Abstract

This work is divided in three main parts. In the first part, we introduce quasi-infinitely divisible (QID) random measures, provide explicit examples and formulate a spectral representation. In the second part, we introduce QID stochastic integrals together with integrability conditions and continuity properties. In the last part, we introduce QID stochastic processes, whose definition is: a process $X$ is QID if and only if there exists two ID processes $Y$ and $Z$ such that $X + Y \overset{d}{=} Z$ with $Y$ independent of $X$. The class of QID processes is strictly larger than the class of ID processes. We present a spectral representation of discrete parameters QID processes and provide Lévy-Khintchine formulations for general QID processes and certain subclasses.
1 Introduction

Infinitely divisible distributions represents one of the main class of probability distributions. Their development goes back to the work of Lévy and De Finetti. Concerning ID processes, one of the most pivotal work in the field is given by the Rajput and Rosinski paper in 1989 [18]. This work provides extremely useful results concerning the spectral representation discrete and centred continuous of ID process.

Recently, a series of work shed new lights on a broader class of distributions called quasi-ID (QID) distributions. In few words, the main difference is that Lévy measure is now a signed measure (thus taking negative values). The recent work of Lindner, Pan and Sato [13] represents one of the most important paper of this series. It shows for example that the set of QID distributions is dense in the set of all probability distributions with respect to weak convergence. Moreover, they proved that a distribution concentrated on the integers is QID if and only if its characteristic function does not have zeroes. This last result is extended in [1] to distributions which can be written as the sum of a distribution concentrated on the integers and a distribution which is absolutely continuous w.r.t. the Lebesgue measure. An interesting result shown in [1] states that it does not exist a distribution which has a Lévy measure with complex values.

QID distributions has shown to have link to the field of prime numbers as well. Indeed, in [15] it is shown that $\xi(\sigma - it)$ has a characteristic function which is QID, but not ID, for $\sigma > 1$, where $\xi$ is the complete Riemann zeta function and $t, \sigma \in \mathbb{R}$. Moreover, QID distributions turn out to be of importance in mathematical physics too, as seen in [7] and [4].

Therefore, it appears natural to see to what extent the results in [18] extend to the QID case. This is part of the content of the present work. Indeed, this work extends the results of the celebrated 1989 paper by Rajput and Rosinski to the QID framework. However, we also investigate general questions like: What is the Lévy-Khintchine representation of general QID processes? What is the Lévy-Khintchine representation when the QID process belongs to $l_2$? What is the connection between QID processes and Lévy processes?

This work is structured as follows. Chapter 2 is about notation and preliminaries, including the introduction of QID random measures. Chapter 3 shows the connections between QID and ID random measures. In Chapter 4 we present general results on QID random measures and show explicit cases in which it arises. In Chapter 5 QID stochastic integrals are defined. We extend to the signed measure framework the measure theoretical results at the base of the Rajput and Rosinski’s 1989 paper ([18]) and provide a Lévy-Khintchine representation and integrability conditions for QID stochastic integrals. In Chapter 6 we show a continuity property of the QID stochastic integral. Finally, in Chapter 7 quasi-Lévy measures on $l_2$ (the Hilbert space of square summable sequences) and QID processes are defined, a Lévy-Khintchine representation for certain QID on $l_2$ is provided and a spectral representation for general discrete parameters QID processes à la Rajput and Rosinski is shown. In addition, a Lévy-Khintchine representation for general (and for a subclass of) QID processes is provided. In the last two sections of 7 further results on QID processes and some examples are provided and we discuss the atomless condition of random measures in the QID framework.

2 Notation and Preliminaries

In this section we introduce the notation and the preliminaries needed in this work. We start with some notation.

By a measure on a measurable space $(X, \mathcal{G})$ we always mean a positive measure on $(X, \mathcal{G})$, namely an $[0, \infty]$-valued $\sigma$-additive set function on $\mathcal{G}$ that assigns the value 0 to the empty set. Given a
non-empty set \( X \), the symbol \( \mathcal{B}(X) \) stands for the Borel \( \sigma \)-algebra of \( X \), unless stated differently. The law and the characteristic function of a random variable \( X \) will be denoted by \( \mathcal{L}(X) \) and by \( \hat{X}(X) \), respectively. We will use term measure for a positive measure and the term signed measure for a signed measure. Finally, due to their frequent use we abbreviate the following words: random variable by r.v., random measure by r.m., characteristic function by c.f. and characteristic triplet by c.t..

Given the importance of signed measures in this work we recall now the definition and some properties.

**Definition 2.1** (signed measure). Given a measurable space \( (X, \Sigma) \), that is, a set \( X \) with a \( \sigma \)-algebra \( \Sigma \) on it, an extended signed measure is a function \( \mu : \Sigma \to \mathbb{R} \cup \{\infty, -\infty\} \) such that \( \mu(\emptyset) = 0 \) and \( \mu \) is sigma additive, that is, it satisfies the equality \( \mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n) \)
where the series on the right must converge absolutely, for any sequence \( A_1, A_2, \ldots \) of disjoint sets in \( \Sigma \).

As a consequence any extended signed measure can take plus or minus infinity as value but not both. Recall also that the total variation of a signed measure \( \mu \) is defined as the measure \( |\mu| : \Sigma \to [0, \infty] \) defined by

\[
|\mu|(A) := \sup \sum_{j=1}^{\infty} |\mu(A_j)|
\]

where the supremum is taken over all the partitions of \( \{A_j\} \) of \( A \in \Sigma \). The total variation \( |\mu| \) is finite if and only if \( \mu \) is finite. By the Hahn decomposition theorem, for a finite signed measure \( \mu \), there exist disjoint Borel sets \( C^+ \) and \( C^- \) with \( C^+ \cup C^- = X \) and \( C^+ \cap C^- = \emptyset \) such that \( \mu(A \cap C^+) \geq 0 \) and \( \mu(A \cap C^-) \leq 0 \) for every \( A \in \Sigma \). In the following, we present the definition of mutually singular measures and the Jordan decomposition theorem.

**Definition 2.2.** Two measures \( \mu \) and \( \nu \) on \( (X, \Sigma) \) are said to be mutually singular if there are disjoint sets \( A, B \in \Sigma \) with \( X = A \cup B \) and \( \mu(A) = 0 \) while \( \nu(B) = 0 \). In this case, we write \( \mu \perp \nu \).

**Theorem 2.3** (Jordan Decomposition Theorem). Let \( \mu \) be a signed measure on \( (X, \Sigma) \). Then there exist two mutually singular positive measures \( \mu^+ \) and \( \mu^- \) such that \( \mu = \mu^+ - \mu^- \). Furthermore, if \( \lambda \) and \( \nu \) are any two positive measures with \( \mu = \lambda - \nu \), then for each \( E \in \Sigma \) we have \( \lambda(E) \geq \mu^+(E) \) and \( \nu(E) \geq \mu^-(E) \). Finally, if \( \lambda \perp \nu \), then \( \lambda = \mu^+ \) and \( \nu = \mu^- \).

Recall that while the Jordan decomposition is unique the Hahn decomposition is only essential unique, indeed \( \mu \)-null sets can be transferred from \( C^+ \) to \( C^- \) and vice versa.

Now, we introduce the concept of quasi-Lévy type measure. Although it is called measure it is not always a measure. This explains the need of the following definitions, which we recall from [13]:

**Definition 2.4.** Let \( \mathcal{B}_r(\mathbb{R}) := \{ B \in \mathcal{B}(\mathbb{R}) | B \cap (r, r) = \emptyset \} \) for \( r > 0 \) and \( \mathcal{B}_0(\mathbb{R}) := \bigcup_{r \geq 0} \mathcal{B}_r(\mathbb{R}) \) be the class of all Borel sets that are bounded away from zero. Let \( \nu : \mathcal{B}_0(\mathbb{R}) \to \mathbb{R} \) be a function such that \( \nu|_{\mathcal{B}_r(\mathbb{R})} \) is a finite signed measure for each \( r > 0 \) and denote the total variation, positive and negative part of \( \nu|_{\mathcal{B}_r(\mathbb{R})} \) by \( |\nu|_{\mathcal{B}_r(\mathbb{R})} \), \( \nu^+_{\mathcal{B}_r(\mathbb{R})} \) and \( \nu^-_{\mathcal{B}_r(\mathbb{R})} \) respectively. Then the total variation \( |\nu| \), the positive part \( \nu^+ \) and the negative part \( \nu^- \) of \( \nu \) are defined to be the unique measures on \( (\mathbb{R}, \mathcal{B}(\mathbb{R})) \) satisfying

\[
|\nu\{0\}| = \nu^+\{0\} = \nu^-\{0\} = 0
\]

and

\[
|\nu|(A) = |\nu|_{\mathcal{B}_r(\mathbb{R})}|, \quad \nu^+(A) = \nu^+_{\mathcal{B}_r(\mathbb{R})}(A), \quad \nu^-(A) = \nu^-_{\mathcal{B}_r(\mathbb{R})}(A),
\]

for \( A \in \mathcal{B}_r(\mathbb{R}) \), for some \( r > 0 \).
As mentioned in [13], $\nu$ is not a a signed measure because it is defined on $\mathcal{B}_0(\mathbb{R})$, which is not a $\sigma$-algebra. In the case it is possible to extend the definition of $\nu$ to $\mathcal{B}(\mathbb{R})$ such that $\nu$ will be a signed measure then we will identify $\nu$ with its extension to $\mathcal{B}(\mathbb{R})$ and speak of $\nu$ as a signed measure.

Throughout this work we define the centering function $\tau$ (on a general Hilbert space) as

$$\tau(x) := \begin{cases} x & \text{if } \|x\| \leq 1, \\ \frac{x}{\|x\|} & \text{if } \|x\| > 1. \end{cases}$$

where $\|\cdot\|$ is the norm of the Hilbert space considered. Notice that this centering function satisfies equation (1) in [13]. The following definition has been introduced for the first time in [13].

**Definition 2.5** (quasi-Lévy type measure, quasi-Lévy measure). A quasi-Lévy type measure is a function $\nu : \mathcal{B}_0(\mathbb{R}) \to \mathbb{R}$ satisfying the condition in Definition 2.4 and such that its total variation $|\nu|$ satisfies

$$\int_{\mathbb{R}} (1 \wedge x^2)|\nu|(dx) < \infty.$$ 

Let $\mu$ be a probability distribution on $\mathbb{R}$. We say that $\mu$ is quasi-infinitely divisible if its characteristic function has a representation

$$\hat{\mu}(\theta) = \exp\left(i\theta \gamma - \frac{\theta^2}{2} a + \int_{\mathbb{R}} e^{i\theta x} - 1 - i\theta \tau(x) \nu(dx)\right)$$

where $a, \gamma \in \mathbb{R}$ and $\nu$ is a quasi-Lévy type measure. The characteristic triplet $(a, \nu, \gamma)$ of $\mu$ is unique (see [21], Exercise 12.2), and $a$ is called the Gaussian variance of $\mu$.

A quasi-Lévy measure $\nu$ is called quasi-Lévy measure, if additionally there exist a quasi-infinitely divisible distribution $\mu$ and some $a, \gamma \in \mathbb{R}$ such that $(a, \nu, \gamma)$ is the characteristic triplet of $\mu$. We call $\nu$ the quasi-Lévy measure of $\mu$.

The above definition extend to the $\mathbb{R}^d$ case (for $d > 1$) as shown in Remark 2.4 in [13].

A quasi-Lévy measure is always a quasi-Lévy type measure, while the converse it is not true as pointed out in Example 2.9 of [13]. Moreover, we say that a function $f$ is integrable with respect to quasi-Lévy type measure $\nu$ if it is integrable with respect to $|\nu|$. Then, we define:

$$\int_B f \, d\nu := \int_B f \, d\nu^+ - \int_B f \, d\nu^-, \quad B \in \mathcal{B}(\mathbb{R}).$$

Give the above discussions, it appears clear why sometimes it is useful to work with the characteristic pair $(\zeta, \gamma)$ instead of $(a, \nu, \gamma)$ where $\zeta$ is a signed measure defined as:

$$\zeta(B) = a\delta_0(B) + \int_B 1 \wedge x^2 \nu(dx), \quad B \in \mathcal{B}(\mathbb{R})$$

Then, the representation of c.f. of $\mu$ becomes:

$$\hat{\mu}(\theta) = \exp\left(i\theta \gamma + \int_{\mathbb{R}} g_\tau(x, \theta) \zeta(dx)\right)$$

where $g_\tau : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ defined by

$$g_\tau(x, \theta) = \begin{cases} (e^{i\theta x} - 1 - i\theta \tau(x))/(1 \wedge x^2), & x \neq 0, \\ -\theta^2, & x = 0. \end{cases}$$

The function $g_\tau(\cdot, \theta)$ is bounded for each fixed $\theta \in \mathbb{R}$ and is continuous at zero (see [13] for further details).

We introduce now the framework needed to work with ID (and QID) processes. Throughout the paper, we denote, by $S$, an arbitrary non-empty set and, by $\mathcal{S}$, a $\sigma$-algebra on $S$ and assume that there are sets $S_1, S_2, \cdots \in \mathcal{S}$ s.t. $\bigcup_{n \in \mathbb{N}} S_n = S.$
Definition 2.6 (Quasi-ID Lévy random measure). Let \( \Lambda = \{ \Lambda(A) : A \in \mathcal{S} \} \) be a real stochastic process defined on some probability space \((\Omega, \mathcal{F}, \mathbb{P})\). We call \( \Lambda \) to be an independently scattered r.m., if, for every sequence \( \{ A_n \} \) of disjoint sets in \( \mathcal{S} \), the r.v. \( \Lambda(A_n) \), \( n = 1, 2, \ldots \), are independent, and, if \( \bigcup_{n=1}^{\infty} A_n \), belong to \( \mathcal{S} \), then we also have \( \Lambda(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \Lambda(A_n) \) a.s., where the series is assumed to converge almost surely. In addition, if \( \Lambda(A) \) is a ID (QID) r.v., for every \( A \in \mathcal{S} \) then we call \( \Lambda \) a ID (QID) r.m..

In this work \( \Lambda = \{ \Lambda(A) : A \in \mathcal{S} \} \) will denote a QID r.m.. Since, for every \( A \in \mathcal{S} \), \( \Lambda(A) \) is a QID r.v., its c.f. can be written in the Lévy-Khintchine form:

\[
\hat{L}(\Lambda(A))(\theta) := \mathbb{E}(e^{i\theta \Lambda(A)}) = \exp \left( i\theta \nu_0(A) - \frac{\theta^2}{2} \nu_1(A) + \int_{\mathbb{R}} e^{i\theta x} - 1 - i\theta x F_A(dx) \right)
\]

(i) the function \( A \rightarrow M(A, B) \) is a signed measure on \( \mathcal{S} \) for every \( B \in \mathcal{G} \),
(ii) the function \( B \rightarrow M(A, B) \) is a signed measure on \( \mathcal{G} \) for every \( A \in \mathcal{S} \).

For a signed bimeasure \( M \), we denote \( M^+ \) and \( M^- \) the Jordan decomposition of \( M(A, B) \) for fixed \( B \in \mathcal{G} \), and \( M_+ \) and \( M_- \) the Jordan decomposition of \( M(A, B) \) for fixed \( A \in \mathcal{S} \).

We conclude this section with the following remark on the condition of a quasi-Lévy measure to be a signed measure.

From the discussion done in this section we have not mentioned when a quasi-Lévy measure is a signed measure, namely when we can extend \( \nu \) from \( \mathcal{B}(\mathbb{R}) \) to the \( \sigma \)-algebra \( \mathcal{B}(\mathbb{R}) \). The idea behind is that \( \nu \) is the difference of two Lévy measures \( \mu_1 \) and \( \mu_2 \) and when these measures are both infinite then we cannot find a signed measure such that \( \nu = \mu_1 - \mu_2 \). This is because we get the form \( \infty - \infty \), which is not defined. However, as soon as one of the two Lévy measures is finite then we \( \nu = \mu_1 - \mu_2 \) defines a signed measure. Finally, we do not make any general assumption about it but we remark that almost all the interesting cases considered in the existing literature, mainly in [13], quasi-Lévy measures are signed measures.

3 The connection between ID and QID random measures

In this section, we investigate the connection between ID and QID random measures. We start with a result, which represents a partial QID analogue of Proposition 2.1 (ii) of [18].

Lemma 3.1. Let \( \nu_0 : \mathcal{S} \mapsto \mathbb{R} \) be a signed measure, \( \nu_1 : \mathcal{S} \mapsto \mathbb{R} \) be a measure, \( G_A \) be a Lévy measure on \( \mathbb{R} \) for every \( A \in \mathcal{S} \) and \( \mathcal{S} \ni A \mapsto G_A(B) \in [0, \infty) \) be a measure for every \( B \in \mathcal{B}(\mathbb{R}) \) such that \( \nu_0(B) \notin \mathbb{B}(\mathbb{R}) \). Let \( M \) defined as \( G \) and additionally assume \( M_A \) to be finite for every \( A \in \mathcal{S} \). Let \( F_A(B) = G_A(B) - M_A(B) \) for every \( B \in \mathcal{B}(\mathbb{R}) \) and \( A \in \mathcal{S} \). Then, \( F_A \) is a quasi-Lévy measure for every \( A \in \mathcal{S} \) and \( \mathcal{S} \ni A \mapsto F_A(B) \in (\infty, -\infty) \) is a signed measure for every \( B \in \mathcal{B}(\mathbb{R}) \) such that \( \nu_0(B) \notin \mathbb{B}(\mathbb{R}) \). Further, if \( (\nu_0(A), \nu_1(A), F_A) \) is the characteristic triplet of a QID random variable \( \forall A \in \mathcal{S} \). Then there exists a unique (in the sense of finite-dimensional distributions) QID random measure \( \Lambda \) such that (4) holds.
Further, the condition that $\Lambda(A) + \Lambda(B) = \Lambda(A \cup B)$ follows by a standard application of Kolmogorov extension theorem. To prove that $\Lambda$ is countable additive let $A_n \searrow \emptyset$ with $A_n \in S$. Then, by definition of (signed) measures, $\nu_0(A_n) \to 0$, $\nu_1(A_n) \to 0$, $G_{A_n}(B) \to 0$ and $M_{A_n}(B) \to 0$ for every $B \in \mathcal{B}(\mathbb{R})$. Notice that $G_A(B) \geq F_{A_n}^+(B)$ and $M_A(B) \geq F_{A_n}^-(B)$ for all $B \in \mathcal{B}(\mathbb{R})$. Then, by applying the arguments of the proof of Proposition 2.1 (ii) of [18] we obtain that

$$\int_{\mathbb{R}} (1 \wedge x^2) |F_{A_n}|(dx) \leq \int_{\mathbb{R}} (1 \wedge x^2) G_{A_n}(dx) + \int_{\mathbb{R}} (1 \wedge x^2) M_{A_n}(dx) \to 0 \text{ as } n \to \infty$$

Now, by the Lévy continuity theorem we obtain that $\Lambda(A_n) \overset{d}{\to} 0 \Rightarrow \Lambda(A_n) \overset{d}{\to} 0$, which implies the countable additivity of $\Lambda$. The uniqueness follows by the one to one relation between $(\nu_0(A), \nu_1(A), F_A)$ and $\Lambda(A)$, for every $A \in S$. \hfill \Box

Now, we present the first link between ID and QID random measures.

**Proposition 3.2.** For every $A \in S$. Let $\Lambda(A)$ be a random variable and let two ID random measures $\Lambda_1$ and $\Lambda_2$ with $\Lambda(A) + \Lambda_2(A) \overset{d}{=} \Lambda_1(A)$ such that $\Lambda_2(A)$ is independent of $\Lambda(A)$ and with finite Lévy measure. Then, there exists a unique QID random measure $\Lambda = \{\Lambda(A) : A \in S\}$. In this case, we say that the r.m. $\Lambda$ is generated by two ID random measures.

**Remark 3.3.** The above result can be restated using instead of the condition that $\Lambda(A)$ is a random variable and there exist two ID random measures $\Lambda(A) + \Lambda_2(A) \overset{d}{=} \Lambda_1(A)$, such that $\Lambda_2(A)$ is independent of $\Lambda(A)$, $\forall A \in S$, the equivalent alternative conditions:

(i) $\forall A \in S$ and $\theta \in \mathbb{R}$, $\mathcal{L}(\Lambda(A))$ is a distribution and

$$\hat{\mathcal{L}}(\Lambda(A))(\theta) = \frac{\hat{\mathcal{L}}(\Lambda_1(A))(\theta)}{\mathcal{L}(\Lambda_2(A))(\theta)},$$

(ii) $\forall A \in S$ $\mathcal{L}(\Lambda(A))$ is a distribution and the c.t. of $\Lambda(A)$ can be rewritten as the difference of the c.t. of $\Lambda_1(A)$ and $\Lambda_2(A)$.

**Proof.** Let $\nu_0^{(j)}$, $\nu_1^{(j)}$ and $F^{(j)}$ the measures corresponding to $\Lambda_j$, for $j = 1, 2$ (see Proposition 2.1 point (i) and (ii) of [18]). Let $\nu_0(A) := \nu_0^{(1)}(A) - \nu_0^{(2)}(A)$, $\nu_1(A) := \nu_1^{(1)}(A) - \nu_1^{(2)}(A)$ for all $A \in S$. Moreover, let $F_A$ be the quasi-Lévy measure defined as the difference between the Lévy measures $F_A^{(1)}$ and $F_A^{(2)}$, namely $F_A(B) := F_A^{(1)}(B) - F_A^{(2)}(B)$ for any $B \in \mathcal{B}(\mathbb{R})$, for all $A \in S$. Then $\nu_0$, $\nu_1$ and $F$ satisfies the conditions of Lemma 3.1.

Further, the condition that $\Lambda(A)$ is random variable together with $\Lambda(A) + \Lambda_2(A) \overset{d}{=} \Lambda_1(A)$ imply that $\mathcal{L}(\Lambda(A))$ is QID with c.t. $(\nu_0(A), \nu_1(A), F_A)$, hence, by Lemma 2.7 in [18] we have $\nu_1(A) \geq 0$, $\forall A \in S$.

Then, by Lemma 3.1 we obtain the stated result (including the statement that $\Lambda$ is uniquely determined). \hfill \Box

In the following we present two results, where the second represent a partial only if result of Proposition 3.2.
Lemma 3.4. (i) Let $\Lambda$ be a QID random measure with $F^+_A(B) : A \mapsto F^+_A(B)$ a measure for every $B \in \mathcal{B}(\mathbb{R})$. Then $\nu_0 : \mathcal{S} \mapsto \mathbb{R}$ is a signed measure, $\nu_1 : \mathcal{S} \mapsto [0, \infty)$ is a measure, $F_A$ is a quasi-Lévy measure on $\mathbb{R}$, for every $A \in \mathcal{S}$, and $\mathcal{S} \ni A \mapsto F_A(B) \in (-\infty, \infty)$ is a signed measure, for every $B \in \mathcal{B}(\mathbb{R})$ whenever $0 \notin \overline{B}$. Moreover, we have that $\mathcal{S} \ni A \mapsto F^{-}_A(B) \in [0, \infty)$ is a measure, for every $B \in \mathcal{B}(\mathbb{R})$ whenever $0 \notin \overline{B}$.

Proof. (i) Let $\{A_k\}_{k=1}^n$ be pairwise disjoint sets in $\mathcal{S}$. By uniqueness of the Lévy-Khintchine representation of a quasi-ID distribution, it follows, using $\hat{\mathcal{L}}(\Lambda(\cup_{k=1}^n A_k)) = \prod_{k=1}^n \hat{\mathcal{L}}(\Lambda(A_k))$, that all three set functions $\nu_0$, $\nu_1$ and $F(B)$ (for every fixed $B \in \mathcal{B}(\mathbb{R})$) are finitely additive. Let now $A_n \in \mathcal{S}$, $A_n \searrow \emptyset$. Then $\nu_0(A_n) \to 0$, $\nu_1(A_n) \to 0$. Concerning $F_{A_n}(B)$, we have that $F^{-}_{A_n}(B) \to 0$ by assumption and

$$
\int_{\mathbb{R}} 1 \wedge x^2 F_{A_n}(dx) \to 0 \quad \text{as } n \to \infty.
$$

Therefore, we obtain that

$$
\int_{\mathbb{R}} 1 \wedge x^2 F^{-}_{A_n}(dx) \to 0 \quad \text{as } n \to \infty.
$$

and applying the arguments of the proof of Proposition 2.1 point (i) in [18] we obtain that $F^{-}_{A_n}(B) \to 0$ for every $B \in \mathcal{B}(\mathbb{R})$, whenever $0 \notin \overline{B}$. Therefore, we have $F_{A_n}(B) \to 0$, and thus $F_{A_n}(B)$ is a signed measure, for every $B \in \mathcal{B}(\mathbb{R})$ whenever $0 \notin \overline{B}$. To obtain the second statement it is sufficient to notice that for every $B \in \mathcal{B}(\mathbb{R})$ whenever $0 \notin \overline{B}$ we have $F_A = F^+_A(B) - F_A(B)$. \hfill \Box

Remark 3.5. Notice that we can use the assumption $F^{-}(\mathbb{R}) := A \mapsto F^{-}_A(\mathbb{R})$ is a measure, instead. Then, the arguments would be the same.

Proposition 3.6. Let $\Lambda$ be QID random measure with $F^+_A(B)$ being a measure for every $B \in \mathcal{B}(\mathbb{R})$. Then, there exist two ID random measures $\Lambda_1$ and $\Lambda_2$ with $\Lambda(A) + \Lambda_2(A) \overset{d}{=} \Lambda_1(A)$ such that $\Lambda_2(A)$ is independent of $\Lambda(A)$ and has zero Gaussian part, $\forall A \in \mathcal{S}$.

Proof. Define $F^{(1)}_A := F^+_A$, $F^{(2)}_A := F^-_A$, $\nu_0^{(1)} := \nu^{0}_A$, $\nu_1^{(1)} := \nu^1_A$, and $\nu_0^{(2)} := 0$. By Lemma 3.4 we obtain that $F^{(j)}_A(B) \in [0, \infty)$ for any $B \in \mathcal{B}(\mathbb{R})$ whenever $0 \notin \overline{B}$, $\nu_0^{(j)} : \mathcal{S} \to [0, \infty)$ and $\nu_1^{(j)} : \mathcal{S} \to [0, \infty)$ are all measures, for $j = 1, 2$. In addition, since $F_A$ is a quasi-Lévy measure on $\mathbb{R}$ then $F^{(j)}_A$ are Lévy measures for $j = 1, 2$.

Then, by Proposition 2.1 point (ii) of [18] we know that there exist two unique ID random measures $\Lambda_1$ and $\Lambda_2$ such that, for every $A \in \mathcal{S}$, they have c.t. given by $(\nu_0^{(1)}(A), \nu_1^{(1)}(A), F^{(1)}_A)$ and $(\nu_0^{(2)}(A), \nu_1^{(2)}(A), F^{(2)}_A)$, respectively.

Finally, it is possible to see that the following relation hold: for every $A \in \mathcal{S}$ and $\theta \in \mathbb{R}$, $\hat{\mathcal{L}}(\Lambda(A))(\theta) = \frac{\hat{\mathcal{L}}(\Lambda_1(A))(\theta)}{\hat{\mathcal{L}}(\Lambda_2(A))(\theta)}$. \hfill \Box

Remark 3.7. The reason why $\Lambda_1$ and $\Lambda_2$ are not determined uniquely by $\Lambda$ is because given $\nu_0$ we cannot exclude the possibility to find two measures $\tilde{\nu}_0^{(1)}$ and $\tilde{\nu}_0^{(2)}$ such that $\nu_0(A) = \tilde{\nu}_0^{(1)}(A) - \tilde{\nu}_0^{(2)}(A)$, where $\tilde{\nu}_0^{(1)}(A) \geq F^+_0(A)$ and $\tilde{\nu}_0^{(2)}(A) \geq F^-_0(A)$. This lead to two measures $\Lambda_1$ and $\Lambda_2$ such that they satisfies the same conditions as $\Lambda_1$ and $\Lambda_2$. Moreover, the same reasoning may also applies to $F_A$.

We would like to have the previous result without the condition on $F^+_A(B)$ being a measure, but the following example shows that this is not always possible even if we restrict to random measure $\Lambda$ with $\mathcal{L}(\Lambda(A))$ concentrated on $\mathbb{Z}$, $\forall A \in \mathcal{S}$.  

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Example 3.8. Consider $F_A = \sum_{l \in \mathbb{Z}, l \neq 0} b_l^A \delta_l$ then $|F_A| = \sum_{l \in \mathbb{Z}, l \neq 0} |b_l^A| \delta_l$. Such a quasi-Lévy measure arises when we consider distribution supported on the integers (see Theorem 8.1 in [13]). While $F_A^+$ and $F_A^-$ are Lévy measure for every fixed $A \in S$ they are not always measures for fixed $B \in \mathcal{B}(\mathbb{R})$. Indeed, consider $A = A_1 \cap A_2$ with $A_1 \cap A_2 = \emptyset$ we know by the fact that $F_A$ is a measure that $b_{k_1} A_1 \cup A_2 = b_{k_1} A_1 + b_{k_2} A_2$. However, it might happen that $b_{k_1}^A$ and $b_{k_2}^A$ have different sign. This implies that $F_A^+ A_1 \cup A_2((k)) \neq F_A^+ A_1 (B) + F_A^+ A_2 (B)$ and $F_A^+ A_1 \cup A_2((k)) \neq F_A^+ A_1 ((k)) + F_A^+ A_2 ((k))$.

On the other hand, this example shows that if we consider $F_A = \sum_{l \in \mathbb{Z}, l \neq 0} b_l^A \delta_l$ such that $b_l^A$ do not change sign for all $A \in S$ then this represents an example where the theory presented so far applies.

We conclude this section with the following lemma, which is related to the results and discussion in this section and throughout this work.

Lemma 3.9. Let $M(A, B)$ be a signed bimeasure and let $M^+(A, B)$ be a measure. Then, $M^+(A, B)$ and $M^-(A, B)$ are measures for any $B \in \mathcal{B}(\mathbb{R})$.

Proof. We first prove the finite additivity. Let $k \in \mathbb{N}$ and consider $A = \bigcup_{i=1}^k A_i$, where $A_i$ are pairwise disjoint sets. Recall that we have by Jordan decomposition $M^+(A, B) = M(A, E_A^+)$ where $E_A^+$ is the subset of $\mathbb{R}$ such that $M(A, B \cap E_A^+) \geq 0$ for any $B \in \mathcal{B}(\mathbb{R})$. Then, since $M^+(A, \mathbb{R})$ is a measure we have $M^+(A, \mathbb{R}) = \sum_{i=1}^k M^+(A_i, \mathbb{R})$; in addition, since $M(A, E_A^+)$ is a signed measure for fixed $E_A^+$ then $M(A, E_A^+) = \sum_{i=1}^k M(A_i, E_A^+)$ and $M(A_i, \mathbb{R}) = \sum_{i=1}^k M(A_i, E_A^+)$ which implies that $M(A_i, E_A^+) = M^+(A_i, \mathbb{R}) = M(A_i, E_A^+)$. This shows that, for fixed $A_i$, $M(A_i, \mathbb{R})$ is not strictly negative on any of the subset of $E_A^+$ and, in particular, it is strictly positive only on a subset of $E_A^+$. Now, consider a set $B \in \mathcal{B}(\mathbb{R})$, then $M^+(A, B) = M^+(A, B \cap E_A^+) = \sum_{i=1}^k M(A_i, B \cap E_A^+) = \sum_{i=1}^k M(A_i, B \cap E_A^+) = \sum_{i=1}^k M(A_i, B)$. Since $M^-(A, \mathbb{R}) = M^+(A, \mathbb{R}) - M(A, \mathbb{R})$, then $M^-(A, \mathbb{R})$ is a measure and applying the same argument we obtain the finite additivity of $M^-(A, B)$ for any $B \in \mathcal{B}(\mathbb{R})$.

Let $A_n \rightarrow 0$. Then, by assumption $M^+(A_n, \mathbb{R}) \rightarrow 0$ and, by the previous sentence, $M^-(A_n, \mathbb{R}) \rightarrow 0$. Now since for each $A_n$ we have that $M^+(A_n, \mathbb{R}) \geq M^+(A_n, B)$ then $M^+(A_n, B) \rightarrow 0$ for any $B \in \mathcal{B}(\mathbb{R})$; and the same applies to $M^+(A_n, B)$. \hfill \qed

4 QID random measure

In the previous section we have seen that under certain conditions two ID random measure generates a QID random measure and vice versa. In this section we are going to explore other cases where necessary and sufficient conditions for existence and uniqueness of QID random measure arise.

4.1 Extension of the result of Chapter 3

It might be possible to generalise the Lemmas presented in Chapter 3 as we do in the following proposition.

Proposition 4.1. Let $\nu_0 : S \rightarrow \mathbb{R}$ be a signed measure, $\nu_1 : S \rightarrow \mathbb{R}$ be a measure, $F_A$ be a quasi-Lévy measure on $\mathbb{R}$ for every $A \in S$, $S \ni A \mapsto F_A(B) \in (-\infty, \infty)$ be a signed measure for every $B \in \mathcal{B}(\mathbb{R})$ such that $0 \notin B$, and such that $(\nu_0(A), \nu_1(A), F_A)$ is the characteristic triplet of a QID random variable $\forall A \in S$. For any $A_n \in S$ s.t. $A_n \rightarrow A$, assume that $\int_{\mathbb{R}} 1 \wedge x^2 |F_{A_n}|(dx)$ is uniformly bounded and that $\lim_{\epsilon \rightarrow 0} \lim_{n \rightarrow 0} \int_{|x| < \epsilon} e^{i\theta x} - 1 - i\theta x F_{A_n}(dx) = 0$ for all $\theta \in \mathbb{R}$. Then there
exists a unique (in the sense of finite-dimensional distributions) QID random measure Λ such that \( (\text{1}) \) holds.

**Proof.** The existence of a finitely additive independently scattered random measure Λ = \{Λ(A) : A ∈ S\} follows by a standard application of Kolmogorov extension theorem. To prove that Λ is countable additive let \( A_n \setminus \emptyset \) with \( A_n \in S \). Then, by definition of (signed) measures, \( \nu_0(A_n) \to 0, \nu_1(A_n) \to 0 \) and \( F_{A_n}(B) \to 0 \) for every \( B ∈ B(\mathbb{R}) \) such that \( 0 \notin B \). Now, observe that

\[
\int_{\mathbb{R}} e^{iθx} - 1 - iθτ(x)F_{A_n}(dx) = \int_{|x| ≥ ε} e^{iθx} - 1 - iθτ(x)F_{A_n}(dx) + \int_{|x| < ε} e^{iθx} - 1 - iθτ(x)F_{A_n}(dx)
\]

For the first integral observe that we can replace \( F_{A_n} \) by the signed measure \( F_{A_n|B_r} \). Since \( F_{A_n|B_r(\mathbb{R})} \) converges setwise \( (F_{A_n|B_r(\mathbb{R})}(B) \to 0 \) for every \( B ∈ B_r(\mathbb{R}) \)) and since the sequence is uniformly bounded by assumption then we have weak convergence for our sequence (see page 175 and 244 of \([9]\)). In particular, we obtain that the first integral converges to zero for every \( θ ∈ \mathbb{R} \):

\[
\int_{|x| ≥ ε} e^{iθx} - 1 - iθτ(x)F_{A_n}(dx) \to 0, \text{ as } n → \infty
\]

For the second integral, the needed convergence follows by assumption by taking \( ε ≤ ε_0 \). Then, for every \( θ ∈ \mathbb{R} \) we have that:

\[
\int_{\mathbb{R}} e^{iθx} - 1 - iθτ(x)F_{A_n}(dx) \to 0, \text{ as } n → \infty
\]

Now, by the Lévy continuity theorem we obtain that \( Λ(A_n) \xrightarrow{d} 0 \Rightarrow Λ(A_n) \xrightarrow{P} 0 \), which implies the countable additivity of Λ.

We would like to generalise also the result for the other direction, but not general results are known. We will see some special cases in the next sections.

### 4.2 The distribution concentrated on the integers

In this section we are going to investigate the case where \( \text{supp}(\mathcal{L}(Λ(A))) \subset \mathbb{Z} \). In other words, we consider \( Λ(\mathbb{A}) = \sum_{m∈\mathbb{Z}} a_m δ_m \). We start by showing the equivalent of Proposition 2.1 of \([18]\).

**Proposition 4.2.** (i) Let Λ be a QID random measure with \( \text{supp}(\mathcal{L}(Λ(A))) \subset \mathbb{Z}, \forall A ∈ S \). Then for each \( A ∈ S \), \( ν_1(A) = 0 \), the drift of \( \hat{Λ}(Λ(A)) ∈ \mathbb{Z} \) and \( F_A \) is a quasi-Lévy measure concentrated on \( \mathbb{Z} \setminus \{0\} \) and finite. Moreover, \( ν_0 : S \mapsto \mathbb{R} \) is a signed measure, \( ν_1 : S \mapsto \mathbb{R} \) is the zero measure, \( F_A \) is a quasi-Lévy measure on \( \mathbb{R} \), for every \( A ∈ S \), and \( S ∋ A ⇒ F_A(B) ∈ (−∞,∞) \) is a signed measure, for every \( B ∈ B(\mathbb{R}) \).

(ii) Let \( ν_0, ν_1 \) and \( F \) be as in (i) and such that \( (\text{1}) \) is the c.f. of a distribution and has no zeros, for each \( A ∈ S \). Then there exists a unique (in the sense of finite-dimensional distributions) quasi-ID random measure Λ such that \( (\text{1}) \) holds.

**Proof.** (i) The first assertion is the content of Theorem 8.1 of \([13]\). For the finite additivity of \( ν_0, ν_1 \) and \( F(B) \) we follows the first part of the proof of Proposition 2.1 of \([13]\). Let \( \{A_k\}_{k=1}^p \) be pairwise disjoint sets in \( S \). By the uniqueness of Lévy-Khintchine representation of the c.f. of a QID distribution and by \( \hat{Λ}(Λ(\bigcup_{k=1}^p A_k)) = \prod_{k=1}^p \hat{Λ}(Λ(A_k)) \), we obtain the finitely additivity property. Now, let \( A_n ∈ S \), \( A_n \setminus \emptyset \). Since Λ(A_n) → 0 then \( \hat{Λ}(Λ(A_n)) → δ_0 \) and by Theorem 8.4 of \([13]\) we obtain that \( ν_0(A_n) → 0, ν_1(A_n) → 0 \) and \( \sum_{t∈\mathbb{Z}} |F_{A_n}(\{t\})| → 0 \). Now, observe that
we can write \( F_{A_n} = \sum_{l \in \mathbb{Z}, l \neq 0} b_{n,l} \delta_l \) for some \( b_{n,l} \in \mathbb{R} \), hence \( |F_{A_n}| = \sum_{l \in \mathbb{Z}, l \neq 0} |b_{n,l}||\delta_l| \), and since \( F_{A_n}(\{l\}) = b_{n,l} \) we obtain that \( |F_{A_n}|(\mathbb{R}) \to 0 \). Therefore, we deduce that \( F_{A_n}(B) \to 0 \) for every \( B \in \mathcal{B}(\mathbb{R}) \).

(ii) First, from Theorem 8.5 of [13] we obtain that \( (\nu_0(A), \nu_1(A), F_A) \) is the characteristic triplet of a QID random variable \( \forall A \in \mathcal{S} \). The existence of a finitely additive independently scattered random measure \( \Lambda = \{\lambda(A) : A \in \mathcal{S}\} \) follows by a standard application of Kolmogorov extension theorem. To prove that \( \Lambda \) is countable additive let \( A_n \searrow \emptyset \) with \( A_n \in \mathcal{S} \). Then, by definition of (signed) measures, \( \nu_0(A_n) \to 0, \nu_1(A_n) \to 0 \) and \( F_{A_n}(B) \to 0 \) for every \( B \in \mathcal{B}(\mathbb{R}) \) such that \( 0 \notin B \). From the last limit and since \( F_{A_n} \) is concentrated on the integers, we obtain that \( |F_{A_n}|(\mathbb{R}) \to 0 \). Therefore, we have that \( \Lambda(A_n) \to 0 \), hence \( \lambda \) is countably additive.

\[ \square \]

**Remark 4.3.** Thanks to Corollary 8.2 in [13], it is possible to extend the above result to the case of distributions concentrated on a lattice of the form \( r + h\mathbb{Z} \) where \( r \in \mathbb{R} \) and \( h > 0 \).

### 4.3 The distribution with an atom of mass greater than \( 1/2 \)

In this section we are going to explore the case where a random measure is such that \( \mathcal{L}(\Lambda(A))(\{k_A\}) > 1/2 \) for some \( k_A \in \mathbb{R} \). We have the following result for \( k_A = 0 \).

**Proposition 4.4.** (i) Let \( \Lambda \) be an independent scattered random measure with \( p_{A} := \mathcal{L}(\Lambda(A))(\{0\}) > 1/2, \forall A \in \mathcal{S} \). Then \( \Lambda \) is a QID random measure and, for each \( A \in \mathcal{S} \), \( \nu_1(A) = 0 \), the drift of \( \mathcal{L}(\Lambda(A)) \) equal zero and \( F_A \) is a finite quasi-Lévy measure given by

\[
F_A = \left( \sum_{m=1}^{\infty} \frac{1}{m} (-1)^{m+1} \left( \frac{1 - p_A}{p_A} \right)^m (\delta_0 * \sigma_A)^m \right)_{\mathbb{R}\setminus\{0\}}
\]

where \( \sigma_A = (1 - p_A)^{-1}(\mathcal{L}(\Lambda(A)) - p_A \delta_0) \).

Moreover, \( \nu_0 : \mathcal{S} \to \mathbb{R} \) and \( \nu_1 : \mathcal{S} \to \mathbb{R} \) are the zero measure, \( F_A \) is a quasi-Lévy measure on \( \mathbb{R} \), for every \( A \in \mathcal{S} \), and \( \mathcal{S} \ni A \mapsto F_A(B) \in (-\infty, \infty) \) is a signed measure, for every \( B \in \mathcal{B}(\mathbb{R}) \).

(ii) Let \( \nu_0 \) and \( \nu_1 \) be the zero measures, let \( \mu_A \) be a distribution with c.t. \( (0,0,F_A) \) where \( F_A \) is a quasi-Lévy measure on \( \mathbb{R} \), for every \( A \in \mathcal{S} \), and \( \mathcal{S} \ni A \mapsto F_A(B) \in (-\infty, \infty) \) is a signed measure, for every \( B \in \mathcal{B}(\mathbb{R}) \). In particular, let

\[
F_A = \left( \sum_{m=1}^{\infty} \frac{1}{m} (-1)^{m+1} \left( \frac{1 - p_A}{p_A} \right)^m (\delta_0 * \sigma_A)^m \right)_{\mathbb{R}\setminus\{0\}}
\]

where \( p_A = \mu_A(\{0\}) > 1/2 \) and \( \sigma_A = (1 - p_A)^{-1}(\mu_A - p_A \delta_0) \), for each \( A \in \mathcal{S} \). Then there exists a unique (in the sense of finite-dimensional distributions) quasi-ID random measure \( \Lambda \) such that \( \mathcal{L}(\Lambda(A)) = \mu_A \), for every \( A \in \mathcal{S} \).

**Proof.** (i) The first statement is the content of Theorem 4.3.7 in [5] (see also Theorem 3.1 of [13]).

Now, the finite additivity follows from the same arguments used before (see the proof of Proposition 4.2). For countable additivity, let \( A_n \in \mathcal{S}, A_n \searrow \emptyset \). Since \( \Lambda(A_n) \to 0 \) then \( \mathcal{L}(\Lambda(A_n)) \to \delta_0 \), which implies that \( p_{A_n} \to 1 \) and from [13] and using the fact that \( F_{A_n} \) is finite we get \( F_{A_n}(B) \to 0 \) for every \( B \in \mathcal{B}(\mathbb{R}) \) such that \( 0 \notin B \).

(ii) Notice that existence and finite additivity of \( \Lambda \) are straightforward. Concerning countable additivity, notice that if \( F_{A_n}(B) \to 0 \) for every \( B \in \mathcal{B}(\mathbb{R}) \) such that \( 0 \notin B \) then \( p_{A_n} \to 1 \), which implies that \( \mathcal{L}(\Lambda(A_n)) \to \delta_0 \). \[ \square \]
4.4 The strictly negative Lévy measure

Under certain conditions, it is possible to have QID distributions with strictly negative quasi-Lévy measure. An example of such conditions are presented in Lemma 2.8 in [13]. Thus, we have the following corollary.

**Corollary 4.5.** All the results presented in Section II of [13] apply mutatis mutandis to the case of QID random measures with quasi-Lévy measure which take only non-positive values.

**Proof.** Consider $F_A(\cdot)$ to be our non-positive quasi-Lévy measure, where $A \in S$. Then all the results in Section II of [13] hold for the total variation $|F_A(\cdot)|$. But since $F_A = -|F_A|$ we obtain the stated result. □

5 QID Stochastic integrals

5.1 The existing two ID random measure case

In this and in the following sections of this chapter we investigate different results concerning QID stochastic integrals, including the necessary and sufficient conditions for their existence. In this section we focus on the case seen in Chapter 3 where the r.m. $\Lambda$ is generated by two ID r.m. (see Proposition 3.2). In order to simplify the presentation of the results, we make the assumption that the Lévy measures of two random measures are mutually singular. However, we point out that this assumption can be eliminated and all the main results presented in this section would still hold.

The first result regards the construction of the control measure of $\Lambda$.

**Proposition 5.1.** Let $\nu_0, \nu_1$ and $F$ be as in the Lemma 3.4 and define

$$\lambda(A) = |\nu_0|(A) + \nu_1(A) + \int_\mathbb{R} (1 \wedge x^2)|F_A|(dx), \quad A \in S. \quad (4)$$

Then $\lambda : S \mapsto [0, \infty)$ is a measure such that $\lambda(A_n) \to 0$ implies $\Lambda(A_n) \to 0$ in probability for every $\{A_n\} \subset S$.

**Proof.** Notice, first, that $\lambda$ is a measure. This follows from the fact that $|\nu_0|, \nu_1$ are measures and $|F_A|(B)$ which is a measure for any $B \in \mathcal{B}(\mathbb{R})$ whenever $0 \notin \overline{B}$ can be extended uniquely to $\mathbb{R}$. Moreover, using Theorem 4.3 point (a) it is straightforward to see that if $\lambda(A_n) \to 0$ then $\Lambda(A_n) \to 0$ in probability. □

**Definition 5.2.** Since $\lambda(S_n) < \infty, n = 1, 2, \ldots$ we extend $\lambda$ to a $\sigma$-finite measure on $(S, \mathcal{S})$; we call $\lambda$ the control measure of $\Lambda$.

The above definition extends the concept of control measure introduced in Definition 2.2 of [13] to quasi-ID random measure, and the two definitions are the same for ID random measures. The following lemma extend Lemma 2.3 of [13].

**Lemma 5.3.** Let $F$ be as in Lemma 3.4. Then there exists a unique $\sigma$-finite measure $F$ on $S \otimes \mathcal{B}(\mathbb{R})$ such that

$$F(A \times B) = F_A(B), \quad \text{for all } A \in S, B \in \mathcal{B}(\mathbb{R}).$$

Moreover, there exists a function $\rho : S \times \mathcal{B}(\mathbb{R}) \mapsto [-\infty, \infty]$ such that

(i) $\rho(s, \cdot)$ is a quasi-Lévy type measure on $\mathcal{B}(\mathbb{R})$, for every $s \in S$;

(ii) $\rho(\cdot, B)$ is a Borel measurable function, for every $B \in \mathcal{B}(\mathbb{R})$;

(iii) $\int_{S \times \mathbb{R}} h(s, x)F(ds, dx) = \int_{S} \int_{\mathbb{R}} h(s, x)\rho(s, dx)\lambda(ds)$, for every $S \otimes \mathcal{B}(\mathbb{R})$-measurable function $h : S \times \mathbb{R} \mapsto [0, \infty]$. This equality can be extended to real and complex-valued functions $h$. 

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Proof. Let
\[ Q_0^+(A, B) := \int_B (1 \wedge x^2) F_A^+(dx) \quad \text{and} \quad Q_0^-(A, B) := \int_B (1 \wedge x^2) F_A^-(dx), \quad A \in S, B \in B(\mathbb{R}). \]

Recall that \( \Lambda \) is generated by two ID random measure with respective Lévy measure \( F_A^+ \) and \( F_A^- \) for every \( A \in S \) then \( Q_0^+(\cdot, B) \) and \( Q_0^-(\cdot, B) \) are two measures.

Now, it is possible to observe that the assumptions of Proposition 2.4 of [15] are satisfied with \( (T, A) = (S, S) \) and \( (X, B) = (\mathbb{R}, B(\mathbb{R})) \). Therefore, there exists measures \( Q^+ \) and \( Q^- \) on the product \( \sigma \)-algebra \( S \otimes B(\mathbb{R}) \) such that
\[ Q^+(A \times B) = Q_0^+(A, B) = \int_A q^+(s, B) \lambda_0^+(ds) \quad \text{and} \quad Q^-(A \times B) = Q_0^-(A, B) = \int_A q^-(s, B) \lambda_0^-(ds) \]
where \( \lambda_0^+(A) = Q_0^+(A, \mathbb{R}) \), \( \lambda_0^-(A) = Q_0^-(A, \mathbb{R}) \), and both \( q^+ \) and \( q^- \) satisfies (d) and (e) of Proposition 2.4. Observe that \( \lambda_0^+ \ll \lambda \) and \( \lambda_0^- \ll \lambda \) because we have \( \lambda_0^+(A) \leq \lambda(A) \) and \( \lambda_0^-(A) \leq \lambda(A) \), for every \( A \in S \). Consider now
\[ \rho^+ (s, dx) := \frac{d \lambda_0^+}{d \lambda}(s) (1 \wedge x^2)^{-1} q^+(s, dx), \quad \rho^- (s, dx) := \frac{d \lambda_0^-}{d \lambda}(s) (1 \wedge x^2)^{-1} q^-(s, dx) \]
and \( \rho(s, dx) := \rho^+(s, dx) - \rho^-(s, dx) \).

Then (ii) is satisfied. Now notice that \( \rho^+(s, \cdot) \) and \( \rho^-(s, \cdot) \) forms the Jordan decomposition of \( \rho(s, \cdot) \) because from the definition of \( q^+(s, \cdot) \) and \( q^-(s, \cdot) \), and the mutual singularity of \( Q_0^+(A, \cdot) \) and \( Q_0^-(A, \cdot) \), we have that \( \rho^+(s, \cdot) \) and \( \rho^-(s, \cdot) \) are mutually singular and their difference is equal to \( \rho(s, \cdot) \).

Further, we have
\[ \int_{\mathbb{R}} (1 \wedge x^2) |\rho|(s, dx) = \frac{d \lambda_0^+}{d \lambda}(s) \int_{\mathbb{R}} q^+(s, dx) + \frac{d \lambda_0^-}{d \lambda}(s) \int_{\mathbb{R}} q^-(s, dx) = \frac{d \lambda_0^+}{d \lambda}(s) + \frac{d \lambda_0^-}{d \lambda}(s) \leq 2 \]
where the last inequality comes from the fact that we can always assume that \( \frac{d \lambda_0^+}{d \lambda}(s) \leq 1 \) for all \( s \) (and the same for \( \lambda_0^+ \)). This proves (i).

Let
\[ F(C) = \int_S \int_{\mathbb{R}} 1_C(s, x) \rho(s, dx) \lambda(ds), \quad (5) \]
where \( C \in S \otimes B(\mathbb{R}) \), then \( F \) is a well defined signed measure that satisfies, for every \( A \in S \) and \( B \in B(\mathbb{R}) \),
\[ F(A \times B) = \int_A \int_B \rho(s, dx) \lambda(ds) \]
\[ = \int_B \int_A (1 \wedge x^2)^{-1} q^+(s, dx) \lambda_0^+(ds) - \int_A \int_B (1 \wedge x^2)^{-1} q^-(s, dx) \lambda_0^-(ds) \]
\[ = \int_{A \times B} (1 \wedge x^2)^{-1} Q^+(ds, dx) - \int_{A \times B} (1 \wedge x^2)^{-1} Q^-(ds, dx) \]
\[ = \int_B (1 \wedge x^2)^{-1} Q_0^+(A, dx) - \int_B (1 \wedge x^2)^{-1} Q_0^-(A, dx) \]
\[ = F_A^+(B) - F_A^-(B) = F_A(B). \]

Therefore, (iii) follows from [15] by standard arguments for functions integrated with respect to signed measures. The uniqueness comes from the uniqueness of the Jordan decomposition. \( \square \)
The following result is a consequence of the main assumption made in this section, namely $F^+(\mathbb{R})$ being a measure.

**Corollary 5.4.** Under the same assumption, we have that $F^+(C) = \int_S \int_{\mathbb{R}} 1_C(s, x) \rho^+(s, dx) \lambda(ds)$. In addition, if $C = A \times B$ then $F^+(C) = F^+_A(B)$. The same holds for $F^-(C)$.

**Proof.** The existence, the uniqueness and the form of $F^+(C)$ follows from the same arguments adopted in the proof of Lemma 5.3. The fact that $F^+(C)$ is the Jordan decomposition of $F(C)$ follows by noticing that for $C = A \times B$ we have $F^+(A \times B) \geq F^+_A(B)$ because of their respective definitions (i.e. the former takes the supremum over the countable decompositions of $A \times B$ while the latter only of $B$), however we have also $F^+(A \times B) \leq F^+_A(B)$ because $F^+(C)$ is the positive part of the Jordan decomposition and $F^+_A(B) - F^-_A(B) = F_A(B)$. Another way to prove this result is to see that $F^+_A(B) - F^-_A(B) = F_A(B)$ and noticing that they both are singular measures, namely $F^+_A(B) \perp F^-_A(B)$. Now by uniqueness of extension the arguments hold for general $C$. \qed

We present now one of the key results of this section.

**Proposition 5.5.** The characteristic function (11) of $\Lambda(A)$ can be written in the form:

$$
\mathbb{E}(e^{i\theta \Lambda(A)}) = \exp \left( \int_A K(\theta, s) \lambda(ds) \right), \quad \theta \in \mathbb{R}, A \in S,
$$

where

$$
K(\theta, s) = i\theta a(s) - \frac{\theta^2}{2} \sigma^2(s) + \int_{\mathbb{R}} e^{i\theta x} - 1 - i\theta \tau(x) \rho(s, dx),
$$

$a(s) = \frac{d\alpha}{dx}(s)$, $\sigma^2(s) = \frac{d\mu}{dx}(s)$ and $\rho$ is given by Lemma 5.3. \(\exp(K(\theta, s))\) is the characteristic function of a QID random variable if it exists. Moreover, we have

$$
|a(s)| + \sigma^2(s) + \int_{\mathbb{R}} (1 - x^2)|\rho|(s, dx) = 1, \quad \lambda\text{-a.e.}
$$

**Proof.** The first statement follows from the Lévy-Khintchine formulation (11) and Lemma 5.3. The second statement follows from the fact that for every $A \in S$, we have

$$
\int_A \left(|a(s)| + \sigma^2(s) + \int_{\mathbb{R}} (1 - x^2)|\rho|(s, dx) \right) \lambda(ds)
= |\nu_0|(A) + \nu_1(A) + \int_{\mathbb{R}} (1 - x^2)|F_A|(dx) = \lambda(A) = \int_A d\lambda(ds).
$$

Notice that the equality $\int_A \int_B |\rho|(s, dx) \lambda(ds) = F^+_A(B) + F^-_A(B)$ comes from Corollary 5.4, the same computations done at the end of the proof of Lemma 5.3. \qed

The following definition of the stochastic integral is the same as the one presented in [18] but in the framework of QID random measure. We report it here for the sake of completeness.

**Definition 5.6.** Let $f = \sum_{j=1}^n x_j I_{A_j}$ be a real simple function on $S$, where $A_j \in S$ are disjoint. Then, for every $A \in S$, we define

$$
\int_A f d\Lambda = \sum_{j=1}^n x_j \Lambda(A \cap A_j).
$$
Further, a measurable function $f : (S,S) \to (\mathbb{R},\mathcal{B}(\mathbb{R}))$ is said to be $\Lambda$-integrable if there exists a sequence $\{f_n\}$ of simple functions such that

(i) $f_n \to f$, $\lambda$-a.e.,

(ii) for every $A \in S$, the sequence $\{\int_A f_n d\Lambda\}$ converges in probability as $n \to \infty$.

If $f$ is $\Lambda$-integrable, then we write

$$\int_A f d\Lambda = \mathbb{P} - \lim_{n \to \infty} \int_A f_n d\Lambda$$

where $\{f_n\}$ satisfies (i) and (ii).

In the following result we prove that $\int_A f d\Lambda$ does not depend on the approximating sequence, hence it is well defined. This does not follow from [22] or from [19] since they focus on ID random variables, although we use some of their arguments. In particular, in [22] (which is the work cited in [18]) the random measures considered is atomless (see Section 7.4 for the definition and for further details) while we (and [19]) do not have such a restriction.

**Lemma 5.7.** Let $\Lambda$ be a QID random measure and let $f$ be $\Lambda$-integrable then $\int_A f d\Lambda$ is well defined.

**Proof.** Let $\{f_n\}$ and $\{g_n\}$ be two real simple functions on $S$ as in the Definition 5.6 and satisfying (i) and (ii) with the same limit. Let $h_n = f_n - g_n \ (n = 1, 2, \ldots)$. Then, the sequence $\{h_n\} \to 0$ $\lambda$-a.e. and $\int_A h_n d\Lambda$ converges in probability for every $A \in S$. We need to show that $\int_A h_n d\Lambda$ converges to zero in probability for every $A \in S$.

Let $N_n(A) = \int_A f_n d\Lambda$ then $N_n$ is a measure on $L_0(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R})$, which is the set of all random elements defined on $(\Omega, \mathcal{F}, \mathbb{P})$ with values in $\mathbb{R}$. In particular, $N_n$ is absolutely continuous with respect to $\lambda$ and for every $A \in S$ we have that $N(A) := \lim_{n \to \infty} N_n(A)$ exists. In particular the existence is guaranteed by the convergence of $\int_A h_n d\Lambda$. Applying the Hahn-Saks-Vitali theorem we obtain that $N$ is a measure on $L_0(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R})$ and $N \ll \lambda$.

Since $\{h_n\}$ converges $\lambda$-a.e., we can apply the Egorov (or more correctly Lusin) theorem and obtain that for every set $A$ in $\mathcal{A}$ we have $A = \bigcup_{k=0}^{\infty} A_k$ where $A_k$’s are pairwise disjoint sets belonging to $\mathcal{A}$ such that $\lambda(A_0) = 0$ and $h_n \to 0$ uniformly on every set $A_k$’s for $k > 0$. Then, for every $A_k$ we have

$$\hat{\mathcal{L}}(N(A_k)) = \lim_{n \to \infty} \hat{\mathcal{L}}(N_n(A_k)) = \lim_{n \to \infty} \exp \left( \int_{A_k} K(\theta h_n(s), s) \lambda(ds) \right)$$

$$= \lim_{n \to \infty} \exp \left( \int_{A_k} i\theta h_n(s)a(s) - \frac{\theta^2}{2} h_n^2(s)\sigma^2(s) + \int_{\mathbb{R}} e^{i\theta h_n(s)x} - 1 - i\theta h_n(s)\rho(s, dx)\lambda(ds) \right) = 1,$$

using the dominated convergence theorem and that $\sup_{s \in A_k} h_n(s) \to 0$ as $n \to \infty$. Hence, $N(A_k) = 0$ in probability for every $k > 0$. Then, since $N(A)$ is a measure we have

$$N(A) = \sum_{k=0}^{\infty} N(A_k) = 0$$

for every $A \in S$. \qed

It is possible to give now a representation of the c.f. of $\int_S f d\Lambda$.

**Proposition 5.8.** If $f$ is $\Lambda$-integrable, then $\int_S |K(\theta f(s), s)|\lambda(ds) < \infty$, where $K$ is given in Proposition 5.5 and

$$\hat{\mathcal{L}} \left( \int_S f d\Lambda \right)(\theta) = \exp \left( \int_S K(\theta f(s), s) \lambda(ds) \right), \ \theta \in \mathbb{R}.$$
Proof. The statement follows from the same arguments used in the proof of Proposition 2.6 of [18].

We are now ready to present the main result of this section, which concern the integrability conditions for \( \int_S f d\Lambda \).

**Theorem 5.9.** Let \( f : S \to \mathbb{R} \) be a \( \mathcal{S} \)-measurable function. Then \( f \) is \( \Lambda \)-integrable if the following three conditions hold:

(i) \( \int_S |U(f(s), s)|\lambda(ds) < \infty \),

(ii) \( \int_S |f(s)|^2\sigma^2(s)\lambda(ds) < \infty \),

(iii) \( \int_S V_0(f(s), s)\lambda(ds) < \infty \),

where \( U(u, s) = ua(s) + \int_\mathbb{R} \tau(xu) - u\tau(x)\rho(s, dx) \), \( V_0(u, s) = \int_\mathbb{R} (1 \wedge |x|^2)|\rho|(s, dx) \).

Further, the c.f. of \( \int_S f d\Lambda \) can be written as

(iv) \( \mathcal{L}(\int_S f d\Lambda)(\theta) = \exp \left( i\theta a_f - \frac{1}{2} \theta^2 \sigma^f_2 + \int_\mathbb{R} e^{i\theta x} - 1 - i\theta\tau(x)F_f(dx) \right) \), where

\[
a_f = \int_S U(f(s), s)\lambda(ds), \quad \sigma^f_2 = \int_S |f(s)|^2\sigma^2(s)\lambda(ds), \quad \text{and}
\]

\[
F_f(B) = F((s, x) \in S \times \mathbb{R} : f(s)x \in B \setminus \{0\}), \quad B \in \mathcal{B}(\mathbb{R}).
\]

**Proof.** Assume that (i), (ii), (iii) hold. Let \( A_n = \{ s : |f(s)| \leq n \} \cap S_n \). We have that \( A_n \in \mathcal{S} \) and \( A_n \nearrow S \). Consider a sequence \( (f_n) \) of simple \( \mathcal{S} \)-measurable functions, such that \( f_n(s) = 0 \) if \( s \nmid A_n \), \( |f_n(s) - f(s)| \leq \frac{1}{n} \) if \( s \in A_n \), and \( |f_n(s)| \leq |f(s)| \) for all \( s \in S \). Then, \( f_n \to f \) everywhere on \( S \) as \( n \to \infty \). Further, notice that for every \( A \in \mathcal{S} \) and \( n, m \geq 1 \), \( |(f_n(s) - f_m(s))1_A(s)| \leq 2|f(s)| \).

Then by Lemma 5.10 (see the next result) we derive

\[
|U((f_n(s) - f_m(s))1_A(s), s)| \leq 2|U(f(s), s)| + 27V_0(f(s), s).
\]

Thus, by the dominated convergence theorem, we obtain that for every \( A \in \mathcal{S} \),

\[
\lim_{n,m \to \infty} \int_S U((f_n(s) - f_m(s)), 1_A(s), s)\lambda(ds) = 0
\]

\[
\lim_{n,m \to \infty} \int_S (f_n(s) - f_m(s))^2, 1_A(s)\sigma^2(s)\lambda(ds) = 0
\]

and \( \lim_{n,m \to \infty} \int_S V_0((f_n(s) - f_m(s)), 1_A(s), s)\lambda(ds) = 0 \).

Now, we prove that

\[
\int_\mathbb{R} (1 \wedge x^2)|F_f|(dx) \leq \int_\mathbb{R} \int_\mathbb{R} (1 \wedge |f(s)x|^2)|\rho|(s, dx)\lambda(ds)
\]

where \( |F_f| \) is the total variation of the measure \( F_f \). Define \( E^+ \) and \( E^- \) to be the Hahn decomposition of \( \mathbb{R} \) under the signed measure \( F_f \). Then, we have

\[
\int_\mathbb{R} (1 \wedge x^2)|F_f|(dx) = \int_{E^+} (1 \wedge x^2)F_f(dx) - \int_{E^-} (1 \wedge x^2)F_f(dx) = ...
\]
\[
= \int_S \int_{\{f(s)x \in E\}} (1 \wedge |f(s)x|^2) \rho(s, dx) \lambda(ds) - \int_S \int_{\{f(s)x \in E^c\}} (1 \wedge |f(s)x|^2) \rho(s, dx) \lambda(ds)
\]
\[
\leq \int_S \int_{\{f(s)x \in E^c\}} (1 \wedge |f(s)x|^2) \rho(s, dx) \lambda(ds)
\]
\[
= \int_S \int_{\{f(s)x \in E^c\}} (1 \wedge |f(s)x|^2) \rho(s, dx) \lambda(ds).
\]

Then, from (i), (ii), (iii) it is possible to obtain (iv) for simple measurable functions, hence including \((f_n(s) - f_m(s))1_A\).

Using (iv), we get \(\lim_{n,m \to \infty} \mathbb{E}(f_n(s) - f_m(s))1_A d\Lambda(\theta) = 1\) for every \(\theta \in \mathbb{R}\) and \(A \in \mathcal{S}\). Therefore, the sequence \(\int_A f_n d\Lambda\) converges in probability for every \(A \in \mathcal{S}\), namely \(f\) is \(\Lambda\)-integrable and, then, the c.f. of \(\int_S f d\Lambda\) can be written as in (iv).

\[\square\]

Lemma 5.10. For every \(u \in \mathbb{R}, s \in \mathbb{S} \text{ and } d > 0,\)
\[\sup\{|U(cu, s)| : |c| \leq d\} \leq d|U(u, s)| + (1 + d)^3 V_0(u, s).\]

Proof. Let \(|c| \leq d\). Define \(R(c, u, s) := \int_{\mathbb{R}} \tau(xcu) - cu(x) \rho(s, dx)\). We can rewrite \(U(cu, s)\) as follows
\[U(cu, s) = cu(s) + \int_{\mathbb{R}} \tau(xcu) - cu(x) \rho(s, dx)\]
\[= cu(s) + c \int_{\mathbb{R}} \tau(xu) - u(x) \rho(s, dx) + \int_{\mathbb{R}} \tau(xcu) - cu(x) \rho(s, dx) = eU(u, s) + R(c, u, s).\]

Observe that \(\tau(xcu) - cu(x) = 0\) if \(|ux| \leq (1 \wedge |c|^{-1})\) and \(\tau(xcu) - cu(x)| \leq 1 + d\) otherwise. Thus, we obtain, using Chebychev’s inequality, that
\[|R(c, u, s)| \leq (1 + d) \int_{|ux| > (1 \wedge |c|^{-1})} |\rho(s, dx)| \leq (1 + d)|\rho(s, \{x : (1 \wedge |ux|) \geq (1 \wedge |c|^{-1})\})|\]
\[\leq \frac{1 + d}{(1 \wedge |c|^{-2})} \int_{\mathbb{R}} (1 \wedge |ux|^2) |\rho(s, dx)| \leq (1 + d)^3 V_0(u, s).\]

\[\square\]

The reason why it is not possible to get an “only if” result is because of the following argument. Recall that we have denoted by \(C^+\) and \(C^-\) the Hahn essential decomposition of \((S \times \mathbb{R}, \mathcal{B}(S \times \mathbb{R}), F)\) under the measure \(F\). We have that
\[\int_S \int_{\mathbb{R}} (1 \wedge |f(s)x|^2) |\rho(s, dx)| \lambda(ds)\]
\[= \int_S \int_{\mathbb{R}} (1 \wedge |f(s)x|^2)^+ |\rho(s, dx)| \lambda(ds) + \int_S \int_{\mathbb{R}} (1 \wedge |f(s)x|^2)^- |\rho(s, dx)| \lambda(ds)\]
\[= \int_{\mathbb{R}} (1 \wedge y^2) \bar{F}_f^+(dy) + \int_{\mathbb{R}} (1 \wedge y^2) \bar{F}_f^-(dy) = \int_{\mathbb{R}} (1 \wedge y^2) |\bar{F}_f|(dy)\]
where \(\bar{F}_f^+(B) = F^+(\{(s, x) : f(s)x \in B \setminus \{0\}\}), \text{ for every } B \in \mathcal{B}(\mathbb{R})\). However, notice that \(\bar{F}_f^+(B) \geq F_f^+(B)\) because there might exists two sets \(A_1, A_2 \in \mathcal{S} \cap \mathcal{B}(\mathbb{R})\) with \(A_1 \in C^+\) and \(A_2 \in C^-\) such that \(f(s)x \in B \subset \mathbb{R}\) for every \((s, x) \in A_1\) and \((s, x) \in A_2\). For example, think of \((s, x) \in A_1\) and \((s', x') \in A_2\) it might happen that \(f(s)x = f(s')x'\), especially if the image of \(f\) is the entire real line.

However, by adding a further assumption we have the following result. Recall that under the assumption made in this section there exist two ID random measures \(\Lambda^+\) and \(\Lambda^-\) that generate \(\Lambda\).
**Theorem 5.11.** Let \( f : S \to \mathbb{R} \) be a \( \mathcal{S} \)-measurable function. Then, if \( f \) is integrable with respect to \( \Lambda_1 \) and \( \Lambda_2 \), the three conditions (i), (ii) and (iii) of Theorem 5.8 hold. Further, the c.f. of \( \int_S f d\Lambda \) can be written as in point (iv) of Theorem 5.8.

**Proof.** From the fact that \( |\mathcal{L}(\int_S f d\Lambda_1)|^2 \) and \( |\mathcal{L}(\int_S f d\Lambda_2)|^2 \) are the c.f. of ID-random variables it follows that
\[
\int_{\mathbb{R}} (1 \wedge y^2)|F_f|((dy) < \infty
\]
and we obtain (ii) and (iii). Moreover, since \( |\tau(x) - \sin(x)| \leq 2(1 \wedge x^2) \), (i) follows from noticing that
\[
|U(u, s)| \leq |ua(s) + \int_{\mathbb{R}} \sin(xu) - u\tau(x)\rho(s, dx)| + |\int_{\mathbb{R}} \tau(xu) - \sin(xu)|\rho|(s, dx)|
\]
\[
\leq |\text{Im} K(u, s)| + 2V_0(u, s).
\]
which is finite because of (iii) and of Proposition 5.8. \( \square \)

### 5.2 The bounded case

In this section we will work with the following two assumptions. First, \( F(A, B) \) is a bimeasure on the whole \( S \times \mathcal{B}(\mathbb{R}) \) and, second, \( \sup \sum_{i \in I} |F_{A_i}(B_i)| < \infty \), where the supremum is taken over all the finite sets \( (A_i, B_i)_{i \in I} \) of elements of \( S \times \mathcal{B}(\mathbb{R}) \) such that the rectangles \( A_i \times B_i \) are disjoint.

Let us introduce the set function \( \nu(A) : S \mapsto [0, \infty) \) such that
\[
\nu(A) := \sup_{I_A} \sum_{i \in I} |F_{A_i}(B_i)|
\]
where \( I_A \) is defined as \( I \) but with the constraint that \( A_i \subset A \).

We start with the following proposition on the control measure.

**Proposition 5.12.** Let \( \Lambda \) be a QID random measure. Let \( \nu_0 : S \mapsto \mathbb{R} \) be a signed measure, \( \nu_1 : S \mapsto \mathbb{R} \) be a measure, \( F_A \) be a quasi-Lévy measure on \( \mathbb{R} \) for every \( A \in \mathcal{S}, S \supset A \mapsto F_A(B) \in (-\infty, \infty) \) be a signed measure for every \( B \in \mathcal{B}(\mathbb{R}) \) such that \( (\nu_0(A), \nu_1(A), F_A) \) is the characteristic triplet of \( \Lambda(A), \forall A \in \mathcal{S} \). Define
\[
\lambda(A) = |\nu_0|(A) + \nu_1(A) + \nu(A).
\]
Then \( \lambda : S \mapsto [0, \infty) \) is a measure such that \( \lambda(A_n) \to 0 \) implies \( \Lambda(A_n) \to 0 \) in probability for every \( \{A_n\} \subset \mathcal{S} \).

**Proof.** From Theorem 4 in [8] we have that \( \nu(A) \) defines a measure. Then it is straightforward to see that \( \lambda(A) \) is a measure. Now if \( \nu(A_n) \to 0 \) then \( F_{A_n}^+(B) \to 0 \) and \( F_{A_n}^-(B) \to 0 \) for every \( B \in \mathcal{B}(\mathbb{R}) \), hence we obtain the stated result. \( \square \)

Before presenting one of the main crucial result of this work, which consists of an extension to Proposition 2.4 in [18], we give a remark on its topology.

**Remark 5.13.** Proposition 2.4 in [18] is stated for a standard Borel space, where a measurable space \( (X, \Gamma) \) is said to be standard Borel space if there exists a metric on \( X \) which makes it a complete separable metric space in such a way that \( \Gamma \) is then the Borel \( \sigma \)-algebra \( \mathcal{B}(X) \). However, the following result holds for more general topological spaces: the Lusin measurable spaces. Lusin measurable spaces are measurable spaces isomorphic to a measurable space \( (H, \mathcal{B}(H)) \), where \( H \) is
Then there exists a unique signed measure \( Q \) for every \( \nu \) space. Let Theorem 5.14. In this remark, we do not provide an extensive discussion (indeed, see [6], [8] and [10] for further details), we just mention that \( H \) can be any Polish space (thus, including the real line).

Theorem 5.14. Let \((X, \mathcal{B})\) be a Lusin measurable space and let \((T, \mathcal{A})\) be an arbitrary measurable space. Let \( Q_0(A, B) \) be a (possibly negative) function of \( A \in \mathcal{A}, B \in \mathcal{B} \), satisfying:
(a) for every \( A \in \mathcal{A}, Q_0(A, \cdot) \) is a signed measure on \((X, \mathcal{B})\),
(b) for every \( B \in \mathcal{B}, Q_0(\cdot, B) \) is a signed measure on \((T, \mathcal{A})\),
(c) \( \nu(A) := \sup \sum_{i \in I} |Q_0(A_i, B_i)| < \infty \).

Then there exists a unique signed measure \( Q \) on the product \( \sigma \)-algebra \( \mathcal{A} \otimes \mathcal{B} \) such that
\[
Q(A \times B) = Q_0(A, B) = \int_A q(t, B) \nu(dt),
\]
with Jordan decomposition
\[
Q^+(C) = \int_T \int_X 1_C(x, t)\tilde{q}^+(t, dx)\nu(dt) \quad \text{and} \quad Q^-(C) = \int_T \int_X 1_C(x, t)\tilde{q}^-(t, dx)\nu(dt)
\]
for every \( A \in \mathcal{A}, B \in \mathcal{B}, C \in \mathcal{A} \otimes \mathcal{B} \) where \( q : T \times \mathcal{B} \to [-1, 1] \) fulfills the following conditions:
(d) for every \( t, q(t, \cdot) \) is a signed measure on \( \mathcal{B} \),
(e) for every \( B, q(\cdot, B) \) is \( \mathcal{A} \)-measurable,
and where \( \tilde{q}^+(\cdot, \cdot) \) and \( \tilde{q}^-(\cdot, \cdot) \) are the Jordan decomposition of \( q(t, \cdot) \).
Further, if \( q_1(\cdot, \cdot) \) is some other function satisfying \([\text{iv}], \text{(d)} \) and \( \text{(e)} \), then off a set of \( \nu \)-measure zero, \( q_1(t, \cdot) = q(t, \cdot) \).

Proof. First, recall that \( \nu(\cdot) \) is a measure. Notice that by definition \( \nu(\cdot) \geq Q_0^+(\cdot, B) \) and \( \nu(\cdot) \geq Q_0^-(\cdot, B) \) for every \( B \in \mathcal{B} \), where \( Q_0^+(\cdot, B) \) and \( Q_0^-(\cdot, B) \) are the Jordan decomposition of \( Q_0(\cdot, \cdot) \). Then, by the Radon-Nikodym theorem we have that \( Q_0^+(A, B) = \int_A \tilde{q}^+(x, B)\nu(dx) \) and \( Q_0^-(A, B) = \int_A \tilde{q}^-(x, B)\nu(dx) \). Hence, \( Q_0(A, B) = \int_A q(x, B)\nu(dx) \) and \( q(x, B) = \tilde{q}^+(x, B) - \tilde{q}^-(x, B) \) is \( \nu \)-a.s. uniquely defined (since \( \tilde{q}^+(x, B) \) and \( \tilde{q}^-(x, B) \) are the unique Radon-Nikodym derivatives) and, for every \( B \in \mathcal{B}, q(\cdot, B) \) is \( \mathcal{A} \)-measurable.

Observe also that by definition \( \nu(A) \geq Q_0^+(A, B) \) and \( \nu(A) \geq Q_0^+(A, B) \) for every \( A \in \mathcal{A} \) and every \( B \in \mathcal{B}(\mathbb{R}) \). Then, \( \tilde{q}^+(x, B) \leq 1 \) and \( \tilde{q}^-(x, B) \leq 1 \), \( \nu \)-a.e.. Further, since \( Q(A, \cdot) \) is a (signed) measure for any \( A \in \mathcal{A} \) then \( q(x, \cdot) \) is also a (signed) measure with \( |q(x, B)| \leq 1 \) for every \( B \in \mathcal{B} \). This is possible to see by adapting the arguments of the proof of Theorem 4 in [8] when the function considered is the indicator function. Indeed, using their formalism \( Q(A, 1_B) := \int_X 1_B(y)Q(A, dy) \), hence \( Q(A, 1_B) = Q(A, B) \), and by the uniqueness of the Radon-Nikodym derivative \( q(x, 1_B) = q(x, B) \), \( \nu \)-a.e..

Again by adapting the arguments of Theorem 4 in [8] we get that there exists a signed measure \( Q(A \times B) = \int_A q(x, B)\nu(dx) \). Moreover, by the uniqueness of \( q \) we have that this signed measure is unique.

We prove now the stated Jordan decomposition of \( Q \). From the proof of Theorem 4 in [8] it is shown, using our notation, that \( Q(C) = Q^+(C) - Q^-(C) \) for any \( C \in \mathcal{B} \otimes \mathcal{A} \). However, it is not mentioned that \( Q^+ \) and \( Q^- \) are the Jordan decomposition of \( Q \), however this is indeed the case and we are going to prove it here.

If we apply the arguments of the proof of Proposition 6 in [8] to indicator functions then it is possible to see that the decomposition of \( q(\cdot, 1_B) \) into \( q^+(\cdot, 1_B) \) and \( q^-(\cdot, 1_B) \) in [8] is actually the Jordan decomposition since \( q^+(\cdot, 1_B) \) is defined as
\[
q^+(\cdot, 1_B) := \text{ess sup} \int_{0 \leq g \leq 1_B} q(\cdot, g)
\]

where $g$ are measurable functions uniformly bounded on absolute values on the finite measure space $(T, \mathcal{A}, \nu)$ such that $g(y) \leq 1_{\mathcal{B}}(y)$, $\nu$-a.e.. Now, since we can approximate any such $g$ with indicator functions, namely the set of indicator functions are dense in the set of measurable functions with compact support (in this case $\mathcal{B}$) and bounded by 1, then by standard density arguments we can substitute the $g$’s by indicator functions and using the fact that $q(x,1_B) = q(x,B)$ $\nu$-almost everywhere we have

$$q^+ (\cdot, B) := \sup_{K \subset B} q(\cdot, K)$$

which is the definition of the positive measure of the Jordan decomposition of the signed measure $q(x, \cdot)$. Further, since $q^- (\cdot, B)$ is defined as $q^- (\cdot, B) := q^+ (\cdot, B) - q(\cdot, B)$ then it coincides with the negative measure of the Jordan decomposition.

Now, since $\int_A \int_B 1_C(x,t)\tilde{q}_+(t, dx)\nu(dt)$ and $\int_A \int_B 1_C(x,t)\tilde{q}_-(t, dx)\nu(dt)$ are mutually singular and $Q(C) = \int_A \int_B 1_C(x,t)\tilde{q}_+(t, dx)\nu(dt) - \int_A \int_B 1_C(x,t)\tilde{q}_-(t, dx)\nu(dt)$ then we have the sated Jordan decomposition of $Q$.

**Remark 5.15.** Notice that it is not always true that $q^+ = \tilde{q}_+$ since $Q_{0,+}(A, \cdot)$ and $Q_{0,-}(A, \cdot)$ are not necessarily measures (on the contrary $Q_{0,+}(B, \cdot)$ and $Q_{0,-}(B, \cdot)$ are measures since they are the Jordan decomposition of if $Q(\cdot, B)$). The only thing we know is that $q_+(x,B) - q_-(x,B) = \tilde{q}_+(x,B) - \tilde{q}_-(x,B)$, where all the four functions are positive and with values less or equal than 1. In particular, the fact that $q^+(x,B) \leq 1$ and $q^-(x,B) \leq 1$ for $\nu$-almost every $x \in T$ and for every $B \in \mathcal{B}$ comes from the fact that $|q(x,B)| \leq 1$ for every $B \in \mathcal{B}$.

**Lemma 5.16.** Let $F$ be as in Proposition 5.12 and assume that $\sum_{i \in I} |F_{A_i}(B_i)| < \infty$. Then there exists a unique $\sigma$-finite measure $F$ on $\mathcal{S} \otimes \mathcal{B}(\mathbb{R})$ such that

$$F(A \times B) = F_A(B), \quad \text{for all } A \in \mathcal{S}, B \in \mathcal{B}(\mathbb{R}).$$

Moreover, there exists a function $\rho : S \times \mathcal{B}(\mathbb{R}) \mapsto [-\infty, \infty]$ such that

(i) $\rho(s, \cdot)$ is a quasi-Lévy type measure on $\mathcal{B}(\mathbb{R})$, for every $s \in S$;

(ii) $\rho(\cdot, B)$ is a Borel measurable function, for every $B \in \mathcal{B}(\mathbb{R})$,

(iii) $\int_S \int_{\mathbb{R}} h(s, x)F(ds, dx) = \int_S \int_{\mathbb{R}} h(s, x)\rho(s, dx)\lambda(ds)$, for every $S \otimes \mathcal{B}(\mathbb{R})$-measurable function $h : S \times \mathbb{R} \mapsto [0, \infty]$. This equality can be extended to real and complex-valued functions $h$.

**Proof.** Using Theorem 5.14 we have that $F(A \times B) = \int_A q(t,B)\nu(dt)$ where $q$ satisfies point (d) and (e) in that Proposition. Since $\lambda \ll \nu$, then defining

$$\rho_+(s, dx) := \frac{d\nu}{d\lambda}(s)\tilde{q}_+(s, dx), \quad \rho_-(s, dx) := \frac{d\nu}{d\lambda}(s)\tilde{q}_-(s, dx)$$

and

$$\rho(s, dx) := \rho_+(s, dx) - \rho_-(s, dx)$$

we have that (ii) is satisfied and that

$$\int_{\mathbb{R}} (1 \wedge x^2)|\rho|(s, dx) = \frac{d\nu}{d\lambda}(s) \int_{\mathbb{R}} (1 \wedge x^2)\tilde{q}_+(s, dx) + \frac{d\nu}{d\lambda}(s) \int_{\mathbb{R}} (1 \wedge x^2)\tilde{q}_-(s, dx)$$

$$\leq \frac{d\nu}{d\lambda}(s) \int_{\mathbb{R}} \tilde{q}_+(s, dx) + \frac{d\nu}{d\lambda}(s) \int_{\mathbb{R}} \tilde{q}_-(s, dx)$$

$$\leq \frac{d\nu}{d\lambda}(s) + \frac{d\nu}{d\lambda}(s) \leq 2$$

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where the last inequality comes from the fact that we can always assume that $\frac{df}{ds}(s) \leq 1$ for all $s$ (and the same for $\nu$). This proves (i).

Therefore, we have immediately (iii) since

$$
\int_{S} \int_{R} 1_{C}(s, x) \rho(s, dx) \lambda(ds) = \int_{S} \int_{R} 1_{C}(s, x) q(s, dx) \nu(ds) = F(C).
$$

The extension to real and complex integrand follows by standard arguments, which can be found in [3] for example.

\[\square\]

**Corollary 5.17.** Under the same assumption, we have that $F^+(C) = \int_{S} \int_{R} 1_{C}(s, x) \rho^+(s, dx) \lambda(ds)$ and $F^-(C) = \int_{S} \int_{R} 1_{C}(s, x) \rho^-(s, dx) \lambda(ds)$.

**Proof.** First, notice that $\int_{S} \int_{R} 1_{C}(s, x) \rho^+(s, dx) \lambda(ds)$ and $\int_{S} \int_{R} 1_{C}(s, x) \rho^-(s, dx) \lambda(ds)$ are measures on $S \otimes B(R)$ and that $\hat{F}(C) = \int_{S} \int_{R} 1_{C}(s, x) \rho^+(s, dx) \lambda(ds) - \int_{S} \int_{R} 1_{C}(s, x) \rho^-(s, dx) \lambda(ds)$. Finally, let $E^+_1$ and $E^-_1$ the Hahn decomposition of $\rho(s, \cdot)$ (which is the same as the one of $q(s, \cdot)$). Define $C^+ := \{(s, x) \in S \times R : x \in E^+_1\}$ and $C^- := \{(s, x) \in S \times R : x \in E^-_1\}$. It is possible to see that $C^+$ and $C^-$ form an essential decomposition of $S \times R$ and so the measures $\int_{S} \int_{R} 1_{C}(s, x) \rho^+(s, dx) \lambda(ds)$ and $\int_{S} \int_{R} 1_{C}(s, x) \rho^-(s, dx) \lambda(ds)$ are singular, thus concluding the proof.

\[\square\]

Notice that an alternative proof of the above result follows from Theorem 5.14 and Lemma 5.16.

Using the above results, we obtain the following proposition.

**Proposition 5.18.** The characteristic function $\hat{\Lambda}(A)$ of $\Lambda(A)$ can be written in the form:

$$
\mathbb{E}(e^{i \theta \Lambda(A)}) = \exp \left( \int_{A} K(\theta, s) \lambda(ds) \right), \quad \theta \in \mathbb{R}, A \in \mathcal{S},
$$

where

$$
K(\theta, s) = i \theta a(s) - \frac{\theta^2}{2} \sigma^2(s) + \int_{R} e^{i \theta x} - 1 - i \theta \tau(x) \rho(s, dx),
$$

$a(s) = \frac{d\mu(s)}{d\lambda}(s)$, $\sigma^2(s) = \frac{d\nu(s)}{d\lambda}(s)$ and $\rho$ is given by Lemma 5.16 and $\exp(K(\theta, s))$ is the characteristic function of a QID random variable if it exists. Moreover, we have

$$
|a(s)| + \sigma^2(s) + \frac{d\nu}{d\lambda}(s) = 1 \quad \lambda\text{-a.e.}
$$

**Proof.** The first statement follows from the Lévy-Khintchine formulation (11) and Lemma 5.16.

The second statement follows from the fact that for every $A \in \mathcal{S}$, we have

$$
\int_{A} \left( |a(s)| + \sigma^2(s) + \frac{d\nu}{d\lambda}(s) \right) \lambda(ds) = |\nu_0|(A) + \nu_1(A) + \nu(A) = \Lambda(A) = \int_{A} d\lambda(ds).
$$

\[\square\]

**Proposition 5.19.** If $f$ is $\Lambda$-integrable, then $\int_{S} |K(tf(s), s)| \lambda(ds) < \infty$, where $K$ is given in Proposition 5.18, and

$$
\hat{\mathcal{L}} \left( \int_{S} f d\Lambda \right)(\theta) = \exp \left( \int_{S} K(\theta f(s), s) \lambda(ds) \right), \quad \theta \in \mathbb{R}.
$$
Proof. The statement follows from the same arguments used in the proof of Proposition 2.6 of [18].

We state now the main theorem of this section.

**Theorem 5.20.** Let \( f : S \to \mathbb{R} \) be a \( S \)-measurable function. Then \( f \) is \( \Lambda \)-integrable if and only if the following three conditions hold:

(i) \( \int_S |U(f(s), s)| \lambda(ds) < \infty \),

(ii) \( \int_S |f(s)|^2 \sigma^2(s) \lambda(ds) < \infty \),

(iii) \( \int_S V_0(f(s), s) \lambda(ds) < \infty \),

where \( U(u, s) = \frac{a(s)}{u(s)} + \int_{\mathbb{R}} \tau(xu) - u\tau(x) \rho(s, dx) \), \( V_0(u, s) = \int_{\mathbb{R}} (1 \wedge |xu|^2) |\rho(s, dx)| \).

Further, if \( f \) is \( \Lambda \)-integrable, then the c.f. of \( \int_S f d\Lambda \) can be written as

(iv) \( \hat{L}(\int_S f d\Lambda)(\theta) = \exp \left( i\theta a_f - \frac{1}{2} \theta^2 \sigma_f^2 + \int_{\mathbb{R}} e^{i\theta x} - 1 - i\theta \tau(x) F_f(dx) \right) \), where

\[
a_f = \int_S U(f(s), s) \lambda(ds), \quad \sigma_f^2 = \int_S |f(s)|^2 \sigma^2(s) \lambda(ds),
\]

and

\[
F_f(B) = F(\{(s, x) \in S \times \mathbb{R} : f(s)x \in B \setminus \{0\}\}), \quad B \in \mathcal{B}(\mathbb{R}).
\]

**Proof.** \( \Leftarrow \): Notice that the discussions made for the case in the previous section hold as well in this case. However, the assumption \( \nu(\mathbb{R}) < \infty \) directly implies that \( \int_{\mathbb{R}} 1 \wedge |y|^2 |\hat{F}_f|(dy) < \infty \), hence using Corollary [5.17] we have

\[
\int_S \int_{\mathbb{R}} (1 \wedge |f(s)x|^2) |\rho(s, dx)| \lambda(ds) < \infty.
\]

We conclude this part of the proof with the same arguments used in the proof of Theorem [5.11].

\( \Rightarrow \): It follows from similar arguments used in the proof of Theorem [5.9].

### 5.3 The general case

In this section we extend the results presented in the previous sections. We will work with the assumption that \( F(A, \cdot) \) is a signed measure for every \( A \in \mathcal{S} \) and \( F(\cdot, B) \) is a signed measure \( B \in \mathcal{B}(\mathbb{R}) \) such that \( 0 \notin B \). We define for every \( A \in \mathcal{S} \) and \( B \in \mathcal{B}(\mathbb{R}) \)

\[
J(A, B) := \int_B 1 \wedge x^2 F_A(dx)
\]

and notice that it is a signed bimeasure on \( \mathcal{S} \times \mathcal{B}(\mathbb{R}) \). Further, observe that \( \sup_I \sum_{i \in I} |J(A_i, B_i)| \leq \int_{\mathbb{R}} 1 \wedge x^2 |F_A|(dx) \). Hence, in the case \( F_A \) is a quasi Lévy measure then \( \sup_I \sum_{i \in I} |J(A_i, B_i)| < \infty \). Let us introduce the measure \( \xi(A) : \mathcal{S} \mapsto [0, \infty) \) such that

\[
\xi(A) := \sup_{I_A} \sum_{i \in I} |J(A_i, B_i)|
\]

We start with the following proposition.
Proposition 5.21. Let \( \Lambda \) be a QID random measure. Let \( \nu_0 : S \mapsto \mathbb{R} \) be a signed measure, \( \nu_1 : S \mapsto \mathbb{R} \) be a measure, \( F_A \) be a quasi-Lévy measure on \( \mathbb{R} \) for every \( A \in S \), \( S \ni A \mapsto F_A(B) \in (-\infty, \infty) \) be a signed measure for every \( B \in \mathcal{B}(\mathbb{R}) \) such that \( 0 \notin B \) and such that \((\nu_0(A), \nu_1(A), F_A)\) is the characteristic triplet of \( \Lambda(A) \), \( \forall A \in S \). Define

\[
\lambda(A) = |\nu_0|(A) + \nu_1(A) + \xi(A).
\]

Then \( \lambda : S \mapsto [0, \infty) \) is a measure such that \( \lambda(A_n) \to 0 \) implies \( \Lambda(A_n) \to 0 \) in probability for every \( \{A_n\} \subset S \).

**Proof.** Since by Theorem 4 in [S] we know that \( \xi \) is a measure, it is straightforward to see that \( \lambda(A) \) is a measure. Now, let \( \lambda(A_n) \to 0 \) then we have that \( |\nu_0|, \nu_1 \) and \( \xi \) go to zero. Then since \( \xi(A_n) \to 0 \) implies that \( \int_{\mathbb{R}} 1 \wedge x^2|F_A_n|(dx) \to 0 \), then we have that \( \Lambda(A_n) \to 0 \) in probability for every \( \{A_n\} \subset S \). \( \square \)

We provide now the extension to Proposition 2.3 of [18].

Lemma 5.22. Let \( F \) be as in Proposition 5.21. Then there exists a unique \( \sigma \)-finite measure \( F \) on \( S \otimes \mathcal{B}(\mathbb{R}) \) such that

\[
F(A \times B) = F_A(B), \quad \text{for all } A \in S, B \in \mathcal{B}(\mathbb{R}).
\]

Moreover, there exists a function \( \rho : S \times \mathcal{B}(\mathbb{R}) \mapsto [-\infty, \infty] \) such that

(i) \( \rho(s, \cdot) \) is a quasi-Lévy type measure on \( \mathcal{B}(\mathbb{R}) \), for every \( s \in S \),

(ii) \( \rho(\cdot, B) \) is a Borel measurable function, for every \( B \in \mathcal{B}(\mathbb{R}) \),

(iii) \( \int_{S \times \mathbb{R}} h(s, x)F(ds, dx) = \int_S \int_{\mathbb{R}} h(s, x)\rho(s, dx)\lambda(ds) \), for every \( S \otimes \mathcal{B}(\mathbb{R}) \)-measurable function \( h : S \times \mathbb{R} \mapsto [0, \infty] \). This equality can be extended to real and complex-valued functions \( h \).

**Proof.** First, notice that \( J(A, B) \) satisfies the assumptions of Theorem 5.14 with \( (T, A) = (S, S) \) and \( (X, B) = (\mathbb{R}, \mathcal{B}(\mathbb{R})) \). Therefore, there exists a measure \( Q \) on the product \( \sigma \)-algebra \( S \otimes \mathcal{B}(\mathbb{R}) \) such that

\[
Q(A \times B) = J(A, B) = \int_A q(s, B)\xi(ds) = \int_A \tilde{q}^+(s, B)\xi(ds) - \int_A \tilde{q}^-(s, B)\xi(ds)
\]

where \( q \) satisfies (d) and (e) of Proposition 2.4. Since \( \lambda \gg \xi \), define

\[
\rho_+(s, dx) := \frac{d\xi}{d\lambda}(s)(1 \wedge x^2)^{-1}\tilde{q}^+(s, dx), \quad \rho_-(s, dx) := \frac{d\xi}{d\lambda}(s)(1 \wedge x^2)^{-1}\tilde{q}^-(s, dx)
\]

and

\[
\rho(s, dx) := \rho_+(s, dx) - \rho_-(s, dx) = \frac{d\xi}{d\lambda}(s)(1 \wedge x^2)^{-1}q(s, dx)
\]

we have that (ii) is satisfied and since \( \rho_+(s, \cdot) \) and \( \rho_-(s, \cdot) \) are mutually singular, they indeed the Jordan decomposition of \( \rho(s, \cdot) \), then

\[
\int_{\mathbb{R}} (1 \wedge x^2)|\rho|(s, dx) = \frac{d\xi}{d\lambda}(s) \int_{\mathbb{R}} \tilde{q}^+(s, dx) + \frac{d\xi}{d\lambda}(s) \int_{\mathbb{R}} \tilde{q}^-(s, dx) \leq 2
\]

where the last inequality comes from the fact that we can always assume that \( \frac{d\xi}{d\lambda}(s) \leq 1 \) for all \( s \). This proves (i).

Let

\[
F(C) = \int_S \int_{\mathbb{R}} 1_C(s, x)\rho(s, dx)\lambda(ds),
\]

(10)
where \( C \in \mathcal{S} \otimes \mathcal{B}(\mathbb{R}) \), then \( F \) is a well defined signed measure that satisfies, for every \( A \in \mathcal{S} \) and \( B \in \mathcal{B}(\mathbb{R}) \),
\[
F(A \times B) = \int_A \int_B \rho(s, dx) \lambda(ds) = \int_A \int_B (1 \wedge x^2)^{-1} q(s, dx) \xi(ds) = \int_A \int_B (1 \wedge x^2)^{-1} Q(ds, dx) = F_A(B)
\]
Therefore, (iii) follows from (10) by standard arguments for functions integrated with respect to signed measures (again see [8]).

**Corollary 5.23.** Under the same assumption, we have that 
\[
F^+(C) = \int_S \int_\mathbb{R} 1_C(s, x) \rho^+(s, dx) \lambda(ds) \text{ and } F^-(C) = \int_S \int_\mathbb{R} 1_C(s, x) \rho^-(s, dx) \lambda(ds).
\]

**Proof.** It follows from the same arguments used in the proof of Corollary 5.17.

Using the above results, we obtain the following proposition.

**Proposition 5.24.** The characteristic function \((1)\) of \( \Lambda(A) \) can be written in the form:
\[
\mathbb{E}(e^{i \theta \Lambda(A)}) = \exp \left( \int_A K(\theta, s) \lambda(ds) \right), \quad \theta \in \mathbb{R}, A \in \mathcal{S},
\]
where 
\[
K(\theta, s) = i \theta a(s) - \frac{\theta^2}{2} \sigma^2(s) + \int_\mathbb{R} e^{i \theta x} - 1 - i \theta \tau(x) \rho(s, dx),
\]
a(s) = \( \frac{d\nu_0}{d\lambda}(s) \), \( \sigma^2(s) = \frac{d\nu_1}{d\lambda}(s) \) and \( \rho \) is given by Lemma 5.22 and \( \exp(K(\theta, s)) \) is the characteristic function of a QID random variable if it exists. Moreover, we have 
\[
|a(s)| + \sigma^2(s) + \frac{d\xi}{d\lambda}(s) = 1, \quad \lambda\text{-a.e.}
\]

**Proof.** The first statement follows from the Lévy-Khintchine formulation \((1)\) and Lemma 5.22. The second statement follows from the fact that for every \( A \in \mathcal{S} \), we have 
\[
\int_A \left( |a(s)| + \sigma^2(s) + \frac{d\xi}{d\lambda}(s) \right) \lambda(ds) = |\nu_0|(A) + \nu_1(A) + \xi(A) = \lambda(A) = \int_A d\lambda(ds).
\]

**Proposition 5.25.** If \( f \) is \( \Lambda \)-integrable, then \( \int_S |K(tf(s), s)| \lambda(ds) < \infty \), where \( K \) is given in Proposition 5.24, and 
\[
\hat{\mathcal{L}} \left( \int_S fd\Lambda \right)(\theta) = \exp \left( \int_S K(\theta f(s), s) \lambda(ds) \right), \quad \theta \in \mathbb{R}.
\]

**Proof.** The statement follows from the same arguments used in the proof of Proposition 2.6 of [18].

We state now the main theorem of this section.
**Theorem 5.26.** Let \( f : S \to \mathbb{R} \) be a \( S \)-measurable function. Then \( f \) is \( \Lambda \)-integrable if the following three conditions hold:

i. \( \int_S |U(f(s), s)| \lambda(ds) < \infty \),

ii. \( \int_S |f(s)|^2 \sigma^2(s) \lambda(ds) < \infty \),

iii. \( \int_S V_0(f(s), s) \lambda(ds) < \infty \),

where \( U(u, s) = ua(s) + \int_{\mathbb{R}} \tau(xu) - u\tau(x)\rho(s, dx), \quad V_0(u, s) = \int_{\mathbb{R}} (1 \wedge |xu|^2)|\rho|(s, dx) \).

Further, the c.f. of \( \int_S f d\Lambda \) can be written as

iv. \( \hat{\mathcal{L}}(\int_S f d\Lambda)(\theta) = \exp \left(i\theta a_f - \frac{1}{2}\theta^2 \sigma_f^2 + \int_{\mathbb{R}} e^{i\theta x} - 1 - i\theta \tau(x)F_f(dx)\right) \),

where

\[ a_f = \int_S U(f(s), s) \lambda(ds), \quad \sigma_f^2 = \int_S |f(s)|^2 \sigma^2(s) \lambda(ds), \quad \text{and} \]

\[ F_f(B) = F(\{(s, x) \in S \times \mathbb{R} : f(s)x \in B \setminus \{0\}\}, \quad B \in \mathcal{B}(\mathbb{R}). \]

**Proof.** It follows from similar arguments used in the proof of Theorem 5.9. \( \square \)

### 5.4 The symmetric case

We conclude this chapter with a discussion on the symmetric case, namely the case where \( \bar{\Lambda}(A) = \Lambda(A) - \Lambda'(A) \) and \( \Lambda'(A) \) is an independent copy of \( \Lambda(A) \). We start with the following general lemma.

**Lemma 5.27.** Let \( \mu \) be a QID distribution. Then, the symmetrisation and the dual of \( \mu \) are QID distributions.

**Proof.** The dual is straightforward. Regarding the symmetrisation, we have \( |\mu|^2 = \mu \ast \bar{\mu} \), where \( \bar{\mu} \) denotes the dual of \( \mu \). Since the class of QID distributions is closed under convolution (see Remark 2.6 of [13]), then \( |\mu|^2 \) is QID. \( \square \)

Then, we have the following result.

**Proposition 5.28.** Let \( \lambda(A) = \Lambda(A) - \Lambda'(A) \) where \( \Lambda'(A) \) is an independent copy of \( \Lambda(A) \) for every \( A \in S \). Then, \( \bar{\Lambda} \) is a QID random measure. Further, for an arbitrary function \( f : S \to \mathbb{R} \), \( f \) is \( \Lambda \)-integrable if it is \( \Lambda \)-integrable.

**Proof.** By Lemma 5.27 we have that \( \bar{\Lambda}(A) \) is a QID r.v. for every \( a \in A \). Moreover, using the notations used before, we have that

\[ \hat{\mathcal{L}}(\bar{\Lambda}(A)) (\theta) = \exp \left(-\theta^2 \nu_1(A) + \int_{\mathbb{R}} e^{i\theta x} - 1 - i\theta \tau(x)\bar{F}_A(dx)\right), \]

where \( \bar{F}_A(B) = F_A(B) + F_A(-B) \). It is possible to see that \( \bar{\Lambda} \) is a QID random measure by definition.

Now, notice that indeed we have

\[ \hat{\mathcal{L}}(\bar{\Lambda}(A)) (\theta) = \exp \left(\int_A \left(-\theta^2 \sigma^2(s) + 2 \int_{\mathbb{R}} \cos(\theta x) - 1 \hat{\rho}(s, dx)\right) \lambda(ds)\right), \]

where \( \hat{\rho}(s, B) = \rho(s, B) + \rho(s, -B) \) for every \( B \in \mathcal{B}(\mathbb{R}) \). Then, by applying Theorem 5.9 we obtain the stated result. \( \square \)

**Remark 5.29.** Differently from the ID case (see Proposition 2.9 in [13]) we do not have an “if and only if” result but just an “if” result. This is because \( |\hat{\rho}(s, B)| \leq |\rho(s, B)| + |\rho(s, -B)| \) for every \( B \in \mathcal{B}(\mathbb{R}) \).
6 Continuity of the stochastic integral mapping

In this section we are going to explore the set of $\Lambda$-integrable functions and show a continuity property of the linear the stochastic integral mapping $f \mapsto \int_S f d\Lambda$ from this space into the $L_p$ space (more precisely to $L^p(\Omega, \mathbb{P})$). In particular, the space of integrable function is a subset of the corresponding Musielak-Orlicz modular space defined in Chapter III of [18]. We begin with some preliminaries. Let $q \in [0, \infty)$. Consider the following condition:

$$\mathbb{E}[|\Lambda(A)|^q] < \infty, \quad \text{for all } a \in S.$$ 

Observe that for $q = 0$ every $A$ satisfies this condition. Throughout this section, we shall assume that the above condition is satisfied. Further, we assume that for all $s \in S$

$$\int_{|x|>1} |x|^q \rho^+(s, dx) < \infty. \quad (11)$$

From the arguments in the previous section we have that

$$\int_A \int_{|x|>1} |x|^q \rho^+(s, dx) \lambda(ds) \geq \int_{|x|>1} |x|^q F^+_A(dx)$$

and then by Theorem 6.2 point (a) in [13] the assumption (11) implies that

$$\int_{|x|>1} |x|^q F^+_A(dx) < \infty \quad \text{and} \quad \int_{\mathbb{R}} |x|^q \mathcal{L}(\Lambda(A))(dx) < \infty.$$ 

Observe that while Theorem 6.2 point (a) in [13] is stated for the centering function $\tau(x) = x1_{|x|\leq 1}$, the result holds for any centering function since the proof is based on results on ID distributions with no restrictions on the choice of the centering function (see Theorem 25.3 of [21]).

Now, define, for $0 \leq p \leq q$, $u \in \mathbb{R}$ and $s \in S$,

$$\Phi_p(u, s) = U^*(u, s) + u^2 \sigma^2(s) + V_p(u, s), \quad (12)$$

where

$$U^*(u, s) = \sup_{|c|\leq 1} |U(cu, s)| \quad \text{and} \quad V_p(u, s) = \int_{\mathbb{R}} |xu|^p 1_{|xu|>1}(x) + |xu|^2 1_{|xu|\leq 1}(x)|\rho|(s, dx).$$

In the following lemma, which is an equivalent of Lemma 3.1 in [18] in our framework, we abuse the notation and consider $\lambda$ to take all the different formulations it has taken in this work, namely (4), (7) and (9).

**Lemma 6.1.** The following are satisfied:

(i) for every $s \in S$, $\Phi_p(\cdot, s)$ is a continuous non-decreasing function on $[0, \infty)$ with $\Phi_p(0, s) = 0$,

(ii) $\lambda(\{s : \Phi_p(u, s) = 0 \text{ for some } u = u(s) \neq 0\}) = 0$,

(iii) there exists a constant $C > 0$ such that $\Phi_p(2u, s) \leq C\Phi_p(u, s)$, for all $u \geq 0$ and $s \in S$.

**Proof.** Points (i) and (iii) are proved adapting the same arguments of the proof of the points (i) and (iii) of Lemma 3.1 in [18] to our framework.

To prove point (ii) we proceed similarly as in Lemma 3.1 in [18], but due to the different formulations of $\lambda$ we need to pay particular attention.

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If $\Phi_p(u,s) = 0$ for some $u = u(s) \neq 0$ then $\sigma^2(s) = 0$, $|\rho|(s,\mathbb{R}) = 0$ and $U(u,s) = 0$. The last two equalities imply that $a(s) = 0$. Thus,

$$S_0 := \{ s : \Phi_p(u,s) = 0 \text{ for some } u = u(s) \neq 0 \} = \{ s : a(s) = \sigma^2(s) = |\rho|(s,\mathbb{R}) = 0 \},$$

which shows also that $S_0$ is a measurable set. Let $A$ be any measurable set in $S_0$ and notice that $\nu_0(A) = \int_A a(s)\lambda(ds) = 0$, hence we obtain that $|\nu_0|(S_0) = 0$.

Now, observe that given the above arguments we have that

$$\lambda(s) := 0, \quad \forall s \in S_0.$$

Therefore, we get (in its different formulations) that $\lambda(S_0) = 0$.

**Lemma 6.2.** Let $\{\mu_n\}$ be a sequence of QID probability distributions on $\mathbb{R}$ with c.t. $(a_n, \sigma_n, G_n)$. Let $a_n \to 0$, $\sigma_n^2 \to 0$ ad $\int_{\mathbb{R}} (1 \wedge |x|^2)|G_n|(dx) \to 0$. Assume that $\int_{|x|>1} |x|^b|G_n|(dx) < \infty$ for all $n \in \mathbb{N}$. Then, for every $b > 0$,

$$\int_{|x|>1} |x|^b|G_n|(dx) \to 0 \Rightarrow \int_{\mathbb{R}} |x|^b\mu_n(dx) \to 0.$$

**Proof.** For every $n \in \mathbb{N}$, let $\mu_n^+$ and $\mu_n^-$ the ID distributions with c.t. $(a_n, \sigma_n^2, G_n^+)$ and $(0, 0, G_n^-)$ respectively. Then, from Lemma 3.2 in [18] we obtain that

$$\int_{|x|>1} |x|^bG_n^+(dx) \to 0 \Rightarrow \int_{\mathbb{R}} |x|^b\mu_n^+(dx) \to 0.$$  

and

$$\int_{|x|>1} |x|^bG_n^-(dx) \to 0 \Rightarrow \int_{\mathbb{R}} |x|^b\mu_n^-(dx) \to 0.$$

Now, for every $n \in \mathbb{N}$ let $X_n^+$, $X_n^-$ be the real valued random variables with distributions $\mu_n$, $\mu_n^+$ and $\mu_n^-$, respectively; thus, $X_n + X_n^- \overset{d}{=} X_n^+$ with $X_n$ and $X_n^-$ independent. Notice that for $b \leq 1$ we have

$$E[|X_n|^b] \leq E[|X_n + X_n^-|^b] + E[|X_n^-|^b] = E[|X_n|^b] + E[|X_n^-|^b]$$

while for $b > 1$ we have

$$E[|X_n|^b] \leq [b]\left(E[|X_n + X_n^-|^b] + E[|X_n^-|^b]\right) = [b]\left(E[|X_n^+|^b] + E[|X_n^-|^b]\right)$$

where $[b]$ stands for the lowest natural number greater than $b$.

Finally, by sending $n \to \infty$ we obtain the stated result. \qed

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Before presenting the main result of this section, we need some preliminaries.

Define the Musielak-Orlicz space as in [18]:

\[ L_{\Phi_p}(S; \lambda) = \left\{ f \in L_0(S; \lambda) : \int_S \Phi_p(|f(s)|, s) \lambda(ds) < \infty \right\} \]

The space \( L_{\Phi_p}(S; \lambda) \) is a complete linear metric space with the \( F \)-norm defined by

\[ \|f\|_{\Phi_p} = \inf_{c > 0} \left\{ \int_S \Phi_p(c^{-1}|f(s)|, s) \lambda(ds) \leq c \right\}. \]

Simple functions are dense in \( L_{\Phi_p}(S; \lambda) \) and \( L_{\Phi_p}(S; \lambda) \hookrightarrow L_0(S; \lambda) \) is continuous, where in the present case \( L_0(S; \lambda) \) is equipped with the topology of convergence in \( \lambda \) measure on every set of finite \( \lambda \)-measure. Moreover, \( \|f_n\|_{\Phi_p} \to 0 \iff \int_S \Phi_p(|f(s)|, s) \lambda(ds) \to 0. \)

We remark that the above definitions and results hold both in the case \( \lambda \) is atomless and when it is not (see Chapter III of [18] and Musielak’s book [14]).

**Theorem 6.3.** Let \( 0 \leq p \leq q \) and \( \Phi_p \) defined as in (12). Then

\[ \left\{ f : f \text{ is } \Lambda\text{-integrable and } \mathbb{E} \left[ \left| \int_S f d\Lambda \right|^p \right] < \infty \right\} \subset L_{\Phi_p}(S; \lambda) \]

and the linear mapping

\[ L_{\Phi_p}(S; \lambda) \ni f \mapsto \int_S f d\Lambda \in L_p(\Omega; \mathbb{P}) \]

is continuous.

**Remark 6.4.** When \( p = 0 \) the statement becomes \( \{ f : f \text{ is } \Lambda\text{-integrable} \} \subset L_{\Phi_0}(S; \lambda). \)

**Proof.** It follows from the same arguments of the proof of Theorem 3.3 in [18] adapted to our framework.

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### 7 Spectral representation of QID processes

In this section we are going to introduce QID stochastic processes, give a Lévy-Khintchine representation of discrete parameters QID processes and present a spectral representation of them.

We start with the definition of QID processes.

**Definition 7.1 (QID processes).** Let \( T \) be an arbitrary index set. A stochastic process \( X = \{X_t; t \in T\} \) is said to be a QID process if and only if for every finite set of indices \( t_1, \ldots, t_k \) in the index set \( T \)

\[ X_{t_1, \ldots, t_k} = (X_{t_1}, \ldots, X_{t_k}) \]

is a multivariate QID random variable.

The existence of QID processes is ensured by the Kolmogorov extension theorem. It is possible to see that an equivalent definition is the following: a stochastic process \( X \) is QID if and only if there exists two ID processes \( Y \) and \( Z \) such that \( X + Y \overset{d}{=} Z \) with \( Y \) independent of \( X \).

We call \( Y \) and \( Z \) the generating ID process of \( X \). Notice that the equivalence of the two definitions is ensured by the Kolmogorov extension theorem and by Proposition 3.2 in [9]. From the above definition, it is clear that the class of QID processes is strictly larger than the class of ID processes.

For the rest of this section we will consider the index set \( T \) to be countable. Let \( B_{l^2} \) be the Borel \( \sigma \)-algebra on \( l^2 \). We note that since \( l^2 \) is separable the Borel and cylindrical \( \sigma \)-algebras coincide (see page 38 in [12]). Then, we have the following definition.
Definition 7.2 (quasi-Lévy type measure on \((l_2, \mathcal{B}^{l_2})\)). Let \(\mathcal{B}^{l_2}_r := \{B \in \mathcal{B}^{l_2} \mid B \cap \{x \in l_2 : \|x\| < r\} = \emptyset\}\) and \(\mathcal{B}^{l_2}_0 := \bigcup_{r > 0} \mathcal{B}^{l_2}_r\). Let \(\nu : \mathcal{B}^{l_2}_0 \to \mathbb{R}\) be a set function such that \(\nu|_{\mathcal{B}^{l_2}_r}\) is a finite signed measure for each \(r > 0\) and denote the total variation, positive and negative part of \(\nu|_{\mathcal{B}^{l_2}_r}\) by \(|\nu|, \nu^+|_{\mathcal{B}^{l_2}_r}\) and \(\nu^-|_{\mathcal{B}^{l_2}_r}\) respectively. Then the total variation \(|\nu|\), the positive part \(\nu^+\) and the negative part \(\nu^-\) of \(\nu\) are defined to be the unique measures on \((l_2, \mathcal{B}^{l_2})\) satisfying

\[|\nu(\{0\})| = \nu^+(\{0\}) = \nu^-(\{0\}) = 0\]

and

\[|\nu|(A) = |\nu| \{s \in l_2 \mid \nu^+ \{s\} > 0\}, \nu^+(A) = \nu^+ \{s \in l_2 \mid \nu^+ \{s\} > 0\}, \nu^-(A) = \nu^- \{s \in l_2 \mid \nu^- \{s\} > 0\},\]

for \(A \in \mathcal{B}^{l_2}_r\), for some \(r > 0\).

A quasi-Lévy type measure on \((l_2, \mathcal{B}^{l_2})\) is a function \(\nu : \mathcal{B}^{l_2}_0 \to \mathbb{R}\) such that \(\nu|_{\mathcal{B}^{l_2}_r}\) is a finite signed measure for each \(r > 0\) and \(\int_{l_2} (1 \land \|x\|^2) |\nu|(dx) < \infty\).

Remark 7.3. Notice that \(\mathcal{B}^{l_2}_r\) is a \(\sigma\)-algebra. Indeed, since the space \(Y_r := \{x \in l_2 : \|x\| \geq r\}\) is a non-empty subspace of \(l_2\) then we have \(Z_r := \{B \subseteq Y_r \mid B \in \mathcal{B}^{l_2}\} = \mathcal{B}^{l_2}_r\). Observe now that \(Y_r\) is measurable (namely \(Y_r \in \mathcal{B}^{l_2}\)) because \(l_2\) is a complete separable metric space and the norm is a continuous, then \(Z_r\) is the restriction of \(\mathcal{B}^{l_2}\) on \(Y_r\). Hence, \(\mathcal{B}^{l_2}_r\) is a \(\sigma\)-algebra and the argument applies to any \(r > 0\).

If \(\nu : \mathcal{B}^{l_2}_0 \to \mathbb{R}\) is such that \(\nu|_{\mathcal{B}^{l_2}_r}\) is a finite signed measure for each \(r > 0\), then we have that \(|\nu|_{\mathcal{B}^{l_2}_r}\{A\} = |\nu|_{\mathcal{B}^{l_2}_r}\{A\}\) for every \(A \in \mathcal{B}^{l_2}_r\) with \(0 < s \leq r\) and the same applies to \(\nu^+|_{\mathcal{B}^{l_2}_r}\) and \(\nu^-|_{\mathcal{B}^{l_2}_r}\).

Moreover, since we are in the countable framework (the index set considered \(T\) is countable since \(l_2 \subseteq \mathbb{R}^\infty\) then \(|\nu|, \nu^+\) and \(\nu^-\) are well defined and the condition \(|\nu|\{\{0\}\} = \nu^+\{\{0\}\} = \nu^-\{\{0\}\} = 0\) ensures their uniqueness (see Section 2.1 in [20] for further details).

We have the following representation of a certain class of QID processes.

Theorem 7.4. Let \(X\) be a discrete parameter QID process with values in \(l_2\) such that there exist two generating ID processes with values in \(l_2\). Then there exists an unique triplet \((\omega_0, \mathcal{K}, \nu)\) consisting of \(\omega_0 \in l_2\), a non-negative definite function operator \(\mathcal{K} : l_2 \to \mathbb{R}\) and a quasi-Lévy type measure \(\nu\) on \((l_2, \mathcal{B}^{l_2})\) such that for every fixed \(l, k \in \mathbb{N}\) with \(l \leq k\) and \(\theta_1, \ldots, \theta_k\) real, we have

\[
\mathbb{E} \left[ \exp \left( i \sum_{i \in l} \theta X_i \right) \right] = \exp \left( i \langle \omega_0, y \rangle - \frac{1}{2} \langle y, \mathcal{K} y \rangle + \int_{l_2} \left( e^{i(y,x)} - 1 - i \langle \tau(x), y \rangle \right) \nu(dx) \right)
\]

where \(y = (0, \ldots, 0, \theta_1, \ldots, \theta_k, 0, 0, \ldots)\).

Proof. Let \(\nu^{(1)}\) and \(\nu^{(2)}\) be the Lévy measures of the \(l_2\) valued ID processes \(Z\) and \(Y\), respectively.

Then, we have

\[
\mathbb{E} \left[ \exp \left( i \sum_{i \in l} \theta X_i \right) \right] = \mathbb{E} \left[ \exp \left( i \sum_{i \in l} \theta Z_i \right) \right] \mathbb{E} \left[ \exp \left( i \sum_{i \in l} \theta Y_i \right) \right]^{-1} = \exp \left( i \langle \omega_0, y \rangle - \frac{1}{2} \langle y, \mathcal{K} y \rangle + \int_{l_2} \left( e^{i(y,x)} - 1 - i \langle \tau(x), y \rangle \right) \nu^{(1)}(dx) \right) - \int_{l_2} \left( e^{i(y,x)} - 1 - i \langle \tau(x), y \rangle \right) \nu^{(2)}(dx) \right)
\]

Define \(\nu|_{\mathcal{B}^{l_2}_r} = \nu^{(1)}|_{\mathcal{B}^{l_2}_r} - \nu^{(2)}|_{\mathcal{B}^{l_2}_r}\) for any \(r > 0\). This ensures that \(\nu|_{\mathcal{B}^{l_2}_r}\) is a finite signed measure for each \(r > 0\), hence \(\nu^+, \nu^-\) and \(|\nu|\) are uniquely determined, which implies that \(\nu\) is uniquely determined.
by the difference \( \nu^{(1)} - \nu^{(2)} \). Moreover, since this difference is unique then \( \nu \) is unique, indeed the absence of uniqueness of \( \nu^{(1)} - \nu^{(2)} \) will violate the condition \( X + Y \overset{d}{=} Z \) due to the uniqueness of the Lévy-Khintchine representation on \( l_2 \) (see Theorem 4.10 Chapter 4 in [16]). Observe also that from the fact that \( \nu^{(1)} \) and \( \nu^{(2)} \) are Lévy measures then \( \nu \) is a quasi-Lévy type measures. Hence, we obtain the expression (13). Finally, it directly follows that \( z_0 \) and \( K \) are uniquely determined.

In Theorem 7.4 we need that there exist two generating ID processes with values in \( l_2 \). However, it might be possible that this condition is not needed. Thus, we have the following open question.

**Open Question 7.5.** Let \( X \) be any \( l_2 \) valued random variable. Does it have a Lévy-Khintchine formulation?

### 7.1 Spectral representation of discrete parameters QID processes

The following framework is the QID version of the one presented in Chapter VI in [18]. We denote \( \mathbb{R}^+ \) for \( (0, \infty) \), the space \( l_2 \) as the space of square summable sequences, \( \partial U \) the boundary of the unit ball in \( l_2 \) and we define the map \( \Psi : \partial U \times \mathbb{R}^+ \rightarrow l_2 \setminus \{0\} \) such that \( \Psi(u, x) = xu \). Given a quasi-Lévy measure \( M \) on \( l_2 \), we define the finite signed measure \( \Gamma \) on \( B(\partial U) \times B(\mathbb{R}^+) \) by

\[
\Gamma = M_0 \circ \Psi, \quad \text{where} \quad M_0(\text{d}z) = 1 \wedge \|z\|^2 M(\text{d}z).
\]

Notice that the map \( \Psi \) is surjective and that for every \( A \in B(\partial U) \) and \( B \in B(\mathbb{R}^+) \) we have

\[
\Gamma(A \times B) = M_0 \circ \Psi(A \times B) = \int_{\{z = xu : u \in A, x \in B\}} 1 \wedge \|z\|^2 M(\text{d}z).
\]

Moreover, we associate the finite measure \( \xi \) defined in Section 5.3 to the present framework, that is, for every \( A \in B(\partial U) \),

\[
\xi(A) = \sup_{I : A_i \in A} \sum_{i \in I} |\Gamma(A_i \times B_i)|.
\]

Then, using Theorem 5.14 we can represent \( \Gamma \) as

\[
\Gamma(C) = \int_{\partial U} \int_{\mathbb{R}^+} I_C(u, x) q(u, dx) \xi(\text{d}u), \tag{14}
\]

where \( q : \partial U \times B(\mathbb{R}^+) \rightarrow [-1, 1] \) satisfies the properties stated in Theorem 5.14. Now, we define the signed measure \( \rho(u, \cdot) \) on \( B(\mathbb{R}^+) \), \( F \) on \( B(\partial U \times \mathbb{R}^+) \) and \( F_A(\cdot) \) on \( B(\mathbb{R}^+) \) by

\[
\rho(u, dx) = (1 + x^2)^{-1} q(u, dx), \quad \text{for every } u \in \partial U, \tag{15}
\]

\[
F(C) = \int_{\partial U} \int_{\mathbb{R}^+} I_C(u, x) \rho(u, dx) \xi(\text{d}u), \quad \text{for every } C \in B(\partial U \times \mathbb{R}^+), \tag{16}
\]

and \( F_A(\cdot) = F(A \times \cdot) \), for every \( A \in B(\partial U) \). Further, we extend the signed measure \( \rho(u, \cdot) \) and \( F_A(\cdot) \) to \( \mathbb{R}_0 \) (or to \( \mathbb{R} \)) by setting \( \rho(u, B) = F_A(B) = 0 \) for every \( B \subset \mathbb{R}_0 \setminus \mathbb{R}^+ \) (or \( B \subset \mathbb{R}_0 \setminus \mathbb{R} \)). Notice that \( \rho(u, \cdot) \) and \( F_A(\cdot) \) are quasi type Lévy measure on \( \mathbb{R} \). In particular, following similar arguments as in Lemma 5.22 we have

\[
\int_{\mathbb{R}} 1 + x^2 |\rho|(u, dx) = \int_{\mathbb{R}^+} 1 + x^2 |\rho|(u, dx) \leq 2 \tag{17}
\]
and
\[
\int \mathbb{R} 1 \wedge x^2 |F_A|(dx) = \int_{\mathbb{R}^+} 1 \wedge x^2 |F_A|(dx) \leq \int_{A} \int_{\mathbb{R}^+} 1 \wedge x^2 |\rho_i(u, dx)\xi(du) \leq 2\xi(A) \quad (18)
\]

We are now ready to state the first result of this section, which provides a useful representation of the measure \( M \).

**Proposition 7.6.** Let \( M \) be a quasi Lévy measure on \( l_2 \); then \( F \) is a unique measure on \( \mathcal{B}(\partial U \times \mathbb{R}^+) \) satisfying
\[
M = F \circ \Psi^{-1}.
\]
In particular, for every \( D \in \mathcal{B}(l_2 \setminus \{0\}) \) we have
\[
M(D) = \int_{\partial U} \int_{\mathbb{R}^+} I_D(xu)\rho(u, dx)\xi(du);
\]
more generally
\[
\int_{l_2 \setminus \{0\}} f dM = \int_{\partial U} \int_{\mathbb{R}^+} f(xu)\rho(u, dx)\xi(du)
\]
whenever \( \int_{l_2 \setminus \{0\}} |f| dM < \infty \), where \( f \) might even take complex values.

**Proof.** The first two equations follows by (14), (15), (16), (17), (18), the fact that \( u \in \partial U \) (and hence \( \|u\| = 1 \)) and by the Fubini’s theorem applied to signed measures (see [2] page 186). More explicitly, using standard argument for the push-forward measures we have the following. Consider any set \( D \in \mathcal{B}(l_2 \setminus \{0\}) \) then define \( \Psi^{-1}(D) := \{(u, x) \in \partial U \times \mathbb{R}_+ : \Psi(u, x) \in D\} \) and notice that \( \Psi^{-1}(D) \in \mathcal{B}(\partial U \times \mathbb{R}_+) \). Then, we have
\[
\Gamma(\Psi^{-1}(dz)) = M_0(dz) \Rightarrow M_0(dz) = \int_{\partial U} \int_{\mathbb{R}^+} I_{\Psi^{-1}(dz)}(u, x)\Gamma(du, dx)
\]
\[
= \int_{\partial U} \int_{\mathbb{R}^+} I_{dz}(xu)\Gamma(du, dx) = \int_{\partial U} \int_{\mathbb{R}^+} I_{dz}(xu)q(u, dx)\xi(du)
\]
\[
\Rightarrow M(dz) = (1 \wedge \|z\|^2)^{-1} \int_{\partial U} \int_{\mathbb{R}^+} I_{dz}(xu)q(u, dx)\xi(du)
\]
\[
\Rightarrow M(D) = \int_D (1 \wedge \|z\|^2)^{-1} \int_{\partial U} \int_{\mathbb{R}^+} I_{dz}(xu)q(u, dx)\xi(du)dz
\]
\[
= \int_{\partial U} \int_{\mathbb{R}^+} \int_D (1 \wedge \|z\|^2)^{-1} I_{dz}(xu)dzq(u, dx)\xi(du)
\]
\[
= \int_{\partial U} \int_{\mathbb{R}^+} (1 \wedge \|xu\|^2)^{-1} I_D(xu)q(u, dx)\xi(du)
\]
\[
= \int_{\partial U} \int_{\mathbb{R}^+} (1 \wedge |x|^2)^{-1} I_D(xu)q(u, dx)\xi(du)
\]
\[
= \int_{\partial U} \int_{\mathbb{R}^+} I_D(xu)\rho(u, dx)\xi(du).
\]
The last equations in the statement follows by similar arguments. \( \square \)
Before presenting the main result of this section we need some preliminaries. Let $X = \{X_n; n = 1, 2, \ldots\}$ be a discrete parameters QID process with generating ID processes $V$ and $Z$; let $b_n > 0$ be such that $\{b_n X_n\}, \{b_n V_n\}, \{b_n Z_n\} \in l_2$ almost surely. Let $Y = \{b_n X_n\} \in l_2$ and let $\mu = \mathcal{L}(Y)$ be the QID law of $Y$ on $l_2$ with c.t. $(z_0, K, M)$, where $z_0 \in l_2$, $K$ is the covariance operator and $M$ is the quasi Lévy measure of $\mu$. Now, for every $y \in l_2$,

$$K(y) = \sum_{j=1}^{\infty} \beta_j \langle e_j, y \rangle e_j,$$

where $\beta_j \geq 0$, $\sum_{j=1}^{\infty} \beta_j < \infty$ and $\{e_j\}$ is an orthonormal set in $l_2$. Moreover, let

$$\nu_0 = \begin{cases} \|z_0\| \delta(z_0/\|z_0\|) & \text{if } z_0 \neq 0, \\ 0 & \text{if } z_0 = 0, \end{cases} \quad \text{and} \quad \nu_1 = \sum_{j=1}^{\infty} \beta_j \delta(e_j)$$

to be two finite measures on $B(\partial U)$. Now, recall the measure $\xi$ and the signed measure $F$ associated to $M$ and consider the following definition.

**Definition 7.7.** Let $X$, $\nu_0$, $\nu_1$ and $F$ be as above, then the QID random measure (call it $\Lambda$) on $B(\partial U)$ with parameters $(\nu_0, \nu_1, F)$ will be called the associated QID random measure of $X$.

Observe that the control measure $\lambda$ of $\Lambda$ is given by $\lambda(A) = \nu_0(A) + \nu_1(A) + \xi(A)$ for every $A \in B(\partial U)$. Moreover, denote by $\pi_j$ the $j$-th coordinate projection in $l_2$.

For the sake of completeness, we report here Lemma 4.7 in [18].

**Lemma 7.8** (Lemma 4.7 in [18]). Let $a_1, a_2, \ldots, a_n$ be $n$-real numbers, then

$$\int_{\partial U} \sum_{j=1}^{n} a_j \pi_j(z) \nu_0(dz) = \sum_{j=1}^{n} a_j \pi_j(z_0),$$

$$\int_{\partial U} \left( \sum_{j=1}^{n} a_j \pi_j(z) \right)^2 \nu_1(dz) = \sum_{k=1}^{\infty} \beta_k \left( \sum_{j=1}^{n} a_j \pi_j(e_k) \right)^2 = \langle K(y), y \rangle,$$  \hspace{1cm} (19)

where $y = (a_1, \ldots, a_n, 0, 0, \ldots)$ and

$$\int_{\partial U} \int_{\mathbb{R}^+} (1 \wedge \pi_n^2(u)x^2) \rho(u, dx) \xi(du) = \int_{l_2} (1 \wedge \pi_n^2(z)) M(dz)$$

In addition to the previous Lemma we have the following easy result.

**Lemma 7.9.** Using the same notation as in the previous statement we have

$$\int_{\partial U} \int_{\mathbb{R}^+} (1 \wedge \pi_n^2(u)x^2) |\rho|(u, dx) \xi(du) = \int_{l_2} (1 \wedge \pi_n^2(z)) |M| (dz)$$  \hspace{1cm} (20)

**Proof.** It follows from the definition of $\Gamma$ and the arguments of Proposition 7.6 applied to the Jordan decomposition of $M$. Indeed, $M$ and $M_0$ have the same Hahn decomposition (in essential terms) and this is also true for $\Gamma$ and $F$. \hfill \Box

We can now finally present the main result of this section.
Theorem 7.10. Let $X = \{X_n\}$ be a QID process and let $\Lambda$ be its associated QID random measure with parameters $(\nu_0, \nu_1, F)$ and control measure $\lambda$. Let $f_n = b_n^{-1}\pi_n$; then $f_n$’s are integrable and

$$\{X_n\} \overset{d}{=} \left\{ \int_{\partial U} f_n d\Lambda \right\}.$$

Proof. We prove first that the $\pi_n$’s are $\Lambda$-integrable, which means that we need to satisfy conditions (i), (ii) and (iii) of Theorem 5.26. However, by (19) and (20) we have

$$\int_{\partial U} \int_{\mathbb{R}^+} (1 \wedge \pi_n^2(u)) M|(dz) = \int_{\partial U} \int_{\mathbb{R}^+} (1 \wedge \pi_n^2(u)x^2) \rho|[(u, dx)](du)$$

$$\leq \int_{\partial U} \int_{\mathbb{R}^+} (1 \wedge x^2) \rho|[(u, dx)](du) < \infty$$

using the fact that $\rho$ is a quasi Lévy measure and $|\pi_n(u)| \leq 1$. Then, (ii) and (iii) are satisfied.

Now, observe that for $x < 1$ we have

$$|\tau(\pi_n(u)x) - \pi_n(u)\tau(x)| = |\pi_n(u)x - \pi_n(u)x| = 0$$

and otherwise

$$|\tau(\pi_n(u)x) - \pi_n(u)\tau(x)| \leq 2.$$

Therefore, since $\xi$ is finite we have that

$$\int_{\partial U} \int_{\mathbb{R}^+} |\tau(\pi_n(u)x) - \pi_n(u)\tau(x)| \rho|[(u, dx)](du) \leq \int_{\partial U} \int_{\mathbb{R}^+} \rho|[(u, dx)](du)$$

$$\leq \int_{\partial U} \int_{\mathbb{R}^+} (1 \wedge x^2) \rho|[(u, dx)](du)$$

Since $\pi_n$’s are integrable then we get immediately that $f_n$’s are integrable.

The rest of the proof follows directly from the same arguments used in the proof of Theorem 4.9 in [18].

7.2 Lévy-Khintchine representation of QID processes

In this section we are going to investigate the representation of QID processes. For the first result we will assume that the quasi-Lévy measure of QID processes, whose definition is going to be provided below, is a signed measure. This situation happens when one of the two generating ID processes has a finite Lévy measure (see Definition 2.1 in [20]). This is because in that case $\nu = \nu_1 - \nu_2$ is a signed measure. On the other hand, the second result concerns general QID processes.

We will use the general framework introduced in [20]. Let $T$ be an arbitrary (possibly uncountable) index set, $\mathbb{R}^T$ denote the space of all functions $x : T \to \mathbb{R}$ and $\mathcal{B}^T$ be its cylindrical product $\sigma$-algebra. Given an underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$, for a stochastic process $X = (X_t)_{t \in T}$ we define its law $\mathcal{L}(X)$ as a probability measure on $(\mathbb{R}^T, \mathcal{B}^T)$ such that

$$\mathcal{L}(X)(A) = \mathbb{P}\{\omega \in \Omega : (X_t(\omega))_{t \in T} \in A\}, \quad A \in \mathcal{B}^T$$

We denote by $x_S$ the restriction of $x$ to $S \subset T$ and by $0_S$ the origin of $\mathbb{R}^S$, which depending on the setting is a point or a one-point set. Further, we let $\hat{T}$ to be defined as $\hat{T} := \{I \subset T : 0 < \text{Card}(I) < \infty\}$, let $\pi_S : \mathbb{R}^T \to \mathbb{R}^S$ be the projection from $\mathbb{R}^T$ onto $\mathbb{R}^S$ (namely $\pi_S(x) = x_S$) and let
Definition 7.11. A signed measure $\nu$ on $(\mathbb{R}^T, \mathcal{B}^T)$ is said to be a quasi-Lévy type measure if the following two conditions hold

(QL1) for every $t \in T$ we have $\int_{\mathbb{R}^T} |x(t)|^2 \wedge 1|\nu|(dx) < \infty$,

(QL2) for every $A \in \mathcal{B}^T$ we have $|\nu(A)| = \nu^+(A \setminus 0_T) + \nu^-(A \setminus 0_T)$, where $\nu^+$ and $\nu^-$ are the inner measures of the Jordan decomposition of $\nu$.

We can present now the first main result of this section, namely the Lévy-Khintchine formula of QID processes with quasi-Lévy measure being a signed measure.

Theorem 7.12. Let $T$ be an arbitrary index set and let $X = (X_t)_{t \in T}$ be a QID process such that there exists two of generating ID process with one having finite Lévy measure. Then there exists a unique triplet $(\Sigma, \nu, b)$ where $\Sigma$ is a non-negative definite function on $T \times T$, $\nu$ is a quasi-Lévy type measure on $(\mathbb{R}^T, \mathcal{B}^T)$ and a function $b \in \mathbb{R}^T$ such that for every $I \in \mathcal{T}$ and $\theta \in \mathbb{R}^I$

$$\mathcal{L}(X_I) = \exp \left( i\langle \theta, b_I \rangle - \frac{1}{2} \langle \theta, \Sigma_I \theta \rangle + \int_{\mathbb{R}^T} (e^{i\theta \cdot x_I}) - 1 - i\langle \theta, [x_I] \rangle \nu(dx) \right),$$

(21)

where $\Sigma_I$ is the restriction of $\Sigma$ to $I \times I$. $(\Sigma, \nu, b)$ is called the generating triplet of $X$ and $\nu$ the quasi-Lévy measure of $X$. Therefore, for any consistent system of quasi-Lévy measures $\{\nu_I : I \in \mathcal{T}\}$ there exists a unique quasi-Lévy measure $\nu$ on $(\mathbb{R}^T, \mathcal{B}^T)$ such that

$$\nu^+ \circ \pi_I^{-1} = \nu^+_I \quad \text{and} \quad \nu^- \circ \pi_I^{-1} = \nu^-_I \quad \text{on } \mathcal{B}_0^I, \; I \in \mathcal{T}. \quad (22)$$

where “$+$” and “$-$” stands for the positive and negative Jordan decomposition.

Furthermore, if $\rho$ and $\eta$ are two measures on $(\mathbb{R}^T, \mathcal{B}^T)$ such that $\rho \circ \pi_I^{-1} = \nu^+_I$ and $\eta \circ \pi_I^{-1} = \nu^-_I$ on $\mathcal{B}_0^I$ for every $I \in \mathcal{T}$, then $\nu^+(A) = \rho_*(A \setminus 0_T) \leq \rho(A)$ and $\nu^-(A) = \eta_*(A \setminus 0_T) \leq \eta(A)$ for all $A \in \mathcal{B}^T$. Hence, $\nu$ has the smallest total variation among the measures satisfying (22).

Proof. Let $X^1$ and $X^2$ the ID processes generating $X$, with one having finite Lévy measure, and denote by $\nu^{(1)}$ and $\nu^{(2)}$ the respective Lévy measures on $(\mathbb{R}^T, \mathcal{B}^T)$. Let $\nu^+_I$ and $\nu^-_I$ be Jordan decomposition of $\nu^{(1)}_I - \nu^{(2)}_I$ for every $I \in \mathcal{T}$. Then, by the uniqueness of the characteristic triplet of QID distributions $\nu^+_I$ and $\nu^-_I$ form two consistent systems of Lévy measures. Then, by Theorem 2.8 in [20] we have that there exists two unique Lévy measures on $(\mathbb{R}^T, \mathcal{B}^T)$, call them $\nu^+$ and $\nu^-$, such that (22) holds. Again by Theorem 2.8 in [20] we have that if $\rho$ and $\eta$ are two measures such that $\rho \circ \pi_i^{-1} = \nu^+_i$ and $\eta \circ \pi_i^{-1} = \nu^-_i$ on $\mathcal{B}_0^I$ for every $I \in \mathcal{T}$ then $\nu^+ \leq \rho$ and $\nu^- \leq \eta$.

Now notice that either $\nu^+$ or $\nu^-$ is finite since either $\nu^{(1)}$ or $\nu^{(2)}$ is finite. This is because $\nu^+_I \leq \nu^{(1)}_I$ and $\nu^-_I \leq \nu^{(1)}_I$ for every $I \in \mathcal{T}$. Then the measure $\nu := \nu^+ - \nu^-$ is a well defined measure on $(\mathbb{R}^T, \mathcal{B}^T)$. Furthermore, since $\nu^+$ and $\nu^-$ are orthogonal then they constitute the Jordan decomposition of $\nu$. Indeed, assume that $\nu$ has $\nu^+$ and $\nu^-$ as Jordan decomposition. Then, we have that $\nu^+ \leq \nu^+$ and $\nu^+ \circ \pi_i^{-1} = \nu^+_i$, but then by the property of $\nu^+$ we have that $\nu^+ \leq \nu^+$ which implies that $\nu^+ = \nu^+$. Hence, $|\nu| \leq |\rho|$, where $\rho$ is such that $\rho \circ \pi_i^{-1} = \nu^+_i$ and $\rho \circ \pi_i^{-1} = \nu^-_i$ on $\mathcal{B}_0^I$ for every $I \in \mathcal{T}$.

Then, it is straightforward to see that $\nu$ satisfies (QL1) and (QL2) and, thus, it is a quasi-Lévy type measure on $(\mathbb{R}^T, \mathcal{B}^T)$. 

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Moreover, we have
\[
\hat{L}(X_I) = \frac{\hat{L}(X_I^1)}{\hat{L}(X_I^2)} = \exp \left( i \langle \theta, b_I \rangle - \frac{1}{2} \langle \theta, \Sigma_I \theta \rangle \right) \\
+ \int_{\mathbb{R}^T} (e^{i(\theta, x_i)} - 1 - i \langle \theta, [x_i] \rangle) \nu^{(1)}(dx) - \int_{\mathbb{R}^T} (e^{i(\theta, x_i)} - 1 - i \langle \theta, [x_i] \rangle) \nu^{(2)}(dx)
\]
\[
= \exp \left( i \langle \theta, b_I \rangle - \frac{1}{2} \langle \theta, \Sigma_I \theta \rangle \right) \\
+ \int_{\mathbb{R}^T} (e^{i(\theta, x_i)} - 1 - i \langle \theta, [x_i] \rangle) \nu^+(dx) - \int_{\mathbb{R}^T} (e^{i(\theta, x_i)} - 1 - i \langle \theta, [x_i] \rangle) \nu^-(dx),
\]
\[
= \exp \left( i \langle \theta, b_I \rangle - \frac{1}{2} \langle \theta, \Sigma_I \theta \rangle + \int_{\mathbb{R}^T} (e^{i(\theta, x_i)} - 1 - i \langle \theta, [x_i] \rangle) \nu(dx) \right).
\]

From the arguments used in the previous result we have the following Lévy-Khintchine representation of general QID processes, namely the QID equivalent of Theorem 2.8 in [20].

**Theorem 7.13.** Let $T$ be an arbitrary index set and let $X = (X_t)_{t \in T}$ be a QID process. Then there exists a unique quadruplet $(\Sigma, \nu^+, \nu^-, b)$ where $\Sigma$ and $b$ are as in the Theorem 7.12 and $\nu^+$ and $\nu^-$ are Lévy measure on $(\mathbb{R}^T, \mathcal{B}^T)$ such that for every $I \in \hat{T}$ and $\theta \in \mathbb{R}^I$

\[
\hat{L}(X_I) = \exp \left( i \langle \theta, b_I \rangle - \frac{1}{2} \langle \theta, \Sigma_I \theta \rangle \right) \\
+ \int_{\mathbb{R}^T} (e^{i(\theta, x_i)} - 1 - i \langle \theta, [x_i] \rangle) \nu^+(dx) - \int_{\mathbb{R}^T} (e^{i(\theta, x_i)} - 1 - i \langle \theta, [x_i] \rangle) \nu^-(dx),
\]

(23)

$(\Sigma, \nu^+, \nu^-, b)$ is called the generating quadruplet of $X$. In particular, we have $\nu^+ \circ \pi_I^{-1} = \nu_I^+$ and $\nu^- \circ \pi_I^{-1} = \nu_I^-$ on $\mathcal{B}_0^I$, $I \in \hat{T}$.

where $\nu_I^+$ and $\nu_I^-$ are the positive and negative part of the quasi Lévy measure $\nu_I$ (see Definition 2.4 and Remark 2.4 in [13]).

Furthermore, if $\rho$ and $\eta$ are two measures on $(\mathbb{R}^T, \mathcal{B}^T)$ such that $\rho \circ \pi_I^{-1} = \nu^+_I$ and $\eta \circ \pi_I^{-1} = \nu^-_I$ on $\mathcal{B}_0^I$ for every $I \in \hat{T}$, then $\nu^+(A) = \rho_s(A \setminus 0_T) \leq \rho(A)$ and $\nu^+(A) = \eta_s(A \setminus 0_T) \leq \eta(A)$ for all $A \in \mathcal{B}^T$.

**Proof.** It follows from the same arguments used in the proof of Theorem 7.12. □

Since $\nu := \nu^+ - \nu^-$ might not be defined (because we might have $\infty - \infty$) we cannot reduce the generating quadruplet to a generating triplet unless we provide an extension of the definition of the quasi-Lévy type measure as done in Definition 2.4 or Definition 7.2 but for the measurable space $(\mathbb{R}^T, \mathcal{B}^T)$. Thus, we end this section with the following open question:

**Open Question 7.14.** Is it possible to obtain a generating triplet instead of a generating quadruplet? In other words, how do we extend the definition of quasi-Lévy type measure on $(\mathbb{R}^T, \mathcal{B}^T)$ when the quasi-Lévy type measure is not a signed measure as in the case of Definition 2.4 or Definition 7.2?
7.3 Further results and examples of QID processes

We move now to the presentation of additional results on QID processes and of some examples. Recall that a process $X$ is QID if $X + X' \overset{d}{=} X^{(1)}$ where $X^{(2)}$ and $X^{(1)}$ are ID processes with $X^{2}$ independent of $X$. From this we have many examples. However, let us start with some results.

The first idea that might come to mind is: If $X^{(2)}$ and $X^{(1)}$ are Lévy processes is $X$ a QID process without being a Lévy process? The answer is negative, as shown in the following result.

**Proposition 7.15.** Let $X^{(1)}$ and $X^{(2)}$ be two Lévy processes with generating triplets $(a^{(1)}, \sigma^{(1)}, \nu^{(1)})$ and $(a^{(2)}, \sigma^{(2)}, \nu^{(2)})$. Let $X$ be a stochastic process independent of $X^{(2)}$. The equality $X_t + X^{(2)}_t \overset{d}{=} X^{(1)}_t$ for every $t \geq 0$ holds only if $\sigma^{(1)} \geq \sigma^{(2)}$ and $\nu^{(1)} \geq \nu^{(2)}$. In this case $X$ might be a Lévy process.

**Proof.** If $\sigma^{(1)} < \sigma^{(2)}$ then $X_1$ will not be a random variable. If $\nu^{(1)} < \nu^{(2)}$ then $X_t$ is a QID r.v. for every $t \geq 0$ without being a Lévy. But then, for every $n$ we would have that $X_1 \overset{d}{=} X^{(1)}_1 \overset{d}{=} X^{(2)}_1$ is a QID r.v. hence for every $n \in \mathbb{N}$ we would have that $\hat{L}(X_1) = (\hat{L}(X_1))^n$, which implies that $X_1$ thus a contradiction. However, if $\sigma^{(1)} \geq \sigma^{(2)}$ and $\nu^{(1)} \geq \nu^{(2)}$ then $X_t$ is a ID r.v. for every $t \geq 0$, hence, $X$ might potentially be a Lévy process.

However, we have the following positive general result. We use the representation function $c(\cdot)$ (see [13]) which is a function $c : \mathbb{R} \to \mathbb{R}$ which is bounded, Borel measurable and satisfies $\lim_{x \to 0} (c(x) - x)/x^2 = 0$.

**Proposition 7.16.** Let $X_0$ be any QID random variable and let $(a, \sigma, \nu)$ be its characteristic triplet. Then

$$\hat{L}(X_t) := \exp \left( i \theta h(t) + i \theta a_t f(t) - \frac{1}{2} \sigma_t^2 f(t)^2 \theta^2 + \int_\mathbb{R} e^{i \theta x} - 1 - i \theta c(x) \nu_t(f(t)^{-1} dx) \right)$$

is the characteristic function of a QID random variable for every $t \geq 0$, where $h(\cdot), a_\cdot, f(\cdot)$ and $\sigma$ are real valued functions with $f(t) \neq 0$ and $\sigma_t \geq \sigma$ for all $t \geq 0$, $\nu_t(\cdot)$ is a set function such that $\nu_t(A) \geq \nu(A)$ for every $t \geq 0$ and $A \in B_0(\mathbb{R})$, and $c(\cdot)$ is any representation function.

Moreover, let $k \in \mathbb{N}$ and let $X^{(1)}_0, \ldots, X^{(k)}_0$ be QID random variables. Let $h^{(j)}(\cdot), a^{(j)}_\cdot, f^{(j)}(\cdot), \sigma^{(j)}$ and $\nu^{(j)}_t(\cdot)$ satisfy the above respective properties and similarly define $\hat{L}(X^{(j)}_t)$. Then

$$\hat{L}(Y) := \hat{L}(X^{(1)}_{t_1}) \cdots \hat{L}(X^{(k)}_{t_k})$$

is the characteristic function of a QID random variable for every $t_1, \ldots, t_k \geq 0$.

**Proof.** Let for the moment $f(t) = 1$ and $h(t) = 0$ for every $t \geq 0$. Since for any QID distribution $\mu$ with c.t. $(\gamma, b, \eta)$ if there exists $b' \geq b$ and $\eta'(A) \geq \eta(A)$ for every $A \in B_0(B)$ then the c.t. $(\gamma' - \gamma, b' - b, \eta' - \eta)$, where $\gamma' \in \mathbb{R}$, is the c.t. of a ID distribution, call it $\tilde{\mu}$, (see Remark 2.6 point (a) in [13]) hence since the class of QID distributions is closed under convolution then we have that $\mu' = \mu * \tilde{\mu}$ is QID with c.t. $(\gamma', b', \eta')$ (see Remark 2.6 point (c) in [13]). Thus, we have that

$$\exp \left( a_t \theta - \frac{1}{2} \sigma_t^2 \theta^2 + \int_\mathbb{R} e^{i \theta x} - 1 - i \theta c(x) \nu_t(dx) \right)$$

is the c.f. of a QID distribution. Now, since the class of QID distributions is closed under shifts and dilatation (see Remark 2.6 point (b) in [13]) then

$$\exp \left( i \theta \left( \tilde{h}(t) + \int_\mathbb{R} c(mx) - mc(x) \nu_t(dx) + a_t f(t) \right) - \frac{1}{2} \sigma_t^2 f(t)^2 \theta^2 + \int_\mathbb{R} e^{i \theta x} - 1 - i \theta c(x) \nu_t(f(t)^{-1} dx) \right)$$
is the c.f. of QID distribution, where \( \hat{h} \) is any real valued function. Hence, by setting \( \hat{h} \) such that 
\[
\hat{h}(t) = h(t) + \int_\mathbb{R} c(mx) - mc(x)\nu_1(dx)
\]
we have the first statement.

The second statement follows by the fact that the class of QID distributions is closed under convolution.

\[\square\]

\textbf{Corollary 7.17.} Under the same conditions of Proposition 7.16 we have that for every \( t \geq 0 \) there exist two QID random variables \( Y_t \) and \( Z_t \) such that

\[
\hat{L}(Y_t) = \exp \left( i\theta a_t - \frac{1}{2} \sigma_1^2 \theta^2 + \int_\mathbb{R} e^{i\theta x} - 1 - i\theta c(x)\nu_1(dx) \right)
\]

and

\[
\hat{L}(Z_t) = \exp \left( i\theta h(t) + i\theta a f(t) - \frac{1}{2} \sigma_2^2 f(t)^2 \theta^2 + \int_\mathbb{R} e^{i\theta x} - 1 - i\theta c(x)\nu(f(t)^{-1}dx) \right).
\]

\textbf{Proof.} It follows directly from Proposition 7.16 \[\square\]

From the formulation \( X + X^{(2)} \overset{d}{=} X^{(1)} \) and from the previous results many examples can be constructed. We will not deal with the existence of particular QID stochastic processes in the continuous time since it is a major topic and we leave it for further research.

We briefly mention that, in the discrete case under the assumption that \( (X_n)_{n \in \mathbb{N}} \) is a sequence of independent QID random variables any process \( X \) is QID. For example, consider \( (X_n) \) to be a sequence of independent random variables with distribution given by \( \mathcal{L}(X_n)(dx) = \mu_d(dx) + f(x)\lambda(dx) \) where \( \mu_d \) is a non-zero discrete measure supported on a lattice of the form \( r + h\mathbb{Z} \) for some \( r \in \mathbb{R} \) and \( h > 0 \), \( \lambda \) is the Lebesgue measure on \( \mathbb{R} \) and \( f \in L_1(\mathbb{R},[0,\infty)) \). Then, from Theorem 4.14 in [1] \( X_n \) is a QID r.v., hence \( X \) is a QID process. An example of such distribution is given in Corollary 4.11 in [1] as generalisation of variance mixture models. From that we can build a straightforward explicit QID process:

\[
\mathcal{L}(X_n)(B) = \int_{[t_1,\infty)} \mathcal{L}(L_i^{(n)})(B)\varrho(dt), \quad n \in \mathbb{N} \text{ and } B \in \mathcal{B}(\mathbb{R}),
\]

where \( \varrho \) is a probability distribution on \( \mathbb{R} \) with \( \varrho((-\infty, t_1)) = 0 \) and \( \varrho(\{t_1\}) > 0 \) for some \( t_1 > 0 \), \( L_i^{(n)} \) are independent Lévy processes. In particular, when \( \hat{L}(X_n) \neq 0 \) for every \( n \in \mathbb{N} \) (which happens for example in the case \( \varrho = \sum_{i=1}^k p_i\delta_{t_i} \) where \( p_1,\ldots,p_k \) are weights and \( t_1 < \ldots < t_k \), c.f. Corollary 4.11 in [1]) then \( X \) is a discrete parameter QID process. Moreover, if \( \varrho \) is bounded and non-degenerate then \( X \) cannot be infinitely divisible (see Theorem 2 in [1]).

It is clear that more sophisticated examples can be build, but we leave this topic for further research.

### 7.4 The atomless condition

One of the main properties that independently scattered random measure might satisfy is the atomless condition. Indeed, the work of Prékopa [17] is centred on this condition and in Theorem 2.2 of [17] he proves that if an independently scattered random measure satisfies the atomless condition then it is an ID random measure. For the sake of completeness we write here the mentioned condition (see [17] for further discussions).

\textbf{Definition 7.18.} Let \( \Lambda \) be a completely additive set function defined on a \( \sigma \)-ring \( S \). A set \( A \in S \) is called an atom relative to the set function \( \Lambda \) if for every \( C \subseteq A \) with \( C \in S \) we have either \( \Lambda(C) = 0 \) or \( \Lambda(C) = \Lambda(A) \). Moreover, the completely additive set function \( \Lambda \) will be called atomless if for every atom \( A \) we have \( \Lambda(A) = 0 \).
The atomless condition is for random measures what the continuity in probability is for continuous time stochastic processes. Notice that we have mentioned explicitly continuous time processes because for discrete time ones the condition is meaningless.

Then, we have the following result.

**Proposition 7.19.** It does not exist a QID r.m. which is atomless and which is not ID.

**Proof.** The result is straightforward since any atomless independently scattered r.m. is ID.

In Chapter 3, we have shown the connections between QID and ID random measure. From this discussion it appears clear that in case the ID random measures considered are atomless the generated QID r.m. is indeed an ID r.m.. This lead us to the following question: is it true that ID r.m. are always atomless? The answer is negative. This is because in case either the drift $\nu_0$ or the Gaussian part $\nu_1$ or the measure $F(B)$ for some $B \in B(\mathbb{R})$ with $0 \notin \overline{B}$ have at least one atom then the corresponding ID measure is not atomless. Therefore, the results shown in Chapter 3 applies to any possible ID and QID r.m.

We point out that the atomless condition is not relevant for the spectral representation of the discrete parameter QID process presented in Chapter 7. The associated r.m. might potentially have atoms and this is true also in the ID framework.

In addition, as we mentioned in the discussion before Lemma 5.7, the fact that the integral $\int_A f d\Lambda$ is well defined does not depend on $\Lambda$ having atoms or not. Indeed, the work of Rajput and Rosinski [18] does not assume any atomless condition and it holds for any ID random measure (even tough they mentioned that their integral is well defined thanks to a result of [22], which only applies to atomless ID random measures).

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