On Black Hole Effective Potential in
6D/7D $\mathcal{N}=2$ Supergravity

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Abstract

Using the harmonic superspace method and the duality between real and complex representations of hypermultiplets, we compute the explicit scalar field expression of the quaternionic metric $G_{mn}(\varphi)$ of the moduli space $\text{SO}(4,k) \times \text{SO}(1,1)$ of 6D $\mathcal{N}=2$ supergravity with generic $k$ Maxwell supermultiplets. The obtained metric includes the particular case $k=20$ associated with 10D type IIA superstring on K3. Uplifting to 7D $\mathcal{N}=2$ supergravity is described and aspects of 6D/7D black attractor effective potentials are studied.

Key words: Type IIA superstring on K3, Quaternionic geometry, 6D black attractors, Uplifting to 7D.

1 Introduction

The dynamics of 10D type II superstrings compactified on the real four dimensional K3 manifold at Planck scale is described by 6D $\mathcal{N}=2$ supergravity [1]-[5]. One distinguishes two 6D models, A and B, depending on whether one started from 10D type IIA or 10D type IIB superstrings. These models are respectively given by the usual 6D non chiral $\mathcal{N}=(1,1)$ and 6D chiral $\mathcal{N}=(2,0)$ supersymmetric models and have different moduli
spaces $M_{\text{IIA/K3}} = \frac{SO(4,20) \times SO(1,1)}{SO(4)\times SO(20)}$ and $M_{\text{IIB/K3}} = \frac{SO(5,21)}{SO(5)\times SO(21)}$ with real dimensions $	ext{dim} M_{\text{IIA/K3}} = 80 + 1$ and $	ext{dim} M_{\text{IIB/K3}} = 105$ respectively.

In this paper, we focus our attention on the study the interacting dynamics of the 81 scalar fields $\varphi^m = \varphi^m(x)$ of the non chiral 6D $\mathcal{N} = 2$ supergravity. Our interest in this issue has been motivated by looking for the explicit field expression of the 6D black hole effective potential $V_{\text{eff}}^6,\mathcal{N}=2(\varphi)$ which depends on the scalar fields self couplings. Generally, the interacting dynamics of the $\varphi^m$'s is described by the following typical non linear sigma model field action,

$$S_b[\varphi] = \int d^6x \sqrt{-\det g} \left( \sum_{\mu, \nu=0}^5 g^{\mu\nu} \left[ \sum_{m,n=1}^{\text{dim} M} \partial \varphi^m \partial \varphi^n \hat{G}_{mn}\right] \right), \quad (1.1)$$

where $M$ stands for the scalar manifold $M_{\text{IIA/K3}}$. The above field action $S_b[\varphi]$ appears in 6D $\mathcal{N} = 2$ supergravity as the scalar field part of the action $S_{\text{6D sugra}}[\varphi,...]$ describing the dynamics of all the degrees of freedom of the theory.

The main objective of this study is to use the link between quaternionic geometry and 6D supersymmetry to determine the explicit scalar field expression of the self couplings matrix,

$$\hat{G}_{mn} = \hat{G}_{mn}(\varphi), \quad (1.2)$$

of the field action $S_b[\varphi]$. We also take this opportunity to give results regarding the family of the quaternionic scalar manifolds $\frac{SO(4,k)}{SO(4)\times SO(k)}$ with $k \geq 1$, concerning generic 6D $\mathcal{N} = 2$ supergravity models as well as the cousin family $\frac{SO(3,k)}{SO(3)\times SO(k)}$ dealing with uplifting to 7D space time.

Notice that besides the fact that the computation of $\hat{G}_{mn}$ is by itself an interesting question, the knowledge of its explicit relation in terms of the field variables $\{\varphi^m\}$ is particularly important for the study of black attractors in 6D and 7D space time dimensions [6]-[12]; see also [14]-[16]. The role of the metric $\hat{G}_{mn}$ in this matter should be compared with the role played by the Kahler metric

$$g_{\bar{z}z} \sim \frac{\partial^2 \mathcal{K}(z, \bar{z})}{\partial z^i \bar{\partial} \bar{z}^j}, \quad \mathcal{K} = \mathcal{K}(z, \bar{z}), \quad (1.3)$$

in the study of the BPS and non BPS attractors in 4D $\mathcal{N} = 2$ supergravity [17]-[21]; and, up on imposing some constraint eqs, in the uplifting to $\mathcal{N} = 2$ supergravity in 5D space time [22]-[29].

To fix the ideas, recall that in 10D type IIA superstring on K3, which is dual to the

\footnote{Following [13] and using the decomposition $so(4,20) = so(3,19) + so(1,1) + (3,19)^+ + (3,19)^-$, the quaternionic manifold $\frac{SO(4,20)}{SO(4)\times SO(20)}$ can be realized as a fibration over the submanifold $\frac{SO(3,19)}{SO(3)\times SO(19)}$.}
10D heterotic superstring on the real 4-torus $T^4$, the scalar manifold $M$ is given by the following particular non compact real space

$$M = Q_{80} \times SO(1,1) \quad \text{dim } M = 81,$$

where the factor $Q_{80}$ is the real eighty dimensional quaternionic space,

$$Q_{80} = \frac{SO(4,20)}{SO(4) \times SO(20)} \quad \text{dim } Q_{80} = 4 \times 20,$$

with the isotropy symmetry $SO(4) \times SO(20)$.

The real one dimensional $SO(1,1)$ factor is parameterized by the dilaton $\sigma$; which generally appears in the analysis as a multiplicative factor $e^{-m\sigma}$ with some number $m$. If freezing the dynamics of $\sigma$ by imposing the constraint relation,

$$d\sigma = 0,$$

then the dynamics of the scalars of the 6D supergravity theory reduces to that of the eighty scalars $\{\phi^m(x)\}$ parameterizing $Q_{80}$. In this case, the $\hat{G}_{mn}(\sigma,\phi)$ coupling matrix of eq(1.1) reduces as well to the restricted field coupling $G_{mn} = G_{mn}(\phi)$ which is nothing but the field metric of $Q_{80}$, that is:

$$dl^2 = \sum_{m,n=1}^{\text{dim } Q} G_{mn} d\phi^m d\phi^n.$$

For later use, it is useful to rewrite the above relation in a more convenient way by taking advantage of the isometries of the scalar manifold $Q_{80}$ \cite{9,6}. Then, we have,

$$dl^2 = \sum_{a,b=1}^{4} \sum_{I,J=1}^{20} G^{IJ}_{ab} d\phi^a_I d\phi^b_J,$$

where the real 80 field variables $\phi^a_I$ are in the bi-fundamental of $SO(4) \times SO(20)$ isotropy symmetry and where $G^{IJ}_{ab} = G^{IJ}_{ab}(\phi)$ is the metric of $Q_{80}$ that we want to determine explicitly; but now expressed in the local coordinate frame $\{\phi^a_I\}$.

As noticed above, the knowledge of the explicit field expression of the coupling $G^{IJ}_{ab}$ is crucial in the study of the BPS and non BPS black attractors in 6D supergravity. There, the black hole (and up on using electric/magnetic duality the black membrane) effective potential $V_{eff}^{6D,N=2}(\phi)$ has the form \cite{9,8},

$$V_{eff}^{6D,N=2}(\phi) = \sum_{a,b=1}^{4} K^{ab} \left( Z_a Z_b + \sum_{I,J=1}^{20} G^{IJ}_{ab} Z_I Z_J \right) \geq 0,$$

where $Z_a = Z_a(\phi)$ and $Z_I = Z_I(\phi)$ are respectively the so called geometric and matter central charges of the $6D \ N = 2$ supersymmetric algebra and where the factor $G^{IJ}_{ab}$ is as
The above effective potential (1.7) should be compared with the well known potential of the black hole in 4D $\mathcal{N} = 2$ supergravity \[30\],

$$V_{\text{eff}}^{4D,N=2}(z,\bar{z}) = e^K \left( |Z|^2 + \sum_{i,j=1}^{m} g_{ij} Z_i \bar{Z}_j \right) \geq 0 , \quad (1.8)$$

where $K$ and $g_{ij}$ are as in eq(1.3).

Moreover, by focusing on eq(1.7), we see that the knowledge of $V_{\text{eff}}^{6D,N=2}$ requires, in addition to $Z_a, Z_I$, and the matrix potential $K^{ab}$, the metric $G_{IJ}^{ab}$ in terms of the fields $\phi^a$. With the explicit field expression of eq(1.7) at hand, one can write down the explicit field expression of the attractor eqs of 6D black attractor. These eqs are given by the criticality condition,

$$\frac{\partial V_{\text{eff}}(\phi)}{\partial \phi^a} = 0 .$$

The explicit fields coupling $G_{IJ}^{ab}$ is also useful for elaborating generalizations beyond the 6D supergravity limit of 10D type IIA superstring on K3; in particular for the two following issues:

(a) reducing or adding extra quaternionic dimensions:
(i) the dimension reduction of $Q_{80}$ down to $Q_{4k}$ with $k < 20$ can be done by looking at particular realizations of the compact K3 manifold. This restriction corresponds to setting part of the moduli to zero

$$\{\phi^m\}_{1 \leq m \leq k-1} , \quad \{\phi^m\}_{k \leq m \leq 20} \rightarrow 0 ,$$

leading then to K3’s with singularities \[31\]-\[35\].

(ii) the extension of $Q_{80}$ up to higher dimensional $Q_{4k}$ with $k > 20$ can be done by going beyond the 6D supergravity models that follow from 10D type IIA superstring on K3. In this case, we have, in addition to the eighty field variables, extra quaternionic moduli. These two situations correspond to 6D supergravity theories with generic moduli spaces

$$Q_{4k} = \frac{SO(4,k)}{SO(4) \times SO(k)} \quad k \geq 1 ,$$

and make the results to be derived throughout this study more general.

Moreover, seen that the scalar manifolds of the 6D $\mathcal{N} = 2$ supergravity models are real $4n$ dimensional manifolds, one may use other Lie group representations such as the real $4n$ dimensional symplectic coset $\frac{SP(2,2n)}{SP(2) \times SP(2n)}$. However, it turns out to be more interesting to, instead of real field coordinates $\{\phi\}$, use the $2n$ local complex coordinates $\{f_1^A, f_1^{\bar{A}}; f_2^A, f_2^{\bar{A}}\}_{A=1,\ldots,n}$ by thinking about the real scalar manifolds $Q_{4n}$ as follows,

$$H_{2n} = \frac{SU(2,n)}{SU(2) \times U(n)} , \quad \dim_R H_{2n} = 2 \times (2n) . \quad (1.9)$$
The quaternionic manifolds $H_{2n}$, which are contained in $Q_{4n}$, define as well an infinite family of real $4n$- (complex $2n$-) dimensional spaces.

Notice that for the leading $n = 1$ term of this series, the manifold $H_2$ has an $SU(2) \times U(1)$ isotropy symmetry. This isometry can be remarkably interpreted as the same isometry that we have in the real 4 dimensional Taub-Newman-Unti-Tamburino (Taub-NUT) metric [36, 37, 38, 39]. The latter has been extensively studied in literature from several views; it will be revisited in section 4 from the view of 6D black hole perspective.

(b) Uplifting to 7D

By borrowing the idea of 4D/5D correspondence of [23], the results obtained for the case 6D black attractors could a priori be used to derive their 7D counterpart. The uplifting to 7D is too particularly interesting in dealing with the effective potential $V_{\text{eff}}^7$ of black attractors in 7D $\mathcal{N} = 2$ supergravity and the classification of 7D BPS and non BPS solutions along the line of [23]. The first step in this way is the determination of the metric $(G_{\alpha\beta})_{1 \leq \alpha, \beta \leq 3}$ of the generic moduli space,

$$M_{7D}^n = \frac{SO(3,n) \times SO(1,1)}{SO(3) \times SO(n)}, \quad n \geq 1. \quad (1.10)$$

Recall that for $n = 19$, the 7D $\mathcal{N} = 2$ supergravity is a limit of 11D M- theory on K3. This theory can be also recovered by uplifting the 6D $\mathcal{N} = 2$ supergravity to 7D.

The moduli space $M_{7D}^{19}$ can be obtained from eq(1.14) by switching off the 22 fluxes of the NS-NS B- field over the 2-cycles of K3; together with a constraint on the volume of K3. Using this property, one can a priori determine the explicit field metric $(G_{\alpha\beta})_{1 \leq \alpha, \beta \leq 3}$ of the generic spaces $M_{7D}^n$ by imposing appropriate constraint eqs on the metric $G_{ab}^{IJ}$ of the moduli space $M_{6D}^n$ of the 6D supergravity. A comment regarding this issue will be made in the discussion section.

The organization of this paper is as follows: In section 2, we review briefly the fields content of 10D type IIA on K3 and recall useful aspects of harmonic superspace (HSS) method. In section 3, we focus on the scalars of the 6D theory. We develop a duality transformation mapping real coordinates $\phi^{ia}_k$ complex ones $f^{iA}$ and use it to study the abelian gauge invariance of the Maxwell-matter sector as well as their self couplings by using results on 6D supersymmetry in harmonic superspace. In section 4, we compute the explicit expression of the metric components $h^{iA}_{kB}$, $g^{iAkB}$ and $\overline{g}_{iAkB}$ ($G_{ab}^{IJ}$ for short) of the scalar manifolds $H_{2n}$ ([1.9]). In section 5, we give the conclusion and make two discussions; one on the explicit field expression of the 6D black hole potential and the

\[\text{below, we shall think about } Q_{4n}, \text{with real field coordinates } \phi^i, \text{and } H_{2n}, \text{with complex fields } f^{iA}, \text{as roughly referring to the same scalar manifold of the 6D } \mathcal{N} = 2 \text{ supergravity. The two field coordinates are related by duality transformations } (3.65). \text{ The metric of } Q_{4n} \text{ is denoted as } G_{ab}^{IJ} \text{ and often used to refer to the metric of } H_{2n} \text{ which has three component blocks } (h^{iA}_{kB}, g^{iAkB}, \overline{g}_{iAkB}) \text{ as in eq(3.50).} \]
other regarding the uplifting to 7D. In the appendices 6 and 7, we give respectively some useful tools on harmonic superspace and on the geometric approach of $\hat{F}_4$ supergravity.

# 2 Fields in 10D type IIA on K3

In this section, we review briefly some useful results on 10D type IIA superstring on K3. We begin by recalling the two sectors of the spectrum of the 10D type IIA superstring: (1) the perturbative sector containing the following 10D IIA supergravity massless fields,

\[
\begin{align*}
NS-NS \text{ fields} & : g_{MN}, B_{MN}, \phi_{\text{dil}}, \\
RR \text{ fields} & : A_M, C_{MNK},
\end{align*}
\]

where the index $M. = 0, \cdots, 9$ captures the 10- vectors of $SO(1,9)$.

Along with these bosonic fields, which carry a total number of 128 on shell degrees of freedom, we also have two 10D- gravitinos and two 10D- gauginos.

(2) the non perturbative sector containing D- branes carrying RR charges. They are collected in the following table together with the associated gauge invariant field strengths,

\[
\begin{align*}
\text{Type IIA D- branes} & : D0, D2, D4, D6, \\
\text{Field strenghts} & : \mathcal{F}_2, \mathcal{F}_4, *\mathcal{F}_4, *\mathcal{F}_2,
\end{align*}
\]

where $*$ stands for the Hodge dual.

## 2.1 Compactification on K3

Under the compactification of 10D type IIA superstring on K3, the $SO(1,9)$ space-time symmetry breaks down to the subgroup $SO(1,5) \times SU_R(2)$; which is contained in $SO(1,5) \times SO(4)$. Moreover, the initial the 32 conserved supersymmetric charges get reduced down to 16.

The degrees of freedom of 10D type IIA superstring on K3 describe, at the gravity level, a non chiral $6D \mathcal{N} = 2$ supergravity theory and appear in two irreducible $\mathcal{N} = 2$ supersymmetric representations namely the gravity supermultiplet and Maxwell ones. Below, we describe these supermultiplets.

(1) **6D $\mathcal{N} = 2$ gravity supermultiplet**

The bosonic fields of the six dimensional $\mathcal{N} = 2$ gravity supermultiplet contains 32 on shell degrees of freedom distributed as

\[
\begin{align*}
g_{\mu\nu}, & B_{\mu\nu}, \sigma, \\
\mathcal{A}^{(ij)}_{\mu}, & \mathcal{C}_{\mu\nu\rho},
\end{align*}
\]

\[
\]
where \( \mu, \nu = 0, ..., 5 \) stands for the space-time indices and \( i, j = 1, 2 \) are the isospin 1/2 indices of the \( SU_R (2) \) symmetry.

The \( g_{\mu\nu} \) is the space time metric and \( B_{\mu\nu} \) the 6D antisymmetric 2-form. The gauge fields \( A_{\mu}^{(ij)} \) and \( C_{\mu\nu\rho} \) can be thought of as four gravi-photons.

Besides the real 3-form \( \mathcal{H}_3 = dB \), we also have the following the gauge invariant field strengths,
\[
\mathcal{F}_2^{(ij)} = dA_1^{(ij)}, \\
\mathcal{F}_4^0 = dC_3,
\]
and their_duals
\[
\mathcal{F}_2^{(ij)} = *\mathcal{F}_2^{(ij)}, \\
\mathcal{F}_4^0 = *\mathcal{F}_4^0.
\]

Notice the two following features:
(a) the 6D fields \( A_{\mu}^{(ij)} \), \( \mathcal{F}_2^{(ij)} \) and \( *\mathcal{F}_2^{(ij)} \) are \( SU_R (2) \) isotriplets while \( C_3 \), \( \mathcal{F}_4^0 \) and \( \mathcal{F}_2^0 \) are isosinglets.
(b) the 6D gauge field \( C_3 \) is dual to a 6D Maxwell field \( A_0^0 \).

So eqs (2.4-2.5) can be also exhibited in terms of quartets as follows
\[
A_{\mu}^{ij} = A_0^{ij} \varepsilon^{ij} + A_\mu^{(ij)},
\]
and
\[
\mathcal{F}_2^{ij} = dA^{ij}, \\
\mathcal{F}_4^{ij} = *\mathcal{F}_2^{ij},
\]
with \( \varepsilon^{ij} = -\varepsilon^{ji} \) and \( \varepsilon^{12} = 1 \). Notice that the fields \( \mathcal{F}_2^{ij} \) and \( \mathcal{F}_4^{ij} \) can be also decomposed as in eq (2.6).

(2) 6D \( \mathcal{N} = 2 \) Maxwell supermultiplets: \( (V_{6D}^{N=2})^I \)

The Maxwell-matter sector of the 6D \( \mathcal{N} = 2 \) supergravity theory embedded in 10D type IIA superstring on K3 involves twenty 6D Maxwell supermultiplets
\[
V_{6D,N=2}^I : \quad I = 1, ..., 20.
\]

Each supermultiplet \( V_{6D,N=2} \) has \( (8 + 8) \) on shell degrees of freedom. The eight bosonic degrees of freedom are captured by a 6D gauge field \( A_\mu \) and four real scalars \( \phi^{ij} \). The eight fermionic degrees of freedom are captured by two spinors \( \lambda^1 \) and \( \lambda^2 \):
\[
[V_{6D,N=2}]_{\text{Bose}} = A_\mu \oplus \phi^{ij}, \\
[V_{6D,N=2}]_{\text{Fermi}} = \lambda^1_\alpha \oplus \lambda^2_\alpha.
\]

These fields transform in different representations of the \( SU_R (2) \) symmetry. The gauge field \( A_\mu \) is a isosinglet, the two gauginos \( \lambda^i_\alpha \) form an isodoublet and the four scalars form
as a reducible quartet; that is
\[
4 = 1 + 3 . \tag{2.10}
\]
In the 6D field theory set up, the four scalars are described by the sum of a singlet \( \phi^0 \) and a triplet \( \phi^{(ij)} \) as shown below,
\[
\phi^{ij} = \phi^0 \varepsilon^{ij} + \phi^{(ij)} . \tag{2.11}
\]
We will need this property later on when we consider the geometric interpretation of the \( \phi^{ij} \)'s as periods of a quaternionic 2-form \( J^{ij} \) to be introduced at appropriate time.

Notice that the Maxwell supermultiplet \( V_{6D,N=2} \) contains scalar fields that allow to make a formal correspondence with the Coulomb branch in the \( \mathcal{N} = 2 \) supergravity theory in 4D space time.

Notice also that generally, the twenty Maxwell supermultiplets \( V_{6D,N=2}^I \); in particular their Bosonic sector
\[
[ V_{6D,N=2}^I ]_{\text{Bose}} = A^I_\mu \oplus \phi^{ijI} , \tag{2.12}
\]
the field components have quantum numbers with respect to the \( SO(4) \times SO(20) \) isotropy symmetry of the moduli space \([1.4]\). We have
\[
A^I_\mu \simeq (1,20) , \quad \phi^{ijI} \simeq (4,20) , \tag{2.13}
\]
where the \( SO(4) \) isotropy is thought of as \( SU_R(2) \times SU_R(2) \). We also have for the gauge field strengths
\[
F^I_2 = dA^I \simeq (1,20) , \quad *F^I_2 \simeq (1,20) . \tag{2.14}
\]

### 2.2 \( \mathcal{N} = 1 \) formalism in 6D

To study the geometry of the scalar manifold \([1.4]\), it is enough to focus the attention on the scalar fields \( \{ \phi^{ijI} \} \). This can be nicely done by using 6D \( \mathcal{N} = 1 \) supersymmetric representations by splitting the 6D \( \mathcal{N} = 2 \) gauge multiplet \( V_{6D}^{N=2} \) as the sum of two 6D \( \mathcal{N} = 1 \) multiplets; namely a vector multiplet \( V_{6D}^{N=1} \) and a hypermultiplet \( H_{6D}^{N=1} \);
\[
V_{6D}^{N=2} = V_{6D}^{N=1} \oplus H_{6D}^{N=1} , \tag{2.15}
\]
with
\[
V_{6D}^{N=1} = (A_\mu, \lambda_\alpha) \equiv (1, \frac{1}{2})_{6D} , \quad H_{6D}^{N=1} = (\phi^{ij}, \psi_\alpha) \equiv (0^4, \frac{1}{2})_{6D} , \tag{2.16}
\]
where 1, \( \frac{1}{2} \) and 0 stand for the space time spin of the component fields and the powers for their numbers.
Notice the three following features:

First, the decomposition (2.15) is a general property of $\mathcal{N} = 2$ supersymmetry in any space-time dimension. Irreducible supermultiplets ($R_{\mathcal{N}=2}$) can be usually split into pairs of $\mathcal{N} = 1$ irreducible representations as given below,

$$ R_{\mathcal{N}=2} = R_{\mathcal{N}=1} \oplus R'_{\mathcal{N}=1}. \quad (2.17) $$

Second, the vector supermultiplet $V_{6D}^{\mathcal{N}=1}$ has a gauge field but no scalars; while the hypermultiplet $H_{6D}^{\mathcal{N}=1}$ has no vector field but four scalars capturing the quaternionic structure of the Coulomb branch of $6D \mathcal{N} = 2$ supersymmetry.

Finally, it is interesting to note that there is a remarkable parallel between the reductions of the irreducible vector representations of $\mathcal{N} = 2$ supersymmetry in $6D$ and $4D$ space times,

$$ \mathcal{N} = 2 \to \mathcal{N} = 1 \oplus \mathcal{N}' = 1. \quad (2.18) $$

Concerning the underlying geometries of the associated Coulomb branches, we have

| vector multiplet | matter multiplet | scalar manifold |
|------------------|-----------------|----------------|
| $4D \mathcal{N} = 2$ | $\to$ | $\mathcal{N} = 1$ chiral matter | Kahler |
| $6D \mathcal{N} = 2$ | $\to$ | $\mathcal{N} = 1$ hyper matter | quaternionic |

(2.19)

In superspace formulation of $\mathcal{N} = 1$ supersymmetry in $4D$, the scalar multiplet $\Phi_{4D}^{\mathcal{N}=1}$ is described by a chiral superfield $\Phi = \Phi (x, \theta)$ with dynamics described by the superspace Lagrangian density,

$$ L_{4D}^{\mathcal{N}=1} = \int d^4\theta \ K (\Phi, \bar{\Phi}) + \int d^2\theta \ W (\Phi) + \int d^2\bar{\theta} \bar{W} (\bar{\Phi}), \quad (2.20) $$

where $K (\Phi, \Phi)$ is the Kahler potential and $W (\Phi)$ the chiral superpotential.

In the harmonic superspace (HSS) formulation of $\mathcal{N} = 1$ supersymmetry in $6D$, the hypermultiplet is described by the superfield $\Phi^+ = \Phi^+ (x, \theta^+, u^\pm)$ with dynamics governed by the HSS Lagrangian density

$$ L_{6D}^{\mathcal{N}+4} = \int d^4\theta^+ du \left[ \Phi^+ D^{++} \Phi^+ + L_{\text{int}}^{\mathcal{N}+4} (\Phi^+, \bar{\Phi}^+) \right], \quad (2.21) $$

where

$$ D^{++} = \partial^{++} - 2 \theta^{+\alpha} \theta^{+\beta} \partial_{[\alpha\beta]}, \quad (2.22) $$

$$ \partial^{++} = u^{i+} \frac{\partial}{\partial u_i^+}, $$

is the HSS covariant derivatives and $\partial_{[\alpha\beta]} = \frac{\partial}{\partial x^{[\alpha\beta]}} \sim \frac{\partial}{\partial x^\mu}$. The superfield $L_{\text{int}}^{\mathcal{N}+4}$ is the HSS potential which can be thought of as the quaternionic superpotential that specify the geometry of the quaternionic scalar manifold.
Below, we will refer to $\mathcal{L}_{\text{int}}^{4+4}$ as the quaternionic potential.

Notice that the charges $q$ carried by the HSS superfunction $F^q$ are the charges of the $U_C (1)$ Cartan subgroup of the $SU_R (2)$ symmetry. We have

$$[D^0, F^q] = qF^q$$

where $D^0$ is the generators of $U_C (1)$. The operator $D^0$ together with the covariant derivatives $D^{++}$ of eq (2.22) and its adjoint $D^{--}$ are the generators of the $SU_R (2)$ symmetry satisfying the usual commutation relation

$$[D^0, D^{++}] = +2D^{++},$$
$$[D^0, D^{--}] = -2D^{--},$$
$$[D^{++}, D^{--}] = +D^0.$$

More details can be found in the original works on HSS [40, 41]; some useful relations are collected in the appendix of this paper.

3 $U^{20} (1)$ invariance and quaternionic potential

In this section, we consider the two main points:

(1) the field theoretic implementation of the $U^{20} (1)$ gauge invariance of the matter sector of the 10D type IIA superstring on K3.

(2) the derivation of the explicit expression of the quaternionic potential $\mathcal{L}_{\text{int}}^{4+4}$ of the scalar manifold $H_{2n}$ (1.9).

To achieve these goals, we shall use:

(a) tools on the real second homology/cohomology of K3;

(b) known results on the HHS method for hyperKahler metrics building.

This section is organized in three subsections:

In the first subsection, we first study the two following things:

(i) develop two dual descriptions of 6D hypermultiplets;

the first description involves real scalars $\{\phi^\alpha I \sim \phi^{i\beta I}\}$ and is adapted to deal with the manifold $\frac{SO(4,20)}{SO(4) \times SO(20)}$.

the other realization uses complex fields $f^I_A$ and $\bar{T}_{iA}$ concerns the complex scalar manifold $H_{2n}$.

(ii) give an heuristic derivation the quaternionic potential $\mathcal{L}_{\text{int}}^{4+4}$ of the scalar manifold of the 10D type IIA superstring on K3.

In the two other subsections 3.2 and 3.3, we give rigorous details an the refining of the results given in subsection 3.1.
3.1 Duality relation and quaternionic potential

Our interest in exhibiting explicitly the $U^{20}(1)$ symmetry is because of the central role it plays in computing the explicit field expression of the quaternionic metric $G_{ab}^{IJ}$. Though the existence of this symmetry is directly identified from the spectrum of the Maxwell-matter sector of the 10D type IIA superstring on K3,

$$A_{\mu}^I, \chi_{\alpha}^{ij}, \phi^{ijI}; \quad I = 1, ..., 20,$$

the field theory implementation of this gauge invariance is a little bit subtle. The point is that the scalar moduli $\phi_{kI}^{ij}$ describing the 6D matter are real fields

$$\left(\phi_{kI}^{ij}\right) = \phi_{kI}^{ij},$$

and so neutral under $U^{20}(1)$,

$$U^{20}(1): \phi_{kI}^{ij} \rightarrow \left(\phi_{kI}^{ij}\right)' = \phi_{kI}^{ij}.$$ (3.3)

It is then interesting to look for a dual complex description where the twenty quartets of scalar fields $\phi_{kI}^{ij} = \phi_{kI}(x)$ are put into twenty complex isodoublets

$$f^{iA} = f^{iA}(x), \quad A = 1, ..., 20.$$ (3.4)

These complex fields allow the following phases changes

$$f^{iA'} = e^{i\lambda_{iA}}(x),$$ (3.5)

with $\lambda$ being a real diagonal matrix which can be expanded as

$$\lambda = \sum_{i=1}^{20} \lambda_i T^I.$$ (3.6)

In above expansion, the $T^I$'s are 20 commuting matrices

$$\{T^I\}_{I=1, \ldots, 20}, \quad T^IT^J = T^JT^I,$$ (3.7)

generating the $U^{20}(1)$ gauge invariance of the Maxwell sector of the 6D supergravity theory.

**Duality relation and its superfield extension**

The duality relation that maps the real fields $\phi_{kI}^{ij}$ into the complex isodoublets $f^{iA}$ and $f_{iA}$ is given by

$$\phi_{kI}^{ij} = \check{T}_k T^I f_i = \sum_{A,B=1}^{20} \check{T}_{kB} \left(T^I\right)^B_A f^{iA},$$ (3.8)
This relation has been motivated from the relation existing between real 4-vectors and the hermitian $2 \times 2$ matrices; it will be described with some details in a moment. But before coming to that, let us recall that the $6D \mathcal{N} = 2$ supergravity embedded in $10D$ type IIA superstring on K3 has 24 Maxwell gauge fields:

- *Four* of these gauge fields, denoted above as $A^i_{\mu}$, belongs to the supergravity multiplet and are not directly our target here. Nevertheless, keep in mind that the associated gauge invariant charges are involved in the effective potential $\mathcal{V}_{eff}^{6D}$ of the black attractor.

- *Twenty* other gauge fields, denoted as $A^I_{\mu}$, belong to the Maxwell-matter sector (3.1). They transform under the change eq(3.5) as follows,

$$U^{20} (1) : \quad A^I_{\mu} \rightarrow A^I_{\mu} + \partial \vartheta_I , \quad (3.9)$$

where the $\vartheta_I$'s are gauge parameters.

It is these 20 gauge fields and their supersymmetric partners (3.1) that we are interested in here.

Notice that the duality relation (3.8) concerning the component scalar fields, is very suggestive.

First, it can be extended to a superfield duality relation involving the superfields

$$H^I_k = H^I_k (x, \theta) , \quad (3.10)$$

and

$$\Phi^{iA} = \Phi^{iA} (x, \theta) , \quad (3.11)$$

with leading $\theta$- components

$$\phi^I_k = (H^I_k)_{\theta=0} , \quad \phi^i_A = (\Phi^{iA})_{\theta=0} . \quad (3.12)$$

The superfield extension of the duality relation (3.8) reads as follows

$$H^I_k = \Phi_k T^I \Phi^i . \quad (3.13)$$

Second, eq(3.13) has a nice description in the $6D \mathcal{N} = 1$ harmonic superspace formalism. There, the superfields $H^{ijI}$ and $\Phi^{iA}$ are mapped to HSS superfields

$$H^{++I} = H^{++I} (x, \theta^+, u^\pm) , \quad (3.14)$$

with leading components

$$(H^{++I})_{\theta=0} = \sum_{ij=1}^2 u^{+}_i u^{+}_j \phi^{ij} , \quad (3.15)$$

$$(\Phi^{iA})_{\theta=0} = \sum_{i=1}^2 u^{+}_i \phi^{i}_A .$$
Thus, the superfield duality relation (3.13) reads in HSS like,

$$H^{++I} = \tilde{\Phi}^+ T^I \Phi^+ ,$$  \hspace{0.5cm} (3.16)\]

Notice in passing that the $\Phi^+_A$'s are the superfields that describe hypermultiplets in harmonic superspace.

Third, eq(3.16) obey the HSS relation,

$$D^{++} H^{++I} = 0 ,$$  \hspace{0.5cm} (3.17)\]

where the harmonic covariant derivative $D^{++}$ is as in eq(2.21). These relations have interpretation in HSS formulation as the conservation laws of Noether HSS currents $J^{++I}(x, \theta, u^\pm) = \tilde{\Phi}^+ T^I \Phi^+$.

**Quaternionic superpotential**

Using the above tools, we can give the explicit superfield expression of the quaternionic potential $L^{+4}_{int}$ associated with the scalar manifold $Q_{80}$. It reads in the 6D $\mathcal{N} = 1$ harmonic superspace formalism as follows,

$$L^{+4}_n \simeq \frac{\lambda}{2} \sum_{I,J=1}^{n} \left( \tilde{\Phi}^+ T^I \Phi^+ \right) d_{IJ} \left( \tilde{\Phi}^+ T^J \Phi^+ \right) ,$$  \hspace{0.5cm} (3.18)\]

with $n = 20$. In this relation, the real symmetric tensor $d_{IJ} = d_{JI}$ is a coupling metric which can be interpreted in terms of intersections of 2-cycles of K3.

Notice that eq(3.18) is valid for $n = 20$; but also for generic integers $n$; in particular for $n = 1$ where the above quaternionic potential reduces to,

$$L^{+4}_1 \simeq \frac{\lambda}{2} \left( \tilde{\Phi}^+ \Phi^+ \right)^2 .$$  \hspace{0.5cm} (3.19)\]

which, according to [39], is nothing but the quaternionic potential of the real 4-dimensional Taub-NUT geometry.

**3.2 Geometric and stringy interpretations**

First notice that the real *eighty* scalars $\phi_a^I = \phi_a^I (x)$ of the 6D $\mathcal{N} = 2$ supergravity are in the $(4,20)$ bi-fundamental of the $SO(4) \times SO(20)$ isotropy symmetry of the moduli space $Q_{80}$. These fields can be also written as

$$\phi_a^I (x) = \sum_{i,j=1}^{2} \sigma^a_{ij} \phi_i^j (x) , \hspace{0.5cm} I = 1, ..., 20 , $$  \hspace{0.5cm} (3.20)\]

where $\sigma^a$, $a = 1, 2, 3$, are the usual $2 \times 2$ Pauli matrices and $\sigma^0 \equiv I_{id}$ is the identity matrix. Sometimes we also refer to $\sigma^0$ as $\sigma^4$. 

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Viewed from the 10D type IIA superstring on K3, the 80 scalar fields $\phi^{ij}_I$ have two origins:

- 58 geometric moduli having as well an interpretation in 7D $\mathcal{N} = 2$ supergravity.
- 22 stringy origin moduli; but having no analogue in 7D $\mathcal{N} = 2$ supergravity.

Let us give some details about these scalars.

(a) Geometric moduli

The 58 scalars of the 80 moduli decompose as

$$58 = 1 + 3 \times 19 \ , \quad (3.21)$$

and have a geometric interpretation in terms of the Kahler and complex deformations of the metric of K3. In general, we have $(1+19)$ real moduli and 19 complex ones. These fields can be denoted altogether as

$$\phi^0 \oplus \phi^{(ij)}_I \ , \ I = 1,...,19 \ , \quad (3.22)$$

and belong to two kinds of representations of the $SU_R(2)$ symmetry. The field variable $\phi^0$ is an isosinglet; it is interpreted as the volume of K3. The fields $\phi^{(ij)}_I$ describe nineteen isotriplets combining the Kahler and complex deformations.

(b) Stringy moduli

The remaining 22 field moduli decompose as

$$22 = 3 + 1 \times 19 \ , \quad (3.23)$$

and have stringy interpretation in terms of the NS-NS 2-form periods. These fields can be denoted like,

$$\chi^{(ij)} \oplus \chi^0_I \ , \ I = 1,...,19 \ , \quad (3.24)$$

that is an isotriplet $\chi^{(ij)}$ and nineteen isosinglets $\chi^0_I$.

(c) Comments

We give three comments.

(i) Eqs (3.22) and (3.23) combine altogether into twenty quartets as follows

$$\chi^{ij} \oplus \phi^{ij}_I \ , \ I = 1,...,19 \ , \quad (3.25)$$

and will be read now on like,

$$\phi^{ij}_I \ , \ I = 1,...,20 \ , \quad (3.26)$$

with $\chi^{ij} = \phi^{ij}_{20} = \phi^0 \varepsilon^{ij} + \chi^{ij}$.

(ii) The fields moduli $\chi^0_I$ and $\phi^{(ij)}_I$ can be interpreted in terms of the periods

$$\chi^0_I = \int_{C_I} B_{NS} \ , \ I = 1,...,19 \ , \quad (3.27)$$

and

$$\phi^{(ij)}_I = \int_{C_I} J^{(ij)} \ , \quad (3.27)$$
where the nineteen $C^I$'s form 2-cycles sub-basis of the 22 dimensional second real homology of K3.

Now, let us introduce the quaternionic 2-form

$$J^{ij} = J^{[ij]} + J^{(ij)}, \quad (3.28)$$

where $J^{[ij]}$ is an isosinglet 2-form and $J^{(ij)}$ an isotriplet 2-form and use the $SU_R(2)$ quantum numbers of the moduli $(3.22)$ to identify the two irreducible components of $(3.28)$. We distinguish two representations of $J^{ij}$ depending on the 2-cycles of K3. To that purpose, it is interesting use the $(3,19)$ signature of $H_2(K3,\mathbb{R})$ to split the real 22 dimensional basis

$$\{B^\Lambda\} = \{B^1,\ldots,B^{22}\}, \quad (3.29)$$

like

$$\{B^\Lambda\} \equiv \{C^I\} \oplus \{D^a\}, \quad (3.30)$$

where, roughly,

$$\{C^I\} \sim \{B^1,\ldots,B^{19}\},$$

$$\{D^a\} \sim \{B^{20},B^{21},B^{22}\}. \quad (3.30)$$

By duality

$$\int_{B^\Lambda} \alpha^\Sigma = \delta^\Sigma_\Lambda, \quad (3.31)$$

the real 2-forms basis $\alpha^\Sigma$ can be also split as

$$\{\alpha^\Lambda\} = \{\gamma_I\}_{1 \leq I \leq 19} \oplus \{\delta_a\}_{1 \leq a \leq 3}. \quad (3.32)$$

Then we have:

(a) Case of the 2-cycles $\{C^I\}_{1 \leq I \leq 19}$

For the 19 dimensional sub-basis $\{C^I\}$, the quaternionic 2-form $J^{ij}$ reads as

$$J^{ij} = B_{NS} \varepsilon^{ij} + J^{(ij)}, \quad (3.33)$$

and eqs $(3.27)$ are just the periods,

$$\phi^{ijI} = \int_{C^I} J^{ij}, \quad I = 1,\ldots,19. \quad (3.34)$$

In eq $(3.33)$, $B_{NS}$ is the NS-NS 2-form B-field and $J^{(ij)}$ the real isotriplet 2-form standing for the hyperkahler 2-form on K3

$$J^{(ij)} = \left( \begin{array}{c} \Omega^{(2,0)} \\ \Omega^{(1,1)} \\ \Omega^{(0,2)} \end{array} \right), \quad (3.35)$$
where $\Omega^{(1,1)}$ is the usual Kahler 2-form while $\Omega^{(2,0)}$ and $\Omega^{(2,0)}$ are the holomorphic and anti-holomorphic 2-forms on K3.

$(\beta)$ Case of the 2- cycles $\{D^a\}_{1 \leq a \leq 3}$

For the 3 dimensional sub- basis $\{D^a\}$, the quaternionic 2- form $J^{ij}$ reads as

$$J^{ij} = \Omega^{ij} + B^{(ij)\ns} \quad .$$  

(3.36)

where $\Omega = \Omega^{(1,1)}$ is the Kahler 2-form and the isotriplet $B^{(ij)\ns}$ as described below. Indeed, using eqs (3.22-3.24), it follows that the twentieth quartet $\phi^{ij}_{20} = \chi^{ij} = (\phi^0, \chi^{(ij)})$ can be written as the periods,

$$\phi^0 = \int_{B^{20}} \Omega^{(1,1)} ,$$

$$\chi^a = \int_{D^a} B^{NS} \quad ,$$  

(3.37)

from which we learn that $\chi^a \sim \chi^{(ij)}$ is an isotriplet as required by the signature of $H_2(K3,R)$. This property allows to set,

$$\int_{D^a} B^{NS} = \int_{C^{20}} B^{(c)\ns}_{NS} \quad ,$$  

(3.38)

and think about the $19 + 1 = 20$ quartets as $(\phi^{ij}_I)_{1 \leq I \leq 20}$ with the interpretation

$$\phi^{ij}_I = \int_{C^I} J^{ij} \quad , \quad I = 1, ..., 20 \quad ,$$  

(3.39)

where $(C^I)_{1 \leq I \leq 19}$ as in eq (3.30) and $\int_{C^{20}} \left( \Omega^{ij} + B^{(ij)\ns} \right)$ as in eqs (3.37,3.38).

### 3.3 Deriving the quaternionic potential (3.18)

The real 80 fields moduli $\phi^{ij}_I = \phi^{ij}_I(x)$ we have been using are six dimensional real scalar field variables obeying the reality condition (3.2). These fields parameterize the coset manifold,

$$Q_{80} = \frac{SO(4,20)}{SO(4) \times SO(20)} \quad .$$  

(3.40)

The reality condition (3.2) is required by $\mathcal{N} = 2$ supersymmetric gauge theory which demands that gauge vector supermultiplets should be in the (real) adjoint representation of gauge groups (3.3).

As shown by eqs (3.4,3.5,3.8), to deal with the hyper- matter $\phi^{ij}$ it is more convenient to use the complex isodoublet field variables,

$$\phi^{iA} = (\phi^{1A}, \phi^{2A}) \quad , \quad \bar{\phi}_{iA} = (\bar{\phi}_{1A}, \bar{\phi}_{2A}) \quad .$$  

(3.41)

Recall that besides the relation $SO(4) \simeq SU(2) \times SU(2)$, the switch from the real $\phi^{i}_k$ to the complex coordinates $\phi^{i}$ has been as well motivated from the two following:

(i) the wish to exhibit manifestly the $U^{20}(1)$ invariance, and

(ii) the objective to use the harmonic superspace method for building quaternionic metrics.
3.3.1 Harmonic superspace method

Here, we review briefly some useful tools on HSS method; in particular those aspects concerning the superfields $\Phi_A^+$, that describe off shell hypermultiplets, the quaternionic potential $L_{\text{int}}^{4+} = L_{\text{int}}^{4+} \left( \Phi^+, \tilde{\Phi}^+ \right)$, and the way to get the quaternionic metrics $G^{IJ}_{ab}$ in this superfield theory set up.

6D $\mathcal{N} = 1$ harmonic superspace formalism

Like in the case of 4D $\mathcal{N} = 2$ supersymmetry, the $\mathcal{N} = 2$ vector representation $V_{\mathcal{N}=2}^{4D}$ splits into the sum of a $\mathcal{N} = 1$ vector representation $V_{\mathcal{N}=1}^{4D}$ and a hypermultiplet $H_{\mathcal{N}=1}^{4D}$; eq(2.15). This splitting allows here also to use the $\mathcal{N} = 1$ superspace formalism to study hypermultiplet interactions.

In HSS with supercoordinates, $Z = \{ (x, \theta^+, u^\pm) ; \theta^- \}$, (3.42)
a generic hypermultiplet $H_{\mathcal{N}=1}^{4D}$ is described by complex (graded analytic) superfields $\Phi_A^+ = \Phi_A^+ (x, \theta^+, u)$, (3.43)
with no dependence in $\theta^-$ and the typical $\theta^+$- expansion

$$
\Phi_A^+ = \phi_A^+ + \theta^{\alpha} \theta^{\beta} B_{[\alpha \beta]} A + \theta^{+4} \Delta^{---}_{A} + \theta^{\alpha} \psi_A + \theta^{+\alpha} \theta^{\beta} \theta^{+\gamma} \chi_{[\alpha \beta \gamma]} A .
$$

(3.44)

The 6D spinor $\theta = (\theta^{+\alpha})$ is the usual superspace Grassmann variables. Moreover, each component field

$$
F_A^q = F_A^q (x, u) , \quad q \in \mathbb{Z} ,
$$

(3.45)
of the development (3.44) is a function of the space time coordinates $x = (x^\mu)$ and the harmonic variables $u^\pm = (u_i^\pm)$ with the harmonic expansion

$$
F_A^q = \sum_{n+q \geq 0} u^+_i u^+_j ... u^+_m u^-_n ... u^-_j F_A^{(i_1 ... i_n + j_1 ... j_n)} (x) .
$$

(3.46)

In particular, we have for the two leading components,

$$
\phi^+ (x, u) = u^+_i \phi^i (x) + u^+_j u^+_k \phi^{ijk} (x) + ... ,
$$

(3.47)
where $\phi^i (x)$ is precisely the scalar field given by eq(3.41).

The extra fields $\phi^{(i_1 ... i_m)}$, $m > 2$ of eq(3.47) as well as the following,

$$
B_{[\alpha \beta]} \sim B_{[\alpha \beta]} (x, u) , \quad \Delta^{---} = \Delta^{---} (x, u) ,
$$

(3.48)

3There are two main ways to describe hypermultiplets in terms of HSS superfields. One is hermitian and the other is complex as in 3.43.
are auxiliary fields required by off shell supersymmetry. They play a crucial role in the
determination of the quaternionic metric we are looking for.
Notice that the determination of the explicit field expression of these auxiliary fields in
terms of the physical degrees of freedom $f^{\pm A}$ and $\overline{f}_{iA}$ is one of the main difficult step in
using $HSS$ method.
As we will show later on, this difficulty can be overcome in the present case by help of
the $U^{20}$ (1) symmetry.

*From the superfield action to the metric*

Generally, the superfield action $S$, describing hypermultiplet interactions, reads in rigid
harmonic superspace as follows

$$S ≃ \int d^6x d^4\theta^+ \left[ \Phi^+ D^{++} \Phi^+ + \mathcal{L}_{int}^{++} \left( \Phi^+, \bar{\Phi}^+ \right) \right], \quad (3.49)$$

where $D^{++}$ is the harmonic derivative whose basic properties are collected in the ap-
pendix.

The superfield equation of motion of the hypermultiplet $\Phi^+$ reads as

$$\left( D^{++} \Phi^+ + \frac{\partial \mathcal{L}_{int}^{++}}{\partial \Phi^+} \right) = 0. \quad (3.50)$$

Eq (3.50) describes the dynamics of the degrees of freedom $f^{\pm A}$ and $\overline{f}_{iA}$; but gives also
constraint eqs on the auxiliary fields $\phi^{(i_1...i_m)}$, $B^{-\mu}_{[\alpha\beta]}$ and $\Delta^{-\ldots-}$.

For example, the equation of motion of the auxiliary field $\Delta^{-\ldots-}$ corresponding to the
highest $\theta^+$- term in the expansion $\Phi$ (3.47), reads as follows.

$$\left( u^{++} \phi^+ + \frac{\partial \mathcal{L}_{int}^{++}}{\partial \phi^+} \right) = 0, \quad (3.51)$$

This relation is a constraint equation that fix the auxiliary fields of eq (3.47) as follows,

$$\phi^+ = \phi^+ \left( f^{+A}, f^{-A} \right), \quad (3.52)$$

where

$$f^{\pm A} (x, u) = \sum_{i=1}^{2} u^\pm_i f^{iA} (x), \quad A = 1, ..., n. \quad (3.53)$$

A quite similar relation to eq (3.51), can be also written down for $B^-_{\mu}$; and its solution
gives the expression of $B^-_{\mu}$ in terms of the scalar moduli $f^{\pm A}$; i.e

$$B^-_{\mu} = B^-_{\mu} \left( f^{+A}, f^{-A} \right). \quad (3.54)$$

With eqs (3.52)(3.54) in mind; and integrating eq (3.49) with respect to the Grassmann
variables $\theta^+\alpha$, we can bring the superfield action to the following remarkable form,

$$S = \int d^6x du \left( B^-_{\mu} \partial^\mu \phi^+ - B^-_{\mu} \partial^\mu \bar{\phi}^+ \right). \quad (3.55)$$
Substituting $\phi^+$ and $B^-$ by their expression (3.52-3.54) and integrating with respect to the harmonic variables $u^{\pm i}$, we can further bring (3.55) to the form

$$S = \int d^6x \left( 2h^{iA}_{\mu} \partial^i \partial^{\mu} \bar{I}_j B + g_{iA,jB} \partial^i \partial^{\mu} \bar{f}_j B + g^{iA,jB} \partial^i \bar{I}_j B \right),$$

(3.56)

from which we read the expression of $\bar{g}_{iA,jB}$, $g_{iA,jB}$ and $h^{iA,jB}$ associated with eq(3.18).

Notice that bringing the action $S$ from its expression (3.49) into the form (3.56) is in fact a very complicated task; except for some special situations where there are symmetries. As we will see for the case at hand, it is possible to put (3.49) with eq(3.18) into the form (3.56); thanks to the $U^{20}(1)$ invariance that we want to study below.

### 3.3.2 Complex fields and $U^{20}(1)$ symmetry

First we study the field duality mapping the real field coordinates $\phi^{ij}$ into the complex ones $\phi^i$ and $\bar{\phi}^i$. Then, we describe the $U^{20}(1)$ invariance of eq(3.49).

**From real $\phi^{ij}$ to complex $\phi^i$ fields**

A typical field change that relates the four real variables, described by the SU(2) rank 2- tensor $\phi^{ij}$, to the complex isodoublets $(f^i)$ and $(\bar{f}^i)$ is given by,

$$\phi^i_j = \bar{I}_j f^i , \quad (\phi^i_j) = \bar{\phi}^i_j .$$

(3.57)

The complex scalar fields $f^i = (f^1, f^2)$ capture two complex degrees of freedom and can be expressed in terms of $\phi^i_j$ as follows

$$f^i = \sum_{k=1}^{2} u^k \phi^i_k ,$$

$$\bar{f}^i = \sum_{k=1}^{2} \bar{u}^k \phi^i_k .$$

(3.58)

Multiplying both sides of eq(3.57) by $f^j$ and by using the following relations,

$$\phi^i_j = \frac{1}{2} \delta^i_j + \frac{1}{2} \sum_{k=1}^{2} \varepsilon_{jk} \phi^{(ik)} ,$$

(3.59)

and

$$\phi^0 = f^i \bar{I}_j \equiv f^i \bar{f}^j ,$$

$$\phi^{(ik)} = f^i \bar{f}^j \bar{I}_k + f^k \bar{f}^i \bar{I}_j ,$$

(3.60)

we obtain eq(3.58) with

$$u^k = \frac{f^k}{f} = \frac{\bar{f}^k}{\bar{f}} ,$$

$$\bar{u}^k = \frac{\bar{f}^k}{f} = \frac{\bar{f}^k}{\bar{f}} .$$

(3.61)
satisfying the identities
\[
\sum_{k=1}^{2} \bar{\nu}_k \nu^k = 1 , \quad \sum_{k,l=1}^{2} \varepsilon_{kl} \nu^k \nu^l = 0 , \quad \sum_{k,l=1}^{2} \varepsilon^{kl} \bar{\nu}_k \bar{\nu}_l = 0 .
\]

(3.62)

Implementing the $U^{20}(1)$ gauge symmetry

The above field variable change is remarkable and exhibits the following properties:

(i) Eq.(3.57) has a manifest $U(1)$ abelian symmetry,
\[
\begin{align*}
t^k & \rightarrow \phi^k = e^{i\vartheta t^k} , \\
\bar{t}_k & \rightarrow \bar{\phi}_k = e^{-i\vartheta \bar{t}_k} ,
\end{align*}
\]
where the local real function $\vartheta$ is the gauge parameter of the $U(1)$ invariance.

This $U(1)$ symmetry tells us that the fields $t^k$ involved in the duality relation (3.57) are not uniquely defined. Under the abelian gauge transformation (3.63), we have
\[
\phi^i = \bar{t}_j f^i = \bar{\phi}_j \phi^i ,
\]
showing that $\phi^i$ can be interpreted as conserved quantity; that is a conserved Noether current in the field theory set up. This conserved quantity is precisely the one given by eq(3.17).

(ii) The duality relation (3.57) can be generalized as follows,
\[
\phi^i = \bar{t}_j f^i = \bar{\phi}_j \phi^i ,
\]
where now the matter fields $f^{iA}$ are in the $(2,n)$ bi-fundamental of $SU(2) \times U(n)$. For the case $n = 20$, we have,
\[
f^{iA} \simeq (2,20) \in SU(2) \times U(20) ,
\]
where the $20 \times 20$ commuting matrices $T^I$ are as in eqs(3.7). They are the commuting Cartan generators of the $U(20)$ unitary group.

Notice that the duality relation (3.65) has the following manifest abelian $U^{20}(1)$ gauge symmetry
\[
\begin{align*}
t^{kA} & \rightarrow \phi^{kA} = (e^{i\vartheta t^k})^A , \\
\bar{t}_{kA} & \rightarrow \bar{\phi}_{kA} = (e^{-i\vartheta \bar{t}_k})_A .
\end{align*}
\]
These transformations read more explicitly like,
\[
\begin{align*}
t^{kA} & \rightarrow \phi^{kA} = \sum_{C=1}^{20} (e^{i\vartheta})^A_C t^{kC} , \\
\bar{t}_{kB} & \rightarrow \bar{\phi}_{kB} = \sum_{D=1}^{20} (e^{-i\vartheta})^D_B \bar{t}_{kD} .
\end{align*}
\]

(3.68)
with \( \vartheta \) given by the expansion

\[
\vartheta = \sum_{I=1}^{20} \vartheta_I T^I,
\]

where the \( \vartheta_I \)'s are the 20 gauge parameters of the \( U^{20}(1) \) abelian invariance. It is not difficult to check that we have the identity,

\[
\phi^I_k = \bar{\phi}^I_k T^I \phi^I_k = \bar{\phi}^I_k \phi^I_k,
\]

showing that \( \phi^I_k \) is a conserved quantity; thanks to eqs (3.7).

In HSS formalism, the relation (3.70) corresponds to the leading term of the \( \theta^+ \) expansion of the HSS relation \( H^{++I} = \Phi^+ T^I \Phi^+ \) as given by eq (3.16). These conserved quantities (\( D^{++H^{++I}} = 0 \)) play a determinant role in the solving the underlying constraint eqs that lead to the computation of the metric of the moduli space \( Q_{80} \).

(c) Comments

Below, we make three comments regarding the use of the complex field variables \( (f^{kA}, \bar{f}_{kA}) \) rather than the \( \phi^I_k \) real ones.

(i) the complex fields \( f^{kA} \) are in the bi-fundamental of \( SU(2) \times U(20) \). They parameterize the complex 40 dimensional coset manifold,

\[
H_{20} = \frac{SU(2,20)}{SU(2) \times U(20)},
\]

which is contained in \( \frac{SO(4,20)}{SO(4) \times SO(20)} \). The manifold \( H_{20} \) has a richer isotropy symmetry. The isosinglet \( \phi^0_i \) and isotriplet \( \phi^{(kl)}_I \) are now given by

\[
\phi^0_i = - \bar{\phi} T^I \phi^I_i,
\]

\[
\phi^{(kl)}_I = \bar{\phi} T^I \phi^I_k + \bar{\phi} T^I \phi^I_l.
\]

By using the quaternionic form \( J^{kl} \) introduced previously, we can also rewrite the above relations collectively like,

\[
\bar{\phi} T^I \phi^I = \int_C I J^{kl}, \quad I = 1, \ldots, 20.
\]

(ii) Eqs (3.72) appear in the HSS method as the lowest component of the \( \theta^+ \) expansion of the analytic superfield

\[
H^{++I} = \Phi^+ T^I \Phi^+,
\]

that is

\[
\sum_{i,j=1}^{2} u^+_i u^-_j \left( \bar{\phi} T^I \phi^I \right) = \sum_{i,j=1}^{2} u^+_i u^-_j \left[ \Phi^+ T^I \Phi^+ \right]_{\theta=0}.
\]

Eq (3.74) obeys \( D^{++H^{++I}} = 0 \) and solved like

\[
H^{++I} = \sum_{i,j=1}^{2} u^+_i u^+_j H^{ijI}.
\]
They may be interpreted "periods" of some 10\(D\) superspace 2-form \(J^{++} = u_i^+ u_j^+ J^{ij}\) as

\[
H^{ijI} = \int_{C^I} J^{ij}, \quad I = 1, \ldots, 20, \quad (3.77)
\]

or equivalently

\[
H^{++I} = \int_{C^I} J^{++}, \quad (3.78)
\]

with lowest \(\theta^+\)-component as in eq(3.73).

(iii) Using eq(3.78), we can compute the HSS potential that describe the moduli space of 10\(D\) type IIA superstring on K3

\[
\mathcal{L}_{int}^{++} \simeq \int_{K3} J^{++} \wedge J^{++}. \quad (3.79)
\]

Upon integration, we get precisely the relation (3.18).

With these tools at hand, we are in position to compute the explicit field expression of the quaternionic metric (3.56).

### 4 Quaternionic metric

In this section, we use the complex coordinates \((f^{iA}, \bar{f}_{iA})\) and derive the explicit expression of the quaternionic metric \(G_{ab}^{IJ} = G_{ab}^{IJ} (f, \bar{f})\) of the scalar manifold of the 6\(D\) \(\mathcal{N} = 2\) supergravity.

Since the result we give here is valid for the real \(4n\) dimensional manifolds

\[
H_{2n} = \frac{SU(2,n)}{SU(2) \times U(n)}, \quad \dim H_{2n} = 4n. \quad (4.1)
\]

and in order to be as much as general, we will proceed as follow:

After, showing how the dilaton factorizes, we focus on the derivation of the quaternionic metric of the scalar manifold \(H_2\) for the case of 6\(D\) \(\mathcal{N} = 2\) supergravity multiplet coupled to a \(n = 1\) Maxwell supermultiplet. There, the scalar manifold \(H_2\) is given by,

\[
H_2 = \frac{SU(2,1)}{SU(2) \times U(1)}, \quad (4.2)
\]

and corresponds to the real 4-dimensional Taub-NUT model.

Then, we consider the computation of the metric for the generic real \(4n\) dimensional scalar manifold (4.1). The 6\(D\) \(\mathcal{N} = 2\) supergravity embedded in 10\(D\) type IIA superstring on K3 is obtained by setting \(n = 20\).

#### 4.1 Taub-NUT geometry

This geometry concerns the scalar manifold \(H_2\) involved in the the moduli space

\[
SO(1,1) \times H_2 \subset SO(1,1) \times \frac{SO(4,1)}{SO(4)}, \quad (4.3)
\]
of the $\mathcal{N} = 2$ gravity supermultiplet coupled to one Maxwell supermultiplet; i.e: $n = 1$.

**Field theory set up**

In the real field coordinate frame $\{\sigma, \phi^a\}$ of the moduli space $SO(1,1) \times \frac{SO(4,1)}{SO(4)}$, the component field action $S$ describing the underlying non linear sigma model reads as follows

$$S = \int d^6x \sqrt{-\text{det} g} \left( R - g^{\mu\nu} \partial_\mu \sigma \partial_\nu \sigma \right) + S_1 + \ldots \ . \quad (4.4)$$

Here $\mathcal{R}$ is the 6D space time scalar curvature, $\sigma$ the dilaton associated with the factor $SO(1,1)$ and

$$S_1 = \int d^6x \sqrt{-\text{det} g} \left( e^{-\sigma} g^{\mu\nu} \left( \partial_\mu \phi^a \partial_\nu \phi^b G_{ab} \right) \right) + \ldots \ . \quad (4.5)$$

The dots in the above relations stand for the extra terms required by supersymmetry and $G_{ab} = G_{ab}(\phi)$ is the metric of the factor $\frac{SO(4,1)}{SO(4)}$.

Below, we focus on the study of the scalar field contribution in $S_1$ by using the complex fields $f^i$ and $\overline{f}_k$ variables given by the field coordinates change

$$\phi^i_k = \overline{f}_k f^i \ , \quad (\phi^i_k) = \phi^k_i \ , \quad (4.6)$$

together with the $U(1)$ gauge symmetry $(3.63)$. The complex isodoublet $f^i$ parameterize the manifold $H_2$ given by eq$(1.1)$.

To get the expression of the scalar field part $S_1^{scalar}$ of the field action $(4.5)$, we first substitute,

$$\partial_\nu \phi^i_k = \overline{f}_k \partial_\nu f^i + f^i \partial_\nu \overline{f}_k \ , \quad (4.7)$$

which allows to bring $S_1^{scalar}$ to,

$$S_1^{scalar} \simeq \int d^6x \sqrt{-\text{det} g} L_1 \ , \quad (4.8)$$

with

$$L_1 = \left( B^i_\mu \partial^\mu f^i + \overline{B}_{\mu i} \partial_\mu f^i \right) \ , \quad (4.9)$$

where we have set $d\sigma = 0$. In this relation, the factor $B^i_\mu$ is a function of the physical degrees of freedom $f^i$,

$$B^i_\mu = B^i_\mu (f, \overline{f}) \ , \quad \overline{B}_{\mu i} = \overline{B}_{\mu i} (f, \overline{f}) \ . \quad (4.10)$$

The $B^i_\mu$ captures the scalar fields coupling and should be compared with eq$(3.54)$. Notice that the Lagrangian $(4.8-4.9)$ has the same structure as $(3.55)$. This property is just the manifestation of the fact that $L_1$ is nothing but the bosonic part of the HSS Lagrangian,

$$\mathcal{L}^{4+}_1 = \Phi^+ D^{++} \Phi^+ - \frac{\lambda}{2} \left( \Phi^+ \Phi^+ \right)^2 \ , \quad (4.11)$$

23
where $\Phi^+$ as in eq (3.44), $\lambda$ is a real coupling constant and $(\Phi^+ \Phi^+)^2$ is the Tub-NUT potential (3.19).

The HSS Lagrangian density (4.11) has the $U(1)$ symmetry

$$\Phi^+ \rightarrow \Phi^+ = e^{i\vartheta} \Phi^+ ,$$

with group parameter $\vartheta$. This symmetry should be associated with eq (3.63).

Moreover, using covariance under the $SU(2) \times U(1)$ isotropy symmetry of the scalar manifold, we can also put $B^i_\mu$ and $\mathcal{B}_{\mu i}$ in the form,

$${B^i}_\mu = h^i_k \partial_\mu f^k + g^{ik} \partial_\mu f^k ,$$

$$\mathcal{B}_{\mu i} = h^i_k \partial_\mu f^k + \overline{g}_{ik} \partial_\mu f^k ,$$

with

$${h^i}_k = h^i_k \left( f, \overline{f} \right) ,$$

$${g^{ik}} = g^{ik} \left( f, \overline{f} \right) ,$$

$${h^i}_k = \frac{h^i_k}{\left( f, \overline{f} \right)} ,$$

$${g^{ik}} = \frac{g^{ik}}{\left( f, \overline{f} \right)} .$$

The same covariance argument allows as well to factorize the metric components $h^i_k$, $g^{ik}$ and $\overline{g}_{ik}$ like,

$${h^i}_k = \delta^i_k \left( 1 + \xi \right) + \alpha \overline{f}_k f^i ,$$

$${g^{ik}} = \beta \overline{f}_i f^j ,$$

$${\overline{g}_{ik}} = \overline{\beta} \overline{f}_k f^j ,$$

with

$$\alpha = \alpha \left( \rho \right) ,$$

$$\xi = \xi \left( \rho \right) ,$$

$$\beta = \beta \left( \rho \right) ,$$

$$\rho = \lambda f^i \overline{f}^j ,$$

The coupling constant $\lambda$ is same as above and may be also interpreted, in the $6D$ black hole physics, as the area of the $AdS_2 \times S^4$ near horizon geometry.

Substituting (4.13) back into (4.9), we obtain

$$L_1 = \left( 2h^i_k \partial_\mu f^i \partial^\mu \overline{f}^k + g^{ij} \partial_\mu f^i \partial^\mu f^j + g^{ij} \partial_\mu \overline{f}^i \partial_\mu \overline{f}^j \right) ,$$

which should be compared with eq (3.56).

On the other hand, following [39, 47], the integration of (4.11) with respect to the Grassmann variables $\theta^{+\alpha}$ and the harmonic variables $u^i_\pm$ leads to the following,

$${h^i}_j = \delta^i_j \left( 1 + \lambda f^i \overline{f}^j \right) - \frac{\lambda}{2} \frac{(2 + \lambda f^i \overline{f}^j)}{\left( 1 + \lambda f^i \overline{f}^j \right)} f^i \overline{f}^j ,$$

$${g^{ij}} = \frac{\lambda}{2} \frac{(2 + \lambda f^i \overline{f}^j)}{\left( 1 + \lambda f^i \overline{f}^j \right)} f^i \overline{f}^j ,$$

$${\overline{g}^{ij}} = \frac{\lambda}{2} \frac{(2 + \lambda f^i \overline{f}^j)}{\left( 1 + \lambda f^i \overline{f}^j \right)} f^i \overline{f}^j .$$
From these relations, we can easily read the explicit field expressions of the functions $\alpha$, $\xi$ and $\beta$ given by eqs (4.16).

Now we turn to derive the quaternionic metric for the generic models (4.1).

### 4.2 Generic quaternionic metric

In the generic case where the $6D N = 2$ gravity supermultiplet is coupled to $n$ Maxwell multiplets, the HSS potential $\mathcal{L}_{n}^{4+}$ has the structure (3.18). The superspace Lagrangian density reads then as follows:

$$
\mathcal{L}_{n}^{4+} = \int d^{4}\theta d\bar{\theta} \left[ \tilde{\Phi}^{+} D^{++} \Phi^{+} - \lambda \left( \tilde{\Phi}^{+} T^{I} \Phi^{+} \right) d_{I,J} \left( \tilde{\Phi}^{+} T^{J} \Phi^{+} \right) \right].
$$

(4.19)

In this relation the $n \times n$ hermitian matrices $T^{I}$ are the $U(n)$ Cartan generators of the group $U(n)$; and the symmetric matrix $d_{I,J}$ is a coupling constant matrix which, for the case $n = 20$, has an interpretation in terms of intersections of 2-cycles of K3.

The equations of motion following from $\mathcal{L}_{n}^{4+}$ read as

$$
\left[ D^{++} - \lambda \left( \tilde{\Phi}^{+} T^{J} \Phi^{+} \right) d_{I,J} T^{I} \right] \Phi^{+} = 0,
$$

$$
\left[ D^{++} + \lambda \left( \tilde{\Phi}^{+} T^{J} \Phi^{+} \right) d_{I,J} T^{I} \right] \tilde{\Phi}^{+} = 0.
$$

(4.20)

The HSS Lagrangian density (4.19) and the equations of motion are invariant under the $U(n)$ gauge symmetry,

$$
\Phi^{+}_{A} = (\epsilon^{iA})_{B}^{f} \Phi^{+}_{B} , \quad \Lambda = \sum_{I=1}^{n} \Lambda_{I} T^{I}.
$$

(4.21)

The conserved HSS Noether currents corresponding to the gauge parameters $\Lambda_{I}$ are precisely

$$
H^{++}_{I} = \tilde{\Phi}^{+}_{I} T^{I} \Phi^{+} ,
$$

(4.22)

and obey the HSS conservation laws $D^{++} H^{++}_{I} = 0$.

Moreover, performing the integration of eq (4.19) with respect to the Grassmann variables $\theta^{+}$ and $\bar{\theta}^{+}$, we obtain the following,

$$
\mathcal{L}_{n} = \frac{1}{2} \int d\bar{\theta} \left( B_{\mu}^{-A} \partial^{\mu} \phi^{+}_{A} - \bar{B}_{\mu}^{-A} \partial^{\mu} \phi^{+}_{A} \right).
$$

(4.23)

To put $\mathcal{L}_{n}$ in the form,

$$
\mathcal{L}_{n} = \frac{1}{2} \left( 2h_{i_{A}i_{B}} \partial_{\mu}^{i_{A}} \partial^{\mu} \phi^{+}_{i_{B}} + \bar{g}_{i_{A}j_{B}} \partial_{\mu}^{i_{A}} \partial^{\mu} \bar{j}_{j_{B}} + g^{i_{A}j_{B}} \partial_{\mu}^{i_{A}} \partial^{\mu} \bar{j}_{j_{B}} \right),
$$

we have to perform the two following steps:

1. determine the explicit field dependence of $\phi^{+}_{A}$ and $B_{\mu}^{-A}$ in terms of the physical
\( f_A^\pm = u_i^\pm f_A^i \).

(2) Integrate eq(4.23) with respect the harmonic variables \( u_i^\pm \).

Concerning the first point, we have to solve the constraint eqs on the auxiliary fields \( \Delta A^{-} \) and \( B^{-}_{\mu A} \). These calculations are technical and lengthy. Below, we give the main lines.

For the case of the auxiliary field \( \Delta A^{-} \), the constraint eq reads as follows

\[
\left[ \partial^{++} - \lambda \left( \tilde{\phi}^+ T^I \phi^+ \right) T_I \right] \phi^+ = 0 ,
\]

(4.24)

where \( \partial^{++} \) is as in eq(2.22) and where we have set \( d_{IJ} T^J = T_I \). Eq(4.24) can be easily solved as

\[
\phi_A^+ (x,u) = u_i^+ \left( e^{\lambda \zeta T_I} \right) ,
\]

(4.25)

with

\[
\zeta = \sum_{I=1}^n \zeta_I T_I ,
\]

(4.26)

and

\[
\zeta_I = u_i^+ u_k^+ \tilde{T}_I f^k .
\]

(4.27)

Thanks to the conservation laws which equate \( \bar{\phi}^+ T_I \phi^+ = \bar{T}^+ T_I f^+ \) and makes eq(4.24) solvable.

For the case of the auxiliary field \( B^{-}_{\mu A} \), the constraint eq reads as follows

\[
\left[ \partial^{++} - \lambda \left( \tilde{\phi}^+ T^I \phi^+ \right) T_I \right] B^{-}_{\mu A} - \lambda K^\mu \phi_A^+ = 2 \partial^\mu \phi^+ ,
\]

(4.28)

where we have set

\[
K^\mu = \left( \bar{B}^{-}_{\mu T} \phi^+ + \bar{\phi}^+ T^I B^{-}_{\mu} \right) T_I .
\]

(4.29)

Eq(4.29) satisfies the useful property

\[
\partial^{++} K^\mu = 2 i T_I \partial^\mu \left( \bar{\phi}^+ T^I \phi^+ \right) .
\]

(4.30)

To solve eq(4.28), we first set

\[
B^{-}_{\mu} = e^{\lambda \zeta C^{-}_{\mu}} ,
\]

(4.31)

with \( \zeta \) as in eqs(4.26,4.27), and then look for \( C^{-}_{\mu} \). Lengthy, but straightforward, calculations lead to,

\[
C^{-\mu A} = 2 \partial^\mu f^{-A} + \lambda F^I_{\mu} f^I_{\mu} Q^{-A} + \lambda Q^+_{\mu A} Q^{-B} \left( \partial^\mu f^{-B} \right) + \lambda Q^+_{\mu A} Q^{-B} \left( \partial^\mu \tilde{f}^{-B} \right) ,
\]

where we have set

\[
Q^i_{ja} = (T_I)^A_{\mu} f^i_{\mu} , \quad \bar{Q}^j_{jB} = \tilde{T}_j D^j (T^I)^D_B .
\]

(4.32)
Putting the expression of $\phi^+$ and $B^-_{\mu}$ back into (4.23) and integrating with respect to the harmonic variables, we end with the following result:

$$2h_{kC}^{ID} = +2\delta_k^l \left( \delta_C^D + \frac{1}{2} \tilde{T}_{kA} f_{I} d_{JJ} \left( T^I \right)_C^A \left( T^J \right)_B^D \right)$$

$$-\lambda \mathcal{F}_f^{ij} \left[ \left( \delta_{I}^{A} \tilde{T}_{kA} f_{I} d_{kL} \left( T^I \right)_C^A \left( T^L \right)_B^D \right) \right], \quad (4.33)$$

and

$$g_{kC}^{ID} = +\frac{1}{2} \mathcal{F}_f^{ij} \left[ f^{A} f_{kA} \left( d_{IJ} \left( T^I \right)_C^A \left( T^J \right)_B^D \right) \right]$$

$$+\frac{1}{2} \mathcal{F}_f^{ij} \left( f^{A} f_{kA} \left( \bar{\mathcal{E}}_{I}^{K} d_{kL} \left( T^I \right)_C^A \left( T^K \right)_B^D \right) \right) \left( T^J \right)_C^A \left( T^L \right)_B^D \right), \quad (4.34)$$

as well as

$$\overline{g}_{kC}^{ID} = +\frac{1}{2} \mathcal{F}_f^{ij} \left[ f^{A} f_{kA} \left( d_{IJ} \left( T^I \right)_C^A \left( T^J \right)_B^D \right) \right]$$

$$+\frac{1}{2} \mathcal{F}_f^{ij} \left( f^{A} f_{kA} \left( \bar{\mathcal{E}}_{I}^{K} d_{kL} \left( T^I \right)_C^A \left( T^K \right)_B^D \right) \right) \left( T^J \right)_C^A \left( T^L \right)_B^D \right) \varepsilon^{kl} \varepsilon_{ij} \right]. \quad (4.35)$$

In these relations, the space time scalars $f^{A}_i$ are in the bi-fundamental of $SU(2) \times U(n)$ and the matrices $\mathcal{E}_f^i$ and $\mathcal{F}_f^k$ are given by

$$\mathcal{E}_f^i = \left[ \delta^i_j + \lambda \left( \bar{T}^j I T^J f \right) \right], \quad (4.36)$$

$$\mathcal{E}_f^i \mathcal{F}_f^k = \delta^k_i. \quad (4.37)$$

Notice that for the leading case $n = 1$, eq.(4.36) reduces to

$$\mathcal{E} = 1 + \lambda \bar{f}, \quad \mathcal{F} = \frac{1}{1 + \lambda \bar{f}}. \quad (4.37)$$

It is not difficult to check that the metric components $h_{kC}^{ID}$, $g_{kC}^{ID}$ and $\overline{g}_{kC}^{ID}$ reduce exactly to the metric terms $h^I_k$, $g^{kl}$ and $\overline{g}_{kl}$ of the real four dimensional Taub-NUT geometry. The explicit computations and the technical details from eq.(4.19) to eq.(4.35) as well as other results will be reported elsewhere.

## 5 Conclusion and discussions

In this paper, we have derived the explicit field expression of the metric $\hat{G}_{ab}^{IJ} = \hat{G}_{ab}^{IJ} (\sigma, \phi)$ of the scalar manifold $M_n^{6D, N=2}$ of generic non chiral $N = 2$ supergravity in six dimensional space time. Generally, the moduli space $M_n^{6D, N=2}$ is given by

$$SO(1, 1) \times \frac{SO(4, n)}{SO(4) \times SO(n)}, \quad n \geq 1. \quad (5.1)$$
It is a generic real \((1+4n)\)-dimensional manifold parameterized by the local real field coordinates \((\sigma, \phi_I^a)\) where
\[
\phi_I^a \sim \sum_{i,k=1}^2 \sigma_i^{ak} \phi_{ik}^I, \quad \overline{\phi_I^a} = \phi_I^a,
\] (5.2)
with \(a = 1, 2, 3, 4\) and \(I = 1, ..., 20\).

To get the explicit field expression of the metric \(\hat{G}_{ab}^{IJ}\), we have to work a little bit harder as we need various tools and several steps. To that purpose, we have first reviewed some specific aspects on \(10D\) type IIA superstring on K3 and developed useful ingredients to approach \(\hat{G}_{ab}^{IJ}\); such as the duality relation (3.8) and the quaternionic potential (3.18).

One of the basic tools that we have used to determine \(\hat{G}_{ab}^{IJ}\) is the harmonic superspace (HSS) method; which is known to be a powerful method for building quaternionic metrics [39, 46, 47, 48]. In this regards, recall that like in the case of \(4D\) \(\mathcal{N} = 1\) supersymmetry and Kahler geometry, the HSS metrics building method relies on the link between \(6D\) \(\mathcal{N} = 1\) supersymmetry and quaternionic geometry.

Below, we summarize the main steps that we have used to derive \(\hat{G}_{ab}^{IJ}\) or equivalently, the components \(h_{ijB}^A\), \(\mathcal{g}_{iAb}^J\) and \(g_{iAb}^J\):

1. First, start from the non linear sigma model field action \(S_b\) associated with the scalar manifold \(\frac{SO(1,1) \times SO(4,n)}{SO(4) \times SO(n)}\).
\[
S_b \sim \int d^6x \sqrt{-\text{det} gg^{\mu\nu}} \left( \partial_\mu \sigma \partial_\nu \sigma - e^{-\sigma} \partial_\mu \phi_i^a \partial_\nu \phi_j^b \hat{G}_{ab}^{IJ} \right), \quad (5.3)
\]
where the scalar field \(\sigma\) is the dilaton parameterizing the factor \(SO(1,1)\); and where the metric component
\[
G_{ab}^{IJ} = G_{ab}^{IJ} (\phi), \quad (5.4)
\]
has no dependence in \(\sigma\). The metric \(G_{ab}^{IJ}\) concerns then the real \(4n\) dimensional quaternionic manifold \(\frac{SO(4,n)}{SO(4) \times SO(n)}\). The minus sign in front of the second term of the right hand side is required by the flat limit \(G_{ab}^{IJ} \rightarrow -\delta^{IJ} \delta_{ab}\).

2. Second, focus on the term \(G_{ab}^{IJ}\) (5.4) by freezing the dilaton \(\sigma\) in eq(5.3); that is by setting \(d\sigma = 0\) in \(S_b\). This restriction allows to use rigid supersymmetry in \(6D\) to deal with the real \(4n\) dimensional quaternionic metric \(G_{ab}^{IJ}\).

3. Third, use the following duality relation,
\[
\phi_i^k = \bar{T}_k T_i^f f^i, \quad (5.5)
\]
extending the typical relation \(\phi_i^k = \bar{T}_k f^i\). This remarkable duality relation maps the real 4- vector \(\phi_i^k\) (and in general \(\phi_i^{kI}\)) to the hermitian \(2 \times 2\) matrix \(\bar{T}_i^k f^i\) (resp. \(\bar{T}_k T_i^f f^i\)).

The complex fields \(f^i^A\) are in the \((2, n)\) bi-fundamental of \(SU(2) \times U(n)\) and the \(T_i^f\)s are the commuting Cartan generators of \(U(n)\).
The price to pay for the field change \((5.5)\) is the symmetries of the moduli since the scalar manifold \(\frac{SO(4,n)}{SO(4) \times SO(n)}\) gets mapped to,
\[
\frac{SU(2,n)}{SU(2) \times U(n)} \subset \frac{SO(4,n)}{SO(4) \times SO(n)}.
\] (5.6)
The use of the field coordinates \(f^A_i\) and \(\bar{f}_i^A\) allows to split the metric \(G_{ab}^I(\phi)\) into the following form,
\[
G = \left( \begin{array}{cc} h_{iA}^{jB} & g_{iA}^{jB} \\ g_{iA}^{jB} & h_{iA}^{jB} \end{array} \right).
\] (5.7)
Eq\((5.5)\) permits as well to exhibit manifestly the \(U_n(1)\) gauge invariance of the Maxwell-matter sector of the 6D \(\mathcal{N} = 2\) supergravity. The \(U_n(1)\) gauge change
\[
f^i \rightarrow e^{i\vartheta} f^i, \quad \vartheta = \sum_{I=1}^{20} \vartheta_I T^I,
\] (5.8)
leaves invariant eq\((5.5)\); thanks to the relation \(T^I e^{i\vartheta} = e^{i\vartheta} T^I\).

(4) Then, use the rigid 6D harmonic superspace formalism with \(d\sigma = 0\); and think about eq\((5.3)\) as the bosonic part of the following HSS superfield action,
\[
S_n = \int d^6x d^4\theta^+ du \left[ \tilde{\Phi}^+ D^{++} \Phi^+ - \frac{\lambda}{2} \left( \tilde{\Phi}^+ T^I \Phi^+ \right) d_{IJ} \left( \tilde{\Phi}^+ T^J \Phi^+ \right) \right],
\] (5.9)
where the interaction term
\[
\mathcal{L}_{int}^{+4} = \frac{\lambda}{2} \sum_{I,J=1}^{20} \left( \tilde{\Phi}^+ T^I \Phi^+ \right) d_{IJ} \left( \tilde{\Phi}^+ T^J \Phi^+ \right),
\] (5.10)
has been derived in section 3; eq\((3.18)\). In the above relation, the HSS superfield \(\tilde{\Phi}^+\) is the off shell representation of the hypermultiplet; and \(\lambda\) is a real coupling constant which may be interpreted as the \(\mathbb{S}^4\) area of the \(AdS_2 \times \mathbb{S}^4\) near horizon geometry of the 6D black hole.

The next steps are to perform the following:

(i) integrate the superfield action \(S_n\) with respect to the Grassmann variables \(\theta^+\). This brings the action \(S_n\) to the form
\[
S_n = \int d^6x du \mathcal{L} \left( \phi^+, B^-, \Delta^{--}, u^\pm \right),
\] (5.11)
(ii) eliminate the auxiliary fields \(B^-\) and \(\Delta^{--}\) of the off shell hypermultiplets \(\tilde{\Phi}^+\) through their eqs of motion. This reduces \(S_n\) further to
\[
S_n = \int d^6x du \mathcal{L} \left( f^A_i, u^\pm_i \right),
\] (5.12)
(iii) then integrate the above relation with respect to the harmonic variables \(u^\pm_i\).

After doing all these steps, we end with the following component fields action
\[
S_n = \frac{1}{2} \int d^6x \left( 2h_{iA}^{jB} \partial_{\mu} f^A_i \partial^\mu \bar{T}^j_B + \bar{g}_{iA}^{jB} \partial_{\mu} f^A_i \partial^\mu \bar{T}^j_B + g_{iA}^{jB} \partial_{\mu} \bar{T}^i_A \partial^\mu \bar{T}^j_B \right),
\] (5.13)
from which we read the metric components

\[ h_{ij}^B = h_{ij}(f, \bar{f}) \]
\[ g_{iA}^B = g_{iA}(f, \bar{f}) \]
\[ g_{iA}^{AB} = g_{iA}(f, \bar{f}) \] \hspace{0.5cm} (5.14)

These relations are explicitly given by eqs (4.33-4.34-4.35).

We end this study by making two comments: one regarding the effective potential of the 6D black hole. The other concerns the uplifting to 7D.

### 5.1 Effective potential \( V_{6D,N=2}^{BH} \)

With this metric (5.14) at hand, we can use it to study the black hole effective potential (1.7) which we rewrite in the form,

\[ V_{6D,N=2}^{BH} = \sum_{a,b=1}^{4} K^{ab} \left( e^{-2\sigma} \left[ Z_a Z_b + \sum_{I,J=1}^{n} G_{iA}^{IJ} Z_I Z_J \right] \right) \geq 0 \hspace{0.5cm} (5.15) \]

where we have use the factorization

\[ K_{ab}(\sigma, \phi) = e^{-2\sigma} K_{ab}(\phi). \] \hspace{0.5cm} (5.16)

The above relation can be also rewritten by using the complex coordinates \( f^{iA} \) and \( \bar{f}_{iA} \); that is \( V_{6D,N=2}^{BH} = V_{BH}(f^{iA}, \bar{f}_{iA}) \).

Below, we will mainly focus our attention on describing the way to compute the quantities \( Z_a, Z_I \) and \( K_{ab} \) involved in \( V_{6D,N=2}^{BH} \) for the case of the black hole in 10D type IIA superstring on K3.

The four \( Z_a \)'s are the geometric central charges of the 6D \( \mathcal{N} = 2 \) supersymmetry. They are the dressed charges \( [6, 7, 8] \) associated with the fluxes of the four gauge field strengths of the supergravity multiplet (2.3). A way to define the \( Z_a \)'s is as follows

\[ Z_a = \int_{K_3} \mathcal{H}_2 \wedge J_a \hspace{0.5cm} (5.17) \]

where \( J_a \sim J_{ij} \) is the quaternionic 2-form given by eqs (3.33-3.36) and the real 2-form \( \mathcal{H}_2 \) is given by,

\[ \mathcal{H}_2 = \int_{S^2} \mathcal{F}_4 \hspace{0.5cm} (5.18) \]

with \( \mathcal{F}_4 \) being the the gauge invariant real 4-form field strength \( dC_3 \) of 10D type IIA superstring. The real 2-sphere \( S^2 \) belongs to the 6D space time.

The twenty dressed charges \( Z_I \) are the so called matter central charges associated with the fluxes of the twenty gauge field strengths of the Maxwell sector of the supergravity theory. Like for \( Z_a \), the \( Z_I \)'s may be also defined as \( \int_{K_3} \mathcal{H}_2 \wedge J_I \) where the real 2-form
captures the stringy and the geometric deformation moduli. Moreover, as in the case of special Kahler geometry, the $Z_I$'s can be also given by the covariant derivatives,

$$Z_I = D^a_I Z_a. \quad (5.19)$$

More explicit relations regarding the structure of the covariant derivatives can be found in [30, 9].

Concerning the real symmetric matrix $K_{ab}$; it is given by the following intersections

$$K_{ab} = \int_{K^3} J_a \wedge J_b, \quad a, b = 1, ..., 4. \quad (5.20)$$

It factorizes as $K_{ab} (\sigma, \phi) = e^{-2\sigma} K_{ab} (\phi)$ where $e^{-2\sigma}$ is associated with the factor $SO(1, 1)$ and the real symmetric field matrix $K_{ab} (\phi)$ describing the contribution of quaternionic manifold $SO(4, n) \times SO(1, 1)$.

The next steps are to compute the explicit expression of these quantities in the complex coordinate frame; write down the attractor eqs of the 6D black hole (black membrane) and then look for their solutions.

### 5.2 Uplifting to 7D

Here we develop a way to get the explicit field expression of the metric $G_{UV}^{(7D)} = G_{UV} (\sigma, \xi), U, V = 1, ..., 58$, of the generic scalar manifolds

$$M_{7D,N=2}^{n+1} = \frac{SO(4, n+1)}{SO(4) \times SO(n+1)} \times SO(1, 1), \quad (5.21)$$

of the 7D $N = 2$ supergravity models with $n$ Maxwell supermultiplets. The factor $SO(1, 1)$ is parameterized by the field isosinglet $\sigma$ and the factor $SO(4, n) \times SO(3) \times SO(n)$ by the 19 real isotriplets field coordinates

$$\xi^{\alpha u}, \quad \alpha = 1, 2, 3, \quad u = 1, ..., 19. \quad (5.22)$$

so that the metric of $M_{7D,N=2}^{n+1}$ reads as

$$dl^2 = (d\sigma)^2 - e^{-2\sigma} \sum_{\alpha, \beta=1}^{3} \sum_{u,v=1}^{19} G_{\alpha\beta}^{uv} d\xi^{\alpha u} d\xi^{\beta v}. \quad (5.23)$$

Notice that the $SO(3)$ 3- vectors $\xi^{\alpha u}$ can be also put in the form of $SU(2)$ isotriplets $\xi^{(ij)u}$ by using the homomorphism $SO(3) \simeq SU(2)$.

To determine the explicit field expression of the metric component $G_{\alpha\beta}^{uv}$, we start from our result [133, 134, 135] on the metric $G_{ab}^{IJ}$ of

$$M_{n+1}^{6D,N=2} = \frac{SO(4, n+1)}{SO(4) \times SO(n+1)} \times SO(1, 1). \quad (5.24)$$
and impose the appropriate constraint eqs that map
\[ M^{6D,N=2}_n \rightarrow M^{7D,N=2}_n. \] (5.25)

Below, we focus on the case \( n = 19 \) corresponding to embedding 7D (resp. 6D) supergravity in 11D M-theory (resp. 10D type IIA superstring) on K3.

First consider the local coordinates \((\sigma, \phi^{ij}_I)\) of the moduli space \( M^{6D,N=2}_{20} \) used in section 3. Then notice that the field coordinate variables \( \phi^{ij}_I \) can be usually decomposed into a symmetric and antisymmetric parts as follows,
\[
\phi^{ij}_I = \phi^{[ij]}_I + \phi^{(ij)}_I, \quad I = 1, \ldots, 19, \tag{5.26}
\]
with
\[
\phi^{[ij]}_I = \chi^0_{I E} \varepsilon^{ij}, \quad I = 1, \ldots, 19, \tag{5.27}
\phi^{[ij]}_I = \phi^0_{I E} \varepsilon^{ij}, \quad I = 20.
\]

Notice also that from the view of the real dimensions of the scalar manifolds \( M^{6D,N=2}_{19} \) and \( M^{7D,N=2}_{19} \) namely,
\[
dim M^{6D,N=2}_{20} = 1 + (4 \times 19 + 4), \tag{5.28}
dim M^{7D,N=2}_{19} = 1 + 3 \times 19,
\]
the uplifting from 6D to 7D requires imposing 23 constraint eqs on the local field variables \( \phi^{ij}_I \) \( \text{(5.26)} \). This number should be thought of as
\[
23 = 19 + 3 + 1, \tag{5.29}
= 22 + 1,
\]
as given by eqs\(\text{(3.22)-3.25}\).

Physically, the constraint eqs we have to impose correspond to:
(i) switching off the 22 moduli associated with the B-field fluxes. The 22 undesired moduli are precisely given by the 19 isosinglets \( \chi^0_I \) and the isotriplet \( \chi^{(ij)} \) of eq\(5.26\).
(ii) Identifying the volume of K3, captured by the isosinglet \( \phi^0 \) as in eq\(3.22\), with the field \( \sigma \) parameterizing the \( SO(1,1) \) factor.

A way to put these constraint eqs is to simply set
\[
\chi^0_I = 0, \quad I = 1, \ldots, 19, \tag{5.30}
\chi^{(ij)} = 0,
\]

\[ \phi^0 \sim e^\sigma. \tag{5.31} \]
Substituting into eq(5.26), we get

$$
\xi^{(ij)}_{u} = \phi^{(ij)}_{u} , \quad u = 1, \ldots, 19 .
$$

The constraint eqs(5.30-5.31) can be also read and solved in terms of the complex field coordinates \( f^{iA} \) and \( \overline{f}_{iA} \). We have,

$$
\chi^{0}_{I} = \sum_{A,B=1}^{20} \overline{f}_{iA} (T_{I})^{A}_{B} f^{iB} = 0 , \quad I = 1, \ldots, 19 ,
$$

(5.32)

and

$$
\phi^{0} = \lambda \sum_{A=1}^{20} \overline{f}_{iA} f^{iA} \sim e^{\sigma} .
$$

(5.33)

Eqs(5.32) can be solved by specifying the \( U^{n}(1) \) matrix generators \( T_{I} \) of the \( U(n) \) symmetry. Taking the \( T_{I} \)'s as

$$
T_{I} = \varrho_{I} - \varrho_{I+1} , \quad I = 1, \ldots, 19 ,
$$

$$
Tr (T_{I}) = 0 , \quad I = 1, \ldots, 19 ,
$$

(5.34)

and

$$
T_{20} \simeq \varrho_{1} + \ldots + \varrho_{20} ,
$$

(5.35)

with \( \varrho_{I} \) being the matrix projectors \( \varrho^{2}_{I} = \varrho_{I} \) on the vector basis \( \{ e_{I} \} \) of the underlying 20- dimensional space \( (\varrho_{K} e_{I} = \delta_{KI} e_{I}) \), it follows from the constraint eqs \( \chi^{0}_{I} = 0 \) and the realization \( \chi^{0}_{I} = 0 \) that

$$
\overline{f}_{i1} f^{i1} = \overline{f}_{i2} f^{i2} = \ldots = \overline{f}_{i20} f^{i20} .
$$

(5.36)

These relations tell us that the uplifting to 7D requires that all complex fields \( f^{iA} \) should have the same norm.

Putting this solution back into the second relation of \( \chi^{0}_{I} = 0 \), we obtain for each complex isodoublet \( f^{iA} \) the following norm,

$$
\lambda \overline{f}_{iA} f^{iA} = \frac{1}{20} e^{\sigma} .
$$

(5.37)

To conclude, the metric \( G_{\alpha\beta}^{uv} \) of the moduli space \( \frac{SO(3,19)}{SO(3) \times SO(19)} \times SO(1,1) \) of the 7D \( N = 2 \) supergravity is recovered from eqs(5.14) by imposing the constraint relations \( \chi^{0}_{I} = 0 \) and \( \phi^{0} \).

This result applies as well for the generic scalar manifolds \( \frac{SO(3,n)}{SO(3) \times SO(n)} \times SO(1,1) \) with \( n \geq 1 \).

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6 Appendix: Generalities on HSS method

Harmonic superspace method has been first introduced for solving the problem of a manifestly off shell superspace formulation of $4D \mathcal{N} = 2$ extended supersymmetric Yang Mills [40] and $4D \mathcal{N} = 2$ supergravity theories; for a review see [41], see also [42, 43]. This method has been extended as well to other dimensions and for different purposes, in particular to 2D supersymmetric field theories with eight supercharges [44]. It has been used also for approaching different matters; in particular for studying $4D$ Euclidean Yang-Mills and gravitational instantons [45] and refs therein, in the HK metrics building [39, 46, 47, 48], in dealing with the analysis of singularities of so called HyperKahler Calabi-Yau manifolds used in type II superstring compactifications [49] and in topological string on conifold [43].

In the HSS formulation of $4D \mathcal{N} = 2$ extended (or equivalently $6D \mathcal{N} = 1$) supersymmetric theory, the ordinary superspace with $SU(2)$ R-symmetry,

$$z^M = (x^\mu, \theta^a, \overline{\theta}_a), \quad i = 1, 2 \quad a, \dot{a} = 1, 2,$$

gets mapped into the harmonic superspace

$$z^M = (Y^m, \theta^+_a, \overline{\theta}^+_a, u^\pm_i),$$

with analytic supercoordinates $Y^m$ as

$$Y^m = (y^\mu, \theta^+_a, \overline{\theta}^+_a),$$

and

$$y^\mu = x^\mu + i \left( \theta^+ \sigma^\mu \overline{\theta}^- + \theta^- \sigma^\mu \overline{\theta}^+ \right),$$

$$\theta^+_i = u^+_i \theta^i_a, \quad \overline{\theta}^+_a = u^+_i \overline{\theta}^a_i,$$

where $u^+_i$ are the harmonic variables satisfying, amongst others, the relation $u^-_i u^{+i} = 1$. In HSS, the $4D \mathcal{N} = 2$ (or equivalently the $6D \mathcal{N} = 1$) hypermultiplets are represented by the analytic harmonic superfunction

$$\Phi^+ = \Phi^+ (Y, u).$$

This HSS function carry one positive Cartan-Weyl charge

$$[D^0, \Phi^+] = \Phi^+,$$

where the U(1) charge operator $D^0$ is given by

$$D^0 = \partial^0 + \left( \theta^+ \frac{\partial}{\partial \theta^+} + \overline{\theta}^+ \frac{\partial}{\partial \overline{\theta}^+} \right) - \left( \theta^- \frac{\partial}{\partial \theta^-} + \overline{\theta}^- \frac{\partial}{\partial \overline{\theta}^-} \right).$$

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The superfield $\Phi^+$ satisfies as well the following analyticity condition,

$$D_a^+ \Phi^+ = 0, \quad \overline{D}_a^+ \Phi^+ = 0, \quad a, \; \dot{a} = 1, 2, \quad (6.8)$$

where $D_a^+ = u_i^+ D_i^a$ and $\overline{D}_a^+ = u_i^+ \overline{D}_i^a$ are the supersymmetric covariant derivatives of the $4D$ $N = 2$ superalgebra. They can be thought of as

$$D_a^+ = \frac{\partial}{\partial \theta_a^+}, \quad \overline{D}_a^+ = \frac{\partial}{\partial \overline{\theta}_a^+}. \quad (6.9)$$

The $\theta^+$- expansion of $\Phi^+$ reads as

$$\Phi^+ (Y, u) = \phi^+ + \theta^{+2} F^- + \overline{\theta}^{+2} G^- + i \theta^a \overline{\theta}^{+\dot{a}} B_{a \dot{a}}^- + \theta^{+2} \overline{\theta}^{+2} \Delta^{--}, \quad (6.10)$$

where we have dropped out fermions for simplicity.

Notice that the components,

$$F^- = F^- (x, u), \quad G^- = G^- (x, u), \quad B_{a \dot{a}}^- = B_{a \dot{a}}^- (x, u), \quad (6.11)$$

are auxiliary fields scaling as a mass squared; i.e mass$^2$. In the 6D HSS formulation, these auxiliary fields combine altogether into a complex 6D vector $B^-_{[a \dot{a}]}$. The remaining one namely,

$$\Delta^{--} = \Delta^{--} (x, u), \quad (6.12)$$

is also an auxiliary field; but scaling as mass$^3$. All these fields are needed to have off shell supersymmetric theory. We also have

$$\tilde{\Phi}^+ (Y, u) = \tilde{\phi}^+ + \theta^{+2} \tilde{G}^- + \overline{\theta}^{+2} \tilde{F}^- + i \theta^a \overline{\theta}^{+\dot{a}} \tilde{B}_{a \dot{a}}^- + \theta^{+2} \overline{\theta}^{+2} \Delta^{--} \quad (6.13)$$

where ($\sim$) stands for the twild conjugation preserving the harmonic analyticity \[40\]. ($\sim$) = ($\overline{\pi}$) is a combination of the complex conjugation (bar) and the ($*$)- conjugation of the $U(1)$ subsymmetry of $SU(2)$ R-symmetry. Moreover, the component fields

$$F^q = F^q (y, u), \quad (6.14)$$

with Cartan charge $q$, can be also expanded in a harmonic series as follows:

$$F^q (y, u) = \sum_{n=0}^{\infty} u^+_{(i_1} u^+_{i_2} \ldots u^+_{i_{n+q}} u^-_{j_1} \ldots u^-_{j_n)} F^{(i_1 \ldots i_{n+q} j_1 \ldots j_n)} (y), \quad (6.15)$$

where we have taken $q \geq 0$. Notice in passing that one of the key difficulties in HSS method is to determine the right dependence of the component field $F^q (y, u)$ in terms of the dynamical scalars $f^i (y)$ and $\overline{f}_i (y)$ of the hypermultiplets.
A quite similar formulation is valid for the gauge supermultiplet $V_{4D}^{N=2}$ (or equivalently $6D\ N=1$ vector supermultiplet $V_{6D}^{+N=1}$). It is described by the analytic prepotential

$$
V^{++} = V^{++}(Y,u),
D^{a}V^{++} = 0,
\overline{D}_{a}V^{++} = 0, 
$$

(6.16)

carrying two positive Cartan-Weyl charge,

$$
[D^{0}, V^{++}] = 2V^{++}.
$$

(6.17)

The $\theta^{+}$-expansion of $V^{++}$ reads, in the Wess-Zumino gauge as follows

$$
V^{++}(Y,u) = \theta^{+2}C + \overline{\theta}^{+2}\overline{C} + \theta^{+a}\overline{\theta}^{+\dot{a}}A_{a\dot{a}} + \theta^{+2}\overline{\theta}^{+2}P^{--}.
$$

(6.18)

Notice that in the 6D HSS, the field components $C, \overline{C}$ and $A_{a\dot{a}}$ combine altogether to form a 6D gauge field $A_{[\alpha\beta]}$.

Besides describing the gauge dynamics, the gauge superfield $V^{++}$ is also used to covariantize the HSS derivative,

$$
D^{++} = \partial^{++} - 2i\theta^{+}\sigma^{\mu}\overline{\theta}^{+}\partial_{\mu} - 2i\theta^{+2}\frac{\partial}{\partial \tau} - 2i\overline{\theta}^{+2}\frac{\partial}{\partial \tau},
$$

(6.19)

where we have set $\tau = x^{4} + ix^{5}$. The gauge covariant harmonic derivative reads as

$$
\mathcal{D}^{++} = D^{++} - i\check{V}^{++}, \quad \check{V}^{++} = V^{++}.
$$

(6.20)

Under HSS gauge transformation of hypermultiplet superfields

$$
\Phi^{+\prime} = e^{i\Lambda}\Phi^{+},
$$

(6.21)

where $\Lambda$ is a real analytic gauge superparameter, U(1) gauge covariance

$$
\mathcal{D}^{++}\Phi^{+\prime} = e^{i\Lambda}\mathcal{D}^{++}\Phi^{+}
$$

(6.22)

requires that

$$
V^{+\prime} = V^{++} + D^{++}\Lambda.
$$

(6.23)

Notice that setting $\theta^{+} = 0$, the harmonic derivative $D^{++}$ and its partners

$$
D^{--} = \partial^{--} - 2i\theta^{-}\sigma^{\mu}\overline{\theta}^{-}\partial_{\mu} - 2i\theta^{-2}\frac{\partial}{\partial \tau} - 2i\overline{\theta}^{-2}\frac{\partial}{\partial \tau},
D^{0} = [D^{++}, D^{-}]
$$

(6.24)

coincide exactly with the usual $\partial^{++}, \partial^{--}$ and $\partial^{0}$ generators and obey the same $su(2)$ algebra (4.16) namely,

$$
[D^{0}, D^{++}] = 2D^{++},
[D^{0}, D^{--}] = -2D^{--},
[D^{++}, D^{--}] = D^{0}.
$$

(6.25)
Notice finally that $D^{++}$, $D^{--}$ and $D^0$ are real under the conjugation $\sim$, i.e $\tilde{D}^{++} = D^{++}$, $\tilde{D}^0 = D^0$ and $\tilde{D}^{--} = D^{--}$ and they play a crucial role in the HSS formulation of $4D$ $\mathcal{N} = 2$ ($6D$ $\mathcal{N} = 1$) supersymmetric field theory.

7 Appendix B: Geometrical approach

In this appendix, we review briefly the main lines of an alternative geometric method for dealing with the field dynamics in non chiral $\mathcal{N} = 2$ supergravity in $6D$, including the scalar field couplings considered in this paper. This geometrical construction, which has been developed in a series of papers [50]-[54], is a powerful method based on generalized Maurer-Cartan equations and the solving of the superspace Bianchi identities of the so called $\tilde{F}_4$ supergravity containing the $6D$ $\mathcal{N} = 2$ superalgebra as a subsymmetry. Though beyond the objective of our paper, we think that the geometric approach deserves a comment as it is an alternative way to the harmonic superspace method that we have developed in the present study.

Before going into technical details, it is instructive to start by describing the main result. According to the studies [50]-[53], the geometric method leads to the following supersymmetric gauge invariant component field action,

$$S_{D=6}^{\mathcal{N}=2} = \int d^6x \sqrt{-g} [\mathcal{L}_{\text{kin}} + \mathcal{L}_{\text{cs}}] + \int d^6x \sqrt{-g} \mathcal{L}_p + \text{higher spinor terms} \quad . \quad (7.1)$$

The kinetic term $\mathcal{L}_{\text{kin}}$ and the Chern-Simons one $\mathcal{L}_{\text{cs}}$ are given by,

$$\mathcal{L}_{\text{kin}} + \mathcal{L}_{\text{CS}} = -\frac{i}{4} R_6 - \frac{1}{8} e^{2\sigma} \mathcal{N}_{\Lambda \Sigma} F^\Lambda_{\mu \nu} F^\Sigma_{\mu \nu} + \frac{3}{64} e^{4\sigma} H_{\mu \rho \sigma} H^{\mu \rho \sigma} + \frac{1}{2} \bar{\psi}_{\mu} \gamma^{\mu \rho \sigma} D_{\nu} \psi^\rho - 2 i \bar{\chi}_{\mu} \gamma^\mu D_{\nu} \chi^\nu + \frac{i}{8} \bar{\lambda}_{\mu} \gamma^\mu D_{\nu} \lambda^\nu$$

$$- \frac{1}{64} e^{\mu \rho \sigma \lambda \tau} B_{\mu \nu} F^\Lambda_{\rho \sigma} F^\Sigma_{\lambda \tau} \eta_{\Lambda \Sigma} \quad . \quad (7.2)$$

The Pauli term $\mathcal{L}_p$ as well as the higher fermionic terms are lengthy and highly nontrivial; their explicit expression can be found in [50]. In the Lagrangian density (7.2), one recognizes the pure scalar fields term

$$\left( P_{\mu}^{\dagger} P_{\mu} + \frac{1}{6} \sum_{y,z=1}^{4n} \left( P_{y}^{\dagger} P_{10z} + \sum_{r=1}^{3} P_{y}^{\dagger} P_{1rz} \right) \right) \partial_\mu \phi^y \partial_\mu \phi^z,$$

with $P_{\mu} = \sum_{y=1}^{4n} \bar{P}_{y}^{\dagger} d^{\mu}$ being the vielbein of the coset $\frac{SO(4,n)}{SO(4) \times SO(n)}$, and should be then compared with eq(1.1). The gauge field coupling $\mathcal{N}_{\Lambda \Sigma}$ reads, in terms of the field matrix $L_{\Lambda \Sigma}$ parameterizing the duality group $SO(4,n)$, as follow:

$$\mathcal{N}_{\Lambda \Sigma} = \sum_{\Lambda = 1}^{4} L_{\alpha \Lambda} L_{\Sigma}^\alpha - \sum_{I = 1}^{n} \tilde{L}_{I \Lambda} L_{\Sigma}^I \quad . \quad (7.3)$$

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The other fields appearing in (7.2) are the usual gauge invariant components of the non chiral $\mathcal{N} = 2$ supergravity fields; some of them were already discussed in previous sections of our paper, others will be defined below.

Notice that the geometrical approach we are describing here has been first considered in [54] for the case of pure supergravity. It has been further developed in [50, 51, 52] by implementing the matter couplings and gaugings. The geometrical approach has been also used for the study of special features of AdS/CFT correspondence; in particular the points regarding the two following issues: (i) the relation between the $SU(2)$ gauging coupling constant $g$ and the inverse $AdS_6$ radius $4m$; see eq(7.7). (ii) the description of the Higgs phenomenon by which the gravitational two-form $B_{\mu\nu}$ becomes massive; see eq(7.10). Below, we give a general sketch of this construction by following the analysis given in the above mentioned works.

(1) Pure supergravity

Using the same convention notations as in [50, 51], the field content of the pure 6D non chiral $\mathcal{N} = 2$ supergravity multiplet in a Poincaré background reads as,

$$V_\mu^a, B_{\mu\nu}, A_\mu^\alpha, e^\sigma;\quad \psi_\mu^A, \bar{\psi}_\mu^{\dot{A}}, \chi_\mu^A, \chi_\mu^{\dot{A}},$$

with $V_\mu^a$ being the 6D space time vielbein, with indices $a, b = 0, ..., 5$ being the Lorentz flat indices, $\mu, \nu = 0, ..., 5$ the corresponding world indices and:

the fields $\psi_\mu^A$, $\bar{\psi}_\mu^{\dot{A}}$ are respectively the left-handed and the right-handed four- component gravitino fields with $A, \dot{A} = 1, 2$ transforming under the two factors of the $R$- symmetry group $O(4) \simeq SU(2)_L \times SU(2)_R$.

$B_{\mu\nu}$ is the antisymmetric field with field strength $H_{\mu\nu\rho}$.

$A_\mu^\alpha$ ($\alpha = 0, 1, 2, 3$), are the four real 6D vector gauge fields with field strength $F_{\mu\nu}^\alpha$.

$\chi_\mu^A, \chi_\mu^{\dot{A}}$ describing respectively the left-handed and the right-handed spin $\frac{1}{2}$ four components dilatinos, and finally $e^\sigma$ denotes the dilaton.

Unified description of Poincaré and AdS backgrounds

With AdS$_6$/CFT$_5$ correspondence in mind, it is interesting to start by describing some useful tools. The point is that the description of the spinors of the gravity multiplet in terms of left-handed and right-handed projection holds only in a Poincaré background. There, the $SO(1, 5)$ Weyl spinors are 4-dimensional (4-dim) with $R$-symmetry group as $SU(2)_L \times SU(2)_R$. In the AdS$_6$ background, the chiral projection cannot however be defined; one is then restricted to use rather 8-dim pseudo-Majorana spinors. The $SO(2, 5)$ spinors are 8-dim pseudo real and the above $R$-symmetry gets reduced to the $SU(2)$ diagonal subgroup of $SU(2)_L \times SU(2)_R$.

To study in a unique setting both Poincaré and AdS$_6$ vacua, it is then convenient to use from the 8-dim pseudo-Majorana spinors even in a Poincaré framework. The pseudo-Majorana condition on the gravitino 1-forms $\psi_A = \psi_\mu^A dx^\mu$ is as follows $(\psi_A)^\dagger \gamma^0 = \frac{1}{2}$
\( \overline{\psi_A} = \epsilon^{AB} \psi_B^T \). The indices \( A, B = 1, 2 \) of the spinor fields \( \psi_A, \chi_A \) transform in the fundamental of the diagonal subgroup \( SU(2) \) contained in \( SU(2)_L \times SU(2)_R \). We also have the following properties for the 8-dim antisymmetric gamma matrices: \( \gamma^7 = i \gamma^0 \gamma^1 \gamma^2 \gamma^3 \gamma^4 \gamma^5 \), \( \gamma_7^T = -\gamma_7 \) and \( (\gamma_7)^2 = -1 \).

\( \hat{F}_4 \) superalgebra

Using the fact that the \( \hat{F}_4 \) supergroup [53]-[57] has as bosonic subsymmetry the \( SO(2, 5) \times SU(2) \) group product and simply following the studies of refs [50, 51, 52], we can construct the \( \hat{F}_4 \) supergravity by using supergeometry techniques. To that purpose, we start by considering the 1-form gauge fields,

\[
\begin{align*}
V^a &= V^a_\mu dx^\mu, \\
\omega^{ab} &= \omega^{ab}_\mu dx^\mu
\end{align*}
\]

respectively associated to the \( AdS_6 \) and \( SU(2) \) algebras. The 1-forms \( V^a \) and \( \omega^{ab} \) are dual to the \( so(2, 5) \) generators \( (P_a, M_{ab}) \) and \( A^r \) to the \( su(2) \) generators \( T_r \). The \( (P_a, M_{ab}) \) matrices satisfy the following \( AdS_6 \) commutation relations,

\[
\begin{align*}
[P_a, P_b] &= 8m^2 M_{ab}, \\
[M_{ab}, P_c] &= \frac{1}{2} (\eta_{ac} P_b - \eta_{bc} P_a), \\
[M_{ab}, M_{cd}] &= \frac{1}{2} (\eta_{bc} M_{ad} + \eta_{ad} M_{bc} - \eta_{bd} M_{ac} - \eta_{ac} M_{bd})
\end{align*}
\]

while the \( T_r \)'s obey:

\[
\begin{align*}
[T^s, T^t] &= ig \varepsilon^{rst} T_r, \\
[T^s, P_a] &= [T^s, M_{ab}] = 0.
\end{align*}
\]

In above relations, the number \( g \) is the gauge coupling constant of \( SU(2) \) and \( m \) is related to the \( AdS_6 \) radius of \( \frac{SO(2, 5)}{SO(1, 5)} \) by

\[
m = (2R_{AdS})^{-1}.
\]

To construct the full \( \hat{F}_4 \) superalgebra, the minimal extension of the conformal group \( SO(2, 5) \) or equivalently of the \( AdS_6 \) group, one introduces the pseudo-Majorana spinor charges \( Q_A = (Q_{A1}, ..., Q_{A8}) \) and enlarges the

\[
so(2, 5) \oplus su(2)
\]

algebra to the full \( \hat{F}_4 \) superalgebra. The simplest way to get \( \hat{F}_4 \) is to proceed as follows:

(i) Start from the Maurer-Cartan equation of \( SO(2, 5) \times SU(2) \) gauge symmetry,

\[
\begin{align*}
DV^a &\equiv dV^a - \omega^{ab} V_b = 0, \\
\mathcal{R}^{ab} + 4m^2 V^a V^b &= 0, \\
dA^r + \frac{g}{2} \varepsilon^{rst} A_s A_t &= 0.
\end{align*}
\]
with $R^{ab} \equiv d\omega^{ab} - \omega^{ac} \wedge \omega^c_b$. (ii) Enlarge these eqs by implementing the spinor 1-forms $\psi^A = \psi^A_{\mu} dx^\mu$ dual to the fermionic generators $Q^{Aa}$.

Following [50 51 53], the minimal extension of (7.5) is given by:

$$
\begin{align*}
DV^a &= \frac{i}{2} \bar{\psi}_A \gamma_a \psi^A, \\
R^{ab} + 4m^2V^aV^b &= -m\bar{\psi}_A \gamma_{ab} \psi^A, \\
dA^r + \frac{g}{2} f_{rst} A_s A_t &= i\bar{\psi}_A \psi_B \sigma^{rAB}, \\
D\psi_A &= i\gamma_a \psi^A V^a,
\end{align*}
$$

where $D$ is the $SO(1,5) \times SU(2)$ covariant derivative which acts on spinors as follows,

$$
D\psi_A = D\psi_A - \frac{i}{2} \sigma_{AB} A_r \psi^B,
$$

with $\sigma^{r(AB)} = \frac{1}{2} \epsilon^{BC} \sigma^{rA}_C$ with $\sigma^{rA}_B$ denoting the usual Pauli matrices. One of the remarkable properties of eqs (7.6) is that their closure under $d$-differentiation is ensured only if the following relation holds

$$g = 3m.
$$

The graded commutation relation of the $\tilde{F}_4$ supergroup are obtained by using the standard identity $d\omega(X,Y) = \frac{1}{2} (X(\omega(Y)) - Y(\omega(X)) - \omega[X,Y])$ and the duality relations

$$
\begin{align*}
\omega^{ab}(M_{cd}) &= \delta^{ab}_{cd}, \\
V^a(P_b) &= \delta^a_b, \\
\psi^A = (Q_{B\beta}) &= \delta^a_a \delta^\beta_\beta,
\end{align*}
$$

The super- commutation relations read, in addition to eqs(7.4), as follows

$$
\begin{align*}
\{ \bar{Q}_{Aa}, Q_{B\beta} \} &= -i\epsilon_{AB} (\gamma^a)_{a\beta} P_a + 4i\delta_{a\beta} T_{(AB)} + m\epsilon_{AB} (\gamma^{ab})_{a\beta} M_{ab}, \\
[M_{ab}, Q_{B\beta}] &= -\frac{i}{2} \bar{Q}_{Aa} (\gamma_{ab})_{a\beta}, \\
[P_a, \bar{Q}_{Aa}] &= im\bar{Q}_{Aa} (\gamma_a)_{a\beta}, \\
[T_{(AB)}, \bar{Q}_{C\alpha}] &= \frac{i}{2} g (\bar{Q}_{Aa} \delta_{BC} + \bar{Q}_{Ba} \delta_{AC}),
\end{align*}
$$

where $T_{(AB)}$ stands for $T_r \sigma^{r(AB)}$ and obviously $g = 3m$ as an outcome of the graded Jacobi identities.

**Free Differential Algebra (FDA.)**

Using results obtained in [51 53 58], the supersymmetric Maurer-Cartan eqs(7.6) keep the same form when we pass from the $\tilde{F}_4$ supergroup to the superspace coset $SO(1,5) \times SU(2)$, which contains $AdS_6$ as bosonic submanifold. The previous 1-forms

$$
V^a, \omega^{ab}, \psi_A, A^r,
$$

40
become now *superfield 1-forms* describing the vacuum configuration in superspace whose bosonic subspace is $AdS_6$. Notice that on the ordinary space-time, recovered by setting odd superspace dimension $\theta = 0$ in the superfields 1-forms, the background vacuum fields have precisely as $dx^\mu$ components the following expressions:

$$V^a_\mu = \delta^a_\mu \ , \ \psi_{A\mu} = 0 \ , \ (\omega^{ab}_\mu, A^r_\mu) = \text{pure gauge} \ .$$

Notice also that because of the absence of the 2-form $B$ and of the 1-form $A^0$ superfields, eqs (7.6) cannot describe the supersymmetric vacuum of the full $\hat{F}_4$ supergravity theory. To overcome this difficulty, we use the *Free Differential Algebra* (FDA) \[58\] obtained from the $\hat{F}_4$ Maurer-Cartan eqs (7.6) by adding two more equations as given below,

$$D V^a - \frac{i}{2} \bar{\psi}_A \gamma_a \psi^A = 0 \ ,$$

$$\mathcal{R}^{ab} + 4m^2 V^a V^b + m \bar{\psi}_A \gamma_{ab} \psi^A = 0 \ ,$$

$$dA^r + \frac{g}{2} \varepsilon^{rst} A_s A_t - i \bar{\psi}_A \psi_B \sigma^{rAB} = 0 \ ,$$

$$D \psi_A - im \gamma_a \psi^A V^a = 0 \ ,$$

and

$$dA - mB - i \bar{\psi}_A \gamma^r \psi^A = 0 \ ,$$

$$dB + 2 \bar{\psi}_A \gamma_a \psi^A V^a = 0 .$$

(7.8)

One of the very remarkable features of the FDA eqs, describing the full supersymmetric vacuum configuration, is the appearance of the combination

$$dA^0 - mB = \frac{1}{2} dx^\mu \wedge dx^\nu (\partial[\mu A^0_\nu] - m B[\mu\nu]) \ .$$

(7.10)

At the dynamical level, this relation implies a Higgs phenomenon where the 2-form $B$ *eats* the 1-form $A^0$ and acquires a non vanishing mass $m$.

Setting $m = g = 0$, one reduces the $\hat{F}_4$ superalgebra to the $6D \mathcal{N} = 2$ superalgebra existing only in a super- Poincaré background. In this case, the four gauge fields

$$A^a \equiv (A^0, A^r)$$

transforms in the fundamental of the $R$-symmetry group $SO(4)$ and the pseudo-Majorana spinors $\psi_A, \chi_A$ decomposes in two chiral spinors in such a way that all the resulting FDA is invariant under $SO(4)$.

Moreover it is no difficult to see that no FDA exists for the cases

$$\begin{cases} 
  m = 0, & g \neq 0 \\
  m \neq 0, & g = 0 
\end{cases}$$

since the corresponding FDA eqs do not close anymore under $d$- differentiation. For a supersymmetric vacuum to exist, the gauging of $SU(2)$,

$$g \neq 0,$$
must be necessarily accompanied by the presence of the parameter \( m \) which makes the closure of the supersymmetric algebra consistent as far as the condition (7.7) holds. Now we turn to discuss matter couplings.

(2) Implementing matter vector multiplets

The vector multiplets of the 6D non chiral supergravity are given by the multiplets \((A^I_\mu, \lambda^I_\alpha, \phi^{\alpha I})\) where \( \alpha = 0, \ldots, 3 \) and \( I = 1, \ldots, n \) labeling an arbitrary number \( n \) of \((A^I_\mu, \lambda^I_\alpha, \phi^{\alpha I}).\) The \( 4n \) scalars \( \phi^{\alpha I} (\equiv \phi^y) \) with \( y = 1, \ldots, 4n, \) together with \( e^\sigma \) of the pure supergravity multiplet, parameterize the coset manifold

\[
\frac{SO(4,n)}{SO(4) \times SO(n)} \times SO(1,1), \quad G = SO(4,n)
\]

To perform the matter coupling we use the geometrical procedure whose main lines are as follows: First, introduce the coset representative \( L^\Lambda_{\Lambda^\Sigma} \) of the matter coset manifold with \( \Lambda, \Sigma = 0, \ldots, 3 + n; \) that is a \((n + 4) \times (n + 4)\) matrix \( L_{(n+4)\times(n+4)} = (L_{\Lambda\Sigma}).\) Then, decompose the \( SO(4,n) \) index \( \Sigma \) with respect to \( H = SO(4) \times SO(n) \) and put \( L^\Lambda_{\Lambda^\Sigma} \) like:

\[
L^\Lambda_{\Lambda^\Sigma} = (L^\Lambda_0, L^\Lambda_1)
\]  

(7.11)

To gauge the \( SU(2) \) diagonal subgroup of \( SO(4) \) as in pure supergravity, it is useful to further decompose \( L^\Lambda_\alpha \) as

\[
L^\Lambda_\alpha = (L^\Lambda_0, L^\Lambda_1), \quad r = 1, 2, 3
\]

The \((4 + n)\) gravitational and matter vectors transform in the fundamental of \( SO(4,n), \) the corresponding superspace curvatures are then labeled by the index \( \Lambda \) and the covariant derivatives acting on the spinor fields will now contain the composite connections of the \( SO(4,n) \) duality group. Moreover, the \( SO(4,n) \) left- invariant 1-form \( \Omega^\Lambda_{\Sigma} = (L^{-1})^\Lambda_{\Gamma} \, dL^\Gamma_{\Sigma} \) satisfying the usual Maurer-Cartan equation,

\[
d\Omega^\Lambda_{\Sigma} + \Omega^\Lambda_{\Gamma} \wedge \Omega^\Gamma_{\Sigma} = 0,
\]

can be decomposed as follows

\[
R^r_s = -P^r_s \wedge P^s_t, \quad R^0_0 = -P^t_t \wedge P^0_0, \\
R^1_j = -P^r_r \wedge P^j_j - P^t_t \wedge P^0_0, \quad \nabla P^r_r = \nabla P^t_t = 0
\]

where we have set \( P^I_\alpha = (P^I_0, P^I_1) \equiv (\Omega^I_0, \Omega^I_1) \) and where

\[
R^{rs} = d\Omega^r_r + \Omega^r_r \wedge \Omega^s_s + \Omega^r_r \wedge \Omega^s_0, \\
R^{r0} = d\Omega^r_r + \Omega^r_r \wedge \Omega^0_0, \\
R^{Ij} = d\Omega^I_i + \Omega^I_i \wedge \Omega^K_j
\]
Notice that \((P_I^0, P_I^r)\) are the 1-form vielbeins of the coset, \((\Omega^r, \Omega^0)\) are the connections of \(SO(4)\) decomposed with respect to the diagonal subgroup \(SU(2) \subset SO(4)\) and \((R^r, R^0)\) are the corresponding curvatures. The superspace curvatures of the matter coupled theory read as follows

\[
T^A = D V^a - \frac{i}{2} \bar{\psi}_A \gamma^a \psi^A V^a = 0 , \quad R^{ab} = R^{ab} , \quad H = dB + 2 e^{-2\sigma} \bar{\psi}_A \gamma^A \psi^A V^a , \quad R(\sigma) = d\sigma , \quad F^A = F^A - i e^\sigma \bar{\psi}_A (L^A_0 \epsilon^{AB} \gamma_7 + \sigma L^r_0 \sigma^{rAB}) \psi_B , \quad R^0_0 (\phi) = P^I_{0y} d\phi^y , \quad \rho_A = D \psi_A + \frac{i}{2} \sigma_{rAB} (\frac{1}{2} \epsilon^{rst} \Omega_{st} + i \gamma_7 \Omega_{r0}) \psi^B , \quad R^I_r (\phi) = P^I_{ry} d\phi^y ,
\]

together with

\[
D \chi_A = D \chi_A + \frac{i}{2} \sigma_{rAB} (\frac{1}{2} \epsilon^{rst} \Omega_{st} + i \gamma_7 \Omega_{r0}) \chi^B , \quad \nabla \lambda_{IA} = D \lambda_{IA} + \frac{i}{2} \sigma_{rAB} (\frac{1}{2} \epsilon^{rst} \Omega_{st} + i \gamma_7 \Omega_{r0}) \lambda^B_I + \Omega^I_J \lambda_{JA} ,
\]

These relations extend the usual curvatures

\[
T^a = D V^a - \frac{i}{2} \bar{\psi} \gamma^a \psi^A V^a = 0 , \quad R^{ab} = R^{ab} , \quad H = dB + 2 e^{-2\sigma} \bar{\psi} \gamma^A \psi^A V^a , \quad \rho_A = D \psi_A , \quad F = dA - i e^\sigma \bar{\psi} \gamma_7 \psi^A , \quad R(\chi_A) \equiv D \chi_A , \quad F^r = dA^r - i e^\sigma \bar{\psi} \psi_B \sigma^{rAB} , \quad R(\sigma) \equiv d\sigma ,
\]

associated with the pure supergravity case \([54]\). Notice in passing that in eq \((7.12)\) there appear, in the vector field strengths \(F^A\), the G/H coset representatives \(L^A_0\), which intertwine between the \(R\)-symmetry indices \(A, B\) of the gravitinos and the indices \(\Lambda, \Sigma\) of the \((4+n)\)-dimensional \(G\) representation.

The Bianchi identities read as \(D R^{ab} = d^2 \sigma = D P^I_{AB} = 0\) together with

\[
R^{ab} V_b - i \bar{\psi}_A \gamma^a \rho_B e^{AB} = 0 , \quad D^2 \lambda^A_A + \frac{1}{4} R^{ab} \sigma_{ab} \lambda^I_I - \frac{i}{2} \sigma_{rAB} (\frac{1}{2} \epsilon^{rst} R^r + i \gamma_7 R_{r0}) \lambda^B_I - R^I_J \lambda_{JA} = 0 , \quad dH + 4 e^{-2\sigma} d\sigma \bar{\psi}_A \gamma_7 \gamma_a \psi_B e^{AB} V^a + 4 e^{-2\sigma} \bar{\psi}_A \gamma_7 \gamma_a \rho_B e^{AB} V^a = 0 , \quad D \rho_A + \frac{1}{4} R^{ab} \gamma_a \psi_B - \frac{i}{2} \sigma_{rAB} (\frac{1}{2} \epsilon^{rst} R^r + i \gamma_7 R_{r0}) \psi^B = 0 , \quad D^2 \chi_A + \frac{1}{4} R^{ab} \gamma_a \chi_B - \frac{i}{2} \sigma_{rAB} (\frac{1}{2} \epsilon^{rst} R^r + i \gamma_7 R_{r0}) \chi^B = 0 ,
\]

as well as

\[
D F^A + id\sigma e^\sigma \bar{\psi}_A \gamma_7 \psi_B L^A_{[AB]} + id\sigma e^\sigma \bar{\psi}_B \psi_A L^A_{(AB)} - 2 i e^\sigma \bar{\psi}_A \gamma_7 \rho_B L^A_{[AB]} - 2 i e^\sigma \bar{\psi}_A \rho_B L^A_{(AB)} + i e^\sigma L^A \bar{\psi}_A \psi_B P^I_{AB} + i e^\sigma L^A \bar{\psi}_A \gamma_7 \psi_B P^I_{AB} = 0 ,
\]

where \(P^I_{AB} = P^I_0 \epsilon_{AB} + P^I_r \sigma^{rAB}\).

**Supersymmetric transformations**

The solution of these Bianchi identities is highly non trivial, especially the one regarding
gravitino 1-forms where one needs cubic fermionic terms of the form $\psi\psi\chi$; explicit results may be found in [50, 51, 53]. Below, we content ourself to quote the solution in terms of the supersymmetric transformations of the physical fields which, as is known, can be written down once the parameterizations of the supercurvatures in superspace are identified. The result is

$$
\delta V^\mu_a = -i\bar{\psi}_{A_a}A^\mu \varepsilon^A,
\delta B_{\mu\nu} = 4i\varepsilon^{A} \bar{\chi}_A \gamma_7 \gamma_{\mu\nu} \varepsilon^A - 4\varepsilon^{A} \bar{\psi}_A \gamma_7 \gamma_{\mu\nu} \varepsilon^A,
\delta A^\lambda_\mu = 2\varepsilon^{A} \left( L_0^A \varepsilon_{AB} \gamma_7 + L_{AB} \varepsilon_{rAB} \right) \gamma_\lambda \gamma_\mu \gamma^B - \varepsilon^{A} \gamma^B L_0^A \varepsilon_{AB} \gamma_\mu \gamma^B \varepsilon^A
+ 2\varepsilon^{A} \left( \varepsilon_{AB} L_0^A \gamma_7 + L_{AB} \sigma_{rAB} \right) \psi_B
$$

and

$$
\delta \psi_{A_\mu} = \mathcal{D}_\mu \varepsilon_A + \frac{i}{16} \varepsilon^{A} \left[ T_{[AB]\mu\nu} \gamma_7 - T_{(AB)\mu\nu} \right] (\gamma_\mu \varepsilon^{A} \gamma_\nu \varepsilon^A) + \frac{\varepsilon^{A}}{32} H_{\mu\nu\lambda} \gamma_7 \gamma_\mu \gamma_\nu \varepsilon^A
\quad + \frac{i}{2} H_{\mu\nu\lambda} \gamma_7 \gamma_\mu \gamma_\nu \varepsilon^A + \frac{i}{4} \varepsilon^{A} \bar{\psi}_{A_\mu} \gamma_7 \varepsilon^A
\quad - \frac{i}{4} \varepsilon^{A} \bar{\psi}_{A_\mu} \gamma_7 \varepsilon^A,
$$

as well as

$$
\delta \chi_A = \frac{i}{2} \gamma_\mu \partial_\mu \varepsilon_A + \frac{i}{16} \varepsilon^{A} \left( T_{[AB]\mu\nu} \gamma_7 - T_{(AB)\mu\nu} \right) \gamma_\mu \varepsilon^{A} \gamma_\nu \varepsilon^A + \frac{\varepsilon^{A}}{32} H_{\mu\nu\lambda} \gamma_7 \gamma_\mu \gamma_\nu \varepsilon^A,
\delta \sigma = \frac{\varepsilon^{A}}{X_A \gamma_7} \varepsilon^A,
\delta \lambda_{IA} = \left( P_{0y} \varepsilon_{AB} \gamma_7 - i P_{ry} \sigma_{rAB} \right) \gamma_\mu \varepsilon_B \partial_\mu \phi^y + \frac{i}{2} \varepsilon^{A} \bar{\psi}_{A_\mu} \gamma_\mu \gamma_7 \varepsilon^A
\quad + \frac{i}{2} \varepsilon^{A} \bar{\psi}_{A_\mu} \gamma_7 \varepsilon^A,
\delta \psi_{A_\mu} = \mathcal{D}_\mu \varepsilon_A,
\delta \chi_A = 0,
\delta \lambda_{IA} = 0
$$

where

$$
T_{[AB]\mu\nu} = \varepsilon_{AB} L_{0A}^{-1} F_{\mu\nu}^A,
T_{(AB)\mu\nu} = \sigma_{rAB} L_{rA}^{-1} F_{\mu\nu}^A,
T_{1\mu\nu} = L_{1A}^{-1} F_{\mu\nu}^A,
$$

stands for dressed vector field strengths.

Notice that in the transformation of the fermions, the cubic fermionic terms type $\chi^3$, $(\lambda^2 \chi)$, $(\chi^2 \lambda)$ have been omitted. An important property of the solution presented above is that no supersymmetric $AdS_6$ background exists. In the Poincaré vacuum, where all the field strengths are zero and the scalar $\sigma$ takes an arbitrary constant value, one has,

$$
\delta \psi_{A_\mu} = \mathcal{D}_\mu \varepsilon_A,
\delta \chi_A = 0,
\delta \lambda_{IA} = 0,
$$

The solutions of the Bianchi identities give as well the equations of motion of the physical fields which allow in turn to reconstruct the space-time Lagrangian. The obtained component field Lagrangian is precisely the one given by eq(7.1).
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