Neutrino mass matrix in the standard parametrization with texture two zeros

R. Mohanta\textsuperscript{1}, G. Kranti\textsuperscript{1}, A. K. Giri\textsuperscript{2}

\textsuperscript{1} School of Physics, University of Hyderabad, Hyderabad - 500 046, India
\textsuperscript{2} Department of Physics, Punjabi University, Patiala - 147 002, India

Abstract

We study the texture two zeros neutrino mass matrices using the standard parametrization for the neutrino mixing matrix. We find that if the origin of CP violation in the leptonic sector is not due to the Dirac-type complex phase of the mixing matrix but because of some non-standard phenomena then some of the possible texture two mass matrices, which are allowed by standard parametrization, are found to be unsuitable to accommodate the observed data in the neutrino sector. Furthermore, incorporating nonzero Dirac phase in our analysis we find that many of them do not exhibit normal hierarchy.
The study of neutrino physics is now one of the hotly pursued areas of High Energy Physics research. The recent experiments on solar, atmospheric, reactor and accelerator neutrinos [1] have provided us an unambiguous evidence that neutrinos are massive and lepton flavors are mixed. Within the standard model neutrinos are strictly massless. Thus the non-vanishing neutrino mass is the first clear evidence of new physics beyond the standard model. Since neutrinos are massive, there will be flavor mixing in the charged current interaction of the leptons and a leptonic mixing matrix will appear analogous to the CKM mixing matrix for the quarks. Thus, the three flavor eigenstates of neutrinos ($\nu_e$, $\nu_\mu$, $\nu_\tau$) are related to the corresponding mass eigenstates ($\nu_1$, $\nu_2$, $\nu_3$) by the unitary transformation

$$
\begin{pmatrix}
\nu_e \\
\nu_\mu \\
\nu_\tau
\end{pmatrix}
= \begin{pmatrix}
V_{e1} & V_{e2} & V_{e3} \\
V_{\mu1} & V_{\mu2} & V_{\mu3} \\
V_{\tau1} & V_{\tau2} & V_{\tau3}
\end{pmatrix}
\begin{pmatrix}
\nu_1 \\
\nu_2 \\
\nu_3
\end{pmatrix},
$$

(1)

where $V$ is the $3 \times 3$ unitary matrix known as PMNS matrix [2], which contains three mixing angles and three CP violating phases (one Dirac type and two Majorana type). In general $V$ can be written as $V = UP$, where $U$ is the unitary matrix analogous to the quark mixing matrix and $P$ is a diagonal matrix containing two Majorana phases, i.e., $P = \text{diagonal} (e^{i\rho}, e^{i\sigma}, 1)$. The presence of the leptonic mixing, analogous to that of quark mixing, has opened up the possibility that CP violation could also be there in the lepton sector as it exists in the quark sector. In the standard parametrization (PDG) the mixing matrix is given as

$$
U = \begin{pmatrix}
c_x c_z & s_x c_z & s_z e^{-i\delta} \\
-s_x c_y - c_x s_y s_z e^{i\delta} & c_x c_y - s_x s_y s_z e^{i\delta} & s_y c_z \\
 s_x s_y - c_x c_y s_z e^{i\delta} & -c_x s_y - s_x c_y s_z e^{i\delta} & c_y c_z
\end{pmatrix},
$$

(2)

where $\theta_{(x,y,z)} \equiv \theta_{(12,23,13)}$ and $s_x \equiv \sin \theta_x$, $c_x \equiv \cos \theta_x$, and so on.

Several analyses have been performed in order to understand the form of the neutrino mixing matrix and the pattern of lepton mixing appears to be understood. The 2-3 mixing is consistent with maximal, 1-2 mixing is large but not maximal, 1-3 mixing is small and appears to be close to zero. It is thus inferred from the current experimental data that the mixing matrix $U$ involves two large mixing angles ($\theta_{12} \sim 30^\circ$ and $\theta_{23} \sim 45^\circ$) and one small angle ($\theta_{13} < 12^\circ$) [3]. The best-fit values [4] of the mixing angles with $2\sigma$ errors are found to be $\theta_x = 34^{+3.5}_{-2.9}^\circ$, $\theta_y = 41.6^{+10.4}_{-5.7}^\circ$, $\theta_z = 5.4^{+4.9}_{-5.4}^\circ$. On the other hand, the three CP violating phases $\delta$ (Dirac type), $\rho$ and $\sigma$ (Majorana type) are totally unrestricted.
The study of CP violation in the leptonic sector is also very important for a complete understanding of the neutrino masses and mixing as it is intimately related to the mixing matrix. Furthermore, there appears to be no reason why CP violation should not be there in the leptonic sector keeping in mind the fact that large CP violation has already been established in the quark sector. CP violation in the leptonic sector occurs in the neutrino oscillation due to the non vanishing Dirac type phase $\delta$ or due to some symmetry breaking at very high energy. If one considers the effect of CP violation is due to the neutrino flavor-mixing one can then obtain the rephasing invariant quantity [5]

$$J = \text{Im} \left( U_{\alpha i} U_{\beta j}^* U_{\alpha j}^* U_{\beta i}^* \right) = \frac{1}{8} \sin 2\theta_x \sin 2\theta_y \sin 2\theta_z \cos \theta_z \sin \delta .$$

(3)

Using the current experimental data on the mixing angles, one thus obtains $J \sim \mathcal{O}(10^{-2}) \sin \delta$. Therefore, unless $\delta$ is very small the CP violation effect could be observable in the long baseline experiments. However, since CP violation is not observed so far in the lepton sector, $\delta$ is expected to be negligibly small. In our analysis, therefore, we would like to see the effect on the neutrino mass matrix when the Dirac type phase happens to be zero and also when it is non-zero.

One of the main objectives of neutrino physics research is to identify the form and the origin of neutrino mass matrix [6]. Unfortunately, so far, we have been able to infer only the mass difference squares for the neutrinos but not the individual ones apart from the maximal (23), large but not maximal (12) and small (13) mixing angles. Furthermore, there is another important issue which needs to be settled regarding whether neutrinos respect the normal hierarchy, as in case of quarks, or to that of inverted hierarchy apart from the very fact that it is not yet established whether neutrinos are of Dirac type or of Majorana nature. Dedicated neutrino experiments have already provided us with the first ever clear evidence of physics beyond the standard model in the form of non-zero neutrino mass squares. Therefore, it is a challenging time for the theoretical community to settle down some of the issues, mentioned above, at the earliest possibility.

Studies based on mass matrices can help us to understand the nature of neutrinos where one can obtain relations among the individual neutrino masses and the mixing angles, and those findings in turn, alongwith the inputs form the data, can guide us to unravel the true nature of the neutrino mass matrix. There exist many studies in the literature regarding the textures in neutrinos as well as in the quark sector. These studies help us to identify with the flavor symmetry and are also shown to be related to the physics at higher scale, e.g.,
TeV scale physics. The spirit of lepton-quark universality motivates one to assume that the lepton mass matrices might have the same texture zeros as the quark mass matrices. Such an assumption is indeed reasonable in some specific models of grand unified theory (GUT) in which mass matrices of leptons and quarks are related to each other by a new kind of flavor symmetry. It is well known that the texture two zero quark mass matrices for $M_u$ and $M_d$ are more successful than the corresponding three-zero textures to interpret the strong hierarchy of the quark masses and the smallness of flavor mixing angles. That is why two-zero texture of charged-lepton and neutrino mass matrices have been considered as a typical example in some model buildings. Furthermore, the texture two zero neutrino mass matrices have more free parameters than texture three zeros, which are quite suitable to interpret the observed bi-large pattern of lepton flavor mixing. Recently, Frampton, Glashow and Marfatia [7] have examined the possibility that the lepton mass matrices with texture two zeros may describe the current experimental data and obtained seven acceptable forms. Considering the Fritzsch type parametrization Xing [8] has carried out the investigation and obtained the expressions for neutrino mass ratios and calculated the Majorana-type CP-violating phases for all seven possible textures.

In this paper, we first study the effects of vanishing Dirac type phase on the texture two neutrino mass matrices. We then consider the PDG standard parametrization for the mixing matrix and obtain the ratios of different neutrino masses. We find that out of the seven possible forms only three are allowed by the current experimental data, if the Dirac type CP violating phase happens to be zero. We thereafter study the case of non-zero Dirac phase and obtain interesting results.

In the flavor basis, where the charged lepton mass matrix is diagonal, the neutrino mass matrix can be written as

$$M = V \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} V^T = U \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} U^T,$$

where $m_i$ (for $i = 1, 2, 3$) denote the real and positive neutrino masses, and $\lambda_i$ are the complex neutrino mass eigenvalues which include the two Majorana-type CP-violating phases

$$\lambda_1 = m_1 e^{2i\rho}, \quad \lambda_2 = m_2 e^{2i\sigma}, \quad \lambda_3 = m_3.$$

Since $M$ is symmetric with two texture zeros one can immediately obtain the constraint
mass ratios as follows:

\[ \sum_{i=1}^{3} (U_{ai} U_{bi} \lambda_i) = 0, \quad \sum_{i=1}^{3} (U_{ai} U_{bi} \lambda_i) = 0, \]  

(6)

where each of the four subscripts run over \( e, \mu \) and \( \tau \), but \( (\alpha, \beta) \neq (a, b) \). Solution of Eq. (6) yields

\[ \frac{\lambda_1}{\lambda_3} = \frac{U_{a3} U_{b3} U_{\alpha2} U_{\beta2} - U_{a2} U_{b2} U_{\alpha3} U_{\beta3}}{U_{a2} U_{b2} U_{\alpha1} U_{\beta1} - U_{a1} U_{b1} U_{\alpha2} U_{\beta2}}, \]  

(7)

and

\[ \frac{\lambda_2}{\lambda_3} = \frac{U_{a1} U_{b1} U_{\alpha3} U_{\beta3} - U_{a3} U_{b3} U_{\alpha1} U_{\beta1}}{U_{a2} U_{b2} U_{\alpha1} U_{\beta1} - U_{a1} U_{b1} U_{\alpha2} U_{\beta2}}. \]  

(8)

Now comparing Eqs. (7) and (8) with Eq. (5), one can obtain the expressions of neutrino mass ratios as follows:

\[ \frac{m_1}{m_3} = \frac{|U_{a3} U_{b3} U_{\alpha2} U_{\beta2} - U_{a2} U_{b2} U_{\alpha3} U_{\beta3}|}{|U_{a2} U_{b2} U_{\alpha1} U_{\beta1} - U_{a1} U_{b1} U_{\alpha2} U_{\beta2}|}, \]  

(9)

and the two Majorana phases are found to be

\[ \rho = \frac{1}{2} \arg \left[ \frac{U_{a3} U_{b3} U_{\alpha2} U_{\beta2} - U_{a2} U_{b2} U_{\alpha3} U_{\beta3}}{U_{a2} U_{b2} U_{\alpha1} U_{\beta1} - U_{a1} U_{b1} U_{\alpha2} U_{\beta2}} \right], \]

\[ \sigma = \frac{1}{2} \arg \left[ \frac{U_{a1} U_{b1} U_{\alpha3} U_{\beta3} - U_{a3} U_{b3} U_{\alpha1} U_{\beta1}}{U_{a2} U_{b2} U_{\alpha1} U_{\beta1} - U_{a1} U_{b1} U_{\alpha2} U_{\beta2}} \right]. \]  

(10)

Furthermore, the ratio of the mass square differences, which is basically the ratio of solar and atmospheric mass square differences is give as [4]

\[ R_\nu \equiv \left| \frac{m_2^2 - m_1^2}{m_3^2 - m_2^2} \right| = \frac{\Delta m_{\text{sun}}^2}{\Delta m_{\text{atm}}^2} \approx 0.033 \pm 0.008. \]  

(11)

The Majorana nature of the neutrinos allows us to probe one element of the mass matrix directly. The decay width for the neutrinoless double \( \beta \) decay, i.e., \((A, Z) \to (A, Z + 2) + 2e^-\), a second order weak process, is proportional to the effective mass given as

\[ |M_{ee}| = m_3 \left| \frac{m_1}{m_3} U_{e1}^2 e^{2i\rho} + \frac{m_2}{m_3} U_{e2}^2 e^{2i\sigma} + U_{e3}^2 \right|. \]  

(12)

Thus, the \( ee \) element of the mass matrix \( M \) can be directly obtained from the experiment.
Now we evaluate the above quantities using the standard parametrization and with Dirac type phase as zero for the flavor mixing matrix $U$:

$$U = \begin{pmatrix} c_x c_z & s_x c_z & s_z \\ -s_x c_y - c_x s_y s_z & c_x c_y - s_x s_y s_z & s_y c_z \\ s_x s_y - c_x c_y s_z & -c_x s_y - s_x c_y s_z & c_y c_z \end{pmatrix}.$$  \hspace{1cm} (13)

**Pattern A\textsubscript{1}:** $M_{ee} = M_{e\mu} = 0$ (i.e., $a = b = e$; $\alpha = e$ and $\beta = \mu$). By use of Eqs. (7)–(12), we obtain the mass ratios as

$$\frac{\lambda_1}{\lambda_3} = \frac{s_z}{c_z} [t_x t_y - s_z],$$
$$\frac{\lambda_2}{\lambda_3} = -\frac{s_z}{c_z} [t_y t_x + s_z].$$  \hspace{1cm} (14)

Since the 1-3 mixing angle ($\theta_z$) is very small, it is appropriate to take the limit $s_z^2 \ll 1$ and $c_z^2 \to 1$. In this limit, one can explicitly obtain the different mass ratios and the Majorana type phases as

$$\frac{m_1}{m_3} \approx s_z t_x t_y, \quad \frac{m_2}{m_3} \approx s_z \frac{t_y}{t_x}; \quad \rho \approx 0, \quad \sigma \approx \frac{\pi}{2};$$
$$R_\nu \approx \frac{t_y^2}{t_x^2} \left| 1 - t_x^4 \right| s_z^2, \quad |M_{ee}| = m_3 s_z^2.$$  \hspace{1cm} (15)

Now using the central values of the mixing angles from [4] i.e., $\theta_x = 34^\circ$, $\theta_y = 42^\circ$ and $\theta_z = 5^\circ$, we obtain the values of the mass ratios as

$$\frac{m_1}{m_3} \approx 0.053, \quad \frac{m_2}{m_3} \approx 0.116, \quad R_\nu \approx 0.01, \quad |M_{ee}|/m_3 = 0.0076.$$  \hspace{1cm} (16)

Thus, from Eq. (16), it can be seen that this pattern of mass matrix corresponds to the normal hierarchy case i.e., $m_1 < m_2 < m_3$. The ratio of mass square differences $R_\nu$ is found to be $O(10^{-2})$, which is consistent with the ratio of solar to atmospheric squared mass differences.

**Pattern A\textsubscript{2}:** $M_{ee} = M_{e\tau} = 0$ (i.e., $a = b = e$; $\alpha = e$ and $\beta = \tau$). In this case the mass ratios are given as

$$\frac{\lambda_1}{\lambda_3} = -\frac{s_z}{c_z^2} \left[ t_x t_y + s_z \right],$$
$$\frac{\lambda_2}{\lambda_3} = \frac{s_z}{c_z^2} \left[ \frac{1}{t_x t_y} - s_z \right].$$  \hspace{1cm} (17)
As done for A1, in the lowest-order approximation, we explicitly obtain
\[
\frac{m_1}{m_3} \approx \frac{t_x}{t_y} s_z , \quad \frac{m_2}{m_3} \approx \frac{1}{t_x t_y} s_z ; \quad \rho \approx \pm \frac{\pi}{2} , \quad \sigma \approx 0 ;
\]
\[
R_\nu \approx \frac{1}{t_x^2 t_y^2} \left| 1 - t_x^4 s_z^2 \right| , \quad |M_{ee}| = m_3 s_z^2 . \tag{18}
\]
Again using the values of the mixing angles, as given above, we obtain the numerical values of different mass ratios as
\[
\frac{m_1}{m_3} \approx 0.065 , \quad \frac{m_2}{m_3} \approx 0.143 , \quad R_\nu \approx 0.016 , \quad |M_{ee}|/m_3 = 0.0076 . \tag{19}
\]
This pattern gives results almost similar to pattern A1 and corresponds to normal hierarchy nature of neutrino masses. It is very difficult to differentiate between these two patterns from the experimental data

**Pattern B1:** \( M_{\mu\mu} = M_{ee} = 0 \) (i.e., \( a = b = \mu; \alpha = e \) and \( \beta = \tau \)). Here, we obtain
\[
\begin{align*}
\lambda_1 &= s_x s_y c_x (2 s_y^2 c_y^2 - s_x^2 c_y^2) - c_y s_z (c_x^2 c_y^2 + s_x^2 s_y^2) , \\
\lambda_3 &= s_x s_y c_x c_y^2 + c_y^2 s_z (s_x^2 - c_x^2) + s_x s_y s_z^2 c_x (1 + c_y^2) , \\
\lambda_2 &= s_x s_y c_x (2 s_x^2 c_y^2 - s_y^2 c_y^2) + c_y s_z (s_x^2 c_y^2 + s_y^2 c_y^2) , \\
\lambda_3 &= s_x s_y c_x c_y^2 + c_y^2 s_z (s_x^2 - c_x^2) + s_x c_x s_y s_z^2 (1 + c_y^2) .
\end{align*}
\tag{20}
\]
The smallness of \( s_z^2 \) allows us to make a similar analytical approximation as before. To lowest order, we find
\[
\frac{m_1}{m_3} \approx \frac{m_2}{m_3} \approx t_y^2 ; \quad \rho \approx \sigma \approx \pm \frac{\pi}{2} ; \quad R_\nu \approx \frac{1 + t_x^2}{t_x} t_{2y} s_z , \quad |M_{ee}| \approx m_3 \left[ t_y^2 \frac{1 - t_x^2}{t_x t_y} s_z \right] , \tag{21}
\]
where \( t_{2y} \equiv \tan 2\theta_y \). It should be noted that \( m_1 \) and \( m_2 \) are not exactly degenerate and their difference is given as
\[
\frac{m_2}{m_3} - \frac{m_1}{m_3} \approx \frac{4 s_z}{s_{2y} s_{2x}} . \tag{22}
\]
Again using the values of the mixing angles we obtain
\[
\frac{m_1}{m_3} \approx \frac{m_2}{m_3} \approx 0.81 , \quad R_\nu \approx 1.79 , \quad |M_{ee}|/m_3 \approx 0.89 . \tag{23}
\]
As seen from above equation, this pattern corresponds to \( m_1 \approx m_2 \approx m_3 \) and the ratio of mass difference square \( R_\nu \) is found to be \( \mathcal{O}(1) \). If we vary the mixing angles \( \theta_x \) and \( \theta_y \) within
their 2σ range (i.e., $31.1^\circ \leq \theta_x \leq 37.5^\circ$ and $35.9^\circ \leq \theta_y \leq 52^\circ$) and $\theta_z$ between $(1^\circ - 12^\circ)$, the allowed region in the $R_\nu - s_z$ parameter plane is shown in Figure-1. It should be noted here that we have ignored the case of $\theta_z=0$, since it will give $R_\nu=0$ and we know from the data that $R_\nu$ is non-zero. Second, there is no compelling reason (keeping in mind the quark mixing angles) so as to take it to be zero, although in the literature one can find the explanation that it could be zero under certain symmetry condition (e.g., $\mu$-$\tau$ symmetry). But again this symmetry has to be broken to incorporate CP violation in the neutrino sector. The minimum value of $R_\nu$ is found to be 0.15 for this case which is nearly five times larger the observed $R_\nu$ value. Thus, this pattern is ruled out by the current experimental data.

Figure 1: The allowed region in the $R_\nu - s_z$ parameter plane.
Pattern B$_2$: \( M_{e\mu} = M_{\tau\tau} = 0 \) (i.e., \( a = b = \tau; \alpha = e \) and \( \beta = \mu \)). Here, the mass ratios are given as
\[
\begin{align*}
\lambda_1 &= \frac{s_x c_x c_y (2 s_x^2 s_y^2 - c_y^2 c_z^2)}{s_x c_x c_y s_y^2 - (s_x^2 - c_z^2) s_y^2 s_z + s_x c_x c_y s_z^2 (1 + s_y^2)}, \\
\lambda_3 &= \frac{c_x s_x c_y (2 s_x^2 s_y^2 - c_y^2 c_z^2) - s_y s_z (s_x^2 s_y^2 + c_y^2 c_z^2)}{s_x c_x c_y s_y^2 - (s_x^2 - c_z^2) s_y^2 s_z + s_x c_x c_y s_z^2 (1 + s_y^2)}. 
\end{align*}
\] (24)

In the lowest-order approximation, we explicitly obtain
\[
\begin{align*}
\frac{m_1}{m_3} &\approx \frac{m_2}{m_3} \approx \frac{1}{t_y^2}; \quad \rho \approx \sigma \approx \pm \frac{\pi}{2}; \quad R_\nu \approx \frac{1 + t_x^2}{t_x} t_y s_z, \\
|M_{ee}| &\approx m_3 \left[ \frac{1}{t_y^2} - \frac{s_z t_y}{t_x} (1 - t_x^2) \right], \quad \frac{m_1}{m_3} - \frac{m_2}{m_3} \approx \frac{4 s_z}{s_2 y s_2 x} s_z. 
\end{align*}
\] (25)

Numerically they are found to be
\[
\frac{m_1}{m_3} \approx \frac{m_2}{m_3} \approx 1.23, \quad R_\nu \approx 1.79, \quad |M_{ee}|/m_3 \approx 1.79. \quad (26)
\]

This pattern is also similar to B1 and hence not acceptable by the current data.

Pattern B$_3$: \( M_{\mu\mu} = M_{e\mu} = 0 \) (i.e., \( a = b = \mu; \alpha = e \) and \( \beta = \mu \)). We obtain
\[
\begin{align*}
\lambda_1 &= -s_y \left( \frac{s_x s_y - c_x c_y s_z}{c_x c_y + c_x s_y s_z} \right), \\
\lambda_3 &= -s_y \left( \frac{c_x s_y + s_x c_y s_z}{c_x c_y - s_x s_y s_z} \right). 
\end{align*}
\] (27)

The approximate expressions for the neutrino mass ratios, the Majorana phases and the observables \( R_\nu \) and \( |M_{ee}| \) turn out to be
\[
\begin{align*}
\frac{m_1}{m_3} &\approx \frac{m_2}{m_3} \approx t_y^2; \quad \rho \approx \sigma \approx \pm \frac{\pi}{2}; \quad R_\nu \approx \frac{1 + t_x^2}{t_x^2} t_y |t_{2y}| s_z, \\
|M_{ee}| &\approx m_3 \left[ t_y^2 - \frac{(1 - t_x^2)}{t_x} t_y s_z \right], \quad \frac{m_1}{m_3} - \frac{m_2}{m_3} \approx \frac{4 s_z t_y^2}{s_2 y s_2 x}. 
\end{align*}
\] (28)

Substituting the values of the mixing angles, we obtain the numerical values as
\[
\frac{m_1}{m_3} \approx \frac{m_2}{m_3} \approx 0.81, \quad R_\nu \approx 1.45 \quad \text{and} \quad |M_{ee}|/m_3 = 0.75. \quad (29)
\]

B3 is also similar to B1 with \( R_\nu = \mathcal{O}(1) \). The allowed region in the parameter space for this case (where \( \theta_x \) and \( \theta_y \) are varied within their 2$\sigma$ ranges) is also shown in figure-1 and the
minimum $R_\nu$ value obtainable in this case is 0.08, which is nearly two times greater than the observed $R_\nu$. Thus, this form is also not acceptable by the current experimental data.

**Pattern B** : $M_{\tau\tau} = M_{e\tau} = 0$ (i.e., $a = b = \tau$; $\alpha = e$ and $\beta = \tau$). We obtain

$$\frac{\lambda_1}{\lambda_3} = -\frac{c_y}{s_y} \left( \frac{s_xc_y + c_xs_ys_z}{s_xs_y - c_xc_ys_z} \right),$$

$$\frac{\lambda_2}{\lambda_3} = -\frac{c_y}{s_y} \left( \frac{c_xc_y - s_xs_ys_z}{c_xs_y + s_xc_ys_z} \right).$$  \hspace{1cm} (30)

To lowest order, we get the following approximate results:

$$\frac{m_1}{m_3} \approx \frac{m_2}{m_3} \approx \frac{1}{t_y^2}; \quad \rho \approx \sigma \approx \pm \frac{\pi}{2}; \quad R_\nu \approx \frac{1 + t_x^2}{t_xt_y^2} |t_{2y}| s_z,$$

$$|M_{ee}| \approx m_3 \left[ \frac{1}{t_y^2} + \frac{1 - t_x^2}{s_x t_y s_z} \right], \quad \frac{m_1}{m_3} - \frac{m_2}{m_3} \approx \frac{4s_z}{t_{2y} s_2 y s_{2x}}. \hspace{1cm} (31)$$

Using the central values of the mixing angles we obtain

$$\frac{m_1}{m_3} \approx \frac{m_2}{m_3} \approx 1.23, \quad R_\nu \approx 2.2 \quad \text{and} \quad |M_{ee}| / m_3 \approx 1.31. \hspace{1cm} (32)$$

This pattern is also similar to the earlier B’s and allowed region in the parameter space is also shown in Figure-1. In this case the minimum $R_\nu$ value is found to be 0.1, and hence, this pattern is also unacceptable.

**Pattern C** : $M_{\mu\mu} = M_{\tau\tau} = 0$ (i.e., $a = b = \mu$; $\alpha = \beta = \tau$). We obtain

$$\frac{\lambda_1}{\lambda_3} = -\frac{c_x^2}{s_x} \cdot \frac{c_x(s_y^2 - c_y^2) + 2s_x s_y c_y s_z}{2s_x c_x s_y c_y - s_x(s_x^2 - s_y^2)(c_y^2 - s_y^2) + 2s_x c_x s_y c_y s_z^2},$$

$$\frac{\lambda_2}{\lambda_3} = \frac{s_x c_x}{s_x} \cdot \frac{s_x(s_y^2 - c_y^2) - 2c_x s_y c_y s_z}{2s_x c_x s_y c_y - (c_x^2 - s_x^2)(c_y^2 - s_y^2)s_z + 2s_x c_x s_y c_y s_z^2}. \hspace{1cm} (33)$$

Assuming $s_z^2 << 1$, we obtain

$$\frac{m_1}{m_3} \approx \left( 1 - \frac{1}{t_xt_2 y s_z} \right), \quad m_2 / m_3 \approx \left( 1 + \frac{t_x}{t_2 y s_z} \right), \quad \rho = \sigma = \pm \pi / 2,$$

$$R_\nu \approx \frac{1 + t_x^2}{t_{2x}^2 t_{2y}^2} \left[ \frac{2}{t_{2x}} \cdot \frac{1 - t_{2x} t_{2y} s_z}{t_{2x}^2 + 2s_t z t_{2y}} \right], \quad |M_{ee}| \approx m_3 \left[ 1 - \frac{2}{t_{2x} t_{2y} s_z} \right]. \hspace{1cm} (34)$$

We find that, in this case $R_\nu$ is very sensitive to the value of $\theta_z$ and is found to be $\mathcal{O}(1)$ for $\theta_z = 5^\circ$ (the other mixing angles being same as before). However, $R_\nu$ is found to be
acceptable-one for $\theta_z \approx 2.5^\circ$. The numerical values of the mass ratios are given as

$$\frac{m_1}{m_3} \approx 2.57, \quad \frac{m_2}{m_3} \approx 2.62, \quad R_\nu \approx 0.05, \quad |M_{ee}|/m_3 \approx 0.95.$$  \hspace{1cm} (35)

This pattern corresponds to the mass structure $m_1 \approx m_2 > m_3$. Thus, although this pattern is allowed by the current experimental data, it is very sensitive for model building.

Thus, we find that out of the seven possibilities, which are allowed when $\delta \neq 0$, only three of them are found to be allowed for $\delta = 0$, which is our main result. The order of magnitude of the mass matrix $M$ (4) for the three acceptable forms for $\delta = 0$ are given in Table-1.

Now we will repeat the above procedure for $\delta \neq 0$. Although the same has been done by Xing for the Fritzsch type parametrization, we will redo it here for the exact standard parametrization (PDG parametrization) and would like to see if there could be any differences that can arise due to the difference in the position of the Dirac phase.

Pattern A1: $M_{ee} = M_{e\mu} = 0$ (i.e., $a = b = e$; $\alpha = e$ and $\beta = \mu$). We obtain

$$\frac{\lambda_1}{\lambda_3} = \frac{s_z}{c_z^2} \left[ t_x t_y e^{i\delta} - s_z \right] e^{-2i\delta},$$

$$\frac{\lambda_2}{\lambda_3} = \frac{s_z}{c_z^2} \left[ t_y t_x e^{i\delta} + s_z \right] e^{-2i\delta}. \hspace{2cm} (36)$$

Again in the lowest order approximation we obtain the mass ratios as

$$\frac{m_1}{m_3} \approx s_z t_x t_y, \quad \frac{m_2}{m_3} \approx s_z \frac{t_y}{t_x}; \quad \rho \approx -\frac{\delta}{2}, \quad \sigma \approx -\frac{\delta}{2} \pm \frac{\pi}{2};$$

$$R_\nu \approx \frac{t_y^2}{t_x^2} \left| 1 - t_x^4 s_z^2 \right|, \quad |M_{ee}| = m_3 s_z^2. \hspace{2cm} (37)$$

The numerical values of the mass ratios are same as that of the pattern A1 of without Dirac phase. So we are not presenting them here. Only the Majorana phases differ in these two cases.

Pattern A2: $M_{ee} = M_{e\tau} = 0$ (i.e., $a = b = e$; $\alpha = e$ and $\beta = \tau$). We obtain

$$\frac{\lambda_1}{\lambda_3} = -\frac{s_z}{c_z^2} \left[ t_x e^{i\delta} + s_z \right] e^{-2i\delta},$$

$$\frac{\lambda_2}{\lambda_3} = \frac{s_z}{c_z^2} \left[ \frac{1}{t_x t_y} e^{i\delta} - s_z \right] e^{-2i\delta}. \hspace{2cm} (38)$$
which in the limit \( s_z^2 \ll 1 \), reduces to
\[
\frac{m_1}{m_3} \approx \frac{t_y}{t_z} s_z, \quad \frac{m_2}{m_3} \approx \frac{1}{t_x t_y} s_z; \quad \rho \approx -\frac{\delta}{2} + \frac{\pi}{2}, \quad \sigma \approx -\frac{\delta}{2};
\]
\[
R_\nu \approx \frac{1}{t_x^2 t_y} \left| 1 - t_x^4 \right| s_z^2, \quad |M_{ee}| = m_3 s_z^2. \quad (39)
\]

The ratio of mass parameters are also same as that of with \( \delta = 0 \) case.

**Pattern B₁:** \( M_{\mu\mu} = M_{e\tau} = 0 \) (i.e., \( a = b = \mu; \alpha = e \) and \( \beta = \tau \)). We obtain
\[
\frac{\lambda_1}{\lambda_3} = \frac{s_x c_x s_y (2 s_z^2 c_y^2 - s_y^2 c_z^2) - c_y s_z (c_x^2 c_y^2 e^{-i\beta} + s_x^2 s_y^2 e^{i\beta})}{s_x c_x s_y c_y^2 + c_y s_z (s_x^2 - c_x^2) e^{i\beta} + s_x c_x s_y s_z^2 (1 + c_y^2) e^{2i\beta}};
\]
\[
\frac{\lambda_2}{\lambda_3} = \frac{s_x c_x s_y (2 s_z^2 c_y^2 - s_y^2 c_z^2) + c_y s_z (c_x^2 c_y^2 e^{i\beta} + s_x^2 s_y^2 e^{-i\beta})}{s_x c_x s_y c_y^2 + c_y s_z (s_x^2 - c_x^2) e^{i\beta} + s_x c_x s_y s_z^2 (1 + c_y^2) e^{2i\beta}}. \quad (40)
\]

To the lowest order, we find
\[
\frac{m_1}{m_3} \approx \frac{m_2}{m_3} \approx t_y^2; \quad \rho \approx \sigma \approx \pm \frac{\pi}{2}; \quad R_\nu \approx \frac{1 + t_x^2}{t_x} |t_{2y} c_\delta| s_z, \quad |M_{ee}| \approx m_3 \left[ t_y^2 + \frac{c_\delta s_z}{t_x t_y} \left( (1 - t_x^2)(1 + t_y^2) \right) \right], \quad (41)
\]
where \( c_\delta \equiv \cos \delta \). Also,
\[
\frac{m_1}{m_3} - \frac{m_2}{m_3} \approx \frac{4 s_z c_\delta}{s_{2y} s_{2x}}, \quad \sigma - \rho \approx \frac{2 s_z s_\delta}{s_{2x} t_{2y} t_y^2}. \quad (42)
\]

As seen from (41), \( R_\nu \) is proportional to \( \cos \delta \) and therefore unless \( \cos \delta \) is very small (i.e., \( \delta \) is very close to \( \pi/2 \)) this pattern will not accommodate the observed data. Thus using \( \delta = 89^\circ \), we obtain
\[
\frac{m_1}{m_3} \approx \frac{m_2}{m_3} \approx 0.81 \quad R_\nu \approx 0.03, \quad |M_{ee}|/m_3 \approx 0.81,
\]
\[
\frac{m_1}{m_3} - \frac{m_2}{m_3} \approx 0.007, \quad \sigma - \rho \approx 0.024. \quad (43)
\]

Thus this pattern corresponds to the situation \( m_1 \approx m_2 < m_3 \) and accommodates the observed data on \( R_\nu \) for \( \delta \) close to \( \pi/2 \).

**Pattern B₂:** \( M_{\mu\tau} = M_{e\tau} = 0 \) (i.e., \( a = b = \tau; \alpha = e \) and \( \beta = \mu \)). We obtain
\[
\frac{\lambda_1}{\lambda_3} = \frac{s_x c_x c_y (2 s_z^2 s_y^2 - c_y^2 c_z^2) + s_y s_z (c_x^2 s_y^2 e^{-i\beta} + c_y^2 s_z^2 e^{i\beta})}{s_x c_x c_y s_y^2 - (s_z^2 - c_z^2) s_y s_z e^{i\beta} + s_x c_x c_y s_z^2 (1 + s_y^2) e^{2i\beta}};
\]
\[
\frac{\lambda_2}{\lambda_3} = \frac{c_x c_x c_y (2 s_z^2 s_y^2 - c_y^2 c_z^2) - s_y s_z (c_x^2 s_y^2 e^{i\beta} + s_y^2 s_z^2 e^{-i\beta})}{s_x c_x c_y s_y^2 - (s_z^2 - c_z^2) s_y s_z e^{i\beta} + s_x c_x c_y s_z^2 (1 + s_y^2) e^{2i\beta}}. \quad (44)
\]
In the lowest-order approximation, we explicitly obtain

\[
\frac{m_1}{m_3} \approx \frac{m_2}{m_3} \approx \frac{1}{t_y^2}; \quad \rho \approx \sigma \approx \pm \frac{\pi}{2}; \quad R_\nu \approx \frac{1 + t_x^2}{t_x} |t_{2y} \ c_\delta| \ s_z ,
\]

\[
|M_{ee}| \approx m_3 \left[ \frac{1}{t_y^2} - \frac{c_\delta s_z}{t_xt_y} \left( (1 - t_x^2)(1 + t_y^2) \right) \right] ,
\]

\[
\frac{m_2}{m_3} - \frac{m_1}{m_3} \approx \frac{4s_zc_\delta}{s_{2y}s_{2x}} , \quad \sigma - \rho \approx \frac{2t_y^2s_zs_\delta}{s_{2x}t_{2y}} .
\]  \hspace{1cm} (45)

As in B1, this case will also give acceptable solution for \( \delta \) close to \( \pi/2 \). Numerically the values are found for \( \delta = 89^\circ \) as

\[
\frac{m_1}{m_3} \approx \frac{m_2}{m_3} \approx 1.23 \quad R_\nu \approx 0.03 , \quad |M_{ee}|/m_3 \approx 1.23 ,
\]

\[
\frac{m_1}{m_3} - \frac{m_2}{m_3} \approx 0.007 , \quad \sigma - \rho \approx 0.02 .
\]  \hspace{1cm} (46)

This pattern is similar to B1 and can accommodate the observed data for \( \delta \) close to \( 90^\circ \). This corresponds to the mass pattern as \( m_1 \approx m_2 > m_3 \).

**Pattern B3:** \( M_{\mu\mu} = M_{e\mu} = 0 \) (i.e., \( a = b = \mu; \alpha = e \) and \( \beta = \mu \)). We obtain

\[
\lambda_1 = \frac{s_y \ s_x s_y - c_x c_y s_z e^{-i\delta}}{c_y} , \quad \lambda_3 = \frac{s_y \ c_x s_y + c_x c_y s_z e^{-i\delta}}{c_y} ,
\]

\[
\lambda_2 = \frac{s_y \ s_x c_y - s_x c_y s_z e^{-i\delta}}{c_x c_y} , \quad \lambda_3 = \frac{s_y \ c_x c_y - s_x s_y s_z e^{-i\delta}}{c_x c_y} .
\]  \hspace{1cm} (47)

The approximate expressions for the neutrino mass ratios, the Majorana phases and the observables \( R_\nu \) and \( |M_{ee}| \) turn out to be

\[
\frac{m_1}{m_3} \approx \frac{m_2}{m_3} \approx t_y^2 ; \quad \rho \approx \sigma \approx \pm \frac{\pi}{2}; \quad R_\nu \approx \frac{1 + t_x^2}{t_x} t_{2y} \ c_\delta| \ s_z ,
\]

\[
|M_{ee}| \approx m_3 \left[ t_y^2 - \frac{c_\delta s_z}{t_xt_y} \left( (1 - t_x^2)(1 + t_y^2) \right) \right] ,
\]

\[
\frac{m_2}{m_3} - \frac{m_1}{m_3} \approx \frac{4s_zt_y^2c_\delta}{s_{2y}s_{2x}} , \quad \rho - \sigma \approx \frac{2s_zs_\delta}{s_{2x}t_{2y}} .
\]  \hspace{1cm} (48)

The mass ratios are same as that of \( B_3 \) with \( \delta = 0 \). However, using \( \delta = 89^\circ \) the ratio of mass square difference \( R_\nu \) and \( |M_{ee}| \) are found to be

\[
R_\nu = 0.025 , \quad |M_{ee}|/m_3 = 0.81 , \quad \frac{m_1}{m_3} - \frac{m_2}{m_3} \approx 0.005 , \quad \rho - \sigma \approx 0.024 .
\]  \hspace{1cm} (49)
To the lowest order, we get the mass ratios

Thus, the pattern which was not acceptable for $\delta = 0$ can accommodate the observed data for $\delta = 89^\circ$.

**Pattern B.** $M_{\tau\tau} = M_{e\tau} = 0$ (i.e., $a = b = \tau; \alpha = \tau$ and $\beta = \tau$). We obtain

$$\frac{\lambda_1}{\lambda_3} = -\frac{c_y}{s_y} \frac{s_x c_y + c_x s_y s_z e^{-i\delta}}{s_x s_y - c_x c_y s_z e^{i\delta}},$$

$$\frac{\lambda_2}{\lambda_3} = -\frac{c_y}{s_y} \frac{c_x c_y - s_x s_y s_z e^{-i\delta}}{c_x c_y + s_x s_y s_z e^{i\delta}}.$$

To lowest order, we get the following approximate results:

$$\frac{m_1}{m_3} \approx \frac{m_2}{m_3} \approx \frac{1}{t_y^2}; \quad \rho \approx \sigma \approx \pm \frac{\pi}{2}; \quad R_\nu \approx \frac{1 + t_y^2}{t_x t_y^2} |t_{2y} c_\delta| s_z,$$

$$|M_{ee}| \approx m_3 \left[ \frac{1}{t_y^2} + \frac{c_\delta s_z}{t_x t_y} \left( (1 - t_x^2) (1 + t_y^2) \frac{1}{t_y^2} \right) \right],$$

$$\frac{m_1}{m_3} - \frac{m_2}{m_3} \approx \frac{4s_c c_\delta}{t_y s_{2y} s_{2x}}, \quad \rho - \sigma \approx \frac{2s_c s_\delta}{s_{2x} t_{2y}}.$$

Thus, the numerical values of the mass parameters are given for $\delta = 89^\circ$ as

$$R_\nu \approx 0.04, \quad |M_{ee}|/m_3 \approx 1.24, \quad \frac{m_1}{m_3} - \frac{m_2}{m_3} \approx 0.008, \quad \rho - \sigma \approx 0.024.$$

Thus, the ratio of the square of mass difference is found to be $O(10^{-2})$ as observed by the current experiments.

**Pattern C.** $M_{\mu\mu} = M_{\tau\tau} = 0$ (i.e., $a = b = \mu; \alpha = \beta = \tau$). We obtain

$$\frac{\lambda_1}{\lambda_3} = -\frac{c_x c_z^2}{s_z} \frac{c_x (s_y^2 - c_y^2) e^{-i\delta} + 2s_x s_y c_y s_z}{2s_x c_x s_y c_y - (s_x^2 - c_x^2)(s_y^2 - c_y^2) s_z e^{i\delta} + 2s_x c_x s_y c_y s_z^2 e^{2i\delta}},$$

$$\frac{\lambda_2}{\lambda_3} = \frac{s_x c_z^2}{s_z} \frac{s_x (s_y^2 - c_y^2) e^{-i\delta} - 2c_x s_y c_z s_z}{2s_x c_x s_y c_y - (s_x^2 - c_x^2)(s_y^2 - c_y^2) s_z e^{i\delta} + 2s_x c_x s_y c_y s_z^2 e^{2i\delta}},$$

To the lowest order, we get the mass ratios

$$\frac{m_1}{m_3} \approx \sqrt{\frac{1}{1 - \frac{2c_\delta}{t_x t_{2y} s_z}} + \frac{1}{\frac{t_x^2 t_{2y} s_z^2}{2}}}, \quad \frac{m_2}{m_3} \approx \sqrt{1 + \frac{2t_x c_\delta}{t_{2y} s_z} + \frac{t_y^2}{t_{2y}^2 s_z^2}},$$

$$\rho = \pm \frac{\pi}{2} + \frac{1}{2} \tan^{-1} \left( \frac{s_\delta}{t_x t_{2y} s_z - c_\delta} \right), \quad \sigma = \pm \frac{\pi}{2} - \frac{1}{2} \tan^{-1} \left( \frac{t_x s_\delta}{t_{2y} s_z + t_c_\delta} \right),$$
\[ R_\nu \approx \frac{1 + t_x^2}{t_x^2} \left( \frac{2}{t_x^2} \frac{1 - t_{2x} t_{2y} c_\delta s_z}{t_x + 2 s_z c_\delta t_{2y}} \right), \]
\[ |M_{ee}| \approx m_3 \sqrt{1 - \frac{4 c_\delta}{t_x t_{2y} s_z} + \frac{4}{t_x^2 t_{2y}^2 s_z^2}}. \]

(54)

In this case \( R_\nu \) is very sensitive to \( \delta \). Using \( \theta_z = 5^\circ \) and \( \delta = 60^\circ \), we obtain the numerical values of the mass parameters as

\[ \frac{m_1}{m_3} \approx 1.55, \quad \frac{m_2}{m_3} \approx 1.57, \quad R_\nu \approx 0.04, \quad |M_{ee}|/m_3 \approx 0.99. \]

(55)

Thus, the pattern also gives acceptable solutions for \( \delta = 60^\circ \). The order of magnitude of the mass matrix \( M \) (4) for the seven possible forms for \( \delta \neq 0 \) are presented in Table-2.

To summarize, in this paper we have reanalyzed the seven possible forms of texture two neutrino mass matrices in the light of current neutrino data. We found that, with standard parametrization, if the Dirac type CP violating phase in the neutrino mixing matrix turns out to be zero, then out of the seven possible forms only two forms (A1, A2) are allowed by the current experimental data, which corresponds to normal hierarchy. Furthermore, if we allow a slight variation in \( \theta_z \) then pattern C is also allowed but with inverted hierarchy. For nonzero \( \delta \) (Dirac CP phase) we have derived the expressions for the different mass ratios using the standard parametrization of the neutrino mixing matrix, which are different from those obtained in [8]. The mass matrices are also found to be slightly different. Interestingly, when the Dirac phase is nonzero all the possible forms are allowed by the current data and most of the structures (except \( A_1 \) and \( A_2 \), which are insensitive to the Dirac phase and also follow normal hierarchy) do not exhibit normal hierarchy and prefer Dirac phase close to \( \pi/2 \). In future, with more theoretical studies and with more accurate data, we hope to understand better the true nature of the neutrino mass matrices.

References

[1] Super-Kamiokande Collaboration, S. Fukuda et al., Phys. Rev. Lett. 86, 5651 (2001); Phys. Rev. Lett. 86, 5656 (2001); SNO Collaboration, Q.R. Ahmad et al., Phys. Rev. Lett. 87, 071301 (2001); Super-Kamiokande Collaboration, Y. Fukuda et al., Phys. Lett. B 467, 185 (1999); S. Fukuda et al., Phys. Rev. Lett. 86, 5651 (2001); Phys. Rev. Lett. 85, 3999 (2000); CHOOZ Collaboration, M. Apollonio et al., Phys. Lett. B 420, 397 (1998); Palo Verde Collaboration, F. Boehm et al., Phys. Rev. Lett. 84, 3764 (2000).
Table 1: The allowed three patterns of the neutrino mass matrix $M$ with two texture zeros, for $\delta = 0$ in the mixing matrix. The order-of-magnitude of $M$ is given for illustration for of $\theta_x = 34^\circ$, $\theta_y = 42^\circ$, $\theta_z = 5^\circ$ for A1, A2 and $\theta_z = 2.5^\circ$ for pattern C.

| Pattern | Texture of $M$ | Order of Magnitude |
|---------|----------------|-------------------|
| A1      | $\begin{pmatrix} 0 & 0 & \times \\ 0 & \times & \times \\ \times & \times & \times \end{pmatrix}$ | $\sim m_3 \begin{pmatrix} 0 & 0 & 0.12 \\ 0 & 0.41 & 0.53 \\ 0.12 & 0.53 & 0.51 \end{pmatrix}$ |
| A2      | $\begin{pmatrix} 0 & \times & 0 \\ \times & \times & \times \\ 0 & \times & \times \end{pmatrix}$ | $\sim m_3 \begin{pmatrix} 0 & 0.13 & 0 \\ 0.13 & 0.47 & 0.46 \\ 0 & 0.46 & 0.59 \end{pmatrix}$ |
| C       | $\begin{pmatrix} \times & \times & \times \\ \times & 0 & \times \\ \times & \times & 0 \end{pmatrix}$ | $\sim m_3 \begin{pmatrix} 0.95 & 1.79 & 1.61 \\ 1.79 & 0 & 1.0 \\ 1.61 & 1.0 & 0 \end{pmatrix}$ |

[2] B. Pontecorvo, Sov. Phys. JETP 7, 172 (1958); Z. Maki, M. Nakagawa and S. Sakata, Prog. Theor. Phys. 28, 870 (1962).

[3] For a recent review see, R. N. Mohapatra and A. Y. Smirnov, hep-ph/0603118.

[4] G. L. Fogli, E. Lisi, A. Marrone and A. Palazzo, Prog. Part. Nucl. Phys. 57, 742 (2006).

[5] C. Jarlskog, Phys. Rev. Lett. 55, 1039 (1985); Z. Phys. C 29, 491 (1985).

[6] R. N. Mohapatra et al., hep-ph/0412099, hep-ph/0510213.

[7] P. H. Frampton, S. L. Glashow and D. Marfatia, Phys. Lett. B 536, 79 (2002).

[8] Z. Z. Xing, Phys. Lett. B 530, 159 (2002).
Table 2: The allowed seven patterns of the neutrino mass matrix $M$ with two texture zeros, for $\delta \neq 0$ in the mixing matrix. The order-of-magnitude of $M$ is given for illustration for of $\theta_x = 34^\circ$, $\theta_y = 42^\circ$, $\theta_z = 5^\circ$, $\delta = 90^\circ$ for A1 and A2, $\delta = 89^\circ$ for B1, B2, B3, B4 and $\delta = 60^\circ$ for C.

| Pattern | Texture of $M$ | Order of Magnitude |
|---------|----------------|---------------------|
| A₁      | 0 0 ×          | $m_3 0 0 .12$      |
|         | 0 × ×          |                     |
|         | × × ×          | ~ $m_3 0 .46 .49$  |
| A₂      | × × 0          |                     |
|         | × × ×          | ~ $m_3 .12 .49 .54$|
| B₁      | × × 0          |                     |
|         | × 0 ×          | ~ $m_3 .81 .02 0$  |
|         | 0 × ×          |                      |
| B₂      | × 0 ×          | ~ $m_3 1.2 0 .03$  |
|         | × × ×          |                      |
|         | × 0 ×          | ~ $m_3 0 .03 .11$  |
| B₃      | × 0 ×          | ~ $m_3 .81 0 .02$  |
|         | 0 0 ×          |                      |
|         | × × ×          | ~ $m_3 1.2 .03 0$  |
| B₄      | × × ×          | ~ $m_3 .99 .89 .80$|
|         | 0 × 0          |                      |
|         | × × 0          | ~ $m_3 .89 0 1.0$  |
| C       | × × ×          | ~ $m_3 .80 1.0 0$  |