PARAMETER ROBUST PRECONDITIONING BY CONGRUENCE FOR MULTIPLE-NETWORK PoroELASTICITY *

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Abstract. The mechanical behaviour of a poroelastic medium permeated by multiple interacting fluid networks can be described by a system of time-dependent partial differential equations known as the multiple-network poroelasticity (MPET) equations or multi-porosity/multi-permeability systems. These equations generalize Biot’s equations, which describe the mechanics of the one–network case. The efficient numerical solution of the MPET equations is challenging, in part due to the complexity of the system and in part due to the presence of interacting parameter regimes. In this paper, we present a new strategy for efficiently and robustly solving the MPET equations numerically. In particular, we introduce a new approach to formulating finite element methods and associated preconditioners for the MPET equations. The approach is based on designing transformations of variables that simultaneously diagonalize (by congruence) the equations’ key operators and subsequently constructing parameter-robust block-diagonal preconditioners for the transformed system. Our methodology is supported by theoretical considerations as well as by numerical results.

1. Introduction. In this paper, we consider the preconditioned iterative solution of finite element discretizations of the multiple-network poroelasticity (MPET) equations. These equations traditionally originate in geomechanics where they are also known under the term multi-porosity/multi-permeability systems [1]. The MPET equations generalize Biot’s equations [2] from the one network to the multiple network case, and multi-compartment Darcy (MPT) equations [17] from a porous (but rigid) to a poroelastic medium. Over the last decade, the MPT and MPET equations have seen a surge of interest in biology and physiology; e.g. to model perfusion in the heart [17, 12], cancer [21] brain [11] or liver [3], or to model the interaction between elastic deformation and fluid flow and transport in the brain [5, 6, 18, 23, 24, 25].

Concretely, the quasi–static MPET equations read as follows [1]: for a given number of networks $J \in \mathbb{N}$, find the displacement $u$ and the network pressures $p_j$ for $j = 1, \ldots, J$ such that

\begin{align}
(1.1a) & \quad -\text{div}(2\mu\varepsilon(u) + \lambda \text{div} u I) + \sum_j \alpha_j \nabla p_j = f, \\
(1.1b) & \quad s_j \dot{p}_j + \alpha_j \text{div} \dot{u} - \text{div} K_j \nabla p_j + \sum_i \xi_{j\rightarrow i}(p_j - p_i) = g_j,
\end{align}

where $u = u(x,t)$ and $p_j = p_j(x,t)$ for $x \in \Omega \subset \mathbb{R}^d$ ($d = 1, 2, 3$) and for $t \in (0, T]$. Physically, the equations (1.1) describe a porous and elastic medium permeated by a number of fluid networks under the assumptions that the solid matrix can be modeled
### Table 1: Sample parameter values for hydraulic conductivities and exchange coefficients with reference to (1.1) and/or (1.2).

| Parameter                               | Unit         | Value               | Reference |
|-----------------------------------------|--------------|---------------------|-----------|
| Hydraulic conductivities \((K_j)\)     | mm² (kPa s)⁻¹|                     |           |
| Brain gray matter                       |              | 2.0 × 10⁻³          | [22]      |
| Brain white matter                      |              | 2.0 × 10⁻²          | [22]      |
| Cardiac arteries                        |              | 1.0                 | [17]      |
| Cardiac capillaries                     |              | 2.0                 | [17]      |
| Cardiac veins                           |              | 10.0                | [17]      |
| Brain vasculature                       |              | 3.75 × 10¹          | [25]      |
| Brain fluid exchange                    |              | 1.57 × 10⁻²         | [25]      |

| Exchange coefficients \((\xi_j\leftarrow i)\) | (kPa s)⁻¹  |                     |           |
|-----------------------------------------------|------------|---------------------|-----------|
| Brain capillary-vasculature                  |            | 1.5 × 10⁻¹⁶        | [25]      |
| Brain capillary-tissue fluid                 |            | 2.0 × 10⁻¹⁶        | [25]      |
| Brain tissue fluid-veins                    |            | 2.0 × 10⁻¹⁰        | [25]      |
| Cardiac capillary-arteries                   |            | 2.0 × 10⁻²         | [17]      |
| Cardiac capillary-veins                     |            | 5.0 × 10⁻²         | [17]      |

The MPT equations represents a reduced version of (1.1) that result from ignoring the elastic contribution of the solid matrix. These equations then read as follows: for a given number of networks \(J \in \mathbb{N}\), find the network pressures \(p_j\) for \(j = 1, \ldots, J\) such that

\[
- \text{div} K_j \nabla p_j + \sum_{i=1}^{J} \xi_j\leftarrow i (p_j - p_i) = g_j,
\]

where for \(i, j = 1, \ldots, J, p_j = p_j(x)\) for \(x \in \Omega \subset \mathbb{R}^d (d = 1, 2, 3)\), the parameters \(K_j\) and \(\xi_j\leftarrow i\) remain the hydraulic conductivity and exchange coefficients, respectively, and \(g_j\) again represents other sources (or sinks) in each network.

The relative size of the conductivities \(K_j\) and the exchange coefficients \(\xi_j\leftarrow i\) may vary tremendously in applications. Large parameter variation is certainly present in applied problems of a physiological nature; a selection of representative parameter values, from research literature, is given in Table 1.

Here, we see that the hydraulic conductivities span four orders of magnitude while the exchange coefficients span fourteen orders of magnitude. Hence, there is a need for preconditioners that are robust with respect to variations in parameters. Physiological applications, in particular, can benefit from preconditioners which are
robust with respect to $K_j$, $\xi_{j+1}$ and $\lambda$ as in (1.1) and (1.2).

With this in mind, parameter-robust numerical approximations and solution algorithms for (1.1) is currently an active research topic. In the nearly incompressible case $\lambda \gg 1$, the standard two-field variational formulation of (1.1) is not robust. To address this challenge, we introduced and analyzed a mixed finite element method for the MPET equations based on a total pressure formulation in [15]. We note that the total pressure in case of one network was presented in [19, 14]. Hong et al. [7] shortly thereafter presented a three-field-type formulation involving the displacement, the network fluid fluxes and the network pressures targeting a range of parameter regimes. Uzawa and splitting schemes was further developed for the three-field-type formulations in [9, 8]. Alternatives to these fully coupled approaches in the form of splitting schemes has been analyzed by Lee [13]. Regarding the iterative solution and preconditioning of the coupled formulations, a robust preconditioner for Biot’s equations (the case for $J = 1$) was presented by Lee et al. [14]. Hong et al. [7] presented both theoretical results and numerical examples regarding parameter–robust preconditioners.

In this paper, we present a new approach for preconditioning linear systems of equations resulting from a conforming finite element discretization of the total pressure variational formulation of the MPET equations. The key idea, as introduced for the MPT equations in [20], to design a parameter-dependent transformation of the pressure variables $p = (p_1, \ldots, p_J)$ into a set of transformed variables $\tilde{p}$. The transformation should be such that the originally coupled exchange operator decouples while the originally decoupled diffusion operator stays decoupled (diagonal). The design of such a transform hinges on the concept of diagonalization by congruence and associated matrix theory. After transformation, we then define a natural block diagonal preconditioner for the transformed system of equations. This strategy yields a parameter–robust preconditioner, which we both prove theoretically and demonstrate numerically.

This manuscript is organized as follows. We introduce notation and review relevant preconditioning and matrix theory in Section 2. We briefly consider the reduced case of the MPT equations in Section 3 before turning to the analysis of the preconditioner for the MPET equations in Section 4. Finally, we present some conclusions and outlook in Section 5.

2. Preliminaries. In this section, we briefly review preconditioning of parameter-dependent systems in Section 2.1, matrix theory for diagonalization by congruence in Sections 2.2–2.3 and notation for the remainder of the manuscript in Section 2.4.

2.1. Preconditioning of parameter-dependent systems. In this paper, we consider the preconditioning of discretizations of the systems (1.1) and (1.2) under large parameter variations. Therefore, we begin by summarizing core aspects of the theory of parameter–robust preconditioning as presented in [16]. We will apply this theory for formulations of the MPT equations (1.2) and MPET equations (1.1) in the subsequent sections.

Let $X$ be a separable, real Hilbert space with inner product $\langle \cdot, \cdot \rangle_X$, norm $\| \cdot \|_X$ and dual space $X^*$. Let $A : X \to X$ be an invertible, symmetric isomorphism on $X$ such that $A \in \mathcal{L}(X, X^*)$ where $\mathcal{L}(X, X^*)$ is the set of bounded linear operators mapping $X$ to its dual. Given $f \in X^*$ consider the problem of finding $x \in X$ such that

\begin{equation}
Ax = f.
\end{equation}
The preconditioned problem reads as follows

$$B Ax = B f,$$

where $B \in \mathcal{L}(X^*, X)$ is a symmetric isomorphism defining the preconditioner. The convergence rate of a Krylov space method for this problem can be bounded in terms of the condition number $\kappa(BA)$ where

$$\kappa(BA) = \|BA\|_{\mathcal{L}(X,X)}((BA)^{-1})_{\mathcal{L}(X,X)}.$$

Here, the operator norm $\|A\|_{\mathcal{L}(X,X)}$ is defined by

$$\|A\|_{\mathcal{L}(X,X)} = \sup_{x \in X} \frac{\|Ax\|}{\|x\|}.$$

Now, for a parameter $\varepsilon$ (or more generally a set of parameters $\varepsilon$) consider the parameter-dependent operator $A_\varepsilon$ and its preconditioner $B_\varepsilon$. Assume that we can choose appropriate spaces $X_\varepsilon$ and $X_\varepsilon^*$ such that the norms

$$\|A_\varepsilon\|_{\mathcal{L}(X_\varepsilon,X_\varepsilon^*)} \text{ and } \|A_\varepsilon^{-1}\|_{\mathcal{L}(X_\varepsilon^*,X_\varepsilon)}$$

are bounded independently of $\varepsilon$. Similarly, we assume that we can find a preconditioner $B_\varepsilon$ such that the norms $\|B_\varepsilon\|_{\mathcal{L}(X_\varepsilon,X_\varepsilon^*)}$ and $\|B_\varepsilon^{-1}\|_{\mathcal{L}(X_\varepsilon^*,X_\varepsilon)}$ are bounded independently of $\varepsilon$. Given these assumptions, the condition number $\kappa(B_\varepsilon A_\varepsilon)$ will be bounded independently of $\varepsilon$. We will refer to such a preconditioner as robust in (or with respect to) $\varepsilon$.

### 2.2. Simultaneous diagonalization of matrices by congruence

The following definitions and results from matrix theory [10], in particular concepts related to simultaneously diagonalization of matrices by congruence, will be used in Sections 3 and 4.

By definition, a matrix $C \in \mathbb{C}^{n \times n}$ is diagonalizable if $C$ is similar to a diagonal matrix i.e. there exists an invertible $P \in \mathbb{C}^{n \times n}$ such that $P^{-1}CP$ is diagonal [10, Definition 1.3.6]. On the other hand, a matrix $A$ is diagonalizable by congruence if there exists a $P$ such that $P^TAP$ is diagonal. Further, matrices $A, B \in \mathbb{C}^{n \times n}$ are simultaneously diagonalizable by congruence if there exists a $P$ such that $P^TAP$ and $P^TBP$ are both diagonal [10].

Assume that $A, B \in \mathbb{C}^{n \times n}$ are symmetric and that $A$ is non-singular. Then $A, B$ are simultaneously diagonalizable by congruence if and only if $C = A^{-1}B$ is diagonalizable [10, Theorem 4.5.17a-b p. 287]. In our case $A$ will represent a positive definite, real and symmetric matrix; thus $A^{-1}$ exists and is also symmetric. Moreover, $B$ will be real and symmetric as well; therefore $C = A^{-1}B$ will satisfy the identity

$$A^{1/2}CA^{-1/2} = A^{-1/2}BA^{-1/2}.$$

The above equation shows that $C$ is similar to $A^{-1/2}BA^{-1/2}$. Since $A^{-1/2}BA^{-1/2}$ is real and symmetric there exists an orthonormal matrix $Q$, i.e. $Q^{-1} = Q^T$, and a diagonal matrix $D$ such that

$$Q \left(A^{-1/2}BA^{-1/2}\right) Q^{-1} = D.$$

Thus, the matrix $S^{-1} = QA^{1/2}$ provides a similarity transformation diagonalizing $C$; i.e. $S^{-1}CS = D$. It follows from [10, Theorem 4.5.17] that such an $A$ and $B$ are therefore similar by congruence.
The construction of the congruence relation $P$, yielding both $P^T AP = D_1$ and $P^T B P = D_2$, is straightforward for the case when $C = A^{-1}B$ has $n$ distinct eigenvalues. In this case $C$ has $n$ linearly independent eigenvectors; if $\{v_1, \ldots, v_n\}$ denote these eigenvectors then $P = [v_1, \ldots, v_n]$ is the matrix whose $j$-th column is $v_j$. When the eigenvalues of $C$ are not distinct: $P$ can be realized as the product of block-wise eigenvector matrices. The general procedure for this case is discussed in [10] and we will present an example in Section 3.

2.3. Change of variables versus diagonalization by congruence. Let $A : Q \to Q^*$, $B : Q \to Q^*$ be symmetric linear operators over a Hilbert space $Q$ with $n$ components. Consider the following variational problem: find $p = (p_1, \ldots, p_n) \in Q$ such that

$$\langle Ap, q \rangle + \langle Bp, q \rangle = \langle f, q \rangle$$

for all $q \in Q$, a given source $f \in Q^*$ and inner product $\langle \cdot, \cdot \rangle$.

Now, introduce a change of variables of $p$ into a new set of variables $\tilde{p}$. We denote the (inverse) change of variables transform by $P$; that is

$$p = P\tilde{p}, \quad \tilde{p} = P^{-1}p.$$ 

Inserting the change of variables into (2.4), we obtain the alternative system of equations in terms of $\tilde{p}$:

$$\langle P^T AP \tilde{p}, \tilde{q} \rangle + \langle P^T BP \tilde{p}, \tilde{q} \rangle = \langle P^T f, \tilde{q} \rangle,$$

for all $\tilde{q}$ where $q = P\tilde{q}$. Thus, if there exists an invertible transform $P$ that simultaneously diagonalizes $A$ and $B$ by congruence, the system (2.4) in the new variables (2.5) decouples and reduces to a diagonal system of equations:

$$\langle D_A \tilde{p}, \tilde{q} \rangle + \langle D_B \tilde{p}, \tilde{q} \rangle = \langle P^T f, \tilde{q} \rangle,$$

where $D_A = P^T AP$ and $D_B = P^T BP$ are diagonal linear operators. The matrix theory summarized in Section 2.2 gives precisely the conditions for the existence and construction of such a $P$.

2.4. Notation. In the subsequent manuscript, we use the following notation. Let $\Omega$ be an open, bounded domain in $\mathbb{R}^d$, $d = 2, 3$, with Lipschitz polyhedral boundary $\partial \Omega$. We denote by $L^2(\Omega)$ the space of square integrable functions on $\Omega$ with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. We denote by $H^m(\Omega)$ the standard Sobolev space with norm $\|\cdot\|_{H^m}$ and semi-norm $|\cdot|_{H^m}$ for $m \geq 1$ and $H^m(\Omega; \mathbb{R}^d)$ the corresponding $d$-vector fields. We use $H^m_0$ to denote the subspace of $H^m(\Omega)$ with vanishing trace on the boundary $\Gamma \subset \partial \Omega$. We introduce the parameter-dependent $L^2$-inner product and norm:

$$\|p\|^2_\alpha = \langle p, p \rangle_\alpha = \langle \alpha p, p \rangle$$

for $\alpha \in L^\infty(\Omega)$, $\alpha(x) > 0$, and $p \in L^2(\Omega)$ (and similarly for vector or tensor fields).

3. Preconditioning the MPT equations via diagonalization. We begin by summarizing our novel approach to variational formulations and associated preconditioning of the MPT equations. This approach was introduced in [20]. The core idea is to reformulate the MPT (and MPET) equations using a change of pressure variables
In particular, we aim to find a transformation of the variables \( p \mapsto \tilde{p} \) such that the transformed system of pressure equations decouple. Here, we briefly illustrate the core idea, formulation of the MPT equations and resulting preconditioner, and refer to [20] for more details. This approach is then extended to the MPET equations in Section 4.

### 3.1. The MPT equations in operator form

We consider the MPT equations as defined by (1.2). We further impose homogeneous Dirichlet boundary conditions for all pressures: \( p_j = 0 \) on \( \partial \Omega \) for \( 1 \leq j \leq J \). We write \( p = (p_1, p_2, \ldots, p_J) \), and \( g = (g_1, g_2, \ldots, g_J) \). Define \( \xi_j = \sum_{i=1}^J \xi_{i\leftarrow i} \) for each \( 1 \leq j \leq J \). The system (1.2) can be expressed in operator form as

\[
A_{\text{MPT}} p = g \quad \text{with} \quad A_{\text{MPT}} = -K \Delta + E,
\]

where

\[
K = \begin{pmatrix}
K_1 & 0 & \cdots & 0 \\
0 & K_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & K_J
\end{pmatrix}, \quad E = \begin{pmatrix}
\xi_1 & -\xi_{1\leftarrow 2} & \cdots & -\xi_{1\leftarrow J} \\
-\xi_{1\leftarrow 2} & \xi_2 & \cdots & -\xi_{2\leftarrow J} \\
\vdots & \vdots & \ddots & \vdots \\
-\xi_{1\leftarrow J} & -\xi_{2\leftarrow J} & \cdots & \xi_J
\end{pmatrix}.
\]

We note that \( K \) is real, positive definite and diagonal (and thus invertible), and that \( E \) is real, symmetric and (weakly row) diagonally dominant by definition. In particular, \( E \) is symmetric positive semi-definite because of the identity

\[
w E w^T = \sum_{1 \leq i, j \leq J} \xi_{i\leftarrow j} (w_i - w_j)^2,
\]

for \( w = (w_1, w_2, \ldots, w_J) \) with the convention \( \xi_{i\leftarrow i} = 0 \).

A naive block diagonal preconditioner \( B_{\text{MPT}} \) can be constructed by taking the inverse of the diagonal blocks of \( A_{\text{MPT}} \). However, as we demonstrated in [20], the resulting preconditioner is not robust with respect to variations in the conductivity and exchange parameters. In fact, the condition numbers increased linearly with the ratio between the exchange and conductivity coefficients.

### 3.2. Change of variables using diagonalization by congruence

In this section we discuss a new formulation for the MPT equations, which in turn easily offers a parameter-robust preconditioner.

Let \( P \in \mathbb{R}^{J \times J} \) be an invertible linear transformation defining a change of variables and let \( \tilde{p} \) and \( \tilde{q} \) be the new set of variables such that

\[
p = P \tilde{p}, \quad q = P \tilde{q},
\]

with \( q = (q_1, q_2, \ldots, q_J) \) and similarly for \( \tilde{q}, \tilde{p} \). In light of the discussion in Section 2.3, we aim to find a transformation \( P \) that simultaneously diagonalizes \( K \) and \( E \) by congruence. We observe that \( C = K^{-1} E \) is diagonalizable since \( K \) is positive definite and diagonal, and \( E \) is symmetric. By matrix analysis theory, see Section 2.2 and references therein, there exists indeed such a \( P \).

Substituting (3.4) in (3.1) and multiplying by \( P^T \) we can obtain a new formulation for the MPT equations that reads as follows: find the transformed pressures \( \tilde{p} = (\tilde{p}_1, \ldots, \tilde{p}_J) \) such that

\[
\tilde{A}_{\text{MPT}} \tilde{p} = (\tilde{K} \Delta + \tilde{E}) \tilde{p} = P^T g,
\]

where \( \tilde{K} = P^T K P \) and \( \tilde{E} = P^T E P \) are diagonal with

\[
\tilde{K} = \text{diag}(\tilde{K}_1, \ldots, \tilde{K}_J), \quad \tilde{E} = \text{diag}(\tilde{\xi}_1, \ldots, \tilde{\xi}_J).
\]
3.3. Preconditioning the transformed MPT system. As in [20], we can immediately identify the following preconditioner for the transformed system (3.5):

\[
\tilde{B}_{\text{MPT}} = \begin{pmatrix}
(\hat{K}_1 \Delta + \hat{\xi}_1)^{-1} & 0 & \ldots & 0 \\
0 & (\hat{K}_2 \Delta + \hat{\xi}_2)^{-1} & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & (\hat{K}_J \Delta + \hat{\xi}_J)^{-1}
\end{pmatrix},
\]

and the associated norm:

\[
\|\tilde{p}\|_{\tilde{B}_{\text{MPT}}}^2 = \sum_{j=1}^{J} \langle \hat{K}_j \nabla \tilde{p}_j, \nabla \tilde{p}_j \rangle + \langle \hat{\xi}_j \tilde{p}_j, \tilde{p}_j \rangle.
\]

Clearly, \(\tilde{A}_{\text{MPT}} = \tilde{B}_{\text{MPT}}^{-1}\) are trivially spectrally equivalent. We refer to [20] for numerical experiments comparing the standard and transformed formulation and preconditioners.

3.4. Finding the transformation matrix. The number of distinct eigenvalues of \(C = K^{-1}E\) will depend on the material parameter values \(K_j\) and \(\xi_{j,i}\) for \(1 \leq i, j \leq J\). In the common case where \(C\) has \(J\) distinct eigenvalues, the transformation matrix is easily defined as follows. Let \(\lambda_1, \ldots, \lambda_J\) be the real eigenvalues of \(C\), and let \(v_1, \ldots, v_J\) be the corresponding normalized eigenvectors. Then,

\[
P = [v_1, \ldots, v_J],
\]

will diagonalize \(K\) and \(E\) by congruence. In [20], we presented numerical examples for the case of \(J\) distinct eigenvalues (with \(J = 2\)).

However, cases with repeated eigenvalues are also easily constructed. For these cases, the transform \(P\) can be constructed by repeated application of block-wise eigenvector matrices, see [10] for the general procedure. In Example 3.1 below, we present an example on how to obtain the transformation matrix \(P\) in the case where one of the eigenvalues has algebraic multiplicity 2 with \(J = 3\).

**Example 3.1.** In this example we show how to obtain the transformation matrix \(P\) for a three–network case when one of the eigenvalues of \(K^{-1}E\) has algebraic multiplicity 2. We remark that in this example \(P\) is not normalized. Let

\[
K = \begin{pmatrix}
1.0 & 0 & 0 \\
0 & 0.0001 & 0 \\
0 & 0 & 0.01
\end{pmatrix}, \quad E = \begin{pmatrix}
1.01 & -0.01 & -1.0 \\
-0.01 & 0.0101 & -0.0001 \\
-1.0 & -0.0001 & 1.0001
\end{pmatrix}.
\]

By definition

\[
C = K^{-1}E = \begin{pmatrix}
1.01 & -0.01 & -1.0 \\
-100 & 101 & -1.0 \\
-100 & -0.01 & 100.01
\end{pmatrix}.
\]

The eigenvalues \(\lambda_1, \lambda_2, \lambda_3\) and eigenvectors \([v_1, v_2, v_3] = P_1\) of \(C\) are then:

\[
\lambda_1 = 0, \quad \lambda_2 = \lambda_3 = 101.01;
\]

\[
P_1 = \begin{pmatrix}
-0.5773 & -0.0071 & -0.0091 \\
-0.5773 & 0.7070 & -0.4031 \\
-0.5773 & 0.7070 & 0.9150
\end{pmatrix},
\]
In this specific case the eigenvalues $\lambda_2, \lambda_3$ have algebraic multiplicity 2 and geometrical multiplicity 1. If we try to diagonalize $K$ and $E$ by congruence via $P_1$, we obtain

$$P_1^T K P_1 = \begin{pmatrix} 3.3670 \times 10^{-1} & 0 & 0 \\ 0 & 5.1007 \times 10^{-3} & 6.5069 \times 10^{-3} \\ 0 & 6.5069 \times 10^{-3} & 8.4729 \times 10^{-3} \end{pmatrix},$$

(3.13)

$$P_1^T E P_1 = 101.01 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 5.1007 \times 10^{-3} & 6.5069 \times 10^{-3} \\ 0 & 6.5069 \times 10^{-3} & 8.4729 \times 10^{-3} \end{pmatrix}.$$

In this case, the resulting matrices are block diagonal. The lower right blocks are multiples of each other. We can diagonalize the lower right blocks by computing the eigendecomposition of either of these. The lower right block of $P_1^T K P_1$ is

$$\begin{pmatrix} 5.1007 \times 10^{-3} & 6.5069 \times 10^{-3} \\ 6.5069 \times 10^{-3} & 8.4729 \times 10^{-3} \end{pmatrix}$$

and its eigenpairs are

$$\lambda_1 = 6.4967 \times 10^{-5}, \lambda_2 = 1.3508 \times 10^{-2};$$

(3.14)

$$P_2 = \begin{pmatrix} -0.79083 & -0.6120 \\ 0.6120 & -0.7908 \end{pmatrix},$$

(3.15)

The final transformation matrix $P$ that diagonalizes $K$ and $E$ by congruence is then:

$$P = P_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & P_2 \end{pmatrix} = \begin{pmatrix} -5.7735 \times 10^{-1} & 7.1935 \times 10^{-5} & 1.1575 \times 10^{-2} \\ -5.7735 \times 10^{-1} & -8.0594 \times 10^{-1} & -1.1391 \times 10^{-1} \\ -5.7735 \times 10^{-1} & 8.6590 \times 10^{-4} & -1.1564 \end{pmatrix},$$

(3.16)

and the diagonalized matrices are as follows

$$\tilde{K} = P^T K P = \begin{pmatrix} 3.3670 \times 10^{-1} & 0 & 0 \\ 0 & 6.4967 \times 10^{-5} & 0 \\ 0 & 0 & 1.3508 \times 10^{-2} \end{pmatrix},$$

(3.17)

$$\tilde{E} = P^T E P = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 6.5623 \times 10^{-3} & 0 \\ 0 & 0 & 1.3645 \end{pmatrix}.$$

4. Preconditioning the MPET equations via diagonalization. In this section, we present a change of variables for the total pressure formulation of the time-discrete MPET equations and propose and analyze a preconditioning strategy for the resulting variational formulation. The change of MPET variables is guided by the change of MPT variables presented in the previous section.

4.1. Total pressure formulation of the MPET equations. The total pressure formulation of Biot’s equations [14] and more generally the MPET equations [15] is a robust mixed variational formulation targeting the nearly incompressible case and incompressible limit ($\lambda \gg 1$). The total pressure is defined as:

$$p_0 = \lambda \div u - \alpha \cdot p,$$

(4.1)
where $\alpha = (\alpha_1, \ldots, \alpha_J)$, $p = (p_1, \ldots, p_J)$ and $\alpha \cdot p = \sum_{i=1}^J \alpha_i p_i$. The total pressure formulation of (1.1) then reads as follows: for $t \in (0, T]$, find the displacement vector field $u$ and the pressure scalar fields $p_0$ and $p_j$ for $j = 1, \ldots, J$ such that

\begin{align}
(4.2a) & \quad - \text{div} \left( 2\mu \varepsilon(u) + p_0 I \right) = f, \\
(4.2b) & \quad \text{div} u - \lambda^{-1} p_0 - \lambda^{-1} \alpha \cdot p = 0, \\
(4.2c) & \quad \lambda^{-1} p_0 + s_j \dot{p}_j - \text{div} (K_j \nabla p_j) + \alpha_j \lambda^{-1} \alpha \cdot \dot{p} + \sum_{i=1}^J \xi_{j-i} (p_j - p_i) = g_j,
\end{align}

for $j = 1, \ldots, J$.

### 4.2. Variational formulation of the time-discrete MPET equations

We consider an implicit Euler discretization in time of the total pressure formulation of the time-dependent MPET equations (4.2) and examine the resulting stationary problem at each time step. The resulting time-discrete version of (4.2) with time step $\tau > 0$ reads as follows: find the displacement $u$ and the pressures $p_j$ for $0 \leq j \leq J$ such that

\begin{align}
(4.3a) & \quad - \text{div} \left( 2\mu \varepsilon(u) + p_0 I \right) = f, \\
(4.3b) & \quad \text{div} u - \lambda^{-1} p_0 - \lambda^{-1} \alpha \cdot p = 0, \\
(4.3c) & \quad - s_j p_j - \alpha_j \lambda^{-1} p_0 - \alpha_j \lambda^{-1} \alpha \cdot p + \tau \text{div} (K_j \nabla p_j) - \tau \sum_{i=1}^J \xi_{j-i} (p_j - p_i) = g_j,
\end{align}

for $1 \leq j \leq J$ where the new right hand sides $g_j$ for $j = 1, \ldots, J$ have been negated and contain also terms from the previous timestep. Again, we impose homogeneous Dirichlet boundary conditions for all network pressures: $p_j = 0$ on $\partial \Omega$ for $1 \leq j \leq J$.

Let $V = H^1_0(\Omega)^d$, $Q_0 = L^2(\Omega)$ and $Q_j = H^1_0(\Omega)$ for $1 \leq j \leq J$ and $\Omega \in \mathbb{R}^d$. Let $Q = Q_1 \times \cdots \times Q_J$. As in Section 3, we write $p = (p_1, \ldots, p_J)$, $q = (q_1, \ldots, q_J)$, and $g = (g_1, \ldots, g_J)$. Multiplying by test functions, and integrating second-order derivatives by parts, we obtain the following variational formulation of (4.3): find $u \in V$ and $p_i \in Q_i$ for $i = 0, \ldots, J$ such that

\begin{align}
(4.4a) & \quad a(u, v) + b(v, p_0) = \langle f, v \rangle \quad \forall v \in V, \\
(4.4b) & \quad b(u, q_0) - c_1(p_0, q_0) - c_2(q_0, p) = 0 \quad \forall q_0 \in Q_0, \\
(4.4c) & \quad - c_2(p_0, q) - c_3(p, q) = \langle g, q \rangle \quad \forall q \in Q.
\end{align}

The bilinear forms $a : V \times V \to \mathbb{R}$ and $b : V \times Q_0 \to \mathbb{R}$ are defined as:

\begin{align}
(4.5) & \quad a(u, v) = \langle 2\mu \varepsilon(u), \varepsilon(v) \rangle, \\
(4.6) & \quad b(v, q_0) = \langle \text{div} v, q_0 \rangle,
\end{align}

while $c_1 : Q_0 \times Q_0 \to \mathbb{R}$, $c_2 : Q_0 \times Q \to \mathbb{R}$, and $c_3 : Q \times Q \to \mathbb{R}$ are defined as:

\begin{align}
(4.7) & \quad c_1(p_0, q_0) = \langle \lambda^{-1} p_0, q_0 \rangle, \\
(4.8) & \quad c_2(p_0, q) = \langle \lambda^{-1} \alpha \cdot q, p_0 \rangle, \\
(4.9) & \quad c_3(p, q) = k(p, q) + s(p, q) + \varepsilon(p, q) + l(p, q).
\end{align}

\footnote{Note that we start counting at 1 in the definition of $p$ here and throughout, in contrast to e.g. in [20].}
The subforms $k$, $e$, $s$ and $l$ defining $c_2$ are given by

$$k(p, q) = \tau \sum_{j=1}^J \langle K_j \nabla p_j, \nabla q_j \rangle,$$

$$s(p, q) = \sum_{j=1}^J \langle s_j p_j, q_j \rangle,$$

$$e(p, q) = \tau \sum_{j=1}^J \sum_{i=1}^J \langle \xi_{j-i}(p_j - p_i), q_j \rangle,$$

$$l(p, q) = \langle \lambda^{-1} \alpha \cdot p, \alpha \cdot q \rangle.$$

We define the composite $b : V \times (Q_0 \times Q) \rightarrow \mathbb{R}$ and $c : (Q_0 \times Q) \times (Q_0 \times Q) \rightarrow \mathbb{R}$ via

$$c((p_0, p), (q_0, q)) = c_1(p_0, q_0) + c_2(p_0, q) + c_2(q_0, p) + c_3(p, q),$$

$$b(v, (q_0, q)) = b(v, q_0).$$

4.2.1. MPET as a parameter-dependent saddle point system. The system (4.3) or equivalently (4.4) can be viewed as a parameter-dependent saddle point problem with a penalty term (given by $c$). Thus, the equations fit well into Brezzi saddle point theory [4] and into the parameter-dependent preconditioning framework [16] summarized in Section 2.1. However, as we shall see, the structure is non-trivial.

In operator form, we can express (4.4) as

$$A_{\text{MPET}} \begin{pmatrix} u \\ p_0 \\ p \end{pmatrix} = \begin{pmatrix} -2\mu \text{div} \varepsilon & -\nabla & 0 \\ -\lambda^{-1} & -C_2^T & -C_3 \\ 0 & -C_2 & -C_3 \end{pmatrix} \begin{pmatrix} u \\ p_0 \\ p \end{pmatrix} = \begin{pmatrix} f \\ 0 \\ g \end{pmatrix},$$

where $C_i$ for $i = 2, 3$ are given by the terms

$$C_2 = \lambda^{-1} \alpha^T, \quad C_3 = -\tau K \Delta + S + \tau E + L,$$

with $K$ and $E$ given by (3.2), and

$$S = \text{diag}(s_1, \ldots, s_J), \quad L = \lambda^{-1} \alpha \alpha^T.$$

We can rewrite $A_{\text{MPET}}$ of (4.10) in the standard saddle point form

$$A_{\text{MPET}} = \begin{pmatrix} A & B_0^T \\ B_0 & -C \end{pmatrix}$$

by considering the product space grouping $V \times (Q_0 \times Q)$ and identifying

(4.12) $A = -2\mu \text{div} \varepsilon, \quad B_0 = (\text{div}, 0)^T,$

and

$$C = \begin{pmatrix} \lambda^{-1} & C_2^T \\ C_2 & C_3 \end{pmatrix}.$$

A good block preconditioner for this system, cf. Section 2.1, is of the form

$$\begin{pmatrix} A^{-1} & 0 \\ 0 & (C + B_0 A^{-1} B_0)^{-1} \end{pmatrix}.$$
in our case. In detail, by definition (4.12), it is clear that the kernel of $B_0$, and hence that of $B_0AB_0^T$, is large. In particular $Q_0 \times Q_0 \supseteq \ker(B_0) = (0, p_1, \ldots, p_J)$ for $p_j \in Q_j, 1 \leq j \leq J$. On the other hand, $C$ is not obviously uniformly coercive in the material parameters. We note that $-\tau K \Delta$ is coercive on $\ker(B_0)$ but that the operator degenerates as the hydraulic conductivities $K_j$ (or time step $\tau$) decrease. Further, if the exchange coefficients are large relative to the hydraulic conductivities, the off-diagonal terms in $E$ and thus $C$ may become dominant. Furthermore, both $E$ and $L$ are non-definite: $E$ has an (at least) one-dimensional kernel and $L$ is rank 1.

A natural first attempt preconditioner (a direct extension of the preconditioner in [14]) for this system, based on the diagonal blocks, is:

$$(4.13) \quad B_{mpet} = \begin{pmatrix} (-\mu \Delta)^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & D \end{pmatrix}, \quad D = \text{diag} \left( \{(-\tau K_j \Delta + s_j + \tau \xi_j + \alpha_j^2 \lambda^{-1})^{-1}\}_{j=1}^{J} \right)$$

However, this preconditioner is not robust with respect to the material parameters in general, and the hydraulic conductivity and the exchange coefficients in particular. We illustrate this in Example 4.1.

**Example 4.1.** Let $\Omega = [0, 1]^2 \subset \mathbb{R}^2$, and consider a structured triangulation $T_\Delta$ of $\Omega$ constructed by dividing $\Omega$ into $N \times N$ squares and then subdividing each square by a fixed diagonal. Let $J = 2$. Consider a finite element discretization of (4.4) using the lowest order Taylor–Hood-type elements i.e. continuous piecewise quadratics for each displacement component, and continuous piecewise linear for all pressures [15]. Let $\tau = 1.0, \mu = 1.0, s_j = 1.0, \alpha_j = 0.5$ for $j = 1, 2$ and $K_1 = 1.0$, and consider ranges of values for $\lambda, \xi_{j=2}$ and $K_2$. Starting from an initial random guess, we consider a MinRes solver of the resulting linear system of equations with an algebraic multigrid (Hypre AMG) preconditioner of the form (4.13). The convergence criterion used was

$$(Br_k, r_k)/(Br_0, r_0) \leq 10^{-6}$$

where $r_k$ is the residual of the $k$-th iteration. The resulting number of Krylov iterations are shown in Table 2 for $\xi_{J=2} = 10^6$ and ranges of $K_2$ and $\lambda$. We observe that the number of iterations is moderate ($\approx 30$) for $K_2$ of comparable magnitude ($10^6$) to $\xi_{J=2}$. The number of iterations increase with decreasing $K_2$: up to $\approx 10000$ for $K_2 = 1$. For large $K_2$, the number of iterations seems independent of the mesh resolution $N$. In contrast, for smaller $K_2$ (relative to $\xi_{J=2}$), the number of iterations also increase with the mesh resolution. We note that the iteration counts do not vary substantially with $\lambda$.

**4.3. Change of variables for the MPET equations.** In this section, we will explore to what extent it is beneficial to employ linear combinations of the multiple pressures as unknowns rather than the pressures themselves. In other words, we will explore the benefit of the approach introduced for the MPT equations in Section 3.2 for the MPET equations.

Again, we aim to find an invertible transformation $\mathbb{R}^{d+J+1} \rightarrow \mathbb{R}^{d+J+1}$ of the unknowns that leads to a (partial) diagonalization of the system of equations. We choose to keep the displacement and total pressure fixed, and consider transformations of the network pressures only. More precisely, we consider an invertible linear map $P \in \mathbb{R}^{J \times J}$ such that

$$(p_1, \ldots, p_J) = P(p_1, \ldots, p_J)$$
Table 2: Number of MinRes iterations (c.f. Example 4.1): (4.4) as discretized with Taylor-Hood type elements and an algebraic multigrid preconditioner of the form (4.13). Of note is the fact that the number of iterations grow for $K_2$ decreasing relative to $\xi_2 \rightarrow 1 = 10^6$, and for increasing $N$.

| $K_2$ | $\lambda$ | $N = 16$ | 32 | 64 | 128 |
|-------|-----------|----------|----|----|-----|
| $10^0$ |           | 738 | 1756 | 1938 |
| $10^2$ |           | 1024 | 1679 | 1631 |
| $10^4$ |           | 1028 | 1666 | 1628 |
| $10^6$ |           | 1004 | 1677 | 1633 |
| $10^0$ |           | 396  | 406  | 353  |
| $10^2$ |           | 337  | 351  | 333  |
| $10^4$ |           | 364  | 348  | 332  |
| $10^6$ |           | 345  | 361  | 328  |
| $10^0$ |           | 65   | 62   | 60   |
| $10^2$ |           | 64   | 56   | 55   |
| $10^4$ |           | 62   | 57   | 55   |
| $10^6$ |           | 63   | 58   | 55   |
| $10^0$ |           | 30   | 30   | 28   |
| $10^2$ |           | 34   | 29   | 29   |
| $10^4$ |           | 32   | 31   | 29   |
| $10^6$ |           | 33   | 31   | 29   |

Applying this transformation of variables to the semi-discretized total pressure variational formulation of the MPET equations (4.4), we obtain the following variational formulation for the transformed variables: find the displacement $u \in V$, the total pressure $p_0 \in Q_0$ and the transformed pressures $\tilde{p} = (\tilde{p}_1, \ldots, \tilde{p}_J) \in Q$ such that

\begin{align}
\tag{4.14a}
a(u, v) + b(v, p_0) &= \langle f, v \rangle \quad \forall v \in V, \\
\tag{4.14b}b(u, q_0) - c_1(p_0, q_0) - c_2(q_0, \tilde{p}) &= 0 \quad \forall q_0 \in Q_0, \\
\tag{4.14c}-c_2(p_0, P\tilde{\tilde{p}}) - c_3(P\tilde{\tilde{p}}, P\tilde{\tilde{q}}) &= \langle g, \tilde{\tilde{q}} \rangle \quad \forall \tilde{\tilde{q}} \in Q.
\end{align}

The operator form of the transformed system (4.14) then reads as:

\begin{align}
\tag{4.15}\tilde{A}_{\text{MPET}} \begin{pmatrix} u \\ p_0 \\ \tilde{p} \end{pmatrix} &= \begin{pmatrix} f \\ 0 \\ \tilde{g} \end{pmatrix}, \\
\tilde{A}_{\text{MPET}} &= \begin{pmatrix} A & B^T & 0 \\ B & -\lambda^{-1} & -\tilde{C}_2^T \\ 0 & -\tilde{C}_3 & -\tilde{C}_3 \end{pmatrix},
\end{align}

where $A = -2\mu \text{div } \varepsilon$ as before, we write $B = \text{div}$ here and onwards, $\tilde{C}_i = P^T C_i P$ for $i = 2, 3$ and $\tilde{g} = P^T g$. By inserting (4.11) and reordering, we note that

\begin{align*}
\tilde{C}_2 &= P^T C_2 = \lambda^{-1} P^T \alpha^T = \lambda^{-1} \tilde{\alpha}^T \\
\tilde{C}_3 &= P^T C_3 P = -\tau \Delta P^T K P + P^T (S + \tau E + L) P
\end{align*}

As used above, we will write

\begin{align}
\tilde{\alpha}^T &= P^T \alpha^T. 
\end{align}
Following the approach of Subsection 2.3, we aim to identify a transform $P$ that simultaneously diagonalizes $K$ and $S + \tau E + L$ by congruence yielding

$$\tilde{K} = P^T K P = \text{diag}(\tilde{K}_1, \ldots, \tilde{K}_J),$$

$$\tilde{\Gamma} = P^T (S + \tau E + L) P = \text{diag}(\tilde{\gamma}_1, \ldots, \tilde{\gamma}_J),$$

respectively. Since $K$ is diagonal and invertible and $S + \tau E + L$ is symmetric, such a transform indeed exists and can be constructed as described in Section 2.2. Thus,

$$\tilde{C}_2 \equiv \lambda^{-1} \tilde{\alpha}^T,$$

$$\tilde{C}_3 \equiv -\tau \Delta \tilde{K} + \tilde{\Gamma}.$$ 

The following lemma summarizes the useful properties of the transformed matrix operators. These properties will be used to demonstrate the spectral equivalence of a preconditioner in Subsection 4.5.

**Lemma 4.2.** $\tilde{\Gamma}$ defined in (4.18) is a positive definite and diagonal matrix. Furthermore, $\tilde{\Gamma}$ is spectrally equivalent to $P^T (S + \tau E) P$. In fact,

$$P^T (S + \tau E) P \preceq \tilde{\Gamma} \preceq P^T (S + \tau E) P.$$ 

**Proof.** By (3.3), and the definitions of $L$ and $S$, $E$ and $L$ are positive semi-definite and $S$ is positive definite. Hence $\tilde{\Gamma} = P^T (S + \tau E + L) P$ is positive definite as $P$ has full rank. Furthermore, since $\lambda^{-1} \leq c_0 \min_j s_j$ and $\alpha_j \in (0, 1]$, $L = \lambda^{-1} \alpha \alpha^T \leq c \lambda S$ holds with a uniform constant $c > 0$ depending on $J$ and $c_0$. Hence, $\tilde{\Gamma} = P^T (S + \tau E + L) P \preceq P^T (S + \tau E) P$. Finally, clearly $P^T (S + \tau E) P \preceq \tilde{\Gamma} = P^T (S + \tau E + L) P$ because $L$ is positive semi-definite. Hence, $\tilde{\Gamma}$ and $P^T (S + \tau E) P$ are spectrally equivalent with the given bounds. 

### 4.4. Preconditioning the transformed MPET system.

Taking Schur complements, one might consider the following block diagonal operator as a point of departure for preconditioning (4.15):

$$
\begin{pmatrix}
A^{-1} & 0 & 0 & 0 \\
0 & ((2\mu)^{-1} + \lambda^{-1} + \tilde{C}_2 \tilde{C}_3^{-1} \tilde{C}_2)^{-1} & 0 & 0 \\
0 & 0 & \tilde{C}_3^{-1} & 0 \\
\end{pmatrix}.
$$

In particular, the second block is the Schur complement of $-\lambda^{-1}$. Observe that we can rearrange this second block as (the inverse of) a summation:

$$
(2\mu)^{-1} + \lambda^{-1} + \tilde{C}_2 \tilde{C}_3^{-1} \tilde{C}_2 = \sum_{k=0}^{J+1} (a_k - b_k \Delta)^{-1},
$$

where

$$a_0 = 2\mu, b_0 = 0, \quad a_1 = \lambda, b_1 = 0, \quad a_j + 1 = \frac{\lambda^2}{\alpha_j^2} \tilde{\gamma}_j, \quad b_j + 1 = \frac{\lambda^2}{\alpha_j^2} \tilde{K}_j, \quad j = 1, \ldots, J.$$ 

The right-hand-side sum in (4.19) can be interpreted as taking a harmonic mean. The harmonic mean is dominated by the minimum of its arguments when its arguments are positive. In other words, consider the spectral decomposition of $-\Delta$ such
that \(-\Delta u_l = \lambda_l u_l\). Then for a concrete eigenvector \(u_l\) we obtain
\[
(\sum_{k=0}^{J+1} (a_k - b_k \Delta)^{-1})^{-1} u_l = (\sum_{k=0}^{J+1} (a_k + b_k \lambda_k)^{-1})^{-1} u_l \leq ((\min_{k=0,\ldots,J+1} (a_k + b_k \lambda_k))^{-1})^{-1} u_l.
\]
Here, \(\lambda_l\) is an increasing sequence of eigenvalues. Hence, the first elements of the sum, i.e., \(a_0 = 2\mu\) and \(b_0 = 0\) is a reasonable guess for the smallest value (as long as \(\tau\) is not very small). This observation leads to the idea that a weighted mass matrix can suffice for defining a parameter robust preconditioner of (4.15). Concretely, we thus propose the following preconditioner for the transformed MPET system:

\[
\tilde{B}_{\text{MPET}} = \begin{pmatrix}
A^{-1} & 0 & 0 \\
0 & (2\mu)^{-1} & 0 \\
0 & 0 & \tilde{C}_3'\gamma_j^{-1}
\end{pmatrix}
\]

where \(\tilde{C}_3'\gamma_j^{-1}\) is a diagonal operator such that:

\[
\tilde{C}_3' = -\tau \tilde{K} \Delta + \Gamma'
\]

and where we have defined the diagonal matrix \(\Gamma'\) as a partial version of \(\tilde{\Gamma}\):

\[
\Gamma' = \text{diag}(\gamma'_1, \ldots, \gamma'_J), \quad \gamma'_j = \left\{ P^T (S + \tau E) P \right\}_{jj}.
\]

We note that by this definition

\[
\tilde{\gamma}_j = \frac{\gamma'_j}{\lambda_j + \lambda^{-1} \tilde{\alpha}^2_j}.
\]

By Lemma 4.2, we note that \(\gamma'_j > 0\) and \(\tilde{\gamma}_j > 0\) for \(j = 1, \ldots, J\).

Finally, we define the following norm associated with the preconditioner (4.20) over \(V \times Q_0 \times Q_0\):

\[
\|(u, p_0, \tilde{p})\|^2_B = \langle \tilde{B}_{\text{MPET}}(u, p_0, \tilde{p}), (u, p_0, \tilde{p}) \rangle
\]
or more explicitly

\[
\|(u, p_0, \tilde{p})\|^2_B = \sqrt{\mu} \|\varepsilon(u)\|^2 + \|p_0\|^2_{(2\mu)^{-1}} + \sum_{j=1}^{J} \|\nabla \tilde{p}_j\|^2_{\tau \tilde{K}_j} + \sum_{j=1}^{J} \|\tilde{p}_j\|^2_{\gamma'_j}.
\]

**Example 4.3.** For concreteness, we here illustrate the form of the MPET equations and of the proposed preconditioner in a specific example. We consider the simple case of two networks with \(K_1 = K_2 = 1.0\), \(s_1 = s_2 = 1.0\), \(\alpha_1 = \alpha_2 = 0.5\), \(\lambda = 1.0\), \(\xi_{1\rightarrow 2} = 0.0\), and \(\tau = 1.0\). The transformation matrix in this case is

\[
P = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.
\]

We remark that \(P\) is not normalized. The associated transformed MPET operator (expanded), cf. (4.15) and associated definitions, is then

\[
\tilde{A}_{\text{MPET}} = \begin{pmatrix}
-2\mu \text{div} \varepsilon & -\nabla & 0 & 0 \\
\text{div} & -\lambda^{-1} & -((\sqrt{2}) \lambda)^{-1} & 0 \\
0 & -((\sqrt{2}) \lambda)^{-1} & -\Delta + \frac{3}{2} & 0 \\
0 & 0 & 0 & -\Delta + 1
\end{pmatrix},
\]
and the proposed preconditioner will be in the following form:

\[
\mathcal{B}_{\text{MPET}} = \begin{pmatrix}
(2\mu)^{-1} & 0 & 0 \\
0 & (2\mu)^{-1} & 0 \\
0 & 0 & (-\Delta + 1)^{-1}
\end{pmatrix}.
\]

The objective of this example was to illustrate the layout of the operators in a simple case. The results for more general numerical examples will be presented later.

### 4.5. Norm equivalence.

We remark that \( \mathcal{A}_{\text{MPET}} \) is an indefinite operator while \( \mathcal{B}_{\text{MPET}} \) is positive definite. Hence, the two operators are not spectrally equivalent. However, we can prove that \( \mathcal{A}_{\text{MPET}} \) is norm equivalent to the preconditioner \( \mathcal{B}_{\text{MPET}} \) defined by (4.20) in the sense that \( |\mathcal{A}_{\text{MPET}}| \) is bounded by the norm (4.25) under certain assumptions on the material parameters. These bounds are otherwise numerical and material parameter independent, i.e. \( \mathcal{B}_{\text{MPET}} \) defines a parameter-robust preconditioner for \( \mathcal{A}_{\text{MPET}} \).

**Lemma 4.4 (Continuity).** Let \( \tilde{\mathcal{A}}_{\text{MPET}} \) be defined by (4.15), and consider the norm defined by (4.25) induced by the preconditioner \( \mathcal{B}_{\text{MPET}} \). Consider the assumptions of Lemma 4.2, and additionally assume that \( 2\mu \lesssim \lambda \), and that \( \lambda^{-1} \lesssim \min_j s_j \). Then, there exists a constant \( C > 0 \), dependent on the number of networks \( J \) but independent of other material parameters, such that

\[
\langle \tilde{\mathcal{A}}_{\text{MPET}}(u, p_0, \tilde{p}), (v, q_0, \tilde{q}) \rangle \leq C \|(u, p_0, \tilde{p})\|_\mathcal{B} \|(v, q_0, \tilde{q})\|_\mathcal{B},
\]

for all \((u, p_0, \tilde{p}), (v, q_0, \tilde{q}) \in V \times Q_0 \times Q\).

**Proof.** By definition, redistributing the material parameter weights, and the Cauchy-Schwarz inequality, we obtain the preliminary upper bound

\[
\langle \tilde{\mathcal{A}}_{\text{MPET}}(u, p_0, \tilde{p}), (v, q_0, \tilde{q}) \rangle \leq Z_1 + Z_2 + Z_3 =: Z,
\]

where

\[
\begin{align*}
Z_1 &= \|\varepsilon(u)\|_{2\mu} \|\varepsilon(v)\|_{2\mu} + ||p_0||_{(2\mu)^{-1}} \|\text{div } v\|_{2\mu} + \|q_0\|_{(2\mu)^{-1}} \|\text{div } u\|_{2\mu}, \\
Z_2 &= \|p_0\|_{(2\mu)^{-1}} + \|q_0\|_{\lambda^{-1}} + \|\tilde{p}\|_{\lambda^{-1}} + \|\tilde{q}\|_{\lambda^{-1}}, \\
Z_3 &= \sum_{j=1}^{J} \left( \|\nabla \tilde{p}_j\|_{\mathcal{K}_j} \|\nabla \tilde{q}_j\|_{\mathcal{K}_j} + \|\tilde{p}_j\|_{\mathcal{K}_j} \|\tilde{q}_j\|_{\mathcal{K}_j} \right).
\end{align*}
\]

Since \( \|\text{div } u\| \leq \|\varepsilon(u)\| \) and by the assumption that \( 2\mu \lesssim \lambda \) (and thus that \( \lambda^{-1} \lesssim (2\mu)^{-1} \)), it follows that

\[
Z \lesssim \left( \|\varepsilon(u)\|_{2\mu} + ||p_0||_{(2\mu)^{-1}} + \|\tilde{p}\|_{\lambda^{-1}} + \sum_{j=1}^{J} \left( \|\nabla \tilde{p}_j\|_{\mathcal{K}_j} \|\tilde{q}_j\|_{\mathcal{K}_j} \right) \right).
\]

Next, by the assumptions that \( \lambda^{-1} \leq \min_j s_j = s_{\min} \) and \( 0 < \alpha_j \leq 1 \), we note that

\[
\|\alpha \cdot \tilde{p}\|_{\lambda^{-1}} \lesssim \sum_{j=1}^{J} \|\tilde{p}_j\|_{s_{\min}}.
\]

Further, from Lemma 4.2 we have that

\[
\|\tilde{p}_j\|_{\mathcal{K}_j}^2 = \langle \{P^T (S + \tau E + L) P\} \tilde{p}_j, \tilde{p}_j \rangle \leq \langle \{P^T (2S + \tau E) P\} \tilde{p}_j, \tilde{p}_j \rangle \leq 2\|\tilde{p}_j\|_{s_{\min}}^2.
\]

This concludes the proof.
Lemma 4.5 (Inf-sup condition). Let $\mathcal{A}_{\text{MPET}}$, $\mathcal{B}_{\text{MPET}}$ and all assumptions be as in Lemma 4.4. Then, there exists a constant $C > 0$, dependent on the number of networks $J$ but independent of other material parameters, such that

\begin{equation}
\inf_{(u,p_0,p)} \sup_{(v,q_0,q)} \frac{\langle \mathcal{A}_{\text{MPET}}(u,p_0,p), (v,q_0,q) \rangle}{\| (u,p_0,p) \|_\mathcal{B} \| (v,q_0,q) \|_\mathcal{B}} \geq C,
\end{equation}

where the inf and sup are taken over the non-vanishing elements in $V \times Q_0 \times Q$.

Proof. Consider any $(u,p_0,\tilde{p}) \in V \times Q_0 \times Q$, and choose $\tilde{q} = -\tilde{p}$, and $q_0 = -p_0$. Let $w \in H^1(\Omega; \mathbb{R}^d)$ be such that

\begin{equation}
\langle \text{div} \, w, p_0 \rangle = \| p_0 \|^2_{(2\mu)^{-1}}, \quad \| \varepsilon(w) \|_{2\mu} \leq C_0 \| p_0 \|^2_{(2\mu)^{-1}}.
\end{equation}

for a $C_0 > 0$ depending on the domain $\Omega$ via Korn’s inequality, and next choose $v = u + \delta w$ for $\delta > 0$ to be further specified. We note that, with this choice of $v, q_0,$ and $q$,

\[ \| (v,q_0,\tilde{q}) \|_\mathcal{B} \lesssim \| (u,p_0,\tilde{p}) \|_\mathcal{B}, \]

with inequality constant depending only on the domain $\Omega$ and the choice of $\delta$ since

\[ \| \varepsilon(v) \|_{2\mu} \leq \| \varepsilon(u) \|_{2\mu} + \delta C_0 \| p_0 \|^2_{(2\mu)^{-1}} \lesssim \| (u,p_0,\tilde{p}) \|_\mathcal{B}. \]

Therefore, it suffices to show that

\begin{equation}
\langle \mathcal{A}_{\text{MPET}}(u,p_0,\tilde{p}), (v,q_0,\tilde{q}) \rangle \gtrsim \| (u,p_0,\tilde{p}) \|^2_\mathcal{B}.
\end{equation}

Using the definition of $\mathcal{A}_{\text{MPET}}$ together with (4.32), we find that

\begin{align*}
\langle \mathcal{A}_{\text{MPET}}(u,p_0,\tilde{p}), (v,q_0,\tilde{q}) \rangle &= \| \varepsilon(u) \|^2_{2\mu} + \delta (\varepsilon(u), \varepsilon(w) )_{2\mu} + \delta \| p_0 \|^2_{(2\mu)^{-1}} \\
&\quad + 2 \langle \tilde{\alpha} \cdot \tilde{p}, p_0 \rangle_{\lambda^{-1}} + \| p_0 \|^2_{\lambda^{-1}} + \sum_{j=1} J \| \nabla \tilde{p}_j \|^2_{\tau K_j} + \| \tilde{p}_j \|^2_{\gamma_j}.
\end{align*}

Given (4.23), the following algebraic identity and subsequent bound hold

\[ \| p_0 \|^2_{\lambda^{-1}} + 2 \langle \tilde{\alpha} \cdot \tilde{p}, p_0 \rangle_{\lambda^{-1}} + \sum_{j=1} J \| \tilde{p}_j \|^2_{\gamma_j} = \| p_0 + \tilde{\alpha} \cdot \tilde{p} \|^2_{\lambda^{-1}} + \sum_{j=1} J \| \tilde{p}_j \|^2_{\gamma_j} \]

\[ \geq \sum_{j=1} J \| \tilde{p}_j \|^2_{\gamma_j}. \]

Thus,

\begin{equation}
\langle \mathcal{A}_{\text{MPET}}(u,p_0,\tilde{p}), (v,q_0,\tilde{q}) \rangle \gtrsim \| (u,p_0,\tilde{p}) \|^2_\mathcal{B}.
\end{equation}

On the other hand, the Cauchy-Schwarz inequality, the definition of $w$, and Young’s inequality give that

\begin{equation}
\delta |\langle \varepsilon(u), \varepsilon(w) \rangle_{2\mu} | \leq \delta C_0 \| \varepsilon(u) \|_{2\mu} \| p_0 \|^2_{(2\mu)^{-1}} \leq \frac{1}{2} \| \varepsilon(u) \|^2_{2\mu} + \frac{1}{2} \delta^2 C_0^2 \| p_0 \|^2_{(2\mu)^{-1}}.
\end{equation}

Inserting the negation of (4.35) as a lower bound in (4.34), we thus obtain that

\begin{align*}
\langle \mathcal{A}_{\text{MPET}}(u,p_0,\tilde{p}), (v,q_0,\tilde{q}) \rangle &\geq \frac{1}{2} \| \varepsilon(u) \|^2_{2\mu} + \delta (1 - \frac{1}{2} \delta C_0) \| p_0 \|^2_{(2\mu)^{-1}} + \sum_{j=1} J \| \nabla \tilde{p}_j \|^2_{\tau K_j} + \| \tilde{p}_j \|^2_{\gamma_j}.
\end{align*}

By choosing $\delta$, in particular e.g. by letting $\delta < 2/C_0^2$, the estimate (4.33) follows. \qed
4.6. Numerical performance.

Example 4.6. In this final example we demonstrate the robustness of the block diagonal preconditioner (4.20) for a mixed finite element discretization of the transformed total pressure MPET equations (4.15). We consider the same test case, discretization and solver set-up as described in Example 4.1; the new preconditioner is the only modification. Parameter ranges are as follows: $K_2 \in [10^{-6}, 10^6]$, $\xi_{1\rightarrow 2} \in [10^{-6}, 10^6]$ and $\lambda \in [1, 10^6]$.

The resulting number of iterations are shown in Figure 1 for $K_2 \in [10^{-6}, 1]$ and $\xi_{1\rightarrow 2} \in [1, 10^6]$; omitted values demonstrated similar behaviour. Each of the subplots in Figure 1 represent a fixed choice of $K_2$ and $\xi_{1\rightarrow 2}$. In each subplot four curves are shown; these curves show the number of MinRes iterations corresponding to different values of $\lambda$, indicated by their respective symbols, at discretization levels $N = 16, 32, 64$ and 128. The stopping criterion was

$$\frac{(\tilde{B}r_k, r_k)}{(\tilde{B}r_0, r_0)} \leq 10^{-6}$$

where $r_k$ is the residual of the $k$-th iteration. We observe that the number of iterations is moderate in general. Moreover, the number of iterations does not grow for smaller $K_2$'s relative to larger $\xi_{1\rightarrow 2}$ or larger $N$ – in contrast to what was observed for Example 4.1.

5. Conclusions. In this paper, we have presented a new strategy for decoupling the total-pressure variational formulation of the multiple-network poroelasticity equations. The decoupling strategy is based on a transformation via a change of variables, allowing for simultaneous diagonalization by congruence of the equation operators. In particular, the transformed equations are readily amenable for block-diagonal preconditioning. Moreover, we have proposed a block-diagonal preconditioner for the transformed system and shown theoretically that the preconditioner and the equation operator are norm equivalent, independently of the material parameters, under reasonable parameter assumptions. The theoretical results are supported by numerical examples. Combined, these results allow the efficient iterative solution of the multiple-network poroelasticity equations, even in the case of nearly incompressible materials.

We note that our strategy is based on spatially constant material parameters. The applicability of this approach for spatially varying parameters has not yet been considered.

REFERENCES

[1] M. Bai, D. Elsworth, and J.-C. Roegiers, Multiporosity/multipermeability approach to the simulation of naturally fractured reservoirs, Water Resources Research, 29 (1993), pp. 1621–1633.

[2] M. A. Biot, General theory of three-dimensional consolidation, Journal of Applied Physics, 12 (1941), pp. 155–164.

[3] J. Brašnová, V. Lukeš, and E. Rohan, Identification of multi-compartment Darcy flow model material parameters, in 20th International Conference Applied Mechanics 2018: April 9-11, 2018, Myslovice, Czech republic: conference proceedings, 2019, pp. 9–13.

[4] F. Brezzi, On the existence, uniqueness and approximation of saddle-point problems arising from Lagrange multipliers, Revue française d'automatique, informatique, recherche opérationnelle. Analyse numérique, 8 (1974), pp. 129–151.

[5] D. Chou, J. C. Vardakis, L. Guo, B. J. Tully, and Y. Ventikos, A fully dynamic multi-compartmental poroelastic system: Application to aqueductal stenosis, Journal of Biomechanics, 49 (2016), pp. 2306–2312.
Fig. 1: Number of MinRes iterations: (4.4) discretized with Taylor-Hood type elements and algebraic multigrid. $K_2$ varies along the horizontal axis while the vertical axis shows variations in $\xi_{1-2}$ for $K_2$ fixed. Each subplot contains four piecewise linear curves; each curve is decorated by a symbol indicating a corresponding value of $\lambda$ and corresponds to results for discretizations $N = 16, 32, 64$ and 128.

[6] A. Eisenträger and I. Sobey, Multi-fluid poroelastic modelling of CSF flow through the brain, in Poromechanics V: Proceedings of the Fifth Biot Conference on Poromechanics, 2013, pp. 2148–2157.

[7] Q. Hong, J. Kraus, M. Lymbery, and F. Philo, Conservative discretizations and parameter-robust preconditioners for Biot and multiple-network flux-based poroelasticity models, Numerical Linear Algebra with Applications, 26 (2019), p. e2242.

[8] Q. Hong, J. Kraus, M. Lymbery, and F. Philo, Parameter-robust Uzawa-type iterative methods for double saddle point problems arising in Biot’s consolidation and multiple-network poroelasticity models, arXiv preprint arXiv:1910.05883, (2019).

[9] Q. Hong, J. Kraus, M. Lymbery, and M. F. Wheeler, Parameter-robust convergence anal-
ysis of fixed-stress split iterative method for multiple-permeability poroelasticity systems, arXiv preprint arXiv:1812.11809, (2018).

[10] R. A. Horn and C. R. Johnson, Matrix analysis, Cambridge university press, 2nd ed., 1990.

[11] T. Józsa, W. El-Bouri, R. Padmos, S. Payne, and A. Hoekstra, A cerebral circulation model for in silico clinical trials of ischaemic stroke. In the proceedings of CompBioMed Conference 2019.

[12] J. Lee, A. Cookson, R. Chabinik, S. Rivolo, E. Hyde, M. Sinclair, C. Michler, T. Sochi, and N. Smith, Multiscale modelling of cardiac perfusion, in Modeling the heart and the circulatory system, Springer, 2015, pp. 51–96.

[13] J. J. Lee, Unconditionally stable second order convergent partitioned methods for multiple-network poroelasticity, arXiv preprint arXiv:1901.06078, (2019).

[14] J. J. Lee, K.-A. Mardal, and R. Winther, Parameter-robust discretization and preconditioning of Biot’s consolidation model, SIAM Journal on Scientific Computing, 39 (2017), pp. A1–A24.

[15] J. J. Lee, E. Piersanti, K.-A. Mardal, and M. E. Rognes, A mixed finite element method for nearly incompressible multiple-network poroelasticity, SIAM Journal on Scientific Computing, 41 (2019), pp. A722–A747.

[16] K.-A. Mardal and R. Winther, Preconditioning discretizations of systems of partial differential equations, Numerical Linear Algebra with Applications, 18 (2011), pp. 1–40.

[17] C. Michler, A. Cookson, R. Chabinik, E. Hyde, J. Lee, M. Sinclair, T. Sochi, A. Goyal, G. Vigueras, D. Nordsetletten, et al., A computationally efficient framework for the simulation of cardiac perfusion using a multi-compartment Darcy porous-media flow model, International Journal for Numerical Methods in Biomedical Engineering, 29 (2013), pp. 217–232.

[18] M. J. M. Mokhtarudin and S. J. Payne, The study of the function of AQP4 in cerebral ischaemia–reperfusion injury using poroelastic theory, International Journal for Numerical Methods in Biomedical Engineering, 33 (2017), p. e02784.

[19] R. Oyarzúa and R. Ruiz-Baier, Locking-free finite element methods for poroelasticity, SIAM Journal on Numerical Analysis, 54 (2016), pp. 2951–2973.

[20] E. Piersanti, M. E. Rognes, and K.-A. Mardal, Parameter robust preconditioning for multi–compartmental Darcy equations, in , 2019. (in review for EnuMath 2019).

[21] R. Shipley, P. Sweeney, S. Chapman, and T. Roose, A four-compartment multiscale model of fluid and drug distribution in vascular tumours, International Journal for Numerical Methods in Biomedical Engineering, (2020), p. e3315.

[22] K. H. Støverud, M. Alnæs, H. P. Langtangen, V. Haughton, and K.-A. Mardal, Poroelastic modeling of syringomyelia—a systematic study of the effects of pia mater, central canal, median fissure, white and gray matter on pressure wave propagation and fluid movement within the cervical spinal cord, Computer Methods in Biomechanics and Biomedical Engineering, 19 (2016), pp. 686–698.

[23] B. Tully and Y. Ventikos, Modelling normal pressure hydrocephalus as a two-hit disease using multiple-network poroelastic theory, in ASME 2010 Summer Bioengineering Conference, American Society of Mechanical Engineers Digital Collection, 2013, pp. 877–878.

[24] B. J. Tully and Y. Ventikos, Cerebral water transport using multiple-network poroelastic theory: application to normal pressure hydrocephalus, Journal of Fluid Mechanics, 667 (2011), pp. 188–215.

[25] J. C. Vardakis, D. Chou, B. J. Tully, C. C. Hung, T. H. Lee, P.-H. Tsui, and Y. Ventikos, Investigating cerebral oedema using poroelasticity, Medical Engineering & Physics, 38 (2016), pp. 48–57.