BOUNDARY SPIKE OF THE SINGULAR LIMIT OF AN ENERGY MINIMIZING PROBLEM

XINFU CHEN, HUIQIANG JIANG AND GUOQING LIU
Department of Mathematics, University of Pittsburgh
301 Thackeray Hall, Pittsburgh, PA 15260, USA

Abstract. In this paper, we consider the singular limit of an energy minimizing problem which is a semi-limit of a singular elliptic equation modeling steady states of thin film equation with both Van der Waals force and Born repulsion force. We show that the singular limit of energy minimizers is a Dirac mass located on the boundary point with the maximum curvature.

1. Introduction and main results. The equation

$$u_t = \nabla (M(u) \nabla p)$$

(1.1)

has been used to model the dynamics of long wave unstable thin films of viscous fluids. Here $u$ is the thickness of the thin film. The nonlinear mobility is given by

$$M(u) = u^3 + \lambda u^b$$

with $\lambda \geq 0$ and $b \in (0, 3)$ where $\lambda = 0$ corresponds to the no-slip boundary condition. And we assume the pressure

$$p = u^{-1-\alpha} - \epsilon^\beta u^{-1-\alpha-\beta} - \Delta u$$

(1.2)

is a sum of contributions from disjoining pressure due to an attractive van der Waals force, a Born repulsion force and a linearized curvature term corresponding to surface tension effects. Here $\alpha > 0$, $\beta > 0$ and $\epsilon > 0$. Abundant research on thin film equations including the stability of the solutions has been done by lots of authors [6, 7, 8, 9, 10, 11, 22, 23, 24, 25, 26, 27, 28, 33, 35, 36].

Following [8, 15, 21], we consider viscous fluids in a cylindrical container whose bottom is represented by $\Omega$, a bounded smooth domain in $\mathbb{R}^n$, $n \geq 2$ where $n = 2$ is the physically meaningful dimension. Since there is no flux across the boundary, we have the Neumann boundary condition

$$\frac{\partial p}{\partial \nu} = 0 \text{ on } \partial \Omega$$

(1.3)

where $\nu$ is the unit outer normal to $\partial \Omega$. We also ignore the wetting or nonwetting effect, and assume that the fluid surface is perpendicular to the boundary of the container, i.e.,

$$\frac{\partial u}{\partial \nu} = 0 \text{ on } \partial \Omega.$$  

(1.4)
We can associate (1.1) with energy
\[ E_\epsilon (u) = \frac{1}{2} \int_\Omega |\nabla u|^2 - \frac{1}{\alpha} \int_\Omega u^{-\alpha} + \epsilon^\beta \frac{1}{\alpha + \beta} \int_\Omega u^{-\alpha - \beta}. \] (1.5)
Then, using (1.3), (1.4), we have formally
\[
\frac{d}{dt} E_\epsilon (u) = \hat{\Omega} \left( -\Delta uu_t + u^{-1-\alpha} u_t - \epsilon^\beta u^{-1-\alpha-\beta} u_t \right)
= \int_\Omega \nabla (M (u) \nabla p) \cdot p = -\int_\Omega M (u) |\nabla p|^2.
\]
Hence, for a thin film fluid at rest, \( p \) must be a constant, and \( u \) satisfies (1.2), i.e., \( u \) is a critical point for the energy \( E_\epsilon (u) \). If we prescribe the total volume of viscous fluids in the thin film, we obtain an energy minimizing problem with volume constraint.

Rewriting the energy in the following form,
\[
E_\epsilon [u] = \int_\Omega \left\{ \frac{1}{2} |\nabla u|^2 + \frac{1}{\epsilon^\alpha} F \left( \frac{u-\epsilon}{\epsilon} \right) \right\} \, dx - \frac{F_*}{\epsilon^\alpha}
\]
where
\[ F (s) = \frac{(1+s)^{-\alpha-\beta}}{\alpha + \beta} - \frac{(1+s)^{-\alpha}}{\alpha} + F_*, \quad F_* = \frac{\beta}{\alpha (\alpha + \beta)}, \]
the authors in [15, 21] considered the singular limit of the energy as \( \epsilon \to 0 \) with volume constraint and proved that the energy minimizing solutions converge to the limiting profile which is a Dirac measure located on the boundary. Such behavior has also been verified in one dimensional space in [8].

Our goal in this paper is to understand the location of Dirac measure in the limit. If we let \( \epsilon \) approach 0 in the energy \( E_\epsilon [u], F \left( \frac{u-\epsilon}{\epsilon} \right) \) tends to \( F_* \chi_{\{u>0\}} \) formally which leads to the well-known energy functional
\[
E_* [u] := \int_\Omega \left\{ \frac{1}{2} |\nabla u|^2 + \frac{1}{\epsilon^\alpha} \chi_{\{u>0\}} \right\} \, dx \] (1.6)
where \( \epsilon = \epsilon^{\alpha/2} \) and we have dropped the constant term \(-\frac{F_*}{\epsilon^\alpha}\). Adding the mass constrain, we consider the admissible function space
\[
\mathcal{H} (M) = \left\{ u \in H^1 (\Omega) : u \geq 0 \text{ a.e. in } \Omega \text{ and } \int_\Omega u \, dx = M \right\}
\]
where \( M > 0 \) is a given constant.

Such energy functional without mass constraint has been extensively studied. Caffarelli and Alt [2] showed the Lipschitz continuity for the minima and proved singularities cannot occur for minimizer in two dimensional space. Later, Alt, Caffarelli and Friedman [4] extended the result to the case with two phases using monotonicity formula and developed the full regularity theory of the free boundary \( \partial \{u > 0\} \) in dimension 2 and partial regularity theory in higher dimension. In 1999, Weiss [34] claimed the existence of critical dimension \( k \) such that the free boundary is smooth if \( n < k \). Later, Caffarelli, Jerison and Kenig [13] proved the full regularity result in dimension 3. And in 2015, Jerison and Savin [20] extended the result to dimension 4, hence \( k > 4 \). Moreover, the work completed by De Silva and Jerison pointed out \( k < 7 \) since in 7-dimensional space the singular axisymmetric critical point of the functional is an energy minimizer. Till now, the cases \( 5 \leq k \leq 6 \) remain open. Also more general energy functionals have been studied in [1, 3, 14, 16, 29].
The existence of energy minimizers of $E_\varepsilon$ in $\mathcal{H}(M)$ follows from the direct method in calculus of variation. Moreover, we have the following asymptotic behavior of the energy minimizers:

**Theorem 1.** Let $M > 0$ and $\{\varepsilon_k\}_{k=1}^\infty$ be a positive sequence converging to 0. For each $k \geq 1$, let $u_{\varepsilon_k} \in \mathcal{H}(M)$ be an energy minimizer of $E_{\varepsilon_k}$ in $\mathcal{H}(M)$. Then up to a subsequence if necessary, $\{u_{\varepsilon_k}\}_{k=1}^\infty$ approaches a Dirac mass supported on the boundary; that is, there exists $p \in \partial \Omega$ such that

$$\lim_{k \to \infty} \int_{\Omega} u_{\varepsilon_k} (x) \varphi(x) \, dx = M \varphi(p) \quad \forall \varphi \in C(\overline{\Omega}).$$

Next, we want to understand the microscopic structure of the energy minimizer near its concentration point. Let $u_{\varepsilon} \in \mathcal{H}(M)$ be a minimizer of $E_{\varepsilon}$. Let $x_{\varepsilon} \in \Omega$ be a point where $u_{\varepsilon}$ attains its maximum and $p_{\varepsilon} \in \partial \Omega$ be such that

$$|p_{\varepsilon} - x_{\varepsilon}| = \inf_{p \in \partial \Omega} |p - x_{\varepsilon}|. \quad (1.7)$$

Let $\delta = \varepsilon^{1/(n+1)}$. We define

$$\Omega_\delta = \left\{ \frac{x - p_{\varepsilon}}{\delta} : x \in \Omega \right\} \quad (1.8)$$

and

$$v_{\delta} (y) = \delta^n u_{\varepsilon} (x) \quad \text{where} \quad y = \frac{x - p_{\varepsilon}}{\delta} \in \Omega_\delta. \quad (1.9)$$

Then one can verify that $v_{\delta}$ is an energy minimizer of

$$E_\delta[v] := \int_{\Omega_\delta} \left\{ \frac{1}{2} |\nabla v|^2 + \chi_{\{v > 0\}} \right\} \, dy \quad (1.10)$$

in the space

$$\mathcal{H}_\delta(M) = \left\{ v \in H^1(\Omega_\delta) : v \geq 0 \, \text{a.e. in} \, \Omega_\delta \, \text{and} \, \int_{\Omega_\delta} v \, dy = M \right\}.$$

**Theorem 2.** Under the assumption of Theorem 1, passing to a subsequence if necessary, as $k \to \infty$, $p_{\varepsilon_k} \to p$ for some point $p \in \partial \Omega$ and $v_{\varepsilon_k} \to v^*$, locally uniformly in

$$\mathbb{R}^+_{\nu(p)} := \left\{ y \in \mathbb{R}^n \mid y \cdot \nu(p) < 0 \right\}$$

where $\delta_k = \varepsilon_k^{1/(n+1)}$ and $\nu(p)$ is the unit exterior normal of $\partial \Omega$ at $p$. Here

$$v^*(y) = A^* \max\{0, R^* - |y|^2\}$$

where

$$R^* := \left( \frac{1}{2} \right)^{-\frac{n+2}{2n+1}} \left( \frac{(n+2)M}{\omega_n} \right)^{-\frac{1}{n+1}}, \quad A^* := \left( \frac{1}{2} \right)^{\frac{n+2}{2n+1}} \left( \frac{(n+2)M}{\omega_n} \right)^{-\frac{1}{n+1}}$$

and $\omega_n$ denotes the volume of the unit ball in $\mathbb{R}^n$.

Note that $v^*$ is the global minimizer of

$$E^*[v] := \int_{\mathbb{R}^n_{+}} \left\{ \frac{1}{2} |\nabla v|^2 + \chi_{\{v > 0\}} \right\} \, dy$$

in the space

$$\mathcal{H}^*(M) := \left\{ v \in H^1(\mathbb{R}^n_{+}) : \int_{\mathbb{R}^n_{+}} v \, dy = M \, \text{and} \, v \geq 0 \right\}.$$
Now, we are about to investigate the location of the boundary spike. After translation and rotation if necessary, we could assume the concentration point $p$ is the origin. Locally the boundary of $\partial \Omega$ can be written as

$$x_n = \psi(x'), \quad x' = (x_1, \cdots, x_{n-1}), \quad |x'| \leq \eta$$

where $\psi(0') = 0$, $\nabla_x \psi(0') = 0'$. Consequently, the boundary of $\Omega_\delta$ near the origin can be expressed as

$$y_n = \frac{1}{\delta} \psi(\delta y'), \quad y' = (y_1, \cdots, y_{n-1}), \quad |y'| \leq \frac{\eta}{\delta}.$$ 

Based on the limit profile of $v^*$, we apply the asymptotic analysis and assume the energy minimizer has the asymptotic expansion as follows,

$$
\begin{aligned}
D &= \left\{ y \in \mathbb{R}^n : y_n > \psi(\delta y')/\delta, \quad |y| < R + \delta R_1 \left( \frac{x}{|x|} \right) + O(\delta^2) \right\}, \\
v &= \frac{\lambda}{2n}[R^2 - |y|^2] + \delta v_1(y) + O(\delta^2) \quad \forall y \in \hat{D}.
\end{aligned}
$$

Here

$$
D = \{ y : v(y) > 0 \},
$$

$v$ and $R$ are some constants depending on $\delta$, and $\lambda$ depending on $\delta$ through $R$ is Lagrange multiplier corresponding to the mass constraint. We know, in general, the solution does not necessarily have its mass concentrated near original point so some additional constraints have to be added:

$$\int_{\Omega_\delta} y_i v dy = 0 \quad \forall i = 1, \cdots, n-1.$$ 

Therefore, $v_1$, $R_1$ satisfy

$$
\begin{aligned}
-\Delta v_1 &= 0 \quad \text{in } B_R \cap \mathbb{R}^{n+} =: B_R^+, \\
v_1 &= R \partial_\nu v_1 \quad \text{on } \partial B_R \cap \mathbb{R}^{n+} =: \Gamma_R, \\
\partial_{y_n} v_1 &= -\frac{1}{R} \sum_{i=1}^{n-1} \kappa_i y_i^2 \quad \text{on } B_R^+ \times \{0\}, \\
R_1(y/|y|) &= n \partial_\nu v_1(y)/\lambda \quad \forall y \in \Gamma_R
\end{aligned}
$$

(1.11)

where $\nu$ is the unit exterior normal vector and $\kappa_i$ is the principal curvature. We are able to show the above linear system has a unique pair of solution. Moreover, we have

**Theorem 3.** The energy of the Quasi-stationary solution $(v, D)$ has the asymptotic expansion

$$
E_\delta[v] \equiv \int_D \left\{ \frac{1}{2} |\nabla v|^2 + 1 \right\} = E^*[v^*] - c(n) M R \delta + O(\delta^2)
$$

where

$$
c(n) = \frac{(n-1)(n+2)(n+7) \omega_{n-1}}{\sqrt{2}(n+1)(n+3) \omega_n}
$$

is a positive constant.

The above formula implies the peak should be situated near the most curved part of $\partial \Omega$. This type of behavior has been seen before in [30, 31] where Ni and Takagi proved that a type of semilinear elliptic equation with homogeneous Neumann boundary condition admits a least energy solution which attains exactly one peak on the boundary with the maximum of the peak uniformly bounded. We remark here that the maximum of our solutions tends to be unbounded. Later, related
results for the semilinear Dirichlet problem have been obtained by Ni and Wei in [32].

This paper is organized as follows: After introduction, we prove the existence of energy minimizing solution to (1.6) and then present some preliminary results about the energy bound for (1.10). We then derive corresponding Euler equation and prove some regularity results in next three sections. The limit profiles in the theorem 1 and 2 are obtained in section 7. We perform asymptotic analysis in section 8 and prove the existence of the solution to the linearization problem (1.11). Finally in section 9, we derive the energy expansion formula in the theorem 3.

2. Existence of energy minimizing solutions. Let $\Omega$ be a bounded smooth domain in $\mathbb{R}^n$, $n \geq 2$. We consider the energy functional

$$E_\varepsilon [u] := \int_\Omega \left\{ \frac{1}{2} |\nabla u|^2 + \frac{1}{\varepsilon^2} \chi_{\{u > 0\}} \right\} \, dy$$

(2.1)

in the admissible space

$$\mathcal{H}(M) = \left\{ u \in H^1(\Omega) : u \geq 0 \text{ a.e. in } \Omega \text{ and } \hat{\Omega} ud\gamma = M \right\}$$

where $\varepsilon > 0$ and $M > 0$ are given constants. The existence of energy minimizers follows from the direct method in the calculus of variations:

**Theorem 4.** For any given $\varepsilon > 0$, and $M > 0$, there exists at least one energy minimizer of (2.1) in the admissible space $\mathcal{H}(M)$.

**Proof.** Since $E_\varepsilon [u] \geq 0$ for any $u \in \mathcal{H}(M)$ and

$$E_\varepsilon [u_0] = \frac{\hat{\Omega}}{\varepsilon^2} < \infty$$

where the constant function $u_0 \equiv \frac{M}{\hat{\Omega}} \in \mathcal{H}(M)$, $\inf_{u \in \mathcal{H}(M)} E_\varepsilon [u]$ is finite and there exists a minimizing sequence $\{u_k\}_{k=1}^\infty \subset \mathcal{H}(M)$ such that

$$\lim_{k \to \infty} E_\varepsilon [u_k] = \inf_{u \in \mathcal{H}(M)} E_\varepsilon [u].$$

The boundedness of $\{E_\varepsilon [u_k]\}$ implies that the sequence $\{\nabla u_k\}$ is uniformly bounded in $L^2(\Omega)$. Now Poincaré inequality implies

$$\int_\Omega \left| u_k - \frac{M}{\hat{\Omega}} \right|^2 = \int_\Omega \left| u_k - \frac{1}{\hat{\Omega}} \int_\Omega u_k \right|^2 \leq C \int_\Omega |\nabla u_k|^2,$$

hence $\{u_k\}$ is a bounded sequence in $H^1(\Omega)$. Passing to a subsequence if necessary, we can assume that $\{u_k\}_{k=1}^\infty$ converges weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$ to some $u^* \in H^1(\Omega)$. The strong convergence in $L^2(\Omega)$ implies

$$\int_\Omega u^* = \lim_{k \to \infty} \int_\Omega u_k = M.$$

and $u^* \geq 0$ a.e. in $\Omega$. Hence $u^* \in \mathcal{H}(M)$. The weak convergence in $H^1(\Omega)$ implies

$$\int_\Omega |\nabla u^*|^2 \leq \liminf_{k \to \infty} \int_\Omega |\nabla u_k|^2.$$

Since $\{u_k\}_{k=1}^\infty$ converges to $u$ almost everywhere in $\Omega$, we also have

$$\int_\Omega \chi_{\{u^*>0\}} \leq \liminf_{k \to \infty} \int_\Omega \chi_{\{u_k>0\}}.$$
Hence,
\[
E_\varepsilon [u^*] = \int_\Omega \left\{ \frac{1}{2} |\nabla u^*|^2 + \frac{1}{\varepsilon^2} \chi_{\{u^* > 0\}} \right\} \, dy \\
\leq \liminf_{k \to \infty} \int_\Omega |\nabla u_k|^2 + \frac{1}{\varepsilon^2} \liminf_{k \to \infty} \int_\Omega \chi_{\{u_k > 0\}} \\
\leq \lim_{k \to \infty} \int_\Omega \left\{ \frac{1}{2} |\nabla u_k|^2 + \frac{1}{\varepsilon^2} \chi_{\{u_k > 0\}} \right\} \, dy = \inf_{u \in H(M)} E_\varepsilon [u]
\]
which implies \(E_\varepsilon [u^*] = \inf_{u \in H(M)} E_\varepsilon [u]\), i.e. \(u^*\) is an energy minimizer in of \(E_\varepsilon [u]\) in \(H(M)\).

The goal of the paper is to understand the asymptotic behavior of the energy minimizers as \(\varepsilon \to 0^+\). On the other hand, the only energy minimizer for sufficiently large \(\varepsilon\) is the constant function.

**Theorem 5.** When \(\varepsilon\) is large enough, the only energy minimizer of \(E_\varepsilon [u]\) is the constant function \(u_0 \equiv \frac{M}{|\Omega|} \in H(M)\).

**Proof.** For any \(u \in H(M)\), the Poincaré inequality implies
\[
\int_\Omega \left| u - \frac{M}{|\Omega|} \right|^2 \leq c \int_\Omega |\nabla u|^2,
\]

hence
\[
E_\varepsilon [u] = \int_\Omega \left\{ \frac{1}{2} |\nabla u|^2 + \frac{1}{\varepsilon^2} \chi_{\{u > 0\}} \right\} \, dy \\
\geq \int_\Omega \left\{ \frac{1}{2c} \left| u - \frac{M}{|\Omega|} \right|^2 + \frac{1}{\varepsilon^2} \chi_{\{u > 0\}} \right\} \, dy \\
\geq \int_\Omega \frac{1}{2c} \left| u - \frac{M}{|\Omega|} \right|^2 \chi_{\{u = 0\}} \, dy + \frac{1}{\varepsilon^2} \int_\Omega \chi_{\{u_k > 0\}} \\
= \int_\Omega \left[ \frac{M^2}{2c |\Omega|^2} - \frac{1}{\varepsilon^2} \right] \chi_{\{u = 0\}} \, dy + \frac{|\Omega|}{\varepsilon^2} \geq \frac{|\Omega|}{\varepsilon^2} = E_\varepsilon [u_0]
\]
if we choose \(\varepsilon\) large enough such that
\[
\frac{M^2}{2c |\Omega|^2} - \frac{1}{\varepsilon^2} > 0,
\]
i.e.,
\[
\varepsilon > \sqrt{\frac{2c |\Omega|}{M}}.
\]
Hence \(u_0 \equiv \frac{M}{|\Omega|}\) is an energy minimizer. Moreover, \(E_\varepsilon [u] = E_\varepsilon [u_0]\) implies that \(\int_\Omega \chi_{\{u = 0\}} = 0\) and \(u = \frac{M}{|\Omega|}\) a.e. when \(u > 0\), hence \(u \equiv u_0\) a.e. in \(\Omega\). So the energy minimizer is unique when \(\varepsilon > \sqrt{\frac{2c |\Omega|}{M}}\). \(\square\)

3. **Properties of energy.** Given \(\varepsilon > 0\). Let \(u_\varepsilon \in H(M)\) be an energy minimizer of \(E_\varepsilon\). Let \(x_\varepsilon \in \Omega\) be a point where \(u_\varepsilon\) attains its maximum and \(p_\varepsilon \in \partial \Omega\) be such that
\[
|p_\varepsilon - x_\varepsilon| = \inf_{p \in \partial \Omega} |p - x_\varepsilon|.
\]
Let $\delta = \varepsilon^{1/(n+1)}$. We define

$$\Omega_\delta = \left\{ \frac{x - p_\varepsilon}{\delta} : x \in \Omega \right\}$$

and

$$v_\delta(y) = \delta^n u_\varepsilon(x) \text{ where } y = \frac{x - p_\varepsilon}{\delta} \in \Omega_\delta.$$  

Then one can verify that $v_\delta$ is an energy minimizer of

$$E_\delta[v] := \int_{\Omega_\delta} \left\{ \frac{1}{2} |\nabla v|^2 + \chi_{\{v > 0\}} \right\} dy$$

in the space

$$H_\delta(M) = \left\{ v \in H^1(\Omega_\delta) : v \geq 0 \text{ a.e. in } \Omega_\delta \text{ and } \int_{\Omega_\delta} v dy = M \right\}.$$

Let

$$e_\delta(M) = E_\delta[v_\delta] = \inf_{v \in H_\delta(M)} E_\delta[v].$$

The goal in this section is to understand the dependence of energy $e_\delta(M)$ on $\delta$ and mass constraint $M$.

Formally, letting $\delta \to 0$, we obtain the limit energy

$$E^*[v] = \int_{\mathbb{R}^n} \left\{ \frac{1}{2} |\nabla v|^2 + \chi_{\{v > 0\}} \right\} dy$$

in the limit admissible class

$$H^*(M) := \left\{ v \in H^1(\mathbb{R}_+^n) : \int_{\mathbb{R}_+^n} v(y) dy = M \text{ and } v \geq 0 \right\}.$$

Chen and Jiang [15] have proved the following result:

**Proposition 1.** Up to a translation, any global minimizer of $E^*$ in $H^*(M)$ is of the form

$$v^*(y) = A^* \max\{0, R^*^2 - |y|^2\}$$

where

$$R^* := \left( \frac{1}{2} \right)^{-\frac{n+2}{n+1}} \left( \frac{(n+2)M}{\omega_n} \right)^{\frac{n+1}{n+3}},$$

$$A^* := \left( \frac{1}{2} \right)^{\frac{n+2}{n+1}} \left( \frac{(n+2)M}{\omega_n} \right)^{-\frac{1}{n+1}}$$

and $\omega_n$ denotes the volume of unit ball in $\mathbb{R}^n$. Moreover, the minimum energy

$$e^*(M) = \inf_{v \in H^*(M)} E^*[v] = 2(n+1)(\frac{\omega_n}{n+2})^{\frac{n+1}{n+3}} \left( \frac{1}{2} \right)^{\frac{n+2}{n+1}} M^{\frac{n+1}{n+3}}.$$  

We start with the dependence of minimum energy $e_\delta(M)$ on $M$.

**Lemma 1.** For $0 < \delta < 1$, $0 < M_1 \leq M_2$,

$$e_\delta(M_1) \leq e_\delta(M_2) \leq \left( \frac{M_2}{M_1} \right)^2 e_\delta(M_1).$$

In particular, $e_\delta(M)$ is continuous in $M$. 

Proof. Assuming \( v_1 \) is a minimizer for \( E_\delta[v] \) in \( H_\delta(M_1) \), we have
\[
\frac{M_2}{M_1} v_1 \in H_\delta(M_2)
\]
and
\[
e_\delta(M_2) \leq E_\delta\left[ \frac{M_2}{M_1} v_1 \right] = \int_{\Omega_\delta} \left\{ \frac{1}{2} \frac{M_2}{M_1}^2 |\nabla v_1|^2 + \chi_{\{v_1 > 0\}} \right\}
\]
\[
\leq \left( \frac{M_2}{M_1} \right)^2 \int_{\Omega_\delta} \left\{ \frac{1}{2} |\nabla v_1|^2 + \chi_{\{v_1 > 0\}} \right\}
\]
\[
= \left( \frac{M_2}{M_1} \right)^2 e_\delta(M_1).
\]
Assuming \( v_2 \) is a minimizer for \( E_\delta[v] \) in \( H_\delta(M_2) \), we define
\[
v_1 = \begin{cases} v_2 & \text{if } v_2 \leq \eta, \\ \eta & \text{if } v_2 > \eta \end{cases}
\]
where \( \eta > 0 \) is chosen so that
\[
\int_{\Omega_\delta} v_1 = M_1.
\]
Therefore,
\[
v_1 \in H_\delta(M_1)
\]
and
\[
e_\delta(M_1) \leq E_\delta[v_1] = \int_{\Omega_\delta} \left\{ \frac{1}{2} |\nabla v_1|^2 + \chi_{\{v_1 > 0\}} \right\}
\]
\[
\leq \int_{\Omega_\delta} \left\{ \frac{1}{2} |\nabla v_2|^2 + \chi_{\{v_2 > 0\}} \right\} = e_\delta(M_2).
\]
Now we check the continuity of \( e_\delta(M) \) on \( M \). For any given \( M \) and \( t \), then if \( t > 1 \),
\[
e_\delta(M) \leq e_\delta(tM) \leq t^2 e_\delta(M).
\]
If \( t < 1 \),
\[
t^2 e_\delta(M) \leq e_\delta(tM) \leq e_\delta(M).
\]
Hence,
\[
\lim_{t \to 1} e_\delta(tM) = e_\delta(M).
\]

Next, we establish an upper bound of \( e_\delta(M) \) for small \( \delta > 0 \).

**Lemma 2.** For small \( \delta > 0 \),
\[
e_\delta(M) \leq e^*(M)[1 + O(\delta)].
\]

**Proof.** Up to a translation and rotation, we can assume \( p \in \partial \Omega \) is the origin and the unit exterior normal of \( \partial \Omega \) at \( p \) is \((0, \cdots, 0, -1)\). In a small neighborhood of \( p \) we express the boundary of \( \Omega \) as
\[
x_n = \psi(x'), \quad x' = (x_1, \cdots, x_{n-1}), \quad |x'| \leq \eta
\]
where \( \psi(0') = 0, \nabla x' \psi(0') = 0' \) and \( \psi_{x_ix_j}(0') = \kappa_i \delta_{ij}, 1 \leq i, j \leq n - 1 \). Here \( \kappa_i \) is the principal curvature and
\[
\kappa = \sum_{i=1}^{n-1} \kappa_i/(n-1)
\]
is the mean curvature of $\Omega$ at $p$. Consequently, the boundary of $\Omega_\delta$ near the origin can be expressed as

$$y_n = \frac{1}{\delta} \psi(\delta y'), \quad y' = (y_1, \cdots, y_{n-1}), \quad |y'| \leq \frac{\eta}{\delta}.$$ 

Denoting by $B_r$ the ball of radius $r$ centered at the origin, using the Taylor expansion

$$\frac{1}{\delta} \psi(\delta y') = \frac{\delta}{2} \kappa_i y_i^2 + O(\delta^2),$$

we can conclude, for $r \in (0, R]$ with fixed $R$ independent of small $\delta$,

$$|\partial B_r \cap \Omega_\delta| - \frac{1}{2} |\partial B_r| = -\frac{(n-1) \omega_{n-1} R^{n-1} \kappa}{2} + O(\delta^2),$$

$$|B_r \cap \Omega_\delta| - \frac{1}{2} |B_r| = -\frac{(n-1) \omega_{n-1} R^{n+1} \kappa}{2(n+1)} + O(\delta^2).$$

Let $v = A \left( R^2 - |y|^2 \right)_+$ where

$$A = \frac{A^* \int_{B_{R^*} \cap \Omega_\delta} \left( R^* - |y|^2 \right) dy}{\int_{B_{R^*} \cap \Omega_\delta} \left( R^2 - |y|^2 \right) dy},$$

we have

$$A = A^* \left[ 1 + O(\delta) \right] \text{ and } \int_{\Omega_\delta} v = M.$$ 

Consequently, $v \in H_\delta(M)$ implies

$$e_\delta(M) \leq E_\delta[v] = \int_{\Omega_\delta} \left\{ \frac{1}{2} |\nabla v|^2 + \chi_{\{v > 0\}} \right\} dy \leq \int_{\Omega_\delta} \frac{1}{2} \left( \frac{A}{A^*} \right)^2 |\nabla v|^2 dy + |B_{R^*} \cap \Omega_\delta| = e^*(M) \left[ 1 + O(\delta) \right].$$

**Remark 1.** The above estimate implies that the boundedness of $e_\delta(M)$ is uniform in $\delta$. We can pick up $\delta$ small such that $e_\delta(M) \leq 2e^*(M)$. Note that

$$e_\delta(M) \geq \int_{\Omega_\delta} \chi_{\{v_\delta > 0\}} dy.$$ 

The upper bound for $e_\delta(M)$ implies that for $\delta$ small enough, the minimum for $v_\delta(y)$ is equal to 0. Meanwhile, the measure of set $\{x : v_\delta(x) > 0\}$ is bounded above by $2e^*(M)$.

**Lemma 3.** When $0 < \delta < \frac{M}{|\Omega_\delta|}$,

$$\max_{y \in \Omega_\delta} v_\delta(y) \geq \frac{M}{e_\delta(M)}.$$

**Proof.** Since

$$e_\delta(M) \geq \int_{\{v_\delta > 0\}} \frac{1}{\max_{y \in \Omega_\delta} v_\delta} \geq \int_{\{v_\delta > 0\}} \frac{v_\delta(y) dy}{\max_{y \in \Omega_\delta} v_\delta} = \frac{M}{\max_{y \in \Omega_\delta} v_\delta},$$

we have

$$\max_{y \in \Omega_\delta} v_\delta(y) \geq \frac{M}{e_\delta(M)}.$$
Analogously to [15], we use a rearrangement argument to establish a lower bound for $e_{\delta}(M)$ and then obtain the limit of $e_{\delta}(M)$. For convenience, we provide the detail after little revision here.

**Theorem 6.** For any $M > 0$,

$$\liminf_{\delta \to 0^+} e_{\delta}(M) \geq e^*(M).$$

Moreover,

$$\lim_{\delta \to 0^+} e_{\delta}(M) = e^*(M).$$

**Proof.** Let $v$ be a minimizer of $E_{\delta}$ in $H_{\delta}(M)$ and

$$\bar{v} = \max_{y \in \Omega_{\delta}} v(y) \quad \text{and} \quad \underline{v} = \min_{y \in \Omega_{\delta}} v(y).$$

For $\delta$ small enough, from the above remark, we know $\underline{v} = 0$. We define for any $t \in [0, \infty)$,

$$D(t) = \{ x \in \Omega_{\delta} : v(x) > t \}, \quad \Gamma(t) = \partial D(t) \cap \Omega_{\delta}$$

and

$$\mu(t) = |D(t)|, \quad \ell(t) = |\Gamma(t)|.$$

For any open interval $(a, b)$, we have from the coarea formula [17],

$$-\mu'(t) = \int_{\Gamma(t)} \frac{1}{|\nabla v(y)|} dH^{n-1}(y) \quad (3.7)$$

and

$$\int_{\{x \in \Omega_{\delta}, a < v < b\}} |\nabla v|^2 \, dy = \int_a^b \int_{\Gamma(t)} |\nabla v(y)| \, dH^{n-1}(y) \, dt.$$ 

Using

$$|\ell(t)|^2 = \left( \int_{\Gamma(t)} 1 \, dH^{n-1} \right)^2 \leq \int_{\Gamma(t)} \frac{1}{|\nabla v(y)|} \, dH^{n-1} \int_{\Gamma(t)} |\nabla v(y)| \, dH^{n-1},$$

we derive from (3.7),

$$\int_{\Gamma(t)} |\nabla v(y)| \, dH^{n-1}(y) \geq \frac{|\ell(t)|^2}{-\mu'(t)}.$$

Thus,

$$\int_{\{x \in \Omega_{\delta}, a < v < b\}} |\nabla v|^2 \, dy \geq \int_a^b \frac{|\ell(t)|^2}{-\mu'(t)} \, dt.$$

Let $P(\cdot)$ be the best constant of isoperimetric inequality:

$$P(\alpha) := \inf_{D \subset \Omega_{\delta}, |D| \leq \alpha} \frac{|\partial D \cap \Omega_{\delta}|}{\frac{2}{\omega_n} \frac{|D|}{\frac{2}{\omega_n}}}.$$

$P(\alpha)$ is decreasing in $\alpha$ since the infimum is taken on a bigger set for larger $\alpha$. Now for small $\epsilon > 0$, since

$$e_{\delta}(M) \geq \int_{\Omega_{\delta}} \chi_{\{v > \epsilon\}} = \mu(\epsilon),$$

we have

$$\mu(\epsilon) \leq e^*(M) \left[ 1 + O(\delta) \right].$$
Also as $\Omega_\delta$ has almost flat and smooth boundary $\partial\Omega_\delta = \partial\Omega/\delta$, we see that

$$P(\epsilon) = 1 + O(\delta).$$

Hence,

$$\int_{\Omega_\delta} |\nabla v|^2 \, dy \geq \int_\epsilon^{\bar{v}} \frac{|\ell(t)|^2}{-\mu'(t)} \, dt$$

$$\geq \int_\epsilon^{\bar{v}} \left[ P(\mu(t)) \right]^2 \left( \frac{2\mu(t)}{\omega_n} \right) \frac{2-\frac{2}{n}}{-4\mu'(t)} \, dt$$

$$\geq (P(\epsilon))^2 \left( \frac{\omega_n}{2} \right)^\frac{n}{2} \int_\epsilon^{\bar{v}} \frac{\mu(t)}{|\mu'(t)|} \, dt.$$

Now define the symmetric decreasing rearrangement function $w$ by

$$r(t) = \left( \frac{2\mu(t)}{\omega_n} \right)^\frac{1}{n}, \quad w(r(t)) = t, \quad t \in [0,\bar{v}].$$

Then

$$\mu(t) = \frac{\omega_n r(t)^n}{2}, \quad w'(r(t)) r'(t) = 1,$n

$$\mu'(t) = \frac{n\omega_n r^{n-1} r'(t)}{2} = \frac{n\omega_n r^{n-1}}{2w_r}, \quad dt = \frac{dr}{r'(t)} = w_r dr.$$

It then follows that

$$\int_{\Omega_\delta} |\nabla v|^2 \, dx \geq (P(\epsilon))^2 \left( \frac{\omega_n}{2} \right)^\frac{n}{2} \int_\epsilon^{\bar{v}} \frac{\mu(t)}{|\mu'(t)|} \, dt$$

$$\geq (P(\epsilon))^2 \left( \frac{\omega_n}{2} \right)^\frac{n}{2} \int_0^{r(\epsilon)} \left( \frac{\omega_n r^{n-1}}{2w_r} \right)^{2-\frac{2}{n}} w_r \, dr$$

$$= \frac{1 + O(\delta)}{2} \int_0^{r(\epsilon)} w'^2 n\omega_n r^{n-1} \, dr.$$

And then,

$$e_\delta(M) = \int_{\Omega_\delta} \left\{ \frac{1}{2} |\nabla v|^2 + \chi_{\{v>0\}} \right\}$$

$$\geq \frac{1 + O(\delta)}{2} \int_0^{r(\epsilon)} \left\{ \frac{1}{2} w'^2 + 1 \right\} n\omega_n r^{n-1} \, dr.$$

Finally, we define

$$\hat{w}(r) = \begin{cases} w(r) & \text{if } r \in [0, r(\epsilon)], \\ \epsilon + r(\epsilon) - r & \text{if } r \in [r(\epsilon), r(\epsilon) + \epsilon], \\ 0 & \text{if } r \in [r(\epsilon) + \epsilon, \infty). \end{cases}$$

Then we have

$$\int_0^{\infty} \hat{w}'^2 r^{n-1} \, dr - \int_0^{r(\epsilon)} w'^2 r^{n-1} \, dr = \int_{r(\epsilon)}^{r(\epsilon)+\epsilon} \hat{w}'^2 r^{n-1} \, dr.$$
\[ \leq \epsilon |r(\epsilon) + \epsilon|^{n-1} = \epsilon \left( \frac{2\mu(\epsilon)}{\omega_n} \right)^{\frac{1}{n}} + \epsilon |n-1| = O(\epsilon). \]

Meanwhile
\[ \int_{r(\epsilon)}^{r(\epsilon)+\epsilon} r^{n-1}dr - \int_{r(\epsilon)}^{r(\epsilon)+\epsilon} r^{n-1}dr \leq \epsilon |r(\epsilon) + \epsilon|^{n-1} = O(\epsilon) \]

and
\[ \hat{M} := \frac{n\omega_n}{2} \int_{0}^{\infty} \hat{w} r^{n-1}dr + \frac{n\omega_n}{2} \int_{r(\epsilon)}^{r(\epsilon)+\epsilon} \hat{w} r^{n-1}dr + \int_{\{v > \epsilon\}} v(x) dx \geq M - \int_{\{v \leq \epsilon\}} v(x) dx \geq M - \epsilon |\Omega_\delta|. \]

Thus, we obtain
\[ e_{\delta}(M) \geq \frac{1 + O(\delta)}{2} \int_{0}^{r(\epsilon)} \left\{ \frac{1}{2} w'^2 + 1 \right\} n\omega r^{n-1}dr \geq [1 + O(\delta)] \left\{ \int_{\mathbb{R}^n_+} \left\{ \frac{1}{2} |\nabla \hat{w}|^2 + \chi_{\{\hat{w} > 0\}} \right\} - O(\epsilon) \right\} \geq [1 + O(\delta)] \left\{ e^* \left( \hat{M} \right) - O(\epsilon) \right\} \geq [1 + O(\delta)] \left\{ e^* \left( M - \epsilon |\Omega_\delta| \right) - O(\epsilon) \right\}. \]

Letting \( \epsilon \to 0 \), we obtain
\[ e_{\delta}(M) \geq [1 + O(\delta)] e^*(M). \]

Taking \( \delta \to 0 \),
\[ \lim_{\delta \to 0+} \inf e_{\delta}(M) \geq e^*(M). \]

The assertion is completely proved due to Lemma 2.

4. Euler-Lagrange equation. In this section we are going to derive the Euler-Lagrange equation for the minimizer \( \nu_\delta \). Firstly, we prove that \( \nu_\delta \) is continuous inside \( \Omega_\delta \).

**Theorem 7.** For any compact set \( K \subset \Omega_\delta \), there exists a constant \( C \) such that
\[ |\nu_\delta(x) - \nu_\delta(y)| \leq C |x - y| \log \left( \frac{1}{|x - y|} \right) \] (4.1)
if \( x, y \in K, |x - y| < r_0 \) with \( r_0 \) small.

**Proof.** Let \( B_r(y) \subset \Omega_\delta \) be any ball of radius \( r \) with center \( y \) and \( u \in H^1(\Omega_\delta) \) be the unique function satisfying
\[ \Delta u = 0 \text{ in } B_r(y) \text{ and } u = \nu_\delta \text{ in } \Omega_\delta \setminus \bar{B}_r(y). \]
Then let $M_r = \int_{\Omega_{\delta}} (v_{\delta} - u)$ then $\int_{\Omega_{\delta}} u = M - M_r$ and
\[
|M_r| \leq \int_{B_r(y)} |v_{\delta} - u| \\
\leq C r^{n/2} (\int_{B_r(y)} |v_{\delta} - u|^2)^{1/2} \\
\leq C r^{n/2+1} (\int_{B_r(y)} |\nabla v_{\delta} - \nabla u|^2)^{1/2} \\
= C r^{n/2+1} (\int_{B_r(y)} |\nabla v_{\delta}|^2 - \int_{B_r(y)} |\nabla u|^2)^{1/2} \\
\leq C r^{n/2+1} (\mathbf{E}_{\delta}(M))^{1/2} \leq C r^{n/2+1}
\]
where $C = C(M, n)$ is some constant depending on the total mass $M$ and dimension $n$. Here (4.2) follows from the Poincaré Inequality and the last step (4.4) holds according to Lemma 2. We choose $r$ small such that $|M_r| \leq \frac{M}{2}$. Define $\tilde{u} = ku$, where $k = \frac{M}{M - M_r}$. Note that $\frac{2}{3} \leq k \leq 2$. Since $\tilde{u} \in H_0(M)$, we can derive,
\[
0 \leq \mathbf{E}_{\delta}(\tilde{u}) - \mathbf{E}_{\delta}(v_{\delta}) \\
= \int_{\Omega_{\delta}} \left\{ \frac{1}{2} k^2 |\nabla u|^2 + \chi\{u > 0\} \right\} - \int_{\Omega_{\delta}} \left\{ \frac{1}{2} |\nabla v_{\delta}|^2 + \chi\{v_{\delta} > 0\} \right\} \\
= \int_{B_r(y)} (\chi\{u > 0\} - \chi\{v_{\delta} > 0\}) + \frac{k^2 - 1}{2} \int_{\Omega_{\delta}} |\nabla v_{\delta}|^2 + \frac{k^2}{2} \int_{B_r(y)} (|\nabla u|^2 - |\nabla v_{\delta}|^2) \\
\leq C_1 r^n + C_2 (k^2 - 1) - \frac{1}{2} k^2 \int_{B_r(y)} |\nabla (u - v_{\delta})|^2
\]
where $C_1, C_2 = C_2(M, n)$ are constants following Lemma 2. Therefore,
\[
\int_{B_r(y)} |\nabla u - \nabla v_{\delta}|^2 \leq C_1 \frac{2 k^2}{k^2} r^n + C_2 \frac{(k^2 - 1)}{k^2} \leq \frac{9}{2} C_1 r^n + \frac{27}{M} C_2 M r.
\]
Plug (4.2) into (4.5), we obtain
\[
\int_{B_r(y)} |\nabla (u - v_{\delta})|^2 \leq \frac{9}{2} C_1 r^n + \frac{27}{M} C_2 r^{n/2+1} \left( \int_{B_r(y)} |\nabla (u - v_{\delta})|^2 \right)^{1/2}.
\]
Consequently, solving the above quadratic equation yields
\[
\int_{B_r(y)} |\nabla v_{\delta}|^2 - \int_{B_r(y)} |\nabla u|^2 = \int_{B_r(y)} |\nabla u - \nabla v_{\delta}|^2 \leq C r^n
\]
where $C = C(M, n)$ is a constant. Proceeding as [4] Theorem 2.1, we have finished the proof of the above estimate (4.1) which indicates the interior continuity of the energy minimizer. \( \square \)

For convenience, we will suppress the subscript $\delta$ here and let $v = v_{\delta}$. $D := \{ y \in \Omega_{\delta} : v > 0 \}$ is an open set as a result of the continuity. By the standard calculus of variation, for $\forall \zeta \in C_0^\infty (D)$ with $\int_D \zeta dy = 0$ and $\varepsilon$ sufficiently small so that $v + \varepsilon \zeta > 0$ in $D$,
\[
0 = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} (\mathbf{E}_{\delta}[v + \varepsilon \zeta] - \mathbf{E}_{\delta}[v]) = \int_D \nabla v \cdot \nabla \zeta dy.
\]
We can derive that

$$\Delta v = -\lambda \delta \text{ in } D$$

where $\lambda \delta$ is the Lagrange multiplier.

The next theorem shows that, in a generalized sense, on the free boundary $\partial D \cap \Omega_\delta$,

$$\partial_\nu v = -\sqrt{2}.$$

**Theorem 8.** If $v = v_\delta$ is a minimizer of $E_\delta[v]$, then

$$\lim_{\epsilon \to 0} \int_{\partial \{v > \epsilon\}} (|\nabla v|^2 - 2) \eta \cdot \nu d\mathcal{H}^{n-1} = 0$$

for every $\eta \in C_0^\infty(\Omega, \mathbb{R}^n)$ where $\nu$ is the outer normal vector.

**Proof.** For $\eta \in C_0^\infty(\Omega, \mathbb{R}^n)$, define $\tau_\epsilon(y) = x + \epsilon \eta(y)$ for $\epsilon > 0$ small. Then, it follows that

$$D\tau_\epsilon = I + \epsilon D\eta \quad \text{and} \quad \det D\tau_\epsilon = 1 + \epsilon \nabla \cdot \eta + O(\epsilon^2).$$

Let $v_\epsilon(\tau_\epsilon(y)) = v(y)$. The mass $M_\epsilon$ of $v_\epsilon$ is obtained by

$$M_\epsilon = \int_{\Omega_\delta} v_\epsilon(x) dx = \int_D v(y) dy \det D\tau_\epsilon dy = M + \epsilon \int_D \nabla \cdot \eta dy + O(\epsilon^2).$$

Since $\frac{M}{M_\epsilon} v_\epsilon \in H_\delta(M)$,

$$0 \leq E_\delta \left( \frac{M}{M_\epsilon} v_\epsilon \right) - E_\delta[v]$$

$$= \int_D \left[ \frac{1}{2} \left( \frac{M}{M_\epsilon} \right)^2 |\nabla v(D\tau_\epsilon)|^2 + 1 \right] \det D\tau_\epsilon dy - \int_D \left[ \frac{1}{2} |\nabla v|^2 + 1 \right] dy$$

$$= \int_D \left[ \frac{1}{2} \left( \frac{M}{M_\epsilon} \right)^2 |\nabla v(I + \epsilon D\eta)|^2 + 1 \right] (1 + \epsilon \nabla \cdot \eta) dy - \int_D \left[ \frac{1}{2} |\nabla v|^2 + 1 \right] dy$$

$$= \frac{1}{2} \left( 1 - \frac{\epsilon \int_D \nabla \cdot \eta}{M} \right) \left( \int_D |\nabla v|^2 - 2 \epsilon \nabla v \cdot D\eta \cdot \nabla v \right) (1 + \epsilon \nabla \cdot \eta) dy$$

$$- \int_D \left[ \frac{1}{2} |\nabla v|^2 + 1 \right] dy + \epsilon \int_D \nabla \cdot \eta dy$$

$$= \epsilon \left( \frac{1}{2} \int_D \left[ -2 \nabla \cdot D\eta \cdot \nabla v + |\nabla v|^2 \nabla \cdot \eta + 2 \nabla \cdot \eta - 2 \nabla \cdot \eta \frac{\int_D |\nabla v|^2 dy}{M} \right] dy \right).$$

The linear term in $\epsilon$ must vanish, giving

$$0 = \int_D \left[ -2 \nabla \cdot D\eta \cdot \nabla v + |\nabla v|^2 \nabla \cdot \eta + 2 \nabla \cdot \eta - 2 \nabla \cdot \eta \frac{\int_D |\nabla v|^2 dy}{M} \right] dy$$

$$= \lim_{\epsilon \to 0} \left( \int_{\{v > \epsilon\}} 2(\Delta v + \lambda) \nabla v \cdot \eta dy - \int_{\partial \{v > \epsilon\}} (|\nabla v|^2 - 2 + 2 \lambda v) \eta \cdot \nu d\mathcal{H}^{n-1} \right)$$

$$= \lim_{\epsilon \to 0} \left( \int_{\{v > \epsilon\}} 2(\Delta v + \lambda) \nabla v \cdot \eta dy - \int_{\partial \{v > \epsilon\}} (|\nabla v|^2 - 2 + 2 \lambda) \eta \cdot \nu d\mathcal{H}^{n-1} \right)$$
where \( \lambda = \frac{\int_D |\nabla v|^2 \, dy}{M} \). Therefore,

\[
\lim_{\epsilon \to 0} \int_{\partial \{ v > \epsilon \}} (|\nabla v|^2 - 2) \eta \cdot \nu \, d\mathcal{H}^{n-1} = 0.
\]

Using standard variation argument, we also have \( \partial_{\nu} v = 0 \) on \( \partial \Omega_\delta \). Hence, \( v = v_\delta \) is a weak solution of the Euler-Lagrange equation

\[
\begin{align*}
\Delta v &= -\lambda_\delta \quad \text{in} \quad D = \{ y \in \Omega_\delta : v > 0 \} \\
\partial_{\nu} v &= 0 \quad \text{on} \quad \partial D \cap \Omega_\delta, \\
\int_D v(y) \, dy &= M,
\end{align*}
\]

where \( \nu \) is the unit outer normal and the constant \( \lambda_\delta \) is the Lagrange multiplier such that for \( \delta \) small,

\[
\lambda_\delta = \frac{\int_D |\nabla v|^2 \, dy}{M} \leq \frac{2e_\delta(M)}{M} \leq \frac{4e^*(M)}{M}.
\]

5. **Uniform Hölder Continuity.** In this section, we prove the Uniform Hölder Continuity for the minimizer \( v_\delta \). we need the following uniform Poincaré inequality.

**Lemma 4.** For any open connected domain \( \Omega \) in class \( \Theta \) defined by

\[
\Theta = \{ \Omega(a,f) \mid 0 \leq a \leq 1, f \in C^2(B_1^{n-1}), \| f \|_{C^2} \leq \varepsilon < \frac{1}{2}, f(0) = 0, \nabla f(0) = 0 \}
\]

where

\[
\Omega(a,f) = \{ \forall x = (x',x_n) \in B_1(0), x_n < a + f(x') \},
\]

there exists a uniform \( C \) such that

\[
(\int_{B_r(0) \cap \Omega} u^2)^{1/2} \leq C r (\int_{B_r(0) \cap \Omega} |\nabla u|^2)^{1/2}
\]

for all \( u \in H^1(\Omega) \) with \( u = 0 \) on \( \partial B_r(0) \cap \Omega \) and \( r < 1 \).

**Proof.** Apply the Corollary 3 in [12]. One can check that \( B_1(0) \cap \Omega \) satisfies the \( \varepsilon \)-cone property. Then there exists a uniform \( C \) such that

\[
(\int_{B_1(0) \cap \Omega} u^2)^{1/2} \leq C (\int_{B_1(0) \cap \Omega} |\nabla u|^2)^{1/2}
\]

for all \( \Omega \) in class \( \Theta \) and \( \forall u \in H^1(\Omega) \) with \( u = 0 \) on \( \partial B_1(0) \cap \Omega \). Rescaling argument shows that

\[
(\int_{\tilde{B}_r(0) \cap \tilde{\Omega}} u^2)^{1/2} = (r^n) (\int_{B_1(0) \cap \Omega} u(ry)^2)^{1/2} \leq C (r^{n+2} (\int_{B_1(0) \cap \Omega} |\nabla u(ry)|^2)^{1/2} = C r (\int_{B_1(0) \cap \Omega} |\nabla u|^2)^{1/2}
\]

where \( \tilde{\Omega} \) is the transformation of \( \Omega \) after scaling which still belongs to class \( \Theta \). □

It is ready to make a comparison with a harmonic function in any small ball and obtain the growth of local integrals.
Lemma 5. For any \( y \in \Omega_\delta \) and \( r > 0 \),
\[
\int_{B_r(y) \cap \Omega_\delta} |\nabla v_\delta|^2 \leq \int_{B_r(y) \cap \Omega_\delta} |\nabla v|^2 + C r^n
\]  
holds for any \( v \in H^1(\Omega_\delta) \) satisfying that \( v \) is harmonic in \( B_r(y) \cap \Omega_\delta \) and \( v = v_\delta \) in \( \Omega_\delta \setminus B_r(y) \). Here \( C \) is a constant depending on \( M \).

Proof. Let \( B_r(y) \) be any ball of radius \( r \) centered at a point \( y \) in \( \Omega_\delta \) and define function \( v \in H^1(\Omega_\delta) \) satisfying
\[
\Delta v = 0 \text{ in } B_r(y) \cap \Omega_\delta \text{ and } v = v_\delta \text{ in } \Omega_\delta \setminus B_r(y).
\]
Then let \( M_r = \int_{\Omega_\delta} (v_\delta - v) \) then \( \int_{\Omega_\delta} v = M - M_r \) and
\[
|v_\delta - v| \leq \int_{B_r(y) \cap \Omega_\delta} |v_\delta - v| \leq C r^{n/2} \left( \int_{B_r(y) \cap \Omega_\delta} |v_\delta - v|^2 \right)^{1/2} \leq C r^{n/2 + 1} \left( \int_{B_r(y) \cap \Omega_\delta} |\nabla v_\delta - \nabla v|^2 \right)^{1/2}
\]
where \( C = C(M) \) is some constant. \((5.2)\) follows from the uniform Poincaré Inequality (Lemma 4) due to the fact that the smooth boundary of \( \Omega_\delta \) is almost flat if we take \( r \) so small. The last step \((5.3)\) holds according to Lemma 2. We choose \( \delta \) to be small so that \( |M_r| \leq \frac{M}{2} \). Define \( \bar{v} = kv \), where \( \frac{3}{2} \leq k = \frac{M}{M - M_r} \leq 2 \). Since \( \bar{v} \in H_\delta(\Omega) \), we can derive,
\[
0 \leq \mathbf{E}_\delta(\bar{v}) - \mathbf{E}_\delta(v_\delta)
\]
\[
= \int_{\Omega_\delta} \left\{ \frac{1}{2} k^2 |\bar{v}|^2 + \chi_{\{v_\delta > 0\}} \right\} - \int_{\Omega_\delta} \left\{ \frac{1}{2} |\nabla v_\delta|^2 + \chi_{\{v_\delta > 0\}} \right\}
\]
\[
= \int_{B_r(y) \cap \Omega_\delta} \left( \chi_{\{v_\delta > 0\}} - \chi_{\{v_\delta > 0\}} \right) + \frac{k^2 - 1}{2} \int_{\Omega_\delta} |\nabla v_\delta|^2 + \frac{k^2}{2} \int_{B_r(y) \cap \Omega_\delta} \left( |\nabla v|^2 - |\nabla v_\delta|^2 \right)
\]
\[
\leq C_1 r^n + C_2 (k^2 - 1) - \frac{1}{2} k^2 \int_{B_r(y) \cap \Omega_\delta} |\nabla v - \nabla v_\delta|^2
\]
where \( C_1, C_2 = C_2(M) \) are constants following Lemma 2. Therefore,
\[
\int_{B_r(y) \cap \Omega_\delta} |\nabla v - \nabla v_\delta|^2 \leq C_1 \frac{2}{k^2} r^n + C_2 \frac{2(k^2 - 1)}{k^2}
\]
\[
\leq \frac{9}{2} C_1 r^n + \frac{27}{M} C_2 M_r.
\]
Plug \((5.2)\) into \((5.4)\), we obtain
\[
\int_{B_r(y) \cap \Omega_\delta} |\nabla v - \nabla v_\delta|^2 \leq \frac{9}{2} C_1 r^n + \frac{27}{M} C_2 r^{n/2 + 1} \left( \int_{B_r(y) \cap \Omega_\delta} |\nabla v - \nabla v_\delta|^2 \right)^{1/2}.
\]
Consequently, solving the above quadratic equation yields
\[
\int_{B_r(y) \cap \Omega_\delta} |\nabla v_\delta|^2 - \int_{B_r(y) \cap \Omega_\delta} |\nabla v|^2 = \int_{B_r(y) \cap \Omega_\delta} |\nabla v - \nabla v_\delta|^2 \leq C r^n
\]
where \( C = C(M) \) is a constant only depending on \( M \). \( \Box \)
Lemma 6. Let \( 0 < \varepsilon \leq 1 \). For any \( \hat{\alpha} \in (0,1) \), there exist \( r_0 > 0 \) and \( K_{\hat{\alpha}} > 1 \) such that for any \( y \in \Omega_{\delta} \) and \( r \in (0,r_0] \) and for any \( v \) satisfying
\[
\Delta v = 0 \text{ in } \Omega_{\delta} \cap B_r(y), \quad \partial_{\nu} v = 0 \text{ on } \partial \Omega_{\delta} \cap B_r(y),
\]
we have for any \( \sigma \in (0,1) \),
\[
\int_{B_{r}(y)\cap\Omega_{\delta}} |\nabla v|^2 \leq K_{\hat{\alpha}}\sigma^{n-2+2\hat{\alpha}} \int_{B_{r}(y)\cap\Omega_{\delta}} |\nabla v|^2. \tag{5.5}
\]
Combining (5.1) and (5.5) gives the core lemma regarding the growth of the Dirichlet integral for \( v_{\delta} \). This is the key step to show \( C^\alpha \) continuity.

Lemma 7. Let \( 0 < \delta \leq 1 \) and \( \alpha \in (0,1) \). There exists \( r_0 > 0 \) such that for any \( y \in \Omega_{\delta} \) and \( r \in (0,r_0] \),
\[
\int_{B_{r}(y)\cap\Omega_{\delta}} |\nabla v_{\delta}|^2 \leq C_3r^{n-2+2\alpha}.
\]
Here we can take
\[
C_3 = \inf_{\hat{\alpha}\in(\alpha,1), \ 0<\sigma<K_{\hat{\alpha}}} \left\{ \sqrt{Cutschenstein_{\hat{\alpha}}} \sigma^{1-\alpha-n/2} \left( \frac{2\epsilon_\delta(M)}{(\sigma r_0)^{n-2+2\alpha}} \right)^{1/2} \right\}.
\]

Proof. For any \( \alpha \in (0,1) \), let \( \hat{\alpha} \in (\alpha,1) \). Let \( r_0 \) be defined in Lemma 6. For any \( y \in \Omega_{\delta} \) and \( r \in (0,r_0] \), for simplicity we denote \( \tilde{B}_r = B_r(y) \cap \Omega_{\delta} \). Let \( v \) be the unique harmonic function in \( \tilde{B}_r \) satisfying
\[
v = v_{\delta} \text{ in } \Omega_{\delta} \cap \partial B_r(y), \quad \frac{\partial v}{\partial \nu} = 0 \text{ on } \partial \Omega_{\delta} \cap B_r(y).
\]
We have from Lemma 5,
\[
\int_{B_{r}} |\nabla (v_{\delta} - v)|^2 = \int_{\tilde{B}_r} |\nabla v_{\delta}|^2 - \int_{\tilde{B}_r} |\nabla v|^2 \leq C r^n.
\]
For any \( \sigma \in (0,1) \), we have
\[
\left( \int_{B_{\sigma r}} |\nabla v_{\delta}|^2 \right)^{1/2} \leq \left( \int_{B_{\sigma r}} |\nabla (v_{\delta} - v)|^2 \right)^{1/2} + \left( \int_{B_{\sigma r}} |\nabla v|^2 \right)^{1/2} \\
\leq \left( \int_{B_{r}} |\nabla (v_{\delta} - v)|^2 \right)^{1/2} + \left( K_{\hat{\alpha}}\sigma^{n-2+2\hat{\alpha}} \int_{B_{r}} |\nabla v|^2 \right)^{1/2} \\
\leq (C r^n)^{1/2} + \left( K_{\hat{\alpha}}\sigma^{n-2+2\hat{\alpha}} \int_{B_{r}} |\nabla v_{\delta}|^2 \right)^{1/2}.
\]
Here in the second inequality, we have applied Lemma 6 to the second term on the right-hand side. Divide both sides by \( (\sigma r)^{n/2-1-\alpha} \) and define
\[
\phi(r) = \left( \frac{1}{r^{n-2+2\alpha}} \int_{B_{r}} |\nabla v_{\delta}|^2 \right)^{1/2}.
\]
Theorem 9. Let $M > 0$, $\delta > 0$ and $v_\delta$ be a minimizer of $E_\delta$ in $H_\delta(M)$. There exists a constant $C$ such that for given small $\delta_0$ and any $0 < \delta \leq \delta_0$,

$$\|v_\delta\|_{C^{\alpha}(\Omega_\delta)} \leq C.$$  

Therefore, $v_\delta$ is uniformly bounded in $\delta$.

Proof. Applying Poincaré’s inequality, we have for any $y \in \Omega_\delta$ and $r \in (0, r_0]$,

$$\inf_{c \in \mathbb{R}} \int_{B_r(y) \cap \Omega_\delta} |v_\delta - c|^2 \leq C r^2 \int_{B_r(y) \cap \Omega_\delta} |\nabla v_\delta|^2 \leq C_4 r^{n+2\alpha}.$$

Hence, $v_\delta$ is in Campanato space $L^{2,n+2\alpha}(\Omega_\delta)$. Following from Theorem 1.2 in page 70 of [18], $v_\delta$ is Hölder continuous with $\alpha$th Hölder seminorm bounded by constant $C_4$ which is independent of $\delta$. Since

$$2e^*(M) \geq e_\delta(M) \geq \int_{\Omega_\delta} \chi_{\{v_\delta > 0\}} dy$$

we are able to choose $R \geq \left(\frac{2e^*(M)}{\omega_n}\right)^{1/n}$ such that $\forall y \in D = \{y : v_\delta > 0\}$, there exists $z \in B_R(y) \cap D^c$, then $v_\delta(z) = 0$.

$$v_\delta(y) \leq v_\delta(z) + C_4 R^\alpha = C_4 R^\alpha.$$ 

Therefore, sup $v_\delta$ is uniformly bounded which ends the proof of the theorem. \qed
6. **Uniform Lipschitz continuity.** The main goal of this section is to prove the uniform Lipschitz continuity of \( v_\delta \). We remark here that uniform Hölder continuity obtained in the previous section is sufficient for the results of this paper, but we think such an estimate may be useful in our future research in this topic. The idea is based on the work by Caffarelli and H.W.Alt \[2\] with Dirichlet boundary setting. The mass constraint is the new technical difficulty here and we also require uniform global estimate involving boundary under Neumann boundary setting which is not covered in \[2\]. In this section, we assume \( \delta \) is small and all the constants are independent of such uniformly small \( \delta \).

**Theorem 10.** Let \( M > 0, \delta > 0 \) and \( v_\delta \) be a minimizer of \( E_\delta \) in \( H_\delta(M) \). There exists a constant \( C = C(M, n) \) such that, for \( \delta_0 \) sufficiently small and any \( 0 < \delta \leq \delta_0 \),

\[
\| \nabla v_\delta \|_{L^\infty(\Omega_\delta)} \leq C.
\]

Firstly, we prove the following lemma.

**Lemma 8.** Let \( v \in H_\delta(M) \) be a minimizer of \( E_\delta[v] \). Then for any small ball \( B_r \subset \Omega_\delta \),

\[
\frac{1}{r} \int_{\partial B_r} v \geq C \quad \text{implies} \quad v > 0 \quad \text{in} \quad B_r,
\]

where \( C \) is positive constant independent of \( \delta \).

**Proof.** Take harmonic function \( u \) such that \( \Delta u = 0 \) in \( B_r \) and \( u = v \) in \( \Omega_\delta \setminus B_r \).

**Case 1.** \( \int_{B_r} u \geq \int_{B_r} v \). Since \( e_\delta(M) \) is increasing in \( M \), we have

\[
\int_{\Omega_\delta} \left\{ \frac{1}{2} |\nabla v|^2 + \chi_{\{v > 0\}} \right\} dy \leq \int_{\Omega_\delta} \left\{ \frac{1}{2} |\nabla u|^2 + \chi_{\{u > 0\}} \right\} dy
\]

It follows that

\[
\int_{B_r} \frac{1}{2} |\nabla (v - u)|^2 dy \leq \int_{B_r} \chi_{\{v = 0\}} dy.
\] (6.1)

Since (6.1) is scaling invariant, we just take \( B_r = B_1(0) \). For \( |z| \leq \frac{1}{2} \), define

\[
v_z(x) = v((1 - |z|)z + x)
\]

and

\[
u_z(x) = u((1 - |z|)z + x).
\]

Note that the map \( x \mapsto (1 - |z|)z + x \) is an isomorphism from \( B_1(0) \) to itself. Also for \( \forall \xi \in \partial B_1 \), define

\[
r_\xi := \inf \{ r | \frac{1}{8} \leq r \leq 1 \text{ and } v_z(r\xi) = 0 \}
\]

if the set is nonempty and \( r_\xi = 1 \) if the set is empty. For almost all \( \xi \in \partial B_1 \),

\[
u_z(r_\xi) = \int_{r_\xi} \frac{d}{dr} (v_z - u_z)(r\xi) dr \leq \sqrt{1 - r_\xi^2} \int_{r_\xi} |\nabla (v_z - u_z)(r\xi)|^2 dr.
\] (6.2)

Also using Green function \( G(x, y) \) with \( \Delta G(x, y) = \delta_x \) for \( x \in B_1(0) \),

\[
u_z(r_\xi) = \int_{\partial B_1} \frac{\partial G(r_\xi, y)}{\partial y} u_z(y) dS(y) \geq c(n)(1 - r_\xi) \int_{\partial B_1} v.
\] (6.3)
Combining (6.2) and (6.3), we have,

\[ \int_{r_\xi}^{1} |\nabla (v_z - u_z)(r\xi)|^2 dr \geq C(n)(1 - r_\xi)(\int_{\partial B_1} v)^2. \]

Integrating over \( \xi \) and then integrating over \( z \),

\[ C(n) \int_{B_1} |\nabla (v_z - u_z)|^2 \geq \int_{B_1} \chi_{\{v = 0\}} (\int_{\partial B_1} v)^2. \]  

(6.4)

Together with (6.1), we obtain,

\[ C(n) \int_{B_1} \chi_{\{v = 0\}} dy \geq \int_{B_1} \chi_{\{v = 0\}} (\int_{\partial B_1} v)^2. \]  

(6.5)

**Case 2.** \( \int_{B_r} u < \int_{B_r} v \). We define

\[ \tilde{u} = u + \lambda \frac{2n}{r^2 - |x|^2} \]

where \( \lambda > 0 \) is chosen such that \( \int_{B_r} \tilde{u} = \int_{B_r} v \). It is easy to check that

\[ \int_{B_r} |\nabla v|^2 - |\nabla \tilde{u}|^2 = \int_{B_r} |\nabla v - \nabla \tilde{u}|^2 + 2 \int_{B_r} \nabla (v - \tilde{u}) \nabla \tilde{u} \]

\[ = \int_{B_r} |\nabla v - \nabla \tilde{u}|^2 - 2 \int_{B_r} (v - \tilde{u}) \Delta \tilde{u} \]

\[ = \int_{B_r} |\nabla v - \nabla \tilde{u}|^2 + 2\lambda \int_{B_r} (v - \tilde{u}) = \int_{B_r} |\nabla v - \nabla \tilde{u}|^2. \]  

(6.6)

Then (6.1) follows. Repeat the process in Case 1. It remains to check (6.3). By the definition of Green function, \( G(x, y) < 0 \) for \( x, y \in B_1(0) \).

\[ \tilde{u}_z(r_\xi) = \int_{\partial B_1} \frac{\partial G(r_\xi, y)}{\partial y} \tilde{u}_z dy + \int_{B_1} \Delta \tilde{u}_z G(r_\xi, y) \]

\[ \geq C(n)(1 - r_\xi) \int_{\partial B_1} \tilde{u}_z - \lambda \int_{B_1} G(r_\xi, y) \]

\[ \geq C(n)(1 - r_\xi) \int_{\partial B_1} v. \]  

(6.7)

Therefore for both cases, we have (6.5), that is,

\[ C(n) \int_{B_1} \chi_{\{v = 0\}} dy \geq \int_{B_1} \chi_{\{v = 0\}} (\int_{\partial B_1} v)^2. \]  

(6.8)

If \( \int_{\partial B_1} v > C(n) \), then \( \int_{B_1} \chi_{\{v = 0\}} = 0 \) which indicates \( v > 0 \) in \( B_1 \).

Recall that \( D = \{ x \in \Omega_\delta : v(x) > 0 \} \) and the measure of \( D \) is bounded above by \( 2e^*(M) \). Define \( \Sigma = \{ x \in \Omega_\delta : v(x) = 0 \} \). We have the following interior estimate,

**Lemma 9.** For any \( x \in \Omega_\delta \setminus \Sigma \) such that \( \text{dist}(x, \Sigma) \leq \text{dist}(x, \partial \Omega_\delta) \), we have

\[ |\nabla v(x)| \leq C \]

where \( C = C(M, n) \) is a positive constant.
Proof. For any $x \in \Omega_{\delta}$, take the maximal ball $B_r(x) \subset D = \Omega_{\delta} \setminus \Sigma$. Since the measure of $D$ is bounded above, then $r \leq C(M,n)$. Since $\Delta v = -\lambda_{\delta}$ in $B_r(x)$, then we can rewrite $v$ as

$$v = v_* + \frac{\lambda_{\delta} r^2 - |x|^2}{2n}$$

where $\Delta v_* = 0$ in $B_r(x)$. (6.9)

For $\text{dist}(x, \Sigma) < \text{dist}(x, \partial \Omega_{\delta})$, $\partial B_r(x)$ does not touch $\partial \Omega_{\delta}$. Then for arbitrary small $\epsilon$, $B_{r+\epsilon}(x) \cap D$ is nonempty which follows,

$$\frac{1}{r+\epsilon} \int_{\partial B_{r+\epsilon}(x)} v \leq C(n).$$

Take $\epsilon \to 0$, then

$$\frac{1}{r} \int_{\partial B_r(x)} v_* = \frac{1}{r} \int_{\partial B_r(x)} v \leq C(n).$$

Consequently,

$$|\nabla v| \leq |\nabla v_*| + \frac{\lambda_{\delta}}{n} r \leq \frac{1}{r} \int_{\partial B_r} v_* + \frac{\lambda_{\delta}}{n} r \leq C(n, M).$$

For $\text{dist}(x, \Sigma) = \text{dist}(x, \partial \Omega_{\delta})$, we see $x$ as the limit of a sequence of points $\{x_n\}_{n=1}^{\infty}$ with $\text{dist}(x_n, \Sigma) < \text{dist}(x_n, \partial \Omega_{\delta})$. By applying the continuity of $|\nabla v|$ in $D$, we will be able to finish our proof for this lemma.

In order to prove the uniform Lipschitz continuity, we have to make some boundary estimates and prove the boundedness of $|\nabla v(x)|$ for $\text{dist}(x, \Sigma) > \text{dist}(x, \partial \Omega_{\delta})$. So we divide into two cases $\text{dist}(x, \Sigma) \geq r_0$ and $r_0 > \text{dist}(x, \Sigma) > \text{dist}(x, \partial \Omega_{\delta})$ for fixed small $r_0$.

For any given small $r_0 > 0$ and $x \in D$, if $\text{dist}(x, \Sigma) \geq r_0$, we consider the following elliptic problem in $B_{r_0}(x) \cap \Omega_{\delta}$,

$$\begin{cases}
\Delta v = -\lambda_{\delta} & \text{in } B_{r_0}(x) \cap \Omega_{\delta}, \\
\partial_{\nu} v = 0 & \text{on } B_{r_0}(x) \cap \partial \Omega_{\delta}.
\end{cases}$$

(6.10)

Since $\partial \Omega$ is smooth, Neumann boundary condition allows us to perform the standard even reflection of $v$. Denote $d(y) := \text{dist}(y, \partial \Omega) = \text{dist}(y, \Omega_{\delta})$ and $n(y) := n(p_y)$ where $p_y \in \partial \Omega$. Then the resulting function

$$\tilde{v}(y) = \begin{cases} v(y) & \text{for } y \in B_{r_0}(x) \cap \Omega_{\delta}, \\
v(y) - 2v(y)d(y) & \text{for } y \in B_{r_0}(x) \setminus \Omega_{\delta}.
\end{cases}$$

(6.11)

satisfies

$$\partial_{\nu}(a_{ij}(y)\partial_{y_i} \tilde{v}) = -\lambda_{\delta} \text{ in } B_{r_0}$$

where $|a_{ij} - \delta_{ij}|$ is uniformly small. Applying Lemma 6.5 and Theorem 6.6 in [19],

$$|\nabla \tilde{v}(x)| \leq C(||\tilde{v}||_{L^\infty} + |\lambda_{\delta}|).$$

According to Theorem 9, $v$ is uniform bounded by a constant depending on total mass $M$ and dimension $n$. Moreover, $\lambda_{\delta}$ is bounded by $\frac{4e(M)}{M}$. Therefore, there exists a constant $C = C(M, n, r_0)$ such that for $x \in D$ with $\text{dist}(x, \Sigma) \geq r_0$,

$$|\nabla v(x)| \leq C.$$

(6.12)

The remaining case is $r_0 > \text{dist}(x, \Sigma) > \text{dist}(x, \partial \Omega)$. For simplicity, denote $R = \text{dist}(x, \Sigma)$. Considering $B_R(x) \cap \Omega_{\delta}$, we define this rescaled function

$$\tilde{v}(z) = \frac{1}{R} v(y)$$

(6.13)
where \( z \in B_1(x) \) and \( y = x + R(z - x) \). Hence,

\[
|\nabla \hat{v}(z)| = |\nabla v(y)| \quad \text{and} \quad \Delta \hat{v}(z) = R \Delta v(y) = -R \lambda_3.
\]

Also, \( \partial \tilde{\Omega}_\delta = \{ z : y \in \partial \Omega_\delta \} \) is almost flat. Take the ball \( B_\varrho(x_0) \) with center point \( x_0 = p_x - \frac{1}{2}(1 + \bar{R})n_x \) and radius \( \varrho = \frac{1}{2}(1 + \bar{R}) \) where \( \bar{R} = \text{dist}(x, \partial \tilde{\Omega}_\delta) \). Roll \( B_\varrho(x_0) \) along \( \partial \tilde{\Omega}_\delta \) towards \( \Sigma \) until it touches \( \Sigma \) which results a ball \( B_\varrho(x_1) \) satisfying,

\[
\varrho = \text{dist}(x_1, \Sigma) \leq \text{dist}(x_1, \tilde{\Omega}_\delta).
\]

Since the boundary is almost flat,

\[
\text{dist}(x_1, x_0) < 1.
\]

According to Lemma 9, there exists some positive constant \( C \) independent of \( \delta \), \( |\nabla \hat{v}(x_1)| \leq C \). It yields that \( |\hat{v}(x_1)| \leq C \). Similarly as above, we extend \( \hat{v} \) to \( w \) by even reflection,

\[
w(y) = \begin{cases} 
\hat{v}(y) & \text{for } y \in B_1(x) \cap \tilde{\Omega}_\delta \\
\hat{v}(y - 2\nu(y)d(y)) & \text{for } y \in B_1(x) \setminus \tilde{\Omega}_\delta.
\end{cases}
\] (6.14)

\( w \) satisfies

\[
\partial_{y_j}(a_{ij}(y)\partial_{y_i} w(y)) = -\lambda_3 R,
\]

where \( |a_{ij} - \delta_{ij}| \) is small and \( a_{ij} = \delta_{ij} \) for \( y \in B_1(x) \cap \tilde{\Omega}_\delta \). We take finite series of ball \( B_\varrho(y_i) \) with \( 1 \leq i \leq N \) such that \( \text{dist}(x_0, y_1) = \text{dist}(y_1, y_2) = \cdots = \text{dist}(y_N, x_1) \leq \frac{1}{4} \).

Apply the Harnack inequality for \( \hat{v} \) in each ball,

\[
|\hat{v}(x_0)| \leq C|\hat{v}(y_1)| \leq C^2|\hat{v}(y_2)| \leq \cdots \leq C^{N+1}|\hat{v}(x_1)|.
\]

Note that the Harnack inequality we used here is for function \( \Delta u = -\lambda_3 R \leq 0 \) in a ball \( B_R(0) \). The proof is to take the classical the Harnack inequality on harmonic function \( u^* = u - \frac{\lambda_3 R}{2n}(R^2 - |x|^2) \). Then,

\[
|\hat{v}(x_0)| \leq C.
\]

Now take ball \( B_{\frac{1}{2}}(x) \) and then \( x_0 \in B_{\frac{1}{2}}(x) \). Apply the Harnack inequality again in \( B_{\frac{1}{2}}(x) \),

\[
\|w\|_{L^\infty(B_{\frac{1}{2}}(x))} \leq C.
\]

Therefore, apply the same estimate as above for \( w(x) \) in \( B_{\frac{1}{2}}(x) \)

\[
|\nabla v(x)| = |\nabla \hat{v}(x)| = |\nabla w(x)| \leq C\|w\|_{L^\infty(B_{\frac{1}{2}}(x))} + |\lambda_3 R| =: \check{C}
\]

where \( \check{C} \) only depends on \( M, n \) and \( r_0 \).

**Remark 2.** One referee suggested that with the estimates we obtained, it is possible to show that the free boundary has uniform \( C^{1,\alpha} \) regularity following Alt-Caffarelli theory. We decide to focus on the singular limit of the variational problem and will not pursue the free boundary regularity here.
7. Singular limit profile. Given the total mass $M > 0$, let \( \{ \varepsilon_k \}_{k=1}^{\infty} \subset (0, \frac{M}{M}) \) be a sequence such that \( \lim_{k \to \infty} \varepsilon_k = 0 \). Let \( u_{\varepsilon_k} \in \mathcal{H}_M \) be an energy minimizer of \( \mathcal{E}_{\varepsilon_k} \) in \( \mathcal{H}_M \). For simplicity, we will suppress the \( k \) subscript whenever there is no confusion.

Let \( x_\varepsilon \in \Omega \) be a point where \( u_\varepsilon \) attains its maximum and \( p_\varepsilon \in \partial \Omega \) be such that

\[
|p_\varepsilon - x_\varepsilon| = \min_{p \in \partial \Omega} |p - x_\varepsilon|.
\]

Passing to a subsequence if necessary, we can assume

\[
\lim_{k \to \infty} p_\varepsilon_k = p^* \in \partial \Omega
\]

and we denote \( \nu^* = \nu(p^*) \), the unit outer normal of \( \partial \Omega \) at \( p^* \). Let \( \Omega_\delta \) and \( v_\delta \) be defined in (1.8) and (1.9). Then \( v_\delta \) is a minimizer of \( E_\delta \) in \( \mathcal{H}(M, \Omega_\delta) \) and as \( k \to \infty \),

\[
\Omega_\delta \to \mathbb{R}^n_+ := \{ y \in \mathbb{R}^n \mid y \cdot \nu^* < 0 \}.
\]

For simplicity, after a rotation if necessary, we assume \( \nu^* = (0, \cdots, 0, -1) \) and hence \( \mathbb{R}^n_+ = \mathbb{R}^n_+ \).

**Proposition 2.** There exist constants \( C_1, C_2, C_3 > 0 \) such that for any \( k \in \mathbb{N} \),

\[
\max_{y \in \Omega_\delta} v_{\delta_k}(y) \geq C_1, \|v_{\delta_k}\|_{C^{0,\alpha}(\Omega_\delta)} \leq C_2 \text{ and } \|\nabla v_{\delta_k}\|_{L^\infty(\Omega_\delta)} \leq C_3.
\]

**Proof.** When \( k \) is sufficiently large, we have \( \delta_k \leq \frac{m}{2^m} \) and from Theorem 6, \( e_\delta(M) \leq 2e^*(M) \), hence Lemma 3 implies

\[
\max_{y \in \Omega_\delta} v_{\delta_k}(y) \geq \frac{M}{e_\delta(M)} \geq \frac{M}{2e^*(M)}.
\]

On the other hand, the uniform Hölder norm of \( v_{\delta_k} \) follows from Theorem 9 and uniform Lipschitz continuity follows from Theorem 10. Note that the constant is independent of \( \delta \).

Due to the uniform bound of Hölder continuity, passing to a subsequence if necessary, we can assume \( v_\delta \) converges locally uniformly to a limit \( v^* \) in \( \mathbb{R}^n_+ \). The main goal in this section is to show that \( v^* \) is the unique energy minimizer of \( E^* \) with \( \int_{\mathbb{R}^n_+} v^*(x) \, dx = M \).

**Lemma 10.** There exists a constant \( C > 0 \), such that for any \( k \in \mathbb{N} \),

\[
|p_{\varepsilon_k} - x_{\varepsilon_k}| \leq \frac{C}{\delta_k}.
\]

**Proof.** If such constant doesn’t exist, passing to a subsequence if necessary, we can assume

\[
\lim_{k \to \infty} \frac{|p_{\varepsilon_k} - x_{\varepsilon_k}|}{\delta_k} = \infty.
\]

For simplicity, we again suppress the \( k \) subscript. We define a blow up sequence along \( x_\varepsilon \) by

\[
\tilde{\Omega}_\delta = \left\{ \frac{x - x_\varepsilon}{\delta} : x \in \Omega \right\}.
\]

Correspondingly,

\[
\tilde{v}_\delta(y) = \delta^n u_\varepsilon(x) \quad \text{where} \quad y = \frac{x - x_\varepsilon}{\delta} \in \tilde{\Omega}_\delta.
\]
Then $\tilde{v}_\delta$ is a minimizer of $E_\delta$ in the space

$$H\left(M; \tilde{\Omega}_\delta\right) := \left\{ v \in H^1(\tilde{\Omega}_\delta) : v \geq 0 \text{ a.e. and } \int_{\tilde{\Omega}_\delta} v dy = M \right\}$$

where in the definition of energy $E_\delta$, $\Omega_\delta$ is replaced by $\tilde{\Omega}_\delta$. Since $\|v_\delta - x_n\| \to \infty$, we have $\tilde{\Omega}_\delta \to \mathbb{R}^n$ as $k \to \infty$. Noticing that for each $k$, $\tilde{v}_\delta$ is a translation of $v_\delta$, the uniform bound of H"older norms of $v_\delta$ implies that, passing to a subsequence if necessary, $\tilde{v}_\delta \to v^*$ locally uniformly in $\mathbb{R}^n$ as $k \to \infty$, which implies

$$M^* = \int_{\mathbb{R}^n} v^* dy \leq M.$$

Since

$$\tilde{v}_\delta(0) = \max_{y \in \Omega_{\sigma K}} v_{\delta_k}(y) \geq C_1,$$

the uniform H"older continuity of $\tilde{v}_\delta$ implies

$$M^* = \int_{\mathbb{R}^n} v^* dy > 0.$$

For any $\sigma > 0$ sufficiently small, we can choose $R_0 > 0$, such that

$$\int_{B_{R_0}(0)} v^* dx \geq M^* - \sigma.$$

Let $N = \left[\frac{1}{\delta}\right] + 1$. For small $\delta > 0$, we have $B_{R_0 + N} \subset \tilde{\Omega}_\delta$. Since

$$\int_{B_{R_0 + N} \setminus B_{R_0}} \left\{ \frac{1}{2} |\nabla \tilde{v}_\delta|^2 + \chi_{\{\tilde{v}_\delta > 0\}} \right\} + \int_{B_{R_0 + N} \setminus B_{R_0}} \left\{ |\tilde{v}_\delta|^2 + |\tilde{\nabla} \tilde{v}_\delta| \right\} \leq e_\delta(M) + M \max \tilde{v}_\delta + M \leq K$$

for some $K > 0$ which is independent of small $\epsilon$, we can choose $1 \leq l \leq N$ so that

$$\int_{B_{R_0 + l} \setminus B_{R_0 + l-1}} \left\{ \frac{1}{2} |\nabla \tilde{v}_\delta|^2 + \chi_{\{\tilde{v}_\delta > 0\}} + |\tilde{v}_\delta|^2 + |\tilde{\nabla} \tilde{v}_\delta| \right\} \leq \frac{K}{N} \leq \sigma K.$$

Now let $\eta \in C^\infty(\mathbb{R}^n)$ be a cutoff function, such that

$$\eta(x) = 1 \text{ if } x \in B_{R_0 + l}(0); \eta(x) = 0 \text{ if } x \notin B_{R_0 + l-1}(0);$$

$$\eta \in [0,1] \text{ and } |\nabla \eta| \leq 2 \text{ for any } x \in \mathbb{R}^n.$$

We have

$$\tilde{v}_\delta = \eta \tilde{v}_\delta + (1 - \eta) \tilde{v}_\delta \equiv \tilde{v}_\delta^1 + \tilde{v}_\delta^2.$$

Direct calculation yields

$$\frac{1}{2} \int_{\tilde{\Omega}_\delta} |\nabla \tilde{v}_\delta|^2 + \frac{1}{2} \int_{\tilde{\Omega}_\delta} |\nabla \tilde{v}_\delta^2|^2 - \frac{1}{2} \int_{\tilde{\Omega}_\delta} |\nabla \tilde{v}_\delta^1|^2$$

$$= \frac{1}{2} \int_{\tilde{\Omega}_\delta} \left\{ \left( \eta^2 + (1 - \eta)^2 - 1 \right) |\nabla \tilde{v}_\delta|^2 + 2 |\nabla \eta|^2 |\tilde{v}_\delta|^2 + (-2 + 4\eta) \tilde{v}_\delta \nabla \tilde{v}_\delta \nabla \eta \right\}$$

$$\leq \frac{1}{2} \int_{B_{R_0 + l}(0) \setminus B_{R_0 + l-1}(0)} \left\{ \left[ 8 |\tilde{v}_\delta|^2 + 4 |\tilde{v}_\delta \nabla \tilde{v}_\delta| \right] \right\}$$

$$\leq \int_{B_{R_0 + l}(0) \setminus B_{R_0 + l-1}(0)} \left( |\nabla \tilde{v}_\delta|^2 + 5 |\tilde{v}_\delta|^2 \right)$$

$$\leq 7\sigma K,$$
and
\[
\int_{\Omega_{\delta}} \chi\{\tilde{v}_1^+ > 0\} + \chi\{\tilde{v}_2^+ > 0\} \, dy - \int_{\Omega_{\delta}} \chi\{\tilde{v}_3 > 0\} \\
\leq \int_{B_{R_0+1}(0)} \chi\{\tilde{v}_3 > 0\} \, dy < \sigma K.
\]
Hence, we conclude
\[
e_{\delta} (m) \geq E_{\delta} [\tilde{v}_1^+] + E_{\delta} [\tilde{v}_2^+] - 8\sigma K.
\]
Since
\[
\lim_{k \to \infty} \int_{\Omega_{\delta}} \tilde{v}_1^k = \lim_{k \to \infty} \int_{\mathbb{R}^n} n u^* \in [M^* - \sigma, M^*],
\]
when \( k \) sufficiently large, we have
\[
\int_{\Omega_{\delta}} \tilde{v}_1^k \in [M^* - 2\sigma, M^* + \sigma] \text{ and } \int_{\Omega_{\delta}} \tilde{v}_2^k \in [M - M^* - \sigma, M - M^* + 2\sigma].
\]
Letting \( k \to \infty \), we have
\[
e^* (M) \geq 2e^* \left( \frac{M^* - 2\sigma}{2} \right) + e^* (M - M^* - \sigma) - 8\sigma K
\]
where
\[
\liminf_{k \to \infty} E_{\delta_k} [\tilde{v}_1^k] \geq 2e^* \left( \frac{M^* - 2\sigma}{2} \right)
\]
follows from the fact that \( \tilde{v}_1^k \) is compactly supported. Letting \( \sigma \to 0 \), we have
\[
e^* (M) \geq 2e^* \left( \frac{M^*}{2} \right) + e^* (M - M^*).
\]
From the formula (3.6), \( e^* (M) \) is a strictly concave function and
\[
2e^* \left( \frac{M^*}{2} \right) + e^* (M - M^*) \leq e^* (M^*) + e^* (M - M^*) \leq e^* (M)
\]
with the first identity holds only when \( M^* = 0 \) which yields a contradiction if \( M^* \in (0, M] \).

Next, we show there is no loss of mass in the limiting process.

**Lemma 11.** \( \int_{\mathbb{R}^n_+} v^* (x) \, dx = M. \)

**Proof.** Let
\[
M^* = \int_{\mathbb{R}^n_+} v^* (y) \, dy.
\]
Since \( \frac{|x - x_0|}{\delta} \) is uniformly bounded, the uniform Hölder bound for \( v_\delta \) and uniformly positive lower bounds for
\[
v_\delta \left( \frac{x - p_k}{\delta} \right) = \max_{y \in \Omega_{\delta}} v_\delta (y)
\]
implies
\[
M^* > 0.
\]
Similar argument as Lemma 10 will imply
\[
e^* (M) \geq e^* (M^*) + e^* (M - M^*)
\]
Now \( M^* > 0 \) implies \( M^* = M. \)
Now we are ready to prove Theorem 2.

**Proof of Theorem 2.** Up to a subsequence, we also assume \( v_{\delta k} \) converges to \( v^* \) weakly in \( H^1_{loc}(\mathbb{R}^n_+) \) as \( k \to \infty \) and hence, the lower semi-continuity of norms implies

\[
\frac{1}{2} \int_{\mathbb{R}^n_+} |\nabla v^*|^2 \, dy \leq \liminf_{k \to \infty} \frac{1}{2} \int_{\Omega_k} |\nabla v_{\delta k}|^2 \, dy. \tag{7.3}
\]

On the other hand, let

\[
|\{ v^* > 0 \}| = \mu^* > 0.
\]

For each \( \sigma > 0 \), there exists \( N > 0 \) such that

\[
\left| \left\{ v^* > \frac{1}{N} \right\} \cap B_N(0) \right| \geq \mu^* - \sigma.
\]

Now since \( v_{\delta} \) converges to \( v^* \) uniformly on \( \left| \left\{ v^* > \frac{1}{N} \right\} \cap B_N(0) \right| \), we conclude

\[
\lim_{k \to \infty} \int_{\left\{ v^* > \frac{1}{N} \right\} \cap B_N(0) \cap \Omega} \chi_{\{v^*_\delta > 0\}} = \left| \left\{ v^* > \frac{1}{N} \right\} \cap B_N(0) \right| \geq \mu^* - \sigma.
\]

Since \( \sigma \) is arbitrary,

\[
\liminf_{k \to \infty} \int_{\Omega} \chi_{\{v^*_\delta > 0\}} \geq \mu^*. \tag{7.4}
\]

Combining (7.3) and (7.4), we have

\[
E^*[v^*] \leq \lim_{k \to \infty} E_{\delta_k}(M) = E^*(M).
\]

On the other hand, since \( \int_{\mathbb{R}^n_+} v^*(x) \, dx = M \), we have \( E^*[v^*] \geq E^*(M) \) and hence \( E^*[v^*] = E^*(M) \). Our choice of \( p_{\epsilon} \) guarantees that \( \max v^* \) is assumed on the vertical line passing through the origin. So the theorem follows from the uniqueness up to a translation of the global energy minimizer for \( E^* \).

The convergence of the blow up sequence \( v_{\delta} \) implies the convergence of \( u_{\epsilon} \).

**Proof of Theorem 1.** Since \( \{ u_{\epsilon_k} \}_{k=1}^\infty \) is a sequence of positive function with total mass \( m \), there exists a measure \( \mu \) on \( \Omega \) such that passing to a subsequence if necessary

\[
u_{\epsilon_k} \rightharpoonup \mu \quad \text{in the weak star topology as } k \to \infty.
\]

Passing to a subsequence if necessary, we also have the blow up sequence \( u_{\delta} \to v^* \) locally uniformly as \( k \to \infty \) and \( \int_\Omega v^* = M \). Hence

\[
\lim_{k \to \infty} \int_{B_{R^*}(0) \cap \Omega} v_{\delta}(y) \, dy = M
\]

which implies

\[
\lim_{k \to \infty} \int_{B_{R^*}(p_{\epsilon}) \cap \Omega} u_{\epsilon}(x) \, dx = M.
\]

Since \( \int_\Omega u_{\epsilon}(x) \, dx = M \) and \( p_{\epsilon} \to p^* \), we conclude \( u_{\epsilon} \rightharpoonup \mu = M \delta_{p^*} \) as \( k \to \infty \).

The above theorem implies when \( \epsilon \) approaches zero, the energy minimizer converges to a Dirac measure concentrated on the boundary. We are going to show next that the Dirac measure should be located near the point with maximal mean curvature.
8. Linearization. To understand the location of the boundary spike, we consider the free boundary problem (4.6) associated to the scaled energy minimizing problem:

\[
\begin{aligned}
\Delta v &= -\lambda &\quad \text{in } &D = \{ y \in \Omega_\delta : v > 0 \} \\
v &= 0 \text{ and } \partial_v v &= -\sqrt{2} &\quad \text{on } &\partial D \cap \Omega_\delta, \\
\partial_v v &= 0 &\quad \text{on } &\partial D \cap \partial \Omega_\delta, \\
\int_D v(y) \, dy &= M.
\end{aligned}
\]  

(8.1)

Since true solution should have spikes near specific boundary points, here for a fixed point \( p \in \partial \Omega \), we seek a pair \( (v, D) \) such that the “center” of \( D \) is the origin and that \( (v, D) \) only approximately solves the free boundary problem (8.1), e.g., having error \( O(\delta^2) \). Then we compare the energy when \( p \) is moving around the boundary.

By shifting and rotation, we assume that \( p = 0 \) and the unit normal of \( \partial \Omega \) at \( p \) is \( (0', -1) \). The boundary near \( p \) is represented in local coordinates as

\[ x_n = \psi(x_1, \cdots, x_{n-1}), \quad \psi(0') = 0, \quad \psi_{x_i}(0') = 0, \quad \psi_{x_ix_j}(0') = \kappa_i \delta^i. \]

We call \( \kappa_i \) the principal curvature of \( \partial \Omega \) at \( p \) and denote by \( \kappa = \sum_{i=1}^{n-1} \kappa_i / (n-1) \) the mean curvature of \( \partial \Omega \) at \( p \). Locally the boundary of \( \partial \Omega_\delta \) near \( q := p / \delta \) is expressed as

\[ \delta y_n = \psi_i' (y') = \frac{\delta^2}{2} \sum_{i=1}^{n-1} \kappa_i y_i^2 + O \left( \delta^3 ; |y'|^3 \right). \]

In general, (8.1) does not have a solution that has mass concentrated near \( q \). To overcome this difficulty, we add an extra constraint in the class of minimization to ensure that the mass is near \( q \). Hence we consider the minimization of \( E^*_\delta \) in the space

\[ H(q, M) = \left\{ v \in H^1(\Omega_\delta) : v \geq 0 \text{ a.e. in } \Omega_\delta, \quad \int_{\Omega_\delta} v = M \text{ and } \int_{\partial \Omega_\delta} (y - q)v \parallel N(q) \right\} \]

where \( N(q) \) is the normal vector of \( \Omega_\delta \) at \( q \) and the symbol \( \parallel \) represents parallel relation between two vectors. In the current notation, the second set of constraints mean

\[ \int_{\partial \Omega_\delta} y_i v \, dy = 0 \quad \forall i = 1, \cdots, n - 1. \]  

(8.2)

The corresponding free boundary problem can be written as

\[
\begin{aligned}
\Delta v &= -\lambda - \sum_{i=1}^{n-1} \lambda_i y_i &\quad \text{in } &D = \{ y \in \Omega_\delta : v > 0 \} \\
v &= 0 \text{ and } \partial_v v &= -\sqrt{2} &\quad \text{on } &\partial D \cap \Omega_\delta, \\
\partial_v v &= 0 &\quad \text{on } &\partial D \cap \partial \Omega_\delta, \\
\int_D v(y) \, dy &= M.
\end{aligned}
\]  

(8.3)

where \( \lambda, \lambda_1, \cdots, \lambda_{n-1} \) are Lagrange multipliers.

We search a solution of (8.3) that can be expanded in the \( \delta \)-power series as follows

\[
\begin{aligned}
D &= \left\{ y \in \mathbb{R}^n : y_n > \psi(\delta y') / \delta, |y| < R + \delta R_1 \left( \frac{y}{|y|} \right) + O(\delta^2) \right\}, \\
v &= R^2 - |y|^2 + \delta v_1(y) + O(\delta^2) \quad \forall y \in D, \quad v(y) = 0 \quad \forall y \in \Omega_\delta \setminus D, \\
\lambda_i &= O(\delta^3) \quad \forall i = 1, \cdots, n - 1
\end{aligned}
\]

where \( R \) and \( \lambda \) are constants depending on \( \delta \), \( R_1 \) and \( v_1 \) are unknown functions that depend on \( \delta \) only through the constants \( \lambda \) and \( R \).

We derive the equations for \( (R, \lambda, v_1, R_1) \) as follows.
(1) The free boundary condition \( v = 0 \) on the free boundary implies

\[
0 = v(y) = \frac{\lambda}{2n} \left[ R^2 - R + \delta R_{1} \left( \frac{y}{|y|} \right) \right] + \delta v_1(y) + O(\delta^2)
\]

\[
= \delta \left( -\frac{\lambda RR}{n} + v_1(y) \right) + O(\delta^2)
\]

which is equivalent to

\[
v_1(y) = \frac{\lambda R}{n} R_1 \left( \frac{y}{|y|} \right) \quad \forall y \in \Gamma := \partial B_R \cap \mathbb{R}^{n+}.
\]

(2) The normal of the free boundary is

\[
N = (N^1, \cdots, N^n), \quad N^i = \frac{y^i}{|y|} - \delta \sum_{j=1}^{n} \frac{\partial R_1}{\partial y_j} \left( \delta^{ji} \frac{y^j}{|y|^2} \right) + O(\delta^2),
\]

\[
|N| = \sqrt{\sum_{i=1}^{n} \left( \frac{y^i}{|y|} - \delta \sum_{j=1}^{n} \frac{\partial R_1}{\partial y_j} \left( \frac{\delta^{ji}}{|y|} - \frac{y^j y^i}{|y|^3} \right) \right)^2} = \sqrt{1 + O(\delta^2)}.
\]

The free boundary condition \( \partial_n v = -\sqrt{2} \) becomes

\[
-\sqrt{2} F_*= n \cdot \nabla v
\]

\[
= \frac{1}{\sqrt{1 + O(\delta^2)}} \sum_{i=1}^{n} \left[ \frac{y^i}{|y|} - \delta \sum_{j=1}^{n} \frac{\partial R_1}{\partial y_j} \left( \frac{\delta^{ji}}{|y|} - \frac{y^j y^i}{|y|^3} \right) \right] \left( -\frac{\lambda y^i}{n} + \delta \partial_i v_1 \right)
\]

\[
= \sum_{i=1}^{n} \left[ \frac{y^i}{|y|} - \delta \sum_{j=1}^{n} \frac{\partial R_1}{\partial y_j} \left( \frac{\delta^{ji}}{|y|} - \frac{y^j y^i}{|y|^3} \right) \right] \left( -\frac{\lambda y^i}{n} + \delta \partial_i v_1 \right) (1 + O(\delta^2))
\]

\[
= \left( -\frac{\lambda (R + \delta R_{1})}{n} + \delta \frac{y}{|y|} \cdot \nabla v_1 \right) + O(\delta^2)
\]

which can be achieved by setting

\[
\lambda = \frac{\sqrt{2n}}{R},
\]

and

\[
\partial_n v_1 = \frac{\lambda R}{n} = \frac{v_1}{R} \quad \text{on} \quad \Gamma.
\]

(3) Finally, using

\[
y_n = \frac{\delta}{2} \sum_{i=1}^{n-1} \kappa_i y_i^2 + O \left( \delta^2 |y'|^3 \right),
\]

we have

\[
N = (\delta \kappa_1 y_1, \ldots, \delta \kappa_{n-1} y_{n-1}, -1) + O \left( \delta^2 \right),
\]

and the boundary condition \( \partial_n v = 0 \) on \( \partial \Omega_5 \) can be written as

\[
0 = N \cdot \nabla v
\]

\[
= \delta \sum_{i=1}^{n-1} \kappa_i y_i \left( -\frac{\lambda y_i}{n} + \delta \partial_i v_1 \right) - \left( -\frac{\lambda y_n}{n} + \delta \partial_n v_1 \right) + O(\delta^2)
\]
\[
\delta \sum_{i=1}^{n-1} \kappa_i y_i \left( -\frac{\lambda y_i}{n} + \delta \partial_{y_i} v_1 \right) - \left( -\frac{\lambda \delta}{2n} \sum_{i=1}^{n-1} \kappa_i y_i^2 + \delta \partial_{y_i} v_1 \right) + O(\delta^2)
\]
\[
\delta \left( -\frac{\lambda}{2n} \sum_{i=1}^{n-1} \kappa_i y_i^2 - \partial_{y_i} v_1 \right) + O(\delta^2).
\]
This can be achieved only by setting
\[
\frac{\partial v_1}{\partial y_n}(y',0) = -\frac{\lambda}{2n} \sum_{i=1}^{n-1} \kappa_i y_i^2 \quad \forall \ y' \in B_R' := \{ y' \in \mathbb{R}^{n-1} | |y'| < R \}.
\]
Thus we see that \((v_1, R_1)\) needs to be a solution of the linearized problem given by
\[
\begin{align*}
-\Delta v_1 &= 0 \quad \text{in } B_R \cap \mathbb{R}^n =: B_R^+, \\
v_1 &= R \partial_{\nu} v_1 \quad \text{on } \partial B_R \cap \mathbb{R}^n =: \Gamma_R, \\
\partial_{y_n} v_1 &= -\frac{\lambda}{2n} \sum_{i=1}^{n-1} \kappa_i y_i^2 \quad \text{on } B_R^+ \times \{0\}, \\
R_1(y/|y|) &= n \partial_{y_n} v_1(y)/\lambda \quad \forall \ y \in \Gamma_R. 
\end{align*}
\]
It is sufficient to consider only the equation for \(v_1\). Note that
\[
\hat{D}_{y_i} (R^2 - |y|^2)dy = O(\delta^2) \quad \forall \ i = 1, \ldots, n - 1.
\]
We derive from (8.1) that
\[
\int_{B_R^+} y_i v_1(y)dy = 0 \quad \forall \ i = 1, \ldots, n - 1. \tag{8.5}
\]

**Theorem 11.** The mixed boundary condition problem (8.4) with the constraint (8.5) admits a unique solution.

First we establish the lemma for Robin boundary condition problem on a ball.

**Lemma 12.** Assume \(f \in L^2(S_R)\), the Robin boundary condition problem on a ball with radius \(R\) given by
\[
\begin{align*}
\Delta u &= 0 \quad \text{in } B_R, \\
\frac{n}{R} - \partial_{\nu} u &= f \quad \text{on } \partial B_R =: S_R. 
\end{align*}
\]
admits a solution \(u \in H^1(B_R)\) if and only if \(f\) satisfies the compatibility condition
\[
\int_{\partial B_R} y_i f(y)d\mathcal{H}^{n-1} = 0, \quad \forall \ i = 1, \ldots, n.
\]
The solution is unique if we add the constraints
\[
\int_{B_R} y_i u dy = 0, \quad \forall \ i = 1, \ldots, n.
\]

**Proof.** Firstly, let \(u \in H^1(B_R)\) be a solution to (8.6). \(\forall \ i = 1, \ldots, n, \)
\[
0 = -\int_{B_R} \nabla y_i \nabla u dy + \int_{S_R} y_i \partial_{\nu} u d\mathcal{H}^{n-1}
\]
\[
= -\int_{S_R} \partial_{\nu} y_i u d\mathcal{H}^{n-1} + \int_{B_R} \Delta y_i u dy + \int_{S_R} y_i \partial_{\nu} u d\mathcal{H}^{n-1}
\]
\[
= -\int_{S_R} y_i \left( \frac{u}{R} - \partial_{\nu} u \right) d\mathcal{H}^{n-1}.
\]
So we have
\[ \int_{S_R} y_i f(y) dH^{n-1} = 0, \quad \forall i = 1, \cdots, n. \]
Secondly, suppose \( f \in L^2(S_R) \) satisfying the compatibility condition. Let \( H_m(\mathbb{R}^n) \) denote the subspace of all the homogeneous harmonic polynomials on \( \mathbb{R}^n \) of degree \( m \) and \( H_m(S_R) \) represent the subspace of all the homogeneous harmonic polynomials in \( H_m(\mathbb{R}^n) \) with restriction to \( S_R \) of degree \( m \). Since \( L^2(S_R) = \bigoplus_{m=0}^{\infty} H_m(S_R) \) (Theorem 5.12 and Theorem 5.29 in [5]),
\[ f = \sum_{m=0}^{\infty} p_m(y) \]
where \( p_m(y) \in H_m(S_R) \) satisfying
\[ \int_{S_R} p_m(y) p_k(y) dH^{n-1} = 0, \quad \forall m \neq k. \]
Using the homogeneity, we see,
\[ \int_{S_r} p_m(y) p_k(y) dH^{n-1} = \int_{S_R} \left( \frac{R}{r} \right)^{m+k+n-1} p_m(y) p_k(y) dH^{n-1} = 0, \quad \forall r > 0. \]
Furthermore,
\[ \int_{B_R} p_m(y) p_k(y) dy = \int_0^R \int_{S_r} p_m(y) p_k(y) dH^{n-1} dr = 0 \]
and each component in \( \nabla p_m(y) \) belongs to \( H_{m-1}(\mathbb{R}^n) \) implies
\[ \int_{B_R} \nabla p_m(y) \cdot \nabla p_k(y) dy = 0. \]
Suppose solution \( u \) has expansion
\[ u = \sum_{m=0}^{\infty} d_m p_m(y) \]
where \( d_m \) is to be determined. Formal calculation gives
\[ \frac{u}{R} - \partial_r u = \sum_{m=0}^{\infty} \frac{d_m p_m(y)}{R} - \sum_{m=0}^{\infty} \frac{m d_m p_m(y)}{R} = \sum_{m=0}^{\infty} \frac{(1-m)d_m p_m(y)}{R}. \]
According to the Robin boundary condition, we define
\[ d_m = \frac{R}{1-m} \quad \text{and} \quad u_M = \sum_{m \geq 0, m \neq 1} \frac{R}{1-m} p_m(y). \]
Then \( u_M \) is harmonic in \( B_R \) and for \( N > M > 1 \),
\[ \| u_N - u_M \|_{L^2(B_R)} = \left\| \sum_{m>M}^{N} \frac{R}{1-m} p_m(y) \right\|_{L^2(B_R)} \]
\[ = \left( \int_0^R \int_{S_r} \left( \sum_{m>M}^{N} \frac{R}{1-m} p_m(y) \right)^2 dH^{n-1} dr \right)^{1/2} \]
\[
\begin{align*}
= \left( \int_0^R \int_{S_R} \sum_{m > M} \frac{R^2}{(1-m)^2} p_m(y)^2 \frac{R^{2m+n-1}}{R^{2m+n-1}} \, dH^{n-1} \right)^{1/2} \\
= \left( \int_{S_R} \sum_{m > M} \frac{R^2}{(m-1)^2} p_m(y)^2 \frac{R^{2m+n}}{(2m+n)R^{2m+n-1}} \, dH^{n-1} \right)^{1/2} \\
\leq \frac{R^2}{(M-1)^2} \left\| \sum_{m > M} p_m(y) \right\|_{L^2(S_R)}.
\end{align*}
\]

Moreover,
\[
\left\| \nabla u_N - \nabla u_M \right\|_{L^2(B_R)} = \left\| \sum_{m > M} \frac{R}{1-m} \nabla p_m(y) \right\|_{L^2(B_R)}
= \sum_{m > M} \frac{R}{m-1} \left\| \nabla p_m(y) \right\|_{L^2(B_R)}
= \sum_{m > M} \frac{R}{m-1} \left( \int_{S_R} p_m(y) \cdot \frac{\partial p_m(y)}{\partial n} \, dH^{n-1} \right)^{1/2}
= \sum_{m > M} \frac{R}{m-1} \left( \frac{m}{R} \right)^{1/2} \left\| p_m(y) \right\|_{L^2(S_R)}
\leq \frac{\sqrt{R}}{\sqrt{M} - 1} \left\| \sum_{m > M} p_m(y) \right\|_{L^2(S_R)}.
\]

Therefore, \( u_M \) is a Cauchy sequence in \( L^2(B_R) \) and \( \nabla u_M \) is a Cauchy sequence in \( (L^2(B_R))^n \). Then let
\[
u = \sum_{m \neq 1} \frac{R}{1-m} p_m(y)
\]
we can obtain that as the limit of \( u_M \) in \( H^1(B_R) \), \( u \in H^1(B_R) \) is harmonic in \( B_R \).

Regarding the uniqueness, we consider solution \( u \in H^1(B_R) \) to the homogeneous system,
\[
\begin{align*}
\Delta u &= 0 \quad \text{in } B_R, \\
\frac{u}{R} - \partial_n u &= 0 \quad \text{on } S_R.
\end{align*}
\]

Using Spherical Harmonic Functions (for example corollary 5.34 of [5]),
\[
u = \sum_{m=0}^{\infty} p_m(y)
\]
where \( p_m(y) \in H_m(\mathbb{R}^n) \). Applying the robin boundary condition, we obtain,
\[
0 = \frac{u}{R} - \partial_n u = \sum_{m=0}^{\infty} \frac{p_m(y)}{R} - \sum_{m=0}^{\infty} \frac{mp_m(y)}{R} = \sum_{m=0}^{\infty} \frac{(1-m)p_m(y)}{R}.
\]

Due to the orthogonality of \( H_m(\mathbb{R}^n) \) on \( S_R \) in sense of \( L^2 \) inner product, then,
\[
\begin{align*}
p_m(y) &= 0, \quad \forall \ m \neq 1.
\end{align*}
\]
We have,

$$u = \sum_{i=1}^{n} c_i y_i$$

where $c_i$ are arbitrary constants.

Therefore, the constraints $\int_{B_R^c} y_i u dy = 0$ implies $c_i = 0$ which is the uniqueness of the solution. \(\Box\)

Now we are ready to prove the existence of the solution to the non-homogeneous problem on half ball.

**Theorem 12.** Given $f \in L^2(B^+_R)$, $g \in L^2(\Gamma_R)$ and $h \in L^2(B'_R \times \{0\})$, the mixed boundary condition problem given by

$$\begin{cases}
\Delta w = f & \text{in } B^+_R, \\
\frac{w}{R} - \partial_n w = g & \text{on } \Gamma_R, \\
\partial_{y_n} w = h & \text{on } B'_R \times \{0\}
\end{cases} \tag{8.7}$$

admits a solution $w \in H^1(B^+_R)$, if and only if $(f,g,h)$ satisfies the compatibility conditions

$$\int_{B^+_R} y_i f(y) dy + \int_{\Gamma_R} y_i g(y) d\mathcal{H}^{n-1} + \int_{B'_R} y_i h(y') dy' = 0 \quad \forall i = 1, \ldots, n-1. \tag{8.8}$$

If there is a solution $w_{sp}$, then the general solution is given by

$$w(y) = w_{sp}(y) + \sum_{i=1}^{n-1} c_i y_i$$

where $c_1, \ldots, c_{n-1}$ are arbitrary constants. The solution is unique if we require

$$\int_{B'_R} y_i w(y) dy = 0 \quad \forall i = 1, \ldots, n-1.$$

**Proof.** Let $w \in H^1(B^+_R)$ be a solution to (8.7). For $i = 1, \ldots, n-1$,

$$\int_{B^+_R} y_i \Delta w dy = \int_{\Gamma_R} y_i \partial_n w d\mathcal{H}^{n-1} - \int_{B'_R} y_i \partial_{y_n} w dy' - \int_{\Gamma_R} w(y) \partial_n y_i d\mathcal{H}^{n-1}$$

$$- \int_{B'_R} w(y',0) \partial_{y_n} y_i dy' + \int_{B^+_R} w \Delta y_i dy$$

$$= \int_{\Gamma_R} y_i \partial_n w d\mathcal{H}^{n-1} - \int_{B'_R} y_i \partial_{y_n} w dy' - \int_{\Gamma_R} \frac{w}{R} y_i d\mathcal{H}^{n-1}$$

$$= \int_{\Gamma_R} y_i (\partial_n w - \frac{w}{R}) d\mathcal{H}^{n-1} - \int_{B'_R} y_i \partial_{y_n} w dy'$$

$$= - \int_{\Gamma_R} y_i g(y) d\mathcal{H}^{n-1} - \int_{B'_R} y_i h(y') dy'.$$

That is the compatibility condition (8.8).

We first consider the homogeneous system

$$\begin{cases}
\Delta w = 0 & \text{in } B^+_R, \\
\frac{w}{R} - \partial_n w = 0 & \text{on } \Gamma_R, \\
\partial_{y_n} w = 0 & \text{on } B'_R \times \{0\}.
\end{cases}$$

Due to the Neumann boundary condition on $B'_R \times \{0\}$, even reflection gives

$$\begin{cases}
\Delta w = 0 & \text{in } B_R, \\
\frac{w}{R} - \partial_n w = 0 & \text{on } \partial B_R.
\end{cases}$$
Applying Lemma 12 and the fact that $w$ is even in $y_n$, we have the general solutions for the homogeneous system are given by

$$w = \sum_{i=1}^{n-1} c_i y_i.$$ 

Next, for the non-homogeneous problem, we choose functions $F \in H^2(B^+_R)$ and $H \in H^1(B^+_R)$ such that

$$\begin{cases} 
\Delta F = f & \text{in } B^+_R, \\
F = 0 & \text{on } \partial B^+_R
\end{cases}$$

and

$$\begin{cases} 
\Delta H = 0 & \text{in } B^+_R, \\
\partial_{y_n} H = h - \partial_{y_n} F & \text{on } B'_R \times \{0\}.
\end{cases}$$

Set $u = w - F - H$,

$$\begin{cases} 
\Delta u = 0 & \text{in } B^+_R, \\
\frac{\pi}{n} - \partial_n u = G = g + \partial_n F - \frac{H}{R} + \partial_n H & \text{on } \Gamma_R, \\
\partial_{y_n} u = h - \partial_{y_n} F - (h - \partial_{y_n} F) = 0 & \text{on } B'_R \times \{0\}.
\end{cases}$$

Here $G \in L^2(\Gamma_R)$. Similarly, applying even reflection for $u$ and making use of Lemma 12, solution $u \in H^1(\Gamma_R)$ exists if and only if for any $i = 1, \ldots, n-1$,

$$\int_{\partial B_R} y_i G dy = 0.$$ 

Hence, for any $i = 1, \ldots, n-1$,

$$0 = \int_{\Gamma_R} y_i G dy$$

$$= \int_{\Gamma_R} y_i (g + \partial_n F - \frac{H}{R} + \partial_n H) d\mathcal{H}^{n-1}$$

$$= \int_{\Gamma_R} (y_i g + y_i \partial_n F - y_i \frac{H}{R} + y_i \partial_n H) d\mathcal{H}^{n-1}$$

$$= \int_{\Gamma_R} y_i g d\mathcal{H}^{n-1} + \int_{B^+_R} y_i \Delta F dy + \int_{B'_R} y_i \partial_{y_n} F(y', 0) dy' + \int_{\Gamma_R} -y_i H \frac{H}{R} d\mathcal{H}^{n-1}$$

$$+ \int_{B'_R} y_i (h - \partial_{y_n} F) dy' + \int_{\Gamma_R} \frac{\partial y_i}{\partial n} H d\mathcal{H}^{n-1} + \int_{B'_R} \frac{\partial y_i}{\partial n} H(y', 0) dy'$$

$$= \int_{\Gamma_R} y_i g d\mathcal{H}^{n-1} + \int_{B^+_R} y_i f dy + \int_{B'_R} y_i \partial_{y_n} F(y', 0) dy' + \int_{\Gamma_R} -y_i \frac{H}{R} d\mathcal{H}^{n-1}$$

$$+ \int_{B'_R} y_i (h - \partial_{y_n} F) dy' + \int_{\Gamma_R} \frac{y_i}{R} H d\mathcal{H}^{n-1} + \int_{B'_R} \frac{\partial y_i}{\partial n} H(y', 0) dy'$$

$$= \int_{\Gamma_R} y_i g d\mathcal{H}^{n-1} + \int_{B^+_R} y_i f dy + \int_{B'_R} y_i h dy'.$$

In order to obtain the explicit solution, we can compute the basis for $H_m(\mathbb{R}^n)$ using zonal harmonics (See Chapter 5 in [5]). Then given compatibility condition, the special solution for $u$ can be calculated using inner product. Therefore it gives the special solution $w_{sp} = u + F + H$. The general solution is given by

$$w(y) = w_{sp}(y) + \sum_{i=1}^{n-1} c_i y_i.$$
It is easy to see problem (8.4) together with the constraint (8.5) is a special case of (8.7). Then the proof of Theorem 11 naturally follows from theorem 12. Hence, $v_1$ is uniquely solvable.

9. **Location of spike.** Though it is hard to find explicit solution of (8.4) and (8.5), we can still proceed to find quantities of our interest. In this section, we will focus on the energy expansion which helps to locate the position of the spike. Applying the asymptotic analysis, we consider the effect of the mass constraint. Later as the theorem 3 stated, the energy of the Quasi-stationary solution $(v, D)$ has the asymptotic expansion

$$E_δ[v] \equiv \int_D \left\{ \frac{1}{2} |\nabla v|^2 + 1 \right\} = E^* [v^*] - c(n) M \kappa \delta + O(\delta^2)$$

where

$$c(n) = \frac{(n-1)(n+2)(n+7)}{\sqrt{2}(n+1)(n+3)} \omega_{n-1}.$$

Hence, the spike should locate on the boundary point with the maximum curvature. We begin with computing

$$\int_{B_R^+} v_1(y) dy = \int_{B_R^+} v_1(y) \Delta (|y|^2 + R^2) dy$$

$$= \frac{1}{n} \int_{\partial B_R^+ \cap \mathbb{R}^n} \{ R v_1 - R^2 \partial_n v_1 \} + \frac{1}{2n} \int_{B_R^+} \left\{ (|y'|^2 + R^2) \partial_{y'} v_1 \right\}$$

$$= \frac{1}{2n} \int_{B_R^+} \left( |y'|^2 + R^2 \right) \left[ -\frac{\lambda}{2n} \sum_{i=1}^{n-1} \kappa_i y_i^2 \right] dy'$$

$$= -\frac{\lambda}{4n^2} \sum_{i=1}^{n-1} \kappa_i \int_{B_R^+} (|y'|^2 + R^2) y_i^2 dy'.$$

Note that, by symmetry,

$$\int_{B_R^+} (|y'|^2 + R^2) y_i^2 dy' = \int_{B_R^+} (|y'|^2 + R^2) \frac{|y'|^2}{n-1} dy'$$

$$= \int_0^R (r^2 + R^2) \frac{r^2}{n-1} (n-1) \omega_{n-1} r^{n-2} dr$$

$$= \frac{2(n+2)}{(n+1)(n+3)} \omega_{n-1} R^{n+3}$$

where $\omega_{n-1}$ is the volume of the unit ball in $\mathbb{R}^{n-1}$. Recalling the definition of mean curvature, we then obtain

$$\int_{B_R^+} v_1(y) dy = -\frac{\lambda}{2n^2} \frac{(n-1)(n+2) \omega_{n-1} R^{n+3}}{(n+1)(n+3) \omega_{n-1} R^{n+3}}$$

$$= -\frac{\sqrt{2}(n-1)(n+2) R^{n+2}}{2n(n+1)(n+3)} \omega_{n-1} \kappa.$$

Similarly, we can estimate

$$\int_D \frac{\lambda}{2n} (R^2 - |y|^2) dy$$
Hence, the mass constraint implies

$$\int_{B_R^+} \frac{\lambda}{2n} \left( R^2 - |y|^2 \right) dy - \int_{B_R^+} \left[ \int_0^\frac{\lambda}{2n} \left( R^2 - |y|^2 \right) dy_n \right] dy'$$

$$= \frac{\lambda\omega_n R_n^{n+2}}{2n(n+2)} - \frac{n-1}{2n} \delta \lambda \sum_{i=1}^{n-1} \kappa_i y_i^2 \left( R^2 - |y'|^2 \right) dy'$$

$$= \lambda\omega_n R_n^{n+2}$$

$$= \frac{\lambda\omega_n R_n^{n+2}}{2n(n+2)} - \delta \sum_{i=1}^{n-1} \kappa_i y_i^2 \left( n - \frac{1}{2} n \right) \omega_{n-1} R_n^{n+2}$$

$$= \lambda\omega_n R_n^{n+2}$$

$$= \frac{\lambda\omega_n R_n^{n+2}}{2n(n+2)} - \delta \sum_{i=1}^{n-1} \kappa_i y_i^2 \left( n - \frac{1}{2} n \right) \omega_{n-1} R_n^{n+3} + O(\delta^3).$$

Here we used

$$\int_0^{\frac{\lambda}{2n}} \sum_{i=1}^{n-1} \kappa_i y_i^2 \left( R^2 - |y|^2 \right) dy_n$$

$$= \frac{\lambda}{2n} \int_0^{\frac{\lambda}{2n}} \sum_{i=1}^{n-1} \kappa_i y_i^2 \left( R^2 - |y|^2 - |y'|^2 \right) dy_n$$

$$= \frac{\lambda}{2n} \left( R^2 - |y|^2 \right) \delta \sum_{i=1}^{n-1} \kappa_i y_i^2 + O(\delta^3).$$

Now the mass constraint \( \int_D v = M \) is equivalent to

$$M = \int_D \frac{\lambda}{2n} \left( R^2 - |y|^2 \right) dy + \delta \int_D v_1 + O(\delta^2)$$

$$= \int_D \frac{\lambda}{2n} \left( R^2 - |y|^2 \right) dy + \delta \int_{B_R^+} v_1 + O(\delta^2)$$

$$= \frac{\lambda\omega_n R_n^{n+2}}{2n(n+2)} - \frac{\delta \lambda (n-1)}{2n(n+1)(n+3)} \omega_{n-1} R_n^{n+3}$$

$$= \frac{\lambda\omega_n R_n^{n+2}}{2n(n+2)} - \frac{\delta \lambda (n-1)}{2n^2(n+1)(n+3)} \omega_{n-1} R_n^{n+3} + O(\delta^2)$$

$$= \frac{\lambda\omega_n R_n^{n+2}}{2n(n+2)} - \frac{\delta \lambda (n-1)}{n^2(n+3)} \omega_{n-1} R_n^{n+3} + O(\delta^2)$$

$$= \frac{\sqrt{2n}}{R} \left( \frac{\omega_n R_n^{n+2}}{2n(n+2)} - \delta \frac{(n-1)}{n^2(n+3)} \omega_{n-1} R_n^{n+3} \right) + O(\delta^2)$$

$$= \frac{\sqrt{2n}}{R} \left( \frac{\omega_n R_n^{n+1}}{2n(n+2)} - \delta \frac{(n-1)}{n^2(n+3)} \omega_{n-1} R_n^{n+2} \right) + O(\delta^2).$$

Hence, the mass constraint implies

$$R = R^* \left\{ 1 + \frac{(n-1)}{n(n+3)} \omega_{n-1} R^* \frac{\omega_{n-1} R^*}{(n+1)\pi^2(n+2)} \delta + O(\delta^2) \right\}$$
where
\[
\left( R^* \right)^{n+1} = \frac{2(n+2) M}{\omega_n \sqrt{2}}.
\]

For the solution of (8.3), (8.2), we can compute its energy as follows:
\[
e (q, M) = \int_D \left\{ \frac{1}{2} |\nabla v|^2 + 1 \right\} = -\frac{1}{2} \int_D v \Delta v \, dy + |D|
\]
\[
= \frac{\lambda M}{2} + \left\{ \frac{\omega_n R^2}{2} + \delta \int_{\Gamma^R} R_1 - \delta \int_{B_R^*} \sum_{i=1}^{n-1} \kappa_i v_i^2 \, dy' + O(\delta^2) \right\}.
\]

Finally,
\[
\int_{\Gamma^R} R_1 = \frac{n}{\lambda} \int_{\Gamma^R} \partial_n v_1 = \frac{n}{\lambda} \int_{B_R^*} \Delta v_1 + \frac{n}{\lambda} \int_{B_R^*} \partial_y v_1(y', 0) \, dy'
\]
\[
= -\frac{1}{2} \int_{B_R^*} \sum_{i=1}^{n-1} \kappa_i v_i^2 \, dy' = -\frac{(n-1) \kappa \omega_{n-1} R^{n+1}}{2(n+1)}.
\]

Thus, using
\[
R = R^* \left\{ 1 + \frac{2(n+2)(n-1) \omega_{n-1} \kappa R^* \delta + O(\delta^2)}{2(n+1)(n+3) \omega_n} \right\} = R^* \left\{ 1 + A \delta + O(\delta^2) \right\},
\]
we have
\[
e(q, M) - e^*(M)
\]
\[
= \frac{M \sqrt{2} n}{2 R^*} + \frac{\omega_n R^2}{2} - \frac{3(n-1) \kappa \omega_{n-1} R^{n+1}}{2(n+1)} \delta + O(\delta^2) - e^*(M)
\]
\[
= \frac{M \sqrt{2} n}{2 R^*} (1 - A \delta) + \frac{\omega_n (R^*)^2}{2} (1 + n \delta) - \frac{3(n-1) \kappa \omega_{n-1} (R^*)^{n+1}}{2(n+1)} \delta
\]
\[
+ O(\delta^2) - e^*(M)
\]
\[
= -\delta \left( \frac{M \sqrt{2} n}{2 R^*} A - \frac{\omega_n (R^*)^2}{2} n A + \frac{3(n-1) \kappa \omega_{n-1} (R^*)^{n+1}}{2(n+1)} \right) + O(\delta^2)
\]
\[
= -\delta \frac{M \sqrt{2}}{2 R^*} \frac{(n+2)(n-1) \omega_{n-1} R^*}{(n+1)(n+3) \omega_n} + \delta (R^*)^n \frac{(n+2)(n-1) \omega_{n-1} R^*}{(n+1)(n+3)}
\]
\[
= \delta (R^*)^{n+1} \frac{3(n-1) \kappa \omega_{n-1}}{2(n+1)} + O(\delta^2)
\]
\[
= \delta (R^*)^{n+1} \omega_n R^{n+1} \frac{(n+2)(n-1)}{(n+1)(n+3) \omega_n} + \delta (R^*)^n \frac{3(n-1)}{2(n+1)}
\]
\[
= \delta (R^*)^{n+1} \omega_n R^{n+1} \frac{(n+2)(n-1)}{(n+1)(n+3) \omega_n} + \delta (R^*)^n \frac{3(n-1)}{2(n+1)} + O(\delta^2)
\]
\[ = -\delta M \sqrt{2} \left[ \frac{(n+2)(n-1)\omega_{n-1}}{(n+1)(n+3)}\kappa + \frac{(n+2)}{2(n+1)(n+3)} \right] + O(\delta^2) \]

\[ = -\frac{\delta M \kappa \sqrt{2} \omega_{n-1}}{\omega_n} \left[ \frac{(n+2)(n-1)}{(n+1)(n+3)} + \frac{(n+2)(n-1)(n+5)}{2(n+1)(n+3)} \right] + O(\delta^2) \]

\[ = -c(n) M \kappa \delta + O(\delta^2) \]

where

\[ c(n) = \frac{(n-1)(n+2)(n+7) \omega_{n-1}}{\sqrt{2(n+1)(n+3)} \omega_n} \]

is a positive constant. It then follows that energy minimizer should be concentrated near the point of maximal mean curvature. Theorem 3 has been proved.

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E-mail address: xinfu@pitt.edu
E-mail address: hqjiang@pitt.edu
E-mail address: gul8@pitt.edu