Dressing Transformations and the Algebraic–Geometrical Solutions in the Conformal Affine $sl(2)$ Toda Model

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Abstract

It is shown that the algebraic–geometrical (or quasiperiodic) solutions of the Conformal Affine $sl(2)$ Toda model are generated from the vacuum via dressing transformations. This result generalizes the result of Babelon and Bernard which states that the $N$–soliton solutions are generated from the vacuum by dressing transformations.
1 Introduction

The dressing group \cite{1}, \cite{2} is a symmetry of the nonlinear equations which admit a zero–curvature representation. It acts via gauge transformations which leave invariant the form of the Lax connection and hence the dressing symmetry acts on the space of solutions of the corresponding integrable model. An important property of the dressing group is that its action is a Poisson–Lie action. This means that in order to ensure the covariance of the Poisson brackets one has to introduce a nontrivial bracket on the dressing group. It was argued in \cite{2} that the dressing group appears as a semiclassical limit of the quantum group symmetry of the two–dimensional integrable quantum field theories.

In this paper we study the relation between the dressing transformations and the algebraic–geometrical solutions (or finite–gap solutions) of the Conformal Affine Toda (CAT) $sl(2)$ model which is a conformally invariant extension of the sinh–Gordon equation \cite{3}. The algebraic–geometrical solutions of the sine–Gordon equation (which differs from the sinh–Gordon by a trivial renormalization) are studied extensively in the literature \cite{4}–\cite{8}. The $N$–gap sinh–Gordon solutions are related to hyperelliptic Riemann surfaces of genus $2N − 1$ with a fixed–point–free automorphism. This relation can be expressed explicitly by the evolution of the $N$–gap sinh–Gordon system (3.2a), (3.2b) or equivalently by the flow equations (3.3). In the singular limit when the length of all branch cuts of the underlying hyperelliptic Riemann surface tend to zero, the $N$–gap solutions coincide with the $N$–soliton solutions \cite{6}, \cite{9}, \cite{10}. The solitons, as it was shown by Babelon and Bernard \cite{11}, belong to the dressing group orbit of the vacuum. The dressing group elements which generate the soliton solutions from the vacuum are found. The $N$–soliton solutions of the sine–Gordon model are considered as a relativistically invariant $N$–body problem in \cite{12}.

We outline the content of the paper. Sec. 2 is a review of the dressing symmetries of the $sl(2)$ CAT and sinh–Gordon models. The sec. 3 deals with the algebraic–geometrical solutions of the sine–Gordon equation and the related to them dressing problem. The dressing group elements which transform the vacuum into the $N$–gap solutions are constructed starting from the explicit expression for the transport matrix (3.7). Sec. 4 is devoted to the dressing problem for the algebraic–geometrical solutions of the $sl(2)$ CAT model.

2 Dressing transformations in the $sl(2)$ CAT and the sinh–Gordon models

Both the $sl(2)$ CAT and the sinh–Gordon models appear as a zero–curvature condition of Lax connections the components of which belong to the affine $sl(2)$ algebra and the
\(\tilde{sl}(2)\) loop algebra respectively. We introduce some Lie algebra notations. Denote by \(E^\pm\) and \(H\) the three generators of the \(sl(2)\) Lie algebra with the commutators

\[
[H, E^\pm] = \pm 2E^\pm \quad [E^+, E^-] = H
\]

In the fundamental (two–dimensional) representation we shall fix these generators by normalizing \(H = \text{diag}(1, -1)\). The loop algebra \(\tilde{sl}(2)\) is the Lie algebra of the traceless \(2 \times 2\) matrices whose entries are Laurent series in the spectral parameter \(\lambda\); moreover, in the principal gradation, the diagonal elements contain only even powers of \(\lambda\) while the off–diagonal entries contain only odd powers of the spectral parameter. The loop algebra \(\tilde{sl}(2)\) has a central extension which is the affine \(\hat{sl}(2)\) algebra. As a basis of \(\hat{sl}(2)\) in the principal gradation one can choose the elements:

\[
H_n = \lambda^n H \quad E^\pm_n = \lambda^n E^\pm \\
\hat{d} \quad \hat{c}
\]

and \(\hat{c}\) commutes with all the generators. The derivation \(\hat{d}\) defines a gradation in \(\hat{sl}(2)\); the element \(X_n = \lambda^n X\), \(X \in sl(2)\) has grade \(n\). We also recall the Cartan decomposition: \(\hat{sl}(2) = \mathcal{N}_+ \oplus \mathcal{N}_- \oplus \mathcal{H}\) where \(\mathcal{N}_+\) (\(\mathcal{N}_-\)) is spanned on the positive (negative) grade elements and \(E^+_0\) (\(E^-_0\)); the Cartan subalgebra \(\mathcal{H}\) has three independent elements: \(H\), \(\hat{d}\) and \(\hat{c}\). The Borel subalgebras are \(\mathcal{B}_\pm = \mathcal{H} \oplus \mathcal{N}_\pm\). The highest weight vectors \(|\Lambda\rangle\) are annihilated by \(\mathcal{N}_+\) and are eigenvectors of the elements of the Cartan subalgebra. In what follows we shall frequently use the notation \(\mathcal{E}_\pm = \lambda^{\pm 1}(E^+ + E^-)\).

The equations of motion of the \(sl(2)\) CAT model follow from the flatness of the connection

\[
\mathcal{D}_{x^\pm} = \partial_{x^\pm} + A_{x^\pm} = \partial_{x^\pm} \mp \partial_{x^\pm} \Phi + me^{\pm ad\Phi} \mathcal{E}_\pm \quad \partial_{x^\pm} = \frac{\partial}{\partial x^\pm} 
\]

\[
\Phi = \frac{1}{2} \varphi H + \eta \hat{d} + \frac{1}{4} \zeta \hat{c} 
\]

The zero–curvature representation of the sinh–Gordon equation has the same form as (2.3a) with \(A_{x^\pm}\) being in the loop algebra \(\tilde{sl}(2)\) and \(\Phi = \frac{1}{2} \varphi H\). Inserting (2.3a) and (2.3b) in \(\{\mathcal{D}_{x^+}, \mathcal{D}_{x^-}\} = 0\) one obtains the \(sl(2)\) CAT equations

\[
\partial_{x^+} \partial_{x^-} \varphi = m^2 e^{2\eta} \left(e^{2\varphi} - e^{-2\varphi}\right) \\
\partial_{x^+} \partial_{x^-} \eta = 0 \\
\partial_{x^+} \partial_{x^-} \zeta = m^2 e^{2\eta} \left(e^{2\varphi} + e^{-2\varphi}\right)
\]
where the grade of the elements $X$ solution $\Phi$. The first few equations are obtained infinite set of equations which characterize the dressing group orbit of the components on the Cartan subgroup $\ast$ which reflects the requirement that the transformation $T$ by using either $g$ element of grade $\pm$ one uses only the grade analysis. Note that (2.8a) fixes $X$ parameter freedom in while in the loop algebra due to the absence of a central element (2.8a) leaves a one–parameter freedom in $X_{\pm 1}$. Further, due to the fact that the unique independent element of grade $\pm 2$ is $\lambda^{\pm 2}H$, the equation (2.8b) implies a new restriction on $X_{\pm 1}$ which together with (2.4) gives

\begin{equation}
(\partial_{\pm} \varphi)^2 - \partial_{\pm}^2 \zeta^g = (\partial_{\pm} \varphi)^2 - \partial_{\pm}^2 \zeta
\end{equation}

*more precisely the elements $g_{\pm}$ in the affine group are fixed up to the factors $\exp\{f_{\pm}(x^\pm)\dot{c}\}$ where $f_{\pm}$ are arbitrary functions.
and hence the quantity \((\partial_x \pm \varphi)^2 - \partial_{x \pm}^2 \zeta\) is an invariant of the dressing group orbit. An important observation done by Babelon and Bernard in \[11\] is that the central element \(\hat{c}\) appears only in (2.8a) (and yields the solution (2.9)) and all the other restrictions on \(X_{\pm i}\) can be calculated in the loop algebra. Therefore, if \(\tilde{g}_\pm\) is a solution of the dressing problem (2.5) for the sinh–Gordon solutions \(\varphi\) and \(\varphi^g\), then the elements of the affine \(\hat{SL}(2)\) group

\[
\hat{g}_\pm(x^+, x^-, \lambda) = e^{\frac{\varphi^g - \varphi}{2} \hat{g}_\pm(x^+, x^-, \lambda)}
\]  

(2.11)

define a dressing transformation which relates the solutions \((\varphi, \zeta)\) and \((\varphi^g, \zeta^g)\) of the \(sl(2)\) CAT model. Note that \(\zeta\) and \(\zeta^g\) are determined from (2.9) with \(X_{\pm 1}\) derived from the grade expansions (2.7) of the loop group elements \(\tilde{g}_\pm\). We shall use this remark in the demonstration that the algebraic–geometrical solutions of the \(sl(2)\) CAT model are in the dressing group orbit of the vacuum.

The dressing group action on the fields \(\xi_\Lambda = \langle \Lambda | e^\Phi, \bar{\xi} = T^{-1} e^{-\Phi} | \Lambda \rangle\) where \(\Lambda\) is a highest weight is extremely simple

\[
\xi^g_\Lambda(x^+, x^-, \lambda) = \xi(x^+, x^-, \lambda) \cdot g_\pm^{-1}(0, 0, \lambda) \quad \bar{\xi}^g_\Lambda(x^+, x^-, \lambda) = g_\pm(0, 0, \lambda) \cdot \bar{\xi}_\Lambda(x^+, x^-, \lambda)
\]

The last equation gives a nice relation between the tau–functions \(\tau_\Lambda(\Phi) = e^{-2\Lambda(\Phi)}\) of the solutions \(\Phi\) and \(\Phi^g\) \[13\]

\[
e^{-2\Lambda(\Phi_g(x^+, x^-))} = \xi_\Lambda(x^+, x^-, \lambda) \cdot g \cdot \bar{\xi}_\Lambda(x^+, x^-, \lambda)
\]

\[
e^{-2\Lambda(\bar{\Phi}(x^+, x^-))} = \xi_\Lambda(x^+, x^-, \lambda) \cdot g \cdot \bar{\xi}(x^+, x^-, \lambda)
\]

\[
g = g_\pm^{-1}(0, 0, \lambda) \cdot g_\pm(0, 0, \lambda)
\]  

(2.12)

The above relations show that the multiplication in the dressing group is different from the multiplication in the initial loop or affine group: the product of two elements \(g = g_- g_+\) and \(h = h_-^{-1} h_+\) is \(g \times h = (g_- h_-)^{-1} \cdot g_+ h_+\). We note also that the correspondence \(g \to (g_-, g_+)\) is one–to–one since due to (2.7) \(g_-\) and \(g_+\) have inverse components on the Cartan subgroup.

3 The dressing problem for the algebraic–geometrical solutions of the sinh–Gordon equation

The algebraic–geometrical solutions of the sinh–Gordon equation are expressed in terms of theta functions on hyperelliptic Riemann surfaces of odd genus \(2N - 1\) which possess a fixed–point–free automorphism of order two \[3\]. There is another expression \[4\], \[5\], \[8\] based on theta functions on arbitrary hyperelliptic Riemann surfaces. In \[7\] it is demonstrated that these two expressions are in fact equivalent. In this paper we shall use another description: in the spirit of \[12\], the algebraic–geometrical solutions
of the sinh–Gordon equation will be represented as solutions of an integrable system with finite number of degrees of freedom. From this point of view the hyperelliptic Riemann surfaces appear as characteristic equations of the corresponding Lax operators which turn out to be polynomials of finite order on the spectral parameter \([8], [14]\). Let \(\hat{C}_N\) be the hyperelliptic Riemann surface of the algebraic function \(s_N = s_N(\mu)\) given by the equation

\[
s_N^2 = \prod_{p=1}^{2N} \left( \lambda^2 - \mu^2_p \right)
\]

(3.1)

where \(\pm \mu_p, p = 1\ldots 2N\) are the ramification points. The genus of \(\hat{C}_N\) is \(2N - 1\). We note that \(\hat{C}_N\) has a free involution \(T : (\lambda, s) \rightarrow (-\lambda, -s)\). The algebraic–geometrical solutions of the sinh–Gordon corresponding to (3.1) are introduced in terms of a positive divisor of degree \(N\) on \(\hat{C}_N\)

\[
D(x) = \sum_{j=1}^{N} P_j(x)
\]

where \(x = (x^+, x^-)\) and \(P_j(x) = (\epsilon_j(x), s(\epsilon_j(x)))\). The evolution of \(D(x)\) is determined by

\[
\sum_{j=1}^{N} \int_{P_j(0)}^{P_j(x)} \frac{\mu^2 d\mu}{s_N(\mu)} = \frac{m}{\prod_{p=1}^{2N} \mu_p} x^- \delta_{1,0} + (-)^{N-1} m x^+ \delta_{l,N-1}
\]

(3.2a)

\[
\partial_x \epsilon_j = ms_N(\epsilon_j) \prod_{l \neq j} \frac{1}{\epsilon_l^2 - \epsilon_j^2}
\]

\[
\partial_x^- \epsilon_j = \frac{m}{\prod_{p=1}^{2N} \mu_p} s_N(\epsilon_j) \prod_{l \neq j} \frac{\epsilon_l^2}{\epsilon_l^2 - \epsilon_j^2}
\]

(3.3)

which is integrable for arbitrary function \(s_N\). The sinh–Gordon equation \(\partial_x^+ \partial_x^- \varphi_N = 2m^2 s h 2 \varphi_N\) is equivalent to the identity

\[
\sum_{j=1}^{N} \left( \frac{1}{\epsilon_j^2} - \sum_{p=1}^{2N} \frac{1}{\epsilon_j^2 - \mu_p^2} - \sum_{k \neq j} \frac{1}{\epsilon_k^2 - \epsilon_j^2} \right) s_N^2(\epsilon_j) \prod_{l \neq j} \frac{\epsilon_l^2}{(\epsilon_l^2 - \epsilon_j^2)^2} =
\]

\[
= \frac{\prod_{p=1}^{2N} \mu_p^2}{\prod_{j=1}^{N} \epsilon_j^2} - \prod_{j=1}^{N} \epsilon_j^2
\]

(3.4)

We sketch its proof: first we notice that both sides of (3.4) are symmetric meromorphic functions on the variables \(\epsilon_1^2 \ldots \epsilon_N^2\); the left hand side has no poles at the points \(\epsilon_l^2 = \epsilon_j^2\) ( \(l \neq j\)) and \(\epsilon_l^2 = \mu_p^2\). Due to the symmetry one can consider the two sides of (3.4) as functions of, say \(\epsilon_1^2\). It is easy to see that their residues at \(\epsilon_1^2 = 0, \infty\) coincide and
therefore, their difference is a constant on $\epsilon_j^2$'s. In order to show that this constant is zero we set $x_j = \mu_j^2$, and $y_j = \mu_j^2 + N$ ($j = 1 \ldots N$) in the identity

$$
\sum_{j=1}^N \prod_{k=1}^N (x_j - y_k) \prod_{l \neq j} \frac{x_l}{(x_i - x_j)^2} = x_1 \ldots x_N - y_1 \ldots y_N
$$

(3.5)

In the singular limit $\mu_{2j-1} \to \mu_{2j} = \alpha_j$, $j = 1 \ldots N$ in (3.1) the integrals (3.2a) are expressed by logarithms and the corresponding sinh–Gordon solution (3.2b) is identical to the $N$–soliton solution. The parameters $\alpha_1 \ldots \alpha_N$ are the soliton rapidities. The soliton limit of the finite–gap solutions will be discussed in details in [15].

The hyperelliptic curve (3.1) is a two–sheeted covering of the Riemann sphere by the spectral parameter $\lambda$ with ramification points at $\pm \mu_1 \ldots \pm \mu_{2N}$. We shall assume in what follows that the ramification points are all finite and different from zero. Then $\hat{C}_N$ is represented as a couple of Riemann spheres $\hat{C}_N^+$ and $\hat{C}_N^-$ glued along 2N cuts connecting the branching points. For example, as cuts one can choose the lines between $\pm \mu_{2j-1}$ to $\pm \mu_{2j}$. Denote the infinity and the zero points on $\hat{C}_N^\pm$ as $\infty_{\pm}$ and $\mathcal{O}_{\pm}$ respectively. Directly from (3.1) one gets the expansions

$$
s_N = \pm \lambda^{2N} \left( 1 + O \left( \frac{1}{\lambda^2} \right) \right) \quad (\lambda, s) \to \infty_{\pm}
$$

(3.6a)

$$
s_N = \pm (-)^N \prod_{p=1}^{2N} \mu_p \left( 1 + O (\lambda^2) \right) \quad (\lambda, s) \to \mathcal{O}_{\pm}
$$

(3.6b)

We continue with the problem of construction of a dressing transformation which generates the solution $\varphi_N$ from the vacuum $\varphi_0 = 0$. In [15] it will be shown that the matrix

$$
\mathcal{T}_N(x, \lambda, s) = e^{\frac{2\pi i}{N} H} \cdot \\
\prod_{j=1}^N \frac{\lambda^{+(+)^N \epsilon_j(x)}(\lambda^{+1})^{-\epsilon_j(0)}}{\lambda^{-(+)^N \epsilon_j(x)}(\lambda^{+1})^{-\epsilon_j(0)}} e^{A_N}
$$

$$
\prod_{j=1}^N \frac{s_N(\lambda) - l_N(\lambda)}{(\lambda^{+1} \epsilon_j(x))(\lambda^{+1} \epsilon_j(0)) e^{A_N}} - \prod_{j=1}^N \frac{s_N(\lambda) + l_N(\lambda)}{(\lambda^{+1} \epsilon_j(x))(\lambda^{+1} \epsilon_j(0)) e^{-A_N}}
$$

(3.7)

where

$$
\mathcal{A}_N = (-)^{N-1} s_N(\lambda) \left( \frac{m_N}{\lambda \prod_{j=1}^{2N} \mu^p_j} + \lambda \sum_{j=1}^N \int_{P_j(0)} \frac{d\mu}{(\mu^2 - \lambda^2)^s(\mu)} \right) + \frac{(-)^N}{2} l_N \prod_{j=1}^N \frac{(\lambda - \epsilon_j(x))(\lambda + \epsilon_j(0))}{(\lambda + \epsilon_j(x))(\lambda - \epsilon_j(0))}
$$

(3.8a)

$$
l_N(\lambda) = (-)^N \lambda \sum_{j=1}^N \frac{s(\epsilon_j(x))}{\epsilon_j(x)} \prod_{k \neq j} \frac{\lambda^2 - \epsilon_k^2(x)}{\epsilon_j^2(x) - \epsilon_k^2(x)}
$$

(3.8b)

*In order to get a non–singular hyperelliptic Riemann surface one should also assume that $\mu_i \neq \mu_j$ for $i \neq j$ and $\mu_i + \mu_j \neq 0$
is a non–normalized transport matrix corresponding to the $N$ gap solution (3.2b). Its determinant

$$
\det T_N = -2 \frac{s_N(\lambda)}{\prod_{j=1}^{N}(\lambda^2 - \epsilon_j^2(0))}
$$

vanishes at the ramification points. In analysing the singularities of (3.7) we first note that due to (3.6a), the integrated $x^+$ and $x^-$ flows of the system (3.2a) are equivalent to the expansions $A_N = \mp m\lambda x^+ + O(\lambda^2)$ around the infinity points $\infty_{\pm}$ while from (3.6b) one immediately obtains $A_N = \mp \frac{m\lambda x^+}{\lambda} + O(\lambda)$ at the vicinity of the zero points $O_{\pm}$. The integrand of the second term in (3.8b) is a differential of a third kind with simple poles at the points $(\lambda, \pm s)$ and $(-\lambda, \pm s)$. Therefore, one expects $A_N$ to have logarithmic singularities at the points whose projections on the sphere by $\lambda$ are $\pm \epsilon_j(x)$ or $\pm \epsilon_j(0)$. Note that if such a singularity exists at some of these points belonging to $\hat{C}_N$ (or $\hat{C}_N$) then at the point on $\hat{C}_N$ the same value of the spectral parameter $A_N$ is regular. Taking into account these remarks it is easy to see that the elements of (3.7) are functions on $\hat{C}_N$ with simple poles which does not depend on $x^+$ and $x^-$ and exponential singularities at the infinity and the zero points. Fixing all the points $P_j(0)$, $j = 1 \ldots N$ of the divisor $D(0)$ to belong to $\hat{C}_N$ it is easy to see that on $\hat{C}_N$ the elements of the first (the second) column of $T_N$ have simple poles at the points $\lambda = (-)^{N+1}\epsilon_j(0)$ ($\lambda = (-)^N\epsilon_j(0)$); in order to find the positions of the poles on $\hat{C}_N$ it is sufficient to note that the change $s \to -s$ in (3.1) exchanges the columns of the transport matrix. We have thus shown that the entries of (3.7) have the following property: if at certain point of $\hat{C}_N$ which is not a ramification point there is a simple pole of a given matrix element, then at the point with the same value of the spectral parameter but with the opposite sign of $s$ the same matrix element has no singularity. Similar property was established for the Bloch functions associated with the finite–gap solutions of the KdV equation which are also known to correspond to hyperelliptic Riemann surfaces $10$. In (3) the transport matrix (3.7) was expressed in terms of theta functions on $\hat{C}_N$.

The explicit knowledge of the transport matrix $T_N$ allows to find a dressing transformation which generates the solution (3.2b) from the vacuum. In doing that we express the normalized transport matrix $\tilde{T}_N$ corresponding to the solution $\varphi_N$ as

$$
T_N(x, \lambda, s) = \tilde{T}_N(x, \lambda, s)\tilde{T}_N^{-1}(0, \lambda, s) = f_N(x, \lambda, s)T_0(x, \lambda)f_N^{-1}(0, \lambda, s) \quad (3.10a)
$$

$$
f_N(x, \lambda, s) = \tilde{T}_N(x, \lambda, s)\tilde{T}_0^{-1}(x, \lambda)S(\lambda, s) \quad (3.10b)
$$

$$
\tilde{T}_0(x, \lambda) = \begin{pmatrix}
  e^{-m(\lambda x^+ + \frac{\lambda^2}{2})} & e^{m(\lambda x^+ + \frac{\lambda^2}{2})} \\
  e^{-m(\lambda x^- + \frac{\lambda^2}{2})} & e^{m(\lambda x^- + \frac{\lambda^2}{2})}
\end{pmatrix}
$$

$$
S(\lambda, s) = \begin{pmatrix}
  a(\lambda, s) & b(\lambda, s) \\
  b(\lambda, s) & a(\lambda, s)
\end{pmatrix}
$$

$$
a^2 - b^2 = \frac{\prod_{j=1}^{N}(\lambda^2 - \epsilon_j^2(0))}{s(\lambda)}
$$

$$
T_0(x, \lambda) = \tilde{T}_0(x, \lambda)\tilde{T}_0^{-1}(0, \lambda) = e^{-mx^+\varepsilon_+}e^{-mx^-\varepsilon_-} \quad (3.10c)
$$
and hence, the gauge transformation induced by the element $f_N$ transforms the vacuum transport matrix $T_0$ into $T_N$. Note that the $x^\pm$ independent matrix $S$ commutes with the connection (2.3a) corresponding to the vacuum solution $\Phi = 0$. We can choose this matrix in such a way that the element $f_N$ has no singularities on $\hat{g}_N^\pm$ except of simple poles at the ramification points. The explicit expression for such $f_N$ is

$$ f_N = \frac{e^{\frac{\pi i}{2} H + B_N}}{2 \sqrt{\prod_{\mu=1}^{2N}(\lambda - \mu \rho)}} \left( \begin{array}{cc} \prod_{j=1}^{N}(\lambda + (-)^N \epsilon_j(x)) & \prod_{j=1}^{N}(\lambda + (-)^N \epsilon_j(x)) \\ \frac{s_N(\lambda) - t_N(\lambda)}{\prod_{j=1}^{N}(\lambda - (-)^N \epsilon_j(x))} & \frac{s_N(\lambda) - t_N(\lambda)}{\prod_{j=1}^{N}(\lambda - (-)^N \epsilon_j(x))} \end{array} \right) + \frac{e^{\frac{\pi i}{2} H - B_N}}{2 \sqrt{\prod_{\mu=1}^{2N}(\lambda + \mu \rho)}} \left( \begin{array}{cc} \prod_{j=1}^{N}(\lambda - (-)^N \epsilon_j(x)) & \prod_{j=1}^{N}(\lambda - (-)^N \epsilon_j(x)) \\ -\frac{s_N(\lambda) + t_N(\lambda)}{\prod_{j=1}^{N}(\lambda + (-)^N \epsilon_j(x))} & \frac{s_N(\lambda) + t_N(\lambda)}{\prod_{j=1}^{N}(\lambda + (-)^N \epsilon_j(x))} \end{array} \right) $$

where

$$ B_N = m \lambda x^+ + \frac{m x^-}{\lambda} + A_N $$

Substituting the asymptotic expansion for $A_N$ around the infinity and the zero points into (3.11) and taking into account (3.2a), (3.6a) and (3.6b) one gets the expansions

$$ f_N(x, \lambda, s) = e^{\frac{\pi i}{2} H} \left( 1 + X_{-1} + O\left(\frac{1}{\lambda^2}\right) \right) (\lambda, s) \rightarrow \infty_+ $$

(3.12a)

$$ f_N(x, \lambda, s) = e^{-\frac{\pi i}{2} H} \left( 1 + X_1 + O(\lambda^2) \right) (\lambda, s) \rightarrow \mathcal{O}_+ $$

(3.12b)

from which together with (2.7) it follows that the expansion of $f_N$ around the points $\infty_+$ and $\mathcal{O}_+$ can be identified with the loop group elements $\hat{g}_-$ and $\hat{g}_+$ which by gauge transformations generate the $N$-gap solution $\varphi_N$ from the vacuum. In [13] we shall show that in the soliton limit (3.11) reproduces the elements constructed in [11].

## 4 The dressing problem in the affine $SL(2)$ group

In this section we shall demonstrate that there exists a dressing transformation $\hat{g} = \hat{g}_-^{-1} \hat{g}_+$ which relates the vacuum solution ($\varphi_0 = 0, \zeta_0 = 2m^2 x^+ x^-$) to the $N$-gap solution ($\varphi_N, \zeta_N$) of (2.4a)–(2.4c). We recall that for both solutions the field $\eta$ vanishes and that $\varphi_N$ is the $N$-gap sinh–Gordon solution (3.2b) considered in the previous section. The field $\zeta_N$ which satisfies (2.4c) has still to be determined. We outline our strategy. First, in view of the observation done at the end of the previous section, the expansions of the element (3.11) around the infinity point $\infty_+$ and the zero point $\mathcal{O}_+$ produce the loop group elements $\hat{g}_-$ and $\hat{g}_+$ which determine the dressing transformation $\varphi_0 \rightarrow \varphi_N$. Further, comparing (3.12a) and (3.12b) with (2.9) we shall get a first order differential
system for the field $\zeta_N$. Finally we use (2.11) to find $\hat{g}_\pm$. In order to guarantee the consistency of this procedure we check that (2.10) is satisfied.

From (2.7) and (3.12a)–(3.12b) it is seen that in order to get $X_{-1}(X_1)$ one should expand $f_N$ up to the terms proportional to $\lambda^{-1}(\lambda)$ when $(\lambda, s) \to \infty_+$. Inserting (3.8a) into (3.11) and using the expansions (3.6a), (3.6b) as well as the equations (3.2a), (3.2b) we get

$$
\lambda B_N = \frac{m}{2} x + \frac{m x^+}{2} \sum_{p=1}^{2N} \frac{\mu_p^2}{\lambda^2} + \left( - \right)^N \sum_{j=1}^{N} \left( \frac{1}{\epsilon_j(x)} - \frac{1}{\epsilon_j(0)} \right) - \frac{2N}{\lambda} \mu_p \sum_{j=1}^{N} \int_{P_j(0)} P_j(x) d\mu_m - O\left( \lambda^2 \right) \quad (\lambda, s) \to \infty_+ \quad (4.1)
$$

On the other hand (3.8b) is a polynomial of degree $2N - 1$ on the spectral parameter

$$
l_N(\lambda) = \lambda^{2N-1} \frac{\partial x_+^{\varphi_N}}{m} + \ldots + (-1)^{N+1} \lambda \prod_{p=1}^{2N} \frac{\partial x_+^{\varphi_N}}{m}
$$

Substituting the expansions (4.1) and (4.2) in (3.11) one obtains

$$
X_{-1} = \left( m x + \frac{1}{2} \sum_{p=1}^{2N} \left( m x^+ \mu_p + 1 \right) \mu_p - \left( - \right)^N \sum_{j=1}^{N} \int_{P_j(0)} P_j(x) \frac{\mu^2 N d\mu}{s_N(\mu)} + \left( - \right)^N \sum_{j=1}^{N} \epsilon_j(0) \right) E_- + \frac{\partial x_+ \varphi_N}{m} \lambda^{-1} E_- \quad (4.3a)
$$

$$
X_1 = \left( m x + \frac{1}{2} \left( \frac{m x^+}{\mu} + 1 \right) - \prod_{p=1}^{2N} \mu_p \sum_{j=1}^{N} \int_{P_j(0)} P_j(x) \frac{d\mu_m}{\mu^2 m_s N(\mu)} + \sum_{j=1}^{N} \frac{1}{\epsilon_j(0)} \right) E_+ + \frac{\partial x_- \varphi_N}{m} \lambda E_- \quad (4.3b)
$$

which compared with (2.9) produce the system

$$
\partial x_+ \zeta_N = \partial x_+ \varphi_N - m \sum_{p=1}^{2N} \left( m x^+ \mu_p + 1 \right) \mu_p - 2m \left( - \right)^N \sum_{j=1}^{N} \int_{P_j(0)} P_j(x) \frac{\mu^2 N d\mu}{s_N(\mu)} - 2m(-)^N \sum_{j=1}^{N} \epsilon_j(0) \quad (4.4a)
$$
\[
\partial_x \zeta_N = -\partial_x \varphi_N - m \sum_{p=1}^{2N} \left( \frac{m x^-}{\mu_p} + 1 \right) \frac{1}{\mu_p} + 2m \prod_{p=1}^{2N} \mu_p \sum_{j=1}^{N} \int_{P_j(0)} d\mu \frac{d \mu}{\mu^2 S_N(\mu)} - 2m \sum_{j=1}^{N} \frac{1}{\epsilon_j(0)}
\]

(4.4b)

The integrand of the third term of (4.4a) is a differential of a second kind on \( \hat{C}_N \) whose unique singularities are second–order poles at the two infinity points \( \infty_\pm \); the differential in the corresponding term of (4.4b) is again a differential of a second kind with second–order poles at the zero points \( O_\pm \). The \( x^- \)–derivative of (4.4a) and the \( x^+ \)–derivative of (4.4b) can be calculated explicitly with the help of (3.3). The result is

\[
m^2 \partial_x^{-} \partial_x^{+} \zeta_N = \prod_{p=1}^{2N} \mu_p^{-1} \left( 2 \sum_{j=1}^{N} \epsilon_{2j}^N(x) \prod_{l \neq j} \frac{\epsilon_l^2(x)}{\epsilon_l^2(x) - \epsilon_j^2(x)} - \prod_{l=1}^{N} \frac{N}{\epsilon_l^2(x)} \right) + \\
+ \frac{\prod_{j=1}^{N} \epsilon_j^2(x)}{\prod_{p=1}^{2N} \mu_p}
\]

(4.5a)

\[
m^2 \partial_x^{+} \partial_x^{-} \zeta_N = \prod_{p=1}^{2N} \mu_p \left( 2 \sum_{j=1}^{N} \frac{1}{\epsilon_j^2(x)} \prod_{l \neq j} \frac{1}{\epsilon_l^2(x) - \epsilon_j^2(x)} - \prod_{l=1}^{N} \frac{N}{\epsilon_l^2(x)} \right) + \\
+ \frac{\prod_{j=1}^{N} \epsilon_j^2(x)}{\prod_{p=1}^{2N} \mu_p}
\]

(4.5b)

Note that under the change

\[
\epsilon_j \rightarrow \frac{1}{\epsilon_j}, \quad \mu_p \rightarrow \frac{1}{\mu_p}
\]

(4.6)

the r. h. s. of (4.5a) transforms into the r. h. s. of (4.5b). In order to show that (4.4a), (4.4b) is integrable, we set in the identity

\[
\prod_{l=1}^{N} \frac{1}{x_l} = \sum_{j=1}^{N} \frac{1}{x_j} \prod_{l \neq j} \frac{1}{x_l - x_j}
\]

(4.7)

\( x_j = \epsilon_j^{\pm 2} \). Comparing (4.5a) and (4.5b) we get \( \partial_x^{+} \partial_x^{-} \zeta_N = \partial_x^{-} \partial_x^{+} \zeta_N = m^2(e^{2\varphi_N} + e^{-2\varphi_N}) \) and hence \((\varphi_N, \zeta_N)\) is an algebraic–geometrical solution of the \( sl(2) \) CAT model.

To demonstrate that this solution is in the orbit of the vacuum one still has to check the identities (2.10). A direct calculation based on (3.3) and (4.4a), (4.4b) shows that (2.10) reduces to

\[
\sum_{j=1}^{N} \left( \sum_{p=1}^{2N} \frac{1}{\epsilon_j^2 - \mu_p^2} + 2 \sum_{k \neq j} \frac{1}{\epsilon_k^2 - \epsilon_j^2} \right) s_N^2(\epsilon_j) \prod_{l \neq j} \frac{1}{\epsilon_l^2 - \epsilon_j^2} = 0
\]

10
\[ -\sum_{p=1}^{2N} \mu_p^2 + 2(-)^{N+1} \sum_{j=1}^{N} \epsilon_j^{2N} \prod_{l \neq j} \frac{1}{\epsilon_l^2 - \epsilon_j^2} \]

\[ \frac{1}{\prod_{p=1}^{2N} \mu_p^2} \sum_{j=1}^{N} \left( \frac{2}{\epsilon_j^2} - \sum_{p=1}^{2N} \frac{1}{\epsilon_j^2 - \mu_p^2} - 2 \sum_{k \neq j} \frac{1}{\epsilon_k^2 - \epsilon_j^2} \right) \frac{s_N^2(\epsilon_j)}{\prod_{l \neq j} (\epsilon_l^2 - \epsilon_j^2)^2} = \]

\[ -\sum_{p=1}^{2N} \frac{1}{\mu_p^2} - 2 \sum_{j=1}^{N} \frac{1}{\epsilon_j^2} \prod_{l \neq j} \frac{\epsilon_l^2}{\epsilon_l^2 - \epsilon_j^2} \]

The above two identities are equivalent since the substitution (4.6) exchanges them and therefore it is sufficient to check only (4.8a) which can be done using the same approach as in the proof of (3.4). This concludes the demonstration of the existence a dressing group transformation which transforms the vacuum into the \( N \)-gap solution \((\varphi_N, \zeta_N)\). The corresponding dressing group element \( \hat{g} = \hat{g}_-\hat{g}_+ \) is given by (2.11) where \( \hat{g}_- \) and \( \hat{g}_+ \) are the expansions of (3.11) around the infinity point \( \infty_+ \) and the zero point \( \mathcal{O}_+ \) of \( \hat{C}_N \) respectively.

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