Primary operations in differential cohomology

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Abstract

We characterize primary operations in differential cohomology via stacks, and illustrate by differentially refining Steenrod squares and Steenrod powers explicitly. This requires a delicate interplay between integral, rational, and mod $p$ cohomology, as well as cohomology with $U(1)$ coefficients and differential forms. Along the way we develop computational techniques in differential cohomology, including a Künneth decomposition, that should also be useful in their own right, and point to applications to higher geometry and mathematical physics.

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1 Introduction

Cohomology operations with coefficients in a group $G$ are natural transformations of the form $H^n(\cdot; G) \to H^m(\cdot; G)$. By Brown representability and the Yoneda lemma this is equivalent to calculating the universal cohomology group of Eilenberg-MacLane spaces $H^m(K(G,n); G)$, which is in turn equivalent to calculating the homotopy classes of maps $[K(G,n), K(G,m)]$. Thus all cohomology operations of fixed degree are accounted for by calculating the cohomology of the Eilenberg-MacLane space $K(G,n)$. To account for all cohomology operations one obviously has to vary both $m$ and $n$. See [MT68] [St62] for detailed accounts.

We are interested in differential cohomology (see [CS85] [Fr00] [HS05] [SS08] [Bu12] [BS10] [Sc13] [BB14]). What replaces Eilenberg-MacLane spaces are various stacks of higher $U(1)$-bundles ($n$-bundles) with connections. Thus, cohomology operations will involve the differential cohomology of such stacks, and this process can be described via mapping spaces of stacks. For differential refinements we will need to study morphisms of stacks

$$\hat{\theta} : B^nU(1) \to B^mU(1),$$

(1.1)
where $B^nU(1)_\nabla$ represents the moduli stack of $n$-bundles equipped with connection, studied in \cite{FSS12, FSS13, FSS15a}. The homotopy classes of such morphisms will in turn be describe the differential cohomology group

$$\tilde H^{k+1}(B^nU(1)_\nabla; \mathbb{Z}) := \pi_0 \text{Map}(B^nU(1)_\nabla, B^kU(1)_\nabla).$$

One of the main goals of this paper is to characterize this group for various values of $k$ and $n$. This in turn will lead to a full characterization of primary cohomology operations in differential cohomology.

Since differential cohomology operations, as we will see, involve various coefficients, we find it useful to point out the interrelations that already exist between these (we found the discussion in \cite{FFGS6} particularly useful). This should also help us develop some intuition for the full differential case. Note that for coefficients being one of $\mathbb{Z}, \mathbb{Z}/p$ or $\mathbb{Q}$ particularly useful). This should also help us develop some intuition for the full differential case. Note that for coefficients being one of $\mathbb{Z}, \mathbb{Z}/p$ or $\mathbb{Q}$ i.e. an abelian group, then the set of all cohomology operations $H^m(K(G, n); G)$, where $G$ and $G'$ are from the above set, will also be abelian.

**Operations from $\mathbb{Z}/p$ to $\mathbb{Q}$**. We know that $H^q(K(G, n); \mathbb{Q}) = 0$ for all $q > 0$ when $G$ is a finite abelian group, i.e. for us $\mathbb{Z}/p$. This shows that there are nontrivial cohomology operations from $\mathbb{Z}/p$-coefficients to $\mathbb{Q}$-coefficients.

**Operations from $\mathbb{Z}$ to $\mathbb{Q}$**. We will distinguish the odd and even cases. For the first, $H^q(K(\mathbb{Z}, 2n+1); \mathbb{Q})$ is nonzero only for $q = 2n + 1$, where it is equal to $\mathbb{Q}$, with generator the image of the fundamental class $\iota$ under the homomorphism $r: H^{2n+1}(K(\mathbb{Z}, 2n+1); \mathbb{Z}) \to H^{2n+1}(K(\mathbb{Z}, 2n+1); \mathbb{Q})$ induced by the natural embedding $\mathbb{Z} \hookrightarrow \mathbb{Q}$. Thus, every operation from an odd-dimensional integral class to rational cohomology preserves the dimension, i.e. takes $\alpha \in H^{2n+1}(X; \mathbb{Z})$ to $\lambda r(\alpha) \in H^{2n+1}(X; \mathbb{Q})$ for some fixed rational number $\lambda$ corresponding to the operation. In the even case, $H^q(K(\mathbb{Z}, 2n); \mathbb{Q}) = \mathbb{Q}[r(\iota)]$, so that every operation from even integral cohomology to rational cohomology is given as the power $\alpha \mapsto \lambda \alpha^k$, where $k \in \mathbb{Z}$, $\lambda \in \mathbb{Q}$ are determined by the operation.

**Operations from $\mathbb{Q}$ to $\mathbb{Q}$**. The group $H^n(K(\mathbb{Q}, m); \mathbb{Q})$ can be straightforwardly calculated via e.g. the Serre spectral sequence. In this case, one has that any cohomology operation assigns to an element $\alpha \in H^n(X; \mathbb{Q})$ the element $\lambda \alpha^k \in H^{nk}(X; \mathbb{Q})$, where $k \in \mathbb{Z}$, $\lambda \in \mathbb{Q}$ both fixed by the operation.

**Operations from $H^*(-)_{\text{dR}}$ to $H^*(-)_{\text{dR}}$**. On the other hand, operations in de Rham cohomology can be deduced from those on rational cohomology via the de Rham theorem. Hence, de Rham operations should be systematically characterized. The de Rham cohomology groups and the rational cohomology groups have the same underlying algebraic structure.

**Operations from $\mathbb{Z}/p$ to $\mathbb{Z}/p$**. From an algebraic and homotopic point of view, these are perhaps most studied. We start with degree-preserving operations. From $H^n(K(\mathbb{Z}/p; n); \mathbb{Z}/p) = \mathbb{Z}/p$ it follows that any degree-preserving such operation is multiplication by a scalar in $\mathbb{Z}/p$. Then from $H^{n+1}(K(\mathbb{Z}/p, n); \mathbb{Z}/p) = \mathbb{Z}/p$ it follows that there exist a unique operation raising the degree by one generating all such operation. This generator is given by the connecting homomorphism $\beta_p$, i.e. the Bockstein homomorphism for the the coefficients sequence $\mathbb{Z}/p \xrightarrow{\times p} \mathbb{Z}/p^2 \xrightarrow{\beta_p} \mathbb{Z}/p$. 


Note that since $H^q(K(Z/p,n);Z/p) = 0$ for $n+1 < q < n+2p-2$, there are no operations that raise the degree by $2, 3, 4, \ldots, 2p-3$. The next degree where a nonzero operation exits is in dimension $2p-1$, where there is a unique operation corresponding to $H^{n+2p-2}(K(Z/p,n);Z/p) \cong Z/p$ for $n > p$, which is the reduced Steenrod power $P^1$. The next degrees $H^{n+2p-1}(K(Z/p,n);Z/p) = Z/p$ and $H^{n+2p}(K(Z/p,n);Z/p) = Z/p \oplus Z/p$ correspond to combinations of the operations $\beta_p P^1, P^1 \beta_p$ and $\beta_p P^1 \beta_p$. The next nontrivial degree is $4p-4$ corresponding to the reduced power $P^2$.

The complete classification of these operations for $Z/p$ is given by studying the mod $p$ Steenrod algebra $A_p$ (see [St62], [Ca54], [Mi58], [Ma70]).

**Operations from $Z$ to $Z/p$.** The main example here is $\rho_p : Z \to Z/p$, the mod $p$ reduction, for $p$ a prime number. This induces an operation of the same name on cohomology $\rho_p : H^n(-;Z) \to H^n(-;Z/p)$.

**Operations from $Z/p$ to $Z$.** Consider $\beta$, the connecting homomorphisms, i.e. the Bockstein homomorphism, for the the coefficients sequence $Z \xrightarrow{x^p} Z \xrightarrow{\rho_p} Z/p$. For a class $x \in H^a(X;Z/p)$, the class $\beta(x)$ is an integral element of $H^{a+1}(X;Z/p)$, i.e. it belongs to the image of the mod $p$ reduction homomorphism $\rho_p : H^{a+1}(X;Z) \to H^{a+1}(X;Z/p)$. Note that all operations from $Z/p$ to $Z$ and from $Z$ to $Z/p$ are built from combinations of $\rho_p$ or $\beta$ with Steenrod powers (or squares for $p = 2$).

As differential cohomology is built out of integral cohomology and differential form data, cohomology operations in both of these settings are essential for our construction of the refined cohomology operations. Somewhat surprisingly, we found that neither has been studied to the extent that one might expect from such classical notions.

**Operations from $Z$ to $Z$.** Integral cohomology operations $K(Z,n) \to K(Z,m)$ have been studied starting with Cartan [Ca54]. The algebraic structure has been investigated in [Ma70], [Ko82], [Pe04]. However, there does not seem to be a complete characterization, at least in the unstable case, and explicit calculations do not seem to be available in all cases. An exception is perhaps [Pe10], where the Leray-Serre spectral sequence for the path-loop fibration $K(Z,n) \to PK(Z,n+1) \to K(Z,n+1)$ is used to calculate the groups $H^m(K(Z,n);Z)$ for $2 \leq n \leq 7$ and $2 \leq m \leq 13$, which should be useful for applications. So, aside from arriving at $Z$-operations via the Steenrod algebra, not much seems to be known.

**Operations from $\Omega^n$ to $\Omega^m$.** Unlike all the above, these operations are not at the level of cohomology, but rather occur at the level of differential forms. For compact manifolds, linear operations on differential forms $\Omega^n(X) \to \Omega^m(X)$ which commute with diffeomorphisms have been considered in Palais [Pa59] from the point of view of functional analysis. This was extended to the noncompact case in [Jo71]. More general operations in a much broader context are studied in [KMS93], but the operations relevant to us are still linear (and we are interested in nonlinear ones as well); there it is shown that all operations that raise the form degree by one are multiples of the exterior derivative, and linearity follows from naturality. More recently, operations (both linear and nonlinear) acting on 1-forms (connections) were considered in [FH13], and generalized to differential forms of all degrees in [NS15]. We will make use of this for our construction of cohomology operations on closed differential forms $\Omega^3$ in stacks.

Cohomology operations need not be homomorphisms. Indeed, the power map $H^n(X;G) \to H^{2n}(X;G)$ is a cohomology operation by naturally, but is obviously not a homomorphism. However,
this map becomes a homomorphism when $G = \mathbb{Z}/p$, and it is the example of a top degree Steenrod operations.

One might wonder whether there is anything at all to be gained by considering differential cohomology operations, as after all we are considering operations that emerge from $\mathbb{Z}/p$ coefficients, while the additional data in differential cohomology is that of de Rham forms. On the other hand, one would think that there must be some effect of differential refinement on the cohomology operations, as after all there are differential refinements of characteristic cohomology classes that led to a considerable amount of utility and applications (see [HS05] [SSS12] [FSSt12] [FSS13] [FSS15a] [FSS15b]). It turns out that the general result will fall somewhere in between. For instance, we find that

- Even Steenrod squares cannot be differentially refined.
- Odd Steenrod squares refine as $\tilde{Sq}^{2m+1} = j\Gamma_p Sq^{2m+1}$, where
  
  - $I : \tilde{H}^*(-; \mathbb{Z}) \to H^*(-; \mathbb{Z})$ the integration map, corresponding to ‘unrefinement’.
  - $\Gamma_p$ is induced by a representation $\mathbb{Z}/p \hookrightarrow U(1)$ as the roots of unity
  - $j$ is the flat inclusion $H^*(-, U(1)) \hookrightarrow \tilde{H}^*(-; \mathbb{Z})$, i.e. inclusion of flat bundles into bundles with connection.

Note that the special case of Steenrod squares in degree one less than the top degree were considered in Gomi [Go08] (see also [Bu12] Sec. 3.4) and related to the Deligne-Beilinson cup product. Thus, a portion of our work can be viewed as a generalization of this relationship to all degrees.

The paper is organized as follows. The first two subsections of Sec. 2 are meant to give a directed overview of the two main ingredients that we aim to coherently merge together, namely Steenrod operations and stacks. In Sec. 2.1 we recall the definition of Steenrod squares and Steenrod powers from two points of view: via (co)chains and via symmetric group actions. We present these in such a way that helps the reader conceptually follow the constructions in later sections. Here we also recall the integral lifts of Steenrod squares, which are needed for differential refinements. In Sec. 2.2 we set up the machinery of stacks, adapted to the context of differential cohomology, that we need in order to formulate the differential refinements. The main general results are presented in Sec. 2.3, where we present the characterization theorem (Theorem 6) of general differential cohomology operations. This requires an interplay between integral cohomology operations and operations on differential forms.

We apply this formulation to the Steenrod operations in Sec. 3, where the even case is given in Proposition 9 while the odd case is given in Corollary 11. In Sec. 3.1 we investigate whether or not the differential Steenrod squares are related to the homotopy commutativity of the Deligne-Beilinson cup product [De71] [Be86] (see also [Br93]), refining the classical point of view on the Steenrod squares presented in Sec. 2.1. This leads to a generalization of [Go08] to Steenrod square of all degrees and at the level of stacks. Along the way we prove a Kunneth decomposition for differential cohomology (Proposition 16) which should be interesting in its own right as a general computational tool. The properties of the refined Steenrod operations are given in Sec. 3.2. Most of the properties of the classical operations continue to hold with the exception of the identity and the Cartan formula, both of which can be traced to the fact that the even Steenrod squares do not refine. Finally we present applications of our operations in Sec. 3.3, where we also introduce a special version of stability for the operations (Proposition 24) and end with tantalizing applications to physics via higher geometry.
2 Formulation of primary operations in differential cohomology

2.1 Classical cohomology operations via (co)chains and via symmetric group actions

The material in this section is standard ([MT68] [St62]), but we include it as it helps in the conceptual understanding of our constructions later, due to the similarity of the structure involved.

Steenrod squares are initially meant to square a class \( x \), \(|x| = n \) of the same degree as the operation, i.e. \( Sq^i(x) = x^2 \) when \( i = n \). The lower degree operations \( Sq^i \), \( 0 \leq i \leq n - 1 \), measure in a precise way the extent to which homotopy commutativity of the cup product deviates from strict commutativity. The cup product is not (graded)-commutative at the chain level, and the obstruction is measured by the Steenrod operations. We will closely follow the presentation in [Br] (Ch. 3) for an illuminating illustration of how the commutativity vs. homotopy commutativity arise.

The diagonal map \( \Delta : X \to X \times X \) leads to a strictly commutative triangle at the chain level

\[
\begin{array}{ccc}
C_\ast(X) & \xrightarrow{\Delta} & C_\ast(X \times X) \\
\downarrow{\Delta_*} & & \downarrow{\tau_*} \\
C_\ast(X \times X) & , & \\
\end{array}
\]

where \( \tau_* \) is the map induced from the exchange map \( \tau : X \times X \to X \times X \) given by \( \tau(x, y) = (y, x) \).

Now the Alexander-Whitney map \( C_\ast(X \times X) \xrightarrow{\text{AW}} C_\ast(X) \otimes C_\ast(X) \), defines an equivalence of chain complexes but is only homotopy commutative and not strictly so (see [Mc95]). Dualizing to cochains, we can define the cup product operation by \( (a \cup b)(x) = (a \otimes b)(\text{AW}(\Delta_*(x))) \).

The effect of the homotopy commutativity of the Alexander-Whitney map propagates to the cup product and we get a diagram

\[
\begin{array}{ccc}
C^\ast(X) \otimes C^\ast(X) & \xrightarrow{\cup} & C^\ast(X) \\
\downarrow{\tau} & & \downarrow{\cup} \\
C^\ast(X) \otimes C^\ast(X) & , & \\
\end{array}
\]

which is only homotopy commutative and not strictly commutative. This then induces the commutative diagram at the level of mod 2 cohomology

\[
\begin{array}{ccc}
H^\ast(X; \mathbb{Z}_2) & \xrightarrow{\mu} & H^\ast(C^\ast(X) \otimes C^\ast(X); \mathbb{Z}_2) \\
\downarrow{\tau^*} & & \downarrow{\mu} \\
H^\ast(C^\ast(X) \otimes C^\ast(X); \mathbb{Z}_2) & . & \\
\end{array}
\]

This is not yet a multiplication, for which one needs the Künneth isomorphism \( H^\ast(X \times X; \mathbb{Z}/2) \cong H^\ast(C^\ast(X) \otimes C^\ast(X); \mathbb{Z}_2) \cong H^\ast(X; \mathbb{Z}_2) \otimes H^\ast(X; \mathbb{Z}_2) \).

Having homotopy commutativity then allows for a lot of structure, arising from the chain homotopies and then homotopies on these, all the way up until the dimensions are exhausted. At the first level, one gets a chain homotopy \( \cup_1 \) between \( \cup \tau \) and \( \cup \) corresponding to the homotopy \( \cup \tau \simeq \cup \), such that \( a \cup \tau(a \otimes b) - \cup(a \times b) = d \cup_1 (a \otimes b) + \cup_1 d(a \otimes b) \). Now considering
the case \( b = a \), we get \( a \otimes a - a \otimes a = 0 = \partial_1 (a \otimes a) + \partial_1 (a \otimes a) \). If, furthermore, \( a \) is taken to be a cocycle, i.e. \( da = 0 \), then \( d(a \otimes a) = 0 \) as well. Then we are left with one factor, \( \partial_1 (a \otimes a) = 0 \), which defines a cohomology class at the next lower level

\[
Sq^{n-1}(a) := a \cup_1 a \in H^{2n-1}(X; \mathbb{Z}_2) .
\]

The lower Steenrod squares are obtained as the higher chain homotopies, obtained by iterating the above process to \( \cup_{i+1} : \cup_i \tau \simeq \cup_i \) for each \( i \geq 0 \). These give the remaining Steenrod squares

\[
Sq^{n-i}(a) := a \cup_i a \in H^{2n-i}(X; \mathbb{Z}_2) .
\]

The process stops at after \( n \) steps, when we reach \( Sq^0 \), which is the identity.

Steenrod powers \( P^i \), at a prime \( p \), work similarly by replacing \( \tau \) with the cyclic permutation operation on the product of \( p \)-fold copies of the space \( X \). This then gets translated analogously to a power map on cohomology classes.

Note that one does not necessarily need to deal with chain complexes in order to construct the Steenrod operations. In fact, there is an analogous construction in topological spaces, i.e in the category \( \mathcal{S} \text{Top} \), which makes use of the representability of cohomology via Eilenberg-MacLane spaces (see e.g. [Ha02]). As in the above description, one begins with a homotopy commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\Delta} & X \times X \\
\downarrow \Delta & & \downarrow \tau \\
X \times X & \xrightarrow{\tau} & X \times X
\end{array}
\]

\[
\begin{array}{ccc}
& & K(\mathbb{Z}/2, n) \times K(\mathbb{Z}/2, n) \xrightarrow{\cup} K(\mathbb{Z}/2, n) \\
\end{array}
\]

describing the homotopy commutativity of the cup product. Since we are concerned with the square cup product, we choose the maps to the Eilenberg-MacLane spaces to be given by the same map on each factor in the product, that is,

\[
x \times x : X \times X \to K(\mathbb{Z}/2, n) \times K(\mathbb{Z}/2, n) .
\]

Then a homotopy from this map to itself represents a loop: that is a map (analogous to the chain homotopy \( \cup_1 \) in the above approach)

\[
h : X \times X \times S^1 \to K(\mathbb{Z}/2, n) ,
\]

defined via the equivalence \( \text{Map}(S^1, \text{Map}(X \times X, K(\mathbb{Z}/2, n))) \simeq \text{Map}(S^1 \times X \times X, K(\mathbb{Z}/2)) \). Choosing \( h \) to be nontrivial, one can iterate this process and extend this map to the infinite-dimensional sphere \( S^\infty \) (a process that is analogous to choosing the higher homotopies in the above approach). Using the symmetry of the cup product, one can choose this map in such a way that it commutes with the \( \mathbb{Z}/2 \)-action on \( X \times X \) (given by transposing the factors) and the \( \mathbb{Z}/2 \)-action on \( S^\infty \) (given by the antipodal action). The map \( h \) then descends to a map on the quotient of the diagonal action

\[
h : X \times X \times_{\mathbb{Z}/2} S^\infty \to K(\mathbb{Z}/2, n) .
\]

Taking the trivial \( \mathbb{Z}/2 \)-action on \( X \), we see that the diagonal map \( \Delta : X \to X \times X \) is equivariant with respect to this action. The same map then induces a map on the corresponding homotopy orbits and the entire construction can be represented diagrammatically as follows. The map \( h \) is the universal map filling the homotopy commutative diagram
In fact, this diagram can be summarized in terms of \((\infty, 1)\)-colimits by identifying the map \(h\) as arising from the universal property of \((\infty, 1)\)-colimits. Although the construction of the map \(h\) is quite classical, its interpretation as the universal map filling the homotopy commutative diagram seems to be a new idea; one which we consider to have a distinct conceptual advantage. We will make use of this type of construction explicitly later in this paper (see Proposition 15).

To get to the construction of the Steenrod squares, we notice that the composite map \(hQ(\Delta) : X \times \mathbb{R}P^\infty \to K(\mathbb{Z}/2, 2n)\) can be identified with an element in \(Sq \in H^*(X \times \mathbb{R}P^\infty; \mathbb{Z}/2)\). The Künneth formula allows us to expand this element as a polynomial in powers of the first Stiefel-Whitney class. Define the Steenrod squares to be the coefficients of the polynomial

\[
Sq = Sq^n + Sq^{n-1} \otimes w_1 + Sq^{n-2} \otimes w_1^2 + \ldots + Sq^0 \otimes w_1^n. \tag{2.1}
\]

One can think of the various powers of \(w_1\) as indexing the cells of a CW structure, built from the transposition map. The coefficients attach homotopies involved in the cup product to these cells.

The Steenrod squares satisfy several desirable properties [St62] [MT68] [MS74] [Ha02]. To avoid unnecessary redundancies, we will not record these. The differential refinement of these cohomology operations will satisfy some of the same properties, generalized appropriately, but with marked differences. Hence, we prefer that these classical properties be deduced from the refined one (see Sec. 3.2).

We need to discuss integral cohomology operations on the path to arriving at differential cohomology operations. In particular, we would like to consider integral lifts of the Steenrod squares, that is we seek a diagram

\[
\begin{array}{ccc}
H^\bullet(-; \mathbb{Z}) & \xrightarrow{Sq^j} & H^{\bullet+j}(-; \mathbb{Z}) \\
\rho_2 & & \rho_2 \\
H^\bullet(-; \mathbb{Z}/2) & \xrightarrow{Sq^i} & H^{\bullet+i}(-; \mathbb{Z}/2)
\end{array}
\]

where \(\bullet, *, i, j\) are degrees to be determined. In the case where \(i\) is even, it is impossible to find such a lift. This can be immediately deduced using the long exact Bockstein sequence corresponding to the short exact sequence \(\mathbb{Z} \xrightarrow{x} \mathbb{Z} \xrightarrow{\rho_2} \mathbb{Z}/2\). Indeed, if such an integral refinement existed, we would have an integral operation \(\theta\) such that \(\rho_2(\theta) = Sq^{2i}\). But by the long exact sequence, this would imply \(\beta(Sq^{2i}) = 0\) (where \(\beta\) is the Bockstein corresponding to the above sequence). However, this is not the case as \(\rho_2 \beta(Sq^{2i}) = Sq^{2i+1}\). In contrast, for the odd Steenrod squares, the above formula allows us to define an integral refinement

\[
Sq^{2i+1}_\mathbb{Z} := \beta(Sq^{2i}). \tag{2.2}
\]

Our results on differential Steenrod squares will end up having a similar pattern. The even case will be presented in Proposition 9 while the odd case is given in Corollary 11.
2.2 Stacks associated to differential cohomology

In this section, we review some of the stacks with which we will be working, and highlight some of their properties that are useful for us. A more complete study can be found in [FSSt12] [SSS12] [FSS13] [FSS15a].

To start, in order to properly account for our stacks we have the following.

**Definition 1.** Let $\text{CartSp}$ denote the category whose objects are open subsets $U \subset \mathbb{R}^n$ that are diffeomorphic to $\mathbb{R}^n$, for some $n \in \mathbb{N}$, and whose morphisms are smooth maps $f : U \to V$. This category admits the structure of a Grothendieck site, with topology generated by good open covers (contractible finite intersections).

We denote the $\infty$-category of $\infty$-presheaves on $\text{CartSp}$ by $\text{PSh}_\infty(\text{CartSp})$. We denote the $\infty$-category of $\infty$-sheaves on $\text{CartSp}$ by $\text{Sh}_\infty(\text{CartSp})$.

**Remark 1.** The category of $\infty$-sheaves on $\text{CartSp}$ admits a presentation by model categories. More precisely, taking the homotopy coherent nerve on fibrant/cofibrant objects in the Čech local, projective (or injective) simplicial model structure on simplicial presheaves on $\text{CartSp}$ yields a presentation. This allows us to present $\infty$-sheaves via projectively fibrant simplicial presheaves that satisfy homotopy descent.

**Definition 2.** A local object $G \in \text{Sh}_\infty(\text{CartSp})$ is called a smooth stack (or higher stack). In the presentation by simplicial presheaves, such an object can be identified with a functor

$$G : \text{CartSp} \to \text{Kan} \subset s\text{Set},$$

that satisfies homotopy descent with respect to Čech covers (see [Du01] [Lu09] [Sc13] for review).

An important functor that allows for passage between simplicial sets and chain complexes in positive degrees is the Dold-Kan functor

$$DK : \text{Ch}^+ \longrightarrow \text{Kan} \subset s\text{Set}. \quad (2.3)$$

The stack that we will use most frequently in this paper is the moduli stack of $n$-bundles (or gerbes) with connection: $B^nU(1)_\nabla$. This stack arises as a pullback of stacks¹

$$\begin{array}{ccc}
B^nU(1)_\nabla & \longrightarrow & \Omega^{n+1}_{\text{cl}} \\
\downarrow & & \downarrow \\
B^{n+1}\mathbb{Z} & \longrightarrow & b_{\text{dR}}B^{n+1}U(1)
\end{array} \quad (2.4)$$

If we forget about the connection on the these $n$-bundles, we obtain the bare moduli stack of $n$-gerbes $B^nU(1)$. Explicitly, this stack is obtained by applying the Dold-Kan functor to the sheaf of chain complexes $C^\infty(-,U(1))[n]$: the sheaf of smooth $U(1)$-valued functions in degree $n$. The other stacks related to $B^nU(1)_\nabla$ in the above diagram are defined via the Dold-Kan correspondence (2.3) as follows (see [FSSt12] [FSS13] [FSS15a] [Sc13]):

- The stack $B^{n+1}\mathbb{Z}$ is defined to be the smooth stack obtained by applying the Dold-Kan functor to the sheaf of chain complexes $\mathbb{Z}[n]$, with the sheaf of locally constant $\mathbb{Z}$-valued functions in degree $n$.

¹Whenever we say pullback, pushout, limit or colimit, we mean these operations in the $(\infty,1)$-sense. With an appropriate choice of model structure, these can be thought of as the homotopy pullback.
- The stack representing the truncated de Rham complex $♭dR^nU(1)$ is obtained by applying Dold-Kan to the truncated de Rham sheaf of chain complexes
\[
\Omega^{\leq n} := \cdots 0 \cdots \Omega^0 \to \Omega^1 \to \cdots \Omega^n_{cl} .
\]

- The stack of closed $n$-forms $\Omega^n_{cl}$ is defined to be the stack obtained by applying Dold-Kan to the sheaf of closed $n$-forms. This is also discussed in [HS05] and [Bu12] (Problem 4.42).

The differential cohomology diagram [SS08] lifts to a diagram of stacks [Bu12] [Sc13]

\[
\begin{array}{ccc}
\Omega^{\leq n} & \xrightarrow{d} & \Omega_{cl}^{n+1} \\
♭dR^nU(1) & \xrightarrow{a} & B^nU(1) \vee \\
♭B^nU(1) & \xrightarrow{β} & B^{n+1}Z \\
\end{array}
\]

where the diagonals are fiber sequences. Moreover, the maps $a$, $I$ and $R$ induce homomorphisms in cohomology. The functors which produce these surrounding stacks are part of an $\infty$-adjunction called a cohesive adjunction [BNV16] [Sc13].

It is shown in [Sc13] that the $\infty$-category of smooth higher stacks $\text{Sh}_\infty(\text{CartSp})$ admits a quadruple $\infty$-categorical adjunction ($\Pi \dashv \text{disc} \dashv \Gamma \dashv \text{codisc}$)

\[
\begin{array}{c}
\text{Sh}_\infty(\text{CartSp})
\end{array}
\xrightarrow{\Pi, \text{disc}, \Gamma, \text{codisc}}
\begin{array}{c}
\text{sSet}
\end{array}
\]

where the fundamental groupoid functor $\Pi$ preserves finite $\infty$-limits, and the discretization and co-discretization functors disc and codisc are fully faithful. In our context the quadruple adjunction is presented by Quillen functors, where on the left we take the Čech local projective model structure.

One implication of this is that $\text{sSet}$ embeds into $\text{Sh}_\infty(\text{CartSp})$ in two different ways as a reflexive $\infty$-subcategory. From the reflectors we can produce two monads and one comonad defined as follows:

\[
\Pi := \Pi \circ \text{disc}, \quad b := \text{disc} \circ \Gamma, \quad ♯ := \text{codisc} \circ \Gamma .
\]

These monads fit into a triple adjunction ($\Pi \dashv b \dashv ♯$) which is called a cohesive adjunction. We remark that the adjoint functors can be presented by derived functors in the appropriate Quillen model structure (either local injective or local projective) on simplicial presheaves (see [Sc13] for details).

**Remark 2.** Each monad in the cohesive adjunction picks out a different part of the nature of a smooth stack. This nature is perhaps best exemplified by how the adjoints behave on smooth manifolds (viewed as stacks). More precisely, if $M$ is a smooth manifold then, for instance,

(i) the comonad $♭$ (flat) takes the underlying set of points of the manifold and then embeds this set back into stacks as a discrete object. This functor therefore misses the smooth structure of the manifold and treats it instead as a discrete object.
(ii) The monad $\Pi$ essentially takes the singular nerve of the manifold using smooth paths and higher smooth simplices on the manifold. It therefore retains the geometry of the manifold and “knows” that the points of the manifold ought to be connected together in a smooth way.

As discussed in [Sc13], the bare stack $B^n\mathbb{Z}$ representing integral cohomology is equivalent to $\Pi B^nU(1)_\nabla$, while the discrete stack $B^nU(1)_{\delta}$ is equivalent to the moduli stack of flat bundles $♭B^nU(1)_\nabla$. Thus, one can rewrite the diamond diagram (2.5) using only the monads $\Pi$ and $♭$. In this context the “unrefinement map” $I$ arises as the unit of the monad, i.e. a natural transformation

$$I : \text{id} \to \Pi,$$

where $\text{id}$ is the identity functor. In the stable case, the characterization of differential cohomology theories using these monads is due to [BNV16].

We now prove a few properties which we will use later.

**Proposition 3.** Let $A$ be a discrete abelian stack. That is $A \simeq \text{disc}(B)$ form some $B \in s\text{Ab}$. Then

$$♭B^nA \simeq B^nA.$$

**Proof.** Since $\Gamma$ and disc are right adjoints, they commute with looping, as this operation is an example of an $\infty$-limit. Consequently they also commute with delooping, as this is defined using looping. It follows by definition (see (2.7)) that $♭$ commutes with delooping. Then we have

$$♭B^nA \simeq B^n♭A = B^n(\text{disc} \circ \Gamma)(A) \simeq B^n(\text{disc} \circ \Gamma \circ \text{disc}(B)) \simeq B^n(\text{disc}(B)) \simeq B^nA.$$ 

In the step before last we used the fact that $\Gamma \circ \text{disc} = \text{id}$, because disc is given by the stackification of the constant functor $U \mapsto A$, and then $\Gamma$ evaluates that at a point $U = \mathbb{R}^0$, giving back the original object $A$. $\square$

The moduli stacks $B^nU(1)_\nabla$, as $n$ varies, represent differential cohomology in the sense that the functor

$$\pi_0\text{Map}(−, B^nU(1)_\nabla) : \text{Sh}_\infty(\text{CartSp}) \to \text{Ab}$$

assigns to every smooth manifold $X$ (embedded in stacks via the sheaf of smooth plots $C^\infty(−, X)$) the differential cohomology group $\tilde{H}^{n+1}(X; \mathbb{Z})$. This follows almost immediately from the presentation of ordinary differential cohomology as Deligne cohomology, along with the Dold-Kan correspondence [FSS12] [Sc13]. The unrefinement morphism $I : \text{id} \to \Pi$ then induces a natural transformation

$$I_* : \pi_0\text{Map}(−, B^nU(1)_\nabla) \to \pi_0\text{Map}(−, \Pi B^nU(1)_\nabla) \simeq \pi_0\text{Map}(−, B^{n+1}\mathbb{Z}).$$

We can equivalently write this map as a map

$$I_* : \tilde{H}^{n+1}(−; \mathbb{Z}) \to H^{n+1}(−; \mathbb{Z}),$$

which is the familiar “integration map” which relates differential cohomology to its underlying cohomology theory.
We will frequently need to use representability in our calculations and indeed we are able to pass from the stacks to underlying theories somewhat seamlessly. For this reason, we remind the reader of the various theories that are represented by the stacks in the refined diamond diagram (2.5) (see [FSS13] [FSS15a] [FSS15b] [Sc13] for details). Below we use the notation \( \text{Map}(-,-) \) for the derived simplicial hom.

- The stack \( \mathbb{B}^{n+1} \mathbb{Z} \) represents ordinary integral cohomology in degree \( n+1 \). That is, we have a natural isomorphism
  \[ \pi_0 \text{Map}(-, \mathbb{B}^{n+1} \mathbb{Z}) \simeq H^{n+1}(-; \mathbb{Z}). \]

- The stack \( \mathbb{B}\mathbb{B}^n \mathbb{U}(1) \) represents cohomology with \( \mathbb{U}(1) \)-coefficients:
  \[ \pi_0 \text{Map}(-,\mathbb{B}\mathbb{B}^n \mathbb{U}(1)) \simeq H^n(-; \mathbb{U}(1)). \]

- The stack \( \mathbb{B}\mathbb{dR} \mathbb{B}^n \mathbb{U}(1) \) represents de Rham cohomology in degree \( n+1 \),
  \[ \pi_0 \text{Map}(-,\mathbb{B}\mathbb{dR} \mathbb{B}^n \mathbb{U}(1)) \simeq H^{n+1}_{\mathbb{dR}}(-). \]
  Equivalently, by de Rham’s theorem, this stack also represents cohomology with real coefficients \( H^{n+1}(-; \mathbb{R}) \).

- The stack \( \Omega^n_{\text{cl}} \) represents the sheaf of closed differential \( n \)-forms:
  \[ \pi_0 \text{Map}(-, \Omega^n_{\text{cl}}) \simeq \Omega^n_{\text{cl}}(-). \]

In some of the proofs, we will use the properties of discrete stacks when calculating homotopy classes of maps. More precisely, we have an adjunction

\[
\begin{array}{ccc}
\text{sSet} & \xrightarrow{\pi_0} & \text{Set} \\
\downarrow \text{sk}_0 & & \\
\text{Set} & \xleftarrow{\pi_0} & \text{sSet}
\end{array}
\]

with \( \pi_0 \) takes the connected components of the simplicial set. The right adjoint \( \text{sk}_0 \) simply embeds a set as a discrete stack. To illustrate how one uses the adjunction in practice, observe that both a manifold \( X \) and the sheaf of closed \( n \)-forms \( \Omega^n_{\text{cl}} \) are represented by discrete objects in the category of stacks. Then we have

\[ \pi_0 \text{Map}(X, \Omega^n_{\text{cl}}) \simeq \pi_0 \text{Map}(X, \text{sk}_0(\Omega^n_{\text{cl}})) \simeq \text{hom}(\pi_0(X), \Omega^n_{\text{cl}}) \simeq \Omega^n_{\text{cl}}(X). \]

Here we have used the adjunction between \( \text{sk}_0 \) and \( \pi_0 \), passing to the category of sheaves, and finally used the Yoneda lemma.

### 2.3 General differential Cohomology operations

We consider differential cohomology operations from a general point of view, and then specialize in later sections. We will consider these operations in the context of the stacks approach to differential cohomology.

As indicated in the Introduction, we will need to study morphisms of stacks

\[ \hat{\theta} : \mathbb{B}^n \mathbb{U}(1)_\nabla \to \mathbb{B}^m \mathbb{U}(1)_\nabla. \]

The goal of this section will be to establish the general properties of these maps and to provide a characterization theorem describing the general form of all such operations. In order to prove the
theorem, we will need to understand how differential cohomology operations refine classical ones. The cohesive adjoints \((2.6)\) will be extremely useful in our discussion on this point. Essentially, this boils down to the fact that these functors pick out different aspects of the moduli stack \(\mathcal{B}^nU(1)_{\nabla}\). Hence, when studying the maps \((2.10)\), we can use these functors to isolate various parts of the source or target stack (depending on the situation). We can then use the properties of these functors to arrive at isomorphisms between hom sets in the homotopy category which would otherwise require a lot of work to establish.

In what follows, we will continue to denote the differential cohomology group of a differentiable manifold \(X\) in degree \(n\) by \(\hat{H}^n(X;\mathbb{Z})\) and identify it with the contravariant functor

\[
\hat{H}^n(-;\mathbb{Z}) := \pi_0 \text{Map}(-, \mathcal{B}^nU(1)_{\nabla}),
\]

restricted to the subcategory of smooth manifolds.

**Definition 4.** A differential cohomology operation is a natural transformation of functors

\[
\hat{\theta} : \hat{H}^n(-;\mathbb{Z}) \to \hat{H}^m(-;\mathbb{Z})
\]

or, equivalently, a homotopy class of maps between stacks

\[
\hat{\theta} : \mathcal{B}^nU(1)_{\nabla} \to \mathcal{B}^mU(1)_{\nabla}.
\]

At this stage, we can already see one of the advantages provided by the stacky approach to differential cohomology operations. This is, we can describe these operations as elements in the set \(\pi_0 \text{Map}(\mathcal{B}^nU(1)_{\nabla}, \mathcal{B}^mU(1)_{\nabla})\), just as integral cohomology operations are elements in \(H^m(K(\mathbb{Z}, n);\mathbb{Z}) \simeq \pi_0 \text{Map}(K(\mathbb{Z}, n), K(\mathbb{Z}, m))\). This allows us to do constructions at the universal level, which is not possible without the use of stacks.

Now since the stack \(\mathcal{B}^nU(1)_{\nabla}\) arises as the pullback \((2.4)\), the universal property of pullbacks ensures that a map \(\hat{\theta}\) of the type \((2.10)\) is induced by a homotopy commutative diagram involving operations \(\tau\) on closed differential forms, \(\alpha\) on de Rham cohomology, and \(\theta\) on singular cohomology

\[
\begin{array}{cccc}
\Omega^{n+1}_{\text{cl}} & \xrightarrow{\tau} & \Omega^{m+1}_{\text{cl}} \\
\downarrow c & & \downarrow c \\
\mathcal{B}^nU(1)_{\nabla} & \xrightarrow{\delta_{\text{dR}}} & \mathcal{B}^mU(1)_{\nabla} \\
\downarrow i & & \downarrow i \\
\mathcal{B}^{n+1}Z & \xrightarrow{\theta} & \mathcal{B}^{m+1}Z.
\end{array}
\]

Hence, we see that a triple \((\theta, \alpha, \tau)\) which makes the above diagram commute up to a choice of 2-morphism immediately induces a differential cohomology operation \(\hat{\theta}\). This point of view emphasizes that differential cohomology operations are really a compatible combination of operations on differential forms and operations on integral cohomology. Furthermore, these operations are required to be homotopic in the de Rham stack \(\delta_{\text{dR}}\mathcal{B}^mU(1)\).

If a differential cohomology operation \(\hat{\theta}\) is induced from such a triple, we say that \(\hat{\theta}\) refines the integral cohomology operation \(\theta\) and the operation \(\tau\), on differential forms. On general abstract grounds, the converse of the above statement may not be true. That is, every morphism \(\hat{\theta}\) as in Definition \(\square\) need not be induced by such a triple. However, because of the special nature of the particular pullback involving \(\mathcal{B}^nU(1)_{\nabla}\), we now see that this is indeed the case.
Proposition 5. Every differential cohomology operation $\hat{\theta}$ refines an integral operation

$$\theta : \mathbb{B}^{n+1} \mathbb{Z} \to \mathbb{B}^{m+1} \mathbb{Z}$$

and an operation on forms

$$\tau : \Omega_{cl}^{n+1} \to \Omega_{cl}^{m+1}$$

which fit into a homotopy commuting triple $(\theta, \alpha, \tau)$ as in (2.11). Moreover, at the level of homotopy classes, we have

$$\tau R = R \hat{\theta} \quad \text{and} \quad \theta I = I \hat{\theta}.$$ 

Diagrammatically, we have homotopy commutativity

\[
\begin{tikzcd}
\mathbb{B}^{n+1} \mathbb{Z} \ar[r, \theta] & \mathbb{B}^{m+1} \mathbb{Z} \\
\Omega^{n+1}_{cl} \ar[u, \tau] \ar[dr, c] & \Omega^{m+1}_{cl} \ar[u, I] \ar[dl, c] \\
& b_{dR} \mathbb{B}^{n+1} U(1) \ar[r, \alpha] \ar[ur, \beta] & b_{dR} \mathbb{B}^{m+1} U(1) \ar[u, i] \\
\mathbb{B}^{n+1} U(1) \ar[r, R] & \mathbb{B}^{m+1} U(1) \ar[u, \hat{\theta}]
\end{tikzcd}
\]

(2.12)

Proof. Let $\hat{\theta}$ be a differential cohomology operation. Then $I \hat{\theta} : \mathbb{B}^{n+1} U(1) \mathbb{V} \to \mathbb{B}^{m+1} \mathbb{Z}$. Since $\mathbb{Z}$ is discrete, we have $\mathbb{B}^{n+1} \mathbb{Z} \simeq b\mathbb{B}^{n+1} \mathbb{Z}$. This fact, along with the cohesive adjunction, imply that we have an equivalence

$$\text{Map}(\mathbb{B}^{n+1} U(1) \mathbb{V}, \mathbb{B}^{m+1} \mathbb{Z}) \simeq \text{Map}(\mathbb{B}^{n+1} U(1) \mathbb{V}, b\mathbb{B}^{m+1} \mathbb{Z}) \quad \text{(by above)}$$

$$\simeq \text{Map}(\Pi(\mathbb{B}^{n+1} U(1) \mathbb{V}), \mathbb{B}^{m+1} \mathbb{Z}) \quad \text{(by (2.7))}$$

$$\simeq \text{Map}(\mathbb{B}^{n+1} \mathbb{Z}, \mathbb{B}^{m+1} \mathbb{Z})$$

Since the composite equivalence between the first and third line is induced by precomposition with $I$ (see the discussion around eq. (2.8)), we have an operation $\theta : \mathbb{B}^{n+1} \mathbb{Z} \to \mathbb{B}^{m+1} \mathbb{Z}$ such that $\theta I = I \hat{\theta}$ at the level of homotopy classes.

To prove that $\hat{\theta}$ refines an operation on forms, we observe that $R \hat{\theta} : \mathbb{B}^{n+1} U(1) \mathbb{V} \to \Omega_{cl}^{m+1}$ and since $\Omega_{cl}^{m+1}$ is a discrete object, using the adjunction (2.9), we have an isomorphism

$$\text{Map}(\mathbb{B}^{n+1} U(1) \mathbb{V}, \Omega_{cl}^{m+1}) \simeq \text{hom}(\pi_0(\mathbb{B}^{n+1} U(1) \mathbb{V}), \Omega_{cl}^{m+1}) \simeq \text{hom}(\Omega^n/\text{im}(d), \Omega_{cl}^{m+1}).$$

(2.13)

Here we have used the isomorphisms

$$\pi_0(\mathbb{B}^{n+1} U(1) \mathbb{V}) \simeq \pi_0(DK(\mathbb{Z}_\mathbb{D}^\infty(n+1)))$$

$$\simeq H_0(\mathbb{Z}_\mathbb{D}^\infty(n+1))$$

$$\simeq H_0[\mathbb{Z} \to \Omega^0 \to \cdots \to \Omega^{n-1} \to \Omega^n]$$

$$\simeq \Omega^n/\text{im}(d),$$

13
The exterior derivative induces an isomorphism on sheafification (by Poincaré lemma)
\[ d : L(\Omega^n/\text{im}(d)) \to \Omega^{n+1}_\text{cl}, \]
(2.14)
where \( L \) is the sheafification functor. Therefore, the right hand side of eq. (2.13) is isomorphic to \( \text{hom}(\Omega^{n+1}_\text{cl}, \Omega^n_{\text{cl}+1}) \). The isomorphism is exactly precomposition with the curvature (eq. (2.14)), and therefore there is an operation \( \tau \) on forms such that \( \tau R = R\theta \) at the level of homotopy classes. The homotopy commutativity follows from the homotopy commutativity of the pullback diagram (2.11). The homotopy commutativity of diagram (2.12) gives a homotopy \( cR\theta \to c\hat{\theta} \), which we can identify with \( \alpha \). Making this identification explicit is not particularly illuminating and we leave such details to the interested reader. \( \square \)

Remark 3. Thinking about elements of \( \hat{H}^n(X; \mathbb{Z}) \) as higher line bundles with connection, the previous proposition makes it explicit how a differential cohomology operation can be interpreted as an operation on bundles. Moreover, the curvature and underlying integral class of the resulting bundle is obtained via some singular cohomology operation and an operation on forms.

At this stage, the reader might wish to see some examples of differential cohomology operations. Indeed, we have the following two examples which, as we will see explicitly in Lemma 7, are essentially the only examples which give classes with nonzero curvature.

**Example 1** (Dixmier-Douady class). The homotopy class of the identity morphism
\[ DD := \text{id} : B^n U(1)_\nabla \to B^n U(1)_\nabla \]
is a differential cohomology operation called the (higher) Dixmier-Douady class, as this corresponds topologically to the fundamental cohomology class \( \iota_n \) in \( H^n(K(G, n); G) \). These higher classes are amplified in [FSS13] [FSS15a]. It is easy to see that this class refines the operations
\[ \mu_{n+1} := \text{id} : \Omega^{n+1}_\text{cl} \to \Omega^{n+1}_\text{cl}, \]
\[ \iota_{n+1} := \text{id} : B^{n+1} \mathbb{Z} \to B^{n+1} \mathbb{Z}, \]
where \( \mu_{n+1} \) and \( \iota_{n+1} \) are both the identity map, thought of as fundamental classes for the corresponding stacks. The latter is perhaps familiar from the representability of singular cohomology via Eilenberg-MacLane spaces. For the former, note that in stacks we can think of the stack of closed \( n \)-forms as representing a cohomology theory as well, but now in a more general sense.

**Example 2** (Power operation). Let \( m \) be a positive integer. We will consider the Deligne-Beilinson cup product \( \cup_{\text{DB}} \) on stacks (see [FSS13] [FSS15a] [GS16a]). Then the \( m \)-fold power gives a morphism of stacks
\[ DD^m := \cup_{\text{DB}} \cdots \cup_{\text{DB}} : B^n U(1)_\nabla \to B^{m(n+1)-1} U(1)_\nabla \]
as described in [FSS13]. The homotopy class of this map is, by definition, a differential cohomology operation. The cup product morphism refines the singular cup product and the wedge product of forms [FSS13][GS16a]. As a consequence, we immediately see that this operation refines the wedge product power and the cup product power, respectively, viewed as powers of the fundamental classes we encountered in example 1. Explicitly,
\[ \mu_{n+1}^m = \bigwedge_{m-\text{times}}^{\cdots} \bigwedge_{m-\text{times}} : \Omega^{n+1}_\text{cl} \to \Omega^{m(n+1)}_\text{cl}, \]
\[ \iota_{n+1}^m = \bigcup_{m-\text{times}}^{\cdots} \bigcup_{m-\text{times}} : B^{n+1} U(1) \to B^{m(n+1)} U(1). \]
The remainder of this section will be devoted to proving the following main classification theorem for differential cohomology operations.

**Theorem 6** (Characterization theorem). Let \( \widehat{\theta} \) be a differential cohomology operation. Then exactly one of the following holds:

1. \( \widehat{\theta} = n \text{DD} \), for some \( n \in \mathbb{Z} \).
2. \( \widehat{\theta} = n \text{DD}^m \), for some \( n \in \mathbb{Z} \).
3. \( \widehat{\theta} \) factorizes as
   \[
   \widehat{\theta} = j \phi I ,
   \]
where \( j : b\mathbb{B}^m U(1) \hookrightarrow \mathbb{B}^m U(1) \) is the flat inclusion, \( \phi : \mathbb{B}^{n+1} \mathbb{Z} \to b\mathbb{B}^m U(1) \) is an operation from singular cohomology to cohomology with \( U(1) \)-coefficients, and \( I \) is the canonical morphism
   \[
   I : \mathbb{B}^n U(1) \hookrightarrow \mathbb{B}^{n+1} \mathbb{Z}.
   \]

**Remark 4.** (i) This theorem is analogous to cohomology of integral Eilenberg-MacLane spaces being finite or not, depending on the degree.

(ii) Note that the morphisms \( j \), \( \phi \), and \( I \) can be described more classically along the lines of the presentation in the Introduction, and in fact generalizing those. Indeed, \( j \) is an operation from \( U(1) \cong \mathbb{R}/\mathbb{Z} \)-coefficients to differential cohomology (which can be viewed in a precise sense ‘as’ the Deligne complex \( \mathbb{Z}^{\infty}_{\mathbb{D}} \)), \( \phi \) is a map from \( \mathbb{Z} \)-coefficients to \( U(1) \)-coefficients, and \( I \) is a map from differential cohomology (as \( \mathbb{Z}^{\infty}_{\mathbb{D}} \)) to \( \mathbb{Z} \)-coefficients.

To prove the theorem, we will need to understand the rational and integral operations along with the operations on forms. We begin with a brief recollection of integral and rational operations.

Recall that the only operations that arise rationally are the identity and the power operations. Indeed, the rational cohomology ring of a rational Eilenberg-MacLane space is generated by a single generator (see e.g. [GM13] Lemma 8.5) and is a \( \mathbb{Q} \)-polynomial algebra or a \( \mathbb{Q} \)-exterior algebra, depending on parity,

\[
H^n(K(\mathbb{Q},n);\mathbb{Q}) = \begin{cases} 
\mathbb{Q}[\iota_{2m}], & n = 2m \text{ even} \\
\Lambda_{\mathbb{Q}}[\iota_{2m+1}], & n = 2m + 1 \text{ odd},
\end{cases} \tag{2.15}
\]

where \( \iota_q \) is the \( q \)th fundamental class.

The case of singular cohomology is of course more complicated. However, the situation is made much more tractable by the above rational considerations. In fact, the above implies that \( H^{n+q}(K(\mathbb{Z},n);\mathbb{Z}) \) must be finite when \( n \nmid q \). Otherwise, the rationalization would be nonzero in these degrees, which is not the case.

**Remark 5.** Summarizing this, along with other properties of these groups, we have (see e.g. [Ca54] [Pa66] [FFG86])

1. \( H^{n+q}(K(\mathbb{Z},n);\mathbb{Z}) \) is finite and independent of \( n \) for \( 0 < q < n \).
2. When \( n \nmid q \) then this is infinite cyclic generated by powers of the fundamental class.
3. The p-primary part of \( H^{n+q}(K(\mathbb{Z},n);\mathbb{Z}) \) is zero for \( 0 < q < 2p - 1 \)
4. If \( n < 2p - 1 \), the p-primary part of \( H^{n+2p-1}(K(\mathbb{Z},n);\mathbb{Z}) \) is cyclic of order \( p \), generated by the operation \( (\beta_p P^1_p)(u) \), where \( u \) is the fundamental class, \( P^1_p \) is the 1st \( P \) operation at the prime \( p \) and \( \beta_p \) is the Bockstein homomorphism for the mod-\( p \) sequence.
We now turn to the possible operations on forms, which turn out to be in harmony with the operations in rational cohomology.

**Lemma 7.** Let $\mu_n : \Omega^n_{cl} \to \Omega^n_{cl}$ denote the identity morphism on forms (thought of as a fundamental class for the sheaf of closed $n$-forms). The set $\pi_0 \text{Map}(\Omega^n_{cl}, \Omega^*_{cl})$ forms a graded algebra and we have

$$\pi_0 \text{Map}(\Omega^n_{cl}, \Omega^*_{cl}) = \begin{cases} \mathbb{R}[\mu_{2m}], & n = 2m \text{ even} \\ \Lambda_{\mathbb{R}}[\mu_{2m+1}], & n = 2m + 1 \text{ odd}, \end{cases}$$

where the asterisk $*$ on the left hand side is a grading that is determined by the powers on the right hand side.

**Proof.** Let $f : \Omega^n \to \Omega^*$ be a natural transformation of sheaves. In [NS15] it was shown that any assignment of differential forms $\omega \mapsto f(\omega)$, which is natural with respect to pullback, is given by a polynomial in $\omega$ and its derivative $d\omega$. Hence, if we restrict $f$ to the sheaf of closed forms, we see that $f$ must assigns each section $\omega \in \Omega^n_{cl}$, a polynomial in $\omega$. The claim is simply a restatement of this fact. \[\square\]

The next lemma will be needed in the proof of the main theorem, Theorem 6. Essentially, the lemma shows that the only differential cohomology operations detected by de Rham cohomology are the rational ones. The method of proof again appeals to the cohesive adjunction, which extracts the relevant information from the full moduli stack of $n$-bundles.

**Lemma 8.** We have

$$\pi_0 \text{Map}(B^nU(1)\nabla, b_{dR}B^mU(1)) = \begin{cases} \mathbb{R} & \text{if } n = 2k \mid m \text{ or } n = m = 2k + 1, \\ 0 & \text{otherwise.} \end{cases}$$

**Proof.** By de Rham's theorem, we have $b_{dR}B^mU(1) \simeq B^{m+1}\mathbb{R}$. Since $\mathbb{R}$ is discrete (as a stack), we have $b_{dR}B^{m+1}\mathbb{R} \simeq B^{m+1}\mathbb{R}$. By cohesion, we have equivalences

$$\text{Map}(B^nU(1)\nabla, b_{dR}B^mU(1)) \simeq \text{Map}(B^nU(1)\nabla, bB^{m+1}\mathbb{R}) \quad \text{by above}$$

$$\simeq \text{Map}(\Pi(B^nU(1)\nabla), B^{m+1}\mathbb{R}) \quad \text{(by (2.7))}$$

$$\simeq \text{Map}(B^{n+1}\mathbb{Z}, B^{m+1}\mathbb{R})$$

The claim then follows from the properties of Eilenberg-MacLane spaces, specifically the structure in (2.15) and the first part of Remark 5. \[\square\]

We are now ready to prove Theorem 6 We will find that the first case is straightforward due to the appeal to de Rham theory, while for the second case things become very subtle due to torsion.

**Proof.** Let $\hat{\theta}$ be a differential cohomology operation. By Lemma 7 we have two possibilities for the corresponding operation $\tau$ on forms.

(i) $\tau = \lambda \mu^q_n$, $q \geq 0$, $\lambda \in \mathbb{R}$

First consider the case $\lambda = 1$. Then $\tau$ admits at least one refinement, since $D^n \theta$ refines this operation. To see that this is the only possibility, let $\hat{\theta}$ be another operation refining $\tau$. Then since $n + nq$ is a multiple of $n$, we have $H^{n+nq}(K(\mathbb{Z}, n); \mathbb{Z})$ is infinite cyclic, generated by the
The homotopy commutativity of (2.11) forces the underlying singular cohomology operation to be \( \theta = \iota^n \) and therefore \( \hat{\theta} \) is a refinement of both the wedge power and cup product. It is known (see e.g. [Bu12]) that the Deligne-Beilinson cup product is the unique refinement of these operations (up to homotopy) and \( \hat{\theta} = DD^q \).

For arbitrary \( \lambda \), recall that the curvature map is surjective onto closed forms with integral periods. Hence, for \( \lambda \not\in \mathbb{Z} \), the operation \( \lambda \mu^n R \) is not in the image of \( R \) and therefore \( \tau \) does not admit a differential refinement. For \( \lambda \in \mathbb{Z} \), \( \lambda DD^q \) defines a refinement and is again unique up to homotopy.

(ii) \( \tau = 0 \)

Let \( \hat{\theta} \) be a refinement of \( \tau = 0 \). Then \( R\hat{\theta} \simeq \tau R \simeq 0 \). Since \( b\mathcal{B}^mU(1) \) is the fiber of \( R \), \( \hat{\theta} \) must factor through the flat inclusion \( j : b\mathcal{B}^mU(1) \hookrightarrow \mathcal{B}^mU(1)_{\nabla} \). Call the factorizing map \( \phi' \). By the homotopy commutativity of the diagram (2.11) and using Proposition 5, we also have that

\[
\beta \phi I = \theta I \quad (\theta = \beta \phi \text{ from above})
\]

\[
= I \hat{\theta} \quad \text{(Prop. 5)}
\]

\[
= I j \phi' \quad \text{(factorization through flat inclusion)}
\]

\[
= \beta \phi' \quad \text{(stacky diamond (2.5))},
\]

which implies \( \phi I - \phi' \) is in the kernel of \( \beta \). By exactness in the stacky diamond (2.5), this implies that \( \phi I - \phi' \) is in the image of

\[
\exp : \pi_0 \Map(\mathcal{B}^n U(1)_{\nabla}, b\mathcal{B}^m U(1)) \to \pi_0 \Map(\mathcal{B}^n U(1)_{\nabla}, b\mathcal{B}^m U(1)) .
\]

If \( m \neq kn \), for some \( k > 0 \), then the group on the left is zero by Lemma 8. Hence, \( \phi I = \phi' \) and

\[
j \phi I = j \phi' = \hat{\theta} .
\]

If \( m = kn \), then the group on the left of (2.16) is isomorphic to \( \mathbb{R} \), again by Lemma 8. In this case, let us recall from Prop. 3 that we have an equivalence

\[
b\mathcal{B}^n U(1) \simeq \mathcal{B}^n U(1)^{\delta} \simeq b\mathcal{B}^m U(1)^{\delta} .
\]
Then, by cohesion, we have
\[\pi_0 \text{Map}(\mathcal{B}^n U(1) \vee \mathcal{B}^{kn+1} U(1)) \simeq \pi_0 \text{Map}(\Pi \mathcal{B}^n U(1) \vee, \mathcal{B}^{kn} U(1) \downarrow) \quad (\text{by above and (2.7)})\]
\[\simeq \pi_0 \text{Map}(\mathcal{B}^{n+1} \mathbb{Z}, \mathcal{B}^{kn+1} U(1)) \quad (\text{by (2.7)})\]
\[\simeq \pi_0 \text{Map}(\mathcal{B}^{n+1} \mathbb{Z}, \mathcal{B}^{kn} U(1)) \quad (\text{by above}). \quad (2.17)\]

Again, the isomorphism is provided by precomposing with \(I : \text{id} \to \Pi\) (eq. (2.8)). Now let \(\varphi\) be such that \(\phi I - \phi' = \exp(\varphi)\). By the above isomorphism (2.17), it follows that there is \(\phi'' \in \pi_0 \text{Map}(\mathcal{B}^{n+1} \mathbb{Z}, \mathcal{B}^{kn+1} U(1))\), such that \(\exp(\varphi) = \phi'' I\). Hence, \(\phi I - \phi' = \exp(\varphi) = \phi'' I\), so that \(\phi' = (\phi - \phi'') I\), since \(I\) is a linear operation. Applying \(j\) to the latter equation gives the result. 

\[\square\]

3 Differential Steenrod operations

We now would like to apply and specialize the discussion in the previous section to describe differential cohomology operations which refine the classical Steenrod squares (see Sec. 2.1). That is, we seek differential cohomology operations \(\tilde{\theta}_i\) such that
\[\rho I \tilde{\theta}_k = \text{Sq}^k \rho I. \quad (3.1)\]

Here, \(\rho_2 : \mathbb{Z} \to \mathbb{Z}/2\) denotes the mod 2 reduction morphism. Rephrasing this diagrammatically, we aim for a commutative diagram

\[
\begin{array}{ccc}
\mathcal{B}^n U(1) \vee & \to & \mathcal{B}^{n+1} \mathbb{Z} \\
\theta_k & \downarrow & \downarrow \rho_2 \\
\mathcal{B}^{n+k} U(1) \vee & \to & \mathcal{B}^{n+1+k} \mathbb{Z} \\
\end{array}
\]
\[\text{Sq}^k \quad \rho_2 \quad \text{Sq}^k \]

(3.2)

In the previous section, we saw that every differential cohomology operation refines a singular operation. Therefore, we can fill in the middle vertical arrow and ask for the entire diagram to commute up to homotopy.

However, as we saw in the Introduction, the homotopy commutativity of the right square is too much to ask in general. That is, not every \(\mathbb{Z}/2\) operation admits an integral refinement. For example, the operations
\[\text{Sq}^{2k} : H^n (-; \mathbb{Z}/2) \to H^{n+2k} (-; \mathbb{Z}/2)\]
cannot have an integral refinement. Otherwise the operation would be in the image of the mod 2 reduction map and, by exactness, the Bockstein \(\beta(\text{Sq}^{2m})\) (relating integral to mod 2 coefficients) would vanish. This is not the case, however, since the Adem relations imply
\[0 \neq \text{Sq}^{2k+1} = \text{Sq}^{2} \text{Sq}^{2m} = (\rho_2 \beta) \text{Sq}^{2k}.\]

It therefore does not make sense to refine the even Steenrod squares.

**Proposition 9.** The even Steenrod squares do not admit differential refinements.
Put another way, when refining mod 2 operations (or mod \( p \) in general), one first needs an integral refinement. If such an integral lift exists, then one can ask for a differential refinement.

Given the characterization theorem, Theorem 6 established in the previous section, we can identify what these classes must be for odd Steenrod squares relatively easily.

**Lemma 10.** The odd integral Steenrod operations \( Sq_{2k}^{2k+1} : H^n(-; \mathbb{Z}) \to H^{n+2k+1}(-; \mathbb{Z}) \) factorizes uniquely as

\[
\theta : H^n(-; \mathbb{Z}) \xrightarrow{\rho_2} H^n(-; \mathbb{Z}/2) \xrightarrow{Sq_{2k}} H^{n+2k}(-; \mathbb{Z}/2) \xrightarrow{\Gamma_2} H^{n+2k}(-; U(1)) \xrightarrow{\tilde{\beta}} H^{n+2k+1}(-; \mathbb{Z}) ,
\]

where \( \Gamma_2 \) is induced by the representation \( \mathbb{Z}/2 \to U(1) \) as the square roots of unity, and \( \tilde{\beta} \) is the Bockstein corresponding to the exponential sequence \( \mathbb{Z} \to \mathbb{R} \to U(1) \).

**Proof.** Recall that \( Sq_{2k}^{2k+1} \) is defined as the operation \( \beta Sq_{2k} \rho_2 \), where \( \beta \) is the Bockstein corresponding to the sequence

\[
\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\rho_2} \mathbb{Z}/2 .
\]

Now consider the morphism of short exact sequences

\[
\begin{array}{cccccc}
0 & \to & \mathbb{Z} & \xrightarrow{\times 2\pi i} & \mathbb{R} & \xrightarrow{\text{exp}} & U(1) & \to & 0 \\
\downarrow \text{id} & & \downarrow \pi i & & \downarrow \Gamma_2 & & & & \\
0 & \to & \mathbb{Z} & \xrightarrow{\times 2} & \mathbb{Z}/2 & \xrightarrow{\rho_2} & 0 .
\end{array}
\]

This morphism induces a morphism of long fibration sequences involving the Bockstein homomorphisms

\[
\begin{array}{cccccc}
\ldots & \to & B^{n-1} \mathbb{Z} & \xrightarrow{\times 2\pi i} & B^{n-1} \mathbb{R} & \xrightarrow{\text{exp}} & B^{n-1} U(1) & \xrightarrow{\tilde{\beta}} & B^n \mathbb{Z} & \to & \ldots \\
\uparrow \text{id} & & \uparrow \times \pi i & & \uparrow \Gamma_2 & & \uparrow \beta & & \uparrow \text{id} & & \\
\ldots & \to & B^{n-1} \mathbb{Z} & \xrightarrow{\times 2} & B^{n-1} \mathbb{Z} & \xrightarrow{\rho_2} & B^{n-1} \mathbb{Z}/2 & \xrightarrow{\beta} & B^n \mathbb{Z} & \to & \ldots .
\end{array}
\]

The homotopy commutativity of the right square gives the desired factorization. Uniqueness follows from the definition of \( Sq_{2k}^{2k+1} \), along with the fact that every stable operation \( \phi : B^{n-1} \mathbb{Z}/2 \to B^{n-1} U(1) \) is induced by a representation \( \Gamma_2 : \mathbb{Z}/2 \to U(1) \) (of which there is only 1). \(\square\)

As a corollary of Proposition 10 and the characterization theorem, Theorem 6 we have the following.

**Corollary 11.** Let \( \tilde{\theta}_{2k+1} \) be a cohomology operation refining the odd integral Steenrod square \( Sq_{2k}^{2k+1} \). Then we have

\[
\tilde{\theta}_{2k+1} = j \Gamma_2 Sq_{2k} \rho_2 I ,
\]

so that we can define the refinement as \( \tilde{Sq}_{2k}^{2k+1} := \tilde{\theta}_{2k+1} \). Diagrammatically, we have

\[
\begin{array}{c}
B^n U(1) \\
\downarrow \text{id} \\
B^{n+1} \mathbb{Z} \\
\downarrow \rho_2 \\
B^{n+1} \mathbb{Z}/2 \\
\downarrow \Gamma_2 \\
B^{n+2k} \mathbb{Z}/2
\end{array}
\xrightarrow{\partial} \begin{array}{c}
B^{n+2k} U(1) \\
\downarrow j \\
\partial B^{n+2k} U(1)
\end{array}.
\]
**Proof.** Since $\hat{\theta}$ refines $Sq^{2k+1}_\mathbb{Z}$, which takes values in torsion, we have $i(Sq^{2k+1}_\mathbb{Z}) = 0$. The homotopy commutativity of diagram (2.11) implies the corresponding operation on forms $\tau = 0$. By theorem 6, we must have that $\hat{\theta}_{2k+1} = j\phi I$, for some operation $\phi : B^{n+1} \mathbb{Z} \to \mathbb{B}_n^{n+2k}U(1)$. By proposition 10, Proposition 10 implies that $\phi$ must be $\Gamma_2 Sq^{2k} \rho_2$. 

3.1 Relationship with the Deligne-Beilinson cup product

In the previous section we established that the only differential refinement of the odd Steenrod squares is given by the operation $\hat{\theta}_{2k+1} := j\Gamma_2 Sq^{2k} \rho_2 I$. Hence, from the point of view of refinement our work is done. However classically, we know that the Steenrod squares are related to the homotopy commutativity of the cup product. One could ask whether or not the differential Steenrod squares are related to the homotopy commutativity of the Deligne-Beilinson cup product. This section can be viewed as a refinement of the second classical point of view on the Steenrod squares presented in Sec. 2.1.

In fact, it is already known [Go08] that if $\hat{x}$ is a differential cohomology class of degree $2n + 1$, then the Deligne-Beilinson square cup $\hat{x}^2$ is related to the image of the Steenrod square $Sq^{n-1}_\mathbb{Z}$ in differential cohomology via the map $j\Gamma_2$, introduced in the previous section. At the end of the section, we generalize the result of Gomi [Go08].

Let $X$ be a manifold. As outlined in [FSS13], we have a cup product morphism in differential cohomology

$$X \xrightarrow{\Delta} X \times X \xrightarrow{\hat{x} \times \hat{x}} B^n \mathbb{U}(1)_\nabla \times B^n \mathbb{U}(1)_\nabla \xrightarrow{\cup_{\text{DB}}} B^{2n+1} \mathbb{U}(1)_\nabla.$$ 

As in the classical case, the Deligne-Beilinson cup product is not strictly graded commutative, but is graded commutative up to homotopy. That is, we have a homotopy commutative diagram in stacks

If we choose homotopies and higher coherence homotopies filling the diagram, we can equivalently express this by saying that $B^{2n+1} \mathbb{U}(1)_\nabla$ is an $(\infty, 1)$-cocone over the the diagram given by the $\mathbb{Z}/2$-action (call it $\psi$) on $X \times X$ via the above transposition map. If we take the colimit over this $\mathbb{Z}/2$-action, then the universal property of the colimit will ensure that there is a map (unique up to homotopy) from this colimit to $B^{2n+1} \mathbb{U}(1)_\nabla$. That is, we have the following.

**Lemma 12.** The colimit of the $\mathbb{Z}_2$-action $\psi$ (described above) sits a homotopy commuting diagram.
Remark 6. The colimit here serves to extract the homotopies involved in the \( \mathbb{Z}/2 \)-action. The map \( \hat{\lambda} \) attaches homotopies involved with the cup product to these homotopies.

Although there may be several ways to compute this colimit, we will make use of the cohesive structure on smooth stacks to perform the calculation.

Proposition 13. Let \( Y \) be a stack equipped with an action of \( \mathbb{Z}/2 \), that is, a functor \( \psi : \mathbb{Z}/2 \to \mathcal{S}h_\infty(\text{CartSp}) \) sending the unique object \( * \in \mathbb{Z}/2 \) to the stack \( Y \). The colimit over this functor is computed as

\[
hocolim(\psi) \simeq E\mathbb{Z}/2 \times_\psi Y ,
\]

where \( E\mathbb{Z}/2 = \text{disc}(S^\infty) \) is the discrete universal principal \( \mathbb{Z}/2 \)-bundle over the discrete stack \( B\mathbb{Z}/2 = \text{disc}(\mathbb{R}P^\infty) \).

Proof. Since the prestack category \([\text{CartSp}, s\text{Set}]\) is combinatorial and simplicial, the homotopy colimit in prestacks is presented by the local homotopy colimit

\[
hocolim_{\text{local}}(\psi) = \int^{* \in \mathbb{Z}/2} \mathcal{N}((\mathbb{Z}/2)/*) \odot \psi(*) .
\]

Here, \( \mathcal{N} \) denotes the nerve while \( \odot \) denotes the tensoring of a stack and a simplicial set. To compute the right hand side, we observe that the tensoring of a prestack \( Y \) and a simplicial set \( X \) is provided by taking the product with the constant stack

\[
X \odot Y := \text{const}(X) \times Y .
\]

Then the coend is computed as

\[
\int^{* \in \mathbb{Z}/2} \mathcal{N}((\mathbb{Z}/2)/*) \odot \psi(*) = \int^{* \in \mathbb{Z}/2} \text{const}(\mathbb{Z}/2) \times Y \equiv \text{const}(\mathbb{Z}/2) \times_\psi Y .
\]

The homotopy colimit was computed in prestacks. Since the stackification functor is a left \( \infty \)-adjoint, it preserves homotopy colimits and we need only compute the stackification of the prestack \( \text{const}(\mathbb{Z}/2) \times_\psi Y \). Since \( Y \) was assumed to be a stack, this is \( \text{disc}(\mathbb{Z}/2) \times_\psi Y \), as claimed. \( \square \)

Corollary 14. For the trivial action \( \psi \) on a stack \( Y \), we have

\[
hocolim(\psi) \simeq \text{disc}(B\mathbb{Z}/2) \times Y \simeq B\mathbb{Z}/2 \times Y .
\]
Returning to our discussion, we can now unravel the homotopies contained in the $\mathbb{Z}/2$-action.

**Proposition 15.** The stacky cup product map $X \to X \times X \to B^{2n+1}U(1)\nabla$ can be extended to a map $\hat{\lambda}$ making the diagram

\[
\begin{array}{ccc}
X \times B\mathbb{Z}/2 & \xrightarrow{Q(\Delta)} & X \times X \times \mathbb{Z}/2 \\
\uparrow & & \uparrow \\
X & \xrightarrow{\Delta} & X \times X \xrightarrow{\cup} B^{2n+1}U(1)\nabla
\end{array}
\]

commute up to homotopy. Moreover, given choices of homotopies and higher homotopies filling the diagram, $\hat{\lambda}$ is uniquely determined up to homotopy. Here, the two vertical maps are canonical sections of the projection $p_X : X \times B/\mathbb{Z}/2 \to X$ and $Q(\Delta)$ is yet to be determined.

**Proof.** Equip $X$ with the trivial $\mathbb{Z}/2$-action, and equip $X \times X$ with the action given by transposing the two factors. Then the diagonal

$\Delta : X \to X \times X$

defines a natural transformation of $\mathbb{Z}/2$-actions, and hence induces a map $Q(\Delta)$ on the corresponding homotopy colimits. Moreover, by the homotopy commutativity of the cup product, the map

$X \times X \xrightarrow{\hat{\lambda} \times \hat{\lambda}} B^nU(1)\nabla \times B^nU(1)\nabla \xrightarrow{\cup DB} B^{2n+1}U(1)\nabla$

commutes (up to homotopy) with the $\mathbb{Z}/2$-action. Given a choice of homotopies and higher homotopies, the universal property for $(\infty, 1)$-colimits produces a map $\hat{\lambda}$, defined uniquely up to homotopy, making the diagram commute. $\square$

To extract the Steenrod squares from this diagram, we will need to choose homotopies filling the diagram and study the composite map $\hat{\lambda}Q(\Delta)$. This is analogous to the classical case, where one produces such a diagram and then used the Künneth formula to compute the degree $2n$ cohomology of $X \times \mathbb{R}P^\infty$. The coefficients are then defined to be the Steenrod squares.

**Remark 7.** It is interesting to note that our method seems conceptually much simpler than the classical construction. However, we emphasize the fact that the explicit construction of the higher coherence homotopies would be just as complicated as in the classical case. Fortunately, we will be able to use the classical construction to our advantage for the choice of homotopies.

As indicated, we will need to make use of a Künneth-type theorem for differential cohomology. Although it is likely that such a theorem follows from a more general theorem for sheaf hypercohomology, this particular case does not require such machinery and we can prove the claim directly.

**Proposition 16.** (Künneth decomposition for differential cohomology) Let $X$ be a smooth manifold and let $Y = \text{disc}(Y')$, where $Y'$ is a simplicial set of finite type (only finitely many simplices in

\footnote{In the journal version of this article, the formulation of the Künneth theorem is false. The correct hypothesis on $Y$ is that it is a discrete simplicial presheaf of finite type. We have corrected this error in the present version.}
Then we have a natural short exact sequence

\[
0 \rightarrow \tilde{H}^n(X; \mathbb{Z}) \oplus \bigoplus_{i=1}^n H^{n-i-1}(X; U(1)) \otimes H^i(Y; \mathbb{Z}) \rightarrow \tilde{H}^n(X \times Y; \mathbb{Z}) \rightarrow \text{Tor}(\tilde{H}^n(X; \mathbb{Z}), H^1(Y; \mathbb{Z})) \oplus \bigoplus_{i=1}^n \text{Tor}(H^{n-i-1}(X; U(1)), H^{i+1}(Y; \mathbb{Z})) \rightarrow 0.
\]

Moreover, the sequence splits (but not naturally).

**Proof.** Let \( \{U_i\} \) be a good open covers of \( X \) and let \( \mathcal{C}'(\{U_i\}) \) be the Čech nerve of the cover. Since \( Y = \text{disc}(Y') \) for some simplicial set \( Y' \), it is cofibrant in the projective model structure on simplicial presheaves. Since this model category is cartesian, the product \( \mathcal{C}'(\{U_i\}) \times Y \) is a projective resolution of \( X \times Y \).

The total complex of the Čech-Deligne double complex of \( X \times Y \) is, by definition, the hom in unbounded chain complexes

\[
tot(\mathcal{C}_\bullet(\mathcal{C}'(\{U_i\}) \times Y); \mathbb{Z}_D^\infty(n + 1)) := \text{hom}_{\text{ch}}(\mathcal{C}_\bullet(\mathcal{C}'(\{U_i\}) \times Y)), \mathbb{Z}_D^\infty(n + 1)).
\]

where \( \mathcal{C}_\bullet : \text{PSh}_\infty(\text{CartSp}) \rightarrow \text{PSh}(\text{CartSp}; \text{sAb}) \rightarrow \text{PSh}_\infty(\text{CartSp}; \text{Ch}_+ \mathbb{Z}) \) is the (prolongation to presheaves) of the composition of the free functor and the Moore complex functor. Now the Eilenberg-Zilber map gives a chain homotopy equivalence

\[
\nabla : \mathcal{C}_\bullet(\mathcal{C}'(\{U_i\})) \otimes \mathcal{C}_\bullet(Y) \xrightarrow{\sim} \mathcal{C}_\bullet(\mathcal{C}'(\{U_i\}) \times Y).
\]

Taking the hom in chain complexes to the Deligne complex it follows that we have an equivalence

\[
\nabla : \text{hom}_{\text{ch}}(\mathcal{C}_\bullet(\mathcal{C}'(\{U_i\}) \times Y), \mathbb{Z}_D^\infty(n + 1)) \xrightarrow{\sim} \text{hom}_{\text{ch}}(\mathcal{C}_\bullet(\{U_i\}) \otimes \mathcal{C}_\bullet(Y), \mathbb{Z}_D^\infty(n + 1)).
\]

Since \( Y' \) is finitely generated in each simplicial degree, the appropriate finiteness assumption is satisfied and the canonical map

\[
\text{hom}_{\text{ch}}(\mathcal{C}_\bullet(\{U_i\}), \mathbb{Z}_D^\infty(n + 1)) \otimes \text{hom}_{\text{ch}}(\mathcal{C}_\bullet(Y), \mathbb{Z}) \rightarrow \text{hom}_{\text{ch}}(\mathcal{C}_\bullet(\{U_i\}) \otimes \mathcal{C}_\bullet(Y), \mathbb{Z}_D^\infty(n + 1))
\]

is a quasi isomorphism.

Write

\[
C^j(Y; \mathbb{Z}) = \text{hom}_{\text{ch}}(\mathcal{C}_\bullet(Y), \mathbb{Z})_{-j}
\]

for the Čech cochain complex of \( Y \). For the total Čech Deligne complex, we have

\[
tot_{n-j}(\mathcal{C}_{\bullet\bullet}(\{U_i\}; \mathbb{Z}_D^\infty(n + 1))) = \text{hom}_{\text{ch}}(\mathcal{C}_\bullet(\{U_i\}), \mathbb{Z}_D^\infty(n + 1))_{j}.
\]

Then

\[
\bigoplus_{i+j=0} H_i(\text{hom}_{\text{ch}}(\mathcal{C}_\bullet(\{U_i\}), \mathbb{Z}_D^\infty(n + 1)) \otimes H_j(\text{hom}_{\text{ch}}(\mathcal{C}_\bullet(Y), \mathbb{Z}))
\]

\[
= \bigoplus_{i+j=0} H^{n-i-1}(X; \mathbb{Z}_D^\infty(n + 1)) \otimes H^{-j}(Y; \mathbb{Z})
\]

\[
= \tilde{H}^n(X; \mathbb{Z}) \oplus \bigoplus_{i=1}^n H^{n-i-1}(X; U(1)) \otimes H^i(Y; \mathbb{Z})
\]
and
\[ \bigoplus_{i+j=-1} \text{Tor} \left( H_i(\hom_{ch}(C_\bullet\{U_i\}), \mathbb{Z}_p(n+1)), H_j(\hom_{ch}(C_\bullet(Y), \mathbb{Z})) \right) \]
\[ = \bigoplus_{i+j=-1} \text{Tor} \left( H^{n-i}(X; \mathbb{Z}_p(n+1)), H^{-j}(Y; \mathbb{Z}) \right) \]
\[ = \text{Tor}(\hat{H}^n(X; \mathbb{Z}), H^1(Y; \mathbb{Z})) \oplus \bigoplus_{i=1}^n \text{Tor} \left( H^{n-i-1}(X; U(1)), H^{i+1}(Y; \mathbb{Z}) \right) . \]

Now the result follows from the usual K"unneth formula. \( \square \)

We now adapt the general K"unneth decomposition to the case directly related to Steenrod squares.

**Proposition 17.** For a smooth manifold \( X \), we have
\[ \hat{H}^{2n}(X \times \mathbb{B}Z/2; \mathbb{Z}) \simeq \hat{H}^{2n}(X; \mathbb{Z}) \oplus \bigoplus_{j < 2n \text{ even}} T_j^2, \]
where \( T_j^2 \) is the 2-torsion subgroup of \( \hat{H}^i(X; \mathbb{Z}) \).

**Proof.** Applying Proposition 16 with \( Y = \mathbb{B}Z/2 \), we have
\[ \hat{H}^{2n}(X \times \mathbb{B}Z/2; \mathbb{Z}) \cong \hat{H}^{2n}(X; \mathbb{Z}) \oplus \bigoplus_{i=1}^n H^{2n-i-1}(X; U(1)) \otimes H^i(\mathbb{B}Z/2; \mathbb{Z}) \oplus \]
\[ \bigoplus \text{Tor} \left( \hat{H}^{2n}(X; \mathbb{Z}), H^1(\mathbb{B}Z/2; \mathbb{Z}) \right) \oplus \bigoplus_{i=1}^{2n} \text{Tor} \left( H^{2n-i-1}(X; U(1)), H^{i+1}(\mathbb{B}Z/2; \mathbb{Z}) \right) . \]

It remains to identify these groups. We first note that we have an isomorphism
\[ H^i(X, U(1)) \otimes \mathbb{Z}/2 \simeq 0 \] (3.3)
for all \( i \). To see this, consider the short exact sequence \( \mathbb{Z} \xrightarrow{x^2} \mathbb{Z} \to \mathbb{Z}/2 \). Tensoring with \( H^i(X; U(1)) \) leads to the sequence
\[ H^i(X, U(1)) \overset{x^2}{\longrightarrow} H^i(X; U(1)) \to H^i(X; U(1)) \otimes \mathbb{Z}/2 \to 0 . \]
But since we have \( U(1) \) coefficients, the map \( x^2 \) is surjective and the first isomorphism theorem confirms the claim (3.3).

Now recall the integral cohomology groups of the classifying space \( K(\mathbb{Z}_2, 1) = \mathbb{RP}^\infty \)
\[ H^i(\mathbb{RP}^\infty; \mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0, \\ \mathbb{Z}/2 & i \text{ even } \neq 0, \\ 0 & \text{otherwise.} \end{cases} \]
Combining this with equation (3.3) gives
\[ \hat{H}^{2n}(X; \mathbb{Z}) \oplus \bigoplus_{i=1}^n H^{2n-i-1}(X; U(1)) \otimes H^i(\mathbb{B}Z/2; \mathbb{Z}) \simeq \hat{H}^{2n}(X; \mathbb{Z}) . \]
Then the Tor groups are easily computed

\[
\text{Tor}(\hat{H}^{2n}(X;\mathbb{Z}), H^1(\mathbb{BZ}/2; \mathbb{Z})) \oplus \bigoplus_{i=1}^{2n} \text{Tor} \left( H^{2n-i-1}(X; U(1)), H^{i+1}(\mathbb{BZ}/2; \mathbb{Z}) \right) \\
\cong \bigoplus_{1 \leq i \leq 2n, \text{ odd}} \text{Tor} \left( H^{2n-i-1}(X; U(1)), \mathbb{Z}/2 \right) \\
\cong \bigoplus_{j < 2n \text{ even}} T^j_2.
\]

In fact, we can be a bit more precise about what the torsion groups \(T^j_2\) actually look like. To that end, let us start by recalling the following.

**Lemma 18.** As a ring, the integral cohomology of \(\mathbb{BZ}/2\) takes the form

\[
H^*(\mathbb{BZ}/2; \mathbb{Z}) \cong H^*(\mathbb{RP}^\infty; \mathbb{Z}) \cong \mathbb{Z}[x]/\langle 2x \rangle,
\]

where \(x\) is an integral lift of \(w_1\).

The statement of the lemma is classical and can be established in various ways. For example, one can view \(\mathbb{RP}^\infty\) as a Grassmannian \(G_{2m+1}(\mathbb{R}^\infty)\) with \(m = 0\), for which the Bockstein exact sequence

\[
\ldots \longrightarrow H^j(-; \mathbb{Z}) \xrightarrow{\times 2} H^j(-; \mathbb{Z}) \xrightarrow{\rho_2} H^j(-; \mathbb{Z}/2) \xrightarrow{\beta} H^{j+1}(-; \mathbb{Z}) \longrightarrow \ldots
\]

implies that the integral cohomology \(H^*(G_{2m+1}(\mathbb{R}^\infty); \mathbb{Z})\) splits additively as the direct sum of a polynomial \(\mathbb{Z}[p_1, \ldots, p_m]\) and the image of \(\beta\) (see [MS74] Problem 15-C). For \(m = 0\) this then gives that the integral cohomology of \(\mathbb{RP}^\infty\) is the image of \(\beta\). This can also be deduced via chain complexes (see [Hat02] p. 222). A third way is to consider the Gysin sequence for integral cohomology corresponding to the circle bundle \(S^1 \to S^\infty \xrightarrow{\pi} \mathbb{RP}^\infty\), that is

\[
\ldots \longrightarrow H^n(S^\infty; \mathbb{Z}) \xrightarrow{\pi_*} H^{n-1}(\mathbb{RP}^\infty; \mathbb{Z}) \xrightarrow{\cup e} H^{n+1}(\mathbb{RP}^\infty; \mathbb{Z}) \xrightarrow{\pi_*} H^{n+1}(S^\infty; \mathbb{Z}) \longrightarrow \ldots,
\]

where \(\pi^*\) is pullback, \(\pi_*\) is pushforward, and \(e\) is the Euler class of the circle bundle. The latter gives an isomorphism between all even degree cohomology groups of \(\mathbb{RP}^\infty\). Then the Euler class \(e\), being 2-torsion, gives the desired result.

Going back to the 2-torsion subgroup \(T^j_2\), for even \(j\), the torsion pairing is given explicitly as

\[
\text{Tor} \left( H^{2n-i-1}(X; U(1)), H^{i+1}(\mathbb{BZ}/2; \mathbb{Z}) \right) \cong \text{Tor} \left( H^{2n-j}(X; U(1)), \mathbb{Z}/2 \langle x^j \rangle \right).
\]

Now the sequence

\[
0 \to \mathbb{Z} \langle x^j \rangle \xrightarrow{\times 2} \mathbb{Z} \langle x^j \rangle \to \mathbb{Z}/2 \langle x^j \rangle \to 0
\]

is a projective resolution. Therefore, the torsion pairing is given as the kernel of the map

\[
\times 2 : H^{2n-j}(X; U(1)) \otimes \mathbb{Z} \langle x^j \rangle \to H^{2n-j}(X; U(1)) \otimes \mathbb{Z} \langle x^j \rangle
\]

which is spanned by elements of the form \(y \otimes x^j\), with \(y\) a 2-torsion element in \(H^{2n-j}(X; U(1))\).

Finally, let us return to our analysis of the map

\[
\hat{\lambda}Q(\Delta) : X \times \mathbb{BZ}/2 \to \mathbb{B}^{2n}U(1)_\nabla,
\]

25
defined in the diagram of Proposition 15. By Proposition 17 and the above discussion, we see that we can expand the class of the map \( \hat{\lambda}Q(\Delta) \) as a homogeneous polynomial

\[
[\hat{\lambda}Q(\Delta)] = \hat{s}_n^{2n} + \hat{s}_n^{2n-2} \otimes x + \ldots + \hat{s}_n^2 \otimes x^{2n-2} + \hat{s}_n^0 \otimes x^{2n}, \tag{3.4}
\]

where each \( \hat{s}^{2k} \) represents a differential cohomology operation. Although we know the general form that the map \( \hat{\lambda}Q(\Delta) \) takes, the homotopy class of this map still depends on an explicit choice of homotopies and higher homotopies. For now, we leave these choices implicit and return to this point later.

**Definition 19.** Define the operations/classes \( \hat{s}_n^{2k} : B^nU(1) \to B^kU(1) \) by the expansion \( (3.4) \).

**Remark 8.** (i) Notice that naturality of the cup product implies that the classes \( \hat{s}_n^{2k} \) are natural with respect to pullback and hence define differential cohomology operations.

(ii) Notice that for \( k < n \), the operation \( \hat{s}_n^{2k} \) must represent a trivial class. Indeed, since \( \hat{s}_n^{2k} \) has image in 2-torsion the curvature vanishes

\[
2R(\hat{s}_n^{2k}) = R(2\hat{s}_n^{2k}) = R(0) = 0,
\]

indicating that the class \( \hat{s}_n^{2k} \) takes values in flat bundles. It follows that the map \( \hat{s}_n^{2k} \) factorizes through the stack \( B^kU(1)^{\delta} \), representing cohomology with \( U(1) \)-coefficients. Since there are no degree-decreasing cohomology operations with \( U(1) \)-coefficients, the class of \( \hat{s}_n^{2k} \) must be trivial in this case.

It remains to identify the classes \( \hat{s}_n^{2k} \) for \( k > n \). We start with the top class.

**Proposition 20.** The class \( \hat{s}_n^{2n}(\hat{x}) \) defined by the polynomial expression \( (3.4) \) can be identified with the cup product \( \hat{x} \cup \hat{x} \).

**Proof.** By the homotopy commutativity of the diagram \( (15) \), the pullback of the class \([\hat{\lambda}Q(\Delta)]\) by the canonical section \( X \to X \times BZ/2 \) is the cup product. This pullback simply restricts a class in \( \bar{H}^{2n}(X \times B/Z/2; \mathbb{Z}) \) to \( X \). From the polynomial expansion of \([\hat{\lambda}Q(\Delta)]\), it is apparent that this class is \( \hat{s}_n^{2n}(\hat{x}) \).

Recalling that the classes \( \hat{s}_n^{2l} \) were left undetermined, we have the following.

**Proposition 21.** There is a choice of homotopy commutative diagram \( (15) \) such that

\[
\hat{s}_n^{2k} = \hat{S}^{2k+1}q = j\hat{\Gamma}_2\hat{S}q^{2k}\rho_2 I.
\]

Moreover, when \( n \) is odd, these homotopies uniquely refine the homotopies involved in the classical case.

**Proof.** Since the map \( \hat{\lambda}Q(\Delta) \) was left ambiguous, we simply define \( \hat{s}_n^{2k} \) as needed. The homotopy class of this map then determines a homotopy commutative diagram \( (15) \).
To see how these relate to the homotopies involved in the classical case. Observe that since the DB cup product refines the singular cup product, we have a homotopy commutative diagram

\[
\begin{array}{ccc}
X \times B\mathbb{Z}/2 & \xrightarrow{Q(\Delta)} & X \times X \times_{\mathbb{Z}/2} E\mathbb{Z}/2 \\
\downarrow \Delta & & \downarrow \hat{\Delta} \\
X \times X & \xrightarrow{\hat{\delta} \times \hat{\delta}} & B^{n-1}U(1)_{\text{conn}} \times B^{n-1}U(1)_{\nabla} \\
\downarrow \rho_2 \circ I \times \rho_2 \circ I & & \downarrow \rho_2 \circ I \\
B^n\mathbb{Z}/2 \times B^n\mathbb{Z}/2 & \xrightarrow{\cup_{\text{DB}}} & B^{2n}\mathbb{Z}/2.
\end{array}
\]

Using the polynomial expansion of \([\hat{\lambda}Q(\Delta)]\), we can write the homotopy class of the upper top-left to bottom-right composite as

\[
\rho_2 I[\hat{\lambda}Q(\Delta)] = [\rho_2 I\hat{\lambda}Q(\Delta)]
\]

\[
= \rho_2 I \left( \hat{s}_n^{2n} + \hat{s}_n^{2n-2} \otimes x + \ldots + \hat{s}_n^2 \otimes x^{n-1} + \hat{s}_n^0 \otimes x^n \right)
\]

\[
= \rho_2 I \left( \tilde{s}_n^{2n} + \tilde{s}_n^{2n-2} \otimes x + \ldots + \tilde{s}_n^{n+1} \otimes x^{(n-1)/2} + \tilde{s}_n^n \otimes x^{n/2} + \ldots \right)
\]

\[
= (\rho_2 I\tilde{s}_n^{2n}) + (\rho_2 I\tilde{s}_n^{2n-2}) \otimes w_1^2 + \ldots + (\rho_2 I\tilde{s}_n^{n+2}) \otimes w_1^{n-2} + (\rho_2 I\tilde{s}_n^n) \otimes w_1^n.
\]

Now the using the classical construction of the Steenrod squares discussed in we recall that there is a map from the top-left corner to the bottom-right given by

\[
[Q(\Delta)\lambda] = Sq^n\rho_2 I + Sq^{n-1}\rho_2 I \otimes w_1 + \ldots + Sq^1\rho_2 I \otimes w_1^{n-1} + w_1^n.
\]

We would like to compare this polynomial with the previous one to identify the coefficients. Unfortunately, the map \(Q(\Delta)\lambda\) can not be homotopic to \(\rho_2 I\hat{\lambda}Q(\Delta)\). This is immediately clear from the fact that \(Sq^k\) is not in the image of the mod 2-reduction for even \(k\). This also reflects the fact that we cannot choose homotopies and higher homotopies filling the top diagram which are mapped to the right homotopies in the classical, outer diagram. However, we can split the map \(Q(\Delta)\lambda\) into two parts depending on the parity of the exponent of \(Sq^k\). That is, we define

\[
\lambda_0 = \sum_{k \text{ even}, k \leq n} Sq^k \otimes w_1^{n-k} \quad \text{and} \quad \lambda_1 = \sum_{k \text{ odd}, k \leq n} Sq^k \otimes w_1^{n-k}.
\]

Now recall that for an odd Steenrod square \(Sq^{2k+1}\), we have \(Sq^{2k+1} = Sq^1 Sq^k\). Since \(Sq^1 Sq^1 = 0\), we have that \(Sq^{2k+1}\) is in the kernel of \(Sq^1\). The equation \(Sq^1 = \rho_2 \beta\) relating \(Sq^1\) to the Bockstein \(\beta\) for the mod 2 reduction \(\mathbb{Z} \to \mathbb{Z}/2\) implies that \(Sq^{2k+1}\) must be in the image of the mod 2 reduction \(\rho_2\). Moreover, since \(n\) is odd, \(w_1^{n-k}\) is an even power when \(k\) is odd and \(w_1^{n-k}\) is in the image of the mod 2-reduction. Factorizing the map \(\lambda_1\) through the mod 2-reduction \(\rho_2\) and integration map \(I\), we can write

\[
\lambda_1 = \rho_2 I \alpha.
\]

By the universal property, \(\alpha\) determines a homotopy commutative diagram and, by expression \(3.4\), is a map of the form

\[
\alpha = \tilde{s}_n^{2n} + \tilde{s}_n^{2n-2} \otimes x + \ldots + \tilde{s}_n^2 \otimes x^{n-1} + \tilde{s}_n^0 \otimes x^n.
\]
Setting $\hat{\lambda}Q(\Delta) := \alpha$, we have
\[ \rho_2 I[\hat{\lambda}Q(\Delta)] = [\lambda_1] . \]

Comparing coefficients, we see that
\[ Sq^{2k+1} \rho_2 I = Sq^1 Sq^{2k} \rho_2 I = \rho_2 \beta Sq^{2k} \rho_2 I = \rho_2 I j Sq^{2k} \rho_2 I = \rho_2 I \hat{s}^{2k} . \]

Since both $Sq^{2k} \rho_2 I$ and $\hat{s}^{2k}$ take values in 2-torsion, we must have $j Sq^{2k} \rho_2 I = \hat{s}^{2k}$. Alternatively, since $\hat{s}^{2k}$ refines $Sq^{2k}$, we could use Theorem 8 to conclude that $\hat{s}^{2k}$ has the desired form. \(\square\)

In [Go08], it was observed that for an odd degree differential cohomology class $\hat{x}$, the Deligne-Beilinson square is given by the inclusion of the $(n-1)$st Steenrod square $Sq^{n-1} \rho_2 I(\hat{x})$ into differential cohomology via the representation of $\Gamma_2 : \mathbb{Z}/2 \hookrightarrow U(1)$ as the primitive square roots of unity. We provide another proof of this fact to highlight the power of the stacky perspective.

**Proposition 22.** For each $n > 0$, the Steenrod square $Sq^{2n}$ fits into a homotopy commutativity of the diagram:

\[ \begin{array}{ccc}
\mathbb{B}^{2n} U(1) \cup_2 & \to & \mathbb{B}^{4n+1} U(1) \\
\downarrow & & \downarrow \\
\mathbb{B}^{2n+1} \mathbb{Z} & \searrow & \mathbb{B}^{4n+1} \mathbb{Z}/2 \\
\downarrow \rho_2 & & \downarrow \beta \\
\mathbb{B}^{2n+1} \mathbb{Z}/2 & \to & \mathbb{B}^{4n+1} \mathbb{Z}/2
\end{array} \]

**Proof.** Let $DD_{2n+1}$ denote the Dixmier-Douady class in degree $2n+1$. Since this class has odd degree, the homotopy commutativity of the Deligne-Beilinson (DB) cup product implies that its square is 2-torsion, $2DD^2 = 0$. Consequently, the curvature obeys
\[ R(2DD^2) = 2R(DD)^2 = 0 . \]

This, in turn implies $R(DD)^2 = 0$. It follows that the square cup factorizes as
\[ DD^2 : \mathbb{B}^{2n} U(1) \cup_2 \to \mathbb{B}^{4n+1} U(1) \to \mathbb{B}^{4n+1} U(1) \cup_2 . \]

Since $DD^2$ is 2-torsion, it is killed by the $\times 2$ map. Given the Bockstein sequence associated to
\[ 0 \to \mathbb{Z}/2 \to U(1) \xrightarrow{\times 2} U(1) \to 0 , \]
we see that we have a further factorization
\[ DD^2 : \mathbb{B}^{2n} U(1) \cup_2 \to \mathbb{B}^{4n+1} \mathbb{Z}/2 \to \mathbb{B}^{4n+1} U(1) \to \mathbb{B}^{4n+1} U(1) \cup_2 . \]

Since the DB-cup product refines the classical cup product we can extend this map to a homotopy commutative diagram

\[ \begin{array}{cccc}
\mathbb{B}^{2n} U(1) \cup_2 & \to & \mathbb{B}^{4n+1} \mathbb{Z}/2 & \to & \mathbb{B}^{4n+1} U(1) \\
\downarrow \rho_2 I & & \downarrow \beta & & \downarrow \rho_2 \beta \\
\mathbb{B}^{2n+1} \mathbb{Z}/2 & \to & \mathbb{B}^{4n+1} \mathbb{Z}/2 & \to & \mathbb{B}^{4n+2} \mathbb{Z}/2 ,
\end{array} \]
where, recall, $\tilde{\beta}$ is the Beckstein corresponding to the exponential sequence. By Proposition 10 (or Theorem 6) there is an operation $\varphi : B^{2n}Z/2 \to B^{4n+1}Z/2$ which fills the left corner

\[
\begin{array}{ccc}
B^{2n}U(1)_{\text{conn}} & \xrightarrow{\rho_2 I} & B^{4n+1}U(1) \cong B^{2n}U(1)_{\text{conn}} \\
\xrightarrow{\varphi} & & \xrightarrow{\beta_2} \\
B^{2n+1}Z/2 & \xrightarrow{\gamma^2} & B^{4n+2}Z/2
\end{array}
\]

such that everything commutes up to homotopy. The homotopy commutativity of the bottom triangle, along with the fact that $Sq^{2n+1}(\iota) = \iota^2$, implies that $\varphi = Sq^{2n}$. The homotopy commutativity of the top part of the diagram proves the claim. \qed

### 3.2 Properties of the differential Steenrod operations

We now discuss general properties of the differential Steenrod squares. These properties can be directly deduced from the general form of these operations as

$$\tilde{Sq}^{2m+1} = j \Gamma_2 Sq^{2m} \rho_2 I,$$

but we make them explicit for the sake of completeness.

**Theorem 23** (Properties of differential Steenrod squares). The operations $\tilde{Sq}$ satisfy the following:

1. **Refinement**: The mod 2 reduction of the integral class $I \tilde{Sq}^{2m+1}$ is $Sq^{2m+1} \rho_2 I$.

2. **Torsion**: $\tilde{Sq}^{2m+1}$ takes values in 2-torsion.

3. **Connectivity**: $\tilde{Sq}^{2m+1} = 0$, $q < 0$.

4. **Linearity**: $\tilde{Sq}^{2m+1}(\tilde{x} + \tilde{y}) = \tilde{Sq}^{2m+1}(\tilde{x}) + \tilde{Sq}^{2m+1}(\tilde{y})$.

5. **Squaring**: For $\tilde{x}$ of degree $2n + 1$, we have $\tilde{Sq}^{2n+1}(\tilde{x}) = \tilde{x}^2$.

6. **Finiteness**: $\tilde{Sq}^{2m+1}(\tilde{x}) = 0$ when $q > n$.

7. **Adem relations**: For even integers $a$ and $b$, we have

$$\tilde{Sq}^a \tilde{Sq}^b = \sum_{c \text{ odd}} \binom{b - c - 1}{a - 2c} \tilde{Sq}^{a+b-c} \tilde{Sq}^c.$$ 

**Proof.** We have already proven properties (1), (2), (3), (5) and (6). For property (4), we have

$$\tilde{Sq}^{2m+1}(\tilde{x} + \tilde{y}) = j \Gamma_2 Sq^{2m} \rho_2 I(\tilde{x} + \tilde{y}).$$

Since the morphisms $I$, $\rho_2$, $\Gamma_2$ and $j$ are induced from homomorphisms of abelian groups, they represent linear operations. Since the classical Steenrod squares are linear, we immediately deduce that the right hand side is

$$j \Gamma_2 Sq^{2m} \rho_2 I(\tilde{x} + \tilde{y}) = j \Gamma_2 Sq^{2m} \rho_2 I(\tilde{x}) + j \Gamma_2 Sq^{2m} \rho_2 I(\tilde{y}).$$
which gives the result. To prove property (7), we have that, by definition, \( \hat{S}^a_q = j \Gamma_2 S^{a-1} \rho_2 I \). Since \( j \Gamma_2 \) is linear and takes values in 2-torsion, we have

\[
j \Gamma_2 \left( \sum_c \binom{b-c-1}{a-2c-1} S^{a+b-c-1} \right) S^a_q \rho_2 I = \sum_c \binom{b-c-1}{a-2c-1} \rho_2 I j \Gamma_2 S^{a+b-c-1} \rho_2 I \]

\[
= \sum_{c \, \text{odd}} \binom{b-c-1}{a-2c-1} j \Gamma_2 S^{a+b-c-1} \rho_2 I j \Gamma_2 S^{c-1} \rho_2 I 
\]

Recall that we have the relation on binomial coefficients

\[
\binom{b-c-2}{a-2c} = \binom{b-c-1}{a-2c-1} + \binom{b-c-1}{a-2c}.
\]

The left hand side is easily seen to be 0 mod 2. Hence

\[
\binom{b-c-1}{a-2c-1} = \binom{b-c-1}{a-2c} \mod 2.
\]

\[ \blacksquare \]

**Remark 9** (No identity). *Note that there is no refined Steenrod square which acts as an identity. One might be tempted to say \( \hat{S}^1_q \) is \( \text{Id} \) by looking at the definition (see expression (3.5)) and the fact that the identity on the classical Steenrod square is \( S^0 \). However, the effect of other morphisms acting on this 'classical identity' from both sides will affect it nontrivially. Indeed, from the diagram in Proposition 22, we have the following effect on a differential class \( \hat{x} \):

\[
\hat{x} \xrightarrow{\rho_2 I(\hat{x})} S^0 q(\hat{x}) \xrightarrow{\beta_2 = \rho_2 I \circ j \Gamma_2} \beta_2 \rho_2 I(\hat{x}) \, ,
\]

which is equal to \( S^1 q(\rho_2 I(\hat{x})) \), evidently not the identity. Of course this is just a confirmation that \( \hat{S}^1_q \) is a refinement of \( S^1 \). This lack of identity is one of several reasons why there should be no notion of a differential Steenrod algebra. One could also make such a statement already at the level of integral cohomology where the identity is certainly not 2-torsion.*

**Remark 10** (No Cartan formula). *A Cartan formula, which would expand \( \hat{S}(\hat{x} \cup DB \hat{y}) \) in terms of the product of \( \hat{S}(\hat{x}) \) and \( \hat{S}(\hat{y}) \), with appropriate combinations of degrees, does not seem to exist. There are several reasons for this. The most immediate is that the even Steenrod squares admit no refinements (Proposition 9). Explicitly, in expanding an odd Steenrod square into a sum of products, each one of the summands will be necessarily a product that involves an even differential Steenrod square, which does not exist. That is, it boils down to the basic fact that an odd number (the degree of the Steenrod square) cannot be split into a sum of two odd numbers.*

**Remark 11** (Steenrod reduced powers). *Similarly, for odd primes one has an analogue of Theorem 23 for the classical Steenrod reduced powers. We do not spell this out, as the proofs will be very similar to those of the above theorem, with obvious changes to coefficients.*
3.3 Applications

In this section, we offer several applications. It will be particularly useful to describe the differential Steenrod squares in terms of bundle data as done in \[\text{GS16}\] for the refined Massey products. In order to do this, we will need to make use of the stability of differential Steenrod squares under delooping. Note that differential cohomology does not obey the strict suspension isomorphism, i.e. in general

\[
\hat{H}^n(X; \mathbb{Z}) \not\cong \hat{H}^{n+1}(\Sigma X; \mathbb{Z}).
\]

However, we will see that there is a particular sense of stability which the refined Steenrod operations enjoy. Let us consider this in more precise terms. For an odd integer \(2n + 1\), the squaring operation is equal to the top Steenrod square

\[
\hat{Sq}^{2n+1} = \cup^2: \mathcal{B}^{2n+1}U(1)_{\nabla} \to \mathcal{B}^{4n+1}U(1)_{\nabla}.
\]

Now we can deloop this map \(k\) times to get an operation

\[
\mathcal{B}^k(\cup^2): \mathcal{B}^k(\mathcal{B}^{2n+1}U(1)_{\nabla}) \to \mathcal{B}^k(\mathcal{B}^{4n+1}U(1)_{\nabla}) \simeq \mathcal{B}^{4n+1+k}U(1)_{\nabla}.
\]

Note that delooping does not commute with having a connection in general, and this is indicated by the parentheses. However, the two operations commute when the stack is flat, as is indicated in the equivalence on the right hand side. Explicitly, we can write down what this map does on sections (at the level of chain complexes) using the formula for the DB-cup product; the map \(\mathcal{B}^k(\cup^2)\) is given by

\[
\mathcal{B}^k(\cup^2)(\alpha) = \begin{cases} 
\alpha^2 & \text{if deg}(\alpha) = 2n + k + 1, \\
\alpha \wedge d\alpha & \text{if deg}(\alpha) = 2n, \\
0 & \text{otherwise}.
\end{cases}
\]

Now the integration map \(I\) and mod 2 reduction \(\rho_2\) clearly commute with delooping. Furthermore, since the differential Steenrod squares refine the classical one, we have

\[
\rho_2 I \mathcal{B}^k(\cup^2) = \rho_2 I \mathcal{B}^k(\hat{Sq}^{2n+1}) = \mathcal{B}^k(\rho_2 I \hat{Sq}^{2n+1}) = \mathcal{B}^k(\hat{Sq}^{2n+1} \rho_2 I).
\]

Then stability of the classical Steenrod squares implies that the right hand side is just \(\hat{Sq}^{2n+1} \rho_2 I\). Hence, the stacky delooping \(\mathcal{B}^k(\hat{Sq}^{2n+1})\) refines the classical operation \(\hat{Sq}^{2n+1}\). However, the source of this operation is not the stack \(\mathcal{B}^{2n+1+k}U(1)_{\nabla}\). To remedy this, we simply note that we have a (in fact, strictly) commutative diagram

\[
\begin{array}{ccc}
\mathcal{B}^{2n+k}U(1)_{\nabla} & \xrightarrow{pr} & \mathcal{B}^k(\mathcal{B}^{2n}U(1)_{\nabla}) \\
\downarrow I & & \downarrow I \\
\mathcal{B}^{2n+k+1}Z & & \mathcal{B}^{2n+k+1}Z
\end{array}
\]

where \(pr\) is the projection map induced from the morphism of chain complexes

\[
\begin{array}{cccccccc}
Z & \xrightarrow{\Omega^0} & \cdots & \xrightarrow{\Omega^{2n}} & \Omega^{2n+1} & \cdots & \Omega^{2n+k} \\
\downarrow & & & & & & & \\
Z & \xrightarrow{\Omega^0} & \cdots & \xrightarrow{\Omega^{2n}} & 0 & \cdots & 0
\end{array}
\]

\[\text{This view on stacks has been used in } \text{[FRS13]}, \text{ where interesting results on geometric quantization arise by considering e.g. } \mathcal{B}(\mathcal{B}U(1)_{\text{conn}}) \text{ as the 2-stack whose sections are } U(1)-\text{bundle gerbes with connective structure but without curving, and in } \text{[CS16]} \text{ to succinctly get results on String structures that otherwise require considerable buildu}.
\]
Proposition 24. The refined Steenrod squares are stable, in the sense that they give morphisms of stacks
\[
\hat{Sq}^{2n+1} = B^k(\hat{Sq}^{2n+1}) \text{pr} : B^{2n+k}U(1)\nabla \to B^{4n+k+1}U(1)\nabla.
\]

The next example will present the general case. Then we will narrow our scope to specific cases arising from physics, which can be seen as a continuation and extension of the discussions in [FSS13] [FSS15a] [FSS15b] [GS16a].

Example 3. Let
\[
\hat{x} : X \to B^{2n+k}U(1)\nabla
\]
be a \((2n+k-1)\)-bundle representing a differential cohomology class and let \(\{U_\alpha\}\) be a good open cover of \(X\). Then, from [FSS12], \(\hat{x}\) determines the following data:

- An integral Čech cocycle \(n_{\alpha_0...\alpha_{2n+k+1}}\) on \((2n+k+1)\)-fold intersections.
- Differential \(j\)-forms, \(j \leq 2n+k\), \(A_j\) on \((2n+k-j)\)-fold intersections such that
  \[(-1)^k \delta A_j = dA_{j-1} .\]

Post-composing the map \(\hat{x}\) with the map (3.6), gives the refined degree \((2n+1)\) Steenrod square
\[
\hat{Sq}^{2n+1}(\hat{x}) : X \to B^{2n+k}U(1)\nabla \to B^k(B^{2n}U(1)\nabla) \to B^{4n+k+1}U(1)\nabla.
\]

In terms of bundle data, this composite is fairly straightforward to write down explicitly. The combinatorics involved can be deduced from a rather tedious (but straightforward) calculation involving the analogue of the map in unbounded sheaves of chain complexes. Explicitly, we get the following bundle data.

- An integral Čech cocycle
  \[n_{\alpha_0...\alpha_{2n+k}}n_{\alpha_{2n}...\alpha_{2n+k+1}}\]
on \((4n+k+1)\)-fold intersections.
- Differential \(j\)-forms, \(0 \leq j \leq 2n\), \(n_{\alpha_0...\alpha_{2n+k}}A_j\) on \((4n+k+1-j)\)-fold intersections satisfying the Čech-Deligne cocycle condition.
- Differential \((j+2n+1)\)-forms, \(0 \leq j \leq 2n\), \(A_j \wedge dA_{2n}\) on \((2n+k-j)\)-fold intersections satisfying the Čech-Deligne cocycle condition.
- \(0\) on all other intersections.

The following two examples are directed related, but we split for ease of presentation. We will calculate \(\hat{Sq}^1\) and \(\hat{Sq}^3\) in terms of bundle data.

Example 4. Let \(H\) be a closed 3-form. Suppose that \(H\) admits a differential refinement \(\tilde{H}\). Then we can identify \(\tilde{H}\) with a bundle given by the following data:

- An integral Čech cocycle \(n_{\alpha_0...\alpha_3}\) on quadruple intersections.
• A smooth, real valued function $f_{\alpha\beta\gamma}$ on triple intersections (satisfying the Čech-Deligne cocycle condition).

• A differential 1-form $A_{\alpha\beta}$ on intersections (satisfying the Čech-Deligne cocycle condition).

• A differential 2-form $B_{\alpha}$ on open sets (satisfying the Čech-Deligne cocycle condition).

The refined Steenrod square $\hat{Sq}^3$, being in top degree, is the usual Deligne-Beilinson square given by the data

• An integral Čech cocycle $n_{\alpha\beta\gamma\delta\epsilon\eta\xi}$ on 7-fold intersections.

• A smooth, real valued function $n_{\alpha\beta\gamma\delta} f_{\delta\epsilon\eta}$ on 6-fold intersections.

• A differential 1-form $n_{\alpha\beta\gamma\delta} A_{\delta\epsilon}$ on 5-fold intersections.

• A differential 2-form $n_{\alpha\beta\gamma\delta} B_{\delta}$ on 4-fold intersections.

• $f_{\alpha\beta\gamma} \wedge H$ on 3-fold intersections.

• $A_{\alpha\beta} \wedge H$ on intersections.

• $B_{\alpha} \wedge H$ on open sets.

Example 5. Continuing Example 4, $\hat{Sq}^1(\hat{H})$ is given by

• An integral Čech cocycle $n_{\alpha\beta\gamma\delta\epsilon\eta\xi}$ on 5-fold intersections.

• A smooth, real valued function $n_{\alpha\beta\gamma\delta} f_{\beta\gamma\delta}$ on 4-fold intersections.

• $f_{\alpha\beta\gamma} \wedge df_{\alpha\beta\gamma}$ on 3-fold intersections.

• 0 on all other intersections.

Remark 12 (Global patterns). (i) In each of the bundles in the last two examples, it is interesting to note the form of the first nonzero value. These are

$$f_{\alpha\beta\gamma} \wedge df_{\gamma\epsilon\eta}, \quad B_{\alpha} \wedge H = B_{\alpha} \wedge dB_{\alpha},$$

(3.7)

respectively. From left to right these correspond to the first, second, and third refined Steenrod operation, respectively.

(ii) In fact, an interesting pattern emerges when one looks at the first non-zero information in the Steenrod squares. This is summarized in the following table.

| Degree of curvature form | $\deg(H) = 1$ | $\deg(H) = 3$ | $\deg(H) = 5$ | $\deg(H) = 7$ |
|--------------------------|--------------|--------------|--------------|--------------|
| Degree of square of $H$  | 2            | 6            | 10           | 14           |
| Degree of first nonzero form in $\hat{Sq}^1(\hat{H})$ | 1-form | 1-form | 1-form | 1-form |
| Dimension of bundle      | 1            | 3            | 5            | 7            |
| Degree of first nonzero form in $\hat{Sq}^3(\hat{H})$ | N/A         | 5-form | 5-form | 5-form |
| Dimension of bundle      | N/A          | 5           | 7           | 9            |
| Degree of first nonzero form in $\hat{Sq}^5(\hat{H})$ | N/A         | N/A         | 9-form | 9-form |
| Dimension of bundle      | N/A          | N/A         | 9           | 11           |
| Degree of first nonzero form in $\hat{Sq}^7(\hat{H})$ | N/A         | N/A         | N/A         | 13-form |
| Dimension of bundle      | N/A          | N/A         | N/A         | 13           |
Remark 13 (Physical action functionals). The proper way to formulate action functionals is to do so with a view towards quantization. That is, the exponentiated action functional, which is the integrand in a path integral over the configuration/moduli space needed for quantization, should be well-defined. The above expressions, are of Chern-Simons type and automatically take values in $\mathbb{R}/\mathbb{Z}$ by our constructions, hence give rise to well-defined path integrands. These are also secondary classes associated with flat bundles. The discussions in [GS16a], being about Massey products in differential cohomology, can then be viewed as a secondary formulation of a secondary formulation. The action functionals reflect that effect. In both formulations the main point, as far as action functionals go, is that the cup product being zero does not end the story, but rather opens up the possibility for a considerable amount of geometry and dynamics to be captured by resorting to secondary considerations.

We close by noting that there are many possible further applications to geometry and topology. Some of these discussions will appear soon. Indeed, the first application will be to the Atiyah-Hirzebruch spectral sequence in differential generalized cohomology theories [GS16b].

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