ON $\mathbb{F}_p$-ROOTS OF THE HILBERT CLASS POLYNOMIAL MODULO $p$

MINGJIE CHEN, JIANGWEI XUE

Abstract. The Hilbert class polynomial $H_O(x) \in \mathbb{Z}[x]$ attached to an order $O$ in an imaginary quadratic field $K$ is the monic polynomial whose roots are precisely the distinct $j$-invariants of elliptic curves over $\mathbb{C}$ with complex multiplication by $O$. Let $p$ be a prime inert in $K$ and strictly greater than $|\text{disc}(O)|$. We show that the number of $\mathbb{F}_p$-roots of $H_O(x) \pmod{p}$ is either zero or $|\text{Pic}(O)[2]|$ by exhibiting a free and transitive action of $\text{Pic}(O)[2]$ on the set of $\mathbb{F}_p$-roots of $H_O(x) \pmod{p}$ whenever it is nonempty. We also provide a concrete criterion for the existence of $\mathbb{F}_p$-roots. A similar result was first obtained by Xiao et al. [XLD] and generalized much further by Li et al. [LLO21] (that covers the current result) with a different approach.

1. Introduction

Let $O$ be an order in an imaginary quadratic field $K$, and $\text{Pic}(O)$ be the Picard group of $O$, i.e. the group of isomorphism classes of invertible fractional $O$-ideals under multiplication. The Hilbert class polynomial $H_O(x)$ attached to $O$ is defined to be

$$H_O(x) = \prod_{[a] \in \text{Pic}(O)} (x - j(C/a)),$$

where $[a]$ denotes the isomorphism class of the invertible fractional $O$-ideal $a$, and $j(C/a)$ stands for the $j$-invariant of the complex elliptic curve $C/a$. It is well known that $H_O(x)$ has integral coefficients, and it is irreducible over $\mathbb{Q}$ (see Cox [1989] §13 and Lam [1987] Chapter 10, App., p.144).

Let $p \in \mathbb{N}$ be a prime number, and $\tilde{H}_O(x) \in \mathbb{F}_p[x]$ be the polynomial obtained by reducing $H_O(x) \in \mathbb{Z}[x]$ modulo $p$. Suppose that $p$ is non-split in $K$ so that the roots of $\tilde{H}_O(x)$ are supersingular $j$-invariants, which are known to lie in $\mathbb{F}_p^2$. It’s natural to ask how many of them are actually in $\mathbb{F}_p$. Castryck, Panny, and Vercauteren answered this question in [CPV20] Theorem 26 for special cases when $p \equiv 3 \pmod{4}$, $K$ is of the form $\mathbb{Q}(\sqrt{-l})$ with $l$ prime, $l < (p + 1)/4$ and $O$ is an order containing $\sqrt{-l}$. Their method as in [CPV20] Section 5.2 counts the $\mathbb{F}_p$-roots by constructing supersingular elliptic curves over $\mathbb{F}_p$. We take a different approach here by reinterpreting the $\mathbb{F}_p$-roots in terms of quaternion orders, which allows us to answer the question in more generality.

Our main result is as follows.

Theorem 1.1. Let $K$ be an imaginary quadratic field and $O$ be an order in $K$. Let $p$ be a prime inert in $K$ and strictly greater than $|\text{disc}(O)|$, and $\mathcal{H}_p$ be set of $\mathbb{F}_p$-roots of $H_O(x)$. If $\mathcal{H}_p$ is nonempty, then it admits a regular (i.e. free and transitive) action by the $2$-torsion subgroup $\text{Pic}(O)[2] \subset \text{Pic}(O)$. In particular, the number of $\mathbb{F}_p$-roots of $\tilde{H}_O(x)$ is either zero or $|\text{Pic}(O)[2]|$.

Moreover, $\mathcal{H}_p \neq \emptyset$ if and only if for every prime factor $\ell$ of $\text{disc}(O)$, either condition (i) or (ii) below holds for $\ell$ depending on its parity:

(i) $\ell \neq 2$ and the Legendre symbol $(\frac{-p}{\ell}) = 1$;
(ii) $\ell = 2$ and one of the following conditions holds:
(a) $p \equiv 7 \pmod{8}$;
(b) $-p + \frac{\text{disc}(O)}{4} \equiv 0, 1$ or $4 \pmod{8}$;
(c) $-p + \text{disc}(O) \equiv 1 \pmod{8}$.

Date: February 14, 2022.
2020 Mathematics Subject Classification. 14H52, 11G20, 11R52, 11G15.
Key words and phrases. Hilbert class polynomial, supersingular elliptic curve, endomorphism ring, quaternion algebra, Picard group.

Mingjie Chen was supported by NSF grants DMS-1844206, DMS-1802161.
The assumption that \(| \text{disc}(O) | < p \) immediately implies that \( p \) does not divide the discriminant of \( H_K(x) \) by an influential work of Gross and Zagier [GZ89]. Therefore, \( H_K(x) \) has no repeated roots. We provide an alternative proof of this fact under the current assumptions in Corollary 4.2.

**Remark 1.2.** After the first version of this manuscript appeared on the web, Jianing Li kindly informed us that a similar result to Theorem 1.1 has firstly been obtained in [XLD, Theorem 1.1] under the assumption by this purpose, we first describe more concretely the reduction of singular moduli with complex multiplication.

altitude proof of this fact under the current assumptions in Corollary 2.7.

**Theorem 1.3** ([Cox89 Proposition 3.11]). Let \( r \) be the number of odd primes dividing \( \text{disc}(O) \). Define the number \( \mu \) as follows: if \( \text{disc}(O) \equiv 1 \pmod 4 \), then \( \mu = r \), and if \( \text{disc}(O) \equiv 0 \pmod 4 \), then \( \text{disc}(O) = -4n \), where \( n > 0 \), and \( \mu \) is determined as follows:

\[
\mu = \begin{cases} 
  r & \text{if } n \equiv 3 \pmod 4; \\
  r + 1 & \text{if } n \equiv 1, 2 \pmod 4; \\
  r + 1 & \text{if } n \equiv 4 \pmod 8; \\
  r + 2 & \text{if } n \equiv 0 \pmod 8. 
\end{cases}
\]

Then \( |\text{Pic}(O)[2]| = 2^{\mu - 1} \).

This paper is organized as follows. In section 2, we give a reinterpretation of \( H_p \) in terms of quaternion orders. In section 3, we show that there is a regular action of \( \text{Pic}(O)[2] \) on \( H_p \), whenever \( H_p \neq 0 \), and provide a nonemptiness criterion for \( H_p \). Throughout the paper, the prime \( p \in \mathbb{N} \) is assumed to be non-split in \( K \). The notation \( B_{p, \infty} \) is reserved for the unique quaternion \( \mathbb{Q} \)-algebra ramified precisely at \( p \) and infinity. Given a set \( X \) and an equivalence relation on \( X \), the equivalence class of an element \( x \in X \) is denoted by \([x]\).

**Acknowledgements.** The first author would like to thank WIN5 research group members Sarah Arpin, Kristin Lauter, Renate Scheidler, Katherine Stange and Ha Tran for helpful comments on some initial ideas on this problem. The first author would also like to thank Kiran Kedlaya and Rachel Pries for helpful suggestions, and Nandagopal Ramachandran for helpful discussions. The second author thanks Chia-Fu Yu for helpful discussions.

## 2. Reinterpretation of the \( \mathbb{F}_p \)-roots

As mentioned before, we are going to reinterpret the \( \mathbb{F}_p \)-roots of \( H_K(x) \) in terms of quaternion orders. For this purpose, we first describe more concretely the reduction of singular moduli with complex multiplication by \( O \). Assume that the prime \( p \) is non-split in \( K \). For the moment, we make no assumption on the discriminant of the order \( O \subset K \).

Let \( \ell \text{Ell}(O) \) be the set of isomorphism classes of elliptic curves over \( \mathbb{Q} \) with complex multiplication by \( O \). It is canonically identified with the singular \( j \)-invariants with complex multiplication by \( O \) (i.e. the roots of \( H_K(x) \in \mathbb{Z}[x] \)). The Picard group \( \text{Pic}(O) \) acts regularly on \( \ell \text{Ell}(O) \) via \( \alpha \)-transformation [Shi93 §7] and [Mil07 §1]:

\[
\text{Pic}(O) \times \ell \text{Ell}(O) \rightarrow \ell \text{Ell}(O), \quad ([\alpha], E) \mapsto E^\alpha.
\]

More concretely, if we pick \( \alpha \) to be an integral ideal of \( O \) and write \( E[\alpha] \) for the finite group scheme \( \cap_{\alpha \in E} E[\alpha] \), then \( E^\alpha = E/E[\alpha] \) by [Wat69 Corollary A.4]. Here \( E[\alpha] = \ker(E \rightarrow E) \). See [Mil07 Proposition 1.26] and [Wat69 Appendix] for the functorial characterization of \( E^\alpha \). Alternatively, since \( \alpha \) is an invertible \( O \)-ideal, \( E^\alpha \) can also be identified canonically with the Serre tensor construction \( \alpha^{-1} \otimes_O E \) (see [AK18 §1] and
Fix a member $E_0 \in \mathcal{E}\ell(\mathcal{O})$. The regular action in [2] gives rise to a $\text{Pic}(\mathcal{O})$-equivariant bijection $\xi : \mathcal{E}\ell(\mathcal{O}) \to \text{Pic}(\mathcal{O})$ that sends $E_0$ to the identity element $[\mathcal{O}] \in \text{Pic}(\mathcal{O})$.

Similarly, let $\mathcal{E}\ell^\times / \mathbb{F}_p$ be the set of isomorphism classes of supersingular elliptic curves over $\mathbb{F}_p$, which is canonically identified with the set of supersingular $j$-invariants in $\mathbb{F}_p$. From [Sil09, Theorem V.3.1], an elliptic curve $E/\mathbb{F}_p$ is supersingular if and only if its endomorphism algebra $\text{End}(E) := \text{End}(E) \otimes \mathbb{Q}$ is a quaternion $\mathbb{Q}$-algebra. Assume that this is the case. Then $\text{End}(E)$ coincides with the unique quaternion $\mathbb{Q}$-algebra $B_{p,\infty}$ ramified precisely at $p$ and infinity, and $\text{End}(E)$ is a maximal order in $\text{End}(E)$ by [Wat69, Theorem 4.2].

For simplicity, put $B := B_{p,\infty}$ and let $\text{Typ}(B)$ be the type set of $B$, that is, the set of isomorphism (i.e. $B^\times$-conjugacy) classes of maximal orders in $B$. We obtain the following canonical map, which is known to be surjective [Voi21, Corollary 42.2.21]:

$$\rho : \mathcal{E}\ell^\times / \mathbb{F}_p \to \text{Typ}(B), \quad E \mapsto [\text{End}(E)].$$

Let $\mathcal{R}$ be a maximal order in $B$, and $\text{Cl}(\mathcal{R})$ be its left ideal class set, that is, the set of isomorphism (i.e. right $B^\times$-equivalent) classes of fractional left ideals of $\mathcal{R}$ in $B$. Given a fractional left ideal $I$ of $\mathcal{R}$, we write $\mathcal{R}_r(I)$ for the right order of $I$, which is defined as follows:

$$\mathcal{R}_r(I) := \{x \in B \mid IX \subseteq I\}.$$

Sending a fractional left $\mathcal{R}$-ideal to its right order induces a surjective map

$$\Upsilon : \text{Cl}(\mathcal{R}) \to \text{Typ}(B), \quad [I] \mapsto [\mathcal{R}_r(I)].$$

The Deuring correspondence [Voi21, Corollary 42.3.7] establishes a bijection between $\text{Cl}(\mathcal{R})$ and $\mathcal{E}\ell^\times / \mathbb{F}_p$. One direction of this correspondence goes as follows. From the surjectivity of $\rho$, we may always fix $E_\mathcal{R} \in \mathcal{E}\ell^\times / \mathbb{F}_p$ such that $\text{End}(E_\mathcal{R}) = \mathcal{R}$. Then the member of $\mathcal{E}\ell^\times / \mathbb{F}_p$ corresponding to a left ideal class $[I] \in \text{Cl}(\mathcal{R})$ is the $I$-transform $E^I_\mathcal{R}$ of $E_\mathcal{R}$. If $I$ is chosen to be an integral left ideal of $\mathcal{R}$, then $E^I_\mathcal{R}$ can be identified with the quotient $E_\mathcal{R} / E_\mathcal{R}[I]$ by [Wat69, Corollary A.4] again. From [Voi21, Corollary 42.3.7], we have

$$\text{End}(E^I_\mathcal{R}) \simeq \mathcal{R}_r(I).$$

Let $\mathfrak{P}$ be a place of $\overline{\mathbb{Q}}$ lying above $p$, and $r_{\mathfrak{P}} : \mathcal{E}\ell(\mathcal{O}) \to \mathcal{E}\ell^\times / \mathbb{F}_p$ be the reduction map modulo $\mathfrak{P}$. For each $E \in \mathcal{E}\ell(\mathcal{O})$, we write $\overline{E}$ for the reduction of $E$ modulo $\mathfrak{P}$. From [Lan87, §9.2], reducing $E_0$ modulo $\mathfrak{P}$ gives rise to an embedding $\iota : \mathcal{O} \hookrightarrow \mathcal{R}_0 := \text{End}(\overline{E}_0)$. By an abuse of notation, we still write $\iota$ for both of the following two induced maps:

$$K \to B \quad \text{and} \quad \text{Pic}(\mathcal{O}) \xrightarrow{[\alpha] \to [\mathcal{R}_0 \iota(\alpha)]} \text{Cl}(\mathcal{R}_0).$$

For simplicity, we identify $K$ with its image in $B$ via $\iota$ and write $\mathcal{R}_0 a$ for $\mathcal{R}_0 \iota(a)$.

Now we are ready to give a concrete description of $r_{\mathfrak{P}} : \mathcal{E}\ell(\mathcal{O}) \to \mathcal{E}\ell^\times / \mathbb{F}_p$.

**Proposition 2.1.** The reduction map $r_{\mathfrak{P}}$ fits into a commutative diagram as follows:

$$\begin{array}{c}
\mathcal{E}\ell(\mathcal{O}) \xrightarrow{r_{\mathfrak{P}}} \mathcal{E}\ell^\times / \mathbb{F}_p \\
\xi \simeq \chi \simeq \delta \xrightarrow{\rho} \text{Typ}(B).
\end{array}$$

Here $\xi$ is the $\text{Pic}(\mathcal{O})$-equivariant bijection that sends the fixed member $E_0 \in \mathcal{E}\ell(\mathcal{O})$ to $[\mathcal{O}] \in \text{Pic}(\mathcal{O})$, and $\delta$ is the Deuring correspondence obtained by taking $E_{\mathcal{R}_0} = \overline{E}_0$.

**Proof.** According to [Shi98, Proposition 15, §11], $\alpha$-transforms are preserved under good reductions\footnote{A prior, the statement of [Shi98, Proposition 15, §11] requires that $\mathcal{O} = \mathcal{O}_K$, the maximal order of $K$. Nevertheless, the result here holds for general $\mathcal{O}$ here since $\alpha$ is an invertible $\mathcal{O}$-ideal by our assumption.}. This implies that for every $[\alpha] \in \text{Pic}(\mathcal{O})$, we have

$$\overline{E}_0^\alpha = (\overline{E}_0)^\alpha = (\overline{E}_0)_{\mathcal{R}_0 a},$$

so the left square commutes. The right triangle commutes because of $[\alpha]$. \hfill $\square$

**Corollary 2.2.** For any $[\alpha] \in \text{Pic}(\mathcal{O})$, we have $\text{End}(\overline{E}_0^\alpha) \simeq \alpha^{-1} \mathcal{R}_0 a$. 
Proof. This follows directly from Proposition 2.1 since the right order of \( \mathcal{R}_0 \mathfrak{a} \) is precisely \( \mathfrak{a}^{-1} \mathcal{R}_0 \mathfrak{a} \).

**Remark 2.3.** Let \( \mathcal{O}_K \) be the ring of integers of \( K \), and \( f \) be the conductor of \( \mathcal{O} \) so that \( \mathcal{O} = \mathbb{Z} + f \mathcal{O}_K \). Write \( f = p^m f' \) with \( p \mid f' \), and put \( \mathcal{O}' := \mathbb{Z} + f' \mathcal{O}_K \). According to [Oka21] Lemma 3.1, \( \iota(K) \cap \mathcal{R}_0 = \iota(\mathcal{O}') \).

For any invertible fractional ideal \( \mathfrak{a} \) of \( \mathcal{O} \), we have \( \mathcal{R}_0 \mathfrak{a} = (\mathcal{R}_0 \mathfrak{a})^{\mathfrak{a}} = \mathcal{R}_0 (\mathcal{O}' \mathfrak{a}) \). It follows that the map \( \iota : \text{Pic}(\mathcal{O}) \rightarrow \text{Cl}(\mathcal{R}_0) \) factors through the following canonical homomorphism

\[
\varpi : \text{Pic}(\mathcal{O}) \rightarrow \text{Pic}(\mathcal{O}'), \quad [\mathfrak{a}] \mapsto [\mathcal{O}' \mathfrak{a}].
\]

From this, one easily deduces that \( \tilde{H}_\mathcal{O}(x) = (\tilde{H}_{\mathcal{O}'}(x))^{[\ker(\varpi)]} \).

Now assume that \( \mathcal{O} \) is maximal at \( p \) (i.e. \( p \mid f' \)). From Remark 2.3, \( \iota : \mathcal{O} \rightarrow \mathcal{R}_0 \) is an optimal embedding of \( \mathcal{O} \) into \( \mathcal{R}_0 \), that is, \( \iota(K) \cap \mathcal{R}_0 = \iota(\mathcal{O}) \).

Given an arbitrary maximal order \( \mathcal{R} \) of \( B \), we write \( \text{Emb}(\mathcal{O}, \mathcal{R}) \) for the set of optimal embeddings of \( \mathcal{O} \) into \( \mathcal{R} \). The unit group \( \mathcal{R}^\times \) acts on \( \text{Emb}(\mathcal{O}, \mathcal{R}) \) by conjugation, and there are only finitely many orbits. Put \( m(\mathcal{O}, \mathcal{R}, \mathcal{R}^\times) := |\mathcal{R}^\times \setminus \text{Emb}(\mathcal{O}, \mathcal{R})| \), the number of \( \mathcal{R}^\times \)-conjugacy classes of optimal embeddings from \( \mathcal{O} \) into \( \mathcal{R} \). We recall below a precise formula by Elkies, Ono and Yang for the cardinality of each fiber of the reduction map \( r_\mathfrak{p} : \mathcal{E}(\mathcal{O}) \rightarrow \mathcal{E}(\mathcal{O})_{/\mathfrak{p}} \).

**Lemma 2.4 ([EOY05, Lemma 3.3]).** Suppose that \( \mathcal{O} \) is maximal at \( p \). Then for any member \( E \in \mathcal{E}(\mathcal{O})_{/\mathfrak{p}} \), we have

\[
|r_\mathfrak{p}^{-1}(E)| = \varepsilon \cdot m(\mathcal{O}, \mathcal{R}, \mathcal{R}^\times),
\]

where \( \mathcal{R} = \text{End}(E) \), and \( \varepsilon = 1/2 \) or 1 according as \( p \) is inert or ramified in \( K \).

A priori, [EOY05, Lemma 3.3] is only stated for the maximal order \( \mathcal{O}_K \). Nevertheless, the same proof applies more generally to quadratic orders maximal at \( p \). Alternatively, using Proposition 2.1 and the Deuring lifting theorem [Lan87, Theorem 14, §13.5] ([GZ85, Proposition 2.7]), one easily sees that Lemma 2.3 is equivalent to the following purely arithmetic result, whose independent proof will be left for the interested reader.

**Lemma 2.5.** Keep \( \mathcal{O} \) and \( \varepsilon \) as in Lemma 2.4. Let \( \mathcal{R} \) be a maximal order in \( B \), and \( \varphi : \mathcal{O} \rightarrow \mathcal{R} \) be an optimal embedding. Denote the induced map \( \text{Pic}(\mathcal{O}) \rightarrow \text{Cl}(\mathcal{R}) \) by \( \varphi \) as well. Then for each \( [I] \in \text{Cl}(\mathcal{R}) \), we have

\[
|\varphi^{-1}([I])| = \varepsilon \cdot m(\mathcal{O}, \mathcal{R}, I, \mathcal{R}_r(I)^\times).
\]

We immediately obtain the following corollaries from Lemma 2.4.

**Corollary 2.6.** Suppose that \( \mathcal{O} \) is maximal at \( p \). The \( j \)-invariant of a supersingular elliptic curve \( E_{/\mathfrak{p}} \) is a root of \( \tilde{H}_\mathcal{O}(x) \) if and only if \( \mathcal{O} \) can be optimally embedded into \( \text{End}(E) \).

This matches well with Corollary 2.2. Indeed, a classical result of Chevalley, Hasse and Noether ([Has22, §4]) says that any maximal order of \( B \) that contains a copy of \( \mathcal{O} \) optimally is isomorphic to \( \mathfrak{a}^{-1} \mathcal{R}_0 \mathfrak{a} \) for some \( [\mathfrak{a}] \in \text{Pic}(\mathcal{O}) \).

**Corollary 2.7.** If \( p > |\text{disc}(\mathcal{O})| \), then the reduction map \( r_\mathfrak{p} : \mathcal{E}(\mathcal{O}) \rightarrow \mathcal{E}(\mathcal{O})_{/\mathfrak{p}} \) is injective. In particular, \( \tilde{H}_\mathcal{O}(x) \) has no repeated roots.

We give a simple proof that is independent of the result of Gross and Zagier ([GZ85]).

**Proof.** Since \( p \) does not split in \( K \) and is strictly greater than \( |\text{disc}(\mathcal{O})| \), it is necessarily inert in \( K \). From Lemma 2.4 it suffices to show that \( |\text{Emb}(\mathcal{O}, \mathcal{R})| \leq 2 \) for any maximal order \( \mathcal{R} \) in \( B \). Since \( p > |\text{disc}(\mathcal{O})| \), Kaneko’s inequality [Kan89, Theorem 2] forces any two optimal embeddings \( \varphi, \varphi' : \mathcal{O} \rightarrow \mathcal{R} \) to have the same image. On the other hand, \( \varphi \) and \( \varphi' \) share the same image if and only if \( \varphi' = \varphi \) or \( \bar{\varphi} \), the complex conjugate of \( \varphi \). The desired inequality \( |\text{Emb}(\mathcal{O}, \mathcal{R})| \leq 2 \) follows immediately.

**Remark 2.8.** In another direction, Elkies, Ono and Yang ([EOY05, Theorem 1.4]) showed that there exists a bound \( N_p \) such that the reduction map \( r_\mathfrak{p} : \mathcal{E}(\mathcal{O}_K) \rightarrow \mathcal{E}(\mathcal{O})_{/\mathfrak{p}} \) is surjective whenever \( |\text{disc}(\mathcal{O}_K)| > N_p \). This bound is first effectivized by Kane ([Kan09] conditionally upon the generalized Riemann hypothesis. Liu et al. further improved this bound in [LY15, Corollary 1.3].

\[ \text{[Here the Deuring lifting theorem guarantees that the optimal embedding } \iota : \mathcal{O} \rightarrow \mathcal{R}_0 \text{ is “non-special”, that is, every optimal embedding } \varphi : \mathcal{O} \rightarrow \mathcal{R} \text{ is realizable as } \text{End}(E) \rightarrow \text{End}(E) \text{ for some } E \in \mathcal{E}(\mathcal{O}). \]
Let us return to the task of interpreting $F_p$-roots of $\tilde{H}_O(x) \in F_p[x]$ in terms of maximal orders in $B$. For the rest of this section, we keep the additional assumption that $p > |\text{disc}(O)|$. We recall from [DG16] Proposition 2.4 a classical result on supersingular elliptic curves in characteristic $p$.

**Lemma 2.9.** Let $p > 3$ and let $E$ be a supersingular elliptic curve over $\overline{F}_p$. Then $j(E) \in F_p$ if and only if there exists $\psi \in \text{End}(E)$ such that $\psi^2 = -p$.

Recall that $\mathcal{H}_p$ denotes the set of $F_p$-roots of $\tilde{H}_O(x)$, which can be identified canonically with a subset of $\mathcal{E}\ell\ell_{/F_p}$.

**Lemma 2.10.** The map $\rho : \mathcal{E}\ell\ell_{/F_p} \to \text{Typ}(B)$ in (3) induces a bijection between $\mathcal{H}_p$ and the following subset $\mathcal{T}_p \subseteq \text{Typ}(B)$:

(7) \quad $\mathcal{T}_p := \{[R] \in \text{Typ}(B) \mid \text{Emb}(O, R) \neq \emptyset, \text{ and } \exists \alpha \in R \text{ such that } \alpha^2 = -p\}$.

**Proof.** Combining Corollary 2.6 and Lemma 2.9, we see that $\rho(\mathcal{H}_p) = \mathcal{T}_p$. Now it follows from [Voi21] Lemma 42.4.1 that $\rho : \mathcal{H}_p \to \mathcal{T}_p$ is injective, and hence bijective. \qed

We give another characterization of $\mathcal{T}_p$ by presenting the quaternion algebra $B = B_{p,\infty}$ more concretely. Let $d \in \mathbb{N}$ be the unique square-free positive integer such that $K = \mathbb{Q}(\sqrt{-d})$. The assumption that $p$ is inert in $K$ amounts to the equality $(\frac{d}{p}) = -1$. Let $\left(\frac{-d,-p}{Q}\right)$ be the quaternion $\mathbb{Q}$-algebra with standard basis $\{1, i, j, k\}$ such that

(8) \quad $i^2 = -d, \quad j^2 = -p \quad \text{and} \quad k = ij = -ji$.

We identify $K = \mathbb{Q}(\sqrt{-d})$ with $\mathbb{Q}(i)$, and $O$ with the corresponding order in $\mathbb{Q}(i)$. Put $\Lambda := O + jO$, which is an order (of full rank) in the above quaternion algebra. Consider the following finite set of maximal orders:

(9) \quad $S^{opt} := \left\{ R \subseteq \left(\frac{-d,-p}{Q}\right) \mid R \text{ is a maximal order containing } \Lambda \text{ and } R \cap \mathbb{Q}(i) = O \right\}$.

Here the superscript “opt” stands for “O-optimal”.

**Proposition 2.11.** Let $R$ be a maximal order in $B$. We have $[R] \in \mathcal{T}_p$ if and only if $R \simeq R$ for some $R \in S^{opt}$. In particular, $\mathcal{H}_p \neq \emptyset$ if and only if $\left(\frac{-d,-p}{q}\right) \simeq B$ and $S^{opt} \neq \emptyset$.

**Proof.** Clearly, if $R \simeq R$ for some $R \in S^{opt}$, then $[R] \in \mathcal{T}_p$. Conversely, suppose that $[R] \in \mathcal{T}_p$, that is, $R$ contains a copy of $O$ optimally, and there exists $\alpha \in R$ with $\alpha^2 = -p$. Then $R\alpha$ is the unique two sided prime ideal of $R$ lying above $p$. From [Vig80] Exercise I.4.6, $R$ is normalized by $\alpha$, which implies that $O_\alpha := \alpha O \alpha^{-1}$ is still a quadratic order optimally embedded in $R$. If $O_\alpha \neq O$, then $\text{disc}(O) \geq p$ by Kaneko’s inequality [Kan89] Theorem 2], contradicting to our assumption that $\text{disc}(O) < p$. Thus $O_\alpha = O$, and conjugation by $\alpha$ induces an automorphism $\sigma \in \text{Aut}(O)$. If $\sigma$ is the identity, then $\alpha$ lies in the centralizer of $O$ in $B$, which is just $K$. This contradicts to the assumption $\text{disc}(O) < p$ again. It follows that $\sigma$ is the unique nontrivial automorphism of $O$, i.e. the complex conjugation. We conclude that $\Delta_R := O + \alpha O \subset R$ is isomorphic to $\Lambda$, and $B = \Lambda_R \otimes \mathbb{Z} \mathbb{Q} \simeq \Lambda \otimes \mathbb{Z} \mathbb{Q} = \left(\frac{-d,-p}{Q}\right)$. Consequently, $R$ is isomorphic to some member of $S^{opt}$. The last statement follows from the bijection $\mathcal{H}_p \simeq \mathcal{T}_p$ in Lemma 2.11. \qed

**Lemma 2.12.** The isomorphism $\left(\frac{-d,-p}{q}\right) \simeq B$ holds if and only if $\left(\frac{-p}{q}\right) = 1$ for every odd prime factor $\ell$ of $d$.

**Proof.** For the moment, let $\ell$ be either a prime number or $\infty$. Write $(-d,-p)_\ell$ for the Hilbert symbol of $-d$ and $-p$ relative to $\mathbb{Q}_\ell$ (where $\mathbb{Q}_\infty = \mathbb{R}$). From [Vig80] Corollaire II.1.2, $\left(\frac{-d,-p}{q}\right)$ is split at $\ell$ if and only if $(-d,-p)_\ell = 1$. Clearly, $(-d,-p)_\infty = -1$.

Now assume that $\ell$ is an odd prime. By our assumption, $p$ is an odd prime satisfying $\left(\frac{-p}{q}\right) = -1$. From [Ser73] Theorem 1, §III.1], we easily compute that

\[
(-d,-p)_\ell = \begin{cases} 
1 & \text{if } \ell \nmid (dp); \\
-1 & \text{if } \ell = p; \\
\left(\frac{-p}{q}\right) & \text{if } \ell | d.
\end{cases}
\]
Therefore, if \((\frac{-d-x}{q}) \simeq B\), then necessarily \((\frac{-p}{q}) = 1\) for every odd prime factor \(\ell\) of \(d\).

Conversely, if \((\frac{-p}{q}) = 1\) for every odd prime factor \(\ell\) of \(d\), then \((-d,-p)_2 = 1\) by the product formula. Hence this condition is also sufficient for the isomorphism \((\frac{-d-x}{q}) \simeq B\). \(\square\)

### 3. The \(\text{Pic}(O)[2]\)-action on \(\mathcal{H}_p\) and the Nonemptiness Criterion

Throughout this section, we assume that \(p\) is inert in \(K = \mathbb{Q}(\sqrt{-d})\) and strictly greater than \(|\text{disc}(O)|\).

Assume further that the quaternion \(\mathbb{Q}\)-algebra \((\frac{-d-x}{q})\) ramified precisely at \(p\) and infinity, for otherwise \(\mathcal{H}_p = \emptyset\). Denote \((\frac{-d-x}{q})\) simply by \(B\) henceforth and let \(\{i,j,k\}\) be the standard basis of \(B\) as in [8]. We identify \(K\) with the subfield \(\mathbb{Q}(i)\) of \(B\). Then conjugation by \(j\) stabilizes \(K\) and sends each \(x \in K\) to its complex conjugate \(\bar{x}\). Let \(\Lambda = \mathcal{O} + j\mathcal{O}\), and \(S_{\text{opt}}\) be the set of maximal orders in \(\mathcal{H}_p\).

First, we assume that \(\mathcal{H}_p \neq \emptyset\) and exhibit a regular action of \(\text{Pic}(O)[2]\) on \(\mathcal{H}_p\). Since the reduction map \(r_p : \mathbb{E}\ell(O) \to \mathbb{E}\ell(\mathcal{H}_p)\) is injective by Corollary 2.7, the regular action of \(\text{Pic}(O)[2]\) on \(\mathbb{E}\ell(O)\) induces a regular action of \(\text{Pic}(O)[2]\) on the image \(r_p(\mathbb{E}\ell(O))\) (or equivalently, on the full set of roots of \(\bar{H}_O(x)\)). We show that this action restricts to a regular \(\text{Pic}(O)[2]\)-action on \(\mathcal{H}_p\).

**Proposition 3.1.** Let \(E_0 \in \mathbb{E}\ell(O)\) be a member satisfying \(j(E_0) \in \mathbb{F}_p\). Given \([a] \in \text{Pic}(O)\), we have \(j(E_0^a) \in \mathbb{F}_p\) if and only if \([a]\) is a 2-torsion. In particular, \(\text{Pic}(O)[2]\) acts regularly on \(\mathcal{H}_p\).

**Proof.** Put \(R_0 := \text{End}(E_0)\) and \(R := a^{-1}R_0a\) so that \(\text{End}(E_0^a) \simeq R\) by Corollary 2.22. From Lemma 2.10 it is enough to show that there exists \(a \in R\) with \(a^2 = -p\) if and only if \([a] \in \text{Pic}(O)[2]\). By Proposition 2.11 we may assume that \(R_0 \in S_{\text{opt}}\), that is, \(R_0\) is a maximal order in \(B\) satisfying \(R_0 \supseteq \mathcal{O} + j\mathcal{O}\) and \(R_0 \cap K = \mathcal{O}\).

Then

\[
R \cap K = a^{-1}(R_0 \cap K)a = \mathcal{O}, \quad \text{and} \quad R \supseteq a^{-1}ja = a^{-1}\tilde{a}j.
\]

First, suppose that \([a] \in \text{Pic}(O)[2]\). Then \(a^{-1}\tilde{a} = a\tilde{a}\) for some \(a \in K^\times\). Moreover, \(N_{K/Q}(o_a) = N_{K/Q}(a\tilde{a}) = \mathbb{Z}\), so \(N_{K/Q}(a) = 1\). Therefore \(a = aj \in R\) satisfies that \(a^2 = a\tilde{a}j^2 = -p\).

Conversely, suppose that \(a \in R\) is an element satisfying \(a^2 = -p\). From the proof of Proposition 2.11 we must have \(ax = \tilde{a}x\) for every \(x \in \mathcal{O}\). Thus \(j^{-1}\alpha\) centralizes \(\mathcal{O}\), so there exists \(a \in K^\times\) such that \(\alpha = ja\). Moreover, \(N_{K/Q}(a) = 1\) since \(a^2 = j^2\tilde{a}a\). Now we have

\[
R \supseteq a^{-1}\tilde{a}j \cdot \alpha = a^{-1}\tilde{a}j \cdot ja = -pa^{-1}\tilde{a}.
\]

We claim that \(R \supseteq aa^{-1}\tilde{a}\) if and only if \(aa^{-1}\tilde{a} \subseteq \mathcal{O}\). This suffices to show that \(\mathcal{H}_p \neq \emptyset\) if and only if \(S_{\text{opt}} \neq \emptyset\). Therefore, \(a = a\tilde{a}\), so \([a] \in \text{Pic}(O)[2]\). \(\square\)

Now we drop the assumption that \(\mathcal{H}_p \neq \emptyset\) and derive a non-emptiness criterion for \(\mathcal{H}_p\). From Proposition 2.11 \(\mathcal{H}_p \neq \emptyset\) if and only if \(S_{\text{opt}} \neq \emptyset\) (as we have already assumed that \((\frac{-d-x}{q}) \simeq B_{p,\infty}\)). For each prime \(\ell \in \mathbb{N}\), let us put

\[
S_{\text{opt}}^\ell := \{R_\ell \subseteq B_\ell \mid R_\ell \text{ is a maximal order containing } \Lambda_\ell \text{ and } R_\ell \cap K_\ell = \mathcal{O}_\ell\}.
\]

The local-global correspondence of lattices [CR91, Proposition 4.21] establishes a bijection between \(S_{\text{opt}}\) and \(\prod_{\ell} S_{\text{opt}}^\ell\), where the product runs over all prime \(\ell\). Since the reduced discriminant of \(B\) is \(p\) and the reduced discriminant of \(\Lambda\) is \(p\text{disc}(O)\) by [LXY21, Lemmas 2.7 and 2.9], \(\Lambda\) is maximal at every prime \(\ell\) coprime to \(\text{disc}(O)\). Moreover, for each such \(\ell\), the maximal order \(\Lambda_\ell\) automatically satisfies the condition \(\Lambda_\ell \cap K_\ell = \mathcal{O}_\ell\).
by its definition \( \Lambda_\ell = \mathcal{O}_\ell + j \mathcal{O}_\ell \). Hence for \( \ell \nmid \text{disc}(\mathcal{O}) \), the set \( S^\text{opt}_\ell \) has a single element \( \Lambda_\ell \), and the bijection above simplifies as

\[
S^\text{opt} \longleftrightarrow \prod_{\ell \mid \text{disc}(\mathcal{O})} S^\text{opt}_\ell.
\]

**Lemma 3.2.** Let \( \ell \) be a prime factor of \( \text{disc}(\mathcal{O}) \). Then \( S^\text{opt}_\ell \neq \emptyset \) if and only if \( -p \in N_{K/Q}(\mathcal{O}^\times_\ell) \). Moreover, if \( S^\text{opt}_\ell \neq \emptyset \), then there is a regular action of \( H^1(K/Q, \mathcal{O}^\times_\ell) \) on \( S^\text{opt}_\ell \), so any fixed member of \( S^\text{opt}_\ell \) gives rise to a bijection \( S^\text{opt}_\ell \approx H^1(K/Q, \mathcal{O}^\times_\ell) \).

The Galois cohomological description of \( S^\text{opt}_\ell \) is nice to know but not used elsewhere in this paper.

**Proof.** By our assumption, \( \text{disc}(\mathcal{O}) \) is coprime to \( p \), so \( B \) splits at the prime \( \ell \). This allows us to identify \( B_\ell \) with the matrix algebra \( M_2(\mathbb{Q}_\ell) \). Let \( V_\ell = \mathbb{Q}_\ell^2 \) be the unique simple \( B_\ell \)-module. Every maximal order \( \mathcal{R}_\ell \) in \( B_\ell \) is of the form \( \text{End}_{\mathbb{Z}_\ell}(L_\ell) \) for some \( \mathbb{Z}_\ell \)-lattice \( L_\ell \subseteq V_\ell \), and \( L_\ell \) is uniquely determined by \( \mathcal{R}_\ell \) up to \( \mathbb{Q}_\ell^\times \)-isometry. In other words, \( \text{End}_{\mathbb{Z}_\ell}(L_\ell) = \text{End}_{\mathbb{Z}_\ell}(L'_\ell) \) if and only if \( L_\ell = cL'_\ell \) for some \( c \in \mathbb{Q}_\ell^\times \). If \( \mathcal{R}_\ell \in S^\text{opt}_\ell \), then the inclusion \( \Lambda_\ell \subseteq \mathcal{R}_\ell \) puts a \( \Lambda_\ell \)-module structure on \( L_\ell \). Moreover, the \( \Lambda_\ell \)-lattice \( L_\ell \) is \( \mathcal{O}_\ell \)-optimal in the sense that \( \text{End}_{\mathbb{Z}_\ell}(L_\ell) \cap K_\ell = \mathcal{O}_\ell \). Conversely, if \( M_\ell \) is an \( \mathcal{O}_\ell \)-optimal \( \Lambda_\ell \)-lattice in \( V_\ell \), then \( \text{End}_{\mathbb{Z}_\ell}(M_\ell) \) is a member of \( S^\text{opt}_\ell \). We have established the following canonical bijection

\[
S^\text{opt}_\ell \longleftrightarrow \mathcal{M} := \{ \mathcal{O}_\ell \text{-optimal } \Lambda_\ell \text{-lattices } L_\ell \subset V_\ell \}/\mathbb{Q}_\ell^\times.
\]

Recall that \( \Lambda_\ell = \mathcal{O}_\ell + j \mathcal{O}_\ell \), where \( j^2 = -p \) and \( jx = jy \) for any \( x, y \in \mathcal{O}_\ell \). If there exists \( a \in \mathcal{O}^\times_\ell \) satisfying \( aa = -p \), then we can put a \( \Lambda_\ell \)-module structure on \( \mathcal{O}_\ell \) as follows:

\[
(x + jy) \cdot z = xz + j\bar{y}a, \quad \forall x, y, z \in \mathcal{O}_\ell.
\]

Since \( B_\ell = \Lambda_\ell \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \), this also puts a \( B_\ell \)-module structure on \( K_\ell = \mathcal{O}_\ell \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \). Consequently, it identifies \( K_\ell \) with the unique simple \( B_\ell \)-module \( V_\ell \), and in turn identifies \( \mathcal{O}_\ell \) with a \( \Lambda_\ell \)-lattice \( L_\ell \) in \( V_\ell \). Necessarily, \( L_\ell \) is \( \mathcal{O}_\ell \)-optimal since \( \text{End}_{\mathbb{Z}_\ell}(L_\ell) \cap K_\ell = \text{End}_{\mathbb{O}_\ell}(L_\ell) = \text{End}_{\mathcal{O}_\ell}(\mathcal{O}_\ell) = \mathcal{O}_\ell \). We have shown that if \( -p \in N_{K/Q}(\mathcal{O}^\times_\ell) \), then \( S^\text{opt}_\ell \neq \emptyset \).

Conversely, suppose that \( S^\text{opt}_\ell \neq \emptyset \) and let \( M_\ell \) be an \( \mathcal{O}_\ell \)-optimal \( \Lambda_\ell \)-lattice in \( V_\ell \). The inclusion \( \mathcal{O}_\ell \subset \Lambda_\ell \) equips \( M_\ell \) with an \( \mathcal{O}_\ell \)-module structure satisfying \( \text{End}_{\mathcal{O}_\ell}(M_\ell) = \mathcal{O}_\ell \). Being a quadratic \( \mathbb{Z}_\ell \)-order, \( \mathcal{O}_\ell \) is both Gorenstein and semi-local. It follows from [UT13, Characterization B 4.2] that \( M_\ell \) is a free \( \mathcal{O}_\ell \)-module of rank one. Pick a basis \( e \) so that \( M_\ell = \mathcal{O}_\ell e \). Since \( M_\ell \) is at the same time a module over \( \Lambda_\ell \), we have \( je = ae \) for some \( a \in \mathcal{O}_\ell \). Necessarily, \( aa = -p \) because

\[
-p e = j^2 e = j(je) = j(ae) = \bar{a} je = \bar{a} ae.
\]

This also implies that \( a \in \mathcal{O}^\times_\ell \) since \( \ell \neq p \). Therefore, \( S^\text{opt}_\ell \neq \emptyset \) if and only if \( -p \in N_{K/Q}(\mathcal{O}^\times_\ell) \).

Had we picked a different basis \( e' \) for \( M_\ell \), then \( e' = ue \) for some \( u \in \mathcal{O}^\times_\ell \). It follows that

\[
je' = j(ue) = \bar{u}je = \bar{u} ae = u^{-1} \bar{u} ae.
\]

Correspondingly, \( a \) is changed to \( u^{-1} \bar{u} a \). Therefore, we have defined the following map:

\[
\Phi : \mathcal{M} \to \{ a \in \mathcal{O}^\times_\ell \mid aa = -p \}/\sim,
\]

where \( a \sim a' \) if and only if there exists some \( u \in \mathcal{O}^\times_\ell \) such that \( a' = a(\bar{u}/u) \). We have already seen that \( \Phi \) is surjective. Suppose that \( \Phi([M_1]) = \Phi([M_2]) \) for \( [M_\ell] \in \mathcal{M} \) with \( r = 1, 2 \). By the above discussion, we can choose suitable \( \mathcal{O}_\ell \)-base \( e_r \) for \( M_r \) such that they give rise to the same \( a \in \mathcal{O}^\times_\ell \). Then the \( \mathcal{O}_\ell \)-linear map sending \( e_1 \) to \( e_2 \) defines a \( \Lambda_\ell \)-isomorphism between \( M_1 \) and \( M_2 \). Since \( \text{Aut}_{B_\ell}(V_\ell) = \mathcal{O}^\times_\ell \), it follows that \( M_1 \) and \( M_2 \) are \( \mathcal{O}^\times_\ell \)-isomorphic, so \( \Phi \) is injective as well.

Lastly, if the right hand side of (13) is nonempty, then it admits a regular action by \( H^1(K/Q, \mathcal{O}^\times_\ell) \) via multiplication. The second part of the lemma follows by combining the bijections (13) and (14) with the above action.

**Lemma 3.3.** Let \( \ell \) be a prime factor of \( \text{disc}(\mathcal{O}) \). Then \( -p \in N_{K/Q}(\mathcal{O}^\times_\ell) \) if and only if either condition (i) or (ii) below holds for \( \ell \) depending on its parity:

(i) \( \ell \neq 2 \) and \( (\frac{\ell}{p}) = 1 \);
(ii) \( \ell = 2 \) and one of the following conditions holds:

(a) \( p \equiv 7 \) (mod 8);
(b) \( -p + \frac{\text{disc}(\mathcal{O})}{4} \equiv 0, 1 \) or 4 (mod 8);
(c) \(-p + \text{disc}(\mathcal{O}) \equiv 1 \) (mod 8).

Proof. For simplicity, put \( D := \text{disc}(\mathcal{O}) \) and \( \delta = \frac{1}{2}\sqrt{D} \). We claim that \( \mathcal{O}_\ell = \mathbb{Z}_\ell + \mathbb{Z}_\ell \delta \). It is well known that \( \mathcal{O} = \mathbb{Z} + \mathbb{Z}(D + \sqrt{D})/2 \). The claim is obviously true if \( 4 \mid D \). If \( 4 \nmid D \), then \( \ell \neq 2 \), so the claim is true in this case as well. Given an element \( a + b\delta \in \mathcal{O}_\ell \) with \( a, b \in \mathbb{Z}_\ell \), we have \( N_{\mathbb{K}/\mathbb{Q}}(a + b\delta) = a^2 - b^2D/4 \). Therefore, \(-p \in N_{\mathbb{K}/\mathbb{Q}}(\mathcal{O}_\ell^+) \) if and only if the equation

\[
(15) \quad x^2 - y^2\frac{D}{4} = -p
\]

has a solution in \( \mathbb{Z}_\ell^2 \).

First, suppose that \( \ell \) is odd. Then equation \((15)\) is solvable in \( \mathbb{Z}_\ell^2 \) if and only if \( \left(\frac{-p}{\ell}\right) = 1 \). Indeed, suppose \( \left(\frac{-p}{\ell}\right) = 1 \) so that \(-p\) is a square in \( \mathbb{F}_\ell \). By Hensel’s lemma \([\text{Voi21}, \text{Lemma 12.2.17}]\), the equation \( x^2 = -p \) has a solution \( x_0 \in \mathbb{Z}_\ell \). Hence \((x_0, 0)\) is a solution of \((15)\) in \( \mathbb{Z}_\ell^2 \). Conversely, suppose \((15)\) has a solution \((x_0, y_0) \in \mathbb{Z}_\ell^2 \). Reducing \((15)\) modulo \( \ell \) shows that \( x_0 (\text{mod } \ell) \) is a square root of \(-p\) in \( \mathbb{F}_\ell \), i.e. \( \left(\frac{-p}{\ell}\right) = 1 \).

For the rest of the proof we assume that \( \ell = 2 \), which implies that \( 4 \mid D \). First, suppose that \((x, y) \in \mathbb{Z}_2^2 \) is a solution of \((14)\). Since \( x^2, y^2 \equiv 0, 1 \) or 4 (mod 8) and at least one of \( x, y \) lies in \( \mathbb{Z}_2^2 \) because \( p \) is odd, we see that the pair \((x^2, y^2)\) takes on five possibilities modulo 8:

\[
(x^2, y^2) \equiv (0, 1), (1, 0), (1, 1), (1, 4) \text{ or } (4, 1) \pmod{8}.
\]

Each possibility puts the following respective constraint on \( p \) and \( D \):

\[
-p + \frac{D}{4} \equiv 0 \pmod{8}, \quad -p \equiv 1 \pmod{8}, \quad -p + \frac{D}{4} \equiv 1 \pmod{8},
\]

\[
-p + D \equiv 1 \pmod{8}, \quad -p + \frac{D}{4} \equiv 4 \pmod{8}.
\]

We have proved the necessity part of the lemma for the case \( \ell = 2 \).

Conversely, let us show that the above congruence conditions are also sufficient. From the discussion above, each of these conditions guarantees the existence of a solution \((\tilde{x}, \tilde{y})\) of equation \((15)\) in \((\mathbb{Z}/8\mathbb{Z})^2 \) such that either \( \tilde{x} \) or \( \tilde{y}D/4 \) lies in \((\mathbb{Z}/8\mathbb{Z})^\times \). Now from a multivariate version of Hensel’s lemma \([\text{Voi21}, \text{Lemma 12.2.8}]\), the pair \((\tilde{x}, \tilde{y})\) lifts to a solution of \((15)\) in \( \mathbb{Z}_2^2 \). The sufficiency is proved.

Therefore, \(-p \in N_{\mathbb{K}/\mathbb{Q}}(\mathcal{O}_\ell^+) \) if and only if one of the following conditions holds:

(a) \( p \equiv 7 \) (mod 8);
(b) \( -p + \frac{\text{disc}(\mathcal{O})}{4} \equiv 0, 1 \) or 4 (mod 8);
(c) \(-p + \text{disc}(\mathcal{O}) \equiv 1 \) (mod 8).

\( \square \)

Proof of Theorem 1.1. If \( \mathcal{H}_p \neq \emptyset \), then there is a regular action of \( \text{Pic}(\mathcal{O})[2] \) on \( \mathcal{H}_p \) by Proposition 3.1. The criterion for the nonemptiness of \( \mathcal{H}_p \) follows from combining Proposition 2.11 with equation (12) and Lemmas 2.12, 3.2 and 3.3.

\( \square \)

REFERENCES

[AK18] Zavosh Amir-Khosravi, Serre’s tensor construction and moduli of abelian schemes, Manuscripta Math. 156 (2018), no. 3-4, 409–456. MR 3811797

[CCO14] Ching-Li Chai, Brian Conrad, and Frans Oort, Complex multiplication and lifting problems, Mathematical Surveys and Monographs, vol. 195, American Mathematical Society, Providence, RI, 2014. MR 3137398

[Cox89] David A. Cox, Primes of the form \( x^2 + ny^2 \): Fermat, class field theory and complex multiplication, Wiley, New York, 1989.

[CPV20] Wouter Castryck, Lorenz Panny, and Frederik Vercauteren, Rational isogenies from irrational endomorphisms, pp. 523–548, 05 2020.

[CR90] Charles W. Curtis and Irving Reiner, Methods of representation theory. Vol. I, Wiley Classics Library, John Wiley & Sons, Inc., New York, 1990, With applications to finite groups and orders, Reprint of the 1981 original, A Wiley-Interscience Publication. MR 1038525 (90k:20001)

[DG16] Christina Delfs and Steven D. Galbraith, Computing isogenies between supersingular elliptic curves over \( \mathbb{F}_p \), Designs, Codes and Cryptography 78 (2016), no. 425-440.
[EOY05] Noam Elkies, Ken Ono, and Tonghai Yang, *Reduction of CM elliptic curves and modular function congruences*, Int. Math. Res. Not. (2005), no. 44, 2695–2707. MR 2181309

[GZ85] Benedict H. Gross and Don B. Zagier, *On singular moduli*, J. Reine Angew. Math. **355** (1985), 191–220. MR 772491

[Ibu82] Tomoyoshi Ibukiyama, *On maximal orders of division quaternion algebras over the rational number field with certain optimal embeddings*, Nagoya Math. J. **88** (1982), 181–195. MR 683249

[Kan89] Masanobu Kaneko, *Supersingular j-invariants as singular moduli mod p*, Osaka J. Math. **26** (1989), no. 4, 849–855. MR 1040429

[Kan09] Ben Kane, *CM liftings of supersingular elliptic curves*, J. Théor. Nombres Bordeaux **21** (2009), no. 3, 635–663. MR 2605537

[Lan87] Serge Lang, *Elliptic functions*, second ed., Graduate Texts in Mathematics, vol. 112, Springer-Verlag, New York, 1987. With an appendix by J. Tate. MR 890960

[LLY21] Jianing Li, Songsong Li, and Yi Ouyang, *Factorization of hilbert class polynomials over prime fields*, 2021, https://arxiv.org/abs/2108.00168.

[LMY15] Sheng-Chi Liu, Riad Masri, and Matthew P. Young, *Rankin-Selberg L-functions and the reduction of CM elliptic curves*, Res. Math. Sci. **2** (2015), Art. 22, 23. MR 3402822

[LXY21] Qun Li, Jiangwei Xue, and Chia-Fu Yu, *Unit groups of maximal orders in totally definite quaternion algebras over real quadratic fields*, Trans. Amer. Math. Soc. **374** (2021), no. 8, 5349–5403. MR 4293775

[Mil07] J. S. Milne, *The fundamental theorem of complex multiplication*, 2007, https://arxiv.org/abs/0705.3446.

[Onu21] Hiroshi Onuki, *On oriented supersingular elliptic curves*, Finite Fields Appl. **69** (2021), 101777, 18. MR 4170779

[Ser73] J.-P. Serre, *A course in arithmetic*, Graduate Texts in Mathematics, No. 7, Springer-Verlag, New York-Heidelberg, 1973, Translated from the French. MR 0344216

[Shi98] Goro Shimura, *Abelian varieties with complex multiplication and modular functions*, Princeton Mathematical Series, vol. 46, Princeton University Press, Princeton, N.J., 1998. MR 1492449 (99e:11076)

[Sil90] Joseph H. Silverman, *The arithmetic of elliptic curves*, second ed., Graduate Texts in Mathematics, vol. 106, Springer, Dordrecht, 2009. MR 2514094 (2010b:11055)

[UT15] Christian U. Jensen and Anders Thorup, *Gorenstein orders*, Journal of Pure and Applied Algebra **219** (2015), no. 3, 551–562.

[Vig80] Marie-France Vignéras, *Arithmétique des algèbres de quaternions*, Lecture Notes in Mathematics, vol. 800, Springer, Berlin, 1980. MR 580949 (82i:12016)

[Voi21] John Voight, *Quaternion algebras*, Graduate Texts in Mathematics, vol. 288, Springer, Cham, [2021] ©2021. MR 4279905

[Wat69] William C. Waterhouse, *Abelian varieties over finite fields*, Ann. Sci. École Norm. Sup. (4) **2** (1969), 521–560. MR 0265369 (42 #279)

[XLD] Guanju Xiao, Lixia Luo, and Yingpu Deng, *Supersingular j-invariants and the class number of Q(−p)*, International Journal of Number Theory, https://doi.org/10.1142/S1793042122500555.

(Chen) DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA SAN DIEGO, 9500 GILMAN DRIVE, LA JOLLA, CA 92093-0112  
Email address: mic181@ucsd.edu

(Xue) COLLABORATIVE INNOVATION CENTER OF MATHEMATICS, SCHOOL OF MATHEMATICS AND STATISTICS, WUHAN UNIVERSITY, LUOHUASHAN, 430072, WUHAN, HUBEI, P.R. CHINA

(Xu) HUBEI KEY LABORATORY OF COMPUTATIONAL SCIENCE (WUHAN UNIVERSITY), WUHAN, HUBEI, 430072, P.R. CHINA.  
Email address: xue_j@whu.edu.cn