A TENTATIVE EXPRESSION OF THE KÁROLYHÁZY
UNCERTAINTY OF THE SPACE-TIME STRUCTURE
THROUGH VACUUM SPREADS IN QUANTUM
GRAVITY

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Abstract
In the existing expositions of the Károlyházy model, quantum mechan-
ic uncertainties are mimicked by classical spreads. It is shown how to
express those uncertainties through entities of the future unified theory
of general relativity and quantum theory.

1 Introduction
In his pioneering work [1, 2] on the stochastic modification of the Schrödinger
time evolution, known as the K model (for a recent review, see [3]), Károly-
házy relates the loss of coherence and the breakdown of the superposition
principle to the uncertainty of the Einsteinian space-time structure, caused
by the quantum mechanical uncertainties of the position and of the momen-
tum of material objects. The K model leads to sound results concerning the
occurrence (or the absence) of the breakdown of the superposition principle
for various physical systems. It should be noted that in obtaining those re-
results there is no room for maneuvers, because there are no free parameters in
the K model. However, the theoretical reliability of the results is weakened
by the fact that in the calculations the quantum mechanical uncertainty of
the space-time structure has been replaced by a classical spread. In [1, 2] (see
also [4, 5]) the single space-time $S$, with a quantum mechanical uncertainty
in its structure, has been mimicked by a stochastic set $\{S_\beta\}$ of classical
space-times $S_\beta$, with appropriately chosen metric tensors $(g_{\mu\nu}(x,t))_\beta$. In-
stead of propagating on the single but “hazy” space-time $S$, the quantum
mechanical wave function had to propagate on the concise, but different
from each other space-times $S_\beta$, and the spread of the relative phases of the
wave function over the set $\{S_\beta\}$ was supposed to be equal to the amount of
the quantum mechanical uncertainty of the relative phase, induced by the uncertain structure of the single space-time $S$. In \cite{6, 7} an other stochastic space-time model, leading to the same result as the original model, has been used. In this second model $g_{\mu\nu}$ remains concisely Minkowskian, but the moments of time are randomized via the introduction of an appropriate random set $\{t_\beta(x, t)\}$ of moments of time $t_\beta(x, t)$.

Károlyházy was well aware of the shortcomings of such space-time models. In \cite{4} he wrote: “To avoid possible misunderstanding, we would like to stress that . . . no physical significance should be attached to the individual members of the family $\{(g_{\mu\nu})_\beta\}$. The only role of the family is to provide us with a mathematical model of a single physical space-time with smeared metric . . .”

Until recently, I thought that because of the lack of a unified theory of general relativity and quantum theory, the use of a classically stochastic space-time model is inevitable. It turns out that this is not so. As shown in the present paper, with the help of two entities of the future unified theory of general relativity and quantum theory, one can obtain the known results of the K model without relying on such a space-time model. In addition, the calculations are simpler than those carried out with the help of the $(g_{\mu\nu})_\beta$’s in \cite{2} (outlined also in \cite{3, 5}). There, many expressions contain $k$-space Fourier sums or integrals. In the formulas for the physical quantities these sums or integrals are convergent, but in some intermediary expressions they diverge. In the present work the calculations are carried out in $x$-space, $k$-space integrals do not appear at all, and it is easy to see that the claims that the K model needs a cutoff parameter to make divergent $k$-space integrals finite \cite{8} are not justified.

In Sections 2 and 3 two relations exhibiting the uncertainty of the space-time structure are given and discussed. In Section 4 these relations are expressed with the help of entities of the future unified theory of general relativity and quantum theory. In Sections 5 and 6 a new derivation of the formulas for two basic quantities of the K model — the quantum mechanical spread $\Delta \Phi$ of the relative phases and the cell length $a_c$ — is presented. The characterisation of quantum and of classical behavior, and the transition between them are also discussed in Section 6. Section 7 is devoted to the presentation of the stochastically modified Schrödinger evolution. In Section 8 a few concluding remarks are made.
2 Uncertainty in the Structure of the Einsteinian Space-Time and in the Relative Phases of the Wave Functions

2.1 The Lower Bound of the Uncertainty of the Length of Time Intervals

Investigating the uncertainties of the Einsteinian space-time structure induced by the quantum mechanical uncertainties in the position and in the momentum of various quantum objects, Károlyházy discovered that the uncertainty of the length $T$ of a time interval has a lower bound $\Delta L_T$. The relation between $T$ and $\Delta L_T$ is simple:

$$\Delta L_T \approx T^{2/3}P^{1/3},$$

where

$$T_P = \frac{\Lambda}{c} = \sqrt{\frac{Gh}{c^5}} = 5.3 \times 10^{-44} \text{sec}$$

is the Planck time,

$$\Lambda = 1.6 \times 10^{-33}$$

being the Planck length and $G$ the constant of gravitation.

The approximate equality sign $\approx$ in relation (1) and in further equations takes into account that because of the absence of a unified theory of general relativity and quantum theory, numerical factors, unlike the $1/2$ in Heisenberg’s relation $\Delta x \cdot \Delta p \geq \hbar/2$, could not be fixed. In the context of the present paper such factors are unimportant. Indeed, as we shall see, the value of the relevant parameter — of the cell length — characterizing the coherence changes by tens of orders of magnitude while going from quantum behavior to classical behavior. Compared to this change, a shift by a factor between $10^{-1}$ and $10$, or even between $10^{-2}$ and $10^2$, is irrelevant. However, in a prospective experimental search for the anomalous Brownian motion predicted by the K model [2, 4], the said loose factors may cause problems.

Taking advantage of the regrettable looseness in the basic relation (1), we shall often lump into the symbol $\approx$ known but neglected factors, e.g. when rounding numerical values or dropping $\pi$’s.

Two restrictions should be made concerning the applicability of relation (1).

(i) In agreement with the fact that the K model is a model for non-relativistic quantum mechanics, in relation (1) $T$ refers to a time interval
along a world line of a body slowly moving (or standing) in a reference frame in which the 2.7° K background radiation is isotropic.

(ii) When

\[ T \lesssim T_P, \]  

the very concept of space-time becomes questionable, and relation (1) may lose its physical meaning. Therefore, this relation should be applied only to time intervals for which

\[ T \gg T_P. \]  

This restriction means that besides being a fundamental parameter of the K model, \( T_P \) is also a physical cutoff parameter. It is the only inherent cutoff parameter in the model. It is needed in order to keep out from those very small space-time domains inside which the physical laws are not known, and not in order to make divergent mathematical expressions finite. Of course, similarly to regular non-relativistic quantum mechanics, the predictions of the K model become unreliable (but not divergent) when the realm of high energy particle physics is reached, that is already for time intervals of the order of \( 10^{-24} \) sec and for spatial distances of the order of \( 10^{-13} \) cm, much larger than \( T_P \) and \( \Lambda \), respectively.

Notice that for \( T = 1 \) sec, \( \Delta_L T \) is only of the order of \( 10^{-29} \) sec. On the other hand, \( \Delta_L T \) is an absolute, inescapable lower bound. For a given \( T \), it cannot be diminished at the expense of some other quantity, like \( \Delta x \) at the expense of \( \Delta p \) in Heisenberg’s uncertainty relation.

The extreme smallness of \( \Delta_L T \) is due to the fact that when deriving relation (1), only the basic laws of general relativity and of quantum mechanics should be respected, all other theoretical, as well as all practical limitations should be ignored. As a result, the minimal time uncertainty of actual physical processes of duration \( T \) is much larger than the lower bound \( \Delta_L T \) for that value of \( T \). \( \Delta_L T \) could be reached only in processes with bodies of unrealistically high density, although this density is still much-much smaller than the Planck density \( \rho_P = m_P/\Lambda^3 \approx 10^{94} \text{ g/cm}^3 \). Thus, the existence of such irrealistic bodies would not contradict the basic laws of general relativity and of quantum mechanics.

Only the existence and the order of magnitude of the lower bound \( \Delta_L T \) is exploited in the K model, the possibility of reaching it is not necessary.
2.2 The Uncertainty of the Space-Time Structure and the Breakdown of the Superposition Principle

Attention should be paid to the rather remarkable fact that all the physical quantities referring to a particular body (like its mass, its velocity, etc.) dropped out from relation (1), the relation between \( T \) and \( \Delta L_T \) involves only the universal constant of nature \( T_P = \sqrt{G\hbar/c^5} \). Now, a space-time relation independent of any particular property of matter can be, perhaps even must be attributed to space-time itself, therefore also to the empty space-time. Accordingly, Károlyházy proposed to regard relation (1) as an expression of the uncertainty of the structure of space-time. \( \Delta L_T \) gives then the measure of the limitation of the sharpness of the Einsteinian space-time structure, imposed by quantum mechanics. As shown in [1, 2], this tiny uncertainty of the space-time structure induces uncertainties in the relative phases of the wave function of any isolated system, and thereby limits, in return, the sharpness of the phase relations. The amount of the uncertainty of the relative phases turns out to be negligible in the case of microsystems, but it is large enough to destroy the coherence of the wave function of a macrosystem, in agreement with the observed breakdown of the superposition principle.

3 The Structural Uncertainty of Synchronization

The structural uncertainty \( \Delta L_T \) of the length of the time intervals along the \( |v| \ll c \) worldlines produces a structural uncertainty of the same order of magnitude in the synchronization of the times between two such worldlines \( W_1 \) and \( W_2 \) at a distance \( r \) from each other. Let the times along these worldlines be synchronized. A light signal emitted on \( W_1 \) arrives back to \( W_1 \) from \( W_2 \) after a time \( 2T \), where

\[
T = \frac{r}{c}.
\]

In the space-time of Károlyházy, the time interval of length \( 2T \) along \( W_1 \) has the structural uncertainty \( \Delta L_T \) given by relation (1). (The uncertainties of \( 2T \) and of \( T \) are of the same order of magnitude.) Consequently, the moment of arrival of a signal to \( W_2 \) suffers from the same uncertainty relative to the time along \( W_1 \). This means that the uncertainty of the synchronization of the times along two \( |v| = 0 \) worldlines at a distance \( r = cT \)
from each other has a structural lower bound

\[ (\Delta L T)_{\text{syn}} \approx T^3 \left( \frac{\tau}{c} \right)^{1/3} = \frac{\Lambda^{2/3} \tau^{1/3}}{c}. \]  

(7)

This lower bound is by many orders of magnitude smaller than the uncertainty in the synchronization carried out by realistic quantum clocks. But again, we shall rely only on the existence of relation (7), the impossibility of actually reaching the lower bound \((\Delta L T)_{\text{syn}}\) is not important.

4 The Expression of the Uncertainties \(\Delta L T\) and \((\Delta L T)_{\text{syn}}\) in Terms of Entities of the Future Unified Theory of General Relativity and Quantum Theory

Since in relations (1) and (7) \(\Delta L T\) and \((\Delta L T)_{\text{syn}}\) are uncertainties of the empty space-time, in the future unified theory they will presumably take the form of vacuum spreads of appropriate operators. The vacuum state \(|V\rangle\) should “know” about general relativity, i.e., about gravitons, and if the K model is sound, then in the non-relativistic approximation \(|V\rangle\) should represent the empty space-time of Károlyházy, instead of the empty Minkowskian space-time. As far as the appropriate operators are concerned, they should refer to time because relations (1) and (7) are time uncertainty relations, and since these relations do not contain quantities describing particular objects, the operators should not refer to particular objects either. An effective local time operator \(\hat{t}(x, t)\) meets the above requirements. We call this operator effective, because it may well be that similarly to non-relativistic quantum mechanics, there will be no time operator at the basic level of the unified theory. The reason for considering the operator \(\hat{t}(x, t)\) is that, as we shall see presently, the uncertainties \(\Delta L T\) and \((\Delta L T)_{\text{syn}}\) can be expressed in a simple way with the help of \(\hat{t}(x, t)\) and \(|V\rangle\). Also, \(\hat{t}(x, t)\) exhibits two important features of the K model. It states that time is not a global, but a local quantity, in agreement with the involvement of general relativity in the K model, and says that the values of the moments of time have an uncertainty, corresponding to the uncertainty of the space-time structure.

We assume that the vacuum expectation value (the “vev”) of \(\hat{t}(x, t)\) is independent of \(x\) and equals \(t\),

\[ \langle \hat{t}(x, t) \rangle_V = t, \]  

(8)
and we write \( \hat{t} \) in the convenient form

\[
\hat{t}(x, t) = t + \hat{\tau}(x, t),
\]

(9)

where, due to (8),

\[
\langle \hat{\tau}(x, t) \rangle_V = 0.
\]

(10)

Let us now look at relation (1). In this relation \( T \) stands for the length of a time interval belonging to a segment \([ (x, t), (x, t') ]\) of a \(|v| = 0\) worldline in the empty space-time of Károlyházy. It is therefore reasonable to identify \( T \) with the vev of the operator difference

\[
\hat{t}(x, t') - \hat{t}(x, t).
\]

(11)

From (8) and (9) we see that \( T \) is equal to the Minkowskian length \( t' - t \) of our time interval (we take \( t' > t \)):

\[
T := \langle \hat{t}(x, t') - \hat{t}(x, t) \rangle_V = t' - t.
\]

(12)

Since \( \Delta_L T \) is the uncertainty of \( T \) in the empty Károlyházy space-time, it should be given by the vacuum spread of the operator difference in (11):

\[
(\Delta_L T)^2 := \langle (\hat{t}(x, t') - \hat{t}(x, t) - T)^2 \rangle_V.
\]

(13)

With equations (8) and (12) one finds that

\[
\langle \hat{t}(x, t') - \hat{t}(x, t) \rangle_V = 0,
\]

(17)

therefore

\[
(\Delta_L T)^2_{\text{syn}} = \langle (\hat{\tau}(x', t) - \hat{\tau}(x, t))^2 \rangle_V.
\]

(18)
and relation (7) takes the form

\[ \langle (\hat{\tau}(x', t) - \hat{\tau}(x, t))^2 \rangle_V \approx \frac{T_P^{4/3} r^{2/3}}{c^2}, \]  

(19)

where

\[ r = |x' - x|. \]  

(20)

A mathematical remark should be made here. The left-hand side of relation (19) can be written in the form

\[ \langle \hat{\tau}^2(x', t) + \hat{\tau}^2(x, t) - \hat{\tau}(x', t)\hat{\tau}(x, t) - \hat{\tau}(x, t)\hat{\tau}(x', t) \rangle_V, \]  

(21)

involving vev’s of bilinear products of \( \hat{\tau} \)’s taken at equal time. It is well known from quantum field theory that such vev’s of local field operators are, as a rule, divergent, and become finite only after renormalization. In the absence of a detailed theory, one cannot evaluate the individual vev’s in (21). However, if relation (7) of the K model, leading to (19), is correct, then the divergences in (21) should cancel and the finite part should give the right-hand side of (19). A similar remark applies to relation (14).

5 Uncertainties in the Phases of the Quantum States and the Spread of the Relative Phases

In regular non-relativistic quantum mechanics, the pure state of an isolated physical system, constituted by \( N \) microparticles with masses \( M_1, M_2, \ldots, M_N \), is usually represented by a Schrödinger wave function

\[ \Psi(x, t) = \exp \left( -\frac{i}{\hbar} \hat{H} t \right) \cdot \Psi(x, 0), \]  

(22)

where \( \hat{H} \) stands for the Hamiltonian of the system, and the evolution of \( \Psi(x, t) \) takes place on the Minkowskian space-time. Since the Schrödinger evolution is deterministic, \( \Psi \), and consequently also its relative phases between any pairs of points

\[ x = (x_1, \ldots, x_N) \]  

(23)

and \( x' \) of the configuration space are sharply determined. We shall call such a wave function “perfectly coherent”.

\(^1\)Spin variables are omitted, because they do not play a role in the K model.
In the K model, the quantum state has to propagate on the Károlyházy space-time having the discussed uncertainty in its structure. As we have seen, this uncertainty can be taken into account by substituting the effective time operator \( \hat{\tau}(x, t) \) for the global Minkowskian time \( t \):

\[
t \to \hat{\tau}(x, t) = t + \hat{\tau}(x, t).
\]  

(24)

At this point we have to recall that on both sides of Equation (22) the rest energy phase factor

\[
\exp(i\Phi(t)) := \exp\left(-\frac{i}{\hbar} \sum_{\ell=1}^{N} M_\ell c^2 t \right)
\]

(25)

is omitted, because being independent of \( x \) and of \( p = -i\hbar(\nabla_1, \ldots, \nabla_N) \), it drops out from all the observables. However, under the substitution (24), one has to put in \( \Phi(t) \)

\[
M_\ell c^2 t \to M_\ell c^2 \hat{\tau}(x_\ell, t),
\]

(26)

because the coordinate of the \( \ell \)-th particle is \( x_\ell \). This implies that

\[
\Phi(t) \to \Phi(t) + \hat{\Phi}(x, t),
\]

(27)

where

\[
\hat{\Phi}(x, t) = -\frac{c^2}{\hbar} \sum_{\ell=1}^{N} M_\ell \hat{\tau}(x_\ell, t).
\]

(28)

The rest energy phase \( \Phi(t) \) can be omitted again, but the \( x \)-dependent rest energy phase operator \( \hat{\Phi}(x, t) \) should be kept.

The substitution \( t \to \hat{\tau}(x, t) \) should have been carried out in the Schrödinger wave function (22), too. However, the non-relativistic matrix elements of \( \hat{H} \) are much smaller than the rest energies of the particles. For solid bodies their contribution has been estimated in \([2]\) in the framework of the \((g_{\mu\nu})_\beta\) model. In the present paper we shall neglect it.

Keeping only the contribution of the rest energy phase, one realizes that in the K model the Schrödinger wave function acquires an operator phase factor

\[
\exp(i\hat{\Phi}(x, t)).
\]

In other words, with any Schrödinger wave function \( \psi(x, t) \) one has to associate a Károlyházy state

\[
\hat{\Psi}_K(x, t) = \exp(i\hat{\Phi}(x, t)) \cdot \Psi(x, t).
\]

(29)

Unlike \( \Psi(x, t) \), the K state \( \hat{\Psi}_K(x, t) \) is not perfectly coherent. Through the operator phase factor it feels the uncertainty of the space-time structure. Its
departure from perfect coherence between two points \( x, x' \) of the configuration space can be assessed by the amount of the uncertainty of its relative phase, i.e. by the vacuum spread \( \Delta \Phi(x, x', t) \) of the relative phase operator

\[
\hat{\Phi}_R(x, x', t) = \hat{\Phi}(x', t) + \varphi(x', t) - \hat{\Phi}(x, t) - \varphi(x, t)
\]

of the K state between those points. Here

\[
\varphi(x, t) = \arg \Psi(x, t)
\]
denotes the phase of \( \Psi(x, t) \).

The \( x, x' \) dependence of \( \Delta \Phi \) can be found with the help of relation (19). From Equations (28) and (10) it follows that the vev of \( \hat{\Phi}_R \) is equal to the Schrödingerian relative phase,

\[
\left\langle \hat{\Phi}_R(x, x', t) \right\rangle_V = \varphi(x', t) - \varphi(x, t),
\]

which drops then out from the vacuum spread of \( \hat{\Phi}_R \):

\[
\Delta^2 \Phi(x, x', t) = \left\langle \left( \hat{\Phi}_R(x, x', t) - \varphi(x', t) - \varphi(x, t) \right)^2 \right\rangle_V = \left\langle \left( \hat{\Phi}(x', t) - \hat{\Phi}(x, t) \right)^2 \right\rangle_V.
\]

With Equation (28) one finds

\[
\Delta^2 \Phi(x, x', t) = \frac{c^4}{\hbar^2} \sum_{i, \ell=1}^N M_i M_\ell \left\langle \left( \tilde{\tau}(x'_i, t) - \tilde{\tau}(x_i, t) \right) \left( \tilde{\tau}(x'_\ell, t) - \tilde{\tau}(x_\ell, t) \right) \right\rangle_V.
\]

The vev in the last expression is identically equal to

\[
\frac{1}{2} \left\langle \left( \tilde{\tau}(x'_i, t) - \tilde{\tau}(x_i, t) \right)^2 + \left( \tilde{\tau}(x'_\ell, t) - \tilde{\tau}(x_\ell, t) \right)^2
- \left( \tilde{\tau}(x'_i, t) - \tilde{\tau}(x'_\ell, t) \right)^2 - \left( \tilde{\tau}(x_i, t) - \tilde{\tau}(x_\ell, t) \right)^2 \right\rangle_V,
\]

and with relation (19) one obtains for the spread of the relative phase the formula

\[
\Delta^2 \Phi(x, x') \approx \Lambda^{4/3} \frac{c^2}{\hbar^2} \sum_{i, \ell=1}^N M_i M_\ell \left( |x'_i - x_i|^{2/3} - \frac{1}{2} |x_i - x_\ell|^{2/3} - \frac{1}{2} |x'_i - x'_\ell|^{2/3} \right)^2.
\]
The time argument of $\Delta \Phi$ has been omitted since $\Delta \Phi$ turned out to be time independent, although $\hat{\Phi}(x, t)$ may depend on $t$.

According to Equation (36), $\Delta \Phi$ increases with the masses and with the number of the microparticles constituting the system, and for a given system it increases with the distances $|x'_i - x_\ell|$, that is with the separation between the points $x, x'$ in the configuration space. These are encouraging features concerning the expected loss of coherence between “macroscopically distinct” components of the quantum state of a macroscopic body.

With the notations of the present paper, the formula for $\Delta \Phi$ derived in [2] (see also [3]) with the help of the $(g_{\mu\nu})_3$'s reads

$$\Delta_\Phi^2(x, x') \approx \Lambda^{4/3} c^2 \int \frac{d^3k}{k^{11/3}} |\mu_k(x') - \mu_k(x)|^2,$$

where

$$\mu_k(x) = \sum_{\ell} M_\ell e^{ikx_\ell} $$

is the Fourier transform of the mass distribution

$$\rho(x) = \sum_{\ell=1}^N M_\ell \delta(x - x_\ell)$$

of $N$ pointlike particles of masses $M_1, \ldots, M_N$ in the configuration $x = [x_1, \ldots, x_N]$. From Equation (37) one sees that the uncertainty of the relative phase increases with the difference (of the absolute values of the Fourier transform) of the mass distribution of the $N$ particles in the configurations $x$ and $x'$.

The formula (36) for $\Delta \Phi$ has been obtained previously with the help of the $\{t_\beta\}$ model by the present author [7], who realized then that the Fourier integral in the original formula (37) can be calculated analytically and is equal to the sum in formula (36). So, if the expressions (13) and (18) for $\Delta LT$ and $(\Delta LT)_{syn}$ are reliable, then the influence of the single, quantum mechanically uncertain space-time on the relative phases has been correctly mimicked by both classical space-time models.

6 Coherence Properties of the K states, Coherence Cells and Cell Lengths
6.1 K States with Nearly Perfect and with Destroyed Coherence. The Coherence Cell

One sees from formula (37) that for any pairs of points $x' \neq x$, $\Delta \phi > 0$. This means that a K state $\hat{\Psi}_K$, normalized to 1, is never perfectly coherent.

Concerning the norm of $\hat{\Psi}_K$, from (29) one gets

$$\hat{\Psi}_K^+(x,t)\hat{\Psi}_K(x,t) = |\Psi(x,t)|^2, \quad (40)$$

because the unitary operator phase factor drops out from the product. Therefore, the weight $w_\Omega$ of a K state in a domain $\Omega$ of the configuration space is equal to the weight of the Schrödinger wave function associated with $\hat{\Psi}_K$,

$$w_\Omega = \int_{\Omega} dx |\Psi(x,t)|^2, \quad (41)$$

and $\hat{\Psi}_K$ is normalized to 1 together with $\Psi$. (Pedantically, in (40) one should consider the vev of $\hat{\Psi}_K^+(x,t)\hat{\Psi}_K(x,t)$, but it is obviously equal to the product itself.)

Let us now see how the coherence of the K states can be characterized.

(1) If $\hat{\Psi}_K$ occupies a domain $\Omega$ of the configuration space such that

$$\Delta \phi(x,x') \ll \pi \quad \text{for all} \quad x,x' \in \Omega, \quad (42)$$

then in good approximation the uncertainties of the relative phases of $\hat{\Psi}_K$ can be neglected, and the relative phases of $\hat{\Psi}_K$ are practically equal to those of $\Psi$. In such a case we shall say that the coherence of $\hat{\Psi}_K$ is nearly perfect.

(2) If $\hat{\Psi}_K$ occupies a domain containing non-overlapping subdomains $\Omega$, $\Omega'$ such that

$$\Delta \phi(x,x') \geq \pi \quad \text{for all} \quad x \in \Omega \quad \text{and all} \quad x' \in \Omega', \quad (43)$$

then the relative phases of $\hat{\Psi}_K$ between these subdomains are completely uncertain, the coherence between the components of $\hat{\Psi}_K$ belonging to these subdomains is destroyed. Notice, however, that within smaller subdomains inside which $\Delta \phi < \pi$, a certain degree of coherence persists, and inside

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2As a rule, $\Psi$ is different from zero almost everywhere. Therefore, strictly speaking, $\Psi$, and then $\hat{\Psi}_K$ too, occupy the whole configuration space. In our terminology the “domain occupied by $\Psi$” is the smallest domain in which the weight $w$ of $\Psi$ is close to the maximal weight 1 (e.g. $w = 1 - 10^{-4}$). The expansion and the shrinking (or the localization) of $\Psi$ means that this domain expands or shrinks.
sufficiently small subdomains even $\Delta \Phi \ll \pi$ holds, so that within such a small domain $\hat{\Psi}_K$ is near to perfect coherence.

The maximal domains $\Omega_c$ of the configuration space such that

$$\Delta \Phi(x, x') \leq \pi \text{ for all } x, x' \in \Omega_c, \quad (44)$$

have been called “coherence cells” in [2]. A K state occupying a single cell is still not incoherent, but if it occupies non-overlapping cells, then it is incoherent (it has incoherent components of non-negligible weights).

The size and the shape of the coherence cell depends strongly on the composition of the physical system considered. Below we shall look at the cells of microparticles and of a class of solid objects. As we shall see, the cells of these systems are spherical, and can therefore be characterized by a single parameter, the diameter of the cell.

### 6.2 The Coherence Cell and the Cell Length of a Single Microparticle

For a single microparticle of mass $M = M_1$, one easily finds from formula (36) that

$$\Delta \Phi(a) \approx \frac{\Lambda^{2/3}}{L} a^{1/3}, \quad (45)$$

where (with $x = x_1$)

$$a = |x' - x|, \quad (46)$$

and

$$L = \frac{\hbar}{Mc} \quad (47)$$

is the Compton wavelength of the particle. Notice that $\Delta \Phi$ increases monotonically with the distance $a$. The coherence cell is therefore a sphere of diameter $a_c$ in the configuration space $x = (x)$, where $a_c$ is equal to the value of $a$ at which $\Delta \Phi$ reaches the value $\pi$:

$$\Delta \Phi(a_c) = \pi \approx 1. \quad (48)$$

From (45) one finds that

$$a_c \approx \left( \frac{L}{\Lambda} \right)^2 L. \quad (49)$$

So, for a microparticle the coherence cell is characterized by a single parameter, the cell length $a_c$. In [2] the term “cell diameter” has been used.
However, there are physical systems the coherence cell of which is not spherical, and then $a_c$ is not a diameter.

For the electron $L \approx 10^{-11}$ cm, and

$$a_c \approx 10^{33} \text{ cm.}$$

(50)

The cell length of the electron, and also of the other microparticles, has a supraastronomical value. Therefore, any realistic Schrödinger wave function $\Psi$ of an isolated microparticle occupies only a tiny part of a coherence cell, a part inside which $\Delta \Phi \ll \pi$, and the K state associated with $\Psi$ is always practically perfectly coherent. Due to the very small masses of the microparticles, the uncertainty of the space-time structure has no appreciable effect on a single microparticle. For the same reason this is also true for any isolated microsystem (for a system consisting of a few microparticles, free or bound).

### 6.3 The Coherence Cell and the Cell Length of Spherical, Homogeneous Solid Objects

In a homogeneous object there are $N \gg 1$ identical microscopic constituents (e.g. molecules). Formula (36) for $\Delta \Phi$ then becomes

$$\Delta_\Phi^2(x, x') \approx \frac{L_{\text{micro}}^{4/3}}{\Lambda} \sum_{i, \ell=1}^{N} \left( |x'_i - x_\ell|^2/2 - \frac{1}{2} |x_i - x_\ell|^2/3 - \frac{1}{2} |x'_i - x'_\ell|^2/3 \right) ,$$

(51)

where

$$L_{\text{micro}} = \frac{\hbar}{M_{\text{micro}} c}$$

(52)

is the Compton wavelength of a constituent of mass $M_{\text{micro}}$.

In a solid object the constituents are at, or very close to, their equilibrium positions. Consequently, the Schrödinger wave function of an isolated solid object is practically zero everywhere, except in such points $x = (x_1, \ldots, x_N)$ of the configuration space, in which the constituents are at their equilibrium positions. Therefore, in the case of a homogeneous, spherical solid object of radius $R$, in $\Delta \Phi(x, x')$ the $x_i$ coordinates belonging to $x$ are distributed uniformly in the volume of a sphere of radius $R$, and the $x'$ coordinates belonging to $x'$ fill uniformly an other such sphere in the three dimensional $XYZ$ space.

For a solid object of arbitrary shape, two equilibrium configurations $x, x'$ differ from each other by a translation of their center of mass (c.m.) and
by a rotation leaving the c.m. fixed. For a spherical homogeneous object, 
$\Delta \Phi(x, x')$ does not change appreciably under a rotation, because $\Delta \Phi$ is in-
variant under any permutation of the $x$ coordinates among themselves and
of the $x'$ coordinates among themselves. Therefore, we have to consider only
such configurations which differ from each other by a translation

$$x'_i = x_i + a,$$  \hspace{1cm} (53)

where

$$a = x'_{\text{c.m.}} - x_{\text{c.m.}}.$$  \hspace{1cm} (54)

is the vector joining the centers of the spheres, which are also the centers of
mass of the objects in the two configurations.

With (53) the sum $\sum$ in (51) becomes

$$\sum = \sum_{i,\ell=1}^{N} \left( |x_i - x_\ell + a|^{2/3} - |x_i - x_\ell|^{2/3} \right).$$  \hspace{1cm} (55)

The $x_i$ and the $x_\ell$ coordinates fill a sphere uniformly, $N \gg 1$, and the
expression under the sum is a continuous, slowly varying function of the
coordinates. Therefore, the double sum in (55) can be replaced, in a good
approximation, by a double integral. With $x_i \rightarrow r$, $x_\ell \rightarrow r'$, one gets

$$\sum = \frac{N^2}{V^2} \int_V d^3r \int_V d^3r' \left( |r - r' + a|^{2/3} - |r - r'|^{2/3} \right),$$  \hspace{1cm} (56)

where the integrals have to be taken over the volume of the same sphere of
radius $R$. $\sum$ can be calculated for any value of $a$, but it is more enlightening
to evaluate it in two extreme situations, namely when $|a| \equiv a \gg R$ and when
$a \ll R$.

a) The case $a \gg R$

In this case one has also $a \gg |r - r'|$, because in (56) $|r - r'| \leq 2R$. Therefore, in good approximation,

$$|r - r' + a|^{2/3} - |r - r'|^{2/3} = a^{2/3},$$  \hspace{1cm} (57)

and

$$\sum = N^2 a^{2/3}.$$  \hspace{1cm} (58)

Noticing that

$$\frac{N}{L_{\text{micro}}} = \frac{1}{L},$$  \hspace{1cm} (59)
where \( L \) is the Compton wavelength corresponding to the mass \( M = N M_{\text{micro}} \) of the object considered, Equation (51) becomes
\[
\Delta \Phi(x, x') \approx \frac{\Lambda^2}{L} a^{1/3}, \quad a \gg R.
\] (60)

This formula is formally identical with the one obtained for a single microparticle, but \( a \) stands now for the distance between the centers of mass of the object in the two configurations \( x, x' \). The coherence cell is again a sphere of diameter
\[
a_c \approx \left( \frac{L}{\Lambda} \right)^2 L, \quad a_c \gg R,
\] (61)
but now in the center of mass coordinate subspace of the configuration space.

b) The case \( a \ll R \)

In this case in (56) \( a \ll |r - r'| \), except for a small subdomain of the integration domain. The detailed calculation shows that one can forget about the violation of the condition \( a \ll |r - r'| \). Expanding \( |r - r' + a|^{2/3} \) in powers of \( a \), one finds that the leading contribution to \( \sum \) is of the order \( a^2 \). Since the dimension of \( \sum \) is \( cm^{3/2} \), and apart from \( a \) the only length parameter in (56) is \( R \), one finds that
\[
\sum \approx N^2 \frac{a^2}{R^{4/3}}, \quad a \ll R.
\] (62)

Equation (51) takes now the form
\[
\Delta \Phi(x, x') \approx \left( \frac{\Lambda}{R} \right)^{2/3} \frac{a}{L}, \quad a \ll R.
\] (63)

\( \Delta \Phi \) increases again monotonically with \( a \), and reaches the value \( \pi \approx 1 \) when \( a \) is equal to
\[
a_c \approx \left( \frac{R}{\Lambda} \right)^{2/3} L, \quad a_c \ll R.
\] (64)

Formulas (61) and (64) for the cell length of spherical, homogeneous objects have been presented in [2] as “the most important results” of the model. Their physical meaning has been discussed in due detail in [2, 4, 3]. Here we recall only that \( a_c \gg R \) is the region where quantum behavior, \( a_c \ll R \) — the region where classical behavior dominates. Indeed, if \( a_c \gg R \), then the Schrödinger wave function \( \Psi(x, t) \) may have an uncertainty of the order of \( a_c \) in the position of the c.m., much larger than the geometrical size \( R \) of
the object, without making the K state associated with $\Psi$ incoherent. Large coherent uncertainties correspond to quantum behavior. On the contrary, when $a_c \ll R$, the K state becomes incoherent when the uncertainty of the position of the c.m. is still much smaller than the geometrical size $R$ of the object. In other words, when $a_c \ll R$, the coherent, quantum mechanical uncertainty of the c.m. is much smaller than $R$. Small quantum mechanical positional uncertainty is a characteristic of classical behavior.

6.4 Tiny Grains, Macroscopic Bodies and the Transition Region between Quantum and Classical Behavior

It can be shown that formulas (61) and (64) hold, with small corrections absorbable in the symbol $\approx$, in the whole region $a_c \geq R$ and $a_c \leq R$, respectively. In particular, they hold also in the region $a_c \approx R$. It follows from the discussion in the preceding subsection that the latter region is, for spherical homogeneous objects, the transition region between quantum and classical behavior. In this region the maximal quantum mechanical uncertainty of the position of the c.m. is of the same order of magnitude as the geometrical size of the object. For usual terrestrial densities $\rho \approx 1 \text{ g/cm}^3$, one easily finds from Equation (61) (as well as from (64), of course) that

$$a_c^{\text{tr}} \approx R^{\text{tr}} \approx 10^{-5} \text{ cm},$$

$$M^{\text{tr}} = \frac{4\pi}{3} \rho (R^{\text{tr}})^3 \approx 10^{-14} \text{ g}.$$  \hspace{1cm} (65) (66)

Thus, the transition mass region is the region of the colloidal grains and dust particles. The region $a_c \gg R$ of quantum behavior corresponds to tiny grains with masses much smaller than $M^{\text{tr}}$. To give an example, for a grain of $R \approx 10^{-6} \text{ cm}$ and of mass $M \approx \rho R^3 \approx 10^{-18} \text{ g}$, one finds from (61) that $a_c \approx 10^5 \text{ km}$, indeed much larger than $R$. If the wave function of such an isolated grain expanded, say, over $10^3 \text{ km}$ in the c.m. coordinate subspace, its associated K state would remain still very coherent. On the contrary, for a ball of $M \approx 1 \text{ g}$ and radius $R \approx 1 \text{ cm}$, one finds from (64) that

$$a_c \approx 10^{-16} \text{ cm},$$

3The estimate $a_c \approx 10 \text{ km}$, quoted in [3], is erroneous.

a value much smaller indeed than $R$. The K model states that two positions of this ball, with separation $2a_c \approx 2 \cdot 10^{-16} \text{ cm}$ between the centers of mass, correspond already to an incoherent K state, these positions are already
“macroscopically distinct”. We are in the region of predominantly classical behavior, with $a_c \ll R, M \gg M^u$. It should be noted that formula (64) for $a_c$ is applicable only to bodies of moderate size. Above $R = 1$ m the elastic vibrations of the body are not negligible.

7 The Stochastic Modification of the Schrödinger Evolution of Isolated Systems

The mode of the combination of a stochastic process with the Schrödinger evolution is suggested by the behavior of the $K$ state

$$\hat{\Psi}_K(x,t) = \exp(i\hat{\Phi}(x,t)) \cdot \Psi(x,t)$$ (68)

during the Schrödinger evolution of its associated $\Psi$ function. As it is well known, during that evolution $\Psi$ expands\(^4\), at least in the c.m. subspace of the configuration space. According to Equations (68) and (40), $\hat{\Psi}_K$ expands (or shrinks) exactly in the same way as $\Psi$. However, while $\Psi$ remains perfectly coherent, the coherence of $\hat{\Psi}_K$ deteriorates during the expansion, and when it occupies a domain larger than a coherence cell, it has already incoherent components.

The basic idea\(^1, 2\) for the introduction of the stochastic element is that the $K$ state $\hat{\Psi}_K$ counterbalances its loss of coherence by stochastically and instantaneously localizing itself, of course together with $\Psi$, to one of the coherence cells lying in the domain to which the state expanded. After a localization, $\Psi$ expands again under the Schrödinger equation, until the situation gets ripe for a new localization, and so on.

There are many, essentially equivalent ways to specify these expansion–localization cycles. In\(^2\), for an isolated system the coherence cell of which is characterized by a single cell diameter $a_c$, an instantaneous random localization occurs whenever the quantum state occupies a domain of diameter $2a_c$. The localizations occur then at discrete moments of time, at intervals

$$\tau_c \approx \frac{Ma_c^2}{\hbar},$$ (69)

the time needed for the Schrödinger wave function of an isolated system to expand from size $a_c$ to $2a_c$.

It has been shown, among others by Ghirardi, Rimini and Weber\(^10\) and by Caves and Milburn\(^11\), that the stochastic localizations from size $2a_c$ for appropriately awkward initial wave functions, a transient shrinking precedes the steady expansion.
to $a_c$ occurring at equally spaced discrete moments of time, can be replaced
by infinitesimal stochastic localizations from size $a_c + da_c$ to $a_c$ occurring
continuously in time, blended with the continuous Schrödinger evolution.
The application of the infinitesimal GRW localizations to the K model, in
the case of a single cell length $a_c$, has been carried out in [9].

Let us now look at the expansion–localization cycles of various physical
systems. Since for the electron $a_c \approx 10^{33}$ cm, a realistic wave function of an
electron occupies only a tiny part of a coherence cell. Therefore, the Schrö-
dinger evolution is not interrupted by localizations. However, if a ball of 1 g
could be isolated, then as soon as its $\Psi$ function would expand to a domain of
diameter $2a_c \approx 2 \cdot 10^{-16}$ cm (in the c.m. subspace), a stochastic localization
onto a single cell of diameter $a_c$ would take place. So, according to the
K model, the localization of the c.m. of the ball would remain practically
pointlike even if the ball were isolated. Notice that from $\Delta x \cdot \Delta v \approx \hbar/M$
one finds $\Delta v \approx 10^{-11}$ cm/sec. The uncertainty of the velocity would be very
small, too.

Of course, a macroscopic ball cannot be perfectly isolated. The inter-
play of the Károlyházy law of time evolution with the effects caused by the
surroundings has been discussed in [2, 4, 9].

Károlyházy succeeded in applying the basic idea of his time evolution law
to systems of many quasi independent degrees of freedom. In p articular, he
worked out the process of the decay of the superposition of tracks in a cloud
chamber. The detailed discussion is given only in [2]. For a short outline,
see [5, 6, 3].

8 Concluding Remarks

The expression of the structural uncertainties $\Delta_L T$ and $(\Delta_L T)_{syn}$
through $\hat{\tau}$ and $|V\rangle$ and the derivation of the formula for the spread $\Delta_\phi$
of the relative phases without the use of a space-time model, is a tentative step towards
turning the K model into a theory. A further step could concern the law of
time evolution described in the preceding section. There, the phase operator
$\Phi$ producing the spread of the relative phases acts as a “hop-master”. It
tells when $\hat{\Psi}$, and with it $\hat{\Psi}_K$, should jump in order to prevent the loss of
coherence of $\hat{\Psi}_K$. It would be much better if one could set up a general
stochastic differential equation for $\hat{\Psi}_K$, presumably of Itô type, and let the
equation work. However, this does not seem to be possible without knowing
more about the phase operator $\Phi$, that is about the local time operator
$\hat{\tau}(x, t)$ entering into $\Phi$. The basic relations (1) and (3) made possible only to
evaluate the vacuum spreads (15) and (19), involving $\tilde{\tau}$’s. This was sufficient to derive the formulas for $\Delta \Phi$ and then for $a_c$, and to formulate with their help the hop-master’s rule. Of course, it is possible, after having calculated the cell length $a_c$ (or the cell lengths $a_c(1), a_c(2), \ldots$) of a given system, to set up an Itô equation containing these $a_c$’s, which would smoothly reproduce the results of the bumpy hop-master’s rule. What we have in mind is a general equation for $\tilde{\Psi}_K$, not containing the parameters $a_c(i)$ of the particular system to which the equation is applied. It is probable that such an equation cannot be found without knowing more about the unified theory of general relativity and quantum theory.

Another open problem is the relativistic generalization of the K model. This is a common open problem of the existing models with stochastic modification of the Schrödinger evolution. In the case of the K model, a specific task arises. One should find, first of all, a covariant description for the structural uncertainty of the Károlyházy space-time. Again, there is little hope for progress without the unified theory.

A direction into which progress can be made is the derivation of the basic relation (1) between $T$ and $\Delta L_T$. This relation has been deduced by Károlyházy in three different ways [1, 2; 2, 3; 4]. On the one hand, it is reassuring that various approaches lead to the same result. On the other hand, each of these approaches, taken separately, has loose ends. For instance, in [2] and [3] relation (1) has been obtained from the study of the quantum behavior of the hand of a clock, and one can legitimately ask what would happen if the dial also entered the game. The answer [12] is that the dial would not upset relation (1). This will be shown in a forthcoming paper, where the derivation(s) of relation (1) will be scrutinized and a comparison of Károlyházy’s clock with a Wigner–Salecker clock [13] will also be made.

Relations (1) and (7) have been recently rediscovered by Ng and Van Dam [14]. These authors obtain also the formula for the spread of the relative phases for a single particle (our formula (15)), but they do not consider composite systems. Having only $\Lambda$, $a$ and $L$ at their disposal, they had to choose ($\Delta \Phi \approx 1, a \lesssim L$) as the condition for classical behavior, which leads then to the condition $m \gtrsim m_P \approx 10^{-5}$ g, $m_P$ being the Planck mass.

The authors remark that this condition is only a sufficient condition for classical behavior. Indeed, objects with masses much smaller than $10^{-5}$ g are known to behave classically. As we have seen, in the K model the transition mass depends not only on $m_P$ (i.e. on $\Lambda$), and for homogeneous spherical solid objects the sufficient and necessary condition for classical behavior is $a_c \ll R$. It should be noted that in [14] there are many interesting relations,
not overlapping with the K model.

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