Uniqueness in Harper’s vertex-isoperimetric theorem

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Abstract

Harper’s vertex-isoperimetric theorem states that, to minimise the $t$-neighbourhood $N^t(A)$ of a subset $A$ of the hypercube $Q_n$, one should take an initial segment of the simplicial order. Aubrun and Szarek asked the following question: if $N^t(A)$ and $N^t(A^c)$ are minimal for all $t > 0$, does it follow that $A$ is isomorphic to an initial segment of the simplicial order?

Our aim is to give a counterexample. Perhaps surprisingly, it turns out that there is no counterexample that is a Hamming ball. We also classify all counterexamples, and prove some related results.

Keywords: Harper’s theorem, isoperimetric inequality

1 Introduction

The $n$-dimensional hypercube $Q_n$ has vertex-set the power set $\mathcal{P}\{1, \ldots, n\}$ with metric $d(x, y) = |x \Delta y|$. For a subset $A$ of the hypercube $Q_n$ define the *neighbourhood* of $A$ to be the set $N(A) = \{x \in Q_n : d(x, A) \leq 1\}$, where $d(x, A) = \min_{y \in A} d(x, y)$. Also more generally for each $t > 0$ define $N^t(A) = \{x \in Q_n : d(x, A) \leq t\}$.

In order to state Harper’s vertex-isoperimetric theorem we need a few definitions. For any $n$ and $0 \leq r \leq n$ define the *lexicographic order* on $[n]^{(r)} = \{A : A \subseteq \{1, \ldots, n\}, |A| = r\}$ to be given by $A <_{lex} B$ if $\min(A \Delta B) \in A$ and define the *simplicial order* on $Q_n$ to be given by $A <_{sim} B$ if

$$ |A| < |B| \text{ or } (|A| = |B| \text{ and } A <_{lex} B) $$

Theorem 1 (Harper, [5]). Let $A$ be a subset of $Q_n$ and let $B$ be an initial segment of the simplicial order with $|A| = |B|$. Then $|N(A)| \geq |N(B)|$. □

It turns out that the sets for which Harper’s theorem holds with equality are not in general unique. As a trivial example, any subset of $Q_2$ of size 2 has minimal vertex boundary and not all such sets are isomorphic. There are more interesting and less trivial examples as well.

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It is easy to verify that if $A$ is an initial segment of the simplicial order, then so is $N(A)$. Hence Harper’s theorem implies that an initial segment of the simplicial order minimises $N^t(A)$ for all $t > 0$. For a general introduction to the vertex-isoperimetric theorem, see e.g. Bollobás (Chapter 16 in [2]).

In this paper we will consider the following question of Aubrun and Szarek [1, Exercise 5.66]: If $A \subseteq Q_n$ for which $N^t(A)$ and $N^t(A^c)$ are minimal for all $t > 0$, does it follow that $A$ is isomorphic to an initial segment of the simplicial order? For convenience, we say that $A$ is extremal if $N^t(A)$ and $N^t(A^c)$ are minimal for all $t > 0$.

Define the exact Hamming ball of radius $r$ centred at $x$ to be $B(x, r) = \{y \in Q_n : d(x, y) \leq r\}$, and define set $A$ to be a Hamming ball if there exists $x$ and $r$ such that $B(x, r) \subseteq A \subseteq B(x, r + 1)$. Note that $B(\emptyset, r)$ is the initial segment of the simplicial order of length $\sum_{i=0}^r \binom{n}{i}$, and every initial segment of the simplicial order is a Hamming ball.

If $A$ is an initial segment of simplicial order then $N(A)$ is also an initial segment of simplicial order, and $A^c$ is isomorphic to an initial segment of simplicial order. Hence initial segments of simplicial order are always extremal. On the other hand, requiring only $N^t(A)$ to be extremal for all $t > 0$ is not strong enough condition to guarantee that $A$ should be isomorphic to the initial segment of simplicial order. Indeed, one could take for example $A = B(x, r) \setminus \{x\}$ for $r \geq 1$. Then $N^t(A) = B(x, r + t)$ for all $t > 0$ and hence $N^t(A)$ is always extremal, yet $A$ is not isomorphic to the initial segment.

It turns out that the answer to the question is negative, and we will present a counterexample in Section 2. Rather surprisingly, it turns out that the only Hamming balls which are extremal are the initial segments of the simplicial order. However, it turns out that all the extremal sets are contained between two exact Hamming balls with same centre and radius differing by 2, i.e. there exists $x$ and $r$ such that $B(x, r) \subseteq A \subseteq B(x, r + 2)$.

The second aim of this paper is to classify all the extremal sets $A$ up to isomorphism. In order to state the result, we need some notation. We write $X = [n] = \{1, \ldots, n\}$, $[n]^{(r)} = \{A \subseteq [n] : |A| = r\}$, $[n]^{(2 \cdot r)} = \{A \subseteq [n] : |A| \geq r\}$, $X_i = [n] \setminus \{i\}$ and $X_{i, j} = [n] \setminus \{i, j\}$.

Define the colexicographic order on $[n]^{(r)} := \{A : A \subseteq \{1, \ldots, n\}, |A| = r\}$ to be given by $A <_{\text{colex}} B$ if $\max(A \Delta B) \in B$. For $k \leq \binom{n-1}{r-1}$ let $A \subseteq [n]^{(r)}$ be the initial segment of the colexicographic order of size $k$. For each $1 \leq i \leq n$, let $A_{i,0} = \{B \in X_i^{(r-1)} : B \cup \{i\} \in A\}$ and $A_{i,1} = \{B \in X_i^{(r)} : i \notin B, B \in A\}$ be the $i$-sections of $A$. For each $i$ set $A_i = X^{(2r+1)} \cup A_{i,1} \cup A_{i,0}$. Note that $A_n = X^{(2r+1)} \cup A$ is isomorphic to an initial segment of the simplicial order, and that some of $A_i$ might be isomorphic to each other.

Now we are ready to give the classification of all extremal sets.

**Theorem 2 (Classification of extremal sets).** Let $A \subseteq Q_n$ with $|X^{(\leq r)}| < |A| \leq |X^{(\leq r)} \cup \{B \in X^{(r+1)} : 1 \in B\}|$ for some $r$. Let $A_1, \ldots, A_n$ be defined as above with $|A_i| = |A|$. Then $A$ is extremal if and only if $A$ is isomorphic to some $A_i$. 

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It is known that the exact Hamming balls are uniquely extremal sets for Harper’s inequality. That is, for \(|A| = |X^{(\leq r)}|\), if \(N(A)\) is minimal then \(A = B(x, r)\) for some \(x\). Thus if \(|A| = |X^{(\leq r)}|\) for some \(r\), and \(A\) is extremal, it certainly follows that \(A\) has to be isomorphic to an initial segment of the simplicial order. Note that if \(A\) is extremal then so is \(A^c\) as the conditions in the definition of extremality are symmetric under taking complements. Set \(G_r = X^{(\leq r)} \cup \{B \in X^{(r+1)} : 1 \in B\}\). It is easy to check that \(|G_r| + |G_{n-r-2}| = 2^n\). Thus provided \(|A| \neq |X^{(\leq r)}|\) for all \(r\), at least one of \(A\) and \(A^c\) satisfies \(|X^{(\leq r)}| < |A| \leq |G_r|\) or \(|X^{(\leq r)}| < |A^c| \leq |G_r|\) for some \(r\). Hence Theorem 2 together with these observations covers the classification of all extremal sets.

The plan of the paper is as follows. In Section 2 we will construct an extremal set which is not isomorphic to an initial segment of the simplicial order. In Section 3 we will prove Theorem 2.

In Section 4 we will discuss how the results presented in Section 3 will change if the conditions of extremality are weakened to requiring only \(N(A)\) and \(N(A^c)\) to be minimal. In this case there are extremal sets \(A\) for which there does not exist \(x\) and \(r\) with \(B(x, r) \subseteq A \subseteq B(x, r+2)\). In fact the situation is not even bounded, as it turns out that the constant 2 cannot be replaced by any finite number. However, it remains true in the weaker version as well that all extremal Hamming balls are isomorphic to the initial segment.

Recall that exact Hamming balls are uniquely extremal sets for Harper’s inequality. In Section 5 we will prove another near-uniqueness result: we will show that there exists only one set \(B_r\) of size \(|G_r|\), apart from the initial segment, which is extremal for Harper’s inequality. In fact the set \(B_r\) is also an extremal set, and we will describe it in Section 2.

For convenience we will write \(f_r = n_r = |X^{(\leq r)}| = \sum_{j=0}^{r} \binom{n}{j}\) and \(g_r = g_{n,r} = |G_r| = \sum_{j=0}^{r} \binom{n}{j} + \binom{n-r-1}{r+1}\). In both cases the dependence on \(n\) will not be highlighted if \(n\) is clear from the context.

## 2 Construction of an example

In this section we will give a family of counterexamples \(B_r \subseteq Q_n\), with \(|B_r| = g_r\) and \(B_r\) extremal for all \(r\). The initial segment of the simplicial order of size \(g_r\) is \(C_r = X^{(\leq r)} \cup \{\{1\} + X_1^{(r+1)}\}\) and hence it follows that \(N^t(C_r) = C_{r+t}\), which has size \(g_r+t\) for all \(t\). Also \(C_r^c = [n]^{(\geq r+2)} \cup X_1^{(r+1)}\) and hence \(N^t(C_r^c) = C_{r+t}^c\), which has size \(g_{n-2-(r+1)} = g_{n-2-r+1}\) as \(g_r + g_{n-r-2} = 2^n\).

For \(i \in [n]\) and \(A \subseteq Q_n\) define \(A_+ = \{B \subseteq [n] : i \notin B, B \cup \{i\} \in A\}\) and \(A_- = \{B \subseteq [n] : i \notin B, B \in A\}\) to be the \(i\)-sections of \(A\). Note that \(A_+\) depends on the choice of \(i\), but since the choice of \(i\) is usually clear this dependence will not be highlighted in the notation. Now \(A_+\) and \(A_−\) are subsets of \(Q_{n-1} = \mathcal{P}([n] \setminus \{i\})\), and it is easy to verify that \(N(A)_+ = N(A_+) \cup A_-\) and \(N(A)_- = \)

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those indification, and this is possible due to the following much stronger result.

Define $A$ by taking $i = 1, A_+ = B(\{2\},r)$ and $A_- = B(\emptyset,r)$, i.e. $A = (\{1\} + A_+) \cup A_-$. Note that the set $A$ constructed in this way is the union of two exact Hamming balls of same radius $r$ with centres at $\emptyset$ and $\{1, 2\}$, which are points of distance 2 apart from each other. Now $|A| = 2f_{n-1,r} = g_{n,r}$.

Since $d(\emptyset,\{2\}) = 1$ it follows that

$$N^{t-1}(A_+) = B(\{2\},r+t-1) \subseteq B(\emptyset,r+t) = N^t(A_-)$$

and also $N^{t-1}(A_-) \subseteq N^t(A_+)$. Thus $|N^t(A)| = 2f_{n-1,r+t} = g_{n,r+t}$ which proves that $N^t(A)$ is minimal for all $t > 0$. The minimality of $N^t(A^c)$ for all $t > 0$ follows similarly by observing that $(A^c)_+ = B(\{3,\ldots,n\},(n-1) - r - 1)$, $(A^c)_- = B(\{2,\ldots,n\},(n-1) - r - 1)$ and $d(\{3,\ldots,n\},\{2,\ldots,n\}) = 1$. Thus we can take $B_r = A$.

Note that it can be checked that the $B_r$ obtained in this way is

$$B_r = X^{(\leq r)} \cup \left\{ B : B \in X^{(r+1)} \cup X^{(r+2)}, \{1, 2\} \subseteq B \right\}$$

### 3 Classifying all extremal sets

Recall that $f_r = \sum_{i=0}^{r} \binom{r}{i}$ is the size of an exact Hamming ball of radius $r$ and $g_r = \sum_{i=0}^{r} \binom{r}{i} + \binom{n-1}{r}$ is the size of the initial segment $X^{(\leq r)} \cup \left( \{1\} + X^{(r)} \right)$, where $X_i = [n] \setminus \{i\}$. It is convenient to exclude sets of size $f_r$ from the classification, and this is possible due to the following much stronger result.

**Proposition 3.** Let $|A| = f_r$ for which $N(A)$ is minimal. Then $A = B(x,r)$ for some $x \in Q_n$.

Since this is a well-known fact, the proof is omitted. It can be deduced by induction on $n$ and applying Lemma 6 of Katona from [6]. A similar technique will be used in the proof of Claim 1 in Theorem 13 in Section 5.

Since $|A| = f_r$ for some $r$ is covered by Proposition 3, it is enough to consider the case $f_r < |A| < f_{r+1}$. Furthermore, since $g_r + g_{n-1-r} = 2^n$ and $f_{r+1} + f_{n-1-r} = 2^n$, by considering $A^c$ if necessary it is enough to classify just those $A$ with $f_r < |A| \leq g_r$ for some $r$. Hence from now on we will assume that $f_r < |A| \leq g_r$.

**Lemma 4.** Let $A$ be an extremal set with $f_r < |A| \leq g_r$. Then there exist distinct points $x, y, z \in Q_n$ such that $B(x,r) \subseteq A$, $B(y,n-r-2) \subseteq A^c$ and
\(B(z, n - r - 2) \subseteq A^c\). Furthermore, it follows that \(B(x, r) \subseteq A \subseteq B(x, r + 2)\) and \(d(y, z) \leq 2\).

The aim of this Lemma is to show that the structure of extremal sets is quite restricted, as the only interesting behaviour occurs only on two layers of the cube, namely on those which are distance \(r + 1\) and \(r + 2\) apart from \(x\). It also gives some inside on why it is convenient to assume that \(f_r < |A| \leq g_r\), rather than \(f_r < |A| < f_{r+1}\), as the condition \(f_r < |A| < f_{r+1}\) would not be strong enough to guarantee existence of both \(y\) and \(z\).

**Proof.** Since \(|A| \leq g_r\), it follows from the minimality of \(N^r(A)\) that \(|N^{n-r-2}(A)| \leq g_{n-2} = 2^n - 2\). Thus there exists distinct points \(y\) and \(z\) such that \(B(y, n - r - 2) \subseteq A^c\) and \(B(z, n - r - 2) \subseteq A^c\). Since \(|A^c| < f_{n-r-1}\), it follows that \(|N^r(A^c)| \leq f_{n-1} = 2^n - 1\), and hence there exists \(x\) with \(B(x, r) \subseteq A\). Since \(B(x, r) \cap B(y, n - r - 2) \subseteq A \cap A^c = \emptyset\) we must have \(d(x, y) \geq r + (n - r - 2) + 1 = n - 1\) and thus \(d(x^c, y) = d(x, y^c) \leq 1\). Similarly \(d(x^c, z) \leq 1\). Thus \(B(x, r) \subseteq A \subseteq B(y^c, r + 1) \subseteq B(x, r + 2)\) and the triangle inequality implies that \(d(y, z) \leq d(y, x^c) + d(x^c, z) \leq 2\) as required. \(\square\)

Given this result, we can split the rest of the classification into two parts: considering those \(A\) which are Hamming balls, i.e. for which \(B(x, r) \subseteq A \subseteq B(x, r + 1)\), and considering those \(A\) for which no such \(x\) and \(r\) exists. It turns out that all the examples apart from the initial segment appear in the second case. This is proved in the Proposition 6 but before that we need a short preliminary lemma.

**Lemma 5.** For all \(r \geq 1\) and \(x \neq y\) we have \(|B(x, r) \cup B(y, r)| \geq g_r\), with equality if and only if \(d(x, y) \leq 2\).

**Proof.** Let \(A = B(x, r) \cup B(y, r)\), we may assume that \(x = \emptyset\). For any \(i \in y\) we have \(A_+ = B(\emptyset, r) \cup B(y \setminus \{i\}, r - 1)\) and \(A_+ = B(\emptyset, r - 1) \cup B(y \setminus \{i\}, r)\). Hence \(|A| = |A_-| + |A_+| \geq 2f_{n-1, r} = g_r\) and the equality holds if and only if \(B(y \setminus \{i\}, r - 1) \subseteq B(\emptyset, r)\) and \(B(\emptyset, r - 1) \subseteq B(y \setminus \{i\}, r)\). Thus the equality holds if and only if \(d(\emptyset, y \setminus \{i\}) \leq 1\), i.e. if and only if \(d(x, y) \leq 2\).\(\square\)

**Proposition 6.** Suppose that \(A \subseteq Q_n\) is an extremal set for which there exists \(t \in Q_n\) and \(r\) such that \(B(t, r) \subseteq A \subseteq B(t, r + 1)\). Then \(A\) is isomorphic to an initial segment of the simplicial order.

**Proof.** Proof is by induction on \(n\). When \(n = 2\) it is easy to verify that the claim is true.

Suppose that the claim holds for \(n - 1\). If \(|A| = f_r\) then the claim follows from Proposition 3. Otherwise, by taking complements if necessary, we may assume that \(|A| \leq g_r\). Hence by Lemma 4 there exists distinct \(x, y, z \in Q_n\) with \(d(z, y) \leq 2\), \(B(x, r) \subseteq A\), \(B(y, n - r - 2) \subseteq A^c\) and \(B(z, n - r - 2) \subseteq A^c\).
Case 1. $|A| = g_r.$

By Lemma 5 $|A^c| \geq |B(y, n - r - 2) \cup B(z, n - r - 2)| \geq g_{n-r-2} = |A^c|$ so the equality must hold throughout, and hence $A^c = B(y, n - r - 2) \cup B(z, n - r - 2).$

Thus $A = B(y^c, r - 1) \cup B(z^c, r - 1).$ Without loss of generality set $y^c = \emptyset.$

Then $d(y^c, z^c) \leq 2$ implies that we may assume $z^c = \{1\}$ or $z^c = \{1, 2\}$ (corresponding to $d(y, z) = 1$ and $d(y, z) = 2$ respectively).

In the first case $A = X^{(\leq r)} \cup \{1\} + X^{(r)}_1$ which is an initial segment of the simplicial order.

In the second case

$$A = X^{(\leq r)} \cup \left(\{1, 2\} + \left(X^{(r-1)}_{1,2} \cup X^{(r)}_{1,2}\right)\right)$$

It is straightforward to check that $B(w, r) \subseteq A$ implies $w = \emptyset$ or $w = \{1, 2\}$ and in both cases $A \nsubseteq B(w, r + 1)$ contradicting the assumption on the existence of $t$ with $B(t, r) \subseteq A \subseteq B(t, r + 1).$ This completes the proof of the case $|A| = g_r.$

Case 2. $|A| < g_r$

Case 2.1. Suppose there exists $y_1$ and $z_1$ with $B(y_1, n - r - 2) \subseteq A^c,$ $B(z_1, n - r - 2) \subseteq A^c$ and $d(y_1, z_1) = 1.$

Recall from the proof of Lemma 4 that these points also satisfy $d(x^c, y_1) \leq 1$ and $d(x^c, z_1) \leq 1.$ Together with $d(y_1, z_1) = 1$ it follows that $x^c \in \{y_1, z_1\}$ as $Q_n$ is triangle free. Hence we may assume that $x = \emptyset,$ $y_1 = \{1, \ldots, n\}$ and $z_1 = \{2, \ldots, n\}.$ Thus $B(x, r) \subseteq A$ implies that $X^{(\leq r)} \subseteq A.$ Also $A \subseteq B(y_1, r + 1) \cap B(z_1, r + 1) = B(\emptyset, r + 1) \cap B(\{1\}, r + 1)$ and hence $A \subseteq X^{(\leq r)} \cup \left(\{1\} + X^{(r)}_1\right).$ Let $A = X^{(\leq r)} \cup \left(\{1\} + A\right)$ with $A \subseteq X^{(r)}_1.$

Consider $A_{\pm}$ in the direction $i = 1.$ Recall from (1) that $|N^t(A)|$ is given by

$$|N^t(A)| = |N^t(A_+) \cup N^{t-1}(A_-)| + |N^t(A_-) \cup N^{t-1}(A_+)|$$

Following Bollobás and Leader [3], let $C_+$ and $C_-$ be initial segments of the simplicial order of same sizes as $A_+$ and $A_-$ respectively, and set $C = (C_+ + \{1\}) \cup C_-.$ Note that $|C_+| = f_{n-1,r-1} + |A| \in [f_{n-1,r-1}, f_{n-1,r})$ and $|C_-| = f_{n-1,r}.$ Note that initial segments are nested, and the $t$-neighbourhood of an initial segment is also an initial segment and therefore the $t$–neighbourhood of an initial segment is also minimal. Hence it follows that

$$|N^t(C_{\pm})| = |N^t(C_{\pm}) \cup N^{t-1}(C_{\mp})| \leq |N^t(A_{\pm}) \cup N^{t-1}(A_{\mp})|$$

Adding up the inequalities corresponding to both choices of $+$ and $-$ yields

$$|N^t(C)| = |N^t(C_+) \cup N^{t-1}(C_-)| + |N^t(C_-) \cup N^{t-1}(C_+)| \leq |N^t(A_+) \cup N^{t-1}(A_-)| + |N^t(A_-) \cup N^{t-1}(A_+)| = |N^t(A)|$$
By the minimality of $|N^t(A)|$ the equality has to hold throughout and hence
$|N^t(C_{±})| = |N^t(A_{±})| = |N^t(A_{±})| + |N^{t−1}(A_{±})|$ for all $t$.

Since $C_+$ and $C_−$ are initial segments it follows that

$$|N^t(A_{±})| ≥ |N^t(C_{±})| = |N^t(A_{±})| + |N^{t−1}(A_{±})| ≥ |N^t(A_{±})|$$

Hence $|N^t(A_{±})| = |N^t(C_{±})|$ for all $t$ and in particular both $N^t(A_{±})$ and $N^t(A_{−})$ are minimal for all $t > 0$. By similar argument $N^t(A_{+})$ and $N^t(A_{−})$ are minimal for all $t > 0$ as well.

Note that $A_+ = X_1(1 ≤ r−1) ∪ A$ and hence $B(∅, r−1) ⊆ A_+ ⊆ B(∅, r)$. Since $N^t(A_{±})$ and $N^t(A_{−})$ are minimal for all $t > 0$, it follows by induction that $A_+$ is isomorphic to an initial segment of the simplicial order. Hence $A$ is isomorphic to an initial segment of the lexicographic order in $[n]$. Thus $A$ is isomorphic to an initial segment of the simplicial order which completes the proof of Case 2.1.

**Case 2.2.** Suppose that every $y_1, z_1$ with $B(y_1, n − r − 2) ⊆ A^c$ and $B(z_1, n − r − 2) ⊆ A^c$ satisfies $d(y_1, z_1) = 2$.

Since $d(y_1, x^c) ≤ 1$ and $d(z_1, x^c) ≤ 1$ it follows that $d(y_1, x^c) ≠ 0$ as otherwise $d(z_1, y_1) < d(z_1, x^c) + d(y_1, x^c) = 1$ which is a contradiction. Thus $x^c ≠ y_1$ and similarly $x^c ≠ z_1$. Without loss of generality let $x = ∅, y = \{2, \ldots, n\}$ and $z = \{1, 3, \ldots, n\}$. Note that if there exists $w_1 ≠ w_2$ with $B(w_1, r) ∪ B(w_2, r) ⊆ A$ then Lemma 5 would imply that $|A| ≥ g_r$, contradicting the assumption of Case 2.

Thus it follows that $t = x$ is the unique point of $Q_n$ for which $B(t, r) ⊆ A$. Recall that by assumption there exists $t$ for which $B(t, r) ⊆ A ⊆ B(t, r + 1)$. Therefore we have $A ⊆ B(∅, r + 1)$ and thus $B(\{1, \ldots, n\}, n − r − 2) ⊆ A^c$. But $d(\{1, \ldots, n\}, y_1) = 1$ so in fact Case 2.2 cannot ever occur, which completes the proof. □

As usual we define the lower shadow of $A$ by $\partial_{−}A = \{B : B ∪ \{i\} ∈ A$ for some $i\}$, and the iterated lower shadow by $\partial_{−}^{-1}A = \partial(\partial_{−}^{-1−1}A)$. Similarly define the upper shadow of $A$ by $\partial_{+}A = \{B ∪ \{i\} : i ∈ [n], B ∈ A\}$, and the iterated upper shadow by $\partial_{+}^{t}A = \partial_{+}(\partial_{+}^{t−1}A)$. Note that $\partial_{+}A$ depends on the ground set, which will be $[n]$ unless otherwise highlighted in the notation.

Now Proposition 6 has the following straightforward corollary.

**Corollary 7.** Let $A ⊆ X^{(r)}$. Suppose that $\partial^{−t}A$ and $\partial^{+t}A$ are minimal for all $t > 0$. Then $A$ is isomorphic to an initial segment of the lexicographic order.

**Proof.** By considering $A' = A^c : A ∈ B$ if necessary, and using $|\partial^{−t}A'| = |\partial^{−t}B|$ and $|\partial^{+t}X^{(r)} \setminus A'| = |\partial^{+t}A|$, we may assume that $|A| ≤ \binom{n−1}{r−1}$. Set $A = X^{(2r+1)} \setminus A$, then $f_{n−r} < |A| ≤ g_{n−r+1}$. Since $\partial^{−t}A$ and $\partial^{+t}B$ are minimal
for all \( t > 0 \) it follows that \( N^t(A) \) and \( N^t(A^c) \) are minimal for all \( t > 0 \), and hence \( A \) is extremal. Thus Proposition 6 implies that \( A \) is isomorphic to an initial segment of the order given by \( A < B \) if and only if

\[
|A| > |B| \text{ or } (|A| = |B| \text{ and } A < B \text{ in colexicographic order})
\]

Indeed this follows from the fact that the order defined above is isomorphic to the simplicial order in \( Q_n \) via taking complements and permuting the ground set.

Denote the isomorphism by \( \theta \). If \( |A| < g_{n-r+1} \) then \( B(\{1, \ldots, n\}, n-r-1) \) is the unique exact Hamming ball of radius \( n-r-1 \) inside \( A \), so the isomorphism must fix \( \{1, \ldots, n\} \) and hence \( \theta(A) = \theta(A) \cap X^{(r)} \), which is an initial segment of the colexicographic order.

If \( |A| = g_{n-r+1} \) it follows that \( \theta(\{1, \ldots, n\}) = \{1, \ldots, n\} \) or \( \theta(\{1, \ldots, n\}) = \{1, \ldots, n-1\} \). In the first case we’re done as above. Note that \( \theta(A) = X^{(2r+1)} \cup \{1, \ldots, n-1\}^{(r)} \) and hence \( \theta(A) \) is preserved under \( \tau(X) = X \Delta \{n\} \), which is an isomorphism of \( Q_n \). Also \( \tau \theta(\{1, \ldots, n\}) = \{1, \ldots, n\} \) so replacing \( \theta \) by \( \tau \theta \) we obtain that \( A \) is isomorphic to an initial segment of the colexicographic order. \( \Box \)

Recall that the sets \( A_1, \ldots, A_n \) were defined in the introduction as follows. For \( k \leq \binom{n-1}{r} \) let \( A \subseteq [n]^{(r)} \) be the initial segment of the colexicographic order of size \( k \). For each \( 1 \leq i \leq n \) define \( A_{i,0} = \{B \in X^{(r-1)}_i: B \cup \{i\} \in A\} \) and \( A_{i,1} = \{B \in X^{(r)}_i: i \notin B, B \in A\} \). For each \( i \) set \( A_i = X^{(2r+1)} \cup A_{i,1} \cup A_{i,0} \).

The motivation behind \( k \leq \binom{n-1}{r} \) follows from the fact that \( g_{n-r-1} = \sum_{i=0}^{n-r-1} \binom{n}{i} + \binom{n-1}{n-r-1} = \sum_{i=r+1}^{n} \binom{n}{i} + \binom{n-1}{r} \) so \( k \leq \binom{n-1}{r} \) corresponds exactly to \( f_{n-r-1} < |A| \leq g_{n-r-1} \). Note that we have turned our attention into sets of the form \( X^{(2r)} \cup A \) instead and the reason is the fact that the notation is slightly simpler in terms of lower shadows.

For the convenience, we restate Theorem 2:

**Theorem 2 (Classification of extremal sets).** Let \( A \subseteq Q_n \) with \( |X^{(\leq r)}| < |A| \leq |X^{(\leq r)} \cup \{B \in X^{(r+1)}: 1 \in B\}| \) for some \( r \). Let \( A_1, \ldots, A_n \) be defined as above with \( |A_i| = |A| \). Then \( A \) is extremal if and only if \( A \) is isomorphic to some \( A_i \)

**Proof**

**Case 1.** \( |A| = g_r \)

As noticed in the proof of Proposition 6, such set \( A \) has to be of the form

\[
A = X^{(\leq r)} \cup \{1\} + X^{(r)}_1
\]

or

\[
A = X^{(\leq r)} \cup \{1, 2\} + \left(X^{(r-1)}_1 \cup X^{(r)}_{1,2}\right)
\]

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In the first case it can be checked that $A$ is isomorphic to $A_n = X(≥n-r) ∪ X_n^{(n-r-1)}$, and in the second case $A$ is isomorphic to $A_1 = X(≥n-r) ∪ X_{1,n}^{(n-r-2)}$, which completes the proof of Case 1.

Case 2. $|A| < g_r$

Set $k = n - r$ and let such $A$ be given. By Lemma 4 there exists $x, y, z$ with $d(y, z) ≤ 2$, $d(x, y) ≤ 1$, $d(x, z) ≤ 1$, $B(x, r) ⊆ A$, $B(y, k - 2) ⊆ A^c$ and $B(z, k - 2) ⊆ A^c$. If $d(y, z) = 1$ then the Case 2.1 in the proof of Proposition 6 implies that $A$ is isomorphic to an initial segment of the simplicial order. Hence it suffices to only consider the case $d(y, z) = 2$.

Without loss of generality let $y = \{n - 1\}$ and $z = \{n\}$. Then $d(x, y) ≤ 1$ and $d(x, z) ≤ 1$ implies that $x = [n]$ or $x = \{1, \ldots, n - 2\}$. Since everything up to this point is preserved under the map $A → AΔ \{n - 1, n\}$, we may assume that $x = [n]$. Taking complements from $B(y, k - 2) ⊆ A^c$ and $B(z, k - 2) ⊆ A^c$ it follows that

$$A ⊆ B([1, \ldots, n - 1], r + 1) \cap B([1, \ldots, n - 2], r + 1) = X(≥k) \cup \{B : |B| \in \{k - 1, k - 2\}, B \cap \{n - 1, n\} = \emptyset\}$$

Hence $A = X(≥k) \cup A_1 \cup A_0$ with $A_i \subseteq [n - 1]^{(k-i-1)}$ for $i \in \{0, 1\}$ (in fact these are subsets of $[n - 2]^{(k-i-1)}$ but them being subsets of $[n - 1]^{(k-i-1)}$ is enough).

Set $A = (A_0 + \{n\}) \cup A_1 \subseteq [n]^{(k-1)}$. Now $|A| = |A_0| + |A_1|$ and

$$∂^{-t}A = (∂^{-t}A_0 + \{n\}) \cup (∂^{-(t-1)}A_0 ∪ ∂^{-t}A_1) \quad (2)$$

On the other hand

$$N^t(A) = X(≥k-t) \cup (∂^{-t}A_1 ∪ ∂^{-(t-1)}A_0) \cup ∂^{-t}A_0 \quad (3)$$

and hence combining (2) and (3) yields

$$|N^t(A)| = |X(≥k-t)| + |∂^{-t}A| \quad (4)$$

Let $B_i = [n - 1]^{(k-1-i)} \setminus A_i$ and $B = [n]^{(k-1)} \setminus A = (B_0 + \{n\}) \cup B_1$. In this notation

$$A^c = X(≤k-3) \cup B_0 \cup B_1 \cup \{B : |B| \in \{k - 2, k - 1\}, n \in B\}$$

Let $∂^+_n$ be the upper shadow operator with respect to the ground set $\{1, \ldots, n - 1\}$ and $∂^+_1$ be the usual upper shadow operator (i.e. with ground set $[n]$). Note that

$$∂^{+t}A = (\left(∂^{+t}B_0 ∪ ∂^{+(t-1)}B_1\right) + \{n\}) \cup ∂^{+t}B_1 \quad (5)$$

Now

$$N^t(A^c) = X(≤k+t-3) \cup (∂^{+t}B_0 ∪ ∂^{+(t-1)}B_1) \cup ∂^{+t}B_1 \quad (6)$$
and note that each of
\[ X^{(\leq k+t-3)} , \left( \partial_n^{k+t} B_0 \cup \partial_n^{(t-1)} B_1 \right) , \partial_n^{k+t} B_1 \]
and
\[ \{ B : |B| \in \{ k + t - 2, k + t - 1 \} , n \in B \} \]
are pairwisely disjoint set systems. Hence it follows from (6) that
\[
\begin{align*}
|N^t(A^c)| &= |X^{(\leq k+t-3)}| + |\partial_n^{k+t} B_0 \cup \partial_n^{(t-1)} B_1| + |\partial_n^{k+t} B_1| \\
&+ |\{ B : |B| \in \{ k + t - 2, k + t - 1 \} , n \in B \}|
\end{align*}
\]
(7)
From (5) it can be deduced that
\[
\begin{align*}
|\partial_n^{k+t} B_0 \cup \partial_n^{(t-1)} B_1| + |\partial_n^{k+t} B_1| = |\partial^n B|
\end{align*}
\]
(8)
and finally by counting we have
\[
\begin{align*}
&\{ B : |B| \in \{ k + t - 2, k + t - 1 \} , n \in B \} \\
&= \left( \frac{n-1}{k+t-3} \right) + \left( \frac{n-1}{k+t-2} \right) = \left( \frac{n}{k+t-2} \right)
\end{align*}
\]
(9)
By using (8) and (9), (7) simplifies to
\[
\begin{align*}
|N^t(A^c)| &= |X^{(\leq k+t-3)}| + |\partial^n B| + \left( \frac{n}{k+t-2} \right) = |X^{(\leq k+t-2)}| + |\partial^n B| \quad \text{ (10)}
\end{align*}
\]
Let \( C \subseteq X^{(k-1)} \) be the initial segment of the colexicographic order of size \(|A|\), and set \( C = X^{(\geq k)} \cup C \). Since \(|C| = |A|\) and \( C \) is isomorphic to an initial segment of the simplicial order, we must have \(|N^t(C)| = |N^t(A)|\) and \(|N^t(C^c)| = |N^t(A^c)|\) for all \( t > 0 \), as \( N^t(A) \) and \( N^t(A^c) \) are minimal. Setting \( D = X^{(k-1)} \setminus C \) it is straightforward to verify that \( N^t(C) = X^{(\geq k+t)} \cup \partial^n C \) and \( N^t(C^c) = X^{(\leq k+t-2)} \cup \partial^n D \). Hence it follows that \( |\partial^n A| = |\partial^n C| \) and \( |\partial^n B| = |\partial^n D| \) for all \( t > 0 \). Thus Corollary 7 implies that \( A \) has to be isomorphic to an initial segment of the colexicographic order. Hence \( A \) is isomorphic to some \( A_i \).

The extremality of \( A_i \)'s follows immediately from (4) and (10) as when \( A = A_i \), it is immediate from the definition of \( A_i \) that \( A \) is isomorphic to an initial segment of the colexicographic order. □

**Corollary 8.** For all \( n \) and \( k \not\in \{ f_0, \ldots, f_n \} \) there exists extremal set \( A \subseteq Q_n \) which is not isomorphic to any initial segment of the lexicographic order.

**Proof.** When \(|A| = g_r \) the claim is true, so by the same argument as before we may assume that \( f_r < |A| < g_r \). Let \( A \subseteq X^{(n-r-1)} \) be the initial segment of the colexicographic order of size \(|A| - f_r \) and take \( i \in \bigcup_{A \in A} A \). Then
\[ A = (A_0 + \{ i \}) \cup A_1 \] with \( A_0 \neq \emptyset \). Note that since \( |A| < g_r \) it follows from Lemma 5 that \( B(x, r) \subseteq A_i \) implies \( x = |n| \) and thus it is easy to see that \( A_i \) is not isomorphic to an initial segment of the simplicial order as \( B([n], r) \subseteq A_i \), but \( A_i \not\subseteq B([n], r + 1) \). \( \Box \)

It is natural to ask: when are \( A_i \) and \( A_j \) isomorphic as subsets of \( Q_n \)? Let \( A \) be the initial segment from which \( A_i \)'s are obtained. If \( \sigma = (ij) \in S_n \) satisfies \( \sigma(A) = A \) then certainly \( A_i \) and \( A_j \) are isomorphic, where \( \sigma(A) = \{ \sigma(b_1), \ldots, \sigma(b_t) \} : \{ b_1, \ldots, b_t \} \subseteq A \). The aim of the following lemma is to prove that this is the only way the isomorphism can occur.

**Lemma 9.** \( A_i \) and \( A_j \) are isomorphic if and only if \( \sigma(A) = A \) for \( \sigma = (ij) \)

**Proof.** If \( |A| = g_r \) then \( A = \{1, \ldots, n-1\}^{(n-r-1)} \) and clearly \( A_i \) and \( A_j \) are isomorphic for all \( i, j \in [n-1] \). Also note that \( A_{n-1} = B([n], r) \cup B([n-1], r) \) and \( A_n = B([n], r) \cup B([n-1], r) \), so in particular \( A_{n-1} \) is union of two exact Hamming balls of radius \( r \) whose centres are distance 2 apart, and \( A_n \) is union of two exact Hamming balls of radius \( r \) whose centres are distance 1 apart. Thus they are not isomorphic.

Now suppose that \( f_r < |A| < g_r \). Thus each \( A_i \) contains an unique exact Hamming ball of radius \( r \), which is by construction centred at \( |n| \). Suppose \( i < j \) and that \( \theta : A_i \to A_j \) is an isomorphism. Since \( \theta \) must fix the centre of the unique exact Hamming ball of radius \( r \), we must have \( \theta([n]) = [n] \) and hence \( \theta(\emptyset) = \emptyset \).

It is easy to verify that \( \text{Stab}(\emptyset) = S_n \) is given by \( \theta_{\sigma}(A) = \{ \sigma(a) : a \in A \} \) for \( \sigma \in S_n \). Hence \( \theta \) maps \( A_{i,0} \) to \( A_{j,0} \) and \( A_{i,1} \) to \( A_{j,1} \) so in particular \( |\{ A \in A : i \in A \}| = |\{ A \in A : j \in A \}| \). Since \( A \) is an initial segment, it is left compressed so for all \( A \in A \) if \( i \notin A, \ j \in A \) we must have \( (A \setminus \{ i \}) \cup \{ j \} \subseteq A \). Thus \( |\{ A \in A : i \in A \}| = |\{ A \in A : j \in A \}| \) implies that the converse must hold as well, that is for all \( A \in A \) if \( j \in A, \ i \in A \) we must have \( (A \setminus \{ i \}) \cup \{ j \} \subseteq A \) and hence \( \sigma(A) = A \) for \( \sigma = (ij) \). \( \Box \)

From Lemma 9 it follows that for all \( s \) there exists \( n, k \) such that there are at least \( s \) pairwise non-isomorphic extremal sets \( A_1, \ldots, A_s \) of size \( k \) in \( Q_n \). Indeed, this follows by taking \( n = 2s + 3, \ k = |X(2s+2)| + \sum_{i=2}^{s+1} \binom{2(i-1)}{i} \). If \( A \subseteq X^{(s+1)} \) is the initial segment of the colexicographic order of size \( |A| = \sum_{i=2}^{s+1} \binom{2(i-1)}{i} \) it is clear that \( (ij)A \neq A \) for any distinct even integers \( 1 \leq i, j \leq 2s \).

## 4 The weak version

In this section we consider how the results in Section 3 change if we only require \( N(A) \) and \( N(A') \) to be minimal. First of all we will prove that no such result as Lemma 4 can hold in the weak version. That is, we will prove that there is no constant \( k \) such that the extremal sets are contained between two exact
Hamming balls with same centre and whose radius differ by at most $k$.

**Proposition 10.** For any positive integer $s$ there exists $n$ and a set $A \subseteq Q_n$ for which $N^t(A)$ is minimal for all $t > 0$, $N(A^c)$ is minimal, and for all $x \in Q_n$, $t \in \mathbb{Z}_+$ at least one of $B(x, t) \subseteq A$ or $A \subseteq B(x, t + s)$ is violated.

**Proof.** Let $n = 2s + 8$, $r = s + 4$, $k = s + 2$ and

$$A = X^{(\geq r)} \setminus \{1, \ldots, r + i\} : 0 \leq i \leq k - 1$$

That is, we take $A$ to be $X^{(\geq r)}$ but we exclude the set $\{1, \ldots, r + i\} \in X^{(r+i)}$ for all $0 \leq i \leq k - 1$. Now $A^c = X^{(\leq r-1)} \cup \{1, \ldots, r + i\} : 0 \leq i \leq k - 1$.

Thus

$$N(A^c) = X^{(\leq r)} \cup \{1, \ldots, r + i, a_i\} : 0 \leq i \leq k - 1, r + i + 1 \leq a_i \leq n$$

and hence $|N(A^c)| = |X^{(\leq r)}| + \sum_{i=0}^{k-1} (n - r - i)$.

Let $C \subseteq X^{(r)}$ be the initial segment of the lexicographic order of size $k$. Since $k + r = 2s + 6 < n + 1$, it follows that $C = \{1, \ldots, r - 1, i\} : r \leq i \leq r + k - 1$.

By definition $B = X^{(\leq r-1)} \cup C$ is an initial segment of the lexicographic order with $|B| = |A^c|$, and $N(B) = X^{(\leq r)} \cup \partial^+ C$, so in order to verify that $N(A^c)$ is minimal it suffices to show that $|\partial^+ C| = \sum_{i=0}^{k-1} (n - r - i)$.

But $\partial^+ C = \{A \in X^{(r+1)} : \{1, \ldots, r - 1\} \subseteq A, \{r, \ldots, r + k - 1\} \cap A \neq \emptyset\}$ and hence we can identify $\partial^+ C$ as the set $\{r, \ldots, n\}^{(2)} \setminus \{r + k, \ldots, n\}^{(2)}$ via $A \rightarrow A \setminus \{1, \ldots, r - 1\}$. Thus

$$|\partial^+ C| = \left(\binom{n - r + 1}{2} - \binom{n - r - k + 1}{2}\right)$$

$$= \sum_{i=1}^{n-r} i - \sum_{i=1}^{n-r-k} i = \sum_{i=n-r-(k-1)}^{n-r} i = \sum_{i=0}^{k-1} (n - r - i)$$

(11)

as required. Hence (11) shows that $N(A^c)$ is minimal.

Let $D = X^{(r)} \setminus C$. Note that $|D| = |X^{(r)}| - k$. For any given $A \in X^{(r-1)}$, there are $n - (r - 1) = n - r + 1$ sets $B \in X^{(r)}$ such that $A \subseteq B$. Since $k < n - r + 1$ it follows that for all $A \in X^{(r-1)}$ there exists $B \in D$ such that $A \subseteq B$, so in particular $\partial^D = X^{(r-1)}$. Thus $N(B^c) = X^{(\geq r-1)}$. Since $N(B^c)$ is minimal, $|B^c| = |A|$, and $N(A) \subseteq X^{(\geq r-1)}$ it follows that $N(A) = X^{(\geq r-1)}$ and hence $N(A)$ is minimal. But since $N(A)$ is isomorphic to an initial segment of the simplicial order, it follows that $N^t(A)$ is minimal for all $t > 0$.

To finish the proof, note that it suffices to prove that if $B(x, d) \subseteq A^c$ and $A^c \subseteq B(x, f)$ then $f - d \geq k - 1$. Indeed, supposing that this holds, then $B(x^c, n - f - 1) \subseteq A$ and $A \subseteq B(x^c, n - d - 1)$ with $(n - d - 1) - (n - f - 1) = f - d \geq k - 1 > 0$ which completes the proof of the claim.

Suppose that $B(x, d) \subseteq A^c$ with $|x| = t$. Since $\{1, \ldots, r + 1\}$ is the only $r+1$-set in $A^c$ it follows that $d + t \leq r$ (as $x$ is contained as a subset in strictly more
than 1 set in $X^{(r+1)}$. Also $A^c$ contains a $r + k - 1$-set $y = \{1, \ldots, r + k - 1\}$ so $y \in B(x, f)$ implies that $f \geq |y \Delta x| \geq |y| - |x| = r + k - 1 - t$. Thus $f - d \geq (r + k - 1 - t) - (r - t) = k - 1 > s$ as required. □

Recall that Proposition 6 states that if $A$ is extremal and is contained between two consecutive layers of the cube, i.e. $X^{(\leq r)} \subseteq A \subseteq X^{(\leq r+1)}$, it follows that $A$ has to be isomorphic to an initial segment of the simplicial order. It turns out that this still remains true in the weak version, and in fact the following theorem by Füredi and Griggs reduces the proof of this fact to Corollary 7.

**Theorem 11 (Füredi, Griggs - Theorem 2.1 in [4]).** Suppose $A \in X^{(r)}$ for which $|\partial A|$ is minimal. Then $|\partial^t A|$ is minimal for all $t > 0$. □

**Corollary 12 (Proposition 6 for the weak version).** Suppose $A \subseteq Q_n$ for which there exists $r$ such that $X^{(\geq r)} \subseteq A \subseteq X^{(\geq r - 1)}$, and suppose that $N(A)$ and $N(A^c)$ are minimal. Then $A$ is isomorphic to an initial segment of the simplicial order.

**Proof.** Set $A = X^{(\geq r)} \cup A$ with $A \subseteq X^{(r-1)}$ and let $Q = X^{(r-1)} \setminus A$, and $B = \{T^c : T \in Q\} \subseteq X^{(n-r+1)}$. Since $N(A)$ and $N(A^c)$ are minimal it follows that $\partial^- A$ and $\partial^+ Q$ are both minimal. Note that $|B| = |Q|$ and $|\partial^{-t} B| = |\partial^{+t} Q|$ for all $t > 0$.

Since $\partial^+ Q$ is minimal, so is $\partial^- B$ as $|\partial^- B| = |\partial^+ Q|$. Thus Theorem 11 implies that $\partial^{-t} A$ and $\partial^{-t} B$ are minimal for all $t > 0$. Hence $\partial^{+t} Q$ is minimal for all $t > 0$ as $|\partial^{-t} B| = |\partial^{+t} Q|$ for all $t > 0$. Thus Corollary 7 implies that $A$ is isomorphic to an initial segment of the colexicographic order and hence $A$ is isomorphic to an initial segment of simplicial order. □

Since the classification of all extremal sets was done for the stronger version in which we required $N^t(A)$ and $N^t(A^c)$ to be minimal for all $t > 0$, one could also ask whether it could be done when we require only $N(A)$ and $N(A^c)$ to be minimal. This seems to be much harder, as in the stronger version one of the key observations was that any extremal set $A$ satisfies $B(x, r) \subseteq A \subseteq B(x, r + 2)$ for some $x \in Q_n$ and $r$ which already restricts the structure of $A$ - but as it was shown in Proposition 10, a similar result cannot be proved with the weaker conditions on $N(A)$ and $N(A^c)$.

However we are able to show that for sets of size $g_n$ the sets presented in Section 2 together with the initial segment are the only extremal sets for the weak version as well. In fact, the proof even shows that the sets $B_r$ introduced in Section 2 are the only sets of size $g_r$, together with the initial segment, for which $N(A)$ is minimal. This result is presented in the following section.
5 A uniqueness result for certain sizes

Recall that \( f_{n,r} \) and \( g_{n,r} \) are defined by \( f_{n,r} = \sum_{i=0}^{r} \binom{n}{i} \) and \( g_{n,r} = f_{n,r} + \binom{n-1}{r-1} \).

It is easy to verify that \( g_{n,r} = g_{n-1,r-1} + g_{n-1,r} \) and \( g_{n,r} = 2f_{n-1,r-1} \). For \( k \in \mathbb{Z}_+ \) let \( C \) be the initial segment of the simplicial order of size \( k \) in \( Q_n \). Set \( N(k) = |N(C)| \) for convenience - note that this depends on \( n \), but the dependence will not be highlighted in the notation as the value of \( n \) is clear from the context.

**Theorem 13.** Let \( |A| = g_r \) for which \( N(A) \) is minimal. Then either \( A \) is isomorphic to an initial segment of the simplicial order or \( A \) is isomorphic to \( B_r \).

Note that this proves that the only extremal families of size \( g_r \) for the weak version are \( B_r \) and the initial segment of size \( g_r \).

**Proof.** The main idea of the proof is to carefully analyse the codimension 1 compressions. Let \( |A| = g_r \) be a set for which \( N(A) \) is minimal. Let \( I = \{ i : |A_i| > |A_i^-| \} \). By considering \( A \Delta I = \{ B \Delta I : B \in A \} \) if necessary we may assume that \( |A_i| \leq |A_i^-| \) for all directions \( i \) - note that clearly \( A \) is isomorphic to \( A \Delta I \).

Choose a direction \( i \), and again similarly as in [3] let \( C_+ \) and \( C_- \) be initial segments of the simplicial order with \( |C_+| = |A_+| \) and \( |C_-| = |A_-| \), and define \( C = C_- \cup \{ \langle i \rangle \} \cup C_+ \). Since initial segments are nested, we have \( |C_+ \cup N(C_+)| = \max(|C_+|, |N(C_+)|) \). Also recall that for all sets \( A \subseteq Q_n \), by (1) we have

\[
|N(A)| = |N(A_+) \cup A_-| + |A_+ \cup N(A_-)|
\]

and thus as in the proof of Proposition 6 it follows that \( |N(C)| = |N(A)| \) and so \( N(C) \) is also minimal.

**Claim 1.** \( |A_+| = g_{n-1,r-1} \) or \( |A_+| = f_{n-1,r} \)

**Proof of Claim 1.** By the definition of \( C_+ \) it is equivalent to prove same assertion for \( |C_+| \) instead of \( |A_+| \). If \( |C_+| < g_{n-1,r-1} \) then \( |C_-| > g_{n,r} - g_{n-1,r-1} = g_{n-1,r} \) and also \( |N(C_-)| \geq N(g_{n-1,r}) = g_{n-1,r+1} \) so \( |N(C)| > g_{n-1,r} + g_{n-1,r+1} = g_{n,r+1} \) which contradicts the minimality of \( N(C) \) as \( N(g_{n,r}) = g_{n,r+1} \). Thus \( |C_+| \geq g_{n-1,r-1} \) and on the other hand \( |C_+| \leq \frac{1}{2}g_{n,r} = f_{n-1,r} \).

Similarly \( |C_-| \geq f_{n-1,r} \) and \( |C_+| \leq g_{n-1,r} \).

Note that an initial segment of size \( g_{n-1,r-1} \) in \( P(X_i) \) is \( X_{i}^{(\leq r-1)} \cup \{ A : |A| = r, s \in A \} \) where \( s \) is the smallest element of \( X_i \) (i.e. \( s = 1 \) if \( i \neq 1 \) and \( s = 2 \) if \( i = 1 \)). Hence it follows from \( g_{n-1,r} \leq |C_+| \leq f_{n-1,r} \) that \( C_+ = X_i^{(\leq r-1)} \cup \left( 1 + X_{i,2}^{(r-1)} \right) \cup A_+ \) where \( A_+ \subseteq X_{i,1}^{(r)} \). Similarly the initial segment of size \( g_{n-1,r} \) is \( X_i^{(\leq r-1)} \cup \{ A : |A| = r, 1 \in A \} \) and it follows that \( C_+ = X_i^{(\leq r)} \cup \left( 1 + A_- \right) \) where \( A_- \subseteq X_{i,1}^{(r)} \), and

\[
|A_+| + |A_-| = g_{n,r} - f_{n-1,r} - f_{n-1,r-1} - \binom{n-2}{r-1} = \binom{n-2}{r}
\]

(12)
We have
\[ N(C_{-}) = X_{1}^{(\leq r+1)} \cup \{1\} \cup \partial_{1,i}^{+}A_{-} \]
and
\[ N(C_{+}) = X_{1}^{(\leq r)} \cup \{1\} \cup X_{1,i}^{(r)} \cup \partial_{1,i}^{+}A_{+} \]
where \( \partial_{1,i}^{+} \) is the upper shadow operator with respect to the ground set \( X_{1,i} \).

The Local LYM inequality for upper shadows states that if \( A \subseteq X^{(r)} \) then
\[ \frac{|\partial^{+}A|}{\binom{n}{r}} \geq \frac{|A|}{\binom{n}{r}} \]  
(13)
and the equality holds if and only if \( A = X^{(r)} \) or \( A = \emptyset \).

Applying (13) to \( A_{\pm} \subseteq X_{1,i}^{(r)} \), adding these inequalities together and using (12) yields
\[ |\partial_{1,i}^{+}A_{-}| + |\partial_{1,i}^{+}A_{+}| \geq \frac{(n-2)}{n} \left( |A_{+}| + |A_{-}| \right) = \frac{(n-2)}{r+1} \]  
(14)

It follows from (14) that
\[ |N(C)| \geq f_{n-1,r+1} + f_{n-1,r} + \left( \frac{n-2}{r} \right) \left( n - \frac{n-2}{r+1} \right) = 2f_{n-1,r+1} = g_{n,r+1} \]  
(15)

Since \( N(C) \) is minimal, it follows that the equality must hold in (15) and hence the equality must hold in both applications of (13) - that is, \( A_{\pm} = \emptyset \) or \( A_{\pm} = X_{1,i}^{(r)} \). Since \( |A_{+}| + |A_{-}| = \binom{n-2}{r} \) it follows that exactly one of \( A_{\pm} \) is \( \emptyset \) and the other one is \( X_{1,i}^{(r)} \). Thus \( |C_{+}| = g_{n-1,r-1} \) (if \( A_{+} = \emptyset \)) or \( |C_{+}| = f_{n-1,r} \) (if \( A_{+} = X_{1,i}^{(r)} \)) which completes the proof of claim. \( \square \)

**Claim 2.** There exists \( i \) for which \( |A_{i,+}| = f_{n-1,r} \).

**Proof of Claim 2.** Suppose that such \( i \) does not exist. From Claim 1 it follows that \( |A_{i,+}| = g_{n-1,r-1} \) for all \( i \). Note that by definition \( A_{i,+} = \{|B \in A : i \in B\}| \), and hence by double counting
\[ \sum_{B \in A} |B| = \sum_{i=1}^{n} |A_{i,+}| = ng_{n-1,r-1} \]  
(16)

For given \( |A| \), the quantity \( \sum_{B \in A} |B| \) is minimal when \( A \) is an initial segment of the simplicial order (or in particular any set of the form \( A = X^{(\leq r)} \cup A \) for suitable \( r \) and for any \( A \subseteq X^{(r+1)} \) of appropriate size). Hence if \( |C| = g_{n,r} \) then
\[ \sum_{B \in C} |B| \] is minimal for \( C = X^{(\leq r)} \cup \{1\} \cup X_{1}^{(r)} \).

Note that \( |C_{1,+}| = f_{n-1,r} \) and \( |C_{i,+}| = g_{n-1,r-1} \) for all \( i \neq 1 \). Thus
\[ \sum_{B \in C} |B| = (n-1)g_{n-1,r-1} + f_{n-1,r} \]  
(17)

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But $f_{n-1,r} > g_{n-1,r-1}$ for all $n > 2$ so (17) together with the fact that $C$ minimises $\sum_{B \in C} |B|$ contradicts (16). Thus there exists $i$ with $|A_{i,+}| = f_{n-1,r}$.

$\square$

In order to finish the proof, choose $i$ such that $|A_{i}| = f_{n-1,r}$. Since $|A_{i} \cup N(A_{i})| = |C_{i} \cup N(C_{i})|$ and initial segments are nested, it follows that $A_{i} \subseteq N(A_{i})$ and $N(A_{i})$ are minimal. But since exact Hamming balls are uniquely minimal for the vertex isoperimeter (Proposition 3), it follows that $A_{i} = B(x_{i},r)$ for some $x_{i} \in Q_{n-1}$, say with $x_{i} = \emptyset$. Now $A_{i} \subseteq N(A_{i})$ implies that $d(x_{i},x_{i}) \leq 1$.

If $x_{i} = x_{i} = \emptyset$ then $A$ is isomorphic to an initial segment of the simplicial order (and the isomorphism is given by any $f_{\sigma}$ with $\sigma(i) = 1$). If $x_{i} \neq x_{i}$, we have $x_{i} = \{j\}$ for some $j \neq i$. It is easy to verify that $A = X(\leq r) \cup \left(\{i,j\} + X_{i,j}^{(r-1)} \cup X_{i,j}^{(r)}\right)$ which is isomorphic to $B_{r}$ (via $f_{\sigma}$ for any $\sigma$ with $\sigma(i) = 1$, $\sigma(j) = 2$).

$\square$

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