On duality symmetry in perturbative quantum theory

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Abstract

Non-compact symmetries of extended 4d supergravities involve duality rotations of vectors and thus are not manifest off-shell invariances in standard “second-order” formulation. To study how such symmetries are realised in the quantum theory we consider examples in 2 dimensions where vector-vector duality is replaced by scalar-scalar one. Using “doubled” formulation, where fields and their momenta are treated on an equal footing and the duality becomes a manifest symmetry of the action (at the expense of the Lorentz symmetry), we argue that the corresponding on-shell quantum effective action or S-matrix are duality symmetric as well as Lorentz invariant. The simplest case of discrete $Z_2$ duality corresponds to a symmetry of the S-matrix under flipping the sign of the negative-chirality scalars in 2 dimensions or phase rotations of chiral (definite-helicity) parts of vectors in 4 dimensions. We also briefly discuss some 4d vector models and comment on implications of our analysis for extended supergravities.

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1 Introduction

Recent discussions of potential divergences in 4d supergravities brought to light the question of how the non-compact symmetries [1, 2, 3] of classical $\mathcal{N} \geq 4$ supergravity equations of motion (involving transformation of scalars combined with a 4d duality rotation of vectors) extend to quantum level and which constraints on counterterms they impose (see [4, 5, 6, 7, 8, 9, 10] and refs. therein). Duality symmetries involving vectors are specific to 4d extended supergravities and have their origin (which still remains to be fully clarified) in the dimensional reduction from higher dimensional supergravities.

A prototypical example of a relevant scalar-vector subsector of ($\mathcal{N} \geq 4$) supergravity theory is provided by the following (Minkowski-space) action

$$S = -\frac{1}{2} \int d^4x \left[ (\partial_m \phi)^2 + e^{2\phi} (\partial_m \chi)^2 + \frac{1}{2} e^{-2\phi} F_{mn}^2 + \frac{1}{2} \chi F^*_{mn} F^{mn} \right],$$

(1.1)

where $F^{*kl} \equiv \frac{1}{2} \epsilon^{klmn} F_{mn}$ and $k, l, m, n = 0, 1, 2, 3$. The scalar part here is $SO(1,2)/SO(2)$ sigma model; its global invariance under $SL(2,\mathbb{R}) \approx SO(1,2)$ is promoted to the invariance of the full equations of motion (written in first-order form) when combined with vector-vector duality transformation. The simplest example of such a transformation is at $\chi = 0$, when $\phi \rightarrow -\phi, A_m \rightarrow \tilde{A}_m$ with $\tilde{F}_{mn} = e^{-2\phi} F^*_{kl}$.

As this symmetry is not of a standard type, i.e. it is not a manifest local symmetry of the action (1.1), one may wonder how it is reflected in the corresponding quantum effective action or S-matrix. An early application of the expected symmetry under "$F \rightarrow F^*$" (i.e. $F \rightarrow \cos \epsilon F + \sin \epsilon F^*$) vector-vector duality was to constrain the structure of the leading one-loop divergent terms in the on-shell effective action or S-matrix of the Einstein-Maxwell theory [12] and the $\mathcal{N} = 1$ supergravity coupled to a vector multiplet [13] (where the duality rotation is a symmetry if combined with a chiral rotation of the fermion in the vector multiplet, $\delta F_{mn} = \epsilon F_{mn}^*, \delta \lambda = \epsilon \gamma_5 \lambda$). It was argued that the observed absence of all $F_{mn}$-dependent one-loop UV divergent terms except for the $T_{mn}T^{mn}$ one (where $T_{mn}$ is the vector stress tensor) may be attributed to the fact that this is the only duality-invariant term at this order.

As was pointed out in [14], the duality symmetry of Maxwell theory is naturally realised in the phase space formulation (at the expense of manifest Lorentz symmetry). It was noted in [15] that the duality invariance of the stress tensor $T_{mn}$ and thus of the corresponding Hamiltonian [14] should imply the invariance of the S-matrix (but this point was not elaborated on).

The duality symmetry acting also on scalars turns out to be anomalous on a curved background due to fermion [11] and vector [3, 16, 17, 6] contributions, i.e. already the bosonic theory [11] and also $\mathcal{N} = 4$ supergravity [3] have $U(1) \subset SU(1,1)$ anomalies. Such anomalies do not affect, however, the invariance of the leading UV divergent terms but produce finite

1 More general actions contain $G/H$ scalar part and several abelian vectors coupled to the scalars.

2 Here the phrase "local symmetry" is used for a symmetry whose transformation rules of fundamental fields do not involve inverse derivatives. The duality exchanging field strength with its dual does not act locally on vector potential.

3 Free quantum Maxwell theory in curved background has a duality symmetry anomaly reflected in $\langle FF^* \rangle = \frac{1}{48\pi^2} RR^*$ [16]. Once coupled also to scalars, it leads to duality-noninvariant terms in the effective action containing an $RR^*$ factor.
non-invariant terms in the quantum effective action.

If one integrates out the vector field, i.e. if one considers the effective action $\Gamma$ depending only on the scalars, then it is expected to be $SL(2)$ invariant. Indeed, performing the vector-vector $A_m \rightarrow \tilde{A}_m$ duality in the path integral (by adding, as usual, the Lagrange multiplier term, etc. [18]) one finds the same partition function with $\tilde{A}_m$ coupled to $SL(2)$ transformed scalars, implying that integrating out the vector should give an invariant functional of the scalars [1].

A non-trivial question is what happens if one keeps both the scalars and the vector as arguments of the effective action or external states in the S-matrix. A natural expectation is that this duality symmetry (in its form as defined on the classical equations of motion) should be present in the quantum effective action evaluated on the equations of motion or in the off-shell S-matrix. The precise meaning of the action of this symmetry on the S-matrix (beyond its purely-scalar part [11,15,18]) is, however, not immediately clear and is to be defined. A far less obvious possibility (discussed in [10,19,20]) is that the quantum effective equations derived from an off-shell effective action should be covariant under a deformed version of this duality [6].

A motivation behind the present paper is to try to clarify these questions. The vector-vector duality in 4 dimensions, or more generally the $p$-form – $p$-form duality in $d = 2p + 2$ dimensions, naturally acts on the phase space: the corresponding first-order action is duality-invariant [14]. Replacing momenta by spatial derivative of a new (dual) field, one can rewrite the phase-space action as an action for a “doubled” set of fields in which the duality acts locally (without inverse spatial derivatives) and is a manifest off-shell symmetry. This is achieved at the expense of Lorentz invariance; in its standard form, the Lorentz invariance is recovered on the equations of motion. Such a manifestly duality invariant action was first written down in 2 dimensions (following earlier work on chiral scalars [22] generalized to chiral $p$-forms in [23]) in [24]. This action for the “doubled” set of fields is describing the same number of degrees of freedom as the original action (and an equivalent quantum theory) but is more suitable for addressing the above questions about realization of duality at the quantum level. The corresponding “doubled” action for vectors in 4 dimensions was constructed in [26] and was generalized to $N = 8$ supergravity to obtain a manifestly $E_7(7)$ invariant action in [27]. Detailed study of symmetry aspects of quantum theory based on this “doubled” version of $N = 8$ supergravity (in particular, the absence of $SU(8)$ anomaly) appeared in [6] though the crucial question of preservation of 4d Lorentz symmetry at the quantum level was not addressed.

As the general issues with quantum realization of a symmetry involving the duality are the same in any number $d = 2p + 2$ of dimensions here we shall concentrate on a technically simpler

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4 This is no longer true in general on a curved 4d background: one may get an “anomalous” local curvature coupling analogous to the dilaton shift under scalar-scalar duality in 2d case (see Appendix A).

5 Keeping all the fields on equal footing may be natural in supersymmetric theories.

6 This condition is the same as the form-invariance (“self-duality”) of the effective action under the “Legendre” transformation from the original to the dual variables. Due to $U(1)$ gauge invariance the effective action should depend only on the field strength and thus one can formally apply the duality transformation but, in contrast to the case of the classical theory, it is not clear a priori why this should lead to “self-duality” of the effective action, i.e. to a duality-covariant set of equations of motion. For a discussion of classical duality covariant non-linear (supersymmetric) theories see [21] and references there.

7 In string theory context the scalar-scalar duality has a target-space interpretation as T-duality acting on coupling functions. The $O(n,n)$ duality-symmetric form of sigma model equations of motion based on doubling of coordinates was considered in [25].
but still non-trivial case \( d = 2 \), \( p = 0 \), i.e. the case when the 4d vectors are replaced by scalars. The 2d analog of the action (1.1) is

\[
S = -\frac{1}{2} \int d^2 \sigma \left[ (\partial_a \phi)^2 + e^{4\phi} (\partial_a \chi)^2 + e^{-2\phi} (\partial_a x_s)^2 + \epsilon^{ab} \epsilon_{rs} \partial_a x_r \partial_b x_s \right],
\]

(1.2)

where \( a, b = 0, 1 \) and \( r, s = 1, 2 \). Like (1.1), this model has the \( SL(2) \) symmetry of the \((\phi, \chi)\) sector extended to the full set of equations of motion provided it is combined with 2d duality transformation on the scalars \( x_s \). We need at least \( n = 2 \) scalars \( x_s \) to have the \( O(n, n) \) duality group (acting on \( x_s \) and their “momenta”) big enough to contain the \( SL(2) \) acting on \((\phi, \chi)\).\(^8\)

Similar sigma models with an “external” \( y_i = (\phi, \chi) \) and “internal” \( x_s \) sectors were considered in the string theory context in [28]. Integrating out \( x_s \) one finds the \( SL(2) \) invariant quantum theory for \((\phi, \chi)\) modulo local dilaton shift term [30, 31], but the realization of the duality symmetry on the full set of fields at the quantum level is a priori non-trivial.

We shall consider the “doubled” formulation [24] in which the duality in \( x_s \) sector and thus the \( SL(2) \) symmetry of (1.2) is manifest but the Lorentz symmetry is not (but is recovered on-shell)\(^9\). It turns out to be natural to split \( x_s \) into its chiral parts so that in the simplest case of \( \chi = 0 \) the duality symmetry of the S-matrix translates into a symmetry under flipping the sign of the anti-chiral part and the sign of \( \phi \). Similar transformation will apply to higher-dimensional models, e.g., in 4d one would need to flip the sign of the anti-chiral part of the vector field.

Below we shall consider examples of computations of quantum effective actions for simple sigma models with structure similar to (1.2) or its truncations and demonstrating how the duality is realised at the quantum level. We shall concentrate on the most non-trivial discrete subgroup of the duality \((\phi \rightarrow -\phi)\) for \( \chi = 0 \) as generalization to continuous transformations does not bring new conceptual problems. Starting with the manifestly duality symmetric formulation we shall check the presence of the 2d Lorentz invariance in the quantum on-shell effective action or in the S-matrix.

In section 2 we shall consider an example of a 2d model analogous to (1.2) – the sigma model with euclidean AdS target space. We shall describe its duality-symmetric “doubled” action and discuss the action of duality symmetry on the corresponding quantum effective action. We shall argue that the corresponding S-matrix is both duality-symmetric and Lorentz-invariant. In section 3 we shall consider a different example of duality-symmetric theory with a non-linear action for a single 2d scalar that can be written in a standard form by introducing a coupling (similar to the \( \phi \)-coupling in (1.2)) to a non-propagating auxiliary field. We shall compute the corresponding 1-loop effective action and demonstrate its duality symmetry. We shall also explain that the quantum-corrected effective action remains duality-covariant if treated in loop perturbation theory. In section 4 we shall discuss the 4d action (1.1) and also the Born-Infeld

\(^8\)Note that this sigma model has constant 3-form strength \( H_{xrs} = \epsilon_{rs} \) and its metric has only one non-zero component of the Ricci tensor: \( R_{\phi \phi} = -\frac{3}{2} \). This model is not conformal (the beta-function cannot be cancelled by a linear in \( \phi \) dilaton background).

\(^9\)One may try to regularize the theory in such a way that the Noether current of the modified off-shell Lorentz symmetry [29, 24] (reducing on-shell to the standard one) is conserved in the presence of quantum corrections. Assuming such a regularization exists (i.e. if the modified Lorentz symmetry is non-anomalous) the on-shell quantum corrections should be invariant under the standard Lorentz transformations.
(BI) action and briefly discuss implications of our analysis for $\mathcal{N} = 8$ supergravity. In Appendix A we shall point out the possible presence of duality-non-invariant local terms when the theory is defined on a curved background. Appendix B will describe some details of the calculation of the 1-loop effective action for the scalar theory of section 3.

## 2 AdS sigma model: an example of duality-invariant theory in 2 dimensions

Let us consider the following extension of the $\chi = 0$ truncation of the model (1.2): a sigma-model based on euclidean $AdS_{n+1}$ metric ($s = 1, \ldots, n$)

$$ds^2 = d\phi^2 + e^{-2\phi} dx_s dx_s.$$  \hspace{1cm} (2.1)

2d duality for all $x_s$ maps this sigma-model into itself provided one also does the coordinate transformation $\phi \to -\phi$ \cite{32}. This transformation interchanges manifest (Noether) charges with an equivalent subset of hidden charges (conserved due to the integrability of the model) \cite{33}; this is the origin of “dual conformal symmetry” \cite{34, 35}.

### 2.1 Classical theory

Writing the sigma model action for (2.1) in “first-order” form ($a = 0, 1$; $(\partial_a x)^2 = -\dot{x}_s^2 + x'^2$)

$$S(\phi, x) = \frac{1}{2} \int d^2 \sigma \left[ - (\partial_\phi)^2 - e^{-2\phi} (\partial_a x_s)^2 \right]$$  \hspace{1cm} (2.2)

$$\rightarrow S(\phi, p, x) = \frac{1}{2} \int d^2 \sigma \left[ - (\partial_\phi)^2 + 2p_s \dot{x}_s - e^{-2\phi} x_s^2 - e^{2\phi} p_s^2 \right]$$  \hspace{1cm} (2.3)

and introducing a new field $\tilde{x}_s$ such that $p_s = \dot{\tilde{x}}_s$ we get the following duality-invariant action \cite{24}

$$\hat{S}(\phi, x, \tilde{x}) = \frac{1}{2} \int d^2 \sigma \left[ - (\partial_\phi)^2 + \dot{x}_s \dot{\tilde{x}}_s + \ddot{x}_s \ddot{\tilde{x}}_s - e^{-2\phi} x_s^2 - e^{2\phi} \dot{x}_s^2 \right]$$  \hspace{1cm} (2.4)

$$= -\frac{1}{2} \int d^2 \sigma \left[ (\partial_\phi)^2 - \Omega_{IJ} \dot{X}^I \dot{X}^J + M_{IJ} X'^I X'^J \right],$$  \hspace{1cm} (2.5)

$$X = \begin{pmatrix} x \\ \tilde{x} \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad M = \begin{pmatrix} e^{-2\phi} & 0 \\ 0 & e^{2\phi} \end{pmatrix},$$  \hspace{1cm} (2.6)

where we used integration by parts and $I, J = 1, \ldots, 2n$. This action describes the same number of degrees of freedom as (2.2) and is manifestly invariant under the duality transformation

$$\phi \to -\phi, \quad x_s \to \tilde{x}_s, \quad \tilde{x}_s \to x_s.$$  \hspace{1cm} (2.7)

Note that the original sigma model action corresponding to (2.1) is invariant under $SO(1, n + 1)$, the dual action for $\phi, \tilde{x}$ is also invariant under another copy of $SO(1, n + 1)$ (which may be interpreted as part of hidden symmetry of the original model). The interpolating “doubled”
action \((2.4)\) does not have manifest \(SO(1, n + 1) \times SO(1, n + 1)\) symmetry but it has of course an equivalent integrable structure (Lax pair)\(^{10}\) and thus the same set of conserved charges as the original \(AdS_{n+1}\) model\(^{11}\).

Let us note that the analog of the doubled action \((2.5)\) corresponding to \((1.2)\) is given by

\[
\hat{S}(\phi, x, \tilde{x}) = -\frac{1}{2} \int d^2 \sigma \left[ (\partial_a \phi)^2 + e^{4\phi} (\partial_a \chi)^2 - \Omega_{IJ} \dot{X}^I \dot{X}^J + M_{IJ} \dot{X}^I \dot{X}^J \right],
\]

\[
X = \left( \begin{array}{c} x \\ \tilde{x} \end{array} \right), \quad \Omega = \left( \begin{array}{cc} 0 & I \\ I & 0 \end{array} \right), \quad M = \left( \begin{array}{cc} G - BG^{-1}B & BG^{-1} \\ -G^{-1}B & G^{-1} \end{array} \right),
\]

\[(G - BG^{-1}B)_{rs} = (e^{-2\phi} + 4 \chi^2 e^{2\phi}) \delta_{rs}, \quad (BG^{-1})_{rs} = 2 \chi e^{2\phi} e_{rs}, \quad G^{-1} = e^{2\phi} \delta_{rs}\]

The symmetry of the full model is the \(SO(1, 2)\) subgroup of \(O(2, 2)\) duality transformations on \(M\) that can be compensated by \(SL(2)\) transformations on \((\phi, \chi)\).

The classical equations for \(x_s\) and \(\tilde{x}_s\) following from \((2.4)\) may be written as

\[
(\dot{x}_s - e^{2\phi} \tilde{x}_s')' = 0, \quad (\dot{\tilde{x}}_s - e^{-2\phi} x_s')' = 0
\]

\[
\rightarrow \dot{x}_s - e^{2\phi} \tilde{x}_s = 0, \quad \dot{\tilde{x}}_s - e^{-2\phi} x_s = 0,
\]

where as in \([24]\) we dropped \(\tau\)-dependent integration functions assuming they are absent at the boundaries of spatial interval – this ensures that we recover the standard equation of motion for \(x_s\)\(^{12}\).

Introducing the combinations (that become free chiral scalars for \(\phi = 0\))

\[
x_s = x_s^+ + x_s^-, \quad \tilde{x}_s = x_s^+ - x_s^-, \quad x_s^\pm = \frac{1}{2} (x_s \pm \tilde{x}_s),
\]

we can rewrite \((2.4)\) as \((\partial_\pm = \pm \partial_0 + \partial_1)\)

\[
\hat{S}(\phi, x^+, x^-) = -\int d^2 \sigma \left[ \frac{1}{2} (\partial_a \phi)^2 + x_s^+ \partial_+ x_s^+ + x_s^- \partial_- x_s^- 
\right.

\[
\left. + f_1(\phi) \left( x_s^+ r^s + x_s^- r^s \right) - 2 f_2(\phi) x_s^+ x_s^- \right],
\]

\[
f_1 = 2 \sinh^2 \phi, \quad f_2 = \sinh 2\phi.
\]

The corresponding duality symmetry of this action is

\[
\phi \rightarrow -\phi, \quad x_s^+ \rightarrow x_s^+, \quad x_s^- \rightarrow -x_s^-.
\]

\(^{10}\)The corresponding flat currents in Poincaré-patch parametrization of \(AdS_{n+1}\) were discussed in \([33]\). The Lax pair corresponding to the 1st-order form of the equations of motion was written down in \([35]\). It is not, however, manifestly symmetric under \((2.4)\) as this duality should be accompanied by a transformation of the spectral parameter or \(Z_2\) automorphism of the symmetry algebra \([35]\) that implies certain transformation on the set of conformal charges. In particular, the Noether symmetry charges of the original model are mapped into hidden charges of the dual model and vice versa.

\(^{11}\)The invariance under \((2.4)\) corresponds to \(X \rightarrow \Omega X, \ M \rightarrow \Omega M\Omega\). This is a special case \(\Lambda = \Omega\) of more general \(O(n, n)\) transformations \(X \rightarrow Ax, \ \Lambda^T \Omega \Lambda = \Omega\), that preserve the structure of the action provided also \(M \rightarrow \Lambda^{-T} M \Lambda^{-1}\) but this change of \(M\) cannot be in general compensated by a redefinition of \(\phi\). In the case of \((2.4)\) \((2.5)\) the continuous part of \(O(1, 1)\) duality symmetry can be realized by rotation of \(X\) combined with translations of \(\phi\).

\(^{12}\)Note that the action is invariant under \(\delta x_s = f_s(\tau)\).
Like (2.10) the equations of motion for $x^\pm$ are Lorentz-invariant \footnote{The “free” action in (2.4),(2.12) is invariant under the Lorentz-type symmetry: $\delta x_s = \tau x'_s + \sigma x''_s$, $\delta \tilde{x}_s = \tau \tilde{x}'_s + \sigma \tilde{x}''_s$, or $\delta x^\pm = (\tau \pm \sigma) x'^\pm$. An analog of this symmetry exists also for non-zero $\phi$ \cite{[24]}. This symmetry becomes standard Lorentz symmetry on the equations of motion.}

Note that the action (2.12) with any even $f_1$ and odd $f_2$ functions would be invariant under (2.14) but it will be Lorentz-invariant only for a special choice of $f_1, f_2$: integrating $\tilde{x}$ out one gets $-(f_1 + f_2 + 1)^{-1} \dot{x}^2 + (f_1 - f_2 + 1)x^2$ which is Lorentz invariant only if $(f_1 + f_2 + 1)^{-1} = f_1 - f_2 + 1$. Note also that $x^+_s$ and $x^-_s$ do not decouple in (2.12); they do if one considers $AdS_3$ and introduces a particular antisymmetric tensor coupling which leads to the $SL(2)$ WZW model.

\section*{2.2 Quantum theory}

Let us now turn to the quantum theory. The original (2.2) and the doubled theory (2.4) are expected to be equivalent quantum mechanically when probed with common observables. An example of such observables are scattering amplitudes of $x$ fields, in which $\tilde{x}$ fields enter only through loops. Indeed, integrating out $\tilde{x}$ in (2.4) gives back (2.2). The doubled theory allows, however, for a larger set of observables, e.g. scattering amplitudes of both $x$ and $\tilde{x}$ fields which have duality acting as a standard symmetry. Given the duality symmetry (2.7),(2.14) of the classical action (2.4), (2.12) one expects to find the same symmetry in the quantum effective action, i.e.

$$\Gamma[\phi, x, \tilde{x}] = \Gamma[-\phi, \tilde{x}, x], \quad \text{i.e.} \quad \Gamma[\phi, x^+, x^-] = \Gamma[-\phi, x^+, -x^-]. \quad (2.15)$$

For this to happen one should maintain the symmetry at the quantum level by a proper choice of quantization prescription (i.e. regularization and path integral measure). This may not be automatic if other fields and symmetries are also present. For example, the simplest special case to consider would be effective action depending just on $\phi$ (found by integrating out both $x_s$ and $\tilde{x}_s$) which should be invariant (by standard path integral transformation argument \cite{30,31}) under just $\phi \to -\phi$. As we shall discuss in Appendix A, maintaining this duality depends on assumptions about preservation of other symmetries (like target space diffeomorphism invariance), i.e. on the choice of measure and regularization scheme.

The central question, however, is if, like the classical action, the quantum effective action or $S$-matrix will be Lorentz-invariant on-shell. This on-shell invariance may a priori have two different interpretations:

(I) $\Gamma[\phi, x, \tilde{x}]$ should be Lorentz-invariant once evaluated on a solution of the equations of motion, i.e. to the leading order, (2.10); \footnote{We will be assuming for simplicity that the relevant classical solutions do not get non-trivial quantum corrections. In a simple case of a field like $\tilde{x}$ entering the action only quadratically we may expect that the corresponding equation of motion does not receive quantum corrections. In general, given a field theory with a classical action $S[\varphi]$, the corresponding quantum $S$-matrix generating functional is given by $\hat{S}[\varphi_{in}] = \Gamma[\varphi(\varphi_{in})]$ where $\Gamma[\varphi]$ is the quantum effective action and $\varphi(\varphi_{in})$ is the solution of the quantum equations of motion $\frac{\delta}{\delta \varphi} = 0$ with “scattering” boundary conditions, $\varphi = \varphi_{in} + ...$, $(\partial^2 + m^2)\varphi_{in} = 0$ (see, e.g. \cite{36} and references therein). While both should be invariant under the standard Lorentz transformations, $\Gamma$ evaluated on the classical solution may differ from $\Gamma$ evaluated on the solution of the effective equation $\frac{\delta}{\delta \varphi} = 0$ starting with 2-loop order.}$

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(II) the quantum equations of motion following from \( \Gamma[\phi, x, \tilde{x}] \) should be Lorentz-invariant. The property (I) should indeed be expected given that the classical equations of motion are Lorentz-invariant and that integrating over \( \tilde{x} \) in (2.4) leads us back to the Lorentz-invariant action (2.2). This may be verified explicitly at 1-loop order, i.e. by expanding (2.4) to quadratic order in fluctuations near a classical solution \( (\phi(0), x(0), \tilde{x}(0)) \) and restoring the Lorentz invariance of the fluctuation Lagrangian by a field redefinition of the fluctuation fields which makes the Lorentz invariance of the resulting effective action manifest.\(^{15}\)

The property (II) is likely to be true too if understood in a perturbative sense, i.e. that the effective equations of motion are solved order by order in loop expansion: then given the classical Lorentz invariance, the 1-loop corrected equations of motion should also be Lorentz-invariant, etc. However, (II) is far from obvious if considered as an exact property of the effective action: it is not a priori clear if one should expect (some deformed version of) Lorentz invariance to apply to the full quantum equations of motion. (II) is essentially equivalent to an assumption that quantum equations of motion derived from the original Lorentz-covariant action (2.2) should admit an analog of the duality symmetry (2.7)\(^{16}\).

Since \( \Gamma[\phi, x, \tilde{x}] \) evaluated on a general classical solution with “in” (plane-wave) initial conditions at zero coupling is the generating functional for the S-matrix, (I) is equivalent to the condition of Lorentz invariance of the the S-matrix for \( \{\phi, x^+, x^-\} \) following from (2.12) (whose manifest duality invariance (2.14) is expected due to the structure of the interaction terms with \( f_1, f_2 \) in (2.13) but 2d Lorentz invariance is a priori non-trivial). The key fact is that the on-shell conditions for the chiral scalars implied by (2.12) are Lorentz-invariant

\[ \partial_- x^+_s = 0, \quad \partial_+ x^-_s = 0. \quad (2.16) \]

To demonstrate Lorentz invariance of the S-matrix one should note also that:

(i) to any loop order, \( x^\pm \) lines cannot “terminate”: they are either open (i.e. \( x^\pm \) coming into a diagram eventually exits it) or closed (representing a loop of \( x^\pm \) fields with \( \phi \) lines attached to it); (ii) a tree-level Green’s function with on-shell \( x^\pm \) and off-shell \( \phi \)’s is Lorentz invariant;\(^{17}\) (iii) the determinant of the \( x^\pm \)-quadratic fluctuation operator depending on an off-shell \( \phi \) is Lorentz-invariant.\(^{18}\)

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\(^{15}\)In view of the direct relation between (2.3) and (2.4) this is basically the same as the expectation that in a semiclassical expansion with a phase-space action one should end up with the same 1-loop effective action as found in the usual second-order Lagrangian formulation.

\(^{16}\)The assumption that such modified duality should apply to quantum counterterms in 4d supergravity was made in [9, 10]. It is not clear, however, why this property should apply only to local (divergent) part of the effective action. Also, if one is prepared to consider subsets of local terms in \( \Gamma \) forming duality-invariant combinations (containing terms of all orders in fields/derivatives like BI action and thus containing all powers of logarithm of a UV cutoff) one may as well consider a possibility that UV divergent terms go away after a resummation of loop expansion.

\(^{17}\)To find tree-level S-matrix one may solve classical equations of motion with “in” initial conditions, i.e. \( x^\pm = x^\pm_{\text{in}} + ... \), with \( \partial_\mp x^\pm_{\text{in}} = 0 \), and substitute the result into the classical action. This gives a generating functional \( \hat{S}(\phi, x^\pm_{\text{in}}, x^-_{\text{in}}) \) for the corresponding tree S-matrix elements. Solving the classical equation for the combination \( \tilde{x} = x^+_s - x^-_s \) and substituting the result back into the action leads us to (2.2) with \( x = x^+_s + x^-_s \). The resulting functional of \( x^+_s, x^-_s \) is then obviously Lorentz-invariant (but of course is no longer manifestly duality-invariant).

\(^{18}\)This follows again from the fact that integrating out \( \tilde{x} \) in (2.4) leads us back to the Lorentz-invariant action (2.2). Note that this would not be the case for generic functions \( f_1, f_2 \) in (2.12).
The observation (i) allows us to break up any S-matrix element into parts of two types that appear in (ii) and (iii) which are connected by (Lorentz-invariant) \( \phi \) propagators. Since each part is Lorentz invariant, the whole S-matrix element is then also invariant.

The same observations (i) and (ii) will apply of course in the 4d vector case of \( (1.1) \) where we can split \( F_{mn} \) into selfdual and anti-selfdual parts that should correspond at the S-matrix level to positive and negative helicity photons; the (discrete part of) duality symmetry of the S-matrix will then mean a symmetry analogous to \( (2.14) \).

We may also compute the 1-loop S-matrix elements explicitly and check their invariance under the duality transformations \( (2.14) \) as well as their Lorentz invariance. Albeit with a singular momentum configuration, the simplest on-shell matrix elements are

\[
A(\phi(p_1), x_s^+(p_2), x_s^+(p_3)) = A(\phi(p_1), x_s^-(p_2), x_s^-(p_3)) = 0 ,
A(\phi(p_1), x_s^-(p_2), x_s^+(p_3)) \propto p_2 \cdot p_3 \lg \Lambda + \text{finite} ,
\]

where \( \Lambda \) is a UV cutoff. As expected, they are Lorentz-invariant and renormalize the trilinear interaction in \( (2.12) \)\(^{19}\). The matrix elements with four external \( x_s^\pm \) are the simplest ones with non-singular external momentum configurations. It is easy to see that the Feynman rules following from the action \( (2.12) \) do not allow four-point scattering amplitudes with an odd number of external \( x_s^- \). For the ones with an even number of external \( x_s^- \) lines we find

\[
A(x_s^+(p_1), x_s^+(p_2), x_s^+(p_3), x_s^+(p_4))
\]

\[
= \int \left[ \frac{d^2l}{(2\pi)^2} \right] \left[ \frac{(p_1 + l)_- + (p_2 + l)_+}{(p_1 + l)_+ + (p_2 + l)_-} \right] \left[ \frac{(p_3 - l)_- + (p_4 - l)_+}{(p_3 - l)_+ + (p_4 - l)_-} \right] \frac{p_1 + p_2 + p_3 + p_4}{l^2(l + p_1 + p_2)^2}
\]

\[
A(x_s^+(p_1), x_s^+(p_2), x_s^-(p_3), x_s^-(p_4))
\]

\[
= \int \left[ \frac{d^2l}{(2\pi)^2} \right] \left[ \frac{(p_1 + l)_- + (p_2 + l)_+}{(p_1 + l)_+ + (p_2 + l)_-} \right] \left[ \frac{(p_3 - l)_+ + (p_4 - l)_-}{(p_3 - l)_- + (p_4 - l)_+} \right] \frac{p_1 + p_2 + p_3 - p_4}{l^2(l + p_1 + p_2)^2}
\]

\[
A(x_s^-(p_1), x_s^-(p_2), x_s^-(p_3), x_s^-(p_4))
\]

\[
= \int \left[ \frac{d^2l}{(2\pi)^2} \right] \left[ \frac{(p_1 + l)_- + (p_2 + l)_+}{(p_1 + l)_+ + (p_2 + l)_-} \right] \left[ \frac{(p_3 - l)_+ + (p_4 - l)_-}{(p_3 - l)_- + (p_4 - l)_+} \right] \frac{p_1 - p_2 - p_3 - p_4}{l^2(l + p_1 + p_2)^2} .
\]

As expected, these are also Lorentz-invariant provided that the regularization scheme included in the integration measure is chosen to preserve Lorentz symmetry.

Let us mention a potentially subtle issue of the choice of UV cutoff (or, more generally, the choice of regularization scheme) in a theory without manifest Lorentz symmetry. While we do not expect a genuine Lorentz anomaly in a theory with balance of chiral spinors and self-dual tensors, there may nevertheless exist a spurious breaking of Lorentz symmetry due to an unfortunate choice of UV cutoff. In general, one may of course use an arbitrary UV cutoff and then attempt to add local counterterms to satisfy the Ward identities of the required symmetries. We expect the same philosophy should be applicable to the present case of on-shell Lorentz symmetry.

To try to check the possibility (II), i.e. that the quantum effective action may have a nonlinear analog of the tree-level duality (cf. BI action vs. Maxwell action) one may try to compute the

\(^{19}\)We ignored the overall coupling constant in \( (2.2) \) which is logarithmically running in AdS sigma model.
1-loop $\Gamma = \Gamma_1$ for a classically invariant theory like (1.1) or (2.2) in some approximation (e.g. keeping only field strength dependence but ignoring dependence on its derivatives). For example, starting with (1.1) it is easy to find $\Gamma_1[\frac{F(0)}{0}]$ for $\frac{F(0)}{0}=$const. To have a consistent classical solution we will need to require that $\phi = \phi(0) =$const and thus $(\frac{F(0)}{0})_{mn}(\frac{F(0)}{0})^{mn} = 0$. In this case $\Gamma_1$ will depend only on the traceless stress tensor or $T_{kn} = e^{-2\phi(0)}(\frac{F(0)}{0})_{mn}(\frac{F(0)}{0})^{mn}$ and thus (an even) function of only one invariant $e^{-2\phi(0)}(\frac{F(0)}{0})_{mn}(\frac{F(0)}{0})^{mn}$. It is then invariant under the classical duality symmetry but this approximation is not sufficient to address the question about possible duality symmetry of the quantum equations following from $\Gamma$: for that we need to know the dependence of $\Gamma$ on both $F_{mn}F^{mn}$ and $F_{mn}F^{*mn}$ invariants. A similar remark applies to the action (2.2) where we may consider a classical solution with $\phi = \phi(0) =$const, $\partial_+x(0) =$const and $(\partial_+x(0))^2 = 0$ (i.e. $\partial_+x(0) = 0$ or $\partial_+x(0) = 0$). We shall discuss such a computation (and also its generalization) on the example of a slightly different scalar model in the next section.

### 3 An example of nonlinear scalar action

Let us now consider a superficially different but, in fact, related example of a non-linear scalar theory depending only on $(\partial x)^2$. It has classical duality symmetry and we shall study if this theory has also a generalization of the duality at the quantum level. The corresponding action is (here $x$ is a single scalar field and we ignore a dimensionfull coupling constant)

$$S = \int d^2\sigma \ L(x), \quad L(x) = -\sqrt{1 + (\partial_a x)^2}.$$  \hfill (3.1)

Finding the momentum $p$ conjugate to $x$ and setting $p \equiv \tilde{x}'$ we get the corresponding phase-space or “doubled” Lagrangian which is the manifestly duality-invariant analog of (2.4)

$$\tilde{L}(x, \tilde{x}) = \tilde{x}'\tilde{x} - \sqrt{1 + x'^2}\sqrt{1 + \tilde{x}'^2}. \hfill (3.2)$$

Note that here the integral over $\tilde{x}$ (or the momentum) is non-gaussian so the quantum theories defined by (3.1) and (3.2) are equivalent only in the leading semiclassical approximation of the integral over $\tilde{x}$.

Semiclassically, (3.1) is equivalent to the following Lagrangian

$$L(x, G) = -\frac{1}{2} \left[ G(\partial_a x)^2 + G + G^{-1} \right], \quad G = e^{-2\phi}, \hfill (3.3)$$

where $G$ (or $\phi$) is an auxiliary 2d field. The corresponding “doubled” action is then the same as in (2.4)

$$\tilde{L}(x, \tilde{x}) = \tilde{x}'\tilde{x} - \frac{1}{2}G(1 + x'^2) - \frac{1}{2}G^{-1}(1 + \tilde{x}'^2). \hfill (3.4)$$

The duality symmetry of the equations of motion for (3.1) ($x \rightarrow \tilde{x}$ with $\epsilon^{ab}\partial_b \tilde{x} = [1 + (\partial_a x)^2]^{-1/2}\partial^a x$) corresponds to $x \rightarrow \tilde{x}$, $G \rightarrow G^{-1}$ which is the manifest symmetry of (3.4). Solving for $G$ in (3.4) leads again to (3.2), while integrating out $\tilde{x}$ gives back (3.3).

---

$^{20}$This representation is a simple analog of replacing the Nambu action with the “Polyakov” action with an independent 2d metric. Analogous “polynomial” representations using auxiliary scalars exist for other similar actions like Born-Infeld one [37, 38].
We thus get an analog of (2.2) but with a potential instead of a kinetic term for \( \phi \). The Lagrangian that generalizes both (2.2) and (3.3)

\[
L = -\frac{1}{2}(\partial_a \phi)^2 - \frac{1}{2}e^{-2\phi}(\partial_a x)^2 - \cosh 2\phi ,
\]

also represent a duality-covariant theory.

In view of non-polynomiality of (3.1) it is natural (as in the Nambu \rightarrow Polyakov action case) to define the corresponding quantum theory by the path integral with the action (3.3) or the equivalent “doubled” action (3.4). If we start with (3.3) and integrate out \( x \), we get an effective action for \( \phi \) which is invariant under \( \phi \rightarrow -\phi \). If we keep a background for \( x \) and evaluate the effective action on shell, we should get again a duality-symmetric result as in (2.15). The classical solution for \( x(0), G(0) \) in (3.3) satisfies

\[
G(0)n^a = \epsilon^{ab}\tilde{n}_b , \quad n_a \equiv \partial_a x(0) , \quad G(0) = (1 + n^2)^{-1/2} = (1 + \tilde{n}^2)^{1/2} = \tilde{G}^{-1}(0) ,
\]

where \( \tilde{n}_a, \tilde{G}(0) \) is the classical solution following from the action dual to (3.3)

\[
\tilde{L}(\tilde{x}, G) = -\frac{1}{2} \left[ G^{-1}(0) \partial_a \tilde{x} \right]^2 + G + G^{-1} .
\]

To compute the effective action we need to expand near the classical solution, \( x = x(0) + \eta, G = G(0)(1 + \xi) \). If we first treat \( G \) as an external background and perform the path integral duality transformation with respect to the fluctuation \( \eta \), we end up with (3.7) with \( \tilde{x} = \tilde{x}(0) + \tilde{\eta} \), where \( \tilde{\eta} \) is dual to \( \eta \). Integrating then over \( \eta, \xi \) in (3.3) or over \( \tilde{\eta}, \xi \) in (3.7) we should get the same result for the effective action.

Let us demonstrate this more explicitly. Expanding (3.3) we find the following quadratic-fluctuation action

\[
L_2(\xi, \eta) = -\frac{1}{2}G(0)(\partial_a \eta)^2 - G(0)n^a\xi\partial_a \eta - \frac{1}{2}G^{-1}(0)\xi^2 .
\]

Integrating out \( \xi \) gives

\[
L_2(\eta) = -\frac{1}{2}G(0)(\partial_a \eta)^2 + \frac{1}{2}G^3(0)(n^a \partial_a \eta)^2 .
\]

We may now perform the standard path integral duality over \( \eta \) by replacing (3.9) with

\[
L(B, \tilde{\eta}) = -\frac{1}{2}G(0)B^2_a + \frac{1}{2}G^3(0)(n^a B_a)^2 + \epsilon^{ab}B_a \partial_b \tilde{\eta} ,
\]

Performing the Gaussian integral over the auxiliary vector field \( B_a \) we get the following action for the dual fluctuation field \( \tilde{\eta} \)

\[
\tilde{L}_2(\tilde{\eta}) = -\frac{1}{2}\tilde{G}^{-1}(0)(\partial_a \tilde{\eta})^2 + \frac{1}{2}\tilde{G}^3(0)(\tilde{n}^a \partial_a \tilde{\eta})^2 .
\]

This is exactly the same quadratic-fluctuation action that follows if one starts with (3.7). This shows again that the resulting effective action \( \Gamma_{1} \) which is a functional of \( x(0), G(0) \), i.e. (in view of (3.6)) a functional of \( n_a = \partial_a x(0) \) is duality-symmetric, i.e. invariant under \( n_a \rightarrow \tilde{n}_a, G(0) \rightarrow \tilde{G}(0) = \tilde{G}^{-1}(0) \).
This formal argument ignored local measure-like factors (like a “determinant” term in \( \Gamma_1 \) proportional to \( \ln G^{-1}_{(0)} \) coming from integration over \( \xi \)) which are absent in a regularization ignoring power divergences. In general, the duality invariance of the resulting effective action depends on a choice of measure/regularization. If we use a dimensionfull (e.g. proper-time) UV cutoff \( \Lambda \) then the effective action is sensitive to the contribution of the measure. The standard choice \( \left[ dx \right] = dx \sqrt{G} \) (corresponding to the trivial measure in the phase-space path integral or path integral corresponding to the “doubled” action (3.4)) ensures the duality invariance of the result.\[21\]

Let us consider the special case of \( n_a = \partial_a x(0) = \text{const.} \)\[22\] Taking into account the “determinant” term from the \( \xi \) integration and the measure contribution for \( \eta \) the resulting 1-loop on-shell effective action may be written as

\[ \Gamma_1 = \frac{1}{2} \ln \det K \, , \quad K = G^{-1}_{(0)} \partial^a \partial_a - G_{(0)} (n^a \partial_a)^2 \, . \] (3.12)

Using that \( \epsilon^{ab} \tilde{n}_b = G_{(0)} n^a \), \( G_{(0)} = (1 + n^2)^{-1} = 1 + \tilde{n} \) where \( \tilde{n}_a = \partial_a \tilde{x} \), one can see that

\[ K = G_{(0)} \partial^a \partial_a - G_{(0)}^{-1} (n^a \partial_a)^2 \, , \] (3.13)

and thus \( \Gamma_1 \) is duality invariant under \( x \to \tilde{x}, \, G \to G^{-1} \). The resulting (classical plus one-loop) effective action for constant \( n_a = \partial_a x \) has the form (see Appendix B)

\[ \Gamma = \int d^2 \sigma \left[ - \sqrt{1 + (\partial_a x)^2} + \Lambda^2 F\left( \sqrt{1 + (\partial_a x)^2} \right) \right] \, , \] (3.14)

where the function \( F(y) \) (whose argument on-shell is \( G_{(0)}^{-1} \)) is

\[ F(y) = \ln\left[ \frac{1}{2} \left( y^{1/2} + y^{-1/2} \right) \right] \, . \] (3.15)

Its symmetry \( F(y) = F(y^{-1}) \) makes the duality invariance of the 1-loop effective action manifest.

Assuming one starts directly with (3.1) let us now study whether the tree-level action plus the part of the 1-loop effective action which depends only on the first derivative of \( x \),

\[ \Gamma(\partial x) = - \int d^2 \sigma \sqrt{1 + (\partial_a x)^2} + h \Gamma_1(\partial x) + \mathcal{O}(h^2) \, , \] (3.16)

leads to the duality-covariant quantum equations of motion to the relevant leading order in the loop expansion, i.e. is “self-dual” under the “Legendre” transform from \( x \) to the dual variable. From the above discussion the resulting \( \Gamma_1 \) is the same as found by starting from (3.3) and it depends on \( \partial_a x \) only through \( G_{(0)}^{-1} = \sqrt{1 + (\partial_a x)^2} \). To check the duality covariance one is to carry out the “Legendre” transform from the original to the dual variable while keeping all the

\[ ^{21}\text{We also assume that the fundamental variable is } \phi = \frac{1}{2} \ln G, \text{ i.e. the measure of integration over } \phi \text{ is trivial.} \]

\[ ^{22}\text{Below for notational simplicity we shall omit subindex } (0) \text{ on } x \text{ and } \tilde{x}. \]
relevant $O(\hbar)$ terms. Replacing $\partial_a x$ by an independent field strength $n_a$ and introducing the dual variable $\tilde{x}$ through the Lagrange multiplier term we get

$$\hat{\Gamma}(n, \partial \tilde{x}) = - \int d^2 \sigma \sqrt{1 + n_a^2} + \hbar \Gamma_1(n) + O(h^2) + \int d^2 \sigma \, \epsilon^{ab} n_a \partial_b \tilde{x}.$$  

(3.17)

Solving the resulting effective equation for $n_a$

$$\frac{n^a}{\sqrt{1 + n^2}} - \hbar \frac{\delta \Gamma_1}{\delta n_a} + O(h^2) = \epsilon^{ab} \partial_b \tilde{x}$$  

(3.18)

perturbatively in $\hbar$ we get

$$n^a = n^a_{(0)} + \hbar n^a_{(1)} + O(h^2),$$  

(3.19)

$$n^a_{(0)} = \frac{\epsilon^{ab} \partial_b \tilde{x}}{\sqrt{1 + (\partial_a \tilde{x})^2}},$$  

(3.20)

$$n^a_{(1)} = \frac{n^a_{(0)} (n_{(0)} \cdot n_{(1)})}{(1 + n^2_{(0)})^{3/2}} + \left( \frac{\delta \Gamma_1}{\delta n_a} \right)_{n_{(0)}} = 0,$$  

(3.21)

$$n_{(0)} \cdot n_{(1)} = (1 + n^2_{(0)})^{3/2} n^a_{(0)} \left( \frac{\delta \Gamma_1}{\delta n_a} \right)_{n_{(0)}}, \quad n^a_{(1)} = (1 + n^2_{(0)})^{1/2} \left( n^a_{(0)} n^b_{(0)} - \eta^{ab} \right) \left( \frac{\delta \Gamma_1}{\delta n^b} \right)_{n_{(0)}}$$

The dual action, i.e. $\hat{\Gamma}(\partial \tilde{x})$ expressed in terms of $\partial \tilde{x}$ is then

$$\hat{\Gamma}(\partial \tilde{x}) = - \int d^2 \sigma \frac{1}{\sqrt{1 + n^2_{(0)}}} - \hbar \int d^2 \sigma \frac{n_{(0)} \cdot n_{(1)}}{(1 + n^2_{(0)})^{3/2}} + \hbar \left( \Gamma_1 - n_a \frac{\delta \Gamma_1}{\delta n_a} \right)_{n_{(0)}} + O(h^2)$$

$$= - \int d^2 \sigma \sqrt{1 + (\partial_a \tilde{x})^2} + \hbar \Gamma_1 \bigg|_{n_a \to \epsilon^{ab} \partial_b \tilde{x}} + O(h^2).$$  

(3.22)

As we have shown above, the 1-loop effective action $\Gamma_1$ is invariant under the classical duality transformation ($\Gamma_1(\partial x) = \Gamma_1(\partial \tilde{x})$) so that the $\hat{\Gamma}(\partial \tilde{x})$ has the same form as $\Gamma(\partial x)$, up to $O(h^2)$ terms,

$$\hat{\Gamma}(\partial \tilde{x}) = - \int d^2 \sigma \sqrt{1 + (\partial_a \tilde{x})^2} + \hbar \Gamma_1 \bigg|_{n_a \to \epsilon^{ab} \partial_b \tilde{x}} + O(h^2).$$  

(3.23)

Since this argument used only the duality-invariance of $\Gamma_1$, it follows, quite generally, that if the leading quantum correction to a classically “self-dual” (in the sense of the above “Legendre” transform) action is duality-invariant, then the resulting effective action is “self-dual” up to higher-order corrections. The relation between the original and dual fields receives, in general, loop corrections (cf. (3.21)).

One may attempt to extend the above discussion by including higher-loop corrections $\Gamma_n$ to the effective action and finding the constraints $\Gamma_n$ must satisfy for the complete effective action to be “self-dual” through the required order. For example, the two-loop effective action should be a solution of

$$\Gamma_2(\partial \tilde{x}) = \Gamma_2(n_{(0)}) + \int d^2 \sigma \frac{1}{(1 + n^2_{(0)})^{1/2}} \left[ n^2_{(1)} - \frac{(n_{(0)} \cdot n_{(1)})^2}{1 + n^2_{(0)}} \right].$$  

(3.24)

It is not clear a priori why $\Gamma_2$ should obey this constraint.
4 Four-dimensional vector models

Duality symmetries (in 2d or 4d) discussed above are on-shell symmetries; it is not possible to promote them to manifest symmetries of the action while preserving all the other symmetries of the theory, in particular Lorentz invariance. The “doubled” formalism provides a framework in which the duality symmetry becomes a manifest off-shell symmetry; while the “doubled” action is not invariant under the Lorentz transformations, it nevertheless exhibits a symmetry which becomes the standard Lorentz symmetry on shell. The advantage of the “doubled” formalism is that details of the duality group are not important for its quantum realization – the main features are the same for discrete or continuous duality symmetries. Since the “doubled” action is manifestly duality invariant it should be possible to maintain it in the presence of a UV regularization. If the regularization also preserves the off-shell Lorentz-type symmetry present in the classical action then the on-shell observables, such as S-matrix elements, should exhibit both the duality and the Lorentz invariance.

While the discussion in the previous sections focused mainly on two-dimensional examples, it is straightforward to extend it to 4 dimensions. For example, it is straightforward to construct the “doubled” action for the theory \( \mathcal{N} > 4 \) supergravity in 4 dimensions. We may either follow the strategy described in sec. 2.1 and start with the first-order phase-space action or construct a first-order “master action” (as discussed e.g. in Appendix A of [38]) whose gauge symmetry may then be fixed in a convenient way. Dropping total derivatives (and fixing \( A_0 = 0 \)), the result is [26] \((i = 1, 2, 3)\)

\[
\hat{S} = \int d^4x \left[ -\frac{1}{2}(\partial_\alpha \phi)^2 - \frac{1}{2} e^{4\phi}(\partial_\alpha \chi)^2 + \hat{L}(A, \tilde{A}; \phi, \chi) \right], \quad (4.1)
\]

\[
\hat{L} = \frac{1}{2} \left( E_i^T \hat{\Omega} B_i - B_i^T M B_i \right), \quad (4.2)
\]

where\(^{23}\)

\[
E_i = \partial_0 A_i , \quad B_i = \epsilon_{ijk} \partial_j A_k , \quad A_i = \begin{pmatrix} A_i \\ \tilde{A}_i \end{pmatrix}, \quad (4.3)
\]

\[
\hat{\Omega} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad M = \begin{pmatrix} e^{-2\phi} + 4\chi^2 e^{2\phi} & -2\chi e^{2\phi} \\ -2\chi e^{2\phi} & e^{2\phi} \end{pmatrix}. \quad (4.4)
\]

In the case of \( \chi = 0 \), this action is invariant under the \( Z_2 \) duality transformation \((\hat{\Omega}^T = -\hat{\Omega}, \hat{\Omega}^2 = -I)\)

\[
A'_i = \hat{\Omega} A_i , \quad M' = \hat{\Omega}^T M \hat{\Omega} , \quad \text{i.e.} \quad A'_i = \tilde{A}_i , \quad \tilde{A}'_i = -A_i , \quad \phi' = -\phi. \quad (4.5)
\]

The corresponding equations of motion are \( E_i - e^{2\phi} \tilde{B}_i = 0 , \quad \tilde{E}_i + e^{-2\phi} B_i = 0 \). The action (4.2) has also a modified Lorentz-type symmetry [26] which becomes the standard Lorentz symmetry on the equations of motion; as in the 2d examples, the S-matrix elements should exhibit this symmetry simultaneously with being invariant under (4.5).

\(^{23}\)This action is similar the 2d one in (2.8). Note that in \( d = 2p + 2 \) dimensions \( \hat{\Omega} = \begin{pmatrix} 0 & 1 \\ (-1)^p & 0 \end{pmatrix} \).
Given that $\hat{\Omega}^2 = -I$ it is natural to introduce the complex combinations

$$A_i^\pm = A_i \pm i\tilde{A}_i, \quad \tilde{A}_i^\pm = A_i^-,$$

(4.6)

which transform under the duality (4.5) as

$$(A_i^\pm)' = \mp iA_i^\pm, \quad \phi' = -\phi.$$  

(4.7)

The classical equations written in terms of derivatives of $A_i^\pm$ take the form

$$E^+ + i(B^+ \cosh 2\phi - B^- \sinh 2\phi) = 0, \quad E^- - i(B^- \cosh 2\phi - B^+ \sinh 2\phi) = 0.$$  

(4.8)

For $\phi = 0$ they become the (anti)self-duality conditions: $F_{+}^{\pm} = \pm i\epsilon_{mn}^{kl}F_{kl}^\pm$, where as before $k, l, m, n = 0, 1, 2, 3$. $A_i^\pm$ will thus describe on shell photons of definite helicity (see [10] and below). The Lagrangian $\hat{L}$ in (4.2) written in terms of $A_i^\pm$ (leading to (4.8)) becomes

$$\hat{L} = \frac{1}{4} \left[ i(E_i^+ B_i^- - E_i^- B_i^+) - 2 \cosh 2\phi B_i^+ B_i^- - \sinh 2\phi (B_i^+ B_i^- + B_i^- B_i^+) \right],$$

(4.9)

and is obviously invariant under (4.7).

This symmetry implies that the S-matrix elements without external $\phi$ lines labeled by $A_i^\pm$ fields, $(A^+)^{n_+}(A^-)^{n_-}$, pick up a phase $i^{-n_+ + n_-}$ under the duality (4.7); since they must be invariant, they are nonvanishing only if $n_+ - n_- = 4k$.

One may repeat the above discussion for the 4d Born-Infeld theory (for simplicity we set coupling to scalars $\phi, \chi$ to zero)

$$L(A) = -\sqrt{1 + \frac{1}{2}F_{mn}^*F_{mn} - \frac{1}{16}(F_{mn}^*F_{mn})^2},$$

(4.10)

which is semiclassically equivalent to the following quadratic in $F_{mn}$ Lagrangian [38] involving two real auxiliary fields $U, V$

$$L(A; U, V) = -\frac{1}{4} \left( V F_{mn}^*F_{mn} - U F_{mn}^*F_{mn}^* \right) - \frac{1}{2} \left( V + V^{-1} + V^{-1}U^2 \right).$$

(4.11)

The “doubled” action for (4.11) is found in the same way as (4.2) and is quantum-equivalent to (4.11). The “doubled” action for the BI theory (4.10) found from its phase-space formulation follows also from the “doubled” action for (4.11) upon eliminating $U, V$ through their equations of motion. The resulting non-polynomial “doubled” Lagrangian corresponding to (4.10) written in terms of derivatives of $A_i^\pm = A_i + i\tilde{A}_i$ fields is [38]

$$\hat{L} = \frac{1}{2} \left( E_i \tilde{B}_i - \tilde{E}_i B_i \right) - \sqrt{1 + B_i^2 + \tilde{B}_i^2 + B_i^2 \tilde{B}_i^2 - (B_i \tilde{B}_i)^2}
= \frac{1}{4} \left( E_i^+ B_i^- - E_i^- B_i^+ \right) - \sqrt{1 + B_i^+ B_i^- + \frac{1}{4} (B_i^+ B_i^-)^2 - \frac{1}{4} (B_i^+ B_i^-)(B_i^- B_i^+)}. \quad (4.12)$$

---

24 Note that the natural role of complex combinations and the different structure of this action as compared to (2.12), (2.13) is related to the fact that real solutions of self-duality equations or chiral forms exist in dimensions $d = 4q + 2 = 2, 6, 10, \ldots$.

25 As in the scalar case (cf.5.1, 5.3), one may prefer to use the “polynomial” action (4.11) in the path integral definition of the quantum BI theory.
Expanding the square root, one finds that to quadratic order this is the same as the “doubled” action (4.9) (with $\phi = 0$). Let us note that as in the non-linear scalar theory case (cf. (3.2) and (3.3)) the Lagrangian (4.12) is a priori equivalent to (4.11) only semiclassically (i.e. at the tree and one-loop level) since the integral over $\tilde{A}_i$ (or $\tilde{B}_i$) here is non-gaussian.

The Lagrangian (4.12) is invariant under the same duality transformation (4.5). The consequences of this symmetry for the scattering amplitudes are also the same: the difference between the number of positive and negative helicity photons, $n_+ - n_-$, must be a multiple of 4 (in particular, zero). In fact, the S-matrix corresponding to (4.12) is actually helicity-conserving (see [42, 43] for earlier demonstrations of this for the BI theory). Indeed, every term in the expansion of (4.12) has an equal number of $B^+$ and $B^-$ factors, and since the propagator connects $B^+$ and $B^-$, in every Feynman graph (at any loop order) the numbers of external positive and negative helicity photons (i.e. the numbers of on-shell $B^+$ and $B^-$ fields) are equal.

The fact that the structure of the amplitudes appears to be more constrained than required by the duality (4.9) is not too surprising: since there exist infinitely many generalizations of the Maxwell’s action exhibiting duality (and thus having an S-matrix obeying the $n_+ - n_- = 4k$ rule), the additional constraints implying helicity conservation reflect the special property of the Born-Infeld action.

It is interesting to contrast the emergence of helicity conservation in the “doubled” theory (4.12) and in the standard (Lorentz-covariant) description of the Born-Infeld theory in which the action (4.10) contains interacting terms of the type $(F^+)^m(F^-)^n$ with arbitrary $m \geq 1$ and $n \geq 1$. Here $F^\pm$ are the self-dual and anti-self-dual components of the field strength which, similarly to $B^\pm$, reduce on shell to the positive/negative helicity photons. At the tree-level, the contribution of the $m \neq n$ terms to helicity non-conserving amplitudes is cancelled [43] by Feynman graphs with contact (momentum-independent) propagators $\langle F^+(p)F^+(-p) \rangle$ and $\langle F^-(p)F^-(-p) \rangle$. In the “doubled” theory (4.12) all these cancellations are built into the action. The remaining graphs, in which all propagators are $\langle F^+(p)F^+(-p) \rangle \sim \frac{1}{p^2}$, generate only helicity-conserving amplitudes.

Finally, let us comment on some implications for extended supergravities. The $\mathcal{N} = 8$ supergravity may be obtained by compactifying type IIB 10d supergravity on a 6-torus; the corresponding $O(6,6)$ symmetry is a subgroup of the full $E_{7(7)}$ duality group. The physical states of the theory are in one-to-one correspondence [46] with the states of the doubleton multiplet of the maximally extended superconformal group in 4 dimensions, $SU(2,2|8)$. While interactions break this symmetry to the maximally-extended Poincaré superalgebra, the physical fields continue to transform in representations of $SU(8)$ (it is possible to argue that all four-dimensional

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26Note that the form of the on-shell relation between the dual and original field strengths is of course modified, so in this sense one may say that BI equations of motion are covariant with respect to “deformed” version of Maxwell theory duality. This distinction is absent in the “doubled” description.

27Note that helicity conservation is a feature of some supersymmetric theories [40, 41]; the BI action admits a natural supersymmetrization [44].

28Apart from the discrete duality (4.5), the action (4.12) exhibits also the continuous symmetry $(A_\pm^\pm)' = e^{\pm i\alpha} A_\pm^\pm$, with (4.5) being included as the special case of $\alpha = -\pi^2$. One may then understand [42] the additional restriction $n_+ - n_- = 4k = 0$ as a consequence of the infinitesimal part of this symmetry. This infinitesimal symmetry is not present for the action in (4.9). Let us mention also that the action of linearized $SL(2, R)$ invariance on S-matrix of scalars and vectors for D3-brane in tree-level open/closed string theory was studied in [45].
generalized unitarity cuts of all S-matrix elements to any loop order are $SU(8)$-invariant; one may therefore expect that the S-matrix has this symmetry). It turns out[29] that the complete duality group is the closure of the six commuting $Z_2$ subgroups of $O(6,6)$ together with the $SL(2,R)$ symmetry of the type IIB supergravity and this $SU(8)$ symmetry. While our discussion in this paper focussed on a simple example of discrete $Z_2$ duality similar considerations should apply also to the full duality symmetry of the $\mathcal{N} = 8$ supergravity[30].

Our discussion in this paper suggests that the S-matrix and the associated on-shell counterterms computed in perturbative loop expansion[31] should be invariant under the $E_{7(7)}$ transformations of the scalar fields together with duality transformations acting on the chiral (anti/self-dual) parts of the vector fields. The latter symmetries are manifest in the “doubled” formulation of the theory, where the action is not invariant under the standard (tangent-space) Lorentz symmetry; nevertheless, as we have argued in examples above, the on-shell effective action and/or the S-matrix should have this symmetry along with the duality symmetry. It remains an open question whether there may be some additional implications of the $E_{7(7)}$ duality for the structure of potential counterterms, as conjectured in [9].

Acknowledgments

We would like to thank R. Kallosh for illuminating discussions and explanations. RR is grateful to M. Gunaydin for useful discussions on symmetries of $\mathcal{N} = 8$ supergravity. AAT is grateful to J. Buchbinder and N. Pletnev for a discussion related to 4d case in Appendix A. The work of RR was supported by the National Science Foundation under grant PHY-08-55356. The work of AAT was supported by the ERC Advanced grant No.290456.

[29] We thank Murat Gunaydin for discussions on this point.

[30] Let us note that as was argued in [3] and more explicitly in [6] the $SU(8)$ chiral anomalies that would imply a breakdown of $E_{7(7)}$ duality in the quantum theory (the $E_{7(7)}$ anomaly is determined by the $SU(8)$ anomaly through a Wess-Zumino consistency condition [6]) actually cancel out. Our focus here was not on the possible anomaly aspect but rather on the realization of the duality in the quantum theory. Let us note also that while the global $SU(8)$ is only an on-shell symmetry, it is possible to formulate $\mathcal{N} = 8$ supergravity in such a way that the Lagrangian has $SL(8,R)$[1] or $SU^*(8)$[47] off-shell symmetry. The $E_{7(7)}$ duality group is the closure of either one of these groups together with the 10-dimensional $SL(2,R)$ symmetry of type IIB supergravity and six abelian generators (2 non-compact and 4 compact). Considerations similar to ours should apply, e.g., to the $SL(2,R)$ symmetry as well as to the two non-compact of the extra six abelian generators (these are also generators of $O(6,6)$); see [48, 49] for the relevant change of basis for $E_{7(7)}$ generators.

[31] Note that to relate the leading divergence of the effective action $\Gamma = \sum_n \Gamma_n$ (first appearing at some $n$-th loop order) to to the corresponding divergent term in the S-matrix one needs to evaluate $\Gamma_n$ on a “scattering” solution of just classical (un-corrected) equations. For the same reason, being interested only in the duality properties of the leading counterterms, one does not need to worry about modification of the duality transformation by finite quantum corrections.
Appendix A: Issue of quantum $\phi \to -\phi$ invariance on curved background

Integrating $x, \tilde{x}$ out in (2.2), (2.4) one expects to find the $\phi \to -\phi$ symmetry in the resulting effective action. This is not, however, automatic if other fields and symmetries are present and may depend on a regularization prescription (reflected in a choice of finite local counterterms). To illustrate this, let us consider the following example with just one 2d scalar field $x$ coupled to an external 2d scalar $\phi$ and 2d metric $g_{ab}$

$$\Gamma[\phi, g_{ab}] = -\ln \int [dx] e^{-\frac{1}{2} \int d^2 \sigma \sqrt{g} g^{ab} G \partial_a x \partial_b x} , \quad G \equiv e^{-2\phi} . \quad (A.1)$$

In the string or 2d sigma model context one may think of $G$ as a component of a target space metric in isometric direction $x$ (cf. (2.1)). 2d on-shell duality implies $G \to G^{-1}, \ x \to \tilde{x}$, with $G \sqrt{g} g^{ab} \partial_a x = ie^{ab} \partial_b \tilde{x}$. As was shown in [31], the definition of $\Gamma$ in (A.1) implies

$$\Gamma[\phi, g_{ab}] - \Gamma[-\phi, g_{ab}] = \frac{1}{8\pi} \int d^2 x \sqrt{g} \phi R , \quad (A.2)$$

where $R$ is the curvature of $g_{ab}$. This means, in particular, that under T-duality $G \to G^{-1}$ the target-space dilaton get shifted [30] by $\phi = -\frac{1}{2} \ln G$ term. In the present context we may interpret (A.2) as an anomaly (present only in a curved 2d background) of the $\phi \to -\phi$ symmetry. More precisely, since (A.2) is a local term, one may interpret it not as a genuine anomaly but as a finite local counterterm required for preservation of some other symmetry – target space reparametrization invariance in this 2d sigma model context. As this counterterm breaks $\phi \to -\phi$ symmetry, that means that both symmetries – the 2d duality and the target space reparametrization invariance – cannot be manifest at the same time.\(^{33}\)

Indeed, let us recall the assumptions that went into the derivation of (A.2) in [30, 39, 31]. It was assumed that the path integral measures used for all of the fields – the original scalar, the dual scalar and the 2d auxiliary vector $n_a$ (needed to perform the duality transformation at the path integral level) contain the same factors of $G$, i.e. are covariant with respect to the target space metric. This follows from the requirement of target space reparametrization invariance which is natural in the sigma model context. As the right-hand side of (A.2) is a local term, it depends effectively on a choice of a quantization scheme or regularization prescription. Indeed, one way to get this expression is to notice [30] that the path integral over $n_a$ with action

\[^{32}\text{In this Appendix we assume the world-sheet signature to be euclidean.}\]

\[^{33}\text{To recall, anomalous, i.e. symmetry violating terms are usually (i) nonlocal and (ii) depend on which symmetry of two or more one wants to preserve, i.e. depend on a quantization prescription. In some cases anomalous terms may be local, but then they are ambiguous, i.e. can be altered by adding finite local counterterms. For example, in 4d conformal anomaly case the stress-energy tensor trace $T^m_m$ may contain a total derivative term $D^2 R$ that corresponds to non-Weyl-invariant (finite) local term $R^2$ in the effective action. As such term can be cancelled by a local counterterm, the $D^2 R$ term in the stress tensor anomaly is ambiguous (may depend on a gauge choice in the vector field case, etc). In 2 dimensions there are similar terms that are cancelled by introducing a local $R$-counterterm, i.e. the dilaton coupling $\int d^2 \sigma \sqrt{g} R \Phi(x)$ which cancels the derivative terms in $T^m_m$. In general, one is required to introduce all possible local counterterms and try to satisfy Ward identities of required symmetries; in the anomalous case this can be done only for a subset of all classical symmetries.}\]
\[
\int d^2x \sqrt{g} \, G n^a n_a \text{ gives the following contribution to } \Gamma
\]

\[
\frac{1}{2} \text{tr} \ln(G e^{-\Lambda^2 \Delta_1}) = \frac{1}{2} \int d^2x \sqrt{g} \, \ln G \left[ 2 \left( \Lambda^2 + \frac{1}{6} R \right) - R \right], \quad (A.3)
\]

where \((\Delta_1)_{ab} = -g_{ab} \nabla^2 + R_{ab}\) is a natural operator on vectors used to regularize the local "\(\delta(0)\)" factor and \(\Lambda\) is a UV cutoff. \(2(\Lambda^2 + \frac{1}{6} R)\) term cancels against similar terms from the scalar and dual scalar contributions to \(\Gamma\) but \(-R\) term survives and leads to (A.2).

Let us now consider the corresponding 4d example (1.1) on a curved 4d background, i.e. define

\[
\Gamma[\phi, g_{mn}] = -\ln \int [dA] \, e^{-\frac{1}{4} \int d^4x \sqrt{g} \, g^{mn} g^{pq} \, e^{-2\phi} F_{mp} F_{nq}}. \quad (A.4)
\]

Here the classical equations of motion have symmetry under \(A \to \tilde{A}, \phi \to -\phi\) with \(e^{-2\phi} \ast dA = d\tilde{A}\) and one may ask if this classical symmetry becomes symmetry of the effective action (A.4), i.e. \(\Gamma[\phi, g_{mn}] = \Gamma[-\phi, g_{mn}]\). This is, of course, expected from formal path integral transformation argument implying that \(\Gamma\) should depend only on derivatives of \(\phi\) and only on even powers of \(\phi\). For example, if \(\Gamma\) were to contain a local term \(\Gamma' = a_0 \int d^4x \sqrt{g} \, \phi \, C^2_{mnkl}\), where \(C_{mnkl}\) is the Weyl tensor then this term would change under the constant shift of \(\phi\) but such shift in (A.4), but in view of the above 2d example one may avoid this objection by replacing \(C^2_{mnkl}\) by the 4d Euler density combination.

Indeed, this is what we find if we follow the same steps that in 2d case led to (A.2). In general, torison of an elliptic complex \(^{35}\) will be given by a combination of Seeley coefficients (the one appearing in \(t^0\) power in expansion of \(\text{Tr} e^{-t\Delta} = \sum_k B_k t^k\)). It is straightforward to repeat the analysis in \(^{31}\) for the torsion \(\frac{1}{2} \sum_{n=0}^d (-1)^n (n+1) \ln \det \Delta_n\) of the 4d elliptic complex (scalar, vector and 2-tensor operators). This will give the analog of (A.2) originating from the corresponding Seeley coefficients \(b_4 \sim R^2_{mnkl} + \ldots\). Under the same assumption as in the 2d case that all the measure factors are the same for all the operators in the complex, this computation was done in \(^{51}\) and led to the direct analog of (A.2) with the 4d Euler density replacing the 2d one:

\[
\Gamma[\phi, g_{mn}] - \Gamma[-\phi, g_{mn}] = -\frac{1}{32\pi^2} \int d^2x \sqrt{g} \, \phi (R^2_{mnkl} - 4R^2_{mn} + R^2). \quad (A.5)
\]

Given that the Euler density is a total derivative, this expression depends only on \(\partial \phi\) (assuming trivial topology). As this is a local term, one may interpret its presence as reflecting the desire to preserve some other symmetry at the expense of the duality \(\phi \to -\phi\). Since, in contrast to 2d sigma model case, in the 4d vector case of (1.1) we do not have target space diffeomorphisms acting on vectors we may instead insist on the preservation of the duality by fine-tuning the coefficient of this term to zero, i.e. by canceling it by a local counterterm \(^{36}\).

\(^{34}\) SL(2, R) invariance of the conformal anomaly (controlling logarithmically UV divergent part of the effective action and thus its finite Weyl-anomalous part) resulting from integrating over the vector field coupled to \((\phi, \chi)\) as in (1.1) was explicitly demonstrated in \(^{50}\).

\(^{35}\) This is a set of operators like the three ones – scalar, vector, and dual scalar – mentioned in 2d case.

\(^{36}\) This may be required in the context of coupling this model to gravity, but in the supergravity context one
Let us also note that coupling an $\mathcal{N} = 4$ vector supermultiplet (SYM) to $\mathcal{N} = 4$ conformal supergravity \cite{52} and integrating out the $\mathcal{N} = 4$ vector multiplet fields leads to an effective action \cite{53} whose UV divergent part should be the same as the action of $\mathcal{N} = 4$ conformal supergravity \cite{54,55} which should have manifest off-shell $SL(2)$ duality invariance involving the dilaton $\phi$ and its pseudoscalar partner $\chi$, with $SU(4)$ vectors not transforming (here there is no scalar-vector coupling, in contrast to the case of $\mathcal{N} = 4$ Poincaré supergravity). The invariance of the full local part of the resulting quantum effective action under $\phi \rightarrow -\phi$ will thus hold provided the “anomalous” term in (A.5) is cancelled by a local counterterm.

As was noted in the introduction, in addition to the above local non-invariant terms that can be removed by local counterterms, the effective action of a theory like (1.1) contains also genuine non-local duality non-invariant (anomalous) terms containing scalars and curvature-dependent $RR^*$ factor (cf. \cite{16}). They cancel only in $\mathcal{N} = 8$ supergravity \cite{3}.

Appendix B: One-loop effective action corresponding to scalar theory (3.3)

The quadratic operator in eq. (3.12) has constant coefficients ($n_a = \partial_a x = \text{const}$, $G(0) = (1 + n^2)^{-1/2} = \text{const}$) so that we find ($V_2$ is the 2d space-time volume factor)

\[
\Gamma_1 = \frac{1}{2} V_2 \int \frac{d^2 q}{(2\pi)^2} \ln \left( G(0)^{-1} q^2 - G(0)(n \cdot q)^2 \right) = \frac{1}{2} V_2 \int \frac{d^2 q}{(2\pi)^2} \left( -\ln G(0) + \ln q^2 + \ln \left[ 1 - G(0)^2 \frac{(n \cdot q)^2}{q^2} \right] \right). \tag{B.1}
\]

The second term here may be dropped as it is independent of the classical background. Explicitly,

\[
\frac{1}{2} \int \frac{d^2 q}{(2\pi)^2} \ln \left[ 1 - G(0)^2 \frac{(n \cdot q)^2}{q^2} \right] = -\frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{k} (2k-1)!! \left( G(0)^2 n^2 \right)^k I = \ln \left[ \frac{1}{2} (1 + G(0)) \right] I, \tag{B.2}
\]

where we used that

\[
\int \frac{d^2 q}{(2\pi)^2} \left( \frac{(n \cdot q)^2}{q^2} \right)^k = \frac{(2k-1)!!}{(2k)!!} (n \cdot n)^k I, \quad I \equiv \int \frac{d^2 q}{(2\pi)^2} = \Lambda^2, \tag{B.3}
\]

and that $G_0 = (1 + n^2)^{-1/2}$. Then the resulting combination in the integrand of (B.1) is

\[
-\frac{1}{2} \ln G(0) + \ln \left[ \frac{1}{2} (1 + G(0)) \right] = \ln \left[ \frac{1}{2} (G(0)^{1/2} + G(0)^{-1/2}) \right], \tag{B.4}
\]

so that the function $F$ in (3.14) is

\[
F = \ln \left[ \frac{1}{2} (y^{1/2} + y^{-1/2}) \right], \quad F(y) = F(y^{-1}). \tag{B.5}
\]

may expect measure-related factors to cancel. This may, of course, also depend on a choice of field redefinitions relating classically equivalent supergravity theories. In fact, similar $\int d^2 x \sqrt{g} \phi R^2_{mnkl}$ local counterterm previously appeared in \cite{56} when discussing the quantum equivalence of the $SO(4)$ and $SU(4)$ versions of $\mathcal{N} = 4$ supergravity. It is directly related to the fact that the required field redefinition \cite{2} produces the duality-anomalous Jacobian.
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