A TRINOMIAL ANALOGUE OF BAILEY’S LEMMA
AND N = 2 SUPERCONFORMAL INVARIANCE

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Dedicated to Dora Bitman on her 70th birthday

Abstract. We propose and prove a trinomial version of the celebrated Bailey’s lemma. As an application we obtain new fermionic representations for characters of some unitary as well as nonunitary models of $N = 2$ superconformal field theory (SCFT). We also establish interesting relations between $N = 1$ and $N = 2$ models of SCFT with central charges $\frac{3}{2} \left(1 - \frac{2(2-4\nu)^2}{2(4\nu)}\right)$ and $3 \left(1 - \frac{2}{4\nu}\right)$. A number of new mock theta function identities are derived.

1. Brief review of Bailey’s method and its generalizations.

It may come as a surprise that Manchester, England was an ideal setting for pure mathematics during the height of World War II. However, a variety of historical coincidences conspired to make this the case. In particular, mathematics that would later prove extremely valuable in the development of statistical mechanics and conformal field theory (CFT) flourished there.

Essentially, Bailey, extending the original ideas of Rogers, came up with a new method \cite{1,2} of deriving Rogers-Ramanujan type identities during the winter 1943–44. Hardy who was then editor for the Journal of London Mathematical Society sent a referee report with Dyson’s name on it back to Bailey. Bailey’s reply was immediate. A charming account of Dyson-Bailey collaboration appears in Dyson’s article, A Walk Through Ramanujan’s Garden \cite{3}.

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A few years later, Slater, in a study building on Bailey’s work, systematically derived 130 identities of Rogers-Ramanujan type [4,5]. In the last decade, Bailey’s technique was streamlined and generalized by Andrews [6] and further extended by Agarwal, Andrews and Bressoud [7,8].

Bailey’s method may be summarized as follows. Let

\[ \alpha_r = \{ \alpha_r \}_{r \geq 0}, \beta_L = \{ \beta_L \}_{L \geq 0}, \]

be sequences related by identities

\[ \beta_L = \sum_{r=0}^{\infty} \frac{\alpha_r}{(q)_{L-r}(aq)_{L+r}}, \quad L \in \mathbb{Z}_{\geq 0} \]  

(1.1)

and \( \gamma = \{ \gamma_L \}_{L \geq 0}, \delta = \{ \delta_r \}_{r \geq 0} \) be another pair of sequences related by

\[ \gamma_L = \sum_{r=L}^{\infty} \frac{\delta_r}{(q)_{r-L}(aq)_{r+L}}. \]  

Then the new identity

\[ \sum_{L=0}^{\infty} \alpha_L \gamma_L = \sum_{L=0}^{\infty} \beta_L \delta_L \]  

(1.4)

holds. A pair of sequences \((\alpha, \beta)\) that satisfies (1.1) is called a Bailey pair relative to \(a\). Analogously, a pair of sequences \((\gamma, \delta)\) subject to (1.3) is referred to as conjugate Bailey pair relative to \(a\). In [2], Bailey proved that

\[ \gamma_L = \frac{(\rho_1, \rho_2)_L(aq/\rho_1 \rho_2)_L}{(aq/\rho_1, aq/\rho_2)_L} \frac{1}{(q)_{M-L}(aq)_{M+L}}, \]  

(1.5)

\[ \delta_L = \frac{(\rho_1, \rho_2)_L(aq/\rho_1 \rho_2)_L}{(aq/\rho_1, aq/\rho_2)_M} \frac{1}{(q)_{M-L}}. \]  

(1.6)

with \((a_1, a_2)_L \equiv (a_1)_L(a_2)_L, L \leq M \in \mathbb{Z}_{\geq 0}\) satisfy (1.3) for any choice of parameters \(\rho_1, \rho_2\). Combining (1.4, 1.5, 1.6) yields

\[ \sum_{L=0}^{\infty} \frac{(\rho_1, \rho_2)_L}{(aq/\rho_1, aq/\rho_2)_M} \left( \frac{aq}{\rho_1 \rho_2} \right)_L \frac{(aq/\rho_1 \rho_2)_{M-L}}{(q)_{M-L}} \beta_L = \]

\[ \sum_{L=0}^{\infty} \frac{(\rho_1, \rho_2)_L(aq/\rho_1 \rho_2)_L}{(aq/\rho_1, aq/\rho_2)_L} \frac{\alpha_L}{(q)_{M-L}(aq)_{M+L}}. \]  

(1.7)

From the last equation, we deduce immediately:
(Bailey’s lemma) Sequences \((\alpha', \beta')\) defined by
\[
\begin{align*}
\alpha'_L &= \frac{(\rho_1, \rho_2)_L (aq/\rho_1 \rho_2)^L}{(aq/\rho_1, aq/\rho_2)_L} \alpha_L \\
\beta'_L &= \sum_{r=0}^{\infty} \frac{(\rho_1, \rho_2)_r (aq/\rho_1 \rho_2)^r (aq/\rho_1 \rho_2)^{L-r}}{(aq/\rho_1, aq/\rho_2)_L (q)_L} \beta_r
\end{align*}
\]
form again a Bailey pair relative to \(a\). Obviously, Bailey’s lemma can be iterated and infinitum leading to a “Bailey chain” [6,9] of new identities
\[
(\alpha, \beta) \rightarrow (\alpha', \beta') \rightarrow (\alpha'', \beta'') \rightarrow (\alpha''', \beta''') \rightarrow \ldots
\]
with parameter \(a\) remaining unchanged throughout the chain. The notion of a Bailey chain was upgraded to a “Bailey lattice” in [7,8] where it was shown how to pass from a Bailey pair with given parameter \(a\) to another pair with arbitrary new parameter.

Further important developments have taken place in the last few years. In [10], Milne and Lilly found higher-rank generalizations of Bailey’s lemma. Many new polynomial identities of Rogers-Ramanujan type were discovered in [11-18] as result of recent progress in CFT and Statistical Mechanics initiated by the Stony Brook group [19-21]. Following observation made by Foda-Quano [22], these identities were recognized as new \((\alpha, \beta)\) pairs. New \((\gamma, \delta)\) pairs were discovered in [23,24]. Intriguing connections between Bailey’s lemma and the so-called renormalization group flows connecting different models at CFT were discussed in [17,25-27].

This paper is intended as the first step towards a multinomial (or higher-spin) generalization of Bailey’s lemma. Here we concentrate on the trinomial case. Our main assertion is

**Theorem 1.** (Trinomial analogue of Bailey’s lemma)

If for \(L \geq 0, a = 0, 1\)
\[
\tilde{\beta}_a(L) = \sum_{r=0}^{L} \tilde{\alpha}_a(r) \frac{T_a(L, r, q)}{(q)_L}
\]
then for \(M \in \mathbb{Z}_{\geq 0}\)
\[
\sum_{L=0}^{\infty} (-1)_L q^{\frac{L(L-1)}{2}} \tilde{\beta}_0(L) = \sum_{r=0}^{\infty} \tilde{\alpha}_0(r) \frac{(-1)^{M+1}}{q^{\frac{r(r-1)}{2}} + q^{-\frac{r(r-1)}{2}}} \frac{T_1(M, r, q)}{(q)_M}
\]
and
\[
\sum_{L=0}^{\infty} (-q^{-1}) L q^L \tilde{\beta}_1(L) = \sum_{r=0}^{\infty} \hat{\alpha}_1(r) \frac{(-1)^M}{(q)_M} \left\{ T_1(M, r, q) - \frac{(1-q^M)}{1+q^{-1-r}} T_1(M-1, r+1, q) \right. \\
- \left. \frac{(1-q^M)}{1+q^{r-1}} T_1(M-1, r-1, q) \right\}
\]
where \( T_a(L, r, q) \) are \( q \)-trinomial coefficients [28] to be defined in the next section. The pair of sequences \( (\hat{\alpha}_a, \tilde{\beta}_a) \) that satisfies identities (1.11) will be called a trinomial Bailey pair.

The rest of this paper is organized as follows. In Section 2 we shall collect the necessary background on \( q \)-trinomials and then prove Theorem 1. In Section 3 we shall exploit this theorem to derive a number of new \( q \)-series identities related to characters of \( N=2 \) SCFT. We conclude with a brief discussion of the physical significance of our results and some comments about possible generalizations.

2. \( q \)-Trinomial Coefficients and a Trinomial Analogue of Bailey’s Lemma.

2.1 Preliminaries.

Before turning our attention to the \( q \)-trinomial coefficients, let us briefly recall the ordinary trinomials \( \binom{L}{A}_2 \) [28], defined by
\[
\left( x + 1 + \frac{1}{x} \right)^L = \sum_{A=-L}^{L} \binom{L}{A}_2 x^A
\]
\[
\binom{L}{A}_2 = 0, \quad |A| > L.
\]
By applying the binomial theorem twice to (2.1) we find by coefficient comparison that
\[
\binom{L}{A}_2 = \sum_{j \geq 0} \frac{L!}{j!(j+A)!(L-2j-A)!}.
\]
Furthermore, it is easy to deduce from (2.1) the following recurrences
\[
\binom{L}{A} = \binom{L-1}{A-1} + \binom{L-1}{A} + \binom{L-1}{A+1}.
\]
which along with (2.2) and
\[
\binom{0}{0}_2 = 1
\] (2.5)
specify trinomials uniquely. Equations (2.4) and (2.5) lead to the Pascal-like triangle for \((L_A)_2\) numbers
\[
\begin{array}{ccccccc}
1 \\
1 & 1 & 1 \\
1 & 2 & 3 & 2 & 1 \\
1 & 3 & 6 & 7 & 6 & 3 & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{array}
\]

The straightforward \(q\)-deformation of (2.3) is as follows
\[
\binom{L; B; q}{A}_2 = \sum_{j \geq 0} \frac{q^{j(j+B)}(q)^L}{(q)_j(q)_jA(q)_L_{-2jA}}.
\] (2.6)

Let us further define another useful \(q\)-analogue of (2.3):
\[
T_n(L, A, q) = q^{\frac{L(L-n) - A(A-n)}{2}} \binom{L; A - n; q^{-1}}{A}_2, \quad n \in \mathbb{Z}.
\] (2.7)

Polynomials \(T_n(L, A, q)\) are symmetric under \(A \rightarrow -A\)
\[
T_n(L, A, q) = T_n(L, -A, q)
\] (2.8)

and vanish for \(|A| > L\):
\[
T_n(L, A, q) = 0 \quad \text{if} \quad |A| > L.
\] (2.9)

The generalization of Pascal-triangle type recurrences (2.4) found in [31, 33] is
\[
T_n(L, A, q) = T_n(L - 1, A - 1, q) + T_n(L - 1, A + 1, q) +
q^{L - \frac{n+1}{2}} T_n(L - 1, A, q) + (q^{L-1} - 1)T_n(L - 2, A, q).
\] (2.10)
Additionally, there are four more identities needed

\[ T_n(L, A, q) = T_{n+2}(L, A, q) + (q^L - 1)q^{-\frac{1+n}{2}}T_n(L - 1, A, q) \]  
\[ q^{\frac{L+A}{2}}T_{n+1}(L, A, q) = T_n(L, A, q) + (q^L - 1)T_n(L - 1, A + 1, q) \]  
\[ T_1(L, A, q) - T_1(L - 1, A, q) = q^{\frac{L+A}{2}}T_0(L - 1, A + 1, q) + q^{\frac{L-A}{2}}T_0(L - 1, A - 1, q) \]  
\[ T_1(L, A, q) + T_1(L - 1, A, q) = T_{-1}(L - 1, A + 1, q) + T_{-1}(L - 1, A - 1, q) + 2T_{-1}(L - 1, A, q). \]

Identities (2.11), (2.12), (2.13) follow from equations (2.24), (2.23), (2.16) of [28] and identity (2.14) is equation (4.5) of [33].

Next we shall require the limiting formula

\[ \lim_{L \to \infty} T_1(L, A, q) = \frac{(-q)_{\infty}}{(q)_{\infty}} \]  

which is equation (2.51) of [28].

Let us combine (2.11) and (2.12) with \( n = -1 \) to obtain

\[ q^{\frac{L+A}{2}}T_0(L, A, q) - T_1(L, A, q) = (q^L - 1)\{T_1(L - 1, A + 1, q) + T_{-1}(L - 1, A, q)\}. \]

We now replace \( A \) by \(-A\) in the above equation to get, with the help of (2.8),

\[ q^{\frac{L-A}{2}}T_0(L, A, q) - T_1(L, A, q) = (q^L - 1)\{T_1(L - 1, A - 1, q) + T_{-1}(L - 1, A, q)\}. \]

If we add (2.16) and (2.17) and use (2.14), the result is

\[ q^{\frac{L}{2}} \left( q^{\frac{A}{2}} + q^{-\frac{A}{2}} \right) T_0(L, A, q) - 2T_1(L, A, q) = (q^L - 1)\{T_1(L, A, q) + T_1(L - 1, A, q)\} \]

which may be conveniently rewritten as

\[ T_0(L, A, q) = \frac{q^{\frac{L}{2}}}{q^{\frac{A}{2}} + q^{-\frac{A}{2}}} (1 + q^L)T_1(L, A, q) \]  
\[ - \frac{q^{\frac{L}{2}} (1 - q^L)}{q^{\frac{A}{2}} + q^{-\frac{A}{2}}} T_1(L - 1, A, q). \]
2.3 Proof of Theorem 1.

We shall prove Theorem 1 in two steps. First, we find a trinomial analogue of a conjugate Bailey pair and then use a standard Bailey Transform argument. To this end let us introduce an auxiliary function \( \phi(L, q) \)

\[
\phi(L, q) = q^{\frac{L}{2}} (\frac{-1}{q})^L ( \frac{q}{L} )^L
\]  

(2.20)

with the easily verifiable property

\[
\phi(L + 1, q) = \sqrt{q} \frac{1 + q^L}{1 - q^{L+1}} \phi(L, q)
\]  

(2.21)

Next we multiply both sides of (2.19) by \( \phi(L, q) \) and sum both extrems of the result on \( L \) from \( A \) to \( M \) to obtain with the aid of (2.21) the following

\[
\sum_{L=A}^{M} q^{\frac{L}{2}} (-1)^L T_0(L, A, q) = (-1)^{M+1} T_1(M, A, q) \frac{T_0(M, A, q)}{(q)_M} \frac{1}{q^{\frac{M}{2}} + q^{-\frac{M}{2}}}
\]  

(2.22)

which can be restated as a trinomial analogue of the conjugate relation (1.3)

\[
\tilde{\gamma}_0(A, M) = \sum_{L=A}^{\infty} \tilde{\delta}_0(L, M) \frac{T_0(L, A, q)}{(q)_L}
\]  

(2.23)

with conjugate pair \((\tilde{\gamma}_0, \tilde{\delta}_0)\) defined as

\[
\tilde{\gamma}_0(A, M) = \frac{(-1)^{M+1} T_1(M, A, q)}{q^{\frac{M}{2}} + q^{-\frac{M}{2}}} \frac{T_0(M, A, q)}{(q)_M}
\]  

(2.24)

\[
\tilde{\delta}_0(L, M) = \theta(L \leq M) q^{\frac{L}{2}} (\frac{-1}{q})^L
\]  

(2.25)

where

\[
\theta(a \leq b) = \begin{cases} 
1 & \text{if } a \leq b \\
0 & \text{otherwise .}
\end{cases}
\]  

(2.26)

The proof of the first statement of Theorem 1 (1.12) now easily follows by a Bailey Transform argument

\[
\sum_{r=0}^{\infty} \tilde{\alpha}_0(r) \tilde{\gamma}_0(r, M) = \sum_{r=0}^{\infty} \tilde{\alpha}_0(r) \sum_{L=r}^{\infty} \tilde{\delta}_0(L, M) \frac{T_0(L, r, q)}{(q)_L} \frac{T_0(L, r, q)}{(q)_L}
\]

(2.27)

\[
= \sum_{L=0}^{\infty} \tilde{\delta}_0(L, M) \sum_{r=0}^{L} \tilde{\alpha}_0(r) \frac{T_0(L, r, q)}{(q)_L} = \sum_{L=0}^{\infty} \tilde{\delta}_0(L, M) \tilde{\beta}_0(L).
\]
Substituting (2.24) and (2.25) into (2.27) we arrive at the desired result (1.12).

Similar to the binomial case, identity (1.12) can be interpreted as a defining relation for new trinomial Bailey pair \((\tilde{\alpha}_1, \tilde{\beta}_1)\). However, unlike the binomial case, the second analogue of (1.3)

\[
\tilde{\gamma}_1(A, M) = \sum_{L=A}^{\infty} \tilde{\delta}_1(L, M) \frac{T_1(L, A, q)}{(q)_L}
\]

(2.28)
is now needed to iterate further.

To find a \((\tilde{\gamma}_1, \tilde{\delta}_1)\) pair we multiply equation (2.22) by \(q^{\frac{A}{2}} (q^{-\frac{A}{2}})\) and then replace \(A\) by \(A + 1 (A - 1)\) to get

\[
\sum_{L=A+1}^{M} \frac{(-1)^L}{(q)_L} q^{\frac{L+1}{2}} T_0(L, A+1, q) = \frac{(-1)^{M+1}}{(q)_M} \frac{T_1(M, A+1, q)}{1 + q^{-1-A}}
\]

(2.29)

\[
\sum_{L=A-1}^{M} \frac{(-1)^L}{(q)_L} q^{\frac{L-1}{2}} T_0(L, A-1, q) = \frac{(-1)^{M+1}}{(q)_M} \frac{T_1(M, A-1, q)}{1 + q^{-1+A}}.
\]

(2.30)

Adding (2.29) and (2.30) and using (2.9), (2.13) gives

\[
\sum_{L=A-1}^{M} \frac{(-1)^L}{(q)_L} \{T_1(L+1, A, q) - T_1(L, A, q)\} = \frac{(-1)^{M+1}}{(q)_M} \left\{ \frac{T_1(M, A+1, q)}{1 + q^{-1-A}} + \frac{T_1(M, A-1, q)}{1 + q^{-1+A}} \right\}.
\]

(2.31)

Next we treat the sum in (2.31) as follows

\[
\sum_{L=A-1}^{M} \frac{(-1)^L}{(q)_L} \{T_1(L+1, A, q) - T_1(L, A, q)\} = \sum_{L=A-1}^{M} \left\{ \frac{(-1)^L}{(q)_L} - \frac{(-1)^{L+1}}{(q)_{L+1}} \right\} T_1(L+1, A, q) + \sum_{L=A-1}^{M} \left\{ \frac{(-1)^{L+1}}{(q)_{L+1}} T_1(L+1, A, q) - \frac{(-1)^L}{(q)_L} T_1(L, A, q) \right\}
\]

\[
= -\sum_{L=A}^{M+1} (-q^{-1})^L T_1(L, A, q) \frac{q^L}{(q)_L} + \frac{(-1)^{M+1}}{(q)_{M+1}} T_1(M+1, A, q).
\]

(2.32)

Combining (2.31), (2.32) and replacing \(M\) by \(M - 1\) yields

\[
\sum_{L=A}^{M} (-q^{-1})^L q^L \frac{T_1(L, A, q)}{(q)_L} = \frac{(-1)^M}{(q)_M} \left\{ T_1(M, A, q) \right\} - \frac{1 - q^M}{1 + q^{-1-A}} T_1(M-1, A+1, q)
\]

\[
- \frac{1 - q^M}{1 + q^{-1+A}} T_1(M-1, A-1, q).
\]

(2.33)
which is nothing else but (2.28) with
\[
\tilde{\delta}_1(L, M) = \theta(L \leq M) \left( -\frac{1}{q} \right)_L q^L
\] (2.34)
\[
\tilde{\gamma}_1(L, M) = \frac{(-1)_M}{(q)_M} \left\{ T_1(M, A, q) \right. \\
- \frac{(1 - q^M)}{1 + q^{-1 - A}} T_1(M - 1, A + 1, q) \\
- \frac{(1 - q^M)}{1 + q^{-1 + A}} T_1(M - 1, A - 1, q) \left\}
\] (2.35)

The proof of the second statement of Theorem 1 (1.13) follows again by the Bailey Transform argument (2.27) (with subindex 0 replaced by 1, everywhere). Unlike (1.12), equation (1.13) does not appear to be a defining relation for the new Bailey pair and therefore can not be iterated further.

Finally, letting \( L \) tend to infinity in (1.12), (1.13) and using the limiting formula (2.15) gives:

\[\text{Theorem 2.} \quad \text{If a pair of sequences } (\tilde{\alpha}_{a=0,1}, \tilde{\beta}_{a=0,1}) \text{ is subject to identities (1.11) then}
\]
\[
\sum_{L=0}^{\infty} (-1)_L q^{\frac{L}{2}} \tilde{\beta}_0(L) = \frac{(-1)_\infty (-q)_\infty}{(q^2)_\infty} \sum_{r=0}^{\infty} \frac{\tilde{\alpha}_0(r)}{q^r + q^{-r}}
\] (2.36)
\[
\sum_{L=0}^{\infty} (-q^{-1})_L q^L \tilde{\beta}_1(L) = \frac{(-1)_\infty (-q)_\infty}{(q^2)_\infty} \sum_{r=0}^{\infty} \tilde{\alpha}_1(r) \left\{ \frac{1}{1 + q^{r+1}} - \frac{1}{1 + q^{r-1}} \right\}
\] (2.37)

hold.

3. Applications.

3.1 Preliminaries.

Recently it was shown [25] that Bailey’s lemma “connects” \( M(p, p + 1) \) models of CFT with \( N = 1 \) \( SM(p + 1, p + 3) \) and \( N = 2 \) \( SM(p + 1, 1) \) models of SCFT\(^\dagger\). In

\(^\dagger\)Throughout this paper notations \( M(p, p') \), \( N = 1 \) \( SM(p, p') \) \( N = 2 \) \( SM(p, p') \) stand for models of CFT and SCFT with central charges
\[
1 - \frac{6(p' - p)^2}{pp'} \cdot \frac{3}{2} \left( 1 - \frac{2(p - p')^2}{pp'} \right), \quad 3 \left( 1 - \frac{2p'}{p} \right)
\] respectively.
this section we shall demonstrate that Theorem 2 leads to very different relations
between these models. We begin by collecting necessary definitions and formulas.

For $A, B \in \mathbb{Z}$ $q$-binomial coefficients $\left[\begin{array}{c} A \\ B \end{array}\right]_q$ are defined as

$$\left[\begin{array}{c} A \\ B \end{array}\right]_q = \left\{ \begin{array}{ll} \frac{(q)_A}{(q)_B(q)_{A-B}} & 0 \leq B \leq A \\
0 & \text{otherwise.} \end{array} \right. \tag{3.1}$$

The following properties of $q$-binomials

$$\left[\begin{array}{c} A \\ B \end{array}\right]_{1/q} = q^{B(A-B)} \left[\begin{array}{c} A \\ B \end{array}\right]_q \tag{3.2}$$

$$\lim_{A \to \infty} \left[\begin{array}{c} A \\ B \end{array}\right]_q = \frac{1}{(q)_B} \tag{3.3}$$

are well known. Next we state some bosonic character formulas

$$M(p, p')[35 - 37]: \chi^{p, p'}_{r,s}(q) = \frac{1}{(q)_\infty} \sum_{j=\infty}^{\infty} \left\{ q^{j(p'+r-p'-sp)} - q^{j(p+r)(p'+s)} \right\} \tag{3.4}$$

where $p' > p \geq 2$ are positive coprime integers and $r \in \{1, 2, \ldots, p - 1\}$, $s \in \{1, 2, \ldots, p' - 1\}$ are labels of irreducible highest weight representations

$$N = 1 \text{ SM}(\hat{p}, \hat{p}')[37 - 40]: \tilde{\chi}^{\hat{p}, \hat{p}'}_{\hat{r}, \hat{s}}(q) = \frac{(-q^{\hat{r} - \hat{s}})}{(q)_\infty} \sum_{j=\infty}^{\infty} \left\{ q^{j(p'p' + r' - sp)} - q^{j(p + r)(p' + s)} \right\} \tag{3.5}$$

where

$$\hat{p}' > \hat{p} \geq 2 \text{ are positive integers (with } \frac{p' - p}{2} \text{ and } p \text{ being coprime integers) and } \hat{r} \in \{1, 2, \ldots, \hat{p} - 1\}, \hat{s} \in \{1, 2, \ldots, \hat{p}' - 1\}. \tag{3.6}$$

$$N = 2 \text{ SM}(\tilde{p}, 1)[41 - 43]: \tilde{\chi}^{\tilde{p}, 1}_{\tilde{r}, \tilde{s}}(q, y) = \frac{(-q^{\tilde{r} + \tilde{s}})}{(q)_{\infty}^2} \sum_{j=\infty}^{\infty} q^{j^2\tilde{p} + j(\tilde{r} + \tilde{s})} \frac{1 - q^{2\tilde{p} j + \tilde{r} + \tilde{s}}}{(1 + y^{-1}q^{\tilde{p} + r})(1 + yq^{\tilde{p} + s})} \tag{3.7}$$

where $\tilde{p} \geq 2$ is a positive integer and

(1) in the $A$ sector, $\tilde{r}, \tilde{s}$ are half integers with $0 < \tilde{r}, \tilde{s}, \tilde{r} + \tilde{s} \leq \tilde{p} - 1; \tilde{\xi} = 1/2$

(2) in the $B$ sector, $\tilde{r}, \tilde{s}$ are integers with $0 < \tilde{r}, \tilde{s}, \tilde{r} + \tilde{s} \leq \tilde{p} - 1; \tilde{\xi} = 1$. 


All $N = 2$ $SM(\tilde{p}, \tilde{p} > 1)$ characters were calculated by Ahn et al [45] in terms of fractional level string functions. However, for the vacuum sector for $N = 2$ $SM(\tilde{p}, \tilde{p} > 1)$ model (with $\tilde{p} > \tilde{p}' \geq 2; \tilde{p}, \tilde{p}'$ coprime) character formula similar to (3.7)

$$\tilde{\chi}_{\tilde{r}, \tilde{s}}^{\tilde{p}, \tilde{p}'}(q, y) = \frac{(-q^{1/2}y)_{\infty}(-q^{1/2}y^{-1})_{\infty}}{(q)_{\infty}^{2}}$$

$$\sum_{j=-\infty}^{\infty} \frac{q^{-j^{2}\tilde{p}\tilde{p}'+j(\tilde{r}+\tilde{s})\tilde{p}'}}{(1+y^{-1}q^{\tilde{p}j+\tilde{r}})(1+yq^{\tilde{p}j+\tilde{s}})}; \tilde{r} = \tilde{s} = 1/2$$

was recently found in [46]. Presumably, (3.8) also holds for sufficiently small $\tilde{r}, \tilde{s} \in Z + \frac{1}{2}$, such that the embedding diagram is the same as in vacuum case $\tilde{r} = \tilde{s} = 1/2$. There are many important differences between $N = 2$ $SM(p, 1)$ and $N = 2$ $SM(p, p' > 1)$ models. In particular, in contrast to the $N = 2$ $SM(p, 1)$ case $N = 2$ $SM(p, p' > 1)$ models are neither unitary nor rational [45, 46]. Moreover, while characters (3.7) have nice modular properties [47], those of $N = 2$ $SM(p, p' > 1)$ do not. Nevertheless, one can show that characters (3.8) are, in fact, mock theta functions‡, i.e. they exhibit sharp asymptotic behaviour when $q(|q| < 1)$ tends to a rational point of unit circle.

While bosonic characters (3.4, 3.5, 3.7) were known for quite some time, new fermionic expressions for these characters became available only in the last few years. Existence of the fermionic representations suggests that the Hilbert space of (S)CFT can be described in terms of quasi-particles obeying Pauli’s exclusion principle. The equivalence of the bosonic and fermionic character formulas gives rise to many new $q$-series identities of Rogers-Ramanujan type. Remarkably, in many known cases these identities admit polynomial analogues which can be written as defining relations (1.11) for trinomial Bailey pairs.

### 3.2 Polynomial analogues of generalized Göllnitz-Gordon identities.

$N = 1$ $SM(2, 4\nu)$, $N = 2$ $SM(4\nu, 1)$ relation.

1) See [44] for the latest discussion regarding (3.7).

‡ Notion of mock theta function was introduced by Ramanujan in his last letter to Hardy, dated January 1920.
Many polynomial Fermi-Bose character identities for $N = 1 \text{ SM}(2,4\nu), \nu \geq 2$ were derived in [33, 34]. Not to overburden our narrative with cumbersome notations we shall consider here only the simplest of these identities

\[
\sum_{n_1, \ldots, n_\nu = 0}^{\infty} \frac{q^{n_1^2 - n_1 N_2 + \sum_{j=2}^{\nu} n_j^2}}{q} \left[ \frac{N_2}{n_1} \right] \prod_{q, i = 2}^{\nu} \left[ n_i + L + n_1 - 2 \sum_{j=2}^{i} n_j \right] = \sum_{j=-\infty}^{\infty} (-1)^j q^{\nu j^2 + j/2} \{ T_0(L, 2\nu j, q) + T_0(L, 2\nu j + 1, q) \}
\]

where

\[
N_j = n_j + n_{j+1} + \cdots + n_\nu
\]

Letting $L$ in (3.9) tend to infinity yields

\[
\sum_{n_1, \ldots, n_\nu = 0}^{\infty} \frac{q^{n_1^2 - n_1 N_2 + \sum_{j=2}^{\nu} n_j^2}}{q} \left[ \frac{N_2}{n_1} \right] \prod_{q, i = 2}^{\nu} \left[ n_i + L + n_1 - 2 \sum_{j=2}^{i} n_j \right] = \frac{(-q^{1/2})_\infty}{(q)_\infty} \sum_{j=-\infty}^{\infty} (-1)^j q^{\nu j^2 + j/2} = \chi_{1,2\nu-1}^{2,4\nu}(q),
\]

where we used (3.3) and a limiting formula

\[
\lim_{L \to \infty} \{ T_0(L, A, q) + T_0(L, A + 1, q) \} = \frac{(-q^{1/2})_\infty}{(q)_\infty}
\]

proven in [28]. Identity (3.11) is nothing else but Andrews generalization of Göllnitz-Gordon identities [48]. A moment’s reflection shows that (3.9) is in the form (1.11) with

\[
\tilde{\alpha}_0(r) = \begin{cases} 
(-1)^j q^{\nu j^2} (q^{j/2} + q^{-j/2}) & \text{for } r = 2\nu j, \ j > 0 \\
1 & \text{for } r = 0 \\
(-1)^j q^{\nu j^2 + j/2} & \text{for } r = 2\nu j + 1, \ j \geq 0 \\
(-1)^j q^{\nu j^2 - j/2} & \text{for } r = 2\nu j - 1, \ j \geq 1
\end{cases}
\]

\[
\tilde{\beta}_0(L) = \frac{1}{(q)_L} \sum_{n_1, \ldots, n_\nu = 0}^{\infty} \frac{q^{n_1^2 - n_1 N_2 + \sum_{j=2}^{\nu} n_j^2}}{q} \left[ \frac{N_2}{n_1} \right] \prod_{q, i = 2}^{\nu} \left[ n_i + L + n_1 - 2 \sum_{j=2}^{i} n_j \right] q
\]

Substituting (3.13), (3.14) into (2.36) gives

\[
\sum_{L, n_1, \ldots, n_\nu = 0}^{\infty} \frac{(-1)_L}{(q)_L} \frac{L+n_1^2}{q} \frac{n_1 N_2 + \sum_{j=2}^{\nu} n_j^2}{n_1} \left[ \frac{N_2}{n_1} \right] \prod_{q, i = 2}^{\nu} \left[ n_i + L + n_1 - 2 \sum_{j=2}^{i} n_j \right] q = \frac{(-1)_\infty}{(q)_\infty^2} \sum_{j=-\infty}^{\infty} (-1)^j q^{\nu j^2 + j/2} \left\{ \frac{q^{\nu j}}{1 + q^{2\nu j}} + \frac{q^{\nu j+1/2}}{1 + q^{2\nu j+1}} \right\}
\]

\[
= \chi_{1/2,2\nu+1/2}^{4\nu,1}(q, q^{1/2}) + q^{1/2} \chi_{3/2,2\nu-1/2}^{4\nu,1}(q, q^{1/2})
\]
which establishes advertised relation between $N = 1 \, SM(2, 4\nu)$ and $N = 2 \, SM(4\nu, 1)$ models of SCFT. Moreover, left hand side of equation (3.15) provides new fermionic companion form for $N = 2 \, SM(4\nu, 1)$ characters. This form is quite different from known fermionic representation [21,25] given in terms of $D_{4\nu}$-Cartan matrix.

3.3. Trinomial Bailey flow from $M(3, 4)$ (Ising) model to $N = 2 \, SM(6, 1)$ model of SCFT.

In [29] the following polynomial identity
\[
\sum_{j=0}^{L} \left[ \begin{array}{c} L \\ j \end{array} \right] q^{j^2/2} = \sum_{j=-\infty}^{\infty} q^{6j^2+j} (T_0(L, 6j, q) + T_0(L, 6j + 1, q))
\]
\[
- \sum_{j=-\infty}^{\infty} q^{6j^2+5j+1} (T_0(L, 6j + 2, q) + T_0(L, 6j + 3, q))
\]
was proven. One may check that in the limit $L \to \infty$ this identity reduces to Fermi-Bose character identity for $M(3, 4)$ (Ising) model
\[
\sum_{j \geq 0} q^{j^2/2} (q)_j = (-q^{1/2})_\infty \sum_{j=-\infty}^{\infty} (q^{6j^2+j} - q^{6j^2+5j+1})
\]
\[
= \chi_{3,4}^{1,1}(q) + q^{1/2} \chi_{3,4}^{2,1}(q).
\]
(3.17)

The middle expression in (3.17) is remarkably similar to (3.5) with $\hat{p} = 3$, $\hat{p}' = 4$. This similarity suggests an interpretation of (3.17) as a character of some extended Virasoro algebra.

It is straightforward to verify that (3.16) is the defining relation (1.11) for trinomial pair
\[
\tilde{\alpha}_0(r) = \begin{cases} 
q^{6j^2}(q^j + q^{-j}) & \text{for } r = 6j, \ j > 0 \\
1 & \text{for } r = 0 \\
q^{6j^2+j} & \text{for } r = 6j + 1, j \geq 0 \\
q^{6j^2-j} & \text{for } r = 6j - 1, j > 0 \\
-q^{6j^2+5j+1} & \text{for } r = 6j + 2 \text{ and } r = 6j + 3, \ j \geq 0 \\
-q^{6j^2-5j+1} & \text{for } r = 6j - 2 \text{ and } r = 6j - 3, \ j > 0
\end{cases}
\]
(3.18)
\[
\tilde{\beta}_0(L) = \frac{1}{(q)_L} \sum q^{L^2} \left[ \begin{array}{c} L \\ j \end{array} \right].
\]
(3.19)
Next we apply Theorem 2 to the pair (3.18, 3.19), the result is

\[
\sum_{L,j \geq 0} q^{\frac{L+2j}{2}} \frac{(-1)^L}{(q)_L} \left[ \begin{array}{c} L \\ j \end{array} \right]_q = \frac{(-1)^\infty}{(q)_{\infty}^2} \left\{ \sum_{j=-\infty}^{\infty} q^{6j^2+j} \left( \frac{q^{3j}}{1+q^{6j}} + \frac{q^{3j+1/2}}{1+q^{6j+1}} \right) - \sum_{j=-\infty}^{\infty} q^{6j^2+5j+1} \left( \frac{q^{3j+1}}{1+q^{6j+2}} + \frac{q^{3j+3/2}}{1+q^{6j+3}} \right) \right\} = \tilde{\chi}_{1,1}^6(q, \frac{q^2}{\overline{q}}) + \frac{q}{2} \chi_{1,2}^6(q, \frac{q^2}{\overline{q}}).
\]

(3.20)

Recently, Warnaar proposed polynomial identities similar to (3.16) for all models \( M(p, p+1), p \geq 3 \) [16]. \( p = 3 \) Case is the one treated above. We have checked that his conjecture implies the following identities

\[
\sum_{L,m \geq 0} q^{\frac{L+m-2}{2}} \frac{(-1)^L}{(q)_L} \left[ \begin{array}{c} L \\ m \end{array} \right]_q = \frac{(-1)^\infty}{(q)_{\infty}^2} \left\{ \sum_{j=-\infty}^{\infty} q^{j^2+5j+1} \left( \frac{q^{4pj+2}}{(1+q^{2pj})(1+q^{2pj+2})} \right) \right\} = \chi_{1,2-a}^2(q); a = 0, 1
\]

where \( m_{p-1} \equiv 0 \). Therefore Trinomial Bailey flow for \( p \equiv 1 \) (mod 2) is

\[
M(p, p+1) \leftrightarrow N = 2 SM \left( 2p, \frac{p-1}{2} \right).
\]

(3.22)

This is to be contrasted with Bailey flow discussed in [25] where one has

\[
M(p, p+1) \leftrightarrow N = 2 SM(p+1, 1).
\]

(3.23)

### 3.4 Results related to Rogers-Ramanujan identities.

It is well known that Rogers-Ramanujan identities

\[
\sum_{j \geq 0} \frac{q^{j(j+1)}}{(q)_j} = \frac{1}{(q)_{\infty}} \sum_{j=-\infty}^{\infty} \left\{ q^{j(10j+1+2a) - q^{2j+1}(5j+2-a)} \right\} = \chi_{1,2-a}^{2,5}(q);
\]

(3.24)

admit polynomial analogues

\[
\sum_{j \geq 0} q^{j(j+a)} \left[ \begin{array}{c} 2L-j-a \\ j \end{array} \right]_q = \sum_{j=-\infty}^{\infty} \left\{ q^{j(10j+1+2a)} \left[ \begin{array}{c} 2L \\ L-5j-a \end{array} \right]_q - q^{(2j+1)(5j+2-a)} \left[ \begin{array}{c} 2L \\ L-5j-2 \end{array} \right]_q \right\}
\]

(3.25)
which reduce to (3.24) as $L \to \infty$. It is rather surprising that polynomials appearing in (3.25) have $q$-trinomial representation as well [29]. In particular, for $a = 0$, one has

$$\sum_{j \geq 0} q^{j^2} \left[ \frac{2L - j}{j} \right] = \sum_{j = -\infty}^{\infty} \left\{ q^{60j^2 - 4j} \left( \frac{L, 10j; q^2}{10j} \right)_2 \right.\
\left. - q^{60j^2 + 44j + 8} \left( \frac{L, 10j + 4; q^2}{10j + 4} \right)_2 \right.\
\left. + q^{60j^2 + 16j + 1} \left( \frac{L, 10j + 1; q^2}{10j + 1} \right)_2 \right.\
\left. - q^{60j^2 + 64j + 17} \left( \frac{L, 10j + 5; q^2}{10j + 5} \right)_2 \right\} \right.$$  

(3.26)

Identity (3.26) is not of the form (1.11). However, if we replace $q$ by $\frac{1}{\sqrt{q}}$ in (3.26) and multiply the result by $q^{L^2/2}$ we obtain with the help of (2.6), (3.2)

$$\sum_{j \geq 0} q^{j^2} \left[ \frac{L + j}{2j} \right] = \sum_{j = -\infty}^{\infty} q^{20j^2 + 2j} (T_0(L, 10j, q) + T_0(L, 10j + 1, q))\
- \sum_{j = -\infty}^{\infty} q^{20j^2 + 18j + 4} (T_0(L, 10j + 4, q) + T_0(L, 10j + 5, q))$$  

(3.27)

which gives rise to trinomial Bailey pair

$$\tilde{\tilde{\beta}}_0(L) = \frac{1}{(q)_L} \sum_{j \geq 0} q^{j^2} \left[ \frac{L + j}{2j} \right] \sqrt{q}$$  

(3.28)

$$\tilde{\alpha}_0(r) = \begin{cases} q^{20j^2} (q^{2j} + q^{-2j}) & \text{for } r = 10j, j > 0 \\ 1 & \text{for } r = 0 \\ q^{20j^2 + 2j} & \text{for } r = 10j + 1, j \geq 0 \\ q^{20j^2 - 2j} & \text{for } r = 10j - 1, j > 0 \\ -q^{20j^2 + 18j + 4} & \text{for } r = 10j + 4 \text{ and } r = 10j + 5, j \geq 0 \\ -q^{20j^2 - 18j + 4} & \text{for } r = 10j - 4 \text{ and } r = 10j - 5, j > 0 \end{cases}$$  

(3.29)

Next we apply Theorem 2 to derive

$$\sum_{L, j \geq 0} q^{L + j^2} \frac{(-1)_L}{(q)_L} \left[ \frac{L + j}{2j} \right] \sqrt{q}$$

$$= (1)_{\infty} (-q)_{\infty} \left\{ \sum_{j = -\infty}^{\infty} q^{20j^2 + 2j} \left( \frac{q^{5j}}{1 + q^{10j}} \right) + \frac{q^{5j + \frac{1}{2}}}{1 + q^{10j + 1}} \right\}$$

$$- \sum_{j = -\infty}^{\infty} q^{20j^2 + 18j + 4} \left( \frac{q^{5j + 2}}{1 + q^{10j + 4}} + \frac{q^{5j + \frac{5}{2}}}{1 + q^{10j + 5}} \right)$$  

(3.30)
We note that the expression on the right hand side of (3.30) bears a strong resemblance to formula (3.8) with \( \hat{p} = 10, \hat{p}' = 2 \). It is also similar in form to the \( \Phi(q) \) and \( \Psi(q) \) considered by Ramanujan in his development of the fifth-order mock theta functions [49].

Expressions in (3.30) are not modular functions. Nevertheless, using Poisson summation formula one can show that asymptotic behaviour of (3.30) can be neatly expressed in terms of exponential forms. For instance, when \( q = e^{-t} \) and \( t \to 0 \) we have for (3.30)

\[
\sqrt{\frac{2}{5\pi t}} \cos \frac{\pi}{10} e^{\frac{2\pi^2}{20t}}
\]  

(3.31)

Proof of (3.31) along with asymptotic analysis at nonunitary characters (3.8) will be given elsewhere.

4. Discussion.

It is widely believed that different fermionic expressions for the (super) conformal character are related to different integrable perturbations of the same (super) conformal model. Thus, it would be interesting to identify perturbations which correspond to the new fermionic representations for \( N = 2 \ SM(4\nu,1) \) characters found in Section 3.2.

Furthermore, following [25, 26], it is tempting to interpret the relations

\[
N = 1 \ SM(2, 4\nu) \leftrightarrow N = 2 \ SM(4\nu, 1) \\
M(p, p + 1) \leftrightarrow N = 2 \ SM\left(2p, \frac{p - 1}{2}\right), \ p = 1 \ (\text{mod} \ 2)
\]  

(4.1)

established here as massless renormalization group flows. If such an interpretation is indeed correct, then one should be able to carry out Thermodynamic Bethe Ansatz (TBA) analysis of these flows along the lines of [50]. We expect that related TBA systems will have the same incidence structure as that of fermionic forms discussed in section 3. Also, we would like to point out that a “folding in half” relation between \( N = 2 \ SM(4\nu, 1) \) and \( N = 1 \ SM(2, 4\nu) \) models has already been noticed in [51,52]. Partition theoretical interpretation of our results will undoubtedly lead to construction of subtractionless bases for \( N = 2 \) super Virasoro modules.
From the mathematical point of view it is highly desirable to find an appropriate $q$-hypergeometric background for the Trinomial analogue of Bailey’s lemma. Recall that the classical Bailey’s lemma is intimately related to the $q$-Pfaff-Saalschütz formula

$$
\sum_{j=0}^{n} \frac{(q^{-n})_j (c/a)_j (c/b)_j}{(q)_j(c)_{j}(cq^{1-n}/ab)_j} \cdot q^j = \frac{(a)_n(b)_n}{(c)_{n}(ab/c)_n} \quad (4.2)
$$

which was first derived by Jackson [53]. In this direction we have already determined that

$$
\sum_{L=0}^{\infty} (-q^{-n})_L q^{\frac{(1+2n)L}{2}} \beta_0(L) = \frac{(-q^{n+1})_2^2}{(q)_{\infty}(q^{2n+1})_{\infty}} \sum_{r=0}^{\infty} \frac{q^{\frac{1+2n}{2} r} (-q^{-n})_r}{(-q^{n+1})_r} \tilde{\alpha}_0(r) \quad (4.3)
$$

with $n = 0, 1, 2, 3, \ldots$. Identity (4.3) can be derived from (2.12, 2.13, 2.15, 2.22) after a bit of labour. It immediately follows that $q^n$ in (4.3) may be replaced by an arbitrary parameter, say $\rho$. Details will be given elsewhere [54].

Building on a proposal made in [31], Schilling and Warnaar defined and extensively studied $q$-multinomials [55–57]. One may wonder if these new objects will lead to additional generalizations of Bailey’s lemma. We strongly believe that the answer is “yes” and hope to say more about it in a subsequent paper.

**Note Added.**

Soon after this paper was completed, Warnaar provided a simple and elegant proof of the conjecture from [16] used in deriving (3.21). Moreover, he has shown that each ordinary Bailey pair gives rise to a trinomial Bailey pair. In particular, he demonstrated that the trinomial Bailey pair (3.18–19) is a “descendant” of the A(1) and A(2) Bailey pairs of Slater’s list [4].

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