Identification of strong and weak hyperplanes in data envelopment analysis

N. Aghayi* and Z. Gheleyj Beigi

Department of Mathematics, Ardabil Branch, Islamic Azad University, Ardabil, Iran

Department of Mathematics, Mobarakeh Branch, Islamic Azad University, Mobarakeh, Isfahan, Iran.

Abstract
Finding of the strong and weak) defining hyperplanes of the production possibility set (PPS) is an important subject in data envelopment analysis (DEA). The concept of strong defining hyperplanes is useful in dealing with the status of returns to scale (RTS), sensitivity and stability analysis and so on. Clearly, if the optimal solutions of the envelopment form for extreme efficient DMUs are degenerate, the multiplier form may have alternative optimal solutions which yield various supporting hyperplanes at the PPS. In this study, it is shown that the strong (weak) defining hyperplane is supporting and there exists, at least, one affine independent set with \( m + s \) elements of extreme efficient DMUs (extreme efficient and weak efficient virtual DMUs) where \( m \) and \( s \) are the number of inputs and outputs, respectively. This paper aims at obtaining all the defining hyperplanes of the PPS. Moreover, the characterization of the stability region preserving the RTS classification and efficiency classification is an arguable issue which is easily obtained by strong defining hyperplanes.

Keywords: Data Envelopment Analysis, Production Possibility Set, Strong Defining Hyper-plane, Affine Independent.

1 Introduction

Efficiency and productivity measurement in organizations has enjoyed a great deal of interest among researchers studying performance analysis. The DEA is the non-parametric method of measuring
the efficiency of a DMU such as a firm or a public sector agency. The DEA was originally developed to measure the relative efficiency of peer DMUs in multiple input-multiple output settings. In this respect, Charnes et al. [1] proposed the first DEA model known as the CCR model, which satisfies four axioms: free disposability, feasibility of observed data, convexity and constant returns to scale. The efficient units in the DEA are those with maximum output levels for given input levels or with minimum input levels for given output levels. Banker et al. [2] proposed the BCC model in which the constant returns to scale condition is removed.

Finding strong defining hyperplanes is useful for the following applications:
- Identification of the RTS classification of DMUs. If we know DMUs lie on which strong defining hyperplanes then we can characterize the RTS classification of DMUs. Banker et al. [3] estimated the RTS using different DEA models such as: CCR, BCC and additive models.
- Characterizing the stability region preserving the efficiency classifications.
- Characterizing the stability region preserving the RTS classifications.

2 Preliminaries

In the DEA, each observed DMU is represented by the pair of non-negative input and output vectors \((x_j, y_j) \in \mathbb{R}^{m+j}_+, j = 1, \ldots, n\). the minimum PPS \(T\) can be stated as:

\[
T = \left\{ (x, y) \mid x \geq \sum_{j=1}^{n} \lambda_j x_j, \ y \leq \sum_{j=1}^{n} \lambda_j y_j, \ \sum_{j=1}^{n} \lambda_j = 1, \ \lambda_j \geq 0, j = 1, \ldots, n \right\}.
\] (1)

Jahanshahloo et al. [4] proposed two following models to identify extreme BCC efficient and weak efficient virtual DMUs.

\[
\begin{align*}
\min & \quad \left( \sum_{i=1}^{m} S^+_i + \sum_{i=1}^{m} S^-_i \right) \\
\text{s.t.} & \quad \sum_{j=1, j \neq k}^{n} \lambda_j^k x_{ij} + S^-_i = \theta^k w_{ik}, \quad i = 1, \ldots, m, \ i \neq l,\\
& \quad \sum_{j=1, j \neq k}^{n} \lambda_j^k x_{ij} + S^-_i = x_{ik}, \quad i = 1, \ldots, m, \ i \neq l,\\
& \quad \sum_{j=1, j \neq k}^{n} \lambda_j^k y_{rj} - S^+_r = y_{rk}, \quad r = 1, \ldots, s,\\
& \quad \sum_{j=1, j \neq k}^{n} \lambda_j^k = 1, \ \theta^k \text{ free, } \ S^-_i \geq 0, \\
& \quad \lambda_j^k \geq 0, \ S^-_i \geq 0, \ S^+_r \geq 0, \quad j = 1, \ldots, n, \ j \neq k, \ i = 1, \ldots, m, \ r = 1, \ldots, s.
\end{align*}
\] (2)
max $\varphi_q^* + \varepsilon \left( \sum_{i=1}^{m} t_i^- + \sum_{r=1}^{s} t_r^+ \right)$

s.t. $\sum_{j=1, j \neq k}^{n} \mu_j^k y_{kj} + t_k^- = w_{0k}$,

$\sum_{j=1, j \neq k}^{n} \mu_j^k y_{kj} - t_q^+ = \varphi_q^* y_{qk},$ (3)

$\sum_{j=1, j \neq k}^{n} \mu_j^k y_{kj} - t_r^- = y_{rk},$ $r = 1, \ldots, s, \quad r \neq q,$

$\sum_{j=1, j \neq k}^{n} \mu_j^k = 1, \varphi_q^* + \varepsilon, t_q^+ \geq 0,$

$\mu_j^k \geq 0, \quad t_k^- \geq 0, \quad t_r^- \geq 0, \quad j = 1, \ldots, n, \quad j \neq k, \quad k = 1, \ldots, m.$

where $\varepsilon$ is a non-Archimedean number. Suppose that the optimal objective of Models (2) and (3) are $\eta_l^k = \theta_l^k + \varepsilon \left( \sum_{i=1}^{m} s_i^+ + \sum_{r=1}^{s} s_r^+ \right)$ and $\mu_q^k = \varphi_q^k + \varepsilon \left( \sum_{i=1}^{m} t_i^- + \sum_{r=1}^{s} t_r^+ \right)$, respectively.

**Theorem 1:** If Model (2) (Model(3)) for at least one $l(q)$ is infeasible then, $DM U_k$ lies on the weak defining hyperplanes.

**Theorem 2:** If extreme efficient $DM U_k$ is on the infinite edge, which is parallel to the $l^{th}$ axis of input ($q^{th}$ axis of output), then Model (2) (Model (3)) is infeasible.

**Theorem 3:** $DM U_k$ is extreme efficient iff Model (2) (or Model (3)) is infeasible for at least one $l(q)$ or $\eta_l^k > 1(\mu_q^k < 1)$ for at least one $l(q)$.

According to Theorems 1 and 2, Jahanshahloo et al. [4] defined $I_I$ and $O_q$ to obtain the weak efficient virtual DMU as follows:

$I_I = \{DM U_k |$ Model (2) is infeasible $\} \forall l$, $O_q = \{DM U_k |$ Model (3) is infeasible $\} \forall q$.

The weak efficient virtual DMU corresponding to each $DM U_j (j \in I_I$ or $j \in O_q)$ is defined as follows:

$DM U_j^f$ = $(x_{1j}, x_{2j}, \ldots, x_{l-1j}, x_{lj} + 1, x_{l+1j}, \ldots, x_{mj}, Y_{1j}, Y_{2j}, \ldots, Y_{(q-1)j}, Y_{qj}, Y_{(q+1)j}, \ldots, Y_{sj})$

$DM U_j$ = $(x_{1j}, x_{2j}, \ldots, x_{l-1j}, x_{lj}, x_{l+1j}, \ldots, x_{mj}, Y_{1j}, Y_{2j}, \ldots, Y_{(q-1)j}, Y_{qj} - 1, Y_{(q+1)j}, \ldots, Y_{sj})$.

3 Finding all defining hyperplanes of PPS

**Definition 1:** Hyperplane $H(u^*, v^*, w^*) = \{(x,y) | u^*y - v^*x + w^* = 0 \}$ is a supporting hyperplane of the PPS on $(x_v, y_v)$ if

(i) $\langle x_v, y_v \rangle \in \{(x,y) | u^*y - v^*x + w^* = 0 \} \cap T_v$,

(ii) $\forall (x,y) \in T_v, u^*y - v^*x + w^* \leq 0$,

(iii) $(u^*, v^*) \neq 0$.

**Definition 2:** Hyperplane $H(u, v, w_0) = \{(x,y) | u^*y - v^*x + w_0 = 0 \}$ is a strong defining hyperplane of $T_v$ if it is supporting and there is at least one affine independent set with $m+s$ elements of extreme efficient DMUs that lie on $H$. On the other hand, $d_{sm}(T_v \cap H) = m+s-1$.

**Definition 3:** Hyperplane $H(u, v, w_0) = \{(x,y) | u^*y - v^*x + w_0 = 0 \}$ is a weak defining hyperplane of $T_v$ if it is supporting and there is at least one affine independent set with $m+s$ elements of extreme efficient and weak efficient virtual DMUs that lie on $H$. On the other hand, $d_{sm}(T_v \cap H) = m+s-1$. 

3 Finding all defining hyperplanes of PPS

**Definition 1:** Hyperplane $H(u^*, v^*, w^*) = \{(x,y) | u^*y - v^*x + w^* = 0 \}$ is a supporting hyperplane of the PPS on $(x_v, y_v)$ if

(i) $\langle x_v, y_v \rangle \in \{(x,y) | u^*y - v^*x + w^* = 0 \} \cap T_v$,

(ii) $\forall (x,y) \in T_v, u^*y - v^*x + w^* \leq 0$,

(iii) $(u^*, v^*) \neq 0$.

**Definition 2:** Hyperplane $H(u, v, w_0) = \{(x,y) | u^*y - v^*x + w_0 = 0 \}$ is a strong defining hyperplane of $T_v$ if it is supporting and there is at least one affine independent set with $m+s$ elements of extreme efficient DMUs that lie on $H$. On the other hand, $d_{sm}(T_v \cap H) = m+s-1$.

**Definition 3:** Hyperplane $H(u, v, w_0) = \{(x,y) | u^*y - v^*x + w_0 = 0 \}$ is a weak defining hyperplane of $T_v$ if it is supporting and there is at least one affine independent set with $m+s$ elements of extreme efficient and weak efficient virtual DMUs that lie on $H$. On the other hand, $d_{sm}(T_v \cap H) = m+s-1$. 

3 Finding all defining hyperplanes of PPS

**Definition 1:** Hyperplane $H(u^*, v^*, w^*) = \{(x,y) | u^*y - v^*x + w^* = 0 \}$ is a supporting hyperplane of the PPS on $(x_v, y_v)$ if

(i) $\langle x_v, y_v \rangle \in \{(x,y) | u^*y - v^*x + w^* = 0 \} \cap T_v$,

(ii) $\forall (x,y) \in T_v, u^*y - v^*x + w^* \leq 0$,

(iii) $(u^*, v^*) \neq 0$.

**Definition 2:** Hyperplane $H(u, v, w_0) = \{(x,y) | u^*y - v^*x + w_0 = 0 \}$ is a strong defining hyperplane of $T_v$ if it is supporting and there is at least one affine independent set with $m+s$ elements of extreme efficient DMUs that lie on $H$. On the other hand, $d_{sm}(T_v \cap H) = m+s-1$.

**Definition 3:** Hyperplane $H(u, v, w_0) = \{(x,y) | u^*y - v^*x + w_0 = 0 \}$ is a weak defining hyperplane of $T_v$ if it is supporting and there is at least one affine independent set with $m+s$ elements of extreme efficient and weak efficient virtual DMUs that lie on $H$. On the other hand, $d_{sm}(T_v \cap H) = m+s-1$.
To the best of the authors’ knowledge, there are only limited researches that have been undertaken on the subject of finding all the efficient frontier. In this section, we propose an effective method for finding all defining hyperplanes of $T_v$.

First, to illustrate the procedure, we explain our method geometrically. The strong hyperplanes of figure(1) consist of AB and BC and its weak hyperplanes are OA and CO. We know that at least one affine independent set with $m + s = 2$ elements of extreme efficient DMUs lies on the strong hyperplane, and at least one affine independent set with $m + s = 2$ elements of extreme efficient DMUs and weak efficient virtual DMUs lies on the weak hyperplane. Therefore, for finding all hyperplanes, we should find all efficient and weak efficient virtual DMUs. It means that we should find $E = \{A,B,C,D,E\}$. Then we should find all affine independent sets with 2 elements of E, i.e. $T = \{\{A,B\}, \{A,C\}, \{A,D\}, \{A,E\}, \{B,C\}, \{B,D\}, \{B,E\}, \{C,D\}, \{C,E\}, \{D,E\}\}$. If one affine independent set with $m + s = 2$ elements of extreme efficient DMUs and weak efficient virtual DMUs lies on strong or weak hyperplanes, then the strict convex combination of the affine set lies on strong or weak hyperplanes.

We determine all the defining hyperplanes by Algorithm 1:

Algorithm 1:

Step 0: Find all extreme efficient DMUs and weak efficient virtual using Models (2) and (3). Define $E_1 = \{DMU_1|DMU_1$ is extreme efficient} and $E_2 = \{DMU_2|DMU_2$ is weak efficient virtual}. Let $E = E_1 \cup E_2$ and $|E| = k$.

Step 1: Calculate $O_{m+s}^k = r$. Define $S_l \subseteq E$, $l = 1, ..., r$, $|S_l| = m + s$ and set $T = \{S_1, S_2, ..., S_r\}$.

Step 2: Choose $S_p \in T$ which is the affine independent.

Step 3: Set $DMU_p = (x_p, y_p) = \{(1/m+s) \sum_{j=1}^{m+s} x_{pj} + (1/m+s) \sum_{j=1}^{m+s} y_{pj} | (x_{pj}, y_{pj}) \in S_p\}$, $p = 1, ..., r$.

Step 4: 4-1: Evaluate $DMU_p$, $p = 1, ..., r$, using the following model when it is as the convexity composition of the member set $E_1$:

$$
\text{max} \sum_{j=1}^{m+s} u_j y_{jp} + u_0
$$

s.t.
$$
\sum_{j=1}^{m+s} u_j x_{jp} = 1,
\sum_{j=1}^{m+s} u_j y_{jp} - \sum_{i=1}^{m} u_i x_{ip} + u_0 \leq 0, \quad j \in E_p,
\sum_{j=1}^{m+s} u_j y_{jp} - \sum_{i=1}^{s} u_i x_{ip} + u_0 \leq 0,
\sum_{r=1}^{m} u_r \geq 0, \quad u_i \geq 0, \quad r = 1, ..., s, \quad i = 1, ..., m.
\tag{4}
$$

Put $\bar{E}_1 = \{DMU_p|DMU_p$ is efficient by Model (4)} and $\bar{E}_2 = \{DMU_p|DMU_p$ is inefficient by Model (4)}. 

Put $\bar{E}_1 = \{DMU_p|DMU_p$ is efficient by Model (4)} and $\bar{E}_2 = \{DMU_p|DMU_p$ is inefficient by Model (4)}.
Finding stability region by strong defining hyperplanes

In order to find the stability region of efficiency and the RTS for the IRS (DRS) efficient DMU, assume that \( \{ H_1, H_2, ..., H_p \} \) are all strong defining hyperplanes so that all IRS (DRS) efficient DMUs lie on. Then, put all efficient DMUs on \( H_i \) in set \( Z_i \) such that \( |Z_i| = M_i(i = 1, ..., p) \).

Hence, we define \( \mathcal{Z}_i = \{(x, y)\} = \{(\sum_{j=1}^{b_i} \lambda_j x_{ij}, \sum_{j=1}^{b_i} \lambda_j y_{ij}) : \sum_{j=1}^{b_i} \lambda_j = 1, \lambda_j \in [0, 1), (x_{ij}, y_{ij}) \in Z_i, i = 1, ..., p\} \), in fact, \( \mathcal{Z}_i \) is \( H_i \cap \text{PPS} \). Finally, \( I = \bigcup_{i=1}^{p} \mathcal{Z}_i \) gives the stability region of efficiency and the RTS for the IRS (DRS) DMU. In order to find the stability region of efficiency and the RTS of the CRS efficient DMUs, the same approach used for the IRS efficient DMUs is used. The only difference is that we consider \( \{ H_1, H_2, ..., H_p \} \) as all strong defining hyperplanes with \( u_0 = 0 \) and that each CRS DMU lies on them. Generally, Algorithm 2 gives the stability region of efficiency and RTS classifications of efficient DMUs.

4 Finding stability region by strong defining hyperplanes

In order to find the stability region of efficiency and the RTS for the IRS (DRS) efficient DMU, assume that \( \{ H_1, H_2, ..., H_p \} \) are all strong defining hyperplanes so that all IRS (DRS) efficient DMUs lie on. Then, put all efficient DMUs on \( H_i \) in set \( Z_i \) such that \( |Z_i| = M_i(i = 1, ..., p) \).

Hence, we define \( \mathcal{Z}_i = \{(x, y)\} = \{(\sum_{j=1}^{b_i} \lambda_j x_{ij}, \sum_{j=1}^{b_i} \lambda_j y_{ij}) : \sum_{j=1}^{b_i} \lambda_j = 1, \lambda_j \in [0, 1), (x_{ij}, y_{ij}) \in Z_i, i = 1, ..., p\} \), in fact, \( \mathcal{Z}_i \) is \( H_i \cap \text{PPS} \). Finally, \( I = \bigcup_{i=1}^{p} \mathcal{Z}_i \) gives the stability region of efficiency and the RTS for the IRS (DRS) DMU. In order to find the stability region of efficiency and the RTS of the CRS efficient DMUs, the same approach used for the IRS efficient DMUs is used. The only difference is that we consider \( \{ H_1, H_2, ..., H_p \} \) as all strong defining hyperplanes with \( u_0 = 0 \) and that each CRS DMU lies on them. Generally, Algorithm 2 gives the stability region of efficiency and RTS classifications of efficient DMUs.
Algorithm 2:

Step 0: Determine all strong defining hyperplanes by Algorithm 1.

Step 1: Evaluate $n$ DMUs by Model (2). Put all efficient DMUs in $\hat{E}$.

Step 2: Characterize RTS classifications by Theorem 4.

Step 3: For all DMUs in $\hat{E}$ that are the IRS (DRS), find all defining hyperplanes and pass it. For all DMUs in $\hat{E}$ that are the CRS, find all defining hyperplanes with $w^*_i = 0$ and pass it. Suppose that $H_1, \ldots, H_p$ are the related hyperplanes. Let $Z_\zeta = \{DMU_j | DMU_j \in H_\zeta \}, \ | Z_\zeta | = M_\zeta , \zeta = 1, \ldots, p$.

Step 4: Define

$$\bar{Z}_\zeta = \{(x_i,y_i) | \sum_{j=1}^{M_\zeta} \lambda_j x_j = \sum_{j=1}^{M_\zeta} \lambda_j y_j , \sum_{j=1}^{M_\zeta} \lambda_j = 1, \lambda_j \in [0,1], (x_j,y_j) \in Z_\zeta \}, \ \zeta = 1, \ldots, p.$$  

Step 5: $I_j = \bigcup_{\zeta=1}^{p} \bar{Z}_\zeta$, $j \in \hat{E}$ is the stability region. Namely, after changing DMU under evaluation in this region, the DMU preserves the efficiency classification and belongs to the same class in the RTS classification.

5 Conclusion

In this paper, we proposed an algorithm for finding all the defining hyperplanes of the PPS. It was shown that a strong (weak) hyperplane contains $(m + s)$ extreme efficient DMUs (extreme efficient DMUs and weak efficient virtual DMUs) which are affine independent with dimension $m + s - 1$ which is defining hyperplane of PPS for DMUs with $m$ inputs and $s$ outputs. It was also presented that the supporting hyperplane of PPS with aforesaid characteristic is unique. Consequently, the obtained hyperplane is the defining hyperplane of the PPS. Using strong defining hyperplanes, we characterized the stability region preserving returns to scale and efficiency classifications. Namely, after changing the DMU being evaluated in this region, the DMU preserves the efficiency classification and belongs to the same class in the RTS classification. It should be mentioned that the presented method for obtaining the stability region does not offer the largest stability region. Also, the complexity of the proposed algorithm is high therefore we suggest to find a simpler algorithm for future work.

References

[1] A. Charnes, W.W. Cooper, E. Rodes, Measuring the efficiency of decision making units, Eur. J. Oper. Res. 2 (6) (1978) 429-444.

[2] R.D. Banker, A. Charnes, W.W. Cooper, Some models for estimating technical and scale inefficiencies in data envelopment analysis, Manage. Sci. 30 (1984) 1078-1092.
[3] R.D. Banker, W.W. Cooper, L.M. Seiford, R.M. Thrall, J. Zhu, Returns to scale in different DEA models, Eur. J. Oper. Res. 154 (2004) 345-362.

[4] R.D. Banker, R.M. Thrall, Estimation of returns to scale using data envelopment analysis. Eur. J. Oper. Res. 62 (1992) 74-84.

[5] G.R. Jahanshahloo, F. Hosseinzadeh Lotfif, D. Akbarian, Finding weak defining hyperplanes of PPS of the BCC model, Appl. Math. Model. 34 (2010) 3321-3332.