Monoidal structure of the category of $u_q^+$-modules

Elísabet Gunnlaugsdóttir

1 Introduction.

We consider the half-quantum group $u_q^+ (\mathfrak{sl}_2 (\mathbb{C}))$ at a root of unity which order is not 4. This non quasi-cocommutative Hopf algebra is the upper triangular sub-Hopf algebra of $u_q (\mathfrak{sl}_2 (\mathbb{C}))$, quotient of the quantized enveloping algebra at a root of unity $q$ (see [5]). Half quantum groups provide universal $R$-matrices through the Drinfeld double and hence solutions to the Yang-Baxter equation. Furthermore they appear of interest in knot theory and 3-manifold invariants. For a simple Lie algebra $G$, a presentation of $u_q^+ (G)$ by quiver and relations has been established by Cibils in [3], showing that only $u_q^+$ is of finite representation type, the others being of tame or wild representation type.

In order to study more deeply the representation theory of $u_q^+$, we consider the particular family of indecomposable modules on $u_q^+$ which are $u_q^+$-modules as well. We call them “extendable modules”. They form a subring of the Grothendieck ring of $u_q^+$, and their study leads to a Clebsch-Gordan-like formula for the decomposition of the tensor product, taken on the ground field, of two indecomposable $u_q^+$-modules. The extendable modules, together with the $R$-matrix of $u_q$ and the action of the Auslander-Reiten transpose (see [1]) on the category of modules, complete the proof which was not achieved in [2]. As a consequence the tensor product commutes, despite the non quasi cohommutativity of $u_q^+$. Moreover we obtain explicit isomorphisms between $M \otimes N$ and $N \otimes M$ for any two $u_q^+$-modules and we can observe that these canonical isomorphisms have the properties of morphisms in a braided category (see [3]), except of course that they are not natural.

We also consider tensor products of simple modules over the entire $u_q$. The crucial observation is that extendable non-projective $u_q^+$-modules are the simple modules on $u_q$. A connection between the decomposition formulas over $u_q^+$ and $u_q$ is established. We thus derive formulas previously obtained

---

2000 Mathematical subject classification, primary 20G42 ; secondary 18D10.

Keywords : Monoidal categories ; Representations of quantum groups ; Half-quantum groups at a root of unity
by Reshetikhin and Turaev in [8] for the tensor product of simple \( u_q \)-modules in a new way. The proof we obtain is new and entirely based on basic properties of extendable modules.

Furthermore we establish a totally different proof of the decomposition formula for \( u_q^+ \)-modules which actually includes the three situations \( u_q^+ \), the universal enveloping algebra \( U(\mathfrak{sl}_2) \) of \( \mathfrak{sl}_2 \) and the quantum universal enveloping algebra \( U_q(\mathfrak{sl}_2) \) of \( \mathfrak{sl}_2 \) when \( q \) is not a root of unity. The proof consists in a fairly simple axiomatisation on the Grothendieck ring of these Hopf algebras.

## 2 The Hopf algebras \( u_q(\mathfrak{sl}_2(\mathbb{C})) \) and \( u_q^+ \).

We recall definitions and known facts about the above algebras, choosing Kassel’s (see [5]) presentation of \( u_q \), originally from Lusztig (see [7]). Let \( q \) be a primitive \( n \)-th root of unity in \( \mathbb{C} \), \( n \) different from 4, and set

\[
d = \begin{cases} 
n & \text{if } n \text{ is odd} \\
n/2 & \text{if } n \text{ is even}
\end{cases}
\]

**Definition 2.1** The Hopf algebra \( u_q(\mathfrak{sl}_2(\mathbb{C})) \) is defined over \( \mathbb{C} \) by the generators \( E, F, K \) and the relations :

\[
E^d = F^d = 0, \quad K^d = 1, \quad KE = q^2EK, \quad KF = q^{-2}FK
\]

and \( EF - FE = \frac{K - K^{-1}}{q - q^{-1}} \).

It admits a Poincar-Birkhoff-Witt type basis in the set \( \{ E^iK^jF^l \} \) for \( 0 \leq i, j, l \leq d - 1 \) (see [5]).

The coalgebra structure is given on the generators as follows : the comultiplication \( \Delta : u_q \to u_q \otimes u_q \) is defined by

\[
\Delta(E) = 1 \otimes E + E \otimes K \\
\Delta(F) = K^{-1} \otimes F + F \otimes 1 \\
\Delta(K) = K \otimes K
\]

the counit \( \epsilon : u_q \to k \) by

\[
\epsilon(E) = \epsilon(F) = 0 \quad \epsilon(K) = 1
\]

and the antipode, \( S : u_q \to u_q \), is given by

\[
S(E) = -EK^{-1}, \quad S(F) = -KF, \quad S(K) = K^{-1}
\]

2
We have the following formulas for the comultiplication:

\[ \Delta(E)^r = \sum_{k=0}^{j} q^{-k(r-k)} \left[ \begin{array}{c} r-k \\ r \end{array} \right]_q E^k \otimes K^k E^{r-k} \quad \text{and} \]

\[ \Delta(F)^r = \sum_{k=0}^{r} q^{k(r-k)} \left[ \begin{array}{c} r-k \\ r \end{array} \right]_q F^k K^{-(r-k)} \otimes F^{r-k} \]

Where \( \left[ \begin{array}{c} x \\ y \end{array} \right] = \frac{[y]!}{[x]! [y-x]!} \) with \([x]! = [1][2] \ldots [x]\) and \([x] = \frac{q^x - q^{-x}}{q - q^{-1}}\). A formula which calculates the commutators \([E^m, F^m]\) with \(m \in \{0, \ldots, d-1\}\) will be needed (see [5]):

\[ E^m F^m = \sum_{h=0}^{m} c_h E^{m-h} E^{m-h} \prod_{j=0}^{h-1} \frac{K q^{-j} - K^{-1} q^{j}}{q - q^{-1}} \]

where \(c_h\) is a nonzero coefficient. It is well known that this Hopf algebra is quasi-triangular (see [5], [6]). Its \(R\)-matrix has the following expression:

\[ R = \frac{1}{d^2} \sum_{0 \leq i,j,k \leq d-1} (q - q^{-1})^k q^{k(k-1)/2 + 2k(i-j) - 2ij} E^i K^j \otimes F^j K^i \]

**Remark 2.1**
1) Hopf algebras have the property that the tensor product over the ground field of two left modules is still a left module. Indeed, for a Hopf algebra \(H\), restricting the natural action of \(H \otimes H\) to \(H\) through the comultiplication \(\Delta\) yields a left \(H\)-module structure.

2) Recall that the \(R\)-matrix satisfies in particular the relation \(\Delta^{op} = R \Delta R^{-1}\), where \(\Delta^{op} = \tau \Delta\) and \(\tau\) is the flip, \(\tau(a \otimes b) = b \otimes a\) for \(a, b \in u_q\). This relation is equivalent to the existence of a family of natural isomorphisms between \(M \otimes N\) and \(N \otimes M\) for any \(u_q\)-modules \(M\) and \(N\). The isomorphisms are given by the action of \(\tau R\).

The upper triangular sub-algebra of \(u_q\) generated by \(E\) and \(K\) is a sub-Hopf algebra, denoted by \(u_q^+\); indeed

\[ \Delta(u_q^+) \subset u_q^+ \otimes u_q^+ \]

\[ S(u_q^+) \subset u_q^- \]

The dimension over \(k\) of \(u_q^+\) is \(d^2\). The set \(\{E^i K^j\}_{0 \leq i,j \leq d-1}\) is a basis of \(u_q^+\) (see [7]).

**Remark 2.2** In [2] it has been shown that \(u_q^+\) is isomorphic to a quotient of a path algebra endowed with a Hopf algebra structure. It is our reference for the following remarks as well as for the representations of \(u_q^+\).

1) As an associative algebra \(u_q^+\) is uniserial, meaning that each indecomposable module on \(u_q^+\) has a unique decomposition series. As a consequence \(u_q^+\) is of finite representation type.

2) The Jacobson radical of \(u_q^+\) is generated by \(E\).
We have the following proposition:

**Proposition 2.1** If $q$ is an $n$-th root of unity with $n \neq 4$, the Hopf algebra $u_q^+$ is not quasi-cocommutative.

**Proof:** Suppose there exists an invertible element $R \in u_q^+ \otimes u_q^+$ such that $\Delta^{op} = R\Delta R^{-1}$. Then $R$ is of the form $R = \sum a_{i,j,k,l} E^i K^j \otimes E^K K^l$ where the $a_{i,j,k,l}$ belong to $k$. We have in particular $\Delta^{op}(K) = R\Delta(K) R^{-1}$, i.e. $K \otimes KR = RK \otimes K$, implying that $a_{i,j,k,l} q^{2(i+k)} = a_{i,j,k,l}$. Hence the expression of $R$ must reduce to $R = \sum a_{i,j,k,l} E^i K^j \otimes E^K K^l + \sum_{0 \leq j,l \leq n-1} a_{0,j,0,l} K^j \otimes K^l$. In order to show that the coefficients $a_{i,j,o,l} = a_{j,l}$ are 0 we use the identity $\Delta^{op}(E) R = R \Delta(E)$ and obtain the relations $a_{j,l} = q^{2j} a_{j,l-1}$ and $a_{j,l} = q^{-2l} a_{j-1,l}$ whenever they make sense. As a consequence, $a_{j,o} = a_{0,l} = 0$, implying that $q^{2j} a_{1,0} = q^{-2l} a_{0,1}$, hence $a_{1,1} = 0$ and $a_{j,o} = 0$. We infer $a_{i,j} = 0$ for all $0 \leq j,l \leq d-1$, and $R$ is therefore reduced to $R = \sum a_{i,j,l} E^i K^j \otimes E^K K^l$. Finally, we note that $\Delta^{op}(E^{d-1})$ must be different from zero, and then develop the expression $R \Delta(E^{d-1})$. Writing $\Delta(E^{d-1}) = \sum b_{x,y,z} E^x K^y \otimes E^{d-1-x} K^{z}$ with $b_{x,y,z} \in \mathbb{C}$, we obtain $R \Delta(E^{d-1}) = \sum c_{i,j,l,x,y,z} E^{i+j} K^{l+y} \otimes E^{2d-1-x-1} K^{l+x}$. Since either $i+x \geq d$ or $2d-i-x-1 \geq d$, we necessarily have $R \Delta(E^{d-1}) = 0$. We thus arrive to the contradiction $R \Delta(E^{d-1}) R^{-1} = 0$ and $R \Delta(E^{d-1}) R^{-1} = \Delta^{op}(E^{d-1}) \neq 0$. \[ \square \]

**Remark 2.3** The case $n = 4$ yields a quasi-cocommutative Hopf algebra (see [2]). An alternative proof of Proposition 2.1 is provided in [2] using the presentation of $u_q^+$ by quiver and relations.

### 2.1 Modules.

The isomorphism classes of the modules described below constitute the complete list of isomorphism classes of indecomposable $u_q^+$-modules; they are all non-isomorphic. To each couple $(i, u)$, where $i \in \mathbb{Z}/d \mathbb{Z}$ and $0 \leq u \leq d-1$, corresponds a $u_q^+$-module, denoted by $M_i^u$, of dimension $u+1$. It admits a basis $\{e_i^0, e_i^1, \ldots, e_i^u\}$ over $K$ such that the action of $u_q^+$ on the basis vectors is given by

\[
\begin{align*}
K e_i^j &= q^{2(i+j)} e_i^j \\
E e_i^j &= e_i^{j+1} & \text{for } 0 \leq j \leq u-1 \\
E e_i^u &= 0 
\end{align*}
\]

Note that $e_i^0$ is a generator of $M_i^u$ over $u_q^+$. The indecomposable projective modules are those of dimension $d-1$, and we denote them by $P_i = M_i^{d-1}$. The simple modules are the one-dimensional modules, and we denote them by $S_i = M_i^0$. 

4
**Notations** : The length of a vector \( v \) belonging to a \( u_q^+ \)-module is an integer \( 0 \leq m \leq d - 1 \), minimal for the property \( E^{m+1}v = 0 \). In particular, the length of a basis vector of the type \( e_j^i \) is \( u - j \). For \( r, s \in \mathbb{Z} \) let \( E(r/s) \) be the entire part of \( r/s \).

### 3 Axiomatisation of the tensor product of modules.

The tensor product of modules on \( u_q^+ \) has decomposition formulas which are similar to those for the universal enveloping algebra of \( sl_2(\mathbb{C}) \), and for the quantum universal enveloping algebra of \( sl_2(\mathbb{C}) \) when \( q \) is not a root of unity. The following axiomatisation unifies the proofs of these formulas leaving behind the concrete decomposition.

**Remark 3.1** Recall that the Grothendieck group of a ring \( \Lambda \), denoted by \( K(\Lambda) \), is the quotient of the free abelian group with basis the isomorphism classes \([X] \) of modules \( X \) on \( \Lambda \) by the subgroup generated by elements \([X_2] - [X_1] - [X_3] \) provided by each split exact sequence \( X_1 \to X_2 \to X_3 \) of \( \Lambda \)-modules. Moreover if \( \Lambda \) is a Hopf algebra, the free abelian group is endowed with a ring structure through the tensor product of modules. The functor induced by tensoring over the ground field is flat, implying that the subgroup above is an ideal, and hence the quotient \( K(\Lambda) \) is still a ring.

If \( \Lambda \) is a finite dimensional algebra, its Grothendieck group is a free abelian group with basis given by the isomorphism classes of indecomposable modules.

Let \( I \) be the set \( \{0\} \) or \( \mathbb{Z}/d\mathbb{Z} \). To \( m \) belonging to \( \mathbb{N} - \{0\} \), where \( \mathbb{N} = \mathbb{N} \cup \infty \), put \( J_m \) to be the set \( \{0, \ldots, m - 1\} \) if \( m \in \mathbb{N} \) and \( J_m = \mathbb{N} \) if \( m = \infty \). Consider the free commutative group generated by the elements \([i,u] \), where \((i, u) \) belong to \( I \times J_m \). Suppose now that this group is equipped with an extra multiplicative structure, making it into a ring. Denote by \( \oplus \) the addition law and \( \otimes \) the multiplication law. We need to put \([i, u] = 0 \) if \( u < 0 \). We have the proposition :

**Proposition 3.1** Assume the relations below hold and are symmetric with respect to \( \otimes \) :

\[
[i, 0] \otimes [j, 0] = [i + j, 0], \quad [0, 1] \otimes [j, v] = [j, v + 1] \oplus [j + 1, v - 1] \quad \text{for } 0 \leq v \leq m - 2,
\]

and \([0, 1] \otimes [j, m - 1] = [j, m - 1] \oplus [j + 1, m - 1] \) where \((i, j) \in I \times J_m \) and \( u, v \in J_m \).

Then the following decomposition formulas are true:

1. \([i, u] \otimes [j, v] = \oplus_{l=0}^{\min(u,v)} [i + j + l, u + v - 2l] \) for \( u + v \leq m - 1 \)

2. \([i, u] \otimes [j, v] = \oplus_{l=0}^{\min(u,v)} [i + j + l, m - 1] \oplus \oplus_{l=e+1}^{\min(u,v)} [i + j + l, u + v - 2l] \) for \( u + v \geq m - 1 \) where \( e = u + v - (m - 1) \)
Proof: We proceed by double induction. First we prove that
\[ [i, u] \otimes [j, 0] = [i + j, u] \] for all \( i, j \in I \) and \( u \leq m - 1 \) by induction on \( u \). By assumption it is true for \( u = 0 \). Suppose it is valid up to a rank \( 0 < u < m - 1 \) and let’s show it for \( u + 1 \). For this purpose we look at \( [0, 1] \otimes [i, u] \otimes [j, 0] \). Developing the left and right side respectively we obtain the equality:
\[
([i, u + 1] \otimes [j, 0]) \oplus ([i + 1, u - 1] \otimes [j, 0]) = [0, 1] \otimes [i + j, u]
\]
that is \( ([i, u + 1] \otimes [j, 0]) \oplus [i + j + 1, u - 1] = [i + j, u + 1] \oplus [i + j + 1, u - 1] \), and as a consequence \([i, u + 1] \otimes [j, 0] = [i + j, u + 1]\).

Next, we take an arbitrary \( u \), and show the formulas by induction on \( v \). Suppose they hold up to a rank \( v \geq 1 \), then we have two situations to consider, either \( u + v + 1 \leq m - 1 \) or \( u + v + 1 \geq m - 1 \). Developing \( [0, 1] \otimes [i, u] \otimes [j, v] \) on the left and right hand side respectively easily solves the first case. For the second more care is needed. Set \( e = u + v - (m - 1) \) and let \( \bigoplus_{l=0}^{\min(u,v)}[x_l, y_l] = 0 \) if \( a \leq 0 \) with \((x_l, y_l \in I \times J_m)\). We proceed as before by developing the left and right sides of \([i, u] \otimes [j, v] \otimes [0, 1]\) and thus obtaining the equality
\[
(\bigoplus_{l=0}^{\min(u,v)}[i + j + l, m - 1] \oplus \bigoplus_{l=e+1}^{\min(u,v)}[i + j + l, u + v - 2l + 1]) \otimes [0, 1] = [i, u] \otimes ([j, v + 1] \oplus [j + 1, v - 1]).
\]
Developing this gives us the identity
\[
(\bigoplus_{l=0}^{\min(u,v)}[i + j + l, m - 1] \oplus \bigoplus_{l=e+1}^{\min(u,v)}[i + j + l + 1, m - 1]) \oplus \\
\bigoplus_{l=e+1}^{\min(u,v)}([i + j + l + 1, u + v - 2l + 1] \oplus [i + j + l + 1, u + v - 2l - 1]) = [i, u] \otimes \bigoplus_{l=0}^{e+1}([i + j + l, m - 1] \oplus \bigoplus_{l=e+2}^{\min(u,v+1)}[i + j + l + 1, u + v - 2l])
\]
Therefore \([i, u] \otimes [j, v + 1] = \bigoplus_{l=0}^{e+1}([i + j + l, m - 1] \oplus \bigoplus_{l=e+2}^{\min(u,v+1)}[i + j + l, u + v - 2l]) \). □

Remark 3.2 The Grothendieck ring of the Hopf algebra \( u_q^+ \) corresponds to \( I = \mathbb{Z}/d\mathbb{Z} \) and \( m = d \) where we replace the formal writing \([i, u]\) by the isomorphism class of the indecomposable module \([M_i^u]\). This observation leads us to the next result.

Theorem 3.1 Let \( M_i^u \) and \( M_j^v \) be indecomposable \( u_q^+ \)-modules for \( i, j \in \mathbb{Z}/d\mathbb{Z} \) and \( 0 \leq u, v \leq d - 1 \). There are isomorphisms:

1. If \( u + v \leq d - 1 \)
\[ M_i^u \otimes M_j^v \cong \bigoplus_{l=0}^{\min(u,v)} M_i^{u+v-2l} \]

2. If \( u + v \geq d - 1 \), set \( e = u + v - (d - 1) \), then
\[ M_i^u \otimes M_j^v \cong \bigoplus_{l=0}^{e} P_{i+j+l} \oplus \bigoplus_{l=e+1}^{\min(u,v)} M_i^{u+v-2l} \]

6
Proof: In view of the previous remark we can apply the proposition. We need to check that \( S_i \otimes S_j \cong S_{i+j} \) and \( M^1_i \otimes S_j \cong M^1_{i+j} \) as well as \( M^1_i \otimes M^v_j \cong M^v_j \otimes M^1_i \cong M^{v+1}_{i+j} \oplus M^{v-1}_{i+j+1} \) and finally that \( M^1_i \otimes P_j \cong P_j \otimes M^1_i \cong P_{i+j} \oplus P_{i+j+1} \). The first two isomorphisms are simply given by letting \( e^0_i \otimes e^0_j \) go to a non-zero multiple of \( e^0_{i+j} \).

To prove the third assertion (we treat the case \( M^1_i \otimes M^v_j \cong M^{v+1}_{i+j} \oplus M^{v-1}_{i+j+1} \)) we need to ensure that \( E \) is not a root of unity. Indeed this implies the existence of \( M^{v+1}_{i+j} \) and \( M^{v-1}_{i+j+1} \) as submodules of \( M^1_i \otimes M^v_j \), as well as their sum which is necessarily direct.

For dimension reasons we therefore obtain the required isomorphism.

Remark 3.3 We will see that the theorem can be obtained in a totally different way, by means of extendable \( u_q^+ \)-modules.

Next we consider two different cases where our axiomatisation applies.

Proposition 3.2 Taking \( I = 0 \) and \( m = \infty \) leads to Clebsch-Gordan formulas for \( U(\mathfrak{sl}_2(\mathbb{C})) \) and \( U_q(\mathfrak{sl}_2(\mathbb{C})) \) when \( q \) is not a root of unity.

Proof: 1) Recall the irreducible representations of \( U(\mathfrak{sl}_2(\mathbb{C})) \). To each integer \( n \) corresponds a simple \( U(\mathfrak{sl}_2(\mathbb{C})) \)-module \( V(n) \) of dimension \( n + 1 \). It admits a basis \( \{ v_0, \ldots, v_n \} \) over \( k \) such that the action of \( U(\mathfrak{sl}_2(\mathbb{C})) \) is given by

\[
\begin{align*}
X v_i &= (n - i + 1) v_{i-1} \\
Y v_i &= (i + 1) v_{i+1} \\
H v_i &= (n - 2i) v_i
\end{align*}
\]

where \( v_i = 0 \) for \( i \not\in \{0, \ldots, n\} \).
and we have the Clebsch-Gordan formula for the decomposition of the tensor product of two such modules: $V(n) \otimes V(m) \cong \oplus_{l=0}^{\min(n,m)} V(n+m-2l)$. In view of the preceding results, this formula can be obtained by checking the following isomorphisms of $U(\mathfrak{sl}_2(\mathbb{C}))$: $V(0) \otimes V(0) \cong V(0)$ and $V(1) \otimes V(n) \cong V(n+1) \oplus V(n-1)$ for $n \geq 1$.

The first is trivial, the second is obtained by giving an explicit decomposition as it was done for $u_q^+$. Indeed, let $\{v_0, v_1\}$ and $\{v'_0, \ldots, v'_n\}$ be the basis of $V(1)$ and $V(n)$ respectively. Then the vectors $v_0 \otimes v'_0$ and $v_0 \otimes v'_1 - mv_1 \otimes v'_0$ are generators of the modules $V(n+1)$ and $V(n-1)$ respectively. Their sum is a direct sum and comparing the dimensions leads to the desired isomorphism.

2) The case of $U_q(\mathfrak{sl}_2(\mathbb{C}))$ when $q$ is not a root of unity is similar. Let $\epsilon = \pm 1$. To each integer $n$ correspond two modules $V_{1,n}$ and $V_{-1,n}$ who admit bases $\{v_{\epsilon,0}, v_{\epsilon,1}, \ldots, v_{\epsilon,n-1}\}$ such that the action of $U_q(\mathfrak{sl}_2(\mathbb{C}))$ is given by

$$
\begin{align*}
Ev_{\epsilon,i} &= \epsilon |n-i+1| v_{\epsilon,i-1} \\
Fv_{\epsilon,i} &= \epsilon (i+1) v_{\epsilon,i+1} \quad \text{where } v_{\epsilon,i} = 0 \text{ for } i \notin \{0, \ldots, n\} \\
Kv_{\epsilon,i} &= q^{\epsilon n-2} v_{\epsilon,i}.
\end{align*}
$$

The Clebsch-Gordan formula is: $V_{\epsilon,n} \otimes V_{\epsilon',m} \cong \oplus_{l=0}^{\min(n,m)} V_{\epsilon\epsilon',n+m-2l}$. One easily reduces to the case of modules of type $V_{1,n}$ and as in the former situations the isomorphism between $V_{1,1} \otimes V_{1,n}$ and $V_{1,n+1} \oplus V_{1,n-1}$ for $n \geq 1$ is guaranteed by the two vectors $v_0 \otimes v'_0$ and $v_0 \otimes v'_1 - [m]q^{-m}v_1 \otimes v'_0$ (we assume that the vectors $v_i$ and $v'_j$ form bases for $V_{1,1}$ and $V_{1,n}$ respectively).

\[ \square \]

**Remark 3.4** Considering the simple $u_q^+$-modules, we can observe that they form a multiplicative group for the tensor product, isomorphic to the cyclic group of order $n$. Actually, the isomorphism classes of simple modules over a basic and split Hopf algebra always provide a group (see for instance [2]). Now this group acts on the category of $u_q^+$-modules via the tensor product and it is interesting to note that the action of the generator $S_1$ on an indecomposable module yields the dual transpose (see [2]).

## 4 Extendable modules.

It is obvious that a $u_q^+$-module is not in general issued from a $u_q$-module, in the sense that it is not obtained by restricting the action of $u_q$ to $u_q^+$. Nevertheless we can consider the subfamily of $u_q^+$-modules on which indeed there exists an action of $u_q$ such that the original action of $u_q^+$ is respected. We call these modules extendable. They have the property that the $R$-matrix of $u_q$ provides isomorphisms making the tensor product of two such modules commutative. Restricting our study to this family gives some information on the decomposition of $u_q^+$-modules, as well as on simple $u_q$-modules. We
need the following notation:

**Notation**: Let \( u \in \mathbb{N} \), then \( \pi \) is the representative element of the class of \( u \) modulo \( d \) contained in the set \( \{0, \ldots, d-1\} \).

**Theorem 4.1** The extendable indecomposable modules are:

1. The indecomposable modules of type \( M_i^{2k} \) for \( 0 \leq i \leq d-1 \). These modules extend in a unique way and provide all the simple \( u_q \)-modules.

2. The projective indecomposable modules \( P_i \) for \( 0 \leq i \leq d-1 \). These modules extend in two non-isomorphic ways, except \( P_{d+1} \) when \( d \) is odd.

**Proof**: We proceed in the following way: First we consider an arbitrary indecomposable \( u_q \)-module, and we try to define an action of \( F \in u_q \) on its basis elements, such that the original action of \( u_q^+ \) is preserved, and the algebra structure of \( u_q \) is respected. We thus infer the necessary conditions for an indecomposable module to be extendable.

Consider a module \( M_i^u \) with \( i \in \mathbb{Z}/d\mathbb{Z} \) and \( 0 \leq u \leq d-1 \). It is generated over \( u_q^+ \) by the element \( e_i^0 \), and the set \( \{e_i^j\}_{0 \leq j \leq u} \) is a basis over \( k \). The action of \( u_q^+ \) is given by \( Fe_i^j = e_i^{j+1} \) for \( 0 \leq j \leq u-1 \), \( Fe_i^u = 0 \) and \( Ke_i^j = q^{2(i+j)}e_i^j \).

Suppose we have an action of \( F \) given by \( Fe_i^j = \sum_{0 \leq h \leq u} \lambda_{i,j}^he_i^h \) where \( \lambda_{i,j}^h \in \mathbb{C} \). The relation \(KF = q^{-2}FK \) implies \(KF e_i^j = \sum_{0 \leq h \leq u} \lambda_{i,j}^h q^{2(i+h)}e_i^h \)

It follows that \( \lambda_{i,j}^h q^{2(i+h)} = q^{-2} q^{2(i+j)} \lambda_{i,j}^h \) and therefore:

\[
Fe_i^j = \lambda_{i,j}^{j-1} e_i^{j-1} = \lambda_{i,j}^{j-1} e_i^{j-1} \quad \text{for all } 1 \leq j \leq u \text{ and } Fe_i^0 = \lambda_{i,j}^{d-1} e_i^{d-1},
\]

where \( \lambda_{i,j}^{d-1} = 0 \) if \( u \leq d-2 \). Since \( EF - FE = \frac{K-K^{-1}}{q-q^{-1}} \), we must have the following:

\[
\lambda_{i,j}^0 = -[2i] + \lambda_{i,j}^{d-1}
\]

We next proceed by induction and obtain

\[
\lambda_j = \sum_{0 \leq h \leq j} -[2(i+h)] + \lambda_{i,j}^{d-1}.
\]

The remaining relations are now \( Ee_i^u = 0 \) and \( Fd = 0 \). From the first one we deduce:

\[
(EF - FE)e_i^u = E\lambda_i^{u-1} e_i^{u-1} = \lambda_i^{u-1} e_i^u = \frac{K-K^{-1}}{q-q^{-1}} e_i^u = \frac{q^{2(i+u)} - q^{-2(i+u)}}{q-q^{-1}} e_i^u.
\]

On the other hand, \( \lambda_{u-1} = \sum_{0 \leq h \leq u-1} -[2(i+h)] + \lambda_{i,j}^{d-1} \). The equality is automatically realized when dealing with a projective module. Otherwise, that is when \( u \leq d-2 \), we need
The equality is true when \(2(u + 1) = 0 \mod n\) and \(2u = -4i \mod n\). For \(n\) odd the first case is never realized, and for \(n\) even it corresponds to the projective modules. Otherwise we need the condition \(u = -2i \mod d\).

The last condition on the \(\lambda_i^j\) coming from \(F^d = 0\) is \(\lambda_{d+1}^{-1} = 0\) for \(d\) odd.

Hence the indecomposable modules for which the action of \(u_q^+\) extends to \(u_q\) are the projectives and the modules of the type \(M_i^{-\frac{2i}{q}}\). For \(i \in \{0, \ldots, d - 1\}\), it is easy to check that the modules obtained on \(u_q\) from the modules \(M_i^{-\frac{2i}{q}}\) are simple, and we thus obtain all the simple modules on \(u_q\) up to isomorphism (the list of simple \(u_q\)-modules is given in [3]). □

**Remark 4.1** The projective modules are examples of modules extendable to \(u_q\)-modules in two non-isomorphic ways. We are therefore allowed to imagine the case of an extendable module whose indecomposable components are not extendable. This turns out to be impossible.

**Proposition 4.1** A \(u_q^+\)-module is extendable if and only if it is a direct sum of indecomposable extendable modules.

**Proof:** Let \(X\) be an arbitrary \(u_q^+\)-module, decomposable into \(M_i^u \oplus (\bigoplus_{l \in L} \bigoplus_{v \in V} M_i^v)\), where \(L\) is a finite set. We examine the possible actions of \(F\) on the basis \(\{e_i^j\}\) of \(M_i^v\). Using a simple induction and the relation \(EF - FE = \frac{K - K^{-1}}{q - q^{-1}}\), we find that an action must be of the form:

\[
F^{j-1} = \lambda_i^{j-1} e_i^{j-1} + \text{linear combination of } \{e_i^k\} \text{ for } j \leq k.
\]

The action of \(E\) on \(e_i^u\) given by \(E e_i^u = 0\) requires that

\[
(EF - FE) e_i^u = \frac{q^{2(i+u)} - q^{-2(i+u)}}{q - q^{-1}} e_i^u
= EF(e_i^u)
= (\lambda_i^u e_i^u + \text{linear combination of } \{e_i^k\}_{u+1} \leq k)
\]

This implies that \(\lambda_i^{u-1} = \frac{q^{2(i+u)} - q^{-2(i+u)}}{q - q^{-1}}\), i.e. that \(M_i^u\) is an extendable module. □

**Remark 4.2** There may be more than one way to extend a direct sum of non projective, indecomposable, extendable modules. As an example we can give the \(u_q^+\)-module \(M_i^1 \oplus S_0\) in the case \(n = 3\). Indeed the possible
actions of $F$ are easily found to be: $Fe^0_1 = ce^0_0$, $Fe^1_1 = \lambda^0_1e^0_1$ and $Fe^0_0 = 0$ where $c$ belongs to $\mathbb{C}$. Considering the options $c = 0$ and $c \neq 0$ respectively, the result is two non-isomorphic representations of $u_q$.

**Remark 4.3** - For $d$ odd there is exactly one indecomposable extendable module per dimension $m$, where $1 \leq m \leq d-1$.

- For $d$ even there are exactly two indecomposable extendable modules per dimension $2m + 1$, where $0 \leq m \leq \frac{d}{2} - 1$.

The following result provides a characterisation of self-dual indecomposable modules in terms of extendable ones. We recall that the dual $\text{Hom}_k(M, k)$ of a module $M$ over a Hopf algebra $H$ over a field $k$ can be provided with a left $H$-module structure by means of the antipode $S$ (see [10]) (we denote this left $H$-module by $^*M$):

$$\lambda.f(x) = f(S(\lambda)x) \text{ for } \lambda \in H , \ f \in \text{Hom}_k(M, k) \text{ and } x \in M.$$

**Proposition 4.2** Let $M$ be a $u^+_q$-module. Then the following are equivalent

1. The module $M$ is indecomposable and self-dual.

2. The module $M$ is indecomposable and extendable of type $M^u_i$ with $u \equiv -2i$.

**Proof:** We consider an arbitrary indecomposable module $M^u_i$. Let $\{(e^i_j)^*\}$ be the dual basis of $^*M^u_i$; then $(e^u_i)^*$ is a generator of this module and we have another basis given by the elements $\{E^j(e^u_i)^*\}_{0 \leq j \leq u-1}$. The action of $K$ on $E^j(e^u_i)^*$ is the following:

$$KE^j(e^u_i)^* = q^{2j}E^j(K(e^u_i)^*) = q^{2(j-i-u)}E^j(e^u_i)^*.$$

We deduce an isomorphism between $^*M^u_i$ and $M^u_{n-i-u}$. The explicit isomorphism is

$$M^u_i \rightarrow ^*M^u_{n-i-u},$$

$$e^i_j \mapsto (-1)^j q^{j(j+2i+1)}e^u_{n-i-u}.$$

Consequently $M^u_i$ is selfdual iff $u \equiv -2i$. □

**Remark 4.4** The extendable modules provide a different proof of the Clebsch-Gordan-like formula for $u^+_q$ stated before. We sketch the proof briefly.

**Proof:** The first step does not involve the extendable modules (see [3] where the proof is incomplete). It consists in showing that the tensor product of two arbitrary indecomposable $u^+_q$-modules must decompose as follows: $M^u_i \otimes M^u_j \cong \oplus_v M^2_{i+j+l}$ where $u - d \leq x_l \leq u + v - d$ and $u + v \leq d - 1$ (we request the latter condition here in order to simplify, and we suppose that $v \leq u$). This is done by considering the dimension of each $K$-eigenspace and the action of $E$ on those. Indeed the $K$-eigenvalues are $q^{2(i+j+l)}$ with $0 \leq l \leq u + v$, and the dimensions are distributed as follows: to $q^{2(i+j+l)}$ with $0 \leq l \leq v$ corresponds a vector space of dimension $l + 1$, moreover the
vector space morphism induced by $E$ between the eigenspace of eigenvalue $q^{2(i+j+l)}$ and the one of eigenvalue $q^{2(i+j+l+1)}$ is injective. To the same situation with $v \leq l \leq u$ corresponds a vector space of dimension $v + 1$ and the morphism induced by $E$ is one to one. Finally, for $u \leq l \leq u + v$ the dimension is $u + v - l + 1$, and $E$ induces a surjective morphism whose kernel is one-dimensional. As a consequence the quotient by the action of the Jacobson radical, $\top(M^i_u \otimes M^j_v)$, is $\oplus_{l=0}^{\min(u,v)} S_{i+j+l}$ and we conclude by uniseriality.

Now in the specific case of two indecomposable and extendable $u_q^+$-modules, necessarily $x_l = u + v - 2l$, which is the result we want in the general case. Indeed, to each index $i + j + l$ corresponds one and only one extendable indecomposable module. Moreover the tensor product of two extendable modules is still extendable, hence it decomposes into a direct sum of indecomposable extendable modules, and leaves only one choice for the value of $x_l$. Denote by $\phi$ the resulting isomorphism.

This observation on the extendable modules immediately leads to the solution of the general case. Let $X$ and $Y$ be the indecomposable extendable modules of dimension $u + 1$ and $v + 1$ respectively, and let $S$ be the simple module s.t. $M_i^u \otimes M_j^v \cong S \otimes X \otimes Y$. Then the morphism $id \otimes \phi$ realizes the required decomposition isomorphism. □

Remark 4.5 The $R$-matrix of $u_q$ provides isomorphisms through the action of $\tau R$ between $M_i^u \otimes M_j^v$ and $M_j^v \otimes M_i^u$ when these are extendable modules. For any simple module $S_l$, induced isomorphisms are given between $S_l \otimes M_i^u \otimes M_j^v$ and $S_l \otimes M_j^v \otimes M_i^u$ by $id_{S_l} \otimes \tau R$. Hence explicit isomorphisms are obtained, which make the tensor product of any two modules lying in the orbit of the extendable modules under the action of the structure group commutative (see remark 3.4). We let $\text{Indu}_q^+$ denote the set of indecomposable finite dimensional $u_q^+$-modules, and we have the following corollary.

**Corollary 4.1**

1. When $d$ is odd, the orbit, under the action of the structure group, of the extendable indecomposables is $\text{Indu}_q^+$, hence isomorphisms are obtained in all cases.

2. When $d$ is even the orbit covers all the indecomposables whose dimension over $k$ is odd. Hence isomorphisms are given between $M_i^u \otimes M_j^v$ and $M_j^v \otimes M_i^u$ when $u$ and $v$ are even.

The explicit isomorphisms obtained when $d$ is odd are not natural, since $u_q^+$ is not quasi-cocommutative. Nevertheless they satisfy the other relations defining a braided module category (see [5]). Denote by $c_{U,V}$ the isomorphism between $U \otimes V$ and $V \otimes U$, where $U,V$ are $u_q^+$-modules. Then we have the following:
Then we define the following action of $F$.

We denote the resulting modules by $\bar{q}$.

Proof: The proof is based on the study of indecomposable modules on $u_q$ and decomposition formulas are established here by a totally different approach, involving modules and autoinjective modules as well as exact sequences of these. These are obtained here in a similar way. There exist extendable modules $M_1, M_2$ and $M_3$ together with a simple module $S$ and isomorphisms:

1. \[ \phi_1 : U \otimes V \otimes W \cong S \otimes M_1 \otimes M_2 \otimes M_3 \]
2. \[ \phi_2 : V \otimes W \otimes U \cong S \otimes M_2 \otimes M_3 \otimes M_1 \]
3. \[ \phi_3 : V \otimes U \otimes W \cong S \otimes M_2 \otimes M_1 \otimes M_3. \]

Then

\[ c_{U,V \otimes W} = \phi_2^{-1}(id_S \otimes c_{M_1,M_2,M_3})\phi_1 \]
\[ = \phi_2^{-1}(id_S \otimes (id_{M_2} \otimes c_{M_1,M_3}) \circ id_S \otimes (c_{M_1,M_3} \otimes id_{M_3}))\phi_1 \]
\[ = \phi_2^{-1}((\phi_2 \circ id_V \otimes c_{U,V} \circ \phi_3^{-1}) \circ (\phi_3 \circ c_{U,V} \otimes id_W \circ \phi_3^{-1}))\phi_1 \]
\[ = (id_V \otimes c_{U,W})(c_{U,V} \otimes id_W). \]

Remark 4.6: The underlying isomorphism of vector spaces $M_i^u \otimes M_j^v \cong M_j^v \otimes M_i^u$ does not depend on $i$ and $j$, therefore we obtain no new solution to the Yang-Baxter equation.

5 Tensor product of simple $u_q$-modules.

Recall that the simple $u_q$-modules are obtained from indecomposable extendable $u_q^+$-modules (see proposition 4.1). We denote by $M_i$, where $i \in \mathbb{Z}/n\mathbb{Z}$ and $0 \leq u \leq d-1$, a simple module over $u_q$. We need to recall a family of indecomposable finite dimensional $u_q$-modules, which are both projective and injective (see § and §). To begin with, take the direct sum of the projective indecomposable $u_q^+$-modules $P_i \oplus P_{-2i}$, where $i \in \{0, \ldots, E((d-1)/2)\}$.

Then we define the following action of $F$ on its basis elements, making it into a $u_q$-module:

\[ Fe^i_j = \lambda_i^{-1}e_i^{j-1} \] and \[ Fe^{-2i}_j = e_i^{-2j-1} + \lambda_i^{-1}e_i^{-2j-1} \]

where $j \in \{0, \ldots, \overline{4i-1}\}$ and $Fe_j^{-2i} = \lambda_j^{-1}e_j^{-1}$ for $j \in \{\overline{4i-1+1}, \ldots, d-1\}$.

We denote the resulting modules by $\bar{P}_i$. In § Reshetikhin and Turaev give decomposition formulas for the tensor product of simple $u_q$-modules. The proof is based on the study of indecomposable modules on $u_q$; the Verma modules and autoinjective modules as well as exact sequences of these. These decomposition formulas are established here by a totally different approach, using the preceding results obtained on $u_q^+$-modules.

Theorem 5.1 Let $\overline{M_i^u}$ and $\overline{M_j^v}$ be simple $u_q$-modules for $i, j \in \mathbb{Z}/d\mathbb{Z}$, $0 \leq u, v \leq d-1$ and $u+v \leq d-1$. Suppose $v \leq u$. There is an isomorphism

\[ \overline{M_i^u} \otimes \overline{M_j^v} \cong \bigoplus_{l=0}^{v} \overline{M_{i+j+l}^{u+v-2l}}. \]
Theorem 5.2 Let $\overline{M}_i^u$ and $\overline{M}_j^v$ be simple $u_q$-modules for $i, j \in \mathbb{Z}/d\mathbb{Z}$, $0 \leq u, v \leq d - 1$ and $u + v \geq d - 1$. There is an isomorphism

$$\overline{M}_i^u \otimes \overline{M}_j^v \cong \bigoplus_{l=0}^{E(e/2)} \overline{P}_{i+j+l} \oplus \bigoplus_{l=e+1}^{\min(u,v)} \overline{M}_{i+j+l}^{u+v-2l}.$$

Proof: We can observe three cases:

$$\begin{align*}
\begin{cases}
u = d - 2i, \\
v = d - 2j,
\end{cases}
\begin{cases}
u = 2d - 2i, \\
v = d - 2j
\end{cases}
\begin{cases}
u = 2d - 2i, \\
v = 2d - 2j.
\end{cases}
\end{align*}$$

We restrict ourselves to the first case since the only difference between these is of elementary computational order. We furthermore assume that $\min(u,v) = v$. 

Proof: We simply show that there’s a unique way extending the direct sum $X = \bigoplus_{l=0}^{e} M_{i+j+l}^{u+v-2l}$, that is by extending each module separately.

Recall that $M_{i+j+l}^{u+v-2l}$ is generated by $e_{i+j+l}^0$ as a $u_q$-module and admits the set $\{e_{i+j+l}^0\}_{0 \leq k \leq u+v-2l}$ as a basis over $k$.

Recall also that the unique extended action of $u$ on $M_{i+j+l}^{u+v-2l}$ is given by $F e_{i+j+l}^k = \lambda_{i+j+l}^{k-1} e_{i+j+l}^{k-1}$.

In order to extend $X$, we study the possible actions of $F$ on the basis elements. They are entirely determined by the action of $F$ on the generators of each indecomposable module. Indeed, $F e_{i+j+l}^k = \lambda_{i+j+l}^{k-1} e_{i+j+l}^{k-1} + F^k(F e_{i+j+l}^0)$. Let us first show that $F e_{i+j+l}^0$ is necessarily a linear combination of elements of the set $\{e_{i+j+l-k}^{l-1}\}_{k \leq l-1}$. Suppose $F e_{i+j+l}^0$ is a linear combination of elements $e_{i+j+k}^m$ with $0 \leq k \leq v$ and $0 \leq m \leq u + v - 2k$.

Applying the identity $KF = q^{-2}FK$, we find that $m_k$ is congruent to $l - k - 1$ modulo $d$. Therefore $m_k = l - k - 1 + pd$ with $p \in \mathbb{Z}$. Since $0 \leq m_k \leq u + v - 2k$, necessarily $p = 0$ and $m_k = l - k - 1$. Consequently we can write $F e_{i+j+l}^0 = \sum_{k=1}^{l-1} a_k e_{i+j+l-k}^{k-1}$ with $a_k \in k$. Using the relation $E F - F E = \frac{K - K^{-1}}{q - q^{-1}}$, our previous observation on the action of $F$ on an arbitrary basis element implies

$$F e_{i+j+l}^m = \lambda_{i+j+l}^{m-1} e_{i+j+l}^{m-1} + \sum_{k=1}^{l-1} a_k e_{i+j+l-k}^{k-1}.$$

Finally, since $e_{i+j+l}^{u+v-2l} = 0$, we must have that $E F e_{i+j+l}^{u+v-2l} = \lambda_{i+j+l}^{u+v-2l-1} e_{i+j+l}^{u+v-2l-1}$. This implies

$$\sum_{k=1}^{l-1} e_{i+j+l-k}^{u+v-2l+k} = 0.$$ But for $0 \leq m \leq u + v - 2(l - k)$ we have

$$e_{i+j+l-k}^m \neq 0,$$

and since $0 \leq u + v - 2l + k \leq u + v - 2l + 2k$, we find that $a_k = 0$ for $1 \leq k \leq l - 1$. Hence $F e_{i+j+l}^0 = 0$ and $F e_{i+j+l}^m = \lambda_{i+j+l}^{m-1} e_{i+j+l}^{m-1}$.

Theorem 5.2 Let $\overline{M}_i^u$ and $\overline{M}_j^v$ be simple $u_q$-modules for $i, j \in \mathbb{Z}/d\mathbb{Z}$, $0 \leq u, v \leq d - 1$ and $u + v \geq d - 1$. There is an isomorphism

$$\overline{M}_i^u \otimes \overline{M}_j^v \cong \bigoplus_{l=0}^{E(e/2)} \overline{P}_{i+j+l} \oplus \bigoplus_{l=e+1}^{\min(u,v)} \overline{M}_{i+j+l}^{u+v-2l}.$$
Step 5.2.1 The tensor product decomposes in the following sum:
\[ M^l_j \otimes M^{l+1}_j \cong \bigoplus_{i=0}^{e-1} \bigoplus_{l=0}^{i+j+l} \bigoplus_{k=e+1}^{i+j+l} M^{u+v-2k}_{i+j+l}. \]

Proof: As in the preceding proof, \( F e^0_{i+j+k} \) is a linear combination of \( K \)-eigenvectors with \( K \)-eigenvalue equal to \( q^{2(i+j+k-1)} \), and \( F e^l_{i+j+k} = \lambda^{l-1}_{i+j+k} e^l_{i+j+k} + E^l F e^0_{i+j+k} \). First we consider the decomposition as a \( u^+_q \)-modules decomposition and show that the element \( F e^0_{i+j+l} \) is not in \( \bigoplus_{k=e+1}^{i+j+l} M^{u+v-2k}_{i+j+l} \)
for \( 0 \leq l \leq e \). Indeed, for \( 1 \leq l \leq e \) the \( K \)-eigenvalue of the vector \( F e^0_{i+j+l} \) is \( q^{2(i+j+l-1)} \) (note that \( F e^0_{i+j} = 0 \)), whereas for \( e+1 \leq k \leq v \) and \( 0 \leq m_k \leq u+v-2k \) the \( K \)-eigenvalue for the vector \( e^{m_k}_{i+j+k} \) is \( q^{2(i+j+k+m_k)} \).

Asking \( 2(i+j+k+m_k) \) to be congruent to \( 2(i+j+l-1) \) modulo \( n \) is equivalent to require that \( k + m_k \equiv l - 1 \mod d \). But \( k + m_k \in \{ e+1, \ldots, d-2 \} \) and \( l - 1 \in \{ 0, \ldots, e-1 \} \), therefore this congruence is impossible. On the other hand, a computation similar to that of the proof of the preceding proposition shows that \( F e^0_{i+j+e+l} = 0 \) for \( l = 1, \ldots, v \). Hence the first step.

Step 5.2.2 There exists a \( u^+_q \)-decomposition of \( M^l_j \otimes M^l_j \) such that for \( 0 \leq k \leq E(e/2) \), the action of \( F \) on the generators \( e^0_{i+j+k} \) of the \( u^+_q \)-modules \( P_{i+j+k} \) is zero.

Proof: We show that there exists a \( K \)-eigenvector with eigenvalue \( q^{2(i+j+k)} \) (unique up to scalar multiples) for \( 0 \leq k \leq e \), s.t. \( F \) acts on this vector as zero. Furthermore, we show that for \( 0 \leq k \leq E(e/2) \), this vector is of length \( d-1 \) and hence generates a projective \( u^+_q \)-module.

The list of basis-vectors with \( K \)-eigenvalue equal to \( q^{2(i+j+k)} \) is given by the following set of \( e+1 \) vectors:
\[ \{ e^0_i \otimes e^0_j, e^1_i \otimes e^{k-1}_j, \ldots, e^k_i \otimes e^0_j, e^u_i \otimes e^d+k-u_j, e^{u-1}_i \otimes e^{d+k-u+1}_j, \ldots, e_i^{u-(e-k-1)} \otimes e^1_j \}. \]

The action of \( F \) induces a vector space morphism between the vector space generated by the above vectors and the vector space generated by the \( e+1 \) vectors of \( K \)-eigenvalue \( q^{2(i+j+k-1)} \). The action of \( F \) is described by
\[ F e^m_i \otimes e^{k-m}_j = q^{-2(i+m)} \lambda^{k-m-1}_j e^m_i \otimes e^{k-m-1}_j + \lambda^{m-1}_i e^{m-1}_i \otimes e^{k-m}_j \]
and
\[ F e^u_i \otimes e^{d+k-u+m}_j = q^{-2(u-m)} \lambda^{d+k-u+m-1}_j e^u_i \otimes e^{d+k-u+m-1}_j \]
and
\[ \lambda^u_i e^{u-m-1}_i \otimes e^{d+k-u+1}_j, \]
and the corresponding matrix has the following entries:

\[
\begin{cases}
  a_{p,p} = q^{-2(i+p-1)} \lambda^{k-p}_j & \text{for } 1 \leq p \leq k-1 \\
  a_{p,p+1} = \lambda^{k-1}_j & \text{for } 1 \leq p \leq k-1 \\
  a_{p,p} = 0 & \text{for } k + 2 \leq p \leq e + 1 \\
  a_{k+1,p} = 0 & \text{for } p \neq k + 2 \\
  0 & \text{otherwise.}
\end{cases}
\]

We can make the following remarks: 1) The matrix is of rank \( e \) and consequently the kernel of the morphism is one-dimensional, which gives a unique
vector (up to scalar multiples), which we denote by \( v_k \), s.t. \( F v_k = 0 \).

2) This vector \( v_k \) is a linear combination of the basis vectors \( e^m_i \otimes e^k_j \), which all appear with a non-zero coefficient. We can therefore put \( v_k = e^k_i \otimes e^0_j + w_k \), where \( w_k \) is a linear combination of \( e^m_i \otimes e^k_j \) for \( 1 \leq m \leq k \).

3) The vectors \( e^m_i \otimes e^{k-m}_j \) for \( m \in \{0, \ldots, k-1\} \) are all in the image of this morphism.

What remains to be satisfied is that \( P \) is congruent to \( 2(i + j + k) \) mod \( d \) (thm. 4.1.). We need to exclude the second possibility. Suppose that \( m \) is congruent to \( -2(i + j + k) \); this means that \( m = d - 2(i + j) - 2k = e - 2k - 1 \) for \( 0 \leq k \leq E((e - 1)/2) \). If \( e \) is even and \( k = e/2 \), then \( m = d - 1 \), and the two situations coincide. Observing that \( u - k \geq e - k > e - 2k - 1 \), we compute \( E^{u-k} v_k = q^{2(u-k)} e^u \otimes e^0 + ( \text{vectors linearly independent with } e^u \otimes e^0 ) \). Necessarily \( m > u - k \), which is a contradiction, and so \( m = d - 1 \).

In \( P_{i+j+k} \) with \( k \in \{0, \ldots, E((e - 1)/2)\} \), we put \( l_k = d - 2(i + j + k) + 1 = e - 2k \), and we have \( F e^l_{i+j+k} = 0 \) (see proof of theorem 4.1).

**Step 5.2.3** There exists a vector \( \alpha_{l_k} \) such that \( F \alpha_{l_k} = e^l_{i+j+k} \). Furthermore, the \( u^+_q \)-module generated by \( \alpha_{l_k} \) is isomorphic to \( P_{i+j+k+l_k} \).

**Proof:** We observe that \( l_k \in \{e((e + 1)/2), \ldots, e\} \), and since \( e^0_{i+j+k} \) is a linear combination of the vectors \( e^0_i \otimes e^k_j, \ldots, e^k_i \otimes e^0_j \), we have that \( e^k_i \otimes e^j_{l_k} \) is a linear combination of the vectors \( e^0_i \otimes e^k_{l_k}, \ldots, e^k_i \otimes e^0_{l_k} \). Therefore, considering the third remark in step 5.2.2, there exists a vector \( \alpha_{l_k} \) with \( K \)-eigenvalue equal to \( q^{2(i+j+k+l_k)} \) s.t. \( F \alpha_{l_k} = e^{l_k}_{i+j+k} \). We now look at the \( u^+_q \)-module generated by \( \alpha_{l_k} \). There are two things to prove:

1) The module \( u^+_q \alpha_{l_k} \) is extendable. First of all, the sum \( P_{i+j+k} + u^+_q \alpha_{l_k} \) of \( u^+_q \)-modules is a direct sum. In order to prove this, we show that the vectors \( e^m_{i+j+k} \) for \( m \in \{0, \ldots, d-1\} \) are linearly independent. Considering their \( K \)-eigenvalues, this means that we must have \( E^{d-m} \alpha_{l_k} \neq a_m e^l_{i+j+k} \) and \( E^s \alpha_{l_k} \neq a_s e^l_{i+j+k} \) for \( m \in \{0, \ldots, l_k - 1\} \) and \( s \in \{l_k, \ldots, d-1\} \).

Indeed, if we suppose \( E^{d-m} \alpha_{l_k} = a_m e^l_{i+j+k} \), where \( a_m \) is a nonzero coefficient, it implies \( 0 = E^m E^{d-m} = a_m \), which is a contradiction. In the same way, assume that \( E^s \alpha_{l_k} = a_s e^l_{i+j+k} \); this means that \( E^{s+1} E^s \alpha_{l_k} = 0 \), and therefore, in view of remark 1) in step 5.2.2, we have \( F^s E^s \alpha_{l_k} = b_s e^l_{i+j+k} \), where \( b_s \in \mathbb{C} \). Applying the formula (see section 2) for the commutator \( [E^s, F^s] \), we arrive to the conclusion that \( \alpha_{l_k} = c_s e^l_{i+j+k} \), which is impossible.

2) Now the module over \( u^+_q \) generated by \( \alpha_{l_k} \) is an extension of \( P_{i+j+k} \oplus u^+_q \alpha_{l_k} \),
hence they are both compelled to be extendable (see proposition 4.2). As in the proof of theorem 5.1, \( u^+_q \alpha l_k \) must be isomorphic to \( M^m_{i+j+k+l_k} \), with \( m = d - 1 \) or \( m \equiv -2(i + j + k + l_k) \mod d \). In order to exclude the second possibility, we suppose that \( l_k = d - 2(i + j + k) + 1 \); this means that \( m \equiv -2(i + j + k + d - 2(i + j + k) + 1) \equiv -2(d - (i + j + k) + 1) \equiv 2(i + j + k) - 2 \mod d \). In this case, the vectors \( E^m \alpha l_k \) and \( e^{d-1}_{i+j} \) are in the kernel of the morphism induced by the action of \( E \) on the vector spaces concerned. The fact that the kernel is one-dimensional gives a contradiction and therefore \( u^+_q \alpha l_k = P_{i+j+k+l_k} \).

**Step 5.2.4** The \( u_q \)-module \( P_{i+j+k} \oplus P_{i+j+k+l_k} \) is indecomposable.

**Proof**: Suppose it admits a non trivial decomposition \( P_{i+j+k} \oplus P_{i+j+k+l_k} = A \oplus B \), with \( A \) and \( B \) non zero. This implies that as \( u^+_q \)-modules (as such we denote them by \( A \) and \( B \)) \( A \) or \( B \) is equal to \( P_{i+j+k} \), and \( B \) or \( A \) is equal to \( P_{i+j+k+l_k} \) (by the Krull-Schmidt theorem). Hence \( A \) and \( B \) are extended \( u_q^+ \)-projective modules, which is excluded. □

**References**

[1] Auslander, M.; Reiten, I.; Smal, S., (1995), *Representation Theory of Artin Algebras*, Cambridge studies in advanced mathematics 36, Cambridge University Press, Cambridge.

[2] Cibils, C., (1993), *A Quiver Quantum Group*, Commun. Math. Phys. 157, 459-477.

[3] Cibils, C., (1997), *Half quantum groups at roots of unity, path algebras and representation theory*, International Mathematical Research Notices 12, 541-553.

[4] Cibils, C., (1999), *The Projective Class Ring of basic and split Hopf algebras*, K-Theory 17, 383-391.

[5] Kassel, C., (1995), *Quantum Groups*, Graduate Texts in Mathematics 155, Springer, New York.

[6] Kirby, R.; Melvin, P., (1991), *The 3-manifold invariants of Witten and Reshetikhin-Turaev for \( sl_2 \) (\( \mathbb{C} \))*. Invent. Math. 105, 473-545.

[7] Lusztig, G., (1992), *Finite dimensional Hopf algebras arising from quantum groups*, J. Amer. Soc. 3, 257-296.

[8] Reshetikhin, N. Yu.; Turaev, V.G., (1991) *Invariants of 3-manifolds via link polynomials and quantum groups*, Invent. Math. 103, 547-597.
[9] Suter, R., (1994), *Modules over $U_q(\mathfrak{sl}_2(\mathbb{C}))*, Commun. in Math. 163, 359-393.

[10] Sweedler, M.E., (1969), *Hopf Algebras*, Benjamin, New York.

Département de Mathématiques, GTA (CNRS ESA 5030), Université Montpellier II, case 51, 34095 MONTPELLIER, FRANCE. Email: beta@math.univ-montp2.fr