To the memory of O.N. Vvedenskii

Abelian varieties, homogeneous spaces and
duality. I
(with Mass Formulas, Formal Groups and Shtukas)

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Abstract. The article is dedicated to the memory of Oleg Nikolaevich
Vvedenskii (1937-1981). The results obtained by O. N. Vvedenskii are pre-
sented, as well as selected new results of the authors, which develop the study
of arithmetic algebraic geometry in the directions of crystalline cohomology,
fundamental groups of schemes, torsors, dualities. Elements of ontology of
Vvedenskii’s research are also given. A continuation of the review of Vveden-
skii’s results, as well as a review of new selected results, including variants
of Smith–Minkowski–Siegel mass formula and Drinfeld shtukas, will be pre-
sented in the second part of the paper.

Keywords: Abelian variety, Picard variety, local field, duality, etale
(étale) topology, fundamental group of a scheme, formal group

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“I remember well, for example, stories about a seminar led by A.O. Gel-
fond, B.N. Delone and L.G. Shniroleman, where they tried to understand the
class field theory and came to the conclusion that it was hopeless. Later,
as a student, I participated in a seminar by B.N. Delone and A.G. Kurosh
on the same topic, which ended with the same result. With regard to al-
gebraic geometry, and especially the works of the Italian school, even such
attempts were not made - the belief was widespread that it was impossible
to understand them.” (I. Shafarevich, 1990)
1 Introduction

The article is dedicated to the memory of Oleg Nikolaevich Vvedenskii (1937-1981). O.N. Vvedenskii was a student of Academician I.R. Shafarevich. Vvedenskii’s research and his received results are connected with duality in elliptic curves and with the corresponding Galois cohomology over local fields, with the Shafarevich-Tate pairing, and with other pairings, with local and quasi-local class field theories of elliptic curves, with the theory of Abelian varieties of dimension greater than 1, with the theory of commutative formal groups over local fields and over quasi local fields, with M. Artin effect in abelian varieties. Both results obtained by O. N. Vvedenskii and selected new results of authors are presented that develop the study of arithmetic algebraic geometry in the directions of crystalline cohomology, fundamental groups of schemes, torsors, dualities. A continuation of the review of Vvedenskii’s works, as well as a review of new selected results, including variants of Smith–Minkowski–Siegel mass formula and Drinfeld shtukas, will be presented in the second part of the paper. Let $S$ be a scheme. An Abelian scheme over $S$ is a $S$-group scheme $A \to S$ which proper, flat, finitely presented, and which has smooth and connected geometric fibers.

Professor I. Dolgachev is the I. Shafarevich’s student. Vvedenskii met with I. Dolgachev. Prof. Dolgachev supported the work of Vvedenskii’s students and acted as an opponent in the defense of their dissertations. The monograph [9] by Dolgachev and co-authors contains references to the works of Vvedenskii. We also note that the monograph [9] gives some answer to the last phrase the above quote by I. Shafarevich.

In the monograph by J. Milne [28], translated into Russian, edited by I.R. Shafarevich on the initiative of O. N. Vvedenskii (who was also one of the translators), along with the terminology of principal homogeneous spaces, the concept of a torsor, or, more precisely, a $G$-torsor, is used. Note that the well-known monograph “Arithmetic Duality Theorems” by Milne [29] lists the main works of O.N. Vvedenskii on the arithmetic duality theory. Since presently the presentation of the corresponding results in the language of torsors has become generally accepted [15, 16, 19, 20, 21, 22], and we use, along with the term ”principal homogeneous space”, the language of torsors. Elements of ontology of Vvedenskii’s research are also given.

A continuation of the review of Vvedenskii’s results and result of his students [33, 34, 35, 36, 37, 38, 39, 40], as well as a review of new selected results, including variants of Smith–Minkowski–Siegel mass formula and Drin-
feld shtukas, will be presented in the second part of the paper

2 Elements of Ontology of O.N. Vvedenskii’s research

2.1 List of symbols that O.N. Vvedenskii have used

$K$ a local field
$\bar{K}$ algebraic closure of $K$
$C$ the completion of $\bar{K}$
$U_K$ group of units of $K$
$\mathfrak{D}_K$ ring of integers of $K$
t (sometimes $T$) prime element of $K$
k residue field of $K$
k the algebraic closure of $k$
$K\bar{k}$ compositum of fields or composite field (if exists),
p > 3 the characteristic of the residue field
$G_a$ additive group of the residue field $k$
$G_k$ multiplicative group of the residue field $k$
$A$ an Abelian variety
$A_K$ an Abelian variety defined over $K$
$\bar{A}$ the Picard variety of the variety $A$
$A'$ reduction of abelian variety mod $t$
$A_K^0$ subgroup of points of $A_K$ which reduced to non singular points of $A'$
$\pi_0(A_K) = A_K/A_K^0$ zero dimensional homotopy group
$\pi_1(A_K)$ fundamental group of the pro algebraic group of $A_K$
$\Gamma_K$ the kernel of the epimorphism of the reduction $A_K \to A'$
$\Gamma_K^1 \supset \Gamma_K^2 \supset \ldots, \bigcap \Gamma_K^n = 0$ filtration on $\Gamma_K$
$L$ a finite normal extension of $K$
$\hat{\mathbb{Z}}$ profinite completion of the ring of integers
$\mathfrak{g}$ the Galois group of the extension $L/K$
$\mathfrak{S}_K$ the Galois group of the maximal abelian extension of the local field $K$
$\mathfrak{S}_K$ the Galois group of the algebraic closure of $K$
$H^1(K,A)$ the group of principal homogeneous spaces over $A$, $K$ the quasi local field.
$H^0(\mathfrak{g},L)$ the zero cohomology group of the group $\mathfrak{g}$ with coefficients in $L$, modified by Tate
$H^1(\mathfrak{G}, L)$ the first cohomology group of the group $\mathfrak{G}$ with coefficients in $L$, modified by Tate

**Remark.** In papers by O. Vvedenskii these symbols can have and another meaning. In the latter case, the meaning of the symbol is specified.

### 2.2 Concepts and some definitions

Finite groups and finite group schemes
Pontryagin duality
Cartier duality
Algebraic, quasi algebraic and pro algebraic groups
Elliptic curves
Neron model
Hasse invariant
Finite, local, quasi local, quasi finite, global and quasi global fields
Principal homogeneous spaces and abelian varieties
Duality in elliptic curves over a local field
On the Galois cohomology of elliptic curves defined over a local field
Abelian $l$-adic representations and elliptic curves (by Serre)
Local class field theory
Quasi local class field theory
Artin effect in elliptic curves and in abelian varieties.
Abelian varieties and formal groups
$\mathbb{G}_m(R)$ - multiplicative group over a commutative ring $R$.
$S$ or $\mathcal{S}$ - complete defining set of the group $G$
$\lim\ inv G/H$ - inverse, or projective limit of groups $G/H$ ($H \in S$).
Etale (étale) topology
Etale (étale) sheaf

A local field $K$, i.e., a discretely normed field complete with respect to the topology induced by the norm. Below we mainly will consider non archimedean local fields with finite residue field $k$ and with normalized discrete valuation $\nu$ that is the homomorphism $\nu : K^* \rightarrow \mathbb{Z}$ is surjective.
Denote by $\#S$ the number of elements of a finite set $S$. Put $q = p^n = \#k$.
There are
a) non archimedean local fields $K$ of characteristic 0: these are finite exten-
sions of $p$-adic fields $\mathbb{Q}_p$; if $[K : \mathbb{Q}_p] = n$ then $n = f \cdot e$ where $f$ is the residue degree and $e$ is the ramification index $\nu(p)$; and

b) the equal characteristic case, when $\text{char } K = p > 0$ and $K$ is isomorphic to a field $k((T))$ of formal power series, where $T$ is a uniformization parameter.

**Remark.** Besides with this definition of the local field O.N. Vvedenskii (and other researchers) subdivide non archimedean local fields on two classes: non archimedean local fields, if the residue field is finite, and quasi local fields when the residue field is algebraically closed.

O.N. Vvedenskii also uses and investigates in the framework of his research common local fields - a complete discrete valued fields with a quasi-finite residue field of positive characteristic. Recall that a field $k$ is called quasi-finite if it is perfect and if $\text{Gal } k \cong \hat{\mathbb{Z}}$ where $\text{Gal } k$ is the Galois group of the algebraic closure $k_c$ over $k$ and $\hat{\mathbb{Z}}$ is the completion of the additive group of the rational integers.

The Hasse or Hasse–Witt invariant $H$ is the rank of the Hasse–Witt matrix of a non-singular algebraic curve over a finite field. In the case of elliptic curves it is equal 0, $(H = 0)$ if the elliptic curve is super singular and $H = 1$ if the elliptic curve is ordinary.

According to Weil [1], the principal homogeneous space over $A$ is the algebraic variety $V$ on which $A$ acts as a group of regular mappings, and the following conditions are satisfied:

1) For any $u, v \in V$ equation

$$au = v$$

has a unique solution $a \in A$.

2) Mapping that associates a pair $u$ and $v$ solution $a$ equation (1), is a regular mapping of $V \times V$ into $A$.

Let $G$ be a group acting on a set $X$. The action is said to be simply transitive if it is transitive and for all $x, y \in X$ there is a unique $g \in G$ such that $g \cdot x = y$.

Let $G$ a smooth algebraic group. A $G$-torsor or a principal $G$-bundle $P$ over a scheme $X$ is a scheme with an action of $G$ that is locally trivial in the given Grothendieck topology.
G-torsor as a principal homogeneous space: a G-torsor $P$ on a scheme $X$ is a principal homogeneous space for the group scheme $G_X = X \times G$ (i.e., $G_X$ acts simply transitively on $P$.)

2.2.1 Divisor equivalences

Linear equivalence of divisors ([23], p.57) (connected with Picard group of algebraic variety $X$). Two divisors $D$ and $D'$ are said to be linearly equivalent, written $D \sim D'$, if $D - D'$ is a principal divisor.

Algebraic equivalence of divisors ([23], p. 140) $D \sim_{alg} D'$, or $D \equiv D'$ (although $\equiv$ is used more often for numerical equivalence of divisors).

Numerical equivalence of divisors ([23], p.364) $\equiv$

2.3 Problems and conjectures

Vvedenskii’s works [10, 11, 12, 14, 15, 16, 17, 18, 19] are related to the study of the following problems and hypotheses:

Duality in elliptic curves over a local field

Galois cohomology of elliptic curves defined over a local field

Elliptic class field theory.

In a number of works by J. Tate, I. R. Shafarevich, J. Cassels and other authors, it turned out that for elliptic curves (Abelian varieties of arbitrary dimension) must take place some analog of the classical class field theory of the multiplicative group, the core of which is the duality between the group of principal homogeneous spaces over an elliptic curve (an Abelian variety) and some "arithmetic" object associated with this curve (variety).

"Universal norms" of formal groups defined over the ring of a local field

Duality in elliptic curves over a quasilocal field

Pairings in elliptic curves over global fields
The Artin effect in elliptic curves.

Let \( A \) be an Abelian variety over a quasi-global field \( K \) (i.e. over the field of algebraic functions of one variable with an algebraically closed field of constants \( k \)). Let \( \text{char } K = \text{char } k = p > 0 \). Let \( st \) be the well-known Shafarevich-Tate group corresponding to \( A \) considered over \( K \). Further, for an Abelian group \( X \) and a prime number \( q \), denote \( X_q = \text{Ker}(X \rightarrow X) \). Similar notation is then retained for the cases when \( X \) is a commutative group scheme or an Abelian sheaf on some Grothendieck topology. From the works of I. R. Shafarevich, A. P. Ogg and A. Grothendieck it is known that \( st_q \) is a finite group for all prime \( q \neq p \). M. Artin obtained the result that the group \( st_p \) can be infinite. Vvedenskii called this result the Artin effect.

2.3.1 Shafarevich conjecture

\[
H^1(k, A) \simeq \text{Hom}_{ct}(\pi_1(A_k), \mathbb{Q}/\mathbb{Z})
\]

In some cases conjecture proved by Vvedenskii (the cohomologies on the right are taken over continuous cochains).

3 Groups and group scheme

3.1 Elements of the theory of algebraic groups and group schemes.

Let \( R \) be a commutative ring with identity. It is known what the affine scheme \( \text{Spec } R \) is [8]. We recall here, and briefly explain, following [8, 23], some concepts related to the class of varieties that are generated by the reduced separated smooth schemes \((X, \mathcal{O}_X)\) of finite type over an algebraically closed field. The important notion of separable scheme is defined through the concept of the product of schemas and their clousernes. In turn, the product of schemes is defined as the product of objects in the category of schemes, but in terms of morphisms of schemes over a basic scheme \( S \) (for example, if \( S \) is an algebraically closed field) as a fiber product of these morphisms. A morphism of schemes \( \varphi : X \rightarrow Y \) is called a closed embedding if every point \( x \in X \) has such an affine neighborhood \( U \) such that the scheme \( \varphi^{-1}(U) \) is affine and the homomorphism \( \varphi^* : \mathcal{O}_X \rightarrow \mathcal{O}_Y \) epimorphic. In the category
of schemes over $S$, there is a morphism $(1, 1) : X \to X \times_S X$, which is called a diagonal. A scheme $X$ is called closed if the morphism of its diagonal is a closed embedding, and a scheme over a ring $R$ if the morphism of schemes $R \to \text{Spec} R$ is given. A finite group scheme, or a finite group of order $m$ over $R$, is a group scheme locally free of rank $m$ over $R$. Such a group scheme $G$ is defined by a sheaf of locally free algebras $A$ of rank $m$ over $R$. In the works Serre’s quasi-algebraic and pro-algebraic groups are constructed and studied. In the definition of quasi-algebraic and pro-algebraic groups according to Serre, the concept of structure is used. The structure $St$ of a group scheme, or group structure, is given by homomorphisms 1) $\mu : X \times_R X \to X$ (group law), 2) $p : X \to X$, $p(x) = e$ (unit), 3) $i : X \to X$ (taking the inverse), satisfying the axioms: a) $\mu \ast (\mu \times 1) = \mu \ast (1 \times \mu)$ (associativity), b) $\mu \ast (1, i) = (i, 1) \ast \mu$ c) $\mu \ast (1, p) = (p, 1) \ast \mu = 1$, specifying, respectively, the group law, taking the inverse element and unity which satisfy the known axioms. If in an algebraic group $G$, or, more generally, in a group scheme, the group structure $St$ is fixed, then we reveal this through $G_{St}$.

### 3.2 Quasi-algebraic and pro-algebraic groups by Serre.

In what follows, all groups, unless otherwise stated, are assumed to be commutative. In this section, the letter $K$ denotes a perfect field (algebraically closed field), the letter $p$ denotes its characteristic exponent, that is, $p = 1$ if the characteristic $K$ is equal to zero, characteristic exponent $= p$ if characteristic $K$ is equal to $p > 0$. All algebraic varieties are considered defined over $K$. As is known, in the category of algebraic groups over $K$ there are bijective morphisms that are not morphisms in the sense of algebraic groups. In other words, the category of algebraic groups is additive but not abelian. Let us recall a well-known example. Let $K = k$ be an algebraically closed field of characteristic $p > 0$. The topological spaces of algebraic groups $X$ and $Y$, which we denote by the same letters, are given by the condition $X = Y = k$. Let the group operation of each of the groups be additive and given by the mapping $\mu(x, y) = x + y, x, y \in k$. Consider the homomorphism $\varphi : X \to Y$ of algebraic groups $X$ and $Y$ given by the condition $\varphi(x) = x^p$. As a point mapping it is one-to-one and as a mapping of abstract groups it is an isomorphism, but as a regular mapping of manifolds it is not an isomorphism, since the corresponding ring homomorphism $\varphi^* : k[Y] \to k[X], \varphi^*(T) = T^p, k[Y] = k[X] = k[T], \varphi^*(k[Y]) = k[T^p] \neq k[T]$ not is an isomorphism.
3.2.1 Quasi-algebraic groups.

Let \( V \) be an algebraic variety and \( \mathcal{O} \) be the sheaf of functions on \( V \). If \( q = p^n, n \in \mathbb{Z} \) denote by \( \mathcal{O}^q \) the sheaf whose sections over open sets \( U \subset V \) are the \( q \)-th powers of the sections of the sheaf \( \mathcal{O} \) over \( U \).

The concept of a quasi-algebraic group \([5]\) combines into one class algebraic groups between which there are bijections that may not be isomorphisms of algebraic groups. Let \( G_{St} \) be an algebraic group and \( \mathcal{O}_{St} \) be a sheaf of functions on \( G_{St} \). If \( q = p^n, n \in \mathbb{Z} \), then we denote by \( \mathcal{O}^q_{St} \) the sheaf whose sections over open sets \( U \subset G \) are the \( q \)-th powers of the sections of the sheaf \( \mathcal{O}_{St} \) over \( U \). Follow to Serre we also put \( \mathcal{O}^{p^{-\infty}} = \bigcup_{n \in \mathbb{Z}} \mathcal{O}^{pn} \). If \( q > 1 \), then the variety \( G^q \) corresponding to the sheaf \( \mathcal{O}^q \) is an algebraic group.

**Proposition 1.** Let \( f : G_1 \to G_2 \) be a bijective morphism of algebraic groups. Then there is a positive power \( q \) of \( p \), and morphism \( g : G_2 \to G_1^q \) such that their composition:

\[
G_1 \xrightarrow{f} G_2 \xrightarrow{g} G_1^q
\]

gives the identity mapping \( i : G_1 \to G_1^q \).

Let \( G \) be a group. If \( St \) is an algebraic group structure on \( G \) compatible with the group structure, then \( G_{St} \) denotes the corresponding algebraic group, and \( T_{St} \) and \( \mathcal{O}_{St} \) respectively denote the topology and the sheaf of rings.

**Proposition 2.** Let \( St_1 \) and \( St_2 \) be two algebraic group structures on \( G \) compatible with the group structure. The following conditions are equivalent:

(i) There is structure \( St_3 \) such that the identity mappings \( G_{St_3} \to G_{St_1} \) and \( G_{St_3} \to G_{St_2} \) are morphisms.

(ii) There is structure \( St_4 \) such that the identity mapping \( G_{St_4} \to G_{St_1} \) and \( G_{St_4} \to G_{St_2} \) are morphisms.

(iii) There exists for an arbitrary positive power \( q \) of the number \( p \) the identity mapping \( G_{St_1} \to G_{St_2} \) which is a morphism of algebraic groups.

(iv) \( T_{St_1} = T_{St_2} \) and \( \mathcal{O}^{p^{-\infty}}_{St_1} = \mathcal{O}^{p^{-\infty}}_{St_2} \)

**Definition 1** Let \( G \) be a group and let \( St_1 \) and \( St_2 \) be two structures of algebraic group on \( G \) compatible with the group structure. Following Serre we say that \( St_1 \) and \( St_2 \) are equivalent if they satisfy Proposition \([2]\).
Note that, in characteristic zero, equivalence reduces to equality, since $\mathcal{O}^{p_{-\infty}} = \mathcal{O}$.

**Definition 2** A group $G$ is called a quasi-algebraic group if it is endowed with the class of equivalence (in the sense of definition 1) algebraic group structures agreed with its group structure.

**Proposition 3**. Let $G$ and $G_1$ be quasi-algebraic groups and let $f : G \to G_1$ be a homomorphism. The following conditions are equivalent:

a) There are algebraic structures $S$ and $S_1$ on $G$ and $G_1$ respectively, which are compatible with their quasi-algebraic structures and such that $f : G \to G_1$ is a morphism of algebraic groups.

b) The mapping $f$ is continuous and if $\varphi$ is a section of $\mathcal{O}^{p_{-\infty}}_{G_1}$ on an open $U_1$, $\varphi \circ f$ is a section of $\mathcal{O}^{p_{-\infty}}_G$ on the open $f^{-1}(U_1)$.

c) The graph of $f$ is a closed subgroup in $G \times G_1$.

**Definition 3** Let $G$ and $G_1$ be two quasi-algebraic groups. A morphism from $G$ to $G_1$ is any homomorphism $f : G \to G_1$ satisfying the equivalent conditions of Proposition 3.

**Proposition 4**. Let $f : G \to G_1$ be a morphism of quasi-algebraic groups such that $N$ is the kernel of $f$ and $I$ is its image. Then $N$ and $I$ are closed respectively in $G$ and in $G_1$, and $f$ determines after passing to the factors an isomorphism of $G/N$ with $I$.

Follow to Serre denote by $Z$ the category formed by quasi-algebraic groups and their morphisms.

**Proposition 5**. The category $Z$ is Abelian and the notion of a subobject coincides with the notion of a closed subgroup.

**Remark 1** Recall in connection with Proposition 4 axioms that turn an additive category to abelian:

AB 1) Every morphism has a kernel and a cokernel.

AB 2) For every morphism $f$, the canonical morphism from $\text{coim} f$ to $\text{im} f$ is an isomorphism.

Recall also that if $\text{char } k = 0$ then the category $Z$ is identical with the category $A$ of algebraic groups.
Proposition 6. Every object of the category $\mathcal{Z}$ is artinian.

Example 1. Let $AR$ be an Artinian local ring with algebraically closed field $k$ of characteristic $p > 0$. By M. Atiyah, I. Macdonald, a Noetherian local ring with maximal ideal $m$ is an Artin local ring if $m^n = 0$ for some $n$.

Let $K$ be a quasi local field, i.e., a discretely normed field complete with respect to the topology induced by the norm and with the algebraically closed residue field $k$, $A$ its ring of valuation, and let $m$ its maximal ideal. Let $U = A - m$ be the group of units of $A$ and let $U^n = 1 + m^n$. The ring $A/m^n$ is Artinian, whose group of units is identified with the quotient $U/U^{(n)}$.

Let $W_n(k)$ be the ring of Witt vectors of length $n$ over $k$; if $n$ is large enough, we can lift $W_n(k) \to k$ into a homomorphism $W_n(k) \to AR$ that makes $AR$ a $W_n(k)$-module of finite type. As a module, $AR$ is isomorphic to a direct sum of modules $W_{n_i}(k)$, $n_i \leq n$.

As each of the $W_{n_i}(k)$ has a natural structure of algebraic variety on $k$, we can transport this structure to $AR$, and the structure thus obtained does not depend on the choice of the isomorphism.

By applying the above, we therefore obtain an algebraic group structure on $U/U^{(n)}$, and $U$ is the projective limit of the groups $U/U^{(n)}$.

The quotient $U/U^{(1)}$ is identified with the multiplicative group $G_m$. For $n \geq 1$, the quotient $U^{(n)}/U^{(n+1)}$ is identified with the additive group $G_a$.

Recall result about the structure of quasi-algebraic groups.

Theorem 1. Every quasi-algebraic group has a composition series whose successive quotients are isomorphic, either to the group $G_a$ or to the group $G_m$, or to an abelian variety, or to a finite group.

Remark 2. The definition of a quasi-algebraic group can also be given in terms of group schemes. Let us briefly recall this construction. We extend the category of algebraic groups over $K$ to the category of group schemes over $K$. Since here in what follows we consider only commutative groups, we restrict ourselves to the category of commutative group schemes $\mathcal{CG}_K$ over $K$.

Let $H$ and $G$ lie in $\mathcal{CG}_K$ and $\varphi: H \to G$ be a purely non-separable isogeny from $H$ to $G$. Let $H$ and $G$ be equivalent if there exists a group scheme $F \in \mathcal{CG}_K$ and purely non-separable isogenies $\psi: F \to H$, $\tau: F \to G$. Then a quasi-algebraic group will be a class of equivalent (in the above sense) group schemes.
3.2.2 Pro-algebraic groups.

Let \( G \) be a group with the neutral element \( o \), and let \( X \) be a homogeneous space on \( G \). We will say that \( X \) is principal if the isotropy subgroup of a point \( x \in X \) is reduced to \( o \); the choice of \( x \) then defines a bijection from \( G \) onto \( X \).

**Definition 4** Follow to Serre and others we call proalgebraic group a group \( G \) endowed with a non-empty family \( S \) of subgroups and, for all \( H \in S \), with a quasi-algebraic group structure on \( G/H \), these data satisfying the following axioms:

\((P_1)\) \( H, H_1 \in S \Rightarrow H \cap H_1 \in S \).

\((P_2)\) If \( H \in S \), the subgroups \( H_1 \) containing \( H \) which belong to \( S \) are the resiproced images of the closed subgroups of \( G/H \).

\((P_3)\) If \( H, H_1 \in S \) and if \( H \subset H_1 \), the homomorphism \( G/H \to G/H_1 \) is the morphism of quasi algebraic groups.

\((P_4)\) The natural map \( G \to \lim \text{ inv } G/H \) (inverse, or projective limit) is a bijection of \( G \) onto the projective limit of groups \( G/H \) (\( H \in S \)).

Denote by \( \mathcal{P}G_K \) the category of proalgebraic groups over \( K \).

**Definition 5** By Serre and others a group \( G \in \mathcal{P}G_K \) is the dimension zero if for any definitive subgroup \( H \), the quotient \( G/H \) is a finite group.

**Example 2** For any prime \( p \) the group \( \mathbb{Z}_p \) has the dimension zero.

4 Fundamental groups of schemes

4.1 Homotopy groups.

In this subsection, we follow to Serre \[5\] and to Grothendieck et al. \[7\]. Let \( G \) be a quasi-algebraic group. Denote by \( G^0 \) the connected component of the unit of \( G \). Further \( G^0 \) is called the connected component of the group \( G \). Suppous that \( G \) is the proalgebraic group with complete defining set \( S \); for \( H \in S \) connected component \( (G/H)^0 \) of the factor group \( G/H \) is closed subgroup in \( G/H \), and, if \( H_1 \subset H \) then the image \( G/H_1 \) in \( G/H \) is \( (G/H)^0 \). In view of this, one can put \( G/G^0 = \lim \text{ inv } (G/H)/(G/H)^0 \).
**Definition 6** Factor group $G/G^0$ denoted by $\pi_0(G)$ and is called the 0th homotopy group of the proalgebraic group $G$.

**Remark 3** Factorization operation

$$\pi_0(G) = G/G^0$$

defines a functor

$$\pi_0 : \mathcal{P}G_K \to \mathcal{P}G^0_K$$

from category $\mathcal{P}G_K$ to category $\mathcal{P}G^0_K$ proalgebraic groups of dimension zero.

**Definition 7** The left derived functors of the functor $\pi_0$ are called the $i$th homotopy groups of the proalgebraic group $G$ and denoted by $\pi_i(G)$.

**Remark 4** The presence of a sufficient number of projective objects in the category $\mathcal{P}G_K$ makes Definition 7 correct.

**Definition 8** Let $G \in \mathcal{P}G_K$. The first homotopy group $\pi_1(G)$ of the group $G$ is called the fundamental group of the group $G$.

**Definition 9** A group $G \in \mathcal{P}G_K$ is called connected if $G = G_0$. Group $G \in \mathcal{P}G_K$ is called singly connected if $\pi_1(G) = 0$.

**Proposition-Definition 1** Let $G \in \mathcal{P}G_K$. There is the connected and singly connected proalgebraic group $\overline{G}$ and a morphism $u : \overline{G} \to G$ such that the kernel and cokernel of $u$ are proalgebraic groups of dimension zero. The pair $(\overline{G}, u)$ is unique, up to isomorphism. The pair $(\overline{G}, u)$ is called the universal covering group of the group $G$.

Recall the example of the first étale homotopy group (the étale fundamental group) $\pi_1(G)$ of the group $G$.

**Example 3** Let $\mathbb{A}^1(F)$ be the affine line over an algebraically closed field $F$ of characteristic zero and $m$ be the geometric point of $\mathbb{A}^1(F)$. Then

$$\pi_1(\mathbb{A}^1 \setminus 0, m) = \lim \ inv \ \mu_n(F) \simeq \hat{\mathbb{Z}}.$$
4.2 Fundamental groups of fields

In his works, which relate to the arithmetic of number fields and rings, O. N. Vvedenskii used and developed the results of S. Lichtenbaum [24]. In recent works, C. Lichtenbaum [25, 26] defined the Weil étale topology and the Weil étale site and studied the fundamental groups associated with their. Let us recall Lichtenbaum’s considerations. For the function field $K$ of a curve over a finite field $k$ let $G_K$ be the Galois group of $\overline{K}$ over $K$. The group $\text{Gal}(K\overline{K}/K)$ is isomorphic to $\hat{\mathbb{Z}}$. There is a natural surjection $\pi : G_K \to \text{Gal}(K\overline{K}/K)$. The Weil group $W_K \simeq \pi^{-1}(\mathbb{Z})$ where $\mathbb{Z}$ is the subgroup of $\text{Gal}(K\overline{K}/K)$. Weil’s site and topos are defined in a natural way.

Example 4 Let $X$ be an algebraic variety over $\mathbb{F}_q$ and $\overline{X} = X \times_{\mathbb{F}_q} \mathbb{F}_q$. Let $\overline{g} = \text{Gal}(\overline{\mathbb{F}_q} : \mathbb{F}_q)$ be the corresponding Galois group. An étale sheaf on $X$ corresponds to a sheaf on $\overline{X}$ together with a continues action of $\overline{g}$.

In the cited paper [30], Morin defines the fundamental group underlying the étale Weil cohomology of a number ring. By Molin, corresponding Weil étale topos is determined as a refinement of the Weil’s site according to Lichtenbaum. The author of [30] demonstrates the naturalness of his definition in the case of a smooth projective curve and further defines the étale fundamental Weil group of an open subscheme of the spectrum of a number ring. This fundamental group is a projective system of locally compact topological groups that represents the first cohomology with coefficients in a locally compact Abelian group. This result is used in [30] to calculate cohomology groups of small degrees and to check that the Weil étale topos satisfies the expected properties of the Lichtenbaum topos. Let $Y$ be an open subscheme of a smooth projective curve over a finite field $k$, and let $S_{\text{et}}(W_k, \overline{Y})$ be the topos $W_k$-equivariant étale sheaves on the projective curve $\overline{Y} = Y \otimes_k \overline{k}$.

Theorem M1([30]) There is an equivalence $Y_{\text{et}}^{sm} \simeq S_{\text{et}}(W_k, \overline{Y})$ where $Y_{\text{et}}^{sm}$ is the (small) Weil-étale topos defined in this paper.
For a connected étale $\overline{X}$-scheme $\overline{U}$ author defines its Weil-étale topos as the slice topos $\mathcal{U}_W := \mathcal{X}_W/\gamma^*\mathcal{U}$. Let $K$ be the number field corresponding to the generic point of $\overline{U}$, and let $q:\text{Spec}(K) \to \overline{U}$ be a geometric point. Similarly to the definition of the étale fundamental group as a (strict) projective system of finite quotients of the Galois group $G_K$, author defines the analogous (strict) projective system $\mathcal{W}(\mathcal{U}, q)$ of locally compact quotients of the Weil group $W_K$.

Theorem M2(30) The Weil-étale topos $\mathcal{U}_W$ is connected and locally connected over the topos $T$ of locally compact spaces. The geometric point $q\overline{\sigma}$ defines a $T$-valued point $p\overline{\sigma}$ of the topos $\mathcal{U}_W$, and we have an isomorphism $\pi_1(\mathcal{U}_W, p\overline{\sigma}) \simeq \mathcal{W}(\mathcal{U}, q\overline{\sigma})$ of topological pro-groups.

The fundamental group and the fundamental scheme can be defined in terms of the Tannakian category approach (see [31, 32] and references therein).

5 On Local Fields and Local Class Field Theory

5.1 On local fields

Let $L$ be a finite extension of the local field $K$, $l, k$— their residue fields, $p = \text{char } k$, and $e_{L/K}$— ramification index of $L$ over $K$.

An extension $L/K$ is called unramified if

a) $e_{L/K} = 1$;

b) the extension $l/k$ is separable.

An extension $e_{L/K}$ is said to be weakly ramified if

a) $p \nmid e_{L/K}$;

b) the extension $l/k$ is separable.

An extension $L/K$ is said to be wildly ramified if $e_{L/K} = [L : K] = (\text{char } k)^s, s \geq 1$.

Further, by $\text{Tr}_{L/K}$ and $\text{Norm}_{L/K}$ we denote, respectively, the trace and the norm $L/K$ extensions, omitting indices when it is clear which extension is in question.

Denote by $K_{nr}$ the maximal unramified extension of the field $K$ (in a fixed algebraic closure of the field $K$) with the residue field $k_s$, which is the algebraic closure of the field $k$. 2

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Recall that a local field with an algebraically closed residue field is called quasilocal.

**Lemma 1** If local field $K$ contains a primitive root $\xi_p$ $p$-th degree of unity, then $\nu_K(\xi_p - 1) = e_1 = \frac{e}{p-1}$ (i.e.) $e_1$ is an integer. Here $e = \nu_K(p), p = \text{char } k$.

**Proof.** If $\xi_p$ is a primitive $p$- root of unity, then $\xi_p^p - 1 = 0$ and $\xi_p$ is a root of a polynomial $x^{(p-1)} + x^{(p-2)} + \ldots + x + 1$ irreducible over $K$. Then $\xi_p - 1$ is the root of the polynomial $(x+1)^{(p-1)} + (x+1)^{(p-2)} + \ldots + (x+1) + 1 = x^{(p-1)} + p(x+\ldots)+p$. The value of the exponent $p$ at the root of such a polynomial is $\frac{e}{p-1}$, i.e. $\nu_K(\xi_p - 1) = e = \frac{e}{p-1}$. The lemma is proven.

**Corollary.** Under the conditions of Lemma 1, $e = \nu_K(p)$ is divisible by $p$.

**Ramification groups.** Let $L/K$ be a finite Galois extension with Galois group $G = \text{Gal}(L/K)$. Let $\mathcal{O}_K$ be the ring of integers in the field $K$.

We define ramification groups $\mathfrak{G}_i (i = -1, 0, 1, \ldots)$ by $\mathfrak{G}_i = \{\sigma \in G | \nu_L(\sigma a - a) \geq i + 1 \text{ for all } a \in \mathcal{O}_L\}$.

It is easy to check that the groups $\mathfrak{G}_i$ are normal subgroups of the group $\mathfrak{G}_0, \mathfrak{G}_{i+1} \subset \mathfrak{G}_i, \mathfrak{G}_{-1} = G$ and for sufficiently large $m$ $\mathfrak{G}_m = 1$.

Let us introduce a lower and an upper (Herbrand) numbering of ramification groups. Let $x$ denote a real variable that is $\geq -1$. Let’s put $\mathfrak{G}_x = \mathfrak{G}_i$ where $i$ is the smallest integer that is $\geq x$. We introduce the notation $g_i := (\text{the order of the group } \mathfrak{G}_i)$. Let $x$ be a real number and $m$ the integer part of the number $x$.

Define the function

$$\varphi(x) = \begin{cases} (x, if \ -1 \leq x \leq 0) \\ \frac{1}{90}(g_1 + \ldots + g_m + (x-m)g_{m+1}), if \ x \geq 0. \end{cases}$$

The function $\varphi(x)$ is continuous, strictly increasing, and therefore has an inverse function $\psi(y)$, which is also continuous and strictly increasing ($-1 \leq y$). The new, ‘top’ numbering of the ramification groups is now given as follows:

$\mathfrak{G}_{\varphi(x)} = \mathfrak{G}_{\psi(y)}$, where $y = \varphi(x)$ and $x = \psi(y)$.

**Different.** Denote by $\pi_K$ the uniformizing element of the field $K$, that is, such element $\pi_K$ such that $\nu_K(\pi_K) = 1$. 
Denote by $m_K = \pi_K \cdot \mathcal{O}_K$ the maximal ideal of the ring $\mathcal{O}_K$. Let $L/K$ be a wildly ramified extension of prime degree $p$. Let us define the different $\mathcal{D}$ of the extension $L/K$ by the formula

$$\mathcal{D} = (f'(\pi_L)), \quad (2)$$

where $f(x)$ is the minimal polynomial for $\pi_L$ over $K$.

Note that

$$\mathcal{D} \subset \mathfrak{d} \cdot \mathcal{D}^{-1} \iff Tr(\mathcal{D}) \subset \mathfrak{d}, \quad (3)$$

where $\mathfrak{d}$ is a fractional ideal in $\mathcal{O}_K$ and $\mathcal{D}$ is a fractional ideal in $\mathcal{O}_L$.

5.2 On local class field theory

Let us illustrate the elements of the local class field theory and its application on the example of the invariance of the Hodge-Tate decompositions according to Serre [6]. Let $K$ be a local field of characteristic zero with perfect residue field $k$ of characteristic $p > 0$. For the completion $\mathbb{C}$ of $\overline{K}$ the Galois group $\mathfrak{G}_K$ acts continuously on $\mathbb{C}$.

In the locally compact case, when $K$ is the finite extension of $\mathbb{Q}_p$, by the local class field theory it is possible to identify $\mathfrak{G}_K$ with completion of $\hat{K}^*$ of $K^*$ and the inertia subgroup of $\mathfrak{G}_K$ with the group of units of $K$.

6 Vvedenskii's research

6.1 Duality in elliptic curves over a local field

When $k$ is finite, it is known from Tate results [2] that the group of principal homogeneous spaces over $A$ is dual to the group $\overline{A}$ the Picard variety of the variety $A$ except for the $p$-component, where $p$ is the characteristic of $K$.

When $k$ is algebraically closed, it was shown by Shafarevich [3] and independently by Ogg [4] that the group of principal homogeneous spaces over $A$ is dual to the group $\pi_1(\overline{A}_K)$, i.e., the fundamental group of the pro-algebraic group $\overline{A}_K$ in the sense by Serre [5] except for the $p$-component, where $p$ is the characteristic of $k$.

It was conjectured that the duality holds also for the $p$-component. In the present work [10] this conjecture is proved for the special case when $A$ is an elliptic curve whose reduction has a Hasse invariant other than 0.
Vvedenskii remarks that he do not find explicitly a natural pairing between $\pi_1(\mathbb{A}_K)$ and the group of principal homogeneous spaces, although the proof of the duality, which is done in a purely computational way, permits him to deduce that one exists.

References

[1] Weil, A. On Algebraic Groups and Homogeneous Spaces, American Journal of Mathematics, Jul., Vol. 77, No. 3 (Jul., 1955), pp. 493-512, 1955.

[2] Tate J. Duality theorems in Galois cohomology over number fields, Proc. Internat. Congr. Mathematicians (Stockholm, 1962), pp. 288–295, Mittag-Leffler, Djursholm, 1963.

[3] Shafarevich I.R., Principal homogeneous spaces defined over the field of functions, Sochineniya, T. 3, ch. 2. M.: Fizmatlit, 637 p. 1996.

[4] Ogg A. P., Cohomology of abelian varieties over function fields, Ann.of Math., 76, № 2, 18–212, 1962.

[5] Serre J.P. Groupes proalgebriques, Publications mathematiques IHES. no. 7, 65 p., 1960.

[6] Serre J.P. Abelian $l$-Adic Representations and Elliptic Curves, Addison-Wesley Publishing Company, 1988.

[7] Grothendieck A., Artin M., VerdierJ.L., Théorie des Topos et cohomologie étale des schémas (SGA4), Lecture Notes in Math. Berlin-N.Y.: Springer-Verlag, Vol. 269, 270, 305, 1972.

[8] Shafarevich I. R. Fundamentals of algebraic geometry. In 2 vols. M.: Nauka, 1988.

[9] Cossec F., Dolgachev I., Liedtke C. With Appendix by S. Kondo, Enriques Surfaces I, www.math.lsa.umich.edu/idolga/EnriquesOne.pdf, 2021.
[10] Vvedenskii, O.N. Duality in elliptic curves over a local field. I., Izv. Akad. Nauk SSSR Ser. Mat., 28, 1091–1112, 1964.

[11] Vvedenskii, O.N., Duality in elliptic curves over a local field. II, Izv. Akad. Nauk SSSR Ser. Mat., 30, Issue 4, 891–922, 1966.

[12] Vvedenskii, O.N., On the Galois cohomology of elliptic curves defined over a local field, Mat. Sb., vol, 83(125), no. 3(11), 474–484, 1970.

[13] Vvedenskii, O.N., On local “class fields” of elliptic curves, Izv. Akad. Nauk SSSR Ser. Mat., 37, issue 1, 20–88, 1973.

[14] Vvedenskii, O.N., On the “universal norms” of formal groups defined over the ring of a local field, Izv. Akad. Nauk SSSR Ser. Mat., 37, Issue 4, 737–751, 1973.

[15] Vvedenskii, O.N., On duality in elliptic curves over a quasilocal field, Dokl. AN SSSR, 219:6, 1291–1293, 1974.

[16] Vvedenskii, O.N., On pairings in elliptic curves over global fields, Izv. Akad. Nauk SSSR Ser. Mat., 12:2, 225–246, 1976.

[17] Vvedenskii, O.N., The Artin effect in elliptic curves. I, Izv. Math., 43:5 (1979), 1042–1053, 1979.

[18] Vvedenskii, O.N., Frequency of occurrence of the Artin–Milne effect in elliptic curves, Dokl. AN SSSR, 245:4, 780–781.

[19] Vvedenskii, O.N., The Artin effect in Abelian varieties. II, Izv. Math., Volume 45:1, 23-46, 1981.

[20] Tate J. Duality theorems in Galois cohomology over number fields, International Congress, Stockholm, 1962.

[21] Cassels J. W. S, Arithmetic on curves of genus one. VII. The dual exact sequence, J. reine angew. Math., 216, 150—158, 1964.

[22] Greenberg Marvin J. Schemata Over Local Rings, Annals of Math., 73, no. 3, 624-648, 1961.

[23] Hartshorne R., Algebraic Geometry, Springer Science+Business Media, Inc., 1977.
[24] Lichtenbaum S., The Period-Index Problem for Elliptic Curves, Amer. J. of Math., vol. 90, no. 4, 1209-1223 1968.

[25] Lichtenbaum S., THE WEIL-ETALE TOPOLOGY, Preprint, Braun University, 1988.

[26] Lichtenbaum S. The Weil-etale topology for number rings, Ann. of Math., Vol. 170, № 2, P. 657-683. 2009.

[27] Lutz E., Sur l'équation \(y^2 = x^3 - Ax - B\) dans les corps \(p\)-adiques, Journ. fur Math., 177, 238-247, 1937.

[28] Milne, J., Etale Cohomology, Princeton Univ. Press, Princeton, 1980.

[29] Milne J.S., Arithmetic Duality Theorems, BookSurge, LLC, viii+339 p., 2006.

[30] Morin B., The Weil-\textasciitilde etale fundamental group of a number field . II, Sel. Math., New Ser. 17, no. 1. pp. 67—137, 2011.

[31] Nori M., On the representations of the fundamental group, Compos. Math. Vol. 33, no. 1, 29-41, 1976.

[32] Biswas I., J.P. dos Santos, Abelianization of the \(F\)-divided fundamental group scheme, Proc. Indian Acad. Sci., Math. Sci. Vol. 127, no. 2, 281—287, 2017.

[33] Glazunov N. M., On norm subgroups of one-dimensional formal groups defined over the ring of integers of a local field (in Ukraine), Dopovidi AN Ukr.SSR, Ser. A, 11, 965-968, 1973.

[34] Glazunov N. M. Remarks on \(n\)-dimensional commutative formal groups over the ring of integers in the field of \(p\)-adic numbers, Ukrainian Math. Journl., vol. XXV, no. 3, 352 - 354, 1973.

[35] Konovalov G. T., Multidimensional theorem of Shafarevich and Serre, Math. Notes, 13:4, 346-348, 1973.

[36] Konovalov G. T. Triviality of universal norms on formal groups over a local field, Math. Notes, 18:5, 1015-1018, 1975.
[37] Glazunov N. M. On Langlands program, global fields and shtukas, Chebyshevskii Sb., Volume 21, Issue 3, 68–83, 2020.

[38] Glazunov N.M., p-adic $L$-functions and p-adic multiple zeta values, Chebyshevskii Sb., Volume 20, Issue 1, 112–130, 2019.

[39] Glazunov N.M., Methods to Justifying of Arithmetic Hypotheses and Computer Algebra, Programmirovanie, N 3, P.2-8, 2006.

[40] Glazunov N.M., On moduli spaces, equidistribution, bounds and rational points of algebraic curves, Ukrainian Math. Journal, 53:9, 1407-1418, 2001.