A mathematical proof that the transition to a superconducting state is a second-order phase transition

Shuji Watanabe
Division of Mathematical Sciences
Graduate School of Engineering, Gunma University
4-2 Aramaki-machi, Maebashi 371-8510, Japan
Email: watanabe@fs.aramaki.gunma-u.ac.jp

Abstract

We deal with the gap function and the thermodynamical potential in the BCS-Bogoliubov theory of superconductivity, where the gap function is a function of the temperature $T$ only. We show that the squared gap function is of class $C^2$ on the closed interval $[0, T_c]$ and point out some more properties of the gap function. Here, $T_c$ stands for the transition temperature. On the basis of this study we then give, examining the thermodynamical potential, a mathematical proof that the transition to a superconducting state is a second-order phase transition. Furthermore, we obtain a new and more precise form of the gap in the specific heat at constant volume from a mathematical point of view.

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1 Introduction

Let $\varepsilon > 0$ be small enough and let us fix it unless otherwise stated. Let $k_B > 0$ and $\omega_D > 0$ stand for the Boltzmann constant and for the Debye frequency, respectively. We denote Planck’s constant by $h (> 0)$ and set $\hbar = h/(2\pi)$. Let $\mu > 0$ stand for the chemical potential. Let $N(\xi) \geq 0$ stand for the density of states per unit energy at the energy $\xi (-\mu \leq \xi < \infty)$ and let $N_0 = N(0) > 0$. Here, $N_0$ stands for the density of states per unit energy at the Fermi surface ($\xi = 0$). Let $U_0 > 0$ be a constant.

It is well known that superconductivity occurs at temperatures below the temperature $T_c > 0$ called the transition temperature. We now define it.

Definition 1.1. The transition temperature is the temperature $T_c > 0$ satisfying

$$\frac{1}{U_0 N_0} = \int_{\varepsilon}^{\infty} \frac{\hbar \omega_D/(2k_B T_c)}{\eta} \frac{\tanh \eta}{\eta} \, d\eta.$$
Generally speaking, the gap function is a function both of the temperature $T$ and of wave vector. In this paper we however regard the gap function as a function of the temperature $T$ only, and denoted it by $\Delta(T)$ ($\geq 0$). Such a situation is considered in the BCS-Bogoliubov theory [1, 3], and is accepted widely in condensed matter physics (see e.g. [6] (7.118), p. 250, [12] (11.45), p. 392). See also [9] and [10] for related material.

The gap function satisfies the following nonlinear integral equation called the gap equation (c.f. [1]):

$$1 = U_0 N_0 \int \frac{\hbar \omega_D}{2k_B T c \epsilon} \frac{1}{\sqrt{\epsilon^2 + f(T)}} \tanh \frac{\sqrt{\epsilon^2 + f(T)}}{2k_B T} d\epsilon.$$  \hspace{1cm} (1.1)

Here, for later convenience, the squared gap function is denoted by $f$, i.e., $f(T) = \Delta(T)^2$.

Remark 1.2. We introduce the cutoff $\epsilon$ in Definition (1.1) and in the gap equation (1.1). When $\epsilon = 0$, Definition (1.1) and the gap equation (1.1) reduce to those in the BCS-Bogoliubov theory [1, 3]. Furthermore, when $\epsilon = 0$, the thermodynamical potential $\Omega$ in Definition (1.4) below reduces to that in the BCS-Bogoliubov theory (see also (1.2) and (1.3) below). See e.g. Niwa [6, sec. 7.7.3, p. 255].

The gap equation (1.1) is a simplified one, and the gap equation with a more general potential is studied extensively. Odeh [7] and Billard and Fano [2] established the existence and uniqueness of the positive solution to the gap equation with a more general potential in the case $T = 0$. In the case $T \geq 0$, Vansevenant [8] and Yang [11] determined the transition temperature and showed that there is a unique positive solution to the gap equation with a more general potential. Recently Hainzl, Haenza, Seiringer and Solovej [4], and Hainzl and Seiringer [5] proved that the existence of a positive solution to the gap equation with a more general potential is equivalent to the existence of a negative eigenvalue of a certain linear operator to show the existence of a transition temperature.

Let $f(T)$ be as in (1.1) and set

$$\Omega_S(T) = \Omega_N(T) + \delta(T),$$

$$\Omega_N(T) = -2N_0 \int \frac{\hbar \omega_D}{2k_B T c \epsilon} \xi d\epsilon - 4N_0 k_B T \int \frac{\hbar \omega_D}{2k_B T c \epsilon} \ln \left(1 + e^{-\xi/(k_B T)}\right) d\epsilon + V(T), \hspace{1cm} T > 0,$$

$$\delta(T) = \frac{f(T)}{U_0} - 2N_0 \int \frac{\hbar \omega_D}{2k_B T c \epsilon} \left\{\sqrt{\xi^2 + f(T)} - \xi\right\} d\epsilon$$

$$-4N_0 k_B T \int \frac{\hbar \omega_D}{2k_B T c \epsilon} \ln \left(1 + e^{-\sqrt{\xi^2 + f(T)/(k_B T)}}\right) d\epsilon,$$ $$0 < T \leq T_c,$$

$$V(T) = 2 \int_{-\mu}^{-\hbar \omega_D} \xi N(\xi) d\xi - 2k_B T \int_{-\mu}^{-\hbar \omega_D} N(\xi) \ln \left(1 + e^{\xi/(k_B T)}\right) d\xi$$

$$-2k_B T \int_{-\mu}^{\infty} N(\xi) \ln \left(1 + e^{-\xi/(k_B T)}\right) d\xi, \hspace{1cm} T > 0.$$  \hspace{1cm} (1.4)

Remark 1.3. Since $N(\xi) = O(\sqrt{\xi})$ as $\xi \to \infty$, the integral on the right side of (1.4)
is well defined for $T > 0$.

**Definition 1.4.** Let $\Omega_S(T)$ and $\Omega_N(T)$ be as above. The thermodynamical potential $\Omega$ is defined by

$$\Omega(T) = \begin{cases} 
\Omega_S(T) & (0 < T \leq T_c), \\
\Omega_N(T) & (T > T_c). 
\end{cases}$$

**Remark 1.5.** Generally speaking, the thermodynamical potential $\Omega$ is a function of the temperature $T$, the chemical potential $\mu$ and the volume of our physical system. Fixing the values of $\mu$ and of the volume of our physical system, we deal with the dependence of $\Omega$ on the temperature $T$ only.

**Remark 1.6.** Hainzl, Hamza, Seiringer and Solovej [4], and Hainzl and Seiringer [5] studied the gap equation with a more general potential examining the thermodynamic pressure.

**Definition 1.7.** We say that the transition to a superconducting state at the transition temperature $T_c$ is a second-order phase transition if the following conditions are fulfilled:

(a) The thermodynamical potential $\Omega$, regarded as a function of $T$, is of class $C^1$ on $(0, \infty)$.

(b) The second-order derivative $(\partial^2 \Omega/\partial T^2)$ is continuous on $(0, \infty) \setminus \{T_c\}$ and is discontinuous at $T = T_c$.

**Remark 1.8.** Condition (a) implies that the entropy $S = - (\partial \Omega/\partial T)$ is continuous on $(0, \infty)$ and that, as a result, no latent heat is observed at $T = T_c$. On the other hand, (b) implies that the specific heat at constant volume, $C_V = -T (\partial^2 \Omega/\partial T^2)$, is discontinuous at $T = T_c$. See Proposition 2.4 below, which gives a new and more precise form of the gap $\Delta C_V$ in the specific heat at constant volume at $T = T_c$ from a mathematical point of view.

From a physical point of view, it is pointed out that the transition from a normal state to a superconducting state is a second-order phase transition. But a mathematical proof of this statement has not been given yet as far as we know. In this paper we first show that there is a unique solution: $T \mapsto f(T)$ of class $C^2$ on the closed interval $[0, T_c]$ to the gap equation (1.1) and point out some more properties of the gap function. Examining the thermodynamical potential $\Omega$, we then give a mathematical proof that the transition to a superconducting state at the transition temperature $T_c$ is a second-order phase transition. Furthermore, we obtain a new and more precise form of the gap in the specific heat at constant volume from a mathematical point of view.

The paper proceeds as follows. In section 2 we state our main results without proof. In sections 3 and 4 we study some properties of the function $F$ defined by (2.1) below. On the basis of this study, in sections 5 and 6, we prove our main results in a sequence of lemmas.
2 Main results

Let

\[ h(T, Y, \xi) = \begin{cases} \frac{1}{\sqrt{\xi^2 + Y}} \tanh \frac{\sqrt{\xi^2 + Y}}{2k_B T} & (0 < T \leq T_c, \ Y \geq 0), \\ \frac{1}{\sqrt{\xi^2 + Y}} & (T = 0, \ Y > 0) \end{cases} \]

and set

\[ F(T, Y) = \int_{2k_B T_c \varepsilon}^{\hbar_0 T} h(T, Y, \xi) d\xi - \frac{1}{U_0 N_0}. \]

Set also

\[ \Delta_0 = \frac{\hbar_0 \varepsilon}{\sinh \frac{1}{U_0 N_0}}, \ \Delta = \sqrt{\left\{ \hbar_0 \varepsilon - 2k_B T_c \varepsilon e^{1/(U_0 N_0)} \right\} \left\{ \hbar_0 \varepsilon - 2k_B T_c \varepsilon e^{-1/(U_0 N_0)} \right\} \sinh \frac{1}{U_0 N_0}}. \]

Since \( \varepsilon > 0 \) is small enough, it follows that \( \Delta_0 > \Delta \).

We consider the function \( F \) on the following domain \( W \subset \mathbb{R}^2 \):

\[ W = W_1 \cup W_2 \cup W_3 \cup W_4, \]

where

\[
\begin{align*}
W_1 &= \{(T, Y) \in \mathbb{R}^2 : 0 < T < T_c, \ 0 < Y < 2 \Delta_0^2\}, \\
W_2 &= \{(0, Y) \in \mathbb{R}^2 : 0 < Y < 2 \Delta_0^2\}, \\
W_3 &= \{(T, 0) \in \mathbb{R}^2 : 0 < T \leq T_c\}, \\
W_4 &= \{(T_c, Y) \in \mathbb{R}^2 : 0 < Y < 2 \Delta_0^2\}.
\end{align*}
\]

**Remark 2.1.** The gap equation (1.1) is rewritten as \( F(T, Y) = 0 \), where \( Y \) corresponds to \( f(T) = \Delta(T)^2 \).

Let \( g \) be given by

\[ g(\eta) = \begin{cases} \frac{1}{\eta^2} \left( \frac{1}{\cosh^2 \eta} - \frac{\tanh \eta}{\eta} \right) & (\eta > 0), \\ -\frac{2}{3} & (\eta = 0). \end{cases} \]

Note that \( g(\eta) < 0 \).

Our main results are the following.

**Proposition 2.2.** Let \( F \) be as in (2.1) and \( \Delta \) as in (2.2). Then there is a unique solution:

\[ T \mapsto Y = f(T) \] of class \( C^2 \) on the closed interval \([0, T_c]\) to the gap equation \( F(T, Y) = 0 \) such that the function \( f \) satisfies \( f(0) = \Delta^2 \) and \( f(T_c) = 0 \), and is monotonically decreasing on \([0, T_c]\):

\[ f(0) = \Delta^2 > f(T_1) > f(T_2) > f(T_c) = 0, \quad 0 < T_1 < T_2 < T_c. \]
Furthermore, the value of the derivative $f'$ at $T = T_c$ is given by

$$f'(T_c) = 8 k_B^2 T_c \int_{\xi}^{\infty} \frac{\hbar \omega_D/(2k_B T_c)}{\cosh^2 \eta} \frac{d\eta}{g(\eta)} < 0.$$  

**Theorem 2.3.** The transition to a superconducting state at the transition temperature $T_c$ is a second-order phase transition, and the following relation holds at the transition temperature $T_c$:

$$\lim_{T \uparrow T_c} \frac{\partial^2 \Omega}{\partial T^2}(T) - \lim_{T \downarrow T_c} \frac{\partial^2 \Omega}{\partial T^2}(T) = \frac{2 N_0 f'(T_c)}{T_c} \left( \frac{1}{1 + e^{2\epsilon}} - \frac{1}{1 + e^{\hbar \omega_D/(k_B T_c)}} \right),$$

where $f'(T_c)$ is given by Proposition 2.2.

Setting $\epsilon = 0$ in the results of Proposition 2.2 and Theorem 2.3 immediately yields the following.

**Proposition 2.4.** Let $T_c$ satisfy

$$\frac{1}{U_0 N_0} = \int_0^\infty \frac{\hbar \omega_D/(2k_B T_c) \tanh \eta}{\eta} d\eta$$

and let $f'(T_c)$ be given by

$$f'(T_c) = 8 k_B^2 T_c \int_0^{\infty} \frac{\hbar \omega_D/(2k_B T_c)}{\cosh^2 \eta} \frac{d\eta}{g(\eta)} < 0.$$  

Then the gap $\Delta C_V$ in the specific heat at constant volume, $C_V = -T (\partial^2 \Omega/\partial T^2)$, at the transition temperature $T_c$ is given by the form

$$\Delta C_V = -N_0 f'(T_c) \tanh \frac{\hbar \omega_D}{2k_B T_c} > 0.$$

**Remark 2.5.** A form similar to (2.4) has already been obtained by a different method in the context of theoretical condensed matter physics, but it is an approximate one. However, the form (2.4) is a more precise one obtained in the context of mathematics.

### 3 The first-order partial derivatives of the function $F$

In this section, we deal with the first-order partial derivatives of the function $F$ and show that $F$ is of class $C^1$ on $W$.

A straightforward calculation gives the following.
Lemma 3.1. Let $g$ be as in (2.3). Then the function $g$ is of class $C^1$ on $[0, \infty)$ and satisfies

$$g(\eta) < 0, \quad g'(0) = 0, \quad \lim_{\eta \to \infty} g(\eta) = \lim_{\eta \to \infty} g'(\eta) = 0.$$  

Lemma 3.2. The partial derivatives $\frac{\partial F}{\partial T}$ and $\frac{\partial F}{\partial Y}$ exist on $W$, and are given as follows. At $(T, Y) \in W \setminus W_2$,

$$\begin{cases} 
\frac{\partial F}{\partial T}(T, Y) = -\frac{1}{2k_BT^2} \int_{2k_BTc}^{\hbar \omega_D} \frac{d\xi}{\cosh^2 \sqrt{\xi^2 + Y} / 2k_BT}, \\
\frac{\partial F}{\partial Y}(T, Y) = \frac{1}{2(2k_BT)^3} \int_{2k_BTc}^{\hbar \omega_D} g \left( \frac{\sqrt{\xi^2 + Y}}{2k_BT} \right) d\xi
\end{cases}$$

and at $(0, Y) \in W_2$,

$$\begin{cases} 
\frac{\partial F}{\partial T}(0, Y) = 0, \\
\frac{\partial F}{\partial Y}(0, Y) = -\frac{1}{2} \int_{2k_BTc}^{\hbar \omega_D} \frac{d\xi}{(\sqrt{\xi^2 + Y})^3}.
\end{cases}$$

Lemmas 3.1 and 3.2 immediately give the following.

Lemma 3.3. At $(T, Y) \in W \setminus W_2$,

$$\frac{\partial F}{\partial T}(T, Y) < 0, \quad \frac{\partial F}{\partial Y}(T, Y) < 0.$$  

We now study the continuity of the functions $F$, $(\partial F/\partial T)$ and $(\partial F/\partial Y)$ on $W$.

Lemma 3.4. The partial derivatives $\frac{\partial F}{\partial T}$ and $\frac{\partial F}{\partial Y}$ are continuous on $W_1$. Consequently, the function $F$ is of class $C^1$ on $W_1$.

Proof. It is enough to show that the functions: $(T, Y) \mapsto I_1(T, Y)$ and $(T, Y) \mapsto I_2(T, Y)$ (see Lemma 3.2) are continuous at $(T_0, Y_0) \in W_1$. Here,

$$I_1(T, Y) = \int_{2k_BTc}^{\hbar \omega_D} \frac{d\xi}{\cosh^2 \eta}, \quad I_2(T, Y) = \int_{2k_BTc}^{\hbar \omega_D} g(\eta) d\xi, \quad \eta = \frac{\sqrt{\xi^2 + Y}}{2k_BT}. $$

Set $\eta_0 = \frac{\sqrt{\xi^2 + Y_0}}{2k_BT_0}$. Since $(T, Y) \in W_1$ is close to $(T_0, Y_0) \in W_1$, it follows that $T >$
\( T_0/2 \). Then
\[
\left| I_1(T, Y) - I_1(T_0, Y_0) \right| \\
\leq \int_{2k_BT_c \varepsilon}^{\hbar \omega_D} \left| \left( \frac{1}{\cosh \eta} + \frac{1}{\cosh \eta_0} \right) \frac{\cosh \eta - \cosh \eta_0}{\cosh \eta \cosh \eta_0} \right| d\xi \\
\leq 2\hbar \omega_D \sinh \frac{\sqrt{\hbar^2 \omega_D^2 + 2 \Delta_0^2}}{k_BT_0} \left( \frac{\sqrt{\hbar^2 \omega_D^2 + 2 \Delta_0^2}}{k_BT_0^2} |T - T_0| + \frac{|Y - Y_0|}{k_BT_0\sqrt{Y_0}} \right), \\
\left| I_2(T, Y) - I_2(T_0, Y_0) \right| \\
\leq \int_{2k_BT_c \varepsilon}^{\hbar \omega_D} |g(\eta) - g(\eta_0)| d\xi \\
\leq \hbar \omega_D \max_{\eta \geq 0} |g'(\eta)| \left( \frac{\sqrt{\hbar^2 \omega_D^2 + 2 \Delta_0^2}}{k_BT_0^2} |T - T_0| + \frac{|Y - Y_0|}{k_BT_0\sqrt{Y_0}} \right).
\]
Thus the functions: \((T, Y) \mapsto I_1(T, Y)\) and \((T, Y) \mapsto I_2(T, Y)\), and hence \((\partial F/\partial T)\) and \((\partial F/\partial Y)\) are continuous at \((T_0, Y_0) \in W_1\).

**Lemma 3.5.** The function \(F\) is continuous on \(W\).

**Proof.** Note that \(F\) is continuous on \(W_1\) by Lemma 3.4. We then show that \(F\) is continuous on \(W_2\).

Let \((0, Y_0) \in W_2\) and let \((T, Y) \in W_1 \cup W_2\). Since \((T, Y)\) is close to \((0, Y_0)\), it follows that \(Y > Y_0/2\). Then, by (2.1),
\[
|F(T, Y) - F(0, Y_0)| \\
\leq \int_{2k_BT_c \varepsilon}^{\hbar \omega_D} \left\{ \frac{1 - \tanh \frac{\sqrt{\xi^2 + Y}}{2k_BT}}{\sqrt{\xi^2 + Y}} + \left| \frac{1}{\sqrt{\xi^2 + Y}} - \frac{1}{\sqrt{\xi^2 + Y_0}} \right| \right\} d\xi \\
\leq \hbar \omega_D \left\{ \frac{1}{\sqrt{Y_0}} \left( 1 - \tanh \frac{\sqrt{Y_0/2}}{2k_BT} \right) + \frac{2|Y - Y_0|}{(\sqrt{2} + 1)Y_0^{3/2}} \right\}.
\]
Thus \(F\) is continuous on \(W_2\). Similarly we can show the continuity of \(F\) on \(W_3\), and on \(W_4\).

**Lemma 3.6.** The partial derivatives \(\frac{\partial F}{\partial T}\) and \(\frac{\partial F}{\partial Y}\) are continuous on \(W\). Consequently, the function \(F\) is of class \(C^1\) on \(W\).

**Proof.** Note that \((\partial F/\partial T)\) and \((\partial F/\partial Y)\) are continuous on \(W_1\) by Lemma 3.4. We then show that \((\partial F/\partial T)\) and \((\partial F/\partial Y)\) are continuous at \((T_c, 0) \in W_3\). We can show the continuity of those functions at other points in \(W\) similarly.

**Step 1.** Let \((T, Y) \in W_1\). We show
\[
\frac{\partial F}{\partial T}(T, Y) \rightarrow \frac{\partial F}{\partial T}(T_c, 0), \quad \frac{\partial F}{\partial Y}(T, Y) \rightarrow \frac{\partial F}{\partial Y}(T_c, 0) \quad \text{as} \quad (T, Y) \rightarrow (T_c, 0).
\]
Since \((T, Y)\) is close to \((T_c, 0)\), it then follows that 
\[ T_c/2 < T < T_c. \]

Set 
\[ \eta_0 = \sqrt{\frac{\hbar \omega^2_D + 2 \Delta^2_0}{k_B T_c}}. \]

Then
\[
|T - T_c| (\cosh \eta_0 + \eta_0 \sinh \eta_0) + \frac{\sqrt{Y}}{4k_B} \sinh \eta_0 \leq 
\]
and hence 
\[(\partial F / \partial T)(T, Y) - (\partial F / \partial T)(T_c, 0) \to 0 \quad \text{as} \quad (T, Y) \to (T_c, 0).\]

Step 2. When \((T, Y) = (T, 0) \in W_3 \) and \((T, Y) = (T_c, Y) \in W_4\), an argument similar to that in Step 1 gives
\[
\frac{\partial F}{\partial Y}(T, 0) \to \frac{\partial F}{\partial Y}(T_c, 0), \quad \frac{\partial F}{\partial Y}(T, 0) \to \frac{\partial F}{\partial Y}(T_c, 0) \quad \text{as} \quad (T, 0) \to (T_c, 0)
\]

and hence 
\[(\partial F / \partial Y)(T, Y) - (\partial F / \partial Y)(T_c, 0) \to 0 \quad \text{as} \quad (T, Y) \to (T_c, 0).\]

4 The second-order partial derivatives of the function \(F\)

In this section we deal with the second-order partial derivatives of the function \(F\) and show that \(F\) is of class \(C^2\) on \(W_1\).

Let \(G\) be given by
\[
G(\eta) = \begin{cases} 
\frac{1}{\eta^2} \left\{ 3 g(\eta) + 2 \frac{\tanh \eta}{\eta \cosh^2 \eta} \right\} & (\eta > 0), \\
\frac{16}{15} & (\eta = 0).
\end{cases}
\]

A straightforward calculation gives the following.

**Lemma 4.1.** Let \(G\) be as in (4.1) and \(g\) as in (2.3). Then the function \(G\) is of class \(C^1\) on \([0, \infty)\) and satisfies
\[
g'(\eta) = -\eta G(\eta), \quad G'(0) = 0, \quad \lim_{\eta \to \infty} G(\eta) = \lim_{\eta \to \infty} G'(\eta) = 0.
\]
Lemma 4.2. The values of the partial derivatives \( \frac{\partial^2 F}{\partial T^2}, \frac{\partial}{\partial Y} \left( \frac{\partial F}{\partial T} \right), \) 
\( \frac{\partial}{\partial T} \left( \frac{\partial F}{\partial Y} \right) \) and \( \frac{\partial^2 F}{\partial Y^2} \) exist at each point in \( W_1 \). Furthermore,
\[
\frac{\partial}{\partial Y} \left( \frac{\partial F}{\partial T} \right) = \frac{\partial}{\partial T} \left( \frac{\partial F}{\partial Y} \right) \quad \text{on} \quad W_1.
\]

Proof. Let \((T, Y) \in W_1\). Then there is a \( \theta \) \((0 < \theta < 1)\) satisfying \( \theta T_c < T < T_c \). Set 
\[
\eta = \sqrt{\frac{\xi^2 + Y^2}{2k_BT}}.
\]
Then
\[
\left| \frac{\partial}{\partial T} \frac{1}{\cosh^2 \eta} \right| \leq \sqrt{\frac{\hbar^2 \omega_D^2 + 2 \Delta_0^2}{k_BT^2}},
\]
where the right side is integrable on \([2k_BT_c \varepsilon, \hbar \omega_D]\). So the function: \((T, Y) \mapsto I_1(T, Y)\) (see (3.1)), and hence \((\partial F/\partial T)\) (see Lemma 3.2) is differentiable with respect to \( T \) on \( W_1 \), and the second-order partial derivative is given by
\[
\frac{\partial^2 F}{\partial T^2}(T, Y) = \frac{1}{k_BT^3} \left\{ I_1(T, Y) - \int_{2k_BT_c \varepsilon}^{\hbar \omega_D} \frac{\eta \tanh \eta}{\cosh^2 \eta} d\xi \right\}, \quad \eta = \sqrt{\frac{\xi^2 + Y^2}{2k_BT}}.
\]
Similarly we can show that \((\partial F/\partial T)\) is differentiable with respect to \( Y \) on \( W_1 \), that \((\partial F/\partial Y)\) is differentiable with respect to \( T \) on \( W_1 \), and that \((\partial F/\partial Y)\) is differentiable with respect to \( Y \) on \( W_1 \). The corresponding second-order partial derivatives are given as follows:
\[
\frac{\partial}{\partial Y} \left( \frac{\partial F}{\partial T} \right)(T, Y) = \frac{\partial}{\partial T} \left( \frac{\partial F}{\partial Y} \right)(T, Y) = \int_{2k_BT_c \varepsilon}^{\hbar \omega_D} \frac{\tan \eta}{\eta \cosh^2 \eta} d\xi,
\]
\[
\frac{\partial^2 F}{\partial Y^2}(T, Y) = -\frac{1}{4(2k_BT)^3} \int_{2k_BT_c \varepsilon}^{\hbar \omega_D} G(\eta) d\xi, \quad \eta = \sqrt{\frac{\xi^2 + Y^2}{2k_BT}}.
\]
Here, \( G \) is that in Lemma 4.1 (see also (4.1)).

Lemma 4.3. The partial derivatives \( \frac{\partial^2 F}{\partial T^2}, \frac{\partial}{\partial Y} \left( \frac{\partial F}{\partial T} \right) \) and \( \frac{\partial^2 F}{\partial Y^2} \) are continuous on \( W_1 \). Consequently, \( F \) is of class \( C^2 \) on \( W_1 \).

Proof. We show that \((\partial^2 F/\partial Y^2)\) is continuous on \( W_1 \). Similarly we can show the continuity of other second-order partial derivatives.

By (4.2), it suffices to show that the function: \((T, Y) \mapsto I_3(T, Y)\) is continuous at \((T_0, Y_0) \in W_1\). Here,
\[
I_3(T, Y) = \int_{2k_BT_c \varepsilon}^{\hbar \omega_D} G(\eta) d\xi, \quad \eta = \frac{\sqrt{\xi^2 + Y^2}}{2k_BT}.
\]
Since \((T, Y)\) is close to \((T_0, Y_0)\), it then follows that \(T_0/2 < T\). A straightforward calculation then gives
\[
|I_3(T, Y) - I_3(T_0, Y_0)| \leq \hbar \omega_D \max_{\eta \geq 0} |G'(\eta)| \left( \frac{\sqrt{\hbar^2 \omega_D^2 + 2 \Delta_0^2}}{k_B T_0^2} |T - T_0| + \frac{|Y - Y_0|}{k_B T_0 Y_0} \right).
\]
Hence the function: \((T, Y) \mapsto I_3(T, Y)\) is continuous at \((T_0, Y_0) \in W_1\).

\section{Proof of Proposition 2.2}

In this section we prove Proposition 2.2 in a sequence of lemmas.

\textbf{Remark 5.1.} One may prove Proposition 2.2 on the basis of the implicit function theorem. In this case, an interior point \((T_0, Y_0)\) of the domain \(W\) satisfying \(F(T_0, Y_0) = 0\) need not to exist. But there are the two points \((0, \Delta^2)\) and \((T_c, 0)\) in the boundary of \(W\) satisfying
\[
F(0, \Delta^2) = F(T_c, 0) = 0.
\]
So one can not apply the implicit function theorem in its present form.

\textbf{Lemma 5.2.} There is a unique solution: \(T \mapsto Y = f(T)\) to the gap equation \(F(T, Y) = 0\) such that the function \(f\) is continuous on the closed interval \([0, T_c]\) and satisfies \(f(0) = \Delta^2\) and \(f(T_c) = 0\).

\textit{Proof.} By Lemmas 3.3, 3.6 and (5.1), the function: \(Y \mapsto F(T_c, Y)\) is monotonically decreasing and there is a \(Y_1\) \((0 < Y_1 < 2\Delta_0^2)\) satisfying \(F(T_c, Y_1) < 0\). Note that \(Y_1\) is arbitrary as long as \(0 < Y_1 < 2\Delta_0^2\). Hence, by Lemma 3.6, there is a \(T_1\) \((0 < T_1 < T_c)\) satisfying \(F(T_1, Y_1) < 0\). Hence, \(F(T, Y_1) < 0\) for \(T_1 \leq T \leq T_c\). On the other hand, by Lemmas 3.3, 3.6 and (5.1), the function: \(T \mapsto F(T, 0)\) is monotonically decreasing and there is a \(T_2\) \((0 < T_2 < T_c)\) satisfying \(F(T_2, 0) > 0\). Note that \(T_2\) is arbitrary as long as \(0 < T_2 < T_c\). Hence, \(F(T, 0) > 0\) for \(T_2 \leq T < T_c\).

Let \(\max(T_1, T_2) \leq T < T_c\) and fix \(T\). It then follows from Lemmas 3.3 and 3.6 that the function: \(Y \mapsto F(T, Y)\) with \(T\) fixed is monotonically decreasing on \([0, Y_1]\). Since \(F(T, 0) > 0\) and \(F(T, Y_1) < 0\), there is a unique \(Y\) \((0 < Y < Y_1)\) satisfying \(F(T, Y) = 0\). When \(T = T_c\), there is a unique value \(Y = 0\) satisfying \(F(T_c, Y) = 0\) (see (5.1)).

Since \(F\) is continuous on \(W\) by Lemma 3.6 there is a unique solution: \(T \mapsto Y = f(T)\) to the gap equation \(F(T, Y) = 0\) such that the function \(f\) is continuous on \([\max(T_1, T_2), T_c]\) and \(f(T_c) = 0\).

Since \((\partial F/\partial Y)(0, Y) < 0\) \((0 < Y < 2\Delta_0^2)\) by Lemma 3.2 there is a unique value \(Y = \Delta^2\) satisfying \(F(0, Y) = 0\). Combining Lemma 3.6 with Lemma 3.3 therefore implies that the function \(f\) is continuous on \([0, T_c]\) and that \(f(0) = \Delta^2\) and \(f(T_c) = 0\). \qed

\textbf{Lemma 5.3.} The function \(f\) given by Lemma 5.2 is of class \(C^1\) on \([0, T_c]\), and the derivative \(f'\) satisfies
\[
f'(0) = 0, \quad f'(T_c) = 8 k_B^2 T_c \int_\epsilon^\infty \frac{\hbar \omega_D/(2 k_B T_c)}{\cosh^2 \eta} \, d\eta \int_\epsilon^\infty \frac{\hbar \omega_D/(2 k_B T_c)}{g(\eta)} \, d\eta.
\]
Proof. Lemma 3.6 immediately implies that the function $f$ is of class $C^1$ on the interval $[0, T_c]$ and that its derivative is given by

$$f'(T) = -\frac{F_T(T, f(T))}{F_Y(T, f(T))}.$$  

(5.2)

The values of $f'(0)$ and $f'(T_c)$ are derived from (5.2). 

Combining (5.2) with Lemma 3.3 immediately yields the following.

**Lemma 5.4.** The function $f$ given by Lemma 5.2 is monotonically decreasing on $[0, T_c]$:

$$f(0) = \Delta^2 > f(T_1) > f(T_2) > f(T_c) = 0, \quad 0 < T_1 < T_2 < T_c.$$  

Let $\phi$ be a function of $\eta$ and let $\eta$ be a function of $\xi$. Set

$$I [\phi(\eta)] = \int_{2k_BT_c \varepsilon}^{\hbar \omega_D} \phi(\eta) d\xi.$$  

(5.3)

**Lemma 5.5.** Let $f$ be given by Lemma 5.2 and let $I [\cdot]$ be as in (5.3). Then the function $f$ is of class $C^2$ on $[0, T_c]$ and the second derivative $f''$ satisfies

$$f''(0) = 0 \quad \text{and} \quad f''(T_c) = 16 k_B^2 \left[ \frac{I [\eta_0 \tanh \eta_0 - 1]}{\cosh^2 \eta_0} \right] - 32 k_B^2 \left[ \frac{I [g(\eta_0)]}{\cosh^2 \eta_0} \right]^2$$

$$+ 8 k_B^2 \left[ \frac{I [g(\eta_0)]}{\cosh^2 \eta_0} \right] I [G(\eta_0)]^3,$$  

where $\eta_0 = \xi / 2k_BT_c$. 

Proof. Lemma 4.3 implies that $f$ is of class $C^2$ on the open interval $(0, T_c)$ and that

$$f''(T) = -\frac{F_{TT}F_Y^2 + 2F_{TY}F_TF_Y - F_{YY}F_T^2}{F_Y^3}, \quad 0 < T < T_c.$$  

(5.4)

So we have only to deal with $f$ and its derivatives at $T = 0$ and at $T = T_c$.

**Step 1.** We show that $f'$ is differentiable at $T = 0$ and that $f''$ is continuous at $T = 0$.

Note that $f'(0) = 0$ by Lemma 5.3. Since $T$ is close to $T = 0$, the inequality $f(T) > \Delta^2 / 2$ holds. It then follows from (5.2) and Lemma 3.2 that

$$\left| \frac{f'(T) - f'(0)}{T} \right| \leq 4 \frac{\sqrt{\hbar^2 \omega_D^2 + 2 \Delta^2}}{k_B T^3 \left( \tanh \eta_1 - \frac{\eta_1}{\cosh^2 \eta_1} \right)} \exp \left( -\frac{\sqrt{\Delta^2 / 2}}{k_B T} \right) \to 0 \quad (T \downarrow 0).$$

Here, $\eta_1 = \sqrt{\xi^2 + f(T)} / 2k_BT \to \infty$ as $T \downarrow 0$ (since $2k_BT_c \varepsilon < \xi < \hbar \omega_D$). Hence $f'$ is differentiable at $T = 0$ and $f''(0) = 0$. 

11
By (5.4), a similar argument gives \( \lim_{T \to 0} f''(T) = 0 \). Hence \( f'' \) is continuous at \( T = 0 \).

**Step 2.** We show that \( f' \) is differentiable at \( T = T_c \) and that \( f'' \) is continuous at \( T = T_c \).

Note that

\[
f'(T_c) = 8 k_B^2 T_c \left[ \frac{I \left[ \frac{1}{\cosh^2 \eta_0} \right]}{I \left[ g(\eta_0) \right]} \right], \quad \eta_0 = \frac{\xi}{2 k_B T_c}
\]

by Lemma 5.3. It follows from (5.2) and Lemma 3.2 that

\[
f'(T) = 8 k_B^2 T \left[ \frac{I \left[ \frac{1}{\cosh^2 \eta} \right]}{I \left[ g(\eta) \right]} \right], \quad \eta = \sqrt{\xi^2 + f(T)}
\]

Hence

\[
\frac{f'(T_c) - f'(T)}{T_c - T} = 8 k_B^2 \left[ \frac{I \left[ \frac{1}{\cosh^2 \eta_0} \right]}{I \left[ g(\eta_0) \right]} \right] + 8 k_B^2 T \frac{I \left[ \frac{1}{\cosh^2 \eta_0} \right]}{T_c - T} \left\{ I \left[ g(\eta) \right] - I \left[ g(\eta_0) \right] \right\}
\]

\[
+ \frac{8 k_B^2 T_c I \left[ g(\eta_0) \right]}{T_c - T} \left\{ \frac{I \left[ \frac{1}{\cosh^2 \eta_0} \right]}{I \left[ g(\eta) \right]} - \frac{I \left[ \frac{1}{\cosh^2 \eta} \right]}{I \left[ g(\eta) \right]} \right\}.
\]

Note that \( g(\eta) - g(\eta_0) = (\eta - \eta_0)g'(\eta_1) \) and \( \cosh \eta - \cosh \eta_0 = (\eta - \eta_0) \sinh \eta_2 \). Here,

\[
\eta_0 = \frac{\xi}{2 k_B T_c} < \eta_i < \eta = \frac{\sqrt{\xi^2 + f(T)} + \xi}{2 k_B T_c}, \quad i = 1, 2
\]

and

\[
\eta - \eta_0 = \frac{1}{2 k_B T_c} \left\{ \frac{f(T)}{\sqrt{\xi^2 + f(T)} + \xi} + \xi \frac{T_c - T}{T_c} \right\}.
\]

Since \( T \) is close to \( T_c \), the inequality \( T > T_c/2 \) holds. Therefore, by Lemma 4.1

\[
\left| \frac{g'(\eta_1)}{\sqrt{\xi^2 + f(T)} + \xi} \right| \leq \frac{1}{k_B T_c} \max_{\eta \geq 0} |G(\eta)|
\]

and

\[
\left| \frac{\sin \eta_2}{\sqrt{\xi^2 + f(T)} + \xi} \right| \leq \frac{1}{k_B T_c} \max_{0 \leq \eta \leq M} \left| \frac{\sin \eta}{\eta} \right|, \quad M = \frac{\sqrt{\hbar^2 \omega_D^2 + \Delta^2}}{k_B T_c}.
\]

So \( f' \) is differentiable at \( T = T_c \), and it is easy to see that the form of \( f''(T_c) \) is exactly the same as that mentioned just above.

Furthermore, it follows from (5.3) that \( f'' \) is continuous at \( T = T_c \).
6 Proof of Theorem 2.3

In this section we prove Theorem 2.3 in a sequence of lemmas. Fixing the values of the chemical potential $\mu$ and of the volume of our physical system, we deal with the dependence of the thermodynamical potential $\Omega$ on the temperature $T$ only.

**Lemma 6.1.** Let $V$ be as in (1.4). Then $V$ is of class $C^2$ on $(0, \infty)$.

**Proof.** For each $T > 0$, there are a $\theta_1$ ($0 < \theta_1 < 1$) and a $\theta_2$ ($\theta_2 > 1$) satisfying $\theta_1 T_c < T < \theta_2 T_c$. Then

\[
\left| \frac{\partial}{\partial T} \ln \left( 1 + e^{-|\xi|/(k_B T)} \right) \right| \leq \frac{|\xi| e^{-|\xi|/(k_B \theta_2 T_c)}}{k_B \theta_1^2 T_c^2},
\]

where the right side is integrable on $[-\mu, -\hbar \omega_D]$ and on $[\hbar \omega_D, \infty)$ since $N(\xi) = O(\sqrt{\xi})$ as $\xi \to \infty$ (see Remark 1.3). Hence $V$ is differentiable on $(0, \infty)$ and

\[
\frac{\partial V}{\partial T}(T) = -2k_B \int_{[-\mu, -\hbar \omega_D] \cup [\hbar \omega_D, \infty)} N(\xi) \ln \left( 1 + e^{-|\xi|/(k_B T)} \right) d\xi - \frac{2}{T} \int_{[-\mu, -\hbar \omega_D] \cup [\hbar \omega_D, \infty)} N(\xi) \frac{|\xi|}{1 + e^{\xi}/(k_B T)} d\xi.
\]

A similar argument gives

\[
\left| \frac{\partial}{\partial T} \frac{1}{1 + e^{\xi}/(k_B T)} \right| \leq \frac{|\xi| e^{-|\xi|/(k_B \theta_2 T_c)}}{k_B \theta_1^2 T_c^2},
\]

where the right side is integrable on $[-\mu, -\hbar \omega_D]$ and on $[\hbar \omega_D, \infty)$. Therefore, $(\partial V/\partial T)$ is again differentiable on $(0, \infty)$ and

\[
\frac{\partial^2 V}{\partial T^2}(T) = -\frac{2}{k_B T^3} \int_{[-\mu, -\hbar \omega_D] \cup [\hbar \omega_D, \infty)} N(\xi) \frac{|\xi|^2 e^{\xi}/(k_B T)}{(1 + e^{\xi}/(k_B T))^2} d\xi.
\]

Clearly, $(\partial^2 V/\partial T^2)$ is continuous on $(0, \infty)$.

**Lemma 6.2.** Let $\Omega_N$ be as in (1.2). Then $\Omega_N$ is of class $C^2$ on $(0, \infty)$.

**Proof.** An argument similar to that in the proof of Lemma 6.1 gives that $\Omega_N$ is of class $C^2$ on $(0, \infty)$ and that the derivatives are given by

\[
\frac{\partial \Omega_N}{\partial T}(T) = -4N_0 k_B \int_{2k_B T_c \epsilon}^{\hbar \omega_D} \ln \left( 1 + e^{-\xi/(k_B T)} \right) d\xi - \frac{4N_0}{T} \int_{2k_B T_c \epsilon}^{\hbar \omega_D} \frac{\xi}{1 + e^{\xi}/(k_B T)} d\xi + \frac{\partial V}{\partial T}(T),
\]

\[
\frac{\partial^2 \Omega_N}{\partial T^2}(T) = -\frac{4N_0}{k_B T^3} \int_{2k_B T_c \epsilon}^{\hbar \omega_D} \frac{\xi^2 e^{\xi}/(k_B T)}{(1 + e^{\xi}/(k_B T))^2} d\xi + \frac{\partial^2 V}{\partial T^2}(T).
\]
Lemma 6.3. Let $\delta$ be as in (1.3). Then $\delta$ is of class $C^2$ on $(0, T_c]$.

Proof. Note that the squared gap function $f$ is of class $C^2$ on $[0, T_c]$ by Lemma 5.5 and that

\begin{equation}
\delta(T_c) = 0
\end{equation}

since $f(T_c) = 0$ (see Lemma 5.2). A straightforward calculation gives that $\delta$ is continuous on $(0, T_c]$ and that

$$
\left| \frac{\partial}{\partial T} \sqrt{\xi^2 + f(T)} \right| \leq \frac{\max_{0 \leq T \leq T_c} |f'(T)|}{4k_B T_c \epsilon},
$$

where the right side is integrable on $[2k_B T_c \epsilon, \hbar \omega_D]$. By an argument similar to that in the proof of Lemma 6.1, $\delta$ is differentiable on $(0, T_c]$ and the derivative is given by

$$
\frac{\partial \delta}{\partial T} (T_c) = f'(T_c) \left\{ \frac{1}{U_0} - N_0 \int_{2k_B T_c \epsilon}^{\hbar \omega_D} \frac{1}{2k_B T_c} \tanh \frac{\sqrt{\xi^2 + f(T)}}{2k_B T} d\xi \right\}
$$

$$
-4N_0 k_B \int_{2k_B T_c \epsilon}^{\hbar \omega_D} \ln \frac{1 + e^{-\sqrt{\xi^2 + f(T)}/(k_B T)}}{1 + e^{\xi/(k_B T)}} d\xi
$$

$$
+ \frac{4N_0}{T} \int_{2k_B T_c \epsilon}^{\hbar \omega_D} \left\{ \frac{\xi}{1 + e^\xi/(k_B T)} - \frac{1}{1 + e^{\sqrt{\xi^2 + f(T)}/(k_B T)}} \right\} d\xi,
$$

where the first term on the right side is equal to 0 by the gap equation (1.1). Note that

\begin{equation}
\frac{\partial \delta}{\partial T} (T_c) = 0.
\end{equation}

An argument similar to that in the proof of Lemma 6.1 gives that $(\partial \delta/\partial T)$ is again differentiable on $(0, T_c]$ and the second-order derivative is given by

$$
\frac{\partial^2 \delta}{\partial T^2} (T_c)
$$

$$
= \frac{4N_0}{k_B T^3} \int_{2k_B T_c \epsilon}^{\hbar \omega_D} \frac{\xi^2 e^\xi/(k_B T)}{(1 + e^\xi/(k_B T))^2} d\xi
$$

$$
- \frac{4N_0}{k_B T^3} \int_{2k_B T_c \epsilon}^{\hbar \omega_D} \frac{e^{\sqrt{\xi^2 + f(T)}/(k_B T)}}{(1 + e^{\sqrt{\xi^2 + f(T)}/(k_B T)})^2} \left\{ \frac{\xi^2 + f(T) - T f'(T)}{2} \right\} d\xi,
$$

which is also continuous on $(0, T_c]$. Thus $\delta$ is of class $C^2$ on $(0, T_c]$, and

\begin{equation}
\frac{\partial^2 \delta}{\partial T^2} (T_c) = \frac{2N_0 f'(T_c)}{T_c} \left( \frac{1}{1 + e^{2\epsilon}} - \frac{1}{1 + e^{\hbar \omega_D/(k_B T_c)}} \right).
\end{equation}

We now give a proof of Theorem 2.3.
Lemma 6.4. Let \( f'(T_c) \) be given by Lemma 5.3 and let \( \Omega \) be the thermodynamical potential given by Definition 1.4.

(i) The thermodynamical potential \( \Omega \), regarded as a function of \( T \), is of class \( C^1 \) on \((0, \infty)\).

(ii) The second-order derivative \( (\partial^2 \Omega/\partial T^2) \) is continuous on \((0, \infty) \setminus \{T_c\}\).

(iii) 
\[
\lim_{T \uparrow T_c} \frac{\partial^2 \Omega}{\partial T^2}(T) - \lim_{T \downarrow T_c} \frac{\partial^2 \Omega}{\partial T^2}(T) = \frac{2N_0 f'(T_c)}{T_c} \left( \frac{1}{1 + e^{2\varepsilon}} - \frac{1}{1 + e^{\hbar \omega_D/(k_B T_c)}} \right).
\]

Proof. Note that \( \delta(T_c) = (\partial \delta/\partial T)(T_c) = 0 \) (see (6.1) and (6.2)). Hence both (i) and (ii) follow immediately from Lemmas 6.1, 6.2 and 6.3. Since
\[
\lim_{T \uparrow T_c} \left( \frac{\partial^2 \Omega}{\partial T^2}(T) - \lim_{T \downarrow T_c} \frac{\partial^2 \Omega}{\partial T^2}(T) \right) = \frac{\partial^2 \delta}{\partial T^2}(T_c),
\]
(iii) follows immediately from (6.3). \( \square \)

Remark 6.5. This lemma implies that the transition to a superconducting state at the transition temperature \( T_c \) is a second-order phase transition.

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