The lattice Burnside rings

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Abstract

We introduce the concept of lattice Burnside ring for a finite group $G$ associated to a family of nonempty sublattices $L_H$ of a finite $G$-lattice $L$ for $H \leq G$. The slice Burnside ring introduced by S. Bouc [6] is isomorphic to a lattice Burnside ring. Any lattice Burnside ring is isomorphic to an abstract Burnside ring. The ring structure of a lattice Burnside ring is explored on the basis of the fundamental theorem of abstract Burnside rings. We study the units and the primitive idempotents of a lattice Burnside ring. There are certain rings called partial lattice Burnside rings. Any partial lattice Burnside ring, which is isomorphic to an abstract Burnside ring, consists of elements of a lattice Burnside ring; it is not necessarily a subring. The section Burnside ring introduced by S. Bouc [6] is isomorphic to a partial lattice Burnside ring.

1 Introduction

Let $G$ be a finite group, and let $\mathcal{L}$ be a finite $G$-lattice, that is, $\mathcal{L}$ is a finite lattice on which $G$ acts and the binary relation in $\mathcal{L}$ is invariant under the action of $G$. The purpose of this paper is to introduce the concept of lattice Burnside ring associated to a family of nonempty sublattices $\mathcal{L}_H$ of $\mathcal{L}$ for $H \leq G$, together with

*This work was supported by JSPS KAKENHI Grant Number JP16K05052.

2010 Mathematics Subject Classification. Primary 19A22; Secondary 16U60.

Keywords. Abstract Burnside ring, finite lattice, Green functor, Mackey functor, plus construction, primitive idempotent, representation ring, section Burnside ring, slice Burnside ring, unit group.
its ring structure. The slice Burnside ring of $G$ introduced by S. Bouc [6], which arises from morphisms of finite $G$-sets, inspired us to study lattice Burnside rings.

In Section 2, we first introduce the concept of monoid functor $M = (M, \text{con}, \text{res})$ assigning each $H \leq G$ a monoid $M(H)$. Let $M = (M, \text{con}, \text{res})$ be a monoid functor, and let $H \leq G$. There is an additive contravariant functor $T^M_H$ from the category of finite left $H$-sets to the category of monoids such that for any finite left $H$-set $J$, $T^M_H(J)$ is the set of $H$-equivariant maps $\pi : J \rightarrow \bigcup_{K \leq H} M(K)$ with $\pi(x) \in M(H_x)$, where $H_x$ is the stabilizer of $x$ in $H$. A pair $(J, \pi)$ of a left $H$-set $J$ and $\pi \in T^M_H(J)$ is called an element of $T^M_H$. The $M$-Burnside ring $\Omega(H, M)$ is defined to be the Grothendieck ring of the category of elements of $T^M_H$ (cf. [12, 14, 15]).

Any finite lattice is considered as a monoid with the binary operation given by ‘meet’, so that $\mathcal{L}$ is viewed as a finite monoid. We introduce the concept of a monoid functor $M_{\mathcal{L}} = (M_{\mathcal{L}}, \text{con}, \text{res})$ assigning each $H \leq G$ a nonempty sublattice $\mathcal{L}_H$ of $\mathcal{L}$ (see Proposition 4.2), and call $\Omega(G, M_{\mathcal{L}})$ the lattice Burnside ring associated to the family of nonempty sublattices $\mathcal{L}_H$ of $\mathcal{L}$ for $H \leq G$. If $\sup \mathcal{L}$ is $G$-invariant, then there is a monoid functor $C_{\mathcal{L}} = (C_{\mathcal{L}}, \text{con}, \text{res})$ assigning each $H \leq G$ the set of $H$-invariants in $\mathcal{L}$, and the lattice Burnside ring $\Omega(G, C_{\mathcal{L}})$ is isomorphic to the crossed Burnside ring associated to $\mathcal{L}$. We consider the set $\mathcal{F} := \mathcal{F}(G)$ of subgroups of $G$ to be a finite $G$-lattice with the binary relation given by inclusion and the action of $G$ given by conjugation. There is a monoid functor $M_{\mathcal{F}} = (M_{\mathcal{F}}, \text{con}, \text{res})$ assigning each $H \leq G$ the set of subgroups of $G$ containing $H$, and the lattice Burnside ring $\Omega(G, M_{\mathcal{F}})$ is isomorphic to the slice Burnside ring of $G$. We also obtain a monoid functor $M_{\mathcal{F}}^0 = (M_{\mathcal{F}}^0, \text{con}, \text{res})$ assigning each $H \leq G$ the set of normal subgroups of $H$, and direct our attention to the deference between $\Omega(G, M_{\mathcal{F}}^0)$ and $\Omega(G, M_{\mathcal{F}})$.

The lattice Burnside rings are extensions of the ordinary Burnside ring of $G$. There are various attempts to generalize the theory of ordinary Burnside ring of $G$. Among others, the concept of abstract Burnside ring was introduced in [25], and any lattice Burnside ring is isomorphic to an abstract Burnside ring. By using the fundamental theorem of abstract Burnside rings, we determine the primitive idempotents of the $\mathbb{Q}$-algebra $\mathbb{Q} \otimes_{\mathbb{Z}} \Omega(G, M_{\mathcal{F}})$ in Section 4, and establish a criterion for the units of $\Omega(G, M_{\mathcal{F}})$ which generalize [23, Proposition 6.5] in Section 5.

The theory of abstract Burnside rings provides a principle of determining the primitive idempotents of the lattice Burnside rings. In Section 6, we present a characterization of solvable groups for a certain class of lattice Burnside rings as a sequel to the study of primitive idempotents of $\Omega(G, M_{\mathcal{F}})$, which extend Dress’ characterization of solvable groups for the ordinary Burnside ring of $G$ (cf. [8]).

In Section 7, we introduce the concept of partial lattice Burnside ring on the basis of results shown in [24], together with its ring structure. Any partial lattice Burnside ring, which is isomorphic to an abstract Burnside ring, consists of elements of a lattice Burnside ring; it is not necessarily a subring. The section Burnside ring of $G$ introduced by S. Bouc [6] is isomorphic to a partial lattice Burnside ring.
Notation Let \( G \) be a finite group. The rational integers, the rational numbers, and the complex numbers are denoted by \( \mathbb{Z} \), \( \mathbb{Q} \), and \( \mathbb{C} \), respectively. We denote by \( \epsilon \) the identity of \( G \). The subgroup generated by an element \( g \) of \( G \) is denoted by \( \langle g \rangle \). For subgroups \( H \) and \( K \), we write \( K \leq H \) if \( K \) is a subgroup of \( H \). Let \( H \leq G \). We denote by \( H\text{-set} \) the category of finite left \( H \)-sets and \( H \)-equivariant maps. Given \( J, G \in \text{G-set} \) and \( x \in J, H_x \) denotes the stabilizer of \( x \) in \( H \). We set \( ^rH = rHr^{-1} \) and \( ^rg = rgr^{-1} \) for all \( g, r \in G \). For each \( K \leq H \), \( H/K \) denotes the set of left cosets \( hK, h \in H, \) of \( K \) in \( H \). Given a pair \((U, K)\) of subgroups \( U \) and \( K \) of \( H \), we denote by \( U\setminus H/K \) the set of \((U, K)\)-double cosets \( UhK, h \in H, \) in \( H \). The identity map on a set \( \Sigma \) is denoted by \( \text{id}_\Sigma \). For a finite set \( \Sigma \), \(|\Sigma|\) denotes the cardinality \( |\Sigma| \) of \( \Sigma \). For each positive integer \( m \), we set \( \{m\} = \{1, 2, \ldots, m\} \).

2 \( M \)-Burnside ring functors

We first review the Mackey functors and the Green functors (cf. [4, 11, 20, 21]). Let \( G \) be a finite group, and let \( k \) be a commutative unital ring. A Mackey functor for \( G \) over \( k \) is a quadruple \( X = (X, \text{con}, \text{res}, \text{ind}) \) consisting of a family of \( k \)-modules \( X(H) \) for \( H \leq G \) and a family of \( k \)-module homomorphisms

\[
\begin{align*}
\text{con}^g_H & : X(H) \to X(\langle g \rangle H), & \text{the conjugation maps,} \\
\text{res}^H_K & : X(H) \to X(K), & \text{the restriction maps,} \\
\text{ind}^H_K & : X(K) \to X(H), & \text{the induction maps,}
\end{align*}
\]

for \( g \in G \) and \( K \leq H \leq G \), satisfying the axioms

\[
\begin{align*}
\text{(G.1)} & \quad \text{con}^g_H \circ \text{con}^r_H = \text{con}^{gr}_H, & \text{con}^h_H = \text{id}_{X(H)}, \\
\text{(G.2)} & \quad \text{res}^K_L \circ \text{res}^H_K = \text{res}^L_H, & \text{res}^H_H = \text{id}_{X(H)}, \\
\text{(G.3)} & \quad \text{con}^g_K \circ \text{res}^H_K = \text{res}^{gh}_K \circ \text{con}^g_H, \\
\text{(G.4)} & \quad \text{ind}^H_K \circ \text{ind}^L_K = \text{ind}^H_L, & \text{ind}^H_H = \text{id}_{X(H)}, \\
\text{(G.5)} & \quad \text{con}^g_H \circ \text{ind}^H_K = \text{ind}^{gh}_K \circ \text{con}^g_K, \\
\text{(G.6) (Mackey axiom)} & \quad \text{res}^H_K \circ \text{ind}^H_U = \sum_{K \cap hU \subseteq K \setminus H/U} \text{ind}^K_{K \cap hU} \circ \text{res}^h_{K \cap hU} \circ \text{con}^h_U
\end{align*}
\]

for all \( U \leq H \leq G \), \( L \leq K \leq H \), \( g, r \in G \), and \( h \in H \), which was introduced by Green [11]. A Green functor for \( G \) over \( k \) is a Mackey functor \( X = (X, \text{con}, \text{res}, \text{ind}) \) for \( G \) over \( k \) such that the \( k \)-modules \( X(H) \) for \( H \leq G \) are \( k \)-algebras, the conjugation maps and the restriction maps are \( k \)-algebra homomorphisms, and the axioms

\[
\begin{align*}
\text{(G.7) (Frobenius axioms)} & \quad x \cdot \text{ind}^H_K(y) = \text{ind}^H_K(\text{res}^H_K(x) \cdot y), & \text{ind}^H_K(y) \cdot x = \text{ind}^H_K(y \cdot \text{res}^H_K(x))
\end{align*}
\]
are satisfied for all $K \leq H$, $x \in X(H)$, and $y \in X(K)$.

A semigroup with identity is called a monoid. We denote by $\epsilon_A$ the identity of a monoid $A$. A map $f : A \to B$ between monoids is called a (monoid) homomorphism if $f(\epsilon_A) = \epsilon_B$ and $f(x \cdot y) = f(x) \cdot f(y)$ for all $x, y \in A$.

**Definition 2.1** A monoid functor for $G$ is a triple $M = (M, \text{con}, \text{res})$ consisting of a family of monoids $M(H)$ for $H \leq G$ and a family of monoid homomorphisms

\[
\begin{align*}
\text{con}^g_H : M(H) &\to M(gH), & \text{the conjugation maps}, \\
\text{res}^H_K : M(H) &\to M(K), & \text{the restriction maps},
\end{align*}
\]

for $g \in G$ and $K \leq H \leq G$, satisfying the axioms

\[
\begin{align*}
\text{(M.1)} \; \text{con}^g_H \circ \text{con}^r_H &= \text{con}^{gr}_H, & \text{con}^h_H &= \text{id}_{M(H)}, \\
\text{(M.2)} \; \text{res}^K_L \circ \text{res}^H_K &= \text{res}^L_K, & \text{res}^H_K &= \text{id}_{M(H)}, \\
\text{(M.3)} \; \text{con}^q_K \circ \text{res}^H_K &= \text{res}^q_K \circ \text{con}^q_H
\end{align*}
\]

for all $L \leq K \leq H \leq G$, $g, r \in G$, and $h \in H$.

Henceforth, we suppose that $M$ is a monoid functor for $G$. Let $H \leq G$, and set

\[
\widetilde{M}(H) = \bigcup_{K \leq H} M(K).
\]

We consider $\widetilde{M}(H)$ to be a left $H$-set with the action given

\[
h_s = \text{con}^h_K(s)
\]

for all $h \in H$ and $s \in M(K)$ with $K \leq H$. Given $J, J_0 \in H\text{-set}$, let $\text{Map}_H(J_0, J)$ be the set of $H$-equivariant maps from $J_0$ to $J$. There exists an additive contravariant functor $T_H^M : H\text{-set} \to \text{Mon}$, where $\text{Mon}$ is the category of monoids, such that for each $J \in H\text{-set}$, $T_H^M(J)$ is defined to be the monoid

\[
\{\pi \in \text{Map}_H(J, \widetilde{M}(H)) \mid \pi(x) \in M(H_x) \text{ for all } x \in J\}
\]

with pointwise multiplication, where $H_x$ is the stabilizer of $x$ in $H$, and the morphism $T_H^M(\lambda) : T_H^M(J) \to T_H^M(J_0)$ with $J, J_0 \in H\text{-set}$ and $\lambda \in \text{Map}_H(J_0, J)$ is defined by

\[
T_H^M(\lambda)(\pi) : J_0 \to \widetilde{M}(H), \quad x \mapsto \text{res}^{H_{\lambda(x)}}_{H_x}(\pi(\lambda(x)))
\]

for all $\pi \in T_H^M(J)$ (cf. [12, §2]). We call a pair $(J, \pi)$ of $J \in H\text{-set}$ and $\pi \in T_H^M(J)$ an element of $T_H^M$. The morphisms $\lambda : (J_0, \pi_0) \to (J, \pi)$ between elements $(J_0, \pi_0)$ and $(J, \pi)$ of $T_H^M$ are defined to be the $H$-equivariant maps $\lambda : J_0 \to J$ such that $T_H^M(\lambda)(\pi) = \pi_0$. Thus we obtain the category of elements of $T_H^M$ (cf. [15, (2.10)]).
Definition 2.2 Let $H \leq G$. For each element $(J, \pi)$ of $T_H^M$, let $\{J, \pi\}$ denote the isomorphism class of elements of $T_H^M$ containing $(J, \pi)$. We define a ring $\Omega(H, M)$ to be the ring consisting of all $\mathbb{Z}$-linear combinations of isomorphism classes of elements of $T_H^M$ with addition and multiplication given by

$$[J_1, \pi_1] + [J_2, \pi_2] = [J_1 \cup J_2, \pi_1 + \pi_2] \quad \text{and} \quad [J_1, \pi_1] \cdot [J_2, \pi_2] = [J_1 \times J_2, \pi_1 \cdot \pi_2],$$

where

$$\pi_1 + \pi_2 : J_1 \cup J_2 \to \tilde{M}(H), \quad x \mapsto \pi_1(x) \quad \text{if} \quad x \in J_1, \quad x \mapsto \pi_2(x) \quad \text{if} \quad x \in J_2$$

and

$$\pi_1 \cdot \pi_2 : J_1 \times J_2 \to \tilde{M}(H), \quad (x_1, x_2) \mapsto \pi_1(x_1) \cdot \pi_2(x_2),$$

where

$$\pi_1(x_1) \cdot \pi_2(x_2) = \text{res}_{H_{x_1} \cap H_{x_2}}^H(\pi_1(x_1)) \cdot \text{res}_{H_{x_1} \cap H_{x_2}}^H(\pi_2(x_2)),$$

for all elements $(J_1, \pi_1)$ and $(J_2, \pi_2)$ of $T_H^M$, and call it the $M$-Burnside ring. This ring is the $F$-Burnside ring with $F = T_H^M$ introduced by Jacobson [12] (cf. [14]).

For each $H \leq G$, let $\mathbb{Z}M(H)$ be the $\mathbb{Z}$-algebra consisting of all $\mathbb{Z}$-linear combinations of elements of $M(H)$ with multiplication given by the binary operation of $M(H)$, which is called a monoid ring. Then the family of monoid rings $\mathbb{Z}M(H)$ for $H \leq G$ defines an algebra restriction functor $\mathbb{Z}M = (\mathbb{Z}M, \text{con}, \text{res})$ with conjugation maps and restriction maps given by those of the monoid functor $M = (M, \text{con}, \text{res})$ (cf. [4, 19]). The ring $\Omega(H, T_H^{\mathbb{Z}M})$ defined in [19, §3] coincides with $\Omega(H, M)$.

Let $H \leq G$. The Burnside ring $\Omega(H)$ is the commutative ring consisting of all $\mathbb{Z}$-linear combinations of isomorphism classes $[J]$ for $J \in H\text{-set}$ with disjoint union for addition and cartesian product for multiplication (cf. [7, §80]). When $\tilde{M}(H)$ is the monoid $\{e\}$ consisting of only the identity $e$, we regard $\Omega(H, M)$ with $\Omega(H)$.

The Burnside ring functor $\Omega = (\Omega, \text{con}, \text{res}, \text{ind})$, which is a Green functor, is defined to be the family of $\mathbb{Z}$-algebras $\Omega(H)$ for $H \leq G$, together with conjugation maps $\text{con}^H_H : \Omega(H) \to \Omega(H)$, where $g \in G$, restriction maps $\text{res}^H_K : \Omega(H) \to \Omega(K)$, and induction maps $\text{ind}^H_K : \Omega(K) \to \Omega(H)$, where $K \leq H$ in both cases, arising from the usual conjugation $\text{con}^H_H(J)$, restriction $\text{res}^H_K(J)$, and induction $\text{ind}^H_K(V)$, where $J \in H\text{-set}$ and $V \in K\text{-set}$ (cf. [21, Example 2.11]).

Following [19, §3], we define the $M$-Burnside ring functor. Let $g \in G$, and let $K \leq H \leq G$. Given an element $(J, \pi)$ of $T_H^M$ and an element $(V, \varpi)$ of $T_K^M$, we define $\varpi^g \in T_H^M(\text{con}^g_H(J))$, $\pi|_K \in T_K^M(\text{res}^H_K(J))$, and $\varpi^H \in T_H^M(\text{ind}^H_K(V))$ by

$$(\varpi^g)(g \otimes x) = \varpi(x), \quad \pi|_K(x) = \text{res}^H_K(\pi(x)), \quad \text{and} \quad \varpi^H(h \otimes y) = h\varpi(y)$$

for all $x \in J, y \in V$, and $h \in H$ (see [19, p. 97]).
Definition 2.3 For each $H \leq G$, let $\Omega^M(H)$ denote the $M$-Burnside ring $\Omega(H, M)$. We define a Green functor $\Omega^M = (\Omega^M, \text{con}, \text{res}, \text{ind})$ to be the family of $\mathbb{Z}$-algebras $\Omega^M(H)$ for $H \leq G$, together with conjugation maps $\text{con}^g_H : \Omega^M(H) \to \Omega^M(gH)$, where $g \in G$, restriction maps $\text{res}_K^H : \Omega^M(H) \to \Omega^M(K)$, and induction maps $\text{ind}_K^H : \Omega^M(K) \to \Omega^M(H)$, where $K \leq H$ in both cases, given by
\[
\text{con}^g_H([J, \pi]) = [\text{con}^g_H(J), g\pi],
\]
\[
\text{res}_K^H([J, \pi]) = [\text{res}^H_K(J), \pi|_K],
\]
\[
\text{ind}_K^H([V, \varpi]) = [\text{ind}^H_K(V), \varpi^H]
\]
for all elements $(J, \pi)$ of $T^M_H$ and for all elements $(V, \varpi)$ of $T^M_K$.

We call the Green functor $\Omega^M = (\Omega^M, \text{con}, \text{res}, \text{ind})$ the $M$-Burnside ring functor, which is a $G$-functor version of the $F$-Burnside ring functor defined in [12, 14].

Let $H \leq G$, and set $\mathcal{S}(H, M) = \{(K, s) \mid K \leq H, s \in M(K)\}$. For each $(K, s) \in \mathcal{S}(H, M)$, we define an $H$-equivariant map $\pi_s : H/K \to \tilde{M}(H)$ by

$hK \mapsto h^s \in M(hK)$

for all $h \in H$, and set $[(H/K)_s] = [H/K, \pi_s] \in \Omega(H, M)$.

Definition 2.4 Given $H \leq G$, we define an action of $H$ on $\mathcal{S}(H, M)$ by

$h.K(s) = (hK, h^s)$

for all $h \in H$ and $(K, s) \in \mathcal{S}(H, M)$, and denote by $\mathcal{R}(H, M)$ a complete set of representatives of $H$-orbits in $\mathcal{S}(H, M)$ such that $K \in \mathcal{C}(H)$, where $\mathcal{C}(H)$ is a full set of nonconjugate subgroups of $H$, for each $(K, s) \in \mathcal{R}(H, M)$.

Given $(K, s), (U, t) \in \mathcal{S}(H, M)$, it is easily verified that $(H/K)_s \simeq (H/U)_t$ if and only if $(K, s) = h(U, t)$ for some $h \in H$ (cf. Example 5.2).

Proposition 2.5 Let $H \leq G$. The elements $[(H/K)_s]$ for $(K, s) \in \mathcal{R}(H, M)$ form a $\mathbb{Z}$-basis of $\Omega(H, M)$, and multiplication on $\Omega(H, M)$ is given by

\[
[(H/K)_s] \cdot [(H/U)_t] = \sum_{K \cap H \in K \cap H/U} \left( (H/(K \cap hU)) \times_{\text{res}_{K \cap hU}^K} (K/(K \cap hU)) \times_{\text{res}_{K \cap hU}^U} (K/(K \cap hU)) \right)
\]

for all $(K, s), (U, t) \in \mathcal{R}(H, M)$. Moreover,

\[
\text{con}^g_H([(H/U)_t]) = [(gH/gU)_s],
\]
\[
\text{res}_K^H([(H/U)_t]) = \sum_{K \cap H \in K \cap H/U} \left( (K/(K \cap hU)) \times_{\text{res}_{K \cap hU}^K} (K/(K \cap hU)) \times_{\text{res}_{K \cap hU}^U} (K/(K \cap hU)) \right),
\]
\[
\text{ind}_K^H([(K/L)_s]) = [(H/L)_s]
\]
for all $g \in G$, $(U, t) \in \mathcal{R}(H, M)$, $K \leq H$, and $(L, s) \in \mathcal{R}(K, M)$. 

Proof. The proof is straightforward. (See also the proof of \cite[Proposition 3.1]{19}.)

Example 2.6 Let $A$ be a finite abelian $G$-group, that is, a finite abelian group on which $G$ acts as automorphisms of $A$ (cf. \cite[Chapter 1, Definition 8.1]{17}). Given $g \in G$ and $a \in A$, the effect of $g$ on $a$ is denoted by $ga$. Let $H \leq G$. By restriction of operators from $G$ to $H$, we view $A$ as an $H$-group. A map $\sigma : H \to A$ is called a 1-cocycle or a crossed homomorphism if

$$\sigma(h_1h_2) = \sigma(h_1)h_i\sigma(h_2)$$

for all $h_1, h_2 \in H$ (cf. \cite[I, p. 243]{17}). The set $Z^1(H, A)$ of 1-cocycles from $H$ to $A$ is an abelian group with the product operation given by

$$\sigma \cdot \tau(h) = \sigma(h)\tau(h)$$

for all $\sigma, \tau \in Z^1(H, A)$ and $h \in H$. Let $\sigma \in Z^1(H, A)$. For each $K \leq H$, let $\sigma|_K : K \to A$ denote the 1-cocycle obtained by restriction of $\sigma : H \to A$ from $H$ to $K$. Given $g \in G$ and $a \in A$, we define 1-cocycles $g\sigma : gH \to A$ and $\sigma^a : H \to A$ by

$$(g\sigma)(ghg^{-1}) = g\sigma(h) \quad \text{and} \quad \sigma^a(h) = a^{-1}\sigma(h)h$$

for all $h \in H$, respectively. Let $\sigma$ denote the $A$-orbit $\{\sigma^a \mid a \in A\}$ in $Z^1(H, A)$ containing $\sigma$. We denote by $H^1(H, A)$ the set of $A$-orbits in $Z^1(H, A)$, that is,

$$H^1(H, A) = \{\sigma \mid \sigma \in Z^1(H, A)\},$$

and make it into an abelian group by defining

$$\overline{\sigma} \cdot \overline{\tau} = \overline{\sigma \cdot \tau}$$

for all $\sigma, \tau \in Z^1(H, A)$. Observe that $\overline{(g\sigma)} = \overline{(g\sigma)^g} = \overline{g\sigma}$ for all $g \in G$ and $a \in A$.

We define a monoid functor $H_A = (H_A, \text{con}, \text{res})$ for $G$ by

$$H_A(H) = H^1(H, A),$$

$$\text{con}^g_H : H^1(H, A) \to H^1(gH, A), \overline{\sigma} \mapsto \overline{g\sigma},$$

$$\text{res}^K_H : H^1(H, A) \to H^1(K, A), \overline{\sigma} \mapsto \overline{\sigma|_K}$$

for all $g \in G$ and $K \leq H \leq G$. The $H_A$-Burnside ring $\Omega(G, H_A)$ is isomorphic to the ring of monomial representations of $G$ with coefficients in $A$ introduced by Dress \cite{9} which is also called the monomial Burnside ring for $G$ with fibre group $A$ in \cite{2}.
Example 2.7 Let $S$ be a finite $G$-monoid, that is, a finite monoid on which $G$ acts as monoid homomorphisms (cf. [15, (2.1)]). Given $g \in G$ and $s \in S$, $g s$ denotes the effect of $g$ on $s$. We define a monoid functor $C_S = (C_S, \text{con}, \text{res})$ for $G$

$$C_S(H) = \{ s \in S \mid h s = s \quad \text{for all} \quad h \in H \},$$

$$\text{con}^g_H : C_S(H) \to C_S(gH), \quad s \mapsto g s,$$

$$\text{res}^H_K : C_S(H) \to C_S(K), \quad s \mapsto s$$

for all $g \in G$ and $K \leq H \leq G$. The crossed Burnside ring functor defined in [16, §4] is isomorphic to the $C_S$-Burnside ring functor. The ring $\Omega(G, C_S)$ is isomorphic to the crossed Burnside ring of $G$ associated to $S$ (cf. [5, 15]).

3 A fundamental theorem of $M$-Burnside rings

Following [15, §4], we present a fundamental theorem for $M$-Burnside rings. The results in this section are special cases of those in [19, §9].

Let $H \leq G$. For any $g \in G$, there is a map $\text{con}^g_H : ZM(H) \to ZM(gH)$ given by

$$\sum_j \ell_j s_j \mapsto \sum_j \ell_j g s_j$$

for all $\sum_j \ell_j s_j \in ZM(H)$ with $\ell_j \in \mathbb{Z}$ and $s_j \in M(H)$.

We define a subring $\mathfrak{U}(H, M)$ of $\prod_{U \leq H} ZM(U)$ by

$$\mathfrak{U}(H, M) = \left\{ (x_U)_{U \leq H} \in \prod_{U \leq H} ZM(U) \mid \text{con}^h_U (x_U) = x_U \quad \text{for all} \quad h \in H \right\}.$$ 

There is a ring homomorphism $\rho_H : \Omega(H, M) \to \mathfrak{U}(H, M)$ given by

$$[(H/K)_s] \mapsto \left( \sum_{hK \in H/K, U \leq hK} \text{res}^{hK}_U (h s) \right)_{U \leq H}$$

for all $(K, s) \in \mathcal{R}(H, M)$ (cf. [4, 2.3]), which is called the mark homomorphism.

Definition 3.1 We define a Green functor $\mathfrak{U}^M = (\mathfrak{U}^M, \text{con}, \text{res}, \text{ind})$ for $G$ by

$$\mathfrak{U}^M(H) = \mathfrak{U}(H, M),$$

$$\text{con}^g_H ((x_U)_{U \leq H}) = (\text{con}^g_U (x_U))_{U \leq gH},$$

$$\text{res}^H_K ((x_U)_{U \leq H}) = (x_U)_{U \leq K},$$

$$\text{ind}^H_K ((y_U)_{U \leq K}) = \sum_{hK \in H/K} (a^h_L)_{L \leq H}$$
for all \( g \in G, K \leq H \leq G, (x_U)_{U \leq H} \in \mathfrak{U}(H, M), \) and \((y_U)_{U \leq K} \in \mathfrak{U}(K, M), \) where

\[
a^h_L = \begin{cases} 
\text{con}_{\mathcal{G}}(y_U) & \text{if } L = hU \text{ with } U \leq K, \\
0 & \text{otherwise}.
\end{cases}
\]

**Remark 3.2** Let us remind the plus constructions \(-_+ : \text{Res}_{\text{alg}}(G) \to \text{Green}(G) \) and \(-^+ : \text{Con}_{\text{alg}}(G) \to \text{Green}(G), \) which are defined in [4]. The Green functors \( \mathbb{Z}M_+ \) and \( \mathbb{Z}M^+ \) are isomorphic to \( \Omega^M \) and \( \Omega^M \), respectively (cf. [19, Proposition 3.1]), and there is a morphism of Green functors \( \rho : \Omega^M \to \Omega^M \) defined to be a family of the mark homomorphisms \( \rho_H \) for \( H \leq G, \) which is called the mark morphism.

**Definition 3.3** For each \( H \leq G, \) we define an additive group \( \bar{\Omega}(H, M) \) to be

\[
\bar{\Omega}(H, M) = \coprod_{(K, s) \in \mathcal{R}(H, M)} \mathbb{Z}_s
\]

and define an additive map \( \varphi_H : \Omega(H, M) \to \bar{\Omega}(H, M) \) by

\[
[(H/K)_s] \mapsto (\{ hK \in H/K \mid U \leq ^hK \text{ and } t = \text{res}_U^hK(h_s) \})_{(U, t) \in \mathcal{R}(H, M)}
\]

for all \((K, s) \in \mathcal{R}(H, M)\) (cf. [19, p. 130]).

Let \( H \leq G. \) Given \((U, t) \in \mathcal{S}(H, M)\) and \((x_{(K, s)})_{(K, s) \in \mathcal{R}(H, M)} \in \bar{\Omega}(H, M), \) we set \( x_{(U, t)} = x_{(K, s)} \) if \( h(U, t) = (K, s) \in \mathcal{R}(H, M) \) for some \( h \in H. \)

There exists an additive map \( \varsigma_H : \bar{\Omega}(H, M) \to \Omega(H, M) \) given by

\[
(x_{(K, s)})_{(K, s) \in \mathcal{R}(H, M)} \mapsto \sum_{(U, t) \in \mathcal{S}(H, M)} x_{(U, t)} \sum_{L \leq U} |L| \mu(L, U) \left( (H/L)_{\text{res}^L_U(t)} \right)
\]

for all \((x_{(K, s)})_{(K, s) \in \mathcal{R}(H, M)} \in \bar{\Omega}(H), \) where \( \mu \) is the Möbius function of the partially ordered set consisting of all subgroups of \( G \) with the binary relation \( \leq \) (cf. [1]).

**Proposition 3.4** For any \( H \leq G, \)

\[
\varsigma_H \circ \varphi_H = |H| \cdot \text{id}_{\Omega(H, M)} \quad \text{and} \quad \varphi_H \circ \varsigma_H = |H| \cdot \text{id}_{\bar{\Omega}(H, M)}.
\]

**Proof.** Let \( H \leq G. \) For each \((K, s) \in \mathcal{R}(H, M), \) we have

\[
\varsigma_H \circ \varphi_H ([(H/K)_s])
\]

\[
= \sum_{(U, t) \in \mathcal{S}(H, M)} \varphi_{(U, t)}([(H/K)_s]) \sum_{L \leq U} |L| \mu(L, U) \left( (H/L)_{\text{res}^L_U(t)} \right)
\]

\[
= \sum_{hK \in H/K} \sum_{L \leq hK \leq U \leq hK} |L| \mu(L, U) \left( (H/L)_{\text{res}^L_U(h_s)} \right)
\]

\[
= |H| \cdot [(H/K)_s],
\]
where
\[ \varphi_{(U,t)}([\langle H/K \rangle]_s) = \{ hK \in H/K \mid U \leq hK \quad \text{and} \quad t = \text{res}_{hK}^h(hs) \}. \]

Thus \( \varsigma_H \circ \varphi_H(x) = |H| \cdot x \) for any \( x \in \Omega(H, M) \). Let \( \delta \) denote the Kronecker delta. For each \( (x(K,s))(K,s) \in R(H,M) \in \tilde{\Omega}(H, M) \), we have
\[
\varphi_H \circ \varsigma_H((x(K,s))(K,s) \in R(H,M)) \\
= \sum_{(U,t) \in S(H,M)} x(U,t) \sum_{L \leq U} |L| \mu(L,U) \left( \sum_{h \in H/L, K \leq hL} \delta_{s \text{res}_{hK}^h(hU)} \right) \left( (K,s) \in R(H,M) \right) \\
= \sum_{h \in H} \left( \sum_{L \leq U} \mu(hL, U) \sum_{t \in M(U)} x(hU, t) \delta_{s \text{res}_{hK}^h(hU))} \right) \left( (K,s) \in R(H,M) \right) \\
= \sum_{h \in H} \left( \delta_K^hU x(K,s) \right)_{(K,s) \in R(H,M)} \\
= |H| \cdot (x(K,s))(K,s) \in R(H,M),
\]
Hence \( \varphi_H \circ \varsigma_H(y) = |H| \cdot y \) for any \( y \in \tilde{\Omega}(H, M) \). This completes the proof. \( \Box \)

Given \( H \leq G \) and \( (K,s) \in R(H,M) \), we set
\[ N_H(K,s) = \{ h \in H \mid hK = K \quad \text{and} \quad hs = s \}, \]
which is a subgroup of the normalizer \( N_H(K) \) of \( K \) in \( H \), and set
\[ W_H(K,s) = N_H(K,s)/K. \]

**Proposition 3.5** Let \( H \leq G \). Given \( (K,s) \in R(H, M) \) and \( L \leq H \), define
\[ y^{(K,s)}_L := \begin{cases} \sum_{hN_H(K,s) \in N_H(K,s)/N_H(K,s) \mid \text{if } L = rK \text{ for some } r \in H,}^{rh_s} & \\
0 & \text{otherwise}. \end{cases} \]

The elements \( (y^{(K,s)}_L)_{L \leq H} \) of \( \mathcal{U}(H, M) \) for \( (K,s) \in R(H, M) \) form a free \( \mathbb{Z} \)-basis of \( \mathcal{U}(H, M) \), and the additive map \( \kappa_H : \Omega(H, M) \to \mathcal{U}(H, M) \) given by
\[ \langle (\delta_{\varsigma_H(\varsigma_H(x))}(U,t))(U,t) \in R(H,M) \rangle \mapsto (y^{(K,s)}_L)_{L \leq H} \]
for all \( (K,s) \in R(H, M) \) is an isomorphism. Moreover, the diagram
is commutative, and $\rho_H$ and $\varphi_H$ are injective.

**Proof.** The proof of the first assertion is straightforward (cf. [19, §9]). We have

$$\kappa_H \circ \varphi_H([(H/K)_s]) =$$

$$= \left( \sum_{(U,t) \in \mathcal{R}(H,M)} \sum_{h_K \in H/K, U \leq h_K} \delta_{1 \text{res}^h_K(h_s)} \gamma^U_L^{(U,t)} \right)_{L \leq H}$$

$$= \left( \sum_{h_K \in H/K} \sum_{(U,t) \in \mathcal{R}(H,M)} \sum_{h_N \in N_H(U, t) \in N_H(U)/N_H(U,t)} z_{L}^{\{h_s, h_1 t\}} \right)_{L \leq H}$$

$$= \left( \sum_{h_K \in H/K, L \leq h_K} \text{res}_L^{h_K}(h_s) \right)_{L \leq H}$$

$$= \rho_H([(H/K)_s]),$$

where

$$z_{L}^{\{h_s, h_1 t\}} = \begin{cases} \frac{r h_1 t}{L} & \text{if } L = rU \leq rK \text{ for some } r \in H \text{ and if } t = \text{res}^h_U(h_s), \\ 0 & \text{otherwise,} \end{cases}$$

for all $(K, s) \in \mathcal{R}(H, M)$, and thus $\kappa_H \circ \varphi_H = \rho_H$. Moreover, by Proposition 3.4, $\varphi_H$ is injective, and so is $\rho_H$ (cf. [4, 2.4]). This completes the proof. □

**Proposition 3.6** For each $H \leq G$,

$$\bar{\Omega}(H, M) = \bigoplus_{(U,t) \in \mathcal{R}(H,M)} \frac{1}{|W_H(U,t)|} \varphi_H([(H/U)_t]) \mathbb{Z}.$$  

**Proof.** The proof is analogous to that of [7, (80.15) Proposition]. □

**Definition 3.7** For each $H \leq G$, we define an additive group $\text{Obs}(H, M)$ to be

$$\prod_{(U,t) \in \mathcal{R}(H,M)} \mathbb{Z}/|W_H(U,t)| \mathbb{Z}.$$
Lemma 3.8 For each $H \leq G$,

$$\tilde{\Omega}(H, M)/\text{Im}\varphi_H \cong \text{Obs}(H, M).$$

Proof. By Propositions 2.5 and 3.5, we have

$$\text{Im}\varphi_H = \bigoplus_{(U,t) \in \mathcal{R}(H,M)} \varphi_H([(H/U)_t])\mathbb{Z}.$$ 

Hence the assertion follows from Proposition 3.6. □

Let $p$ be a prime, and let $\infty$ be just a symbol. For each $\mathbb{Z}$-module $R$, we set $R(p) = \mathbb{Z}(p) \otimes \mathbb{Z} R$, where $\mathbb{Z}(p)$ is the localization of $\mathbb{Z}$ at $p$, and $R(\infty) = R$.

Let $H \leq G$. Given $(U,t) \in \mathcal{S}(H,M)$, we denote by $W_H(U,t)_p$ a Sylow $p$-subgroup of $W_H(U,t)$, and set $W_H(U,t)_\infty = W_H(U,t)$. By Proposition 2.5, the elements $[(G/K)_s]$ for $(K,s) \in \mathcal{R}(H,M)$ are supposed to form a free $\mathbb{Z}(p)$-basis of $\Omega(H,M)(p)$. We identify $\Omega(H,M)(p)$ and $\text{Obs}(H,M)(p)$ with

$$\prod_{(K,s) \in \mathcal{R}(H,M)} \mathbb{Z}(p) \text{ and } \prod_{(U,t) \in \mathcal{R}(H,M)} \mathbb{Z}(p)/|W_G(U,t)_p|\mathbb{Z}(p),$$

respectively. Let $\varphi_H^{(p)}$ denote the $\mathbb{Z}(p)$-module homomorphism from $\Omega(H,M)(p)$ to $\tilde{\Omega}(H,M)(p)$ determined by $\varphi_H$. Then it follows from Lemma 3.8 that

$$\tilde{\Omega}(H, M)(p)/\text{Im}\varphi_H^{(p)} \cong \text{Obs}(H, M)(p). \quad (3.1)$$

Henceforth, we denote by $p$ a prime or the symbol $\infty$, and set $\varphi^{(\infty)}_H = \varphi_H$. The expression ‘$a$ mod $b$’ with $a, b \in \mathbb{Z}(p)$ denotes the coset of $a + b\mathbb{Z}(p)$ of $b\mathbb{Z}(p)$ in $\mathbb{Z}(p)$ containing $a$. Given $(U,t) \in \mathcal{S}(H,M)$ and $(x_{(K,s)})_{(K,s) \in \mathcal{R}(H,M)} \in \tilde{\Omega}(H,M)(p)$, we set $x_{(U,t)} = x_{(K,s)}$ if $h(U,t) = (K,s) \in \mathcal{R}(H,M)$ for some $h \in H$.

We define a $\mathbb{Z}(p)$-module homomorphism $\psi_H^{(p)} : \tilde{\Omega}(H,M)(p) \rightarrow \text{Obs}(H,M)(p)$ by

$$(x_{(K,s)})_{(K,s) \in \mathcal{R}(H,M)} \mapsto \left( \sum_{s \in M((r)U)_t} x_{((r)U,s)} \mod |W_H(U,t)_p| \right)_{(U,t) \in \mathcal{R}(H,M)},$$

where

$$M((r)U)_t = \{s \in M((r)U) \mid \text{res}_{U}^{(r)U}(s) = t\},$$

for all $(x_{(K,s)})_{(K,s) \in \mathcal{R}(H,M)} \in \tilde{\Omega}(H,M)$ (cf. [19, p. 133]).

Lemma 3.9 For each $H \leq G$, $\psi_H^{(p)}$ is surjective.
Proof. The lemma follows from [19, Lemma 9.3]. (See also the proof of Lemma 4.9.) \(\square\)

Let \(H \leq G\), and let \((K, s) \in \mathcal{R}(H, M)\). We have

\[
\psi_H^{(p)} \left( \varphi_H^{(p)} \left( \left[ (H/K)_s \right] \right) \right) = \sum_{rU \in W_H(U, t)p} \left| \text{inv}_{(r)U}((H/K)_s)_{(U, t)} \right| \mod |W_H(U, t)_p|, \quad (U, t) \in \mathcal{R}(H, M)
\]

where

\[
\text{inv}_{(r)U}((H/K)_s)_{(U, t)} = \{ hK \in H/K \mid (r)U \leq hK \text{ and } \text{res}_{(r)U}^hK(hs) = t \}.
\]

Given \((U, t) \in \mathcal{R}(H, M)\) and \(F \leq W_H(U, t)\), it follows from [19, Lemma 9.2] that

\[
\sum_{rU \in F} \left| \text{inv}_{(r)U}((H/K)_s)_{(U, t)} \right| \equiv 0 \pmod{|F|}.
\]

(See also the proof of Lemma 4.10). Hence we have

\[
\text{Im} \varphi_H^{(p)} \subseteq \text{Ker} \psi_H^{(p)}.
\]  \( (3.2) \)

The following theorem is a generalization of [15, (4.4) Theorem].

**Theorem 3.10 (Fundamental theorem)** For each \(H \leq G\), the sequence

\[
0 \rightarrow \Omega(H, M)_{(p)} \xrightarrow{\varphi_H^{(p)}} \tilde{\Omega}(H, M)_{(p)} \xrightarrow{\psi_H^{(p)}} \text{Obs}(H, M)_{(p)} \rightarrow 0
\]

of \(\mathbb{Z}_{(p)}\)-modules is exact.

**Proof.** The theorem is a special case of [19, Theorem 9.4]. We give a standard and concise proof of the theorem under the assumption that all the monoids \(M(H)\) for \(H \leq G\) are finite. Let \(H \leq G\). By Proposition 3.5 and Lemma 3.9, it is enough to verify that \(\text{Im} \varphi_H^{(p)} = \text{Ker} \psi_H^{(p)}\). Since \(\psi_H^{(p)}\) is surjective, it follows that

\[
\tilde{\Omega}(H, M)_{(p)}/\text{Ker} \psi_H^{(p)} \simeq \text{Obs}(H, M)_{(p)}.
\]

Moreover, by Eqs.\((3.1)\) and \((3.2)\), there is a sequence

\[
\text{Obs}(H, M)_{(p)} \xrightarrow{\sim} \tilde{\Omega}(H, M)_{(p)}/\text{Im} \varphi_H^{(p)} \rightarrow \tilde{\Omega}(H, M)_{(p)}/\text{Ker} \psi_H^{(p)}
\]

of finite groups, where the first arrow is an isomorphism and the second one is a natural surjection. Hence we have \(\text{Im} \varphi_H^{(p)} = \text{Ker} \psi_H^{(p)}\), completing the proof. \(\square\)
4 Lattice Burnside ring functors

Let $\mathcal{L}$ be a complete lattice with the binary relation $\leq$ (cf. [3]). For each subset $\Sigma$ of $\mathcal{L}$, $\inf \Sigma$ denotes the greatest lower bound of $\Sigma$ in $\mathcal{L}$, and $\sup \Sigma$ denotes the least upper bound of $\Sigma$ in $\mathcal{L}$. Given $x, y \in \mathcal{L}$, we set $x \wedge y = \inf \{x, y\}$ and $x \vee y = \sup \{x, y\}$. We consider $\mathcal{L}$ as a monoid with the binary operation given by $x \cdot y = x \wedge y$ for all $x, y \in \mathcal{L}$. (Of course, the binary operation $\wedge$ is associative.) The identity of the monoid $\mathcal{L}$ is the greatest element of $\mathcal{L}$. We call $\mathcal{L}$ a finite $G$-lattice if $\mathcal{L}$ is a finite left $G$-set and the binary relation $\leq$ is invariant under the action of $G$. Assume that $\mathcal{L}$ is a finite $G$-lattice. Then it turns out that $\mathcal{L}$ is a finite $G$-monoid (see Example 2.7). Given $g \in G$ and $s \in \mathcal{L}$, $g s$ denotes the effect of $g$ on $s$.

Example 4.1 The set $\Lambda$ of subsets of a finite left $G$-set is considered as a finite $G$-lattice with the binary relation given by inclusion and the action of $G$ given by $g I = \{g x \mid x \in I\}$ for all $g \in G$ and $I \in \Lambda$; the binary operations $\wedge$ and $\vee$ are $\cap$ and $\cup$, respectively.

Proposition 4.2 Let $\mathcal{L}$ be a finite $G$-lattice. For each $H \leq G$, let $\mathcal{L}_H$ be a nonempty sublattice of $\mathcal{L}$. Suppose that the family of nonempty sublattices $\mathcal{L}_H$ of $\mathcal{L}$ for $H \leq G$ satisfies the following conditions.

1. $\mathcal{L}_{g H} = \{g s \mid s \in \mathcal{L}_H\}$ for any $g \in G$.
2. $h s = s$ for any $h \in H$ and $s \in \mathcal{L}_H$.
3. $s \wedge \sup \mathcal{L}_K \in \mathcal{L}_K$ for any $K \leq H$ and $s \in \mathcal{L}_H$.
4. $\sup \mathcal{L}_K \leq \mathcal{L}_H$ for any $K \leq H$.

Then there exists a monoid functor $M_{\mathcal{L}} = (M_\mathcal{L}, \mathrm{con}, \mathrm{res})$ for $G$ given by

$M_{\mathcal{L}}(H) = \mathcal{L}_H$,

$\mathrm{con}_H^g : \mathcal{L}_H \to \mathcal{L}_{g H}, \; s \mapsto g s,$

$\mathrm{res}_K^H : \mathcal{L}_H \to \mathcal{L}_K, \; s \mapsto s \wedge \sup \mathcal{L}_K$

for all $g \in G$ and $K \leq H \leq G$.

Proof. The proof is straightforward. □

Let $\mathcal{L}$ be a finite $G$-lattice, and let $M_{\mathcal{L}} = (M_\mathcal{L}, \mathrm{con}, \mathrm{res})$ be the monoid functor given in Proposition 4.2. The $M_{\mathcal{L}}$-Burnside ring functor is called the lattice Burnside ring functor on $M_{\mathcal{L}}$, and $\Omega(G, M_{\mathcal{L}})$ is called the lattice Burnside ring of $G$ associated to the family of nonempty sublattices $\mathcal{L}_H$ of $\mathcal{L}$ for $H \leq G$. 
Example 4.3 If $\sup \mathcal{L}$ is $G$-invariant, then the $C_{\mathcal{L}}$-Burnside ring functor given in Example 2.7 with $S = \mathcal{L}$ is the lattice Burnside ring functor on $C_{\mathcal{L}}$. In particular, the $C_{\Lambda}$-Burnside ring functor, where $\Lambda$ is the finite $G$-lattice given in Example 4.1, is the lattice Burnside ring functor on $C_{\Lambda}$, because $\sup \Lambda$ is $G$-invariant.

Let $\mathcal{S}(G)$ denote the set of subgroups of $G$. We consider $\mathcal{S}(G)$ to be a finite $G$-lattice with the binary relation given by inclusion and the action of $G$ given by conjugation, and write $\mathcal{S} = \mathcal{S}(G)$ for the sake of shortness.

Example 4.4 For each $H \leq G$, we define a nonempty sublattice $\mathcal{S}_{\geq H}$ of $\mathcal{S}$ to be the set consisting of all subgroups of $G$ containing $H$. By Proposition 4.2, there exists a monoid functor $M_{\mathcal{S}} = (M_{\mathcal{S}}, \con, \res)$ for $G$ given by

\[
M_{\mathcal{S}}(H) = \mathcal{S}_{\geq H}, \quad \con^g_H : \mathcal{S}_{\geq H} \rightarrow \mathcal{S}_{\geq gH}, \quad E \mapsto gE, \quad \res^K_H : \mathcal{S}_{\geq H} \rightarrow \mathcal{S}_{\geq K}, \quad E \mapsto E
\]

for all $g \in G$ and $K \leq H \leq G$. Let $H \leq G$. Given $E, F \in \mathcal{S}_{\geq H}$,

\[
E \cdot F = E \cap F.
\]

By Proposition 2.5, multiplication on $\Omega(H, M_{\mathcal{S}})$ is given by

\[
[(H/K)_E] \cdot [(H/U)_F] = \sum_{KhU \in K \setminus H/U} [(H/(K \cap hU))_{E \cap hF}]
\]

for all $(K, E), (U, F) \in \mathcal{R}(H, M_{\mathcal{S}})$. The lattice Burnside ring $\Omega(G, M_{\mathcal{S}})$ is isomorphic to the slice Burnside ring $\Xi(G)$ of $G$ introduced by S. Bouc [6].

Almost all results on the ring structure of lattice Burnside rings are relative to the extension of the ring structure of slice Burnside rings. There is another example.

Example 4.5 For each $H \leq G$, we define a nonempty sublattice $\mathcal{S}_{\triangleleft H}$ of $\mathcal{S}$ to be the set consisting of all normal subgroups of $H$. By Proposition 4.2, there exists a monoid functor $M_{\mathcal{S}}^\circ = (M_{\mathcal{S}}^\circ, \con, \res)$ for $G$ given by

\[
M_{\mathcal{S}}^\circ(H) = \mathcal{S}_{\triangleleft H}, \quad \con^g_H : \mathcal{S}_{\triangleleft H} \rightarrow \mathcal{S}_{\triangleleft gH}, \quad L \mapsto gL, \quad \res^K_H : \mathcal{S}_{\triangleleft H} \rightarrow \mathcal{S}_{\triangleleft K}, \quad L \mapsto L \cap K
\]

for all $g \in G$ and $K \leq H \leq G$. Let $H \leq G$. Given $L, N \in \mathcal{S}_{\triangleleft H}$,

\[
L \cdot N = L \cap N.
\]
By Proposition 2.5, multiplication on $\Omega(H, M^\circ)$ is given by
\[
[(H/K)_L] \cdot [(H/U)_N] = \sum_{K \leq U \leq K \cap H \cap U} [(H/(K \cap hU))_{L \cap hN}] \tag{4.1}
\]
for all $(K, L), (U, N) \in \mathcal{R}(H, M^\circ)$.

The lattice Burnside ring $\Omega(G, M^\circ)$ is isomorphic to an abstract Burnside ring (see Theorem 5.5). But, the fundamental theorem of $\Omega(H, M^\circ)$ (see Theorem 3.10) is not derived from that of an abstract Burnside ring (see Theorem 5.1). So we need to make an adjustment of Theorem 3.10 with $M = M^\circ$. Recall that $p$ is a prime or the symbol $\infty$. For each $H \leq G$, we consider $\Omega(H, M^\circ)(p)$ as the ring
\[
\prod_{(K, s) \in \mathcal{R}(H, M)} \mathbb{Z}(p),
\]
and provide complementary maps from $\Omega(H, M^\circ)(p)$ to itself.

**Definition 4.6** Given $U \leq G$, let $\mu_U$ denote the Möbius function of the partially ordered set $\mathcal{L}_U$ with the binary relation $\leq$. For each $H \leq G$, we define maps $\alpha_H^{(p)} : \Omega(H, M^\circ)(p) \to \Omega(H, M^\circ)(p)$ and $\beta_H^{(p)} : \Omega(H, M^\circ)(p) \to \Omega(H, M^\circ)(p)$ by
\[
\alpha_H^{(p)}(\{(x(K, s))(K, s) \in \mathcal{R}(H, M^\circ)\} = \left( \sum_{t \leq s \in \mathcal{L}_U} x(u, s) \right)_{(U, t) \in \mathcal{R}(H, M^\circ)}
\]
and
\[
\beta_H^{(p)}((x(K, s))(K, s) \in \mathcal{R}(H, M^\circ)) = \left( \sum_{t \leq s \in \mathcal{L}_U} \mu_U(t, s) x(u, s) \right)_{(U, t) \in \mathcal{R}(H, M^\circ)}
\]
for all $(x(K, s))(K, s) \in \mathcal{R}(H, M^\circ) \in \Omega(H, M^\circ)(p)$, respectively.

**Lemma 4.7** For each $H \leq G$, $\beta_H^{(p)} \circ \alpha_H^{(p)} = \alpha_H^{(p)} \circ \beta_H^{(p)} = \text{id}_{\Omega(H, M^\circ)(p)}$.

**Proof.** Let $H \leq G$, and let $(x(K, s))(K, s) \in \mathcal{R}(H, M^\circ) \in \Omega(H, M^\circ)(p)$. We have
\[
\beta_H^{(p)} \circ \alpha_H^{(p)}((x(K, s))(K, s) \in \mathcal{R}(H, M^\circ))
\]
\[
= \left( \sum_{t \leq s_2 \in \mathcal{L}_U} \mu_U(t, s_2) \sum_{s_2 \leq s_1 \in \mathcal{L}_U} x(u, s_1) \right)_{(U, t) \in \mathcal{R}(H, M^\circ)}
\]
\[
= \left( \sum_{t \leq s_1 \in \mathcal{L}_U} x(u, s_1) \sum_{t \leq s_2 \in \mathcal{L}_U} \mu_H(t, s_2) \right)_{(U, t) \in \mathcal{R}(H, M^\circ)}
\]
\[
= (x(u, t))_{(U, t) \in \mathcal{R}(H, M^\circ)},
\]
Lemma 4.9

For each $H \leq G$, let $\alpha_H^{(p)} \circ \beta_H^{(p)}((x_{(K,s)})(K,s) \in \mathcal{R}(H, M_{\mathcal{L}}))$ be a partially order $L \leq s_1$, and

$$
\alpha_H^{(p)} \circ \beta_H^{(p)}((x_{(K,s)})(K,s) \in \mathcal{R}(H, M_{\mathcal{L}}))
= \left( \sum_{t \leq s_2 \in \mathcal{L}_U} \sum_{s_2 \leq s_1 \in \mathcal{L}_U} \mu_U(s_2, s_1) x_{(U,s_1)} \right)_{(U,t) \in \mathcal{R}(H, M_{\mathcal{L}})}
= \left( \sum_{t \leq s_1 \in \mathcal{L}_U} x_{(U,s_1)} \sum_{t \leq s_2 \in \mathcal{L}_U} \mu_U(s_2, s_1) \right)_{(U,t) \in \mathcal{R}(H, M_{\mathcal{L}})}
= (x_{(U,t)})(U,t) \in \mathcal{R}(H, M_{\mathcal{L}}).
$$

This completes the proof. $\square$

Definition 4.8

Let $H \leq G$. Given $(U, t) \in \mathcal{R}(H, M)$ and $rU \in W_H(U, t)$, set

$$
\mathcal{L}_{(r)U}^{\geq t} = \{ s \in \mathcal{L}_{(r)U} \mid s \geq t \}.
$$

We define a $\mathbb{Z}_{(p)}$-module homomorphism $\tilde{\psi}_H^{(p)} : \tilde{\Omega}(H, M_{\mathcal{L}})(p) \to \text{Obs}(H, M_{\mathcal{L}})(p)$ by

$$
(x_{(K,s)})(K,s) \in \mathcal{R}(H, M_{\mathcal{L}}) \mapsto \left( \sum_{rU \in W_H(U, t), s \in \mathcal{L}_{(r)U}^{\geq t}} x_{(r,U,s)} \mod |W_H(U, t)| \right)_{(U,t) \in \mathcal{R}(H, M)}.
$$

The proof of the following lemma is analogous to that of [18, Lemma 4.3].

Lemma 4.9

For each $H \leq G$, $\tilde{\psi}_H^{(p)}$ is surjective.

Proof. Set $\tilde{\delta}_{(K,s)} = (\delta_{(K,s)}(U,t) \mod |W(U,t)|)(U,t) \in \text{Obs}(H, M_{\mathcal{L}})(p)$ for each $(K,s) \in \mathcal{R}(H, M_{\mathcal{L}})$. Obviously, the elements $\tilde{\delta}_{(K,s)}$ for $(K,s) \in \mathcal{R}(H, M_{\mathcal{L}})$ form a $\mathbb{Z}_{(p)}$-basis of $\text{Obs}(H, M_{\mathcal{L}})(p)$. Now set

$$
\mathcal{R}_0 = \{ (K,s) \in \mathcal{R}(H, M_{\mathcal{L}}) \mid \tilde{\delta}_{(K,s)} \notin \text{Im}\tilde{\psi}_H^{(p)} \}.
$$

We define a partially order $\leq_H$ on $\mathcal{R}(H, M_{\mathcal{L}})$ by the rule that

$$
(U, t) \leq_H (K, s) \iff U \leq hK \text{ and } t \leq hs \text{ for some } h \in H.
$$

Suppose that $\mathcal{R}_0 \neq \emptyset$, and let $(K,s)$ be a minimal element of $\mathcal{R}_0$ with respect to $\leq_H$. Then no element $(U,t)$ of $\mathcal{R}_0 - \{(K,s)\}$ satisfies $(U,t) \leq_H (K,s)$, and thus

$$
\tilde{\psi}_H^{(p)}(\delta_{(K,s)}(U,t))(U,t) \in \mathcal{R}(H, M) = (y_{(U,t)})(U,t) \in \mathcal{R}(H, M),
$$
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where

\[ y(U, t) = \begin{cases} 
1 \mod |W_H(U, t)_p| & \text{if } (U, t) = (K, s), \\
0 \mod |W_H(U, t)_p| & \text{if } (U, t) \in \mathcal{R}_0 - \{(K, s)\}.
\]

But, \( \Phi(U, t) \in \text{Im} \tilde{\psi}_H^{(p)} \) for any \((U, t) \notin \mathcal{R}_0\), which yields \( \Phi(K, s) \in \text{Im} \tilde{\psi}_H^{(p)} \). This is a
contradiction. Consequently, we have \( \mathcal{R}_0 = \emptyset \), completing the proof. \( \square \)

The proof of the following lemma is analogous to that of [19, Lemma 9.2].

**Lemma 4.10** Let \( H \leq G \), and let \((K, s), (U, t) \in \mathcal{R}(H, M_{\mathcal{V}})\). Set
\[ \text{inv}_U(\langle H/K \rangle_{s \geq t}) = \{hK \in H/K \mid U \leq hK \text{ and } t \leq h\}
\]
Then for any \( F \leq W_H(U, t) \),
\[ \sum_{rU \in F} |\{hK \in \text{inv}_U(\langle H/K \rangle_{s \geq t}) \mid \langle r \rangle U \leq hK\}| \equiv 0 \pmod{|F|}.
\]

**Proof.** Let \( F \leq W_H(U, t) \), and make \( \text{inv}_U(\langle H/K \rangle_{s \geq t}) \) into a left \( F \)-set by defining
\[ rUhK = rhK \]
for all \( rU \in F \) and \( hK \in \text{inv}_U(\langle H/K \rangle_{s \geq t}) \). Then for any \( rU \in F \),
\[ \{hK \in \text{inv}_U(\langle H/K \rangle_{s \geq t}) \mid \langle r \rangle U \leq hK\} = \{hK \in \text{inv}_U(\langle H/K \rangle_{s \geq t}) \mid rUhK = hK\},
\]
which is the set of \( F \)-invariants in \( \text{inv}_U(\langle H/K \rangle_{s \geq t}) \). Let \( F \setminus \text{inv}_U(\langle H/K \rangle_{s \geq t}) \), denote the set of \( F \)-orbits in \( \text{inv}_U(\langle H/K \rangle_{s \geq t}) \). By [24, Lemma 2.7], we have
\[ \sum_{rU \in F} |\{hK \in \text{inv}_U(\langle H/K \rangle_{s \geq t}) \mid \langle r \rangle U \leq hK\}| = |F| \cdot |F \setminus \text{inv}_U(\langle H/K \rangle_{s \geq t})| \equiv 0 \pmod{|F|},
\]
completing the proof. \( \square \)

Let \( H \leq G \). Given \( x \in \Omega(H, M_{\mathcal{V}})_{(p)} \), it follows from Lemmas 4.7 and 4.10 that
\[ (\tilde{\psi}_H^{(p)} \circ \rho_H^{(p)}) \circ (\alpha_H^{(p)} \circ \varphi_H^{(p)})(x) = \tilde{\psi}_H^{(p)} \circ \varphi_H^{(p)}(x) \equiv 0 \pmod{|W_H(U, t)_p|} \quad (4.2)
\]
We define a \( \mathbb{Z}_{(p)} \)-module homomorphism \( \tilde{\alpha}_H^{(p)} : \Omega(H, M_{\mathcal{V}})_{(p)} \to \tilde{\Omega}(H, M_{\mathcal{V}})_{(p)} \) by
\[ [(H/K), s] \mapsto [(\{hK \in H/K \mid U \leq hK \text{ and } t \leq h\})_{(U, t) \in \mathcal{R}(H, M_{\mathcal{V}})}]
\]
for all \((K, s) \in \mathcal{R}(H, M_{\mathcal{V}})\). Given \((U, t) \in \mathcal{R}(H, M_{\mathcal{V}})\) and \( rU \in W_H(U, t)_p \), set
\[ s(r, t) = \inf \mathcal{L}_{(r)}^{\geq t} \cdot \]
We define a $\mathbb{Z}_p\langle p \rangle$-module homomorphism $\tilde{\beta}_H^{(p)} : \tilde{\Omega}(H, M_{\mathcal{L}})_{(p)} \to \text{Obs}(H, M_{\mathcal{L}})_{(p)}$ by

$$
(x_{(K,s)\in \mathcal{R}(H,M)}) \mapsto \left( \sum_{rU \in W_H(U,t)_p} x_{((r)U,s(r,t))} \mod |W_H(U,t)_p| \right)_{(U,t) \in \mathcal{R}(H,M_{\mathcal{L}})}
$$

for all $(x_{(K,s)\in \mathcal{R}(H,M)}) \in \tilde{\Omega}(H, M_{\mathcal{L}})_{(p)}$.

**Theorem 4.11** For each $H \leq G$, the sequence

$$
0 \longrightarrow \Omega(H, M_{\mathcal{L}})_{(p)} \xrightarrow{\alpha_H^{(p)} \circ \varphi_H^{(p)}} \tilde{\Omega}(H, M_{\mathcal{L}})_{(p)} \xrightarrow{\tilde{\varphi}_H^{(p)} \circ \tilde{\beta}_H^{(p)}} \text{Obs}(H, M_{\mathcal{L}})_{(p)} \longrightarrow 0
$$

of $\mathbb{Z}_p\langle p \rangle$-modules is exact; moreover, $\tilde{\alpha}_H^{(p)} = \alpha_H^{(p)} \circ \varphi_H^{(p)}$, and $\tilde{\beta}_H^{(p)} = \tilde{\varphi}_H^{(p)} \circ \beta_H^{(p)}$.

**Proof.** Let $H \leq G$. By Theorem 3.10, Lemma 4.9, and Eq.(4.2), the sequence

$$
0 \longrightarrow \Omega(H, M_{\mathcal{L}})_{(p)} \xrightarrow{\varphi_H^{(p)}} \tilde{\Omega}(H, M_{\mathcal{L}})_{(p)} \xrightarrow{\tilde{\varphi}_H^{(p)}} \text{Obs}(H, M_{\mathcal{L}})_{(p)} \longrightarrow 0
$$

of $\mathbb{Z}_p\langle p \rangle$-modules is exact (see the proof of Theorem 3.10). Thus it follows from Lemma 4.7 that Eq.(4.3) is exact. Obviously, $\tilde{\alpha}_H^{(p)} = \alpha_H^{(p)} \circ \varphi_H^{(p)}$. Suppose that

$$
\tilde{\varphi}_H^{(p)} \circ \beta_H^{(p)}((x_{(K,s)\in \mathcal{R}(H,M)}) = (y_{(U,t)})_{(U,t) \in \mathcal{R}(H,M)} \in \text{Obs}(H, M_{\mathcal{L}})_{(p)}
$$

with $y_{(U,t)} \in \mathbb{Z}_p\langle p \rangle/W_H(U,t)_p\mathbb{Z}_p$. Then for each $(U, t) \in \mathcal{R}(H, M)$,

$$
y_{(U,t)} = \sum_{rU \in W_H(U,t)_p} \sum_{s_1 \leq s_2 \in \mathcal{L}_{(r)U}} \mu(r)U(s_1, s_2)x_{((r)U,s_1, s_2)} \mod |W_H(U,t)_p|
$$

$$
= \sum_{rU \in W_H(U,t)_p} \sum_{s_1 \leq \mathcal{L}_{(r)U}} \sum_{s_2 \geq \mathcal{L}_{(r)U}} \mu(r)U(s_1, s_2)x_{((r)U,s_1, s_2)} \mod |W_H(U,t)_p|
$$

$$
= \sum_{rU \in W_H(U,t)_p} x_{((r)U,s(r,t))} \mod |W_H(U,t)_p|.
$$

Hence we have $\tilde{\beta}_H^{(p)} = \tilde{\varphi}_H^{(p)} \circ \beta_H^{(p)}$, completing the proof. □

For each $H \leq G$, the primitive idempotents of $\mathbb{Q} \otimes \mathbb{Z} \Omega(H)$ are the elements

$$
e_{K}^{(H)} := \frac{1}{|NH(K)|} \sum_{U \leq K} |U| \mu(U, K)[H/U]
$$

of $\mathbb{Q} \otimes \mathbb{Z} \Omega(H)$ for $K \in \mathcal{C}(H)$ which is given in [10, 22]. In addition, the primitive idempotents of $\mathbb{Q} \otimes \mathbb{Z} \Xi(G)$ are given in [6, Corollary 5.5], where $\Xi(G)$ is the slice Burnside ring of $G$ (see Example 4.4). We are now successful in finding the primitive idempotents of $\mathbb{Q} \otimes \mathbb{Z} \Omega(H, M_{\mathcal{L}})$ for $H \leq G$. 

Theorem 4.12 Let $H \leq G$, and define a map $\zeta_H : \Omega(H, M_\mathcal{L}) \rightarrow \widetilde{\Omega}(H, M_\mathcal{L})$ by

$$
\left( \sum_{s \in L_K} \ell_{(K,s)}^s \right)_{K \leq G} \mapsto \left( \sum_{t \leq s \in L_U} \ell_{(U,t)}^s \right)_{(U,t) \in \mathcal{R}(H, M_\mathcal{L})}
$$

for all $\ell_{(K,s)} \in \mathbb{Z}$ with $(K,s) \in \mathcal{S}(H, M_\mathcal{L})$. Let $\alpha_H$ denote $\alpha_H^{(\infty)}$. Then the diagram

$$
\begin{array}{ccc}
\Omega(H, M_\mathcal{L}) & \xrightarrow{\varphi_H} & \widetilde{\Omega}(H, M_\mathcal{L}) \\
\downarrow{\rho_H} & & \downarrow{\zeta_H} \\
\mathcal{O}(H, M_\mathcal{L}) & & \mathcal{O}(H, M_\mathcal{L})
\end{array}
$$

is commutative. Moreover, the map $\alpha_H \circ \varphi_H : \Omega(H, M_\mathcal{L}) \rightarrow \widetilde{\Omega}(H, M_\mathcal{L})$ is a ring monomorphism, and the primitive idempotents of $\mathbb{Q} \otimes \mathbb{Z} \Omega(H, M_\mathcal{L})$ are the elements

$$
e_{(K,s)}^{(H)} := \frac{1}{|N_H(K,s)|} \sum_{t \in \mathcal{L}_K^{\leq s}} \mu_K(t,s) \sum_{U \leq K} |U| \mu(U,K)[|H/U|_{t \wedge \sup \mathcal{L}_U}],
$$

where $\mathcal{L}_K^{\leq s} = \{ t \in \mathcal{L}_K \mid t \leq s \}$, for $(K,s) \in \mathcal{R}(H, M_\mathcal{L})$.

Proof. By Proposition 3.5, $\kappa_H \circ \varphi_H = \rho_H$. Moreover, $\alpha_H = \zeta_H \circ \kappa_H$, because

$$
\alpha_H((\delta_{(K,s)}((U,t))(U,t) \in \mathcal{R}(H,M))
= (\delta_{KU} \{ s_1 \in \mathcal{L}_K \mid t \leq s_1 \text{ and } h s = s_1 \text{ for some } h \in N_H(K) \})_{(U,t) \in \mathcal{R}(H,M)}
= \zeta_H \circ \kappa_H((\delta_{(K,s)}((U,t))(U,t) \in \mathcal{R}(H,M)))
$$

for all $(K,s) \in \mathcal{R}(H, M_\mathcal{L})$. Hence $\alpha_H \circ \varphi_H = \zeta_H \circ (\kappa_H \circ \varphi_H) = \zeta_H \circ \rho_H$. Obviously, $\zeta_H$ is a ring isomorphism. Since $\rho_H$ is a ring monomorphism (cf. [4, 2.4]), so is $\alpha_H \circ \varphi_H$. This, combined with Proposition 3.4 and Lemma 4.7, shows that the primitive idempotents of $\mathbb{Q} \otimes \mathbb{Z} \Omega(H, M_\mathcal{L})$ are the elements

$$
\frac{1}{|H|} \zeta_H \circ \beta_H((\delta_{(K,s)}((U,t))(U,t) \in \mathcal{R}(H,M_\mathcal{L}))
$$

for $(K,s) \in \mathcal{R}(H, M_\mathcal{L})$, where $\beta_H = \beta_H^{(\infty)}$. Given $(K,s) \in \mathcal{R}(H, M_\mathcal{L})$, we have

$$
\beta_H((\delta_{(K,s)}((U,t))(U,t) \in \mathcal{R}(H,M_\mathcal{L})) = \left( \delta_{KU} \sum_{t \leq s_1 \in \mathcal{L}_K^{(s)}} \mu_K(t,s_1) \right)_{(U,t) \in \mathcal{R}(H,M_\mathcal{L})}
$$
where $L^{(s)}_K = \{ s_1 \in L_K \mid s_1 = h s \text{ for some } h \in N_H(K) \}$, and

$$
\zeta_H \circ \beta_H ( ((\delta(s,t))_{(U,t)} )_{(U,t) \in \mathcal{R}(H,M_L)} )
$$

$$
= \frac{|H|}{|N_H(K)|} \sum_{t \in L^{(s)}_K} \sum_{s \leq s_1 \in L^{(s)}_K} \mu_K(t, s_1) \sum_{U \leq K} [U|\mu(U,K)] [H/U]_{t \sup \mathcal{L}_U}
$$

$$
= \frac{|H|}{|N_H(K)|} \sum_{s_1 \in L^{(s)}_K} \sum_{t \in L^{(s)}_K} \mu_K(t, s_1) \sum_{U \leq K} [U|\mu(U,K)] [H/U]_{t \sup \mathcal{L}_U}
$$

$$
= \frac{|H|}{|N_H(K)|} \sum_{t \in L^{(s)}_K} \mu_K(t, s) \sum_{U \leq K} [U|\mu(U,K)] [H/U]_{t \sup \mathcal{L}_U},
$$

where $L^{(s)}_K = \{ t \in L_K \mid t \leq h s \text{ for some } h \in N_H(K) \}$. This completes the proof.

The exact sequence (4.3) with $H = G$ is derived from the exact sequence (5.2) for abstract Burnside rings (see the next section). Also, the primitive idempotents $e^{(G)}_{(K,s)}$ of $\mathbb{Q} \otimes \mathcal{O}(H, M_L)$ for $(K, s) \in \mathcal{R}(H, M_L)$, which are given in Theorem 4.12, are explained in terms of the Möbius function of $S(G, M_L)$ (see Remark 7.4).

5 Units of lattice Burnside rings

We show that any lattice Burnside ring is isomorphic to an abstract Burnside ring, and explore the units of a lattice Burnside ring.

Let $\Gamma$ be an essentially finite category, that is, a category equivalent to a finite category. The set of isomorphism classes of objects of $\Gamma$ is denoted by $\Gamma/\simeq$. Given objects $I$ and $J$ of $\Gamma$, the set of morphisms from $I$ to $J$ is denoted by $\text{Hom}(I, J)$ or $\Gamma(I, J)$. Since $\Gamma$ is essentially finite, it follows that $\Gamma/\simeq$ and $\Gamma(I, J)$ are finite sets.

As before, $p$ denotes a prime or the symbol $\infty$. Let $\mathbb{Z}\Gamma$ be the free abelian group on $\Gamma/\simeq$, and set $\mathbb{Z}_{(p)}\Gamma = \mathbb{Z}_{(p)} \otimes \mathbb{Z}\Gamma$. The isomorphism class containing an object $I$ of $\Gamma$ is denoted by $[I]$. The ghost ring $\mathbb{Z}_{(p)}\Gamma$ of $\mathbb{Z}_{(p)}\Gamma$ is defined to be

$$
\mathbb{Z}_{(p)}\Gamma = \prod_{[I] \in \Gamma/\simeq} \mathbb{Z}_{(p)}
$$

We define a $\mathbb{Z}_{(p)}$-module homomorphism $\varphi_{\Gamma}^{(p)} : \mathbb{Z}_{(p)}\Gamma \rightarrow \mathbb{Z}_{(p)}\Gamma$ by

$$
[J] \mapsto \langle (|\Gamma(I, J)|)_{[I] \in \Gamma/\simeq} \rangle
$$

for all $[J] \in \Gamma/\simeq$, and call it the Burnside map.

The $\mathbb{Z}_{(p)}$-module $\mathbb{Z}_{(p)}\Gamma$ is called an abstract Burnside ring if it has a $\mathbb{Z}_{(p)}$-algebra structure such that $\varphi_{\Gamma}^{(p)}$ is an injective $\mathbb{Z}_{(p)}$-algebra homomorphism (cf. [25]).
Let $I$ be an object of $\Gamma$. We denote by $\text{Aut}(I)$ the group of automorphisms of $I$. For each $\sigma \in \text{Aut}(I)$, a morphism $c_{\sigma} : I \to I/\sigma$ in $\Gamma$ is said to be a coequalizer of $\sigma$ and $\text{id}_I$ if $c_{\sigma} \circ \sigma = c_{\sigma}$ and it is universal with this property; that is, for any morphism $f : I \to J$ in $\Gamma$ which satisfies that $f \circ \sigma = f$, there exists a unique morphism $f_1 : I/\sigma \to J$ in $\Gamma$ such that $f = f_1 \circ c_{\sigma}$:

\[
\begin{array}{ccc}
I & \xrightarrow{\text{id}_I} & I \\
\downarrow{\sigma} & & \downarrow{c_{\sigma}} \\
I/\sigma & \xrightarrow{f} & J \\
\downarrow{f_1} & & \text{f}_1 \\
\end{array}
\]

Given an object $I$ of $\Gamma$, let $\text{Aut}(I)_p$ denote a Sylow $p$-subgroup of $\text{Aut}(I)$, and let $\text{Aut}(I)_\infty$ denote $\text{Aut}(I)$. For an epi-mono factorization system of the category $\Gamma$, we refer to [25, 1.4]. The assertions of the following theorem are presented in [25, Theorems 2.4 and 2.6] under a slightly weaker assumption.

**Theorem 5.1 (Fundamental theorem)** Assume that $\Gamma$ has an epi-mono factorization system and that for any object $I$ of $\Gamma$, any element $\sigma$ of $\text{Aut}(I)_p$ has a coequalizer $c_{\sigma} : I \to I/\sigma$ of $\sigma$ and $\text{id}_I$. Then there exists an exact sequence

\[
0 \to \mathbb{Z}(p)_\Gamma \xrightarrow{\varphi(p)_{\Gamma}} \mathbb{Z}(p)_\Gamma \xrightarrow{\psi(p)_{\Gamma}} \text{Obs}(p)(\Gamma) \to 0
\]

(5.2)
of $\mathbb{Z}(p)$-modules, where $\text{Obs}(p)(\Gamma)$ is the group of obstruction of $\Gamma$ defined to be

\[
\text{Obs}(p)(\Gamma) = \prod_{[I] \in \Gamma/\simeq} \mathbb{Z}(p)/|\text{Aut}(I)_p|\mathbb{Z}(p),
\]

and $\psi(p)_{\Gamma} : \mathbb{Z}(p)_\Gamma \to \text{Obs}(p)(\Gamma)$ is the Cauchy-Frobenius map given by

\[
(x_I)_{[I] \in \Gamma/\simeq} \mapsto \left( \sum_{\sigma \in \text{Aut}(I)_p} x_{I/\sigma} \mod |\text{Aut}(I)_p| \right)_{[I] \in \Gamma/\simeq}
\]

for all $(x_I)_{[I] \in \Gamma/\simeq} \in \mathbb{Z}(p)_\Gamma$. Moreover, $\mathbb{Z}(p)_\Gamma$ is an abstract Burnside ring.

If $\Gamma$ satisfies the assumptions of Theorem 5.1, we consider $\mathbb{Z}(p)_\Gamma$ to be the abstract Burnside ring which has a $\mathbb{Z}(p)$-algebra structure such that $\varphi(p)_{\Gamma}$ is an injective $\mathbb{Z}(p)$-algebra homomorphism. The origin of abstract Burnside rings is $\Omega(G)$. 
Example 5.2 Let $\text{trans}_G^S$ be the category of transitive $G$-sets $G/H$ for $H \leq G$ and $G$-equivariant maps. Given $U, H \leq G$, we have

$$\text{Hom}(G/U, G/H) = \{ \lambda_g : G/U \to G/H, rU \mapsto rgH \mid gH \in \text{inv}_U(G/H) \},$$

where $\text{inv}_U(G/H) = \{ gH \in G/H \mid U \leq gH \}$, so that $G/U \simeq G/H$ if and only if $U$ is a conjugate of $H$ (cf. [7, (80.5) Proposition]). Let $H \leq G$. Then

$$\text{Aut}(G/H) = \{ \sigma_g : G/H \to G/H, rH \mapsto rgH \mid gH \in N_G(H)/H \}.$$

Given $g \in N_G(H)$, there exists a coequalizer $c_{\sigma_g} : G/H \to (G/H)/\sigma_g$ of $\sigma_g$ and $\text{id}_{G/H}$ given by $(G/H)/\sigma_g = G/(gH)$ and $c_{\sigma_g}(rH) = r(g)H$ for all $r \in G$. Hence $\text{trans}_G^S$ satisfies the assumption of Theorem 5.1 with $\Gamma = \text{trans}_G^S$. The abstract Burnside ring $\mathbb{Z}\text{trans}_G^S$ is isomorphic to the Burnside ring $\Omega(G)$.

Let $R(G)$ be the ring of virtual $\mathbb{C}$-characters, and let $R_{\mathbb{Q}}(G)$ be the subring of $R(G)$ generated by the characters afforded by $\mathbb{Q}G$-modules. There exists a ring homomorphism $\text{char}_G : \Omega(G) \to R_{\mathbb{Q}}(G)$ given by

$$[G/H] \mapsto 1_{H}^G$$

for all $H \leq G$, where $1_{H}^G$ is the character induced from the trivial character $1_H$ of $H$ (or the permutation character of $G$ on $G/H$) defined by

$$1_{H}^G(g) = \{ rH \in G/H \mid g \in r^{-1}H \}$$

for all $g \in G$. Obviously, if $u$ is a unit of $\Omega(G)$, then $\text{char}_G(u)$ is a unit of $R_{\mathbb{Q}}(G)$.

For any unital ring $R$, we denote by $R^\times$ the unit group of $R$. Let $\text{Hom}(G, \langle -1 \rangle)$ be the set of homomorphisms from $G$ to the subgroup $\langle -1 \rangle$ of $\mathbb{C}^\times$.

Lemma 5.3 For any $\chi \in R_{\mathbb{Q}}(G)^\times$, $\chi(\epsilon)\chi \in \text{Hom}(G, \langle -1 \rangle)$.

Proof. Obviously, $\chi(g) \in \langle -1 \rangle$ for all $g \in G$, and the assertion follows from the first orthogonality relation (cf. [7, (9.21), (9.26) Proposition]). \qed

Let $I$ be an object of $\Gamma$. We define an additive map $\omega_I : \mathbb{Z}\Gamma \to \Omega(\text{Aut}(I))$ by

$$[J] \mapsto [\Gamma(I, J)]$$

for all objects $J$ of $\Gamma$, where the left action of $\text{Aut}(I)$ on $\Gamma(I, J)$ is given by

$$\sigma f = f \circ \sigma^{-1}$$

for all $\sigma \in \text{Aut}(I)$ and $f \in \Gamma(I, J)$.

There is a criteria for the units of an abstract Burnside ring, which is a generalization of that for the units of $\Omega(G)$ (cf. [23, Proposition 6.5]).
Proposition 5.4  Keep the assumptions of Theorem 5.1, and assume further that for any object $I$ of $\Gamma$, the map $\omega_I: \mathbb{Z} \Gamma \to \Omega(\text{Aut}(I))$ is a ring homomorphism. Let $\bar{x} = (x_I)_{I \in \Gamma^\sim} \in \mathbb{Z}^{\Gamma^\times}$. For any $[I] \in \Gamma/\sim$, define a map $\gamma_I^x : \text{Aut}(I) \to \langle -1 \rangle$ by
\[
\sigma \mapsto x_I x_{I/\sigma}
\]
for all $\sigma \in \text{Aut}(I)$. Then there exists an element $x$ of $\mathbb{Z} \Gamma$ such that $\bar{x} = \varphi(x)$, where $\varphi_I = \varphi_{\Gamma}^{(\infty)}$, if and only if $\gamma_I^x \in \text{Hom}(\text{Aut}(I), \langle -1 \rangle)$ for any $[I] \in \Gamma/\sim$.

Proof. Let $\bar{x} = (x_I)_{I \in \Gamma^\sim} \in \mathbb{Z}^{\Gamma^\times}$. If $\gamma_I^x \in \text{Hom}(\text{Aut}(I), \langle -1 \rangle)$ for any $[I] \in \Gamma/\sim$, then it follows from Theorem 5.1 that $\bar{x} = \varphi(x)$ for some $x \in \mathbb{Z} \Gamma$, because
\[
\langle \gamma_I^x, 1_{\text{Aut}(I)} \rangle_{\text{Aut}(I)} = \frac{1}{|\text{Aut}(I)|} \sum_{\sigma \in \text{Aut}(I)} x_I x_{I/\sigma} \in \{0, 1\},
\]
where $1_{\text{Aut}(I)}$ is the trivial character of $\text{Aut}(I)$ and $\langle \gamma_I^x, 1_{\text{Aut}(I)} \rangle_{\text{Aut}(I)}$ is the inner product of $\gamma_I^x$ and $1_{\text{Aut}(I)}$, for any $[I] \in \Gamma/\sim$. Conversely, assume that $\bar{x} = \varphi(x)$ for some $x \in \mathbb{Z} \Gamma$. Let $I$ be an object of $\Gamma$. Since the map $\omega_I : \mathbb{Z} \Gamma \to \Omega(\text{Aut}(I))$ is a ring homomorphism, it follows that $\text{char}_{\text{Aut}(I)}(\omega_I(x)) \in R_{Q}(\text{Aut}(I))^\times$. By the definition of coequalizer, we have $x_{I/\sigma} = \text{char}_{\text{Aut}(I)}(\omega_I(x))(\sigma)$ for all $\sigma \in \text{Aut}(I)$ (cf. [25, (2.28)]). Consequently, it follows from Lemma 5.3 that $\gamma_I^x \in \text{Hom}(\text{Aut}(I), \langle -1 \rangle)$. The proof is now complete. □

As before, we suppose that $\mathscr{L}$ is a finite $G$-lattice and $M_\mathscr{L} = (M_\mathscr{L}, \text{con}, \text{res})$ is the monoid functor given in Proposition 4.2. Let $\text{trans}_G^{\mathscr{L}}$ be the category given in Example 5.2. There is an additive contravariant functor $T_G^M : \text{trans}_G^{\mathscr{L}} \to \text{Mon}$ inherited from $T_G^M : G\text{-set} \to \text{Mon}$. For each $K \leq G$, $T_G^M(G/K)$ consists of the $G$-equivariant maps $\pi_s : G/K \to \tilde{M}_\mathscr{L}(G)$, $rK \mapsto r^s$ for $s \in \mathcal{L}_K$. We call a pair $(G/K, \pi_s)$ of $G/K$ and $\pi_s$ with $(K, s) \in \mathcal{S}(G, M_\mathscr{L})$ an element of $T_G^M$. The morphisms $\lambda : (G/U, \pi_t) \to (G/K, \pi_s)$ between elements $(G/U, \pi_t)$ and $(G/K, \pi_s)$ of $T_G^M$ are defined to be the $G$-equivariant maps $\lambda_g : G/U \to G/K$, $rU \mapsto rgK$ for $gK \in \text{inv}(G/K)$ (see Example 5.2) such that $t \leq g\pi_s$. Thus we obtain the category $\text{El}(T_G^M)$ of elements of $T_G^M$. Note that the above definition of morphisms is different from that in the category of elements of $T_G^M$.

For each $(U, t) \in \mathcal{S}(G, M_\mathscr{L})$, the group $\text{Aut}((G/U, \pi_t))$ of automorphisms of $(G/U, \pi_t)$ consists of all maps $\sigma_g : G/U \to G/U$ for $gU \in W_G(U, t)$ given by
\[
rU \mapsto rgU
\]
for all $rU \in G/U$, which is identified with $\text{Aut}((G/U)_t)$.

We are now in a position to prove that $\mathbb{Z}_{(p)} \Gamma$ with $\Gamma = \text{El}(T_G^M)$ is an abstract Burnside ring. By the following theorem, the lattice Burnside ring $\Omega(G, M_\mathscr{L})_{(p)}$, which is called a $p$-local lattice Burnside ring, is isomorphic to $\mathbb{Z}_{(p)} \Gamma$. 
**Theorem 5.5** Suppose that $\mathbf{\Gamma} = \text{EI}(\mathcal{M}_G^G)$. Then $\mathbf{\Gamma}$ satisfies the assumptions of Theorem 5.1, and the abstract Burnside ring $\mathbb{Z}(\mathbf{\Gamma})$ is isomorphic to the $p$-local lattice Burnside ring $\Omega(G, M_\mathcal{G})(p)$. In this connection,

$$
(G/U, \pi_t)/\sigma_g = (G/(g)U), \pi_{s_g(t)}),
$$

where $s_g(t) = \inf \mathcal{L}^t_{(g)U}$, for all $(U, t) \in S(G, M_\mathcal{G})$ and $gU \in W_G(U, t)$. Moreover,

$$
|\text{Hom}((G/U, \pi_1), (G/K, \pi_s))| = |\{gK \in G/K \mid U \leq gK \text{ and } t \leq gK\}|
$$

for all $(K, s), (U, t) \in S(G, M_\mathcal{G})$ (cf. Eq. (5.1)).

**Proof.** Since all the morphisms of $\mathbf{\Gamma}$ are epimorphisms, it follows that $\mathbf{\Gamma}$ has an epimono factorization system. Let $(U, t) \in S(G, M_\mathcal{G})$, and let $g \in N_G(U, t)$. Observe that there exists a coequalizer $c_{s_g} : (G/U, \pi_t) \rightarrow (G/U, \pi_t)/\sigma_g$ of $\sigma_g$ and $\text{id}_{(G/U, \pi_t)}$ given by $(G/U, \pi_t)/\sigma_g = (G/(g)U), \pi_{s_g(t)}$ and $c_{s_g}(rU) = r(g)U$ for all $r \in G$. Thus Eq.(5.3) holds, and $\mathbf{\Gamma}$ is a finite category which satisfies the assumptions of Theorem 5.1. Given $(K, s), (U, t) \in S(G, M_\mathcal{G})$, $(G/K, \pi_s) \simeq (G/U, \pi_t)$ if and only if $(K, s) = g(U, t)$ for some $g \in G$. Moreover, Eq.(5.4) is obvious. Consequently, it follows from Proposition 2.5 and Theorems 4.11 and 4.12 that there is a ring isomorphism $\Omega(G, M_\mathcal{G})(p) \simeq \mathbb{Z}(\mathbf{\Gamma})$ given by

$$
[(G/K)_s] \mapsto [(G/K, \pi_s)]
$$

for all $(K, s) \in S(G, M_\mathcal{G})$. This completes the proof. $\square$

While the ring structure of $\Omega(G, M_\mathcal{G})$ is given in Proposition 2.5, the proof of the following corollary to Theorem 5.5 makes it clear by a Burnside map.

**Corollary 5.6** Suppose that $\mathbf{\Gamma} = \text{EI}(\mathcal{M}_G^G)$, and let $I$ be an object of $\mathbf{\Gamma}$. Then the map $\omega_I : \mathbb{Z}\mathbf{\Gamma} \rightarrow \Omega(\text{Aut}(I))$ is a ring homomorphism.

**Proof.** Let $(U, t), (K_i, s_i) \in S(G, M_\mathcal{G})$ with $i = 1, 2$, and define a map

$$
\Phi : \mathbf{\Gamma}((G/U, \pi_t), (G/K_1, \pi_{s_1})) \times \mathbf{\Gamma}((G/U, \pi_1), (G/K_2, \pi_{s_2}))
$$

$$
\rightarrow \bigcup_{K_1 \cap K_2 \in K_1 \cap G/K_2} \Gamma((G/U, \pi_t), (G/(K_1 \cap K_2), \pi_{s_1 \cap r_2}))
$$

by

$$(\lambda_{g_1}, \lambda_{g_2}) \mapsto \lambda_{g_1 r_1} \in \Gamma((G/U, \pi_t), (G/(K_1 \cap K_2), \pi_{s_1 \cap r_2})),$$

where $g_1^{-1} g_2 K_2 = r_1 r K_2$ with $r_1 \in K_1$, for all $g_i K_i \in \text{inv}_U(G/K_i)$ with $i = 1, 2$ such that $t \leq g_2 s_i$. Then $\Phi$ is an isomorphism of $\text{Aut}((G/U, \pi_t))$-sets. Hence it follows from Theorem 5.5 that $\omega_I$ is a ring homomorphism. $\square$
In the proof of Theorem 5.5, we are aware that the exact sequence (5.2) with $I = \text{El}(\tilde{T}^M_{G,X})$ is identified with the exact sequence (4.3) with $H = G$, namely,

$$0 \longrightarrow \Omega(G, M_{X})(p) \xrightarrow{\alpha_G^{(p)} \circ \varphi_G^{(p)}} \tilde{\Omega}(G, M_{X})(p) \xrightarrow{\gamma_G^{(p)} \circ \beta_G^{(p)}} \text{Obs}(G, M_{X})(p) \longrightarrow 0.$$  

(By Theorems 4.11 and 4.12 with $H = G$, $\alpha_G^{(p)} = \alpha_G^{(p)} \circ \varphi_G^{(p)}$, $\beta_G^{(p)} = \gamma_G^{(p)} \circ \beta_G^{(p)}$, and the map $\alpha_G \circ \varphi_G$ is a ring monomorphism, where $\alpha_G = \alpha_G^{(\infty)}$.)

We are now ready to give a criteria of the units of $\Omega(G, M_{X})$.

**Proposition 5.7** Let $\tilde{x} = (x(U,t))_{(U,t) \in R(G, M_{X})} \in \tilde{\Omega}(G, M_{X})^\times$. Then $\tilde{x}$ is contained in the image of the ring monomorphism $\alpha_G \circ \varphi_G : \Omega(G, M_{X}) \rightarrow \tilde{\Omega}(G, M_{X})$ if and only if, for each $(U, t) \in R(G, M_{X})$, the map $\gamma_{(U,t)} : \text{Aut}((G/U)t) \rightarrow \langle -1 \rangle$ given by

$$\sigma_g \mapsto x(U,t)^{x((g)U, s_{(g,t)})}$$

for all $gU \in W_G(U, t)$ is a group homomorphism.

**Proof.** The proposition follows from Proposition 5.4 and Corollary 5.6. □

Let $M_{X}$ be the monoid functor given in Example 4.4. Then Proposition 5.7 with $M_{X} = M_{X}$ is equivalent to [6, Theorem 8.4].

We give an example of $\Omega(G, M_{X})^\times$ (see also [6, Theorem A.13]).

**Proposition 5.8** Suppose that $G$ is abelian. Let $M^\varphi_{X}$ be the monoid functor given in Example 4.5. Then $\Omega(G, M^\varphi_{X})^\times$ is generated by $-[(G/G)_{1}]$ and the elements $\{(G/H)_{U} - [(G/G)_{G}]$ for $U \leq H \leq G$ with $|G : H| = 2$. In particular, $\Omega(G, M^\varphi_{X})^\times$ is an elementary abelian 2-group of order $2^{\vartheta(G)+1}$, where

$$\vartheta(G) = |\{(H,U) \in S(G, M^\varphi_{X}) | |G : H| = 2\}|.$$

**Proof.** Let $\tilde{x} = (x(K,L))_{(K,L) \in R(G, M^\varphi_{X})} \in \tilde{\Omega}(G, M^\varphi_{X})^\times$, and suppose that $\tilde{x}$ is contained in the image of the ring monomorphism $\alpha_G \circ \varphi_G : \Omega(G, M_{X}) \rightarrow \tilde{\Omega}(G, M_{X})$ with $M_{X} = M^\varphi_{X}$. Then by Proposition 5.7, we have

$$x(K,L)^{x((g)K,L)}x((r)K,L) = x((g)K,L)$$

for all $(K,L) \in R(G, M^\varphi_{X})$ and $gK, rK \in W_G(K, L)$. This implies that for each $(K,L) \in R(G, M^\varphi_{X})$, if $|G : K| > 2$, then the value $x(K,L)$ is determined by the values $x(H,L)$ for $H \leq G$ with $K < H$ (cf. [6, p. 904]). Hence we have

$$|\Omega(G, M^\varphi_{X})^\times| \leq 2^{\vartheta(G)+1}. \tag{5.5}$$

If $G$ is of odd order, then the assertion clearly holds. Assume that $G$ is of even order, and let $H_1, H_2, \ldots, H_m$ be the subgroups of index 2 in $G$. For each integer
with $1 \leq i \leq m$, let $\mathcal{U}(H_i)$ denote the subgroup of $\Omega(G, M_\varphi)^\times$ generated by the elements $[(G/H_i)\mathbb{U} - (G/G)_G]$ for $U \leq H_i$. By Eq.(4.1), we have

$$(\mathcal{U}(H_1) \cdots \mathcal{U}(H_i)) \cap \mathcal{U}(H_{i+1}) = \{[(G/G)_G]\}$$

for each integer $i$ with $1 \leq i \leq m - 1$ and

$$(\mathcal{U}(H_1) \cdots \mathcal{U}(H_m)) \cap (-[(G/G)_G]) = \{[(G/G)_G]\}.$$

Thus $(-(G/G)_G)\mathcal{U}(H_1) \cdots \mathcal{U}(H_m) = -[(G/G)_G] \times \mathcal{U}(H_1) \times \cdots \times \mathcal{U}(H_m)$. Likewise, for each integer $i$ with $1 \leq i \leq m$, $\mathcal{U}(H_i)$ is the direct product of the subgroups $\langle [(G/H_i)_\mathbb{U}] - (G/G)_G \rangle$ for $U \leq H_i$. By these facts, $\Omega(G, M_\varphi)^\times$ contains the direct product of the subgroups $\langle [(G/H_i)_\mathbb{U}] - (G/G)_G \rangle$ for $i = 1, 2, \ldots, m$ and $U \leq H_i$. Combining this fact with Eq.(5.5), we conclude that the assertion holds. □

Remark 5.9 There is an embedding $\Omega(G) \hookrightarrow \Omega(G, M_\varphi)$ given by

$$[G/H] \mapsto [(G/H)_H]$$

for all $H \leq G$. By Proposition 5.8, $\Omega(G)^\times$ is generated by $-[G/G]$ and the elements $[G/H] - [G/G]$ for $H \leq G$ with $|G : H| = 2$, and $|\Omega(G)^\times| = 2^{\text{Hom}(G, (-1))}$ (see also [6, Remark A.14] and [23, Lemma 7.1]), which is due to Matsuda [13, Example 4.5].

6 Primitive idempotents of lattice Burnside rings

Let $\Gamma$ be a finite category, and suppose that the assumptions of Theorem 5.1 hold. Let $\text{id}_Q \otimes \varphi_\Gamma : Q \otimes Z \Gamma \to Q \otimes Z \Gamma$ be the algebra monomorphism determined by $\varphi_\Gamma^{(\infty)}$. By Theorem 5.1, the primitive idempotents of $Q \otimes Z \Gamma$ are the elements $e_I$ for $[I] \in \Gamma/\simeq$ such that $\text{id}_Q \otimes \varphi_\Gamma(e_I) = (\delta_{\text{id}}[I], [I] \in \Gamma/\simeq)$.

Let $\sim_p$ be the equivalence relation on the set $\Gamma/\simeq$ generated by

$$[I/\sigma] \sim_p [I] \quad \text{with} \quad \sigma \in \text{Aut}(I)_p.$$ 

(Note that $[I] = [I/\text{id}] \sim_p [I]$ for any object $I$ of $\Gamma$.) We define an equivalence relation $\sim_p$ on the set of objects of $\Gamma$ by letting

$$I \sim_p J \quad \text{if and only if} \quad [I] \sim_p [J].$$

Let $C_p(\Gamma)$ be a complete set of representatives of equivalence classes with respect to the equivalence relation $\sim_p$ on $\Gamma$. For each $I \in C_p(\Gamma)$, we define

$$e_I^{(p)} := \sum_{[I] \sim_p [J] \in \Gamma/\simeq} e_J,$$

where the sum is taken over all $[J] \in \Gamma/\simeq$ such that $I \sim_p J$.

There is a generalization of [10, Lemma 2] and [22, Theorem 3.1]:
Lemma 6.3 Let \( H \subset G \) be subgroups of \( G \). Then we define an equivalence relation \( \sim_p \) on \( S(G) \) by letting \((K, s) \sim_p (U, t)\) if and only if \((G/K, \pi_s) \sim_p (G/U, \pi_t)\). Given \((U, t) \in S(G)\) and \(gU \in W_G(U, t)\), it follows from Theorem 5.5 that \((G/\langle g \rangle U, \pi_s(t_{g,U})) = (G/U, \pi_t)/\sigma_g \sim_p (G/U, \pi_t)\) with \(s_{g,U} = \inf \mathcal{L}^{t_{g,U}}\), whence \((\langle g \rangle U, s_{g,U}) \sim_p (U, t)\). We often use this basic fact.

Let \( K \leq G \). When \( p \) is a prime, we denote by \( O^p(K) \) the smallest normal subgroup of \( K \) such that \( K/O^p(K) \) is a \( p \)-group. Suppose that

\[ K = K^{(0)} \geq K^{(1)} \geq K^{(2)} \geq \cdots \geq K^{(i)} \geq \cdots \]

is the derived series of \( K \) (cf. [17, Chapter 2, Definition 3.11]). Then we define \( O^\infty(K) := \bigcap_{i=1}^\infty K^{(i)} \). A subgroup \( K \) of \( G \) is said to be \( p \)-perfect if \( K = O^p(K) \).

Lemma 6.2 Let \((K, s), (U, t) \in S(G)\), and assume that \((K, s) \sim_p (U, t)\). Then the subgroup \( O^p(K) \) of \( K \) is a conjugate of the subgroup \( O^p(U) \) of \( U \) in \( G \).

Proof. We may assume that \( K = \langle g \rangle U \) for some \( g \in N_G(U, t)_p \). If \( p \) is a prime, then \( U \geq O^p(U) \geq O^p(K) \), and thus \( O^p(U) = O^p(K) \). Suppose that \( p = \infty \). We have \( U^{(i-1)} \geq K^{(i)} \geq U^{(i)} \) for any \( i \geq 1 \). If \( U^{(i-1)} = U^{(i)} \) for some \( i \), then \( U^{(i-1)} = K^{(i)} = U^{(i)} \). Hence we have \( O^\infty(K) = O^\infty(U) \), completing the proof. \( \square \)

Lemma 6.3 Let \( U \) and \( H \) be subgroups of \( G \), and suppose that \( O^p(U) \) is a conjugate of \( O^p(H) \) in \( G \). Then \((H, \inf \mathcal{L}H) \sim_p (U, \inf \mathcal{L}U)\).

Proof. We may assume that \( H = \langle g \rangle U \) for some \( gU \in W_G(U, \inf \mathcal{L}U)_p \). By definition, \( \inf \mathcal{L}(g)_U = \inf \mathcal{L}(g)_U \). Hence we have \( \langle g \rangle U, \inf \mathcal{L}(g)_U \sim_p (U, \inf \mathcal{L}U) \), completing the proof. \( \square \)

Given \( K \leq G \) and \( s_1, s_2 \in \mathcal{L}_K \) with \( s_1 > s_2 \), the phrase ‘\( s_1 \) covers \( s_2 \) in \( \mathcal{L}_K \)’ means that there is no element \( t \) of \( \mathcal{L}_K \) satisfying the condition \( s_1 > t > s_2 \).

We now extend Dress’ characterization of solvable groups for \( \Omega(G) \) (cf. [8]).
Theorem 6.4 Assume that, given \( s \in \mathcal{L} \) with \( s \in \mathcal{L}_K \) for some \( K \leq G \), there exist subgroups \( H \) and \( U \) of \( G \) with \( U \leq H \leq N_G(U) \) satisfying the following conditions:

(i) The set of subgroups \( K \) of \( H \) containing \( U \) coincides with \( \{ K \leq G \mid s \in \mathcal{L}_K \} \).

(ii) If \( s \) covers \( t \) in \( \mathcal{L}_U \), then \( s = \inf \mathcal{L}_U^\leq t \) for some \( gU \in W_G(U,t)_p \) with \( g \not\in U \).

Then \( G \) is a \( p \)-group, where an \( \infty \)-group is a solvable group, if and only if the prime spectrum of \( \Omega(G,M_{\mathcal{S}})_p \) is connected in the Zariski topology, that is, if and only if 0 and 1 are the only idempotents of \( \Omega(G,M_{\mathcal{S}})_p \).

Proof. By Proposition 6.1, 0 and 1 are the only idempotents of \( \Omega(G,M_{\mathcal{S}})_p \) if and only if \( (K,s) \sim_p (U,t) \) for all \( (K,s), (U,t) \in \mathcal{S}(G,M_{\mathcal{S}}) \). If \( \mathcal{S}(G,M_{\mathcal{S}}) \) has only one equivalence class with respect to \( \sim_p \), then by Lemma 6.2, \( O^p(G) = \{e\} \), which forces \( G \) to be a \( p \)-group. Conversely, assume that \( G \) is a \( p \)-group. Let \( K \leq G \), and let \( s \in \mathcal{L}_K \). By the condition (i), there exist subgroups \( H \) and \( U \) of \( G \) such that \( U \leq H \leq N_G(U,s) \) and \( \{ K \leq G \mid s \in \mathcal{L}_K \} \) coincides with the set of subgroups \( K \) of \( H \) containing \( U \). We have \( (K,s) \sim_p (U,s) \), because \( K/U \) is a \( p \)-group. If \( s \) covers \( t \) in \( \mathcal{L}_U \), then by the condition (ii), \( s = \inf \mathcal{L}_U^\leq t \) and \( (gU)_p \sim_p (U,t) \) for some \( gU \in W_G(U,t)_p \) with \( g \not\in U \), which implies that \( (U,s) \sim_p (gU,s) \sim_p (U,t) \). Hence either \( s = \inf \mathcal{L}_U \) or \( (K,s) \sim_p (U,t) \) for some \( t \in \mathcal{L}_U \) with \( t < s \). By repeating this argument, we can choose elements \( (U_0,t_0) := (K,s), (U_1,t_1), \ldots, (U_\ell,t_\ell) \) of \( \mathcal{S}(G,M_{\mathcal{S}}) \) with \( t_\ell = \inf \mathcal{L}_U \) such that \( (U_i,t_i) \sim_p (U_{i+1},t_{i+1}) \) and \( t_{i+1} \leq t_i \in \mathcal{L}_U \) with \( i = 0, 1, \ldots, \ell - 1 \). Moreover, \( (U_\ell, \inf \mathcal{L}_U) \sim_p (\{e\}, \inf \mathcal{L}_U) \) by Lemma 6.3. Thus we have \( (K,s) \sim_p (U_\ell, \inf \mathcal{L}_U) \sim_p (\{e\}, \inf \mathcal{L}_U) \). Consequently, \( \mathcal{S}(G,M_{\mathcal{S}}) \) has only one equivalence class with respect to \( \sim_p \). This completes the proof. \( \square \)

Example 6.5 Keep the notation of Example 4.4, and assume further that \( G \) is a \( p \)-group. Let \( E \leq G \). Then \( \{ K \leq G \mid E \in \mathcal{S}_\geq K \} \) is the set of subgroups of \( E \). Suppose that \( E \neq \{e\} \) and \( E \) covers \( F \) in \( \mathcal{S}_\geq \{e\} \). Since \( G \) is a \( p \)-group, it turns out that \( F \) is a normal maximal subgroup of \( E \). We can take an element \( g \in N_E(F) \) for which \( E = \langle g \rangle F \). Then it is obvious that \( E = \inf \mathcal{S}_\geq \{g \} F \). Hence the assumption of Theorem 6.4 with \( M_{\mathcal{S}} = M_{\mathcal{S}} \) and \( s = E \in \mathcal{S} \) holds. Consequently, 0 and 1 are the only idempotents of \( \Omega(G,M_{\mathcal{S}})_p \). The assertion of Theorem 6.4 with \( M_{\mathcal{S}} = M_{\mathcal{S}} \) is given in [6, Theorem 7.9], together with the primitive idempotents of \( \Omega(G,M_{\mathcal{S}}) \).

This section ends with a deference between \( \Omega(G,M_{\mathcal{S}})_p \) and \( \Omega(G,M_{\mathcal{S}})_p \).

Theorem 6.6 Let \( M_{\mathcal{S}} \) be the monoid functor given in Example 4.5. Then \( G \) is a \( p \)-group if and only if the number of primitive idempotents of \( \Omega(G,M_{\mathcal{S}})_p \) is \( |C(G)| \), where \( C(G) \) is a full set of nonconjugate subgroups of \( G \).

Proof. Let \( (H,U) \in \mathcal{S}(G,M_{\mathcal{S}}) \), and let \( gU \in W_G(H,U)_p \). Since \( \langle g \rangle H \leq N_G(U) \), we have \( (\langle g \rangle H,U) \in \mathcal{S}(G,M_{\mathcal{S}}) \) and \( U = \inf \mathcal{S}_{\leq (g)_H}^\geq U \). Hence, if \( (K,L) \sim_p (H,U) \)
for some \((K,L) \in \mathcal{S}(G,M)\), then \(L\) is a conjugate of \(U\) in \(G\). Assume that the number of primitive idempotents of \(\Omega(G,M)_{(p)}\) is \(|C(G)|\). By Proposition 6.1 and the above fact, we have \((G,\{e\}) \sim_p (\{e\},\{e\})\). Thus it follows from Lemma 6.2 that \(O^p(G) = \{e\}\), which forces \(G\) to be a \(p\)-group. Conversely, assume that \(G\) is a \(p\)-group. Then by the previous fact, \((H,U) \sim_p (U,U)\) for any \(U \leq H \leq N_G(U)\). Moreover, for any subgroups \(K\) and \(U\) of \(G\), if \((K,K) \sim_p (U,U)\), then \(K\) is a conjugate of \(U\) in \(G\). Consequently, it follows from Proposition 6.1 that the number of primitive idempotents of \(\Omega(G,M)_{(p)}\) is \(|C(G)|\). This completes the proof. \(\square\)

7 Partial lattice Burnside rings

We define a partially order \(\leq\) on \(\mathcal{S}(G,M)\) by the rule that

\[(U,t) \leq (K,s) \iff U \leq K \text{ and } t \leq s.\]

Let \(\mathfrak{X}\) be a subset of \(\mathcal{S}(G,M)\) closed under the action of \(G\) (see Definition 2.4), and suppose that the following condition holds.

\[(A)\] Given \((U,t) \in \mathfrak{X}\) and \(gU \in W_G(U,t)\), the set

\[\{(K,s) \in \mathfrak{X} \mid (K,s) \geq ((g)U,s_{(g,t)})\},\]

where \(s_{(g,t)} = \inf \mathcal{L}_{(g)U}^\geq\), has a unique minimal element, denoted by \((\overline{(g)U},\overline{s_{(g,t)}})\), with respect to the partially order \(\leq\) on \(\mathcal{S}(G,M)\).

We denote by \(\text{El}(\tilde{T}_G^M)_{\mathfrak{X}}\) the full subcategory of \(\text{El}(\tilde{T}_G^M)\) whose objects are the elements \((G/K,\pi_s)\) for \((K,s) \in \mathfrak{X}\), and write \(\Gamma_{\mathfrak{X}} = \text{El}(\tilde{T}_G^M)_{\mathfrak{X}}\).

**Lemma 7.1** The category \(\Gamma_{\mathfrak{X}}\) is a finite category which satisfies the assumptions of Theorem 5.1, so that \(Z_{(p)}\Gamma_{\mathfrak{X}}\) is an abstract Burnside ring. Moreover,

\[(G/U,\pi_t)/\sigma_g = (G/(\overline{(g)U}),\pi_{\overline{s_{(g,t)}}})\]

for all \((U,t) \in \mathfrak{X}\) and \(gU \in W_G(U,t)\).

**Proof.** Let \((U,t) \in \mathfrak{X}\), and let \(gU \in W_G(U,t)\). By the condition \((A)\), there exists a coequalizer \(c_{\sigma_g} : (G/U,\pi_t) \to (G/U,\pi_t)/\sigma_g\) of \(\sigma_g\) and \(\text{id}_{(G/U,\pi_t)}\) given by \((G/U,\pi_t)/\sigma_g = (G/(\overline{(g)U}),\pi_{\overline{s_{(g,t)}}})\) and \(c_{\sigma_g}(rU) = r(g)U\) for all \(r \in G\), as desired. \(\square\)

Set \(\overline{\mathfrak{X}} = \mathcal{R}(G,M) \cap \mathfrak{X}\). We denote by \(\Omega(G,\mathfrak{X})_{(p)}\) the \(Z_{(p)}\)-submodule of \(\Omega(G,M)_{(p)}\) generated by the elements \([\overline{(G/K)_{s}}]\) for \((K,s) \in \overline{\mathfrak{X}}\), and define

\[\tilde{\Omega}(G,\mathfrak{X})_{(p)} := \prod_{(K,s) \in \overline{\mathfrak{X}}} Z_{(p)}\] and \(\text{Obs}(G,\mathfrak{X})_{(p)} := \prod_{(K,s) \in \overline{\mathfrak{X}}} Z_{(p)}/|W_G(K,s)|Z_{(p)}\).
The \( \mathbb{Z}(p) \)-module \( \Omega(G, \mathfrak{X})(p) \) is identified with the abstract Burnside ring \( \mathbb{Z}(p) \Gamma_{\mathfrak{X}} \). We define a \( \mathbb{Z}(p) \)-module homomorphism \( \varphi_{\mathfrak{X}}^{(p)} : \Omega(G, \mathfrak{X})(p) \to \tilde{\Omega}(G, \mathfrak{X})(p) \) by

\[
[(G/H)_s] \mapsto (\{gH \in G/H \mid U \leq ^gH \text{ and } t \leq ^g s\}) \quad (U, t) \in \mathfrak{X}
\]

for all \((H, s) \in \mathfrak{X}\), and define a map \( \psi_{\mathfrak{X}}^{(p)} : \tilde{\Omega}(G, \mathfrak{X})(p) \to \text{Obs}(G, \mathfrak{X})(p) \) by

\[
(x_{(H,s)})_{(H,s) \in \mathfrak{X}} \mapsto \left( \sum_{(gU \in W_G(U, t))_p} x_{(gU, s_{(gU, t)})} \mod |W_G(U, t)_p| \right) \quad (U, t) \in \mathfrak{X}
\]

for all \((x_{(H,s)})_{(H,s) \in \mathfrak{X}} \in \tilde{\Omega}(G, \mathfrak{X})(p)\); these maps are identified with the Burnside map and the Cauchy-Frobenius map, respectively (see Theorem 5.5 and Lemma 7.1).

**Theorem 7.2** Under the above notation, the sequence

\[
0 \to \Omega(G, \mathfrak{X})(p) \xrightarrow{\varphi_{\mathfrak{X}}^{(p)}} \tilde{\Omega}(G, \mathfrak{X})(p) \xrightarrow{\psi_{\mathfrak{X}}^{(p)}} \text{Obs}(G, \mathfrak{X})(p) \to 0
\]

of \( \mathbb{Z}(p) \)-modules is exact. Moreover, the \( \mathbb{Z}(p) \)-module \( \Omega(G, \mathfrak{X})(p) \) has a \( \mathbb{Z}(p) \)-algebra structure such that \( \varphi_{\mathfrak{X}}^{(p)} \) is an injective \( \mathbb{Z}(p) \)-algebra homomorphism.

**Proof.** The theorem follows from Theorem 5.1 and Lemma 7.1. \( \Box \)

**Corollary 7.3** Let \( \mu_{\mathfrak{X}} \) denote the Möbius function of the partially ordered set \( \mathfrak{X} \) with the binary relation \( \leq \). The primitive idempotents of \( \mathbb{Q} \otimes \mathbb{Z} \Omega(G, \mathfrak{X}) \) are the elements

\[
\varepsilon_{(K,s)} := \frac{1}{|N_G(K, s)|} \sum_{(U,t) \in \mathfrak{X}, (U,t) \leq (K,s)} |U| \mu_{\mathfrak{X}}((U, t), (K, s)) [(G/U)_t]
\]

for \((K, s) \in \mathfrak{X}\), where the sum is taken over all \((U, t) \in \mathfrak{X}\) with \((U, t) \leq (K, s)\).

**Proof.** Set \( \varphi_{\mathfrak{X}} = \varphi_{\mathfrak{X}}^{(\infty)} \). For each \((K, s) \in \mathfrak{X}\), we have

\[
\varphi_{\mathfrak{X}}(|G| \cdot \varepsilon_{(K,s)})
\]

\[
= \frac{|G|}{|N_G(K, s)|} \sum_{(U,t) \in \mathfrak{X}, (U,t) \leq (K,s)} |U| \mu_{\mathfrak{X}}((U, t), (K, s)) \varphi_{\mathfrak{X}}([G/U]_t)
\]

\[
= \frac{|G|}{|N_G(K, s)|} \sum_{(U,t) \in \mathfrak{X}, (U,t) \leq (K,s)} |U| \mu_{\mathfrak{X}}((U, t), (K, s))
\]

\[
\quad \cdot \left( \sum_{g \in G} \mu_{\mathfrak{X}}((U, t), (K, s)) \right) \quad (U, t) \in \mathfrak{X}
\]

\[
= \frac{|G|}{|N_G(K, s)|} \sum_{(U,t) \in \mathfrak{X}, (U,t) \leq (K,s)} |U| \cdot \left( \sum_{g \in G} \mu_{\mathfrak{X}}((U, t), (K, s)) \right) \quad (U, t) \in \mathfrak{X}
\]

\[
= |G| \cdot (\delta_{(K,s), (U,t)}) \quad (U, t) \in \mathfrak{X},
\]
where \( x^{(K,s)}_{(g^{-1}U_1, s^{-1}t_1)} = \{(U, t) \in X \mid (U_1, t_1) \leq (gU, gt) \leq (gK, gs)\} \). Hence the assertion is an immediate consequence of Theorem 7.2. □

**Remark 7.4** From Theorem 4.12 with \( H = G \) and Corollary 7.3 with \( X = S(G, M_{\mathcal{I}}) \), we know that the elements \( e^{(G)}_{(K,s)} \) of \( \mathbb{Q} \otimes \mathbb{Z} \Omega(G, M_{\mathcal{I}}) \) for \( (K, s) \in \mathcal{R}(G, M_{\mathcal{I}}) \) are the primitive idempotents, and so are the elements \( \varepsilon_{(K,s)} \) of \( \mathbb{Q} \otimes \mathbb{Z} \Omega(G, M_{\mathcal{I}}) \) for \( (K, s) \in \mathcal{R}(G, M_{\mathcal{I}}) \). Let \( (K, s) \in \mathcal{R}(G, M_{\mathcal{I}}) \). Given \((U, t) \in S(G, M_{\mathcal{I}})\), we define

\[
\tilde{\mu}((U, t), (K, s)) := \begin{cases}
\sum_{t_1 \in \mathcal{L}^{\leq s}_K, \ t=t_1 \wedge \sup \mathcal{L}_U} \mu_K(t_1, s)\mu(U, K) & \text{if } (U, t) \leq (K, s), \\
0 & \text{otherwise},
\end{cases}
\]

where the sum is taken over all \( t_1 \in \mathcal{L}^{\leq s}_K \) with \( t = t_1 \wedge \sup \mathcal{L}_U \). Observe that

\[
\varepsilon_{(K,s)} = \frac{1}{|N_G(K,s)|} \sum_{(U, t) \in \mathcal{X}_{(K,s)}} |U|\mu_X((U, t), (K, s))[(G/U)_t]
\]

and

\[
e^{(G)}_{(K,s)} = \frac{1}{|N_G(K,s)|} \sum_{(U, t) \in S(G, M_{\mathcal{I}}), \ (U, t) \leq (K, s)} |U|\tilde{\mu}((U, t), (K, s))[(G/U)_t].
\]

Set \( X = S(G, M_{\mathcal{I}}) \). We have \( \mu_X((U, t), (K, s)) = \tilde{\mu}((U, t), (K, s)) \), or equivalently,

\[
\sum_{(U, t) \leq (H, s_1) \leq (K, s)} \tilde{\mu}((U, t), (H, s_1)) = \begin{cases}
1 & \text{if } (U, t) = (K, s), \\
0 & \text{otherwise},
\end{cases}
\]

for all \((U, t) \in \mathcal{X}\), by which \( \varepsilon_{(K,s)} \) coincides with \( e^{(G)}_{(K,s)} \). In fact, it follows that

\[
\sum_{(U, t) \leq (H, s_1) \leq (K, s)} \tilde{\mu}((U, t), (H, s_1)) = \sum_{(U, t) \leq (H, s_1) \leq (K, s)} \sum_{t_1 \in \mathcal{L}^{\leq s}_H, \ t=t_1 \wedge \sup \mathcal{L}_U} \mu_H(t_1, s_1)\mu(U, H)
\]

\[
= \sum_{U \leq H \leq K} \sum_{t_1 \in \mathcal{L}^{\leq s}_H, \ t=t_1 \wedge \sup \mathcal{L}_U} \mu(U, H)
\]

\[
= \begin{cases}
\sum_{U \leq H \leq K} \mu(U, H) & \text{if } t = s \wedge \sup \mathcal{L}_U, \\
0 & \text{otherwise}
\end{cases}
\]

\[
= \begin{cases}
1 & \text{if } (U, t) = (K, s), \\
0 & \text{otherwise}
\end{cases}
\]

for all \((U, t) \in \mathcal{X}\). If \( M_{\mathcal{I}} = M_{\mathcal{I}} \) (see Example 4.4), then this equality is stated in

[6, Proposition 5.3], and the primitive idempotent \( e^{(G)}_{(K,s)} \) is due to [6, Corollary 5.5].
Assume that \((K \cap U, s \wedge t) \in X\) for all \((K, s), (U, t) \in X\) and \((G, \sup \mathcal{L}_G) \in X\). By Proposition 2.5 and Theorem 7.2, \(\Omega(G, X)\) is a subring of \(\Omega(G, M_{\mathcal{L}})\), which is called the partial lattice Burnside ring relative to \(X\).

We identify \(\Omega(G, X)\) with \(\mathbb{Z}G_X\), and turn to a criteria for the units of \(\Omega(G, X)\).

**Proposition 7.5** Assume that \((K \cap U, s \wedge t) \in X\) for all \((K, s), (U, t) \in X\) and \((G, \sup \mathcal{L}_G) \in X\). Let \(\tilde{x} = (x_{(U,t)})_{(U,t) \in X} \in \Omega(G, X)\). Then \(\tilde{x}\) is contained in the image of the ring homomorphism \(\varphi_X : \Omega(G, X) \rightarrow \tilde{\Omega}(G, X)\), where \(\varphi_X = \varphi_X^{(\infty)}\), if and only if, for each \((U, t) \in X\), the map \(\gamma_{(U,t)} : \text{Aut}((G/U)_t) \rightarrow \langle -1 \rangle\) given by

\[
\sigma_g \mapsto x_{(U,t)}\sigma_{(gU, x_{(gU,t)})}
\]

for all \(gU \in W_G(U, t)\) is a group homomorphism.

**Proof.** Since \(\Omega(G, X)\) is a subring of \(\Omega(G, M_{\mathcal{L}})\), the proposition is a consequence of Proposition 5.4, Corollary 5.6, and Lemma 7.1. \(\square\)

**Example 7.6** We set \(X = \{(K, E) \mid K \trianglelefteq E \leq G\}\), where \(K \trianglelefteq E\) denotes that \(K\) is a normal subgroup of \(E\). Then \(X\) is a subset of \(\mathcal{S}(G, M_{\mathcal{L}})\) closed under the action of \(G\), where \(M_{\mathcal{L}}\) is given in Example 4.4. Let \((U, F) \in X\), and let \(gU \in W_G(U, F)\). We denote by \(\langle (g)U \rangle^{\trianglelefteq (g)F}\) the normal closure of \(\langle g \rangle U\) in \(\langle g \rangle F\). If \((K, E) \geq (\langle g \rangle U, s_{(g,F)})\) with \((K, E) \in X\), then \(E \geq s_{(g,F)} = \langle g \rangle F\) and \(K \geq \langle (g)U \rangle^{\trianglelefteq (g)F} \geq \langle g \rangle U\), which implies that \(\langle (g)U, x_{(gU,F)} \rangle = \langle (\langle g \rangle U)^{\trianglelefteq (g)F}, \langle g \rangle F \rangle\). Hence the condition (A) holds. The partial lattice Burnside ring \(\Omega(G, X)\) is isomorphic to the section Burnside ring \(\Gamma(G)\) introduced by S. Bouc [6]. The primitive idempotents of \(\mathbb{Q} \otimes_{\mathbb{Z}} \Gamma(G)\), together with the description of \(\mu_X\) in terms of the Möbius function of \(\mathcal{S}(G)\), are given in [6, Corollary 12.5]. Obviously, \((K \cap U, E \cap F) \in X\) for all \((K, E), (U, F) \in X\) and \((G, G) \in X\), so that Proposition 7.5 in this case is equivalent to [6, Theorem 15.3].

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