An Eigenvalue Inclusion Set for Matrices with a Constant Main Diagonal Entry

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Abstract: A set to locate all eigenvalues for matrices with a constant main diagonal entry is given, and it is proved that this set is tighter than the well-known Geršgorin set, the Brauer set and the set proposed in (Linear and Multilinear Algebra, 60:189-199, 2012). Furthermore, by applying this result to Toeplitz matrices as a subclass of matrices with a constant main diagonal, we obtain a set including all eigenvalues of Toeplitz matrices.

Keywords: eigenvalue; matrices with a constant main diagonal; Toeplitz; inclusion set

1. Introduction

Eigenvalue localization is an important topic in Matrix theory and its applications. Many eigenvalue inclusion sets for a matrix $A = [a_{ij}] \in \mathbb{C}^{n \times n}$ [1–11] have been established, such as the well-known Geršgorin set [5,11] and the Brauer set [1,11]. However, as Melman [9] pointed out, for the special class of matrices with a constant main diagonal (c.m.d.), both the Geršgorin and Brauer sets each consists of a single disc, a rather uninteresting outcome. In fact, if a matrix $A = [a_{ij}] \in \mathbb{C}^{n \times n}$ satisfies $a_{11} = a_{22} = \cdots = a_{nn} = \bar{a}$, then both $\Gamma(A)$ and $K(A)$ reduce, respectively, to the following forms:

$$\Gamma(A) = \{z \in \mathbb{C} : |z - \bar{a}| \leq \max_{i \in N} r_i(A)\},$$

and

$$K(A) = \left\{z \in \mathbb{C} : |z - \bar{a}| \leq \max_{i \in N, j \neq i} \sqrt{r_i(A)r_j(A)} \right\},$$

where $r_i(A) = \sum_{j \neq i} |a_{ij}|$ and $N = \{1, 2, \ldots, n\}$. Obviously, the Geršgorin and Brauer sets are just discs [9].

To localize all eigenvalues of matrices with a c.m.d. more precisely, Melman also [9] gave an eigenvalue inclusion set (see Theorem 1), which is tighter than $\Gamma(A)$ and $K(A)$.

**Theorem 1** ([9] Theorem 2.1). Let $A = [a_{ij}] \in \mathbb{C}^{n \times n}$ with $a_{ii} = \bar{a}$ for all $i \in N, n \geq 2$. Let $\sigma(A)$ be the spectrum of the matrix $A$, that is, $\sigma(A) = \{\lambda \in \mathbb{C} : \det(\lambda I - A) = 0\}$. Then,

$$\sigma(A) \subseteq \Omega(A) = \bigcup_{i \in N} \Omega_i(A),$$

where $A_0 = A - \bar{a}I$, $(A_0^2)_{ij}$ denotes the $(i,j)$th entry of $A_0^2$ and

$$\Omega_i(A) = \left\{z \in \mathbb{C} : \left|z - \bar{a} - \sqrt{(A_0^2)_{ii}} \right| \leq r_i(A_0^2) \right\}.$$
Furthermore, $\Omega(A) \subseteq K(A) \subseteq \Gamma(A)$.

In [7], Li and Li provided two tighter sets including all eigenvalues of a matrix with a c.m.d. (see Theorems 2 and 3).

**Theorem 2 ([7] Theorem 2.4).** Let $A = [a_{ij}] \in \mathbb{C}^{n \times n}$ with $a_{ii} = \bar{a}$ for all $i \in \mathbb{N}, n \geq 2$. Then,

$$\sigma(A) \subseteq \Omega_1(A) = \bigcap_{0 \leq \alpha \leq 1} \bigcup_{i \in \mathbb{N}} \Omega_{\alpha}^{ia}(A),$$

where

$$\Omega_{\alpha}^{ia}(A) = \left\{ z \in \mathbb{C} : \left| z - \bar{a} - \sqrt{(A_{0}^\alpha)_{ii}} \right| \leq \alpha \left| r_i \left( A_{0}^\alpha \right) \right| \right\}.$$

**Theorem 3 ([7] Theorems 2.5 and 2.7).** Let $A = [a_{ij}] \in \mathbb{C}^{n \times n}$ with $a_{ii} = \bar{a}$ for all $i \in \mathbb{N}, n \geq 2$. Then,

$$\sigma(A) \subseteq \Omega_2(A) = \bigcap_{0 \leq \alpha \leq 1} \bigcup_{i \in \mathbb{N}} \Omega_{\alpha}^{2a}(A),$$

where

$$\Omega_{\alpha}^{2a}(A) = \left\{ z \in \mathbb{C} : \left| z - \bar{a} - \sqrt{(A_{0}^\alpha)_{ii}} \right| \leq \left( r_i \left( A_{0}^\alpha \right) \right)^{1-\alpha} \right\}.$$

Furthermore,

$$\Omega_2(A) \subseteq \Omega_1(A) \subseteq \left( \Omega(A) \cap \Omega(A^T) \right) \subseteq (K(A) \cap K(A^T)) \subseteq (\Gamma(A) \cap \Gamma(A^T)).$$

In this paper, we first give a sufficient condition for non-singular matrices, which leads to a new set including all eigenvalues of matrices with a c.m.d. As an application, in Section 3, we apply the result obtained in Section 2 to Toeplitz matrices as a subclass of matrices with a c.m.d. and obtain a new eigenvalue inclusion set. All the new eigenvalue inclusion sets are proved to be tighter than those in [9].

2. A New Eigenvalue Inclusion Set for Matrices with a c.m.d.

In this section, we present a new eigenvalue inclusion set for matrices with a c.m.d. First, a sufficient condition for non-singular matrices is given.

**Lemma 1.** For any $A = [a_{ij}] \in \mathbb{C}^{n \times n}$ with $a_{ii} = \bar{a}$ for all $i \in \mathbb{N}, n \geq 2$, if

$$|\bar{a}^2 - (A_{0})_{ii}||\bar{a}^2 - (A_{0})_{jj}| > r_i(A_{0}^2)r_j(A_{0}^2),$$

(1)

where $A_{0} = A - aI$, then $A$ is non-singular.

**Proof.** Suppose on the contrary that $A = [a_{ij}] \in \mathbb{C}^{n \times n}$ satisfies Inequality (1) and is singular, then there is an $x = [x_1, x_2, \ldots, x_n]^T \in \mathbb{C}^n$, with $x \neq 0$, such that $Ax = 0$. Let

$$0 < |x_i| \geq |x_j| \geq \max\{ |x_k| : k \in \mathbb{N}, k \neq s, k \neq t \}.$$

Note that $A_{0} = A - aI$. Then, $A_{0}x = -\bar{a}x$, which leads to $A_{0}^2x = \bar{a}^2x$, equivalently, $(A_{0}^2 - \bar{a}^2)x = 0$. This implies that for all $i \in \mathbb{N},$

$$((A_{0}^2)_{ii} - \bar{a}^2)x_i = -\sum_{j \in \mathbb{N}, j \neq i} (A_{0}^2)_{ij}x_j.$$
Hence,

\[(A_0^2)_{ij} - a^2 ||x_i|| \leq \sum_{j \in N, j \neq i} ||(A_0^2)_{ij}|| ||x_j||, \forall i \in N.\] (2)

Taking \(i = t\), Inequality (2) becomes

\[(A_0^2)_{tt} - a^2 ||x_t|| \leq \sum_{j \in N, j \neq t} ||(A_0^2)_{ij}|| ||x_j|| \leq r_t(A_0^2)||x_t||.\] (3)

If \(|x_s| = 0\), then Inequality (3) reduces to \(||(A_0^2)_{tt} - a^2 ||x_t|| = 0\), implying that \(||(A_0^2)_{tt} - a^2|| = 0\). However, this contradicts Inequality (1). Hence, \(|x_s| > 0\). We now take \(i = s\) in Inequality (3), and obtain similarly

\[(A_0^2)_{ss} - a^2 ||x_s|| \leq r_s(A_0^2)||x_s||.\]

On multiplying the above inequality with Inequality (3), then

\[(A_0^2)_{tt} - a^2||(A_0^2)_{ss} - a^2|| ||x_t|| ||x_s|| \leq r_t(A_0^2)r_s(A_0^2)||x_t|| ||x_s||.\] (4)

Note that \(|x_i||x_s| > 0\), then

\[(A_0^2)_{tt} - a^2||(A_0^2)_{ss} - a^2|| \leq r_t(A_0^2)r_s(A_0^2),\] (5)

which contradicts Inequality (1). Therefore, \(A\) is non-singular. \qed

From Lemma 1, we can obtain a new eigenvalue inclusion set for matrices with a c.m.d.

**Theorem 4.** Let \(A = [a_{ij}] \in \mathbb{C}^{n \times n}\) with \(a_{ii} = \bar{a}\) for all \(i \in N\), and \(n \geq 2\). Then,

\[\sigma(A) \subseteq \Omega(A) = \bigcup_{i,j \in N, i \neq j} \Omega_{ij}(A),\]

where

\[\Omega_{ij}(A) = \{z \in \mathbb{C} : |z - \bar{a} - \sqrt{(A_0^2)_{ii}}||z - \bar{a} + \sqrt{(A_0^2)_{ii}}||z - \bar{a} - \sqrt{(A_0^2)_{jj}}||z - \bar{a} + \sqrt{(A_0^2)_{jj}}| \leq r_i(A_0^2)r_j(A_0^2)\},\] (6)

\[|z - \bar{a} + \sqrt{(A_0^2)_{jj}}| \leq r_i(A_0^2)r_j(A_0^2)),\] (7)

and \(A_0 = A - \bar{a}I\).

**Proof.** Suppose that \(\lambda \in \sigma(A)\), then \(\lambda I - A\) is singular. If \(\lambda \not\in \Omega(A)\), then \(\lambda \not\in \Omega_{ij}(A)\) for any \(i, j \in N, i \neq j\), which leads to that for any \(i, j \in N, i \neq j\),

\[|z - \bar{a} - \sqrt{(A_0^2)_{ii}}||z - \bar{a} + \sqrt{(A_0^2)_{ii}}||z - \bar{a} - \sqrt{(A_0^2)_{jj}}||z - \bar{a} + \sqrt{(A_0^2)_{jj}}| > r_i(A_0^2)r_j(A_0^2),\]

that is,

\[(z - \bar{a})^2 - (A_0^2)_{ii}||z - \bar{a})^2 - (A_0^2)_{jj} > r_i(A_0^2)r_j(A_0^2).\]

From Lemma 1, we have that \(\lambda I - A\) is non-singular. This contradicts that \(\lambda I - A\) is singular. Hence, \(\lambda \in \Omega(A)\). \qed

We now give a comparison between the new eigenvalue set \(\Omega(A)\) and the set \(\Omega(A)\) in Theorem 1.

**Theorem 5.** Let \(A = [a_{ij}] \in \mathbb{C}^{n \times n}\) with \(a_{ii} = \bar{a}\) for any \(i \in N\), and \(n \geq 2\). Then,

\[\Omega(A) \subseteq \Omega(A).\]
Proof. Suppose that \( z \in \tilde{\Omega}(A) \), then there exist \( i, j \in N \) with \( i \neq j \) and \( z \in \tilde{\Omega}_{ij}(A) \), that is,

\[
|z - a - \sqrt{(A^2_0)_{ii}}| |z - a + \sqrt{(A^2_0)_{ii}}||z - a - \sqrt{(A^2_0)_{jj}}| \\
|z - a + \sqrt{(A^2_0)_{jj}}| \leq r_i(A^2_0) r_j(A^2_0).
\]

Equivalently,

\[
| (z - a)^2 - (A^2_0)_{ii} | | (z - a)^2 - (A^2_0)_{jj} | \leq r_i(A^2_0) r_j(A^2_0). \tag{8}
\]

If \( r_i(A^2_0) r_j(A^2_0) = 0 \), then \( (z - a)^2 = (A^2_0)_{ii} \) or \( (z - a)^2 = (A^2_0)_{jj} \). We can get \( z \in \Omega_i(A) \) or \( z \in \Omega_j(A) \) and hence \( z \in \Omega_i(A) \cup \Omega_j(A) \). If \( r_i(A^2_0) r_j(A^2_0) > 0 \), we have from Inequality (8),

\[
\left( \frac{|(z - a)^2 - (A^2_0)_{ii}|}{r_i(A^2_0)} \right) \left( \frac{|(z - a)^2 - (A^2_0)_{jj}|}{r_j(A^2_0)} \right) \leq 1,
\]

that is, \( |(z - a)^2 - (A^2_0)_{ii}| \leq r_i(A^2_0) \) or \( |(z - a)^2 - (A^2_0)_{jj}| \leq r_j(A^2_0) \). Hence, \( z \in \Omega_i(A) \) or \( z \in \Omega_j(A) \), consequently, \( z \in \Omega_i(A) \cup \Omega_j(A) \) and

\[
\tilde{\Omega}_{ij}(A) \subseteq \Omega_i(A) \cup \Omega_j(A). \tag{9}
\]

As Equation (9) holds for any \( i \) and \( j \) \((i \neq j)\) in \( N \), therefore \( \tilde{\Omega}(A) \subseteq \Omega(A) \).

Example 1. Consider the matrix \( A \) (the matrix \( A_4 \) in [9]),

\[
A = \begin{bmatrix}
2 & i & -3 & -i \\
0 & 2 & 1 & -5i \\
4 & 1 & 2 & 2 \\
i & -1 & 1 & 2
\end{bmatrix}.
\]

the sets \( \Gamma(A) \), \( \mathcal{K}(A) \), \( \Omega(A) \), and \( \tilde{\Omega}(A) \) are shown in Figure 1, where \( \Gamma(A) \) is represented by the outside boundary, \( \mathcal{K}(A) \) by the middle, \( \Omega(A) \) by the inner, and \( \tilde{\Omega}(A) \) is filled. The exact eigenvalues are plotted with asterisks. It is easy to see that

\[
\tilde{\Omega}(A) \subset \Omega(A) \subset \mathcal{K}(A) \subset \Gamma(A).
\]

This example shows that the new eigenvalue inclusion set in Theorem 4 is tighter than the Geršgorin set \( \Gamma(A) \), the Brauer set \( \mathcal{K}(A) \) and the set \( \Omega(A) \) obtained in [9].

![Figure 1. \( \tilde{\Omega}(A) \subset \Omega(A) \subset \mathcal{K}(A) \subset \Gamma(A) \).](image-url)
Remark 1. From Theorems 3 and 5, we have that
\[
\overline{\Omega}(A) \subseteq \Omega(A), \quad \overline{\Omega}(A^T) \subseteq \Omega(A^T), \quad \left(\overline{\Omega}(A) \cap \overline{\Omega}(A^T)\right) \subseteq \left(\Omega(A) \cap \Omega(A^T)\right)
\]
and
\[
\Omega^2(A) \subseteq \Omega^1(A) \subseteq \left(\Omega(A) \cap \Omega(A^T)\right).
\]

Note here that \(\Omega^1(A) = \Omega^1(A^T)\) and \(\Omega^2(A) = \Omega^2(A^T)\). However, the sets \(\Omega^2(A)\) and \(\Omega(A) \cap \Omega(A^T)\) (also \(\Omega^1(A)\) and \(\Omega(A) \cap \Omega(A^T)\)) cannot be compared with each other. In fact, we also consider the matrix \(A\) in Example 1, and draw \(\Omega^2(A)\), and \(\Omega(A) \cap \Omega(A^T)\) in Figures 2 and 3. It is not difficult to see that
\[
\Omega^2(A) \nsubseteq \Omega(A) \cap \Omega(A^T)
\]
and
\[
\Omega(A) \cap \Omega(A^T) \nsubseteq \Omega^2(A).
\]

Figure 2. \(\Omega^2(A)\).

Figure 3. \(\Omega(A) \cap \Omega(A^T)\).
3. Eigenvalue Inclusion Set for Toeplitz Matrices

Toeplitz matrices, a subclass of matrices with a c.m.d., arise in many fields of application [12–18], such as probability and statistics, signal processing, differential and integral equations, Markov chains, Padé approximation, etc. For example, consider an assigned Lebesgue integrable function \( f \) defined on the fundamental interval \( I = [\pi, -\pi] \) and periodically extended to the whole real axis, and the Fourier coefficients \( a_k \) of \( f \) that is

\[
a_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx} dx, (i^2 = -1)
\]

where \( k \) is an integer number. From the coefficients \( a_k \) one can build the infinite dimensional Toeplitz matrix \( T_n(f) \) with entries \( (T_n(f))_{ij} = a_{i-j} \), \( i, j = 0, 1, \ldots, n - 1 \) [12,13,16].

Toeplitz matrices are constant along all their NW-SE diagonals [7,9], i.e., a Toeplitz matrix

\[
T_n = \begin{bmatrix}
t_0 & t_1 & t_2 & \cdots & t_{n-1} \\
t_{-1} & t_0 & t_1 & \cdots & t_{n-2} \\
& \ddots & \ddots & \ddots & \vdots \\
t_{2-n} & \cdots & t_{-1} & t_0 & t_1 \\
t_{1-n} & \cdots & t_{-2} & t_{-1} & t_0 \end{bmatrix}
\]

Indeed, if \( f \) is a real valued function, we have \( a_k = \bar{a}_{-k} \) and, consequently, \( T_n(f) \) is Hermitian; moreover, if \( f(x) = f(-x) \), then the coefficients \( a_k \) are real and \( T_n(f) \) is symmetric. The following result can be found in [12,19] and in a multilevel setting in [16,17].

**Theorem 6** ([17,19]). Let \( \lambda_{j}^{(n)} \) be the eigenvalues of \( T_n(f) \) sorted in nondecreasing order, and \( m_f = \text{ess inf } f \), \( M_f = \text{ess sup } f \).

- **a.** If \( m_f < M_f \), then \( \lambda_{j}^{(n)} \in (m_f, M_f) \) for every \( j \) and \( n \); if \( m_f = M_f \), then \( f \) is constant and trivially \( T_n(f) = m_f I_n \) with \( I_n \) identity of size \( n \);
- **b.** The following asymptotic relationships hold: \( \lim_{n \to \infty} \lambda_{1}^{(n)} = m_f \), \( \lim_{n \to \infty} \lambda_{n}^{(n)} = M_f \).

Furthermore, there exist further results establishing precisely how fast the convergence holds [13,17]. Since in applications (differential and fractional operators/equations, shift-invariant integral operators/equations, signal and image processing etc.) often the underlying Toeplitz matrices have large size \( n \), then the results in [12,13,16,17] are difficult to beat and improved. When \( f \) is complex-values the theory is more complicated and in that case the convex hull of the essential range of \( f \) plays a role (see [13,18]). Obviously, a Toeplitz matrix is persymmetric. Here, we call \( A \) persymmetric if \( A \) is symmetric with respect to the main anti-diagonal [9]. Furthermore, the square of a Toeplitz matrix \( T \) is not necessary Toeplitz, but it is persymmetric.

In [9], Melman applied the eigenvalue inclusion Theorem (Theorem 1) of matrices with a c.m.d. to Toeplitz matrices, and obtained the following simpler form of the eigenvalue inclusion set.

**Theorem 7** ([9] Theorem 3.1). Let \( T = [t_{ij}] \in C^{n \times n} \) be a Toeplitz matrix and \( t_{ii} = \bar{t}_i \), \( n \geq 2 \). Then,

\[
\sigma(T) \subseteq \Omega(T) = \bigcup_{i=1}^{\lfloor \frac{n}{2} \rfloor} \Omega_i(T),
\]

where

\[
\Omega_i(T) = \{ z \in C : |z - \bar{t} - \sqrt{(T_0^2)_{ii}}||z - \bar{t} + \sqrt{(T_0^2)_{ii}}| \leq v_i(T_0^2) \},
\]

\[
T_0 = T - \bar{I}, v_i(T_0^2) = \max \{ r_i(T_0^2), r_{n-i+1}(T_0^2) \},
\]
where
\[
\left\lfloor \frac{n}{2} \right\rfloor = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even,} \\ \frac{n+1}{2}, & \text{if } n \text{ is odd.} \end{cases}
\]

Furthermore, \( \Omega(T) \subseteq \mathcal{K}(T) \subseteq \Gamma(T) \).

Next, by applying Theorem 4 to Toeplitz matrices, we obtain a new eigenvalue inclusion set.

**Theorem 8.** Let \( T = [t_{ij}] \in \mathbb{C}^{n \times n} \) be a Toeplitz matrix with \( t_{11} = 1 \) and \( n \geq 2 \). Then,

\[
\sigma(T) \subseteq \Omega(T) = \left( \bigcup_{i,j \in \left[ \frac{n}{2} \right], i \neq j} \Omega_{ij}^T(T) \right) \bigcup \left( \bigcup_{i \in \left[ \frac{n}{2} \right]} \Omega_i(T) \right),
\]

where

\[
\Omega_{ij}^T(T) = \left\{ z \in \mathbb{C} : |z - t_0 - \sqrt{(T_0^2)_{ii}}| |z - t_0 + \sqrt{(T_0^2)_{ii}}| \leq V_i(T_0^2) V_j(T_0^2) \right\},
\]

\[
\Omega_i(T) = \left\{ z \in \mathbb{C} : \left| |z - t_0 - \sqrt{(T_0^2)_{ii}}| |z - t_0 + \sqrt{(T_0^2)_{ii}}| \right|^2 \leq r_i(T_0^2) r_{n-i+1}(T_0^2) \right\},
\]

\[
V_i(T_0^2) = \max \{ r_i(T_0^2), r_{n-i+1}(T_0^2) \}, \text{ and } T_0 = T - t_0 I.
\]

**Proof.** Since \( T \) is Toeplitz and \( T_0 = T - I \), we have that \( T_0 \) is also Toeplitz and \( T_0^2 \) is persymmetric. Therefore, the main diagonal of \( T_0^2 \) has at most \( \left\lfloor \frac{n}{2} \right\rfloor \) distinct values, and \( (T_0^2)_{ij} = (T_0^2)_{n-i+1,n-j+1} \) for \( i = 1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor \). Hence, by Theorem 4 and Equation (6), for any \( \lambda \in \sigma(T), \lambda \in \Omega(T) = \bigcup_{i,j \in \mathbb{N}, j \neq i} \Omega_{ij}(T) \).

For the case \( i, j \in \{1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor \}, i \neq j \), we have

\[
|\lambda - t_0|^2 - (T_0^2)_{ii} ||\lambda - t_0|^2 - (T_0^2)_{jj}| \leq r_i(T_0^2) r_j(T_0^2). \tag{10}
\]

For the case \( i \in \{1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor \}, j \in \mathbb{N}\backslash\{1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor \}, j \neq n-i+1 \), we have

\[
|\lambda - t_0|^2 - (T_0^2)_{ii} ||\lambda - t_0|^2 - (T_0^2)_{n-j+1,n-j+1}| \leq r_i(T_0^2) r_{n-j+1}(T_0^2). \tag{11}
\]

Note that \( (T_0^2)_{jj} = (T_0^2)_{n-j+1,n-j+1} \), then

\[
|\lambda - t_0|^2 - (T_0^2)_{ii} ||\lambda - t_0|^2 - (T_0^2)_{jj}| \leq r_i(T_0^2) r_{n-j+1}(T_0^2). \tag{11}
\]

From Inequalities (10) and (11), we can get that

\[
|\lambda - t_0|^2 - (T_0^2)_{ii} ||\lambda - t_0|^2 - (T_0^2)_{jj}| \leq V_i(T_0^2) V_j(T_0^2), \tag{12}
\]

where \( V_j(T_0^2) = \max \{ r_j(T_0^2), r_{n-j+1}(T_0^2) \} \). Similarly, we obtain

\[
|\lambda - t_0|^2 - (T_0^2)_{n-i+1,n-i+1} ||\lambda - t_0|^2 - (T_0^2)_{jj}| \leq r_{n-i+1,n-i+1}(T_0^2) V_j(T_0^2). \tag{13}
\]

From \( (T_0^2)_{ii} = (T_0^2)_{n-i+1,n-i+1} \), Inequalities (12) and (13), we could easily get, for any \( i, j \in \{1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor \} \) and \( j \neq i \),

\[
|\lambda - t_0|^2 - (T_0^2)_{ii} ||\lambda - t_0|^2 - (T_0^2)_{jj}| \leq V_i(T_0^2) V_j(T_0^2). \tag{14}
\]
Furthermore, for any $i \in \{1, 2, \ldots, \left\lceil \frac{n}{2} \right\rceil \}$, $j = n - i + 1$,

$$|(\lambda - \bar{t})^2 - (T_0^2)_{ii}| \leq r_i(T_0^2) r_{n-i+1}(T_0^2),$$

which is equivalent to

$$|(\lambda - \bar{t})^2 - (T_0^2)_{ii}| \leq r_i(T_0^2) r_{n-i+1}(T_0^2),$$

that is,

$$\left( |z - t_0 - \sqrt{(T_0^2)_{ii}}| z - t_0 + \sqrt{(T_0^2)_{ii}} \right)^2 \leq r_i(T_0^2) r_{n-i+1}(T_0^2).$$ (15)

The conclusion follows from Inequalities (14) and (15). □

From Theorems 5, 7 and 8, we can obtain easily the comparison results as follows.

**Theorem 9.** Let $T = [t_{ij}] \in \mathbb{C}^{n \times n}$ be a Toeplitz matrix with $t_{11} = \bar{t}$ and $n \geq 2$. Then,

$$\bar{\Omega}(T) \subseteq \Omega(T) \subseteq K(T) \subseteq \Gamma(T).$$

**Example 2.** Consider the Toeplitz matrix $Q$ in [9]:

$$Q = \begin{bmatrix}
6 & 1 & -1 & -2i \\
0 & 6 & 1 & -1 \\
-1 & 0 & 6 & 1 \\
4 & -1 & 0 & 6
\end{bmatrix}.$$

In Figure 4, the sets $\Gamma(Q)$, $K(Q)$, $\Omega(Q)$, and $\bar{\Omega}(Q)$ are shown, where $\Gamma(Q)$ is represented by the outside boundary, $K(Q)$ by the middle, $\Omega(Q)$ by the inner, and $\bar{\Omega}(Q)$ is filled. The exact eigenvalues are plotted with asterisks. As we can see,

$$\bar{\Omega}(Q) \subset \Omega(Q) \subset K(Q) \subset \Gamma(Q).$$

This example shows that the new eigenvalue inclusion set in Theorem 8 is tighter than the set obtained in [9], the Geršgorin set and the Brauer set for a Toeplitz matrix.

**Figure 4.** $\bar{\Omega}(Q) \subset \Omega(Q) \subset K(Q) \subset \Gamma(Q)$. 
4. Conclusions

In this paper, we obtain a new eigenvalue inclusion set for matrices with a c.m.d. We then apply this result to Toeplitz matrices, and get a set including all eigenvalues of Toeplitz matrices. Although they needs more computations to obtain the new eigenvalue sets than those in [9], the new sets capture all eigenvalues more precisely than those in [9].

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References

1. Brauer, A. Limits for the characteristic roots of a matrix II. *Duke Math. J.* 1947, 14, 21–26. [CrossRef]
2. Cvetković, L. H-matrix theory vs. eigenvalue localization. *Numer. Algorithms* 2006, 42, 229–245. [CrossRef]
3. Cvetković, L.; Kostić, V.; Bru, R.; Pedroche, F. A simple generalization of Geršgorin’s theorem. *Adv. Comput. Math.* 2011, 35, 271–280. [CrossRef]
4. Cvetković, L.; Kostić, V.; Varga, R.S. A new Geršgorin-type eigenvalue inclusion set. *Electron. Trans. Numer. Anal.* 2004, 18, 73–80.
5. Geršgorin, S. Über die Abgrenzung der Eigenwerte einer Matrix. *Izv. Akad. Nauk SSSR Ser. Mat.* 1931, 1, 749–754.
6. Li, C.Q.; Li, Y.T. Generalizations of Brauer’s eigenvalue localization theorem. *Electron. J. Linear Algebra* 2011, 22, 1168–1178. [CrossRef]
7. Li, C.Q.; Li, Y.T. New regions including eigenvalues of Toeplitz matrices. *Linear Multilinear Algebra* 2014, 62, 229–241. [CrossRef]
8. Melman, A. Modified Gersgorin Disks for Companion Matrices. *Siam Rev.* 2012, 54, 355–373.
9. Melman, A. Ovals of Cassini for Toeplitz matrices. *Linear Multilinear Algebra* 2012, 60, 189–199. [CrossRef]
10. Sang, C.L.; Zhao, J.X. Eventually DSDD Matrices and Eigenvalue Localization. *Symmetry* 2018, 10, 448. [CrossRef]
11. Varga, R.S. *Geršgorin and His Circles*; Springer: Berlin, Germany, 2004.
12. Di Benedetto, F.; Fiorentino, G.; Serra, S. CG preconditioning for Toeplitz matrices. *Comput. Math. Appl.* 1993, 25, 35–45. [CrossRef]
13. Böttcher, A.; Grudsky, S. On the condition numbers of large semi-definite Toeplitz matrices. *Linear Algebra Appl.* 1998, 279, 285–301. [CrossRef]
14. Bunch, J.R. Stability of methods for solving Toeplitz systems of equations. *SIAM J. Sci. Stat. Comput.* 1985, 6, 349–364. [CrossRef]
15. Pourahmadi, M. Remarks on extreme eigenvalues of Toeplitz matrices. *Int. J. Math. Math. Sci.* 1988, 11, 23–26. [CrossRef]
16. Serra, S. Preconditioning strategies for asymptotically ill-conditioned block Toeplitz systems. *BIT* 1994, 34, 579–594. [CrossRef]
17. Serra, S. On the Extreme Eigenvalues of Hermitian (Block) Toeplitz Matrices. *Linear Algebra Appl.* 1998, 270, 109–129. [CrossRef]
18. Capizzano, S.S.; Tili, P. Extreme singular values and eigenvalues of non-Hermitian block Toeplitz matrices. *J. Comput. Appl. Math.* 1999, 108, 113–130. [CrossRef]
19. Grenander, U.; Szegö, G. *Toeplitz Forms and Their Applications*, 2nd ed.; Chelsea: New York, NY, USA, 1984.