Heat kernels of non-symmetric jump processes with exponentially decaying jumping kernel

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Abstract

In this paper we study the transition densities for a large class of non-symmetric Markov processes whose jumping kernels decay exponentially or subexponentially. We obtain their upper bounds which also decay at the same rate as their jumping kernels. When the lower bounds of jumping kernels satisfy the weak upper scaling condition at zero, we also establish lower bounds for the transition densities, which are sharp.

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1 Introduction

Let $d \in \mathbb{N}$, $\mathbb{R}^d$ be the $d$-dimensional Euclidian space and $\mathbb{R}_+ = \{x \in \mathbb{R}^1 : x > 0\}$. Define

$$\mathcal{L}^\kappa f(x) := \lim_{\varepsilon \downarrow 0} \mathcal{L}^{\kappa, \varepsilon} f(x) := \lim_{\varepsilon \downarrow 0} \int_{\{z \in \mathbb{R}^d : |z| > \varepsilon\}} \left( f(x + z) - f(x) \right) \kappa(x, z) J(|z|) dz$$

where $\kappa : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_+$ is a Borel function satisfying the following conditions: there exist positive constants $\kappa_0, \kappa_1, \kappa_2$ and $\delta \in (0, 1)$ such that

$$\kappa_0 \leq \kappa(x, z) \leq \kappa_1, \quad \kappa(x, z) = \kappa(x, -z) \quad \text{for all } x, z \in \mathbb{R}^d$$

and

$$|\kappa(x, z) - \kappa(y, z)| \leq \kappa_2 |x - y|^{\delta} \quad \text{for all } x, y, z \in \mathbb{R}^d.$$

The operator $\mathcal{L}^\kappa$ can be regarded as the non-local counterpart of elliptic operators in non-divergence form. In this context, the Hölder continuity of $\kappa(\cdot, z)$ in (1.3) is a natural assumption.

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In [CZ16], Zhen-Qing Chen and Xicheng Zhang studied $L^κ$ and its heat kernel when $J(r) = r^{−d−α}$, $r > 0$ and $α ∈ (0, 2)$. They proved the existence and uniqueness of the heat kernel and its sharp two-sided estimates, cf. [CZ16, Theorem 1.1] for details. The methods in [CZ16] are quite robust and have been applied to non-symmetric and non-convolution operators (see [BSV, CHXZ, CZa, CZb, KSV18, KS17, Jin] and references therein). In particular, the first named author, jointly with Renming Song and Zoran Vondraček in [KSV18], studied the operator $L^κ$ and its heat kernel when $J$ is comparable to jumping kernels of subordinate Brownian motions and its Lévy exponent satisfying a weak lower scaling condition at infinity. In this paper we consider the case that $J(r)$ decays exponentially or subexponentially when $r → ∞$ and we obtain sharp two-sided estimates for the heat kernel of $L^κ$.

Throughout this paper, we assume $d ∈ \mathbb{N}$, and that $J : \mathbb{R}_+ → \mathbb{R}_+$ is continuous and non-increasing function satisfying that there exist a continuous and strictly increasing function $φ : [0, 1] → \mathbb{R}_+$ with $φ(0) = 0$, and constants $b > 0$, $0 < β ≤ 1$ and $a ≥ 1$ such that
\[
\frac{a−1}{r^dφ(r)} ≤ J(r) ≤ \frac{a}{r^dφ(r)}, \quad 0 < r ≤ 1 \quad \text{and} \quad J(r) ≤ a \exp(−br^β), \quad r > 1. \tag{1.4}
\]
In addition, we assume that $J$ is differentiable in $\mathbb{R}_+$ and
\[
r \mapsto − J'(r) \quad \text{is non-increasing in } \mathbb{R}_+. \tag{1.5}
\]
Our main assumption on $φ$ is the following weak lower scaling condition at zero: there exist $α_1 ∈ (0, 2]$ and $a_1 > 0$ such that
\[
a_1 \left(\frac{R}{r}\right)^{α_1} ≤ \frac{φ(R)}{φ(r)}, \quad 0 < r ≤ R ≤ 1. \tag{1.6}
\]
Since we allow $α_1$ to be 2, to guarantee that $J$ is to be a Lévy density, we also need the following integrability condition for $φ$ near zero:
\[
\int_0^1 \frac{s}{φ(s)} ds := C_0 < ∞. \tag{1.7}
\]
The monotonicity of $J(r)$ and (1.7) ensure the existence of an isotropic unimodal Lévy process in $\mathbb{R}^d$ with the Lévy measure $J(|x|)dx$, which is infinite because of (1.6) and the lower bound in (1.4).

Our goal is to obtain estimates of the heat kernel for $L^κ$. First we introduce the function $G(t, x)$ which plays an important role for the estimates of heat kernel. Let us define the function $Φ$ and $θ$ as
\[
Φ(r) := \begin{cases} 
\frac{r^2}{\int_0^r \frac{s}{φ(s)} ds}, & 0 < r ≤ 1 \\
Φ(1)r^2, & r > 1
\end{cases} \tag{1.8}
\]
and
\[
θ(r) := \begin{cases} 
\frac{1}{r^dΦ(r)}, & r ≤ 1, \\
\exp(−br^β)1_{\{0 < β < 1\}} + r^{−d−1}\exp(−\frac{b}{β}r)1_{\{β = 1\}}, & r > 1.
\end{cases}
\]
By (1.7), \( \int_0^r \frac{\kappa}{\phi(s)} ds \) is integrable so that \( \Phi \) is well-defined. Note that \( \Phi(1) = \left( 2 \int_0^1 \frac{s}{\phi(s)} ds \right)^{-1} = (2C_0)^{-1} \) is determined by \( C_0 \). Also, by Lemma 2.1 we will see that \( \Phi \) is a strictly increasing function in \( \mathbb{R}_+ \) and \( \lim_{r \to 0} \Phi(r) = 0 \), which imply that there exists an inverse function \( \Phi^{-1} : \mathbb{R}_+ \to \mathbb{R}_+ \). For \( t > 0 \) and \( r > 0 \) define \( \mathcal{G}(t, r) \) by

\[
\mathcal{G}(t, r) = \mathcal{G}(d)(t, r) := \frac{1}{t\Phi^{-1}(t)d} \wedge \theta(r)
\]

where \( a \wedge b := \min\{a, b\} \). By an abuse of notation we also define

\[
\mathcal{G}(t, x) = \mathcal{G}(d)(t, x) := \frac{1}{t\Phi^{-1}(t)d} \wedge \theta(|x|), \quad t > 0, \ x \in \mathbb{R}^d, \quad (1.9)
\]

so \( \mathcal{G}(t, x) = \mathcal{G}(t, |x|) \). Note that the definition of \( \theta(r) \) for \( \beta = 1 \) is simply technical and it is harmless for readers to regard \( \theta(r) \) as \( \frac{1}{r^\beta \Phi(r)} 1_{\{r \leq 1\}} + \exp\left( -\frac{b}{6} r \right) 1_{\{r > 1\}} \) as the upper bound of heat kernel for \( \beta = 1 \) in Theorems 1.1-1.3 below.

Let us compare \( \mathcal{G} \) with the following function defined by

\[
\mathcal{G}(t, x) = \mathcal{G}(t, |x|) := \frac{1}{t\Phi^{-1}(t)d} \wedge \frac{1}{|x|d\Phi(|x|)}. \quad (1.10)
\]

By [KSV18] Proposition 2.1] and our Lemma 3.3 below we see that \( \mathcal{G} \) is the function used for the upper heat kernel estimate in [KSV18] (see Remark 2.3 for details). It is easy to see that \( \mathcal{G}(t, x) \leq c \mathcal{G}(t, x) \) (see Lemma 2.2 below). Here is our main result.

**Theorem 1.1** Let \( L^\kappa \) be the operator in (1.1). Assume that jumping kernel \( J \) satisfies (1.4) and (1.5), that \( \phi \) satisfies (1.6) and (1.7), and that \( \kappa \) satisfies (1.2) and (1.3). Then, there exists a unique jointly continuous function \( p^\kappa(t, x, y) \) on \( \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d \) solving

\[
\partial_t p^\kappa(t, x, y) = L^\kappa p^\kappa(t, \cdot, y)(x), \quad x \neq y,
\]

and satisfying the following properties:

(i) *(Upper bound)* For every \( T \geq 1 \), there is a constant \( c_1 > 0 \) such that for all \( t \in (0, T] \) and \( x, y \in \mathbb{R}^d \),

\[
p^\kappa(t, x, y) \leq c_1 t \mathcal{G}(t, x - y). \quad (1.12)
\]

(ii) *(Fractional derivative)* For any \( x, y \in \mathbb{R}^d \) with \( x \neq y \), the map \( t \mapsto L^\kappa p^\kappa(t, \cdot, y)(x) \) is continuous, and for each \( T \geq 1 \), there exists a constant \( c_2 > 0 \) such that for all \( t \in (0, T] \), \( \varepsilon \in [0, 1] \) and \( x, y \in \mathbb{R}^d \),

\[
|L^{\kappa, \varepsilon} p^\kappa(t, \cdot, y)(x)| \leq c_2 \mathcal{G}(t, x - y). \quad (1.13)
\]

(iii) *(Continuity)* For any bounded and uniformly continuous function \( f : \mathbb{R}^d \to \mathbb{R} \),

\[
\lim_{t \downarrow 0} \sup_{x \in \mathbb{R}^d} \left| \int_{\mathbb{R}^d} p^\kappa(t, x, y) f(y) dy - f(x) \right| = 0. \quad (1.14)
\]


Furthermore, such unique function $p^\kappa(t, x, y)$ satisfies the following lower bound: for every $T \geq 1$, there exists a constant $c_3, c_4 > 0$ such that for all $t \in (0, T)$,

$$p^\kappa(t, x, y) \geq c_3 \begin{cases} \Phi^{-1}(t)^{-d}, & |x - y| \leq c_4 \Phi^{-1}(t) \\ tJ(|x - y|), & |x - y| > c_4 \Phi^{-1}(t) \end{cases}$$

(1.15)

The constants $c_i, i = 1, \ldots, 4$, depend only on $d, T, a, a_1, b, \beta, C_0, \delta, \kappa_0, \kappa_1$ and $\kappa_2$.

The upper bound of the fractional derivative of $p^\kappa$ in (1.13), which is a counterpart of [KSV18, (1.12)], will be used to prove the uniqueness of heat kernel.

We emphasize here that unlike [KSV18, (1.21)] we obtain (1.15) without any upper weak scaling condition on $\phi$. The estimates in (1.12) and (1.15) in Theorem 1.1 are not sharp in general. However, when the jumping kernel $J$ satisfies

$$J(r) \geq a_1 \exp(-b_1 r^{\beta_1}), \quad r > 1,$$

and $\phi$ satisfies upper weak scaling condition at zero, that is,

$$\frac{\phi(R)}{\phi(r)} \leq a_2 \left(\frac{R}{r}\right)^{\alpha_2}, \quad 0 < t \leq R \leq 1$$

(1.17)

where $a_2 > 0$ and $\alpha_2 \in (0, 2)$, then the lower bound in (1.15) is comparable to that in [GKK15, Theorem 1.2], which is lower heat kernel estimates for symmetric Hunt process with exponentially decaying jumping kernel. Note that $\phi$ is comparable to $\Phi$ under (1.6) and (1.17). Therefore, under additional assumptions (1.17) and (1.16) we have the following corollary.

**Corollary 1.2** Let $L^\kappa$ be the operator in (1.1). Assume that jumping kernel $J$ satisfies (1.4), (1.5) and (1.16), that $\phi$ satisfies (1.6) and (1.17), and that $\kappa$ satisfies (1.2) and (1.3). Then, the heat kernel $p^\kappa(t, x, y)$ for $L^\kappa$ satisfies the following estimates: for every $T \geq 1$, there is a constant $c_1 > 0$ such that for all $t \in (0, T]$ and $x, y \in \mathbb{R}^d$,

$$c_1^{-1} \left( \phi^{-1}(t)^{-d} \wedge \frac{t}{|x-y|^d \phi(|x-y|)} \right) \leq p^\kappa(t, x, y) \leq c_1 \left( \phi^{-1}(t)^{-d} \wedge \frac{t}{|x-y|^d \phi(|x-y|)} \right), \quad |x - y| \leq 1,$$

and

$$c_1^{-1} t \exp(-b_1 |x-y|^{\beta_1}) \leq p^\kappa(t, x, y) \leq c_1 t \theta(|x-y|), \quad |x - y| > 1.$$

The constant $c_1$ depend on $d, T, a, a_1, a_2, \alpha_1, \alpha_2, b, b_1, \beta, \beta_1, \beta_2, C_0, \delta, \kappa_0, \kappa_1$ and $\kappa_2$.

Comparing to [KSV18], Corollary 1.2 provides further precise heat kernel estimates for the operator (1.1) with exponential decaying function $J$. We remark here that, when $\beta > 1$, the estimates of $p^\kappa(t, x, y)$ are different and so the result in Corollary 1.2 does not hold even for symmetric Lévy processes. See [CKK11, Szt17]. We will address this interesting case somewhere else.

More properties of the heat kernel $p^\kappa(t, x, y)$ are listed in the following theorems.
Theorem 1.3 Suppose that the assumptions of Theorem 1.1 are satisfied.

(1) (Conservativeness) For all \((t, x) \in \mathbb{R}_+ \times \mathbb{R}^d\),
\[
\int_{\mathbb{R}^d} p^\kappa(t, x, y) \, dy = 1.
\] (1.18)

(2) (Chapman-Kolmogorov equation) For all \(s, t > 0\) and \(x, y \in \mathbb{R}^d\),
\[
\int_{\mathbb{R}^d} p^\kappa(t, x, z) p^\kappa(s, z, y) \, dz = p^\kappa(t + s, x, y).
\] (1.19)

(3) (Hölder continuity) For every \(T \geq 1\) and \(\gamma \in (0, \alpha_1) \cap (0, 1]\), there is a constant \(c_1 > 0\) such that for all \(0 < t \leq T\) and \(x, x', y \in \mathbb{R}^d\) with either \(x \neq y\) or \(x' \neq y\),
\[
|p^\kappa(t, x, y) - p^\kappa(t, x', y)| \leq c_1 |x - x'|^\gamma t \Phi^{-1}(t)^\gamma \left(\mathcal{G}(t, x - y) \vee \mathcal{G}(t, x' - y)\right).
\] (1.20)

The constant \(c_1\) depends only on \(d, T, a, a_1, \alpha_1, \beta, C_0, \gamma, \delta, \kappa_0, \kappa_1\) and \(\kappa_2\).

For \(t > 0\), define the operator \(P^\kappa_t\) by
\[
P^\kappa_t f(x) = \int_{\mathbb{R}^d} p^\kappa(t, x, y) f(y) \, dy, \quad x \in \mathbb{R}^d,
\] (1.21)

where \(f\) is a nonnegative (or bounded) Borel function on \(\mathbb{R}^d\), and let \(P^\kappa_0 = \text{Id}\). Then by Theorems 1.3, \((P^\kappa_t)_{t \geq 0}\) is a Feller semigroup with the strong Feller property. Let \(C^{2,\varepsilon}_b(\mathbb{R}^d)\) be the space of bounded twice differentiable functions in \(\mathbb{R}^d\) whose second derivatives are uniformly Hölder continuous.

Theorem 1.4 (1) (Generator) Let \(\varepsilon > 0\). For any \(f \in C^{2,\varepsilon}_b(\mathbb{R}^d)\), we have
\[
\lim_{t \downarrow 0} \frac{1}{t} (P^\kappa_t f(x) - f(x)) = \mathcal{L}^\kappa f(x),
\] (1.22)

and the convergence is uniform. (2) (Analyticity) The semigroup \((P^\kappa_t)_{t \geq 0}\) of \(\mathcal{L}^\kappa\) is analytic in \(L^p(\mathbb{R}^d)\) for every \(p \in [1, \infty)\).

In this paper, we defined the function \(\mathcal{G}(t, x)\) from the conditions on \(J\) directly, while in [KSV18] the function \(\rho(t, x)\) is defined by the characteristic exponent of an isotropic unimodal Lévy process with jumping kernel \(J(x)dx\). The reason is that, in our situation, it is more convenient than using characteristic exponent to describe exponential decaying jumping kernel. See Remark 2.3 below for the connections between two definitions.

As [KSV18], the approach in this paper is based on the method originally developed in [CZ16]. In Section 2, we introduce basic setup and scaling inequalities. In addition, we obtain some convolution inequalities at Proposition 2.8 in Section 2.2. The results in Section 2.2 is similar to [KSV18] Lemma 2.6], although our function \(\mathcal{G}(t, x)\) is smaller than that in [KSV18].
In Section 3, we discuss gradient estimates for the heat kernel of isotropic unimodal Lévy process with jumping kernel $J(|x|)dx$, which follows from the results in [KS15, KS17]. We only use Proposition 3.2 in the proof of our main theorem, but Proposition 3.1 itself is of independent interest.

In Section 4, we obtain some useful estimates on functions involving the heat kernel for the isotropic Lévy process whose jumping kernel is $J(|x|)dx$. In Section 4.1, we improve inequalities in Propositions 3.2 and 4.1–4.2 for the symmetric Lévy processes whose jumping kernel is $K(x)J(|x|)dx$, where $K(x)$ is symmetric and bounded between two positive constants. As [KSV18, Section 3], we also observe the continuous dependency of the heat kernel $p^R$ with respect to the jumping kernel $K(x)J(|x|)dx$.

In Section 5, we follow the Levi’s construction in [KSV18, Section 4]. Note that as in [KSV18, Section 4], many results in Section 5 are derived from the estimates in Sections 2 and 4 so that we can follow [KSV18] for the most of proofs. Finally we provide the proofs of Theorems 1.1, 1.3 and 1.4 in Section 6.

In this paper, we use the following notations. We will use “:=” to denote a definition, which is read as “is defined to be”. For any two positive functions $f$ and $g$, $f \asymp g$ means that there is a positive constant $c \geq 1$ such that $c^{-1}g \leq f \leq cg$ on their common domain of definition. Denote $\text{diam}(A) = \sup\{|x-y| : x, y \in A\}$ and $\sigma(dz) = \sigma_d(dz)$ be a uniform measure in the sphere $\{z \in \mathbb{R}^d : |z| = 1\}$. For a function $f : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$, we define $f(t, x \pm z) = f(t, x + z) + f(t, x - z)$ and

$$\delta_f(t, x; z) := f(t, x + z) + f(t, x - z) - 2f(t, x) = f(t, x \pm z) - 2f(t, x). \quad (1.23)$$

Throughout the rest of this paper, the positive constants $T, a, a_1, b, \beta, \delta, \kappa_0, \kappa_1, \kappa_2$ and $C_i, i = 0, 1, 2, \ldots,$ can be regarded as fixed. In the statements of results and the proofs, the constants $c_i = c_i(a, b, c, \ldots), i = 0, 1, 2, \ldots,$ denote generic constants depending on $a, b, c, \ldots,$ whose exact values are unimportant. They start anew in each statement and each proof.

2 Preliminaries

In this section we first study some elementary properties of $\Phi$ defined in (1.8).

**Lemma 2.1** Assume that $\phi$ satisfies (1.6) and (1.7). Then, $\Phi$ is continuous and strictly increasing in $(0, 1]$, and satisfies

$$\Phi(r) \leq \phi(r), \quad 0 < r \leq 1 \quad (2.1)$$

and

$$a_1 \left(\frac{R}{r}\right)^{\alpha_1} \leq \frac{\Phi(R)}{\Phi(r)} \leq \left(\frac{R}{r}\right)^2, \quad 0 < r \leq R, \quad (2.2)$$

where $a_1 > 0$ and $\alpha_1 \in (0, 2]$ are constants in (1.6). In particular, (2.1) implies $\lim_{r \downarrow 0} \Phi(r) = 0$. 

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Proof. Since \( \phi \) is continuous in \((0, 1]\), \( \Phi \) is continuous in \( \mathbb{R}_+ \) by definition. Also, since \( \phi \) is strictly increasing, we have

\[
\Phi(r) = \frac{r^2}{2 \int_0^r \frac{s}{\phi(s)} ds} \leq \frac{r^2}{2 \int_0^r \frac{s}{\phi(r)} ds} = \phi(r).
\]

To show that \( \Phi \) is strictly increasing, it suffices to observe that for \( 0 < r < 1 \),

\[
\left( \frac{1}{2\Phi(r)} \right)' = \left( r^2 \int_0^r \frac{s}{\phi(s)} ds \right)' = 2r^3 \int_0^r s \left( \frac{1}{\phi(r)} - \frac{1}{\phi(s)} \right) ds < 0.
\]

Now we prove (2.2). Clearly, by the definition of \( \Phi \), (2.2) holds for \( 1 \leq r < R \).

For \( 0 < r < 1 \), we have \( R^2/\Phi(R) = \int_0^R \left( s/\phi(s) \right) ds \geq \int_0^R \left( s/\phi(s) \right) ds = r^2/\Phi(r) \), which implies the second inequality in (2.2). Also, by change of variables and (1.6)

\[
\frac{\Phi(R)}{\Phi(r)} = \frac{2\Phi(R)}{r^2} \int_0^r \frac{s}{\phi(s)} ds = \frac{2\Phi(R)}{r^2} \int_0^R \frac{(r/R)t}{\phi((r/R)t)} (r/R) dt
\]

\[
= \frac{2\Phi(R)}{R^2} \int_0^R \frac{t}{\phi((r/R)t)} dt = a_1 \left( \frac{R}{r} \right)^{\alpha_1}.
\]

For \( R \geq 1 \geq r > 0 \), using \( \Phi(R) = \Phi(1)R^2 \) and above estimates we have

\[
a_1 \left( \frac{R}{r} \right)^{\alpha_1} \leq \frac{R^2}{r^{\alpha_1}} \leq \frac{\Phi(R)}{\Phi(r)} = \frac{\Phi(R) \Phi(1)}{\Phi(1) \Phi(r)} \leq \frac{R^2}{r^2}.
\]

Note that our main results hold for all \( t \leq T \), while the definition of \( \mathcal{G} \) in (1.9) is independent of \( T \). To make our proofs simpler, we introduce a family of auxiliary functions which will be used mostly in proofs.

Let \( T \geq \Phi(1) \) and define \( \mathcal{G}_T : (0, T] \times (0, \infty) \to (0, \infty) \) by

\[
\mathcal{G}_T(t, r) = \begin{cases} 
\frac{1}{t\Phi^{-1}(t)^d}, & r \leq \Phi^{-1}(t), \\
\frac{1}{r^d\Phi(r)}, & \Phi^{-1}(t) < r \leq \Phi^{-1}(T), \\
C_T \exp(-br^{\beta_1})1_{0<\beta<1} + \frac{C_T}{r^{d+1}} \exp(-\frac{b}{\beta} r) \mathbf{1}_{\beta=1}, & r > \Phi^{-1}(T)
\end{cases}
\]

where \( C_T := T^{-1}\Phi^{-1}(T)^{-d} \exp(b\Phi^{-1}(T)^{\beta_1})1_{0<\beta<1} + T^{-1}\Phi^{-1}(T) \exp(\frac{b}{4} \Phi^{-1}(T)) \mathbf{1}_{\beta=1} \). Note that \( r \mapsto \mathcal{G}_T(t, r) \) is continuous and non-increasing (due to such choice of \( C_T \)).

Recall that \( \mathcal{G}(t, r) \) is defined in (1.10). In the following lemma we show that \( \mathcal{G}_T \) and \( \mathcal{G}(t, x) \) are comparable and less than \( \mathcal{G}(t, r) \).

Lemma 2.2 (a) Let \( T \geq \Phi(1) \). Then, there exists a constant \( c_1 = c_1(T) > 0 \) such that

\[
c_1^{-1} \mathcal{G}_T(t, r) \leq \mathcal{G}(t, r) \leq c_1 \mathcal{G}_T(t, r)
\]
for any $t \in (0,T]$ and $r > 0$.

(b) There exists a constant $c_2 > 0$ such that

$$G(t,r) \leq c_2 G(t,r).$$

for any $t > 0$ and $r > 0$. The constant $c_1$ depends on $d, b, T, \Phi^{-1}(T), \beta$ and $C_0$, and $c_2$ depends on $d, b, \beta$ and $C_0$.

**Proof.** (a) Define

$$\theta_T(r) := \begin{cases} r^{-d} \Phi(r)^{-1}, & r \leq \Phi^{-1}(T), \\ C_T \exp(-b\beta) \mathbf{1}_{0<\beta<1} + C_T r^{-d-1} \exp(-\frac{b}{5}r) \mathbf{1}_{\beta=1}, & r > \Phi^{-1}(T). \end{cases}$$

Note that $r \mapsto \theta_T(r)$ is strictly decreasing and $\theta_T(\Phi^{-1}(t)) = \frac{1}{t \Phi^{-1}(t)^d}$ for any $0 < t \leq T$. Thus we can obtain

$$\theta_T(r) \leq \frac{1}{t \Phi^{-1}(t)^d} \quad \text{if and only if} \quad t \leq \Phi(r).$$

By (2.3) and (2.6) we have

$$G_T(t,r) = \frac{1}{t \Phi^{-1}(t)^d} \wedge \theta_T(r).$$

Let

$$M_T := \begin{cases} \sup_{1 \leq r \leq \Phi^{-1}(T)} \frac{1}{r \Phi(r)} \exp(b \beta) & \text{for} \quad 0 < \beta < 1, \\
\sup_{1 \leq r \leq \Phi^{-1}(T)} \frac{r}{\Phi(r)} \exp(b \beta) & \text{for} \quad \beta = 1 \end{cases}$$

and

$$m_T := \begin{cases} \inf_{1 \leq r \leq \Phi^{-1}(T)} \frac{1}{r \Phi(r)} \exp(b \beta) & \text{for} \quad 0 < \beta < 1, \\
\inf_{1 \leq r \leq \Phi^{-1}(T)} \frac{r}{\Phi(r)} \exp(b \beta) & \text{for} \quad \beta = 1 \end{cases}$$

Then, for $0 < \beta < 1$,

$$\theta(r) = \begin{cases} \frac{1}{r \Phi(r)} = \theta_T(r), & r \leq 1, \\
\exp(-br\beta) \geq M_T^{-1} \frac{1}{r \Phi(r)} = M_T^{-1} \theta_T(r), & 1 < r \leq \Phi^{-1}(T), \\
\exp(-br\beta) \leq m_T^{-1} \theta_T(r), & 1 < r \leq \Phi^{-1}(T), \\
\exp(-br\beta) = C_T^{-1} \theta_T(r), & r > \Phi^{-1}(T) \end{cases}$$

and for $\beta = 1$,

$$\theta(r) = \begin{cases} \frac{1}{r \Phi(r)} = \theta_T(r), & r \leq 1, \\
\frac{1}{r \Phi(r)} \exp(-\frac{b}{5}r) \geq M_T^{-1} \frac{1}{r \Phi(r)} = M_T^{-1} \theta_T(r), & 1 < r \leq \Phi^{-1}(T), \\
\exp(-br\beta) \leq m_T^{-1} \theta_T(r), & 1 < r \leq \Phi^{-1}(T), \\
\frac{1}{r \Phi(r)} \exp(-\frac{b}{5}r) = C_T^{-1} \theta_T(r), & r > \Phi^{-1}(T) \end{cases}$$

Thus, for any $0 < \beta \leq 1$ and $r > 0$,

$$(1 \wedge M_T^{-1} \wedge C_T^{-1}) \theta_T(r) \leq \theta(r) \leq (1 \vee m_T^{-1} \wedge C_T^{-1}) \theta_T(r).$$
Using this and (2.7) we arrive (2.4).

(b) Clearly we have \( G(t,r) = G(t,r) \) for \( r \leq 1 \). For any \( r > 1 \) and \( 0 < \beta < 1 \) we have

\[
\tilde{G}(t,r) = \frac{1}{r^d \Phi(r)} \geq \left( \sup_{s \geq 1} s^d \Phi(s) \exp(-bs^\beta) \right)^{-1} \exp(-br^\beta) = c(\beta)G(t,r).
\]

Similarly, for \( r > 1 \) and \( \beta = 1 \)

\[
\tilde{G}(t,r) = \frac{1}{r^{d+2} \Phi(1)} \geq \left( \sup_{s \geq 1} \frac{\Phi(s)}{s} \exp(-b/s) \right)^{-1} \frac{1}{r^{d+1}} \exp(-b/r) = c(1)G(t,r).
\]

Combining above estimates with (2.4) we arrive (2.5) with

\[ c_2 = c(\beta) \wedge c_1^{-1}. \]

\[ \blacksquare \]

In the following remark we will see that our \( \tilde{G}(t,x) \) and the function \( \rho(t,x) \) in [KSV18] are comparable.

**Remark 2.3** Let \( r(t,r) := \psi^{-1}(t^{-1}d \wedge [t\psi(r^{-1})r^{-d}]) \) as in [KSV18], where \( \psi \) is the characteristic exponent with respect to the \( \text{Lévy} \) process whose jumping kernel is \( J(|y|)dy \). By Lemma 3.3 below we have \( \psi(r^{-1})^{-1} \asymp \Phi(r) \) for all \( r > 0 \), which implies that \( r(t,r)/t \asymp \tilde{G}(t,r) \) for all \( r > 0 \). Thus, by [KSV18, Proposition 2.1] we conclude that \( \tilde{G}(t,x) \) is comparable to the function \( \rho(t,x) \) in [KSV18].

### 2.1 Basic scaling inequalities.

We start with weak scaling condition for the inverse \( ) \) of \( \Phi \). In this subsection we assume that \( \phi \) satisfies (1.6).

**Lemma 2.4** For any \( 0 < r \leq R \),

\[
\left( \frac{R}{r} \right)^{1/2} \leq \frac{\Phi^{-1}(R)}{\Phi^{-1}(r)} \leq a_1^{-1/\alpha_1} \left( \frac{R}{r} \right)^{1/\alpha_1}
\]

(2.8)

where \( a_1 \) and \( \alpha_1 \) are constants in (1.6).

**Proof.** Letting \( (r,R) = (\Phi^{-1}(r),\Phi^{-1}(R)) \) in (2.2), we have that for \( 0 < r \leq R \),

\[
a_1 \left( \frac{\Phi^{-1}(R)}{\Phi^{-1}(r)} \right)^{\alpha_1} \leq \frac{R}{r} = \frac{\Phi(\Phi^{-1}(R))}{\Phi(\Phi^{-1}(r))} \leq \left( \frac{\Phi^{-1}(R)}{\Phi^{-1}(r)} \right)^2,
\]

which implies (2.8). \[ \blacksquare \]

Now we introduce some scaling properties of \( \mathcal{G} \) which will be used throughout this paper.

**Lemma 2.5** Let \( T \geq 1 \) and \( 0 < \varepsilon \). Then, there exist constants \( c_1, c_2 > 0 \) such that for any \( 0 < t \leq T \), \( x \in \mathbb{R}^d \) and \( z \in \mathbb{R}^d \) satisfying \( \Phi(|z|) \leq t \),

\[
\mathcal{G}(\varepsilon t, x) \leq c_1 \mathcal{G}(t, x)
\]

(2.9)

and

\[
\mathcal{G}(t, x + z) \leq c_2 \mathcal{G}(t, x),
\]

(2.10)

where \( c_1 \) depends only on \( d, a_1, \alpha_1, \varepsilon \), and \( c_2 \) depends only on \( d, T, a_1, \alpha_1, b, \beta \) and \( C_0 \).
Proof. (a) Since $t \mapsto G(t, x)$ is non-increasing, we can assume $\varepsilon < 1$ without loss of generality. By (2.8), there is a constant $c_1(\varepsilon) > 1$ satisfying
\[
\frac{1}{\varepsilon t \Phi^{-1}(\varepsilon t)^d} \leq \frac{c_1}{t \Phi^{-1}(t)^d}
\]
for $t \leq T$. Thus, we arrive
\[
G(\varepsilon t, x) = \frac{1}{\varepsilon t \Phi^{-1}(\varepsilon t)^d} \wedge \theta(|x|) \leq \frac{c_1}{t \Phi^{-1}(t)^d} \wedge \theta(|x|) \leq c_1 G(t, x).
\]

(b) We claim that (2.10) holds with the ) $G_T(t, x)$. In other words, there exists a constant $c_2 > 0$ such that
\[
G_T(t, x + z) \leq c_2 G_T(t, x), \quad x \in \mathbb{R}^d, \; \Phi(|z|) \leq t.
\]
Since $r \mapsto G_T(t, r)$ is non-increasing, it suffices to show that there exists $c_3 > 0$ such that for any $0 < t \leq T$ and $r > 0$,
\[
G_T(t, r) \leq c_3 \exp(b \Phi^{-1}(t)^{\beta}) G_T(t, r + \Phi^{-1}(t)). \tag{2.11}
\]
Indeed, since $0 < t \leq T$, (2.11) implies our claim with $c_2 = c_3 \exp(b \Phi^{-1}(T)^{\beta})$.

We prove (2.11) by considering several cases separately. Firstly when $r \leq \Phi^{-1}(T) - \Phi^{-1}(t)$, using (2.3) we have
\[
G_T(t, r + \Phi^{-1}(t)) = \frac{1}{(r + \Phi^{-1}(t))^d \Phi(r + \Phi^{-1}(t))} \geq \frac{1}{(2 \Phi^{-1}(t))^d \Phi(2 \Phi^{-1}(t))} \wedge \frac{1}{2^d \Phi(2r)} \geq c_4 \left( \frac{1}{t \Phi^{-1}(t)^d} \wedge \frac{1}{r^d \Phi(r)} \right) = c_4 G_T(t, r),
\]
The second line above follows from (2.2).

When $r \geq \Phi^{-1}(T)$ and $0 < \beta < 1$, using (2.3) and triangular inequality $r^\beta + \Phi^{-1}(t)^\beta \geq (r + \Phi^{-1}(t))^{\beta}$ we get
\[
G_T(t, r + \Phi^{-1}(t)) = \exp(-b(r + \Phi^{-1}(t))^\beta) \geq \exp(-b \Phi^{-1}(t)^\beta - b r^\beta) = \exp(-b \Phi^{-1}(t)^\beta) G_T(t, r).
\]
Similarly, for $r \geq \Phi^{-1}(T)$ and $\beta = 1$ we have
\[
G_T(t, r + \Phi^{-1}(t)) = C_T \frac{1}{(r + \Phi^{-1}(t))^{d+1}} \exp(-\frac{b}{5}(r + \Phi^{-1}(t))) \geq C_T \frac{1}{(2r)^{d+1}} \exp(-\frac{b}{5} \Phi^{-1}(t) - \frac{b}{5} r) 
\]
\[
= 2^{-d-1} \exp\left(-\frac{b}{5} \Phi^{-1}(t)\right) G_T(t, r) \geq 2^{-d-1} \exp(-b \Phi^{-1}(t)) G_T(t, r).
\]
When $\Phi^{-1}(T) - \Phi^{-1}(t) \leq r \leq \Phi^{-1}(T)$, combining above estimates we arrive
\[
G_T(t, r + \Phi^{-1}(t)) \geq 2^{-d-1} \exp(-b \Phi^{-1}(t)^\beta) G_T(t, \Phi^{-1}(T)) \geq c_5 \exp(-b \Phi^{-1}(t)^\beta) G_T(t, r).
\]
Note that $r \mapsto G_T(t, r)$ is continuous at $r = \Phi^{-1}(T)$. Therefore, we conclude (2.11). Applying (2.4) for (2.11) we arrive our desired estimate (2.10). \hfill \Box
2.2 Convolution inequalities

In this section, we obtain some convolution inequalities for \( \mathcal{G}(t,x) \) which will be used for Levi’s method in Section 5. To get these inequalities we will use some estimates in \( \text{KSV}18 \) Section 2. Note that by Remark 2.3 we already have convolution inequalities for Levi’s method in Section 5. For \( a,b > 1 \) Lemma 2.6 we skip the proof.

Lemma 2.6 Assume that \( \phi \) satisfies (1.6) and \( \gamma, \delta \geq 0, \eta, \theta \in \mathbb{R} \) are constants satisfying \( 1_{\gamma \geq 0}(\gamma/2) + 1_{\gamma < 0}(\gamma/\alpha) + \delta/2 + 1 - \eta > 0. \) Then for every \( t > 0, \) we have

\[
\int_0^t s^{-\eta} \Phi^{-1}(s)^{\alpha \Phi^{-1}(t)} ds \leq B(\delta/2 + 1 - \theta, \gamma/2 + 1 - \eta) t^{1-\eta-\delta} \Phi^{-1}(t)^{\gamma+\delta} . \tag{2.12}
\]

For \( 0 \leq s \leq t, \) let \( g(s) := t^\beta + (2^\beta - 1)s^\beta - (t+s)^\beta. \) Then we can easily check that \( g(0) = g(t) = 0 \) and

\[
g'(s) = \beta ((2^\beta - 1)s^{\beta-1} - (t+s)^{\beta-1}) \begin{cases} \geq 0, & s \in [0,kt], \\ < 0, & s \in [kt,t], \end{cases}
\]

where \( k := ((2^\beta - 1)^{\beta-1} - 1)^{-1} \in (0,1) \) is the constant satisfying \( g'(kt) = 0. \) Thus, we conclude that \( g(s) \geq 0 \) for any \( 0 \leq s \leq t, \) which implies

\[
t^\beta + s^\beta \geq (t+s)^\beta + (2 - 2^\beta)(t^\beta \wedge s^\beta) , \quad \text{for all } 0 < \beta < 1 \text{ and } t,s > 0 . \tag{2.13}
\]

Using (2.13) we prove the following lemma, which we need for our convolution inequalities.

Lemma 2.7 (a) Let \( 0 < \beta < 1 \text{ and } b > 0. \) Then, there exists a constant \( c_1 > 0 \) such that for any \( x \in \mathbb{R}^d, \)

\[
\int_{\mathbb{R}^d} \exp(-b|x-z|^\beta - b|z|^\beta)dz \leq c_1 \exp(-b|x|^{\beta}). \tag{2.14}
\]

(b) There exists a constant \( c_2 > 0 \) such that for any \( x \in \mathbb{R}^d \) with \( |x| \geq 1, \)

\[
\int_{\mathbb{R}^d} (|x-z|^{d-1} \wedge 1)(|z|^{d-1} \wedge 1)dz \leq c_2 |x|^{-d-1} . \tag{2.15}
\]

The constant \( c_1 \) depends only on \( b, d \text{ and } \beta, \text{ and } c_2 \) depends only on \( d. \)

Proof. (a) Let

\[
c_1 = 2 \int_{\mathbb{R}^d} \exp(-b(2-2^\beta)|z|^\beta)dz < \infty.
\]

Using (2.13) for the second line, we arrive

\[
\int_{\mathbb{R}^d} \exp(-b|x-z|^\beta - b|z|^\beta)dz \leq \int_{\mathbb{R}^d} \exp(-b|x|^{\beta}) \exp(-b(2-2^\beta)(|z|^\beta \wedge |x-z|^\beta))dz
\]

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\[
\leq \exp(-b|x|^\beta) \left( \int_{\mathbb{R}^d} \exp(-b(2-2\beta)|z|\beta)dz + \int_{\mathbb{R}^d} \exp(-b(2-2\beta)|x-z|^\beta)dz \right)
= c_1 \exp(-b|x|^\beta).
\]

This proves (2.14).
(b) Using \(|x-z|^{-1} \wedge |z|^{-1} \leq 2|x|^{-1}\), we have
\[
\int_{\mathbb{R}^d} (|x-z|^{-d-1} \wedge 1)(|z|^{-d-1} \wedge 1)dz
\leq \left( \frac{2}{|x|} \right)^{d+1} \left( \int_{|x-z|\geq |z|} (|z|^{-d-1} \wedge 1)dz + \int_{|x-z|<|z|} (|x-z|^{-d-1} \wedge 1)dz \right)
\leq \left( \frac{2}{|x|} \right)^{d+1} \left( \int_{\mathbb{R}^d} (|z|^{-d-1} \wedge 1)dz + \int_{\mathbb{R}^d} (|x-z|^{-d-1} \wedge 1)dz \right) := c_2|x|^{-d-1}.
\]
This concludes the lemma. \(\square\)

For \(\gamma, \delta \in \mathbb{R}, t > 0\) and \(x \in \mathbb{R}^d\) we define
\[
\mathcal{G}_\gamma^\delta(t, x) := \Phi^{-1}(t)^\gamma(|x|^{\delta} \wedge 1) \mathcal{G}(t, x) \quad \text{and} \quad \mathcal{G}_\gamma(t, x) := \Phi^{-1}(t)^\gamma(|x|^{\delta} \wedge 1) \mathcal{G}(t, x).
\]
Note that \(\mathcal{G}_0^0(t, x) = \mathcal{G}(t, x)\), and \(\mathcal{G}_\gamma^\delta(t, x)\) is comparable to the function \(\rho_\gamma^\delta(t, x)\) in [KSV18] by Remark 2.3. Also, we can easily check that for \(T \geq \Phi(1)\),
\[
\mathcal{G}_\gamma^\delta(t, x) \leq \Phi^{-1}(T)^{\gamma_1-\gamma_2}\mathcal{G}_\gamma^\delta(t, x), \quad (t, x) \in (0, T] \times \mathbb{R}^d, \quad \gamma_2 \leq \gamma_1, \quad (2.16)
\]
\[
\mathcal{G}_\gamma^\delta(t, x) \leq \mathcal{G}_\gamma^\delta(t, x), \quad (t, x) \in (0, \infty) \times \mathbb{R}^d, \quad 0 \leq \delta_2 \leq \delta_1. \quad (2.17)
\]
We record the following inequality which immediately follows from (2.16) and (2.17): for any \(T \geq \Phi(1), \delta \geq 0\) and \((t, x) \in (0, T] \times \mathbb{R}^d\),
\[
(\mathcal{G}_0^\delta + \mathcal{G}_0^0)(t, x) \leq (\Phi^{-1}(T)^{\delta} + 1) \mathcal{G}(t, x) \leq 2\Phi^{-1}(T)^{\delta} \mathcal{G}(t, x). \quad (2.18)
\]

Now we are ready to introduce convolution inequalities for \(\mathcal{G}(t, x)\).

**Proposition 2.8** Assume that \(\phi\) satisfies (1.3). Let \(T \geq 1\) and \(0 < \alpha < \alpha_1\).
(a) There exists a constant \(c = c(d, T, \alpha_1, \alpha, \alpha_1) > 0\) such that for any \(0 < t \leq T, \delta \in [0, \alpha]\) and \(\gamma \in \mathbb{R}\),
\[
\int_{\mathbb{R}^d} \mathcal{G}_\gamma^\delta(t, x)dx \leq ct^{-1}\Phi^{-1}(t)^{\gamma+\delta}. \quad (2.19)
\]
(b) There exists \(C = C(\alpha, T) = C(d, T, \alpha_1, \alpha, \alpha_1, b, \beta) > 0\) such that for all \(x \in \mathbb{R}^d, \delta_1, \delta_2 \geq 0\) with \(\delta_1 + \delta_2 \leq \alpha, \gamma_1, \gamma_2 \in \mathbb{R}\) and \(0 < s < t \leq T\),
\[
\int_{\mathbb{R}^d} \mathcal{G}_\gamma^\delta(t-s, x-z)\mathcal{G}_\gamma^\delta(s, z)dz \leq C \left( (t-s)^{-1}\Phi^{-1}(t-s)^{\gamma_1+\delta_1+\delta_2} \Phi^{-1}(s)^{\gamma_2} \mathcal{G}(t, x) + \Phi^{-1}(t-s)^{\gamma_1} s^{-1} \Phi^{-1}(s)^{\gamma_2+\delta_1+\delta_2} \mathcal{G}(t, x) \right).
\]
\[ + (t-s)^{-1}\Phi^{-1}(t-s)^{\gamma_1+\delta_1}\Phi^{-1}(s)^{\gamma_2}\mathcal{G}_0^{\delta_2}(t,x) \]
\[ + \Phi^{-1}(t-s)^{\gamma_1}s^{-1}\Phi^{-1}(s)^{\gamma_2+\delta_2}\mathcal{G}_0^{\delta_1}(t,x). \]  
(2.20)

In particular, letting \( \gamma_1 = \gamma_2 = \delta_1 = \delta_2 = 0 \) in (2.20) we have
\[ \int_{\mathbb{R}^d} \mathcal{G}(t-s,x-z)\mathcal{G}(s,z)dz \leq 2C(s^{-1} + (t-s)^{-1})\mathcal{G}(t,x). \]  
(2.21)

(c) For all \( x \in \mathbb{R}^d \), \( 0 < t \leq T \), \( \delta_1, \delta_2 \geq 0 \) and \( \theta, \eta \in [0, 1] \) satisfying \( \delta_1 + \delta_2 \leq \alpha \), \( 1_{\gamma_1 \geq 0}(\gamma_1/2) + 1_{\gamma_1 < 0}(\gamma_1/\alpha_1) + \delta_1/2 + 1 - \theta > 0 \) and \( 1_{\gamma_2 \geq 0}(\gamma_2/2) + 1_{\gamma_2 < 0}(\gamma_2/\alpha_1) + \delta_2/2 + 1 - \eta > 0 \), we have a constant \( C_2 > 0 \) satisfying
\[ \int_0^t \int_{\mathbb{R}^d} (t-s)^{1-\theta}G_{\gamma_1}^\delta(t-s,x-z)s^{1-\eta}G_{\gamma_2}^\delta(s,z)dzds \leq C_2^\theta \int_{\mathbb{R}^d} (t-s)^{1-\theta}G_{\gamma_1}^\delta(t-s,x-z)s^{1-\eta}G_{\gamma_2}^\delta(s,z)dzds \]  
(2.22)

for any \( 0 < t \leq T \) and \( x \in \mathbb{R}^d \). Moreover, when \( \gamma_1, \gamma_2 \geq 0 \) we further have
\[ C_2 = 4CB\left(\frac{\gamma_1 + \delta_1}{2} + 1 - \theta, \frac{\gamma_2 + \delta_2}{2} + 1 - \eta\right). \]  
(2.23)

**Proof.** (a) See [KSV18, Lemma 2.6(a)].

(b) By (2.4), it suffices to show (2.20) with the function \( (\mathcal{G}_T)^\delta(t, x) := \Phi^{-1}(t)^\gamma(|x|^\delta - 1)\mathcal{G}(t, x) \). Without loss of generality we assume \( T \geq \Phi(1) \) and for notational convenience we drop \( T \) in the notations so we use \( \mathcal{G}(t, x) \) and \( \mathcal{G}_T^\delta(t, x) \) instead of \( \mathcal{G}_T(t, x) \) and \( (\mathcal{G}_T)^\delta(t, x) \) respectively.

First let \( |x| \leq \Phi^{-1}(T) \). By Remark 2.3 and [KSV18, Lemma 2.6(b)], we already have that there exists \( c_1 > 0 \) satisfying
\[ \int_{\mathbb{R}^d} G_{\gamma_1}^\delta(t-s,x-z)G_{\gamma_2}^\delta(s,z)dz \leq c_1 \left( (t-s)^{1-\theta}\Phi^{-1}(t-s)^{\gamma_1+\delta_1+\delta_2}\Phi^{-1}(s)^{\gamma_2}\mathcal{G}(t,x) \right. \]
\[ + \left. \Phi^{-1}(t-s)^{\gamma_1}s^{-1}\Phi^{-1}(s)^{\gamma_2+\delta_2}\mathcal{G}(t,x) \right) \]
\[ + (t-s)^{-1}\Phi^{-1}(t-s)^{\gamma_1+\delta_1}\Phi^{-1}(s)^{\gamma_2}\mathcal{G}_0^{\delta_2}(t,x) \]
\[ + (t-s)^{-1}\Phi^{-1}(t-s)^{\gamma_1+\delta_1}\Phi^{-1}(s)^{\gamma_2+\delta_2}\mathcal{G}_0^{\delta_1}(t,x). \]

Note that \( \mathcal{G}(t, x) = \tilde{\mathcal{G}}(t, x) \) by (2.3) since \( |x| \leq \Phi^{-1}(T) \). Using (2.3) for the left-hand side and \( \mathcal{G}(t, x) = \tilde{\mathcal{G}}(t, x) \) for the right-hand side, we obtain (2.20) for \( |x| \leq \Phi^{-1}(T) \).

Now assume \( |x| > \Phi^{-1}(T) \) and observe that
\[ \int_{\mathbb{R}^d} G_{\gamma_1}^\delta(t-s,x-z)G_{\gamma_2}^\delta(s,z)dz \]
\[ = \left( \int_{|z| > \Phi^{-1}(T)} + \int_{|z| = \Phi^{-1}(T)} + \int_{|z| \leq \Phi^{-1}(T), |z-x| > \Phi^{-1}(T)} + \int_{|z| \leq \Phi^{-1}(T), |z-x| \leq \Phi^{-1}(T)} \right) G_{\gamma_1}^\delta(t-s,x-z)G_{\gamma_2}^\delta(s,z)dz \]

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\[ := I_1 + I_2 + I_3 + I_4. \]

First we assume \( 0 < \beta < 1 \) and obtain upper bounds for \( I_i, i = 1, \ldots, 4 \). For \( I_1 \), using \( \Phi^{-1}(T) \geq 1 \) we have

\[
I_1 = \int_{|x-z|>\Phi^{-1}(T),|z|>\Phi^{-1}(T)} \mathcal{G}_t^\delta_1 (t-s, x-z) \mathcal{G}_s^\varepsilon_2 (s,z) \, dz
\]

\[
= \int_{|x-z|>\Phi^{-1}(T),|z|>\Phi^{-1}(T)} \Phi^{-1}(t-s)^\gamma_1 (|x-z|_1^\delta_1 \wedge 1) \mathcal{G}(t-s, x-z) \Phi^{-1}(s)^\gamma_2 (|z|_2^\delta_2 \wedge 1) \mathcal{G}(s,z) \, dz
\]

\[
= \Phi^{-1}(t-s)^{\gamma_1} \int_{|x-z|>\Phi^{-1}(T),|z|>\Phi^{-1}(T)} \exp(-b|x-z|^{\beta} - b|z|^{\beta}) \, dz. \tag{2.24}
\]

By (2.14) we obtain

\[
I_1 \leq c_1 \Phi^{-1}(t-s)^{\gamma_1} \Phi^{-1}(s)^{\gamma_2} \exp(-b|x|^{\beta}) = c_1 \Phi^{-1}(t-s)^{\gamma_1} \Phi^{-1}(s)^{\gamma_2} \mathcal{G}(t,x)
\]

\[
\leq c_2 (t-s)^{-1} \Phi^{-1}(t-s)^{\gamma_1 + \delta_1 + \delta_2} \Phi^{-1}(s)^{\gamma_2} \mathcal{G}(t,x),
\]

where we used \( \delta_1, \delta_2 \geq 0 \) and \( t-s \leq T \) for the last line. For the estimates of \( I_2, I_3 \) and \( I_4 \) we omit counterpart of the last line above.

For \( I_2 \), using (2.3) we have

\[
I_2 = \int_{|x-z|\leq\Phi^{-1}(T),|z|>\Phi^{-1}(T)} \mathcal{G}_t^\delta_1 (t-s, x-z) \mathcal{G}_s^\varepsilon_2 (s, z) \, dz
\]

\[
= \Phi^{-1}(t-s)^{\gamma_1} \Phi^{-1}(s)^{\gamma_2} \int_{|x-z|\leq\Phi^{-1}(T),|z|>\Phi^{-1}(T)} \mathcal{G}_0^\delta_1 (t-s, x-z) \exp(-b|z|^{\beta}) \, dz.
\]

Since \( |x-z| \leq \Phi^{-1}(T) \), using triangular inequality we have

\[
\exp(-b|z|^{\beta}) \leq \exp(-b|x|^{\beta}) \exp(b|x-z|^{\beta}) \leq \exp(b\Phi^{-1}(T)^{\beta}) \exp(-b|x|^{\beta}). \tag{2.25}
\]

Thus by (2.19),

\[
I_2 \leq \Phi^{-1}(t-s)^{\gamma_1} \Phi^{-1}(s)^{\gamma_2} \int_{\mathbb{R}^d} \mathcal{G}_0^\delta_1 (t-s, x-z) \, dz
\]

\[
\leq c_3 (t-s)^{-1} \Phi^{-1}(t-s)^{\gamma_1 + \delta_1} \Phi^{-1}(s)^{\gamma_2} \mathcal{G}(t,x).
\]

By the similar way, we obtain

\[
I_3 \leq c_3 s^{-1} \Phi^{-1}(t-s)^{\gamma_1} \Phi^{-1}(s)^{\gamma_2 + \delta_2} \mathcal{G}(t,x).
\]

When \( |x| \geq 2\Phi^{-1}(T) \), we have \( I_4 = 0 \). So we can assume \( |x| < 2\Phi^{-1}(T) \) without loss of generality for the estimate of \( I_4 \). By (2.25) We have

\[
I_4 \leq \int_{\mathbb{R}^d} \mathcal{G}_t^\delta_1 (t-s, x-z) \mathcal{G}_s^\varepsilon_2 (s, z) \, dz \leq c_4 \mathcal{G}(t,x).
\]
Using $\mathcal{G}(t, x) \leq \mathcal{G}(t, \Phi^{-1}(T)) = \mathcal{G}(t, \Phi^{-1}(T)) \leq \exp(b\Phi^{-1}(T)^\beta)\mathcal{G}(t, 2\Phi^{-1}(T)) \leq \exp(b\Phi^{-1}(T)^\beta)\mathcal{G}(t, x)$, we can obtain desired estimates. Combining estimates for $I_1, I_2, I_3$ and $I_4$, we arrive \((2.20)\) for $0 < \beta < 1$.

For the case $\beta = 1$, estimate for $I_4$ is same as above. For $I_2$ and $I_3$, instead of \((2.25)\) we argue as the following: using $|x - z| \leq \Phi^{-1}(T)$ and $|x|, |z| \geq \Phi^{-1}(T)$, we have

$$
\frac{1}{|z|^{d+1}} \exp(-\frac{b}{5}|z|) \leq \frac{2^{d+1}}{|x|^{d+1}} \exp(\frac{b}{5}\Phi^{-1}(T)) \exp(-\frac{b}{5}|x|).
$$

For $I_1$, following \((2.24)\) and using \((2.14)\) for the fourth line and \((2.8)\) for the fifth line we have

$$
I_1 = \Phi^{-1}(t - s)^{\gamma_1} \Phi^{-1}(s)^{\gamma_2} \int_{|x - z| > \Phi^{-1}(T), |z| > \Phi^{-1}(T)} \frac{1}{|x - z|^{d+1}|z|^{d+1}} \exp(-\frac{b}{5}|x - z| - \frac{b}{5}|z|)dz
$$

$$
\leq c_1 \Phi^{-1}(t - s)^{\gamma_1} \Phi^{-1}(s)^{\gamma_2} \exp(-\frac{b}{5}|x|) \int_{|x - z| > 1, |z| > 1} \frac{1}{|x - z|^{d+1}|z|^{d+1}} dz
$$

$$
\leq c_1 \Phi^{-1}(t - s)^{\gamma_1} \Phi^{-1}(s)^{\gamma_2} \exp(-\frac{b}{5}|x|) \int_{\mathbb{R}^d} (1 \wedge |x - z|^{d-1})(1 \wedge |z|^{d-1})dz
$$

$$
\leq c_2 \Phi^{-1}(t - s)^{\gamma_1} \Phi^{-1}(s)^{\gamma_2} \frac{1}{|x|^{d+1}} \exp(-\frac{b}{5}|x|) = c_2 \Phi^{-1}(t - s)^{\gamma_1} \Phi^{-1}(s)^{\gamma_2} \mathcal{G}(t, x)
$$

$$
\leq c_3 (t - s)^{-\gamma_1} \Phi^{-1}(t - s)^{\gamma_1+\delta_1+\delta_2} \Phi^{-1}(s)^{\gamma_2} \mathcal{G}(t, x).
$$

(c) Integrating \((2.20)\) with respect to $s$ from 0 to $t$. With \((2.12)\), we can follow the proof of \([KSV18, Lemma 2.6(c)]\).

\[\square\]

3 Heat kernel estimates for Lévy processes

Following the framework of \([CZ16, KSV18]\), we need estimates of derivatives of the heat kernel for the symmetric Lévy process whose jumping kernel is $J(|y|)$ (see, for example, \([KSV18, Proposition 3.2]\)). To be more precise, in our case, to get the upper bound of heat kernel for non-symmetric operator of the form \((1.1)\), we need correct upper bounds of the first and second order derivatives of the heat kernel for unimodal Lévy processes. In this section, we will prove that \((1.4)\) and \((1.5)\) are sufficient condition for the estimates of the second order derivatives in Proposition \(3.2\) which decay exponentially or subexponentially.

3.1 Settings

In this section, we fix $T \leq [1, \infty)$ and let $\nu(dy) = \nu(|y|)dy$ be an isotropic measure in $\mathbb{R}^d$ satisfying $\int_{\mathbb{R}^d} (1 \wedge |y|^2) \nu(dy) < \infty$. Throughout this section we further assume that $\nu: \mathbb{R}_+ \to \mathbb{R}_+$ is non-increasing, differentiable function.

Here are our goals in this section.
Proposition 3.1 Let $X$ be an isotropic unimodal Lévy process in $\mathbb{R}^d$ with Lévy measure $\nu(|y|)dy$ satisfying the following assumptions: $\phi$ is a nondecreasing function with $\phi(0) = 0$ satisfying (1.6) and (1.7), and there exist constants $a > 0$ and $0 < \beta \leq 1$ such that

$$
\frac{a^{-1}}{r^d \phi(r)} \leq \nu(r) \leq \frac{a}{r^d \phi(r)}, \quad 0 < r \leq 1 \quad \text{and} \quad \nu(r) \leq a \exp(-br^\beta), \quad r > 1. \tag{3.1}
$$

Then its transition density $x \mapsto p_t(x)$ is in $C_0^\infty(\mathbb{R}^d)$ and satisfies gradient estimates

$$
|\nabla^k x p_t(x)| \leq ct \Phi^{-1} \left( \frac{1}{t \Phi^{-1}(t)^d} \wedge \theta(|x|) \right), \quad k = 0, 1 \tag{3.2}
$$

for any $0 < t \leq T$ and $x \in \mathbb{R}^d$. The constant $c$ depends only on $k, d, T, a, a_1, \alpha_1, b, \beta$ and $C_0$.

With the above result, we can obtain the second gradient estimate for the isotropic unimodal Lévy process whose jumping kernel satisfies (1.4) and (1.5).

Proposition 3.2 Suppose that $\phi$ is a nondecreasing function with $\phi(0) = 0$ satisfying (1.6) and (1.7), and that Lévy measure $J(|y|)dy$ satisfies (1.4) and (1.5) with $0 < \beta \leq 1$. Then, its corresponding transition density $x \mapsto p(t, x)$ is in $C_0^\infty(\mathbb{R}^d)$ and satisfies gradient estimates

$$
|\nabla^k p(t, x)| \leq ct \Phi^{-1} \left( \frac{1}{t \Phi^{-1}(t)^d} \wedge \theta(|x|) \right), \quad k = 0, 1, 2 \tag{3.3}
$$

for any $0 < t \leq T$ and $x \in \mathbb{R}^d$. The constant $c$ depends only on $k, d, T, a, a_1, \alpha_1, b, \beta$ and $C_0$.

In the next subsection, we prove Propositions 3.1 and 3.2.

3.2 Proof of Propositions 3.1 and 3.2.

In this subsection, we will combine some results in [KR16, KS15, KS17] to prove Proposition 3.1. Recall that we have assumed that $\nu : \mathbb{R}_+ \to \mathbb{R}_+$ is non-increasing differentiable function satisfying $\int_{\mathbb{R}^d} (1 \wedge |y|^2) \nu(|y|)dy < \infty$. In this subsection, instead of the function $\Phi$, we mainly use

$$
\phi(r) := \begin{cases} 
\frac{r^2}{\int_0^r s^{d+1} \nu(s)ds}, & 0 < r \leq 1, \\
\phi(1)r^2, & r > 1,
\end{cases} \tag{3.4}
$$

Note that the integral $\int_0^r s^{d+1} \nu(s)ds$ above is finite because of our assumption $\int_{\mathbb{R}^d} (1 \wedge |y|^2) \nu(|y|)dy < \infty$.

To prove Propositions 3.1 and 3.2 at once, we need to consider the following conditions on Lévy measure $\nu(|y|)dy$ which is slightly more general than (3.1). We assume that there exist constants $a > 0, 0 < \beta \leq 1$ and $\ell \geq 0$ such that

$$
\nu(r) \leq ar^{-\ell} \exp(-br^\beta), \quad r > 1. \tag{3.5}
$$
Thus, letting 

\[ a_3 \left( \frac{R}{r} \right)^{a_3} \leq \frac{\varphi(R)}{\varphi(r)}, \quad 0 < r \leq R < \infty. \]  

(3.6)

For instance, when \( X \) is an isotropic Lévy process in Proposition 3.1 we have \( \frac{s}{\alpha_0(s)} \leq \nu(s)s^{d+1} \leq \frac{\alpha}{\phi(s)} \), which implies \( \varphi(r) \asymp \Phi(r) \). Using this and Lemma 2.1 we obtain (3.6) with \( \alpha_3 = \alpha_1 \). Thus, the conditions in Proposition 3.1 imply (3.5) and (3.6).

Under (3.6), we have \( \varphi(r) \leq cr^{\alpha_3} \) for \( r \leq 1 \) so that

\[
c^{-1}r^{-\alpha_3} \leq \int_0^r \frac{s^{d+1}}{r^2} \nu(s)ds \leq \int_0^r s^{d-1} \nu(s)ds \leq \int_0^1 s^{d-1} \nu(s)ds, \quad r \leq 1.
\]

Thus, letting \( r \downarrow 0 \) we obtain \( \int_0^1 s^{d-1} \nu(s)ds = \infty \). Now we record the counterpart of Lemma 2.1. Following the proof of Lemma 2.1 we obtain

\[
\frac{\varphi(R)}{\varphi(r)} \leq \left( \frac{R}{r} \right)^2, \quad 0 < r \leq R.
\]

(3.7)

In addition, since \( \nu \) is non-increasing, we have

\[
\varphi(r)^{-1} = r^{-2} \int_0^r s^{d+1} \nu(s)ds \geq r^{-2} \int_0^r s^{d+1} \nu(r)dr = \frac{r^d \nu(r)}{d+2}, \quad r < 1.
\]

(3.8)

In this subsection except the proofs of Propositions 3.1 and 3.2 we will always assume that \( \nu \) satisfies (3.5) and (3.6). Let \( X \) be the Lévy process with Lévy measure \( \nu(|y|)dy \), and \( \xi \mapsto \psi(|\xi|) \) be the characteristic exponent of \( X \). First note that \( \nu(\mathbb{R}^d) = \int_{\mathbb{R}^d} \nu(|y|)dy = \infty \) because \( \int_0^1 s^{d-1} \nu(s)ds = \infty \). Also, since \( X \) is isotropic, characteristic exponent of \( X \) is also isotropic function. Define \( \Psi(r) := \sup_{|y| \leq r} \psi(|y|) \) and let \( \mathcal{P}(r) := \int_{\mathbb{R}^d} (1 \wedge \frac{|y|^2}{r^2}) \nu(|y|)dy \) be the Prüf function for \( X \). By [BGR14, Lemma 1 and Proposition 2], we have that for \( r > 0 \),

\[
\frac{2}{\pi^2 d} \mathcal{P}(r^{-1}) \leq \psi(r) \leq \Psi(r) \leq \pi^2 \psi(r) \leq 2\pi^2 \mathcal{P}(r^{-1}), \quad r > 0.
\]

(3.9)

Using (3.9), we can prove the following lemma.

**Lemma 3.3** Assume that \( \nu(|y|)dy \) satisfies (3.5) and (3.6). Then, \( \Psi(r) \) is comparable to \( \varphi(r)^{-1} \), i.e., there exists a constant \( c > 0 \) such that

\[
c^{-1} \varphi(r)^{-1} \leq \Psi(r) \leq c \varphi(r)^{-1}, \quad r > 0.
\]

(3.10)

**Proof.** We claim that

\[
\mathcal{P}(r) \asymp \varphi(r)^{-1} \quad \text{for} \quad r > 0.
\]

(3.11)

First assume \( r \leq 1 \) and observe that

\[
\mathcal{P}(r) = \int_{\mathbb{R}^d} (1 \wedge \frac{|z|^2}{r^2}) \nu(z)dz
\]
implies (3.11) for \( r > \frac{3}{2} \) where we used (3.10), (3.6) and (3.7) we obtain the following weak scaling condition for \( \Psi \): there exists a constant \( c > 0 \) such that

\[
c^{-1} \left( \frac{R}{r} \right)^{\alpha_3} \leq \frac{\Psi(R)}{\Psi(r)} \leq c \left( \frac{R}{r} \right)^2, \quad 0 < r \leq R < \infty.
\]

Using (3.10), (3.6) and (3.7) we obtain the following weak scaling condition for \( \Psi \): there exists a constant \( c > 0 \) such that

\[
c^{-1} \left( \frac{R}{r} \right)^{\alpha_3} \leq \frac{\Psi(R)}{\Psi(r)} \leq c \left( \frac{R}{r} \right)^2, \quad 0 < r \leq R < \infty.
\]

By the definition of \( \varphi \) we have \( I_1 = \varphi(r)^{-1} \). To estimate \( I_2 \), let us define \( k := \left\lfloor \frac{\log r}{\log 2} \right\rfloor \), the largest integer smaller than or equal to \( \frac{\log r}{\log 2} \). Then we have

\[
0 \leq I_2 \leq \sum_{i=0}^{k} \int_{2^i r}^{2^{i+1} r} s^{-d-1} \nu(s)ds =: \sum_{i=0}^{k} I_{2i}.
\]

Using (3.6), we have

\[
I_{2i} \leq (2^i r)^{-2} \int_{2^i r}^{2^{i+1} r} s^{d+1} \nu(s)ds \leq (2^i r)^{-2} \int_{2^i r}^{2^{i+1} r} s^{d+1} \nu(s)ds = 4 \varphi(2^i r)^{-1} \leq a_3 2^{2-\alpha_3(i+1)} \varphi(r)^{-1}.
\]

Thus,

\[
0 \leq I_2 \leq \sum_{i=0}^{k} I_{2i} \leq \frac{2^{2-\alpha_3}}{\varphi(r)} \sum_{i=0}^{k} 2^{-\alpha_3 i} \leq \frac{c_1}{\varphi(r)}.
\] (3.12)

Also, using (3.5) and (3.6) we obtain

\[
0 \leq I_3 \leq a \int_{1}^{\infty} s^{d-\ell-1} \exp(-bs^\beta)ds = c_2 \leq \frac{c_2 \varphi(1)}{a_3 \varphi(r)}.
\]

where we used \( a_3 \leq a_3 \left( \frac{1}{2} \right)^{\alpha_3} \leq \frac{\varphi(1)}{\varphi(r)} \) for the last inequality. Combining estimates of \( I_1, I_2 \) and \( I_3 \) we have proved the claim (3.11) for \( r \leq 1 \).

Now assume \( r > 1 \). Then we have

\[
\mathcal{P}(r) = \int_{\mathbb{R}^d} \left( 1 \wedge \frac{|z|^2}{r^2} \right) \nu(z)dz
\]

\[
= c(d) \left( r^{-2} \int_{0}^{1} s^{d+1} \nu(s)ds + \int_{1}^{\infty} \left( 1 \wedge \frac{s^2}{r^2} \right) s^{d-\ell-1} \nu(s)ds \right)
\]

\[
:= c(d) \left( \varphi(r)^{-1} + I_4 \right).
\]

Also, using (3.5) we have

\[
0 \leq I_4 \leq \int_{1}^{\infty} \frac{s^2}{r^2} s^{d-1} \nu(s)ds \leq a r^{-2} \int_{1}^{\infty} s^{d-\ell+1} \exp(-bs^\beta)ds \leq c_3 r^{-2}.
\]

Using \( \varphi(r) = \varphi(1) r^2 \) for \( r \geq 1 \) we obtain that \( \mathcal{P}(r) \leq r^{-2} \varphi(r)^{-1} \) for \( r > 1 \), which implies (3.11) for \( r > 1 \). Therefore, (3.11) holds for any \( r > 0 \). Combining (3.11) and (3.9) we conclude the lemma. \( \square \)
Let \( p_t(x) \) be a transition density of \( X \). Since \( X \) is isotropic, \( x \mapsto p_t(x) \) is also isotropic function for any \( t > 0 \). By an abuse of notation we also denote the radial part of the heat kernel \( p_t(x) \) of \( X \) as \( p_t(r) \), \( r > 0 \).

To obtain gradient estimate for \( p_t(x) \), we first follow the proof of [KR16, Proposition 3.1] to construct a \((d + 2)\)-dimensional Lévy process \( Y \) whose heat kernel estimate implies gradient estimate of \( X \).

**Lemma 3.4** Assume that isotropic unimodal Lévy measure \( \nu \) satisfies (3.5) and (3.6). Then there exists an isotropic Lévy process \( Y \) in \( \mathbb{R}^{d+2} \) such that its characteristic exponent is \( \xi \mapsto \psi(|\xi|), \xi \in \mathbb{R}^{d+2} \). Let \( \nu_1(|x|) \) and \( q_t(|x|) \) be the jumping kernel and heat kernel of \( Y \), respectively. Then for any \( r > 0 \),

\[
q_t(r) = -\frac{1}{2\pi r} \frac{d}{dr} p_t(r) \tag{3.14}
\]

and

\[
\nu_1(r) = -\frac{1}{2\pi r} \nu'(r). \tag{3.15}
\]

**Proof.** The existence of \( Y \) and (3.14) are immediately followed by [KR16, Proposition 3.1]. Note that using (3.9) and (3.13) we have

\[
\lim_{\rho \to \infty} \frac{\psi(\rho)}{\log \rho} \geq \lim_{\rho \to \infty} \frac{\Psi(\rho)}{\pi^2 \log \rho} \geq \lim_{\rho \to \infty} \frac{c_1 \rho^{\alpha_1}}{\log \rho} = \infty,
\]

which is one of the conditions in [KR16, Proposition 3.1]. For (3.15), we just need to follow the corresponding part in the proof of [KR16, Theorem 1.5]. Here we provide a brief sketch for the proof for reader’s convenience; As in the proof of [KR16, Theorem 1.5], without using the assumption that \(-\nu'(r)/r\) is non-increasing, one can show that there exists an isotropic Lévy process \( X^{(d+2)} \) in \( \mathbb{R}^{d+2} \) with jumping kernel \( \nu_1(dy) \) and that the characteristic exponent of \( X^{(d+2)} \) is \( \psi(r) \). Thus, \( X^{(d+2)} \) and \( Y \) are identical in law, which concludes the proof. To show this, only [KR16, (8) and (9)] are used, which follow directly from the fact that \( \nu \) is isotropic, unimodal measure satisfying \( \int_{\mathbb{R}^d} (|y|^2 \wedge 1) \nu(dy) < \infty \).

We emphasize here that we don’t impose the condition (1.5) on \( \nu \). Thus the function \( r \to \nu_1(r) \) in the above lemma may not be non-increasing.

Now we are going to establish heat kernel estimates for the process \( Y \) obtained in Lemma 3.4, which will imply heat kernel estimate and gradient estimate of \( X \) as a consequence of (3.14). To do this, we will check conditions (E), (D), (P) and (C) (when \( \beta < 1 \)) in [KS17, Theorem 4] for the process \( X \) and \( Y \), and apply [KS17, Theorem 1] and [KS15, Theorem 1].

First, we verify the condition (E) in [KS17]. Recall \( \Psi(r) = \sup_{|y| \leq r} \psi(|y|) \).

**Lemma 3.5** Assume that isotropic unimodal Lévy measure \( \nu \) satisfies (3.5) and (3.6). Then for any \( n, m \in \mathbb{N} \), there exists a constant \( c = c(n, m) > 0 \) such that

\[
\int_{\mathbb{R}^n} e^{-\psi(|z|)} |z|^m dz \leq c \Psi^{-1}(t^{-1})^{n+m}, \quad t > 0.
\]
Proof. By \textbf{(3.9)} and \textbf{(3.13)} we have that for $0 < t$,

\[
\int_{\mathbb{R}^n} e^{-t\psi(|z|)} |z|^m dz \leq c_1 \int_0^{\Psi^{-1}(t-1)} r^{n+m-1} dr + c_1 \int_{\Psi^{-1}(t-1)}^{\infty} e^{-\pi^{-2}t\Psi(r)} r^{n+m-1} dr
\]

\[
\leq c_2 \Psi^{-1}(t-1)^{n+m} + c_1 \int_{\Psi^{-1}(t-1)}^{\infty} e^{-c_3t\Psi^{-1}(t-1)(r/\Psi^{-1}(t-1))^{\alpha_3} r^{n+m-1} dr}
\]

\[
= \left( c_2 + c_1 \int_1^{\infty} e^{-c_3s} s^{n+m-1} ds \right) \Psi^{-1}(t-1)^{n+m} = c_4 \Psi^{-1}(t-1)^{n+m},
\]

where we have used the change of variables with $s = \frac{r}{\Psi^{-1}(t-1)}$ in the last line. \hfill \Box

Note that Lemma \textbf{3.5} for $(n, m) = (d, 1)$ and $(n, m) = (d + 2, 1)$ implies the condition \textbf{(E)} in [KS17] for the process $X$ and $Y$, respectively.

For $0 < \beta < 1$ and $\ell \geq 0$, we define non-increasing functions $f$ and $\tilde{f}$ by

\[
f(r) := \begin{cases} \frac{\varphi(1)}{r^d \varphi(r)}, & r \leq 1, \\ r^{-\ell-1} \exp(-br^\beta), & r > 1 \end{cases}
\]

and

\[
\tilde{f}(r) := \begin{cases} \frac{\varphi(1)}{r^d \varphi(r)}, & r \leq 1, \\ r^{-\ell} \exp(-br^\beta), & r > 1 \end{cases}
\]

The functions $f$ and $\tilde{f}$ above are non-increasing since for any $0 < r < R \leq 1$,

\[
\frac{1}{r^d \varphi(r)} = r^{-1} \int_0^r \left( \frac{s}{r} \right)^{d+1} \nu(s) ds = \int_0^1 t^{d+1} \nu(rt) dt \geq \int_0^1 t^{d+1} \nu(Rt) dt = \frac{1}{R^d \varphi(R)}.
\]

Here we used that $\nu$ is nonincreasing. Note that by \textbf{3.5} and \textbf{3.8},

\[
\frac{\nu(r)}{r} \leq cf(r) \quad \text{and} \quad \nu(r) \leq \tilde{f}(r) \quad \text{for} \quad r > 0
\]

In the next lemma we verify the condition \textbf{(D)} in [KS17] for both $X$ and $Y$. In fact, we are going to verify \textbf{(D)} for $X$ with the above $f$ and $\gamma = d$, while we use $f$ and $\gamma = d + 1$ to verify \textbf{(D)} for $Y$. Let $B_d(x,r) := \{ y \in \mathbb{R}^d : |x-y| < r \}$ and recall that $\text{diam}(A) = \sup\{|x-y| : x, y \in A\}$ and $\nu_1(r) = -\frac{1}{2\pi} \nu'(r)$.

\textbf{Lemma 3.6} Assume that $\nu$ satisfies \textbf{3.5} and \textbf{3.6}. Then both $\nu(\mathbb{R}^d)$ and $\nu_1(\mathbb{R}^{d+2}) = \int_{\mathbb{R}^{d+2}} \nu_1(|x|) dx$ are infinite, and there exists $c > 0$ such that

\[
\nu(A) \leq c \tilde{f}(\delta(A)) [\text{diam}(A)]^d, \quad A \in \mathcal{B}(\mathbb{R}^d).
\]

and

\[
\nu_1(A) = \int_A \nu_1(|x|) dx \leq c \tilde{f}(\delta(A)) [\text{diam}(A)]^{d+1}, \quad A \in \mathcal{B}(\mathbb{R}^{d+2}).
\]

for some $c > 0$, where $\delta(A) := \inf\{|y| : y \in A\}$.
Proof. We have already showed that $\nu(B_d(0,1)) = \nu(\mathbb{R}^d) = \infty$. For any $A \in \mathcal{B}(\mathbb{R}^d)$, using (3.17) we have

$$\nu(A) = \int_A \nu(|y|)dy \leq \nu(\delta(A))[\text{diam}(A)]^d \leq c\tilde{f}(\delta(A))[\text{diam}(A)]^d.$$  

This concludes (3.18).

Using $\nu'(r) \leq 0$, (3.1), the integration by parts and the fact $\nu(B_d(0,1)) = \infty$ we have

$$\nu_1(\mathbb{R}^{d+2}) \geq \int_{B_{d+2}(0,1)} \nu_1(|y|)dy = c(d) \int_0^1 r^{d+1} \nu_1(r)dr = c_1 \liminf_{\varepsilon \downarrow 0} \int_\varepsilon^1 -r^d \nu'(r)dr$$

$$= c_1 \liminf_{\varepsilon \downarrow 0} \left( -[r^d \nu(r)]_\varepsilon^1 + d \int_\varepsilon^1 r^{d-1} \nu(r)dr \right)$$

$$= c_1 \liminf_{\varepsilon \downarrow 0} \left( \varepsilon^d \nu(\varepsilon) + d \int_\varepsilon^1 r^{d-1} \nu(r)dr - \nu(1) \right) \geq c_2 \nu(B_d(0,1)) - c_1 \nu(1) = \infty.$$  

Now it remains to prove (3.19). First observe that using the integration by parts, we have that for any $0 < r < R$,

$$\int_r^R s^{d+1} \nu_1(s)ds = -\frac{1}{2\pi} \int_r^R s^{d} \nu'(s)ds = \frac{1}{2\pi} \left( -[s^d \nu(s)]_r^R + d \int_r^R s^{d-1} \nu(s)ds \right)$$

$$\leq \frac{1}{2\pi} \left( r^d \nu(r) + \nu(r) d \int_r^R s^{d-1}ds \right) = \frac{1}{2\pi} \nu(r)R^d$$

(3.20)  

where we used that $\nu$ is non-increasing. Now denote $r := \delta(A)$ and $l := \text{diam}(A)$.

When $l \geq r/2$, using $A \subset \{y \in \mathbb{R}^{d+2} : r \leq |y| \leq r + l\}$ we obtain

$$\nu_1(A) \leq \nu_1(\{y : r \leq |y| \leq r + l\}) = c(d) \int_r^{r+l} s^{d+1} \nu_1(s)ds$$

$$\leq \frac{c(d)}{2\pi} \nu(r)(r + l)^d \leq c_1 \frac{\nu(r)}{r} l^{d+1} \leq c_3 f(r) l^{d+1},$$

where we used (3.20) and (3.17) for the last line.

When $l < r/2$, choose a point $y_0 \in \bar{A}$ with $|y_0| = r$. Since $A \subset B_{d+2}(y_0,l)\setminus B_{d+2}(0,r)$, there exists $c_4 = c_4(d) > 0$ such that

$$\int_{|y|=s} 1_A(y)\sigma(dy) \leq c_4 l^{d+1}$$

for any $s \in [r,r+l]$. Thus, by (3.20) and (3.17) we have

$$\nu_1(A) \leq \nu_1(B(y_0,l)\setminus B(0,r)) \leq c_5 \int_r^{r+l} l^{d+1} \nu_1(s)ds \leq c_5 \frac{l^{d+1}}{r^{d+1}} \int_r^{r+l} s^{d+1} \nu_1(s)ds$$

$$\leq c_5 \frac{l^{d+1}}{2\pi r^{d+1}} (r + l)^d \nu(r) \leq c_6 l^{d+1} \frac{\nu(r)}{r} \leq c_7 f(r) l^{d+1},$$

which proves (3.19). \qed

Recall $\Psi(r) = \sup_{|y|\leq r} \psi(|y|)$.  

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Lemma 3.7 Assume that $\nu$ satisfies (3.5) and (3.6). For every $\kappa < 1$, there exists $c = c(\kappa) > 0$ such that

$$\int_{\{y \in \mathbb{R}^d : |y| > r\}} \exp (bk|y|^{\beta}) \nu(dy) \leq c \Psi(\frac{1}{r}), \quad r > 0$$

(3.21)

and

$$\int_{\{y \in \mathbb{R}^{d+2} : |y| > r\}} \exp (bk|y|^{\beta}) \nu_1(dy) \leq c \Psi(\frac{1}{r}), \quad r > 0$$

(3.22)

Proof. Since (3.21) can be derived directly from the estimate of $I_2$ below, we only prove (3.22) here. Using the integration by parts, we have

$$\int_{|y| > r} \exp (bk|y|^{\beta}) \nu_1(dy) = c(d) \int_r^\infty \exp (bk^t \beta) t^d (-\nu'(t)) dt$$

$$= c(d) \left( \int_r^\infty \exp (bk^t \beta) t^d (-\nu(t)) \right)_r^\infty + \int_r^\infty (\exp (bk^t \beta) t^d)^' \nu(t) dt$$

$$:= c(d) (I_1 + I_2).$$

For $I_1$, by (3.17) $\lim_{t \to \infty} \exp (bk^t \beta) t^d \nu(t) \leq \lim_{t \to \infty} a \exp (-b(1-\kappa)t^\beta)^{t^d-\ell} = 0$ so $I_1 = \exp (bk^r \beta) r^d \nu(r) \leq c_1 \varphi(r)^{-1}$. Now let us estimate $I_2$. First we observe that

$$\frac{d}{dt} \left( \exp (bk^t \beta) t^d \right) \leq c_2 \begin{cases} t^{d-1}, & t \leq 1 \\ \exp (bk^t \beta) t^{d+\beta-1}, & t > 1. \end{cases}$$

Thus, for $r \geq 1$ we have

$$\int_r^\infty \exp (bk^t \beta) t^d \nu'(t) dt \leq c_2 \int_r^\infty \exp (-b(1-\kappa)t^\beta) t^{d-\ell+\beta-1} dt \leq c_3 r^{\ell-2} = \frac{c_3 \varphi(1)}{\varphi(r)}.$$

For $r < 1$, using above estimate, (3.12) and (3.6) we get

$$\int_r^\infty \exp (bk^t \beta) t^d \nu'(t) dt = \left( \int_r^1 + \int_1^\infty \right) \exp (bk^t \beta) t^d \nu'(t) dt$$

$$\leq c_2 \left( \int_r^1 t^{d-1} \nu(t) dt + \int_1^\infty \exp (-b(1-\kappa)t^\beta) t^{d-\ell+\beta-1} dt \right)$$

$$\leq \frac{c_4}{\varphi(r)} + c_3 \leq \frac{c_5}{\varphi(r)}.$$

Combining above two inequalities and (3.10), we obtain $I_1 + I_2 \leq c_6 \Psi(\frac{1}{r})$. Therefore, we have proved the lemma.

Using Lemma 3.7, we verify the condition (P) in [KS17] for both $X$ and $Y$. We continue to use the non-increasing functions $f$ and $\tilde{f}$ defined in (3.16).
Lemma 3.8 Assume that isotropic unimodal Lévy measure $\nu$ satisfies (3.5) and (3.6). Then, there exists $c > 0$ such that

$$\int_{\{y \in \mathbb{R}^d : |y| > r\}} \tilde{f} \left( s \vee |y| - \frac{|y|}{2} \right) \nu(dy) \leq c \tilde{f}(s) \Psi\left( \frac{1}{r} \right), \quad r, s > 0$$  \hspace{1cm} (3.23)

and

$$\int_{\{y \in \mathbb{R}^{d+2} : |y| > r\}} f \left( s \vee |y| - \frac{|y|}{2} \right) \nu(dy) \leq c f(s) \Psi\left( \frac{1}{r} \right), \quad r, s > 0$$  \hspace{1cm} (3.24)

Proof. We only prove (3.24) here, since (3.23) can be verified similarly. We claim that for any $0 < \beta \leq 1$, there exists $c_1 > 0$ such that for any $s, t > 0$,

$$f(s \vee t - \frac{t}{2}) \leq c_1 f(s) \exp(b \kappa t^\beta)$$  \hspace{1cm} (3.25)

where $\kappa = \frac{1}{2}(2^{-\beta} + 1)$. First we define

$$f_1(r) := \begin{cases} \frac{\varphi(1)}{r^{\beta+1} \varphi(r)}, & r \leq 1 \\ \frac{\beta}{r^{\beta+1} \varphi(r)} \exp(-br^\beta), & r > 2. \end{cases}$$

Then, since $f(r) = f_1(r)$ for $r \in (0, 1] \cup (2, \infty)$ we have

$$c_2^{-1} f(r) \leq f_1(r) \leq c_2 f(r), \quad r > 0.$$  \hspace{1cm} (3.26)

Now assume $s \vee t > 2$. Then, using $1 \vee \frac{s}{2} \leq s \vee t - \frac{t}{2}$ and triangular inequality,

$$f(s \vee t - \frac{t}{2}) = (s \vee t - \frac{t}{2})^{-\ell-1} \exp\left(-b(s \vee t - \frac{t}{2})^\beta\right) \leq (1 \vee \frac{s}{2})^{-\ell-1} \exp\left(-b(1 \vee \frac{s}{2})^\beta\right) \leq c_3 f(s) \exp(b(\frac{t}{2})^\beta).$$

Here in the last inequality we used $\ell \geq 0$ and $\exp(-bs^\beta) \leq c_3 f(s)$ for $0 < s \leq 2$. When $s \leq 2$ and $t \leq 2$, using (3.26), (3.6) and (3.7) with $s \vee t - \frac{t}{2} \geq \frac{s}{2}$ we obtain

$$f(s \vee t - \frac{t}{2}) \leq c_2 f_1(s \vee t - \frac{t}{2}) = \frac{c_2 \varphi(1)}{(s \vee t - \frac{t}{2})^{d+1} \varphi(s \vee t - \frac{t}{2})} \leq \frac{c_4 \varphi(1)}{s^{d+1} \varphi(s)} \leq c_4 f_1(s) \leq c_5 f(s).$$

Here we used $\frac{\varphi(s)}{\varphi(s \vee t - \frac{t}{2})} = \frac{\varphi(s)}{\varphi(s/2)} \leq 4a_3^{-1}$ which follows from (3.6) and (3.7). Thus, we conclude (3.25). Combining (3.25) and Lemma 3.7 we have proved the lemma. \qed

Now we obtain a priori heat kernel estimates for the process $X$ and $Y$. To state the results, we need to define generalized inverse of $\varphi$ by $\varphi^{-1}(t) := \inf\{s > 0 : \varphi(s) \geq t\}$. Using (3.6) and [BGR14, Remark 4], we obtain

$$c^{-1} \left( \frac{R}{r} \right)^{1/2} \leq \frac{\varphi^{-1}(R)}{\varphi^{-1}(r)} \leq c \left( \frac{R}{r} \right)^{1/\alpha_3}$$  \hspace{1cm} (3.27)

and

$$a_3^{-1} \varphi(\varphi^{-1}(r)) \leq r \leq a_3 \varphi(\varphi^{-1}(r)), \hspace{1cm} (3.28)$$

which are counterparts of (2.8). First we apply [KS15, Theorem 3] to obtain the regularity of the transition density $p_t(x)$ of $X$.  \hspace{1cm} 23
Proposition 3.9 Let $X$ be an isotropic unimodal Lévy process in $\mathbb{R}^d$ with jumping kernel $\nu(|y|) dy$ satisfying (3.5) and (3.6) with $0 < \beta \leq 1$. Then $x \to p_t(x) \in C_b^\infty(\mathbb{R}^d)$ and for any $k \in \mathbb{N}_0$ there exists $c_k > 0$ such that

$$|\nabla^k_x p_t(x)| \leq c_k \varphi^{-1}(t)^{-k} \left( \varphi^{-1}(t)^{-d} \wedge \frac{t}{|x|^d \varphi(|x|)} \right)$$

(3.29)

for any $t > 0$ and $x \in \mathbb{R}^d$.

Proof. Define $h(t) := \frac{1}{\Psi^{-1}(t)}$ as in [KS15]. Note that by (3.10) and (3.28) we have

$$h(t) \asymp \varphi^{-1}(t), \quad t > 0.$$  

(3.30)

Applying [KS15, Theorem 3] for the process $X$, $p_t(x) \in C_b^\infty(\mathbb{R}^d)$ and for any $k \in \mathbb{N}, \gamma \in [1, d]$ and $n > \gamma$ we have constants $c_{k,n}$ satisfying

$$|\nabla^k_x p_t(x)| \leq c_{k,n}(h(t))^{-d-k} \min \left\{ 1, \frac{t[h(t)]^\gamma}{|x|^{\gamma \varphi(|x|)}} e^{b(|x|/4)^\gamma} + \left( 1 + \frac{|x|}{h(t)} \right)^{-n} \right\}.$$  

Note that we already verified [KS15, (8)] at Lemma 3.5. Thus, using $h(t) \asymp \varphi^{-1}(t)$ we obtain

$$|\nabla^k_x p_t(x)| \leq \tilde{c}_{k,n} \varphi^{-1}(t)^{-d-k}.$$  

Also, taking $\gamma = d, n = d + 2$ and using $h(t) \asymp \varphi^{-1}(t)$ we get

$$|\nabla^k_x p_t(x)| \leq c_k \left( h(t)^{-k} \frac{t}{|x|^d \varphi(|x|)} e^{b(|x|/4)^\gamma} + h(t)^{-k} |x|^d \left( 1 + \frac{|x|}{h(t)} \right)^{-2} \right)$$  

$$\leq c \varphi^{-1}(t)^{-k} \left( \frac{t}{|x|^d \varphi(|x|)} + \left( \frac{\varphi^{-1}(t)}{|x|} \wedge 1 \right)^2 \frac{1}{|x|^d} \right)$$  

$$\leq \tilde{c}_{k,n} \varphi^{-1}(t)^{-k} \frac{t}{|x|^d \varphi(|x|)}.$$  

The last inequality is straightforward when $|x| < \varphi^{-1}(t)$ and it follows from (3.27) and (3.28) when $|x| \geq \varphi^{-1}(t)$. Therefore, we conclude that

$$|\nabla^k_x p_t(x)| \leq c_k \varphi^{-1}(t)^{-k} \left( \varphi^{-1}(t)^{-d} \wedge \frac{t}{|x|^d \varphi(|x|)} \right).$$  

□

Note that the gradient estimates in Proposition 3.9 is same as the ones in [KSV18, Proposition 3.2] except that the gradient estimates in [KSV18, Proposition 3.2] is for $t \leq T$ (see Remark 2.3).

Combining above estimates with Lemmas 3.5, 3.6 and 3.8 we can apply [KS15, Theorem 1] for the process $X$ and $Y$. Here is the result.
Lemma 3.10 Assume that $\nu$ satisfies (3.5) and (3.6) and $\beta = 1$. Then for any $T \geq 1$, there exists a constant $c > 0$ such that

$$p_t(x) \leq ct \exp(-\frac{b}{4} |x|) \quad \text{and} \quad q_t(x) \leq ct\varphi^{-1}(t)^{-1} \exp(-\frac{b}{4} |x|) \quad (3.31)$$

for any $0 < t \leq T$ and $|x| > \varphi^{-1}(T)$.

Proof. Define $h(t) := \frac{1}{\varphi^{-1}(t^{-1})}$ as in [KS15] and denote $q_t(|x|) = q_t(x)$. Applying Lemmas 3.5, 3.6 and 3.8 to [KS15, Thoerem 1] for the process $Y$ in Lemma 3.4, we have

$$q_t(r) \leq c_1 h(t)^{-1} \left( h(t)^{-d-1} \wedge \left[ tf(r/4) + h(t)^{-d-1} \exp \left( - c_2 \frac{r}{h(t)} \log(1 + \frac{r}{h(t)}) \right) \right] \right)$$

$$\leq c_3 \varphi^{-1}(t)^{-1} \left( h(t)^{-d-1} \wedge \left[ tf(r/4) + \varphi^{-1}(t)^{-d-1} \exp \left( - c_4 \frac{r}{\varphi^{-1}(t)} \log(1 + c_5 \frac{r}{\varphi^{-1}(t)}) \right) \right] \right)$$

for any $t, r > 0$.

First observe that using $f(\frac{r}{4}) = (\frac{r}{4})^{-\ell-1} \exp(-\frac{r}{4})$ for $r > 4$, we obtain

$$t \varphi^{-1}(t)^{-1} f(\frac{r}{4}) \leq c_6 t r^{-\ell-1} \exp(-\frac{b}{4} r) \leq c_7 t \exp(-\frac{b}{4} r), \quad r > 4. \quad (3.32)$$

Let $c(T) > 4$ be a constant which is large enough to satisfy

$$\frac{c_4}{2 \varphi^{-1}(T)} \log(1 + c_5 \frac{c(T)}{\varphi^{-1}(T)}) \geq \frac{b}{4}, \quad r > 1.$$ 

Then using (3.27) in the second inequality, for any $0 < t \leq T$ and $r > c(T)$ we have

$$\varphi^{-1}(t)^{-d-1} \exp \left( - c_4 \frac{r}{\varphi^{-1}(t)} \log(1 + c_5 \frac{r}{\varphi^{-1}(t)}) \right)$$

$$\leq \varphi^{-1}(t)^{-d-1} \exp \left( - \frac{c_4 r}{c_5 \varphi^{-1}(t)} \log(1 + c_5 \frac{c(T)}{\varphi^{-1}(T)}) \right) \exp \left( - \frac{c_4 r}{2 \varphi^{-1}(T)} \log(1 + c_5 \frac{c(T)}{\varphi^{-1}(T)}) \right)$$

$$\leq \varphi^{-1}(t)^{-d-1} \exp \left( - \frac{c_4 r}{2 \varphi^{-1}(T)} \log(1 + c_5 \frac{c(T)}{\varphi^{-1}(T)}) \right) \exp \left( - \frac{c_4 r}{c_5 \varphi^{-1}(T)} \log(1 + c_5 \frac{c(T)}{\varphi^{-1}(T)}) \right)$$

$$\leq \varphi^{-1}(t)^{-d-1} \exp(-c_8 \frac{r}{\varphi^{-1}(t)} - \frac{b}{4} r) \leq \frac{c_9 t}{r^{d+1} \varphi(r)} \exp(-\frac{b}{4} r) \leq c_9 t \exp(-\frac{b}{4} r)$$

where $c_9 = \sup_{s \geq 1} s^{d+1} \exp(-c_8 s) < \infty$. Thus,

$$q_t(r) \leq c_{10} t \varphi^{-1}(t)^{-1} \exp(-\frac{b}{4} r), \quad r > c(T) = \varphi^{-1}(c(T)).$$

Meanwhile, by (3.14) and (3.29) we have

$$q_t(r) = \frac{1}{2\pi r} \left| \frac{r}{dr} p_t(r) \right| \leq \frac{c_{11} t \varphi^{-1}(t)^{-1}}{r^{d+1} \varphi(r)} \leq c_{12} t \varphi^{-1}(t)^{-1} \exp(-\frac{b}{4} r + \frac{bc(T)}{4}).$$

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for $\varphi^{-1}(T) < r \leq c(T)$. Therefore, combining above two estimates we conclude the estimate on $q$ in (3.31).

Note that, applying Lemmas 3.5, 3.6 and 3.8 to [KS15] Theorem 1] for the process $X$ and using $h(t) \asymp \varphi^{-1}(t)$ we have

$$p_t(r) \leq c_{t_0} \left( \varphi^{-1}(t)^{-d} \wedge \left[ t f(r/4) + \varphi^{-1}(t)^{-d} \exp\left(-c_{t_1} \frac{r}{\varphi^{-1}(t)} \log(1 + c_{t_2} \frac{r}{\varphi^{-1}(t)})\right) \right] \right).$$

(3.33) for any $t, r > 0$. Using (3.33), the estimate on $p$ in (3.31) can be verified similarly. □

Now we check condition (C) in [KS17] with $r_0 = 1$, $t_p = \infty$ and $\gamma = d$ for $X$ ($\gamma = d + 1$ for $Y$, respectively). We need additional condition $0 < \beta < 1$ to verify the condition (C).

**Lemma 3.11** Assume $\nu$ satisfies (3.2) and (3.6) with $0 < \beta < 1$. Then, there exists constant $c > 0$ such that for every $|x| \geq 2$ and $r \in (0, 1]$,

$$\tilde{f}(r) \leq cr^{-d} \Psi\left(\frac{1}{r}\right), \quad \int_{\{|y| \geq 1, |y| > r\}} \tilde{f}(|x - y|) \nu(dy) \leq c \Psi\left(\frac{1}{r}\right) \tilde{f}(|x|)$$

(3.34) and

$$f(r) \leq cr^{-d-1} \Psi\left(\frac{1}{r}\right), \quad \int_{\{|y| \geq 1, |y| > r\}} f(|x - y|) \nu_1(dy) \leq c \Psi\left(\frac{1}{r}\right) f(|x|).$$

(3.35)

**Proof.** The first inequalities in (3.34) and (3.35) immediately follow from (3.10) and (3.16).

Let us show the second inequality in (3.35). When $|x - y| \geq \frac{|x|}{2}$, using (2.13) and triangular inequality, we have $|x|^\beta \leq |x - y|^\beta + (2^\beta - 1)|y|^\beta$. Thus, using this inequality and Lemma 3.7 we obtain

$$\int_{|y| > r} f(|x - y|) \nu_1(dy) = \int_{|y| > r} |x - y|^{-\ell - 1} \exp(-b|x - y|^\beta) \nu_1(dy) \leq \left(\frac{|x|}{2}\right)^{-\ell - 1} \int_{|y| > r} \exp(-b|x|^\beta) \exp(b(2^\beta - 1)|y|^\beta) \nu_1(dy)$$

$$= f(|x|) \int_{|y| > r} \exp(b(2^\beta - 1)|y|^\beta) \nu_1(dy) \leq c_1 f(|x|) \Psi\left(\frac{1}{r}\right).$$

So, it suffices to show that there exists a constant $c_2 > 0$ such that for every $|x| \geq 2$,

$$\int_{1 \leq |x - y| \leq \frac{|x|}{2}} f(|x - y|) \nu_1(dy) \leq c_2 f(|x|).$$

(3.36)

To show this, we will divide the set $D := \{y : 1 \leq |x - y| \leq \frac{|x|}{2}\}$ into cubes with diameter 1. Let $x = (x_1, \ldots, x_{d+2})$. For $(a_1, \ldots, a_{d+2}) \in \mathbb{Z}^{d+2}$, we define $a := (\sqrt{d+2})^{-1}(a_1, \ldots, a_{d+2})$, and let

$$C_a := \prod_{i=1}^{d+2} \left[ x_i + \frac{2a_i - 1}{2\sqrt{d+2}}, x_i + \frac{2a_i + 1}{2\sqrt{d+2}} \right]$$

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be a cube with length \((\sqrt{d+2})^{-1}\). Since diam\((C_a) = 1\) and \(x + a\) is the center of cube \(C_a\), for any \(|a| \leq \frac{|x|+1}{2}\) we have \(c_5 > 0\) independent of \(a\) such that

\[
\nu_1(C_a \cap D) \leq c_3 f(\delta(C_a \cap D)) \leq c_3 f\left(\left(|x| - |a| - \frac{1}{2}\right) \vee \frac{|x|}{2}\right)
\]

\[
\leq c_4 (|x| - |a|)^{-\ell - 1} \exp\left(-b| |x| - |a| \right) \leq c_5 |x|^{-\ell - 1} \exp\left(-b( |x| - |a| )^\beta \right)
\]

where we used Lemma \(3.6\) for the first inequality and triangular inequality for the second line. Thus, using \(|a| - \frac{1}{2} \leq |x - y|\) on \(C_a\) and

\[
D \subset \bigcup_{1 \leq |a| \leq \frac{|x|+1}{2}} C_a,
\]

we obtain

\[
\int_{1 \leq |x - y| \leq \frac{|x|}{2}} f(|x - y|) \nu_1(dy) \leq \sum_{1 \leq |a| \leq \frac{|x|+1}{2}} \int_{C_a \cap D} |x - y|^{-\ell - 1} \exp\left(-b| |x - y| \right) \nu_1(dy)
\]

\[
\leq \sum_{1 \leq |a| \leq \frac{|x|+1}{2}} \left(|a| - \frac{1}{2}\right)^{-\ell - 1} \exp\left(-b\left(|a| - \frac{1}{2}\right)^\beta \right) \nu_1(C_a \cap D)
\]

\[
\leq c_6 |x|^{-\ell - 1} \sum_{1 \leq |a| \leq \frac{|x|+1}{2}} \left|a\right|^{-\ell - 1} \exp\left(-b\left(|a| \right)^\beta \right) \exp\left(-b\left(|x| - |a| \right) \right)
\]

Since \(|a| \leq \frac{|x|+1}{2}\), by (2.13) we have \(|a|^\beta + (|x| - |a|)^\beta + 1 \geq |a|^\beta + (|x| + 1 - |a|)^\beta \geq |x|^\beta + (2 - 2\beta)|a|^\beta\). Thus,

\[
|x|^{-\ell - 1} \int_{1 \leq |a| \leq \frac{|x|+1}{2}} \exp\left(-b\left(|a| \right)^\beta \right) \exp\left(-b\left(|x| - |a| \right) \right)
\]

\[
\leq c_7 |x|^{-\ell - 1} \exp\left(-b| |x| \right) \sum_{a \in \mathbb{Z}^d \setminus \{0\}} |a|^{-\ell - 1} \exp\left(-b(2 - 2\beta)|a|^{\beta}\right) \leq c_8 f(|x|).
\]

Combining above inequalities and using (3.10), we arrive (3.36). Therefore, we conclude that the second inequality in (3.34) holds.

The second inequality in (3.34) can be verified similarly so skip the proof.

Now we have that conditions (E), (D) and (C) in [KS17] holds for the process \(Y\) when \(\nu\) satisfies (3.5) and (3.6) with \(0 < \beta < 1\). Thus, we can apply [KS17] Theorem 4 for both \(X\) and \(Y\).

**Lemma 3.12** Let \(T \geq 1\) and assume that \(\nu\) satisfies (3.5) and (3.6) with \(0 < \beta < 1\). Then, there exists a constant \(c > 0\) such that

\[
p_t(r) \leq ctr^{-\ell} \exp(-br^\beta)
\]

(3.37)
and

$$\left| \frac{d}{dr} p_t(r) \right| \leq ct\varphi^{-1}(t)^{-1}r^{-\ell}\exp(-br^\beta)$$  \hspace{1cm} (3.38)

for any $0 < t \leq T$ and $r \geq 4$.

**Proof.** Applying [KS17, Theorem 4] for $Y$ and (3.10) we have that for $0 < t \leq t_p = T$ and $r \geq 4r_0 = 4$,

$$q_t(r) \leq c_1t\varphi^{-1}(t)^{-1}f(r) = c_1t\varphi^{-1}(t)^{-1}r^{-\ell-1}\exp(-br^\beta).$$

Combining this with (3.14), $|\frac{d}{dr} p_t(r)| \leq 2\pi rq_t(r) \leq c_2t\varphi^{-1}(t)^{-1}r^{-\ell}\exp(-br^\beta)$. This concludes (3.38). (3.37) immediately follows from applying [KS17, Theorem 4] for $X$. \hfill $\Box$

For reader’s convenience, we put the heat kernel estimates and gradient estimates in Proposition 3.9 and Lemmas 3.10 and 3.12 together into one proposition.

**Proposition 3.13** Let $X$ be an isotropic unimodal Lévy process in $\mathbb{R}^d$ with jumping kernel $\nu(|y|)dy$ satisfying (3.3) and (3.6). Then, $x \mapsto p_t(x) \in C^\infty_b(\mathbb{R}^d)$ and the following holds.

(a) There exists a constant $c_1 > 0$ such that

$$|\nabla^k p_t(x)| \leq c_1\varphi^{-1}(t)^{-k} \left( \varphi^{-1}(t)^{-d} \wedge \frac{t}{\beta d\varphi(r)} \right), \quad t > 0, x \in \mathbb{R}^d \text{ and } k \in \mathbb{N}_0.$$

(b) Assume $\beta = 1$. Then for any $T \geq 1$, there exists a constant $c_2 > 0$ such that

$$|\nabla^k p_t(x)| \leq c_2t\varphi^{-1}(t)^{-k}\exp\left(-\frac{b}{4}r\right), \quad t \in (0, T], |x| > \varphi^{-1}(T) \text{ and } k = 0, 1.$$

(c) Assume $0 < \beta < 1$. Then for any $T \geq 1$, there exists a constant $c_3 > 0$ such that

$$|\nabla^k p_t(x)| \leq c_3t\varphi^{-1}(t)^{-k}r^{-\ell}\exp(-br^\beta), \quad t \in (0, T], |x| > \varphi^{-1}(T) \text{ and } k = 0, 1.$$

**Proof.** (a) and (b) immediately follow from Proposition 3.9 and Lemma 3.10 respectively. (c) Observe that for any $t \in (0, T]$, $\varphi^{-1}(T) < |x| \leq 4$ and $k = 0, 1$,

$$|\nabla^k p_t(x)| \leq c_1\varphi^{-1}(t)^{-k} \frac{t}{\beta d\varphi(r)} \leq c_4t\varphi^{-1}(t)^{-k}r^{-\ell}\exp(-br^\beta).$$

This and Lemma 3.12 finish the proof. \hfill $\Box$

Now we are ready to prove Propositions 3.1 and 3.2.

**Proof of Proposition 3.1** Now assume that $X$ is an isotropic Lévy process in Proposition 3.1 with Lévy measure $\nu(|y|)dy$. Recall that $\varphi(r) \approx \Phi(r)$, and $\nu$ satisfies (3.3) with $\ell = 0$ and (3.6). Therefore, we can apply results in Proposition 3.13 with the function $\Phi$ instead of $\varphi$. Using Proposition 3.13 and (2.4), we conclude that for any $t \in (0, T]$ and $x \in \mathbb{R}^d$

$$|\nabla^k p_t(x)| \leq c_1t\varphi^{-1}(t)^{-k}\mathcal{G}_T(t, x) \leq c_2t\mathcal{G}_0(t, x), \quad k = 0, 1.$$
Proof of Proposition 3.2. \((3.3)\) for \(k = 0, 1\) and that \(t \mapsto p(t, x)\) is in \(C^\infty_b(\mathbb{R}^d)\) immediately follow from Proposition 3.1.

Now it suffices to prove \((3.3)\) when \(k = 2\). Let \(X\) be an isotropic unimodal Lévy process with jumping kernel \(J(|x|)dx\) satisfying \((3.1)\) with \(0 < \beta \leq 1\) and \((1.5)\), and let \(\psi(|x|) = \psi(x)\) be a characteristic exponent of \(X\). By Lemma \(3.4\) there exists an isotropic Lévy process \(Y\) in \(\mathbb{R}^{d+2}\) with characteristic exponent \(\psi(r)\) satisfying \((3.14)\) and \((3.15)\). In particular, by \((1.5)\) and \((3.15)\), \(Y\) is unimodal. Denote \(J_1(|x|)dx\) and \(q_t(|x|)\) be the Lévy density and transition density of \(Y\) respectively. Using \((3.14)\) we have

\[
2\pi \int_s^r J_1(t)dt = - \int_s^r \frac{J'(t)}{t} dt = - \frac{\int_s^r J(t)}{t}dr = - \int_s^r \frac{J(t)}{s-t^2} dt = \frac{J(s)}{s} - \frac{J(r)}{r} - \int_s^r \frac{J(t)}{t^2} dt.
\]

Since \(J_1\) is non-increasing by \((1.5)\), we obtain that for any \(0 < s \leq r\),

\[
(r - s)J_1(r) \leq \int_s^r J_1(t)dt \leq \frac{J(s)}{2\pi s} \quad (3.39)
\]

and

\[
(r - s)J_1(s) \geq \int_s^r J_1(r)dr \geq \frac{1}{2\pi} \left( \frac{J(s)}{s} - \frac{J(r)}{r} - \int_s^r \frac{J(t)}{t^2} dt \right) = \frac{1}{2\pi} r (J(s) - J(r)). \quad (3.40)
\]

We claim that there exists a constant \(c > 0\) such that

\[
\frac{c^{-1}}{r^{d+2} \phi(r)} \leq J_1(r) \leq \frac{c}{r^{d+2} \phi(r/2)}; \quad r \leq 1 \quad \text{and} \quad J_1(r) \leq cr^{-1} \exp(-br^\beta), \quad r > 1. \quad (3.41)
\]

For \(r \leq 1\), letting \(s = \frac{r}{2}\) in \((3.39)\) we have \(J_1(r) \leq \frac{J(r/2)}{5\pi r^2} \leq \frac{c_1}{r^{d+2} \phi(r/2)}\) by \((1.4)\). Also, taking \((r, s) = (Cr, r)\) with constant \(C = \left(\frac{2}{d+2}\right)^{2/d} > 1\) in \((3.40)\) we have

\[
J_1(r) \geq \frac{J(r) - J(Cr)}{2\pi C(C-1)r^2} \geq \frac{1}{2\pi C(C-1)r^2} \left( \frac{1}{r^d \phi(r)} - \frac{1}{a(Cr)^d \phi(Cr)} \right)
\]

\[
\geq \frac{1}{C(C-1)} \left( \frac{1}{a C^d} - \frac{1}{r^{d+2} \phi(r)} \right) = \frac{a}{2C(C-1)} \frac{1}{r^{d+2} \phi(r)},
\]

where we used \(\phi(Cr) \geq \phi(r)\) and \(a - \frac{1}{a C^d} = \frac{d}{2}\) in the second line.

When \(r > 1\), letting \(s = r - 1\) in \((3.39)\) we have

\[
J_1(r) \leq \frac{J(r - 1)}{r} \leq \frac{1}{r} \exp(-b(r - 1)^\beta) \leq e^b \frac{1}{r} \exp(-br^\beta),
\]

where we used the assumptions \(r > 1\) and \(0 < \beta \leq 1\) for the last inequality. We have proved \((3.41)\).
Let $\varphi$ be the function \((3.4)\) with $\nu = J_1$ and the dimension $d+2$ (instead of $d$). Note that \((3.41)\) implies that $\varphi$ satisfies
\[
c_1 \Phi(r) \leq 2c^{-1} \Phi(r/2) = \frac{c^{-1}r^2}{\int_0^r s \varphi(s/2) \, ds} \leq \varphi(r) = \frac{r^2}{\int_0^r s^{d+3} J_1(s) \, ds} \leq \frac{cr^2}{\int_0^r s \varphi(s) \, ds} = 2c\Phi(r), \ r < 1.
\]
Thus, $J_1$ satisfies \((3.5)\) with $\ell = 0$ and \((3.6)\) since $\Phi$ satisfies \((2.2)\). Combining $\varphi(r) \asymp \Phi(r)$ and Proposition \((3.13)\) for the process $Y$, we have that there is a constant $c_2 > 0$ satisfying
\[
\frac{-1}{2\pi r} \frac{\partial}{\partial r} p(t,r) = q_t(r) \leq c_2 t \mathcal{G}^{(d+2)}(t,r) \quad \text{and} \quad \left| \frac{d}{dr} q_t(r) \right| \leq c_2 t \Phi^{-1}(t)^{-1} \mathcal{G}^{(d+2)}(t,r)
\]
for any $0 < t \leq T$ and $r > 0$. From now on, assume $t \in (0,T]$ and $x \in \mathbb{R}^d$. Also, let $r = |x|$. Combining above inequalities and \((3.4)\) we have
\[
|\nabla_x^2 p(t,x)| = \left| \frac{\partial^2}{\partial r^2} p(t,r) + \frac{d - 1}{r} \frac{\partial}{\partial r} p(t,r) \right| = 2\pi \left| \frac{d}{dr} \left( -r q_t(r) + (d-1)q_t(r) \right) \right|
\leq 2\pi d \left( q_t(r) + r \left| \frac{d}{dr} q_t(r) \right| \right) \leq c_3 t \left( 1 + r \Phi^{-1}(t)^{-1} \right) \mathcal{G}^{(d+2)}(t,r)
\]
where we used \((2.4)\) for the last line. Thus, using \((2.3)\) we obtain
\[
|\nabla_x^2 p(t,x)| \leq c_4 t \mathcal{G}^{(d+2)}(t,r) \leq c_4 \Phi^{-1}(t)^{-d-2} = c_4 \Phi^{-1}(t)^{-2} \mathcal{G}_T(t,r), \quad r \leq \Phi^{-1}(t).
\]
Also, for $\Phi^{-1}(t) < r \leq \Phi^{-1}(T)$ we have
\[
|\nabla_x^2 p(t,x)| \leq \frac{2c_4 tr}{\Phi^{-1}(t)^{-d-2}} \mathcal{G}^{(d+2)}(t,r) \leq \frac{2c_4 tr^2}{\Phi^{-1}(t)^2} \mathcal{G}^{(d+2)}(t,r)
\leq 2c_4 \Phi^{-1}(t)^{-2} \frac{t}{r^d \Phi(r)} = 2c_4 \Phi^{-1}(t)^{-2} \mathcal{G}_T(t,r).
\]
Note that above estimates are valid for any $0 < \beta < 1$.

Now assume $0 < \beta < 1$. Let us recall that $J_1$ satisfies \((3.5)\) with $\ell = 1$ and \((3.6)\). Applying Proposition \((3.13)(c)\) for the process $Y$ we have
\[
q_t(r) \leq c_5 tr^{-1} \exp(-br^\beta) \quad \text{and} \quad \left| \frac{d}{dr} q_t(r) \right| \leq c_5 t \Phi^{-1}(t)^{-1} r^{-1} \exp(-br^\beta)
\]
for $r > \Phi^{-1}(T)$. Thus, by \((3.42)\)
\[
|\nabla_x^2 p(t,x)| \leq 2\pi d \left( q_t(r) + r \left| \frac{d}{dr} q_t(r) \right| \right)
\leq c_6 t \left( r^{-1} + t \Phi^{-1}(t)^{-1} \right) \exp(-br^\beta) \leq c_7 t \Phi^{-1}(t)^{-2} \mathcal{G}_T(t,r)
\]
for $r > \Phi^{-1}(T)$. Combining this with \((3.43)\), \((3.44)\) and \((2.4)\) we obtain
\[
|\nabla_x^2 p_t(x)| \leq c_8 t \Phi^{-1}(t)^{-2} \mathcal{G}_T(t,x) \leq c_9 \mathcal{G}_0^{\beta}(t,x), \quad 0 < t \leq T, x \in \mathbb{R}^d.
\]
This concludes (3.3) for $0 < \beta < 1$.

Similarly, for $\beta = 1$ using Proposition 3.13(b) for the process $Y$ we have

$$q_t(r) \leq c_{10} t \exp(-\frac{b}{4} r) \quad \text{and} \quad \left| \frac{d}{dr} q_t(r) \right| \leq c_{10} t \Phi^{-1}(t)^{-1} \exp(-\frac{b}{4} r)$$

for $r > \Phi^{-1}(T)$. Thus,

$$|\nabla^2 p(t, x)| \leq 2\pi d \left( q_t(|x|) + |x| \left| \frac{d}{dr} q_t(|x|) \right| \right) \leq c_{11} t \left( \Phi^{-1}(t)^{-1} + r \right) \exp(-\frac{b}{4} r)$$

$$\leq c_{12} t \Phi^{-1}(t)^{-2} \exp(-\frac{b}{5} |x|) \leq c_{13} t \Phi^{-1}(t)^{-2} \mathcal{G}(t, r), \quad r > \Phi^{-1}(T).$$

Hence, combining this with (3.43), (3.44) and (2.4) we obtain

$$|\nabla^2 p(t, x)| \leq c_{14} t \Phi^{-1}(t)^{-2} \mathcal{G}(t, x) \leq c_{15} t \mathcal{G}(t, x), \quad 0 < t \leq T, x \in \mathbb{R}^d,$$

which is our desired result for $\beta = 1$. \qed

4 Further properties of heat kernel for isotropic Lévy process

In this section we assume that $J$ satisfies (1.4) with $0 < \beta \leq 1$ and (1.5), and that nondecreasing function $\phi$ satisfies (1.6) and (1.7). As in the previous section, let $X$ be an isotropic unimodal Lévy process with jumping kernel $J(|y|)dy$ and $p(t, x)$ be the transition density of $X$. Also, let $L$ be an infinitesimal generator of $X$.

Recall that $\delta_f$ is defined in (1.23). The following results are counterpart of [KSV18, Proposition 3.3].

**Proposition 4.1** For every $T \geq 1$, there exists a constant $0 < c = c(d, T, a, a_1, \alpha_1, b, \beta, C_0)$ such that for every $t \in (0, T]$ and $x, y, z \in \mathbb{R}^d$,

$$|p(t, x) - p(t, y)| \leq c \left( \frac{|x - y|}{\Phi^{-1}(t)} \wedge 1 \right) t \left( \mathcal{G}(t, x) + \mathcal{G}(t, y) \right), \quad (4.1)$$

and

$$|\delta_p(t, x; z)| \leq c \left( \frac{|z|}{\Phi^{-1}(t)} \wedge 1 \right)^2 t \left( \mathcal{G}(t, x \pm z) + \mathcal{G}(t, x) \right), \quad (4.2)$$

**Proof.** (a) Since (4.1) is clearly true when $\Phi^{-1}(t) \leq |x - y|$ by (3.3), we assume that $\Phi^{-1}(t) \geq |x - y|$. Let $\alpha(\theta) = x + \theta(y - x), \theta \in [0, 1]$ be a segment from $x$ to $y$. Then, for any $\theta \in [0, 1]$ we have

$$|\alpha(\theta)| \geq |x| - |x - \alpha(\theta)| \geq |x| - |x - y| \geq |x| - \Phi^{-1}(t),$$
here we used $|x - y| \leq \Phi^{-1}(t)$ for the last inequality. Thus, we obtain

$$|p(t, x) - p(t, y)| = |\int_0^1 \alpha'(\theta) \cdot \nabla_x p(t, \alpha(\theta)) \, d\theta| \leq \int_0^1 |\alpha'(\theta)||\nabla_x p(t, \alpha(\theta))| \, d\theta$$

$$\leq c_1 \int_0^1 |\alpha'(\theta)|\Phi^{-1}(t)^{-2}t\mathcal{G}(t, x) \, d\theta \leq c_1|x - y|\Phi^{-1}(t)^{-1}t\mathcal{G}(t, |x| - \Phi^{-1}(t))$$

$$\leq c_2 |x - y|\Phi^{-1}(t)^{-1}t\mathcal{G}(t, x).$$

Here we used \((\ref{3.3})\) with $k = 1$ for the second line and \((\ref{2.10})\) for the last line. This concludes \((\ref{4.1})\).

Note that using \((\ref{3.3})\) for $k = 2$ and following the same argument as the above we can estimate $|\nabla p(t, x) - \nabla p(t, y)|$. Hence, we have a constant $c_3 > 0$ satisfying

$$|\nabla p(t, x) - \nabla p(t, y)| \leq c_3 |x - y|\Phi^{-1}(t)^{-2}t(\mathcal{G}(t, x) + \mathcal{G}(t, y))$$

\((\ref{4.3})\)

for $0 < t \leq T$ and $|x - y| \leq \Phi^{-1}(t)$.

(b) \((\ref{4.2})\) is clearly true when $\Phi^{-1}(t) \leq 2|z|$. Now assume $\Phi^{-1}(t) \geq 2|z|$. Let $\alpha(\theta) = x + \theta z$, $\theta \in [-1, 1]$ be a segment from $x - z$ to $x + z$. Then, for any $\theta \in [-1, 1]$ we have $|\alpha(\theta)| \geq |x| - \Phi^{-1}(t)/2$, hence

$$|\delta p(t, x; z)| = |(p(t, x) - p(t, x - z)) - (p(t, x + z) - p(t, x))|$$

$$= |\int_0^1 \alpha'(\theta) \cdot \nabla p(t, \alpha(\theta)) - \alpha'(-\theta) \cdot \nabla p(t, \alpha(-\theta)) \, d\theta|$$

$$= |\int_0^1 z \cdot (\nabla p(t, \alpha(\theta)) - \nabla p(t, \alpha(-\theta))) \, d\theta|$$

$$\leq 4c_3 |z|\Phi^{-1}(t)^{-2} \left(|z|t\mathcal{G}(t, |x| - \Phi^{-1}(t))\right)$$

$$\leq c_4 \Phi^{-1}(t)^{-2}|z|^2t\mathcal{G}(t, x).$$

Here we used $|\alpha(\theta) - \alpha(-\theta)| \leq 2|z| \leq \Phi^{-1}(t)$ and \((\ref{4.3})\) for the first inequality, and \((\ref{2.10})\) for the second one.

\[ \square \]

**Proposition 4.2** For every $T \geq 1$, there exist constants $c_i = c_i(d, T, a, a_1, \alpha_1, b, \beta, C_0) > 0$, $i = 1, 2$, such that for any $t \in (0, T]$ and $x \in \mathbb{R}^d$,

$$\int_{\mathbb{R}^d} |\delta p(t, x; z)| J(|z|) \, dz \leq c_1 \int_{\mathbb{R}^d} \left(\frac{|z|}{\Phi^{-1}(t)}\right)^2 t \left(\mathcal{G}(t, x \pm z) + \mathcal{G}(t, x)\right) J(|z|) \, dz \leq c_2 \mathcal{G}(t, x)$$

\((\ref{4.4})\)

**Proof.** By \((\ref{4.2})\) we have

$$\int_{\mathbb{R}^d} |\delta p(t, x; z)| J(|z|) \, dz$$

$$\leq c_1 \int_{\mathbb{R}^d} \left(\frac{|z|}{\Phi^{-1}(t)}\right)^2 t \left(\mathcal{G}(t, x \pm z) + \mathcal{G}(t, x)\right) J(|z|) \, dz$$

$$\leq c_2 \left(\int_{\mathbb{R}^d} \left(\frac{|z|}{\Phi^{-1}(t)}\right)^2 t\mathcal{G}(t, x \pm z) J(|z|) \, dz + t\mathcal{G}(t, x)P(\Phi^{-1}(t))\right)$$

$$=: c_2 (I_1 + I_2)$$

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Clearly, by (3.11) we have
\[ I_2 \leq c_3 \mathcal{G}(t, x). \] (4.6)

To estimate \( I_1 \), we divide \( I_1 \) into two parts as
\[
I_1 = \int_{|z| \leq \Phi^{-1}(t)} \left( \frac{|z|}{\Phi^{-1}(t)} \right)^2 t \mathcal{G}(t, x + z) J(|z|) \, dz + \int_{|z| > \Phi^{-1}(t)} t \mathcal{G}(t, x + z) J(|z|) \, dz =: I_{11} + I_{12}.
\]

By using (2.10) in the first inequality below and (3.11) in the third, we have
\[
I_{11} \leq c_4 t \mathcal{G}(t, x) \int_{|z| \leq \Phi^{-1}(t)} \left( \frac{|z|^2}{\Phi^{-1}(t)^2} \wedge 1 \right) J(|z|) \, dz \leq c_4 t \mathcal{G}(t, x) \mathcal{P}((\Phi^{-1}(t)) \leq c_5 \mathcal{G}(t, x).
\]

For the estimates of \( I_{12} \), we will use
\[
J(|z|) \leq c_6 \theta(|z|) = c_6 \mathcal{G}_T(t, z), \quad |z| > \Phi^{-1}(t),
\]
which follows from (1.4) and (2.1). Using (2.4), (4.7) and (2.21), we arrive
\[
I_{12} \leq c_6 at \int_{|z| > \Phi^{-1}(t)} \mathcal{G}(t, x - z) \mathcal{G}(t, z) \, dz \leq c_7 \mathcal{G}(t, x).
\]

Here we used (2.9) for the last inequality. The lemma follows from the estimates of \( I_{11}, I_{12} \) and \( I_2 \). \( \square \)

### 4.1 Dependency of \( p^K \) in terms of \( \mathfrak{K} \)

Recall that
\[
\mathcal{L}^\kappa f(x) = \lim_{\epsilon \downarrow 0} \int_{|z| > \epsilon} (f(x + z) - f(x)) \kappa(x, z) J(|z|) \, dz
\]
where \( J : \mathbb{R}_+ \to \mathbb{R}_+ \) is a non-increasing function satisfying (1.4) and (1.5) with strictly increasing function \( \phi \) satisfying (1.6) and (1.7).

Let \( \mathfrak{K} : \mathbb{R}^d \to (0, \infty) \) be a symmetric function satisfying
\[
\kappa_0 \leq \mathfrak{K}(z) \leq \kappa_1 \quad \text{for all } z \in \mathbb{R}^d
\]
where \( \kappa_0 \) and \( \kappa_1 \) are constants in (1.2). We denote \( Z^\mathfrak{K} \), symmetric Lévy process whose jumping kernel is given by \( \mathfrak{K}(z) J(|z|), z \in \mathbb{R}^d \). Then the infinitesimal generator of \( Z^\mathfrak{K} \) is a self-adjoint operator in \( L^2(\mathbb{R}^d) \) and is of the following form:
\[
\mathcal{L}^\mathfrak{K} f(x) = \lim_{\epsilon \downarrow 0} \int_{|z| > \epsilon} (f(x + z) - f(x)) \mathfrak{K}(z) J(|z|) \, dz
\]
\[
= \frac{1}{2} \lim_{\epsilon \downarrow 0} \int_{|z| > \epsilon} (f(x + z) + f(x - z) - 2f(x)) \mathfrak{K}(z) J(|z|) \, dz. \quad (4.9)
\]
\( (1.4) \) implies that when \( f \in C^2_b(\mathbb{R}^d) \), it is not necessary to take the principal value in the last line in \( (4.9) \). The transition density of \( Z^\Phi \) (i.e., the heat kernel of \( \mathcal{L}^\Phi \)) will be denoted by \( p^\Phi(t, x) \). In this section, we will observe further properties of \( p^\Phi(t, x) \).

**Remark 4.3** The operator \( (1.1) \) satisfies all conditions in \( [KSV18] \) with respect to the function \( ˜G(t, x) \) and \( \Phi(r^{-1})^{-1} \) except \( [KSV18, (1.7)] \): Recall from Remark 2.3 that \( ˜G(t, x) \) is comparable to the function \( \rho(t, x) \) in \( [KSV18] \). Moreover, by Lemma 3.3, the characteristic exponent of any symmetric Lévy process whose jumping kernel comparable to \( J(|z|) \), is comparable to \( \Phi(r^{-1})^{-1} \). Clearly \((2.2)\) and \([KSV18, (1.4)]\) imply \([KSV18, (1.4), (1.5)\) and \((1.9)]\). Also, we obtain gradient estimates with respect to \( ˜G(t, x) \) in Proposition 3.9, which are same as the gradient estimates in \([KSV18, Proposition 3.2]\). Under these observations, one can follow the proofs of \([KSV18, Theorems 1.1–1.3]\) using \((1.4)\) instead of the condition \([KSV18, (1.7)]\) and see that \([KSV18, Theorems 1.1–1.3]\) hold under our setting.

Using the above Remark 4.3, from the remainder this paper we use \([KSV18, Theorems 1.1–1.3]\) without any further remark.

Let \( \hat{\mathcal{R}} := \mathcal{R} - \frac{\kappa_0}{2} \). Then, \( \frac{\kappa_0}{2} \leq \hat{\mathcal{R}}(z) \leq \kappa_1 \). Let \( p^\hat{\mathcal{R}} \) be the heat kernel of symmetric Lévy process \( Z^\hat{\mathcal{R}} \) whose jumping kernel is \( \hat{\mathcal{R}}(z) J(|z|)dz \) and \( p^{\kappa_0}(t, x) = p\left(\frac{\kappa_0}{2} t, x\right) \) be the heat kernel of symmetric Lévy process \( Z^{\kappa_0} \) whose jumping kernel is \( \frac{\kappa_0}{2} J(|z|)dz \). Without loss of generality, we can assume that \( Z^\hat{\mathcal{R}} \) and \( Z^{\kappa_0} \) are independent. By \([GKK15, Theorem 1.2]\), there exists a constant \( c = c(T) = c(d, T, a, a_1, \alpha_1, b, \beta, C_0, \kappa_0, \kappa_1) > 0 \) such that

\[
P^\hat{\mathcal{R}}(t, x) \leq ct \Phi(t^{-1})^{-1} G(t, x) \quad \text{for all} \quad 0 < t \leq T, \ x \in \mathbb{R}^d
\]

for every \( \mathcal{R} \) satisfying \((4.8)\). Also, by Remark 4.3 we have \([KSV18, (3.21)]\). We record this for the readers:

\[
\frac{\partial p^\mathcal{R}(t, x)}{\partial t} = \mathcal{L}^\mathcal{R} p^\mathcal{R}(t, x), \quad \lim_{t \downarrow 0} p^\mathcal{R}(t, x) = \delta_0(x).
\]

(4.11)

Since \( Z^\hat{\mathcal{R}} \) and \( Z^{\kappa_0} \) are independent, \( Z^\hat{\mathcal{R}} \) and \( Z^\hat{\mathcal{R}} + Z^{\kappa_0} \) have same distributions. Thus, we have

\[
\begin{align*}
p^\hat{\mathcal{R}}(t, x) &= \int_{\mathbb{R}^d} p^{\kappa_0}(t, x - y)p^\hat{\mathcal{R}}(t, y)dy \\
&= \int_{\mathbb{R}^d} p\left(\frac{\kappa_0}{2} t, x - y\right)p^\hat{\mathcal{R}}(t, y)dy.
\end{align*}
\]

(4.12)

First we extend Propositions 3.2 and \((4.1)\) \((4.2)\)

**Proposition 4.4** There exists a constant \( c = c(d, T, a, a_1, \alpha_1, b, \beta, C_0, \kappa_0, \kappa_1) > 0 \) such that for any \( t \in (0, T] \) and \( x, y, z \in \mathbb{R}^d \),

\[
|\nabla_x p^\mathcal{R}(t, x)| \leq ct \Phi(t^{-1})^{-1} G(t, x)
\]

(4.13)
\[ |p^R(t, x) - p^R(t, y)| \leq ct(\Phi^{-1}(t)^{-1}|x - y| \wedge 1)(G(t, x) + G(t, y)) \] (4.14)

\[ |\delta_{p^R}(t, x; z)| \leq ct((\Phi^{-1}(t)^{-1}|z|)^2 \wedge 1)(G(t, x \pm z) + G(t, x)) \] (4.15)

\[
\int_{\mathbb{R}^d} |\delta_{p^R}(t, x; z)| J(|z|) dz \leq cG(t, x)
\] (4.16)

**Proof.** (a) Using (4.12), (4.11), (3.3), (2.21) and (2.9) for each line, we obtain

\[
|\nabla_x p^R(t, x)| = |\nabla_x \int_{\mathbb{R}^d} p(\frac{K_0}{2} t, x - y)p^\hat{A}(t, y) dy| 
\leq |\int_{\mathbb{R}^d} \nabla_x p(\frac{K_0}{2} t, x - y) tG(t, y) dy| 
\leq c_1 \int_{\mathbb{R}^d} t \Phi^{-1}(t)^{-1} G(\frac{K_0}{2} t, x - y) \times tG(t, y) dy 
\leq c_2 t \Phi^{-1}(t)^{-1} G((1 + \frac{K_0}{2}) t, x) 
\leq c_3 t \Phi^{-1}(t)^{-1} G(t, x).
\]

(b) Using (4.12), (4.11), (4.1), (2.21) and (2.9) we obtain

\[
|p^R(t, x) - p^R(t, y)| \leq \int_{\mathbb{R}^d} |p(\frac{K_0}{2} t, x - z) - p(\frac{K_0}{2} t, y - z)| p^\hat{A}(t, z) dz 
\leq c_1 \int_{\mathbb{R}^d} t(\frac{|x - y|}{\Phi^{-1}(t)} \wedge 1)(G(\frac{K_0}{2} t, x - z) + G(\frac{K_0}{2} t, y - z)) \times tG(t, z) dz 
\leq c_2 t(\frac{|x - y|}{\Phi^{-1}(t)} \wedge 1)(G((1 + \frac{K_0}{2}) t, x) + G((1 + \frac{K_0}{2}) t, y)) 
\leq c_3 t(\frac{|x - y|}{\Phi^{-1}(t)} \wedge 1)(G(t, x) + G(t, y)).
\]

(c) We use (4.12), (4.11), (4.2), (2.21) and (2.9) for each line to estimate \( |\delta_{p^R}(t, x; z)| \).

\[
|\delta_{p^R}(t, x; z)| \leq \int_{\mathbb{R}^d} |\delta_{p}(\frac{K_0}{2} t, x - y; z)| p^\hat{A}(t, y) dy 
\leq c_1 \int_{\mathbb{R}^d} t(\Phi^{-1}(t)^{-1}|z|)^2 \wedge 1)(G(\frac{K_0}{2} t, x - y \pm z) + G(\frac{K_0}{2} t, y - z)) \times tG(t, y) dy 
\leq c_2 t(\Phi^{-1}(t)^{-1}|z|)^2 \wedge 1)(G((1 + \frac{K_0}{2}) t, x \pm z) + G((1 + \frac{K_0}{2}) t, y)) 
\leq c_3 t(\Phi^{-1}(t)^{-1}|z|)^2 \wedge 1)(G(t, x \pm z) + G(t, x)).
\]

(d) We use (4.12), Fubini’s theorem, (4.11), (4.1), (2.21) and (2.9) for each line to estimate \( \int_{\mathbb{R}^d} |\delta_{p^R}(t, x; z)| J(|z|) dz. \)

\[
\int_{\mathbb{R}^d} |\delta_{p^R}(t, x; z)| J(|z|) dz \leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\delta_{p}(\frac{K_0}{2} t, x - y; z)| p^\hat{A}(t, y) dy J(|z|) dz 
= \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |\delta_{p}(\frac{K_0}{2} t, x - y; z)| J(|z|) dz \right) p^\hat{A}(t, y) dy
\]
Next, we obtain continuity of transition density with respect to $\mathcal{F}$. This is the counterpart of [KSV18, Theorem 3.5].

**Theorem 4.5** There exists a constant $c = c(d, T, a, a_1, \alpha_1, b, \beta, C_0, \kappa_0, \kappa_1) > 0$ such that for any two symmetric functions $\mathcal{F}_1$ and $\mathcal{F}_2$ in $\mathbb{R}^d$ satisfying (4.8), any $t \in (0, T]$ and $x \in \mathbb{R}^d$, we have

\[
\begin{align*}
&\left| p^{\mathcal{F}_1}(t, x) - p^{\mathcal{F}_2}(t, x) \right| \leq c\|\mathcal{F}_1 - \mathcal{F}_2\|_{\infty} \Phi(t, x), \\
&\left| \nabla p^{\mathcal{F}_1}(t, x) - \nabla p^{\mathcal{F}_2}(t, x) \right| \leq c\|\mathcal{F}_1 - \mathcal{F}_2\|_{\infty} \Phi^{-1}(t)^{-1}\Phi(t, x)
\end{align*}
\]

and

\[
\int_{\mathbb{R}^d} \left| \delta_{p^{\mathcal{F}_1}}(t, x; z) - \delta_{p^{\mathcal{F}_2}}(t, x; z) \right| J(|z|)dz \leq c\|\mathcal{F}_1 - \mathcal{F}_2\|_{\infty} \Phi(t, x).
\]

**Proof.** (a) $p^{\mathcal{F}_1}(s, y)$ is uniformly bounded on $s \in [t/2, t]$ by (4.10) and $\lim_{s \to t} p^{\mathcal{F}_2}(t, x - y) = \delta_0(x - y)$ by (4.11). Thus, we have

\[
\lim_{s \to t} \int_{\mathbb{R}^d} p^{\mathcal{F}_1}(s, y)p^{\mathcal{F}_2}(t - s, x - y) dy = p^{\mathcal{F}_1}(t, x).
\]

By the similar way, we get

\[
\lim_{s \to 0} \int_{\mathbb{R}^d} p^{\mathcal{F}_1}(s, y)p^{\mathcal{F}_2}(t - s, x - y) dy = p^{\mathcal{F}_2}(t, x).
\]

Hence, for $t \in (0, T]$ and $x \in \mathbb{R}^d$,

\[
\left| p^{\mathcal{F}_1}(t, x) - p^{\mathcal{F}_2}(t, x) \right| = \left| \int_0^t \frac{d}{ds} \left( \int_{\mathbb{R}^d} p^{\mathcal{F}_1}(s, y)p^{\mathcal{F}_2}(t - s, x - y) dy \right) ds \right|.
\]

Using (4.11) in the second line, the fact that $\mathcal{L}^{\mathcal{F}_1}$ is self-adjoint in the third line and (4.19) in the fourth line, we have

\[
\begin{align*}
&\int_0^{t/2} \frac{d}{ds} \left( \int_{\mathbb{R}^d} p^{\mathcal{F}_1}(s, y)p^{\mathcal{F}_2}(t - s, x - y) dy \right) ds \\
&= \int_0^{t/2} \left( \int_{\mathbb{R}^d} \left( \mathcal{L}^{\mathcal{F}_1} p^{\mathcal{F}_1}(s, y)p^{\mathcal{F}_2}(t - s, x - y) - p^{\mathcal{F}_1}(s, y)\mathcal{L}^{\mathcal{F}_2} p^{\mathcal{F}_2}(t - s, x - y) \right) dy \right) ds \\
&= \int_0^{t/2} \left( \int_{\mathbb{R}^d} p^{\mathcal{F}_1}(s, y) \left( \mathcal{L}^{\mathcal{F}_1} - \mathcal{L}^{\mathcal{F}_2} \right) p^{\mathcal{F}_2}(t - s, x - y) dy \right) ds \\
&= \frac{1}{2} \int_0^{t/2} \left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} \delta_{p^{\mathcal{F}_2}}(t - s, x - y; z)(\mathcal{F}_1(z) - \mathcal{F}_2(z))J(|z|)dz \right) dy \right) ds.
\end{align*}
\]
Therefore, we arrive

\[
\left| \int_0^{t/2} \frac{d}{ds} \left( \int_{\mathbb{R}^d} p_{\mathcal{R}_1}(s, y)p_{\mathcal{R}_2}(t-s, x-y) \, dy \right) \, ds \right| \\
\leq \frac{1}{2} \| \hat{\mathcal{R}}_1 - \hat{\mathcal{R}}_2 \|_\infty \int_0^{t/2} \left( \int_{\mathbb{R}^d} p_{\mathcal{R}_1}(s, y) \left( \int_{\mathbb{R}^d} |\delta_{p,p}(t-s, x-y; z)| J(|z|) \, dz \right) \, dy \right) \, ds \\
\leq c_1 \| \hat{\mathcal{R}}_1 - \hat{\mathcal{R}}_2 \|_\infty \int_0^{t/2} \int_{\mathbb{R}^d} sG(s, y) \mathcal{G}(t-s, x-y) \, dy \, ds \\
\leq c_2 \| \hat{\mathcal{R}}_1 - \hat{\mathcal{R}}_2 \|_\infty \int_0^{t/2} \int_{\mathbb{R}^d} (s^{-1} + (t-s)^{-1}) \mathcal{G}(t, x) \, ds \leq c_3 \| \hat{\mathcal{R}}_1 - \hat{\mathcal{R}}_2 \| t \mathcal{G}(t, x),
\]

for all \( t \in (0, T] \) and \( x \in \mathbb{R}^d \). By the similar way, we also obtain

\[
\left| \int_{t/2}^t \frac{d}{ds} \left( \int_{\mathbb{R}^d} p_{\mathcal{R}_1}(s, y)p_{\mathcal{R}_2}(t-s, x-y) \, dy \right) \, ds \right| \\
= \frac{1}{2} \left| \int_{t/2}^t \left( \int_{\mathbb{R}^d} p_{\mathcal{R}_2}(t-s, y) \left( \int_{\mathbb{R}^d} \delta_{p,p}(s, x-y; z)(\hat{\mathcal{R}}_1(z) - \hat{\mathcal{R}}_2(z))J(|z|) \, dz \right) \, dy \right) \, ds \right| \\
\leq c_1 \| \hat{\mathcal{R}}_1 - \hat{\mathcal{R}}_2 \|_\infty \int_{t/2}^t \int_{\mathbb{R}^d} (t-s)G(s, y) \mathcal{G}(t-s, x-y) \, dy \, ds \\
\leq c_3 \| \hat{\mathcal{R}}_1 - \hat{\mathcal{R}}_2 \|_\infty t \mathcal{G}(t, x).
\]

Therefore, we arrive

\[
|p_{\mathcal{R}_1}(t, x) - p_{\mathcal{R}_2}(t, x)| \\
\leq \int_0^{t/2} \frac{d}{ds} \left( \int_{\mathbb{R}^d} p_{\mathcal{R}_1}(s, y)p_{\mathcal{R}_2}(t-s, x-y) \, dy \right) \, ds + \int_{t/2}^t \frac{d}{ds} \left( \int_{\mathbb{R}^d} p_{\mathcal{R}_1}(s, y)p_{\mathcal{R}_2}(t-s, x-y) \, dy \right) \, ds \\
\leq 2c_3 \| \hat{\mathcal{R}}_1 - \hat{\mathcal{R}}_2 \|_\infty t \mathcal{G}(t, x).
\]

(b) Set \( \hat{\mathcal{R}}_i(z) := \mathcal{R}_i(z) - \kappa_0/2, \ i = 1, 2 \). Using (4.12), (3.3), (4.17), (2.21) and (2.9), we have that for all \( t \in (0, T] \) and \( x \in \mathbb{R}^d \),

\[
|\nabla p_{\mathcal{R}_1}(t, x) - \nabla p_{\mathcal{R}_2}(t, x)| = \left| \int_{\mathbb{R}^d} \nabla p \left( \frac{\kappa_0}{2} t, x-y \right) \left( p_{\mathcal{R}_1}(t, y) - p_{\mathcal{R}_2}(t, y) \right) \, dy \right| \\
\leq c_1 \| \hat{\mathcal{R}}_1 - \hat{\mathcal{R}}_2 \|_\infty \hat{\Phi}^{-1}(t)^{-1} t^2 \int_{\mathbb{R}^d} \mathcal{G} \left( \frac{\kappa_0}{2} t, x-y \right) \mathcal{G}(t, y) \, dy \\
\leq c_2 \| \hat{\mathcal{R}}_1 - \hat{\mathcal{R}}_2 \|_\infty \hat{\Phi}^{-1}(t)^{-1} t^2 t^{-1} \mathcal{G}(1 + \frac{\kappa_0}{2} t, x) \\
\leq c_3 \| \hat{\mathcal{R}}_1 - \hat{\mathcal{R}}_2 \|_\infty \hat{\Phi}^{-1}(t)^{-1} t \mathcal{G}(t, x).
\]

(c) By using (4.12), (4.2), (4.17), (2.21) and (2.9) we have that for any \( t \in (0, T] \) and \( x, z \in \mathbb{R}^d \),

\[
|\delta_{p_{\mathcal{R}_1}}(t, x; z) - \delta_{p_{\mathcal{R}_2}}(t, x; z)| = \left| \int_{\mathbb{R}^d} \delta_p \left( \frac{\kappa_0}{2} t, x-y \right) \left( p_{\mathcal{R}_1}(t, y) - p_{\mathcal{R}_2}(t, y) \right) \, dy \right| \
\]

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\[
\begin{align*}
&\leq c_1\|\mathcal{R}_1 - \mathcal{R}_2\|_\infty (\Phi^{-1}(t)^{-1}|z| \wedge 1)^2 t^2 \int_{\mathbb{R}^d} \left(\mathcal{G}(\frac{K_0}{2} t, x - y \pm z) + \mathcal{G}(t, x - y)\mathcal{G}(t, y) dy \right) \\
&\leq c_2\|\mathcal{R}_1 - \mathcal{R}_2\|_\infty (\Phi^{-1}(t)^{-1}|z| \wedge 1)^2 t^{-1} \mathcal{G}((1 + \frac{K_0}{2}) t, x \pm z) + \mathcal{G}((1 + \frac{K_0}{2}) t, x) \\
&\leq c_3\|\mathcal{R}_1 - \mathcal{R}_2\|_\infty (\Phi^{-1}(t)^{-1}|z| \wedge 1)^2 t (\mathcal{G}(t, x \pm z) + \mathcal{G}(t, x)).
\end{align*}
\]

Integrating above inequality we obtain that
\[
\int_{\mathbb{R}^d} |\delta_{p^{\kappa_1}}(t, x; z) - \delta_{p^{\kappa_2}}(t, x; z)| J(|z|) dz \\
\leq c_3 t\|\mathcal{R}_1 - \mathcal{R}_2\|_\infty \int_{\mathbb{R}^d} \left(\frac{|z|}{\Phi^{-1}(t)} \wedge 1\right)^2 (\mathcal{G}(t, x \pm z) + \mathcal{G}(t, x)) J(|z|) dz \\
\leq c_4\|\mathcal{R}_1 - \mathcal{R}_2\|_\infty \mathcal{G}(t, x),
\]
where the last inequality follows from Proposition 4.2.

Estimates in this section are almost same with [KSV18, Section 2 and 3] except these: First of all, the function \(\mathcal{G}\) is different from [KSV18], hence our estimates are more precise than estimates in [KSV18]. However, we don’t have estimates for third derivatives in terms of \(\mathcal{G}\) of the heat kernel in Proposition 3.2. Thus, we do not have the improvements on [KSV18, (3.14) and (3.18)], which are used for the gradient estimate of the function \(p^{\kappa}(t, x, y)\) in Theorems 1.1-1.4, for instance, [KSV18, Theorem 1.1(2) and 1.4(4)].

5 Estimates of \(p^{\kappa}(t, x, y)\)

For the remainder of this paper, we always assume that \(\kappa : \mathbb{R}^d \times \mathbb{R}^d \to (0, \infty)\) is a Borel function satisfying (1.2) and (1.3), that \(J\) satisfies (1.4)-(1.5) with the function \(\phi\) satisfying (1.6) and (1.7).

For a fixed \(y \in \mathbb{R}^d\), let \(\mathcal{R}_y(z) = \kappa(y, z)\) and let \(\mathcal{L}^{\mathcal{R}_y}\) be the freezing operator defined by
\[
\mathcal{L}^{\mathcal{R}_y} f(x) = \lim_{\varepsilon \downarrow 0} \int_{|z| > \varepsilon} \delta_f(x; z) \kappa(y, z) J(|z|) dz.
\]

Let \(p_y(t, x) := p^{\mathcal{R}_y}(t, x)\) be the heat kernel of the operator \(\mathcal{L}^{\mathcal{R}_y}\). Note that \(\mathcal{R}_y\) satisfies (1.8) so that there exists a constant \(c > 0\) such that
\[
p_y(t, x) \leq c t \mathcal{G}(t, x) \quad \text{for all } x, y \in \mathbb{R}^d, t \in (0, T].
\]

By Remark 1.3 and [KSV18, Theorem 1.1], we have a continuous function \(p^{\kappa}(t, x, y)\) on \((0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d\) solving (1.11) and it satisfies
\[
p^{\kappa}(t, x, y) \leq c t \mathcal{G}(t, x - y), \quad 0 < t \leq T \text{ and } x \in \mathbb{R}^d.
\]

In this section, we will investigate further estimates and regularity of \(p^{\kappa}(t, x, y)\). We first recall the construction of \(p^{\kappa}\) from [KSV18, section 4]. For \(t > 0\) and \(x, y \in \mathbb{R}^d\), define
\[
g_0(t, x, y) := \frac{1}{2} \int_{\mathbb{R}^d} \delta_{p_y}(t, x - y; z)(\kappa(x, z) - \kappa(y, z)) J(|z|) dz = (\mathcal{L}^{\mathcal{R}_y} - \mathcal{L}^{\mathcal{R}_y}) p_y(t, \cdot)(x - y).
\]
For $n \in \mathbb{N}$, we inductively define the function $q_n(t, x, y)$ by

$$q_n(t, x, y) := \int_0^t \int_{\mathbb{R}^d} q_0(t-s, x, z)q_{n-1}(s, z, y)dzds$$ (5.4)

and

$$q(t, x, y) := \sum_{n=0}^{\infty} q_n(t, x, y).$$ (5.5)

Finally we define

$$\phi_y(t, x) := \int_0^t \phi_y(t, x, s)ds = \int_0^t \int_{\mathbb{R}^d} p_z(t-s, x-z)q(s, z, y)dzds$$ (5.6)

and

$$p^\kappa(t, x, y) := p_y(t, x-y) + \phi_y(t, x) = p_y(t, x-y) + \int_0^t \int_{\mathbb{R}^d} p_z(t-s, x-z)q(s, z, y)dzds.$$ (5.7)

As [KSV18, section 4], the definitions in (5.3)–(5.7) are well-defined. In other words, each integrand in (5.3)–(5.7) is integrable and series in (5.5) is absolutely converge on $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$.

In the next lemma, we will establish the upper bounds of $p^\kappa$.

**Theorem 5.1** For every $T \geq 1$ and $\delta_0 \in (0, \delta] \cap (0, \alpha_1/2)$, there are constants $c_1$ and $c_2$ such that for any $t \in (0, T]$ and $x, y \in \mathbb{R}^d$,

$$|\phi_y(t, x)| \leq c_1 t \left( G_0^{\delta_0 + \delta_0} \right)(t, x-y)$$ (5.8)

and

$$p^\kappa(t, x, y) \leq c_2 t G(t, x-y).$$ (5.9)

The constants $c_1$ and $c_2$ depend on $d, T, a, \alpha_1, b, \beta, C_0, \delta_0, \delta, \kappa_0, \kappa_1$ and $\kappa_2$.

**Proof.** We first claim that for $n \in \mathbb{N}_0$,

$$|q_n(t, x, y)| \leq d_n \left( G_0^{\delta_0} + G_0^{\delta_0} \right)(t, x-y)$$ (5.10)

with

$$d_n := (16C(\delta_0, T)c_2)^{n+1} \prod_{k=1}^{n} B(\delta_0/2, k\delta_0/2) = (16C_0c_2)^{n+1} \frac{\Gamma(\delta_0/2)^{n+1}}{\Gamma((n+1)\delta_0/2)}$$

where $C = C(\delta_0, T)$ is the constant in (2.20). Without loss of generality, we assume that $C \geq 1/16$.

For $n = 0$, using (5.3), (1.2), (1.3) and (4.16) we have

$$|q_0(t, x, y)| \leq \frac{1}{2} \int_{\mathbb{R}^d} |\delta_{p_y}(t, x-y; z)(\kappa(x, z) - \kappa(y, z))]J(|z|)dz$$

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\[ \leq c_1 (|x - y|^\delta_0 \wedge 1) \int_{\mathbb{R}^d} |\delta_p(t, x - y; z)| J(|z|)dz \]
\[ \leq c_2 (|x - y|^\delta_0 \wedge 1) \mathcal{G}(t, x - y) = c_2 \mathcal{G}^\delta_0 (t, x - y). \]

Suppose that (5.10) is valid for \( n \). Then for \( t \leq T \),
\[ |q_{n+1}(t, x, y)| \leq \int_0^t \int_{\mathbb{R}^d} |q_0(t - s, x, z)q_n(s, z, y)| dz ds \]
\[ \leq c_2 d_n \int_0^t \int_{\mathbb{R}^d} \mathcal{G}^\delta_0(t - s, x - z) (\mathcal{G}^0_{(n+1)} + \mathcal{G}^\delta_0_{n\delta_0})(s, z - y) dz ds \]
\[ \leq 16 C c_2 d_n B(\delta_0/2, (n + 1)\delta_0/2) (\mathcal{G}^0_{(n+2)\delta_0} + \mathcal{G}^\delta_0_{(n+1)\delta_0})(t, x - y) \]
\[ = d_{n+1}(\mathcal{G}^0_{(n+1)\delta} + \mathcal{G}^\delta_0_{(n+1)\delta_0})(t, x - y) \]

Here we used induction in the second line, and used (2.22) and (2.23) in the last line. For the third line, we need the following: let \( \theta = \eta = 1, \gamma_1 = \delta_1 = 0, \delta_1 = \delta_0, \gamma_2 = (n + 1)\delta_0 \) which satisfy conditions in Lemma 2.8(c) since \( \delta_0 \in (0, \alpha_1/2) \). Then, by (2.22) we have
\[ \int_0^t \int_{\mathbb{R}^d} \mathcal{G}^\delta_0(t - s, x - z) \mathcal{G}^0_{(n+1)\delta_0}(s, z - y) dz ds \]
\[ \leq 4 C B(\delta_0/2, (n + 1)\delta_0/2) (\mathcal{G}^0_{(n+2)\delta_0} + \mathcal{G}^\delta_0_{(n+1)\delta_0} + \mathcal{G}^0_{(n+2)\delta_0})(t, x - y) \]
\[ \leq 8 C B(\delta_0/2, (n + 1)\delta_0/2) (\mathcal{G}^0_{(n+2)\delta_0} + \mathcal{G}^\delta_0_{(n+1)\delta_0})(t, x - y). \]

Also, letting \( \theta = \eta = 1, \gamma_1 = 0, \delta_1 = \delta_2 = \delta_0 \) and \( \gamma_2 = \delta_0 \) which satisfy conditions in Lemma 2.8(c),
\[ \int_0^t \int_{\mathbb{R}^d} \mathcal{G}^\delta_0(t - s, x - z) \mathcal{G}^\delta_0_{n\delta_0}(s, z - y) dz ds \]
\[ \leq 4 C B(\delta_0/2, (n + 1)\delta_0/2) (\mathcal{G}^0_{(n+2)\delta_0} + \mathcal{G}^\delta_0_{(n+1)\delta_0} + \mathcal{G}^\delta_0_{(n+1)\delta_0})(t, x - y) \]
\[ \leq 8 C B(\delta_0/2, (n + 1)\delta_0/2) (\mathcal{G}^0_{(n+1)\delta_0} + \mathcal{G}^\delta_0_{(n+1)\delta_0})(t, x - y). \]

Thus, (5.10) is valid for all \( n \in \mathbb{N}_0 \). Note that
\[ \sum_{n=0}^{\infty} d_n \Phi^{-1}(T)^{\delta_0} := C_1(\delta_0, T) < \infty \] (5.11)
since \( \frac{d_{n+1} \Phi^{-1}(T)^{\delta_0}(n+1)\delta_0}{d_n \Phi^{-1}(T)^{n\delta_0}} = 16 C c_2 \Phi^{-1}(T)^{\delta_0} B(\delta_0/2, (n + 1)\delta_0/2) \to 0 \) as \( n \to \infty \). So, by using (2.10) in the second line we obtain
\[ \sum_{n=0}^{\infty} |q_n(t, x, y)| \leq \sum_{n=0}^{\infty} d_n (\mathcal{G}^0_{(n+1)\delta_0} + \mathcal{G}^\delta_0_{n\delta_0})(t, x - y) \]
\[ \leq \sum_{n=0}^{\infty} d_n \Phi^{-1}(T)^{n\delta_0} (\mathcal{G}^\delta_0 + \mathcal{G}^\delta_0_0)(t, x - y) = C_1(\mathcal{G}^0 + \mathcal{G}^\delta_0)(t, x - y) \]
for $t \leq T$. Therefore, for every $t \in (0, T]$ and $x, y \in \mathbb{R}^d$,

$$|q(t, x, y)| \leq C_1(\mathcal{G}^0_{b_0} + \mathcal{G}^\delta_{b_0})(t, x - y). \quad (5.12)$$

To obtain (5.8) and (5.9), we calculate that

$$|\phi_y(t, x)| \leq \int_0^t \int_{\mathbb{R}^d} p_z(t - s, x - z)|q(s, z, y)| \, dz \, ds$$

$$\leq c_3 \int_0^t \int_{\mathbb{R}^d} (t - s)\mathcal{G}(t - s, x - z) (\mathcal{G}^0_{b_0} + \mathcal{G}^\delta_{b_0}) (s, z - y) \, dz \, ds$$

$$\leq c_4 t (\mathcal{G}^0_{b_0} + \mathcal{G}^\delta_{b_0}) (t, x - y)$$

$$\leq 2c_4 \Phi^{-1}(T)^\delta_0 t \mathcal{G}(t, x - y) = c_5 t \mathcal{G}(t, x - y), \quad \text{for all } t \in (0, T].$$

Here we used (4.111) and (5.12) for the second line, (2.22) for the third line and (2.18) for the last line. Therefore, using (4.11) we arrive $p^\kappa(t, x, y) \leq p_y(t, x - y) + |\phi_y(t, x)| \leq c_6 t \mathcal{G}(t, x - y)$.

□

We concludes this section with some fractional estimates on $p^\kappa(t, x, y)$.

**Lemma 5.2** For every $T \geq 1$ and $\gamma \in (0, 1] \cap (0, \alpha_1)$, there exists a constant $c_3$ such that for any $t \in (0, T]$ and $x, x', y \in \mathbb{R}^d$,

$$|p^\kappa(t, x, y) - p^\kappa(t, x', y)| \leq c_3 |x - x'|^\gamma t \left( \mathcal{G}^0_{-\gamma}(t, x - y) + \mathcal{G}^0_{-\gamma}(t, x' - y) \right).$$

The constant $c_3$ depends on $d, T, a, a_1, b, \alpha_1, b, \beta, c_0, \gamma, \delta, \kappa_0, \kappa_1$ and $\kappa_2$.

**Proof.** Assume that $x, x', y \in \mathbb{R}^d$ and $t \in (0, T]$. By (4.14) and the fact that $\gamma \leq 1$, we have

$$|p_z(s, x - z) - p_z(s, x' - z)| \leq c_1 |x - x'|^\gamma s \Phi^{-1}(s)^{-\gamma} (\mathcal{G}(s, x - z) + \mathcal{G}(s, x' - z))$$

$$\leq c_1 |x - x'|^\gamma s (\mathcal{G}^0_{-\gamma}(s, x - z) + \mathcal{G}^0_{-\gamma}(s, x' - z)). \quad (5.13)$$

for any $0 < s \leq T$ and $z \in \mathbb{R}^d$. Thus, by (5.12), (5.13) and a change of the variables, we further have that for $\delta_0 := (\delta \wedge \alpha_1/4) \in (0, \delta] \cap (0, \alpha_1/2)$,

$$|\phi_y(t, x) - \phi_y(t, x')| \leq \int_0^t \int_{\mathbb{R}^d} |p_z(t - s, x - z) - p_z(t - s, x' - z)| |q(s, z, y)| \, dz \, ds$$

$$\leq c_2 |x - x'|^\gamma \int_0^t \int_{\mathbb{R}^d} (t - s) (\mathcal{G}^0_{-\gamma}(t - s, x - z) + \mathcal{G}^0_{-\gamma}(t - s, x' - z)) (\mathcal{G}^\delta_{b_0} + \mathcal{G}^0_{b_0}) (s, z - y) \, dz \, ds$$

$$\leq c_3 |x - x'|^\gamma t \left( \mathcal{G}^0_{-\gamma + \delta_0}(t, x - y) + \mathcal{G}^0_{-\gamma}(t, x - y) + \mathcal{G}^0_{-\gamma + \delta_0}(t, x' - y) + \mathcal{G}^\delta_{b_0}(t, x' - y) \right)$$

$$\leq 2c_3 \Phi^{-1}(T)^\delta_0 |x - x'|^\gamma t \left( \mathcal{G}^0_{-\gamma}(t, x - y) + \mathcal{G}^0_{-\gamma}(t, x' - y) \right), \quad \text{for all } t \in (0, T].$$

Since $\gamma < \alpha_1$, the penultimate line follows from (2.222) (with $\theta = 0$), and the last line by (2.16) and (2.17). The lemma follows by combining above two estimates and (5.7). □
6 Proof of Theorems 1.1–1.4

In this section we prove the main theorems in Section 1. We first prove that the function \( p^\kappa(t, x, y) \) defined by (5.7) satisfies all statements in Theorems 1.1–1.4, then we show that \( p^\kappa(t, x, y) \) is the unique solution to (1.11) satisfying (i)–(iii) in Theorem 1.1.

Proof of Theorems 1.3 and 1.4. It follows from Remark 4.3 that we can apply the results in [KSV18, Theorem 1.1–1.4] for operator (1.1) with the function \( \tilde{G}(t, x) \). Note that the function \( p^\kappa(t, x, y) \) in [KSV18, Theorems 1.1–1.4] is constructed by the same way as (5.7). Therefore, Theorems 1.3 and 1.4 except (1.20) immediately follow from Remarks 2.3 and 4.3, and [KSV18, Theorem 1.1(iii), 1.2 and 1.3]. Finally (1.20) is proved in Lemma 5.2.

Now we prove the lower bound estimates in Theorem 1.1 and Corollary 1.2 for the function \( p^\kappa(t, x, y) \) in (5.7). By Theorems 1.3 and 1.4, we have that \( (P^\kappa_t)_{t \geq 0} \) defined by \( p^\kappa(t, x, y) \) in (5.7) with (1.21) is a Feller semigroup and there exists a Feller process \( X = (X_t, \mathbb{P}_x) \) corresponding to \( (P^\kappa_t)_{t \geq 0} \). Moreover, by (1.22) for \( f \in C^2_b(\mathbb{R}^d), \)

\[
f(X_t) - f(x) - \int_0^t L^\kappa f(X_s) \, ds
\]  

is a martingale with respect to the filtration \( \sigma(X_s, s \leq t) \). Therefore, by the same argument as that in [CZ16, Section 4.4], we have the following Lévy system formula: for every function \( f : \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty) \) vanishing on the diagonal and every stopping time \( S \),

\[
\mathbb{E}_x \sum_{0 < s \leq S} f(X_{s-}, X_s) = \mathbb{E}_x \int_0^S f(X_s, y) J_X(X_s, dy) \, ds,
\]  

where \( J_X(x, y) := \kappa(x, y - x) J(|x - y|) \). For \( A \in \mathcal{B}(\mathbb{R}^d) \) we define \( \tau_A := \inf\{t \geq 0 : X_t \notin A\} \) be the exit time from \( A \).

Using (6.1), (1.2) and (3.11), the proof of the following result is the same as the one in [KSV18, Lemma 5.7]. We skip the proof.

Lemma 6.1 Let \( T \geq 1 \). For each \( \varepsilon \in (0, 1) \) there exists \( \lambda = \lambda(\varepsilon) > 0 \) such that for every \( 0 < r \leq \Phi^{-1}(T), \)

\[
\sup_{x \in \mathbb{R}^d} \mathbb{P}_x \left( \tau_{B(x, r)} \leq \lambda \Phi(r) \right) \leq \varepsilon.
\]  

We record that by (6.3), for any \( x \in \mathbb{R}^d \) and \( 0 < r \leq \Phi^{-1}(T) \) we have

\[
\mathbb{E}_x[\tau_{B(x, r)}] \geq \lambda(1/2) \Phi(r) \mathbb{P}_x(\tau_{B(x, r)} > \lambda \Phi(r)) \geq \frac{\lambda}{2} \Phi(r) = c \Phi(r).
\]

Now we are ready to prove the lower bound in (1.15).

Lemma 6.2 The function \( p^\kappa(t, x, y) \) in (5.7) satisfies (1.15).
Proof. Fix $T \geq 1$. Let $p_y(t,x)$ be the heat kernel of the freezing operator in (5.1), and $J_y(z) := \kappa(y, z)J(|z|)$ and $\psi_y(z)$ be the corresponding Lévy measure and characteristic exponent, respectively. By [KS15, Theorem 2], there exist constants $C_1, C_2 > 0$ such that

$$p_y(t,x) \geq C_1 \Phi^{-1}(t)^{-d}, \quad t \in (0, T], y \in \mathbb{R}^d \text{ and } |x| \leq C_2 \Phi^{-1}(t).$$ \hspace{1cm} (6.5)

Indeed, $J_y(z)dz$ is symmetric and infinite Lévy measure by (1.2) and that $J(|z|)dz$ is infinite Lévy measure. To check the condition [KS15, (3)], we need to show that there exists a constant $c > 0$ such that

$$\int_{\mathbb{R}^d} e^{-t\psi(z)}|z|dz \leq ch_y(t)^{-d-1}, \quad 0 < t, y \in \mathbb{R}^d$$ \hspace{1cm} (6.6)

where $h_y(t) := \frac{1}{\Psi_y^{-1}(t)}$ and $\Psi_y(z) := \sup_{|z| \leq r} \psi_y(z)$. Let $\mathcal{P}(r) := \int_{\mathbb{R}^d} (1 \wedge \frac{|z|^2}{r^2})J(|z|)dz$ and $\mathcal{P}_y(r) := \int_{\mathbb{R}^d} (1 \wedge \frac{|z|^2}{r^2})J_y(|z|)dz$. Then, by [KS15, (11)] we have

$$c_1 \kappa_0 \mathcal{P}(r^{-1}) \leq c_1 \mathcal{P}_y(r^{-1}) \leq \Psi_y(r) \leq 2 \mathcal{P}_y(r^{-1}) \leq 2 \kappa_1 \mathcal{P}(r^{-1}), \quad r > 0.$$ \hspace{1cm} (6.7)

On the other hand, by the symmetry of $J_y$ and [KS15, (10)], we have

$$\psi_y(z) \geq (1 - \cos 1) \int_{|\xi| \leq 1/|z|} |\xi \cdot z|^2 J_y(d\xi) \geq \kappa_0 (1 - \cos 1) \int_{|\xi| \leq 1/|z|} |\xi \cdot z|^2 J(|\xi|)d\xi.$$ \hspace{1cm} (6.8)

Since by a rotation

$$\int_{|\xi| \leq 1/|z|} |\xi \cdot z|^2 J(|\xi|)d\xi = |z|^2 \int_{|\xi| \leq 1/|z|} \xi_i^2 J(|\xi|)d\xi, \quad i = 1, \ldots, d,$$

we have

$$\psi_y(z) \geq d^{-1} \kappa_0 (1 - \cos 1) |z|^2 \int_{|\xi| \leq 1/|z|} |\xi|^2 J(|\xi|)d\xi.$$ \hspace{1cm} (6.9)

Thus, when $|z| \leq 1$ we have

$$\psi_y(z) \geq d^{-1} \kappa_0 (1 - \cos 1) |z|^2 \int_{|\xi| \leq 1} |\xi|^2 J(d\xi) \geq c_2 |z|^2 = c_3 \Phi(|z|^{-1}),$$

whereas by (1.4) we have

$$\psi_y(z) \geq d^{-1} \kappa_0 (1 - \cos 1) |z|^2 \int_{|\xi| \leq 1/|z|} |\xi|^2 J(d\xi) \geq c_4 |z|^2 \int_0^{1/|z|} \frac{s}{\phi(s)} ds = c_4 \Phi(|z|^{-1})$$

for $|z| \geq 1$. Therefore, using (3.11) and (3.9) we obtain

$$\psi_y(z) \geq c_5 \Phi(|z|^{-1}) \geq c_6 \mathcal{P}(|z|) \geq (c_6/2) \psi(|z|).$$ \hspace{1cm} (6.8)
Moreover, (3.11) and (6.7) also imply that \( h_y(t) \asymp \Phi^{-1}(t) \asymp h(t) := \frac{1}{\sqrt{1-(t-1)}} \). From this and (6.8) we can follow the proof of Lemma 3.5 and obtain (6.6) as

\[
\int_{\mathbb{R}^d} e^{-\psi_y(z)} |z| dz \leq \int_{\mathbb{R}^d} e^{-ca\psi(z)/2} |z| dz \leq c_7 h(t)^{-d-1} \leq c_8 h_y(t)^{-d-1}, \quad 0 < t, y \in \mathbb{R}^d. \tag{6.9}
\]

Note that every constant above is independent of \( y \). Therefore, letting \( f(r) \equiv 0 \) we obtain all conditions in [KS15, Theorem 2] so we have (6.5) where \( C_1 > 0 \) is independent of \( y \).

The rest of the proof is almost identical to the one of [KSV18, Theorem 1.4]. Note that there is minor gap in [KSV18] (5.36). We provide the full details here including the correction of [KSV18] (5.36).

Choose \( t_0 \in (0, T] \) small enough to satisfy \( 2c_1 \Phi^{-1}(t_0)^{\delta_0} \leq C_1/2 \) where \( c_1 \) and \( \delta_0 \) are constants in [5.8]. Then, using (5.8) and (2.18) we have that for any \( 0 < t \leq t_0 \) and \( x, y \in \mathbb{R}^d \),

\[
|\phi_y(t, x)| \leq c_1 t (\mathcal{G}_{\delta_0} + \mathcal{G}_{\delta_0}^0)(t, x-y) \leq 2c_1 \Phi^{-1}(t_0)^{\delta_0} t \mathcal{G}(t, x-y) \leq 2c_1 \Phi^{-1}(t_0)^{\delta_0} \Phi^{-1}(t)^{-d} \leq \frac{C_1}{2} \Phi^{-1}(t)^{-d}.
\]

Thus, combining above inequality and (6.5) we obtain

\[
p^k(t, x, y) = p_y(t, x-y) + \phi_y(t, x) \geq p_y(t, x-y) - |\phi_y(t, x)| \geq \frac{C_1}{2} \Phi^{-1}(t)^{-d}
\]

for \( 0 < t \leq t_0 \) and \( x, y \in \mathbb{R}^d \) with \( |x-y| \leq C_2 \Phi^{-1}(t) \). By (1.19) and iterating at most \( n_0 := \left\lceil \frac{T}{t_0} \right\rceil + 1 \) times, we obtain the following near-diagonal lower bound

\[
p^k(t, x, y) \geq C_3 \Phi^{-1}(t)^{-d} \quad \text{for all } t \in (0, T] \text{ and } |x-y| \leq C_4 \Phi^{-1}(t) \tag{6.10}
\]

for some constants \( C_3, C_4 > 0 \). Indeed, for \( t \in (0, T] \) and \( x, y \in \mathbb{R}^d \) with \( |x-y| \leq C_4 \Phi^{-1}(t) \) where \( C_4 := C_2(n_0/a_1)^{1/a_1} \in (0, 2^{-d}) \) is a sufficiently small constant satisfying

\[
C_4 \Phi^{-1}(t) \leq C_2 \Phi^{-1}(\frac{t}{n_0})
\]

by (2.8). Let \( n = \left\lceil \frac{T|x-y|}{C_4 t_0 \Phi^{-1}(t_0)} \right\rceil + 1 \) and \( z_1, \ldots, z_{n-1} \) be the points in the segment from \( x \) to \( y \) satisfying \( |z_1-x| = |z_{i+1}-z_i| = |y-z_{n-1}| = \frac{|x-y|}{n} \). Note that \( n \leq n_0 \) since \( |x-y| \leq C_4 \Phi^{-1}(t) \).

Then, by (1.15)

\[
p^k(t, x, y) = \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} p^k(t/n, x, w_1)p^k(t/n, w_1, w_2) \cdots p^k(t/n, w_{n-1}, y) dw_1 \cdots dw_{n-1}
\]

\[
\geq \int_{B(z_1, \frac{|x-y|}{n})} \cdots \int_{B(z_{n-1}, \frac{|x-y|}{n})} p^k(t/n, x, w_1)p^k(t/n, w_1, w_2) \cdots p^k(t/n, w_{n-1}, y) dw_1 \cdots dw_{n-1}
\]

\[
\geq \omega_d^{n-1} \left( \frac{|x-y|}{3n} \right)^{n-1} \cdot \frac{C_1}{2^n} \Phi^{-1}(\frac{t}{n})^{1/2} \geq C_3 \Phi^{-1}(t)^{-d}.
\]

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Here we used \( \frac{|x-y|}{n} \geq \frac{T}{C_{1}^{n}} \Phi_{-1}(t) \), \( n \leq n_{0} \) and (2.8) for the last line. Therefore, we obtain (6.10). Now we assume \( |x-y| > C_{4} \Phi^{-1}(t) \) and let \( \tau > 0 \) be the constant in Lemma 6.1 for \( \varepsilon = 1/2 \) and \( \tau(z, c, t) := \tau_{B(x, C \Phi^{-1}(t))} \). Then for every \( 0 < t \leq T \),

\[
\sup_{z \in \mathbb{R}^{d}} \mathbb{P}_{z} (|z, 2^{-2}C_{4}t| \leq \lambda t) \leq \frac{1}{2}.
\]  

(6.11)

Let \( \sigma = \inf \{ s \geq 0 : X_{s} \in B(y, 2^{-2}C_{4} \Phi^{-1}(t)) \} \) be the hitting time of \( B(y, 2^{-2}C_{4} \Phi^{-1}(t)) \). By the strong Markov property and (6.11) we have

\[
\mathbb{P}_{x} \left( X_{\lambda t} \in B(y, 2^{-2}C_{4} \Phi^{-1}(t)) \right) \geq \mathbb{P}_{x} \left( \sigma \leq \lambda t, \sup_{s \in [\sigma, \lambda t]} |X_{s} - X_{\sigma}| < 2^{-2}C_{4} \Phi^{-1}(t) \right) = \mathbb{E}_{x} \left( \mathbb{P}_{x} \left( \sup_{s \in [0, \lambda t]} |X_{s} - X_{0}| < 2^{-2}C_{4} \Phi^{-1}(t); \sigma \leq \lambda t \right) \right) \geq \inf_{z \in B(y, \Phi^{-1}(t))} \mathbb{P}_{z} (\tau(z, 2^{-2}C_{4}, t) > \lambda t) \mathbb{P}_{z} (\sigma \leq \lambda t) \geq \frac{1}{2} \mathbb{P}_{x} (\sigma \leq \lambda t) \geq \frac{1}{2} \mathbb{P}_{x} \left( X_{(\Lambda \wedge \tau)(x, 2^{-3}C_{4}, t)} \in B(y, 2^{-2}C_{4} \Phi^{-1}(t)) \right) .
\]

Since \( |x-y| > C_{4} \Phi^{-1}(t) \), we have

\[
X_{s} \notin B(y, 2^{-2}C_{4} \Phi^{-1}(t))^{c} \subset B(x, 2^{-3}C_{4} \Phi^{-1}(t))^{c}, \quad s < \lambda t \wedge \tau(x, 2^{-3}C_{4}, t).
\]

Thus,

\[
1_{\{X_{(\Lambda \wedge \tau)(x, 2^{-3}C_{4}, t)} \in B(y, 2^{-2}C_{4} \Phi^{-1}(t))\}} = \sum_{s \leq \Lambda \wedge \tau(x, 2^{-3}C_{4}, t)} 1_{\{X_{s} \in B(y, 2^{-2}C_{4} \Phi^{-1}(t))\}}.
\]

Therefore, by the Lévy system formula in (6.2) we obtain

\[
\mathbb{P}_{x} \left( X_{(\Lambda \wedge \tau)(x, 2^{-3}C_{4}, t)} \in B(y, 2^{-2}C_{4} \Phi^{-1}(t)) \right) = \mathbb{E}_{x} \left[ \int_{0}^{\Lambda \wedge \tau(x, 2^{-3}C_{4}, t)} \int_{B(y, 2^{-2}C_{4} \Phi^{-1}(t))} J_{x}(X_{s}, u) \, du \, ds \right] \geq \mathbb{E}_{x} \left[ \int_{0}^{\Lambda \wedge \tau(x, 2^{-3}C_{4}, t)} \int_{B(y, 2^{-2}C_{4} \Phi^{-1}(t))} \kappa_{0} J(|X_{s} - u|) 1_{\{u : |X_{s} - u| \leq |x-y|\}} \, du \, ds \right] .
\]  

(6.13)

Let \( w \) be the point in the segment from \( x \) to \( y \) satisfying \( |w - y| = 3 \cdot 2^{-4}C_{4} \Phi^{-1}(t) \). Since \( |x - X_{s}| \leq 2^{-3}C_{4} \Phi^{-1}(t) \), we have that for any \( u \in B(w, 2^{-4}C_{4} \Phi^{-1}(t)) \),

\[
|X_{s} - u| \leq |x - X_{s}| + |x - w| + |w - u| \leq |x - y|
\]

so that \( B(w, 2^{-4}C_{4} \Phi^{-1}(t)) \subset B(y, 2^{-2}C_{4} \Phi^{-1}(t)) \cap \{ u : |X_{s} - u| \leq |x - y| \} \) for every \( s < \lambda t \wedge \tau(x, 2^{-3}C_{4}, t) \). Thus,

\[
\mathbb{E}_{x} \left[ \int_{0}^{\Lambda \wedge \tau(x, 2^{-3}C_{4}, t)} \int_{B(y, \Phi^{-1}(t))} \kappa_{0} J(|X_{s} - u|) 1_{\{u : |X_{s} - u| \leq |x-y|\}} \, du \, ds \right] \geq \kappa_{0} \mathbb{E}_{x} \left[ \int_{0}^{\Lambda \wedge \tau(x, 2^{-3}C_{4}, t)} \int_{B(w, 2^{-4}C_{4} \Phi^{-1}(t))} J(|x - y|) \, du \, ds \right] \geq c_{1} \Phi^{-1}(t)^{d} J(|x - y|) \mathbb{E}_{x}[\Lambda t \wedge \tau(x, 2^{-3}C_{4}, t)] \geq c_{2} t \Phi^{-1}(t)^{d} J(|x - y|),
\]  

(6.14)
where we used (6.4) and (2.2) for the last line.

Therefore, combining (6.12), (6.13) and (6.14) we arrive

\[ p^\kappa(t, x, y) \geq \int_{B(y, \Phi^{-1}(t))} p^\kappa(\lambda t, x, z) p^\kappa((1 - \lambda)t, z, y) \, dz \]

\[ \geq \inf_{z \in B(y, 2^{-1}C_4 \Phi^{-1}(t))} p^\kappa((1 - \lambda)t, z, y) \int_{B(y, 2^{-1}C_4 \Phi^{-1}(t))} p^\kappa(\lambda t, x, z) \, dz \]

\[ \geq c_3 t \Phi^{-1}(t)^{-d} \Phi^{-1}(t)^d \, J(|x - y|). \]

for all \(0 < t \leq T\) and \(x, y \in \mathbb{R}^d\) with \(|x - y| > C_4 \Phi^{-1}(t)\).

\[\Box\]

**Proof of Theorem 1.1.** By Remarks 2.3 and 4.3, \(p^\kappa(t, x, y)\) defined in (5.7) satisfies (1.11), (1.13) and (1.14). Also, (1.12) and (1.15) follow from Theorem 5.1 and Lemma 6.2 respectively. It remains to show the uniqueness part of Theorem 1.1. Recall that we observe in Remark 4.3 that \([KSV18, (1.9)]\) holds. Thus all results in \([KSV18, Sections 5.1 and 5.2]\) hold for our case. Since properties (i)–(iii) are stronger than ones in \([KSV18, Theorem 1.1]\), we now see that the proof of the uniqueness part of Theorem 1.1 is exactly same as the one of the uniqueness part of \([KSV18, Theorem 1.1]\).

\[\Box\]

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