The manipulation of quantum entanglement has found enormous potential for improving performances of devices such as gyroscopes, clocks, and even computers. Similar improvements have been demonstrated for lithography and microscopy. We present an overview of some aspects of enhancement by quantum entanglement in imaging and metrology.

In state-of-the-art optical-lithographic, semiconductor etching techniques, the Rayleigh diffraction limit puts a lower bound on the feature size that can be printed on a chip. This limit states that the minimal resolvable feature is on the order of $\lambda/4$, where $\lambda$ is the wavelength of the light used. Classically, if you want to etch features of size 50 nm and smaller, you will be forced to use optical radiation with wavelengths less than 200 nm. Hence, in the optical lithographic community great efforts are put into producing commercial lithographic schemes that utilize wavelengths in the hard UV and X-ray regimes. However, such an approach introduces severe technological and commercial difficulties. For example, mirrors and lenses that are cheap and well understood in the optical regime are much less common and much harder (and more expensive) to make in the UV and X-ray regions of the spectrum, and the problems become worse the shorter you go. Recently, it has been shown, however, that the Rayleigh diffraction limit in optical lithography can be circumvented by the use of path-entangled photon number states.

Fundamentally, optical light beams used for lithography are quantum-mechanical in nature: they are superpositions of photon-number states. This quantum language allows us to tailor non-classical states of light, in which it is possible to program nonlocal correlations between photons. Typically, one photon contains information about the location and the momentum of other photons in the stream. If the proper nonlocal correlations are employed, you can actually manipulate the location at which the light strikes the lithographic substrate, such that features of size $\lambda/4N$ can be etched using $N$ photons of wavelength $\lambda$. The use of such an effect was also proposed in sub-natural spectroscopy.

This remarkable property of quantum-correlated photons has been recognized for some years in the context of quantum optical interferometers. In a typical optical interferometer in which ordinary coherent) laser light enters via only one port, the phase sensitivity in the shot noise limit scales as $\Delta \varphi = 1/\sqrt{n}$ where $\bar{n}$ is the mean number of photons. It would seem that
any desired sensitivity $\Delta \varphi$ could be attained by simply increasing the laser power. However, when the intensity of the laser ($\bar{n}$) becomes too large, the power fluctuations at the interferometer’s mirrors introduce additional noise terms that limit the device’s overall sensitivity. Much of the early interest in squeezed light empathized overcoming this signal-to-noise barrier. In the early 1980s it was demonstrated that squeezing the vacuum in the unused input port of the interferometer causes the phase sensitivity to beat the standard shot-noise limit. The total laser power required for a given amount of phase sensitivity $\varphi$ is thus greatly reduced.

In 1986, Yurke and collaborators, as well as Yuen, considered the question of phase noise reduction using correlated particles in number states, incident upon both input ports of a Mach-Zehnder interferometer. They showed that if $N$ quantum particles entered into each input port of the interferometer in nearly equal numbers (and in a highly entangled fashion), then, for large $N$, it was indeed possible to obtain an asymptotic phase sensitivity of $1/N$, instead of $1/\sqrt{N}$. This is the best you can do using number states in only one input port, and it indicates that the photon counting noise does not originate from the intensity fluctuations of the input beam. Similar observations were made by many authors for optical interferometers as well as Ramsey-type atom interferometers. Wineland and co-workers, in particular, have shown that the optimal frequency measurement can be achieved by using maximally entangled states. These maximally entangled states are of a particular interest, since they have a similar form as the ones required for quantum lithography.

Let us take a look at the quantum enhancement due to maximally entangled states, using standard parameter estimation. Consider an ensemble of $N$ two-state systems in the state:

$$|\varphi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{i\varphi}|1\rangle)$$

(1)

where $|0\rangle$ and $|1\rangle$ denote the two basis states. The phase information can be obtained by measurement of an observable $\hat{A} = |0\rangle\langle 1| + |1\rangle\langle 0|$. The expectation value of $\hat{A}$ is then given by

$$\langle \varphi | \hat{A} | \varphi \rangle = \cos \varphi.$$  

(2)

When we repeat this experiment $N$ times, we obtain $\langle \varphi_R | \hat{A}_R | \varphi_R \rangle = N \cos \varphi$, where $|\varphi_R\rangle = |\varphi\rangle_1 \ldots |\varphi\rangle_N$, and $\hat{A}_R = \bigoplus_{k=1}^N \hat{A}^{(k)}$. Since $\hat{A}_R^2 = 1$, the variance of $\hat{A}_R$, given $N$ samples, is readily computed to be $(\Delta A_R)^2 = N(1 - \cos^2 \varphi) = N \sin^2 \varphi$. According to estimation theory, we have

$$\Delta \varphi_{SL} = \frac{\Delta A_R}{|d(\hat{A}_R)/d\varphi|} = \frac{1}{\sqrt{N}}.$$  

(3)

This is the standard variance in the parameter $\varphi$ after $N$ trials. In other
words, the uncertainty in the phase is inversely proportional to the square root of the number of trials. This is called the shot-noise limit.

Now consider an entangled state

\[ |\varphi_N\rangle \equiv \frac{1}{\sqrt{2}} |N, 0\rangle + e^{iN\varphi} |0, N\rangle , \] (4)

where \( |N, 0\rangle \) and \( |0, N\rangle \) are collective states of \( N \) particles, defined as

\[
\begin{align*}
|N, 0\rangle &= |0\rangle_1 |0\rangle_2 \cdots |0\rangle_N, \\
|0, N\rangle &= |1\rangle_1 |1\rangle_2 \cdots |1\rangle_N. 
\end{align*}
\] (5)

The relative phase \( e^{iN\varphi} \) is accumulated when each particle in state \( |1\rangle \) acquires a phase shift of \( e^{i\varphi} \). An important question now is what we need to measure in order to extract the phase information. Recalling the single-particle case of \( \hat{A} = |0\rangle\langle 1| + |1\rangle\langle 0| \), we need an observable that does what the operator \( |0, N\rangle\langle N, 0| + |N, 0\rangle\langle 0, N| \) does. For the given state of Eq. (4), we can see that this can be achieved by an observable, \( \hat{A}_N = \bigotimes_{k=1}^N \hat{A}^{(k)} \). The expectation value of \( \hat{A}_N \) is then

\[
\langle \varphi_N | \hat{A}_N | \varphi_N \rangle = \cos N\varphi .
\] (6)

Again, \( \hat{A}_N^2 = 1 \), and \((\Delta A_N)^2 = 1 - \cos^2 N\varphi = \sin^2 N\varphi \). Using Eq. (6) again, we obtain the so-called Heisenberg limit (HL) of the minimal detectable phase:

\[
\Delta \varphi_{\text{HL}} = \frac{\Delta A_N}{|d\langle \hat{A}_N \rangle/d\varphi|} = \frac{1}{N}.
\] (7)

The precision in \( \varphi \) is increased by a factor \( \sqrt{N} \) over the standard noise limit. Of course, the preparation of a quantum state such as Eq. (4) is essential to the given protocol.

In quantum lithography, we exploit the \( \cos N\varphi \) behavior, exhibited by Eq. (6), to draw closely spaced lines on a suitable substrate. Entanglement-enhanced frequency measurements and gyroscopes exploit the \( \sqrt{N} \) increased precision given by Eq. (7). The physical interpretations of \( A_N \) and the phase \( \varphi \) will differ in the different protocols. Three distinct physical representations of this construction are of particular interest.

First, in a Mach-Zehnder interferometer [as depicted in Fig. 1(a)] the input light field is divided into two different paths by a beam splitter, and recombined by another beam splitter. The phase difference between the two paths is then measured by balanced detection of the two output modes. Secondly, in Ramsey-type spectroscopy, atoms are put in a superposition of the ground state and an excited state with a \( \pi/2 \)-pulse [see Fig. 1(b)]. After a relative phase shift is accumulated by the atomic states during the free evolution, the second \( \pi/2 \)-pulse is applied and the internal state of the outgoing atom is measured. The third system is given by a qubit that undergoes a Hadamard transform \( H \), then picks up a relative phase and is transformed
back with a second Hadamard transformation [Fig. 1(c)]. This representation is more mathematical than the previous two, and it allows us to extract the unifying mathematical principle that underlies the three systems.

In all three protocols, the initial state is transformed by a discrete Fourier transform (beam splitter, $\pi/2$-pulse or Hadamard), then picks up a relative phase, and is transformed back again. The Hadamard transform is the standard (two-dimensional) “quantum” finite Fourier transform, such as used in the implementation of Shor’s algorithm\textsuperscript{21}. The phase shift, which is hard to measure directly, is applied to the transformed basis. The result is a bit flip in the initial basis $\{0\}, \{1\}$, and this is readily measured. We call the formal equivalence between these three systems the \textit{quantum Rosetta stone}\textsuperscript{22}. In discussing quantum computer circuits with researchers from the fields of quantum optics or atomic clocks, we find the “quantum Rosetta stone” a useful tool. For example, in a Ramsey-type atom interferometer, noting that
\[ \hat{A} \equiv \sigma_x = H\sigma_z H, \]

\[ \hat{A}_N = \bigotimes_{k=1}^N \hat{A}^{(k)} = \left( \bigotimes_{k=1}^N H^{(k)} \right) \hat{A}_N' \left( \bigotimes_{k=1}^N H^{(k)} \right), \]

we need to measure \( \hat{A}_N' = \bigotimes_{k=1}^N \sigma_z^{(k)} \) after the second beam splitter (see Fig. 2). On the other hand, a direct optical measurement of \( \hat{A}_N \) corresponds to an \( N \)-photon absorption scheme in quantum lithography.

Such an enhancement of a factor of \( \sqrt{N} \) in quantum metrology is known to be the best possible precision permitted by the uncertainty principle. It is also interesting to see that exploiting quantum correlations yields a \( \sqrt{N} \) enhancement in Grover’s search algorithm, which has also been shown to be optimal. Is this \( \sqrt{N} \) enhancement over the classical protocols universal? It certainly does seem so. But then Shor’s algorithm shows its exponential improvement over the best known classical algorithm.

We like to end this article with the “Williams-Dowling Inverse-Shor Conjecture”. Assume that the best known classical factoring algorithm is the best possible one. Then, according to Conjecture 1, the quantum Rosetta stone tells us that there must exist an interferometric measurement strategy for phase sensitivity that is exponentially better than the shot-noise limit (the best classical strategy). However, we know that this is false. Conjecture 2, therefore, tells us that according to the quantum Rosetta stone, Shor’s algorithm is a \( \sqrt{N} \) improvement over the best, but unknown classical protocol. Thus, there exists a classical algorithm that is exponentially faster than the
best known one, though quadratically slower than the quantum algorithm!

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