HIGHEST WEIGHT CATEGORIES OF $\mathfrak{gl}(\infty)$-MODULES

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ABSTRACT. We study a category of modules over $\mathfrak{gl}(\infty)$ analogous to category $\mathcal{O}$. We fix adequate Cartan, Borel and Levi-type subalgebras $\mathfrak{h}, \mathfrak{b}$ and $\mathfrak{l}$ with $\mathfrak{l} \cong \mathfrak{gl}(\infty)^n$, and define $\mathcal{O}_L^1 \mathfrak{gl}(\infty)$ to be the category of $\mathfrak{h}$-semisimple, $n$-nilpotent modules that satisfy a large annihilator condition as $\mathfrak{l}$-modules. Our main result is that these are highest weight categories in the sense of Cline, Parshall and Scott. We compute the simple multiplicities of standard objects and the standard multiplicities in injective objects, and show that a form of BGG reciprocity holds in $\mathcal{O}_L^1 \mathfrak{gl}(\infty)$. We also give a decomposition of $\mathcal{O}_L^1 \mathfrak{gl}(\infty)$ into irreducible blocks.

Keywords: Lie algebras, representations, $\mathfrak{gl}(\infty)$, Large annihilator condition.

1. Introduction

The algebra $\mathfrak{gl}(\infty)$ is the most basic example of a complex locally-finite Lie algebra, i.e. a limit of finite-dimensional complex Lie algebras. Its representation theory has been a very active area of study for the last ten years. See for example [PS11a, PS11b, DCPS16, HPS19, GP18, PS19]. Beyond its intrinsic interest the representation theory of $\mathfrak{gl}(\infty)$ has connections to two other areas of Lie theory: the stable representation theory of the family $\mathfrak{gl}(d, \mathbb{C})$, and the classical representation theory of the Lie superalgebra $\mathfrak{gl}(n|m)$.

The relation between representations of $\mathfrak{gl}(\infty)$ and $\mathfrak{gl}(n|m)$ goes back to Brundan [Bruo3], who used a categorical action of the quantized enveloping algebra of $\mathfrak{gl}(\infty)$ on category $\mathcal{O}_{\mathfrak{gl}(n|m)}$ to compute the characters of atypical finite dimensional modules. This approach was expanded by Brundan, Losev and Webster in [BLW17] and by Brundan and Stroppel in [BS12]. Also, Hoyt, Penkov and Serganova studied the $\mathfrak{gl}(\infty)$-module structure of the integral block $\mathcal{O}_{\mathfrak{gl}(n|m)}^{\mathfrak{g}}$ in [HPS19]. Recently Serganova has extended this to a categorification of Fock-modules of $\mathfrak{sl}(\infty)$ through the Deligne categories $\mathcal{V}_t$ in [Ser21].

Let us now turn to the connections of $\mathfrak{gl}(\infty)$ with stable representation theory. Informally this refers to the study of sequences of representations $V_d$ of $\mathfrak{gl}(d, \mathbb{C})$, compatible in a suitable sense, as $d$ goes to infinity. For example, set $W_d = \mathbb{C}^d$ and $V_d(p,q) = W_d^p \otimes (W_d^*)^q$. For each $d \geq 1$ the decomposition of this module into its simple components is a consequence of Schur-Weyl duality, and we can

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ask whether this decomposition is in some way compatible with the obvious inclusion maps \( V_d(p,q) \hookrightarrow V_{d+1}(p,q) \). One way to solve this problem is regard the limit \( T^{p,q} = \lim_{d \to \infty} V_d(p,q) \) as a \( gl(\infty) \)-module and study its decomposition, as done by Penkov and Styrkas in \[PS11\]. A more categorical approach to this problem appeared independently in the articles by Sam and Snowden \[SS15\] and by Dan-Cohen, Penkov and Serganova in \[DCPS16\]. The former studies the category \( \text{Rep}(GL(\infty)) \) of stable algebraic representation theory of the family of groups \( GL_d(\mathbb{C}) \), and the latter is focused on the category \( T_{gl(\infty)^0} \) of modules that arise as subquotients of the modules \( T^{p,q} \). As mentioned in the introduction to \[SS15\], these categories are equivalent.

By \[DCPS16\], objects in the category \( T_{gl(\infty)^0}^{\text{fin}} \) are the integrable finite length \( gl(\infty) \)-modules that satisfy the so-called large annihilator condition (LAC from now on). This category is analogous in many ways to the category of integrable finite dimensional representations of \( gl(n,\mathbb{C}) \), and it forms the backbone of the representation theory of \( gl(\infty) \). Since this category is well understood by now, it is natural to look for analogues of category \( O \) for \( gl(\infty) \). Several alternatives have been proposed such as those by Nampaïsarn \[Nam17\], Coulembier and Penkov \[CP19\], and Penkov and Serganova \[PS19\].

The fact that there is no one obvious analogue of \( O \) is due to two reasons. First, Cartan and Borel subalgebras of \( gl(\infty) \) behave in a much more complicated way than in the finite dimensional case, in particular different choices of these will produce non-equivalent categories of highest weight modules. The second problem is that the enveloping algebra of \( gl(\infty) \) is not left-noetheran, so its finitely generated modules do not form an abelian category. The first problem can be solved (or rather swept under the rug) by fixing adequate choices of Cartan and Borel subalgebras; this is the road followed in this paper. The second however does not have an obvious solution.

In \[PS19\] Penkov and Serganova propose replacing this condition with the LAC. They study the category \( O_LA \) consisting of \( h \)-semisimple and \( n \)-nilpotent modules satisfying the LAC, and in particular show that it is a highest weight category. On the other hand, the categorifications arising in \[HPS19, Ser21\] above only satisfy the LAC when seen as modules over a Levi-type subalgebra \( l \subset gl(\infty) \). Given the importance of the work relating representations of \( gl(\infty) \) to Lie superalgebras, this motivates the study of analogues of category \( O \) where the condition of being finitely generated is replaced by this weaker version of the LAC. Thus in this paper we study the category \( O^l_{LA}gl(\infty) \) of \( h \)-semisimple, \( n \)-torsion modules satisfying the LAC with respect to various Levi-type subalgebras \( l \).

As mentioned above our main result is that \( O^l_{LA}gl(\infty) \) is a highest weight category. Let us give some details on the result. We introduce the notion of eligible weights, which are precisely those that can appear in a module satisfying the LAC. Admissible weights are endowed with an interval-finite order that has maximal but no minimal elements. Simple objects of \( O^l_{LA}gl(\infty) \) are precisely the simple highest weight modules indexed by eligible weights. Standard objects are given by projections of dual Verma modules to \( O^l_{LA}gl(\infty) \) and have infinite length. Their simple multiplicities are given.
in terms of weight multiplicities of a (huge) representation of $\mathfrak{gl}(\infty)$. Finally, injective envelopes have finite standard filtrations that satisfy a form of BGG reciprocity. We point out that $O_{\mathfrak{LA}}^{1} \mathfrak{gl}(\infty)$ does not have enough projectives, and hence no costandard modules.

The article is structured as follows. Section 2 contains some general notation. In section 3 we introduce several presentations of $\mathfrak{gl}(\infty)$, each of which highlights some particular features and subalgebras. Section 4 deals with various matters related to representation theory of Lie algebras, and some specifics regarding $\mathfrak{gl}(\infty)$. In section 5 we discuss some basic categories of representations of $\mathfrak{gl}(\infty)$ in its various incarnations. We begin our study of $O_{\mathfrak{LA}}$ in section 6 where we classify its simple objects, prove some general categorical properties, and show that simple multiplicities of a general object can be computed in terms of simple multiplicities for category $O_s$. In section 7 we prove the existence of standard modules and compute their simple multiplicities. This last result depends on a long technical computation given in an appendix to the section. Finally, in section 8 we wrap up the proof that $O_{\mathfrak{LA}}$ is a highest weight category with an analysis of injective modules. We also prove an analogue of BGG reciprocity and give a decomposition of $O_{\mathfrak{LA}}$ into irreducible blocks.

The usual zoo of Cartan, Borel, parabolic, and Levi subalgebras, along with their nilpotent ideals, is augmented in each case by their corresponding exhaustions and the subalgebras spanned by finite-root spaces. To help keep track of these wild variety, most of these subalgebras are introduced at once in subsection 3.6 along with a visual device to describe them.

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2. Generalities

2.1. Notation. For each $r \in \mathbb{N}$ we set $[r] = \{1, 2, \ldots, r\}$. Throughout we denote by $\mathbb{Z}^\times$ the set of nonzero integers endowed with the following, not quite usual order

$$1 < 2 < 3 < \cdots < -3 < -2 < -1.$$ 

The symbol $\delta_{ij}$ denotes the Kronecker delta.

We denote by Part the set of all partitions, and we identify each partition with its Young diagram. Given $\lambda \in \text{Part}$ we will the corresponding Schur functor by $S_{\lambda}$. For any $\lambda, \mu, \nu$ we denote by $c^\nu_{\lambda, \mu}$ the corresponding Littlewood-Richardson coefficient.

Vector spaces and unadorned tensor products are always taken over $\mathbb{C}$ unless explicitly stated. Given a vector space $V$ we denote its algebraic dual by $V^*$. Given $n \in \mathbb{Z}_{>0}$ we will denote by either $V^n$ or $nV$ the direct sum of $n$ copies of $V$. Given $d \in \mathbb{Z}_{>0}$ we will denote by $S^d(V)$ the $d$-th symmetric power of $V$, by $S^*(V)$ its symmetric algebra,
and by $S^d(V)$ the direct sum of all $S^{d'}(V)$ with $d' \leq d$. We will often use that given a second vector space $W$ we have

$$S^\bullet(V \otimes W) \cong \bigoplus_{\lambda \in \text{Part}} S_\lambda(V) \otimes S_\lambda(W).$$

2.2. **Locally finite Lie algebras.** A locally finite Lie algebra $g$ is one where any finite set of elements is contained in a finite dimensional subalgebra. If $g$ is countable dimensional then this is equivalent to the existence of a chain of finite dimensional subalgebras $g_1 \subset g_2 \subset \cdots \subset g = \bigcup_{r \geq 0} g_r$. Any such chain is called an *exhaustion* of $g$. We say $g$ is locally root-reductive if each $g_r$ can be taken to be reductive, and the inclusions send root spaces to root spaces for a fixed choice of nested Cartan subalgebras $h_r \subset g_r$ satisfying $h_r = h_{r+1} \cap g_r$. In general we will say that $g$ is locally reductive, nilpotent, etc. if there is has an exhaustion such that each $g_r$ is of the corresponding type.

We denote by $\mathfrak{gl}(\infty)$ the direct limit of the sequence of Lie algebras $\mathfrak{gl}(1, \mathbb{C}) \hookrightarrow \mathfrak{gl}(2, \mathbb{C}) \hookrightarrow \cdots \mathfrak{gl}(n, \mathbb{C}) \hookrightarrow \cdots$

where each map is given by inclusion in the upper left corner. This restricts to inclusions of $\mathfrak{sl}(n, \mathbb{C})$ into $\mathfrak{sl}(n + 1, \mathbb{C})$, and we denote the limit of these algebras by $\mathfrak{sl}(\infty)$, which is clearly a subalgebra of $\mathfrak{gl}(\infty)$. This is a locally finite Lie algebra.

Fix a locally finite Lie algebra $\mathfrak{g}$ and an exhaustion $\mathfrak{g}_r$. Suppose also that we have a sequence $(M_r, j_r)$ where every $M_r$ is a $\mathfrak{g}_r$-module (which makes every $M_s$ with $s \geq r$ a $\mathfrak{g}_r$-module), and $j_r : M_r \rightarrow M_{r+1}$ is an injective morphism of $\mathfrak{g}_r$-modules. The limit vector space $M = \varinjlim M_r$ has the structure of a $\mathfrak{g}_r$-module for each $r$, and these structures are compatible in the obvious sense so $M$ is a $\mathfrak{g}$-module. We refer to $M_r$ as an exhaustion of $M$. Any $\mathfrak{g}$-module $M$ has an exhaustion: if $X$ is a generating set of $M$, we can take as $M_r$ the $\mathfrak{g}_r$-submodule generated by $X$.

Concepts such as Cartan subalgebras, root systems, Borel and parabolic subalgebras relative to these systems, weight modules, etc. are defined for locally finite Lie subalgebras. In the context of this paper these objects will behave as in the finite dimensional case, but there are some subtle differences which we will point out when relevant. For details we refer the reader to the monograph [PH23] and the references therein.

We will write $g = \mathfrak{b} \oplus \mathfrak{a}$ if $\mathfrak{b}$ is a subalgebra of $g$ and $\mathfrak{a}$ is an ideal that is also a vector space complement for $\mathfrak{b}$. Suppose we have a locally finite Lie algebra $g$ with fixed splitting Cartan subalgebra $\mathfrak{h}$ (i.e. such that $g$ is a semisimple $\mathfrak{h}$-module through the adjoint action), and Borel (i.e. maximal locally solvable) subalgebra $\mathfrak{b}$ of the form $\mathfrak{h} \oplus \mathfrak{n}$ for some subalgebra $\mathfrak{n}$. Given any weight $\lambda \in \mathfrak{h}^*$, we can define Verma modules as usual by $\text{Ind}_\mathfrak{b}^g \mathbb{C}_\lambda$. We will denote the Verma module by $M_\mathfrak{g}(\lambda)$ or simply $M(\lambda)$ when $g$ is clear from the context. We will also denote by $L(\lambda) = L_\mathfrak{g}(\lambda)$ the corresponding unique simple quotient of $M_\mathfrak{g}(\lambda)$.
3. The many faces of \( \mathfrak{gl}(\infty) \)

In this section we will review some standard facts about the Lie algebra \( \mathfrak{gl}(\infty) \) and introduce several different ways in which this arises. While this amounts to choosing different exhaustions, we will take a different approach to these various “avatars” of \( \mathfrak{gl}(\infty) \), which brings to the fore some non-obvious structures in this Lie algebra.

3.1. The Lie algebra \( \mathfrak{g}(\mathbb{V}) \). Let \( I \) be any infinite denumerable set and let \( \mathbb{V} = \langle v_i \mid i \in I \rangle \) and \( \mathbb{V}_s = \langle v'_i \mid i \in I \rangle \), which we endow with a perfect pairing

\[
\text{tr} : \mathbb{V}_s \otimes \mathbb{V} \to \mathbb{C}
\]

\[
v'_i \otimes v_j \mapsto \delta_{ij}.
\]

The vector space \( \mathbb{V} \otimes \mathbb{V}_s \) is a nonunital associative algebra with \( v' \otimes v \cdot w' \otimes w = \text{tr}(v \otimes w')v' \otimes w \). We denote the Lie algebra associated to this algebra by \( \mathfrak{g}(\mathbb{V}, \mathbb{V}_s, \text{tr}) \), or simply by \( \mathfrak{g}(\mathbb{V}) \). For \( i, j \in I \) we set \( E_{ij} = v'_i \otimes v_j \).

Take for example \( I = \mathbb{N} \). Given \( r \in \mathbb{Z}_{>0} \) write \( \mathfrak{g}_r = \langle E_{ij} \mid i, j \in [r] \rangle \). Then \( \mathfrak{g}_r \) is a subalgebra of \( \mathfrak{g}(\mathbb{V}) \) isomorphic to \( \mathfrak{gl}(r, \mathbb{C}) \). The inclusions \( \mathfrak{g}_r \subset \mathfrak{g}_{r+1} \) correspond to the injective Lie algebra morphisms \( \mathfrak{gl}(r, \mathbb{C}) \to \mathfrak{gl}(r+1, \mathbb{C}) \) given by embedding a \( r \times r \) matrix into the upper left corner of a \( (r+1) \times (r+1) \) matrix with its last row and column filled with zeroes. Thus

\[
\mathfrak{g}(\mathbb{V}) = \bigcup_{r \geq 1} \mathfrak{g}_r \cong \varprojlim \mathfrak{gl}(r, \mathbb{C}) \cong \mathfrak{gl}(\infty).
\]

The choice of \( I \) does not change the isomorphism type of this algebra, so in general every algebra of the form \( \mathfrak{g}(\mathbb{V}, \mathbb{V}_s, \text{tr}) \) is isomorphic to \( \mathfrak{gl}(\infty) \).

Notice that \( \mathbb{V} \) and \( \mathbb{V}_s \) are \( \mathfrak{g}(\mathbb{V}) \)-modules with \( E_{ij}e_k = \delta_{jk}e_i \) and \( E_{ij}e^k = -\delta_{kj}e^i \). A simple comparison shows that \( \mathbb{V} = \lim V_r \), where \( V_r \) is the natural representation of \( \mathfrak{gl}(r, \mathbb{C}) \) and the maps \( V_r \to V_{r+1} \) are uniquely determined up to isomorphism by the fact that they are \( \mathfrak{gl}(r, \mathbb{C}) \)-linear. In a similar fashion, \( \mathbb{V}_s \cong \varprojlim V_r^* \).

3.2. Cartan subalgebra and root decomposition. The definition and study of Cartan subalgebras of \( \mathfrak{gl}(\infty) \) is an interesting and subtle question, which was taken up in [NP03]. General Cartan subalgebras can behave quite differently from Cartan subalgebras of finite dimensional reductive Lie algebras, for example they may not produce a root decomposition of \( \mathfrak{gl}(\infty) \). We will sidestep this problem by choosing one particularly well-behaved Cartan subalgebra as “the” Cartan subalgebra of \( \mathfrak{g}(\mathbb{V}) \), namely the obvious maximal commutative subalgebra \( \mathfrak{h}(\mathbb{V}) = \langle E_{ij} \mid i \in I \rangle \), and freely borrow notions from the theory of finite root systems that work for this particular choice.

The action of the Cartan subalgebra \( \mathfrak{h}(\mathbb{V}) \) on \( \mathfrak{g}(\mathbb{V}) \) is semisimple. Denoting by \( \varepsilon_i \) the unique functional of \( \mathfrak{h}(\mathbb{V})^* \) such that \( \varepsilon_i(E_{jj}) = \delta_{ij} \), the root system of \( \mathfrak{g}(\mathbb{V}) \) with respect to \( \mathfrak{h}(\mathbb{V}) \) is

\[
\Phi = \{ \varepsilon_i - \varepsilon_j \mid i, j \in I \}.
\]
with the component of weight $\varepsilon_i - \varepsilon_j$ spanned by $E_{i,j}$, and hence 1-dimensional. Also $g(V)_0 = h(V)$, which is of course infinite dimensional.

3.3. Positive roots, simple roots and Borel subalgebras. If we fix a total order $\preceq$ on $I$ (as we did implicitly when setting $I = \mathbb{N}$) we can see an element of $g(V)$ as an infinite matrix whose rows and columns are indexed by the set $I$. A total order also allows us to define a set of positive roots

$$\Phi_I^+ = \{ \varepsilon_i - \varepsilon_j \mid i \prec j \}$$

and a corresponding Borel subalgebra $b_I = h(V) \oplus \bigoplus_{\alpha \in \Phi_I^+} g(V)_\alpha$.

Again, Borel subalgebras of $g(\infty)$ behave quite differently than Borel subalgebras of $g(l,\mathbb{C})$, and their behaviour is tied to the order type of $I$. One striking difference is the following: simple roots can be defined as usual, namely as those positive roots which can not be written as the sum of two positive roots, but it may happen that simple roots span a strict subspace of the root space. In the extreme case $I = \mathbb{Q}$ there are no simple roots. In this article we will only consider a few well-behaved examples, the reader interested in the general theory of Borel subalgebras of $g(\infty)$ can consult [DPo4, DC08, DCPSo7].

Up to isomorphism, the only cases where simple roots span the full root space are $I = \mathbb{Z}_{>0}, \mathbb{Z}_{<0}$ and $\mathbb{Z}$ with their usual orders. These are called the Dynkin cases, and the corresponding Borel subalgebras are said to be of Dynkin type. An example of non Dynkin type comes from the choice $I = \mathbb{Z}^\times$, where the simple roots are

$$\{ \varepsilon_i - \varepsilon_{i+1} \mid i \in \mathbb{Z}^\times \setminus \{-1\} \},$$

but the root $\varepsilon_1 - \varepsilon_{-1}$ is not in their span. We get a basis of the root space by adding this extra root to the set of simple roots. A root is called finite if it is in the span of simple roots, otherwise it is called infinite. By definition we are in a Dynkin case if and only if every root is finite.

3.4. Further subalgebras and visual representation. For the rest of this article we will assume that the bases of $V$ and $V_\bullet$ are indexed by $\mathbb{Z}^\times$. Thus elements of $g(V)$ can be seen as infinite matrices with rows and columns indexed by $\mathbb{Z}^\times$ and finitely many nonzero entries. The choice of $\mathbb{Z}^\times$ as index set means that it makes sense to speak of the $k$-to-last row or column of an infinite matrix $g \in g(V)$ for any $k \in \mathbb{Z}_{>0}$.

We denote by $b(V)$ Borel subalgebra corresponding to $I = \mathbb{Z}^\times$, which is non-Dynkin since it has an infinite root in its support. We also denote by $n(V)$ the commutator subalgebra $[b(V), b(V)]$, so $b(V) = h(V) \oplus n(V)$. Notice that $n(V)$ is not nilpotent but only locally nilpotent.

We now introduce some useful subalgebras of $g(V)$. Fix $r \in \mathbb{Z}_{>0}$ and set

$$g(V)_r = \langle E_{i,j} \mid i, j \in \pm[r] \rangle$$

This is a finite dimensional subalgebra of $g(V)$, isomorphic to $g(2r, \mathbb{C})$. Clearly $g(V)_r \subset g_{r+1}(V)$ and $g(V) = \bigcup_{r \geq 1} g(V)_r$, so this is an exhaustion of $g(V)$. We also
set
\[ \mathbb{V}[r] = \langle v_i \mid r + 1 \leq i \leq -r - 1 \rangle \subset \mathbb{V}, \quad \mathbb{V}_s[r] = \langle v'_i \mid r + 1 \leq i \leq -r - 1 \rangle \subset \mathbb{V}_s. \]

The map \( \text{tr} \) restricts to a non-degenerate pairing between these two subspaces and we write \( g(\mathbb{V})[r] = g(\mathbb{V}[r], \mathbb{V}_s[r], \text{tr}) \). This Lie algebra is isomorphic to \( g(\mathbb{V}) \), and it is the centraliser of \( g_k(\mathbb{V}) \) inside \( g(\mathbb{V}) \).

We will meet many more subalgebras of \( g(\mathbb{V}) \), so we now introduce a visual aid to recall them. We will represent elements of \( g(\mathbb{V}) \) as squares, and we will represent a subalgebra by shading in gray the region where nonzero entries can be found, while unshaded areas will always be filled with zeroes. The following examples should help clarify this idea.

\[
\begin{array}{cccc}
\mathfrak{g}(\mathbb{V}) & \mathfrak{b}(\mathbb{V}) & \mathfrak{n}(\mathbb{V}) & \mathfrak{g}(\mathbb{V})[r] \\
\mathfrak{g}_k(\mathbb{V})
\end{array}
\]

Notice that with these representations each corner of the square contains finitely many entries, and that the centre of the square concentrates infinitely many entries.

3.5. **The Lie algebra** \( g(\mathbb{V}^n) \). Fix \( n \in \mathbb{N} \). We denote by \( e_i \) the \( i \)-th vector of the canonical basis of \( \mathbb{C}^n \), and by \( e' \) the \( i \)-th vector in the dual canonical basis of \( (\mathbb{C}^n)^* \). Set \( \mathbb{V}^n = \mathbb{V} \otimes \mathbb{C}^n, \mathbb{V}_s = \mathbb{V}_s \otimes (\mathbb{C}^n)^* \) and set
\[
\text{tr}^n : \mathbb{V}_s^o \otimes \mathbb{V}^n \to \mathbb{C},
\]
\[
(v^i \otimes e^k) \otimes (v'_j \otimes e'_l) \mapsto \delta_{i,j} \delta_{k,l}.
\]

We can form the Lie algebra \( g(\mathbb{V}^n) = g(\mathbb{V}^n, \mathbb{V}_s^n, \text{tr}^n) \) as before. Since \( \mathbb{V} \) and \( \mathbb{V}_s \) have bases indexed by \( \mathbb{Z}^\times \), the vector spaces \( \mathbb{V}^n \) and \( \mathbb{V}_s^n \) have bases indexed by \( \mathbb{Z}^\times \times [n] \). Since the isomorphism type of \( g(\mathbb{V}) \) is independent of the indexing set as long as it is infinite and denumerable, any bijection between \( \mathbb{Z}^\times \) and \( \mathbb{Z}^\times \times [n] \) will induce an isomorphism \( g(\mathbb{V}^n) \cong g(\mathbb{V}) \). Notice that under any such isomorphism \( h(\mathbb{V}^n) \) is mapped to \( h(\mathbb{V}) \).

On the other hand there is an obvious isomorphism of vector spaces \( g(\mathbb{V}^n) \cong g(\mathbb{V}) \otimes M_n(\mathbb{C}) \). Thus we can see elements of \( g(\mathbb{V}^n) \) as \( n \times n \) block matrices, with each block an infinite matrix from \( g(\mathbb{V}) \). We set for each \( k \in [n], r \in \mathbb{Z}_{>0} \)
\[
\mathbb{V}^{(k)} = \mathbb{V} \otimes \langle e_k \rangle \quad \mathbb{V}_s^{(k)} = \mathbb{V}_s \otimes \langle e_k \rangle;
\]

\[
\mathbb{V}^{(k)}[r] = \mathbb{V}[r] \otimes \langle e_k \rangle \quad \mathbb{V}_s^{(k)}[r] = \mathbb{V}_s[r] \otimes \langle e_k \rangle.
\]

so \( g(\mathbb{V}^n) = \bigoplus_{k,l \in [n]} (\mathbb{V}^{(k)} \otimes \mathbb{V}^{(l)}) \).

We highlight this particular avatar of \( \mathfrak{gl}(\infty) \) since it reveals some internal structure which is not obvious in its usual presentation. The first example is the Borel subalgebra
\( \mathfrak{b}(\mathbb{V}^n) \), which is awkward to handle as a subalgebra of \( \mathfrak{gl}(\infty) \) with the usual presentation. We also obtain a non-obvious exhaustion by setting \( \mathfrak{g}_k(\mathbb{V}^n) = \mathfrak{g}_k(\mathbb{V}) \otimes M_n(\mathbb{C}) \) for each \( k \in \mathbb{Z}_{>0} \).

Another feature of \( \mathfrak{gl}(\infty) \) which becomes clear by looking at its avatar \( \mathfrak{g}(\mathbb{V}^n) \) is that it has a \( \mathbb{Z}^n \)-grading compatible with the Lie algebra, inherited from the weight grading of \( M_n(\mathbb{C}) \) as \( \mathfrak{gl}(n, \mathbb{C}) \)-module. Thus for \( k, l \in [n] \) the direct summand \( \mathbb{V}^{(k)} \otimes \mathbb{V}^{(l)}_* \) is contained in the homogeneous component of degree \( e_k - e_l \). This grading highlights some interesting subalgebras of block diagonal and block upper-triangular matrices, namely

\[
I = \bigoplus_{i=1}^n \mathbb{V}^{(k)} \otimes \mathbb{V}^{(l)}_*; \quad U = \bigoplus_{i<j} \mathbb{V}^{(k)} \otimes \mathbb{V}^{(l)}_*; \quad P = \bigoplus_{i\leq j} \mathbb{V}^{(k)} \otimes \mathbb{V}^{(l)}_*,
\]

which are the subalgebras corresponding to 0, strictly positive, and non-negative \( \mathfrak{gl}(n, \mathbb{C}) \)-weights, respectively. Notice the obvious Levi-type decomposition \( P = I \oplus U \), where \( I \) is not reductive but rather locally reductive, and \( U \) is \( n-1 \)-step nilpotent.

### 3.6. Subalgebras of \( \mathfrak{g}(\mathbb{V}^n) \) and their visual representation

As before, we will use some visual aids to describe the various subalgebras of \( \mathfrak{g}(\mathbb{V}^n) \). We will take \( n = 3 \) for these visual representations, and hence represent a subalgebra of \( \mathfrak{g}(\mathbb{V}^n) \) by shading regions in a three-by-three grid of squares. We refer to these as the pictures of the subalgebras. The following examples should clarify the idea.
Recall the subalgebras $g(\mathbb{V}^n)$, for all $r \in \mathbb{Z}_{>0}$. We denote by $g(\mathbb{V}^n)[r]$ the subalgebra of $g(\mathbb{V}^n)$, invariants inside $g(\mathbb{V}^n)$, and by $l[r]$ its intersection with $l$.

We also introduce the parabolic subalgebra $p(r) = l[r] + b$. This algebra again has a Levi-type decomposition $p(k) = l[r]^+ \oplus u(k)$, where $l[r]^+ = l[r] + h$ and $u(r)$ is the unique ideal giving this decomposition.

Another family of algebras we will study are related to the finite and infinite roots of $g(\mathbb{V}^n)$. We denote by $s$ the subalgebra spanned by all finite root spaces of $g(\mathbb{V}^n)$, and by $q$ the parabolic subalgebra $s + b(\mathbb{V}^n)$. This subalgebra has a Levi-type decomposition $q = s \oplus m$, where $m$ is the subspace spanned by all root spaces corresponding to positive infinite roots.

We will also occasionally need the exhaustion of $s$ given by $s_r = s \cap g(\mathbb{V}^n)_r$, whose picture is left for the interested reader.

### 3.7. Transpose automorphism.
Denote by $E_{ij}^{(k,l)}$ the element $E_{ij} \otimes e_k \otimes e_l$. The Lie algebra $g(\mathbb{V}^n)$ has an anti-automorphism $\tau$, analogous to transposition in $gl(r, \mathbb{C})$, given by $\tau(E_{ij}^{(k,l)}) = E_{ji}^{(l,k)}$. Given a subalgebra $\mathfrak{t} \subset g(\mathbb{V}^n)$ its image by $\tau$ will also be denoted by $\mathfrak{t}$. Thus we have further subalgebras $\overline{\mathfrak{b}}, \overline{\mathfrak{p}}, \overline{\mathfrak{u}}$, etc. Notice that the picture of the image of a subalgebra by $\tau$ is the reflection of the picture of the subalgebra by the
3.8. Roots, weights and eligible weights. We denote by \( \varepsilon_i^{(k)} \) the functional in \( h(\mathbb{V}^n)^* \) given by \( \varepsilon_i^{(k)}(E_{i,j}^{(l)}) = \delta_{ij} \delta_{k,l} \). By a slight abuse of notation we can represent any element of \( h(\mathbb{V}^n)^* \) as an infinite sum of the form \( \sum_{i,k} a_i^{(k)} \varepsilon_i^{(k)} \) with \( a_i^{(k)} \in \mathbb{C} \). We denote by \( \omega^{(k)} \) the functional \( \omega^{(k)} = \sum_{i \in \mathbb{Z}^n} \varepsilon_i^{(k)} \). We set \( h(\mathbb{V}^n)^r \) to be the span of \( \{ \varepsilon_i^{(k)}, \omega^{(k)} \mid i \in \mathbb{Z}^n, k \in [n] \} \), and refer to its elements as eligible weights. A weight will be called \( r \)-eligible if it is a linear combination of the \( \omega^{(k)} \) and the \( \varepsilon_i^{(k)} \) with \( i \in \pm [r] \).

As we will see in section 4, \( r \)-eligible weights parametrise the simple finite dimensional representations of \( l[r]^+ \), which are all 1-dimensional. The importance of these representations will be discussed in that same section.

As usual, \( \Lambda \) denotes the \( \mathbb{Z} \)-span of the roots of \( g(\mathbb{V}^n) \) inside \( h(\mathbb{V}^n)^* \). It is a free \( \mathbb{Z} \)-module spanned by the families of roots

\[
\begin{align*}
\varepsilon_i^{(k)} - \varepsilon_{i+1}^{(k)} & \quad k \in [n], i \in \mathbb{Z}^n \setminus \{-1\}; \\
\varepsilon_{i-1}^{(k)} - \varepsilon_i^{(k+1)} & \quad k \in [n-1]; \\
\varepsilon_{i-1}^{(k)} - \varepsilon_i^{(k)} & \quad k \in [n].
\end{align*}
\]

The first two families consist of finite roots and any finite root is a linear combination of them, while the roots in the last family are infinite roots.

We denote by \( \Lambda_C^+ \) the \( \mathbb{C} \)-span of the \( \varepsilon_i^{(k)} \). There is a pairing \( (-,-) : h(\mathbb{V}^n)^* \otimes \Lambda_C^+ \rightarrow \mathbb{C} \) with \( (\lambda, \varepsilon_i^{(k)}) = \lambda(E_{i,j}^{(k)}) \). This allows us to define for any root \( \alpha \) the reflection

\[
s_\alpha : h(\mathbb{V}^n)^* \rightarrow h(\mathbb{V}^n)^* \\
\lambda \mapsto \lambda - \frac{2(\lambda, \alpha)}{(\alpha, \alpha)} \alpha.
\]

The group generated by these reflections is denoted by \( \mathcal{W} \), and by analogy with the finite dimensional case is called the **Weyl group** of \( g(\mathbb{V}^n) \).

Set \( h_r = h(\mathbb{V}^n) \cap g(\mathbb{V}^n)_r \) and for \( \lambda \in h(\mathbb{V}^n) \) denote by \( \lambda |_r \) its restriction to \( h_r \). Set also \( \mathcal{W}_r \) to be the group generated by the reflections \( s_\alpha \) with \( \alpha \) a root of \( g(\mathbb{V}^n)_r \). Clearly \( \mathcal{W} \) is the union of the \( \mathcal{W}_r \), and hence the direct limit of the Weyl groups of the Lie algebras in the exhaustion \( \{ g(\mathbb{V}^n)_r \}_{r \geq 0} \). Furthermore, if \( \sigma \in \mathcal{W}_r \) and \( \lambda \in h(\mathbb{V}^n)_r \) then \( \sigma(\lambda)|_r = \sigma(\lambda |_r) \). The group \( \mathcal{W} \) is isomorphic to the group of bijections of \( \mathbb{Z}^n \times [n] \) that leave a cofinite set fixed, and its action on \( h(\mathbb{V}^n)^* \) is given by the corresponding permutation of the \( \varepsilon_i^{(k)} \), suitably extended.

There is also an analogue of the dot action of the Weyl group for \( h(\mathbb{V}^n)^* \). Set \( \rho = \sum_{i,k} -i \varepsilon_i^{(k)} \), and set for every \( \sigma \in \mathcal{W} \) and every \( \lambda \in h(\mathbb{V}^n)^* \)

\[
\sigma \cdot \lambda = \sigma(\lambda + \rho) - \rho.
\]
Both the usual and dot action send eligible weights to eligible weights but the dot action of the Weyl group of \( g(\mathbb{V}^n) \) does not restrict to the dot action of \( g(\mathbb{V}^n) \). However, if we denote by \( \mathcal{W}(s) \) and \( \mathcal{W}(s_r) \) the groups generated by reflections associated to finite roots of \( g(\mathbb{V}^n) \) and \( g(\mathbb{V}^n) \), respectively, then \( \mathcal{W}^\infty \) is the union of the \( \mathcal{W}(s_r) \) and the dot action of \( \sigma \in \mathcal{W}(s_r) \) on \( \mathfrak{h}(\mathbb{V}^n)^* \) does satisfy that \( (\sigma \cdot \lambda)|_r = \sigma \cdot (\lambda|_r) \).

We define an order \( <_{\text{fin}} \) on weights defined as follows: \( \mu <_{\text{fin}} \lambda \) if and only if \( \mu = \sigma \cdot \lambda \) for some \( \sigma \in \mathcal{W}(s) \) and \( \lambda - \mu \) is a sum of positive roots, which necessarily will be finite. The following lemma will be used several times in the sequel. The argument originally appeared in the proof of [PS10, Lemma 4.10].

**Lemma 3.1.** Let \( \lambda, \mu \) be \( r \)-eligible weights, with \( \lambda \geq \mu \). If \( \lambda|_s \) and \( \mu|_s \) are linked for all \( s \geq r \) then \( \mu = \sigma \cdot \lambda \), with \( \sigma \in \mathcal{W}(s_{r+1}) \). Thus \( \lambda - \mu \) is a sum of positive finite roots of \( g(\mathbb{V}^n) \), and in particular \( \mu <_{\text{fin}} \lambda \).

**Proof.** Denote by \( \sigma_s \) the element in the Weyl group of \( g(\mathbb{V}^n)_s \) such that \( \sigma_s \cdot \lambda|_s = \mu|_s \), and by \( \rho \) the half-sum of the positive roots of \( g \). If \( \alpha \) is a simple root of \( g(\mathbb{V}^n)_s \) that is also a root of \( \mathfrak{h}(\mathbb{V}^n) \) then \( (\mu, \alpha) = 0 \), so \( \sigma_s \) can only involve reflections \( s_\beta \) with \( \beta \) a root of \( g(\mathbb{V}^n)_{r+1} \). It follows then that

\[
0 = (\lambda|_s - \sigma_s(\mu|_s), \alpha) = (\sigma_s(\rho_s) - \rho_s, \alpha)
\]

for all \( s \) and this is only possible if \( \sigma_s \) only involves reflection through simple roots, i.e. if \( \sigma_s \in \mathcal{W}(s_{r+1}) \).

4. Representation theory

Throughout this section \( g \) is a Lie algebra with a fixed splitting Cartan subalgebra \( \mathfrak{h} \), and \( \mathfrak{t} \) is a root-subalgebra of \( g \) containing \( \mathfrak{h} \).

4.1. Socle filtrations. Let \( M \) be a \( g \)-module. Recall that the socle of \( M \) is the largest semisimple \( g \)-submodule contained in \( M \) and is denoted by \( \text{soc} M \). The socle filtration of \( M \) is the filtration defined inductively as follows: \( \text{soc}^{(0)} M = \text{soc} M \), and \( \text{soc}^{(n+1)} M \) is the preimage of \( \text{soc}(M/\text{soc}^{(n)} M) \) by the quotient map \( M \to M/\text{soc}^{(n)} M \). The layers of the filtration are the semisimple modules \( \overline{\text{soc}}^{(n)} M = \text{soc}^{(n)} M/\text{soc}^{(n-1)} M \).

4.2. Categories of weight modules. Given a \( g \)-module \( M \) and \( \lambda \in \mathfrak{h}^* \) we denote by \( M_\lambda \) the subspace of \( \mathfrak{h} \)-eigenvectors of eigenvalue \( \lambda \), and refer to the set of all \( \lambda \) with \( M_\lambda \neq 0 \) as the support of \( M \). We say \( M \) is a weight module if \( M = \bigoplus \lambda M_\lambda \). We denote by \( (g, \mathfrak{h}) \)-Mod the full subcategory of \( g \)-Mod whose objects are weight modules.

The inclusion of \( (g, \mathfrak{h}) \)-Mod in \( g \)-Mod is an exact functor with right adjoint \( \Gamma_\mathfrak{h} : g \)-Mod \( \to (g, \mathfrak{h}) \)-Mod that assigns to each module \( M \) the largest \( \mathfrak{h} \)-semisimple submodule it contains. By standard homological algebra, this functor is left exact and sends injective objects to injective objects and direct limits to direct limits. In particular \( (g, \mathfrak{h}) \)-Mod has enough injective objects. Given two weight modules \( M, N \) we denote by \( \text{Hom}_{g, \mathfrak{h}}(M, N) \) the space of morphisms in the category of weight modules, and by \( \text{Ext}^*_{g, \mathfrak{h}}(N, M) \) the corresponding derived functors.
Given any \( \mathfrak{g} \)-modules \( M, N \) the space \( \text{Hom}_\mathfrak{g}(M, N) \) is a \( \mathfrak{h} \)-module in a natural way, and we denote by \( \text{Hom}_\mathfrak{g}(M, N) \) the subspace spanned by its \( \mathfrak{h} \)-semisimple vectors. If \( M \) is a weight representation then a map \( \varphi : M \to N \) is semisimple of weight \( \lambda \) if and only if \( \varphi(M_\mu) \subset N_{\lambda + \mu} \) for any weight \( \mu \).

4.3. **Induction and coinduction for weight modules.** The restriction functor \( \text{Res}^\mathfrak{g} : (\mathfrak{g}, \mathfrak{h})\text{-Mod} \to (\mathfrak{t}, \mathfrak{h})\text{-Mod} \) has a left adjoint, given by the usual induction functor \( \text{Ind}^\mathfrak{g} : (\mathfrak{g}, \mathfrak{h})\text{-Mod} \to (\mathfrak{t}, \mathfrak{h})\text{-Mod} \); both functors are exact. We often write just \( N \) instead of \( \text{Res}^\mathfrak{g} N \). Restriction also has a right adjoint \( \text{Coind}^\mathfrak{g} : (\mathfrak{g}, \mathfrak{h})\text{-Mod} \to (\mathfrak{t}, \mathfrak{h})\text{-Mod} \), given by

\[
\text{Coind}^\mathfrak{g} M = \text{Hom}_\mathfrak{g}(U(\mathfrak{g}), M)
\]

By definition a morphism \( \varphi : U(\mathfrak{g}) \to M \) will be in \( (\text{Coind}^\mathfrak{g} M)_\mu \) if it is \( \mathfrak{t} \)-linear and maps the weight space \( U(\mathfrak{g})_\lambda \) to \( M_{\lambda + \mu} \). In particular the semisimple coinduction of \( M \) only depends on \( \Gamma^\mathfrak{g}(M) \).

Unlike usual coinduction, semisimple coinduction is not exact. However it is left exact, sends injective objects to injective objects and direct limits to direct limits. This follows from the fact that it is right adjoint to an exact functor.

4.4. **Semisimple duals.** Suppose we have fixed an antiautomorphism \( \tau \) of \( \mathfrak{g} \) which restricts to the identity over \( \mathfrak{h} \) (in all our examples \( \mathfrak{g} \) will be \( \mathfrak{gl}(r, \mathbb{C}) \) or \( \mathfrak{gl}(\infty, \mathbb{C}) \), and \( \tau \) will be the transposition map). Denote by \( T \) the antiautomorphism of \( U(\mathfrak{g}) \) induced by \( \tau \) and by \( \overline{T} \) the image of \( \mathfrak{t} \) by \( \tau \). If \( M \) is a \( \mathfrak{t} \)-module with structure map \( \rho : \mathfrak{t} \to \text{End}(M) \) we can define a \( \overline{T} \)-module \( \overline{T}M \) whose underlying vector space is \( M \) and whose structure map is \( \rho \circ \tau \). If \( M \) is a \( \mathfrak{t} \)-module with structure map \( \rho : \mathfrak{t} \to \text{End}(M) \) we can define a \( \overline{T} \)-module \( \overline{T}M \) whose underlying vector space is \( M \) and whose structure map is \( \rho \circ \tau \). This assignation is functorial and commutes with the functor \( \Gamma^\mathfrak{g} \). The **semisimple dual** of a \( \mathfrak{t} \)-module \( M \) is the \( \overline{T} \)-module \( M^\vee = \overline{T} \Gamma^\mathfrak{g}(M^*) \). The definition is analogous to the duality operator of category \( \mathcal{O} \) for finite dimensional reductive Lie algebras. Indeed, we have \( (M^\vee)_\lambda = (M_\lambda)^* \), and if \( f \in M^\vee_\lambda \) then \( (u \cdot f)(m) = f(T(u)m) \) for all \( u \in U(\mathfrak{t}) \) and \( m \in M \). We have the following result relating semisimple duals with induction and semisimple coinduction functors.

**Proposition 4.4.** Let \( \mathfrak{t} \subset \mathfrak{g} \) be a subalgebra containing \( \mathfrak{h} \), and let \( M \) be a weight \( \mathfrak{t} \)-module. There is a natural isomorphism

\[
(\text{Ind}^\mathfrak{g}_\mathfrak{t} M)^\vee \cong \text{Coind}^\mathfrak{g}_\mathfrak{t} M^\vee.
\]

In particular semisimple coinduction is exact over the image of the semisimple dual functor.

**Proof.** By [Dix77, 5.5.4 Proposition] there exists a natural isomorphism \( (\text{Ind}^\mathfrak{g}_\mathfrak{t} M)^* \cong \text{Coind}^\mathfrak{g}_\mathfrak{t} M^* \). If we apply \( \Gamma^\mathfrak{g} \) and then twist by the automorphism \( T \) we get

\[
(\text{Ind}^\mathfrak{g}_\mathfrak{t} M)^\vee \cong \Gamma^\mathfrak{g} \left( \overline{T} \text{Hom}_\mathfrak{t}(U(\mathfrak{g}), M^*) \right).
\]

The automorphism \( T \) defines an equivalence between the categories of weight \( \mathfrak{t} \) and \( \overline{T} \)-modules, so there is a natural isomorphism \( \text{Hom}_\mathfrak{t}(U(\mathfrak{g}), M^*) \cong \text{Hom}_{\overline{T}}(\overline{T}U(\mathfrak{g}), \overline{T}M^*) \).
Twisting the left \( g \)-action on this module by \( T \) corresponds to twisting the right \( g \)-action of \( U(\mathfrak{g}) \) by \( T \), so

\[
\Gamma_b^{}(\operatorname{Hom}_\mathfrak{g}(\mathfrak{g}, M^*)) \cong \operatorname{Hom}_\mathfrak{g}(\Gamma_b^{}(U(\mathfrak{g}), M^*)),
\]

where the last isomorphism comes from the fact that \( T : U(\mathfrak{g}) \to \Gamma_b^{}(U(\mathfrak{g})) \) defines an isomorphism of \( U(\mathfrak{g}) \)-\( \mathfrak{g} \)-bimodules. Thus

\[
\Gamma_b^{}(\operatorname{Hom}_\mathfrak{g}(\mathfrak{g}, M^*)) \cong \operatorname{Hom}_\mathfrak{g}(U(\mathfrak{g}), \Gamma_b^{}(T M^*)) = \operatorname{Coind}_\mathfrak{g}^{} M^\vee.
\]

The last statement follows from the fact that induction and semisimple duality are exact functors.

4.5. **Torsion and inflation.** Let \( M \) be a \( \mathfrak{g} \)-module. We say that \( m \in M \) is \( \mathfrak{t} \)-integrable if for any \( g \in \mathfrak{t} \) the dimension of the span of \( \{g^r m \mid r \geq 0 \} \) is finite. This happens in particular if \( g^r m = 0 \) for \( r \gg 0 \) depending on \( g \), and in this case we say that \( M \) is \( \mathfrak{t} \)-locally nilpotent. We say that \( m \) is \( \mathfrak{t} \)-torsion if \( g^r m = 0 \) and \( r \) can be chosen independently of \( g \). We say that \( M \) is \( \mathfrak{t} \)-integrable, nilpotent, or torsion if each of its elements is.

The subspace of \( \mathfrak{t} \)-torsion vectors of a \( \mathfrak{g} \)-module \( M \) is again a \( \mathfrak{g} \)-module, which we denote by \( \Gamma_\mathfrak{t}^{}(M) \). This is a left exact functor, as can be easily checked. Denoting by \( I \), the right ideal generated by \( \mathfrak{t} \) inside \( U(\mathfrak{g}) \), we see there exists a natural isomorphism

\[
\Gamma_{\mathfrak{n}} \cong \lim_{\longrightarrow} \operatorname{Hom}_{\mathfrak{g}}(U(\mathfrak{g})/I, -).
\]

Now suppose \( \mathfrak{g} = \mathfrak{t} \oplus \mathfrak{r} \). The projection \( \mathfrak{g} \to \mathfrak{g}/\mathfrak{r} \cong \mathfrak{t} \) induces a functor

\[
\mathcal{T}^\mathfrak{g}_{\mathfrak{t}} : (\mathfrak{t}, \mathfrak{h}) \text{-Mod} \to (\mathfrak{g}, \mathfrak{h}) \text{-Mod}.
\]

We refer to this as the **inflation** functor. It can be identified with \( \operatorname{Hom}_{\mathfrak{g}}(U(\mathfrak{t}), -) \) where \( U(\mathfrak{t}) \) is seen as a right \( \mathfrak{g} \)-module, which implies it is exact and has a left adjoint \( M \mapsto U(\mathfrak{t}) \otimes_{\mathfrak{g}} M \cong M/U(\mathfrak{g}) \mathfrak{r} M \). We will write \( M \) for \( \mathcal{T}^\mathfrak{g}_{\mathfrak{t}}(M) \) when the context makes clear that we are seeing \( M \) as a \( \mathfrak{g} \)-module.

4.6. **Weights, gradings and parabolic subalgebras.** Given any morphism of abelian groups \( \varphi : \mathfrak{h}^* \to A \), we can turn an semisimple \( \mathfrak{h} \)-module into an \( A \)-graded vector space by setting \( M_\lambda^{} = \bigoplus_{\varphi(\lambda) = \sigma^} M_\lambda^{} \). We will denote this \( A \)-graded vector space by \( M^\varphi \), though we will often omit the superscript when it is clear from context. The assignation \( M^\varphi \) is clearly functorial, and in particular turns \( \mathfrak{g} \) into an \( A \)-graded Lie algebra, and any weight module into an \( A \)-graded \( \mathfrak{g} \)-module.

Suppose now that \( A = \mathbb{C}^n \) and that \( \varphi(a) \in \mathbb{Z}^n \) for each root. For \( \chi, \xi^{} \in \mathbb{C}^n \) we write \( \chi \succeq \xi^{} \) if \( \chi - \xi^{} \in \mathbb{Z}_{\geq 0}^n \). We get a decomposition \( \mathfrak{g} = \mathfrak{g}_{\geq 0}^{} \oplus \mathfrak{g}_{\geq 0}^{} \oplus \mathfrak{g}_{\geq 0}^{} \). If we have defined a set of positive roots \( \Phi^+ \) along with a corresponding Borel subalgebra \( \mathfrak{b} \), and if \( \varphi(a) \succeq 0 \) for every positive root, then the subalgebra \( \mathfrak{g}_{\geq 0}^{} \) is a parabolic subalgebra of \( \mathfrak{g} \).

**Example.** Fix \( n \in \mathbb{Z}_{> 0} \) and set \( \mathfrak{g} = \mathfrak{g}(\mathbb{V}^n) \). 

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We fix $n \in \mathbb{N}$ and $\mathfrak{h}^\circ = \mathfrak{h}(\mathbb{C}^n)^\circ$. Consider the $\mathbb{C}$-linear map $\varphi : \mathfrak{h}(\mathbb{C}^n)^\circ \to \mathbb{C}^n$ given by $\varphi(e_i^{(k)}) = \varphi(\omega_i^{(k)}) = e_i$, and extend to $\mathfrak{h}^\circ$ arbitrarily. Since roots are mapped to vectors in $\mathbb{Z}^n$, this map induces a $\mathbb{Z}^n$-grading on $\mathfrak{gl}(\infty)$, which coincides with the one introduced in [DGK82]. The corresponding parabolic subalgebra is $\mathfrak{gl}_0 = \mathfrak{p}$ with $\mathfrak{g}_0^0 = \mathfrak{l}$ and $\mathfrak{g}_{\geq 0}^0 = \mathfrak{u}$.

Fix $r > 0$ and denote by $\rho_a$ the sum of all positive roots of $s_r$. Set $\theta : \mathfrak{h}^\circ \to \mathbb{Z}$ to be the map $\alpha \mapsto (\alpha, \rho_a) + (\varphi(\alpha), \rho_n)$, where we are seeing $\rho_n$ as a vector of $\mathbb{C}^n$ in the usual way. The corresponding parabolic subalgebra is $\mathfrak{p}(r)$ with $\mathfrak{g}_{\geq 0} = \{[r]\}$ and $\mathfrak{g}_{\geq 0}^0 = \mathfrak{u}(r)$.

Let $\psi : \mathfrak{h}(\mathbb{C}^n)^\circ \to \mathbb{C}$ be the map $\psi(e_i^{(k)}) = \frac{1}{2}(n + 1 - 2k + \text{sg}(i))$ (here $\text{sg}(i)$ is 1 if $i$ is a positive integer and $-1$ if it is negative) and $\psi(\omega_i^{(k)}) = 0$. This formula has the nice property that if $\alpha$ is a finite root then $\psi(\alpha) = 0$, while the infinite roots $e_{i-1}^{(k)} - e_i^{(k)}$ are mapped by $\psi$ to 1. The corresponding parabolic subalgebra is $\mathfrak{q}$ with $\mathfrak{g}_{\geq 0} = \mathfrak{s}$ and $\mathfrak{g}_{\geq 0}^0 = \mathfrak{m}$.

These maps and the induced gradings will reappear throughout the rest of the article, so we fix the notations $\varphi, \theta$ and $\psi$ for them.

### 4.7. Local composition series

In general it is hard to establish a priori whether a $\mathfrak{g}$-module has a composition series. For this reason we introduce the following notion taken from [DGK82] Proposition 3.2].

**Definition 4.2.** Let $M$ be a $\mathfrak{g}$-module and let $\lambda \in \mathfrak{h}^\circ$. We say that $M$ has a *local composition series at* $\lambda$ if there exist a finite filtration

$$M = F_0 M \supset F_1 M \supset \cdots \supset F_t M = \{0\}$$

and a finite set $J \subset \{0, 1, \cdots, t - 1\}$ such that

(i) if $j \in J$ then $F_j M / F_{j+1} M \cong L(\lambda_j)$ for some $\lambda_j \geq \lambda$;

(ii) if $j \notin J$ and $\mu \geq \lambda$ then $(F_{j-1} M / F_j M)_\mu = 0$.

We say that $M$ has local composition series, or LCS for short, if it has a local composition series at all $\lambda \in \mathfrak{h}^\circ$.

It follows from the definition that a local composition series at $\lambda$ induces a local composition series at $\lambda'$ for any $\lambda' \geq \lambda$, possibly with a different set $J$. A standard argument shows that if $M$ has two local composition series at $\lambda$ then the multiplicities of any simple object $L(\mu)$ in each series coincide. If $M$ has LCS then we denote by $[M : L(\mu)]$ this common multiplicity. The class of modules having LCS is closed under submodules, quotients and extensions, and multiplicity is additive for extensions.

### 4.8. Finite dimensional representations of $\mathfrak{gl}(\infty)$

For each $a \in \mathbb{C}$ the algebra $\mathfrak{gl}(\infty)$ has a one-dimensional representation $\mathbb{C}_a$, where $g \in \mathfrak{gl}(\infty)$ acts by multiplication by $a \text{tr}(g)$; in particular $\mathbb{C}_a$ is a trivial $\mathfrak{sl}(\infty)$-module. These are the only simple finite dimensional representations of $\mathfrak{gl}(\infty)$, and any finite dimensional weight representation is a finite direct sum of these. In order to find more interesting representations, we
need to look for infinite dimensional modules. Finitely generated modules of $\mathfrak{gl}(\infty)$ do not form an abelian category, since $U(\mathfrak{gl}(\infty))$ is not left-noetherian, so we need to look for an alternative notion of a “small” $\mathfrak{gl}(\infty)$-module.

4.9. The large annihilator condition. Let $\mathfrak{t} \subseteq \mathfrak{gl}(\infty)$ be a subalgebra. Let $M$ be a $\mathfrak{gl}(\infty)$-module and let $m \in M$. We say that $m$ satisfies the large annihilator condition (LAC from now on) with respect to $\mathfrak{t}$ if there exists a finite dimensional subalgebra $\mathfrak{t} \subseteq \mathfrak{t}$ such that $[\mathfrak{t}^\ell, \mathfrak{t}^\ell]$, the derived subalgebra of the centraliser of $\mathfrak{t}$ in $\mathfrak{t}$, acts trivially on $\mathbb{C}m$. In other words, the annihilator of $m$ contains the “large” subalgebra $[\mathfrak{t}^\ell, \mathfrak{t}^\ell]$. We say that $M$ satisfies the LAC if every vector in $M$ satisfies the LAC.

The adjoint representation of $\mathfrak{gl}(\infty)$ satisfies the LAC with respect to itself, and hence with respect to any other subalgebra $\mathfrak{t}$. If $m \in M$ satisfies the LAC with respect to $\mathfrak{t}$ then the $\mathfrak{gl}(\infty)$ module generated by $m$ also satisfies the LAC. The tensor product of two representations satisfying the LAC again satisfies the LAC.

Denote by $(\mathfrak{gl}(\infty), \mathfrak{h})$-$\text{Mod}^\text{LA}$ the full subcategory of modules satisfying the LAC with respect to $\mathfrak{t}$. The natural inclusion of this category in $(\mathfrak{gl}(\infty), \mathfrak{h})$-$\text{Mod}$ has a right adjoint, which we denote by $\Phi_a$, or just $\Phi$ for simplicity. Being right adjoint to an exact functor, this functor is left exact and sends direct limits to direct limits and injective objects to injective objects.

Let us consider the case $\mathfrak{t} = \mathfrak{gl}(\infty)$ in more detail. For simplicity we fix the exhaustion $\mathfrak{g}(\mathcal{V})_r$ associated to the order $\mathfrak{l} = \mathbb{Z}_+^n$. If $M$ satisfies the large annihilator condition and $m \in M$, the finite dimensional Lie algebra $\mathfrak{t}$ is contained in $\mathfrak{g}(\mathcal{V})_r$, for some $r \gg 0$, so we might as well take $\mathfrak{t} = \mathfrak{g}(\mathcal{V})_r$. Now the algebra $\mathfrak{gl}(\infty)^\mathfrak{t}$ is the subalgebra we denoted by $\mathfrak{g}(\mathcal{V})_r$ and is isomorphic to $\mathfrak{gl}(\infty)$. Thus $\mathbb{C}m$ is a 1-dimensional representation of $\mathfrak{g}(\mathcal{V})$, and hence isomorphic as such to $\mathbb{C}_a$ for some $a \in \mathbb{C}$.

4.10. The large annihilator condition over $\mathfrak{l}$. Recall we have set $\mathfrak{l} = \mathfrak{g}_0$ for the $\mathbb{Z}^n$-grading introduced in subsection 4.5. In the sequel we will mostly consider modules satisfying the large annihilator condition with respect to $\mathfrak{l}$, so we now focus on this condition. Since $\mathfrak{l} = \mathfrak{g}(\mathcal{V}^{(1)}) \oplus \cdots \oplus \mathfrak{g}(\mathcal{V}^{(n)})$, its one-dimensional representations are external tensor products of the form $\mathbb{C}_{a_1} \otimes \cdots \otimes \mathbb{C}_{a_n}$ with $\chi = (a_1, \ldots, a_n) \in \mathbb{C}^n$.

A vector $m \in M$ satisfying the LAC with respect to $\mathfrak{l}$ must span a one-dimensional module over $[\mathfrak{l}] \cong \mathfrak{l}$, for some $r \gg 0$. This forces the weight of $m$ to be $r$-eligible and hence in $h(\mathcal{V}^\chi)^\circ$. In other words, the weight must be of the form

$$\sum_{k=1}^n a_k \omega^{(k)} + \sum_{k=1}^n \sum_{i \in \mathbb{Z}^r} b_i^{(k)} \xi_i^{(k)}.$$ 

We call the $n$-tuple $\chi$ the level of $m$, or rather of its weight. Any vector in the module spanned by $m$ has the same level as $m$, and denoting by $M^\chi$ the space of all vectors of level $\chi$ we see that $M = \bigoplus_{\chi \in \mathbb{C}^n} M^\chi$. Thus every vector in an indecomposable module $M$ satisfying the LAC with respect to $\mathfrak{l}$ must have the same level, and we define this to be the level of $M$.  

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For each \( r \geq 1 \) we set \( \Phi_r : (\mathfrak{g}, \mathfrak{h}) \)-Mod \( \rightarrow (\mathfrak{g}_r, \mathfrak{h}_r) \)-Mod to be the functor that assigns to a module \( M \) the space of invariants \( M^{[r]} \). If \( m \in M_\lambda \) spans a trivial \( [r]^+ \)-module then \( \mathbb{C} m \cong \mathbb{C}_\lambda \) as \( [r]^+ \)-modules. We can use this observation to get natural isomorphisms of functors

\[
\Phi_r \cong \bigoplus_\lambda \text{Hom}_{\mathfrak{g}}(U(\mathfrak{g}) \otimes [r]^+, \mathbb{C}_\lambda, -); \quad \Phi \cong \bigoplus_{\lambda \in \mathfrak{h}^*} \lim_{\lambda \in \mathfrak{h}^*} \text{Hom}_{\mathfrak{g}}(U(\mathfrak{g}) \otimes [r]^+, \mathbb{C}_\lambda, -),
\]

where the first sum is taken over the space of all \( r \)-eligible weights.

5. Some categories of representations of \( \mathfrak{gl}(\infty) \)

Fix \( n \in \mathbb{N} \) and set \( \mathfrak{g} = \mathfrak{g}(\mathbb{V}^n) \). We omit the symbol \( \mathbb{V}^n \) from this point on and simply write \( b, s, \mathfrak{g}_r, \) etc. for the subalgebras introduced in Subsection 3.6.

5.1. Category \( \mathcal{O} \) for Dynkin Borel algebras. In this subsection we review some results from Nampaisarn’s thesis [Nam17], where he introduces and studies category \( \mathcal{O} \). We will state his results for the Dynkin subalgebra \( s \subset \mathfrak{g} \) since this will be the only example we will need. We set \( b_s = b \cap s \) and \( n_s = [b_s, b] \). For each \( r \in \mathbb{Z}_{>0} \) we set \( s_r = s \cap \mathfrak{g}_r \), which is the Lie subalgebra of \( \mathfrak{g}_r \) spanned by finite roots, and is isomorphic to \( \mathfrak{gl}(r, \mathbb{C})^2 \oplus \mathfrak{gl}(2r, \mathbb{C})^{n-1} \), in particular it is reductive.

**Definition 5.1.** An \( s \)-module \( M \) lies in \( \mathcal{O} = \mathcal{O}^\mathfrak{g}_{b_s} \) if it is \( \mathfrak{h} \)-semisimple, \( n_s \)-torsion, and \( \dim M_\lambda < \infty \) for each \( \lambda \in \mathfrak{h}^* \).

The definition mimics that of \( \mathcal{O} \) for finite dimensional reductive algebras. It makes sense for arbitrary locally reductive algebras with a fixed Borel subalgebra and many results from category \( \mathcal{O}_s \) extend to \( \mathcal{O} \) in a straightforward manner. On the other hand the category lacks several obvious modules, for example the adjoint representation; if we waive the Dynkin hypothesis then Verma modules are also excluded. Still, it contains many interesting objects, and turns out to be an important stepping stone in the study of further categories of representations.

Since \( b_s \) is a Dynkin Borel subalgebra, given \( \lambda \in \mathfrak{h}^* \) the Verma module \( M_\lambda(\lambda) \) and its simple subquotient \( L_s(\lambda) \) belong to \( \mathcal{O} \). The category is also closed under semisimple duals, so the dual Verma module \( M(\lambda)^\vee \) also belongs to \( \mathcal{O} \). A weight \( \lambda \in \mathfrak{h}^* \) is integral if \( (\lambda, \alpha) \in \mathbb{Z} \) for any simple root \( \alpha \) of \( s \), dominant if \( (\lambda + \rho, \alpha) \notin \mathbb{Z}_{<0} \), and almost dominant if \( (\lambda + \rho, \alpha) \in \mathbb{Z}_{<0} \) for only finitely many simple roots \( \alpha \).

**Theorem 5.2** ([Nam17] Theorem 6.7, Proposition 8.11]). Let \( \lambda, \mu \in \mathfrak{h}^* \).

(a) If \( M(\mu) \subset M(\lambda) \) then \( \lambda \succeq \mu \) and there exists \( \sigma \in \mathcal{W}(s) \) such that \( \mu = \sigma \cdot \lambda \).

(b) If \( M \) is a highest weight module with highest weight vector \( m \in M_\lambda \) then for \( r \gg 0 \)

\[
[M : L_s(\mu)] = [U(s_r)m : L_{s_r}(\mu)].
\]

It follows that \( m(\lambda, \mu) = [M_s(\lambda) : L_s(\mu)] \) coincides with the Kazhdan-Lusztig multiplicities of the corresponding Verma modules for the finite-dimensional reductive algebra \( s_k \).
Let $\lambda$ be a dominant weight and denote by $\mathcal{O}[\lambda]$ the subcategory of the modules in $\mathcal{O}$ whose support is contained in $\{\lambda - \mu \mid \mu \geq 0\}$. It follows from the previous theorem that $\mathcal{O}[\lambda]$ is a block of category $\mathcal{O}$. By [Nam17, Theorem 9.9] this block has enough injectives, and by [Nam17, Proposition 9.21] these injective objects have finite filtrations by dual Verma modules (notice Nampaisarn refers to dual Verma modules as costandard modules), and their multiplicities are given by BGG-reciprocity. We put this down as a theorem for future reference.

**Theorem 5.3 ([Nam17, Theorem 9.9, Proposition 9.21]).** Let $\lambda$ be an almost dominant weight. Then $L_\lambda(\lambda)$ has an injective envelope $L_\lambda(\lambda)$ in $\mathcal{O}$. The injective envelope has a finite filtration whose layers are modules of the form $M(\lambda^{(i)})$ with $i = 0, \ldots, m$, and such that $\lambda^{(0)} = \lambda$ and $\lambda^{(k)} \geq 6\lambda$. Furthermore, the multiplicity of $M(\mu)^{\vee}$ in this filtration equals $m(\lambda, \mu)$.

### 5.2. The categories $T_0$ and $T_1$

We now return to our study of representations of $\mathfrak{g}(\infty)$. Throughout this section we identify $\mathfrak{gl}(\infty)$ with $\mathfrak{g}(\mathbb{V})$, and so $\mathbb{V}$ and $\mathbb{V}_s$ are $\mathfrak{gl}(\infty)$-modules. A tensor module of $\mathfrak{gl}(\infty)$ is a subquotient of a finite direct sum of modules of the form $\mathbb{V}^p \otimes \mathbb{V}^q$ for $p, q \in \mathbb{N}_0$. Tensor modules were first studied by Penkov and Styrkas in [PS11b] and later by Dan-Cohen, Penkov and Serganova in [DCPS16]; in [SS15], Sam and Snowden study the equivalent category $\text{Rep}^0 \mathfrak{gl}(\infty)$. It follows from [DCPS16, section 4] that a module $\mathbb{V}$ is a tensor module if and only if it is an integrable module of finite length satisfying the LAC with respect to $\mathfrak{gl}(\infty)$ and of level 0. This category is denoted by $T_0^{\mathfrak{gl}(\infty)}$.

The simple tensor modules are parametrised by pairs of partitions $(\lambda, \mu)$. The corresponding simple module, which we denote by $L(\lambda, \mu)$, is the simple highest weight module with respect to the Borel subalgebra $\mathfrak{b}(\mathbb{V})$ with highest weight $\lambda_1 \varepsilon_1 + \cdots + \lambda_n \varepsilon_n - \mu_1 \varepsilon_1 - \cdots - \mu_1 \varepsilon_1$. The module $L(\lambda, \mu) = S_\lambda(\mathbb{V}) \otimes S_\mu(\mathbb{V}_s)$ belongs to $T_0^{\mathfrak{gl}(\infty)}$ and is injective in this category [DCPS16, Corollary 4.6]. The layers of its socle filtration are given by [PS11b, Theorem 2.3]

$$\text{soc}(r+1) L(\lambda, \mu) = \bigoplus_{\gamma + r' \lambda = \gamma'} c_{\lambda', \gamma'}^{\lambda, \mu} L(\lambda', \mu').$$

In particular $L(\lambda, \mu)$ is the injective envelope of $L(\lambda, \mu)$ in $T_0^{\mathfrak{gl}(\infty)}$.

**Lemma 5.4.** Let $M$ be an integrable weight $\mathfrak{gl}(\infty)$-module satisfying the large annihilator condition with respect to $\mathfrak{gl}(\infty)$. Then $M$ has finite length if and only if $M$ is finitely generated.

**Proof.** We will use the isomorphism $\mathfrak{gl}(\infty) \cong \mathfrak{g}(\mathbb{V})$ to see $M$ as a $\mathfrak{g}(\mathbb{V})$-module. If $M$ has finite length then it is finitely generated. To prove the reverse implication it is enough to prove it when $M$ is cyclic, so we assume that $M$ is generated by $v \in M^0_\lambda$ for $\lambda \in \mathfrak{h}^*$ and $a \in \mathbb{C}$.

Take $r \gg 0$ so $v$ spans a 1-dimensional $\mathfrak{g}(\mathbb{V})[r]$-module isomorphic to $\mathbb{C}_a$. Take $\mathfrak{r}$ to be the locally nilpotent ideal of $\mathfrak{g}(\mathbb{V})[r] + \mathfrak{b}$, and $\mathfrak{r}$ to be the opposite ideal, so
\[ \mathfrak{gl}(\infty) = \mathfrak{t} \oplus \mathfrak{g}(\mathcal{V})[r]^+ \oplus \mathfrak{f}, \] where \( \mathfrak{g}(\mathcal{V})[r]^+ = \mathfrak{g}(\mathcal{V})[r] + \mathfrak{h}. \)

As \( \mathfrak{g}(\mathcal{V})[r]-\)module, \( M \) is isomorphic to a quotient of \( S^*(\mathfrak{t}) \otimes S^*(\mathfrak{r}) \otimes \mathbb{C}_d \), while \( \mathfrak{t} \) and \( \mathfrak{r} \) decompose as \( \mathcal{V}[r]^+ \oplus \mathcal{V}_*[r]^+ \oplus \mathbb{C}^{2r-\tau} \). By [DCPS16, Lemma 4.1] there exists a finite dimensional subspace \( X \subset \mathfrak{t} \) such that \( S^p(\mathfrak{r}) \) is generated over \( \mathfrak{g}(\mathcal{V})[r] \) by \( S^p(\mathfrak{X}) \) for all \( p \geq 0 \). On the other hand, since \( M \) is integrable every element of \( X \) acts nilpotently on \( M \), and so \( S^p(\mathfrak{X})m = 0 \), and hence \( S^p(\mathfrak{r})m = 0 \), for \( p \gg 0 \). By an analogous reasoning, \( S^p(\mathfrak{t})m = 0 \), for \( p \gg 0 \).

It follows that \( M \) is in fact isomorphic as \( \mathfrak{g}(\mathcal{V})[r]-\)module to a quotient of \( S^{\leq p}(\mathfrak{t}) \otimes S^{\leq p}(\mathfrak{r}) \) for \( p \gg 0 \). Since \( S^{\leq p}(\mathfrak{t}) \otimes S^{\leq p}(\mathfrak{r}) \) is a tensor \( \mathfrak{g}(\mathcal{V})[r]-\)module it has finite length as \( \mathfrak{g}(\mathcal{V})[r]-\)module. Thus \( M \) has finite length over \( \mathfrak{g}(\mathcal{V})[r] \) and hence over \( \mathfrak{g}(\mathcal{V}). \)

We now turn back to the case \( \mathfrak{g} = \mathfrak{g}(\mathcal{V}^n) \). We say that an \( l \)-module is a tensor module if it is a subquotient of the tensor algebra \( T(\mathcal{V}^{(1)} \oplus \mathcal{V}^{(2)}_1 \oplus \cdots \oplus \mathcal{V}^{(n)} \oplus \mathcal{V}^{(n)}_1 \). An indecomposable tensor module is isomorphic to an external tensor product \( M_1 \otimes M_2 \otimes \cdots \otimes M_n \) with each \( M_i \) a tensor \( \mathfrak{g}(\mathcal{V}^{(k)}) \)-module. Simple tensor modules are parametrised by pairs \( (\lambda, \mu) \) with \( \lambda = (\lambda_1, \ldots, \lambda_n) \) and \( \mu = (\mu_1, \ldots, \mu_n) \) \( n \)-tuples of partitions, and

\[
L(\lambda, \mu) \cong L(\lambda_1, \mu_1) \otimes L(\lambda_2, \mu_2) \otimes \cdots \otimes L(\lambda_n, \mu_n).
\]

Tensor \( l \)-modules have finite filtrations whose layers are simple tensor modules, and hence they have finite length. It is also clear they are integrable and satisfy the LAC by the characterisation of \( \mathfrak{gl}(\infty) \) tensor modules given above.

**Definition 5.5.** The category \( \mathcal{T}_l \) is the full subcategory of integrable \( \mathfrak{g} \)-modules of finite length satisfying the LAC. For each \( \chi \in \mathcal{C}^n \) we define \( \mathcal{T}^\chi_l \) to be the subcategory of \( \mathcal{T}_l \) formed by modules of level \( \chi \).

Tensor \( l \)-modules belong to \( \mathcal{T}^0_l \). Each \( \mathcal{T}^\chi_l \) is a block of \( \mathcal{T}_l \). Also \( \mathcal{C}_\chi \subset \mathcal{T}^\chi_l \), and tensoring with \( \mathcal{C}_\chi \) gives an equivalence between \( \mathcal{T}^0_l \) and \( \mathcal{T}^\chi_l \).

**Proposition 5.6.** Let \( M \) be an object of \( \mathcal{T}_l \).

(a) If \( M \) is simple then it is isomorphic to \( \mathcal{C}_\chi \otimes L(\lambda, \mu) \) for \( \lambda, \mu \in \mathcal{P}^n \).

(b) If \( M \) is a highest weight module with highest weight \( \lambda \) and \( \alpha \) is a positive root of \( \mathfrak{l} \) then \( (\lambda, \alpha) \in \mathbb{Z}_{\geq 0} \).

**Proof.** Since tensoring with a fixed one-dimensional module gives an equivalence between blocks of \( \mathcal{T}_l \), to prove (a) it is enough to prove that a simple object \( L \) in \( \mathcal{T}^0_l \)
is a simple tensor module. Take $v \in L$, and set $L_k$ to be the $g(\mathcal{V}^{(k)})$-submodule of $L$ spanned by $v$. By definition $L_k$ is integrable and satisfies the LAC with respect to $g(\mathcal{V}^{(k)})$, and so by Lemma 5.4, it is a tensor $g(\mathcal{V}^{(k)})$-module. Since the $g(\mathcal{V}^{(k)})$ commute with each other inside $l$, there is a surjective map $L_1 \boxtimes L_2 \boxtimes \cdots \boxtimes L_n \to L$, so $L$ is a quotient of a tensor module, and hence a simple tensor module. To prove (b), notice that if $M$ is a highest weight module in $\mathcal{T}_1$ then its unique simple quotient also lies in $\mathcal{T}_1$ and so it must be of the form $\mathbb{C}_\lambda \otimes L$ with $L$ a simple tensor module. The statement is easily proved for the highest weight of this simple module. \qed

5.3. The category $\mathcal{T}_{[r]}$. Recall that weights in $\mathfrak{h}^0$ are called eligible weights, and a weight is $r$-eligible if it is in the span of $\{\omega_i^{(k)} e_i^{(k)} \mid k \in \mathbb{N}, i \in \pm \mathbb{Z}[r]\}$.

Let $\lambda \in \mathfrak{h}^*$. We denote by $D(\lambda)$ the set of all weights $\lambda - \mu$ with $\mu \geq 0$. Let $M$ be a weight $g$-module. A weight $\lambda$ is said to be extremal if $M_\lambda \neq 0$ and $M_\mu = 0$ for all $\mu \succ \lambda$. If a module $M$ has finitely many extremal weights $\lambda_1, \ldots, \lambda_k$ then its support is contained in the union of the $D(\lambda_i)$.

**Definition 5.7.** The category $\mathcal{T}_{[r]}$ is the full subcategory of $([r]^+, \mathfrak{h})$-Mod whose objects are modules $M$ satisfying the following conditions.

(i) $M$ has finitely many extremal weights.
(ii) For each $\mu \in \mathfrak{h}^*$ the $[r]$-module generated by $M_{\geq \mu} = \bigoplus_{\nu \geq \mu} M_\nu$ lies in $\mathcal{T}_{[r]}$.

The following lemma is an easy consequence of the definition.

**Lemma 5.8.** The category $\mathcal{T}_{[r]}$ is closed under direct summands, finite direct sums and finite tensor products.

**Example.** We have decompositions

$$\mathcal{V}^{(k)} = V_{-,r}^{(k)} \oplus \mathcal{V}^{(k)}[r] \oplus V_{+,r}^{(k)}$$
$$\mathcal{V}^{(k)*} = \left(V_{-,r}^{(k)*}\right) \oplus \mathcal{V}^{(k)}[r] \oplus \left(V_{+,r}^{(k)*}\right)$$

where $V_{\pm,r}^{(k)} = \langle v_i \otimes e_k \mid i \in \pm \mathbb{Z}[r]\rangle$. Each of these modules is a tensor $l[r]$-module: $\mathcal{V}^{(k)}[r]$ is the natural representation of $g(\mathcal{V}^{(k)}[r])$, while $\mathcal{V}^{(l)}[r]$ is the conatural representation of $g(\mathcal{V}^{(l)}[r])$, and the other four spaces are finite direct sums of 1-dimensional $\mathfrak{h}$-modules with trivial $l[r]$ action. Since tensor modules are closed by finite direct sums, it follows that both are tensor modules over $l[r]$. 19
Now consider \( \mathcal{V}^{(k,l)} = \mathcal{V}^{(k)} \otimes \mathcal{V}^{(l)}_s \). From the previous example we get a decomposition of this space into nine summands, illustrated in the picture below.

\[ \mathcal{V}^{(k)} \otimes \mathcal{V}^{(l)}_s \text{ decomposes as } \mathfrak{l}[r]-
\]

module as a direct sum of:

- \( \mathcal{V}^{(k,l)}[r] = \mathcal{V}^{(k)}[r] \otimes \mathcal{V}^{(l)}_s[r] \);
- 2\( r \) copies of \( \mathcal{V}^{(k)}[r] \);
- 2\( r \) copies of \( \mathcal{V}^{(l)}_s[r] \);
- a trivial module of dimension \( 4r^2 \).

The fact that \( \mathfrak{T}_{\mathfrak{l}[r]} \) is closed by direct sums, direct summands and tensor products implies that \( \mathcal{S}^p(\mathcal{V}^{(k,l)}) \) is again in \( \mathfrak{T}_{\mathfrak{l}[r]} \). We claim that if \( k > l \) then \( \mathcal{S}^\bullet(\mathcal{V}^{(k,l)}) \) is in \( \mathfrak{T}_{\mathfrak{l}[r]} \).

Indeed, its support is contained in \( D(0) \), and furthermore for each \( \lambda \in \mathfrak{h}^\circ \) we have \( \mathcal{S}^\bullet(\mathcal{V}^{(k,l)})_{\geq \lambda} \subset \bigoplus_{p=1} S^p(\mathcal{V}^{(k,l)}) \), which lies in \( \mathfrak{T}_{\mathfrak{l}[r]} \); here \( \varphi \) is the map introduced in Example 4.6. On the other hand if \( k \leq l \) then the symmetric algebra is not in \( \mathfrak{T}_{\mathfrak{l}[r]} \) as it has no extremal weights.

We will mostly be interested in studying \( \mathfrak{g} \)-modules whose restriction to \( \mathfrak{l}[r]^+ \) lies in \( \mathfrak{T}_{\mathfrak{l}[r]} \). We now show that such a module has LCS. The statement and proof are similar to [DGK82] Proposition 3.2, but we replace the dimension of \( M_{\leq \lambda} \), which could be infinite in our case, with the length of the \( \mathfrak{l}[r]- \)

module it spans.

**Proposition 5.9.** Let \( M \) be a \( \mathfrak{g} \)-module lying in \( \mathfrak{T}_{\mathfrak{l}[r]} \). Then \( M \) has LCS.

**Proof.** Fix \( \mu \) in the support of \( M \) and denote by \( N(\mu) \) the \( \mathfrak{l}[r]- \)

submodule spanned by \( M_{\leq \mu} \). The proof proceeds by induction on the length of \( N(\mu) \), which we denote by \( \ell \). If \( \ell = 0 \) then \( 0 \subset M \) is the desired composition series. In any other case we have \( \mu \prec \lambda \) for some \( \lambda \) which is maximal in the support of \( M \). Taking \( v \in M_{\lambda} \) we see that \( V = U(\mathfrak{g})v \) is a highest weight module and hence has a unique maximal submodule \( V' \). We then have a filtration

\[ \begin{align*}
(1) \quad 0 \subset V' \subset V \subset M.
\end{align*} \]

Since \( v \in N(\mu) \) by hypothesis, \( N'(\mu) = N(\mu) \cap V' \subset N \) has length strictly less than \( l \). Using this module and the induction hypothesis we know there exists an LCS at \( \mu \) for \( V' \). The same applies to \( N(\mu)/(N(\mu) \cap V) \subset M/V \), so \( M/V \) also has a LCS at \( \mu \).

These LCS can be used to refine the filtration \( V' \subset V \) into an LCS of \( M \) at \( \mu \). \( \square \)

5.4. The symmetric algebras of parabolic radicals. Recall that we have introduced for each \( r \geq 0 \) the subalgebra \( \mathfrak{p}(r) = \mathfrak{l}[r] + \mathfrak{b} \). This subalgebra has a Levi-type decomposition \( \mathfrak{p}(r) = \mathfrak{l}[r]^+ \oplus \mathfrak{u}(r), \) which in turn induces a decomposition \( \mathfrak{g} = \mathfrak{u}(r) \oplus \mathfrak{l}[r]^+ \oplus \mathfrak{u}(r). \)
The pictures of these subalgebras are given in subsection 3.4. We will now study the structure of $S^\bullet(\bar{\pi}(r))$ as $l[r]_\pi$-module, as this will be essential in the sequel.

Using the the decomposition of $V^{(k,l)}$ given in Example 5.3 it is easy to see that $\pi(r)$ is isomorphic as $l[r]$-module to

$$Z \oplus \left( \bigoplus_{k=1} \bigoplus_{l=1} V^{(k,l)}[r] \right)$$

for some trivial $l[r]$-modules $X_k, Y_l, Z$, which we will study in more detail.

The above decomposition of $\pi(r)$ is in fact a decomposition into indecomposable $\xi_r \oplus l[r]$-modules, where $\xi_r = \xi \cap g_r$. Indeed, $Z$ coincides with the subalgebra $\pi_r = \pi \cap g_r$. The modules $X_k$ and $Y_l$ can be described recursively as

$$X_n = V^{(n)}_{+r} \quad X_{k-1} = X_k \oplus V^{(k-1)}_{-r} \oplus V^{(k-1)}_{+r}$$

$$Y_1 = (V^{(1)}_{+r})^* \quad Y_{k+1} = Y_k \oplus (V^{(k-1)}_{-r})^* \oplus (V^{(k)}_{+r})^*.$$

It follows from this that $X_k$ is isomorphic to the $\xi_r$-submodule of the conatural representation of $g_r$ spanned by the unique vector of weight $-\epsilon_r^{(k)}$, and $X_l$ is isomorphic to the $\xi_r$-submodule of the natural representation of $g_r$ spanned by the unique vector of weight $\epsilon_r^{(l)}$.

**Proposition 5.10.** The symmetric algebra $S^\bullet(\pi(r))$ is an object of $T_{l[r]}$.

**Proof.** To study the symmetric algebra $S^\bullet(\pi(r))$ we look at the symmetric algebra of each summand in the decomposition of $\pi(r)$ as $\xi_r \oplus l[r]$-module separately.

$$S^\bullet(\pi_r) = \bigoplus_{k \geq 0} S^k(\pi_r)$$

$$S^\bullet \left( V^{(k)}[r] \otimes Y_k \right) = \bigoplus_{\nu_k} S_{\nu_k} \left( V^{(k)}[r] \right) \otimes S_{\nu_k} (Y_k) \quad (k = 1, \ldots, n)$$

$$S^\bullet \left( X_l \otimes V^{(l)}_+[r] \right) = \bigoplus_{\eta_l} S_{\eta_l} \left( X_l \right) \otimes S_{\eta_l} \left( V^{(l)}_+[r] \right) \quad (l = 1, \ldots, n)$$

$$S^\bullet \left( V^{(k)}[r] \otimes V^{(l)}_+[r] \right) = \bigoplus_{\rho_{kl}} S_{\rho_{kl}} \left( V^{(k)}[r] \right) \otimes S_{\rho_{kl}} \left( V^{(l)}_+[r] \right) \quad (1 \leq l < k \leq n)$$

By Lemma 5.8 it is enough to see that each symmetric algebra lies in $T_{l[r]}$, and in each case we will show that if $\mu \in h^*$ is in the support then it is in the support of finitely many of the indecomposable summands.

**Case 1:** $S^\bullet(\pi_r)$. If $\mu$ is any weight in the support then $S^\bullet(\pi_r)_{\geq \mu}$ must be finite dimensional. In particular it can only intersect $S^k(\pi_r)$ for finitely many $k \in \mathbb{Z}_{\geq 0}$.

**Case 2:** $S^\bullet \left( V^{(k)}[r] \otimes Y_k \right)$. Fix a weight $\mu$ in the support. Recall that $\mu|_r$ and $\mu[r]$ denote the restriction of $\mu$ to $h_r$ and $h[r]$, respectively. Then

$$\left( S_{\nu} \left( V^{(k)}[r] \right) \otimes S_{\nu} (Y_k) \right)_\mu = S_{\nu} \left( V^{(k)}[r] \right)_{\mu|_r} \otimes S_{\nu} (Y_k)_{\rho_{kl}}.$$
Using the decomposition of the image of a direct sum by a Schur functor we get
the LAC, so $g^\vee$ duals. For example $h$.
Since the support of $X_i$ consists entirely of positive weights, and the argument is analogous to the previous case.

**Proof.** Set $V_i = \bigoplus_{\lambda} V^{(\lambda)}[\lambda]$.
Now the support of $X_i$ consists entirely of positive weights, and this is an isomorphism of $\mathbb{T}_r \oplus [r]$-modules.

**Lemma 5.11.** The large annihilator dual of $I(\lambda, \mu)$ is isomorphic to

$$\bigoplus_{a, \beta} \bigoplus_{\gamma} \left( \prod_{k} c_{a_k, \gamma_k} c_{\beta_k, \gamma_k} \right) I(\alpha, \beta),$$

where the sum runs over all $n$-tuples of partitions $\alpha, \beta$ and $\gamma$. In particular $I(\lambda, \mu)^\#$ lies in $\mathbb{T}^0_1$.

**Proof.** Set $I_r = I \cap g_r$. There exist decompositions of $I_r \oplus [r]$-modules

$$\mathbb{V}(r) = V^{(r)} \oplus \mathbb{V}(r)[r]; \quad \mathbb{V}_*(r) = \left( V^{(r)}_r \right)^* \oplus \mathbb{V}_*(r)[r].$$

Using the decomposition of the image of a direct sum by a Schur functor we get

$$I(\lambda, \mu) = \bigoplus_{a, \beta, \gamma} c_{a, \gamma} c_{\beta, \gamma} S_{a, \gamma}(V^{(r)}_r) \otimes S_{\beta, \gamma} \left( \left( V^{(r)}_r \right)^* \otimes I(\gamma, \delta)[r] \right),$$

where $I(\gamma, \delta)[r] = S_\gamma(\mathbb{V}(r)[r]) \otimes S_\delta(\mathbb{V}_*(r)[r])$, and this is an isomorphism of $I_r \oplus [r]$-modules.
Notice that twisting the modules $\mathcal{V}^{(k)}$ and $\mathcal{V}^{*(k)}$ by the automorphism $-\tau$ produces isomorphic $l$-modules. Since twisting by $-\tau$ commutes with Schur functors, it follows that $-\tau I(\lambda, \mu) \cong I(\lambda, \mu)$. Computing the semisimple duals we get

$$(I(\lambda, \mu))^* \cong \text{Hom}_C(-\tau I(\lambda, \mu), \mathbb{C}) \cong \text{Hom}_C(I(\lambda, \mu), \mathbb{C}).$$

Now writing $I(\lambda, \mu)$ as the tensor product of the $I(\lambda_i, \mu_i)$ and using the above decomposition, we get

$$(I(\lambda, \mu))^* \cong \text{Hom}_C \left( \bigotimes_{k=1}^n T^{(k)}_{a_k, \beta_k, \gamma_k, \delta_k} \otimes I(\gamma_k, \delta_k), \mathbb{C} \right)$$

where

$$T^{(k)}_{a_k, \beta_k, \gamma_k, \delta_k} = \bigoplus_{\gamma_k, \delta_k} c_{\alpha_k, \delta_k}^{\mu_k} \left( S_{\alpha_k} (V^{(k)}_r) \otimes S_{\beta_k} ((V^{(k)}_r)^*) \right).$$

This last module is a semisimple finite dimensional $l$-module, and hence isomorphic to its semisimple dual. Using the fact that duals distribute over tensor products if one of the factors is finite dimensional, we obtain an isomorphism of $l_r \oplus l[r]$-modules

$$I(\lambda, \mu) \cong \bigoplus_{a, \beta, \gamma, \delta} \left( \bigotimes_{k=1}^n T^{(k)}_{a_k, \beta_k, \gamma_k, \delta_k} \right) \otimes \text{Hom}_C(I(\gamma, \delta)[r], \mathbb{C}).$$

Now to compute the image of this module by $\Phi_r$ only the $l[r]$-module structure is relevant, and

$$\Phi_r(\text{Hom}_C(I(\gamma, \delta)[r], \mathbb{C})) \cong \text{Hom}_{l[r]}(I(\gamma, \delta)[r], \mathbb{C}) \cong \text{Hom}_r(I(\gamma, \delta), \mathbb{C}).$$

As mentioned in the preamble this is the zero vector space unless $\gamma = \delta$, in which case it is one-dimensional. Thus $\Phi_r(I(\lambda, \mu))^*$ is isomorphic to

$$\bigoplus_{a, \beta, \gamma} \left( \bigotimes_{k=1}^n T^{(k)}_{a_k, \beta_k, \gamma_k} \right) = \bigoplus_{a, \beta, \gamma} \left( \bigotimes_{k=1}^n c_{\alpha_k, \gamma_k}^{\mu_k} \left( S_{\alpha_k} (V^{(k)}_r) \otimes S_{\beta_k} ((V^{(k)}_r)^*) \right) \right).$$

Taking the limit as $r$ goes to infinity, we see that $I(\lambda, \mu)^*$ is isomorphic to

$$\bigoplus_{a, \beta, \gamma} \left( \bigotimes_{k=1}^n c_{\alpha_k, \gamma_k}^{\mu_k} \left( S_{\alpha_k} (\mathcal{V}^{(k)}) \otimes S_{\beta_k} (\mathcal{V}^{*(k)}) \right) \right),$$

and this precisely the module in the statement. \hfill \Box

The following is an immediate consequence of the last two results.

**Proposition 5.12.** The large annihilator dual of $S^*(\mathbb{P}(r))$ lies in $\mathbb{P}_{l[r]}$.

6. **Category $\mathcal{O}_{LA}$: first definitions**

As in the previous section we fix $n \in \mathbb{Z}_{>0}$ and denote by $\mathfrak{g}$ the Lie algebra $\mathfrak{g}(\mathcal{V}^n)$. We will also continue to omit $\mathcal{V}^n$ from the notation for subalgebras of $\mathfrak{g}$. 


6.1. **Introduction to** $O_{LA}$. We now introduce a category of representations of $g$ that serves as an analogue of category $O$. The definition is analogous to the usual definition of category $O$ for the reductive Lie algebra $gl(r, C)$ but, since $U(gl(\infty))$ is not noetherian, we need to replace finite generation with the LAC with respect to $\mathfrak{l}$.

**Definition 6.1.** The category $O_{LA}^g$ is the full subcategory of $g$-Mod whose objects are the $g$-modules $M$ satisfying the following conditions.

(i) $M$ is $h$-semisimple.

(ii) $M$ is $n$-torsion.

(iii) $M$ satisfies the LAC with respect to $\mathfrak{l}$.

We often write $O_{LA}$ for $O_{LA}^g$. If $M$ is an object of $O_{LA}$ then so is any subquotient of $M$. It follows from the definition that the support of an object $M$ in $O_{LA}$ must be contained in $\mathfrak{h}^\circ$. The following lemma shows that finitely generated objects in $O_{LA}$ have LCS and hence well-defined Jordan-Holder multiplicities. Recall from Example 4.6 that there is a map $\psi : h^\circ \to C$ that sends the finite roots to 0, induces a $\mathbb{Z}$-grading on $g$, and turns a weight module $M$ into a graded module $M^\psi$.

**Lemma 6.2.** Let $M$ be any $g$-module and let $v \in M$ be an $h$-semisimple and $n$-torsion vector satisfying the LAC with respect to $\mathfrak{l}$. Then the submodule $N = U(\mathfrak{g})v$ lies in $O_{LA}$, has LCS, and $N^\psi = 0$ for $z \gg 0$.

**Proof.** Take $r \gg 0$ such that $Cv$ is a trivial $l[r]$-module. Using the PBW theorem we have a surjective map of $l[r]$-modules

$$p : S^*(\mathfrak{p}(r)) \otimes S^*(u(r)) \otimes S^*(l[r]^+) \otimes \mathbb{C}_\lambda \to N$$

$$u \otimes u' \otimes l \otimes 1_\lambda \mapsto uu'lv.$$

Since $v$ is $n$-nilpotent and satisfies the LAC we have $S^*(l[r]^+)v = Cv$ and $S^k(u(r))v = 0$ for $k \gg 0$. Thus the restriction

$$p' : S^*(\mathfrak{p}(r)) \otimes S^\leq k(u(r)) \otimes \mathbb{C}_\lambda \to N$$

is surjective. By Propositions 5.9 and 5.12 the domain of this last map lies in $\mathfrak{p}_{l[r]}$ and has LCS. Since $u(r)^\mathfrak{p}$ has a right-bounded grading, the same holds for the domain of $p'$ and hence for $N$. By the PBW theorem

$$n \cdot S^i(\mathfrak{p}(r)) \otimes S^j(u(r)) \subset \bigoplus_{i', j', t'} S^i(\mathfrak{p}(r)) \otimes S^j(u(r)) \otimes S^{t'}(l[r]^+)$$

where $i' + j' + t' = i + j + 1$ and $i' \leq i$. Induction on $i$ shows that $p'(S^i(\mathfrak{p}(r)) \otimes S^j(u(r)) \otimes \mathbb{C}_\lambda)$ is annihilated by a large enough power of $n$, and so $N$ is $n$-torsion. □

This lemma has a very useful consequence: every module in $(g, h)\text{-Mod}$ has a largest submodule contained in $O_{LA}$, namely the submodule spanned by its $n$-torsion elements satisfying the LAC with respect to $\mathfrak{l}$. Categorically, this means that the inclusion functor of $O_{LA}$ in $(g, h)\text{-Mod}$ has a right adjoint $\pi_{LA}$. We will come back to this observation later on, when we look at categorical properties of $O_{LA}$.
6.2. **Simple objects in $\mathcal{O}_{LA}$:** Let $\lambda \in h^*$. The Verma module $M(\lambda)$ does not belong to $\mathcal{O}_{LA}$ since the highest weight vector does not satisfy the LAC. However, if $\lambda$ is $r$-eligible there exists a 1-dimensional $l[r]^+$-module $C_\lambda$, which we can inflate it to a $p(r)$-module setting the action of $u(r)$ to be zero. Set $M_r(\lambda) = \text{Ind}^{p(r)}_{l[r]} C_\lambda$, which is clearly a highest weight module of weight $\lambda$.

Since the highest weight vector of $M_r(\lambda)$ satisfies the LAC, Lemma 6.2 implies that $M_r(\lambda)$ lies in $\mathcal{O}_{LA}$. Notice also that we have surjective maps $M_{r+1}(\lambda) \to M_r(\lambda)$, that given a weight $\mu$ in the support the restriction $M_{r+1}(\lambda)_{\mu} \to M_r(\lambda)_{\mu}$ is an isomorphism for $r \gg 0$, and that $M(\lambda)$ is the inverse limit of this system. This shows in particular that $\mathcal{O}_{LA}$ is not closed under inverse limits. With these parabolic Verma modules in hand, we are ready to prove our first result.

**Theorem 6.3.** The simple objects in $\mathcal{O}_{LA}$ are precisely the highest weight simple modules whose highest weight is in $h^*$.

**Proof.** Suppose $L$ is a simple object in $\mathcal{O}_{LA}$. Since $L$ is $n$-torsion it has a highest weight vector $v$ of weight $\lambda \in h^*$, so $L \cong L(\lambda)$. On the other hand, if $\lambda$ is an $r$-eligible weight then $M_r(\lambda)$ lies in $\mathcal{O}_{LA}$ and so does its unique simple quotient $L(\lambda)$.

**Remark.** A more subtle difference between the definition of $\mathcal{O}_{LA}$ and that of category $\mathcal{O}$ for finite dimensional reductive Lie algebras is that we ask for $M$ to be $n$-torsion and not just locally $n$-nilpotent. In the finite-dimensional case, and even in the case $n = 1$, these two conditions are equivalent (see [PS10], Proposition 4.2] for $\mathfrak{sl}(\infty)$, the proof is the same of $\mathfrak{gl}(\infty)$).

This is no longer true as soon as $n \geq 2$. Indeed, when $n = 2$ the simple lowest weight module with lowest weight $\lambda = -\omega^{(1)} + \omega^{(2)}$ is the limit of the simple finite dimensional lowest weight modules $L(\lambda|_k)$, where each embeds in the next by sending the lowest weight vector to the lowest weight vector. Thus $\tilde{L}(\lambda) = \lim_{\leftarrow} L(\lambda|_k)$ is generated by a weight vector satisfying the LAC, and the construction shows that $\tilde{L}(\lambda)$ is locally $n$-nilpotent, but not $n$-torsion. Notice that this module can not be a highest weight module: indeed, if it had a highest weight vector $v$ then $v$ would belong to $\tilde{L}(\lambda|_k)$ for all $k \gg 0$ and be a highest weight vector, but the highest weight vectors of $\tilde{L}(\lambda|_k)$ are never sent to highest weight vectors of $\tilde{L}(\lambda|_{k+1})$.

6.3. **Highest weight modules in $\mathcal{O}_{LA}$:** As a consequence of Lemma 6.2 a highest weight module $M$ in $\mathcal{O}_{LA}$ has finite Jordan-Holder multiplicities, and furthermore if the highest weight vector generates a 1-dimensional $l[r]$-module then $M$ lies in $\mathcal{T}_r l[r]$. We will now show that highest weight modules have finite length.

Recall that we denote by $s$ and $m$ the subalgebras of $\mathfrak{g}$ spanned by root spaces corresponding to finite and positive infinite roots, respectively, and that $s = \mathfrak{g}_0$ and $m = \mathfrak{g}_{\geq 0}$. If $M$ is an object of $\mathcal{O}_{LA}$ we can see it as a $\mathbb{Z}$-graded module through the map $\psi$, and each homogeneous component is a $s$-module. If the grading on $M$ is right-bounded, for example if $M$ is finitely generated, we will denote by $M^+$ the top nonzero homogeneous component.
We have also set \( q = s \oplus m \). An \( s \)-module can be inflated into a \( q \)-module by imposing a trivial \( m \)-action. The following result shows that the problem of computing Jordan-Holder multiplicities of a highest weight module in \( O_{LA} \) reduces to computing Jordan-Holder multiplicities of highest weight \( s \)-modules in \( O_s \). This is very useful since weight components of finite length \( s \)-modules are finite dimensional, and hence characters can be given in terms of these dimensions.

**Proposition 6.4.** Let \( M \in O_{LA} \) be a highest weight module of highest weight \( \lambda \).

(i) If \( N \subset M \) is a nontrivial module then \( N^+ = N \cap M^+ \).

(ii) \( M \) is simple if and only if \( M^+ \) is a simple \( s \)-module.

(iii) \( M = \text{Ind}_{q}^g \mathcal{I}_s^g M^+ \).

(iv) If \( \mu \) is an eligible weight and \( [M : L(\mu)] \neq 0 \) then \( \lambda - \mu \) is finite, and furthermore \( [M : L(\mu)] = [M^+ : L_\phi(\mu)] \leq m(\lambda, \mu) \).

**Proof.** Denote by \( v \) the highest weight vector of \( M \). If \( N \) is a nontrivial submodule of \( M \) then it contains a highest weight vector, say \( w \) of weight \( \mu \). For each \( r \geq 0 \) denote by \( M_r \) the \( g_r \)-module generated by \( v \). Then \( M_r \) is a highest weight \( g_r \)-module and for \( r \) large enough \( w \in M_r \), and it is a highest weight vector. Thus \( \mu \mid_r \) and \( \lambda \mid_r \) are linked for all large \( r \). Lemma 3.3 tells us that \( \lambda - \mu \) must then be a finite root, and so \( w \in N \cap M^+ = \emptyset \) and \( N^+ = N \cap M^+ \). This proves the first item.

If \( M \) is simple then \( M^+ \) is a \( U(g)_0^\emptyset \) simple module. Using the PBW theorem we have a decomposition \( U(g)_0^\emptyset = \bigoplus_{k \in \mathbb{N}} U(m)^\psi \mathcal{I}_k U(s)U(\mathfrak{m})^\psi \), and since \( m \) acts trivially on \( M^+ \) it follows that \( M^+ \) is a simple \( U(s) \)-module. Conversely, if \( M^+ \) is simple then any nonzero submodule \( N \subset M \) must have \( N^+ = M^+ \) by the first item, so it must contain the highest weight vector. Thus \( N = M \) and \( M \) is simple.

Denote by \( K \) be the kernel of the natural map \( \text{Ind}_{q}^g M^+ \rightarrow M \). Then by construction \( K \cap M^+ = 0 \), and the first item implies \( K = 0 \). In particular this shows that \( L(\mu) = \text{Ind}_{q}^g \mathcal{I}_s^g L_\phi(\mu) \). Finally, the module \( M^+ \) is a highest weight \( s \)-module and its highest weight \( \lambda \) is almost dominant since this holds for all eligible weights. Thus \( M_\phi(\lambda) \) has a composition series and if we apply the exact functor \( \text{Ind}_{q}^g \circ \mathcal{I}_s^g \) to this filtration we get a filtration of \( M \). By the previous item the layers of this filtration are simple modules, and hence it is again a composition series and we can use it to compute

\[
[M : L(\mu)] = [M^+ : L_\phi(\mu)].
\]

Finally, by the universal property of Verma modules \( M^+ \) is a quotient of \( M_\phi(\lambda) \), so \( [M^+ : L_\phi(\mu)] \leq m(\lambda, \mu) \).

We will now show that parabolic Verma modules have finite length, and compute their Jordan-Holder multiplicities in terms of multiplicities of Verma modules over \( g_k \).

**Corollary 6.5.** Let \( \lambda, \mu \in \mathfrak{h}^\circ \) with \( \lambda \) \( r \)-eligible. If \( L(\mu) \) is a simple constituent of \( M_\phi(\lambda) \) then the following hold.

(i) \( \mu \) lies in the dot orbit of \( \lambda \) by \( \mathcal{W}(s_{r+1}) \).

(ii) If \( \mu \) is \( r \)-eligible then \( [M_\phi(\lambda) : L(\mu)] = m(\lambda, \mu) \).
In particular $M_r(\lambda)$ has finite length.

Proof. We have already seen in Proposition 6.4 that $\mu$ and $\lambda$ must be linked. Since $M_r(\lambda)$ is an object of $\mathbb{T}_i[r]$ so is $L(\mu)$, and in particular $\mu|_r$ must be the highest weight of a highest weight module in $\mathbb{T}_i[r]$. Thus $(\lambda, \alpha)$ and $(\mu, \alpha)$ are positive for any finite root $\alpha$ of $[r]$, and Lemma 5.3 implies that $\mu = \sigma \cdot \lambda$ for $\sigma \in W(s_{r+1})$. In particular $\mu$ belongs to a finite set, so $M_r(\lambda)$ has finite length.

We also know from Proposition 6.4 that $[M_r(\lambda) : L(\mu)] = [M_r(\lambda)^+, L_s(\mu)]$. The surjective map of $s$-modules $M_s(\lambda) \to M_r(\lambda)^+$ restricts to a bijection of the weight components $\preceq \mu$, and so

$$\dim M_r(\lambda)^+ \mu = \sum_{\mu \preceq \nu} [M_r(\lambda)^+: L_s(\nu)] \dim L_s(\nu) \mu = \dim M_s(\lambda) \mu = \sum_{\mu \preceq \nu} m(\lambda, \nu) \dim L_s(\nu) \mu$$

Thus $[M_r(\lambda)^+: L_s(\mu)] = m(\lambda, \mu)$.

6.4. The structure of a general object in $\mathcal{O}_{LA}$. We now turn to more general objects of $\mathcal{O}_{LA}$. First we show that finitely generated objects in $\mathcal{O}_{LA}$ must have finite length.

Proposition 6.6. Let $M$ be an object of $\mathcal{O}_{LA}$. The following are equivalent.

(a) $M$ is finitely generated.
(b) $M$ has a finite filtration whose layers are highest weight modules.
(c) $M$ has finite length.

Proof. To prove (a) $\Rightarrow$ (b) suppose first that $M$ is cyclic, and its generator is a weight vector $v$ of weight $\lambda$ such that $[r]^i v = 0$. As seen in Lemma 6.2, $M$ is in $\mathbb{T}_i[r]$, so we can proceed by induction on the length of the $[r]$-module spanned by $M_{\preceq \lambda}$. This submodule contains a highest weight vector, say $w$, and we set $N = U(\mathfrak{g}) w$. Clearly $N$ is a highest weight module and $M/N$ has a filtration by highest weight modules by hypothesis, so the same holds for $M$. The general case now follows by induction on the number of generators of $M$. The implication (b) $\Rightarrow$ (c) follows from Corollary 6.5 and (c) $\Rightarrow$ (a) is obvious.

We now give a general approach to compute the Jordan-Holder multiplicities of an arbitrary object of $\mathcal{O}_{LA}$. We will use the following tool.

Definition 6.7. Let $M \in \mathcal{O}_{LA}$. For each $i \in \mathbb{Z}$ we set $F_i M$ to be the submodule of $M$ generated by $\bigoplus_{j \geq i} M^j$. The family $\{F_i M \mid i \in \mathbb{Z}\}$ is the $\psi$-filtration of $M$. We also define $S_i M$ to be the $s$-module $(F_i M/ F_{i-1} M)_i$, which is the top component of the corresponding layer of the filtration.

By construction the layers of the $\psi$-filtration of any module are generated by their top degree component, and hence their multiplicities can be computed using Proposition 6.4.
Proposition 6.8. Let \( M \) be an object in \( O_{\text{LA}} \) with LCS and let \( \lambda \) be an eligible weight with \( \psi(\lambda) = p \). Then \([M : L(\lambda)] = [S_p M : L_s(\lambda)]\).

Proof. It is enough to show the result for finitely generated objects, and by Proposition 6.6 it is enough to show it for finite length objects. Let \( M \) be a finite length object, and take \( q \in \mathbb{Z} \) such that \( M^+ = M^q \). A simple induction on \( p - q \) shows that \([S_p M : L_s(\lambda)]\) is an additive function on \( M \), with the base case a consequence of Proposition 6.4. Since both \([M : L(\lambda)]\) and \([S_p M : L_s(\lambda)]\) are additive functions on \( M \) it is enough to show they are equal when \( M \) is simple, which again follows from Proposition 6.4. \(\square\)

6.5. Categorical properties of \( O_{\text{LA}} \). We now focus on the general categorical properties of \( O_{\text{LA}} \). Let \( M \) be a \( h \)-semisimple \( g \)-module. By Lemma 6.2 the submodule spanned by all its \( n \)-torsion vectors satisfying the large annihilator condition is an object of \( O_{\text{LA}} \), and is in fact the largest submodule of \( M \) lying in \( O_{\text{LA}} \). We thus have a diagram of functors, where each arrow from left to right is an embedding of categories and each right to left arrow is a left adjoint:

\[
O^1_{\text{LA}} \begin{array}{c} \Gamma_n \end{array} \begin{array}{c} \Phi \end{array} \begin{array}{c} \Gamma_g \end{array} \quad (g, h) \text{-Mod}_{\text{LA}} \quad (g, h) \text{-Mod} \quad g \text{-Mod}
\]

It follows that we have a functor \( \pi_{\text{LA}} = \Gamma_n \circ \Phi = \Phi \circ \Gamma_n : (g, h) \text{-Mod} \rightarrow O_{\text{LA}} \), which is right adjoint to the exact inclusion functor \( O_{\text{LA}} \rightarrow (g, h) \text{-Mod} \). In particular it preserves direct limits and sends injectives to injectives.

Recall that an abelian category \( A \) is locally artinian if every object is the limit of its finite length objects. Also, \( A \) has the Grothendieck property if it has direct limits, and for every object \( M \), every subobject \( N \subset M \) and every directed family of subobjects \((A_\alpha)_{\alpha \in I}\) of \( M \) it holds that \( \bigcap_{\alpha} A_\alpha = \lim_{\alpha} N \cap A_\alpha \). The following result is an easy consequence of the properties of \( \pi_{\text{LA}} \).

Theorem 6.9. Category \( O_{\text{LA}} \) is a locally artinian category with direct limits, enough injective objects, and the Grothendieck property.

7. Category \( O_{\text{LA}} \): Standard objects

In this section we will introduce the standard objects of \( O_{\text{LA}} \). We then compute their simple multiplicities through their \( \psi \)-filtrations.

7.1. Large-annihilator dual Verma modules. We now introduce the modules that will play the role of standard objects on \( O_{\text{LA}} \). In the finite dimensional case this role is played by the semisimple duals of Verma modules, so it is natural to consider the “best approximation” to these modules in \( O_{\text{LA}} \).

Definition 7.1. For every \( \lambda \in h^* \) we set \( A(\lambda) = M(\lambda)^\# \).

The module \( A(\lambda) \) can be hard to grasp. For example, it is not clear at first sight that it lies in \( O_{\text{LA}} \). As a first approximation, set \( A_r(\lambda) \) to be the LA dual of the parabolic Verma \( M_r(\lambda) \). The Verma module \( M(\lambda) \) is the inverse limit of the \( M_r(\lambda) \), and since
LA duality sends inverse limits to direct limits, $A(\lambda) = \lim A_r(\lambda)$. Also, the natural maps $A_r(\lambda) \to A(\lambda)$ are injective, and for any $\mu$ in the support of $A(\lambda)$ and $r \gg 0$ the map $A(\lambda)_{>\mu} \to A_r(\lambda)_{>\mu}$; this follows from the dual facts for $M_r(\lambda)$ and $M(\lambda)$. This approach allows us to prove that the $A(\lambda)$ are indeed in $O_{LA}$.

**Proposition 7.2.** For every eligible weight $\lambda$ and every $r \geq 0$ the modules $A(\lambda), A_r(\lambda)$ lie in $O_{LA}$ and have LCS. Furthermore, we have $[A(\lambda) : L(\mu)] = [A_r(\lambda) : L(\mu)]$ for $r \gg 0$.

**Proof.** By definition the space $M_r(\lambda)^g$ is an $g$-semisimple module satisfying the LAC. Also it is isomorphic to $S^\bullet(\mathfrak{g}(r))^g \otimes \mathbb{C}_\lambda$ as $l[r]^+$-module, and hence lies in $\mathbb{T}_{l[r]}$ by Proposition 5.12 and by Lemma 5.2 it has LCS. To see that it is $n$-torsion, first observe that $n(r) = l[r] \cap n$-torsion by virtue of being in $\mathbb{T}_{l[r]}$. Now recall the map $\theta$ from Example 4.6. It induces a right-bounded grading on $A_r(\lambda)$, and since $g_{\geq 0}^g = u(r)$, it follows that this subalgebra acts nilpotently on $A(\lambda)$ and hence $A(\lambda)$ is $n = n(r) \geq u(r)$-torsion. This completes the proof for $A_r(\lambda)$.

Since $O_{LA}$ is closed by direct limits it follows that $A(\lambda)$ belongs to $O_{LA}$. Given a weight $\mu$ we build an LCS for $A(\lambda)$ at $\mu$ as follows: start with an LCS at $\mu$ for $A_r(\lambda)$ such that $A_r(\lambda)_\mu \to A(\lambda)_\mu$ is an isomorphism, and use the natural map to obtain a filtration of $A(\lambda)$. Adding $A(\lambda)$ at the top of the filtration gives us the desired LCS, and shows that the multiplicities coincide as desired.

This result shows that $A(\lambda)$ is the projection of $M(\lambda)^\vee$ to $O_{LA}$. By Proposition 4.1 we get that $A(\lambda) \cong \Phi(\text{Coind}^g C_\lambda) \cong \pi_{LA}(\text{Coind}^g C_\lambda)$. The following theorem shows that the $A(\lambda)$ have the usual properties associated to standard objects in highest weight categories.

**Theorem 7.3.** For each $\lambda \in \mathfrak{h}^\circ$ the following hold.

(i) $A(\lambda)$ is indecomposable and $\text{soc} A(\lambda) \cong L(\lambda)$.

(ii) The composition factors of $A(\lambda)/L(\lambda)$ are of the form $L(\mu)$ with $\mu \prec \lambda$.

(iii) For each $\mu \in \mathfrak{h}^\circ$ we have $\dim \text{Hom}_{O_{LA}}(A(\mu), A(\lambda)) < \infty$.

**Proof.** Let $\mu \in \mathfrak{h}^\circ$. Since $\pi_{LA}$ is right adjoint to the inclusion of $O_{LA}$ in $(\mathfrak{g}, h)$-Mod there are isomorphisms

$$\text{Hom}_{O_{LA}}(L(\mu), A(\lambda)) \cong \text{Hom}_{\mathfrak{g}, h}(L(\mu), \text{Coind}^g C_\lambda) \cong \text{Hom}_{\mathfrak{g}, h}(L(\mu), \mathbb{C}_\lambda),$$

and since $L(\mu)$ has $C_\mu$ as its unique simple quotient as $\mathfrak{g}$-module, it follows that this space has dimension 1 when $\lambda = \mu$ and zero otherwise. Hence $L(\lambda)$ is the socle of $A(\lambda)$, which implies item (i). Since the support of $A(\lambda)/L(\lambda)$ is contained in the set of weights $\mu \prec \lambda$, item (iii) follows.

Since $A(\mu)$ has a local composition series at $\lambda$, there exists a finite length $\mathfrak{g}$-module $N \subset A(\mu)$ such that $(A(\mu)/N)_\lambda = 0$. Now consider the following exact sequence

$$0 \to \text{Hom}_{O_{LA}}(A(\mu)/N, A(\lambda)) \to \text{Hom}_{O_{LA}}(A(\mu), A(\lambda)) \to \text{Hom}_{O_{LA}}(N, A(\lambda)).$$

Since any nonzero map to $A(\lambda)$ must contain $L(\lambda)$ in its image, the first Hom-space in the long exact sequence is zero and the map $\text{Hom}_{O_{LA}}(A(\mu), A(\lambda)) \to \text{Hom}_{O_{LA}}(N, A(\lambda))$
is injective. Now a simple induction shows that \( \dim \Hom_{\mathcal{O}_\Lambda}(X, A(\lambda)) \) is finite for any \( X \) of finite length. This proves (iii). \qed

From now on we will refer to the \( A(\lambda) \) as standard modules of \( \mathcal{O}_\Lambda \). We point out that the order \( \prec \) on \( \mathfrak{h}^\circ \) is not interval-finite, and so it is not yet clear that the \( A(\lambda) \) are standard in the sense of highest weight categories. In the coming subsections we will compute the Jordan-Holder multiplicities of the standard modules explicitly, and in the process find an interval-finite order for its simple constituents.

7.2. The submodules \( T_r(\lambda) \). Our computation of the simple multiplicities of the module \( A(\lambda) \) is rather long and technical. The idea is to find an exhaustion of \( \Lambda(\lambda) \) which will allow us to compute the layers of the \( \psi \)-filtration of \( A(\lambda) \). Set \( T_r(\lambda) = \Phi_r(M_r(\lambda)^\psi) \); this is clearly a \( g_\lambda \)-module, and since it is spanned by \( \mathfrak{h} \)-semisimple vectors it is in fact a \( g_\lambda^+ \) = \( g_\lambda + \mathfrak{h} \)-module, and through \( \psi \) we can see it as a graded module.

**Lemma 7.4.** For each \( r \gg 0 \) there are injective maps of \( \psi \)-graded \( g_\lambda^+ \)-modules \( T_r(\lambda) \hookrightarrow T_{r+1}(\lambda) \) and \( T_r(\lambda) \hookrightarrow T(\lambda) \). Furthermore \( A(\lambda) \cong \varprojlim T_r(\lambda) \) as \( g_\lambda \)-modules.

**Proof.** Identifying \( M_r(\lambda)^\psi \) with its image inside \( M(\lambda)^\psi \) it is clear that \( M_r(\lambda)^\psi \subset M_{r+1}(\lambda)^\psi \) and that
\[
T_r(\lambda) = \Phi_r(M_r(\lambda)^\psi) \subset \Phi_{r+1}(M_r(\lambda)^\psi) \subset \Phi_{r+1}(M_{r+1}(\lambda)^\psi) = T_{r+1}(\lambda).
\]
The desired maps are given by the inclusions, which trivially satisfy the statement. Now if \( m \in \Phi_r(M(\lambda)^\psi) \) then there exists \( s \geq r \) such that \( m \in \Phi_r(M_s(\lambda)^\psi) \subset T_s(\lambda) \), so \( A(\lambda) = \bigcup_{r \geq 0} T_r(\lambda) \subset M(\lambda)^\psi \). \qed

The definition of \( \psi \)-filtrations given in [6, 7] is easily adapted to \( \psi \)-graded \( g_\lambda^+ \)-modules. The fact that \( A(\lambda) \) is the direct limit of the \( T_r(\lambda) \) in the category of \( \psi \)-graded \( g_\lambda^+ \)-modules implies that the \( p \)-th module in the \( \psi \)-filtration of \( A(\lambda) \) is the limit of the \( p \)-th module in the \( \psi \) filtrations of the \( T_r(\lambda) \). Furthermore, since direct limits are exact we can recover the \( s \)-module \( S_p(A(\lambda)) \) as the direct limit of the \( s \)-modules \( S_p(T_r(\lambda)) \). This will allow us to compute their characters and, using Proposition 6.8 the simple multiplicities of \( A(\lambda) \).

**Remark.** Notice that it is not true that the Jordan-Holder multiplicities of the \( T_r(\lambda) \) can be recovered from the multiplicities of the \( s \)-modules \( S_p(T_r(\lambda)) \). Indeed, Proposition 6.8 follows from Proposition 6.4 and the analogous statement is clearly false for \( g_\lambda \).

Set \( R_i = g_{\leq i}^\psi \) i.e. the subspace of \( g \) spanned by all root-spaces corresponding to roots \( \alpha \) with \( \psi(\alpha) \leq -i \), and set \( R_{i, \lambda} = R_i \cap g_\lambda \). We see \( R_i \) as \( \mathfrak{h} \)-module through the isomorphism \( R_i \cong g / g_{> -i}^\psi \). We set \( \mathcal{R} = \bigoplus_{k=1}^n 2^{k-1} R_k \) and \( \mathcal{R}_\lambda = \bigoplus_{k=1}^n 2^{k-1} R_{k, \lambda} \).

**Lemma 7.5.** There exists an isomorphism of \( s_\lambda \)-modules
\[
S_p T_r(\lambda) \cong \left( \text{Ind}_{s_\lambda \cap \mathfrak{h}}^{s_\lambda} S^*(\mathcal{R}_r)_{\rho - \psi(\lambda)} \otimes \mathbb{C}_\lambda \right)^\psi.
\]
The proof of this statement is quite technical and not particularly illuminating, so we postpone it until the end of this section. It is, however the main step to compute the Jordan-Hölder multiplicities of standard modules

**Theorem 7.6.** Let $\lambda$ and $\mu$ be eligible weights. Then

$$[A(\lambda) : L(\mu)] = m(\lambda, v, \mu) \dim \mathbf{S}^*(\mathcal{R}_{\gamma})_v.$$  

In particular this is nonzero if and only if $\mu$ is of the form $\sigma \cdot (\lambda + v)$ for a weight $v$ in the support of $\mathbf{S}^*(\mathcal{R}_{\gamma})$ and $\sigma \in \mathcal{W}^0$.

**Proof.** Since $A(\lambda)^{(p)} = \bigcup_{r \geq 0} T_r(\lambda)^{(p)}$, it follows that $F_p A(\lambda) = \lim_{r \to \infty} F_p T_r(\lambda)$. Exactness of direct limits of vector spaces implies that

$$\frac{F_p A(\lambda)}{F_{p-1} A(\lambda)} \approx \lim_{r \to \infty} \frac{F_p T_r(\lambda)}{F_{p-1} T_r(\lambda)}$$

and so taking top degree components we get $S_p A(\lambda) = \lim_{r \to \infty} S_p T_r(\lambda)$. It follows that $[A(\lambda) : L(\mu)] = [T_r(\lambda) : L_s(\mu)]$ for large $r$.

Using Lemma 7.5 and standard results on category $\mathcal{O}$ for the reductive algebra $s_{\gamma}$, see for example [Hum92, 3.6 Theorem], $S_p T_r(\lambda)$ has a filtration by dual Verma modules of the form $M_s(\lambda + v)^{\gamma}$, and each of these modules appears with multiplicity $\dim \mathbf{S}^*(\mathcal{R}_{\gamma})_v$. Thus

$$[S_p T_r(\lambda) : L_s(\mu)] = [M_s(\lambda + v)^{\gamma} : L_s(\mu)] \dim \mathbf{S}^*(\mathcal{R}_{\gamma})_v,$$

and for large $r$ this is $m(\lambda + v, \mu) \dim \mathbf{S}^*(\mathcal{R}_{\gamma})_v$, as desired. \qed

### 7.3. An interval finite order on $\mathfrak{h}^0$

We are now ready to show that there is an interval finite order for the simple constituents in standard objects of $\mathcal{O}_{\lambda \Lambda}$. In view of Theorem 7.6 there is only one reasonable choice.

**Definition 7.7.** Let $\lambda, \mu \in \mathfrak{h}^0$. We write $\mu <_{\inf} \lambda$ if there exist $v$ in the support of $W$ and $\sigma \in \mathcal{W}(s)$ such that $\mu = \sigma \cdot (\lambda + v) \preceq \lambda$.

**Lemma 7.8.** The order $<_{\inf}$ is interval finite.

**Proof.** Let us write $\mu \preceq \lambda$ if either

(i) there is a simple root $\alpha$ such that $\mu = s_\alpha \cdot \lambda \preceq \lambda$ or

(ii) $\mu = \lambda + v$ with $v$ a root such that $\psi(v) < 0$.

Notice that $\mu <_{\inf} \lambda$ implies $\mu \preceq \lambda$. Since the support of $\mathcal{R}$ is the $\mathbb{Z}_{>0}$-span of the space of roots of $g^\phi_{<0}$, it follows that $\preceq$ is a subrelation of $<_{\inf}$, and in fact this order is the reflexive transitive closure of $\preceq$. To prove the statement, it is enough to show that there are only finitely many weights $\gamma$ such that $\mu < \gamma <_{\inf} \lambda$, and that there is a global bound on the length of chains of the form $\mu < \lambda_1 < \cdots < \lambda_l < \lambda$.

If we are in case (i) then $\gamma = s_\alpha \cdot \mu$, and since $\gamma \prec \lambda$ it follows that $\alpha \in \mathcal{W}(s_{\gamma})$, so there are only finitely many options in this case. Notice also that $(\mu, \rho_{r+1}) < (\gamma, \rho_{r+1}) < (\lambda, \rho_{r+1})$, where $\rho_{r+1}$ is the sum of all positive roots of $s_{r+1}$. Thus in a chain as above there can be at most $(\lambda - \mu, \rho_{r+1})$ inequalities of type (i).
If we are in case (ii), then \( \lambda - \mu > 0 \) and \( \gamma = \mu - \nu \) for some \( \psi \)-negative root \( \nu \). Since \( \lambda - \gamma = \lambda - \mu - \nu \geq 0 \), it is enough to check that there are only finitely many \( \nu \) such that the last inequality holds. Now since \( \lambda \) and \( \mu \) are \( r \)-eligible, \( (\lambda - \mu, \alpha) = 0 \) for all roots outside of the root space of \( g_{r+1} \). This means that \( \nu \) must be a negative root of \( g_{r+1} \), of which there are finitely many. Notice also that in this case \( \psi(\mu) < \psi(\gamma) \leq \psi(\lambda) \), and hence in any chain as above there are at most \( \psi(\lambda - \mu) \) inequalities of type (ii). Thus any chain is of length at most \( \psi(\lambda - \mu) + (\lambda - \mu, \rho_{r+1}) \) and we are done. \( \Box \)

**Appendix: A proof of Lemma 7.5**

### 7.4. A result on Schur functors.

We begin by setting some notation for Schur functors and Littlewood-Richardson coefficients. Recall that given two vector spaces \( U, W \) and partitions \( \lambda, \mu \) we have isomorphisms, natural in both variables,

\[
S_\lambda(V) \otimes S_\mu(V) = \bigoplus_{\nu} c_{\lambda, \mu}^\nu S_\nu(V)
\]

\[
S_\lambda(V \oplus W) = \bigoplus_{\alpha, \beta} c_{\alpha, \beta}^\gamma S_\alpha(V) \otimes S_\beta(W).
\]

Given an \( m \)-tuple of partitions \( \alpha = (\alpha_1, \ldots, \alpha_m) \) and an \( m \)-tuple of vector spaces \( U_1, \ldots, U_m \) we set \( S_\alpha(U_1, \ldots, U_m) = S_{\alpha_1}(U_1) \otimes \cdots \otimes S_{\alpha_m}(U_m) \). It follows that for each \( \gamma \in \text{Part} \) there exists \( c^\gamma_\alpha \in \mathbb{Z}_{\geq 0} \) such that the following isomorphisms hold

\[
S_\gamma(U_1 \oplus \cdots \oplus U_m) \cong \bigoplus_{\alpha} c^\gamma_\alpha S_\alpha(U_1, \ldots, U_m);
\]

\[
S_\alpha(U, U_2, \ldots, U) \cong \bigoplus_{\gamma} c^\gamma_\alpha S_\gamma(U).
\]

These easily imply the following lemma.

**Lemma 7.9.** Fix \( \alpha \in \text{Part}^m \). Given vector spaces \( U_1, \ldots, U_m \) there is an isomorphism, natural in all variables,

\[
\bigoplus_{\gamma \in \text{Part}} \bigoplus_{\beta \in \text{Part}^m} c^\gamma_\alpha c^\gamma_\beta S_\gamma(U_1, U_2, \ldots, U_m) \cong S_\alpha(U, U_2, \ldots, U_m)
\]

where \( U = U_1 \oplus U_2 \oplus \cdots \oplus U_m \).

### 7.5. The module \( \mathcal{H} \).

In order to understand the module \( T_r(\lambda) \) we will need to look under the hood of the dual modules \( M_r(\lambda) \). For this subsection only we write \( \mathcal{W} = \mathcal{V}[r] \), so in particular \( \mathcal{W}^{(k)} = \mathcal{V}^{(k)}[r] \) and \( \mathcal{W}^{(l)} = \mathcal{V}^{(l)}[r] \), etc.

We begin by decomposing the parabolic algebra \( \mathcal{P}(r) \) as \( \mathcal{A}[r] \oplus \mathcal{B}_r \oplus \mathcal{F}_r \), where \( \mathcal{B}_r = \mathcal{B} \cap g_r, \mathcal{A}[r] = \bigoplus_{k<l} \mathcal{W}^{(k)} \otimes \mathcal{W}^{(l)} \), and \( \mathcal{F}_r \) is the unique root subspace completing the
The picture of \( f_r \). Notice that each rectangle is a direct summand of \( f_r \) as right \( l[r] \oplus s_r \)-module.

Using these spaces we can decompose \( g = g[r] \oplus (f_r \oplus \overline{f}_r) \oplus g_r \). This is a decomposition of \( g \) as \( g[r] \oplus g_r \)-module, since \( f_r \oplus \overline{f}_r \) is stable by the adjoint action of this subalgebra and isomorphic to \((W^n \otimes (V_r)^*) \oplus (V_r \otimes W_n^*)\). We also obtain a decomposition of \( \overline{f}_r \oplus l[r] \oplus \overline{a}[r] \)-modules

\[
\mathfrak{P}(r) = \overline{B}_r \oplus \overline{f}_r \oplus l[r] \oplus \overline{a}[r].
\]

Set \( q_r = s_r + b_r \). The subspaces \( a[r] \) and \( \overline{a}[r] \) are trivial \( q_r \)-submodules of \( g \). The subspace \( f_r \) is also a \( q_r \)-submodule of \( g \), and we endow \( \overline{f}_r \) with the structure of \( q_r \)-modules through the vector space isomorphism \( \overline{f}_r = f_r \oplus \overline{f}_r / f_r \). We set \( H = \mathbf{S}^*(\overline{f}_r \oplus \overline{a}[r]) \). By the PBW theorem we have an isomorphism of \( g_r \oplus l[r] \)-modules

\[
M_r(\lambda) \cong U(g) \otimes_{\mathfrak{g}(r)} \mathbb{C}_\lambda \cong U(g_r) \otimes_{b_r} (H \otimes \mathbb{C}_\lambda) = \text{Ind}^H_{g_r} H \otimes \mathbb{C}_\lambda.
\]

where the \( l[r] \)-module structure is determined by that of \( H \). Thus when computing the \( l[r]' \)-invariant vectors of the semisimple dual of this module we get

\[
T_r(\lambda) = \Phi_r(M_r(\lambda)^{\vee}) \cong \text{Coind}^H_{g_r} \Phi_r(H^{\vee}) \otimes \mathbb{C}_\lambda.
\]

7.6. The invariants of \( H^{\vee} \). Our task is thus to compute \( \Phi_r(H^{\vee}) \), and for this we will need a finer description of \( \overline{f}_r \). For each \( k \in \llbracket 0, n \rrbracket \) denote by \( B_k = V_{r+}^{(k)} \oplus V_{r-}^{(k-1)} \) and \( A_k = B_k^* \). These are \( s_r \)-modules and we have a decomposition of \( l[r] \oplus s_r \)-modules

\[
\overline{f}_r = \bigoplus_{k \geq l} (W^{(k)} \otimes A_l) \oplus (B_k \otimes W_l^{(l)}),
\]

\[
\mathbf{S}^*(\overline{f}_r) \cong \bigoplus_{a, \beta} \bigotimes_{k \geq l} S_{\alpha_{k,l}}(A_l) \otimes S_{\beta_{k,l}}(B_k) \otimes S_{\alpha_{l,k}}(W^{(k)}) \otimes S_{\beta_{l,k}}(W_l^{(l)}),
\]

where the sum runs over all families \( \alpha = (a_{k,l})_{1 \leq k \leq l \leq n} \) and \( \beta = (\beta_{k,l})_{1 \leq k \leq l \leq n} \). On the other hand we have decompositions

\[
\overline{a}[r] = \bigoplus_{k \geq l} W^{(k)} \otimes W_l^{(l)}; \quad \mathbf{S}^*(\overline{a}[r]) = \bigoplus_{\gamma} \bigotimes_{k \geq l} S_{\gamma_{k,l}}(W^{(k)}) \otimes S_{\gamma_{l,k}}(W_l^{(l)});
\]

where again the sum runs over all families of partitions \( \gamma = (\gamma_{k,l})_{1 \leq l \leq k \leq n} \). We extend these families by setting \( \gamma_{k,k} = \emptyset \), so \( \gamma_{k,l} \) is defined for all \( k \geq l \).

We now introduce the following notation: given a family of partitions \( \alpha = (a_{k,l})_{1 \leq l \leq k \leq n} \) we set \( \alpha_{k,*} = (a_{k,l})_{1 \leq k \leq n} \) and \( \alpha_{*,l} = (a_{k,l})_{1 \leq k \leq l} \). By collecting all terms of the form
\[ S_{\eta_1}(\mathbb{W}^{(k)}) \otimes S_{\nu_1}(\mathbb{W}^{(k)}) \] we obtain the following isomorphism of \( s_r \oplus [r]-\)modules

\[ \mathcal{H} \cong \bigoplus_{\alpha, \beta, \gamma, \eta, k} S_{\eta_1}(\mathbb{W}^{(k)}) \otimes S_{\eta_1}(\mathbb{W}^{(k)}) \]

where the sum is taken over all collections of partitions and

\[ Q(\alpha, \beta, \gamma, \eta, k) = S_{\alpha, \beta}(A_1, A_2, \ldots, A_k) \otimes S_{\beta, \gamma}(B_k, B_{k+1}, \ldots, B_n). \]

Since this module is contained in \( S^*(\mathbb{V}(r)) \), it must lie in \( \mathcal{T}_{[r]} \). It follows that each weight component intersects at most finitely many of these direct summands. Thus semisimple duality will commute with both the direct sum and the large tensor product. Furthermore, since each \( Q(\alpha, \beta, \gamma, \eta, k) \) has finite dimensional components we have

\[ \mathcal{H}^\vee \cong \bigoplus_{\alpha, \beta, \gamma, \eta, k} S_{\eta_1}(\mathbb{W}^{(k)}) \otimes S_{\eta_1}(\mathbb{W}^{(k)}) \]

Now recall that \( \Phi_r(I(\eta, \nu)\langle r \rangle^\vee) \cong \text{Hom}_r(I(\eta, \nu)\langle r \rangle^\vee, \mathbb{C}) \cong \delta_{\eta, \nu}. \) Thus

\[ \Phi_r(\mathcal{H}^\vee) \cong \bigoplus_{\alpha, \beta, \gamma, \eta, k} S_{\eta_1}(\mathbb{W}^{(k)}) \otimes S_{\eta_1}(\mathbb{W}^{(k)}) \]

Denote the module inside the semisimple dual by \( Q \), and let us analyse each factor of the tensor product separately. When \( k = 1 \) the Littlewood-Richardson coefficient \( c_{\eta_1, \nu_1}^{\eta} \) simplifies to \( c_{\alpha, \beta}^{\eta} \) and the only way for this to be nonzero is that \( \eta_1 = \alpha, \beta \). Thus the first factor in the tensor product has the form

\[ c_{\beta_1, \gamma_1}^{\eta_1} S_{\eta_1}(A_1) \otimes S_{\beta_1}(B_1, B_2, \ldots, B_n). \]

We can sum all these factors over \( \eta_1 \) and \( \beta_1 \) for every \( l \geq 1 \) since these do not appear in any other factors. Using Lemma \[ z \] we obtain

\[ \bigoplus_{\beta_1, \gamma_1} c_{\beta_1, \gamma_1}^{\eta_1} S_{\eta_1}(A_1) \otimes S_{\beta_1}(B_1, B_2, \ldots, B_n) \]

\[ \cong \bigoplus_{\beta_1} S_{\beta_1}(A_1, A_1, \ldots, A_1) \otimes S_{\gamma_1}(A_1, A_1, \ldots, A_1) \otimes S_{\beta_1}(B_1, B_2, \ldots, B_n) \]

\[ \cong S^*(A_1 \otimes (B_1 \oplus B_2 \oplus \cdots \oplus B_n)) \otimes S_{\gamma_1}(A_1, A_1, \ldots, A_1). \]

We see thus that \( Q \) is isomorphic to

\[ S^*(A_1 \otimes (B_1 \oplus B_2 \oplus \cdots \oplus B_n)) \otimes \bigoplus_{\alpha, \beta, \gamma, \eta, k} S_{\eta_1}(\mathbb{W}^{(k)}) \otimes S_{\eta_1}(\mathbb{W}^{(k)}) \]

where the sum is now restricted to the subsequences where \( k \geq 2 \).
Let us now focus on the new factor corresponding to \( k = 2 \). We can again sum over \( \eta_2 \) and \( \beta_{\ast,2} \) to get
\[
\bigoplus_{\eta_2, \beta_{\ast,2}} c_{\eta_2, \beta_{\ast,2}}^p A_{\ast,2, \gamma_2, \delta_2, \tau_2, s_{\tau_2}} (A_1) \otimes S_{\delta_2, \gamma_2, \tau_2, s_{\tau_2}} (A_2) \otimes S_{\beta_{\ast,2}} (B_2, \ldots, B_n)
\]
\[
\cong \bigoplus_{\beta_{\ast,2}} S_{\beta_{\ast,2}} (A_2) \otimes S_{\gamma_2, \tau_2} (A_2) \otimes S_{\beta_{\ast,2}} (B_2, \ldots, B_n)
\]
\[
\cong S^\ast (A_2 \otimes (B_2 \oplus \cdots \oplus B_n)) \otimes S_{\gamma_2, \tau_2} (A_2).
\]
where \( \overline{A}_2 = A_1 \oplus A_1 \oplus A_2 \). After \( n \) steps of this recursive process we obtain
\[
Q \cong \bigotimes_{k=1}^n S^\ast (A_k \otimes B_k) \cong S^\ast \left( \bigoplus_{k=1}^n \overline{A}_k \otimes B_k \right)
\]
where
\[
B_k = B_k \oplus B_{k+1} \oplus \cdots \oplus B_n
\]
and
\[
\overline{A}_k = A_1 \oplus A_2 \oplus \cdots \oplus A_k \oplus \overline{A}_1 \oplus \overline{A}_2 \oplus \cdots \oplus \overline{A}_{k-1}
\]
\[
= 2^{k-1} A_1 \oplus 2^{k-2} A_2 \oplus \cdots \oplus A_k.
\]
Thus in the direct sum the term \( A_k \otimes B_l \) appears \( 2^{l-k} + 2^{l-k-1} + \cdots + 2 + 1 = 2^{l-k+1} - 1 \) times. Now the sum of all the \( A_k \otimes B_l \) with \( l - k + 1 = i \) fixed is the space \( R_{l,r} \) introduced in the preamble of Lemma 7.5 and so \( Q \cong S^\ast (R_r) \). The naturality of Schur functors implies that this is an isomorphism of \( s_r \)-modules.

Now recall that \( T_\ast (\lambda) = \text{Coind}^p_{\psi_\lambda} Q^\vee \otimes C_\lambda \). The \( p \)-th module of the \( \psi \)-filtration of \( T_\ast (\lambda) \) is thus given by \( \text{Coind}^p_{\psi_\lambda} Q^\vee_{\geq p-\psi(\lambda)} \otimes C_\lambda \), and by exactness of semisimple coinduction on duals, the \( p \)-th layer of the filtration is isomorphic to \( \text{Coind}^p_{\psi_\lambda} Q^\vee_{p-\psi(\lambda)} \otimes C_\lambda \), whose top layer is in turn isomorphic to \( \text{Coind}^p_{\psi_\lambda} Q^\vee_{p-\psi(\lambda)} \otimes C_\lambda \). Finally, the isomorphism we found before tells us that \( Q^\vee_{\geq p-\psi(\lambda)} \otimes C_\lambda \cong S^\ast (R^\vee)_{p-\psi(\lambda)} \) as \( s_r \cap \overline{\mathfrak{p}} \)-modules, so we are done.

8. Category \( O_{LA} \): injective objects and further categorical properties

8.1. Large annihilator coinduction. In previous sections we have used inflation functors to turn objects in \( \mathcal{O}_s \) into \( q \)-modules and then used induction to construct objects in \( O_{LA} \). In this section we will use a dual construction. Recall that \( I_\ast (\lambda) \) denotes the injective envelope of \( L_\ast (\lambda) \) in \( \mathcal{O}_s \).

**Definition 8.1.** We denote by \( C : \mathcal{O}_s \rightarrow (q, h) \)-Mod the functor \( C = \Phi \circ \text{Coind}^{-\mathfrak{p}}_{\psi} \circ I_{\overline{\mathfrak{p}}} \). For each \( \lambda \in h^\circ \) we set \( I(\lambda) = C(I_\ast (\lambda)) \).

Notice that \( C \) does not automatically fall in \( O_{LA} \). However from the definition and Proposition 4.4 it follows that \( C(M(\lambda))^\vee = A(\lambda) \), so standard objects from the category \( \mathcal{O}_s \) are sent to standard objects of \( O_{LA} \). We will show that the same holds for injective
modules, and that in fact the standard filtrations of injective modules in $\overline{O}_s$ also lift to standard filtrations in $O_{LA}$.

8.2. **Injective objects.** We now begin our study of injective envelopes in $\overline{O}_{LA}$. Set $J(\lambda) = \text{Coind}_\theta^\mu \overline{\mathcal{I}}^\lambda I_s(\lambda)$. We will show that, just as for standard modules, we have $I(\lambda) = \Phi(J(\lambda)) = \pi_{LA}(J(\lambda))$, and that this is the injective envelope of the simple module $L(\lambda).

**Lemma 8.2.** Let $\lambda \in \mathfrak{h}^0$ and let $I_s(\lambda)$ be the injective envelope of $L_s(\lambda)$ in $\overline{O}_s$. Then for any object $M$ in $O_{LA}$ we have $\text{Ext}^1_{\overline{O}_s}(M, J(\lambda)) = 0$.

**Proof.** It is enough to prove the result for $M = L(\mu)$ with $\mu$ an eligible weight. It follows from Proposition 6.4 that $L(\mu) \cong \text{Ind}_\theta^\mu \overline{\mathcal{I}}^\lambda$ as $\overline{\mathfrak{g}}$-module. Since $I_s(\lambda)$ is an injective object it is acyclic for the functor $\text{Coind}_\theta^\mu \circ \overline{\mathcal{I}}^\lambda$ so by standard homological algebra

$$\text{Ext}^1_{\overline{O}_s}(L(\mu), J(\lambda)) \cong \text{Ext}^1_{\overline{O}_s}(U(\mathfrak{g}) \otimes \overline{\mathfrak{g}} L(\mu), I_s(\lambda)) = \text{Ext}^1_{\overline{O}_s}(L_s(\lambda), I_s(\lambda)).$$

Since $\overline{O}_s$ is closed under semisimple extensions, this last Ext-space must be 0. \qed

The class of $\Phi$-acyclic modules is closed by extensions, and so is the subclass of $\Phi$-acyclic modules $M$ such that $\Phi(M) = \pi_{LA}(M)$. The following lemma states that any module with a filtration by dual Verma modules lies in this class.

**Lemma 8.3.** Let $M$ be a weight $\mathfrak{g}$-module. Suppose $M$ has a finite filtration $F_1M \subset F_2M \subset \cdots F_nM$ such that its $i$-th layer is isomorphic to $M(\lambda_i)^\vee$ for $\lambda_i \in \mathfrak{h}^0$. Then $\Phi(M) = \pi_{LA}(M)$, and $\pi_{LA}(F_iM) / \pi_{LA}(F_{i-1}M) \cong A(\lambda_i).

**Proof.** The proof reduces to showing that $M(\lambda)^\vee$ is acyclic for both $\Phi$ and $\Gamma_n$. Recall that $\Phi$ can be written as the direct limit of functors $\Phi_r$ taking $l[r]^r$-invariant submodules, and that there are natural isomorphisms

$$\Phi_r \cong \bigoplus_\mu \text{Hom}_{\mathfrak{g}|_r}(U(\mathfrak{g}) \otimes_{l[r]^r} \mathbb{C}_\mu, -)$$

where the sum is taken over all $r$-eligible weights $\mu \in \mathfrak{h}^0$. Since derived functors commute with direct limits, and in particular with direct sums, it is enough to show that if $\lambda$ and $\mu$ are $r$-eligible then $\text{Ext}^1_{\mathfrak{g},h}(U(\mathfrak{g}) \otimes_{l[r]^r} \mathbb{C}_\mu, M(\lambda)^\vee) = 0$.

Notice that the result is trivial if $\lambda$ and $\mu$ have different levels, so we may assume that both have level $\chi = \chi_1 \omega^{(1)} + \cdots + \chi_n \omega^{(n)}$ and so $\lambda = \lambda_r + \chi$ and $\mu = \mu_r + \chi$. We can rewrite $M(\lambda)^\vee$ as follows

$$M(\lambda)^\vee \cong \text{Coind}_\theta^\mu \mathcal{F}^{(r)}_{l[r]^r} \text{Coind}_\theta^\mu_{l[r]^r} (\mathbb{C}_{\chi_1} \boxtimes \cdots \boxtimes \mathbb{C}_{\chi_n}) \otimes \mathbb{C}_{\lambda_r}$$

$$\cong \text{Coind}_\theta^\mu \mathcal{F}^{(r)}_{l[r]^r} (M_1^\vee \boxtimes \cdots \boxtimes M_n^\vee) \otimes \mathbb{C}_{\lambda_r},$$

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where $M_i = M_i[r](\chi_i \omega(i)[r])$. Using that inflation functors are exact and semisimple coinduction is exact on semisimple duals, standard homological algebra implies

$$\text{Ext}_{\mathfrak{g},h}^1(U(\mathfrak{g}) \otimes I[r] + C_{\mu}, M(\lambda) ) $$

$$\cong \text{Ext}_{\mathfrak{g},h}^1(U(I[r])^+ \otimes T(r) U(\mathfrak{g}) \otimes I[r] + C_{\mu}, M_1^\vee \otimes \cdots \otimes M_n^\vee \otimes C_{\lambda} ) $$

$$\cong \text{Ext}_{\mathfrak{g},h}^1( (\mathfrak{S}^+(u(r))) \otimes C_{\mu-\lambda}, M_1^\vee \otimes \cdots \otimes M_n^\vee ). $$

The left hand side of this Ext-space decomposes as a direct sum of tensor $I[r]$-modules, so the Ext-space decomposes as a tensor product of spaces of the form

$$\text{Ext}_{\mathfrak{g}(\mathfrak{V}(I[r]),h)}^1(T_k, M_k^\vee )$$

with $T_k$ a tensor module, and hence in $O_{LA}^{|I[r]|}$. Now each $M_i^\vee$ is injective in the corresponding category $\mathfrak{O}$, since it is the dual Verma module corresponding to a weight that is maximal in its dot-orbit. By Lemma 8.2, each of these Ext-spaces is 0, and hence $M(\lambda)^\vee$ is $\Phi$-acyclic.

Now take $J_r$ to be the left ideal of $U(\mathfrak{g})$ generated by $n^r$. There exists a natural isomorphism

$$\Gamma_n \cong \lim_{\rightarrow} \bigoplus_{\lambda \in \mathfrak{h}^r} \text{Hom}_{\mathfrak{g},h}(U(\mathfrak{g})/J_r \otimes h, C_{\lambda}, -), $$

so a similar argument works in this case. Finally since $\pi_{LA} = \Gamma_n \circ \Phi$, we are done. □

We are now ready to prove the main result of this section.

**Theorem 8.4.** Let $\lambda \in \mathfrak{h}^r$. The module $I(\lambda)$ lies in $O_{LA}$ and it is an injective envelope for $L(\lambda)$. Furthermore it has a finite filtration whose first layer is $A(\lambda)$ and whose higher layers are isomorphic to $A(\mu)$ with $\mu >_{\text{fin}} \lambda$.

**Proof.** Since inflation is exact and semisimple coinduction is exact over semisimple duals, the costandard filtration of $I_\mu(\lambda)$ induces a finite filtration over $I(\lambda)$, whose first layer is isomorphic to $M(\lambda)^\vee$ and whose higher layers are isomorphic to $M(\mu)^\vee$ for $\mu > \lambda$. Applying Lemma 8.3, it follows that $I(\lambda) = \Phi(I(\lambda)) = \pi_{LA}(I(\lambda))$ and hence belongs to $O_{LA}$, and that this module has the required filtration. Finally, reasoning as in the proof of Lemma 8.2, we have isomorphisms

$$\text{Ext}_{O_{LA}}^1(L(\mu), I(\lambda)) \cong \text{Ext}_{\mathfrak{g},h}^1(L(\mu), I(\lambda)) \cong \text{Ext}_{\mathfrak{g},h}^1(L(\mu), I_\mu(\lambda)). $$

Taking $i = 0$ and varying $\mu$ we see that $L(\lambda)$ is the socle of $I(\lambda)$, and taking $i = 1$ it follows that $I(\lambda)$ is injective in $O_{LA}$. □

### 8.3. Highest weight structure and blocks.

We are now ready to prove the main result of this paper, namely that $O_{LA}$ is a highest weight category in the sense of Cline, Parshall and Scott [CPS88].

**Theorem 8.5.** Category $O_{LA}$ is a highest weight category with indexing set $(\mathfrak{h}^r, <_{\text{inf}})$. Simple modules are simple highest weight modules $L(\lambda)$, the family of standard objects $A(\lambda)$ have infinite length, and the corresponding injective envelopes $I(\lambda)$ have finite standard filtrations.
Thus the desired multiplicity is the same as the multiplicity of $M_\mu(\lambda)$ by those modules in $O_{\Lambda A}$ whose support is contained in $\Lambda$. Notice in particular that each class has a fixed level, and so the decomposition $O_{\Lambda A} = \prod_\kappa O_{\Lambda A}(\kappa)$ refines the decomposition by levels. We now show that these are the indecomposable blocks of $O_{\Lambda A}$.

**Corollary 8.6.** Let $\lambda, \mu \in \mathfrak{h}^\circ$. The multiplicity of $A(\mu)$ in any standard filtration of $I(\lambda)$ equals $m(\lambda, \mu)$.

**Proof.** It is enough to check the result for one standard filtration. We can obtain a standard filtration of $I(\lambda)$ by applying the functor $C$ to a standard filtration of $I_s(\lambda)$. Thus the desired multiplicity is the same as the multiplicity of $M_\mu(\lambda)$ in $I_s(\lambda)$. By Theorem 7.3 this multiplicity equals $[M_\mu(\lambda) : L_s(\lambda)] = m(\lambda, \mu)$.

We finish with a description of the irreducible blocks of $O_{\Lambda A}$. Recall that we denote by $\Lambda$ the root lattice of $\mathfrak{g}$. For each $\kappa \in \mathfrak{h}^\circ / \Lambda$ set $O_{\Lambda A}(\kappa)$ to be the subcategory formed by those modules in $O_{\Lambda A}$ whose support is contained in $\kappa$. In particular that each class has a fixed level, and so the decomposition $O_{\Lambda A} = \prod_\kappa O_{\Lambda A}(\kappa)$ refines the decomposition by levels. We now show that these are the indecomposable blocks of $O_{\Lambda A}$.

**Proposition 8.7.** The blocks $O_{\Lambda A}(\kappa)$ are indecomposable.

**Proof.** The support of $\mathfrak{g}_\kappa \subset \mathcal{R}$ spans the root lattice, and from Theorem 7.6 that $[A(\lambda) : L(\lambda)] = [A(\lambda) : L(\lambda + \mu)]$ for any root in $\mathfrak{g}_\kappa$. Thus whenever two weights $\lambda, \mu$ lie in the same class modulo the root lattice, there is a finite chain of weights $\lambda = \lambda_1, \lambda_2, \ldots, \lambda_m = \mu$ such that $L(\lambda_{i+1})$ is a simple constituent of the indecomposable module $A(\lambda_i)$. This shows that all simple modules in the block $O_{\Lambda A}(\kappa)$ are linked and hence the block is indecomposable.

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