SECOND-ORDER NECESSARY CONDITIONS FOR OPTIMAL CONTROL OF SEMILINEAR ELLIPTIC EQUATIONS WITH LEADING TERM CONTAINING CONTROLS

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Dedicated to Professor Eduardo Casas at his 60th birthdate

ABSTRACT. An optimal control problem for a semilinear elliptic equation of divergence form is considered. Both the leading term and the semilinear term of the state equation contain the control. The well-known Pontryagin type maximum principle for the optimal controls is the first-order necessary condition. When such a first-order necessary condition is singular in some sense, certain type of the second-order necessary condition will come in naturally. The aim of this paper is to explore such kind of conditions for our optimal control problem.

1. Introduction. Let \( \Omega \subseteq \mathbb{R}^n \) \((n \geq 2)\) be a bounded domain with a smooth boundary \( \partial \Omega \). Consider the following controlled elliptic partial differential equation (PDE, for short):

\[
\begin{aligned}
- \nabla \cdot (A(x, u(x)) \nabla y(x)) &= f(x, y(x), u(x)), \quad \text{in } \Omega, \\
y|_{\partial \Omega} &= 0,
\end{aligned}
\] (1.1)

where \( A : \Omega \times U \to \mathbb{S}_+^n \) and \( f : \Omega \times \mathbb{R} \times U \to \mathbb{R} \), with \( \mathbb{S}_+^n \) being the set of all \((n \times n)\) positive definite matrices, \( U \) being a separable (nonempty) metric space (a typical example is \( U = \{0, 1\} \)). In the above, \( u(\cdot) \) is the control which belongs to the set \( \mathcal{U} \) of all admissible controls defined by the following:

\[ \mathcal{U} \equiv \{ u : \Omega \to U \mid u(\cdot) \text{ is measurable} \}. \]

Under some mild conditions (see our assumptions \((S2)-(S3)\) a little later), for any \( u(\cdot) \in \mathcal{U}, \) (1.1) admits a unique weak solution \( y(\cdot) \equiv y(\cdot ; u(\cdot)) \) which is called the

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state (corresponding to the control $u(\cdot)$). The performance of the control $u(\cdot)$ is measured by the following cost functional

$$J(u(\cdot)) = \int_\Omega f^0(x, y(x), u(x))dx$$

for some given map $f^0 : \Omega \times \mathbb{R} \times U \rightarrow \mathbb{R}$. Our optimal control problem can be stated as follows.

**Problem (C).** Find a $\bar{u}(\cdot) \in U$ such that

$$J(\bar{u}(\cdot)) = \inf_{u(\cdot) \in U} J(u(\cdot)).$$

Any $\bar{u}(\cdot) \in U$ satisfying (1.3) is called an optimal control, and the corresponding $\bar{y}(\cdot) \equiv y(\cdot; \bar{u}(\cdot))$ is called an optimal state. The pair $(\bar{y}(\cdot), \bar{u}(\cdot))$ is called an optimal pair.

Let us make some rough observations. Suppose $(\bar{y}(\cdot), \bar{u}(\cdot))$ is an optimal pair of Problem (C). For any given $u(\cdot) \in U$, let $u^\delta(\cdot) \in U$ be a suitable perturbation of $\bar{u}(\cdot)$ determined by $u(\cdot)$ with a parameter $\delta > 0$ (for example, a convex type perturbation, or a spike type variation), so that $\bar{\rho}(u^\delta(\cdot), \bar{u}(\cdot)) = O(\delta)$ with $\bar{\rho}(\cdot, \cdot)$ being a suitable metric on the set $U$, and the following holds:

$$J(u^\delta(\cdot)) = J(\bar{u}(\cdot)) + \delta J^1(\bar{u}(\cdot), u(\cdot)) + o(\delta), \quad \text{as } \delta \to 0.$$  

Here, $J^1(u(\cdot), u(\cdot))$ is some functional of $(\bar{u}(\cdot), u(\cdot))$. The above can be called the first-order Taylor expansion of $J(\cdot)$ at $\bar{u}(\cdot)$, and $J^1(\bar{u}(\cdot), u(\cdot))$ can be regarded as the “directional derivative” of $J(\cdot)$ at $\bar{u}(\cdot)$ in the “direction” $u(\cdot)$. Hence, the minimality of $\bar{u}(\cdot)$ implies

$$J^1(\bar{u}(\cdot), u(\cdot)) \geq 0, \quad \forall u(\cdot) \in U.$$  

This is called the first-order necessary condition for $\bar{u}(\cdot)$, which is essentially the Pontryagin’s maximum principle for our Problem (C). Now, suppose that there is a set $U_0 \subseteq U$, which is different from the singleton $\{\bar{u}(\cdot)\}$, such that the following holds:

$$J^1(\bar{u}(\cdot), u(\cdot)) = 0, \quad \forall u(\cdot) \in U_0.$$  

Then $\bar{u}(\cdot)$ is said to be singular on the set $U_0$. For convenience, we call $U_0$ a singular set of $\bar{u}(\cdot)$. Let

$$U_0(\bar{u}(\cdot)) = \left\{ u(\cdot) \in U \mid J^1(\bar{u}(\cdot), u(\cdot)) = 0 \right\},$$

which is called the maximum singular set of $\bar{u}(\cdot)$. When $U_0(\bar{u}(\cdot)) = U$, we say that $\bar{u}(\cdot)$ is fully singular (or simply singular); When $U_0(\bar{u}(\cdot)) = \{\bar{u}(\cdot)\}$, we say that $\bar{u}(\cdot)$ is nonsingular; And, more interestingly, when $\{\bar{u}(\cdot)\} \subset U_0(\bar{u}(\cdot)) \subset U$, we say that $\bar{u}(\cdot)$ is partially singular. The notion of singular controls was introduced by Gabasov–Kirillova in [16], where our partial singularity was called “the singularity in the sense of Pontryagin’s maximum principle”, and our full singularity was called “the singularity in the classical sense”. We prefer to use our shorter names. Now, suppose $\bar{u}(\cdot)$ is partially singular. Then one should expect that the following (comparing with (1.4))

$$J(u^\delta(\cdot)) = J(\bar{u}(\cdot)) + \delta J^2(\bar{u}(\cdot), u(\cdot)) + o(\delta^2), \quad \forall u(\cdot) \in U_0(\bar{u}(\cdot)),$$  

for some functional $J^2(\bar{u}(\cdot), u(\cdot))$ of $(\bar{u}(\cdot), u(\cdot))$. The above can be called the second-order Taylor expansion of $J(\cdot)$ at $\bar{u}(\cdot)$ in the direction of $u(\cdot) \in U_0(\bar{u}(\cdot))$, and $J^2(\bar{u}(\cdot), u(\cdot))$ can be regarded as the “second order directional derivative” at $\bar{u}(\cdot)$.
in the “direction” of \( u(\cdot) \in \mathcal{U}_0(\bar{u}(\cdot)) \). Then the minimality of \( \bar{u}(\cdot) \) leads to the following:
\[
J^2(\bar{u}(\cdot), u(\cdot)) \geq 0, \quad \forall u(\cdot) \in \mathcal{U}_0(\bar{u}(\cdot)).
\]
This is referred to as the second-order necessary condition of \( \bar{u}(\cdot) \). We emphasize that the above holds only for all \( u(\cdot) \in \mathcal{U}_0(\bar{u}(\cdot)) \), the maximum singular set of \( \bar{u}(\cdot) \). To get some more feeling, let us look at the following simple example, consisting of three situations.

**Example 1.1.** Let \( U = [-1, 1] \times [-1, 1] \).

(i) Let \( J(u) = u^2 = u_1^2 + u_2^2 \) with \( u = (u_1, u_2) \in U \). Then \( u \mapsto J(u) \) is differentiable and the minimum is attained at \( \bar{u} = (0, 0) \), an interior point of \( U \), with \( J(\bar{u}) = 0 \). Therefore, for any \( u = (u_1, u_2) \in U \) and \( \delta \in (0, 1) \), we have \( u^\delta = \bar{u} + \delta(u - \bar{u}) = \delta u \in U \), and
\[
J(u^\delta) = \delta^2 u^2 = J(\bar{u}) + \delta J^1(\bar{u}, u) + \delta^2 J^2(\bar{u}, u).
\]
Consequently,
\[
J^2(\bar{u}, u) = 0, \quad \forall u \in U.
\]
This means that the maximum singular set \( \mathcal{U}_0(\bar{u}) \) of \( \bar{u} \) coincides with \( U \), and \( \bar{u} \) is fully singular. Hence,
\[
J^2(\bar{u}, u) = u^2 \geq 0, \quad \forall u \in \mathcal{U}_0(\bar{u}) = U,
\]
which is the classical second-order necessary condition for \( \bar{u} \).

(ii) Let \( J(u) = u_1^2 + u_2^3 \) with \( u = (u_1, u_2) \in U \). The minimum is attained at \( \bar{u} = (-1, -1) \). Then, for any \( u = (u_1, u_2) \in U \) and any \( \delta \in (0, 1) > 0 \), one has \( u^\delta = \bar{u} + \delta(u - \bar{u}) = (-1, -1) + \delta(u_1 + 1, u_2 + 1) \in U \), and \( J(u^\delta) \)
\[
= \left[ -1 + \delta(u_1 + 1) \right]^3 + \left[ -1 + \delta(u_2 + 1) \right]^3
= -2 + 3\delta(u_1 + u_2 + 2) - 3\delta^2[(u_1 + 1)^2 + (u_2 + 1)^2] + 3\delta^3[(u_1 + 1)^3 + (u_2 + 1)^3]
= J(\bar{u}) + \delta J^1(\bar{u}, u) + \delta^2 J^2(\bar{u}, u) + \delta^3 J^3(\bar{u}, u).
\]
Thus,
\[
J^1(\bar{u}, u) = 3(u_1 + u_2 + 2) \neq 0, \quad \forall u = (u_1, u_2) \in U \setminus \{\bar{u}\}.
\]
This means that \( \bar{u} \) is nonsingular. In this case, there is no second-order necessary condition for \( \bar{u} \).

(iii) Let \( J(u) = u_1^3 + u_2^2 \) with \( u = (u_1, u_2) \in U \). The minimum is attained at \( \bar{u} = (-1, 0) \). For any \( u = (u_1, u_2) \in U \), let the perturbation \( u^\delta \) be defined by the following:
\[
u^\delta = \bar{u} + \delta(u - \bar{u}) = (-1, 0) + \delta(u_1 + 1, u_2) = (-1 + \delta(u_1 + 1), \delta u_2) \in U.
\]
Then we have
\[
J(u^\delta) = \left[ -1 + \delta(u_1 + 1) \right]^3 + \delta^2 u_2^2
= J(\bar{u}) + 3\delta(u_1 + 1) + \delta^2[-3(u_1 + 1)^2 + u_2^2] + \delta^3(u_1 + 1)^3
\equiv J(\bar{u}) + \delta J^1(\bar{u}, u) + \delta^2 J^2(\bar{u}, u) + \delta^3 J^3(\bar{u}, u).
\]
Hence, the first-order necessary condition is
\[
J^1(\bar{u}, u) \equiv 3(u_1 + 1) \geq 0, \quad \forall u = (u_1, u_2) \in U,
\]
and $\bar{u}$ is partially singular with $U_0(\bar{u}) = \{(-1, u_2) \mid u_2 \in [-1, 1]\}$. The second-order necessary condition is

$$J^2(\bar{u}, u) \equiv u_2^2 \geq 0, \quad \forall u \in U_0(\bar{u}).$$

However, we do not have (see (1.9))

$$J^2(\bar{u}, u) \equiv -3(u_1 + 1)^2 + u_2^2 \geq 0, \quad \forall u \in U.$$

The above example shows that in general, fully singular, partially singular, and nonsingular all can happen for a minimum of a function. Note that the above is for scalar functions. It is expected that the case of optimal control problems should be much more complicated. In Lou [24], second-order necessary and sufficient conditions for partially singular optimal controls of ordinary differential equations were established with general control domain. On the other hand, for convex control domains (mainly interval type), and mainly for fully singular cases, second-order necessary/sufficient optimality conditions have been studied for PDEs by many authors. We mention just a few of them here: Casas–Tröltzsch [9, 10, 11], Casas–Tröltzsch–Unger [14], Raymond–Tröltzsch [28], Mittelmann [27], Casas–Mateos [8], Rösch–Tröltzsch [29, 30], Wang–He [31], Casas–Los Reye–Tröltzsch [7], and Bonnans–Herman [3, 4]. For some earlier works on ODEs, see Kelly [18], Kopp–Moyer [20], Gabasov–Kirillova [16], Krener [21], and Knobloch [19].

For problems of elliptic PDEs with control appearing in the leading term, Casas [5] studied the first-order necessary conditions for the case $A(x, u) = uI$ with quadratic cost functional and with the control being Lipschitz continuous. General case were treated by Lou–Yong in [26], and analogous results for parabolic and hyperbolic cases were given by Lou in [25] and Li–Lou in [22]. If the leading term of the state equation (1.1) does not contain controls, i.e., $A(x, u) \equiv A(x)$, then one can establish the second-order necessary conditions for partially singular optimal controls following similar arguments of [24]. However, if the leading term of the equation contains the control, we will see that it is much more complicated, even in defining the partial singularity of the optimal control. It turns out that the construction of a proper family of perturbations is much more difficult than the case without having control in the leading term, in order to have the first-order term disappear in the Taylor type expansion. The difficult will be overcome by introducing the notion of weak singularity which involves a proper vector field. Consequently, the results obtained will have some big difference compared to those for the problems without having the control in the leading term.

Let us briefly mention the existence issue of optimal control for our Problem (C). Since in general, we do not have the convexity for the map $u(\cdot) \mapsto J(u(\cdot))$, even for the case that the leading term does not contain the control, it is not trivial to get the existence of optimal controls for Problem (C). However, there are some ways to approach the problem.

- If the family $\{A(\cdot, u(\cdot)) \mid u(\cdot) \in U\}$ is closed in the sense of $H$-convergence (see Allaire [1, 2]), then the problem can be similarly treated as the one that the leading term does not contain the control.

- If introduce relaxed controls $\sigma(\cdot)$ (which are the probability measures on $U$), then under proper conditions, one can get the existence of optimal relaxed controls. If one could further characterize the optimal relaxed control as some probability measure that concentrates at some singletons, then Problem (C) will admit an optimal control.
Necessary conditions of optimal controls could help to exclude many controls that are not optimal, and leave some candidates for further checking, just like in standard calculus.

One of the points that we would like to make is that the problems of existence of optimal controls and the necessary conditions for optimal controls are two independent problems, both of them are nontrivial and the study of them might have to be carried out separately. The purpose of this paper is to investigate the second order necessary conditions for Problem (C), and we will not discuss the existence of optimal controls.

The rest of the paper is organized as follows. In Section 2, we will introduce the notions of singularity and weak singularity of the optimal controls. The main result of the paper will be stated, together with a couple of corollaries. Section 3 will be devoted to a review of the proof for the first-order necessary condition for Problem (C), which will inspire the second-order necessary condition. Section 4 is devoted to a proof of a result crucial for the proof our main result. A proof of the second-order necessary condition will be presented in Section 5.

2. The main result. For any differentiable function \( \varphi : \Omega \to \mathbb{R} \), its gradient is denoted by \( \nabla \varphi = (\frac{\partial \varphi}{\partial x_1}, \ldots, \frac{\partial \varphi}{\partial x_n})^\top : \Omega \to \mathbb{R}^n \equiv \mathbb{R}^{n \times 1} \); For any differentiable vector-valued function \( f = (f^1, f^2, \ldots, f^n)^\top : \Omega \to \mathbb{R}^n \), its Jacobean matrix \( f_x \) is denoted by

\[
f_x = \begin{pmatrix}
\frac{\partial f^1}{\partial x_1} & \frac{\partial f^2}{\partial x_1} & \cdots & \frac{\partial f^n}{\partial x_1} \\
\frac{\partial f^1}{\partial x_2} & \frac{\partial f^2}{\partial x_2} & \cdots & \frac{\partial f^n}{\partial x_2} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial f^1}{\partial x_n} & \frac{\partial f^2}{\partial x_n} & \cdots & \frac{\partial f^n}{\partial x_n}
\end{pmatrix}
\equiv \nabla f = \left( \nabla f^1, \nabla f^2, \cdots, \nabla f^n \right).
\]

Compatible with the above notation, we also will use

\[
\nabla^\top F = (\nabla \cdot F^1, \nabla \cdot F^2, \ldots, \nabla \cdot F^n)
\]

for \( F = (F^1, F^2, \ldots, F^n) : \Omega \to \mathbb{R}_+^{n \times n} \). A function \( g : \mathbb{R}_+^n \to \mathbb{R} \) is said to be \([0,1]^n\)-periodic if it admits a period 1 in every coordinate direction \( x_i \), \( i = 1, 2, \ldots, n \). Denote \( W^{1,2}_0([0,1]^n; \mathbb{R}^n) \) the space of all \([0,1]^n\)-periodic vector-valued functions in \( W^{1,2}_loc(\mathbb{R}^n; \mathbb{R}^n) \) and \( W^{1,2}_0([0,1]^n; \mathbb{R}^n) / \mathbb{R}^n \) the corresponding quotient space.

Next, let us introduce the following assumptions.

(S1) Set \( \Omega \) is a bounded domain in \( \mathbb{R}^n \) \((n \geq 2)\) with a smooth boundary \( \partial \Omega \), and metric space \((U, \rho)\) is separable.

(S2) Function \( A : \Omega \times U \to \mathbb{S}^n_+ \), (recall that \( \mathbb{S}^n_+ \) is the set of all \((n \times n)\) (symmetric) positive definite matrices), for which \( x \mapsto A(x, v) \) is measurable, and \( v \mapsto A(x, v) \) is continuous. Further, there exist constants \( \lambda \geq \lambda > 0 \) such that

\[
\lambda |\xi|^2 \leq \langle A(x, v)\xi, \xi \rangle \leq \Lambda |\xi|^2, \quad \forall (\xi, v) \in \mathbb{R}_+ \times U, \; \text{a.e.} \; x \in \Omega. \tag{2.1}
\]

(S3) Function \( f : \Omega \times \mathbb{R} \times U \to \mathbb{R} \) has the following properties: \( x \mapsto f(x, y, v) \) is measurable, \((y, v) \mapsto f(x, y, v) \) is continuous for almost all \( x \in \Omega \), and \( y \mapsto f(x, y, v) \) continuously differentiable. Moreover,

\[
f_y(x, y, v) \leq 0, \quad \forall (y, v) \in \mathbb{R} \times U, \; \text{a.e.} \; x \in \Omega \tag{2.2}
\]
and for any \( R > 0 \), there exists an \( M_R > 0 \) such that
\[
|f(x, y, v)| + |f_y(x, y, v)| \leq M_R, \quad \forall v \in U, \quad |y| \leq R, \quad \text{a.e. } x \in \Omega. \tag{2.3}
\]

(S4) Function \( f^0 : \Omega \times \mathbb{R} \times U \rightarrow \mathbb{R} \) has the following properties: \( x \mapsto f^0(x, y, v) \) is measurable, \( (y, v) \mapsto f^0(x, y, v) \) is continuous for almost all \( x \in \Omega \), and \( y \mapsto f^0(x, y, v) \) is continuously differentiable. Moreover, for any \( R > 0 \), there exists a \( K_R > 0 \) such that
\[
|f^0(x, y, v)| + |f^0_y(x, y, v)| \leq K_R, \quad \forall v \in U, \quad |y| \leq R, \quad \text{a.e. } x \in \Omega. \tag{2.4}
\]

It is standard that under (S1)–(S3), for any \( u(\cdot) \in U \), state equation (1.1) admits a unique weak solution \( y(\cdot) = y(\cdot; u(\cdot)) \in H^1_0(\Omega) \cap C(\Omega) \) and the following estimate holds:
\[
\|y(\cdot)\|_{H^1_0(\Omega)} + \|y(\cdot)\|_{L^\infty(\Omega)} \leq K, \tag{2.5}
\]
for some constant \( K > 0 \). Therefore, if, in addition, (S4) is also assumed, then the cost functional is well-defined. Consequently, Problem (C) is well-formulated. The following was established in [26].

**Theorem 2.1.** Let (S1)–(S4) hold. Let \((\bar{y}(\cdot), \bar{u}(\cdot))\) be an optimal pair of Problem (C), and \( \bar{v}(\cdot) \) be the weak solution of the following adjoint equation:
\[
\begin{align*}
\left\{
\begin{array}{ll}
-\nabla \cdot (A(x, \bar{u}(x)) \nabla \bar{v}(x)) = f_y(x, \bar{y}(x), \bar{u}(x)) \bar{v}(x) - f^0_y(x, \bar{y}(x), \bar{u}(x)), & \text{in } \Omega, \\
\bar{v}\big|_{\partial \Omega} = 0.
\end{array}
\right.
\end{align*}
\tag{2.6}
\]

Then
\[
H(x, \bar{y}(x), \bar{v}(x), \nabla \bar{y}(x), \nabla \bar{v}(x), \bar{u}(x)) - H(x, \bar{y}(x), \bar{v}(x), \nabla \bar{y}(x), \nabla \bar{v}(x), v) \\
\geq \max_{\mu \in S_{n-1}} \left\{ \left[ A(x, \bar{u}(x)) - A(x, v) \right] \nabla \bar{y}(x), \mu \right\} \left\{ \left[ A(x, \bar{u}(x)) - A(x, v) \right] \nabla \bar{v}(x), \mu \right\} \\
= \frac{1}{2} \left| A(x, v)^{-\frac{1}{2}} (A(x, \bar{u}(x)) - A(x, v)) \nabla \bar{y}(x) \right|^2 \\
+ \frac{1}{2} \left| A(x, v)^{-\frac{1}{2}} (A(x, \bar{u}(x)) - A(x, v)) \nabla \bar{v}(x) \right|^2 \\
\geq 0, \quad \forall v \in U, \quad \text{a.e. } x \in \Omega, \tag{2.7}
\]

where
\[
H(x, y, \psi, \xi, \eta, v) = \psi f(x, y, v) - f^0(x, y, v) - \langle A(x, v) \xi, \eta \rangle, \quad \forall (x, y, \psi, \xi, \eta, v) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times U, \tag{2.8}
\]

which is called the Hamiltonian, and \( S_{n-1} \) is the unit sphere in \( \mathbb{R}^n \).

The equality in (2.7) follows from the following simple fact (see [26], Lemma 2.3, for a proof).
\[
\max_{\mu \in S_{n-1}} \langle \mu, \xi \rangle \langle \mu, \eta \rangle = \frac{||\xi||^2 + ||\eta||^2}{2}, \quad \forall \xi, \eta \in \mathbb{R}^n, \quad n \geq 2, \tag{2.9}
\]

and the last inequality in (2.7) is due to the Cauchy-Schwartz inequality. Therefore, (2.7) implies (and might be a little stronger than) the following:
\[
H(x, \bar{y}(x), \bar{v}(x), \nabla \bar{y}(x), \nabla \bar{v}(x), \bar{u}(x)) \\
= \max_{v \in U} H(x, \bar{y}(x), \bar{v}(x), \nabla \bar{y}(x), \nabla \bar{v}(x), v), \quad \text{a.e. } x \in \Omega. \tag{2.10}
\]
Let
\[ \mathcal{L} = \{ \ell : \Omega \to S^{n-1} \mid \ell(\cdot) \text{ is measurable} \}. \]

We now introduce the following definition.

**Definition 2.2.** Let \( \bar{u}(\cdot) \in \mathcal{U} \) be an admissible control.

(i) Let \((u(\cdot), \ell(\cdot)) \in \mathcal{U} \times \mathcal{L}\) satisfy the following:
\[
H(x, \bar{y}(x), \bar{v}(x), \nabla \bar{y}(x), \nabla \bar{v}(x), \bar{u}(x)) - H(x, \bar{y}(x), \bar{v}(x), \nabla \bar{y}(x), \nabla \bar{v}(x), u(x)) = \langle [A(x, \bar{u}(x)) - A(x, u(x))]\nabla \bar{v}(x), \ell(x) \rangle,
\]
\[
\{A(x, u(x))\ell(x), \ell(x)\}, \quad \text{a.e. } x \in \Omega.
\]

Then we say that the control \( \bar{u}(\cdot) \) is weakly singular at \((u(\cdot), \ell(\cdot))\).

(ii) Let \( u(\cdot) \in \mathcal{U} \) such that
\[
H(x, \bar{y}(x), \bar{v}(x), \nabla \bar{y}(x), \nabla \bar{v}(x), \bar{u}(x))
\]
\[
= H(x, \bar{y}(x), \bar{v}(x), \nabla \bar{y}(x), \nabla \bar{v}(x), u(x)), \quad \text{a.e. } x \in \Omega.
\]

Then we say that the control \( \bar{u}(\cdot) \) is singular at \( u(\cdot) \).

(iii) Denote
\[
\mathcal{V}_0(\bar{u}(\cdot)) = \{(u(\cdot), \ell(\cdot)) \in \mathcal{U} \times \mathcal{L} \mid \bar{u}(\cdot) \text{ is weakly singular at } (u(\cdot), \ell(\cdot))\},
\]
\[
\mathcal{U}_0(\bar{u}(\cdot)) = \{u(\cdot) \in \mathcal{U} \mid \bar{u}(\cdot) \text{ is singular at } u(\cdot)\}.
\]

If \( \mathcal{V}_0(\bar{u}(\cdot)) = \mathcal{U} \times \mathcal{L} \), we say that \( \bar{u}(\cdot) \) is fully weakly singular; if
\[
\{\bar{u}(\cdot)\} \times \mathcal{L} \subset \mathcal{V}_0(\bar{u}(\cdot)) \subset \mathcal{U} \times \mathcal{L},
\]
then we say that the control \( \bar{u}(\cdot) \) is partially weakly singular; if \( \mathcal{V}_0(\bar{u}(\cdot)) = \{\bar{u}(\cdot)\} \times \mathcal{L} \), then we say that the control \( \bar{u}(\cdot) \) is weakly nonsingular.

Likewise, we may define a control \( \bar{u}(\cdot) \) to be fully singular, partially singular, and nonsingular, respectively, when \( \mathcal{U}_0(\bar{u}(\cdot)) = \mathcal{U} \), \{\bar{u}(\cdot)\} \subset \mathcal{U}_0(\bar{u}(\cdot)) \subset \mathcal{U} \) and \( \mathcal{U}_0(\bar{u}(\cdot)) = \{\bar{u}(\cdot)\} \), respectively.

Let us make some observations on the above notions.

- If optimal control \( \bar{u}(\cdot) \) is weakly singular at \((u(\cdot), \ell(\cdot))\) (with \( u(\cdot) \neq \bar{u}(\cdot) \)), then comparing (2.11) with (2.7), we see that for almost all \( x \in \Omega \), \((u(x), \ell(x))\) is a maximum of the map
\[
(v, \mu) \mapsto \frac{\langle [A(x, \bar{u}(x)) - A(x, v)]\nabla \bar{v}(x), \mu \rangle}{\langle A(x, v)\mu, \mu \rangle}
\]
over \( U \times S^{n-1} \). Note that such a maximum point might not be unique, in general.

- If optimal control \( \bar{u}(\cdot) \) is singular at \( u(\cdot) \neq \bar{u}(\cdot) \), then by (2.7), we see that
\[
\max_{\mu \in S^{n-1}} \frac{\langle [A(x, \bar{u}(x)) - A(x, u(x))]\nabla \bar{v}(x), \mu \rangle}{\langle A(x, u(x))\mu, \mu \rangle}
\]
\[
= \frac{1}{2} |A(x, u(x))|^{-\frac{1}{2}} |A(x, \bar{u}(x)) - A(x, u(x))| \nabla \bar{v}(x)|
\]
\[
+ \frac{1}{2} |A(x, \bar{u}(x))^{-\frac{1}{2}} |A(x, u(x)) - A(x, u(x))| \nabla \bar{v}(x)|
\]
\[
+ \frac{1}{2} \langle A(x, u(x))^{-\frac{1}{2}} |A(x, \bar{u}(x)) - A(x, u(x))| \nabla \bar{v}(x), A(x, u(x))^{-\frac{1}{2}} |A(x, \bar{u}(x)) - A(x, u(x))| \nabla \bar{v}(x) \rangle = 0, \quad \text{a.e. } x \in \Omega.
\]
Therefore, by the compactness of $S^{n-1}$, together with Filippov’s measurable selection lemma ([23]), we have some $\ell(\cdot) \in L$ such that
\[
\langle [A(x, \bar{u}(x)) - A(x, u(x))] \nabla \bar{y}(x), \ell(x) \rangle \\
\cdot \langle [A(x, \bar{u}(x)) - A(x, u(x))] \nabla \bar{\psi}(x), \ell(x) \rangle = 0, \quad \text{a.e. } x \in \Omega. \tag{2.14}
\]
Hence, the optimal control $\bar{u}(\cdot)$ is weakly singular at $(u(\cdot), \ell(\cdot))$ for some $\ell(\cdot) \in L$.

- If optimal control $\bar{u}(\cdot)$ is weakly singular at $(u(\cdot), \ell(\cdot))$ such that (2.14) holds, then $\bar{u}(\cdot)$ must be singular at $u(\cdot)$. Further, it follows from the last equality in (2.13) that the equality holds in Cauchy–Schwartz inequality. Therefore,
\[
[A(x, \bar{u}(x)) - A(x, u(x))] \nabla \bar{y}(x) \quad \text{and} \quad [A(x, \bar{u}(x)) - A(x, u(x))] \nabla \bar{\psi}(x)
\]
must be linearly dependent and have opposite directions. Consequently, (2.14) implies that
\[
\langle [A(x, \bar{u}(x)) - A(x, u(x))] \nabla \bar{y}(x), \ell(x) \rangle \\
=\langle [A(x, \bar{u}(x)) - A(x, u(x))] \nabla \bar{\psi}(x), \ell(x) \rangle = 0, \quad \text{a.e. } x \in \Omega. \tag{2.15}
\]
The above can be summarized as follows.

**Proposition 2.3.** Let (S1)–(S4) hold. Suppose $\bar{u}(\cdot)$ is an optimal control of Problem (C). If $\bar{u}(\cdot)$ is singular at $u(\cdot) \in U \setminus \{\bar{u}(\cdot)\}$, then there exists an $\ell(\cdot) \in L$ such that (2.15) holds and $\bar{u}(\cdot)$ is weakly singular at $(u(\cdot), \ell(\cdot))$. Conversely, if $\bar{u}(\cdot)$ is weakly singular at $(u(\cdot), \ell(\cdot)) \in U \times L$ such that (2.15) holds, then $\bar{u}(\cdot)$ is singular at $u(\cdot)$.

If $A(x, v)$ is independent of $v \in U$, then the right hand side of (2.11) is automatically zero, and (2.12) is true. Thus, in such a case, weak singularity is equivalent to singularity, and
\[
V_0(\bar{u}(\cdot)) = U_0(\bar{u}(\cdot)) \times L.
\]

To state our main result of the current paper, the second-order necessary condition for optimal control of Problem (C), we need the following further assumption.

**S5** Function $y \mapsto (f(x, y, v), f^0(x, y, v))$ is twice continuously differentiable. Moreover, for any $R > 0$, there exists a $K_R > 0$ such that
\[
|f_{yy}(x, y, v)| + |f^0_{yy}(x, y, u)| \leq K_R, \quad \forall v \in U, \quad |y| \leq R, \quad \text{a.e. } x \in \Omega. \tag{2.16}
\]

We point out that, unlike most of the literature on PDE controls that we cited, no differentiability condition is assumed for the map $u \mapsto (f(x, y, u), f^0(x, y, u))$. Actually, our $U$ is just a metric space which does not have a linear structure, in general. In particular, no convexity condition is assumed for $U$. Now, we state our main result of this paper.

**Theorem 2.4.** Let (S1)–(S5) hold and $(\bar{y}(\cdot), \bar{u}(\cdot))$ be an optimal pair of Problem (C). Let $\bar{u}(\cdot)$ be partially weakly singular and $(u(\cdot), \ell(\cdot)) \in V_0(\bar{u}(\cdot))$ with $u(\cdot) \neq \bar{u}(\cdot)$. Then the following holds:
\[
\int_\Omega \left( H(x, \bar{y}(x), \bar{\psi}(x), \nabla \bar{y}(x), \nabla \bar{\psi}(x), \bar{u}(x)) \\
- H(x, \bar{y}(x), \bar{\psi}(x), \nabla \bar{y}(x), \nabla \bar{\psi}(x), u(x)) \right) \frac{\ell(x)^T A(x, \bar{u}(x)) \ell(x)}{f(x)^T A(x, u(x)) \ell(x)}
\]

\[
+ \int_\Omega \left( H(x, \bar{y}(x), \bar{\psi}(x), \nabla \bar{y}(x), \nabla \bar{\psi}(x), u(x)) \\
- H(x, \bar{y}(x), \bar{\psi}(x), \nabla \bar{y}(x), \nabla \bar{\psi}(x), \bar{u}(x)) \right) \frac{\ell(x)^T A(x, u(x)) \ell(x)}{f(x)^T A(x, \bar{u}(x)) \ell(x)}
\]

\[
\geq \int_\Omega \left( H(x, \bar{y}(x), \bar{\psi}(x), \nabla \bar{y}(x), \nabla \bar{\psi}(x), \bar{u}(x)) \\
- H(x, \bar{y}(x), \bar{\psi}(x), \nabla \bar{y}(x), \nabla \bar{\psi}(x), u(x)) \right) \frac{\ell(x)^T A(x, \bar{u}(x)) \ell(x)}{f(x)^T A(x, u(x)) \ell(x)}
\]

\[
+ \int_\Omega \left( H(x, \bar{y}(x), \bar{\psi}(x), \nabla \bar{y}(x), \nabla \bar{\psi}(x), u(x)) \\
- H(x, \bar{y}(x), \bar{\psi}(x), \nabla \bar{y}(x), \nabla \bar{\psi}(x), \bar{u}(x)) \right) \frac{\ell(x)^T A(x, u(x)) \ell(x)}{f(x)^T A(x, \bar{u}(x)) \ell(x)}
\]
where $\tilde{\psi}(\cdot)$ is the weak solution to the adjoint equation (2.6), $H(\cdot)$ is the Hamiltonian defined by (2.8), and $Y(\cdot)$ is the weak solution to the following variational equation:

$$\begin{cases}
- \nabla \cdot (A(x, u(x)) \nabla Y(x)) = f_y(x, \tilde{y}(x), \tilde{u}(x)) Y(x) + \nabla \cdot (\Theta(x) \nabla \tilde{y}(x)) \\
Y|_{\partial \Omega} = 0,
\end{cases}$$

with

$$\Theta(x) = A(x, u(x)) - A(x, \tilde{u}(x))$$

$$\frac{1}{2} \int_{\Omega} \left[ \left( H_y(x, \tilde{y}(x), \tilde{\psi}(x), \nabla \tilde{y}(x), \nabla \tilde{\psi}(x), \tilde{u}(x)) \right) Y(x) \\
- \frac{1}{2} H_{yy}(x, \tilde{y}(x), \tilde{\psi}(x), \nabla \tilde{y}(x), \nabla \tilde{\psi}(x), \tilde{u}(x)) |Y(x)|^2 \\
+ \langle \left( A(x, u(x)) - A(x, \tilde{u}(x)) \nabla \tilde{\psi}(x), \nabla Y(x) \right) \rangle \right] \, dx \geq 0,$$

where $\tilde{\psi}(\cdot)$ and $H(\cdot)$ are the same as those in Theorem 2.4, and $Y(\cdot)$ is the weak solution to the following variational equation:

$$\begin{cases}
- \nabla \cdot (A(x, \tilde{u}(x)) \nabla Y(x)) = f_y(x, \tilde{y}(x), \tilde{u}(x)) Y(x) \\
+ \nabla \cdot (A(x, u(x)) \nabla \tilde{y}(x)) + f(x, \tilde{y}(x), u(x)), \quad \text{in } \Omega, \\
Y|_{\partial \Omega} = 0.
\end{cases}$$

**Proof.** By Proposition 2.3, we know that since $\tilde{u}(\cdot)$ is singular at $u(\cdot)$, (2.15) holds for some $\ell(\cdot) \in \mathcal{L}$. Then

$$\nabla \cdot (\Theta(s) \nabla \tilde{y}(x)) = \nabla \cdot \left( [A(x, u(x)) - A(x, \tilde{u}(x))] \nabla \tilde{y}(x) \right)$$

$$= \nabla \cdot \left( A(x, u(x)) \nabla \tilde{y}(x) \right) + f(x, \tilde{y}(x), \tilde{u}(x)).$$

Hence, (2.18) becomes (2.21), and (2.17) becomes (2.20) (making use the singularity of $\tilde{u}(\cdot)$ at $u(\cdot)$, see (2.12)). Therefore, our conclusion follows.

The following gives the situation that the leading term does not contain the control, whose proof is pretty straightforward.
Corollary 2.6. Let (S1)–(S5) hold with \( A(x,v) \equiv A(x) \) independent of \( v \). Let \( (\bar{y}()\bar{u}()) \) be an optimal pair of Problem (C), for which \( \bar{u}() \) is singular at \( u() \in U_0(\bar{u}()) \setminus \{\bar{u}()\} \). Then
\[
\int_{\Omega} \left[ \left( \mathcal{H}_y(x, \bar{y}(x), \bar{\psi}(x), \bar{u}(x)) - \mathcal{H}_y(x, \bar{y}(x), \bar{\psi}(x), u(x)) \right) \right] Y(x) dx
\end{equation}
where
\[
\mathcal{H}(x,y,v) = \langle \psi, f(x,y,v) \rangle - f^0(x,y,v), \quad (x,y,v) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \times U, \quad (2.23)
\]
\[
\mathcal{H}(x,y,v) = \langle \psi, f(x,y,v) \rangle - f^0(x,y,v), \quad (x,y,v) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \times U, \quad (2.23)
\]
and \( \bar{\psi}() \) and \( Y() \) are the weak solutions to the following adjoint equation and variational equation, respectively:
\[
\begin{cases}
- \nabla \cdot (A(x) \nabla \bar{\psi}(x)) = f_y(x, \bar{y}(x), \bar{u}(x)) \bar{\psi}(x) - f_y^0(x, \bar{y}(x), u(x)), \quad \text{in } \Omega, \\
\bar{\psi} \mid_{\partial \Omega} = 0.
\end{cases}
\]
\[
\begin{cases}
- \nabla \cdot (A(x) \nabla Y(x)) = f_y(x, \bar{y}(x), \bar{u}(x)) Y(x) + f(x, \bar{y}(x), u(x)) - f(x, \bar{y}(x), u(x)), \quad \text{in } \Omega, \\
Y \mid_{\partial \Omega} = 0.
\end{cases}
\]

3. The first-order necessary condition revisited. In this section, we briefly recall the proof of Theorem 2.1, from which we will find a correct direction approaching the second-order necessary condition for the optimal control. To this end, we first recall the following lemma ([1]).

Lemma 3.1. Let \( G : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}_+^n \) be measurable. Assume that
(i) \( z \mapsto G(x,z) \) is \([0,1]^n\)-periodic;
(ii) There exists two constants \( \Lambda > \lambda > 0 \), such that
\[
\lambda |\xi|^2 \leq \langle G(x,z)\xi,\xi \rangle \leq \Lambda |\xi|^2, \quad \forall (x,z) \in \Omega \times \mathbb{R}^n, \quad \xi \in \mathbb{R}^n;
\]
(iii) The following holds:
\[
\lim_{\varepsilon \to 0^+} \int_{\Omega} |G(x, \frac{x}{\varepsilon})|^2 \frac{dx}{\varepsilon} = \int_{\Omega} \int_{[0,1]^n} |G(x,s)|^2 ds dx.
\]

Let \( g \in H^{-1}(\Omega) \) and \( y_{\varepsilon}() \) be the solution of
\[
\begin{cases}
- \nabla \cdot (G(x, \frac{x}{\varepsilon}) \nabla y_{\varepsilon}(x)) = g, \quad \text{in } \Omega, \\
y_{\varepsilon} \mid_{\partial \Omega} = 0.
\end{cases}
\]
Then \( y_{\varepsilon}() \) converges weakly to \( y() \) in \( H_0^1(\Omega) \) where \( y() \) solves
\[
\begin{cases}
- \nabla \cdot (G(x) \nabla y(x)) = g, \quad \text{in } \Omega, \\
y \mid_{\partial \Omega} = 0,
\end{cases}
\]
and \( \tilde{G}(x) \equiv (\tilde{G}_{ij}(x)) \) is given by
\[
\tilde{G}_{ij}(x) = \int_{[0,1]^n} \langle G(x,z) \nabla z [\phi_1(x,z) + z_i], \nabla z [\phi_2(x,z) + z_j] \rangle dz,
\]
with \( \phi_1(x,z) \in W^{1,2}([0,1]^n; \mathbb{R}^n) \) being the unique solution of
\[
- \nabla_z \cdot \left( G(x,z) \nabla z [\phi_1(x,z) + z_i] \right) = 0, \quad 1 \leq i \leq n.
\]
Observe that
\[
\int_{[0,1]^n} \langle G(x, z) \nabla_z [\phi_i(x, z) + z_i], \nabla_z [\phi_j(x, z) + z_j] \rangle dz = \\
= \int_{[0,1]^n} \left( \langle \nabla_z \phi_i(x, z), G(x, z) [\nabla_z \phi_j(x, z) + z_j] \rangle + \langle G(x, z) e_i, \nabla_z \phi_j(x) + e_j \rangle \right) dz \\
= \int_{[0,1]^n} \left( \langle G(x, z) e_i, e_j \rangle + \langle G(x, z) e_i, \nabla_z \phi_j(x) \rangle \right) dz \\
= \int_{[0,1]^n} \left( G_{ij}(x, z) + (G(x, z) \nabla_z \phi(x, z)^\top)_{ij} \right) dz.
\]

Hence,
\[
\bar{G}(x) = \int_{[0,1]^n} G(x, z) \left( I + \nabla_z \phi(x, z)^\top \right) dz. \tag{3.7}
\]

Also, (3.6) can be written as
\[
- \nabla_z^\top \left[ G(x, z) (I + \nabla_z \phi)^\top \right] = 0. \tag{3.8}
\]

Note that in general \( G(x, \tilde{z}) \) does not necessarily converge strongly in \( L^2(\Omega) \) (as \( \varepsilon \downarrow 0 \)). Therefore, the above lemma is by no means trivial or obvious. On the other hand, the following result is much easier, which will also be used later, for different situations.

**Lemma 3.2.** Let \( \varepsilon > 0 \) and \( G_\varepsilon(\cdot) \in L^\infty(\Omega; \mathbb{S}_+^n) \). Assume that there exist two constants \( \Lambda > \lambda > 0 \), such that
\[
\lambda |\xi|^2 \leq \langle G_\varepsilon(x) \xi, \xi \rangle \leq \Lambda |\xi|^2, \quad \forall x \in \Omega, \xi \in \mathbb{R}^n;
\]
and \( G_\varepsilon(\cdot) \) converges to \( G(\cdot) \) strongly in \( L^2(\Omega) \). Let \( g \in H^{-1}(\Omega) \) and \( y_\varepsilon(\cdot) \) be the solution of
\[
\begin{cases}
- \nabla \cdot \left( G_\varepsilon(x) \nabla y_\varepsilon(x) \right) = g, & \text{in } \Omega, \\
y_\varepsilon \big|_{\partial \Omega} = 0.
\end{cases}
\]

Then \( y_\varepsilon(\cdot) \) converges strongly to \( y(\cdot) \) in \( H_0^1(\Omega) \), as \( \varepsilon \downarrow 0 \), where \( y(\cdot) \) solves
\[
\begin{cases}
- \nabla \cdot \left( G(x) \nabla y(x) \right) = g, & \text{in } \Omega, \\
y \big|_{\partial \Omega} = 0.
\end{cases}
\]

To prove the above lemma, one can first easily get that \( y_\varepsilon(\cdot) \) converges weakly to \( y(\cdot) \) in \( H_0^1(\Omega) \), as \( \varepsilon \downarrow 0 \). The strong convergence follows from
\[
\lim_{\varepsilon \downarrow 0} \int_\Omega \langle G_\varepsilon(x) (\nabla y_\varepsilon(x) - \nabla y(x)), \nabla y_\varepsilon(x) - \nabla y(x) \rangle dx \\
= \lim_{\varepsilon \downarrow 0} \int_\Omega \langle G_\varepsilon(x) \nabla y_\varepsilon(x), \nabla y_\varepsilon(x) - \nabla y(x) \rangle dx \\
= \lim_{\varepsilon \downarrow 0} \int_\Omega (y_\varepsilon(x) - y(x)) g dx = 0.
\]

We now recall the proof of Theorem 2.1 (see [26] for technical details). Let \( \tilde{u}(\cdot) \in \mathcal{U} \) be an optimal control and \( u(\cdot) \in \mathcal{U} \) be an arbitrary fixed control. Pick
any \( \mu \in S^{n-1} \). Define a two-parameter spike variation \( u^{\alpha, \varepsilon} (\cdot; \mu) \) of the control \( \bar{u}(\cdot) \) associated with \( u(\cdot) \) and \( \mu \) as follows:

\[
\begin{align*}
    u^{\alpha, \varepsilon}(x; \mu) &= \begin{cases} 
        u(x), & \text{if } \left\{ \frac{x}{\varepsilon} \right\} \in [0, \alpha), \\
        \bar{u}(x), & \text{if } \left\{ \frac{x}{\varepsilon} \right\} \in [\alpha, 1),
    \end{cases}
\end{align*}
\]

where \( \{ a \} \equiv a - [a] \) denotes the decimal part of the real number \( a \). Then \( u^{\alpha, \varepsilon}(\cdot; \mu) \in \mathcal{U} \). Here, the dependence on \( \mu \) is emphasized. We should keep in mind that \( u^{\alpha, \varepsilon}(\cdot; \mu) \) also depends on the selected control \( u(\cdot) \) (which is fixed). Let \( y^{\alpha, \varepsilon}(\cdot; \mu) = y(\cdot; u^{\alpha, \varepsilon}(\cdot; \mu)) \) be the state corresponding to the control \( u^{\alpha, \varepsilon}(\cdot; \mu) \). Then by Lemma 3.1, as \( \varepsilon \downarrow 0 \), \( y^{\alpha, \varepsilon}(\cdot; \mu) \) converges to \( y^\alpha(\cdot; \mu) \), weakly in \( H^1_0(\Omega) \) and strongly in \( L^2(\Omega) \), where \( y^\alpha(\cdot; \mu) \) solves the following PDE, which is called a relaxed state equation:

\[
\begin{align*}
    \begin{cases}
        - \nabla \cdot (A^\alpha(x; \mu) \nabla y^\alpha(x; \mu)) = (1 - \alpha)f(x, y^\alpha(x; \mu), \bar{u}(x)) + \alpha f(x, y^\alpha(x; \mu), u(x)), & \text{in } \Omega, \\
        y^\alpha(x; \mu) = 0, & \text{on } \partial \Omega,
    \end{cases}
\end{align*}
\]

with

\[
A^\alpha(x; \mu) = \alpha A(x, u(x)) + (1 - \alpha)A(x, \bar{u}(x))
\]

\[
- \frac{\alpha(1 - \alpha)[A(x, u(x)) - A(x, \bar{u}(x))]\mu \mu^\top [A(x, u(x)) - A(x, \bar{u}(x))] - (1 - \alpha)\mu \mu^\top A(x, u(x))\mu}{(1 - \alpha)\mu \mu^\top A(x, u(x))\mu + \alpha \mu \mu^\top A(x, \bar{u}(x))\mu}.
\]

Define

\[
Y^\alpha(\cdot; \mu) = \frac{y^\alpha(\cdot; \mu) - \bar{y}(\cdot)}{\alpha}.
\]

Then, as \( \alpha \downarrow 0 \), \( Y^\alpha(\cdot; \mu) \) converges to \( Y(\cdot; \mu) \) weakly in \( H^1_0(\Omega) \) and strongly in \( L^2(\Omega) \), where \( Y(\cdot; \mu) \) is the weak solution to the following:

\[
\begin{align*}
    \begin{cases}
        - \nabla \cdot (A(x, \bar{u}(x)) \nabla Y(x; \mu)) = f_g(x, \bar{y}(x), \bar{u}(x)) Y(x; \mu) + \nabla \cdot (\Theta(x; \mu) \nabla \bar{y}(x)) + f(x, \bar{y}(x), u(x)) - f(x, \bar{y}(x), \bar{u}(x)), & \text{in } \Omega, \\
        Y(x; \mu) = 0, & \text{on } \partial \Omega,
    \end{cases}
\end{align*}
\]

with

\[
\Theta(x; \mu) = A(x, u(x)) - A(x, \bar{u}(x)) - \frac{[A(x, u(x)) - A(x, \bar{u}(x))]\mu \mu^\top [A(x, u(x)) - A(x, \bar{u}(x))]}{\mu \mu^\top A(x, u(x))\mu}.
\]

Consequently, as \( \alpha \downarrow 0 \), \( y^\alpha(\cdot; \mu) \) converges to \( \bar{y}(\cdot) \) weakly in \( H^1_0(\Omega) \) and strongly in \( L^2(\Omega) \). On the other hand, by the convergence of \( y^{\alpha, \varepsilon}(\cdot; \mu) \to y^\alpha(\cdot; \mu) \) (strongly in \( L^2(\Omega) \), as \( \varepsilon \downarrow 0 \)), one has

\[
\begin{align*}
    \lim_{\varepsilon \downarrow 0} J(u^{\alpha, \varepsilon}(\cdot; \mu)) &= J^\alpha(u(\cdot), \mu) = \alpha \int_{\Omega} f_0(x, y^\alpha(x; \mu), u(x)) \, dx \\
    &+ (1 - \alpha) \int_{\Omega} f^0(x, y^\alpha(x; \mu), \bar{u}(x)) \, dx.
\end{align*}
\]

Further, (suppressing \( x \))

\[
\begin{align*}
    J^\alpha(u(\cdot), \mu) - J(\bar{u}(\cdot))
    &= \int_{\Omega} \left[ \alpha \left( f^0(y^\alpha, u) - f^0(y^\alpha, \bar{u}) \right) + f^0(y^\alpha, \bar{u}) - f^0(\bar{y}, \bar{u}) \right] \, dx.
\end{align*}
\]
\[ = \alpha \int_{\Omega} \left[ \left( f^0(y^\alpha, u) - f^0(y^\alpha, \bar{u}) \right) + \left( \int_0^1 f_y^0(\bar{y} + \alpha t Y^\alpha, \bar{u}) dt \right) Y^\alpha \right] dx. \] (3.15)

Thus,

\[ \lim_{\alpha \downarrow 0} \frac{J(\alpha(u(\cdot), \mu)) - J(\bar{u}(\cdot))}{\alpha} = \int_{\Omega} \left[ f^0(x, \bar{y}(x), u(x)) - f^0(x, \bar{y}(x), \bar{u}(x)) + f_y^0(x, \bar{y}(x), \bar{u}(x)) Y(x; \mu) \right] dx. \] (3.16)

Consequently,

\[ J(\alpha^{\alpha,\varepsilon}(\cdot; \mu)) - J(\bar{u}(\cdot)) = J^\alpha(u(\cdot), \mu) - J(\bar{u}(\cdot)) + r_\varepsilon \]

\[ = \alpha \int_{\Omega} \left[ f^0(x, \bar{y}(x), u(x)) - f^0(x, \bar{y}(x), \bar{u}(x)) + f_y^0(x, \bar{y}(x), \bar{u}(x)) Y(x; \mu) \right] dx + \alpha \rho_\alpha + r_\varepsilon, \] (3.17)

where

\[ \lim_{\varepsilon \downarrow 0} r_\varepsilon = 0, \quad \lim_{\alpha \downarrow 0} \rho_\alpha = 0. \]

By the duality, we obtain

\[ J(u^{\alpha,\varepsilon}(\cdot; \mu)) = J(\bar{u}(\cdot)) \]

\[ + \alpha \int_{\Omega} \left[ H(x, \bar{y}(x), \bar{\psi}(x), \nabla \bar{y}(x), \nabla \bar{\psi}(x), \bar{u}(x)) \right. \]

\[ - H(x, \bar{y}(x), \bar{\psi}(x), \nabla \bar{y}(x), \nabla \bar{\psi}(x), u(x)) \]

\[ - \frac{1}{\langle A(x, u(x)) \mu, \mu \rangle} \left( [A(x, \bar{u}(x)) - A(x, u(x))] \nabla \bar{y}(x), \mu \right) \]

\[ \cdot \left( [A(x, \bar{u}(x)) - A(x, \bar{v}(x))] \nabla \bar{\psi}(x), \mu \right) dx + \alpha \rho_\alpha + r_\varepsilon \]

\[ \equiv J(\bar{u}(\cdot)) + \alpha J^1(\bar{u}(\cdot); u(\cdot), \mu) + \alpha \rho_\alpha + r_\varepsilon. \] (3.18)

By the optimality of \( \bar{u}(\cdot) \), the above leads to the following:

\[ J^1(\bar{u}(\cdot); u(\cdot), \mu) \geq 0, \]

which implies

\[ H(x, \bar{y}(x), \bar{\psi}(x), \nabla \bar{y}(x), \nabla \bar{\psi}(x), \bar{u}(x)) - H(x, \bar{y}(x), \bar{\psi}(x), \nabla \bar{y}(x), \nabla \bar{\psi}(x), v) \]

\[ - \langle [A(x, \bar{u}(x)) - A(x, v)] \nabla \bar{y}(x), \mu \rangle \langle [A(x, \bar{u}(x)) - A(x, v)] \nabla \bar{\psi}(x), \mu \rangle \geq 0, \] (3.19)

\[ \forall v \in U, \quad \mu \in S^{n-1}, \quad \text{a.e.} \quad x \in \Omega. \]

Hence, (2.7) follows, proving the first-order necessary condition.

Note that in the above result, \( \mu \in S^{n-1} \) is a given fixed direction. Whereas, when an optimal control \( \bar{u}(\cdot) \) is weakly singular at \((u(\cdot), \ell(\cdot)) \in V_0(\bar{u}(\cdot)), \ell(\cdot) \in L \) might not be a fixed \( \mu \). Therefore, we need to extend (3.18), allowing \( \mu \) to be replaced by \( \ell(\cdot) \in L \). More precisely, we hope to have the following result.

**Proposition 3.3.** Let (S1)-(S4) hold and \((\bar{y}(\cdot), \bar{u}(\cdot))\) be an optimal pair of Problem (C). Let \((u(\cdot), \ell(\cdot)) \in U \times L, \) and let \( y^\alpha(\cdot; \ell(\cdot)) \) be the weak solution to the following
equation:
\[
\begin{align*}
&\left\{\begin{array}{l}
-\nabla \cdot \left( A^\alpha(x; \ell(\cdot)) \nabla y^\alpha(x; \ell(\cdot)) \right) = (1 - \alpha)f(x, y^\alpha(x; \ell(\cdot)), \bar{u}(x)) \\
y^\alpha(x; \ell(\cdot)) = 0,
\end{array}\right. & \text{in } \Omega,
\end{align*}
\]
with
\[
A^\alpha(x; \ell(\cdot)) = \alpha A(x, u(x)) + (1 - \alpha)A(x, \bar{u}(x))
\]
\[
- \alpha(1 - \alpha)[A(x, u(x)) - A(x, \bar{u}(x))] \ell(x)\ell(x)^T [A(x, u(x)) - A(x, \bar{u}(x))] (1 - \alpha)\ell(x)^T A(x, u(x)) \ell(x).
\]

Define
\[
J^\alpha(u(\cdot); \ell(\cdot)) = \alpha \int_\Omega f^0(s, y^\alpha(x; \ell(\cdot)), u(x)) \, dx \\
+ (1 - \alpha) \int_\Omega f^0(x, y^\alpha(x; \ell(\cdot)), \bar{u}(x)) \, dx.
\]

Then
\[
J^\alpha(u(\cdot), \ell(\cdot)) = J(\bar{u}(\cdot)) + \alpha \int_\Omega [H(x, \bar{y}(x), \bar{\psi}(x), \nabla \bar{y}(x), \nabla \bar{\psi}(x), \bar{u}(x))
- H(x, \bar{y}(x), \bar{\psi}(x), \nabla \bar{y}(x), \nabla \bar{\psi}(x), u(x))
- \frac{1}{\langle A(x, u(x)) \ell(x), \ell(x) \rangle} \left\langle [A(x, \bar{u}(x)) - A(x, u(x))] \nabla \bar{y}(x), \ell(x) \right\rangle \\
\cdot \left\langle [A(x, \bar{u}(x)) - A(x, u(x))] \nabla \bar{\psi}(x), \ell(x) \right\rangle \, dx + \alpha \rho_x,
\]
and
\[
J^\alpha(u(\cdot), \ell(\cdot)) - J(\bar{u}(\cdot)) \geq 0.
\]

If the above result holds true, then in the case that \(\bar{u}(\cdot)\) is weakly singular at \((u(\cdot), \ell(\cdot))\), the above (3.23) will become the following in which the first order term disappears
\[
J^\alpha(u(\cdot), \ell(\cdot)) = J(\bar{u}(\cdot)) + \alpha \rho_x.
\]

To further characterize the optimal control, the second-order necessary condition will be needed.

To prove Proposition 3.3, it is natural to try a modification of (3.9) as follows:
\[
u^{\alpha, \varepsilon}(x) = \left\{\begin{array}{ll}
u(x), & \text{if } \left\{\frac{x, \ell(x)}{\varepsilon}\right\} \in [0, \alpha), \\
\bar{u}(x), & \text{if } \left\{\frac{x, \ell(x)}{\varepsilon}\right\} \in [\alpha, 1),
\end{array}\right.
\]
and wish that \(y^{\alpha, \varepsilon}(\cdot) = y(\cdot; u^{\alpha, \varepsilon}(\cdot))\) converges to \(y^\alpha(\cdot)\) weakly in \(H^1_0(\Omega)\) (as \(\varepsilon \downarrow 0\)) with \(y^\alpha(\cdot)\) being the weak solution to (3.20). However, (3.25) does not work as we expected. For example, for \(n = 2\), let
\[
\ell(x_1, x_2) = \frac{(-x_2, x_1)}{\sqrt{x_1^2 + x_2^2}}, \quad \forall x \neq 0.
\]

Then (3.25) implies
\[
u^{\alpha, \varepsilon}(x) = u(x), \quad \forall x \in \Omega.
\]
Thus, in this case, \( u^{a, \varepsilon}(-) \) is not a proper perturbation of \( \bar{u}(-) \) that we expected since for any proper metric \( \bar{\rho} \) on \( U \),

\[
\bar{\rho}(u^{a, \varepsilon}(-), \bar{u}(-)) = \bar{\rho}(u(-), \bar{u}(-)),
\]

which will not go to zero as \( a, \varepsilon \downarrow 0 \). Nevertheless, in the next section, we will prove that Proposition 3.3 is true, by a different method.

4. **Proof of Proposition 3.3.** In this section, we will present a proof of Proposition 3.3. Let us begin with the following lemma.

**Lemma 4.1.** Let \( B_1, B_2 \in S^n_+ \) and

\[
\lambda |\xi|^2 \leq \langle B_i \xi, \xi \rangle \leq \Lambda |\xi|^2, \quad \forall i = 1, 2, \quad \xi \in \mathbb{R}^n
\]

for some constants \( \Lambda \geq \lambda > 0 \). Let \( a \in (0, 1), \mu \in \mathbb{R}^n \setminus \{0\} \) and

\[
G = aB_1 + (1-a)B_2 - \alpha(1-\alpha)(B_2 - B_1)C^{-1}(B_2 - B_1)\mu^\top [\alphaB_2 + (1-\alpha)B_1]\mu.
\]

Then

\[
(aB_1^{-1} + (1-a)B_2^{-1})^{-1} \leq G \leq aB_1 + (1-a)B_2.
\] (4.1)

**Proof.** The lemma is a consequence of Theorem 1.3.14 and Lemma 1.3.32 in [2]. Here we give a direct proof of it. Let \( C = \alpha B_2 + (1-a)B_1 = B_1 + \alpha (B_2 - B_1) = B_2 - (1-a)(B_2 - B_1). \)

Then \( C \in S^n_+ \), and

\[
C - B_1 = \alpha (B_2 - B_1), \quad C - B_2 = -(1-a)(B_2 - B_1).
\]

Thus,

\[
\begin{align*}
\alpha B_1 + (1-a)B_2 - \alpha(1-\alpha)(B_2 - B_1)C^{-1}(B_2 - B_1) \\
= aB_1 + (1-a)B_2 + (C - B_1)C^{-1}(C - B_2) \\
= aB_1 + (1-a)B_2 + C - B_1 - B_2 + B_1 C^{-1}B_2 \\
= B_1[\alpha B_2 + (1-a)B_1]^{-1}B_2 = [aB_1^{-1} + (1-a)B_2^{-1}]^{-1}.
\end{align*}
\] (4.2)

Consequently, we get (4.1) since

\[
0 \leq \frac{\mu \mu^\top}{\mu^\top [\alphaB_2 + (1-a)B_1]\mu} = \frac{\mu \mu^\top}{\mu^\top C\mu} = C^{-\frac{1}{2}} \frac{C^\frac{1}{2} \mu}{\mu^\top (\frac{C^\frac{1}{2} \mu}{\mu^\top})} C^{-\frac{1}{2}} \leq C^{-1},
\]

proving our conclusion. \( \square \)

The following lemma will play an interesting role below.

**Lemma 4.2.** Let \( \nu = (\nu_1, \cdots, \nu_n)^\top \in \mathbb{Z}^n \setminus \{0\} \) (where \( \mathbb{Z} \) is the set of all integers). Then for any \( a \in (0, 1) \),

\[
\int_{[0,1]^n} \chi_{[0, \alpha]}(\langle \nu, z \rangle)dz = \alpha.
\]

**Proof.** Recall that \( \{a\} = a - [a] \) is the decimal part of the real number \( a \). Note that

\[
\langle \nu, z \rangle = \sum_{k=1}^{n} \nu_k z_k,
\]

If some of integers \( \nu_k \) are zero, we could drop the corresponding terms and reduce the dimension of \( z \). Thus, we assume all \( \nu_k \) are non-zero. Also, if some \( \nu_k < 0 \), we
may replace corresponding \( z_k \) by \((1 - z_k)\). Therefore, we may let all \( \nu_k > 0 \). Next, we observe the following (noting the \([0,1]^n\)-periodicity of the maps \( z \mapsto \{ \langle \nu, z \rangle \} \)):

\[
\int_{[0,1]^n} \chi_{[0,\alpha)}(\{\langle \nu, z \rangle\}) \, dz = \int_0^1 dz_1 \int_0^{z_1} \cdots \int_0^{z_{n-1}} \chi_{[0,\alpha)}(\{\sum_{k=1}^n \nu_k z_k\}) \, dz_n
\]

\[
= \nu_1 \nu_2 \cdots \nu_n \int_0^1 dz_1 \int_0^{z_1} \cdots \int_0^{z_{n-1}} \chi_{[0,\alpha)}(\{\sum_{k=1}^n \nu_k z_k\}) \, dz_n
\]

\[
= \int_0^1 dz_1 \int_0^{z_1} \cdots \int_0^{z_{n-1}} \chi_{[0,\alpha)}(\{\sum_{k=1}^n z_k\}) \, dz_n.
\]

Hence, we need to prove the following:

\[
\int_0^1 dz_1 \int_0^{z_1} \cdots \int_0^{z_{n-1}} \chi_{[0,\alpha)}(\{\sum_{k=1}^n z_k\}) \, dz_n = \alpha.
\]

Let us use induction. For \( n = 1 \), the above is clearly true. Suppose the above holds for \( n - 1 \). Then, for the \( n \)-dimensional case, we observe the following: For \( z_1 \in [0,1) \),

\[
\left\{ \sum_{k=1}^n z_k \right\} = \left\{ z_1 + \sum_{k=2}^n z_k \right\} = \begin{cases} z_1 + \sum_{k=2}^n z_k, & \text{if } 0 \leq z_1 + \sum_{k=2}^n z_k < 1, \\ z_1 + \sum_{k=2}^n z_k - 1, & \text{if } 1 \leq z_1 + \sum_{k=2}^n z_k < 2. \end{cases}
\]

Then for \( z_1 \in [0, \alpha) \), the following holds

\[
\chi_{[0,\alpha)}(\left\{ z_1 + \sum_{k=2}^n z_k \right\}) = 1,
\]

if and only if either

\[
0 \leq z_1 + \sum_{k=2}^n z_k < \alpha \quad \text{and} \quad 0 \leq z_1 + \sum_{k=2}^n z_k < 1.
\]

or

\[
0 \leq z_1 + \sum_{k=2}^n z_k - 1 < \alpha \quad \text{and} \quad 1 \leq z_1 + \sum_{k=2}^n z_k < 2.
\]

That is, either

\[
0 \leq \sum_{k=2}^n z_k < \alpha - z_1,
\]

or

\[
1 - z_1 \leq \sum_{k=2}^n z_k \leq (\alpha + 1 - z_1) \land 1 = 1.
\]

Note that the above two cases are mutually exclusive (since \( \alpha - z_1 < 1 - z_1 \)). On the other hand, for \( z_1 \in [\alpha, 1) \), if \( 0 \leq z_1 + \sum_{k=2}^n z_k < 1 \), then

\[
\left\{ z_1 + \sum_{k=2}^n z_k \right\} = z_1 + \sum_{k=2}^n z_k \geq \alpha \quad \Rightarrow \quad \chi_{[0,\alpha)}(z_1 + \sum_{k=2}^n z_k) = 0.
\]
Thus the following holds:

\[ \chi_{[0, \alpha)} \left( \left\{ z_1 + \sum_{k=2}^{n} z_k \right\} \right) = 1, \]

if and only if

\[ 0 \leq z_1 + \left\{ \sum_{k=2}^{n} z_k \right\} - 1 < \alpha, \quad 1 \leq z_1 + \left\{ \sum_{k=2}^{n} z_k \right\} < 2. \]

That is,

\[ 0 \leq 1 - z_1 \leq \left\{ \sum_{k=2}^{n} z_k \right\} \leq \alpha + 1 - z_1 \leq 1. \]

Hence, by induction hypothesis,

\[ \int_{[0,1]^n} \chi_{[0, \alpha)} \left( \left\{ z_1 + \sum_{k=2}^{n} z_k \right\} \right) \, dz \]

\[ = \int_{0}^{\alpha} dz_1 \int_{[0,1]^{n-1}} \chi_{[0, \alpha-z_1)} \left( \left\{ \sum_{k=2}^{n} z_k \right\} \right) dz_2 \cdots dz_n \]

\[ + \int_{0}^{\alpha} dz_1 \int_{[0,1]^{n-1}} \chi_{[1-z_1, 1)} \left( \left\{ \sum_{k=2}^{n} z_k \right\} \right) dz_2 \cdots dz_n \]

\[ + \int_{\alpha}^{1} dz_2 \int_{[0,1]^{n-1}} \chi_{[1-z_1, \alpha+1-z_1)} \left( \left\{ \sum_{k=2}^{n} z_k \right\} \right) dz_3 \cdots dz_n \]

\[ = \int_{0}^{\alpha} (\alpha - z_1) \, dz_1 + \int_{0}^{\alpha} z_1 \, dz_1 + \int_{\alpha}^{1} \alpha \, dz_1 = \alpha. \]

This completes the proof. \( \square \)

The following gives a crucial convergence of the weak solution to the state equation under a suitable perturbation of the leading coefficient.

**Lemma 4.3.** Let (S1) hold. Let \( B(\cdot), C(\cdot) \in L^\infty(\Omega; \mathbb{S}_+^n) \) such that

\[ \lambda |\xi|^2 \leq \langle B(x) \xi, \xi \rangle \leq \Lambda |\xi|^2, \]

\[ \lambda |\xi|^2 \leq \langle C(x) \xi, \xi \rangle \leq \Lambda |\xi|^2, \quad \forall (\xi, x) \in \mathbb{R}^n \times \Omega, \quad (4.3) \]

for some \( \Lambda \geq \lambda > 0 \). Let \( \alpha \in (0, 1), \mu_k = \left( \frac{r_{k1}}{p_{k1}}, \frac{r_{k2}}{p_{k2}}, \ldots, \frac{r_{kn}}{p_{kn}} \right) \in \mathbb{Q}^n \setminus \{0\}, \mathbb{Q} \) is the set of all rational numbers, with all \( r_{kj} \) being integers, and \( p_{kj} \) being positive integers, \( 1 \leq k \leq m, \, 1 \leq j \leq n \). Let \( E_1, E_2, \ldots, E_m \) be mutually disjoint measurable sets such that \( \bigcup_{k=1}^{m} E_k = \Omega \). Let \( h(\cdot) \in L^2(\Omega) \), and

\[ G(x, z) = \begin{cases} B(x), & \text{if } \{z, \mu_k\} \in [0, \alpha), \\ C(x), & \text{if } \{z, \mu_k\} \in [\alpha, 1), \end{cases} \quad x \in E_k, \, 1 \leq k \leq m. \quad (4.4) \]

For \( \varepsilon > 0 \), let \( y^\varepsilon(\cdot) \in H^1_0(\Omega) \) be the weak solution of the following:

\[ \begin{cases} -\nabla \cdot \left( G(x, \frac{y^\varepsilon}{\varepsilon}) \nabla y^\varepsilon(x) \right) = h(x), & \text{in } \Omega, \\ y^\varepsilon |_{\partial \Omega} = 0. \end{cases} \quad (4.5) \]

Then, as \( \varepsilon \downarrow 0 \),

\[ y^\varepsilon(\cdot) \to y(\cdot), \quad \text{weakly in } H^1_0(\Omega), \quad (4.6) \]
Thus, (4.6)–(4.7) with
\[
\hat{G}(\cdot) \in L^\infty(\Omega; \mathbb{S}_+^n) \quad \text{given by}
\]
\[
\hat{G}(x) = \alpha B(x) + (1 - \alpha) C(x) - \frac{\alpha(1 - \alpha)[B(x) - C(x)] \ell(x)\ell(x)^\top [B(x) - C(x)]}{(1 - \alpha)\ell(x)^\top B(x)\ell(x) + \alpha\ell(x)^\top C(x)\ell(x)}, \quad x \in \Omega,
\]
and
\[
\ell(x) = \sum_{k=1}^m \mu_k \chi_{E_k}(x), \quad x \in \Omega.
\]

**Proof.** The proof is essentially inspired by that of Lemma 1.3.32 of [2]. Let \( P \) be a common multiple of \( p_{kj}, k = 1, 2, \cdots, m, j = 1, 2, \cdots, n \). Then one can verify that 
\( G(x, Pz) \) is \([0, 1]^n\)–periodic in \( z \), for any \( \varepsilon > 0 \), \( G\left(x, \frac{Pz}{\varepsilon}\right) \) is measurable, and
\[
\lambda|\xi|^2 \leq \langle G(x, Pz)\xi, \xi \rangle \leq \Lambda|\xi|^2, \quad \forall (x, z) \in \Omega \times \mathbb{R}^n, \quad \xi \in \mathbb{R}^n.
\]
Moreover, using Riemann-Lebesgue’s Theorem (see Ch. II, Theorem 4.15 in [33]),
\[
\lim_{\varepsilon \downarrow 0} \int_{\Omega} |G\left(x, \frac{Pz}{\varepsilon}\right)|^2 \, dx = \lim_{\varepsilon \downarrow 0} \sum_{k=1}^m \int_{E_k} |G\left(x, \frac{Pz}{\varepsilon}\right)|^2 \, dx
\]
\[
= \lim_{\varepsilon \downarrow 0} \sum_{k=1}^m \int_{E_k} |B(x)\chi_{[0,\alpha)}(\{\frac{Pz}{\varepsilon}, \mu_k\})| + C(x)\chi_{[0,1)}(\{\frac{Pz}{\varepsilon}, \mu_k\})|^2 \, dx
\]
\[
= \lim_{\varepsilon \downarrow 0} \sum_{k=1}^m \int_{E_k} [|B(x)|^2 \chi_{[0,\alpha)}(\{\frac{Pz}{\varepsilon}, \mu_k\}) + |C(x)|^2 \chi_{[0,1)}(\{\frac{Pz}{\varepsilon}, \mu_k\})] \, dx
\]
\[
= \sum_{k=1}^m \left[ \int_{E_k} |B(x)|^2 \, dx \int_{[0,1]^n} \chi_{[0,\alpha)}(\{Pz, \mu_k\}) \, dz + \int_{E_k} |C(x)|^2 \, dx \int_{[0,1]^n} \chi_{[0,1)}(\{Pz, \mu_k\}) \, dz \right]
\]
\[
= \sum_{k=1}^m \left[ \int_{[0,1]^n} |B(x)| \chi_{[0,\alpha)}(\{Pz, \mu_k\}) + C(x) \chi_{[0,1)}(\{Pz, \mu_k\}) \right]^2 \, dx
\]
\[
= \int_{\Omega \times [0,1]^n} |G(x, Pz)|^2 \, dz \, dx.
\]
Thus, \((x, z) \mapsto G(x, Pz)\) satisfies conditions of Lemma 3.1. Using Lemma 3.1, we get that (4.6)–(4.7) with
\[
\hat{G}(x) = \int_{[0,1]^n} G(x, Pz)
\]
\[
I + \nabla_z \phi(x, z)^\top \right] dz,
\]
with \( \phi \in W^{1,2}_{\#}([0, 1]^n; \mathbb{R}^n) / \mathbb{R}^n \) being the unique solution of
\[
- \nabla_z^\top \left[ G(x, Pz) \left( I + \nabla_z \phi(x, z)^\top \right) \right] = 0.
\]
That is, for $x \in E_k$,
\[
\nabla_z^\top \left[ \left( \chi_{[0,\alpha)} \left( \{ \langle Pz, \mu_k \rangle \} \right) B(x) + \chi_{[\alpha,1)} \left( \{ \langle Pz, \mu_k \rangle \} \right) C(x) \right) \left( I + \nabla_z \phi(x, z)^\top \right) \right] = 0. \tag{4.11}
\]

To solve (4.11), we fix $k$, denote $\bar{\mu} = \frac{\mu_k}{|\mu_k|}$ and define
\[
\varphi(x, z) = P|\mu_k| \phi \left( x, \frac{z}{P|\mu_k|} \right), \quad (x, z) \in \mathbb{R}^n \times \mathbb{R}^n.
\]
Then
\[
\nabla_z \varphi(x, z)^\top = \nabla_z \phi \left( x, \frac{z}{P|\mu_k|} \right)^\top,
\]
which leads to
\[
\nabla_z \phi(s, z)^\top = \nabla_z \varphi(x, P|\mu_k|z)^\top.
\]
Using $\varphi(\cdot)$, equation (4.11) can be written as
\[
\nabla_z^\top \left[ \left( \chi_{[0,\alpha)} \left( \{ \langle z, \bar{\mu} \rangle \} \right) B(x) + \chi_{[\alpha,1)} \left( \{ \langle z, \bar{\mu} \rangle \} \right) C(x) \right) \left( I + \nabla_z \varphi(x, z)^\top \right) \right] = 0,
\]
which is equivalent to the following:
\[
\nabla_z^\top \left[ \left( \chi_{[0,\alpha)} \left( \{ \langle z, \bar{\mu} \rangle \} \right) B(x) + \chi_{[\alpha,1)} \left( \{ \langle z, \bar{\mu} \rangle \} \right) C(x) \right) \left( I + \nabla_z \varphi(x, z)^\top \right) \right] = 0.
\]
Since $\bar{\mu}$ is a unit vector, there exists an orthogonal matrix $Q$ such that
\[
\bar{\mu} = Q^T e_1, \quad \text{with} \quad e_1 = (1, 0, \ldots, 0)^\top \in \mathbb{R}^n.
\]
Let
\[
\tilde{z} = (\tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_n)^\top = Qz, \quad \bar{\varphi}(x, \tilde{z}) = \varphi(x, Q^\top \tilde{z}) \equiv \varphi(x, z).
\]
Then
\[
\tilde{z}_1 = \langle \tilde{z}, e_1 \rangle = \langle Qz, Q\bar{\mu} \rangle = \langle z, \bar{\mu} \rangle,
\]
and
\[
\nabla_{\tilde{z}} \varphi(x, z)^\top = Q^T \nabla_z \bar{\varphi}(x, Q^\top \tilde{z})^\top \equiv Q^T \nabla_z \bar{\varphi}(x, \tilde{z})^\top.
\]
Thus,
\[
\nabla_{\tilde{z}}^\top \left[ Q \left( \chi_{[0,\alpha)} \left( \{ \langle \tilde{z}_1 \rangle \} \right) B(x) + \chi_{[\alpha,1)} \left( \{ \langle \tilde{z}_1 \rangle \} \right) C(x) \right) \left( I + Q^T \nabla_{\tilde{z}} \bar{\varphi}(x, \tilde{z})^\top \right) \right] = 0. \tag{4.12}
\]
Since $Q \left( \chi_{[0,\alpha)} \left( \{ \langle \tilde{z}_1 \rangle \} \right) B(x) + \chi_{[\alpha,1)} \left( \{ \langle \tilde{z}_1 \rangle \} \right) C(x) \right)$, the coefficient of the above equation, is independent of $\tilde{z}_2, \tilde{z}_3, \ldots, \tilde{z}_n$, by the uniqueness, the solution $\bar{\varphi}(x, \cdot)$ of equation (4.12) must be independent of $\tilde{z}_2, \tilde{z}_3, \ldots, \tilde{z}_n$. Thus (4.12) further implies
\[
\frac{\partial}{\partial \tilde{z}_1} Q \left( \chi_{[0,\alpha)} \left( \{ \langle \tilde{z}_1 \rangle \} \right) B(x) + \chi_{[\alpha,1)} \left( \{ \langle \tilde{z}_1 \rangle \} \right) C(x) \right) \left( I + \tilde{\mu} \frac{\partial \bar{\varphi}}{\partial \tilde{z}_1}(x, \tilde{z}_1 e_1)^\top \right) = 0.
\]
Hence, there exists a constant vector $X \in \mathbb{R}^n$ such that (note $Q^T e_1 = \bar{\mu}$)
\[
X^\top = e_1^T Q \left( \chi_{[0,\alpha)} \left( \{ \langle \tilde{z}_1 \rangle \} \right) B(x) + \chi_{[\alpha,1)} \left( \{ \langle \tilde{z}_1 \rangle \} \right) C(x) \right) \left[ I + \tilde{\mu} \left( \frac{\partial \bar{\varphi}}{\partial \tilde{z}_1}(x, \tilde{z}_1 e_1)^\top \right) \right] = \bar{\mu}^T \left( \chi_{[0,\alpha)} \left( \{ \langle \tilde{z}_1 \rangle \} \right) B(x) + \chi_{[\alpha,1)} \left( \{ \langle \tilde{z}_1 \rangle \} \right) C(x) \right) + \tilde{\mu}^T \left( \chi_{[0,\alpha)} \left( \{ \langle \tilde{z}_1 \rangle \} \right) B(x) + \chi_{[\alpha,1)} \left( \{ \langle \tilde{z}_1 \rangle \} \right) C(x) \right) \tilde{\mu} \left( \frac{\partial \bar{\varphi}}{\partial \tilde{z}_1}(x, \tilde{z}_1 e_1)^\top \right).
Consequently,
\[
\frac{\partial \tilde{\varphi}}{\partial \tilde{z}_1}(x, \tilde{z}_1 e_1) = X - \left(\chi_{[0,\alpha)}(\{\tilde{z}_1\})B(x) + \chi_{[\alpha,1)}(\{\tilde{z}_1\})C(x)\right)\hat{\mu} = X - B(x)\hat{\mu} + X - C(x)\hat{\mu} = X - B(x)\hat{\mu} + X - C(x)\hat{\mu}.
\]

Since \(\phi(x, \cdot) \in W^{1,2}_{\#}(\mathbb{R}^n; \mathbb{R}^n)\), for \(\tilde{z} = (P^T[k^2, \tilde{z}_2, \ldots, \tilde{z}_n]^{\top}\), we have
\[
\tilde{\varphi}(x, \tilde{z}) = \tilde{\varphi}(x, P^T[k^2 e_1]) = \varphi(x, P^T[k^2 Q] e_1) = \varphi(x, P^T[k^2 \hat{\mu}]) = P[k, \Phi(x, P_k)] = \varphi(x, 0) = \tilde{\varphi}(x, 0).
\]

Hence,
\[
0 = \frac{\tilde{\varphi}(x, P^T[k^2 e_1]) - \tilde{\varphi}(x, 0)}{P^T[k^2]} = \frac{1}{P^T[k^2]} \int_0^{P^T[k^2]} \left[X - B(x)\hat{\mu} \chi_{[0,\alpha)}(\{\tilde{z}_1\}) + X - C(x)\hat{\mu} \chi_{[\alpha,1)}(\{\tilde{z}_1\})\right] d\tilde{z}_1 = \left(\frac{\alpha}{\hat{\mu}} B(x) + 1 - \alpha \right) \frac{X - B(x)\hat{\mu}}{\hat{\mu}} + \alpha \frac{X - C(x)\hat{\mu}}{\hat{\mu}}.
\]

This yields
\[
X = \frac{\alpha \hat{\mu}^T C(x)\hat{\mu} B(x)\hat{\mu} + (1 - \alpha) \hat{\mu}^T B(x)\hat{\mu} C(x)\hat{\mu} \hat{\mu}}{\hat{\mu}^T [\alpha C(x) + (1 - \alpha) B(x)] \hat{\mu}},
\]
and for any \(\tilde{z} \in \mathbb{R}^n\) with \(\tilde{z}_1 \in [0, \alpha)\),
\[
\tilde{\varphi}(x, \tilde{z}) = \tilde{\varphi}(x, \tilde{z}_1 e_1) = \tilde{\varphi}(x, 0) + \int_0^{\tilde{z}_1} \frac{\partial \tilde{\varphi}}{\partial \tilde{z}_1}(x, \tau e_1) d\tau = \tilde{\varphi}(x, 0) + \int_0^{\tilde{z}_1} \frac{X - B(x)\hat{\mu}}{\hat{\mu}} d\tau = \tilde{\varphi}(x, 0) + \tilde{z}_1 \frac{X - B(x)\hat{\mu}}{\hat{\mu}}.
\]

For \(\tilde{z}_1 \in [\alpha, 1)\), we have
\[
\tilde{\varphi}(x, \tilde{z}) = \tilde{\varphi}(x, \alpha e_1) = \tilde{\varphi}(x, 0) + \alpha \frac{X - B(x)\hat{\mu}}{\hat{\mu}} + \int_\alpha^{\tilde{z}_1} \frac{\partial \tilde{\varphi}}{\partial \tilde{z}_1}(x, \tau e_1) d\tau = \tilde{\varphi}(x, 0) + \alpha \frac{X - B(x)\hat{\mu}}{\hat{\mu}} + \int_\alpha^{\tilde{z}_1} \frac{X - C(x)\hat{\mu}}{\hat{\mu}} d\tau = \tilde{\varphi}(x, 0) + \alpha \frac{X - B(x)\hat{\mu}}{\hat{\mu}} + (\tilde{z}_1 - \alpha) \frac{X - C(x)\hat{\mu}}{\hat{\mu}}.
\]

As a result,
\[
\tilde{\varphi}(x, e_1) - \tilde{\varphi}(x, 0) = \frac{\alpha X - B(x)\hat{\mu}}{\hat{\mu}} + (1 - \alpha) X - C(x)\hat{\mu} \bigg(\frac{\alpha B(x)\hat{\mu}}{\hat{\mu}} + (1 - \alpha) C(x)\hat{\mu}\bigg) = 0.
\]
Thus, $\tilde{z}_1 \mapsto \tilde{\varphi}(x, \tilde{z}_1 e_1)$ is 1-periodic. On the other hand, for any $z \in [0, 1]^n$,

$$
\phi(x, z) = \frac{1}{P[\mu_k]} \varphi(x, P[\mu_k]|z) = \frac{1}{P[\mu_k]} \tilde{\varphi}(x, P[\mu_k]|Qz)
$$

$$
= \frac{1}{P[\mu_k]} \tilde{\varphi}(x, P[\mu_k]|(e_1^T Qz)e_1) = \frac{1}{P[\mu_k]} \tilde{\varphi}(x, P[\mu_k]|(\mu^T z)e_1)
$$

$$
= \frac{1}{P[\mu_k]} \tilde{\varphi}(x, P[\mu_k]|(e_1 \mu^T z)e_1).
$$

Hence,

$$
\nabla_z \phi(x, z)^T = \tilde{\mu} e_1^T \nabla_z \tilde{\varphi}(x, (\mu^T z)P[\mu_k]|e_1) \tilde{\mu} = \tilde{\mu} \frac{\partial \tilde{\varphi}}{\partial \tilde{z}_1}(x, P(\mu^T z)e_1)^T
$$

$$
= \tilde{\mu} \frac{X^T - \tilde{\mu}^T \chi[0, \alpha](\{P[\mu_k], z\})B(x) + \chi[1, (\{P[\mu_k], z\})C(x)]}{\tilde{\mu}}
$$

$$
= \tilde{\mu}[X^T - \tilde{\mu}^T B(x)] \frac{\nabla_z \chi[0, \alpha](\{P[\mu_k], z\})}{\nabla_z \chi[1, \alpha](\{P[\mu_k], z\})}.
$$

Therefore, making use of Lemma 4.2, one has

$$
\tilde{G}(x) = \int_{[0, 1]^n} G(x, Pz) \left[I + \nabla_z \phi(x, z)^T\right] dz
$$

$$
= \int_{[0, 1]^n} \left[B(x)\chi[0, \alpha](\{P[\mu_k], z\}) + C(x)\chi[1, \alpha](\{P[\mu_k], z\})\right]
$$

$$
+ \left[I + \frac{\tilde{\mu}[X^T - \tilde{\mu}^T B(x)]}{\tilde{\mu}}\chi[0, \alpha](\{P[\mu_k], z\})\right]
$$

$$
+ \left[I + \frac{\tilde{\mu}[X^T - \tilde{\mu}^T B(x)]}{\tilde{\mu}}\chi[0, \alpha](\{P[\mu_k], z\})\right]
$$

$$
+ \frac{\tilde{\mu}[X^T - \tilde{\mu}^T C(x)]}{\tilde{\mu}} \chi[1, \alpha](\{P[\mu_k], z\})
$$

$$
= \alpha B(x) + (1 - \alpha) C(x) + \alpha \frac{B(x)[X^T - \tilde{\mu}^T C(x)]}{\tilde{\mu}^T B(x)}
$$

$$
+ (1 - \alpha) \frac{C(x)[X^T - \tilde{\mu}^T C(x)]}{\tilde{\mu}^T C(x)}.
$$

Now, we simplify the expression of $\tilde{G}(x)$, suppressing $x$,

$$
B\tilde{\mu}[X^T - \tilde{\mu}^T B] = B\tilde{\mu}[\tilde{\mu}^T C\tilde{\mu}]B + (1 - \alpha) B\tilde{\mu}[X^T - \tilde{\mu}^T B]
$$

$$
= \frac{(1 - \alpha)[\tilde{\mu}^T B\tilde{\mu}][\tilde{\mu}^T C - B]}{\tilde{\mu}^T [\alpha C + (1 - \alpha) B]}.
$$
Likewise,
\[
C\hat{\mu}[X - \hat{\mu}^\top C] = C\hat{\mu}\frac{\alpha[\hat{\mu}^\top C]\hat{\mu}^\top B + (1 - \alpha)[\hat{\mu}^\top B]\hat{\mu}^\top C}{\hat{\mu}^\top [\alpha C + (1 - \alpha)B] \hat{\mu}} - C\hat{\mu}\hat{\mu}^\top C
\]
\[= \frac{\alpha[\hat{\mu}^\top C]\hat{\mu}^\top (B - C)}{\hat{\mu}^\top [\alpha C + (1 - \alpha)B] \hat{\mu}}.
\]
Hence,
\[
\hat{G} = \alpha B + (1 - \alpha)C + \alpha \frac{(1 - \alpha)B\hat{\mu}^\top (C - B)}{\hat{\mu}^\top [\alpha C + (1 - \alpha)B] \hat{\mu}} + (1 - \alpha) \frac{\alpha C\hat{\mu}^\top (B - C)}{\hat{\mu}^\top [\alpha C + (1 - \alpha)B] \hat{\mu}}.
\]
This means that for any \(x \in E_k\),
\[
\hat{G}(x) = \alpha B(x) + (1 - \alpha)C(x) - \alpha (1 - \alpha) \left( C(x) - B(x) \right) \frac{1}{\hat{\mu}^\top [\alpha C(x) + (1 - \alpha)B(x)] \hat{\mu}} \mu_k(x) + (1 - \alpha) \frac{\alpha C\hat{\mu}^\top (B - C)}{\hat{\mu}^\top [\alpha C + (1 - \alpha)B] \hat{\mu}}.
\]
The proof is completed. 

By Lemma 4.1, we know that the function \(G : \Omega \rightarrow S^n_+\) appears in the above lemma satisfies the following:
\[
\lambda |\xi|^2 \leq \langle G(x)\xi, \xi \rangle \leq \Lambda |\xi|^2, \quad \forall x \in \Omega, \; \xi \in S^n_+.
\]

Lemma 4.4. Let (S1)-(S5) hold and \((\bar{u}(\cdot), \bar{u}(\cdot))\) be an optimal pair of Problem (C). Let \(u(\cdot) \in U, \; \mu_1, \mu_2, \ldots, \mu_m \in \mathbb{R}^n \setminus \{0\}\) and \(E_1, E_2, \ldots, E_m\) be mutually disjoint measurable sets such that \(\bigcup_{i=1}^m E_i = \Omega\). Define
\[
\ell(\cdot) = \sum_{k=1}^m \mu_k \chi_{E_k}(\cdot).
\]
Let \(\alpha \in (0, 1)\) and \(y^\alpha(\cdot; \ell(\cdot))\) be the weak solution of
\[
\begin{cases}
- \nabla \cdot (\hat{G}(x)\nabla y^\alpha(x; \ell(\cdot))) = \alpha f(x, y^\alpha(x; \ell(\cdot)), u(x)) + (1 - \alpha) f(x, y^\alpha(x; \ell(\cdot)), u(x)), & \text{in } \Omega, \\
y^\alpha(x; \ell(\cdot)) = 0, & \text{on } \partial \Omega,
\end{cases}
\]
with \(\hat{G}(\cdot) \in L^\infty(\Omega; S^n_+)\) being given by
\[
\hat{G}(x) = \alpha A(x, u(x)) + (1 - \alpha)A(x, \bar{u}(x)) - \alpha (1 - \alpha) \left[ A(x, u(x)) - A(x, \bar{u}(x)) \right] \ell(x) \ell(x)^\top \left[ A(x, u(x)) - A(x, \bar{u}(x)) \right],
\]
for all \(x \in \Omega\).
Then
\[
J^\alpha(u(\cdot), \ell(\cdot)) \equiv \int_{\Omega} \left( \alpha f^0(x, y^\alpha(x; \ell(\cdot)), u(x)) + (1 - \alpha) f^0(x, y^\alpha(x; \ell(\cdot)), u(x)) \right) dx \geq J(\bar{u}(\cdot)).
\]
Proof. We split the proof into two steps.

**Step I.** First, let all the components of \( \mu_1, \mu_2, \ldots, \mu_m \in \mathbb{Q}^n \setminus \{0\} \). Let \( P \) be a positive integer such that all the components of \( P\mu_1, P\mu_2, \ldots, P\mu_m \) are integers.

For \( \varepsilon > 0 \), define

\[
u^{\alpha,\varepsilon}(x) = \begin{cases} u(x), & \text{if } \left\{ \frac{x}{\varepsilon}, \mu_k \right\} \in [0,\alpha), \ x \in E_k, \\ \bar{u}(x), & \text{if } \left\{ \frac{x}{\varepsilon}, \mu_k \right\} \in [\alpha,1), \ x \in E_k. \end{cases}
\] (4.16)

Let \( y^{\alpha,\varepsilon}(\cdot) = y(\cdot; \nu^{\alpha,\varepsilon}(\cdot)) \) be the solution to the state equation (1.1) corresponding to the control \( u^{\alpha,\varepsilon}(\cdot) \). It is standard that \( y^{\alpha,\varepsilon}(\cdot) \) is uniformly bounded in \( H_0^1(\Omega) \) and \( L^\infty(\Omega) \). Thus, along a subsequence \( \varepsilon \downarrow 0 \), \( y^{\alpha,\varepsilon}(\cdot) \) converges to some \( z^\alpha(\cdot) \) weakly in \( H_0^1(\Omega) \) and strongly in \( L^2(\Omega) \). Let \( \bar{y}^{\alpha,\varepsilon}(\cdot) \in H_0^1(\Omega) \) be the weak solution of

\[
\begin{cases} -\nabla \cdot (A(x, u^{\alpha,\varepsilon}(x))\nabla \bar{y}^{\alpha,\varepsilon}(x)) = \alpha f(x, z^{\alpha}(x), u(x)) + (1-\alpha)f(x, z^{\alpha}(x), u(x)), & \text{in } \Omega, \\ \bar{y}^{\alpha,\varepsilon}|_{\partial \Omega} = 0, \end{cases}
\] (4.17)

Then \( \bar{y}^{\alpha,\varepsilon}(\cdot) \equiv y^{\alpha,\varepsilon}(\cdot) - \bar{y}^{\alpha,\varepsilon}(\cdot) \) satisfies

\[
\begin{cases} -\nabla \cdot (A(x, u^{\alpha,\varepsilon}(x))\nabla \bar{y}^{\alpha,\varepsilon}(x)) = f(x, y^{\alpha,\varepsilon}(x), u^{\alpha,\varepsilon}(x)) - \alpha f(x, z^{\alpha}(x), u(x)) - (1-\alpha)f(x, z^{\alpha}(x), u(x)), & \text{in } \Omega, \\ \bar{y}^{\alpha,\varepsilon}|_{\partial \Omega} = 0, \end{cases}
\] (4.18)

By (S3) and the boundedness of \( y^{\alpha,\varepsilon}(\cdot) \) in \( L^\infty(\Omega) \), we can see that

\[
f(\cdot, y^{\alpha,\varepsilon}(\cdot), u^{\alpha,\varepsilon}(\cdot)) - f(\cdot, z^{\alpha}(\cdot), u^{\alpha,\varepsilon}(\cdot))
\]

converges strongly in \( L^2(\Omega) \). In addition, by Riemann-Lebesgue’s Theorem, we see that

\[
f(\cdot, y^{\alpha,\varepsilon}(\cdot; \ell(\cdot)), u^{\alpha,\varepsilon}(\cdot)) \to \alpha f(\cdot, z^{\alpha}(\cdot), u(\cdot)) + (1-\alpha)f(\cdot, z^{\alpha}(\cdot), u(\cdot)),
\]

weakly in \( L^2(\Omega) \) as \( \varepsilon \to 0 \).

Then, (S2) and (4.18) imply that \( \bar{y}^{\alpha,\varepsilon}(\cdot) \) converges weakly to zero in \( H_0^1(\Omega) \) (as \( \varepsilon \downarrow 0 \)). Consequently, along a subsequence \( \varepsilon \downarrow 0 \), \( y^{\alpha,\varepsilon}(\cdot) \) converges to \( z^{\alpha}(\cdot) \) weakly in \( H_0^1(\Omega) \). Combining this with Lemma 4.3, we get \( z^{\alpha}(\cdot) = y^{\alpha,\varepsilon}(\cdot; \ell(\cdot)) \), where \( y^{\alpha,\varepsilon}(\cdot; \ell(\cdot)) \) is the weak solution of (4.13). This implies \( y^{\alpha,\varepsilon}(\cdot) \) itself converges to \( y^{\alpha,\varepsilon}(\cdot; \ell(\cdot)) \) weakly in \( H_0^1(\Omega) \), strongly in \( L^2(\Omega) \) (as \( \varepsilon \downarrow 0 \)). Hence,

\[
J(\bar{u}(\cdot)) \leq \lim_{\varepsilon \downarrow 0} \int_{\Omega} f^0(x, y^{\alpha,\varepsilon}(x), u^{\alpha,\varepsilon}(x)) \, dx
= \lim_{\varepsilon \downarrow 0} \int_{\Omega} f^0(x, y^\alpha(x; \ell(\cdot)), u^{\alpha,\varepsilon}(x)) \, dx
= \int_{\Omega} \left( \alpha f^0(x, y^\alpha(x; \ell(\cdot)), u(x)) + (1-\alpha)f^0(x, y^\alpha(x; \ell(\cdot)), \bar{u}(x)) \right) \, dx
= J^\alpha(u(\cdot), \ell(\cdot)),
\] (4.19)

proving (4.15) for the case of \( \ell(\cdot) \) valued in rational numbers.

**Step II.** Now, let \( \mu_1, \cdots, \mu_m \in \mathbb{R}^n \setminus \{0\} \). We can select \( \mu_1^\lambda, \mu_2^\lambda, \ldots, \mu_m^\lambda \in \mathbb{Q}^n \setminus \{0\} \) such that \( \mu_k^\lambda \to \mu_k \) as \( \lambda \to +\infty \). By Step I, one has

\[
\int_{\Omega} \left( \alpha f^0(x, y^{\alpha,\varepsilon}(x), u(\cdot)) + (1-\alpha)f^0(x, y^{\alpha,\varepsilon}(x), u(\cdot)) \right) \, dx \geq J(\bar{u}(\cdot)),
\] (4.20)
where
\[
\begin{aligned}
- \nabla \cdot (G_\lambda(x) \nabla y_\lambda(x)) &= \alpha f(x, y_\lambda(x), u(x)) \\
+ (1 - \alpha) f(x, y_\lambda(x), u(x)), & \quad \text{in } \Omega, \\
y_\lambda|_{\partial \Omega} &= 0
\end{aligned}
\]  
\tag{4.21}

with \( G_\lambda(\cdot) \in L^\infty(\Omega; S_+^n) \) given by
\[
G_\lambda(x) = \alpha A(x, u(x)) + (1 - \alpha) A(x, \bar{u}(x)) - \frac{\alpha(1 - \alpha)}{\|\mu_\lambda\|}\left[A(x, u(x)) - A(x, \bar{u}(x))\right] + \frac{(1 - \alpha)\mu_\lambda^\top A(x, u(x))\mu_\lambda + \alpha\mu_\lambda^\top A(x, \bar{u}(x))\mu_\lambda}{\|\mu_\lambda\|^2}, \\
\forall x \in E_\lambda.
\]
\tag{4.22}

By Lemma 4.1, \( G_\lambda(x) \in S_+^n \) for almost all \( x \in \Omega \). Clearly, \( G_\lambda(\cdot) \) converges to \( G(\cdot) \) strongly in \( L^\infty(\Omega) \). Then by Lemma 3.2 and a standard argument, we have the convergence of \( y_\lambda(\cdot) \) to \( y^\alpha(\cdot) \) strongly in \( H_0^1(\Omega) \). Then (4.15) follows.

We now turn to a proof of Proposition 3.3.

**Proof of Proposition 3.3.** By Luzin’s Theorem, for any integer \( \lambda \geq 1 \), there exists a closed subset \( F_\lambda \) of \( \Omega \), such that \( \ell(\cdot) \) is continuous on \( F_\lambda \) and \( |\bar{\Omega} \setminus F_\lambda| \leq \frac{1}{\lambda} \), where \( |S| \) stands for the Lebesgue measure of the set \( S \). Since \( F_\lambda \) is also bounded, \( \ell(\cdot) \) is uniformly continuous on \( F_\lambda \). Thus, there exist disjoint measurable sets \( E_{\lambda j} \), \( E_{\lambda 2} \), \ldots, \( E_{\lambda m_\lambda} \) such that
\[
\bigcup_{k=1}^{m_\lambda} E_{\lambda k} = F_\lambda, \quad \sup_{x, \bar{x} \in E_{\lambda k}} |\ell(x) - \ell(\bar{x})| \leq \frac{1}{\lambda}, \quad \forall j = 1, 2, \ldots, m_\lambda.
\]
Choosing arbitrary \( x^{\lambda k} \) from \( E_{\lambda k} \) and \( x^{\lambda 0} \) from \( E_{\lambda 0} = \bar{\Omega} \setminus F_\lambda \), we define
\[
\mu_{\lambda k} = \ell(x^{\lambda k}), \quad k = 0, 1, 2, \ldots, m_\lambda.
\]

By Lemma 4.3,
\[
\int_\Omega \left( \alpha f^0(x, y_\lambda^0(x), u(x)) + (1 - \alpha) f^0(x, y_\lambda^0(x), u(x)) \right) dx \geq J(\bar{u}(\cdot)),
\]
\tag{4.23}

where \( y_\lambda^0(\cdot) \) is the weak solution of the following:
\[
\begin{aligned}
- \nabla \cdot (G_\lambda^0(x) \nabla y_\lambda^0(x)) &= \alpha f(x, y_\lambda^0(x), u(x)) \\
(1 - \alpha) f(x, y_\lambda^0(x), u(x)), & \quad \text{in } \Omega, \\
y_\lambda^0|_{\partial \Omega} &= 0
\end{aligned}
\]  
\tag{4.24}

with \( G_\lambda^0(\cdot) \in L^\infty(\Omega; S_+^n) \) being given by
\[
G_\lambda^0(x) = \alpha A(x, u(x)) + (1 - \alpha) A(x, \bar{u}(x)) - \frac{\alpha(1 - \alpha)}{\|\mu_\lambda^0\|}\left[A(x, u(x)) - A(x, \bar{u}(x))\right] + \frac{(1 - \alpha)\mu_\lambda^0 A(x, u(x))\mu_\lambda + \alpha\mu_\lambda^0 A(x, \bar{u}(x))\mu_\lambda}{\|\mu_\lambda^0\|^2}, \\
\forall x \in E_{\lambda k}, \quad k = 0, 1, 2, \ldots, m_\lambda.
\]
\tag{4.25}

Obviously, \( G_\lambda^0(\cdot) \) converges to \( A^\alpha(\cdot) \) strongly in \( L^2(\Omega) \), as \( \lambda \to \infty \), where \( A^\alpha(\cdot) \) is defined by (3.21). Thus it follows from (S2)–(S3) and Lemma 3.2 that \( y_\lambda^0(\cdot) \) converges to \( y^\alpha(\cdot; l(\cdot)) \), as \( \lambda \to \infty \), strongly in \( H_0^1(\Omega) \). We then obtain (3.24).
5. Second-order necessary conditions. In this section, we are going to prove Theorem 2.4. For readers' convenience, we will rewrite the relevant equations when needed. We first establish the following lemma.

**Lemma 5.1.** Let (S1)–(S5) hold. Let \( \bar{u}(\cdot), u(\cdot) \in \mathcal{U} \) and \( \ell(\cdot) \in \mathcal{L} \). For any \( \alpha \in (0, 1) \), define

\[
A^\alpha(x; \ell(\cdot)) = \alpha A(x, u(x)) + (1 - \alpha) A(x, \bar{u}(x)) \quad (5.1)
\]

Let \( \bar{y}(\cdot), y^\alpha(\cdot; \ell(\cdot)) \) be the weak solutions of the following equations:

\[
\begin{align*}
- \nabla \cdot (A(x, \bar{u}(x)) \nabla \bar{y}(x)) &= f(x, \bar{y}(x), \bar{u}(x)), \quad \text{in } \Omega, \\
\bar{y}|_{\partial \Omega} &= 0,
\end{align*}
\]

and

\[
\begin{align*}
- \nabla \cdot (A^\alpha(x; \ell(\cdot)) \nabla y^\alpha(x; \ell(\cdot))) &= (1 - \alpha)f(x, y^\alpha(x; \ell(\cdot)), \bar{u}(x)) \\
&\quad + \alpha f(x, y^\alpha(x; \ell(\cdot)), u(x)), \quad \text{in } \Omega, \quad (5.3)
\end{align*}
\]

respectively. Then, as \( \alpha \downarrow 0 \), \( Y^\alpha = y^\alpha(\cdot; \ell(\cdot)) - \bar{y}(\cdot) \) converges to \( Y(\cdot) \) weakly in \( H^1_0(\Omega) \) with \( Y(\cdot) \) being the weak solution of

\[
\begin{align*}
- \nabla \cdot (A(x, \bar{u}(x)) \nabla Y(x)) &= f(x, \bar{y}(x), \bar{u}(x)) Y(x) + \nabla \cdot (\Theta(x) \nabla \bar{y}(x)) \\
&\quad + f(x, \bar{y}(x), u(x)) - f(x, \bar{y}(x), \bar{u}(x)), \quad \text{in } \Omega, \\
Y|_{\partial \Omega} &= 0, \quad (5.4)
\end{align*}
\]

where

\[
\Theta(x) = A(x, u(x)) - A(x, \bar{u}(x)) - \frac{\ell(x)}{\ell(x)^\top A(x, u(x)) \ell(x)} [A(x, u(x)) - A(x, \bar{u}(x))]. \quad (5.5)
\]

**Proof.** Let \( y^\alpha(\cdot) = y^\alpha(\cdot; \ell(\cdot)) \). We have (suppressing \( x \))

\[
\begin{align*}
- \nabla \cdot (A(\bar{u}) \nabla Y^\alpha) &= \frac{f(y^\alpha, \bar{u}) - f(y^\alpha, \bar{u})}{\alpha} \\
&\quad + f(y^\alpha, \bar{u}) - f(y^\alpha, \bar{u}) - \nabla \cdot \left( \frac{A(\bar{u}) - A^\alpha(\ell)}{\alpha} \nabla y^\alpha \right) \\
&= \left( \int_0^1 f_y(\bar{y} + t(y^\alpha - \bar{y}), \bar{u}) dt \right) Y^\alpha + f(y^\alpha, \bar{u}) - f(y^\alpha, \bar{u}) \\
&\quad + \nabla \cdot \left( (A(u) - A(\bar{u})) \nabla y^\alpha \right) \\
&\quad - (1 - \alpha) \nabla \cdot \left( \frac{[A(u) - A(\bar{u})] \ell \top [A(u) - A(\bar{u})]}{(1 - \alpha) \ell \top A(u) \ell + \alpha \ell \top A(\bar{u}) \ell} \nabla y^\alpha \right). \quad (5.6)
\end{align*}
\]

By Proposition 3.3 and (S2)–(S3), one can see that \( y^\alpha(\cdot) \) is bounded uniformly in \( H^1_0(\Omega) \). Thus, we can prove that \( Y^\alpha(\cdot) \) is bounded uniformly in \( H^1_0(\Omega) \). Consequently, as \( \alpha \downarrow 0 \), \( y^\alpha(\cdot) \) converges to \( \bar{y}(\cdot) \) strongly in \( H^1_0(\Omega) \) (making use of the definition of \( Y^\alpha(\cdot) \)), and \( Y^\alpha(\cdot) \) converges to \( Y(\cdot) \) weakly in \( H^1_0(\Omega) \) with \( Y(\cdot) \) being the weak solution of (5.4). \( \square \)
Let us define
\[ Z^\alpha(\cdot) \equiv \frac{Y^\alpha(\cdot) - Y(\cdot)}{\alpha}. \] (5.7)
Then it is natural to expect that as \( \alpha \downarrow 0 \), \( Z^\alpha(\cdot) \to Z(\cdot) \), weakly in \( H^1_0(\Omega) \), with \( Z(\cdot) \) being the weak solution of the following equation:
\[
\begin{cases}
- \nabla \cdot (A(x, \bar{u}(x))\nabla Z(x)) = f_y(x, \bar{y}(x), \bar{u}(x))Z(x) \\
+ \frac{1}{2}f_{yy}(x, \bar{y}(x), \bar{u}(x))|Y(x)|^2 + \left( f_y(x, \bar{y}(x), u(x)) - f_y(x, \bar{y}(x), \bar{u}(x)) \right)Y(x) \\
+ \nabla \cdot \left( (A(x, u(x)) - A(x, \bar{u}(x)))\nabla Y(x) \right) + \nabla \cdot \left( Y(x)\nabla \bar{y}(x) \right), \quad \text{in } \Omega,
\end{cases}
\] (5.8)
where
\[
Y(x) = \left[ A(x, u(x)) - A(x, \bar{u}(x)) \right] \ell(x)\ell(x)^T \\
\cdot \left[ A(x, u(x)) - A(x, \bar{u}(x)) \right] \ell(x)^T A(x, \bar{u}(x))\ell(x) \\
\frac{\ell(x)^T A(x, \bar{u}(x))\ell(x)}{|\ell(x)^T A(x, u(x))\ell(x)|^2}.
\]
However, this seems not to be true when \( n \) is a large. The main reason is that \( A(\cdot, \bar{u}(\cdot)) \) is only bounded and measurable so that we have \( W^{1,p}(\Omega) \) estimates of \( Z^\alpha(\cdot) \) only for \( p \) near \( 2 \). Thus, the weak maximum principle for \( Y(\cdot) \) fails. Hence, we could not get the uniform boundedness for \( Y(\cdot) \) unless \( n \leq 2 \). Then, generally, \( Y(\cdot)^2 \) times an \( L^\infty \) function might not be in \( H^{-1}(\Omega) \) unless \( n \leq 6 \). Hence, \( Z(\cdot) \) is probably not well-defined for large \( u \). Fortunately, we have the following relation, which will be sufficient for the proof of our main result—Theorem 2.4. Recalling the definition of \( Z^\alpha(\cdot) \), we have
\[
- \nabla \cdot (A(\bar{u})\nabla Z^\alpha) = \frac{1}{\alpha} \left[ \left( \int_0^1 f_y(\bar{y} + t(y^\alpha - \bar{y}), \bar{u}) \, dt \right)Y^\alpha - f_y(\bar{y}, \bar{u}) \right] \\
+ \frac{f(y^\alpha, \bar{u}) - f(\bar{y}, \bar{u})}{\alpha} + \frac{f(\bar{y}, \bar{u}) - f(\bar{y}, \bar{u})}{\alpha} + \frac{1}{\alpha} \left[ \nabla \cdot \left( (A(u) - A(\bar{u}))\nabla y^\alpha \right) \right] \\
- (1 - \alpha) \nabla \cdot \left( \frac{[A(u) - A(\bar{u})] \ell\ell^T [A(u) - A(\bar{u})]}{(1 - \alpha)\ell\ell^T A(u)\ell + \alpha \ell\ell^T A(\bar{u})\ell} \nabla y^\alpha \right) \\
- \nabla \cdot \left( \frac{\ell\ell^T [A(u) - A(\bar{u})]}{(1 - \alpha)\ell\ell^T A(u)\ell + \alpha \ell\ell^T A(\bar{u})\ell} \nabla y^\alpha \right)
\]
\[= f_y(\bar{y}, \bar{u}) \cdot Z^\alpha + \left( \int_0^1 f_y(\bar{y} + t(y^\alpha - \bar{y}), \bar{u}) - f_y(\bar{y}, \bar{u}) \, dt \right) Y^\alpha \\
+ \frac{f(y^\alpha, \bar{u}) - f(\bar{y}, \bar{u})}{\alpha} + \nabla \cdot \left( (A(u) - A(\bar{u}))\nabla Y^\alpha \right) \\
+ \left( \frac{\ell\ell^T [A(u) - A(\bar{u})]}{(1 - \alpha)\ell\ell^T A(u)\ell + \alpha \ell\ell^T A(\bar{u})\ell} \nabla y^\alpha \right) \\
- \nabla \cdot \left( \frac{\ell\ell^T [A(u) - A(\bar{u})]}{(1 - \alpha)\ell\ell^T A(u)\ell + \alpha \ell\ell^T A(\bar{u})\ell} \nabla y^\alpha \right)
\] (5.9)
\[
= f_y(\bar{y}, \bar{u}) \cdot Z^\alpha + \left( \int_0^1 f_y(\bar{y} + t(y^\alpha - \bar{y}), \bar{u}) \, dt \right) Y^\alpha \\
+ \left( \int_0^1 f_y(\bar{y} + t(y^\alpha - \bar{y}), \bar{u}) - f_y(\bar{y} + t(y^\alpha - \bar{y}), \bar{u}) \, dt \right) Y^\alpha \\
+ \nabla \cdot \left( (A(u) - A(\bar{u}))\nabla Y^\alpha \right) \\
+ \nabla \cdot \left( \frac{\ell\ell^T [A(u) - A(\bar{u})]}{(1 - \alpha)\ell\ell^T A(u)\ell + \alpha \ell\ell^T A(\bar{u})\ell} \nabla y^\alpha \right).
\]
Now, we are ready to prove our main result.
Proof of Theorem 2.4. Let \((u(\cdot), \ell(\cdot)) \in \mathcal{V}_0(\tilde{u}(\cdot))\) and \(\alpha \in (0, 1)\). Let \(y^\alpha(\cdot)\) be the weak solution to (3.20), with \(A^\alpha(\cdot)\) and \(J^\alpha(u(\cdot), \ell(\cdot))\) be defined by (3.21) and (3.22), respectively. Let \(Y^\alpha(\cdot) = \frac{y^\alpha(\cdot) - \tilde{y}(\cdot)}{\alpha}\). Then \(Y^\alpha(\cdot)\) satisfies (5.6). We have

\[
\begin{align*}
\int_\Omega \left(f^0(\bar{y}, u) - f^0(\bar{y}, \bar{u}) + f^0_y(\bar{y}, \bar{u})Y\right) dx \\
= \int_\Omega \left(f^0(\bar{y}, u) - f^0(\bar{y}, \bar{u}) + f^0_y(\bar{y}, \bar{u})\right) dx + \int_\Omega \left(f_y(\bar{y}, \bar{u}) \bar{\psi} + \nabla \cdot (A(\bar{u})\nabla \bar{\psi})Y\right) dx \\
= \int_\Omega \left(f^0(\bar{y}, u) - f^0(\bar{y}, \bar{u}) + f^0_y(\bar{y}, \bar{u})\right) dx + \int_\Omega \left(f_y(\bar{y}, \bar{u}) \bar{\psi} + \nabla \cdot (A(\bar{u})\nabla \bar{\psi})\right) dx \\
= \int_\Omega \left[f^0(\bar{y}, u) - f^0(\bar{y}, \bar{u}) - \left(f(\bar{y}, u) - f(\bar{y}, \bar{u}) + \nabla \cdot (\Theta \nabla \bar{\psi})\right)\right] dx \\
= \int_\Omega \left\{H(\bar{y}, \bar{\psi}, \nabla \bar{y}, \nabla \bar{\psi}, \bar{u}) - H(\bar{y}, \bar{\psi}, \nabla \bar{y}, \nabla \bar{\psi}, u) \right. \\
\left. - \frac{\langle A(u) - A(\bar{u})\rangle \nabla \bar{y}, \Theta \nabla \bar{\psi}, \ell \rangle}{\langle A(u)\ell, \Theta \nabla \bar{\psi}, \ell \rangle}\right\} dx = 0,
\end{align*}
\]

where \(\Theta(\cdot)\) is given by (5.5). In fact,

\[
\int_\Omega \left(f^0(\bar{y}, u) - f^0(\bar{y}, \bar{u}) + f^0_y(\bar{y}, \bar{u})Y(x)\right) dx = \lim_{\alpha \downarrow 0} \frac{J^\alpha(u(\cdot), \ell(\cdot)) - J(\bar{u}(\cdot))}{\alpha} = 0,
\]

due to the partial singularity of \(\bar{u}(\cdot)\) at \((u(\cdot), \ell(\cdot)) \in \mathcal{V}_0(\tilde{u}(\cdot))\). This leads to

\[
\int_\Omega \left(f^0(\bar{y}, u) - f^0(\bar{y}, \bar{u})\right) dx = - \int_\Omega f^0_y(\bar{y}, \bar{u})Y dx.
\]

Then, using (5.10), (2.6), (5.6), and (5.9), we have (suppressing \(x\) whenever no confusion would be caused, for notational simplicity)

\[
\begin{align*}
J^\alpha(u(\cdot), \ell(\cdot)) - J(\bar{u}(\cdot)) &= \alpha \int_\Omega f^0(y^\alpha, u) dx + (1 - \alpha) \int_\Omega f^0(y^\alpha, \bar{u}) dx - \int_\Omega f^0(\bar{y}, \bar{u}) dx \\
&= \alpha \int_\Omega \left(f^0(y^\alpha, u) - f^0(y^\alpha, \bar{u})\right) dx + \int_\Omega \left(f^0(y^\alpha, \bar{u}) - f^0(\bar{y}, \bar{u})\right) dx \\
&= \alpha \int_\Omega \left(f^0(y^\alpha, u) - f^0(y^\alpha, \bar{u}) + f^0(\bar{y}, \bar{u}) + f^0(\bar{y}, u) - f^0(\bar{y}, \bar{u})\right) dx \\
&\quad + \int_\Omega \left(f^0(y^\alpha, \bar{u}) - f^0(\bar{y}, \bar{u})\right) dx \\
&= \alpha^2 \int_\Omega \int_0^1 \left(f^0_y(\bar{y} + \alpha tY^\alpha, u) - f^0_y(\bar{y} + \alpha tY^\alpha, \bar{u})\right) dt \right] Y^\alpha dx \\
&\quad + \alpha \int_\Omega \left[\int_0^1 f^0_y(\bar{y} + \alpha tY^\alpha, \bar{u}) dt\right] Y^\alpha - f^0_y(\bar{y}, \bar{u})\right] dx \\
&= \alpha^2 \int_\Omega \int_0^1 \left(f^0(\bar{y} + \alpha tY^\alpha, u) - f^0(\bar{y} + \alpha tY^\alpha, \bar{u})\right) dt \right] Y^\alpha dx \\
&\quad + \alpha \int_\Omega \left[\int_0^1 f^0(\bar{y} + \alpha tY^\alpha, \bar{u}) dt - f^0_y(\bar{y}, \bar{u})\right] Y^\alpha + \alpha f^0_y(\bar{y}, \bar{u})Z^\alpha] dx \\
&= \alpha^2 \int_\Omega \left[\int_0^1 \left(f^0(\bar{y} + \alpha tY^\alpha, u) - f^0(\bar{y} + \alpha tY^\alpha, \bar{u})\right) dt\right] Y^\alpha dx \\
&\quad + \alpha \int_\Omega \left[\int_0^1 f^0(\bar{y} + \alpha tY^\alpha, \bar{u}) dt - f^0_y(\bar{y}, \bar{u})\right] Y^\alpha + \alpha f^0_y(\bar{y}, \bar{u})Z^\alpha] dx \\
&= \alpha^2 \int_\Omega \left[\int_0^1 \left(f^0(\bar{y} + \alpha tY^\alpha, u) - f^0(\bar{y} + \alpha tY^\alpha, \bar{u})\right) dt\right] Y^\alpha dx.
\end{align*}
\]
\[ + \alpha^2 \int \Omega \left[ \int_0^1 \left( \int_0^1 t f_{yy}^0 (\bar{y} + at Y^\alpha, \bar{u}) dt \right) \right] Y^\alpha \, dx \]
\[ + \alpha^2 \int \Omega \left( f_y (\bar{y}, \bar{u}) \bar{\psi} + \nabla : (A(\bar{u}) \nabla \bar{\psi}) \right) Z^\alpha \, dx \]
\[ = \alpha^2 \int \Omega \left[ \int_0^1 \left( f_y^0 (\bar{y} + at Y^\alpha, u) - f_y^0 (\bar{y} + at Y^\alpha, \bar{u}) \right) dt \right] Y^\alpha \, dx \]
\[ + \alpha^2 \int \Omega \left[ \int_0^1 \left( \int_0^1 t f_{yy}^0 (\bar{y} + at Y^\alpha, \bar{u}) dt \right) \right] \bar{\psi} Y^\alpha \, dx \]
\[ + \int \Omega \langle (A(u) - A(\bar{u})) \nabla Y^\alpha, \nabla \bar{\psi} \rangle \, dx \]
\[ + \int \Omega \left\langle \frac{\ell^T A(\bar{u}) \{ \ell^T [A(u) - A(\bar{u})] \nabla y^\alpha] \} \nabla Y^\alpha}{\ell^T A(u) \ell \ell^T [A(u) - A(\bar{u})] \ell + \alpha \ell^T A(\bar{u}) \ell} \, \bar{\psi} \right\rangle \, dx \]
\[ = \alpha^2 \left\{ \int \Omega \left[ \int_0^1 \left( H_y (\bar{y} + at Y^\alpha, \bar{\psi}, \nabla \bar{\psi}, \bar{u}) \right) dt \right] Y^\alpha \, dx \right\}
\[ - \int \Omega \left[ \int_0^1 \left( \int_0^1 t H_{yy} (\bar{y} + at Y^\alpha, \bar{\psi}, \nabla \bar{\psi}, \bar{u}) dt \right) \right] \bar{\psi} Y^\alpha \, dx \]
\[ + \int \Omega \langle (A(u) - A(\bar{u})) \nabla Y^\alpha, \nabla \bar{\psi} \rangle \, dx \]
\[ + \int \Omega \left\{ \frac{\ell^T A(u) \ell \{ \ell^T [A(u) - A(\bar{u})] \nabla Y^\alpha]\} \nabla \bar{\psi}}{\ell^T A(u) \ell \ell^T [A(u) - A(\bar{u})] \ell + \alpha \ell^T A(\bar{u}) \ell} \, dx \right\} \].

In the above, (5.9) has played an important role. Consequently,

\[ 0 \leq \lim_{\alpha \downarrow 0} \frac{J^\alpha (u(\cdot), \ell(\cdot)) - J(\bar{u}(\cdot))}{\alpha^2} \]
\[ = \int \Omega \left[ \left( H_y (x, \bar{y}(x), \bar{\psi}(x), \nabla \bar{\psi}(x), \bar{u}(x)) \right) \right] Y(x) \]
\[ - \frac{1}{2} H_{yy} (x, \bar{y}(x), \bar{\psi}(x), \nabla \bar{\psi}(x), \bar{u}(x)) Y(x)^2 \]
\[ + \ell(x)^T [A(x, u(x)) - A(x, \bar{u}(x))] \nabla \bar{\psi}(x), \nabla Y(x) \]
\[ + \ell(x)^T [A(x, u(x)) - A(x, \bar{u}(x))] \nabla \bar{\psi}(x) \]
Consider the following controlled equation:

\[
\ell(x)^T [A(x, u(x)) - A(x, \bar{u}(x))] \nabla \bar{y}(x) \frac{[\ell(x)^T A(x, \bar{u}(x)) \ell(x)]}{[\ell(x)^T A(x, u(x)) \ell(x)]^2} \, dx
\]

\[
= \int_{\Omega} \left( \left( H_y(x, \bar{y}(x), \bar{\psi}(x), \nabla \bar{y}(x), \nabla \bar{\psi}(x), \bar{u}(x)) \right) Y(x)
- \frac{1}{2} H_{yy}(x, \bar{y}(x), \bar{\psi}(x), \nabla \bar{y}(x), \nabla \bar{\psi}(x), \bar{u}(x)) |Y(x)|^2
+ \langle (A(x, u(x)) - A(x, \bar{u}(x)) \nabla \bar{\psi}(x), \nabla Y(x) \rangle
+ \left( H(x, \bar{y}(x), \bar{\psi}(x), \nabla \bar{y}(x), \nabla \bar{\psi}(x), \bar{u}(x)) \right) \frac{[\ell(x)^T A(x, \bar{u}(x)) \ell(x)]}{[\ell(x)^T A(x, u(x)) \ell(x)]^2} \, dx.
\]

This completes the proof.

6. Illustrative examples. In this section, we present two examples. They are just for the purpose of illustration. We do not have an intention to present complicated (and non-trivial) examples.

Example 6.1. Consider the following controlled equation:

\[
\begin{aligned}
\left\{ \begin{array}{l}
-\nabla \cdot \left( A(u(x)) \nabla y(x) \right) = u(x), \quad \text{in } \Omega, \\
y|_{\partial \Omega} = 0.
\end{array} \right.
\]
\tag{6.1}
\]

The cost functional is defined to be

\[
J(u(\cdot)) = \int_{\Omega} u(x) y(x) \, dx.
\tag{6.2}
\]

The control domain is \( U = \{0, 1\} \), and \( A : U \rightarrow \mathbb{S}^n \) with \( A(u) > 0 \) for \( u = 0, 1 \). Note that

\[
J(u(\cdot)) = \int_{\Omega} u(x) y(x) \, dx = - \int_{\Omega} \nabla \cdot \left( A(u(x)) \nabla y(x) \right) y(x) \, dx
= \int_{\Omega} \langle A(x) \nabla y(x), \nabla y(x) \rangle \, dx = \int_{\Omega} |A(x)^{\frac{1}{2}} \nabla y(x)|^2 \, dx.
\]

Clearly, \( \bar{y}(\cdot) = 0 \) and \( \bar{u}(\cdot) = 0 \) is an optimal pair. The corresponding adjoint state \( \bar{\psi}(\cdot) \) satisfies

\[
\left\{ \begin{array}{l}
-\nabla \cdot \left( A(\bar{u}(x)) \nabla \bar{\psi}(x) \right) = -\bar{u}(x), \quad \text{in } \Omega, \\
\bar{\psi}|_{\partial \Omega} = 0,
\end{array} \right.
\]

which reads

\[
\left\{ \begin{array}{l}
-\nabla \cdot \left( A(0) \nabla \bar{\psi}(x) \right) = 0, \quad \text{in } \Omega, \\
\bar{\psi}|_{\partial \Omega} = 0.
\end{array} \right.
\]

Hence, \( \bar{\psi}(\cdot) = 0 \). The Hamiltonian is given by

\[
H(x, y, \psi, \xi, \eta) = \psi v - vy - \langle A(v) \xi, \eta \rangle.
\]

Thus,

\[
H(x, \bar{y}(x), \bar{\psi}(x), \nabla \bar{y}(x), \nabla \bar{\psi}(x), \bar{u}(x)) - H(x, \bar{y}(x), \bar{\psi}(x), \nabla \bar{y}(x), \nabla \bar{\psi}(x), u(x)) = 0.
\]
That is $\bar{u}(\cdot)$ is singular at any $u(\cdot) \in U$, and the first order necessary condition (the Pontryagin's maximum principle) trivially holds. Now, for any $u(\cdot) \in U$, the variational equation for $Y(\cdot)$ reads

$$
\begin{cases}
-\nabla \cdot (A(0)\nabla Y(x)) = u(x), & \text{in } \Omega, \\
Y|_{\partial \Omega} = 0,
\end{cases}
$$

and the second order necessary condition reads

$$
0 \leq \int_{\Omega} u(x)Y(x)dx = \int_{\Omega} \langle A(0)\nabla Y(x), \nabla Y(x) \rangle dx.
$$

which is true.

Example 6.2. Consider state equation (6.1) with $U = \{0, 1\}$ and with the following cost functional

$$
J(u(\cdot)) = -\int_{\Omega} y(x)^2 dx. \tag{6.3}
$$

Let $y^*(\cdot) = 0$ and $u^*(\cdot) = 0$. Then $(y^*(\cdot), u^*(\cdot))$ is an admissible pair. The corresponding adjoint equation reads

$$
\begin{cases}
-\nabla \cdot (A(u^*(x))\nabla \psi^*(x)) = -2y^*(x), & \text{in } \Omega, \\
\psi^*|_{\partial \Omega} = 0,
\end{cases}
$$

which takes the following form:

$$
\begin{cases}
-\nabla \cdot (A(0)\nabla \psi^*(x)) = 0, & \text{in } \Omega, \\
\psi^*|_{\partial \Omega} = 0.
\end{cases}
$$

Hence, $\psi^*(\cdot) = 0$. In the current case, the Hamiltonian is given by

$$
H(x, y, \psi, \xi, \eta, v) = \psi v + y^2 - \langle A(v)\xi, \eta \rangle.
$$

Thus, the first order necessary condition reads

$$
H(x, y^*(x), \psi^*(x), \nabla y^*(x), \nabla \psi^*(x), u^*(x))
- H(x, y^*(x), \psi^*(x), \nabla y^*(x), \nabla \psi^*(x), u(x)) = 0, \quad x \in \Omega,
$$

which is true. This also means that $u^*(\cdot)$ is singular at every $u(\cdot) \in U$. Now, for any $u(\cdot) \in U$, the corresponding variational system reads:

$$
\begin{cases}
-\nabla \cdot (A(0)\nabla Y(x)) = -u(x), & \text{in } \Omega, \\
Y|_{\partial \Omega} = 0,
\end{cases}
$$

and the second order necessary condition reads:

$$
0 \leq -\int_{\Omega} Y(x)^2 dx.
$$

Clearly, as long as $u(\cdot)$ is non-zero, the above fails. Hence, $u^*(\cdot)$ is not optimal.

7. **Concluding remarks.** We have established the second-order necessary conditions for the optimal controls of Problem (C). There are some challenging problems left open. We list some of them here, for which we are still working on with our great efforts.

- Construction of more general (non-trivial) examples for which our second-necessary conditions could lead to some optimal solutions.
- The second-order necessary conditions that we obtained looks complicated. Is it possible to have some better forms?
- Extension to fully non-linear elliptic equations.
We hope to be able to report some further results before long. Also, any participation of other interested researchers are welcome.

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