On the complexity of learning a language: An improvement of Block’s algorithm

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Abstract

Language learning is thought to be a highly complex process. One of the hurdles in learning a language is to learn the rules of syntax of the language. Rules of syntax are often ordered in that before one rule can be applied one must apply another. It has been thought that to learn the order of \( n \) rules one must go through all \( n! \) permutations. Thus to learn the order of 27 rules would require 27! steps or \( 1.08889 \times 10^{28} \) steps. This number is much greater than the number of seconds since the beginning of the universe! In an insightful analysis the linguist Block ([Block 86], pp. 62-63, p.238) showed that with the assumption of transitivity this vast number of learning steps reduces to a mere 377 steps. We present a mathematical analysis of the complexity of Block’s algorithm. The algorithm has a complexity of order \( n^2 \) given \( n \) rules. In addition, we improve Block’s results exponentially, by introducing an algorithm that has complexity of order less than \( n \log n \).

Key Words: Language learning, rules of language, complexity, learning algorithms, evolution of language.

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1 Introduction

Language learning is thought to be a highly complex process. One of the hurdles in learning a language is to learn the rules of syntax of the language. Rules of syntax are often ordered in that before one can apply one rule one must apply another. It has been thought that to learn the order of \( n \) rules one must go through all \( n! \) permutations. Thus to learn the order of 27 rules would require 27! steps or \( 1.08889 \times 10^{28} \) steps. This number is much greater than the number of seconds since the beginning of the universe! In a brilliant analysis the linguist Block ([Block 86], pp. 62-63, p.238) showed that with the assumption of transitivity this vast number of learning steps reduces to a mere 377 steps.

We present a mathematical analysis of the complexity of Block’s algorithm. The algorithm has a complexity of order \( n^2 \) given \( n \) rules. In addition, we improve on Block’s algorithm, by introducing an algorithm that has complexity of order less than \( n \log n \). For example, given 27 rules our method requires 104 steps.
2 \hspace{1em} \textbf{Block’s algorithm}

Given \( n \) rules \( R_1 \ldots R_n \) we are to guess (learn) an unknown ordering of these rules. Starting with any given rule \( Y \), we are to position a rule \( X \). To do this we test (query, learn) if \( X \) is before \( Y \), if so we place it before \( Y \) to give the sequence \( XY \), else we place it after \( Y \) resulting in \( YX \). Now, assume we have ordered \( i \) rules \( Y_1 \ldots Y_i \) and we have to a new place rule \( X \), Block suggests we test for each \( j \) if \( X < Y_j \), if so all we need do is to place \( X \) before \( Y_j \) in our sequence \( Y_1 \ldots Y_j \ldots Y_i \) to give \( Y_1 \ldots XY_j \ldots Y_i \). The reason we can do this is because the rules are linearly ordered and hence by transitivity we know \( X \) is less than all rules beyond \( Y_j \). Hence, we need test no further. If \( X \) is not less than \( Y_j \) we continue until we find such a \( j \). If we don’t find such a \( j \), then we know that \( X \) is greater than all all the given rules and we place \( X \) at the end to give \( Y_1 \ldots Y_i X \). Thus starting with one rule we gradually build up our knowledge about the order of the rules.

2.1 Complexity of Block’s algorithm

At any stage, given we know the order of rules \( Y_1 \ldots Y_i \), it takes at most \( i \) comparisons to place our new rule \( X \). Hence, to learn the total rule set \( R_1 \ldots R_n \) it takes at most \( 1 + 2 + \ldots + i + \ldots + n \) steps. This worst case is how computer scientists measure the complexity of an algorithm.

\[
S(n) = \sum_{i=1}^{n} i - 1 = \sum_{i=2}^{n} i
\] (1)

We subtract 1 (or equivalently start with 2) because the first rule does not have to be tested or compared. Thus, we have a measure of the complexity of Block’s algorithm in defined in terms of the number of queries or steps needed to order \( n \) rules.

2.2 A simpler formula measuring Block’s algorithm’s complexity

There is a simpler and shorter formula that eliminates the need to sum up the steps and thereby can easily be done on a calculator. It gives an exact characterization of the steps needed to order \( n \) rules:
Exact complexity of Block’s algorithm

\[ S(n) = \frac{(n^2 - n)}{2} + n - 1 \]  

Since, the exponent of 2 ultimately overpowers any addition or division by a constant, the complexity of Block’s algorithm is \( O(n^2) \).

For example, for 27 rules it takes at most \( \sum_{i=1}^{27} i = 27 + 26 + \ldots + 2 = 377 \) This is, of course, a very modest number when compared to the estimates of \( 27! = 1.08889 \times 10^{28} \) steps by some linguists. For practical, purposes one would think this is enough and we should be satisfied with such an astounding reduction of steps needed to learn some 27 rules.

However, if we increase the number of rules to hundreds, we quickly face numbers that may again seem insurmountable for the developing brain of a child. The method we present here improves on Blocks algorithm significantly since it is exponentially more efficient. As a result, even the ordering of thousands of rules can be learned relatively quickly.

For example, for one thousand rules, the number of steps required to determine the ordering of the rules using Block’s algorithm is 500,499 steps. Below we will show that for this case we can improve this to less than 8,977 steps which is about 55 times faster.

3 A fast algorithm to learn the order of \( n \) rules

Block’s algorithm moves linearly from start to end to find and test if the rule is before or after the given rule. Instead we will adapt binary search to start at the middle of the sequence of rules and test the rule there. This will give us an exponential improvement over Block’s method.

3.1 A faster rule ordering algorithm

Given \( n \) rules and a rule \( X \) that needs to be placed in its proper order, choose the middle rule, call it \( Y \). This can be done since the rules are linearly ordered. Test to see if the rule is applied before \( Y \) if so test the middle rule between \( Y \) and the first rule, else test the middle rule between the end rule and rule \( Y \). Repeat until done.
3.2 Complexity of the fast rule ordering algorithm

In effect we use binary search to find the proper place to insert our new rule $Y$ given $n$ rules. And, this process takes at most $n \log n$ steps. Hence, given we have $n$ rules to insert into at most $n$ rules, we have at most $n \log n$ steps to find the completely ordered sequence of $n$ rules.

Actually, the process takes fewer steps. Since, work incrementally starting with $n = 1$, call this one rule $Y$ and one rule $X$ to insert we just have to make one test to see if $X < Y$ or not. If it is less than $Y$ we insert $X$ before $Y$ giving $XY$ else we insert it after $Y$ resulting in the partial sequence $YX$. We then pick the next rule $Z$ and insert it. Given a sequence of $n$ rules we insert $X$ at each stage according to the binary search algorithm above.

Hence, for $n$ rules we require at most

$$B(n) = \lceil \log(n) \rceil + \lceil \log(n - 1) \rceil + \lceil \log(n - 2) \rceil + \ldots + \lceil \log(n - (n - 1)) \rceil$$  \hspace{1cm} (3)

steps to order them.

**Notation** Throughout this article we use log to mean the logarithm base 2. The notation, $\lceil \alpha \rceil$ denotes the smallest integer $i$ greater than or equal to the number $\alpha$. The log of a number will in general not be an integer. This promotes a real (floating point) number to the next integer since we are counting steps in an algorithm and they are discrete.

Expressed more succinctly we have:

**Complexity of the fast algorithm**

$$B(n) = \sum_{i=0}^{n-1} \lceil \log(n - i) \rceil$$  \hspace{1cm} (4)

Note, that $B(n) < n \log n$. Thus, the complexity is less than $O(n \log n)$.

3.3 An shorter but less accurate complexity function

Looking at it from a different perspective, we can shorten the description of the complexity function by using factorial.
Thus, we can get an approximation to the function $B(n)$ in equation 4 using the following:

$$B_f(n) = \lceil \log(n!) \rceil$$  \hspace{1cm} (5)

However, this formulation has the disadvantage that the size of the factorial quickly expands to be computationally impractical. In addition, the result is not as accurate in that it underestimates the complexity because the \( \log \) is used instead of \( \lceil \log \rceil \) when \( \log(n!) \) is calculated. If we add \( n \) to the result, giving \( \log(n!) + n \) we get an overestimate. Thus,

$$\log(n!) < B(n) < \log(n!) + n$$  \hspace{1cm} (6)

4 Examples comparing the performance of the algorithms

For example, for 27 rules Block’s algorithm requires 377 steps. We need only 104 steps which is which is 3.6 times faster. However, since there is an exponential improvement using the fast algorithm, the differences become more dramatic the larger the number of rules that are to be ordered. For example, for 1,000 rules, some linguists would tell us we need to search through 1000! possible combinations. Using Block’s algorithm, we need only 500,499 steps, a much much smaller number. With our improved fast algorithm we need less than 8,977 steps which is more than 55 times faster (smaller) in this case. So, for instance, if a child would learn to order two rules per day (into its given set of learned rules), it would take 685 years to learn the ordering of 1,000 rules using Block’s method. With our fast method it would take about 12 years.

There is the question to what extent such considerations are relevant to learning a language. The above presupposed that the rules are linearly ordered. That means each rule stands in a unique linear relationship to all the other rules. Partially ordered rules are yet another matter.
References

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