MICROLOCAL SHEAF CATEGORIES AND THE $J$-HOMOMORPHISM

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ABSTRACT. Let $X$ be a smooth manifold and $k$ be a ring spectrum, either commutative or at least $E_2$. Given a smooth exact Lagrangian $L \hookrightarrow T^*X$, the microlocal sheaf theory (following Kashiwara–Schapira) naturally assigns a locally constant sheaf of categories on $L$ with fiber equivalent to the category of $k$-spectra $\text{Mod}(k)$. We show that the classifying map for the local system of categories factors through the stable Gauss map $L \to U/O$ and the delooping of the $J$-homomorphism $U/O \to B\text{Pic}(S)$.

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1. Introduction

Let $X$ be a smooth manifold and $k$ be a ring spectrum, either commutative or at least $E_2$. Given a smooth exact Lagrangian $L \hookrightarrow T^* X$, the microlocal sheaf theory (developed by Kashiwara–Schapira) naturally assigns a locally constant sheaf of categories on $L$ with fiber equivalent to the category of $k$-spectra $\text{Mod}(k)$. It is a standard fact that such a locally constant sheaf of categories is determined (up to homotopy) by a classifying map $L \to B\text{Aut}_k(\text{Mod}(k)) \simeq B\text{Pic}(k)$.

The goal of this paper is to give a solution to the following fundamental problem:

**Problem 1.1.** Give a topological description of the classifying map for the sheaf of microlocal categories on $L$.

**Theorem 1.2** (see Theorem 4.1 for the precise statement). The classifying map (1.1) can be factored, up to homotopy, as

$$L \xrightarrow{\gamma} U/O \xrightarrow{BJ} B\text{Pic}(S) \to B\text{Pic}(k),$$

where $\gamma$ is the stable Gauss map, $BJ$ is the delooping of the $J$-homomorphism in stable homotopy theory, and the last map is the tautological one as the sphere spectrum $S$ is the initial object in the $\infty$-category of all ring spectra.

The above answer was predicted in [JiTr], and here we give a full proof. We remark that for $k$ an ordinary ring, the classifying map factors through $U/O \to B\text{Pic}(Z) \simeq B\mathbb{Z} \times B^2(\mathbb{Z}/2\mathbb{Z})$, and the obstruction classes given by (1.1) for $k = \mathbb{Z}$ are exactly the Maslov class and the second relative Stiefel–Whitney class. The case for an ordinary ring $k$ has been obtained in [Gui].

1.1. Motivations and applications. In this subsection, we give a brief overview of microlocal sheaf approaches to symplectic topology, leading to motivations of Problem 1.1 and applications of Theorem 1.2 as an answer to it.

Microlocal sheaf theory was developed by Kashiwara–Schapira [KaSc] in the 90s, and it has been substantially applied to the study of symplectic topology, especially making connections to Floer cohomology, after the work of Nadler–Zaslow [NaZa], [Nad] and Tamarkin [Tam1].

For any topological space $X$ and a commutative ring $k$ (will be generalized to ring spectra later), let $\text{Shv}(X; k)$ be the (large) dg-category (equivalently stable $\infty$-category) of sheaves of $k$-modules on $X$. A key notion in microlocal sheaf theory is the singular support of a sheaf $\mathcal{F} \in \text{Shv}(X; k)$, denoted by $SS(\mathcal{F})$, which is a closed conic subset in...
\[ T^*X = T^*X \setminus \{ \text{zero-section} \}, \]
that collects all the covectors along which non-propagation of sections of \( \mathcal{F} \) occurs. A deep result of Kashiwara–Schapira is that the singular support of any sheaf is coisotropic, which is in some sense a mathematical shadow of the Heisenberg Uncertainty Principle that one can only observe the position and momentum of a particle along a coisotropic subvariety in the cotangent bundle, but not smaller than that.

Among the coisotropic subvarieties in \( T^*X \), the class of Lagrangian submanifolds, which is having the smallest possible dimension (\( = \dim X \)), has drawn particular interest in the study of symplectic topology, due to their nice intersection theory through Floer theory, simple moduli property by Arnold’s Nearby Lagrangian conjecture for exact Lagrangians and the analysis for special Lagrangians, and easier quantization results (than other coisotropic varieties) by microlocal sheaves or holonomic \( D \)-modules.

The approach of microlocal sheaf theory to the study of Lagrangian submanifolds (or more generally subvarieties) goes as follows. For any (exact) Lagrangian \( L \subset T^*X \), there is a standard procedure to transform \( L \) into a conic Lagrangian \( L \) inside \( T^*c^0(X \times \mathbb{R}_t) \), where \( T^*c^0(X \times \mathbb{R}_t) \) is the open locus in \( T^*(X \times \mathbb{R}_t) \) consisting of covectors whose pairing with \( \partial_t \) is strictly negative. Now one can study the category of sheaves with singular support contained in \( L \) (more precisely some localization of it), denoted by \( \text{Shv}_L^c(X \times \mathbb{R}_t; \mathbb{k}) \). This can be regarded as sheaf quantizations of \( L \). Since it is invariant under Hamiltonian isotopies of \( L \), the sheaf category is an interesting symplectic invariant of \( L \). It is closely related to the Fukaya category of \( T^*X \) (cf. [Tam1], [Gui]).

The sheaf quantization has a generalization over ring spectra (cf. [JiTr]). It is broadly expected that working over ring spectra and making connections to stable homotopy theory would help in understanding the topology of Lagrangian submanifolds, especially shed new light on Arnold’s Nearby Lagrangian conjecture (cf. [AbKr] for work in a similar direction as Floer homotopy types).

The first step to understand \( \text{Shv}_L^c(X \times \mathbb{R}_t; \mathbb{k}) \) is to understand the sheaves microlocally along \( L \). Namely, following the pioneering work of Kashiwara–Schapira, one can assign a sheaf of microlocal sheaf categories along \( L \) (sometimes referred as Kashiwara–Schapira stack), denoted by \( \mu \text{Shv}_L \), whose sections over an open set \( V \subset L \) form a certain sheaf category that only involves the local geometry of \( V \) (inside a tubular neighborhood). There is a canonical functor

\[ \text{Shv}_L^c(X \times \mathbb{R}_t; \mathbb{k}) \rightarrow \Gamma(L, \mu \text{Shv}_L) \]

that can be roughly interpreted as taking microlocal stalks.

If \( L \) is smooth, then \( \mu \text{Shv}_L \) is locally constant with fiber equivalent to \( \text{Mod}(\mathbb{k}) \), and this is the focus of this paper as it appears in Problem [1]. In nice situations, e.g. \( L \) is a closed (embedded) exact Lagrangian submanifold, then knowing \( \Gamma(L, \mu \text{Shv}_L) \) is enough to recover \( \text{Shv}_L^c(X \times \mathbb{R}_t; \mathbb{k}) \) and some twisted version if \( \mu \text{Shv}_L \) has interesting monodromy (cf. [Gui], [JiTr]). So to understand \( \text{Shv}_L^c(X \times \mathbb{R}_t; \mathbb{k}) \), it is crucial to first understand \( \mu \text{Shv}_L \), captured by the classifying map [1].

The understanding of the classifying map for \( \mu \text{Shv}_L \) is also crucial in the definition of a microlocal sheaf category for a Lagrangian skeleton \( \Lambda \) (e.g. inside a Weinstein manifold). In general, there is no canonical symplectic framing as in the cotangent bundle case, so
basically one first takes the (stable) symplectic frame bundle (or a thickening of it) of the exact symplectic neighborhood of Λ, which records the auxiliary data of how one can view local pieces of Λ inside cotangent bundles, and define a sheaf of microlocal categories there, denoted by \( \mu \mathcal{Shv}_\Lambda \). The stable symplectic group, which is homotopy equivalent to the stable unitary group \( U \), acts on the symplectic frame bundle, inducing a convolution action

\[
\mu \mathcal{Shv}_U \boxtimes \mu \mathcal{Shv}_\Lambda \longrightarrow \mu \mathcal{Shv}_\Lambda
\]

where \( U \) is viewed as the stable version of its action Lagrangian graph inside \( \lim_{N \to \infty} T^*U(N) \times (T^*R^N)^- \times T^*R^N \). This exhibits \( \mu \mathcal{Shv}_\Lambda \) as a twisted equivariant sheaf of categories with respect to the character \( U \longrightarrow B\text{Pic}(k) \) that classifies \( \mu \mathcal{Shv}_U \). This character is easy to write down using Theorem 1.2, and this enables one to define a twisted sheaf of microlocal categories on Λ through a twisted equivariant descent. More details of this approach will appear in a forthcoming paper. There is an alternative construction of \( \mu \mathcal{Shv}_\Lambda \) using h-principle [She], whose complete version over ring spectra is also based on Theorem 1.2.

1.2. Sketch of the proof of Theorem 1.2. In this subsection, we give a sketch of the proof of the main theorem, and highlight the underlying geometry and key tools. Here for simplicity, we will assume \( X = R^N \). For a general manifold \( X \), everything can be reduced to the Euclidean case by embedding \( X \) into \( R^N \), \( N \gg 0 \) (see Subsection 4.5).

The first ingredient is the class of correspondences, which we call (local) Morse transformations, that gives local trivializations of \( \mu \mathcal{Shv}_L \). Roughly speaking, the Morse transformations for a given small open set \( V \subset L \) is the class of local symplectomorphisms of \( T^*R^N \) such that

- its graph in \( (T^*R^N)^- \times T^*R^N \) admits a generating function in the base variables \( (q_0, q_1) \) of \( R^N \times R^N \) (i.e. its projection to the base is a local isomorphism),
- it transforms \( V \) to a Lagrangian graph in \( T^*R^N \) (i.e. admits a generating function in \( q_1 \)).

The Morse transformations have appeared as a generic class of contact transformations in [KaSc], and they are closely related to stratified Morse functions as in stratified Morse theory developed by Goresky–MacPherson [GMac]. For more discussions in this aspect, see [Jin].

The second ingredient is the connection between Morse transformations for a germ of smooth Lagrangian \( (L, x) \) and the space of paths from \( \ell_x =: T_xL \) as an affine Lagrangian in \( T^*R^N \) to the zero-section. The upshot is that these two spaces, after stabilizations, are homotopy equivalent up to a “group completion”.

More explicitly, at the tangent space level, a Morse transformation is doing a Fourier transform, denoted by \( FT \), then followed by the time-1 map of a quadratic Hamiltonian depending only on \( p \) (the momentum coordinates) that transforms \( FT(\ell_x) \) into a graph-type affine Lagrangian. If we view the open contractible locus \( \mathcal{B} \) of graph-type Lagrangians in the stable (affine) Lagrangian Grassmannian as a “fat” base point, then each Morse transformation of \( (L, x) \) can be assigned a path starting from \( \ell_x \) and ending at \( \mathcal{B} \) as the concatenation of a path \( FT_t(\ell_x), 0 \leq t \leq \frac{\pi}{2} \) (where one can take the Hamiltonian flow
of $\frac{1}{2}(|q|^2 + |p|^2)$ for $\text{FT}_t$ and the image of $\text{FT}(\ell_x)$ under the flow of a certain quadratic Hamiltonian in $p$. From this point of view, the space of such quadratic Hamiltonians involved (after stabilization) is homotopy equivalent to a torsor of
\begin{equation}
\bigoplus_n \{\text{nonnegative quadratic forms in } p \text{ of rank } n\} \simeq \bigoplus_n \text{BO}(n),
\end{equation}
as a topological monoid. The Hamiltonian flows of the latter give based loops of $U/O$ at $B$, and this is an incidence of Bott periodicity.

Now we can explain the key point of the proof. Let $P_L \to L$ be the pullback of the universal principal $\Omega(U/O) \simeq \mathbb{Z} \times \text{BO}$-bundle over $U/O$ through the stable Gauss map $\gamma : L \to U/O$. Each Morse transformation for an open set $V \subset L$ gives a correspondence

$$
H \xrightarrow{p_1} \mathbb{R}^N \times \mathbb{R}_{t_1} \xrightarrow{p_0} \mathbb{R}^N \times \mathbb{R}_{t_0},
$$

where $H \subset \mathbb{R}^N \times \mathbb{R}_{t_0} \times \mathbb{R}^N \times \mathbb{R}_{t_1}$ is a smooth hypersurface, that establishes an equivalence of categories
\begin{equation}
(p_1)_* p_0^! : \mu \mathcal{Sh}_L(V; k) \xrightarrow{\sim} \text{Loc}(V; k),
\end{equation}
i.e. a trivialization of $\mu \mathcal{Sh}_L$ on $V$. Now for simplicity, let us take $V$ sufficiently small and contractible around a point $x \in L$ (so that $\text{Loc}(V; k) \simeq \text{Mod}(k)$), then from the above remarks, we can (almost) think of $P_L|_V$ as parametrizing the Morse transformations, hence we get a family of trivializations (1.3). Now for each object $\mathcal{F} \in \mu \mathcal{Sh}_L(V; k)$, we get a family of $k$-modules, assembled into a local system on $P_L|_V$. The key proposition that we establish is the following.

**Proposition 1.3** (see Proposition 4.10 and Proposition 4.13). For $k = \mathcal{S}$ (the sphere spectrum), there is a natural equivalence of categories
\begin{equation}
\mu \mathcal{Sh}_L(V; \mathcal{S}) \xrightarrow{\sim} \text{Loc}(P_L|_V; \mathcal{S})^{J\text{-equiv}},
\end{equation}
that is compatible with restrictions along open inclusions $V' \hookrightarrow V$.

Here $P_L|_V$ is viewed as a torsor over $\Omega(U/O)$ in the $\infty$-category of spaces $\text{Spc}$, and the $J$-homomorphism $J : \Omega(U/O) \to \text{Pic}(\mathcal{S})$ is viewed as a character so that one can talk about $J$-equivariant local systems on $P_L|_V$. Then it is easy to see that Proposition 1.3 immediately implies our main result.

The proof of Proposition 1.3 is essentially contained in [Jin], though the part of compatibility with restrictions under open inclusions was not discussed there (due to the non-flexibility of the model representing (1.2) that we chose there). The idea is to quantize the Hamiltonian action of the space of nonnegative quadratic Hamiltonians depending only on $p$ as in (1.2), which we referred as the Hamiltonian $\bigoplus_n \text{BO}(n)$-action in loc. cit.

Then the tautological vector bundle on $\bigoplus_n \text{BO}(n)$ shows up, which gives rise to the $J$-homomorphism.
Since there is a nontrivial amount of structures and functorialities involved in Proposition 1.3, one needs to adopt the right machinery to write down a clean argument. Here we employ the $(\infty, 2)$-category of correspondences developed by Gaitsgory–Rozenblyum [GaRo], which is designed to treat the six-functor formalism in a clean and functorial way. With our inputs on concrete constructions of (commutative) algebra objects and their modules together with (right-lax) morphisms between them in the category of correspondences $\text{Corr}(\mathcal{C})$, for a Cartesian symmetric monoidal $(\infty, 1)$-category $\mathcal{C}$, in [Jin] (further upgraded in Appendix A), we are able to choose the appropriate models (explained in Section 4) to achieve the functorial results.

1.3. Organization. The organization of the paper goes as follows. In Section 2, we give some account of the geometry of $U/O$ and review one step of Bott periodicity $\Omega(U/O) \simeq \mathbb{Z} \times BO$. The point of view is essentially following [Har]. In Section 3, we briefly review microlocal sheaf theory over ring spectra and define the sheaf of microlocal categories $\mu_{\text{Shv}}L$ on a smooth Lagrangian $L$ in $T^*X$. We present two equivalent versions of $\mu_{\text{Shv}}L$ with the latter version (denoted by $\mu_{\text{Shv}}L$) adopted in later arguments, for it is convenient to run arguments with correspondences using this version. Then we recall Morse transformations and relate them to paths in $U/O$, which illustrates the geometry underlying the proof of the main theorem. We finish the section by recalling the category of correspondences ([GaRo]) and the functor $\text{ShvSp}^L$. In Section 4, we give the proof of the main result. As explained before, the key idea of quantizing a Hamiltonian $\coprod BO(n)$-action is already contained in [Jin]. The additional ingredient here is the construction of an $\infty$-category $\text{QHam}_L(U/O)$ over $\text{Open}(L)^{op}$ that roughly plays the role of a “principal $\coprod BO(n)$-bundle” to approximate the pullback of the universal principal $\Omega(U/O)$-bundle along the Gauss map. Finally, in Appendix A, we give the construction of a canonical functor from a correspondence category of $\text{Fin}^*$-objects of a Cartesian symmetric monoidal $\infty$-category $\mathcal{C}$ to the category of commutative algebra objects in $\text{Corr}(\mathcal{C})$ with right-lax homomorphisms (and also a module version). This is an upgraded version of the results that we have obtained in [Jin] about constructions of (commutative) algebra and their modules in $\text{Corr}(\mathcal{C})$, and it is frequently (sometimes implicitly) used in the arguments throughout Section 4. We also include a list of notations and conventions for categories in Appendix B.

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2. Geometry of $U/O$ and Bott periodicity

2.1. Spectral decomposition for linear Lagrangians. Let $\omega_0$ be the standard symplectic form on $T^*\mathbb{R}^N$, and let $J_0$ be the standard complex structure on $T^*\mathbb{R}^N$ compatible with $\omega_0$ determined by $g_0(\cdot, \cdot) = \omega_0(\cdot, J_0(\cdot))$ is the Euclidean metric on $\mathbb{R}^{2N}$. For any linear Lagrangian $\ell \in T^*\mathbb{R}^N$, it determines a quadratic form $A_\ell$ on its projection to the
zero-section, given by \( A_\ell(u, v) = \tilde{u}(v) \), where \( \tilde{u} \) is any lifting of \( u \) in \( \ell \) under the projection to be base. Note that \( A_\ell \) is twice of a primitive of \( \ell \). Let \( \ell_{[\infty]} \) be the cotangent fiber at 0.

**Lemma 2.1.** For any linear Lagrangian \( \ell \subset T^*\mathbb{R}^N \), the subspace \( J_0(\ell \cap \ell_{[\infty]}) \) is the orthogonal complement to \( \text{proj}_{\mathbb{R}^N}(\ell) \).

**Proof.** For any \( u \in \text{proj}_{\mathbb{R}^N}\ell \) and \( v \in \ell \cap \ell_{[\infty]} \), we have
\[
g_0(J_0v, u) = \omega_0(v, u) = \omega_0(v, \tilde{u}) = 0,
\]
where \( \tilde{u} \) is any lifting of \( u \) in \( \ell \) as before. So we see that \( J_0(\ell \cap \ell_{[\infty]}) \) is orthogonal to \( \text{proj}_{\mathbb{R}^N}(\ell) \). By dimension reason, they are orthogonal complements to each other. □

By Lemma 2.1 any linear Lagrangian \( \ell \subset T^*\mathbb{R}^N \) is determined by a subspace \( J_0(\ell \cap \ell_{[\infty]}) \subset \mathbb{R}^N \) and a linear Lagrangian graph
\[
(2.1) \quad \ell_{\text{fin}} = \Gamma(\frac{1}{2}dA_\ell) \subset T^*(J_0(\ell \cap \ell_{[\infty]}))^\perp,
\]
and we have \( \ell = J_0(\ell \cap \ell_{[\infty]}) \oplus \ell_{\text{fin}} \) under the orthogonal splitting \( T^*\mathbb{R}^N = T^*(J_0(\ell \cap \ell_{[\infty]})) \oplus T^*(J_0(\ell \cap \ell_{[\infty]}))^\perp \). By this observation, we can view \( \ell \) as a *generalized quadratic form* in the sense that it has eigenvalues range in \( \mathbb{R} \cup \{\infty\} \cong S^1 \), and its spectral decomposition is equal to the sum of \( A_\ell \) and the eigenspace of \( \infty \) given by \( J_0(\ell \cap \ell_{[\infty]}) \). Such a point of view will be very useful to us. First, it will give a convenient way to describe and prove one step of Bott periodicity: \( \Omega(U/O) \cong \mathbb{Z} \times BO \). Second, we will often use it to give an easy description of a small neighborhood of any \( \ell \in U(N)/O(N) \), namely, those linear Lagrangians given by small deformations of the spectral decomposition of \( \ell \).

Regarding the stabilization process \( U/O = \lim_{\rightarrow N} U(N)/O(N) \), where the inclusions \( U(N)/O(N) \hookrightarrow U(N + 1)/O(N + 1) \) are given by direct sum with the zero-section in the extra dimension, a *stable* linear Lagrangian has spectral decomposition given by a finite collection of nonzero (possibly including \( \infty \)) distinct eigenvalues \( \lambda_1, \ldots, \lambda_k \), and finite dimensional eigenspaces \( V_{\lambda_1}, \ldots, V_{\lambda_k} \), and the orthogonal complement of \( \bigoplus_{j=1}^k V_{\lambda_j} \) (which is infinite dimensional) serves as the eigenspace of 0.

### 2.2. One step of Bott periodicity: \( \Omega(U/O) \cong \mathbb{Z} \times BO \)

Recall the ordinary 1-category \( \Gamma \) defined in [Seg74], which is canonically equivalent to \( \text{Fin}_{\ast}^p \), and a \( \Gamma \)-space is equivalent to a commutative algebra object in \( \text{Spc} \). To be consistent with notations used in the rest of the paper, we will use \( \text{Fin}_{\ast}^p \) instead of \( \Gamma \). If we have fixed an identification \( \langle n \rangle \cong \{1, \ldots, n, \ast\} \), then one has a natural functor \( \Delta \to \text{Fin}_{\ast}^p \), taking \([n] \in \Delta \to \langle n \rangle \) and a morphism \( f : [n] \to [m] \) to \( \hat{f} : \langle m \rangle \to \langle n \rangle \) determined by requiring \( \hat{f}^{-1}(k) = \{f(k - 1) + 1, \ldots, f(k)\}, k \in \langle n \rangle^\circ \). Using this functor, one can turn a \( \text{Fin}_{\ast} \)-space into a simplicial space, and one takes its geometric realization as the *geometric realization* of the original \( \text{Fin}_{\ast} \)-space.
Recall the definition of the delooping functor in *loc. cit.*

\[
\mathcal{B} : \text{CAlg}(\mathcal{Sp}) \to \text{CAlg}(\mathcal{Sp})
\]

where \( \mathcal{B}A : N(\text{Fin}_*) \to \mathcal{Sp} \) takes \( \langle n \rangle \) to the geometric realization of the \( \text{Fin}_* \)-object \( \langle m \rangle \mapsto A(\langle m \rangle \wedge \langle n \rangle) \), induced from the canonical functor

\[
N(\text{Fin}_*) \times N(\text{Fin}_*) \to N(\text{Fin}_*)
\]

\[
\langle n \rangle \wedge \langle m \rangle \cong (\langle n \rangle^o \times \langle m \rangle^o) \cup \{ \ast \}.
\]

Since \( \text{CAlg}(\mathcal{Sp}) \) is presentable and \( \mathcal{B} \) preserves colimits, it admits a right adjoint \( \Omega \) which is the based loop functor. The natural map \( A \to \Omega \mathcal{B}A \) from adjunction, for any \( A \in \text{CAlg}(\mathcal{Sp}) \), serves as the group completion for \( A \) (cf. [Qui]).

Now following the construction in [Har], let \( G^* \) be the \( \text{Fin}_* \)-space defined as follows:

\[
G^{(0)} = \text{pt},
\]

\[
G^{(n)} := \{(E_j)_{j \in \langle n \rangle^o} : E_j \text{ linear } \lim_{N \to \infty} \mathbb{R}^N, \dim E_j < \infty, E_j \perp E_k \text{ for } j \neq k \};
\]

for any \( f : \langle n \rangle \to \langle m \rangle \),

\[
f_* : G^{(n)} \to G^{(m)}
\]

\[
(E_j)_{j \in \langle n \rangle} \mapsto \left( \bigoplus_{j \in f^{-1}(k)} E_j \right)_{k \in \langle m \rangle^o}.
\]

There is a natural homeomorphism from \( BG^{(1)} \) to \( U/O \) induced from

\[
\prod_n (G^{(n)} \times |\Delta^n|) \to U/O
\]

(2.2)

\[
((E_j)_{1 \leq j \leq n}, (0 \leq t_1 \leq \cdots \leq t_n \leq 1)) \mapsto (E_j, -\tan(\pi t_j))_{1 \leq j \leq n}.
\]

Here we have fixed an identification between \( \langle n \rangle \) and \( \{1, \cdots, n, \ast \} \), and \((E_j, -\tan(\pi t_j))_{1 \leq j \leq n} \) denotes for the spectral decomposition of the image Lagrangian. Here we adopt the convention that the orthogonal complement of \( \bigoplus_{j=1}^n E_j \) has eigenvalue 0. Since the map (2.2) is compatible with the gluing map for the geometric realization, it factors through \( BG^{(1)} \) and gives the desired homomorphism. The group completion \( G^{(1)} \to \Omega BG^{(1)} \simeq \Omega(U/O) \) takes \( E \in G^{(1)} \) to the based loop \( \ell(t), t \in [0, 1] \), where \( \ell(t) \) has spectral decomposition \( (E, -\tan(\pi t)) \).

To see the equivalence \( B(Z \times BO) \simeq U/O \), one considers the following \( \text{Fin}_* \)-topological categories (the objects form a space rather than just a set)

\[
\tilde{G}^{(0)} = \text{pt},
\]

\[
\tilde{G}^{(n)} := \left\{ \begin{array}{ll}
\text{Objects : } (E_j, E'_j)_{j \in \langle n \rangle^o} : E_j, E'_j \subset \lim_{N \to \infty} \mathbb{R}^N \text{ finite dimensional, } E_j \perp E'_j, E'_j \perp E'_k \text{ for } j \neq k \\
\text{Hom}((E_j, E'_j), (F_j, F'_j)) = \left\{ \begin{array}{ll}
\ast, \text{ if } \exists W_j \perp E_j, E'_j \text{ with } E_j \oplus W_j = F_j, E'_j \oplus W_j = F'_j \\
\emptyset, \text{ otherwise}
\end{array} \right. 
\end{array} \right.
\]

\( \tilde{G}^{(n)} \) denoted for the spectral decomposition of the image Lagrangian. Here we adopt the convention that the orthogonal complement of \( \bigoplus_{j=1}^n E_j \) has eigenvalue 0. Since the map (2.2) is compatible with the gluing map for the geometric realization, it factors through \( BG^{(1)} \) and gives the desired homomorphism. The group completion \( G^{(1)} \to \Omega BG^{(1)} \simeq \Omega(U/O) \) takes \( E \in G^{(1)} \) to the based loop \( \ell(t), t \in [0, 1] \), where \( \ell(t) \) has spectral decomposition \( (E, -\tan(\pi t)) \).

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\emptyset, \text{ otherwise}
\end{array} \right. 
\end{array} \right.
\]
Taking the geometric realization of the Fin$_*$-space $|\tilde{G}^\bullet|$, where each $|\tilde{G}^{(n)}|$ means the classifying space of the topological category in the sense of [Seg68], i.e. the geometric realization of its nerve, gives $B|\tilde{G}^{(1)}| \simeq B(\mathbb{Z} \times BO)$. There is a homomorphism from the geometric realization of $|\tilde{G}^\bullet|$ to $(U/O \times U/O)/U/O \simeq U/O$, where $U/O$ acts on the two factors by the diagonal action of taking direct sum of Lagrangian subspaces.

3. Microlocal sheaf categories for Lagrangian submanifolds

In this section, we first recall the construction of the sheaf of microlocal categories on a smooth Lagrangian following [KaSc], which generalizes to the ring spectra setting without much change (for more details see [JiTr]), then we recall the notion of Morse transformations [Jin] and make connections to the space of paths in $U/O$ ending at the zero-section. Lastly, we briefly review the category of correspondences and collect the results that we will need.

3.1. Basics in microlocal sheaf theory. For any smooth manifold $Y$ and a commutative (or at least $E_2$) ring spectrum $k$, let $\text{Shv}(Y; k)$ be the stable $\infty$-category of all sheaves valued in $k$-spectra on $Y$. The notion of singular support for any sheaf over an ordinary ring defined in [KaSc] can be extended in the spectra setting without change.

Definition 3.1. [KaSc, Definition 5.1.2] For any sheaf $\mathcal{F} \in \text{Shv}(Y; k)$, a covector $(x, \xi) \in \tilde{T}^*Y = T^*Y \setminus T^*_Y$ is not in the singular support of $\mathcal{F}$, which is denoted by $SS(\mathcal{F})$, if there exists an open neighborhood $U$ of $(x, \xi)$ such that for any covector $(\tilde{x}, \tilde{\xi}) \in U$ and any smooth function $f : X \to \mathbb{R}$ satisfying $f(\tilde{x}) = 0$, $df(\tilde{x}) = \tilde{\xi}$, we have

$$\Gamma_{\{f \geq 0\}}(\mathcal{F})|_{\tilde{x}} \simeq 0.$$ 

It is clear from the definition that $SS(\mathcal{F})$ is a closed conic subset of $\tilde{T}^*Y$. A deep theorem of Kashiwara–Schapira [KaSc] asserts that $SS(\mathcal{F})$ is always coisotropic. For any closed conic (coisotropic) subset $C \subset \tilde{T}^*Y$, define

$$\text{Shv}_C(Y; k) := \{ \mathcal{F} \in \text{Shv}(Y; k) : SS(\mathcal{F}) \subset C \} \subset \text{Shv}(Y; k),$$

which is a full stable subcategory of $\text{Shv}(Y; k)$ closed under infinite direct sums.

We remark that the parts of microlocal sheaf theory that we will rely on throughout the paper are the functorial properties of singular support under the six functors, which are contained in [KaSc, Section 5.4] (see also [Tam1, Appendix 2]). Although the original proofs are for sheaves over ordinary rings and for the bounded derived category, all the arguments can be carried out for the spectra setting without essential change. In the case when $C = \Lambda$ is a conic Lagrangian, $\text{Shv}_\Lambda(Y; k)$ is relatively easy to understand, which only has weakly constructible sheaves, and this is the situation that we will mostly focus on.

3.2. The sheaf of microlocal categories on a smooth Lagrangian. Let $X$ be a smooth manifold. The cotangent bundle $T^*X$ is an exact symplectic manifold, in the sense that the symplectic form $\omega = dq \wedge dp = \sum_j dq_j \wedge dp_j$ has a primitive $\alpha = pdq$ (up to a negative sign), where $(q, p)$ is any local Darboux coordinate. For a possibly non-conic
exact Lagrangian \( L \subset T^*X \), i.e. \( \alpha|_L \) is an exact 1-form, there is a standard way to turn it into a conic Lagrangian in a larger cotangent bundle as follows.

Let \( T^*_{<0}(X \times \mathbb{R}_t) \) denote for the negative half of the cotangent bundle \( T^*(X \times \mathbb{R}_t) \) consisting of covectors that have strictly negative coefficients in the \( dt \)-factor. For a smooth exact Lagrangian \( L \subset T^*\mathbb{R}^N \), fix a primitive \( f_L \) for \( \alpha|_L \), i.e. \( \alpha|_L = df_L \). Let \( L \) be the associated conic Lagrangian lifting in \( T^*_{<0}(\mathbb{R}^N \times \mathbb{R}_t) \) defined by

\[
L =: \{(q, p; t, \tau) : (q, -p/\tau) \in L, t = f_L(q, p), \tau < 0\} \subset T^*\mathbb{R}^N \times \mathbb{R}_t \times \mathbb{R}_{\tau<0}.
\]

It is usually convenient to view \( L \) as the cone over the Legendrian \( L_{\tau=1} \) in the 1-jet space \( (T^*\mathbb{R}^N \times \mathbb{R}_t, -dt + \alpha) \). For any open subset \( U \subset \mathbb{R}^N \), we can embed it into \( \mathbb{R}^N \times \mathbb{R}_t \times \mathbb{R}_{\tau<0} \).

For simplicity, in the following discussions, we will restrict to the case where \( x = \mathbb{R}^N \) (or denoted by \( V \) as a vector space). Eventually, the proof of our main theorem reduces to the \( \mathbb{R}^N \) case, since for any smooth manifold \( X \), we can embed it into \( \mathbb{R}^N, N \gg 0 \) and the case for \( X \) follows easily then (see Subsection 4.1.5 for the details).

Following the work [KaSc], one can define a sheaf of microlocal sheaf categories on \( L \) (over some fixed ring spectrum \( k \)) as follows. For any open subset \( \mathcal{U} \subset L \), set

\[
\mu\text{Shv}^\text{pre}_L(\mathcal{U}; k) := \lim_{\Omega \in \text{ConicOpen} \{ (T^*(R^N \times \mathbb{R}_t)) \}} \text{Shv}_{\text{Cone}(\mathcal{U}); \Omega}(R^N \times \mathbb{R}_t; k)/\text{Shv}_{\Omega}(R^N \times \mathbb{R}_t; k),
\]

and the restriction maps are the natural ones. By the invariance of microlocal sheaf categories under quantized contact transformations, for sufficiently small \( \mathcal{U} \subset L \), we have a non-canonical equivalence \( \mu\text{Shv}^\text{pre}_L(\mathcal{U}; k) \simeq \text{Loc}(\mathcal{U}; k) \) compatible with restrictions (see Section 3.4 for more details). This implies that the sheafification of \( \mu\text{Shv}^\text{pre}_L \), denoted by \( \mu\text{Shv}_L \), is a locally constant sheaf of categories \( \mu\text{Shv}_L \) on \( L \) with fiber equivalent to \( \text{Mod}(k) \).

Note that since the construction of \( \mu\text{Shv}_L \) comes from local data, one can drop the exact condition on the Lagrangian, and the definition also makes sense for a smooth Lagrangian immersion.

3.3. An alternative construction of \( \mu\text{Shv}_L \). We present an equivalent construction of \( \mu\text{Shv}_L \) for a smooth Lagrangian \( L \subset T^*\mathbb{R}^N \) based on weighted blow up, which has the advantage of avoiding the quotient by \( \text{Shv}_{\Omega}(\mathbb{R}^N \times \mathbb{R}_t; k) \) in (3.1), and which later gives us a convenient and precise setting to perform Morse transformations (cf. Section 3.4). The approach presented here is largely inspired by a course given by D. Tarmarkin in 2016.

Let \( V = \mathbb{R}^N \). Let

\[
\mathcal{V}_L = L \times V \times \mathbb{R}_t, \quad \text{and} \quad \pi_L : \mathcal{V}_L \to L, \quad \pi_{V \times \mathbb{R}_t} : \mathcal{V}_L \to V \times \mathbb{R}_t
\]
be the obvious projections. Assume as before that $L$ is exact and fix a primitive $f_L$, and let
\[
\iota_{f_L} : L \hookrightarrow \mathcal{V}_L
\]
\[(q, p) \mapsto (q, p; f_L(q, p))
\]
be the “diagonal” embedding. Now we do a version of deformation to the normal cone with respect to the following $\mathbb{R}_+$-action on the normal bundle of $\iota_{f_L}$, whose fiber at $\iota_{f_L}(q_0, p_0)$ is canonically identified with $V \times \mathbb{R}_t$ with shifted center at $(q_0, f_L(q_0, p_0))$:
\[
s \cdot (q - q_0, t - f_L(q_0, p_0)) = (s(q - q_0), s p_0 \cdot (q - q_0) + s^2(t - f_L(q_0, p_0) - p_0 \cdot (q - q_0)),
\]
for $(q, t) \in V \times \mathbb{R}_t$, $s \in \mathbb{R}_+$. The outcome is
\[
\tilde{\mathcal{V}}_L = \mathcal{B}_{\iota_{f_L}(L) \times \{0\}}^{\text{weighted}}(\mathcal{V}_L \times \mathbb{R}_{\geq 0}) - (\mathcal{V}_L - \iota_{f_L}(L)) \times \{0\} \to \mathbb{R}_{\geq 0}
\]
with an open embedding
\[
j : \mathcal{V}_L \times \mathbb{R}_{>0} \hookrightarrow \tilde{\mathcal{V}}_L,
\]
and a closed embedding
\[
i_0 : \mathcal{V}_L \hookrightarrow \tilde{\mathcal{V}}_L
\]
induced from the natural identification between the central fiber and $\mathcal{V}_L$.

Taking the tangent space of $L$ for each $x \in L$ gives the (affine) Gauss map from $L$ to the affine Lagrangian Grassmannian
\[
L \to \text{LagGr}_{\text{aff}}(T^*V)
\]
\[x \mapsto \ell_x = T_x L.
\]
By choosing the primitive of $\alpha|_{\ell_x}$ such that its value at $x$ equals $f_L(x)$, we get a family of Legendrian liftings of the affine Lagrangians $\ell_x, x \in L$. Let
\[
L_{\text{Gauss}} \subset T^{*<0}(L \times V \times \mathbb{R}_t)
\]
be the cone over the smooth closed Legendrian in the 1-jet bundle $T^*L \times T^*V \times \mathbb{R}_t$ from this family of Legendrian liftings.

For any smooth manifold $Y$ and closed conic set $C \subset T^{*<0}(Y \times \mathbb{R}_t)$, following [Tam1], define
\[
\text{Shv}^{\geq 0}(Y \times \mathbb{R}_t; k) =: \text{Shv}_{T^{*<0}(Y \times \mathbb{R}_t)}(Y \times \mathbb{R}_t; k),
\]
\[
\text{Shv}^{<0}(Y \times \mathbb{R}_t; k) =: \text{Shv}((Y \times \mathbb{R}_t; k)/\text{Shv}^{\geq 0}(Y \times \mathbb{R}_t; k),
\]
\[
\text{Shv}_{C}^{<0}(Y \times \mathbb{R}_t; k) =: \{ F \in \text{Shv}^{<0}(Y \times \mathbb{R}_t; k) : SS(F) \cap T^{*<0}(Y \times \mathbb{R}_t) \subset C \} \subset \text{Shv}^{<0}(Y \times \mathbb{R}_t; k).
\]
Here we set $\text{Shv}^{<0}(Y \times \mathbb{R}_t; k)$ as the right orthogonal complement of $\text{Shv}^{\geq 0}(Y \times \mathbb{R}_t; k)$ (see Theorem [128]).

Let $\pi_{V \times \mathbb{R}_t} : \mathcal{V}_L \times \mathbb{R}_{>0} \to V \times \mathbb{R}_t$ be the projection to the $V \times \mathbb{R}_t$ factor.
Lemma 3.2. For each open $\mathcal{U} \subset L$, there is a natural equivalence
\begin{equation}
\iota_0^* j_* \pi_!^* : \mu \text{Shv}_L(\mathcal{U}; k) \rightarrow \text{Shv}_{L_{\text{Gaus}}}^< (\mathcal{U} \times V \times R_t; k).
\end{equation}

Proof. First, both sides are invariant under compactly supported Hamiltonian isotopies in $T^*V$ (cf. [GKS]), so without loss of generality, we may assume that $L$ is in general position, i.e. its projection to $V$ is finite-to-one. Then we can cut out small open balls $\Omega_\alpha$ in $\mathcal{U} \subset L$ by neighborhoods of the form $B_\alpha \times I_\alpha \subset V \times R_t$, where $B_\alpha \subset V$ and $I_\alpha \subset R_t$ are open balls and intervals respectively (cf. [JiTr]). Then we have
\[ \mu \text{Shv}_L(\Omega_\alpha; k) \simeq \text{Shv}_{\Omega_\alpha}(B_\alpha \times I_\alpha; k)/\text{Loc}(B_\alpha \times I_\alpha; k). \]
Using the above identification, the functor (3.2) with $\mathcal{U}$ replaced by $\Omega_\alpha$ is an equivalence of categories (with both sides non-canonically equivalent to Mod($k$)). Since $\mathcal{U}$ has a complete cover by such $\Omega_\alpha$, and the equivalences are compatible with restrictions for any inclusion $\Omega_\beta \subset \Omega_\alpha$, it induces an equivalence of categories for $\mathcal{U}$.

Let
\[ \mu \text{Shv}_L(\mathcal{U}; k) := \text{Shv}_{L_{\text{Gaus}}}^< (\mathcal{U} \times V \times R_t; k), \]
for any open $\mathcal{U} \subset L$, which is clearly a sheaf of categories on $L$. Then Lemma 3.2 assures that $\mu \text{Shv}_L \simeq \mu \text{Shv}_L$. It is also immediate from the construction and the $R$-equivariance of $\text{Shv}_{L_{\text{Gaus}}}^< (\mathcal{U} \times V \times R_t; k)$ under any shift of the primitive $f_L|_{\mathcal{U}}$ by a constant, that $\mu \text{Shv}_L(\mathcal{U}; k)$ is also well defined for any smooth (not necessarily exact) Lagrangian immersion. In the following, we will constantly use $\mu \text{Shv}_L$ as our definition for the sheaf of microlocal categories on $L$.

3.4. Morse transformations and paths in $U/O$. Recall the notion of Morse transformations (cf. [Jin Section 5.1]). Thanks to the construction in Subsection 3.3, we only need to focus on Morse transformations for any conic lifting of an affine Lagrangian $\ell \subset T^*R^N$ that are the negative conormal bundles of smooth hypersurfaces in $R^N_{q_0} \times R^N_{t_0} \times R^N_{q_1} \times R^N_{t_1}$ of the form (up to some shifts in coordinates)
\[ t_1 - t_0 - (Q + \frac{1}{2} A_\ell)|q_0| - q_0 \cdot q_1 = 0, \]
for some nondegenerate quadratic form $Q$ on $\pi(\ell)$ and $A_\ell$ is as in Subsection 2.1 (after parallel shifting $\ell$ to be a linear Lagrangian). Here the negative conormal bundle means the part of conormal bundle of the hypersurface inside $(T^*<0(R^N_{q_0} \times R^N_{t_0}))^{-} \times T^*<0(R^N_{q_1} \times R^N_{t_1})$. One should think of each Morse transformation corresponding to a smooth Lagrangian in $T^*(R^N_{q_0} \times R^N_{q_1})$ of the form
\begin{equation}
\text{Graph}(d((Q + \frac{1}{2} A_\ell)|q_0| + q_0 \cdot q_1)),
\end{equation}
which is a symplectomorphism that has a generating function in $q_0, q_1$. We stick to the convention that for a quadratic form $Q$, thought of as a symmetric matrix, $Q[q] = q^T \cdot Qq$ means the value of the quadratic function, that is to distinguish from the matrix-vector product $Qq$ (which will also show up).
The key point of considering Morse transformations is that they provide local trivializations of $\mu_{Shv_L}$. For any open $U \subset L$, if there is a Morse transformation $L_{01}$ for $U$ with underlying smooth hypersurface $H$, then the correspondence
\begin{align*}
\xymatrix{ U \times H & \ar[l]_p U \times R_q^N \times R_{t_1} & \ar[r]^{p_0} U \times R_q^N \times R_{t_0} }
\end{align*}
induces an equivalence of categories
\begin{equation}
(3.4) \quad (p_1)_* p_! : \mu_{Shv_L}(U; k) \longrightarrow Shv^<_{0}((\Delta^T U) \times L_{01}) \circ L_{Gauss} (U \times R_q^N \times R_{t_1}; k).
\end{equation}

Since $((\Delta^T U) \times L_{01}) \circ L_{Gauss}|_U$ is the negative conormal bundle of a smooth hypersurface whose projection to $U$ is a trivial fibration, the right-hand-side of (3.5) is naturally identified with $Loc(U; k)$.

Now we will explain some close relation between Morse transformations and paths in $U/O$ ending at the base point (the stable zero-section). Technically, we should use the stable affine Lagrangian Grassmannian instead of $U/O$, but since the difference is trivial at the topological level, we will always ignore such issues. It is clear from (3.3) that the underlying symplectomorphism of a Morse transformation is the composition of the Fourier transform, denoted by $FT$, and the time-1 Hamiltonian map of the quadratic Hamiltonian function purely in $p$, given by $(Q + \frac{1}{2} A_\ell)[p]$.

Let
\begin{equation}
(3.6) \quad FT_t = \phi^{2t}_\frac{1}{2}(p^2 + q^2), 0 \leq t \leq 1
\end{equation}
be the path of linear symplectomorphism with $FT_0 = id$ and $FT_1 = FT$, given by the Hamiltonian flow of the function $\frac{1}{2}(p^2 + q^2)$. If we think of the contractible locus of graph like linear Lagrangians in $U/O$ as a “fat” base point, then we can assign each Morse transformation the following path in $U(\mathbb{N})/O(\mathbb{N}) \subset U/O$ starting from $\ell$ and ending at the base point:
\begin{equation*}
(\phi_{Q + \frac{1}{2} A_\ell} \bullet FT_t)(\ell) := \begin{cases} 
FT_{2t}, & 0 \leq t \leq \frac{1}{2} \\
\phi_{Q+ \frac{1}{2} A_\ell}^{2t-1} \circ FT, & \frac{1}{2} \leq t \leq 1.
\end{cases}
\end{equation*}
This point of view is a key ingredient in our proof for the main theorem.

3.5. **The category of correspondences and the functor $ShvSp^I$.** In this subsection, we recall some basic facts about the correspondence category defined in [GaRo], and the symmetric monoidal functor $ShvSp^I$ that we will use later in the proof of the main theorem. In [Jin], we proved based on [GaRo] and [Lu1] that there is a canonically defined
By the same approach, one gets the canonical symmetric monoidal functor
\[
\text{ShvSp}_! : \text{Corr}(S_{LCH})_{\text{all, all}}^{\text{prop}} \to (\text{Pr}_{\text{st}}^L)^{2-\text{op}}
\]
\[
X \mapsto \text{Shv}(X; \text{Sp})
\]
\[
\begin{array}{ccc}
Z & \xrightarrow{p} & X \\
\downarrow{q} & & \downarrow{q} \\
Y & & \\
\end{array}
\]
\[
\left( Z \xrightarrow{p} X \right) \mapsto \left( q_! p^* : \text{Shv}(X; \text{Sp}) \to \text{Shv}(Y; \text{Sp}) \right).
\]

The second functor is the one that we will use later.

Recall that in [Jin, Theorem 2.6, Theorem 2.11, Theorem 2.21, Proposition 2.22], we provided a list of constructions of commutative algebra objects, their modules and (right-lax) homomorphisms among them in \(\text{Corr}(\mathcal{C})\), for \(\mathcal{C}\) a Cartesian symmetric monoidal \(\infty\)-category, out of concrete data including \(\text{Fin}^*\)-objects, \(\text{Fin}_{s, !}\)-objects and correspondences among them. We will frequently use these, through the functoriality of \(\text{ShvSp}_!\), to present symmetric monoidal structures on certain microlocal sheaf categories and (right-lax) homomorphisms among them. Note that in this setting, tensor product in a sheaf category should be taken as \(\otimes\). On the other hand, (almost) all the sheaf categories that we are interested in will be some full subcategories with singular support condition in conic Lagrangians, restricted to which the functor \(q_! p^!\) are continuous, thus (almost) all the categories and functors involved will be lying in \(\text{Pr}_{\text{st}}^L\).

4. Proof of the main theorem

First, let us give a more precise statement of the main theorem and make some remarks.

**Theorem 4.1.** For any smooth immersed Lagrangian \(L\) in \(T^*X\), the classifying map for the sheaf of microlocal categories \(\mu \text{Shv}_L\) over a ring spectrum \(\mathbf{k}\) (commutative or at least \(E_2\)) is homotopic to the composition
\[
L \xrightarrow{\partial \gamma} U/O \xrightarrow{BJ} B\text{Pic}(S) \to B\text{Pic}(\mathbf{k}),
\]
where \(\gamma\) is the stable Gauss map, \(\partial\) is the canonical involution of \(U/O\), and \(BJ\) is the the delooping of the \(J\)-homomorphism.

Here the canonical involution \(\partial\) on \(U/O\) appears due to our presentation of the universal principal \(\Omega(U/O)\)-bundle on \(U/O\) as the space of paths ending at the base point, whose
classifying map differs from $\gamma$, which is the classifying map for the space of paths starting at the base point, by $\vartheta$.

Now let us start the proof of the theorem. For any affine Lagrangian $\ell \subset T^*\mathbb{R}^N$, we fix a Legendrian lifting of $\ell$ in $T^*\mathbb{R}^N \times \mathbb{R}$, denoted by $\Lambda_\ell$. For any quadratic form $Q$ on $T^*\mathbb{R}^N$ depending only on $p$, such that $\varphi_Q(FT(\ell))$ is graph like, it determines a Morse transformation for $\Lambda_\ell$ given by the negative conormal bundle (negative in $dt_1$) of the smooth hypersurface

$$-t_1 + t_0 + Q[q_0] + q_0 \cdot q_1 = 0$$

in $\mathbb{R}^N_q \times \mathbb{R}^N_{q_1} \times \mathbb{R}_{t_0} \times \mathbb{R}_{t_1}$. Let $L_{01,Q}^N$ denote for the corresponding Morse transformation.

4.1. A commutative monoid $G_{Q_0,\ell}$ arising from a Morse transformation $L_{01,Q_0}$. For any (stable) affine Lagrangian $\ell$, let $\text{Spectr}(\ell)$ be the spectral decomposition of $\ell$ (after a parallel shifting to be a linear Lagrangian), i.e. the sum of that of $A_\ell$ and the eigenspace of $\infty$, and let $\text{Spectr}^-\ell)$ (resp. $\text{Spectr}^+\ell)$, $\text{Spectr}^∞\ell)$, $\text{Spectr}^1\ell)$ be the negative (resp. positive, $\infty$, $I \subset \mathbb{R} \cup \{\infty\}$) part of the spectral decomposition. The same notation (except for $\text{Spectr}^∞$) applies to (usual) quadratic forms, i.e. for a quadratic form $Q$, $\text{Spectr}^-Q)$ (resp. $\text{Spectr}^+Q)$) denotes for its negative (resp. positive) spectral part. For a nonnegative quadratic form $Q$, we also call the underlying vector space of $\text{Spectr}^1Q)$ the support of $Q$, and denote it by $\text{Supp}(Q)$. For the corresponding underlying vector space of a spectral part, we replace $\text{Spectr}$ in the notation by $\text{Spectr}$.

We fix the following notions. We say a quadratic form $Q_0$ is negatively (resp. nonpositively, etc.) stable on $\lim R^N \cong \lim R^N^*$ (under the Euclidean metric), if there exists $-\epsilon$ with $\epsilon > 0$(resp. $\epsilon \geq 0$, etc.) such that for $N \gg 0$,

$$Q_0|_{R^{N+k} = Q_0|_{R^N} \oplus (-\epsilon)I_{(R^N)^k}, \forall k \geq 0}.$$

For any negatively stable $Q_0$ and any affine Lagrangian $\ell$ such that $\varphi_{Q_0} \circ FT(\ell)$ is graph like, we assign the stable Morse transformation $L_{01,Q_0}$ for $A_\ell$, whose underlying symplectomorphism is $\varphi_{Q_0} \circ FT$. Here for any quadratic function $Q$ on $\lim R^N \cong \lim R^N^*$,

$$\varphi_Q = \text{ the time-1 map of the Hamiltonian flow of } Q[p].$$

The following constructions can be thought of as a more formal treatment of the relation between Morse transformations and paths in $U/O$, as discussed in Subsection 3.4.

For any stable Morse transformation $L_{01,Q_0}$ for $A_\ell$, let

$$G_{Q_0,\ell,N}^{(0)} = pt,$$

$$G_{Q_0,\ell,N}^{(n)} = \{(Q_i)_{i \in (n)^o} : \text{each } Q_i \text{ is a nonnegative quadratic form supported on } \mathbb{R}^N;$$

$$\text{rank}(\text{Spectr}^- \varphi \sum_{i \in (n)^o} Q_i \circ \varphi_{Q_0}(FT(\ell)))) = \sum_{i \in (n)^o} \text{rank}(Q_i) + \text{rank}(\text{Spectr}^- \varphi_{Q_i}(FT(\ell))))}.$$
For any nonnegative quadratic form $Q$, we have
\[
\text{rank}(\text{Spectr}^-(\varphi_{Q_1} \circ \varphi_{Q_2}(\text{FT}(\ell)))) \leq \text{rank}(Q_1) + \text{rank}(\text{Spectr}^-(\varphi_{Q_2}(\text{FT}(\ell)))).
\]
This in particular implies that for any $(Q_i)_{i \in (n)^{\circ}} \in G_{Q_1,\ell:N}^{(n)}$, we have
\[
\text{rank}(\text{Spectr}^-\varphi_{\sum_{i \in \mathcal{S}} Q_i} \circ \varphi_{Q_2}(\text{FT}(\ell)))) = \sum_{i \in \mathcal{S}} \text{rank}(Q_i) + \text{rank}(\text{Spectr}^-(\varphi_{Q_2}(\text{FT}(\ell))))
\]
for any subset $\mathcal{S} \subset (n)^{\circ}$. So we can organize $G_{Q_1,\ell:N}^{(n)}$, $n \in \mathbb{Z}_{\geq 0}$ (resp. $V G_{Q_2,\ell:N}^{(n)}$, $n \in \mathbb{Z}_{\geq 0}$) into a $	ext{Fin}_*\text{-}object$ in the way that for any $f : \langle n \rangle \to \langle m \rangle$, we associate
\[
G_{Q_1,\ell:N}^{(n)} \longrightarrow G_{Q_2,\ell:N}^{(m)}
\]
and similarly for $V G_{Q_1,\ell:N}^{(n)}$, $n \in \mathbb{Z}_{\geq 0}$.

**Lemma 4.2.**

(a) For any nonnegative quadratic form $Q$ on $\mathbb{R}^N$ of rank $k$ and for any partition $(k_1, \cdots, k_n)$ of $k$ of size $n$, the space of $n$-tuples of nonnegative quadratic forms $(Q_1, \cdots, Q_n)$ satisfying that
\[
\sum_{i=1}^{n} Q_i = Q, \quad \text{rank}(Q_i) = k_i, \quad i = 1, \cdots, n,
\]
is isomorphic to the partial flag variety $\text{Fl}_{k_1,k_1+k_2,\cdots}(E_Q)$.

(b) For any active map $\langle n \rangle \to \langle m \rangle$, the map $G_{Q_1,\ell:N}^{(n)} \to G_{Q_2,\ell:N}^{(m)}$ is a proper fiber bundle over a subset of connected components of $G_{Q_2,\ell:N}^{(m)}$, and the fiber over any $(Q_i) \in G_{Q_2,\ell:N}^{(m)}$ in the image is diffeomorphic to the product of disjoint unions of some partial flag varieties of the underlying subspace $\text{Supp}(Q_i)$.

**Proof.** (a) Up to congruent relation, we may assume that $Q$ has eigenvalues 0 and 1, and is supported on $E \subset \mathbb{R}^N$. Then it is easy to see that the decomposition in (4.3) is the same as decomposing $E$ into mutually orthogonal subspaces $E_i$ with $\dim E_i = k_i$, $1 \leq i \leq n$. So the lemma follows.

(b) follows directly from (a). \qed

An immediate corollary of Lemma 4.2 and [Jin, Theorem 2.6] is

**Proposition 4.3.** The $	ext{Fin}_*\text{-}objects$ $G^*_{Q_1,\ell:N}$ and $V G^*_{Q_2,\ell:N}$ represent commutative algebra objects in Corr($S_{\text{LCH}}^{\text{fib,all}}$).
Recall the following notions from [Jin]:
\[ \langle n \rangle_1 = \langle n \rangle \cup \{ \dagger \}; \]
a morphism
\[ f : \langle n \rangle_1 \to \langle m \rangle_1 \]
is a map satisfying \( f(*) = *, f(\dagger) = \dagger \); these together defines an ordinary category \( \text{Fin}_{*,\dagger} \) with objects \( \langle n \rangle_1, n \geq 0 \) and morphisms as above.

Now we endow \( G_{Q_b,\ell;N} \times R^N_{q_1} \times R^\ell_{t_1} \), with the following module structure over \( VG_{Q_b,\ell;N} \) through [Jin] Theorem 2.21:

\[ (G_{Q_b,\ell;N} \times R^N_{q_1} \times R^\ell_{t_1})^\langle (n) \rangle_1 = \{(Q_1, p_1, \cdots, Q_n, p_n; Q_1, q_1, t_1) : (Q_j)_{\bullet \in \langle n \rangle_1^o} \in G_{Q_b,\ell;N}^{(n+1)}; p_i \in \text{Supp}(Q_i), i \in \langle n \rangle_1^o \}, \]
and for any \( f : \langle n \rangle_1 \to \langle m \rangle_1 \),
\[ (G_{Q_b,\ell;N} \times R^N_{q_1} \times R^\ell_{t_1})^\langle (n) \rangle_1 \longrightarrow (G_{Q_b,\ell;N} \times R^N_{q_1} \times R^\ell_{t_1})^\langle (m) \rangle_1 \]

\[ ((Q_1, p_1, \cdots, Q_n, p_n; Q_1, q_1, t_1)) \mapsto (( \sum_{j \in f^{-1}(i)} Q_j, \sum_{j \in f^{-1}(i)} p_j)_{i \in \langle m \rangle_1^o}; Q_1 + \sum_{j \in f^{-1}(\{\dagger\}) \setminus \{\dagger\}} Q_j, q_1 - 2 \sum_{j \in f^{-1}(\{\dagger\}) \setminus \{\dagger\}} p_j), \]

where \( p_j \) is any vector satisfying that \( Q_j p_j' = p_j \). By the assumption on \( Q_j, j \in \langle n \rangle_1^o \), there exists a vector \( p' \) satisfying \( Q_j p' = p_j, j \in f^{-1}(\{\dagger\}) \setminus \{\dagger\} \), hence we have \( \sum_{j \in f^{-1}(\{\dagger\})} Q_j[p_j'] = (\sum_{j \in f^{-1}(\{\dagger\})} Q_j)[p'] \). This shows that the module structure (4.6) is well defined.

For any \( G_N \subset G_{Q_b,\ell;N} \) (note that \( G_N \) here is not specific to the particular model of \( \bigsqcup_{n \leq N} BO(n) \) as in [Jin]), if for any inert map \( \langle n \rangle \to \langle k \rangle \) in \( \text{Fin}_{\ast} \), the induced map
\[ G^\langle (n) \rangle_{Q_b,\ell;N} \to G^\langle (k) \rangle_{Q_b,\ell;N} \]
has image in \( G^\langle (n) \rangle_{G_{Q_b,\ell;N}} \times G^\langle (k) \rangle_{G_{Q_b,\ell;N}} \)

where the map \( G^\langle (n) \rangle_{Q_b,\ell;N} \to G_{Q_b,\ell;N} \) is through the unique active map \( \langle n \rangle \to \langle 1 \rangle \) then we can define the \( \text{Fin}_{\ast} \)-object
\[ G^\bullet_N = G_N \times_{G^\langle (n) \rangle_{G_{Q_b,\ell;N}}} G^\langle (k) \rangle_{G_{Q_b,\ell;N}}; \]

which is automatically a commutative algebra object in \( \text{Corr}(S_{LCH})_{\text{fib}} \), equipped with an algebra homomorphism \( G_N \to G_{Q_b,\ell;N} \). Similarly, the \( \text{Fin}_{\ast} \)-object \( VG^\bullet_N = G_N \times_{G^\langle (n) \rangle_{G_{Q_b,\ell;N}}} VG^\langle (k) \rangle_{G_{Q_b,\ell;N}} \)

\( VG^\bullet_{Q_b,\ell;N} \) is a commutative algebra object in \( \text{Corr}(S_{LCH})_{\text{fib}} \), and we can uniquely associate a module structure on \( G_N \times R^N_{q_1} \times R^\ell_{t_1} \) over \( VG_N \) as above in \( \text{Corr}(S_{LCH}) \).
If we choose \( Q_♭ \) to be a negatively stable quadratic Hamiltonian, then we let

\[
G_{Q_♭, \ell} = \bigcup_N G_{Q_♭, \ell; N}, \quad VG_{Q_♭, \ell} = \bigcup_N VG_{Q_♭, \ell; N}.
\]

Though these no longer give objects in \( \text{Corr}(S_{LCH}) \) (they are like ind-objects), a standard way to view and study them is through the truncations at finite levels \( N \).

4.2. The fibrant simplicial category \( \text{QHam}(U/O) \).

**Definition 4.4.** Consider the following fibrant simplicial category, denoted by \( \text{QHam}(U/O) \):

(i) The objects are \( (U, Q_♭, G^\bullet, M^\bullet; \dagger) \), where \( U \subset U/O \), \( Q_♭ \) is a negatively stable quadratic Hamiltonian depending only on \( p \), and

\[
(G^\bullet = \bigcup_{N \geq 0} G^\bullet_N, M^\bullet; \dagger = \bigcup_{N \geq 0} M^\bullet_N)
\]

is a pair of \( \text{Fin}_s \) and \( \text{Fin}_{s, \dagger} \) objects in the ordinary category of paracompact Hausdorff spaces, with a filtration given by \( \text{Fin}_s \) and \( \text{Fin}_{s, \dagger} \) objects \( (G^\bullet_N, M^\bullet; \dagger_N) \) in \( S_{LCH} \), that determine \( \text{Fun}(Z_{\geq 0}, \text{Mod}(\text{Corr}(S_{LCH}))) \).

They need to satisfy the following conditions.

ia) The Hamiltonian map \( \varphi_{Q_♭} \) takes the Fourier transform of all elements in \( U \) to graph type linear Lagrangians.

ib) We have \( G^\bullet \subset \bigcap_{\ell \in U} G^\bullet_{Q_♭, \ell}, \quad G^\bullet_N = G^\bullet \cap \bigcap_{\ell \in U} G^\bullet_{Q_♭, \ell; N} \) and the diagram

\[
\begin{array}{ccc}
G^{(n)} & \longrightarrow & G^{(n)}_{Q_♭, \ell} \\
\downarrow & & \downarrow \\
G^{(1)} & \longrightarrow & G^{(1)}_{Q_♭, \ell}
\end{array}
\]

is Cartesian for every \( n \) and \( \ell \in U \). Equivalently, this means the embeddings \( G^\bullet_N \subset G^\bullet_{Q_♭, \ell; N} \) induce commutative algebra homomorphisms from the latter to the former in \( \text{CAlg}(\text{Corr}(S_{LCH})) \). We require that \( G \hookrightarrow G_{Q_♭, \ell} \) is a homotopy equivalence and the positive eigenvalues of elements in \( G \) have a uniform finite upper bound, and the above implies that

\[
\prod_{j \in (n)^\circ} G^{(j, \ast)}_{Q_♭, \ell} \hbox{h.e.} \simeq G^{(n)}_{Q_♭, \ell} \hbox{h.e.} \simeq G_{Q_♭, \ell} \hbox{h.e.} \prod_{j \in (n)^\circ} G^{(j, \ast)}_{Q_♭, \ell}.
\]

ic) For any \( \ell \in U \), let \( V^-_{\ell, Q_♭} = \text{Spectr}^- (\varphi_{Q_♭}(\text{FT}(\ell))) \). We require that for any subspace \( E \subset V^-_{\ell, Q_♭} \) the space of subspaces that can be realized as graphs of linear functions from \( E \) to \( E^\perp \) (in \( \lim \frac{R^N}{N!} \)) and that intersect trivially with the support of every element in \( G \) is contractible.
(ii) A morphism \(j_{Q_{12}}: (U_1, Q_1^{(1)}, G_1^*, M_1^{* \dagger}) \to (U_2, Q_2^{(2)}, G_2^*, M_2^{* \dagger})\), depending on a nonnegative quadratic form \(Q_{12} \in G_2\), consists of the following data.

\(\text{iiia) } U_2 \subset U_1, \ G_1^* \text{ is included in } G_2^*, \text{ and } j_{Q_{12}} \text{ determines a morphism } (G_1^*, M_1^{* \dagger}) \to (G_2^*, M_2^{* \dagger}) \text{ in Mod(Corr}(S_{LCH}))\), sending \(Q_1^{(1)}\) to \(Q_2^{(2)} + Q_{12}\).

\(\text{iiib) we have } \bar{Q} = Q_2^{(2)} + Q_{12} - Q_1^{(1)}, \)

is nonnegative and it satisfies

\[ \dim \Spectr^-(\varphi_{Q_1^{(1)} + Q_2^{(2)}}(\text{FT}(\ell))) = \dim \Spectr^-(\varphi_{Q_1^{(1)}}(\text{FT}(\ell))), \]

for any \(\ell \in U_2\). Note that the nonnegativity of \(\bar{Q}\) implies that it has no contribution to any rank equality, i.e.

\[ \dim \Spectr^-(\varphi_{Q_1^{(1)} + Q_2^{(2)}}(\text{FT}(\ell))) = \dim \Spectr^-(\varphi_{Q_1^{(1)}}(\text{FT}(\ell))) = \dim \Spectr^-(\varphi_{Q_1^{(1)}}(\text{FT}(\ell))) + \text{rank}(Q_1), \quad \forall Q_1 \in G_1, \ \ell \in U_2. \]

From this, we see that \(j_{Q_{12}}\) (uniquely) exists if and only if \(G_2 \supset G_1 \cup (G_1 + Q_{12})\), where \(G + Q_{12} = \{Q_1 + Q_{12} : Q_1 \in G\}\).

(iii) For any two objects \((U_1, Q_1^{(1)}, G_1^*, M_1^{* \dagger})\) and \((U_2, Q_2^{(2)}, G_2^*, M_2^{* \dagger})\), we take the space of \(j_{Q_{12}}\) endowed with the induced topology from the space of quadratic forms in \(p\). The composition of morphisms are defined in the most natural way: for any \(j_{Q_{i+1}}: (U_i, Q_i^{(i)}, G_i^*, M_i^{* \dagger}) \to (U_{i+1}, Q_{i+1}^{(i+1)}, G_{i+1}^*, M_{i+1}^{* \dagger}), \ i = 1, 2\), the composition \(j_{Q_{23}} \circ j_{Q_{12}}\) is equal to \(j_{Q_{12} + Q_{23}}\). It is not hard to see that \(j_{Q_{12} + Q_{23}}\) is well defined.
Let $\tilde{Q}_i = Q_s^{(i+1)} + Q_{s,i+1} - Q_s^{(i)}$. Then $\tilde{Q}_1 + \tilde{Q}_2$ satisfies condition $[4,11]$. Indeed,
\[
\dim \text{Spectr}^-(\varphi_{\tilde{Q}_1+\tilde{Q}_2+Q_s^{(i)}}(\text{FT}(\ell))) = \dim \text{Spectr}^-(\varphi_{Q_s^{(2)}+Q_{12}+\tilde{Q}_2}(\text{FT}(\ell)))
\]
\[
= \dim \text{Spectr}^-(\varphi_{Q_s^{(2)}+Q_{12}}(\text{FT}(\ell))) = \dim \text{Spectr}^-(\varphi_{\tilde{Q}_1+Q_s^{(1)}}(\text{FT}(\ell))) = \dim \text{Spectr}^-(\varphi_{Q_s^{(1)}}(\text{FT}(\ell))).
\]

Taking the singular simplicial complex of each mapping space defines the mapping simplicial sets and the composition rules for them.

In the following, unless otherwise specified, we will equally view $Q_{\text{Ham}}(U/O)$ as an $\infty$-category through the coherent nerve functor.

**Lemma 4.5.** For any $\ell_0 \in U/O$, there exists an object $(\mathcal{U}, Q_s, G^\bullet, M^\bullet, \dagger)$ in $Q_{\text{Ham}}(U/O)$ such that $\mathcal{U}$ contains $\ell_0$.

**Proof.** Given any $\ell_0 \in U/O$, choose a negatively stable $Q_s$ such that $\varphi_{Q_s}(\text{FT}(\ell_0))$ is of graph type and has no negative spectrum part. Up to congruent equivalences, we may assume that $\varphi_{Q_s}(\text{FT}(\ell_0))$ has only eigenvalues 0 and/or 1. The Fourier transform of $\varphi_{Q_s}(\text{FT}(\ell_0))$ has spectrum concentrated in $-1$ and $\infty$. Take an open neighborhood $\tilde{\mathcal{U}}$ of $\text{FT}(\varphi_{Q_s}(\text{FT}(\ell_0)))$ in $U/O$ by choosing (i) a small neighborhood $I_{-1}$ and $I_\infty$ of $-1$ and $\infty$ respectively on the spectrum circle in Figure 1; (ii) an open neighborhood $\mathcal{V}$ of $b_0^+$ in $\text{Gr}(\dim b_0^+, \infty)$—then $\tilde{\mathcal{U}}$ consists of linear Lagrangians $\tilde{\ell}$ with $\text{Spectr}^{\infty}\tilde{\ell} \in \mathcal{V}$ and $\text{Spectr}(\tilde{\ell}) = \text{Spectr}^{I_{-1}}(\tilde{\ell}) \oplus \text{Spectr}^{I_{\infty}}(\tilde{\ell})$. Let $\mathcal{U} = \text{FT} \circ \varphi_{-Q_s} \circ \text{FT}(\tilde{\mathcal{U}})$.

![Figure 1. The spectrum of $\text{FT}(\varphi_{Q_s}(\text{FT}(\ell_0)))$](image)

For any fixed $K > 0$, $\epsilon > 0$, let
\[
(4.12) \quad G_{K,\epsilon,R,b_0} = \{Q_1 \in G_{\varphi_{Q_s}, \epsilon_0} : \forall v \in \text{Spectr}^+(Q_1), |\text{proj}_{b_0^+} v| \leq K|\text{proj}_{b_0} v|; Q_1|_{b_0} \geq (1/2 + \epsilon)I_{\text{proj}_{b_0} (\text{Supp}(Q_1))}, Q_1 \leq R \cdot I\}.
\]

Since $G_{K,\epsilon,R,b_0}$ satisfies $(4,7)$, it uniquely determines a commutative algebra (ind-)object in $\text{Corr}(S_{\text{LCH}})$. Let $M^\bullet_{K,\epsilon,R,b_0}$ be the free module of $G^\bullet_{K,\epsilon,R,b_0}$ generated by the element $Q_s$. We show that for appropriate choices of $I_{-1}, I_\infty$ and $\mathcal{V}$, on the above defined $\mathcal{U}$ containing $\ell_0$, $\varphi_{Q_{1+Q_s}}(\text{FT}(\ell)))$ satisfies the rank equality
\[
\dim \text{Spectr}^-(\varphi_{Q_{1+Q_s}}(\text{FT}(\ell))) = \dim \text{Spectr}^-(\varphi_{Q_s}(\text{FT}(\ell))) + \text{rank}(Q_1).
\]

for all $Q_1 \in G_{K,\epsilon,R,b_0}$ and $\ell \in \mathcal{U}$. Moreover, the eigenvalues of $\text{FT}(\varphi_{Q_{1+Q_s}}(\text{FT}(\ell)))$ are away from a fixed open interval containing 0.
For any \( \ell \in \U \), we have
\[
\text{Spectr}(\text{FT}(\varphi_{Q_1 + Q_2}(\text{FT}(\ell)))) = \text{Spectr}(\tilde{\ell} + 2Q_1),
\]
where \( \tilde{\ell} = \text{FT} \circ \varphi_{Q_2} \circ \text{FT}(\ell) \) and \( \tilde{\ell} + 2Q_1 \) means the sum as generalized quadratic forms. We just need to show that for sufficiently small \( I_1, I_\infty \) and \( \mathcal{V} \), we can make
\[
\text{dim Spectr}^+(\tilde{\ell} + 2Q_1) \geq \text{dim Spectr}^+(\tilde{\ell}) + \text{rank}(Q_1),
\]
and
\[
\text{dim Spectr}^-(\tilde{\ell} + 2Q_1) \geq \text{dim Spectr}^-(\tilde{\ell}) - \text{rank}(Q_1).
\]
Note that consequently, the above inequalities are equalities. Consider the subspace \( \text{Spectr}^+(\tilde{\ell}) \oplus \text{proj}_{\text{Spectr}^+(\tilde{\ell})}^1(\text{Spectr}^+(Q_1)) \). By shrinking \( \mathcal{V} \) and \( I_1 \) if necessary, we can find \( \delta > 0 \) such that the restriction of \( Q_1 \in G_{K,\varepsilon,R,b_0} \) to \( \text{proj}_{\text{Spectr}^+(\tilde{\ell})}^1(\text{Spectr}^+(Q_1)) \), \( \tilde{\ell} \in \tilde{U} \), has eigenvalues all greater than \( 1/2 + \delta \), and then \( \tilde{\ell} + 2Q_1 |_{\text{proj}_{\text{Spectr}^+(\tilde{\ell})}^1(\text{Spectr}^+(Q_1))} \) has eigenvalues greater than \( 2\delta \). Since for any \( u \in \text{Spectr}^+(\tilde{\ell}) \) and \( v \in \text{proj}_{\text{Spectr}^+(\tilde{\ell})}^1(\text{Spectr}^+(Q_1)) \),
\[
(\tilde{\ell} + 2Q_1)(u + v, u + v) = (\tilde{\ell} + 2Q_1)(u, u) + (\tilde{\ell} + 2Q_1)(v, v) + 2(\tilde{\ell} + 2Q_1)(u, v)
\geq \text{R}|u|^2 + 2\delta|v|^2 - 4\text{R}|u||v| > 0,
\]
for sufficiently large \( \text{R}' \) achievable by shrinking \( \mathcal{V}, I_1 \) and \( I_\infty \). This shows that \( (\tilde{\ell} + 2Q_1) \) is positive definite on \( \text{Spectr}^+(\tilde{\ell}) \oplus \text{proj}_{\text{Spectr}^+(\tilde{\ell})}^1(\text{Spectr}^+(Q_1)) \), and this confirms (4.13).

In a similar way, we can show that \( \tilde{\ell} + 2Q_1 \) is negative definite on \( \text{Spectr}^-(\tilde{\ell}) \oplus (\text{Spectr}^-(\tilde{\ell}) \cap \text{Spectr}^+(\tilde{\ell}))^\perp \), and so (4.14) is established as well. Along the way, we also see that the eigenvalues of \( \text{FT}(\varphi_{Q_1 + Q_2}(\text{FT}(\ell))) \) are away from the open interval \( (-2\delta, 2\delta) \). The above confirms that \( (\U, Q_0, G_{K,\varepsilon,R,b_0}^*, M_{K,\varepsilon,R,b_0}^*) \) satisfies ia), ib) and ic).

Now we show that condition ic) holds for \( (\U, Q_0, G_{K,\varepsilon,R,b_0}^*, M_{K,\varepsilon,R,b_0}^*) \), provided that \( \U \) has been chosen very small. Let \( B_K(b_{t_0}, b_{t_0}^1) \) be the space of vectors \( v \) satisfying \( |\text{proj}_{b_{t_0}} v| \leq \text{K}|\text{proj}_{b_{t_0}} v| \), i.e. it is the union of the vector subspaces in \( G_{K,\varepsilon,R,b_0}^* \). Given any finite dimensional vector subspace \( E \) very close to \( b_{t_0}^1 \), at any finite level (i.e. \( \text{dim } b_{t_0} < \infty \)), we have the following natural diffeomorphisms
\[
B_K(b_{t_0}, b_{t_0}^1) \cong \text{Cone}(S^{|\text{dim } b_{t_0} - 1|} \times D_K^{|\text{dim } b_{t_0}^1|})
\]
\[
E^\perp \cap B_K(b_{t_0}, b_{t_0}^1) \cong \text{Cone}(S^{|\text{dim } b_{t_0} - 1|} \times D_K^{|\text{dim } E^\perp - \text{dim } b_{t_0}|}),
\]
where one can identify \( D_K^{|\text{dim } E^\perp - \text{dim } b_{t_0}|} \) as the intersection of an affine subspace in \( b_{t_0}^1 \) with \( D_K^{|\text{dim } b_{t_0}^1|} \) (see the illustration in Figure 2). Now to show that ic) holds, we just need to show that there is a homotopy equivalence between the space of vector subspaces of the same dimension as \( E \) that intersect every element in \( G_{K,\varepsilon,R,b_0}^* \) trivially, denoted by \( \text{Gr}(|\text{dim } E; \text{rel } G_{K,\varepsilon,R,b_0}^*|) \), and the open locus \( \text{Hom}(E, E^\perp) \subset \text{Gr}(|\text{dim } E; \infty|) \). This is
Figure 2. The red vector field determines a 1-parameter subgroup $\varphi_t$ in $GL(\mathbb{R}^N)$, compatible with stabilization on $N$, that deformation retracts $B_K(b_{t_0}, b_{t_0}^\perp) \cup E^\perp$ onto $E^\perp$. This induces a homotopy equivalence between $\text{Gr}(\dim E; \text{rel } G_{K,\epsilon,R,b_{t_0}})$ and the open locus $\text{Hom}(E,E^\perp) \subset \text{Gr}(\dim E; \infty)$.

obvious by defining a contracting vector field on $E \oplus (\text{proj}_{E^\perp} b_{t_0})^\perp E^\perp$ that generates a one-parameter subgroup of $GL(\mathbb{R}^N)$.

Therefore, for $U$ sufficiently small, we have $(U, Q, G^\bl, M^\bl, M^{\sharp}) \in \text{QHam}(U/O)$. □

The above lemma shows that $\text{QHam}(U/O)$ is nonempty and its projection to $\text{Open}(U/O)^{op}$ form a basis for the topology of $U/O$.

**Lemma 4.6.** For any $\ell_0 \in U/O$ and any object $(U, Q, G^\bl, M^\bl, M^{\sharp}) \in \text{QHam}(U/O)$ with $\ell_0 \in U$, there exists an object $(U_1, Q^{(1)}_b, G^\bl_1, M^{\sharp}_1)$ and a morphism $j_{Q_1}$ from the former to the latter, such that $\ell_0 \in U_1$ and $\varphi_{Q_1^{(1)}}(\text{FT}(\ell_0))$ has no negative spectral part.

**Proof.** Using congruent equivalences, we may assume that $\text{FT}(\varphi_Q(\text{FT}(\ell_0)))$ has spectral decomposition concentrated in $\pm 1$ and $\infty$. Let $V_1$ (resp. $V_{-1}, V_{\infty}$) be the eigenspace of 1 (resp. $-1$ and $\infty$) of $\text{FT}(\varphi_Q(\text{FT}(\ell_0)))$. Our first observation about $G$ is that for any $Q_1 \in G$, the restriction of $Q_1|_{V_{-1}}$ must have positive eigenvalues bounded below by $1/2 + \epsilon$ for some fixed $\epsilon > 0$, followed from assumption id). Therefore, by the boundedness of the eigenvalues of elements in $G$, we must have the support of all $Q_1 \in G$ contained in the region

$$B_K(V_{-1}, V_1 \oplus V_{\infty}) = \{ v \in \mathbb{R}^N : |\text{proj}_{V_1 \oplus V_{\infty}} v| \leq K |\text{proj}_{V_{-1}} v| \},$$
for some $K > 0$ (shown as the blue cone in Figure 3). Now take $Q_{12}$ to be the nonnegative quadratic form supported on $V_1$ and with eigenvalue 1, then $\varphi_{Q_{12}}(FT(\ell_0))$ is graph like and has spectral decomposition concentrated at 1 and $\infty$.

![Figure 3](image)

**Figure 3.** The blue cone is $B_K(V_1, V_1 \oplus V_\infty)$, and the red cone, with the filled region showing one slice along $V_1 \oplus V_\infty$, is $B_K(V_1 \oplus V_1, V_\infty)$. Consider the sum $G + Q_{12} = \{Q + Q_{12} : Q \in G\}$. Since the support of $Q_{12}$ intersects $\text{Supp}(Q), Q \in G$ trivially, we have the rank equality holds $\text{rank}(Q + Q_{12}) = \text{rank}(Q) + \text{rank}(Q_{12})$ for all $Q \in G$. On the other hand, the forms in $G + Q_{12}$ have eigenvalues bounded above by some $R > 0$, and their support are contained in the cone $B_K(V_1 \oplus V_1, V_\infty)$ (shown as the red cone in Figure 3).

By the relation $b_\ell_0 = V_1 \oplus V_1$, choose sufficiently small $U_1$ containing $\ell_0$, and let $(U_1, Q_1^{(1)}, G_1^*, M_1^{\dagger})$ be $(U_1, Q_0 - Q_{12}, G_{K,e,R,b_0}^*, M_{K,e,R,b_0}^{\dagger})$ as defined in the proof of Lemma 4.5 (more specifically (4.12)), then the morphism

$$j_{Q_{12}} : (U, Q_b, G^*, M^{\dagger}) \to (U_1, Q_1^{(1)}, G_1^*, M_1^{\dagger})$$

satisfies all the conditions in Definition 4.4 (ii). This completes the proof. □

For any $L \subset T^*\mathbb{R}^N$, let $\gamma_L : L \to U/O$ be the stable Gauss map. Let $Q\text{Ham}_L(U/O)$ be the fibrant simplicial category, whose

(i) Objects are $(U \overset{\text{open}}{\subset} U/O, V \overset{\text{open}}{\subset} L, Q_b, G^*, M^{\dagger})$ subject to the condition that $\gamma_L(V) \subset U$ and $(U, Q_b, G^*, M^{\dagger}) \in Q\text{Ham}(U/O)$;

(ii) Mapping simplicial sets are defined by

$$\text{Maps}_{Q\text{Ham}_L(U/O)}((U_1, V_1, Q_1^{(1)}, G_1^*, M_1^{\dagger}), (U_2, V_2, Q_2^{(2)}, G_2^*, M_2^{\dagger})) = \begin{cases} \text{Maps}_{Q\text{Ham}(U/O)}((U_1, Q_1^{(1)}, G_1^*, M_1^{\dagger}), (U_2, Q_2^{(2)}, G_2^*, M_2^{\dagger})), & V_1 \supset V_2; \\ \emptyset, & V_1 \not\supset V_2. \end{cases}$$
We also view $\text{QHam}_L(U/O)$ as an $\infty$-category. There is an obvious functor $p_L : \text{QHam}_L(U/O)^{op} \to \text{Open}(L)$ by the projection to the $\mathcal{V}$ factor. It is a Cartesian fibration: for any object $(\mathcal{U}, \mathcal{V}, Q_0, G^*, M^{\bullet})$ in $\text{QHam}_L(U/O)^{op}$ and any morphism $\mathcal{V}' \to \mathcal{V}$ in $\text{Open}(L)$, the Cartesian morphism over the inclusion is the unique morphism $(\mathcal{U}, \mathcal{V}', Q_0, G^*, M^{\bullet}) \to (\mathcal{U}, \mathcal{V}, Q_0, G^*, M^{\bullet})$.

4.3. A canonical functor $F_L : \text{QHam}_L(U/O)^{op} \to \text{Fun}(\Delta^1, \text{Pr}_N^H)$. We will construct a canonical functor $F_L : \text{QHam}_L(U/O)^{op} \to \text{Fun}(\Delta^1, \text{Pr}_N^H)$ through a set of correspondences. The construction will heavily rely on the main results in [Jin, Section 2] and Appendix [A].

4.3.1. The first set of correspondences. For any $(\mathcal{U}, \mathcal{V}, Q_0, G^*, M^{\bullet})$ in $\text{QHam}_L(U/O)$, let

$$\mathcal{H}_{MN} = \{(q_0, q_1, t_0, t_1, Q_0 + Q_t) : t_1 - t_0 - (Q_0 + Q_t)(q_0) - q_1 \cdot q_0 \leq 0\}$$

$$\subset R^N \times R^N \times R \times R \times M_N.$$

Then $\mathcal{H}_{MN}$ is naturally endowed with a $VG_N$-module structure:

- $\mathcal{H}_{(n)}^{(\mathcal{H})_N} = \{(Q_i, q_{i})_{i \in \langle n \rangle^o} : (q_{0}, q_{1}, t_{0}, t_{1}, Q_0 + Q_t) \in M_{N}^{\langle \mathcal{H} \rangle_{N}}\}$
- $\subset VG_N^\langle\mathcal{H}\rangle_{N} \times \mathcal{H}_{MN}$
- For any $f : \langle m \rangle_{\mathcal{H}} \to \langle n \rangle_{\mathcal{H}}$ in $N(\text{Fin}_{\mathcal{H}})$,

$$\mathcal{H}_{(m)}^{(\mathcal{H})_N} \longrightarrow \mathcal{H}_{(n)}^{(\mathcal{H})_N}$$

$$((Q_i, q_{i})_{i \in \langle m \rangle^o} : (q_{0}, q_{1}, t_{0}, t_{1}, Q_0 + Q_t)) \mapsto ((\sum_{i \in f^{-1}(j)} Q_i, \sum_{i \in f^{-1}(j)} p_i)_{j \in \langle m \rangle^o} : (q_{0}, q_{1} - 2 \sum_{i \in f^{-1}(\ell) \\cap \{\ell\}} p_i, t_0, t_1 - \sum_{i \in f^{-1}(\ell) \\cap \{\ell\}} Q_i[p_i], Q_0 + Q_t + \sum_{i \in f^{-1}(\ell) \\cap \{\ell\}} Q_i),$$

where $p_i'$ is any solution solving $Q_i p_i' = p_i$. It is easy to check that the map is well defined.

- The morphism $\mathcal{H}_{G^*N,M} \to VG^* \circ \pi_{\mathcal{H}}$ is the natural one induced from the projection of $\mathcal{H}_{(n)}^{(\mathcal{H})_N}$ to $VG_N^n$ for each $n$.

It is straightforward to check that $(VG^*, \mathcal{H}_{(n)}^{(\mathcal{H})_N})$ satisfies the conditions in [Jin, Theorem 2.21], so it is naturally an object in $\text{Mod}^{N(\text{Fin}_{\mathcal{H}})}(S_{LCH})$.

For any $Y = (\mathcal{U}, \mathcal{V}, Q_0, G, M) \in \text{QHam}_L(U/O)$, let $S_Y$ be the space of stable nonnegative quadratic form $\tilde{Q}$ such that $\varphi_{Q_0 + Q_t + \tilde{Q}}(\text{FT}(\ell))$ is graph like and satisfies that

$$\text{rank}(\text{Spectr}^-(\varphi_{Q_0 + Q_1 + \tilde{Q}}(\text{FT}(\ell)))) = \text{rank}(\text{Spectr}^-(\varphi_{Q_0 + Q_1}(\text{FT}(\ell))))$$

for all $\ell \in \mathcal{U}$ and $Q_0 + Q_1 \in M$. It is clear that $S_Y$ is contractible, for it admits a deformation retraction to the point $\{\tilde{Q} = 0\}$ through sending $\tilde{Q}$ to $t\tilde{Q}$, $0 \leq t \leq 1$. In the following and later subsections, the contractibility of $S_Y$ will play a role in the definition of the canonical functor $F_L$ [4.35].
For any \(m\)-simplex \(\tau_m\) in \(S_Y\), which is a family \(\tilde{Q}_t\) parametrized by \(v \in |\Delta^m|\), we have the following correspondence of pairs in \(\text{Mod}^{N(\text{Fin}_\ast)}(\text{Corr}(S_{LCH}))\) for any \(N \gg 0\)

\[
\begin{array}{c}
    (V G_N, \mathcal{V} \times \mathcal{H}_{M_N} \times |\Delta^m|) \\
p_1, \tau_m \\
\end{array}
\]

\[
\begin{array}{c}
    (V G_N, \mathcal{V} \times M_N \times R_{q_1}^N \times R_{t_1} \times |\Delta^m|) \\
p_0 \\
\end{array}
\]

(4.19)

\[ (p_t, \mathcal{V} \times R_{q_0}^N \times R_{t_0} \times |\Delta^m|) \]

where \(p_0\) is the obvious projection, and \(p_{1, \tau_m}\) is the map that does the identity on the factor \(V G_N\) and on \(\mathcal{V} \times \mathcal{H}_{M_N} \times |\Delta^m|\) it sends

\( (x; Q_o + Q_t, q_0, q_1, t_0, t_1; v) \mapsto (x; Q_o + Q_t, q_1 - 2\tilde{Q}_t q_0, t_1 - \tilde{Q}_t [q_0]; v) \)

**Lemma 4.7.** The correspondence (4.19) canonically determines a morphism from \((p_t, \mathcal{V} \times R_{q_0}^N \times R_{t_0})\) to \((V G_N, \mathcal{V} \times M_N \times R_{q_1}^N \times R_{t_1} \times |\Delta^m|)\) in \(\text{Mod}^{N(\text{Fin}_\ast)}(\text{Corr}(S_{LCH}))_{\text{prop}}\) right-lax.

**Proof.** Since everything is constant on \(\mathcal{V}\), we will omit it in the bulk of the proof. According to [Jin Proposition 2.22], we just need to check that diagram (4.20) is Cartesian, and the map (4.21) for the unique active map \(\langle m \rangle_t \rightarrow \langle 0 \rangle_t\) is proper (making the 2-morphism lying in \text{prop}).

\[
\begin{array}{c}
    \mathcal{H}_{M_N}^{(t)} \times |\Delta^m| \\
p_{1, \tau_m} \\
\end{array}
\]

\[
\begin{array}{c}
    \prod_{j \in (k)^o} V G^{(j, \ast)} \times \mathcal{H}^{(\{1\}, \ast)}_{M_N} \times |\Delta^m| \\
\end{array}
\]

\[
\begin{array}{c}
    M_N^{(t)} \times R_{q_1}^N \times R_{t_1} \times |\Delta^m| \\
\end{array}
\]

\[
\begin{array}{c}
    \prod_{j \in (k)^o} V G^{(j, \ast)} \times M_N^{(\{1\}, \ast)} \times R_{q_1}^N \times R_{t_1} \times |\Delta^m| \\
\end{array}
\]

(4.20)

\[
\pi_k : \mathcal{H}_{M_N}^{(t)} \longrightarrow \mathcal{H}_{M_N}
\]

\((Q_j, p_j)_{j \in (k)^o}; (q_0, q_1, t_0, t_1, Q_o + Q_t) \mapsto (q_0, q_1 - 2 \sum_{j \in (k)^o} p_j, t_0, t_1 - \sum_{j \in (k)^o} Q_j[p_j], Q_o + Q_t + \sum_{j \in (k)^o} Q_j)\)

The first one is obvious. Regarding the second one, for any \((q_0, \tilde{q}_1, t_0, \tilde{t}_1, Q_o + \tilde{Q}_t) \in \mathcal{H}_{M_N}\), which by assumption satisfies

\[
\tilde{\mu} := \tilde{t}_1 - t_0 - (Q_o + \tilde{Q}_t)[q_0] - \tilde{q}_1 \cdot q_0 \leq 0
\]

the fiber \(\pi_k^{-1}(q_0, \tilde{q}_1, t_0, \tilde{t}_1, Q_o + \tilde{Q}_t)\) admits a proper map to the space of splittings of \(\tilde{Q}_t\) into \(k + 1\) ones in \(G_N^{(k) \cup \{1\}}\) with fiber at \((Q_j)_{j \in (k)^o}, Q_t\) identified with the space of \((p_j)_{j \in (k)^o}\) satisfying

\[
\sum_{j \in (k)^o} Q_j[p_j'] - 2 \sum_{j \in (k)^o} p_j \cdot q_0 + \sum_{j \in (k)^o} Q_j[q_0] = \sum_{j \in (k)^o} Q_j[p_j' - q_0] \leq \tilde{\mu},
\]

hence \(\pi_k^{-1}(q_0, \tilde{q}_1, t_0, \tilde{t}_1, Q_o + \tilde{Q}_t)\) is compact. On the other hand \(\pi_k\) is clearly closed, so it is proper as desired. \(\Box\)
4.3.2. The second set of correspondences. Again, for any \((U, V, Q^{(1)}_o, G^\bullet, M^{\bullet^*})\) in \(\text{QHam}_L(U/O)\), we consider the following correspondence(s) for localizing sheaves on \(V \times M_N \times R^{\omega}_{q_i} \times R_{t_i}\) along the full subcategory \(Shv^{\geq 0}(V \times M_N \times R^{\omega}_{q_i} \times R_{t_i})\):

\[
(VG_N, V \times M_N \times R^{\omega}_{q_i} \times R_{t_i}) \xrightarrow{\alpha} (V \times M_N \times R^{\omega}_{q_i} \times R_{t_i} \times [0, \infty)) \xrightarrow{p} (VG_N, V \times M_N \times R^{\omega}_{q_i} \times R_{t_i})
\]

where \(p\) is the obvious projection and \(\alpha\) is the addition map on the last two factors \(R_{t_i} \times [0, \infty)\). The functor \(a_{\pi}p^!\) on \(Shv(V \times M_N \times R^{\omega}_{q_i} \times R_{t_i}; k)\) is the convolution with \(\omega_{[0,\infty)}\), and will be denoted as \(\mathfrak{a}_\omega\). In general, we have the following small variant of \(\text{[Tam1]}\) Proposition 2.1 and 2.2.

**Theorem 4.8** (\text{[Tam1]}). For any smooth manifold \(X\), the convolution functor

\[
\mathfrak{a}_\omega : Shv(X \times R_{t_i}; k) \to Shv(X \times R_{t_i}; k)
\]

gives a projector

\[
Shv(X \times R_{t_i}; k) \to Shv(X \times R_{t_i}; k)/Shv^{\geq 0}(X \times R_{t_i}; k) := Shv^{< 0}(X \times R_{t_i}; k),
\]

with the right-hand-side identified with the right orthogonal complement of \(Shv^{\geq 0}(X \times R_{t_i}; k)\).

Let

\[
\mathcal{H}^+_{V,M_N} = \{(x, Q_o + Q^+_1, q_i, t_i) : t_i - \frac{1}{2}(A_{\ell_x} + 2Q_o + 2Q^+_1)^{FT}[q_i] > 0\}
\]

\[
\subset V \times M_N \times R^{\omega}_{q_i} \times R_{t_i},
\]

\[
\mathcal{H}^0_{V,M_N} = \{(x, Q_o + Q^+_1, q_i, t_i) : t_i - \frac{1}{2}(A_{\ell_x} + 2Q_o + 2Q^+_1)^{FT}[q_i] = 0\}
\]

\[
\subset V \times M_N \times R^{\omega}_{q_i} \times R_{t_i}
\]

where \(\ell_x\) is the tangent space \(T_x L\) as an affine Lagrangian in \(T^*R^N\) and for any generalized symmetric matrix \(Q\), \(Q^{FT}\) denotes for its Fourier transform, i.e. changing eigenvalues \(\lambda\) to \(-1/\lambda\) but keeping the eigenspaces. For any closed smooth hypersurface \(H\) in \(R^N \times R_t\), let \(\Lambda_H\) denote the negative conormal bundle of \(H\), i.e. the part of the conormal bundle inside \(T^*_{\geq 0}(R^N \times R_t)\). Note that by the definition of \(\text{[Tam1]}\) \((U, V, Q^{(1)}_o, G^\bullet, M^{\bullet^*})\), \((A_{\ell_x} + 2Q_o + 2Q^+_1)^{FT}\) in \((4.23)\) is a usual symmetric matrix. Similarly to \(\mathcal{H}_{M_N} (4.17)\), one can upgrade \(\mathcal{H}^+_{V,M_N}\) to a \(VG_N\) module in \(Corr(S_{\text{LCH}})\) (see Lemma 4.9 below).
Consider the following collection of correspondences

\[(4.25)\]
\[
\begin{array}{c}
(V_G, \mathbb{H}^+_{V,M_N}) \\
\quad \quad \downarrow id \\
(V_G, \mathbb{H}^+_{V,M_N})
\end{array}
\]
\[
\begin{array}{c}
(V_G, \mathbb{H}^+_{V,M_N}) \\
\quad \quad \downarrow j \downarrow M_N
\end{array}
\]

\[(4.26)\]
\[
\begin{array}{c}
(V_G, \mathbb{H}^+_{V,M_N}) \\
\quad \quad \downarrow id \\
(V_G, \mathbb{H}^+_{V,M_N})
\end{array}
\]
\[
\begin{array}{c}
(V_G, \mathbb{H}^+_{V,M_N}) \\
\quad \quad \downarrow p_{\phi}
\end{array}
\]

\[(4.27)\]
\[
\begin{array}{c}
(V_G, V \times M_N) \\
\quad \quad \downarrow p_{V,G_N} \\
(G, V \times M_N)
\end{array}
\]

where \(j_{V,M_N}\) is the natural inclusion, \(p_{\phi}\) is the projection to the first two factors, \(p_{V,G_N}\) is the natural projection from the free \(V_G\)-module \(V \times M_N\) to the free \(G_N\)-module \(V \times M_N\).

**Lemma 4.9.**

(a) One can upgrade \(\mathbb{H}^+_{V,M_N}\) to a \(V_G\)-module in \(\text{Corr}(S_{LCH})_{\text{fib,all}}\);

(b) All three correspondences \((4.25), (4.26)\) (resp. \((4.27)\)) canonically induce right-lax morphisms (resp. morphisms) in \(\text{Mod}^{N\text{(Fin+)}(\text{Corr}(S_{LCH})_{\text{fib,all}})}\). Moreover, they induce a canonical functor

\[(4.28)\]
\[
(p_{V,G_N})^* \circ (p_{\phi})^{-1} \circ j_{V,M_N}^* :
\]
\[
(\text{Loc}(V_G; Sp), \text{Shv}_{\text{Str}^{\times}}^{0}, (V \times M_N \times R_{q_1}^N \times R_{n_1}; Sp)) \rightarrow (\text{Loc}(G; Sp), \text{Loc}(V \times M_N; Sp))
\]

in \(\text{Mod}^{N\text{(Fin+)}(\text{Pr}_{\text{Str})}}\), where \((p_{\phi})^{-1}\) means the inverse of

\[(4.29)\]
\[
p_{H}^* : (\text{Loc}(V_G; Sp), \text{Loc}(H^+_{V,M_N}; Sp)) \sim (\text{Loc}(V_G; Sp), \text{Loc}(V \times M_N; Sp)).
\]

**Proof.** For (a), we set

\[
(H^+_{V,M_N})^{(n)}_{\downarrow} = M_{N}^{(n)}_{\downarrow} \times H^+_{V,M_N},
\]

where \(M_{N}^{(n)}_{\downarrow} \rightarrow M_{N}^{(0)}_{\downarrow}\) is the map corresponding to the unique inert map \(n_{\downarrow} \rightarrow 0_{\downarrow}\). We just need to prove that for any active map \(n_{\downarrow} \rightarrow 0_{\downarrow}\), the map

\[
(H^+_{V,M_N})^{(n)}_{\downarrow} \rightarrow (H^+_{V,M_N})^{(0)}_{\downarrow}
\]

\[
((Q_j, p_j)_{j \in \{n\}^0}; x, Q_b + Q_{\downarrow} + Q_1, t_1) \mapsto (x, Q_b + Q_{\downarrow} + \sum_j Q_j, q_1 - 2 \sum_j p_j, t_1 - \sum_j Q_j[p_j])
\]

is well defined, i.e. the image satisfies the condition \((4.22)\). Note that it suffices to show the case for \(n = 1\).
Spelling things out, we need to show the following inequality
\[(4.30) \quad t_1 - Q_1[p_1^\prime] - \frac{1}{2}(A_{\ell_x} + 2Q_b + 2Q_\dagger + 2Q_1)^{FT}[q_1 - 2p_1] > 0\]
under the condition
\[t_1 - \frac{1}{2}(A_{\ell_x} + 2Q_b + 2Q_\dagger)^{FT}[q_1] > 0.\]
Let $P : b_{\ell_x} \hookrightarrow \mathbb{R}^N$ be the embedding of the projection of $\ell_x$ and $P^T : \mathbb{R}^N \to b_{\ell_x}$ be the orthogonal projection. Since $A_{\ell_x} + 2Q_b + 2Q_\dagger + 2Q_1$ is the same as $A_{\ell_x} + 2Q_b + 2Q_\dagger + 2PP^TQ_1PP^T$, we may assume without loss of generality that $Q_1$ is supported on $b_{\ell_x}$ and the geometry is happening on $T^*b_{\ell_x}$ (that is we forget about the $\infty$-spectral part of $A_{\ell_x}$).

Fixing $\tilde{q}_1 = q_1 - 2p_1$, and let $p_1$ vary in Supp($Q_1$), we have
\[
\partial_{p_1}(\frac{1}{2}(A_{\ell_x} + 2Q_b + 2Q_\dagger)^{FT}[q_1] - Q_1[p_1^\prime])
= (A_{\ell_x} + 2Q_b + 2Q_\dagger)^{FT}(\frac{1}{2}\tilde{q}_1 + p_1) + (2Q_1)^{FT}p_1 = 0
\]
if and only if
\[\varphi_{-Q_1[p]}^1(\tilde{q}_1, (A_{\ell_x} + 2Q_b + 2Q_\dagger + 2Q_1)^{FT}\tilde{q}_1) = (q_1, p_1^\prime).\]
Moreover, the Hessian
\[(A_{\ell_x} + 2Q_b + 2Q_\dagger)^{FT} + (2Q_1)^{FT}|_{\text{Supp}(Q_1)}\]
is strictly positive, which follows from the rank equality (4.9). Therefore,
\[\frac{1}{2}(A_{\ell_x} + 2Q_b + 2Q_\dagger)^{FT}[q_1] - Q_1[p_1^\prime] \]
obtains a minimum at $\varphi_{-Q_1[p]}^1(\tilde{q}_1, (A_{\ell_x} + 2Q_b + 2Q_\dagger + 2Q_1)^{FT}\tilde{q}_1)$. In that case,
\[\frac{1}{2}(A_{\ell_x} + 2Q_b + 2Q_\dagger)^{FT}[q_1] - Q_1[p_1^\prime] - \frac{1}{2}(A_{\ell_x} + 2Q_b + 2Q_\dagger + 2Q_1)^{FT}[q_1 - 2p_1] = 0,
\]
and (4.30) follows.

Furthermore, we see from the above that the fiber of $\mathbb{H}^{(1)}_1 \to \mathbb{H}^{(0)}_1$ over a fixed $h = (x, Q_b + Q_\dagger, \tilde{q}_1, \tilde{t}_1)$ is isomorphic to the region in
\[\{(Q_1, p_1; x, Q_b + Q_\dagger) : Q_\dagger + Q_1 = \hat{Q}_1\}\]
cut out by
\[t_1 - \frac{1}{2}(A_{\ell_x} + 2Q_b + 2Q_\dagger)^{FT}[q_1] > 0\]
\[\Leftrightarrow \quad \frac{1}{2}(A_{\ell_x} + 2Q_b + 2Q_\dagger)^{FT}[q_1] - Q_1[p_1^\prime] - \frac{1}{2}(A_{\ell_x} + 2Q_b + 2Q_\dagger + 2Q_1)^{FT}[q_1 - 2p_1]
< \tilde{t}_1 - \frac{1}{2}(A_{\ell_x} + 2Q_b + 2Q_\dagger)^{FT}[\tilde{q}_1].\]
So the fiber is isomorphic to an open disc bundle over the space of $Q_1$. This shows that the vertical arrows in defining $H^+_{V,M_N}$ as a $VG_N$-module in Corr($S_{LCH}$) are all locally trivial fibrations.

For part (b), one can show the three correspondences all induce right-lax morphism in $\text{Mod}^{N[\text{Fin}^*]}(\text{Corr}(S_{LCH})^\text{open})$ right-lax in the same way as the proof of Lemma 4.7 (the last one induces a genuine morphism). In particular, they respectively induce left-lax functors $j^!_{V,M_N}: (\text{Shv}(VG_N;Sp), \text{Shv}(V \times M_N \times R^N_{q,t};Sp)) \to (\text{Shv}(VG_N;Sp), \text{Shv}(H^+_{V,M_N};Sp))$
$p^!_{\|}: (\text{Shv}(VG_N;Sp), \text{Shv}(V \times M_N;Sp)) \to (\text{Shv}(VG_N;Sp), \text{Shv}(H^+_{V,M_N};Sp))$
and a functor $(p_{VG_N})_* : (\text{Shv}(VG_N;Sp), \text{Shv}(V \times M_N;Sp)) \to (\text{Shv}(G;Sp), \text{Shv}(V \times M_N;Sp))$.

Now it suffices to show that $j^!_{V,M_N}$, $p^!_{\|}$ and $(p_{VG_N})_*$ restricted to the relevant full subcategories as in (4.28) and (4.29) are equivalences. This is almost evident, as

- $\text{Shv}^{N[0]}_{N[0],N} (V \times M_N \times R^N_{q,t})$ can be identified with the essential image of $(j_{V,M_N})!: \text{Loc}(H^+_{V,M_N};Sp) \to \text{Shv}(V \times M_N \times R^N_{q,t};Sp)$,
- for any active map $\langle n \rangle \to \langle m \rangle$ in $N(\text{Fin}^*)$

$$f: H^+_{V,M_N}(\langle n \rangle) \to H^+_{V,M_N}(\langle m \rangle) \times (V \times M_N)(\langle n \rangle) \to (V \times M_N)(\langle m \rangle)$$

the map $f$ is an open embedding that induces a homotopy equivalence between locally trivial fibrations over $(H^+_{V,M_N})(\langle m \rangle)$.

The composition of the first set (1.19) and the second set (1.22), (1.25), (1.26), (1.27) of correspondences\footnote{More precisely, one should produce the second set of correspondences everywhere by $|\Delta^m|$ and modify $H^+_{V,M_N} \times |\Delta^m|$ by changing $Q_s + Q_1$ to $Q_s + Q_1 + \tau_m(u), u \in |\Delta^m|, \tau_m \in \Delta_y$.} for all $N \gg 0$, together with Lemma 4.9 and results in Appendix A yield the following.

**Proposition 4.10.** Given any $y = (U, V, Q_y, G, M) \in \text{QHam}_L(U/O)$, and any $m$-simplex $\tau_m$ in $\Delta_y$, the composition of the first and the second set of correspondences induces a right-lax functor

$$(\text{Sp}^\circ, \lim_{\leftarrow N} \text{Shv}_{\text{LcGaus}}^0 (V \times R^N_{q,t};Sp)) \to (\lim_{\leftarrow N} \text{Loc}(G_N;Sp)^\circ, \lim_{\leftarrow N} \text{Loc}(V \times M_N \times |\Delta^m|;Sp)),$$
that further induces an equivalence
\[
\begin{align*}
F_{rm} & : \lim_{N} \mathrm{Shv}^0_{L_{\text{Gauss}}} (\mathcal{V} \times \mathbb{R}^N_{q_0} \times \mathbb{R}_{t_0}; \mathrm{Sp}) \xrightarrow{\sim} \lim_{N} \mathrm{Loc}(\mathcal{V} \times M_N \times |\Delta^m|; \mathrm{Sp})^{J\text{-equiv}} \\
& \xrightarrow{\sim} \mathrm{Loc}(\mathcal{V} \times M; \mathrm{Sp})^{J\text{-equiv}},
\end{align*}
\]
where the last equivalence is from \(!\)-pushforward along the projection to \(\mathcal{V} \times M\).

4.3.3. Construction of \(F_L\). Given objects \(y_i = (U_i, V_i, Q^{(i)}_\bullet, G^{(i)}, M^{(i)}_\bullet), i = 1, 2\), for any \(n\)-simplex \(\sigma_n\) in \(\mathrm{Maps}_{Q_{\mathrm{Ham}, U/O}}(y_1, y_2)\) defined by a family of \(Q_{12,u} \in G(2), u \in |\Delta^n|\) (the geometric realization of any \(n\)-simplex will be identified with \(\sum_{j=1}^{n+1} t_j = 1\) in \(\mathbb{R}^{n+1}\) such that \(\tilde{Q}_u = Q_2^{(2)} + Q_{12,u} - Q_1^{(1)} \geq 0\), and an \(n\)-simplex \(\tau_n : |\Delta^n| \to S_{y_2}\) (defined in Subsection 4.3.1), consider the following commutative diagram for \(N \gg 0\) (which can be viewed as an object in
\[
\mathrm{Fun}(\Delta^1 \times \Delta^1, \mathrm{Corr}(\mathrm{Fun}^\circ(N(\mathrm{Fin}_{*1}), (\mathrm{S}_{\mathrm{LCH}}))), \mathrm{inert, all}),
\]
with the vertex \((0, 0)\) and \((1, 1)\) be respectively the bottom right corner and the top left corner in the diagram)
\[
\begin{align*}
\begin{array}{ccc}
(VG_{(1), N}, V_1 \times M_{(1), N} \times \mathbb{R}^N_{q_1}, \mathbb{R}^N_{t_1} \times |\Delta^n|) & \xrightarrow{p_1}\ & (pt, V_1 \times \mathbb{R}^N_{q_0} \times \mathbb{R}_t) \\
(VG_{(1), N}, V_2 \times M_{(1), N} \times \mathbb{R}^N_{q_1}, \mathbb{R}^N_{t_1} \times |\Delta^n|) & \xrightarrow{p_2}\ & (pt, V_2 \times \mathbb{R}^N_{q_0} \times \mathbb{R}_t)
\end{array}
\end{align*}
\]
\[
\begin{align*}
(VG_{(1), N}, V_1 \times M_{(1), N} \times \mathbb{R}^N_{q_1}, \mathbb{R}^N_{t_1} \times |\Delta^n|) & \xrightarrow{j_{\sigma_n}}\ & (pt, V_2 \times \mathbb{R}^N_{q_0} \times \mathbb{R}_t)
\end{align*}
\]
\[
\begin{align*}
(VG_{(2), N}, V_2 \times M_{(2), N} \times \mathbb{R}^N_{q_1}, \mathbb{R}^N_{t_1} \times |\Delta^n|) & \xrightarrow{i_{\sigma_n}}\ & (pt, V_2 \times \mathbb{R}^N_{q_0} \times \mathbb{R}_t)
\end{align*}
\]
where \(i_{\sigma_n}\) is the embedding induced from the morphisms \((VG_{(1), N}, M_{(1), N}) \to (VG_{(2), N}, M_{(2), N})\) determined by \(\sigma_n\) and the identity map on the other factors,
\[
i_{\sigma_n} : (Q_1, p; q_0, q_1, t_0, t_1, Q_1^{(1)}, u) \mapsto (Q_1, p; q_0, q_1 - 2\tilde{Q}_u q_0, t_0, t_1 - \tilde{Q}_u (q_0), Q_2^{(2)} + Q_{12,u}, u),
\]
and \(\tilde{\tau}_n\) is the \(n\)-simplex in \(S_{y_1} = S_{(t_{(1)}, Q^{(1)}_1, G^{(1)}, M^{(1)})}\) defined by
\[
\tilde{\tau}_n(u) = Q_2^{(2)} + Q_{12,u} - Q_1^{(1)} + \tau_n(u).
\]
The top vertical arrows are all obvious inclusions induced from \(V_2 \hookrightarrow V_1\).
These define a natural transformation
\[(4.33)\]
\[
\begin{array}{ccc}
\text{Sing}_n(S_{y_2} \times \text{Maps}_{\text{QHam}_L(U/O)}(y_1, y_2)) & \xrightarrow{pr_1} & \text{Sing}_n(S_{y_2}) \\
\downarrow g_n & & \downarrow f_{y_2,n} \\
\text{Sing}_n(S_{y_1}) & \xrightarrow{f_{y_1,n}} & \text{Fun}(\Delta^1 \times Z_{\geq 0}, \text{Corr}(\text{Fun}^\circ(N(\text{Fin}_{st}), (S_{\text{LCH}})))_{\text{inert,all}})
\end{array}
\]

where \(Z_{\geq 0}\) is the indexed category for the dimensions \(N\), \(g_n(\tau_n, \sigma_n) = \tau_n, f_{y_2,n}\) is sending \(\tau_n \in \text{Sing}_n(S_{y_2})\) to the bottom correspondence in (4.32) and \(f_{y_1,n}\) is sending \(\tau_n \in \text{Sing}_n(S_{y_1})\) to the top correspondence in (4.32). Ideally, we would like to extend diagram (4.33) by further composing with the functor
\[(4.34)\]
\[
\text{Fun}(\Delta^1, \text{Corr}(\text{Fun}^\circ(N(\text{Fin}_{st}), (S_{\text{LCH}})))_{\text{inert,all}}) \to \text{Fun}(\Delta^1, \text{Pr}_{\text{st}}^L)
\]
from first applying \(\text{ShvSp}_i^t\) to \(\text{Fun}(\Delta^1, \text{Mod}^{N(\text{Fin}_{st})}(\text{Pr}_{\text{st}}^R))\) (note that the horizontal correspondences in (4.33) are exactly the first set of correspondences in Subsubsection 4.3.1), then applying the second set of correspondences in Subsubsection 4.3.2 (restricting to the microlocal sheaf categories \(\text{Shv}^{<0}_L(\mathcal{V}_i \times \mathcal{R}_{q0}^N \times \mathcal{R}_t \times |\Delta^n|, i = 1, 2)\), and finally getting to \(F_{\tau_n} (4.31)\) by Proposition 4.10 after taking \(\lim_{\to N}\) everywhere. The goal is to construct a canonical functor
\[(4.35)\]
\[
F_L : \text{QHam}_L(U/O)^{op} \to \text{Fun}(\Delta^1, \text{Pr}_{\text{st}}^L).
\]
from these data. A small problem about doing this is that if we do \(*\)-pushforward along the upper vertical arrows in (4.32), we will create more singular support than required (so not well defined). To remedy this, we will do the followings:

- First, we revert all arrows of the form
\[(4.36)\]
\[
(\mathcal{U}_1, \mathcal{V}_1, Q, G^\ast, M^{\ast, \dagger}) \to (\mathcal{U}_2, \mathcal{V}_2, Q, G^\ast, M^{\ast, \dagger})
\]
in \(\text{QHam}_L(U/O)\), i.e. \(\mathcal{U}_1 \supset \mathcal{U}_2\) and \(\mathcal{V}_1 \supset \mathcal{V}_2\) but the latter three components are identical and the morphism is identity on those factors. The new \(\infty\)-category, which has the same object as \(\text{QHam}_L(U/O)\) but requiring the reversed containment relation on \(\mathcal{V}_i\) and \(\mathcal{U}_i\) in the definition of morphism spaces (4.15), is denoted by \(\text{QHam}_L^{\infty}(U/O)\).

- We replace \(\text{QHam}_L(U/O)\) by \(\text{QHam}_L^{\infty}(U/O)\) in (4.33) and composing with the functor (4.34) described above, then we get a natural transformation between two functors
\[
F_{y_{2,n}} \circ pr_1 \Rightarrow F_{y_{1,n}} \circ g_n : \text{Sing}_n(S_{y_2} \times \text{Maps}_{\text{QHam}_L^{\infty}(U/O)}(y_1, y_2)) \to \text{Fun}(\Delta^1, \text{Pr}_{\text{st}}^L)
\]
- We will then construct a canonical functor
\[
(\text{QHam}_L^{\infty}(U/O))^{op} \to \text{Fun}(\Delta^1, \text{Pr}_{\text{st}}^L).
\]
Let \(\text{opinc}\) be the class of morphisms in \(\text{QHam}_L^{\infty}(U/O)\) coming from reversing the class in (4.36). The above functor sends \(\text{opinc}\) to left adjointable functors in \(\text{Fun}(\Delta^1, \text{Pr}_{\text{st}}^L)\) and satisfies the right Beck-Chevalley condition with respect to
**vert** = opincl (cf. [GaRo, Chapter 7, Definition 3.1.5]), so we can revert them by taking their left adjoints and get the desired functor (1.35). More explicitly, we can view $\text{QHam}_L(U/O)$ as the correspondence category of $\text{QHam}^{\text{op}}_L(U/O)$ with vertical arrows given by opincl and horizontal arrows given by the class with $U_1 = U_2, V_1 = V_2$ (every morphism has such a unique factorization), then apply\(^2\) [GaRo, Chapter 7, Theorem 3.2.2].

By the compatibility of the diagram (4.33) with respect to degeneration and face maps, the diagrams together with (4.34) assemble to define a natural transformation

$$
\text{Sing}_*(S_{y_2} \times \text{Maps}_{\text{QHam}_L^{\text{op}}(U/O)}(Y_1, Y_2)) \xrightarrow{pr_1} \text{Sing}_*(S_{y_2}) \xrightarrow{F_{y_2}} \text{Fun}(\Delta^1, P_{\text{st}}^L)
$$

For simplicity, in the following we will not distinguish a space with its singular simplicial complex (or further taking geometric realization). Let $F_L(Y_i)$ be the left Kan extension of $F_{y_i}$ along the map $S_{y_i} \to pt$. Since $S_{y_i}$ is contractible, $F_L(Y_i)$ is canonically isomorphic to the object

$$(F_{\tau_n} : \lim_{\tau_n} \Shv^{\text{op}}_{\text{Lem}}(V_i \times R^N_{q_0} \times R_{t_0}; \text{Sp}) \xrightarrow{\sim} \text{Loc}(V_i \times M; \text{Sp})^{t-\text{equiv}}),$$

for any $\tau_n \in \text{Sing}_n(S_{y_i})$, and this gives a canonical morphism (up to a contractible space of choices)

$$
(4.37) \quad F_L(Y_1, Y_2) : \text{Maps}_{\text{QHam}^{\text{op}}_L(U/O)}(Y_1, Y_2) \to \text{Maps}_{\text{Fun}(\Delta^1, P_{\text{st}})}(F_L(Y_2), F_L(Y_1)).
$$

To see that the morphism (4.37) is compatible with compositions, note that for three objects $Y_i, i = 1, 2, 3$, we have the following diagram for every $n \geq 0$:

$$
\begin{array}{c}
\text{Sing}_n(S_{y_3} \times \text{Maps}(Y_1, Y_2) \times \text{Maps}(Y_1, Y_2)) \xrightarrow{pr_1} \text{Sing}_n(S_{y_3}) \\
\text{Sing}_n(S_{y_2} \times \text{Maps}(Y_1, Y_3)) \xrightarrow{pr_1} \text{Sing}_n(S_{y_2}) \\
\text{Sing}_n(S_{y_2} \times \text{Maps}(Y_1, Y_2)) \xrightarrow{pr_1} \text{Sing}_n(S_{y_2}) \\
\text{Sing}_n(S_{y_2}) \xrightarrow{F_{y_2}} \text{Fun}(\Delta^1 \times Z^{\text{op}}_{\text{Corr}(\text{Fun}(N(\text{Fin}_{\ast, 1}), S_{\text{LCH}))}_{\text{inert, all}}}
\end{array}
$$

where left square is strictly commutative and there is a strict equality between the composite natural transformation

$$
\bar{f}_{y_3} \circ pr_1 \Rightarrow \bar{f}_{y_2} \circ pr_1 \circ g_{3, 2} \Rightarrow \bar{f}_{y_1} \circ g_{2, 1} \circ g_{3, 2},
$$

\(^2\)In applying the theorem, use horiz = all, then restrict to the so defined subclass of horizontal arrows.
and the natural transformation

\[ f_{y_3} \circ pr_1 \Rightarrow f_{y_1} \circ g_{3,1} \circ \text{comp}. \]

Applying ShvSp\textsuperscript{\dagger} and restricting to the microlocal sheaf categories, then applying Proposition 4.10 and using the compatibility with face and degeneracy maps as above, we get a contractible space of commutative diagrams

(4.39)

\[ F_L(Y_1, Y_2, Y_3) : \text{Maps}(Y_2, Y_3) \times \text{Maps}(Y_1, Y_2) \xrightarrow{\text{Maps}(\Delta^2, \text{Fun}(\Delta^1, \text{Pr}_{st}^L)))} \]

\[ \lim_{I \subseteq \{0,1,2\}} \prod_{i < j \in I} \text{Maps}(Y_{3-j}, Y_{3-i}) \xrightarrow{(F_L(Y_{3-i}, Y_{3-j}), F_L(Y_k))} \text{Maps}(\partial \Delta^2, \text{Fun}(\Delta^1, \text{Pr}_{st}^L))) \]

whose restriction to the second row gives a trivial Kan fibration to the (contractible) space of maps defined by \( F_L(Y_1, Y_2, Y_3) \) and \( F_L(Y_1, Y_2) \) are compatible with degenerations, e.g. if \( Y_1 = Y_2 \), then \( F_L(Y_1, Y_1, Y_3) \mid \{\text{id}_{Y_1}\} \times \text{Maps}(Y_1, Y_3) \) canonically factors through \( \text{Maps}(\Delta^{[0,1]}, \text{Fun}(\Delta^1, \text{Pr}_{st}^L)) \). More systematically, one adds the degeneration maps into diagram (4.37) and (4.39) if some adjacent \( Y_j \)'s are equal.

By induction on the number of \( Y_i \)'s involved, we get a canonical functor (up to a contractible space of choices) in \( \text{Spc}^{\Delta^{op}} \)

(4.40)

\[ F_L : (\prod_{(Y_1, \ldots, Y_{\bullet+1})} \text{Maps}(Y_\bullet, Y_{\bullet+1}) \times \cdots \times \text{Maps}(Y_1, Y_2)) \xrightarrow{F_L(Y_1, \cdots, Y_{\bullet+1})} \text{Maps}(\Delta^\bullet, \text{Fun}(\Delta^1, \text{Pr}_{st}^L))). \]

Since the left-hand-side is equivalent to \( \text{Seq}^{\bullet}(\text{QHam}_L^{\nu-op}(U/O)^{op}) \), the functor \( F_L \) (4.40) corresponds to a functor in 1-Cat, denoted by \( F_L \) as well

(4.41)

\[ F_L : \text{QHam}_L^{\nu-op}(U/O)^{op} \longrightarrow \text{Fun}(\Delta^1, \text{Pr}_{st}^L). \]

As remarked before, by reversing the arrows in \( \text{opincl} \) through sending to the left adjoints (for this we need to replace \( \text{Fun}(\Delta^1, \text{Pr}_{st}^L) \) by \( \text{Fun}(\Delta^1, \text{Pr}_{st}^L) \)), and by some abuse of notations, we get the desired functor \( F_L \) (4.35).

4.4. The main diagram of categories and functors. The main diagram of categories and functors is the following (in which the lower right square and the upper rightmost
triangle are commutative):

\[ (4.42) \]

\[
\begin{array}{c}
\text{\text{QHam}_L(U/O)}^{\text{op}} \\
\text{\text{\text{pr}}_L} \\
\text{\text{\text{ev}_1}} \\
\text{\text{\text{\text{pr}}_L}} \\
\end{array}
\]

\[
\begin{array}{c}
\text{\text{Open}(L)} \\
\text{\text{\text{\text{RKan}}}(\text{\text{\text{pr}}}_L, L)} \\
\text{\text{\text{\text{RKan}}}(\text{\text{\text{ev}_0} \circ F_L}, L)} \\
\end{array}
\]

where

(i) \( (G, M) \) is the functor sending any object \((\mathcal{U}, \mathcal{V}) \in \mathcal{L}, G^*, M^* \) to the pair \((G, M) \) viewed as an object in \( \text{Mod}^{\text{N(Fin}_*\text{)}}(\text{Spc})^{\text{op}} \), and taking any morphism to the correspondence embedding;

(ii) the functor \( F_L : \text{QHam}_L(U/O)^{\text{op}} \to \text{Fun}(\Delta^1, \text{Pr}_L^L) \) has been constructed in Section 4.3 as a small modification of (4.41) with some abuse of notations. Its evaluation at \( 0 \in \Delta^1 \) gives \( \mu \text{Shv}_L(-; \text{Sp}) \), and its evaluation at \( 1 \in \Delta^1 \) gives \( \text{Loc}(-; \text{Sp})^{\text{J-equiv}} \);

(iii) We have the right Kan extension of the functor \((G, M) \to \text{Loc}(-; \text{Sp})^{\text{J-equiv}}, F_L \; \text{and} \; \text{ev}_0 \circ F_L \) along \( p_L \), and denote it by \( \text{RKan}_{p_L}(G, M) \) (resp. \( \text{RKan}_{p_L}^{\mu}(G, M) \)), \( \text{RKan}_{p_L}(F_L) \) and \( \text{RKan}_{p_L}^{\mu}(\text{ev}_0 \circ F_L) \).

For any topological space \( X \), let \( \text{PcoShv}(X; \text{Pr}_k^L) \) (resp. \( \text{coShv}(X; \text{Pr}_k^L) \)) be the \((\infty, 1)\)-category of pre-cosheaves (resp. cosheaves) of (presentable) \( k \)-linear categories on \( X \), with continuous (i.e. colimit-preserving) corestriction functors.

**Lemma 4.11.** For any topological space \( X \), given a presheaf of spaces \( \mathcal{F}^{\text{pre}} : \text{Open}(X)^{\text{op}} \to \text{Spc} \) over \( X \), let \( \text{Loc}^{\text{pre}} : \text{Open}(X)^{\text{op}} \to \text{Spc} \) be the \((\infty, 1)-\text{equivalence} \) of spaces \( \text{Loc}^{\text{pre}} \) and \( \text{Loc}_\mathcal{F} \). Equivalently, this means for any cosheaf \( \mathcal{E} \in \text{coShv}(X; \text{Pr}_k^L) \), we have an isomorphism of spaces \((\infty-\text{groupoids})\)

\[
\text{Maps}_{\text{PcoShv}(X; \text{Pr}_k^L)}(\mathcal{E}, \text{Loc}^{\text{pre}}) \simeq \text{Maps}_{\text{coShv}(X; \text{Pr}_k^L)}(\mathcal{E}, \text{Loc}_\mathcal{F}).
\]

**Proof.** We have the following isomorphism of spaces

\[
\text{Maps}_{\text{PcoShv}(X; \text{Pr}_k^L)}(\mathcal{E}, \text{Loc}^{\text{pre}}) \simeq \text{Maps}_{\text{PShv}(X; \text{Spc})}(\mathcal{F}^{\text{pre}}, \text{Fun}_k^L(\mathcal{E}(-), \text{Mod}(k))^{\text{Spc}})
\]

\[
\simeq \text{Maps}_{\text{Shv}(X; \text{Spc})}(\mathcal{F}, \text{Fun}_k^L(\mathcal{E}(-), \text{Mod}(k))^{\text{Spc}}) \simeq \text{Maps}_{\text{coShv}(X; \text{Pr}_k^L)}(\mathcal{E}, \text{Loc}_\mathcal{F}),
\]

where \( \text{PShv}(X; \text{Spc}) \) is the \( \infty \)-category of presheaves of \( \infty \)-category \( \mathcal{E} \), \( \mathcal{E}^{\text{Spc}} \) is the image obtained from removing all non-invertible morphisms in \( \mathcal{E} \) (equivalently, the image under the right adjoint of the full embedding \( \text{Spc} \to \text{1-Cat} \)).
Lemma 4.12. For any $x \in L$, let $C(x)$ be the full subcategory of $QHam_L(U/O)$ consisting of $(U, V, Q_b, G^*, M^{\bullet \dagger})$, $x \in V$, and let $D(x)$ be the full subcategory of $C(x)$, whose objects satisfy $\varphi_{Q_b}(F\ell(\ell_x))$ has no negative spectral part. Then

(a) $D(x)$ is a filtered poset;
(b) the inclusion $D(x) \hookrightarrow C(x)$ is cofinal.

Proof. (a). The partial ordering $(\mathcal{U}_1, \mathcal{V}_1, Q_b^{(1)}, G^{*, (1)}, M^{\bullet \dagger}_{(1)})) < (\mathcal{U}_2, \mathcal{V}_2, Q_b^{(2)}, G^{*, (2)}, M^{\bullet \dagger}_{(2)})$ in $D(x)$ is given by $\mathcal{U}_1 \supset \mathcal{U}_2, \mathcal{V}_1 \supset \mathcal{V}_2, Q_b^{(2)} - Q_b^{(1)}$ is nonnegative (recall that the difference would contribute trivially to any rank equality), and $G^{*, (1)} < G^{*, (2)}$. Given any two objects $(\mathcal{U}_i, \mathcal{V}_i, Q_b^{(i)}, G^{*, (i)}, M^{\bullet \dagger}_{(i)}), i = 1, 2$ in $D(x)$, let $Q'_b$ be a (negatively stabilized) quadratic form supported on $(\text{the stabilization of}) b_{\epsilon_x}$ such that $Q'_b - Q_b^{(i)} |_{b_{\epsilon_x}}$ are strictly positive for $i = 1, 2$, and have eigenvalues bounded below by some $\epsilon > 0$. Let $Q''_b$ be a positive quadratic form supported on $b_{\epsilon_x}^+$ such that its eigenvalues are greater than some $R > 0$.

Let $\hat{Q}_b = Q'_b + Q''_b$. We claim that for $R$ sufficiently large, $\hat{Q}_b \geq Q_b^{(i)}, i = 1, 2$. Indeed, since $Q_b^{(i)}, i = 1, 2$ both have bounded eigenvalues, for any $u \in b_{\epsilon_x}$ and $v \in b_{\epsilon_x}^+$, we have

$$ (\hat{Q}_b - Q_b^{(i)})(u + v) = (Q'_b(u) - Q_b^{(i)}(u)) + (Q''_b(v) - Q_b^{(i)} (v)) - 2Q_b^{(i)}(u, v) $$

$$ \geq (Q'_b(u) - Q_b^{(i)}(u)) + (Q''_b(v) - Q_b^{(i)}(v)) - 2K|u||v| $$

for some $K > 0$ depending only on $Q_b^{(i)}, i = 1, 2$. Thus for $R$ sufficiently large, (4.43) is always nonnegative. Now we fix such $Q'_b$ and $Q''_b$ as above such that $\varphi_{Q_b}(F\ell(\ell_x))$ is of graph type and has no negative spectral part. For sufficiently large $K, R' > 0$ and sufficiently small $\epsilon' > 0$, we can choose $(\hat{U}, \hat{V}, \hat{Q}_b, G^{*, (K, K', R, b_{\epsilon_x}), M^{\bullet \dagger}})$ as defined in Lemma 4.5 such that it dominates both $(\mathcal{U}_i, \mathcal{V}_i, Q_b^{(i)}, G^{*, (i)}, M^{\bullet \dagger}_{(i)}), i = 1, 2$.

(b). We show that $D(x) \hookrightarrow C(x)$ is cofinal, which means that for any $(\mathcal{U}, \mathcal{V}, Q_b, G^*, M^{\bullet \dagger}) \in C(x)$, the $\infty$-category $D(x) \times C(x)|_{(\mathcal{U}, \mathcal{V}, Q_b, G^*, M^{\bullet \dagger})}$ is contractible. Since $D(x) \times C(x)|_{(\mathcal{U}, \mathcal{V}, Q_b, G^*, M^{\bullet \dagger})}$ is a left fibration over $D(x)$ and $D(x)$ is a filtered poset, the geometric realization $|D(x) \times C(x)|_{(\mathcal{U}, \mathcal{V}, Q_b, G^*, M^{\bullet \dagger})}|$ can be calculated by the (homotopy) colimit

$$ \lim_{j \to \infty} \text{Maps}_{QHam_L(U/O)}((\mathcal{U}, \mathcal{V}, Q_b, G^*, M^{\bullet \dagger}), (\mathcal{U}_j, \mathcal{V}_j, Q_b^{(j)}, G^{*, j}, M_j^{\bullet \dagger})) $$

over any cofinal sequence $(\mathcal{U}_j, \mathcal{V}_j, Q_b^{(j)}, G^{*, j}, M_j^{\bullet \dagger})$ in $D(x)$ if it exists (we will see the existence below). Since the morphisms in the above sequence are all inclusions, the colimit is taking infinite unions.

Without loss of generality (up to congruent relations), we may assume that $F\ell(\varphi_{Q_b}(F\ell(\ell_x)))$ has spectral decomposition concentrated in $\pm 1$ and $\infty$, and let $V_1, V_{-1}$ and $V_{\infty}$ be the corresponding eigenspaces (note that $V_{-1}$ is the stabilized part). First, Lemma 4.6 says
that the fiber product $D(x) \times C(x)(u,v,Q_{(n)},G^{\bullet},M^{\bullet\downarrow})$ is nonempty. Second, let

\[(4.44)\quad Q''_{b,R} = Q_{b}|_{b_x^{-}} + R \cdot I_{b_x^{-}}, \quad \tilde{Q}_{b,R} = Q''_{b,R} + Q_{b,R}, \quad R > 0\]

then $Q_{b} - \tilde{Q}_{b,R}$ is of the form

\[
\begin{pmatrix}
(1/2 + 1/R)I_{V_{1}} & 0 & Y_{13} \\
0 & (-1/2 + 1/R)I_{V_{-1}} & Y_{23} \\
Y_{13}^{T} & Y_{23}^{T} & -R \cdot I_{b_x^{-}}
\end{pmatrix}
\]

where $Y_{13}$ and $Y_{23}$ are independent of $R$ and has finite norm. Let

\[(4.45)\quad (U_{n}, V_{n}, \tilde{Q}_{b,n}, G^{\bullet}_{(n)}, M^{\bullet\downarrow}_{(n)}), n \in \mathbb{Z}_{\geq 0}\]

be a sequence of objects in $D(x)$, where $\tilde{Q}_{b,n}$ is defined as in (4.44) with $R = n$; $G^{\bullet}_{(n)} = G^{\bullet}_{K_{n},\epsilon_{n},nR_{n},b_{x}}$, which has been defined in the proof of Lemma (4.4) with $K_{n} \uparrow +\infty$, $\epsilon_{n} \downarrow 0$, $R_{n} \uparrow +\infty$; $U_{n}$ and $V_{n}$ are decreasing and $\bigcap_{n} U_{n} = \bigcap_{n} V_{n} = \{\ell_{x}\}$. Then we have the following observations

1. The sequence (4.45) is cofinal in $D(x)$;
2. For any $\epsilon > 0$, consider the space of quadratic forms $Q_{12}$ with $Q_{12}|_{V_{1}} > (1/2 + \epsilon)I_{V_{1}}$ of rank $\dim V_{1}$ (so automatically positive) satisfying
   2a) the support intersects trivially with that of every $Q_{1} \in G$;
   2b) The eigenvalues of $Q_{12}$ are less than some $R' > 0$.

Then it satisfies that $Q_{12} \geq Q_{b} - \tilde{Q}_{b,n}$ and $G, G + Q_{12} \subset G_{(n)}$ for $n$ large enough, and this space embeds into the mapping space from $(U_{n}, V_{n}, \tilde{Q}_{b,n}, G^{\bullet}_{(n)}, M^{\bullet\downarrow}_{(n)})$ to $(U_{n}, V_{n}, \tilde{Q}_{b,n}, G^{\bullet}_{(n)}, M^{\bullet\downarrow}_{(n)})$ as an open subset for $n \gg 0$. 
Let $F$ be the functor $\gamma$ where $\mu$ is the constant sheaf of commutative topological monoid $\mathbb{Z}$.

The functor $\gamma$ is the constant sheaf of commutative topological monoid $\mathbb{Z}$.

Proof. By assumption ic) on $(\mathcal{U}, Q_b, G^\bullet, M^\bullet)$, the sublocus $Z_{V_1}$ of $\text{Hom}(V_1, V_{-1} \oplus V_{\infty})$ as an open subset in $\text{Gr}(\dim V_1, \infty)$ consisting of $E$ with $E \cap \text{Supp}(Q_1) = \{0\}$, $\forall Q_1 \in G$, is contractible. On the other hand, the space $(4.46)$ is a trivial fibration over $Z_{V_1}$, therefore it is also contractible. □

Recall that we have the universal principal $\Omega(U/O)$-bundle

$$
\begin{array}{ccc}
\mathbb{Z} \times BO & \simeq & \Omega(U/O) \\
\downarrow & \nearrow \eta_{U/O} & \\
B(\mathbb{Z} \times BO) & \simeq & U/O
\end{array}
$$

and the Gauss map $\gamma : L \to U/O$. A classical realization for $E(\mathbb{Z} \times BO)$ is $\text{Path}_*(U/O)$, the path space on $U/O$ with ending point at the monoidal unit $* \in U/O$.

**Proposition 4.13.**

(a) The functor $L\text{Kan}_{p}^{op}(G, M) : \text{Open}(L)^{op} \to \text{Mod}^{N(\text{Fin}_{\ast})}((\text{Sp}))$ determines a pair of presheaves of spaces $(\Phi_{\text{alg}}^{pre}, \Phi_{\text{mod}}^{pre}) \in \text{Mod}^{N(\text{Fin}_{\ast})}(\text{PShv}(L; \text{Sp}))$, whose sheafification $(\Phi_{\text{alg}}, \Phi_{\text{mod}})$ has its group completion isomorphic to the pair $((\mathbb{Z} \times BO)_{L}, \gamma^{-1}(\eta_{U/O}))$, where $(\mathbb{Z} \times BO)_{L}$ is the constant sheaf of commutative topological monoid $\mathbb{Z} \times BO$.

(b) Let $\mathfrak{S}_L$ denote for the locally constant sheaf of $(\text{stable } \infty\text{-})$categories on $L$ defined by

$$
\text{Open}(L)^{op} \to \text{Pr}_{\text{st}}^{L}
$$

$\mathcal{V} \mapsto \text{Loc}(\gamma^{-1}(\eta_{U/O})|_{\mathcal{V}}; \text{Sp})^{J\text{-equiv}},$

where $\gamma^{-1}(\eta_{U/O})|_{\mathcal{V}}$ means the total space as a module of $\mathbb{Z} \times BO$. There is a canonical equivalence of locally constant sheaf of categories $\mu\text{Shv}_{L} \simeq \mathfrak{S}_L$.

Proof.

(a) First, let $\mathcal{F}^{\text{pre}}_{(G, M)}$ denote for the presheaf on $Q\text{Ham}_{L}(U/O)$ defined by the functor $(G, M)$ in (4.42). For each object $(\mathcal{U}, \mathcal{V}, Q_b, G^\bullet, M^\bullet)$ in $Q\text{Ham}_{L}(U/O)$, the pair $(G, M)$ specifies a commutative submonoid of $\Omega(U/O)$ acting on a space of paths in $U/O$ starting from any point in $\mathcal{U}$ to the base point, i.e. a space of based loops acting on a space of sections of the fibration $\text{Path}_*(U/O) \to U/O$ over $\mathcal{U}$. This induces a map of presheaves on $Q\text{Ham}_{L}(U/O)$

$$
(4.47) \quad \overline{\tau} : \mathcal{F}^{\text{pre}}_{(G, M)} \to (p_{L}^{op})^{-1}((\mathbb{Z} \times BO)_{L}, \gamma^{-1}(\eta_{U/O})).
$$
More explicitly, for each morphism
\[ j_{Q_{12}} : (\mathcal{U}_1, \mathcal{V}_1, Q_1^{(1)}, G_{(1)}^{\bullet}, M_{(1)}^{\ast \dagger}) \rightarrow (\mathcal{U}_2, \mathcal{V}_2, Q_2^{(2)}, G_{(2)}^{\bullet}, M_{(2)}^{\ast \dagger}), \]
in \( \mathbf{QHam}_L(U/O) \), the following diagram

\[
\begin{array}{ccc}
(G_{(1)}, M_{(1)}) & \longrightarrow & (\text{Maps}(\mathcal{V}_1, \mathbb{Z} \times BO), \Gamma(\mathcal{V}_1, \gamma^{-1}(\eta_{U/O}))) \\
j_{Q_{12}} & & \\
(G_{(2)}, M_{(2)}) & \longrightarrow & (\text{Maps}(\mathcal{V}_2, \mathbb{Z} \times BO), \Gamma(\mathcal{V}_2, \gamma^{-1}(\eta_{U/O})))
\end{array}
\]

is commutative up to a contractible space of homotopies between concatenated paths \( \varphi_{Q_{12}}^t \cdot FT_t(\ell_x) \) and \( \varphi_{Q_{12}}^t \cdot \varphi_{Q_2}^t \cdot FT_t(\ell_x) \) for each \( x \in \mathcal{V} \), relative to their endpoints. We require the space of homotopies happen in the space of time-dependent Hamiltonian flows of nonpositively stabilized quadratic functions in \( \mathbf{p} \), denoted by \( \varphi_{Q_{12}}^t, 0 \leq t \leq 1 \) (so each gives a path \( \varphi_{Q_{12}}^t \cdot FT_t(\ell_x) \) with the endpoint graph like), such that the difference \( \Delta Q =: \int_0^1 Q_t dt - Q_0^{(1)} \geq 0 \) and it satisfies the equality (4.11) replacing \( \tilde{Q} \) (which means it has no contribution to any rank equalities as explained in Definition 4.4), hence contractible. It is clear that the space of homotopies are compatible with composition of morphisms in \( \mathbf{QHam}_L(U/O) \).

Now by the functoriality of left Kan extension along \( \mathbf{p}_L^{op} \), \( \tau (4.47) \) determines a morphism of presheaves
\[ \tau : (\Phi_{alg}^{pre}, \Phi_{mod}^{pre}) \longrightarrow ((\mathbb{Z} \times BO)_L, \gamma^{-1}(\eta_{U/O})). \]
To see that \( \tau \) induces an isomorphism of pairs after sheafification and group completion, we just need to show that \( \tau \) induces such isomorphisms at the stalk level. For each \( x \in \mathcal{L} \), the stalk of \( (\Phi_{alg}^{pre}, \Phi_{mod}^{pre}) \) at \( x \) is isomorphic to
\[
\lim_{(\mathcal{U}, \mathcal{V}, Q_s, G^{\bullet}, M^{\ast \dagger}) \in D(x)} (G, M)
\]
where \( D(x) \) is the same as in Lemma 4.12 and it can be calculated by a cofinal sequence in \( D(x) \) (cf. the proof of Lemma 4.12). It is then easy to see that this is isomorphic to the pair \( \prod_n BO(n) \) together with its torsor generated by the concatenated path \( \rho_{(Q_s, \ell_x)} := \varphi_{Q_s}^t \cdot FT_t(\ell_x) \), for any \( Q_s \) satisfying \( \varphi_{Q_s}(FT(\ell_x)) \) is of graph type and has no negative spectral part (here \( FT_t \) is as in (3.6)), and the induced morphism
\[ \tau_x : \lim_{(\mathcal{U}, \mathcal{V}, Q_s, G^{\bullet}, M^{\ast \dagger}) \in D(x)} (G, M) \simeq (\prod_n BO(n), \prod_n BO(n)(\rho_{(Q_s, \ell_x)})) \]
\[ \longrightarrow (\mathbb{Z} \times BO, \eta_{U/O}|_{\ell_x}) \]
becomes a weak homotopy equivalence after taking group completion on the left-hand-side.
In the following, we view \( G \) as a cosheaf by taking the left adjoint of the restriction functors. Let \( G^\text{pre} \) be the pre-cosheaf of categories on \( L \) defined by

\[
\begin{align*}
\text{Open}(L) & \to \text{Pr}^\text{st}_\text{st} \\
\mathcal{V} & \mapsto \text{Loc}(\Gamma(\mathcal{V}, \gamma^{-1}(\eta_{U/O})); \text{Sp})^{J\text{-equiv}}.
\end{align*}
\]

Then it is clear that \( G \) is a cosheafification of \( G^\text{pre} \), and in particular there is a canonical functor of pre-cosheaves \( G_L \to G_L^\text{pre} \). Let \( F_L^\text{pre} \) denote for the pre-cosheaf on \( \text{QHam}_L(U/O)^\text{op} \) defined by the functor \( \text{Loc}(-; \text{Sp})^{J\text{-equiv}} \) in (4.42). From part (a), we see that there is a canonical functor of pre-cosheaves of categories

\[
G_L \to G^\text{pre}_L \to (p_L)_*F^\text{pre}_L \cong (\text{RKan}_{p_L}(\text{Loc}(-; \text{Sp})^{J\text{-equiv}})).
\]

In the following, we will view \( \mu \text{Shv}_L \) as a cosheaf of categories. The functor \( F_L \) in (4.42) determines a canonical equivalence

\[
(4.48) \quad p^{-1}_L \mu \text{Shv}_L \sim F^\text{pre}_L.
\]

and therefore we get a functor of pre-cosheaves

\[
(4.49) \quad G_L \to (p_L)_*(F^\text{pre}_L) \to (p_L)_*p^{-1}_L(\mu \text{Shv}_L) \cong \text{RKan}_{p_L}(\text{ev}_0 \circ F_L).
\]

Let \( pt_{\text{QHam}_L(U/O)} \) denote for the constant functor \( \text{QHam}_L(U/O) \to \text{Spc} \) that maps every object to \( pt \). The left Kan extension \( \text{LKan}_{p_L}(pt_{\text{QHam}_L(U/O)}) \) determines a presheaf of spaces on \( L \), denoted by \((p_L)_!(pt_{\text{QHam}_L(U/O)})^\text{pre}\). Let \((p_L)_!pt_{\text{QHam}_L(U/O)}\) be the sheafification of \((p_L)_!(pt_{\text{QHam}_L(U/O)})^\text{pre}\). Applying Lemma 4.11 and the functor in (4.48), we get a functor of cosheaves:

\[
(4.50) \quad G_L \otimes \mu \text{Shv}_L^\gamma \to \text{Loc}_L((p_L)_!pt_{\text{QHam}_L(U/O)}),
\]

where \( \text{Loc}_L((p_L)_!pt_{\text{QHam}_L(U/O)}) \) is the cosheaf of categories on \( L \) defined as in Lemma 4.11. Since \( \mu \text{Shv}_L \) is locally constant with cofiber equivalent to \( \text{Sp} \), we can tensor with the dual local system of categories of \( \mu \text{Shv}_L \), denoted by \( \mu \text{Shv}_L^\gamma \) on both sides of (4.49), and get a functor of cosheaves

\[
(4.51) \quad T^*X \xleftarrow{f_!} T^*R^N \times_{R^N} X \xrightarrow{f^*_*} T^*R^N.
\]

4.5. The case for a general base manifold \( X \). For a general smooth manifold \( X \), there is a standard treatment to reduce it to the \( R^N \) case. One can choose a smooth embedding \( X \hookrightarrow R^N \) for some large \( N \) and denote the associated embedding \( X \times t \hookrightarrow R^N \times R_t \) by \( t \). For any smooth (immersed) Lagrangian \( L \) in \( T^*X \), let \( \tilde{L} \) be its image under the canonical correspondence

\[
(4.52) \quad T^*X \leftrightarrow T^*R^N \times_{R^N} X \xrightarrow{f_*} T^*R^N.
\]
which is a smooth (immersed) Lagrangian in $T^*\mathbb{R}^N$. There is an obvious fibration
$$\pi_{\tilde{L}} : \tilde{L} \rightarrow L,$$
whose fiber at $(x, \xi) \in L$ is a torsor over the fiber of the conormal bundle of $X$ in $\mathbb{R}^N$ at $x$.

The pushforward functor $\iota_*$ on microlocal sheaf categories induces an equivalence of sheaves of categories
$$\pi^{-1}\mu\text{Shv}_L \rightarrow \mu\text{Shv}_{\tilde{L}}.$$
So the classifying map for $\mu\text{Shv}_L$ is homotopic to
$$L \simeq \tilde{L} \xrightarrow{\gamma_{\tilde{L}}} U/O \rightarrow B\text{Pic}(k).$$
It is not hard to see that the composition of the inverse of the homotopy equivalence $\pi_{\tilde{L}}$ and $\gamma_{\tilde{L}}$ is homotopic to the stable Gauss map $L \xrightarrow{\gamma_L} U/O$. Recall that the stable Gauss map for $L$ is induced from the trivialization of the bundle of stable Lagrangian Grassmannian over $T^*X$ by the section $\sigma_X$ of taking tangent spaces of cotangent fibers. Then we have a commutative diagram

$$\begin{array}{ccc}
\pi^{-1}_L(\text{LagGr}_{T^*X}|_L), & \pi^{-1}_L(\sigma_X|_L) \xrightarrow{\sim} & \pi^{-1}_L(\text{LagGr}_{T^*\mathbb{R}^N}|_{\tilde{L}}, \sigma_{\mathbb{R}^N}|_{\tilde{L}}), \\
\downarrow & & \downarrow \\
\tilde{L} & \rightarrow & L
\end{array}$$

where the top rows consist of pairs of principal $U/O$-bundles together with a reference section, and the connecting map is doing the Lagrangian correspondence \[4.51\] on the tangent space level. Relating the tautological sections (induced from $L$ on the left) determines a homotopy between $\gamma_{\tilde{L}}$ and $\gamma_L \circ \pi_{\tilde{L}}$.

**Proof of Theorem 4.1.** The theorem follows directly from Proposition \[4.13\] (b) and the discussion above. \[\square\]

**Appendix A. The Canonical Functor**

$$\text{Corr}(\text{Fun}^\circ(N(\text{Fin}_*), \mathcal{C}^\times))_{\text{inert,all}} \rightarrow \text{CAlg}(\text{Corr}(\mathcal{C}^\times))^{\text{right-lax}}$$

Let $\mathcal{C}$ be an $\infty$-category that admits finite products. Then the Cartesian symmetric monoidal structure on $\mathcal{C}$ determines a canonical symmetric monoidal structure on $\text{Corr}(\mathcal{C})$. In [Jin], we gave several concrete constructions of commutative algebra objects, their modules, and (right-lax) morphisms among them using $\text{Fin}_*$-objects in $\mathcal{C}$ and correspondences among them, under easy-to-check conditions. In this appendix, we upgrade these constructions to a canonical functor from a correspondence category of a certain functor category $\text{Fun}^\circ(N(\text{Fin}_*), \mathcal{C}^\times)$ to the category of commutative algebras in $\text{Corr}(\mathcal{C}^\times)$ with right-lax morphisms. The main results are Theorem \[A.2\]. With a completely similar argument, one can get a version for the category of pairs of commutative algebras and their modules in $\text{Corr}(\mathcal{C})$ and a version for associative algebras and their modules (cf. Theorem \[A.4\]).
A.1. **An explicit model for the Gray tensor product with** \([n]\). First, for any ordinary 1-category \(J\) we present an explicit model \([\mathcal{P}]\) for \(\text{Seq}_* (J \otimes [n])\). For any \(\lambda \in \text{Seq}_k ([n])\), we will represent it by \(0 \leq \lambda(0) \leq \cdots \lambda(k) \leq n\). Then the 1-category \(\text{Seq}_k (J \otimes [n]) \times \text{Seq}_k ([n])\) has the same objects as

\[
\text{Seq}_k J \times \prod_{0 \leq i < k} \text{Seq}_1 (J) \times \prod_{0 \leq i < k} \text{Seq}_{\lambda(i+1)-\lambda(i)+1} (J),
\]

where the morphism \(\text{Seq}_k J \to \prod_{0 \leq i < k} \text{Seq}_1 (J)\) is induced from the \(k\) inert morphisms \([1] \cong \{i - 1, i\} \hookrightarrow [k]\), and the morphism \(\text{Seq}_{\lambda(i+1)-\lambda(i)+1} (J) \to \text{Seq}_1 J\) is induced from the unique active morphism \([1] \to [\lambda(i + 1) - \lambda(i) + 1]\). We will represent any object of \(\text{Seq}_k (J \otimes [n]) \times \text{Seq}_k ([n])\) by a pair \((\alpha, (\beta_i)_{0 \leq i < k})\), using the identification with objects in \(\text{Seq}_k (J \otimes [n])\). For two objects \((\alpha^{(j)}, (\beta_i^{(j)})_{0 \leq i < k})\), \(j = 0, 1\), in \(\text{Seq}_k (J \otimes [n]) \times \text{Seq}_k ([n])\), we have

\[
\text{Maps}((\alpha^{(0)}, (\beta_i^{(0)})), (\alpha^{(1)}, (\beta_i^{(1)}))) = \begin{cases} 
\prod_{0 \leq i < k} \text{Maps}(1 \times [1], J) \times \prod_{0 \leq i < k} \text{Maps}(\lambda(i) \times [1], J) \\
\prod_{j=0,1} \text{Maps}(j \times [\lambda(i) \times \lambda(i+1)+1], J) \{((\beta_i^{(0)}), (\beta_i^{(1)}))\}, & \text{if } \alpha^{(0)} = \alpha^{(1)}; \\
\emptyset, & \text{otherwise},
\end{cases}
\]

where \(\alpha^{(0)}\) is also viewed as the functor \([1] \times [k] \to [k] \xrightarrow{\alpha^{(0)}} J\) (see Figure 3). Pictorially,

![Figure 4](image)

each object is represented by a path decorated with two kinds of nodes in which the circle nodes indicate \(\alpha(j)\) and the blue cross node indicates \(\beta_i(s)\) as shown in Figure 5.

---

3We remark that there is a difference between our Gray tensor product \(X \otimes Y\) and the traditional definition up to a swap of \(X\) and \(Y\). The difference is caused by our convention to put \(X\) vertically (as indexing rows) and \(Y\) horizontally (as indexing columns). In particular, we have

\[
\text{Maps}_{\mathcal{2}\text{-Cat}} (X \otimes Y, W) \simeq \text{Maps}_{\mathcal{2}\text{-Cat}} (Y, \text{Fun}(X, W)_{\text{right-lax}}), \\
\text{Maps}_{\mathcal{2}\text{-Cat}} (Y \otimes X, W) \simeq \text{Maps}_{\mathcal{2}\text{-Cat}} (Y, \text{Fun}(X, W)_{\text{left-lax}}),
\]

where \(\text{Fun}(X, W)_{\text{right-lax}}\) (resp. \(\text{Fun}(X, W)_{\text{left-lax}}\)) consists of genuine functors between \((\infty, 2)\)-categories and has 1-morphisms right-lax (resp. left-lax) natural transformations.
Equivalently, we can view each $\beta_i$ as a functor

$$\beta_i : (\lambda(i) - 1, \lambda(i) + 1) \times (\lambda(i), \lambda(i+1)) \to (\beta_i : J \times [n])$$

such that $\beta_i : J$ maps every horizontal morphism to an identity morphism in $J$, and $\beta_i : [n]$ is taking the projection to the second factor and then including it to $[n]$ in the obvious way.

We take

$$\text{Seq}_k(J \otimes [n]) = \prod_{\lambda \in \text{Seq}_k([n])} \text{Seq}_k(J \otimes [n]) \times \{\lambda\}$$

The rules for the compositions are obvious. For any degenerate map $[k] \to [\ell]$ in $\Delta$, the functor $\text{Seq}_k(J \otimes [n]) \to \text{Seq}_k(J \otimes [n])$ is the obvious one, and for any active map $f : [k] \to [\ell]$, the functor $\text{Seq}_k(J \otimes [n]) \to \text{Seq}_k(J \otimes [n])$ is defined by joining $(\beta_v : f(i) \leq v < f(i+1)$ via the presentation (A.2) into a single $\beta_{f(i),f(j)}$

$$\text{Seq}_k(J \otimes [n]) \to \text{Seq}_k(J \otimes [n]) \to \text{Seq}_k(J \otimes [n])$$

(A.3) that removes all the joint circle nodes (see Figure 5). In the following, we will also use the notation

$$\beta^+_{i,j} : ([\lambda(i) - 1, \lambda(i + 1) - 1]) \times [\lambda(i), \lambda(i + 1)]) \to J \times [n].$$

(A.4)
to denote the joining of \((\beta^+_v)_{i\leq v\leq j}\) without removing the joint circle nodes. Here a joint circle node is marked by \((x_-, y)\) (resp. \((x_+, y)\)) if it has an outgoing edge (resp. incoming edge) connecting to (resp. from) \((x, y)\); the poset \([a_-, b_+]\) is equivalent to \([a - 1, b + 1]\) with \(a - 1\) and \(b + 1\) marked by \(a_-\) and \(b_+\) respectively, in particular the operation \(y - x\) is defined by identifying \(a_-\) (resp. \(b_+)\) with \(a - 1\) (resp. \(b + 1\)); in the formation of the coproduct, we have

\[
\begin{align*}
\{&(\lambda(v) - 1/2, \lambda(v))\} \leftrightarrow \left[\left[\lambda(v - 1), (\lambda(v) - 1)\right] \times \left[\lambda(v - 1), \lambda(v)\right]\right]_{0 \leq y - x \leq 1} \\
&(\lambda(v) - 1/2, \lambda(v)) \leftrightarrow ((\lambda(v) - 1)_+, \lambda(v)), \\
\{&(\lambda(v) - 1/2, \lambda(v))\} \leftrightarrow \left[\left[\lambda(v), (\lambda(v) + 1)\right] \times \left[\lambda(v), \lambda(v + 1)\right]\right]_{0 \leq y - x \leq 1} \\
&(\lambda(v) - 1/2, \lambda(v)) \leftrightarrow (\lambda(v), \lambda(v)).
\end{align*}
\]

For simplicity, by some abuse of notations, we will denote the domain of \(\beta^+_\{i,j\}\) in (A.4) by

\[(A.5) \quad ([\lambda(i)_-, (\lambda(j) - 1)_{+}] \times [\lambda(j - 1), \lambda(j)])_{0 \leq y - x \leq 1}.
\]

In a similar fashion, assume for each \(x \in [\lambda(v)_-, (\lambda(v + 1) - 1)_{+}], i \leq v < j\) we assign a value \(\kappa(x) \geq 0\), then we use

\[(A.6) \quad ([\lambda(i)_-, (\lambda(j) - 1)_{+}] \times [\lambda(j - 1), \lambda(j)])_{0 \leq y - x \leq \kappa(x)}
\]

to denote the coproduct as in the domain of \(\beta^+_\{i,j\}\) (A.4) with \(0 \leq y - x \leq 1\) replaced by \(0 \leq y - x \leq \kappa(x)\). We also set

\[
\beta^+_\{i,j\} : J = \text{proj}_J \circ \beta^+_\{i,j\}.
\]

We remark that the above model of \(\text{Seq}_k(J \otimes [n])\) also works for any \(\infty\)-category \(J\).

A.2. Definition of the canonical functor. Define \(\text{Fun}^\circ(N(\text{Fin}_*), \mathcal{C})\) to be the full subcategory of \(\text{Fun}(N(\text{Fin}_*), \mathcal{C})\) consisting of \(\text{Fin}_*\)-objects satisfying the condition in [Jin Theorem 2.6] (it was denoted by \(\text{Fun}^\circ(N(\text{Fin}_*), \mathcal{C})\) in loc. cit.). Let \(\text{inert}\) (resp. \(\text{active}\)) be the class of morphisms satisfying that the diagrams (2.4.9) (resp. (2.4.11)) in loc. cit. are Cartesian squares. Then \(\text{Seq}_k(\text{Corr}(\text{Fun}^\circ(N(\text{Fin}_*), \mathcal{C}^\times)))_{\text{inert, all}}\) is the full subcategory of \(\text{Maps}([[n] \times [n]^{op})_{\geq \text{dgnl}}, \text{Fun}(N(\text{Fin}_*), \mathcal{C}))\)

consisting of functors \(F\) satisfying

(i) Each square

\[
\begin{array}{ccc}
F(i, j + 1) & \longrightarrow & F(i, j) \\
\downarrow & & \downarrow \\
F(i + 1, j + 1) & \longrightarrow & F(i + 1, j)
\end{array}
\]

is a Cartesian square.

(ii) For each \((i, j) \in ([n] \times [n]^{op})\), \(F(i, j)\) as a \(\text{Fin}_*\)-object in \(\mathcal{C}\) satisfies the conditions in [Jin Theorem 2.6 (ii)], that is it defines a commutative algebra object in \(\text{Corr}(\mathcal{C})\);
(iii) For each fixed $j \in [n]^\text{op}$, the vertical morphisms between $\text{Fin}_\ast$-objects satisfy the condition that the diagram (2.4.9) in [Jin] is Cartesian for every $n$.

We are going to construct a canonical functor in $\mathcal{Spc}^{\Delta^\text{op}}$

$$P_{\mathcal{X}, \bullet} : \text{Seq}(\text{Corr}(\text{Fun}^\circ(N(\text{Fin}_\ast), \mathcal{C}^\times)))_{\text{inert,all}} \longrightarrow \text{Maps}_{\text{(1-Cat)}^{\Delta^\text{op}}/\text{Seq}_\bullet(N(\text{Fin}_\ast))}(\text{Seq}_\bullet(N(\text{Fin}_\ast) \otimes \bullet), \text{Seq}_\bullet(\text{Corr}(\mathcal{C}^\times) \otimes \text{Fin}_\ast)).$$

A.2.1. Step 1: the case for $\bullet = n$. For any ordinary 2-category (including 1-categories) $K$, let $I_K \rightarrow N(\Delta)^\text{op}$ be the coCartesian fibration from the Grothendieck construction of the functor $\text{Seq}_\bullet : N(\Delta)^{\text{op}} \rightarrow \text{1-Cat}^{\text{ord}}$.

For any $([k], (\alpha, (\beta_i), \lambda)) \in I_{N(\text{Fin}_\ast) \otimes [n]}$, let

$$\pi_{([k], (\alpha, (\beta_i), \lambda))} : \Gamma_{([k], (\alpha, (\beta_i), \lambda))} \longrightarrow ([k] \times [k]^{\text{op}})^{\geq \text{dgnl}}$$

be the Cartesian fibration from the Grothendieck construction for the natural functor

$$(([k] \times [k]^{\text{op}})^{\geq \text{dgnl}})^{\text{op}} \longrightarrow \text{1-Cat}^{\text{ord}}$$

$$(i, j) \mapsto ([\lambda(i) - (\lambda(j) - 1)]_{+} \times [\lambda(i), \lambda(j)]^{\text{op}})^{0 \leq y - x \leq \kappa_{\beta, \lambda}(x)}$$

(the right-hand-side was introduced in (A.6)), where for each $x \in [\lambda(v)_{-}, (\lambda(v + 1) - 1)_{+}]$, we define $\kappa_{\beta, \lambda}(x) = 1$ (resp. 0) if $x = \lambda(v)_{-}$ (resp. $x = (\lambda(v + 1) - 1)_{+}$) and in the other cases $\kappa_{\beta, \lambda}(x)$ is the largest integer $r$ such that the restriction

$$\beta^+_{[v, v + 1] ; N(\text{Fin}_\ast)} : ([x, x + r - 1] \times [x, x + r])^{0 \leq y_2 - y_1 \leq 1} \rightarrow N(\text{Fin}_\ast)$$

is a constant functor ($\beta^+_{[v, v + 1]}$ was introduced in (A.4) with $J = N(\text{Fin}_\ast)$). For any morphism $(i_0, j_0) \rightarrow (i_1, j_1)$ in $([k] \times [k]^{\text{op}})^{\geq \text{dgnl}}$, i.e. $i_0 \geq i_1$, $j_0 \leq j_1$, the corresponding functor is uniquely determined by the natural inclusion. For any $x \in [\lambda(i)_{-}, (\lambda(j) - 1)_{+}]$, let

$$x^+ = \begin{cases} x, & \text{if } x \neq \lambda(v)_{-}, (\lambda(v + 1) - 1)_{+} \text{ for any } i \leq v < j; \\ \lambda(v), & \text{if } x = \lambda(v)_{-}; \\ \lambda(v + 1), & \text{if } x = (\lambda(v + 1) - 1)_{+}. \end{cases}$$

Define

$$F_{([k], (\alpha, (\beta_i), \lambda))} : \Gamma_{([k], (\alpha, (\beta_i), \lambda))} \longrightarrow ([n] \times [n]^{\text{op}})^{\geq \text{dgnl}}$$

$$(i, j; x, y) \mapsto (x^+, y)$$

that takes the Cartesian arrows $(i_1, j_1; x, y) \rightarrow (i_0, j_0; x, y)$ to the identity morphisms.

Let $T^{\text{Comm}}$ be the ordinary 1-category defined by

- **Objects**: pairs $(\langle n \rangle, j \in \langle n \rangle^\circ), \langle n \rangle \in N(\text{Fin}_\ast)$,
- **Morphisms**: a morphism $(\langle n \rangle, j) \rightarrow (\langle m \rangle, k)$ is given by a morphism $\alpha : \langle m \rangle \rightarrow \langle n \rangle$ in $N(\text{Fin}_\ast)$ satisfying the condition $\alpha(k) = j$.

There is a natural projection $T^{\text{Comm}} \rightarrow N(\text{Fin}_\ast)^{\text{op}}$. See [Jin] Subsection 2.2.1 for the role of $T^{\text{Comm}}$ (and its associative version $T$) in the definition of a Cartesian fibration $\mathcal{C}^\times, (\text{Fin}_\ast)^{\text{op}} \rightarrow N(\text{Fin}_\ast)^{\text{op}}$ that exhibiting the Cartesian symmetric monoidal (and plain...
monoidal) structure of $\mathcal{C}$. Recall the definition of a Cartesian fibration $\mathcal{T}^{\text{Comm}} \to I_{N(\text{Fin}^*)}$ in \textit{loc. cit.} determined by the functor

$$
I^{\text{op}}_{N(\text{Fin}^*)} \to (1\text{-Cat})^{\text{ord}}
$$

$$
([k], \alpha) \mapsto ([k] \times [k]^{\text{op}})^{\geq \text{dgnl}} \times \mathcal{T}^{\text{Comm}}
$$

where on the right-hand-side the functor $([k] \times [k]^{\text{op}})^{\geq \text{dgnl}} \to N(\text{Fin}^*)^{\text{op}}$ is given by first projecting to $[k]^{\text{op}}$ and then applying $\alpha^{\text{op}}$.

Consider the natural functor

$$
I^{\text{op}}_{N(\text{Fin}^*) \boxtimes [n]} \to 1\text{-Cat}^{\text{ord}}
$$

$$
([k], \alpha, (\beta_i), \lambda)) \mapsto \Gamma([k], \alpha, (\beta_i), \lambda) \times \mathcal{T}^{\text{Comm}}
$$

where the functor $\Gamma([k], \alpha, (\beta_i), \lambda) \to N(\text{Fin}^*)^{\text{op}}$ is from the composition of the projection to $[k]^{\text{op}}$ and applying $\alpha^{\text{op}}$, whose Grothendieck construction gives a Cartesian fibration

(A.11)

$$
\pi_{T^+, n} : \mathcal{T}^+_N(\text{Fin}^*) \boxtimes [n] \to I_N(\text{Fin}^*) \boxtimes [n].
$$

It is clear that the functors $F_{([k], \alpha, (\beta_i), \lambda)}$ (A.11) induces a functor

$$
F_{T^+, n} : \mathcal{T}^+_N(\text{Fin}^*) \boxtimes [n] \to ([n] \times [n]^{\text{op}})^{\geq \text{dgnl}}.
$$

Now for any $[n] \in N(\Delta)^{\text{op}}$, we have well defined functors

(A.12)

$$
\xymatrix{ I_N(\text{Fin}^*) \boxtimes [n] \ar[r]_-{p_n}^{= p_n(1)} \ar[d]_-{q_n} & ([n] \times [n]^{\text{op}})^{\geq \text{dgnl}} \times N(\text{Fin}^*) \ar[r]^-{\mathcal{T}^{\text{Comm}}} & \mathcal{T}^{\text{Comm}}}
$$

where $p_n(2)$ is the canonical functor that sends

$$
([k], \alpha, (\beta_i), \lambda; (i_0, j_0), x, y); u \in \alpha(j_0)^0) \mapsto (\beta^+_{(i_0, j_0); N(\text{Fin}^*)}(x \to (\lambda(j_0) - 1) + 1)(u) \cup \{\ast\}
$$

For any morphism in $([k], \alpha, (\beta_i), \lambda; \Gamma_{([k], \alpha, (\beta_i), \lambda)} \times \mathcal{T}^{\text{Comm}})$ whose projection to $([k] \times [k]^{\text{op}})^{\geq \text{dgnl}}$ is a horizontal (resp. vertical) morphism, $p_n(2)$ sends it to the obvious inert (resp. active) morphism. It is not hard to check (cf. [Jin Lemma 2.10]) that $p_n(2)$ canonically determines a functor from $\mathcal{T}^+_N(\text{Fin}^*) \boxtimes [n]$ to $N(\text{Fin}^*)$. The functor $q_n$ is determined by the projections $\pi_{([k], \alpha, (\beta_i), \lambda)}$ (A.8) and it is a Cartesian fibration.

Diagram (A.12) defines a sequence of canonical functors

$$
\xymatrix{ \text{Maps}([n] \times [n]^{\text{op}})^{\geq \text{dgnl}}, \text{Fun}(N(\text{Fin}^*), \mathcal{C})) \ar[r]^-{\text{op}_{p_n}} & \text{Maps}(\mathcal{T}^+_N(\text{Fin}^*) \boxtimes [n], \mathcal{C})}
$$

$$
\xymatrix{ \text{Maps}(I_N(\text{Fin}^*) \boxtimes [n], I_{N(\text{Fin}^*)}) \ar[r]^-{\text{RKan}^{p_n}} & \text{Maps}(I_N(\text{Fin}^*) \boxtimes [n], I_{N(\text{Fin}^*)}, \mathcal{C})}
$$
whose essential image lies in the full subcategory
\[
\text{Maps}_{I_N(Fin^\circ)}(\mathcal{I}_N^2, (\mathcal{C} \times I_N(Fin^\circ))) \cong \text{Maps}_{(1-Cat)/\text{Seq}(N(Fin^\circ))}(\text{Seq}(N(Fin^\circ)[n]), \mathcal{C})
\] Here \(\mathcal{I}_N^2\) and \(\mathcal{C} \times I_N(Fin^\circ)\) (resp. \((\mathcal{C} \times I_N(Fin^\circ))\)) are marked by the coCartesian (resp. Cartesian) edges over \(I_N(Fin^\circ)\), and \(\mathcal{C}
\]

Now we take the composite functor of the above and restrict it to the full subcategory
\[
\text{Seq}_n(\text{Corr}(\text{Fun}^\circ(N(Fin^\circ), \mathcal{C}))_{\text{inert, all}}) \subset \text{Maps}(((n) \times [n]^{op})_{\geq \text{dgnl}}, \text{Fun}(N(Fin^\circ), \mathcal{C}))
\]
(see the beginning of this subsection). It is straightforward to check that the functor canonically factors through the full embedding
\[
\text{Maps}_{(1-Cat)/\text{Seq}(N(Fin^\circ))}(\text{Seq}(N(Fin^\circ)[n]), \mathcal{C}) \hookrightarrow \text{Maps}_{(1-Cat)/\text{Seq}(N(Fin^\circ))}(\text{Seq}(N(Fin^\circ)[n]), \mathcal{C})
\]

Then we let
\[
P_{\mathcal{C} \times n} : \text{Seq}_n(\text{Corr}(\text{Fun}^\circ(N(Fin^\circ), \mathcal{C}))_{\text{inert, all}}) \rightarrow \text{Maps}_{(1-Cat)/\text{Seq}(N(Fin^\circ))}(\text{Seq}(N(Fin^\circ)[n]), \mathcal{C})
\]
be the resulting functor.

A.2.2. Step 2: Definition of \(P_{\mathcal{C} \times n}\). We just need to show the functoriality of \(P_{\mathcal{C} \times n}\) with respect to \(n\).

In the following, with some abuse of notation, let
\[
\beta_{i,J} : [\lambda(i)_{-}, (\lambda(i)_{-} + 1)] \rightarrow J
\]
replace for \(\beta_{i,J}\) defined in [A.2], which contains the same amount of information. For any \(\sigma : [n] \rightarrow [m]\) and \(\lambda : [k] \rightarrow [n]\) in \(N(\Delta)\) and \(\beta_{i,J}\) as above, define
\[
\sigma \circ \lambda(i)_{-}, (\sigma \circ \lambda(i)_{-} + 1) : J
\]
as follows. For each \(s \in [(\sigma \circ \lambda(i))_{-}, (\sigma \circ \lambda(i)_{-} + 1)]\), let
\[
r_{i}(s) = \begin{cases} 
\lambda(i)_{-}, & \text{if } s = (\sigma \circ \lambda(i))_{-}, \\
\max\{v \in [\lambda(i), \lambda(i)_{-} + 1] : (\sigma \circ \lambda(i))_{-} \leq s\}, & \text{if } \sigma \circ \lambda(i)_{-} \leq s < \sigma \circ \lambda(i)_{-} + 1, \\
\lambda(i)_{-} + 1, & \text{if } s = (\sigma \circ \lambda(i)_{-} + 1)_{+}.
\end{cases}
\]

Then for each \(s_1 \leq s_2\) in \([(\sigma \circ \lambda(i))_{-}, (\sigma \circ \lambda(i)_{-} + 1)]\), it is sent via \(\sigma \circ \beta_{i,J}\) to the composite morphism of \(\beta_{i,J} : [r_{i}(s_1), r_{i}(s_2)]\).
Now we can give a formula for the natural morphism in $1\text{-Cat}^\Delta^{op}$ associated to $\sigma : [n] \rightarrow [m]$

$$
\text{Seq}_\bullet(N(\text{Fin}_*) \otimes [n]) \rightarrow \text{Seq}_\bullet(N(\text{Fin}_*) \otimes [m])
$$
(A.13)

$$
([k], \alpha, (\beta_i), \lambda) \mapsto ([k], \sigma_* \alpha = \alpha, (\sigma_* \beta_i), \sigma \circ \lambda).
$$
The formula extends naturally to a commutative diagram of functors between ordinary 1-categories

$$
\begin{array}{ccc}
\mathcal{T}^+_N(\text{Fin}_*) \otimes [n] & \xrightarrow{f_{T,\sigma}} & \mathcal{T}^+_N(\text{Fin}_*) \otimes [m] \\
q_m \downarrow & & \downarrow q_m \\
I_N(\text{Fin}_*) \otimes [n] \times \mathcal{T}^{\text{Comm}} & \xrightarrow{f_{I,\sigma}} & I_N(\text{Fin}_*) \otimes [m] \times \mathcal{T}^{\text{Comm}}
\end{array}
$$
(A.14)

where $f_{I,\sigma}$ is the combination of (A.13) and the identity functor on $\mathcal{T}^{\text{Comm}}$, and $f_{T,\sigma}$ is defined as follows.

(i) The functor $f_{T,\sigma}$ sends each fiber of $q_m$ over $([k], \alpha, (\beta_i), \lambda; i, j; u \in \alpha(j))$ to the fiber of $q_m$ over $([k], \sigma_* \alpha, (\sigma_* \beta_i), \sigma \circ \lambda; i, j; u \in \sigma_* \alpha(j))$:

$$
([k], \alpha, (\beta_i), \lambda; i, j; x, y; u \in \alpha(j)) \mapsto ([k], \sigma_* \alpha, (\sigma_* \beta_i), \sigma \circ \lambda; i, j; \hat{\sigma}_\lambda(x), \sigma(y); u \in \sigma_* \alpha(j) = \alpha(j))
$$
where

$$
\hat{\sigma}_\lambda(x) = \begin{cases} 
\sigma(x), & \text{if } x \in (\lambda(v)_-, (\lambda(v + 1) - 1)_+) \text{ and } \sigma(x) < \sigma \circ \lambda(v + 1) \\
(\sigma \circ \lambda(v))_-, & \text{if } x = \lambda(v)_-, \\
(\sigma \circ \lambda(v + 1) - 1)_+, & \text{if } x = (\lambda(v + 1) - 1)_+ \text{ or } \sigma(x) = \sigma \circ \lambda(v + 1).
\end{cases}
$$

(ii) For each morphism in $I_N(\text{Fin}_*) \otimes [n] \times \mathcal{T}^{\text{Comm}}$ induced from $c : [\ell] \rightarrow [k]$ in $N(\Delta)$:

$$
([k], \alpha, (\beta_i), \lambda; i, j; u \in \alpha(j)) \mapsto ([\ell], c^* \alpha, (\beta'_i), \lambda \circ c; i', j'; u' \in \alpha(c(j'))) \text{ with a morphism } c^* (\beta_i) \rightarrow (\beta'_i), i \leq c(i') \leq c(j') \leq j \text{ and } (\alpha(c(j') \rightarrow j))(u') = u,
$$
any Cartesian morphism over it is of the form

$$
([k], \alpha, (\beta_i), \lambda; i, j; x, y; u \in \alpha(j)) \mapsto ([\ell], c^* \alpha, (\beta'_i), \lambda \circ c; i', j'; x, y; u' \in \alpha(c(j'))),
$$
where $x \in [\lambda \circ c(i')_-, (\lambda \circ c(j') - 1)_+]$ and $y \in [\lambda \circ c(i'), \lambda \circ c(j')]$, and if $x = \lambda \circ c(\gamma)_-$ or $(\lambda \circ c(\gamma + 1) - 1)_+$ on the right-hand-side, then it is understood as $x = \lambda(\gamma)_-$ or $(\lambda(\gamma + 1) - 1)_+$ respectively on the left-hand-side. The functor $f_{T,\sigma}$ sends the morphism (A.15) to

$$
([k], \sigma_* \alpha, (\sigma_* \beta_i), \sigma \circ \lambda; i, j; \hat{\sigma}_\lambda(x), \sigma(y); u \in \alpha(j)) \mapsto 
([\ell], c^* (\sigma \alpha), (\sigma \beta'_i), \sigma \circ \lambda \circ c; i', j'; \hat{\sigma}_{\lambda \circ c}(x), \sigma(y); u' \in \alpha(c(j'))) \text{.}
$$

Note that (A.16) is in general not a Cartesian morphism, for example if there exists $c(v) < q < c(v + 1)$ such that $\lambda \circ c(v) \leq \lambda(q) < \lambda(q + 1)$, then $\lambda \circ c(v + 1)$ is not in the interval $(\lambda(q) - 1, \lambda(q + 1))$.
and $\sigma(\lambda(q) - 1) = \sigma(\lambda(q))$, then
\[
\hat{\sigma}_\lambda(\lambda(q) - 1) = \sigma \circ \lambda(q)_-,
\]
\[
\hat{\sigma}_{\lambda \circ c}(\lambda(q) - 1) = \sigma \circ \lambda(q),
\]
but there is always a unique arrow $\hat{\sigma}_\lambda(x) \to \hat{\sigma}_{\lambda \circ c}(x)$ that makes (A.16) well defined.

The assignment (A.16) is clearly well behaved under compositions of Cartesian morphisms over $q_m$.

(iii) According to [Jin, Lemma 2.10], to see that the data (i) and (ii) give a well defined $f_{\mathcal{J}_+; \sigma}$, we just need to check that for each commutative diagram
\[
\begin{array}{ccc}
(k, \alpha, (\beta_\mu)), \lambda; i, j; x, y, u \in \alpha(j)^o & \xrightarrow{(f, c^* \alpha, (\beta_\mu'), \lambda \circ c'i', j'; \sigma)} & (k, \alpha, (\beta_\mu'), \lambda; i, j; x, y, u' \in \alpha(c(j'))^o) \\
(\hat{\sigma}_\lambda(x), \sigma(y), u \in \alpha(j)^o) & \xrightarrow{(f, \sigma \circ c \alpha, (\sigma, \beta_\mu'), \lambda \circ c'i'; j'; \sigma)} & (\hat{\sigma}_{\lambda \circ c}(x), \sigma(y), u' \in \alpha(c(j'))^o)
\end{array}
\]
with $x \leq z \leq w \leq y$, we have the following diagram commute
\[
\begin{array}{ccc}
(k, \sigma \circ \alpha, (\sigma \circ \beta_\mu), \lambda; i, j; x, y, u \in \alpha(j)^o) & \xrightarrow{(f, \sigma \circ c \alpha, (\sigma \circ \beta_\mu'), \lambda \circ c'i'; j'; \sigma)} & (k, \sigma \circ c \alpha, (\sigma \circ \beta_\mu'), \lambda \circ c'i'; j'; \sigma)
\end{array}
\]
but this is obvious.

The diagrams (A.14) for different morphisms in $N(\Delta)$ determines a functor
\[
(A.17) \quad q^\bullet : \mathcal{J}_N(\text{Fin}_*)@[\bullet] \longrightarrow I_N(\text{Fin}_*)@[\bullet] \times \mathcal{J}_{\text{Comm}}
\]
in $(\text{1-Cat}^{\text{ord}})^\Delta$.

Now we have the following diagrams of functors (between ordinary 1-categories) and natural transformations
\[
\begin{array}{ccc}
\mathcal{J}_+^+\!_{N(\text{Fin}_*)@[n]} & \xrightarrow{p_n} & ([n] \times [n]^{op})^{\geq \text{dgln}} \times N(\text{Fin}_*) \\
\mathcal{J}_+^+\!_{N(\text{Fin}_*)@[m]} & \xleftarrow{p_m} & ([m] \times [m]^{op})^{\geq \text{dgln}} \times N(\text{Fin}_*)
\end{array}
\]
\[
(A.18) \quad f_M ; \sigma, \delta_{\sigma} \quad (\sigma, \sigma^{op}; \text{id})
\]
where for each object $A = ([k], \alpha, (\beta_\mu), \lambda; i, j; x, y; u \in \alpha(j)^o)$, the corresponding morphism
\[
\delta_{\sigma}(A) : (\sigma, \sigma^{op}; \text{id}) \circ p_n(A) = ((\sigma, \sigma^{op})(F_{[k], \alpha, (\beta_\nu), \lambda}(x, y)); \beta_{[i, j]}^+(x \to (\lambda(j) - 1)_+^{-1}(u) \cup \{\ast\})
\]
\[
\to p_m \circ f_{\mathcal{J}_+; \sigma}(A) = (F_{[k], \alpha, (\sigma \circ \beta_\mu), \sigma \circ \lambda}(\hat{\sigma}_\lambda(x), \sigma(y)); (\sigma, \beta)^+_{[i, j]}(\hat{\sigma}_\lambda(x) \to (\sigma(\lambda(j)) - 1)_+^{-1}(u) \cup \{\ast\})
is the identity morphism in the first factor and is the obvious active map in the second factor. It is straightforward to check that (A.18) for different morphisms in \(N(\Delta)\) assemble to be a functor

(A.19)  
\[ N(\Delta) \otimes [1] \longrightarrow 1\text{-Cat}^{\text{ord}}. \]

For any \(\infty\)-category \(\mathcal{C}\) that admits finite products, the composition of (A.19) with \(\text{Fun}(-, \mathcal{C})\) gives a functor

(A.20)  
\[ p_{N(\Delta)}^{\text{op}; \mathcal{C}} : [1] \otimes N(\Delta)^{\text{op}} \longrightarrow 1\text{-Cat} \]

representing a left-lax natural transformation between two functors \(N(\Delta)^{\text{op}} \to 1\text{-Cat}\)

\[ \text{Fun}(\mathcal{C}[\bullet] \times \mathcal{C}[\bullet]', \mathcal{C}) \xrightarrow{\text{Fun}(\mathcal{T}_{N(\text{Fin}_*)}^+ \otimes \bullet))} \text{Fun}(I_{N(\text{Fin}_*)} \otimes \bullet, T^{\text{Comm}}, \mathcal{C}) \]

Composing with the right Kan extension along \(q^*\) (A.17), we get another left-lax natural transformation

(A.21)  
\[ \text{Fun}(\mathcal{C}[\bullet] \times \mathcal{C}[\bullet]', \mathcal{C}) \xrightarrow{\text{Fun}(\mathcal{T}_{N(\text{Fin}_*)}^+ \otimes \bullet))} \text{Fun}(I_{N(\text{Fin}_*)} \otimes \bullet, T^{\text{Comm}}, \mathcal{C}). \]

**Lemma A.1.** In fact \(\hat{p}_{\bullet, \mathcal{C}}\) (A.21) is a genuine natural transformation.

**Proof.** It suffices to check the following: given a functor  
\(Y : ([m] \times [m])^\text{dgnl} \times N(\text{Fin}_*) \to \mathcal{C}\)

and \(\sigma : [n] \to [m] \in N(\Delta)\), the induced morphism from \(\delta_\sigma\) on limits

\[ \lim_{(x, y) \in \Gamma_{[n], \alpha, (\beta_\nu), \lambda \mid (i, j)}} Y(\sigma(x^+), \sigma(y); \hat{\beta}_{[i, j]}^+(x \to (\lambda(j) - 1)_+)^{-1}(u) \cup \{\ast\}) \to Y(\sigma(x^+), \sigma(y); \sigma_\ast \hat{\beta}_{[i, j]}^+(x \to (\sigma \circ \lambda(j) - 1)_+)^{-1}(u) \cup \{\ast\}) \]

is an isomorphism, for all \(0 \leq i \leq j \leq k\) and \(u \in \alpha(j)^\circ\). Due to the functoriality on \(\sigma\), one can factor \(\sigma\) as a composition of face maps and degeneracy maps, and reduce to the cases when \(\sigma\) is either a face map or a degeneracy map, which can be checked directly. \qed

Thanks to the establishment of Lemma A.1, we can apply the same argument as in Step 1 to show that the restriction of \(\hat{p}_{\bullet, \mathcal{C}}\) to \(\text{Seq}_\bullet(\text{Corr}(\text{Fun}(\mathcal{C})^\circ(N(\text{Fin}_*))); \mathcal{C})_{\text{inert, all}}\) factors through \(\text{Maps}(1\text{-Cat})^{\otimes p}_{\text{Seq}_\bullet(N(\text{Fin}_*)))} \text{Seq}_\bullet(N(\text{Fin}_*) \otimes [\bullet]), \text{Seq}_\bullet(\text{Corr}(\mathcal{C})^\circ, \text{Fin}_*))\), and this defines \(P_{\mathcal{C}[\bullet]}\) (A.7).

**Theorem A.2.** There are canonically defined functors

(A.22)  
\[ \text{Corr}(\text{Fun}(\mathcal{C})^\circ(N(\text{Fin}_*), \mathcal{C}^\times))_{\text{inert, all}} \longrightarrow \text{CAlg}(\text{Corr}(\mathcal{C}^\times))^{\text{right-lax}}, \]

(A.23)  
\[ \text{Corr}(\text{Fun}(\mathcal{C})^\circ(N(\text{Fin}_*), \mathcal{C}^\times))_{\text{inert, active}} \longrightarrow \text{CAlg}(\text{Corr}(\mathcal{C}^\times)). \]

For any \(c \in \mathcal{C}\), let \(\text{Seg}(c)\) denote for the category of Segal objects \(C^\bullet\) in \(\mathcal{C}\) with \(C^0 \simeq c\). An immediate corollary is

**Corollary A.3** (Chapter 9, Corollary 4.4.5[GaRo]). There is a canonically defined functor

\[ \text{Seg}(c) \longrightarrow \text{Alg}(\text{Corr}(\mathcal{C})). \]
It is natural to expect that the canonical functors \( \text{(A.22)} \) and \( \text{(A.23)} \) are equivalences. Since we will not use this in this paper, we will pursue it in another note.

Recall the definition \( N(\text{Fin}_{n}) \) in [Jin] as the ordinary 1-category whose objects are finite sets of the form \( \langle n \rangle = \langle n \rangle \sqcup \{ \} \) and whose morphisms \( \langle n \rangle \rightarrow \langle m \rangle \) are those maps between finite sets that send \(*\) to \(*\) and \( \uparrow \) to \( \uparrow \). There is a natural functor \( \pi_{1} : N(\text{Fin}_{n}) \rightarrow N(\text{Fin}_{*}) \), and let \( I_{\pi_{1}} \) be the coCartesian fibration over \( \Delta^{1} \) by viewing \( \pi_{1} \) as a functor from \( \Delta^{1} \) to \( 1\text{-Cat}^{\text{ord}} \). By the almost same proof for Theorem \( \text{A.2} \) one gets the following theorem.

**Theorem A.4.** There are canonically defined functors

\[
\text{(A.24)} \quad \text{Corr}(\text{Fun}_{\times}(I_{\pi_{1}}, e^{\times}))_{\text{inert,all}} \rightarrow \text{Mod}^{N(\text{Fin}_{*})}(\text{Corr}(e^{\times}))^{\text{right-lax}},
\]

\[
\text{(A.25)} \quad \text{Corr}(\text{Fun}_{\times}(I_{\pi_{1}}, e^{\times}))_{\text{inert,active}} \rightarrow \text{Mod}^{N(\text{Fin}_{*})}(\text{Corr}(e^{\times})).
\]

**APPENDIX B. A LIST OF NOTATIONS AND CONVENTIONS FOR CATEGORIES**

We list a few notations of categories that we use, mostly following the literatures [GaRo] and [Lu2].

- A category in boldface means it is an \((\infty, 2)\)-category, and its underlying \((\infty, 1)\)-category (by removing all non-invertible 2-morphisms) is denoted by the same notation but not in boldface.
- \(1\text{-Cat}\) (resp. \(2\text{-Cat}\)): the \((\infty, 1)\)-category of \((\infty, 1)\)-categories (resp. \((\infty, 2)\)-categories); \(1\text{-Cat}^{\text{ord}}\): the \((\infty, 2)\)-category of \((\infty, 1)\)-categories; \(1\text{-Cat}^{\text{ord}}\): the full subcategory of \(1\text{-Cat}\) consisting of ordinary 1-categories, i.e. the morphism spaces are weakly homotopy equivalent to discrete sets; \(\text{Pr}^{L}_{\text{st}}\) (resp. \(\text{Pr}^{R}_{\text{st}}\)): the \((\infty, 2)\)-category of presentable stable \((\infty, 1)\)-categories with colimit-preserving (resp. limit-preserving) functors.
- \(\text{Corr}(e)^{\text{adm}}_{\text{vert,horiz}}\): the \((\infty, 2)\)-category of correspondences for an \(\infty\)-category \(e\) with the vertical (resp. horizontal) arrow in a correspondence (i.e. a 1-morphism) in the class \(\text{vert}\) (resp. \(\text{horiz}\)) of morphisms in \(e\), and 2-morphisms in the class \(\text{adm}\) of morphisms in \(e\).
- \(\text{S}_{\text{LCH}}\): the ordinary 1-category of locally compact Hausdorff spaces; \(\text{fib}\) (resp. \(\text{prop}, \text{open}\)): the class of morphisms in \(\text{S}_{\text{LCH}}\) of locally trivial fibrations (resp. proper maps, open embeddings); \(\text{Spc}\): the \((\infty, 1)\)-category of topological spaces (equivalently \(\infty\)-groupoids).
- \(\text{CAlg}(C)^{\text{right-lax}}\) (resp. \(\text{CAlg}(C^{\otimes})\)): the \((\infty, 1)\)-category of commutative algebra objects in a symmetric monoidal \((\infty, 2)\)-category \(C\) (resp. \((\infty, 1)\)-category \(C\)), with right-lax (resp. genuine) homomorphisms; \(\text{Mod}^{N(\text{Fin}_{*})}(C^{\otimes})^{\text{right-lax}}\) (resp. \(\text{Mod}^{N(\text{Fin}_{*})}(C^{\otimes})\)): similar as before but with commutative algebras replaced by pairs of a commutative algebra and its module.
- \(\text{Seq} \quad : \quad 2\text{-Cat} \rightarrow (1\text{-Cat})^{\text{adm}}\): the full embedding of 2-Cat into simplicial \(\infty\)-categories; \(\Delta^{n}\) and \([n]\) both mean the ordinary 1-category \(0 \rightarrow 1 \cdots \rightarrow n\) (though \(\Delta^{n}\) usually means the nerve of \([n]\), but we do not distinguish them here).
- \(\text{Shv}(X; \text{Sp}) = \text{Shv}(X; \text{S})\): both mean the category of sheaves on \(X\) valued in spectra.
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