LATTICE UNIFORMITIES INDUCING UNBOUNDED CONVERGENCE

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Abstract. A net \((x_\gamma)_{\gamma \in \Gamma}\) in a locally solid Riesz space \((X, \tau)\) is said to be unbounded \(\tau\)-convergent to \(x\) if \(|x_\gamma - x| \wedge u \xrightarrow{\tau} 0\) for all \(u \in X_+\). We recall that there is a locally solid linear topology \(u\tau\) on \(X\) such that unbounded \(\tau\)-convergence coincides with \(u\tau\)-convergence, and moreover, \(u\tau\) is characterised as the weakest locally solid linear topology which coincides with \(\tau\) on all order bounded subsets. It is with this motivation that we introduce, for a uniform lattice \((L, u)\), the weakest lattice uniformity \(u^*\) on \(L\) that coincides with \(u\) on all the order bounded subsets of \(L\). It is shown that if \(u\) is the uniformity induced by the topology of a locally solid Riesz space \((X, \tau)\), then the \(u^*\)-topology coincides with \(u\tau\). This allows comparing results of this paper with earlier results on unbounded \(\tau\)-convergence. It will be seen that despite the fact that in the setup of uniform lattices most of the machinery used in the techniques of \([22]\) is lacking, the concept of ‘unbounded convergence’ well fittingly generalizes to uniform lattices. We shall also answer Questions 2.13, 3.3, 5.10 of \([22]\) and Question 18.51 of \([21]\).

1. Introduction

Uniform lattices were introduced in \([24, 26]\) as a generalization of topological Boolean rings (= Boolean rings endowed with an FN-topology) and of locally solid Riesz spaces or, more general, of locally solid \(\ell\)-groups. A uniform lattice is a lattice with a lattice uniformity, i.e. a uniformity making the lattice operations \(\lor\) and \(\land\) uniformly continuous. The uniformity induced by a group topology \(\tau\) on a Boolean ring is a lattice uniformity iff \(\tau\) is an FN-topology (in the sense of \([8]\)); the right (left or two-side) uniformity induced by a group topology \(\tau\) on an \(\ell\)-group is a lattice uniformity iff \(\tau\) is locally solid (see \([24\text{, Propositions 1.1.7 and 1.1.8}]\)).

Recently several authors have studied in a locally solid Riesz space \((X, \tau)\) the concept of ‘unbounded \(\tau\)-convergence’ of a net \((x_\alpha)_{\alpha \in A}\) to \(x\), i.e. \(|x_\alpha - x| \land u \xrightarrow{\tau} 0\) for all \(u \in X_+\). ‘Unbounded convergence’ was studied in \([7, 13, 14]\) when \(\tau\) is the norm topology on a Banach lattice, and in \([22, 6]\), more generally, when \(\tau\) is a locally solid linear topology on a Riesz space. Taylor \([22\text{, Theorem 2.3}]\) showed that there is a locally solid linear topology \(u\tau\) on \(X\) such that unbounded \(\tau\)-convergence coincides with \(u\tau\)-convergence. It can be shown that \(u\tau\) is the weakest locally solid linear topology which coincides with \(\tau\) on all order bounded subsets (cf. Corollary \([13\text{, Theorem 4.8}]\)).

This is the motivation to introduce here for a uniform lattice \((L, u)\) the weakest lattice uniformity \(u^*\) on \(L\) which coincides with \(u\) on all order bounded subsets of \(L\). The \(u^*\)-topology is then the weakest lattice topology (i.e. making \(\lor\) and \(\land\) continuous) which coincides with the \(u\)-topology on all
order bounded subsets of $L$. If $u$ is the uniformity induced by the topology of a locally solid Riesz space $(X, \tau)$, then the $u^*$-topology coincides with $u\tau$ (see Theorem 4.9). This allows comparing results of this paper with earlier results on unbounded $\tau$-convergence. Apart from showing how the concept of ‘unbounded convergence’ well fittingly generalizes to uniform lattices, we shall also answer Questions 2.13, 3.3, 5.10 of [22] and Question 18.51 of [21].

The paper is organized as follows. In Section 2 we consider two modes of order convergence (Definition 2.1) and prove a result (Theorem 2.3) that allows us to answer [21] Question 18.51 and [22] Question 5.10] (see Corollaries 2.5 and 2.6). Section 3 is a summary of tools on uniform lattices that are most relevant to our work. In Section 4 we extend the notion of unbounded convergence to uniform lattices and show that our definition is a ‘faithful’ generalization to the one that has been studied in the existing literature on locally solid Riesz spaces. Section 5 is devoted to the study of the uniformity $u^*$ on sublattices. In particular, Theorems 5.7 and 5.8 answer Questions 3.3 and 2.13 of [22].

2. Order convergence, Order topology and Order continuous functions

Let $(P, \leq)$ be a partially ordered set. A subset $D$ of $P$ is directed provided it is nonempty and every finite subset of $D$ has an upper bound in $D$. Dually, a nonempty subset $F$ of $P$ is filtered if every finite subset of $F$ has a lower bound in $F$. If every subset of $P$ that is bounded from above has a supremum and every subset of $P$ that is bounded from below has an infimum, $P$ is said to be Dedekind complete (or conditionally complete). $P$ is complete if every of its subsets has a supremum and infimum.

Let $(x_\gamma)_{\gamma \in \Gamma}$ be a net in a set $X$. If $\Gamma'$ is some directed set and $\varphi : \Gamma' \to \Gamma$ is increasing and final, then the net $(x_{\varphi(\gamma')})_{\gamma' \in \Gamma'}$ is called a subnet of $(x_\gamma)_{\gamma \in \Gamma}$. Let $P$ be a general property of nets. If there exists a $\gamma_0 \in \Gamma$ such the subnet $(x_\gamma)_{\Gamma \ni \gamma \geq \gamma_0}$ has the property $P$, then we say that $(x_\gamma)_{\gamma \in \Gamma}$ satisfies $P$ eventually. If $(x_\gamma)_{\gamma \in \Gamma}$ is increasing, its supremum exists and equals $x$, we write $x_\gamma \uparrow x$. Dually, $x, \downarrow x$ means that the net $(x_\gamma)_{\gamma \in \Gamma}$ is decreasing with infimum equal to $x$.

There are distinct definitions for order convergence that were used for various applications. We shall here be concerned with the following two modes of order convergence, namely $O_1$- and $O_2$-convergence. For every $a, b \in P$ define $[a, b] := \{x \in P : a \leq x \leq b\}$.

Definition 2.1. Let $(x_\gamma)_{\gamma \in \Gamma}$ be a net and $x$ a point in a poset $P$.

(i) $(x_\gamma)_{\gamma \in \Gamma}$ is said to $O_1$-converge to $x$ in $P$ if there exist two nets $(y_\gamma)_{\gamma \in \Gamma}, (z_\gamma)_{\gamma \in \Gamma}$ in $P$ such that eventually $y_\gamma \leq x_\gamma \leq z_\gamma, y_\gamma \uparrow x$ and $z_\gamma \downarrow x$.

(ii) $(x_\gamma)_{\gamma \in \Gamma}$ is said to $O_2$-converge to $x$ in $P$ if there exists a directed subset $M \subseteq P$, and a filtered subset $N \subseteq P$, such that $\sup M = \inf N = x$, and for every $(m, n) \in M \times N$ the net is eventually contained in $[m, n]$.

The notion of $O_1$-convergence can be traced back to Birkhoff [31] and Kantorovich [15]. In contrast to $O_1$-convergence, in $O_2$-convergence, the controlling nets are not indexed by the same directed set of the original net. $O_2$-convergence is due to McShane [17]. It is easy to see for $i \in \{1, 2\}$ that:

\[\text{i.e. for every } \gamma \in \Gamma \text{ there exists } \gamma' \in \Gamma' \text{ such that } \varphi(\gamma') \geq \gamma\]
• For a net \((x_\gamma)_{\gamma \in \Gamma}\), and point \(x\), \(x_\gamma \xrightarrow{O_1} x\) \(\Rightarrow\) \(x_\gamma \xrightarrow{O_2} x\).

• Every \(O_1\)-convergent net is eventually order bounded and every eventually constant net is \(O_1\)-convergent to its eventual value.

• Every subnet of an \(O_1\)-convergent net is \(O_1\)-convergent to the same limit.

• For a monotone net the two types of order convergence coincide.

• When \(P\) is a Dedekind complete lattice, the expression ‘\((x_\gamma)_{\gamma \in \Gamma}\) \(O_1\)-converges to \(x\)’ conveys the intuitive meaning\(^2\)

\[
\lim \inf x_\gamma = \sup \inf_{\gamma \in \Gamma} x_{\gamma'} = x = \inf \sup_{\gamma \in \Gamma} x_{\gamma'} = \lim \sup x_\gamma.
\]

The following example shows that – even when the poset is a Boolean algebra or a Riesz space – \(O_2\)-convergence does not imply \(O_1\)-convergence. This example is credited to D. Fremlin [20, Ex. 2, pg. 140]. We include it for completeness.

**Example 2.2** (\(O_2\)-convergence is not the same as \(O_1\)-convergence). Let \(X\) be an uncountable set and \(A\) the algebra of all finite and cofinite subsets of \(X\) (partially ordered by inclusion). Let \(x_1, x_2, x_3, \ldots\) be a sequence of distinct elements of \(X\) and \(A_n := \{x_n\}\). Then the sequence \((A_n)_{n \in \mathbb{N}}\) \(O_2\)-converges in \(A\) to \(\emptyset\) since the filtered family of all cofinite subsets of \(X\) has infimum \(\emptyset\), but is not \(O_1\)-convergent since any \(O_1\)-convergent sequence in \(A\) is eventually constant.

Let \(S(A) := \text{span}\{\chi_A : A \in A\}\) be the Riesz space of \(\mathbb{R}\)-valued \(A\)-simple functions. Then the sequence \((\chi_{A_n})_{n \in \mathbb{N}}\) \(O_2\)-converges to 0, but is not \(O_1\)-convergent in \(S(A)\).

A subset \(X\) of \(P\) is said to be \(O_1\)-closed if there is no net in \(X\) that is \(O_1\)-converging to a point outside of \(X\). In [1, Theorem 5] it was shown that \(X\) is \(O_1\)-closed iff it is \(O_2\)-closed. The collection of all \(O_1\)-closed (= \(O_2\)-closed) subsets of \(P\) forms the order topology \(\tau_O(P)\). It is easily seen that \([a, b]\) is \(O_1\)-closed for every \(a \leq b \in P\).

Motivated by the notion of unbounded order convergence (uo-convergence for short), studied for Riesz spaces in [10, 9, 11, 12], let us propose the following definition.

**Definition 2.3.** Let \(P\) be a poset and let \(F \subseteq P^P\). Let \(i \in \{1, 2\}\).

(i) The net \((x_\gamma)_{\gamma \in \Gamma}\) is said to \(F O_i\)-converge to \(x\) in \(P\) if \((f(x_\gamma))_{\gamma \in \Gamma}\) is \(O_i\)-convergent to \(f(x)\) for every \(f \in F\).

(ii) A subset \(X \subseteq P\) is said to be \(F O_i\)-closed if there is no net in \(X\) that is \(F O_i\)-converging to a point outside of \(X\).

Clearly, \(x_\gamma \xrightarrow{F O} x \Rightarrow x_\gamma \xrightarrow{F O_i} x\), for every net \((x_\gamma)_{\gamma \in \Gamma}\) and \(x\) in the poset \(P\), and \(F \subseteq P^P\). Although \(F O_1\)-convergence is different from \(F O_2\)-convergence, we shall prove the following result that will play an important role in order to answer [22, Question 5.10].

**Theorem 2.4.** Let \(P\) be a poset and \(F \subseteq P^P\). Let \((x_\gamma)_{\gamma \in \Gamma}\) be a net in \(P\) that \(F O_2\)-converges to \(x \in P\). Then \((x_\gamma)_{\gamma \in \Gamma}\) has a subnet that \(F O_1\)-converges to \(x\).

\(^2\)Note that for an \(O_i\)-convergent net in a Dedekind complete lattice, the suprema/infima involved in (1) exist eventually.
In particular, if a net \((x_\gamma)_{\gamma \in \Gamma}\) of \(P\) is \(O_2\)-convergent to \(x \in P\), then \((x_\gamma)_{\gamma \in \Gamma}\) has a subnet that \(O_1\)-converges to \(x\).

**Proof.** For every \(f \in F\) let \(M_f\) and \(N_f\) be the directed, and respectively, the filtered subsets of \(P\) ‘witnessing’ the \(O_2\)-convergence of \((f(x_\gamma))_{\gamma \in \Gamma}\) to \(f(x)\), i.e. \(M_f\) and \(N_f\) satisfy: (i) \(\sup M_f = \inf N_f = f(x)\), and (ii) \((f(x_\gamma))_{\gamma \in \Gamma}\) is eventually in \([m, n]\) for every \(m \in M_f\) and \(n \in N_f\).

For every finite subset \(F_0\) of \(F\), let \(M_{F_0}\) denote the set of all \(g \in P^{F_0}\) satisfying \(g(f) \in M_f\) for every \(f \in F_0\); and similarly, let \(N_{F_0}\) denote the set of all \(h \in P^{F_0}\) satisfying \(h(f) \in N_f\) for every \(f \in F_0\). Define \(\Upsilon := \{(F_0, g, h) : F_0 \subseteq F, |F_0| < \infty, g \in M_{F_0}, h \in N_{F_0}\}\).

It can easily be verified that the projection \(\varphi : (F_0, g, h) \mapsto \gamma\) is a directed set when equipped with the partial order inherited from \(\Upsilon \times \Gamma\). One can easily verify that the projection \(\varphi : (F_0, g, h) \mapsto \gamma\) is isotone and final; i.e. the net \((x_\varphi(\lambda))_{\lambda \in \Lambda}\) is a subnet of \((x_\gamma)_{\gamma \in \Gamma}\).

We show that the net \((x_\varphi(\lambda))_{\lambda \in \Lambda}\) is \(FO_1\)-convergent to \(x\). Fix an arbitrary \(f \in F\). Let \(\Lambda_f := \{(F_0, g, h, \gamma) \in \Lambda : f \in F_0\}\) and for every \((F_0, g, h, \gamma) \in \Lambda_f\) define \(m(F_0, g, h, \gamma) := g(f)\) and \(n(F_0, g, h, \gamma) := h(f)\). By construction, the net \((m(\lambda))_{\lambda \in \Lambda_f}\) is increasing, the net \((n(\lambda))_{\lambda \in \Lambda_f}\) is decreasing and \(m(\lambda) \leq f(x(\lambda)) \leq n(\lambda)\) for all \(\lambda \in \Lambda_f\).

Therefore \((f(x_\varphi(\lambda)))_{\lambda \in \Lambda_f}\) is \(O_1\)-convergent to \(f(x)\). Since \(\{\lambda \in \Lambda : \lambda \geq \lambda_0\} = \{\lambda \in \Lambda_f : \lambda \geq \lambda_0\}\) for \(\lambda_0 \in \Lambda_f\), also \((f(x_\varphi(\lambda)))_{\lambda \in \Lambda_f}\) is \(O_1\)-convergent to \(f(x)\). This shows that \((x_\varphi(\lambda))_{\lambda \in \Lambda}\) is \(FO_1\)-convergent to \(x\). In particular, by setting \(F\) equal to the singleton set containing just the identity function on \(P\), one obtains that every net of \(P\) that is \(O_2\)-convergent to some point admits a subnet that is \(O_1\)-convergent to the same point. \(\square\)

**Corollary 2.5.** Let \(P\) be a poset and \(F \subseteq P^P\).

(i) For \(X \subseteq P\), denote by \(\hat{X}^{FO_1}\) the set of \(x \in P\) such that there is a net in \(X\) which \(FO_1\)-converges to \(x\). Then \(\hat{X}^{FO_1} = \hat{X}^{FO_2}\).

(ii) A subset \(X \subseteq P\) is \(FO_1\)-closed iff it is \(FO_2\)-closed. (In particular, by setting \(F\) equal to the singleton set containing the identity map on \(P\), one recovers [1, Theorem 6].)

(iii) The collection of all \(FO_1\)-closed (=\(FO_2\)-closed) subsets of \(P\) forms a topology on \(L\). Denote this topology by \(\tau_{FO}(P)\).

(iv) Let \((Y, \tau)\) be a topological space. For every \(\varphi : P \to Y\) the following assertions are equivalent:
Lemma 2.7. Let \((G, +)\) be a commutative \(\ell\)-group. For every \(s, x, y \in G\) and \(a \in G_+\)

\[ |f_{s,s+a}(x) - f_{s,s+a}(y)| \leq |x - y| \land a = |f_{y,y+a}(x) - f_{y,y+a}(y)| + |f_{y,y-a}(x) - f_{y,y-a}(y)|. \]

Proof. To prove the inequality on the left-hand side observe that \(|f_{s,s+a}(x) - f_{s,s+a}(y)| \leq a\) follows since \(f_{s,s+a}(x), f_{s,s+a}(y) \in [s, s + a]\), and the inequality \(|f_{s,s+a}(x) - f_{s,s+a}(y)| \leq |x - y|\) follows by [4, p. 296].

Let us prove the equality on the right-hand side. First observe that

\[
\begin{align*}
f_{y,y-a}(x) - f_{y,y-a}(y) &= (x \land (y - a)) \lor y - (y \land (y + a)) \lor y, \\
&= (x \land (y + a)) \lor y - y \\
&= (x - y) \land a.
\end{align*}
\]
and

\[
\begin{align*}
    f_{y-a,y}(x) - f_{y-a,y}(y) &= (x \land y) \lor (y - a) - y \\
    &= ((x - y) \land 0) \lor (-a) \\
    &= -(x - y)^- \lor (-a) \\
    &= -(x - y)^- \land a.
\end{align*}
\]

Therefore, since \((x - y)^+ \land a\) and \((x - y)^- \land a\) are disjoint, we obtain

\[
\begin{align*}
    |f_{y+y+a}(x) - f_{y+y+a}(y)| + |f_{y-a,y}(x) - f_{y-a,y}(y)| &= ((x - y)^+ \land a) + ((x - y)^- \land a) \\
    &= ((x - y)^+ \land a) \lor ((x - y)^- \land a) \\
    &= |x - y| \land a.
\end{align*}
\]

\[\square\]

Let us recall that in an \(\ell\)-group \(O_i\)-convergence is: (i) additive, i.e. if \(x_\gamma \overset{O_i}{\rightarrow} x\) and \(y_\gamma \overset{O_i}{\rightarrow} y\), then \(x_\gamma + y_\gamma \overset{O_i}{\rightarrow} x + y\); and (ii) ‘commutes with taking inverses’, i.e. \(x_\gamma \overset{O_i}{\rightarrow} x\) iff \(\bar{x}_\gamma \overset{O_i}{\rightarrow} -x\). Making use of the decomposition \(x = x^+ - x^-\), one can easily deduce that \(x_\gamma \overset{O_i}{\rightarrow} 0\) iff \(|x_\gamma| \overset{O_i}{\rightarrow} 0\). Hence, the following proposition immediately follows by Lemma \ref{lem:2.7}.

**Proposition 2.8.** Let \((G, +, \tau)\) be a commutative \(\ell\)-group and \(A\) a solid subgroup of \(G\). Let

\[
F := \{ f_{s,t} : s, t \in G, t - s \in A_+ \}.
\]

Then \(x_\gamma \overset{uG_{O_i}}{\rightarrow} x\) iff \(x_\gamma \overset{uO_i}{\rightarrow} x\), for every net \((x_\gamma)_{\gamma \in \Gamma}\) and \(x\) in \(G\).

Proposition \ref{prop:2.8} shows that, for \(A = G\), \(u_G O_i\)-convergence coincides with \(uO_i\)-convergence. For \(A = G\) and \(i = 2\), Proposition \ref{prop:2.8} coincides with [19, Proposition 7.2].

**Corollary 2.9.** Let \((G, +)\) be a commutative \(\ell\)-group. Let \((x_\gamma)_{\gamma \in \Gamma}\) be a net in \(G\), \(x \in G\) and \(x_\gamma \overset{uG}{\rightarrow} x\). Then \((x_\gamma)_{\gamma \in \Gamma}\) has a subnet that \(uO_1\)-converges to \(x\).

**Proof.** This follows at once by Proposition \ref{prop:2.8} and Theorem \ref{thm:2.4} \(\square\)

**Corollary 2.10.** Let \((G, +)\) be a commutative \(\ell\)-group and \(A\) a solid subgroup of \(G\). Let \((Y, \tau)\) be a topological space and \(\varphi\) a function from \(G\) into \(Y\). Then the following conditions for \(\varphi\) are equivalent:

\[
\begin{itemize}
    \item \(x_\gamma \overset{u_{aO_1}}{\rightarrow} x \Rightarrow \varphi(x_\gamma) \overset{\tau}{\rightarrow} \varphi(x)\),
    \item \(x_\gamma \overset{u_{aO_2}}{\rightarrow} x \Rightarrow \varphi(x_\gamma) \overset{\tau}{\rightarrow} \varphi(x)\).
\end{itemize}
\]

**Proof.** This follows by Proposition \ref{prop:2.8} and Corollary \ref{cor:2.5} \(\square\)

In view of Proposition \ref{prop:2.8} Corollary \ref{cor:2.5} i) answers [21, Question 18.51].
3. Lattice uniformities – setting up the basics

Let $L$ be a lattice and $\Delta := \{(x, x) : x \in L\}$ the diagonal of $L \times L$. A subset $C$ of $L$ is called convex if $[a, b] \subseteq C$ whenever $a, b \in C$ and $a \leq b$. A lattice uniformity is a uniformity on a lattice making the lattice operations $\vee$ and $\wedge$ uniformly continuous. A uniform lattice is a lattice endowed with a lattice uniformity. One easily verifies:

**Proposition 3.1** ([24, Proposition 1.1.2]). Let $u$ be a uniformity on the lattice $L$. Then the following statements are equivalent:

(i) $u$ is a lattice uniformity;
(ii) for every $U \in u$ there exists $V \in u$ satisfying $V \vee V \subseteq U$ and $V \wedge V \subseteq U$;
(iii) for every $U \in u$ there exists $V \in u$ satisfying $V \vee \Delta \subseteq U$ and $V \wedge \Delta \subseteq U$.

It follows that the supremum of lattice uniformities on $L$ (built in the complete lattice of all uniformities on $L$) is also a lattice uniformity. We remark that – although it is more delicate to see – the analogous statement holds true also for the infimum of lattice uniformities, (see [27, Remark 2.6]).

The topology induced by a lattice uniformity $u$ on $L$, the $u$-topology, is a locally convex lattice topology, i.e. $\vee$ and $\wedge$ are continuous and every point of $L$ has a neighbourhood base consisting of convex sets.

As mentioned in the introduction, uniform lattices generalize topological Boolean rings on one-hand and locally solid $\ell$-groups (and therefore locally solid Riesz spaces) on the other. More specifically, we recall that (see [24, Proposition 1.1.7 & 1.1.8]) the uniformity induced by

- an FN-topology (i.e. a locally convex group topology) on a Boolean ring, or
- a locally solid group topology on an $\ell$-group,

is a lattice uniformity.

For the lattice uniformity $u$ we let $N(u) := \cap_{U \in u} U$. We note that $u$ is Hausdorff iff $N(u) = \Delta$. We recall [24 Proposition 1.1.3] that if $u$ is a lattice uniformity on $L$, every $U \in u$ contains a $V \in u$ such that the rectangle $[a \wedge b, a \vee b]^2 \subseteq V$ for all pairs $(a, b) \in V$.

**Proposition 3.2.** Let $u$ and $v$ be two lattice uniformities on $L$ that agree on every order bounded subset of $L$. Then $N(u) = N(v)$. In particular, $u$ is Hausdorff iff $v$ is Hausdorff.

**Proof.** If $(a, b) \in N(u)$, then the rectangle $[a \wedge b, a \vee b]^2 \subseteq N(u)$ and so, since the uniformity $u\mid_{[a \wedge b, a \vee b]}$ induced by $u$ on $[a \wedge b, a \vee b]$ agrees with $v\mid_{[a \wedge b, a \vee b]}$, we get $[a \wedge b, a \vee b]^2 \subseteq N(v)$, i.e. $(a, b) \in N(v)$. This shows $N(u) \subseteq N(v)$ and therefore $N(u) = N(v)$ by symmetry. □

A lattice uniformity $u$ on $L$ is said to be:

- **order continuous** (o.c.) if $(x_\gamma)_{\gamma \in \Gamma}$ converges to $x$ in $(L, u)$ whenever $(x_\gamma)_{\gamma \in \Gamma}$ is a monotone net in $L$ order converging to $x$;
- **exhaustive** if every monotone net in $L$ is Cauchy;

3A subset $A$ of an $\ell$-group $G$ is solid if for every for every $a \in A$ and $x \in G$ satisfying $|x| \leq |a|$, it holds that $x \in A$. A group topology on an $\ell$-group $G$ is locally solid if $0$ has a neighbourhood base consisting of solid sets.
• locally exhaustive if \( u_{|a,b]} \) is exhaustive for every \( a \leq b \) in \( L \).

It is easy to see that a lattice uniformity \( u \) is o.c. iff the \( u \)-topology is coarser than the order topology \( \tau_\mathcal{O}(L) \): Let \( u \) be o.c. and \( C \) be closed in \((L, u)\). If \((x_\gamma)_{\gamma \in \Gamma} \) is a net in \( C \), \( x \in L \) and \((y_\gamma)_{\gamma \in \Gamma}, (z_\gamma)_{\gamma \in \Gamma} \) are nets in \( L \) such that \( y_\gamma \uparrow x, z_\gamma \downarrow x \) and \( y_\gamma \leq x_\gamma \leq z_\gamma \) for all \( \gamma \in \Gamma \), then both, \((y_\gamma)\) and \((z_\gamma)\), converge to \( x \) w.r.t. \( u \), therefore \((x_\gamma)\) converges to \( x \) w.r.t. \( u \) since the \( u \)-topology is locally convex, hence \( x \in C \), i.e. \( C \) is order-closed. The other implication, that \( u \) is o.c. if the \( u \)-topology is coarser than \( \tau_\mathcal{O}(L) \), is obvious.

By [26, Proposition 6.1], \( u \) is exhaustive iff every monotone sequence in \( L \) is Cauchy.

**Theorem 3.3 (26, Proposition 6.3).** Let \((L, u)\) be a Hausdorff uniform lattice. Then the following conditions are equivalent:

(i) \((L, u)\) is complete (as a uniform space) and \( u \) is exhaustive;

(ii) \((L, \leq)\) is a complete lattice and \( u \) is o.c..

**Proposition 3.4 (26, Proposition 6.14).** Let \( S \) be a dense sublattice of a uniform lattice \((L, u)\). If \( u|_S \) is (locally) exhaustive, then \( u \) is (locally) exhaustive, too.

We recall that every Hausdorff uniform lattice \((L, u)\) is a sublattice and a dense subspace of a Hausdorff uniform lattice \((\tilde{L}, \tilde{u})\), which is complete as a uniform space (see [24, Proposition 1.3.1]). \((\tilde{L}, \tilde{u})\) is called the completion of \((L, u)\).

From Theorem 3.3 and Proposition 3.4 immediately follows

**Corollary 3.5.** Let \((L, u)\) be a (locally) exhaustive Hausdorff uniform lattice and \((\tilde{L}, \tilde{u})\) its completion. Then \((\tilde{L}, \leq)\) is a (Dedekind) complete lattice and \( \tilde{u} \) is o.c..

In Theorem 3.7 we need for a uniform lattice \((L, u)\) the following Condition (C) stronger than order continuity.

**Condition (C):** For every increasing (and resp. decreasing) order bounded net \((x_\gamma)_{\gamma \in \Gamma}\) in \( L \) and every \( U \in u \), there exists an upper bound (resp. lower bound) \( x \in L \) of \((x_\gamma)_{\gamma \in \Gamma}\), such that \((x_\gamma, x) \in U \) eventually.

**Proposition 3.6 (26, Proposition 7.1.2).** (i) Any lattice uniformity satisfying Condition (C) is o.c. and locally exhaustive.

(ii) If \( u \) is a lattice uniformity on a Dedekind complete lattice, then \( u \) satisfies Condition (C) iff \( u \) is o.c..

**Theorem 3.7 (26, Corollary 7.2.4).** Let \( u \) and \( v \) be exhaustive lattice uniformities on the lattice \( L \) satisfying the Condition (C). If \( N(u) = N(v) \), then the \( u \)-topology coincides with the \( v \)-topology.

Let us briefly consider locally solid \( \ell \)-groups (in particular, locally solid Riesz spaces). Following [2] (for locally solid Riesz spaces) we call a locally solid group topology a Lebesgue topology (resp. pre-Lebesgue topology) when the induced uniformity is o.c. (resp. locally exhaustive). In [26, Proposition 7.1.5] it is shown that the uniformity induced by a locally solid group topology on an Archimedean \( \ell \)-group satisfies Condition (C) iff it is o.c.. Therefore, by Proposition 3.6 (a), if an
Proposition 3.8 ([24], Proposition 1.2.4). Let \((L, u)\) be a uniform lattice. Define the equivalence relation \(\sim_u\) by setting \(a \sim_u b\) iff \((a, b) \in N(u)\). Let \(\hat{\alpha}\) denote the equivalence class containing the element \(a \in L\). Then the operations \(\hat{\alpha} \lor \hat{\beta} := a \lor b\) and \(\hat{\alpha} \land \hat{\beta} := a \land b\) turn the quotient \(\hat{L} := L/\sim_u\) into a lattice, and the family \(\hat{U} := \{\hat{(a, b)} : (a, b) \in U\}\), is a Hausdorff lattice uniformity on \(\hat{L}\). For any closed subset \(U\) of \((L, u)^2\) we have \((a, b) \in U\) iff \((\hat{a}, \hat{b}) \in \hat{U}\).

Lattice uniformities can be generated by systems of particular semimetrics. If \(q > 1\) and \(D\) is a system of semimetrics satisfying

\[ d(x \lor z, y \lor z) \leq q \cdot d(x, y) \quad \text{and} \quad d(x \land z, y \land z) \leq q \cdot d(x, y) \quad \text{for every} \ x, y, z \in L \ \text{and} \ d \in D, \]

then the uniformity generated by \(D\) on \(L\) is a lattice uniformity. Vice versa we have:

Proposition 3.9 ([25], Corollaries 1.4 and 1.6). Let \((L, u)\) be a uniform lattice and \(q > 1\). Then \(u\) is generated by a system \(D\) of semimetrics satisfying

\[ d(x \lor z, y \lor z) \leq q \cdot d(x, y) \quad \text{and} \quad d(x \land z, y \land z) \leq q \cdot d(x, y) \quad \text{for every} \ x, y, z \in L \ \text{and} \ d \in D. \]

If \(L\) is distributive, then one can here choose also \(q = 1\).

Let \((L, u)\) be a uniform lattice and \(F\) a set of lattice homomorphisms \(f : L \to (L, u)\). The sets \(U_f := \{(x, y) \in L^2 : (f(x), f(y)) \in U\}\), where \(U \in u\) and \(f \in F\), form a subbase of the initial uniformity of \(F\), i.e. of the coarsest uniformity on \(L\) making \(f\) uniformly continuous for all \(f \in F\) (see [25, Proposition 4 of II.2.3]). Using Proposition 3.1, one sees that this uniformity is a lattice uniformity: If \(U, V \in u\) with \(V \lor \Delta \subseteq U\) and \(V \land \Delta \subseteq U\), then \(V_f \lor \Delta \subseteq U_f\) and \(V_f \land \Delta \subseteq U_f\) for all \(f \in F\).

We are here interested in the initial uniformity w.r.t. functions of the type \(f_{a,b}(x) := (x \land b) \lor a\) or \(g_{a,b}(x) := (x \lor a) \land b\). Proposition 3.10(a) is the reason that later on we only consider distributive lattices. Proposition 3.10(b) shows that it is indifferent whether we study the initial uniformity w.r.t. functions of the type \(f_{a,b}\) or of the type \(g_{a,b}\); moreover, we can hereby always assume that \(a \leq b\).

Proposition 3.10. For a lattice \(L\) the following statements are equivalent:

(i) \(L\) is distributive.
(ii) \(f_{a,b}\) is a lattice homomorphism for all \(a, b \in L\).

Semimetrics are assumed to be real-valued.
(iii) \( g_{a,b} \) is a lattice homomorphism for all \( a, b \in L \).

(b) If \( L \) is distributive, then \( f_{a,b} = f_{a,a \lor b} = g_{a,a \lor b} \) and \( g_{a,b} = g_{a \lor b,b} = f_{a \lor b,b} \) for every \( a, b \in L \).

We omit the proof which is routine.

4. The uniformity \( u^* \) inducing ‘unbounded \( u \)-convergence’.

In this section, let \( u \) be a lattice uniformity on a distributive lattice \( L \).

For \( a, b \in L \), let \( u_{a,b} \) be the initial uniformity of \( f_{a,b} \). Since \( f_{a,b} : (L, u) \to (L, u) \) is uniformly continuous, \( u_{a,b} \subseteq u \) and, as explained before, \( u_{a,b} \) is a lattice uniformity since \( L \) is assumed to be distributive. Moreover, \( u_{a,b} \subseteq u_{a,b} \) if \( v \) is a lattice uniformity on \( L \) coarser than \( u \).

**Proposition 4.1.** Let \( a, b, c, d \in L \). Then

\[
u_{a \lor (b \land c), b \land d} \subseteq (u_{a,b})_{c,d}.
\]

In particular:

(i) If \( a \leq c \leq d \leq b \), then \( u_{c,d} \subseteq u_{a,b} \), and

(ii) \( (u_{a,b})_{a,b} = u_{a,b} \).

**Proof.** First observe that for \( x \in L \)

\[
f_{a,b} \circ f_{c,d}(x) = (((x \land d) \lor c) \land b) \lor a
\]

\[
= (x \land d \land b) \lor (c \land b) \lor a
\]

\[
= (x \land (d \land b)) \lor (a \lor (b \land c)) = f_{a \lor (b \land c), b \land d}(x),
\]

i.e. \( f_{a,b} \circ f_{c,d} = f_{a \lor (b \land c), b \land d} \). The functions \( f_{c,d} : (L, (u_{a,b})_{c,d}) \to (L, u_{a,b}) \) and \( f_{a,b} : (L, u_{a,b}) \to (L, u) \) are uniformly continuous, and therefore so is the function

\[
f_{a \lor (b \land c), b \land d} = f_{a,b} \circ f_{c,d} : \left( L, (u_{a,b})_{c,d} \right) \to (L, u).
\]

This shows that \( u_{a \lor (b \land c), b \land d} \subseteq (u_{a,b})_{c,d} \).

In particular, if \( a \leq c \leq d \leq b \), then \( a \lor (b \land c) = c \) and \( b \land d = d \), and therefore \( u_{c,d} \subseteq (u_{a,b})_{c,d} \subseteq u_{a,b} \), and by setting \( a = c \) and \( b = d \) one also obtains \( u_{a,b} = (u_{a,b})_{a,b} \). \( \Box \)

As observed before, it is enough to consider the uniformities \( u_{a,b} \) for \( a \leq b \). Define \( J(L) := \{ (a, b) \in L^2 : a \leq b \} \). If \( \emptyset \neq J \subseteq J(L) \) then the initial uniformity \( u_J \) of \( \{ f_{a,b} : (a, b) \in J \} \) is the uniformity having as a subbase the sets of the form

\[
U_{a,b} := \{ (x, y) \in L^2 : (f_{a,b}(x), f_{a,b}(y)) \in U \} \quad ((a, b) \in J, \ U \in u).
\]

Note that \( u_J \) is the supremum of the uniformities \( u_{a,b} \) where \( (a, b) \in J \), and thus \( u_J \) is again a lattice uniformity. Let \( u^* := u_{J(L)} \).

**Proposition 4.2.** Let \( J \) and \( J' \) be non-empty subsets of \( J(L) \).

(i) \( v_J \subseteq u_J \subseteq u \) for every lattice uniformity \( v \) coarser than \( u \).

(ii) If for every \( (a, b) \in J \) there exists \( (a', b') \in J' \) satisfying \( a' \leq a \leq b \leq b' \), then \( u_J \subseteq u_{J'} \) and \( (u_J)_{J'} = (u_{J'})_J = u_J \).
(iii) $u_J$ is the weakest lattice uniformity which agrees with $u$ on $[a, b]$ for all $(a, b) \in J$.

(iv) The $u_J$-topology is the weakest lattice topology which agrees with the $u$-topology on $[a, b]$ for all $(a, b) \in J$.

Proof. (i) is obvious.

(ii) The first assertion follows by (i) of Proposition 4.1 and by the fact that $u_J$ is the supremum of the uniformities $u_{a,b}$ for all $(a, b) \in J$.

For the second assertion, we first note that $(u_J)^r \subseteq u_J$ by (i). Using (i) and Proposition 4.1 we get $u_{a,b} = (u_J)^r_{a,b} \subseteq (u_J)^{r_J}$. Hence $u_J \subseteq (u_J)^{r_J}$ since $u_J$ is the supremum of the uniformities $u_{a,b}$, $(a, b) \in J$. Similarly, one can prove $u_J = (u_J)^r$.

(iii) Let us first show that $u_J$ agrees with $u$ on $[a, b]$ for all $(a, b) \in J$. Let $(a, b) \in J$, $U \in u$ and $U_{a,b} := \{(x, y) \in L^2 : (f_{a,b}(x), f_{a,b}(y)) \in U\}$. Then $U_{a,b} \cap [a, b]^2 = U \cap [a, b]^2$ since $f_{a,b}(x) = x$ for $x \in [a, b]$. Therefore $u_{a,b}$ and $u$ agree on $[a, b]$. It follows that $u_J$ and $u$ agree on $[a, b]$ since $u_{a,b} \subseteq u_J \subseteq u$. Now, let $v$ be a lattice uniformity on $L$ which agrees with $u$ on $[a, b]$ for all $(a, b) \in J$. We shall show that $u_{a,b} \subseteq v$ for $(a, b) \in J$. Since $v$ is a lattice uniformity, $f_{a,b} : (L, v) \to (L, u)$ is uniformly continuous. Since $f_{a,b}(L) \subseteq [a, b]$ and $v|_{[a, b]} = u|_{[a, b]}$, also $f_{a,b} : (L, v) \to (L, u)$ is uniformly continuous. Therefore $u_{a,b} \subseteq v$.

The proof of (iv) is similar to that of (iii). \qed

Corollary 4.3. (i) $(u^*)^* = u^*$.

(ii) $u^*$ is the weakest lattice uniformity which agrees with $u$ on any order bounded subset of $L$.

(iii) The $u^*$-topology is the weakest lattice topology which agrees with the $u$-topology on any order bounded subset of $L$.

In contrast to Corollary 4.3, in general, $u^*$ is not the weakest uniformity which agrees with $u$ on every order bounded subset of $L$, and the $u^*$-topology is not the weakest topology which agrees with the $u$-topology on any order bounded subset of $L$:

Example 4.4. Let $u$ be the usual uniformity on $\mathbb{R}$ induced by the absolute value.

(a) The infimum $v$ of all uniformities on $\mathbb{R}$ which agree with $u$ on all bounded subsets of $\mathbb{R}$ is the trivial uniformity.

(b) The sets $U_n := \{(x, y) \in \mathbb{R}^2 : |x - y| \leq \frac{1}{n}$ or $x, y \geq n$ or $x, y \leq -n\}$, $n \in \mathbb{N}$, form a base for $u^*$.

Proof. (a) For $a \in \mathbb{R}$, and $n \in \mathbb{N}$ let $V_{n,a}$ be the subset of $\mathbb{R}^2$ defined by

$$
(x, y) \in V_{n,a} \iff \begin{cases} 
|x - y| \leq \frac{1}{n}, & \text{or} \\
|x|, |y| \geq n, & \text{or} \\
|x| \geq n, |y - a| \leq \frac{1}{n}, & \text{or} \\
|y| \geq n, |x - a| \leq \frac{1}{n}.
\end{cases}
$$

The family $\{V_{n,a} : n \in \mathbb{N}\}$ form a base for a uniformity $v_a$ on $\mathbb{R}$ which agrees with $u$ on all bounded subsets of $\mathbb{R}$. Then $n \to a (v_a)$ and therefore $n \to a (v)$ since $v \subseteq v_a$. Let now $V \in v$ and $a, b \in \mathbb{R}$. Then $n \to a (v)$ and $n \to b (v)$, hence $(a, b) \in V$. This shows $V = \mathbb{R}^2$.

(b) immediately follows from the description of $u^*$ as initial uniformity. \qed
Remark 4.5. Note that in Example 4.4, the $u$-topology coincides with the $u^*$-topology; $u$ is complete whereas $u^*$ is not complete; the completion of $(\mathbb{R}, u^*)$ is $[-\infty, +\infty]$ with the usual compact uniformity.

Proposition 4.6. Let $S$ be a sublattice of $L$ and $I := J(S) = \{(a, b) \in S^2 : a \leq b\}$. Then $u_I$ is Hausdorff iff $u$ is Hausdorff and $x = \sup_{s \in S} s \wedge x = \inf_{s \in S} s \vee x$ for every $x \in L$.

Proof. Assume that $u_I$ is Hausdorff. Then also $u$ is Hausdorff since $u_I \subseteq u$. Let $x \in L$; we show that $\sup_{s \in S} s \wedge s = x$. Let $y \leq x$ with $x \wedge s \leq y$ for every $s \in S$. Then for every $(a, b) \in I$ we have $x \wedge b = y \wedge b$, therefore $(x \wedge b) \vee a = (y \wedge b) \vee a$ and hence $(x, y) \in N(u_I) = \Delta$, i.e. $x = y$. The proof of $\inf_{s \in S} x \vee s = x$ is similar.

Conversely, suppose that $u$ is Hausdorff and that $x = \sup_{s \in S} s \wedge x = \inf_{s \in S} s \vee x$ for every $x \in L$. Let $(x, y) \in N(u_I)$. Then, since $u$ is Hausdorff, we get that $(x \wedge b) \vee a = (y \wedge b) \vee a$ for every $(a, b) \in I$. Then,

$$
(x \wedge b) \vee a = (x \wedge (a \vee b)) \vee a = (y \wedge (a \vee b)) \vee a = (y \wedge b) \vee a,
$$

for arbitrary $a$ and $b$ in $S$, and therefore

$$
x = \sup_{b \in S} x \wedge b = \sup_{b \in S} \inf_{a \in S} (x \wedge b) \vee a = \sup_{b \in S} \inf_{a \in S} (y \wedge b) \vee a = \sup_{b \in S} y \wedge b = y.
$$

This shows that $u_I$ is Hausdorff. \qed

Corollary 4.7. $u$ is Hausdorff iff $u^*$ is Hausdorff.

Corollary 4.7 also follows from Corollary 4.3 (ii) and Proposition 3.2.

We now consider the situation when $u$ is the uniformity induced by the topology $\tau$ of a locally solid Riesz space, or more general, of a locally solid commutative $\ell$-group. We will compare the $u_J$-topology and $u^*$-topology, respectively, with $u_{A^\tau}$ and $u^\tau$ introduced by Taylor [22]. The following theorem was formulated by Taylor [22] Theorem 2.3 and §9] in the case that $\tau$ is a locally solid linear topology on a Riesz space, but in [21] it is mentioned a possible generalization for locally solid $\ell$-groups.

Theorem 4.8. Let $(G, +, \tau)$ be a Hausdorff locally solid commutative $\ell$-group, $\mathfrak{U}$ its 0-neighbourhood system and $A$ a solid subgroup of $G$. Then the sets

$$
\{x \in G : |x| \wedge a \in U\} \ (a \in A_+, U \in \mathfrak{U})
$$

form a 0-neighbourhood base for a locally solid group topology $u_{A^\tau}$ on $G$.

A net $(x_\alpha)$ converges to $x$ w.r.t. $u_{A^\tau}$ iff $|x_\alpha - x| \wedge a \xrightarrow{\tau} 0$ for any $a \in A_+$.

If $(G, \tau)$ is a locally solid Riesz space, then $(G, u_{A^\tau})$ is a locally solid Riesz space, too.

Following Taylor [22] we write $u^\tau$ instead of $u_{A^\tau}$ if $A = G$. Convergence w.r.t. $u^\tau$ is called in [22] unbounded $\tau$-convergence. Moreover, in the terminology of Taylor [22], $\tau$ is called unbounded iff $u^\tau = \tau$.

Theorem 4.9. Let $(G, +, \tau)$ be a Hausdorff locally solid commutative $\ell$-group and $A$ a solid subgroup of $G$. Let $u$ be the uniformity induced by $\tau$ and $J = \{(a, b) \in G^2 : b - a \in A_+\}$. Then the $u_J$-topology coincides with $u_{A^\tau}$. In particular, the $u^*$-topology coincides with $u^\tau$. 

Proof. Let \( \mathfrak{U} \) be a 0-neighbourhood base for \((G, \tau)\) consisting of solid sets. For \( s, t \in L \) with \( t - s \in A_+ \) and \( U \in \mathfrak{U} \) let
\[
\hat{U}(x_0, s, t) = \{ x \in G : f_{s,t}(x) - f_{s,t}(x_0) \in U \}.
\]
Then \( \{ \hat{U}(x_0, s, t) : U \in \mathfrak{U}, t - s \in A_+ \} \) is a neighbourhood subbase of \( x_0 \) w.r.t. the \( u_J \)-topology. We recall that the family \( \{ \hat{U}(x_0, a) : U \in \mathfrak{U}, a \in A_+ \} \), where
\[
\hat{U}(x_0, a) := \{ x \in G : |x - x_0| \wedge a \in U \},
\]
is a neighbourhood base of \( x_0 \) w.r.t. \( u_{A\tau} \).

Let \( U \in \mathfrak{U} \) and \( s, t \in G \) with \( a := t - s \in A_+ \). We show that \( \hat{U}(x_0, a) \subseteq \hat{U}(x_0, s, t) \). Let \( x \in \hat{U}(x_0, a) \). By (ii) of Lemma 2.7, one obtains
\[
|f_{s,t}(x) - f_{s,t}(x_0)| \leq |x - x_0| \wedge a \in U,
\]
i.e. \( x \in \hat{U}(x_0, s, t) \). This shows that the \( u_J \)-topology is coarser than \( u_{A\tau} \).

Let us show the converse. Let \( a \in A_+ \) and \( U \in \mathfrak{U} \). Choose \( V \in \mathfrak{U} \) with \( V + V \subseteq U \), and set \( r := x_0 - a \), \( s := x_0 \) and \( t := x_0 + a \). We show that \( \hat{V}(x_0, s, t) \cap \hat{V}(x_0, r, s) \subseteq \hat{U}(x_0, a) \). For \( x \in \hat{V}(x_0, s, t) \cap \hat{V}(x_0, r, s) \) one has by Lemma 2.7,
\[
|x - x_0| \wedge a = |f_{r,s}(x) - f_{r,s}(x_0)| + |f_{s,t}(x) - f_{s,t}(x_0)| \in V + V \subseteq U,
\]
hence \( x \in \hat{U}(x_0, a) \).

Under the assumption of Theorem 4.9 let \( I := \{ (-a, a) : a \in A_+ \} \). Then obviously \( u_I \subseteq u_J \). The following example shows that the \( u_I \)-topology can be strictly coarser the \( u_J \)-topology (\( = u_{A\tau} \)).

Example 4.10. Let \( G = \mathbb{R}^N \) and \( \tau \) be the locally solid group topology induced by \( \| (x_n) \| := \sum_{n=1}^{\infty} |x_n| \). Let \( A := \ell_\infty \), \( I := \{ (-a, a) : a \in A_+ \} \) and \( \sigma \) the \( u_I \)-topology. Then \( \sigma \) is strictly coarser than \( u_{A\tau} \).

Proof. Let \( x = (1, 2, 3, \ldots) \), \( e = (1, 1, 1, \ldots) \), \( y_n = \frac{1}{n} e \), \( x_n = x + y_n \).

We show that \( x_n \to x \) (\( \sigma \)), but \( x_n \not\to x \) (\( u_{A\tau} \)).

Let \( a \in A \), \( k \in \mathbb{N} \) with \( e \leq a \leq ke \). Then \( |(x_n \wedge a) \vee (-a) - (x \wedge a) \vee (-a)| \leq \frac{k}{n} \chi_{[1,k]} \) where \( \chi_{[1,k]} \) denotes the characteristic function of \( \{1, 2, \ldots, k\} \). Therefore \( \|(x_n \wedge a) \vee (-a) - (x \wedge a) \vee (-a)\| \leq \frac{k}{n} \to 0 \) (\( n \to \infty \)).

On the other hand, \( |x_n - x| \wedge a = y_n \), \( \|y_n\| = +\infty \) and therefore \( (x_n) \) does not converge to \( x \) w.r.t. \( u_{A\tau} \).

5. The uniformity \( u^* \) on sublattices

In this section, let \( u \) be a lattice uniformity on a distributive lattice \( L \).

Let \( D(L) \) denote the set of semimetrics \( d \) on \( L \) satisfying
\[
d(x \vee z, y \vee z) \leq d(x, y) \quad \text{and} \quad d(x \wedge z, y \wedge z) \leq d(x, y) \quad \text{for every} \; x, y, z \in L.
\]

By Proposition 3.9 \( u \) is generated by a subset of \( D(L) \), and therefore by the set \( D_u \) of all \( d \in D(L) \) which are uniformly continuous w.r.t. \( u \).
Remark 5.1. Let $J \subseteq J(L)$. For every $(a, b) \in J$ and $d \in D(L)$ denote by $d_{a,b}$ the semimetric on $L$ defined by $d_{a,b}(x, y) := d(f_{a,b}(x), f_{a,b}(y))$. It is easy to see then that the lattice uniformity $u_J$ is generated by the family $\{d_{a,b} : d \in D_J, (a, b) \in J\}$.

Lemma 5.2. Let $d \in D(L)$.

(i) Let $P_n : L^n \rightarrow L$ be given inductively by $P_1(x) := x$ and

\[
P_n(x_1, \ldots, x_n) := \begin{cases} P_{n-1}(x_1, \ldots, x_{n-1}) \lor x_n, & \text{or} \\ P_{n-1}(x_1, \ldots, x_{n-1}) \land x_n. & \end{cases}
\]

Then $d(P_n(x_1, \ldots, x_n), P_n(y_1, \ldots, y_n)) \leq \sum_{i=1}^n d(x_i, y_i)$.

(ii) $d(f_{a,b}(x), f_{a,b}(y)) \leq d(f_{c,d}(x), f_{c,d}(y)) + 2d(a, c) + 2d(b, d)$, for every $a, b, c, d, x, y \in L$.

Proof. The proof of (i) is by induction. The assertion is trivially true when $n = 1$. Suppose that $d(P_{n-1}(x_1, \ldots, x_{n-1}), P_{n-1}(y_1, \ldots, y_{n-1})) \leq \sum_{i=1}^{n-1} d(x_i, y_i)$. Let $\circ \in \{\land, \lor\}$. Then,

\[
d(P_n(x_1, \ldots, x_n), P_n(y_1, \ldots, y_n)) \\
= d(P_{n-1}(x_1, \ldots, x_{n-1}) \circ x_n, P_{n-1}(y_1, \ldots, y_{n-1}) \circ y_n) \\
\leq d(P_{n-1}(x_1, \ldots, x_{n-1}) \circ x_n, P_{n-1}(y_1, \ldots, y_{n-1}) \circ x_n) \\
+ d(P_{n-1}(y_1, \ldots, y_{n-1}) \circ x_n, P_{n-1}(y_1, \ldots, y_{n-1}) \circ y_n) \\
\leq d(P_{n-1}(x_1, \ldots, x_{n-1}), P_{n-1}(y_1, \ldots, y_{n-1})) + d(x_n, y_n) \\
\leq \sum_{i=1}^n d(x_i, y_i).
\]

To prove (ii) one first writes

\[
d(f_{a,b}(x), f_{a,b}(y)) \leq d(f_{a,b}(x), f_{c,d}(x)) + d(f_{c,d}(x), f_{c,d}(y)) + d(f_{c,d}(y), f_{a,b}(y)),
\]

and then apply (i) to obtain

\[
d(f_{a,b}(x), f_{a,b}(y)) \leq d(f_{c,d}(x), f_{c,d}(y)) + 2d(a, c) + 2d(b, d).
\]

Proposition 5.3. Let $\emptyset \neq J \subseteq J(L)$ and let $\bar{J} := \bar{J} \cap J(L)$, where $\bar{J}$ is the closure of $J$ in $(L \times L, u \times u)$. Then $u_J = u_{\bar{J}}$.

Proof. The inclusion $u_J \subseteq u_{\bar{J}}$ follows by Proposition 4.3(ii). For the reverse inclusion, suppose that $W \in u_{\bar{J}}$. We show that there exists $U \in u_J$ such that $U \subseteq W$. In view of Remark 5.1 there exist $d \in D_J$, $(a, b) \in \bar{J}$ and $\varepsilon > 0$, such that $(x, y) \in W$ for every $(x, y) \in L^2$ satisfying $d_{a,b}(x, y) < 5\varepsilon$. Let $(a', b') \in J$ with $d(a, a') < \varepsilon$ and $d(b, b') < \varepsilon$. Then, $U_{a', b'} := \{(x, y) \in L^2 : d_{a', b'}(x, y) < \varepsilon\} \in u_J$ and by Lemma 5.2(ii)

\[
d_{a,b}(x, y) \leq d_{a', b'}(x, y) + 2d(a, a') + 2d(b, b') < 5\varepsilon,
\]

for every $(x, y) \in U_{a', b'}$. This implies that $U_{a', b'} \subseteq W$.

Corollary 5.4. Let $S$ be a dense sublattice of $(L, u)$ and $v = u|_S$. Then $u^*_S = v^*$.
Proof. Let $J := J(S) = \{(a, b) \in S^2 : a \leq b\}$. First observe that $J(L) \subseteq \tilde{J}$: If $(a, b) \in J(L)$ and $(x_a)$, $(y_a)$ are nets in $S$ converging in $(L, u)$ to $a$ and $b$, respectively, then $J \ni (x_a \wedge y_a, x_a \vee y_a) \to (a, b)$ (w.r.t. $u$), hence $(a, b) \in \tilde{J}$.

Applying Proposition 5.3 we obtain $u^* = u_J$. Obviously $u_J|_S = v_J$.

Combining, we get $u^*|_S = u_J|_S = v_J = v^*$.

We now use besides Corollary 5.4 the following general observation:

**Lemma 5.5** ([23], pg. 381). Let $Y$ be a dense subspace of a uniform space $(X, v)$.

(i) Then $w \mapsto w|_Y$ defines a lattice isomorphism from the lattice of all uniformities on $X$ coarser than $v$ onto the lattice of all uniformities on $Y$ coarser than $v|_Y$.

(ii) Let $w$ be a uniformity on $X$ coarser than $v$ and $y_0 \in Y$. Then

$$\{\tilde{W}^v : W \text{ is a neighbourhood of } y_0 \text{ in } (Y, w|_Y)\}$$

is a neighbourhood base of $y_0$ in $(X, w)$; here $\tilde{W}^v$ denotes the closure of $W$ in $(X, v)$.

**Theorem 5.6.** Let $S$ be a dense sublattice of $(L, u)$ and $v = u|_S$.

(i) Then $u^* = u$ iff $v^* = v$.

(ii) If the $u^*$-topology coincides with the $u$-topology, then $v^*$-topology coincides with the $v$-topology.

**Proof.** (i) By Lemma 5.5, $u^* = u$ iff $u^*|_S = u|_S$. Now observe that $u|_S = v$ by definition and $u^*|_S = v^*$ by Corollary 5.4.

(ii) If the $u^*$-topology coincides with the $u$-topology, then the $u^*|_S$-topology coincides with the $u|_S$-topology, i.e., in view of Corollary 5.4, the $v^*$-topology coincides with the $v$-topology.

We don’t know whether in (ii) of Theorem 5.6 the converse implication is true. Lemma 5.5 (ii) and Corollary 5.4 only imply that every $s \in S$ has the same neighbourhood system in $(L, u^*)$ and in $(L, u)$ if $v^*$-topology coincides with the $v$-topology. But this is enough to answer the question of Taylor (see [T19, Question 3.3]) whether the property ‘unbounded’ passes from the Hausdorff locally solid Riesz space $(X, \tau)$ to its completion $(\tilde{X}, \tilde{\tau})$.

**Theorem 5.7.** Let $H$ be a dense solid subgroup of a locally solid commutative $\ell$-group $(G, \tau)$ and let $\sigma = \tau|_H$. Then $u\tau = \tau$ iff $u\sigma = \sigma$.

**Proof.** Let $u$ and $v$ be the uniformities induced, respectively, by $\tau$ and $\sigma$. We use that the $u^*$-topology and the $v^*$-topology coincide, respectively, with $u\tau$ and $u\sigma$ (see Theorem 4.9) and therefore $u\tau|_H = u\sigma$ (see Corollary 5.4).

If $u\tau = \tau$, then $u\sigma = \sigma$ by Theorem 5.6 (b). Vice versa, if $u\sigma = \sigma$, then $u\tau$ and $\tau$ have the same 0-neighbourhood system by Lemma 5.5 (ii). Therefore $u\tau = \tau$ since both, $u\tau$ and $\tau$, are group topologies.

The following Theorem generalizes [22 Proposition 2.12] and answers [22 Question 2.13].

**Theorem 5.8.** Let $S$ be a sublattice of the uniform lattice $(L, u)$. Then $S$ is (sequentially) closed w.r.t. $u^*$ iff it is (sequentially) closed w.r.t. $u$. 

Proof. For an infinite cardinal $\kappa$ let us say that a subset $U$ of a topological space is $\kappa$-closed when $U$ contains the limit of every convergent net $(x_\alpha)_{\alpha \in A}$ where $|A| \leq \kappa$, that is contained in $U$. We show that $S$ is $\kappa$-closed w.r.t. $u^*$ iff it is $\kappa$-closed w.r.t. $u$. Since $u^* \subseteq u$, clearly every $\kappa$-closed subset w.r.t. $u^*$ is $\kappa$-closed w.r.t. $u$. For the converse, assume that $S$ is $\kappa$-closed w.r.t. $u$ and suppose that $(s_\alpha)_{\alpha \in A}$ is a net in $S$ such that $|A| \leq \kappa$ and convergent to $x$ w.r.t. $u^*$. For every $s \in S$ and $z \in L$ satisfying $z \geq s$ we observe that $(s_\alpha \lor s) \land z \rightarrow (x \lor s) \land z$ w.r.t. $u$. In particular, if we fix $\alpha' \in A$ and set $z := s_{\alpha'} \lor s$, we obtain – by the hypothesis that $S$ is $\kappa$-closed w.r.t. $u$ – that $(x \lor s) \land (s_{\alpha'} \lor s) \in S$. Since $\alpha'$ was arbitrary, and since $s_\alpha \rightarrow x$ w.r.t. $u^*$, this in turn implies that
\[(x \lor s) \land (s_\alpha \lor s) \rightarrow (x \lor s) \land (x \lor s) = x \lor s \quad \text{w.r.t. } u.\]

Using again the fact that $S$ is $\kappa$-closed w.r.t. $u$ we deduce that

1. $x \lor s \in S$ for every $s \in S$.
2. Dually one obtains that $x \land s' \in S$ for every $s' \in S$.

Let now $s \in S$. Then $s' := x \lor s \in S$ by (i) and $x = x \land s' \in S$ by (ii). \qed

Corollary 5.9. Let $S$ be a sublattice of the uniform lattice $(L, u)$. Then the closures of $S$ in $L$ w.r.t. $u$ and $u^*$ coincide.

By an obvious modification of the proof of Theorem 5.8 one obtains an analogous result for unbounded order convergence, which generalizes [10, Proposition 3.15].

Proposition 5.10. Suppose that $x_\alpha \uparrow x$ implies $x_\alpha \land y \uparrow x \land y$ and $x_\alpha \downarrow x$ implies $x_\alpha \lor y \downarrow x \lor y$ for every net $(x_\alpha)$ in $L$ and $x, y \in L$ (i.e. $L$ is continuous in the sense of von Neumann – cf. [10, Definition 2.14]). Then a sublattice $S$ of $L$ is $uO$-closed iff it is $O$-closed.

Proof. $\Leftarrow$ can be proved as $\Leftarrow$ of Theorem 5.8 replacing $u^*$-convergence by $uO$-convergence and $u$-convergence by $O$-convergence. For $\Rightarrow$ observe that the additional assumption exactly means that $O$-convergence implies $uO$-convergence. Therefore any $uO$-closed subset of $L$ is $O$-closed. \qed

Proposition 5.11. The following conditions are equivalent:

(i) $u$ is locally exhaustive;
(ii) $u^*$ is locally exhaustive;
(iii) $u^*$ is exhaustive.

Proof. Since $u$ and $u^*$ coincide on bounded sets by Corollary 4.3 we have (i)$\Leftrightarrow$(ii). (iii)$\Rightarrow$(ii) is obvious.

(i)$\Rightarrow$(iii): By definition, a net $(x_\gamma)_{\gamma \in \Gamma}$ is $u^*$-Cauchy iff $(f_{a,b}(x_\gamma))_{\gamma \in \Gamma}$ is $u$-Cauchy for every $(a, b) \in J(L)$. Let now $(x_\gamma)_{\gamma \in \Gamma}$ be a monotone net in $L$ and $(a, b) \in J(L)$. Then $(f_{a,b}(x_\gamma))_{\gamma \in \Gamma}$ is monotone and bounded, hence $u$-Cauchy by (i), i.e. $(x_\gamma)_{\gamma \in \Gamma}$ is $u^*$-Cauchy. \qed

Theorem 5.12. Let $u$ and $v$ be Hausdorff lattice uniformities on $L$ satisfying Condition (C). Then:

1. the $u^*$-topology and the $v^*$-topology are equal;
2. $\overline{S}^u = \overline{S}^v$ for every sublattice $S$ of $L$. 
Proof. (i) First observe that the lattice uniformities $u^*$ and $v^*$ also satisfy Condition (C) since $u^* \subseteq u$ and $v^* \subseteq v$. Therefore $u^*$ and $v^*$ are locally exhaustive by Proposition 3.6 hence exhaustive in view of Proposition 5.11. Moreover, Corollary 4.7 yields that $u^*$ and $v^*$ are Hausdorff. So, (i) follows by Theorem 3.7.

(ii) follows from (i) and Corollary 5.3.

Note that (ii) of the above theorem generalizes [22, Theorem 5.11].

The following lemma is needed in the proof of Theorem 5.14 to reduce it to the Hausdorff case.

**Lemma 5.13.** Let $(\hat{L}, \hat{u}) = (L/\sim_u, \bar{u})$ be the Hausdorff uniform lattice associated with $(L, u)$ according to Proposition 3.8. Then

$$(\hat{L}, \hat{u}^*) = \left(\hat{L}, (\hat{u})^*\right).$$

**Proof.** Observe first that $N(u) = N(u^*)$ by Proposition 3.2 i.e. $L/\sim_u = L/\sim_u^*$. We show that $(\hat{u})^* = (\hat{u}^*)$. We shall make use of the following fact that follows immediately by Proposition 3.8:

If $v$ is a lattice uniformity on $L$ satisfying $N(u) = N(v)$, then

$$V \cap [a, b]^2 = \hat{V} \cap [\hat{a}, \hat{b}]^2,$$

for every closed subset $V$ of $(L, v)^2$ and $a \leq b$ in $L$.

Hence, for every $a \leq b$ in $L$ we have

$$\hat{u}^*|_{[\hat{a}, \hat{b}]} = \hat{u}^*|_{[a, b]} = u|_{[a, b]} = \hat{u}|_{[\hat{a}, \hat{b}]}$$

and therefore $\hat{u}^* \supset \hat{u}^*$. Conversely, if $v$ is a lattice uniformity on $L$ with $N(v) = N(u)$ and $\hat{v} = \hat{u}^*$, then $\hat{v}|_{[\hat{a}, \hat{b}]} = \hat{u}|_{[\hat{a}, \hat{b}]}$, and therefore $v|_{[a, b]} = u|_{[a, b]}$ for every $a \leq b$. This implies that $v \supset u^*$ and therefore $\hat{u}^* = \hat{v} \supset \hat{u}^*$. □

Let $u$ be Hausdorff and $(\hat{L}, \hat{u})$ the completion of $(L, u)$. It is easy to see that $\hat{L}$ is again distributive, when $L$ is distributive. Moreover, both properties of local exhaustivity and exhaustivity pass directly to this completion by Proposition 3.4.

**Theorem 5.14.** Let $(L, u)$ be a uniform lattice. If $u$ is exhaustive, then $u = u^*$.

**Proof.** (i) Let us first suppose that $u$ is Hausdorff. Let $(\hat{L}, \hat{u})$ denote the completion of $(L, u)$. Then $\hat{L}$ is a complete lattice by Corollary 3.5 and therefore obviously $\hat{u} = (\hat{u})^*$. On the other hand, $\hat{u}^*|_S = u^*$ by Corollary 3.4. Combining we get $u = \hat{u}|_S = (\hat{u})^*|_S = u^*$.

(ii) In general, if $u$ is exhaustive, $\hat{u}$ is Hausdorff and exhaustive, and therefore $\hat{u} = (\hat{u})^*$ by (i). Therefore, by Lemma 5.13 we obtain $\hat{u} = \hat{u}^*$. Since $N(u) = N(u^*)$, this implies that $u = u^*$. □

**Corollary 5.15.** Let $(L, u)$ be a uniform lattice and let $u$ be locally exhaustive. Then $u^*|_S = (u|_S)^*$ for every sublattice $S$ of $L$.

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\(^5\)It is easy to see and follows from Proposition 3.8 that if $u$ and $v$ are two lattice uniformities on $L$ satisfying $N(u) = N(v)$, then the induced uniformities $\hat{u}$ and $\hat{v}$ on the common quotient $L/\sim_u = L/\sim_v$ are equal iff $u$ and $v$ are equal.
Proof. By Proposition 5.11 $u^*$ is exhaustive. Therefore $u^*|_S$ is exhaustive, hence $(u^*|_S)^* = u^*|_S$ by Theorem 5.14. Since $u|_S \supseteq u^*|_S$, we have $(u|_S)^* \supseteq (u^*|_S)^*$. Moreover, $u^*|_S \supseteq (u|_S)^*$ since $u^*|_S$ agrees with $u|_S$ on order-bounded subsets of $S$ (c.f. Proposition 4.2 (iii)). We have seen:

$$(u|_S)^* \supseteq (u^*|_S)^* = u^*|_S \supseteq (u|_S)^*,$$

$u^*|_S = (u|_S)^*$. □

Since for locally solid Riesz spaces the Lebesgue property implies the pre-Lebesgue property, Corollary 5.15 generalizes [22, Lemma 9.7] (therefore [13, Corollary 4.6]).

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