HOMOTOPY GERSTENHABER ALGEBRAS ARE STRONGLY HOMOTOPY COMMUTATIVE

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ABSTRACT. We show that any homotopy Gerstenhaber algebra is canonically a strongly homotopy commutative (shc) algebra in the sense of Stasheff–Halperin with a homotopy-associative structure map. In the presence of certain additional operations corresponding to a $\cup_1$-product on the bar construction, the structure map becomes homotopy-commutative, so that one obtains an shc algebra in the sense of Munkholm.

1. Introduction

Let $A$ and $B$ be augmented dgas over a commutative ring $k$. Recall that an $A_\infty$ map $f: A \Rightarrow B$ is a map of differential graded coalgebras $BA \to BB$ between the bar constructions of $A$ and $B$. According to Stasheff–Halperin [11, Def. 8], the dga $A$ is a strongly homotopy commutative (shc) algebra if

(i) the multiplication map $\mu_A: A \otimes A \to A$ extends to an $A_\infty$ morphism $\Phi: A \otimes A \to A$.

Munkholm [9, Def. 4.1] additionally requires the following:

(ii) The map $\eta_A: k \to A, 1 \mapsto 1$ is a unit for $\Phi$, that is,

$$\Phi \circ (1_A \otimes \eta_A) = \Phi \circ (\eta_A \otimes 1_A) = 1_A.$$ 

(iii) The $A_\infty$ map $\Phi$ is homotopy associative, that is,

$$\Phi \circ (\Phi \otimes 1_A) \simeq \Phi \circ (1_A \otimes \Phi).$$

(iv) The map $\Phi$ is homotopy commutative, that is,

$$\Phi \circ T_{A,A} \simeq \Phi.$$

Note that we write $1_A$ for the identity map of $A$. Also, the compositions and tensor products above are those of $A_\infty$ maps.

Using acyclic models, Munkholm [9, Prop. 4.7] has constructed a natural shc structure on the normalized singular cochain complex $C^*(X)$ of a space $X$. Such cochain complexes are the main example of homotopy Gerstenhaber algebras (hgas) besides the Hochschild cochains of associative algebras. Recall that an hga is essentially an augmented dga $A$ with maps

$$E_k: A \otimes A^{\otimes k} \to A$$

for $k \geq 1$ that induce a dga structure on the bar construction $BA$ compatible with the diagonal. See Section [5, 1] for a reformulation in terms of the identities the operations $E_k$ have to satisfy. Kadeishvili [5] has defined certain additional

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operations on an hga $A$ that allow to define a $\cup_1$-product on $BA$. We call such an hga \textit{extended}, see Section 3.2. Singular cochain algebras are extended hgas.

As pointed out by Kadeishvili [6, Sec. 1.6], it follows from general considerations that any hga admits an shc structure in the sense of Stasheff–Halperin such that the composition of $B\Phi$ with the shuffle map,

$$B(A \otimes A) \xrightarrow{\nabla} B(A \otimes A) \xrightarrow{B\Phi} BA,$$

is homotopic to the product in $BA$ determined by the hga structure.

Our main result is the following. In the companion paper [4] we apply it to determine the cohomology ring of a large class of homogeneous spaces.

\textbf{Theorem 1.1.} Let $A$ be an hga.

(i) There is a canonical shc structure on $A$ satisfying properties (i), (ii) and (iii) of the definition above.

(ii) If $A$ is extended, then the shc structure additionally satisfies property (iv).

(iii) All structure maps commute with morphisms of (extended) hgas.

(iv) The composition (1.1) is exactly the multiplication map on $BA$.

The $A_{\infty}$ map $\Phi$ is defined in Section 4 and the homotopies $h^a$ and $h^c$ in Sections 5 and 6. The verification that they satisfy the required identities is an elementary, but very lengthy computation that has been relegated to the appendices. The claimed naturality will be obvious from the construction. We conclude in Section 7 with results about $A_{\infty}$ maps and shc maps from polynomial algebras to dgas and hgas, respectively.

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2. Preliminaries

From now on, we refer to $A_{\infty}$ maps as \textit{strongly homotopy multiplicative (shm) maps} because this terminology pairs better with “shc algebras”. We refer to the companion paper [4, Secs. 2–4] for our general conventions, the definitions of twisting cochains, twisting cochain homotopies and the corresponding families as well as those of shm maps and their compositions and tensor products. We work over an arbitrary commutative ring $k$ with unit.

We recall from [4, Sec. 2] our “$\kappa$” notation which alleviates us from explicitly specifying signs coming from the Koszul sign rule. For example, if we define a map $F: A \otimes B \otimes C \to A' \otimes B' \otimes C'$ between complexes by

$$F(a, b, c) \overset{\kappa}{=} f(c) \otimes g(a, b),$$

then we mean

$$F(a, b, c) = (-1)^{|a||b|+|g|} f(c) \otimes g(a, b).$$

Composition of maps is distributed over tensor products. For instance,

$$G(a, b) \overset{\kappa}{=} f_1(f_2(a)) \otimes g_1(g_2(b))$$

defines the element

$$G = f_1 f_2 \otimes g_1 g_2 = (-1)^{|f_2| |g_1|} (f_1 \otimes g_1) (f_2 \otimes g_2)$$
in the endomorphism operad.
3. Homotopy Gerstenhaber algebras

Homotopy Gerstenhaber algebras were introduced by Voronov–Gerstenhaber \[12, \S 8\]. For the convenience of the reader, we reproduce the definition of an (extended) homotopy Gerstenhaber algebra from \[4, \text{Sec. 6}\].

3.1. Definition of an hga. A homotopy Gerstenhaber algebra (homotopy G-algebra, hga) is an augmented dga $A$ with certain operations

\[
E_k : A \otimes A^\otimes k \to A, \quad a \otimes b_1 \otimes \cdots \otimes b_k \mapsto E_k(a; b_1, \ldots, b_k)
\]
of degree $|E_k| = -k$ for $k \geq 1$. It is often convenient to use the additional operation $E_0 = A$. These operations satisfy the following properties.

(i) All $E_k$ with $k \geq 1$ take values in the augmentation ideal $\bar{A}$ and vanish if any argument is equal to 1.

(ii) $d(E_k)(a; b_\bullet) \overset{\text{def}}{=} b_1 E_{k-1}(a; b_\bullet) + \sum_{m=1}^{k-1} (-1)^m E_{k-1}(a; b_\bullet, b_m b_{m+1}, b_\bullet)
+ (-1)^k E_{k-1}(a; b_\bullet) b_k.$

for all $k \geq 1$ and all $a, b_1, \ldots, b_k \in A$.

(iii) $E_k(a_1 a_2; b_\bullet) \overset{\text{def}}{=} \sum_{k_1 + k_2 = k} E_{k_1}(a_1; b_\bullet) E_{k_2}(a_2; b_\bullet)$

for $k \geq 0$ and all $a_1, a_2, b_1, \ldots, b_k \in A$, where the sum is over all decompositions of $k$ into two non-negative integers.

(iv) $E_l(E_k(a; b_\bullet); c_\bullet) \overset{\text{def}}{=} \sum_{i_1 + \cdots + i_k + j_1 + \cdots + j_k = l} (-1)^{i_1} E_n(a; c_{i_0}, E_{i_1}(b_\bullet; c_{i_1}), \ldots, c_{i_k}, E_{i_{k+1}}(b_\bullet; c_{i_{k+1}}), c_{i_k})$,

for all $k, l \geq 0$ and all $a, b_1, \ldots, b_k, c_1, \ldots, c_l \in A$, where the sum is over all decompositions of $l$ into $2k + 1$ non-negative integers,

\[
n = k + \sum_{t=0}^{k} j_t
\]

and

\[
\varepsilon = \sum_{s=1}^{k} i_s \left(k + \sum_{t=s}^{k} j_t\right) + \sum_{t=1}^{k} t j_t.
\]

A morphism of hgas is a morphism $f : A \to B$ of augmented dgas that is compatible with the hga operations in the obvious way.

Remark 3.1. By the properties (iii) and (iv) as well as multilinearity, we can rewrite any expression formed within an hga as a linear combination of terms $W$ such that no sums or scalar multiples occur inside $W$ and such that the first argument of any operation $E_k$ appearing in $W$ is a single variable and neither a product nor another hga operation. When we speak of the terms appearing in some expression within an hga, we mean the terms appearing in such an expansion.

An expansion of this kind is actually unique and corresponds to a $\mathfrak{k}$-basis for the operad $F_3 \mathcal{X}$ governing homotopy Gerstenhaber algebras, compare \[7, \text{Sec. 4}\] and \[3, \S 1.6.6\].
3.2. Extended hgas. In [5] Kadeishvili introduced the notion of an ‘extended hga’ as an hga $A$ defined over $k = \mathbb{Z}_2$ that admits certain additional operations $E_{kl}$. Based on this he constructed $\cup_l$-products on $BA$ for all $i \geq 1$. We will only need the family $F_{kl} = E^1_{kl}$, but for coefficients in any $k$. We therefore define an hga to be extended if it comes with a family of operations

$$F_{kl} : A^\otimes k \otimes A^\otimes l \to A$$

of degree $|F_{kl}| = -(k+l)$ for $k, l \geq 1$, satisfying the following conditions: The values of all operations $F_{kl}$ lie in the augmentation ideal $A$ and vanish if any argument is equal to $1 \in A$. The differential of $F_{kl}$ is given by

$$d(F_{kl})(a_\bullet, b_\bullet) = A_{kl} + (-1)^k B_{kl}$$

for all $a_1, \ldots, a_k, b_1, \ldots, b_l \in A$, where

$$A_{kl} \equiv a_1 F_{k-1, l}(a_\bullet; b_\bullet) + \sum_{i=1}^{k-1} (-1)^i F_{k-1, l}(a_\bullet, a_i a_{i+1}, a_\bullet; b_\bullet)$$

$$+ \sum_{j=1}^l (-1)^k F_{k-1, j}(a_\bullet; b_\bullet) E_{l-j}(a_k; b_\bullet)$$

for $k \geq 2$, and

$$B_{kl} \equiv -E_k(b_1; a_\bullet),$$

$$B_{kl} \equiv \sum_{i=0}^{k-1} E_i(b_1; a_\bullet) F_{k-i, l-1}(a_\bullet; b_\bullet) + \sum_{j=1}^{l-1} (-1)^j F_{k, l-1}(a_\bullet; b_\bullet, b_j b_{j+1}, b_\bullet)$$

$$+ (-1)^l F_{k, l-1}(a_\bullet; b_\bullet) b_l$$

for $l \geq 2$, cf. [5, Def. 2].

The operation $\cup_2 = -F_{11}$ is a $\cup_2$-product for $A$ in the sense that

$$d(\cup_2)(a; b) = a \cup_1 b + (-1)^{|a||b|} b \cup_1 a$$

for all $a, b \in A$. This implies that the Gerstenhaber bracket in $H^*(A)$ is trivial, compare [4, eq. (6.16)].

A morphism of extended hgas is a morphism of hgas that commutes with all operations $F_{kl}$, $k, l \geq 1$.

Cochain algebras of simplicial sets, in particular singular cochain algebras of topological spaces, are naturally extended hgas, compare [4, Sec. 8.2].

4. The $shm$ map

We define the family of maps

$$\Phi_{(n)} : (A \otimes A)^\otimes n \to A$$

by

$$\Phi_{(n)}(a_\bullet \otimes b_\bullet) \equiv (-1)^{n-1} \sum_{j_1 + \cdots + j_n = n-1} E_{j_1}(a_1; b_\bullet) \cdots E_{j_n}(a_n; b_\bullet) b_n,$$
where the sum is over all decompositions of $n - 1$ into $n$ non-negative integers such that

$$\forall \, 1 \leq s \leq n \quad j_1 + \cdots + j_s < s.$$  

This condition means that the arguments of any term $E_{j_s}(a_s; \ldots)$ are $b$-variables with indices strictly smaller than $s$. It implies $j_1 = 0$, so that each summand starts with the variable $a_1$. Omitting the arguments $a_s \otimes b_s$, the components of $\Phi$ look as follows in small degrees:

$$\Phi_{(1)} = a_1 b_1,$$

$$\Phi_{(2)} = -a_1 E_1(a_2; b_1) b_2,$$

$$\Phi_{(3)} = a_1 E_1(a_2; b_1) E_1(a_3; b_2) b_3 + a_1 a_2 E_2(a_3; b_1, b_2) b_3.$$

**Proposition 4.1.** The $\Phi_{(n)}$ assemble to an shm map $\Phi : A \otimes A \Rightarrow A$ that satisfies properties (i) and (ii) of the definition of an shc algebra.

**Proof.** The verification of property (i) is a direct computation, see Appendix A. For property (ii) we observe that the normalization condition for the hga operations implies that $\Phi_{(n)}$ vanishes for $n > 1$ if all $a_i$ or all $b_j$ equal 1. Similarly, $\Phi_{(n)}$ vanishes for $n > 1$ if $a_i \otimes b_1 = 1 \otimes 1$ for some $i$, as required by the definition of a twisting family, see [4, eq. (3.3)]: For $i < n$ we would have $b_i = 1$ as an argument to some $E_k$-term with $k \geq 1$. For $i = n$ the term $E_{j_n}(a_n; \ldots)$ vanishes since $a_n = 1$ and $j_n \geq 1$ as there is at least one more argument, namely $b_{n-1}$. □

**Proposition 4.2.** The composition

$$BA \otimes BA \xrightarrow{\nabla} B(A \otimes A) \xrightarrow{B\Phi} BA$$

coincides with the product on $BA$ given by the hga structure of $A$.

**Proof.** We verify that the twisting cochain associated to the composition given above equals the twisting cochain $E$ defined in [4, eq. (6.9)]. Let us consider the components

$$A^\otimes k \otimes A^\otimes l \xrightarrow{(s^{-1})^\otimes n} B_k A \otimes B_l A \xrightarrow{\nabla} B_{k+l}(A \otimes A) \xrightarrow{\Phi} A$$

with $k, l \geq 0$ and $n = k + l$.

A look at the formula for $\Phi_{(1)}$ shows that (4.7) is the identity map of $A$ if $(k, l) = (1, 0)$ or $(0, 1)$. Moreover, the map is zero if $k \neq 1$ and $l = 0$, or if $k = 0$ and $l \neq 1$ since for $n \geq 2$ at least one term $E_m$ with $m \geq 1$ is contained in $\Phi_{(n)}$ and this term vanishes if any argument equals 1. These cases are therefore verified.

Now assume $k, l \geq 1$, and let

$$c' = \pm \left[ a_1' \otimes b_1' \mid \cdots \mid a_n' \otimes b_n' \right]$$

be a term appearing in $\nabla (c)$ such that $\Phi (c')$ is non-zero, where

$$c = (s^{-1})^\otimes n (a_1 \otimes \cdots \otimes a_k \otimes b_1 \otimes \cdots \otimes b_l).$$

Because $b_1', \ldots, b_{n-1}'$ become arguments to $E$ terms, they cannot equal 1. Hence $a_1' = \cdots = a_{n-1}' = 1$ and $i_{n-1}' = 1$ in (4.5). This implies

$$c' = \pm \left[ 1 \otimes b_1 \mid \cdots \mid 1 \otimes b_1 \mid a_1 \otimes 1 \right],$$
hence
\begin{equation}
\Phi \nabla(c) = \begin{cases} 
\pm E_n(a_1; b_1, \ldots, b_l) & \text{if } k = 1, \\
0 & \text{if } k > 1.
\end{cases}
\end{equation}

It remains to verify that the sign is +1 in the case \( k = 1 \). Write \( B = A \otimes A \) and
\begin{equation}
c'' = (1 \otimes b_1) \otimes \cdots \otimes (1 \otimes b_l) \otimes (a_1 \otimes 1) \in B^{\otimes(l+1)}.
\end{equation}
The summand of \( \nabla(c) \) that is not annihilated by \( \Phi \) is
\begin{equation}
c' = T_{s^{-1}B, (s^{-1}B)^{\otimes l}} \big((s^{-1} \otimes (s^{-1})^{\otimes l}) (c)\big)
= (-1)^l (s^{-1} \otimes (s^{-1})^{\otimes l}) T_{B, B^{\otimes l}} (c) = (-1)^{\varepsilon} (s^{-1} \otimes (l+1)) (c'')
\end{equation}
where
\begin{align}
\varepsilon &= l \otimes |b_1| + \cdots + |b_l|,
\end{align}
It is mapped to
\begin{equation}
\Phi \nabla(c) = (-1)^{\varepsilon} \Phi (s^{-1} \otimes (l+1)) (c'')
= (-1)^{\varepsilon} \Phi(n) (c'') = E_l(a_1; b_1, \ldots, b_l),
\end{equation}
as desired. \( \square \)

**Corollary 4.3.** The following diagram commutes for any \( n \geq 0 \):

\begin{center}
\begin{tikzpicture}
\node (A) at (0,0) {\( (BA)^{\otimes n} \)};
\node (B) at (3,0) {\( B(A^{\otimes n}) \)};
\node (C) at (0,-1) {\( BA \)};
\node (D) at (3,-1) {\( B(\Phi^{[n]}(BA)) \)};
\draw[->] (A) to node {\( \nabla^{[n]} \)} (B);
\draw[->] (A) to node {\( \mu^{[n]} \)} (C);
\draw[->] (B) to node {\( \Phi^{[n]} \)} (D);
\end{tikzpicture}
\end{center}

Here we have written \( \nabla^{[n]} \) and \( \mu^{[n]} \) for the \( n \)-fold iterations of the shuffle map and the multiplication on \( A \), which are both associative. See [4, eq. (5.6)] for the definition of the iterations of the shm map \( \Phi \).

**Proof.** We proceed by induction. For \( n \leq 1 \) there is nothing to show, and the case \( n = 2 \) has been done above. For the induction step we observe that the parallelogram in the diagram

\begin{center}
\begin{tikzpicture}
\node (A) at (0,0) {\( (BA)^{\otimes n} \otimes BA \)};
\node (B) at (3,0) {\( B(A^{\otimes n}) \otimes BA \)};
\node (C) at (0,-1) {\( BA \otimes BA \)};
\node (D) at (3,-1) {\( B(A \otimes A) \)};
\draw[->] (A) to node {\( \nabla^{[n]} \otimes 1 \)} (B);
\draw[->] (A) to node {\( \mu^{[n]} \otimes 1 \)} (C);
\draw[->] (B) to node {\( \Phi^{[n]} \otimes 1 \)} (D);
\draw[->] (C) to node {\( \mu \)} (D);
\end{tikzpicture}
\end{center}

commutes by [4, Lemma 4.4] and therefore the outer triangle by induction. This establishes the claim for \( n + 1 \) and completes the proof. \( \square \)
5. Homotopy associativity

The goal of this section is to establish a homotopy $h^a$ between the twisting cochains $\Phi \circ (\Phi \otimes 1)$ and $\Phi \circ (1 \otimes \Phi)$. To state our definition, we need to introduce some terminology.

A $c$-product is a product of one or more variables $c_b$; it is called proper if it has more than one factor. A $b$-term is a term of the form $E_m(b_j; \ldots)$ with $m \geq 0$ where all remaining arguments are $c$-products. A $bc$-product is a product of one or more $b$-terms, say ending with $E_m(b_j; \ldots)$, followed by the variable $c_j$. An $a$-term is a term of the form $E_m(a_i; \ldots)$ with $m \geq 0$ where all remaining arguments are $b$-terms, $c$-products or $bc$-products. An $ab$-product is a product of one or more $a$-terms and possibly $b$-terms that ends with an $a$-term. If we want to more precise about the first variable of an $E$-term, we call it an $a_i$-term or a $b_j$-term.

To motivate our formula, we observe the following: By [4, eq. (4.4)] we have

$$ (1 \otimes \Phi)(a_\bullet \otimes b_\bullet \otimes c_\bullet) = a_1 \cdots a_n \otimes \Phi_{(n)}(b_\bullet \otimes c_\bullet). $$

(5.1)

Note that $\Phi_{(n)}(b_\bullet \otimes c_\bullet)$ is a $bc$-product. To compute $(\Phi \circ (1 \otimes \Phi))_{(n)}(a_\bullet \otimes b_\bullet \otimes c_\bullet)$ we use [4, eq. (3.17)]. Taking property (iii) of the definition of an hga into account, we see that we obtain a sum of terms $\pm U V$ where $V$ is a $bc$-product and $U$ a product of $a$-terms having only $bc$-products as arguments. A similar argument, combined with the associativity condition [iv], shows that each term appearing in $(\Phi \circ (\Phi \otimes 1))_{(n)}(a_\bullet \otimes b_\bullet \otimes c_\bullet)$ is of the form $\pm U b_n W$ where $U$ is an $ab$-product and $W$ a $c$-product. Moreover no $bc$-products appear inside $a$-terms in this case, and the final variable of each $c$-product, say $c_j$, corresponds to a $b_j$-term that appears as a factor of the top-level product and not as an argument to some $a$-term. The homotopy $h^a$ has to interpolate between the two kinds of terms we have described.

We set $h^a_{(0)} = \eta_A$. For $n \geq 1$, we define

$$ h^a_{(n)} \equiv \sum (-1)^x U V $$

(5.2)

where the sum is over all $ab$-products $U$ and all $bc$-products $V$ satisfying the following conditions:

(i) Each of the $3n$ variables $a_1, \ldots, c_n$ appears exactly once in $UV$. The $a_i$'s appear in ascending order, as do the $b_i$'s and the $c_i$'s. Moreover, $a_i$ precedes $b_i$ and $b_i$ precedes $c_i$ for each $i$.

(ii) The first argument to any $E$-term in $U V$ has a larger index than the remaining arguments. (The definitions above imply that the first argument also has a smaller letter than the remaining arguments, where $a < b < c$.)

(iii) Any top-level $b$-term appearing in $U$, say with first argument $b_i$, comes right after the $a$-term with first argument $a_i$.

(iv) Define

$$ J_a = \{ j \mid b_j \text{ appears in a } b \text{-term that is argument to an } a \text{-term} \}, $$

(5.3)

$$ J_b = \{ j \mid b_j \text{ appears in a top-level } b \text{-term inside the } ab \text{-product } U \}, $$

(5.4)

$$ J_c = \{ j \mid b_j \text{ appears in a } bc \text{-product} \}. $$

(5.5)

By construction, $\{1, \ldots, n\}$ is the disjoint union of $J_a$, $J_b$ and $J_c$. Note that $J_c$ cannot be empty as $V$ is a $bc$-product. We additionally require

$$ J_a \neq \emptyset, \quad J_a \cup J_b = \{1, \ldots, \nu\}, $$

(5.6)
(5.7) \( J_a \setminus \{ \nu \} = \{ j \mid c_j \) appears in a \( c \)-product, but not as last factor \}

where we have written \( \nu = \max J_a \). If \( c_j \) does not appear in a proper product, then it is considered to be last factor of a \( c \)-product with a single factor. Note that we have \( J_e = \{ \nu + 1, \ldots, n \} \).

The sign exponent \( \varepsilon \) in (5.2) is defined recursively. Write \( \mu = \min J_a \). If \( \mu = \nu \), that is, if \( J_a = \{ \nu \} \), then

(5.8) \[ \varepsilon = n + \text{contribution of the } b \text{-term } E(b_\nu; \ldots) \]

+ contribution of each \( bc \)-product occurring inside an \( a \)-term.

The contributions are as follows: Consider a \( bc \)-product

(5.9) \[ E_{q_1}(b_1; \ldots) \cdots E_{q_k}(b_k; \ldots) c_k \]

occurring in some term \( E_p(a_1; \ldots) \) as \( m \)-th argument (with \( a_i \) being at position 0 and the last argument at position \( p \)). The contribution of such a term is

(5.10) \[ (q_j + \cdots + q_k)(p - m + 1). \]

Note that \( q_j + \cdots + q_k \) is the degree of the \( bc \)-product, considered as a function of its arguments, and \( p - m + 1 \) is the number of arguments of \( E_p(a_1; \ldots) \) from the \( bc \)-product (including) to the end. If \( E_q(b_\nu; \ldots) \) occurs in the term \( E_p(a_1; \ldots) \) as \( m \)-th argument, then its contribution is

(5.11) \[ \hat{\varepsilon} + m + q(p - m) \]

where \( \hat{\varepsilon} \) is the degree of the terms preceding the \( E(a_1; \ldots) \)-term, again considered as a function of their arguments. For example, if \( UV \) is

(5.12) \[ a_1 b_1 a_2 E_1(b_2; c_1) a_3 a_4 a_5 E_2(a_6; E_1(b_3; c_2), b_4 E_2(b_5; c_3, c_4) c_5) b_6 c_6, \]

then the contribution of the \( bc \)-product inside the \( a_6 \)-term is \( 2 \cdot 1 = 2 \), and that of the \( b_3 \)-term is \( 1 + 1 + 1 \cdot 1 = 3 \).

If \( J_a \) is not a singleton, then we compare (5.2) to a summand \((-1)^{c'} UV'\) with \( J'_a = J_a \setminus \{ \mu \} \) and \( J'_b = J_b \cup \{ \mu \} \). More precisely: We can write \( UV \) as

(5.13) \[ E_{p_1}(a_1; c_*) E_{q_1}(b_1; c_*) \cdots E_{q_{p_{\mu-1}}}(b_{\mu-1}; c_*) E_{p_{\mu}}(a_\mu; c_*) \]

\[ \cdot E_{p_{\mu+1}}(a_{\mu+1}; c_*) \cdots E_{p_\nu}(a_1; c_*, E_q(b_\nu; c_\nu), \ldots) \cdots c_n \]

where the \( b_\mu \)-term is the \( m \)-th argument of the \( a_\nu \)-term. We define \( UV' \) as

(5.14) \[ E_{p_1}(a_1; c_*) E_{q_1}(b_1; c_*) \cdots E_{q_{p_{\mu-1}}}(b_{\mu-1}; c_*) E_{p_{\mu} + \cdots + p_{\nu-1} + m-1}(a_\mu; c_*) \]

\[ \cdot E_q(b_\mu; c_*) a_{\mu+1} \cdots a_{\nu-1} E_{p_{\nu}-m}(a_\nu; \ldots) \cdots c_n \]

where all \( c_\nu \)-variables \( c_* \) appearing between \( a_{\mu+1} \) and \( b_\mu \) have been moved as additional arguments to the term \( E(a_\mu; \ldots) \). Moreover, the proper \( c \)-product starting with \( c_\mu \) is split into \( c_\mu \) and the remaining product. These two arguments replace the original \( c \)-product, wherever it appears in \( UV \). For example, if \( UV \) is

(5.15) \[ a_1 b_1 a_2 b_2 a_3 E_1(a_4; c_1) E_2(a_5; E_1(b_3, c_2), b_4 E_1(a_6; b_5) E_1(b_6; c_3, c_4, c_5) c_6, \]

then \( UV' \) equals

(5.16) \[ a_1 b_1 a_2 b_2 E_1(a_3; c_1) E_1(b_3, c_2) a_4 E_1(a_5; b_4) E_1(a_6; b_5) E_2(b_6; c_3, c_4, c_5) c_6. \]

The sign exponents \( \varepsilon \) and \( \varepsilon' \) are related by

(5.17) \[ \varepsilon' - \varepsilon = \hat{\varepsilon} + m + q(p - m) + \hat{\varepsilon}, \]
The degree of the expression preceding $E(a; \ldots)$ as a function of its arguments, and $\hat{\varepsilon}$ is the sign exponent for the summand of $d(U'V')$ that recombines $c_\mu$ and the following $c$-product to the original one. In the example (5.10) we have $\hat{\varepsilon} = 5$. Since $\hat{\varepsilon}$ is a summand of $\varepsilon$, the difference $\hat{\varepsilon} - \varepsilon'$ is actually independent of $\hat{\varepsilon}$.

Omitting the arguments, $b_\bullet \otimes b_\bullet \otimes c_\bullet$, the components of $h^\kappa$ look as follows in small degrees. We also indicate the values of $J_a$, $J_b$ and $J_c$ for each term. Note that the vanishing of $h^\kappa_{(1)}$ reflects the identity $(\Phi \circ (\Phi \otimes 1))_{(1)} = (\Phi \circ (1 \otimes \Phi))_{(1)} = \mu^{[3]}$.

\begin{equation}
(5.19) \quad h^\kappa_{(1)} = 0,
\end{equation}

\begin{equation}
(5.20) \quad h^\kappa_{(2)} \cong \begin{array}{l}
- a_1 E_2(a_2; b_1, c_1) b_2 c_2 \\
- a_1 E_1(a_2; b_1) E_1(b_2; c_1) c_2
\end{array} \begin{array}{l}
\{1\} \otimes \{2\}
\end{array}
\end{equation}

\begin{equation}
(5.21) \quad h^\kappa_{(3)} \cong \begin{array}{l}
+ a_1 E_2(a_2; b_1, c_1) E_1(a_3; b_2 c_2) b_3 c_3 \\
- a_1 E_1(a_2; b_1) E_1(a_3; b_2 c_2) b_3 c_3 \\
+ a_1 E_1(a_2; b_1) a_3 E_1(b_2; c_1) E_1(b_3; c_2) c_3 \\
+ a_1 E_1(a_2; b_1) a_3 b_2 E_2(b_3; c_1, c_2) c_3 \\
+ a_1 a_2 E_2(a_2; b_1, c_1) b_2 E_1(b_3; c_2) c_3 \\
+ a_1 a_2 E_2(a_3; b_1, c_1) b_2 E_1(b_3; c_2) c_3 \\
- a_1 a_2 E_1(a_3; b_1) E_1(b_2; c_1) E_1(b_3; c_2) c_3 \\
+ a_1 b_1 E_1(a_2; c_1) E_1(a_3; b_2) E_1(b_3; c_2) c_3 \\
+ a_1 b_1 a_2 E_3(a_3; b_2, c_1, c_2) b_3 c_3 \\
- a_1 b_1 a_2 E_3(a_3; b_1, b_2, c_2) b_3 c_3 \\
- a_1 b_1 a_2 E_2(a_3; E_1(b_2; c_1), c_2) b_3 c_3 \\
+ a_1 b_1 a_2 E_2(a_3; b_2, c_1) E_1(b_3; c_2) c_3 \\
- a_1 b_1 a_2 E_2(a_3; c_1, b_2) E_1(b_3; c_2) c_3 \\
+ a_1 b_1 a_2 E_1(a_3; E_1(b_2; c_1)) E_1(b_3; c_2) c_3 \\
+ a_1 b_1 a_2 E_1(a_3; b_2) E_2(b_3; c_1, c_2) c_3 \\
- a_1 E_1(a_2; b_1) E_2(a_3; b_2, c_1) c_2) b_3 c_3 \\
- a_1 E_1(a_2; b_1) E_2(a_3; b_2, c_1) c_2) b_3 c_3 \\
- a_1 a_2 E_3(a_3; b_1, b_2, c_1) b_3 c_3 \\
- a_1 a_2 E_2(a_3; b_1, b_2) E_1(b_3; c_1 c_2) c_3 \\
- a_1 a_2 E_2(a_3; b_1, b_2) E_1(b_3; c_1 c_2) c_3.
\end{array}
\end{equation}
Remark 5.1. The number of terms seems to grow rapidly with $n$, by a factor close to 10 with each degree. There are no terms in $h_{(1)}^a$, 2 terms in $h_{(2)}^a$, 25 terms in $h_{(3)}^a$, 254 terms in $h_{(4)}^a$, 2421 terms in $h_{(5)}^a$, 22,522 terms in $h_{(6)}^a$, 207,682 terms in $h_{(7)}^a$, 1,911,954 terms in $h_{(8)}^a$ and 17,635,830 terms in $h_{(9)}^a$.

Proposition 5.2. The maps $h_{(n)}^a$ assemble to a homotopy $h^a$ from $\Phi \circ (\Phi \otimes 1)$ to $\Phi \circ (1 \otimes \Phi)$.

Proof. This is a very long direct computation, see Appendix B. We remark that this proof is the only place in this paper where we use associativity condition (iv) for the hga structure, besides assuming that the product on $BA$ is associative.

Let us verify here the normalization condition [13, eq. (3.11)] for twisting homotopy families. Consider a term $U \in \Phi$ and assume that $a_i = b_i = c_i = 1$ for some $i$.

If $i \in J_\alpha$, then the $a$-term containing $b_i$ vanishes and therefore also $U$. In the case $i \in J_\beta$ the $b_i$-term is top-level and there is a $c$-product ending in $c_i$. If $i > 1$ and $b_{i-1}$ is not top level, then it must be an argument to the $a_i$-term, which therefore vanishes. Otherwise, the $c$-product containing $c_i$ is just $c_i = 1$ itself. Since it is argument to some $a$-term or $b$-term, the whole expression is again 0.

We finally consider the case $i \in J_\gamma$. Again, the $c$-product containing $c_i$ is $c_i = 1$ itself. If the $bc$-term containing $b_i$ does not end in $c_i$, then $c_i$ is argument to some later $bc$-term in the same $bc$-product, so that $U$ vanishes. If the $bc$-product ends in $c_i$, we look at $b_{i-1}$. It it appears in the same $bc$-product, then $c_{i-1}$ is an argument of the $b_i$-term. If $i = \nu$ or if $b_{i-1}$ is part of an earlier $bc$-product, then it appears inside the $a_i$-term, which once again forces $U$ to vanish.

In the remainder of this section we collect some observations that will be used in Appendix B. We start by noting the following variant of property (II) of the definition of an hga,

$$d(E_p)(a; E_q(b; c_\bullet), c_\bullet) \vDash (-1)^{q(p-1)} E_q(b; c_\bullet) E_{p-1}(a; c_\bullet) + \cdots$$

valid for all $k, q \geq 0$ and all $a, b, c_\bullet \in A$. This is a consequence of the convention (2.3) because we have permuted the operations $E_{p-1}$ and $E_q$. Similarly, property (II) implies

$$E_k(E_{p_1}(a_1; b_\bullet) E_{p_2}(a_2; b_\bullet); c_\bullet) \vDash \sum_{k_1+k_2=k} (-1)^{p_1k_2} E_{k_1}(E_{p_1}(a_1; b_\bullet); c_\bullet) E_{k_2}(E_{p_2}(a_2; b_\bullet); c_\bullet)$$

for all $k, p_1, p_2 \geq 0$ and all $a_1, a_2, b_\bullet, c_\bullet \in A$.

Let us write $\Phi' = \Phi \circ (\Phi \otimes 1)$. We note that each term appearing $\Phi'(a_\bullet \otimes b_\bullet \otimes c_\bullet)$ contains only one $bc$-product (at the very end) and that no two top-level $b$-terms are adjacent.

Lemma 5.3. Let $(-1)^{\varepsilon} W$ be a summand appearing in $\Phi'_{(n)}(a_\bullet \otimes b_\bullet \otimes c_\bullet)$. Assume that it has at least one $b$-term inside an $a$-term, and let $\mu$ be the smallest index of such a $b$-variable. Then a term $E_{q}(b_\mu; \ldots)$ appears as, say, the $n$-th argument of a term $E_{p_1}(a_1; \ldots)$, and $c_\mu$ as the first variable inside a proper $c$-product $c_\mu \tilde{c}$, $\tilde{c}$. Let $W'$ be obtained from $W$ in the same way as in (5.13), and let $\varepsilon'$ be the sign exponent of $W'$ in $\Phi'(a_\bullet \otimes b_\bullet \otimes c_\bullet)$. We have

$$\varepsilon' - \varepsilon \equiv \frac{\varepsilon + m + q(p_1 - 1) + \tilde{\varepsilon}}{2}$$
Here $\varepsilon$ is the degree of all $E$-operations in front of the $a_i$-term in $W$, and $\tilde{\varepsilon}$ is the sign exponent of the part of the differential $d(W')$ that recombines $c_\mu$ and $\tilde{c}$ to $c_\mu \tilde{c}$.

In other words, the rule used in the recursive sign definition (5.17) for $h^a$ also applies to $\Phi'$. Note that $c_\mu \tilde{c}$ may be the trailing $c$-product in the case of $\Phi'$, which is impossible for $h^a$. This will be important for the pair $\Phi(1) = \Phi(1)$ in Appendix B.2.

**Proof.** We can perform the modification $W \rightarrow W'$ in steps: We move $c$-variables in front of the $b_j$-term to the preceding $a$-term (as in the pair $[13]$, in Appendix B.1), we move the $b_j$-term to the preceding $a$-term (as in the pair $[13]$), or out of the current $a$-term if it is contained in the $a_{\mu+1}$-term (as in $[2]$ and $[4, 6]$).

In each step, one could verify the signs directly, keeping track of
1. the sign in the definition (4.2) of $\Phi$,
2. the sign given by [4, eq. (3.18)] that arises from the composition of the twisting cochains $\Phi$ and $\Phi \otimes 1$,
3. the sign that accounts for distributing composition of maps over tensor products as in (2.3),
4. the product of the signs that appears when the first argument in each term $E_i(A; c_\bullet)$ is split into its factors as in (5.23) where $A$ is a summand of some $\Phi[\mu](a_\bullet \otimes b_\bullet) \otimes 1$ given by (4.2), and
5. the product of the signs appearing each time a term $E_i(E_j(a; b_\bullet); c_\bullet)$ is expanded according to the associativity rule (4).

Alternatively, we can argue as follows: Since $\Phi'$ is a twisting cochain, the corresponding family satisfies the defining equation [4, eq. (3.5)]. We observe that the terms $W$ and $W'$ share exactly one term in their differentials with respect to the $\mathbb{B}$-basis for $F_2\mathcal{A}(n)$ described in Remark 6.4. Moreover, this common term is not produced by the differential of any other term appearing in $\Phi'[(n)](a_\bullet \otimes b_\bullet \otimes c_\bullet)$, nor by any term appearing on the right-hand side of the defining equation. Hence the two terms in question must cancel out, which leads to the claimed sign rule. $\Box$

## 6. Homotopy Commutativity

Let $\Phi : A \otimes A \Rightarrow A$ be the shm map constructed in Section 4 or, more generally for the moment, any shm map extending the multiplication in $A$ such that Proposition 4.2 holds. Then

\[(6.1) \quad \Phi[2](a \otimes 1, 1 \otimes b) + (-1)^{|a||b|} \Phi[2](1 \otimes b, a \otimes 1) = a \cup_1 b\]

for all $a, b \in A$, cf. [9] Prop. 4.8. (For our $\Phi$ this can be read off from (4.5).) Now assume that $h$ is an shm homotopy from $\Phi$ to $\Phi \circ \varphi$, as required by property (iv) of an shm algebra. A straightforward computation shows that

\[(6.2) \quad a \cup_2 b = (-1)^{|a||b|} h[2](1 \otimes b, a \otimes 1) - h[2](a \otimes 1, 1 \otimes b) + (-1)^{|a|} a \cup_1 h[1](1 \otimes b) + h[1](a \otimes 1) \cup_1 b\]

is a $\cup_2$-product for $A$ in the sense that it satisfies (6.1). As remarked in Section 6.2 a non-trivial Gerstenhaber bracket in $H^\ast(A)$ is an obstruction to the existence of a $\cup_2$-product and therefore to the homotopy commutativity of $\Phi$.

\footnote{The second “$a$” appearing in the argument of $\Phi$ in [9] Prop. 4.8 should read “$\tilde{a}$”. We also remark that twisting cochain condition [4, eq. (2.14)] implies $\Phi[2](a \otimes 1, 1 \otimes b) = 0$.}
In order to proceed, we put as an additional assumption in this section that \( A \) be extended. Let us define \( h^c(0) = \eta_A \) and

\[
h^c_{(n)}(a_\bullet \otimes b_\bullet) \overset{\Delta}{=} \sum_{j_1 + \cdots + j_n = n} E_{j_1}(a_1; b_\bullet) \cdots E_{j_n}(a_n; b_\bullet)
- \sum E_{i_1}(b_1; a_\bullet) \cdots E_{i_q}(b_q; a_\bullet) F_{i_3}(a_\bullet; b_\bullet) E_{j_1}(a_{q+1}; b_\bullet) \cdots E_{j_p}(a_n; b_\bullet)
\]

for \( n \geq 1 \). The first sum is over all decompositions of \( n \) into \( n \) non-negative integers. The second sum is over all positive integers \( p, q, k \) and all non-negative integers \( i_1, \ldots, i_q, j_1, \ldots, j_p \) such that

\[
q + p = n, \quad \forall 1 \leq t \leq q \quad i_1 + \cdots + i_t < t
\]

and

\[
i_1 + \cdots + i_q + k = q, \quad j_1 + \cdots + j_p + l = p.
\]

Omitting the argument \( a_\bullet \otimes b_\bullet \), the formula for \( h^c \) looks as follows in small degrees.

\[
h^c_{(1)} = E_1(a_1; b_1),
\]

\[
h^c_{(2)} \overset{\Delta}{=} a_1 E_2(a_2; b_1, b_2) + E_1(a_1; b_1) E_1(a_2; b_2) + E_2(a_1; b_1, b_2) a_2
- b_1 F_{1,1}(a_1; b_2) a_2,
\]

\[
h^c_{(3)} \overset{\Delta}{=} a_1 a_2 E_3(a_3; b_1, b_2, b_3) + a_1 E_1(a_2; b_1) E_2(a_3; b_2, b_3)
+ a_1 E_2(a_2; b_1, b_2) E_1(a_3; b_3) + a_1 E_3(a_2; b_1, b_2, b_3) a_3
+ E_1(a_1; b_1) a_2 E_2(a_3; b_2, b_3) + E_1(a_1; b_1) E_1(a_2; b_2) E_1(a_3; b_3)
+ E_1(a_1; b_1) E_2(a_2; b_2, b_3) a_3 + E_2(a_1; b_1, b_2) a_2 E_1(a_3; b_3)
+ E_2(a_1; b_1, b_2) E_1(a_2; b_3) a_3 + E_3(a_1; b_1, b_2, b_3) a_3
- b_1 F_{1,1}(a_1; b_2) a_2 E_1(a_3; b_3) - b_1 F_{1,1}(a_1; b_2) E_1(a_2; b_3) a_3
- b_1 F_{1,1}(a_1; b_2) F_{1,1}(a_3; b_3) - b_1 b_2 F_{2,1}(a_1, b_2; b_3) a_3
- b_1 F_{1,2}(a_1, b_2; b_3) a_2 a_3.
\]

**Proposition 6.1.** Assume that the hga \( A \) is extended. The maps \( h^c_{(n)} \) assemble to a homotopy \( h^c \) from \( \Phi \circ T \) to \( \Phi \).

Note that \( h^c \circ T \) is a homotopy in the other direction.

**Proof.** This is a yet another lengthy direct verification, see Appendix \( \Box \) It is helpful to observe the following: Consider a term appearing in the second sum of \([6.3]\), and let \( 1 \leq m \leq n \). Then \( b_m \) appears in the leading group of \( E \)-terms if and only if \( a_m \) appears in the same group or in the \( F \)-term. Equivalently, \( a_m \) appears in the trailing group of \( E \)-terms if and only if \( b_m \) appears in the same group or in the \( F \)-term.

This in particular shows that \( h^c \) satisfies the normalization condition for twisting homotopy families because it is impossible for any term in the second sum and any \( m \) that \( b_m \) appears before the \( F \)-term and \( a_m \) after it. That the first sum vanishes if some \( b_m = 1 \) is clear. \( \square \)

Using the homotopy \( h^c \), we can generalize the formula for the \( \cup_1 \)-product on the bar construction given by Kadeishvili [5, Prop. 2] for \( k = \mathbb{Z}_2 \). Earlier, Baues [1 [\S 2.9] obtained the dual formula for the cobar construction \( \Omega C(X) \) of a 1-reduced simplicial set \( X \) and any \( k \) (without using the surjection operad explicitly).
Corollary 6.2. The composition $Bh^c \nabla_{A,A}$ is a coalgebra homotopy from the product with commuted factors to the regular product on $BA$. The associated twisting cochain homotopy $F$ is given by

$$F([a_1|\ldots|a_k] \otimes [b_1|\ldots|b_l]) = \begin{cases} 1 & \text{if } k = l = 0, \\ \mp F_{kl}(a_1, \ldots, a_k, b_1, \ldots, b_l) & \text{if } k \geq 1 \text{ and } l \geq 1, \\ 0 & \text{otherwise}, \end{cases}$$

where the “$\mp$” indicates a minus sign combined with the sign from [4, eq. (3.10)].

Proof. Assume $n = k + l > 0$ and consider a term

$$c = \pm \left[ a'_1 \otimes b'_1 | \ldots | a'_n \otimes b'_n \right]$$

appearing in $\nabla([a_1|\ldots|a_k] \otimes [b_1|\ldots|b_l])$. Any such term is mapped to 0 by the first sum in the definition of $h^c_{(n)}$. Assume that $b'_1$ occurs in the leading group of $E$-term in a non-zero contribution to the second sum. Then $a'_i$ appears also in the leading group or in the $F$-term. The normalization conditions for the $E$-operations and the $F$-operations imply $a'_i \neq 1$, so that we have $b'_i = 1$. By looking at the trailing group of $E$-term, we similarly conclude $a'_i = 1$. Hence $p = k$ and $q = l$ in the definition of $h^c_{(n)}$,

$$c = \pm \left[ a_1 \otimes 1 | \ldots | a_k \otimes 1 | 1 \otimes b_1 | \ldots | 1 \otimes b_l \right],$$

and all variables $i_t$ and $j_s$ are 0. \qed

7. Polynomial algebras

Let $A$ and $B$ be augmented dgas, and let $b \lhd B$ be a differential ideal. Recall from [4, eq. (3.9)] that an shm map $f : A \Rightarrow B$ is called $b$-strict if we have $f_{(n)} \equiv 0 \pmod{b}$ for all components with $n \geq 2$. Similarly, a homotopy $h$ between $f$ and another shm map $g$ is called $b$-trivial if $h_{(n)} \equiv 0 \pmod{b}$ for $n \geq 1$, see [4, eq. (3.15)].

We write $k[x]$ for a polynomial algebra on a generator $x$ of even degree. We start with an analogue of the first part of [9, Prop. 6.2]. The second part will be addressed by Proposition 7.2.

Proposition 7.1. Let $f, g : k[x] \Rightarrow A$ be a $b$-strict shm maps. If there is a $b \in A$ such that $db = f_1(x) - g_1(x)$, then $f$ and $g$ are homotopic via an $a$-trivial homotopy.

Proof. We assume first that $g$ is strict with $g(x) = f_1(x) =: a$. It is a direct calculation to verify that an $a$-trivial homotopy from $f$ to $g$ is given by the family

$$h_{(n)}(x^{k_1}, \ldots, x^{k_n}) = (-1)^{n-1} \sum_{k' + k'' = k_n-1} f_{(n+1)}(x^{k_1}, \ldots, x^{k_{n-1}}, x^{k'}, x) x^{k''}$$

for $n \geq 1$, see Appendix [3]. The decomposition into $k_n - 1 = k' + k''$ is into non-negative integers. Note that $h_{(n)}(x^{k'})$ vanishes for any $n \geq 1$ if $k_n \leq 1$ or $k_m = 0$ for some $m < n$. This in particular shows that $h_{(n)}$ satisfies the normalization condition for twisting homotopy families.

As a consequence of this and [4, Lemma 2.2], we can assume that both $f$ and $g$ are strict in order to prove the general case. Then

$$h(x^{k}) = \sum_{k' + k'' = k-1} f(x^{k'}) b g(x^{k''})$$
is an algebra homotopy from \( f \) to \( g \) (in the sense of [9, §1.11]) taking values in \( a \). It gives rise to an \( a \)-trivial shm homotopy \( h \) from \( f \) to \( g \) with \( h^{(1)} = h \) and \( h^{(n)} = 0 \) for \( n \geq 2 \).

The following is a variant of [9, Lemma 7.3] with an explicit homotopy. Recall that any hga is canonically an shc algebra in the sense of Stasheff–Halperin, and in the sense of Munkholm if it is extended. We refer to [4, Sec. 5] for the definition of an \( (a\text{-natural}) \) shc map.

**Proposition 7.2.** Let \( A \) be an hga such that all operations \( E_k, k \geq 2 \), take values in a common ideal \( a \triangleleft A \). Let \( a \in A \) be a cocycle of even degree and assume that there is a \( b \in a \) such that \( db = E_1(a; a) \). Then the dga map

\[
f : k[x] \to A, \quad x^k \mapsto a^k
\]

is an \( a \)-natural shc map.

**Proof.** We have to show that the dga map

\[
k[x] \otimes k[x] \xrightarrow{\mu(x)} k[x] \xrightarrow{f} A
\]

is homotopic to the shm map

\[
k[x] \otimes k[x] \xrightarrow{f \otimes f} A \otimes A \xrightarrow{\Phi_{A,a}} A
\]

via an \( a \)-strict homotopy. Such a homotopy from the latter map to the former is given by the following twisting homotopy family, where we write \( x^{k_1} \otimes x^{l_1}, \ldots, x^{k_n} \otimes x^{l_n} \):

\[
(7.5) \quad h^{(1)}(x^{k_1} \otimes x^{l_1}) = 0,
\]

\[
(7.6) \quad h^{(2)}(x^{k_1} \otimes x^{l_1}) = \sum_{k_1^1+k_1^{l_1} = k_1^1+k_1^{l_1}} a^{k_1} E_2(a^{k_1}; a^{l_1}, a) a^{l_1^{(1)}} + l_1^{(1)}
\]

\[
- \sum_{k_1^1+k_1^{l_1} = k_1^1+k_1^{l_1}} a^{k_1^1+k_1^{l_1}, b^1 a^{l_1^{(1)}}, l_1^{(1)} + l_1^{(2)}},
\]

\[
(7.7) \quad h^{(n)}(x^{k_1} \otimes x^{l_1}) = \sum_{\sum l_1^{(2)}(n-1)} E_{i_1}(a^{k_1}; a^{l_1}) \cdots E_{i_{n-1}}(a^{k_{n-1}}; a^{l_{n-1}})
\]

\[
\cdot E_{i_n+1}(a^{k_n}; a^{l_n}, a) a^{l_n^{(2)}} + l_n^{(2)}
\]

\[
- \sum_{\sum l_1^{(2)}(n-1)} E_{i_1}(a^{k_1}; a^{l_1}) \cdots E_{i_{n-1}}(a^{k_{n-1}}; a^{l_{n-1}})
\]

\[
\cdot E_{i_n+1}(a^{k_n}; a^{l_n}, a) a^{l_n^{(2)}}, b^1 a^{l_n^2}, l_n^{(2)} + l_n^{(3)} + l_n^{(4)}
\]

\[
- \sum_{\sum l_1^{(2)}(n-1)} E_{i_1}(a^{k_1}; a^{l_1}) \cdots E_{i_{n-1}}(a^{k_{n-1}}; a^{l_{n-1}})
\]

\[
\cdot E_{i_n+1}(a^{k_n}; a^{l_n}, a) a^{k_n^{(2)}} + l_n^{(3)} + l_n^{(4)} + l_n^{(5)}
\]

for \( n \geq 3 \). The first sum in the first group of (7.7) extends over all decompositions \( n - 1 = i_1 + \cdots + i_n \) into \( n \) non-negative integers such that

\[
(7.8) \quad \forall \ 1 \leq s \leq n \quad i_1 + \cdots + i_s < s.
\]
and the first sums in the other two groups of (7.7) similarly over all decompositions $n - 2 = i_1 + \cdots + i_{n-1}$ into $n - 1$ non-negative integers such that

\begin{equation}
\forall \ 1 \leq s \leq n - 1 \ i_1 + \cdots + i_s < s.
\end{equation}

The decompositions into $k' + k''$ and $l' + l''$ also involve non-negative integers. Note that the formula for $n = 2$ is the same as the one for $n \geq 3$ with the sum over $l' + l'' = l_{n-2} - 1$ omitted. Also observe that $b$ is of even degree $2a - 2$ and that $i_n + 1 \geq 2$ in the first group of (7.7) so that $h$ is indeed $a$-trivial. The verification of the homotopy property is a lengthy computation, see Appendix E.

That the normalization condition for twisting homotopy families is satisfied follows by direct inspection: Assume that $k_i = l_i = 0$ for some $i$. It is clear that each term in the sum for $n = 2$ vanishes if $k_2 = 0$ or $l_1 = 0$. Similarly, each term in the first two sums for $n \geq 3$ vanishes if $k_n = 0$ or $l_{n-1} = 0$ or $l_m = 0$ for $m \leq n - 2$. For each term in the third sum we get this for $k_n = 0$ or $l_{n-2} = 0$ or $l_m = 0$ for $m \leq n - 3$. In the case $k_{n-1} = 0$ we finally use that the corresponding $E$-term has another argument since $i_{n-1} \geq 1$.

\[ \square \]

8. Concluding Remark

It would certainly be desirable to have a more conceptual proof of Theorem 1.1 than our explicit construction. We point out, however, that the operad $F_2\mathcal{X}$ governing hgas has non-trivial homology in the degrees we are interested in. In fact, since $F_2\mathcal{X}(n)$ models the configuration space of $n$ points in the plane, its homology $H_k(F_2\mathcal{X}(n))$ is non-zero for all $0 \leq k < n$, see the references given in Remark 3.1 as well as [10]. The Gerstenhaber bracket is a non-trivial element in $H_1(F_2\mathcal{X}(n))$ for $n \neq 0$, and its iterations give non-zero elements of higher degree.

A direct way to see this is the following: The Hochschild cohomology of an algebra $A$ is an algebra over $H(F_2\mathcal{X})$. If $A$ is commutative, then $HH^0(A) = A$, and $HH^1(A) = \text{Der}(A)$ are the derivations of $A$. The Gerstenhaber bracket of $a \in A$ and $D \in \text{Der}(A)$ is given by $[D, a] = D(a) \in A$, cf. [2] Props. 19, 22, Example 52]. If $A = k[x_1, \ldots, x_n]$ is a polynomial algebra, then the expression in $n$ elements $H_1^k(A)$ involving a $k$-fold iterated bracket,

\begin{equation}
[\partial_1, \cdots, [\partial_{k-1}, [\partial_k, x_1 \cdots x_k]] \cdots] \cdot 1 \cdots 1 = 1,
\end{equation}

shows $H_k(F_2\mathcal{X}(n)) \neq 0$ for any $0 \leq k < n$.

Appendix A. Proof of Proposition 4.1

In this and the following appendices we outline several proofs that are elementary, but lengthy computations. The main difficulties are to distinguish between the many cases to consider and to keep track of the signs. We write \( X \rightarrow Y \) to indicate that the terms for case $X$ cancel with or result in the terms for case $Y$.

Terms produced by $d(\Phi_{(m)})$

1. $b_j$-variable moved out of an $E(a_i;\ldots)$-term to the left
   In this case we have $i > 1$.
   1.1. $j < i - 1 \rightarrow 2.1.$
   1.2. $j = i - 1 \rightarrow 6.$
2. $b$-variable moved out of an $E(a_i;\ldots)$-term to the right
   2.1. $i < n \rightarrow 1.1.$
2.2. $i = n \rightarrow [5.1]$
In this case we have $j = n - 1$.

3. Two $b$-variables multiplied together in an $E$-term $\rightarrow [4.1]$
In this case the corresponding $a$-variables are further to the left.

Terms appearing in $\Phi_{(n-1)}(\ldots, a_ia_{i+1} \otimes b_i b_{i+1}, \ldots)$

4. $i < n - 1 \rightarrow [3.1]$
5. $i = n - 1 \rightarrow [2.2]$

Terms appearing in $\Phi_{(k)} \Phi_{(n-k)}$

6. all such terms $\rightarrow [1.2]$

APPENDIX B. PROOF OF PROPOSITION 5.2

We write $\Phi' = \Phi \circ (\Phi \otimes 1)$ and $\Phi'' = \Phi \circ (1 \otimes \Phi)$. Because of the recursive definition of the sign for each term appearing in $h^a$, we first pair up the terms appearing in the equation under consideration,

\[(B.1) \quad d(h^a_{(n)})(a_\bullet \otimes b_\bullet \otimes c_\bullet) = \sum_{k=1}^{n-1} (-1)^k h^a_{(n-1)}(a_\bullet \otimes b_\bullet \otimes c_\bullet, a_k a_{k+1} \otimes b_k b_{k+1} \otimes c_k c_{k+1}, a_\bullet \otimes b_\bullet \otimes c_\bullet)
+ \sum_{k=0}^{n} (\Phi'_{(k)}(a_\bullet \otimes b_\bullet \otimes c_\bullet) h^a_{(n-k)}(a_\bullet \otimes b_\bullet \otimes c_\bullet) \Psi'_a_{(n-k)}) (a_\bullet \otimes b_\bullet \otimes c_\bullet) \Phi''_{(n-k)}(a_\bullet \otimes b_\bullet \otimes c_\bullet)) \).

In a second step we show that the signs work out the way they should.

B.1. Pairing the terms. We assume $n \geq 2$. Recall that we write $\nu$ for the maximal index $j$ such that $b_j$ does not appear in a $bc$-product.

Terms produced by $d(h^a_{(n)})$

1. Terms coming from an $a_i$-term
   In this case we have $i > 1$.

1.1. First argument moved out to the left
   1.1.1. $b_j$-term moved out to the left
      In this case the previous term is an $a_{i-1}$-term.
      1.1.1.1. $j < i - 1 \rightarrow [1.2.1.1]$
      1.1.1.2. $j = i - 1$
      In this case either $c_j c_{j+1}$ appears in some $c$-product or $j = \nu$.
      1.1.1.2.1. $c_j c_{j+1}$ appears in a $c$-product
         Note that this $c$-product may contain other factors.
         1.1.1.2.1.1. $c_j c_{j+1}$ appears in a $c$-product inside a $b$-term that is argument to an $a$-term $\rightarrow [1.4.1.3]$
         1.1.1.2.1.2. $c_j c_{j+1}$ appears in a $c$-product inside a $bc$-product that is argument to an $a$-term $\rightarrow [1.4.2.3.2]$
      1.1.1.2.1.3. $c_j c_{j+1}$ appears in a $c$-product that is argument to an $a$-term $\rightarrow [1.3.3.3]$. 
1.1.1.2.1.4. $c_jc_{j+1}$ appears in a $c$-product inside a top-level $b$-term before the $a_n$-term $\rightarrow 2.3.$

1.1.1.2.1.5. $c_jc_{j+1}$ appears in a $c$-product inside final $bc$-product $\rightarrow 3.3.2.$

1.1.1.2.2. $j = \nu$

In this case $c_\nu c_{\nu+1}$ does not appear in a $c$-product.

1.1.1.2.2.1. $b_{\nu+1}$ is first $b$-variable in $bc$-product inside an $a$-term $\rightarrow 1.3.1.3.2.1.2.$

1.1.1.2.2.2. $b_{\nu+1}$ is the only $b$-variable in this $bc$-product $\rightarrow 3.3.2.$

1.1.1.2.2.2.1. $b_{\nu+1}$ is not the only $b$-variable in this $bc$-product $\rightarrow 1.2.1.3.2.1.$

This case means $\nu < n - 1.$

1.1.1.2.2.2.2. $b_{\nu+1}$ is the only $b$-variable in this $bc$-product $\rightarrow 8.1.$

This case means $\nu = n - 1.$

1.1.2. $bc$-product moved out to the left

Let’s write the final $c$-variable of the $bc$-product as $c_j$.

1.1.2.1. $j < i - 1 \rightarrow 1.2.2.1.$

1.1.2.2. $j = i - 1 \rightarrow 10.$

All $c_k$ with $k > j$ appear as single arguments, not inside a proper $c$-product.

1.1.3. $c$-product moved out to the left

Let’s write the final $c$-variable as $c_j$.

1.1.3.1. Previous term is $a_{i-1}$-term $\rightarrow 1.2.3.1.$

In this case we have $j < i - 1.$

1.1.3.2. Previous term is (top-level) $b_{i-1}$-term

1.1.3.2.1. $j < i - 1 \rightarrow 2.2.$

1.1.3.2.2. $j = i - 1 \rightarrow 7.$

Since $b_j$ is top-level, we cannot have $j = \nu.$

1.2. Last argument moved out to the right

1.2.1. $b_j$-term moved out to the right

1.2.1.1. Next term is $a_{i+1}$-term $\rightarrow 1.1.1.1.$

1.2.1.2. Next term is top-level $b_i$-term inside $ab$-product $\rightarrow 5.$

In this case we have $j = i - 1,$ and $c_{i-1}c_i$ appears in a $c$-product (ending in $c_i$).

1.2.1.3. Next term is first $b$-term of final $bc$-product

In this case we have $i = n$ and $j = \nu.$

1.2.1.3.1. $\nu = 1 \rightarrow 9.1.$

1.2.1.3.2. $\nu > 1$

1.2.1.3.2.1. $c_{\nu-1}c_\nu$ does not appear in a $c$-product (inside the final $bc$-product) $\rightarrow 1.1.1.2.2.2.1.$

1.2.1.3.2.2. $c_{\nu-1}c_\nu$ appears in a $c$-product (inside the final $bc$-product) $\rightarrow 3.3.1.$

1.2.2. $bc$-product moved out to the right

In this case the next term cannot be a top-level $b$-term.

1.2.2.1. Next term is $a_{i+1}$-term $\rightarrow 1.1.2.1.$

1.2.2.2. Next term is first $b$-term of final $bc$-product $\rightarrow 3.1.2.2.$

1.2.3. $c$-product moved out to the right

1.2.3.1. Next term is $a_{i+1}$-term $\rightarrow 1.1.3.1.$
1.2.3.2. Next term is top-level $b_i$-term inside $ab$-product $\rightarrow 2.1.$
1.2.3.3. Next term is first $b$-term of final $bc$-product $\rightarrow 3.1.1.$

1.3. Two arguments multiplied together

1.3.1. First argument is a $b_j$-term

1.3.1.1. Second argument is a $b_{j+1}$-term $\rightarrow 4.$
   In this case we have $c_jc_{j+1}$ inside some $c$-product.

1.3.1.2. Second argument is a $bc$-product
   In this case we have $j = \nu.$
   1.3.1.2.1. $\nu = 1$ $\rightarrow 9.2.2.$
   1.3.1.2.2. $\nu > 1$
      1.3.1.2.2.1. $b_{\nu-1}$ appears in a top-level $b$-term $\rightarrow 1.1.1.2.2.1.2.$
      In this case $c_{\nu-1}c_{\nu}$ does not appear in any $c$-product.
      1.3.1.2.2.2. $b_{\nu-1}$ does not appear in a top-level $b$-term $\rightarrow 1.4.2.3.1.$
      In this case $c_{\nu-1}c_{\nu}$ appears in a $c$-product inside the given $bc$-product.

1.3.1.3. Second argument is a $c$-product
   Let’s write $c_k$ for the final $c$-variable in the $c$-product.
   1.3.1.3.1. $k < j$ $\rightarrow 1.4.1.2.$
   1.3.1.3.2. $k = j$
      1.3.1.3.2.1. $c$-product is the single variable $c_j$
         1.3.1.3.2.1.1. $k = j = 1$ $\rightarrow 9.2.1.$
         1.3.1.3.2.1.2. $k = j > 1$ $\rightarrow 1.1.1.2.1.1.$
         In this case $b_{j-1}$ is top-level.
      1.3.1.3.2.2. $c$-product is proper $c$-product $\rightarrow 1.4.2.2.2.2.$
      In this case we have $j = k = \nu.$

1.3.2. First argument is a $bc$-product $\rightarrow 1.4.2.1.2.2.$
In this case the second argument is also a $bc$-product.

1.3.3. First argument is a $c$-product
   Let’s write $c_k$ for the last variable appearing in the $c$-product.
   1.3.3.1. Second argument is a $b$-term $\rightarrow 1.4.1.1.$
   1.3.3.2. Second argument is a $bc$-product $\rightarrow 1.4.2.1.1.$
   1.3.3.3. Second argument is a $c$-product $\rightarrow 1.1.1.2.1.3.$
   In this case we have $k < \nu$ since all $c_l$ with $l > \nu$ appear in $bc$-products.
Hence $b_k$ appears in a top-level $b$-term.

1.4. Terms produced by one of the arguments of the $a$-term

1.4.1. Terms produced by a $b$-term as argument
   Let’s write the $b$-variable as $b_j$.
   1.4.1.1. $c$-product moved out to the left $\rightarrow 1.3.3.1.$
   1.4.1.2. $c$-product moved out to the right $\rightarrow 1.3.3.1.$
   1.4.1.3. Two $c$-products multiplied together $\rightarrow 1.1.1.2.1.1.$
   Let’s write $c_k$ for the final $c$-variable of the first $c$-product. Then $k < j \leq \nu$, hence $b_k$ appears in a top-level $b$-term.

1.4.2. Terms produced by a $b$-term inside a $bc$-product
   Let’s write the $b$-variable as $b_j$.
   1.4.2.1. $c$-product moved out to the left
      Let’s write the final $c$-variable as $c_k$.
   1.4.2.1.1. $b$-term is the first $b$-term in the $bc$-product $\rightarrow 1.3.3.2.$
   1.4.2.1.2. $b$-term is not the first $b$-term in the $bc$-product
   1.4.2.1.2.1. $k < j - 1$ $\rightarrow 1.4.2.2.1.$
1.4.2.1.2.2. \( k = j - 1 \rightarrow 1.3.2. \)
In this case the \( c \)-product is a single variable \( c_{i-1} \).

1.4.2.2. \( c \)-product moved out to the right

Let’s write the final \( c \)-variable as \( c_k \).

1.4.2.2.1. \( b_j \) is not the last \( b \)-variable in the \( bc \)-product \( \rightarrow 1.4.2.1.2.1. \)

1.4.2.2.2. \( b_j \) is the last \( b \)-variable in the \( bc \)-product
In this case we have \( k = j - 1 \).

1.4.2.2.2.1. \( \nu < j - 1 \rightarrow 6.1.2. \)
In this case the \( c \)-product is the single variable \( c_{j-1} \). Also, \( b_{j-1} \) is part of the same \( bc \)-product for otherwise \( c_{n-1} \) would be at the end of the previous \( bc \)-product.

1.4.2.2.2.2. \( \nu = j - 1 \rightarrow 1.3.1.3.2.2. \)

1.4.2.3. Two \( c \)-products multiplied together

Let’s write \( c_k \) for the final \( c \)-variable of the first \( c \)-product.

1.4.2.3.1. \( b_k \) appears in a \( b \)-term inside an \( a \)-term \( \rightarrow 1.3.1.2.2.2. \)
In this case we have \( k = \nu \).

1.4.2.3.2. \( b_k \) appears in a top-level \( b \)-term \( \rightarrow 1.1.1.2.1.2. \)

1.4.2.3.3. \( b_k \) appears in a \( bc \)-product (inside some \( a \)-term) \( \rightarrow 6.1.1. \)
In this case the second \( c \)-product is the single variable \( c_{k+1} \), and \( b_{k+1} \) appears in the same \( bc \)-product as \( b_k \).

2. Terms coming from a top-level \( b \)-term in the \( ab \)-product

2.1. \( c \)-product moved out to the left \( \rightarrow 1.2.3.2. \)

2.2. \( c \)-product moved out to the right \( \rightarrow 1.1.3.2.1. \)

2.3. Two \( c \)-products multiplied together \( \rightarrow 1.1.1.2.1.4. \)
Let’s write \( c_j \) for the final \( c \)-variable of the first \( c \)-product. Then \( j < i \), hence \( b_j \) appears in a top-level \( b \)-term.

3. Terms coming from a \( b_i \)-term in the final \( bc \)-product

3.1. \( c \)-product moved out to the left
Let’s write the final \( c \)-variable as \( c_j \).

3.1.1. \( b_l \) is the first \( b \)-variable in the \( bc \)-product \( \rightarrow 1.2.3.3. \)

3.1.2. \( b_l \) is not the first \( b \)-variable in the \( bc \)-product

3.1.2.1. \( j < i - 1 \rightarrow 3.2.1. \)

3.1.2.2. \( j = i - 1 \rightarrow 1.2.2.2. \)
In this case the \( c \)-product is a single variable \( c_{i-1} \).

3.2. \( c \)-product moved out to the right
Let’s write the final \( c \)-variable as \( c_j \).

3.2.1. \( i < n \rightarrow 3.1.2.1. \)

3.2.2. \( i = n \)
In this case we have \( j = n - 1 \).

3.2.2.1. \( \nu < n - 1 \rightarrow 6.2.2. \)
In this case the \( c \)-product is the single variable \( c_{n-1} \). Also, \( b_{n-1} \) is part of the same \( bc \)-product for otherwise \( c_{n-1} \) would be at the end of the previous \( bc \)-product.

3.2.2.2. \( \nu = n - 1 \rightarrow 8.2. \)

3.3. Two \( c \)-products multiplied together
Let’s write \( c_j \) for the final \( c \)-variable of the first \( c \)-product. Then \( b_j \) appears either in a top-level \( b \)-term or in a \( bc \)-product.

3.3.1. \( b_j \) appears in an \( a \)-term \( \rightarrow 1.2.1.3.2.2. \)
In this case we have \( j = \nu \).

3.3.2. \( b_j \) appears in a top-level \( b \)-term \( \rightarrow [1.1.2.1.5] \).

3.3.3. \( b_j \) appears in a \( bc \)-product \( \rightarrow [6.2.1] \).

In this case the second \( c \)-product is the single variable \( c_{j+1} \), and \( b_{j+1} \) appears in the same \( bc \)-product as \( b_j \).

**Terms appearing in** \( h^a_{(n-1)}(\ldots, a_i a_{i+1} \otimes b_i b_{i+1} \otimes c_i c_{i+1}, \ldots) \)

4. \( b_i b_{i+1} \) in a \( b \)-term that is argument to an \( a \)-term \( \rightarrow [1.3.1.1] \).
5. \( b_i b_{i+1} \) in a top-level \( b \)-term in the \( ab \)-product \( \rightarrow [1.2.1.2] \).
6. \( b_i b_{i+1} \) in a \( b \)-term appearing in a \( bc \)-product
   6.1. \( c_i c_{i+1} \) in a \( bc \)-product inside an \( a \)-term
       6.1.1. \( c_i c_{i+1} \) is argument to a \( b \)-term in the \( bc \)-product \( \rightarrow [1.4.2.3.3] \).
       6.1.2. \( c_i c_{i+1} \) at the end of the \( bc \)-product \( \rightarrow [1.4.2.2.2.1] \).
   6.2. \( c_i c_{i+1} \) in final \( bc \)-product
       6.2.1. \( c_i c_{i+1} \) is argument to a \( b \)-term in the \( bc \)-product \( \rightarrow [3.3.3] \).
       6.2.2. \( c_i c_{i+1} \) at the end of the \( bc \)-product \( \rightarrow [3.2.2.1] \).

7. \( k < n \) \( \rightarrow [1.1.3.2.2] \).
8. \( k = n \)
   8.1. \( b_{n-1} \) appears before \( a_n \)-term in \( \Phi' \) \( \rightarrow [1.1.2.2.2] \).
       In this case the final \( c \)-product in \( \Phi' \) is \( c_n \) only.
   8.2. \( b_{n-1} \) appears inside \( a_n \)-term in \( \Phi' \) \( \rightarrow [3.2.2.2] \).
       In this case \( \Phi' \) ends with a proper \( c \)-product.

**Terms appearing in** \( (\Phi \circ (\Phi \otimes 1))_{(k)} \) \( h^a_{(n-k)} \)

Note that all \( b \)'s and \( c \)'s in \( \Phi'' \) appear inside \( bc \)-products with no proper \( c \)-products.

9. \( k = 0 \)
   9.1. There is only the final \( bc \)-product in \( \Phi'' \) (preceded by the product of all \( a \)-variables) \( \rightarrow [1.2.1.3.1] \).
   9.2. There is a \( bc \)-product inside some \( a \)-term in \( \Phi'' \)
       9.2.1. The first \( bc \)-product is \( b_1 c_1 \) \( \rightarrow [1.3.1.3.2.1] \).
       9.2.2. The first \( bc \)-product contains \( b_2 \) \( \rightarrow [1.3.1.2.1] \).
10. \( k > 0 \) \( \rightarrow [1.1.2.2.2] \).

B.2. Checking the signs.

1. Pair \([1.1.1.1] \leftrightarrow [1.2.1.1] \).
   1.1. \( \mu = j = \nu \)
   1.2. \( \mu = j < \nu \)
   1.3. \( \mu < j \leq \nu \)
   1.3.1. \( b_{\mu} \) appears before \( a_{i-1} \)-term
       1.3.1.1. \( c_{\mu} \) appears before \( a_{i-1} \)-term
       1.3.1.2. \( c_{\mu} \) appears in \( c \)-product that is argument to \( a_{i-1} \)-term
       1.3.1.3. \( c_{\mu} \) appears inside a \( b \)-term that is argument to \( a_{i-1} \)-term
       1.3.1.4. \( c_{\mu} \) appears in \( c \)-product that is argument to \( a_i \)-term
       1.3.1.5. \( c_{\mu} \) appears inside \( b_j \)-term (that is argument to \( a_i \)-term)
1.3.1.6. \( c_\mu \) appears inside another \( b \)-term or \( bc \)-product that is argument to \( a_i \)-term

1.3.1.7. \( c_\mu \) appears after \( a_i \)-term

1.3.2. \( b_\mu \) appears inside \( a_{i-1} \)-term

1.3.2.1. \( c_\mu \) appears in \( c \)-product that is argument to \( a_{i-1} \)-term

1.3.2.2. \( c_\mu \) appears inside a \( b \)-term that is argument to \( a_{i-1} \)-term

1.3.2.3. \( c_\mu \) appears in \( c \)-product that is argument to \( a_i \)-term

1.3.2.4. \( c_\mu \) appears inside \( b_j \)-term (that is argument to \( a_i \)-term)

1.3.2.5. \( c_\mu \) appears inside another \( b \)-term or \( bc \)-product that is argument to \( a_i \)-term

1.3.2.6. \( c_\mu \) appears after \( a_i \)-term

2. Pair \( 1.1.1.2.1.1 \leftrightarrow 1.4.1.3 \) and the following pairs

In \( 1.1.1.2.1.1 \), \( b_j \)-term is the first argument to \( a_i \)-term.

2.1. \( \mu = j < \nu \)

This case is built into the recursive sign formula.

2.2. \( \mu < j < \nu \)

2.2.1. \( c_\mu \) appears before \( a_i \)-term

2.2.2. \( c_\mu \) appears in \( c \)-product that is argument to \( a_i \)-term

2.2.3. \( c_\mu \) appears inside \( b_j \)-term (that is argument to \( a_i \)-term)

2.2.4. \( c_\mu \) appears inside another \( b \)-term or \( bc \)-product that is argument to \( a_i \)-term

2.2.5. \( c_\mu \) appears after \( a_i \)-term

3. Pair \( 1.1.1.2.1.2 \leftrightarrow 1.4.2.3.2 \)

Subsumed under case 2.

4. Pair \( 1.1.1.2.1.3 \leftrightarrow 1.3.3.3 \)

Subsumed under case 2.

5. Pair \( 1.1.1.2.1.4 \leftrightarrow 2.3 \)

Subsumed under case 2.

6. Pair \( 1.1.1.2.1.5 \leftrightarrow 3.3.2 \)

Subsumed under case 2.

7. Pair \( 1.1.1.2.2.1.1 \leftrightarrow 1.3.1.3.2.1.2 \) and the following pair

7.1. \( \mu = j = \nu \)

7.1.1. \( b_{\nu+1} \)-term appears in \( a_i \)-term

7.1.2. \( b_{\nu+1} \)-term appears in later \( a \)-term

7.2. \( \mu < j = \nu \)

7.2.1. \( c_\mu \) appears before \( a_i \)-term

7.2.2. \( c_\mu \) appears in \( c \)-product that is argument to \( a_i \)-term

7.2.3. \( c_\mu \) appears inside \( b_{\nu} \)-term (that is argument to \( a_i \)-term)

7.2.4. \( c_\mu \) appears inside \( bc \)-product starting with \( b_{\nu+1} \) (if it is argument to \( a_i \)-term)

7.2.5. \( c_\mu \) appears after \( a_i \)-term

8. Pair \( 1.1.1.2.2.1.2 \leftrightarrow 1.3.1.2.2.1 \)

Subsumed under case 7.

9. Pair \( 1.1.1.2.2.2.1 \leftrightarrow 1.2.1.3.2.1 \)

9.1. \( \mu = j = \nu \)

9.2. \( \mu < j = \nu \)

9.2.1. \( c_\mu \) appears before \( a_i \)-term

9.2.2. \( c_\mu \) appears in \( c \)-product that is argument to \( a_i \)-term
9.2.3. \( c_\mu \) appears inside \( b_\nu \)-term (that is argument to \( a_i \)-term)

9.2.4. \( c_\mu \) appears after \( a_i \)-term

10. Pair \[ \text{1.1.1.2.2.2.2.} \leftrightarrow \text{8.1.} \]

10.1. \( \mu = \nu \)

10.2. \( \mu < \nu \)

This case uses Lemma \[5.3\]

10.2.1. \( c_\mu \) appears before the \( a_i \)-term

10.2.2. \( c_\mu \) appears in \( b_n-1 \)-term

10.2.3. \( c_\mu \) appears in \( c \)-product that is argument to \( a_i \)-term

10.2.4. \( c_\mu \) appears in \( b_n \)-term

11. Pair \[ \text{1.1.2.1.} \leftrightarrow \text{1.2.2.1.} \]

11.1. \( \mu = \nu \)

11.2. \( \mu < \nu \)

This case is analogous to \[1.3.]\) (with some subcases omitted).

12. Pair \[ \text{1.1.2.2.} \leftrightarrow \text{10.} \]

12.1. \( \mu = \nu \)

12.2. \( \mu < \nu \)

12.2.1. \( c_\mu \) appears before \( a_i \)-term

12.2.2. \( c_\mu \) appears in \( bc \)-product that is first argument to \( a_i \)-term

13. Pair \[ \text{1.1.3.1.} \leftrightarrow \text{1.2.3.1.} \]

This case is analogous to case \[11.\) (by considering a \( c \)-product as a \(bc\)-product with no \( b \)-terms and several trailing \( c \)-variables”) with the following additional case for the recursive step.

13.1. \( c_\mu \) appears inside the \( c \)-product that is first argument to \( a_i \)-term

14. Pair \[ \text{1.1.3.2.1.} \leftrightarrow \text{2.2.} \]

This case is analogous to case \[13.\]

15. Pair \[ \text{1.1.3.2.2.} \leftrightarrow \text{7.} \]

Note that \( k \neq \mu \).

15.1. \( k < \mu \leq \nu \)

15.1.1. \( \mu = \nu \)

15.1.2. \( \mu < \nu \)

15.2. \( \mu < k < \nu \)

Using Lemma \[5.3\], this case is recursively reduced to \[15.1.\]

16. Pair \[ \text{1.2.1.2.} \leftrightarrow \text{5.} \]

16.1. \(| J_a | = 2 \), that is, \( J_a = \{ \mu, \nu \} \)

Let’s say that \( b_\nu \) appears in the \( a_k \)-term.

16.1.1. \( c_\mu \) appears before \( a_k \)-term

16.1.2. \( c_\mu \) appears in \( c \)-product that is argument to \( a_k \)-term before \( b_\nu \)-term

16.1.3. \( c_\mu \) appears inside \( b_\nu \)-term

16.1.4. \( c_\mu \) appears in \( c \)-product that is argument to \( a_k \)-term after \( b_\nu \)-term, but before any \( bc \)-product (if \( a_k \)-term has such arguments)

16.1.5. \( c_\mu \) appears inside first \( bc \)-product that is argument to \( a_k \)-term (if \( a_k \)-term has such an argument)

16.2. \(| J_a | > 2 \)

16.2.1. \( \mu = j < \nu - 1 \)

Let \( k = \min(J_a \setminus \{ \mu \}) \). There are only top-level \( b \)-terms between the \( a_k \)-term and the \( a \)-term containing the \( b_k \)-term. Using the cases \[1.\) and \[13.\) we
can take the $b_k$-term (together with the preceding $c$-products) out of the $a$-term and move it right after the $a_k$-term. The sign change is analogous to the recursive definition \[(5.17)\] of the sign. We then use induction with $J_a \setminus \{k\}$ instead of $J_a$ until we reach the case 16.1.

16.2.2. $\mu < j \leq \nu - 1$
- 16.2.2.1. $c_\mu$ appears before $a_i$-term
- 16.2.2.2. $c_\mu$ appears inside $b_{i-1}$-term
- 16.2.2.3. $c_\mu$ appears inside other $b$-term that is argument to $a_i$-term
- 16.2.2.4. $c_\mu$ appears in $c$-product that is argument to $a_i$-term
- 16.2.2.5. $c_\mu$ appears inside $b_i$-term
- 16.2.2.6. $c_\mu$ appears after $b_i$-term

17. Pair 12.1.3.1. $\leftrightarrow$ 9.1.

18. Pair 12.1.3.2.2. $\leftrightarrow$ 3.3.1

18.1. $|J_a| = 2$, that is, $J_a = \{\mu, \nu\}$ with $\mu = \nu - 1$
- 18.1.1. $b_\mu$ appears before $a_n$-term
- 18.1.2. $b_\mu$ appears inside $a_n$-term
- 18.2. $|J_a| > 2$
- 18.2.1. $c_\mu$ appears before $a_n$-term
- 18.2.2. $c_\mu$ appears inside $b_i$-term
- 18.2.3. $c_\mu$ appears inside other $b$-term that is argument to $a_n$-term
- 18.2.4. $c_\mu$ appears in $c$-product that is argument to $a_n$-term
- 18.2.5. $c_\mu$ appears in final $bc$-product

19. Pair 12.2.2. $\leftrightarrow$ 3.1.2.2.

19.1. $J_a = \{\nu\}$
- 19.2. $|J_a| > 1$
- 19.2.1. $b_\mu$ appears before $a_i$-term
- 19.2.1.1. $c_\mu$ appears before $a_n$-term
- 19.2.1.2. $c_\mu$ appears in $c$-product that is argument to $a_n$-term
- 19.2.1.3. $c_\mu$ appears in $b$-term that is argument to $a_n$-term
- 19.2.1.4. $c_\mu$ appears in $bc$-product that is argument to $a_n$-term, but not the last argument
- 19.2.1.5. $c_\mu$ appears in final $bc$-product inside $a_n$-term (but not as trailing $c$-variable)
- 19.2.2. $b_\mu$ appears inside $a_i$-term
- 19.2.2.1. $c_\mu$ appears in $c$-product that is argument to $a_n$-term
- 19.2.2.2. $c_\mu$ appears in $b$-term that is argument to $a_n$-term
- 19.2.2.3. $c_\mu$ appears in $bc$-product that is argument to $a_n$-term, but not the last argument
- 19.2.2.4. $c_\mu$ appears in final $bc$-product inside $a_n$-term (but not as trailing $c$-variable)

20. Pair 12.3.2. $\leftrightarrow$ 2.1.

This case is analogous to 19. (again by considering a $c$-product as a “$bc$-product with no $b$-terms and several trailing $c$-variables”) with the following additional subcases for the case 19.2. and with some subcases omitted. Note that $\mu \neq i$.

20.1. $b_\mu$ appears before $a_i$-term
- 20.1.1. $c_\mu$ appears in final $c$-product inside $a_i$-term
- 20.1.2. $c_\mu$ appears in or after $b_i$-term
- 20.2. $b_\mu$ appears inside $a_i$-term
20.2.1. $c_\mu$ appears in final $c$-product inside $a_i$-term
20.2.2. $c_\mu$ appears in or after $b_i$-term
20.3. $b_\mu$ appears after $b_i$-term
21. Pair $\ [1.2.3.3. \leftrightarrow 3.1.1. \]
This case is analogous to $\ [19. \leftrightarrow 20. \]$ with some subcases omitted.
22. Pair $\ [1.3.1.1. \leftrightarrow 4. \]
We have $\mu \leq j \leq \nu - 1$.
22.1. $\mu = j = \nu - 1$, that is, $J_a = \{\nu - 1, \nu\}$
22.1.1. $c_\mu c_{\mu+1}$ is argument to $a_i$-term
22.1.2. $c_\mu c_{\mu+1}$ appears after $a_i$-term and before any $bc$-products
22.1.3. $c_\mu c_{\mu+1}$ appears in (first) $bc$-product inside $a_i$-term
22.1.4. $c_\mu c_{\mu+1}$ appears in (first) $bc$-product inside later $a_i$-term
22.1.5. $c_\mu c_{\mu+1}$ appears in final $bc$-product
This case is analogous to $\ [22.1.2. \]$
22.2. $\mu = j < \nu - 1$
Let $k = \min(J_a \setminus \{\mu, \mu + 1\})$.
22.2.1. $i \leq k$
This case is analogous to $\ [16.2.1. \]$. We can take the $b_k$-term out of the $a_i$-term where it appears.
22.2.2. $i > k$
Using the cases $\ [14. \text{and} 13. \]$ we can move the $b_\mu$-term and the $b_{\mu+1}$-term (together with the preceding $c$-products) to the $a_{i-1}$-term. This inductively reduces this case to $\ [22.2.1. \]$
22.3. $\mu < j \leq \nu - 1$
22.3.1. $b_\mu$ appears before $a_i$-term
22.3.1.1. $c_\mu$ appears before $a_i$-term
22.3.1.2. $c_\mu$ appears inside $a_i$-term before $b_j$
22.3.1.3. $c_\mu$ appears inside $b_j$-term
22.3.1.4. $c_\mu$ appears inside $b_{j+1}$-term
22.3.1.5. $c_\mu$ appears after $b_{j+1}$-term
22.3.2. $b_\mu$ appears inside $a_i$-term
22.3.2.1. $c_\mu$ appears inside $a_i$-term before $b_j$
22.3.2.2. $c_\mu$ appears inside $b_j$-term
22.3.2.3. $c_\mu$ appears inside $b_{j+1}$-term
22.3.2.4. $c_\mu$ appears after $b_{j+1}$-term
23. Pair $\ [1.3.1.2.1. \leftrightarrow 9.2.2. \]$
This case uses $\ [4. \text{eq. (3.18)} \]$ for the sign of the composition of two shm maps and the convention $\ |[2.3.| \]$ for our "\&" notation. Note that the sign exponent $n - 1$ in $\ [1.2.| \]$ leads to the sign exponent $n - 1$ in $(\Phi \circ (1 \otimes \Phi))_{(n)}$.
24. Pair $\ [1.3.1.2.2.2. \leftrightarrow 1.4.2.3.1. \]$
24.1. $\mu < \nu - 1$
24.1.1. $c_\mu$ appears before $a_i$-term
24.1.2. $c_\mu$ appears in $c$-product that is argument to $a_i$-term before $b_\nu$
24.1.3. $c_\mu$ appears in $c$-product inside $b_\nu$-term
24.1.4. $c_\mu$ appears between $b_{\nu+1}$ and $c_\nu$
24.2. $\mu = \nu - 1$
24.2.1. $b_\mu$ appears before $a_i$-term
24.2.2. $b_\mu$ appears inside $a_i$-term
25. Pair $\text{1.3.1.3.1} \leftrightarrow \text{1.4.1.2}$

25.1. $\mu = j = \nu$

25.2. $\mu = j < \nu$

25.2.1. $c_\mu$ appears inside $a_i$-term

25.2.2. $c_\mu$ appears after $a_i$-term

25.3. $\mu < j \leq \nu$

25.3.1. $b_\mu$ appears before $a_i$-term

25.3.1.1. $c_\mu$ appears before $a_i$-term

25.3.1.2. $c_\mu$ appears inside $a_i$-term before $b_\nu$-term

25.3.1.3. $c_\mu$ appears inside $b_\nu$-term

25.3.1.4. $c_\mu$ appears in $c$-product following $b_\nu$-term (together with $c_k$)

25.3.1.5. $c_\mu$ appears inside $a_i$-term after $c_k$

25.3.1.6. $c_\mu$ appears after $a_i$-term

25.3.2. $b_\mu$ appears inside $a_i$-term

25.3.2.1. $c_\mu$ appears inside $a_i$-term before $b_\nu$-term

25.3.2.2. $c_\mu$ appears inside $b_\nu$-term

25.3.2.3. $c_\mu$ appears in $c$-product following $b_\nu$-term (together with $c_k$)

25.3.2.4. $c_\mu$ appears inside $a_i$-term after $c_k$

25.3.2.5. $c_\mu$ appears after $a_i$-term

26. Pair $\text{1.3.1.3.2.1.1} \leftrightarrow \text{9.2.1}$

This case is analogous to 23.

27. Pair $\text{1.3.1.3.2.2} \leftrightarrow \text{1.4.2.2.2.2}$

27.1. $J_a = \{ \mu = \nu - 1, \nu \}$

27.1.1. $b_\mu$ appears before $a_i$-term

27.1.2. $b_\mu$ appears inside $a_i$-term

27.2. $|J_a| > 2$

27.2.1. $b_\mu$ appears before $a_i$-term

27.2.1.1. $c_\mu$ appears before $a_i$-term

27.2.1.2. $c_\mu$ appears inside $a_i$-term before $b_\nu$-term

27.2.1.3. $c_\mu$ appears inside $b_\nu$-term

27.2.1.4. $c_\mu$ appears inside $a_i$-term in $c$-product following $b_\nu$-term

This case uses 25.

27.2.2. $b_\mu$ appears inside $a_i$-term

27.2.2.1. $c_\mu$ appears inside $a_i$-term before $b_\nu$-term

27.2.2.2. $c_\mu$ appears inside $b_\nu$-term

27.2.2.3. $c_\mu$ appears inside $a_i$-term in $c$-product following $b_\nu$-term

This case uses 25.

28. Pair $\text{1.3.2} \leftrightarrow \text{1.4.2.1.2.2}$

28.1. $J_a = \{ \nu \}$

28.2. $|J_a| > 1$

28.2.1. $b_\mu$ appears before $a_i$-term

28.2.1.1. $c_\mu$ appears before $a_i$-term

28.2.1.2. $c_\mu$ appears inside $a_i$-term before first $bc$-product

28.2.1.3. $c_\mu$ appears inside first $bc$-product

28.2.2. $b_\mu$ appears inside $a_i$-term

28.2.2.1. $c_\mu$ appears inside $a_i$-term before first $bc$-product

28.2.2.2. $c_\mu$ appears inside first $bc$-product

29. Pair $\text{1.3.3.1} \leftrightarrow \text{1.4.1.1}$
29.1. $J_a = \{ \nu \}$

29.2. $|J_a| > 1$
Let’s write $b_j$ for the $b$-term following $c_k$.

29.2.1. $b_j$ appears before $a_i$-term

29.2.1.1. $c_\mu$ appears before $a_i$-term
29.2.1.2. $c_\mu$ appears inside $a_i$-term before $c$-product containing $c_k$
29.2.1.3. $c_\mu$ appears in $c$-product containing $c_k$
29.2.1.4. $c_\mu$ appears in $b_j$-term
29.2.1.5. $c_\mu$ appears inside $a_i$-term after $b_j$-term
29.2.1.6. $c_\mu$ appears after $a_i$-term

29.2.2. $b_j$ appears inside $a_i$-term before $c$-product containing $c_k$

29.2.2.1. $c_\mu$ appears before $b_j$-term
29.2.2.2. $c_\mu$ appears inside $b_j$-term
29.2.2.3. $c_\mu$ appears inside or after $b_j$-term
29.2.2.4. $c_\mu$ appears inside $a_i$-term after $b_j$-term
29.2.2.5. $c_\mu$ appears after $a_i$-term

29.2.3. $\mu = j$
This case uses \[20\].

30. Pair $\mathbf{1.3.3.2.} \leftrightarrow \mathbf{1.4.2.1.1.}$
This case is analogous to \[28\] with the following additional cases.

30.1. $|J_a| > 1$, $b_j$ appears before $a_i$-term, $c_\mu$ appears inside (formerly second) $bc$-product

30.2. $|J_a| > 1$, $b_j$ appears inside $a_i$-term, $c_\mu$ appears inside (formerly second) $bc$-product

31. Pair $\mathbf{1.4.2.1.2.1.} \leftrightarrow \mathbf{1.4.2.2.1.}$
Let’s write the two $b$-variables in question as $b_j$ and $b_{j+1}$.

31.1. $J_a = \{ \nu \}$
31.2. $|J_a| > 1$

31.2.1. $c_\mu$ appears before $b_j$-term
31.2.2. $c_\mu$ appears inside $b_j$-term
31.2.3. $c_\mu$ appears inside or after $b_{j+1}$-term

32. Pair $\mathbf{1.4.2.2.2.1.} \leftrightarrow \mathbf{6.1.2.}$
32.1. $J_a = \{ \nu \}$
32.2. $|J_a| > 1$

33. Pair $\mathbf{1.4.2.3.3.} \leftrightarrow \mathbf{6.1.1.}$
33.1. $J_a = \{ \nu \}$
33.2. $|J_a| > 1$

34. Pair $\mathbf{3.1.2.1.} \leftrightarrow \mathbf{3.2.1.}$
This case is analogous to \[31\].

35. Pair $\mathbf{3.2.2.1.} \leftrightarrow \mathbf{6.2.2.}$
35.1. $J_a = \{ \nu \}$
35.2. $|J_a| > 1$
This case is analogous to \[32\].

36. Pair $\mathbf{3.2.2.2.} \leftrightarrow \mathbf{8.2.}$
36.1. $\mu = \nu$
36.2. $\mu < \nu$
This case uses Lemma \[5.3\].

36.2.1. $c_\mu$ appears before last argument of $b_n$-term
36.2.2. $c_n$ appears in $c$-product that is last argument of $b_n$-term

37. Pair $3.3.3.1 \leftrightarrow 6.2.1.$

37.1. $J_a = \{\nu\}$

37.2. $|J_a| > 1$

This case is analogous to $33.2.$

Appendix C. Proof of Proposition 6.1

Terms produced by $d(h^\epsilon_{(a)})$

Recall that there are leading $E$-terms if and only if there is an $F$-term. If there is an $F$-term, the index of each $b$-variable appearing in it is larger than the index of any $a$-variables appearing in it.

1. Terms produced by group of trailing $E$-terms

The trailing $E$-terms may or may not be preceded by a leading $E$-group and an $F$-term.

1.1. $b$-variable moved out of a trailing $E_m(a_i; \ldots)$-term to the left

1.1.1. This $E$-term is the first $E$-term of the trailing group

1.1.1.1. There is an $F$-term $\rightarrow 3.2.$

1.1.1.2. There is no $F$-term

1.1.2. This $E$-term is a later $E$-term of the trailing group $\rightarrow 1.2.1.$

1.1.2. $b$-variable moved out of a trailing $E$-term to the right

1.1.2.1. This $E$-term is not final $E$-term $\rightarrow 1.1.2.$

1.1.2.2. This $E$-term is final $E$-term $\rightarrow 8.$

In this case we can split up the trailing $E$-terms in a unique way such that the second part is a valid term in $\Phi$

1.3. Two $b$-variables multiplied together in a trailing $E$-term $\rightarrow 4.1.$

2. Terms produced by group of leading $E$-terms

2.1. $a_i$-variable moved out of a leading $E(b_j; \ldots)$-term to the left

In this case we have $j > 1$.

2.1.1. $i < j - 1 \rightarrow 2.2.2.$

2.1.2. $i = j - 1 \rightarrow 7.$

2.2. $a$-variable moved out of a leading $E$-term to the right

2.2.1. This $E$-term is the last $E$-term of the leading group $\rightarrow 3.3.$

2.2.2. This $E$-term is an earlier $E$-term of the leading group $\rightarrow 2.1.1.$

2.3. Two $a$-variables multiplied together in a leading $E$-term $\rightarrow 4.1.$

In this case the corresponding $b$-variables appear further to the left.

3. Terms produced by $F$-term

3.1. $E(b_j; \ldots)$-term split off $F$-term to the left $\rightarrow 3.4.$

Note the first $a_i$-variable of the trailing $E$-terms has index $i \geq j$.

3.2. $b$-variable moved out of $F$-term to the right $\rightarrow 1.1.1.$

3.3. $a$-variable moved out of $F_{(b_j; \ldots; b_i; \ldots)}$-term to the left $\rightarrow 2.2.1.$

In this case we have $l > 1$, so that $i < j - 1$.

3.4. $E(a_i; \ldots)$-term split off $F$-term to the right $\rightarrow 3.1.$

Note the last $b$-variable of the leading $E$-terms has index $j \geq i$.

3.5. $F_{(b_i)}$-term converted into $E_l(b_i; \ldots)$-term with $l \geq 1$

3.5.1. The next term is an $a$-variable $\rightarrow 6.2.$

3.5.2. The next term is an $E_m(a_i; \ldots)$-term with $m \geq 1 \rightarrow 3.6.2.$
3.6. $F_{k_1}$-term converted into $E_k(a_i; \ldots)$-term with $k \geq 1$

3.6.1. The previous $E$-term is an $b$-variable \( \to 1.1.1.2.2 \).

In this case the $b$-variable is the only leading $E$-term because otherwise $a_{i-1}$ would appear in the $F$-term.

3.6.2. The previous $E$-term is an $E_m(b; \ldots)$-term with $m \geq 1$ \( \to 3.5.2 \).

In this case the corresponding $a$-variables occur in trailing $E$-terms

3.7. Two $b$-variables multiplied together \( \to 5.2 \).

In this case the corresponding $a$-variables occur in trailing $E$-terms

3.8. Two $a$-variables multiplied together \( \to 4.2 \).

In this case the corresponding $b$-variables occur in leading $E$-terms

Terms appearing in $h^c_{(n-1)}(\ldots, a_ia_{i+1} \otimes b_ib_{i+1}, \ldots)$

Either the repeated $b$-terms appear in a leading $E$-term or the repeated $a$-terms appear in a trailing $E$-term.

4. The repeated $b$-terms appear in a leading $E$-term

4.1. The repeated $a$-terms appear in a leading $E$-term \( \to 2.3 \).

4.2. The repeated $a$-terms appear in the $F$-term \( \to 3.8 \).

5. The repeated $a$-terms appear in a trailing $E$-term

5.1. The repeated $b$-terms appear in the trailing $E$-term \( \to 1.3 \).

5.2. The repeated $b$-terms appear in the $F$-term \( \to 3.7 \).

Terms appearing in $\Phi'(m) h^c_{(n-m)}$

Here $\Phi' = \Phi \circ T$.

6. $h^c_{(n-m)}$ does not contain an $F$-term

6.1. $m = 1$ \( \to 1.1.1.2.1 \).

6.2. $m > 1$ \( \to 3.5.1 \).

7. $h^c_{(n-m)}$ contains an $F$-term \( \to 2.1.2 \).

Terms appearing in $h^c_{(m)} \Phi_{(n-m)}$

8. all such terms \( \to 1.2.2 \).

Appendix D. Proof of Proposition 7.1

We assume $n \geq 1$.

Terms produced by $d(h_{(n)})$

1. Two arguments of $f_{(n+1)}$ multiplied together

1.1. At position $m \leq n - 2$ (if $n \geq 3$) \( \to 3 \).

1.2. At position $m = n - 1$ (if $n \geq 2$) \( \to 4.1 \).

1.3. At position $m = n$

1.3.1. $k'' = 0$ \( \to 6 \).

1.3.2. $k'' > 0$ \( \to 2.2.2 \).

2. $f_{(n+1)}$ split into two terms

2.1. At position $m \leq n - 1$ (if $n \geq 2$) \( \to 5 \).

2.2. At position $m = n$

2.2.1. $k' = 0$ \( \to 7 \).

By the normalization condition, this term vanishes unless $n = 1$.

2.2.2. $k' > 0$ \( \to 1.3.2 \).
Terms appearing in $h_{n-1}(\ldots, x^{k_m+k_{m+1}}, \ldots)$

3. $m \leq n - 2$ (if $n \geq 3$) → 1.1.
4. $m = n - 1$ (if $n \geq 2$)
   4.1. $k'' < k_n$ → 1.2.
   4.2. $k'' \geq k_n$ → 8.

Terms appearing in $f(m) h(n-m)$

5. $m < n$ (if $n \geq 2$) → 2.1.
6. $m = n$ → 1.3.1.

Terms appearing in $h(m) g(n-m)$

Since $g$ is strict, the only non-zero contribution is for $m = n - 1$.

7. $n = 1$ → 2.2.1.
8. $n \geq 2$ → 4.2.

Appendix E. Proof of Proposition 7.2

We write $F = \mu A \circ (f \otimes f)$ and $g = f \mu_{[k]}$. There is nothing to show for $n = 1$ since $F(1) = g$. Below we assume $n \geq 2$. Terms appearing only for $n \geq 3$ are marked “*” and those appearing only for $n \geq 4$ are marked “**”.

Terms produced by $d(h(n))$

1. Terms produced by the first group of sums
   1.1. Argument moved out of some $E(a^{k_i}; \ldots)$ to the left
      In this case we have $i > 1$.
      1.1.1.* $i \leq n - 1$
         In this case the argument is some $a^{l_j}$.
      1.1.1.1.* $j < i - 1$ → 1.2.1.
      1.1.1.2.* $j = i - 1$ → 7.1.
      1.1.2. $i = n$
      1.1.2.1. $i_n = 1$ → 2.4.
      1.1.2.2.* $i_n > 1$ → 1.2.2.
         In this case the argument is some $a^{l_j}$.
      1.2. Argument moved out of some $E(a^{k_i}; \ldots)$ to the right
      1.2.1.* $i \leq n - 2$ → 1.1.1.1.
      1.2.2.* $i = n - 1$ → 1.1.2.2.
      1.2.3. $i = n$ → 1.3.3.2.
         We may assume $l' \geq 1$ in this case.
      1.3. Two arguments of some $E(a^{k_i}; \ldots)$ multiplied together
      1.3.1.* Arguments are $a^{l_j}$ and $a^{l_{j+1}}$ with $j \leq n - 3$ → 4.1.
      1.3.2.* Arguments are $a^{l_{n-2}}$ and $a^{l'}$ → 4.2.1.
         In this case we have $i_n > 1$.
      1.3.3. Arguments are $a^{l''}$ and $a$
      1.3.3.1. $l'' = 0$ → 8.
      1.3.3.2. $l'' > 0$ → 1.2.3.

2. Terms produced by the second group of sums
   2.1.* Argument $a^{l_j}$ moved out of some $E(a^{k_i}; \ldots)$ to the left
      In this case we have $i > 1$. 

2.1.1.** $j < i - 1 \rightarrow 2.2.1.$
2.1.2.** $j = i - 1 \rightarrow 7.2.$

2.2. Argument moved out of some $E(a_{k_{j}}; \ldots)$ to the right

2.2.1.** $i \leq n - 2 \rightarrow 2.1.1.$
2.2.2.** $i = n - 1$

2.2.2.1. $i_{n-1} = 1 \rightarrow 5.3.1.$
2.2.2.2. $i_{n-1} > 1 \rightarrow 5.2.1.$

2.3.** Two arguments $a^{j}$ and $a^{j+1}$ of some $E(a_{k_{j}}; \ldots)$ multiplied together

→ 5.1.
In this case we have $j \leq n - 3$.

2.4. Term produced by $b \rightarrow 1.1.2.1.$

We have $\sum_{k' + k'' = k_{n-1}} a^{k'} E_1(a; a) a^{k''} = E_1(a_{k_{n}}; a)$.

3.* Terms produced by the third group of sums

3.1.* Argument moved out of some $E(a_{k_{j}}; \ldots)$ to the left

In this case with have $i > 1$.

3.1.1.** $i \leq n - 2$

In this case the argument is some $a^{j}$.

3.1.1.1.** $j < i - 1 \rightarrow 3.2.1.$
3.1.1.2.** $j = i - 1 \rightarrow 7.3.$

3.1.2.* $i = n - 1$

3.1.2.1.* $i_{n-1} = 1 \rightarrow 3.2.3.1.$
3.1.2.2.* $i_{n-1} > 1 \rightarrow 3.2.2.$

In this case the argument is some $a^{j}$.

3.2.* Argument moved out of some $E(a_{k_{j}}; \ldots)$ to the right

3.2.1.** $i \leq n - 3 \rightarrow 3.1.1.1.$
3.2.2.* $i = n - 2 \rightarrow 3.1.2.2.$
3.2.3.* $i = n - 1$

3.2.3.1.* $i_{n-1} = 1 \rightarrow 3.1.2.1.$
3.2.3.2.* $i_{n-1} > 1 \rightarrow 5.2.2.$

3.3.** Two arguments of some $E(a_{k_{j}}; \ldots)$ multiplied together

3.3.1.** Arguments are $a^{j}$ and $a^{j+1}$ with $j \leq n - 4 \rightarrow 6.1.$
3.3.2.** Arguments are $a^{k_{j} - 3}$ and $a^{j'} \rightarrow 6.2.1.$

In this case we have $i_{n-1} > 1$.

3.4.* Term produced by $b \rightarrow 4.3.2.$

We have $\sum_{k' + k'' = k_{n-1}} a^{k'} E_1(a; a) a^{k''} = E_1(a_{k_{n}}; a)$.

Terms appearing in $h_{(n-1)}(\ldots, a^{k_{i} + k_{i+1}} \otimes a^{l_{i} + l_{i+1}}, \ldots)$

4.* Terms produced by the first group of sums

4.1.** $i \leq n - 3 \rightarrow 1.3.1.$
4.2.* $i = n - 2$

We have either $l' \geq l_{n-2}$ or $l'' \geq l_{n-1}$.

4.2.1.* $l' \geq l_{n-2} \rightarrow 1.3.2.$
4.2.2.* $l'' \geq l_{n-1} \rightarrow 4.3.3.$

4.3.* $i = n - 1$

We decompose the term $E_{i_{n-1} + 1}(a_{k_{n-1} + k_{n}}; \ldots, a_{l_{n-3}}, a^{j'}, a)$ into the following three kinds of terms.

4.3.1.* $E_{i_{n-1} + 1}(a_{k_{n-1}}; \ldots, a_{l_{n-3}}, a^{j'}, a) a^{k_{n}} \rightarrow 9.$
4.3.2.* $E_{i_{n-1}}(a_{k_{n-1}}; \ldots, a_{l_{n-3}}, a^{j'}) E_1(a_{k_{n}}; a) \rightarrow 3.4.$
4.3.3.* $E_{i_n-1}(a^{k_{n-1}}; a^*) E_{i_{n+1}}(a^{k_n}; \ldots, a^{l_n-3}, a^{l'}, a)$ with $i_n \geq 1 \rightarrow 4.2.2.$

5.* Terms produced by the second group of sums

5.1.** $i \leq n - 3 \rightarrow 2.3.$

5.2.* $i = n - 2$

We have either $l' \geq l_{n-2}$ or $l'' \geq l_{n-1}$.

5.2.1.* $l' \geq l_{n-2} \rightarrow 2.2.2.2.$

5.2.2.* $l'' \geq l_{n-1} \rightarrow 3.2.3.2.$

5.3.* $i = n - 1$

We have either $k' \geq k_{n-1}$ or $k'' \geq k_n$.

5.3.1.* $k' \geq k_{n-1} \rightarrow 2.2.2.1.$

5.3.2.* $k'' \geq k_n \rightarrow 1.1.$

6.** Terms produced by the third group of sums

6.1.** $i \leq n - 4 \rightarrow 3.3.1.$

6.2.** $i = n - 3$

We have either $l' \geq l_{n-3}$ or $l'' \geq l_{n-2}$.

6.2.1.** $l' \geq l_{n-3} \rightarrow 3.3.2.$

6.2.2.** $l'' \geq l_{n-2} \rightarrow 6.3.2.$

6.3.** $i = n - 2$

We decompose the term $E_{i_{n-2}}(a^{k_{n-2}}; a^{k_{n-3}}; \ldots, a^{l_n-4}, a^{l'})$ into the following two kinds of terms.

6.3.1.** $E_{i_{n-2}}(a^{k_{n-2}}; \ldots, a^{l_n-4}, a^{l'}) a^{k_{n-1}} \rightarrow 6.4.1.$

6.3.2.** $E_{i_{n-2}}(a^{k_{n-2}}; a^*) E_{i_{n-1}}(a^{k_{n-1}}; \ldots, a^{l_n-4}, a^{l'})$ with $i_{n-1} \geq 1 \rightarrow 6.2.2.$

6.4.** $i = n - 1$

We have either $k' \geq k_{n-1}$ or $k'' \geq k_n$.

6.4.1.** $k' \geq k_{n-1} \rightarrow 6.3.1.$

6.4.2.** $k'' \geq k_n \rightarrow 1.1.$

Terms appearing in $F_m(h_{n-m})$

Recall that $h_{(1)} = 0$.

7.* $1 \leq m \leq n - 2$

7.1.* Terms produced by the first group of sums in $h \rightarrow 1.1.1.2.$

7.2.* Terms produced by the second group of sums in $h \rightarrow 2.1.2.$

7.3.** Terms produced by the third group of sums in $h \rightarrow 3.1.1.2.$

8. $m = n \rightarrow 1.3.3.1.$

Terms appearing in $h_{(n-1)}g$

9.* Terms produced by the first group of sums in $h \rightarrow 4.3.1.$

10.* Terms produced by the second group of sums in $h \rightarrow 5.3.2.$

11.** Terms produced by the third group of sums in $h \rightarrow 6.4.2.$

References

[1] H.-J. Baues, The cobar construction as a Hopf algebra, Invent. Math. 132 (1998), 467–489; doi:10.1007/s002220050231
[2] P. Belmans, Hochschild (co)homology, and the Hochschild–Kostant–Rosenberg decomposition, course notes (2018), available at http://pbelmans.ncag.info/teaching/hh-2018
[3] C. Berger, B. Fresse, Combinatorial operad actions on cochains, Math. Proc. Camb. Philos. Soc. 137 (2004), 135–174; doi:10.1017/s0305004103007138
[4] M. Franz, The cohomology rings of homogeneous spaces, preprint (2019)
[5] T. Kadeishvili, Cochain operations defining Steenrod $\smash{\smile}_r$-products in the bar construction, *Georgian Math. J.* 10 (2003), 115–125; available at [http://www.emis.de/journals/GMJ/vol10/v10n1-9.pdf](http://www.emis.de/journals/GMJ/vol10/v10n1-9.pdf)

[6] T. Kadeishvili, Measuring the noncommutativity of DG-algebras, *J. Math. Sci. (N. Y.)* 119 (2004), 494–512; doi:10.1023/B:JOTH.0000009373.66600.b4

[7] J. E. McClure, J. H. Smith, Multivariable cochain operations and little $n$-cubes, *J. Amer. Math. Soc.* 16 (2003), 681–704; doi:10.1090/S0894-0347-03-00419-3

[8] A. Meurer et al., SymPy: symbolic computing in Python, *PerrJ Computer Science* e103 (2017); doi:10.7717/peerj.cs.103; software available at [http://www.sympy.org](http://www.sympy.org)

[9] H. J. Munkholm, The Eilenberg–Moore spectral sequence and strongly homotopy multiplicative maps, *J. Pure Appl. Algebra* 5 (1974), 1–50; doi:10.1016/0022-4049(74)90002-4

[10] D. P. Sinha, The (non-equivariant) homology of the little disks operad, pp. 253–279 in: J.-L. Loday, B. Vallette (eds.), Operads 2009 (Luminy, 2009), Société Mathématique de France, Paris 2013

[11] J. Stasheff, S. Halperin, Differential algebra in its own rite, pp. 567–577 in: Proceedings of the Advanced Study Institute on Algebraic Topology (Aarhus 1970), vol. 3, Various Publ. Ser. 13, Mat. Inst., Aarhus Univ., Aarhus 1970

[12] A. A. Voronov, M. Gerstenhaber, Higher operations on the Hochschild complex, *Funct. Anal. Appl.* 29, 1–5 (1995); doi:10.1007/BF01077036

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