MEASURING CLUB–SEQUENCES WITH LARGE CONTINUUM

DAVID ASPERÓ AND MIGUEL ANGEL MOTA

Abstract. Measuring, as defined by J. Moore, says that for every sequence \((C_\delta)_{\delta<\omega_1}\) with each \(C_\delta\) being a closed subset of \(\delta\) there is a club \(C \subseteq \omega_1\) such that for every \(\delta \in C\), a tail of \(C \cap \delta\) is either contained in or disjoint from \(C_\delta\). We answer a question of Moore by building a forcing extension satisfying measuring together with \(2^{\aleph_0} > \aleph_2\). The construction works over any model of ZFC and can be described as a finite support forcing iteration with systems of countable structures as side conditions and with symmetry constraints.

1. Introduction

One of the most frustrating problems faced by set theorists working with iterated proper forcing is the lack of techniques for producing models in which the continuum has size greater than the second uncountable cardinal. In this paper we solve this problem in the specific case of measuring, a very strong negation of Club Guessing at \(\omega_1\) introduced by Justin Moore (see [3]). This work is a natural continuation of our previous work in [1] (see also [2]), where we showed \(2^{\aleph_0} > \aleph_2\) to be consistent together with a number of consequences of the Proper Forcing Axiom (PFA).

Our approach in [1] consisted in building, starting from CH, a certain type of finite support forcing iteration of length \(\kappa\) (in a general sense of ‘forcing iteration’) using what one may describe as finite ‘symmetric systems’ of countable elementary substructures of a fixed \(H(\kappa)\) as side conditions. These systems of structures were added at the first stage of the iteration. Roughly speaking, the fact that the supports of the conditions in the iteration were finite ensured that the inductive proofs of the relevant facts – mainly that the iteration has the \(\aleph_2\)–chain

\[\text{2000 Mathematics Subject Classification. 03E50, 03E35, 03E05.}\]

Mota was supported by the Austrian Science Fund FWF Project P22430. Both authors were also partially supported by Ministerio de Educación y Ciencia Project MTM2008–03389 (Spain) and by Generalitat de Catalunya Project 2009SGR–00187 (Catalonia).

\[\text{1This } \kappa \text{ is exactly the value that } 2^{\aleph_0} \text{ attains at the end of the construction.}\]
condition and that it is proper – went through. The use of the sets of structures as side conditions was crucial in the proof of properness.\footnote{For more on the motivation for this type of construction see \cite{1} and \cite{2}.}

In the present paper we add a higher degree of ‘local’ symmetry in the ‘single step’ forcing notions involved and use it to build a model of measuring.

**Definition 1.1.** (Moore, \cite{3}) *Measuring* is the following statement: Let $C = (C_\delta)_{\delta < \omega_1}$ be such that each $C_\delta$ is a closed subset of $\delta$ (where $\delta$ is endowed with the order topology). Then there is a club $C \subseteq \omega_1$ which measures $C_\delta$ for every $\delta \in C$. Specifically, this means that for every $\delta \in C$ there is some $\alpha < \delta$ with either $(C \cap \delta) \setminus \alpha \subseteq C_\delta$ or $(C \setminus \alpha) \cap C_\delta = \emptyset$. We will also say that $C$ measures $C$.

Measuring is of course equivalent to its restriction to club–sequences, where $(C_\delta)_{\delta < \omega_1}$ is a club–sequence if every $C_\delta$ is a club of $\delta$. Also, measuring clearly implies that for every ladder system $(C_\delta)_{\delta \in \text{Lim} \cap \omega_1}$ there is a club $C \subseteq \omega_1$ such that $C \cap C_\delta$ is finite for all $\delta \in C$. Finally, it is easily seen to follow from PFA, and even from its bounded form BPFA.\footnote{\begin{itemize}
\item\footnote{(C_\delta)_{\delta \in \text{Lim} \cap \omega_1} \text{ is a ladder system if each } C_\delta \text{ is a cofinal subset of } \delta \text{ of order type } \omega. \text{ The above statement is the negation of what is usually called Weak Club Guessing (for } \omega_1).}
\item\footnote{This is the negation of a statement Moore calls $\emptyset$ (mho) (see \cite{6}).}
\end{itemize}}

Our main theorem is the following.

**Theorem 1.2.** (CH) Let $\kappa$ be a cardinal such that $2^{<\kappa} = \kappa$ and $\kappa^{\aleph_1} = \kappa$. There is a proper poset $\mathcal{P}$ with the $\aleph_2$–chain condition such that the following statements hold after forcing with $\mathcal{P}$.

1. $2^{\aleph_0} = \kappa$
2. Measuring

The rest of the paper is devoted to proving Theorems 1.2 and is organized as follows: Section 2 starts with the central notion of symmetric system of structures. We then proceed, in Subsection 2.1, to the definition of a sequence $(\mathcal{P}_\alpha)_{\alpha \leq \kappa}$ of partial orders ($\mathcal{P}_\kappa$ will be shown to witness Theorem 1.2). In Section 3 we give the basic analysis of our construction; in particular, we prove that it has the $\aleph_2$–chain condition, its properness, that $\mathcal{P}_\alpha$ embeds completely in $\mathcal{P}_\beta$ if $\alpha < \beta \leq \kappa$, and
that $\mathcal{P}_\kappa$ forces $2^{\aleph_0} = \kappa$ (Lemmas 3.3, 3.7, 3.2 and 3.10). These results, together with the final lemma (Lemma 3.11), establish Theorem 1.2.

For the most part our notation follows set-theoretic standards as set forth for example in [4] and in [5]. If $N$ is a set whose intersection with $\omega_1$ is an ordinal, then $\delta_N$ will denote this intersection. If $N$ is a set, $\mathbb{P}$ is a partial order and $G$ is a ($\mathbb{V}$–generic) filter of $\mathbb{P}$, $N[G]$ will denote $\{\tau_G : \tau \in N, \tau$ a $\mathbb{P}$–name$\}$, where $\tau_G$ denotes the interpretation of $\tau$ by $G$. Also, $G$ is $N$–generic if $G \cap A \cap N \neq \emptyset$ for every maximal antichain $A$ of $\mathbb{P}$ belonging to $N$. A condition $p$ in $\mathbb{P}$ is ($N, \mathbb{P}$) generic if $p$ forces that $G$ is $N$–generic in the above sense. Note that we are not assuming that $\mathbb{P}$ is in $N$ in any of these two sentences.

If $T$ is a predicate (i.e., a subset) of some $H(\theta)$ and $\mathcal{N} = \langle N, T \cap N \rangle$ is a substructure of $\langle H(\theta), T \rangle$, we also denote $\mathcal{N}$ by $\langle N, T \rangle$. Recalling that the elementary diagram of a structure $\langle N, T \rangle$ is the collection of sentences with parameters holding in $\langle N, T \rangle$.

Finally, we will consider the following natural notion of rank: Given two sets $N, M$, we define the Cantor–Bendixson rank of $N$ with respect to $M$, $\text{rank}(N, M)$, by specifying that $\text{rank}(N, M) \geq 1$ if and only if for every $a \in N$ there is some $M \in N \cap M$ such that $a \in M$ and, for each ordinal $\mu \geq 1$, that $\text{rank}(N, M) > \mu$ if and only if for every $a \in N$ there is some $M \in N \cap M$ such that $a \in M$ and $\text{rank}(N, M) \geq \mu$.

### 2. Proving Theorem 1.2: The construction

The proof of Theorem 1.2 will be given in a sequence of lemmas in this and the next section.

Let $\kappa \geq \omega_2$ be a cardinal such that $2^{\omega_2} = \kappa$ and $\kappa^{\aleph_1} = \kappa$, and let $\Phi : \kappa \rightarrow H(\kappa)$ be a surjection such that for every $x$ in $H(\kappa)$, $\Phi^{-1}\{\{x\}\}$ is unbounded in $\kappa$.

As anticipated in the introduction, $\mathcal{P}$ will be the final member $\mathcal{P}_\kappa$ of a certain sequence $\langle \mathcal{P}_\alpha : \alpha \leq \kappa \rangle$ of forcing notions. Together with $\langle \mathcal{P}_\alpha : \alpha \leq \kappa \rangle$ we will also define a sequence $\langle T^\alpha : \alpha < \kappa \rangle$ of subsets of $H(\kappa)$ and a corresponding sequence $\langle M^\alpha : \alpha < \kappa \rangle$ of clubs of $[H(\kappa)]^{\aleph_0}$. 

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6For example, $\langle N, \in \rangle$ will denote the structure $\langle N, \in \cap N \times N \rangle$.

7Note that if $X$ is a set of ordinals and $\delta$ is an ordinal, then $\text{rank}(X, \delta) \geq 1$ if and only if $\delta$ is a limit point of ordinals in $X$ and, for each ordinal $\mu \geq 1$, $\text{rank}(X, \delta) > \mu$ if and only if $\delta$ is a limit of ordinals $\epsilon$ with $\text{rank}(X, \epsilon) \geq \mu$.

8We will see the $T^\alpha$’s as truth predicates.
2.0.1. **Symmetric systems of structures.** One central notion in our construction will be that of ‘symmetric system of structures’. We start by defining what we mean by this.

**Definition 2.1.** Let $\chi$ be an uncountable cardinal, $\vec{P}$ a sequence $(P^\xi)_{\xi<\beta}$ of subsets of $H(\chi)$ (for some ordinal $\beta$), $\mathcal{M}$ a club of $[H(\chi)]^{\aleph_0}$, and $\mathcal{N}$ a finite set. We will say that $\mathcal{N}$ is a $\vec{P}$–symmetric $\mathcal{M}$–system if the following conditions hold.

- $(\alpha)$ $\mathcal{N} \subseteq \mathcal{M}$.
- $(\beta)$ For all $N, N' \in \mathcal{N}$ and all $\xi \in N \cap \beta$, if $\delta_N = \delta_{N'}$ and $\xi' := \Psi_{N,N'}(\xi) < \beta$, then there is a unique isomorphism $\Psi_{N,N'}$ between $\langle N, P^\xi \rangle$ and $\langle N', P^{\xi'} \rangle$.

Furthermore, we ask that $\Psi_{N,N'}$ be the identity on $\chi \cap N \cap N'$.  
- $(\gamma)$ For all $N_0, N_1$ in $\mathcal{N}$, if $\delta_{N_0} < \delta_{N_1}$, then there is some $N_2 \in \mathcal{N}$ such that $\delta_{N_2} = \delta_{N_1}$ and $N_0 \subseteq N_2$.
- $(\delta)$ For all $N_0, N_1, N_2$ in $\mathcal{N}$, if $N_0 \in N_1$ and $\delta_{N_1} = \delta_{N_2}$, then $\Psi_{N_1,N_2}(N_0) \in \mathcal{N}$.

We may omit mentioning suitable parameters $\vec{P}$ and $\mathcal{M}$ when they are not relevant. If $H(\chi)$ is understood or irrelevant we call $\mathcal{N}$ a symmetric system (of structures).

Throughout this paper, if $N$ and $N'$ are such that there is a unique isomorphism from $N$ into $N'$, then we denote this isomorphism by $\Psi_{N,N'}$. The following facts are easy consequences from the above definition.

**Fact 2.2.** Let $\chi$, $\vec{P} = (P^\xi)_{\xi<\beta}$ and $\mathcal{M}$ be as in Definition 2.1. Let $\mathcal{N}$ be a $\vec{P}$–symmetric $\mathcal{M}$–system, let $N, N' \in \mathcal{N}$ be such that $\delta_N = \delta_{N'}$, and let $\xi, \xi'$ be ordinals in $\beta$ such that $\xi \in N$ and $\Psi_{N,N'}(\xi) = \xi'$. Then $\Psi_{N,N'}$ is an isomorphism between the structures $\langle N, \epsilon, P^\xi, \mathcal{N} \cap N \rangle$ and $\langle N', \epsilon, P^{\xi'}, \mathcal{N} \cap N' \rangle$.

**Fact 2.3.** Let $\vec{P} = (P^\xi)_{\xi<\beta}$ and $\mathcal{M}$ be as in Definition 2.1. Let $\mathcal{N}_0 = \{N^0_i : i < m\}$ and $\mathcal{N}_1 = \{N^1_i : i < m\}$ be $\vec{P}$–symmetric $\mathcal{M}$–systems of structures. Suppose that $(\bigcup \mathcal{N}_0) \cap (\bigcup \mathcal{N}_1) = X$, that there is an isomorphism

$$
\Psi : \langle \bigcup_{i<m} N^0_i, \epsilon, X \rangle \rightarrow \langle \bigcup_{i<m} N^1_i, \epsilon, X \rangle
$$

fixing $X$, and that for all $\xi \in \beta \cap \bigcup_{i<m} N^0_i$, if $\xi' = \Psi(\xi) \in \beta$, then $\langle \bigcup_{i<m} N^0_i, \epsilon, P^\xi, X, N^0_i \rangle$, and $\langle \bigcup_{i<m} N^1_i, \epsilon, P^{\xi'}, X, N^1_i \rangle$ are isomorphic structures. Then $\mathcal{N}_0 \cup \mathcal{N}_1$ is a $\vec{P}$–symmetric $\mathcal{M}$–system of structures.
Proof. The proof is a routine verification. Let us show for example that if $i_0, i_1 < m$ are such that $\delta_{N_{i_0}} = \delta_{N_{i_1}}$, then $\Psi_{N_{i_0}, N_{i_1}}$ fixes $\text{Ord} \cap N_{i_0} \cap N_{i_1}$: Let $\Psi$ be the isomorphism between $\langle \bigcup_{i < m} N_{i}, \in, X, N_{i}^{1} \rangle_{i < m}$ and $\langle \bigcup_{i < m} N_{i}, \in, X, N_{i}^{1} \rangle_{i < m}$. If $\gamma \in \text{Ord} \cap N_{i_0} \cap N_{i_1}$, then $\gamma \in X \cap N_{i_0}$, which implies that $\Psi(\gamma) = \gamma \in N_{i_0} \cap N_{i_1}$. But then $\gamma \in N_{i_0} \cap N_{i_1}$ and $\Psi$ is an isomorphism between the structures $\langle \bigcup_{i < m} N_{i}, \in, X, N_{i}^{1} \rangle_{i < m}$ and $\langle \bigcup_{i < m} N_{i}, \in, X, N_{i}^{1} \rangle_{i < m}$, which implies that $\Psi_{N_{i_0}, N_{i_1}}(\gamma) = \gamma$ and hence that $(\Psi \restriction N_{i_1} \circ \Psi_{N_{i_0}, N_{i_1}})(\gamma) = \Psi_{N_{i_0}, N_{i_1}}(\gamma) = \gamma$.  

\[ \square \]

2.1. The definition of $\langle P_{\alpha} : \alpha \leq \kappa \rangle$. Now we proceed to the definition of $\langle P_{\alpha} : \alpha \leq \kappa \rangle$ and $\langle T_{\alpha} : \alpha < \kappa \rangle$.

For every $\alpha < \kappa$, $\mathcal{M}_{\alpha}$ will be the set of $N \in \lceil H(\kappa) \rceil$ such that $\langle N, T_{\alpha} \rangle \subseteq \langle H(\kappa), T_{\alpha} \rangle$. We let $T_{0} \subseteq H(\kappa)$ code $\Phi$ together with the restriction of $\in$ to $H(\kappa)$. For every nonzero $\beta < \kappa$, if $T_{\alpha}$ has been defined for all $\alpha < \beta$, we let $T_{\beta} = (T_{\alpha})_{\alpha < \beta}$ and let $T_{\beta} \subseteq H(\kappa)$ code the elementary diagram of $\langle H(\kappa), T_{\alpha} \rangle_{\alpha < \beta}$. As we will see, each $P_{\alpha}$ will be lightface definable in the structure $\langle H(\kappa), T_{\alpha+1} \rangle$. We let also $\mathcal{M}_{0}^{\kappa} = \{ N \in \lceil H(\kappa) \rceil : N \in \bigcap_{\xi \in N \cap \beta} \mathcal{M}_{\xi} \}$.

We start with the definition of $P_{0}$: A condition in $P_{0}$ will be a pair $(\emptyset, \Delta)$, where

(A1) $\Delta$ is a countable set of pairs of the form $(N, 0)$.

(A2) $\text{dom}(\Delta)$ is a $T_{0}$–symmetric $\mathcal{M}_{0}$–system of countable substructures of $H(\kappa)$.

Given $P_{0}$–conditions $q_{\epsilon} = (\emptyset, \Delta_{\epsilon})$ for $\epsilon \in \{0, 1\}$, $q_{1}$ extends $q_{0}$ if and only if

(B) $\Delta_{0} \subseteq \Delta_{1}$

In the definition of $P_{0}$–condition we have used 0 twice in a completely vacuous way. These (vacuous) 0’s are there to ensure that the (uniformly defined) operation of restricting a condition in a (further) $P_{\beta}$ to an ordinal $\alpha < \beta$ yields a condition in $P_{0}$ when applied to any condition in any $P_{\beta}$ and to $\alpha = 0$.

Given $\alpha \leq \kappa$ for which $P_{\alpha}$ has been defined, let $\dot{G}_{\alpha}$ be a canonical $P_{\alpha}$–name for the generic object and let $\mathcal{N}_{\dot{G}_{\alpha}}$ be a canonical $P_{\alpha}$–name for the set $\bigcup \{ \Delta_{r}^{-1}(\alpha) : r \in \dot{G}_{\alpha} \}$.

Let $\beta < \kappa, \beta \neq 0$, and suppose that for all $\alpha < \beta$

(c) we have defined $T_{\alpha}$ and $P_{\alpha}$, and

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9We will actually make use of $T_{3}^{\beta}$ only for successor $\beta$.

10As we will see, conditions in $P_{\alpha}$ will be pairs $r = (s, \Delta_{r})$, with $\Delta_{r}$ a set of pairs $(N, \gamma), \gamma$ an ordinal.
(c) $\mathcal{P}_\alpha \subseteq H(\kappa)$ is a partial order with the $\aleph_2$–chain condition consisting of pairs $r = (s, \Delta_r)$, with $\Delta_r$ a set of pairs of the form $(N, \gamma)$, with $\gamma$ an ordinal.

The $\aleph_2$–c.c. of $\mathcal{P}_\alpha$ will follow from Lemma 3.3.

We will use a certain poset in $V^{\mathcal{P}_\alpha}$ for measuring a given club–sequence.

2.1.1. A forcing notion for measuring a club–sequence. Suppose $\dot{\mathcal{C}}$ is a $\mathcal{P}_\alpha$–name for a club–sequence $(C_\delta)_{\delta < \omega_1}$. We define next a forcing $\Theta_{\dot{\mathcal{C}}}$, in $V^{\mathcal{P}_\alpha}$, for adding a club measuring $\dot{\mathcal{C}}$:

Conditions in $\Theta_{\dot{\mathcal{C}}}$ are triples $(f, b, O)$ with the following properties.

1. $O \subseteq \dot{\mathcal{N}}_{\dot{\mathcal{G}}_\alpha}$ is a $\mathcal{T}^{\alpha+2}$–symmetric $\mathcal{M}_{\alpha+2}^\ast$–system of structures.
2. $f$ is a finite function that can be extended to a normal function $F : \omega_1 \rightarrow \omega_1$ such that $F(\omega) = \omega$. Moreover, for every $\nu \in \text{dom}(f) \setminus (\omega + 1)$,
   
   (2.1) $f(\nu) \in \{ \delta_N : N \in O \}$, and
   
   (2.2) $\text{rank}(\mathcal{N}_{\dot{\mathcal{G}}_\alpha} \cap \mathcal{M}_{\alpha+2}^\ast, N) \geq \nu$ for every $N \in O$ such that $f(\nu) = \delta_N$.
3. $b$ is a function with domain included in $\text{dom}(f) \setminus (\omega + 1)$. Moreover, the following holds for all $\nu \in \text{dom}(b)$:
   
   (3.1) $b(\nu) < \nu$ and $b(\nu) + 1 \in \text{dom}(f)$.
   
   (3.2) If $\nu_0 \in \text{dom}(f)$ is such that $b(\nu) < \nu_0 < \nu$, then $f(\nu_0) \notin C_f(\nu)$. Furthermore, if $\nu_1 \in \text{dom}(f)$ is such that $\nu_0 + 1 < \nu_1 < \nu$, then $[f(\nu_0), f(\nu_1)] \cap C_f(\nu) = \emptyset$.
   
   (3.3) $\text{rank}(\{ M \in \mathcal{N}_{\dot{\mathcal{G}}_\alpha} \cap \mathcal{M}_{\alpha+2}^\ast : \delta_M \notin C_f(\nu) \}, N) \geq \nu$ for every $N \in O$ such that $f(\nu) = \delta_N$.

Given $\Theta_{\dot{\mathcal{C}}}$–conditions $(f_0, b_0, O_0)$ and $(f_1, b_1, O_1)$, $(f_1, b_1, O_1)$ extends $(f_0, b_0, O_0)$ in case $f_0 \subseteq f_1$, $b_0 \subseteq b_1$, and $O_0 \subseteq O_1$.

The forcing $\Theta_{\dot{\mathcal{C}}}$ is meant to add a club $C$ measuring $\dot{\mathcal{C}}$. This club is the range of the union of all functions $f$ coming from conditions in the generic filter. The fact that this function is continuous and has domain $\omega_1$ is ensured essentially by condition (2.2) in our definition. The function $b$ represents the commitment to avoid a certain member $C_\delta$ of $\dot{\mathcal{C}}$ on a tail of $C \cap \delta$ for every $\delta$ in its domain. This, together with the conditions that must hold in case we make this commitment, is expressed in condition (3). By density, every $\nu$ will eventually be in the domain of a relevant $b$ – in other words, we will promise that $C$ avoids a tail of $f(\nu)$ – unless we cannot keep that promise. If we cannot keep that promise at a given $\nu$, then a density argument using essentially condition (2) and the symmetry of $O$ will show that a tail
of \(C \cap f(\nu)\) will automatically go into \(C_{f(\nu)}\) (see the proof of Lemma 3.11).

The following lemma is immediate.

**Lemma 2.4.** Suppose \(\hat{C}\) is a \(\mathcal{P}_\alpha\)-name for a club–sequence. If \((f, b, \mathcal{O}_0)\) and \((f, b, \mathcal{O}_1)\) are conditions in \(\Theta_{\hat{C}}\) and \(\mathcal{O}_0 \cup \mathcal{O}_1\) is a \(T_{\alpha+2}\)-symmetric \(\mathcal{M}_{*+2}\)-system, then \((f, b, \mathcal{O}_0 \cup \mathcal{O}_1)\) is a condition in \(\Theta_{\hat{C}}\) stronger than both \((f, b, \mathcal{O}_0)\) and \((f, b, \mathcal{O}_1)\).

2.1.2. **Resuming the construction.** We are now in a position to define \(\mathcal{P}_\beta\) in general for any \(\beta > 0, \beta \leq \kappa\) (assuming \(\mathcal{P}_\alpha\) defined for all \(\alpha < \beta\)).

If \(\alpha < \kappa\) and \(\mathcal{P}_\alpha\) is defined, we let \(\Phi^*(\alpha)\) be a \(\mathcal{P}_\alpha\)-name for (say) the sequence \((\delta)_{\beta < \omega_1}\) if \(\Phi(\alpha)\) is not a \(\mathcal{P}_\alpha\)-name for a club–sequence, and let \(\Phi^*(\alpha)\) be \(\Phi(\alpha)\) if \(\Phi(\alpha)\) is a \(\mathcal{P}_\alpha\)-name for a club–sequence.

Assume first that \(\beta < \kappa\). Conditions in \(\mathcal{P}_\beta\) are pairs of the form \(q = (p, \Delta)\) with the following properties.

\((C0)\) \(p\) is a finite function such that \(\text{dom}(p) \subseteq \beta\) and \(\Delta\) is a set of pairs \((N, \gamma)\) with \(\gamma \leq \beta\).

\((C1)\) \(\Delta^{-1}(\beta)\) is a \(T_{\bar{\beta}+1}\)-symmetric \(\mathcal{M}_{\beta+1}\)-system.

\((C2)\) For every \(\alpha < \beta\), the restriction of \(q\) to \(\alpha\) is a condition in \(\mathcal{P}_\alpha\). This restriction is defined as

\[ q|_{\alpha} := (p|\alpha, \{(N, \min\{\alpha, \gamma\}) : (N, \gamma) \in \Delta\}) \]

\((C3)\) If \(\alpha \in \text{dom}(p)\), then \(p(\alpha)\) is of the form \((f_{p,\alpha}^{\beta}, b_{p,\alpha}^{\beta}, \mathcal{O}_{p,\alpha}^{\beta})\) and

\((C3.1)\) \(\Delta^{-1}_{\alpha+1}(\alpha + 1) \cap \mathcal{M}_{*+2} \subseteq \mathcal{O}_{p,\alpha}^{\beta} \subseteq \Delta^{-1}_{\alpha}(\alpha),\)

\((C3.2)\) \(\mathcal{P}_\alpha \upharpoonright q|_{\alpha}\) forces that \((f_{p,\alpha}^{\beta}, b_{p,\alpha}^{\beta}, \mathcal{O}_{p,\alpha}^{\beta})\) is a \(\Theta_{\Phi^*(\alpha)}\)-condition, and

\((C3.3)\) for all \(N \in \Delta^{-1}_{\alpha+1}(\alpha + 1), \text{ if } \alpha \in N, \text{ then } \delta_N \in \text{dom}(f_{p,\alpha}^{\beta})\) and \(f_{p,\alpha}^{\beta}(\delta_N) = \delta_N\).

Given conditions

\[ q_\epsilon = (p_\epsilon, \Delta_\epsilon) \]

(for \(\epsilon \in \{0, 1\}\)) in \(\mathcal{P}_\beta\), we will say that \(q_1 \leq q_0\) if and only if the following holds.

\((D1)\) \(q_1|_{\alpha} \leq q_0|_{\alpha}\) for all \(\alpha < \beta\),

\((D2)\) \(\text{dom}(p_0) \subseteq \text{dom}(p_1)\),

\((D3)\) \(f_{p_0,\alpha}^{\beta} \subseteq f_{p_1,\alpha}^{\beta}, b_{p_0,\alpha}^{\beta} \subseteq b_{p_1,\alpha}^{\beta}\) and \(\mathcal{O}_{p_0,\alpha}^{\beta} \subseteq \mathcal{O}_{p_1,\alpha}^{\beta}\) for all \(\alpha \in \text{dom}(p_0)\),

\((D4)\) \(\Delta^{-1}_{\alpha}(\beta) \subseteq \Delta^{-1}_{\alpha}(\beta)\).

^11It will follow from our definition that, for all \(\alpha < \beta\), a \(\mathcal{P}_\alpha\)-name is also a \(\mathcal{P}_\beta\)-name (literally).

^12It follows of course that \(q_1|_{\alpha}\) forces that \((f_{p_1,\alpha}^{\beta}, b_{p_1,\alpha}^{\beta}, \mathcal{O}_{p_1,\alpha}^{\beta})\) extends \((f_{p_0,\alpha}^{\beta}, b_{p_0,\alpha}^{\beta}, \mathcal{O}_{p_0,\alpha}^{\beta})\) in \(\Theta_{\Phi^*(\alpha)}\) whenever \(\alpha \in \text{dom}(p_0)\).
Note that if $\beta < \kappa$ is a nonzero limit ordinal and $q = (p, \Delta)$ satisfies condition $(C0)$, then $q \in \mathcal{P}_\beta$ iff $q$ satisfies $(C1)$ and $(C2)$. Note also that for all $\beta < \kappa$, $\mathcal{P}_\beta$ is definable in $\langle H(\kappa), T^{\beta+1} \rangle$.

We will use the following easy lemma.

**Lemma 2.5.** For all $\beta < \kappa$ and all $R \subseteq H(\kappa)$, if $M$ is such that $\langle M, T^{\beta+1}, R \rangle \preccurlyeq \langle H(\kappa), T^{\beta+1}, R \rangle$, then $\mathcal{P}_\beta$ forces $\langle M[G], \mathcal{G}_\beta, R \rangle \preccurlyeq \langle H(\kappa)^V[H], \mathcal{G}_\beta, R \rangle$.

Finally we give the definition of the forcing $\mathcal{P}_\kappa$. Conditions in $\mathcal{P}_\kappa$ are pairs of the form $q = (p, \Delta)$ with the following properties.

$(E0)$ $p$ is a finite function such that $\text{dom}(p) \subseteq \kappa$ and $\Delta$ is a set of pairs $(N, \gamma)$ with $\gamma < \kappa$.

$(E1)$ For every $\alpha < \kappa$, the restriction of $q$ to $\alpha$ is a condition in $\mathcal{P}_\alpha$. This restriction is defined as

$q|\alpha := (p \upharpoonright \alpha, \{(N, \min\{\alpha, \gamma\}) : (N, \gamma) \in \Delta\})$

Given conditions

$q_\epsilon = (p_\epsilon, \Delta_\epsilon)$

(for $\epsilon \in \{0, 1\}$) in $\mathcal{P}_\kappa$, we will say that $q_1 \leq_\kappa q_0$ if and only if the following holds.

$(F1)$ $q_1|\alpha \leq_\alpha q_0|\alpha$ for all $\alpha < \kappa$.

3. Proving Theorem 1.2: The actual proof

In this section we prove the main facts about the forcings $\mathcal{P}_\alpha$. Theorem 1.2 will follow immediately from them.

Our first lemma is immediate from the definitions.

**Lemma 3.1.** $\mathcal{P}_\kappa = \bigcup_{\beta < \kappa} \mathcal{P}_\beta$, and $\emptyset \neq \mathcal{P}_\alpha \subseteq \mathcal{P}_\beta$ for all $\alpha \leq \beta \leq \kappa$.

Lemma 3.2 shows in particular that $\langle \mathcal{P}_\alpha : \alpha \leq \kappa \rangle$ is a forcing iteration in a broad sense.

**Lemma 3.2.** Let $\alpha \leq \beta \leq \kappa$. If $q = (p, \Delta_q) \in \mathcal{P}_\alpha$, $s = (r, \Delta_s) \in \mathcal{P}_\beta$ and $q \leq_\alpha s|\alpha$, then $(p\gamma(r \upharpoonright [\alpha, \beta]), \Delta_q \cup \Delta_s)$ is a condition in $\mathcal{P}_\beta$ extending $s$. Therefore, any maximal antichain in $\mathcal{P}_\alpha$ is a maximal antichain in $\mathcal{P}_\beta$ and $\mathcal{P}_\alpha$ is a complete suborder of $\mathcal{P}_\beta$.

**Proof.** It suffices to note that if the pair $(N, \gamma)$ is in $\Delta_q$, then the marker $\gamma$ (which bounds the influence of the side condition $N$ in clauses $(C1)$ and $(C3)$) is at most $\alpha$. \qed
The following lemma shows that all forcings $\mathcal{P}_\beta$ are $\aleph_2$–Knaster, and so in particular have the $\aleph_2$–chain condition.\footnote{A forcing $P$ is $\mu$–Knaster if every subset of $P$ of cardinality $\mu$ includes a subset of cardinality $\mu$ of pairwise compatible conditions.} The proof uses standard $\Delta$–system arguments\footnote{This is the only place where we make use of CH.} (see Fact 2.3 and Lemma 2.4).

Lemma 3.3. For every $\beta \leq \kappa$ and every set $\{ (p_\xi, \Delta_\xi) : \xi < \omega_2 \}$ of $\mathcal{P}_\beta$–conditions there is $I \subseteq \omega_2$ of size $\aleph_2$ such that for all $\xi, \xi'$ in $I$:

\begin{enumerate}
\item if $\gamma \leq \beta$ and $\gamma < \kappa$, then $\Delta_\gamma^{-1}(\gamma) \cup \Delta_\gamma^{-1}(\gamma)$ is a $\check{T}^{\gamma+1}$–symmetric $\mathcal{M}_0^{\gamma+1}$–system of structures,
\item if $\alpha \in \text{dom}(p_\xi) \cap \text{dom}(p_{\xi'})$, $(f^{p_\xi, \alpha}, b^{p_\xi, \alpha}) = (f^{p_{\xi'}, \alpha}, b^{p_{\xi'}, \alpha})$ and $\mathcal{O}^{p_\xi, \alpha} \cup \mathcal{O}^{p_{\xi'}, \alpha}$ is a $\check{T}^{\alpha+2}$–symmetric $\mathcal{M}_0^{\alpha+2}$–system of structures, and
\item letting $p^*(\alpha) = (f^{p_\xi, \alpha} \cup f^{p_{\xi'}, \alpha}, b^{p_\xi, \alpha} \cup b^{p_{\xi'}, \alpha}, \mathcal{O}^{p_\xi, \alpha} \cup \mathcal{O}^{p_{\xi'}, \alpha})$ for all $\alpha \in \text{dom}(p_\xi \cup p_{\xi'})$, $(p^*, \Delta_\xi \cup \Delta_{\xi'})$ is a $\mathcal{P}_\beta$–condition extending both $(p_\xi, \Delta_\xi)$ and $(p_{\xi'}, \Delta_{\xi'})$.\footnote{If $\alpha$ is not in the domain of $p_\xi$, then by $(f^{p_\xi, \alpha}, b^{p_\xi, \alpha}, \mathcal{O}^{p_\xi, \alpha})$ we will mean $(\emptyset, \emptyset, \emptyset)$ and similarly for $p_{\xi'}$.}
\end{enumerate}

In particular, $\mathcal{P}_\beta$ is $\aleph_2$–Knaster.

Corollary 3.4. If $\check{C}$ is a $\mathcal{P}_\alpha$–name for a club–sequence, then $\mathcal{P}_\alpha$ forces $\Theta_{\check{C}}$ to have the $(\aleph_2)^V$–chain condition.\footnote{By lemma 3.7 we will see that $(\aleph_2)^V = (\aleph_2)^{V_{\mathcal{P}_\alpha}}$.} Hence, every maximal antichain of $\Theta_{\check{C}}$ is forced to be a member of $H(\kappa)$.

Proof. Suppose $\check{A}$ is a $\mathcal{P}_\alpha$–name for a maximal antichain of $\Theta_{\check{C}}$ of size $(\aleph_2)^V$. For each $\zeta \in \omega_2$, let $p_\zeta$ be a $\mathcal{P}_\alpha$–condition forcing $\check{a}_\zeta = \check{b}_\zeta$ for some $b_\zeta$ in $V$, where $\check{a}_\zeta$ denotes the $\zeta$-th element of $\check{A}$. By the above lemma, we may assume that all the $p_\zeta$’s are pairwise $\mathcal{P}_\alpha$–compatible. Using Fact 2.3 and Lemma 2.4 we can also assume that any common extension of $p_\zeta$ and $p_{\zeta'}$ forces that $\check{b}_\zeta$ and $\check{b}_{\zeta'}$ are $\Theta_{\check{C}}$–compatible, but that is a contradiction. \hfill $\Box$

Strictly speaking, the following lemma will not be needed in the rest of the paper. However, its proof is a convenient warm-up for the proof of conclusion (2)$_\beta$ of Lemma 3.7.

Lemma 3.5. Let $\beta < \kappa$, and suppose $q = (p, \Delta_q)$ is $(M, \mathcal{P}_\beta)$–generic whenever $q \in \mathcal{P}_\beta$ and $M \in \Delta_q^{-1}(\beta) \cap \mathcal{M}^{\beta+1}$.\footnote{We will see in Lemma 3.7 that this hypothesis is true.} If $\check{C}$ is a $\mathcal{P}_\beta$–name for a club–sequence $(C_\delta)_{\delta < \omega_1}$, then $\mathcal{P}_\beta$ forces that $\sigma_0 = (f_0, b_0, \mathcal{O}_0)$ is $(N[G_{\check{C}}], \Theta_{\check{C}})$–generic whenever $\sigma_0 \in \Theta_{\check{C}}$, $N \in \mathcal{O}_0 \cap \mathcal{M}^{\beta+2}$, $\delta_N \in \text{dom}(f_0)$, and $f_0(\delta_N) = \delta_N$.\footnote{By lemma 3.7 we will see that $(\aleph_2)^V = (\aleph_2)^{V_{\mathcal{P}_\alpha}}$.}
Proof. Let us work in $V^{P_\beta}$. Let $A \in N[\dot{G}_\beta]$ be a maximal antichain of $\Theta_{\dot{\mathcal{C}}}$. By extending $\sigma_0$ if necessary we may assume that $\sigma_0$ extends a condition $\sigma^\dagger$ in $A$. We want to show of course that $\sigma^\dagger$ is in $N[\dot{G}_\beta]$, and for this it will suffice to find a member of $A \cap N[\dot{G}_\beta]$ compatible with $\sigma_0$.

We may assume that $\delta_N \in dom(b)$, as otherwise the proof is slightly simpler. Let $\mu = max(range(f_0 \upharpoonright \delta_N)) + 1$\footnote{Note that $range(f_0 \upharpoonright \delta_N) \neq \emptyset$ by condition (3.1) in the definition of $\Theta_{\dot{\mathcal{C}}}$.} let $\dot{A} \in N[\dot{G}_\beta]$ be the (partially defined) function sending each $\sigma \in A$ to the first $\Theta_{\dot{\mathcal{C}}}$-condition $(f', b', \mathcal{O}')$ extending $\sigma$ (in some canonical well-ordering given by $\Phi$) and such that $f_0 \upharpoonright \delta_N \subseteq f'$, $range(f') \cap \mu = range(f_0) \cap \mu$, $b_0 \upharpoonright \delta_N \subseteq b'$ and $\mathcal{O}_0 \cap N \subseteq \mathcal{O}'$ (whenever this is possible)\footnote{$\dot{A}$ is in $N[\dot{G}_\beta]$ since $(f_0 \upharpoonright \delta_N, b_0 \upharpoonright \delta_N, \mathcal{O}_0) \in N$ and $\dot{A}$ is definable in the structure $\langle H(\kappa)^{V[\dot{G}_\beta]}, \dot{G}_\beta, T^{\beta+2} \rangle$ from $(f_0 \upharpoonright \delta_N, b_0 \upharpoonright \delta_N, \mathcal{O}_0)$ (by Lemma 2.5).} and let $M \in N \cap N_{\dot{G}_\beta} \cap \mathcal{M}_{\dot{\mathcal{C}}}^{\beta+2}$ be such that $\dot{A} \in M[\dot{G}_\beta]$, $\beta + 1 \in M$, and $\delta_M \notin C_{\delta_N}$. Such an $M$ exists by condition (3.3) in the definition of $\Theta_{\dot{\mathcal{C}}}$. Let $\eta < \delta_M$ be such that $[\eta, \delta_M] \cap C_{\delta_N} = \emptyset$ (this $\eta$ exists by openness of $\delta_N \setminus C_{\delta_N}$). Now, in $M[\dot{G}_\beta]$ there is $\sigma_* = (f_*, b_*, \mathcal{O}_*)$ such that

- $(a)$ $\sigma_* \in range(\dot{A})$, and
- $(b)$ $min(range(f_*)) \mu > \eta$.

Note that $max(range(f_*)) < \delta_M$ since $\delta_M[\dot{G}_\beta] = \delta_M$ by our assumption on $N_{\dot{G}_\beta} \cap \mathcal{M}_{\dot{\mathcal{C}}}^{\beta+1}$. Let

$$\mathcal{O}' = \mathcal{O}_0 \cup \{ \Psi_{N, N'}(M) : M \in \mathcal{O}_*, N' \in \mathcal{O}_0, \delta_{N'} = \delta_N \}$$

Now it is easy to check that $(f_* \cup f_0, b_* \cup b_0, \mathcal{O}')$ is a common extension of $\sigma_*$ and $\sigma_0$ in $\Theta_{\dot{\mathcal{C}}}$ (this uses condition (C1) in the definition of $P_\beta$ for the verification of conditions (1), (2.2) and (3.3) in the definition of $\Theta_{\dot{\mathcal{C}}}$. Letting $\overline{\sigma} \in A \cap N[\dot{G}_\beta]$ be the unique $\sigma \in A$ such that $\dot{A}(\sigma) = \sigma_*$\footnote{Note that $\dot{A}$ is one-to-one.} it follows that $\overline{\sigma} = \sigma^\dagger$. □

The following lemma can be proved easily by induction on $\alpha$. It will be used in the proof of Lemma 3.7.

**Lemma 3.6.** For all $\alpha < \kappa$, $v \in P_\alpha$, and $\delta < \omega_1$ there is $w = (\overline{w}, \Delta_w) \in P_\alpha$ extending $v$ together with $\eta < \delta$ such that for all $M \in N$ and all $\xi \in dom(\overline{w}) \cap N$, if $M$ and $N$ are both in $\Delta^{-1}_{\overline{w}|\xi+1}(\xi+1)$, $\delta_N = \delta \in dom(\overline{b}^\overline{w}|\xi)$ and $b^\overline{w}|\xi(\delta_N) < \delta_M = f^\overline{w}|\xi(\delta_M)$, then $w|\xi \parallel P_\xi \dot{G}_\delta \cap [\eta, \delta_M] = \emptyset$.

The properness of all $P_\beta$ ($\beta < \kappa$) is an immediate consequence of the following lemma.
Lemma 3.7. Suppose $\beta < \kappa$ and $N \in \mathcal{M}^{\beta+1}$. Then the following conditions hold.

$(1)_\beta$ For every $q \in N \cap \mathcal{P}_\beta$ there is $q' \leq^\beta q$ such that $N \in \Delta^-_q(\beta)$.

$(2)_\beta$ If $q \in \mathcal{P}_\beta$ and $N \in \Delta^-_q(\beta)$, then $q$ is $(N, \mathcal{P}_\beta)$-generic.

Proof. The proof of $(2)_\beta$ will be the same in all cases, and the proof of $(1)_\beta$ will be by induction on $\beta$. The proof of $(1)_0$ is trivial: It suffices to set $q' = q \cup \{(N, 0)\}$.

The proof of $(1)_\beta$ when $\beta = \alpha + 1$ is as follows. Let $q = (p, \Delta_q)$. By $(1)_\alpha$ we may assume that there is a condition $t = (u, \Delta_t) \in \mathcal{P}_\alpha$ extending $q|_\alpha$ and such that $N \in \Delta^-_t(\alpha)$. This condition $t$ clearly forces (in $\mathcal{P}_\alpha$) that $N \in \mathcal{N}_t$. So, $t$ forces that for every $x \in N$, there is $M \in \mathcal{N}_t \cap \mathcal{M}^{\alpha+1}$ such that $x \in M$.

Let us work in $V^{\mathcal{P}_\alpha \upharpoonright t}$. Since, by Lemma 2.3, $\langle N[\hat{G}_\alpha], \hat{G}_\alpha, T^{\alpha+2}, H(\kappa)^V \rangle$ is an elementary substructure of $\langle H(\kappa)[\hat{G}_\alpha], \hat{G}_\alpha, T^{\alpha+2}, H(\kappa)^V \rangle$, there exists an $M$ as above in $N[\hat{G}_\alpha] \cap V$ (where $V$ denotes the ground model). We can also assume that $M \in N$, since $N[\hat{G}_\alpha] \cap V = N$ (which follows from $(2)_\alpha$ applied to $N$ and $t$). This shows that $t$ forces $\text{rank}(\mathcal{N}_t \cap \mathcal{M}^{\alpha+1}, N) \geq 1$. In fact, a similar argument shows that $t$ forces $\text{rank}(\mathcal{N}_t \cap \mathcal{M}^{\alpha+1}, N) > \mu$ for every $\mu < \delta_N$. In view of these considerations, it suffices to define $q'$ as the condition $(u', \Delta_q \cup \Delta_t \cup \{(N, \beta)\})$, where $u'$ extends $u$ and sends the ordinal $\alpha$ to the triple $(f^{p, \alpha} \cup \{(\delta_N, \delta_N)\}, b^{p, \alpha}, \mathcal{O}^{p, \alpha} \cup \{N\})$.

The proof of $(1)_\beta$ when $\beta$ is a nonzero limit ordinal is trivial using $(1)_\alpha$ for all $\alpha < \beta$, together with the fact if $q = (p, \Delta) \in \mathcal{P}_\beta$, then the domain of $p$ is bounded in $\beta$.

Now let us proceed to the proof of $(2)_\beta$ for general $\beta$. Let $A \subseteq \mathcal{P}_\beta$ be a maximal antichain in $\mathcal{N}^\times$, and suppose $q = (p, \Delta_q)$ extends a condition in $A$. We want to see that $q$ is compatible with a condition in $A \cap N$.

Let $\xi_0$ be the maximum of the set $X$ of $\xi \in \text{dom}(p)$ such that $\xi \in N'$ for some pair $(N', \gamma) \in \Delta_q$ with $\xi < \gamma$ and $\delta_N = \delta_N'$. Let $(N', \gamma) \in \Delta_q$ witness $\xi_0 \in X$. By extending $q$ further if necessary, we may assume that there is some $M' \in N'$ such that $(M', \xi_0) \in \Delta_q$ and such that $\Psi_{N, N'}(x) \in M'$ for all relevant $x \in N$. Let $M = \Psi_{N', N}(M')$. If $\delta_N \in \text{dom}(b^{p, \xi_0})$, we may assume that there is $\eta < \delta_M$ such that $q^{\xi_0} \forces \check{C}^\xi_{\delta_N} \cap [\eta, \delta_M] = \emptyset$. By Lemma 3.8 we may further assume that $q^{\xi} \forces \check{C}^\xi_{\delta_N} \cap [\eta, \delta_M] = \emptyset$ whenever $\xi \in X$, $\delta_N = \delta_N' \in \text{dom}(b^{p, \xi})$, $\bar{M} \in N$, $\delta_M = \delta_M$, $\bar{M}, N$ are both in $\Delta^-_{q^{\xi_0}}(\xi_0+1)$, and $b^{p, \xi}(\delta_N) < \delta_M = f^{p, \xi}(\delta_M)$.

Claim 3.8. Let $\xi \in X$. 

(a) If $\xi = \xi_0$ and $\delta_N \in \text{dom}(b^{p,\xi_0})$, then $q|_{\xi_0}$ forces $\hat{C}_{\delta_N}^{\xi_0} \cap [\eta, \delta_M] = \emptyset$.

(b) If $\xi \neq \xi_0$, $(\overline{N}, \overline{\gamma}) \in \Delta_q$, $\delta_N = \delta_N^{\overline{\gamma}}$, $\xi \in \overline{N} \cap \overline{\gamma} \cap \Psi_{N,N}(M)$ and
$
\delta_N \in \text{dom}(b^{p,\xi}),
$ then $(\Psi_{N,N}(M), \xi + 1) \in \Delta_{q_{\xi+1}}$ and $\delta_M$ is a fixed point of $f^{p,\xi}$. In particular, these hypothesis imply that $q|_{\xi}$ forces $\hat{C}_{\delta_N}^{\xi} \cap [\eta, \delta_M] = \emptyset$.

**Proof.** In order to prove (b), it is enough to note that $\Psi_{N,N}(M) = \Psi_{N,N}(M')$ and to apply clauses (C1) and (C3.3) to condition $q|_{\xi+1}$. □

Let $\{\delta_0, \ldots, \delta_{l-1}\} = \{\delta_{N'}, : N' \in \Delta_{q_0}^{-1}(0)\} \cap \delta_M$. By correctness of $M$ and since $M$ contains all relevant objects, there is a condition $t = (\overline{T}, \Delta_t) \in M$ satisfying the following properties (in $V$).

1. $t \in A$.
2. For all $W \in \Delta_{q_0}^{-1}(0) \cap M$ and for all $\zeta \in \beta + 1$, if $\zeta \in W$ and $W \in \Delta_{q_{\zeta}}^{-1}(\zeta)$, then $W \in \Delta_{t_{\zeta}}^{-1}(\zeta)$.[21]
3. For all $\zeta \in \text{dom}(\overline{T})$, if $\zeta \in \text{dom}(p)$, then
   (3.1) $(f^{T,\zeta}, \overline{F}, \overline{\zeta})$ and $(f^{p,\zeta}, \overline{b^{p,\zeta}}, \overline{O^{p,\zeta}})$ are forced by $q|_{\zeta}$ to be compatible as $\Theta^{(\zeta)}$–conditions, and
   (3.2) the least point in $\text{dom}(f^{T,\zeta})$ above $\text{dom}(f^{p,\zeta})$ is above $\eta$.
4. For all $i < l$, for all $\xi \in \text{dom}(p)$ and for all $(N', \gamma) \in \Delta_q$ with $\xi < \gamma$ and $\delta_{N'} = \delta_N$, if there is no $W$ such that $\xi \in W$, $W' \in \Delta_{q_{\zeta}}^{-1}(\xi + 1)$ and $\delta_W = \delta_t$, then letting $\eta = \Psi_{N',N}(\xi)$ and $\rho = \min((OR \cap M) \setminus \xi)$, there is no $W$ such that $\eta \in W$, $W \in \Delta_{t_{\rho}}^{-1}(\rho)$ and $\delta_W = \delta_t$.[22]
5. If $(M', \gamma) \in \Delta_t$ and $\delta_{M'} \notin \{\delta_0, \ldots, \delta_{l-1}\}$, then $\delta_{M'} > \eta$.

Using Claim 3.8, it is easy to check that one can amalgamate $q$ and $t$ into a condition $q^*$ with $\Delta_{q^*}$ being the union of $\Delta_q$ with the set of all pairs $(\Psi_{N,N'}(W), \min\{\rho, \gamma\})$ such that $(W, \rho) \in \Delta_t$, $(N', \gamma) \in \Delta_q$, and $\delta_{N'} = \delta_N$. The closure of $\Delta_{q^*}$ under isomorphisms does not interfere with the elements of $X$ because of condition (4).

**Corollary 3.9.** For every $\beta \leq \kappa$, $\mathcal{P}_\beta$ is proper.

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21Note that for such a $W$, the set of those $\zeta$ in $M \cap (\beta + 1)$ such that $\zeta \in W$ and $W \in \Delta_{q_{\zeta}}^{-1}(\zeta)$ can be correctly computed (in $M$) by means of a formula using as parameters the structure $W$ and the minimum ordinal in $M$ which is at least the maximum of those $\zeta$ such that $(W, \zeta) \in \Delta_q$.

22By clause (C1) applied to condition $q|_{\xi+1}$, the existence of a $W$ such that $\Psi_{N',N}(\xi) \in W \in \Delta_{t_{\rho}}^{-1}(\rho)$ would imply that $\xi = \Psi_{N',N}(\Psi_{N',N}(\xi)) \in \Psi_{N,N}(W) \in \Delta_{q_{\xi+1}}^{-1}(\xi + 1)$.
Proof. For \( \beta < \kappa \) the conclusion follows immediately from Lemma 3.7.

The remaining case follows from the corresponding conclusions for \( \beta < \kappa \) together with the \( \aleph_2 \)-c.c. of \( \mathcal{P}_\kappa \) and \( cf(\kappa) \geq \omega_2 \). \( \square \)

For every \( \beta < \kappa \) let \( \dot{F}_\beta \) and \( \dot{B}_\beta \) be \( \mathcal{P}_\kappa \)-names for, respectively, the union of all functions \( f \) for which there is a condition \( q = (p, \Delta) \in \dot{G}_\kappa \) such that \( p(\beta) = (f, b, \mathcal{O}) \) for some \( b \) and \( \mathcal{O} \), and the union of all \( b \) for which there is a condition \( q = (p, \Delta) \in \dot{G}_\kappa \) such that \( p(\beta) = (f, b, \mathcal{O}) \) for some \( f \) and \( \mathcal{O} \).

**Lemma 3.10.** \( \mathcal{P}_\kappa \) forces \( 2^{\aleph_0} = \kappa \).

**Proof.** In order to prove that \( \mathcal{P}_\kappa \) forces \( 2^{\aleph_0} \geq \kappa \), it suffices to note that if \( \beta < \kappa \), then the restriction of \( \dot{F}_\beta \) to \( \omega \) is forced to be a Cohen real (recall that \( \dot{F} \) is a name for a normal function having \( \omega \) as a fixed point).

The other inequality follows from counting nice names for subsets of \( \omega \) using the fact that \( \kappa^{\omega_1} = \kappa \) together with Lemma 3.3. \( \square \)

**Lemma 3.11.** For all \( \beta < \kappa \), \( \mathcal{P}_\kappa \) forces that \( \text{range}(\dot{F}_\beta) \) is a club of \( \omega_1 \) measuring \( \Phi^*(\beta) \).

**Proof.** Let \( \dot{C}_\delta \) be, for each \( \delta \), a \( \mathcal{P}_\beta \)-name for the \( \delta \)-th member of \( \Phi^*(\beta) \). We want to show that the following conditions hold in \( V^{\mathcal{P}_\kappa} \):

(A) \( \dot{F}_\beta \) is a normal function with domain \( \omega_1 \).

(B) For each \( \nu < \omega_1 \),

(\( B1 \)) if \( \nu \in \text{dom}(\dot{B}_\beta) \), then \( \text{range}(\dot{F}_\beta \upharpoonright (\dot{B}_\beta(\nu), \nu)) \) is disjoint from \( \dot{C}_{f(\nu)} \), and

(\( B2 \)) if \( \nu \notin \text{dom}(\dot{B}_\beta) \), then a tail of \( \text{range}(\dot{F}_\beta \upharpoonright \nu) \) is included in \( \dot{C}_{f(\nu)} \).

Showing (A) is easy, so here we will only show (B). Note that for every \( q = (p, \Delta) \in \mathcal{P}_\kappa \), if \( p(\beta) = (f, b, \mathcal{O}) \) and \( \nu \in \text{dom}(f) \), then there is some \( q' = (p', \Delta') \) extending \( q \) such that, letting \( p'(\beta) = (f', b', \mathcal{O}') \), either

(\( a \)) \( \nu \in \text{dom}(b') \), or

(\( b \)) \( q' \) forces \( \text{rank}(\{ M \in N_{\dot{G}_\beta} \cap \mathcal{M}_s^{\beta+2} : \delta_M \notin \dot{C}_{f(\nu)} \}, N) = \nu' \) for some given \( \nu' < \nu \) for every (equivalently, for some) \( N \in \mathcal{O}' \) such that \( \delta_N = f(\nu) \).

It is enough to assume (b) and show that \( q' \) forces that a tail of \( \text{range}(\dot{F}_\beta \upharpoonright \nu) \) is included in \( \dot{C}_{f(\nu)} \). For this, fix an \( N \) as in (b) and, extending \( q' \) if necessary, fix also \( x \in N \) such that \( q'|_{\beta} \) forces that if \( M \in N \) is such that \( x \in M \) and \( \text{rank}(N_{\dot{G}_\beta} \cap \mathcal{M}_s^{\beta+2}, M) > \nu' \), then \( \delta_M \in \dot{C}_{f(\nu)} \). By further extending \( q' \) if necessary we may assume that
For this, note that \( q''\beta \) forces that \( f^{p'', \beta}(\nu_o) \) is \( \delta_{M_o} \) for some \( M_o \in \mathcal{O}^{p'', \beta} \subseteq \mathcal{M}^{\beta+2}_\beta \cap \mathcal{N}_{G_\beta} \) such that \( \text{rank}(\mathcal{N}_{G_\beta} \cap \mathcal{M}^{\beta+2}_\beta, M_o) \geq \nu_o \). By symmetry of \( \mathcal{O}^{p'', \beta} \) and since \( \delta_{M_o} > \delta_M \) there is then some \( M'_o \in \mathcal{O}^{p'', \beta} \) such that \( M \in M'_o \) and \( \delta_{M'_o} = \delta_{M_o} \). Since, by symmetry, \( q''\beta \) forces \( \text{rank}(\mathcal{N}_{G_\beta} \cap \mathcal{M}^{\beta+2}_\beta, M'_o) = \text{rank}(\mathcal{N}_{G_\beta} \cap \mathcal{M}^{\beta+2}_\beta, M_o) \geq \nu_o \) and since \( x \in M'_o \), it follows that \( q''\beta \) forces \( f^{p'', \beta}(\nu_o) = \delta_{M'_o} \in \mathcal{C}_{f(\nu)} \), which is what we wanted. \( \square \)

**Corollary 3.12.** \( P_\kappa \) forces measuring.

The above corollary follows from Lemmas 3.11, 3.3 and 3.7 (since \( \Phi \) was chosen to be a book-keeping function), and finishes the proof of Theorem 1.2.

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23Specifically, by condition (\( \gamma \)) in the definition of symmetric system.
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David Asperó, Institute of Discrete Mathematics and Geometry, TU Wien, Wiedner Hauptstrasse 8-10/104, 1040 Wien, Austria
E-mail address: david.aspero@tuwien.ac.at

Miguel Angel Mota, Kurt Gödel Research Center for Mathematical Logic, Währinger Straße 25, 1090 Wien, Austria
E-mail address: motagaytan@gmail.com