WEAK GALERKIN METHODS BASED MORLEY ELEMENTS ON GENERAL POLYTOPAL PARTITIONS

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Abstract. A new weak Galerkin method based on the weak tangential derivative and weak second order partial derivative is proposed to extend the well-known Morley element for the biharmonic equation from triangular elements to general polytopal elements. The Schur complement of the weak Galerkin scheme not only enjoys the same degrees of freedom as the Morley element on the triangular element but also extends the Morley element to any general polytopal element. The error estimates for the numerical approximation are established in the energy norm and the usual $L^2$ norms. Several numerical experiments are demonstrated to validate the theory developed in this article.

Key words. weak Galerkin, finite element methods, Morley element, biharmonic equation, weak tangential derivative, weak Hessian, polytopal partitions, Schur complement.

AMS subject classifications. Primary 65N30, 65N12, 65N15; Secondary 35B45, 35J50.

1. Introduction. This paper is concerned with the development of the Morley element for the biharmonic equation by using the weak Galerkin (WG) method. For simplicity, we consider the following biharmonic model equation

\[
\Delta^2 u = f, \quad \text{in } \Omega, \\
u = g, \quad \text{on } \partial \Omega, \\
\frac{\partial u}{\partial n} = \nu, \quad \text{on } \partial \Omega,
\]

(1.1)

where $\Omega$ is a bounded polytopal domain in $\mathbb{R}^d (d = 2, 3)$ and the vector $n$ is an unit outward normal direction to the boundary of $\Omega$.

It is well known that the construction of $C^1$ continuous finite element requires higher order polynomial functions which leads to the complexity in numerical implementation. To overcome this complexity, the nonconforming finite element method is naturally an attractive alternative. One well-known nonconforming finite element is the Morley element [7], which has the least degrees of freedom but is limited to the triangular partitions. Since then, researchers have been devoted to the development of the Morley element for general polytopal partitions. In [14], the Morley element was generalized to $\mathbb{R}^d (d \geq 3)$. [18] proposed an extension of the Morley element to the quasi-conforming tetrahedral element in $\mathbb{R}^3$. [16] extended the Morley...
element to any dimensions. [19] [17] constructed the $d$-rectangle Morley elements in $\mathbb{R}^d (d \geq 2)$. [13] proposed the quadrilateral Morley element. The Morley element was also extended to the polygonal meshes by the nonconforming virtual element method in [28, 29]. Besides the aforementioned literature, many numerical methods have been developed for solving the biharmonic equation including discontinuous Galerkin methods [5, 6, 12, 27], virtual element methods [11, 3], and weak Galerkin methods [2, 26, 9, 10]. To the best of our knowledge, there has not been any finite element that not only enjoys the least degrees of freedoms but also works on general polyhedral partitions in the literature.

This paper aims to propose an extension of the Morley element to general polytopal meshes by using the WG method. WG method was first developed by Junping Wang and Xin Ye for second order elliptic problems in [25]. The weak Galerkin methods was proposed for the biharmonic equation based on the weak Hessian operator where the error estimates were established in [22, 23]. In order to reduce the degrees of freedom, a new WG method is proposed in this paper where the degrees of freedom is the same as the Morley element on general polytopal elements thanks to the high flexibility of the WG method. The Schur complement of this WG scheme has $NE+NF$ degrees of freedom on general polytopal partitions, where $NE$ and $NF$ are the numbers of $(d-2)$-dimensional sub-polytopes and $(d-1)$-dimensional sub-polytopes of $d$-dimensional polytopal elements, respectively. The main contributions of this paper are summarized as follows: 1) the new WG method extends the Morley element to general polytopal partitions; 2) the new WG method can be applied to solve a wide range of model problems when the corresponding variational forms are based on the Hessian operator; 3) the error estimates for the numerical approximation in the energy norm and the usual $L^2$ norms are established.

This paper is organized as follows. In Section 2, we introduce the definitions of the weak tangential derivative and the weak second order partial derivative. In Section 3, the weak Galerkin scheme for the biharmonic equation (1.1) as well as its Schur complement are presented. In Section 4, we derive an error equation for the proposed WG scheme. In Section 5, some technical results are derived. Section 6 is devoted to establishing the error estimates for the numerical approximation in the energy norm and the usual $L^2$ norms. Finally, a series of numerical results are reported to validate the theory developed in the previous sections.

The standard notations are adopted throughout this paper. Let $D$ be any open bounded domain with Lipschitz continuous boundary in $\mathbb{R}^d$ for $d = 2, 3$. We use $(\cdot, \cdot)_{s,D}$, $|\cdot|_{s,D}$ and $\|\cdot\|_{s,D}$ to denote the inner product, semi-norm and norm in the Sobolev space $H^s(D)$ for any integer $s \geq 0$, respectively. For simplicity, the subscript $D$ is dropped from the notations of the inner product and norm when the domain $D$ is chosen as $D = \Omega$. For the case of $s = 0$, the notations $(\cdot, \cdot)_{0,D}$, $|\cdot|_{0,D}$ and $\|\cdot\|_{0,D}$ are simplified as $(\cdot, \cdot)_D$, $|\cdot|_D$ and $\|\cdot\|_D$, respectively. The notation “$A \leq CB$” refers to the inequality “$A \leq CB$” where $C$ presents a generic constant independent of the meshsize or the functions appearing in the inequality.

2. Weak Partial Derivatives. The goal of this section is to review the definitions of weak tangential derivative [20] and weak second order partial derivative [22, 23]. To this end, let $T_h$ be a polytopal partition of $\Omega \subset \mathbb{R}^d$ that satisfies the shape regular assumptions described in [21]. Let $T$ be any $d$-dimensional polytopal element in $T_h$. Let $\partial T$ be the boundary of $T$ that is a set of $(d-1)$-dimensional
polytopal elements denoted by $\mathcal{F}$. Let $\partial \mathcal{F}$ be the boundary of $\mathcal{F}$ that is a set of the $(d - 2)$-dimensional polytopal elements denoted by $e$. Denote by $\mathcal{F}_h$ and $\mathcal{E}_h$ the sets of all $(d - 1)$-dimensional polytopal elements $\mathcal{F}$ and all $(d - 2)$-dimensional polytopal elements $e$ in $\mathcal{T}_h$, respectively. Moreover, denote by $\mathcal{E}_h^0 = \mathcal{E}_h \setminus \partial \Omega$ and $\mathcal{F}_h^0 = \mathcal{F}_h \setminus \partial \Omega$. For each element $T \in \mathcal{T}_h$, denote by $h_T$ the diameter of $T$ and $h = \max_{T \in \mathcal{T}_h} h_T$ the meshsize of $\mathcal{T}_h$, respectively.

We introduce a triplet $v = \{v_0, v_b, v_g\}$ such that $v_0 \in L^2(T)$, $v_b \in L^2(\partial \mathcal{F})$ and $v_g \in [L^2(\partial T)]^d$. The first component $v_0$ and the second component $v_b$ represent the values of $v$ in the interior of $T$ and on $\partial \mathcal{F}$, respectively. The third component $v_g$ represents the value of $\nabla v$ on the boundary $\partial T$. Note that $v_b$ is defined on $\partial \mathcal{F}$ that is different from the case when $v_b$ is defined on the boundary $\partial T$ as proposed in [22, 23, 24]. We introduce a set of assigned unit normal vectors to $\mathcal{F}_h$ given by

$$N_h = \{n_F : n_F is the assigned unit normal vector to \mathcal{F}, \ \forall \mathcal{F} \in \mathcal{F}_h\}.$$ 

The third component $v_g$ can be decomposed into the sum of its normal and tangential components; i.e.,

$$v_g = (\nabla v \cdot n_F)n_F + n_F \times (\nabla v \times n_F). \quad (2.1)$$

Denote by $v_n$ the value of $\nabla v \cdot n_F$ on $\partial T$. We introduce the set of all weak functions denoted by $W(T)$; i.e.,

$$W(T) = \{v = \{v_0, v_b, v_n\} : v_0 \in L^2(T), v_b \in L^2(\partial \mathcal{F}), v_n \in L^2(\partial T)\}.$$ 

Let $r \geq 0$ be an integer. Let $P_r(\partial T)$ and $P_r(T)$ be the sets of polynomials with degrees no greater than $r$ on the boundary $\partial T$ and on the element $T$, respectively.

**Definition 2.1.** [20] (Discrete weak tangential derivative) A discrete weak tangential derivative for any weak function $v \in W(T)$, denoted by $\nabla_{w,\mathbf{r};0,T}v$, is defined as a vector-valued polynomial in $[P_0(\mathcal{F})]^d$ such that

$$\langle \nabla_{w,\mathbf{r};0,T}v, \psi \times n_F \rangle_{\mathcal{F}} = \langle v_b, \psi \cdot \mathbf{r} \rangle_{\partial \mathcal{F}}, \ \forall \psi \in [P_0(\mathcal{F})]^d. \quad (2.2)$$

Here, $\mathcal{F} \subset \partial T$ and the vector $\mathbf{r}$ is the tangential vector on $\partial \mathcal{F}$ that is chosen such that $n_F$ and $\mathbf{r}$ obey the right-hand rule. Note that $\nabla_{w,\mathbf{r};0,T}v$ defined in (2.2) is an approximation of the tangential component $n_F \times (\nabla v \times n_F)$ of $v_g$ defined in (2.1).

**Definition 2.2.** [22] (Discrete weak second order partial derivative) A discrete weak second order partial derivative for any weak function $v \in W(T)$, denoted by $\partial^2_{w,0,T}v$, is defined as a polynomial in $P_0(T)$ satisfying

$$\langle \partial^2_{w,0,T}v, \varphi \rangle_{\mathcal{F}} = \langle v_{gi}, \varphi n_i \rangle_{\partial \mathcal{F}}, \ \forall \varphi \in P_0(T), \quad (2.3)$$

where $v_{gi}$ is $i$-th component of the vector $v_g$ and $n = (n_1, \cdots, n_d)$ is an unit outward normal vector to $\partial T$. From (2.1) and Definition 2.1, we have

$$v_{gi} = (v_n n_F + \nabla_{w,\mathbf{r};0,T}v)_i, \quad (2.4)$$

where $v_n \in P_0(\mathcal{F})$.

**Remark 2.1.** Note that in Definitions 2.1, 2.2 the discrete weak tangential derivative proposed in [20] and the discrete weak second order partial derivative proposed in [22] are discretized by the lowest order polynomials $[P_0(\mathcal{F})]^d$ and $P_0(T)$, respectively. When it comes to the higher order polynomial approximations in $[P_r(\mathcal{F})]^d$ and $P_r(T)$ for an integer $r \geq 1$, Definitions 2.1, 2.2 need to be redesigned accordingly.
3. Weak Galerkin Schemes. This section presents the weak Galerkin finite element scheme as well as its Schur complement for the model equation (1.1).

We introduce the local discrete space of the weak functions given by

\[ V(T) = \{ v = (v_0, v_b, v_n) : v_0 \in P_2(T), v_b \in P_0(e), v_n \in P_0(\mathcal{F}), e \subset \partial \mathcal{F}, \mathcal{F} \subset \partial T \}. \]

A global weak finite element space \( V_h \) is obtained by patching the local finite element \( V(T) \) over all the elements \( T \in \mathcal{T}_h \) through the common value \( v_h \) on \( \mathcal{E}_h^0 \) and the common value \( v_n \) on \( \mathcal{F}_h^0 \). Denote by \( V_h^0 \) a subspace of \( V_h \) with homogeneous boundary conditions for \( v_b \) and \( v_n \); i.e.,

\[ V_h^0 = \{ v : v \in V_h, v_0|_e = 0, v_n|_{\mathcal{F}} = 0, e \subset \partial \Omega, \mathcal{F} \subset \partial \Omega \}. \]

For simplicity of notation and without confusion, the discrete weak tangential derivative \( \nabla_{w,0,T} v \) defined by (2.2) and the discrete weak second order partial derivative \( \partial^2_{ij,w,0,T} v \) computed by (2.3) are simplified as follows

\[ (\nabla_{w,0,T} v)|_T = \nabla_{w,0,T}(v|_T), \quad (\partial^2_{ij,w,0,T} v)|_T = \partial^2_{ij,w,0,T}(v|_T), \quad v \in V_h. \]

For any \( w, v \in V_h \), we introduce the following bilinear forms; i.e.,

\[
(\partial^2_w w, \partial^2_w v)_{\mathcal{T}_h} = \sum_{T \in \mathcal{T}_h} \sum_{i,j=1}^d (\partial^2_{ij,w} w, \partial^2_{ij,w} v)_T, \\
s(w, v) = \sum_{T \in \mathcal{T}_h} h_T^{-2} (Q_b w_0 - w_b, Q_b v_0 - v_b)_{\partial F} \\
+ \sum_{T \in \mathcal{T}_h} h_T^{-1} (Q_n(\nabla w_0) \cdot n_F - w_n, Q_n(\nabla v_0) \cdot n_F - v_n)_{\partial T}, \\
a(w, v) = (\partial^2_w w, \partial^2_w v)_{\mathcal{T}_h} + s(w, v),
\]

where \( Q_b \) and \( Q_n \) are the usual \( L^2 \) projection operators onto \( P_0(e) \) and \( P_0(\mathcal{F}) \), respectively.

**Weak Galerkin Algorithm 1.** A WG numerical approximation for the biharmonic model equation (1.1) can be obtained by seeking \( u_h = \{ u_0, u_b, u_n \} \in V_h \) such that \( u_b = Q_b g \) on \( e \subset \partial \Omega \) and \( u_n = Q_n v \) on \( \mathcal{F} \subset \partial \Omega \) satisfying

\[ a(u_h, v) = (f, v), \quad \forall v \in V_h^0. \]

For any \( v \in V_h \), we define a semi-norm induced by the WG scheme (3.1); i.e.,

\[ \| v \|^2 = (\partial^2_{ij,w} v, \partial^2_{ij,w} v)_{\mathcal{T}_h} + s(v, v) \]

**Lemma 3.1.** For any \( v \in V_h^0 \), the semi-norm \( \| v \| \) given by (3.2) defines a norm.

**Proof.** It suffices to verify the positive property for \( \| v \| \). To this end, assume that \( \| v \| = 0 \) for any \( v \in V_h^0 \). It follows from (3.2) that \( (\partial^2_{ij,w} v, \partial^2_{ij,w} v)_{\mathcal{T}_h} = 0 \) and \( s(v, v) = 0 \), which indicates \( \partial^2_{ij,w} v = 0 \) for any \( i, j = 1, \ldots, d \) on each \( T \), \( Q_b v_0 = v_0 \) on each \( \partial \mathcal{F} \) and \( Q_n(\nabla v_0) \cdot n_F = v_n \) on each \( \partial T \). We shall first claim that \( \partial^2_{ij,w} v = 0 \) holds true on
each $T$ for $i, j = 1, \ldots, d$. To this end, for any $\varphi \in P_0(T)$, it follows from (2.3), (2.4) and the usual integration by parts that

$$0 = (\partial_{ij, w}^2, \varphi)_T + ((v_0, n_F + \nabla, \tau) v)_i, \varphi_n)_j]_T$$

(3.3)

$$= (\partial_{ij}^2, \varphi)_T + ((v_0, n_F + \nabla, \tau) v)_i, \varphi_n)_j]_T$$

$$= (\partial_{ij}^2, \varphi)_T - (\partial_i v_0 - (v_0, n_F + \nabla, \tau) v)_i, \varphi_n)_j]_T + (\partial_j, \varphi, v_0 \cdot n_i)_j]_T$$

$$= (\partial_{ij}^2, \varphi)_T - (\partial_i v_0 - (v_0, n_F + \nabla, \tau) v)_i, \varphi_n)_j]_T.$$

We shall claim $\partial_i v_0 - (v_0, n_F + \nabla, \tau) v)_i, \varphi_n)_j]_T = 0$. To this end, we have from the identity $\nabla v_0 = (\nabla v_0, n_F) n_F + n_F \times (\nabla v_0, n_F)$, and the fact $Q_n(\nabla v_0, n_F) n_F = v_0$ on each $\partial T$ that

$$y = (\partial_i v_0 - (v_0, n_F + \nabla, \tau) v)_i, \varphi_n)_j]_T$$

$$=((\nabla v_0 - (v_0, n_F + \nabla, \tau) v)_i, \varphi_n)_j)$$

$$=((Q_n(\nabla v_0, n_F) n_F)_i + (n_F \times (\nabla v_0, n_F))_i - v_0, n_F)_i - (\nabla, \tau) v)_i, \varphi_n)_j]_T$$

$$(\nabla v_0, n_F) n_F)_i + (n_F \times (\nabla v_0, n_F))_i - (\nabla, \tau) v)_i, \varphi_n)_j]_T$$

$$= (n_F \times (\nabla v_0, n_F))_i - Q_n(\nabla v_0, n_F) n_F) n_F)_i, \varphi_n)_j]_T$$

$$= (n_F \times (\nabla v_0, n_F))_i - (n_F \times (\nabla v_0, n_F))_i, \varphi_n)_j]_T$$

$$= 0,$$

where we also used $\nabla, \tau v = Q_n(n_F \times (\nabla v_0, n_F))$ on each $F$. We shall claim $\nabla, \tau v = Q_n(n_F \times (\nabla v_0, n_F))$ on each $F$. To this end, for any $\psi \in [P_0(F)]^d$, it follows from (2.3) and the condition $Q_n v_0 = v_0$ on each $\partial F$ that

$$\langle (\nabla, \tau v - Q_n(n_F \times (\nabla v_0, n_F)), \psi \times n_F) \rangle_F$$

$$= \langle (\nabla, \tau v, \psi \times n_F) \rangle_F - \langle Q_n(n_F \times (\nabla v_0, n_F)), \psi \times n_F) \rangle_F$$

$$= \langle v_0, \psi \times \tau \rangle_{\partial F} - \langle n_F \times (\nabla v_0, n_F), \psi \times n_F) \rangle_F$$

$$= \langle v_0, \psi \times \tau \rangle_{\partial F} - \langle n_F \times (\nabla v_0, n_F), \psi \times n_F) \rangle_F$$

$$= \langle v_0, \psi \times \tau \rangle_{\partial F} + (n_F \times (\nabla v_0, n_F)) \times n_F, \psi \times n_F) \rangle_F$$

$$= \langle v_0, \psi \times \tau \rangle_{\partial F} - (n_F \times (\nabla v_0, n_F)) \times n_F, \psi \times n_F) \rangle_F$$

$$= \langle v_0, \psi \times \tau \rangle_{\partial F} - (Q_n, \psi \times \tau)_{\partial F}$$

$$= \langle v_0, \psi \times \tau \rangle_{\partial F} - (v_0, \psi \times \tau)_{\partial F}$$

$$= \langle v_0, \psi \times \tau \rangle_{\partial F} - (Q_n, \psi \times \tau)_{\partial F}$$

$$= 0,$$

where we used $(\nabla \times (v_0, \psi, n_F) \times (Q_n - (\nabla v_0, n_F))) = (v_0, \psi \times \tau)_{\partial F}$, and further $(\nabla, \tau v, \psi \times n_F, \psi, n_F) = \langle v_0, \psi \times \tau \rangle_{\partial F}$. This leads to $\nabla, \tau v = Q_n(n_F \times (\nabla v_0, n_F))$ on each $F$ since $\psi \times n_F$ is parallel to the vector on $F$.

Using (3.4) gives $\partial_{ij}^2 v_0 = 0$ and further $\partial_{ij}^2 v_0 = 0$ on each $T$ for all $i, j = 1, \ldots, d$. This yields $\nabla v_0 = \text{const}$ on each $T$. We have from $Q_n(n_F \times (\nabla v_0, n_F) = \nabla, \tau v, Q_n(\nabla v_0, n_F) = v_0$ on each $\partial T$ that $\nabla v_0 \in C_0^0(\Omega)$. Using $v_n = 0$ on $F \subset \partial T$ and $Q_n(\nabla v_0, n_F) = v_0$ on each $\partial T$, we have $\nabla v_0 = 0$ in $\Omega$ and further $v_n = 0$ on each $\partial T$. This gives $v_0 = \text{const}$ on each $T$. Furthermore, it follows from
\( Q_h v_0 = v_h \) on each \( \partial \mathcal{F} \) and \( v_0 = \text{const} \) on each \( T \) that \( v_0 = v_h \) on each \( \partial \mathcal{F} \) and further \( v_0 \in C^1(\Omega) \) and \( v_0 = \text{const} \) in \( \Omega \), which, combined with \( v_h = 0 \) on \( \partial \Omega \) gives \( v_0 = 0 \) in \( \Omega \) and \( v_0 = 0 \) on each \( \partial \mathcal{F} \). This completes the proof of the lemma.

\[ \square \]

**LEMMA 3.2.** The WG scheme \((3.1)\) has one and only one numerical approximation.

**Proof.** It suffices to prove that the WG scheme \((3.1)\) has a unique numerical approximation. To this end, assume that \( u_h \) and \( \bar{u}_h \) are the two different solutions of the numerical scheme \((3.1)\). We arrive at

\[
(\partial^2_h (u_h - \bar{u}_h), \partial^2_h v)_{T_h} + s(u_h - \bar{u}_h, v) = 0, \quad \forall v \in V^0_h.
\]

Letting \( v = u_h - \bar{u}_h \in V^0_h \) gives rise to \( \|u_h - \bar{u}_h\| = 0 \). This, together with Lemma 7.4 leads to \( u_h = \bar{u}_h \). This completes the proof of the lemma. \( \square \)

The Schur complement technique could be incorporated into the WG schemes \((3.6)\) to reduce the degrees of freedom. The Schur complement of our WG scheme \((3.6)\) is presented as follows.

**SCHUR COMPLEMENT OF WEAK GALERKIN ALGORITHM 1.** A numerical approximation for \((1.1)\) is obtained by finding \( u_h = \{D(u_h, u_n, f), u_h, u_n\} \in V_h \) satisfying \( u_h = Q_h \phi \) on \( e \subset \partial \Omega \), \( u_n = Q_n \nu \) on \( \mathcal{F} \subset \partial \Omega \) and the following equation

\[
(3.6) \quad a\{D(u_h, u_n, f), u_h, u_n\}, v) = 0, \quad \forall v = \{0, u_h, u_n\} \in V^0_h,
\]

where \( u_0 = D(u_h, u_n, f) \) is obtained by solving the following equation

\[
(3.7) \quad a\{u_0, u_h, u_n\}, v) = (f, v_0), \quad \forall v = \{v_0, 0, 0\} \in V^0_h.
\]

Similar to the proof of Lemma 3.2 it is easy to prove that the Schur complement of WG scheme \((3.6)-(3.7)\) has one and only one solution.

**REMARK 3.1.** The degrees of freedom of the Schur complement of the WG scheme \((3.6)-(3.7)\) are shown in Figure 3.1 for two polygonal elements: a triangle and a pentagon. It is easy to see that the Schur complement of the WG scheme \((3.6)-(3.7)\) not only enjoys the same degrees of freedom as the Morley element on the triangular element but also extends the Morley element to any general polytopal element.

**LEMMA 3.3.** The Schur complement of WG scheme \((3.6)-(3.7)\) and the WG scheme \((3.1)\) have the same numerical approximation.

4. **Error Equation.** This section is devoted to deriving an error equation for the WG scheme \((3.1)\). On each element \( T \in T_h \), denote by \( Q_0 \) be the usual \( L^2 \) projection operator onto \( P_2(T) \). For any \( \phi \in H^2(\Omega) \), we define

\[ Q_h \phi = \{Q_0 \phi, Q_h \phi, Q_n (\nabla \phi)\}, \]

Moreover, denote by \( Q_h \) the \( L^2 \) projection operator onto \( P_0(T) \).

**LEMMA 4.1.** For any \( \phi \in H^2(T) \), the following commutative property holds true; i.e.,

\[ \partial^2_{ij,w}(Q_h \phi) = Q_h(\partial^2_{ij,h} \phi), \quad \forall i, j = 1, \ldots, d. \]
Fig. 3.1. Local degrees of freedom on a triangular element (left) and a pentagonal element (right).

Proof. Using (2.3), the definitions of operators $Q_n$ and $Q_h$, the usual integration by parts, one has that for any $\varphi \in P_0(T)$

$$
(\partial_{ij}^2 (Q_h \varphi), \varphi)_T = ((Q_n (\nabla \varphi)), \varphi_{n_j})_{\partial T} = ((\nabla \varphi), \varphi_{n_j})_{\partial T} = (\partial_i \varphi, \varphi_{n_j})_{\partial T} + (\phi, \partial_{ij}^2 \varphi)_T - (\phi n_i, \partial_j \varphi)_{\partial T} = (\partial_{ij}^2 \varphi, \varphi)_T.
$$

This completes the proof. \[\square\]

**Lemma 4.2.** Let $u$ and $u_h \in V_h$ be the exact solution of the model equation (1.1) and the numerical approximation arising from WG scheme (3.1), respectively. The error function is defined by $e_h = Q_h u - u_h$. Then, the error function $e_h$ satisfies the following error equation

$$
(\partial_w^2 e_h, \partial_w^2 v)_T + s(e_h, v) = \zeta_u(v), \quad \forall v \in V_h^0,
$$

where $\zeta_u(v)$ is given by

$$
\zeta_u(v) = s(Q_h u, v) - \sum_{T \in T_h} \sum_{i,j=1}^d \langle v_0, \partial_j (\partial_{ij}^2 u) \cdot n_i \rangle_{\partial T} + \sum_{T \in T_h} \sum_{i,j=1}^d \langle (\partial_i v_0 - v_{ij}) \cdot n_j, (I - Q_h) (\partial_{ij}^2 u) \rangle_{\partial T}.
$$
Proof. Testing the model equation (1.1) against \( v_0 \) and then using the usual integration by parts, we obtain

\[
(f, v_0) = \sum_{T \in T_h} (\Delta^2 u, v_0)_T
\]

(4.3)  
\[
= \sum_{T \in T_h} \sum_{i,j=1}^d (\partial_{ij}^2 u, \partial_{ij}^2 v_0)_T - \langle \partial_{ij}^2 u, \partial_{ij} v_0 \cdot n_j \rangle_{\partial T} + \langle \partial_{ij} (\partial_{ij}^2 u) \cdot n_i, v_0 \rangle_{\partial T}
\]

\[
= \sum_{T \in T_h} \sum_{i,j=1}^d (\partial_{ij}^2 u, \partial_{ij}^2 v_0)_T - \langle \partial_{ij}^2 u, \partial_{ij} v_0 - v_{g_i} \cdot n_j \rangle_{\partial T} + \langle \partial_{ij} (\partial_{ij}^2 u) \cdot n_i, v_0 \rangle_{\partial T},
\]

where \( v_{g_i} \) \((i = 1, \cdots, d)\) is the \( i \)-th component of the vector \( \mathbf{v}_g \), and we used that \( v_{g_i} \) is single valued on each \( F \in F^0 \) and \( v_{g_i} = 0 \) on \( F \subset \partial \Omega \) since \( v_n = 0 \) and \( \nabla_{w, \tau} v = 0 \) on \( F \subset \partial \Omega \). We shall claim \( \nabla_{w, \tau} v = 0 \) on \( \partial \Omega \). To this end, it follows from (2.2) and \( v_0 = 0 \) on \( \partial \Omega \) that

\[
\langle \nabla_{w, \tau} v, \psi \times n \rangle_{F \subset \partial \Omega} = \langle v_0, (\psi \cdot \tau) \rangle_{\partial F \subset \partial \Omega} = 0, \quad \forall \psi \in [P_0(F)]^d,
\]

which gives \( \nabla_{w, \tau} v = 0 \) on \( \partial \Omega \).

To deal with the first term in the last identity of (4.3), we apply (2.3) with \( \varphi \in P_0(T) \) and the usual integration by parts to obtain

\[
(\partial_{ij}^2 u, \varphi)_{T} = \langle v_{g_i}, \varphi n_j \rangle_{\partial T}
\]

(4.4)  
\[
= \langle v_0, \partial_{ij}^2 \varphi \rangle_{T} + \langle v_{g_i}, \varphi n_j \rangle_{\partial T}
\]

\[
= (\partial_{ij}^2 v_0, \varphi)_{T} + \langle v_0, \partial_{ij} \varphi \cdot n_i \rangle_{\partial T} - \langle \partial_{ij} v_0 - v_{g_i}, \varphi n_j \rangle_{\partial T}
\]

\[
= (\partial_{ij}^2 v_0, \varphi)_{T} - \langle \partial_{ij} \varphi - v_{g_i}, \partial_{ij} v_0 \cdot n_j \rangle_{\partial T}.
\]

Using Lemma 4.1 and taking \( \varphi = Q_h (\partial_{ij}^2 u) \in P_0(T) \) in (4.4), there holds

\[
(\partial_{ij}^2 u, \partial_{ij}^2 (Q_h u))_{T} = (\partial_{ij}^2 u, Q_h (\partial_{ij}^2 u))_{T}
\]

(4.5)  
\[
= \langle \partial_{ij}^2 Q_h v_0, Q_h (\partial_{ij}^2 u) \rangle_{\partial T} - \langle \partial_{ij} \varphi - v_{g_i}, \partial_{ij} v_0 \cdot n_j \rangle_{\partial T}
\]

Combining the equation (4.5) with (4.3) gives

\[
\sum_{T \in T_h} \sum_{i,j=1}^d (\partial_{ij}^2 u, \partial_{ij}^2 (Q_h u))_{T} = (f, v_0) - \sum_{T \in T_h} \sum_{i,j=1}^d \langle v_0, \partial_{ij} (\partial_{ij}^2 u) \cdot n_i \rangle_{\partial T}
\]

(4.6)  
\[
+ \sum_{T \in T_h} \sum_{i,j=1}^d \langle v_{g_i}, \partial_{ij} v_0 \cdot n_j \rangle_{\partial T}, (I - Q_h) (\partial_{ij}^2 u)_{\partial T}.
\]

Finally, subtracting the WG scheme (3.1) from (4.6) gives rise to Lemma 4.2. This completes the proof. \( \square \)

5. Technical Results. This section aims to derive some technical results which are crucial in the following convergence analysis.
Let $\mathcal{T}_h$ be a finite element partition of the domain $\Omega$ that satisfies the shape regular assumptions specified in [21]. Then, for any $\phi \in H^1(T)$, the trace inequality holds true; i.e.,

$$\|\phi\|^2_{\partial T} \lesssim h^{-1}_T \|\phi\|^2_T + h_T \|\nabla \phi\|^2_T. \tag{5.1}$$

Moreover, if $\phi$ is a polynomial on $T \in \mathcal{T}_h$, using the inverse inequality, the trace inequality holds true; i.e.,

$$\|\phi\|^2_{\partial T} \lesssim h^{-1}_T \|\phi\|^2_T. \tag{5.2}$$

**Lemma 5.1.** [21] Let $\mathcal{T}_h$ be a finite element partition of the domain $\Omega$ that satisfies the shape regular assumptions specified in [21]. Let $0 \leq s \leq 2$. Then, for any $\phi \in H^3(\Omega)$, there holds

$$\sum_{T \in \mathcal{T}_h} h^{2s}_T \|\phi - Q_0\phi\|^2_{s,T} \lesssim h^6 \|\phi\|^2_3, \tag{5.3}$$

$$\sum_{T \in \mathcal{T}_h} \sum_{i,j=1}^d h^{2s}_T \|\partial^2_{ij}\phi - Q_h(\partial^2_{ij}\phi)\|^2_{s,T} \lesssim h^2 \|\phi\|^2_3. \tag{5.4}$$

Recall that $F$ is a $(d - 1)$-dimensional polytopal element. Let $(d - 2)$-dimensional polygonal element $e_j$ be the boundary of $F$ for $j = 1, \ldots, M$ where $M$ is the number of $e_j$ on $\partial F$. We introduce a linear operator $S$ mapping $v_b$ from a piecewise constant function to a piecewise linear function through the least-squares approach to minimize

$$\sum_{j=1}^M |S(v_b)(A_j) - v_b(A_j)|^2, \tag{5.5}$$

where $\{A_j\}_{j=1}^M$ are the two end points of $F$ when $d = 2$, and $\{A_j\}_{j=1}^M$ is the set of midpoints of $e_j$ for $j = 1, \ldots, M$ when $d = 3$. Denote by $|e_j|$ the length of edge $e_j$. **Lemma 5.2.** For any $v \in V_h$, there holds

$$\sum_{T \in \mathcal{T}_h} h^{-1}_T \|S(v_b)\|^2_{\partial T} \lesssim \sum_{T \in \mathcal{T}_h} \|v_b\|^2_{\partial F}. \tag{5.6}$$
Proof. It follows from \(5.5\) that

\[
\sum_{T \in T_h} h_T^{-1} \|S(v_b)\|^2_{\partial T} \lesssim \sum_{T \in T_h} \sum_{j=1}^M h_T^{-1} |S(v_b)(A_j)|^2 h_T^{d-1} \\
\lesssim \sum_{T \in T_h} \sum_{j=1}^M h_T^{-1} (|S(v_b)(A_j) - v_b(A_j)|^2 + |v_b(A_j)|^2) h_T^{d-1} \\
\lesssim \sum_{T \in T_h} \sum_{j=1}^M (|v_b(A_j)|^2 h_T^{d-2}) \\
\lesssim \sum_{T \in T_h} \|v_b\|^2_{\partial T}.
\]

This completes the proof of the lemma. \(\Box\)

**Lemma 5.3.** For any \(v \in V_h\), there holds

\[
\left( \sum_{i=1}^d h_T^{-1} \|Q_n(\partial_i v_0) - v_{gh}\|^2_{\partial T} \right)^{\frac{1}{2}} \lesssim \|v\|.
\]

**Proof.** It follows from \(\nabla v_0 = (\nabla v_0 \cdot \mathbf{n}_F) \mathbf{n}_F + \mathbf{n}_F \times (\nabla v_0 \times \mathbf{n}_F), \ (2.4)\) and \((3.2)\) that

\[
\left( \sum_{i=1}^d h_T^{-1} \|Q_n(\partial_i v_0) - v_{gh}\|^2_{\partial T} \right)^{\frac{1}{2}} \\
= \left( \sum_{i=1}^d h_T^{-1} \|Q_n(\nabla v_0 \cdot \mathbf{n}_F) \mathbf{n}_F_i + Q_n(\mathbf{n}_F \times (\nabla v_0 \times \mathbf{n}_F)) \right)_{i} \\
- (v_n \mathbf{n}_F + \nabla_w \mathbf{v})_{i} \|_{\partial T}^{\frac{1}{2}} \\
\lesssim \left( \sum_{T \in T_h} h_T^{-1} \|Q_n(\nabla v_0 \cdot \mathbf{n}_F) - v_n\|^2_{\partial T} \\
+ h_T^{-1} \|Q_n(\mathbf{n}_F \times (\nabla v_0 \times \mathbf{n}_F)) - \nabla_w \mathbf{v}\|^2_{\mathbf{F}} \right)^{\frac{1}{2}} \\
\lesssim \left( \|v\|^2 + \sum_{T \in T_h} h_T^{-1} \|Q_n(\mathbf{n}_F \times (\nabla v_0 \times \mathbf{n}_F)) - \nabla_w \mathbf{v}\|^2_{\mathbf{F}} \right)^{\frac{1}{2}}.
\]

We will estimate the second term on the last line in \((5.6)\), from \((3.5)\), the triangular
inequality, the trace inequality \((5.2)\) and \((3.2)\), there holds
\[
\|Q_n(n_F \times (\nabla v_0 \times n_F)) - \nabla_{w,T} v\|_F = \sup_{\psi \in [P_0(F)]^d} \frac{\langle Q_n(n_F \times (\nabla v_0 \times n_F)) - \nabla_{w,T} v, \psi \times n_F \rangle_F}{\|\psi \times n_F\|_F} \\
= \sup_{\psi \in [P_0(F)]^d} \frac{\langle Q_n v_0 - v_h, \psi \cdot \tau \rangle_{\partial F}}{\|\psi \times n_F\|_F} \\
\lesssim \sup_{\psi \in [P_0(F)]^d} \frac{\|Q_n v_0 - v_h\|_{\partial F} \|\psi \cdot \tau\|_{\partial F}}{\|\psi\|_F} \\
\lesssim \sup_{\psi \in [P_0(F)]^d} \frac{\|Q_n v_0 - v_h\|_{\partial F} h_T^{-\frac{1}{2}} \|\psi\|_F}{\|\psi\|_F} \\
\lesssim h_T^{-\frac{1}{2}} \|Q_n v_0 - v_h\|_{\partial F},
\]
which leads to
\[
(5.7)
\]
Substituting \((5.7)\) into \((5.6)\) completes the proof of the lemma. \(\Box\)

**Lemma 5.4.** For any \(v \in V_h\), there holds
\[
\sum_{T \in T_h} |v_0|^2_{2,T} \lesssim \|v\|^2.
\]

**Proof.** Letting \(\varphi = \partial_{ij}^2 v_0 \in P_0(T)\) in \((4.4)\) gives
\[
(\partial_{ij,w}^2, \partial_{ij}^2 v_0)_T = (\partial_{ij}^2 v_0, \partial_{ij}^2 v_0)_T - ((\partial_{ij} v_0) \cdot n_j, \partial_{ij}^2 v_0)_{\partial T} \\
= (\partial_{ij}^2 v_0, \partial_{ij}^2 v_0)_T - ((Q_n (\partial_i v_0) - v_{gi}) \cdot n_j, \partial_{ij}^2 v_0)_{\partial T}.
\]
It follows from the Cauchy-Schwarz inequality, the trace inequality \((5.2), (3.2)\), and Lemma \((5.3)\) that
\[
\sum_{T \in T_h} |v_0|^2_{2,T} \leq \sum_{T \in T_h} \sum_{i,j=1}^d \left( ((Q_n (\partial_i v_0) - v_{gi}) \cdot n_j, \partial_{ij}^2 v_0)_{\partial T} + (\partial_{ij,w}^2 v_0, \partial_{ij}^2 v_0)_{\partial T} \right) \\
\lesssim \left( \sum_{T \in T_h} \sum_{i,j=1}^d h_T^{-1} \|Q_n (\partial_i v_0) - v_{gi}\|_{\partial T} \|n_j\|_{\partial T} \|\partial_{ij}^2 v_0\|_{\partial T} \right)^{\frac{1}{2}} \cdot \left( \sum_{T \in T_h} \sum_{i,j=1}^d h_T \|\partial_{ij}^2 v_0\|_{\partial T} \right)^{\frac{1}{2}} \\
+ \left( \sum_{T \in T_h} \sum_{i,j=1}^d \|\partial_{ij,w}^2 v_0\|_{\partial T} \right)^{\frac{1}{2}} \cdot \left( \sum_{T \in T_h} \sum_{i,j=1}^d \|\partial_{ij}^2 v_0\|_{\partial T} \right)^{\frac{1}{2}} \\
\lesssim \|v\| \left( \sum_{T \in T_h} |v_0|^2_{2,T} \right)^{\frac{1}{2}}.
\]
This completes the proof of the lemma. \(\Box\)

**6. Error Estimates.** In this section, we shall establish some error estimates for the numerical approximation in the energy norm and the usual \(L^2\) norms.

**Theorem 6.1.** Let \(u \in H^4(\Omega)\) and \(u_h \in V_h\) be the exact solution of the model problem \((1.1)\) and the numerical approximation of the WG scheme \((3.1)\), respectively.
There holds

$$\|e_h\| \lesssim h\|u\|_4.$$

**Proof.** By choosing $v = e_h \in V_0$ in Lemma 4.2 one arrives at

$$\|e_h\|^2 = s(Q_h u, e_h) - \sum_{T \in T_h} \sum_{i,j=1}^d \langle e_0, \partial_j (\partial_i^2 u) \cdot n_i \rangle_{\partial T}$$

$$(6.1)$$

$$+ \sum_{T \in T_h} \sum_{i,j=1}^d \langle (\partial_i e_0 - e_{3i}) \cdot n_j, (I - Q_h)(\partial_i^2 u) \rangle_{\partial T}$$

$$= I_1 + I_2 + I_3,$$

where $I_i (i = 1, 2, 3)$ are defined accordingly.

We shall estimate $I_i (i = 1, 2, 3)$ respectively. For the first term $I_1$, from Cauchy-Schwarz inequality, the trace inequality (5.1) and (5.3), one obtains

$$|s(Q_h u, e_h)|$$

$$= |\sum_{T \in T_h} h_T^{-2} \langle Q_b (Q_0 u) - Q_b u, Q_b e_0 - e_b \rangle_{\partial T}$$

$$+ \sum_{T \in T_h} h_T^{-1} |\langle Q_n (\nabla Q_0 u) \cdot n_T - Q_n (\nabla u) \cdot n_T, Q_n (\nabla e_0) \cdot n_T - e_n \rangle_{\partial T}|$$

$$\lesssim (\sum_{T \in T_h} h_T^{-2} \|Q_0 u - u\|^2_{\partial T}) \cdot (\sum_{T \in T_h} h_T^{-2} \|Q_b e_0 - e_b\|^2_{\partial T})^{\frac{1}{2}}$$

$$+ (\sum_{T \in T_h} h_T^{-1} \|Q_n (\nabla Q_0 u - \nabla u)\|^2_{\partial T}) \cdot (\sum_{T \in T_h} h_T^{-1} \|Q_n (\nabla e_0) \cdot n_T - e_n\|^2_{\partial T})^{\frac{1}{2}}$$

$$(6.2)$$

$$\lesssim (\sum_{T \in T_h} h_T^{-2} h_T^{-1} \|Q_0 u - u\|^2_{\partial T} + h_T^{-2} h_T \|\nabla (Q_0 u - u)\|^2_{\partial T}) \frac{1}{2} \|e_h\|$$

$$+ (\sum_{T \in T_h} h_T^{-1} h_T^{-1} \|\nabla Q_0 u - \nabla u\|^2_T + h_T^{-1} h_T \|\nabla (Q_0 u - \nabla u)\|^2_T) \frac{1}{2} \|e_h\|$$

$$\lesssim (\sum_{T \in T_h} h_T^{-3} h_T^{-1} \|Q_0 u - u\|^2_T + h_T^{-3} h_T \|\nabla (Q_0 u - \nabla u)\|^2_T) \frac{1}{2} \|e_h\|$$

$$+ h \|u\|_3 \|e_h\|$$

$$\lesssim (h^{-4} h^6 \|u\|^2_3 + h^{-2} h^4 \|u\|^2_3 + h^{-2} h^4 \|u\|^2_3 + h^2 \|u\|^2_3) \frac{1}{2} + h \|u\|_3 \|e_h\|$$

$$\lesssim h \|u\|_3 \|e_h\|.$$

For the second term $I_2$, using (5.5), Cauchy-Schwarz inequality, Lemma 5.2 with
\[ v_b = Q_b e_0 - e_b, \] the trace inequality (5.1)-(5.2), and Lemma 5.4, we get

\[
\begin{align*}
| \sum_{T \in \mathcal{T}_h} \sum_{i,j=1}^{d} \langle e_0, \partial_j (\partial_i^2 u) \cdot n_i \rangle_{\partial T} | \\
= | \sum_{T \in \mathcal{T}_h} \sum_{i,j=1}^{d} \langle e_0 - S(e_b), \partial_j (\partial_i^2 u) \cdot n_i \rangle_{\partial T} | \\
= | \sum_{T \in \mathcal{T}_h} \sum_{i,j=1}^{d} \langle e_0 - S(Q_b e_0) + S(Q_b e_0) - S(e_b), \partial_j (\partial_i^2 u) \cdot n_i \rangle_{\partial T} | \\
\lesssim \left( \sum_{T \in \mathcal{T}_h} h_T^{-3} \| e_0 - S(Q_b e_0) \|_{\partial T}^2 + h_T^{-3} \| S(Q_b e_0 - e_b) \|_{\partial T}^2 \right)^{\frac{1}{2}} \\
\cdot \left( \sum_{T \in \mathcal{T}_h} \sum_{i,j=1}^{d} h_T^3 \| \partial_j (\partial_i^2 u) \|_{\partial T}^2 \right)^{\frac{1}{2}} \\
\lesssim \left( \sum_{T \in \mathcal{T}_h} h_T^{-3} \| e_0 - S(e_0) \|_{\partial T}^2 + h_T^{-3} \| S(e_0) - S(Q_b e_0) \|_{\partial T}^2 \\
+ h_T^{-3} \| Q_b e_0 - e_b \|_{\partial T}^2 \right)^{\frac{1}{2}} \cdot \left( \sum_{T \in \mathcal{T}_h} h_T^4 \| u \|_{3,T}^2 + h_T^2 \| u \|_{4,T}^2 \right)^{\frac{1}{2}} \\
(6.3)
\end{align*}
\]

where we used \( e_b = 0 \) on \( \partial \Omega \), the approximation properties of \( S \) and \( Q_b \), and the identity \( h_T^{-2} \sum_{j=1}^{M} \| e_0(A_j) - Q_b e_0 \|_{e_j}^2 = 0 \) due to the definition of \( Q_b \) in two dimensions.

As to the third term \( I_3 \), we have from Cauchy-Schwarz inequality, Lemmas 5.3, 5.4.

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the trace inequality (5.1), and (5.4) that

\[ | \sum_{T \in T_h} \sum_{i,j=1}^d \left( (\partial_i e_0 - e_{gi}) \cdot n_j, (I - Q_h)(\partial_{ij}^2 u) \right)_{\partial T} | \]

\[ \lesssim \left( \sum_{T \in T_h} \sum_{i=1}^d h_T^{-1} ||Q_n(\partial_i e_0) - e_{gi}||^2_{\partial T} + h_T^{-1} ||\partial_i e_0 - Q_n(\partial_i e_0)||^2_{\partial T} \right)^{\frac{1}{2}} \]

\[ \cdot \left( \sum_{T \in T_h} \sum_{i,j=1}^d h_T ||(I - Q_h)(\partial_{ij}^2 u)||^2_{\partial T} \right)^{\frac{1}{2}} \]

\[ \lesssim \left( \|e_h\|^2 + \sum_{T \in T_h} h_T^{-1} h_T ||e_0||^2_{L^2(T)} \right)^{\frac{1}{2}} \]

\[ \cdot \left( \sum_{T \in T_h} \sum_{i,j=1}^d ||(I - Q_h)(\partial_{ij}^2 u)||^2_{T} + h_T^2 ||\nabla(\partial_{ij}^2 u - Q_h \partial_{ij}^2 u)||^2_{T} \right)^{\frac{1}{2}} \]

\[ \lesssim \left( \|e_h\|^2 + \|e_h\|^2 \right)^{\frac{1}{2}} \cdot h \|u\|_3 \]

\[ \lesssim h \|u\|_3 \|e_h\|. \]  

Finally, substituting (6.2), (6.3) and (6.4) into (6.1) gives

\[ \|e_h\|^2 \lesssim h \|u\|_4 \|e_h\|. \]

This completes the proof of the theorem. \( \square \)

We shall establish the error estimate for the numerical approximation in the usual \( L^2 \) norm. To this end, consider the following dual problem

\[ \Delta^2 \Phi = e_0, \quad \text{in } \Omega, \]

\[ \Phi = 0, \quad \text{on } \partial \Omega, \]

\[ \frac{\partial \Phi}{\partial n} = 0, \quad \text{on } \partial \Omega. \]  

(6.6)

Assume that the dual problem (6.6) satisfies the \( H^4 \)-regularity property in the sense that there exists a generic constant \( C \) such that

\[ \|\Phi\|_4 \leq C \|e_0\|. \]  

(6.7)

**Theorem 6.2.** Let \( u \in H^4(\Omega) \) and \( u_h \in V_h \) be the exact solution of the model problem (1.1) and the numerical approximation of the WG scheme (3.1), respectively. Assume the dual problem (6.6) has the \( H^4 \)-regularity property (6.7). The following error estimate holds true

\[ \|e_0\| \lesssim h^2 \|u\|_4. \]

**Proof.** Testing the dual problem (6.6) by \( e_0 \) and using the usual integration by
parts, there yields

\[ \|e_0\|^2 = (\Delta^2 \Phi, e_0) \]

\[ = \sum_{T \in \mathcal{T}_h} \sum_{i,j=1}^{d} (\partial_{ij}^2 \Phi, \partial_{ij}^2 e_0)_{T} - (\partial_{ij}^2 \Phi, \partial_i e_0 \cdot n_j)_{\partial T} + (\partial_j (\partial_{ij}^2 \Phi) \cdot n_i, e_0)_{\partial T} \]

\[ (6.8) \]

\[ = \sum_{T \in \mathcal{T}_h} \sum_{i,j=1}^{d} (\partial_{ij}^2 \Phi, \partial_{ij}^2 e_0)_{T} - (\partial_{ij}^2 \Phi, (\partial_i e_0 - e_{gj}) \cdot n_j)_{\partial T} + (\partial_j (\partial_{ij}^2 \Phi) \cdot n_i, e_0)_{\partial T}, \]

where we used \( e_{gi} = 0 \) on \( \partial \Omega \).

To deal with the first term on the third line in (6.8), it follows from (4.5) with \( u = \Phi \), \( v = e_h \) and the error equation (4.1) with \( v = Q_h \Phi \in V_h^0 \) that

\[ \sum_{T \in \mathcal{T}_h} \sum_{i,j=1}^{d} (\partial_{ij}^2 \Phi, \partial_{ij}^2 w)_{T} = \sum_{T \in \mathcal{T}_h} \sum_{i,j=1}^{d} (\partial_{ij}^2 \Phi, \partial_{ij}^2 (Q_h \Phi))_{T} + \sum_{T \in \mathcal{T}_h} \sum_{i,j=1}^{d} ((\partial_i e_0 - e_{gi}) \cdot n_j, Q_h (\partial_{ij}^2 \Phi))_{\partial T}. \]

\[ (6.9) \]

By inserting (6.9) into (6.8) and then combining the linear functional (4.2), one arrives at

\[ \|e_0\|^2 = - \zeta \Phi (e_h) + \zeta u (Q_h \Phi) \]

\[ = - \zeta \Phi (e_h) + \sum_{i=1}^{3} I_i, \]

\[ (6.10) \]

where the last three terms \( I_i \) for \( i = 1, 2, 3 \) are given by (4.2) with \( v = Q_h \Phi \).

Next, we shall estimate each of four terms on the second line of (6.10). For the first term, using (6.5), Theorem 6.1 and the \( H^4 \) regularity property (6.7), we arrive at

\[ |\zeta \Phi (e_h)| \lesssim h \|\Phi\|_4 \]

\[ \lesssim h^2 \|u\|_4 \|\Phi\|_4 \]

\[ \lesssim h^2 \|u\|_4 \|e_0\|. \]

\[ (6.11) \]

As to the term \( I_1 \), from Cauchy-Schwarz inequality, the trace inequality (5.1),
The Schwarz inequality, the trace inequality (5.1), (5.3) and the $H^1$ regularity property (6.7), there holds

\[
|s(Q_h u, Q_h \Phi)|
= \left| \sum_{T \in T_h} h_T^{-2} \langle Q_h(Q_0 u) - Q_h u, Q_h(Q_0 \Phi) - Q_h \Phi \rangle_{\partial T} \right|
+ \sum_{T \in T_h} h_T^{-1} \langle Q_n(\nabla Q_0 u) \cdot n_T - Q_n(\nabla u \cdot n_T), Q_n(\nabla Q_0 \Phi) \cdot n_T - Q_n(\nabla \Phi \cdot n_T) \rangle_{\partial T} |\n
\leq \left( \sum_{T \in T_h} h_T^{-2} \|Q_0 u - u\|_{\partial T}^2 \right)^{\frac{1}{2}} \cdot \left( \sum_{T \in T_h} h_T^{-2} \|Q_0 \Phi - \Phi\|_{\partial T}^2 \right)^{\frac{1}{2}}
+ \left( \sum_{T \in T_h} h_T^{-1} \|\nabla Q_0 u - \nabla u\|_{T}^2 \right)^{\frac{1}{2}} \cdot \left( \sum_{T \in T_h} h_T^{-1} \|\nabla Q_0 \Phi - \nabla \Phi\|_{T}^2 \right)^{\frac{1}{2}}

\leq \left( \sum_{T \in T_h} h_T^{-2} h_T^{-1} \|Q_0 u - u\|_{T}^2 + h_T^{-2} \|\nabla Q_0 u - \nabla u\|_{T}^2 \right)^{\frac{1}{2}}

\leq \left( \sum_{T \in T_h} h_T^{-2} \|Q_0 u - u\|_{T}^2 + h_T^{-2} \|\nabla Q_0 u - \nabla u\|_{T}^2 \right)^{\frac{1}{2}}

\leq \left( \sum_{T \in T_h} h_T^{-3} h_T^{-1} \|Q_0 u - u\|_{T}^2 + h_T^{-3} \|\nabla Q_0 u - \nabla u\|_{T}^2 \right)^{\frac{1}{2}}

+ h_T^{-1} \|\nabla Q_0 u - \nabla u\|_{T}^2 \left( \sum_{T \in T_h} h_T^{-3} \|Q_0 \Phi - \Phi\|_{T}^2 + h_T^{-3} \|\nabla Q_0 \Phi - \nabla \Phi\|_{T}^2 \right)^{\frac{1}{2}}

+ h_T^{-1} \|\nabla (Q_0 \Phi - \Phi)\|_{T}^2 + h_T^{-1} h_T \|Q_0 \Phi - \Phi\|_{T}^2 \right)^{\frac{1}{2}} + h^2 \|u\|_{3} \|\Phi\|_{3}

\leq h^2 \|u\|_{3} \|\Phi\|_{3}

\leq h^2 \|u\|_{3} \|e_0\|.

To estimate the term $I_2$, using the boundary condition $\Phi = 0$ on $\partial \Omega$, Cauchy-Schwarz inequality, the trace inequality (5.1), (5.3) and the $H^1$ regularity property
(6.7) gives
\[
\left| \sum_{T \in T_h} \sum_{i,j=1}^{d} (Q_0 \Phi, \partial_j (\partial^2_{ij} u) \cdot n_i)_{\partial T} \right|
\]
\[= \left| \sum_{T \in T_h} \sum_{i,j=1}^{d} ((Q_0 \Phi - \Phi) + \Phi, \partial_j (\partial^2_{ij} u) \cdot n_i)_{\partial T} \right|
\]
\[= \left| \sum_{T \in T_h} \sum_{i,j=1}^{d} (Q_0 \Phi - \Phi, \partial_j (\partial^2_{ij} u) \cdot n_i)_{\partial T} \right|
\]
\[\lesssim \left( \sum_{T \in T_h} \|Q_0 \Phi - \Phi\|_{\partial T}^2 \right)^{\frac{1}{2}} \cdot \left( \sum_{T \in T_h} \sum_{i,j=1}^{d} \|\partial_j (\partial^2_{ij} u)\|_{\partial T}^2 \right)^{\frac{1}{2}}
\]
\[\lesssim \left( \sum_{T \in T_h} h_T^{-1} \left( Q_0 \Phi - \Phi \right)_{\partial T} + h_T \|\nabla (Q_0 \Phi - \Phi)\|_{\partial T} \right)^{\frac{1}{2}}
\]
\[\times \left( \sum_{T \in T_h} \sum_{i,j=1}^{d} h_T^{-1} \|\partial_j (\partial^2_{ij} u)\|_{\partial T}^2 + h_T \|\nabla (\partial^2_{ij} u)\|_{\partial T} \right)^{\frac{1}{2}}
\]
\[\lesssim \left( \sum_{T \in T_h} h_T^{-2} |\Phi|_{3,T}^2 \right)^{\frac{1}{2}} \cdot \left( \sum_{T \in T_h} h_T^{-1} |u|_{3,T}^2 + h_T \|u\|_{2,T} \right)^{\frac{1}{2}}
\]
\[\lesssim h^2 \|u\|_4 \|\Phi\|_4
\]
\[\lesssim h^2 \|u\|_4 \epsilon_0
\]

To deal with the last term \(I_3\), from Cauchy-Schwarz inequality, the trace inequality \([5.1],[5.4],[5.3]\), and the \(H^4\) regularity property \([6.7]\), we obtain
\[
\left| \sum_{T \in T_h} \sum_{i,j=1}^{d} \left( \partial_i Q_0 \Phi - (Q_n \nabla \Phi)_i, (I - Q_h) (\partial^2_{ij} u) \cdot n_j \right)_{\partial T} \right|
\]
\[\lesssim \left( \sum_{T \in T_h} \sum_{i=1}^{d} \left( \partial_i Q_0 \Phi - (Q_n \nabla \Phi)_i \right)_{\partial T}^2 \right)^{\frac{1}{2}} \cdot \left( \sum_{T \in T_h} \sum_{i,j=1}^{d} \left( (I - Q_h) \partial^2_{ij} u \right)_{\partial T}^2 \right)^{\frac{1}{2}}
\]
\[\lesssim \left( \sum_{T \in T_h} \|\nabla (Q_0 \Phi - \Phi)\|_{\partial T} \right)^{\frac{1}{2}}
\]
\[\cdot \left( \sum_{T \in T_h} \sum_{i,j=1}^{d} h_T^{-1} \left( (I - Q_h) \partial^2_{ij} u \right)_{\partial T}^2 + h_T \|\nabla (\partial^2_{ij} u)\|_{\partial T} \right)^{\frac{1}{2}}
\]
\[\lesssim \left( \sum_{T \in T_h} h_T^{-2} \|\nabla (Q_0 \Phi - \Phi)\|_{2,T}^2 \right)^{\frac{1}{2}} + h_T \|Q_0 \Phi - \Phi\|_{2,T}^2
\]
\[\cdot \left( \sum_{T \in T_h} h_T^{-1} \|\nabla (Q_0 \Phi - \Phi)\|_{\partial T}^2 + h_T \|u\|_{2,T} \right)^{\frac{1}{2}}
\]
\[\lesssim h^2 \|u\|_3 \|\Phi\|_3
\]
\[\lesssim h^2 \|u\|_3 \epsilon_0
\]

Finally, substituting \([6.11],[6.14]\) into \([6.10]\) completes the proof of the theorem.

\[\square\]
To establish the error estimates for the second component $e_b$ and the third component $e_n$ of the error function $e_h$, we introduce

\[ \|e_b\|_{E_h} = \left( \sum_{T \in \mathcal{T}_h} h_T^2 \|e_b\|_{\partial T}^2 \right)^{1/2}, \]

\[ \|e_n\|_{F_h} = \left( \sum_{T \in \mathcal{T}_h} h_T^2 \|e_n\|_{\partial T}^2 \right)^{1/2}. \]

**Theorem 6.3.** In the assumptions of Theorem 6.2, the following error estimates hold true

\[ \|e_b\|_{E_h} \lesssim h^2 \|u\|_4, \]

\[ \|e_n\|_{F_h} \lesssim h \|u\|_4. \]

**Proof.** From the triangular inequality, the trace inequality (5.2), Theorem 6.1 and Theorem 6.2, we have

\[ \|e_b\|_{E_h} = \left( \sum_{T \in \mathcal{T}_h} h_T^2 \|e_b\|_{\partial T}^2 \right)^{1/2} \]

\[ \lesssim \left( \sum_{T \in \mathcal{T}_h} h_T^2 \|Q_b e_0\|_{\partial T}^2 + \sum_{T \in \mathcal{T}_h} h_T^2 \|e_b - Q_b e_0\|_{\partial T}^2 \right)^{1/2} \]

\[ \lesssim \left( \sum_{T \in \mathcal{T}_h} h_T^2 h_T^{-1} \|e_0\|_{\partial T}^2 + \sum_{T \in \mathcal{T}_h} h_T^2 \|e_h\|^2 \right)^{1/2} \]

\[ \lesssim \left( \sum_{T \in \mathcal{T}_h} h_T h_T^{-1} \|e_0\|_{\partial T}^2 + h^4 h^2 \|u\|_4^2 \right)^{1/2} \]

\[ \lesssim h^2 \|u\|_4, \]

which leads to the first estimate for $e_b$. The similar argument can be applied to derive the error estimate for $e_n$. This completes the proof. \[ \square \]

**7. Numerical Experiments.** Several numerical experiments will be implemented to verify the convergence theory established in previous sections. In our numerical examples, the randomised quadrilateral partition, the hexagonal partition, and the non-convex octagonal partition are generated by PolyMesher package [15](see Figure 7.1 (a)-(c) for initial partitions) and the next level of the partitions are refined by the Lloyd iteration [15] (see Figure 7.1 (d)-(f)). The uniform cubic partition is generated by uniformly refining the initial $2 \times 2 \times 2$ cubic partition of domain $\Omega = [0, 1]^3$ into $2^N \times 2^N \times 2^N$ cubes for $N = 2, \ldots, 5$. The uniform triangular partition and the uniform rectangular partition are obtained similarly.

In addition to computing $\|e_h\|$, $\|e_0\|$, $\|e_b\|_{E_h}$ and $\|e_n\|_{F_h}$, more metrics are employed

\[ \|\nabla w, \tau e_b\|_{F_h} = \left( \sum_{T \in \mathcal{T}_h} h_T^2 \|\nabla w, \tau (Q_b u - u_b)\|_{\partial T}^2 \right)^{1/2}, \]

\[ \|\nabla (u - u_0)\| = \left( \sum_{T \in \mathcal{T}_h} \|\nabla (u - u_0)\|_{\partial T}^2 \right)^{1/2}. \]

**Test Example 1.** Table 7.1 shows some numerical results when the exact solution is given by $u = \cos(x + 1) \sin(2y - 1)$ in the domain $\Omega = [0, 1]^2$ on different types of polygonal partitions shown in Figure 7.1. For the uniform triangular partition and uniform
Fig. 7.1. Level 1: Initial partitions (a) – (c); Level 2: Partitions after one refinement (d)-(f).

rectangular partition, we can see from Table 7.1 that the convergence rates for $\|e_h\|$, $\|e_0\|$, $\|e_n\|_{E_h}$ are consistent with what our theory predicts, and the convergence rate for $\|e_n\|_{F_h}$ is higher than the theoretical prediction of $O(h)$. Moreover, we observe the convergence rates for $\|\nabla w, \tau e_b\|_{F_h}$ and $\|\nabla(u - u_0)\|$ are of order $O(h^2)$ on the uniform triangular partition and uniform rectangular partition, for which the theory has not been developed in this paper. In addition, note that the theory established in previous sections does not cover the polygonal partitions shown in Figure 7.1. However, we compute the convergence rates in various norms on the polygonal partitions shown in Figure 7.1 using the least-square methods [4] and the corresponding convergence rates in various norms are illustrated in Table 7.1.

Test Example 2. Table 7.2 presents the numerical results on the uniform cubic partition in $\Omega = [0, 1]^3$ for the exact solution $u = exp(x + y + z)$. The convergence rates for $\|e_h\|$, $\|e_0\|$ and $\|e_b\|_{E_h}$ consist with our theory. Similar to Test Example 1, we can see a super-convergence rate for $\|e_n\|_{F_h}$ from Table 7.2. In addition, Table 7.2 presents the convergence rates for $\|\nabla w, \tau e_b\|_{F_h}$ and $\|\nabla(u - u_0)\|$ for which no theory is available to support.

Test Example 3. Table 7.3 illustrates the numerical performance on the polygonal partitions shown in Figure 7.1 for a low regularity solution given by $u = r^{5/3} \sin(\frac{2}{3}\theta)$, where $r = \sqrt{x^2 + y^2}$ and $\theta = \arctan(y/x)$. It is easy to check $u \in H^{8/3 - \varepsilon}(\Omega)$ for arbitrary small $\varepsilon > 0$ does not satisfy the regularity assumption $u \in H^4(\Omega)$. We observe from these numerical results that on the uniform triangular partition and uniform rectangular partition, the convergence rates for $\|e_h\|$, $\|e_0\|$, $\|e_n\|_{E_h}$, $\|e_b\|_{F_h}$, $\|\nabla w, \tau e_b\|_{F_h}$, $\|\nabla(u - u_0)\|$ are of orders $O(h^{2/3})$, $O(h^2)$, $O(h^5/3)$, $O(h^{8/3})$, $O(h^{5/3})$, respectively. Moreover, the numerical performance of the WG solution on the polygonal partitions shown in Figure 7.1 is demonstrated in Table 7.3.
Table 7.1
Numerical errors and convergence rates for the exact solution $u = \cos(x + 1) \sin(2y - 1)$ on different polygonal partitions in $\Omega = [0, 1]^2$.

| Level | $\|e_h\|$ | $\|e_0\|$ | $\|e_h\|_{x_h}$ | $\|e_n\|_{x_h}$ | $\|\nabla w, \tau \chi\|_{x_h}$ | $\|\nabla (u - u_0)\|$ |
|-------|-----------|-----------|-----------------|----------------|-----------------------------|-----------------|
| Uniform triangular partitions | | | | | | |
| 1     | 1.58E-01  | 1.54E-03  | 6.04E-04        | 1.44E-02       | 7.43E-03                    | 8.32E-03        |
| 2     | 8.20E-02  | 3.94E-04  | 1.61E-04        | 2.15E-03       | 2.24E-03                    |
| 3     | 4.15E-02  | 9.97E-05  | 4.07E-05        | 5.60E-04       | 5.72E-04                    |
| 4     | 2.08E-02  | 2.50E-05  | 1.02E-05        | 1.41E-04       | 1.44E-04                    |
| 5     | 1.04E-02  | 6.25E-06  | 2.55E-06        | 3.55E-05       | 3.60E-05                    |
| Rate  | 1.00      | 2.00      | 2.00            | 2.00           | 2.00                         |
| Uniform rectangular partitions | | | | | | |
| 1     | 2.23E-01  | 9.10E-04  | 4.24E-04        | 2.59E-02       | 4.91E-03                    | 2.03E-02        |
| 2     | 1.22E-01  | 1.91E-04  | 1.84E-04        | 5.26E-03       | 6.04E-03                    |
| 3     | 6.24E-02  | 4.64E-05  | 1.61E-04        | 2.05E-03       | 1.60E-03                    |
| 4     | 3.15E-02  | 1.15E-05  | 5.21E-04        | 1.81E-04       | 4.06E-04                    |
| 5     | 1.58E-02  | 2.86E-06  | 1.31E-04        | 4.57E-05       | 1.02E-04                    |
| Rate  | 1.00      | 2.01      | 1.99            | 1.98           | 1.99                         |
| Randomised quadrilateral partitions | | | | | | |
| 1     | 4.10E-01  | 7.70E-03  | 6.70E-03        | 1.30E-01       | 2.29E-02                    | 7.12E-02        |
| 2     | 2.88E-01  | 1.90E-03  | 2.07E-03        | 5.66E-02       | 1.15E-02                    |
| 3     | 1.65E-01  | 4.85E-04  | 7.91E-04        | 1.87E-02       | 1.05E-02                    |
| 4     | 8.56E-02  | 1.21E-04  | 5.08E-03        | 1.47E-03       | 2.84E-03                    |
| 5     | 4.40E-02  | 3.19E-05  | 1.35E-03        | 4.36E-04       | 7.52E-04                    |
| Rate  | 0.99      | 2.01      | 1.99            | 1.98           | 1.99                         |
| Hexagonal partitions | | | | | | |
| 1     | 4.82E-01  | 5.82E-03  | 7.97E-03        | 1.07E-01       | 1.08E-01                    | 7.38E-02        |
| 2     | 3.23E-01  | 1.08E-03  | 1.64E-03        | 4.03E-02       | 1.47E-02                    |
| 3     | 1.75E-01  | 2.38E-04  | 4.30E-04        | 1.25E-02       | 1.21E-02                    |
| 4     | 1.03E-01  | 4.53E-05  | 9.46E-05        | 3.51E-03       | 2.43E-03                    |
| 5     | 5.30E-02  | 8.70E-06  | 1.89E-05        | 8.92E-04       | 1.37E-03                    |
| Rate  | 0.86      | 2.38      | 2.25            | 1.90           | 1.57                         |
| Non-convex octagonal partitions | | | | | | |
| 1     | 3.90E-01  | 4.22E-03  | 9.68E-03        | 5.20E-02       | 6.99E-02                    |
| 2     | 2.69E-01  | 9.29E-04  | 2.34E-03        | 2.07E-02       | 3.11E-02                    |
| 3     | 1.54E-01  | 2.36E-04  | 5.92E-04        | 1.01E-02       | 4.58E-03                    |
| 4     | 8.11E-02  | 6.33E-05  | 1.63E-04        | 2.81E-03       | 1.24E-03                    |
| 5     | 4.15E-02  | 1.72E-05  | 4.31E-05        | 4.41E-04       | 3.22E-04                    |
| Rate  | 0.94      | 1.89      | 1.92            | 1.89           | 1.91                         |

Test Example 4. Table 7.4 demonstrates the numerical performance on the uniform cubic partition in $\Omega = [0, 1]^3$ for a low regularity solution given by $u = r^{3/2} \sin(\frac{2}{3}\theta)$, where $r = \sqrt{x^2 + y^2}$ and $\theta = \arctan(y/x)$. The exact solution satisfies $u \in H^{5/2-\varepsilon}(\Omega)$ for arbitrary small $\varepsilon > 0$. We observe that the convergence rates for $\|e_h\|$, $\|e_0\|$, $\|e_h\|_{x_h}$, $\|\nabla w, \tau \chi\|_{x_h}$, $\|\nabla (u - u_0)\|$ are at the rates of $O(h^{1/2})$, $O(h^2)$, $O(h^{1/2})$, $O(h^{3/2})$, $O(h^{3/2})$, respectively. Therefore, we conclude that the numerical performance of the WG method for the model equation (1.1) with the low regularity solution is good although the corresponding mathematical theory has not been
Table 7.2

Numerical errors and convergence rates for the exact solution \( u = \exp(x+y+z) \) on the uniform cubic partition in \( \Omega = [0,1]^3 \).

| Level | \( \|e_h\| \) | Rate | \( \|e_0\| \) | Rate | \( \|e_b\| \) | Rate |
|-------|----------------|------|-----------------|------|-----------------|------|
| 1     | 2.68E-00       | -    | 3.37E-01        | -    | 1.88E-01        | -    |
| 2     | 1.79E-00       | 0.58 | 3.49E-02        | 3.27 | 4.69E-02        | 2.00 |
| 3     | 9.85E-01       | 0.86 | 5.52E-03        | 2.66 | 1.24E-02        | 1.92 |
| 4     | 5.07E-01       | 0.96 | 1.10E-03        | 2.33 | 3.11E-03        | 1.99 |
| 5     | 2.55E-01       | 0.99 | 2.50E-04        | 2.15 | 7.67E-04        | 2.02 |

| Level | \( \|e_n\|_{F_h} \) | Rate | \( \|\nabla_{w,\tau} e_b\|_{F_h} \) | Rate | \( \|\nabla(u - u_0)\| \) | Rate |
|-------|-------------------|------|-----------------------------|------|--------------------------|------|
| 1     | 8.05E-01          | -    | 3.30E-01                    | -    | 6.62E-01                 | -    |
| 2     | 3.15E-01          | 1.35 | 1.41E-01                   | 1.22 | 2.58E-01                 | 1.36 |
| 3     | 8.93E-02          | 1.82 | 5.26E-02                   | 1.42 | 8.04E-02                 | 1.68 |
| 4     | 2.30E-02          | 1.96 | 1.53E-02                   | 1.79 | 2.18E-02                 | 1.88 |
| 5     | 5.76E-03          | 2.00 | 4.00E-03                   | 1.93 | 5.61E-03                 | 1.96 |

established in our paper.
Table 7.3

Numerical errors and convergence rates for the exact solution $u = r^{5/3} \sin\left(\frac{5}{3}\theta\right)$ on the polygonal partitions in $\Omega = [0,1]^2$.

| Level | $\|e_h\|$ | $\|e_0\|$ | $\|\mathcal{E}_h\|$ | $\|\mathcal{F}_h\|$ | $\|\nabla w, \tau e_h\|_{\mathcal{F}_h}$ | $\|\nabla (u - u_0)\|$ |
|-------|------------|------------|----------------|----------------|------------------|------------------|
|       | Uniform triangular partitions | | | | | |
| 1     | 2.99E-02  | 7.31E-04  | 2.82E-04      | 2.02E-03      | 3.55E-03         | 2.15E-03         |
| 2     | 2.00E-02  | 1.86E-04  | 7.44E-05      | 6.61E-04      | 1.20E-03         | 7.02E-04         |
| 3     | 1.31E-02  | 4.53E-05  | 1.83E-05      | 2.12E-04      | 3.90E-04         | 2.25E-04         |
| 4     | 8.39E-03  | 1.10E-05  | 4.48E-06      | 6.74E-05      | 1.25E-04         | 7.16E-05         |
| 5     | 5.35E-03  | 2.71E-06  | 1.11E-06      | 2.14E-05      | 3.97E-05         | 2.27E-05         |
| Rate  | 0.65      | 2.02      | 2.02          | 1.66          | 1.65             | 1.66             |
|       | Uniform rectangular partitions | | | | | |
| 1     | 1.30E-01  | 1.74E-03  | 2.20E-03      | 1.57E-02      | 1.73E-02         | 1.22E-02         |
| 2     | 8.89E-02  | 5.27E-04  | 6.53E-04      | 5.62E-03      | 6.43E-03         | 4.31E-03         |
| 3     | 5.80E-02  | 1.36E-04  | 1.68E-04      | 1.83E-03      | 2.15E-03         | 1.40E-03         |
| 4     | 3.73E-02  | 3.37E-05  | 4.14E-05      | 5.85E-04      | 6.99E-04         | 4.45E-04         |
| 5     | 2.38E-02  | 8.35E-06  | 1.02E-05      | 1.86E-04      | 2.24E-04         | 1.41E-04         |
| Rate  | 0.65      | 2.02      | 2.02          | 1.66          | 1.64             | 1.66             |
|       | Randomised quadrilateral partitions | | | | | |
| 1     | 1.39E-01  | 5.47E-03  | 1.75E-02      | 3.82E-02      | 3.71E-02         | 2.74E-02         |
| 2     | 1.32E-01  | 2.26E-03  | 7.64E-03      | 1.79E-02      | 2.03E-02         | 1.24E-02         |
| 3     | 9.09E-02  | 7.20E-04  | 2.53E-03      | 6.49E-03      | 7.74E-03         | 4.26E-03         |
| 4     | 5.95E-02  | 1.77E-04  | 6.41E-04      | 2.16E-03      | 2.55E-03         | 1.39E-03         |
| 5     | 3.85E-02  | 5.25E-05  | 1.95E-04      | 6.89E-04      | 8.77E-04         | 4.39E-04         |
| Rate  | 0.65      | 1.97      | 1.93          | 1.69          | 1.64             | 1.71             |
|       | Hexagonal partitions | | | | | |
| 1     | 2.45E-01  | 3.22E-03  | 9.78E-03      | 2.30E-02      | 1.07E-01         | 1.89E-02         |
| 2     | 2.02E-01  | 1.34E-03  | 3.36E-03      | 8.79E-03      | 5.30E-02         | 7.82E-03         |
| 3     | 1.30E-01  | 4.49E-04  | 1.02E-03      | 3.60E-03      | 1.72E-02         | 3.28E-03         |
| 4     | 7.75E-02  | 8.34E-05  | 1.86E-04      | 1.17E-03      | 5.30E-03         | 1.07E-03         |
| 5     | 5.27E-02  | 1.99E-05  | 4.41E-05      | 3.89E-04      | 1.76E-03         | 3.17E-04         |
| Rate  | 0.65      | 2.25      | 2.26          | 1.61          | 1.64             | 1.68             |
|       | Non-convex octagonal partitions | | | | | |
| 1     | 1.43E-01  | 1.53E-03  | 4.87E-03      | 1.41E-02      | 3.02E-02         | 1.86E-02         |
| 2     | 1.10E-01  | 6.96E-04  | 1.82E-03      | 5.89E-03      | 1.39E-02         | 6.97E-03         |
| 3     | 7.59E-02  | 2.42E-04  | 6.06E-04      | 2.14E-03      | 5.11E-03         | 2.45E-03         |
| 4     | 5.02E-02  | 6.84E-05  | 1.71E-04      | 7.20E-04      | 1.72E-03         | 8.15E-04         |
| 5     | 3.25E-02  | 1.75E-05  | 4.39E-05      | 2.36E-04      | 5.60E-04         | 2.64E-04         |
| Rate  | 0.61      | 1.89      | 1.89          | 1.59          | 1.59             | 1.61             |

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Table 7.4
Numerical errors and convergence rates for the exact solution \( u = r^{3/2} \sin(\frac{3}{2} \theta) \) on the uniform cubic partition in \( \Omega = [0, 1]^3 \).

| Level | \( \| e_h \| \) | Rate | \( \| e_0 \| \) | Rate | \( \| e_b \|, \| x_h \| \) | Rate |
|-------|-----------------|------|-----------------|------|-----------------|------|
| 1     | 1.43E-01        |      | 3.62E-03        |      | 1.26E-02        |      |
| 2     | 1.32E-01 0.12   | 1.26 | 3.78E-03        | 1.74 | 3.78E-03        | 1.74 |
| 3     | 9.65E-02 0.45   | 1.59 | 1.04E-03        | 1.86 | 1.04E-03        | 1.86 |
| 4     | 6.84E-02 0.50   | 1.86 | 2.70E-04        | 1.95 | 2.70E-04        | 1.95 |

| Level | \( \| e_n \|_{F_h} \) | Rate | \( \| \nabla w, \tau \tau \tau e_b \|_{F_h} \) | Rate | \( \| \nabla (u - u_0) \| \) | Rate |
|-------|-----------------|------|-----------------|------|-----------------|------|
| 1     | 3.27E-02        |      | 4.69E-02        |      | 4.13E-02        |      |
| 2     | 1.58E-02 1.05   | 0.88 | 2.03E-02        | 1.03 | 2.03E-02        | 1.03 |
| 3     | 5.65E-03 1.48   | 1.26 | 8.09E-03        | 1.32 | 8.09E-03        | 1.32 |
| 4     | 1.90E-03 1.57   | 1.45 | 2.97E-03        | 1.45 | 2.97E-03        | 1.45 |

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