When unit groups of continuous inverse algebras are regular Lie groups

by

HELGE GLÖCKNER (Paderborn) and KARL-HERMANN NEEB (Erlangen)

Abstract. It is a basic fact in infinite-dimensional Lie theory that the unit group $A^\times$ of a continuous inverse algebra $A$ is a Lie group. We describe criteria ensuring that the Lie group $A^\times$ is regular in Milnor’s sense. Notably, $A^\times$ is regular if $A$ is Mackey-complete and locally m-convex.

1. Introduction and statement of the main result. A locally convex, unital, associative topological algebra $A$ over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ is called a continuous inverse algebra if its group $A^\times$ of invertible elements is open and the inversion map $\iota: A^\times \to A$, $x \mapsto x^{-1}$, is continuous (cf. [20]). Then $\iota$ is $\mathbb{K}$-analytic and hence $A^\times$ is a $\mathbb{K}$-analytic Lie group [6]. Our goal is to describe conditions ensuring that the Lie group $A^\times$ is well-behaved, i.e., it is a regular Lie group in the sense of Milnor [16].

To recall this notion, let $G$ be a Lie group modelled on a locally convex space $E$, with identity element 1, its tangent bundle $TG$ and the Lie algebra $\mathfrak{g} := T_1G \cong E$. Given $g \in G$ and $v \in T_1G$, let $\lambda_g: G \to G$, $x \mapsto gx$ be left translation by $g$ and $gv := T_1(\lambda_g)(v) \in T_gG$. If $\gamma: [0, 1] \to \mathfrak{g}$ is a continuous map, then there exists at most one $C^1$-map $\eta: [0, 1] \to G$ such that

$$\eta'(t) = \eta(t)\gamma(t) \quad \text{for all } t \in [0, 1], \quad \text{and} \quad \eta(0) = 1.$$ 

If such an $\eta$ exists, it is called the evolution of $\gamma$. The Lie group $G$ is called regular if each $\gamma \in C^\infty([0, 1], \mathfrak{g})$ admits an evolution $\eta_\gamma$, and the map evol: $C^\infty([0, 1], \mathfrak{g}) \to G$, $\gamma \mapsto \eta_\gamma(1)$, is smooth (see [16] and [17], where also many applications of regularity are described). If $G$ is regular, then its modelling space $E$ is Mackey-complete in the sense that every Lipschitz

2010 Mathematics Subject Classification: Primary 22E65; Secondary 34G10, 46G20, 46H05, 58B10.

Key words and phrases: continuous inverse algebra, Q-algebra, Waelbroeck algebra, locally m-convex algebra, infinite-dimensional Lie group, regular Lie group, regularity, left logarithmic derivative, product integral, evolution, initial value problem, parameter dependence.

DOI: 10.4064/sm211-2-1 [95] © Instytut Matematyczny PAN, 2012
curve in $E$ admits a Riemann integral \(^{(1)}\) (as shown in \([10]\)). It is a notorious open problem whether, conversely, every Lie group modelled on a Mackey-complete locally convex space is regular \([17\text{ Problem II.2}];\) cf. \([16]\).

As a tool for the discussion of $A^n$, we let $\mu_n : A^n \to A$ be the $n$-linear map defined via $\mu_n(x_1, \ldots, x_n) := x_1 \cdots x_n$, for $n \in \mathbb{N}$. Given seminorms $p, q : A \to [0, \infty[$, we define $B^q_1(0) := \{ x \in A : q(x) \leq 1 \}$ and

$$\|\mu_n\|_{p,q} := \sup \{ p(\mu_n(x_1, \ldots, x_n)) : x_1, \ldots, x_n \in B^q_1(0) \} \in [0, \infty[.$$

Our regularity criterion now reads as follows:

**Theorem 1.1.** Let $A$ be a Mackey-complete continuous inverse algebra such that the following condition is satisfied:

\(^(*)\) For each continuous seminorm $p$ on $A$, there exists a continuous seminorm $q$ on $A$ and $r > 0$ (which may depend on $p$ and $q$) such that

$$\sum_{n=1}^{\infty} r^n \|\mu_n\|_{p,q} < \infty.$$  

Then $A^\times$ is a regular Lie group in Milnor’s sense.

In fact, $A^\times$ even has certain stronger regularity properties (see Proposition \([4,4]\)). Of course, by Hadamard’s formula for the radius of convergence of a power series, condition \(^(*)\) is equivalent to \(^{(2)}\)

$$\lim sup_{n \to \infty} \sqrt[n]{\|\mu_n\|_{p,q}} < \infty.$$  

It is also equivalent to the existence of $M \in [0, \infty[$ such that $\|\mu_n\|_{p,q} \leq M^n$ for all $n \in \mathbb{N}$.

**Remark 1.2.** The authors do not know whether condition \(^(*)\) can be omitted, i.e., whether $A^\times$ is regular for every Mackey-complete continuous inverse algebra $A$. Here are some preliminary considerations:

If $A$ is a continuous inverse algebra, then the map $\pi_n : A \to A, x \mapsto x^n$, is a continuous homogeneous polynomial of degree $n$, for each $n \in \mathbb{N}_0$. It is known that the analytic inversion map $\iota : A^\times \to A$ is given by Neumann’s series, $\iota(1 - x) = \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} \pi_n(x)$, for $x$ in some 0-neighbourhood of $A$. \([6\text{ Lemma 3.3}].\) Hence, for each continuous seminorm $p$ on $A$, there exists a continuous seminorm $q$ on $A$ and $s > 0$ such that

$$\sum_{n=1}^{\infty} s^n \|\pi_n\|_{p,q} < \infty,$$

\(^{(1)}\) See \([13]\) for a detailed discussion of this property.

\(^{(2)}\) If $\|\mu_n\|_{p,q} < \infty$, then also $\|\mu_k\|_{p,q} < \infty$ for all $k \in \{1, \ldots, n\}$. In fact, $\|\mu_k\|_{p,q} \leq q(1)^{n-k} \|\mu_n\|_{p,q}$ since $\mu_k(x_1, \ldots, x_k) = \mu_n(1, \ldots, 1, x_1, \ldots, x_k)$. 


where \( \|\pi_n\|_{p,q} := \sup \{ p(\pi_n(x)) : x \in B^q_1(0) \} \) (cf. [2] Proposition 5.1 [(3)]).

Let \( S_n \) be the symmetric group of all permutations of \( \{1, \ldots, n\} \) and let \( \mu_n^{\text{sym}} : A^n \to A, (x_1, \ldots, x_n) \mapsto (1/n!)(\sum_{\sigma \in S_n} x_{\sigma(1)} \cdots x_{\sigma(n)}) \) be the symmetrization of \( \mu_n \). Then \( \pi_n(x) = \mu_n^{\text{sym}}(x, \ldots, x) \) and thus \( \|\mu_n^{\text{sym}}\|_{p,q} \leq (n^n/n!)\|\pi_n\|_{p,q} \) by the Polarization Formula (in the form [11, p. 34, (2)]).

Since \( \lim_{n \to \infty} (n/n\sqrt{n}) = e \) is Euler’s constant (as a consequence of Stirling’s Formula), it follows that

\[
\sum_{n=1}^{\infty} t^n \|\mu_n^{\text{sym}}\|_{p,q} < \infty \quad \text{for each } t \in ]0, s/e[.
\]

In general, it is not clear how one could give good estimates for \( \|\mu_n\|_{p,q} \) in terms of \( \|\mu_n^{\text{sym}}\|_{p,q} \). Hence, it does not seem to be clear in general whether (1.1) implies the existence of some \( r > 0 \) with \((*)\).

However, \((*)\) is satisfied in some important cases. Following [14], a topological algebra \( A \) is called \emph{locally m-convex} if its topology arises from a set of seminorms \( q \) which are \emph{submultiplicative}, i.e., \( q(xy) \leq q(x)q(y) \) for all \( x, y \in A \).

**Corollary 1.3.** Let \( A \) be a Mackey-complete continuous inverse algebra. If \( A \) is commutative or locally m-convex, then \( A^\times \) is a regular Lie group.

**Proof.** In fact, if \( A \) is commutative, then \( \mu_n = \mu_n^{\text{sym}} \), whence \((*)\) is satisfied with any \( r \in ]0, s/e[ \) as in (1.1). Therefore Theorem 1.1 applies [(4)]. If \( A \) is locally m-convex, then for every continuous seminorm \( p \) on \( A \), there is a submultiplicative continuous seminorm \( q \) on \( A \) such that \( p \leq q \). Using the submultiplicativity, we see that \( \|\mu_n\|_{p,q} \leq \|\mu_n\|_{q,q} \leq 1 \). Thus \((*)\) is satisfied with any \( r \in ]0, 1[ \), and Theorem 1.1 applies. \( \blacksquare \)

It can be shown that every Mackey-complete, commutative continuous inverse algebra is locally m-convex (cf. [19]).

**Remark 1.4.** We mention that there is a quite direct, alternative proof for the corollary if \( A \) is locally m-convex and \emph{complete} [(5)]. The easier argu-

---

(3) If \( k = \mathbb{R} \), we can apply the proposition to \( A_\mathbb{C} \), which is a complex continuous inverse algebra (see, e.g., [6] Proposition 3.4]).

(4) Alternative proof: \((A, +)\) is regular, as it is Mackey-complete [17 Proposition II.5.6]. Since \( \exp : A \to A^\times \) is a homomorphism of groups (as \( A^\times \) is abelian) and a local diffeomorphism (see [6] Theorem 5.6), it follows that also \( A^\times \) is regular [18 Proposition 3].

(5) Then \( A = \lim_{\leftarrow} A_q \) is a projective limit of Banach algebras (where \( q \) ranges through the set of all submultiplicative continuous seminorms on \( A \)). Being a Banach–Lie group, each \( A_q^\times \) is regular [16]. Then \( C^\infty([0, 1], A) = \lim_{\leftarrow} C^\infty([0, 1], A_q) \) and \( \text{evol}_{A^\times} = \lim_{\leftarrow} \text{evol}_{A^\times_q} \) is a smooth evolution (cf. [11 Lemma 10.3]).
ments fail however if \( A \) is not complete, but merely sequentially complete or Mackey-complete. By contrast, our more elaborate method does not require that \( A \) be complete: Mackey-completeness suffices.

**Remark 1.5.** Our Theorem 1.1 is a variant of the (possibly too optimistic) Theorem IV.1.11 announced in the survey [17], and its proof expands the sketch of proof given there. To avoid the difficulties described in Remark 1.2 we added condition (\( \ast \)).

**Remark 1.6.** Unit groups of Mackey-complete continuous inverse algebras are so-called BCH-Lie groups [6, Theorem 5.6], i.e., they admit an exponential function which is an analytic diffeomorphism around 0 (see [5], [17], [18] for information on such groups). Inspiration for the studies came from an article by Robart [18]. He pursued the (possibly too optimistic) larger goal to show that every BCH-Lie group with Mackey-complete modelling space is regular. However, there seem to be gaps in his arguments (6).

**Remark 1.7.** The following questions are open:

(a) Are there examples of Mackey-complete continuous inverse algebras which satisfy (\( \ast \)) but are not locally m-convex? Or even:

(b) Does every Mackey-complete continuous inverse algebra satisfy (\( \ast \))?  

2. Notation and preparatory results

**Basic notation.** Let \( \mathbb{N} = \{1, 2, \ldots \} \) and \( \mathbb{N}_0 := \mathbb{N} \cup \{0\} \). If \( X \) is a set and \( n \in \mathbb{N} \), we write \( X^n := X \times \cdots \times X \) (with \( n \) factors). If \( f : X \to Y \) is a map, we abbreviate \( f^n := f \times \cdots \times f : X^n \to Y^n \). If \( (E, \| \cdot \|_E) \) and \( (F, \| \cdot \|_F) \) are normed spaces and \( \beta : E^n \to F \) is a continuous \( n \)-linear map, we write \( \| \beta \|_{op} \) for its operator norm, defined as usual as \( \sup \{ \| \beta(x_1, \ldots , x_n) \|_F : x_1, \ldots , x_k \in E, \| x_1 \|_E, \ldots , \| x_n \|_E \leq 1 \} \). If \( E \) is a locally convex space, we let \( P(E) \) be the set of all continuous seminorms on \( E \). If \( p \in P(E) \), we consider the factor space \( E_p := E/p^{-1}(0) \) as a normed space with the norm \( \| \cdot \|_p \) given by \( \| x + p^{-1}(0) \|_p := p(x) \). Then the canonical map \( \pi_p : E \to E_p, x \mapsto x + p^{-1}(0) \), is linear and continuous, with \( \| \pi_p(x) \|_p = p(x) \).

**Weak integrals.** Recall that if \( E \) is a locally convex space, \( a \leq b \) are reals and \( \gamma : [a,b] \to E \) a continuous map, then the weak integral \( \int_a^b \gamma(s) \, ds \) (if it exists) is the unique element of \( E \) such that \( \lambda(\int_a^b \gamma(s) \, ds) = \int_a^b \lambda(\gamma(s)) \, ds \) for each continuous linear functional \( \lambda \) on \( E \). If \( \alpha : E \to F \) is a continuous

---

(6) For example, it is unclear whether the limit \( \gamma_u \) constructed in the proof of [18, Proposition 7] takes its values in \( \text{Aut}(\mathcal{L}) \) (as observed by K.-H. Neeb), and no explanation is given how a smooth curve \( g \) in the local group with \( \text{Ad}(g) = \gamma_u \) can be obtained.
linear map between locally convex spaces and \( \int_a^b \gamma(s) \, ds \) (as before) exists in \( E \), then also \( \int_a^b \alpha(\gamma(s)) \, ds \) exists in \( F \) and is given by
\[
(2.1) \quad \int_a^b \alpha(\gamma(s)) \, ds = \alpha \left( \int_a^b \gamma(s) \, ds \right)
\]
(see, e.g., [10] for this observation). If \( E \) is sequentially complete, then \( \int_a^b \gamma(s) \, ds \) always exists (cf. [2] Lemma 1.1 or [11] 1.2.3)).

**\( C^r \)-curves.** Let \( r \in \mathbb{N}_0 \cup \{\infty\} \). As usual, a \( C^r \)-curve in a locally convex space \( E \) is a continuous function \( \gamma: I \to E \) on a non-degenerate interval \( I \) such that the derivatives \( \gamma^{(k)}: I \to E \) of order \( k \) exist for all \( k \in \mathbb{N} \) with \( k \leq r \), and are continuous (see, e.g., [10] for more details). The \( C^\infty \)-curves are also called smooth curves.

**Smooth maps.** If \( E \) and \( F \) are real locally convex spaces, \( U \subseteq E \) is an open subset and \( r \in \mathbb{N}_0 \cup \{\infty\} \), then a function \( f: U \to F \) is called \( C^r \) if \( f \) is continuous, the iterated directional derivatives \( d^{(k)}f(x, y_1, \ldots, y_k) := (D_{y_k} \ldots D_{y_1}f)(x) \) exist for all \( k \in \mathbb{N} \) such that \( k \leq r \), \( x \in U \) and \( y_1, \ldots, y_k \in E \), and define continuous functions \( d^{(k)}f: U \times E^k \to F \). If \( U \) is not open, but is a convex (or locally convex) subset of \( E \) with dense interior \( U^0 \), we say that \( f \) is \( C^r \) if \( f \) is continuous, \( f|_{U^0} \) is \( C^r \) and \( d^{(k)}(f|_{U^0}): U^0 \times E^k \to F \) has a continuous extension \( d^{(k)}f: U \times E^k \to F \) for each \( k \in \mathbb{N} \) such that \( k \leq r \). \( C^\infty \)-maps are also called smooth. We abbreviate \( df := d^{(1)}f \). It is known that the Chain Rule holds in the form \( d(f \circ g)(x, y) = df(g(x), dg(x, y)) \), and that compositions of \( C^r \)-maps are \( C^r \). Moreover, a \( C^0 \)-curve \( \gamma: I \to E \) is a \( C^r \)-curve if and only if it is a \( C^r \)-map, in which case \( \gamma'(t) = d\gamma(t, 1) \) (see [10] for all of these basic facts; cf. also [15], [16], and [4]).

**Analytic maps.** If \( E \) and \( F \) are complex locally convex spaces and \( n \in \mathbb{N} \), then a function \( p: E \to F \) is called a continuous homogeneous polynomial of degree \( n \in \mathbb{N}_0 \) if \( p(x) = \beta(x, \ldots, x) \) for some continuous \( n \)-linear map \( \beta: E^n \to F \) (if \( n = 0 \), this means a constant function). A map \( f: U \to F \) on an open set \( U \subseteq E \) is called complex-analytic (or \( \mathbb{C} \)-analytic) if it is continuous and for each \( x \in U \), there is a \( 0 \)-neighbourhood \( Y \subseteq E \) with \( x + Y \subseteq U \) and continuous homogeneous polynomials \( p_n: E \to F \) of degree \( n \) such that
\[
(\forall y \in Y) \quad f(x + y) = \sum_{n=0}^\infty p_n(y)
\]
(see [2], [4] and [10] for further information). Following [16], [4] and [10] (but deviating from [2]), given real locally convex spaces \( E, F \), we call a function \( f: U \to F \) on an open set \( U \subseteq E \) real-analytic (or \( \mathbb{R} \)-analytic) if it extends to a complex-analytic map \( V \to F_\mathbb{C} \), defined on some open subset \( V \subseteq E_\mathbb{C} \).
of the complexification of $E$, such that $U \subseteq V$. For $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, it is known that compositions of $\mathbb{K}$-analytic maps are $\mathbb{K}$-analytic. Every $\mathbb{K}$-analytic map is smooth (see, e.g., [10] or [4] for both of these facts).

We shall use the following lemma (proved in Appendix A):

**Lemma 2.1.** Let $E$ and $F$ be complex locally convex spaces, $\tilde{F}$ be a completion of $F$ such that $F \subseteq \tilde{F}$ as a dense vector subspace, and $p_n : E \to F$ be continuous homogeneous polynomials of degree $n$ for $n \in \mathbb{N}_0$. Assume that $f(x) := \sum_{n \in \mathbb{N}_0} p_n(x)$ converges in $\tilde{F}$ for all $x$ in a balanced, open 0-neighbourhood $U \subseteq E$, and $f : U \to \tilde{F}$ is continuous. If $F$ is Mackey-complete, then $f(x) \in F$ for all $x \in U$ and $f : U \to F$ is $\mathbb{C}$-analytic.

**Function spaces.** If $E$ is a locally convex space and $r \in \mathbb{N}_0 \cup \{\infty\}$, let $C^r([0,1],E)$ be the space of all $C^r$-maps from $[0,1]$ to $E$. We endow $C^r([0,1],E)$ with the locally convex vector topology defined by the seminorms $\| \cdot \|_{C^k,p}$ given by

$$\|\gamma\|_{C^k,p} := \max_{j=0,\ldots,k} \max_{t \in [0,1]} p(\gamma^{(j)}(t))$$

for $p$ in the set of continuous seminorms on $E$ and $k \in \mathbb{N}_0$ with $k \leq r$. We abbreviate $C([0,1],E) := C^0([0,1],E)$. Three folklore lemmas concerning these function spaces will be used (the proofs can be found in Appendix A):

**Lemma 2.2.** Let $E$ and $F$ be locally convex spaces, $\alpha : E \to F$ be a continuous linear map, and $r \in \mathbb{N}_0 \cup \{\infty\}$. Then also the map

$$\alpha_* := C^r([0,1],\alpha) : C^r([0,1],E) \to C^r([0,1],F), \quad \gamma \mapsto \alpha \circ \gamma,$$

is continuous and linear. If $\alpha$ is a topological embedding (i.e., a homeomorphism onto its image), then also $\alpha_*$ is a topological embedding.

**Lemma 2.3.** If $E$ is a locally convex space and $r \in \mathbb{N}_0 \cup \{\infty\}$, then the topology on the space $C^r([0,1],E)$ is initial with respect to the mappings $(\pi_p)_* : C^r([0,1],E) \to C^r([0,1],E_p)$, $\gamma \mapsto \pi_p \circ \gamma$, for $p \in P(E)$.

**Lemma 2.4.** If $r \in \mathbb{N}_0 \cup \{\infty\}$ and $E$ is a locally convex space which is complete (resp., Mackey-complete), then also $C^r([0,1],E)$ is complete (resp., Mackey-complete).

### 3. Picard iteration of paths in a topological algebra

**Setting 3.1.** Let $A$ be a locally convex topological algebra over $\mathbb{C}$, i.e., a unital, associative, complex algebra, equipped with a Hausdorff locally
convex vector topology making the map \( A \times A \to A, \ (x, y) \mapsto xy \), continuous. We assume that condition \(*\) from Theorem 1.1 is satisfied (\( \text{cf.} \)).

If \( E \) is a locally convex space, then a function \( \gamma \): \([0, 1] \to E \) is a Lipschitz curve if \( \frac{\gamma(t)-\gamma(s)}{t-s} : s \neq t \in [0, 1] \) is bounded in \( E \) (cf. \([13, \text{p. 9}]\)). For our current purposes, we endow the space \( \text{Lip}([0, 1], E) \) of all such curves with the topology \( \mathcal{O}_{C^0} \) induced by \( C^0([0, 1], E) \).

**Lemma 3.2** (Picard Iteration). Let \( A \) be as in \([3.1] \). If \( A \) is sequentially complete and \( \gamma \in C([0, 1], A) \), we can define a sequence \( (\eta_n)_{n \in \mathbb{N}} \) in \( C^1([0, 1], A) \) via

\[
\eta_0(t) := 1, \quad \eta_n(t) := 1 + \int_0^t \eta_{n-1}(t_n) \gamma(t_n) \, dt_n \quad \text{for } t \in [0, 1] \text{ and } n \in \mathbb{N}.
\]

Then:

(a) The limit \( \eta := \eta := \lim_{n \to \infty} \eta_n \) exists in \( C^1([0, 1], A) \).

(b) \( \eta_n(t) = 1 + \sum_{k=1}^n \left[ \int_0^{t_k} \cdots \int_0^{t_2} \gamma(t_1) \cdots \gamma(t_k) \, dt_1 \cdots dt_k \right] \) for all \( n \in \mathbb{N}_0 \) and \( t \in [0, 1] \), and thus

\[
\eta(t) = 1 + \sum_{n=1}^{\infty} \int_0^t \cdots \int_0^t \gamma(t_1) \cdots \gamma(t_n) \, dt_1 \cdots dt_n.
\]

(c) \( \eta'(t) = \eta(t) \gamma(t) \) and \( \eta(0) = 1 \).

(d) The map \( \Phi: C([0, 1], A) \to C^1([0, 1], A), \ \gamma \mapsto \eta_\gamma \), is \( \mathbb{C} \)-analytic.

If \( A \) is not sequentially complete, but Mackey-complete, then the \( (\eta_n)_{n \in \mathbb{N}_0} \) can be defined and (a)–(c) hold for each \( \gamma \in \text{Lip}([0, 1], A) \). Moreover,

(d)’ \( \Phi: (\text{Lip}([0, 1], A), \mathcal{O}_{C^0}) \to C^1([0, 1], A), \ \gamma \mapsto \eta_\gamma \), is \( \mathbb{C} \)-analytic.

**Proof.** If \( A \) is sequentially complete, set \( X := C([0, 1], A) \); otherwise, set \( X := \text{Lip}([0, 1], A) \). Let \( \tilde{A} \) be a completion of \( A \) such that \( A \subseteq \tilde{A} \). Then the inclusion map \( \phi: C^1([0, 1], A) \to C^1([0, 1], \tilde{A}) \) is a topological embedding (Lemma \([2.2] \) and \( C^1([0, 1], \tilde{A}) \) is complete (Lemma \([2.4] \) Hence also the closure \( Y \subseteq C^1([0, 1], A) \) of the image \( \text{im}(\phi) \) is complete, and thus \( Y \) is a completion of \( C^1([0, 1], A) \).

To prove (a), (b), (d), (d)’, let \( \gamma \in X \). Then all integrals needed to define \( \eta_n \) exist, and each \( \eta_n \) is \( C^1 \), by the Fundamental Theorem of Calculus. A trivial induction shows that

\[
\eta_n(t) = 1 + \sum_{k=1}^n \int_0^t \cdots \int_0^t \gamma(t_1) \cdots \gamma(t_k) \, dt_1 \cdots dt_k
\]

(\( ^{\dagger} \)) Note that \( A \) is not assumed to be a continuous inverse algebra in this section.
(as asserted in (b)). Likewise, if \( n \in \mathbb{N} \) and \( \gamma_1, \ldots, \gamma_n \in X \), then the weak integrals needed to define \( \tau_n(\gamma_1, \ldots, \gamma_n) : [0, 1] \to A \),

\[
t \mapsto \int_0^t \cdots \int_0^{t_n} \gamma_1(t_1) \cdots \gamma_n(t_n) \, dt_1 \cdots dt_n,
\]
exist and \( \tau_n(\gamma_1, \ldots, \gamma_n) \) is a \( C^1 \)-map. Since \( \tau_n : X \to C^1([0, 1], A) \), \( (\gamma_1, \ldots, \gamma_n) \mapsto \tau_n(\gamma_1, \ldots, \gamma_n) \), is an \( n \)-linear mapping, it follows that the map \( \sigma_n : X \to C^1([0, 1], A) \), \( \sigma_n(\gamma) := \tau_n(\gamma, \ldots, \gamma) \), is a homogeneous polynomial of degree \( n \) (and this conclusion also holds for \( n = 0 \), if we define \( \sigma_0(\gamma) := 1 \)). If \( p \in P(A) \), there is \( q \in P(A) \) and \( M \in [0, \infty[ \) such that

\[
(\forall n \in \mathbb{N}) \quad \|\mu_n\|_{p,q} \leq M^n,
\]
as a consequence of condition \((\ast)\). Applying \( p \) to the iterated integral defining \( \sigma_n(\gamma)(t) \), we deduce that

\[
p(\sigma_n(\gamma)(t)) \leq \frac{t^n}{n!} \|\mu_n\|_{p,q} \|\gamma\|_{C^0,q}^n \leq \frac{t^n M^n}{n!} \|\gamma\|_{C^0,q}^n
\]
for each \( t \in [0, 1] \) and thus

\[
(3.3) \quad \|\sigma_n(\gamma)\|_{C^0,p} \leq \frac{M^n}{n!} \|\gamma\|_{C^0,q}^n.
\]

Also, \( \sigma_0(\gamma)' = 0 \), \( \sigma_1(\gamma)'(t) = \gamma(t) \) and

\[
(3.4) \quad \sigma_n(\gamma)'(t) = \int_0^t \cdots \int_0^{t_n} \gamma(t_1) \cdots \gamma(t_n-1) \gamma(t) \, dt_1 \cdots dt_{n-1}
\]
if \( n \geq 2 \), by the Fundamental Theorem of Calculus. Thus \( \sigma_n(\gamma)' = \sigma_{n-1}(\gamma) \cdot \gamma \) for all \( n \in \mathbb{N} \). Using \( \eta_n = \sum_{k=0}^n \sigma_k(\gamma) \), we infer that

\[
(3.5) \quad (\forall n \in \mathbb{N}) \quad \eta_n'(t) = \eta_{n-1}(t) \gamma(t),
\]
which will be useful later. By (3.4), also

\[
p(\sigma_n(\gamma)'(t)) \leq \frac{t^{n-1}}{(n-1)!} \|\mu_n\|_{p,q} \|\gamma\|_{C^0,q}^n
\]
and thus

\[
(3.6) \quad \|\sigma_n(\gamma)'\|_{C^0,p} \leq \frac{M^n}{(n-1)!} \|\gamma\|_{C^0,q}^n.
\]

Combining (3.3) and (3.6), we see that

\[
(3.7) \quad \|\sigma_n(\gamma)\|_{C^1,p} \leq \frac{M^n}{(n-1)!} \|\gamma\|_{C^0,q}^n.
\]

Therefore \( \sigma_n : X \to C^1([0, 1], A) \) is a continuous homogeneous polynomial. Moreover, we obtain
When unit groups are regular Lie groups

\[ \sum_{n=1}^{\infty} \| \sigma_n(\gamma) \|_{C^1,p} \leq \sum_{n=1}^{\infty} \frac{M^n \| \gamma \|_{C^0,q}^n}{(n-1)!} = M \| \gamma \|_{C^0,q} e^{M \| \gamma \|_{C^0,q}} < \infty. \]

This estimate entails that the series \( \sum_{n=0}^{\infty} \sigma_n(\gamma) \) converges absolutely in the completion \( Y \) of \( C^1([0,1], A) \). In particular, the limit

\[ \Phi(\gamma) := \sum_{n=0}^{\infty} \sigma_n(\gamma) = \lim_{n \to \infty} \eta_n \]

exists in \( Y \), and defines a function \( \Phi \): \( X \to Y \). We claim that \( \Phi \) is continuous. If this is true, then we can exploit that \( C^1([0,1], A) \) is Mackey-complete by Lemma 2.4 and each \( \sigma_n \) takes its values inside \( C^1([0,1], A) \). Thus all hypotheses of Lemma 2.1 are satisfied, and we deduce that \( \Phi(\gamma) \in C^1([0,1], A) \) for each \( \gamma \) (entailing (a) and (b)), and that the map \( \Phi \): \( X \to C^1([0,1], A) \) is complex-analytic (establishing (d) and (d)’). To establish the claim, we need only show that \( \Phi \) is continuous as a map to \( C^1([0,1], \tilde{A}) \). Identify \( p \in P(A) \) with its continuous extension to a seminorm on \( \tilde{A} \). Let \( \pi_p : \tilde{A} \to (\tilde{A})_p, \| \cdot \|_p \) be the canonical map. By Lemma 2.3 \( \Phi \) will be continuous if the maps \( h := (\pi_p)_* \circ \Phi : X \to C^1([0,1], (\tilde{A})_p) \) are continuous. It suffices that \( h \) is continuous on the ball \( B_R := \{ \gamma \in X : \| \gamma \|_{C^0,q} < R \} \) for each \( R > 0 \). However,

\[ h(\gamma) = \sum_{n=0}^{\infty} \pi_p \circ \sigma_n(\gamma) \]

for \( \gamma \in B_R \), where

\[ \| \pi_p \circ \sigma_n(\gamma) \|_{C^1,\| \cdot \|_p} = \| \sigma_n(\gamma) \|_{C^1,p} \leq \frac{M^n}{(n-1)!} \| \gamma \|_{C^0,q}^n \leq \frac{M^n}{(n-1)!} R^n \]

for \( n \in \mathbb{N} \), by (3.7). Hence

\[ \sum_{n=0}^{\infty} \sup_{\gamma \in B_R} \{ \pi_p \circ \sigma_n(\gamma) : \gamma \in B_R \} \leq p(1) + M R e^{RM} < \infty, \]

entailing that \( \sum_{k=0}^{\infty} ((\pi_p)_* \circ \sigma_n|_{B_R}) \to h|_{B_R} \) uniformly. Thus \( h|_{B_R} \) is continuous, being a uniform limit of continuous functions.

To prove (c), observe that because \( \eta_n \to \eta \) in \( C^1([0,1], A) \), we have \( \eta_n' \to \eta' \) uniformly (and thus pointwise). Letting \( n \to \infty \) in (3.5), we deduce that \( \eta'(t) = \eta(t) \gamma(t) \).

4. Proof of Theorem 1.1. We establish our theorem as a special case of a more general result (Proposition 4.4). The latter deals with certain strengthened regularity properties (as used earlier in [7] and [3]):

**Definition 4.1.** Let \( G \) be a Lie group modelled on a locally convex space, with Lie algebra \( \mathfrak{g} \), and \( k \in \mathbb{N}_0 \cup \{ \infty \} \).
(a) $G$ is called strongly $C^k$-regular if every curve $\gamma \in C^k([0,1],g)$ admits an evolution $\text{Evol}(\gamma) \subset C^1([0,1],G)$ and the mapping $\text{evol}: C^k([0,1],g) \to G$, $\gamma \mapsto \text{Evol}(\gamma)(1)$, is smooth.

(b) $G$ is called $C^k$-regular if each $\gamma \in C^\infty([0,1],g)$ has an evolution and the map $\text{evol}: (C^\infty([0,1],g), \mathcal{O}_{C^k}) \to G$, $\gamma \mapsto \text{Evol}(\gamma)(1)$, is smooth, where $\mathcal{O}_{C^k}$ denotes the topology induced by $C^k([0,1],g)$ on $C^\infty([0,1],g)$.

The reader is referred to [8] and [9] for a discussion of these regularity properties (and applications depending thereon). Both $C^\infty$-regularity and strong $C^\infty$-regularity coincide with regularity in the usual sense. If $k \leq l$ and $G$ is (strongly) $C^k$-regular, then $G$ is also (strongly) $C^l$-regular.

**Remark 4.2.** If $A$ is a continuous inverse algebra, we identify the tangent bundle $T(A^\times)$ of the open set $A^\times$ with $A^\times \times A$ in the natural way. Let $\eta: [0,1] \to A^\times$ be a $C^1$-curve and $\gamma: [0,1] \to A$ be continuous. Then $\eta'(t) = \eta(t)\gamma(t)$ holds in $T(A^\times)$ (using $\eta': [0,1] \to T(A^\times)$, and identifying the range $A$ of $\gamma$ with $\{1\} \times A \subset T_1(A^\times)$) if and only if $\eta'(t) = \eta(t)\gamma(t)$ holds in $A$ (where the product simply refers to the algebra multiplication, and $\eta': [0,1] \to A$ is the derivative of the $A$-valued $C^1$-curve $\eta$).

The next lemma will help us to see that the $A$-valued map $\eta$ associated to $\gamma$ in Lemma 3.2 actually takes its values in $A^\times$ if $A$ is a continuous inverse algebra. Hence $\eta$ will be the evolution of $\gamma$, by Remark 4.2.

**Lemma 4.3.** Let $A$ be a continuous inverse algebra, $\gamma: [0,1] \to A$ be continuous and $\eta: [0,1] \to A$ as well as $\zeta: [0,1] \to A$ be $C^1$-curves. Assume that $\eta(0) = \zeta(0) = 1$ and

\begin{equation}
\eta'(t) = \eta(t)\gamma(t) \quad \text{and} \quad \zeta'(t) = \zeta(t)\gamma(t) \quad \text{for all } t \in [0,1].
\end{equation}

If $\zeta([0,1]) \subset A^\times$, then $\eta = \zeta$.

**Proof.** Recall from [6] proof of Lemma 3.1 that the differential of the inversion map $\iota: A^\times \to A$ is given by $d\iota(a,b) = -a^{-1}ba^{-1}$ for $a \in A^\times$ and $b \in A$. As a consequence, the derivative of the $C^1$-curve $\iota \circ \zeta: [0,1] \to A^\times$, $t \mapsto \zeta(t)^{-1}$, is given by

\begin{equation}
(\iota \circ \zeta)'(t) = -\zeta(t)^{-1}\zeta'(t)\zeta(t)^{-1}.
\end{equation}

Now consider the $C^1$-curve $\theta: [0,1] \to A$, $\theta(t) := \eta(t)\zeta(t)^{-1}$. Using the Product Rule, (4.2) and (4.1), we obtain

\[
\begin{align*}
\theta'(t) &= \eta'(t)\zeta(t)^{-1} - \eta(t)\zeta(t)^{-1}\zeta'(t)\zeta(t)^{-1} \\
&= \eta(t)\gamma(t)\zeta(t)^{-1} - \eta(t)\zeta(t)^{-1}\zeta(t)\gamma(t)\zeta(t)^{-1} \\
&= \eta(t)\gamma(t)\zeta(t)^{-1} - \eta(t)\gamma(t)\zeta(t)^{-1} = 0.
\end{align*}
\]

Hence $\theta(t) = \theta(0) = \eta(0)\zeta(0)^{-1} = 1$ for all $t \in [0,1]$ and thus $\eta = \zeta$.  

**Proposition 4.4.** Let $A$ be a continuous inverse algebra over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ which satisfies the condition $(\ast)$ described in Theorem 1.1.

(a) If $A$ is sequentially complete, then $A$ is strongly $C^0$-regular and the map $\text{Evol}: C^0([0, 1], A) \to C^1([0, 1], A^\times)$ is $\mathbb{K}$-analytic.

(b) If $A$ is Mackey-complete, then $A$ is $C^0$-regular and strongly $C^1$-regular. Further, each $\gamma \in \text{Lip}([0, 1], A)$ has an evolution $\text{Evol}(\gamma) \in C^1([0, 1], A^\times)$, and $\text{Evol}: (\text{Lip}([0, 1], A), O_{C^0}) \to C^1([0, 1], A^\times)$ is $\mathbb{K}$-analytic.

**Proof.** If $A$ is sequentially complete, let $X := C([0, 1], A)$; otherwise, let $X := (\text{Lip}([0, 1], A), O_{C^0})$.

We assume first that $\mathbb{K} = \mathbb{C}$. Let $\Phi: X \to C^1([0, 1], A)$ be the mapping provided by Lemma 3.2. Note that $C^1([0, 1], A^\times) \subseteq C^1([0, 1], A)$ is an identity neighbourhood, $\Phi(0) = 1$ (cf. (3.1)) and $\Phi$ is $\mathbb{C}$-analytic (see (d) or (d)$'$ of Lemma 3.2) and hence continuous. Therefore, there exists an open 0-neighbourhood $\Omega \subseteq X$ such that $\Phi(\Omega) \subseteq C^1([0, 1], A^\times)$. By Lemma 3.2(c), $\text{Evol}(\gamma) := \Phi(\gamma)$ is an evolution for $\gamma \in \Omega$. Moreover, $\text{evol}: \Omega \to A^\times$, $\gamma \mapsto \text{Evol}(\gamma)(1) = \Phi(\gamma)(1)$, is $\mathbb{C}$-analytic, since $\Phi$ and the continuous linear point evaluation $\text{ev}_1: C^1([0, 1], A) \to A$, $\zeta \mapsto \zeta(1)$, are $\mathbb{C}$-analytic.

If $A$ is sequentially complete, Proposition 1.3.10 in [3] now shows that $A^\times$ is strongly $C^0$-regular [8].

If $A$ is Mackey-complete, we see as in the proof of [3] Proposition 1.3.10 that each $\gamma \in \text{Lip}([0, 1], A)$ has an evolution $\text{Evol}(\gamma) \in C^1([0, 1], A^\times)$.

In either case, we deduce with Lemmas 3.2(c) and 4.3 that $\text{Evol} = \Phi$. As a consequence, $\text{Evol}: X \to C^1([0, 1], A^\times)$ is $\mathbb{C}$-analytic and thus (a) holds. In the situation of (b), note that also evol := $\text{ev}_1 \circ \text{Evol}: \text{Lip}([0, 1], A) \to A^\times$ is $\mathbb{C}$-analytic. The inclusion maps $(C^\infty([0, 1], A), O_{C^0}) \to (\text{Lip}([0, 1], A), O_{C^0})$ and $C^1([0, 1], A) \to (\text{Lip}([0, 1], A), O_{C^0})$ being continuous linear and hence $\mathbb{C}$-analytic, it follows that also the maps evol: $(C^\infty([0, 1], A), O_{C^0}) \to A^\times$ and evol: $C^1([0, 1], A) \to A^\times$ are $\mathbb{C}$-analytic and thus smooth. Hence $A^\times$ is $C^0$-regular and strongly $C^1$-regular.

If $\mathbb{K} = \mathbb{R}$, then also the complexification $A_\mathbb{C}$ of $A$ is a continuous inverse algebra (see, e.g., [6] Proposition 3.4) with the same completeness properties. In (a), we can identify $X_\mathbb{C}$ with $C^0([0, 1], A_\mathbb{C})$; in the situation of (b), we can identify $X_\mathbb{C}$ with $\text{Lip}([0, 1], A_\mathbb{C})$. For $p \in P(A)$, let $p_\mathbb{C} \in P(A_\mathbb{C})$ be the seminorm defined via

$$p_\mathbb{C}(a + ib) := \inf \left\{ \sum_j |z_j|p(x_j) : a + ib = \sum_j z_jx_j, x_j \in A, z_j \in \mathbb{C} \right\}$$

for $a, b \in A$ (which satisfies $\max\{p(a), p(b)\} \leq p_\mathbb{C}(a + ib) \leq p(a) + p(b)$). Then also $A_\mathbb{C}$ satisfies $(\ast)$, as $\|\mu_n\|_{p_\mathbb{C}, q_\mathbb{C}} = \|\mu_n\|_{p, q}$. Let $\Phi: X_\mathbb{C} \to C^1([0, 1], A_\mathbb{C})$

(8) Compare already [13] p. 409 and [18] Lemma 3] for similar arguments.
be the complex-analytic map provided by Lemma 3.2 (applied to $A_C$ in place of $A$). By the complex case just discussed,
\[ \Phi = \text{Evol}_{(A_C)^\times} : X_{C} \to C^1([0,1], (A_C)^\times). \]
If $\gamma \in X$, then $\Phi(\gamma)$ takes only values in the closed vector subspace $A$ of $A_C = A \oplus iA$, as is clear from (3.1). Hence $\Phi(\gamma) \in C^1([0,1], A)$ (see 10 or [11 Lemma 10.1]) and thus $\Phi(\gamma) \in C^1([0,1], A^\times)$, using the fact that $A \cap (A_C)^\times = A^\times$ for any unital algebra. (\textsuperscript{9}) We deduce that the map $\Phi|_X : X \to C^1([0,1], A^\times)$ is the evolution map $\text{Evol}_{A^\times}$ of $A^\times$. Note that $\text{Evol}_{A^\times}$ is $\mathbb{R}$-analytic, because $\Phi : X_C \to C^1([0,1], A)_C$ is a $\mathbb{C}$-analytic extension of $\text{Evol}_{A^\times}$. As $\text{ev}_1 : C^1([0,1], A) \to A$, $\zeta \mapsto \zeta(1)$, is continuous linear and so $\mathbb{R}$-analytic, also $\text{evol}_{A^\times} := \text{ev}_1 \circ \text{Evol}_{A^\times} : X \to A^\times$ is $\mathbb{R}$-analytic (and hence smooth). In the situation of (a), this completes the proof. In (b), compose $\text{evol}_{A^\times}$ with the continuous linear inclusion map $C^1([0,1], A) \to \text{Lip}([0,1], A)$ (resp., $(C^\infty([0,1], A), \mathcal{O}_{C^0}) \to \text{Lip}([0,1], A)$) to see that also the evolution mapping on $C^1([0,1], A)$ (resp., on $(C^\infty([0,1], A^\times), \mathcal{O}_{C^0})$) is $\mathbb{R}$-analytic and hence $C^\infty$. \hfill \blacksquare

**Appendix A. Proofs of the lemmas from Section 2.** It is useful to recall that a locally convex space $E$ is Mackey-complete (in the sense presented in the introduction) if and only if every Mackey–Cauchy sequence in $E$ converges, i.e., every sequence $(x_n)_{n \in \mathbb{N}}$ in $E$ for which there exists a bounded subset $B \subseteq E$ and a double sequence $(r_{n,m})_{n,m \in \mathbb{N}}$ of real numbers $r_{n,m} \geq 0$ such that $x_n - x_m \in r_{n,m}B$ for all $n, m \in \mathbb{N}$, and $r_{n,m} \to 0$ as both $n, m \to \infty$ (cf. [13 Theorem 2.14]).

**Proof of Lemma 2.1.** Given $x \in U$, there exists $r \in ]1, \infty[\text{ such that } rx \in U$. Thus $\sum_{n=0}^{\infty} r^np_n(x)$ converges and hence $C := \{r^np_n(x) : n \in \mathbb{N}_0\}$ is a bounded subset of $F$. Then also the absolutely convex hull $B$ of $C$ is bounded. For all $n, m \in \mathbb{N}_0$, we have
\[
\sum_{k=0}^{n+m} p_k(x) - \sum_{k=0}^{n} p_k(x) = \sum_{k=n+1}^{n+m} p_k(x) = r^{-n-1} \sum_{k=n+1}^{n+m} r^{n+1-k}k^ rp_k(x) \\
\in r^{-n-1} \left( \sum_{j=0}^{m-1} (1/r)^j \right) B \subseteq \frac{r^{-n-1}}{1-1/r} B.
\]
Hence $(\sum_{k=0}^{n} p_k(x))_{n \in \mathbb{N}_0}$ is a Mackey–Cauchy sequence in $F$ and hence convergent. Thus $f(x) \in F$. By [2 Theorems 5.1 and 6.1(i)], $f$ is $\mathbb{C}$-analytic as a map to $\tilde{F}$. Hence, if $x \in U$, then $f(x+y) = \sum_{n=0}^{\infty} (1/n!) \delta^n_x(f)(y)$ for all $y$ in some 0-neighbourhood, where $\delta^n_x f(y) := d^{(n)} f(x, y, \ldots, y)$ is the

(\textsuperscript{9}) If $x, a, b \in A$ and $x(a+ib) = (a+ib)x = 1$, then $xa + ixb = 1$ and $ax + ibx = 1$. Hence $xa = ax = 1$, i.e., $x^{-1} = a \in A$. 

106
H. Glöckner and K.-H. Neeb
nth Gâteaux differential of \( f \) at \( x \). Given \( y \in E \), there is \( s > 0 \) such that \( x + zy \in U \) for all \( z \in \mathbb{C} \) such that \( |z| \leq s \). For each \( n \in \mathbb{N}_0 \), Cauchy’s Integral Formula for higher derivatives now shows that

\[
\delta^n_x(f)(y) = \frac{n!}{2\pi i} \int_0^{2\pi} \frac{f(x + se^{i}y)}{(se^{i})^{n+1}} sie^{i} dt,
\]

which lies in \( F \) since the integrand is a Lipschitz curve in \( F \) and \( F \) is Mackey-complete \(^{(10)}\). Hence each \( \delta^n_x(f) \) is a continuous homogeneous polynomial from \( E \) to \( F \) and thus \( f \) is complex-analytic as a map from \( E \) to \( F \).

**Proof of Lemma 2.3** Let \( p \) be a continuous seminorm on \( F \) and \( k \in \mathbb{N}_0 \) be such that \( k \leq r \). Then \( q := p \circ \alpha \) is a continuous seminorm on \( E \). Let \( \gamma \in C^r([0, 1], E) \). For each \( j \in \mathbb{N} \) such that \( j \leq k \), we have \( (\alpha \circ \gamma)(j) = \alpha \circ \gamma(j) \) and thus \( \| (\alpha \circ \gamma)(j) \|_{C^0,p} = \| \alpha \circ \gamma(j) \|_{C^0,p} = \| \gamma(j) \|_{C^0,p\circ\alpha} = \| \gamma(j) \|_{C^0,q} \), entailing that \( \alpha \circ \gamma \) is continuous. Hence \( \alpha_* \) is a topological embedding.

If \( \alpha \) is an embedding and \( Q \) is a continuous seminorm on \( C^r([0, 1], E) \), then there exists \( k \in \mathbb{N}_0 \) such that \( k \leq r \) and a continuous seminorm \( q \) on \( E \) such that \( Q \leq \| \cdot \|_{C^k,q} \). Since \( \alpha \) is an embedding, there exists a continuous seminorm \( p \) on \( F \) such that \( p(\alpha(x)) \geq q(x) \) for all \( x \in E \) (because \( \alpha^{-1} \) is continuous linear). Hence \( \| (\alpha \circ \gamma)(j) \|_{C^0,p} = \| \gamma(j) \|_{C^0,p\circ\alpha} \geq \| \gamma(j) \|_{C^0,q} \) for each \( j \in \mathbb{N}_0 \) such that \( j \leq k \) and thus \( \| \alpha \circ \gamma \|_{C^k,p} \geq \| \gamma \|_{C^k,q} \geq Q(\gamma) \), entailing that \( \alpha_* \) is a topological embedding.

**Proof of Lemma 2.3** Let \( p \in P(E) \) and \( k \in \mathbb{N}_0 \) be such that \( k \leq r \). Since \( p = \| \cdot \|_{p \circ \pi_p} \), we have

\[
\| (\pi_p \circ \gamma)(j) \|_{C^0,p} = \| \pi_p \circ \gamma(j) \|_{C^0,p} = \| \gamma(j) \|_{C^0,p} = \| \gamma \|_{C^k,p},
\]

for each \( \gamma \in C^r([0, 1], E) \) and \( j \in \{0, 1, \ldots, k\} \), whence \( \| (\pi_p)_*(\gamma) \|_{C^k,p} = \| \gamma \|_{C^k,p} \). The assertion follows.

**Remark A.1.** Before we turn to the proof of Lemma 2.4, it is useful to record some simple observations:

(a) It is clear from the definitions that the map

\[
h: C^k([0, 1], E) \to C([0, 1], E) \times C^{k-1}([0, 1], E), \quad \gamma \mapsto (\gamma, \gamma'),
\]

is linear and a homeomorphism onto its image, for each \( k \in \mathbb{N} \).

(b) The image \( \text{im}(h) \) of \( h \) consists of all pairs \( (\gamma, \eta) \) such that \( \gamma(t) = \gamma(0) + \int_0^t \eta(s) ds \) for each \( t \in [0, 1] \). Since point evaluations and the linear mappings \( \eta \mapsto \int_0^t \eta(s) ds \) (with \( p(\int_0^t \eta(s) ds) \leq \| \eta \|_{C^0,p} \)) are continuous, it follows that \( \text{im}(h) \) is a closed vector subspace of \( C([0, 1], E) \times C^{k-1}([0, 1], E) \).

\(^{(10)}\) The integrand is a \( C^\infty \)-curve in \( \tilde{F} \) and hence a Lipschitz curve in \( \tilde{F} \), with image in \( F \).
Proof of Lemma 2.4. Because direct products of Mackey-complete locally convex spaces are Mackey-complete, and so are closed vector subspaces, also projective limits of Mackey-complete locally convex spaces are Mackey-complete. Since $C^\infty([0, 1], E) = \lim \leftarrow C^k([0, 1], E)$ (with the appropriate inclusion maps as the limit maps), we therefore only need to prove Mackey-completeness if $k := r \in \mathbb{N}_0$. Likewise in the case of completeness.

Case $k = 0$. If $E$ is complete, then also $C([0, 1], E)$ is complete, as is well known (cf. [12, Chapter 7, Theorem 10]). If $E$ is merely Mackey-complete, let $\tilde{E}$ be a completion of $E$ which contains $E$. Then $C([0, 1], \tilde{E})$ is complete. The inclusion map $\phi: C([0, 1], E) \to C([0, 1], \tilde{E})$ is a topological embedding, by Lemma 2.2. If $(\gamma_n)_{n \in \mathbb{N}}$ is a Mackey–Cauchy sequence in $C([0, 1], E)$, then $(\phi \circ \gamma_n)_{n \in \mathbb{N}} = (\gamma_n)_{n \in \mathbb{N}}$ is a Mackey–Cauchy sequence in $C([0, 1], \tilde{E})$, hence convergent to some $\gamma \in C([0, 1], \tilde{E})$. For each $t \in [0, 1]$, the point evaluation $\varepsilon_t: C([0, 1], \tilde{E}) \to \tilde{E}$, $\eta \mapsto \eta(t)$, is continuous and linear. Hence $(\gamma_n(t))_{n \in \mathbb{N}}$ is a Mackey–Cauchy sequence in $E$ and hence convergent in $E$. Since $\gamma_n(t) = \varepsilon_t(\gamma_n) \to \varepsilon_t(\gamma) = \gamma(t)$, we deduce that $\gamma(t) \in E$. Therefore $\gamma \in C([0, 1], E)$ and it is clear that $\gamma_n \to \gamma$ in $C([0, 1], E)$.

Induction step. If $C^{k-1}([0, 1], E)$ is (Mackey-)complete, then so is $C^k([0, 1], E)$, being isomorphic to a closed vector subspace of the (Mackey-)complete direct product $C([0, 1], E) \times C^{k-1}([0, 1], E)$ (see Remark A.1(b)).

Acknowledgements. This research was supported by DFG (grant GL 357/5-1) and the Emerging Field Project “Quantum Geometry” of FAU Erlangen-Nürnberg.

References

[1] W. Bertram, H. Glöckner and K.-H. Neeb, Differential calculus over general base fields and rings, Expo. Math. 22 (2004), 213–282.
[2] J. Bochnak and J. Siciak, Analytic functions in topological vector spaces, Studia Math. 39 (1971), 77–112.
[3] R. Dahmen, Direct limit constructions in infinite-dimensional Lie theory, doctoral dissertation, Univ. Paderborn, 2011; http://nbn-resolving.deurn:nbn:de:hbz:466:2-239.
[4] H. Glöckner, Infinite-dimensional Lie groups without completeness restrictions, in: Banach Center Publ. 55, Inst. Math., Polish Acad. Sci., Warszawa, 2002, 43–59.
[5] H. Glöckner, Lie group structures on quotient groups and universal complexifications for infinite-dimensional Lie groups, J. Funct. Anal. 194 (2002), 347–409.
[6] H. Glöckner, Algebras whose groups of units are Lie groups, Studia Math. 153 (2002), 147–177.
[7] H. Glöckner, Direct limits of infinite-dimensional Lie groups, in: Developments and Trends in Infinite-Dimensional Lie Theory, K.-H. Neeb and A. Pianzola (eds.), Progr. Math. 288, Birkhäuser, Boston, 2011, 243–280.
When unit groups are regular Lie groups

[8] H. Glöckner, Notes on regularity properties of infinite-dimensional Lie groups, arXiv: 1208.0715.

[9] H. Glöckner, Regularity in Milnor’s sense for direct limits of infinite-dimensional Lie groups, manuscript in preparation.

[10] H. Glöckner and K.-H. Neeb, Infinite-Dimensional Lie Groups, Vol. I, book in preparation.

[11] M. Hervé, Analyticity in Infinite-Dimensional Spaces, de Gruyter, Berlin, 1989.

[12] J. L. Kelley, General Topology, Springer, 1955.

[13] A. Kriegl and P. W. Michor, The Convenient Setting of Global Analysis, Amer. Math. Soc., Providence, RI, 1997.

[14] E. A. Michael, Locally multiplicatively-convex topological algebras, Mem. Amer. Math. Soc. 11 (1952).

[15] P. W. Michor, Manifolds of Differentiable Mappings, Shiva Publ., Orpington, 1980.

[16] J. Milnor, Remarks on infinite-dimensional Lie groups, in: Relativité, groupes et topologie II (Les Houches, 1983), B. S. DeWitt and R. Stora (eds.), North-Holland, Amsterdam, 1984, 1007–1057.

[17] K.-H. Neeb, Towards a Lie theory of locally convex groups, Japan. J. Math. 1 (2006), 291–468.

[18] Th. Robart, On Milnor’s regularity and the path-functor for the class of infinite dimensional Lie algebras of CBH type, Algebras Groups Geom. 21 (2004), 367–386.

[19] Ph. Turpin, Une remarque sur les algèbres à inverse continu, C. R. Acad. Sci. Paris Sér. A-B 270 (1970), A1686–A1689.

[20] L. Waelbroeck, Les algèbres à inverse continu, C. R. Acad. Sci. Paris 238 (1954), 640–641.

Karl-Hermann Neeb
FAU Erlangen-Nürnberg
Department Mathematik
Cauerstr. 11
91058 Erlangen, Germany
E-mail: karl-hermann.neeb@math.uni-erlangen.de

Received February 6, 2012
Revised version September 28, 2012
(7422)
