PROPER CONGRUENCE-PRESERVING EXTENSIONS OF LATTICES

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Abstract. We prove that every lattice with more than one element has a proper congruence-preserving extension.

1. Introduction

Let \( L \) be a lattice. A lattice \( K \) is a congruence-preserving extension of \( L \), if \( K \) is an extension and every congruence of \( L \) has exactly one extension to \( K \). (Of course, then, the congruence lattice of \( L \) is isomorphic to the congruence lattice of \( K \).

In \cite{4}, the first author and E. T. Schmidt raised the following question:

Is it true that every lattice \( L \) with more than one element has a proper congruence-preserving extension \( K \)?

Here proper means that \( K \) properly contains \( L \), that is, \( K - L \neq \emptyset \).

The first author and E. T. Schmidt pointed out in \cite{4} that in the finite case this is obviously true, and they proved the following general result:

Theorem 1. Let \( L \) be a lattice. If there exist a distributive interval with more than one element in \( L \), then \( L \) has a proper congruence-preserving extension \( K \).

Generalizing this result, in this paper, we provide a positive answer to the above question:

Theorem 2. Every lattice \( L \) with more than one element has a proper congruence-preserving extension \( K \).

2. Background

Let \( K \) and \( L \) be lattices. If \( L \) is a sublattice of \( K \), then we call \( K \) an extension of \( L \). If \( K \) is an extension of \( L \) and \( \Theta \) is a congruence relation of \( K \), then \( \Theta_L \), the restriction of \( \Theta \) to \( L \) is a congruence of \( L \). If the map \( \Theta \mapsto \Theta_L \) is a bijection between the congruences of \( L \) and the congruences of \( K \), then we call \( K \) a congruence-preserving extension of \( L \). Observe that if \( K \) a congruence-preserving extension of \( L \), then the congruence lattice of \( L \) is isomorphic to the congruence lattice of \( K \) in a natural way.

The proof of Theorem 1 is based on the following construction of E. T. Schmidt \cite{9}, summarized below as Theorem 3. (A number of papers utilize this construction; Date: February 20, 1998.

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see, for instance, E. T. Schmidt [10], [11] and the recent paper G. Grätzer and E. T. Schmidt [5].

Let $L$ be a bounded distributive lattice with bounds 0 and 1, and let $M_3 = \{o, a, b, c, i\}$ be the five-element nondistributive modular lattice. Let $M_3[L]$ denote the poset of triples $\langle x, y, z \rangle \in L^3$ satisfying the condition

$$(S) \quad x \land y = y \land z = z \land x.$$

**Theorem 3.**

Let $D$ be a bounded distributive lattice with bounds 0 and 1.

(i) $M_3[D]$ is a modular lattice.

(ii) The subset \(\overline{M}_3 = \{(0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 1)\}\) of $M_3[D]$ is a sublattice of $M_3[D]$ and it is isomorphic to $M_3$.

(iii) The subposet $\overline{D} = \{ (x, 0, 0) \mid x \in D \}$ of $M_3[D]$ is a bounded distributive lattice and it is isomorphic to $D$; we identify $D$ with $\overline{D}$.

(iv) $\overline{M}_3$ and $D$ generate $M_3[D]$. 

(v) Let $\Theta$ be a congruence relation of $D = \overline{D}$; then there is a unique congruence $\overline{\Theta}$ of $M_3[D]$ such that $\overline{\Theta}$ restricted to $\overline{D}$ is $\Theta$; therefore, $M_3[D]$ is a congruence-preserving extension of $D$.

Unfortunately, $M_3[L]$ fails, in general, to produce a lattice, if $L$ is not distributive. In this paper, we introduce a variant on the $M_3[L]$ construction, which we shall denote as $M_3(L)$. This lattice $M_3(L)$ is a proper congruence-preserving extension of $L$, for any lattice $L$ with more than one element, verifying Theorem 2.

3. The Construction

For a lattice $L$, let us call the triple $\langle x, y, z \rangle \in L^3$ Boolean, if

$$x = (x \lor y) \land (x \lor z),$$
$$y = (y \lor x) \land (y \lor z),$$
$$z = (z \lor x) \land (z \lor y).$$

We denote by $M_3(L) \subseteq L^3$ the poset of Boolean triples of $L$.

Here are some of the basic properties of Boolean triples:

**Lemma 1.** Let $L$ be a lattice.

(i) Every Boolean triple of $L$ satisfies $(S)$, so $M_3(L) \subseteq M_3[L]$.

(ii) $\langle x, y, z \rangle \in L^3$ is Boolean iff there is a triple $\langle u, v, w \rangle \in L^3$ satisfying

$$x = u \land v,$$
$$y = u \land w,$$
$$z = v \land w.$$

(R)

(iii) For every triple $\langle x, y, z \rangle \in L^3$, there is a smallest Boolean triple $\langle x, y, z \rangle \in L^3$ such that $\langle x, y, z \rangle \leq \langle x, y, z \rangle$; in fact,

$$\langle x, y, z \rangle = \langle (x \lor y) \land (x \lor z), (y \lor x) \land (y \lor z), (z \lor x) \land (z \lor y) \rangle.$$
(iv) $M_3(L)$ is a lattice with the meet operation defined as
\[ (x_0, y_0, z_0) \land (x_1, y_1, z_1) = (x_0 \land x_1, y_0 \land y_1, z_0 \land z_1) \]
and the join operation defined by
\[ (x_0, y_0, z_0) \lor (x_1, y_1, z_1) = (x_0 \lor x_1, y_0 \lor y_1, z_0 \lor z_1) . \]

(v) If $L$ has 0, then the subposet $\{ (x, 0, 0) \mid x \in L \}$ is a sublattice and it is isomorphic to $L$.

If $L$ has 0 and 1, then $M_3(L)$ has a spanning $M_3$, that is, a $\{0, 1\}$-sublattice isomorphic to $M_3$, namely,
\[ \{ (0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 1) \} . \]

(vi) If $\langle x, y, z \rangle$ is Boolean, then one of the following holds:
(a) the components form a one-element set, so $\langle x, y, z \rangle = \langle a, a, a \rangle$, for some $a \in L$;
(b) the components form a two-element set and $\langle x, y, z \rangle$ is of the form $\langle b, a, a \rangle$, or $\langle a, b, a \rangle$, or $\langle a, a, b \rangle$, for some $a, b \in L$, $a < b$.
(c) the components form a three-element set and two components are comparable and $L$ has two incomparable elements $a$ and $b$ such that
   
   $\langle x, y, z \rangle$ is of the form $\langle a, b, a \land b \rangle$, or $\langle a, a \land b, b \rangle$, or $\langle a \land b, a, b \rangle$.
(d) the components form a three-element set and the components are pairwise incomparable and $L$ has an eight-element Boolean sublattice $B$ so that the components are the atoms of $B$.

Proof.
(i) If $\langle x, y, z \rangle$ is Boolean, then
\[ x \land y = ((x \lor y) \land (x \lor z)) \land ((y \lor x) \land (y \lor z)) \]
\[ = (x \lor y) \land (y \lor z) \land (z \lor x) , \]
which is the upper median of $x$, $y$, and $z$. So (S) holds.

(ii) If $\langle x, y, z \rangle$ is Boolean, then $u = x \lor y$, $v = x \lor z$, and $w = y \lor z$ satisfy (R).
Conversely, if there is a triple $\langle u, v, w \rangle \in L^3$ satisfying (R), then by Lemma 1.5.9 of [1], the sublattice generated by $x$, $y$, and $z$ is isomorphic to a quotient of $C_3$ (where $C_2$ is the two element chain) and $x$, $y$, and $z$ are the images of the three atoms of $C_3$. Thus $\langle x \lor y \rangle \land \langle x \lor z \rangle = x$, the first part of (B). The other two parts are proved similarly.

(iii) For $\langle x, y, z \rangle \in L^3$, define $u = x \lor y$, $v = x \lor z$, $w = y \lor z$. Set $x_1 = u \land v$, $y_1 = u \land w$, $z_1 = v \land w$. Then $\langle x_1, y_1, z_1 \rangle$ is Boolean by (ii) and $\langle x, y, z \rangle \leq \langle x_1, y_1, z_1 \rangle$ in $L^3$. Now if $\langle x, y, z \rangle \leq \langle x_2, y_2, z_2 \rangle$ in $L^3$ and $\langle x_2, y_2, z_2 \rangle$ is Boolean, then
\[ x_2 = (x_2 \lor y_2) \land (x_2 \lor z_2) = (x_2 \land y_2) \lor (x_2 \land z_2) \]
\[ \geq (x \lor y) \land (x \lor z) \]
\[ = u \lor v = x_1 , \]
and similarly, $y_2 \geq y_1$, $z_2 \geq z_1$. Thus $\langle x_2, y_2, z_2 \rangle \geq \langle x_1, y_1, z_1 \rangle$, and so $\langle x_1, y_1, z_1 \rangle$ is the smallest Boolean triple containing $\langle x, y, z \rangle$.

(iv) $M_3(L) \neq \emptyset$; for instance, for all $x \in L$, the diagonal element $\langle x, x, x \rangle \in M_3(L)$. It is obvious from (ii) that $M_3(L)$ is meet closed. By (iii), $M_3(L)$ is a closure system in $L^3$, from which the formulas of (iv) follow.

The proofs of (v) and (vi) are left to the reader.
4. Proof of the theorem

Let $L$ be a lattice with more than one element. We identify $x \in L$ with the diagonal element $⟨x, x, x⟩ ∈ M_3(L)$, so we regard $M_3(L)$ an extension of $L$. This is an embedding of $L$ into $M_3(L)$ different from the embedding in Lemma 3(v). Moreover, the embedding in Lemma 4(v) requires that $L$ have a zero, while the embedding discussed here always works.

Note that $M_3(L)$ is a proper extension; indeed, since $L$ has more than one element, we can choose the elements $a < b$ in $L$. Then $⟨a, a, b⟩ ∈ M_3(L)$ but $⟨a, a, b⟩$ is not on the diagonal, so $⟨a, a, b⟩ ∈ M_3(L) - L$. In fact, if $L = C_2$, the two-element chain, then this is the only type of nondiagonal element:

$$M_3(C_2) = \{ ⟨0, 0, 0⟩, ⟨1, 0, 0⟩, ⟨0, 1, 0⟩, ⟨0, 0, 1⟩, ⟨1, 1, 1⟩ \}.$$ 

For a congruence $Θ$ of $L$, let $Θ^3$ denote the congruence of $L^3$ defined componentwise. Let $M_3(Θ)$ be the restriction of $Θ^3$ to $M_3(L)$.

**Lemma 2.** $M_3(Θ)$ is a congruence relation of $M_3(L)$.

**Proof.** $M_3(Θ)$ is obviously an equivalence relation on $M_3(L)$. Since $M_3(L)$ is a meet subsemilattice of $L^3$, it is clear that $M_3(Θ)$ satisfies the Substitution Property for meets. To verify for $M_3(Θ)$ the Substitution Property for joins, let $⟨x_0, y_0, z_0⟩, ⟨x_1, y_1, z_1⟩ ∈ M_3(L)$, let

$$⟨x_0, y_0, z_0⟩ \equiv ⟨x_1, y_1, z_1⟩ \quad (M_3(Θ)),$$

(that is,

$$x_0 \equiv x_1 \quad (Θ), \quad y_0 \equiv y_1 \quad (Θ), \quad \text{and} \quad z_0 \equiv z_1 \quad (Θ)$$

in $L$) and let $⟨u, v, w⟩ ∈ M_3(L)$. Set

$$⟨x'_i, y'_i, z'_i⟩ = ⟨x_i, y_i, z_i⟩ \lor ⟨u, v, w⟩$$

(the join formed in $M_3(L)$), for $i = 0, 1$.

Then, using Lemma 3(iii) and (iv) for $x_0 \lor u$, $y_0 \lor v$, and $z_0 \lor w$, we obtain that

$$x'_0 = (x_0 \lor u \lor y_0 \lor v) \land (x_0 \lor u \lor z_0 \lor w)$$

$$\equiv (x_1 \lor u \lor y_1 \lor v) \land (x_1 \lor u \lor z_1 \lor w) = x'_1 \quad (M_3(Θ)),$$

and similarly, $y'_0 \equiv y'_1 \quad (M_3(Θ))$, $z'_0 \equiv z'_1 \quad (M_3(Θ))$, hence

$$⟨x_0, y_0, z_0⟩ \lor ⟨u, v, w⟩ \equiv ⟨x_1, y_1, z_1⟩ \lor ⟨u, v, w⟩ \quad (M_3(Θ)).$$

Since $L$ was identified with the diagonal of $M_3(L)$, it is obvious that $M_3(Θ)$ restricted to $L$ is $Θ$. So to complete the proof of Theorem 2 it is sufficient to verify the following statement:

**Lemma 3.** Every congruence of $M_3(L)$ is of the form $M_3(Θ)$, for a suitable congruence $Θ$ of $L$.

**Proof.** Let $Φ$ be a congruence of $M_3(L)$, and let $Θ$ denote the congruence of $L$ obtained by restricting $Φ$ to the diagonal of $M_3(L)$, that is, $x \equiv y \quad (Θ)$ in $L$ iff $⟨x, x, x⟩ \equiv ⟨y, y, y⟩ \quad (Φ)$ in $M_3(L)$. We prove that $Φ = M_3(Θ)$.

To show that $Φ ⊆ M_3(Θ)$, let

$$⟨x_0, y_0, z_0⟩ \equiv ⟨x_1, y_1, z_1⟩ \quad (Φ).$$

(1)
Define
\( o = x_0 \wedge x_1 \wedge y_0 \wedge y_1 \wedge z_0 \wedge z_1, \)
\( i = x_0 \vee x_1 \vee y_0 \vee y_1 \vee z_0 \vee z_1. \)

Meeting the congruence \( 1 \) with \( \langle i, o, o \rangle \) yields
\( \langle x_0, o, o \rangle \equiv \langle x_1, o, o \rangle \) \((\Phi)\).

Since
\( \langle x_0, o, o \rangle \vee \langle o, o, i \rangle = \langle x_0, o, i \rangle = \langle x_0, x_1, i \rangle, \)
joining the congruence \( 4 \) with \( \langle o, o, i \rangle \) yields
\( \langle x_0, x_0, i \rangle \equiv \langle x_1, x_1, i \rangle \) \((\Phi)\).

Similarly,
\( \langle x_0, o, o \rangle \equiv \langle x_1, i, x_1 \rangle \) \((\Phi)\).

Now we meet the congruences \( 5 \) and \( 6 \) to obtain
\( \langle x_0, y_0, z_0 \rangle \equiv \langle x_1, y_1, z_1 \rangle \) \((\Theta^3)\)
in \( L^3 \), proving that \( \Phi \subseteq M_3(\Theta) \).

To prove the converse, \( M_3(\Theta) \subseteq \Phi \), take
\( \langle x_0, y_0, z_0 \rangle \equiv \langle x_1, y_1, z_1 \rangle \) \((M_3(\Theta))\)
in \( M_3(L) \), that is,
\[ x_0 \equiv x_1 \quad (\Theta), \]
\[ y_0 \equiv y_1 \quad (\Theta), \]
\[ z_0 \equiv z_1 \quad (\Theta) \]
in \( L \). Equivalently,
\( \langle x_0, x_0, x_0 \rangle \equiv \langle x_1, x_1, x_1 \rangle \) \((\Phi)\),
\( \langle y_0, y_0, y_0 \rangle \equiv \langle y_1, y_1, y_1 \rangle \) \((\Phi)\),
\( \langle z_0, z_0, z_0 \rangle \equiv \langle z_1, z_1, z_1 \rangle \) \((\Phi)\)
in \( M_3(L) \).

Now, define \( o, i \) as in \( 2 \) and \( 3 \). Meeting the congruence \( 10 \) with \( \langle i, o, o \rangle \), we obtain
\( \langle x_0, o, o \rangle \equiv \langle x_1, o, o \rangle \) \((\Phi)\).

Similarly, from \( 11 \) and \( 14 \), we obtain the congruences
\( \langle o, y_0, o \rangle \equiv \langle o, y_1, o \rangle \) \((\Phi)\),
\( \langle o, o, z_0 \rangle \equiv \langle o, o, z_1 \rangle \) \((\Phi)\).

Finally, joining the congruences \( 12 \) and \( 14 \), we get
\( \langle x_0, y_0, z_0 \rangle \equiv \langle x_1, y_1, z_1 \rangle \) \((\Phi)\),
that is, \( M_3(\Theta) \subseteq \Phi \). This completes the proof of this lemma and of Theorem \( 2 \). \( \square \)
5. Discussion

Special extensions. We can get a slightly stronger result by requiring that the extension preserve the zero and the unit, provided they exist. To state this result, we need the following concept.

An extension $K$ of a lattice $L$ is extensive, provided that the convex sublattice of $K$ generated by $L$ is $K$.

Note that if $L$ has a zero, 0, then an extensive extension is a $\{0\}$-extension (and similarly for the unit, 1); if $L$ has a zero, 0, and unit 1, then an extensive extension is a $\{0, 1\}$-extension.

**Theorem 4.** Every lattice $L$ with more than one element has a proper congruence-preserving extensive extension $K$.

**Proof.** Indeed, every $\langle x, y, z \rangle \in M_3(L)$ is in the convex sublattice generated by $L$ since $\langle x \land y \land z, x \land y \land z, x \land y \land z \rangle \leq \langle x, y, z \rangle \leq \langle x \lor y \lor z, x \lor y \lor z, x \lor y \lor z \rangle$. □

In Theorem 3.(iii), we pointed out that $M_3[D]$ is a congruence-preserving extension of $D = \{0\}$, where $D$ is an ideal of $M_3[D]$. This raises the question whether Theorem 2 can be strengthened by requiring that $L$ be an ideal in $K$. This is easy to do, if $L$ has a zero, 0, since then we can identify $x \in L$ with $\langle x, 0, 0 \rangle \in M_3(L)$.

**Theorem 5.** Every lattice $L$ with more than one element has a proper congruence-preserving extension $K$ with the property that $L$ is an ideal in $K$.

**Proof.** Take an element $a \in L$ such that $[a]$ (the dual ideal generated by $a$) has more than one element. Then by Lemma 1.(v), $A = M_3([a])$ is a proper congruence-preserving extension of $[a]$ and $I = [a]$ is an ideal in $A$. Now form the lattice $K$ by gluing $L$ with the dual ideal $[a]$ to $A$ with the ideal $I$. It is clear that $K$ is a proper congruence-preserving extension of $L$. □

Modularity and semimodularity. R. W. Quackenbush [8] proved that if $L$ is a modular lattice, then $M_3[L]$ is a semimodular lattice. For our construction, the analogous result fails: $M_3(P)$ is not semimodular, where $P$ is a projective plane (a modular lattice). Indeed, let $a, b, c$ be a triangle in $P$, with sides $l, m, n$, that is, let $l, m, n$ be three distinct lines in the plane $P$, and define the points $a = n \land m$, $b = n \land l$, $c = m \land l$. Let $p$ be a point in $P$ not on any one of these lines. Then $\langle p, \emptyset, \emptyset \rangle$ is an atom in $M_3(P)$, $\langle a, b, c \rangle \in M_3(P)$ but $\langle \{p\}, \emptyset, \emptyset \rangle \lor \langle a, b, c \rangle = \langle p \lor a, b, c \rangle = \langle P, l, l \rangle$ and $\langle a, b, c \rangle < \langle n, b, l \rangle < \langle P, l, l \rangle$, showing that $M_3(P)$ is not semimodular.

Now we characterize when $M_3(L)$ is modular.

**Theorem 6.** Let $L$ be a lattice with more than one element. Then $M_3(L)$ is modular iff $L$ is distributive.

**Proof.** If $L$ is distributive, then $M_3(L) = M_3[L]$, so $M_3(L)$ is modular by Theorem 3. 


Conversely, if $M_3(L)$ is modular, then $L$ is modular since it is a sublattice of $M_3(L)$. Now if $L$ is not distributive, then $L$ contains an $M_3 = \{o, a, b, c, i\}$ as a sublattice. By Lemma (vi), the elements 
\[
\langle o, o, a \rangle, \langle o, c, a \rangle, \langle c, c, i \rangle, \langle i, i, i \rangle, \langle b, o, a \rangle
\]
belong to $M_3(L)$. Obviously, 
\[
\langle o, o, a \rangle < \langle o, c, a \rangle < \langle c, c, i \rangle < \langle i, i, i \rangle
\]
and 
\[
\langle o, o, a \rangle < \langle b, o, a \rangle < \langle i, i, i \rangle.
\]
To prove that these five elements form an $N_5$, it is enough to prove that 
\[
\langle c, c, i \rangle \land \langle b, o, a \rangle = \langle o, o, a \rangle
\]
and 
\[
\langle o, c, a \rangle \lor \langle b, o, a \rangle = \langle i, i, i \rangle.
\]
The meet is obvious. Now the join: 
\[
\langle o, c, a \rangle \lor \langle b, o, a \rangle = \langle b, c, a \rangle = \langle i, i, i \rangle.
\]
So $M_3(L)$ contains $N_5$ as a sublattice, contradicting the assumption that $M_3(L)$ is modular. Therefore, $L$ is distributive.  

Further results. $M_3[L]$ is not a lattice for a general $L$. See, however, G. Grätzer and F. Wehrung [6], where a new concept of $n$-modularity is introduced, for any natural number $n$. Modularity is the same as 1-modularity. 

By definition, $n$-modularity is an identity; for larger $n$, a weaker identity. For an $n$-modular lattice $L$, $M_3[L]$ is a lattice, a congruence-preserving extension of $L$.

For distributive lattices (in fact, for $n$-modular lattices), the construction $M_3[L]$ is a special case of the tensor product construction of two semilattices with zero, see, for instance, G. Grätzer, H. Lakser, and R. W. Quackenbush [2] and R. W. Quackenbush [8]. The $M_3(L)$ construction is generalized in G. Grätzer and F. Wehrung [7] to two bounded lattices; the new construction is called box product. Some of the arguments of this paper carry over to box products.

Problems

Lattices. As usual, let us denote by $T$, $D$, $M$, and $L$ the variety of one-element, distributive, modular, and all lattices, respectively. A variety $V$ is nontrivial if $V \neq T$.

Let us say that a variety $V$ of lattices has the Congruence Preserving Extension Property (CPEP, for short), if every lattice in $V$ with more than one element has a proper congruence-preserving extension in $V$. It is easy to see that no finitely generated lattice variety has CPEP. (Indeed, by Jónsson’s lemma, a nontrivial finitely generated lattice variety $V$ has a finite maximal subdirectly irreducible member $L$; if $K$ is a proper congruence-preserving extension of $L$, then $K$ is also subdirectly irreducible and $|L| > |K|$, a contradiction.) In particular, $D$ does not have CPEP.

Theorem [2] can be restated as follows: $L$ has CPEP.

Problem 1. Find all lattice varieties $V$ with CPEP. In particular, does $M$ have CPEP?
Groups. Let us say that a variety $\mathbf{V}$ of groups has the Normal Subgroup Preserving Extension Property (NSPEP, for short), if every group $G$ in $\mathbf{V}$ with more than one element has a proper supergroup $\overline{G}$ in $\mathbf{V}$ with the following property: every normal subgroup $N$ in $G$ can be uniquely represented in the form $N \cap \overline{G}$, where $N$ is a normal subgroup of $G$.

Not every group variety $\mathbf{V}$ has NSPEP, for instance, the variety $\mathbf{A}$ of Abelian groups does not have NSPEP.

Problem 2. Does the variety $\mathbf{G}$ of all groups have NSPEP? Find all group varieties having NSPEP?

Rings. For ring varieties, we can similarly introduce the Ideal Preserving Extension Property (IPEP, for short). The variety $\mathbf{R}$ of all (not necessarily commutative) rings has IPEP. Indeed, if $R$ is a ring with more than one element, then embed $R$ into $M_2(R)$ (the ring of $2 \times 2$ matrices over $R$) with the diagonal map. The two-sided ideals of $M_2(R)$ are of the form $M_2(I)$, where $I$ is a two-sided ideal of $R$, and $I = M_2(I) \cap R$.

Problem 3. Find all ring varieties having IPEP? In particular, does the variety of all commutative rings have IPEP?

The second author found a positive answer for Dedekind domains: every Dedekind domain with more than one element has a proper ideal-preserving extension that is, in addition, a principal ideal domain.

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