Noncommutative Quantum Anisotropic cosmology in K-essence

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Abstract. We study a canonical noncommutative extension of the Bianchi type I Minisuperspace, with a barotropic perfect fluid, in the context of a simplified form of k-essence known as the Sáez-Ballester theory. Noncommutativity is implemented in the pure gravitational sector. The corresponding noncommutative Wheeler-DeWitt equation is constructed and solved. Quantum solutions are obtained for any value of the barotropic parameter. It is seen that the effect of this particular noncommutativity is that of modulating the amplitude of the commutative wave function.

1. Introduction

In the early 2000’s Armendariz et al proposed a minimally coupled scalar field with non standard kinetic term as a dynamical candidate for dark energy which is known today as K-essence [1] (as opposed to Quintessence which models dark energy through the potential term of a minimally coupled standard scalar field). K-essence is based on the idea of a dynamical attractor solution which causes it to act as a cosmological constant only at the onset of matter domination. More recently, attempts to unify the description of inflation, dark energy and dark matter through one or several k-essence fields have been conducted [2].

Quite ago, in the 1980’s, Saez and Ballester [3] formulated a scalar-tensor theory of gravitation in which the metric is coupled with a dimensionless scalar field with non standard kinetic term, which suggested a possible way to solve the missing matter problem in non-flat FRW cosmologies. The Saez-Ballester (SB) formalism can be seen now as a very particular case of the K-essence paradigm applied to dark matter.

Usually K-essence models are restricted to a Lagrangian density of the form

$$S = \int d^4x \sqrt{-g} \left[ f(\phi) G(X) - V(\phi) \right],$$

where the canonical kinetic energy is given by $G(X) = X = -\frac{1}{2} \nabla_\mu \phi \nabla^\mu \phi$. In particular, the SB formalism is obtained in the case in which the potential term is null and $G(X) = X$.

The quantized version of the K-essence paradigm, and in particular the SB formalism, has not yet been constructed, perhaps due to the difficulties in obtaining the ADM formalism for it. We circumvent this issue by reinterpreting the K-essence field as an exotic part of the matter...
General Relativity. This in turn enables us to implement the ADM formulation [4] and thus achieve the corresponding Wheeler-DeWitt quantization of the model.

One of the simplest K-essence Lagrangian density is

\[ L_{\text{geo}} = \left( \frac{R}{2} + f(\phi)G(X) \right), \]  

(2)

here, R is the scalar curvature, and \( f(\phi) \) is an arbitrary function of the scalar field and the potential for it is absent.

From the Lagrangian (2) we can build the complete action

\[ I = \int_{\Sigma} \sqrt{-g}(L_{\text{geo}} + L_{\text{mat}})d^4x, \]  

(3)

where \( L_{\text{mat}} \) is the matter Lagrangian and g is the determinant of the metric tensor. The field equations for this theory are

\[ G_{\alpha\beta} + f(\phi) \left[ G_{X\phi,\alpha\phi,\beta} + G_{g\alpha\beta} \right] = T_{\alpha\beta}, \]  

(4)

\[ f(\phi) \left[ G_{X\phi,\beta} + G_{XX\phi,\beta} \phi^{\beta} \right] + df \frac{d}{d\phi} \left[ G - 2XG_X \right] = 0, \]  

(5)

where we work in units such that \( 8\pi G = 1 \) and, as usual, the semicolon means a covariant derivative while \( G_X \) denotes differentiation of \( G(X) \) with respect to \( X \).

The same set of equations (4,5) is obtained if we consider the scalar field \( X(\phi) \) as part of the matter content, i.e. \( L_{X,\phi} = f(\phi)G(X) \) with the corresponding energy-momentum tensor

\[ T_{\alpha\beta} = f(\phi) \left[ G_{X\phi,\alpha\phi,\beta} + G(X)g_{\alpha\beta} \right]. \]  

(6)

Making an analogy with a perfect fluid, we notice that the case \( G(X) = X \) corresponds to a stiff fluid.

On the other hand, the idea of Noncommutativity in spacetime is quite old (1947) [5]. It was proposed as an attempt to regularize Quantum Field Theory before the renormalization program was established. Due to its non-local behavior, Noncommutativity was quickly forgotten after renormalization proved to be successful. In the 1970’s M. Flato and co-workers proposed an alternative path to quantization [6], in which a deformation of the Poisson structure of classical phase space is performed and is encoded in the moyal \( \star \)-product [7] and generalizations of it. In the early 1980’s mathematicians led by A. Connes succeeded in formulating what they called Noncommutative Geometry [8], motivated by generalizing a classic theorem characterizing C*-algebras.

Recently, the noncommutative paradigm has resurrected, mainly due to results in String Theory [9], in which Yang-Mills theories in a noncommutative space arise in different circumstances as effective theories when taking certain simple limits (for instance, the low energy limit). This renewed interest has led to a deeper understanding, from the physical and mathematical points of view, of noncommutative field theory (for a review see [10]). It is believed that in a full quantum theory of gravity the continuum picture of spacetime would no longer be consistent at distances comparable to the Planck length \( \ell_p \sim 10^{-35} \text{ cm} \), a quantization of spacetime itself could be in order. Furthermore, since quite ago, it has been argued [11] that measurements of position can not be performed to better accuracies than the Planck length, since spacetime itself would be modified due to the energy required to perform such measurements.
A possible way to model these effects could be via an uncertainty relation for the spacetime coordinates of the form

$$[x^i, x^j] = i\theta^{ij}$$

(7)

This commutation relation is the starting point of noncommutative field theory. Attempts to implement this idea have led to different proposals for a noncommutative theory of gravity [12]. Different incarnations of noncommutative gravity have a common feature, they are highly non-linear theories, which makes them very difficult to work with.

In the study of cosmological models spacetime is highly symmetric, as a consequence, the gravitational field can be described in terms of a finite number of degrees of freedom. We are not dealing with a field theory anymore. A possible way to study noncommutativity effects in the early universe was proposed by García-Compeán et al [13]. They implemented noncommutativity in configuration space (in contrast to noncommutative spacetime), but only after a symmetry reduction of spacetime had been imposed and the Wheeler-DeWitt quantization had been carried, giving rise to a noncommutative quantum cosmology (mathematically similar to noncommutative quantum mechanics [14]). The idea is that perhaps this effective noncommutativity could incorporate novel effects and insights of a full quantum theory of the gravitational field, alongside with providing a simple framework for studying the implications of possible noncommutativity effects in the (early) universe. Some works along this line have been conducted. For instance, the noncommutativity of the Friedmann-Robertson-Walker cosmology has been studied, as well as some of the Bianchi Class A models [15].

Since when gravity is coupled to matter the purely gravitational sector of minisuperspace carries information about pure spacetime, and it is the spacetime noncommutativity effects which in principle one wants to approximate in an effective way, the natural step would be to incorporate noncommutativity only in the gravitational degrees of freedom of the resulting coupled minisuperspace. In the present work we implement this idea for the Bianchi Type I model, with the coupling of a particularly simple K-essence field and ordinary matter being modelled by a perfect barotropic fluid.

The manuscript is organized as follows: In Section II we focus on recalling the emergence of the Moyal $\star$-product within the Weyl quantization prescription and how it can be applied to Quantum Mechanics. Section III deals with the implementation of noncommutativity via the Moyal $\star$ product in the Minisuperspace of the Bianchi type I at the quantum level. Noncommutative quantum general solutions are found by solving the corresponding noncommutative Wheeler-DeWitt equation. Section IV is devoted to point out some conclusions and perspectives of the work.

2. Deformations in Quantum Mechanics

A noncommutativity in classical and quantum phase space can be achieved by deforming the flat Poisson structure in those spaces. One can encode this classical and quantum phase space noncommutativity in a deformation of the product, the so called $\star$-product. This procedure gives rise to Noncommutative Classical Mechanics and Noncommutative Quantum Mechanics. It also provides a third road to quantization, the deformation quantization program. A noncommutative spacetime can also be encoded with the help of a $\star$-product, giving rise to the study of field theory in noncommutative spacetimes.

2.1. Weyl-Wigner-Moyal Correspondence

Noncommutativity can be introduced through a $\star$-product. As stated above, this kind of noncommutativity has been developed in connection with deformation quantization. A $\star$-product is an associative noncommutative product which encodes the noncommutativity but still permits us to work in the usual corresponding commutative space (phase space, spacetime,
We implement the Noncommutative Quantum Mechanics as a deformation quantization of classical phase space, we will follow a different route than the ordinary Quantum Mechanics obtained above. Since we do not view Quantum Mechanics as a deformation quantization, we will not use the commutator $\{ f, g \} = i \hbar (\partial f / \partial x \partial g / \partial p - \partial f / \partial p \partial g / \partial x)$, but encode this deformed Heisenberg algebra in a star-product completely analogous to (11). The difference lies in the star-product, which in this case would be of the form

$$i \begin{pmatrix} \theta \kappa & \hbar I \\ -\hbar I & \kappa K \end{pmatrix}$$

This deformation quantization would lead us to a Noncommutative Quantum Mechanics, rather than the ordinary Quantum Mechanics obtained above. Since we do not view Quantum Mechanics as a deformation quantization of classical phase space, we will follow a different route.

2.2. Noncommutative Quantum Mechanics

We implement the star-product at the quantum level ($f, g$ in (11) are now defined on quantum phase space). We take the canonical Heisenberg algebra (9) as given, then we deform it to (12). The $\omega^{\mu \nu}$ is in this case of the form

$$i \begin{pmatrix} \theta \kappa & 0 \\ 0 & \kappa K \end{pmatrix}$$

Here, the star-bracket $\hat{f} \ast \hat{g} - \hat{g} \ast \hat{f}$ reduces to the ordinary bracket as the noncommutative parameters $\theta, \kappa$ go to zero (and thus the deformed algebra reduces to the usual quantum Heisenberg algebra),

$$\hat{f}(q, p) = \int f(\xi, \eta) e^{\imath (\hat{p}_a \xi^a + \hat{q}_b \eta^b - \frac{\hbar}{2} \imath \sum_{\mu, \nu} \omega^{\mu \nu} (\partial_{\xi^\mu} \xi^\nu - \partial_{\eta^\mu} \eta^\nu}) d\xi^a d\eta^b$$

where $\hat{f}$ is the Fourier transform of $f$, $w$ a weight function, and $\text{dim}(T^* Q) = 2l$; with $\hat{q}^a, \hat{p}_b$ satisfying the canonical Heisenberg algebra,

$$[\hat{q}^a, \hat{p}_b] = i\hbar \delta^a_b; \quad [\hat{q}^a, \hat{q}^b] = 0 = [\hat{p}_a, \hat{p}_b]$$

We define the star-product with the help of this correspondence by

$$\mathcal{W}(f \ast g) = \mathcal{W}(f) \cdot \mathcal{W}(g)$$

that is, $f \ast g$ is the Weyl symbol of $\hat{f} \hat{g}$. For this case of the Heisenberg algebra (9) in and the symmetric ordering ($w(1)$) we have

$$f \ast g = \exp \left( \frac{i\hbar}{2} P(f, g) \right) = f(x) \exp \left( \frac{i\hbar}{2} \partial_{\xi^\mu} \omega^{\mu \nu} \partial_{\eta^\nu} \right) g(x) = fg + \frac{1}{2} \{ f, g \} + \mathcal{O}(\hbar^2)$$

where $P^r(f, g) = \omega^{\mu_1 \nu_1} \cdots \omega^{\mu_r \nu_r} (\partial_{\xi^{\mu_1} \cdots \xi^{\mu_r}} f)(\partial_{\eta^{\nu_1} \cdots \eta^{\nu_r}} g)$ is the rth power of the Poisson bracket bidifferential operator, with $\omega^{\mu \nu}$ the components of the usual flat Poisson structure; $x^a$ are the phase space coordinates, denoted collectively as $x$, the first $n$ being the configuration coordinates $q^a$, the second $n$ being the momenta $p_a$. This is the Moyal star-product [7], it replaces the ordinary pointwise multiplication in the algebra of functions defined in phase space. The Moyal star-bracket $f \ast g - g \ast f$ is thus responsible for realizing the canonical Heisenberg algebra (9). Hence, (11) encodes a deformation of the classical phase space which yields the canonical Heisenberg algebra. This product is the cornerstone of deformation quantization [6].

Now, consider the deformation of classical phase space which yields the algebra

$$[\hat{q}^a, \hat{p}_b] = i\hbar \delta^a_b; \quad [\hat{q}^a, \hat{q}^b] = i\theta \delta^{ab}; \quad [\hat{p}_a, \hat{p}_b] = i\kappa \delta_{ab}$$

with $\theta^{ab}, \kappa_{ab}$ antisymmetric constant real matrices.

In light of the above prescription, we could encode this deformed Heisenberg algebra in a star-product completely analogous to (11). The difference lies in the $\omega^{\mu \nu}$, which in this case would be of the form

$$i \begin{pmatrix} \theta \kappa & \hbar I \\ -\hbar I & \kappa K \end{pmatrix}$$

This deformation quantization would lead us to a Noncommutative Quantum Mechanics, rather than the ordinary Quantum Mechanics obtained above. Since we do not view Quantum Mechanics as a deformation quantization of classical phase space, we will follow a different route.
in contrast with the ⋆-bracket that arises in the context of deformation quantization. There, the canonical Heisenberg algebra (9), or a deformation of it like (12), is realized due to the Poisson bracket in the leading term in the deformation parameter.

The time evolution for a state ket |ψ⟩ in abstract Hilbert space is given by the equation

\[ i\hbar \frac{\partial}{\partial t} |\psi⟩ = \hat{H} |\psi⟩ \quad (15) \]

Now, in the case in which the noncommutativity enters only in the configuration sector of quantum phase space (\( \kappa = 0 \) in (14)), in the position representation (⟨q|\hat{q}|ψ⟩ = q⟨q|ψ⟩), \( \langle q | \exp \left( \frac{i}{\hbar} a \cdot \hat{p} \right) |ψ⟩ = (q + a|ψ⟩) \); with the abstract Hilbert space realized by the function space \( L^2(\mathbb{R}^n, dq^n) \) and for the simple case in which \( \hat{H} = \frac{1}{2m} \hat{p}^2 + V(q) \), Schrödinger’s equation for the wave function \( ψ(q) = (q|ψ⟩ \) takes the form (the momenta and \( \frac{\partial}{\partial t} \) terms remain the same under the ⋆ operation)

\[ i\hbar \frac{\partial}{\partial t} \psi(q) = -\frac{\hbar}{2m} \sum_{a=1}^{n} \partial^2_{q^a} \psi + V(q) \star \psi \quad (16) \]

so that the information about the deformation enters only through the potential term. The ⋆ operation has the effect of shifting the coordinates in the following way [14]

\[ V(q^1, ..., q^n) \star \psi = V(q^1 - \frac{1}{2} \theta^{1b} p_b, ..., q^n - \frac{1}{2} \theta^{ab} p_a) \psi \quad (17) \]

This shifting enables one to encode the noncommutativity by performing the transformation \( q^a = q - \theta^{ab} p_b \) in Schrödinger’s wave equation equation (16) but with the ordinary product, since the new coordinates \( \hat{q}^a \) satisfy the algebra (12) with \( \kappa = 0 \). This is the kind of Noncommutative Quantum Mechanics that has been mostly studied [14].

3. Deformation of the Bianchi Type I Minisuperspace

3.1. Canonical Formulation

Let us recall here the canonical formulation in the ADM formalism of the diagonal Bianchi Class A models. The metric has the form

\[ ds^2 = -(Ndt)^2 + e^{2\Omega(t)} (e^{2\beta(t)})_{ij} \omega^i \omega^j = -dr^2 + e^{2\Omega(t)} (e^{2\beta(t)})_{ij} \omega^i \omega^j, \quad (18) \]

where \( \Omega(t) \) is a scalar, \( N \) the lapse function and \( \beta_{ij}(t) \) a 3x3 diagonal matrix, \( \beta_i = \text{diag}(\beta_+ + \sqrt{3} \beta_-, \beta_+ - \sqrt{3} \beta_-, -2 \beta_+) \), \( \omega^i \) are one-forms that characterize each cosmological Bianchi type model and obey \( d\omega^i = \frac{f}{2} \epsilon_{ijk} \omega^j \wedge C_{jk} \), \( C_{jk} \) the structure constants of the corresponding invariance group. For the Bianchi type I model we have

\[ \omega^1 = dx^1, \quad \omega^2 = dx^2, \quad \omega^3 = dx^3 \]

The total Lagrangian density then reads

\[ \mathcal{L}_I = e^{3\Omega} \left[ \frac{\dot{\Omega}^2}{N} - \frac{6}{N} \dot{\beta}_+^2 - \frac{6}{N} \dot{\beta}_-^2 + \frac{f(\phi)}{N} \dot{\phi}^2 + 16\pi GN \rho \right], \quad (19) \]

using the standard definition of the momenta, \( p_{\phi} = \frac{\partial \mathcal{L}_I}{\partial \dot{\phi}} \), where \( q^a = (\Omega, \beta_+, \beta_-, \phi) \), we obtain

\[ p_\Omega = \frac{12}{N} e^{3\Omega} \dot{\Omega}, \quad \dot{\Omega} = \frac{N}{12} e^{-3\Omega} p_\Omega \quad (20) \]

\[ p_\beta = - \frac{12}{N} e^{3\Omega} \dot{\beta}_+, \quad \dot{\beta}_+ = - \frac{N}{12} e^{-3\Omega} p_\beta \quad (21) \]

\[ p_\beta = - \frac{12}{N} e^{3\Omega} \dot{\beta}_-, \quad \dot{\beta}_- = - \frac{N}{12} e^{-3\Omega} p_\beta \quad (22) \]

\[ p_\phi = \frac{2f}{N} e^{3\Omega} \dot{\phi}, \quad \dot{\phi} = \frac{N}{2f} e^{-3\Omega} p_\phi \quad (23) \]
and introducing them into the Lagrangian density, we obtain the canonical Lagrangian as
\[ L_{\text{can}} = p_\mu \dot{q}^\mu - N \mathcal{H}, \]

so, the ADM action has the form,
\[ I = \int \left( p_+ d\dot{\beta}_+ + p_- d\dot{\beta}_- + p_\Omega d\dot{\Omega} + p_\phi d\dot{\phi} + N \mathcal{H} dt \right), \]

where
\[ \mathcal{H} = e^{-3\Omega} \left( -p_\Omega^2 - \frac{6}{f(\phi)} p_\phi^2 + p_+^2 + p_-^2 + m_\gamma e^{-3(\gamma-1)\Omega} \right), \]

with \( m_\gamma = 384\pi G \mu_\gamma. \)

3.2. Noncommutative Wheeler-DeWitt Equation
As stated in the introduction, it is believed that the notion of the spacetime continuum breaks down at scales in which the quantum nature of spacetime plays a major role. A possible scenario is a noncommutative spacetime, in which the notion of a point is replaced by that of a Planck cell. We thus impose noncommutativity among the gravitational degrees of freedom only, since they carry information about spacetime itself.

We consider the deformation
\[ [p_i, p_j] = 0, [q_i, p_j] = i \delta_{ij}, [\Omega, \beta_+] = i \theta_1, [\Omega, \beta_-] = i \theta_2, \]
\[ [\beta_+, \beta_-] = i \theta_3, [\Omega, \phi] = [\phi, \beta_+] = [\phi, \beta_-] = 0 \]

In view of (17), the noncommutative Wheeler-DeWitt (NCWDW) equation, corresponding to the deformed algebra (27), is therefore (\( p_i \mapsto i \partial_i \))
\[ e^{-3\Omega_{nc}} \left[ \partial_\Omega^2 - \partial_+^2 - \partial_-^2 + \frac{6}{f(\phi)} \partial_\phi^2 + m_\gamma e^{-3(\gamma-1)\Omega_{nc}} \right] \Psi(\Omega, \beta_\pm, \phi) = 0 \]

where we have set \( G_X = 1 \) and
\[ \Omega_{nc} = \Omega - \frac{\theta_1}{2} p_+ - \frac{\theta_2}{2} p_- \]

Employing the semi-general factor ordering due to Hartle and Hawking [18], the NCWDW equation takes the form
\[ \left[ \partial_\Omega^2 - \partial_+^2 - \partial_-^2 + \frac{6}{f(\phi)} \partial_\phi^2 - \frac{6s}{f(\phi)} \partial_\phi - q \partial_\Omega + m_\gamma e^{-3(\gamma-1)\Omega_{nc}} \right] \Psi = 0 \]

We further assume a solution of the form
\[ \Psi(\Omega, \beta_\pm, \phi) = A(\Omega) B(\beta_\pm) C(\phi) \]

where we take the ansatz \( B = e^{\pm i \alpha \beta_\pm} e^{\pm i \beta \beta_-}. \) Employing the Baker-Campbell-Hausdorff (BCH) relation
\[ \exp^X \exp^Y = \exp \left( Y + [X, Y] + \frac{1}{2} [X, [X, Y]] + \frac{1}{3!} [X, [X, [X, Y]]] + ... \right) \]
we find
\[
\frac{1}{\mathcal{A}} \frac{d^2 \mathcal{A}}{d\Omega^2} - \frac{1}{B} \frac{\partial^2 B}{\partial \beta^2} - \frac{1}{B} \frac{\partial^2 B}{\partial \beta^2} + \frac{6}{\mathcal{F}} \frac{\partial \mathcal{C}}{\partial \phi^2} - \frac{6s}{\mathcal{F}} \frac{\partial \mathcal{C}}{\partial \phi} - \frac{q^2}{\mathcal{A}} \frac{\partial \mathcal{A}}{\partial \Omega} = m_\gamma e^{-3(\gamma - 1)\Omega} e^{-\frac{2}{3}(\gamma - 1)(\alpha_1 + b\theta_2)} = 0 \tag{33}
\]
Separating variables we obtain the following set of ordinary differential equations
\[
\frac{d^2 \mathcal{A}}{d\Omega^2} - q \frac{d \mathcal{A}}{d\Omega} + (\bar{m}_\gamma e^{-3(\gamma - 1)\Omega} - \alpha^2) \mathcal{A} = 0 \tag{34}
\]
\[
\frac{\partial^2 \mathcal{B}}{\partial \beta^2} + \frac{\partial^2 \mathcal{B}}{\partial \beta^2} - \kappa^2 \mathcal{B} = 0 \tag{35}
\]
\[
\frac{d^2 \mathcal{C}}{d\phi^2} + s \frac{d \mathcal{C}}{d\phi} + \frac{\partial^2 \mathcal{C}}{\partial \phi} = 0 \tag{36}
\]
where \(\kappa^2 = \alpha^2 + \beta^2\) and \(\bar{m}_\gamma = m_\gamma e^{-\frac{2}{3}(\gamma - 1)(\alpha_1 + b\theta_2)}\) carries information about the noncommutativity.

Now, equation (34) can be solved in the following way: perform the transformation
\[
z = \sqrt{\frac{m_\gamma}{\eta}} e^{-\frac{2}{3}(\gamma - 1)\Omega} \tag{37}
\]
substituting in (34) and assuming a solution of the form
\[
\mathcal{A}(\Omega) = z^\xi \mathcal{F}(z) \tag{38}
\]
we get
\[
\frac{9}{4}(\gamma - 1)^2 z^\xi \left[ z^2 \frac{d^2 \mathcal{F}}{dz^2} + z \left( 1 + 2\xi q + \frac{2}{3} \frac{q}{\gamma - 1} \right) \frac{d \mathcal{F}}{dz} + \left( \frac{4\eta^2}{9(\gamma - 1)^2} \right) z^2 + q^2 \left( \xi^2 + \frac{2}{3} \frac{\xi}{\gamma - 1} \right) \right] \mathcal{F} = 0,
\]
which can be written as
\[
z^2 \frac{d^2 \mathcal{F}}{dz^2} + z \frac{d \mathcal{F}}{dz} + \left[ z^2 - \frac{1}{9(\gamma - 1)^2} (q^2 + 4\alpha^2) \right] \mathcal{F} = 0, \tag{39}
\]
when \(\eta\) and \(\xi\) are fixed to
\[
\xi = -\frac{1}{3(\gamma - 1)}, \quad \eta = \frac{3}{2} |\gamma - 1| \tag{40}
\]
This is the Bessel differential equation for the function \(\mathcal{F}\).

(i) The solution for \(\mathcal{A}(\Omega)\) for the case \(\gamma \neq 1\) is then
\[
\mathcal{A}_\gamma = \left( \frac{2\sqrt{m_\gamma}}{3|\gamma - 1|} e^{-\frac{3}{2}(\gamma - 1)\Omega} \right)^{-\frac{q}{3(\gamma - 1)}} \left( \frac{2\sqrt{m_\gamma}}{3|\gamma - 1|} e^{-\frac{3}{2}(\gamma - 1)\Omega} \right) + d_\gamma Y_\nu \left( \frac{2\sqrt{m_\gamma}}{3|\gamma - 1|} e^{-\frac{3}{2}(\gamma - 1)\Omega} \right), \tag{41}
\]
where \(\nu = \pm \sqrt{\frac{1}{9(\gamma - 1)^2} (q^2 + 4\alpha^2)}\) and \(J_\nu\) and \(Y_\nu\) are the ordinary Bessel functions of the first and second kind, respectively.

(ii) For the case \(\gamma = 1\) equation (34) takes the form
\[
\frac{d^2 \mathcal{A}_1}{d\Omega^2} - q \frac{d \mathcal{A}_1}{d\Omega} + \alpha_1^2 \mathcal{A}_1 = 0, \quad \alpha_1^2 = m_1 - \alpha^2, \tag{42}
\]
whose solution is
\[
\mathcal{A}_1 = A_1 e^{\frac{q + \sqrt{q^2 - 4\alpha_1^2}}{2} \Omega} + A_2 e^{\frac{q - \sqrt{q^2 - 4\alpha_1^2}}{2} \Omega} \tag{43}
\]
Finally, to obtain a solution for equation (36) we need to fix \( f(\phi) \), we take
\[
f(\phi) = \omega e^{m\phi}
\] (44)
in the original proposal of Sáez-Ballester \( f(\phi) = \omega \phi^m \). The solution for (36) is
\[
\mathcal{C}(\phi) = e^{-\frac{\phi}{2}} c\Gamma(1 - \nu) J_{-\nu} \left( \left( \frac{\sqrt{\frac{2}{3}} \sqrt{\beta^2 \omega e^{m\phi}}}{m} \right) \right) + d\Gamma(1 + \nu) J_{\nu} \left( \left( \frac{\sqrt{\frac{2}{3}} \sqrt{\beta^2 \omega e^{m\phi}}} {m} \right) \right)
\] (45)
where \( \nu = \frac{z}{2} \) is the order of the Bessel functions and \( \Gamma(z) \) is Euler’s gamma function.

As expected, this noncommutative quantum solutions reduce to the commutative ones (some of them are reported in Ref. [19]) as the noncommutative parameters \( \theta_i \) go to zero.

4. Final Remarks
As can be seen from the relations (41),(43) and Ref. [19], the noncommutativity does not modify the form of the solutions, the noncommutative parameters contribute only through a multiplicative constant in the term \( \tilde{m}_\phi = e^{-\frac{z}{2}(\gamma - 1)(a\theta_1 + b\theta_2)} \) which takes part in the \( z \) term and in the argument of the Bessel functions in (41). A similar situation occurs regarding solution (43). Therefore, the effect of this particular noncommutativity is to modulate the amplitude of the original commutative wave function of the Bianchi type I universe. We recover the original commutative solution in the limiting case when the noncommutative parameter \( \theta \) vanishes.

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