A prolongation of the conformal-Killing operator on quaternionic-Kähler manifolds

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Abstract: A 2-form on a quaternionic-Kähler manifold \((M, g)\) of dimension \(4n \geq 8\) is called compatible (with the quaternionic structure) if it is a section of the direct sum bundle \(S^2H \oplus S^2E\). We construct a prolongation \(\mathcal{D}\) of the conformal-Killing operator acting on compatible 2-forms. We show that \(\mathcal{D}\) is flat if and only if the quaternionic-Weyl tensor of \((M, g)\) vanishes. Consequences of this result are developed. We construct a skew-symmetric multiplication on the space of conformal-Killing 2-forms on \((M, g)\) and we study its properties in connection with the subspace of compatible conformal-Killing 2-forms.

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1 Introduction

Let \((M^n, g)\) be a Riemannian \(m\)-dimensional manifold. For any \(1 \leq p \leq m\) consider the tensor product bundle \(T^*M \otimes \Lambda^p(M)\) and its irreducible \(O(m)\)-decomposition:

\[
T^*M \otimes \Lambda^p(M) = \Lambda^{p+1}(M) \oplus \Lambda^{p-1}(M) \oplus \mathcal{T}^{p,1}(M),
\]

where the sub-bundle \(\mathcal{T}^{p,1}(M)\) of \(T^*M \otimes \Lambda^p(M)\) is the intersection of the kernels of the wedge product and inner contraction maps. If \(\psi \in \Omega^p(M)\)
is a $p$-form, the covariant derivative $\nabla \psi$ with respect to the Levi-Civita connection $\nabla$ of $g$ is a section of $T^*M \otimes \Lambda^p(M)$ and its projection onto $\Lambda^{p+1}(M)$ and $\Lambda^{p-1}(M)$ is given, essentially, by the exterior derivative $d\psi$ and the codifferential $\delta\psi$, respectively. The $p$-form $\psi$ is called conformal-Killing if the projection of $\nabla \psi$ onto the third component in the decomposition (1) is trivial; equivalently, if the conformal-Killing equation

$$\nabla_Y \psi = \frac{1}{p+1} i_Y d\psi - \frac{1}{m-p+1} Y \wedge \delta\psi, \quad \forall Y \in TM$$

(2)

is satisfied. (Above and often in this note we identify, without mentioning explicitly, tangent vectors with 1-forms by means of the Riemannian duality). A co-closed conformal-Killing form is called Killing. There is a large literature on conformal-Killing forms (for a survey, see e.g. [15]). A 1-form is (conformal)-Killing if the dual vector field is a (conformal)-Killing vector field. Conformal-Killing forms exist on spaces of constant curvature, on Sasaki manifolds, on some classes of Kähler manifolds (like Bochner-flat or conformally-Einstein Kähler manifolds) and on Riemannian manifolds which admit Killing spinors [15]. On a compact quaternionic-Kähler manifold of dimension at least eight, any Killing $p$-form ($p \geq 2$) is parallel (see [13]) and any conformal-Killing 2-form parallel unless the quaternionic-Kähler manifold is isometric to the standard quaternionic projective space, in which case the codifferential defines an isomorphism from the space of conformal-Killing 2-forms to the space of Killing vector fields (see [8]).

This paper is concerned with conformal-Killing 2-forms on (not necessarily compact) quaternionic-Kähler manifolds. In Section 2 devoted to fix notations, we recall basic facts from quaternionic-Kähler geometry.

Section 3 contains the main result of our paper (see Theorem 2). A 2-form on a quaternionic-Kähler manifold $(M, g)$ (always assumed to be connected and of dimension $4n \geq 8$) is called compatible (with the quaternionic structure) if it is a section of the direct sum bundle $S^2H \oplus S^2E$. We find a connection $\mathcal{D}$ on the bundle $S^2H \oplus S^2E \oplus TM$, which is a prolongation of the conformal-Killing operator acting on compatible 2-forms (i.e. the space of $\mathcal{D}$-parallel sections is isomorphic to the space $C_2(M)$ of compatible conformal-Killing 2-forms on $(M, g)$). We compute the curvature of $\mathcal{D}$ and we show that $\mathcal{D}$ is flat if and only if the quaternionic-Weyl tensor $W^Q$ of $(M, g)$ is zero.

Section 4 is devoted to applications of Theorem 2. First, we prove that the dimension of the vector space $C_2(M)$ is less or equal to $(n+1)(2n+3)$ and equality holds on the standard quaternionic projective space $\mathbb{H}P^n$ (see Corollary 7). Next, we assume that the quaternionic-Kähler manifold $(M, g)$ has non-zero scalar curvature and, under this additional assumption, we prove that if $(M, g)$ admits a non-parallel compatible conformal-Killing 2-form then

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the holonomy group of \((M, g)\) coincides with \(\text{Sp}(1)\text{Sp}(n)\) (see Proposition 8). In particular, there are no non-parallel compatible conformal-Killing 2-forms on open subsets of Wolf spaces non-isomorphic to the standard quaternionic projective space. At the end of this section we prove that a compatible conformal-Killing 2-form \(\psi\) on a quaternionic-Kähler manifold \((M, g)\) of non-zero scalar curvature is determined by its \(S^2E\)-part \(\psi^{S2E}\) and we find the image and the inverse of the map \(C_2(M) \ni \psi \rightarrow \psi^{S2E}\) (see Proposition 10).

Finally, in Section 5 we construct a skew-symmetric multiplication \([\cdot, \cdot]\) on the space of conformal-Killing 2-forms on a quaternionic-Kähler manifold \((M, g)\), which preserves the subspace \(C_2(M)\) of compatible conformal-Killing 2-forms. When the scalar curvature of \((M, g)\) is non-zero, \((C_2(M), [\cdot, \cdot])\) is a Lie algebra and the codifferential \(\delta : C_2(M) \rightarrow \text{isom}(M, g)\) is a Lie algebra homomorphism (see Corollary 12).

2 Quaternionic-Kähler manifolds

In this Section we recall basic definitions and results from quaternionic-Kähler geometry, which will be useful in our treatment of conformal-Killing forms.

2.1 Basic definitions

A quaternionic-Kähler manifold is a Riemannian manifold \((M, g)\) of dimension \(4n \geq 8\) with holonomy group included in \(\text{Sp}(1)\text{Sp}(n)\). Equivalently, there is a rank three vector sub-bundle \(Q \subset \text{End}(TM)\) preserved by the Levi-Civita connection \(\nabla\) of \(g\) and locally generated by three anti-commuting almost complex structures \(\{J_1, J_2, J_3\}\) with \(J_3 = J_1J_2\). Such a system of almost complex structures is usually called a local admissible basis of \(Q\). The metric \(g\) is Einstein; moreover, \(g\) is Ricci-flat if and only if \((M, g)\) is locally hyper-Kähler, i.e. in a neighborhood of any point there is an admissible basis of \(Q\) formed by \(\nabla\)-parallel complex structures.

The curvature of \(g\) has the expression

\[
R^g_{X,Y} = -\frac{\nu}{4} \left( X \wedge Y + \sum_{i=1}^{3} J_iX \wedge J_iY + 2 \sum_{i=1}^{3} \omega_i(X,Y) \omega_i \right) + W^Q_{X,Y} \quad (3)
\]

where \(\nu := \frac{k}{4n(n+2)}\) is the reduced scalar curvature (\(k\) being the usual scalar curvature), \(\{J_1, J_2, J_3\}\) is a local admissible basis of \(Q\), with Kähler forms \(\omega_i := g(J_i\cdot, \cdot)\), and \(W^Q\) is the quaternionic-Weyl tensor, which satisfies

\[
W^Q_{JX,JY} = W^Q_{X,Y}, \quad \forall X, Y \in TM, \quad \forall J \in Q, \quad J^2 = -\text{Id}
\]
and belongs to the kernel of the Ricci contraction, i.e.

$$\sum_{k=1}^{4n} W^Q_{X,e_k}(e_k) = 0, \quad \forall X \in TM$$

where \( \{e_k\} \) is a local orthonormal frame of \( TM \) (above "Id" denotes the identity endomorphism).

Let \( E \) and \( H \) be the (locally defined) complex vector bundles over \( M \), of rank \( 2n \) and \( 2 \) respectively, associated to the standard representations of \( \text{Sp}(n) \) and \( \text{Sp}(1) \) on \( \mathbb{C}^{2n} \) and \( \mathbb{C}^{2} \). The bundles \( E \) and \( H \) play the role of spin bundles in conformal geometry, since

$$T\mathbb{C}M = E \otimes H.$$  \hspace{1cm} (4)

Let \( \omega_E \in \Lambda^2(E^*) \) and \( \omega_H \in \Lambda^2(H^*) \) be the complex symplectic forms on \( E \) and \( H \), defined by the standard \( \text{Sp}(n) \)-invariant symplectic form of \( \mathbb{C}^{2n} \), and, respectively, the standard \( \text{Sp}(1) \)-invariant symplectic form of \( \mathbb{C}^{2} \). We shall identify \( E \) with \( E^* \) by means of the isomorphism \( E \ni e \rightarrow \omega_E(e, \cdot) \), and, similarly, \( H \) with \( H^* \) using \( \omega_H \). At the level of 2-forms, (4) induces a decomposition

$$\Lambda^2(T\mathbb{C}M) = S^2H \oplus S^2E \oplus S^2H \otimes \Lambda^2_0E,$$  \hspace{1cm} (5)

where \( \Lambda^2_0E \subset \Lambda^2E \) is the kernel of the natural contraction with \( \omega_E \). The bundles \( S^2H \) and \( S^2E \) are the complexifications of the bundle \( Q \) and, respectively, of the bundle of \( Q \)-Hermitian 2-forms (a 2-form is \( Q \)-Hermitian if it is of type \((1,1)\) with respect to any compatible almost complex structure; an almost complex structure is compatible if it is a section of the bundle \( Q \)).

We shall often identify, implicitly, real bundles with their complexifications. The projections of a 2-form \( \psi \in \Omega^2(M) \) on \( S^2H \) and \( S^2E \) have the following expressions:

$$\psi^{S^2H} = \frac{1}{4n} \sum_{i,k} \psi(e_k, J_i e_k) \omega_i$$  \hspace{1cm} (6)

$$\psi^{S^2E} = \frac{1}{4} \left( \psi + \sum_{i=1}^{3} \psi(J_i, J_i^c) \right),$$  \hspace{1cm} (7)

where \( \{J_1, J_2, J_3\} \) is an admissible basis of \( Q \), with associated Kähler forms \( \omega_i \), and \( \{e_k\} \) is a local orthonormal frame of \( TM \).
2.2 The Penrose operator

Penrose (or twistor) operators appear in the literature on several classes of manifolds. On a conformal 4-manifold a Weyl connection defines a Penrose operator and Penrose operators obtained in this way were studied in [9], [11]. Similarly, any quaternionic connection on a quaternionic manifold defines a Penrose operator (see [7]). An important role in our treatment of compatible conformal-Kähler 2-forms will be played by the Penrose operator on a quaternionic-Kähler manifold \((M,g)\), defined using the Levi-Civita connection (see [14], also [3]). It is a first order differential operator

\[
\bar{D} : \Gamma(S^2H) \to \Gamma(S^3H \otimes E)
\]

whose value on a section \(\sigma\) of \(S^2H\) is obtained by taking the covariant derivative \(\nabla \sigma\) (where \(\nabla\) is the Levi-Civita connection), which is a section of \(T^*_cM \otimes S^2H\) and projecting it onto the second component of the irreducible decomposition

\[
T^*_cM \otimes S^2H \cong E \otimes H \oplus E \otimes S^3H. \tag{8}
\]

Let \(\{J_1, J_2, J_3\}\) be a local admissible basis of \(Q\), with Kähler forms \(\omega_1, \omega_2\) and \(\omega_3\) respectively. According to Section 5 of [3], for any section \(\sigma\) of \(Q\),

\[
\bar{D}\sigma = \nabla \sigma + \frac{1}{3} \sum_{i=1}^{3} (\delta \sigma) \circ J_i \otimes \omega_i. \tag{9}
\]

The (real) solutions of the twistor equation (i.e. sections of the quaternionic bundle \(Q\) which belong to the kernel of \(\bar{D}\)) can be easily described when the reduced scalar curvature \(\nu \neq 0\). In this case the map

\[
isom(M,g) \ni X \to \frac{2}{3\nu}(\nabla X)_{S^2H} \in \text{Ker} \bar{D} \tag{10}
\]

is an isomorphism, with inverse the codifferential (see [3], [14]).

2.3 Killing and quaternionic vector fields

There is a useful criterion to check when a vector field on a quaternionic-Kähler manifold of non-zero scalar curvature is Killing (see [1]). We shall use this criterion in Section 4. It is stated as follows.

Recall first that a vector field, say \(X\), on a quaternionic-Kähler manifold \((M^{4n}, g)\) (with \(4n \geq 8\)) is quaternionic, if its flow preserves the quaternionic bundle \(Q\), or, equivalently, \([\nabla X, Q] \subset Q\). The criterion states that a vector field which is quaternionic and divergence-free is necessarily Killing. This
is a consequence the theory developed in [1] and [2]. For completeness
of our exposition, we include the argument, which goes as follows. Let $X$ be a
quaternionic vector field on $(M,g)$. From [1], page 303, and [2], $L_X \nabla = S^\alpha$,
where $\alpha \in \Omega^1(M)$ is defined by
$$\alpha = df, \quad f := \frac{1}{4(n+1)} \text{Trace}(\nabla X).$$
and $S^\alpha$ is a 1-form with values in $\text{End}(TM)$ defined by
$$S^\alpha_Z := \alpha(Z) \text{Id}_{TM} + \alpha \otimes Z - \sum_{i=1}^3 (\alpha(J_iZ)J_i + (\alpha \circ J_i) \otimes J_iZ), \quad \forall Z \in TM. \quad (11)$$
Now, if $X$ is also divergence-free, then $\alpha = 0$ and $L_X \nabla = 0$. This implies that
$X$ preserves the curvature of $g$ (viewed as a 2-form with values in $\text{End}(TM)$)
and also the Ricci tensor (viewed as a bilinear form on $TM$). Since $g$ is
Einstein with non-zero scalar curvature, $X$ is Killing.

Notations 1. Let $(M,g)$ be a quaternionic-Kähler manifold of dimension
$4n \geq 8$. We shall use the scalar product $\langle \cdot, \cdot \rangle$ on $\Lambda^2(M)$, defined by
$$\langle X \wedge Y, Z \wedge V \rangle = g(X,Z)g(Y,V) - g(X,V)g(Y,Z). \quad (12)$$
With respect to the scalar product (12), the Kähler forms $\{\omega_1, \omega_2, \omega_3\}$ cor-
responding to a local admissible basis $\{J_1, J_2, J_3\}$ of the quaternionic bundle
$Q$ are orthogonal and
$$|\omega_1|^2 = |\omega_2|^2 = |\omega_3|^2 = 2n.$$ 
In our conventions, $\nabla$ will always denote the Levi-Civita connection of $g$.

3 Our main result

Given a linear differential operator $D$, it is sometimes useful to determine a
vector bundle connection (called a prolongation of $D$) whose space of parallel
sections is isomorphic with the kernel of $D$. In general, there are several
connections with this property. However, if one prolongation is flat, then all
are.

This Section contains the main result of this paper - we determine a pr o-
longation $D$ of the conformal-Killing operator acting on compatible 2-forms
on a quaternionic-Kähler manifold $(M,g)$ and we show that $D$ is flat if and
only if the quaternionic-Weyl tensor $W^Q$ of $(M,g)$ is zero. The prolongation
$D$ acts on the direct sum bundle $S^2H \oplus S^2E \oplus TM$. More precisely, we state:
Theorem 2. Let \((M, g)\) be a quaternionic-Kähler manifold of dimension \(4n \geq 8\), reduced scalar curvature \(\nu\) and quaternionic-Weyl tensor \(W^Q\). Define a connection \(D\) on \(S^2H \oplus S^2E \oplus TM\), by

\[
D_Z(\psi, X)^{S^2E \oplus S^2H} = \nabla_Z \psi - \frac{1}{4n - 1} \left( X \wedge Z + \sum_{i=1}^{3} J_iX \wedge J_iZ - \sum_{i=1}^{3} \omega_i(X, Z)\omega_i \right),
\]

\[
D_Z(\psi, X)^{TM} = \nabla_Z X - \frac{4n - 1}{4} i_Z \left( \nu \psi S^2E - 2\nu \psi S^2H + \frac{1}{n + 1} W^Q(\psi) \right),
\]

where \(\{J_1, J_2, J_3\}\) is a local admissible basis of the quaternionic bundle \(Q\), with Kähler forms \(\omega_1, \omega_2, \omega_3\), \(\psi\) is a section of \(S^2H \oplus S^2E\) and \(X, Z\) are vector fields on \(M\). Then \(D\) is a prolongation of the conformal-Killing operator acting on compatible 2-forms. Moreover, \(D\) is flat if and only if \(W^Q = 0\).

We divide the proof of Theorem 2 into three steps. In a first stage, we rewrite the conformal-Killing equation on compatible 2-forms in a way suitable for the prolongation procedure (see Proposition 3). We remark that Proposition 3 has already been proved in [8] in the compact case. We now adapt the argument also to the case when the quaternionic-Kähler manifold is non-compact. In a second stage, we show that the connection \(D\) from Theorem 2 is a prolongation of the conformal-Killing operator acting on compatible 2-forms (see Proposition 4). Finally, we compute the curvature of \(D\) and we show that \(D\) is flat if and only if \(W^Q = 0\) (see Proposition 5). Details are as follows.

Proposition 3. A compatible 2-form \(\psi\) on \((M, g)\) is conformal-Killing if and only if it satisfies

\[
\nabla_Y \psi = \frac{1}{4n - 1} \left( X \wedge Y + \sum_{i=1}^{3} J_iX \wedge J_iY - \sum_{i=1}^{3} \omega_i(X, Y)\omega_i \right), \quad \forall Y \in TM,
\]

where \(\{J_1, J_2, J_3\}\) is a local admissible basis of the quaternionic bundle \(Q\), \(\omega_1, \omega_2, \omega_3\) are the associated Kähler forms and \(X\) is a vector field (necessarily equal to \(\delta \psi\)). In particular, \(\psi S^2H\) satisfies the twistor equation.

Proof. Let \(\psi\) be a compatible conformal-Killing 2-form on \((M, g)\) and \(X = \delta \psi\) its codifferential. Hence \(\psi\) satisfies

\[
\nabla_Y \psi = \frac{1}{3} i_Y d\psi + \frac{1}{4n - 1} X \wedge Y, \quad \forall Y \in TM.
\]
Projecting equation (14) onto $S^2 H$ we obtain

$$\nabla_Y \psi^{S^2 H} = \frac{1}{3} (i Y d \psi)^{S^2 H} + \frac{1}{2n(4n - 1)} \sum_{i=1}^{3} \omega_i (X, Y) \omega_i. \quad (15)$$

Let $\{e_k\}$ be a local orthonormal frame of $TM$. To simplify notations, sometimes we omit below the summation sign over $1 \leq k \leq 4$. Note that

$$(i Y d \psi)^{S^2 H} = \frac{1}{4n} \sum_{i=1}^{3} (d \psi)(Y, e_k, J_i e_k) \omega_i$$

$$= \frac{1}{4n} \sum_{i=1}^{3} \left( (\nabla_Y \psi)(e_k, J_i e_k) - (\nabla_{e_k} \psi)(Y, J_i e_k) + (\nabla_{J_i e_k} \psi)(Y, e_k) \right) \omega_i$$

$$= \nabla_Y \psi^{S^2 H} + \frac{1}{4n} \sum_{i=1}^{3} \left( (\nabla_{J_i e_k} \psi)(Y, e_k) - (\nabla_{e_k} \psi)(Y, J_i e_k) \right) \omega_i.$$

Define

$$E(\psi, Y) := \frac{1}{4n} \sum_{i=1}^{3} \left( (\nabla_{J_i e_k} \psi)(Y, e_k) - (\nabla_{e_k} \psi)(Y, J_i e_k) \right) \omega_i. \quad (16)$$

With this notation, equation (15) becomes

$$\nabla_Y \psi^{S^2 H} = \frac{1}{2} E(\psi, Y) + \frac{3}{4n(4n - 1)} \sum_{i=1}^{3} \omega_i (X, Y) \omega_i. \quad (17)$$

We now compute $E(\psi, Y)$. It is easy to check, using that $\psi^{S^2 E}$ is of type $(1, 1)$ with respect to any compatible complex structure, that

$$E(\psi^{S^2 E}, Y) = \frac{1}{2n} \sum_{i=1}^{3} (\delta \psi^{S^2 E})(J_i Y) \omega_i. \quad (18)$$

On the other hand, for any fixed $i \in \{1, 2, 3\}$,

$$\sum_{k=1}^{4n} \left( (\nabla_{J_i e_k} \psi^{S^2 H})(Y, e_k) - (\nabla_{e_k} \psi^{S^2 H})(Y, J_i e_k) \right)$$

$$= \sum_{k=1}^{4n} \left( (\nabla_{J_i e_k} \psi, (Y \wedge e_k)^{S^2 H}) - (\nabla_{e_k} \psi, (Y \wedge J_i e_k)^{S^2 H}) \right)$$

$$= \frac{1}{n} \sum_{j=1}^{3} (\nabla_{J_i J_j Y} \psi^{S^2 H}, \omega_j).$$
where we used (6). From this and (16) it follows that
\[ E(\psi^{S^2H}, Y) = -\frac{1}{4n^2} \sum_{i,j} \langle \nabla_{J_i Y} \psi^{S^2H}, \omega_j \rangle \omega_i - \frac{1}{n} \nabla_Y \psi^{S^2H}. \] (19)

From (18) and (19), relation (17) becomes
\[ \nabla_Y \psi^{S^2H} = \sum_{i=1}^{3} \alpha(J_i Y) \omega_i \] (20)
where the 1-form \( \alpha \in \Omega^1(M) \) is given by\[
\alpha(Y) := \frac{1}{2(2n+1)} \left( (\delta \psi^{S^2E})(Y) - \frac{3}{4n-1} g(X, Y) - \frac{1}{2n} \sum_j \langle \nabla_{J_j Y} \psi, \omega_j \rangle \right) .
\] (21)

From (20) combined with (21) it is easy to see that \( \alpha = \frac{1}{4n-1} X, \quad \delta \psi^{S^2H} = -\frac{3}{4n-1} X, \quad \delta \psi^{S^2E} = \frac{4n+2}{4n-1} X \) (22)
and thus \( \psi^{S^2H} \) satisfies the twistor equation
\[ \nabla_Y \psi^{S^2H} + \frac{1}{4n-1} \sum_{i=1}^{3} \omega_i(X, Y) \omega_i = 0. \] (23)

Using now (23), an argument like in Lemma 5 of [8] shows that
\[ d\psi = -\frac{3}{4n-1} \left( J_1 X \wedge \omega_1 + J_2 X \wedge \omega_2 + J_3 X \wedge \omega_3 \right) . \] (24)

Substituting (24) into the conformal-Killing equation (14) we get (13), as required. Conversely, it is clear that any solution \( \psi \) of (13) is a conformal-Killing 2-form and \( \delta \psi = X \).

We remark that Proposition 3 implies that any compatible Killing 2-form on \((M, g)\) is parallel, a result previously proved, in the compact case (and also for higher degree Killing forms), in [13].

**Proposition 4.** The map \( \psi \to (\psi, \delta \psi) \)

is an isomorphism from the vector space \( C_2(M) \) of compatible conformal-Killing 2-forms to the vector space of \( D \)-parallel sections (where \( D \) is the connection from Theorem 2).
Proof. Let \( \psi \) be a compatible conformal-Killing 2-form and \( X := \delta \psi \) its codifferential. Since \( g \) is Einstein, \( X \) is a Killing vector field (see [15]). When \( \nu \neq 0 \), \((M, g)\) is irreducible (see Theorem 14.45 of [6]) and thus \( \nabla X \) is a section of \( S^2H \oplus S^2E \), because \( X \) is Killing (see [12], page 246). More generally, when \( \nu \) is arbitrary, a Weitzenböck argument (see [15]) shows that
\[
\frac{2}{3} \Delta \psi - q(R) \psi + \frac{4(n-1)}{3(4n-1)} dX = 0 \tag{25}
\]
where \( \Delta = d\delta + \delta d \) is the Laplace operator and \( q(R) \) is a bundle endomorphism of \( \Lambda^2(M) \), related to \( \Delta \) by \( \Delta = \nabla^* \nabla + q(R) \), where \( \nabla^* \nabla = -\sum_{k=1}^{4n} \nabla^2(\psi)(e_k, e_k) \) and \( \{e_k\} \) is a local orthonormal frame of \( TM \). Both \( \Delta \) and \( q(R) \) preserve the irreducible sub-bundles of \( \Lambda^2(M) \) and \( q(R) \) acts on these sub-bundles by scalar multiplication (see [16], Lemma 2.5). Projecting (25) onto \( S^2H \otimes \Lambda^2_0 E \) and using that \( \psi \) a section of \( S^2H \oplus S^2E \), we get that \( \nabla X \) is a section of \( S^2H \oplus S^2E \), for any \( \nu \).

After these preliminary remarks, we now prove that \((\psi, X)\) is \( D \)-parallel, as follows. From Proposition 3, we know that
\[
[\nabla_X^2, \psi] + \sum_{i=1}^{3} J_i \nabla_X Z \wedge J_i Y - \sum_{i=1}^{3} \omega_i(\nabla_X Y) \omega_i = 0, \forall Z \in TM.
\]
Applying this relation to a vector \( U \), taking the trace over \( Y \) and \( U \) and applying the result to a vector \( V \) we obtain:
\[
12(n+1)(\nabla X)(Z, V) - 4(n+2)(\nabla X)^{S^2H}(Z, V) = (4n-1)g([J_i \nabla_X, \psi](e_k), V) \tag{27}
\]
where we used that \( \nabla X \) is a section of \( S^2H \oplus S^2E \). For any fixed \( i \in \{1, 2, 3\} \), replace in (27) the pair \((Z, V)\) with \((J_i Z, J_i V)\) and sum over \( i \). We obtain:
\[
12(n+1)(\nabla X)(Z, V) - 4(3n+2)(\nabla X)^{S^2H}(Z, V) = (4n-1) \sum_{i=1}^{3} g(J_i [R_{Z, e_k}^\theta, \psi](e_k), V).
\]
Combining this relation with (27) we get
\[ \nabla_Z X = \frac{4n - 1}{16(n + 1)} \left( (3n + 2)[R^g_{Z,e_k}, \psi](e_k) + (n + 2) \sum_{i=1}^{3} J_i [R^g_{J_i Z,e_k}, \psi](e_k) \right). \]

We now compute the right hand side of this expression. We first notice that
\[ (3n + 2)[R^g_{Z,e_k}, \psi^{S^2E}](e_k) + (n + 2) \sum_{i=1}^{3} J_i [R^g_{J_i Z,e_k}, \psi^{S^2E}](e_k) = -4 [R^g_{Z,e_k}, \psi^{S^2E}](e_k). \]

On the other hand, from (3),
\[ [R^g_{Z,e_k}, \psi^{S^2E}](e_k) = -\nu(n + 1) i_Z \psi^{S^2E} + [W^Q_{Z,e_k}, \psi^{S^2E}](e_k). \]
Moreover,
\[ [W^Q_{Z,e_k}, \psi^{S^2E}](e_k) = W^Q_{Z,e_k} (\psi^{S^2E}(e_k)) = -i_Z W^Q(\psi^{S^2E}), \]
where in the first equality (29) we used that \( W^Q \) is Ricci-flat and the second equality (29) follows from the following argument: for any \( Y, Z \in TM \),
\[ g \left( W^Q_{Z,e_k}(\psi^{S^2E}(e_k)), Y \right) = -\psi^{S^2E}(e_k, W^Q_{Z,e_k}(Y)) = -\langle \psi^{S^2E}, e_k \wedge W^Q_{Z,e_k}(Y) \rangle = -\langle \psi^{S^2E}, W^Q_{Z,Y} \rangle = -g(i_Z W^Q(\psi^{S^2E}), Y), \]
for any vector fields \( Y \) and \( Z \). Therefore,
\[ (3n + 2)[R^g_{Z,e_k}, \psi^{S^2E}](e_k) + (n + 2) \sum_{i=1}^{3} J_i [R^g_{J_i Z,e_k}, \psi^{S^2E}](e_k) = 4 \left( \nu(n + 1) i_Z \psi^{S^2E} + i_Z W^Q(\psi) \right). \]

Similarly, we can prove that
\[ (3n + 2)[R^g_{Z,e_k}, \psi^{S^2H}](e_k) + (n + 2) \sum_{i=1}^{3} J_i [R^g_{J_i Z,e_k}, \psi^{S^2H}](e_k) = -8\nu(n + 1) i_Z \psi^{S^2H}. \]

We finally obtain
\[ \nabla_Z X = \frac{4n - 1}{4} i_Z \psi^{S^2E} + \frac{4n - 1}{4(n + 1)} i_Z W^Q(\psi) - \frac{\nu(4n - 1)}{2} i_Z \psi^{S^2H}, \]
which is equivalent to (26). Our claim follows. \( \square \)
In order to conclude the proof of Theorem 2, we still need to compute the curvature of the connection $D$ and to show that $D$ is flat if and only if $W^Q = 0$. This is done in the following proposition.

**Proposition 5.** The curvature $R^D$ of the connection $D$ defined in Theorem 2 has the following expression: for any section $(\psi, X)$ of $S^2H \oplus S^2E \oplus TM$ and vector fields $Y, Z \in \mathcal{X}(M),$

\[
R^D_{Y,Z}(\psi, X)_{S^2H\oplus S^2E} = [W^Q_{Y,Z}, \psi] - \frac{1}{n+1} (W^Q(\psi) \wedge \mathrm{Id})_{Y,Z}^{S^2E} \\
R^D_{Y,Z}(\psi, X)_{TM} = \frac{n+2}{n+1} W^Q_{Y,Z} X + \frac{4n-1}{4(n+1)} C(\psi^{S^2E})_{Y,Z},
\]

where

\[
(W^Q(\psi) \wedge \mathrm{Id})_{Y,Z}^{S^2E} := (i_Y W^Q(\psi) \wedge Z - i_Z W^Q(\psi) \wedge Y)^{S^2E} \tag{31}
\]

and

\[
C(\psi^{S^2E})_{Y,Z} := i_Y (\nabla_Z W^Q(\psi^{S^2E})) - i_Z (\nabla_Y W^Q(\psi^{S^2E})). \tag{32}
\]

In particular, $D$ is flat if and only if $W^Q = 0$.

**Proof.** The $S^2H \oplus S^2E$ component of $R^D$ can be computed as follows. It is straightforward to check that

\[
R^D_{Y,Z}(\psi, X)_{S^2H\oplus S^2E} = [R^q_{Y,Z}, \psi] - \nu \left( i_Y (\psi^{S^2E}) \wedge Z - i_Z (\psi^{S^2E}) \wedge Y \right)^{S^2E} \\
- \frac{1}{n+1} \left( i_Y W^Q(\psi) \wedge Z - i_Z W^Q(\psi) \wedge Y \right)^{S^2E} \\
+ 2\nu \left( i_Y (\psi^{S^2H}) \wedge Z - i_Z (\psi^{S^2H}) \wedge Y \right)^{S^2E} \\
- \frac{\nu}{2} \sum_{i=1}^3 \langle \omega_i, i_Y (\psi^{S^2H}) \wedge Z - i_Z (\psi^{S^2H}) \wedge Y \rangle \omega_i.
\]

On the other hand, the following equalities hold: for any vector fields $Y, Z \in \mathcal{X}(M),$

\[
[R^q_{Y,Z}, \psi^{S^2H}] = \frac{\nu}{2} \sum_{i=1}^3 \langle \omega_i, i_Y (\psi^{S^2H}) \wedge Z - i_Z (\psi^{S^2H}) \wedge Y \rangle \omega_i; \\
[R^q_{Y,Z}, \psi^{S^2E}] = \nu \left( i_Y (\psi^{S^2E}) \wedge Z - i_Z (\psi^{S^2E}) \wedge Y \right)^{S^2E} + [W^Q_{Y,Z}, \psi^{S^2E}]; \\
\left( i_Y (\psi^{S^2H}) \wedge Z - i_Z (\psi^{S^2H}) \wedge Y \right)^{S^2E} = 0.
\]

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From these relations we get
\[ R^D_{Y,Z}(\psi, X)^{S^2H\oplus S^2E} = [W^Q_{Y,Z}, \psi] - \frac{1}{n+1} (W^Q(\psi) \wedge \text{Id})^{S^2E}_{Y,Z} \]
as required. The $TM$ component of $R^D$ can be computed in a similar way. It is obvious now that $D$ is flat if and only if $W^Q = 0$. Our claim follows.

The proof of Theorem 2 is now completed. We end this Section with the following result, which is a consequence of the curvature computation from Proposition 3 and will be used in the proof of Proposition 8 from the next section. We remark that relations similar to (33) and (34) hold also in the Kähler setting, with $u$ replaced by (the trace-free part of) a Hamiltonian 2-form and the quaternionic-Weyl tensor $W^Q$ replaced by the Bochner tensor of the Kähler manifold (see Proposition 9 of [10]).

**Proposition 6.** Let $(M, g)$ be a quaternionic-Kähler manifold of dimension $4n \geq 8$. Let $\psi$ be a compatible conformal-Killing 2-form on $(M, g)$, $u := \psi^{S^2E}$ its $S^2E$-part and $X := \delta \psi$. Then

\[ W^Q_{Y,X} = \frac{4n - 1}{4(n+2)}(\nabla_V W^Q)(u) = \frac{4n - 1}{4(n+1)} \nabla_V (W^Q(u)) , \quad \forall V \in TM. \tag{33} \]

Moreover,

\[ [W^Q(u), v] = (n+1)[W^Q(v), u], \quad \forall v \in S^2E \tag{34} \]

and

\[ [W^Q(u), u] = [(\nabla X)^{S^2E}, u] = 0. \tag{35} \]

**Proof.** From Propositions 4 and 5, $(\psi, X)$ is $D$-parallel and

\[ (n+2)W^Q_{Y,Z}X + (n - \frac{1}{4})C(u)_{Y,Z} = 0, \quad \forall Y, Z \in TM. \tag{36} \]

On the other hand, from the definition of the tensor $C$, for any $Y, Z, V \in TM$,

\[ C(u)_{Y,Z}(V) = \langle (\nabla_Z R^\theta)(u), Y \wedge V \rangle - \langle (\nabla_Y R^\theta)(u), Z \wedge V \rangle \]
\[ = \langle (\nabla_Z R^\theta)(Y \wedge V), u \rangle - \langle (\nabla_Y R^\theta)(Z \wedge V), u \rangle \]
\[ = \langle (\nabla_V R^\theta)(Y \wedge Z), u \rangle \]
since $\nabla_Z R^\theta \in \text{End}(\Lambda^2(M))$ is symmetric and $R^\theta$ satisfies the second Bianchi identity. Relation (36) becomes

\[ (n+2)g(W^Q_{Y,Z}X, V) + (n - \frac{1}{4})\langle (\nabla_V R^\theta)(Y \wedge Z), u \rangle = 0 \tag{37} \]
and implies the first relation (33), because both $W^Q$ and $\nabla_V R^g$ are symmetric endomorphisms of $\Lambda^2 M$. The second relation (33) follows from the first, by using

$$\nabla_V W^Q(u) = \nabla_V \left( W^Q(u) \right) - W^Q(\nabla_V u) = \nabla_V \left( W^Q(u) \right) - \frac{4}{4n - 1} W^Q_{X,V},$$

because

$$\nabla_V u = \frac{1}{4n - 1} \left( X \wedge V + \sum_{i=1}^{3} J_i X \wedge J_i V \right)$$

and $W^Q$ is $J_i$-invariant (as an End$(TM)$-valued form). Relation (33) is proved. In order to prove relation (34) note first that for any $v \in S^2 E$,

$$(W^Q(u) \wedge \text{Id})^{S^2 E}(v) = [W^Q(u), v] \quad (38)$$

(relation (38) is obtained by writing $v = \frac{1}{2} \sum_k e_k \wedge v(e_k)$ with respect to a local orthonormal frame $\{e_k\}$ of $TM$ and using the definition of $(W^Q(u) \wedge \text{Id})^{S^2 E}$ and the $J_i$-invariance of $W^Q(u)$ and $v$). Relation (34) follows now from (38), together with

$$[W^Q_{Y,Z}, u] = \frac{1}{n + 1} (W^Q(u) \wedge \text{Id})^{S^2 E}_{Y,Z}, \quad \forall Y, Z \in TM$$

(see Proposition 5). It remains to prove (35). This is a consequence of (34) and

$$(\nabla X)^{S^2 E} = \frac{(4n - 1)\nu}{4} u + \frac{4n - 1}{4(n + 1)} W^Q(u),$$

(which follows from $D(\psi, X) = 0$).

\[\square\]

### 4 Applications of our main result

In this Section we develop several application of Theorem 2.

#### 4.1 The dimension of the space $C_2(M)$

As a first application of Theorem 2 we determine a sharp estimate for the dimension of the vector space $C_2(M)$ of compatible conformal-Killing 2-forms on a quaternionic-Kähler manifold $(M, g)$. It is known that on an arbitrary Riemannian manifold (not necessarily compact) the space of conformal-Killing forms (of any degree) is finite-dimensional and an upper bound, which is realized on the standard sphere, was found in [15]. For compatible conformal-Killing 2-forms on quaternionic-Kähler manifolds there is the following similar result:
Corollary 7. Let \((M, g)\) be a quaternionic-Kähler manifold of dimension \(4n \geq 8\). Then
\[
\dim C_2(M) \leq (n + 1)(2n + 3).
\]
Equality holds on the standard \(\mathbb{H}P^n\).

Proof. Notice that \((n + 1)(2n + 3)\) is the rank of the bundle \(S^2H \oplus S^2E \oplus TM\) on which the connection \(\mathcal{D}\) is defined. Therefore, inequality (39) follows from Theorem 2. It remains to show that equality holds on the quaternionic projective space \(\mathbb{H}P^n\), with its standard quaternionic-Kähler structure. As proved in [8], any conformal-Killing 2-form on \(\mathbb{H}P^n\) is compatible and the co-differential \(\delta\) defines an isomorphism from \(C_2(\mathbb{H}P^n)\) onto the space of Killing vector fields. The latter has dimension \((n + 1)(2n + 3)\). Our claim follows.

4.2 The case when \(\nu \neq 0\)

Let \((M, g)\) be quaternionic Kähler manifold (as usual, connected and of dimension \(4n \geq 8\)). In this section we assume that the scalar curvature of \((M, g)\) is non-zero. Our main results in this setting are Propositions 8 and 10 (see below).

Proposition 8. If \((M, g)\) has a non-parallel compatible conformal-Killing 2-form, then the holonomy group of \((M, g)\) is \(\text{Sp}(1)\text{Sp}(n)\).

Proof. Let \(\psi\) be a non-parallel compatible conformal-Killing 2-form on \((M, g)\). We will show that the holonomy algebra \(\text{hol}(M, g)\) of \((M, g)\) coincides with \(\text{sp}(1) \oplus \text{sp}(n)\). From Proposition 34 \(X := \delta\psi\) is non-trivial. Recall that \(\text{hol}(M, g)\) contains \((\nabla_Y R^g)_{Y,Z}\) for any \(Y, Z, V \in TM\) (see Chapter 10 of [6]). Therefore, relation (33) implies that \(W^Q_{X,V} \in \text{hol}(M, g)\) for any \(V\). Using this fact, the proof of our claim follows like in Lemma 17 of [8]. For completeness of our exposition, we include the argument. Since \(R^g_{Y,Z}\) belongs to the holonomy algebra as well, and \(\text{sp}(1) \subset \text{hol}(M, g)\) (since the scalar curvature is non-zero, see Lemma 14.46 of [6]), we deduce that \((R^g_{Y,Z})^{S^2E}\) belongs to \(\text{hol}(M, g)\), for any \(Y, Z \in TM\). It follows that
\[
(R^g_{X,V})^{S^2E} - W^Q_{X,V} = -\nu(X \wedge V)^{S^2E} \in \text{hol}(M, g), \quad \forall V
\]
and
\[
(R^g_{J_1X,V})^{S^2E} - W^Q_{J_1X,V} = -\nu(J_1X \wedge V)^{S^2E} \in \text{hol}(M, g), \quad \forall V,
\]
where \(\{J_1, J_2, J_3\}\) is a local admissible basis of \(Q\). We just proved that if \(Y\) or \(U\) belong to \(V := \text{Span}\{X, J_1X, J_2X, J_3X\}\), then \((Y \wedge U)^{S^2E}\) belongs
to hol(M, g). It remains to show that \((Y \wedge U)^{S^2E}\) belongs to the holonomy algebra when both \(Y\) and \(U\) are orthogonal to \(V\). Take such two tangent vectors \(Y\) and \(U\). Since both \((X \wedge Y)^{S^2E}\) and \((X \wedge U)^{S^2E}\) belong to hol(M, g), also their Lie bracket, which is equal to

\[
[(X \wedge Y)^{S^2E}, (X \wedge U)^{S^2E}] = \frac{1}{16} \sum_{i,j=1}^{3} g(J_i Y, J_j U) J_i X \wedge J_j X + \frac{1}{4} g(X, X)(Y \wedge U)^{S^2E},
\]

belongs to hol(M, g), as well as the \(S^2E\)-part of this Lie bracket. Using (40) and (41) we get that \((Y \wedge U)^{S^2E} \in \text{hol}(M, g)\). Our claim follows.

**Corollary 9.** One of the following two statements holds:

i) either any compatible conformal-Killing 2-form on \((M, g)\) is parallel;

ii) or any parallel 2-form is trivial. In particular, the codifferential

\[
\delta : C_2(M) \to \text{isom}(M, g)
\]

is injective.

**Proof.** Suppose that \((M, g)\) admits a non-parallel compatible conformal-Killing 2-form. Then, from Proposition 8, the holonomy group of \((M, g)\) is \(\text{Sp}(1)\text{Sp}(n)\) and therefore \((M, g)\) does not admit non-trivial parallel 2-forms (see e.g. [6], page 306). From Proposition 3 any compatible Killing 2-form is parallel. Thus, the codifferential \(\delta\) defined on \(C_2(M)\) is injective.

**Proposition 10.** The map

\[
C_2(M) \ni \psi \to u := \psi^{S^2E}
\]

is an isomorphism from the vector space \(C_2(M)\) of compatible conformal-Killing 2-forms on \((M, g)\) to the vector space of (real) sections of \(S^2E\) which satisfy

\[
\nabla_Y u = \frac{1}{4n - 1} \left( X \wedge Y + \sum_{i=1}^{3} J_i X \wedge J_i Y \right) \quad \forall Y \in TM,
\]

where \(X \in \mathcal{X}(M)\) is a vector field on \(M\) (necessarily equal to \(\frac{4n - 1}{4n + 2} \delta u\)) and \(\{J_1, J_2, J_3\}\) is an admissible basis of \(Q\). The inverse is the map

\[
u \to \psi := u - \frac{1}{(2n + 1)\nu} (\nabla \delta u)^{S^2H}.
\]
Proof. Let $\psi$ be a compatible conformal-Killing 2-form and $u := \psi^{S^2E}$ its $S^2E$-component. Projecting (13) onto $S^2E$ we obtain (43). Moreover, since $\nu \neq 0$,

$$\psi^{S^2H} = -\frac{1}{(2n+1)\nu}(\nabla \delta u)^{S^2H}$$

because both sides are solutions of the twistor equation, with equal codifferentials (recall relation (22) and our comments from Section 2.2). It follows that

$$\psi = u - \frac{1}{(2n+1)\nu}(\nabla \delta u)^{S^2H}.$$  \hspace{1cm} (45)

It remains to show that the map (42) is onto the space of solutions of (43), i.e. any section $u$ of $S^2E$, which is a solution of (43), is the $S^2E$-part of a compatible conformal-Killing 2-form. For this, let $u$ be a solution of (43), where $X$ is a vector field (necessarily equal to $\frac{4n-1}{2(2n+1)}\delta u$). We will show that $X$ is Killing, or, equivalently, $X$ is quaternionic (being divergence-free and $\nu \neq 0$, see Section 2.2). Taking the covariant derivative of (43) with respect to $Z$ and skew symmetrizing in $Y$ and $Z$ we obtain

$$[R^g_{Z,Y},u] = 4\left(\nabla_Z X \wedge Y - \nabla_Y X \wedge Z\right)^{S^2E}$$  \hspace{1cm} (46)

On the other hand, since $u$ is a section of $S^2E$,

$$[R^g_{Z,Y},u] = [(R^g_{Z,Y})^{S^2E},u] = [R^g_{J_1Z,J_1Y},u]$$  \hspace{1cm} (47)

so the right hand side of (46) remains unchanged if we replace $(Z,Y)$ by $(J_1Z,J_1Y)$. Defining

$$S(Z,X) := [\nabla X, J_1](Z) = \nabla_{J_1Z}X - J_1\nabla_Z X, \hspace{0.5cm} \forall Z \in \mathcal{X}(M)$$

we get:

$$S(Z,X) \wedge J_1Y - J_1S(Z,X) \wedge Y - J_2S(Z,X) \wedge J_3Y + J_3S(Z,X) \wedge J_2Y - S(Y,X) \wedge J_1Z + J_1S(Y,X) \wedge Z + J_2S(Y,X) \wedge J_3Z - J_3S(Y,X) \wedge J_2Z = 0.$$  

Applying this relation to a vector $U$ and taking the trace over $Y$ and $U$ we obtain

$$S(Z,X) = \frac{1}{4n} \sum_{k=1}^{4n} (g(S(e_k,X),J_2e_k)J_2Z + g(S(e_k,X),J_3e_k)J_3Z), \hspace{0.5cm} \forall Z \in TM,$$  \hspace{1cm} (48)
where \( \{e_k\} \) is a local orthonormal frame of \( TM \). Relation (13) implies that \( X \) is quaternionic. Being quaternionic and divergence-free (and \( \nu \neq 0 \)), \( X \) is Killing. Using the Koszul formula for \( X \) and (13), it can be checked that
\[
\psi := u - \frac{2}{\nu(4n-1)}(\nabla X)^{S^2H}
\]
satisfies (13) and hence is conformal-Killing. Obviously, \( \psi \) is a compatible 2-form and its \( S^2E \)-part coincides with \( u \). Our claim follows.

### 5 A bracket on conformal-Killing 2-forms

In this final Section we define a skew-symmetric multiplication on the space of conformal-Killing 2-forms on a quaternionic-Kähler manifold and we study its properties in relation with the subspace of compatible conformal-Killing 2-forms. Such a multiplication can be defined in the more general setting of Einstein manifolds, as follows.

**Proposition 11.** If \( \psi_1 \) and \( \psi_2 \) are conformal-Killing 2-forms on an Einstein manifold, then
\[
[\psi_1, \psi_2] := \frac{1}{2} (L_{\delta \psi_1} \psi_2 - L_{\delta \psi_2} \psi_1)
\]
is also conformal-Killing and
\[
\delta [\psi_1, \psi_2] = [\delta \psi_1, \delta \psi_2].
\]

**Proof.** It is easy to check that the Lie derivative of a conformal-Killing form \( \psi \) (of any degree) with respect to a Killing vector field \( X \) is also conformal-Killing, with codifferential \( L_X (\delta \psi) \). Since on an Einstein manifold the codifferential of a conformal-Killing 2-form is a Killing vector field (see [15]), the bracket \([\psi_1, \psi_2]\) of two conformal-Killing 2-forms \( \psi_1 \) and \( \psi_2 \), as defined in (49), is also conformal-Killing. Relation (50) is straightforward.

Since any quaternionic-Kähler manifold is Einstein, Proposition 11 implies that (49) is a skew-symmetric multiplication on the space of conformal-Killing 2-forms on any quaternionic-Kähler manifold.

**Corollary 12.** i) The bracket (49) preserves the subspace \( \mathcal{C}_2(M) \) of compatible conformal-Killings 2-forms on a quaternionic-Kähler manifold \((M,g)\).
ii) Assume that \((M, g)\) is not Ricci-flat. Then \((C_2(M), [\cdot, \cdot])\) is a Lie algebra and the codifferential
\[
\delta : C_2(M) \rightarrow \text{isom}(M, g)
\]
is a Lie algebra homomorphism.

**Proof.** Let \(\psi_1\) and \(\psi_2\) be two compatible conformal-Killing 2-forms, with codifferentials \(X_1\) and \(X_2\), which are Killing vector fields. From Proposition 11 we know that \([\psi_1, \psi_2]\) is conformal-Killing and we need to show that it is a section of \(S^2H \oplus S^2E\). As already mentioned before, \(\nabla X_i\) \((1 \leq i \leq 2)\) are sections of \(S^2H \oplus S^2E\). It follows that
\[
L_{X_1}\psi_2 = \nabla_{X_1}\psi_2 - [\nabla X_1, \psi_2]
\]
is a section of \(S^2H \oplus S^2E\). A similar argument shows that \(L_{X_2}\psi_1\) is also a section of \(S^2H \oplus S^2E\). Thus \([\psi_1, \psi_2]\) is a compatible conformal-Killing 2-form. This proves the first claim. For the second claim, let \(\psi_1, \psi_2, \psi_3 \in C_2(M)\) and define
\[
\psi := [[\psi_1, \psi_2], \psi_3] + [[\psi_3, \psi_1], \psi_2] + [[\psi_2, \psi_3], \psi_1].
\]
From Proposition 11, \(\psi\) is a Killing 2-form. Being compatible, it is parallel. Recall now, from Corollary 9 that on a quaternionic-Kähler manifold with non-zero scalar curvature, either any parallel 2-form is trivial or any compatible conformal-Killing 2-form is parallel. In both cases, \(\psi = 0\) (note that the bracket \([\psi_i, \psi_j]\) is zero when both \(\psi_i\) and \(\psi_j\) are parallel). Thus \((C_2(M), [\cdot, \cdot])\) is a Lie algebra. From Proposition 11 the map (51) is a Lie algebra homomorphism.

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