Quantum anomaly in a quasi-two-dimensional strongly interacting Fermi gas

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(Dated: June 13, 2018)

A harmonically trapped strongly interacting Fermi gas in two dimensions is anticipated to exhibit a quantum anomaly and possesses a breathing mode at frequencies different from a classical scale invariant value \(\omega_B = 2\omega_\perp\), where \(\omega_\perp\) is the frequency of the trapping potential. This quantum anomaly, however, has never been conclusively confirmed in current experiments [E. Vogt \emph{et al.}, Phys. Rev. Lett. 108, 070404 (2012); Holten \emph{et al.}, arXiv:1805.08879; Peppler \emph{et al.}, arXiv:1804.05102]. Here, we theoretically investigate the universal density equation of state and the breathing mode frequency of a unitary Fermi gas at the dimensional crossover from three to two dimensions. We find that the simple model of a two-dimensional strongly interacting Fermi gas with a single s-wave scattering length is not adequate to describe the experiments, as commonly believed. A more complete description of quasi-two-dimensions leads to a much weaker quantum anomaly in the breathing mode frequency, consistent with the experimental observations.

In strongly interacting quantum many-body systems, scale invariance can lead to non-trivial consequences. An intriguing example is a three-dimensional (3D) unitary Fermi gas with an infinitely large s-wave scattering length \(a_{3D} = \pm \infty\) \cite{1}. At zero energy, the free space eigenstates of a unitary Fermi gas have a scale-invariant form, i.e., under a rescaling of the spatial coordinates \(X \rightarrow X/\lambda\), the scaled wave functions satisfy \(\psi(\lambda X) = \lambda^{-\nu}\psi(X)\) for any scaling factor \(\lambda > 0\). In the presence of an isotropic harmonic trap of frequency \(\omega_\perp\), a set of trap eigenstates can then be constructed from zero-energy states in free space \cite{1}, whose spectrum form a ladder with a step of \(2\hbar \omega_\perp\), indicating the existence of a well-defined quasi-particle (i.e., breathing mode) even in the strongly correlated regime. This non-trivial exact mode can be understood from a hidden \(SO(2,1)\) symmetry in the problem \cite{2}.

Classically, a two-dimensional (2D) atomic gas interacting through a contact interaction is also scale invariant. The hidden \(SO(2,1)\) symmetry under an isotropic trap (of frequency \(\omega_\perp\)) would similarly lead to an exact breathing mode with frequency \(\omega_B = 2\omega_\perp\), for both bosons and fermions \cite{2}. Quantum mechanically, however, the contact interaction needs renormalization and the bare interaction strength should be replaced by a regularized 2D s-wave scattering length \(a_{2D}\) \cite{3}. As a result of this new length scale, scale invariance of 2D quantum gases explicitly breaks down \cite{3} and the breathing mode frequency should depend on \(a_{2D}\). In a 2D weakly interacting Bose gas, the quantum anomaly is too weak to be observed \cite{3,4}. For an interacting 2D Fermi gas, the predicted quantum anomaly, i.e., \(\delta \omega_B/(2\omega_\perp)\), is significant and can reach approximately 10% in the strongly interacting crossover regime at zero temperature \cite{3} \cite{10}, as shown in the inset of Fig. 1 as a function of \(\ln(k_F a_{2D})\), where \(k_F\) is the Fermi wave vector at the trap center. This interesting prediction, unfortunately, has never been conclusively confirmed experimentally. The first experiment measured an anomaly of less than 1% at temperature \(0.42T_F\) \cite{11}, where \(T_F\) is the Fermi temperature. While the discrepancy may be understood as a temperature effect \cite{12,13}, two most recent measurements at Heidelberg University \cite{14} and Swinburne University \cite{15} reported consistently quantum anomaly of about 1.3% and 2.5%, respectively, at temperature as low as \(\sim 0.1T_F\). The large discrepancy of the measurements compared to the predicted anomaly is rather surprising. The purpose of this Letter is to show that the puzzle can be resolved by taking into account the quasi-2D nature of the experimental setup of a strongly interacting Fermi

FIG. 1. (color online). The column density \(n_{2D} = \int dz n(z)\) of an ideal Fermi gas at the dimensional crossover from 2D to 3D. Here, \(a_s = \sqrt{\hbar/(M\omega_\perp)}\). The vertical blue dashed line shows the boundary between the 2D and quasi-2D regimes. The inset shows the predicted breathing mode frequencies of a 2D Fermi gas at \(T = 0\) using QMC EoS (dashed line) \cite{8} and GPF EoS (crosses), and at \(T = 0.42T_F\) using virial expansion (solid line) \cite{13}, compared with the experimental data by Vogt \emph{et al.} (stars, \(0.42T_F\)) \cite{11}, Holten \emph{et al.} (squares, \(0.1T_F\) to \(18.18T_F\)) \cite{14} and Peppler \emph{et al.} (circles, \(0.14 - 0.22T_F\)) \cite{15}.
gas at the dimensional crossover from 3D to 2D, under a tight axial confinement with a trap aspect ratio \( \lambda = \omega_z / \omega_r \sim 250 - 300 \) \([14, 15]\). As \( \lambda \) is extremely large, it is widely accepted that the gas will enter the true 2D regime when the number of atoms \( N \) is sufficiently small, so that only the ground single-particle state in the axial direction is occupied \([16]\). For an ideal Fermi gas, this requires \( N < N_{2D} = \lambda^2 \) or equivalently a chemical potential \( \mu < 1.5 \hbar \omega_z \) (see Fig. 1) \([16]\).

Theoretically, the understanding of a strongly interacting Fermi gas at the dimensional crossover is a highly non-trivial challenge, even at the mean-field level, due to both infrared and ultraviolet divergences at low and high energies, respectively \([17]\). The first mean-field description of dimensional crossover was developed by Martikainen and Törnä in the 2D limit \([17]\), by taking three or four single-particle levels along the axial direction. This mean-field approach was later extended by Fischer and Parish \([18]\), by assuming that Cooper pairs condense into the lowest axial state for their center-of-mass motion.

In this work, we completely solve the zero-temperature dimensional crossover problem at the mean-field level, without any additional assumptions. Furthermore, we take into account strong Gaussian pair fluctuations (GPF) on top of the mean-field solutions and therefore quantitatively determine the equation of state (EoS) and the breathing mode frequency on \( \Delta \propto \frac{\mu}{D} \). The first mean-field description of dimensional crossover was achieved by working out the vertex function \([17]\) and \([18]\), by assuming that Cooper pairs condense into the lowest axial state for their center-of-mass motion.

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Universal EoS at the dimensional crossover. — Using the mean-field theory or GPF theory, we calculate the column density $n_{2D} = n_{2D}^{(MF)}$ or $n_{2D} = n_{2D}^{(MF)} + n_{2D}^{(GPF)}$ at a given local chemical potential $\mu \equiv \mu(\rho)$. Focusing on the unitary limit where $a_{3D} \to \pm \infty$, the zero-temperature results are shown in Fig. 2 by black solid lines. This unitary limit is of particular interest, as the length scale $a_{3D}$ in the interatomic interaction disappears and the system therefore should exhibit universal thermodynamics. In our case, we can express $n_{2D}$ as a function of $\mu/(\hbar \omega_z)$ only and the predicted universal EoS in Fig. 2 could be experimentally determined by a single-shot measurement of the column density at the lowest attainable temperature.

In the 3D limit, where a number of singe-particle levels in the axial direction are occupied, we may use the LDA to handle the axial trap $M \omega_z^2 z^2/2$. This gives rise to 

$$n_{2D} (\mu \gg \hbar \omega_z) = \frac{1}{2\pi^2 3\xi^2} \left( \frac{M \mu^2}{\hbar^3 \omega_z} \right),$$

where $\xi$ is the so-called Bertsch parameter. The mean-field and GPF theories predict $\xi_{BCS} \approx 0.59$ and $\xi_{GPF} \approx 0.4$, respectively. The latter is very close to the latest experimental value $\xi_{exp} = 0.376(5)$ [34]. In the opposite 2D limit, if we use a simple 2D model with contact interactions [26, 30], the mean-field theory provides a simple EoS, $n_{2D}(\mu \to \mu_m) = M(\mu - \mu_m)/(\pi \hbar)$ [26], where $\mu_m = (\hbar \omega_z - \epsilon_B)/2$ is the minimum chemical potential allowed, due to the existence of a two-body bound state with binding energy $\epsilon_B = \hbar^2/(M a_{2D}^2)$ [34, 38]. More accurate EoS in the 2D limit could be obtained using numerically exact quantum Monte Carlo (QMC) simulations [36, 39] or the approximate GPF theory [26], as illustrated in the inset of Fig. 2(b). The relative difference between QMC and GPF results is small (i.e., less than 15%), suggesting that the GPF theory is quantitatively reliable also in the 2D limit [40].

In Fig. 2 we show the anticipated EoSs in the 3D and 2D limits using blue dot-dashed lines and red dashed lines, respectively. Our predicted EoS at the dimensional crossover (black curves), from both mean-field and GPF theories, lies in between and seems to smoothly connect the two limits. However, a close examination of the 2D limit shows that the anticipated 2D EoS cannot fully account for the predicted quasi-2D EoS.

This is clearly seen from the mean-field EoS. In the inset of Fig. 2(a), we highlight the density EoS near the 2D limit. Although the predicted mean-field EoS $n_{2D}^{(MF)}$ shows the expected linear dependence on $M(\mu - \mu_m)/\hbar$, the slope of the curve is significantly larger than $1/\pi$ from the simple 2D model. Therefore, it is evident that the 2D model with a single scattering length $a_{2D}$ or binding energy $\epsilon_B$ fails to adequately describe the EoS near the 2D limit. A hint for this failure actually was already observed in the first measurement of the ground state EoS of an interacting 2D Fermi gas [41], where the definition of $a_{2D}$ should be modified to explain the discrepancy between the experimental data and the QMC prediction. A new effective 2D model Hamiltonian therefore has to be introduced, with additional terms accounting for the enhanced slope in the quasi-2D EoS in the strongly interacting regime. As we shall see below, the breakdown of the simple 2D model is responsible for the much reduced quantum anomaly in the breathing mode frequency.

Breathing mode frequency. — In the strongly interacting regime, the breathing mode can be well-described by a hydrodynamic theory [42], which has been successfully applied to predict a large variety of collective oscillations in both Fermi and Bose gases [43, 44]. Here, it is convenient to use the well-documented sum-rule approach [43], which leads to

$$\hbar^2 \omega_B^2 = -2 \langle \rho^2 \rangle \left[ \frac{d \langle \rho^2 \rangle}{d \langle \omega_B^2 \rangle} \right]^{-1},$$

where $\langle \rho^2 \rangle = N^{-1} \int d^2 \rho |\rho|^2 n_{2D}(\rho)$ is the squared radius.
of the Fermi cloud and the chemical potential $\mu_\mathfrak{F}$ in the local chemical potential $\mu(\rho)$ should be adjusted to satisfy the number equation, $N = \int d^2\rho n_{2D}(\rho)$. We note that, the breathing mode frequency evaluated using the sum-rule approach is exact when the density EoS takes a polytropic form, i.e., $\mu(n_{2D}) \propto (n_{2D})^\gamma$. In that case, the density profile is easy to determine within LDA and one finds $\omega_B/\omega_\perp = \sqrt{2 + 2\gamma}$. \cite{43}.

For a quasi-2D unitary Fermi gas in the 3D limit, the density EoS is precisely described by a polytropic form with $\gamma = 1/2$, as given in Eq. \ref{eq:3}, and we obtain $\omega_B = \sqrt{3}\omega_\perp$. \cite{45,46}. On the contrary, in the 2D limit the mean field theory predicts a classical EoS $\mu(n_{2D}) - \mu_0 = \pi n_{2D}/M$ with $\gamma = 1$, and hence we recover the scale-invariant result $\omega_B = 2\omega_\perp$. Quantum fluctuations upshift the breathing mode frequency and lead to the quantum anomaly \cite{48}.

At the dimensional crossover, we report in Fig. \ref{fig:3} the breathing mode frequency of a unitary Fermi gas as a function of $N/N_{2D}$, calculated using both the mean-field (red dashed line) and GPF theories (black solid line), and compare with the recent measurements at Heidelberg \cite{14} and Swinburne \cite{15}. For comparison, we also show the result obtained using the QMC EoS of the simple 2D model \cite{39} (brown dot-dashed line). The mode frequency found by our quasi-2D calculations and measured by experiments exhibits a strong dependence on $N/N_{2D}$, in sharp contrast to the pure 2D prediction. In particular, the anticipated 2D behavior, i.e., the $\sim 10\%$ upshift of the mode frequency, is already washed out at a small number of atoms $N/N_{2D} \sim 0.2$. At this smallest experimentally attainable number, the observed $1\%$ upshift can be well explained by our GPF results. As the number of atoms increase, we find that the data from the Heidelberg group follows continuously the GPF prediction; however, the measurement at Swinburne agrees better with the mean-field result. This difference may be interpreted as a temperature effect: the temperature in Swinburne’s experiment is consistently larger than that in Heidelberg. It is interesting to note that a similar temperature effect was observed in the first measurement of the breathing mode of a 3D interacting Fermi gas near the unitary limit \cite{47,48}, where the measured mode frequency is better explained using mean-field \cite{49}, instead of the more accurate QMC result \cite{50}.

In Fig. \ref{fig:4} we present the breathing mode frequency...
away from the unitary limit at different values of $a_2/a_{3D}$, calculated using the mean-field (a) and GPF theories (b). The mode frequency is less sensitive on the number of atoms $N$. The mode frequency increases with increasing $N$. This is consistent with our interpretation in terms of finite-temperature effect. In particular, the data at $a_2/a_{3D} = -2.12$ can be well explained by using the EoS of an ideal Fermi gas at the dimensional crossover, indicating that the weak interaction effect on the EoS may be compensated by the finite temperature effect [51].

Conclusions. — In summary, we have developed a strong-coupling theory for an interacting Fermi gas at the dimensional crossover from 3D to 2D. It is surprising to find that in the 2D limit, the quasi-2D Fermi gas can not be adequately described by a simple 2D model, as widely accepted. As a result, the anticipated quantum anomaly in the breathing mode frequency, predicted using the simple 2D model, can not be experimentally observed. This provides a natural explanation for the puzzling small dimensional crossover from 3D to 2D. It is surprising to find that in the 2D limit, the quasi-2D Fermi gas at the dimensional crossover may provide valuable information to clarify the possible existence of the Berezinskii-Kosterlitz-Thouless transition in quasi-2D [28, 52, 54].

We thank Paul Dyke for useful discussions and for sharing the experimental data. Our research was supported by Australian Research Council’s (ARC) Programs FT130100815 and DP170104008 (HH), FT140100003 and DP180102018 (XJL), and the Thousand Young Talent Program of China (LH), and the National Natural Science Foundation of China, Grant No. 11775123 (LH).

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[29] The inhomogeneous BCS Green functions of fermions can be constructed from the quasi-particle wave functions and energies and takes the following form: $G_{11}(\mathcal{X} = (k, i\omega_n)); z, z') = \sum_{\mathbf{q} k} \left[u_{\mathbf{q} k}(z)\bar{u}_{\mathbf{q} k}(z')/(i\omega_n - E_{\mathbf{q} k}) + v_{\mathbf{q} k}(z)v_{\mathbf{q} k}(z')/(i\omega_n + E_{\mathbf{q} k})\right]$, $G_{12}(\mathcal{X} = (k, i\omega_n)); z, z') = \sum_{\mathbf{q} k} \left[u_{\mathbf{q} k}(z)v_{\mathbf{q} k}(z')/(i\omega_n - E_{\mathbf{q} k}) - v_{\mathbf{q} k}(z)\bar{u}_{\mathbf{q} k}(z')/(i\omega_n + E_{\mathbf{q} k})\right]$, $G_{21}(\mathcal{X} = (k, i\omega_n)); z, z') = G_{12}(-\mathcal{X} = (-k, i\omega_n)); z, z')$, and $G_{22}(\mathcal{X} = (k, i\omega_n)); z, z') = G_{11}(-\mathcal{X} = (-k, i\omega_n)); z, z')$. Here, $\omega_n = (\pi + 1/2)\pi k_B T$ with integer $m = 0, 1, \pm 2, \cdots$ is the fermionic Matsubara frequency. The matrix elements of the inverse vertex function $\Gamma^{-1}(\Delta = \{q, i\eta\})$, are found by expanding the effective action of Cooper pairs at the Gaussian level [23, 26] and assuming a pair fluctuation field in the form of $\Delta(z)\varphi(\rho, \tau)$, we obtain $M_{11}(\mathcal{Q}) = -\int dz |\Delta(z)|^2/|U + \sum_{\mathbf{q} k} \int dz dz'G_{22}(\mathcal{X} = (k, i\omega_n)); z, z') G_{11}(\mathcal{X} = (k, i\omega_n)); z, z') \Delta(z)\Delta(z')$, where $\sum_{\mathbf{q} k} \equiv k_B T \sum_{\mathbf{q} k} \Delta(z)$ is a short-hand notation, and $M_{12}(\mathcal{Q}) = \sum_{z, z'} \int dz dz'G_{12}(\mathcal{X} = (k, i\omega_n)); z, z') G_{22}(\mathcal{X} = (k, i\omega_n)); z, z') \Delta(z)\Delta(z')$. In the calculations of $M_{11}$ and $M_{12}$, the summation over the fermionic Matsubara frequency $i\omega_n$ can be carried out analytically, leaving a six-dimensional integral (two for the energy index $\eta$, two for the momentum $k$ and two for the spatial coordinates $z$ and $z'$) that has to be handled numerically.
[30] In this way, we reduce the six-dimensional integral in the calculations of $M_{11}$ and $M_{12}$ to a three-dimensional integral, as one summation over the channel index $\eta$ is removed and the integration over $z$ and $z'$ is eliminated. This makes the numerical calculation tractable. [31] T.-L. Ho, Phys. Rev. Lett. 92, 090402 (2004).
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For a homogeneous unitary Fermi gas at zero temperature, the universal thermodynamics states that $\mu_{\text{hom}} = \xi\hbar^2(3\pi^2 n_{\text{hom}})^{2/3}/(2M)$, where $n_{\text{hom}}$ is the 3D density.

By assuming a local chemical potential for the axial harmonic trapping potential $\mu(z) = \mu - M\omega_z^2 z^2/2$, then, we may invert $\mu(z) = \xi\hbar^2(3\pi^2 n(z))^{2/3}/(2M)$ to obtain $n(z) = [2M\mu(z)]^{3/2}/(3\pi^2\xi^{3/2}\hbar^3) = [2M\mu]^{3/2}(1 - z^2/z_T^2)^{3/2}/(3\pi^2\xi^{3/2}\hbar^3)$, where $z_T = \sqrt{2\mu/(M\omega_z^2)}$.

By integrating over the $z$-axis, we obtain the column density $n_{2D} = \int_{-z_T}^{z_T} dz n(z) = M\mu/(2\pi^2\xi^3/2\hbar^3\omega_z)$.

The two-body bound state in a quasi-2D Fermi gas was first studied by Petrov et al. The binding energy $\epsilon_B$ can be determined using the equation, $a_s/a_{3D} = F[\epsilon_B/(\hbar\omega_z)]$, where $a_s \equiv \sqrt{\hbar/(M\omega_z)}$ is the harmonic oscillator length along the tight confinement direction and the function $F(x)$ is given by, $F(x) = \int_0^\infty du (4\pi u^3)^{-1/2}[1 - e^{-xu}/\sqrt{1 - e^{-2u}}]/(2u)$.

It is highly non-trivial that our quasi-2D calculations can accurately recover the minimal chemical potential $\mu_m = (\hbar\omega_z - \epsilon_B)/2$ in the dilute 2D limit.

For the breathing mode frequency of an interacting 2D Fermi gas, the predictions obtained by using QMC EoS and GPF EoS are compared in the inset of Fig. 1. It is readily seen that the GPF theory provides a quantitative description of the 2D breathing mode frequency in the strongly interacting crossover regime.

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