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Piatetski-Shapiro’s phenomenon
and related problems

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by

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Introduction

This thesis is concerned with some problems in two areas of Fourier Analysis: uniqueness theory of trigonometric expansions, and the theory of translation invariant subspaces in function spaces.

Our main result in the first area extends to $\ell_q$ spaces ($q > 2$) a deep phenomenon found by Piatetski-Shapiro in 1954 for the space $c_0$.

The approach we developed also enabled us to get a result in the second mentioned area, which a priori does not look connected with the first one. The result (maybe, a bit surprising) is: one cannot characterize the functions in $\ell_p(\mathbb{Z})$ or $L^p(\mathbb{R})$, $1 < p < 2$, whose translates span the whole space, by the zero set of their Fourier transform. This should be contrasted against the classical Wiener theorems related to the cases $p = 1, 2$.

Below we give a detailed exposition of the main results proved in this thesis.

1. Piatetski-Shapiro’s Phenomenon

Background. The theory of uniqueness deals with the question: if a function admits a representation by a trigonometric series, $f(t) = \sum_{n=-\infty}^{+\infty} c_n e^{int}$, is this representation unique? The answer depends on the sense in which the convergence of the series is understood. It was proved by Cantor that if the series converges at every point then the coefficients $\{c_n\}$ are uniquely defined. On the other hand, for convergence almost everywhere the uniqueness result fails. This was proved by Menshov who constructed a trigonometric series which converges to zero almost everywhere, but which is not identically zero.

A central concept in this subject is the notion of a set of uniqueness. A set $E$ of measure zero is called a set of uniqueness, if the only trigonometric series which converges to zero everywhere outside $E$ is the series which is identically zero. Thus if a function admits a representation by a trigonometric series which converges everywhere outside a set of uniqueness, then the coefficients of the series are uniquely defined. A set $E$ which is not a set of uniqueness, is called a set of multiplicity. The problem to determine which sets $E$ of measure zero are sets of uniqueness and which are sets of multiplicity, is very difficult. It is well-known that a crucial role is played by the arithmetic properties of the set $\mathbb{1}, \mathbb{44}$.

If the set $E$ is closed then the following criterion is available: $E$ is a set of multiplicity if and only if there is a non-zero Schwartz distribution on the circle, which is supported by $E$, and has Fourier coefficients tending to zero $\mathbb{19}$.
Unfortunately, this criterion is still difficult to check. However, it is common to use this criterion in order to show that a given set $E$ is a set of multiplicity, by constructing a measure which is supported by $E$ and which has Fourier coefficients tending to zero. The existence of such a measure is a special property of a set of multiplicity, and in such a case $E$ is called a set of restricted multiplicity.

In fact, most known sets of multiplicity are sets of restricted multiplicity, and it was long believed that the two notions actually coincide. However, in 1954 Piatetski-Shapiro\cite{37} refuted this hypothesis by constructing an example of a closed set of multiplicity $E$ such that no measure supported by $E$ can have Fourier coefficients tending to zero. This subject was further developed by Körner\cite{25} and Kaufman\cite{22}. During the 1980’s, a strong structural difference between the two notions was found using methods of descriptive set theory\cite{23}.

**Results.** We investigate the following problem. Suppose that $E$ is a closed set on the circle, which supports a Schwartz distribution $S \sim \sum c_n e^{int}$ whose Fourier coefficients $\{c_n\}$ belong to a certain space $\mathcal{X}$. Must then $E$ also support a measure $\mu$ whose Fourier coefficients belong to the same space $\mathcal{X}$? We say that Piatetski-Shapiro’s phenomenon exists in the space $\mathcal{X}$ if the answer is negative; that is, if there is a closed set $E$ which supports a distribution with Fourier coefficients in $\mathcal{X}$, but which does not support such a measure. The theorem of Piatetski-Shapiro thus states that Piatetski-Shapiro’s phenomenon exists in the space $c_0$ of sequences tending to zero. On the other hand, it is known from potential theory that this is not the case in certain weighted $\ell_2$ spaces. Precisely, if a closed set $E$ supports a distribution $S \sim \sum c_n e^{int}$ such that $\sum |n|^{-\alpha}|c_n|^2 < \infty$, $0 < \alpha \leq 1$, then it also supports a positive measure satisfying this property\cite{19}.

What can be said about $\ell_q$ spaces? Only the case $q > 2$ is non-trivial, since only in this case there exist distributions with coefficients in $\ell_q$ which are not measures. In our joint work with A. Olevskii\cite{29} we proved that Piatetski-Shapiro’s phenomenon does exist in these spaces. That is,

**THEOREM.** For any $q > 2$ there is a closed set $E$ such that:

(i) There is a distribution $S \sim \sum c_n e^{int}$ supported by $E$, with $\sum |c_n|^q < \infty$.

(ii) No measure $\mu$ may satisfy this property.

As pointed out in\cite{29}, the property of our set $E$ can also be formulated in the language of the uniqueness problem: there is a non-zero trigonometric series $\sum c_n e^{int}$ with $\sum |c_n|^q < \infty$, which converges to zero everywhere outside $E$, but no Fourier-Stieltjes series may satisfy this property. Our proof is inspired by Kahane’s presentation of the Körner-Kaufman results, see\cite{19} pp. 213–216. The main new ingredients in\cite{29} are Riesz products, and probabilistic concentration estimates.

We continue to study further aspects of Piatetski-Shapiro’s phenomenon. For example, how small can be the set $E$ in the above theorem. In particular, how small can be the Hausdorff dimension of $E$. Using the “shrinking method” of Kaufman we prove the following stronger result: any closed set which supports
a distribution with Fourier coefficients in $\ell_q$, contains a closed subset which also supports such a distribution, but not such a measure. In particular, this result enables us to compute the minimal Hausdorff dimension of $E$. Another question under consideration is whether Piatetski-Shapiro’s phenomenon is “typical” or “rare”. Inspired by Körner’s ideas [27] we define the space of all pairs $(E, S)$ where $E$ is a closed set on the circle, and $S$ is a distribution supported by $E$ with Fourier coefficients in $\ell_q$. We prove that “almost all” pairs $(E, S)$, in the sense of Baire categories, satisfy the above theorem.

Are there spaces other than $c_0$ and $\ell_q$ where Piatetski-Shapiro’s phenomenon could exist? We find a certain class of Orlicz spaces, where this is indeed the case. For example, for any $q > 2$ and $\alpha > 0$, Piatetski-Shapiro’s phenomenon exists in the space of sequences $\{c_n\}$ satisfying $\sum |c_n|^q \log^\alpha (e + |c_n|^{-1}) < \infty$.

2. Generators in $\ell_p$ and Zero Set of Fourier Transform

**Background.** A function $F : \mathbb{Z} \to C$ is called a generator (or a cyclic vector) in the space $\ell_p(\mathbb{Z})$ if the linear span of its translates is dense in $\ell_p$. In other words, an element $F \in \ell_p(\mathbb{Z})$ is a generator if the closed translation invariant linear subspace generated by $F$ is the whole $\ell_p$. How to know whether a given $F$ is a generator, or not? The classical cases are $p = 1, 2$. In these cases, Wiener [43] characterized the generators in terms of the zero set $Z_f$ of the Fourier transform

$$f(t) = \sum_{n \in \mathbb{Z}} F(n)e^{int},$$

as follows:

- $F$ is a generator in $\ell_1$ if and only if $f(t)$ has no zeros.
- $F$ is a generator in $\ell_2$ if and only if $f(t) \neq 0$ almost everywhere.

The same characterization is true for $L^1(\mathbb{R})$ and $L^2(\mathbb{R})$.

The case $1 < p < 2$ is much less understood. “Interpolating” between $p = 1$ and 2, one may expect that the generators in $\ell_p (1 < p < 2)$ could be characterized by the condition that the zero set of the Fourier transform is “small” in a certain sense. In this context various metrical, arithmetical and other properties of the zero set for generators and non-generators have been studied by Beurling [3], Pollard [38], Herz [12], Newman [32] and other authors. However, none of their results provides a complete characterization of the generators.

For example, Beurling proved that if the zero set $Z_f$ is sufficiently small in the sense of Hausdorff dimension, then $F$ is a generator in $\ell_p$. However, this is only under the assumption that $F \in \ell_1$. Also, the converse is not true: one can construct a function $F$ which is a generator in every $\ell_p (1 < p < 2)$, but $Z_f$ has Hausdorff dimension 1.

**Results.** We study the following question: is it possible at all to describe the generators $F$ in $\ell_p (1 < p < 2)$ by the zero set of the Fourier transform? In our
joint work with A. Olevskii [30] we show that, in contrast to the classical cases $p = 1$ and $2$, the characterization of the generators in $\ell_p$ ($1 < p < 2$) by the zero set of their Fourier transforms is impossible in principle.

Our main result is the following theorem.

**Theorem.** Given $1 < p < 2$ one can find two continuous functions $f$ and $g$ on the circle, with the following properties:

(i) They have the same set of zeros.

(ii) The Fourier transforms $F = \hat{f}$ and $G = \hat{g}$ both belong to $\ell_p(\mathbb{Z})$.

(iii) $G$ is a generator in $\ell_p$, but $F$ is not.

A similar result is true for $L^p(\mathbb{R})$ spaces.

We point out the role of the continuity condition in the theorem, which makes precise the concept of the “zero set”. Generally, there is no canonical way to define the zero set of the Fourier transform of an arbitrary element in $\ell_p$ ($1 < p < 2$).

The proof of the above theorem is based on a modifications and development of the approach which allowed us to obtain Piatetski-Shapiro’s phenomenon in $\ell_q$ spaces ($q > 2$). In fact, the result is derived from the construction of a closed set $E$ on the circle, which satisfies a certain strong form of Piatetski-Shapiro’s phenomenon in $\ell_q$, where $q$ is the exponent conjugate to $p$ satisfying $1/p + 1/q = 1$.

More specifically, let $A_r(\mathbb{T})$ ($1 \leq r < \infty$) denote the Banach space of functions or distributions on the circle with Fourier coefficients in $\ell_r(\mathbb{Z})$, endowed with the norm $\|f\|_{A_r} := \|\hat{f}\|_{\ell_r}$. We prove the following theorem.

**Theorem.** Given any $1 < p < 2$ one can construct a closed set $E$ on the circle, and a function $g \in C(\mathbb{T}) \cap A_p(\mathbb{T})$, such that:

(i) $Z_g := \{t : g(t) = 0\} = E$;

(ii) The set $\{P(t)g(t)\}$, where $P$ goes through all trigonometric polynomials, is dense in $A_p$.

(iii) There is a (non-zero) distribution $S$, supported by $E$, which belongs to $A_q$, where $q = p/(p - 1)$ is the exponent conjugate to $p$.

The property (ii) means that $\hat{g}$ is a generator. On the other hand, (iii) is equivalent to the fact that no Fourier transform of a smooth function $f$ vanishing on $E$, could be a generator. The first theorem is therefore a consequence of the second one.

We remark that the set $E$ satisfies Piatetski-Shapiro’s phenomenon in $\ell_q$, since the property (ii) implies that $E$ cannot support a non-zero measure $\mu$ whose Fourier coefficients belong to $\ell_q$. 
CHAPTER 1

Piatetski-Shapiro’s Theorem

In this chapter we present a simplified proof of Piatetski-Shapiro’s theorem.

1.1. Introduction

1.1.1. A compact $K$ on the circle is called a set of uniqueness if the only trigonometric series

$$\sum_{n \in \mathbb{Z}} c(n)e^{int}$$

which converges to zero everywhere outside $K$, is the series which is identically zero. Otherwise, $K$ is called a set of multiplicity. It is well-known that $K$ is a set of multiplicity if and only if there is a non-zero Schwartz distribution $S$, supported by $K$, with Fourier coefficients $\{\hat{S}(n)\}$ tending to zero (see [19], Chapter V). If there is a non-zero measure $\mu$ satisfying these properties, then $K$ is called a set of restricted multiplicity.

It was conjectured that any set of multiplicity is in fact a set of restricted multiplicity. For example, a symmetric Cantor set with constant dissection ratio $\xi$, where $0 < \xi < \frac{1}{2}$, is either a set of uniqueness or a set of restricted multiplicity (see [19], Chapter VI). However, Piatetski-Shapiro showed in [37] that in general this is not true. He proved the following theorem.

THEOREM 1.1 (Piatetski-Shapiro, 1954). There exists a compact $K$ on the circle, which supports a distribution $S$ with Fourier coefficients tending to zero, but which does not support any measure with this property.

It is interesting to remark that Piatetski-Shapiro proved the above theorem by constructing the following concrete example. Let $K_\gamma$ $(0 < \gamma < \frac{1}{2})$ denote the set of real numbers $t$ in the segment $[0, 2\pi)$, written in binary expansion

$$t = 2\pi \sum_{k=1}^{\infty} \varepsilon_k 2^{-k} \quad \varepsilon_k \in \{0, 1\},$$

which satisfy

$$\frac{1}{n} \sum_{k=1}^{n} \varepsilon_k \leq \gamma, \quad n = 1, 2, 3, \ldots.$$

Then Theorem 1.1 holds with $K = K_\gamma$, see [37] or [9, Section 4.4].
1.1.2. Piatetski-Shapiro’s theorem was further extended in the years 1971–73, in the following direction. A compact $K$ is called a Helson set if any function defined and continuous on $K$ can be extended to a function on the circle with an absolutely convergent Fourier series. It was proved by Helson [11] that a Helson set can never be a set of restricted multiplicity. The problem whether it can be a set of multiplicity was solved in the affirmative by Körner in [25]. A simpler proof of Körner’s theorem was later found by Kaufman [22], who also proved a stronger result: any set of multiplicity contains a Helson set of multiplicity.

A compact which is not a set of restricted multiplicity is called a set of extended uniqueness. During the 1980’s, descriptive set theory was used to establish a structural difference between the class $\mathcal{U}$ of the closed sets of uniqueness, and the class $\mathcal{U}_0$ of the closed set of extended uniqueness. Namely, Debs and Saint-Raymond [5] proved that $\mathcal{U}$ has no Borel basis, while Kechris and Louveau [23] proved that $\mathcal{U}_0$ does admit such a basis. For a detailed discussion see [23].

1.1.3. In this chapter, our goal is to present a simplified proof of Piatetski-Shapiro’s theorem. Our approach is based on Kahane’s presentation [19, pp. 213–216] of the Körner-Kaufman results mentioned above. The main new ingredient in the proof presented below is the use of Riesz products, which allows us to simplify certain arguments from [19] related to Fourier series in multiple variables.

1.2. Lemmas

1.2.1. Extended uniqueness. We start with the formulation of a condition on a compact $K$ which implies that it supports no measure with Fourier coefficients tending to zero. In fact, one can relax this condition (see [36], Theorem 5), so the following lemma is not the strongest statement which is possible. However, it will be sufficient for us, and its proof is very simple.

**Lemma 1.1.** Suppose a compact $K$ on the circle satisfies the following condition: for any $\varepsilon > 0$ and any integer $\nu$ there is a real trigonometric polynomial

$$X(t) = \sum_{|n| \geq \nu} \hat{X}(n)e^{int}, \quad \sum_{|n| \geq \nu} |\hat{X}(n)| \leq 2, \quad |1 - X(t)| < \varepsilon \text{ on } K,$$

Then $K$ does not support any non-zero measure $\mu$ such that $\hat{\mu}(n) \to 0$ as $|n| \to \infty$.

**Proof.** Let $\mu$ be a non-zero measure supported by $K$, with Fourier coefficients tending to zero. Clearly we may assume $\int |d\mu| = 1$, and by multiplication of $\mu$ by an appropriate exponential $e^{int}$ we may also assume $\hat{\mu}(0) \neq 0$. Given $\varepsilon > 0$, choose $\nu$ such that $\sup_{|n| \geq \nu} |\hat{\mu}(n)| < \varepsilon/2$, and let $X$ be a real trigonometric polynomial satisfying (1.2.1). Then one has

$$\left| \int_X d\mu \right| = \sum_{|n| \geq \nu} |\hat{X}(n)\hat{\mu}(-n)| \leq 2 \sup_{|n| \geq \nu} |\hat{\mu}(n)| < \varepsilon.$$
On the other hand $\mu$ is supported by $K$, hence
\[ \left| \int_{T} X \, d\mu \right| = \left| \int_{K} X \, d\mu \right| \geq \left| \int_{K} 1 - X \, d\mu \right| = |\hat{\mu}(0)| - \varepsilon. \]
Since $\varepsilon$ was arbitrary, it follows that $\hat{\mu}(0) = 0$, a contradiction. \( \square \)

### 1.2.2. Kahane’s lemma

The following lemma is taken from [19] p. 214. For completeness, we include its proof here.

**Lemma 1.2 (Kahane).** Let $I$ be a closed interval, $I \subset (0, 1)$. Given any $\delta > 0$ there is a (signed) measure $\rho$, supported by a finite subset of $I$, such that
\[ \int d\rho = 1 \quad \text{and} \quad \left| \int s^k \, d\rho(s) \right| < \delta \quad (k = 1, 2, \ldots). \] (1.2.2)

**Remark.** Note that the properties (1.2.2) are clearly satisfied by the Dirac measure at the point 0; however, the idea of Lemma 1.2 is to obtain (1.2.2) under the constraint that the support of $\rho$ lies in $I$. Also note that the measure $\rho$ cannot be chosen positive for arbitrarily small values of $\delta$, as one can prove using a standard application of Helly’s theorem.

**Proof of Lemma 1.2.** In the space $c_0$ consider the collection $S$ of all sequences $(1, s, s^2, s^3, \ldots)$ corresponding to $s \in I$ (these sequences belong to $c_0$, since $0 < s < 1$). Let us show that $S$ forms a complete system in $c_0$. Indeed, a linear functional over $c_0$ which annihilates all elements of $S$ is a sequence $\{a_k\} \in \ell_1$ such that $\sum_{k=0}^{\infty} a_k s^k$ vanishes on $I$, hence $\{a_k\} = 0$. The completeness therefore follows from the Hahn-Banach theorem. In particular, it is possible to approximate the sequence $(1, 0, 0, \ldots)$, in the $c_0$ norm, by linear combinations of elements of $S$. That is, given $\eta > 0$ there exist $s_1, \ldots, s_n \in I$ and $c_1, \ldots, c_n \in \mathbb{R}$ such that
\[ \left| \sum_{j=1}^{n} c_j - 1 \right| < \eta \quad \text{and} \quad \left| \sum_{j=1}^{n} c_j s_j^k \right| < \eta \quad (k = 1, 2, \ldots). \]
If $\eta$ is chosen small enough so that $\eta/(1-\eta) < \delta$, the conditions (1.2.2) are satisfied by the measure
\[ \rho = \frac{\sum_{j=1}^{n} c_j \delta_{s_j}}{\sum_{j=1}^{n} c_j}, \]
where $\delta_s$ denotes the Dirac measure at the point $s$. \( \square \)

### 1.3. Riesz products

The Riesz product, first introduced by Riesz in 1918, is an important tool in Fourier analysis (see [44], Volume I, Chapter V, §7). In this section we use finite Riesz products to simplify the construction from [22].
1.3.1. Notation. Define a finite Riesz product

$$\lambda_s(t) = \prod_{j=1}^{N} \left(1 + 2s \cos \nu^j t\right)$$  \hspace{1cm} (1.3.1)$$

where the parameters satisfy $0 < s < \frac{1}{2}$ and $\nu$ is an integer $\geq 3$. One can write $2 \cos \nu^j t = e^{i\nu^j t} + e^{-i\nu^j t}$, and then expand the product (1.3.1). This would yield

$$\lambda_s(t) = 1 + \sum_{\tau \neq 0} s^{\sum |\tau_j|} e^{i(\tau_1 \nu + \tau_2 \nu^2 + \cdots + \tau_N \nu^N) t},$$  \hspace{1cm} (1.3.2)$$

where the sum is over all non-zero vectors $\tau = (\tau_1, \ldots, \tau_N)$, $\tau_j \in \{-1, 0, 1\}$.

Since $\nu \geq 3$, every integer $n$ admits at most one representation of the form

$$n = \tau_1 \nu + \tau_2 \nu^2 + \cdots + \tau_N \nu^N,$$

therefore the frequencies in the sum (1.3.2) are distinct. So (1.3.2) is the Fourier expansion of $\lambda_s$.

1.3.2. Concentration. The trigonometric polynomial $\lambda_s$ is non-negative and has integral $= 1$. We therefore may consider a probability measure $\lambda_s(t) dt / 2\pi$ on the circle, which we shall also denote (with no risk of ambiguity) by $\lambda_s$. Define a trigonometric polynomial

$$X(t) = \frac{2}{N} \sum_{j=1}^{N} \cos \nu^j t.$$  \hspace{1cm} (1.3.3)$$

We regard $X$ as a random variable with respect to the probability measure $\lambda_s$.

**Lemma 1.3.** Given $\delta > 0$ we have

$$\lambda_s \{ t \in T : |X(t) - 1| > 3\delta \} \longrightarrow 0$$

as $N \to \infty$, uniformly in $s \in \left(\frac{1}{2} - \delta, \frac{1}{2}\right)$.

**Proof.** Denote $X_j(t) = \cos \nu^j t$, then using (1.3.2) one can compute

$$\mathcal{E}(X_j) = s, \quad \mathcal{V}(X_j) = \mathcal{E}(X_j - s)^2 = \frac{1}{2} - s^2,$$

$$\mathcal{E}(X_j - s)(X_k - s) = 0, \quad j \neq k.$$

We therefore have

$$\mathcal{E}(X) = \frac{2}{N} \sum_{j=1}^{N} \mathcal{E}(X_j) = 2s,$$

and the fact that the $X_j$’s are uncorrelated allows us to compute

$$\mathcal{V}(X) = \mathcal{E}(X - 2s)^2 = \frac{4}{N^2} \sum_{j=1}^{N} \mathcal{V}(X_j) = \frac{4}{N} \left(\frac{1}{2} - s^2\right) < \frac{2}{N}.$$
If \( s \in \left( \frac{1}{2} - \delta, \frac{1}{2} \right) \), then by Chebyshev’s inequality
\[
\lambda_s \{ |X(t) - 1| > 3\delta \} \leq \lambda_s \{ |X(t) - 2s| > \delta \} \leq \delta^{-2} \mathcal{V}(X) \to 0
\]
as \( N \to \infty \), uniformly in \( s \). \( \square \)

### 1.4. Proof of Piatetski-Shapiro’s theorem

#### 1.4.1. We can now prove

**Lemma 1.4.** Let \( \varepsilon > 0 \) and an integer \( \nu \geq 3 \) be given. Then there exist a \( C^\infty \) function \( f : \mathbb{T} \to \mathbb{C} \) and an integer \( N \) such that

(i) \( f \) is supported by the compact
\[
K = \{ t \in \mathbb{T} : |X(t) - 1| \leq \varepsilon \}, \quad (1.4.1)
\]
where \( X \) is the trigonometric polynomial defined by (1.3.3).

(ii) \( f(t) = 1 + \sum_{n \neq 0} \hat{f}(n)e^{int} \), where \( |\hat{f}(n)| < \varepsilon \) for all \( n \neq 0 \).

**Proof.** The proof follows a similar line as in [19, p. 213]. Given \( \delta > 0 \), choose a measure \( \rho \) supported by the interval \( \left( \frac{1}{2} - \delta, \frac{1}{2} \right) \) according to Lemma 1.2. Define \( \lambda(t) = \int \lambda_s(t) \, d\rho(s) \).

By (1.3.2) and (1.2.2), \( \lambda \) is a trigonometric polynomial with Fourier expansion
\[
\lambda(t) = 1 + \sum_{\tau \neq 0} \left\{ \int s^{\Sigma |\tau_j|} \, d\rho \right\} e^{i(\Sigma \tau_j \nu_j)t},
\]
so \( |\hat{\lambda}(n)| < \delta \) for all \( n \neq 0 \). Set
\[
K' = \{ t : |X(t) - 1| \leq 3\delta \},
\]
and use Lemma 1.3 to choose \( N \) such that
\[
\int_{\mathbb{T} \setminus K'} \lambda_s(t) \, dt = \lambda_s \{ t \in \mathbb{T} : |X(t) - 1| > 3\delta \} < \frac{\delta}{\|\rho\|_M}
\]
for every \( s \in \left( \frac{1}{2} - \delta, \frac{1}{2} \right) \). The function \( h = \lambda \cdot 1_{K'} \) is then supported by \( K' \), and
\[
\| \hat{\lambda} - \hat{h} \|_\infty \leq \| \lambda - h \|_{L^1(\mathbb{T})} = \| \lambda \|_{L^1(\mathbb{T} \setminus K')} \leq \int \| \lambda_s \|_{L^1(\mathbb{T} \setminus K')} \, d\rho(s) < \delta.
\]
If we choose \( \delta \) so that \( \delta/(1 - \delta) < \varepsilon \), then the function \( g(t) = h(t)/\hat{h}(0) \) satisfies
\[
\hat{g}(0) = 1 \quad \text{and} \quad |\hat{g}(n)| < \varepsilon \quad (n \neq 0).
\]
Finally, to obtain a smooth function \( f \) we take the convolution of \( g \) with a \( C^\infty \) non-negative function \( \psi \) with integral \( = 1 \). The condition \( 3\delta < \varepsilon \) will ensure that, if \( \psi \) is supported by a sufficiently small neighbourhood of \( 0 \), then the support of \( f \) will be contained in the compact \( K \) from (1.4.1). \( \square \)
1.4.2. To finish the proof of Theorem 1.1 we proceed in a similar way to [19]. For each \( j = 1, 2, \ldots \) we define a number \( \varepsilon_j > 0 \) and a positive integer \( \nu_j \geq 3 \), and choose \( f_j, X_j \) and \( K_j \) according to Lemma 1.4 (with \( \varepsilon = \varepsilon_j \) and \( \nu = \nu_j \)). We choose the \( \{ \varepsilon_j \} \) by induction to satisfy
\[
\varepsilon_1 < 2^{-2} \quad \text{and} \quad \| f_1 \cdot f_2 \cdot \ldots \cdot f_j \|_{A^{\varepsilon_{j+1}}} < 2^{-2-j} \quad (j = 1, 2, \ldots).
\]
As in [19, p. 215], this implies that the product \( \prod_{j=1}^\infty f_j \) will converge in the space \( A_\infty \) to a nonzero distribution \( S \) with Fourier coefficients tending to zero. Denote \( K = \bigcap_{j=1}^\infty K_j \), then clearly \( S \) is supported by \( K \). On the other hand, we can choose \( \{ \nu_j \} \) to satisfy \( \nu_j \to \infty \). Since \( \sum |\hat{X}_j(n)| \lesssim 2 \), we deduce by Lemma 1.1 that \( K \) supports no nonzero measure with Fourier coefficients tending to zero.

This concludes the proof of Theorem 1.1.
CHAPTER 2

\( \ell_q \) Spaces

2.1. Introduction

2.1.1. Chapter 1 was concerned with Piatetski-Shapiro’s theorem, which states that there exists a compact \( K \) on the circle, which supports a distribution \( S \) such that \( \widehat{S} \in c_0 \), but which does not support such a measure. We indicated this by saying that Piatetski-Shapiro’s phenomenon exists in the space \( c_0 \) of sequences tending to zero.

It is known from potential theory that no Piatetski-Shapiro phenomenon exists in certain weighted \( \ell_2 \) spaces. Precisely, if a compact \( K \) supports a distribution \( S \) such that \( \sum |n|^{-\alpha} |\hat{S}(n)|^2 < \infty \), for some \( 0 < \alpha \leq 1 \), then \( K \) also supports a positive measure satisfying this condition (see [19], Chapter III).

In this chapter we study Piatetski-Shapiro’s phenomenon in \( \ell_q \) spaces. Only the case \( q > 2 \) is non-trivial, since only in this case there are distributions \( S \) such that \( \widehat{S} \in \ell_q \), but which are not measures. Our main result is that Piatetski-Shapiro’s phenomenon exists in \( \ell_q \) for any \( q > 2 \).

**Theorem 2.1.** For any \( q > 2 \) there is a compact \( K \) on the circle, which supports a distribution \( S \) with \( \widehat{S} \in \ell_q \), but does not support such a measure.

The main property of the compact \( K \) in Theorem 2.1 can also be formulated in the language of uniqueness theory. Namely:

*For any \( q > 2 \) there is a compact \( K \) such that*

(i) There is a non-zero trigonometric series \( \sum c(n) e^{int} \) with coefficients \( \{ c(n) \} \) in \( \ell_q \), converging to zero everywhere outside \( K \).

(ii) No Fourier-Stieltjes series may satisfy this property.

The result was published in [29].

2.1.2. In what follows \( q \) is a fixed number, \( q > 2 \), and let \( p = q/(q-1) \) denote the exponent conjugate of \( q \). We denote by \( A_q(\mathbb{T}) \) the Banach space of Schwartz distributions \( S \) on the circle, endowed with the norm \( \| S \|_{A_q} := \| \widehat{S} \|_{\ell_q} \). We also denote by \( M(K) \) the space of finite (complex) Borel measures supported by \( K \), endowed with the total variation norm.
2.2. Excluding measures

We start by presenting a condition, analogous to that of Lemma 1.1, which implies that a compact $K$ does not support any non-zero measure $\mu \in A_q$.

**Lemma 2.1.** Suppose a compact $K$ on the circle satisfies the following condition: for any positive integer $\nu$ there is a real trigonometric polynomial

$$ X(t) = \sum_{|n| \geq \nu} \hat{X}(n)e^{int}, \quad \|\hat{X}\|_p \leq 1, \quad \frac{1}{100} \leq X(t) \leq 100 \text{ on } K. $$

Then $K$ does not support a non-zero measure $\mu \in A_q$.

In the proof of Lemma 2.1 we will use:

**Lemma 2.2.** Let $\mu \in A_q$ be a measure supported by a compact $K$. Then the measure $|\mu|$ belongs to the closure of $A_q \cap M(K)$ in the $M(K)$ norm.

**Proof.** There exists a Borel function $\phi : \mathbb{T} \to \mathbb{C}$ with $|\phi(t)| = 1$, such that $|d\mu| = \phi \, d\mu$. Given $\varepsilon > 0$, the theorem of Lusin (see [40], Theorem 2.24) allows us to choose a continuous function $\xi : \mathbb{T} \to \mathbb{C}$, $\|\xi\|_{C(T)} \leq 1$ such that

$$ |\mu| \{t \in \mathbb{T} : \phi(t) \neq \xi(t)\} < \varepsilon. $$

Choose a trigonometric polynomial $\psi$ such that $\|\psi - \xi\|_{C(T)} < \varepsilon$, then

$$ |\mu| \{t \in \mathbb{T} : |\phi(t) - \psi(t)| > \varepsilon\} < \varepsilon $$

and $\|\psi\|_{C(T)} < 2$. Thus the measure $d\mu_1 = \psi \, d\mu$ belongs to $A_q \cap M(K)$, and

$$ \| |\mu| - \mu_1 \|_{M(K)} = \int_{\mathbb{T}} |\phi - \psi| \, |d\mu| \leq 3 \varepsilon + \varepsilon \|\mu\|_{M(K)}. \qquad \blacksquare $$

**Proof of Lemma 2.1.** Suppose that $\mu \in A_q \cap M(K)$. Given $\varepsilon > 0$, by Lemma 2.2 one can find a measure $\mu_1 \in A_q \cap M(K)$ such that

$$ \| |\mu| - \mu_1 \|_{M(K)} < \varepsilon. $$

Then

$$ \left| \int_{\mathbb{T}} X \, d\mu_1 \right| = \left| \int_{K} X \, d\mu_1 \right| \geq \int_{K} X \, |d\mu| - 100 \varepsilon \geq \frac{1}{100} \|\mu\|_{M(K)} - 100 \varepsilon. $$

On the other hand,

$$ \left| \int_{\mathbb{T}} X \, d\mu_1 \right| = \left| \sum_{|n| \geq \nu} \hat{X}(n)\hat{\mu}_1(-n) \right| \leq \left( \sum_{|n| \geq \nu} |\hat{\mu}_1(n)|^q \right)^{1/q} < \varepsilon
$$

for sufficiently large $\nu$. Hence $\mu = 0$. \qquad \blacksquare

**Remark.** The reason we need Lemma 2.2 is that given a measure $\mu \in A_q$, it is not necessarily true that also $|\mu| \in A_q$. One should compare this with the case when instead of $\ell_q$ we have the space $c_0$; by the lemma of Milicer-Grużewska, if the Fourier coefficients of $\mu$ are tending to zero then also $|\mu|$ has this property (see [23], II.5, Lemma 4).
2.3. Probabilistic tools

2.3.1. Exponential concentration. In Lemma 1.3 the Chebyshev inequality was used to obtain concentration of a Riesz product probability measure. In the current context a stronger estimate will be needed. We shall use the following classical exponential estimate of S. N. Bernstein. For completeness we also provide its proof.

**Lemma 2.3.** Let $X_1, \ldots, X_N$ be independent real random variables such that $\mathcal{E} X_j = 0$ and $|X_j| \leq M_j$ almost surely, for each $1 \leq j \leq N$. Then

$$
\mathcal{P} \left( \left| \sum_{j=1}^{N} X_j \right| > \alpha \right) \leq 2 \exp \left( - \frac{1}{2} \frac{\alpha^2}{\sum M_j^2} \right).
$$

for every $\alpha > 0$.

**Proof.** Suppose first that $X$ is a real random variable, $\mathcal{E} X = 0$ and $|X| \leq 1$ almost surely. Given any $\lambda \in \mathbb{R}$, by convexity we have $e^{\lambda x} \leq \cosh \lambda + x \sinh \lambda$ for $-1 \leq x \leq 1$. Hence

$$
\mathcal{E} e^{\lambda X} \leq \cosh \lambda = \sum_{k=0}^{\infty} \frac{\lambda^{2k}}{(2k)!} \leq \sum_{k=0}^{\infty} \frac{\lambda^{2k}}{2^k k!} = e^{\lambda^2/2}.
$$

(2.3.1)

Now apply (2.3.1) to $M_j^{-1} X_j$ ($1 \leq j \leq N$), then using Markov’s inequality and the independence we get

$$
\mathcal{P} \left( \sum_{j=1}^{N} X_j > \alpha \right) = \mathcal{P} \left( \prod_{j=1}^{N} e^{\lambda X_j} > e^{\lambda \alpha} \right) \leq e^{-\lambda \alpha} \mathcal{E} \prod_{j=1}^{N} e^{\lambda X_j}
$$

$$
= e^{-\lambda \alpha} \prod_{j=1}^{N} \mathcal{E} e^{\lambda X_j} \leq \exp \left( \frac{1}{2} \lambda^2 \sum_{j=1}^{N} M_j^2 - \alpha \lambda \right).
$$

Choosing $\lambda = \alpha / \sum M_j^2$ we obtain the one sided estimate

$$
\mathcal{P} \left( \sum_{j=1}^{N} X_j > \alpha \right) \leq \exp \left( - \frac{1}{2} \frac{\alpha^2}{\sum M_j^2} \right).
$$

The two-sided estimate can now be easily deduced by symmetry. \qed

It is now easy to pass to the case of non-zero expectation; we omit the proof.

**Lemma 2.4.** Let $X_1, \ldots, X_N$ be independent real random variables such that $|X_j| \leq M$ almost surely, for each $1 \leq j \leq N$. Denote $S = \sum_{j=1}^{N} X_j$, then

$$
\mathcal{P} \left\{ |S - \mathcal{E} S| > \alpha \right\} \leq 2 \exp \left( - \frac{\alpha^2}{8M^2N} \right)
$$

(2.3.2)

for every $\alpha > 0$.

**Remark.** The constant $1/8$ in Lemma 2.4 is not optimal, see Hoeffding [13, Theorem 2]. For further results in this context we refer to Petrov [35].
2.3.2. Stochastic independence. Let \( g \) denote a real \( 2\pi \)-periodic function on \( \mathbb{R} \). Given such \( g \), consider the system of functions
\[
\{ g(\nu^j t) \}_{j=1}^N
\]
on the circle \( \mathbb{T} \). We are interested in stochastic properties of this system.

**Lemma 2.5.** Suppose that \( g \) is constant on each interval
\[
I_k^{(\nu)} = \left( \frac{2\pi(k-1)}{\nu}, \frac{2\pi k}{\nu} \right), \quad 1 \leq k \leq \nu.
\]
Then the system \( \{ g(\nu^j t) \}_{j=1}^N \) is stochastically independent on the circle \( \mathbb{T} \), with respect to the Lebesgue measure.

**Proof.** Each of the functions \( g(\nu^j t) \), \( 1 \leq j \leq N-1 \), is constant on each interval \((2\pi(k-1)\nu^{-N}, 2\pi k\nu^{-N})\), where \( 1 \leq k \leq \nu^N \). On the other hand, the function \( g(\nu^N t) \) has the same distribution, with respect to the Lebesgue measure, on each one of these intervals. To finish the proof one continues by induction. \( \square \)

We next consider stochastic independence of the system \( \{ g(\nu^j t) \}_{j=1}^N \) not with respect to the Lebesgue measure, but rather with respect to a certain Riesz product type measure based on \( g \). Precisely, consider the measure
\[
\prod_{j=1}^N \left( 1 + r_j g(\nu^j t) \right) \frac{dt}{2\pi}, \quad -1 < r_j < 1.
\]

We assume that \( g \) is constant on each interval \( 2.3.3 \), and satisfies the conditions
\[
-1 \leq g(t) \leq 1, \quad \int_0^{2\pi} g(t) \, dt = 0.
\]
The measure \( 2.3.4 \) is therefore positive, and using \( 2.3.5 \) and Lemma 2.5
\[
\int_{\mathbb{T}} \prod_{j=1}^N \left( 1 + r_j g(\nu^j t) \right) \frac{dt}{2\pi} = \prod_{j=1}^N \int_{\mathbb{T}} \left( 1 + r_j g(\nu^j t) \right) \frac{dt}{2\pi} = 1.
\]
That is, \( 2.3.4 \) defines a probability measure on the circle \( \mathbb{T} \). We now have

**Lemma 2.6.** Let \( g \) be constant on each interval \( 2.3.3 \) and satisfy \( 2.3.5 \). Then the system \( \{ g(\nu^j t) \}_{j=1}^N \) is stochastically independent on the circle \( \mathbb{T} \) with respect to the probability measure \( 2.3.4 \).

**Proof.** Consider \( Y_j(t) = g(\nu^j t) \) as random variables with respect to the probability measure \( 2.3.4 \). Let \( \xi_1, \ldots, \xi_N \in \mathbb{R} \) be given. Using the independence with respect to the Lebesgue measure as in Lemma 2.5 one has
\[
\mathcal{E} \exp \left( i \sum_{j=1}^N \xi_j Y_j \right) = \int \prod_{j=1}^N \left( 1 + 2r_j g(\nu^j t) \right) e^{i\xi_j g(\nu^j t)} \frac{dt}{2\pi} = \prod_{j=1}^N \int_{\mathbb{T}} \left( 1 + 2r_j g(t) \right) e^{i\xi_j g(t)} \frac{dt}{2\pi}.
\]
Hence, denoting \( \varphi_j(\xi) = \int_T (1 + 2r_j g(t)) e^{i\xi g(t)} \frac{dt}{2\pi} \) \((1 \leq j \leq N)\), it is seen that
\[
\mathcal{E} \exp \left( i \sum_{j=1}^{N} \xi_j Y_j \right) = \varphi_1(\xi_1) \varphi_2(\xi_2) \cdots \varphi_N(\xi_N).
\]
By properties of the Fourier transform, we conclude that the distribution in \( \mathbb{R}^N \) of \((Y_1, \ldots, Y_N)\) is a product measure, which proves the independence. \(\square\)

### 2.4. Riesz products

#### 2.4.1. Notation.
Define a Riesz product
\[
\lambda_s(t) = \prod_{j=1}^{N} \left( 1 + 2sN^{-1/q} \cos \nu_j t \right).
\]
Here we assume that \( \frac{1}{4} < s < \frac{1}{3} \) and \( \nu \geq 3 \). Expanding the product gives
\[
\lambda_s(t) = 1 + \sum_{\tau \neq 0} (sN^{-1/q}) \sum_{ |\tau_j|} e^{i(\nu_1 + \nu_2 + \cdots + \nu_N)t}, \tag{2.4.1}
\]
where the sum is over all non-zero vectors \( \tau = (\tau_1, \ldots, \tau_N), \tau_j \in \{-1, 0, 1\} \).
As before, the frequencies in the sum \(2.4.1\) are distinct.

#### 2.4.2. “Almost independence”.
As before, we also use \( \lambda_s \) to denote the probability measure \( \lambda_s(t)dt/2\pi \) on the circle. Define a trigonometric polynomial
\[
X(t) = N^{-1/p} \sum_{j=1}^{N} \cos \nu_j t. \tag{2.4.2}
\]
If \( \nu \) is sufficiently large, the members of the polynomial \( X \) are “almost independent” with respect to the measure \( \lambda_s \). Precisely, we can prove the following exponential concentration estimate.

**Lemma 2.7.** *Being given \( N \), for \( \nu \geq \nu(N) \) one has*
\[
\lambda_s \{ t \in \mathbb{T} : |X(t) - s| > \alpha \} \leq 3 \exp \left( -\frac{1}{8} \alpha^2 N^{2/p} - 1 \right) \tag{2.4.3}
\]
*for every \( \alpha > 0 \) and every \( \frac{1}{4} < s < \frac{1}{3} \).*

**Proof.** Define a \( 2\pi \)-periodic function \( g \), constant on each interval \(2.3.3\), by the requirement
\[
\int_{I_k(\nu)} g(t) \, dt = \int_{I_k(\nu)} \cos t \, dt.
\]
Then automatically also (2.3.3) is satisfied. Denote \( \delta = \max_{t \in \mathbb{T}} |g(t) - \cos t| \), then clearly \( \delta \to 0 \) as \( \nu \to \infty \). Consider the probability measure

\[
\gamma_s(t) = \prod_{j=1}^{N} \left( 1 + 2sN^{-1/q} g(\nu^j t) \right) \frac{dt}{2\pi}.
\]

Denote \( Y = N^{-1/p} \sum_{j=1}^{N} g(\nu^j t) \), and let \( \eta = \int_{\mathbb{T}} Y(t) \gamma_s(t) \frac{dt}{2\pi} \) be the expectation of \( Y \) with respect to \( \gamma_s \). Using Lemma 2.5 and the properties of the function \( g \), one can compute that

\[
\eta = 2s \int_{\mathbb{T}} g(t)^2 \frac{dt}{2\pi}.
\]

We now use the following fact: the function \( g(t) \), viewed as an element of \( L^2(\mathbb{T}) \), is obtained by orthogonal projection of \( \cos t \) on the subspace of functions constant on each interval (2.3.3). Thus, by the Pythagorean theorem

\[
s - \eta = 2s \int_{\mathbb{T}} \cos^2 t \frac{dt}{2\pi} - 2s \int_{\mathbb{T}} g(t)^2 \frac{dt}{2\pi} = 2s \int_{\mathbb{T}} (\cos t - g(t))^2 \frac{dt}{2\pi},
\]

which implies \( |\eta - s| \leq \delta^2 \). Since we also have \( |X(t) - Y(t)| \leq \delta N^{1/q} \), we obtain

\[
|X(t) - s| \leq |Y(t) - \eta| + \delta N^{1/q} + \delta^2.
\]

Also, for \( \frac{1}{4} < s < \frac{1}{3} \) we have

\[
\frac{1 + 2sN^{-1/q} \cos t}{1 + 2sN^{-1/q} g(t)} = 1 + \frac{2sN^{-1/q}(\cos t - g(t))}{1 + 2sN^{-1/q} g(t)} \leq \exp \left( \frac{2sN^{-1/q} \delta}{1 - 2sN^{-1/q}} \right) \leq \exp(2\delta N^{-1/q}),
\]

hence \( \lambda_s(t) \leq \gamma_s(t) \exp(2\delta N^{1/p}) \). Setting

\[
\varepsilon = \max \left\{ \exp(2\delta N^{-1/q}) - 1, \delta N^{1/q} + \delta^2 \right\},
\]

we have therefore proved that

\[
|X(t) - s| \leq |Y(t) - \eta| + \varepsilon \quad \text{and} \quad \lambda_s(t) \leq (1 + \varepsilon) \gamma_s(t). \tag{2.4.4}
\]

Now using (2.4.4), Lemma 2.6 and Lemma 2.4 we deduce the estimate

\[
\lambda_s \left\{ t : |X(t) - s| > \alpha \right\} \leq (1 + \varepsilon) \gamma_s \left\{ t : |Y(t) - \eta| > \alpha - \varepsilon \right\} \leq 2(1 + \varepsilon) \exp \left( -\frac{1}{8}(\alpha - \varepsilon)^2 N^{2/p} - 1 \right). \tag{2.4.5}
\]

Note that, since \( |X(t)| \leq N^{1/q} \) and \( \frac{1}{4} < s < \frac{1}{3} \), the left side of (2.4.3) is zero if \( \alpha > N^{1/q} + 1 \). It is therefore enough to obtain (2.4.3) for \( \alpha \leq N^{1/q} + 1 \). However, a straightforward calculation shows that this will follows from (2.4.5) if \( \varepsilon \) is chosen sufficiently small, in a way which depends only on \( N \). Since choosing \( \nu \) sufficiently large will make \( \varepsilon \) arbitrarily small, this proves the claim. \( \square \)
2.4.3. Concentration in $L^2$. We use the probabilistic exponential concentration to prove:

**Lemma 2.8.** Denote

$$K' = \{ t \in \mathbb{T} : \frac{1}{90} \leq X(t) \leq 90 \}.$$  

Given $\delta > 0$, for any $N \geq N(\delta)$ and any $\nu \geq \nu(N)$ one has

$$\int_{T \setminus K'} \lambda_s^2(t) \frac{dt}{2\pi} < \delta$$

for every $\frac{1}{4} < s < \frac{1}{3}$.

**Proof.** We have

$$\lambda_s(t) = \prod_{j=1}^{N} \left( 1 + 2s N^{-1/q} \cos \nu^j t \right) \leq \exp \left( 2s N^{-1/q} \sum_{j=1}^{N} \cos \nu^j t \right),$$

so using (2.4.2) we obtain an estimate

$$\lambda_s(t) \leq \exp \left( 2s N^{2/p - 1} X(t) \right). \quad (2.4.6)$$

Now Lemma 2.7 implies that (2.4.3) holds for any $\nu \geq \nu(N)$. Combining with (2.4.6), it follows that for every $\frac{1}{4} < s < \frac{1}{3}$

$$\int_{\{ t : X(t) \leq \frac{1}{8} \}} \lambda_s^2(t) \frac{dt}{2\pi} \leq \int_{\{ t : X(t) \leq \frac{1}{10} \}} \lambda_s(t) \frac{dt}{2\pi} \cdot \max_{\{ t : X(t) \leq \frac{1}{10} \}} \lambda_s(t)
\leq 3 \exp \left( -\frac{1}{8} (s - \frac{1}{90})^2 N^{2/p - 1} \right) \cdot \exp \left( 2s N^{2/p - 1} \cdot \frac{1}{90} \right)
\leq 3 \exp \left( -2^{-10} N^{2/p - 1} \right),$$

and, for any integer $90 \leq k \leq N^{1/q}$,

$$\int_{\{ t : k < X(t) \leq k+1 \}} \lambda_s^2(t) \frac{dt}{2\pi} \leq \int_{\{ t : k < X(t) \leq k+1 \}} \lambda_s(t) \frac{dt}{2\pi} \cdot \max_{\{ t : k < X(t) \leq k+1 \}} \lambda_s(t)
\leq 3 \exp \left( -\frac{1}{8} (k - s)^2 N^{2/p - 1} \right) \cdot \exp \left( 2s N^{2/p - 1} (k + 1) \right)
\leq 3 \exp \left( -N^{2/p - 1} \right).$$

Hence, keeping in mind that $X(t) \leq N^{1/q}$ for every $t$, we obtain for $\nu \geq \nu(N)$

$$\int_{T \setminus K'} \lambda_s^2(t) \frac{dt}{2\pi} \leq 3 \exp \left( -2^{-10} N^{2/p - 1} \right) + 3 N^{1/q} \exp \left( -N^{2/p - 1} \right) \xrightarrow{N \to \infty} 0$$

uniformly in $s$. \hfill $\Box$

2.5. Proof of main result

In this section we finish the proof of Theorem 2.1.
2.5.1. We start with an analog of Lemma 1.4.

**Lemma 2.9.** Let $\varepsilon > 0$. For $N \geq N(\varepsilon)$ and $\nu \geq \nu(N)$, there exist a $C^\infty$ function $f : \mathbb{T} \to \mathbb{C}$ such that

(i) $f$ is supported by

\[ K = \{ t \in \mathbb{T} : \frac{1}{100} \leq X(t) \leq 100 \}, \quad (2.5.1) \]

where $X$ is defined by (2.4.2).

(ii) $f(t) = 1 + \sum_{n \neq 0} \hat{f}(n)e^{int}$, where \( \left( \sum_{n \neq 0} |\hat{f}(n)|^q \right)^{1/q} < \varepsilon \).

**Proof.** Given $\delta > 0$, according to Lemma 1.2 there is a measure $\rho$ supported by the interval $(\frac{1}{4}, \frac{1}{3})$ such that

\[ \int d\rho = 1 \quad \text{and} \quad \left| \int s^k d\rho(s) \right| < \delta \quad (k = 1, 2, \ldots). \]

Define

\[ \lambda(t) = \int \lambda_s(t) d\rho(s). \]

By (2.4.1), the Fourier expansion of $\lambda$ is

\[ \lambda(t) = 1 + \sum_{\tau \neq 0} \left\{ N^{-1/4} \sum |\tau_j| \int s^{\sum |\tau_j|} d\rho(s) \right\} e^{i\sum \tau_j \nu_j} t. \]

It follows that

\[ \sum_{n \neq 0} |\hat{\lambda}(n)|^q < \delta^q \sum_{\tau \in \{-1,0,1\}^N} N^{\sum |\tau_j|} = \delta^q \left( 1 + \frac{2}{N} \right)^N < e^2 \delta^q. \]

Now use Lemma 2.8 to choose $N$ such that

\[ \left( \int_{\mathbb{T} \setminus K'} \lambda_s^2(t) \frac{dt}{2\pi} \right)^{1/2} < \frac{\delta}{\|\rho\|_M} \]

for any $\frac{1}{4} < s < \frac{1}{3}$. Then $h := \lambda \cdot 1_{K'}$ is supported by $K'$, and

\[ \|\lambda - h\|_{A_q} \leq \|\lambda - h\|_{L^2(T)} = \|\lambda\|_{L^2(T \setminus K')} \leq \int \|\lambda_s\|_{L^2(T \setminus K')} |d\rho(s)| < \delta. \]

If $\delta = \delta(\varepsilon)$ is chosen sufficiently small, the function $g(t) = h(t)/\hat{f}_1(0)$ satisfies

\[ \hat{g}(0) = 1 \quad \text{and} \quad \left( \sum_{n \neq 0} |\hat{g}(n)|^q \right)^{1/q} < \varepsilon. \]

To obtain the smooth function $f$, as before, we take the convolution of $g$ with a $C^\infty$ non-negative function with integral $= 1$, which is supported on a sufficiently small neighborhood of 0 to ensure that $f$ will be supported by $K$. \[\square\]
2.5. PROOF OF MAIN RESULT

2.5.2. We now conclude the proof with a procedure similar to [19, p. 215].

**Proof of Theorem 2.1.** For a sequence \( \{\varepsilon_j\} \) let \( f_j, X_j \) and \( K_j \) be given by Lemma 2.9, for an appropriate choice of \( N_j = N(\varepsilon_j) \) and \( \nu_j = \nu(\varepsilon_j) \). We choose \( \{\varepsilon_j\} \) by induction to satisfy

\[ \varepsilon_1 < 2^{-2} \quad \text{and} \quad \| f_1 \cdot f_2 \cdot \ldots \cdot f_j \|_{A \epsilon_j+1} < 2^{-2-j} \quad (j = 1, 2, \ldots). \]

This implies that the product \( \prod_{j=1}^{\infty} f_j \) will converge in the \( A_q \) norm to a non-zero distribution \( S \in A_q \). Indeed, consider the partial products \( S_0 = 1 \) and \( S_j = f_1 \cdots f_j \), then

\[ \| S_{j+1} - S_j \|_{A_q} = \| f_1 \cdots f_j \cdot (f_{j+1} - 1) \|_{A_q} \leq \| f_1 \cdots f_j \|_{A \epsilon_{j+1}} < 2^{-2-j}, \]

hence \( S_j \) converges in \( A_q \) to a limit \( S \). We have

\[ \| S - 1 \|_{A_q} \leq \sum_{j=0}^{\infty} \| S_{j+1} - S_j \|_{A_q} \leq \sum_{j=0}^{\infty} 2^{-2-j} < 1 \]

and so \( S \neq 0 \). Denote \( K = \bigcap_{j=1}^{\infty} K_j \), then clearly \( S \) is supported on \( K \). We may assume \( \nu_j \to \infty \), and we have \( \| \widehat{X}_j \|_p \leq 1 \), so Lemma 2.1 implies that \( K \) supports no non-zero measure in \( A_q \). This completes the proof of Theorem 2.1. \( \square \)
CHAPTER 3

Generators in $\ell_p$ and Zero Set of Fourier Transform

3.1. Introduction

3.1.1. A function $F : \mathbb{Z} \to \mathbb{C}$ is called a generator, or a cyclic vector, in the space $\ell_p(\mathbb{Z})$ if the linear span of its translates is dense in the space. In other words, $F \in \ell_p(\mathbb{Z})$ is a generator if the closed translation invariant linear subspace generated by $F$ is the whole $\ell_p$. How to know whether a given $F$ is a generator, or not? For $p = 1$ and 2, Wiener characterized the generators by the zero set of the Fourier transform

$$f(t) := \sum_{n \in \mathbb{Z}} F(n) e^{int},$$

as follows:

- $F$ is a generator in $\ell_1$ if and only if $f(t)$ has no zeros.
- $F$ is a generator in $\ell_2$ if and only if $f(t) \neq 0$ almost everywhere.

The same characterization is true for $L^1(\mathbb{R})$ and $L^2(\mathbb{R})$, see [43].

“Interpolating” between $p = 1$ and 2 one may expect that the generators in $\ell_p$ (or $L^p$), $1 < p < 2$, could be characterized by the condition that the zero set of the Fourier transform is “small” in a certain sense. In this context various metrical, arithmetical and other properties of the zero set for generators and non-generators have been studied by Beurling [3], Pollard [38], Herz [12], Newman [32] and other authors (see [42], [8], [24]). However, none of these results provides a complete characterization of the generators.

For example, Beurling proved in [3] the following result:

*Let $F \in \ell_1$. If the Hausdorff dimension of the zero set $Z_f$ is less than $2 - 2/p$ then $F$ is a generator in $\ell_p$ ($1 < p < 2$).*

The converse, however, is not true. Indeed, one can construct $F$ which is a generator in every $\ell_p$, $1 < p < 2$, but $Z_f$ has Hausdorff dimension 1 (see [32]).

3.1.2. In this chapter we study the problem: *is it possible at all* to characterize the generators in $\ell_p$ ($1 < p < 2$) by the zero set of the Fourier transform?

It should be pointed out that for $1 < p < 2$, there is no canonical way to define the zero set of the Fourier transform of a general element in $\ell_p$. First we note that no such a problem arises in the cases $p = 1, 2$. Indeed, if $F \in \ell_1$ then the zero set $Z_f$ is a well-defined closed set, since then $f$ is a continuous function. If $F \in \ell_2$ then $f \in L^2(\mathbb{T})$, so in this case the set $Z_f$ is defined up to measure zero. In the case
1 < p < 2, one may notice that the set $Z_f$ could also be defined up to measure zero, since by the Hausdorff-Young theorem $f \in L^q(\mathbb{T})$, where $q = p/(p - 1)$. However, this approach cannot provide a characterization of the generators by the zero set: there exists $F$ which is not a generator in $\ell_p$ (1 < p < 2), but $f(t) \neq 0$ almost everywhere (see [42]).

Our main result shows that, unlike the classical cases $p = 1$ and 2, the characterization of the generators in $\ell_p$ (1 < p < 2) by the zero set $Z_f$ of the Fourier transform is impossible. Precisely, we prove the following theorem:

**Theorem 3.1.** Given 1 < p < 2 one can find two continuous functions $f$ and $g$ on the circle $\mathbb{T}$, with the following properties:

(i) $\{t : f(t) = 0\} = \{t : g(t) = 0\}$,

(ii) $F := \hat{f}$ and $G := \hat{g}$ are both in $\ell_p(\mathbb{Z})$,

(iii) $G$ is a generator in $\ell_p$, but $F$ is not.

**Remarks.**

1. We emphasize the role of the continuity condition in the theorem, which makes precise the concept of the “zero set”. The result shows that it does not matter how one may define the zero set in general; the characterization of the generators in $\ell_p$ (1 < p < 2) by the zero set $Z_f$ is impossible already in the case when $Z_f$ is well-defined.

2. As we will show, the function $f$ in Theorem 3.1 can in fact be chosen infinitely smooth. However, $f$ and $g$ cannot both be smooth. We discuss this in some more detail below.

3. One may check that Theorem 3.1 is not true for $p > 2$. However, in no way does this mean that the generators in $\ell_p$ (p > 2) could be characterized by the zero set of their Fourier transforms. In fact, the Fourier transform of an element in $\ell_p$ (p > 2) is generally not a function, but rather a distribution, so one cannot even define its zero set in general.

The $L^p(\mathbb{R})$ version of the result is also true:

**Theorem 3.2.** Given 1 < p < 2 one can find two functions $F$ and $G$ in $L^p(\mathbb{R})$ with the following properties. The Fourier transforms $\hat{F}(t)$, $\hat{G}(t)$ are continuous functions on $\mathbb{R}$; they have the same zero set; the set of translates $\{G(x - u)\}$, $u \in \mathbb{R}$, spans the whole space, but $\{F(x - u)\}$ does not.

**Remark.** As we show at the end of this chapter, the functions $F$ and $G$ in Theorem 3.2 could be chosen infinitely smooth, and even as the restrictions to the real line of two entire functions of order 1.

The results of this chapter were published in [30].
3.2. Reformulation

3.2.1. The Fourier transform

\[ F \in \ell_p(\mathbb{Z}) \mapsto f(t) := \sum_{n \in \mathbb{Z}} F(n) e^{int} \]

allows to identify \( \ell_p(\mathbb{Z}) \) with the Banach space \( A_p(\mathbb{T}) \) of functions on the circle with Fourier coefficients in \( \ell_p \), endowed with the norm \( \|f\|_{A_p} = \|\hat{f}\|_{\ell_p} \). A linear combination of translates of \( F \) corresponds via this mapping to a multiplication of \( f \) by a trigonometric polynomial. It is therefore obvious that \( F \) is a generator in \( \ell_p(\mathbb{Z}) \) if and only if the set \( \{P(t)f(t)\} \), where \( P \) goes through all trigonometric polynomials, is dense in the space \( A_p(\mathbb{T}) \).

Theorem 3.1 could be therefore reformulated in the following way: there exist two continuous functions \( f, g \in A_p(\mathbb{T}) \), having the same zero set, such that the set

\[ \{P(t)g(t) : P \text{ is a trigonometric polynomial}\} \quad (3.2.1) \]

is dense in \( A_p(\mathbb{T}) \), but the set \( \{P(t)f(t)\} \) is not dense.

3.2.2. We will prove the following theorem.

**Theorem 3.3.** For any \( 1 < p < 2 \) one can construct a compact \( E \subset \mathbb{T} \), and a function \( g \in C(\mathbb{T}) \cap A_p(\mathbb{T}) \), such that:

(a) \( Z_g := \{t : g(t) = 0\} = E \);
(b) The set \( (3.2.1) \) is dense in \( A_p \);
(c) There is a (non-zero) distribution \( S \), supported by \( E \), which belongs to \( A_q \), where \( q = p/(p-1) \) is the conjugate of \( p \).

As explained above, \( \hat{g} \) is a generator. On the other hand \( \hat{c} \) is equivalent to the fact that no Fourier transform of a smooth function \( f \) vanishing on \( E \), could be a generator. More precisely: if \( F = \hat{f} \in \ell_1 \) is not a generator in \( \ell_p \), then \( Z_f \) must support a non-zero distribution \( S \in A_q \) (see [19], Chapter III, §6); and conversely, if \( f \in \text{Lip}(\alpha), \alpha > 1/p - 1/2 \), and \( Z_f \) supports such a distribution, then \( F \) is not a generator in \( \ell_p \) (see [19], Chapter IX, §6).

Theorem 3.1 is therefore a direct consequence of Theorem 3.3. Moreover, the function \( f \) in Theorem 3.1 can be chosen infinitely smooth: any \( f \) such that \( Z_f = E \) will do. Theorem 3.2 also follows from Theorem 3.3 as we show later on.

3.2.3. Theorem 3.3 is a strengthening of our result from Chapter 2, where we constructed a compact \( K \) which supports a distribution belonging to \( A_q \) (\( q > 2 \)), but which does not support such a measure (Theorem 2.1). Indeed, we will now show that from \( \hat{b} \) it follows that \( E \) cannot support a non-zero measure \( \mu \in A_q \), hence the compact \( E \) from Theorem 3.3 also satisfies the result of Theorem 2.1.
To see this, suppose that $\mu \in A_q$ is a measure supported by $E$. Then we have
\[
\sum_{n \in \mathbb{Z}} \hat{g}(n-k) \hat{\mu}(-n) = \int_T g(t) e^{ikt} d\mu(t) = 0, \quad k \in \mathbb{Z},
\]
since $\mu$ is supported by $Z_g$. The Fourier transform $\hat{\mu}$ is therefore an element of $\ell_q$ (the dual space of $\ell_p$) which annihilates all the translates of $\hat{g}$. But $\hat{g}$ is a generator in $\ell_p$, so we must have $\hat{\mu} = 0$. Hence also $\mu = 0$.

3.3. Riesz-type products

3.3.1. Notation. As before we use finite Riesz product, but instead of the cosine function we now use a certain trigonometric polynomial $\varphi$. We define
\[
\lambda_s(t) = \prod_{j=1}^{N} (1 + s \varphi(\nu^jt)) \quad (3.3.1)
\]
where $N$ and $\nu$ denote positive integers, the parameter $s$ denotes a real number and $\varphi$ is a trigonometric polynomial. We will assume that
\[
\varphi \text{ is real, } \hat{\varphi}(0) = 0, \quad \|\varphi\|_{\infty} \leq 1, \quad \|\varphi\|_{L^2(T)} > \frac{9}{10}, \quad (3.3.2)
\]

3.3.2. Fourier expansion. Replacing $\varphi$ by its Fourier series in (3.3.1) and expanding the product, we get
\[
\lambda_s(t) = 1 + \sum_{k \neq 0} \left\{ s^{l(k)} \prod_{k_j \neq 0} \hat{\varphi}(k_j) \right\} e^{i(k_1\nu+k_2\nu^2+\cdots+k_N\nu^N)t}, \quad (3.3.3)
\]
where the sum is over all non-zero vectors
\[
k = (k_1, \ldots, k_N) \in \mathbb{Z}^N, \quad |k_j| \leq \deg \varphi,
\]
and $l(k) > 0$ is the number of non-zero coordinates of $k$. We will assume the condition
\[
\nu > 2 \deg \varphi, \quad (3.3.4)
\]
which guarantees that every integer $n$ admits at most one representation
\[
n = k_1\nu+k_2\nu^2+\cdots+k_N\nu^N
\]
with $k$ as above. The members of the sum (3.3.3) are therefore exponentials with distinct frequencies, so (3.3.3) is the Fourier expansion of $\lambda_s$.

3.3.3. Probabilistic concentration. For $0 < s < 1$ it follows from (3.3.1), (3.3.2) and (3.3.3) that the function $\lambda_s(t)$ is positive and has integral $= 1$. We thus may consider a probability measure $\lambda_s(t) \, dt/2\pi$ on the circle. Define a trigonometric polynomial
\[
X(t) = \frac{1}{N} \sum_{j=1}^{N} \varphi(\nu^jt), \quad (3.3.5)
\]
and consider $X$ as a random variable with respect to the measure $\lambda_s(t) \, dt/2\pi$. 
Lemma 3.1. For $\nu \geq \nu(N, \varphi)$ one has
\[
\int_{\{t \in \mathbb{T} : |X(t) - \varphi(X)| > \alpha\}} \lambda_s(t) \frac{dt}{2\pi} \leq 3 \exp\left(-\frac{1}{8} \alpha^2 N\right) \tag{3.3.6}
\]
which holds for every $\alpha > 0$ and every $s \in \left(\frac{8}{10}, \frac{9}{10}\right)$.

The proof of Lemma 3.1 is very similar to the proof of Lemma 2.7 and is therefore omitted.

3.3.4. Concentration in $L^2$. Using (3.3.3) and (3.3.4) one can calculate
\[
\mathcal{E}(X) = \int_{\mathbb{T}} X(t) \lambda_s(t) \frac{dt}{2\pi} = s \|\varphi\|_{L^2(\mathbb{T})}^2. \tag{3.3.7}
\]

Lemma 3.2. Given $\delta > 0$ there is $N(\delta)$ with the following property. For each $N \geq N(\delta)$ and each trigonometric polynomial $\varphi$ satisfying (3.3.2) one can find $\nu = \nu(N, \varphi)$ satisfying (3.3.4) such that
\[
\int_{\{t \in \mathbb{T} : X(t) < \frac{1}{40}\}} \lambda_s^2(t) \frac{dt}{2\pi} < \delta \tag{3.3.8}
\]
holds for every $s \in \left(\frac{8}{10}, \frac{9}{10}\right)$.

Proof. Given $N$ and $\varphi$, use Lemma 3.1 to choose $\nu = \nu(N, \varphi)$. We have the estimate
\[
\lambda_s(t) \leq \exp\left(s \sum_{j=1}^{N} \varphi(\nu^j t)\right) = \exp\left(sN X(t)\right). \tag{3.3.9}
\]
From (3.3.7) it follows that $\mathcal{E}(X) > \frac{5}{8}$, so using (3.3.6) and (3.3.9) we get
\[
\int_{\{t : X(t) < \frac{1}{40}\}} \lambda_s^2(t) \frac{dt}{2\pi} \leq \left(\int_{\{t : X(t) < \frac{1}{40}\}} \lambda_s(t) \frac{dt}{2\pi}\right) \left(\sup_{\{t : X(t) < \frac{1}{40}\}} \lambda_s(t)\right)
\leq 3 \exp\left(-\frac{1}{8} (\frac{5}{8} - \frac{1}{30})^2 N\right) \exp\left(sN \cdot \frac{1}{40}\right)
< 3 \exp\left(-\frac{1}{50} N\right),
\]
and so (3.3.8) holds for any sufficiently large $N$. \qed

3.4. Auxiliary function construction

The trigonometric polynomial $\varphi$ used in the Riesz product (3.3.1) will be constructed in the following lemma.

Lemma 3.3. Given $0 < \eta < 1$ there is a real trigonometric polynomial $\varphi = \varphi_\eta$ such that
\[
\hat{\varphi}(0) = 0, \quad \|\varphi\|_\infty < 1, \quad \|\varphi\|_{L^2} > \frac{a}{10}, \quad \|\varphi\|_{A_p} \leq C \eta^{-1}, \quad \|\varphi\|_{A_q} \leq C \eta,
\]
where $C$ is an absolute constant.
We use the common notation $A(\mathbb{T}) = A_1(\mathbb{T})$ for the Banach space of absolutely convergent Fourier series. In order to prove Lemma 3.3 we need:

**Lemma 3.4.** Let $T_1, \ldots, T_m \in A_r(\mathbb{T})$, $1 \leq r < \infty$, $f \in A(\mathbb{T})$, $\hat{f}(0) = 0$. Then

\[
\lim_{\nu \to \infty} \left\| \sum_{j=1}^{m} f(\nu^j t) T_j \right\|_{A_r} = \|f\|_{A_r} \left( \sum_{j=1}^{m} \|T_j\|_{A_r}^r \right)^{1/r}.
\] (3.4.1)

**Proof.** Denote by $P$ and $Q_1, \ldots, Q_m$ trigonometric polynomials obtained as partial sums of the Fourier series of $f$ and $T_1, \ldots, T_m$ respectively. We have

\[
\left\| \sum_{j=1}^{m} f(\nu^j t) T_j - \sum_{j=1}^{m} P(\nu^j t) Q_j \right\|_{A_r} \\
\leq \sum_{j=1}^{m} \left\| f(\nu^j t) (T_j - Q_j) \right\|_{A_r} + \sum_{j=1}^{m} \left\| (f(\nu^j t) - P(\nu^j t)) Q_j \right\|_{A_r} \\
\leq \|f\|_{A_r} \sum_{j=1}^{m} \|T_j - Q_j\|_{A_r} + \|f - P\|_{A_r} \sum_{j=1}^{m} \|T_j\|_{A_r}.
\]

Given $\varepsilon > 0$, we may therefore choose $P$ and $Q_1, \ldots, Q_m$ such that

\[
\left\| \sum_{j=1}^{m} f(\nu^j t) T_j - \sum_{j=1}^{m} P(\nu^j t) Q_j \right\|_{A_r} < \varepsilon.
\] (3.4.2)

If $\nu$ is sufficiently large then each Fourier coefficient of $\sum P(\nu^j t) Q_j$ is a product $\hat{P}(k) \hat{Q}_j(l)$, and it is easy to check that this implies

\[
\left\| \sum_{j=1}^{m} P(\nu^j t) Q_j \right\|_{A_r} = \|P\|_{A_r} \left( \sum_{j=1}^{m} \|Q_j\|_{A_r}^r \right)^{1/r}.
\] (3.4.3)

By an appropriate choice of $P, Q_1, \ldots, Q_m$ the right hand side of (3.4.3) will differ by not more than $\varepsilon$ from the right hand side of (3.4.1). On the other hand, (3.4.2) allows us to replace $P$ and $Q_j$ in the left hand side of (3.4.3) by $f$ and $T_j$, and the $A_r$ norm of the error will be smaller than $\varepsilon$. This proves (3.4.1). □

**Proof of Lemma 3.3.** Choose and fix a function $\Psi(u)$ on $[0, 1]$, infinitely smooth and vanishing in a neighborhood of the points 0 and 1, such that

\[0 \leq \Psi(u) < 1, \quad \int_0^1 \Psi(u)^2 \, du > 9/10.\]

We shall denote by $C_1, C_2, \ldots$ positive constants which depend only on $\Psi$. We associate with each interval $I = [a, a + h]$ a function $\Psi_I$, defined on $I$ by

\[\Psi_I(a + uh) := \Psi(u), \quad u \in [0, 1],\]

and vanishes outside $I$. Let $|I|$ denote the length of $I$, then one may check that

\[\|\Psi_I\|_{A_r} \leq C_1 |I|^{(r-1)/r}, \quad \text{for any interval } I \subset [0, 2\pi] \text{ and } 1 \leq r < \infty.\] (3.4.4)
Define a $2\pi$-periodic function $\Phi(t)$ by
$$
\Phi(t) = \Psi_{[0,\pi]}(t) - \Psi_{[\pi,2\pi]}(t), \quad t \in [0, 2\pi].
$$
Partition the segment $[0, 2\pi]$ into $m$ segments $I_1, \ldots, I_m$ of equal lengths, and set
$$
\varphi_{\nu}(t) := \sum_{j=1}^{m} \Phi(\nu^j t) \Psi_{I_j}(t), \quad t \in [0, 2\pi].
$$
Clearly $\|\varphi_{\nu}\|_\infty < 1$. Now apply Lemma 3.4 and use (3.4.4) to conclude that
$$
\lim_{\nu \to \infty} \|\varphi_{\nu}\|_{A_r} = \|\Phi\|_{A_r} \left( \sum_{j=1}^{m} \|\Psi_{I_j}\|_{A_r}^r \right)^{1/r} \leq C_2 m^{(2-r)/r}
$$
for every $1 \leq r < \infty$. Choose $m$ such that $3^{-1}\eta^{-1} < m^{(2-p)/p} < 3\eta^{-1}$, then
$$
\lim_{\nu \to \infty} \|\varphi_{\nu}\|_{A_p} \leq C_3 \eta^{-1}, \quad \lim_{\nu \to \infty} \|\varphi_{\nu}\|_{A_q} \leq C_3 \eta.
$$
We also have
$$
\lim_{\nu \to \infty} \|\varphi_{\nu}\|_{L^2} = \|\Phi\|_{L^2} \left( \sum_{j=1}^{m} \|\Psi_{I_j}\|_{L^2}^2 \right)^{1/2} = \int_0^1 \Psi(u)^2 \, du > 9/10,
$$
and, since $\widehat{\Phi}(0) = 0$,
$$
\lim_{\nu \to \infty} \widehat{\varphi}_{\nu}(0) = 0.
$$
It is therefore possible to obtain the polynomial $\varphi$ of the lemma by convolving the function $\varphi_{\nu}(t) - \widehat{\varphi}_{\nu}(0)$, where $\nu$ is sufficiently large, with an appropriate Fejér kernel.

3.5. Proof of main result

3.5.1. The main lemma. In this section we prove:

**Lemma 3.5.** Given $\varepsilon > 0$ there is a compact $K$ on the circle, an infinitely differentiable function $F$ and a real trigonometric polynomial $X$ such that:

(i) $F$ is supported by $K$, \quad $\|1 - F\|_{A_q} < \varepsilon$,

(ii) $\|X\|_\infty \leq 1$, \quad $\|X\|_{A_p} < \varepsilon$, \quad $X(t) > \frac{1}{50}$ on $K$.

**Proof.** For $\delta > 0$ to be chosen, use Lemma 1.2 to find a measure $\rho$ supported by the interval $(\frac{8}{10}, \frac{9}{10})$ and satisfying (1.2.2). Define
$$
\lambda(t) = \int \lambda_s(t) \, d\rho(s).
$$
In view of (3.3.3), $\lambda$ is a trigonometric polynomial whose Fourier expansion is
$$
\lambda(t) = \sum_{k \neq 0} \left\{ \int s^{(k)} d\rho(s) \prod_{j \neq 0} \widehat{\varphi}(k_j) \right\} e^{i(k_1 \nu + k_2 \nu^2 + \cdots + k_N \nu^N) t}.
$$
We therefore have \( \hat{\lambda}(0) = 1 \), and
\[
\sum_{n \neq 0} |\hat{\lambda}(n)|^q < \delta \sum \prod_{k \neq 0 \ k_j \neq 0} |\hat{\varphi}(k)|^q = \delta (1 + \|\varphi\|_{A_q}^q)^N.
\] (3.5.1)

Consider the compact 
\[ K' = \{ t \in \mathbb{T} : X(t) \geq \frac{1}{40} \} \]
where \( X \) is given by \((3.3.5)\). By Lemma \((3.2)\) for any \( N \gtrsim N(\delta, \rho) \) and for any \( \varphi \) satisfying \((3.3.2)\) we can find \( \nu = \nu(N, \varphi) > 2 \deg \varphi \), such that
\[
\left( \int_{\mathbb{T} \setminus K'} \lambda_s^2(t) \frac{dt}{2\pi} \right)^{1/2} < \frac{\delta}{\|\rho\|_M}
\]
holds for all \( s \in \text{supp}(\rho) \). We now choose \( \varphi = \varphi_\eta \) to be the trigonometric polynomial given by Lemma \((3.3)\) with
\[
\eta = \varepsilon^{-1} N^{-1/q},
\]
(3.5.2)
where \( N \) is taken large enough such that \( 0 < \eta < 1 \). Then
\[
(1 + \|\varphi\|_{A_q}^q)^N \lesssim \exp(N\|\varphi\|_{A_q}^q) \lesssim \exp(C^q \eta^q N) \lesssim \exp(C^q \varepsilon^{-q}),
\]
so from \((3.5.1)\) it follows that
\[
\|1 - \lambda\|_{A_q} \lesssim C(p, \varepsilon) \delta^{1/q}.
\]
Denote \( H = \lambda \cdot 1_{K'} \), then
\[
\|\lambda - H\|_{A_q} \lesssim \|\lambda - H\|_{L^2(\mathbb{T})} = \|\lambda\|_{L^2(\mathbb{T} \setminus K')} \lesssim \int_{\mathbb{T} \setminus K'} \|\lambda_s(t)\|_{L^2(\mathbb{T} \setminus K')} \, d|\rho| < \delta.
\]
It is therefore clear that if \( \delta = \delta(p, \varepsilon) \) is chosen sufficiently small, we get \( \|1 - H\|_{A_q} < \varepsilon \). We then define the function \( F \) as the convolution of \( H \) with a smooth non-negative function with integral \( = 1 \), which is supported by a sufficiently small neighbourhood of \( 0 \) to ensure that \( X(t) > \frac{1}{50} \) on \( K := \text{supp}(F) \).

Finally, we show that the trigonometric polynomial \( X \) satisfied the required conditions. Indeed, \( \|X\|_\infty \lesssim 1 \), and using the fact that \( \nu > 2 \deg \varphi \) we also have
\[
\|X\|_{A_p} = N^{1/p - 1} \|\varphi\|_{A_p} \lesssim C N^{1/p - 1} \|\varphi\|_{A_p} \lesssim C \varepsilon.
\]
The lemma is thus proved. \( \square \)

**3.5.2. Successive approximations.** For each \( n = 0, 1, 2, \ldots \) we shall construct an infinitely differentiable function \( f_n : \mathbb{T} \to \mathbb{C} \) and two trigonometric polynomials \( g_n \) and \( P_n \), such that the following properties hold. For all \( n \gtrsim 0 \),
\[
(i)_n \ |g_n(t)| \leq (\frac{99}{100})^n \text{ on supp}(f_n),
(ii)_n \ \|1 - P \cdot g_n\|_{A_p} < 2^{-n - 1},
\]
and for all \( n \geq 1 \),
\[
(iii)_n \ |g_{n+1} - g_n|_\infty \leq (\frac{99}{100})^{n-1},
(iv)_n \ |g_{n+1} - g_n|_{A_p} < 2^{-n-1} (1 + \|P_0\|_{A} + \|P_1\|_{A} + \cdots + \|P_{n-1}\|_{A}^{-1}),
\]
(v) \( \| f_{n-1} - f_n \|_{A_q} < 2^{-n-1} \),

(vi) \( \text{supp}(f_n) \subset \text{supp}(f_{n-1}) \).

The construction is done by induction. We define \( f_0 = g_0 = P_0 = 1 \), and note that the properties (i), and (ii), are satisfied.

Suppose now that \( f_k, g_k \) and \( P_k \) have been constructed for all \( 0 \leq k \leq n \). Then \( f_{n+1}, g_{n+1} \) and \( P_{n+1} \) can be constructed as follows. The property (iv), ensures that one can find a trigonometric polynomial \( h \) such that \( \| h \|_{\infty} \leq (\frac{50}{99})^n \) and

\[
|g_n(t) - h(t)| < \frac{1}{100} \cdot (\frac{99}{100})^n, \quad \text{for every } t \in \text{supp}(f_n).
\]

For \( \varepsilon > 0 \) to be chosen, let \( K, F \) and \( X \) be given by Lemma 3.5, and set

\[
f_{n+1} := f_n \cdot F, \quad g_{n+1} := g_n - h \cdot X.
\]

Indeed \( f_{n+1} \) is an infinitely differentiable, and \( g_{n+1} \) is a trigonometric polynomial. Since we have

\[
\|g_n - g_{n+1}\|_{A_p} = \|h \cdot X\|_{A_p} \leq \|h\|_A \|X\|_{A_p} \leq \varepsilon \|h\|_A,
\]

and

\[
\|f_n - f_{n+1}\|_{A_q} = \|f_n \cdot (1 - F)\|_{A_q} \leq \|f_n\|_A \|1 - F\|_{A_q} \leq \varepsilon \|f_n\|_A,
\]

for a sufficiently small \( \varepsilon \) the properties (iv), and (v), will be satisfied. Also

\[
\|g_n - g_{n+1}\|_{\infty} = \|h \cdot X\|_{\infty} \leq \|h\|_{\infty} \|X\|_{\infty} \leq (\frac{50}{99})^n,
\]

which gives (v). Next, note that \( \text{supp}(f_{n+1}) \subset \text{supp}(f_n) \cap K \) and so property (vi) is satisfied. Also, the fact that \( |1 - X(t)| < \frac{49}{50} \) on \( K \), together with (3.5.3), allows to conclude that for every \( t \in \text{supp}(f_{n+1}) \)

\[
|g_{n+1}(t)| = |g_n(t) - h(t)X(t)| = |g_n(t) - h(t)(1 - X(t))h(t)|
\leq |g_n(t) - h(t)| + |1 - X(t)| \cdot |h(t)|
\leq \frac{1}{100} \cdot (\frac{99}{100})^n + \frac{49}{50} \cdot (\frac{99}{100})^n = (\frac{99}{100})^{n+1},
\]

hence (i) holds as well. Finally, note that (iv), (iv), \ldots, (iv) imply that

\[
\|1 - g_{n+1}\|_{A_p} \leq \sum_{k=1}^{n+1} \|g_{k-1} - g_k\|_{A_p} < \sum_{k=1}^{n+1} 2^{-k-1} < 2^{-1},
\]

hence the trigonometric polynomial \( g_{n+1} \) is non-zero. One can therefore find a trigonometric polynomial \( P_{n+1} \) satisfying (ii), (vi), (ii), and (iv), together with \( f_0 = 1 \), imply that the sequence \( f_n \) converges in \( A_q \) to a non-zero distribution \( S \) such that \( \text{supp}(S) \subset \bigcap_{n=1}^{\infty} \text{supp}(f_n) \). The properties (iii), and (iv), imply that the sequence \( g_n \) converges both uniformly and in \( A_p(T) \) to a function \( g \in

### 3.5.3. Conclusion of the proof.

The properties (v), and (vi), together with \( f_0 = 1 \), imply that the sequence \( f_n \) converges in \( A_q \) to a non-zero distribution \( S \) such that \( \text{supp}(S) \subset \bigcap_{n=1}^{\infty} \text{supp}(f_n) \). The properties (iii), and (iv), imply that the sequence \( g_n \) converges both uniformly and in \( A_p(T) \) to a function \( g \in
C(\mathbb{T}) \cap A_p(\mathbb{T})$. From [iii], it is seen that \( g(t) = 0 \) for every \( t \in \text{supp}(S) \). In addition, using [iv], we have
\[
\| P_n \cdot (g_n - g) \|_{A_p} \leq \| P_n \|_A \sum_{k=n+1}^{\infty} \| g_{k-1} - g_k \|_{A_p} \leq \sum_{k=n+1}^{\infty} 2^{-k-1} = 2^{-n-1},
\]
and therefore
\[
\| 1 - P_n \cdot g \|_{A_p} \leq \| 1 - P_n \cdot g_n \|_{A_p} + \| P_n \cdot (g_n - g) \|_{A_p} < 2^{-n-1} + 2^{-n-1} = 2^{-n}.
\]
That is, the function 1 can be approximated in \( A_p(\mathbb{T}) \) by functions of the form \( P \cdot g \), where \( P \) is a trigonometric polynomial. This easily implies that any trigonometric polynomial can be approximated by these functions, which shows that the set \( (3.2.1) \) is dense in \( A_p(\mathbb{T}) \). The function \( g \) therefore satisfies the two conditions given in Theorem 3.3 so the proof is complete.

### 3.6. The \( L^p(\mathbb{R}) \) version

In this section we prove Theorem 3.3.

#### 3.6.1. Being given a Schwartz function \( \gamma(x) \) on \( \mathbb{R} \), consider the linear operator \( T \) which maps each trigonometric polynomial
\[
f(t) = \sum_{|n| \leq N} \hat{f}(n)e^{int}
\]
to the function \( Tf : \mathbb{R} \rightarrow \mathbb{C} \) defined by
\[
(Tf)(x) = \sum_{|n| \leq N} \hat{f}(n) \gamma(x + n), \quad x \in \mathbb{R}.
\]
Clearly \( Tf \) is a Schwartz function, for every trigonometric polynomial \( f \).

**Lemma 3.6.** \( \|Tf\|_{L^p(\mathbb{R})} \leq M \|f\|_{A_p(\mathbb{T})} \), the constant \( M \) not depending on \( f \).

**Proof.** We must show that for any \( N \) and scalars \( \{c(n)\} \) one has
\[
\left\| \sum_{|n| \leq N} c(n) \gamma(x + n) \right\|_{L^p(\mathbb{R})} \leq M \left\| \{c(n)\} \right\|_{\ell^p}
\]
for some \( M = M(\gamma) \). An application of Hölder’s inequality yields
\[
\left| \sum_{|n| \leq N} c(n) \gamma(x + n) \right| \leq \left( \sum_{|n| \leq N} |c(n)|^p \right)^{1/p} \left( \sum_{|n| \leq N} |\gamma(x + n)| \right)^{1/q}.
\]
Choose \( M \) large enough so that
\[
\sup_{x \in \mathbb{R}} \sum_{n \in \mathbb{Z}} |\gamma(x + n)| \leq M.
\]
Then
\[
\left( \sum_{|n| \leq N} |c(n)| \right)^p \leq M^{p-1} \sum_{|n| \leq N} |c(n)|^p |\gamma(x + n)|.
\]
If $M$ is sufficiently large then also $\int_\mathbb{R} |\gamma(x)| \, dx \leq M$, thus integration yields

$$\int_\mathbb{R} \left| \sum_{|n| \leq N} c(n) \gamma(x + n) \right|^p \, dx \leq M^p \sum_{|n| \leq N} |c(n)|^p.$$  

This proves the lemma. \hfill $\Box$

Lemma 3.6 allows to extend the linear operator $T$ by continuity to a bounded operator $T : A_p(\mathbb{T}) \to L^p(\mathbb{R})$. It follows from (3.6.1) that $T$ has the properties

$$T(e^{int}f)(x) = (Tf)(x + n), \quad (3.6.2)$$

and

$$\widehat{(Tf)}(t) = \widehat{\gamma}(t)f(t), \quad t \in \mathbb{R}, \quad (3.6.3)$$

where in (3.6.3) the function $f$ is considered as a $2\pi$-periodic function, and the equality is understood in the sense of distributions, or almost everywhere.

3.6.2. Choose a Schwartz function $\gamma$ whose Fourier transform $\widehat{\gamma}$ has no zeros, and let $g \in C(\mathbb{T}) \cap A_p(\mathbb{T})$ be the function given by Theorem 3.3. Define a function $G \in L^p(\mathbb{R})$ by $G := Tg$. For each trigonometric polynomial $P$,

$$\|\gamma - T(P \cdot g)\|_{L^p(\mathbb{R})} = \|T(1 - P \cdot g)\|_{L^p(\mathbb{R})} \leq \|T\| \cdot \|1 - P \cdot g\|_{A_p(\mathbb{T})}.$$  

The set $\{P(t)g(t)\}$ is dense in $A_p(\mathbb{T})$, so in particular the constant function which is equal to 1 identically, can be approximated by products $P(t)g(t)$. It follows that $\gamma$ can be approximated in $L^p(\mathbb{R})$ by functions of the form $T(P \cdot g)$. Observe that $T(P \cdot g)$ is a finite linear combination of translates of $G$ (in fact, integer translates), according to (3.6.2). The function $\gamma$ therefore belongs to the closed linear subspace of $L^p(\mathbb{R})$ generated by the translates of $G$. But $\gamma$ is a Schwartz function with a non-vanishing Fourier transform, so it is a generator in $L^p(\mathbb{R})$ (see [3]). Hence $G$ is a generator as well.

On the other hand, choose an infinitely differentiable function $f : \mathbb{T} \to \mathbb{C}$ such that $Z_f = Z_g$, and define $F := Tf \in L^p(\mathbb{R})$. By (3.6.3) we have $\widehat{F}(t) = \widehat{\gamma}(t)f(t)$ and $\widehat{G}(t) = \widehat{\gamma}(t)g(t)$, and since $\widehat{\gamma}$ has no zeros it follows that $\widehat{F}$ and $\widehat{G}$ are continuous functions having the same zero set. We also note that $F$ is a Schwartz function. Let us show that $F$ is not a generator. Indeed, the set $Z_f \subset \mathbb{T}$ supports a non-zero distribution $S$ belonging to $A_q(\mathbb{T})$. Since $Z_f$ is not the whole circle, we may identify $S$ with a distribution on $\mathbb{R}$, supported by an interval of length $< 2\pi$, whose Fourier transform belongs to $L^q(\mathbb{R})$ (see [4], Corollary 10.6.6, p. 197). But the zero set of $\widehat{F}$ (i.e. the $2\pi$-periodization of $Z_f$) contains the supports of $S$, hence $F$ cannot be a generator (see [38], Theorem B).

This completes the proof of Theorem 3.2.
3.6.3. We conclude this chapter by clarifying our remark made earlier, that the functions $F$ and $G$ in Theorem 3.2 could in fact be chosen as the restrictions to the real line of two entire functions of order 1.

Indeed, we can choose our function $\gamma$ such that its Fourier transform $\hat{\gamma}$ decays arbitrarily fast. In particular, we may assume that for every positive number $r$,

$$|\hat{\gamma}(t)| = O(e^{-|t|^r}), \quad |t| \to \infty. \quad (3.6.4)$$

In this case the integral

$$F(z) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{F}(t) e^{izt} dt, \quad z = x + iy, \quad (3.6.5)$$

will converge absolutely and uniformly in every strip $|y| \leq y_0$, hence a standard argument shows that $F(z)$ is an analytic function in the whole complex plane. Let us prove that it has order 1. It will be enough to show that $|F(z)| = O(e^{|z|^\alpha})$ for every $\alpha > 1$. Obviously there is the estimate

$$|F(z)| \leq C \int_{\mathbb{R}} |\hat{\gamma}(t)| e^{y|t|} dt, \quad C := \|f\|_\infty.$$

Let now $\beta$ denote the exponent conjugate to $\alpha$ satisfying $1/\alpha + 1/\beta = 1$. Then by Young’s inequality $|y| \cdot |t| \leq \frac{1}{\alpha} |y|^\alpha + \frac{1}{\beta} |t|^\beta$, and consequently

$$|F(z)| \leq C \exp \left(\frac{1}{\alpha} |y|^\alpha\right) \int_{\mathbb{R}} |\hat{\gamma}(t)| \exp \left(\frac{1}{\beta} |t|^\beta\right) dt.$$

The condition (3.6.4) now implies the convergence of the integral on the right hand side, so we get $|F(z)| = O(\exp(\frac{1}{\alpha} |y|^\alpha))$. This shows that $F$ is an entire function of order 1, and by similar arguments the same conclusion is true for the function $G$. 
CHAPTER 4

Related Problems

In this chapter we study further aspects of Piatetski-Shapiro’s phenomenon in $\ell_q$ ($q > 2$), and obtain several strengthenings of Theorem 2.1. First we determine how small could be a compact satisfying Piatetski-Shapiro’s phenomenon, and in particular, how small can be the Hausdorff dimension of such a compact. Also the relation with a certain Fourier-analytic dimension will be discussed. Another question under consideration is whether Piatetski-Shapiro’s phenomenon is “typical” or “rare”. Lastly, we study the relation with sets of interpolation for the class $(A_p \cap C)(\mathbb{T})$, namely $p$-Helson sets.

4.1. Shrinking method, Hausdorff dimension

4.1.1. Introduction. In Section 1.1.2 we referred to the theorem of Kaufman: any set of multiplicity contains a Helson set of multiplicity. In this section we are interested in the following corollary of Kaufman’s theorem:

Let $K$ be a compact which supports a non-zero distribution $S$ with Fourier coefficients tending to zero. Then there is a compact $K_1 \subset K$, which also supports a non-zero distribution $S_1$ with Fourier coefficients tending to zero, but which does not support such a measure.

In other words, any compact of multiplicity can be “shrunk” to a compact of Piatetski-Shapiro type. It follows, for example, that such a compact can be “as small as desired” in the sense of Hausdorff dimension. Indeed, it was proved by Ivashev-Musatov [14] that for any continuous non-decreasing function $h(t)$ defined for $t \geq 0$, $h(0) = 0$, there exists a set of multiplicity (even in the restricted sense) with Hausdorff $h$-measure zero. It then follows that there is a compact $K$ which has this property and which satisfies Piatetski-Shapiro’s theorem.

It is interesting to remark that the Hausdorff dimension of the original Piatetski-Shapiro compact $K_\gamma$ ($0 < \gamma < 1/2$) (see Section 1.1.1) was computed by Besicovitch [2]. It is the number $\alpha \in (0, 1)$ determined by the equation

$$2^\alpha = \frac{1}{\gamma^\gamma(1 - \gamma)^{1-\gamma}}.$$  

Observe that by a convenient choice of $\gamma$ the dimension $\alpha$ can be made arbitrarily small, but it remains positive.

4.1.2. Result. Here we obtain a strengthening of Theorem 2.1 which is analogous to the above mentioned “shrinking theorem”. We prove the following result.
Theorem 4.1. Suppose a compact $K \subset \mathbb{T}$ supports a non-zero distribution $S$ such that $\hat{S} \in \ell_q$ ($q > 2$). Then there is a compact $K_1 \subset K$, which also supports a non-zero distribution $S_1$ with $\hat{S}_1 \in \ell_q$, but which does not support such a measure.

In particular, this allows us to determine the smallest possible Hausdorff dimension of a compact satisfying Theorem 2.1. It is known that a compact $K$ which supports a distribution $S$ with $\hat{S} \in \ell_q$, has Hausdorff dimension $\geq 2/q$ (see [19], Chapter VIII, § 4). On the other hand, there are examples of compacts $K$ of Hausdorff dimension $= 2/q$ which support such a distribution (and even a positive measure); this may be deduced from a result of Körner [26, Theorem 1.2] or, alternatively, in [39, Theorem 5] it is explained how to deduce this from results of Kahane [17]. By applying Theorem 4.1 to such a compact we obtain:

Corollary 4.1. For any $q > 2$ there exists a compact $K$ on the circle, which satisfies the result of Theorem 2.1 and which has Hausdorff dimension $2/q$.

4.1.3. Shrinking lemma. We will use the shrinking method of Kaufman, which corresponds to the multiplication of a distribution $S$ by a function $f$ in order to “shrink” its support, see [23, Chapter VII] and [9, Theorem 4.6.2] for more details. The proof of Theorem 4.1 will depend on the following replacement for Lemma 2.9.

Lemma 4.2. Let $K$ be a compact on the circle, and suppose that $S \in A_q$ is a non-zero distribution supported by $K$. Being given $\varepsilon > 0$ and a positive integer $\nu$, one can find a compact $K_1 \subset K$ with the following properties:

(i) $K_1$ supports a distribution $S_1 \in A_q$, $\|S_1 - S\|_{A_q} < \varepsilon$.

(ii) There is a real trigonometric polynomial

$$X(t) = \sum_{|n| \geq \nu} \hat{X}(n)e^{int}, \quad \|\hat{X}\|_p \leq 1, \quad \frac{1}{100} \leq X(t) \leq 100 \quad \text{on} \quad K_1.$$

In order to prove Lemma 4.2 we shall need the following lemma (also compare with Lemma 3.4).

Lemma 4.3. Let $f$ be a function in $A$, and $S$ be a distribution in $A_q$. For any integer $m$ we denote $f_m(t) = f(mt)$, and consider the product $Sf_m$. Then

$$\lim_{m \to \infty} \|Sf_m\|_{A_q} = \|f\|_{A_q} \|S\|_{A_q}.$$ 

Proof. Given two trigonometric polynomials $P$ and $Q$, consider the product $QP_m$, where we denote $P_m(t) = P(mt)$. It is easy to check that if $m$ is sufficiently large then $\|QP_m\|_{A_q} = \|Q\|_{A_q} \|P\|_{A_q}$, since each non-zero Fourier coefficient of $QP_m$ will be a product $\hat{P}(j)\hat{Q}(k)$.

Given $\varepsilon > 0$, we choose $P$ and $Q$ as partial sums of the Fourier series of $f$ and $S$, respectively. By an appropriate choice we may have the difference between
\[ ||Q||_{A_q} ||P||_{A_q} \text{ and } ||S||_{A_q} ||f||_{A_q} \] be smaller than \( \varepsilon \) by modulus. In addition, by a standard argument we have
\[ ||Sf_m - QP_m||_{A_q} \leq ||S - Q||_{A_q} ||f||_A + ||Q||_{A_q} ||f - P||_A , \]
so we may also satisfy
\[ ||Sf_m - QP_m||_{A_q} < \varepsilon \]
for every \( m \). It follows that for sufficiently large \( m \), the difference between \( ||Sf_m||_{A_q} \) and \( ||f||_{A_q} ||S||_{A_q} \) is smaller than \( 2\varepsilon \) by modulus. \( \square \)

**Proof of Lemma 4.2.** Using Lemma 2.9 one can find a compact \( E \subset \mathbb{T} \), a \( C^\infty \) function \( f : \mathbb{T} \to \mathbb{C} \) and a real trigonometric polynomial \( X \) such that
(i) \( f \) is supported by \( E \), \( ||f - 1||_{A_q} < \varepsilon/||S||_{A_q} \).
(ii) \( X(t) = \sum_{|n| \geq v} \hat{X}(n)e^{int}, ||\hat{X}||_p \leq 1 \) and \( \frac{1}{100} \leq X(t) \leq 100 \) on \( E \).

Define \( f_m(t) = f(mt), X_m(t) = X(mt) \) and let \( E_m \) denote the compact defined by \( 1_{E_m}(t) = 1_{E(mt)} \). Then for every positive integer \( m \) the conditions (i) and (ii) are also satisfied by \( E_m, f_m \) and \( X_m \) instead of \( E, f \) and \( X \). Let \( S_m \) denote the product \( S f_m \), then due to Lemma 4.3
\[ \lim_{m \to \infty} ||S_m - S||_{A_q} = \lim_{m \to \infty} ||S(f_m - 1)||_{A_q} = ||S||_{A_q} ||f - 1||_{A_q} < \varepsilon. \]

Hence for sufficiently large \( m \), the compact \( K \cap E_m \) satisfies the conditions (i) and (ii) of Lemma 4.2. \( \square \)

**4.1.4. Conclusion of the proof.** We now deduce Theorem 4.1 from Lemma 4.2. Let \( K \) be a compact on the circle, and let \( S \) be a non-zero distribution in \( A_q \) supported by \( K \). Choose numbers \( \varepsilon_j > 0 \), \( \sum_{j=1}^{\infty} \varepsilon_j < ||S||_{A_q} \) and positive integers \( \nu_j \) such that and \( \nu_j \to \infty \). (\( j \to \infty \)). Using Lemma 4.2 one may construct by induction compacts \( K_j \) on the circle (\( K_0 = K \)), distributions \( S_j \) in \( A_q \) (\( S_0 = S \)) and real trigonometric polynomials \( X_j \) such that, for every \( j \geq 1 \)

(i) \( K_j \subset K_{j-1} \).
(ii) \( S_j \) is supported by \( K_j \).
(iii) \( ||S_j - S_{j-1}||_{A_q} < \varepsilon_j \).
(iv) \( X_j(t) = \sum_{|n| \geq \nu_j} \hat{X}_j(n)e^{int}, ||\hat{X}_j||_p \leq 1 \) and \( \frac{1}{100} \leq X_j(t) \leq 100 \) on \( K_j \).

It follows that the sequence \( S_j \) converges in the \( A_q \) norm to a non-zero distribution \( S_\infty \) supported by \( K_\infty = \bigcap_{j=1}^{\infty} K_j \). By Lemma 2.1 \( K_\infty \) does not support a non-zero measure \( \mu \) in \( A_q \). We have thus obtained a compact \( K_\infty \) as required, so the theorem is proved.
4.2. Fourier-type dimension

4.2.1. The Fourier dimension of a compact \( K \subset \mathbb{T} \) is usually defined as the supremum of the numbers \( 0 \leq \beta \leq 1 \) for which there is a positive (non-zero) measure \( \mu \), supported by \( K \), such that \( |\hat{\mu}(n)| = O(|n|^{-\beta/2}) \). A well-known property of the Fourier dimension is that it can never exceed the Hausdorff dimension of \( K \) (see [19], Chapter VIII). The Fourier dimension need not coincide with the Hausdorff dimension in general. This is not surprising since the Hausdorff dimension is related to metrical properties of the set, while the Fourier dimension is closely related to its arithmetical properties.

In this section we consider a quantity similar to the Fourier dimension. We define the number \( q_0 = q_0(K) \) to be the infimum of all \( q > 2 \) for which there is a positive measure \( \mu \), supported by \( K \), such that \( \hat{\mu} \in \ell_q \). The number \( q_0 \), like the Fourier dimension, could be used to estimate the Hausdorff dimension from below: the set \( K \) has Hausdorff dimension \( \geq 2/q_0 \) (see Section 4.1).

Rosenblatt and Shuman in [39] used results of Beurling [3], Salem [41] and Kahane [17] to construct compact sets \( K \) with \( q_0 \) being any preassigned number, \( q_0 > 2 \). They constructed such examples in which \( K \) supports a positive measure with Fourier coefficients belonging to \( \ell_{q_0} \), and also other examples in which \( K \) does not even support such a distribution.

Here we construct an example which exhibits a different “threshold behavior”. The following strengthening of Theorem 2.1 is true: there is a compact \( K \) which does not support any measure \( \mu \) with \( \hat{\mu} \in \ell_{q_0} \), \( q_0 = q_0(K) \), but which does support such a distribution. In fact, we prove a bit more than that:

**Theorem 4.2.** For any \( q_0 > 2 \) there exists a compact \( K \) such that

(i) \( K \) supports a positive measure \( \mu \) such that \( \hat{\mu} \in \bigcap_{q > q_0} \ell_q \).

(ii) \( K \) does not support any measure with Fourier coefficients belonging to \( \ell_{q_0} \).

(iii) \( K \) supports a distribution \( S \) with Fourier coefficients belonging to \( \ell_{q_0} \).

4.2.2. In order to prove Theorem 4.2 we add a new ingredient to Lemma 2.9

**Lemma 4.4.** Let \( q > 2 \) be given. For any \( \varepsilon > 0 \) and any positive integer \( \nu \) one can find a compact \( K \) on the circle, two \( C^\infty \) functions \( f, g : \mathbb{T} \rightarrow \mathbb{C} \) and a real trigonometric polynomial \( X \) such that

(i) \( f \) is supported by \( K \).

(ii) \( f(t) = 1 + \sum_{n \neq 0} \hat{f}(n)e^{int}, \|f - 1\|_{A_q} < \varepsilon \).

(iii) \( g \) is supported by \( K \).

(iv) \( g(t) \geq 0, \ t \in \mathbb{T} \).

(v) \( g(t) = 1 + \sum_{n \neq 0} \hat{g}(n)e^{int}, \|g - 1\|_{A_{q+\varepsilon}} < \varepsilon \).

(vi) \( X(t) = \sum_{|n| \geq \nu} \hat{X}(n)e^{int}, \|\hat{X}\|_p \leq 1, \ \frac{1}{100} \leq X(t) \leq 100 \) on \( K \).
Proof. We choose $f$, $K$ and $X$ according to Lemma 2.9. Using the notations of this lemma, we now explain how to choose the function $g$. In view of 2.4.1, for any $\frac{1}{4} < s < \frac{1}{3}$ we have

$$\sum_{n \neq 0} |\lambda_s(n)|^{q+\varepsilon} < \sum_{n \neq \varepsilon \in \{-1,0,1\}^N} \left(\frac{1}{2} N^{-(q+\varepsilon)/q}\right)^{\sum|\varepsilon|} = (1 + N^{-\varepsilon/q})^N - 1,$$

from which we deduce the estimate

$$\sum_{n \neq 0} |\lambda_s(n)|^{q+\varepsilon} \leq \exp\left(N^{-\varepsilon/q}\right) - 1. \quad (4.2.1)$$

Let $g_1(t) = \lambda_s \cdot 1_{K'}$ for some arbitrary choice of $\frac{1}{4} < s < \frac{1}{3}$. Then $g_1$ is a positive function supported by $K'$, and

$$\|\lambda_s - g_1\|_{A^{q+\varepsilon}} \leq \|\lambda_s - g_1\|_{L^2(\mathbb{T})} = \|\lambda_s\|_{L^2(\mathbb{T} \setminus K')} < \delta.$$ 

The right side of (4.2.1) is also smaller than $\delta$, if $N$ is sufficiently large. So, as before, we can define $g_2(t) = g_1(t) \hat{g}_1(0)$ and take $g$ to be the convolution of $g_2$ with an appropriate smooth kernel. \qed

4.2.3. We now conclude the proof with a similar procedure as before.

Proof of Theorem 4.2. As in the proof of Theorem 2.1, we choose by induction a sequence $\{\varepsilon_j\}$ and let $f_j$, $g_j$, $X_j$ and $K_j$ be given by Lemma 4.4 (with $q = q_0$). If the $\{\varepsilon_j\}$ are tending to zero fast enough then the product $\prod_{j=1}^\infty f_j$ will converge in $A_{q_0}$ to a non-zero distribution $S$, which is supported by the compact $K = \bigcap_{j=1}^\infty K_j$. Lemma 2.7 implies that $K$ supports no non-zero measure in $A_{q_0}$.

We define the measure $\mu$ as the product $\prod_{j=1}^\infty g_j$. Denote $q_j = q_0 + \varepsilon_j$, $G_0 = 1$ and $G_j = g_1 \cdots g_j$. Given any $q > q_0$, we have $q_j < q$ for all sufficiently large $j$, which implies

$$\|G_{j+1} - G_j\|_{A_q} \leq \|G_{j+1} - G_j\|_{A_{q_{j+1}}} \leq \|G_j\|_A \cdot \|g_{j+1} - 1\|_{A_{q_{j+1}}}.$$ 

Hence if we choose

$$\varepsilon_1 < 2^{-2} \quad \text{and} \quad \|g_1 \cdot g_2 \cdot \ldots \cdot g_j\|_A \varepsilon_{j+1} < 2^{-2-j} \quad (j = 1, 2, \ldots),$$

then the product $\prod_{j=1}^\infty g_j$ will converge in $A_q$, for all $q > q_0$, to a distribution $\mu$. For each $j$, the function $g_j$ is positive and supported by $K_j$, so $\mu$ is a positive measure supported by $K$. Finally, we have

$$|\hat{G}_{j+1}(0) - \hat{G}_j(0)| \leq \|G_{j+1} - G_j\|_{A_{q_{j+1}}} < 2^{-2-j},$$

and therefore $|\hat{\mu}(0) - 1| < \sum_{j=0}^\infty 2^{-2-j} < 1$, so $\mu$ is non-zero. \qed
4.3. Baire category

In this section we consider the question: is the phenomenon of Theorem 2.1 “typical” or “rare”?

One possible approach to such type of questions is via probability theory. In this context, the introduction of a probability measure space allows one to consider an almost sure event as “typical”, and the complement of such an event as “rare”. Probabilistic methods in the theory of thin sets were used by Kahane and Salem, see [19] and [17].

Another approach is via Baire category. In this case, the introduction of a complete metric space is required. Rare phenomena then correspond to sets of first category, also known as meager sets, while typical phenomena correspond to their complements, namely the residual sets (we recall the precise definitions below). In connection with thin sets, Baire category arguments were used by Kaufman [21], Kahane [15] and Körner [27]. For more details see [18].

Inspired by Körner’s paper [27] we use the Baire category approach to show that the phenomenon of Theorem 2.1 is “typical” in an appropriate sense.

4.3.1. Preliminaries. We briefly recall the notions of Baire category theory. Let $\mathcal{X}$ be a complete metric space. A subset of $\mathcal{X}$ is called nowhere dense if its closure has no interior points. Equivalently, a set is nowhere dense if its complement contains an open, dense set. A subset of $\mathcal{X}$ is called a set of first category, or a meager set, if it is contained in the union of countably many nowhere dense sets. The complement of a set of first category is called a residual set. Equivalently, a set is residual if it contains the intersection of countably many open dense sets. The Baire category theorem reads as follows.

**Theorem** (Baire). *In a complete metric space, any residual set is dense.*

The meager sets in the theory of Baire category play the same role as the sets of measure zero in probability theory. If the set of elements $x \in \mathcal{X}$ satisfying a certain property $P$ is residual, we will say that $P$ is “typical” or “quasi-sure”. It is also common to use statements like “$P$ holds quasi-surely” or “quasi-all elements $x \in \mathcal{X}$ satisfy $P$”.

4.3.2. Result. We denote by $\mathcal{K}(\mathbb{T})$ the collection of all non-empty, compact subsets of $\mathbb{T}$. We define the $\varepsilon$-neighborhood of a compact $K \in \mathcal{K}(\mathbb{T})$ to be the set

$$K_\varepsilon = \bigcup_{t \in K} (t - \varepsilon, t + \varepsilon).$$

In other words, $K_\varepsilon$ is the open set consisting of all points whose distance from $K$ is smaller than $\varepsilon$. Given two compacts $K, K' \in \mathcal{K}(\mathbb{T})$, we define the Hausdorff distance between $K$ and $K'$ to be

$$\delta(K, K') = \inf\{\varepsilon > 0 : K \subset K'_\varepsilon \text{ and } K' \subset K_\varepsilon\}.$$
It is well-known that the space $K(T)$, equipped with the Hasudorff distance, is a complete metric space (see [23], IV.2).

Following the theme and notation of Körner [27], we now introduce the metric space suitable for our purpose, as follows. Being given a number $q > 2$, we define $G_q$ to be the collection of all ordered pairs $(K, S)$ such that $K \in K(T)$ and $S$ is a distribution in $A_q$ supported by $K$. We equip the space $G_q$ with the metric

$$d((K, S), (K', S')) = \delta(K, K') + \|S - S'||_{A_q}. \quad (4.3.1)$$

It is obvious that $G_q$ is a metric space. Moreover, we have

**Proposition 4.5.** The metric space $G_q$ is complete.

The fact that $G_q$ is a complete metric space justifies the use of Baire category. The result of this section is that Piatetski-Shapiro’s phenomenon is “typical” in the space $G_q$. Precisely, we prove:

**Theorem 4.3.** For quasi-all pairs $(K, S) \in G_q$, the following holds:

(i) The distribution $S$ supported by $K$ is non-zero.

(ii) The compact $K$ supports no non-zero measure $\mu \in A_q$.

**4.3.3. Basic facts.** We first describe some basic properties of the space $G_q$ which will be needed. Their proofs are essentially the same as those given in [27]. We start with the proof of the fact that $G_q$ is complete.

**Proof of Proposition 4.5.** The product space $K(T) \times A_q(T)$, endowed with the metric (4.3.1), is certainly a complete metric space. It is therefore enough to show that $G_q$ is a closed subspace of $K(T) \times A_q(T)$.

Let $(K, S) \in K(T) \times A_q(T)$, and suppose that $(K, S) /\not\in G_q$. Then $S$ is not supported by $K$, hence there exists a $C^\infty$ function $\varphi : T \to C$ which vanishes on some open set $U \subset T$ containing $K$, and such that $\langle S, \varphi \rangle \neq 0$. Choose $\varepsilon > 0$ such that $K_\varepsilon \subset U$ and $\varepsilon\|\varphi\|_{A_q} < \left|\left\langle S, \varphi \right\rangle\right|$. We claim that the open ball with center $(K, S)$ and radius $\varepsilon$ is disjoint from $G_q$. Indeed, suppose that $(K', S') \in K(T) \times A_q(T)$ is such that $d((K, S), (K', S')) < \varepsilon$. Then $\delta(K, K') < \varepsilon$, which implies $K' \subset K_\varepsilon$, and therefore $\varphi$ vanishes on an open set containing $K'$. Also $\|S - S'||_{A_q} < \varepsilon$, which implies $\left|\left\langle S, \varphi \right\rangle - \left\langle S', \varphi \right\rangle\right| < \varepsilon\|\varphi\|_{A_q}$, and therefore $\left\langle S', \varphi \right\rangle \neq 0$. This shows that $(K', S') /\not\in G_q$. We have thus confirmed that the complement of $G_q$ is an open set, which proves the claim. □

We will use the following fact about the topology of $G_q$.

**Lemma 4.6.** For any open set $U \subset T$, the sets

$$\{(K, S) \in G_q : K \subset U\} \quad \text{and} \quad \{(K, S) \in G_q : \text{supp}(S) \cap U \neq \emptyset\}$$

are open sets in $G_q$.

The proof of Lemma 4.6 is based on similar arguments as in the proof of Proposition 4.5 so we omit it.
Lemma 4.7. Quasi-all \((K, S) \in \mathcal{G}_q\) have \(K = \text{supp}(S)\).

Proof. Let \((K, S) \in \mathcal{G}_q\). If \(\text{supp}(S) \neq K\), then there is a closed arc \(J \subset \mathbb{T}\) such that \(K \cap J \neq \emptyset\), and such that \(S\) vanishes on the interior of \(J\). Moreover, we can choose \(J\) to be an arc with rational endpoints. The collection of all pairs \((K, S) \in \mathcal{G}_q\) such that \(\text{supp}(S) \neq K\) is therefore contained in a countable union of sets of the form

\[A(J) = \{(K, S) \in \mathcal{G}_q : K \cap J \neq \emptyset, \text{ such that } S \text{ vanishes on the interior of } J\},\]

where \(J\) denotes a closed arc. It is therefore enough to prove that \(A(J)\) is nowhere dense, for any such \(J\).

From Lemma 4.6 it follows that \(A(J)\) is closed, so we only need to show that its complement is dense. To this end, let \((K, S) \in A(J)\) and \(\varepsilon > 0\) be given. Since \(K \cap J \neq \emptyset\), we can choose a closed arc \(I\), of length smaller than \(\varepsilon/2\), which is contained in the interior of \(J\) and such that \(I \subset K_{\varepsilon/2}\). Define \(K' = K \cup I\), then \(\delta(K, K') \leq \varepsilon/2\). Next choose a non-zero \(C^\infty\) function \(\varphi : \mathbb{T} \to \mathbb{C}\), supported by \(I\), such that \(\|\varphi\|_{A_q} < \varepsilon/2\). Define \(S' = S + \varphi\), then \(S'\) does not vanish on the interior of \(J\), and \(\|S - S'|_{A_q} < \varepsilon/2\). We therefore found \((K', S') \in \mathcal{G}_q\) such that \(d((K, S), (K', S')) < \varepsilon\) and \((K', S') \notin A(J)\). This proves that \(A(J)\) is nowhere dense.

An immediate consequence of Lemma 4.7 is the fact that \(S \neq 0\) for quasi-all \((K, S) \in \mathcal{G}_q\).

4.3.4. Proof of main result.

Lemma 4.8. Let \((K, S) \in \mathcal{G}_q\). Given any \(\varepsilon > 0\) and any positive integer \(\nu\), we can find \((K', S') \in \mathcal{G}_q\), \(d((K, S), (K', S')) < \varepsilon\), such that the following holds: there is a real trigonometric polynomial

\[X(t) = \sum_{|n| \geq \nu} \hat{X}(n)e^{int}, \quad \|\hat{X}\|_p \leq 1, \quad \frac{1}{100} \leq X(t) \leq 100 \text{ on } K'.\]

Proof. According to Lemma 4.7, the collection of all pairs \((K, S) \in \mathcal{G}_q\) such that \(K = \text{supp}(S)\) is residual. In particular, by the Baire category theorem, this collection is dense. We may therefore assume that \(S \neq 0\) and \(K = \text{supp}(S)\).

We start with the same argument as in Lemma 4.2. Given \(\eta > 0\), using Lemma 2.3 we can find a compact \(E \subset \mathbb{T}\), a \(C^\infty\) function \(f : \mathbb{T} \to \mathbb{C}\) and a real trigonometric polynomial \(X\) such that

(i) \(f\) is supported by \(E\), \(\|f - 1\|_{A_q} < \eta\).

(ii) \(X(t) = \sum_{|n| \geq \nu} \hat{X}(n)e^{int}, \quad \|\hat{X}\|_p \leq 1\) and \(\frac{1}{100} \leq X(t) \leq 100\) on \(E\).

Define \(f_m(t) = f(mt), X_m(t) = X(mt)\) and let \(E_m\) denote the compact defined by \(1_{E_m}(t) = 1_E(mt)\). Then for every positive integer \(m\) the conditions (i) and (ii) are also satisfied by \(E_m\), \(f_m\) and \(X_m\) instead of \(E\), \(f\) and \(X\).
We claim that by choosing first a sufficiently small \( \eta \), and then a sufficiently large \( m \), the required pair \((K',S')\) could be found by taking \( S' \) to be the product \( Sf_m \), and \( K' = K \cap E_m \).

Indeed, to make the distance between \( S \) and \( S' \) small, we may argue as in Lemma 4.2 that if \( m \) is sufficiently large then \( \| S' - S \|_{A_\eta} \) will be arbitrarily close to \( \| S \|_{A_\eta} \| f - 1 \|_{A_\eta} \). So we may assume \( \| S' - S \|_{A_\eta} < \eta \| S \|_{A_\eta} \). In particular, if \( \eta \) is sufficiently small then \( \| S' - S \|_{A_\eta} < \varepsilon/2 \).

To make the distance between \( K \) and \( K' \) small, we argue as follows. First, we choose a finite set \( T \subset K \) such that \( K \subset T_{\varepsilon/4} \). Then, since \( K = \text{supp}(S) \), for every \( t \in T \) we can choose a \( C^\infty \) function \( \varphi_t : T \to \mathbb{C} \), supported by \( (t - \frac{\varepsilon}{4}, t + \frac{\varepsilon}{4}) \), such that \( \langle S, \varphi_t \rangle \neq 0 \). We have

\[
|\langle S', \varphi_t \rangle - \langle S, \varphi_t \rangle| = |\langle S' - S, \varphi_t \rangle| < \eta \| S \|_{A_\eta} \| \varphi_t \|_{A_\eta},
\]

so if \( \eta \) is sufficiently small, \( \langle S', \varphi_t \rangle \neq 0 \) for every \( t \in T \) (note that \( T \) was chosen before \( \eta \)). It follows that \( \text{supp}(S') \) intersects each of the segments \( (t - \frac{\varepsilon}{4}, t + \frac{\varepsilon}{4}) \), \( t \in T \). Since \( \text{supp}(S') \subset K' \), this shows that \( T \subset K'_{\varepsilon/4} \). It follows that \( K \subset K'_{\varepsilon/2} \). On the other hand, \( K' \subset K \), so we conclude that \( \delta(K', K) \leq \varepsilon/2 \).

We have proved that for an appropriate choice of \( \eta \) and \( m \) we have \( \| S' - S \|_{A_\eta} < \varepsilon/2 \) and \( \delta(K', K) \leq \varepsilon/2 \), hence \( d((K, S), (K', S')) < \varepsilon \) as needed.

\textbf{Proof of Theorem 4.3} For any positive integer \( \nu \), let \( W_\nu \) denote the set of all pairs \((K, S) \in \mathcal{G}_q \) for which there exists a real trigonometric polynomial

\[
X(t) = \sum_{|n| \geq \nu} \hat{X}(n)e^{int}, \quad \| \hat{X} \|_p \leq 1, \quad \frac{1}{200} < X(t) < 200 \text{ on } K.
\]

(4.3.2)

From Lemma 4.6 it follows that \( W_\nu \) is an open set, while Lemma 4.8 implies that \( W_\nu \) is dense. It follows that the intersection \( \bigcap_{\nu=1}^\infty W_\nu \) is a residual set. That is, quasi-all pairs \((K, S) \in \mathcal{G}_q \) satisfy the following condition: for every \( \nu \) there is a real trigonometric polynomial \( X(t) \) satisfying (4.3.2). We may therefore use Lemma 2.1 (the precise constant which appears there is different, but clearly plays no special role), which implies that quasi-surely \( K \) does not support a non-zero measure \( \mu \in A_\eta \). On the other hand, Lemma 4.7 ensures that \( S \neq 0 \) quasi-surely, so the theorem is proved.

\textbf{4.4. \( p \)-Helson sets}

\textbf{4.4.1.} A compact \( K \) on the circle \( T \) is called a \textit{\( p \)-Helson set} if any function, defined and continuous on \( K \), can be extended to a continuous function on the circle with Fourier coefficients belonging to \( \ell_p \).

When \( p = 1 \) these are the usual Helson sets, which have been much studied (see [16]). Helson [11] proved that such a set cannot support a measure with Fourier coefficients tending to zero. On the other hand, Körner [25] proved that a Helson set can support a distribution with this property, that is, there exists a
Helson set of multiplicity. Kaufman \cite{22} strengthened the result by proving that any set of multiplicity contains a Helson set of multiplicity.

We now turn to the case $1 < p < 2$. Olevskii \cite{33,34} constructed a compact set of measure zero, which is not $p$-Helson for any $p < 2$. On the other hand Demenko \cite{6,7} proved that if $K$ is sufficiently small in the sense of Hausdorff dimension, or precisely, if $\dim K < 2 - 2/p$, then $K$ is a $p$-Helson set. We point out that no analog is true for $p = 1$; there exist countable compacts which are not Helson sets. An analog of Helson’s theorem was proved by Gregory in \cite{10}, and by Demenko in \cite{6}. Precisely, they proved that a $p$-Helson set (1 < $p$ < 2) cannot support a measure with Fourier transform belonging to $\ell_q$, where $q = p/(p - 1)$ is the conjugate of $p$. We remark that a $p$-Helson set can support a measure with Fourier transform tending to zero, as shown in \cite{10}.

Could a $p$-Helson set support a distribution with Fourier transform belonging to $\ell_q$? In this section we answer this question affirmatively. We prove the following strengthening of Theorem 2.1, analogous to Körner’s theorem about the existence of a Helson set of multiplicity.

**Theorem 4.4.** For any $1 < p < 2$ there exists a $p$-Helson set which supports a non-zero distribution with Fourier coefficients belonging to $\ell_q$, $q = p/(p - 1)$.

**4.4.2.** Let $p$ be a fixed number, $1 < p < 2$, and let $q = p/(p - 1)$ be the exponent conjugate of $p$. We use the following characterization of $p$-Helson sets, which is due to Gregory \cite{10}.

**Proposition 4.9 (Gregory).** A compact $K$ is a $p$-Helson set if and only if $|\mu|(K) = 0$ for every measure $\mu \in A_q$.

We also need the following lemma:

**Lemma 4.10.** Let $K$ be a compact on the circle, and $\mu$ be a measure in $A_q(\mathbb{T})$. Then the measure $1_K \cdot |\mu|$ belongs to the closure of $A_q \cap M(\mathbb{T})$ in the $M(\mathbb{T})$ norm.

The proof of Lemma 4.10 is very similar to that of Lemma 2.2, so it is omitted.

**Lemma 4.11.** Suppose a compact $K$ on the circle satisfies the following condition: for any $\varepsilon > 0$ there is a real trigonometric polynomial $X$ such that

$$
\|X\|_{\infty} \leq 1, \quad \|X\|_{A_p} < \varepsilon, \quad X(t) > \frac{1}{50} \text{ on } K.
$$

Then $K$ is a $p$-Helson set.

**Proof.** Let $\mu \in A_q \cap M(\mathbb{T})$. Given $\varepsilon > 0$, by Lemma 4.10 one can find a measure $\mu_1 \in A_q(\mathbb{T})$ such that

$$
\|1_K \cdot |\mu| - \mu_1\|_{M(\mathbb{T})} < \varepsilon.
$$

Our assumption on the compact $K$ now allows us to choose a real trigonometric polynomial $X$ satisfying $\|X\|_{\infty} \leq 1, \|X\|_{A_p}, \|\mu_1\|_{A_q} < \varepsilon, X(t) > \frac{1}{50} \text{ on } K$. Then

$$
\left| \int_{\mathbb{T}} XD\mu_1 \right| \geq \int_K X|d\mu| - \varepsilon\|X\|_{\infty} \geq \frac{1}{50}|\mu|(K) - \varepsilon.
$$
On the other hand,

\[ \left| \int_X X d\mu_1 \right| = \left| \sum_{n \in \mathbb{Z}} \hat{X}(n)\hat{\mu}_1(-n) \right| \leq \|X\|_{A_p} \|\mu_1\|_{A_q} < \varepsilon. \]

It follows that \( |\mu|(K) = 0 \), for every measure \( \mu \in A_q \). By Proposition \( 4.9 \) this is equivalent to the fact that \( K \) is a \( p \)-Helson set. \( \square \)

**Proof of Theorem 4.4.** For a sequence \( \{\varepsilon_j\} \) tending to zero, we choose \( K_j, F_j \) and \( X_j \) according to Lemma \( 3.5 \). As before, we can choose the \( \varepsilon_j \) by induction, in such a way that the product \( \prod_{j=1}^{\infty} F_j \) will converge in \( A_q \) to a non-zero distribution \( S \) supported by \( K := \bigcap_{j=1}^{\infty} K_j \). Lemma \( 4.11 \) now implies that \( K \) is a \( p \)-Helson set, so the theorem is proved. \( \square \)
CHAPTER 5

Orlicz Spaces

In this chapter we show how the method developed in Chapter 2 can be adopted to further extend Piatetski-Shapiro’s phenomenon to spaces of sequences more general than \( \ell_q \) spaces. As mentioned before, it is known that no Piatetski-Shapiro phenomenon exists in certain weighted \( \ell_2 \) spaces. A substantial difference between \( \ell_q \) spaces and weighted spaces is that \( \ell_q \) spaces are rearrangement invariant, while weighted spaces are in general not. By “rearrangement invariant space” we mean a space \( \mathcal{X} \) of sequences on \( \mathbb{Z} \) such that, for every sequence \( \{x_n\} \in \mathcal{X} \) and every permutation (i.e. bijection) \( \sigma : \mathbb{Z} \to \mathbb{Z} \), the sequence \( \{x_{\sigma(n)}\} \) also belongs to \( \mathcal{X} \).

Orlicz spaces, whose definition we recall below, is a well-known class of rearrangement invariant spaces of sequences. They are therefore natural candidates for additional spaces where Piatetski-Shapiro’s phenomenon could exist. We show that for a certain class of Orlicz spaces, this is indeed the case.

5.1. Introduction

5.1.1. Orlicz spaces. Let us start by recalling the basics of Orlicz sequence spaces, following [31]. An Orlicz function \( \phi \) is a continuous non-decreasing and convex function on \([0, \infty)\) such that \( \phi(0) = 0 \) and \( \lim_{t \to \infty} \phi(t) = \infty \). To any Orlicz function \( \phi \) there corresponds the so-called Orlicz space \( \ell_\phi \) consisting of all sequences of scalars \( x = \{x_n\}_{n \in \mathbb{Z}} \) such that \( \sum_{n \in \mathbb{Z}} \phi(|x_n|/\rho) < \infty \) for some \( \rho > 0 \). The space \( \ell_\phi \) equipped with the norm

\[
\|x\|_\phi = \inf\{\rho > 0 : \sum_{n \in \mathbb{Z}} \phi(|x_n|/\rho) \leq 1\}
\]

(5.1.1)

becomes a Banach space. Orlicz spaces generalize \( \ell_p \) spaces in a natural way, since \( \ell_p \) is the Orlicz space which corresponds to \( \phi(t) = t^p \).

If \( \phi(t_0) = 0 \) for some \( t_0 > 0 \), that is, \( \phi \) vanishes in some neighborhood of zero, we say that \( \phi \) is a degenerate Orlicz function. For a degenerate Orlicz function \( \phi \), as can be easily checked, the space \( \ell_\phi \) coincides with \( \ell_\infty \). Since this case is not of interest in our context we shall consider from now on only non-degenerate Orlicz functions.

Two Orlicz functions \( \phi_1, \phi_2 \) are said to be equivalent at zero if there exist constants \( C, M \) and \( t_0 > 0 \) such that, for all \( 0 \leq t \leq t_0 \),

\[
C^{-1} \phi_2(M^{-1}t) \leq \phi_1(t) \leq C \phi_2(Mt).
\]

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In this case, $\ell_{\phi_1} = \ell_{\phi_2}$ (i.e. both spaces consist of the same sequences) and the norms $\| \cdot \|_{\phi_1}$ and $\| \cdot \|_{\phi_2}$ are equivalent.

An Orlicz function $\phi$ is said to satisfy the $\Delta_2$-condition at zero if

$$\limsup_{t \to 0} \frac{\phi(2t)}{\phi(t)} < \infty. \quad (5.1.2)$$

It is easily verified that (5.1.2) implies that $\limsup_{t \to 0} \phi(Mt)/\phi(t) < \infty$ for every $M > 0$. The role of the $\Delta_2$-condition is illustrated by the following:

**Proposition 5.1** (see [31], Proposition 4.4.4). For an Orlicz function $\phi$ the following conditions are equivalent:

(i) $\phi$ satisfies the $\Delta_2$-condition at zero.

(ii) $\ell_{\phi}$ is separable.

(iii) The system of unit vectors $e_n = (\ldots, 0, 1, 0, \ldots), n \in \mathbb{Z}$, is complete in $\ell_{\phi}$.

**5.1.2. Result.** If no further conditions on $\phi$ are imposed, then it is easy to exhibit spaces $\ell_{\phi}$ where Piatetski-Shapiro’s phenomenon does exist. For let $K$ be the compact given by Theorem 1.1 and $S$ be a distribution supported by $K$ such that $\hat{S} \in c_0$. Then one may construct a (non-degenerate) Orlicz function $\phi$ such that $\hat{S} \in \ell_{\phi}$ (for this it will be enough that $\hat{\phi}$ has a sufficiently fast decay at zero). Since $\phi$ is non-degenerate we have $\ell_{\phi} \subset c_0$, and it follows immediately that Piatetski-Shapiro’s phenomenon exists in $\ell_{\phi}$.

Our last remark shows that if $\phi$ decays sufficiently fast at zero, then Piatetski-Shapiro’s phenomenon does exist in $\ell_{\phi}$. On the other hand, if $\phi$ decays sufficiently slowly at zero then no Piatetski-Shapiro phenomenon can exist in $\ell_{\phi}$. Precisely, if $\liminf_{t \to 0} \phi(t)/t^2 > 0$ then $\ell_{\phi} \subset \ell_2$, hence any distribution $S$ such that $\hat{S} \in \ell_{\phi}$ is automatically a measure (in fact, an $L^2$ function).

In order to give non-trivial conditions under which Piatetski-Shapiro’s phenomenon exists in $\ell_{\phi}$, we introduce the following definition. We say that an Orlicz function $\phi$ is *submultiplicative at zero* if there exist constants $M$ and $t_0 > 0$ such that, for every $0 < s, t < t_0$ one has

$$\phi(st) \leq M \phi(s) \phi(t). \quad (5.1.3)$$

Submultiplicative Orlicz functions appear in the theory of operators in Orlicz spaces, see [28]. We can now formulate the result of this chapter.

**Theorem 5.1.** Let $\phi$ be a non-degenerate Orlicz function satisfying the following conditions: (i) $\lim_{t \to 0} \phi(t)/t^2 = 0$; (ii) $\phi$ satisfies the $\Delta_2$-condition at zero; and (iii) $\phi$ is submultiplicative at zero. Then there is a compact $K$, which supports a distribution $S$ such that $\hat{S} \in \ell_{\phi}$, but which does not support such a measure.

**Remarks.** 1. As mentioned above, the condition that $\phi$ be non-degenerate is required in order to exclude the case $\ell_{\phi} = \ell_\infty$. 
2. The condition (i) can be relaxed a bit. In fact, as we will show, the theorem remains true if (i) is replaced by the weaker condition (i)' defined by
\[ \liminf_{t \to 0} \phi(t)/t^2 = 0 \quad \text{and} \quad \limsup_{t \to 0} \phi(t)/t^2 < \infty. \] (5.1.4)

It can be checked that (i)' holds if and only if there exists a constant $C$ such that $\| \cdot \|_{\phi} \leq C \| \cdot \|_2$, but the norms are not equivalent. As explained above, the condition $\liminf_{t \to 0} \phi(t)/t^2 = 0$ is crucial in order to exclude the case $\ell_{\phi} \subset \ell_2$.

3. We leave open the question whether the theorem remains true in the case $\limsup_{t \to 0} \phi(t)/t^2 = \infty$, and whether the conditions (ii) and (iii) could be relaxed or not.

4. By choosing the function $\phi(t) = t^q$ ($q > 2$) one can see that Theorem 5.1 is in fact a strengthening of Theorem 2.1, since in this case we have $\ell_{\phi} = \ell_q$.

5.1.3. Example. Let us explicitly describe a family of Orlicz function to which Theorem 5.1 can be applied. Fix two numbers $q > 2$ and $\alpha > 0$. Define
\[ \phi(t) = t^q \log^\alpha \frac{1}{t} \] (5.1.5)
for $t > 0$, and $\phi(0) = 0$. It can be checked that $\phi$ is continuous, non-decreasing and convex on some interval $[0, t_0]$. It is therefore possible to extend $\phi$ to the whole segment $[0, \infty)$ so that it becomes an Orlicz function. Note that the corresponding space $\ell_{\phi}$ will be the same regardless of how $\phi$ was extended, and the norms associated with two distinct extensions might be different but equivalent [31, p. 139]. It can now be checked directly using (5.1.5) that the function $\phi$ satisfies the conditions of Theorem 5.1. It can also be checked that a sequence $\{x_n\}$ belongs to $\ell_{\phi}$ if and only if $\sum' |x_n|^q \log^\alpha (e + |x_n|^{-1}) < \infty$, where the notation $\sum'$ is used to indicate summation only over indices $n$ such that $x_n \neq 0$. In view of these remarks, the conclusion of Theorem 5.1 in this case can be stated as follows.

**Corollary 5.2.** For any $q > 2$ and any $\alpha > 0$ there is a compact $K$ on the circle, which supports a distribution $S$ such that
\[ \sum' |\hat{S}(n)|^q \log^\alpha (e + |\hat{S}(n)|^{-1}) < \infty, \]
but does not support such a measure.

5.2. Lemmas

5.2.1. Notation. From now on we assume that $\phi$ is a non-degenerate Orlicz function satisfying conditions (ii) and (iii) of Theorem 5.1, while instead of condition (i) we will assume the weaker condition (i)' given by (5.1.4). We denote by $A_\phi$ the space of Schwartz distributions $S$ on the circle $\mathbb{T}$ such that $\hat{S} \in \ell_{\phi}$. Equipped with the norm $\|S\|_{A_\phi} := \|\hat{S}\|_\phi$ the space $A_\phi$ is clearly a Banach space.
It will be convenient to extend the definition of the Orlicz norm $\| \cdot \|_\phi$ to sequences defined not only on $\mathbb{Z}$, but on any finite or countable set $I$. Precisely, given a sequence of scalars $\{x_n\}_{n \in I}$ we define

$$
\| \{x_n\} \|_\phi = \inf \{ \rho > 0 : \sum_{n \in I} \phi(|x_n|/\rho) \leq 1 \}. \quad (5.2.1)
$$

In what follows $r$ will denote a positive (small) number. Given $r > 0$, we let $N = N(r)$ denote the unique integer satisfying

$$
\frac{1}{\phi(r)} \leq N < \frac{1}{\phi(r)} + 1. \quad (5.2.2)
$$

Note that $N \to \infty$ as $r \to 0$, since $\phi$ is continuous and $\phi(0) = 0$.

5.2.2. Excluding measures. In Lemma 2.1 we considered a trigonometric polynomial $X(t)$ whose Fourier coefficients satisfy $\| \{\hat{X}(n)\} \|_p \leq 1$. This was used in order to deduce that

$$
\left| \sum c(n)\hat{X}(n) \right| \leq \|\{c(n)\}\|_q
$$

for any choice of scalars $\{c(n)\}$, where $p^{-1} + q^{-1} = 1$. In the context of Orlicz spaces, the analog claim is related to the description of the dual space of $\ell_\phi$. It is known [31, pp. 147–148] that if $\phi$ satisfies the $\Delta_2$-condition at zero then the dual space $\ell_\phi^*$ may be identified with another Orlicz space $\ell_\psi$, where $\psi$ is a certain Orlicz function “complementary” to $\phi$. However, we shall not need such a result. The following lemma, which can be proved directly, would be sufficient for us.

**Lemma 5.3.** Given $r > 0$, let $N = N(r)$ be defined by (5.2.2). Then for every choice of scalars $c_1, \ldots, c_N$ one has

$$
\left| \frac{1}{N^r} \sum_{n=1}^N c_n \right| \leq \|\{c_n\}\|_\phi \quad (5.2.3)
$$

**Proof.** Since both sides of (5.2.3) are homogeneous, it would be enough to prove the claim in the case $\|\{c_n\}\|_\phi = 1$. According to the definition (5.2.1) of the Orlicz norm, and due to the continuity of $\phi$, this is equivalent to $\sum_{n=1}^N \phi(|c_n|) = 1$. By the convexity of $\phi$, and using (5.2.2), we have

$$
\phi\left( \frac{1}{N} \sum_{n=1}^N |c_n| \right) \leq \frac{1}{N} \sum_{n=1}^N \phi(|c_n|) = \frac{1}{N} \leq \phi(r). \quad (5.2.4)
$$

However, $\phi$ is strictly increasing, since it is non-degenerate, so from (5.2.4) it follows that $\frac{1}{N} \sum_{n=1}^N |c_n| \leq r$. That is,

$$
\left| \frac{1}{N^r} \sum_{n=1}^N c_n \right| \leq 1,
$$

as we had to show. \qed
With the aid of Lemma 5.3 we may now prove an analog of Lemma 2.1.

**Lemma 5.4.** Let \( K \) be a compact on the circle. Suppose that for any positive integer \( \nu \) there exists \( r > 0 \) such that the following holds: the trigonometric polynomial defined by

\[
X(t) = \frac{1}{N_r} \sum_{j=1}^{N} \cos \nu^j t,
\]

where \( N = N(r) \) is defined by (5.2.2), satisfies \( \frac{1}{100} \leq X(t) \leq 100 \) on \( K \). Then \( K \) does not support a measure \( \mu \in A_{\phi} \).

**Proof.** Suppose that \( \mu \in A_{\phi} \cap M(K) \). As in Lemma 2.2, given \( \varepsilon > 0 \) we can approximate \( |\mu| \) in the \( M(K) \) norm by a measure \( \mu_1 \in A_{\phi} \). We have

\[
\int_{T} X d\mu_1 = \sum_{n \in \mathbb{Z}} \hat{X}(n) \hat{\mu}_1(-n) = \frac{1}{N_r} \sum_{j=1}^{N} \hat{\mu}_1(\nu^j) + \frac{\hat{\mu}_1(-\nu^j)}{2},
\]

hence it follows by Lemma 5.3 that

\[
\left| \int_{T} X d\mu_1 \right| \leq \left\| \left\{ \frac{\hat{\mu}_1(\nu^j) + \hat{\mu}_1(-\nu^j)}{2} \right\}_{j=1}^{N} \right\|_{\phi}.
\] (5.2.5)

We now use the \( \Delta_2 \)-condition: according to Proposition 5.1, the system of unit vectors \( \{e_n\}_{n \in \mathbb{Z}} \) is complete in \( \ell_{\phi} \), hence the Orlicz norm of the “tail” sequence \( \{\hat{\mu}_1(n)\}_{|n| \geq \nu} \) is arbitrarily small if \( \nu \) is sufficiently large. From this it is easy to deduce that the quantity on the right side of (5.2.5) is smaller than \( \varepsilon \) for sufficiently large \( \nu \). Once this has been established, the proof can then be finished as in Lemma 2.1. \( \square \)

**5.2.3. Riesz products.** The appropriate Riesz product in this context is

\[
\lambda_s(t) = \prod_{j=1}^{N} (1 + 2sr \cos \nu^j t),
\]

where \( 0 < r < 1, N = N(r) \) is defined by (5.2.2), \( \frac{1}{4} < s < \frac{1}{3} \) and \( \nu \geq 3 \). As before, this defines a probability measure \( \lambda_s \) on the circle \( \mathbb{T} \) such that

\[
\tilde{\lambda}_s\left( \sum_{j=1}^{N} \tau_j \nu^j \right) = (sr)^{\sum |\tau_j|}, \quad \tilde{\tau} = (\tau_1, \ldots, \tau_N) \in \{-1, 0, 1\}^N.
\] (5.2.6)

Our corresponding trigonometric polynomial \( X(t) \) will be defined by

\[
X(t) = \frac{1}{N_r} \sum_{j=1}^{N} \cos \nu^j t,
\] (5.2.7)

and a straightforward calculation shows again that the expectation of \( X \) with respect to the measure \( \lambda_s \) is equal to \( s \). We have the following lemma, whose proof we omit, as it is basically the same as the proof of Lemma 2.7.
**Lemma 5.5.** Being given $0 < r < 1$, for $\nu \geq \nu(r)$ one has

$$\lambda_s \{ t \in T : |X(t) - s| > \alpha \} \leq 3 \exp \left( -\frac{1}{8} \alpha^2 N r^2 \right)$$

(5.2.8)

for every $\alpha > 0$ and every $\frac{1}{4} < s < \frac{1}{3}$.

The next lemma is an analog of Lemma 2.8. Note however that the estimate (5.2.9) is given explicitly in terms of the function $\phi$. This will allow us to prove the result under the weaker condition (i)' instead of (i).

**Lemma 5.6.** Denote $K' = \{ t \in T : \frac{1}{90} \leq X(t) \leq 90 \}$. Then for $\nu \geq \nu(r)$ and for every $\frac{1}{4} < s < \frac{1}{3}$,

$$\int_T \lambda_s^2(t) \frac{dt}{2\pi} < 2^{11} \cdot \frac{\phi(r)}{r^2}$$

(5.2.9)

where $C > 0$ is an absolute constant.

**Proof.** We follow the same line as in the proof of Lemma 2.8, but somewhat improve on one of our estimates. We have

$$\lambda_s(t) = \prod_{j=1}^{N} (1 + 2sr \cos \nu_j t) \leq \exp \left( 2sr \sum_{j=1}^{N} \cos \nu_j t \right),$$

so using (5.2.7) we get

$$\lambda_s(t) \leq \exp \left( 2s N r^2 X(t) \right).$$

(5.2.10)

We now apply Lemma 5.5 and for $\nu \geq \nu(r)$ we use (5.2.8) and (5.2.10). The same argument as in the proof of Lemma 2.8 will show that

$$\int_{\{ t : X(t) < \frac{1}{90} \}} \lambda_s^2(t) \frac{dt}{2\pi} \leq 3 \exp \left( -2^{-10} N r^2 \right).$$

Also for any integer $90 \leq k \leq 1/r$,

$$\int_{\{ t : k \leq X(t) \leq k+1 \}} \lambda_s^2(t) \frac{dt}{2\pi} \leq \int_{\{ t : k \leq X(t) \leq k+1 \}} \lambda_s(t) \frac{dt}{2\pi} \cdot \max_{\{ t : k \leq X(t) \leq k+1 \}} \lambda_s(t)$$

$$\leq 3 \exp \left( -\frac{1}{8} (k - s)^2 N r^2 \right) \cdot \exp \left( 2s N r^2 (k + 1) \right)$$

$$\leq 3 \exp \left( -\frac{1}{10} k^2 N r^2 \right),$$

and so, since $X(t) \leq 1/r$ for every $t$,

$$\int_{\{ t : X(t) > 90 \}} \lambda_s^2(t) \frac{dt}{2\pi} \leq 3 \sum_{90 \leq k \leq 1/r} \exp \left( -\frac{1}{10} k^2 N r^2 \right)$$

$$\leq 3 \int_{0}^{\infty} x \cdot \exp \left( -\frac{1}{10} x^2 N r^2 \right) dx = \frac{15}{N r^2}.$$
We thus get
\[
\int_{T \setminus K'} \lambda_s^2(t) \frac{dt}{2\pi} \leq 3 \exp\left(-2^{-10}N r^2\right) + \frac{15}{N r^2} < \frac{2^{11}}{N r^2}.
\]
Using (5.2.2), the result follows. □

Remark. As one can see from the above proof, the estimate (5.2.9) is very rough. However, it will be enough for our purpose.

5.3. Proof of main result

To finish the proof of Theorem 5.1 it will be enough to establish the following analog of Lemma 2.9. Once this is done, the theorem follows by an iteration procedure as in the proof of Theorem 2.1, which we shall not repeat.

Lemma 5.7. Let \( \varepsilon > 0 \). There exists \( r = r(\varepsilon) \), \( 0 < r < \varepsilon \) such that the following holds: for any \( \nu \geq \nu(r) \) there is a \( C^\infty \) function \( f : T \to \mathbb{C} \) satisfying

(i) \( f \) is supported by
\[
K = \{ t \in T : \frac{1}{100} \leq X(t) \leq 100 \} \tag{5.3.1}
\]
where \( X \) is the trigonometric polynomial defined by (5.2.7).

(ii) \( f(t) = 1 + \sum_{n \neq 0} \hat{f}(n)e^{int} \), where \( \|\{\hat{f}(n) : n \neq 0\}\|_\phi < \varepsilon \).

Proof. As in the proof of Lemma 2.9, given \( \delta > 0 \) we choose a measure \( \rho \) supported by the interval \( \left(\frac{1}{4}, \frac{1}{3}\right) \) such that (1.2.2) holds. We then define
\[
\lambda(t) = \int \lambda_s(t) \, d\rho(s),
\]
and so by (5.2.6)
\[
\lambda(t) = 1 + \sum_{\tau \neq 0} \left\{ \tau \sum_{|\tau|} \left| \int s \sum_{|\tau|} \, d\rho(s) \right| \right\} e^{i(\sum_{|\tau|} \nu^j)t}.
\]
It follows that
\[
\sum_{n \neq 0} \phi(|\hat{\lambda}(n)|/\delta) \leq \sum_{0 \neq \tau \in \{-1,0,1\}^N} \phi(r \sum_{|\tau|}). \tag{5.3.2}
\]
We now use the submultiplicativity at zero: there exists \( M \) and \( t_0 > 0 \) such that \( \phi(st) \leq M \phi(s) \phi(t) \) for every \( s, t < t_0 \). It is easy to check that this implies
\[
\phi(r^k) \leq M^{k-1} \phi(r)^k \tag{5.3.3}
\]
for all sufficiently small \( r \). We will assume that \( M \geq 1 \), as we clearly may. We then apply (5.3.3) to (5.3.2) to get
\[
\sum_{n \neq 0} \phi(|\hat{\lambda}(n)|/\delta) \leq \sum_{\tau \in \{-1,0,1\}^N} (M \phi(r))^{\sum_{|\tau|}} = (1 + 2M \phi(r))^N < e^{2MN\phi(r)}.
\]
From (5.2.2) we have $Nφ(r) < 1 + φ(r)$, and since $φ$ is continuous and $φ(0) = 0$ this shows that for sufficiently small $r$, $Nφ(r) ≤ 2$. So we arrive at the estimate
\[
\sum_{n≠0} φ(|λ̂(n)|/δ) ≤ e^{4M}.
\] (5.3.4)

Observe that since $φ$ is convex, $φ(0) = 0$ and $e^{4M} > 1$, we have $φ(e^{-4M}x) ≤ e^{-4M}φ(x)$ for any $x > 0$. Thus using (5.3.4),
\[
\sum_{n≠0} φ(|λ̂(n)|e^{-4M}/δ) ≤ e^{-4M} \sum_{n≠0} φ(|λ̂(n)|/δ) ≤ 1,
\]
and according to the definition of the Orlicz norm we conclude that
\[
\|λ - 1\|_{Aφ} ≤ e^{4M} δ.
\] (5.3.5)

We now use the conditions
\[
\liminf_{t→∞} φ(t)/t^2 = 0 \quad \text{and} \quad \limsup_{t→∞} φ(t)/t^2 < ∞.
\]
First, it is easy to check that from $\limsup_{t→∞} φ(t)/t^2 < ∞$ it follows that a constant $C$ exists such that $\|\{x_n\}\|_φ ≤ C\|\{x_n\}\|_2$ for any sequence $\{x_n\} ∈ ℓ_2$.

Second, from $\liminf_{t→∞} φ(t)/t^2 = 0$ it follows that there exist arbitrarily small values of $r$ such that
\[
\left(2^{11} \cdot \frac{φ(r)}{r^2}\right)^{1/2} < \frac{δ}{C\|ρ\|_M}.
\]
In particular, we may assume that $r$ is small enough so that (5.3.5) holds. Using Lemma 5.6 it follows that for $ν ≥ ν(r)$
\[
\left(\int_{T\setminus K'} \lambda^2_s(t) \frac{dt}{2π}\right)^{1/2} < \frac{δ}{C\|ρ\|_M}
\]
for every $\frac{1}{4} < s < \frac{1}{4}$. The function $h := λ · 1_{K'}$ is then supported by $K'$, and
\[
\|λ - h\|_{Aφ} ≤ C\|λ - h\|_{L^2(T)} = C\|λ\|_{L^2(T\setminus K')} ≤ C \int \|λ_s\|_{L^2(T\setminus K')} |dρ(s)| < δ.
\]
We then continue as in the proof of Lemma 2.9. □
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