New Integrable Coupled Nonlinear Schrödinger Equations

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ABSTRACT

Two types of integrable coupled nonlinear Schrödinger (NLS) equations are derived by using Zakharov-Shabat (ZS) dressing method. The Lax pairs for the coupled NLS equations are also investigated using the ZS dressing method. These give new types of the integrable coupled NLS equations with certain additional terms. Then, the exact solutions of the new types are obtained. We find that the solution of these new types do not always produce a soliton solution even they are the kind of the integrable NLS equations.

1 Introduction

One of the most remarkable discoveries in soliton theory is the inverse scattering method (ISM). It is well-known that there are several types of the coupled nonlinear Schrödinger (NLS) equations whose complete integrability can prove by some methods in the ISM such as AKNS method [1], Wadati method, ZS dressing method, and so on [2-5]. In this paper, we introduce two new types of the integrable coupled NLS equations derived using ZS dressing method. The first type of the equations is

\[ \begin{align*}
  iu_t + \chi u_{xx} &\mp 2\mu \left( \frac{|u|^2 + |w|^2}{|u|^2 |w|^2} \right) u = R_1 \\
  iw_t + \chi w_{xx} &\mp 2\mu \left( \frac{|u|^2 + |w|^2}{|u|^2 |w|^2} \right) w = R_2,
\end{align*} \tag{1.1} \]

where the perturbative terms \( R_1 \) and \( R_2 \), and the real parameters \( \chi \), and \( \mu \) are, respectively, defined as follows

\[ R_1 = \frac{2\chi u^2}{u}, \quad R_2 = \frac{2\chi w^2}{w}, \quad \chi = \frac{\alpha + \delta}{\alpha (\alpha - \delta)}, \quad \mu = \frac{\alpha^2 - \delta^2}{\alpha^2 \delta}. \tag{1.2a,b,c,d} \]

The second type,

\[ \begin{align*}
  iu_t + \chi u_{xx} &\mp \left( 2\mu \left( \frac{|u|^2 + |w|^2}{|u|^2 |w|^2} \right) u \right) \left( \frac{1}{1 + s \left( \frac{|u|^2 + |w|^2}{|u|^2 |w|^2} \right)} \right) = Q_1 \\
  iw_t + \chi w_{xx} &\mp \left( 2\mu \left( \frac{|u|^2 + |w|^2}{|u|^2 |w|^2} \right) w \right) \left( \frac{1}{1 + s \left( \frac{|u|^2 + |w|^2}{|u|^2 |w|^2} \right)} \right) = Q_2,
\end{align*} \tag{1.3} \]

is a coupled NLS equation with complicated additional terms \( Q_1 \) and \( Q_2 \) defined as follows

\[ Q_1 = Q_{11} + Q_{12} + Q_{13} + Q_{14}, \quad Q_2 = Q_{21} + Q_{22} + Q_{23} + Q_{24}, \tag{1.4a,b} \]

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where

\[ Q_{11} = i \left( \frac{s}{2} \right) \frac{\left( \left| u \right|^2 + \left| w \right|^2 \right)_t}{1 + s \left( \left| u \right|^2 + \left| w \right|^2 \right)} u, \quad (1.5a) \]

\[ Q_{12} = (s \chi) \frac{\left( \left| u \right|^2 + \left| w \right|^2 \right)_x}{1 + s \left( \left| u \right|^2 + \left| w \right|^2 \right)} u_x, \quad (1.5b) \]

\[ Q_{13} = Q_{23} = \left( \frac{s \chi}{2} \right) \frac{\left( \left( \left| u \right|^2 + \left| w \right|^2 \right)_{xx} \right)}{1 + s \left( \left| u \right|^2 + \left| w \right|^2 \right)} u, \quad (1.5c) \]

\[ Q_{14} = - \left( \frac{3s^2 \chi}{4} \right) \frac{\left( \left( \left| u \right|^2 + \left| w \right|^2 \right)_x \right)^2}{\left( 1 + s \left( \left| u \right|^2 + \left| w \right|^2 \right) \right)} u, \quad (1.5d) \]

\[ Q_{21} = i \left( \frac{s}{2} \right) \frac{\left( \left| u \right|^2 + \left| w \right|^2 \right)_t}{1 + s \left( \left| u \right|^2 + \left| w \right|^2 \right)} w, \quad (1.5e) \]

\[ Q_{22} = (s \chi) \frac{\left( \left| u \right|^2 + \left| w \right|^2 \right)_x}{1 + s \left( \left| u \right|^2 + \left| w \right|^2 \right)} w_x, \quad (1.5f) \]

and

\[ Q_{24} = - \left( \frac{3s^2 \chi}{4} \right) \frac{\left( \left( \left| u \right|^2 + \left| w \right|^2 \right)_x \right)^2}{\left( 1 + s \left( \left| u \right|^2 + \left| w \right|^2 \right) \right)} w. \quad (1.5g) \]

The parameter \( s \) is an effective saturation parameter.

If we put \( s \to 0 \), then the system (1.3) reduces to the integrable Manakov equation [6]:

\[ iu_t + \chi u_{xx} + 2\mu \left( \left| u \right|^2 + \left| w \right|^2 \right) u = 0 \]

\[ iw_t + \chi w_{xx} + 2\mu \left( \left| u \right|^2 + \left| w \right|^2 \right) w = 0, \quad (1.6) \]

Equation (1.3) can be identified as the similar equation with that which has been derived by Christodoulides et al. [7], and it is described by a system of two coupled nonlinear equations for the normalized beam envelopes, \( u(x,t) \) and \( w(x,t) \) [8]. However, in eq.(1.3), we have extended the additional terms appeared in the related equation provided by Christodoulides et al. into the complicated terms (\( Q_1 \) and \( Q_2 \)).

The present paper consists of the following. In section 2, we perform the ZS dressing method for the general NLS equations. In section 3, we propose and derive two types of the integrable coupled NLS equations using the ZS method. In section 4, we solve the solutions of the coupled NLS equations. The last section, section 5, is devoted to the concluding remarks.

## 2 ZS Dressing Method for the General Coupled NLS Equations

### 2.1 The Integral Operators of The ZS Dressing Method

In 1974, Zakharov and Shabat generalised the Lax method using their ZS dressing method. In this section, we will review how the ZS dressing method works on the general NLS equations and will also show the connection with the work of Lax soon, but first we introduce three integral operators [9]. Let, in general, \( F(x,z) \) and \( k_\pm(x,z) \) be \( N \times N \) matrices where

\[ k_+(x,z) = 0, \quad \text{if} \quad z < x, \quad (2.1a) \]

\[ k_-(x,z) = 0, \quad \text{if} \quad z > x, \quad (2.1b) \]

and let \( \psi(x) \) be an \( N \)-vector. The integral operators \( \Phi_F \) and \( \Phi_\pm \) on \( \psi \) are defined by

\[ \Phi_F(\psi) = \int_{-\infty}^{\infty} F(x,z) \psi(z) dz \quad (2.2) \]

for all integrable \( \psi \), similarly

\[ \Phi_\pm(\psi) = \int_{-\infty}^{\infty} k_\pm(x,z) \psi(z) dz, \quad (2.3) \]

so that

\[ \Phi_+ = \int_{-\infty}^{\infty} k_+(x,z) dz, \quad (2.4a) \]
and
\[ \Phi_- = \int_{-\infty}^{x} k_- (x, z) \, dz. \]  
(2.4b)

We suppose that \( \Phi_F \) and \( \Phi_{\pm} \) are related by the operator identity
\[ (I + \Phi_+) \, (I + \Phi_F) = (I + \Phi_-), \]
where we assume that \( (I + \Phi_+) \) is invertible so that
\[ (I + \Phi_F) = (I + \Phi_+)^{-1} \, (I + \Phi_-), \]
i.e. the operator \( (I + \Phi_F) \) is factorisable and \( I \), as usual, is the unit matrix.

The identity (2.5), on \( \psi \), can be written as
\[
\int_{-\infty}^{\infty} k_+ (x, z) \psi (z) \, dz + \int_{-\infty}^{\infty} F (x, z) \psi (z) \, dz \\
+ \int_{x}^{\infty} k_+ (x, z) \left( \int_{-\infty}^{\infty} F (z, y) \psi (y) \, dy \right) \, dz \\
= \int_{x}^{\infty} k_- (x, z) \psi (z) \, dz,
\]
(2.7)
and if we require that \( \psi (z) = 0 \) for \( z < x \) then the right-hand side is zero. Furthermore, the double integral may be expressed as
\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} k_+ (x, y) F (y, z) \psi (z) \, dy \, dz,
\]
where the 'dummy' variables have been relabelled by interchanging \( y \) and \( z \). Thus equation (2.7) becomes
\[
0 = \int_{x}^{\infty} \left[ k_+ (x, z) + F (x, z) \right] \psi (z) \, dz \\
+ \int_{x}^{\infty} \int_{-\infty}^{\infty} (k_+ (x, y) \, F (y, z)) \psi (z) \, dy, \]  
(2.8)

(note the use here of eq.(2.1a) for all \( \psi (z) \)). Therefore
\[ 0 = k_+ (x, z) + F (x, z) \\
+ \int_{t}^{\infty} k_+ (x, y) \, F (y, z) \, dy, \quad \text{for } z > x. \]  
(2.9)

Equation (2.9) is the matrix Marchenko equation for \( k_+ (x, z) \). Similarly, if we consider \( z < x \) in eq.(2.9), it can be shown that
\[ k_- (x, z) = F (x, z) + \int_{t}^{\infty} k_+ (x, y) \, F (y, z) \, dy, \]  
(2.10)
which defines \( k_- (x, z) \) in terms of \( k_+ \) and \( F \). At this stage we have not restricted the choice of \( F \), and this we will next do for some NLS equations.

2.2 The differential Operators of the ZS Dressing Method

In common with our previous analyses, we extend the definitions of the matrices \( k_{\pm} \) and \( F \), and the vector \( \psi \), so that they all may now depend upon auxiliary variable \( t, y \). We shall describe the evolution of \( k_{\pm} \) and \( F \) (in \( t, y \)), and hence relate them to certain evolution equations by introducing appropriate (linear) differential operators. We define the \( N \times N \) matrix differential operator \( \Delta^{(i)}_0 \), \( i = 1, 2 \) on \( \psi (x; t, y) \) which has only constant coefficients and which commutes with the integral operator \( \Phi_F \), i.e.
\[
\left[ \Delta^{(i)}_0, \Phi_F \right] = \Delta^{(i)}_0 \Phi_F - \Phi_F \Delta^{(i)}_0 = 0. \]  
(2.11)

Note that in term \( \Phi_F \Delta^{(i)}_0 \), \( \Delta^{(i)}_0 \) operates on \( \psi (x; t, y) \) first and then this is evaluated on \( x = z \) for the application of the operator \( \Phi_F \). Further, we introduce an associated differential operator, \( \Delta^{(i)} \), which is defined by the operator identity
\[
\Delta^{(i)} \, (I + \Phi_+) = (I + \Phi_+) \, \Delta^{(i)}_0, \quad i = 1, 2. \]  
(2.12)

It can be shown that eq.(2.12) also holds if \( \Phi_- \) is replaced by \( \Phi_+ \):

*Theorem 1.* If operator \( (I + \Phi_F) \) commutes with the differential operator \( \Delta^{(i)}_0 \), and is invertible then both its Volterra factors \( (I + \Phi_{\pm}) \) transform \( \Delta^{(i)}_0 \) into one and the same operator \( \Delta^{(i)} \).
Proof (of theorem 1).

\[ \Delta^{(i)} = (I + \Phi_+) \Delta_0^{(i)} (I + \Phi_+)^{-1} \]
\[ = \left\{ \left( I + \Phi_- \right) (I + \Phi_F)^{-1} \right\} \times \Delta_0^{(i)} (I + \Phi_F) (I + \Phi_-)^{-1} \]
\[ = (I + \Phi_-) \Delta_0^{(i)} (I + \Phi_-)^{-1}. \quad (2.13) \]

The operator \( \Delta_0^{(i)} \) is sometimes referred to as 'undressed', and \( \Delta^{(i)} \) as the 'dressed' operator.

Before the general development, we shall consider an example which will illuminate the meaning of equations (2.11) and (2.12). Let

\[ \Delta_0^{(i)} = I \left( i\alpha \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} \right), \quad (2.14) \]

where \( \alpha \) is an arbitrary real value, and \( I \) is the 3x3 unit matrix. Thus eq.(2.11), when operated on \( \psi(x; t) \), becomes

\[ 0 = \left( i\alpha \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} \right) \int F(x, z; t) \psi(z; t) dz \]
\[ - \int_{-\infty}^{\infty} F(x, z; t) \left( i\alpha \frac{\partial}{\partial t} - \frac{\partial^2}{\partial z^2} \right) \psi(z; t) dz. \quad (2.15) \]

After integration by parts, it follows that

\[ \int_{-\infty}^{\infty} F \psi_{zz} dz = \int_{-\infty}^{\infty} F_{zz} \psi dz, \quad (2.16) \]

for all bounded continuously twice differentiable \( \psi \) provided \( \psi, \psi_z \to 0 \) as \( |z| \to \infty \). Hence eq.(2.15) can be written as

\[ \int_{-\infty}^{\infty} (i\alpha F_t - F_{xx} + F_{zz}) \psi dz = 0, \quad (2.17) \]

and therefore eq.(2.11) (for \( i = 1 \)) is an operator identity only if \( F(x, z; t) \) satisfies

\[ i\alpha F_t - F_{xx} + F_{zz} = 0. \quad (2.18) \]

The associated operator \( \Delta^{(i)} \) is now obtained from eq.(2.12) (for \( i = 1 \)),

\[ \Delta^{(i)} \left\{ \psi(x; t) + \int_{-\infty}^{\infty} k_++(x, z; t) \psi(z; t) dz \right\} \]
\[ = I (i\alpha \psi_t - \psi_{xx}) \]
\[ + \int_{-\infty}^{\infty} k_++(x, z; t) \left( i\alpha \frac{\partial}{\partial t} - \frac{\partial^2}{\partial z^2} \right) \psi(z; t) dz. \quad (2.19) \]

Again, we integrate by parts to find

\[ \int_{-\infty}^{\infty} k_+ \psi_{zz} dz = -\hat{k}_+ \psi_x + \hat{k}_+ \psi + \int_{-\infty}^{\infty} k_{+zz} \psi dz, \quad (2.20) \]

where \( \hat{k}_+ = k_+(x, z; t) \), and we assume that \( k_+, k_{zz} \to 0 \) as \( z \to +\infty \). It is now convenient to set

\[ \Delta^{(i)} = \Delta_0^{(i)} + V_i, \quad i = 1, 2, \quad (2.21) \]

so that eq.(2.19) becomes (for instance, \( i = 1 \))

\[ V_1 \left( \psi + \int_{-\infty}^{\infty} k_+ \psi dz \right) + i\alpha \int_{-\infty}^{\infty} (k_+ \psi + k_+ \psi_t) dz \]
\[ + \hat{k}_+ \psi_x + \hat{k}_+ \psi - \int_{-\infty}^{\infty} (k_{zz} - k_{zz}) \psi dz \]
\[ = i\alpha \int_{-\infty}^{\infty} k_+ \psi_t dz + \hat{k}_+ \psi_x - \hat{k}_+ \psi - \psi \frac{d}{dx} \hat{k}_+, \quad (2.22) \]

or

\[ 0 = \left( V_1 + 2 \frac{d}{dx} \hat{k}_+ \right) \psi + V_1 \int_{-\infty}^{\infty} k_+ \psi dz \]
\[ + \int_{-\infty}^{\infty} (i\alpha k_+ - k_{zz} + k_{zz}) \psi dz, \quad (2.23) \]

since \( \frac{d}{dx} \hat{k}_+ = \hat{k}_x + \hat{k}_{xx} \), the total derivative in \( x \). Hence if this equation is valid for all continuous \( \psi \), then

\[ V_1 (x; t) = -2 \frac{d}{dx} \hat{k}_+ = -2 \left( \hat{k}_x + \hat{k}_{xx} \right), \quad (2.24) \]
(so that $V_1$ is of degree zero), and $k_+(x, z; t)$ satisfies

$$iak_{x_+} - a_{z_+} + k_{z_+} + V_1 k_+ = 0. \quad (2.25)$$

On the other hand, we choose the undressed operator $\Delta_0^{(2)}$ as follows

$$\Delta_0^{(2)} = \left. \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \delta & 0 \\ 0 & 0 & \delta \end{pmatrix} \right\} \frac{\partial}{\partial x}, \quad (2.26)$$

(where $\alpha$ and $\delta$ are arbitrary real values), and then substitute it into eq. (2.12) : $\Delta^{(2)} (I + \Phi_+) = (I + \Phi_+) \Delta_0^{(2)}$, we obtain

$$\Delta_0^{(2)} = \left. \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \delta & 0 \\ 0 & 0 & \delta \end{pmatrix} \right\} \psi_x - V_2 \left( \psi + \int_{x}^{\infty} k_+ \psi dz \right)$$

$$+ \int_{x}^{\infty} k_+ \left( \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \delta & 0 \\ 0 & 0 & \delta \end{pmatrix} \right) \psi_x dz$$

$$= \left. \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \delta & 0 \\ 0 & 0 & \delta \end{pmatrix} \right\} \psi_x - \tilde{k}_+ \psi$$

$$+ \left. \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \delta & 0 \\ 0 & 0 & \delta \end{pmatrix} \right\} \int_{x}^{\infty} k_+ \psi dz, \quad (2.27)$$

where we have written $\Delta^{(2)} = \Delta_0^{(2)} + V_2$. On integrating by parts in the first term, eq. (2.27) becomes

$$0 = V_2 \int_{x}^{\infty} k_+ \psi dz + \int_{x}^{\infty} \left. \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \delta & 0 \\ 0 & 0 & \delta \end{pmatrix} \right\} \tilde{k}_+ \psi dz$$

$$+ \left. \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \delta & 0 \\ 0 & 0 & \delta \end{pmatrix} \right\} \int_{x}^{\infty} \tilde{k}_+ \psi dz$$

and so we choose

$$V_2 = \left. \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \delta & 0 \\ 0 & 0 & \delta \end{pmatrix} \right\} \left[ \tilde{k}_+ - \widetilde{k}_+ \left( \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \delta & 0 \\ 0 & 0 & \delta \end{pmatrix} \right) + V_2 k_+ = 0. \quad (2.29)$$

We now return to the main development and describe a further important step in the ZS dressing method. This is to introduce two pairs of operators $\Delta^{(1)}$ and $\Delta_0^{(1)}$. A typical choice is in the following forms

$$\Delta^{(1)} = I \alpha \frac{\partial}{\partial t} - M; \quad \Delta^{(2)} = I \beta \frac{\partial}{\partial y} + L, \quad (2.31)$$

and

$$\Delta_0^{(1)} = I \alpha \frac{\partial}{\partial t} - M_0; \quad \Delta_0^{(2)} = I \beta \frac{\partial}{\partial y} + L_0. \quad (2.32)$$

where $\beta$ is constant, and $M$, $L$, $M_0$, and $L_0$ are differential operators in $x$ only. Consistent with our notation, $M_0$ and $L_0$ are comprised of constant coefficients only and so $\Delta_0^{(1)}$, $\Delta_0^{(2)}$ commute. Furthermore, both $\Delta^{(1)}$ and $\Delta_0^{(2)}$ are to commute with the same operator $\Phi_F$ so that eq. (2.11) is valid. On the other hand, operators $\Delta^{(1)}$ are defined according to eq. (2.12), with the same $\Phi_+$. Based on this point, it is instructive to examine the operator

$$P = \Delta^{(1)} \Delta^{(2)} (I + \Phi_+) = \Delta^{(2)} \Delta^{(1)} (I + \Phi_+), \quad (2.32)$$

which, upon the use of eq. (2.12) twice, gives

$$P = \Delta^{(1)} (I + \Phi_+) \Delta^{(2)}_0 - \Delta^{(2)} (I + \Phi_+) \Delta^{(1)}_0$$

$$= (I + \Phi_+) \Delta^{(1)}_0 \Delta^{(2)}_0 - (I + \Phi_+) \Delta^{(2)} \Delta^{(1)}_0$$

$$= (I + \Phi_+) \left[ \Delta^{(1)}_0, \Delta^{(2)}_0 \right]. \quad (2.33)$$

However, $\Delta^{(1)}_0$ and $\Delta^{(2)}_0$ are chosen so that they commute with one another; hence $P = 0$. Thus we obtain

$$P = \left[ \Delta^{(1)}, \Delta^{(2)} \right] (I + \Phi_+) = 0, \quad (2.34)$$
and, since \((I + \Phi_+)_b\) is invertible (an easier assumption which can be checked in specific cases), we get
\[
\left[ \Delta^{(1)}, \Delta^{(2)} \right] = 0, \tag{2.35}
\]
which means that \(\Delta^{(1)}\) commutes with \(\Delta^{(2)}\). If we now introduce the choice given in eq.(2.31a) and eq.(2.31b), eq.(2.35) becomes
\[
0 = \left( I\alpha \frac{\partial}{\partial t} - M \right) \left( I\beta \frac{\partial}{\partial y} + L \right) - \left( I\beta \frac{\partial}{\partial y} + L \right) \left( I\alpha \frac{\partial}{\partial t} - M \right), \tag{2.36}
\]
which simplifies to
\[
i\alpha L_t + \beta M_y + [L, M] = 0. \tag{2.37}
\]
This is a generalisation of Lax pair to two auxiliary variables; the Lax pair is recovered if we put \(\beta = 0\). Equation (2.37) is the general system of the coupled NLS evolution equations which can be solved by the ZS dressing method (ZS scheme). We can also rewrite the connections among the operators \(L\), \(M\), \(M_0\), \(L_0\), \(\Delta^{(1)}\) and \(\Delta^{(2)}\) related to the general coupled NLS equations as follows
\[
L = \Delta^{(2)} = L_0 + V_2, \tag{2.38a}
\]
where
\[
L_0 = \Delta_0^{(2)}, \tag{2.38b}
\]
and
\[
M = M_0 - V_1, \tag{2.38c}
\]
where
\[
M_0 = I \frac{\partial^2}{\partial x^2}. \tag{2.38d}
\]

3 The New Integrable Coupled NLS Equations

In this section, we investigate and derive two types of the integrable coupled NLS equations with their certain perturbative terms and also show the Lax pairs of those equations using the ZS dressing method. For this purpose, we will initially choose certain matrix function \(k_+\), and then apply the ZS dressing method provided in section 2 to derive the equations we want.

3.1 Type I

We start by choosing the following matrix function \(k_+\) related to ZS dressing method as follows
\[
k_+ = \begin{pmatrix}
h_1(x, z; t) & \frac{1}{u} & \frac{1}{u^*} \\
\pm \frac{1}{u} & h_2(x, z; t) & h_3(x, z; t) \\
\frac{1}{u^*} & \pm \frac{1}{u^*} & h_4(x, z; t) & h_5(x, z; t)
\end{pmatrix}, \tag{3.1}
\]
where \(h_1, h_2, h_3, h_4, \) and \(h_5\) are functions which will be calculated using eq.(2.12).

By substituting eq.(3.1) into eq.(2.24), we obtain
\[
V_1(x, t) = -2 \begin{pmatrix}
h_{1x} & -\frac{u_{z}}{u^*} & \frac{u_{z}}{u} \\
\mp \frac{u_{z}}{u} & h_{2x} & h_{3x} \\
\mp \frac{u_{z}}{u^*} & h_{4x} & h_{5x}
\end{pmatrix}. \tag{3.2}
\]

On the other hand, by substituting eq.(3.1) into eq.(2.29), we find
\[
V_2(x, t) = \begin{pmatrix}
0 & \frac{(\alpha - \delta)}{u} & \frac{(\alpha - \delta)}{u^*} \\
\pm \frac{(\beta - \alpha)}{u} & 0 & 0 \\
\pm \frac{(\beta - \alpha)}{u^*} & 0 & 0
\end{pmatrix}. \tag{3.3}
\]

The functions \(h_{1x}, h_{2x}, h_{3x}, h_{4x} \) and \(h_{5x}\) in eq.(3.2) can be found by substituting eq.(3.3) and eq.(3.1)
into eq. (2.30),

\[ h_{1x} = \mp \frac{\alpha - \delta}{\alpha} \left( \frac{|u|^2 + |w|^2}{|u|^2 |w|^2} \right), \]

(3.4a)

\[ h_{2x} = \mp \frac{\delta - \alpha}{\delta} \left( \frac{1}{|u|^2} \right), \]

(3.4b)

\[ h_{3x} = \mp \frac{\delta - \alpha}{\delta} \left( \frac{1}{u^*w} \right), \]

(3.4c)

\[ h_{4x} = \mp \frac{\delta - \alpha}{\delta} \left( \frac{1}{u w^*} \right), \]

(3.4d)

and

\[ h_{5x} = \mp \frac{\delta - \alpha}{\delta} \left( \frac{1}{|w|^2} \right). \]

(3.4e)

Hence, the coupled NLS equation type I (eq. (1.1)) and its complex conjugate can easily be derived by putting equations (2.21), (2.14), (2.26), (3.2), and (3.3) into eq. (2.35).

Finally, we can obtain the Lax pair operators of eq. (1.1) and its complex conjugate using the relations of equations (2.37) and (2.38),

\[ L = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \delta & 0 \\ 0 & 0 & \delta \end{pmatrix} \frac{\partial}{\partial x} + \left( \begin{pmatrix} 0 \\ \frac{\delta - \alpha}{u} \\ \frac{\delta - \alpha}{w} \end{pmatrix} \begin{pmatrix} \frac{\frac{\delta - \alpha}{u}}{u^2} \\ 0 \\ 0 \end{pmatrix} \right), \]

(3.5a)

and

\[ M = I \frac{\partial^2}{\partial x^2} + 2 \begin{pmatrix} h_{1x} & \frac{u x}{u + u^*} \\ h_{2x} & h_{3x} \\ \frac{w x}{w + w^*} & h_{4x} \end{pmatrix} \begin{pmatrix} h_{5x} \end{pmatrix}. \]

(3.5b)

### 3.2 Type II

Following the same procedure as type I, we only choose \( k_+ \) in a different form from that in eq. (3.1),

\[ k_+ = \begin{pmatrix} a(x, z; t) & \frac{u}{u + u^*} \\ \frac{d}{d} & \frac{e}{e} \\ \frac{f}{f} & \frac{g}{g} \end{pmatrix}, \]

(3.6)

\( a, \ d, \ e, \ f \) and \( g \) are the functions which will be investigated using eq. (2.12) and

\[ \mathcal{N}(x, z; t) = \sqrt{1 + s \left( |u|^2 + |w|^2 \right)}. \]

By substituting eq. (3.6) into eq. (2.24), we obtain

\[ V_1(x, t) = -2 \begin{pmatrix} \frac{a_x}{x} & \frac{f}{f} \frac{e_x}{x} \\ \frac{+ u^*}{\delta} & \frac{d_x}{x} \end{pmatrix}. \]

(3.7)

On the other hand, by substituting eq. (3.6) into eq. (2.29), we find

\[ V_2(x, t) = \begin{pmatrix} \begin{pmatrix} \frac{\alpha - \delta}{\delta} u \\ \frac{\alpha - \delta}{\delta} w \end{pmatrix} \\ 0 \end{pmatrix} \frac{\begin{pmatrix} \frac{\alpha - \delta}{\delta} u \\ \frac{\alpha - \delta}{\delta} w \end{pmatrix}}{\delta} \]

(3.8)

The functions \( a_x, \ d_x, \ e_x, \ f_x \) and \( g_x \) in eq. (3.7) can be found by substituting eq. (3.8) and eq. (3.6) into eq. (2.30),

\[ a_x = \mp \frac{\alpha - \delta}{\alpha} \left( \frac{|u|^2 + |w|^2}{1 + s \left( |u|^2 + |w|^2 \right)} \right), \]

(3.9a)

\[ d_x = \mp \frac{\delta - \alpha}{\delta} \left( \frac{|u|^2}{1 + s \left( |u|^2 + |w|^2 \right)} \right), \]

(3.9b)

\[ e_x = \mp \frac{\delta - \alpha}{\delta} \left( \frac{u^* w}{1 + s \left( |u|^2 + |w|^2 \right)} \right), \]

(3.9c)

\[ f_x = \mp \frac{\delta - \alpha}{\delta} \left( \frac{u w^*}{1 + s \left( |u|^2 + |w|^2 \right)} \right), \]

(3.9d)

and

\[ g_x = \mp \frac{\delta - \alpha}{\delta} \left( \frac{|w|^2}{1 + s \left( |u|^2 + |w|^2 \right)} \right). \]

(3.9e)

Hence, the coupled NLS equation type II (eq. (1.3)) and its complex conjugate can be derived by substituting equations (2.21), (2.14), (2.26), (3.7), and (3.8) into eq. (2.35).
following four linear differential equations complex conjugate will be found by solving the linear differential equation type I using the ZS dressing method explained in section 2. By putting eq. (3.1) and eq. (4.1) into eq. (2.9), we get (for \( h_1 \), \( u \), and \( w \))

\[
\begin{align*}
A &= A_0 e^{-\alpha \rho} e^{i \rho (\delta x + \rho (\alpha^2 - \delta^2) t)} , \\
B &= B_0 e^{-\alpha \rho} e^{i \rho (\delta x + \rho (\alpha^2 - \delta^2) t)} , \\
A^* &= A_0^* e^{-\delta \sigma} e^{i \sigma (\alpha x + \sigma (\delta^2 - \alpha^2) t)} , \\
B^* &= B_0^* e^{-\delta \sigma} e^{i \sigma (\alpha x + \sigma (\delta^2 - \alpha^2) t)} ,
\end{align*}
\]  

where \( A_0 \), \( A_0^* \), \( B_0 \) and \( B_0^* \) are arbitrary complex parameters, and \( \rho \) and \( \sigma \) are chosen as imaginary constants in which \( \rho^* = -\sigma \).

### 4.1 The solution of the type I

Here, we derive the solution of the coupled NLS equation type I using the ZS dressing method. We then find

\[
\begin{align*}
A = A_0 e^{-\alpha \rho} e^{i \rho (\delta x + \rho (\alpha^2 - \delta^2) t)} , \\
B = B_0 e^{-\alpha \rho} e^{i \rho (\delta x + \rho (\alpha^2 - \delta^2) t)} , \\
A^* = A_0^* e^{-\delta \sigma} e^{i \sigma (\alpha x + \sigma (\delta^2 - \alpha^2) t)} , \\
B^* = B_0^* e^{-\delta \sigma} e^{i \sigma (\alpha x + \sigma (\delta^2 - \alpha^2) t)} ,
\end{align*}
\]  

The solution of the above equations can be derived by using separable variable method. We then find

\[
\begin{align*}
A = A_0 e^{-\alpha \rho} e^{i \rho (\delta x + \rho (\alpha^2 - \delta^2) t)} , \\
B = B_0 e^{-\alpha \rho} e^{i \rho (\delta x + \rho (\alpha^2 - \delta^2) t)} , \\
A^* = A_0^* e^{-\delta \sigma} e^{i \sigma (\alpha x + \sigma (\delta^2 - \alpha^2) t)} , \\
B^* = B_0^* e^{-\delta \sigma} e^{i \sigma (\alpha x + \sigma (\delta^2 - \alpha^2) t)} ,
\end{align*}
\]

The solution of the above equations can be derived by using separable variable method. We then find

\[
\begin{align*}
A = A_0 e^{-\alpha \rho} e^{i \rho (\delta x + \rho (\alpha^2 - \delta^2) t)} , \\
B = B_0 e^{-\alpha \rho} e^{i \rho (\delta x + \rho (\alpha^2 - \delta^2) t)} , \\
A^* = A_0^* e^{-\delta \sigma} e^{i \sigma (\alpha x + \sigma (\delta^2 - \alpha^2) t)} , \\
B^* = B_0^* e^{-\delta \sigma} e^{i \sigma (\alpha x + \sigma (\delta^2 - \alpha^2) t)} ,
\end{align*}
\]  

where \( A_0 \), \( A_0^* \), \( B_0 \) and \( B_0^* \) are arbitrary complex parameters, and \( \rho \) and \( \sigma \) are chosen as imaginary constants in which \( \rho^* = -\sigma \).

**4 The solutions of the integrable coupled NLS equations**

In this section, we investigate the solutions of both coupled NLS equations (type I and type II). We firstly choose the general matrix function \( F \),

\[
F = \begin{pmatrix} 0 & A(x, z; t) & B(x, z; t) \\ A^*(x, z; t) & 0 & 0 \\ B^*(x, z; t) & 0 & 0 \end{pmatrix},
\]

where the functions \( A(x, z; t) \), \( B(x, z; t) \) and their complex conjugate will be found by solving the linear differential equation in eq. (2.18). Hence, after substituting eq. (4.1) into eq. (2.18), we get the following four linear differential equations

\[
\begin{align*}
i \alpha A_1 + A_{zz} - A_{xx} &= 0, \\
i \alpha B_1 + B_{zz} - B_{xx} &= 0, \\
i \alpha A^*_1 + A^*_{zz} - A^*_{xx} &= 0, \\
i \alpha B^*_1 + B^*_{zz} - B^*_{xx} &= 0.
\end{align*}
\]  

The solution of the above equations can be derived by using separable variable method. We then find

\[
\begin{align*}
A = A_0 e^{-\alpha \rho} e^{i \rho (\delta x + \rho (\alpha^2 - \delta^2) t)} , \\
B = B_0 e^{-\alpha \rho} e^{i \rho (\delta x + \rho (\alpha^2 - \delta^2) t)} , \\
A^* = A_0^* e^{-\delta \sigma} e^{i \sigma (\alpha x + \sigma (\delta^2 - \alpha^2) t)} , \\
B^* = B_0^* e^{-\delta \sigma} e^{i \sigma (\alpha x + \sigma (\delta^2 - \alpha^2) t)} ,
\end{align*}
\]  

where \( A_0 \), \( A_0^* \), \( B_0 \) and \( B_0^* \) are arbitrary complex parameters, and \( \rho \) and \( \sigma \) are chosen as imaginary constants in which \( \rho^* = -\sigma \).
The final solution is work on \( x = z \) as the ZS method works (see the explanation of eq.(2.11) in section 2.2). Hence, by substituting eq.(4.4a) to eq.(4.4b) and eq.(4.4c) we find the solution of the type I:

\[
u (x, t) = -\frac{1 + \mathbb{Q}_1 \mathbb{Q}_2}{A_0 \mathbb{Q}_1}, \]

(4.5a)

and

\[
w (x, t) = -\frac{1 + \mathbb{Q}_1 \mathbb{Q}_2}{B_0 \mathbb{Q}_1}, \]

(4.5b)

where

\[
\mathbb{Q} = \mu \left( |A_0|^2 + |B_0|^2 \right)^{1/2} \left( \frac{\alpha - \delta^2}{\alpha} \right)^{1/2} - \rho - \sigma \right),
\]

(4.6a)

\[
\begin{align*}
\Theta_{11} &= e^{\rho(\delta - \alpha)x}, \\
\Theta_{21} &= e^{-\sigma(\delta - \alpha)x}, \\
\Theta_{12} &= e^{-i\frac{\alpha}{\sqrt{\alpha}}(\delta^2 - \alpha^2)t}, \\
\Theta_{22} &= e^{i\frac{\alpha}{\sqrt{\alpha}}(\delta^2 - \alpha^2)t}, \\
\mathbb{Q}_1 &= \Theta_{11} \Theta_{12}, \\
\mathbb{Q}_2 &= \Theta_{21} \Theta_{22}.
\end{align*}
\]

(4.6g)

Hence, by substituting eq.(4.7a) to eq.(4.7b) and eq.(4.7c) we find the solution of the type II:

\[
u (x, t) = -\frac{1 + s \Theta_1}{1 - s \Theta_1} \frac{A_0 \mathbb{Q}_1}{1 + \mathbb{Q}_1 \mathbb{Q}_2},
\]

(4.8a)

and

\[
w (x, t) = -\frac{1 + s \Theta_2}{1 - s \Theta_2} \frac{B_0 \mathbb{Q}_1}{1 + \mathbb{Q}_1 \mathbb{Q}_2},
\]

(4.8b)

where

\[
\begin{align*}
\Theta_1 &= |A_0|^2 \mathbb{Q}_1, \\
\Theta_2 &= |B_0|^2 \mathbb{Q}_1 \\
\Xi &= 1 + \mathbb{Q}_1 \mathbb{Q}_2 + \mathbb{Q}_1 \mathbb{Q}_2', + \mathbb{Q}_1 \mathbb{Q}_2' \mathbb{Q}_1', \mathbb{Q}_2' \mathbb{Q}_1', \\
\Phi_1 &= \frac{\Theta_1}{1 - s \Theta_2}, \\
\Phi_2 &= \frac{\Theta_2}{1 - s \Theta_1}.
\end{align*}
\]

(4.9c)

The solution of the type II

4.2 The Solution of the Type II

By reviewing the solution’s steps in section 4.1, we can also investigate the solution of the type II. We then start by substituting eq.(3.6) and eq.(4.1) into eq.(2.9), we get (for \( a, u, \) and \( w \))

\[
a = -\int_x \frac{u A_0^*}{\mathcal{N}} e^{i \frac{\alpha}{\sqrt{\alpha}}(\delta^2 - \alpha^2)t} e^{-\delta \sigma z} e^{\alpha \sigma y} dy
\]

(4.7a)

and

\[
-\int_x \frac{w B_0^*}{\mathcal{N}} e^{i \frac{\alpha}{\sqrt{\alpha}}(\delta^2 - \alpha^2)t} e^{-\delta \sigma z} e^{\alpha \sigma y} dy.
\]

(4.7a)
\[ \Theta'_{22} = e^{-\left(\frac{x^2}{2}\right)}(x^2 - \sigma^2)^t, \quad (4.9i) \]

\[ \Box' = \Theta'_1 \Theta'_{12}, \quad (4.9j) \]

and

\[ \Box'_2 = \Theta'_2 \Theta'_{22}. \quad (4.9k) \]

It seems that the solution of the type II in equations (4.8a) and (4.8b) can roughly be identified as a stable soliton solution when parameter \( s \) is small. If we put \( s \to 0 \) either in eq.(1.6) or in equations (4.8) then we will find the one soliton solution of the coupled NLS equation of Manakov type appeared in ref.[6].

We can also choose parameters \( \alpha = 4.5 \) and \( \delta = 1.73 \) in our type II equation and its solutions, respectively. The aim of this choice is to make parameters \( \chi = \mu = \frac{1}{2} \) and to reduce our one soliton (in eq.(4.8a)) and eq.(4.8b)) to be a simple solution.

It is very interesting to expand the discussions related to an equation which has a soliton solution. So, it is undoubtedly that the type II equation is particularly to be discussed in this paper. Even the equation (1.3) with a simple additional term has experimentally been investigated by Christodoulides et al. [7], and Kivshar et al. [8]. Based on their explanations, we suppose that there must be a physical meaning of our type II equation closely related to that in ref.[7, 8].

5 Concluding Remarks

In this paper, we have presented two types of the integrable coupled NLS equation with perturbative terms derived using the ZS dressing method and have also found their solutions. Using the Lax pairs of the two types, we have identified the existence of the integrability of those equations. The very important step to obtain the two types of the coupled NLS equations in the ZS dressing method is the choice of \( k_+ \) in eq.(3.1) and eq.(3.6), respectively. The type I has no soliton solution. However, the type II could exactly have a stable soliton solution when the effective saturation parameter, \( s \) is small \( (s \to 0) \).

In the two types, the coupled NLS equations may describe wave propagations in birefringent optical fibers with nonlinear effects such as the Kerr effect, The Raman Scattering and gain and loss effects.

The type II is very interesting to be discussed because it has a complicated soliton solution. We also propose that this type must also have multisolitons solutions. However, the cases will be discussed in our next investigations [10].

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