Four-Dimensional Asymptotically AdS Black Holes with Scalar Hair

P. A. González

Facultad de Ingeniería, Universidad Diego Portales,
Avenida Ejército Libertador 441, Casilla 298-V, Santiago, Chile.

Eleftherios Papantonopoulos

Department of Physics, National Technical University of Athens, Zografou Campus GR 157 73, Athens, Greece.

Joel Saavedra

Instituto de Física, Pontificia Universidad Católica de Valparaíso, Casilla 4950, Valparaíso, Chile.

Yerko Vásquez

Departamento de Ciencias Físicas, Facultad de Ingeniería,
Ciencias y Administración, Universidad de La Frontera,
Avenida Francisco Salazar 01145, Casilla 54-D, Temuco, Chile. and
Departamento de Física, Facultad de Ciencias,
Universidad de La Serena,
Avenida Cisternas 1200, La Serena, Chile.

(Dated: November 27, 2013)

We present a new family of asymptotically AdS four-dimensional black hole solutions with scalar hair of a gravitating system consisting of a scalar field minimally coupled to gravity with a self-interacting potential. For a certain profile of the scalar field we solve the Einstein equations and we determine the scalar potential. Thermodynamically we show that there is a critical temperature below which there is a phase transition of a black hole with hyperbolic horizon to the new hairy black hole configuration.

PACS numbers:

I. INTRODUCTION

Hairy black holes are interesting solutions of Einstein’s Theory of Gravity and also of certain types of Modified Gravity Theories. These solutions have been extensively studied over the years mainly in connection with the no-hair theorems. The recent developments in string theory and specially the application of the AdS/CFT principle to condense matter phenomena like superconductivity (for a review see [1]), triggered the interest of further study of the behaviour of matter fields outside the black hole horizon [2, 3]. There are also very interesting recent developments in Observational Astronomy. High precision astronomical observations of the supermassive black holes may pave the way to experimentally test the no-hair conjecture [4]. Also there are numerical investigations of single and binary black holes in the presence of scalar fields [5].

These developments put forward the necessity of a better understanding of the behaviour of matter fields all the way from the black hole horizon to asymptotic infinity. The basic physical requirement of a black hole solution with scalar hair is the scalar field to be regular on the horizon and to fall off sufficiently fast at the infinity. Another important problem which is still open is to find a way to characterize the presence of the scalar hair. All the hairy black hole solutions known so far are characterized by secondary hair i.e. parameters which are connected in some way with the physical parameters of the solution. This implies that it is not possible to continuously connect the hairy configuration with mass $M$ and a configuration with the same mass and no scalar field. It is desirable to find hairy black hole solutions in which the scalar hair is characterized by a primary hair, i.e. a conserved charge. This may be archived if we could find a stealth solution in which the scalar sector of the theory is hidden from the gravity sector. Work on this direction is in progress from various groups.

---

*Electronic address: pablo.gonzalez@udp.cl
†Electronic address: lpapa@central.ntua.gr
‡Electronic address: joel.saavedra@ucv.cl
§Electronic address: yerko.vasquez@ufrontera.cl
The first attempts to couple a scalar field to gravity was done in a asymptotically flat spacetime. Then hairy black hole solutions were found \cite{6} but soon it was realized that these solutions were not examples of hairy black hole configurations violating the no-hair theorems because they were not physically acceptable as the scalar field was divergent on the horizon and stability analysis showed that they were unstable \cite{7}. To remedy this a regularization procedure has to be used to make the scalar field finite on the horizon.

The easiest way to make the scalar field regular on the horizon is to introduce a scale in the gravity sector of the theory through a cosmological constant. The resulting black hole solutions with the presence of the cosmological constant have regular scalar field on the horizon and all possible infinities are hidden behind the horizon. Hairy black hole solutions were found with a minimally coupled scalar field and a self-interaction potential in asymptotically dS space \cite{8} and also a numerical solution was presented in \cite{9}, but it was unstable. If the scalar field is non-minimally coupled a hairy black hole configuration was found \cite{10}, but perturbation analysis showed the instability of the solution \cite{11,12}. In the case of a negative cosmological constant, stable solutions were found numerically for spherical geometries \cite{13,14} and an exact solution in asymptotically AdS space with hyperbolic geometry was presented in \cite{15} and generalized later to include charge \cite{16}. In all the above solutions the scalar field was conformally coupled to gravity. A generalization to non-conformal solutions was discussed in \cite{17}. Further hairy solutions in the presence of a cosmological constant were reported in \cite{18,21} with various properties. More recently new hairy black hole solutions, boson stars and numerical rotating hairy black hole solutions were reported \cite{22,26}.

Another way to make the scalar field regular on the horizon is to introduce a scale in the scalar sector. This can be done if in the Einstein-Hilbert action there is a coupling of a scalar field to Einstein tensor. The derivative coupling has the dimension of length square and it was shown that acts as an effective cosmological constant \cite{25,26}. Then in \cite{29} a gravitating system of vanishing cosmological constant consisting of an electromagnetic field and a scalar field coupled to the Einstein tensor was discussed. A Reissner-Nordstrom black hole undergoes a second-order phase transition to a hairy black hole of generally anisotropic hair at a certain critical temperature which we compute. The no-hair theorem is evaded due to the coupling between the scalar field and the Einstein tensor. Spherically symmetric hairy black hole solutions with scalar hair were also found in the presence of this coupling \cite{30}. Finally hairy black hole configurations were found when a charged C-metric is conformally coupled to a scalar field \cite{31,32}. In these models the acceleration parameter plays the role of the cosmological constant giving regularity to the scalar field \cite{11,12}. In \cite{33} a charged C-metric coupled also to an electromagnetic field was considered and hairy black hole solutions were found.

The gauge/gravity duality is a principle which relates strongly coupled systems to the their weak coupled gravity duals. One of the most well studied system in the context of gauge/gravity duality is the holographic superconductor. In its simplest form, the gravity sector is a gravitating system with a cosmological constant, a gauge field and a charged scalar field with a potential (for a review see \cite{34}). The dynamics of the system defines a critical temperature above which the system finds itself in its normal phase and the scalar field does not have any dynamics. Below the critical temperature the system undergoes a phase transition to a new configuration. From the gravity side this is interpreted as the black hole to acquire hair while from boundary conformal field theory side this is interpreted as a condensation of the scalar field and the system enters a superconducting phase.

The whole dynamics of the holographic superconductor relies heavily on how the scalar filed behaves outside the black hole horizon and at infinity. In \cite{2,3} a mechanism was proposed to break an Abelian gauge symmetry near a black hole horizon. The mechanism is similar to the Landau-Ginzburg mechanism in superconductivity but it is more geometrical. The effective mass of the scalar field is $m_{eff}^2 = m^2 + q^2 g^{tt} A_t^2$ where $q$ is the charge of the scalar field and $A_t$ the scalar potential of the gauge field. For large values of the scalar charge the effective mass can become negative signaling the breaking of the gauge symmetry outside the black hole horizon \cite{8}. However, for this mechanism to work the behaviour of the scalar field at infinity is crucial. The scalar field at large distances should find a potential barrier (the cosmological constant can be considered as the constant term of the potential at large distances). This is necessary because for large values of $q$ the electrostatic repulsion overcomes the gravitational attraction and the matter field goes to infinity. If it finds a potential barrier then it bounces back and condenses outside the black hole horizon giving in this way a hairy black hole configuration.

From the above discussion we conclude that it is important to understand the behaviour of a hairy black hole at large distances. In this work we will develop a general formalism to generate asymptotically AdS hairy black holes. We consider a gravitational system which for simplicity consists only of a real scalar field conformally coupled to gravity with a self-interacting potential in which we have incorporated the cosmological constant. We also assume spherical symmetry of the system. We look for solutions with the following characteristics. The scalar field to be

\footnote{Actually there is still an irregularity of the scalar field on one point on the horizon but at least the scalar field does not diverge on the whole horizon.}
regular on the horizon, to fall off at asymptotic infinity and the self-interacting potential to go to the cosmological constant at infinity. The strategy we follow is that we choose a profile of the scalar field. Then the metric functions and the self-interacting potential are calculated analytically. A similar procedure to construct asymptotic hairy black holes was considered in five-dimensions \[^35\].

The paper is organized as follows. In Section II we present the general formalism and we apply it to some of the existing exact black hole solutions with a conformally coupled scalar field with a self-interacting potential. In Section III we discuss a new class of asymptotically AdS black holes with scalar hair. In Section IV, we discuss the thermodynamics, first by using the Euclidean formalism we calculate the mass and the entropy of the solutions and then we show that a phase transition occurs between the asymptotically AdS four-dimensional black hole with scalar hair and a black hole with hyperbolic horizon and finally in Section V we conclude.

II. GENERAL FORMALISM

In this section we will present a general formalism of a neutral scalar field minimally coupled to curvature having a self-interacting potential \(V(\phi)\) and using it we will review the existing hairy black hole solutions. We start with the Einstein-Hilbert action with a negative cosmological constant \(\Lambda = -6\ell^{-2}/\kappa\), where \(\ell\) is the length of the AdS space and \(\kappa = 8\pi G_N\), with \(G_N\) the Newton constant. We will incorporate the cosmological constant in the potential as \(\Lambda = V(0)\) \((V(0) < 0)\).

The action is

\[
S = \int d^4x \sqrt{-g} \left( \frac{1}{2\kappa} R - \frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - V(\phi) \right) .
\]  

(1)

The resulting Einstein equations from the above action are

\[
R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa T_{\mu\nu}^{(\phi)}
\]

(2)

and the energy momentum tensor \(T_{\mu\nu}^{(\phi)}\) for the scalar field is

\[
T_{\mu\nu}^{(\phi)} = \nabla_\mu \phi \nabla_\nu \phi - g_{\mu\nu} \left[ \frac{1}{2} g^{\rho\sigma} \nabla_\rho \phi \nabla_\sigma \phi + V(\phi) \right] .
\]  

(3)

If we use Eqs. (2) and (3) we obtain the equivalent equation

\[
R_{\mu\nu} - \kappa (\partial_\mu \phi \partial_\nu \phi + g_{\mu\nu} V(\phi)) = 0 .
\]  

(4)

We consider the following metric ansatz

\[
d\sigma^2 = -f(r) dt^2 + f^{-1}(r) dr^2 + a^2(r) d\sigma^2
\]

(5)

where \(d\sigma^2\) is the metric of the spatial 2-section, which can have positive, negative or zero curvature. In the case of the metric of Eq. \((5)\), if we use Eq. \((4)\) we find the following three independent differential equations

\[
f''(r) + 2 \frac{a'(r)}{a(r)} f'(r) + 2V(\phi) = 0 ,
\]  

(6)

\[
\frac{a'(r)}{a(r)} f'(r) + \left( \frac{(a'(r))^2}{a^2(r)} + \frac{a''(r)}{a(r)} \right) f(r) - \frac{k}{a^2(r)} + V(\phi) = 0 ,
\]  

(7)

\[
f''(r) + 2 \frac{a'(r)}{a(r)} f'(r) + \left( 4 \frac{a''(r)}{a(r)} + 2(\phi'(r))^2 \right) f(r) + 2V(\phi) = 0 ,
\]  

(8)

where \(k\) is the curvature of the spatial 2-section. All the quantities, in the above equations, have been rendered dimensionless via the redefinitions \(\sqrt{\kappa} \phi \rightarrow \phi\), \(\kappa \ell^{-2} V \rightarrow V\) and \(r/\ell \rightarrow r\).

If we eliminate the potential \(V(\phi)\) from the above equations we obtain

\[
a''(r) + \frac{1}{2} (\phi'(r))^2 a(r) = 0 ,
\]  

(9)
\[ f''(r) - 2 \left( \frac{(a'(r))^2}{a^2(r)} + \frac{a''(r)}{a(r)} \right) f(r) + \frac{2k}{a^2(r)} = 0 \tag{10} \]

where the potential can be determined from Eq. (6) if the functions \( a(r) \) and \( f(r) \) are known.

To find exact hairy black hole solutions the differential equations (6)-(8) have to be supplemented with the Klein-Gordon equation of the scalar field which in general coordinates reads

\[ \Box \phi = \frac{dV}{d\phi} \tag{11} \]

There is a well known solution of the differential equations (6)-(8) and (11), the MTZ solution [15], with the change of coordinates \( \frac{dr'}{r'} = \frac{dr}{a(r)r} \), MTZ metric is given by

\[ ds^2 = B(r') \left( -F(r') \, dt^2 + \frac{1}{F(r')} \, dr'^2 + r'^2 d\sigma^2 \right) \tag{12} \]

\[ B(r') = \frac{r'(r' + 2G_N \mu)}{(r' + G_N \mu)^2} \tag{13} \]

\[ F(r') = \frac{r'^2}{l^2} - \left( 1 + \frac{G_N \mu}{2} \right)^2 \tag{14} \]

The scalar field is given by

\[ \phi = \sqrt{\frac{3}{4\pi G_N}} \text{Arctanh} \frac{G_N \mu}{r' + G_N \mu} \tag{15} \]

where the potential is found to be

\[ V(\phi) = \Lambda \sinh^2 \sqrt{\frac{4\pi G_N}{3}} \phi \tag{16} \]

This is the simplest known hairy black hole solution of a scalar field minimally coupled to the curvature which goes to zero at infinity and it is regular on the horizon. Another interesting feature of this solution is revealed if the solution is transformed to the Jordan frame via a conformal transformation. Then it can be understood that its simplicity relies on the fact that conformal symmetry in the scalar sector allows the energy momentum tensor to be traceless resulting to a simple relation of the scalar curvature \( R \) to the cosmological constant.

If one wants to abandon the conformal coupling of the scalar field to curvature one can still solve the equations (6)-(8) and (11), but the resulting potential from equation (6) is more complicated than the corresponding potential of the MTZ black hole solution. This solution has the form in the coordinates \( \frac{dr'}{r'} = \frac{dr}{a(r)r} \) [17]

\[ B(r') = \frac{r'(r' + 2r'_0)}{(r' + r'_0)^2} \tag{17} \]

with

\[ F(r') = \frac{r'^2}{l^2} - g \frac{r'_0}{l^2} r' - 1 + g \frac{r'^2}{l^2} - \left( 1 - 2g \frac{r'^2}{l^2} \right) \frac{r'_0}{r'} \left( 2 + \frac{r'_0}{r'} \right) + g \frac{r'^2}{2l^2} \ln \left( 1 + \frac{2r'_0}{r'} \right) \tag{18} \]

The scalar field is

\[ \phi = \sqrt{\frac{3}{4\pi G_N}} \text{Arctanh} \frac{r'_0}{r' + r'_0} \tag{19} \]

where \( r'_0 \) is a constant and the potential is given by

\[ V(\phi) = \frac{\Lambda}{4\pi G_N} \sinh^2 \sqrt{\frac{4\pi G_N}{3}} \phi \]

\[ + \frac{g \Lambda}{6\pi G_N} \left[ 2\sqrt{3\pi G_N} \phi \cosh \left( \sqrt{\frac{16\pi G_N}{3}} \phi \right) - \frac{9}{8} \sinh \left( \sqrt{\frac{16\pi G_N}{3}} \phi \right) - \frac{1}{8} \sinh \left( 4\sqrt{3\pi G_N} \phi \right) \right] \tag{20} \]
It is interesting to observe that there is a parameter $g$ in the potential indicating the departure from conformal invariance. If $g = 0$ then we recover the MTZ black hole.

Another interesting solution of the equations (6)-(8) and (11) is provided by the C-metric solution [31]. The metric is given by

$$
\frac{ds^2}{A^2(x-y)^2} = \frac{u(y,x)}{1 + \frac{2\pi \Lambda}{9\alpha} \left( \frac{Am(x-y)}{1 + Am(x+y)} \right)^2},
$$

(21)

with metric functions

$$
u(y,x) = 1 + \frac{2\pi \Lambda}{9\alpha} \left( \frac{Am(x-y)}{1 + Am(x+y)} \right)^2,
$$

$$
F(y) = \frac{\Lambda}{3A^2} + 1 - y^2 - 2mA y^3 - m^2 A^2 y^4,
$$

$$
G(x) = 1 - x^2 - 2mA x^3 - m^2 A^2 x^4.
$$

(22)

The self-interacting potential and the scalar field are given by

$$
V(\Psi) = \frac{\Lambda}{8\pi} \left[ \cosh^4 \left( \sqrt{\frac{4\pi}{3}} \Psi \right) + \frac{9\alpha}{2\pi \Lambda} \sinh^4 \left( \sqrt{\frac{4\pi}{3}} \Psi \right) - 1 \right],
$$

(23)

$$
\phi(y,x) = \sqrt{\frac{3}{4\pi}} \text{Arctanh} \left( \sqrt{-\frac{2\pi \Lambda}{9\alpha}} \frac{Am(x-y)}{1 + Am(x+y)} \right).
$$

(24)

The limit of $\Lambda \to 0$ is obtained upon letting the coupling $\alpha \to 0$ so that $\frac{\alpha}{\Lambda} \sim \text{constant}$. This solution then reduces smoothly to the solution found in [6] for zero acceleration $A = 0$. When $A \neq 0$ but $\Lambda = 0$ the potential drops out giving the solution of [36] at minimal EM-scalar coupling. If finally on the other hand $\Lambda \neq 0$, we obtain the solution found in [10]. A generalization of this solution to Plebanski-Demiański spacetime was discussed in [18] where accelerated, stationary and axisymmetric exact solutions with self-interacting scalar fields in (A)dS were found.

Before ending this section we note that in [37] a similar formalism was developed adding a dilaton field coupled to an U(1) gauge field in the action [1].

### III. A NEW CLASS OF FOUR-DIMENSIONAL ASYMPTOTICALLY ADS BLACK HOLES WITH SCALAR HAIR

In this section we will follow another approach. Instead of looking for an another exact black hole solution of the equations (6)-(8) and (11) we will look for asymptotically AdS solutions with scalar hair. Such solutions were studied in (2+1)-dimensions [38] and in (3+1)-dimensions [39] in connection with conserved charges.

Following the general formalism develop in Section II for a scalar field coupled minimally to gravity, we consider a particular profile of the scalar field. Consider the following ansatz for the scalar field

$$
\phi(r) = b \ln \left( 1 + \frac{\nu}{r} \right),
$$

(25)

where $b$ and $\nu$ are parameters. Then from equation (9) we can determine the metric function $a(r)$ analytically

$$
a(r) = \alpha r^\frac{1}{2}(1 + \sqrt{1 - 2\beta^2}) (r + \nu) \frac{1}{2}(1 - \sqrt{1 - 2\beta^2}) + \beta r^\frac{1}{2}(1 - \sqrt{1 - 2\beta^2}) (r + \nu) \frac{1}{2}(1 + \sqrt{1 - 2\beta^2}) \nu \sqrt{1 - 2\beta^2},
$$

(26)

where $\alpha$ and $\beta$ are integration constants. For simplicity we take $\alpha = 1$, $\beta = 0$ and $b = 1/\sqrt{2}$ and consequently equation (26) can be written as

$$
a(r) = \sqrt{r (r + \nu)}.
$$

(27)

---

2 In [31] there is also an electromagnetic field present in the action.

3 The parameter $\alpha$ is the coupling of the scalar field in the conformal frame [31].
We can also determine the metric function \( f(r) \) analytically using equation (10). We find
\[
f(r) = k + Fr (r + \nu) + \frac{G}{r^3} \left( -\nu (\nu + 2r) + 2r (r + \nu) \ln \left( \frac{r + \nu}{r} \right) \right),
\]
where \( k = -1, 0, 1 \) and \( F, G \) are constants being proportional to the cosmological constant and to the mass respectively.

Using the solution (28) from equation (6) we obtain the following family of self-interacting potentials
\[
V(r) = -\frac{1}{2\nu^3 r (r + \nu)} \left( -6\nu G (2r + \nu) + (6r^2 + 6\nu r + \nu^2) \left( \nu^3 F + 2G \ln \left( \frac{r + \nu}{r} \right) \right) \right).
\]

These potentials act as a barrier to the scalar field at large distances. Note that the asymptotic behaviour of the metric function \( f(r) \) and the scalar field is given by
\[
f(r) = \left. k - F \frac{\nu^2}{4} + F \left( r + \frac{\nu}{2} \right)^2 - \frac{G}{3r} + O \left( \frac{1}{r^2} \right) \right|_{r \to \infty},
\]
\[
\phi(r) = \frac{1}{\sqrt{2}} \left( \frac{\nu}{r} - \frac{1}{2} \frac{\nu^2}{r^2} + O \left( \frac{1}{r^3} \right) \right).
\]
The Klein-Gordon equation (11) is giving the scalar field evolution in its potential and at large distances it is trivially satisfied.

Solving equation (25) for \( r \)
\[
r = \frac{\nu}{e^{\sqrt{2}\phi} - 1}
\]
we can write the potential (29) in terms of the scalar field \( \phi \)
\[
V(\phi) = -F \left( 2 + \cosh \left( \sqrt{2}\phi \right) \right) + \frac{G}{\nu^3} \left( 6 \sinh \left( \sqrt{2}\phi \right) - 2\sqrt{2}\phi \left( 2 + \cosh \left( \sqrt{2}\phi \right) \right) \right),
\]
which has a global maximum at \( \phi = 0 \) where \( V(0) = \Lambda \) as expected and also \( F = -\frac{\Lambda}{3} = \frac{1}{l^2} \). Besides, we know that
\[
V''(\phi = 0) = m^2,
\]
where \( m \) is the scalar field mass. Therefore, we obtain that the scalar field mass is given by
\[
m^2 = \frac{2\Lambda}{3} = -2l^{-2},
\]
which satisfies the Breitenhner-Friedman bound that ensures the perturbative stability of the AdS spacetime.

In Fig. 1 we plot the behaviour of the metric function \( f(r) \) and the potential \( V(r) \) for a choice of parameters \( k = 0, \nu = 1, F = 1 \) and \( G = 1 \). The metric function \( f(r) \) changes sign for low values of \( r \) signaling the presence of an horizon, while the potential asymptotically tends to a negative constant (the cosmological constant), and the scalar field is regular everywhere outside the event horizon and null at spatial infinity. We have also checked the behaviour of the curvature outside the black hole horizon. As it is shown in Fig. 2 there is no curvature singularity outside the horizon for \( k = -1, 0, 1 \).

To understand better the behaviour of the metric solution we have found, when the scalar field goes to zero asymptotically, the metric solution should go to the Schwarzschild anti-de Sitter solution. To show that we make a change of coordinates \( \rho = \sqrt{r^2 + \nu r} \). Then the metric can be written as
\[
ds^2 = -\chi(\rho) dt^2 + \frac{4\rho^2/\nu^2}{4\rho^2/\nu^2 + 1} \frac{1}{\chi(\rho)} d\rho^2 + \rho^2 d\sigma^2,
\]
4 This potential was also found in [40] following a different approach.
5 This happens for \( k = 1 \). For \( k = -1 \) it goes to the topological anti-de Sitter black hole solution.
where
\[ \chi(\rho) = k + F \rho^2 - \frac{G}{\nu} \left( \sqrt{\frac{4\rho^2}{\nu^2} + 1} - 2\frac{\rho^2}{\nu^2} \ln \left( \frac{1 + \sqrt{\frac{4\rho^2}{\nu^2} + 1}}{-1 + \sqrt{\frac{4\rho^2}{\nu^2} + 1}} \right) \right). \] (37)

The scalar field in the new coordinates reads
\[ \phi(\rho) = \frac{1}{\sqrt{2}} \ln \left( \frac{1 + \sqrt{\frac{4\rho^2}{\nu^2} + 1}}{-1 + \sqrt{\frac{4\rho^2}{\nu^2} + 1}} \right). \] (38)

At infinity the scalar field decouples and the metric goes to
\[ \chi(\rho) = k + F \rho^2 - \frac{G}{3\rho} + O \left( \frac{1}{\rho^6} \right). \] (39)

From the above relation we can see that the asymptotic behaviour is the Schwarzschild anti-de Sitter metric with the constant G proportional to the black hole mass and also the constant F proportional to the cosmological constant, as expected.

Now let us see the behaviour of the scalar field at large distances. It was observed in [39] that if the scalar hair has a logarithmic form then its backreaction on the metric changes the behaviour of the metric at large distances having a slower asymptotic variation and in some cases even changing the geometry of an AdS space.

The asymptotic behaviour of the scalar field (38) is given by
\[ \phi(\rho) = \frac{1}{\sqrt{2}} \left( \frac{\nu}{\rho} - \frac{1}{24} \frac{\nu^3}{\rho^6} + O \left( \frac{1}{\rho^8} \right) \right). \] (40)

Then the metric (36) has quite different behaviour at large distances. Calculating its asymptotic \( g_{\rho\rho} \) component one finds
\[ g_{\rho\rho} = \frac{l^2}{\rho^2} + \frac{l^4}{\rho^4} \left( -k - \frac{\nu^2}{4l^2} + O \left( \frac{1}{\rho^2} \right) \right). \] (41)
We see that the geometry exhibits a deviation from AdS at large distances in accordance with the observation in [39]. However, the conserved charges are well defined and finite as we will see in next section, where we will compute the mass and the entropy of our black hole solutions using the Euclidean formalism. We will also study in the next section the phase transitions between the asymptotically AdS four-dimensional black holes with scalar hair and the black hole solutions without hair.

IV. THERMODYNAMICS

A. Mass and entropy from boundary terms

In order to calculate the mass and entropy of our black hole solution using the Euclidean formalism, we will transform the metric (36) to a more suitable form. Performing the change of coordinates
\[ \rho = \frac{\nu e^{\nu/(2r)}}{e^{\nu/r} - 1}, \] (42)
the metric (36) acquires the form
\[ ds^2 = N(r) \left(-B(r) dt^2 + \frac{1}{B(r)} dr^2 + r^2 d\sigma^2\right), \] (43)
where
\[ N(r) = \frac{\nu^2 e^{\nu/r}}{r^2 \left(e^{\nu/r} - 1\right)^2}, \] (44)
\[ B(r) = \frac{k r^2}{\nu^2} \left(e^{\nu/r} + e^{-\nu/r} - 2\right) + F r^2 + \frac{2G}{\nu^2} r - \frac{G r^2}{\nu^3} \left(e^{\nu/r} - e^{-\nu/r}\right). \] (45)
Note that metric (43) is symmetric under the change \( \nu \) by \( -\nu \), and the scalar field in the new coordinates becomes
\[ \phi(r) = \frac{\nu}{\sqrt{2}r}. \] (46)

We go to Euclidean time \( t \to it \) and we consider the action
\[ I = \int \left(\pi^{ij} \dot{g}_{ij} + p \dot{\phi} - NH - N^i H_i\right) d^3 x dt + B_{surf}, \] (47)
where \( \pi^{ij} \) is the conjugate momenta of the metric and \( p \) is the conjugate momenta of the field; \( B_{surf} \) is a surface term. So, by considering the metric
\[ ds^2 = N^2(r) f^2(r) dr^2 + f^{-2}(r) dr^2 + R^2(r) d\sigma^2, \] (48)
where
\[ N(r) = \frac{\nu^2 e^{\nu/r}}{r^2 \left(e^{\nu/r} - 1\right)^2}, \quad f^2(r) = \frac{r^2 \left(e^{\nu/r} - 1\right)^2}{\nu^2 e^{\nu/r}} B(r), \quad R^2(r) = \frac{\nu^2 e^{\nu/r}}{\left(e^{\nu/r} - 1\right)^2}, \] (49)
and with a periodic \( \beta = 1/T \) where \( T \) is the temperature, the action becomes
\[ I = -\beta \sigma \int_{r^+}^{\infty} N(r) H(r) dr + B_{surf}, \] (50)
where \( \sigma \) is the area of the spatial 2-section. We now compute the action when the field equations hold. The condition that the geometries which are permitted should not have conical singularities at the horizon imposes
\[ T = \frac{B'(r_+)}{4\pi}. \] (51)
So, by using the grand canonical ensemble we can fix the temperature. The variation of the surface term yields

$$\delta B_{surf} = \delta B_\phi + \delta B_G ,$$  \hspace{1cm} (52)

where

$$\delta B_G = \beta \sigma \left[ N \left( RR' \delta f^2 - (f^2)' R \delta R \right) + 2 f^2 R \left( N \delta R' - N' \delta R \right) \right]_{r+}^\infty ,$$  \hspace{1cm} (53)

$$\delta B_\phi = \beta \sigma N R^2 f^2 \phi' \delta \phi .$$  \hspace{1cm} (54)

With an asymptotic behaviour of the metric function \( f(r) \) given by

$$f^2 (r) = F r^2 + k + \frac{F \nu^2}{12} - \frac{G}{3r} + O \left( \frac{1}{r^2} \right) ,$$  \hspace{1cm} (55)

we find

$$\delta B_G \infty = \beta \sigma \left( \frac{F \nu}{2} + O \left( \frac{1}{r} \right) \right) \delta \nu + \beta \sigma \left( - \frac{1}{3} + O \left( \frac{1}{r^2} \right) \right) \delta G ,$$  \hspace{1cm} (56)

and

$$\delta B_\phi \infty = \beta \sigma \left( - \frac{F \nu}{2} + O \left( \frac{1}{r} \right) \right) \delta \nu .$$  \hspace{1cm} (57)

From the above expressions we deduce the surface terms at infinity

$$B_{surf \infty} = - \frac{\beta \sigma G}{3} ,$$  \hspace{1cm} (58)

and at the horizon

$$B_{surf r+} = - \frac{\sigma}{4G_N} R^2 (r+) .$$  \hspace{1cm} (59)

Therefore, the Euclidean action reads

$$I = - \frac{\beta \sigma G}{3} + \frac{\sigma}{4G_N} R^2 (r+) ,$$  \hspace{1cm} (60)

and as the Euclidean action is related to the free energy through \( I = - \beta F \), we obtain

$$I = S - \beta M ,$$  \hspace{1cm} (61)

where the mass \( M \) is

$$M = \frac{\sigma G}{3} ,$$  \hspace{1cm} (62)

and the entropy \( S \) is

$$S = \frac{\sigma}{4G_N} R^2 (r+) .$$  \hspace{1cm} (63)

Notice that the mass (62) is proportional to the constant \( G \) as expected, and it is not related to the parameter \( \nu \) of the scalar field. This result is interesting compared to the results obtained from the exact solutions of the action (1) presented in Section 2. In all these solutions the mass and the charge of the scalar field are related. This can be understood from the fact that asymptotically the matter fields decouple from the gravity sector so a kind of stealth mechanism is operating [42] allowing to found asymptotically AdS black holes with primary hair.
B. Phase transitions

Having the temperature, mass and entropy for the asymptotically AdS hairy black hole solutions given by Equations (51), (62), and (63) respectively, we can study possible phase transitions to known black hole solutions without hair. In the absence of a scalar field the action (1) for $k = -1$ has as a solution the topological AdS black hole [43–45] with temperature, entropy and mass given respectively by

$$T = \frac{3}{4\pi l} \left( \frac{\rho_+}{l} - \frac{l}{3\rho_+} \right), \quad S_{TBH} = 2\pi \sigma \rho_+^2, \quad M_{TBH} = \sigma \rho_+ \left( \frac{\rho_+^2}{l^2} - 1 \right).$$

(64)

So, the horizon radius $\rho_+ = \frac{2\pi T}{3} + \sqrt{\left(\frac{2\pi T}{3}\right)^2 + \frac{1}{\frac{3}{}l}}$ can be written as a function of the temperature. To study the dependence of the horizon radius of the hairy black hole with respect to the parameter $\nu$ and to the mass $M$ of the hairy black hole, we plot Fig. 3. This Figure shows that the horizon radius takes the same value for $\nu$ and $-\nu$, which is easily seen from Equation (45). Also, we can see that there exist a minimum value of the horizon when $\nu = 0$ where the metric coincides with the topological black hole. In Fig. 4 we plot the parameter $\nu$ as function of the temperature and the mass $M$ of the hairy black hole. It is shown in this Figure that for a given mass and temperature there are two values of the parameter $\nu$ for which the temperature of the topological black hole is obtained.

![Figure 3](image3.png)

FIG. 3: The behaviour of $r_+$ respect to $\nu$ and to the mass $M$ with $k = -1$, $l = 1$, and $\sigma = 1$.  

![Figure 4](image4.png)

FIG. 4: The behaviour of $\nu$ as function of the temperature and the mass $M$ with $k = -1$, $l = 1$, and $\sigma = 1$.  

The graphics for the Euclidean actions are showed in Fig. 5 and 6 for the black hole with scalar hair and the topological black hole respectively and in Fig. 7 we depict both actions in the same figure, in order to see the range of values of black hole mass for which the phase transitions exist. Thus, from Fig. 7 we can see that there exists a phase transition only for negative masses, and the hairy black hole dominates for small temperatures, while for large temperatures the topological black hole would be preferred. Also, we can observe that the critical temperature at
FIG. 5: The behaviour of Euclidean actions for the hairy black hole as function of the temperature and the mass $M$ with $k = -1$, $l = 1$, and $\sigma = 1$.

FIG. 6: The behaviour of Euclidean actions for the topological black hole as function of the temperature and the mass $M$ with $k = -1$, $l = 1$, and $\sigma = 1$.

which this phase transition takes place depends on the mass of the hairy black hole. It is worth mentioning that the specific heat given by

$$C = -\frac{4\pi \nu^3 e^{\frac{\nu}{r}}}{3(1 + e^{\frac{\nu}{r}})(-2r_+ + 2r_+ e^{\frac{\nu}{r}} - \nu - \nu e^{\frac{\nu}{r}})}$$

is positive and therefore the asymptotically AdS black holes with scalar hair that we have found can always reach thermal equilibrium with their surroundings and hence, are stable against thermal fluctuations.

Carrying out the same analysis for spherical horizons $k = 1$ we found that there is no phase transitions of the hairy asymptotically AdS black holes we found to Schwarzschild AdS black hole. This result agrees with the findings in [17] where only phase transitions of exact hairy black hole solutions to black hole solutions with hyperbolic horizons were found.

V. CONCLUDING COMMENTS

We have considered four-dimensional gravity theories where the scalar field is minimally coupled to gravity along with a self-interacting potential. We have found a new family of four-dimensional asymptotically AdS black holes with scalar hair. These solutions asymptotically give the Schwarzschild anti-de Sitter solution. They characterized by a scalar field with a logarithmic behaviour, being regular everywhere outside the event horizon and null at spatial infinity, and by a self-interacting potential, which tends to the cosmological constant at spatial infinity. Calculating the mass and entropy using the Euclidean formalism we found that the mass of the black hole is not related to the charge of the scalar field indicating that the scalar hair is primary. Also, we have showed that there is a critical
temperature below which the gravitational system undergoes a phase transition to a hairy black hole configuration, while above the critical temperature an AdS black hole with hyperbolic horizon dominates.

In this work we considered a particular profile for the scalar field. The formalism developed in Section 2 allows us to consider other profiles for the scalar fields. It would be interesting to consider also exponential profiles for the scalar field which fall off at infinity quite fast and study the properties and the behaviour of possible asymptotically AdS black holes with scalar hair.

The asymptotically AdS black hole solutions with scalar hair we discussed in this work may help to understand better the generation of hairy black hole solutions and the mechanism of scalar condensation which are necessary for the applications of holography to condensed matter systems. In most of these applications, like the holographic superconductor, the probe limit was considered in which the scalar field does not backreact on the metric. The knowledge of exact or asymptotically AdS hairy black hole solutions allows us to consider fully backreacted systems which may give more information on the boundary theory. It is also interesting to note\textsuperscript{6} that the potential\textsuperscript{33} can be embedded in M-theory, as it arises from a consistent $U(1)^4$ truncation of gauged $N = 8$ supergravity\textsuperscript{47}. This may help to construct a realistic holographic superconductor using the top-down approach.

Acknowledgments

We would like to thank Theodoris Kolyvaris and Minas Tsoukalas for valuable discussions and comments on the draft. This work was funded by Comisión Nacional de Ciencias y Tecnología through FONDECYT Grants 1110076 (JS, EP) and 11121148 (YV) and by DI-PUCV Grant 123713 (JS).

---

\textsuperscript{6} We thank A. Anabalon for pointing out this behaviour of the potential to us.
[5] E. Berti, V. Cardoso, L. Gualtieri, M. Horbatsch and U. Sperhake, “Numerical simulations of single and binary black holes in scalar-tensor theories: circumventing the no-hair theorem,” Phys. Rev. D 87, 124020 (2013) [arXiv:1304.2836 [gr-qc]].

[6] N. Bocharova, K. Bronnikov and V. Melnikov, Vestn. Mosk. Univ. Fiz. Astron. 6, 706 (1970);
J. D. Bekenstein, Annals Phys. 82, 535 (1974);
J. D. Bekenstein, “Black Holes With Scalar Charge,” Annals Phys. 91, 75 (1975).

[7] K. A. Bronnikov and Y. N. Kireyev, “Instability of black holes with scalar charge,” Phys. Lett. A 67, 95 (1978).

[8] G. Zloschachstiev, “On co-existence of black holes and scalar field,” Phys. Rev. Lett. 94, 121101 (2005) [arXiv:hep-th/0408163].

[9] T. Torii, K. Maeda and M. Narita, “No-scalar hair conjecture in asymptotic de Sitter spacetime,” Phys. Rev. D 59, 064027 (1999) [arXiv:gr-qc/9809036].

[10] C. Martinez, R. Troncoso and J. Zanelli, “De Sitter black hole with a conformally coupled scalar field in four-dimensions,” Phys. Rev. D 67, 024008 (2003) [hep-th/0205319].

[11] T. J. T. Harper, P. A. Thomas, E. Winstanley and P. M. Young, “Instability of a four-dimensional de Sitter black hole with a conformally coupled scalar field,” Phys. Rev. D 70, 064023 (2004) [gr-qc/0312104].

[12] G. Dotti, R. J. Gleiser and C. Martinez, “Static black hole solutions with a self interacting conformally coupled scalar field,” Phys. Rev. D 77, 104035 (2008) [arXiv:0710.1735 [hep-th]].

[13] T. Torii, K. Maeda and M. Narita, “Scalar hair on the black hole in asymptotically anti-de Sitter spacetime,” Phys. Rev. D 64, 044007 (2001).

[14] E. Winstanley, “On the existence of conformally coupled scalar field hair for black holes in (anti-)de Sitter space,” Found. Phys. 33, 111 (2003) [arXiv:gr-qc/0205092].

[15] C. Martinez, R. Troncoso and J. Zanelli, “Exact black hole solution with a minimally coupled scalar field,” Phys. Rev. D 70, 084035 (2004) [hep-th/0406111].

[16] C. Martinez, J. P. Stafiorelli and R. Troncoso, “Topological black holes dressed with a conformally coupled scalar field and electric charge,” Phys. Rev. D 74, 044028 (2006) [arXiv:hep-th/0512022]; C. Martinez and R. Troncoso, “Electrically charged black hole with scalar hair.” Phys. Rev. D 74, 064007 (2006) [arXiv:hep-th/0606190].

[17] T. Kolyvaris, G. Koutsoumbas, E. Papantonopoulos and G. Siopsis, “A New Class of Exact Hairy Black Hole Solutions,” Gen. Rel. Grav. 43, 163 (2011) [arXiv:0911.1711 [hep-th]].

[18] A. Anabalon, “Exact Black Holes and Universality in the Backreaction of non-linear Sigma Models with a potential in (A)dS4,” JHEP 1206, 127 (2012) [arXiv:1204.2720 [hep-th]].

[19] A. Anabalon and J. Oliva, “Exact Hairy Black Holes and their Modification to the Universal Law of Gravitation,” Phys. Rev. D 86, 107501 (2012) [arXiv:1205.6012 [gr-qc]].

[20] A. Anabalon and A. Cisterna, “Asymptotically (anti) de Sitter Black Holes and Wormholes with a Self Interacting Scalar Field in Four Dimensions,” Phys. Rev. D 85, 084035 (2012) [arXiv:1201.2008 [hep-th]].

[21] V. Bardoux, M. M. Caldarelli and C. Charmousis, “Conformally coupled scalar black holes admit a flat horizon due to axionic charge,” JHEP 1209, 008 (2012) [arXiv:1205.4025 [hep-th]].

[22] O. J. C. Dias, G. T. Horowitz and J. E. Santos, “Black holes with only one Killing field,” JHEP 1107, 115 (2011) [arXiv:1105.4167 [hep-th]].

[23] S. Stotyn, M. Park, P. McGrath and R. B. Mann, “Black Holes and Boson Stars with One Killing Field in Arbitrary Odd Dimensions,” Phys. Rev. D 85, 044036 (2012) [arXiv:1110.2223 [hep-th]].

[24] O. J. C. Dias, P. Figuera, S. Minwalla, P. Mitra, R. Monteiro and J. E. Santos, “Hairy black holes and solitons in global AdS5,” JHEP 1208, 117 (2012) [arXiv:1112.4447 [hep-th]].

[25] B. Kleihaus, J. Kunz, E. Radu and B. Subagya, “Axially symmetric static scalar solitons and black holes with scalar hair,” Phys. Lett. B 725, 489 (2013) [arXiv:1306.4616 [gr-qc]].

[26] A. Buchel, S. L. Liebling and L. Lehner, “Boson Stars in AdS,” Phys. Rev. D 87, 123006 (2013) [arXiv:1304.4166 [gr-qc]].

[27] L. Amendola, “Cosmology with nonminimal derivative couplings,” Phys. Lett. B 301, 175 (1993) [arXiv:gr-qc/9203010].

[28] S. V. Sushkov, “Exact cosmological solutions with nonminimal derivative coupling,” Phys. Rev. D 80, 103505 (2009) [arXiv:0910.0980 [gr-qc]].

[29] T. Kolyvaris, G. Koutsoumbas, E. Papantonopoulos and G. Siopsis, “Scalar Hair from a Derivative Coupling of a Scalar Field to the Einstein Tensor,” Class. Quant. Grav. 29, 205011 (2012) [arXiv:1111.0263 [gr-qc]].

[30] T. Kolyvaris, G. Koutsoumbas, E. Papantonopoulos and G. Siopsis, “Phase Transition to a Hairy Black Hole in Asymptotically Flat SpaceTime,” arXiv:1308.5280 [hep-th].

[31] C. Charmousis, T. Kolyvaris and E. Papantonopoulos, “Charged C-metric with conformally coupled scalar field,” Class. Quant. Grav. 26, 175012 (2009) [arXiv:0906.5568 [gr-qc]].

[32] A. Anabalon and H. Maeda, “New Charged Black Holes with Conformal Scalar Hair,” Phys. Rev. D 81, 041501 (2010) [arXiv:0907.0219 [hep-th]].

[33] M. Astorino, “C-metric with a conformally coupled scalar field in a magnetic universe,” arXiv:1307.4021 [gr-qc].

[34] G. T. Horowitz, “Theory of Superconductivity,” Lect. Notes Phys. 828, 313 (2011) [arXiv:1002.1722 [hep-th]].

[35] K. Farakos, A. P. Kouretsis and E. Papantonopoulos, “Anti de Sitter 5D black hole solutions with a self-interacting bulk scalar field: A Potential reconstruction approach,” Phys. Rev. D 80, 064020 (2009) [arXiv:0905.1345 [hep-th]].

[36] F. Dowker, J. P. Gauntlett, C. A. Kastor and J. H. Traschen, “Pair creation of dilaton black holes,” Phys. Rev. D 49, 2099 (1994) [hep-th/9309075].

[37] A. Anabalon and D. Astefanesei, “On attractor mechanism of AdS4 black holes,” arXiv:1309.5863 [hep-th].

[38] M. Henneaux, C. Martinez, R. Troncoso and J. Zanelli, “Black holes and asymptotics of 2+1 gravity coupled to a scalar field,” Phys. Rev. D 65, 104007 (2002) [hep-th/0201170].
[39] M. Henneaux, C. Martinez, R. Troncoso and J. Zanelli, “Asymptotically anti-de Sitter spacetimes and scalar fields with a logarithmic branch,” Phys. Rev. D 70, 044034 (2004) [hep-th/0404236].

[40] A. Anabalon, “Exact Hairy Black Holes,” arXiv:1211.2765 [gr-qc].

[41] P. Breitenlohner and D. Z. Freedman, “Stability In Gauged Extended Supergravity,” Annals Phys. 144, 249 (1982).

[42] E. Ayon-Beato, C. Martinez and J. Zanelli, “Stealth scalar field overflying a (2+1) black hole,” Gen. Rel. Grav. 38, 145 (2006) [hep-th/0403228].

[43] R. B. Mann, “Pair production of topological anti-de Sitter black holes,” Class. Quant. Grav. 14, L109 (1997) [gr-qc/9607071].

[44] L. Vanzo, “Black holes with unusual topology,” Phys. Rev. D 56, 6475 (1997) [gr-qc/9705004].

[45] R. B. Mann, “Charged topological black hole pair creation,” Nucl. Phys. B 516, 357 (1998) [hep-th/9705223].

[46] G. Koutsoumbas, E. Papantonopoulos and G. Siopsis, “Exact Gravity Dual of a Gapless Superconductor,” JHEP 0907, 026 (2009) [arXiv:0902.0733 [hep-th]].

[47] M. Cvetic, M. J. Duff, P. Hoxha, J. T. Liu, H. Lu, J. X. Lu, R. Martinez-Acosta and C. N. Pope et al., “Embedding AdS black holes in ten-dimensions and eleven-dimensions,” Nucl. Phys. B 558, 96 (1999) [hep-th/9903214].