Multi-Step Greedy and Approximate Real Time Dynamic Programming

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Abstract

Real Time Dynamic Programming (RTDP) is a well-known Dynamic Programming (DP) based algorithm that combines planning and learning to find an optimal policy for an MDP. It is a planning algorithm because it uses the MDP’s model (reward and transition functions) to calculate a 1-step greedy policy w.r.t. an optimistic value function, by which it acts. It is a learning algorithm because it updates its value function only at the states it visits while interacting with the environment. As a result, unlike DP, RTDP does not require uniform access to the state space in each iteration, which makes it particularly appealing when the state space is large and simultaneously updating all the states is not computationally feasible. In this paper, we study a generalized multi-step greedy version of RTDP, which we call $h$-RTDP, in its exact form, as well as in three approximate settings: approximate model, approximate value updates, and approximate state abstraction. We analyze the sample, computation, and space complexities of $h$-RTDP and establish that increasing $h$ improves sample and space complexity, with the cost of additional offline computational operations. For the approximate cases, we prove that the asymptotic performance of $h$-RTDP is the same as that of a corresponding approximate DP – the best one can hope for without further assumptions on the approximation errors. $h$-RTDP is the first algorithm with a provably improved sample complexity when increasing the lookahead horizon.

1 Introduction

Dynamic Programming (DP) algorithms, e.g., Value Iteration, output an optimal policy of a decision problem given a model of the environment. Their convergence in the presence of multi-step greedy policies (Bertsekas and Tsitsiklis 1996; Efroni et al. 2019a) as well as their performance in different approximate settings (Bertsekas and Tsitsiklis 1996; Munos 2007; Scherrer et al. 2012; Geist and Pietquin 2013; Abel, Hershkowitz, and Littman 2016; Efroni et al. 2018) have been well-studied in the literature. Unfortunately, DP requires simultaneous access to the entire state space at run time, and as such, cannot be used in practice when the number of states is too large.

Real Time Dynamic Programming (RTDP) (Barto, Bradtke, and Singh 1995) is a DP-based algorithm that mitigates the need to access all states simultaneously. It can also be interpreted as a generalization of LRTA* (Korf 1990) to the stochastic setting (Bonet and Geffner 2000; Bulitko and Lee 2006). Much alike DP, RTDP updates are based on the Bellman operator, calculated by accessing the known model of the environment. However, unlike DP, in each step, the Bellman operator is applied only to the current state, by which a greedy action is taken and a new state is sampled based on the interaction with the environment. Thus, unlike DP that returns a policy based on offline planning, RTDP interacts with the environment and uses the gathered experience to learn a better policy, hence its name: a real-time algorithm based on the DP principles.

Despite the popularity and simplicity of RTDP, precise characterization of its convergence was only recently established by Efroni et al. (2019b) for finite-horizon MDPs. Yet, previous analyses have neither addressed the question of how multi-step greedy (or lookahead) policies should be used in RTDP nor studied RTDP’s sensitivity to possible approximation errors. Such errors can arise due to misspecified model, or exist in value function updates, when e.g., function approximation is used. Multi-step greedy policies are expected to improve the quality of convergence in some of these scenarios, as they do for DP (Bertsekas and Tsitsiklis 1996; Efroni et al. 2019a). Yet, to the best of our knowledge, these questions have not been addressed in the literature.

In this paper, we focus on filling this gap and establish convergence guarantees for multi-step greedy and approximate RTDP for finite-horizon MDPs. We start by formulating and analyzing a multi-step greedy RTDP algorithm that generalizes the standard 1-step greedy RTDP in Section 4. We call the algorithm $h$-RTDP, where $h$ controls the lookahead horizon. The analysis of $h$-RTDP reveals improved sample complexity with increasing the lookahead horizon, $h$. To the best of our knowledge, this result is the first guarantee for an improved sample complexity when increasing the lookahead horizon. However, the improved sample complexity comes with the cost of worse computational complexity in $h$ (see Section 4.1). In Section 5, we analyze $h$-RTDP in the presence of approximation: (i) when
an approximate model is used, instead of the true one, (ii) when there is approximation in value updates, and finally (iii) when we use approximate state abstraction. In these settings, a corollary of our results is the following possibly intuitive conclusion (see Section 6): the asymptotic performance of approximate $h$-RTDP is similar to that of approximate DP.

All the proofs rely on a generic recipe for analyzing the convergence of RTDP-like algorithms, inspired by and improving upon the results of Efroni et al. (2019b). The results for the following RTDP-like algorithms are based on using variants of two lemmas, from which a regret bound and a finite sample Uniform-PAC bound are derived. We hope the recipe will be useful in future extensions of RTDP and stimulate further research on this topic.

In a broader context, this work shows that RTDP-like algorithms could be a good alternative to upper confidence trees (UCT) (Kocsis and Szepesvári 2006) and Monte Carlo tree search (MCTS) (Browne et al. 2012) algorithms. They have stronger theoretical guarantees, and can also be applied to scenarios in which MCTS is heavily used, i.e., when the model is available but the problem is too big to be solved by DP, e.g., in the game of Go (Silver et al. 2017). As we establish in this work, RTDP-like algorithms combined with multi-step greedy policies have strong theoretical guarantees. Under no assumptions, other than initial optimistic value, these algorithms converge in polynomial time, with high probability, to an optimal policy, and their approximations inherit the asymptotic performance of approximate DP. Unlike RTDP, MCTS acts by using a $\sqrt{\log N/N}$ bonus term instead of optimistic initialization. However, in the general case, its convergence can be quite poor, even worse than uniform sampling (Munos 2014).

## 2 Notations and Preliminaries

We consider finite-horizon MDPs with time-independent dynamics (Bertsekas and Tsitsiklis 1996). A finite-horizon MDP is the tuple $M = (S, A, r, p, H)$, where $S$ and $A$ are the state and action spaces with cardinals $S$ and $A$, respectively, and the immediate reward for taking an action $a$ at a state $s$ is the scalar $r(s, a) \in [0, 1]$. The transition probability is denoted by $p(s' | s, a)$ and represents the probability of transitioning to state $s'$ upon taking action $a$ at state $s$. The initial state in each episode is arbitrarily chosen and $H \in \mathbb{N}$ is the horizon, number of time-steps in each episode. We define $[N] := \{1, \ldots, N\}$, for all $N \in \mathbb{N}$, and throughout the paper denote by $t \in [H]$ and $k \in [K]$ the time-step within an episode and the episode’s number, respectively.

A deterministic policy $\pi : S \times [H] \rightarrow A$ is a mapping from states and time-step indices to actions. We denote by $a_t := \pi(s, t)$, the action taken at time $t$ at state $s$ according to a policy $\pi$. The quality of a policy $\pi$ from a state $s$ at time $t$ is measured by its value function, i.e.,

$$V_t^\pi(s) := \mathbb{E} \left[ \sum_{t'=t}^{H} r(s_{t'}, \pi(s_{t'}, t')) \mid s_t = s, \pi \right],$$

where the expectation is over all the randomness in the environment. An optimal policy maximizes this value for all states $s$ and time-steps $t$. The optimal value is denoted by $V_0^\pi(s) := \max_{\pi} V_t^\pi(s)$, for all $t \in [H]$ and $s \in S$. The optimal value satisfies the optimal Bellman equation, i.e.,

$$V_t^\pi(s) = T V_{t+1}^\pi(s) := \max_a [r(s, a) + p(s') | V_{t+1}^\pi(s')]$$

$$= \max_a \mathbb{E} \left[ r(s_0, a) + V_{t+1}^\pi(s_1) \mid s_0 = s, a_0 = a \right].$$ (1)

By repeatedly applying the optimal Bellman operator $T$, for any $h \in [H]$, we have

$$V_t^\pi(s) = T^h V_{t+h}^\pi(s) = \max_a \left[ r(s, a) + p(s, a) T^{h-1} V_{t+h}^\pi(s) \right]$$

$$= \max_{a_0, \ldots, a_{h-1}} \mathbb{E} \left[ \sum_{j=0}^{h-1} r(s_j, a_j) + V_{t+h}^\pi(s_h) \mid s_0 = s \right],$$ (2)

where it is implicit that the expectation is conditioned on $\{a_t\}_{t=0}^{h-1}$. We refer to $T^h$ as the $h$-step optimal Bellman operator. Similar Bellman recursion is defined for the value of a given policy, $\pi$, i.e., $V_\pi^\pi(s) = T^h V_{t+h}^\pi(s) := r(s, \pi(s, t)) + p(s, \pi(s, t)) T^{h-1} V_{t+h}^\pi(s)$, where $T^h$ is the $h$-step Bellman operator of policy $\pi$.

We now define a measure that characterizes the hardness of local search from a state in a MDP. Let $S_h(s) = \{s' \mid \exists \pi : p(s' = s' | s, \pi) > 0, p(s' = s' | s, \pi) \geq p(s' = s' | s, \pi)\}$, where $p(s' = s' | s, \pi) = \mathbb{E} [1 \{s_h = s'\} | s_0 = s, \pi]$, be the set of reachable states from a state $s$ in $h \in [H]$ steps. We denote by $S_h := \max_{s \in [H]} \max_{\pi} |S_h(s)|$, the maximal cardinality of $S_h(s)$ on the entire state space and all time-steps up until and including $h$. When this set is small, local search up to horizon $h$ can be done efficiently as the number of reachable states in up to $h$ time-steps from any state is small.

We consider an agent that repeatedly interacts with an MDP in a sequence of episodes $[K]$. We denote by $s_k^t$ and $a_k^t$, the state and action taken at the time-step $t$ of the $k$'th episode. We denote by $F_{k-1}$, the filtration that includes all the events (states, actions, and rewards) until the end of the $(k - 1)$'th episode, as well as the initial state of the $k$'th episode. Throughout the paper, we denote by $\pi_k$, the policy that is executed during the $k$'th episode and assume it is $F_{k-1}$ measurable. The performance of an agent is measured by its regret, defined as $\text{Regret}(K) := \sum_{k=1}^{K} (V_0^\pi(s_k^t) - V_0^{\pi_k}(s_k^t))$, as well as by the Uniform-PAC criterion (Dann, Lattimore, and Brunskill 2017), which we generalize to deal with approximate convergence. Let $\epsilon, \delta > 0$ and $N_{\epsilon, \delta} := \sum_{k=1}^{\infty} 1 \{V_k^\pi(s_k^t) - V_k^{\pi_k}(s_k^t) \geq \epsilon\}$ be the number of episodes in which the algorithm outputs a policy whose value is $\epsilon$-inferior to the optimal value. An algorithm is called Uniform-PAC, if $\Pr(\exists \epsilon > 0 : N_{\epsilon, \delta} \geq F(S, 1/\epsilon, \log 1/\delta, H)) \leq \delta$, where $F(\cdot)$ depends polynomially (at most) on its parameters. Note that Uniform-PAC implies $(\epsilon, \delta)$-PAC, and thus, it is a stronger property. As we analyze algorithms with inherent errors in this paper, we use a more general notion of $\Delta$ Uniform PAC by defining the random variable $N_{\epsilon, \delta}^\Delta := \sum_{k=1}^{\infty} 1 \{V_k^\pi(s_k^t) - V_k^{\pi_k}(s_k^t) \geq \Delta + \epsilon\}$, where $\Delta > 0$.
3 Background: RTDP

RTDP (Barto, Bradtke, and Singh 1995) is a well-known algorithm that solves an MDP when a model of the environment is given. Unlike, e.g., Value Iteration (VI) (Bertsekas and Tsitsiklis 1996) that solves an MDP by offline calculations, RTDP solves an MDP in a real-time manner, based on samples acquired from the environment, and DP Bellman updates from the current state \( s_t \).

Algorithm 1 depicts the pseudocode of RTDP for finite-horizon MDPs. The value function is initialized with an optimistic value, i.e., an upper-bound of the optimal value. At each time-step \( t \in [H] \) and episode \( k \in [K] \), the agent updates the value of \( s_t \) according to the optimal Bellman operator. It then acts from the current state \( s_t \) greedily w.r.t. the current value at the next time-step, \( V_{t+1} \).

Central to the analysis of RTDP in Efroni et al. (2019b) is the following high probability bound on the regret of this policy as the next time-step, \( V_{t+1} \).

Algorithm 1 Real-Time Dynamic Programming (RTDP)

init: \( \forall s \in S, \forall t \in \{0\} \cup [H], \bar{V}_{t+1}^0(s) = H - t \)

for \( k \in [K] \) do

Initialize \( s_k^0 \)

for \( t \in [H] \) do

\[ V_{t+1}^{k}(s_t^k) = T \bar{V}_{t+1}^{k-1}(s_t^k) \]

\[ a_t^k \in \arg \max_a r(s_t^k, a) + \pi(s_t^k) V_{t+1}^{k-1}(s_t^k) \]

Act with \( a_t^k \) and observe \( s_{t+1}^k \)

end for

end for

4 Multi-step Greedy RTDP

In this section, we formulate and analyze a multi-step greedy RTDP algorithm, called \( h \)-RTDP, whose pseudocode is shown in Algorithm 2. Without loss of generality, we assume that \( H/h \in \mathbb{N} \). We divide the horizon \( H \) into \( H/h \) intervals, each of length \( h \)-steps. \( h \)-RTDP stores \( H/h \) values in the memory for the last \( h \)-steps in each interval, i.e., for the time-steps \( \{1, h+1, \ldots, H+1\} \).

Given these values, at state \( s_t^k \), \( h \)-RTDP selects an action as

\[ a_t^k \in \arg \max_{a \in A} \max_{a_1, \ldots, a_{t-c}} \mathbb{E} \left[ \sum_{t'=0}^{t_c} r_t \right] V_{h_c}^{k-1}(s_{t_c}) \mid s_0 = s_t^k \]

where \( h_c \) is the next time-step in which a value is stored in the memory and \( t_c = h_c - t \) is the number of time-steps until \( h_c \). Thus, \( h \)-RTDP uses a varying lookahead distance \( t_c \) that depends on the current time-step (see Figure 1). With some abuse of notation, throughout the paper, we refer to this policy as the \( h \)-greedy policy w.r.t. \( V_{h_c}^{k-1} \), or as the \( h \)-greedy policy. Finally, note that \( h \)-RTDP generalizes RTDP as they are equal for \( h = 1 \).

We are now ready to establish finite-sample performance guarantees for \( h \)-RTDP (see Appendix A for the detailed proofs). We start with two lemmas from which we derive the main convergence result of this section.

Lemma 3. For all \( s \in S \), \( n \in \{0\} \cup \frac{H}{h} \), and \( k \in [K] \), it holds that (i) Boundedness from Below / Optimism: \( \bar{V}_{nh+1}^k(s) \leq \bar{V}_{nh+1}^k(s) \leq V_{nh+1}^k(s) \) and (ii) Non-Increasing: \( \bar{V}_{nh+1}^k(s) \leq V_{nh+1}^k(s) \leq \bar{V}_{nh+1}^k(s) \).

Figure 1: Varying lookahead horizon of an \( h \)-greedy policy in \( h \)-RTDP (see Eq. 3) with \( h = 3 \) and \( H = 6 \). The blue arrows show the lookahead horizon from a specific time-step \( t \), and the red bars are the time-steps for which a value is stored, i.e., \( t \in \{1, h + 1 = 4, 2h + 1 = H + 1 = 7\} \).

Lemma 4 (Optimality Gap and Expected Decrease). The expected cumulative value update at the \( k \)-th episode of \( h \)-RTDP satisfies the following relation:

\[ \bar{V}_{1}^{k}(s_1^k) - V_{1}^{\pi_{\bar{V}}}(s_1^k) = \sum_{n=1}^{H/h} \sum_{s \in S} \bar{V}_{nh+1}^{k-1}(s) - \mathbb{E}[V_{nh+1}^{k-1}(s) \mid F_{k-1}] \]

Properties (i) and (ii) of Lemma 3 show that for any \( s, n, \{ V_{nh+1}^k(s) \}_{k \geq 0} \) is a DBP. Lemma 4 establishes a relation between \( V_{1}^{k}(s_1^k) - V_{1}^{\pi_{\bar{V}}}(s_1^k) \) (LHS) and the expected decrease in \( V_{1}^{k} \) at the \( k \)-th episode (RHS). When the LHS is small, then \( V_{1}^{k}(s_1^k) \approx V_{1}^{\pi_{\bar{V}}}(s_1^k) \), due to the optimism of \( V_{1}^{k} \). Thus, \( h \)-RTDP is close to convergence to the optimal value. For this reason we refer the LHS as the optimality gap.

Using the two lemmas and the regret bound of a DBP, we establish finite-sample convergence guarantee for \( h \)-RTDP.

\(^2\)h-RTDP does not need to store \( V_1 \) and \( V_{H+1} \), they are only used in the analysis.
With probability $\Pr[S_h < \delta]$, it is decreasing, and by Lemma 3 (ii), it is bounded from below. In the next step, we define $s_{k+1}^t = \arg \max_a r(s_k^t, a) + \gamma(\bar{s}_{k+1}^t - s_k^t)$ to act with $s_{k+1}^t$ and observe $s_{k+1}^t$.

**Remark 2.** Since RTDP is an instance of $h$-RTDP, the proof of Theorem 5 generalizes and improves the proof of Efroni et al. (2019b). In the new analysis, (i) $\ln(3SH/\delta)$ is replaced by $\ln(3/\delta)$ and (ii) $H^2$ is replaced by $H(H-h)/h$, which is $H(H-1)$ for RTDP ($h = 1$). Although (ii) is not a real improvement over Efroni et al. (2019b), it gives the correct result for large values of $h$ ($h \to H$), and for this reason is important. For example, for $h = H$, we would expect $\tau_k$ to be optimal from the first episode, since by definition, $h$-greedy policy with $h = H$ is optimal.

### 4.1 Space & Computation Complexity of $h$-RTDP

$h$-RTDP assumes access to a black box that returns an $h$-greedy policy (3). The space and per-episode computational complexity (the total number of off-line computational operations per episode) of $h$-RTDP depend on this solver, and thus, it is important that it can be implemented efficiently. Another requirement for this solver is to only use local information; it should not access the entire state space (e.g., solve the $h$-greedy policy in a DP manner with Backward-Induction).

In this section, we propose a procedure, called Forward-Backward DP, to calculate an $h$-greedy policy. Forward-Backward DP is efficient when $S_h$, i.e., an upper-bound on the number of accessible states for all $t \in [h]$ time-steps (Section 2, for the formal definition) is small and it only uses ‘local’ information. We leave studying $h$-RTDP with approximate $h$-greedy policy solvers, e.g., sparse sampling (Kearns, Mansour, and Ng 2002; Sidford et al. 2018) or optimistic planning (Munos 2014), for future work.

Forward-Backward DP operates in two stages: (i) from the current state $s_k^t$, it finds the set of all accessible states in the next $t_c = h_c - t$ time-steps, and (ii) given this set, it performs DP (Backward Induction) to calculate (3). In Appendix F, we show that the space and per-episode computational complexity of Forward-Backward DP are $O(hS_h)$ and $O(HhAS_hS_h)$, respectively. When $S_h$ is small, and thus, $S_t$ is also small by definition, local search up to horizon $h$ can be done efficiently, as the number of reachable states in $h$ time-steps is small. Note that in the worst case, $S_h$ can be exponential in $h$, but it is always smaller than $S$.

**Remark 3 (Space & Computation Complexity of $h$-RTDP).** As discussed above, the space complexity of the Forward-Backward DP by which an $h$-greedy policy is computed is $O(hS_h)$. Since $h$-RTDP stores additional $O(SH/h)$ value entries, its total space complexity is $O(SH/h + hS_h)$. If $S_h$ is significantly smaller than $S$ (smaller than $SH/(h-1)/h^2$), the space complexity of $h$-RTDP is smaller than that of RTDP. The per-episode computational complexity of $h$-RTDP is $O(HhAS_hS_h)$, and thus, increases with $h$. This is expected since solving a decision problem becomes harder with increasing its horizon $h$. When $S_h \ll S$, the computational complexity of $h$-RTDP is $S$ independent.

### 5 Approximate Multi-step Greedy RTDP

In Section 4, we analyzed $h$-RTDP using a multi-step greedy policy and established its improved sample complexity as $h$ increases. We also showed that this improvement comes with a cost of larger per-episode computation as $h$ increases.

In this section, we consider three approximate versions of $h$-RTDP in which its update deviates from its exact form described in Section 4. The three settings are when there exists error 1) in the model and 2) in the value updates, and when we use 3) approximate state abstraction. Generalizing the techniques from previous section, we prove finite-sample bounds on the performance of $h$-RTDPs in the presence of these approximations. The bounds show that the asymptotic performance of $h$-RTDP in these approximate settings is the same as that of approximate DP.
5.1 h-RTDP with Approximate Model

In this section, we assume that the transition model used by h-RTDP to act and update the values is not exact, but close to the true model in the total variation (TV) norm, \( \forall (s, a) \in S \times A, \| p(\cdot | s, a) - \hat{p}(\cdot | s, a) \| \leq \epsilon_p \), where \( \hat{p} \) denotes the approximate model. Throughout this section and the relevant appendix (Appendix B), we denote by \( \hat{T} \) and \( \hat{V}^* \), the optimal Bellman operator and the optimal value of the approximate model \( \hat{p} \), respectively. Note that \( \hat{T} \) and \( \hat{V}^* \) satisfy (1) and (2) with \( p \) replaced by \( \hat{p} \). The h-RTDP algorithm with approximate model (h-RTDP-AM) is exactly the same as h-RTDP (Algorithm 2) with the model \( p \) and optimal Bellman operator \( T \) replaced by their approximations \( \hat{p} \) and \( \hat{T} \). Because of this close resemblance, we report the pseudocode of this algorithm in Appendix B (Algorithm 5).

Although we are given an approximate model, \( \hat{p} \), we are still interested in the performance of (approximate) h-RTDP on the true MDP, \( p \), and relative to its optimal value, \( V^* \). If we would solve the approximate model and act by its optimal policy, the Simulation Lemma (Kearns and Singh 2002; Strehl, Li, and Littman 2009) suggests that the regret is bounded by \( O(H^2 \epsilon_p K) \). For h-RTDP-AM the situation is more involved, as its updates are based on the approximate model and the samples are gathered by interacting with the true MDP. Nevertheless, by properly adjusting the techniques from Section 4, we derive performance bounds for h-RTDP-AM. These bounds reveal that the asymptotic regret increases by at most \( O(H^2 \epsilon_p K) \), similarly to the regret of the optimal policy of the approximate model. We start by stating two lemmas that replace Lemmas 3 and 4 of Section 4 (proofs are in Appendix B).

**Lemma 6.** For all \( s \in S \), \( n \in \{0\} \cup \{H \in \mathbb{N}\} \), and \( k \in [K] \), it holds that (i) \( \hat{V}^*_t(s) \leq \hat{V}^*_t(s) \) and (ii) \( \hat{V}^*_t(s) \leq \hat{V}^*_t(s) \).

**Lemma 7.** The expected cumulative value update at the \( k \)th episode of h-RTDP-AM satisfies the following relation:

\[
\hat{V}_t^k(s) - \hat{V}_t^k(s) = \frac{H(H-1)}{2} \epsilon_p \\
+ \sum_{n=1}^{n-1} \hat{V}^k(s) - \mathbb{E}[\hat{V}^k(s) | F_{k-1}].
\]

Using a technique similar to Theorem 5, we prove the following performance bound for h-RTDP-AM. We report the proof of Theorem 8 in Appendix B.

**Theorem 8 (Performance of h-RTDP with Approximate Model).** Let \( \epsilon, \delta > 0 \). The following holds for h-RTDP-AM:

1. With probability \( 1 - \delta \), for all \( K > 0 \), we have

\[
\text{Regret}(K) \leq \frac{9SH(H-h)}{h} \ln(3/\delta) + H(H-1)\epsilon_p K.
\]

2. Let \( \Delta_p = H(H-1)\epsilon_p \). Then, we have

\[
\Pr\left\{ \exists k > 0 : N^p_k \geq \frac{9SH(H-h) \ln(3/\delta)}{\epsilon_p h} \right\} \leq \delta.
\]

Unlike for the exact version of h-RTDP, these bounds imply approximate convergence and are due to the presence of approximations. However, the asymptotic performance gaps – both in terms of regret and Uniform PAC – of h-RTDP in the presence of approximate model approaches the performance gaps that an optimal policy of the approximate model would experience. Interestingly, although h-RTDP-AM updates using the approximate model, while interacting with the real model, its convergence rate (to the asymptotic performance) is similar to that of h-RTDP (Theorem 5).

5.2 h-RTDP with Approximate Value Updates

Another important question in analysis of approximate DP algorithms is their performance under approximate value updates, motivated by the need to use function approximation. This is often modeled by an extra noise \( |\epsilon_V(s)| \leq \epsilon_V \) added to the update rule (Bertsekas and Tsitsiklis 1996). In this section, we study the performance of h-RTDP with approximate value updates (h-RTDP-AV), whose pseudocode is shown in Algorithm 3. Similar to the previous section, we start with two lemmas, whose proofs are reported in Appendix C, that generalize Lemmas 3 and 4.

**Lemma 9.** For all \( s \in S, n \in \{0\} \cup \{H \in \mathbb{N}\} \), and \( k \in [K] \), it holds that (i) \( \hat{V}^*_t(s) \leq \hat{V}^*_t(s) + \epsilon_V(H-n) \) and (ii) \( \hat{V}^*_t(s) \leq \hat{V}^*_t(s) \).

**Lemma 10.** The expected cumulative value update at the \( k \)th episode of h-RTDP-AV satisfies the following relation:

\[
\hat{V}_t^k(s) - \hat{V}_t^k(s) \leq \frac{H(H-h)}{h} \epsilon_V + \sum_{n=1}^{n-1} \hat{V}^k(s) - \mathbb{E}[\hat{V}^k(s) | F_{k-1}].
\]

Using a proof similar to Theorem 5, we prove the following performance bound for h-RTDP-AV. We report the proof of Theorem 11 in Appendix C.

**Theorem 11 (Performance of h-RTDP with Approximate Value Updates).** Let \( \epsilon, \delta > 0 \). The following holds for h-RTDP-AV:

1. With probability \( 1 - \delta \), for all \( K > 0 \), we have

\[
\text{Regret}(K) \leq \frac{9SH(H-h)}{h} \left( 1 + \frac{3}{\epsilon_V} \right) + \frac{2H}{h} \epsilon_V K.
\]
2. Let $\Delta_V = 2HC_V$. Then, we have

$$\Pr\left\{ \exists \varepsilon > 0: \frac{\Delta_V}{h \varepsilon} \geq \frac{9SH(H-h)(1+\frac{\Delta_V}{2h}) \ln(\frac{1}{\delta})}{h \varepsilon} \right\} \leq \delta.$$

As in Section 5.1, the results of Theorem 11 exhibit an asymptotic linear regret $O(HC_V/K/h)$. As proven in Proposition 20 in Appendix G, such performance gap exists in ADP with approximate value updates. Furthermore, the convergence rate in $S$ to the asymptotic performance of $h$-RTDP-AV is similar to that of its exact version (Theorem 5). Unlike in $h$-RTDP-AM, the asymptotic performance of $h$-RTDP-AV improves with $h$. This phenomenon is expected in this case, since error exists only in the value updates and the total number of value entries decreases with $h$.

5.3 $h$-RTDP with Approximate State Abstraction

We conclude the analysis of approximate $h$-RTDP with exploring the advantages of combining it with approximate state abstraction (Abel, Hershkowitz, and Littman 2016). The central result of this section establishes that given an approximate state abstraction, $h$-RTDP converges with sample, computation and space complexity independent of the size of the state space $S$, as long as $S_\phi$ is smaller than $S$ (see Remark 3). This comes in contrast to the computational complexity of ADP in this setting, which is still $O(HSA)$ (see Appendix G.3 for further discussion).

State abstraction has been widely investigated in approximate planning (e.g., Dearden and Boutilier1997, Dean, Givan, and Leach1997, Even-Dar and Mansour2003), as a mean to deal with large state space problems. Among the existing approximate abstraction settings (e.g., Abel, Hershkowitz, and Littman2016), we focus on a specific one to demonstrate the advantages of using $h$-RTDP with approximate state abstraction ($h$-RTDP-AA). For any $n \in \{0\} \cup [H/h - 1]$, we define $\phi_{nh+1} : S \rightarrow S_\phi$, to be a mapping from the state space $S$ to a smaller space $S_\phi$, i.e., $S_\phi = |S_\phi| \ll |S|$. Inspired by Definition 3.3 in Li, Walsh, and Littman(2006), we make the following assumption:

**Assumption 1** (Approximate Abstraction). For any $s, s' \in S$ and $n \in \{0\} \cup [H/h - 1]$ for which $\phi_{nh+1}(s) = \phi_{nh+1}(s')$, we have $|V^k_{nh+1}(s) - V^k_{nh+1}(s')| \leq \epsilon_A$.

For clarity, let us denote by $\left\{ \tilde{V}^k_{\phi_{nh+1}} \right\}_{n=0}^{H/h}$ the values stored in memory by $h$-RTDP-AA at the $k$th episode. Unlike previous sections, the value function per time-step contains $S_\phi$ entries, $\tilde{V}^k_{\phi_{nh+1}} \in \mathbb{R}^{S_\phi}$. Note that if $\epsilon_A = 0$ this value function is enough to represent the optimal value in the reduced state space $S_\phi$. However, as $\epsilon_A$ can be positive, exact representation of the optimal value $V^*$ cannot be guaranteed. Nevertheless, the asymptotic performance of $h$-RTDP-AA will be ’close’, up to a term proportional to the quality of abstraction $\epsilon_A$, to the optimal policy.

An important quantity in our analysis is the set of states equivalent to a given state $s$ under $\phi_{nh+1}$.

**Definition 1** (Equivalent Set Under Abstraction). For any $s \in S$ and $n \in \{0\} \cup [H/h - 1]$, we define the set of states equivalent to $s$ under $\phi_{nh+1}$ as $\Phi_{nh+1}(s) := \{s' \in S : \phi_{nh+1}(s) = \phi_{nh+1}(s')\}$.

---

**Algorithm 4** $h$-RTDP with Approximate State Abstraction ($h$-RTDP-AA)

```plaintext```
init: $\forall s_\phi \in S_\phi$, $\forall n \in \{0\} \cup [H/h]$, $V^0_{\phi_{nh+1}}(s_\phi) = H - nh$

for $k \in [K]$ do
  Initialize $s^k_1$
  for $t \in [H]$ do
    if $(t - 1) \bmod h = 0$ then
      $h_c = t + h$
      $V^k_{\phi_{t+h}}(s^k_t) = T^h_{\phi}V^k_{\phi_{h}}(s^k_t)$
      $V^k_{\phi_{t+h}}(s^k_t) = \min\left\{ V^k_{\phi_{t+h}}(s^k_t), V^k_{\phi_{t+h}}(s^k_t) \right\}$
    end if
    $a^k_t \in \arg\max_{a \in A} \mathbb{E}\left[ r(s^k_t, a) + \gamma \sum_{t'=t+1}^{t+h-1} V^k_{\phi_{t+h}}(s^k_{t'}) \mid s^k_0 = s^k_t \right]$.
  end for
end for
```

Furthermore, the definition of the multi-step Bellman operator (2) and $h$-greedy policy (3) are revised, and with some abuse of notation, defined as follows:

$$T^h_{\phi}V^k_{\phi_{h}}(s^k_t) := \max_{a \in A} \mathbb{E}\left[ r(s^k_t, a) + \gamma \sum_{t'=t+1}^{t+h-1} V^k_{\phi_{h}}(s^k_{t'}) \mid s^k_0 = s^k_t \right].$$

$$a^k_t \in \arg\max_{a \in A} \mathbb{E}\left[ r(s^k_t, a) + \gamma \sum_{t'=t}^{t+h-1} V^k_{\phi_{h}}(s^k_{t'}) \mid s^k_0 = s^k_t \right].$$

(5)

In words, the $h$-greedy policy uses the given model, similarly to (3), to plan $h$ time-steps ahead. Differently from (3), the value after $h$ time-steps is the one defined in the reduced state space $S_\phi$. Note that the definition of the $h$-greedy policy for $h$-RTDP-AA in (5) is equivalent to the one used in Algorithm 4, obtained by similar recursion as for the optimal Bellman operator (2).

As in previous sections, we start by proving two lemmas that generalize Lemmas 3 and 4. Then, by similar technique as in the proof of Theorem 5, we establish finite-sample convergence guarantees for $h$-RTDP-AA. To ease the notation in the rest of the paper, we omit $\phi$ from $T^h_{\phi}$ and $\overline{V}^k_{\phi}$, and represent them as $T$ and $\overline{V}^k$. The proofs of the following lemmas and Theorem 14 are reported in Appendix D.

**Lemma 12.** For all $s \in S$, $n \in \{0\} \cup [H/h]$, and $k \in [K]$, it holds that (i) $\max_{s' \in \Phi_{nh+1}(s)} V^k_{nh+1}(s') \leq \overline{V}^k_{nh+1}(\phi_{nh+1}(s)) + \epsilon_A(H/h - n)$, (ii) $\overline{V}^k_{nh+1}(\phi_{nh+1}(s)) \leq \overline{V}^k_{nh+1}(\phi_{nh+1}(s))$, and (iii) $\overline{V}^k_{nh+1}(\phi_{nh+1}(s)) \geq 0$.

**Lemma 13.** The expected cumulative value update at the $k$th episode of $h$-RTDP-AA satisfies the following relation:

$$\overline{V}^k_{\phi}(s^k_t) - V^k_{\phi}(s^k_t) \leq \sum_{k=1}^{K} \sum_{n=1}^{\frac{H}{h} - 1} \sum_{s_\phi \in S_\phi} \overline{V}^{k-1}_{nh+1}(s_\phi) - E[V^k_{nh+1}(s_\phi) \mid F_{k-1}].$$
Theorem 14 (Performance of $h$-RTDP with Approximate State Abstraction). Let $\epsilon, \delta > 0$. The following holds for $h$-RTDP-AA:

1. With probability $1 - \delta$, for all $K > 0$, we have
   \[
   \text{Regret}(K) \leq \frac{9S_\epsilon H(H - h)}{h} \ln(3/\delta) + \frac{H \epsilon_A}{h} K.
   \]

2. Let $\Delta_A = H \epsilon_A$. Then, we have
   \[
   \Pr \left\{ \exists \epsilon > 0 : N_{\epsilon, \phi} \geq \frac{9S_\epsilon H(H - h) \ln(3/\delta)}{h \epsilon} \right\} \leq \delta.
   \]

Table 1: We define $\Delta_P = H(H - 1) \epsilon_P$, $\Delta_V = 2H \epsilon_V$, $\Delta_A = H \epsilon_A$, and $g_{\epsilon_V, H}(h) = (1 + H \epsilon_V / h)$. The table summarizes the regret bounds of the approximate versions of $h$-RTDP and compare them to those of corresponding ADP approaches. The regret of $h$-RTDP is bounded w.p. greater than $1 - \delta$, whereas the ADP regret is bounded a.s., w.p. 1.

### 6 RTDP vs. DP

The results of Sections 4 and 5 established finite-time convergence guarantees for $h$-RTDP for the exact and three approximate settings. In the approximate settings, as expected, the regret has a linear term of the form $\Delta K$, where $\Delta$ is linear in the approximation errors $\epsilon_P$, $\epsilon_A$, and $\epsilon_A$, and thus, the performance is continuous in these parameters, as we would desire. We refer to $\Delta K$ as the asymptotic regret, since it dominates the regret as $K \to \infty$.

A natural measure to evaluate the quality of $h$-RTDP in the approximate settings is comparing its regret to the regret of the corresponding approximate DP (ADP) approach. Table 1 contains the regret of both approximate $h$-RTDP and the corresponding ADP. ADP calculates approximate values $\{V_{h+1}^{n_h}\}_{n_h=0}^{H/h}$ by Backward-Induction. Based on these values, the same $h$-greedy policy by which $h$-RTDP acts is evaluated. In the analysis of ADP, we use standard techniques, yet they result in an improved performance relatively to (Bertsekas and Tsitsiklis 1996). The bound proved there for the case of approximate value updates is of $O(H^3 \epsilon_V)$, whereas our bound is of $O(H \epsilon_V)$, for $h = 1$. Note that the results of (Bertsekas and Tsitsiklis 1996) are proved for the $\gamma$-discounted case, and we translate them to the finite horizon case by $H = (1 - \gamma)^{-1}$. It is an interesting question whether our analysis for ADP with approximate value updates can be generalized to the discounted case.

Note that the $h$-ADP approach is analyzed using the same errors experienced by the corresponding approximate $h$-RTDP version. As Table 1 reveals, the asymptotic regret of $h$-RTDP under approximation is equivalent to the regret of $h$-ADP. More importantly, Theorems 8, 11, and 14 establish (uniform) PAC guarantees that are not shown in the table due to space limit. These PAC guarantees bound the number of times that large performance gap exists between the $h$-RTDP and $h$-ADP policies. It is important to note that the asymptotic error decreases with $h$ for the approximate value updates and approximate abstraction settings. In these settings the error is caused by approximation in the value function. As less such values are used with increasing the lookahead horizon $h$ this phenomenon is expected.

### 7 Conclusions and Future Work

In this paper, we formulated $h$-RTDP, a generalization of RTDP which acts according to a multi-step greedy policy or a lookahead policy, instead by a 1-step greedy policy as RTDP. We analyze its finite-sample performance in its exact version as well as in several approximate settings of special interest. The results indicate that $h$-RTDP converges in a very strong sense, e.g., its regret is independent on the number of episodes number. More importantly, the results reveal an improved sample complexity when increasing the lookahead horizon, $h$. Lastly, the asymptotic performance of $h$-RTDP was shown to be equivalent to the one of approximate DP, which under no further assumption on the approximation error, is the best we can hope for.

We believe this work opens interesting research venues, such as, studying alternatives to the solution of the $h$-greedy policy (see Section F), studying a Receding-Horizon extension of RTDP, RTDP with function approximation, and formulating an alternative Thompson-Sampling-RTDP as current RTDP is an ‘optimistic’ algorithm. As the analysis developed in this work was shown to be quite generic, we hope that it can assist with answering some of these questions. On the experimental side, more needs to be understood, especially in comparing MCTS and RTDP approaches and studying how RTDP can be combined with deep neural networks as the value function approximators.

Acknowledgments

The authors thank Nadav Merlis for helpful comments.
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A Multi-step Greedy Real Time Dynamic Programming

This section contains the full proofs of all the results of Section 4 in chronological order.

**Lemma 3.** For all $s \in S$, $n \in \{0\} \cup \left[ \frac{H}{h} \right]$, and $k \in [K]$, it holds that (i) Boundness from Below / Optimism: $V^*_{n+h+1}(s) \leq \bar{V}^k_{n+h+1}(s)$ and (ii) Non-Increasing: $\bar{V}^k_{n+h+1}(s) \leq \bar{V}^{k-1}_{n+h+1}(s)$.

**Proof.** Both claims are proven using induction.

(i) Let $n \in \{0\} \cup \left[ \frac{H}{h} \right]$. By the initialization, $\forall s, n$, $V^*_n(s) \leq V^0_{n+h+1}(s)$. Assume the claim holds for the first $(k-1)$ episodes. Let $s^k_t$ be the state of the algorithm at a time-step $t$ of the $k$'th episode at which a value update takes place, i.e., $t = nh + 1$, for some $n \in \{0\} \cup \left[ \frac{H}{h} \right]$. By the value update of Algorithm 2 and (2), we have

$$\bar{V}^k_t(s^k_t) = (T^h \bar{V}^{k-1}_{h+1})(s^k_t) = (T^h \bar{V}^{k-1}_{t+h})(s^k_t) \geq (T^h V^*_n)(s^k_t) = V^*_n(s^k_t).$$

The inequality holds by the induction hypothesis and the monotonicity of $T^h$, a consequence of the monotonicity of $T$, the optimal Bellman operator (Bertsekas and Tsitsiklis 1996). The last equality holds by the fact that the recursion is satisfied by the optimal value function (2). Thus, the induction step is proven for the first claim.

(ii) Let $n \in \{0\} \cup \left[ \frac{H}{h} \right]$ and $t = nh + 1$ be a time-step in which a value update takes place. To prove the base case, we use the optimistic initialization. Let $s^k_1$ be the state of the algorithm in the $t$'th time-step of the first episode. By the update rule, we have

$$\bar{V}^1_t(s^k_1) = (T^h \bar{V}^0_{t+h})(s^k_1) = \max_{a_0, \ldots, a_{h-1}} E \left[ \sum_{t'=0}^{h-1} r(s_{t'}, a_{t'}) + \bar{V}^0_{t+h}(s_h) \mid s_0 = s^k_1 \right].$$

(a) holds since $r(s, a) \in [0, 1]$ and by the optimistic initialization.

(b) observe that $h+H - (t+h-1) = H - (t-1) = \bar{V}^0_{t+h}(s^k_1)$.

Assume that the claim holds for the first $(k-1)$ episodes. Let $s^k_t$ be the state of the algorithm at a time-step $t$ of the $k$'th episode at which a value update takes place, i.e., $t = nh + 1$, for some $n \in \{0\} \cup \left[ \frac{H}{h} \right]$. By the value update rule of Algorithm 2, we have $\bar{V}^k_t(s^k_t) = (T^h \bar{V}^{k-1}_{h+1})(s^k_t) = (T^h \bar{V}^{k-1}_{t+h})(s^k_t)$. If $s^k_t$ was previously updated, let $\bar{k}$ be the last episode in which the update occurred, i.e., $\bar{V}^k_t(s^k_t) = (T^h \bar{V}^{k-1}_{t+h})(s^k_t) = \bar{V}^{k-1}_{t+h}(s^k_t)$. By the induction hypothesis, we have that $\forall s, t$, $\bar{V}^{k-1}_{t+h}(s) \geq \bar{V}^{k-1}_{t+h}(s)$. Using the monotonicity of $T^h$, we may write

$$\bar{V}^{k-1}_{t+h}(s^k_t) = (T^h \bar{V}^{k-1}_{t+h})(s^k_t) \leq (T^h \bar{V}^{k-1}_{t+h})(s^k_t) = \bar{V}^{k-1}_{t+h}(s^k_t).$$

Thus, $\bar{V}^{k-1}_{t+h}(s^k_t) \leq \bar{V}^{k-1}_{t+h}(s^k_t)$ and the induction step is proved. If $s^k_t$ was not previously updated, then $\bar{V}^{k-1}_{t+h}(s^k_t) = \bar{V}^0_{t+h}(s^k_t)$. In this case, the induction hypothesis implies that $\forall s', \bar{V}^{k-1}_{t+h}(s') \leq \bar{V}^0_{t+h}(s')$ and the result is proven similarly to the base case. \qed

**Lemma 4** (Optimality Gap and Expected Decrease). The expected cumulative value update at the $k'$th episode of $h$-RTDP satisfies the following relation:

$$\bar{V}^k_t(s^k_t) - V^{k-1}_t(s^k_t) = \sum_{n=1}^{k-1} \sum_{s \in S} \bar{V}^{k-1}_{n+h+1}(s) - E[\bar{V}^{k-1}_{n+h+1}(s) \mid F_{k-1}].$$

**Proof.** Let $n \in \{0\} \cup \left[ \frac{H}{h} \right]$ and $t = nh + 1$ be a time-step in which a value update takes place. By the definition of the update rule, the following holds for the value update at the visited state $s^k_t$:

$$\bar{V}^k_t(s^k_t) = (T^h \bar{V}^{k-1}_{t+h})(s^k_t)$$

$$= (T^{\pi_k}(t) \ldots T^{\pi_k}(t+h-1) \bar{V}^{k-1}_{t+h})(s^k_t) = E \left[ \sum_{t'=t}^{t+h-1} r(s_{t'}, a_{t'}) + \bar{V}^{k-1}_{t+h}(s_{t+h}) \mid \pi_k, s_t = s^k_t \right].$$

(a) $E \left[ \sum_{t'=t}^{t+h-1} r(s_{t'}, a_{t'}) + \bar{V}^{k-1}_{t+h}(s_{t+h}) \mid F_{k-1}, s^k_t \right].$
(a) We prove this passage for each reward element \( r(s_t, a_t) \) in the expectation. The proof for the expectation of \( \bar{V}^{k-1}_{t+h}(s_{t+h}) \) follows in a similar manner. Since the first expectation is w.r.t. the dynamics of the true model, a consequence of updating by the true model, for any \( t' \geq t \), we may write

\[
\mathbb{E}[r(s_{t'}, a_{t'}) \mid \pi_k, s_t = s^k_t] = \sum_{s_{t'} \in S} p(s_{t'} \mid s_t = s^k_t, \pi_k) r(s_{t'}, a_{t'}).
\]

\[
= \sum_{s_{t'} \in S} p(s_{t'} \mid s^k_t, \mathcal{F}_{k-1}) r(s_{t'}, a_{t'}),
\]

where \( p(s_{t'} \mid s^k_t, \pi_k) \) is the probability of starting at state \( s^k_t \) following \( \pi_k \), and reaching state \( s_{t'} \) in \( t' - t \) steps.

(i) We use the fact that \( p(s_{t'} \mid s^k_t, \pi_k) = p(s_{t'} \mid s^k_t, \mathcal{F}_{k-1}) \), in words, given the policy \( \pi_k \) (which is \( \mathcal{F}_{k-1} \) measurable) and \( s^k_t \) the probability for a state \( s^k_{t'} \) with \( t' \geq t \) is independent of the rest of the history.

Now that we proved (a), we take the conditional expectation of (6) w.r.t. \( \mathcal{F}_{k-1} \) and use the tower rule to obtain

\[
\mathbb{E}[\bar{V}^k_t(s^k_t) \mid \mathcal{F}_{k-1}] = \mathbb{E}\left[\sum_{t'=t}^{t+h-1} r(s^k_{t'}, a^k_{t'}) + \bar{V}^{k-1}_{t+h}(s^k_{t+h}) \mid \mathcal{F}_{k-1}\right].
\]

Summing (8) for all \( n \in \{0\} \cup [\frac{H}{h}] \), and using the linearity of expectation and the fact that \( \bar{V}^{k}_{t+h}(s) = 0 \) for all \( s, k \), we have

\[
\sum_{n=0}^{\frac{H}{h} - 1} \mathbb{E}[\bar{V}^k_{n+h+1}(s^k_{nh+1}) \mid \mathcal{F}_{k-1}] = \mathbb{E}\left[\sum_{t=1}^{H} r(s^k_{t}, a^k_{t}) \mid \mathcal{F}_{k-1}\right] + \sum_{n=1}^{\frac{H}{h} - 1} \mathbb{E}[\bar{V}^{k-1}_{n+h+1}(s^k_{nh+1}) \mid \mathcal{F}_{k-1}]
\]

\[
\leftrightarrow \bar{V}^1_k(s^k_1) + \sum_{n=1}^{\frac{H}{h} - 1} \mathbb{E}[\bar{V}^k_{n+h+1}(s^k_{nh+1}) \mid \mathcal{F}_{k-1}] = \mathbb{E}\left[\sum_{t=1}^{H} r(s^k_{t}, a^k_{t}) \mid \mathcal{F}_{k-1}\right] + \sum_{n=1}^{\frac{H}{h} - 1} \mathbb{E}[\bar{V}^{k-1}_{n+h+1}(s^k_{nh+1}) \mid \mathcal{F}_{k-1}]
\]

\[
\leftrightarrow \bar{V}^1_k(s^k_1) - V^\pi_k(s^k_1) = \sum_{n=1}^{\frac{H}{h} - 1} \mathbb{E}[\bar{V}^{k-1}_{n+h+1}(s^k_{nh+1}) - \bar{V}^k_{n+h+1}(s^k_{nh+1}) \mid \mathcal{F}_{k-1}].
\]

The second line holds by the fact that \( s^k_t \) is measurable w.r.t. \( \mathcal{F}_{k-1} \). The third line holds since

\[
V^\pi_k(s^k_t) = \mathbb{E}\left[\sum_{t=1}^{H} r(s^k_{t}, a^k_{t}) \mid s^k_t, \pi_k\right] = \mathbb{E}\left[\sum_{t=1}^{H} r(s^k_{t}, a^k_{t}) \mid \mathcal{F}_{k-1}\right].
\]

Applying Lemma 15 from Appendix E with \( g^k_t = \bar{V}^k_t \) for \( t = nh + 1 \), we obtain

\[
\sum_{n=1}^{\frac{H}{h} - 1} \sum_{s \in S} \bar{V}^{k-1}_{nh+1}(s) - \mathbb{E}[\bar{V}^k_{nh+1}(s) \mid \mathcal{F}_{k-1}],
\]

which concludes the proof. Note that the update of \( V^k_t \) occurs only at the visited state \( s^k_t \) and the update rule uses \( \bar{V}^{k-1}_{t+h} \), i.e., it is measurable w.r.t. \( \mathcal{F}_{k-1} \), and thus, it is valid to apply Lemma 15.

**Theorem 5 (Performance of \( h \)-RTDP).** Let \( \epsilon, \delta > 0 \). The following holds for \( h \)-RTDP:

1. With probability \( 1 - \delta \), for all \( K > 0 \), we have

\[
\text{Regret}(K) \leq \frac{9SH(H - h)}{h} \ln(3/\delta).
\]

2. We have \( \Pr\{\exists \epsilon > 0 \mid N_{\epsilon} \geq \frac{9SH(H - h) \ln(3/\delta)}{h}\} \leq \delta \).

**Proof.** We start by proving Claim (1). We know that the following bounds hold on the regret:

\[
\text{Regret}(K) := \sum_{k=1}^{K} V^\pi_k(s^k_1) - \sum_{k=1}^{K} V^\pi_k(s^k_1) \leq \sum_{k=1}^{K} \bar{V}^k_1(s^k_1) - V^\pi_k(s^k_1)
\]

\[
\leq \sum_{k=1}^{K} \sum_{n=1}^{\frac{H}{h} - 1} \sum_{s \in S} \bar{V}^{k-1}_{nh+1}(s) - \mathbb{E}[\bar{V}^k_{nh+1}(s) \mid \mathcal{F}_{k-1}].
\]
(a) is by the optimisation of the value function (Lemma 3), and (b) is by Lemma 4.

We would like to show that (10) is the regret of a Decreasing Bounded Process (DBP). We start by defining

\[
X_k := \sum_{n=1}^{\frac{H}{h}-1} \sum_{s \in S} \bar{V}_{\eta h + 1}^k (s).
\]

We now prove that \(\{X_k\}_{k \geq 0}\) is a DBP. Note that \(\{X_k\}_{k \geq 0}\)
1. is decreasing, since \(\forall s, t, \bar{V}_t^k (s) \leq \bar{V}_{t-1}^k (s)\) by Lemma 3, and thus, their sum is also decreasing, and
2. is bounded since \(\forall s, t \bar{V}_t^k (s) \geq V_t^* (s) \geq 0\) by Lemma 3, and thus, the sum is bounded from below by 0.

We can show that the initial value \(X_0\) is also bounded as

\[
X_0 = \sum_{n=1}^{\frac{H}{h}-1} \sum_{s \in S} \bar{V}_{\eta h + 1}^0 (s) \leq \frac{H}{h} H (H - h) \frac{H}{h}.
\]

Using the linearity of expectation and the definition (11), we observe that (10) can be written as

\[
\text{Regret}(K) \leq (10) = \sum_{k=1}^{K} X_{k-1} - \mathbb{E}[X_k | \mathcal{F}_{k-1}],
\]

which is regret of a DBP. Applying the bound on the regret of a DBP, Theorem 1, we conclude the proof of the first claim.

We now prove Claim (2). Here we use a different technique than the one used in Efroni et al. (2019b). The technique allows us to prove uniform-PAC bounds for the approximate versions of \(h\)-RTDP described in Section 5. For these approximate versions, the uniform-PAC result is not a corollary of the regret bound and more careful analysis should be used.

For all \(\epsilon > 0\), the following relations hold:

\[
\begin{align*}
\mathbb{I}\{V_1^* (s_1^k) - V_1^{\eta_k} (s_1^k) \geq \epsilon\} & \leq \mathbb{I}\{\bar{V}_1^k (s_1^k) - V_1^{\eta_k} (s_1^k) \geq \epsilon\} \\
& \leq \mathbb{I}\{\bar{V}_1^k (s_1^k) - V_1^{\eta_k} (s_1^k) \geq \epsilon\} (\bar{V}_1^k (s_1^k) - V_1^{\eta_k} (s_1^k)) \\
& \leq \mathbb{I}\{\bar{V}_1^k (s_1^k) - V_1^{\eta_k} (s_1^k) \geq \epsilon\} \left(\sum_{n=1}^{\frac{H}{h}-1} \sum_{s \in S} \bar{V}_{\eta h + 1}^k (s) - \mathbb{E}[\bar{V}_{\eta h + 1}^k (s) | \mathcal{F}_{k-1}]\right) \\
& \leq \mathbb{I}\{\bar{V}_1^k (s_1^k) - V_1^{\eta_k} (s_1^k) \geq \epsilon\} (X_{k-1} - \mathbb{E}[X_k | \mathcal{F}_{k-1}]).
\end{align*}
\]

(a) holds since for all \(t, s, \bar{V}_t^k (s) \geq V_t^* (s)\) by Lemma 3, (b) holds by the indicator function, (c) holds by Lemma 4, (d) holds by the definition of \(X_0\) from (11) and the linearity of expectation.

Let define \(N_\epsilon (K) = \sum_{k=1}^{K} \mathbb{I}\{V_1^* (s_1^k) - V_1^{\eta_k} (s_1^k) \geq \epsilon\}\) as the number of times \(V_1^* (s_1^k) - V_1^{\eta_k} (s_1^k) \geq \epsilon\) at the first \(K\) episodes. For all \(\epsilon > 0\), we may write

\[
N_\epsilon (K) \leq \sum_{k=1}^{K} \mathbb{I}\{V_1^* (s_1^k) - V_1^{\eta_k} (s_1^k) \geq \epsilon\} \leq \sum_{k=1}^{K} \mathbb{I}\{\bar{V}_1^k (s_1^k) - V_1^{\eta_k} (s_1^k) \geq \epsilon\} (X_{k-1} - \mathbb{E}[X_k | \mathcal{F}_{k-1}])
\]

(a) holds by the definition of \(N_\epsilon (K)\), (b) follows from (12), (c) holds because \(\{X_k\}_{k \geq 0}\) is a DBP, and thus, \(X_{k-1} - \mathbb{E}[X_k | \mathcal{F}_{k-1}] \geq 0\) a.s. Therefore, the following relation holds:

\[
\left\{ \forall K > 0: \sum_{k=1}^{K} X_{k-1} - \mathbb{E}[X_k | \mathcal{F}_{k-1}] \leq \frac{9SH(H - h)}{h} \ln \frac{3}{\delta} \right\} \leq \left\{ \forall \epsilon > 0: N_\epsilon (K) \leq \frac{9SH(H - h)}{h} \ln \frac{3}{\delta} \right\},
\]

from which we obtain that for any \(K > 0\),

\[
\text{Pr} \left( \forall \epsilon > 0: N_\epsilon (K) \leq \frac{9SH(H - h)}{h} \ln \frac{3}{\delta} \right) \geq \text{Pr} \left( \forall K > 0: \sum_{k=1}^{K} X_{k-1} - \mathbb{E}[X_k | \mathcal{F}_{k-1}] \leq \frac{9SH(H - h)}{h} \ln \frac{3}{\delta} \right) \geq 1 - \delta.
\]
(a) holds because of the bound on the regret of DBP (see Theorem 1). Equivalently, for any $K > 0$,

$$\Pr\left( \exists \epsilon > 0 : N_\epsilon(K) \epsilon \geq \frac{9SH(H-h)}{h} \ln \frac{3}{\delta} \right) \leq \delta. \quad (13)$$

Note that for all $\epsilon > 0$, $K_1 \geq K_2$, $1\{N_\epsilon(K_2) \epsilon \geq C\} = 1$ implies $1\{N_\epsilon(K_1) \epsilon \geq C\} = 1$, and thus, $1\{N_\epsilon(K) \epsilon \geq C\} \leq \lim_{K \to \infty} 1\{N_\epsilon(K) \epsilon \geq C\}$. Furthermore, $1\{N_\epsilon(K) \epsilon \geq C\} \geq 0$ by definition. Thus, we can apply the Monotone Convergence Theorem to conclude the proof:

$$\Pr\left( \exists \epsilon > 0 : N_\epsilon \epsilon \geq \frac{9SH(H-h)}{h} \ln \frac{3}{\delta} \right) = \Pr\left( \lim_{K \to \infty} \left\{ \exists \epsilon > 0 : N_\epsilon(K) \epsilon \geq \frac{9SH(H-h)}{h} \ln \frac{3}{\delta} \right\} \right)$$

$$= \mathbb{E} \left[ \lim_{K \to \infty} 1\left\{ \exists \epsilon > 0 : N_\epsilon(K) \epsilon \geq \frac{9SH(H-h)}{h} \ln \frac{3}{\delta} \right\} \right] \overset{(a)}{=} \lim_{K \to \infty} \mathbb{E}\left[ 1\left\{ \exists \epsilon > 0 : N_\epsilon(K) \epsilon \geq \frac{9SH(H-h)}{h} \ln \frac{3}{\delta} \right\} \right]$$

$$= \lim_{K \to \infty} \Pr\left( \exists \epsilon > 0 : N_\epsilon(K) \epsilon \geq \frac{9SH(H-h)}{h} \ln \frac{3}{\delta} \right) \overset{(b)}{=} \delta.$$

(a) is by the Monotone Convergence Theorem by which $\mathbb{E}[\lim_{k \to \infty} X_k] = \lim_{k \to \infty} \mathbb{E}[X_k]$, for $X_k \geq 0$ and $X_k \leq \lim_{k \to \infty} X_k$. 

(b) holds by (13).
Algorithm 5: $h$-RTDP with Approximate Model ($h$-RTDP-AM)

| Algorithm 5: $h$-RTDP with Approximate Model ($h$-RTDP-AM) |
|------------------------------------------------------------|
| init: $\forall s \in \mathcal{S}$, $\forall n \in \{0\} \cup [\frac{H}{n}], \hat{V}_{nh+1}^0(s) = H - nh$ |
| for $k \in [K]$ do |
| Initialize $s_k^0$ |
| for $t \in [H]$ do |
| if $(t - 1) \mod h == 0$ then |
| $h_c = t + h$ |
| $\hat{V}_t^k(s_k^t) = \hat{T}^h \hat{V}_{h_c}^{k-1}(s_k^t)$ |
| end if |
| $a_k^t = \arg \max_a r(s_k^t, a) + \hat{\rho}(|s_k^t, a|) \top \hat{T}^{h_c-t-1} \hat{V}_{h_c}^{k-1}$ |
| Act with $a_k^t$ and observe $s_{t+1}^k$ |
| end for |
| end for |

Algorithm 5 contains the pseudocode of $h$-RTDP with approximate model. The algorithm is exactly the same as $h$-RTDP (Algorithm 2) with the model $p$ and optimal Bellman operator $T$ replaced by their approximations $\hat{\rho}$ and $\hat{T}$. Meaning, $h$-RTDP is agnostic whether it uses the true or approximate model.

We now provide the full proofs of all results in Section 5.1 in their chronological order. We use the notation $\mathbb{E}_p$ to denote expectation w.r.t. the approximate model, i.e., w.r.t. the dynamics $\hat{\rho}(s' \mid s, a)$ instead according to $\rho(s' \mid s, a)$.

Lemma 6. For all $s \in \mathcal{S}$, $n \in \{0\} \cup [\frac{H}{n}]$, and $k \in [K]$, it holds that (i) $\hat{V}_t^*(s) \leq \hat{V}_{nh+1}^0(s)$ and (ii) $\bar{V}_{nh+1}^k(s) \leq \bar{V}_{nh+1}^{k-1}(s)$.

**Proof.** Both claims are proven using induction.

(i) Let $n \in [0, \frac{H}{h} - 1]$ and denote $\hat{T}, \hat{V}^*$ as the optimal Bellman operators and optimal value of the approximate MDP $(\mathcal{S}, \mathcal{A}, \hat{\rho}, r, H)$. See that they satisfy usual Bellman equation 2.

By the initialization, $\forall s, t$, $\hat{V}_{t+hn}^*(s) \leq V_{t+hn}^0(s)$. Assume the claim holds for $k - 1$ episodes. Let $s_k^t$ be the state the algorithm is at in the $t = 1 + hn$ time-step of the $k$th episode, i.e., at a time-step in which a value update is taking place. By the value update of Algorithm 5,

$$V_t^k(s_k^t) = (\hat{T}^h \hat{V}_{t+h}^k(s_k^t)) \geq (\hat{T}^h \hat{V}_{t+h}^{k-1}(s_k^{t-1})) = \hat{V}_t^*(s_k^t).$$

The second relation holds by the induction hypothesis and the monotonicity of $\hat{T}$, a consequence of the monotonicity of $\hat{T}$, the optimal Bellman operator (Bertsekas and Tsitsiklis 1996). The third relation holds by the recursion satisfied by the optimal value function (2). Thus, the induction step is proven for the first claim.

(ii) Let $n \in [0, \frac{H}{h} - 1]$ and let $t = 1 + hn$ be a time-step in which a value update is taking place. To prove the base case of the second claim we use the optimistic initialization. Let $s_k^{t-1}$ be the state the algorithm is at in the $t$th time-step of the first episode. By the update rule,

$$\hat{V}_t^k(s_k^t) = (\hat{T}^h \hat{V}_{t+h}^0(s_k^{t-1})) \leq (\hat{T}^h \hat{V}_{t+h}^0(s_k^{t-1})) \leq h + H - (t + h - 1) = H - (t - 1) \Rightarrow V_t^0(s_k^t).$$

Relation (1) is by the update rule (see Algorithm 5), when the expectation is taken place w.r.t. the approximate model $\hat{\rho}$. Relation (2) holds since $r(s, a) \in [0, 1]$ and by the optimistic initialization (see that for $t$ the values at times step $t + h$ were not updated and keep their initial value). For (3) observe that $H - (t - 1)$ is the value of the optimistic initialization.

Assume the second claim holds for $k - 1$ episodes. Let $s_k^t$ be the state that the algorithm is at in the $t$th time-step of the $k$th episode. Again, assume that $t = 1 + h(n + m)$, a time-step in which a value update is being done. By the value update of Algorithm 5, we have

$$\hat{V}_t^k(s_k^t) = (\hat{T}^h \hat{V}_{t+h}^{k-1}(s_k^t)).$$
If \( s_k^t \) was previously updated, let \( \bar{k} \) be the previous episode in which the update occurred. By the induction hypothesis, we have that \( \forall s, t, \bar{V}_{t}^k(s) \geq \bar{V}_{t}^{k-1}(s) \). Using the monotonicity of \( T^h \) (due to the monotonicity of the Bellman operator),

\[
(\bar{T}^h \bar{V}_{t+1}^{k-1})(s_k^t) \leq (\bar{T}^h \bar{V}_{t+1}^{k-1})(s_k^t) = \bar{V}_{t+1}^{k-1}(s_k^t)
\]

Thus, \( \bar{V}_{t}^k(s_k^t) \leq \bar{V}_{t}^{k-1}(s_k^t) \) and the induction step is proved. If \( s_k^t \) was not previously updated, then \( \bar{V}_{t}^{k-1}(s_k^t) = \bar{V}_{t}^0(s_k^t) \). In this case, the induction hypothesis implies that \( \forall s', \bar{V}_{t+1}^{k-1}(s') \leq \bar{V}_{t+1}^0(s') \) and the result is proven similarly to the base case.

**Lemma 7.** The expected cumulative value update at the \( k \)’th episode of h-RTDP-AM satisfies the following relation:

\[
\bar{V}_{t}^k(s_k^t) - \bar{V}_{t}^{k-1}(s_k^t) = \frac{H(H - 1)}{2} \epsilon_P + \sum_{n=1}^{\frac{H^2}{4}-1} \sum_{s \in S} \bar{V}_{t}^{k-1}(s_k^t) - \bar{V}_{t}^{k-1}(s_k^t) - \mathbb{E}[\bar{V}_{t+1}^k(s) | \mathcal{F}_{k-1}].
\]

**Proof.** Let \( n \in [0, \frac{H}{4} - 1] \) and let \( t = 1 + hn \) be a time-step in which a value update is taking place. By the definition of the update rule, the following holds for the update at the visited state \( s_k^t \):

\[
\bar{V}_{t}^k(s_k^t) = (\bar{T}^h \bar{V}_{t+1}^{k-1})(s_k^t)
\]

\[
= (\bar{T}^h \bar{T}^h \bar{V}_{t+2}^{k-1})(s_k^t)
\]

\[
= \bar{E}_{\pi}[\sum_{t'=t}^{t+h-1} r(s_{t'}, a_{t'}) + \bar{V}_{t+1}^{k-1}(s_{t+1}) | \pi_k, s_k]
\]

\[
= \bar{E}[\sum_{t'=t}^{t+h-1} r(s_{t'}, a_{t'}) + \bar{V}_{t+1}^{k-1}(s_{t+1}) | \pi_k, s_k] + \bar{E}_{\pi}[\sum_{t'=t}^{t+h-1} r(s_{t'}, a_{t'}) + \bar{V}_{t+1}^{k-1}(s_{t+1}) | \pi_k, s_k]
\]

\[
\leq \bar{E}[\sum_{t'=t}^{t+h-1} r(s_{t'}, a_{t'}) + \bar{V}_{t+1}^{k-1}(s_{t+1}) | \pi_k, s_k]
\]

Applying Lemma 16 we bound the above by,

\[
(14) \leq \bar{E}[\sum_{t'=t}^{t+h-1} r(s_{t'}, a_{t'}) + \bar{V}_{t+1}^{k-1}(s_{t+1}) | \pi_k, s_k] + \sum_{t'=t}^{t+h-1} (t' - t) \epsilon_P + (H - (t + h - 1)) h \epsilon_P
\]

\[
= \bar{E}[\sum_{t'=t}^{t+h-1} r(s_{t'}, a_{t'}) + \bar{V}_{t+1}^{k-1}(s_{t+1}) | \pi_k, s_k] - \frac{1}{2} (h - 1) h \epsilon_P + (H - t) h \epsilon_P
\]

\[
= \bar{E}[\sum_{t'=t}^{t+h-1} r(s_{t'}, a_{t'}) + \bar{V}_{t+1}^{k-1}(s_{t+1}) | \mathcal{F}_{k-1}, s_k] - \frac{1}{2} (h - 1) h \epsilon_P + (H - t) h \epsilon_P.
\]

(15)
Taking the conditional expectation w.r.t. $\mathcal{F}_{k-1}$ of both (14) and its RHS (15), using the tower property and the fact for all $s$, $\bar{V}_{H+1}(s) = 0$ we get,

\[
\mathbb{E} \left[ \bar{V}_t^k(s_t^k) \mid \mathcal{F}_{k-1} \right] \leq \mathbb{E} \left[ \sum_{t'=t}^{t+h-1} r(s_{t'}, a_{t'}) + \bar{V}_{t+h}^{k-1}(s_{t+h}) \mid \mathcal{F}_{k-1} \right] - \frac{1}{2}(h-1)\epsilon_P + (H-t)\epsilon_P
\]

Let us denote $d_n := -\frac{1}{2}(h-1)\epsilon_P + (H-n)\epsilon_P$. Summing the above relation for all $n \in [\frac{H}{n}] - 1$, using linearity of expectation, and the fact $\bar{V}_{H+1}(s) = 0$ for all $s, k$,

\[
\sum_{n=0}^{\frac{H}{t+1}} \mathbb{E} [\bar{V}_1^{k+nh}(s_1^k) \mid \mathcal{F}_{k-1}] = \mathbb{E} \left[ \sum_{t=1}^H r(s_t^k, a_t^k) \mid \mathcal{F}_{k-1} \right] + \sum_{n=1}^{\frac{H}{n}} \mathbb{E} [\bar{V}_{1+nh}^{k+1}(s_{1+nh}) \mid \mathcal{F}_{k-1}] + \sum_{n=1}^{\frac{H}{n}} d_{1+nh}
\]

By simple algebraic manipulation we get $\sum_{n=0}^{\frac{H}{t+1}} d_{1+nh} = \frac{H}{2}(H-1)\epsilon_P$ (see Lemma 18). Thus, (16) has the following equivalent forms, by which we conclude the proof of this lemma.

\[
\iff \bar{V}_1^k(s_1^k) + \sum_{n=1}^{\frac{H}{n}} \mathbb{E} [\bar{V}_{1+nh}^{k+1}(s_{1+nh}) \mid \mathcal{F}_{k-1}] = \mathbb{E} \left[ \sum_{t=1}^H r(s_t^k, a_t^k) \mid \mathcal{F}_{k-1} \right] + \sum_{n=1}^{\frac{H}{n}} \mathbb{E} [\bar{V}_{1+nh}^{k-1}(s_{1+nh}) \mid \mathcal{F}_{k-1}] + \frac{1}{2}H(H-1)\epsilon_P
\]

\[
\iff \bar{V}_1^k(s_1^k) + \sum_{n=1}^{\frac{H}{n}} \mathbb{E} [\bar{V}_{1+nh}^{k+1}(s_{1+nh}) \mid \mathcal{F}_{k-1}] = \bar{V}_1^k(s_1^k) + \sum_{n=1}^{\frac{H}{n}} \mathbb{E} [\bar{V}_{1+nh}^{k-1}(s_{1+nh}) \mid \mathcal{F}_{k-1}] + \frac{1}{2}H(H-1)\epsilon_P
\]

\[
\iff \bar{V}_1^k(s_1^k) - V^{\pi_k}(s_1^k) = \sum_{n=1}^{\frac{H}{n}} \mathbb{E} [\bar{V}_{1+nh}^{k-1}(s_{1+nh}) - \bar{V}_{1+nh}(s_{1+nh}) \mid \mathcal{F}_{k-1}] + \frac{1}{2}H(H-1)\epsilon_P
\]

\[
\iff \bar{V}_1^k(s_1^k) - V^{\pi_k}(s_1^k) = \sum_{n=1}^{\frac{H}{n}} \sum_{s=1}^{\frac{H}{n}} \bar{V}_{1+nh}^{k-1}(s_{1+nh}) - \mathbb{E} [\bar{V}_{1+nh+1}(s) \mid \mathcal{F}_{k-1}] + \frac{1}{2}H(H-1)\epsilon_P
\]

The second line holds by the fact $s_1^k$ is measurable w.r.t. $\mathcal{F}_{k-1}$, the third line holds since

\[
V_1^{\pi_k}(s_1^k) = \mathbb{E} \left[ \sum_{t=1}^H r(s_t^k, a_t^k) \mid \mathcal{F}_{k-1} \right].
\]

The forth line holds by Lemma 15 with $\bar{V}_t^k = g_t^k$ for $t = nh + 1$. See that the update of $\bar{V}_t^k$ occurs only at the visited state $s_t^k$ and the update rule uses $\bar{V}_{t+1}^{k-1}$, i.e., it is measurable w.r.t. $\mathcal{F}_{k-1}$, and it is valid to apply the lemma.

\[ \square \]

**Theorem 8** (Performance of $h$-RTDP with Approximate Model). Let $\epsilon, \delta > 0$. The following holds for $h$-RTDP-AM:

1. With probability $1 - \delta$, for all $K > 0$, we have,

\[
\text{Regret}(K) \leq \frac{9SH(H-h)}{h} \ln(3/\delta) + H(H-1)\epsilon_P K.
\]

2. Let $\Delta_P = H(H-1)\epsilon_P$. Then, we have,

\[
\Pr \left\{ \exists \epsilon > 0 : N^{\Delta_P}_r \geq \frac{9SH(H-h)\ln(3/\delta)}{h\epsilon} \right\} \leq \delta.
\]
Proof. We start by proving claim (1). The following bounds on the regret hold.

\[
\text{Regret}(K) := \sum_{k=1}^{K} V_{1}^*(s_{k}^1) - V_1^\pi_k(s_{k}^1)
\]

\[
\leq \sum_{k=1}^{K} \tilde{V}_{1}^*(s_{k}^1) - V_1^\pi_k(s_{k}^1) + \frac{H(H-1)}{2} \epsilon_P
\]

\[
\leq \sum_{k=1}^{K} \tilde{V}_1^k(s_{k}^1) - V_1^\pi_k(s_{k}^1) + \frac{H(H-1)}{2} \epsilon_P
\]

\[
= H(H-1)\epsilon_P K + \sum_{k=1}^{K} \sum_{n=1}^{H-1} \sum_{s} \tilde{V}_{nh+1}^k(s) - \mathbb{E}[\tilde{V}_{nh+1}^k(s) | F_{k-1}] \tag{17}
\]

The second relation holds by Lemma 17 which relates the optimal value of the approximate model to the optimal value of the environment. The third relation is by the optimism of the value function (Lemma 6), and the forth relation is by Lemma 7.

We now observe the regret is a regret of a Decreasing Bounded Process. Let

\[
X_k := \sum_{n=1}^{H-1} \sum_{s} \tilde{V}_{nh+1}^k(s), \tag{18}
\]

and observe that \(\{X_k\}_{g \geq 0}\) is a Decreasing Bounded Process.

1. It is decreasing since for all \(s, t\), \(\tilde{V}_t^k(s) \leq \tilde{V}_{t+1}^k(s)\) by Lemma 6. Thus, their sum is also decreasing.

2. It is bounded since for all \(s, t\), \(\tilde{V}_t^k(s) \geq V_t^* (s) \geq 0\) by Lemma 6. Thus, the sum is bounded from below by 0.

See that the initial value can be bounded as follows,

\[
X_0 = \sum_{n=1}^{H-1} \sum_{s} \tilde{V}_{nh+1}^0(s) \leq \sum_{n=1}^{H-1} \sum_{s} H = \frac{SH(H-h)}{h}.
\]

Using linearity of expectation and the definition (18) we observe that (17) can be written,

\[
\text{Regret}(K) \leq (17) = H(H-1)\epsilon_P K + \sum_{k=1}^{K} X_{k-1} - \mathbb{E}[X_k | F_{k-1}],
\]

which is regret of A Bounded Decreasing Process. Applying the regret bound on DBP, Theorem 1, we conclude the proof of the first claim.

We now prove the claim (2) using the proving technique at Theorem 5. Denote \(\Delta_P = H(H-1)\epsilon_P\). The following relations hold for all \(\epsilon > 0\).

\[
\mathbb{I}\left\{ \hat{V}_t^*(s_{t}^1) - V_1^\pi_k(s_{t}^1) \geq \frac{\Delta_P}{2} + \epsilon \right\} \geq \left( \epsilon + \frac{\Delta_P}{2} \right)
\]

\[
\mathbb{I}\left\{ \hat{V}_t^k(s_{t}^1) - V_1^\pi_k(s_{t}^1) \geq \frac{\Delta_P}{2} + \epsilon \right\} \geq \left( \epsilon + \frac{\Delta_P}{2} \right)
\]

\[
\mathbb{I}\left\{ \hat{V}_1^k(s_{1}^1) - V_1^\pi_k(s_{1}^1) \geq \frac{\Delta_P}{2} + \epsilon \right\} \hat{V}_1^k(s_{1}^1) - V_1^\pi_k(s_{1}^1)
\]

\[
= \mathbb{I}\left\{ \hat{V}_1^k(s_{1}^1) - V_1^\pi_k(s_{1}^1) \geq \frac{\Delta_P}{2} + \epsilon \right\} \left( \sum_{n=1}^{H-1} \sum_{s} \tilde{V}_{nh+1}^k(s) - \mathbb{E}[\tilde{V}_{nh+1}^k(s) | F_{k-1}] + \frac{\Delta_P}{2} \right)
\]

The first relation holds since for all \(t, s\), \(\hat{V}_t^k(s) \geq \hat{V}_t^*(s)\) by Lemma 6. The second relation holds by the indicator function and the third relation holds by Lemma 7. The forth relation holds by the definition of \(X_k\) (18) and linearity of expectation. Using an algebraic manipulation the above leads to the following relation,

\[
\mathbb{I}\left\{ \hat{V}_1^*(s_{1}^1) - V_1^\pi_k(s_{1}^1) \geq \frac{\Delta_P}{2} + \epsilon \right\} \leq \mathbb{I}\left\{ \hat{V}_1^k(s_{1}^1) - V_1^\pi_k(s_{1}^1) \geq \frac{\Delta_P}{2} + \epsilon \right\} (X_{k-1} - \mathbb{E}[X_k | F_{k-1}] - \frac{\Delta_P}{2}) \tag{19}
\]
As we wish the final performance to be compared to $V^*$ and not $\tilde{V}$ we use the the first claim of Lemma 17, by which for all $s$, $\hat{V}^*_1(s) \geq V^*_1(s) - \frac{\Delta_P}{2}$. This implies that

$$\mathbb{I}\{V^*_1(s^k_1) - V^*_{\pi_k}(s^k_1) \geq \Delta_P + \epsilon\} \leq \mathbb{I}\{\hat{V}^*_1(s^k_1) - V^*_{\pi_k}(s^k_1) \geq \frac{\Delta_P}{2} + \epsilon\}. \quad (20)$$

Combining all the above, we get

$$\mathbb{I}\{V^*_1(s^k_1) - V^*_{\pi_k}(s^k_1) \geq \Delta_P + \epsilon\}$

$$\leq \sum_{k=1}^K \mathbb{I}\{\hat{V}^*_1(s^k_1) - V^*_{\pi_k}(s^k_1) \geq \frac{\Delta_P}{2} + \epsilon\}(X_{k-1} - \mathbb{E}[X_k \mid \mathcal{F}_{k-1}]). \quad (21)$$

The first relation holds by (20) and the second relation by (19).

Define $N_\epsilon(K) = \sum_{k=1}^K \mathbb{I}\{V^*_1(s^k_1) - V^*_{\pi_k}(s^k_1) \geq \Delta_P + \epsilon\}$ as the number of times $V^*_1(s^k_1) - V^*_{\pi_k}(s^k_1) \geq \Delta_P + \epsilon$ at the first $K$ episodes. Summing the above inequality (21) for all $k \in [K]$ and denote we get that for all $\epsilon > 0$

$$N_\epsilon(K) \epsilon = \sum_{k=1}^K \mathbb{I}\{V^*_1(s^k_1) - V^*_{\pi_k}(s^k_1) \geq \Delta_P + \epsilon\} \epsilon$$

$$\leq \sum_{k=1}^K \mathbb{I}\{\hat{V}^*_1(s^k_1) - V^*_{\pi_k}(s^k_1) \geq \frac{\Delta_P}{2} + \epsilon\}(X_{k-1} - \mathbb{E}[X_k \mid \mathcal{F}_{k-1}])$$

$$\leq \sum_{k=1}^K X_{k-1} - \mathbb{E}[X_k \mid \mathcal{F}_{k-1}].$$

The first relation holds by definition, the second by (21) and the third relation holds as $\{X_k\}_{k \geq 0}$ is a DBP (18) and, thus, $X_{k-1} - \mathbb{E}[X_k \mid \mathcal{F}_{k-1}] \geq 0$ a.s. . Thus, the following relation holds

$$\left\{\forall K > 0 : \sum_{k=1}^K X_{k-1} - \mathbb{E}[X_k \mid \mathcal{F}_{k-1}] \leq \frac{9SH(H-h)}{h} \ln \frac{3}{\delta}\right\} \subseteq \left\{\forall \epsilon > 0 : N_\epsilon(K) \epsilon \leq \frac{9SH(H-h)}{h} \ln \frac{3}{\delta}\right\},$$

from which we get that for any $K > 0$

$$\Pr\left(\forall \epsilon > 0 : N_\epsilon(K) \epsilon \leq \frac{9SH(H-h)}{h} \ln \frac{3}{\delta}\right)$$

$$\geq \Pr\left(\forall K > 0 : \sum_{k=1}^K X_{k-1} - \mathbb{E}[X_k \mid \mathcal{F}_{k-1}] \leq \frac{9SH(H-h)}{h} \ln \frac{3}{\delta}\right) \geq 1 - \delta,$$

and the third relation holds the bound on the regret of DBP, Theorem 1. Equivalently, for any $K > 0$,

$$\Pr\left(\exists \epsilon > 0 : N_\epsilon(K) \epsilon \geq \frac{9SH(H-h)}{h} \ln \frac{3}{\delta}\right) \leq \delta. \quad (22)$$

Applying the Monotone Convergence Theorem as in the proof of Theorem 5 we conclude the proof.
C Multi-step Greedy Real Time Dynamic Programming with Approximate Value updates

Lemma 9. For all \( s \in S, n \in \{0\} \cup \left\lceil \frac{H}{h} \right\rceil, \) and \( k \in [K], \) it holds that (i) \( V^*_{t+h}(s) \leq V^k_{t+h}(s) + \epsilon V \left( \frac{H}{h} - n \right) \) and (ii) \( V^k_{t+h+1}(s) \leq V^k_{t+h+1}(s) \).

Proof. We prove the first claim by induction. The second claim holds by construction.

(i) Let \( n \in \{0\} \cup \left\lceil \frac{H}{h} \right\rceil. \) By the optimistic initialization, \( \forall s, n, V^*_{t+h}(s) - \epsilon V \left( \frac{H}{h} - n \right) \leq V^*_{t+h}(s) \leq V^k_{t+h}(s) \). Assume the claim holds for \( k - 1 \) episodes. Let \( s^k \) be the state the algorithm is at in the \( t = 1 + hn \) time-step of the \( k \)th episode, i.e., at a time-step in which a value update is taking place. Let \( e \in \mathbb{R}^S \) be the constant vector of ones. By the value update of Algorithm 3,

\[
\tilde{V}^k_t(s^k_t) = \min \left\{ \epsilon V(s^k_t) + T^h \tilde{V}^k_{t+h}(s^k_t), \tilde{V}^k_{t+1}(s^k_t) \right\}.
\]

If the minimal value is \( \tilde{V}^k_{t+1}(s^k_t) \) then \( \tilde{V}^k_t(s^k_t) \) satisfies the induction hypothesis by the induction assumption. If \( \epsilon V(s^k_t) + T^h \tilde{V}^k_{t+h}(s^k_t) \) is the minimal value in (23), then the following relation holds,

\[
\epsilon V(s^k_t) + T^h \tilde{V}^k_{t+h}(s^k_t) \\
\geq - \epsilon V + T^h \tilde{V}^{*k}_{t+h}(s^k_t) \\
\geq - \epsilon V + T^h (V^*_{t+h} - \epsilon V \left( \frac{H}{h} - n - 1 \right))(s^k_t) \\
= - \epsilon V + T^h V^*_{t+h}(s^k_t) - \epsilon V \left( \frac{H}{h} - n - 1 \right) \\
= T^h V^*_{t+h}(s^k_t) - \epsilon V \left( \frac{H}{h} - n \right) \\
= V^*_{t+h}(s^k_t) - \epsilon V \left( \frac{H}{h} - n \right).
\]

The second relation holds by the assumption \( |\epsilon V(s^k_t)| \leq \epsilon V \). The third relation by the induction hypothesis and the monotonicity of \( T^h \). The forth relation holds since for any constant \( \alpha \in \mathbb{R} \) and \( V \in \mathbb{R}^S, T(V + \alpha e) = TV + \alpha e \) (e.g., Bertsekas and Tsitsiklis1996) and thus \( T^h (V + \alpha e) = T^h V + \alpha \). Lastly, the fifth relation holds by the Bellman equations (2).

(ii) The second claim holds by construction of the update rule \( \tilde{V}^k_t(s^k_t) \leftarrow \min \{ \tilde{V}^k_t(s^k_t), \tilde{V}^k_{t+1}(s^k_t) \} \) which enforces \( \tilde{V}^k_t(s) \leq \tilde{V}^k_{t+1}(s) \) for every updated state, and thus for all \( s \) and \( t \).

Lemma 10. The expected cumulative value update at the \( k \)th episode of \( h \)-RTDP-AV satisfies the following relation:

\[
\tilde{V}^k_t(s^k_t) - V^k_t(s^k_t) \\
\leq \frac{H}{h} \epsilon V + \sum_{k=1}^{K} \sum_{n=1}^{\left\lceil \frac{H}{h} \right\rceil} \sum_{s \in S} \tilde{V}^{k-1}_{t+h+1}(s) - \mathbb{E}[\tilde{V}^k_{t+h+1}(s) | F_{k-1}].
\]

Proof. Let \( n \in \{0\} \cup \left\lceil \frac{H}{h} \right\rceil \) and let \( t = 1 + hn \) be a time-step in which a value update is taking place. By the definition of the update rule, the following holds for the update at the visited state \( s^k_t \):

\[
\tilde{V}^k_t(s^k_t) = \epsilon V(s^k_t) + (T^h \tilde{V}^{k-1}_{t+h})(s^k_t) \\
\leq \epsilon V + (T^h \epsilon V \cdots T^h \epsilon V)(s^k_t) \\
= \epsilon V + \mathbb{E} \left[ \sum_{t'=t}^{t+h-1} r(s^k_{t'}, a^k_{t'}) + \tilde{V}^{k-1}_{t+h}(s^k_{t+h}) | F_{k-1}, s^k_t \right].
\]

Where the third relation holds by the same argument as in (7). Taking the conditional expectation w.r.t. \( F_{k-1} \), using the tower property and the fact for all \( s, \tilde{V}^k_{t+h+1}(s) = 0 \) we get,

\[
\mathbb{E} \left[ \tilde{V}^k_t(s^k_t) | F_{k-1} \right] \leq \epsilon V + \mathbb{E} \left[ \sum_{t'=t}^{t+h-1} r(s^k_{t'}, a^k_{t'}) + \tilde{V}^{k-1}_{t+h}(s^k_{t+h}) | F_{k-1} \right].
\]
Summing the above relation for all \( n \in \{0\} \cup \frac{H}{h} - 1 \), using linearity of expectation, and the fact \( \bar{V}^k_{H+1}(s) = \) for all \( s, k \),

\[
\sum_{n=0}^{\frac{H}{h} - 1} \mathbb{E}[\bar{V}^k_{1+nh}(s^k) \mid \mathcal{F}_{k-1}] \leq \frac{H}{h} \epsilon_V + \mathbb{E} \left[ \sum_{t=1}^{H} r(s^t, a^t) \mid \mathcal{F}_{k-1} \right] + \frac{H}{h} \epsilon_V + \sum_{n=1}^{\frac{H}{h} - 1} \mathbb{E}[\bar{V}^k_{1+nh}(s^k_{1+nh}) \mid \mathcal{F}_{k-1}]
\]

\[
\iff \bar{V}^k_1(s^k_1) + \frac{H}{h} \epsilon_V + \mathbb{E} \left[ \sum_{t=1}^{H} r(s^t, a^t) \mid \mathcal{F}_{k-1} \right] + \frac{H}{h} \epsilon_V + \sum_{n=1}^{\frac{H}{h} - 1} \mathbb{E}[\bar{V}^k_{1+nh}(s^k_{1+nh}) \mid \mathcal{F}_{k-1}]
\]

\[
\iff \bar{V}^k_1(s^k_1) + \frac{H}{h} \epsilon_V + \mathbb{E} \left[ \sum_{t=1}^{H} r(s^t, a^t) \mid \mathcal{F}_{k-1} \right] + \frac{H}{h} \epsilon_V + \sum_{n=1}^{\frac{H}{h} - 1} \mathbb{E}[\bar{V}^k_{1+nh}(s^k_{1+nh}) \mid \mathcal{F}_{k-1}]
\]

The second line holds by the fact \( s^k_1 \) is measurable w.r.t. \( \mathcal{F}_{k-1} \), and the third line holds since

\[
V^\pi_1(s^k_1) = \mathbb{E} \left[ \sum_{t=1}^{H} r(s^t, a^t) \mid \mathcal{F}_{k-1} \right]
\]

The fifth line holds by Lemma 15 with \( \bar{V}^k_t = g^k_t \) for \( t = nh + 1 \). See that the update of \( \bar{V}^k_t \) occurs only at the visited state \( s^k_t \) and the update rule uses \( \bar{V}^{k-1}_{t+1} \), i.e., it is measurable w.r.t. to \( \mathcal{F}_{k-1} \), and it is valid to apply the lemma.

\[ \Box \]

**Theorem 11 (Performance of \( h \)-RTDP with Approximate Value Updates).** Let \( \epsilon, \delta > 0 \). The following holds for \( h \)-RTDP-AV:

1. With probability \( 1 - \delta \), for all \( K > 0 \), we have

\[
\text{Regret}(K) \leq \frac{9SH(H-h)}{h} \left( 1 + \frac{H}{h} \epsilon_V \right) \ln\left( \frac{3}{\delta} \right) + \frac{2H}{h} \epsilon_V K.
\]

2. Let \( \Delta_V = 2H \epsilon_V \). Then, we have

\[
\Pr \left\{ \exists \epsilon > 0 : N_{\epsilon} \Delta_V \geq \frac{9SH(H-h)(1 + \frac{H}{2h}) \ln(\frac{3}{\delta})}{h \epsilon} \right\} \leq \delta.
\]

**Proof.** We start by proving **claim (1)**. The following bounds on the regret hold.

\[
\text{Regret}(K) := \sum_{k=1}^{K} V^*_1(s^k_1) - V^\pi_1(s^k_1)
\]

\[
\leq \sum_{k=1}^{K} \bar{V}^k_1(s^k_1) - V^\pi_1(s^k_1) + \frac{H}{h} \epsilon_V
\]

\[
= \frac{2H}{h} \epsilon_V K + \sum_{k=1}^{K} \sum_{n=1}^{\frac{H}{h} - 1} \mathbb{E}[\bar{V}^k_{1+nh+1}(s) \mid \mathcal{F}_{k-1}]
\]

The second relation is by the approximated optimism of the value function when approximate value updates are used (Lemma 9). The third relation is by Lemma 10.

We now observe the regret is a regret of a Decreasing Bounded Process. Let

\[
X_k := \sum_{n=1}^{\frac{H}{h} - 1} \sum_{s} \bar{V}^k_{nh+1}(s),
\]

and observe that \( \{X_k\}_{q \geq 0} \) is a Decreasing Bounded Process.
1. It is decreasing since for all \( s, t \) \( \tilde{V}_t^k(s) \leq \tilde{V}_t^{k-1}(s) \) by Lemma 9. Thus, their sum is also decreasing.
2. It is bounded since for all \( s, n \in \left[ \frac{H}{n} \right] - 1, \)
\[
\tilde{V}_{t+n}^k(s) \geq V_{t+n}^*(s) - \epsilon_V \left( \frac{H}{n} - n \right) \geq -\epsilon_V \left( \frac{H}{n} - n \right) \geq -\epsilon_V \frac{H}{h}
\]
by Lemma 9. Thus, \( X_0 \) which is a sum of the above terms is bounded from below by \(-\epsilon_V \frac{SH(H-h)}{h}\).

See that the initial value can be bounded as follows,
\[
X_0 = \sum_{n=1}^{\frac{H}{n}} \sum_s \tilde{V}_{n+1}^0(s) \leq \sum_{n=1}^{\frac{H}{n}} \sum_s H = \frac{SH(H-h)}{h}.
\]

Using linearity of expectation and the definition (11) we observe that (24) can be written,
\[
\text{Regret}(K) \leq (24) = \frac{2H}{h} \epsilon_V K + \sum_{k=1}^{K} X_k - \mathbb{E}[X_k | \mathcal{F}_{k-1}],
\]
which is regret of A Bounded Decreasing Process. Applying the regret bound on DBP, Theorem 1 we conclude the proof of the first claim.

We now prove claim (2) using the proving technique at Theorem 5. Denote \( \Delta_V = 2H \epsilon_V \). The following relations hold for all \( \epsilon > 0 \).}

\[
\mathbb{I} \left\{ \tilde{V}_t^k(s^k_t) - V_1^*(s^k_t) \geq \frac{\Delta_V}{2h} + \epsilon \right\} (\epsilon + \frac{\Delta_V}{2h})
\]

\[
\leq \mathbb{I} \left\{ \tilde{V}_1^k(s^k_t) - V_1^*(s^k_t) \geq \frac{\Delta_V}{2h} + \epsilon \right\} (\tilde{V}_1^k(s^k_t) - V_1^*(s^k_t))
\]

\[
= \mathbb{I} \left\{ \tilde{V}_1^k(s^k_t) - V_1^*(s^k_t) \geq \frac{\Delta_V}{2h} + \epsilon \right\} \left( \sum_{n=1}^{\frac{H}{n}} \sum_s \tilde{V}_{n+1}^{k-1}(s) - \mathbb{E}[\tilde{V}_{n+1}^{k-1}(s) | \mathcal{F}_{k-1}] + \frac{\Delta_V}{2h} \right)
\]

\[
= \mathbb{I} \left\{ \tilde{V}_1^k(s^k_t) - V_1^*(s^k_t) \geq \frac{\Delta_V}{2h} + \epsilon \right\} \left( X_k - \mathbb{E}[X_k | \mathcal{F}_{k-1}] + \frac{\Delta_V}{2h} \right).
\]

The first relation holds by the indicator function and the second relation by Lemma 10. The third relation holds by the definition of \( X_k \) (25) and linearity of expectation. Using an algebraic manipulation the above leads to the following relation,
\[
\mathbb{I} \left\{ \tilde{V}_1^k(s^k_t) - V_1^*(s^k_t) \geq \frac{\Delta_V}{2h} + \epsilon \right\} \leq \mathbb{I} \left\{ \tilde{V}_1^k(s^k_t) - V_1^*(s^k_t) \geq \frac{\Delta_V}{2h} + \epsilon \right\} (X_k - \mathbb{E}[X_k | \mathcal{F}_{k-1}]) (26)
\]

As we wish the final performance to be compared to \( V^* \) we use the the first claim of Lemma 9, by which for all \( s, k, \)
\( \tilde{V}_1^k(s) \geq V_1^*(s) - \frac{\Delta_V}{2h}. \) This implies that
\[
\mathbb{I} \left\{ V_1^*(s^k_t) - V_1^*(s^k_t) \geq \frac{\Delta_V}{h} + \epsilon \right\} \leq \mathbb{I} \left\{ V_1^*(s^k_t) - V_1^*(s^k_t) \geq \frac{\Delta_V}{2h} + \epsilon \right\} \leq \mathbb{I} \left\{ V_1^*(s^k_t) - V_1^*(s^k_t) \geq \frac{\Delta_V}{2h} + \epsilon \right\} (X_k - \mathbb{E}[X_k | \mathcal{F}_{k-1}]), (27)
\]

Combining the above we get
\[
\mathbb{I} \left\{ V_1^*(s^k_t) - V_1^*(s^k_t) \geq \frac{\Delta_V}{h} + \epsilon \right\} \leq \mathbb{I} \left\{ V_1^*(s^k_t) - V_1^*(s^k_t) \geq \frac{\Delta_V}{2h} + \epsilon \right\} \leq \mathbb{I} \left\{ V_1^*(s^k_t) - V_1^*(s^k_t) \geq \frac{\Delta_V}{2h} + \epsilon \right\} (X_k - \mathbb{E}[X_k | \mathcal{F}_{k-1}]). (28)
\]

The first relation is by (27) and the second relation by (26).
Define \( N_\epsilon(K) = \sum_{k=1}^{K} 1 \{ V_1^*(s^k) - V_1^{\pi_k}(s^k) \geq \frac{\Delta_V}{h} + \epsilon \} \) as the number of times \( V_1^*(s^k) - V_1^{\pi_k}(s^k) \geq \frac{\Delta_V}{h} + \epsilon \) at the first \( K \) episodes. Summing the above inequality (28) for all \( k \in [K] \) and denote we get that for all \( \epsilon > 0 \)

\[
N_\epsilon(K) \epsilon = \sum_{k=1}^{K} 1 \left\{ V_1^*(s^k) - V_1^{\pi_k}(s^k) \geq \frac{\Delta_V}{h} + \epsilon \right\} \epsilon
\]

\[
\leq \sum_{k=1}^{K} 1 \left\{ \bar{V}_1^k(s^k) - V_1^{\pi_k}(s^k) \geq \frac{\Delta_V}{2h} + \epsilon \right\} (X_{k-1} - E[X_k | \mathcal{F}_{k-1}])
\]

\[
\leq \sum_{k=1}^{K} X_{k-1} - E[X_k | \mathcal{F}_{k-1}].
\]

The first relation holds by definition, the second by (28) and the third relation holds as \( \{X_k\}_{k \geq 0} \) is a DBP (25) and, thus, \( X_{k-1} - E[X_k | \mathcal{F}_{k-1}] \geq 0 \) a.s. Thus, the following relation holds

\[
\forall K > 0 : \sum_{k=1}^{K} X_{k-1} - E[X_k | \mathcal{F}_{k-1}] \leq \frac{9SH(H - h)}{h} (1 + \frac{H}{h} \epsilon_V) \ln \frac{3}{\delta}
\]

\[
\subseteq \left\{ \forall \epsilon > 0 : N_\epsilon(K) \epsilon \leq \frac{9SH(H - h)}{h} (1 + \frac{H}{h} \epsilon_V) \ln \frac{3}{\delta} \right\},
\]

from which we get for any \( K > 0 \)

\[
\Pr \left( \forall \epsilon > 0 : N_\epsilon(K) \epsilon \leq \frac{9SH(H - h)}{h} (1 + \frac{H}{h} \epsilon_V) \ln \frac{3}{\delta} \right) \geq \Pr \left( \forall K > 0 : \sum_{k=1}^{K} X_{k-1} - E[X_k | \mathcal{F}_{k-1}] \leq \frac{9SH(Hh)}{h} (1 + \frac{H}{h} \epsilon_V) \ln \frac{3}{\delta} \right) \geq 1 - \delta,
\]

and the third relation holds the bound on the regret of DBP, Theorem 1. Equivalently, for any \( K > 0 \),

\[
\Pr \left( \exists \epsilon > 0 : N_\epsilon(K) \epsilon \geq \frac{9SH(H - h)}{h} (1 + \frac{H}{h} \epsilon_V) \ln \frac{3}{\delta} \right) \leq \delta.
\] (29)

Applying the Monotone Convergence Theorem as in the proof of Theorem 5 we conclude the proof.
D Multi-step Greedy Real Time Dynamic Programming with Approximate State Abstraction

In this section we analyze the performance of $h$-RTDP performance which uses approximate abstraction. For clarity we restate the assumption we make on the approximate abstraction and the definition of equivalent set under abstraction.

**Assumption 1 (Approximate Abstraction).** For any $s, s' \in S$ and $n \in \{0\} \cup \lceil \frac{H}{n} \rceil - 1$ for which $\phi_{nh+1}(s) = \phi_{nh+1}(s')$, we have $|V^*_{nh+1}(s) - V^*_{nh+1}(s')| \leq \epsilon_A$.

**Definition 1 (Equivalent Set Under Abstraction).** For any $s \in S$ and $n \in \{0\} \cup \lceil \frac{H}{n} \rceil - 1$, we define the set of states equivalent to $s$ under $\phi_{nh+1}$ as $\Phi_{nh+1}(s) := \{s' \in S : \phi_{nh+1}(s) = \phi_{nh+1}(s')\}$.

Before we supply with the proof we emphasize an important difference in the definition of the value function $\bar{V}^k_t$ when using abstraction. Unlike the usual definition of $\bar{V}^k_t : S \rightarrow \mathbb{R}$, in case of abstraction $\bar{V}^k_t$ is a mapping from the abstract state space to the reals, i.e., $\bar{V}^k_{\phi_t} : \Phi_0 \rightarrow \mathbb{R}$. Meaning, $\bar{V}^k_{\phi_t}$ is defined on the abstract state space. Given a state $s \in S$ we need to query $\phi_t$ to obtain its value at time $t$ by $\bar{V}^k_t(\phi_t(s))$.

**Lemma 12.** For all $s \in S$, $n \in \{0\} \cup \lceil \frac{H}{n} \rceil$, and $k \in [K]$, it holds that (i) $\max_{s' \in \Phi_{nh+1}(s)} V^*_{nh+1}(s') \leq \bar{V}^k_{nh+1}(\phi_{nh+1}(s)) + \epsilon_A(\frac{H}{n} - n)$, (ii) $\bar{V}^k_{nh+1}(\phi_{nh+1}(s)) \leq \bar{V}^{k-1}_{nh+1}(\phi_{nh+1}(s))$, and (iii) $\bar{V}^k_{nh+1}(\phi_{nh+1}(s)) \geq 0$.

**Proof.** We prove the first claim by induction. The second and third claims hold by construction.

(i) Let $n \in \{0\} \cup \lceil \frac{H}{n} \rceil - 1$. By the optimistic initialization, $\forall s, n$, $\bar{V}^1_{nh+1}(s) - \epsilon_A(\frac{H}{n} - n) \leq \bar{V}^0_{nh+1}(s) \leq V^0_{\phi_{nh+1}}(s)$.

Assume the claim holds for $k - 1$ episodes. Let $s^k_t$ be the state the algorithm is at in the $t = 1 + nh$ time-step of the $k$th episode, i.e., at a time-step in which a value update is taking place. By the value update of Algorithm 4,

$$\bar{V}^k_t(\phi_t(s^k_t)) = \min \{T^h \bar{V}^{k-1}_{nh+1}(s^k_t), \bar{V}^{k-1}_t(\phi(s^k_t))\}.$$  \hfill (30)

If the minimal value is $\bar{V}^{k-1}_{nh+1}(\phi(s^k_t))$ then $\bar{V}^k_t(\phi(s^k_t))$ satisfies the induction hypothesis by the induction assumption. If $T^h \bar{V}^{k-1}_{nh+1}(s^k_t)$ is the minimal value in (30), then the following relation holds,

$$\bar{V}^k_t(\phi_t(s^k_t)) = \max_{s_0 = s^k_t} \sum_{t' = 0}^{h-1} r(s^k_t, \pi_t(s_{t'}^k)) + \bar{V}^{k-1}_t(\phi(s^k_t))$$

$$\geq \max_{s_0 = s^k_t} \sum_{t' = 0}^{h-1} r(s_t^k, \pi_t(s_{t'}^k)) + \max_{s_0 = s^k_t} V^*_{t+1,h}(s_{t+1}) - \epsilon_A \left( \frac{H}{h} - n - 1 \right)$$

$$= V^*_{t}(s^k_t) - \epsilon_A \left( \frac{H}{h} - n - 1 \right)$$

(ii) The second claim holds by construction of the update rule $\bar{V}^k_t(s^k_t) \leftarrow \min \{\bar{V}^{k-1}_t(s^k_t), \bar{V}^{k-1}_t(s^k_t)\}$ which enforces $\bar{V}^k_t(s) \leq \bar{V}^{k-1}_t(s)$ for every updated state, and thus for all $s$ and $t$. 

The first relation is the definition of the update rule. The second relation holds by the monotonicity of the max operator together with the induction assumption. The third relation as the extracted term out of the max is constant. The forth relation holds by the definition of the max operation. The fifth relation by the Bellman equations $V^*_{t}$ satisfies (2), and the sixth relation by Assumption 1.
(iii) The third claim holds since \( V_t^k(\phi_t(s)) \) is initialized with positive elements and is updated by itself and positive elements, as \( r(s, a) \geq 0 \). Thus, it remains positive a.s.

Lemma 13. The expected cumulative value update at the \( k' \)th episode of \( h \)-RTDP-AA satisfies the following relation:

\[
\bar{V}_t^k(\phi_t(s_k^t)) - V_0^\pi_k(s_k^t) \\
\leq \sum_{k=1}^K \frac{H}{k} \sum_{n=1}^{H-1} \sum_{s \in S_\phi} \bar{V}_{n+1}^{k-1}(s) - \mathbb{E}[\bar{V}_{n+1}^{k}(s) | \mathcal{F}_{k-1}].
\]

Proof. Let \( n \in \{0\} \cup \left[ \frac{H}{k} - 1 \right] \) and let \( t = 1 + nh \) be a time-step in which a value update is taking place. By the definition of the update rule, the following holds for the update at the visited state \( s_k^t \):

\[
\bar{V}_t^k(\phi_t(s_k^t)) \leq \mathbb{E}\left[ \sum_{t'=t}^{t+h-1} r(s_{t'}, a_{t'}) + \bar{V}_{t+h}^{k-1}(\phi_{t+h}(s_{t+h})) | \pi_k, s_k^t \right]
\]

\[
= \mathbb{E}\left[ \sum_{t'=t}^{t+h-1} r(s_{t'}, a_{t'}) + \bar{V}_{t+h}^{k-1}(\phi_{t+h}(s_{t+h})) | \mathcal{F}_{k-1}, s_k^t \right]
\]

where the last relation follows by the same argument as in (7).

Taking the conditional expectation w.r.t. \( \mathcal{F}_{k-1} \) and using the tower property we get,

\[
\mathbb{E}[\bar{V}_t^k(\phi_t(s_k^t)) | \mathcal{F}_{k-1}] \leq \mathbb{E}\left[ \sum_{t'=t}^{t+h-1} r(s_{t'}, a_{t'}) + \bar{V}_{t+h}^{k-1}(\phi_{t+h}(s_{t+h})) | \mathcal{F}_{k-1} \right].
\]

Denote \( s_{\phi,t}^k := \phi_t(s_k^t) \). Summing the above relation for all \( n \in \{0\} \cup \left[ \frac{H}{k} - 1 \right] \), using linearity of expectation, and the fact \( \bar{V}_{H+1}^k(\phi_{H+1}(s)) = 0 \) for all \( s, k \),

\[
\frac{H}{k} \sum_{n=0}^{H-1} \mathbb{E}[\bar{V}_1^{k+1}(s_{\phi,1+n+h}) | \mathcal{F}_{k-1}] \leq \mathbb{E}\left[ \sum_{t=1}^{H} r(s_t^k, a_t^k) | \mathcal{F}_{k-1} \right] + \frac{H}{k} \sum_{n=1}^{H-1} \mathbb{E}[\bar{V}_1^{k-1}(s_{\phi,1+n+h}) | \mathcal{F}_{k-1}]
\]

\[
\iff \bar{V}_1^k(s_{\phi,1}) + \frac{H}{k} \sum_{n=1}^{H-1} \mathbb{E}[\bar{V}_1^{k-1}(s_{\phi,1+n+h}) | \mathcal{F}_{k-1}] \leq \mathbb{E}\left[ \sum_{t=1}^{H} r(s_t^k, a_t^k) | \mathcal{F}_{k-1} \right] + \mathbb{E}[\bar{V}_1^{k-1}(s_{\phi,1+n+h}) | \mathcal{F}_{k-1}]
\]

\[
\iff \bar{V}_1^k(s_{\phi,1}) - \mathbb{E}[\bar{V}_1^{k-1}(s_{\phi,1+n+h}) | \mathcal{F}_{k-1}] \leq \mathbb{E}\left[ \sum_{t=1}^{H} r(s_t^k, a_t^k) | \mathcal{F}_{k-1} \right] - \mathbb{E}[\bar{V}_1^{k-1}(s_{\phi,1+n+h}) | \mathcal{F}_{k-1}]
\]

The second line holds by the fact \( s_k^t \) is measurable w.r.t. \( \mathcal{F}_{k-1} \), the third line holds since

\[
\bar{V}_{H+1}^k(s_{\phi}^k) = \mathbb{E}\left[ \sum_{t=1}^{H} r(s_t^k, a_t^k) | \mathcal{F}_{k-1} \right].
\]

The fifth line holds by Lemma 15 with \( \bar{V}_t^k = \frac{H}{k} \) for \( t = nh + 1 \). Furthermore, we set \( \mathcal{S} \) of Lemma 15 to be \( S_\phi \). See that the update of \( \bar{V}_t^k \) occurs only at the visited state \( s_{\phi,t}^k = \phi(s_k^t) \) of the abstracted state space. Furthermore, the update rule uses \( \bar{V}_{\phi,t+1}^{k-1} \), i.e., it is measurable w.r.t. to \( \mathcal{F}_{k-1} \), and it is valid to apply the lemma.

\[\square\]

Theorem 14 (Performance of \( h \)-RTDP with Approximate State Abstraction). Let \( \epsilon, \delta > 0 \). The following holds for \( h \)-RTDP-AA:
1. With probability $1 - \delta$, for all $K > 0$, we have

$$\text{Regret}(K) \leq \frac{9S_H H(H - h)}{h} \ln(3/\delta) + \frac{H \epsilon}{h} K.$$ 

2. Let $\Delta_A = H \epsilon_A$. Then, we have

$$\Pr \left\{ \exists \epsilon > 0 : N_{\epsilon A}^{\Delta_A} \geq \frac{9S_H H(H - h) \ln(3/\delta)}{h \epsilon} \right\} \leq \delta.$$

Before supplying with the proof observe the following remark.

Proof. We start by proving \textbf{claim (1)}. The following bounds on the regret hold.

$$\text{Regret}(K) := \sum_{k=1}^{K} V_1^* \left( s_1^k \right) - V_1^{\pi_k} \left( s_1^k \right) \leq \sum_{k=1}^{K} \max_{s \in \Phi_k(s_1^k)} V_1^* (s) - V_1^{\pi_k} (s_1^k) \leq \sum_{k=1}^{K} \tilde{V}_1^{\pi_k} (s_1^k) - V_1^{\pi_k} (s_1^k) + \epsilon_A \frac{H}{h} \leq \epsilon_A \frac{H}{h} K + \sum_{k=1}^{K} \sum_{n=1}^{\frac{H}{h} - 1} \sum_{s_\phi \in S_\phi} \tilde{V}_{nh+1}^{k-1} (s_\phi) - \mathbb{E}[\tilde{V}_{nh+1}^{k} (s_\phi) | \mathcal{F}_{k-1}] (31)$$

The second relation holds the definition of the $\max$ operator and since $s_1^k \in \phi_k(s_1^k)$ (by definition we have that $s \in \Phi_k(s_1^k)$, as $\phi_t(s) = \phi_t(s)$ for any $t$). The third relation holds by the approximate optimism of the value function (Lemma 12), and the forth relation is by Lemma 13.

We now observe the regret is a regret of a Decreasing Bounded Process. Let

$$X_k := \sum_{n=1}^{\frac{H}{h} - 1} \sum_{s_\phi \in S_\phi} \tilde{V}_{nh+1}^{k} (s_\phi), \quad (32)$$

and observe that $\{X_k\}_{g \geq 0}$ is a Decreasing Bounded Process.

1. It is decreasing since for all $s_\phi \in S_\phi$, $\tilde{V}_1^{k} (s_\phi) \leq \tilde{V}_1^{k-1} (s_\phi)$ by Lemma 12. Thus, their sum is also decreasing.
2. It is bounded since for all $s \in S_\phi$, $\tilde{V}_1^{k} (s_\phi) \geq 0$ by Lemma 12. Thus, the sum is bounded from below by 0.

See that the initial value can be bounded as follows,

$$X_0 = \sum_{n=1}^{\frac{H}{h} - 1} \sum_{s_\phi \in S_\phi} \tilde{V}_{nh+1}^{0} (s_\phi) \leq \sum_{n=1}^{\frac{H}{h} - 1} \sum_{s_\phi \in S_\phi} \frac{H}{h} = S_H H (H - h).$$

Using linearity of expectation and the definition (11) we observe that (31) can be written,

$$\text{Regret}(K) \leq (31) = \epsilon_A \frac{H}{h} K + \sum_{k=1}^{K} X_{k-1} - \mathbb{E}[X_k | \mathcal{F}_{k-1}],$$

which is regret of A Bounded Decreasing Process. Applying the bound on the regret of a DRP, Theorem 1, we conclude the proof of the first claim.

We now prove \textbf{claim (2)} using the proving technique at Theorem 5. Denote $\Delta_A = H \epsilon_A$. The following relations hold for all $\epsilon > 0$.

$$\mathbb{I} \left\{ \tilde{V}_1^{k} (\phi_1(s_1^k)) - V_1^{\pi_k} (s_1^k) \geq \epsilon \right\} \leq \mathbb{I} \left\{ \tilde{V}_1^{k} (\phi_1(s_1^k)) - V_1^{\pi_k} (s_1^k) \geq \epsilon \right\} \left( \tilde{V}_1^{k} (\phi_1(s_1^k)) - V_1^{\pi_k} (s_1^k) \right)$$

$$\leq \mathbb{I} \left\{ \tilde{V}_1^{k} (\phi_1(s_1^k)) - V_1^{\pi_k} (s_1^k) \geq \epsilon \right\} \left( \sum_{n=1}^{\frac{H}{h} - 1} \sum_{s_\phi \in S_\phi} \tilde{V}_{nh+1}^{k-1} (s_\phi) - \mathbb{E}[\tilde{V}_{nh+1}^{k} (s_\phi) | \mathcal{F}_{k-1}] \right)$$

$$= \mathbb{I} \left\{ \tilde{V}_1^{k} (s_1^k) - V_1^{\pi_k} (s_1^k) \geq \epsilon \right\} (X_{k-1} - \mathbb{E}[X_k | \mathcal{F}_{k-1}]). \quad (33)$$
The first relation holds by the indicator function and the second relation holds by Lemma 13. The forth relation holds by the definition of $X_k$ (32) and linearity of expectation.

As we wish the final performance to be compared to $V^*$ we use the first claim of Lemma 12, by which for all $s, k$, $V^k_1(\phi_1(s)) \geq V^*_1(s) - \frac{\Delta A}{h}$. This implies that
\[
\mathbb{1}\left\{ V^*_1(s^k_1) - V^{\pi_k}_1(s^k_1) \geq \frac{\Delta A}{h} + \epsilon \right\} \leq \mathbb{1}\left\{ \bar{V}^k_1(\phi_1(s^k_1)) - V^{\pi_k}_1(s^k_1) \geq \epsilon \right\}. \tag{34}
\]

Combining the above we get
\[
\mathbb{1}\left\{ V^*_1(s^k_1) - V^{\pi_k}_1(s^k_1) \geq \frac{\Delta A}{h} + \epsilon \right\} \leq \mathbb{1}\left\{ \bar{V}^k_1(\phi_1(s^k_1)) - V^{\pi_k}_1(s^k_1) \geq \epsilon \right\}(X_{k-1} - \mathbb{E}[X_k \mid \mathcal{F}_{k-1}]). \tag{35}
\]
The first relation is by (34) and the second relation by (33).

Define $N_\epsilon(K) = \sum_{k=1}^{K} \mathbb{1}\left\{ V^*_1(s^k_1) - V^{\pi_k}_1(s^k_1) \geq \frac{\Delta A}{h} + \epsilon \right\}$ as the number of times $V^*_1(s^k_1) - V^{\pi_k}_1(s^k_1) \geq \frac{\Delta A}{h} + \epsilon$ at the first $K$ episodes. Summing the above inequality (35) for all $k \in [K]$ and denote we get that for all $\epsilon > 0$
\[
N_\epsilon(K) = \sum_{k=1}^{K} \mathbb{1}\left\{ V^*_1(s^k_1) - V^{\pi_k}_1(s^k_1) \geq \frac{\Delta A}{h} + \epsilon \right\} \leq \left( X_{k-1} - \mathbb{E}[X_k \mid \mathcal{F}_{k-1}] \right).
\]
The first relation holds by definition, the second by (35) and the third relation holds as $\{X_k\}_{k \geq 0}$ is a DBP (32) and, thus, $X_{k-1} - \mathbb{E}[X_k \mid \mathcal{F}_{k-1}] \geq 0$ a.s. . Thus, the following relation holds
\[
\left\{ \forall K > 0 : \sum_{k=1}^{K} X_{k-1} - \mathbb{E}[X_k \mid \mathcal{F}_{k-1}] \leq \frac{9SH(H-h)}{h} \ln \frac{3}{\delta} \right\} \subseteq \left\{ \forall \epsilon > 0 : N_\epsilon(K) \leq \frac{9SH(H-h)}{h} \ln \frac{3}{\delta} \right\},
\]
from which we get that for any $K > 0$
\[
\Pr\left( \forall \epsilon > 0 : N_\epsilon(K) \leq \frac{9SH(H-h)}{h} \ln \frac{3}{\delta} \right) \geq \Pr\left( \forall K > 0 : \sum_{k=1}^{K} X_{k-1} - \mathbb{E}[X_k \mid \mathcal{F}_{k-1}] \leq \frac{9SH(H-h)}{h} \ln \frac{3}{\delta} \right) \geq 1 - \delta,
\]
and the third relation holds the bound on the regret of DBP, Theorem 1. Equivalently, for any $K > 0$,
\[
\Pr\left( \exists \epsilon > 0 : N_\epsilon(K) \geq \frac{9SH(H-h)}{h} \ln \frac{3}{\delta} \right) \leq \delta. \tag{36}
\]
Applying the Monotone Convergence Theorem as in the proof of Theorem 5 we conclude the proof. \qed
E Useful Lemmas

The following lemma is a generalization of Lemma 34 in Efroni et al. (2019b).

Lemma 15 (On Trajectory Regret to Uniform Regret). For any \( t \in [H] \), let \( \{s_t^k, \mathcal{F}_k\}_{k \geq 0} \) be a random process where \( \{s_t^k\}_{k \geq 0} \) is adapted to the filtration \( \{\mathcal{F}_k\}_{k \geq 0} \) and \( s_t^k \in \hat{S} \) where \( \hat{S} \) is a finite set of all possible realizations of \( s_t^k \) with cardinality \( \hat{S} := |\hat{S}| \). Let \( g_t^k \in \mathbb{R}^\hat{S} \) and denoting the \( s \in \hat{S} \) entry of the vector as \( g_t^k(s) \). Furthermore, let \( g_t^k(s) \) be updated only at the state \( s_t^k \) by an update rule which is \( \mathcal{F}_{k-1} \) measurable, i.e.,

\[
g_t^k(s) = \begin{cases} f_t^{k-1}(s), & \text{if } s = s_t^k; \\ g_t^{k-1}(s), & \text{o.w..} \end{cases}
\]

Where \( f_t^{k-1}(s) \) is an update rule \( \mathcal{F}_{k-1} \) measurable. Then,

\[
\sum_{k=1}^K \mathbb{E}[g_t^{k-1}(s_t^k) - g_t^k(s_t^k) \mid \mathcal{F}_{k-1}] = \sum_{k=1}^K \sum_{s \in \hat{S}} g_t^{k-1}(s) - \mathbb{E}[g_t^k(s) \mid \mathcal{F}_{k-1}]
\]

Proof. The following relations hold.

\[
\sum_{k=1}^K \sum_{t=1}^H \mathbb{E}[g_t^{k-1}(s_t^k) - g_t^k(s_t^k) \mid \mathcal{F}_{k-1}] = \sum_{k=1}^K \sum_{t=1}^H \sum_{s \in \hat{S}} \mathbb{E}[\mathbb{1}\{s = s_t^k\} g_t^{k-1}(s) - \mathbb{1}\{s = s_t^k\} g_t^k(s) \mid \mathcal{F}_{k-1}]
\]

\[
= \sum_{k=1}^K \sum_{t=1}^H \sum_{s \in \hat{S}} \mathbb{E}[\mathbb{1}\{s = s_t^k\} f_t^{k-1}(s) - \mathbb{1}\{s = s_t^k\} g_t^k(s) \mid \mathcal{F}_{k-1}]
\]

\[
= \sum_{k=1}^K \sum_{t=1}^H \sum_{s \in \hat{S}} \mathbb{E}[\mathbb{1}\{s = s_t^k\} f_t^{k-1}(s) + \mathbb{1}\{s = s_t^k\} g_t^k(s) \mid \mathcal{F}_{k-1}]
\]

\[
= \sum_{k=1}^K \sum_{t=1}^H \sum_{s \in \hat{S}} \mathbb{E}[g_t^{k-1}(s) - \mathbb{E}[g_t^k(s) \mid \mathcal{F}_{k-1}]].
\] (37)

Relation (1) holds since for \( s = s_t^k \) the vector \( g_t^k \) is updated according by \( f_t^{k-1} \). Relation (2) holds by adding and subtracting \( \mathbb{1}\{s \neq s_t^k\} g_t^{k-1}(s) \) while using the linearity of expectation. (3) holds since for any event \( \mathbb{1}\{A\} + \mathbb{1}\{A^c\} = 1 \) and since \( g_t^{k-1} \) is \( \mathcal{F}_{k-1} \) measurable. (4) holds by the definition of the update rule,

\[
\mathbb{E}[\mathbb{1}\{s = s_t^k\} f_t^{k-1}(s) + \mathbb{1}\{s = s_t^k\} g_t^k(s) \mid \mathcal{F}_{k-1}]
\]

\[
= \mathbb{E}[\mathbb{1}\{s = s_t^k\} \mid \mathcal{F}_{k-1}] f_t^{k-1}(s) + \mathbb{E}[\mathbb{1}\{s = s_t^k\} \mid \mathcal{F}_{k-1}] g_t^k(s)
\]

\[
= \mathbb{P}(s_t^k = s \mid \mathcal{F}_{k-1}) f_t^{k-1}(s) + \mathbb{P}(s_t^k = s \mid \mathcal{F}_{k-1}) g_t^k(s).
\]

Where we used that \( g_t^{k-1}(s) \) is \( \mathcal{F}_{k-1} \) measurable and the assumption that \( f_t^{k-1}(s) \) is \( \mathcal{F}_{k-1} \) measurable in the first relation.

The following lemma is a variant of a well known error propagation analysis in case of an approximate model.

Lemma 16 (Model Error Propagation). Let \( \|P(\cdot \mid s, a) - \hat{P}(\cdot \mid s, a)\| \leq \epsilon_P \) for any \( s, a \). Then, for any policy \( \pi \),

\[
\forall s_1 \in S, \sum_s \left| P(\pi(s_n \mid s_1)) - \hat{P}(\pi(s_n \mid s_1)) \right| \leq n \epsilon_P
\]
Proof. We prove the claim by induction. For the base case \( n = 1 \) we get that for any \( s_1 \in \mathcal{S} \)

\[
\sum_{s_2} \left| P^\pi(s_2 \mid s_1) - \hat{P}^\pi(s_2 \mid s_1) \right|
\]

\[
= \sum_{s_2} \sum_a \pi(a \mid s_1) \left| P(s_2 \mid s_1, a) - \hat{P}^\pi(s_2 \mid s_1, a) \right|
\]

\[
\leq \sum_a \pi(a \mid s_1) \sum_{s_2} \left| P(s_2 \mid s_1, a) - \hat{P}^\pi(s_2 \mid s_1, a) \right|
\]

\[
= \sum_a \pi(a \mid s_1) \left| P(\cdot \mid s_1, a) - P(\cdot \mid s_1, a) \right|_1 \leq \epsilon_P.
\]

Assume the induction step, i.e., assume the claim holds for \( k = n - 1 \). We now prove the induction step, i.e., for \( k = n \)

\[
\sum_{s_n} \left| P^\pi(s_n \mid s_1) - \hat{P}^\pi(s_n \mid s_1) \right|
\]

\[
= \sum_{s_n} \sum_{s_2} P^\pi(s_n \mid s_2) P^\pi(s_2 \mid s_1) - \hat{P}^\pi(s_n \mid s_2) \hat{P}^\pi(s_2 \mid s_1)
\]

\[
\leq \sum_{s_n} \sum_{s_2} P^\pi(s_n \mid s_2) P^\pi(s_2 \mid s_1) - \hat{P}^\pi(s_n \mid s_2) \hat{P}^\pi(s_2 \mid s_1)
\]

\[
\leq \sum_{s_n} \sum_{s_2} P^\pi(s_n \mid s_2) P^\pi(s_2 \mid s_1) - \hat{P}^\pi(s_n \mid s_2) \hat{P}^\pi(s_2 \mid s_1)
\]

\[
+ \hat{P}^\pi(s_n \mid s_2) \hat{P}^\pi(s_2 \mid s_1) - P^\pi(s_2 \mid s_1)
\]

\[
\leq \sum_{s_2} \sum_{s_n} P^\pi(s_2 \mid s_1) \left( \max_{s_2'} \sum_{s_n} P^\pi(s_n \mid s_2') - \hat{P}^\pi(s_n \mid s_2') \right)
\]

\[
+ \sum_{s_2} \left( \sum_{s_n} \hat{P}^\pi(s_n \mid s_2) \right) \left| \hat{P}^\pi(s_2 \mid s_1) - P^\pi(s_2 \mid s_1) \right|
\]

\[
= \max_{s_2} \sum_{s_n} \left| P^\pi(s_n \mid s_2) - \hat{P}^\pi(s_n \mid s_2) \right| + \sum_{s_2} \left| \hat{P}^\pi(s_2 \mid s_1) - P^\pi(s_2 \mid s_1) \right|
\]

By the induction hypothesis and the base case,

\[
\max_{s_2} \sum_{s_n} \left| P^\pi(s_n \mid s_2') - \hat{P}^\pi(s_n \mid s_2') \right| \leq \epsilon(n - 1)
\]

\[
\sum_{s_2} \left| \hat{P}^\pi(s_2 \mid s_1) - P^\pi(s_2 \mid s_1) \right| \leq \epsilon_P,
\]

from which we prove the induction step,

\[
\forall s_1 \in \mathcal{S}, \left\| P^\pi(\cdot \mid s_1) - \hat{P}^\pi(\cdot \mid s_1) \right\| = \sum_{s_n} \left| P^\pi(s_n \mid s_1) - \hat{P}^\pi(s_n \mid s_1) \right| \leq n \epsilon_P.
\]

□
Lemma 17. Let $V^*_t(s), \hat{V}^*_t(s)$ be the optimal values on the MDP $\mathcal{M}, \hat{\mathcal{M}}$, respectively, and let $V^*_\pi(s), \hat{V}^*_\pi(s)$ be the value of a fixed policy $\pi$ on the MDP $\mathcal{M}, \hat{\mathcal{M}}$, respectively. Then,

\begin{align*}
&i) \|V^*_t - \hat{V}^*_t\|_\infty \leq \frac{H(H - 1)}{2} \epsilon_P, \\
&ii) \forall \pi, \|V^*_\pi - \hat{V}^*_\pi\|_\infty \leq \frac{H(H - 1)}{2} \epsilon_P.
\end{align*}

Proof. Both claims follow standard techniques based on the Simulation Lemma (Kearns and Singh 2002; Strehl, Li, and Littman 2009).

(i) Let $\Delta_t(s) := \hat{V}^*_t(s) - V^*_t(s), \Delta_t = \max_s |\Delta_t(s)|$. For $t = H$ we have that for all $s$

$$
\Delta_H(s) = \max_a r(s, a) + \sum_{s'} \hat{P}(s' | s, a)V^*_{t+1}(s') - \max_a r(s, a) + \sum_{s'} P(s' | s, a)V^*_{t+1}(s') = \max_a r(s, a) + \sum_{s'} \hat{P}(s' | s, a)\hat{V}^*_{t+1}(s') - r(s, a^*) + \sum_{s'} P(s' | s, a^*)V^*_{t+1}(s')
$$

$$
= \sum_{s'} \hat{P}(s' | s, a^*)\hat{V}^*_{t+1}(s') - P(s' | s, a^*)V^*_{t+1}(s')
$$

$$
\leq \sum_{s'} \hat{P}(s' | s, a^*)|\hat{V}^*_{t+1}(s') - V^*_{t+1}(s')| + |P(s' | s, a^*) - \hat{P}(s' | s, a^*)|V^*_{t+1}(s')
$$

$$
\leq \sum_{s'} \hat{P}(s' | s, a^*)|\Delta_{t+1}(s')| + (H - t)\epsilon_P
$$

$$
\leq \Delta_{t+1} \sum_{s'} \hat{P}(s' | s, a^*) + (H - t)\epsilon_P = \Delta_{t+1} + (H - t)\epsilon_P
$$

The second relation holds by choosing $a^*$ to maximize the first term first. The forth relation by adding and subtracting $\hat{P}(s' | s, a^*)\hat{V}^*_{t+1}(s')$ and standard inequalities. The fifth relation by the fact $V^*_{t+1}(s) \leq H - t$ and the assumption that for all $s, a \|P(\cdot | s, a) - \hat{P}(\cdot | s, a)\| \leq \epsilon_P$. The sixth by the fact $\hat{P}(\cdot | s, a)$ is a probability distribution and thus sums to 1.

Lower bounding $\Delta_t(s)$ using similar technique with opposite inequalities yields,

$$
\Delta_t(s) \geq -|\Delta_{t+1} + (H - t)\epsilon_P|
$$

and thus,

$$
|\Delta_t(s)| \leq \Delta_{t+1} + (H - t)\epsilon_P.
$$

As the above holds for any $s$ it holds for the maximizer. Thus,

$$
\Delta_t \leq \Delta_{t+1} + (H - t)\epsilon_P.
$$

Iterating on this relation while using $\Delta_H(s) = 0$ by (38),

$$
\|V^*_t - \hat{V}^*_t\|_\infty = \Delta_1 \leq \sum_{t=1}^{H} (H - t)\epsilon_P = \epsilon_P \sum_{t=1}^{H-1} t = \frac{H(H - 1)}{2} \epsilon_P.
$$
The proof of the second claim follows the same proof of the first claim, without while replacing the max operator with the a fixed policy π.

**Lemma 18** (Total Contribution of Approximate Model Errors). Let \( d_n := -\frac{1}{2}(h - 1)h\epsilon_P + (H - n)h\epsilon_P. \) Then,

\[
\sum_{n=0}^{H-1} d_{1+nh} = \frac{1}{2}H(H - 1)\epsilon_P.
\]

**Proof.** The following relations hold.

\[
\begin{align*}
\sum_{n=0}^{H-1} d_{1+nh} &= -\frac{1}{2}H(h - 1)\epsilon_P + \sum_{n=0}^{H-1} (H - n)h\epsilon_P \\
&= -\frac{1}{2}H(h - 1)\epsilon_P + H(H - 1)\epsilon_P - h^2\epsilon_P \sum_{n=0}^{H-1} n \\
&= -\frac{1}{2}H(h - 1)\epsilon_P + H(H - 1)\epsilon_P - \frac{1}{2}h^2\epsilon_P(H - \frac{h}{H}) \\
&= -\frac{1}{2}H(h - 1)\epsilon_P + H(H - 1)\epsilon_P - \frac{1}{2}H(H - h)\epsilon_P \\
&= -\frac{1}{2}H(H - 1)\epsilon_P + H(H - 1)\epsilon_P = \frac{1}{2}H(H - 1)\epsilon_P.
\end{align*}
\]
F  Per-Episode Complexity of h-RTDP

In this section, we define and analyze the Forward-Backward DP by which an $h$-greedy policy can be calculated from a current state $s^k_t$ according to (3). Observe that the algorithm is based on a ‘local’ information, i.e., it does not need access to the entire state space, but to a portion of the state space in the ‘vicinity’ of the current state $s^k_t$. Furthermore, it does not assume prior knowledge on this vicinity.

F.1 Forward-Backward Dynamic Programming Approach

Algorithm 6 $t_c$-Forward-Backward Dynamic Programming

| Input: Current State $s^k_t$, transition $p$, reward $r$, lookahead horizon $t_c$, value at the end of lookahead horizon $V^k_{t_c}$ |
| $\{S_{t'}\}_{t'=1}^{t_c} = \text{Forward-Pass}(s^k_t, p, t_c)$ |
| action = Backward-Pass($\{S_{t'}\}_{t'=1}^{t_c}$, $r$, $t_c$, $V^k_{t_c}$) |
| return: action |

Algorithm 7 Forward-Pass

| Input: $s^k_t$, $p$, $t_c$ |
| init: $S_1 = \{s^k_t\}, \forall t' \in [t_c]\{1\}, S_{t'}(s^k_t) = \{\}$ |
| for $t' = 2, 3, \ldots, t_c$ do |
| for $s \in S_{t'-1}$ do |
| # acquire possible next states from $s$ |
| for $a \in A$ do |
| $S_{t'} = \{s : p(s | s, a) > 0\}$ |
| end for |
| $S_{t'} = S_{t'} \cup S(s, a)$ |
| end for |
| end for |
| return: $\{S_{t'}\}_{t'=1}^{t_c}$ |

Algorithm 8 Backward-Pass

| Input: $\{S_{t'}\}_{t'=1}^{t_c}$, $r$, $p$, $t_c$, $V^k_{t_c}$ |
| # initialize values by arbitrary value $C$ |
| init: $\forall t' \in [t_c], \forall s \in S_{t'}$, $V_{t'}(s) = C$ |
| # Assign the value at $t = t_c$ to the current value, $V^k_{t_c}$ |
| for $s \in S_{t_c}$ do |
| $V_{t_c}(s) = V^k_{t_c}(s)$ |
| end for |
| for $t' = t_c - 1, t_c - 2, \ldots, 2$ do |
| for $s \in S_{t'}$ do |
| $V_{t'}(s) = \max_a r(s, a) + \sum_{s'} p(s' | s, a)^T V_{t'+1}(s')$ |
| end for |
| end for |
| return: $\arg \max_a r(s^k_t, a) + \sum_{s'} p(s' | s^k_t, a)^T V_2(s')$ |

The Forward-Backward DP (Algorithm 6) approach is built on the following observation: would we known the accessible state space from $s^k_t$ in next $t_c = h_c - t$ time-steps we could use Backward Induction / Value Iteration on a finite-horizon MDP, with an horizon of $t_c$, and calculate the optimal policy from $s^k_t$. Unfortunately, as we do not assume such a prior knowledge, we have to create this set before applying the Backward Induction step. Thus, Forward-Backward DP first build this set (in the first, ‘Forward’ stage) and later applies standard Backward-Induction (in the ‘Backward’ stage).

Let us first analyze the computational complexity of Algorithm 6 using the following definitions. Let $S_h(s)$ be the set of reachable states from state $s$, formally, $S_h(s) = \{s' | \exists \pi : p^\pi(s_{h} = s' | s_{0} = s, \pi) > 0\}$, where $p^\pi(s_{h} = s' | s_{0} = s, \pi) = E[\{s_{h} = s'\} | s_0 = s, \pi]$. The maximal cardinality of this set after $h$ time-steps is denote by $S_h := \max_{s' \in [h]} \max_{s} |S_h(s)|$. When $S_h$ is small, local search to an horizon of $h$ can be done efficiently with the Forward-Backward DP.

Based on the above definitions we analyze the computational complexity of Forward-Backward DP starting from the Forward-Pass stage. Calculating the set $S(s, a)$ costs $O(S_1)$ as we need to enumerate all possible $O(S_1)$ next-states. We assume that $S_{t'} = S_{t'} \cup S(s, a)$ can be done by $O(S_1)$, e.g., when using a hash-table for saving $S_{t'}$ in memory. Since, by definition, $|S_{t'-1}| < O(S_h)$, we get that the complexity of each iteration of the forward pass is $O(A S_h S_1)$. As we perform $t_c = h_c - t \leq h$ iterations, we conclude that the complexity of the forward pass is bounded by

$$O(h A S_h S_1).$$  \hfill (39)

The computational complexity of the backward passage is the computational complexity of Backward Induction. Since in every time-step of the Backward induction the number of states is bounded by $S_h$ and $t_c = h_c - t \leq h$, we obtain the following bound on the complexity of the Backward-Pass:

$$O(h A S_h).$$  \hfill (40)

Using (39) and (40) we get that for every $t \in [H]$, the computational complexity of calculating an $h$-greedy policy from the current state $s^k_t$ using the Forward-Backward DP is bounded by

$$O(h A S_h + h A S_h N) = O(h A S_h N).$$
Since in each episode there are $H$ time-step, we get that the computational complexity per episode of $h$-RTDP with Forward-Backward DP as implementing the calculation of the $h_c - t$ lookahead policy from every time-step $t \in [H]$ is

$$O(H h A S_h N).$$

Finally, the space complexity of Forward-Backward DP is the space required the save in memory the possible visited states in $h_c - t \leq h$ time-steps (their identity in the Forward-Pass and their values in the Backward-Pass). By definition it is at most

$$O(h S_h).$$
G Approximate Dynamic Programming in Finite-Horizon MDPs

| Algorithm 9 (Exact) h-DP |
|-------------------------|
| init: $\forall s \in S$, $\forall n \in [\frac{H}{h}], V_{nh+1}(s) = H - nh$ |
| for $n = \frac{H}{h} - 1, \frac{H}{h} - 2, \ldots, 1$ do |
| for $s \in S$ do |
| $V_{nh+1}(s) = (T^h V_{(n+1)h+1})(s)$ |
| end for |
| end for |
| return: $\{V_{nh+1}\}_{n=1}^{H/h}$ |

| Algorithm 10 h-DP with Approximate Model |
|-----------------------------------------|
| init: $\forall s \in S$, $\forall n \in [\frac{H}{h}], V_{nh+1}(s) = H - nh$ |
| for $n = \frac{H}{h} - 1, \frac{H}{h} - 2, \ldots, 1$ do |
| for $s \in S$ do |
| $V_{nh+1}(s) = (\tilde{T}^h V_{(n+1)h+1})(s)$ |
| end for |
| end for |
| return: $\{V_{nh+1}\}_{n=1}^{H/h}$ |

In this section, we follow standard analysis (Kearns and Singh 2002; Strehl, Li, and Littman 2009) and establish bounds on the performance of approximate DP algorithms which update by an $h$ multi-step optimal Bellman operator (2). We abbreviate this class of algorithms by $h$-DP. See that unlike previous analyses (Kearns and Singh 2002; Strehl, Li, and Littman 2009), we focus on finite horizon MDPs, which is the setup in which $h$-RTDP is analyzed. The different approximation setting we analyze in this section corresponds to the ones analyzed for $h$-RTDP: approximate model, approximate value update, and approximate state abstraction.

As a reminder and for the sake of completeness, we start by considering $h$-DP Algorithm 9, which is the exact, approximate-free, version of the following $h$-DP algorithms. The algorithm uses Backward-Induction and a multi-step optimal Bellman operator $T^h$ by which it outputs the values $\{V_{nh+1}\}_{n=2}^H$. Notice that it holds $\{V_{nh+1}\}_{n=2}^H = \{V_{nh+1}\}_{n=2}^H$ by standard arguments on the Backward Induction algorithm. Furthermore, $T^h$ can be solved by Backward Induction with the total computational complexity of $O(SAh)$ by using Backward Induction. Thus, the total computational complexity of $h$-DP is $O(SAh)$ similar to the one of standard DP, e.g., Backward Induction.

In terms of space complexity, $h$-DP stores in memory $O(S^2h)$ value entries. Observe that an $h$-greedy policy (3) w.r.t. $\{V_{nh+1}\}_{n=2}^H$ is an optimal policy, as these values are the optimal values as previously observed. Ultimately, one would like using these values to act in the environment by the optimal policy. If one uses the Forward-Backward DP (Section F) to calculate such an $h$-greedy policy, then an extra $O(hS_h)$ space should be used, which results in total $O(S^2h + hS_h)$ space complexity (as in $h$-RTDP) that decrease in $h$ if $S_h$ is not too big (see Remark 3). Furthermore, the computational complexity of such approach is $O(ShAS_hS_1)$ which increases in $h$.

In next sections, we consider approximate settings of $h$-DP and establish that an $h$-greedy policy (3) w.r.t. the output values $\{V_{nh+1}\}_{n=2}^H$ has an equivalent performance to the asymptotic policy by which $h$-RTDP acts.

G.1 h-ADP with an Approximate Model

In the case of an approximate model, we replace the Bellman operator $T$ used in $h$-DP with $\tilde{T}$, the Bellman operator of the approximate model $\tilde{p}$ instead the true one $p$ (we assume $r$ is exactly known, which correspond to the assumption made in Section 5.1). This results in Algorithm 10. Similarly to Section 5.1, we assume $\|p(\cdot \mid s, a) - \tilde{p}(\cdot \mid s, a)\|_{TV} \leq \epsilon_p$, for all $(s, a) \in S \times A$. Furthermore, denote $\pi_{\tilde{p}}^*$ as the optimal policy of the approximate MDP.

Equivalently to $h$-DP, Algorithm 10 returns the optimal values of the approximate model (Algorithm 10 can be interpreted as exact $h$-DP applied on the approximate model). Thus, the $h$-greedy policy w.r.t. the outputs of Algorithm 10 $\{V_{nh+1}\}_{n=2}^H$ is the optimal policy of the approximate MDP, $\pi_{\tilde{p}}^*$. The performance of $\pi_{\tilde{p}}^*$ is measured by relatively to the performance of the optimal policy, i.e., we wish to bound $\|V_1^* - V_1^{\pi_{\tilde{p}}^*}\|_{\infty}$. This term represents the performance gap between the optimal policy of the ‘real’ MDP to the performance of the optimal policy evaluated on the real MDP, and is bounded in the following proposition.

**Proposition 19.** Assume for all $(s, a) \in S \times A : \|p(\cdot \mid s, a) - \tilde{p}(\cdot \mid s, a)\|_{TV} \leq \epsilon_p$ and let $\pi_{\tilde{p}}^*$ be the optimal policy of the approximate MDP. Then,

$$\|V_1^* - V_1^{\pi_{\tilde{p}}^*}\|_{\infty} \leq H(H - 1)\epsilon_p.$$

**Proof.** Let $\tilde{V}_{\pi_{\tilde{p}}^*}$ be the optimal value on the approximate MDP. By using the triangle inequality, the first and second claim of Lemma 17 conclude the proof,

$$\|V_1^* - V_1^{\pi_{\tilde{p}}^*}\|_{\infty} \leq \|V_1^* - \tilde{V}_{\pi_{\tilde{p}}^*}\|_{\infty} + \|\tilde{V}_{\pi_{\tilde{p}}^*} - V_1^{\pi_{\tilde{p}}^*}\|_{\infty} \leq H(H - 1)\epsilon_p.$$
The second relation holds by the updating equation and the third relation by definition (2). Let
and thus,

The second relation holds by linearity of expectation, the third relation by definition, and the fourth by assumption on

Proof.
In the case of approximate value updates Algorithm 9 is replaced by an value updates with added noise $G$.

Algorithm 11 $h$-DP with Approximate Value Updates

```
Algorithm 11 $h$-DP with Approximate Value Updates
init: $\forall s \in S$, $\forall n \in [H/h], V_{nh+1}(s) = H - nh$
for $n = \lfloor H \rfloor - 1, \lfloor H \rfloor - 2, \ldots, 1$ do
  for $s \in S$ do
    $V_k^h(s) = \epsilon_V(s) + (T^h V_{t+h})^k(s)$
  end for
end for
return: $\{V_{nh+1}\}_{n=1}^{H/h}$
```

G.2 $h$-DP with Approximate Value Updates

In the case of an approximate value updates Algorithm 9 is replaced by an value updates with added noise $\epsilon_V(s)$, by which Algorithm 11 is formulated. Similarly to the assumption used for $h$-RTDP with approximate value updates (see Section 5.2) we assume for all $s \in S$, $|\epsilon_V(s)| \leq \epsilon_V > 0$. The following proposition bounds the performance of an $h$-greedy policy w.r.t. the values output by Algorithm 11.

Proposition 20. Assume for all $s \in S$, $|\epsilon_V(s)| \leq \epsilon_V$. Let $\pi^*_V$ be the $h$-greedy policy (3) w.r.t. output Algorithm 11. Then,

$$\|V^*_1 - V^{\pi^*_V}_1\|_\infty \leq \frac{2H}{h} \epsilon_V.$$

Proof. Let $\{\hat{V}_{nh+1}^h\}_{n=1}^{H/h}$ denote the output of Algorithm 11. We establish two claims which are of similarity to the two claims of Lemma 17. Combining the two we prove the result.

(i) The following relations hold for all $s \in S$ and $n \in \{0\} \cup \{H/h - 1\}$.

$$\Delta_{1+nh}(s) := \hat{V}_{1+nh}^*(s) - V_{1+nh}(s)$$

$$= \epsilon_V(s) + T^h \hat{V}_{1+(n+1)h}^*(s) - T^h V_{1+(n+1)h}(s')$$

$$= \epsilon_V(s) + \max_{a_0, \ldots, a_{h-1}} \mathbb{E} \left[ \sum_{t'=0}^{h-1} r(s_{t'}, a_{t'}(s_{t'})) + \hat{V}_{t+h}^*(s_h) \mid s_0 = s \right] - \max_{a_0, \ldots, a_{h-1}} \mathbb{E} \left[ \sum_{t'=0}^{h-1} r(s_{t'}, a_{t'}(s_{t'})) + V_{t+h}^*(s_h) \mid s_0 = s \right]
$$

(41)

The second relation holds by the updating equation and the third relation by definition (2). Let $\{\hat{a}_0, \hat{a}_1, \ldots, \hat{a}_{h-1}\}$ be the set of policies maximizes the second terms, then, by plugging this sequence to the third term we necessarily decrease it. Thus,

$$|\Delta_{1+nh}(s)| \leq \epsilon_V(s) + \mathbb{E} \left[ \sum_{t'=0}^{h-1} r(s_{t'}, a_{t'}(s_{t'})) + \hat{V}_{t+h}^*(s_h) \mid s_0 = s, \{a_{t'}\}_{t'=0}^{h-1} = \{\hat{a}_{t'}\}_{t'=0}^{h-1} \right]$$

$$- \mathbb{E} \left[ \sum_{t'=0}^{h-1} r(s_{t'}, a_{t'}(s_{t'})) + V_{t+h}^*(s_h) \mid s_0 = s, \{a_{t'}\}_{t'=0}^{h-1} = \{\hat{a}_{t'}\}_{t'=0}^{h-1} \right]$$

$$= \epsilon_V(s) + \mathbb{E} \left[ \hat{V}_{t+h}^*(s_h) - V_{t+h}^*(s_h) \mid s_0 = s, \{a_{t'}\}_{t'=0}^{h-1} = \{\hat{a}_{t'}\}_{t'=0}^{h-1} \right]$$

$$= \epsilon_V(s) + \mathbb{E} \left[ T_{1+(n+1)h}(s) \mid s_0 = s, \{a_{t'}\}_{t'=0}^{h-1} = \{\hat{a}_{t'}\}_{t'=0}^{h-1} \right] \leq \epsilon_V + \|\Delta_{1+(n+1)h}\|_\infty.$$

The second relation holds by linearity of expectation, the third relation by definition, and the forth by assumption on $\epsilon_V(s)$ and by the standard bounded $\mathbb{E}[X] \leq \|X\|_\infty$.

Repeating the above arguments while choosing the sequence which maximizes the third term in (41) allows us to lower bound (41) as follows

$$|\Delta_{1+nh}| \leq \epsilon_V + |\Delta_{1+(n+1)h}| \|_\infty.$$

and thus,

$$\|\Delta_{1+nh}\|_\infty \leq \epsilon_V + |\Delta_{1+(n+1)h}| \|_\infty.$$

Solving the recursion while using $\Delta_{H+1}(s) = 0$ for all $s \in S$ we get

$$\|\Delta_1\|_\infty \leq \frac{H}{h} \epsilon_V.$$

(42)
The following relations hold for all \( s \in S \) and \( n \in [\frac{H}{h}] \).

\[
\Delta_{1+nh}^{\pi^*}(s) := \hat{V}_{1+nh}^*(s) - V_{1+nh}^*(s)
\]

\[
= \epsilon_V(s) + \max_{a_0, \ldots, a_{h-1}} \mathbb{E} \left[ \sum_{t'=0}^{h-1} r(s_{t'}, a_{t'}(s_{t'})) + \hat{V}_{1+(n+1)h}^*(s_h) \mid s_0 = s \right] - \mathbb{E} \left[ \sum_{t'=0}^{h-1} r(s_{t'}, a_{t'}(s_{t'})) + V_{1+(n+1)h}^*(s_h) \mid s_0 = s, \pi_V^* \right].
\]

By definition, the sequence which maximizes the second term is \( \pi_V^* \), as it is the \( h \)-greedy policy w.r.t. \( \hat{V}^* \). Using the linearity of expectation we get

\[
\Delta_{1+nh}^{\pi^*}(s) = \epsilon_V(s) + \mathbb{E} \left[ \hat{V}_{1+(n+1)h}^*(s_h) - V_{1+(n+1)h}^*(s_h) \mid s_0 = s, \pi_V^* \right] - \mathbb{E} \left[ \Delta_{1+(n+1)h}^{\pi^*}(s_{1+(n+1)h}) \mid s_0 = s, \pi_V^* \right].
\]

As for all \( s, |\epsilon_V(s)| \leq \epsilon_V \), using the triangle inequality and \( E[X] \leq \|X\|_\infty \) we get the following recursion,

\[
\|\Delta_{1+nh}^{\pi^*} \|_\infty \leq \epsilon_V + \|\Delta_{1+(n+1)h}^{\pi^*} \|_\infty.
\]

Using \( \|\Delta_{1+H}^{\pi^*} \|_\infty = 0 \) we arrive to its solution,

\[
\|\Delta_{1}^{\pi^*} \|_\infty \leq \frac{H}{h} \epsilon_V. \tag{43}
\]

which proves the second needed result.

Finally, using the triangle inequality and the two proven claims, (42) and (43), we conclude the proof.

\[
\|V^*_1 - \hat{V}^*_1\|_\infty \leq \|V^*_1 - \hat{V}^*_1\|_\infty + \|\hat{V}^*_1 - V^*_1\|_\infty = \|\Delta_1\|_\infty + \|\Delta_1^{\pi^*} \|_\infty \leq 2 \frac{H}{h} \epsilon_V.
\]

\[\square\]

G.3 \( h \)-DP with Approximate State Abstraction

When an approximate state abstraction \( \{\phi_{1+nh}\}_{n=0}^{\frac{H}{h}-1} \) is given, Algorithm 9 can be replaced by an exact value update in the reduced state space \( S_\phi \), as given in Algorithm 12. This corresponds to updating a value \( V \in \mathbb{R}^{S_\phi} \), instead of \( \mathbb{R}^S \). An obvious advantage of such an algorithm, relatively to \( h \)-DP, is its reduced space complexity, as it only needs to store \( O(\frac{H}{h} S_\phi) \) value entries, instead of \( O(\frac{H}{h} S) \) as \( h \)-DP.

Yet, as seen in Algorithm 12, its computational complexity remains \( O(SA\phi) \) as it needs to uniformly update on the entire (non-abstracted) state space. Would have we been given a representative from each equivalence classes under \( \phi_{1+nh} \) for every \( n \in \{0\} \cup (\frac{H}{h})^{-1} \) we could suggest an alternative Backward Induction algorithm with computational complexity of \( O(S_\phi AH) \). However, as we do not assume access to this knowledge, we are obliged to scan the entire state space, without further assumptions.

The following proposition bounds the performance of an \( h \)-greedy policy w.r.t. the values output by Algorithm 12. Similarly to the analysis of the corresponding \( h \)-RTDP algorithm (see Section 5.3), we assume \( \{\phi_{1+nh}\}_{n=0}^{\frac{H}{h}-1} \) satisfy Assumption 1.

Proposition 21. Let \( \{\phi_{1+nh}\}_{n=0}^{\frac{H}{h}-1} \) satisfy Assumption 1. Let \( \{\hat{V}^*_{nh+1}\}_{n=1}^{\frac{H}{h}} \) denote the output of Algorithm 12 and let \( \pi_A^* \) be the \( h \)-greedy policy w.r.t. these approximate values (3). Then,

\[
\|V^*_1 - V^*_1\|_\infty \leq \frac{H}{h} \epsilon_A.
\]

**Proof.** We establish two claims which are of similarly to the two claims of Lemma 17 and Proposition 20. Combining the two we prove the result.

\[\square\]
(i) The following relations hold for any \( s \in S \).

\[
\hat{V}_{1+n_h}^s(\phi_{1+n_h}(s)) - V_{1+n_h}^s(s) = T_h \hat{V}_{\phi,1+(n+1)h}(s) - T_h V_{1+(n+1)h}(s) = \max_{a_0, \ldots, a_{h-1}} \mathbb{E} \left[ \sum_{t'=0}^{h-1} r(s_{t'}, a_{t'}(s_{t'})) + \hat{V}_{1+(n+1)h}(\phi_{1+(n+1)h}(s_h)) \mid s_0 = s \right] - \max_{a_0, \ldots, a_{h-1}} \mathbb{E} \left[ \sum_{t'=0}^{h-1} r(s_{t'}, a_{t'}(s_{t'})) + V_{1+(n+1)h}(s_h) \mid s_0 = s \right]
\]

The second and third relation holds by the updating rule of Algorithm 12. Let \( \{\hat{a}_0, \hat{a}_1, \ldots, \hat{a}_{h-1}\} \) be the set of policies which maximizes the first term. Then, by plugging this sequence to the second term we necessarily decrease it, and the following holds.

\[
(44) \leq \mathbb{E} \left[ \sum_{t'=0}^{h-1} r(s_{t'}, a_{t'}(s_{t'})) + \hat{V}_{1+(n+1)h}(\phi_{1+(n+1)h}(s_h)) \mid s_0 = s, \{a_{t'}\}_{t'=0}^{h-1} = \{\hat{a}_{t'}\}_{t'=0}^{h-1} \right] - \mathbb{E} \left[ \sum_{t'=0}^{h-1} r(s_{t'}, a_{t'}(s_{t'})) + V_{1+(n+1)h}(s_h) \mid s_0 = s, \{a_{t'}\}_{t'=0}^{h-1} = \{\hat{a}_{t'}\}_{t'=0}^{h-1} \right] = \mathbb{E} \left[ \hat{V}_{1+(n+1)h}(\phi_{1+(n+1)h}(s_h)) - V_{1+(n+1)h}(s_h) \mid s_0 = s, \{a_{t'}\}_{t'=0}^{h-1} = \{\hat{a}_{t'}\}_{t'=0}^{h-1} \right] (45)
\]

Where the second relation holds by linearity of expectation. By Assumption 1 the following inequality holds,

\[
(45) \leq \mathbb{E} \left[ \hat{V}_{1+(n+1)h}(\phi_{1+(n+1)h}(s_h)) - \max_{\tilde{s}_h \in \Phi_{1+(n+1)h}(s_h)} V_{1+(n+1)h}(\tilde{s}_h) + \epsilon_A \mid s_0 = s, \{a_{t'}\}_{t'=0}^{h-1} = \{\hat{a}_{t'}\}_{t'=0}^{h-1} \right] = \epsilon_A + \mathbb{E} \left[ \hat{V}_{1+(n+1)h}(\phi_{1+(n+1)h}(s_h)) - \max_{\tilde{s}_h \in \Phi_{1+(n+1)h}(s_h)} V_{1+(n+1)h}(\tilde{s}_h) \mid s_0 = s, \{a_{t'}\}_{t'=0}^{h-1} = \{\hat{a}_{t'}\}_{t'=0}^{h-1} \right] \leq \epsilon_A + \max_s \left| \hat{V}_{1+(n+1)h}(\phi_{1+(n+1)h}(s)) - \max_{\tilde{s} \in \Phi_{1+(n+1)h}(s)} V_{1+(n+1)h}(\tilde{s}) \right| (46)
\]

By choosing the sequence of polices which maximizes the second term in (44) and repeating similar arguments to the above we arrive to the following relations.

\[
(44) \geq \mathbb{E} \left[ \hat{V}_{1+(n+1)h}(\phi_{1+(n+1)h}(s_h)) - V_{1+(n+1)h}(s_h) \mid s_0 = s, \{a_{t'}\}_{t'=0}^{h-1} = \{\hat{a}_{t'}\}_{t'=0}^{h-1} \right] \geq \mathbb{E} \left[ \hat{V}_{1+(n+1)h}(\phi_{1+(n+1)h}(s_h)) - \max_{\tilde{s}_h \in \Phi_{1+(n+1)h}(s_h)} V_{1+(n+1)h}(\tilde{s}_h) \mid s_0 = s, \{a_{t'}\}_{t'=0}^{h-1} = \{\hat{a}_{t'}\}_{t'=0}^{h-1} \right] \geq - \max_s \left| \hat{V}_{1+(n+1)h}(\phi_{1+(n+1)h}(s)) - \max_{\tilde{s} \in \Phi_{1+(n+1)h}(s)} V_{1+(n+1)h}(\tilde{s}) \right| (47)
\]

Let \( \Delta_{\phi, 1+n_h}(s) := \hat{V}_{1+n_h}(\phi_{1+n_h}(s)) - \max_{\tilde{s} \in \Phi_{1+n_h}(s)} V_{1+n_h}(\tilde{s}) \). The following upper bound holds,

\[
\Delta_{\phi, 1+n_h}(s) := \hat{V}_{1+n_h}(\phi_{1+n_h}(s)) - \max_{\tilde{s} \in \Phi_{1+n_h}(s)} V_{1+n_h}(\tilde{s}) \leq \hat{V}_{1+n_h}(\phi_{1+n_h}(s)) - V_{1+n_h}(s) \leq \max_s \left| \hat{V}_{1+(n+1)h}(\phi_{1+(n+1)h}(s)) - \max_{\tilde{s} \in \Phi_{1+(n+1)h}(s)} V_{1+(n+1)h}(\tilde{s}) \right| + \epsilon_A
\]

\[
= \| \Delta_{\phi, 1+(n+1)h} \|_{\infty} + \epsilon_A
\]
where the third relation is by (46). Furthermore, the following lower bounds holds,

\[ \Delta_{\phi,1+nh}(s) := \hat{V}_{1+nh}^\ast(\phi_{1+nh}(s)) - \max_{\bar{s} \in \Phi_{1+nh}(s)} V^\ast_{1+nh}(\bar{s}) \]

\[ \geq \hat{V}_{1+nh}^\ast(\phi_{1+nh}(s)) - V^\ast_{1+nh}(s) - \epsilon_A \]

\[ \geq - \max_s \left[ \hat{V}_{1+(n+1)h}^\ast(\phi_{1+(n+1)h}(s)) - \max_{\bar{s} \in \Phi_{1+(n+1)h}(s)} V^\ast_{1+(n+1)h}(\bar{s}) \right] - \epsilon_A \]

\[ = -\|\Delta_{\phi,1+(n+1)h}\|_\infty - \epsilon_A, \]

where the second relation holds by Assumption 1 and the third by (47).

By the upper and lower bounds on \(\Delta_{\phi,1+nh}(s)\) which holds for all \(s\) we conclude that

\[ \|\Delta_{\phi,1+nh}\|_\infty \leq \|\Delta_{\phi,1+(n+1)h}\|_\infty + \epsilon_A. \]

Using \(\|\Delta_{\phi,H+1}\|_\infty = 0\) we solve the recursion and conclude that

\[ \|\Delta_{\phi,1}\|_\infty \leq \frac{H}{h} \epsilon_A. \tag{48} \]

\((ii)\) The following relations hold based on similar arguments as in (41). Let \(\Delta_{\pi^\ast}^\phi_{1+nh} := \max_s \hat{V}_{1+nh}^\ast(\phi_{1+nh}(s)) - V_{1+nh}^\ast(\phi_{1+nh}(s)).\)

For all \(s\) the following relations hold.

\[ \hat{V}_{1+nh}^\ast(\phi(s)) - V_{1+nh}^\ast(\phi(s)) \]

\[ \leq \max_{a_0,\ldots,a_{h-1}} \mathbb{E} \left[ \sum_{t'=0}^{h-1} r(s_{t'}, a_{t'}(s_{t'})) + \hat{V}_{1+(n+1)h}^\ast(\phi(s_{t'})) \mid s_0 = s \right] \]

\[ - \mathbb{E} \pi^\ast \left[ \sum_{t'=0}^{h-1} r(s_{t'}, a_{t'}(s_{t'})) + V_{1+(n+1)h}^\ast(\phi(s_{t'})) \mid s_0 = s \right], \tag{49} \]

the first relation holds by the updating rule which update by the (see Algorithm 12), and since \(V_t^\pi = (T^\pi)^h V^\pi_{t+h},\) similarly to the optimal Bellman operator (2).

By definition, the sequence which maximizes the first term is \(\pi^\ast\) as it is the \(h\)-greedy policy w.r.t. \(\hat{V}^\ast.\) Using the linearity of expectation we get

\[ (49) = \mathbb{E} \pi^\ast \left[ \hat{V}_{1+(n+1)h}^\ast(\phi(s_{t'})) - V_{1+(n+1)h}^\ast(\phi(s_{t'})) \mid s_0 = s \right] \]

\[ \leq \max_s \hat{V}_{1+(n+1)h}^\ast(\phi(s)) - V_{1+(n+1)h}^\ast(\phi(s)) := \Delta_{1+(n+1)h}^\ast. \tag{50} \]

Since (50) for all \(s\) it also holds for the maximum, i.e.,

\[ \Delta_{1+nh}^\pi := \max_s \hat{V}_{1+nh}^\ast(\phi(s)) - V_{1+nh}^\ast(\phi(s)) \leq \Delta_{1+(n+1)h}^\ast. \]

As \(\Delta_{H+1}^\pi = 0\) and iterating on the above recursion we get,

\[ \Delta_{1}^\pi \leq 0. \tag{51} \]

We are now ready to prove the proposition. For any \(s\) the following holds,

\[ V^\ast_1(s) - V_{1+nh}^\ast(s) = V^\ast_1(s) - \hat{V}_1(\phi(s)) + \hat{V}_1(\phi(s)) - V_{1+nh}^\ast(s). \]

By (48)

\[ (A) \leq \max_{\bar{s} \in \Phi_1(s)} V^\ast_1(\bar{s}) - \hat{V}_1(\phi_1(s)) \]

\[ := -\Delta_{\phi,1}(s) \leq \|\Delta_{\phi,1}\|_\infty \leq \frac{H}{h} \epsilon_A. \]
By (51),
\[ \bar{V}_1(\phi_1(s)) - V^\pi_1(s) \leq \max_{s} (\bar{V}_1(\phi_1(s)) - V^\pi_1(s)) = \Delta^\pi_1 \leq 0. \]

Lastly, combining the above and using \( V^* \geq V^\pi \), we get that for all \( s \)
\[ 0 \leq V^*_1(s) - V^\pi_1(s) \leq \frac{H}{h} \epsilon_A. \]
\[ \rightarrow \|V^*_1 - V^\pi_1\|_\infty \leq \frac{H}{h} \epsilon_A. \]