Black Hole Final State Conspiracies

Brett McInnes

National University of Singapore
e-mail: matmcinn@nus.edu.sg

ABSTRACT

The principle that unitarity must be preserved in all processes, no matter how exotic, has led to deep insights into boundary conditions in cosmology and black hole theory. In the case of black hole evaporation, Horowitz and Maldacena were led to propose that unitarity preservation can be understood in terms of a restriction imposed on the wave function at the singularity. Gottesman and Preskill showed that this natural idea only works if one postulates the presence of “conspiracies” between systems just inside the event horizon and states at much later times, near the singularity. We argue that some AdS black holes have unusual internal thermodynamics, and that this may permit the required “conspiracies” if real black holes are described by some kind of sum over all AdS black holes having the same entropy.
1. Preserving Unitarity

The least-understood regions of spacetime are its spacelike “edges”: the beginning of time and its end, whether the end be in Crunches [the end state of a Coleman-De Luccia bubble [1] which nucleates with negative vacuum energy] or inside a black hole. Without an understanding of the laws governing these regions, we cannot arrive at a complete account of our observations regarding the remainder of spacetime.

The principle of unitarity [2] is one of our most powerful tools for probing these regions. For example, Carroll and Chen [3] used it to argue that Inflation alone cannot explain the very unusual conditions which obtained at the earliest times; instead, one needs a theory of inflationary initial conditions [4][5]. Similarly, it has been argued convincingly [6][7] that the AdS/CFT correspondence indicates that unitarity is [somehow] preserved during black hole evaporation. If we can understand how this works, we can hope to probe the final state of a black hole interior.

Horowitz and Maldacena [8] attempted to obtain a detailed understanding and implementation of this last idea by making a proposal for the final quantum state inside a black hole. The projection onto this unique final state allows a peculiar version of quantum “teleportation” [9] to salvage the information that is apparently lost in a black hole. The proposal amounts to a simple concrete expression of the idea that no information must be allowed to leave the interior spacetime through the singularity — or through whatever replaces the singularity in a more complete theory.

The Horowitz-Maldacena proposal is a very natural and elegant approach to the black hole information problem. Furthermore, it has the great virtue of locating the new physics near to the spacetime edge; so it does not ask us to believe that classical general relativity needs a drastic revision at energy scales where it is extremely well-tested. It therefore came as a surprise when a serious objection was raised against it [11]. Gottesman and Preskill argue that, just inside the event horizon — that is, long before the spacetime edge is approached — the collapsing star which forms the black hole will become entangled with the infalling Hawking radiation. This leads to a non-trivial interaction between past and future versions of the relevant information, and this in itself can lead to violations of unitarity. One might hope that a small modification of the Horowitz-Maldacena condition at the spacetime edge can compensate for this, restoring the unitarity of the black hole S-matrix. But Gottesman and Preskill argue that this would require an “implausible conspiracy” [11] between conditions near to the event horizon and the much later state near to the spacetime edge.

The reliance of the Horowitz-Maldacena proposal on an “implausible conspiracy” between conditions at different times and energy scales is very much reminiscent of the problem of understanding the arrow of time, a problem which has recently attracted renewed attention. As we trace the history of the Universe back towards its beginning, we

---

1Henceforth, in order to avoid repeating this long locution, we shall use “spacetime edge” to mean any of the spacelike regions which are singular in classical general relativity but where some other description takes over in a more complete theory. In a Euclidean-gravity description, for example, such edges might be the ones along which the Lorentzian/Euclidean transition occurs. For a dissenting view on the transmissibility of information through black holes, see [10].

2See [12][13][14][15] for general background, [16][17][18] for the point of view advocated here, and [19][20] for recent alternative perspectives.
witness an “implausible conspiracy” on a cosmological scale as the entropy — particularly the gravitational contribution to it \[21\] — steadily declines, ultimately to fantastically small values. Thus, we know that apparently wildly improbable correlations, leading to an extraordinarily non-generic state at a far-off time and a much higher energy scale, do occur in our Universe as this spacetime edge is approached. The question is whether a [perhaps much milder] version of this can occur inside a black hole. If it can, this might shed new light on the status of the Horowitz-Maldacena proposal.

Horowitz and Maldacena were well aware that there is a fundamental relationship between the information loss problem and the arrow of time\[3\]. Similarly, Gottesman and Preskill point out that one way of understanding the Horowitz-Maldacena proposal is in terms of information propagating \textit{backwards in time} from the spacetime edge \[11\]. Nevertheless Horowitz and Maldacena argue that the arrow does \textit{not} completely reverse inside a black hole. Indeed, a complete reversal\[4\] of the arrow inside a realistic black hole, embedded in an external spacetime which itself has an arrow, could lead to serious difficulties, since one might have to impose both initial and final conditions on every system inside the black hole; and it is far from clear that this can always be done in a way consistent with a quantum-mechanical coarse-graining of phase space \[23\].

In the present case, however, we do not need a \textit{complete} reversal of the arrow inside the black hole, since the effect of the entanglement pointed out in \[11\] is extremely small compared to the overall entropy of the hole. In fact, there is no reason to think that reversing the arrow of time inside the black hole has to be an “all-or-nothing” matter; all we need here is that there should be \textit{some} traces of such an effect.

It is now widely accepted that the behaviour of \textit{Anti-de Sitter} black holes will provide the key to an explicit string-theoretic resolution of the black hole information problem. The duality of the thermodynamics of \textit{these} black holes with thermal aspects of a conformal field theory does strongly suggest that their evolution is completely unitary [particularly in the case of eternal black holes \[7\]]; and of course there is ample evidence \[24\] that these objects really do have some deep physical significance. Despite all this, the precise way in which properties of AdS black holes constrain the behaviour of black holes in the real world, where the cosmological constant is positive, is not yet fully understood. To judge by the methods used in \[7\], it seems likely that this has to be done by passing through the Euclidean domain. But since Euclidean methods invariably involve a “sum over geometries” — and this sum is in fact \textit{necessary} for the maintenance of unitarity \[25\] \[26\] — this probably means that we must not expect a direct one-to-one correspondence between AdS black holes and those in de Sitter spacetime. Instead, we should expect to use some kind of “sum” over the Euclidean versions of \textit{all} AdS black holes of a given entropy to probe the de Sitter black hole with that value for the entropy. As we shall explain in detail, the entropy is \textit{far} from being able to specify a unique black hole in AdS, so the correspondence is indeed not one-to-one. [Note that the information loss problem is as serious for eternal black holes as it is for any other variety, since such holes can apparently convert pure states to thermal ones; so eternal black holes must be

\[3\] A very detailed study of this link, taking a different point of view from the one advocated here, has been given in Reference \[22\].

\[4\] By a “reversal” of the arrow, we really just mean that the arrow points in an unexpected direction, that is, \textit{away} from the black hole spacetime edge. This makes sense whether or not there is an arrow in the external spacetime.
included in the “sum”.

With this in mind, it is interesting to ask: are there any AdS black holes inside which the arrow does reverse? If there are, these objects might make a small contribution to the sum over AdS black holes with a given entropy, and this could point the way towards arranging a minor “conspiracy” of the kind described by Gottesman and Preskill. The hope is that this might allow the Horowitz-Maldacena proposal to work.

We begin by discussing the range of possible AdS$_5$ black hole geometries. [For the sake of definiteness, and because this is the case that is best understood, we focus on five-dimensional spacetimes, but this is not essential.] As is well known, this range is much wider than in the case of asymptotically flat or de Sitter spacetimes: one has black holes with event horizons which are not positively curved. These “topological” black holes [27][28] are just as valid as the more familiar spherical ones at the perturbative level in string theory, and we clearly need to know whether all of them are present in the full theory. Fortunately, in string theory one has a simple yet powerful technique which eliminates a large subset of these “topological” black holes: we just have to apply the [non-perturbative] brane pair-production stability criterion of Seiberg and Witten [29]; see [30] and [31] for clear introductions to and applications of this remarkable method. Using this method, we find in Section 2 that a black hole with a negatively curved event horizon is always unstable [in the Seiberg-Witten sense] in string theory. This eliminates one entire class of topological black holes from consideration.

The survivors are the AdS black holes having a torus, or a non-singular quotient of a torus, as their event horizon. In section 3 we study these toral black holes and show not only that they are stable, but that they remain stable as long as the Null Energy Condition holds. This condition is expected to be valid here, because toral black holes always have a positive specific heat and are able to come into equilibrium with their own Hawking radiation: they are eternal, like the black holes studied in [7]. Thus, we have to take toral black holes as seriously as their spherical counterparts.

In section 4, we briefly review the theory of the arrow of time advanced in [16][17][18], and ask whether it implies that the arrow might “reverse” inside an AdS black hole. The answer is that it does indeed imply this for toral black holes, but not for spherical ones. The argument makes use of the deep mathematical work of Gromov and Lawson on “weakly enlargeable” manifolds [see [32].] If a correspondence between realistic black holes and some kind of sum over their AdS counterparts of the same entropy does exist, then one may ultimately be able to use toral AdS black holes to arrange a “conspiracy” of the kind needed to ensure the unitarity of the final state S-matrix.

2. No Negatively Curved Event Horizons in String Theory

Consider an asymptotically AdS$_5$ black hole, with a fixed entropy $S$. In AdS$_5$, the specification of the entropy does not fix a unique spacetime metric, because the field equations...
do not enforce a particular foliation of the spatial hypersurfaces. Suppose that these hypersurfaces outside the event horizon are foliated by sections modelled on some specific compact three-dimensional space, \( C_k \), of constant curvature \( k \), where \( k = \{-1, 0, +1\} \). Then all of the following metrics\(^6\) are *equally valid* as solutions of the field equations [with no matter other than the [negative] vacuum energy] \(^2\):\(^8\):

\[
g(BH_k) = -\left[ \frac{r^2}{L^2} + k - \frac{16\pi M}{3A_k r^2} \right] dt^2 + \frac{dr^2}{r^2 + k - \frac{16\pi M}{3A_k r^2}} + r^2 d\Omega_k^2. \tag{1}
\]

Here \( L \) is the radius of curvature of AdS\(_5\), and \( d\Omega_k^2 \) is a metric of constant curvature \( k \) on the three-dimensional space \( C_k \); then \( A_k \) is the area of this space.

Before we proceed, let us consider a reasonably realistic black hole, which will be asymptotically de Sitter, not asymptotically AdS. Idealised versions of these objects have singularities in the past as well as the future, but the past singularities do not occur in realistic versions. Instead, a real black hole is formed from the collapse of a star; this eliminates the past singularity. In order to encode this important property of real black holes, we shall take an orbifold quotient of each of the spacetimes with metrics given in equation (1), factoring out the obvious “horizontal fold” isometry which maps the past singularity onto the future singularity; in effect, \( t \) runs from zero to infinity outside the event horizon [though in a classical hole it still runs from \(-\infty\) to \(+\infty\) inside, where it is a spatial coordinate]. In each case, then, there is only one spacetime edge to consider. The conformal diagrams are then similar to the upper half of the AdS black hole conformal diagram discussed in \(^4\).

Returning to equation (1): it is important to understand that \( A_k \) is not yet uniquely defined, because for each \( k \) there are *many* spaces of constant curvature \( k \). In the case of \( k = 1 \), \( A_1 \) is fixed by the topology of \( C_1 \); for example if \( C_1 \) is the unit radius three-sphere, it is equal to \( 2\pi^2 \), while for the unit radius real projective space \( \mathbb{R}P^3 \) it would be \( \pi^2 \), and so on for all of the other [infinitely numerous] three-dimensional manifolds of unit positive constant curvature \(^3\). We shall return to this point in Section 5.

In the case where \( k = 0 \), there is again a topological ambiguity [there are six possible topologies in the orientable case \(^6\)], but, in addition, the size and shape of the compact flat space can be freely prescribed. The simplest possibility is to declare that a given value of the coordinate \( r \) corresponds to a *flat cubic torus*, defined as the Riemannian product of three circles each of radius \( Kr \), where \( K \) is a positive dimensionless number. In this special case, the area of the surface \( r = \text{constant} \) at a given time outside the black hole is \( 8\pi^3K^3r^3 \), so \( A_0 = 8\pi^3K^3 \). We shall take this as the definition of \( K \) in general, that is, for arbitrary compact flat three-dimensional manifolds. \( K \) is then a measure of the overall relative size of the space. The presence of such a continuous parameter distinguishes the flat case in a fundamental way from the other two categories.

The case of \( k = -1 \) is the most difficult and potentially troublesome one, because there is a vast set of distinct compact manifolds of unit negative curvature [see \(^7\)]. Fixing the entropy of the black hole now requires having a complete knowledge of the spectrum of possible volumes of these manifolds. If such black holes\(^7\) have to be included in the

\(^6\)Our emphasis on constant curvature is based on the following fact: if we impose the reasonable condition that the local geometry should become indistinguishable from that of AdS\(_5\) when the black hole mass tends to zero, then \(^2\) the event horizon *must* be a space of constant curvature.

\(^7\)The perturbative behaviour of these black holes has been studied in depth; see \(^3\) and its references.
sum over AdS black holes of given entropy, then we will have to confront a formidable mathematical problem; also, these objects might dominate the sum, which would be hard to understand. Fortunately, we shall see that this is not the case if we work within the constraints imposed by string theory.

All of the metrics in equation (1) have Euclidean versions, in which the complexified version of \( t \) has to parametrise a circle; thus the Euclidean metric, \( g(EBH_k) \), is a metric on a manifold with the topology of \( S^1 \times \mathbb{R} \times C_k \). At large distances, the circumference of the circle is a constant multiple of \( r \); the dimensionless constant of proportionality, \( P \), must be chosen so that the Euclidean metric is not singular at \( r_{eh} \), the value of \( r \) at the event horizon.

Now we wish to consider these spaces in the string context, which means that we have to consider the possible nucleation of branes. In the case of a three-brane in any Euclidean asymptotically AdS_5 space, we can write the action in the form [29]

\[
S = \Theta \left\{ \text{Brane Area} \right\} - \mu \left\{ \text{Volume Enclosed by Brane} \right\},
\]

where \( \Theta \) is the tension and \( \mu \) is a constant related to the charge. Non-perturbative instabilities arise [30][31] if the action becomes negative; the BPS case is the most dangerous one. In that case, \( \mu \) is just \( 4\Theta/L \), where \( L \) is the curvature radius of the asymptotic AdS_5, so we can compute the action if we know the volume and area of a brane located at some value of \( r \). In the case of the Euclidean version of the metric given in equation (2), we have, for a brane located at coordinate value \( r \),

\[
S(r; M, L, \Theta, K(k = 0)) = \Theta P L A_k \left\{ r^3 \left[ \frac{r^2}{L^2} + k - \frac{16\pi M}{3A_k r^2} \right]^{1/2} - \frac{r^4}{L} \right\}.
\]

Here the notation means that the action is determined by \( r, M, L, \Theta, k \), and, if \( k = 0 \), also by \( K \). \( r_{eh} \) and \( P \) are determined by \( M, L, \Theta, k \), and, where applicable, \( K \). The simple form of equation (3) allows us to write the action as

\[
S(r; M, L, \Theta, K(k = 0)) = \Theta P L A_k \left\{ \frac{L[kr^2 - \frac{16\pi M}{3A_k}]}{1 + \left[ 1 + \frac{kL^2}{r^2} - \frac{16\pi M L^2}{3A_k r^4} \right]^{1/2}} + \frac{r_{eh}^4}{L} \right\}.
\]

When \( k = 1 \), it is easy to show that this expression is positive for all values of \( r \) larger than \( r_{eh} \), and in fact it diverges to \( +\infty \) for large \( r \). Thus the familiar spherical AdS black holes, and all of their less familiar relatives with event horizons having the topology of some non-singular quotient of the three-sphere, are completely stable against this effect, and all of them must be retained in string theory. We shall return to them later.

For \( k = -1 \), the action is again positive at first. Soon, however, the graph [Figure 1, with representative parameter values] turns over, and in fact the action is unbounded below. Thus, black holes with negatively curved event horizons are non-perturbatively unstable in string theory; that is, they are not solutions of the full theory. The power of the Seiberg-Witten technique is underlined by the fact that a simple calculation entails such a strong conclusion.

\[8\] In this metric, the coefficient of \( dt^2 \) is the reciprocal of the coefficient of \( dr^2 \); the resulting cancellation explains why the volume integral is so easily evaluated.
In fact, Seiberg and Witten showed that this kind of instability will \textit{always} arise if the scalar curvature at infinity\footnote{That is, the scalar curvature of a metric representing the conformal structure of the boundary; this scalar curvature can be taken to be constant, without loss of generality \cite{39}.} is negative. In the case at hand, the conformal structure at infinity is represented by the metric

\begin{equation}
    g(EBH_{-1}, r \to \infty) = dt^2 + L^2 d\Omega^2_{-1},
\end{equation}

which is obviously a metric of constant negative scalar curvature on a compact manifold of topology $S^1 \times C_{-1}$. The basic point is that a negative scalar curvature at infinity amounts to having a negative (mass)$^2$ for certain scalar fields there; since the boundary theory is therefore unstable, the dual theory in the bulk must likewise be unstable.

We see that the requirement of non-perturbative stability gives an enormous reduction in the number of AdS black holes which need to be considered in a string-theoretic approach. But one class of “topological” black holes remains to be considered: those with $k = 0$. Notice that the Seiberg-Witten duality argument does not work here, since the scalar field at infinity simply becomes massless. In fact, asymptotically AdS spaces with Euclidean versions having compact flat boundaries can be either stable or unstable against brane nucleation, depending on the details. [Explicit examples displaying instability were first presented in \cite{40}.] We now proceed to discuss these black holes.

\section{Black Holes with Flat Event Horizons}

The graph of the brane action in this case has an interesting form: see Figure 2. The action increases from zero, but it does not diverge either positively or negatively: it is
Figure 2: Brane Action, Flat Event Horizon.

asymptotic to a fixed constant positive value. Thus, these black holes are stable against brane pair-production, unlike their counterparts with negatively curved event horizons.

The value of $r$ at the event horizon, $r_{eh}$, is given by

$$r_{eh} = \left( \frac{16\pi ML^2}{3A_0} \right)^{1/4} = \left( \frac{2ML^2}{3\pi^2K^3} \right)^{1/4}. \quad (6)$$

Thus $r_{eh}$ actually decreases as $K$ increases. However, the area of the event horizon is

$$A_{eh} = 8 \times [2/3]^{3/4} \times \pi^{3/2} \times [MK]^{3/4}L^{3/2} \approx 32.866 \times [MK]^{3/4}L^{3/2}, \quad (7)$$

which does increase with $K$. Notice that the size of the event horizon is not fixed by the mass and the AdS radius of curvature, as it is in the case of a spherical horizon; the relevant parameter here is $MK$.

It is possible to show [41] [see also [42] [43]] that the entropy of these black holes is, as usual, proportional to the area of the event horizon; thus one way to think about $K$ is as a measure of the black hole entropy for fixed mass. More remarkably, the specific heat of these black holes is also a constant multiple of the area of the event horizon [41] [and is independent of the temperature], so $K$ might alternatively be regarded as a measure of the specific heat for fixed mass.

It is clear from this discussion that the specific heat of a toral black hole is always positive, for all parameter values. Such black holes do not evaporate away completely: they come into equilibrium with their own Hawking radiation, just as “large” black holes do in the spherical case [6]. This means that they are eternal.

Eternal black holes are of great interest in the context of the information loss problem, because they dominate the relevant canonical thermodynamic ensemble in the spherical case, and, perhaps more importantly, because they are the ones which definitely have a
specific CFT dual. They are therefore the ones for which we have the strongest evidence regarding the preservation of unitarity. For example, Witten [6] stresses that the specific heat of the CFT is positive, so the AdS/CFT correspondence is most direct for eternal black holes. Note that a small object in a pure state which is thrown into an eternal black hole will have its mass radiated in a thermal state according to the usual calculation; thus the information loss problem is just as serious for eternal black holes as for any other variety. The precise form taken by the information loss problem in the case of eternal black holes was discussed in Maldacena’s classic study [7].

Later we shall see that eternal black holes are in fact the most important ones for the purposes we have in view in this work. Notice in this connection that, for a small realistic black hole, there may be no eternal AdS black hole of that entropy in the spherical case; however, there will always be an eternal toral AdS black hole for any value of the entropy [10].

The asymptotic value of the action for flat branes is

\[ S(\infty; M, L, \Theta, K) = \frac{8}{3} \pi \Theta PML^2, \]  

and it can be shown [28] that the value of \( P \) [giving the size of the circle defined by Euclidean “time”] is

\[ P = \frac{\pi L}{r_{eh}}, \]  
in this case. Thus we have, from equation (6),

\[ S(\infty; M, L, \Theta, K) = \frac{8}{3} \times \left[\frac{3}{2}\right]^{1/4} \times \pi^{5/2} \times \Theta L^{5/2}[MK]^{3/4} \approx 51.626 \times \Theta L^{5/2}[MK]^{3/4}. \]  

Notice that

\[ \frac{S(\infty; M, L, \Theta, K)}{\Theta} = \frac{\pi L A_{eh}}{2}, \]  

so the asymptotic brane action per unit tension is essentially just the entropy or the specific heat of the black hole. That the branes at large distances contain information about the entropy of the hole might be regarded as an example of holography. We take it as further evidence that unitarity is maintained here, since the [unitary] CFT on the boundary should encode all data carried by branes propagating towards infinity.

While it is clear that vacuum black holes with flat compact event horizons are stable against the nucleation of branes, it is also clear that this stability is not as secure as in the positively curved case. In the latter, the action increases rapidly and diverges towards \(+\infty\), so while the introduction of matter [including the effects of Hawking radiation] into the spacetime will change the details, it is unlikely to change the qualitative behaviour, particularly for large values of \( r \). In the case of flat event horizons, however, the action is always finite, and it is always small if the entropy of the black hole is small. Experience in the cosmological case [17] shows that, in such cases, there can be a danger that the action will become negative at large \( r \). On the other hand, in all cases — including black holes with positively curved event horizons — the action is close to zero near to \( r = r_{eh} \). One might be concerned that, if the spacetime around the black hole is not exactly
empty, then the action might become negative either at very large values of \( r \) or near to the event horizon.

In some cases one can see directly that this will not happen. For example, it does not happen if the black hole carries a reasonable amount of electric charge. Here, “reasonable” means “small compared to the value which causes the black hole to become extremal,” that is, the value beyond which Cosmic Censorship is violated. In fact, it is possible to show that an AdS black hole with a flat event horizon remains completely stable against brane nucleation when charge is added to it, up to the point where the relevant dimensionless parameter is equal to about 92% of its value in the extremal case. The detailed structure of AdS Reissner-Nordström black holes with flat event horizons is of some independent interest, and will be discussed elsewhere.

Again, experience in the cosmological case \[10,17\] leads us to expect that problems with negative brane actions are often associated with violations of energy conditions. We can investigate this possibility in a simple way by studying a “toral star”, that is, a perfect fluid in a spacetime having a similar structure to that of the vacuum toral black hole. The metric has the form

\[
g(TS) = -f(r, M, L, K) dt^2 + h(r, M, L, K) dr^2 + r^2 d\Omega_0^2, \tag{12}
\]

where \( f \) and \( h \) are functions to be determined, and we confine ourselves to values of \( r \) greater than or equal to \( r_{eh} \) so as to be able to make a comparison with the black hole. [We can imagine that the “star” is about to collapse into the black hole, so \( r = r_{eh} \) represents a surface inside the “star”]. We assume that the fluid is distributed so that \( f \) and \( h \) approach their vacuum toral black hole values at large \( r \).

The brane action in this case is

\[
S(r; M, L, \Theta, K) = 8\pi^3 \Theta PLK^3 \left\{ r^3 f^{1/2} - \frac{4}{L} \int_{r_{eh}}^r \rho^3 \left[ f h \right]^{1/2} d\rho \right\}. \tag{13}
\]

Since the “star” is not a black hole, the function \( f \) is strictly positive at \( r_{eh} \) and near to it, so the first term on the right side has the effect of increasing the action away from the value it would have for a vacuum black hole. The dangerous term is the second one: the function \( fh \) is identically equal to unity for a black hole, but not for a fluid \[44\].

Now the radial null vectors in this geometry, which we denote by \( n^\mu \), can be taken to have the form

\[
n^\mu = (h^{1/2}, \pm f^{1/2}, 0, 0, 0)^T, \tag{14}
\]

and one can show \[44\] that, if \( R_{\mu\nu} \) is the Ricci tensor,

\[
R_{\mu\nu} n^\mu n^\nu = \frac{3 \left[ f h \right]'}{2 rh}, \tag{15}
\]

where the dash denotes a derivative with respect to \( r \). But the Null Ricci Condition or NRC is the statement that the Ricci tensor satisfies

\[
R_{\mu\nu} n^\mu n^\nu \geq 0 \tag{16}
\]

for all null vectors. If the Einstein equations hold, the NRC is equivalent to the Null Energy Condition or NEC, the weakest of the classical energy conditions, which just
requires the stress-energy-momentum tensor $T_{\mu\nu}$ to satisfy $T_{\mu\nu} n^\mu n^\nu \geq 0$ for all null vectors. Assuming that these conditions hold, we see that, if $f_h$ is not identically equal to unity, it has always to increase towards its asymptotic value, which is unity; therefore it must always be smaller than its value in the vacuum case, which is identically equal to unity. Since the domain of integration is the same and the integrand becomes smaller if matter satisfying the NEC is considered, the effect of the fluid is to diminish the second term on the right side of equation (13).

We conclude that the right side of (13) increases from its black hole value if the fluid satisfies the NEC, and so the danger of instability due to uncontrolled pair-production of branes is reduced, never increased, under this assumption. The same conclusion holds for black holes with positively curved event horizons.

The reader might object at this point that while all may be well at the classical level, Hawking radiation, with which we are of course directly concerned in this work, is often associated precisely with violations of the NEC. This actually does happen in the most familiar case, in which the black hole shrinks as it radiates. For, in that case, photons which are inside the apparent horizon at one point will find themselves outside it subsequently, leading to an expansion of the corresponding null congruence; this violates the NEC, by the null version of the Raychaudhuri equation [45].

Toral black holes, however, do not shrink to arbitrarily small size: their specific heat, being a positive multiple of the area of the event horizon, is always positive for all mass values. As we discussed earlier, they settle down to a static equilibrium with their own Hawking radiation. Thus we do not need to be concerned about NEC violation for “large” black holes in the case of positively curved event horizons, or for any black hole in the flat case. On the other hand, there will be strong violations of the NEC in the final stages of the evaporation of a “small” spherical AdS black hole, and the brane action could well become negative in that situation. By its very nature, however, this kind of black hole is a transient phenomenon; when the evaporation is complete, the NEC is no longer violated, and the brane action will cease to be negative. Thus we cannot conclude that “small” spherical AdS black holes should be excluded from the sum over black holes [11]. However, eternal black holes are clearly easier to understand, so we shall focus on them for the remainder of this work.

To summarize: “large” AdS black holes with positively curved event horizons, and all AdS black holes with flat event horizons, settle down to a static state which is non-perturbatively stable in string theory. In particular, we conclude that if we need to survey all AdS black holes which are compatible with string theory, we have to include the flat case. As we shall now argue, the geometric and topological differences between AdS black holes with the two kinds of possible event horizons lead to drastically different thermodynamic behaviour in their interiors.

4. Time Inside AdS Black Holes

We now begin our search for traces of the kind of “conspiracy” which Gottesman and Preskill demand if the Horowitz-Maldacena proposal is to be made to work. This raises

\footnote{AdS black holes with negatively curved event horizons are, by contrast, eternal, so the brane action is permanently negative in that case.}
fundamental issues regarding the nature of time inside AdS black holes.

The question as to the direction of time inside black holes has been debated extensively [46]. The point is this. There are strong arguments [18] to the effect that the observed arrow of time arises from some property of the spacetime edge corresponding to the creation of the Universe: the entropy associated with this edge was extraordinarily low, and the arrow is due to the natural evolution towards more generic states. One naturally asks: why should the arrow not point away from all spacetime edges, leading to a “reversal” of the arrow of time inside black holes? Conversely, if one has an argument which “proves” that the arrow cannot point away from the spacetime edge inside a black hole, then one must immediately explain why the same argument does not “prove” that our Universe should have no arrow at all — that is, why the argument cannot be applied to the spacetime edge at which the Universe was created [12].

We see from this that all such questions can only be answered in the context of a specific theory of the arrow of time. In such a theory one might find that an arrow pointing away from the spacetime edge exists for “initial” cosmological edges but not for any black hole edge [as Penrose postulates [21]], or that it occurs in some black holes contributing to a “sum over geometries” but not in a way that dominates the sum. The point is that such conclusions must be a matter of deduction from a specific theory; the question cannot be settled by appealing to standard statistical-mechanical expectations. For we know that “standard statistical-mechanical expectations” [to the effect that high-entropy states are generic] are not realised in the case of the only spacetime edge to which we have some observational access — the one associated with the Big Bang. It is therefore hard to see why they should be realised inside black holes.

One of the main observations of the present work is that the Gottesman-Preskill argument is based on the assumption that the arrow of time can never point away from a spacetime edge inside a black hole. They state, for example, that “the interior of a black hole is a tumultuous place”. This is indeed the case under “normal” circumstances. Gravitational entropy is not fully understood, but it is clearly associated with the degree to which spatial sections are anisotropic and inhomogeneous. If the arrow of time inside a black hole points towards the spacetime edge, then one will find that a small perturbation of the geometry inside the hole will cause the internal spatial sections to become more and more inhomogeneous and anisotropic as the classical “singularity” is approached, a tendency which becomes still more conspicuous as the spatial sections contract [because this increases the entropy density]. This is precisely the behaviour computed from classical general relativity [47], and it is the picture that Gottesman and Preskill have in mind. But if the arrow points away from the black hole spacetime edge, then a small perturbation will become smaller as the edge is approached, by means of what would look like an “implausible conspiracy”.

In the case of a realistic black hole, embedded in an asymptotically de Sitter spacetime which itself has an arrow of time, the assumption made by Gottesman and Preskill is entirely reasonable. For if we try to maintain two independent arrows pointing in opposite directions, producing two different but consistent accounts of the evolution of any given

---

12 If the ultra-low-entropy conditions at the beginning of our Universe were set up by some still earlier state [3][17], then one either applies the same argument to that state, or one has to explain why black hole singularities are resolved in a radically different way to their apparently similar cosmological counterparts.
system which enters the black hole, we will find that this requires fine-tuning on a scale below the Planck volume in phase space \[23\]. In reality, one would expect the system with the larger number of degrees of freedom — the outside world — simply to overwhelm the system with fewer; at most one would find inside the black hole some traces of a reversal. But that is all we need. The problem is to find a way of describing these “traces”; the hope is that AdS black holes may provide this description.

Consider first de Sitter spacetime. This spacetime is said to be globally hyperbolic \[48\], Chapter 8], which essentially means that it has spacelike surfaces [Cauchy surfaces] on which the prescription of initial data determines the evolution of matter and geometry for all subsequent time. Global hyperbolicity is clearly essential for the existence of an arrow of time of the kind we observe, which is apparently ubiquitous in both space and time. Such an arrow arises only because the prescription of ultra-low-entropy conditions on one Cauchy surface sufficed to enforce the second law of thermodynamics for all time. In fact, the existence of a universal arrow of time is the strongest evidence we have for the global hyperbolicity of our Universe.

When we turn to AdS black holes, we find that the environment is very different. Asymptotically AdS spacetimes are not globally hyperbolic; the future is not determined by data on a single Cauchy surface, since information can enter from infinity. It follows that such spacetimes do not, in general, have a universal arrow of time of the kind we observe in our Universe. This is reflected in the fact that asymptotically AdS spacetimes have a timelike Killing vector field defined everywhere except perhaps near and inside black holes.

An asymptotically AdS spacetime, left to its own devices, will therefore not contain stars; for stars are very low-entropy systems which, in our Universe, inherit that property from the systems of still lower entropy which characterized the Big Bang: they owe their existence to the ubiquity of the arrow of time. Thus we should not think of AdS black holes as forming in the same way as realistic black holes. We should instead think of them as systems which we “prepare” or “set up” in a background spacetime with no global arrow of time. Note that, in string theory, this “setting up” cannot be done in an arbitrary way; it has to be performed under the strict constraints imposed by the AdS/CFT correspondence \[34\] [49]. This keeps the data entering from infinity under control. [For reasons explained when we first discussed equation \(1\), we do this “setting up” in such a way that there is only one spacetime edge.]

The situation is particularly clear in the case of eternal AdS black holes: the hole is in equilibrium with a static gas of Hawking radiation, so it is natural to regard the exterior spacetime as having no arrow. Inside the black hole, however, there is no timelike Killing field, and so it becomes possible to imagine that a local arrow could exist there, pointing either towards or away from the spacetime edge.\[13\] If this local arrow points away from the spacetime edge, then it is possible to think of the resulting object as a “white hole”; but, as is well known \[51\], there is little to be gained from such a description in a quantum-mechanical treatment, and those observations are reinforced in this case by AdS/CFT considerations. We shall therefore not use this terminology.

\[13\]A complete AdS/CFT description of the geometrical dynamics inside a black hole is not available; see however \[50\] for a discussion of one possible way in which an arrow of time might arise in singular asymptotically AdS spacetimes.
The real point here is that if we should find a local arrow inside an eternal AdS black hole, pointing away from the spacetime edge, there will be no “clash of arrows”: for there is no arrow outside the event horizon. Thus the problem of “overdetermination” discussed by Zeh [23] does not arise here.

The situation of non-eternal black hole is different. Such a hole has a local arrow defined in its vicinity just outside the event horizon, defined by the very process of its complete evaporation. Note once more that this arrow arises because of the way the hole is “set up”, as above; it is not generated by low-entropy conditions at an initial time. An acceptable theory of the arrow of time should predict that such black holes do not have a “reversed” arrow inside the event horizon, since this will again avoid a “clash of arrows”.

We see that the kind of behaviour we are looking for here is only to be expected in the eternal case; also that, if it does arise in that case, it will not lead to any paradox. If the arrow is “reversed” inside some kind of eternal AdS black hole, it might be possible to show that the Horowitz-Maldacena hypothesis automatically incorporates Gottesman-Preskill “conspiracies”.

In summary: Gottesman and Preskill make a seemingly natural assumption about the internal thermodynamics of black holes, one which is reasonable for de Sitter holes but perhaps not [always] in the AdS case. We shall now investigate this question, in the specific context of the theory advanced in [16][17][18].

We begin by outlining this concrete proposal for the origin of the arrow of time. In this approach, the original universe [which may either be ours or one in which our universe nucleated as a Coleman-De Luccia bubble] comes into existence along a spacetime edge \( \Sigma \) representing a transition from a Euclidean to a Lorentzian space. This is in the manner of theories of “creation from nothing” [52], as updated by Ooguri et al [53]. We assume that \( \Sigma \) is a surface of minimal volume, reflecting the idea that, in a theory exhibiting T-duality, volumes below the value defined by the self-dual length are not permitted.

The possible “initial” data on \( \Sigma \) are sets of objects of the form \((\rho, J^a, K_{ab}, h_{ab})\), where \( \rho \) is a function, \( J^a \) is a vector field, \( K_{ab} \) is a symmetric tensor [which is traceless in our case, because of the minimality condition], and \( h_{ab} \) is a Riemannian metric, all defined on \( \Sigma \). The number of possible distinct sets of this kind is a measure of the initial entropy of the Universe. Crucially, however, the number of initial data sets we need to consider is cut down by the constraints [48, Chapter 10], which require the sets to satisfy

\[
R(h) = K_{ab} K^{ab} + 16\pi \rho \tag{17}
\]

and

\[
D^a K_{ab} = -8\pi J_b, \tag{18}
\]

where \( D^a \) is the covariant derivative operator, and \( R(h) \) is the scalar curvature, defined by \( h_{ab} \).

Now let \( N^\mu \) be the field of inward-pointing unit normal vectors along \( \Sigma \), and let \( T^\mu \) be the corresponding energy-momentum flux vector. Recall that the Horowitz-Maldacena proposal entails a requirement that no information should exit the spacetime through the spacetime edge inside any black hole. That appears to be a reasonable condition to impose also at the creation of the universe; let us do so. The simplest possible mathematical formulation of this is to demand that \( T^\mu \) must not point outwards from spacetime. But when one solves the Einstein equation with initial data as above, it turns out that \( K_{ab} \) is
the extrinsic curvature of $\Sigma$, while $\rho$ is just the component of $T^\mu$ parallel to $N^\mu$; it is in fact the total energy density evaluated on $\Sigma$. [Similarly, $J^a$ is the projection onto $\Sigma$ of $T^\mu$.] Thus the condition that $T^\mu$ must not point outwards from spacetime is just a geometric way of formulating the condition that

$$\rho \geq 0$$

(19)
everywhere on $\Sigma$. Note that if this is inserted into (17), then every term on the right side is non-negative; so if the left side should vanish, so must every term on the right.

In summary: we have to consider all initial data sets of the form $(\rho, J^a, K_{ab}, h_{ab})$ subject to the conditions (17), (18), (19), and that $K_{ab}$ should be traceless with respect to $h_{ab}$. These may seem to be very mild constraints, and, in most cases, they are indeed mild. Generically, then, the initial entropy of a universe created in this way will be large, which is in accord with the usual principles of statistical mechanics.

Remarkably, however, this conclusion is not always valid. In particular, if [as in [53]] the universe is created along a surface $\Sigma$ with the topology of a torus, then the only data sets satisfying the above conditions are those of the form $(0, 0, 0, p_{ab})$, where $p_{ab}$ is a flat metric on the torus. This follows by combining (17), (18), and (19) with the following extremely deep theorem [see [32][16]]:

**THEOREM (Schoen-Yau-Gromov-Lawson-Bourguignon):** Let $h_{ab}$ be a Riemannian metric on a torus or on any non-singular quotient of a torus. If the scalar curvature of $h_{ab}$ is non-negative everywhere, then $h_{ab}$ is exactly flat.

We see that, if $\Sigma$ has the topology of a torus, then the corresponding universe has an entropy on the spacetime edge which is in fact essentially as low as it can be, at least semi-classically. [As we discuss in the next section, flatness does not determine the geometry of a topological torus completely, but the range of possibilities here is obviously infinitesimal compared to the full set of possible Riemannian metrics on the torus.] Such a universe has an arrow of time, pointing away from the edge. This is a realisation of [the cosmological part of] Penrose’s “Weyl Curvature Hypothesis” [21]. The fact that this hypothesis can be derived as the end result of an exceedingly non-trivial mathematical analysis is a very appealing aspect of the present approach.

By contrast, similar techniques show that, if $\Sigma$ has the topology of a sphere, then (17), (18), and (19) are exceedingly weak: one way to see this is as follows. Given any function satisfying (19), let $F$ be any function on $\Sigma$ such that $F \geq 0$ everywhere. Then it is possible to prove [see [16] for a discussion] that the equation

$$R(h) = F + 16\pi\rho$$

(20)

always has a solution for $h_{ab}$ if the topology of $\Sigma$ is spherical, no matter what choices we make for $\rho$ and $F$. One then merely has to solve the algebraic problem of finding all symmetric tensors $K_{ab}$ which are traceless with respect to this $h_{ab}$ and which satisfy $F = K_{ab}K^{ab}$; obviously there are many such. Finally, if we use (18) to define $J^a$, we have an

---

14The vanishing of $\rho$, the total energy density at the edge, means that some kind of negative energy must be present there. In view of the toral topology, the Casimir effect [54][55] is an obvious candidate here.
initial data set which respects all constraints. Since $\rho$ and $F$ are completely arbitrary apart from being non-negative, it is clear that the number of possible initial value data sets is vast. In particular, both $\rho$ and $F$ could be extremely irregular, asymmetric functions, and then the metric which solves (20) will likewise be highly asymmetric, corresponding to an arbitrarily high gravitational entropy. A universe born along a $\Sigma$ with spherical topology will therefore not have an arrow.

Now let us try to apply this theory to the spacetime edge that [presumably] replaces the singularity inside an AdS black hole. We simply repeat the above reasoning in this case, that is, we just consider all possible “initial” data sets and impose (17)(18), and (19) [and require $K_{ab}$ to be traceless] on the edge and nothing more. [To do otherwise would expose us to the criticism [46] that we are building in a past/future distinction from the outset.] Evidently, everything hinges on the global structure of the black hole spacetime edge. This global structure is quite different to that of the cosmological spacetime edge discussed above, so it is entirely possible that the arrow of time behaves differently in this case. This is how we answer the question raised earlier: why should the arrow not point away from all spacetime edges? The point is that the arrow is intrinsically a *global* phenomenon, in the geometric sense.

To determine the global structure of the spacetime edge in the AdS black hole case, we use equation (1) inside the event horizon, letting $r$ and $t$ switch roles as usual [and remembering that $t$ is now allowed to take on negative values]. Whatever resolves the singularity will of course change the metric, and we are not allowed to assume that the spatial geometry is particularly symmetric; but since we are only interested in the global structure at this point, this will not matter. A spatial section near to the singularity will be given by $r = a$, where $a$ is a small constant, and the metric of this spatial section is, from (1),

$$h(r = a) = \left[ -\frac{a^2}{L^2} - k + \frac{16\pi M}{3A_k a^2} \right] dt^2 + a^2 d\Omega^2_k. \quad (21)$$

Note that the coefficients of both terms are positive constants.

If we assume for the moment that $t$ runs from $-\infty$ to $+\infty$, then it is clear that this metric is a *complete* Riemannian metric on a manifold of topology $\mathbb{R} \times C_k$. Here, since the metric is Riemannian, “completeness” can be defined [50] either in terms of the convergence of Cauchy sequences or in terms of inextensibility of geodesics: it essentially just means that the space has not been mutilated by means of arbitrary deletions. Whether the sections are complete or not, the situation here differs from the cosmological one considered earlier, in that the spatial sections here are non-compact and have infinite volume in the complete case. [The *topology* is the same in both cases; only the geometry differs.]

As we know, “small” black holes in the $k = 1$ case will evaporate completely. In this case, the spatial sections are in fact “mutilated” since the spatial sections inside the black hole are no longer able to extend out towards infinity: finiteness in time implies finiteness in space in this case. Thus the coordinate $t$ has a finite range here, and the spatial sections are no longer complete. We see that the completeness of the internal spatial sections allows us to give mathematical expression to the distinction between eternal black holes and those which evaporate completely\[^{15}\]. It does so in a way that does not

---

\[^{15}\]Horowitz and Maldacena [8] suggested that the infinite volume of spatial sections in the eternal case might prove to be important in some way.
commit us to the particular, highly symmetric metric given in (21).

We can now formulate the question of the genericity of the black hole edge metric as follows. We have to consider “initial” data in the form of sets \((\rho, J^a, K_{ab}, h_{ab})\), where \(\rho\), \(J^a\), and \(K_{ab}\) are defined as before, but where now \(h_{ab}\) is a metric defined on a manifold of topology \(\mathbb{R} \times \mathbb{C}^k\), on which the Riemannian structure may be complete or incomplete depending on whether the black hole is eternal or not. The question is whether the restrictions (17), (18), and (19) have any significant impact on the set of all such data.

If we allow the internal spacetime edge to be incomplete, then it is clear that the kind of argument we used earlier to impose powerful restrictions on initial value data sets cannot be made to work. For those restrictions were entirely global: they are due to subtle conditions arising when one tries to extend a solution of an equation like (20) to the entire “initial” surface. If we allow the Riemannian structure to be incomplete, then we can push any difficulties which might arise into some region which can then be conveniently deleted. We conclude that no black hole which evaporates entirely can have an independent, internal arrow of time. Thus there is no “reversal” of the arrow inside a “small” AdS black hole. This is consistent with the fact that the external spacetime has its own local arrow [near to but outside the event horizon] in this case.

We turn now to eternal AdS black holes. In the \(k = 1\) case, we are dealing with complete Riemannian metrics on manifolds of the form \(\mathbb{R} \times [S^3/\Gamma]\), where \(\Gamma\) is some finite group [which could be trivial] selected from a known [infinite] list \([35]\). It can be shown [see \([16]\) for a discussion] that any function on \(S^3/\Gamma\) can be the scalar curvature of some metric on that manifold. Even if we confine our attention to Riemannian products of \(\mathbb{R}\) with \(S^3/\Gamma\), this already yields a vast number of possible initial metrics, and of course there will be many more choices if we allow non-product metrics. The constraints (17), (18), and (19) then place hardly any restrictions on the “initial” energy density, energy flux, extrinsic curvature, or spatial metric. [See our earlier discussion around equation (20).] We conclude that there is no internal arrow of time even in the eternal case if the event horizon is positively curved. [Note that “positively curved” here really refers to the scalar curvature, so a “black ring” event horizon of the form \(S^1 \times S^2\) is “positively curved”. Therefore, if black rings exist in the AdS case — see \([33]\) — then their internal thermodynamics can be expected to be similar to that of AdS black holes with spherical event horizons.]

The last case is that of eternal AdS black holes with toral event horizons. Here we need to consider initial data sets \((\rho, J^a, K_{ab}, h_{ab})\), where now \(h_{ab}\) is a metric defined on a manifold of topology \(\mathbb{R} \times [T^3/\Delta]\), where \(T^3\) is the three-torus and where [in the orientable case] \(\Delta\) is one of six finite groups \([36]\) [including the trivial group]; note again that this space has to be complete with respect to \(h_{ab}\). Now the manifolds \(T^3/\Delta\) have a particular property: they are said to be enlargeable [\([32]\), page 302]. Roughly speaking this means that they can be “enlarged” to an arbitrary extent in all directions by taking a topological covering. This property of enlargeability [together with the fact that tori and their orientable non-singular quotients are spin manifolds] underlies the fact that tori and their quotients are unable to admit metrics of positive scalar curvature; and this, as we saw above, is the ultimate reason for the vast reduction in the set of admissible initial data for toral cosmologies.

Here, however, the situation is more complex, because \(\mathbb{R} \times [T^3/\Delta]\) is not enlargeable.
Thus, even in the case of toral event horizons, it is far from clear that the argument we used in the cosmological case will work again. Once more we stress that the [potential] difference between the behaviour of the arrow of time in the black hole and cosmological cases is not “built in” [46]: it arises simply because the global structures of the spatial sections in the two cases are so different.

Nevertheless, in this particular case we can proceed as follows. We have to work with the concept of weak enlargeability [32, page 318]. This is defined by studying certain maps between Riemannian spaces, which have the effect of changing the “sizes” of curvature two-forms rather than that of the space itself. This is weaker than true enlargeability because — crucially for us — it allows one dimension to escape being enlarged. Because of this, it turns out that [topological] products of the form $\mathbb{R} \times E$ are weakly enlargeable if the space $E$ is enlargeable. Thus, the spaces $\mathbb{R} \times [T^3/\Delta]$ are all weakly enlargeable.

The importance of this is revealed by the following theorem [32, page 319], which involves a delicate application of index theory:

**THEOREM (Gromov-Lawson-Kazdan):** Let $h_{ab}$ be a Riemannian metric of non-negative scalar curvature on a weakly enlargeable manifold, and assume that this manifold is complete with respect to $h_{ab}$. Then the Ricci tensor of $h_{ab}$ must vanish.

Now let us impose the restrictions (17), (18), and (19) on data sets $(\rho, J^a, K_{ab}, h_{ab})$ defined on a spacetime edge of the form $\mathbb{R} \times [T^3/\Delta]$. By the Gromov-Lawson-Kazdan theorem, we see that the left side of (17) has to vanish identically; hence, so must each term on the right, and so, by (18), must $J^a$. Thus we drastically cut down the number of possible data sets: they must be of the form $(0, 0, 0, q_{ab})$, where $q_{ab}$ is a complete metric which is very severely constrained by the requirement that its Ricci tensor vanishes exactly.

In fact, with specific spatial boundary conditions, this restriction will essentially determine $q_{ab}$ uniquely. But since the spatial sections extend to infinity here, the field theory at infinity will supply such boundary conditions. Thus $q_{ab}$ is fixed, and once again we have essentially unique initial data in this case. With the boundary conditions one expects here, $q_{ab}$ will be highly symmetric; see the discussion of the Lemma stated in Section 5 below. It is certainly unique [apart from well-understood ambiguities of the kind to be discussed in the next section], and maximally locally symmetric, for four-dimensional toral AdS black holes. For the spatial sections will then be three-dimensional, and it is an elementary fact that Ricci-flat metrics in that case are exactly flat. [In four dimensions there are of course non-flat Ricci-flat metrics on topologically complicated manifolds such as K3. But the universal cover of $\mathbb{R} \times [T^3/\Delta]$ is just the topologically trivial space $\mathbb{R}^4$, and any metric on $\mathbb{R} \times [T^3/\Delta]$ pulls back to $\mathbb{R}^4$; with the kind of boundary conditions we expect here, it seems very likely that the spatial metric will have to be flat — given that it is complete — in this case also.]

We see that there is an essentially unique “initial” data set for the interiors of toral AdS black holes, just as there is for toral cosmology, despite the non-compactness of the sections in the present case. We conclude that these AdS black holes, and these only, do have an internal arrow of time pointing in the opposite direction to what one would expect; that is, their arrow of time is reversed relative to their counterparts with positively
We note in passing that this part of the theory contradicts the Weyl Curvature Hypothesis [21] [which states that the Weyl curvature should vanish only at the “initial” cosmological spacetime edge].

We see that some terms in the sum over geometries do have unusual internal thermodynamic behaviour, while others do not. Let us turn to a more precise survey of all of the terms in this sum.

5. Different Black Holes with the Same Entropy

The next step is to perform the “sum over geometries” [analogous to the “sums” discussed by Maldacena [7] and Hawking [26]] to see whether the presence of toral black holes in the sum leads to the appropriate modification of the Horowitz-Maldacena proposal. Unfortunately, it is completely unknown how this can be done at this level of detail. We can however specify precisely the full set of AdS black holes over which the sum will have to be performed. We can also point out some hints from global geometry as to which terms in the sum are likely to dominate.

A realistic non-rotating neutral black hole in de Sitter spacetime is determined by its entropy. For there are no topological ambiguities in this case: a black hole formed by the collapse of a star obviously has a topologically spherical event horizon, and its mass is fixed by the area of this horizon.

The [five-dimensional] AdS black holes we have been discussing in this work, by contrast, are by no means specified by their entropy. There are several levels of ambiguity. First, given any value for the entropy, there will be AdS black holes of the same entropy with either positively curved or flat event horizons. Then, within each class, there are still many different black holes with that entropy. Let us consider the positive case first.

The entropy of the hole with metric (11) is proportional to the area of the event horizon, which is given by

\[ A_{eh} = A_k r_{eh}^3, \]  

where \( r_{eh} \) is the value of the radial coordinate at that surface. If \( k = 1 \), \( A_{eh} \) is given by

\[ A_{eh} = 2^{-3/2} L^3 A_1 \left[ -1 + \left( 1 + \frac{64\pi M}{3A_1 L^2} \right)^{1/2} \right]^{3/2}. \]  

For such a black hole, \( A_1 \) is \( 2\pi^2 \) if the topology of the event horizon is that of the [unit] sphere \( S^3 \). But as we know, the event horizon can have the topology of any non-singular quotient of the form \( S^3/\Gamma \), where \( \Gamma \) is a finite group [35]. For example, \( \Gamma \) can be any of the well-known ADE finite subgroups of SU(2) [the cyclic, quaternionic, and binary polyhedral groups], but there are other possibilities. If \( |\Gamma| \) is the order of this group, then the value of \( A_1 \) for \( S^3/\Gamma \) is \( 2\pi^2/|\Gamma| \). Thus \( A_1 \) can have many values extending downwards from \( 2\pi^2 \); because \( |\Gamma| \) is arbitrarily large for the lens spaces \( S^3/\mathbb{Z}_n \), where \( \mathbb{Z}_n \) is the cyclic group of order \( n \), \( A_1 \) can in fact be arbitrarily small. It takes the form \( 2\pi^2/n \), where every integer \( n \) is possible.

If we think of the area of the event horizon as a function of \( A_1 \), as given in equation (23), then it has an interesting behaviour as \( A_1 \) decreases from its maximum value of \( 2\pi^2 \). If the mass of the black hole is small compared to \( L^2 \) [that is, if it has a value typical for a
black hole which evaporates completely], then taking the quotient of the event horizon by small groups $\Gamma$ actually *increases* the entropy of the black hole for a fixed value of $M$. For larger groups, however, the entropy is decreased by taking the quotient. For [sufficiently] “large” black holes [meaning that $M$ is large compared to $L^2$], the effect of taking the quotient is always to decrease the entropy.

More importantly for our purposes: suppose that we fix the entropy, that is, we fix the area of the event horizon. Then the dimensionless quantity $M/L^2$ can be expressed in terms of $A_1$:

$$M/L^2 = \frac{3}{16\pi} \left( \frac{A_{eh}^{2/3}}{L^2} \right) \left( \left( \frac{A_{eh}^{2/3}}{L^2} \right) A_1^{-1/3} + A_1^{1/3} \right).$$ (24)

The fact that $M$ can always be found for any value of $A_1$ means that, if $A_1$ changes [by taking quotients as above], the effects of this can always be compensated by choosing $M$ appropriately. The function on the right side in this equation has a global minimum at a formal value of $A_1 = A_{eh}/L^3$, so if $A_{eh}/L^3$ is much smaller than $2\pi^2$ then for a finite number of steps downwards one has to reduce $M$ in order to keep the entropy constant; subsequently [and always if $A_{eh}/L^3$ is not small] $M$ has to be increased.

We see from this that for each specified value of the entropy, there is a countable infinity of AdS black holes with positively curved event horizons having that entropy. Note that these black holes are not uniquely fixed by their masses [or by the integer $|\Gamma|$], because two different AdS black holes with positively curved event horizons can have the same entropy and the same mass: consider for example the black holes of mass $M$ and with event horizons having the respective geometries of $S^3/\mathbb{Z}_{120}$, $S^3/Q_{120}$, and $S^3/\tilde{I}$, where $Q_{120}$ is the quaternionic [binary dihedral] group of order 120 and $\tilde{I}$ is the binary icosahedral group \[35\]. In all three cases we have $A_1 = \pi^2/60$, so all three holes have the same entropy.

The sum over this sector of AdS black holes therefore takes the form of a discrete “sum” over the candidates for $\Gamma$, with groups of larger order generally corresponding to larger masses. We stress that the candidates for $\Gamma$ are completely and explicitly known: see \[35\].

We now turn to the case of AdS black holes with flat event horizons. Here the situation is quite different: there is a limited range of possible topologies, but many continuous parameters. The possible topologies \[36\] \[57\] in the orientable case [to which we shall confine ourselves in this work] are just the torus or torocosm $T^3$, the dicosm $T^3/\mathbb{Z}_2$, the tricosm $T^3/\mathbb{Z}_3$, the tetracosm $T^3/\mathbb{Z}_4$, the hexacosm $T^3/\mathbb{Z}_6$, and the didicosm or Hantzsche-Wendt space $T^3/[\mathbb{Z}_2 \times \mathbb{Z}_2]$. In each case, there are continuous parameters which distinguish manifolds of the same topology which have different [global] geometries. For example, in the case of the torus, one can cover $\mathbb{R}^3$ with tiles of various shapes and sizes; identifying all of the tiles produces, in every case, a space with the topology of $T^3$. There are six continuous parameters describing the possible shape and size of the fundamental domain for a torus. For the other orientable topologies, the number of continuous parameters is smaller [because some parameters have to be fixed in order to perform the projection to the quotient]. There are four for $T^3/\mathbb{Z}_2$, two for $T^3/\mathbb{Z}_3$, two for $T^3/\mathbb{Z}_4$, two for $T^3/\mathbb{Z}_6$, and three for $T^3/[\mathbb{Z}_2 \times \mathbb{Z}_2]$; see \[36\] for the details.

In each case, one can think of the parameter $K$, used in Section 3, as a function of the continuous parameters; in each case, $K$ ranges continuously from arbitrarily small to arbitrarily large values. If we fix the entropy of the black hole, then from equation (7) we
have
\[
\frac{M}{L^2} = \frac{3}{32\pi^2} \left( \frac{A_{eh}^{4/3}}{L^4} \right) K^{-1}.
\] (25)

Clearly it is always possible to adjust M to keep \( A_{eh} \) fixed, no matter how K varies over the six sets of flat compact orientable three-manifolds; for a given \( A_{eh} \), pick any of the six topologies, choose the parameters, compute K, and use (25) to deduce M. In other words, for each specified value of the entropy, there is an uncountable infinity of AdS black holes with flat event horizons having that entropy. The “sum” over black holes in this case is a discrete sum over the six topological classes, followed by a continuous sum over the size and shape parameters.

In conclusion, then, we have a complicated but complete and explicit procedure for finding all of the stable AdS black holes with a specified value for the entropy: we have two choices for the overall type of hole, discrete choices of the detailed topology in each class, and, in one case, a choice of continuous parameters. The problem now is to understand how to weight the terms in the “sum” and how to evaluate it.

There is an obvious sense in which, for fixed entropy, the black holes with flat event horizons “outnumber” those with positively curved event horizons. This is where the question of weighting the terms in the “sum” is crucial. The following line of thought may be relevant.

Our discussion in Section 3 underlined the fact that black holes with flat event horizons are, in a sense, “closer to being unstable” than their counterparts with positively curved event horizons. One can in fact use the global techniques we have been discussing here to formulate another sense in which these holes are “close to instability”: one can prove, using the theorem of Schoen et al stated above [and other methods discussed in [16]], the following lemma.

**LEMMA:** Let g be a Riemannian metric on a manifold with the topology of a torus or of a non-singular quotient of a torus. Then unless g is conformal to an exactly flat metric, it is conformal to a metric of constant negative scalar curvature.

The Euclidean version of the metric (1) for \( k = 0 \) is defined on a space which, after Euclidean time is compactified, does have a conformal boundary with the topology of a compact flat manifold; and the conformal structure at infinity is indeed that of a perfectly flat metric. But the Lemma means that any non-conformal deformation of the geometry at infinity will result in negative scalar curvature there. As we discussed in Section 2, this would immediately lead to an instability of the kind discussed by Seiberg and Witten. Thus the flat case is indeed “close” to being unstable in this strong [global-geometric] sense. Possibly this reduces the weight of black holes with flat event horizons in the “sum” over AdS black holes, so that they only make a small [but crucial] contribution to a sum which is dominated by spherical holes. [For example, if in general the boundary can be “set free” in the AdS/CFT correspondence [58], the above Lemma apparently implies that this is possible for spherical black holes but not for toral ones. Note that this “rigidity” enforces the geometrically regular boundary conditions discussed above when we applied the Gromov-Lawson-Kazdan theorem.]

One might also wish eventually to consider whether “black rings” — if they exist in the asymptotically AdS case [33] — may also have to be included; also whether unusual
behaviour of the kind described in [59] may have to be taken into account, along with recent, more sophisticated analyses of the nature of Hawking radiation itself [60]. A deeper understanding of the mechanism which prevents the black hole spacetime edge from being singular will of course be necessary; see for example [61] for a recent discussion of one aspect of this issue. Finally, one would of course like to relate any string-theoretic account of black hole evaporation to the well-known analysis of black hole entropy given by Strominger and Vafa [62].

6. Conclusion: The Centrality of the Arrow

The black hole information loss problem is disconcerting, because it apparently requires us to believe that we have failed to understand some fundamental aspect of either quantum mechanics or general relativity at scales where both are exceedingly well-tested. The Horowitz-Maldacena proposal [8] offers an escape from this dilemma. In that sense it is, despite its strangeness, the most conservative approach to the problem, and as such it merits further attention.

One of the main objectives of this work is to point out that the objection made by Gottesman and Preskill [11] to this proposal is essentially thermodynamic in nature. It is therefore based on the assumption that we understand the [gravitational] thermodynamics of systems near to spacetime edges. But in at least one case — the spacetime edge associated with the beginning of our Universe — this understanding is work in progress: the origin of the arrow of time has long been and remains a matter of much debate. It follows that we need to agree on a theory of the arrow before we can decide whether the Gottesman-Preskill objection is fatal to the Horowitz-Maldacena proposal. Conversely, the ability to salvage this proposal would be strong evidence in favour of any particular theory of the arrow.

In this work we have outlined some ideas leading in this direction. First, we note that both Maldacena [7] and Hawking [26] have emphasised that the information loss problem cannot be solved by studying just one geometry: some kind of “sum over geometries” will be required. Next, we observe that the correspondence between five-dimensional de Sitter black holes and their AdS$_5$ counterparts [for which there is strong evidence that unitarity is indeed preserved, particularly in the eternal case] is far from being one-to-one; the range of possible AdS$_5$ holes with a given entropy is large but can be surveyed. One therefore knows exactly which geometries will have to be “summed over”. Crucially, we applied a specific theory of the arrow to deduce that the internal thermodynamics of AdS black holes is “normal” for spherical holes but “reversed” for toral holes. A sum over all AdS$_5$ black holes of the same entropy will therefore sample thermodynamic behaviour of both kinds; and there are hints that the sum will be dominated by the “normal” variety, though not to the total exclusion of the more unusual behaviour found in the toral case. This holds out hope that the Gottesman-Preskill objection can eventually be answered.

Acknowledgement

The author is grateful to Dr. Soon Wannei for useful discussions.
References

[1] Sidney R. Coleman, Frank De Luccia, Gravitational Effects On and Of Vacuum Decay, Phys.Rev. D21 (1980) 3305

[2] Stephen D.H. Hsu, David Reeb, Unitarity and the Hilbert space of quantum gravity, arXiv:0803.4212

[3] Sean M. Carroll, Jennifer Chen, Spontaneous Inflation and the Origin of the Arrow of Time, arXiv:hep-th/0410270

[4] Andreas Albrecht, Cosmic Inflation and the Arrow of Time, in Science and Ultimate Reality: Quantum Theory, Cosmology and Complexity, eds J. D. Barrow, P.C.W. Davies, C.L. Harper, Cambridge University Press (2004), arXiv:astro-ph/0210527

[5] D. H. Coule, Difficulties with inflationary initial conditions, arXiv:0706.0205

[6] Edward Witten, Anti-de Sitter Space, Thermal Phase Transition, and Confinement in Gauge Theories, Adv.Theor.Math.Phys. 2 (1998) 505, arXiv:hep-th/9803131

[7] Juan M. Maldacena, Eternal Black Holes in AdS, JHEP 0304 (2003) 021, arXiv:hep-th/0106112

[8] Gary T. Horowitz, Juan Maldacena, The black hole final state, JHEP 0402 (2004) 008, arXiv:hep-th/0310281

[9] F. Alexander Bais, J. Doyne Farmer, The Physics of Information, arXiv:0708.2837

[10] Lee Smolin, The Status of cosmological natural selection, arXiv:hep-th/0612185

[11] Daniel Gottesman, John Preskill, Comment on “The black hole final state”, JHEP 0403 (2004) 026, arXiv:hep-th/0311269

[12] Lisa Dyson, Matthew Kleban, Leonard Susskind, Disturbing Implications of a Cosmological Constant, JHEP 0210 (2002) 011, arXiv:hep-th/0208013

[13] Robert M. Wald, The Arrow of Time and the Initial Conditions of the Universe, arXiv:gr-qc/0507094

[14] Stephen D. H. Hsu, Brian M. Murray, Thermal Gravity, Black Holes and Cosmological Entropy, Phys.Rev. D73 (2006) 044017, arXiv:hep-th/0512033

[15] T. Banks, Entropy and initial conditions in cosmology, arXiv:hep-th/0701146

[16] Brett McInnes, Arrow of Time in String Theory, Nucl. Phys. B782 (2007) 1, arXiv:hep-th/0611088

[17] Brett McInnes, Initial Conditions for Bubble Universes, Phys. Rev. D 77 (2008) 123530, arXiv:0705.4141

[18] Brett McInnes, The Arrow of Time in the Landscape, to appear in R. Vaas (ed.): Beyond the Big Bang. Springer: Heidelberg 2008 arXiv:0711.1656
[19] Martin Bojowald, Reza Tavakol, Recollapsing quantum cosmologies and the question of entropy, arXiv:0803.4484

[20] Samir D. Mathur, What is the state of the Early Universe?, arXiv:0803.3727

[21] R. Penrose, Singularities and Time-Asymmetry, in General Relativity: An Einstein Centenary Survey, eds S W Hawking, W Israel, Cambridge University Press, 1979

[22] Guido Festuccia, Hong Liu, The Arrow of time, black holes, and quantum mixing of large N Yang-Mills theories, JHEP 0712 (2007) 027, arXiv:hep-th/0611098

[23] H. D. Zeh, The Physical Basis of The Direction of Time, Fifth Edition, Springer, Heidelberg, 2007

[24] D. T. Son, A. O. Starinets, Viscosity, Black Holes, and Quantum Field Theory, Ann.Rev.Nucl.Part.Sci. 57 (2007) 95, arXiv:0704.0240

[25] Andy Strominger, Les Houches lectures on black holes, hep-th/9501071

[26] S.W. Hawking, Information loss in black holes, Phys.Rev.D72 (2005) 084013, arXiv:hep-th/0507171

[27] R.B. Mann, Pair Production of Topological anti de Sitter Black Holes, Class.Quant.Grav. 14 (1997) L109, arXiv:gr-qc/9607071

[28] Danny Birmingham, Topological Black Holes in Anti-de Sitter Space, Class.Quant.Grav. 16 (1999) 1197, arXiv:hep-th/9808032

[29] Nathan Seiberg, Edward Witten, The D1/D5 System And Singular CFT, JHEP 9904 (1999) 017, arXiv:hep-th/9903224

[30] Juan Maldacena, Liat Maoz, Wormholes in AdS, JHEP 0402 (2004) 053, arXiv:hep-th/0401024

[31] M. Kleban, M. Porrati, R. Rabadan, Stability in Asymptotically AdS Spaces, JHEP 0508 (2005) 016, arXiv:hep-th/0409242

[32] H. Blaine Lawson and Marie-Louise Michelsohn, Spin Geometry, Princeton University Press, Princeton, 1984

[33] Roberto Emparan, Harvey S. Reall, Black Holes in Higher Dimensions, arXiv:0801.3471

[34] Lukasz Fidkowski, Veronika Hubeny, Matthew Kleban, Stephen Shenker, The Black hole singularity in AdS/CFT, JHEP 0402:014,2004, arXiv:hep-th/0306170

[35] J.A. Wolf, Spaces of Constant Curvature, Fifth Edition, Publish or Perish Press, 1984

[36] John Horton Conway, Juan Pablo Rossetti, Describing the platycosms, arXiv:math.DG/0311476
[37] Jeffrey R. Weeks, Detecting topology in a nearly flat hyperbolic universe, Mod.Phys.Lett. A18 (2003) 2099, arXiv:astro-ph/0212006

[38] George Koutsoumbas, Eleftherios Papantonopoulos, George Siopsis, Discontinuities in Scalar Perturbations of Topological Black Holes, arXiv:0806.1452

[39] R. Schoen, Conformal deformation of a Riemannian metric to constant scalar curvature. J. Differential Geom. 20 (1984) 479

[40] Brett McInnes, The Phantom Divide in String Gas Cosmology, Nucl.Phys. B718 (2005) 55, arXiv:hep-th/0502209

[41] Sumati Surya, Kristin Schleich, Donald M. Witt, Phase Transitions for Flat adS Black Holes, Phys.Rev.Lett. 86 (2001) 5231, arXiv:hep-th/0101134

[42] G.J. Galloway, S. Surya, E. Woolgar, A uniqueness theorem for the adS soliton, Phys.Rev.Lett. 88 (2002) 101102, arXiv:hep-th/0108170

[43] G.J. Galloway, S. Surya, E. Woolgar, On the Geometry and Mass of Static, Asymptotically AdS Spacetimes, and the Uniqueness of the AdS Soliton, Commun.Math.Phys. 241 (2003) 1, arXiv:hep-th/0204081

[44] Ted Jacobson, When is $g_{tt}g_{rr} = -1$?, arXiv:0707.3222

[45] Andrew Strominger, David Thompson, A Quantum Bousso Bound, Phys.Rev. D70 (2004) 044007, arXiv:hep-th/0303067

[46] Huw Price, Cosmology, Time’s Arrow, and That Old Double Standard, in Time’s Arrows Today, ed S. Savitt, Cambridge University Press 1994, arXiv:gr-qc/9310022, The Thermodynamic Arrow: Puzzles and Pseudo-puzzles, arXiv:physics/0402040

[47] Giovanni Montani, Marco Valerio Battisti, Riccardo Benini, Giovanni Impenente, Classical and Quantum Features of the Mixmaster Singularity, arXiv:0712.3008

[48] Robert M. Wald, General Relativity, Chicago University Press, 1984

[49] Guido Festuccia, Hong Liu, Excursions beyond the horizon: Black hole singularities in Yang-Mills theories (I), JHEP 0604 (2006) 044, arXiv:hep-th/0506202

[50] Thomas Hertog, Gary T. Horowitz, Holographic Description of AdS Cosmologies, JHEP 0504 (2005) 005, arXiv:hep-th/0503071

[51] S.W. Hawking, Black Holes and Thermodynamics, Phys. Rev. D13 (1976) 191

[52] A. Vilenkin, Creation of Universes from Nothing, Phys.Lett.B117 (1982) 25

[53] Hirosi Ooguri, Cumrun Vafa, Erik Verlinde, Hartle-Hawking Wave-Function for Flux Compactifications: The Entropic Principle, Lett.Math.Phys. 74 (2005) 311, arXiv:hep-th/0502211

[54] D.H. Coule, Quantum Cosmological Models, Class.Quant.Grav. 22 (2005) R125, arXiv:gr-qc/0412026
[55] Nima Arkani-Hamed, Sergei Dubovsky, Alberto Nicolis, Enrico Trincherini, Giovanni Villadoro, A Measure of de Sitter Entropy and Eternal Inflation, JHEP 0705 (2007) 055, arXiv:0704.1814

[56] S. Kobayashi, K. Nomizu Foundations of Differential Geometry I, Interscience, 1963

[57] Alain Riazuelo, Jeffrey Weeks, Jean-Philippe Uzan, Roland Lehoucq, Jean-Pierre Luminet, Cosmic microwave background anisotropies in multi-connected flat spaces, Phys.Rev. D69 (2004) 103518, arXiv:astro-ph/0311314

[58] Geoffrey Compe, Don Marolf, Setting the boundary free in AdS/CFT, arXiv:0805.1902

[59] K.A. Bronnikov, E. Elizalde, S.D. Odintsov, O.B. Zaslavskii, Horizons vs. singularities in spherically symmetric space-times, arXiv:0805.1095

[60] Tanmay Vachaspati, Dejan Stojkovic, Quantum radiation from quantum gravitational collapse, Phys. Lett. B663 (2008) 107, arXiv:gr-qc/0701096

[61] Eric Greenwood, Dejan Stojkovic, Quantum gravitational collapse: Non-singularity and non-locality, arXiv:0802.4087

[62] A. Strominger, C. Vafa, Microscopic Origin of the Bekenstein-Hawking Entropy, Phys.Lett. B379 (1996) 99, arXiv:hep-th/9601029