NAVIER–STOKES EQUATIONS, THE ALGEBRAIC ASPECT

V. V. Zharinov*

We present an analysis of the Navier–Stokes equations in the framework of an algebraic approach to systems of partial differential equations (the formal theory of differential equations).

Keywords: Navier–Stokes equations, integrability condition, evolution, constraints, differential algebra, symmetries, cohomology

DOI: 10.1134/S0040577921120011

1. Preliminaries

1.1. Navier–Stokes equations. The Navier–Stokes equations in the physical notation take the form (see, e.g., [1]–[6])

\begin{align*}
    \mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} &= \nu \Delta \mathbf{u} - \nabla p, \\
    \nabla \cdot \mathbf{u} &= 0,
\end{align*}

where \( \mathbf{u} = (u^1, u^2, u^3) \) is the velocity field, \( t \) is the time variable, the dot \( \cdot \) denotes the scalar product, \( \nabla = (\nabla_1, \nabla_2, \nabla_3) \) is the gradient with respect to the spatial variables \( \mathbf{x} = (x^1, x^2, x^3) \), the parameter \( \nu > 0 \) is the viscosity of the flow (not to be confused with the index \( \nu \)), \( \Delta = \nabla_1^2 + \nabla_2^2 + \nabla_3^2 \) is the Laplacian, and \( p \) is the pressure.

Here, we study the Navier–Stokes equations from the algebro-geometrical standpoint. System (1), (2) is not formally integrable, as we need; to obtain an equivalent formally integrable system, we add the trivial differential prolongations (see, e.g., [7])

\begin{align*}
    \nabla \cdot \mathbf{u}_t &= 0, \\
    \nabla (\nabla \cdot \mathbf{u}) &= 0
\end{align*}

and the nontrivial differential prolongation (\textit{hidden integrability condition})

\begin{align*}
    \Delta p + \nabla ((\mathbf{u} \cdot \nabla)\mathbf{u}) &= \Delta p + \nabla \mathbf{u} \cdot \nabla \mathbf{u} = 0
\end{align*}

(we use Eq. (2) here).

\textbf{Remark 1.} Equation (4) is a Poisson equation for the pressure \( p \) with the density \( \rho = -\nabla \mathbf{u} \cdot \nabla \mathbf{u} \). It can be regarded as the \textit{inner differential constraint} for the Navier–Stokes equations.

*Steklov Mathematical Institute, Russian Academy of Sciences, Moscow, Russia, e-mail: zharinov@mi-ras.ru.

Translated from \textit{Teoreticheskaya i Matematicheskaya Fizika}, Vol. 209, No. 3, pp. 397–413, December, 2021. Received October 4, 2021. Revised October 4, 2021. Accepted October 6, 2021.
1.2. Notation. We use a rather sophisticated notation.

- \( \mathbb{Z} = \{0, \pm 1, \pm 2, \ldots \} \supset \mathbb{Z}_+ = \{0, 1, 2, \ldots \} \supset \mathbb{N} = \{1, 2, 3, \ldots \} \);
- \( \mathbb{M} = \overline{1, m} = \{1, 2, \ldots, m\} \), \( \mathbb{N} = \overline{2, m} = \{2, \ldots, m\} \), \( \mathbb{M} = \{1\} \cup \mathbb{N} \);
- \( \mathfrak{I} = \mathbb{Z}^+ = \{i = (i^1, \ldots, i^m) \mid i^\mu \in \mathbb{Z}_+, \mu \in \mathbb{M}\} \), \( |i| = i^1 + \cdots + i^m \);
- \( i + j = (i^1 + j^1, \ldots, i^m + j^m) \), \( i + (\mu) = (i^1, \ldots, i^\mu + 1, \ldots, i^m) \), \( (\mu) = 0 + (\mu) \), for all \( i, j \in \mathfrak{I}, \mu \in \mathbb{M} \);
- \( \mathcal{J} = \mathbb{Z}^N = \{j = (j^2, \ldots, j^m) \mid j^\alpha \in \mathbb{Z}_+, \alpha \in \mathbb{N}\} \);
- \( \mathfrak{I}_0 = \{i \in \mathfrak{I} \mid i^1 = 0\} = \{0\} \times \mathfrak{I}, \mathfrak{I}_1 = \{i \in \mathfrak{I} \mid i^1 = 0, 1\} = \{0, 1\} \times \mathfrak{I} \);
- \( \mathfrak{I}_0' = \mathfrak{I} \setminus \mathfrak{I}_0 = \{i \in \mathfrak{I} \mid i^1 > 0\}, \mathfrak{I}_1' = \mathfrak{I} \setminus \mathfrak{I}_1 = \{i \in \mathfrak{I} \mid i^1 > 1\}, \mathfrak{I} = \mathfrak{I}_0 \cup \mathfrak{I}_0' = \mathfrak{I}_1 \cup \mathfrak{I}_1' \).

We also use the notation

- \( \mathbb{X} = \mathbb{R}^\mathbb{M} = \{x = (x^1, \ldots, x^m) \mid x^\mu \in \mathbb{R}, \mu \in \mathbb{M}\} \);
- \( \mathbb{R}_i^\mu = \{u^\mu \mid u^\mu_i \in \mathbb{R}, \mu \in \mathbb{M}, i \in \mathfrak{I}\} \), \( \mathbb{R}_i^\alpha = \{u^\alpha \mid u^\alpha_i \in \mathbb{R}, \alpha \in \mathbb{N}, i \in \mathfrak{I}\}, \mathbb{R}_i^M = \mathbb{R}_i^1 \times \mathbb{R}_i^N = \mathbb{R}_{i_0}^1 \times \mathbb{R}_{i_0}^N \times \mathbb{R}_i^1 \);
- \( \mathbb{B} = \mathbb{X} \times \mathbb{R}_i^\mathbb{M} \times \mathbb{R}_i, \mathbb{CE} = \mathbb{X} \times \mathbb{R}_{i_0}^1 \times \mathbb{R}_i^N \times \mathbb{R}_i, \mathbb{CPE} = \mathbb{X} \times \mathbb{R}_{i_0}^1 \times \mathbb{R}_i^N \times \mathbb{R}_i \).

We assume the summation over repeated upper and lower indices in suitable limits.

The paper is largely based on calculations, which are straightforward but tiresome, and we omit them in most cases.

1.3. The base components. The base space for the Navier–Stokes equations in the algebraic approach is the infinite-dimensional space \( \mathbb{B} = \mathbb{X} \times \mathbb{R}_i^\mathbb{M} \times \mathbb{R}_i \), where \( \mathbb{X} \) is the set of independent variables (spatial coordinates), \( \mathbb{R}_i^\mathbb{M} \) is the set of differential variables (velocity coordinates and their partial derivatives), and \( \mathbb{R}_i \) is the set of differential variables (pressure and its partial derivatives).

The base algebra in the algebraic approach is the unital commutative associative algebra \( \mathcal{A}(\mathbb{B}) = C_c^\infty(\mathbb{B}) \) of all smooth real functions on the base space \( \mathbb{B} \) of a finite order, i.e., depending on a finite number of variables \( x^\mu, u^\mu_i, p_i \), where \( \mu \in \mathbb{M}, i \in \mathfrak{I} \). In more detail, the integer \( r \in \mathbb{Z}_+ \) is called the \( u \)-order of a function \( f(x, u, p) \in \mathcal{A}(\mathbb{B}) \), and we write \( \text{ord}_u f = r \), if the partial derivative \( \partial_{u^\mu} f \neq 0 \) for some variable \( u^\mu_i, |i| = r \), while partial derivatives \( \partial_{u^\mu} f = 0 \) for all \( |i| > r \). The \( p \)-order is defined similarly.

The base algebra of derivations of the algebra \( \mathcal{A}(\mathbb{B}) \) is the Lie algebra

\[
\mathfrak{D}(\mathbb{B}) = \mathfrak{D}(\mathcal{A}(\mathbb{B})) = \{\zeta = \zeta^\mu \partial_{x^\mu} + \zeta^\mu_i \partial_{u^\mu_i} + \zeta_i \partial_{p_i} \mid \zeta^\mu, \zeta^\mu_i, \zeta_i \in \mathcal{A}(\mathbb{B})\}.
\]

The Lie algebra \( \mathfrak{D}(\mathbb{B}) \) splits into the \textit{vertical} and \textit{horizontal} subalgebras

\[
\mathfrak{D}(\mathbb{B}) = \mathfrak{D}(\mathfrak{V}(\mathbb{B})) \oplus \mathfrak{A}(\mathbb{B}) \mathfrak{D}(\mathfrak{H}(\mathbb{B})),
\]

where
\( \mathfrak{D}_V(B) = \{ \zeta \in \mathfrak{D}(B) \mid \zeta|_{C^\infty(X)} = 0 \} = \{ \zeta = \zeta^\mu_{ij} \partial_{u^\mu} + \zeta_i \partial_p \mid \zeta^\mu_{ij}, \zeta_i \in \mathfrak{A}(B) \} \);

\( \mathfrak{D}_H(B) = \{ \zeta = \zeta^\mu D_\mu \mid \zeta^\mu \in \mathfrak{A}(B) \}, \quad (D_\mu|_{C^\infty(X)} = \partial_{x^\mu}) \)
\[
D_\mu = \partial_{x^\mu} + u^\lambda_{i+} \partial_{u^\lambda} + p_{i+} \partial_p, \quad \lambda, \mu \in M;
\]

\([D_\mu, D_\nu] = 0, \quad [D_\mu, \zeta] = (D_\mu \zeta^\lambda_{i+} - \zeta^\lambda_{i+} \partial_{u^\lambda}) D_\mu \zeta_i - \zeta_{i+} \partial_p, \]
for all \( \lambda, \mu, \nu \in M \), \( \zeta = \zeta^\mu \partial_{u^\mu} + \zeta_i \partial_p \in \mathfrak{D}_V(B) \).

We set
\[
D_i = (D_i)^{i^1} \ldots (D_i)^{i^m} \quad \text{for all} \quad i = (i^1, \ldots, i^m) \in \mathbb{I}.
\]

The pair \((\mathfrak{A}(B), \mathfrak{D}_H(B))\) is called the differential algebra associated with the base space \( B \).

The Lie algebra
\[
\text{Sym}(\mathfrak{A}(B), \mathfrak{D}_H(B)) = \{ \zeta = ev_i \in \mathfrak{D}_V(B) \mid [D_\mu, ev_i] = 0, \quad \mu \in M \}
\]
is the Lie algebra of symmetries of the differential algebra \((\mathfrak{A}(B), \mathfrak{D}_H(B))\), where

- \( f = (f^\mu, f) \in \mathfrak{A}^M(B) \times \mathfrak{A}(B), \quad f^\mu = \zeta^\mu_0, \quad f = \zeta_0, \)
- \( ev_i = D_i f^\mu \cdot \partial_{u^\mu} + D_i f \cdot \partial_p, \quad D_i f^\mu = \zeta^\mu_i, \quad D_i f = \zeta_i. \)

The \( \mathbb{Z} \)-graded \( \mathfrak{A}(B) \)-module \( \Omega^q_H(B) = \oplus_{q \in \mathbb{Z}} \Omega^q_H(B) \) of horizontal differential forms is defined as follows:

\[
\Omega^q_H(B) = \begin{cases} 
0, & q < 0, q > m, \\
\mathfrak{A}(B), & q = 0, \\
\text{Hom}_{\mathfrak{A}(B)}(\wedge^q \mathfrak{D}_H(B); \mathfrak{A}(B)), & 1 \leq q \leq m.
\end{cases}
\]

Here,
\[
\text{Hom}_{\mathfrak{A}(B)}(\wedge^q \mathfrak{D}_H(B); \mathfrak{A}(B)) = \{ \omega^\mu = \omega_{\mu_{i_1} \ldots \mu_i} \mid [\omega^\mu]_{q} \wedge \cdots \wedge [\omega^\mu]_{q} \in \mathfrak{A}(B), +s-s \},
\]

where the abbreviation \(+s-s\) indicates that the components \( \omega_{\mu, i_1 \ldots i_q} \) are skew-symmetric in the indices \( \mu_1, \ldots, \mu_q \in M \).

The horizontal differential \( d_H \in \text{End}_R(\Omega^q_H(B)) \), \( d_H \circ d_H = 0 \), is defined by the rule
\[
d^q_H = d_H|_{\Omega^q_H(B)} : \Omega^q_H(B) \to \Omega^{q+1}_H(B),
\]
\[
\omega_{\mu_1 \ldots \mu_q} \cdot dx^\mu_1 \wedge \cdots \wedge dx^\mu_q \mapsto d_{[\mu_0, \mu_1 \ldots \mu_q]} \cdot dx^\mu_0 \wedge \cdots \wedge dx^\mu_q,
\]
the brackets \([\ldots]\) denote the skew-symmetrization in the indices \( \mu_0, \ldots, \mu_q \in M \), whence \( d^q_H \circ d^q_H = 0, q \in \mathbb{Z} \).

The \( \mathbb{Z} \)-graded linear space \( H(\Omega^q_H(B); d_H) = \oplus_{q \in \mathbb{Z}} \Omega^q_H(B) \) of the cohomology of the differential algebra \((\mathfrak{A}(B); \mathfrak{D}_H(B))\) is defined in the usual way, \( H(\Omega^q_H(B); d_H) = \text{Ker} d_H / \text{Im} d_H \), \( H^q_H(B) = \text{Ker} d^q_H / \text{Im} d^{q-1}_H \), \( q \in \mathbb{Z} \).
The following theorem holds (see, e.g., [8] and the references therein).

**Theorem 1** (the main theorem of the formal calculus of variations). We have the linear spaces

\[
H^q_H(B) = \begin{cases} 
0, & q < 0, q > m, \\
\mathbb{R}, & q = 0, \\
\mathcal{H}(B), & q = m,
\end{cases}
\]

where (in our setting) the Helmholtz linear space is

\[
\mathcal{H}(B) = \{ \chi = (\chi_\mu, \chi) \in \mathcal{A}_M(B) \times \mathcal{A}(B) \mid \chi_\mu = \chi^* \}
\]

and the linear maps \( \chi_\mu, \chi^* : \mathcal{A}_M(B) \times \mathcal{A}(B) \to \mathcal{A}_M(B) \times \mathcal{A}(B) \) act by the rules

\[
\mathcal{A}_M(B) \times \mathcal{A}(B) \ni f = (f^\mu, f) \mapsto \chi_\mu f = g = (g_\mu, g) \in \mathcal{A}_M(B) \times \mathcal{A}(B), \\
g_\mu = \partial_{u^\nu} \chi_\mu \cdot D_1 f^\nu + \partial_{p_\nu} \chi_\mu \cdot D_1 f, \\
\mathcal{A}_M(B) \times \mathcal{A}(B) \ni f = (f^\mu, f) \mapsto \chi^* f = g = (g_\mu, g) \in \mathcal{A}_M(B) \times \mathcal{A}(B), \\
g_\mu = (-D_i) \cdot (f^\nu \cdot \partial_{u^\nu} \chi^* + f \cdot \partial_{u^\nu} \chi), \\
g_\mu = (-D_i) \cdot (f^\nu \cdot \partial_{p_\nu} \chi^* + f \cdot \partial_{p_\nu} \chi).
\]

The linear-space isomorphism \( \delta = (\delta_{u^\nu}, \delta_{p_\nu}) : H^m_H(B) \simeq \mathcal{H}(B) \) is defined by variational derivatives:

\[
\Omega^m_H(B) \ni \omega = L \cdot d^m x \mapsto \chi = \delta L = (\delta_{u^\nu} L, \delta_{p_\nu} L), \\
\delta_{u^\nu} L = (-D_i) \partial_{u^\nu} L, \\
\delta_{p_\nu} L = (-D_i) \partial_{p_\nu} L.
\]

**Remark 2.** We recall that the cohomology is defined up to isomorphisms.

2. Constraints

2.1. The divergence-free space. The continuity equation \( \text{CE} = \{ \partial_{x^\mu} u^\mu = 0 \} \) (see (2)) has the algebraic counterpart

\[
\text{CE} = \{ (x, u, p) \in B \mid \text{CE}_i = u^\mu_{i+1(\mu)} = 0, \ i \in \mathbb{I} \}.
\]

The subset \( \text{CE} \subset B \) is called the solution manifold of the equation \( \text{CE} \) because a smooth function \( \phi(x) = (\phi^\mu(x)) \in C^\infty(X; \mathbb{R}^M) \) is a solution of the equation \( \text{CE} \) if and only if

\[
\text{graph} \, j \phi(x) = \{ (x, u, p) \mid u^\mu_i = \partial_{x^\nu} \phi^\mu(x), \ \mu \in M, \ i \in \mathbb{I} \} \subset CE,
\]

where \( j \phi(x) = \{ \partial_{x^\nu} \phi^\mu(x) \mid i \in \mathbb{I}, \ \mu \in M \} \) is the jet of the function \( \phi(x) \) and the variables \( p = (p_i) \in \mathbb{R}^I \) are parameters, \( \partial_{x^\nu} = (\partial_{x^i})^1 \cdots \partial_{x^m}^m \), \( i \in \mathbb{I} \).

With the solution manifold \( \text{CE} \), we associate the ideal

\[
\mathcal{I} = \{ f \in \mathcal{A}(B) \mid f|_{\text{CE}} = 0 \} = \{ f = P(D)u^\mu_{i+1(\mu)} \mid P(D) \in \mathcal{A}(B)[D] \} = \\
= \{ f = \phi^i \cdot u^\mu_{i+1(\mu)} \mid \phi^i \in \mathcal{A}(B), \ i \in \mathbb{I} + \text{fin} \}
\]

of the algebra \( \mathcal{A}(B) \), where \( \mathcal{A}(B)[D] \) is the unital associative algebra of all polynomials in the variables \( D_1, \ldots, D_m \) with coefficients in \( \mathcal{A}(B) \), i.e.,

\[
\mathcal{A}(B)[D] = \{ P(D) = \phi^i \cdot D_i \mid \phi^i \in \mathcal{A}(B), \ i \in \mathbb{I} + \text{fin} \},
\]

where the abbreviation \( +\text{fin} \) states that \( \phi^i \neq 0 \) for only a finite number of the coefficients. The ideal \( \mathcal{I} \) is differential, i.e., \( D_\mu|_\mathcal{I} : \mathcal{I} \to \mathcal{I}, \ \mu \in M \).
By definition, we set $\mathcal{A}(\mathbf{CE}) = \mathcal{A}(\mathbf{B})/\mathcal{I}$, i.e., we define the algebra $\mathcal{A}(\mathbf{CE})$ of \textit{smooth functions on the space} $\mathbf{CE}$ as the quotient algebra. We let $\bar{f} = f|_{\mathbf{CE}} = f + \mathcal{I}$ denote the restriction of the function $f \in \mathcal{A}(\mathbf{B})$ to the solution manifold $\mathbf{CE}$ and its equivalence class in $\mathcal{A}(\mathbf{CE})$.

We need the following data:

- a subalgebra of the Lie algebra $\mathfrak{D}(\mathbf{B})$,
  \[
  \mathfrak{D}_\mathcal{I}(\mathbf{B}) = \{ \zeta \in \mathfrak{D}(\mathbf{B}) \mid \zeta|_\mathcal{I} : \mathcal{I} \to \mathcal{I} \} = \{ \zeta \in \mathfrak{D}(\mathbf{B}) \mid \zeta|_\mathcal{I} \in \mathfrak{D}(\mathcal{I}) \} = \\
  = \{ \zeta = \zeta^\mu \partial_{x^\mu} + \zeta_i \partial_{u_i^\mu} + \zeta_i \partial_p, \mid \zeta^\mu_i \in \mathcal{I}, i \in \mathbb{I} \},
  \]
  of all derivations of $\mathcal{A}(\mathbf{B})$ leaving the ideal $\mathcal{I}$ invariant;

- a subalgebra of the Lie algebra $\mathfrak{D}(\mathbf{B})$ and an ideal of the Lie algebra, $\mathfrak{D}_\mathcal{I}(\mathbf{B}) \subset \mathfrak{D}(\mathbf{B})$
  \[
  \mathfrak{D}(\mathbf{B}; \mathcal{I}) = \{ \zeta \in \mathfrak{D}(\mathbf{B}) \mid \zeta : \mathcal{A}(\mathbf{B}) \to \mathcal{I} \} = \\
  = \{ \zeta = \zeta^\mu \partial_{x^\mu} + \zeta_i \partial_{u_i^\mu} + \zeta_i \partial_p, \mid \zeta^\mu, \zeta_i \in \mathcal{I} \},
  \]
  of all derivations whose image belongs to the ideal $\mathcal{I}$;

- the quotient Lie algebra $\mathfrak{D}(\mathbf{CE}) = \mathfrak{D}_\mathcal{I}(\mathbf{B})/\mathfrak{D}(\mathbf{B}; \mathcal{I})$ of derivations of the quotient algebra $\mathcal{A}(\mathbf{CE})$.

Here, the map $\mathfrak{D}_\mathcal{I}(\mathbf{B}) \to \mathfrak{D}(\mathbf{CE})$ is defined as

\[
\mathfrak{D}_\mathcal{I}(\mathbf{B}) \ni \zeta \mapsto \bar{\zeta} = \zeta + \mathfrak{D}(\mathbf{B}; \mathcal{I}) : \mathcal{A}(\mathbf{CE}) \to \mathcal{A}(\mathbf{CE}), \\
\bar{f} = f + \mathcal{I} \mapsto \bar{\zeta} = \bar{f},
\]

(we let $\bar{\zeta} = \zeta + \mathfrak{D}(\mathbf{B}; \mathcal{I})$ denote the equivalence class of a derivation $\zeta \in \mathfrak{D}_\mathcal{I}(\mathbf{B})$).

The Lie algebra $\mathfrak{D}(\mathbf{CE})$ splits into the \textit{vertical} and \textit{horizontal} subalgebras:

- $\mathfrak{D}(\mathbf{CE}) = \mathfrak{D}_V(\mathbf{CE}) \oplus_{\mathcal{A}(\mathbf{CE})} \mathfrak{D}_H(\mathbf{CE})$;

- $\mathfrak{D}_V(\mathbf{CE}) = \{ \bar{\zeta} \in \mathfrak{D}(\mathbf{CE}) \mid \zeta \in \mathfrak{D}_V(\mathbf{B}) \cap \mathfrak{D}_\mathcal{I}(\mathbf{B}) \}$;

- $\mathfrak{D}_H(\mathbf{CE}) = \{ \bar{\zeta} \in \mathfrak{D}(\mathbf{CE}) \mid \zeta \in \mathfrak{D}_H(\mathbf{B}) \}$ (note, $\mathfrak{D}_H(\mathbf{B}) \subset \mathfrak{D}_\mathcal{I}(\mathbf{B})$).

The Lie algebra

\[
\text{Sym}(\mathcal{A}(\mathbf{CE}), \mathfrak{D}_H(\mathbf{CE})) = \{ \bar{\zeta} \in \mathfrak{D}_V(\mathbf{CE}) \mid [\bar{D}_\mu, \bar{\zeta}] = \bar{0} = \mathcal{I}, \mu \in \mathbb{M} \} = \\
= \{ \bar{\zeta} \mid \zeta = D_i f^\mu \cdot \partial_{u_i^\mu} + D_i f \cdot \partial_p, \quad D_\mu f^\mu \in \mathcal{I} \} = \\
= \{ \bar{\zeta} \mid \zeta = ev_l, \; f = (f^\mu, f) \in A^\mathbb{M}(\mathbf{B}) \times \mathcal{A}(\mathbf{B}), \; (D_\mu f^\mu)|_{\mathbf{CE}} = 0 \}
\]

is the Lie algebra of symmetries of the differential algebra $(\mathcal{A}(\mathbf{CE}), \mathfrak{D}_H(\mathbf{CE}))$.

The $\mathbb{Z}$-graded $\mathcal{A}(\mathbf{CE})$-module $\Omega^\mathbb{H}_H(\mathbf{CE}) = \oplus_{q \in \mathbb{Z}} \Omega^q_H(\mathbf{CE})$ of \textit{horizontal differential forms} is defined as

\[
\Omega^q_H(\mathbf{CE}) = \begin{cases} \\
0, & q < 0, \; q > m, \\
\mathcal{A}(\mathbf{CE}), & q = 0, \\
\text{Hom}_{\mathcal{A}(\mathbf{CE})}(\wedge^q \mathfrak{D}_H(\mathbf{CE}); \mathcal{A}(\mathbf{CE})), & 1 \leq q \leq m.
\end{cases}
\]

Here,

\[
\text{Hom}_{\mathcal{A}(\mathbf{CE})}(\wedge^q \mathfrak{D}_H(\mathbf{CE}); \mathcal{A}(\mathbf{CE})) = \{ \bar{\omega}^\mathbb{H} = \bar{\omega}_{\mu_1 \ldots \mu_q} \mid [\bar{\omega}^\mathbb{H} \wedge \ldots \wedge [\bar{\omega}^\mathbb{H}]_{\mu_1 \ldots \mu_q} \in \mathcal{A}(\mathbf{CE}), \; +\text{-s-s}, \}
\]

where $d\bar{\omega}^\mu(\bar{D}_\nu) = \delta^\nu \bar{\omega}^\mu + \mathcal{I}$, $\mu, \nu \in \mathbb{M}$.\[\text{1661}\]
The horizontal differential \( d_H \in \text{End}_\mathbb{R}(\Omega_H(CE)) \), \( d_H \circ d_H = 0 \), is defined by the rule

\[
d_H^q = d_H|_{\Omega_H^q(CE)} : \Omega_H^q(CE) \to \Omega_H^{q+1}(CE),
\]

\[
\bar{\omega}_{\mu_1 \ldots \mu_q} : d\bar{\omega}^{\mu_1} \wedge \cdots \wedge d\bar{\omega}^{\mu_q} \mapsto \bar{D}_{\mu_0} (\bar{\omega}_{\mu_1 \ldots \mu_q}) : d\bar{\omega}^{\mu_0} \wedge \cdots \wedge d\bar{\omega}^{\mu_q},
\]

where \( d_H^{q+1} \circ d_H^q = 0, q \in \mathbb{Z} \).

The \( \mathbb{Z} \)-graded linear space \( H(\Omega_H(CE); d_H) = \oplus_{q \in \mathbb{Z}} H_H^q(CE) \) is independent of the variable \( \vartheta \).

\[\text{The} \ H(\Omega_H(CE); d_H) = \text{Ker} d_H / \text{Im} d_H, \quad H_H^q(CE) = \text{Ker} d_H^q / \text{Im} d_H^{q-1}, \quad q \in \mathbb{Z}.\]

To proceed, and to calculate the cohomology spaces in particular, we introduce global coordinates on the solution manifold \( CE \). Namely, we split the linear space \( \mathbb{R}^M_\Omega \) as

\[\mathbb{R}^M_\Omega = \mathbb{R}^1 \times \mathbb{R}^N_\Omega = \mathbb{R}^1 \times \mathbb{R}^1 \times \mathbb{R}^N_\Omega \]

(see Sec. 1.2). Then the global coordinates on \( CE \) are defined by the linear-space isomorphism

\[\begin{align*}
\text{CE} = \{(x^\mu, u_1^\mu, p_1) \in \mathcal{B} & \mid u_1^\mu = 0\} \to X \times \mathbb{R}^1 \times \mathbb{R}^N_\Omega \times \mathbb{R}^1, \\
(x^\mu, u_1^\mu, p_1) & \mapsto (x^\mu, u_1^\mu, p_1),
\end{align*}\]

with the inverse map given by

\[\begin{align*}
X \times \mathbb{R}^1 \times \mathbb{R}^N_\Omega \times \mathbb{R}^1 & \to \{(x^\mu, u_1^\mu, p_1) \in \mathcal{B} & \mid u_1^\mu = 0\} = \text{CE}, \\
(x^\mu, u_1^\mu, p_1) & \mapsto (x^\mu, u_1^\mu, p_1),
\end{align*}\]

where \( u_1^\mu = -u_0^\beta \big|_{\mu_0 = -1, \beta} \), \( \mu = (1, \alpha) \in M, i_0 \in I_0, \alpha, \beta \in N, \iota_1 \in \mathbb{R}^1 \). In these global coordinates, we have

- \( \text{CE} = X \times \mathbb{R}^1 \times \mathbb{R}^N_\Omega \times \mathbb{R}^1, \quad \mathcal{A}(CE) = \mathcal{C}_M^\infty(CE); \)
- \( \mathcal{D}_\mathcal{V}(CE) = \{ \zeta = \zeta_0^{\alpha} \partial_{u_0^{\alpha}} + \zeta_1^{\alpha} \partial_{u_1^{\alpha}} + \zeta_i \partial_{p_i} \mid \zeta_0^{\alpha}, \zeta_1^{\alpha}, \zeta_i \in \mathcal{A}(CE) \}; \)
- \( D_{\mu} = \partial_{x^\mu} + u_1^{\mu} \partial_{y_1^{\mu}} + u_0^{\alpha} \partial_{u_0^{\alpha}} + p_{i(\mu)} \partial_{p_i}, \quad \mu \in M, \)

where \( u_1^{\mu} = -u_0^\alpha \big|_{\mu_0 = -1, \alpha} \);
- \( \text{Sym}(\mathcal{A}(CE), \mathcal{D}_\mathcal{H}(CE)) = \{ \zeta = ev_\mathcal{V} \in \mathcal{D}_\mathcal{V}(CE) \mid [D_{\mu}, ev_\mathcal{V}] = 0, \mu \in M \}, \)

where

\[\begin{align*}
f = (f^\mu, f) & \in \mathcal{A}(CE) \times \mathcal{A}(CE), \quad D_{\mu} f^\mu = 0, \\
ev_\mathcal{V} = D_{i_0} f^1 \cdot \partial_{u_1^{\alpha}} + D_{i_1} f^1 \cdot \partial_{u_1^{\alpha}} + D_{i_0} f^1 \cdot \partial_{p_i};
\end{align*}\]

- \( \Omega_H^q(CE) = \{ \omega = \omega_{\mu_1 \ldots \mu_q} : d\omega^{\mu_1} \wedge \cdots \wedge d\omega^{\mu_q} \mid \omega_{\mu_1 \ldots \mu_q} \in \mathcal{A}(CE), + s-s \}, 1 \leq q \leq m \).

In more detail, the equation \( D_{\mu} f^\mu = 0 \) takes the form

\[D_{\mu} f^\mu = \partial_{x^\mu} f^\mu + u_1^{\mu} \partial_{y_1^{\mu}} f^\mu + u_0^{\alpha} \partial_{u_0^{\alpha}} f^\mu + p_{i(\mu)} \partial_{p_i} f^\mu = 0.
\]

We assume that the function \( f = (f^\mu) \) has the \( p \)-order \( r = \max_{\mu \in M} \text{ord}_p f^\mu \). Then the term \( \partial_{p_{i_1 \ldots i_{r+1}}} f^\mu \cdot f^\mu \) is of the \( p \)-order \( r + 1 \), while all other terms in Eq. (5) have \( p \)-orders \( \leq r \). This leads to the conclusion \( \partial_{p_{i_1 \ldots i_r}} f^\mu = 0, \sum_{i_1 \ldots i_{r+1}} \text{ord}_p f^\mu \leq r - 1 \). By induction, we then obtain \( f^\mu = f^\mu(x, u), \mu \in M \), i.e., \( f \) is independent of the variable \( p_1 \in \mathbb{R}^1 \), and we arrive at the following statement.
Proposition 1. In the above setting, Eq. (5) reduces to

$$\partial_{\nu^r} f^\mu + u^1_{\nu^r + \mu} \partial_{\nu^d} f^\mu + u_{\nu^r + \mu} \partial_{\nu^d} f^\mu = 0, \quad f^\mu(x, u) \in C^\infty_\text{in}(X \times \mathbb{R}^N \times \mathbb{R}^N).$$

Remark 3. If we want to have symmetries that actually depend on the variable $p \in \mathbb{R}_+$, we must restrict to a subvariety of the space $\text{CE}$. Exactly that situation arises when studying the Navier–Stokes equations.

To calculate the cohomology spaces $H^q_\Theta(\text{CE})$, we introduce the auxiliary complex $\{\Theta^q, d^q_\Theta \mid q \in \mathbb{Z}\}$, where

- $\Theta^q = \{\theta^q = \theta_{\alpha_1 \ldots \alpha_q} \cdot dx^{\alpha_1} \land \cdots \land dx^{\alpha_q} \mid \alpha_1, \ldots, \alpha_q \in N, \theta_{\alpha_1 \ldots \alpha_q} \in A(\text{CE})\};$
- $d^q_\Theta : \Theta^{q+1} \to \Theta^q$, $\theta_{\alpha_1 \ldots \alpha_q} \cdot dx^{\alpha_1} \land \cdots \land dx^{\alpha_q} \mapsto D_{[\alpha_0 \alpha_1 \ldots \alpha_q]} \cdot dx^{\alpha_0} \land \cdots \land dx^{\alpha_q};$
- $H^q(\Theta) = \text{Ker} d^q_\Theta / \text{Im} d^{q-1}_\Theta$.

We have the commutative diagram

\[
\begin{array}{ccccccc}
0 & \xrightarrow{\iota^{-1}} & \Omega^0 & \xrightarrow{\pi^0} & \Theta^0 & \xrightarrow{d_0^\Theta} & 0 \\
\downarrow d_0^{-1} & & \downarrow d_1^\Theta & & \downarrow d_0^\Theta & & \\
\vdots & & \vdots & & \vdots & & \\
0 & \xrightarrow{d_{\Theta}^{-1}} & \Theta^{q-1} & \xrightarrow{\iota^{q-1}} & \Omega^q & \xrightarrow{\pi^q} & \Theta^q & \xrightarrow{d_{\Theta}^q} & 0 \\
\downarrow d_{\Theta}^{q-1} & & \downarrow d_{\Theta}^{q-1} & & \downarrow d_{\Theta}^{q-1} & & \\
\vdots & & \vdots & & \vdots & & \\
0 & \xrightarrow{d_{\Theta}^{m-2}} & \Theta^{m-1} & \xrightarrow{\iota^{m-1}} & \Omega^m & \xrightarrow{\pi^m} & \Theta^m & \xrightarrow{d_{\Theta}^m} & 0
\end{array}
\]

where

- $\Omega^q = \Omega^q_\Theta(\text{CE})$, $\Omega^0 = \Theta^0 = A(\text{CE})$;
- $\Omega^q = dx^1 \land \cdots \land dx^q \oplus A(\text{CE}) \Theta^q$;
- $d_\Theta(dx^1 \land \cdots \land dx^q + \theta^q) = dx^1 \land \cdots \land dx^q \land \theta^q + d_\Theta \theta^q$;
- $D_\Theta \theta^q = D_1(\theta_{\alpha_1 \ldots \alpha_q} \cdot dx^{\alpha_1} \land \cdots \land dx^{\alpha_q}) = \theta_{\alpha_1 \ldots \alpha_q} \cdot dx^{\alpha_1} \land \cdots \land dx^{\alpha_q}$;
- $\iota^q : \Theta^q \to \Omega^{q+1}$, $\theta^q \mapsto \omega^{q+1} = (-1)^q dx^1 \land \cdots \land dx^q + 0$;
- $\pi^q : \Omega^q \to \Theta^q$, $\omega^q = dx^1 \land \cdots \land dx^q + \theta^q \mapsto \theta^q$.

According to the general results of homology theory (see, e.g., [9]), this diagram defines a long exact sequence of the cohomology spaces

$$0 \to H^0_\Theta(\text{CE}) \to H^0(\Theta) \xrightarrow{D^0_\Theta} H^0_\Theta(\text{CE}) \to H^1(\Theta) \xrightarrow{D^1_\Theta} \cdots \xrightarrow{D^{m-2}_\Theta} H^{m-2}(\Theta) \to \cdots \xrightarrow{D^{m-2}_\Theta} H^{m-2}(\Theta) \to H^m(\Theta) \xrightarrow{D^{m-1}_\Theta} H^m(\Theta) \to 0.$$
where \( D^0_1 : H^q(\Theta) \to H^q(\Theta) \),

\[
[\theta^q] = \theta^q + \text{Im} \, d_{3q}^{\Theta} \to D^0_1[\theta^q] = [D_1 \theta^q] = D_1 \theta^q + \text{Im} \, d_{3q}^{\Theta}.
\]

By Theorem 1, we have \( H^q(\Theta) = 0 \) for \( q \neq 0 \) and \( q \neq m - 1 \), while

- \( H^0(\Theta) = \{ \phi(x^1) \in C^\infty(\mathbb{R}) \} \);

- \( H^{m-1}(\Theta) = \mathcal{H}(\Theta) = \{ \chi \in \mathcal{F}(\Theta) \mid \chi_+ = \chi_-^* \} \),

\[
\mathcal{F}(\Theta) = \mathcal{A}_n(\mathbf{CE}) \times \mathcal{A}_{N_+}^2(\mathbf{CE}) \times \mathcal{A}_{Z^*}^2(\mathbf{CE}),
\]

where the isomorphism \( \delta : H^{m-1}(\Theta) \cong \mathcal{H}(\Theta) \) is defined as

\[
\Theta^{m-1} \ni \theta = L \cdot dx^2 \wedge \cdots \wedge dx^m \mapsto \delta L = (\delta_{u^1_{\alpha}} L, \delta_{u^1_{\alpha}} L, \delta_{p_1} L),
\]

\[
\delta_{u^1_{\alpha}} L = (\delta L)_{\partial u^1_{\alpha}}, \quad \delta_{u^1_{\alpha}} L = (\delta L)_{\partial u^1_{\alpha}}, \quad \delta_{p_1} L = (\delta L)_{\partial p_1},
\]

with \( j \in \mathbb{J}, \alpha \in \mathbb{N}, i^1 \in \mathbb{Z}_+ \), and, we recall, \( i_0 = (0, j), i = (i^1, j) \) (see Sec. 1.2), whence \( u^1_{i_0} = u^1_{i_0, j}, u^q_1 = u^q_{i^1, j}, p_1 = p^1_{i^1, j} \).

The above long exact sequence defines the short exact sequence

\[
0 \to H^0_H(\mathbf{CE}) \to \text{Ker} \, D^0_1 \to 0, \quad \text{whence} \quad H^0_H(\mathbf{CE}) = \mathbb{R}.
\]

In the same way, we have the exact sequence

\[
0 \to \text{Im} \, D^0_1 \to H^0(\Theta) \to H^1_H(\mathbf{CE}) \to 0 \quad (H^1(\Theta) = 0),
\]

whence \( H^1_H(\mathbf{CE}) = H^0(\Theta) / \text{Im} \, D^0_1 \). Further, for \( 1 \leq q \leq m - 2 \), we have \( H^q(\Theta) = 0 \) and \( H^{q-1}(\Theta) \to H^q_H(\mathbf{CE}) \to H^q(\Theta) \), and hence \( H^q_H(\mathbf{CE}) = 0 \) for \( 2 \leq q \leq m - 2 \). Now, for \( q = m - 1 \), we have

\[
\ldots \to H^{m-2}(\Theta) \to H^{m-1}_H(\mathbf{CE}) \to H^{m-1}(\Theta) \xrightarrow{D_1^{m-1}} \ldots .
\]

Here, \( H^{m-2}(\Theta) = 0 \), and therefore the sequence \( 0 \to H^{m-1}_H(\mathbf{CE}) \to \text{Ker} \, D^{m-1}_1 \to 0 \) is exact, i.e., \( H^{m-1}_H(\mathbf{CE}) = \text{Ker} \, D^{m-1}_1 \) (we recall that the cohomology is defined up to an isomorphism).

**Lemma 1** (See, e.g., [10], [11]). We have the commutative diagram

\[
\begin{array}{ccc}
0 & \xrightarrow{H^{m-1}(\Theta)} & H(\Theta) \\
\downarrow D_1^{m-1} & & \downarrow D_1 \\
0 & \xrightarrow{H^{m-1}(\Theta)} & H(\Theta)
\end{array}
\]

In particular, \( \delta \circ D_1^{m-1} = D_1 \circ \delta \), where \( D_1 = \partial_{x^1} + \text{ev}^1_1 \), \( D_1 = \partial_{x^1} + \text{ev}^1_1 \),

\[
\text{ev}^1_1 = D_1 f^1_{0} \cdot \partial u^1_{i^1, j} + D_1 f^1_{1} \cdot \partial u^1_{i^1, j} + D_1 f^1_{2} \cdot \partial p^1_{i^1, j},
\]

\( j \in \mathbb{J}, \alpha \in \mathbb{N}, i^1 \in \mathbb{Z}_+ \).
In our setting, \( f = (f_0^a, f_2, f_1 = p_{i+1,0}) \). Hence, \( \tilde{D}_1, f^* \in \text{End}_\mathbb{R}(\mathcal{A}_1(\mathbf{CE}) \times \mathcal{A}^2_N(\mathbf{CE}) \times \mathcal{A}^2_D_N(\mathbf{CE})) \),

\[
\chi = (\chi_1^0, \chi_\alpha^i, \chi^i) \mapsto \gamma^i \chi = ((\gamma^i \chi)_1^0, (\gamma^i \chi)_\alpha^i, (\gamma^i \chi)^i),
\]

where

\[
(\gamma^i \chi)_1^0 = 0, \quad (\gamma^i \chi)_\alpha^i = \delta_1^i D_\alpha \chi_1^0 + \chi_\alpha^{i-1}, \quad (\gamma^i \chi)^i = \chi_1^{i-1}.
\]

The system \( (\tilde{D}_1 + \gamma^i) \chi = 0 \) of the equations for the unknown function \( \chi \in \mathcal{A}_1(\mathbf{CE}) \times \mathcal{A}^2_N(\mathbf{CE}) \times \mathcal{A}^2_D_N(\mathbf{CE}) \) with a finite number of the components \( \chi_1^0, \chi_\alpha^i, \chi_1^i \neq 0 \) reduces to

\[
D_1 \chi_1^0 = 0, \quad D_1 \chi_\alpha^i + \delta_0^i D_\alpha \chi_1^0 + \chi_\alpha^{i-1} = 0, \quad D_1 \chi_1^i + \chi_1^{i-1} = 0
\]

and has the general solution \( \chi = (\chi_1^0, 0, 0), \chi_1^0 \in \mathbb{R} \). Now, we can take \( \delta u^i = (1, 0, 0) \); then the linear space \( \text{Ker} \ D_1^{m-1} \) is one-dimensional with the basis \( [u^i d_1 x] \) (we recall that \( d_\mu x = (-1)^{\mu-1} dx^1 \wedge \cdots \wedge dx^m \), the term \( dx^\nu \) is omitted, and hence \( dx^\nu \land d_\mu x = \delta^\nu_\mu \cdot d^m x \), \( \mu, \nu \in \mathcal{M} \), i.e., \( \text{Ker} \ D_1^{m-1} = \mathbb{R} \cdot [u^i d_1 x] \) (cf. [10], [11]). The condition \( \chi_* = \chi^* \) is satisfied trivially, and therefore the following statement holds.

**Proposition 2.** We have the cohomology space \( H_H^{m-1}(\mathbf{CE}) = \mathbb{R} \cdot [u^i d_\mu x] \).

We note that \( H_H^{m-1}(\mathbf{CE}) \ni [u^i d_\mu x] \mapsto [u^i d_1 x] \in \text{Ker} \ D_1^{m-1} \).

**Remark 4.** The cohomology \( [u^i d_\mu x] \) is generated by the constraint \( CE = D_\mu u^\mu = 0 \).

Finally, we have \( H_H^{m-1}(\Theta) \xrightarrow{D_1^{m-1}} H_H^{m-1}(\Theta) \rightarrow H_H^m(\mathbf{CE}) \rightarrow 0, \) i.e.,

\[
0 \rightarrow \text{Im} \ D_1^{m-1} \rightarrow H_H^{m-1}(\Theta) \rightarrow H_H^m(\mathbf{CE}) \rightarrow 0,
\]

whence \( H_H^m(\mathbf{CE}) = H_H^{m-1}(\Theta)/\text{Im} \ D_1^{m-1} \). Thus, the following theorem holds.

**Theorem 2.** The linear spaces of the cohomology of the differential algebra \( (\mathcal{A}(\mathbf{CE}), \mathcal{D}_H(\mathbf{CE})) \) are

\[
H_H^q(\mathbf{CE}) = \begin{cases} 
0, & q < 0, \ 1 \leq q \leq m - 2, \ q > m, \\
\mathbb{R}, & q = 0, \\
\text{Ker} \ D_1^{m-1} = \mathbb{R} \cdot [u^i d_\mu x], & q = m - 1, \\
H_H^{m-1}(\Theta)/\text{Im} \ D_1^{m-1}, & q = m,
\end{cases}
\]

where \( H_H^{m-1}(\Theta)/\text{Im} \ D_1^{m-1} = \mathcal{H}(\Theta)/\text{Im} \tilde{D}_1 \).

### 2.2. An additional constraint

According to Remark 3, to apply the algebraic analysis to the Navier–Stokes equations, which contain an explicit dependence on the pressure \( p \), we need to add an additional constraint in the space \( \mathbf{CE} \). With the specific form of these equations in mind, we choose the constraint defined by the equation

\[
\text{PE} = \Delta p + \nabla u \cdot \nabla u = \delta^\lambda_\mu \partial_\lambda \partial_\mu p + \partial_\lambda u^\mu \cdot \partial_\mu u^\lambda = 0
\]

(see integrability condition (4); we took Eq. (2) into account). Namely, we set

\[
\text{PE} = \{(x, u, p) \in B \mid \text{PE}_i = \Delta p_i + D_i(u^\lambda_{(\mu)}) u^\mu_{(\lambda)} = 0, \ i \in \mathbb{I} \},
\]

where

- \( \Delta = \delta^\lambda_\mu \partial_\lambda \partial_\mu \)
- \( D_i(u^\lambda_{(\mu)}) u^\mu_{(\lambda)} = \sum_{k+i=\mathbb{I}} (\frac{1}{k}) u^\lambda_{k+\mu} u^\mu_{i+\lambda} \).
The subspace \( \text{CPE} = \text{CE} \cap \text{PE} \subset \text{B} \) generates the differential ideal
\[
\mathcal{J} = \{ f \in \mathcal{A}(\text{B}) \mid f|_{\text{CPE}} = 0 \} = \{ f = g^i \cdot \text{CE}_i + h^i \cdot \text{PE}_i \mid g^i, h^i \in \mathcal{A}(\text{B}) \}.
\]

The quotient algebra \( \mathcal{A}(\text{CPE}) = \mathcal{A}(\text{B})/\mathcal{J} \) is the algebra of smooth functions on the space \( \text{CPE} \).

The Lie algebra \( \mathfrak{D}(\text{CPE}) \) of derivations of the algebra \( \mathcal{A}(\text{CPE}) \), the Lie algebra \( \text{Sym}(\mathcal{A}(\text{CPE}), \mathfrak{D}_H(\text{CPE})) \) of the symmetries of the differential algebra \( (\mathcal{A}(\text{CPE}), \mathfrak{D}_H(\text{CPE})) \), the \( \mathbb{Z} \)-graded \( \mathcal{A}(\text{CPE}) \)-module \( \Omega_H(\text{CPE}) = \oplus_{q \in \mathbb{Z}} \Omega^q_H(\text{CPE}) \) of horizontal differential forms, and the horizontal differential \( d_H \in \text{End}_R(\Omega_H(\text{CPE})) \) on the space \( \text{CPE} \) are defined as above (see Sec. 2.1). Namely, we now have

- \( \mathfrak{D}(\mathcal{J})(\text{B}) = \{ \zeta \in \mathfrak{D}(\text{B}) \mid \zeta|_{\mathcal{J}} : \mathcal{J} \to \mathcal{J} \} \);
- \( \mathfrak{D}(\mathcal{J})(\text{B}) = \{ \zeta \in \mathfrak{D}(\text{B}) \mid \zeta : \mathcal{A}(\text{B}) \to \mathcal{J} \} \);
- \( \mathfrak{D}(\text{CPE}) = \mathfrak{D}(\mathcal{J})(\text{B})/\mathfrak{D}(\text{B}) ) \) is the quotient Lie algebra of derivations of the quotient algebra \( \mathcal{A}(\text{CPE}) \);
- \( \mathfrak{D}(\text{CPE}) = \mathfrak{D}(\text{CPE}) + \mathfrak{D}(\text{CPE}) \mathfrak{D}_H(\text{CPE}) \);
- \( \mathfrak{D}_H(\text{CPE}) = \{ \zeta \in \mathfrak{D}(\text{CPE}) \mid \zeta \in \mathfrak{D}_H(\text{B}) \subset \mathfrak{D}(\mathcal{J})(\text{B}) \} \);
- \( \text{Sym}(\mathcal{A}(\text{CPE}), \mathfrak{D}_H(\text{CPE})) = \{ \zeta \in \mathfrak{D}_H(\text{CPE}) \mid [\bar{D}_H, \zeta] = 0 = \mathcal{J}, \mu \in \mathbb{M} \} \).

To write the \( \mathbb{Z} \)-graded \( \mathcal{A}(\text{CPE}) \)-module \( \Omega_H(\text{CPE}) = \oplus_{q \in \mathbb{Z}} \Omega^q_H(\text{CPE}) \) and its cohomology spaces, it suffices to replace \( \text{CE} \) with \( \text{CPE} \) in all relevant formulas. In particular,
\[
\Omega^q_H(\text{CPE}) = \begin{cases} 0, & q < 0, q > m; \\ \mathcal{A}(\text{CPE}), & q = 0; \\ \text{Hom}_{\mathcal{A}(\text{CPE})}(\wedge^n \mathfrak{D}_H(\text{CPE}); \mathcal{A}(\text{CPE})), & 1 \leq q \leq m. \end{cases}
\]

The horizontal differential \( d_H \in \text{End}_R(\Omega_H(\text{CPE})) \), \( d_H \circ d_H = 0 \), is defined by the rule
\[
d_H^q = d_H^q|_{\Omega^q_H(\text{CPE})}: \Omega^q_H(\text{CPE}) \to \Omega^{q+1}_H(\text{CPE}),
\]
\[
\bar{\omega}_{\mu_1...\mu_q} \cdot d\bar{\omega}_{\mu_1...\mu_q} \cdots d\bar{\omega}_{\mu_1...\mu_q} \Rightarrow \bar{D}_{[\mu_1...\mu_q]} \cdot d\bar{\omega}_{\mu_1...\mu_q} \cdots d\bar{\omega}_{\mu_1...\mu_q},
\]
where \( d_H^{q+1} \circ d_H^q = 0, q \in \mathbb{Z} \).

We now define global coordinates on the subspace \( \text{CPE} \subset \text{B} \). Namely, we split \( \mathbb{I} = I_1 \cup I_1' \) (see Sec. 1.2) and obtain

- \( \mathbb{R}_i = \mathbb{R}_{i_1} \times \mathbb{R}_{i_1}' \);
- \( \mathbb{B} = X \times \mathbb{R}_{I_1} \times \mathbb{R}_{I_1}' \times \mathbb{R}^N \times \mathbb{R}_{I_1} \times \mathbb{R}_{I_1}' \);
- \( \text{CPE} = \text{CE} \cap \text{PE} = X \times \mathbb{R}_{I_1} \times \mathbb{R}^N \times \mathbb{R}_{I_1} \), i.e., the subspace \( \text{CPE} \) has the global coordinates \( (x, u, p) = \{ x^\alpha, u^\mu_{i_0}, i_0, p_{i_1} \} \), with the indices \( \mu \in \mathbb{M}, \alpha \in \mathbb{N}, i_0 \in \mathbb{I}_0, i_1 \in \mathbb{I}_1, i \in \mathbb{I} \);
- \( \text{CE} = u^\mu_{(\mu)} = 0 \), \( \text{PE} = \Delta p + u^\lambda_{(\lambda)} u^\mu_{(\mu)} |_{\text{CE}} = p_{2(1)} + \Phi(u, p) = 0 \);
- \( \Phi(u, p) = \Delta' p + u^\lambda_{(\lambda)} u^\mu_{(\mu)} |_{\text{CE}} \), \( \Delta' = \delta_{\alpha \beta} D_\alpha \circ D_\beta \);
- \( u^\mu_{i_0} = -u^\mu_{i_0-1} + (\alpha), i_0 \in \mathbb{I}_0, p_{i_1} = -D_{i_1-2(1)} \Phi(u, p), i_1 \in \mathbb{I}_1 \);
- \( D_i(u^\mu_{(\mu)} v^\lambda_{(\lambda)}) = \sum_{k+1=i}^{(i)} u^\mu_{k+1(\mu)} v^\lambda_{1(\lambda)}, i = i_1 - 2(1) \in \mathbb{I} \).
In these coordinates, we have

- \( \mathcal{A}(\text{CPE}) = C_\infty^0(\text{CPE}) = \{ f \in \mathcal{A}(B) \mid \partial_{u^i_0} f = 0, \partial_{p^i_1} f = 0, i_0 \in I_0, i_1 \in I_1 \} \subset \mathcal{A}(B) \);

- \( \mathcal{D}_V(\text{CPE}) = \{ \zeta = \zeta_{i_0} \partial_{u^i_0} + \zeta_{i_1} \partial_{p^i_1}, \zeta_{i_0}, \zeta_{i_1} \in \mathcal{A}(\text{CPE}), i_0 \in I_0, i_1 \in I_1, i \in I, \alpha \in N \}; \)

- \( D_\mu = \partial_{u^i_0 + (\mu)} \partial_{u^i_0} + u^i_0 (\mu) \partial_{u^i_1} + p^i_1 (\mu) \partial_{p^i_1}, \mu \in M \), where \( u^i_0 + (1) = -u^i_0, p^i_0 + 2(1) = -D_\mu \Phi(u, p), i_0 \in I_0, i_1 = i_0, i_0 + (1); \)

- \( \text{Sym}(\mathcal{A}(\text{CPE}), \mathcal{D}_H(\text{CPE})) = \{ \zeta = ev_t \in \mathcal{D}_V(\text{CPE}) \mid [D_\mu, ev_t] = 0, \mu \in M \}, \)

- \( \Omega_1^0(\text{CPE}) = \{ \omega = \omega_{\mu_1 ... \mu_q} \cdot dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_q} \mid \omega_{\mu_1 ... \mu_q} \in \mathcal{A}(\text{CPE}), + s-s \}, 1 \leq q \leq m; \)

- \( d^n_1 = d_H|_{\Omega_1^0(\text{CPE})} : \Omega_1^0(\text{CPE}) \rightarrow \Omega_1^{n+1}(\text{CPE}), \)

where \( \omega = \omega_{\mu_1 ... \mu_q} \cdot dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_q} \mapsto D_{[\mu_0} \omega_{\mu_1 ... \mu_q]} \cdot dx^{\mu_0} \wedge \cdots \wedge dx^{\mu_q}. \)

Similarly to Sec. 2.1, to calculate the cohomology spaces \( H_1^0(\text{CPE}) \), we introduce an auxiliary complex \( \{ \Theta_1^0, d_1^0 \mid q \in \mathbb{Z} \} \) (see page 1663). The only changes are

- the set \( \text{CE} \) is replaced with the set \( \text{CPE} \);

- the basic derivations \( D_1, D_0 \) are as in this subsection;

- we have the operator \( \tilde{D}_1 = D_1 + f^*; \)

- \( D_1 = \partial_{u^i_0} + ev^*_i, \) where \( f = (f^1_0, f^1_1, f_0, f_1), \)

- \( \tilde{D}_1 = D_1 f^1_0, f^1_1 = f^1_0 f^1_1, f_0 = p_1, f_1 = -\Phi(u, p); \)

- \( \Delta' = \delta^{\alpha \beta} D_{(\alpha)+(\beta)} \), \( u^\alpha_0 u^\beta_0 = u^\alpha_0 u^\beta_0 + 2u^1_1 u^0_1 + u^\beta_1 u^\alpha_1; \)

- \( \chi \mapsto f^* \chi, \chi = (\chi_0, \chi_1, \chi_2, \chi_3), \) where the set \( \chi \) is finite, i.e., only finitely many of its elements are nonzero;

- \( (f^* \chi)^1 = (f^* \chi)^0, (f^* \chi)^0, ), \)

- \( (f^* \chi)^0 = 2D_\alpha (u^1_1 \chi^1), \)

- \( (f^* \chi)^0 = 2D_\alpha (u^1_1 \chi^1), \)

- \( (f^* \chi)^0 = -\Delta' + \chi^0 \), \( (f^* \chi)^1 = \chi^0, \)

where \( \alpha, \beta \in N, \lambda, \mu \in M, i^1 \in \mathbb{Z}_+, \) and \( j \in \mathbb{J}. \)
The resulting system \((D_1 + f^*)\chi = 0\) for the unknown function \(\chi\) with a finite number of nonzero components reduces to

\[
\begin{align*}
D_1\chi_0^0 + (f^*\chi)_1^0 &= 0, & D_1\chi_0^i + (f^*\chi)_1^i &= 0, \\
D_1\chi_0^0 + (f^*\chi)_0^0 &= 0, & D_1\chi_1^1 + (f^*\chi)_1^1 &= 0.
\end{align*}
\]

The equation

\[
D_1\chi_0^i + (f^*\chi)_1^i = D_1\chi_0^i + \delta_1^i D_\alpha \chi_1^1 + \chi_0^i - 1 + 2(\delta_1^i D_\alpha (u_{(\beta)}^\beta \chi_1^1) - \delta_1^i (u_{(\alpha)}^1 \chi_1^1 + \delta_1^i D_\beta (u_{(\alpha)}^\beta \chi_1^1))) = 0
\]

with \(i^1 \geq 2\) gives \(\chi_0^i = 0\) for \(i^1 \geq 1\), \(\chi_0^0 = 2u_{(\alpha)}^1 \chi_1^1\) for \(i^1 = 1\), and takes the form

\[
D_1\chi_0^0 + D_\alpha \chi_1^0 + 2(D_\alpha (u_{(\beta)}^\beta \chi_1^1) + D_\beta (u_{(\alpha)}^\beta \chi_1^1)) = 0
\]

for \(i^1 = 0\). Further, the equation

\[
\Delta \chi_1^1 = 0, \quad \chi_1^1 = 0
\]

gives \(\chi_0 = -D_1 \chi_1^1\), the equation

\[
\Delta \chi_1^1 = 0, \quad \chi_1^1 = 0
\]

gives \(\chi_0 = -D_1 \chi_1^1\), and the last equation

\[
D_1\chi_0^0 + (f^*\chi)_0^0 = 0
\]

and

\[
D_1\chi_0^0 + 2D_\alpha (u_{(1)}^\alpha \chi_1^1) = 0.
\]

After some algebra, we deduce the following statement.

**Lemma 2.** The system \((D_1 + f^*)\chi = 0\), \(\chi = (\chi_0^0, \chi_0^i, \chi_0^0, \chi_1^1)\), reduces to

\[
\chi_0^0 = 2u_{(\alpha)}^1 \chi_1^1, \quad \chi_0^i = 0, \quad \chi_0^0 = -D_1 \chi_1^1, \quad \chi_1^1 = 0, \quad \chi_1^0 \geq 1, \quad \alpha \in \mathbb{N}, \tag{8}
\]

where \(\chi_1^1, \chi_1^0 \) are solutions of the system

\[
\Delta \chi_1^1 = 0, \quad (u_{(1)}^\mu D_\mu D_{(\alpha)} - u_{(\alpha)}^\mu D_\mu D_{(1)}) \chi_1^1 = 0, \quad \alpha \in \mathbb{N}, \tag{9}
\]

\[
D_1\chi_0^0 = -2D_\alpha (u_{(1)}^\alpha \chi_1^1), \quad D_\alpha \chi_1^0 = -2(u_{(\alpha)}^\mu D_{(1)} \chi_1^1 + D_\alpha (u_{(\alpha)}^\beta \chi_1^1)), \quad \alpha \in \mathbb{N}. \tag{10}
\]

**Remark 5.** The right subsystem in system (9) is a nontrivial compatibility condition of system (10).

**Remark 6.** System (9) has the trivial solution \(\chi_1^1 = 0\). In that case, system (10) gives \(\chi_0^0 = \text{const}\), and we obtain the solution \(\chi = (1, 0, 0, 0)\) of the full system \((D_1 + f^*)\chi = 0\) (see a similar result in Sec. 2.1).

**Theorem 3.** The linear spaces of the cohomology of the differential algebra \((A(CPE), D_H(CPE))\)

are

\[
H^q_H(CPE) = \begin{cases}
0, & q < 0, 1 \leq q \leq m - 2, \quad q > m, \\
\mathbb{R}, & q = 0, \\
\text{Ker} D_0^{m-1} = \mathcal{S} \cap \mathcal{H}, & q = m - 1, \\
H^{m-1}(\Theta)/\text{Im} D_1^{m-1}, & q = m
\end{cases}
\]

(cf. Theorems 1 and 2), where

1668
\( S = \text{Sol}(D_t + f^*) \) is the linear space of solutions \( \chi = (\chi_1^0, \chi_\alpha^1, \chi_\alpha^0, \chi^1) \) of the linear system \( D_t \chi + f^* \chi = 0 \) (the map \( f^* \) is defined above);

\( \mathcal{H} = \{ \chi = (\chi_1^0, \chi_\alpha^1, \chi_\alpha^0, \chi^1) \mid \chi_1^* = \chi^* \} \) is the Helmholtz space of the differential algebra \((\mathcal{A}(\text{CPE}), \mathcal{D}_H(\text{CPE}))\);

\( H^{m-1}_H(\text{CPE}) = S \cap \mathcal{H}, \quad [J^\mu \cdot d_\mu x] \mapsto (\delta_{u_0^\alpha} J^1, \delta_{u_1^\alpha} J^1, \delta_{p_0} J^1, \delta_{p_1} J^1) \) (we recall that the cohomology is defined up to an isomorphism).

**Remark 7.** In particular, \( H^{m-1}_H(\text{CPE}) \ni [u^\mu d_\mu x] \) (see Remark 6). Moreover, the additional constraint \( \text{PE} = \Delta p + u_0^\mu u_\lambda^\mu = 0 \) generates the additional cohomology \( [F^\mu d_\mu x] \in H^{m-1}_H(\text{CPE}) \), where

\[
D_\mu F^\mu = -(\Delta p + u_0^\mu u_\lambda^\mu ) + (-u^\lambda D_\lambda u_\mu^\mu + \nu \Delta u_\mu^\mu ) = -\text{PE} + (\nu \Delta - u^\lambda D_\lambda )CE = 0
\]

due to the constraints \( \text{PE} = 0 \) and \( \text{CE} = 0 \). Here, \( p^{(\mu)} = D_\mu p = \delta^{\mu\lambda} D_\lambda p = D_\mu p = p^{(\mu)} \) (the Euclidean metrics).

**Remark 8.** Every symmetry \( ev_t \in \text{Sym}(\mathcal{A}(\text{CPE}), \mathcal{D}_H(\text{CPE})) \), \( f = (f^\mu, f) \), generates the (possibly trivial) cohomology \( [f^\mu d_\mu x] \in H^{m-1}_H(\text{CPE}) \), because in this case \( D_\mu f^\mu = 0 \) by definition.

### 3. Navier–Stokes equations as evolution in the space CPE

#### 3.1. Evolution in the space \text{CPE}

Evolution in the space

\[
\text{CPE} = T \times X \times (\mathbb{R}_{l_0}^1 \times \mathbb{R}_{l_1}^N) \times \mathbb{R}_{l_1} = \{ t, x = (x^\mu), u = (u_0^\alpha, u_1^\alpha), p = (p_i) \}
\]

\((M = 2, 3)\) is governed by the evolutionary derivation \( D_t = \partial_t + ev_E \), where

- we added the time variable \( t \in T = \mathbb{R} \), which can be assumed to have been present from the start as a parameter;
- \( E = (E^\mu, E) \in \mathcal{A}^M(\text{CPE}) \times \mathcal{A}(\text{CPE}) \);
- \( D_\mu E^\mu = 0, \quad \Delta E + 2(u_0^\mu D_\lambda E^\mu) \big|_{u_0^\mu} = 0, \quad \Delta = \delta^{\mu\lambda} D_\lambda + (\mu) \);
- \( ev_E = D_{i_0} E^1 \cdot \partial_{a_{i_0}} + D_1 E^\alpha \cdot \partial_{u_1^\alpha} + D_{i_1} E \cdot \partial_{p_{i_1}} \in \text{Sym}(\mathcal{A}(\text{CPE}), \mathcal{D}_H(\text{CPE})) \)

(see Sec. 2.2 and also Proposition 1 and Remark 3).

**Remark 9.** In particular, \( D_t u^\mu = F^\mu, \mu \in M, \) and \( D_t p = F, \) while

\[
D_t \text{CE} = D_t (D_\mu u^\mu ) = D_t E^\mu, \quad D_t \text{PE} = D_t (\Delta p + u_0^\mu u_\lambda^\mu ) = \Delta E + 2u_0^\mu D_\lambda E^\mu.
\]

The differential algebra \((\mathcal{A}(\text{CPE}), \mathcal{D}_V(\text{CPE}))\) is defined, where

- \( \mathcal{D}(\text{CPE}) = \mathcal{D}_V(\text{CPE}) \oplus \mathcal{A}(\text{CPE}) \mathcal{D}_E(\text{CPE}) \);
- \( \mathcal{D}_V(\text{CPE}) = \{ \zeta = \zeta_{i_0}^1 \partial_{a_{i_0}} + \zeta_i^\alpha \partial_{u_1^\alpha} + \zeta_{i_1} \partial_{p_{i_1}} \mid \zeta_{i_0}^1, \zeta_i^\alpha, \zeta_{i_1} \in \mathcal{A}(\text{CPE}) \} \).
• $\mathfrak{D}_E(CPE)$ has the $A(CPE)$-basis $\{D_t, D_\mu \mid \mu \in M\}$, the basic time derivation $D_t = \partial_t + ev_E$, $[D_t, D_\mu] = 0$, $\mu \in M$, whence

$$\mathfrak{D}_E(CPE) = \{\zeta = \zeta^t D_t + \zeta^\mu D_\mu \mid \zeta^t, \zeta^\mu \in A(CPE)\}.$$ 

The Lie algebra of symmetries here is

$$\text{Sym}(A(CPE), \mathfrak{D}_E(CPE)) = \{\zeta = ev_t \in \text{Sym}(A(CPE), \mathfrak{D}_E(CPE)) \mid [D_t, ev_t] = 0\},$$

where the condition $[D_t, ev_t] = 0$ reduces to the equation $(D_t - E_\tau)f = 0$. In more detail,

• $E_\tau : A(CPE)^M \times A(CPE) \to A(CPE)^M \times A(CPE)$;

• $f = (f^\mu, f) \mapsto (F, f) = ((F, f)^\mu, (F, f))$;

• $(E_\tau f)^\mu = \partial_{u_i} E^\mu \cdot D_i f^1 + \partial_{u_i} E^\mu \cdot D_i f^\alpha + \partial_{p_1} E^\mu \cdot D_i f$;

• $(E_\tau f) = \partial_{u_i} E \cdot D_i f^1 + \partial_{u_i} E \cdot D_i f^\alpha + \partial_{p_1} E \cdot D_i f$.

As above, to study the cohomology spaces $H^q_m(CPE) = \text{Ker} d^q_m/ \text{Im} d^{q-1}_m$, $q \in \mathbb{Z}$, of the differential algebra $(A(CPE), \mathfrak{D}_E(CPE))$, we use the decomposition (see pages 1663 and 1667):

• $\Omega^q_H = \Omega^q_E(CPE) = \{\omega^q_E = \omega_{\mu_1...\mu_q} \cdot dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_q} \mid \omega_{\mu_1...\mu_q} \in A(CPE), + s-s\}$;

• $\Omega^q_E = \Omega^q_E(CPE) = dt \land \Omega^{q-1}_H \oplus A(CPE) \Omega^q_H$, $q \in \mathbb{Z}$;

• $0 \rightarrow \Omega^{q-1}_E \xrightarrow{\iota^{q-1}} \Omega^q_E \xrightarrow{\pi^q} \Omega^q_H \rightarrow 0$,

$\omega^{q-1}_E \mapsto \omega^{q-1}_E = (-1)^q dt \land \omega^{q-1}_H$,

$\omega^q_E = dt \land \omega^{q-1}_H + \omega^q_H \mapsto \pi^q \omega^q_E = \omega^q_H$;

• $d^q_E = d^q_t + d^q_H : \Omega^q_E \rightarrow \Omega^{q+1}_E$, $d_t = dt \land D_t$, $d_H = dx^\mu \land D_\mu$,

$\omega^q_E = dt \land \omega^{q-1}_H + \omega^q_H \mapsto d_E \omega^q_E = dt \land (D_t \omega^q_H - d_H \omega^{q-1}_H) + d_H \omega^q_H$.

These constructions lead to a commutative diagram with the associated long exact sequence of cohomology spaces

$$0 \rightarrow H^0_E(CPE) \rightarrow H^0_H(CPE) \xrightarrow{D^0_t} H^1_E(CPE) \rightarrow H^1_H(CPE) \xrightarrow{D^1_t} \ldots$$

$$\ldots \xrightarrow{D^{m-2}} H^{m-2}_E(CPE) \rightarrow H^{m-2}_H(CPE) \rightarrow H^{m-1}_H(CPE) \xrightarrow{D^{m-1}_t} \ldots$$

$$\xrightarrow{D^{m-1}} H^{m-1}_H(CPE) \rightarrow H^m_E(CPE) \rightarrow H^m_H(CPE) \xrightarrow{D^m_t}$$

$$\xrightarrow{D^m} H^{m+1}_H(CPE) \rightarrow H^{m+1}_E(CPE) \rightarrow 0,$$

where $D^q_t : H^q_H(CPE) \rightarrow H^q_E(CPE)$ by the componentwise rule

$$D^q_t[\omega_{\mu_1...\mu_q} \cdot dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_q}] = [(D_t \omega_{\mu_1...\mu_q}) \cdot dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_q}].$$
Theorem 4. The linear spaces of the cohomology of the differential algebra \((\mathcal{A}(\text{CPE}); \mathfrak{D}_E(\text{CPE}))\) are

\[
H^q_E(\text{CPE}) = \begin{cases} 
0, & q < 0, 1 \leq q \leq m - 2, q > m + 1, \\
\mathbb{R}, & q = 0, \\
\text{Ker} \, D^m_t, & q = m - 1, \\
H^m_{\text{H}}(\text{CPE})/\text{Im} \, D^m_t, & q = m + 1,
\end{cases}
\]

while in the case \(q = m\), we have \(H^m_E(\text{CPE})/\text{Im} \, H^m_{\text{H}}(\text{CPE}) = \text{Ker} \, D^m_t.

Proof. Indeed, the exact subsequence

\[
0 \to H^0_E(\text{CPE}) \to \text{Ker} \, D^0_t \to 0
\]
gives \(H^0_E(\text{CPE}) = \mathbb{R}\) (we recall that we added the time variable \(t \in T = \mathbb{R}\), and therefore now \(H^0_H(\text{CPE}) = T = \{\phi(t) \in C^\infty(\mathbb{R})\}\)). Further, the exact subsequence

\[
H^0_H(\text{CPE}) \xrightarrow{D^0_t} H^0_E(\text{CPE}) \to H^1_E(\text{CPE}) \to H^1_H(\text{CPE}) = 0
\]
gives \(H^1_E(\text{CPE}) = T/\text{Im} \, D^0_t = 0\). Now, the exact subsequence

\[
H^1_H(\text{CPE}) \to H^2_{\text{E}}(\text{CPE}) \to H^2_H(\text{CPE})
\]
gives \(H^2_E(\text{CPE}) = 0\) for \(2 \leq q \leq m - 2\) because in this case \(H^{q-1}_H(\text{CPE}) = H^q_H(\text{CPE}) = 0\). Then the exact subsequence

\[
0 = H^{m-2}_H(\text{CPE}) \to H^{m-1}_E(\text{CPE}) \to H^{m-1}_H(\text{CPE}) \xrightarrow{D^{m-1}_t} \ldots
\]
gives \(H^{m-1}_E(\text{CPE}) = \text{Ker} \, D^{m-1}_t\). Finally, for \(q = m, m+1\), the statements follow from the exact subsequence \(H^{m-1}_H(\text{CPE}) \to H^m_E(\text{CPE}) \to \text{Ker} \, D^m_t \to 0\) and from the exact subsequence \(0 \to \text{Im} \, D^m_t \to H^m_H(\text{CPE}) \to H^m_{E} \to 0\).

Remark 10. To calculate \(\text{Ker} \, D^{m-1}_t\), we can use the technique from Lemmas 1 and 2. The cases \(q = m, m + 1\) are uninformative in this approach and require a special study.

3.2. Navier–Stokes equations as evolution with constraints. We consider Navier–Stokes system (1)–(4) as the evolution process governed by Eq. (1) in the divergence-free space with inner constraint (4).

The algebraic counterpart of Eq. (1) is the symmetry

\[
ev_E = D_{i_1} E^1 \cdot \partial_{u^1_{i_1}} + D_i E^\alpha \cdot \partial_{u^\alpha_i} + D_i E \cdot \partial_{\rho_1} \in \text{Sym}(\mathcal{A}(\text{CPE}), \mathfrak{D}_H(\text{CPE})),
\]

where

- \(E = (E^\mu, E) \in \mathcal{A}(\text{CPE})^M \times \mathcal{A}(\text{CPE});\)
- \(E^\mu = -u^\lambda \nabla_\lambda u^\mu + \nu \Delta u^\mu - \nabla^\mu p = -u^\lambda u^\mu_{(\lambda)} + \nu \Delta u^\mu - p_{(\mu)};\)
- \(u^1_{(1)} = -u^\alpha_{(\alpha)}, \quad \Delta u^\mu = \sum_\lambda u^\mu_{(\lambda)}, \quad \Delta u^1 = -u^1_{(1)} + \sum_\alpha u^1_{(\alpha)};\)
- \(\nabla^\mu = \delta^\mu_\nu \nabla_\nu = \nabla_\mu\) (the Euclidean metric), and hence \(p_{(\mu)} = p_{(\mu)};\)
- \(E\) defined by the condition \(\ev_E \in \text{Sym}(\mathcal{A}(\text{CPE}), \mathfrak{D}_H(\text{CPE})).\)
It is easy to verify that $D_\mu E^\mu = 0$. On the other hand, the condition

$$\Delta E + \text{ev}_E(u^\lambda_{(\mu)} u^\mu_{(\lambda)}) = \Delta E + 2u^\lambda_{(\mu)} D_\lambda E^\mu = 0$$

is the Poisson equation for the component $E$ (cf. Remark 1).

Thus, the technique outlined in Sec. 3.1 can be used for the algebraic analysis of the Navier–Stokes equations.

4. Conclusion

It can be seen from the foregoing that the Navier–Stokes equations are amenable to meaningful analysis in the framework of the algebraic approach to differential equations. The resulting equations for algebraic characteristics of the Navier–Stokes equations, such as their symmetries and cohomologies, are essentially complicated. Hopefully, at least their partial solutions might be found, especially using analytic computation packets (e.g., Mathematica).

Conflicts of interest. The author declares no conflicts of interest.

REFERENCES

1. L. I. Sedov, *A Course in Continuum Mechanics*, Vol. 1, Wolters-Noordhoff Publ., Groningen, The Netherlands (1971).
2. L. D. Landau and E. M. Lifshitz, *Course of Theoretical Physics*, Vol. 6: *Fluid Mechanics*, Elsevier, Amsterdam (2013).
3. T.-P. Tsai, *Lectures on Navier–Stokes Equations*, Graduate Studies in Mathematics, Vol. 192, AMS, Providence, RI (2018).
4. P. G. Lemarié-Rieusset, *The Navier–Stokes Problem in the 21st Century*, Chapman and Hall/CRC, Boca Raton, FL (2016).
5. M. V. Korobkov, K. Pileckas, V. V. Pukhnachov, and R. Russo, “The flux problem for the Navier–Stokes equations,” *Russian Math. Surveys*, 69, 1065–1122 (2014).
6. C. L. Fefferman, J. C. Robinson, and J. L. Rodrigo (eds.), *Partial Differential Equations in Fluid Mechanics*, London Mathematical Society Lecture Note Series, Vol. 452, Cambridge Univ. Press, Cambridge (2018).
7. W. M. Seiler, *Involution: The Formal Theory of Differential Equations and its Applications in Computer Algebra*, Algorithms and Computation in Mathematics, Vol. 24, Springer, Berlin–Heidelberg (2010).
8. P. J. Olver, *Applications of Lie Groups to Differential Equations*, Graduate Texts in Mathematics, Vol. 107, Springer, New York (1993).
9. S. Mac Lane, *Homology*, Springer, Berlin–Heidelberg (1994).
10. V. V. Zharinov, “Conservation laws of evolution systems,” *Theoret. and Math. Phys.*, 68, 745–751 (1986).
11. V. V. Zharinov, “Conservation laws, differential identities, and constraints of partial differential equations,” *Theoret. and Math. Phys.*, 185, 1557–1581 (2015).