Dynamical systems on the Liouville plane and the related strictly contact systems

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Abstract. We study vector fields of the plane preserving the form of Liouville. We present their local models up to the natural equivalence relation, and describe local bifurcations of low codimension. To achieve that, a classification of univariate functions is given, according to a relation stricter than contact equivalence. We discuss, in addition, their relation with strictly contact vector fields in dimension three. Analogous results for diffeomorphisms are also given.

Keywords systems preserving the form of Liouville, strictly contact systems, classification, bifurcations

MSC[2000] Primary 37C15, 37J10, Secondary 58K45, 53D10
Introduction

Dynamical systems preserving a geometrical structure have been studied quite extensively. Especially those systems preserving a symplectic form have attracted a lot of attention, due to their fundamental importance in all kinds of applications. Dynamical systems preserving a contact form are also of interest, both in mathematical considerations (for example, in classifying partial differential equations) and in specific applications (study of Euler equations).

The 1–form of Liouville may be associated both with a symplectic form (by taking the exterior derivative of it) and with a contact form (by adding to it a simple 1–form of a new variable). We wish here to study dynamical systems respecting the form of Liouville. As we shall see, they are symplectic systems which may be extented to contact ones.

To set up the notation, let $M$ be a smooth (which, in this work, means continuously differentiable the sufficient number of times) manifold of dimension $2n + 1$. A contact form on $M$ is a 1-form $\alpha$ such that $\alpha \wedge (d\alpha)^n \neq 0$. A strict contactomorphism is a diffeomorphism of $M$ which preserves the contact form (their group will be denoted as $Diff(M, \alpha)$) while a vector field on $M$ is called strictly contact if its flow consists of strict contactomorphisms (we denote their algebra as $\mathcal{X}(M, \alpha)$). In terms of the defining contact form $\alpha$, we have $f^*\alpha = \alpha$ for a strict contactomorphism $f$ and $\mathcal{L}_X \alpha = 0$ for a strictly contact vector field $X$, where $\mathcal{L}_X \alpha$ denotes the Lie derivative of $\alpha$ in the direction of the field $X$. The classical example of a strictly contact vector field associated to $\alpha$ is the vector field of Reeb, $\mathcal{R}_\alpha$, uniquely defined by the equations $\alpha(\mathcal{R}_\alpha) = 1$ and $d\alpha(\mathcal{R}_\alpha, \cdot) = 0$.

Associated to every contact vector field $X$ is a smooth function $H : M \to \mathbb{R}$, called the contact Hamiltonian of $X$, which is given as $H = \alpha(X)$. Conversely, every smooth function $H$ gives rise to a unique contact vector field $X$, such that $\alpha(X) = H$ and $d\alpha(X, \cdot) = (\mathcal{L}_{\mathcal{R}_\alpha}H)\alpha(\cdot) - dH(\cdot)$. Usually we write $X_H$ to denote the dependence of vector field $X$ on its (contact) Hamiltonian function $H$.

Results concerning the local behavior for systems of this kind may be found in [6, 11, 9, 4], where the authors provide explicit conditions for their linearization, in the neighborhood of a hyperbolic singularity. The study of degenerate zeros, and of their bifurcations, remains, however, far from complete.

Here, in section 1, we recall the form of strictly contact vector fields of $\mathbb{R}^3$, and their relation with symplectic vector fields of the plane. We show that the algebra $\mathcal{X}(\mathbb{R}^2, xdy)$ of plane fields preserving the form of Liouville $xdy$ may be obtained by projecting on $\mathbb{R}^2$ strictly contact fields with constant third component. We begin the classification of vector fields belonging in $\mathcal{X}(\mathbb{R}^2, xdy)$ (we shall call them Liouville vector fields) by introducing the natural equivalence relation, and by showing that the problem of their classification is equivalent to a classification of functions up to a specific equivalence relation.

In section 2, (germs at the origin of) univariate functions are classified up to
this equivalence relation, which we name “restricted contact equivalence”, due to its similarity with the classical contact equivalence of functions. We provide a complete list of normal forms for function germs up to arbitrary (finite) codimension.

In section 3, based on the previous results, we give local models for Liouville vector fields of the plane. We first prove that all such fields are conjugate at points where they do not vanish, then we prove that they can be linearized at hyperbolic singularities, and finally we state the result concerning their finite determinacy, which is based on the finite determinacy theorem obtained in section 2.

In section 4, we first show how to construct a transversal unfolding of a singularity class of Liouville vector fields and then we present transversal unfoldings for singularity classes of codimension 1 and 2. Phase portraits for generic bifurcations of members of $\mathcal{X}(\mathbb{R}^2, xdy)$ are also given.

Next, in section 5, we see that there is only one polynomial member of the group of plane diffeomorphisms preserving the form of Liouville ($\text{Diff}(\mathbb{R}^2, xdy)$ stands for this group). This is the linear Liouville diffeomorphism, and we show the linearization of plane diffeomorphisms of this kind at hyperbolic fixed points.

In section 6, we return to members of $\mathcal{X}(\mathbb{R}^3, a)$ to observe that the models obtained above are members of a specific base of the vector space of homogeneous vector fields. Their linearization is again shown, albeit using classical methods of normal form theory.

Last section contains some observations concerning future directions.

For a classical introduction to symplectic and contact topology the reader should consult [8], while [12] offers a more complete study of the contact case. Singularities of mappings are treated in a number of textbooks; we recommend [7, 3] and [5] (see [15] for a recent application of singularity theory to problems of dynamics).

1. **Strictly contact vector fields and fields of Liouville**

Let $M$ be a closed smooth manifold of dimension $2n+1$ equipped with a contact form $\alpha$. The contact form is called regular if its Reeb vector field, $\mathcal{R}_\alpha$, generates a free $S^1$ action on $M$. In this case, $M$ is the total space of a principal $S^1$ bundle, the so called Boothby-Wang bundle (see [2] for more details):

$$S^1 \xrightarrow{k} M \xrightarrow{\pi} B,$$

where $k : S^1 \to M$ is the action of the Reeb field and $\pi : M \to B$ is the canonical projection on $B = M/S^1$. $B$ is a symplectic manifold with symplectic form $\omega = \pi_* da$. The projection $\pi$ induces an algebra isomorphism between functions on the base $B$ and functions on $M$ which are preserved under the flow of $\mathcal{R}_\alpha$ (such functions are called basic). It also induces a surjective homomorphism between strictly contact vector fields $X$ of $(M, \alpha)$ and hamiltonian vector fields $Y$ of $(B, \omega)$ (that is, fields $Y$ with $\mathcal{L}_Y \omega = 0$), the kernel of which homomorphism is generated by the vector field of Reeb.

In our local, three dimensional, case, things are of course simpler. Using a local Darboux chart, consider the euclidean space $\mathbb{R}^3$ equipped with the standard
contact structure $\alpha = dz + xdy$. Its Reeb vector field, $R_\alpha = \frac{\partial}{\partial z}$, induces the action $\varphi^t(x, y, z) = (x, y, z+t)$, and the quotient of $\mathbb{R}^3$ by this action, that is, the plane $\mathbb{R}^2$ with coordinates $(x,y)$, inherits the symplectic form $\omega = \pi_*d\alpha = dx \wedge dy$. Strictly contact vector fields of $\mathbb{R}^3$ project to Hamiltonian fields on this plane (for a direct analogy with the volume-preserving case the reader should consult [10]).

Basic functions now depend, as one may easily verify, only on the first two variables, while the kernel of the above mentioned projection contains the multiples of $\frac{\partial}{\partial z}$. Studying equation $L_X\alpha = 0$ we get the general expression of $X = X_1\frac{\partial}{\partial x} + X_2\frac{\partial}{\partial y} + X_3\frac{\partial}{\partial z} \in \mathcal{X}(\mathbb{R}^3, \alpha)$:

$$X = (-\frac{\partial}{\partial y} \int X_2(x,y) dx) \frac{\partial}{\partial x} + X_2(x,y) \frac{\partial}{\partial y} + (-x X_2(x,y) + \int X_2(x,y) dx) \frac{\partial}{\partial z}.$$ 

Its contact Hamiltonian is of course $H(x,y,z) = \int X_2(x,y) dx$ (recall that it does not depend on the third variable), thus:

$$X = -\frac{\partial H(x,y,z)}{\partial y} \frac{\partial}{\partial x} + \frac{\partial H(x,y,z)}{\partial x} \frac{\partial}{\partial y} + (H(x,y,z) - x \frac{\partial H(x,y,z)}{\partial x}) \frac{\partial}{\partial z}.$$ 

Observe that all vector fields of the $(x,y)$-plane, preserving the symplectic structure $dx \wedge dy$, may be obtained in this way.

In this work we restrict our attention to those members of $\mathcal{X}(\mathbb{R}^3, \alpha)$, which preserve the form of Liouville $xdy$ (we shall denote their set as $\mathcal{X}_L(\mathbb{R}^3, \alpha)$). The reason for this choice will become clear in section 6. In this case, equation $L_X\alpha = 0$ becomes:

$$X_1(x,y) = -x \frac{dX_2(y)}{dy},$$

while $X_3(x,y) = c \in \mathbb{R}$. Thus, their general form is $-x \frac{d h(y)}{dy} \frac{\partial}{\partial x} + h(y) \frac{\partial}{\partial y} + c \frac{\partial}{\partial z}$, for some univariate function $h(y)$ and a constant $c$. Observe that all vector fields of the plane preserving the form of Liouville may be obtained by projecting the members of $\mathcal{X}_L(\mathbb{R}^3, \alpha)$ on the $z = 0$ plane. We have, therefore, the following:

**1.1 Lemma.** To every $h \in C^k(\mathbb{R}, \mathbb{R})$, $k \geq 2$, corresponds a unique $X \in \mathcal{X}(\mathbb{R}^2, xdy)$, namely $-x \frac{d h(y)}{dy} \frac{\partial}{\partial x} + h(y) \frac{\partial}{\partial y}$. Members of $\mathcal{X}_L(\mathbb{R}^3, \alpha)$ are trivially obtained by adding constant multiples of $\frac{\partial}{\partial z}$ to members of $\mathcal{X}(\mathbb{R}^2, xdy)$.

This lemma provides the general form of the vector fields we are interested in.

Our goal is the classification of these vector fields according to the natural relation defined in the obvious way: two fields $X, Y \in \mathcal{X}(\mathbb{R}^2, xdy)$ are Liouville conjugate if there exists a diffeomorphism of the plane preserving the form of Liouville, $\phi \in Diff(\mathbb{R}^2, xdy)$, such that $\phi_* X = Y$, while two fields $Z, W \in \mathcal{X}(\mathbb{R}^3, \alpha)$ are strictly contact conjugate if a $\psi \in Diff(\mathbb{R}^3, \alpha)$ exists, such that $\psi_* Z = W$. Observe that classifying members of $\mathcal{X}(\mathbb{R}^2, xdy)$ leads to a classification of fields belonging in $\mathcal{X}_L(\mathbb{R}^3, \alpha)$; one needs only to extend $\phi$ to $\mathbb{R}^3$ as $\psi(x,y,z) = (\phi(x,y), z)$.

To proceed with the classification of Liouville vector fields of the plane, we shall exploit their dependence on real valued functions.

**1.2 Lemma.** Let $f$ be a univariate function and $\varphi$ a diffeomorphism of $\mathbb{R}$. The Liouville vector field corresponding to function $f$ may be transformed, via a diffeomorphism respecting the form $xdy$, to the Liouville vector field corresponding to the function $\frac{1}{\varphi'(y)} f(\varphi(y))$. 

Proof. Constructing the fields corresponding to these two functions, according to the recipe given in lemma Lemma 1.1, we conclude that the diffeomorphism accomplishing the desired transformation is \( \psi(x, y) = (\frac{x}{\phi(y)}, \phi(y)) \) which also preserves the Liouville form.

This lemma ensures that the classification of Liouville vector fields, up to diffeomorphisms belonging in \( \text{Diff}(\mathbb{R}^2, xdy) \), reduces to a classification of univariate real functions. In the next section, we turn our attention to this classification.

2. Restricted contact equivalence

Let \( f : (\mathbb{R}, 0) \rightarrow (\mathbb{R}, 0) \) be the germ at the origin of a smooth function. Their ring will be denoted as \( \mathcal{E} \). We introduce the following equivalence relation.

2.1 Definition. Let \( f, g \in \mathcal{E} \). We shall call them restrictively contact equivalent (\( \mathcal{RK} \)-equivalent) if there exists a germ of a smooth diffeomorphism \( \varphi : (\mathbb{R}, 0) \rightarrow (\mathbb{R}, 0) \) such that \( g = \frac{1}{\varphi'}(f \circ \varphi) \).

2.2 Example. Let \( f, g \in \mathcal{E} \), with \( f(x) = x \), \( g(x) = x + x^2 \). Define \( \varphi(x) = \frac{x}{x+1} \). It is easy to check that \( \varphi \) is a local diffeomorphism at \( 0 \in \mathbb{R} \) and \( g = \frac{1}{\varphi'}(f \circ \varphi) \).

Let us recall here that two univariate function germs \( f, g \in \mathcal{E} \) are called contact equivalent if \( f(x) = M(x)g(\varphi(x)) \), for some function germ \( M(x) \) and diffeomorphism \( \varphi \). The equivalence relation we study here requires \( M(x) = \frac{1}{\varphi'(x)} \). This explains why we called the above defined equivalence relation restricted contact.

Suppose now that \( g_s \in \mathcal{E} \) is a curve of \( \mathcal{RK} \)-equivalent germs, depending on the real parameter \( s \), with \( g_0 = f \). There exists thus a curve of local diffeomorphisms \( \varphi_s : (\mathbb{R}, 0) \rightarrow (\mathbb{R}, 0) \), with \( \varphi_s(0) = 0, \forall s \in \mathbb{R} \) and \( \varphi_0(x) = x \), such that \( g_s(x) = \frac{1}{\varphi'_s(x)}f(\varphi_s(x)) \).

Differeniting with respect to \( s \) and evaluating at \( s = 0 \) we get:

\[
\frac{\partial}{\partial s}g_s(x)|_{s=0} = -X'(x)f(x) + f'(x)X(x),
\]

where \( X(x) \) is defined by the relation \( \frac{\partial}{\partial s}\varphi_s(x) = X(\varphi_s(x)) \). Note that \( X(0) = 0 \), thus \( X(x) \in m \), the ideal of \( \mathcal{E} \) generated by \( x \in \mathcal{E} \).

2.3 Lemma. Let \( f \in \mathcal{E} \). The ideal generated from the germs \( -X'(x)f(x) + f'(x)X(x), \ X \in m \), equals \( \langle f(x) \rangle + f'(x)m \).

Proof. It is obvious that, if \( X(x) \in m \), then \( -X'(x)f(x) + f'(x)X(x) \) is a member of \( \langle f(x) \rangle + f'(x)m \). Let us prove the opposite inclusion.

Let \( h \in \langle f(x) \rangle + f'(x)m \). Germs \( g \in \mathcal{E} \) and \( k \in m \) exist, such that \( h(x) = g(x)f(x) + f'(x)k(x) \). We wish to find a germ \( X \in m \) such that:

\[
h(x) = -X'(x)f(x) + X(x)f'(x) \Rightarrow g(x)f(x) + f'(x)k(x) = -X'(x)f(x) + X(x)f'(x).
\]
One may easily check that a solution of the last differential equation is:

\[ X(x) = \begin{cases} 
  k(x) - \int_0^x \frac{g(t)+k'(t)}{f(t)} \, dt & \text{if } x \neq 0 \\
  0 & \text{if } x = 0
\end{cases} \]

which is well defined and smooth in a neighborhood of the origin and, therefore, for every \( h \in \langle f(x) \rangle + f'(x)m \) a \( X \in m \) exists, such that \( h = -X'(x)f(x) + X(x)f'(x) \), hence the conclusion.

Under the light of the lemma above, we proceed to the following:

**2.4 Definition.** The tangent space of \( f \in \mathcal{E} \), with respect to \( RK \)-equivalence, is defined to be \( T_{RK}f := \langle f(x) \rangle + f'(x)m \). The codimension of \( f \) is defined as \( \text{codim}_{RK}(f) := \dim(m/T_{RK}f) \).

**2.5 Example.** We calculate that, if \( f(x) = x \), then \( T_{RK}f = m \), thus \( \text{codim}_{RK}(f) = 0 \), while if \( g(x) = x^2 \), \( T_{RK}g = m^2 \) and \( \text{codim}_{RK}(g) = 1 \).

As usual, the germ \( f \in \mathcal{E} \) is called \( k \)-determined, with \( k \in \mathbb{N} \), if every other \( g \in \mathcal{E} \) having the same \( k \)-jet with \( f \) is \( RK \)-equivalent to \( f \). If such a finite \( k \) does not exist, we say that \( f \) is not finitely determined.

**2.6 Theorem.** The germ \( f \in \mathcal{E} \) is \( k \)-determined, with respect to \( RK \)-equivalence, if \( m^{k+1} \subseteq mT_{RK}f \).

**Proof.** We have to prove that if \( h \in m^{k+1} \subseteq mT_{RK}f \), the germs \( f \) and \( f + h \) are \( RK \)-equivalent.

Towards this end, define \( f_s = f + sh \), \( s \in [0, 1] \). We shall construct diffeomorphisms \( \varphi_s(x) \), defined in a neighborhood of the origin, such that \( f_s = \frac{1}{\varphi'_s}f(\varphi_s(x)) \). Differentiating with respect to \( s \), we get:

\[ h(x) = -\frac{1}{\varphi'_s(x)}X'(\varphi_s(x))f(\varphi_s(x)) + \frac{1}{\varphi'_s(x)}X(\varphi_s(x))f'(\varphi_s(x)). \]

Note that, for \( s = 0 \), we get the relation \( h(x) = -X'(x)f(x) + X(x)f'(x) \), which, by the previous lemma, has a solution \( X(x) \in m \) since \( m^{k+1} \subseteq mT_{RK}f \). We need to show that a solution exists for all \( s \in [0, 1] \).

Consider \( \mathbb{R} \times [0, 1] \), let \( \mathcal{R} \) be the ring of function germs at \( 0 \times [0, 1] \) and denote by \( m_s \) the ideal of \( \mathcal{R} \) consisting of those germs vanishing at \( 0 \times [0, 1] \). We have:

\[
m^{k+1} \subseteq m_s^{k+1} \subseteq m_s(f)_{\mathcal{R}} + f'(x)m_s^2 \\
\subseteq m_s(f_s)_{\mathcal{R}} + m_s(h)_{\mathcal{R}} + f'_s m_s^2 + h' m_s^2 \\
\subseteq m_s(f_s)_{\mathcal{R}} + f'_s m_s^2 + m_s^{k+2} \\
\subseteq m_s(f_s)_{\mathcal{R}} + f'_s m_s^2 + m_s m_s^{k+1} \\
\subseteq m_s((f_s)_{\mathcal{R}} + f'_s m_s),
\]
where the inclusion in the last line holds due to the Nakayama lemma. Thus, for every $s \in [0, 1]$, we have that $h \in m^{k+1} \subseteq m_{s}T_{RK}f$. We can therefore find $X_{s}(x) \in m_{s}$, defining the germ of diffeomorphism $\varphi_{s}$ which, for $s = 1$, establishes an equivalence between $f$ and $f + h$.

The classification of the elements of $\mathcal{E}$ now follows. We begin with germs that either do not vanish at the origin, or have a regular point there.

2.7 Lemma. Let $f \in \mathcal{E}$. If $f(0) \neq 0$, it is $RK$–equivalent to 1, while if $f(0) = 0$ and $f'(0) = a \neq 0$, $f$ is $RK$–equivalent to $ax$.

Proof. Let $f \in \mathcal{E}$, with $f(0) \neq 0$. To show that it is $RK$–equivalent to 1, we must find a local diffeomorphism $k(x)$ such that $\frac{k'(x)}{f'(x)} = f(x)$, which is the same as $k'(x) = \frac{1}{f(x)}$, which is a differential equation with smooth right hand side, at least in a neighborhood of the origin, thus, such a smooth $k(x)$ exists.

On the other hand, let $f(0) = 0$ and $f'(0) = a \neq 0$. It is 1–determined, thus $RK$–equivalent to its linear part $ax$, while, as may be easily verified, the germ $ax$ is $RK$–equivalent to $bx$ only if $a = b$.

Let us now proceed to germs with critical points.

2.8 Lemma. Let $f \in \mathcal{E}$, with $f(0) = f'(0) = \ldots = f^{k-1}(0) = 0$ and $f^{k}(0) \neq 0$, $k > 1$. Then $f$ is $RK$–equivalent to $x^k$, if $k$ is an even number and to $x^k$ or $-x^k$, if $k$ is an odd number.

Proof. If $f$ is such a germ, then, in a neighborhood of the origin, we may write $f(x) = x^kg(x)$, with $g(0) \neq 0$. Thus $T_{RK}f = m^k$, and $f$ is $k$–determined. It is thus $RK$–equivalent to $ax^k$, while, as may easily be verified, the germ of a diffeomorphism $\varphi(x)$ exists such that $\frac{1}{\varphi(x)}a\varphi(x)^k = x^k$, for every $a \in \mathbb{R} \setminus \{0\}$, if $k$ is even, while if $k$ is odd then $ax^k$ is $RK$–equivalent to $-x^k$, for $a < 0$ and to $x^k$, for $a > 0$.

Combining all the above, we may now state the main theorem for the classification of members of $\mathcal{E}$.

2.9 Theorem. If a member of $\mathcal{E}$ does not vanish at the origin it is $RK$–equivalent to the constant function 1. Members of $\mathcal{E}$ having codimension 0 are $RK$–equivalent to $ax$ (a being the value of their derivative there). A member of $\mathcal{E}$ of odd codimension $k$ is $RK$–equivalent to $x^{k+1}$, while if it is of even codimension $k$ it is $RK$–equivalent to $\pm x^{k+1}$, depending on the sign of the value of its first non–vanishing derivative at the origin.

Table 1 contains the local models of members of $\mathcal{E}$ having codimension up to five. We note that there are differences with the classical classification list for right equivalence (in which list the $A_1$, $A_3$ and $A_5$ models may have both negative and positive sign) and for contact equivalence (in which, for example, the $A_0$ model does not depend on the constant $a$, see [7, 5]). The interested reader should consult [14] for a relation of contact and right equivalence, while the equivalence relation studied here provides more models than right and contact equivalence since it is stricter than both.
3. Local models for members of $\mathcal{X}(\mathbb{R}^2, xdy)$

We return now to our study of vector fields of the plane, which preserve the form of Liouville. To construct their local models, we make use of lemma [Lemma 1.2] along with theorem [Theorem 2.9].

3.1 Lemma. (of regular points) Let $X \in \mathcal{X}(\mathbb{R}^2, xdy)$ be such that $X(0) \neq 0$. Then, in a neighborhood of zero, it is conjugate, via a diffeomorphism preserving the form of Liouville, to the constant vector field $\frac{\partial}{\partial y}$.

Proof. Since $X \in \mathcal{X}(\mathbb{R}^2, xdy)$, it is of the form $X(x, y) = -xf'(y)\frac{\partial}{\partial x} + f(y)\frac{\partial}{\partial y}$, for a smooth, real valued, function $f(y)$. Since $X(0, 0) = f(0)\frac{\partial}{\partial y} \neq 0$, we get $f(0) \neq 0$, which means that $f$ is $\mathcal{RK}$–equivalent to the constant function 1, thus, by lemma [Lemma 1.2], a diffeomorphism preserving $xdy$ exists, transforming $X$ to $\frac{\partial}{\partial y}$.

Let us now turn our attention to hyperbolic singularities.

3.2 Lemma. (hyperbolic singularities) Let $X \in \mathcal{X}(\mathbb{R}^2, xdy)$ having a hyperbolic singularity at the origin. Then, in a neighborhood of zero, it is conjugate, via a diffeomorphism preserving the form of Liouville, to the vector field $-ax\frac{\partial}{\partial x} + ay\frac{\partial}{\partial y}$.

Proof. The vector field is of the form $-xf'(y)\frac{\partial}{\partial x} + f(y)\frac{\partial}{\partial y}$, and it is easy to check that the eigenvalues of zero are $-f'(0)$ and $f'(0)$. Thus zero is a hyperbolic singularity if, and only if, $f'(0) \neq 0$, and $f$ is therefore $\mathcal{RK}$–equivalent to $ay$, $a = f'(0)$. The existence of a diffeomorphism transforming $X$ to $-ax\frac{\partial}{\partial x} + ay\frac{\partial}{\partial y}$ is guarantied, by lemma [Lemma 1.2].

We see that, at a hyperbolic singularity, all members of $\mathcal{X}(\mathbb{R}^2, xdy)$ are topologically equivalent: they are of the saddle type. Up to diffeomorphisms respecting the form of Liouville, however, their equivalence classes are classified by a real number.

The lemmata above ensure that the first non–vanishing jet of members of $\mathcal{X}(\mathbb{R}^2, xdy)$ completely determine their local behavior, at least in the simplest cases. Actually, this holds in general.

3.3 Theorem. Let $X, Y \in \mathcal{X}(\mathbb{R}^2, xdy)$. If $j^kX(0) = j^kY(0) = 0, k = 0, \ldots, i - 1$, and $j^iX(0) = j^iY(0) \neq 0$, for some $i \in \mathbb{N} \setminus \{0\}$, there exists a diffeomorphism preserving $xdy$ which conjugates $X$ and $Y$. 

Table 1

| symbol | codimension | function |
|--------|-------------|----------|
| $A_0^+$ | 0           | $ay$     |
| $A_1^+$ | 1           | $y^2$    |
| $A_2^+$ | 2           | $\pm y^3$ |
| $A_3^+$ | 3     | $y^4$    |
| $A_4^+$ | 4           | $\pm y^5$ |
| $A_5^+$ | 5           | $y^6$    |
We omit the proof, since it follows the lines of the lemma classifying the hyperbolic singularities. Using theorem [Theorem 2.9] we give in Table 2 the local models of singularities of members of $\mathcal{X}(\mathbb{R}^2, xy)$, up to codimension 5.

Table 2

| symbol | codimension | local model |
|--------|-------------|-------------|
| $A_0^a$ | 0           | $-ax \frac{\partial}{\partial x} + ay \frac{\partial}{\partial y}$ |
| $A_1$   | 1           | $-2xy \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y}$ |
| $A_2$   | 2           | $-3xy^2 \frac{\partial}{\partial x} + y^3 \frac{\partial}{\partial y}$ |
| $A_3$   | 3           | $-4xy^3 \frac{\partial}{\partial x} + y^4 \frac{\partial}{\partial y}$ |
| $A_4$   | 4           | $-5xy^4 \frac{\partial}{\partial x} + y^5 \frac{\partial}{\partial y}$ |
| $A_5$   | 5           | $-6xy^5 \frac{\partial}{\partial x} + y^6 \frac{\partial}{\partial y}$ |

For the cases $A_{2}^{\pm}$, $A_{4}^{\pm}$ we have omitted writing the vector fields for the negative and the positive sign since one may be obtained from the other after a multiplication with $-1$ (which means that their phase portraits are identical up to a reversal of time).

Except from the hyperbolic model (and the non-vanishing one), they all have an infinity of equilibria (the $x$–axis). Other than that, topologically their behavior is quite simple to analyze, since function $xf(y)$ serves as a first integral.

It remains to analyze the behavior of perturbations of these vector fields.

4. Bifurcations of low codimension

At regular points, members of $\mathcal{X}(\mathbb{R}^2, xy)$ are all conjugate to each other, via a diffeomorphism preserving the form of Liouville. At hyperbolic singularities all such vector fields may be transformed to their linear part; these linear parts are not conjugate to each other, since the eigenvalues there are a conjugacy invariant. However, up to topological equivalence, they are all saddle points, thus hyperbolic singularities are structurally stable.

This is no more the case when we analyze vector fields belonging to the classes $A_k$, $k \geq 1$. To describe their local bifurcations we should first compute their transversal unfoldings.

4.1 Definition. Let $X$ be the germ at the origin of a Liouville vector field. Denote by $S$ its singularity class (that is, the set of all germs at the origin of vector fields of Liouville which are Liouville equivalent to $X$). A transversal unfolding of $X$ consists of a set of germs at the origin of Liouville vector fields, which set intersects $S$ transversally at $X$.

Thus, to construct transversal unfoldings of Liouville vector fields, we must first compute the tangent spaces of singularity classes.

4.2 Theorem. Let $X_f \in \mathcal{X}(\mathbb{R}^2, xy)$ (where $f$ is the function defining $X_f$) and $S$ its singularity class. We have:
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\[ T_{X_f}S = \{ X_g \in \mathcal{X}(\mathbb{R}^2, xdy)/g \in \langle f \rangle \}. \]

**Proof.** Let \( X_f = -xf'(y)\frac{\partial}{\partial x} + f(y)\frac{\partial}{\partial y} \) be the germ at the origin of a Liouville vector field and \( \psi_s(x, y) = (\frac{x}{\varphi'(y)}, \varphi_s(y)) \) the germ at the origin of a family of diffeomorphisms preserving the Liouville form, where \( \varphi_0(y) = y, \varphi_s(0) = 0 \) and \( \varphi'_s(0) \neq 0 \). Define:

\[ X_s = \psi_s X_f = (-\frac{xf'(y)}{\varphi'(y)} - \frac{xf''(y)}{\varphi''(y)}\varphi'_s(y))\frac{\partial}{\partial x} + \varphi'_s(y)f(y)\frac{\partial}{\partial y}. \]

It is a curve of Liouville vector fields belonging to \( S \), and we have \( X_0 = X_f \). To calculate the tangent space \( T_{X_f}S \) we need to evaluate at \( s = 0 \) the derivative with respect to the parameter \( s \) of \( X_s \). It is:

\[
\frac{\partial}{\partial s} X_s|_{s=0} = (-xf'(y)\Phi'(y) - x\Phi''(y)f(y))\frac{\partial}{\partial x} + \Phi'(y)f(y)\frac{\partial}{\partial y}.
\]

We have denoted as \( \Phi(y) \) the vector field defined by \( \frac{\partial}{\partial x}\varphi_s(y) = \Phi(\varphi_s(y)) \). Note that \( \frac{\partial}{\partial s} X_s|_{s=0} \) is a Liouville vector field, corresponding to the function \( \Phi'(y)f(y), \) which belongs to \( \langle f \rangle \mathcal{E} \), since \( \Phi \in \mathcal{M} \). Thus, the tangent space of \( S \) at \( X_f \) consists of those Liouville fields corresponding to functions belonging in the ideal \( \langle f \rangle \mathcal{E} \).

The theorem above allows us to study bifurcations of Liouville vector fields. To illustrate this, we present here such bifurcations of low codimension.

We begin with the singularity class \( A_1 \). The members of this class form a subset of codimension 1 in the set of those members of \( \mathcal{X}(\mathbb{R}^2, xdy) \) vanishing at the origin. To transversally unfold them, we only need to add to their local model, linear terms preserving the form of Liouville. We arrive thus at the vector field \( Q_a(x, y) = (-ax - 2xy)\frac{\partial}{\partial x} + (ay + y^2)\frac{\partial}{\partial y}, \) where \( a \) a real parameter. We have the following:

**4.3 Proposition.** The set of \( X \in \mathcal{X}(\mathbb{R}^2, xdy) \) with \( j^0 X(0) = j^1 X(0) = 0 \) and \( j^2 X(0) \neq 0 \) has codimension 1 in the set of Liouville vector fields vanishing at the origin. Its members are all conjugate to the \( A_1 \) model given above. The curve of vector fields \( Q_a(x, y) \) intersects at \( a = 0 \) this set transversally.

**Proof.** The codimension and the conjugacy to the \( A_1 \) model follows easily from the analysis given in the previous sections. Note that \( Q_0(x, y) \) is the \( A_1 \) model, corresponding to the function \( y^2 \). The intersection is transversal, since:

\[
\frac{\partial}{\partial a} Q_a(x, y)|_{a=0} = -x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}.
\]

This is a Liouville vector field corresponding to the function \( y \) which is the only function (up to a constant) which vanishes at the origin and belongs to \( \mathcal{E}/\langle y^2 \rangle \).

Thus, \( Q_a(x, y) \) is a transversal unfolding of the \( A_1 \) singularity. Vector fields depending on a single parameter undergo, for isolated values of this parameter, the bifurcation depicted in Figure 1; this bifurcation is therefore the codimension 1 bifurcation occurring in vector fields of interest.
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We proceed to bifurcations of codimension two. Consider the $A_2$ model, and add to it terms of lower degree. We arrive at $T_{a,b} = (-ax - 2bxy - 3x^2y) \frac{\partial}{\partial x} + (ay + by^2 + y^3) \frac{\partial}{\partial y}$, where $a$, $b$ real parameters. We have the following:

4.4 Proposition. The set of $X \in \mathcal{X}(\mathbb{R}^2, xdy)$ with $j^0X(0) = j^1X(0) = j^2X(0) = 0$ and $j^3X(0) \neq 0$ has codimension 2 in the set of Liouville vector fields vanishing at the origin. Its members are all conjugate to the $A_2$ model given above. The surface of vector fields $T_{a,b}(x,y)$ intersects at $a = b = 0$ this set transversally.

Its proof goes along the lines of the previous proposition, and it is therefore omitted. In figure 2 we present the bifurcations system $T_{a,b}$ system undergoes, for characteristic parameter values.

Before discussing the diffeomorphism case, let us note that we could study bifurcations of arbitrary, finite, codimension following the exact same approach.
5. Plane diffeomorphisms preserving the form of Liouville

Let us now turn our attention to diffeomorphisms of the plane respecting the form of Liouville. As we saw, they are of the general form \( f(x, y) = \left(\frac{x}{h(y)}, h(y)\right) \). Diffeomorphism \( h(y) \) of \( \mathbb{R} \) uniquely defines such a diffeomorphism.

The unique linear diffeomorphism preserving the form of Liouville (and the origin) is thus \( (x, y) \mapsto (ax, \frac{1}{a} y) \). Aside from this, there are no other polynomial members of \( Diff(\mathbb{R}^2, xdy) \); as a consequence, finite jets (of any order) of Liouville diffeomorphisms studied here do not belong to the same group.

The classification of strict contactomorphisms, according to the natural equivalence relation, is of course our purpose; \( f, g \in Diff(\mathbb{R}^2, xdy) \) are Liouville conjugate if there exists a third Liouville diffeomorphism \( \phi \) such that \( f \circ \phi = \phi \circ g \). To continue, and since we focus on fixed points, we impose the conditions \( f(0) = g(0) = 0 \).

Generically, such diffeomorphisms may be linearized in a neighborhood of the origin.

5.1 Proposition. There exists a codimension zero subset of those members of \( Diff(\mathbb{R}^2, xdy) \) vanishing at the origin, every member of which may be transformed, via a change of coordinates preserving the Liouville form, to its linear part.

Proof. Let us consider the set of Liouville diffeomorphisms having linear part \( (ax, \frac{1}{a} y) \), \( a \neq \pm 1 \). Its codimension is zero (in the set of Liouville diffeomorphisms vanishing at the origin) and its members are of the form \( f(x, y) = \left(\frac{x}{h'(y)}, h(y)\right) \) where \( h(y) = \frac{1}{a} y + \text{h.o.t.} \) a local diffeomorphism (we use h.o.t. as an abbreviation for ”higher order terms”).

We have supposed that \( a \neq \pm 1 \); therefore a local diffeomorphism \( \psi \) of \( \mathbb{R} \) exists such that \( \psi \circ h \circ \psi^{-1} = \frac{1}{a} y \) (this is the content of the Sternberg linearization theorem, see [1]). Using this diffeomorphism define \( \phi(x, y) = \left(\frac{x}{\psi'(y)}, \psi(y)\right) \) and observe that it is a diffeomorphism, preserving the Liouville form, with inverse \( \phi^{-1}(x, y) = \left(\frac{x}{(\psi^{-1}(y))^a}, \psi^{-1}(y)\right) \).

As is easy to confirm, \( \psi \circ f \circ \psi^{-1} = (ax, \frac{1}{a} y) \). \qed

We have thus found the generic model for the mappings under study, that is \( (x, y) \mapsto (ax, \frac{1}{a} y) \). As already remarked, it is actually the unique polynomial model for members of \( Diff(\mathbb{R}^2, xdy) \); thus Liouville diffeomorphisms either may be linearized or are not finitely determined (at least finitely determined under the relation of Liouville conjugacy).

6. Homogeneous members of \( \mathcal{X}(\mathbb{R}^3, a) \) and linearization

Having completed the study of vector fields of Liouville we may now state results for strictly contact vector fields of \( \mathbb{R}^3 \). Indeed, one needs only to add constant multiples of \( \frac{\partial}{\partial z} \) to the local models presented above, to obtain vector fields which preserve both the contact form \( a \) and the form of Liouville.
Our choice of restricting our study to members of $\mathcal{X}_L(\mathbb{R}^3, a)$ stems from the fact that they are the only strictly contact vector fields which may have homogeneous components. Indeed, recall from section 1 the general form of a strictly contact vector field:

$$X = -\frac{\partial H(x,y,z)}{\partial y} \frac{\partial}{\partial x} + \frac{\partial H(x,y,z)}{\partial y} \frac{\partial}{\partial x} + (H(x,y,z) - x \frac{\partial H(x,y,z)}{\partial x}) \frac{\partial}{\partial z}.$$ 

Assuming that $H(x,y,z)$ (remember it does not depend on $z$) is a homogenous polynomial of degree $d$, vector field $X$ above is homogeneous of degree $d-1$ only in case its third component is constant, for $d = 1$, or zero, for $d \geq 2$. Members of $\mathcal{X}_L(\mathbb{R}^3, a)$ are therefore the only homogeneous members of $\mathcal{X}(\mathbb{R}^3, a)$. We shall elaborate in this observation in this section, to show, using classical normal form theory, the linearization of strictly contact vector fields respecting the form of Liouville.

Consider members of $\mathcal{X}(\mathbb{R}^3, a)$ vanishing at the origin. If $X$ is such a field, let $X = X_1 + X_2 + ... + X_k$ be its $k$–jet at zero, for some natural number $k$, where each $X_i$, $i = 1, ..., k$, is a homogeneous field of degree $k$. It is easy to see, equating terms of the same degree in equation $L_X(a) = 0$, that each $X_i$ is itself a member of $\mathcal{X}(\mathbb{R}^3, a)$.

We denote as $\mathcal{X}^d(\mathbb{R}^3, a)$ the subset of $\mathcal{X}(\mathbb{R}^3, a)$, the components of which are homogeneous functions of degree $d$. We easily prove the following:

6.1 Lemma. The vector space $\mathcal{X}^d(\mathbb{R}^3, a)$ is one dimensional. For each $d \in \mathbb{N} \setminus \{0\}$, its base consists of the field $X_d = dxy^{d-1} \frac{\partial}{\partial x} - y^d \frac{\partial}{\partial y}$.

The local models of Table 2 constitute, therefore, the basis generating the fields of interest.

Linear fields (belonging to $\mathcal{X}^1(\mathbb{R}^3, a)$) are of the form $X_1 = ax \frac{\partial}{\partial x} - ay \frac{\partial}{\partial y}$, with $a$ arbitrary constant. In our case, therefore, the existence of hyperbolic singularities is excluded (actually, $X_1$ is also the unique linear member of $\mathcal{X}(\mathbb{R}^3, a)$; strictly contact vector fields do not possess hyperbolic singularities). Despite this fact, fields having non–zero linear part can be linearized, in a neighborhood of the origin. We shall prove it now using an approach different from the one indicated above.

There are $\frac{1}{2}(d^2 + 3d + 2)$ monomials depending on three variables, having degree $d$, as simple counting arguments may assure. Thereupon, the vector space $\mathcal{X}^d(\mathbb{R}^3)$ of homogeneous vector fields of degree $d$ is of dimension $\frac{3}{2}(d^2 + 3d + 2)$, and one may easily verify that the fields appearing in Table 3, being $\frac{3}{2}(d^2 + 3d + 2)$ independent vector fields of degree $d$, constitute a basis of it.

6.2 Remark. Vector fields of interest here belong to this base (to obtain them, just set $m_1 = m_3 = 0$ to the first field of the second class). This base was presented, in the general n–dimensional case, in [13], section 4 of which contains the arguments we shall use to prove the next proposition. The author wishes to thank Prof. J D Meiss for clarifying them to him.
### Table 3

| fields                                      | condition | number |
|---------------------------------------------|-----------|--------|
| $y^{m_1}z^{m_2} \frac{\partial}{\partial x}$ |            |        |
| $x^{m_1}z^{m_2} \frac{\partial}{\partial y}$ |            |        |
| $x^{m_1}y^{m_2} \frac{\partial}{\partial z}$ | $m_1 + m_2 = d$ | $3d + 3$ |

$$(1 + m_2)x^{m_1+1}y^{m_2}z^{m_3} \frac{\partial}{\partial x} - (1 + m_1)x^{m_1}y^{m_2+1}z^{m_3} \frac{\partial}{\partial y}$$

$$(1 + m_3)x^{m_1}y^{m_2+1}z^{m_3} \frac{\partial}{\partial y} - (1 + m_2)x^{m_1}y^{m_2}z^{m_3+1} \frac{\partial}{\partial z}$$

$m_1 + m_2 + m_3 + 1 = d$  

$d^2 + d$

If $X \in \mathcal{X}^d(\mathbb{R}^3)$, the vector field $[X_1, X]$, where $X_1$ is the unique, linear and non-zero, strictly contact vector field presented above, is also homogeneous of degree $d$ (the brackets $[\cdot, \cdot]$ denote the usual commutator of vector fields). We may define therefore the operator $ad_{X_1} : \mathcal{X}^d(\mathbb{R}^3) \to \mathcal{X}^d(\mathbb{R}^3)$, $X \mapsto [X_1, X]$. Vector fields belonging to the base of $\mathcal{X}^d(\mathbb{R}^3)$ are eigenvectors of this operator; thus the subspaces generated by them are invariant under $ad_{X_1}$, ensuring the diagonal form of its matrix.

### 6.3 Proposition.

There exists a codimension zero subset of $\mathcal{X}_L(\mathbb{R}^3, \alpha)$ every member of which may be transformed to its linear part. The linearizing diffeomorphism is close to the identity and preserves the contact form.

**Proof.** The subset we refer to is the set of vector fields of interest with non-zero linearization, and its codimension is easily obtained.

Classical normal form theory ensures that, by changing coordinates, we may discard all terms of $X = X_1 + X_2 + ... \in \mathcal{X}_L(\mathbb{R}^3, \alpha)$ which are not contained in the complement of the range of this operator (an operator which leaves invariant the spaces $\mathcal{X}^d(\mathbb{R}^3, \alpha)$, as well as the subspaces generated by the basic vector fields, the subspace of fields which interest us included).

The matrix of $X_1$ is self-adjoint, so a complement to the range of $ad_{X_1}$ is the kernel of this operator. This kernel however, as may easily be verified, is trivial, providing us with a diffeomorphism which transforms to its (non-zero) linear part every field of $\mathcal{X}_L(\mathbb{R}^3, \alpha)$. This diffeomorphism preserves the 1-form defining the contact structure; this stems from the diagonal form of the matrix of $ad_{X_1}$. 

Strictly contact vector fields project to symplectic fields of the plane; homogeneous strictly contact vector fields project to fields of the plane preserving the form of Liouville. We have studied here the local behavior of the latter; the local study of the first remains a challenging task.
7. Conclusions

Contact systems have a long history, and attract a lot of attention, since they form a valuable tool in topological constructions, in Hamiltonian dynamics and in many physical applications (see [12] for a textbook account of these fields, and further references).

Almost all contact systems possess hyperbolic singularities, as transversality arguments show. In this case, conditions for linearization have been obtained ([11, 9, 4]). Results are much more rare, however, if the singularities are degenerate.

We chose here to consider the simpler case of homogeneous strictly contact systems. This led us to the study of plane systems, preserving the form of Liouville, a subject which has an interest of its own. To study these fields we had to classify univariate functions according to the restricted contact equivalence relation. All these admit generalizations and deserve more study.

Indeed, extending the definition of restricted contact equivalence to arbitrary dimensions we get of course the differential conjugacy relation for vector fields. One could probably reobtain results of normal form theory, using this approach, which would potentially help the problem of classifying vector fields preserving the form of Liouville in any dimension.

And, as already mentioned, the general problem of analyzing the behavior of contact dynamical systems stands, both interesting and difficult. The author hopes to further comment on these subjects in the future.

Acknowledgments

This work is dedicated to my two professors, Tassos Bountis and Spyros Pnevmatikos, on the occasion of their 65th birthday. It is only a pleasure for the author to acknowledge the influence they had on him and to thank them for their constant support.

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