TRAPPED REEB ORBITS DO NOT IMPLY PERIODIC ONES

HANSJÖRG GEIGES, NENA RÖTTGEN, AND KAI ZEHMISCH

Abstract. We construct a contact form on \( \mathbb{R}^{2n+1} \), \( n \geq 2 \), equal to the standard contact form outside a compact set and defining the standard contact structure on all of \( \mathbb{R}^{2n+1} \), which has trapped Reeb orbits, including a torus invariant under the Reeb flow, but no closed Reeb orbits. This answers a question posed by Helmut Hofer.

1. Introduction

In [3, Theorem 2], Eliashberg and Hofer proved a global version of the Darboux theorem for contact forms in dimension 3: Any contact form \( \alpha \) on \( \mathbb{R}^3 \) that equals the standard form

\[
\alpha_{\text{st}} = dz + \frac{1}{2}(x \, dy - y \, dx)
\]

outside a compact set and whose Reeb vector field does not have any periodic orbits, is diffeomorphic to the standard form, i.e. there is a diffeomorphism \( \phi \) of \( \mathbb{R}^3 \) such that \( \phi^* \alpha = \alpha_{\text{st}} \).

Recall that a contact form \( \alpha \) on a \((2n+1)\)-dimensional manifold is a 1-form such that \( \alpha \wedge (d\alpha)^n \) is a volume form. The Reeb vector field of such a contact form is the unique vector field \( R \) satisfying

\[
d\alpha(R, \, .) \equiv 0 \quad \text{and} \quad \alpha(R) \equiv 1.
\]

These defining equations imply that diffeomorphic contact forms have diffeomorphic Reeb vector fields, so if \( \phi^* \alpha = \alpha_{\text{st}} \), then \( T\phi(R_{\text{st}}) = R \), where \( R_{\text{st}} = \partial_z \) is the Reeb vector field of \( \alpha_{\text{st}} \). Thus, the Reeb vector field of a contact form \( \alpha \) on \( \mathbb{R}^3 \) satisfying the assumptions of the Eliashberg–Hofer theorem does not have any orbits that are bounded in forward or backward time (we shall call such orbits ‘trapped’). Phrased contrapositively:

**Theorem 1** (Eliashberg–Hofer). Let \( \alpha \) be a contact form on \( \mathbb{R}^3 \) that equals the standard form \( \alpha_{\text{st}} \) outside a compact set. If the Reeb vector field of \( \alpha \) has a trapped orbit, then it also has a periodic orbit. \( \square \)

By taking the connected sum of \((\mathbb{R}^3, \alpha_{\text{st}})\) with a 3-sphere carrying the standard contact form (all of whose Reeb orbits are closed), one can easily construct a contact form on \( \mathbb{R}^3 \) that equals \( \alpha_{\text{st}} \) outside a compact set but has periodic Reeb orbits (and hence cannot be diffeomorphic to \( \alpha_{\text{st}} \)).

In a talk at the conference on *Recent Progress in Lagrangian and Hamiltonian Dynamics* (Lyon, 2012) and in personal communication to Victor Bangert, Helmut Hofer conjectured the higher-dimensional analogue of Theorem 1 see also [2]. The purpose of this note is to disprove that conjecture by an example.

\begin{footnotesize}
2010 Mathematics Subject Classification. 37C27, 37C70, 53D10.
\end{footnotesize}
We write
\[ \alpha_{st} = dz + \frac{1}{2} \sum_{j=1}^{n} (x_j dy_j - y_j dx_j) \]
for the standard contact form on \( \mathbb{R}^{2n+1} \), and \( \xi_{st} = \ker \alpha_{st} \) for the standard contact structure.

**Theorem 2.** There is a contact form \( \alpha \) on \( \mathbb{R}^{2n+1} \), \( n \geq 2 \), defining the standard contact structure, i.e., \( \ker \alpha = \xi_{st} \), with the following properties:

(i) The Reeb vector field \( R \) of \( \alpha \) has a compact invariant set (and hence orbits bounded in forward and backward time).

(ii) There are Reeb orbits which are bounded in forward time and whose \( z \)-component goes to \(-\infty \) for \( t \to -\infty \).

(iii) \( \alpha \) equals \( \alpha_{st} \) outside a compact set.

(iv) \( R \) does not have any periodic orbits.

A related result in Riemannian geometry is due to Bangert and the second author. In [1], answering a question of Walter Craig, they showed the existence of a Riemannian metric on \( \mathbb{R}^n \), \( n \geq 4 \), equal to the Euclidean metric outside a compact set, that admits bounded geodesics (or ‘trapped bicharacteristics’) but no periodic ones.

A contact form with the Reeb dynamics described in Theorem 2 was first discovered by the second author [5]. In joint work we derived the simple construction of such an example that we are going to present now.

## 2. REEB AND CONTACT VECTOR FIELDS

Let \((M, \xi = \ker \alpha)\) be a contact manifold. A contact vector field is a vector field whose flow preserves the contact structure \( \xi \). Once a contact form \( \alpha \) has been chosen, there is a one-to-one correspondence between smooth functions \( H : M \to \mathbb{R} \) and contact vector fields \( X \), defined as follows (cf. [4, Theorem 2.3.1]): Given \( H \), the corresponding contact vector field \( X \) is given by \( X = HR + Y \), where \( R \) is the Reeb vector field of \( \alpha \) and \( Y \) is the unique vector field tangent to \( \xi \) satisfying
\[ i_Y d\alpha = dH(R)\alpha - dH. \] (1)

Conversely, the Hamiltonian function \( H \) corresponding to a contact vector field \( X \) is given by \( H = \alpha(X) \).

The Reeb vector field \( R \), corresponding to the constant function 1, is a contact vector field whose flow even preserves the contact form \( \alpha \). The following well-known lemma says that any contact vector field positively transverse to \( \xi \) is the Reeb vector field of some contact form for \( \xi \). The proof is a straightforward computation using the defining equations of the Reeb vector field.

**Lemma 3.** The contact vector field corresponding to the positive Hamiltonian function \( H : M \to \mathbb{R}^+ \) is the Reeb vector field of the contact form \( \alpha/H \). \( \square \)

## 3. THE EXAMPLE

We are going to prove Theorem 2 for \( n = 2 \); the higher-dimensional generalisation is straightforward. Thus, \( \alpha_{st} \) now denotes the standard contact form on \( \mathbb{R}^5 \), with Reeb vector field \( R_{st} = \partial_z \). Write \((r_j, \theta_j)\) for the polar coordinates in the \((x_j, y_j)\)-plane, \( j = 1, 2 \). By Lemma 3 it suffices to construct a contact vector field positively transverse to \( \xi_{st} \) with the desired dynamics.
Proposition 4. There is a contact vector field $X$ for $\xi_{st}$ with the following properties:

(X-i) On the Clifford torus

$$T := \{ r_1 = 1, r_2 = 1, z = 0 \}$$

the vector field $X$ equals $\partial_{\theta_1} + s\partial_{\theta_2}$ for some $s \in [0, 1] \setminus \mathbb{Q}$.

(X-ii) The cylinder $T \times [-1, 0]$, i.e.

$$\{ r_1 = 1, r_2 = 1, z \in [-1, 0] \},$$

is mapped to itself under the flow of $X$ in forward time.

(X-iii) Outside a compact neighbourhood of $T$, the vector field $X$ equals $\partial_z$.

(X-iv) On $\mathbb{R}^5 \setminus T$ we have $dz(X) > 0$.

Condition (X-i) guarantees that the Clifford torus $T$ is an invariant set of $X$ without any closed orbits. Then by condition (X-iv) there are no closed orbits whatsoever. Condition (X-iii) ensures that the contact form with Reeb vector field $X$ is the standard form $\alpha_{st}$ outside a compact neighbourhood of $T$. With condition (X-ii) this yields an orbit coming from $-\infty$ and trapped in forward time, since $T$ is attracting for the whole cylinder $T \times [-1, 0]$. Likewise, our construction will yield orbits trapped in backward time and going off to $\infty$.

Proof of Proposition 4. We wish to construct $X$ as the contact vector field corresponding to a Hamiltonian function $H: \mathbb{R}^5 \to \mathbb{R}^+$. To that end, we translate the conditions on $X$ into conditions on $H$.

With $dH(R_{st}) = H_z$, equation (11) for $\alpha = \alpha_{st}$ becomes

$$i_Y d\alpha_{st} = \sum_{j=1}^{2} \left( -\frac{y_j}{2} H_z + H_{x_j} \right) dx_j + \left( \frac{x_j}{2} H_z - H_{y_j} \right) dy_j.$$

The contact structure $\xi_{st}$ is spanned by the vector fields

$$e_j = \partial_{x_j} + \frac{y_j}{2} \partial_z, \quad f_j = \partial_{y_j} - \frac{x_j}{2} \partial_z, \quad j = 1, 2.$$

By writing $Y$ in terms of these vector fields, we find with equation (2) that

$$Y = \sum_{j=1}^{2} \left( \left( \frac{x_j}{2} H_z - H_{y_j} \right) e_j + \left( \frac{y_j}{2} H_z + H_{x_j} \right) f_j \right).$$

Condition (X-i) says that along $T$ we must have

$$H = \alpha_{st}(\partial_{\theta_1} + s\partial_{\theta_2}) = \frac{1 + s}{2}$$

and

$$Y = X - HR_{st} = \partial_{\theta_1} + s\partial_{\theta_2} - \frac{1 + s}{2} \partial_z.$$

With (3) this gives

$$\begin{align*}
H_{x_1} &= x_1 - \frac{y_1}{2} H_z \\
H_{y_1} &= y_1 + \frac{x_1}{2} H_z \\
H_{x_2} &= sx_2 - \frac{y_2}{2} H_z \\
H_{y_2} &= sy_2 + \frac{x_2}{2} H_z
\end{align*}$$

on $T$.

But on $T$ we also have

$$0 = dH(\partial_{y_j}) = x_j H_{y_j} - y_j H_{x_j},$$
which by the previous equations equals $H_z/2$. So in fact we obtain

\[
\begin{align*}
H &= (1 + s)/2 \\
H_{x_1} &= x_1 \\
H_{y_1} &= y_1 \\
H_{x_2} &= s x_2 \\
H_{y_2} &= s y_2 \\
H_z &= 0
\end{align*}
\]

on $T$.

Next we turn to condition (X-ii). For the moment we may disregard the $\partial_z$-component of $X$, as this will be controlled by the condition on $H$ corresponding to (X-iv). By looking at equation (3) we see that $X$ will have the required behaviour (and the similar one for the flow on $T \times [0, 1]$ in backward time) if we stipulate

\[
H = (1 + s)/2 \text{ on the cylinder } \{ r_1 = 1, r_2 = 1, z \in [-1, 1] \}.
\]

Indeed, then $H_z = 0$ on that cylinder, and

\[
0 = H \partial_j = x_j H_{y_j} - y_j H_{x_j}, \quad j = 1, 2,
\]

which implies that $H_x \partial_{y_j} - H_{y_j} \partial_{x_j}$ is proportional to $x_j \partial_{y_j} - y_j \partial_{x_j} = \partial_{\theta_j}$ on that cylinder.

Condition (X-iii) simply translates into

\[
H \equiv 1 \text{ outside a compact neighbourhood of } T.
\]

Finally, from (3) we find that

\[
dz(Y) = -\frac{1}{2} \sum_{j=1}^{2} (x_j H_{x_j} + y_j H_{y_j}),
\]

so condition (X-iv) is equivalent to

\[
H - \frac{1}{2} \sum_{j=1}^{2} (x_j H_{x_j} + y_j H_{y_j}) > 0 \text{ on } \mathbb{R}^5 \setminus T.
\]

We now proceed to construct an explicit function $H$ satisfying properties (H-i) to (H-iv). The basic idea is very simple. We modify the function

\[
(x_1, y_1, x_2, y_2, z) \mapsto \frac{1}{2} (x_1^2 + y_1^2) + \frac{s}{2} (x_2^2 + y_2^2),
\]

which satisfies (H-i), such that conditions (H-ii) to (H-iv) are also satisfied. This essentially amounts to smoothing out this function in such a way that it becomes constant 1 outside a compact neighbourhood of $T$, and such that it has a growth rate in radial direction in the planes $\{z = \text{const.}\}$ smaller than the quadratic growth rate of the function we start with.

Let $f_z : \mathbb{R}_0^+ \to \mathbb{R}, \ z \in \mathbb{R}$, be a smooth family of smooth functions with the following properties:

(i) $f_z(1) = 0$ for all $z$;
(ii) $tf_z'(t) \leq 1$ for all $z$ and $t$, with equality only for $z = 0$ and $t = 1$;
(iii) for $t$ large (uniformly in $z$), $f_z(t) > \log c$ for some constant $c > 2/s > 2$.
In other words, \( f_z \) has the same value as \( \log \) at \( t = 1 \), \( f_0 \) has the same derivative at \( t = 1 \) as \( \log \), for other values of \( z \) or \( t \) the function \( f_z \) grows more slowly than \( \log \). The function

\[
H_0(x_1, y_1, x_2, y_2, z) := \frac{1}{2} \exp(f_z(x_1^2 + y_1^2)) + \frac{s}{2} \exp(f_z(x_2^2 + y_2^2))
\]

satisfies (H-i) and (H-iv), and it satisfies (H-ii) on the whole cylinder (in \( z \)-direction) over \( T \).

Notice that by condition (iii) on \( f_z \), either of the summands in \( H_0 \) is greater than \( sc/2 \) for \( r_1 \) resp. \( r_2 \) sufficiently large. This will be used below when we enforce condition (H-iii).

Let \( g: \mathbb{R}^+ \to \mathbb{R} \) be a smooth monotone increasing function with these properties:

- (i) \( g(t) = \log t \) near \( t = (1 + s)/2 \);
- (ii) \( g(t) = 0 \) for \( t \geq sc/2 \);
- (iii) \( g'(t) \leq 1/t \) for all \( t \).

Then \( H_1 := \exp(g \circ H_0) \) satisfies all requirements bar one: (H-iii) only holds outside a cylinder over a compact neighbourhood of \( T \) in \( \{z = 0\} \).

Finally, we choose a smooth function \( h: \mathbb{R} \to [0, 1] \) with

- (i) \( h(z) = 0 \) for \( z \in [-1, 1] \);
- (ii) \( h(z) = 1 \) for \( |z| \) large.

Then set

\[
H(x_1, y_1, x_2, y_2, z) = (1 - h(z)) \cdot H_1(x_1, y_1, x_2, y_2, z) + h(z).
\]

This positive function \( H \) satisfies conditions (H-i) to (H-iv).

\[ \square \]

**Remark.** Statement (ii) in Theorem 2 is a topological consequence of statements (i) and (iii): Consider a hyperplane \( E = \{z = -z_0\} \) with \( z_0 > 0 \) sufficiently large, such that \( R = \partial z \) along \( E \). The flow of \( R \) (for any given finite time) cannot send \( E \) to the region \( \{z > 0\} \), since this is obstructed by the invariant torus \( T \). Our proof, in addition, gives explicit orbits trapped in one direction of time only.

**Acknowledgements.** We thank Victor Bangert for directing our attention to this question. This note was written during the workshop on Legendrian submanifolds, holomorphic curves and generating families at the Académie Royale de Belgique, August 2013, organised by Frédéric Bourgeois. H. G. and K. Z. are partially supported by DFG grants GE 1245/2-1 and ZE 992/1-1, respectively.

**References**

1. V. Bangert and N. Röttgen, Isoperimetric inequalities for minimal submanifolds in Riemannian manifolds: a counterexample in higher codimension, *Calc. Var. Partial Differential Equations* 45 (2012), 455–466.
2. B. Bramham and H. Hofer, First steps towards a symplectic dynamics, *Surv. Differ. Geom.* 17 (2012), 127–178.
3. Ya. Eliashberg and H. Hofer, A Hamiltonian characterization of the three-ball, *Differential Integral Equations* 7 (1994), 1303–1324.
4. H. Geiges, *An Introduction to Contact Topology*, Cambridge Stud. Adv. Math. 109 (Cambridge University Press, Cambridge, 2008).
5. N. Röttgen, A contact cylinder with standard boundary and a bounded Reeb orbit but no periodic Reeb orbit, preprint (2013).
