A COMMUTATIVE ALGEBRA APPROACH TO MULTIPLICATIVE HOM-LIE ALGEBRAS

YIN CHEN AND RUNXUAN ZHANG

ABSTRACT. Let \( g \) be a finite-dimensional complex Lie algebra and HLie\(_m\)(\( g \)) be the affine variety of all multiplicative Hom-Lie algebras on \( g \). We use a method of computational ideal theory to describe HLie\(_m\)(\( gl_n(C) \)), showing that HLie\(_m\)(\( gl_2(C) \)) consists of two 1-dimensional and one 3-dimensional irreducible components and HLie\(_m\)(\( gl_n(C) \)) = \{ diag(\delta, \ldots, \delta, a) | \delta = 1 or 0, a \in C \} for \( n \geq 3 \). We construct a new family of multiplicative Hom-Lie algebras on the Heisenberg Lie algebra \( h_{2n+1}(C) \) and characterize the affine varieties HLie\(_m\)(\( u_2(C) \)) and HLie\(_m\)(\( u_3(C) \)). We also study the derivation algebra Der\(_D\)(\( g \)) of a multiplicative Hom-Lie algebra \( D \) on \( g \) and, under some hypotheses on \( D \), we prove that the Hilbert series \( \mathcal{H}(\text{Der}_D(g), t) \) is a rational function.

1. INTRODUCTION

In the last fifteen years, Hom-algebra structures have occupied an important place in nonassociative algebras, deformation theory and mathematical physics. Realizing Hom-Lie algebra structures on a vector space has substantial ramifications in the study of representation theory, deformations of infinite-dimensional Lie algebras and generalized Yang-Baxter equations, whereas finding a powerful method to describe these Hom-Lie algebras is indispensable in developing efficient classifying tools. Jin-Li’s Theorem [JL08, Proposition 2.1], proving that all Hom-Lie algebras on complex simple finite-dimensional Lie algebras except for \( \mathfrak{sl}_2(C) \) are trivial, serves as a motivational example. Our primary objective is to describe multiplicative Hom-Lie algebra structures on several typical families of complex finite-dimensional Lie algebras and our approach depends upon techniques from commutative algebra.

Motivated by characterizing algebraic structures of some \( q \)-deformations of the Witt and the Virasoro algebras, [HLS06] originally introduced the notion of a Hom-Lie algebra (on a vector space), showing that these \( q \)-deformations have a Hom-Lie algebra structure. This initial definition of a Hom-Lie algebra was also modified slightly to the current version; see [MS08], [BM14] and [She12]. Recently, the structure and representation theory of Hom-Lie algebras, Hom-associative, and even Hom-Novikov algebras, have been studied extensively; see for example [HLS06], [MS08], [Yau11], [ZHB11] and references therein. We concentrate on Hom-Lie algebra structures on a finite-dimensional complex Lie algebra because the well-developed structure theory of Lie algebras and related representation theory have been demonstrated to be useful in solving such problems; see [Bau99].

Let \( g \) be a finite-dimensional complex Lie algebra. A linear transformation \( D \) on \( g \) is called a Hom-Lie algebra structure on \( g \) if the Hom-Jacobi identity: \([D(x), [y, z]] + [D(y), [z, x]] + [D(z), [x, y]] = 0\) holds for all \( x, y, z \in g \). A Hom-Lie algebra \( D \) on \( g \) is said to be multiplicative if \( D \) is a Lie algebra.
hommomorphism. Inspired by [JL08], we wonder whether there exists a nontrivial (multiplicative) Hom-Lie algebra structure on non-semisimple complex Lie algebras and further, if there exist such Hom-Lie algebras, we also seek a systematic way to describe and classify them up to isomorphism. Consolidating and comparing with existing methods (see [Rem18] and [GDSSV20]), we take a point of view of affine varieties on the set of all Hom-Lie algebras and multiplicative Hom-Lie algebras on $\mathfrak{g}$. This means that techniques from computational ideal theory will be our main source of tools.

**Affine varieties of Hom-Lie algebras.** We use $\dim_{\mathbb{C}}(*)$ and $\dim(*)$ to denote the dimension and the Krull dimension of $*$ as a $\mathbb{C}$-vector space and an affine variety over the complex field $\mathbb{C}$, respectively. Suppose $\dim_{\mathbb{C}}(\mathfrak{g}) = n$ and $M_n(\mathbb{C})$ denotes the affine space of all $n \times n$-matrices over $\mathbb{C}$. With respect to a chosen basis $\{e_1, e_2, \ldots, e_n\}$ of $\mathfrak{g}$, each element of $M_n(\mathbb{C})$ corresponds to a linear transformation on $\mathfrak{g}$. The main objects of study in the present paper are the vector space $HLie(\mathfrak{g}) := \{D \in M_n(\mathbb{C}) \mid D \text{ is a Hom-Lie algebra on } \mathfrak{g}\}$ and the affine variety $HLie_m(\mathfrak{g}) := \{D \in HLie(\mathfrak{g}) \mid D \text{ is multiplicative}\}$. We also refer to an element $D \in HLie_m(\mathfrak{g})$ as a *regular* Hom-Lie algebra on $\mathfrak{g}$ if it is an automorphism of $\mathfrak{g}$, and we refer to $D$ as *involutive* if it is an involution. Let $HLie_r(\mathfrak{g})$ and $HLie_i(\mathfrak{g})$ denote the subsets of all regular Hom-Lie algebras and of all involutive Hom-Lie algebras on $\mathfrak{g}$ respectively. The following set inclusions hold:

$$HLie_i(\mathfrak{g}) \subseteq HLie_r(\mathfrak{g}) \subseteq HLie_m(\mathfrak{g}) \subseteq HLie(\mathfrak{g}).$$

We observe that a Hom-Lie algebra structure $D \in HLie(\mathfrak{g})$ can be determined by finitely many polynomial equations in at most $n^2$ variables, which means that $HLie(\mathfrak{g}), HLie_m(\mathfrak{g}), HLie_r(\mathfrak{g})$ and $HLie_i(\mathfrak{g})$ all can be viewed as affine varieties in an affine space of dimension at most $n^2$. Moreover, it is easy to see that $HLie(\mathfrak{g})$ is a linear variety and thus it is irreducible; $HLie_m(\mathfrak{g}), HLie_r(\mathfrak{g})$ and $HLie_i(\mathfrak{g})$ are not linear in general and thus they may not be irreducible. We also note that $HLie_r(\mathfrak{g}) \subseteq GL(\mathfrak{g}) \cap Aut(\mathfrak{g})$ is a linear algebraic group (not necessarily irreducible), where $GL(\mathfrak{g})$ and $Aut(\mathfrak{g})$ denote the general linear group on $\mathfrak{g}$ and the automorphism group of $\mathfrak{g}$ respectively.

**Main results.** Specifically, we study multiplicative Hom-Lie algebra structures on three important families of finite-dimensional complex Lie algebras: the general linear Lie algebra $gl_n(\mathbb{C})(n \geq 2)$, the Heisenberg Lie algebra $h_{2n+1}(\mathbb{C})(n \geq 1)$ and the Lie algebra $u_n(\mathbb{C})(n \geq 2)$ of upper triangular matrices, which are the most typical examples of reductive, nilpotent and solvable Lie algebras respectively.

Theorem 2.10 is our first theorem that describes the geometric structure of the affine variety $HLie_m(gl_2(\mathbb{C}))$, showing that $HLie_m(gl_2(\mathbb{C}))$ consists of two 1-dimensional and one 3-dimensional irreducible components. The key to our proof is to apply a Gröbner basis method from computational ideal theory in commutative algebra; compared with [XJL15, Corollary 3.4] for the case of $sl_2(\mathbb{C})$. Our second result establishes a complete description of $HLie_m(gl_n(\mathbb{C}))$ for $n \geq 3$. Based on Jin-Li’s theorem on $HLie_m(sln(\mathbb{C}))$ and the close relationship between $sl_n(\mathbb{C})$ and $gl_n(\mathbb{C})$, we prove that if $D \in HLie_m(gl_n(\mathbb{C}))$, then $D$ must be equal to a diagonal matrix $diag\{\delta, \ldots, \delta, a\}$ where $\delta$ is either 1 or 0 and $a \in \mathbb{C}$; see Theorem 3.3 for details. The two results also positively answer the previous question on the existence of nontrivial Hom-Lie algebra structures on $gl_n(\mathbb{C})$.

Hom-Lie algebra structures on a nilpotent or solvable Lie algebra are more complicated than that on a semi-simple or reductive Lie algebra. Proposition 4.2, as our third result, gives a new family of multiplicative Hom-Lie algebras on $h_{2n+1}(\mathbb{C})$. As applications we prove that all the three
containments appeared in (1.1) are strict for the case \( g = h_{2n+1}(\mathbb{C}) \); see Corollaries 4.3–4.5. Our final major result is about the derivation algebra \( \text{Der}_D(g) \) of a Hom-Lie algebra \( D \) on a Lie algebra \( g \) and the corresponding Hilbert series \( \mathcal{H}(\text{Der}_D(g), t) \). The derivation algebra of a Hom-Lie algebra was introduced and studied by [She12], aiming at developing the representation and cohomology theory of Hom-Lie algebras. Inspired by the classical topic on the rationality of a Hilbert series \( H \), we prove in Theorem 5.5 that under some additional hypotheses, \( \mathcal{H}(\text{Der}_D(g), t) \) is a rational function.

**Organization.** Section 2 is devoted to describing three irreducible components of the affine variety \( \text{HLie}_m(gl_2(\mathbb{C})) \), particularly demonstrating that there exists a nontrivial Hom-Lie algebra structure on \( gl_2(\mathbb{C}) \). Section 3 completely characterizes multiplicative Hom-Lie algebra structures on \( gl_n(\mathbb{C}) \) for \( n \geq 3 \). In Section 4, we study Hom-Lie algebra structures on the Heisenberg Lie algebra \( h_{2n+1}(\mathbb{C}) \) and the upper triangular Lie algebra \( u_n(\mathbb{C}) \). Section 5 is mainly concentrated on the rationality of the Hilbert series of the derivation algebra of Hom-Lie algebras.

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### 2. The Affine Variety \( \text{HLie}_m(gl_2(\mathbb{C})) \)

After summarizing the fundamental facts on \( gl_2(\mathbb{C}) \) and some preparations for the vanishing ideal of the affine variety \( \text{HLie}_m(gl_2(\mathbb{C})) \), we capitalize on a Gröbner basis method from computational ideal theory to analyze the geometric structure of \( \text{HLie}_m(gl_2(\mathbb{C})) \).

**Basics for \( gl_2(\mathbb{C}) \).** We start with some basic facts on \( gl_2(\mathbb{C}) \). For \( 1 \leq i, j \leq 2 \), let \( E_{ij} \) be the \( 2 \times 2 \)-matrix in which the \((i, j)\)-entry is 1 and 0 otherwise. For notational brevity, we set \( e_1 := E_{11}, e_2 := E_{12}, e_3 := E_{21} \) and \( e_4 := E_{22} \). Then \( \{e_1, \ldots, e_4\} \) is the standard basis of \( gl_2(\mathbb{C}) \), subject to the following nontrivial relations:

\[
[e_1, e_2] = e_2, [e_1, e_3] = -e_3, [e_2, e_3] = e_1 - e_4, [e_2, e_4] = e_2, [e_3, e_4] = -e_3.
\]

Let \( D \in \text{HLie}_m(gl_2(\mathbb{C})) \) be an arbitrary element. With respect to this standard basis of \( gl_2(\mathbb{C}) \), we always identify \( D \) with a matrix \((a_{ij})_{4 \times 4}^T\) in \( M_4(\mathbb{C}) \). This means that

\[
D(e_i) = \sum_{j=1}^{4} a_{ij} e_j
\]

for \( i = 1, \ldots, 4 \).

**The vanishing ideal of \( \text{HLie}_m(gl_2(\mathbb{C})) \).** Let \( A := \mathbb{C}[x_{ij} \mid 1 \leq i, j \leq 4] \) be the polynomial ring in 16 variables. Then \( A \) can be viewed as the coordinate ring of the affine space \( M_4(\mathbb{C}) \) of all \( 4 \times 4 \)-matrices over \( \mathbb{C} \) in the natural way. To articulate the vanishing ideal of \( \text{HLie}_m(gl_2(\mathbb{C})) \), we need to define the following 23 polynomials in \( A \):

\[
\begin{align*}
 f_1 & := x_{11} - x_{44}, \\
f_2 & := x_{23} x_{32} + x_{41} x_{44} - x_{42} x_{43} - (x_{41}^2 + x_{41} + x_{44}^2 - x_{44})/2, \\
f_3 & := x_{12} + x_{42}, \\
f_4 & := x_{23} x_{41} - x_{23} x_{44} - x_{23} + 2 x_{43}^2, \\
f_5 & := x_{13} + x_{43}, \\
f_6 & := x_{23} x_{42} - (x_{41} x_{43} - x_{43} x_{44} + x_{43})/2.
\end{align*}
\]
and \( f_{23} := x_{41}^3 - 3x_{41}^2x_{44} + 4x_{41}x_{42}x_{43} + 3x_{41}x_{42}^2 - x_{41} - 4x_{42}x_{43}x_{44} - x_{44}^3 + x_{44} \). Throughout this section we let \( I \) be the ideal generated by \( \{ f_i | 1 \leq i \leq 23 \} \) in \( A \), and we will show that \( I \) is exactly the vanishing ideal of HLie_\text{m}(gl_2(\mathbb{C})). \) One might be interested in how to obtain these polynomials \( f_i \). Indeed, the way of constructing these \( f_i \) is quite direct by using the definition of multiplicative Hom-Lie algebra, i.e., choosing a generic matrix \( D \) in HLie_\text{m}(gl_2(\mathbb{C})), the properties of algebraic homomorphism and Hom-Jacobi identity lead to a bunch of polynomial equations in the entries of \( D \). By deleting redundant equations, one will reveal the above 23 polynomials \( f_i \) such that \( f_i(D) = 0 \) for all \( i \); see the proof of Lemma 2.1 below. For any ideal \( J \) of \( A \), we also define \( \mathcal{V}(J) := \{ T \in M_4(\mathbb{C}) | f(T) = 0, \text{ for all } f \in J \} \). Note that \( \mathcal{V}(J) = \mathcal{V}((J) \).

**Lemma 2.1.** HLie_\text{m}(gl_2(\mathbb{C})) \subseteq \mathcal{V}(I).

**Proof.** Suppose \( D = (a_{ij})_{4 \times 4} \in \text{HLie}_\text{m}(gl_2(\mathbb{C})) \) is an arbitrary element. It suffices to show that \( D \in \mathcal{V}(I) \); equivalently the valuation of every \( f_i \) at \( D \) is zero for \( 1 \leq i \leq 23 \). Indeed, since \( D \) is an algebraic homomorphism and it satisfies the Hom-Jacobi identity, we see that \( D([e_i,e_j]) = [D(e_i),D(e_j)] \) and \([D(e_i),[e_j,e_k]] + [D(e_j),[e_k,e_i]] + [D(e_k),[e_i,e_j]] = 0 \), for all \( 1 \leq i,j,k \leq 4 \). Putting these equations together, we can use the relations among \( e_1, \ldots, e_4 \) in (2.1) and the rule in (2.2) to solve them. Deleting redundant equations, we finally obtain 23 polynomial equations in \( a_{ij} \). These \( f_i \) are the corresponding polynomials of the equations. Hence, \( f_i(D) = 0 \) for all \( i \). \( \Box \)

**Remark 2.2.** In fact, we will show that the two affine varieties \( \text{HLie}_\text{m}(gl_2(\mathbb{C})) \) and \( \mathcal{V}(I) \) are equal. By Lemma 2.1, it suffices to show that \( \mathcal{V}(I) \) is contained in \( \text{HLie}_\text{m}(gl_2(\mathbb{C})) \). To achieve this, we need to investigate the geometric structure of \( \mathcal{V}(I) \). More precisely, our first step is to find all irreducible components \( \mathcal{V}(p_1), \ldots, \mathcal{V}(p_k) \) of \( \mathcal{V}(I) \) for some \( k \in \mathbb{N}^+ \), and our last step is to show that every element in each component \( \mathcal{V}(p_i) \) is a multiplicative Hom-Lie algebra structure on \( gl_2(\mathbb{C}) \), where all \( p_i \) are prime ideals of \( A \). \( \diamond \)

**Irreducible components of \( \mathcal{V}(I) \).** We will see that the affine variety \( \mathcal{V}(I) \) has three irreducible components: \( \mathcal{V}(p_1), \mathcal{V}(p_2), \mathcal{V}(p_3) \). To better understand these components, we need to define the following six auxiliary polynomials in \( A \):

\[
\alpha := x_{14} - x_{44}, \quad \beta := x_{41} - x_{44}, \quad h := \beta^2 + \beta.
\]

\[
g_1 := x_{22} - (\beta - 1)/2, \quad g_2 := x_{23}x_{32} + x_{42}x_{43} - (\beta + 1)/2, \quad g_3 := \beta^2 + 4x_{42}x_{43} - 1.
\]

We also define four ideals of \( A \) as follows:

\[
p_1 := (f_1, \alpha, \beta, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{24}, x_{31}, x_{32}, x_{33}, x_{34}, x_{42}, x_{43}),
\]
Lemma 2.3. The ideals $p_1, p_2$ and $p_3$ are prime.

Proof. Clearly, $p_1$ and $p_2$ are prime ideals, since the generators of $p_1$ and $p_2$ are polynomials of degree 1. To show that $p_3$ is prime, it suffices to show that $A/p_3$ is an integral domain. In fact, $A/p_3 \cong \mathbb{C}[x_{23}, x_{32}, x_{41}, \ldots, x_{44}]/(g_2, g_3, f_2i \mid 2 \leq i \leq 5)$. The latter is isomorphic to

$$(\mathbb{C}[\beta, x_{23}, x_{32}, x_{42}, x_{43}]/(g_2, g_3, f_2i \mid 2 \leq i \leq 5))[x_{44}].$$

Thus it suffices to show that $B := \mathbb{C}[\beta, x_{23}, x_{32}, x_{42}, x_{43}]/(g_2, g_3, f_2i \mid 2 \leq i \leq 5)$ is an integral domain.

We claim that the image of $\beta - 1$ in $B$ is not a zero-divisor. Let $J_1 := (g_2, g_3, f_2i \mid 2 \leq i \leq 5)$ and $J_2 := (\beta - 1)$ be ideals of $\mathbb{C}[\beta, x_{23}, x_{32}, x_{42}, x_{43}]$. The feasibility of this claim is equivalent to deciding whether the colon ideal $(J_1 : J_2)$ is equal to $J_1$. By [DK15, Section 1.2.4] we see that $(J_1 : J_2) = (\beta - 1)^{-1}(J_1 \cap J_2)$. We use the lexicographic ordering with $\beta > x_{23} > x_{32} > x_{42} > x_{43}$ in $\mathbb{C}[\beta, x_{23}, x_{32}, x_{42}, x_{43}]$. Applying [DK15, Algorithm 1.1.9], a direct calculation shows that the following eight polynomials

$$\beta - 2x_{23}x_{32} - 2x_{42}x_{43} + 1, \quad x^2_{23}x_{32} + x_{23}x_{42}x_{43} - x_{23} + x^2_{43},$$
$$x^2_{32}x_{42} - x_{23}x_{32} + x^3_{43}, \quad x_{23}x_{32}^2 + x_{32}x_{42}x_{43} - x_{32} + x^2_{42},$$
$$x_{23}x_{32}x_{42} - x_{32}x_{43} + x^2_{42}, \quad x_{23}x_{32}x_{43} - x_{23}x_{42} + x^2_{43},$$
$$x_{23}x_{42}^2 - x_{32}x_{42} + x^2_{43}, \quad x^3_{23}x_{43} - x_{32}x_{42} + x^2_{42}$$

form a Gröbner basis for the ideal $J_1$. Denote this Gröbner basis by $G_1$. To derive a Gröbner basis for $J_1 \cap J_2$, we consider the polynomial ring $\mathbb{C}[t, \beta, x_{23}, x_{32}, x_{42}, x_{43}]$ and use the lexicographic ordering with $t > \beta > x_{23} > x_{32} > x_{42} > x_{43}$. Let $J_{12}$ be the ideal of $\mathbb{C}[t, \beta, x_{23}, x_{32}, x_{42}, x_{43}]$ generated by $(1-t) \cdot J_1 + t \cdot J_2$, where the products are formed by multiplying each generator of $J_1$ and $J_2$ by $1-t$ and $t$ respectively. It follows from [Vas98, Corollary 2.1.1] that $J_1 \cap J_2 = J_{12} \cap \mathbb{C}[\beta, x_{23}, x_{32}, x_{42}, x_{43}]$. Applying [DK15, Algorithm 1.1.9] again we observe that these polynomials:

$$t\beta - t, \quad tx_{23}x_{32} - t + (\beta x_{23}x_{32} + \beta x_{42}x_{43} + \beta - 3x_{23}x_{32} - x_{42}x_{43} + 1)/2, $$
$$tx_{23}x_{42} - tx_{32} + \beta x_{43}/2 - x_{23}x_{42} + x_{43}/2, \quad tx_{32}x_{43} - tx_{42} + \beta x_{42}/2 - x_{32}x_{43} + x_{42}/2, $$
$$tx^2_{42} - \beta x_{32}/2 + x_{32} + x^2_{42}, \quad tx^2_{43} - \beta x_{23}/2 + x_{23} + x^2_{43}, $$
$$tx_{42}x_{43} - (\beta x_{23}x_{32} + \beta x_{42}x_{43} - x_{23}x_{32} + x_{42}x_{43})/2$$

together with $(\beta - 1) \cdot G_1$, form a Gröbner basis for $J_{12}$. Thus [DK15, Algorithm 1.2.1] implies that $(\beta - 1) \cdot G_1$ is a Gröbner basis for $J_1 \cap J_2$. This means that $(J_1 : J_2)$ can be generated by $G_1$ and so $(J_1 : J_2) = J_1$, the claim follows.

Hence, $B$ can be embedded into $B^* := B[\beta^{-1}]$ and we need only to show that $B^*$ is an integral domain. Since the images of $x_{23}$ and $x_{32}$ in $B$ can be expressed by the images of $x_{42}$, $x_{43}$ and $(\beta - 1)^{-1}$, it follows that $B^* \cong (\mathbb{C}[\beta, x_{42}, x_{43}]/(\beta^2 + 4x_{42}x_{43} - 1)[\beta^{-1}])$. As $\beta^2 + 4x_{42}x_{43} - 1$ is irreducible, $B^*$ is isomorphic to a localization of an integral domain. Thus $B^*$ is also an integral domain. The proof is completed. \[\square\]
Lemma 2.3 has the following immediate consequences.

**Corollary 2.4.** Let \( C_a := \begin{pmatrix} a & 0 & 0 & a \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a & 0 & 0 & a \end{pmatrix} \) and \( D_a := \begin{pmatrix} a & 0 & 0 & a-1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ a-1 & 0 & 0 & a \end{pmatrix} \) with \( a \in \mathbb{C} \). Then \( \mathcal{V}(p_1) = \{ C_a \mid a \in \mathbb{C} \} \) and \( \mathcal{V}(p_2) = \{ D_a \mid a \in \mathbb{C} \} \) are 1-dimensional irreducible affine varieties.

**Corollary 2.5.** The affine variety \( \mathcal{V}(p_3) \) is 3-dimensional and irreducible, consisting of all matrices

\[
E_{a,b,c,\xi} := \begin{pmatrix} a - \xi & -c & -b & a \\ -b & \frac{\xi-1}{2} & \frac{-2c^2}{\xi-1} & b \\ -c & \frac{2b^2}{\xi-1} & \frac{-1}{2} & c \\ a & c & b & a - \xi \end{pmatrix},
\]

with \( a, b, c, 1 \neq \xi \in \mathbb{C} \) and \( \xi^2 + 4bc - 1 = 0 \).

**Proof.** We have seen from Lemma 2.3 that \( p_3 \) is a prime ideal, thus the variety \( \mathcal{V}(p_3) \) is irreducible. For a generic element \( E \in \mathcal{V}(p_3) \), the fact that generators of \( p_3 \) evaluated on \( E \) are zero, shows that \( E \) must have the form \( E_{a,b,c,\xi} \) for some \( a, b, c, 1 \neq \xi \in \mathbb{C} \) with the condition \( \xi^2 + 4bc - 1 = 0 \). Hence, \( \mathcal{V}(p_3) \) is consisting of all such matrices \( E_{a,b,c,\xi} \). This fact also shows that the coordinate ring of \( \mathcal{V}(p_3) \) is isomorphic to \( \mathbb{C}[x,y,z,w]/(z^2 + 4xy - 1) \), which has Krull dimension 3.

**Lemma 2.6.** \( \mathcal{V}(p_1) \cup \mathcal{V}(p_2) \cup \mathcal{V}(p_3) \subseteq \mathcal{V}(I) \).

**Proof.** It suffices to show that \( I \subseteq p_1, I \subseteq p_2 \) and \( I \subseteq p_3 \). To show the first containment, it is easy to see that all generators of \( I \) except for \( f_2, f_7, f_{12}, f_{23} \) are zero modulo \( p_1 \). Thus it is sufficient to show that \( f_2, f_7, f_{12}, f_{23} \) are equal to zero modulo \( p_1 \). In fact, working over modulo \( p_1 \), we see that

\[
f_2 \equiv f_{12} \equiv -\frac{1}{4}(\beta^2 + \beta) \equiv 0; \quad f_7 = \alpha - \beta \equiv 0; \quad f_{23} \equiv \beta^3 - \beta \equiv 0.
\]

This proves that \( I \subseteq p_1 \). Similarly, one can show that \( I \subseteq p_2 \) and \( I \subseteq p_3 \).

**Lemma 2.7.** \( p_1 \cdot p_2 \subseteq p \).

**Proof.** By [CLO07, Proposition 6, page 185] we see that the set of products of any two elements that come from the generating sets of \( p_1 \) and \( p_2 \) respectively can generate the product \( p_1 \cdot p_2 \). Hence, we only need to show that every element in \( \mathcal{B}_1 \cdot \mathcal{B}_2 := \{ b_1b_2 \mid b_i \in \mathcal{B}_i, 1 \leq i \leq 2 \} \) belongs to \( p \), where

\[
\mathcal{B}_1 := \{ \alpha, \beta, x_{22}, x_{33} \} \quad \text{and} \quad \mathcal{B}_2 := \{ \alpha + 1, \beta + 1, x_{22} - 1, x_{33} - 1 \}.
\]

As the three elements \( f_7, f_{22} + \beta, x_{33} + \beta \) are contained in \( p \), we see that \( \alpha \equiv \beta \) and \( x_{22} \equiv x_{33} \equiv -\beta \) modulo \( p \). This indicates that it is sufficient to show that \( \beta \cdot \mathcal{B}_2 \) is contained in \( p \). Note that \( h \in p \), so \( \beta(\alpha + 1) + \beta(\beta + 1) = \beta^2 - 1 = 0 \) and \( \beta(x_{22} - 1) \equiv \beta(x_{33} - 1) \equiv -h \equiv 0 \) modulo \( p \).

**Lemma 2.8.** \( \mathcal{V}(I) \subseteq \mathcal{V}(p_1) \cup \mathcal{V}(p_2) \cup \mathcal{V}(p_3) \).

**Proof.** Lemma 2.7, together with the fact that \( \mathcal{V}(p_1) \cup \mathcal{V}(p_2) \cup \mathcal{V}(p_3) = \mathcal{V}(p_1 \cdot p_2 \cdot p_3) \), implies that it suffices to show that the product \( p \cdot p_3 \) is contained in \( I \). By [CLO07, Proposition 6, page 185], we only need to show that every element in \( \mathcal{B} : \mathcal{B}_3 := \{ bb_3 \mid b \in \mathcal{B} \text{ and } b_3 \in \mathcal{B}_3 \} \) belongs to \( I \), where

\[
\mathcal{B} := \{ h, x_{12}, x_{13}, x_{21}, x_{22} + \beta, x_{23}, x_{24}, x_{31}, x_{32}, x_{33} + \beta, x_{34}, x_{42}, x_{43} \} \quad \text{and} \quad \mathcal{B}_3 := \{ g_1, g_2, g_3 \}.
\]

Moreover, as \( f_3, f_5, f_9, f_{11}, f_{15}, f_{17}, f_{21} \in I \), the set \( \mathcal{B} \) can be replaced by \( \mathcal{B}' := \{ h, x_{23}, x_{32}, x_{33} + \).
\(\beta, x_{42}, x_{43}\}\). Now we have to verify that each element of \(B_3 \cdot B'\) belongs to \(I\). Throughout the rest of the proof, we are working over modulo \(I\). By the definition of the generators \(f_i\) of \(I\), we observe that

\[
\begin{align*}
x_{23}x_{32} - x_{42}x_{43} - (\beta^2 + \beta)/2 &= 0, & x_{23}(\beta - 1) + 2x_{43} &= 0, \\
x_{23}x_{42} - x_{43}(\beta + 1)/2 &= 0, & x_{32}(\beta - 1) + 2x_{42} &= 0, \\
x_{32}x_{43} - x_{42}(\beta + 1)/2 &= 0, & x_{23}^2 - x_{42}x_{43} - (\beta^2 - \beta)/2 &= 0, \\
x_{33}(\beta + 1) + 2x_{42}x_{43} &= 0, & x_{33}x_{42} - x_{42}(\beta - 1)/2 &= 0, \\
x_{33}x_{43} - x_{43}(\beta - 1)/2 &= 0, & x_{42}g_3 &= 0, \\
x_{43}g_3 &= 0, & \beta g_3 &= 0.
\end{align*}
\]

We will use these equations to complete the proof. Note that \(g_1h = (x_{33} - (\beta - 1)/2)(\beta^2 + \beta) = (x_{33}(\beta + 1) - (\beta^2 - 1)/2)\beta = (-2x_{42}x_{43} - (\beta^2 - 1)/2)\beta = -(\beta g_3)/2 = 0; g_1x_{23} = (x_{33} - (\beta - 1)/2)x_{23} = x_{23}x_{33} + x_{43}^2 = f_{13} = 0; g_1x_{32} = f_{19} = 0; g_1(x_{33} + \beta) = x_{33}^2 + (\beta + 1)x_{33}/2 - (\beta^2 - \beta)/2 = (x_{33}(\beta + 1) + 2x_{42}x_{43})/2 = 0; g_1x_{42} = x_{33}x_{42} - x_{42}(\beta - 1)/2 = 0; and g_1x_{43} = x_{33}x_{43} - x_{43}(\beta - 1)/2 = 0.\) This shows that \(g_1B' \subseteq I\). Further, \(g_2 = 2x_{42}x_{43} + (\beta^2 - 1)/2 = -(\beta + 1)g_1\) and \(g_3 = (\beta^2 - 1) - 2x_{33}(\beta + 1) = 2(\beta + 1)g_1\). This implies that \(g_2 \cdot B'\) and \(g_3 \cdot B'\) are contained in \(I\). Hence, \(B_3 \cdot B' \subseteq I\) and we are done. \(\Box\)

Lemmas 2.6 and 2.8 combine to obtain

**Corollary 2.9.** \(\forall (I) = \forall (p_1) \cup \forall (p_2) \cup \forall (p_3)\).

**Theorem 2.10.** \(\mathrm{HLie}_m(\mathfrak{gl}_2(\mathbb{C})) = \forall (I) = \forall (p_1) \cup \forall (p_2) \cup \forall (p_3)\).

**Proof.** We have seen in Lemma 2.1 that \(\mathrm{HLie}_m(\mathfrak{gl}_2(\mathbb{C})) \subseteq \forall (I)\). To complete the proof, by Corollary 2.9, it suffices to show that any \(C_a, D_a\) and \(E_{a,b,c,\xi}\) appeared in Corollaries 2.4 and 2.5 belong to \(\mathrm{HLie}_m(\mathfrak{gl}_2(\mathbb{C}))\). One can verify the assertion by a direct calculation. \(\Box\)

**Remark 2.11.** Note that each irreducible component we obtained in Theorem 2.10 is the closure of infinite families of algebras, and thus there are no rigid algebras in the variety \(\mathrm{HLie}_m(\mathfrak{gl}_2(\mathbb{C}))\); see for example [GK96, Chapter 5] for more details about rigid algebras. \(\Diamond\)

### 3. The Affine Varieties \(\mathrm{HLie}_m(\mathfrak{gl}_n(\mathbb{C})) (n \geq 3)\)

This section characterizes multiplicative Hom-Lie algebra structures on \(\mathfrak{gl}_n(\mathbb{C})\) for all \(n \geq 3\). Suppose that \(\{e_1, e_2, \ldots, e_{n^2-1}\}\) is a basis of \(\mathfrak{sl}_n(\mathbb{C})\). By the classical fact that a matrix that commutes with all matrices must be a scalar matrix, we see that the center of \(\mathfrak{gl}_n(\mathbb{C})\) consists of all scalar matrices. We may take a nonzero scalar matrix \(z\) in \(\mathfrak{gl}_n(\mathbb{C})\), and then \(\mathfrak{gl}_n(\mathbb{C}) = \mathfrak{sl}_n(\mathbb{C}) \oplus \mathbb{C} \cdot z\). Moreover, we have the following useful fact.

**Lemma 3.1.** \([\mathfrak{gl}_n(\mathbb{C}), \mathfrak{gl}_n(\mathbb{C})] = \mathfrak{sl}_n(\mathbb{C}).\)

**Proof.** Note that \([\mathfrak{gl}_n(\mathbb{C}), \mathfrak{gl}_n(\mathbb{C})] = [\mathfrak{sl}_n(\mathbb{C}) \oplus \mathbb{C} \cdot z, \mathfrak{sl}_n(\mathbb{C}) \oplus \mathbb{C} \cdot z] = [\mathfrak{sl}_n(\mathbb{C}), \mathfrak{sl}_n(\mathbb{C})]\), which is a nonzero ideal of \(\mathfrak{sl}_n(\mathbb{C})\). However, \(\mathfrak{sl}_n(\mathbb{C})\) is a simple Lie algebra, thus \([\mathfrak{sl}_n(\mathbb{C}), \mathfrak{sl}_n(\mathbb{C})] = \mathfrak{sl}_n(\mathbb{C})\). \(\Box\)

**Corollary 3.2.** Every homomorphism on \(\mathfrak{gl}_n(\mathbb{C})\) restricts to a homomorphism on \(\mathfrak{sl}_n(\mathbb{C})\).
Proof. Let $D$ be a homomorphism from $\mathfrak{gl}_n(\mathbb{C})$ to itself. It suffices to show that $\mathfrak{sl}_n(\mathbb{C})$ is stable under the action of $D$. Indeed, given an element $x \in \mathfrak{sl}_n(\mathbb{C})$, by Lemma 3.1, we may write $x = \sum_{\text{finite}} [x_i, x_j]$ for some $x_i, x_j \in \mathfrak{gl}_n(\mathbb{C})$. Thus $D(x) = \sum_{\text{finite}} D([x_i, x_j]) = \sum_{\text{finite}} [D(x_i), D(x_j)] \in [\mathfrak{gl}_n(\mathbb{C}), \mathfrak{gl}_n(\mathbb{C})] = \mathfrak{sl}_n(\mathbb{C})$. \[ \square \]

We are ready to prove our second result.

**Theorem 3.3.** Let $n \geq 3$ and $D \in \text{HLie}_m(\mathfrak{gl}_n(\mathbb{C}))$ be an arbitrary element. Then with respect to the basis $\{e_1, e_2, \ldots, e_{n^2-1}, z\}$, $D$ is equal to either $\text{diag}\{1, \ldots, 1, a\}$ or $\text{diag}\{0, \ldots, 0, a\}$, where $a \in \mathbb{C}$.

Proof. By [XJL15, Corollary 3.4 (ii)] we see that $\text{HLie}_m(\mathfrak{sl}_n(\mathbb{C}))$ consists of the identity matrix $I_{n^2-1}$ and the zero matrix. On the other hand, Corollary 3.2 implies that $D$ restricts to an element of $\text{HLie}_m(\mathfrak{sl}_n(\mathbb{C}))$. Thus the restriction of $D$ on $\mathfrak{sl}_n(\mathbb{C})$ is either to $I_{n^2-1}$ or $0$.

For the first case, we have $D(x) = x$ for $x \in \mathfrak{sl}_n(\mathbb{C})$. Note that for $i = 1, 2, \ldots, n^2-1$, we have $0 = D(0) = D([e_i, z]) = [D(e_i), D(z)] = [e_i, D(z)]$ and moreover, $[z, D(z)] = 0$. This implies that $D(z)$ is also an element of the center of $\mathfrak{gl}_n(\mathbb{C})$. Thus $D(z) = az$ for some $a \in \mathbb{C}$. Hence, with respect to the basis $\{e_1, e_2, \ldots, e_{n^2-1}, z\}$, we see that $D = \text{diag}\{1, \ldots, 1, a\}$.

For the second case, we have $D(x) = 0$ for $x \in \mathfrak{sl}_n(\mathbb{C})$. Suppose $D(z) = \sum_{i=1}^{n^2-1} a_i e_i + az$, where all $a_i$ and $a$ belong to $\mathbb{C}$. For $x \in \mathfrak{sl}_n(\mathbb{C})$, as $\mathfrak{sl}_n(\mathbb{C}) = [\mathfrak{sl}_n(\mathbb{C}), \mathfrak{sl}_n(\mathbb{C})]$, we may write $x = \sum_{i,j=1}^{n^2-1} a_{ij} [e_i, e_j]$ for $a_{ij} \in \mathbb{C}$. Then

\[
[D(z), x] = \left[ D(z), \sum_{i,j=1}^{n^2-1} a_{ij} [e_i, e_j] \right] \\
= \sum_{i,j=1}^{n^2-1} a_{ij} [D(z), [e_i, e_j]] \quad \text{(by the Hom-Jacobi identity)} \\
= - \sum_{i,j=1}^{n^2-1} a_{ij} ([D(e_i), [e_j, z]] + [D(e_j), [z, e_i]]) = 0.
\]

This fact, together with $[D(z), z] = 0$, implies that $D(z)$ is in the center of $\mathfrak{gl}_n(\mathbb{C})$. Thus $D(z) = az$ for some $a \in \mathbb{C}$. This shows that in this case, $D = \text{diag}\{0, \ldots, 0, a\}$. \[ \square \]

**Remark 3.4.** Note that a Hom-Lie algebra structure on a vector space is determined by two parts: a linear map $D$ and the bracket product $[-, -]$. The two parts both are required to satisfy some conditions. In the present paper, we fix the part of bracket product on the vector space, thus if we embed $\text{HLie}_m(\mathfrak{gl}_n(\mathbb{C}))$ into the whole variety $\text{HLie}_m(\mathbb{C}^{n^2})$ of $n^2$-dimensional multiplicative complex Hom-Lie algebras, Theorem 3.3 says that $\text{HLie}_m(\mathfrak{gl}_n(\mathbb{C}))$ is the union of two lines in $\text{HLie}_m(\mathbb{C}^{n^2})$ when $n \geq 3$. \[ \diamond \]

4. **The Affine Varieties $\text{HLie}_m(\mathfrak{h}_{2n+1}(\mathbb{C}))$ and $\text{HLie}_m(\mathfrak{u}_n(\mathbb{C}))$**

In this section, we study multiplicative Hom-Lie algebra structures on the Heisenberg Lie algebra and the Lie algebra of upper triangular matrices which are the most typical examples of nilpotent and solvable Lie algebras respectively.
**Heisenberg Lie algebras.** Let \( n \in \mathbb{N}^+ \). Recall that the complex Heisenberg Lie algebra \( \mathfrak{h}_{2n+1}(\mathbb{C}) \) of dimension \( 2n + 1 \) generated by the following \((n + 2) \times (n + 2)\)-matrices

\[
x_i = \begin{pmatrix} 0 & e_i^T & 0 \\ 0 & 0_n & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad y_j = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0_n & e_j \\ 0 & 0 & 0 \end{pmatrix}, \quad z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0_n & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]

where \( 1 \leq i, j \leq n \), \( \{e_1, \ldots, e_n\} \) is the standard basis of \( \mathbb{C}^n \), \( e_i^T \) denotes the transpose of \( e_i \) and \( 0_n \) denotes the zero matrix of size \( n \). In \( \mathfrak{h}_{2n+1}(\mathbb{C}) \), there are the following relations:

\[
[x_i, y_j] = \delta_{ij} \cdot z, [x_i, z] = 0 = [y_j, z].
\]

We have already known a complete characterization of multiplicative Hom-Lie algebras on the 3-dimensional complex Heisenberg Lie algebra \( \mathfrak{h}_3(\mathbb{C}) \); see for example [AC19, Corollary 2.3].

**Proposition 4.1.** The affine variety \( \text{HLie}_m(\mathfrak{h}_3(\mathbb{C})) \) is a 6-dimensional irreducible affine variety, consisting of the following matrices:

\[
\begin{pmatrix}
bf - ce & a & d \\
0 & b & e \\
0 & c & f
\end{pmatrix}
\]

where \( a, b, c, d, e, f \in \mathbb{C} \). In particular, there exists a nontrivial involutive Hom-Lie algebra structure on \( \mathfrak{h}_3(\mathbb{C}) \).

**Proof.** See [AC19, Corollary 2.3] for a proof of the first assertion. More description on 3-dimensional Hom-Lie algebras can be found in [GDSSV20] and [Rem18]. For the second assertion, we take \( a = c = d = e = 0, b = f = -1 \) and

\[
D = \begin{pmatrix} 1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1 \end{pmatrix}.
\]

Then \( D \in \text{HLie}_m(\mathfrak{h}_3(\mathbb{C})) \) gives rise to a nontrivial involutive Hom-Lie structure on \( \mathfrak{h}_3(\mathbb{C}) \). \( \square \)

Giving a complete description of multiplicative Hom-Lie algebra structures on \( \mathfrak{h}_{2n+1}(\mathbb{C}) \) is a difficult and challenging task. Here we construct a family of multiplicative Hom-Lie algebra structures on \( \mathfrak{h}_{2n+1}(\mathbb{C}) \), generalizing the construction in Proposition 4.1; and we will see that this new construction leads to some remarkable consequences.

**Proposition 4.2.** Let \( D(a, b, c, d; \alpha) := \begin{pmatrix} ad - bc & \alpha \\
0 & \Theta \end{pmatrix} \) be the block matrix of size \( 2n + 1 \), where \( \alpha = (a_1, b_1, \ldots, a_n, b_n) \), \( \Theta = \text{diag} \{\theta, \ldots, \theta\} \) and \( \theta = \begin{pmatrix} a & b \\
c & d \end{pmatrix} \). Then

\[
D(a, b, c, d; \alpha) \in \text{HLie}_m(\mathfrak{h}_{2n+1}(\mathbb{C})).
\]

**Proof.** For simplicity, we denote \( D(a, b, c, d; \alpha) \) by \( D \) throughout the proof. Then \( D \) acts on the basis \( \{z, x_1, y_1, x_2, y_2, \ldots, x_n, y_n\} \) of \( \mathfrak{h}_{2n+1}(\mathbb{C}) \) as follows:

\[
D(z) = \det(\theta) \cdot z; \quad D(x_i) = a_i z + ax_i + cy_i; \quad D(y_i) = b_i z + bx_i + dy_i
\]

for \( 1 \leq i \leq n \). Note that \( z \) is a central element of \( \mathfrak{h}_{2n+1}(\mathbb{C}) \), thus for \( 1 \leq i, j \leq n \), we have

\[
[D(x_i), D(y_j)] = [a_i z + ax_i + cy_i, b_j z + bx_j + dy_j]
\]
Theorem 4.7. The affine variety $\text{HLie}_m(h_{2n+1}(\mathbb{C}))$ can be decomposed into three 4-dimensional irreducible components $\mathcal{V}(p_1)$, $\mathcal{V}(p_2)$ and $\mathcal{V}(p_3)$, where

$$\mathcal{V}(p_1) = \left\{\begin{pmatrix} a & 0 & c \\ b & 0 & d \\ a & 0 & c \end{pmatrix} \mid a, b, c, d \in \mathbb{C}\right\},$$
Next we need to decompose the ideal \( I \) into three irreducible components \( p_1, p_2, \) and \( p_3 \), where

\[
p_1 = \langle x_{11} - x_{31}, x_{12}, x_{13} - x_{33}, x_{22}, x_{32} \rangle
\]

\[
p_2 = \langle x_{11} - x_{31} - 1, x_{12}, x_{13} - x_{33} + 1, x_{21} + x_{23}, x_{32} \rangle
\]

\[
p_3 = \langle x_{11} + x_{13} - x_{31} - x_{33}, x_{12}, x_{21} + x_{23}, x_{22}, x_{32} \rangle.
\]

Now it is not difficult to see that \( p_1, p_2, \) and \( p_3 \) are prime ideals and elements in each \( \mathcal{V}(p_i) \) are of the matrix form in the statement. One may show that \( \mathcal{V}(I) = \mathcal{V}(p_1) \cup \mathcal{V}(p_2) \cup \mathcal{V}(p_3) \), which together with a direct verification that all elements in \( \mathcal{V}(p_1) \cup \mathcal{V}(p_2) \cup \mathcal{V}(p_3) \) give rise to a multiplicative Hom-Lie algebra structure on \( u_2(\mathbb{C}) \), forces that \( \mathcal{V}(I) \subseteq \text{HLie}_m(\text{u}_2(\mathbb{C})) \). Therefore, \( \text{HLie}_m(\text{u}_2(\mathbb{C})) = \mathcal{V}(I) \). 

A similar strategy can be applied to describe the variety \( \text{HLie}_m(\text{u}_3(\mathbb{C})) \). Note that \( \dim_{\mathbb{C}}(\text{u}_3(\mathbb{C})) \) is equal to 6. Thus each element in \( \text{HLie}_m(\text{u}_3(\mathbb{C})) \) can be expressed by a \( 6 \times 6 \)-matrix over \( \mathbb{C} \), and we need to work in the 36-dimensional affine space. Suppose that the corresponding polynomial ring we need is \( \mathbb{C}[x_{ij} \mid 1 \leq i, j \leq 6] \) and denote by \( I \) the vanishing ideal of \( \text{HLie}_m(\text{u}_3(\mathbb{C})) \). We will give a Gröbner basis for \( I \) with respect to the lexicographic ordering: \( x_{11} > x_{12} > \cdots > x_{66} \) as previously, and decompose \( I \) into three irreducible components \( p_1, p_2, \) and \( p_3 \). To describe elements in \( \mathcal{V}(p_i) \), we first define

\[
P_{a,b,c,d,e,f,g} := \begin{pmatrix}
a & 0 & 0 & d & 0 & f \\
0 & 0 & 0 & 0 & 0 & 0 \\
b & 0 & 0 & e & 0 & g \\
a & 0 & 0 & d & 0 & f \\
c & 0 & 0 & -c & 0 & 0 \\
a & 0 & 0 & d & 0 & f
\end{pmatrix}, \quad Q_{a,b,c,d,e,f,g} := \begin{pmatrix}
a & 0 & 0 & c & 0 & f \\
0 & 0 & 0 & d & 0 & -d \\
b & 0 & 0 & e & 0 & g \\
a & 0 & 0 & c & 0 & f \\
0 & 0 & 0 & 0 & 0 & 0 \\
a & 0 & 0 & c & 0 & f
\end{pmatrix}
\]
and
\[
T_{a,b,c,d,e} := \begin{pmatrix}
ad & 0 & 0 & c & 0 \\
0 & e & 0 & 0 & 0 \\
b & 0 & e & 0 & -b \\
a - 1 & 0 & 0 & c + 1 & 0 \\
0 & 0 & 0 & 1 & 0 \\
a - 1 & 0 & 0 & c & d + 1
\end{pmatrix}.
\]

**Theorem 4.8.** The affine variety $\text{HLie}_m(u_3(\mathbb{C}))$ can be decomposed into two 7-dimensional irreducible components $\mathcal{V}(p_1), \mathcal{V}(p_2)$ and one 5-dimensional irreducible component $\mathcal{V}(p_3)$, where

\[
\mathcal{V}(p_1) = \{ P_{a,b,c,d,e,f,g} | a, b, c, d, e, f, g \in \mathbb{C} \},
\]

\[
\mathcal{V}(p_2) = \{ Q_{a,b,c,d,e,f,g} | a, b, c, d, e, f, g \in \mathbb{C} \},
\]

\[
\mathcal{V}(p_3) = \{ T_{a,b,c,d,e} | a, b, c, d, e \in \mathbb{C} \}.
\]

**Sketch of Proof.** The method of proving this statement is basically the same as the proof sketch in Theorem 4.7. One of the essential points is to determine the generating set for the vanishal ideal $I$. Here we define $I$ to be the ideal generated by $\bigcup_{i=1}^{6} \mathcal{B}_i$, where

\[
\mathcal{B}_1 := \{ x_{11} - x_{55} - x_{61}, x_{12}, x_{13}, x_{14} - x_{64}, x_{15}, x_{16} + x_{55} - x_{66} \}
\]

\[
\mathcal{B}_2 := \{ x_{21}, x_{22} - x_{33}, x_{23}, x_{24} + x_{26}, x_{25}, x_{26} x_{33}, x_{26} x_{34}, x_{26} x_{55} \}
\]

\[
\mathcal{B}_3 := \{ x_{31} x_{33} + x_{31} x_{35} + x_{36} x_{55}, x_{32}, x_{33} x_{34}, x_{33} x_{54}, x_{33} x_{55} - x_{33}, x_{34} x_{55}, x_{35} \}
\]

\[
\mathcal{B}_4 := \{ x_{41} - x_{61}, x_{42} x_{43}, x_{44} - x_{55} - x_{64}, x_{45}, x_{46} + x_{55} - x_{66} \}
\]

\[
\mathcal{B}_5 := \{ x_{51} + x_{54}, x_{52} x_{53}, x_{54} x_{55}, x_{54} x_{55} - x_{55}, x_{56} \}
\]

The union of these $\mathcal{B}_i$ is a Gröbner basis for $I$. Another key point is to determine the generators of the prime ideals $p_1, p_2,$ and $p_3$. In this case, we can see from the matrix form of elements of $\mathcal{V}(p_i)$ that they are generated by polynomials of degree 1. More precisely, $p_1$ is generated by $\{ x_{11} - x_{61}, x_{12}, x_{13}, x_{14} - x_{64}, x_{15}, x_{16} - x_{66}, x_{21}, x_{22}, x_{23}, x_{24}, x_{26}, x_{25}, x_{26}, x_{33}, x_{26}, x_{34}, x_{26}, x_{35} \}$ for $1 \leq i \leq 6$, $p_2$ is generated by $\{ x_{11} - x_{61}, x_{12}, x_{13}, x_{14} - x_{64}, x_{15}, x_{16} - x_{66}, x_{21}, x_{22}, x_{23}, x_{24} + x_{26}, x_{25}, x_{26} x_{33}, x_{26} x_{34}, x_{26} x_{35}, x_{26} x_{34}, x_{26} x_{35}, x_{41} - x_{61}, x_{42}, x_{43}, x_{44} - x_{64}, x_{45}, x_{46} - x_{66}, x_{51} + x_{54}, x_{52}, x_{53}, x_{55}, x_{56}, x_{62}, x_{63}, x_{65} \}$, and the ideal $p_3$ is generated by $\{ x_{11} - x_{61}, x_{12}, x_{13}, x_{14} - x_{64}, x_{15}, x_{16} - x_{66}, x_{21}, x_{22}, x_{23}, x_{24} + x_{26}, x_{25}, x_{26} x_{33}, x_{26} x_{34}, x_{26} x_{35}, x_{26} x_{34}, x_{26} x_{35}, x_{41} - x_{61}, x_{42}, x_{43}, x_{44} - x_{64}, x_{45}, x_{46} - x_{66}, x_{51} + x_{54}, x_{52}, x_{53}, x_{55}, x_{56}, x_{62}, x_{63}, x_{65} \}$ for $1 \leq i \leq 6$. Gathering this information together and following the strategy in Remark 2.2 (or Theorem 4.7), one can prove the statement. \hfill \Box

**Remark 4.9.** Considering Theorem 3.3 and the results of Theorems 4.7 and 4.8, an open problem is to determine whether the affine variety $\text{HLie}_m(u_n(\mathbb{C}))$ (for all $n \geq 2$) always has three irreducible components. \hfill \Diamond

## 5. Derivations of Multiplicative Hom-Lie Algebras

Let $\mathfrak{g}$ be a finite-dimensional complex Lie algebra and $D \in \text{HLie}_m(\mathfrak{g})$ be a multiplicative Hom-Lie algebra on $\mathfrak{g}$. Let $k \in \mathbb{N}$ and recall that a linear transformation $\delta : \mathfrak{g} \rightarrow \mathfrak{g}$ is called a $D^k$-derivation of $D$ if $D \circ \delta = \delta \circ D$ and $\delta([x,y]) = [\delta(x), D^k(y)] + [D^k(x), \delta(y)]$ for every $x, y \in \mathfrak{g}$. Here $D^k$ denotes
the composite map of $k$ copies of $D$, with the convention that $D^0 := I_g$ and $D^1 := D$. We denote by 
$\text{Der}_k(g)$ the space of all $D^k$-derivations of $D$ and call 

$$\text{Der}_D(g) := \bigoplus_{k=0}^{\infty} \text{Der}_k(g)$$
the derivation algebra of $(g,D)$, which is a Lie algebra with the bracket product:

$$[\delta, \tau] := \delta \circ \tau - \tau \circ \delta.$$

Note that $[\delta, \tau] \in \text{Der}_{k+s}(g)$ for $\delta \in \text{Der}_k(g)$ and $\tau \in \text{Der}_s(g);$ see [She12, Section 3] for details. We define the Hilbert series of $\text{Der}_D(g)$ to be the formal series:

$$\mathcal{H}(\text{Der}_D(g), t) := \sum_{k=0}^{\infty} \dim_C(\text{Der}_k(g)) \cdot t^k$$
where $t$ denotes a real indeterminant. In order to make the geometric series $\sum_{k=0}^{\infty} t^k$ convergent, we need to assume that $|t| < 1$, except for the second statement of Theorem 5.5 where we assume that $|t^{m-1}| < 1$. Let $\text{Der}(g)$ be the usual derivation algebra of $g$. We start this section with the following two general properties.

**Proposition 5.1.** The space $\text{Der}_0(g)$ is a Lie subalgebra of $\text{Der}(g)$.

**Proof.** As the space $\text{Der}_0(g)$ consists of all derivations of $g$ that commute with $D$, it follows that $\text{Der}_0(g)$ is a subspace of $\text{Der}(g)$. Taking arbitrary $\delta, \tau \in \text{Der}_0(g)$, we have $[\delta, \tau] \circ D = (\delta \circ \tau - \tau \circ \delta) \circ D = D \circ (\delta \circ \tau - \tau \circ \delta) = D \circ [\delta, \tau]$. Thus $\text{Der}_0(g)$ is a Lie algebra. \hfill $\Box$

**Proposition 5.2.** The left multiplication with $D$ gives rise to a linear map

$$\rho^k_D : \text{Der}_k(g) \rightarrow \text{Der}_{k+1}(g), \quad \delta \mapsto D \circ \delta$$
for every $k \in \mathbb{N}$.

**Proof.** It is immediate to see that $\rho^k_D$ is linear. To complete the proof, it suffices to show that $D \circ \delta \in \text{Der}_{k+1}(g)$ for each $\delta \in \text{Der}_k(g)$. Indeed, $D \circ (D \circ \delta) - (D \circ \delta) \circ D = D \circ (D \circ \delta) - D \circ (\delta \circ D) = D \circ (D \circ \delta - \delta \circ D) = 0$; thus $D$ and $D \circ \delta$ are commutes. For every $x, y \in g$, since $\delta([x,y]) = [\delta(x), D(y)] + [D(x), \delta(y)]$ and $D$ is an algebra homomorphism, it follows that

$$D \circ \delta([x,y]) = [D \circ \delta(x), D^{k+1}(y)] + [D^{k+1}(x), D \circ \delta(y)].$$

Hence, $\rho^k_D$ is a linear map. \hfill $\Box$

**Corollary 5.3.** If $D$ is invertible, then $\rho^k_D$ is a linear isomorphism.

**Proof.** Clearly, the left multiplication with $D^{-1}$ gives rise to a linear map $\rho^{-1}_{D^{-1}} : \text{Der}_{k+1}(g) \rightarrow \text{Der}_k(g)$ and $(\rho^k_D)^{-1} = \rho^{-1}_{D^{-1}}$. \hfill $\Box$

**Theorem 5.4.** Let $D$ be a regular Hom-Lie algebra on $g$. Then

$$\mathcal{H}(\text{Der}_D(g), t) = \frac{\dim_C(\text{Der}_0(g))}{1 - t}.$$
In particular, $\dim_C(\text{Der}_D(g))$ is either zero or infinite.
Thus Theorem 5.5. Let \( D \) be a Hom-Lie algebra on \( \mathfrak{g} \). Let \( \dim \mathcal{C}(\text{Der}_0(\mathfrak{g})) = \dim \mathcal{C}(\text{Der}_1(\mathfrak{g})) = \dim \mathcal{C}(\text{Der}_2(\mathfrak{g})) = \cdots \). Suppose \( \dim \mathcal{C}(\text{Der}_0(\mathfrak{g})) = \ell \). Thus
\[
\mathcal{H}(\text{Der}_D(\mathfrak{g}), t) = \frac{\ell \cdot t^0 + \ell \cdot t^1 + \ell \cdot t^2 + \cdots = \ell \cdot (1 + t + t^2 + \cdots)}{1 - t}.
\]
In particular, if \( \text{Der}_0(\mathfrak{g}) = \{0\} \), then \( \dim \mathcal{C}(\text{Der}_D(\mathfrak{g})) = 0 \); if \( \text{Der}_0(\mathfrak{g}) \neq \{0\} \), then \( \dim \mathcal{C}(\text{Der}_D(\mathfrak{g})) \) is infinite.

**Theorem 5.5.** Let \( D \) be a Hom-Lie algebra on \( \mathfrak{g} \).

1. If \( D \) is nilpotent, then there exists a polynomial function \( f(t) \in \mathbb{Z}[t] \) such that
\[
\mathcal{H}(\text{Der}_D(\mathfrak{g}), t) = \frac{f(t)}{1 - t}.
\]
2. If \( D^m = D \) for some integer \( m \geq 2 \), then there also exists a polynomial function \( f(t) \in \mathbb{Z}[t] \) such that
\[
\mathcal{H}(\text{Der}_D(\mathfrak{g}), t) = \frac{f(t)}{1 - t^{m-1}}.
\]

**Proof.** Since \( D \) is invertible, it follows from Corollary 5.3 that \( \dim \mathcal{C}(\text{Der}_0(\mathfrak{g})) = \dim \mathcal{C}(\text{Der}_1(\mathfrak{g})) = \dim \mathcal{C}(\text{Der}_2(\mathfrak{g})) = \cdots \). Suppose \( \dim \mathcal{C}(\text{Der}_0(\mathfrak{g})) = \ell \). Thus
\[
\mathcal{H}(\text{Der}_D(\mathfrak{g}), t) = \frac{\ell \cdot t^0 + \ell \cdot t^1 + \ell \cdot t^2 + \cdots = \ell \cdot (1 + t + t^2 + \cdots)}{1 - t}.
\]

Example 5.6. We consider the multiplicative Hom-Lie algebra \( C_\alpha \) on \( \mathfrak{gl}_2(\mathbb{C}) \) appeared in Corollary 2.4. We observe that
\[
C_\alpha^k = \begin{pmatrix}
2^{k-1} \alpha^k & 0 & 0 & 2^{k-1} \alpha^k \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
2^{k-1} \alpha^k & 0 & 0 & 2^{k-1} \alpha^k
\end{pmatrix}.
\]
Thus $C^{1/2}_{1/2} = C_{1/2}$ for $k > 1$ and so in order to give an explicit formula for the Hilbert series \( \mathcal{H}(\text{Der}_{C_{1/2}}(\mathfrak{gl}_2(\mathbb{C}), t)) \), it is sufficient to examine the dimensions of \( \text{Der}_0(\mathfrak{gl}_2(\mathbb{C})) \) and \( \text{Der}_1(\mathfrak{gl}_2(\mathbb{C})) \) for \( D = C_{1/2} \). A long but direct calculation shows that the usual derivation algebra \( \text{Der}(\mathfrak{gl}_2(\mathbb{C})) \) is a 4-dimensional irreducible affine variety (also a 4-dimensional vector space over \( \mathbb{C} \)), consisting of all derivations with the following forms:

\[
\begin{pmatrix}
    e & c & b & e \\
    -b & -d & 0 & b \\
    -c & 0 & d & c \\
    e & -c & -b & e
\end{pmatrix}
\]

where \( b, c, d, e \in \mathbb{C} \). Now it is easy to check that each element of \( \text{Der}(\mathfrak{gl}_2(\mathbb{C})) \) commutes with \( C_a \). Hence, \( \text{Der}_0(\mathfrak{gl}_2(\mathbb{C})) = \text{Der}(\mathfrak{gl}_2(\mathbb{C})) \). A direct calculation shows that \( \text{Der}_1(\mathfrak{gl}_2(\mathbb{C})) = \mathcal{V}(p_1) \), where \( p_1 \) is defined as in Corollary 2.4. Therefore,

\[
\mathcal{H}(\text{Der}_{C_{1/2}}(\mathfrak{gl}_2(\mathbb{C}), t)) = 4 + t + t^2 + \cdots = \frac{4 - 3t}{1 - t}
\]

is a rational function.

We close this paper with a remark that might give readers a hint to use the method in other possible related directions.

**Remark 5.7.** It has been shown recently that the method of our article is useful to deal with some linear structures on Lie algebras; see [CCZ21, Section 5]. Except for Hom-Lie algebras, we also note that there exists a relatively new notion of nonassociative algebras, \( \omega \)-Lie algebras, that contain Lie algebras as a subclass and have attracted many researchers’ attention; see for example, [CLZ14, CZ17, CZZZ18, Zha21]. Our method appeared in this paper might be applied to studying these \( \omega \)-algebra structures.

**REFERENCES**

[AC19] María Alejandra Alvarez and Francisco Cartes, *Cohomology and deformations for the Heisenberg Hom-Lie algebras*, Linear Multilinear Algebra 67 (2019), no. 11, 2209–2229.

[Bau99] Oliver Baues, *Left-symmetric algebras for \( \mathfrak{gl}(n) \)*, Trans. Amer. Math. Soc. 351 (1999), no. 7, 2979–2996.

[BM14] Said Benayadi and Abdennacer Makhlouf, *Hom-Lie algebras with symmetric invariant nondegenerate bilinear forms*, J. Geom. Phys. 76 (2014), 38–60.

[CCZ21] Hongliang Chang, Yin Chen, and Runxuan Zhang, *A generalization on derivations of Lie algebras*, Electron. Res. Arch. 29 (2021), no. 3, 2457–2473.

[CLZ14] Yin Chen, Chang Liu, and Runxuan Zhang, *Classification of three-dimensional complex \( \omega \)-Lie algebras*, Port. Math. 71 (2014), no. 2, 97–108.

[CZZZ18] Yin Chen, Ziping Zhang, Runxuan Zhang, and Rushu Zhuang, *Derivations, automorphisms, and representations of complex \( \omega \)-Lie algebras*, Commun. Algebra 46 (2018), no. 2, 708–726.

[CLZ17] Yin Chen and Runxuan Zhang, *Simple \( \omega \)-Lie algebras and 4-dimensional \( \omega \)-Lie algebras over \( \mathbb{C} \)*, Bull. Malays. Math. Sci. Soc. 40 (2017), no. 3, 1377–1390.

[CLO07] David Cox, John Little, and Donal O’Shea, *Ideals, varieties, and algorithms*, 3rd ed., Undergraduate Texts in Mathematics, Springer, New York, 2007.

[DK15] Harm Derksen and Gregor Kemper, *Computational invariant theory*, Second enlarged edition, Encyclopaedia of Mathematical Sciences, vol. 130, Springer, Heidelberg, 2015.

[GDSSV20] R. García-Delgado, G. Salgado, and O. A. Sánchez-Valenzuela, *On 3-dimensional complex Hom-Lie algebras*, J. Algebra 555 (2020), 361–385.
[GK96] M. Goze and Y. Khakimdjanov, *Nilpotent Lie algebras*, Mathematics and its Applications, vol. 361, Kluwer Academic Publishers Group, Dordrecht, 1996.

[HLS06] Jonas T. Hartwig, Daniel Larsson, and Sergei D. Silvestrov, *Deformations of Lie algebras using $\sigma$-derivations*, J. Algebra 295 (2006), no. 2, 314–361.

[JL08] Quanqin Jin and Xiaochao Li, *Hom-Lie algebra structures on semi-simple Lie algebras*, J. Algebra 319 (2008), no. 4, 1398–1408.

[MS08] Abdenacer Makhlouf and Sergei D. Silvestrov, *Hom-algebra structures*, J. Gen. Lie Theory Appl. 2 (2008), no. 2, 51–64.

[NS02] Mara D. Neusel and Larry Smith, *Invariant theory of finite groups*, Mathematical Surveys and Monographs, vol. 94, American Mathematical Society, Providence, RI, 2002.

[OOS19] E. Ongong’a, J. Ongaro, and S. Silvestrov, *Hom-Lie structures on complex 4-dimensional Lie algebras* (2019), 373–381. International Workshop on Lie Theory and Its Applications in Physics.

[Rem18] Elisabeth Remm, *3-dimensional skew-symmetric algebras and the variety of Hom-Lie algebras*, Algebra Colloq. 25 (2018), no. 4, 547–566.

[She12] Yunhe Sheng, *Representations of Hom-Lie algebras*, Algebr. Represent. Theory 15 (2012), no. 6, 1081–1098.

[Vas98] Wolmer V. Vasconcelos, *Computational methods in commutative algebra and algebraic geometry*, Algorithms and Computation in Mathematics, vol. 2, Springer-Verlag, Berlin, 1998.

[XJL15] Wenjuan Xie, Quanqin Jin, and Wende Liu, *Hom-structures on semi-simple Lie algebras*, Open Math. 13 (2015), no. 1, 617–630.

[Yau11] Donald Yau, *Hom-Novikov algebras*, J. Phys. A 44 (2011), no. 8, 085202, 20.

[Zha21] Runxuan Zhang, *Representations of $\omega$-Lie algebras and tailed derivations of Lie algebras*, Internat. J. Algebra Comput. 31 (2021), no. 2, 325–339.

[ZHB11] Runxuan Zhang, Dongping Hou, and Chengming Bai, *A Hom-version of the affinizations of Balinskii-Novikov and Novikov superalgebras*, J. Math. Phys. 52 (2011), no. 2, 023505, 19.

School of Mathematics and Statistics, Northeast Normal University, Changchun, China & Department of Mathematics and Statistics, Queen’s University, Kingston, K7L 3N6, Canada

Email address: ychen@nenu.edu.cn

School of Mathematics and Statistics, Northeast Normal University, Changchun, China

Email address: zhangrx728@nenu.edu.cn