$L_p \to L_q$ BOUNDEDNESS OF FOURIER MULTIPLIERS

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Abstract. We investigate the $L_p \to L_q$ boundedness of the Fourier multipliers. We obtain sufficient conditions, namely, we derive Hormander and Lizorkin type theorems. We also obtain the necessary conditions. For $M$-generalized monotone functions, we obtain a criteria for boundedness of the corresponding Fourier multipliers.

1. Introduction

The study of Fourier multipliers has been attracting attention of researchers for more than a century. This is related to numerous applications in mathematical analysis, in particular, in partial differential equations. One of the important questions in this field is to understand the $L_p \to L_q$ boundedness of a Fourier multipliers.

In case $p = q$, one of the earliest important works was obtained by Marcinkiewicz [18] in 1939, see also [12]. He obtained a sufficient condition for $L_p \to L_p$ boundedness of Fourier series multipliers. An analogue of his result for Fourier transform multipliers also holds, see [17]. Another important result was obtained by Mikhlin [19] in 1956, which was improved by Stain [32] and Hörmander [10]. There were further developments in this topic, we mention works [3, 6, 8, 9, 11, 23] and references therein. We also refer to the work [7] for a short historical overview of the Mikhlin-Hörmander and Marcinkiewicz theorems.

For the case $p \leq q$, there another two classical results available: Hörmander’s multiplier theorem [10] and Lizorkin’s multiplier theorem [16]. There is a fundamental difference between these two results: Hörmander’s theorem does not require any regularity of the symbol and applies to $p$ and $q$ separated by 2, while Lizorkin’s theorem requires weaker conditions on $p$, $q$ but imposes certain regularity conditions on the symbol. For this case, we also mention works [1, 2, 5, 25, 26, 27, 28, 31] and references therein.

In this work we are interested on $L_p(I) \to L_q(I)$ boundedness of a Fourier multipliers in cases $I = \mathbb{R}$ and $I = (0, 1)$. The corresponding higher dimensional cases will be considered in future work.

1.1. Hörmander type theorem. We recall that in [10, Theorem 1.11], Hörmander showed that, for $1 < p \leq 2 \leq q < \infty$, a symbol $\lambda$ and the corresponding Fourier transform multiplier $T_\lambda$ satisfy

\[
\|T_\lambda\|_{L_p(\mathbb{R}) \to L_q(\mathbb{R})} \lesssim \|\lambda\|_{L_{r, \infty}(\mathbb{R})}, \quad 1/r = 1/p - 1/q.
\]

This result was also obtained for the case of interval. It was shown in [5, p. 303] that under the same conditions on $p$, $q$, and $r$, for a sequence of complex numbers

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\( \lambda = \{ \lambda_k \}_{k \in \mathbb{Z}} \) and the corresponding Fourier series multiplier \( T_\lambda \), the estimate holds
\[
\| T_\lambda \|_{L_p(0,1) \to L_q(0,1)} \lesssim \| \lambda \|_{(r,\infty)(\mathbb{Z})}.
\]

There were some other works in this direction. In \([25, 26]\), authors improved the sufficient condition \((1.2)\). Moreover, they obtain a necessary condition. We also mention that Hörmander type theorem was obtained in \([1, 2]\), where author investigate he \( L_p \to L_q \) boundedness of Fourier multipliers in the context of compact Lie groups.

This work partially devoted to further development of Hörmander’s result. We weaken the sufficient conditions \((1.1)\) and \((1.2)\). Additionally, we obtain necessary conditions for \( L_p \to L_q \) boundedness of Fourier multipliers. In the interest of brevity, we present a slightly simplified version of our results:

**Theorem 1.1.** Let \( 1 < p \leq 2 \leq q < \infty \) and \( 1/r = 1/p - 1/q \). Let \( r' \) be the conjugate exponent of \( r \), then, the following statements are true

(i) For a measurable function \( \lambda \), it follows
\[
\sup_{k \in \mathbb{Z}} \sup_{e \in M_k} \frac{1}{|e|^{1/r'}} \left| \int e(\xi) d\xi \right| \lesssim \| T_\lambda \|_{L_p(\mathbb{R}) \to L_q(\mathbb{R})} \lesssim \sup_{k \in \mathbb{Z}} \sup_{e \in \Delta_k} \frac{1}{|e|^{1/r'}} \left| \int e(\xi) d\xi \right|,
\]
where \( M_k \) is the set of intervals containing in
\[
\Delta_k := (-2^{k+1}, -2^k) \cup [2^k, 2^{k+1}).
\]

(ii) For a sequence \( \lambda = \{ \lambda_k \}_{k \in \mathbb{Z}} \), it follows
\[
\sup_{k \in \mathbb{Z}} \sup_{e \in W_k} \frac{1}{|e|^{1/r'}} \left| \sum_{m \in e} \lambda_m \right| \lesssim \| T_\lambda \|_{L_p(0,1) \to L_q(0,1)} \lesssim \sup_{k \in \mathbb{Z}} \sup_{e \in \delta_k} \frac{1}{|e|^{1/r'}} \left| \sum_{m \in e} \lambda_m \right|,
\]
where \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \) and \( W_k \) is the set of all discrete intervals (finite arithmetic progressions with a common difference of 1) containing in
\[
\delta_k := \{-2^{k+1} + 1, \ldots, -2^k\} \cup \{2^k, \ldots, 2^{k+1} - 1\}.
\]

If \( k \in \mathbb{N} \), and \( \delta_0 := \{-1, 0, 1\} \), if \( k = 0 \).

A comparison of our results with those of previous studies is presented in Section 6. It is shown that the sufficient conditions in Theorem 1.1 are strictly weaker than \((1.1)\) and \((1.2)\).

For a large class of functions, we show that the sufficient and necessary conditions we obtained here are equivalent, allowing us to formulate a criteria for the \( L_p \to L_q \) boundedness of Fourier multipliers. Namely, we say that a complex valued function \( \lambda \) on \( \mathbb{R} \) is a \( M \)-generalized monotone function if
\[
\lambda^*(t) \leq C \sup_{e \in M, \| e \| \geq 1} \frac{1}{|e|} \left| \int e(\lambda(x)) dx \right|,
\]
where \( M \) is a set of some measurable subsets in \( \mathbb{R} \) with positive measure, \( \lambda^* \) is a non-increasing rearrangement, and \( C \) is some positive constant depending on \( \lambda \). We show that if \( M \) is the set of all intervals and \( \lambda \) is a \( M \)-generalized monotone function, then \( \lambda \) represents \( L_p \to L_q \) Fourier transform multiplier if and only if
\[
\sup_{e \in M} \frac{1}{|e|^{1/r'}} \left| \int e(\lambda(\xi)) d\xi \right| < \infty.
\]
The dependencies of $C\lambda$ are similar to those of $\lambda$. For a sequence $x$ such that $\parallel x \parallel < p < q < \infty$, Lizorkin [16] showed that for a continuously differentiable function $\lambda$ on $\mathbb{R}$, the corresponding Fourier transform multiplier $T_\lambda$ satisfies

$$
\|T_\lambda\|_{L^p(\mathbb{R}) \to L^q(\mathbb{R})} \lesssim \sup_{\xi \in \mathbb{R}} \left( |\xi|^\frac{1}{r} |\lambda(\xi)| + |\xi|^{\frac{1}{r} + 1} |\lambda'(\xi)| \right), \quad 1/r = 1/p - 1/q.
$$

A similar result holds for intervals as well: For a sequence $\lambda = \{\lambda_k\}_{k \in \mathbb{Z}}$, the corresponding Fourier series multiplier $T_\lambda$ satisfies

$$
\|T_\lambda\|_{L^p(0,1) \to L^q(0,1)} \lesssim \sup_{k \in \mathbb{Z}} \left( |k|^\frac{1}{r} |\lambda_k| + |k|^{\frac{1}{r} + 1} |\lambda_k - \lambda_{k+1}| \right).
$$

These results were generalized in [31] and [27]. Authors derive strictly weaker sufficient conditions.

In this work we make further improvements of these results. We prove the following Lizorkin type theorem:

**Theorem 1.2.** Let $1 < p < q < \infty$, $1/r = 1/p - 1/q$, and $\Delta_k$, $\delta_k$ be the sets defined in Theorem [1.1]. Then the following statements are true

(i) Let $\lambda$ be a real-valued function on $\mathbb{R}$ which is absolutely continuous on $(-\infty, 0]$ and $[0, \infty)$ such that $\lambda(\xi) \to 0$ as $|\xi| \to \infty$. Then the corresponding Fourier transform multiplier satisfies

$$
\|T_\lambda\|_{L^p(\mathbb{R}) \to L^q(\mathbb{R})} \lesssim \sup_{k \in \mathbb{Z}} 2^{k\frac{1}{r}} \int_{\Delta_k} |\lambda'(\xi)| d\xi.
$$

(ii) Let $\lambda = \{\lambda_k\}_{k \in \mathbb{Z}}$ be a sequence of real numbers such that $\lambda_k \to 0$ as $k \to \infty$. Then the corresponding Fourier series multiplier satisfies

$$
\|T_\lambda\|_{L^p(0,1) \to L^q(0,1)} \lesssim \sup_{k \in \mathbb{N}_0} 2^{\frac{k+1}{r} - 1} \sum_{m = 2^k} (|\lambda_{-m} - \lambda_{-m+1}| + |\lambda_m - \lambda_{m-1}|).
$$

We note that the sufficient conditions in Theorem 1.2 are strictly weaker than (1.3) and (1.4), see Examples 6.5 and 6.6. We show that Theorem 1.2 is at least complementary to results in [31] and [27].

The paper has simple structure. In Section 2 we introduce notation and recall some definitions and know results. In Section 3 we prove sufficient conditions, namely, we obtain Hörmander type and Lizorkin type theorems. In Section 4 we obtain necessary conditions. In the next section, we introduce notion of $M$-generalized monotone functions and sequences. We obtain criteria for boundedness of Fourier multipliers corresponding to $M$-generalized monotone functions and sequences. Finally, in Section 6 we derive corollaries, give examples and compare our theorems with some previous results.

2. Preliminaries

In this section we introduce notations and recall some definitions. In our analysis, we often write $x \lesssim y$ or $y \gtrsim x$ to mean that $x \leq Cy$, where $C > 0$ is some constant. The dependencies of $C$ will either be explicitly specified or otherwise, clear from context. By $x \approx y$ we mean that $x \lesssim y$ and $x \gtrsim y$. For two sequences $a = \{a_k\}_{k \in \mathbb{Z}}$
and \( b = \{b_k\}_{k \in \mathbb{Z}} \), we write \( ab := \{a_k b_k\}_{k \in \mathbb{Z}} \). We also use notation \( \mathbb{N}_0 := \mathbb{N} \cup \{0\} \), where \( \mathbb{N} := \{1, 2, \ldots\} \).

Let \((\Omega, \Sigma, \mu)\) be a measure space. We denote by \( L_p(\Omega) \), \( 1 \leq p \leq \infty \), the space of measurable functions in \( \Omega \) with integrable \( p \)th power, and write

\[
\|f\|_{L_p(\Omega)} := \left( \int_{\Omega} |f(x)|^p d\mu(x) \right)^{1/p}, \quad f \in L_p(\Omega).
\]

When \( p = \infty \) this is understood as the essential supremum of \(|f|\). When \( 1 \leq p < \infty \) we use notation \( p' \) for the conjugate exponent defined by \( 1/p + 1/p' = 1 \).

For a measurable function \( f \), by \( d_f \) and \( f^* \) we denote its distribution function and non-increasing rearrangement:

\[
d_f(\sigma) := |\{x \in \Omega : |f(x)| \geq \sigma\}|, \quad f^*(t) := \inf\{\sigma > 0 : d_f(\sigma) < t\}.
\]

**Definition 2.1.** Let \( 0 < p < \infty \) and \( 0 < q \leq \infty \). The Lorentz space, \( L_{p,q}(\Omega) \), is defined as the space of finitely measurable functions \( f \) such that \( \|f\|_{L_{p,q}(\Omega)} \leq \infty \), where

\[
\|f\|_{L_{p,q}(\Omega)} := \left\{ \begin{array}{ll}
\left( \int_0^{\infty} \left( \frac{t}{\int_0^t \! f^*(s) \, ds} \right)^q \frac{dt}{t} \right)^{1/q}, & \text{for } q < \infty, \\
\sup_{t \geq 0} t^{\frac{1}{q}} f^*(t) < \infty, & \text{for } q = \infty.
\end{array} \right.
\]

For \( \Omega = \mathbb{Z} \) with \( \Sigma = 2^\mathbb{N} \) and \( \mu = # \) being the power set of \( \mathbb{Z} \) and counting measure, respectively, the non-increasing rearrangement, \( a^* = \{a_k^*\}_{k \in \mathbb{N}} \), of \( a = \{a_k\}_{k \in \mathbb{Z}} \) can be obtained by permuting \( \{|a_k|\}_{k \in \mathbb{Z}} \) in the non-increasing order. For this case, the definition becomes as follows:

**Definition 2.2.** Let \( 0 < p < \infty \) and \( 0 < q \leq \infty \). The Lorentz sequence space, \( l_{p,q}(\mathbb{Z}) \), is defined as a space of sequences \( a = \{a_k\}_{k \in \mathbb{Z}} \) such that \( \|a\|_{l_{p,q}(\mathbb{Z})} < \infty \), where

\[
\|a\|_{l_{p,q}(\mathbb{Z})} := \left\{ \begin{array}{ll}
\left( \sum_{k \in \mathbb{N}} \left( k^{\frac{1}{p}} a_k^* \right)^q \frac{1}{k^q} \right)^{\frac{1}{q}}, & \text{for } q < \infty, \\
\sup_{k \in \mathbb{N}} k^{\frac{1}{p}} a_k^* < \infty, & \text{for } q = \infty.
\end{array} \right.
\]

Furthermore, for a subset \( B \subset \mathbb{Z} \), we define

\[
\|a\|_{l_{p,q}(B)} := \|\tilde{a}\|_{l_{p,q}(\mathbb{Z})},
\]

where \( \tilde{a} \) is a sequence such that

\[
\tilde{a}_k = \begin{cases} 
a_k & \text{if } k \in B, \\
0 & \text{if } k \in \mathbb{Z} \setminus B.
\end{cases}
\]

**Remark 2.3.** When \( q = \infty \), the definitions above also make sense for \( p = \infty \), so that the spaces \( L_{\infty,\infty} \) and \( l_{\infty,\infty} \) are well defined and they coincide with \( L_\infty \) and \( l_\infty \), respectively.

Let \( S(\mathbb{R}) \) be the space of Schwartz functions on \( \mathbb{R} \). For a function \( \lambda \), the Fourier transform multiplier, \( T_\lambda \), is given by the multiplication on the Fourier transform side, that is

\[
\mathcal{F}T_\lambda f(\xi) = \lambda(\xi) \mathcal{F}f(\xi) \quad \xi \in \mathbb{R}, \quad f \in S(\mathbb{R}),
\]

where \( \mathcal{F} \) is the Fourier transform:

\[
\mathcal{F}f(\xi) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-ix\xi} \, dx.
\]
We also recall the definition of the Fourier series multipliers. We say that the sequence of complex numbers \( \lambda = \{ \lambda_k \}_{k \in \mathbb{Z}} \) represents a Fourier series multiplier, \( T_\lambda \), from \( L_p(0,1) \) to \( L_q(0,1) \) if for any \( f \in L_p(0,1) \) with
\[
f \sim \sum_{k \in \mathbb{Z}} a_k e^{2\pi i k x},
\]
there exists \( f_\lambda \in L_q(0,1) \) with
\[
f_\lambda \sim \sum_{k \in \mathbb{Z}} \lambda_k a_k e^{2\pi i k x}
\]
and the operator \( T_\lambda : f \mapsto f_\lambda \) is bounded from \( L_p(0,1) \) to \( L_q(0,1) \).

By \( M_p^q \) we denote the normed space of Fourier transform multipliers with the norm given by
\[
\| \lambda \|_{M_p^q} := \| T_\lambda \|_{L_p \to L_q}.
\]
Similarly, we denote the normed Fourier series multipliers by \( m_p^q \).

2.1. E.Nursultanov’s NET space. Here we recall the NET space which was introduced by E.Nursultanov in \([21, 22]\).

**Definition 2.4.** Let \( 0 < p < \infty \) and \( 0 < q \leq \infty \). Let \( (\Omega, \Sigma, \mu) \) be a measure space and \( M \) be a family of some measurable sets in \( \Omega \) with finite positive measures. Then, E.Nursultanov’s space, \( N_{p,q}(M) = N_{p,q}(\Omega, M) \), is defined as the space of integrable on each \( e \in M \) functions \( f \) such that \( \| f \|_{N_{p,q}(M)} < \infty \), where
\[
\| f \|_{N_{p,q}(M)} := \left\{ \begin{array}{ll}
\left( \int_0^\infty \left( \frac{1}{t^q} \int f(t, M) \right)^q \frac{dt}{t} \right)^{\frac{1}{q}}, & \text{for } q < \infty, \\
\sup_{t > 0} \left( \frac{1}{t^q} \int f(t, M) \right), & \text{for } q = \infty
\end{array} \right.
\]
and\(^3\)
\[
\tilde{f}(t, M) := \sup_{|e| \geq t, e \in M} \frac{1}{|e|} \left| \int_e f(x) d\mu x \right|.
\]

For the sake of convenience, we repeat this definition for the case \( (\Omega, \Sigma, \mu) = (\mathbb{Z}, 2^\mathbb{N}, \#) \).

**Definition 2.5.** Let \( 0 < p < \infty \) and \( 0 < q \leq \infty \). Let \( W \) be a set of some finite non-empty subsets of \( \mathbb{Z} \), then E.Nursultanov’s sequence space, \( n_{p,q}(W) = n_{p,q}(\mathbb{Z}, W) \), is defined as a space of complex sequences \( a = \{ a_k \}_{k \in \mathbb{Z}} \) such that \( \| a \|_{n_{p,q}(W)} < \infty \), where
\[
\| a \|_{n_{p,q}(W)} := \left\{ \begin{array}{ll}
\left( \sum_{k \in \mathbb{N}} \left( k^q \tilde{a}_k(W) \right)^q \frac{1}{k} \right)^\frac{1}{q}, & \text{for } q < \infty, \\
\sup_{k \in \mathbb{N}} k^q \tilde{a}_k(W), & \text{for } q = \infty
\end{array} \right.
\]
and
\[
\tilde{a}_k(W) := \sup_{|e| \geq k, e \in W} \frac{1}{|e|} \left| \sum_{j \in e} a_j \right|.
\]

**Remark 2.6.** Let us note that if \( M \) is the set of all measurable subsets of \( \Omega \) with finite positive measures, then \( N_{p,q}(\Omega, M) = L_{p,q}(\Omega) \). Similarly, if \( W \) be set of all finite non-empty subsets of \( \mathbb{Z} \), then \( n_{p,q}(\mathbb{Z}, W) = l_{p,q}(\mathbb{Z}) \).

\(^3\)For \( e \in \Sigma \) we write \( |e| := \mu(e) \). From the context, it will be clear if it is the measure or absolute value.
Like Lorentz spaces, E.Nursultanov’s spaces are nested increasingly with respect to the second parameter; see Remark 1 in [24] and Proposition 2 in [22]. More precisely:

**Proposition 2.7.** Let $0 < p < \infty$ and $0 < q_1 \leq q_2 \leq \infty$, then the following statements are true:

(i) Let $M$ be a family of some measurable subsets of $\Omega$ with finite positive measures. Then $N_{p,q_1}(M) \hookrightarrow N_{p,q_2}(M)$, that is

$$\|f\|_{N_{p,q_2}(M)} \lesssim \|f\|_{N_{p,q_1}(M)}, \quad \text{for } f \in N_{p,q_1}(M).$$

(ii) Let $W$ be a family of some finite non-empty sets in $\mathbb{Z}$. Then $n_{p,q_1}(W) \hookrightarrow n_{p,q_2}(W)$, that is

$$\|a\|_{n_{p,q_2}(W)} \lesssim \|a\|_{n_{p,q_1}(W)}, \quad \text{for } a \in n_{p,q_1}(W).$$

Next, we give alternative expressions for the quasi-norms $\|\cdot\|_{N_{p,\infty}}$ and $\|\cdot\|_{n_{p,\infty}}$.

**Proposition 2.8.** Let $(\Omega, \Sigma, \mu)$ be a measure space and $0 < p < \infty$. Let $M \subset \Sigma$ be a fixed set, whose elements have finite positive measure. Then, for a function $f$ integrable over each $e \in M$, it follows

$$\|f\|_{N_{p,\infty}(M)} = \sup_{e \in M} \frac{1}{|e|^{1/p'}} \left| \int_{e} f(x) d\mu(x) \right|.$$  

In particular, if $(\Omega, \Sigma, \mu) = (\mathbb{Z}, 2^\mathbb{N}, \#)$ and $W$ being some fixed set of finite non-empty subsets of $\mathbb{Z}$, then

$$\|a\|_{n_{p,\infty}(W)} = \sup_{e \in W} \frac{1}{|e|^{1/p'}} \left| \sum_{k \in e} a_k \right|.$$  

for any sequence of complex numbers $a = \{a_k\}_{k \in \mathbb{Z}}$.

**Proof.** For any $e_0 \in M$, we estimate

$$\|f\|_{N_{p,\infty}(M)} = \sup_{t > 0} \frac{1}{|t|^{1/p'}} \left| \int_{e_0} f(x) d\mu(x) \right| \leq \left| e_0 \right|^{1/p'} \sup_{|e| \geq |e_0|, e \in M} \frac{1}{|e|} \left| \int_{e} f(x) d\mu(x) \right| \leq \left| e_0 \right|^{1/p'} \left| \int_{e_0} f(x) d\mu(x) \right|.$$

Conversely,

$$\|f\|_{N_{p,\infty}(M)} = \sup_{t > 0} \frac{1}{|t|^{1/p'}} \left| \int_{e} f(x) d\mu(x) \right| \sup_{t > 0} \left| e_0 \right|^{1/p'} \sup_{|e| \geq |e_0|, e \in M} \frac{1}{|e|} \left| \int_{e} f(x) d\mu(x) \right| \leq \sup_{e \in M} \left| e \right|^{1/p'} \left| \int_{e} f(x) d\mu(x) \right|.$$  

From now on, we only consider the cases where $\Omega$ is $\mathbb{R}$ or $\mathbb{Z}$ and $\mu$ being the Lebesgue or counting measure, respectively.

**Lemma 2.9.** Let $0 < p < \infty$ and $0 < q \leq \infty$, then the following statements are true:
(i) Let $M$ be a family of some measurable sets in $\mathbb{R}$ with finite positive measures. Then we have the equivalence
\[
\|f\|_{N_{p,q}(M)} \approx \begin{cases} \left( \sum_{k \in \mathbb{Z}} \left( 2^k \overline{f}(2^k, M) \right)^q \right)^{1/q}, & q \neq \infty, \\
\sup_{k \in \mathbb{Z}} 2^k \overline{f}(2^k, M), & q = \infty. \end{cases}
\]

(ii) Let $W$ be a family of some finite non-empty sets in $\mathbb{Z}$. Then
\[
\|a\|_{n_{p,q}(W)} \approx \begin{cases} \left( \sum_{n \in \mathbb{N}_0} \left( 2^n \overline{a}(2^n, W) \right)^q \right)^{1/q}, & q \neq \infty, \\
\sup_{k \in \mathbb{N}_0} 2^n \overline{a}(2^n, W), & q = \infty. \end{cases}
\]

Proof. By definition,
\[
\|f\|_{N_{p,q}(M)}^q = \int_0^\infty \left( t^{1/p} \overline{f}(t, M) \right)^q dt = \sum_{k \in \mathbb{Z}} \int_{2^k}^{2^{k+1}} \left( t^{1/p} \overline{f}(t, M) \right)^q dt \approx \sum_{k \in \mathbb{Z}} \left( 2^k \overline{f}(2^k, M) \right)^q.
\]

and
\[
\|a\|_{n_{p,q}(W)}^q = \sum_{k \in \mathbb{N}} \left( k^{1/p} \overline{a}(k, W) \right)^q \frac{1}{k} = \sum_{n \in \mathbb{N}_0} \sum_{k=2^n}^{2^{n+1}-1} \left( k^{1/p} \overline{a}(k, W) \right)^q \frac{1}{k} \approx \sum_{n \in \mathbb{N}_0} \left( 2^n \overline{a}(2^n, W) \right)^q.
\]

Similarly, one can obtain the corresponding formulas for the case $q = \infty$. □

Finally, we will state known results which will be used later in this work. The first part of the following theorem was proved in [21] and the second part in [22, Theorem 3]

**Theorem 2.10.** Let $2 \leq p < \infty$ and $0 < q \leq \infty$, then the following statements are true:

(i) Let $M$ be the set of all finite intervals on $\mathbb{R}$. Let $f \in L_{p,q}(\mathbb{R})$, then
\[
\|Ff\|_{N_{p,q}(M)} \lesssim \|f\|_{L_{p,q}(\mathbb{R})}.
\]

(ii) Let $W$ be the set of all finite intervals on $\mathbb{Z}$. Let $f \in L_{p,q}(0,1)$ and $f \sim \sum_{k \in \mathbb{Z}} a_k e^{2\pi ikx}$, then
\[
\|a\|_{n_{p,q}(W)} \lesssim \|f\|_{L_{p,q}(0,1)}.
\]

**Remark 2.11.** Originally, the above theorem was stated for $2 < p < \infty$. However, by careful checking the proof, one can verify that Theorem 2.10 holds also for $1 < p < \infty$.

3. **Sufficient conditions**

In this section we obtain necessary conditions for the $L_p - L_q$ boundedness of Fourier multipliers, which imply the upper bounds in Theorems 1.1 and 1.2.
3.1. **Hörmander type theorems.** We begin by proving Hörmander type theorem for Fourier transform multipliers:

**Theorem 3.1.** Let \( 1 < p \leq 2 \leq q < \infty \) and \( 1/r = 1/p - 1/q \). Assume that \( \lambda \) is a measurable function, then

\[
\| \lambda \|_{M^p_q} \lesssim \sup_{k \in \mathbb{Z}} \| \lambda \|_{L^{r, \infty}(\Delta_k)},
\]

where \( \Delta_k := (-2^{k+1}, -2^k) \cup [2^k, 2^{k+1}) \).

**Proof.** Let \( f \in S(\mathbb{R}) \), then

\[
\| T_\lambda f \|_{L^q} = \| \mathcal{F}^{-1} \lambda \mathcal{F} f \|_{L^q} = \left\| \sum_{k \in \mathbb{Z}} \mathcal{F}^{-1} \chi_{\Delta_k} \mathcal{F} f \right\|_{L^q},
\]

where \( \chi_{\Delta_k} \) is the indicator function of \( \Delta_k \), that is

\[
\chi_{\Delta_k}(\xi) := \begin{cases} 
1 & \text{if } \xi \in \Delta_k, \\
0 & \text{otherwise.}
\end{cases}
\]

By using the Littlewood-Paley inequality, see [32], we write

\[
\| T_\lambda f \|_{L^q} \lesssim \left( \sum_{k \in \mathbb{Z}} \| \mathcal{F}^{-1} \chi_{\Delta_k} \mathcal{F} f \|^2_{L^q} \right)^{1/2}.
\]

Since \( q \geq 2 \) the Minkowski inequality gives

\[
\| T_\lambda f \|_{L^q} \lesssim \left( \sum_{k \in \mathbb{Z}} \| \mathcal{F}^{-1} \lambda \chi_{\Delta_k} \mathcal{F} f \|^2_{L^q} \right)^{1/2}.
\]

Further, the Hardy-Littlewood inequality (if \( q > 2 \)) or the Parseval identity (if \( q = 2 \)) gives

\[
\| T_\lambda f \|_{L^q} \lesssim \left( \sum_{k \in \mathbb{Z}} \| \lambda \chi_{\Delta_k} \mathcal{F} f \|^2_{L^{p', q'}} \right)^{1/2}.
\]

If \( r = \infty \), that is \( p = q = 2 \), we estimate

\[
\| T_\lambda f \|_{L^q} \lesssim \sup_{k \in \mathbb{Z}} \| \lambda \|^2_{L^{r, \infty}(\Delta_k)} \left( \sum_{k \in \mathbb{Z}} \| \chi_{\Delta_k} \mathcal{F} f \|^2_{L^{2, 2}(\Delta_k)} \right)^{1/2}
\]

\[
\lesssim \sup_{k \in \mathbb{Z}} \| \lambda \|^2_{L^{r, \infty}(\Delta_k)} \left( \sum_{k \in \mathbb{Z}} \| \chi_{\Delta_k} \mathcal{F} f \|^2_{L^{p', q'}(\Delta_k)} \right)^{1/2}.
\]

Otherwise, when \( r < \infty \), we use the Hölder inequality to derive the same estimate

\[
\| T_\lambda f \|_{L^q} \lesssim \left( \sum_{k \in \mathbb{Z}} \| \lambda \|^2_{L^{r, \infty}(\Delta_k)} \| \chi_{\Delta_k} \mathcal{F} f \|^2_{L^{p', q'}(\Delta_k)} \right)^{1/2}
\]

\[
\lesssim \sup_{k \in \mathbb{Z}} \| \lambda \|^2_{L^{r, \infty}(\Delta_k)} \left( \sum_{k \in \mathbb{Z}} \| \chi_{\Delta_k} \mathcal{F} f \|^2_{L^{p', q'}(\Delta_k)} \right)^{1/2}.
\]
Since $p \leq q$, it follows that $L_{p', p}(\Delta_k) \rightarrow L_{p', q}(\Delta_k)$ and
\[
\|f\|_{L_{p', p}(\Delta_k)} \leq C\|f\|_{L_{p', q}(\Delta_k)}, \quad \text{for } f \in L_{p', p}(\Delta_k),
\]
where $C > 0$ is independent on $k \in \mathbb{Z}$. Therefore, the penultimate inequality gives
\[
\|T_\lambda f\|_{L_q} \lesssim \sup_{k \in \mathbb{Z}} \|\lambda\|_{L_{r, \infty}(\Delta_k)} \left(\sum_{k \in \mathbb{Z}} \|\chi_{\Delta_k} F f\|_{L_{p', p}(\Delta_k)}^2\right)^{\frac{1}{2}}.
\]
We will repeat our steps in reverse order given that $p \leq 2$. The Hardy-Littlewood inequality (or Parseval identity if $p = 2$) gives
\[
\|T_\lambda f\|_{L_q} \lesssim \sup_{k \in \mathbb{Z}} \|\lambda\|_{L_{r, \infty}(\Delta_k)} \left(\sum_{k \in \mathbb{Z}} \|F^{-1} \chi_{\Delta_k} F f\|_{L_p}^2\right)^{\frac{1}{2}}.
\]
Since $p \leq 2$, by Minkowski inequality, we obtain
\[
\|T_\lambda f\|_{L_q} \lesssim \sup_{k \in \mathbb{Z}} \|\lambda\|_{L_{r, \infty}(\Delta_k)} \left(\sum_{k \in \mathbb{Z}} \|F^{-1} \chi_{\Delta_k} F f\|_{L_p}^2\right)^{\frac{1}{2}}.
\]
Finally, the Littlewood-Paley inequality implies that
\[
\|T_\lambda f\|_{L_q} \lesssim \sup_{k \in \mathbb{Z}} \|\lambda\|_{L_{r, \infty}(\Delta_k)} \|f\|_{L_p}.
\]

Next, we obtain analogue of this theorem but for the Fourier series multipliers:

**Theorem 3.2.** Let $1 < p \leq 2 \leq q < \infty$ and $1/r = 1/p - 1/q$. Let $\lambda = \{\lambda_k\}_{k \in \mathbb{Z}}$ be a sequence of complex numbers, then
\[
\|\lambda\|_{m_{q'}} \lesssim \sup_{k \in \mathbb{N}_0} \|\lambda\|_{l_{r, \infty}(\delta_k)},
\]
where
\[
\delta_k := \begin{cases} 
-2^k + 1, \cdots, -2 \} \cup \{2^k, \cdots, 2^{k+1} - 1\}, & k \in \mathbb{N}, \\
-1, 0, 1, & k = 0.
\end{cases}
\]

**Proof.** Let $f \in L_p(0, 1)$ and $a = \{a_k\}_{k \in \mathbb{Z}}$ be its Fourier coefficients. By using the Littlewood-Paley inequality, we write
\[
\|T_\lambda f\|_{L_q(0, 1)} \lesssim \left\|\sum_{k \in \mathbb{N}_0} \sum_{m \in \delta_k} \lambda_m a_m e^{2\pi i m x} \right\|_{L_q(0, 1)}^{1/2}.
\]
Since $q \geq 2$, by the Minkowski inequality, we obtain
\[
\|T_\lambda f\|_{L_q(0, 1)} \lesssim \left(\sum_{k \in \mathbb{N}_0} \left\|\sum_{m \in \delta_k} \lambda_m a_m e^{2\pi i m x} \right\|_{L_q(0, 1)}^2\right)^{1/2}.
\]
Further, the Hardy-Littlewood inequality (if $q > 2$) or the Parseval identity (if $q = 2$) gives
\[
\|T_\lambda f\|_{L_q(0, 1)} \lesssim \left(\sum_{k \in \mathbb{N}_0} \|\lambda\|_{L_{q'}(\delta_k)}^2\right)^{1/2}.
\]
If \( r = \infty \), that is \( p = q = 2 \), we estimate
\[
\|T_\lambda f\|_{L_q(0,1)} \lesssim \sup_{k \in \mathbb{N}_0} \|\lambda\|_{l_\infty(\delta_k)} \left( \sum_{k \in \mathbb{N}_0} \|a\|_{l_2(\delta_k)}^2 \right)^{1/2} 
\lesssim \sup_{k \in \mathbb{N}_0} \|\lambda\|_{l_{r,\infty}(\delta_k)} \left( \sum_{k \in \mathbb{N}_0} \|a\|_{l_{r,p}(\delta_k)}^2 \right)^{1/2}.
\]

Otherwise, when \( r < \infty \), we use the Hölder inequality to derive the same estimate
\[
\|T_\lambda f\|_{L_q(0,1)} \lesssim \left( \sum_{k \in \mathbb{N}_0} \|\lambda\|_{l_{r,\infty}(\delta_k)}^2 \|a\|_{l_{p',q}(\delta_k)}^2 \right)^{1/2} \lesssim \sup_{k \in \mathbb{N}_0} \|\lambda\|_{l_{r,\infty}(\delta_k)} \left( \sum_{k \in \mathbb{N}_0} \|a\|_{l_{p',q}(\delta_k)}^2 \right)^{1/2}.
\]

Since \( p \leq q \), we know that \( l_{p',p} \rightarrow l_{p',q} \) and the corresponding inequality does not depend on \( k \in \mathbb{N}_0 \). Therefore, the last estimate gives
\[
\|T_\lambda f\|_{L_q(0,1)} \lesssim \sup_{k \in \mathbb{N}_0} \|\lambda\|_{l_{r,\infty}(\delta_k)} \left( \sum_{k \in \mathbb{N}_0} \|a\|_{l_{p',p}(\delta_k)}^2 \right)^{1/2}.
\]

We will repeat our steps in reverse order given that \( p = 2 \). The Hardy-Littlewood inequality (or Parseval identity if \( p = 2 \)) gives
\[
\|T_\lambda f\|_{L_q(0,1)} \lesssim \sup_{k \in \mathbb{N}_0} \|\lambda\|_{l_{r,\infty}(\delta_k)} \left( \sum_{k \in \mathbb{N}_0} \left( \sum_{m \in \delta_k} a_m e^{2\pi i m x} \right)^2 \right)^{1/2}.
\]

By the Minkowski inequality,
\[
\|T_\lambda f\|_{L_q} \lesssim \sup_{k \in \mathbb{N}_0} \|\lambda\|_{l_{r,\infty}(\delta_k)} \left( \sum_{k \in \mathbb{N}_0} \left( \sum_{m \in \delta_k} a_m e^{2\pi i m x} \right)^2 \right)^{1/2}.
\]

Finally, Littlewood-Paley inequality implies
\[
\|T_\lambda f\|_{L_q(0,1)} \lesssim \sup_{k \in \mathbb{N}_0} \|\lambda\|_{l_{r,\infty}(\delta_k)} \|f\|_{L_p(0,1)}.
\]

\[\square\]

### 3.2. Lizorkin type theorems

Here, we obtain Lizorkin type theorems. We start with the Fourier transform multipliers:

**Theorem 3.3.** Let \( 1 < p < q < \infty \) and \( \lambda \) be a real-valued function on \( \mathbb{R} \) which is absolutely continuous on \((-\infty, 0]\) and \([0, \infty)\) such that
\[
\lambda(\xi) \to 0 \quad \text{as} \quad |\xi| \to \infty,
\]
(3.1)
\[
\sup_{k \in \mathbb{Z}} 2^{k\left(\frac{p}{2} - \frac{1}{2}\right)} \int_{\Delta_k} |\lambda'(\xi)| d\xi < A < \infty,
\]
(3.2)
for some constant \( A > 0 \) and
\\[\Delta_k := (-2^{k+1}, -2^k) \cup [2^k, 2^{k+1})\].
Then $\lambda \in M^{q}_{p}$ and
\[ \|\lambda\|_{M^{q}_{p}} \lesssim A. \]

**Proof.** First, we prove that $T_\lambda$ is a bounded operator from $L_{p,1}(\mathbb{R})$ to $L_{q,\infty}(\mathbb{R})$. We estimate
\[
\|T_\lambda\|_{L_{p,1} \rightarrow L_{q,\infty}} = \sup_{\|f\|_{L_{p,1}} = 1} \|T_\lambda f\|_{L_{q,\infty}} \lesssim \sup_{\|f\|_{L_{p,1}} = 1} \|T_\lambda f\|_{L_{q}} = \sup_{\|f\|_{L_{p,1}} = \|g\|_{L_{q}'} = 1} \int_{\mathbb{R}} T_\lambda f(x)g(x)dx.
\]
Then, by the Parseval’s identity, we obtain
\[
(3.3) \quad \|T_\lambda\|_{L_{p,1} \rightarrow L_{q,\infty}} \lesssim \sup_{\|f\|_{L_{p,1}} = \|g\|_{L_{q}'} = 1} \int_{\mathbb{R}} \lambda(\xi) \mathcal{F} f(\xi) \mathcal{F} g(\xi)d\xi.
\]
Let us denote
\[
\phi(\xi) := \int_{0}^{\xi} \mathcal{F} f(\zeta) \mathcal{F} g(\zeta)d\zeta, \quad I_1 := \int_{0}^{\infty} \lambda(\xi) \phi'(\xi)d\xi, \quad I_2 := \int_{-\infty}^{0} \lambda(\xi) \phi'(\xi)d\xi.
\]
By integration by parts, we obtain
\[
|I_1| = \left| \int_{0}^{\infty} \lambda(\xi) \phi'(\xi)d\xi \right| = \left| \int_{0}^{\infty} \lambda'(\xi) \phi(\xi)d\xi \right| = \left| \sum_{k \in \mathbb{Z}} \int_{\Delta_k^+} \lambda'(\xi) 2^{k(\frac{1}{2} - \frac{1}{p})} 2^{k(\frac{1}{q} - \frac{1}{p'})} \phi(\xi)d\xi \right|,
\]
where $\Delta_k^+ := [2^k, 2^{k+1})$. Using the hypothesis of the theorem, we conclude that
\[
|I_1| \lesssim A \sum_{k \in \mathbb{Z}} 2^{k(\frac{1}{q} - \frac{1}{p'})} \sup_{\xi \in \Delta_k^+} \frac{1}{2^k} \|\phi(\xi)\| \lesssim A \sum_{k \in \mathbb{Z}} 2^{k(\frac{1}{q} - \frac{1}{p'})} \sup_{\xi \in \Delta_k^+} \frac{1}{2^k} \left| \int_{0}^{\xi} \mathcal{F} f(\zeta) \mathcal{F} g(\zeta)d\zeta \right|.
\]
Let $r > 0$ be such that $1/r = 1/q + 1/p'$, then Lemma 2.9 implies
\[
|I_1| \lesssim A \|\mathcal{F} f \mathcal{F} g\|_{N_{r,1}(M)} = A \|\mathcal{F}(f * g)\|_{N_{r,1}(M)}.
\]
By Theorem 2.10 and O’Neil inequality, we obtain
\[
|I_1| \lesssim A \|f * g\|_{L_{q,r,1}} \lesssim A \|f\|_{L_{p,1}} \|g\|_{L_{q',\infty}}.
\]
Similarly, one can derive the same upper-bound for $|I_2|$. Putting these inequalities into (3.3) gives
\[
(3.4) \quad \|T_\lambda\|_{L_{p,1} \rightarrow L_{q,\infty}} \lesssim A.
\]
Let us pic $p_0, p_1$ such that $1 < p_0 < p < p_1 < \infty$ and choose $q_0, q_1$ so that
\[
\frac{1}{p_0} - \frac{1}{q_0} = \frac{1}{p_1} - \frac{1}{q_1} = \frac{1}{p} - \frac{1}{q}.
\]
Then, by (3.3), we know that
\[
\|T_\lambda\|_{L_{p_j,1} \rightarrow L_{q_j,\infty}} \lesssim A \quad \text{for } j = 0, 1.
\]
Since $p_0 < p < p_1$, there exists $0 < \theta < 1$ such that
\[
\frac{1}{p} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1}.
\]
and hence, the relation (3.5) gives
\[ \frac{1}{q} = \frac{1 - \theta}{q_0} + \frac{\theta}{q_1}, \]

Therefore, from the Marcinkiewicz-Calderon’s interpolation theorem, it follows that
\[ \|\lambda\|_{M_p^q} = \|T_\lambda\|_{L_p \rightarrow L_q} \lesssim A. \]

\[ \square \]

Analogue of this result holds for Fourier series multipliers:

**Theorem 3.4.** Let \( 1 < p < q < \infty \) and \( 1/r = 1/p - 1/q \). Let \( \lambda = \{\lambda_k\}_{k \in \mathbb{Z}} \) be a sequence such that
\[ \lambda_k \to 0 \quad \text{as} \quad k \to \infty \]
and
\[ \sup_{k \in \mathbb{N}_0} 2^k \sum_{m=2^k}^{2^{k+1}-1} (|\lambda_m - \lambda_{m+1}| + |\lambda_m - \lambda_{m-1}|) \leq A, \]
for some constant \( A > 0 \). Then \( \lambda \in m_p^q \) and \( \|\lambda\|_{m_p^q} \lesssim A. \)

**Proof.** First, we prove that \( T_\lambda \) is a bounded operator from \( L_{p,1}(0,1) \) to \( L_{q,\infty}(0,1) \). We estimate
\[ \|T_\lambda\|_{L_{p,1} \rightarrow L_{q,\infty}} = \sup_{\|f\|_{L_{p,1}} = 1} \|T_\lambda f\|_{L_{q,\infty}} \leq \sup_{\|f\|_{L_{p,1}} = 1} \|T_\lambda f\|_{L_q} \leq \sup_{\|f\|_{L_{p,1}} = \|g\|_{L_{q'}} = 1} \int_0^1 T_\lambda f(x)g(x)dx. \]

Therefore,
\[ \|T_\lambda\|_{L_{p,1} \rightarrow L_{q,\infty}} \leq \sup_{\|f\|_{L_{p,1}} = \|g\|_{L_{q'}} = 1} \left| \sum_{m \in \mathbb{Z}} \lambda_m a_m b_m \right|, \]
where \( \{a_k\} \) and \( \{b_k\} \) are Fourier coefficients of functions \( f \) and \( g \), respectively. Since \( \lambda_k \to 0 \) as \( k \to \infty \), by using Abel transform, we derive
\[ \|T_\lambda\|_{L_{p,1} \rightarrow L_{q,\infty}} \lesssim \sup_{\|f\|_{L_{p,1}} = \|g\|_{L_{q'}} = 1} \left( \sum_{m=1}^\infty |\lambda_m - \lambda_{m-1}| \sum_{l=0}^{m-1} a_l b_l + \sum_{m=1}^\infty |\lambda_m - \lambda_{m+1}| \sum_{l=0}^{m-1} a_{m-l} b_{-l} \right) \]

Then, we estimate
\[ \|T_\lambda\|_{L_{p,1} \rightarrow L_{q,\infty}} \leq \sup_{\|f\|_{L_{p,1}} = \|g\|_{L_{q'}} = 1} \sum_{k=0}^\infty \sup_{e \in \mathbb{W}, 2^k \leq |e| < 2^{k+1}} \sum_{l \in e} a_l b_l \sum_{m=2^k}^{2^{k+1}-1} (|\lambda_m - \lambda_{m-1}| + |\lambda_m - \lambda_{m+1}|) \]

Using the theorem’s conditions, we obtain
\[ \|T_\lambda\|_{L_{p,1} \rightarrow L_{q,\infty}} \leq A \sup_{\|f\|_{L_{p,1}} = \|g\|_{L_{q'}} = 1} \sum_{k=0}^\infty (2^k)^{1-\frac{1}{r}} \sup_{e \in \mathbb{W}, 2^k \leq |e| < 2^{k+1}} \frac{1}{2^k} \sum_{l \in e} a_l b_l \]
\[ \lesssim A \sup_{\|f\|_{L_{p,1}} = \|g\|_{L_{q'}} = 1} \sum_{k=0}^\infty (2^k)^{1-\frac{1}{p}} \sup_{e \in \mathbb{W}, |e| \geq 2^k} \frac{1}{|e|} \sum_{l \in e} a_l b_l. \]
Let \( \tau > 0 \) be a number such that \( 1/\tau = 1 - 1/r \). Using Lemma 2.9, we derive
\[
\|T \lambda\|_{L^p,1^\to L^q,\infty} \lesssim A \sup_{\|f\|_{L^p,1} = \|g\|_{L^q,r}} \|ab\|_{n,r,1}.
\]
From Theorem 2.10, it follows
\[
\|T \lambda\|_{L^p,1^\to L^q,\infty} \lesssim A \sup_{\|f\|_{L^p,1} = \|g\|_{L^q,r}} \|f \ast g\|_{L^q',1}.
\]
Since
\[
1 + \frac{1}{r'} = 1 + 1 - \frac{1}{\tau} = 1 + \frac{1}{p} + \frac{1}{q'},
\]
the O’Neil inequality gives
\[
(3.6) \quad \|T \lambda\|_{L^p,1^\to L^q,\infty} \lesssim A \sup_{\|f\|_{L^p,1} = \|g\|_{L^q,r}} \|f\|_{L^p,1} \|g\|_{L^q',\infty} \lesssim A.
\]

Let us pick \( p_0, p_1 \) such that \( 1 < p_0 < p < p_1 < \infty \) and choose \( q_0, q_1 \) so that
\[
(3.7) \quad \frac{1}{p_0} - \frac{1}{q_0} = \frac{1}{p_1} - \frac{1}{q_1} = \frac{1}{p} - \frac{1}{q}.
\]
Then, by (3.6), we know that
\[
\|T \lambda\|_{L^p,1^\to L^q,\infty} \lesssim A, \quad \text{for } j = 0, 1.
\]
Since \( p_0 < p_1 \), there exists \( 0 < \theta < 1 \) such that
\[
\frac{1}{p} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1},
\]
and hence, the relation (3.7) gives
\[
\frac{1}{q} = \frac{1 - \theta}{q_0} + \frac{\theta}{q_1}.
\]
Therefore, from the Marcinkiewicz-Calderon’s interpolation theorem, it follows that
\[
\|\lambda\|_{m^p_r} = \|T \lambda\|_{L^p \to L^q} \lesssim A.
\]

\[
\square
\]

4. Necessary Conditions

In this section, we derive sufficient condition for \( L^p - L^q \) boundedness for Fourier multipliers. First, we obtain this for Fourier transform multipliers:

**Theorem 4.1.** Let \( 1 < p \leq 2 \leq q < \infty \) and \( 1/r = 1/p - 1/q \). Let \( 0 < \tau \leq \infty \) and \( M \) be the set of all finite intervals in \( \mathbb{R} \). Then, for a measurable function \( \lambda \), it follows
\[
\sup_{e \in M} \frac{1}{|e|^{1/\tau}} \left| \int_{e} \lambda(\xi) d\xi \right| \lesssim \|T \lambda\|_{L^p \to L^q,\tau}.
\]

**Proof.** Let \( e_0 \) be an arbitrary interval, that is \( e_0 \in M \). We choose \( f \) such that \( \mathcal{F} f = \chi_{e_0} \), where \( \chi_{e_0} \) is the indicator function of \( e_0 \). By Theorem 2.10 we obtain
\[
(4.1) \quad \|T \lambda f\|_{L^q,\tau(\mathbb{R})} = \|\mathcal{F}^{-1} \lambda \mathcal{F} f\|_{L^q,\tau(\mathbb{R})} \gtrsim \|\lambda \mathcal{F} f\|_{N_{q',\tau}(M)} \gtrsim \|\lambda \mathcal{F} f\|_{N_{q',\infty}(M)},
\]
where \( N_{p,q}(M) = N_{p,q}(\mathbb{R}, M) \); see Definition 2.4. By Proposition 2.8, we obtain
\[
\|\lambda \mathcal{F} f\|_{N_{q',\infty}(M)} = \sup_{e \in M} \frac{1}{|e|^{1/q}} \left| \int_{e} \lambda(\xi) \mathcal{F} f(\xi) d\xi \right| \geq \frac{1}{|e_0|^{1/q}} \left| \int_{e_0} \lambda(\xi) \mathcal{F} f(\xi) d\xi \right| = \frac{1}{|e_0|^{1/q}} \left| \int_{e_0} \lambda(\xi) d\xi \right|,
\]
so that
\[
(4.2) \quad \|T_\lambda f\|_{L_q,\tau(\mathbb{R})} \geq \frac{1}{|e_0|^{1/q}} \left| \int_{e_0} \lambda(\xi) d\xi \right|.
\]
Since \( \chi_{e_0} \) is a monotone even function (modulo shifting), by Theorem 2.2 in [30], we obtain
\[
\|f\|_{L_p(\mathbb{R})} \approx \|\chi_{e_0}\|_{L_{q',p}(\mathbb{R})} = \left( \int_{0}^{\infty} \left( \frac{1}{t^p} \chi_{e_0}^*(t) \right)^{1/p} dt \right)^{1/p} = \left( \int_{0}^{\frac{1}{|e_0|^2}} \frac{t^{\frac{p}{q}} - 1}{t^{\frac{p}{q}}} dt \right)^{1/p} = |e_0|^\frac{1}{q}.
\]
Therefore, (4.2) implies
\[
\frac{1}{|e_0|^{1/r}} \left| \int_{e_0} \lambda(\xi) d\xi \right| \lesssim \|T_\lambda\|_{L_p \rightarrow L_{q,\tau}}.
\]
Recalling that this is true for an arbitrary \( e_0 \in M \), we complete the proof. \( \Box \)

Now, we prove similar result, but for Fourier series multipliers:

**Theorem 4.2.** Let \( 1 < p \leq 2 \leq q < \infty \) and \( 1/r = 1/p - 1/q \). Let \( 0 < \tau \leq \infty \) and \( W \) be the set of all finite intervals in \( \mathbb{Z} \). Then, for any sequence of complex numbers \( \lambda = \{\lambda_k\}_{k \in \mathbb{Z}} \), it follows
\[
\sup_{e \in W} \frac{1}{|e|^{1/r}} \left| \sum_{k \in e} \lambda_k \right| \lesssim \|T_\lambda\|_{L_p(0,1) \rightarrow L_{q,\tau}(0,1)},
\]

**Proof.** Let \( e_0 \) be an arbitrary interval on \( \mathbb{Z} \), that is \( e_0 \in W \). Then we choose \( f \) with \( f \sim \sum_{k \in \mathbb{Z}} a_k e^{2\pi ikx} \) such that
\[
a_k = \begin{cases} 
1 & \text{for } k \in e_0, \\
0 & \text{for } k \notin e_0.
\end{cases}
\]
By Theorems 2.10 and 2.7, we estimate
\[
\|T_\lambda f\|_{L_q,\tau(0,1)} \sim \|\lambda a\|_{n_{q',\tau}(W)} \gtrsim \|\lambda a\|_{n_{q',\infty}(W)},
\]
where \( n_{q',\tau}(W) = n_{q',\tau}([0,1],W) \) and \( n_{q',\infty}(W) = n_{q',\tau}([0,1],W) \); see Definition 2.4. From Proposition 2.8, it follows
\[
\|\lambda a\|_{n_{q',\infty}(W)} = \sup_{e \in W} \frac{1}{|e|^{1/q}} \left| \sum_{k \in e} \lambda_k a_k \right| \gtrsim \frac{1}{|e_0|^{1/q}} \left| \sum_{k \in e_0} \lambda_k a_k \right|.
\]
Recalling the choice of the sequence \( a = \{a_k\}_{k \in \mathbb{Z}} \), we derive that
\[
(4.3) \quad \|T_\lambda f\|_{L_q,\tau} \geq \frac{1}{|e_0|^{1/q}} \left| \sum_{k \in e_0} \lambda_k \right|.
\]
Since \( a \) is non-increasing and vanishing at infinity, Theorem 4 in [29] gives
\[
\|f\|_{L_p(0,1)} \approx \|a\|_{L^p_{\text{b}}(\mathbb{Z})} \left( \sum_{k=0}^{[e_0]} \left( k^{1/p'} a_k^p \right)^{1/p} \right)^{1/p} \approx \left( \sum_{k=0}^{[e_0]} k^{p-2} \right)^{1/p} \approx |e_0|^{1/p'}.
\]
This and (4.3) give
\[
\|T_mf\|_{L_{p',\tau}(0,1)} \gtrsim \frac{1}{|e_0|^{1/q'}} \left| \sum_{m \in e_0} \lambda_m \right| = \frac{1}{|e_0|^{1/r'}} \left| \sum_{m \in e_0} \lambda_m \right|.
\]
Since, \( e_0 \) was an arbitrary interval, this finishes the proof. \( \square \)

**Remark 4.3.** In case \( \tau = q \), Theorem 4.2 was obtained in [25]. While Theorem 4.1 was obtained only for the case of non-negative symbols, see [20].

## 5. Criteria for the \( L_p - L_q \) boundedness

In this section, we introduce the notion of \( M \)-generalized monotone functions and sequences. For the corresponding Fourier multipliers, we obtain criteria for \( L_p \to L_q \) boundedness.

**Definition 5.1.** Let \( M \) be a set of all finite intervals on \( \mathbb{R} \). We say that \( f : \mathbb{R} \to \mathbb{C} \) is a \( M \)-generalized monotone function if
\[
f^*(t) \leq C \hat{f}(t, M)
\]
holds for some \( C > 0 \) depending on \( f \).

Let \( W \) be a set of all finite intervals in \( \mathbb{Z} \). We say that a sequence of complex numbers \( \{a_k\}_{k \in \mathbb{Z}} \) is \( M \)-generalized monotone if
\[
a_k^* \leq C \tilde{a}_k(W).
\]

This is the simplified version of definition needed for the purpose of this work. For a more general setting we define it as follows:

**Definition 5.2.** Let \( (\Omega, \mu) \) be a measurable space and \( M \) be a set of measurable subsets of \( \Omega \) with finite positive measures. We say that \( f : \Omega \to \mathbb{C} \) is a \( M \)-generalized monotone function if
\[
f^*(t) \leq C \hat{f}(t, M)
\]
holds for some \( C > 0 \) depending on \( f \).

**Theorem 5.3.** Let \( M \) be a set of all finite intervals on \( \mathbb{R} \), \( 1 < p \leq 2 \leq q < \infty \), and \( 1/r = 1/p - 1/q \). Then a \( M \)-generalized monotone function \( \lambda : \mathbb{R} \to \mathbb{C} \) belongs to \( M^q_p \) if and only if
\[
\sup_{e \in M} \frac{1}{|e|^{1/r'}} \left| \int_e \lambda(\xi) d\xi \right| < \infty.
\]

**Proof.** If \( \lambda \in M^q_p \), then Theorem 4.1 gives (5.1). To prove the converse, it suffices to show that the upper bound in Theorem 1.1(i) is finite. To do this, we estimate
\[
\sup_{k \in \mathbb{Z}} \sup_{e \subset \Delta_k} \frac{1}{|e|^{1/r'}} \left| \int_e \lambda(\xi) d\xi \right| \leq \sup_{k \in \mathbb{Z}} \sup_{e \subset \Delta_k} \frac{1}{|e|^{1/r'}} \int_e |\lambda(\xi)| d\xi \leq \sup_{k \in \mathbb{Z}} \sup_{e \subset \Delta_k} \frac{1}{|e|^{1/r'}} \int_0^{e|} \lambda^*(s) ds.
\]
Therefore, since $\lambda$ is a $M$-generalized monotone function, we obtain
\[
\sup_{k \in \mathbb{Z}} \sup_{e \in \Delta_k} \frac{1}{|e|^{1/r'}} \left| \int_0^1 \lambda(\xi) d\xi \right| \leq \sup_{k \in \mathbb{Z}} \sup_{e \in \Delta_k} \frac{1}{|e|^{1/r'}} \int_0^1 \lambda(t, M) dt
\]
\[
\leq \sup_{k \in \mathbb{Z}} \sup_{e \in \Delta_k} \frac{1}{|e|^{1/r'}} \int_0^{|e|} \sup_{e' \in \mathbb{Z}, |e'| \geq |e|} \left| \int_{e'} \lambda(\xi) d\xi \right| dt
\]
\[
\leq \sup_{k \in \mathbb{Z}} \sup_{e \in \Delta_k} \frac{1}{|e|^{1/r'}} \int_0^{|e|} \frac{1}{t^{1-1/r'}} dt \sup_{e' \in \mathbb{Z}} \left| \int_{e'} \lambda(\xi) d\xi \right|
\]
\[
\leq \sup_{e \in \mathbb{Z}} \sup_{e \in \Delta_k} \frac{1}{|e|^{1/r'}} \int_e \lambda(\xi) d\xi
\]
\[
< \infty.
\]
Then, by Theorem 3.2 it follows that $\lambda \in M^q_p$.

Similar result holds for Fourier series multipliers:

**Theorem 5.4.** Let $W$ be a set of all finite intervals on $\mathbb{Z}$, $1 < p \leq 2 \leq q < \infty$, and $1/r = 1/p - 1/q$. Then a $M$-generalized monotone sequence $\lambda = \{\lambda_k\}_{k \in \mathbb{Z}}$ belongs to $m^q_p$ if and only if
\[
\sup_{e \in W} \frac{1}{|e|^{1/r'}} \left| \sum_{j \in e} a_j \right| < \infty. 
\]

**Proof.** Due to Theorems 1.1 and 4.2 it is suffices to prove that the upper bound in Theorem 1.1 (ii) is finite if (5.2) holds. We estimate
\[
\sup_{k \in \mathbb{Z}} \sup_{e \in \Delta_k} \frac{1}{|e|^{1/r'}} \left| \sum_{m \in e} \lambda_m \right| \leq \sup_{k \in \mathbb{Z}} \sup_{e \in \Delta_k} \frac{1}{|e|^{1/r'}} \sum_{j=1}^{|e|} \lambda^*_j
\]

Since $\lambda$ is a $M$-generalized monotone sequence, we obtain
\[
\sup_{k \in \mathbb{Z}} \sup_{e \in \Delta_k} \frac{1}{|e|^{1/r'}} \left| \sum_{m \in e} \lambda_m \right| \leq \sup_{k \in \mathbb{Z}} \sup_{e \in \Delta_k} \frac{1}{|e|^{1/r'}} \sum_{j=1}^{|e|} \sup_{e_0 \in W, |e_0| \geq j} \frac{1}{|e_0|} \sum_{j \in e_0} \lambda_j \leq \sup_{e \in W} \frac{1}{|e|^{1/r'}} \left| \sum_{m \in e} \lambda_m \right|.
\]

This completes the proof.

6. **Examples and corollaries**

In the final section, as a corollary, we will prove Theorem 1.1. We will demonstrate that our results are strictly stronger than Hörmander’s and Lizorkin’s multiplier theorems.

**Proof of Theorem 1.1** The upper bounds in Theorem 1.1 follows from Theorems 3.1 and 3.2. Since $M_k \subset M$ and $W_k \subset W$, by choosing $\tau = q$ in Theorems 3.1 and 4.2 we obtain the lower bounds.

Next, we obtain the following known result.

**Corollary 6.1.**

(i) Let $\lambda$ be a measurable function on $\mathbb{R}$, then
\[
\|\lambda\|_{M^2_p} \approx \|\lambda\|_{L^\infty(\mathbb{R})},
\]
that is $\lambda \in M^2_p$ if and only if $\lambda \in L^\infty(\mathbb{R})$. 

(ii) Let $\lambda$ be a sequence of complex numbers, then

$$\|\lambda\|_{m_2^2} \approx \|\lambda\|_{l_\infty(Z)},$$

that is $\lambda \in m_2^2$ if and only if $\lambda \in l_\infty(Z)$.

**Proof.** The first part follows from Theorems 3.1 and 4.1, while the second one follows from Theorems 3.2 and 4.2. □

For the Fourier series multipliers, we also have the following result:

**Corollary 6.2.** Let $1 < \tau < \infty$, then

$$\|\lambda\|_{m_\tau^\tau} \lesssim \sup_{k \in \mathbb{N}_0} \|\lambda\|_{l_{2(\tau-2)/\tau}}(\delta_k)$$

for a sequence of complex numbers $\{\lambda_k\}_{k \in \mathbb{Z}}$.

**Proof.** The statement follows from Theorem 3.2, by choosing $(p, q) = (2, \tau)$ if $2 \leq \tau$, and choosing $(p, q) = (\tau, 2)$ if $\tau \leq 2$. □

Let us note that Corollary 6.2 and Marcinkiewicz theorem are not equivalent. For the right-hand side of (6.1) to be finite it is necessary that $\lambda \in l_\infty$, which is not needed for Marcinkiewicz theorem. Conversely, if we choose $\lambda = \{\lambda_k\}_{k \in \mathbb{Z}}$ such that $\lambda_0 = 0$ and

$$\lambda_{\pm k} = (-1)^k \frac{1}{k^{\frac{1}{2(\tau-2)/\tau}}}$$

for $k \in \mathbb{N}$.

Then the right-hand side of (6.1) is bounded by 1, while

$$\sup_{n \in \mathbb{N}_0} \sum_{k=2^n+1}^{2^{n+1}-1} |\lambda_k - \lambda_{k-1}| = \infty.$$

Next, we show that the sufficient conditions in Theorem 1.1 are strictly weaker than (1.1) and (1.2).

Let $\chi_{\Delta_k}$ be the indicator function of $\Delta_k$. Then, for the distribution functions of $\lambda \chi_{\Delta_k}$ and $\lambda$, it follows that $d_{\lambda \chi_{\Delta_k}}(\sigma) \leq d_{\lambda}(\sigma)$, and hence,

$$(\lambda \chi_{\Delta_k})^*(t) \leq \lambda^*(t), \quad \text{for } t > 0, \ k \in \mathbb{Z}.$$

Therefore

$$\sup_{k \in \mathbb{Z}} \sup_{c \in \Delta_k} \frac{1}{c^{1/r'}} \left| \int_{c} \lambda(\xi) d\xi \right| \approx \sup_{k \in \mathbb{Z}} \|\lambda\|_{L_{r,\infty}(\Delta_k)} \leq \|\lambda\|_{L_{r,\infty}(\mathbb{R})},$$

so that Theorem 1.1(i) implies the Hörmander’s theorem for Fourier transform multipliers.

Let us consider the following example:

**Example 6.3.** Let $r > 0$ and $\lambda$ be an even function such that

$$\lambda(\xi) = \frac{1}{(\xi - 2^k)^2}, \quad \text{for } \xi \in (2^k, 2^{k+1}), \ k \in \mathbb{Z}.$$

Since $d_{\lambda}(\sigma) = \infty$ for $0 < \sigma < \infty$, we obtain that $\|\lambda\|_{L_{r,\infty}(\mathbb{R})} = \infty$, so that we cannot apply Hörmander’s theorem. However, one can check that

$$d_{\lambda \chi_{\Delta_k}}(\sigma) = \begin{cases} 2^k & \sigma \leq (\frac{1}{2^k})^{1/r}, \\ (\frac{1}{2^k})^r & \sigma > (\frac{1}{2^k})^{1/r}, \end{cases} \quad (\lambda \chi_{\Delta_k})^*(t) = \begin{cases} (\frac{1}{2})^{1/r} & t \leq 2^k, \\ 0 & t > 2^k. \end{cases}$$

Therefore
\[ \|\lambda\|_{L_r(\Delta_{\kappa})} = \sup_{t>0} t^{1/r} (\lambda \chi_{\Delta_{\kappa}})^\ast (t) = \sup_{0<t\leq 2^k} t^{1/r} t^{-1/r} = 1, \]
and hence, by Theorem 1.1(i), it follows that \( \lambda \in M_p^q \).

Similarly, one can show that
\[ \sup_{k \in \mathbb{N}_0} \sup_{\epsilon \in \delta_k} \sum_{m \in \epsilon} |\lambda_m| \approx \sup_{k \in \mathbb{N}_0} \|\lambda\|_{L_r(\delta_k)} \lesssim \|\lambda\|_{L_r(\mathbb{Z})}, \]
so that Theorem 1.1(ii) implies the Hörmander theorem for Fourier series multipliers.

Let us consider the example:

**Example 6.4.** Let \( r > 0 \) and \( \lambda = \{\lambda_k\}_{k \in \mathbb{Z}} \) be the sequence such that
\[ \lambda_j = \begin{cases} \frac{1}{j^{1-2^j}}, & \text{for } j \in \delta_k \text{ and } k \in \mathbb{N}, \\ 0, & \text{if } j \leq 0. \end{cases} \]
Since \( \lambda_j^\ast = 1 \) for \( j \in \mathbb{N} \), we derive
\[ \|\lambda\|_{L_r(\mathbb{Z})} = \sup_{k \geq 0} k^{1/r} \lambda_k^\ast = \infty, \]
while
\[ \|\lambda\|_{L_r(\delta_k)} = 1 < \infty. \]
Therefore, we can not apply the Hörmander’s theorem, however, we can apply Theorem 1.1(ii) to see that \( \lambda \in m_p^q \).

Further, we check that the sufficient conditions in Theorem 1.2 are strictly weaker than (1.3) and (1.4). Indeed, we estimate
\[
2^\frac{k}{r} \int_{\Delta_k} |\lambda'(\xi)| d\xi = 2^\frac{k}{r} \int_{\Delta_k} |\lambda'(\xi)| |\xi|^{\frac{1}{r}+1} |\xi|^{-\frac{1}{r}+1} d\xi 
\leq \sup_{\xi \in \mathbb{R}} |\lambda'(\xi)| |\xi|^{\frac{1}{r}+1} 2^\frac{k}{r} \int_{\Delta_k} |\xi|^{-\frac{1}{r}-1} d\xi
\leq \sup_{\xi \in \mathbb{R}} |\lambda'(\xi)| |\xi|^{\frac{1}{r}+1} r \left( 1 - 2^{-\frac{1}{r}} \right).
\]
Therefore, Theorem 1.2(ii) gives (1.3).

We consider the following example:

**Example 6.5.** Let \( 0 < \alpha < 1 \) and \( \lambda \) be an even function on \( \mathbb{R} \) such that
\[ \lambda(\xi) = \begin{cases} (2-x)^{\alpha}, & x \leq 2, \\ 0, & x > 2. \end{cases} \]
Note that \( \lambda'(\xi) = \alpha (2-x)^{\alpha-1} \) on \([0,2)\), which is not bounded at \( \xi = 2 \). Therefore, the right-hand side of (1.3) is infinite. However, since the singularity of \( \lambda' \) at \( \xi = 2 \) is integrable, we conclude that
\[ \sup_{k \in \mathbb{Z}} 2^\frac{k}{r} \int_{\Delta_k} |\lambda'(\xi)| d\xi < \infty. \]
Moreover, \( \lambda \) is absolutely continuous on \((-\infty,0]\) and \([0,\infty)\), and \( |\lambda(\xi)| \to 0 \) as \( |\xi| \to \infty \). Therefore, by Theorem 1.2(i), \( \lambda \in M_p^q \) for \( 1 < p < q < \infty \).
We repeat these arguments for the second part. We write
\[ 2^k \sum_{m=2^k}^{2^{k+1}-1} (|\lambda_m - \lambda_{m+1}| + |\lambda_m - \lambda_{m-1}|) \]
\[ = 2^k \sum_{m=2^k}^{2^{k+1}-1} (|\lambda_m - \lambda_{m+1}| + |\lambda_m - \lambda_{m-1}|) |m-1|^{1+\frac{1}{r}} |m-1|^{-\frac{1}{r}} , \]
so that
\[ 2^k \sum_{m=2^k}^{2^{k+1}-1} (|\lambda_m - \lambda_{m+1}| + |\lambda_m - \lambda_{m-1}|) \]
\[ \lesssim \sup_{n \in \mathbb{Z}} |\lambda_n - \lambda_{n+1}| n^{1+\frac{1}{r}} \sum_{m=2^k}^{2^{k+1}-1} |m-1|^{-\frac{1}{r}} \lesssim \sup_{n \in \mathbb{Z}} |\lambda_n - \lambda_{n+1}| n^{1+\frac{1}{r}}. \]

The following example demonstrates that the converse inequality does not hold:

Example 6.6. Let \( 1 < p < q < \infty \), \( 1/r = 1/p - 1/q \), and
\[ \gamma = \sum_{j=0}^{\infty} \left( \frac{1}{2^{1/r}} \right)^j . \]
Then, we define recursively
\[ \lambda_0 = \gamma , \]
\[ \lambda_{2^k} = \cdots = \lambda_{2^{k+1}-1} = \lambda_{2^{k}-1} - \left( \frac{1}{2^{1/r}} \right)^k \]
for \( k \in \mathbb{N}_0 \). We also set \( \lambda_{-k} = \lambda_k \) and compute
\[ (2^k - 1)^{\frac{1}{r}} |\lambda_{2^{k-1}} - \lambda_{2^k}| = (2^k - 1)^{\frac{1}{r}} \left( \frac{1}{2^{1/r}} \right)^k \left( \frac{2^k - 1}{2^k} \right)^{1/r} = (2^k - 1) \]
which is unbounded as \( k \to \infty \). Hence, the Lizorkin’s theorem is not applicable. However, we can apply Theorem 1.2. Indeed, by definition of \( \gamma \) and \( \lambda_j \), we obtain
\[ \lambda_{2^k} = \cdots = \lambda_{2^{k+1}-1} = \gamma - \sum_{j=0}^{k} \left( \frac{1}{2^{1/r}} \right)^j \to 0 \]
as \( k \to \infty \). Further, we compute
\[ 2^k \sum_{j=2^k}^{2^{k+1}-1} |\lambda_j - \lambda_{j-1}| = 2^k |\lambda_{2^k} - \lambda_{2^{k-1}}| = 2^k \left( \frac{1}{2^{1/r}} \right)^k = 1. \]
Therefore, by Theorem 1.2, it follows that \( \lambda \in m^q_p \).

Finally, we note that Theorem 1.2 is at least complementary to [31] Theorem 2] and [27] Theorem 1.3], respectively. To see this, consider the following examples.

Example 6.7. For \( 0 < \gamma < 1 \), define a function
\[ \lambda(x) := \begin{cases} 2^{\frac{k}{r}} \left( 2 - |(2^k + 2) - x| \right)^\gamma & x \in [2^k, 2^k + 4] \text{ and } k \geq 2, \\ 0 & \text{otherwise}. \end{cases} \]
First, we check that
\[
\sup_{k \in \mathbb{Z}} 2^k \int_{\Delta_k} |\lambda'(x)|dx = \sup_{k \in \mathbb{Z}} 2^{k+4} \int_{2^k+2}^{2^k+4} (2^k + 4 - x)^{\gamma-1}dx < \infty.
\]
Therefore, by Theorem 1.2, \(\lambda\) represents \(L_p \to L_q\) Fourier multiplier.

Further, for \(\alpha < 1 - 1/r\) and \(\beta = \alpha + 1/r\), the inequality holds
\[
|\lambda'(x)x^\beta| > g(x),
\]
where
\[
g(x) = \begin{cases} 
\gamma 2^{-\frac{\beta}{2}} (2 - |(2^k + 2) - x|)^{\gamma-1} 2^\beta, & x \in [2^k, 2^k+4] \text{ and } k \geq 2, \\
0, & \text{otherwise}.
\end{cases}
\]
Note that \(2^{-k/r}2^\beta k = 2^\alpha k\). One can check that \(g^* = \infty\), therefore, \(\lambda\) does not satisfy conditions of [31, Theorem 2].

Consider the following example:

**Example 6.8.** Let
\[
\lambda_m = \begin{cases} 
2^{-\frac{k}{r}}, & m = 2^k + 1 \text{ and } k \geq 2, \\
0, & \text{otherwise}.
\end{cases}
\]
Then
\[
\sup_{k \in \mathbb{N}_0} 2^k \sum_{m=2^k}^{2^{k+1}-1} |\lambda_m - \lambda_{m-1}| = 2 < \infty.
\]
Therefore, by Theorem 1.2 \(\lambda\) represents \(L_p \to L_q\) Fourier multiplier. However, this can not be seen from [27, Theorem 1.3]. Indeed, let \(\alpha < 1 - 1/r\) and \(\beta = \alpha + 1/r\). Denote \(\eta_k := k^\beta |\lambda_k - \lambda_{k+1}|\).

One can check that \(\eta_{2k} = 2^\alpha k\) for \(k \geq 2\), therefore, \(\eta_k^* = \infty\), so that \(\lambda\) dose not satisfies condition of [27, Theorem 1.3].

**Remark 6.9.** Note that [31, Theorem 2] and [27, Theorem 1.3] are stronger than corresponding Hörmander theorems. In particular, Examples 6.7 and 6.8 do not satisfy (1.1) and (1.2), respectively.

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