Dual of Codes over Finite Quotients of Polynomial Rings

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Abstract

Let \( A = \mathbb{F}[x]/(f(x)) \), where \( f(x) \) is a monic polynomial over a finite field \( \mathbb{F} \). In this paper, we study the relation between \( A \)-codes and their duals. In particular, we state a counterexample and a correction to a theorem of Berger and El Amrani (Codes over finite quotients of polynomial rings, Finite Fields Appl. 25 (2014), 165–181) and present an efficient algorithm to find a system of generators for the dual of a given \( A \)-code. Also we characterize self-dual \( A \)-codes of length 2 and investigate when the \( \mathbb{F} \)-dual of \( A \)-codes are \( A \)-codes.

Keywords: Algebraic coding, Dual of a code, Basis of divisors, Polynomial ring.

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1 Introduction

Throughout this paper \( A = \mathbb{F}[x]/(f(x)) \), where \( f(x) \) is a monic polynomial over a finite field \( \mathbb{F} \). Moreover, \( \deg(f) = m \) and \( |\mathbb{F}| = q \). We consider elements of \( A \) as polynomials of degree \( < m \) where the arithmetic is done modulo \( f(x) \). By a linear \( A \)-code (an \( A \)-code, for short) of length \( l \) we mean an \( A \)-submodule of \( A^l \).

In the case \( f(x) = x^m - 1 \) and \( l = 1 \), \( A \)-codes are the well-known cyclic \( q \)-ary codes. Also if \( l > 1 \) with \( f(x) = x^m - 1 \), then \( A \)-codes represent quasi-cyclic codes over \( \mathbb{F} \) which have recently gained great attention (see, for example [1, 5, 8, 10, 11, 13]). Also in the case that \( f(x) \) is a power of an irreducible polynomial, then \( A \) is a finite chain ring and codes over such rings have attracted a lot of researchers (see for example [2, 4, 14]).

In [11], a canonical generator matrix for quasi-cyclic codes is given, when these codes are viewed as \( A \)-codes with \( f(x) = x^m - 1 \). In [3] these results are generalized to arbitrary \( A \)-codes. Let \( C^\perp = \{ (a_1, \ldots, a_l) \in A^l | \forall c \in C \sum_{i=1}^l a_ic_i = 0 \} \) be the dual of an \( A \)-code \( C \). Section 2.6 of [3] states how to compute a system of generators of \( C^\perp \). In Section 2, we will show that the main theorem of [3] Section 2.6 is not correct and we state a correction of this theorem. Also we present an efficient algorithm to find a generator matrix for \( C^\perp \) (that is, a matrix, rows of which generate \( C^\perp \) as an \( A \)-module).

In Section 3, we apply our results to find all self-dual \( A \)-codes with length \( \leq 2 \) and self-dual \( A \)-codes which have a basis of divisors containing just one element.

Every \( A \)-code \( C \) of length \( l \) could be seen as an \( \mathbb{F} \)-code of length \( ml \) (by replacing \( a(x) \in A \) with the sequence of its coefficients). Therefore we can form the \( \mathbb{F} \)-dual of \( C \). The \( \mathbb{F} \)-dual of
Let \( k \) be the smallest integer such that \( u_i \neq 0 \) and \( L_{\text{coef}}(u) = u L_{\text{ind}}(u) \) is called the leading coefficient of \( u \) (we set \( L_{\text{ind}}(0) = \infty \)). Also by \( L_{\text{ind}}(C) \) we mean \( \min \{ L_{\text{ind}}(u) | u \in C \} \) and \( L_{\text{coef}}(C) \) is the single monic polynomial \( g(x) \) with the minimum degree such that there is a \( c \in C \) with \( L_{\text{ind}}(c) = L_{\text{ind}}(C) \) and \( L_{\text{coef}}(c) = g(x) \). An element \( c \in C \) satisfying this condition is called a leading element of \( C \).

Recursively set \( C^{(1)} = C \) and if \( L_{\text{ind}}(C^{(n)}) \leq l \), then
\[
C^{(n+1)} = \{ c \in C^{(n)} | L_{\text{ind}}(c) > L_{\text{ind}}(C^{(n)}) \}.
\]

Let \( k \) be largest integer such that \( C^{(k)} \neq \{ \} \) and assume that for \( 1 \leq j \leq k \), \( g^{(j)} \) is a leading element of \( C^{(j)} \). Then by Theorem 1 and Proposition 2 of \( \cite{3} \), \( C \) is generated by \( B = (g^{(1)}, \ldots, g^{(k)}) \) (as an \( A \)-module) and \( k \) and \( \deg(L_{\text{coef}}(g^{(i)})) \) are independent of the choice of \( g^{(i)} \)s. Also \( |C| = q^a \) where \( a = km - \sum_{i=1}^{k} \deg(L_{\text{coef}}(g^{(i)})) \). Any \( B \) as above is called a basis of divisors of \( C \).

Now let \( G \) be the matrix whose \( i \)-th row is \( g^{(i)} \). Suppose that \( g_{i,j} \) is the leading coefficient of the \( i \)-th row of \( G \). If \( G \) has the property that \( \deg(g_{t,j}) < \deg(g_{i,j}) \) for all \( 1 \leq i \leq k \) and \( t < i \), then \( G \) is called the canonical generator matrix (CGM, for short) of \( C \) and \( B \) is called the canonical basis of divisors of \( C \). In \( \cite{3} \) Theorem 2 it is shown that every \( A \)-code has a unique CGM. Also they present algorithms to find a basis of divisors and the CGM of a a given \( A \)-code.

**Example 1.1.** Suppose that \( f(x) = x(x^2 + x + 1) \) and \( C \) is the submodule of \( A^3 \) generated by \( g = (x^2, 0, x^2 + 1) \). Then \( L_{\text{ind}}(C) = L_{\text{ind}}(g) = 1 \). By definition \( C^{(1)} = C \). To compute \( C^{(2)} \) we should find all elements of \( C \) whose leading index is greater that \( L_{\text{ind}}(C^{(1)}) = 1 \), that is, all elements of \( C \) with zero on the first component. Since every element \( c \) of \( C \) is of the form \( c = a(x)g \) for some \( a(x) \in A \), we see that \( c_1 = 0 \Leftrightarrow x^2 + x + 1 | a(x) \Leftrightarrow c = a'(x)((x^2 + x + 1)g) \) for some \( a'(x) \in A \) which is equivalent to \( c = a'(x)(0,0,x^2 + x + 1) \). Therefore, \( C^{(2)} \) is the \( A \)-code generated by \( g^{(2)} = (0,0,x^2 + x + 1) \) and \( L_{\text{ind}}(C^{(2)}) = 3 \). Also the only element in \( C^{(2)} \) whose leading index is \( > 3 \), is zero, hence \( C^{(3)} = 0 \) and \( k = 2 \).

It is clear that the (only) leading element of \( C^{(2)} \) is \( g^{(2)} \). But as \( x^2 \nmid f(x) \), \( g \) is not a leading element of \( C^{(1)} \). Indeed, since \( \gcd(x^2, f(x)) = x \) and \( (x-1)g = (x,0,-1) \), we deduce that a leading element of \( C^{(1)} \) is \( g^{(1)} = (x,0,-1) \) (note that this element is not unique, for example \( g^{(1)} + g^{(2)} \) is another leading element of \( C^{(1)} \)). Therefore \( (g^{(1)}, g^{(2)}) \) is a basis of divisors of \( C \) and it follows that \( \begin{pmatrix} x & 0 & -1 \\ 0 & 0 & x^2 + x + 1 \end{pmatrix} \) is the CGM of \( C \). Also \( \dim_{\mathbb{F}}(C) = 6 - \deg(x) - \deg(x^2 + x + 1) = 3 \).
2 A Generator Matrix for the Dual of an A-Code

We start by presenting a counterexample of Theorem 3 and stating a correction of this theorem. Then we use this correction to give an algorithm which generates a generator matrix for the dual of an A-code.

Throughout this section, without any further mention, we assume that \( C \) is an A-code of length \( l \) and that \( g^{(1)} = (g_{1,1}(x), \ldots, g_{1,l}(x)) \) is the first element of its canonical basis of divisors. Also we let \( C' \) be the punctured code of \( C^{(2)} \) in the first position and assume that \( G' \) is the canonical generator matrix of \( C' \). Note that \( G' \) is the matrix obtained by deleting the first row and column of the canonical generator of \( C \). The following theorem is claimed to be proved in [3].

**Incorrect Theorem 2.1** ([3, Theorem 3]). Suppose that \( L_{\text{ind}}(C) = 1 \) and \( h_{1,1}(x) = \frac{f(x)}{g_{1,1}(x)} \) (mod \( f(x) \)). Let \( H' \) be a generator matrix of \( C'^\perp \). Then

\[
H = \begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & & H' & \\
\vdots & & & \\
0 & & &
\end{pmatrix}
\begin{pmatrix}
h_{1,1} & 0 & \cdots & 0 \\
- g_{1,2} & g_{1,1} & \cdots & \\
- g_{1,3} & 0 & g_{1,1} & \cdots \\
\vdots & \vdots & \ddots & 0 \\
- g_{1,l} & 0 & \cdots & g_{1,1}
\end{pmatrix}
\]

is a generator matrix for \( C'^\perp \).

To present a counterexample of (2.1), we need the following result. We say an element of \( A^l \) is monic when its leading coefficient is monic.

**Proposition 2.2.** Let \( G \) be a \( k \times l \) generator matrix for an A-code \( C \). Suppose that \( g^{(i)} = \) the \( i \)-th row of \( G \), is monic. Then \( (g^{(1)}, \ldots, g^{(k)}) \) is a basis of divisors of \( C \) if and only if the following hold.

(i) \( G \) is in echelon form.

(ii) \( L_{\text{cof}}(g^{(i)}) | f(x) \).

(iii) \( h_{i} g^{(i)} \) is an A-linear combination of \( g^{(i+1)}, \ldots, g^{(k)} \) where \( h_{i}(x) = \frac{f(x)}{L_{\text{cof}}(g^{(i)})} \).

Moreover, if we replace (iii) with (iii') below, the assertion remains valid.

(iii') \( \dim_F C = \sum_{i=1}^{k} m - \deg(L_{\text{cof}}(g^{(i)})) \).

**Proof.** (\( \Rightarrow \)): (ii) follows from the definition of \( C^{(i)} \) and \( g^{(i)} \). (iii) follows from the remarks above Definition 5 of [3, p. 170]. Let \( l_i = L_{\text{ind}}(g^{(i)}) \), then by the definition of \( h_{i} \), the \( l_i \)-th entry of \( h_{i} g^{(i)} \) is 0 in \( A \), hence \( h_{i} g^{(i)} \in C^{(i+1)} \) and (iii) follows.

(\( \Leftarrow \)): First we prove that for each \( i \), \( C^{(i)} \) is generated by the set \( B = \{ g^{(i)}, g^{(i+1)}, \ldots, g^{(k)} \} \).

We prove this for \( i = 2 \) and the rest follows by induction. Since \( G \) is in echelon form, \( \langle B \rangle \subseteq C^{(2)} \). Let \( g \) be an arbitrary element of \( C^{(2)} \). Then \( g = \sum_{i=1}^{k} a_i(x) g^{(i)} \). Suppose that \( a'_{1}(x) = a_{1}(x) \) (mod \( h_{1}(x) \)). Then by (iii), \( (a_{1}(x) - a'_{1}(x)) g^{(1)} \in \langle B \rangle \), hence

\[
a'_{1}(x) g^{(1)} + \langle B \rangle = g + \langle B \rangle \subseteq C^{(2)}.
\]
As \( g_{1,i} \) is monic (1,5 as in (\( \Rightarrow \)) and \( \deg(g'_i(x)) < \deg(h_i(x)) \), if \( a'_i(x) \neq 0 \), then \( a'_i(x)g_{1,i} \neq 0 \) and \( L_{\text{ind}}(a'_i(x)g(1)) = l_1 \) contradicting \( a'_i(x)g(1) \in C^{(2)} \). Therefore \( a'_i(x) = 0 \) and \( g \in \langle B \rangle \) as required.

Now it is clear that \( l_i = L_{\text{ind}}(C^{(i)}) \) for each \( i \) and if \( g \in C \) with \( L_{\text{ind}}(g) = l_i \), then \( L_{\text{coef}}(g) = a(x)L_{\text{coef}}(g^{(i)}) \) for some \( a(x) \in A \) with \( \deg(a(x)) < \deg(h_i(x)) \). Therefore \( g^{(i)} \) is a leading element of \( C^{(i)} \).

For the “moreover” statement, note that if \( (g^{(1)}, \ldots, g^{(k)}) \) is a basis of divisors, then by Proposition 2, (iii) holds. Conversely if (i) and (ii) hold, then clearly the combinations of the form \( \sum_{j=1}^{k} z_j g^{(j)} \) for \( z_j \in A \) with \( \deg(z_j) < \deg(h_j) = m - \deg(L_{\text{coef}}(g^{(j)})) \) are mutually different elements of \( C \). So if (iii) also is valid, then these combinations are all elements of \( C \).

In particular, \( c = h_i g^{(i)} \) could be written as such a combination and since for each \( j \leq L_{\text{ind}}(g^{(i)}) \) the \( j \)-th entry of \( c \) is zero, we get \( z_j = 0 \) for \( j \leq i \), as required.

**Example 2.3.** Let \( F = F_2, f(x) = x^2(x^3 + 1) \) and \( C \) be the \( A \)-code of length 3 which is generated by

\[
G = \begin{pmatrix}
x & x & 0 \\
0 & x^2 & 1 \\
0 & 0 & x^3 + 1
\end{pmatrix}.
\]

Using (2.2), we can see that \( G \) is the CGM of \( C \). As \( (x, x, 0) \in C \), we have \( L_{\text{ind}}(C) = 1 \) and the assumptions of Theorem (2.1) are valid. Also \( C' \) is generated by \((x^2, 1)\) and \((0, x^3 + 1)\). One can readily check that a generator matrix for \( C'^\perp \) is \( H' = (1 x^2) \). Thus if \( H \) is as in (2.1), then

\[
H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & x^2 \end{pmatrix} \times \begin{pmatrix} x(x^3 + 1) & 0 & 0 \\ x & x & 0 \\ 0 & 0 & x \end{pmatrix} = \begin{pmatrix} x(x^3 + 1) & 0 & 0 \\ x & x & x^3 \end{pmatrix}.
\]

Clearly \( u = (1, 1, x^2) \in C'^\perp \). But if \( u \) is a linear combination of the rows of \( H \), then for some \( a(x), b(x) \in F[x] \) we have \( a(x)x(x^3 + 1) + b(x)x = 1 \) \( \pmod{f(x)} \) which leads to \( x1 \), a contradiction. Thus \( H \) is not a generator matrix of \( C'^\perp \) and (2.1) is not correct.

To present the correct generator matrix for \( C'^\perp \) we need the following lemma.

**Lemma 2.4.** The code \( C'^\perp \) is the punctured code of \( C'^\perp \) in the first position.

**Proof.** We must show that for each \( c' \in C'^\perp \) there is a \( c_1 \in A \) such that \((c_1|c')\) (the concatenation of \( c_1 \) to \( c' \)) is an element of \( C'^\perp \). If \( L_{\text{ind}}(C) > 1 \), then any \( c_1 \in A \) works. Assume that \( L_{\text{ind}}(C) = 1 \) and let \( c = (a_1, \ldots, a_k) \in A^l \). Define \( \phi(c) = (a_2, \ldots, a_l) \). Then \( \phi : C'^\perp \to C'^\perp \) is a \( F \)-linear map and it suffices to show that \( \phi \) is onto.

Suppose that \((g^{(1)}, \ldots, g^{(k)})\) is the canonical basis of divisors of \( C \) and set \( r_i = \deg(L_{\text{coef}}(g^{(i)})) \). Note that \( \ker \phi = \{ (c_1, 0, 0, \ldots, 0) \in A^l | c_1 g_{1,1} = 0 \} \). If \( h_{1,1} = f/g_{1,1} \), then \( c_1 g_{1,1} = 0 \iff c_1 = c_1' h_{1,1} \) for some \( c_1' \in F[x] \) with \( \deg(c'_1) < \deg(g_{1,1}) \). Thus \( \dim_F \ker \phi = \deg(g_{1,1}) = r_1 \). According to \( \triangleright \), Proposition 2, \( \dim_F C'^\perp = lm - \dim_E C = lm - \sum_{i=1}^{k} (m - r_i) \). Similarly \( \dim_F C'^\perp = (l - 1)m - \sum_{i=2}^{k} (m - r_i) \). Hence

\[
\dim_F \phi(C'^\perp) = \dim_F C'^\perp - \dim_F \ker \phi = \dim_F C'^\perp
\]

and hence \( \phi \) is onto. \( \square \)
Theorem 2.5. Assume that \( L_{\text{ind}}(C) = 1 \), \( l > 1 \) and \( h_{1,1}(x) = \frac{f(x)}{g_{1,1}(x)} \) (mod \( f(x) \)). Let \( H' = (h'_{ij})_{2 \leq i \leq l} \) be a generator matrix of \( C'^\perp \). A generator matrix for \( C'^\perp \) is

\[
H = \begin{pmatrix}
    h_{1,1} & 0 & \cdots & 0 \\
    \alpha_2 & \alpha_3 & \cdots & H' \\
    \vdots & \vdots & \ddots & \vdots \\
    \alpha_k & \alpha_{k+1} & \cdots & \alpha_l
\end{pmatrix},
\]

where

\[
\alpha_i = -\sum_{j=2}^{l} \frac{h'_{ij}g_{1j}}{g_{1,1}} \quad (\text{mod } h_{1,1}).
\]

Proof. First note that by the previous lemma, for each \( 2 \leq i \leq k \), there is an \( \alpha_i \in A \), such that \( (a_i, h'_{i2}, h'_{i3}, \ldots, h'_{il}) \in C'^\perp \). This means that \( a_i g_{1,1} + \sum_{j=2}^{l} h'_{ij}g_{1j} = 0 \) in \( A \). Thus in \( F[x] \) we have \( g_{1,1} \sum_{j=2}^{l} h'_{ij}g_{1j} + b f \) for some \( b \in F[x] \). Therefore \( g_{1,1} \sum_{j=2}^{l} h'_{ij}g_{1j} \) in \( F[x] \) for each \( 2 \leq i \leq k \) and \( \alpha_i \) is well defined.

Denote the \( i \)-th row of \( H \) and \( H' \) by \( h^{(i)} \) and \( h'^{(i)} \), respectively. It is easy to see that \( h^{(i)} \)'s are in \( C'^\perp \). Conversely, let \( c = (c_1, \ldots, c_l) \in C'^\perp \), then \( c' = (c_2, \ldots, c_l) \in C'^\perp \). Thus for some \( \lambda_2, \ldots, \lambda_l \in A \), we have \( c' = \sum_{i=2}^{l} \lambda_i h^{(i)} \). Note that in \( A \), \( g_{1,1} \alpha_i = -\sum_{j=2}^{l} h'_{ij}g_{1j} \). So

\[
g_{1,1} \sum_{i=2}^{k} \lambda_i \alpha_i = -\sum_{i=2}^{k} \lambda_i \sum_{j=2}^{l} h'_{ij}g_{1j} = -\sum_{j=2}^{l} \left( \sum_{i=2}^{k} \lambda_i h'_{ij} \right) g_{1j} = -\sum_{j=2}^{l} c_j g_{1j} = c_1 g_{1,1},
\]

where the last equality follows from \( c \cdot g^{(i)} = 0 \). We conclude that \( g_{1,1} \left( c_1 - \sum_{i=2}^{k} \lambda_i \alpha_i \right) = 0 \), that is, \( h_{1,1} c_1 = \sum_{i=2}^{k} \lambda_i \alpha_i \), say \( \lambda_1 h_{1,1} = c_1 - \sum_{i=2}^{k} \lambda_i \alpha_i \). Consequently, \( c = \sum_{i=1}^{k} \lambda_i h^{(i)} \) and \( H \) is a generator matrix for \( C'^\perp \).

It should be noted that the above theorem is correct when \( C' = 0 \), in which case \( H' = I_{l-1 \times l-1} \). Also if \( l = 1 \), then clearly \( H = (h_{1,1}) \) is the generator matrix of \( C'^\perp \). If \( L_{\text{ind}}(C) > 1 \) and \( C_1 \) is obtained by puncturing \( C \) in the first position, then it can be seen that \( C'^\perp = A \oplus C_1'^\perp \) and hence \( H = \begin{pmatrix} 1 & 0 \\ 0 & H' \end{pmatrix} \) is a generator matrix for \( C'^\perp \), where \( H' \) is a generator matrix for \( C_1'^\perp \). Another fact about the previous theorem that should be mentioned is that if we compute \( \alpha_i \)’s modulo \( f(x) \) instead of \( h_{1,1}(x) \), by the same proof the statement still remains true. The difference is that in the current form we have \( \deg \alpha_i < \deg h_{1,1} \), which will be used in (2.6).

Using (2.5) we get the following recursive algorithm for computing a generator matrix of \( C'^\perp \). In each recursion of this algorithm the length of the input code is reduced by one. Also to construct the matrix \( H \) from \( H' \) (in line 19), since \( H' \) has dimensions \( \leq (l-1) \times (l-1) \), at most \( O(l^2) \) polynomial multiplications in \( A \) are performed. Clearly other computations are also bounded above by this bound. Hence totally, the running time of this algorithm is bounded above by \( O(l^3) \) multiplications in \( A \). Noting that the Gaussian elimination method for solving linear equations (even over a field) has the same time complexity \( O(l^3) \) arithmetic operations in the coefficient ring, we see that Algorithm [1] is in fact an efficient algorithm.

The matrix generated by this algorithm is not in the canonical form. But since we calculated the \( \alpha_i \)'s modulo \( h_{1,1}(x) \) instead of \( f(x) \) in (2.5), this matrix is very similar to the canonical form
Algorithm 1 \text{gen-mat-dual}(G) (Calculates a generator matrix of dual of an A-code $C$)

\textbf{Input}: A generator matrix $G_{k \times l}$ of $C$, rows of which form a basis of divisors for $C$

\textbf{Output}: A generator matrix $H$ of $C^\perp$

1: if the first column of $G$ is zero then
2: \hspace{1em} if $l=1$ then
3: \hspace{2em} return $H = (1)$
4: \hspace{1em} else
5: \hspace{2em} set $G'$ to be $G$ with the first column deleted
6: \hspace{2em} $H' = \text{gen-mat-dual}(G')$
7: \hspace{2em} return $H = \begin{pmatrix} 1 & 0 \\ 0 & H' \end{pmatrix}$
8: \hspace{1em} end if
9: \hspace{1em} else
10: \hspace{2em} if $l=1$ then
11: \hspace{3em} return $H = \begin{pmatrix} f(x) \\ g_{1,1}(x) \end{pmatrix} (\text{mod } f(x))$
12: \hspace{2em} else
13: \hspace{3em} if $k=1$ (that is, $C' = 0$) then
14: \hspace{4em} $H' = I_{(l-1) \times (l-1)}$
15: \hspace{3em} else
16: \hspace{4em} let $G'$ be $G$ with the first row and column deleted
17: \hspace{4em} $H' = \text{gen-mat-dual}(G')$
18: \hspace{3em} end if
19: \hspace{2em} construct and return $H$ as in (2.5)
20: \hspace{1em} end if
21: end if

— we should just delete the zero rows and then look at the rows and columns in the reverse order. More concretely, we have the following theorem.

\textbf{Theorem 2.6.} Let $H_{k \times l}$ be the generator matrix of $C^\perp$ calculated by Algorithm 1 after deleting the possible zero rows. Let $H^R$ be the $k \times l$ matrix with $h^R_{i,j} = h_{k-i+1,l-j+1}$. Then $H^R$ is the CGM of $C^{\perp R}$, the reciprocal dual of $C$ (that is, \{(c_l,\ldots,c_1)|(c_1,\ldots,c_l) \in C^\perp\}).

\textbf{Proof.} It follows easily by induction that if $h_{i,j_i}$ is the last nonzero entry on the $i$-th row of $H$, then $j_1 < j_2 < \cdots < j_k$ and $h_{i,j_i}$ divides $f$ and $\deg(h_{i,j_i}) < \deg(h_{i,j_{i-1}})$ for each $1 \leq i \leq l$ and $j_i < j \leq k$ (this is because os of (2.5) are calculated modulo $h_1$). As computed in the proof of (2.4), $\dim_F C^\perp - \dim_F C'^\perp = \deg(g_{1,1}) = m - \deg(h_{1,1})$, where $h_{1,1}$ is as in (2.5) (this is also true in the case that $\text{Lind}(C) > 1$ and $h_{1,1} = 1$). Thus again by induction we see that $\sum_{i=1}^k \deg(h_{i,j_i}) = \dim_F C^\perp$. Consequently, $H^R$ has properties [I], [II] and [III] of (2.2) and the result follows.

Note that $C^\perp$ and $C^{\perp R}$ are equivalent codes. So by finding parameters and properties of one of these codes, we have found those of the other one. We end this section with an example which applies Algorithm 1 on the code $C$ in Example 2.3.
Example 2.7. Consider the code $C$ generated by $G$ in Example (2.3) over an arbitrary field. If we run Algorithm 1 on $G$, it returns $H = \begin{pmatrix} x(x^3 + 1) & 0 & 0 \\ -x(x^2 + 1) & x^3 + 1 & 0 \\ 1 & -1 & x^2 \end{pmatrix}$ and hence $H^R = \begin{pmatrix} x^2 & -1 & 1 \\ 0 & x^3 + 1 & -x(x^3 + 1) \\ 0 & 0 & x(x^3 + 1) \end{pmatrix}$ is the CGM of $C^\perp R$.

3 Some Self-Dual $A$-Codes

Self-dual codes have both theoretical and practical importance (see for example [16]). A lot of effort has been devoted to classify and enumerate self-dual codes of small or moderate length over different rings. For example in [7], all ternary self-dual codes of length 24 are classified and in [6], it is proved that over any finite commutative Frobenius ring, self-dual codes exist. Also in [15], several classes of self-dual codes over rings $\mathbb{Z}_m$ are classified, including self-dual codes of length 4 and 8 over $\mathbb{Z}_{pq}$, where $p$ and $q$ are distinct primes. The reader is referred to [9], for a survey of classification and enumeration of self-dual codes of small length over $\mathbb{F}_2, \mathbb{F}_3, \mathbb{F}_4, \mathbb{Z}_4$ and $\mathbb{F}_2 + u\mathbb{F}_2$.

A source of importance for self-dual codes over finite fields is the MacWilliams identity (see for example [17, Section 5.4]). But $A = \mathbb{F}[x]/\langle f(x) \rangle$ is a finite principal ideal ring (PIR) and hence a finite Frobenius ring. So by [18, Section 8] MacWilliams identity holds for $A$-codes. Thus many of the important properties of self-dual codes over fields, hold for every $A$-code. Also note that since the weight enumerator of $C^\perp$ is the same as the weight enumerator of $C^\perp R$, the codes which equal their reciprocal dual are also of the same importance. In this section, we use (2.5) to find some self-dual $A$-codes. Our first result considers the case that bases of divisors of $C$ have just one element.

Theorem 3.1. Suppose that $C$ has a basis of divisors consisting of one element, say $g = (g_1, \ldots, g_l)$. Then $C$ is self-dual if and only if either $l = 1$ and $f = g_1^2$ in $\mathbb{F}[x]$ or $l = 2$, $g_1 = 1$ and $g_2 = -1$ in $A$. Also $C = C^\perp R$, if and only if one of the following hold:

(i) $l = 1$ and $f = g_1^2$ in $\mathbb{F}[x]$,

(ii) $l = 2$, $g_1 = 0$ and $g_2 = 1$,

(iii) $l = 2$, $g_1 = 1$ and $g_2 = 0$,

(iv) $l = 2$, $\text{char } \mathbb{F} = 2$, $g_1 = 1$ and $g_2$ is any polynomial in $A$.

Proof. First we consider the statement on self-duality. ($\Leftarrow$): Straightforward. ($\Rightarrow$): If $g_1 = 0$, then clearly $C$ is not self-dual. Also if $l = 1$, then $C^\perp$ is generated by $h = f/g_1$ and hence $C$ is self-dual if and only if $f/g_1 = g_1$. Thus assume that $l > 1$ and $g_1 \neq 0$. It follows from property (2.2) that $hg_i = 0$ for all $i$. This means that $g_i = g_i'g_1$ for some $g_i' \in A$. So according to
a generator matrix for \( C^\perp \) is
\[
H = \begin{pmatrix}
  h & 0 & 0 & \cdots & 0 \\
  -g'_2 & 1 & 0 & \cdots & 0 \\
  -g'_3 & 0 & 1 & \cdots & 0 \\
  \vdots & \vdots & \ddots & \ddots & \vdots \\
  -g'_{l} & 0 & 0 & \cdots & 1
\end{pmatrix}.
\]
Therefore for each \( 2 \leq i \leq l \), the \( i \)-th row of \( H = h^{(i)} \in C^\perp = C \), that is, \( h^{(i)} = a_i g \) for some \( a_i \in A \). Hence \( a_i g_1 = -g'_i \) and \( 1 = a_i g_l = a_i g'_l g_1 = g'_l (-g'_i) \). Consequently, the \( a_i \)'s and \( g'_i \)'s are units in \( A \). As \( a_i g_1 = -g'_i, g_1 \) is also a unit of \( A \). Since \( g_1 | f \), we conclude that \( g_1 = 1 \). If \( l > 2 \) we get \( a_2 g_3 = 0 \) which contradicts \( a_2, g_2 \) being units. Thus \( l = 2 \) and since \( g \cdot g = 0 \) we see that \( g_2^2 = -1 \).

For \( C = C^\perp \), again \((\Leftarrow)\) and also \((\Rightarrow)\) for \( l = 1 \) is easy. Suppose that \( C = C^\perp \) and \( l > 1 \). If \( g_1 = \cdots = g_i = 0 \) for some \( 0 < i \leq l \), then according to (2.6) the CGM of \( C^\perp \) has at least \( i \) rows, one of which is \((0,0,\ldots,0,1)\). Since this CGM should have just one row which is \( g \), we deduce that \( i = 1, l = 2, g_1 = 0 \) and \( g_2 = 1 \). Now assume that \( g_1 \neq 0 \). By (2.6), if \( l > 2 \), then the CGM of \( C^\perp \) has more than one row, a contradiction. Also the first row of the CGM of \( C^\perp \) is the reciprocal of the last row of \( H \) above. So \( g_1 = 1 \) and \( -g_2 = g_2 \). This last condition always holds in characteristic 2 and holds just for \( g_2 = 0 \) in other characteristics.

A code \( C \) is called isodual, when \( C^\perp \) is equivalent to \( C \) by a permutation. The following result together with the previous theorem characterize isodual \( A \)-codes with a basis of divisors consisting of one element.

**Proposition 3.2.** Suppose that \( C \) has a basis of divisors consisting of one element. Then \( C \) is isodual if and only if \( l \leq 2 \) and either \( C \) is self-dual or \( C = C^\perp \).

**Proof.** Clearly we just need to show that if \( C \) is isodual, then \( l \leq 2 \). Suppose \( g = (g_1, \ldots, g_l) \) is the only element of a basis of divisors of \( C \). Assume that \( g_1 = \cdots = g_{l-1} = 0 \) and \( g_l \neq 0 \). Then it follows from (2.6) and Algorithm [11] that a generator matrix for \( C^\perp \) is
\[
H = \begin{pmatrix}
  I_{(l-1) \times (l-1)} & 0 \\
  0 & h
\end{pmatrix},
\]
where \( h = f / g_l \) (mod \( f \)). Since \( H^R \) is a CGM for \( C^\perp \) and by [3, Proposition 2], we deduce that
\[
m > \dim_F C = \dim_F C^\perp = \dim_F C^\perp R > (l-1)m
\]
and hence \( l \leq 2 \), as required.

When the basis of divisors of an isodual \( A \)-code \( C \) has more than one element, then we may have \( l \geq 3 \). For instance, if \( f(x) = x^2 \) and \( C \) has CGM
\[
\begin{pmatrix}
  0 & x & 0 \\
  0 & 0 & 1
\end{pmatrix},
\]
then one can readily verify that \( C = C^\perp \) and is an isodual code with length 3.

Next we present a characterization of self-dual \( A \)-codes of length 2. Consider \( g_1, g_2, g_3 \in A \) and set \( h_i = f / g_i \) (mod \( f \)) (if \( g_i = 0 \), then \( h_i = 1 \)). Using [22], we can distinguish three classes
of A-codes of length 2. Class I: those with CGM of the form \((g_1, g_2)\) with \(0 \neq g_1 f\) and \(h_1 g_2 = 0\) (equivalently, \(g_1|g_2\)). Class II: those with CGM \((0, g_2)\) with \(g_2 f\). Class III: those with CGM \(\left(\frac{g_1}{0}, \frac{g_2}{g_3}\right)\), where \(0 \neq g_1, g_2|f, g_3|h_1 g_2\) and \(\deg(g_2) < \deg(g_3)\). The generator matrix for the dual of these codes calculated by Algorithm [1] is:

\[
\text{class I: } \begin{pmatrix} h_1 & 0 \\ \frac{-g_2}{g_1} & 1 \end{pmatrix}, \text{ class II: } \begin{pmatrix} 1 & 0 \\ 0 & h_2 \end{pmatrix} \text{ and class III: } \begin{pmatrix} h_1 & 0 \\ \frac{-h_3 g_2}{g_1} & h_3 \end{pmatrix}.
\]

Therefore by [2.40], we immediately get the following.

**Proposition 3.3.** An A-code C of length 2 is equal to its reciprocal dual \(C^\perp R\) if and only if either it is of class I and \(g_1 = 1\) and when \(\text{char } \mathbb{F} \neq 2, g_2 = 0\) or it is of class II with \(g_2 = 1\) or it is of class III with \(g_1 g_3 = f\) in \(\mathbb{F}[x]\) and when \(\text{char } \mathbb{F} \neq 2, g_2 = 0\).

The main part of characterizing self-dual A-codes of length 2 is the following.

**Theorem 3.4.** Suppose that C is a class III code of length 2 with CGM \(G = \left(\begin{array}{cc} g_1 & g_2 \\ 0 & g_3 \end{array}\right)\).

Then the following are equivalent.

(i) \(C\) is self-dual.

(ii) \(\deg(g_1) + \deg(g_3) = m, 0 = g_3^2 = g_2 g_3 = g_1^2 + g_2^2 \) (in A) and \(g_1^2|f\) (in \(\mathbb{F}[x]\)).

(iii) There exist \(g', f' \in \mathbb{F}[x]\) with \(g'^2 = rf' - 1\) such that \(f = g_1^2 f', g_3 = g_1 f'\) and \(g_2 = g_1 g'\).

**Proof.** (i) \(\Rightarrow\) (ii): Since \(\dim \mathcal{C} = \dim \mathcal{C}^\perp = 2m - \dim \mathcal{C}\) and as

\[
\dim \mathcal{C} = 2m - \deg(g_1) - \deg(g_3)
\]

by [3. Proposition 2], it follows that \(\deg(g_1) + \deg(g_3) = m\). Let \(h_i = f/g_i\). By assumption \((g_1, g_2) \in C = C^\perp\). Thus according to the above notes, \((g_1, g_2) = \alpha \left(\frac{-g_2}{g_1}, h_3\right) + \beta(h_1, 0)\) for some \(\alpha, \beta \in A\). Therefore, in \(\mathbb{F}[x]\) we have \(g_2 = \alpha \frac{f}{g_1} + k f\) (\(\ast\)) for some \(k \in \mathbb{F}[x]\). Hence \(\frac{f}{g_1} g_2\), that is, \(f|g_2 g_3\). Similarly from \((0, g_3) \in C^\perp\) we deduce that \(f|g_3^2\). Also from (\(\ast\)) we deduce that \(\alpha = \frac{g_2}{g_1} g_3 - g_3 g_2\) and hence

\[
g_1 = -\alpha \frac{h_3 g_2}{g_1} + \beta h_1 = -\left(\frac{g_2 g_3}{g_3 g_1} f g_2 + g_3 g_1 \right) g_1 + \frac{\beta}{g_1} = -\frac{g_2^2}{g_1} + \frac{k f}{g_1} + \frac{\beta f}{g_1}.
\]

Multiplying by \(g_1\) we get \(g_1^2 = -g_2^2 (\mod f)\), as claimed.

On the other hand, \((h_1, 0) \in C^\perp = C\), and hence by [2.3. Theorem 1], in \(\mathbb{F}[x]\) we have \(f/g_1 = h_1 = \alpha' g_1\) for some \(\alpha'\) with \(\deg(\alpha') < \deg(h_1)\). Consequently this equality holds in \(\mathbb{F}[x]\) and \(g_1^2|f\).

(ii) \(\Rightarrow\) (iii): By assumption \(g_1^2|f\). Set \(f' = \frac{f}{g_1^2}\). Also since \(rf = g_1^2 + g_2^2\) for some \(r \in \mathbb{F}[x]\) and \(g_1^2|f\), we deduce that \(g_1^2|g_2^2\), whence \(g_1|g_2\). Let \(g' = \frac{g_2}{g_1}\). Then \(rf' = 1 + g^2\). It remains to show \(g_3 = g_1 g'\).

From the equation \(rf' - g^2 = 1\), we see that \((f', g') = 1\). Also by assumption \(f|g_3 g_2 = g_3 g_1 g'\), hence \(g_1 f' = \frac{f'}{g_1} g_3 g_2\). Since \((f', g') = 1\), we deduce that \(f'|g_3\) and \(g_1|g' g_3\) where \(g_3 = \frac{g_2}{g_1}\). But \(g_1^2|f|g_3^2 = g_3^2 f' g_2^2\). So \(g_1|f' g_3\). Thus \(g_1|(rf' - g^2) g_3 = g_3^2\). Now

\[
\deg(g_3) + \deg(g_1) = m = \deg(f) = \deg(f') + 2 \deg(g_1),
\]
whence \( \deg(g_1) = \deg(g_3) - \deg(f') = \deg(g'_3) \). As both \( g_1 \) and \( g'_3 \) are monic, we conclude that \( g_1 = g'_3 \) and the result follows.

\[ \text{(iii) } \Rightarrow \text{(i):} \] Let \( H = \begin{pmatrix} h_1 & 0 & 0 \\ -h_2g_1 & 0 & h_3 \end{pmatrix} = \begin{pmatrix} g_1f' & 0 \\ -g_1g' & g_1 \end{pmatrix} \) which is a generator matrix for \( C^\perp \) according to \((2.5)\). One can readily check that \( G = \begin{pmatrix} r & g' \\ g' & f' \end{pmatrix} \) \( H \). So \( C \subseteq C^\perp \). But since \( \deg(g_1) + \deg(g_3) = \deg(f) = m \), it follows that \(|C| = |C^\perp| \) and hence \( C = C^\perp \). \( \square \)

**Corollary 3.5.** Suppose that \( C \) is an \( A \)-code of length 2. Then \( C \) is self-dual if and only if its CGM is either \([1 \ g_2] \) with \( g_2^2 = -1 \) or it is a class III code satisfying the equivalent conditions of \((3.4)\).

In particular, we get the following family of self-dual codes.

**Example 3.6.** Let \( 0,1 \neq g_1 \) and \( g' \) be any pair of monic polynomials in \( \mathbb{F}[x] \). Then by \((3.4)\), we see that the \( A \)-code generated by the matrix \( \begin{pmatrix} g_1 & g_1g' \\ 0 & g_1(g^2 + 1) \end{pmatrix} \) is self-dual,

where \( A = \frac{\mathbb{F}[x]}{(g_1(g^2 + 1))} \). Also the \( B \)-code with CGM \( \begin{pmatrix} g_1 & g_1x^3 \\ 0 & g_1(x^4 - x^2 + 1) \end{pmatrix} \) is self-dual with \( B = \frac{\mathbb{F}[x]}{(g_1(x^3 - x^2 + 1))} \).

In the case that \( \text{char } \mathbb{F} = 2 \) we can simplify the characterization of self-dual \( A \)-codes presented in \((3.4)\). First we need a simple lemma.

**Lemma 3.7.** Suppose that \( \mathbb{F} \subseteq \mathbb{F}' \) are finite fields and \( \text{char } \mathbb{F} = p \). If \( g \in \mathbb{F}'[x] \) is such that \( g^p \in \mathbb{F}[x] \), then \( g \in \mathbb{F}[x] \).

**Proof.** Suppose that \( |\mathbb{F}'| = p^n \). Note that \( g(x^{p^n}) = (g(x))^p = ((g(x))^p)^{p^{n-1}} \in \mathbb{F}[x] \). Thus all coefficients of \( g \) are in \( \mathbb{F} \). \( \square \)

In the sequel, by \( \sqrt[p]{h} \) we mean \( p_1^{\lceil \alpha_1/p \rceil} \cdots p_t^{\lceil \alpha_t/p \rceil} \), where \( h = p_1^{\alpha_1} \cdots p_t^{\alpha_t} \) is the prime decomposition of \( h \) in \( \mathbb{F}[x] \). Note that \( g^2 \in (h) \) if and only if \( g \in (\sqrt[p]{h}) \).

**Theorem 3.8.** Suppose that \( \text{char } \mathbb{F} = 2 \) and \( C \) is a class III code of length 2 with CGM \( \begin{pmatrix} g_1 & g_2 \\ 0 & g_3 \end{pmatrix} \). Then \( C \) is self-dual if and only if

(i) either \( g_1 = g_3, g_2 = 0 \) and \( f = g_1^2 \),

(ii) or \( f = g_1^2f', g_1 = g_1f' \) and \( g_2 = g_1(h\sqrt[p]{f'} + 1) \) for some \( f', h \in \mathbb{F}[x] \) with \( \deg(h) < \deg(f') - \deg(\sqrt[p]{f'}) \).

**Proof.** (\( \Leftarrow \)): Follows from \((3.4)\). (\( \Rightarrow \)): Since \( C \) is self-dual the conditions of \((3.4)\) hold. Thus we just need to show that in the notations of \((3.4)\), \( g' = h\sqrt[p]{f'} + 1 \) with \( \deg(h) < \deg(f') - \deg(\sqrt[p]{f'}) \). We have \( g'^2 = rf' - 1 = rf' + 1. \) If \( f = 1 \), then \( g_1 = g_3 \) and \( f_1 = g_1^2 \). Also since \( G \) is a CGM, we have \( \deg(g_2) < \deg(g_3) = \deg(g_1) \). But as \( g_2 \) is a multiple of \( g_1 \) we get \( g_2 = 0 \) and case (i) is valid.

Thus assume that \( f' \neq 1 \). Therefore, \( f' \) has some root, say \( a \), in some extension field \( \mathbb{F}' \) of \( \mathbb{F} \). Therefore in \( \mathbb{F}' \) we have \( g'^2(a) = r(a)f'(a) + 1 = 1 \), whence \( g'(a) = 1 \). So \( g' = g''(x - a) + 1 \)
for some \(g'' \in \mathbb{F}'[x]\) and \(rf' = g'^2 - 1 = g''^2(x - a)^2\). Thus by (3.7), \(g''(x - a) \in \mathbb{F}[x]\). Since \(g''^2(x - a)^2 \in \langle f'\rangle\), we deduce that \(g''(x - a) = h \sqrt{f'}\) for some \(h \in \mathbb{F}[x]\), that is, \(g' = h \sqrt{f'} + 1\).

Noting that \(\deg(g_2) < \deg(g_3)\) and \(g_2 = g'g_1\) and \(g_3 = f'g_1\), the degree condition on \(h\) follows and the proof is complete.

At the end of this section, we pay some attention to some self-dual codes of length \(> 2\).

Suppose that \(C_1\) and \(C_2\) are two \(A\)-codes of length \(l_1\) and \(l_2\), respectively, and denote their direct product by \(C = C_1 \times C_2\). Then according to [4, Lemma 3.2], \(C\) is a self-dual \(A\)-code of length \(l_1 + l_2\). Thus if a self-dual \(A\)-code of length 2 exists, then as in the following example we can construct a self-dual \(A\)-code of any given even length. It is straightforward to check that if \(G_1\) and \(G_2\) are CGMs of \(C_1\) and \(C_2\), respectively, then the block matrix \[
\begin{pmatrix}
G_1 & 0 \\
0 & G_2
\end{pmatrix}
\]

is the CGM of \(C_1 \times C_2\).

**Example 3.9.** In the notations of Example [64] and according to the above notes, the following is the CGM of a self-dual \(B\)-code of length 4:

\[
\begin{pmatrix}
g_1 & g_1x^3 & 0 & 0 \\
0 & g_1(x^4 - x^2 + 1) & 0 & 0 \\
0 & 0 & g_1 & g_1x^3 \\
0 & 0 & 0 & g_1(x^4 - x^2 + 1)
\end{pmatrix}.
\]

Let \(R\) be any finite ring and \(\mathfrak{M}\) one of its maximal ideals. Then the *stability index* of \(\mathfrak{M}\) is the least positive integer such that \(\mathfrak{M}^l = \mathfrak{M}^{l+1} = \cdots\). Using [6, Theorem 3.9], we can deduce the following result on lengths of self-dual \(A\)-codes. For its proof, we use the well-known fact that \(-1\) is a square in a field \(\mathbb{F}\) if and only if \(|\mathbb{F}| = 3 \text{ (mod 4)}\) (it follows, for example, from [12, Exercise 2.13]).

**Theorem 3.10.** If \(l\) is a multiple of 4, then there exist a self-dual \(A\)-code of length \(l\). Also there exist self-dual \(A\)-codes of all lengths if and only if \(f\) is a square. Moreover, suppose \(f = f_1^{e_1} \cdots f_t^{e_t}\) is the prime decomposition of \(f\) in \(\mathbb{F}[x]\) and \(d_i = \deg(f_i)\). Then the following are equivalent.

(i) There exist self-dual \(A\)-codes of all even lengths.

(ii) \(f = g^2f'\) for some \(g, f' \in \mathbb{F}[x]\) such that \(-1\) is a square modulo \(f'\) (that is, a square in \(\mathbb{F}[x]/\langle f'(x)\rangle\)).

(iii) Either \(|\mathbb{F}| \equiv 3 \text{ (mod 4)}\) and for each \(1 \leq i \leq t\), \(d_ie_i\) is even or \(|\mathbb{F}|\) is even or \(|\mathbb{F}| \equiv 1 \text{ (mod 4)}\).

**Proof.** Every finite PIR, including \(A\), satisfies the conditions of [6, Theorem 3.9]. Therefore by that theorem, there exist a self-dual \(A\)-code with length \(l\), if \(4|l\). Also according to the construction mentioned in the previous example and the notes before it, we see that there are self-dual codes of every length if and only if there is a self-dual code of length 1 which is clearly equivalent to \(f\) being a square. To prove the last part of the statement, let \(\mathfrak{M}_i\) be the maximal ideal \(\langle f_i\rangle\) of \(A\) and \(\mathbb{F}_i = A/\mathfrak{M}_i \cong \mathbb{F}[x]/\langle f_i\rangle\). Then the stability index of \(\mathfrak{M}_i\) is \(e_i\) and \(|\mathbb{F}_i| = |\mathbb{F}|^{d_i}\).

\(\Rightarrow\) \(\Rightarrow\) \(\Rightarrow\): There exist a self-dual \(A\)-code \(C\) of length 2. Now the result follows from (3.5) and (3.4), if we set \(g = 1\) and \(f' = f\) in the case that \(C\) is a class II code and \(g = g_1\) and \(f'\) as in (3.4) in the case that \(C\) is a class III code.
\[ \Rightarrow \] Suppose that \( \boldsymbol{I} \) holds but \( \boldsymbol{II} \) does not. Then \( |\mathbb{F}| \equiv 3 \pmod{4} \) and there is an \( i \) such that both \( d_i \) and \( e_i \) are odd. Hence \( f_i | f' \) and since \(-1\) is a square modulo \( f' \), it is also a square modulo \( f_i \). This means that \(-1\) is a square in \( \mathbb{F}_i \) and \( |\mathbb{F}_i| \) is either even or \( |\mathbb{F}_i| \equiv 1 \pmod{4} \). But as \( d_i \) is odd, we deduce that \( |\mathbb{F}_i| = |\mathbb{F}|d_i \equiv 3 \pmod{4} \) and from this contradiction the result follows.

\( \Rightarrow \) This follows \cite{3}, Theorem 3.9(ii).

\[ \square \]

\section{Rings over Which the \( \mathbb{F} \)-Dual of Every Linear Code Is a Linear Code}

A polynomial in \( A \) can be viewed as the vector of its coefficients in \( \mathbb{F} \). Similarly a codeword \((g_1(x), \ldots, g_l(x))\) can be viewed as the vector of length \( lm \) over \( \mathbb{F} \) obtained by concatenating the vectors corresponding to \( g_1(x), \ldots, g_l(x) \). In this way, every \( A \)-code of length \( l \) is also a linear code of length \( lm \) over \( \mathbb{F} \) and its \( \mathbb{F} \)-dual can be computed. As Example 7 of \cite{3} shows, the \( \mathbb{F} \)-dual of every \( A \)-code need not be an \( A \)-code. In this section, we characterize monic polynomials \( f(x) \in \mathbb{F}[x] \) with the property that the \( \mathbb{F} \)-dual of every \( A \)-code is an \( A \)-code, where \( A = \mathbb{F}[x] / \langle f(x) \rangle \).

For simplicity, throughout this section we fix the following notations.

**Notation 4.1.** Let \( g(x) = \sum_{i=0}^{m-1} a_i x^i \in A \). We can regard \( g \) as the row vector \((a_0, \ldots, a_{m-1})\) over \( \mathbb{F} \). We denote this vector by \( g \) or \([g(x)]\) and whenever we want to consider \( g \) as a polynomial, we write \( g(x) \) (not \( g \) alone). Similarly if \( u = (u_0, \ldots, u_{m-1}) \) is a vector over \( \mathbb{F} \), then by \( u(x) \) we mean \( \sum_{i=0}^{m-1} u_i x^i \). Also, as in \cite{3}, we set

\[
M_x = \begin{pmatrix}
0 & 1 & 0 & \ldots & 0 \\
\vdots & 0 & 1 & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \ddots & \ddots & 1 & 0 \\
0 & 0 & \ldots & \ldots & 0 & 1 \\
-f_0 & -f_1 & \ldots & \ldots & -f_{m-1}
\end{pmatrix},
\]

to be the companion matrix of \( f(x) = x^m + \sum_{i=0}^{m-1} f_i x^i \). Moreover, for arbitrary \( g(x) = \sum_{i=0}^{m-1} a_i x^i \in A \), we set \( M_g = g(M_x) = \sum_{i=0}^{m-1} a_i M_x^i \). Furthermore, we write \( C^\perp \) for the \( \mathbb{F} \)-dual of \( C \) and \( C^{\perp \perp} \) for the \( \mathbb{F} \)-dual of \( C \).

Consequently, it follows that \( g(x) h(x) = (gM_h)(x) \) for any \( g(x), h(x) \in A \), (see \cite{3}, Proposition 6). To find out when the \( \mathbb{F} \)-dual of \( A \)-codes are \( A \)-codes, we need the following.

**Proposition 4.2.** Assume that \( m \geq 2 \). There exists \( g(x) \in A \) with \( M_x^T = M_g \) if and only if either \( f(x) = x^m \pm 1 \) or \( m = 2 \) and \( f(x) = x^2 + ax - 1 \) for some \( a \in \mathbb{F} \).

**Proof.** (\( \Rightarrow \)): It is easy to check that for each \( 1 \leq i \leq m - 1 \), the first row of \( M_x^i \) is \( e_i \) (the vector with just one nonzero entry which is a 1 on the \( i \)-th place) and the \( m - i + 1 \)-th row of \( M_x^i \) is \((-f_0, -f_1, \ldots, -f_{m-1})\). So if \( g(x) = \sum_{i=0}^{m-1} a_i x^i \), then the first row of \( M_g = \sum_{i=0}^{m-1} a_i M_x^i \) is \((a_0, a_1, \ldots, a_{m-1})\). On the other hand the first row of \( M_x^T \) is \(-f_0 e_m\). Thus \( a_i = 0 \) for all \( 0 \leq i \leq m - 2 \) and \( a_{m-1} = -f_0 \), that is, \( g(x) = -f_0 x^{m-1} \).
Now it follows from the above notes that the second row of $M_g$ is $-f_0(-f_0,\ldots,-f_{m-1})$, which should be equal to the second row of $M_x^T = (1,0,\ldots,0,-f_1)$. Hence $f_0^2 = 1$ and for each $1 \leq i \leq m-2$ we have $f_i = 0$ and $f_0f_{m-1} = -f_1$. This, if $m > 2$, results to $f_{m-1} = 0$ (that is, $f(x) = x^m \pm 1$) and if $m = 2$, results to either $f_1 = 0$ (that is, $f(x) = x^2 \pm 1$) or $f_0 = -1$ (that is, $f(x) = x^2 + ax - 1$ for some $a \in \mathbb{F}$).

($\Leftarrow$): It is routine to verify that in all cases $M_x^T = M_g$, where $g(x) = -f_0x^{m-1}$.

Now we can state the main theorem of this section.

**Theorem 4.3.** The $\mathbb{F}$-dual of every $A$-code is an $A$-code if and only if either $m = 1$ or $m = 2$ and $f(x) = x^2 + ax - 1$ for some $a \in \mathbb{F}$ or $m \geq 2$ and $f(x) = x^m \pm 1$.

**Proof.** The case $m = 1$ is trivial, so we assume that $m \geq 2$. Here if $C$ is a code of length $l$, $u \in C$ and $M$ is a $m \times m$ $\mathbb{F}$-matrix, we regard $u$ as the vector $(u_1,\ldots,u_l)$ with $u_i$s in $A$ and write $uM$ for the vector $(u_1M,\ldots,u_lM)$.

($\Rightarrow$): Suppose $C$ is an $A$-code and $z \in CM_x^T$. Since $C^\perp$ is an $A$-code, we have $xu(x) \in C^\perp$, for each $u \in C^\perp$. Therefore

$$uz^T = u(CM_x^T)^T = uM_xC^T = [xu(x)]C^T \subseteq C^\perp C^T = 0.$$ 

Consequently, $CM_x^T \subseteq C^\perp C^\perp = C$ for each $A$-code $C$. In particular, if $0 \neq v \in A$, then $vM_x^T \in Av$, where $Av$ is the ideal (or equivalently, the $A$-code of length 1) generated by $v$. This means that $vM_x^T = [v(x)g(x)]$ for some $g(x) \in A$. If $0, v \neq v' \in A$, then $(v,v')M_x^T \in A(v,v')$. Hence for some $g(x) \in A$, we have

$$([g(x)v(x)], [g(x)v'(x)]) = (v,v')M_x^T = (vM_x^T, v'M_x^T) = ([g_v(x)v(x)], [g_{v'}(x)v'(x)]).$$

If we apply this to $v'(x) = 1$, we see that $g(x) = g_1(x)$. Therefore, for arbitrary $v(x) \in A$, we have $vM_{g_1} = [g_1(x)v(x)] = vM_x^T$, that is $M_x^T = M_{g_1}$ and the result follows from (4.2).

($\Leftarrow$): By (4.2), there is a $g \in A$ with $M_x^T = M_g$. Since $C$ is an $A$-code, we see that $CM_g \subseteq C$. Therefore, if $u \in C^\perp$, then

$$[xu(x)]C^T = uM_xC^T = u(CM_x^T)^T = u(CM_g)^T \subseteq uC^T = 0,$$

that is $xu(x) \in C^\perp$, hence $C^\perp$ is an $A$-code.

Next we are going to find a generator matrix for $C^\perp$ over $A$, where $A$ satisfies the conditions of (4.13). For this we need some intermediate results. Recall that if $g \in \mathbb{F}[x]$, then $g^{R}(x)$ is the reciprocal of $g(x)$, that is, $x^{\deg(g)}g(x^{-1})$.

**Lemma 4.4.** Assume that $g(x) \in A$. If $f(x) = ax^2 + 1$ for some $a \in \mathbb{F}$, then $M_g^T = M_1$. If $f(x) = x^m \pm 1$, then $M_g^T = M_{h}$, where $h(x) = g(x^{-1}) = \frac{g^R(x)}{x^{\deg(g)}}$.

**Proof.** In either of the cases, if $g(x) = \sum_{i=0}^{m-1} a_i x^i$, then (by the proof of (4.2))

$$M_g^T = \sum_{i=0}^{m-1} (M_x^T)^i = \sum_{i=0}^{m-1} (-f_0 M_x^{m-1})^i = M_h,$$

where $h(x) = g(-f_0 x^{m-1})$. Now if $f(x) = ax^2 + 1$, then $-f_0 x^{m-1} = 0$ and if $f(x) = x^m \pm 1$, then $-f_0 x^{m-1} = x^{-1}$, thus $h'(x)$ is same as $h(x)$ of the statement.
Assume that $G = (g_{ij}(x))$ is a $k \times l$ matrix over $A$. As in [3, Section 3.5], we set $\psi(G)$ and $\zeta(G)$ to be the $km \times lm$ matrices over $\mathbb{F}$ defined blockwise as follows: the $ij$-th block of $\psi(G)$ is $M_{g_{ij}}$ and the $ij$-th block of $\zeta(G)$ is $M_{g_{ij}}^T$. According to [3, Theorem 4], the code that $\psi(G)$ generates over $\mathbb{F}$ is the same as the code $G$ generates over $A$ and by [3, Theorem 5], $\zeta(G)$ generates $(C^\perp)^{1+}$ over $\mathbb{F}$.

**Corollary 4.5.** For every $A$-code $C$ we have $C^\perp = C^{1+}$ if and only if $m = 1$ or $f(x) = x^2 + ax - 1$ for some $a \in \mathbb{F}$.

**Proof.** ($\Rightarrow$): The case $m = 1$ is trivial, thus assume that $f(x) = x^2 + ax - 1$. Let $G$ be a generator matrix for an arbitrary $A$-code $C$. Then by (1.3), we have $\psi(G) = \zeta(G)$. Therefore, according to Theorems 4 and 5 of [3], $(C^\perp)^{1+} = C$ for every $A$-code $C$. Applying this with $C^\perp$ instead of $C$, we get the desired conclusion.

($\Rightarrow$): Suppose $m \geq 2$ and $f(x) \neq x^2 + ax - 1$ for any $a \in \mathbb{F}$. Then by (1.3), we should have $f(x) = x^m \pm 1$. Assume that $f(x) = x^m - 1$. Then $m > 2$. Consider the code $C$ generated by $(1, x + 1)$ over $A$ or equivalently generated by the vectors

$$([1],[x+1]),([x],[x^2+x]),\ldots,([x^{m-1}],[x^{m-1}(x+1)])$$

over $\mathbb{F}$. Then it is routine to check that the dot product of the vector $([x^{m-1}+1],[-1])$ with any of the above $\mathbb{F}$-generators of $C$ is zero, that is, $(x^{m-1}+1,-1) \in C^{1+}$ but

$$(x^{m-1}+1,-1)(1,x+1)=x^{m-1}-x\neq0,$$

that is, $(x^{m-1}+1,-1) \notin C^\perp$, a contradiction.

Now assume that $f(x) = x^m + 1$. If $m = 2$ and char $\mathbb{F} = 2$, then $f(x)$ is in the required form. Thus we can assume that either $m > 2$ or char $\mathbb{F} \neq 2$. Again consider the code $C$ generated by $(1, x + 1)$ over $A$. Using these assumptions one can readily verify that this time $(x^{m-1} - 1, 1) \notin C^\perp \setminus C^\perp$. \qed

**Proposition 4.6.** Suppose that $f(x) = x^m \pm 1$ and assume that rows of a $k \times l$ matrix $G$ form a basis of divisors for an $A$-code $C$. Let $G' = (\alpha_i^{-1} x^{d_i} g_{ij}(x^{-1}))$, where $d_i = \deg(L_{\text{coef}}(g^{(i)}))$ and $\alpha_i$ is the constant coefficient of $L_{\text{coef}}(g^{(i)})$. Then rows of $G'$ form a basis of divisors for the $A$-code $C' = (C^\perp)^{1+}$.

**Proof.** Let $G'' = (g_{ij}(x^{-1}))$. Then as $x$ is invertible in $A$, $G'$ and $G''$ generate the same code. Now $\psi(G'') = \zeta(G)$ by (1.4). Thus by (3.1) $\psi(G'')$ generates $C'$ over $\mathbb{F}$ and hence by (3.1) $G''$ and hence $G$ generate $C'$ over $A$. Let $h_i(x) = L_{\text{coef}}(g^{(i)})$, then $h_i(x)|f(x)$. Now since $L_{\text{coef}}(g^{(i)}) = \alpha_i^{-1} h_i^R(x)$ and $f^R(x) = \pm f(x)$, we see that $L_{\text{coef}}(g^{(i)})|f(x)$. Thus (2.2 iii) holds and obviously (2.2 ii) also holds. Moreover, since $h_i(x)|f(x)$, we see that $h_i(0)\neq0$ and thus $\deg(h_i(x)) = \deg(h_i^R(x))$. Therefore,

$$\sum_{i=1}^k m - \deg\left(L_{\text{coef}}(g^{(i)})\right) = \sum_{i=1}^k m - \deg\left(L_{\text{coef}}(g^{(i)})\right) = \dim_{\mathbb{F}} C = \dim_{\mathbb{F}} C'$$

and (iii) of (2.2) holds and the result follows. \qed

The matrix $G'$ constructed above need not be a CGM, even if the initial $G$ is a CGM for $C$, as the following example shows.
Example 4.7. Let \( f(x) = x^3 - 1 \) and \( C \) be the \( A \)-code with CGM \[
\begin{pmatrix}
x^2 + x + 1 & -1 \\
0 & x - 1
\end{pmatrix}.
\]
Then \( G' = \begin{pmatrix} x^2 + x + 1 & -x^2 \\
0 & x - 1 \end{pmatrix} \) which is not a CGM. Indeed the CGM for \( C' \) is \( G \) itself and \( C = C' \) in this case.

We say that rows of a matrix \( G \) is a reverse basis of divisors for an \( A \)-code \( C \), when the rows of the matrix obtained by reversing the order of both rows and columns of \( G \) (as in (2.6)) are a basis of divisors for \( C_R \). For example, rows of \( H \) in (2.6) form a reverse basis of divisors for \( C \).

Corollary 4.8. Assume that \( A \) satisfies the conditions of (4.3). Suppose that \( H = (h_{ij}(x)) \) is the matrix obtained by Algorithm 1 for the \( A \)-code \( C \). If \( m = 1 \) or \( f(x) = x^2 + ax - 1 \), then rows of \( H' = (\alpha_i^{-1} x^{d_i} h_{ij}(x)) \) form a reverse basis of divisors for \( C^{\perp_F} \), where \( d_i \) is the degree of the last nonzero entry on the \( i \)-th row of \( H \) and \( \alpha_i \) is the constant coefficient of this entry.

Proof. In the first case clearly \( C^{\perp_F} = C^{\perp} \) by (1.5), so assume that \( f(x) = x^m \pm 1 \). Let \( H^R \) be as in (2.6). If we apply (2.6) with \( G = H^R \), then \( G' = H'^R \) and \( C' = (C^{\perp_R})^{\perp_F} = (C^{\perp})^{\perp_F} = (C^{\perp_F})^R \) (note that in all terms, we are taking reciprocal of codes as \( A \)-codes not \( F \)-codes). Therefore, the rows of \( H'^R \) form a basis of divisors for \( (C^{\perp_F})^R \), as required.

Note that although \( H^R \) above is indeed a CGM for \( C^{\perp_R} \), but \( H'^R \) need not be a CGM. For example, if \( f(x) = x^3 - 1 \) and \( C \) is the code with CGM \[
\begin{pmatrix}
x^2 + x + 1 & 1 \\
0 & x - 1
\end{pmatrix},
\]
then \( H^R \) and \( H'^R \) are \( G \) and \( G' \) in Example (4.7), respectively. Also if we apply this corollary for example to the code generated by \( (1, x + 1) \) with \( f(x) = x^m + 1 \), we see that \( C^{\perp_F} \) is generated by \( (x^{m-1} - 1, 1) \) and is different from \( C^{\perp} \) which is generated by \( (-x - 1, 1) \).

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