GLOBAL SOLUTIONS OF SHOCK REFLECTION PROBLEM FOR THE PRESSURE GRADIENT SYSTEM

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(Communicated by Yachun Li)

Abstract. We are concerned with the shock reflection in gas dynamics for the pressure gradient system. Experimental and computational analysis has shown that two patterns of regular reflection may occur: supersonic and subsonic reflection. In this paper we establish the global existence of solutions for both configurations. The ideas and techniques developed here will be useful for the two-dimensional Riemann problems for hyperbolic conservation laws.

1. Introduction. Shock reflection is a fundamental phenomenon in gas dynamics. When a shock wave passes through an obstacle, the shock reflection occurs generally. E. Mach [25] first studied this problem in 1878. He observed two patterns of shock reflection-diffraction configurations that are now named the regular reflection (RR) and the Mach reflection (MR). In 1943 von Neumann [30] modeled the problem on moving plane shock attacking an inclined ramp. Later, a lot of research works concentrated on the shock reflection mainly by experiments and computations. It is observed that there are various different and complicated shock patterns depending on many factors, including the shape of the obstacle, the parameters of the incident shock, etc., see Courant-Friedrichs [12], Ben [2], Glimm-Majda [14], and also [4, 5, 9, 10, 13, 16, 26, 28] and the references cited therein. Mathematical analysis of shock reflection-diffraction configurations involves several core difficulties in the analysis of nonlinear PDEs, including nonlinear PDEs of mixed hyperbolic-elliptic type, free boundary problem for nonlinear degenerate elliptic PDEs, corner singularities and so on. The shock reflection problem governed by full Euler equations or isentropic Euler equations is still open. Many efforts have been made mathematically for the reflection problem via simplified models, including the potential flow equation [1, 7–9, 26], the unsteady transonic small disturbance equation (UTSD) [18–20], the pressure gradient system [32] and the nonlinear wave system [6, 21, 27]. Moreover, there are many papers devoted to the Mach reflection, see [3, 11, 17].

2010 Mathematics Subject Classification. Primary: 35L50, 35L67, 76H05; Secondary: 35J67.
Key words and phrases. Pressure gradient system, von Neumann conjecture, shock reflection, free boundary, sonic boundary, hyperbolic-elliptic mixed type, Leray-Schauder degree theory, existence, regularity.

The first author is supported by NSF of Yunnan University (No. 2019FY003007). The third author is supported by National Natural Science Foundation of China (No. 11761077) and Key Project of Yunnan Provincial Science and Technology Department and Yunnan University (No. 2018FY001(-014)).

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the importance of shock reflection on aerodynamics, the shock reflection configurations are core configurations in the structure of global entropy solutions of the two-dimensional Riemann problems of hyperbolic conservation laws, see [22,33].

In this paper, we are concerned with the pressure gradient system in two dimensional space

\[
\begin{align*}
&u_t + p_{x_1} = 0, \\
&v_t + p_{x_2} = 0, \\
&E_t + (pu)_{x_1} + (pv)_{x_2} = 0,
\end{align*}
\]

where \( t \geq 0 \) and \( x = (x_1, x_2) \in \mathbb{R}^2 \) are the time and spatial variables. \( p, (u, v) \) and \( E = (u^2 + v^2)/2 + p \) are the pressure, the velocity of the fluid and the energy, respectively. The three equations in (1.1) stand for the conservation of momentum along the \( x_1 \)-direction and \( x_2 \)-direction and the conservation of energy, respectively.

From system (1.1) one can derive a decoupled nonlinear equation for \( p \)

\[
\left( \frac{p_t}{p} \right)_t - \Delta p = 0.
\]

When a plane shock \( S_0 \) in the \((x, t)\)-coordinates with the left state \( U_1 = (p_1, u_1, 0) \) and right state \( U_0 = (p_0, 0, 0) \) hits a symmetric wedge

\[
W := \{ |x_2| < x_1 \tan \theta_w, x_1 > 0 \}
\]

head on, it experiences a reflection-diffraction process. Here \( \theta_w \) is the half wedge angle. Since the solid wedge \( W \) is symmetric with respect to the axis \( x_2 = 0 \), it suffices to consider the shock reflection problem in the upper half-plane \( \{ (x_1, x_2) \in \mathbb{R}^2 : x_2 > 0 \} \). Then we can formulate this problem as follows:

**Problem 1.** (Initial-boundary value problem). Seek a solution of system (1.1) satisfying the initial condition at \( t = 0 \):

\[
(p, u, v) = \begin{cases} (p_0, 0, 0) & \text{for } x_2 > x_1 \tan \theta_w, x_1 > 0, \\
(p_1, u_1, 0) & \text{for } x_1 < 0, x_2 > 0,
\end{cases}
\]

and the slip boundary condition along the wedge boundary \( \partial W = \{ x_2 = x_1 \tan \theta_w, x_1 > 0 \} \) and the symmetric boundary \( \Gamma_{sym} = \{ x_1 < 0, x_2 = 0 \} \):

\[
(u, v) \cdot \nu|_{\partial W \cup \Gamma_{sym}} = 0,
\]

where \( \nu \) is the exterior unit normal.

Since the shock reflection problem has self-similarity feature, we seek self-similar solutions in \( \xi = (\xi, \eta) = (x/t) \) coordinates. System (1.1) takes the form

\[
\begin{align*}
-\xi u_\xi - \eta u_\eta + p_\xi &= 0, \\
-\xi v_\xi - \eta v_\eta + p_\eta &= 0, \\
-\xi p_\xi - \eta p_\eta + pu_\xi + pv_\eta &= 0,
\end{align*}
\]

and equation (1.2) becomes

\[
(p - \xi^2)p_{\xi \xi} - 2\xi \eta p_{\xi \eta} + (p - \eta^2)p_{\eta \eta} + \frac{1}{p} (\xi p_\xi + \eta p_\eta)^2 - 2(\xi p_\xi + \eta p_\eta) = 0.
\]

Equation (1.5) is a second order mixed equation of elliptic-hyperbolic type. It is elliptic when \( \xi^2 + \eta^2 < p \), and hyperbolic when \( \xi^2 + \eta^2 > p \). System (1.4) can be written as the following general form as a system of balance law

\[
\text{div} A(w) + B(w) = 0,
\]
where \( \mathbf{w} = (u, v, p) \), and \( A : \mathbb{R}^3 \to \mathbb{R}^{3 \times 3} \) and \( B : \mathbb{R}^3 \to \mathbb{R}^3 \) are given nonlinear mappings.

Definition 1.1. (Weak solution). A function \( \mathbf{w} \in L^\infty_{\text{loc}}(\Omega) \) for an open region \( \Omega \subset \mathbb{R}^2 \) is a weak solution of (1.4) in \( \Omega \), provided that

\[
\int_\Omega (A(\mathbf{w}(\xi)) \cdot \nabla \varphi(\xi) - B(\mathbf{w}(\xi)) \varphi(\xi)) \, d\xi = 0
\]

for any \( \varphi \in C^1(\Omega) \).

We observe that once \( p \) is known within a region, the first two equations in (1.4) can be used to obtain \((u, v)\), provided \((u, v)\) are known on a suitable boundary and the region is of suitable structure. This is possible for our problem, thus we shall mainly consider (1.5) for \( p \) instead of (1.4).

In the polar coordinates \((r, \theta) = (\sqrt{\xi^2 + \eta^2}, \arctan \frac{\eta}{\xi})\), equation (1.5) changes to

\[
(p - r^2)p_{rr} + \frac{p}{r^2} p_{\theta \theta} + \frac{p}{r} p_r + \frac{1}{p} (rp_r)^2 - 2rp_r = 0. \tag{1.7}
\]

Equation (1.7) is elliptic if \( r^2 < p \) and hyperbolic if \( r^2 > p \).

To connect the two constant states \( U_0 \) and \( U_1 \) with a single forwards shock \( S_0 \), the initial condition should satisfy the relations

\[
u_1 = \frac{p_1 - p_0}{\sqrt{p_{10}}}, \quad p_1 > p_0, \quad \sqrt{p_{10}} := \frac{p_1 + p_0}{2}. \tag{1.8}
\]

In the \((\xi, \eta)\)–coordinates, the shock hits the ramp at the location

\[(\xi_{10}, \eta_{10}) := (\sqrt{p_{10}}, \sqrt{p_{10}} \tan \theta_w).\]

Let

\[\Lambda := \{\xi \leq 0, \eta > 0\} \cup \{\xi > 0, \eta > \xi \tan \theta_w\}.\]

Then Problem 1 in the \((x, t)\)–coordinates can be reformulated as the following boundary value problem in the \((\xi, \eta)\)–coordinates.

Problem 2 (Boundary value problem). Seek a solution of system (1.4) in the self-similar domain \( \Lambda \) with the slip boundary condition

\[
(u, v) \cdot \mathbf{v} |_{\partial W \cup \Gamma_{sym}} = 0,
\]

and the asymptotic boundary conditions at infinity

\[
(p, u, v) = \begin{cases} (p_0, 0, 0) & \text{for } \xi > \xi_{10}, \ \eta > \xi \tan \theta_w, \ \text{as } \xi^2 + \eta^2 \to +\infty, \\ (p_1, u_1, 0) & \text{for } \xi < \xi_{10}, \ \eta > 0. \end{cases}
\]

The organization of this paper is as follows. In Section 2, we discuss the normal shock reflection and then derive the necessary condition for the existence of regular shock reflection. We will also make clear the supersonic regular reflection and subsonic regular reflection, and present the main results. In Section 3, we deal with the supersonic reflection. We first introduce a new coordinate system and obtain some basic estimates for the solutions and the domain. Next we consider the regularity of solutions near and away from the sonic arc, which is a degenerate boundary for \( p \). Then we construct the iteration set and use the Leray-Schauder
theory to prove the existence of admissible solutions. In Section 4 we prove the existence of solutions for the subsonic regular reflection.

2. Mathematical formulation. We first derive the Rankine-Hugoniot condition.

2.1. The Rankine-Hugoniot condition. Consider a piecewise smooth weak solution of (1.4) with a jump in $(p, u, v)$ across a curve $S = \{ \xi = \xi(\eta) \}$ with the slope $\sigma = \xi'(\eta)$. Across $S$ it should satisfy the Rankine-Hugoniot condition

$$
\begin{align*}
(\xi - \sigma \eta)[u] - [p] &= 0, \\
(\xi - \sigma \eta)[v] + \sigma [p] &= 0, \\
(\xi - \sigma \eta)[E] - [pu] + \sigma [pv] &= 0,
\end{align*}
$$

(2.1)

where the bracket $[w]$ denotes the jump of the quantity $w$ across $S$. A discontinuity of $(p, u, v)$ on $S$ is called a shock wave if it further satisfies the physical entropy condition: $p$ increases across $S$.

We can solve (2.1) to obtain contact discontinuity

$$
\sigma = \frac{\xi}{\eta} = \frac{[u]}{[v]}, \quad [p] = 0,
$$

and shocks

$$
\begin{align*}
\frac{d\xi}{d\eta} &= \sigma_{\pm} := \frac{\xi \eta \pm \sqrt{p(\xi^2 + \eta^2 - p)}}{\eta^2 - p}, \\
[u] &= \frac{\xi \bar{p} \pm \eta \sqrt{\bar{p} (\xi^2 + \eta^2 - \bar{p})}}{(\xi^2 + \eta^2) \bar{p}} [p], \\
[v] &= \frac{\eta \bar{p} \pm \xi \sqrt{\bar{p} (\xi^2 + \eta^2 - \bar{p})}}{(\xi^2 + \eta^2) \bar{p}} [p].
\end{align*}
$$

(2.2)

There are two states for shocks, with “+” and “−”, which are called weak reflection and strong reflection, respectively. In the sequel, we only focus on the weak reflected shock, denoted by $\Gamma_{\text{shock}}$. The reason is that the states for weak reflection tend to those for the normal shock reflection as $\theta_w \to \frac{\pi}{2}$, which will be specified later.

From (2.1) and the last two equations in (2.2), we obtain the condition that $p$ should satisfy on $\Gamma_{\text{shock}}$:

$$
\hat{p} \left( \frac{|p|}{4p} (\xi^2 + \eta^2) - (\xi^2 + \eta^2 - \bar{p}) \right) \\
+ (\xi \eta + \sigma (\bar{p} - \eta^2)) \left( (p_\xi - \sigma p_\eta) + (\xi p_\xi + \eta p_\eta) \frac{\sigma \eta - \xi}{p} \right) = 0,
$$

(2.3)

where $|p| = p - p_1$, and $\hat{p} = \sigma p_\xi + p_\eta$ is the tangential differentiation of $p$ along the shock.

In the polar $(r, \theta)$ coordinates, the shock becomes

$$
\frac{dr}{d\theta} = -r \sqrt{\frac{r^2 - \bar{p}}{\bar{p}}} =: g(r(\theta), \theta, p(r, \theta)),
$$

(2.4)

and the condition for $p$ on $\Gamma_{\text{shock}}$ changes to

$$
\mathcal{M}p := \sum_{i=1}^{2} \beta_i D_i p = \beta_1 p_r + \beta_2 p_\theta = 0,
$$

(2.5)
where $\beta = (\beta_1, \beta_2)$ is a function of $(p_1, p, r(\theta), r'(\theta))$ with

$$
\beta_1 = -r' \left( \frac{2(r^2 - p)}{r^2} - \frac{|p|}{2p} + \frac{2p(r^2 - p)}{r^2 p} \right), \quad \beta_2 = -\frac{4(r^2 - p)}{r^2} + \frac{|p|}{2p}.
$$

Thus the obliqueness becomes

$$
\beta \cdot (-1, r'(\theta)) = -\frac{2r'}{r^2} \left[ \frac{p(r - r^2)}{p} + r^2 - p \right] \equiv \mu \geq 0.
$$

Note that $\mu$ becomes zero when $r'(\theta) = 0$, that is $r_2 = p$. When the obliqueness fails, we have

$$
\beta_1 = 0, \quad \beta_2 = \frac{|p|}{2p} > 0,
$$

due to $p > p_1$.

Next, we first discuss the normal shock reflection, then derive the necessary conditions for the existence of regular reflection.

2.2. Normal shock reflection. In this case, the wedge angle is $\pi/2$, and the incident shock reflects normally. The reflected shock is also a plane located at $\xi = \xi^*$. Notice that state (2) satisfies $(u_2^*, v_2^*) = (0, 0)$, from the Rankine-Hugoniot condition we obtain

$$
p_2^* = p_1 + \frac{2p_1(p_1 - p_0)}{p_1 + p_0},
$$

and the location of reflected shock

$$
\xi^* = -\sqrt{\frac{p_1 + p_2^*}{2}}.
$$

Moreover, since $|\xi^*|^2 = \frac{p_1 + p_2^*}{2} < p_2^*$, the reflected shock $S_2^*: = \{\xi = \xi^*\}$ can intersect with the sonic circle $\partial B_{c_2^*}(O)$ in two different points, where $c_2^* = \sqrt{p_2^*}$.

2.3. Local existence theory for shock reflection. The necessary condition for the existence of regular shock reflection is stated as follows, whose proof can be found in Zheng [32].

**Proposition 1** (Regular reflection of the algebraic portion [32]). There exists a critical angle $\theta_w = \theta_w^d \in (0, \pi/2)$, depending only on $p_1/p_0$, given by the formula

$$
\tan^2 \theta_w^d = \frac{8p_1(p_1 - p_0)}{(p_1 + p_0)^2},
$$

such that, for each $\theta_w \in (\theta_w^d, \frac{\pi}{2})$ there exist two states $(p_2, u_2, v_2)$ satisfying the Rankine-Hugoniot condition, given by

$$
p_2 = p_1 + p_{10} \tan^2 \theta_w \pm \tan \theta_w \sqrt{p_{10}^2 \tan^2 \theta_w - 2p_1(p_1 - p_0)},
$$

$$
u_2 = \frac{p_2 - p_0}{\sqrt{p_{10}}} \frac{1}{1 + \tan^2 \theta_w}, \quad v_2 = u_2 \tan \theta_w.
$$

Here, we will take the ‘minus one’ or ‘physical one’ for the pressure $p_2$, which tends to $p_2^*$ as $\theta_w \to \frac{\pi}{2}^-$.

Furthermore, by a direct calculation we have
Proposition 2. The state (2) at the reflection point is supersonic in the sense that
\[ \xi_{10}^2 + \eta_{10}^2 > p^2 \]
when \( \theta_w \in (\theta^s_w, \pi) \), where
\[
\tan^2 \theta^s_w := \frac{p_1 - p_0}{(p_1 + p_0)^2} \left( 4p_1 + \sqrt{16p_1^2 + (p_1 + p_2)^2} \right).
\]
Meanwhile, state (2) is subsonic in the sense that \( \xi_{10}^2 + \eta_{10}^2 < p^2 \) for \( \theta_w \in (\theta^d_w, \theta^s_w) \).

Later, the angles \( \theta^s_w \) and \( \theta^d_w \) are called the sonic angle and the detachment angle, respectively.

Definition 2.1. We call a wedge angle \( \theta_w \in (\theta^d_w, \pi] \)
- supersonic if \( \theta_w \in (\theta^s_w, \pi] \);
- sonic if \( \theta_w = \theta^s_w \);
- subsonic if \( \theta_w \in (\theta^d_w, \theta^s_w) \).

Let \( P_0 \) be the intersection point of the incident shock with the wedge. For different wedge angles, there are two types of configurations for the regular reflection: supersonic and subsonic reflection. Supersonic reflection corresponds to when the point \( P_0 \) is outside the sonic circle for state (2); subsonic reflection corresponds to when \( P_0 \) is either on the boundary or in the interior of the sonic circle for state (2), see Figure 1.

In the supersonic regular shock reflection configuration, let \( P_0P_1 \) be the straight shock which intersects the sonic circle \( C_2 \) of state (2) at \( P_1 \), and \( P_1P_2 \) be the reflected curved shock. Let \( P_4 \) be the intersection point between \( C_2 \) and the wedge. There are three uniform states \( (0)(1)(2) \), and a non-uniform state in domain \( \Omega = P_1P_2P_3P_4 \). The solution is equal to state (0) and (1) ahead of and behind the incident shock \( S_0 \), away from \( P_0P_2P_3 \). The solution is equal to state (2) in subregion \( P_0P_1P_4 \), which is supersonic. The non-uniform state in \( \Omega \) is subsonic. We denote the boundary parts of \( \Omega \)
\[
\Gamma_{\text{sonic}} := P_1P_4, \quad \Gamma_{\text{shock}} := P_1P_2, \quad \Gamma_{\text{sym}} := P_2P_3, \quad \Gamma_{\text{wedge}} := P_3P_4
\]
including their endpoints and \( \partial \Omega = \Gamma_{\text{sonic}} \cup \Gamma_{\text{shock}} \cup \Gamma_{\text{sym}} \cup \Gamma_{\text{wedge}} \).
In the subsonic case, the main difference is that the subregion \( P_0 P_1 P_3 \) shrinks into one point \( P_0 \). Hence there are two uniform states (0)(1) and a non-uniform state in \( \Omega \). And \( p \) should match with the state (2) only at the single point \( P_0 \), and the state is subsonic in \( \Omega \).

2.4. The von Neumann conjecture and main theorems. This paper is devoted to solve the von Neumann Detachment Conjecture on shock reflection: There exists a global regular reflection for any angle \( \theta_w \in (\theta^*_w, \frac{\pi}{2}) \), i.e., the existence of state (2) implies the existence of a regular reflection configuration. Moreover, the type (supersonic or subsonic) of the reflection configuration is determined by the type of the weak state (2) at point \( P_0 \).

According to different wedge angles, Problem 2 can further be reformulated into two problems as follows:

**Problem 3** (Supersonic regular shock reflection problem). For a supersonic wedge angle \( \theta_w \in (\theta^*_w, \frac{\pi}{2}) \), find a shock wave \( \Gamma_{shock} \), and a function \( p \), defined in a region \( \Omega \) such that

(i) \( p \) satisfies equation (1.5) in \( \Omega \);

(ii) Equation (1.5) is strictly elliptic in \( \Omega \), i.e., \( \xi^2 + \eta^2 < p \) in \( \Omega \);

(iii) The Rankine-Hugoniot condition and entropy condition are satisfied on \( \Gamma_{shock} \);

(iv) \( p = p_2 \) on \( \Gamma_{sonic} \);

(v) \( p_{w} = 0 \) on \( \Gamma_{sym} \cup \Gamma_{wedge} \), where \( \nu \) is the interior unit normal to \( \Omega \).

**Problem 4** (Subsonic regular shock reflection problem). For a subsonic wedge angle \( \theta_w \in (\theta^*_w, \theta^*_w) \), find a shock wave \( \Gamma_{shock} \), and a function \( p \), defined in a region \( \Omega \) such that all conditions in Problem 3 are satisfied except that (iv) is replaced of \( p = p_2 \) at \( P_0 \).

In most papers concerning the shock reflection problem, they only considered the supersonic reflection case. Recently, Chen-Feldman [9] and Rigby [27] solved the whole regular shock reflection problem, including subsonic reflection for potential flow and nonlinear wave equations, respectively. In this paper we solve the regular shock reflection problem for the pressure gradient system. We first give a definition of admissible solutions for Problems 3-4.

**Definition 2.2** (Admissible solutions). Given \( (p_1, u_1, 0) \) and \( (p_0, 0, 0) \) satisfying (1.8), and wedge defined by (1.3). A weak solution \( p \in C^{0,1}(\Lambda) \) is called an admissible solution if \( p \) satisfies the following properties:

Case I. For \( \theta_w \in (\theta^*_w, \frac{\pi}{2}) \):

(i) There exists a shock curve \( \Gamma_{shock} = \{ r = r(\theta) \} \) with endpoints \( P_1 \) and \( P_2 \), such that

(i-1) Curve \( \Gamma_{shock} \) satisfies

\[
\Gamma_{shock} \subseteq (\Lambda \setminus \overline{B_{\xi_c}(u_1, 0)}) \cap \{ \xi_{P_2} \leq \xi \leq \xi_{P_1} \} \quad \text{with} \quad c_1 = \sqrt{P_1}.
\]

(i-2) \( r \in C^2((\theta_1, \pi)) \cap C^1((\theta_1, \pi)) \), where \( \theta_1 \) is the \( \theta \)-coordinate of \( P_1 \).

(i-3) Curves \( \Gamma_{shock}, \Gamma_{sonic}, \Gamma_{wedge} \), and \( \Gamma_{sym} \) do not have common points except for \( P_1, P_2, P_3 \) and \( P_4 \). Thus, \( \Gamma_{sonic} \cup \Gamma_{shock} \cup \Gamma_{sym} \cup \Gamma_{wedge} \) is a closed curve without self-intersection. Denote by \( \Omega \) the bounded domain enclosed by this closed curve.

(ii) \( p \) satisfies the following properties

(ii-1) There are three constant states (0)(1)(2).

(ii-2) \( p \in C^2(\Omega) \cap C^1(\Omega \setminus \Gamma_{sonic}) \cap C(\Omega) \).
Moreover, \( p \) satisfies: (a). Eq. (1.5) in \( \Omega \) and the Rankine-Hugoniot condition (2.4)-(2.6) on \( \Gamma_{\text{shock}} \); (b). the slip boundary condition on \( \Gamma_{\text{wedge}} \cup \Gamma_{\text{sym}} \);

(iii) Eq. (1.5) is strictly elliptic in \( \Omega \setminus \Gamma_{\text{sonic}} \), i.e.

\[ |\xi|^2 < p \quad \text{in} \quad \Omega \setminus \Gamma_{\text{sonic}}. \]

(iv) \( p_1 < p \leq p_2 \) in \( \Omega \).

Case II. For \( \theta_w \in (\theta^d_w, \theta^*_w) \). In this case the subregion \( P_0P_1P_4 \) shrinks into one point \( P_0 \), and let \( P_0 = P_1 = P_4 \).

(i) There exists a shock curve \( \Gamma_{\text{shock}} = \{ r = r(\theta) \} \) with endpoints \( P_0 \) and \( P_2 \), such that

(i-1) Curve \( \Gamma_{\text{shock}} \) satisfies

\[ \Gamma_{\text{shock}} \subset (\Lambda \setminus B_{\frac{1}{2}}(u_1,0)) \cap \{ \xi P_2 \leq \xi \leq \xi P_0 \}. \]

(i-2) \( r \in C^2(\theta_w, \pi) \cap C^1(\theta_w, \pi) \).

(i-3) Curves \( \Gamma_{\text{shock}}, \Gamma_{\text{wedge}} \) and \( \Gamma_{\text{sym}} \) do not have common points except for \( P_0, P_2 \) and \( P_3 \). Thus, \( \Gamma_{\text{shock}} \cup \Gamma_{\text{sym}} \cup \Gamma_{\text{wedge}} \) is a closed curve without self-intersection.

Denote by \( \Omega \) the bounded domain enclosed by this closed curve.

(ii) \( p \) satisfies the following properties

(ii-1) There are two constant states \((0)(1)\).

(ii-2) \( p \in C^2(\Omega) \cap C^1(\Omega \setminus \{ P_0 \}) \cap C(\Omega) \).

(ii-3) Moreover, \( p \) satisfies: (a). Eq. (1.5) in \( \Omega \) and the Rankine-Hugoniot condition (2.4)-(2.6) on \( \Gamma_{\text{shock}} \); (b). the slip boundary condition on \( \Gamma_{\text{wedge}} \cup \Gamma_{\text{sym}} \);

(iii) Eq. (1.5) is strictly elliptic in \( \Omega \setminus \{ P_0 \} \), i.e.

\[ |\xi|^2 < p \quad \text{in} \quad \Omega \setminus \{ P_0 \}. \]

(iv) \( p_1 < p \leq p_2 \) in \( \Omega \).

Our main result of this paper is stated as follows.

**Theorem 2.3** (Global existence for regular shock reflection). Given \((p_1, u_1, 0)\) and \((p_0, 0, 0)\) satisfying (1.8), and wedge defined by (1.3). For any \( \theta_w \in (\theta^d_w, \frac{\pi}{2}) \), there exists an admissible solution of Problem 3 and 4 in the sense of Definition 2.2.

The main difficulty is how to link different cases as \( \theta_w \) crosses the sonic angle to the detachment angle continuously. In Section 3 and 4, we will consider Problem 3 and 4, respectively.

### 3. Existence result for the supersonic regular shock reflection

In this section we consider Problem 3. We first introduce a new coordinate system.

#### 3.1. Shifting coordinates

Let

\[ (\xi, \eta)_{\text{new}} := (\xi, \eta) - (u_2, v_2), \]

which shifts the origin in the self-similar plane to the center of the sonic circle of state (2). Without confusion of notation, we will always work in the new coordinates without changing the notation \((\xi, \eta)\).

In the new shifted coordinates, the elliptic domain \( \Omega \) can be expressed as

\[ \Omega = B_{\xi_2}(O) \cap \{ \eta > -v_2 \} \cap \{ f(\eta) < \xi < \eta \cot \theta_w \}, \]

where \( \xi = f(\eta) \) is the position of the reflected shock \( \Gamma_{\text{shock}} \). Let \( S_1 : \xi = l(\eta) \) be the location of the reflected shock of state (2), which is a straight line,

\[ l(\eta) = \eta \cot \theta_s + \xi, \]
where $\theta_s$ and $(\xi,0)$ are the angle and the intersection point between the shock $S_1$ and $\eta = 0$, respectively.

Denote by $P_1(\xi_1, \eta_1)$ with $\eta_1 > 0$ the intersection point of the line $S_1$ and the sonic circle of state (2), i.e.,

$$\xi_1^2 + \eta_1^2 = p_2, \quad \xi_1 = l(\eta_1).$$

From the fact that the reflected shock $\Gamma_{\text{shock}}$ and the straight part of the reflected shock $S_1$ should match at least up to first-order, we have

$$f(\eta_1) = l(\eta_1), \quad f'(\eta_1) = l'(\eta_1) = \cot \theta_s.$$

Furthermore, equation (1.5) and the Rankine-Hugoniot conditions on $\Gamma_{\text{shock}}$ do not change in the shifted coordinates. The remaining boundary conditions on the other parts of $\partial \Omega$ are

$$p = p_2 \text{ on } \Gamma_{\text{sonic}} = \partial \Omega \cap \partial B_{\nu_2}(O), \quad p_\nu = 0 \text{ on } \Gamma_{\text{wedge}} \cup \Gamma_{\text{sym}},$$

here $\Gamma_{\text{wedge}} := \partial \Omega \cap \{\eta = \xi \tan \theta_w\}$, and $\Gamma_{\text{sym}} := \partial \Omega \cap \{\eta = -\nu_2\}$.

3.2. Some basic estimates. In the following, we always assume that $(p,r)$ is an admissible solution of Problem 3 and derive some basic estimates.

3.2.1. Monotonicity of $p$ on $\Gamma_{\text{shock}}$ and convexity of $\Omega$.

**Proposition 3.** Let $(p,r)$ be an admissible solution for Problem 3. Then $p$ is monotone increasing on $\Gamma_{\text{shock}}$ from $P_1$ to $P_2$, and the elliptic region $\Omega$ is convex.

**Proof.** The monotonicity of $p$ along $\Gamma_{\text{shock}}$ can be proved by contradiction method. Let us examine the $C^\alpha$-function $p$ on $\Gamma_{\text{shock}}$. Without confusion, we can label the points along $\Gamma_{\text{shock}}$ by their $\xi$-coordinates, and refer to intervals along $\Gamma_{\text{shock}}$ by the labels. The lack of monotonicity implies that, there exist two points $A$ and $B$ on $\Gamma_{\text{shock}}$ with $P_2 < A < B < P_1$, such that $p(A) < p(B)$. We immediately deduce that

- (a) In $(A, P_1)$, there exists $\tilde{C}$ with $p(\tilde{C}) = \max_{[A, P_1]} p$;
- (b) In $(P_2, \tilde{C})$, there exists $D$ with $p(D) = \min_{[P_2, \tilde{C}]} p$.

Then we can find two points $C$ and $D$ on $\Gamma_{\text{shock}}$ with $D < C$, such that

- (i) $p(D) \leq p \leq p(P_2)$ on $[P_2, D]$;
- (ii) $p(D) \leq p \leq p(C)$ on $[D, C]$;
- (iii) $p(P_1) \leq p \leq p(C)$ on $[C, P_1]$.

Here property (ii) may not hold with $C = \tilde{C}$ because that $p(\tilde{C})$ is the maximum value only on the interval $[D, P_1]$. If $D > A$ and there is a point in $(A, D)$ at which $p > p(\tilde{C})$, then we let $C$ be the point at which $p$ obtains its maximum value in this interval. Otherwise, we let $C = \tilde{C}$. Then all three properties hold.

Now we look at the function $p$ in the domain $\Omega$. The idea is to partition $\Omega$ into three subdomains by two curves $\Gamma_C$ and $\Gamma_D$ from $C$ and $D$ to points $E$ and $F$ on $\Gamma_{\text{wedge}} \cup \Gamma_{\text{sym}}$, respectively, such that $p(E) > p(F)$. We want to deduce that there is a point $X_0$ on $\Gamma_{\text{wedge}}$ at which $p$ achieves a maximum on either the subdomain $\Omega_A$ or the domain $\Omega_B$, which violates the Hopf maximum principle. It suffices to show that $p(X_0)$ is the maximum value of $p$ on the boundary of $\Omega_E$ or $\Omega_F$.

Let

$$\mu = \frac{1}{4} \min \{p(C) - p(P_1), P(C) - P(D), p(P_2) - p(D)\}.$$
Figure 2. Hypothetical curves

We want to construct the Lipschitz curves on which \( p \) satisfies

\[
\begin{align*}
p(E) &\geq p \geq p(C) - \mu \quad \text{on } \Gamma_C, \quad p(E) > p(C), \\
p(F) &\leq p \leq p(D) + \mu \quad \text{on } \Gamma_D, \quad p(F) < p(D).
\end{align*}
\]

(3.1)

Since \( p \in C^\alpha(\overline{\Omega}) \), we have

\[
p(X_1) - p(X_2) \leq M|X_1 - X_2|^\alpha
\]

for some \( M > 0 \) and \( X_1, X_2 \in \overline{\Omega} \). Thus on any ball with radius \( r > 0 \), we have

\[
Osc(p) \leq 2Mr^\alpha.
\]

Let \( R = (\frac{M}{Mr})^{1/\alpha} \), then \( Osc_{B_R\cap\Omega}(p) \leq \mu \). We construct \( \Gamma_C \) as follows. In \( B_R(C)\cap\Omega \), let \( Y_1 \) be a point at which \( p \) attains its maximum value in \( B_R(C) \). Then the first segment of \( \Gamma_C \) is a straight line from \( C \) to \( Y_1 \), and on the segment, we have

\[
p(Y) \geq p(C) - \mu, \quad p(Y) \leq p(Y_1).
\]

Now we continue inductively, forming a sequence of line segments with corners at \( \{Y_i\} \) (take \( Y_0 = C \)), along which \( p \geq p(C) - \mu \), and \( p(Y_1) \leq p(Y_2) < \cdots \). Since the domain is finite, the process must terminate after finite steps when we reach a point \( E \in \partial\Omega \). By construction, \( \Gamma_C \) has the properties indicated in (3.1). Similarly, we construct \( \Gamma_D \) with termination point \( F \in \partial\Omega \).

Next we show that the points \( E \) and \( F \) lie on \( \Gamma_{\text{wedge}} \cup \Gamma_{\text{sym}} \). First the two curves can not cross each other, because at every point on \( \Gamma_C \), \( p(X) \geq p(C) - \mu > p(D) + \mu \), while at every point on \( \Gamma_D \), we have \( p(X) \leq p(D) + \mu \). Also, \( \Gamma_C \) can not terminate at \( \Gamma_{\text{sonic}} \) where \( p > p(P_1) - \mu > p(D) - \mu \). Furthermore, \( \Gamma_C \) can not come back to \( \Gamma_{\text{shock}} \) in \([D, C]\) or \([C, P_1]\) where \( p \leq p(C) \). Finally, \( E \) can not lie in the segment \([P_2, D]\) because this would trap \( \Gamma_D \) in a region where \( p \leq p(D) \), which contradicts the fact that \( D \) is a local minimum in \( \Omega \). Hence \( E \in \Gamma_{\text{wedge}} \cup \Gamma_{\text{sym}} \). Similarly, \( F \) must lie on \( \Gamma_{\text{wedge}} \cup \Gamma_{\text{sym}} \) between \( P_2 \) and \( F \).

Now we find the final contradiction. Since \( p(F) \) is smaller than \( p(P_2) \) and \( p(E) \), then there must be a point \( X_0 \) along the boundary \( P_2OE \) at which \( p \) attains its minimum. Assume that \( X_0 \) is not the origin, then \( X_0 \) can not be a local minimum for the domain \( \Omega \) by the Hopf maximum principle. However, along the entire boundary of the domain \( P_2P_3ECDP_2 \), \( p \geq p(X_0) \), which implies that it is a minimum. This is a contradiction. Now if \( X_0 \) coincides with the origin, the similar maximum point resembling \( E \) can not coincide with the origin. We can find no existence of such maximum point. We conclude that \( C \) and \( D \) do not exist, and hence \( \hat{C} \) and \( \hat{D} \).
do not exist, and $p$ is monotonic on $\Gamma_{\text{shock}}$. Since $\Gamma_{\text{shock}}$ is vertical at $P_2$, and is tangent to $S_1$ at $P_1$, we conclude that $p$ is monotone increasing, pointing from $P_1$ to $P_2$.

Next, we calculate the curvature of $\Gamma_{\text{shock}}$. Let $p_r$ denote the tangential derivative of $p$ along $\Gamma_{\text{shock}}$ in the direction of increasing $\theta$. By the formula for $r'(\theta)$ in (2.4), we have

$$r^2 + 2(r')^2 - rr'' = \frac{r^4}{4(p)^2} \sqrt{\frac{p}{r^2 - p}} p_r > 0,$$

which implies the curvature $k(\theta)$ of $\Gamma_{\text{shock}}$ is positive. Hence $\Gamma_{\text{shock}}$ is concave, so the domain $\Omega$ is convex.

The convexity of $\Omega$ leads to the following corollary:

**Corollary 1.** Let $\nu_w = (-\sin \theta_w, \cos \theta_w)$ be the outer unit normal to $\Gamma_{\text{wedge}}$, and $\nu_{sh}$ the inner unit normal to $\Gamma_{\text{shock}}$ for the admissible solution $(p, r)$. Then

$$\nu_w \cdot \nu_{sh} \leq -\delta,$$

for some $\delta > 0$ depending only on the initial data.

**Proof.** At $P_1$, $\nu_s \cdot \nu_w = \nu_s \cdot \nu_w = \sin(\theta_s - \theta_w) < -C$, where $C$ depends only on the initial data. At $P_2$, $\nu_s \cdot \nu_w = (1, 0) \cdot \nu_w = -\sin \theta_w < -\sin \theta^* > 0$. Taking $\delta = \min\{C, \sin \theta^*_w\}$, we prove (3.2) by the convexity of $\Omega$. \(\square\)

3.2.2. *Bounds for the elliptic region $\Omega$ and $p$ within $\Omega*. We first note that

**Lemma 3.1.** For any admissible solution $(p, r)$, $p$ can only attain its global maximum and minimum on $\Gamma_{\text{shock}} \cup \Gamma_{\text{sonic}}$:

$$\sup_{\Omega} p = \sup_{\Gamma_{\text{shock}} \cup \Gamma_{\text{sonic}}} p, \quad \inf_{\Omega} p = \inf_{\Gamma_{\text{shock}} \cup \Gamma_{\text{sonic}}} p.$$

**Proof.** Since equation (1.5) is strictly elliptic in $\Omega \setminus \Gamma_{\text{sonic}}$, we can consider the domain

$$\Omega_\epsilon := \{ P \in \Omega : \text{dist}(P, \partial \Omega) > \epsilon \} \quad \text{for } \epsilon > 0,$$

in which (1.5) is uniformly elliptic, and then by sending $\epsilon \to 0+$ we conclude that the extremum of $p$ cannot be attained in the interior of $\Omega$. Moreover, for any $P \in \Gamma_{\text{sym}} \cup \Gamma_{\text{wedge}}$, equation (1.5) is uniformly elliptic in some neighborhood of $P$. Due to $p_\nu = 0$ on $\Gamma_{\text{sym}} \cup \Gamma_{\text{wedge}}$, the extremum of $p$ over $\Omega$ cannot be attained on $\Gamma_{\text{sym}} \cup \Gamma_{\text{wedge}}$, unless $p$ is constant in $\Omega$. However, $p$ cannot be a constant. Therefore, the extremum of $p$ must be attained on $\Gamma_{\text{sonic}} \cup \Gamma_{\text{shock}}$. \(\square\)

Moreover,

**Lemma 3.2.** For an admissible solution $(p, r)$ with wedge angle $\theta_w \in (\theta^*_w, \frac{\pi}{2})$, there exists a constant $C < \infty$ such that

$$\Omega \subset B_{c_2}(O),$$

$$p_1 \leq p \leq C \quad \text{in } \Omega,$$

$$\text{Lip}(r) \leq C.$$  

**Proof.** Since $r'(\theta) \leq 0$, so $r(\theta) \leq r(\theta_1) = c_2$ on $\Gamma_{\text{shock}}$, (3.3) holds by the structure of the region. Since $\Gamma_{\text{shock}}$ lies outside the sonic circle of state (1), and by the ellipticity of the equation, we have

$$p \leq r^2 \leq p_2, \quad p_1 < r^2 \leq p,$$
so (3.4) holds. By the explicit formula (2.4) for $r'(\theta)$, we can obtain (3.5) easily. 

The next lemma states the uniform separations of the components of $\partial \Omega$, which implies that we can consider each part of the boundary separately, and piece local estimates together to obtain a global one.

**Lemma 3.3.** Let $\theta_w^* \in (\theta_w^*, \frac{3\pi}{2})$. There exists a positive constant $C < \infty$ depending only on the initial data such that for any admissible solution with wedge angle $\theta_w \in [\theta_w^*, \frac{3\pi}{2})$,

\begin{align}
\text{dist}(\Gamma_{\text{sonic}}, \Gamma_{\text{sym}}) & \geq \frac{1}{C}, \\
\text{dist}(\Gamma_{\text{shock}}, \Gamma_{\text{wedge}}) & \geq \frac{1}{C}.
\end{align}

**Proof.** In the polar coordinate, $\Gamma_{\text{sonic}} = \{r = c_2|\theta \in [\theta_w, \theta_1]\}$ and $\Gamma_{\text{sym}} = \{\theta_2 \leq \theta \leq \theta_2 + \pi\}$. Since $\theta_1 < \theta_2 - 1/C$ for some $C$ depending only on the initial data, so (3.6) holds. Estimate (3.7) follows similarly, because of $\Gamma_{\text{shock}} = \{(r, \theta)|r = r(\theta) > c_1, \theta \in [\theta_1, \theta_2]\}$ and $\theta_1 > \theta_w + 1/C$. 

The next lemma is for a uniform positive lower bound for $p - p_1$, which is critical for deriving ellipticity and obliqueness estimates in $\Omega$.

**Lemma 3.4.** Let $\theta_w^* \in (\theta_w^*, \frac{3\pi}{2})$, there exists $\delta > 0$ depending only on the initial data and $\theta_w^*$, such that for any admissible solution $(p, r)$ with the wedge angle $\theta_w \in [\theta_w^*, \frac{3\pi}{2})$, we have

$$p_1 + \delta < p \leq p_2 \quad \text{in } \Omega, \quad \text{and} \quad r > c_1 + \delta \quad \text{on } \Gamma_{\text{shock}}.$$ 

**Proof.** It suffices to prove $p > p_1 + \delta$. We prove this result by contradiction. Suppose that there exists a sequence of admissible solution $(p_k, r_k)$ with wedge angles $\theta^{(k)}_w$ and points $\xi_k \in \Gamma_{\text{shock}} = \{r = r_k(\theta)|\theta \in [\theta_1, \theta_2]\}$ such that $p_k(P_2) \to p_1$. We can show that $r_k$ converges in $C^{1, \alpha}$ to some limiting $r$, and $\theta^{(k)}_w, p_k$ converge to some $\theta_w, p$, respectively. Consider a neighborhood $\mathcal{N}$ of $P_2$ in $\Omega$. We construct a barrier function of the form

$$w = p_1 + A(r_1 - r)^{\frac{1}{2}} + B(r_1 - r)^{\beta} + C(\theta - \theta_2).$$

Via choosing suitable constants $A, B, C$ and $\beta \in \left(\frac{1}{2}, 1\right)$ we can prove that the optimal regularity of $p$ is $C^{\frac{1}{2}}$ near the shock in $\mathcal{N}$. Next we introduce the coordinates

$$(x, y) = (r_1 - r, \theta - \theta_2)$$

and let $\varphi = p - p_1$. We scale $\varphi$ in $\mathcal{N}$ by defining

$$u(S, T) = S^{-\frac{1}{2}}\varphi(S^{-\frac{1}{2}}x, S^{-\frac{1}{2}}y).$$

It can be verified that due to the $C^{\frac{1}{2}}$ regularity of $p$, $u(S, T)$ satisfies an uniformly elliptic equation in $(S, T)$ coordinates. Finally, according to the Harnack’s inequality for uniformly elliptic equations, for $x_0^{-\frac{1}{2}} < S \leq x_0^{-\frac{1}{2}}$ with small $x_0$, we have

$$\frac{ax_0^{\frac{1}{2}}}{S} \leq \sup_{x_0^{-\frac{1}{2}} < S \leq x_0^{\frac{1}{2}}} \varphi(S, T) \leq C \inf_{x_0^{-\frac{1}{2}} < S \leq x_0^{\frac{1}{2}}} \varphi(S, T) \leq Cx_0^{\frac{1}{2}}$$

for some constants $a, C < \infty$. This implies that $x_0^{-\frac{1}{2}} \leq C$, which is a contradiction if $x_0$ is sufficiently small. Hence $p > p_1 + \delta$. 

\end{proof}
3.2.3. Uniform ellipticity estimates. We have the following uniform ellipticity estimates.

**Proposition 4.** Let $\theta^*_w \in (\theta^*_w, \frac{\pi}{2})$. There exist $\lambda > 0$ and $\delta > 0$ depending only on the initial data and $\theta^*_w$, such that for any admissible solution $(p, r)$ with the wedge angle $\theta^*_w \in [\theta^*_w, \frac{\pi}{2})$,

$$p - r^2 \geq \lambda \text{dist}(\xi, \Gamma_{\text{sonic}}) \quad \text{in } \Omega. \quad (3.8)$$

**Proof.** We first define a smooth approximation to $\xi \rightarrow \text{dist}(\xi, \Gamma_{\text{sonic}})$ as follows. Let $C_2 = \partial B_{r_2}(O)$ be the sonic circle of state (2). Denote by $Q$ another point of intersection of the straight shock $S_1$ and $C_2$ except $P_1$. Denote by $Q' = \frac{P_1 + Q}{2}$ and let $\hat{c} = |OQ'|$. Then $\hat{c} = \sqrt{c_2^2 - \frac{1}{4}|P_1Q|^2} < c_2$. Let $h \in C^\infty([0, \infty))$ satisfy

$$h(s) = \begin{cases} s & \text{if } s \in [0, \frac{1}{2}], \\ 1 & \text{if } s \geq 1, \end{cases} \quad 0 \leq h' \leq 2 \text{ on } [0, \infty).$$

Define

$$g(\xi) = \frac{1}{2}(c_2 - \hat{c})h \left( \frac{\text{dist}(\xi, C_2)}{c_2 - \hat{c}} \right),$$

which is an approximation of $\text{dist}(\xi, \Gamma_{\text{sonic}})$. We consider the function $p - r^2 - \lambda g(\xi)$, where $\lambda > 0$ will be specified later. Applying the ellipticity principle, we can prove that $p - r^2 - \lambda g(\xi)$ cannot attain a minimum in the interior of $\Omega$. Since $p$ satisfies slip boundary condition on $\Gamma_{\text{wedge}}$ and $\Gamma_{\text{sym}}$, we can extend $\Omega$ and $p$ by even reflection, then the boundary $\Gamma_{\text{wedge}}$ and $\Gamma_{\text{sym}}$ can be treated as the interior of the extended domain. Similarly, we can show that $p - r^2 - \lambda g(\xi)$ cannot attain its minimum on $\Gamma_{\text{wedge}} \cup \Gamma_{\text{sym}}$. Next, we turn to the shock boundary. Suppose that $p - r^2 - \lambda g(\xi)$ attain its minimum at $P_{\text{min}} \in \Gamma_{\text{shock}}$. Then $(p - r^2)(P_{\text{min}}) \leq \lambda g(P_{\text{min}}) \leq AC$ for some $C > 0$. Using the boundary condition (2.5)-(2.6), we can choose $\lambda$ sufficiently small, then

$$\beta \cdot \nabla (p - r^2 - \lambda g(\xi))(P_{\text{min}}) < 0,$$

which contradicts the Hopf maximum principle. Thus, $p - r^2 - \lambda g(\xi)$ must attain its minimum on $\Gamma_{\text{sonic}}$, and (3.8) is proved. \qed

3.3. Regularity away from the sonic arc. In the subsection, we will obtain uniform estimates for admissible solutions away from the sonic arc. Define

$$\Omega_\epsilon = \{ \xi \in \Omega | \text{dist}(\xi, \Gamma_{\text{sonic}}) < \epsilon \}.$$

For the estimates in $\Omega \setminus \Omega_\epsilon$ with small $\epsilon > 0$, we should pay more attention to the wedge vertex $P_3$, and the point $P_2$, where the obliqueness fails.

3.3.1. Regularity away from $P_2 \cup \Gamma_{\text{sonic}}$. Notice that for $\epsilon > 0$, equation (1.7) is strictly elliptic in $\Omega \setminus \Omega_\epsilon$ with constant depending on $\epsilon$. For small $\delta > 0$, the boundary condition (2.5)-(2.6) is strictly oblique on $\Gamma_{\text{shock}} \setminus B_\delta(P_2)$ with obliqueness constant depending on $\delta$. Moreover, $r'(\theta)$ is a Lipschitz function of $p$ and $r$, with Lipschitz constant depending on $\delta$ and initial data. Thus, away from $P_3$, we can obtain $C^{1, \alpha}$ estimate depending on the initial data using classical elliptic theory. Near $P_3$, we can use the method in [23] to obtain a $C^{0, \alpha}$ estimate and then extend it to $C^{1, \alpha}$ following the method in [24, 29]. The regularity away from $P_2 \cup \Gamma_{\text{sonic}}$ is stated as follows.
Lemma 3.5. Let $\theta_w^* \in (\theta_w^*, \frac{\pi}{2})$. For any $\epsilon, \delta > 0$, there exist $\alpha \in (0, 1)$ and $C > 0$ depending on $\epsilon, \delta$, the initial data and $\theta_w^*$ such that, for any admissible solution $(p, r)$ corresponding to $\theta_w \in [\theta_w^*, \frac{\pi}{2}]$, we have

\[
\|p\|_{1,\alpha,\Omega \setminus (\Omega \cup B_0(p_2))} \leq C, \\
\|r\|_{2,\alpha,((\theta_1+\epsilon, \theta_2-\delta))} \leq C.
\] (3.9) (3.10)

3.3.2. Regularity near $P_2$. Now we consider the regularity near $P_2$, where the obliqueness fails. Because $r(\theta)$ is Lipschitz continuous with bound depending only on the initial data, from the expression (2.4) of $r(\theta)$, we can easily prove

Lemma 3.6. There exists $C > 0$ depending only on the initial data such that

\[
r^2 - \rho \leq C(\theta_2 - \theta) \quad \text{on } \Gamma_{\text{shock}}, \\
-C(\theta_2 - \theta)^{1/2} \leq r'(\theta) \leq 0 \quad \text{for all } \theta \in [\theta_1, \theta_2].
\]

Next we will show an algebraic growth estimate of $p$ on $\Gamma_{\text{shock}}$ in order to obtain the weighted Hölder norms for admissible solutions.

Lemma 3.7. Let $\theta_w^* \in (\theta_w^*, \frac{\pi}{2})$. Let $(p, r)$ be an admissible solution with $\theta_w \in [\theta_w^*, \frac{\pi}{2})$. Then there exists $R > 0$ depending only on the data and $\theta_w^*$, such that for any integer $m \geq 1$ and $M > 0$, $p - p(P_2) + M(\theta_2 - \theta)^m$ does not attain a minimum on $\Gamma_{\text{shock}} \cap B_R(P_2)$.

Furthermore, there exists $0 < C < \infty$ depending on $R$, $\theta_w^*$ and the initial data such that

\[
|p - p(P_2)| \leq C(\theta_2 - \theta)^m \quad \text{on } \Gamma_{\text{shock}} \cap B_R(P_2).
\] (3.11)

Proof. A minimum point $X \in \Gamma_{\text{shock}} \cap B_R(P_2)$ is either a minimum within $\Omega \cap B_R(P_2)$ or a saddle point. If $X$ is a minimum point, by the explicit formula (2.6), we have

\[
\beta_2 = -\frac{4(p^2 - \rho)}{r^2} + \frac{|p|}{2r} \geq \frac{1}{C}
\]

for some $0 < C < \infty$ depending on the data and $\theta_w^*$. Therefore,

\[
\beta \cdot \nabla (p - p(P_2) + M(\theta_2 - \theta)^m) = -\beta_2 M(\theta_2 - \theta)^m - 1 < 0,
\]

which is a contradiction due to the Hopf maximum principle. It needs more care to handle the saddle case. One can construct a modified function $\psi$ instead of $p - p(P_2) + M(\theta_2 - \theta)^m$, such that $\psi$ can attain a minimum over $\Gamma_{\text{shock}}$ at $X$. Applying the Hopf maximum principle again one can derive a contraction. The construction of $\psi$ can be referred to Lemma 3.2 in [21] for details.

Similarly, we can prove that $p - p(P_2) - M(\theta_2 - \theta)^m$ does not attain a maximum on $\Gamma_{\text{shock}} \cap B_R(P_2)$. Thus, we conclude (3.11). □

We introduce the notation for the weighted Hölder norms. Let $\Omega \subset \mathbb{R}^2$ be an open bounded set and $\Sigma \subset \partial \Omega$. Set

\[
\delta_x := \text{dist}(x, \Sigma), \quad \delta_{x,y} := \min\{\delta_x, \delta_y\} \quad \text{for } x, y \in \Omega.
\]

Then, for $k \in \mathbb{R}$, $\alpha \in (0, 1)$ and $m \in \mathbb{N}$, define

\[
\|u\|^{(k), \Sigma}_{m,0,\Omega} := \sum_{0 \leq |\beta| \leq m} \sup_{x \in \Omega} (\delta_x^{\max\{m+k, 0\}} |D^\beta u(x)|), \\
[u]^{(k), \Sigma}_{m,\alpha,\Omega} := \sum_{|\beta|=m} \sup_{x,y \in \Omega, x \neq y} (\delta_{x,y}^{\max\{m+\alpha+k, 0\}} \frac{|D^\beta u(x) - D^\beta u(y)|}{|x-y|^\alpha}).
\]
Lemma 3.8. Let $\theta_w^* \in (\theta_w, \frac{\pi}{2})$. Let $(p, r)$ be an admissible solution with wedge angle $\theta_w \in \left[\theta_w^*, \frac{\pi}{2}\right)$. Then there exists $\alpha \in (0, 1)$ depending on the initial data and $\theta_w^*$ such that, for $\epsilon > 0$ and $\beta \in (0, 1)$, there exists $C < \infty$ depending on $\epsilon$, $\beta$, the initial data and $\theta_w^*$ such that

$$
\|p\|_{1,\alpha,\Omega;\{P_2\}}^{(-\beta),\{P_1\}} \leq C, \quad (3.12)
$$

$$
\|r\|_{2,\alpha,\{\theta_1+\epsilon, \theta_2\}}^{(-1-\beta/2),\{\theta_2\}} \leq C. \quad (3.13)
$$

Proof. Using the structure of $\Omega$ and the property $r'(\theta)$ we can deduce that for any $a > 0$, the ball $B_a(\xi_2+a, \eta_2)$ is an exterior ball to $\Omega$ at $P_2$. We may employ Theorem 1.1.15 [15] to deduce that for $C < \infty$ depending only on $M$ from Proposition 3.7 and the ellipticity and boundedness of the coefficients in $B_R(P_2)$, that

$$
|p-p(P_2)| \leq C|\xi - \xi_2|^{\frac{\alpha}{\beta}} \leq |\xi - \xi_2|^{\beta} \text{ in } \Omega \cap B_R(P_2),
$$

for $R$ depending on the initial data and $\theta_w^*$. Since there exists $\sigma > 0$ such that

$$
|\beta_1| \leq \sigma, \quad \beta_2 > \frac{1}{C} \text{ on } B_R(P_2) \cap \Gamma_{\text{shock}},
$$

by a standard argument with rescaled functions, we can deduce that there exists $\alpha \in (0, 1)$, depending only on the ellipticity in $B_R(P_2)$, such that for any $\beta \in (0, 1)$, there exists $C(\beta)$ depending only on $\beta$, $\theta_w^*$ and the initial data, such that

$$
\|p\|_{1,\alpha,\Omega;\{P_2\}}^{(-\beta),\{P_1\}} \leq C(\beta). \quad (3.12)
$$

Combining with the former estimate (3.9), we obtain (3.12).

We now turn to (3.13). Consider the function

$$
h(r, \theta) = \frac{r^2(\nu^2 - p)}{\bar{p}},
$$

which is a smooth function of $p$ and $r$. Thus

$$
\|h\|_{1,\alpha,\Omega;\{\Omega, \cup B_{\epsilon/2}(P_2)\}}^{(-\beta),\{P_3\}} \leq C(\beta).
$$

Also, by formula (2.4), one has

$$
\|r\|_{1,\alpha,\{\theta_1+\epsilon, \theta_2\}} \leq C(\beta).
$$

If we choose $1 + \beta/2 > 1 + \alpha$, we can deduce that $H(\theta) := h(r(\theta), \theta)$ satisfies

$$
\|H\|_{1,\alpha,\{\theta_1+\epsilon, \theta_2\}}^{(-\beta)} \leq C(\beta).
$$

Since $r'(\theta) = \sqrt{H(\theta)}$, $r''(\theta) = \frac{H'(\theta)}{2\sqrt{H(\theta)}}$. It can be verified that

$$
|r''(\theta)| \leq C(\theta_2 - \theta)^{3/2} \|H\|_{1,\alpha,\{\theta_1+\epsilon, \theta_2\}}^{(-\beta)}.
$$

Moreover, for $\theta \leq \tilde{\theta}$, we find that

$$
|r''(\theta) - r''(\tilde{\theta})| \leq \left| \frac{H'(\theta) - H'(\tilde{\theta})}{2\sqrt{H(\theta)}} \right| + \left| H'(\theta) \left( \frac{1}{2\sqrt{H(\theta)}} - \frac{1}{2\sqrt{H(\tilde{\theta})}} \right) \right|
\leq C(\theta_2 - \theta)^{2\beta - \alpha - 5/2} |\theta - \tilde{\theta}|^\alpha \|H\|_{1,\alpha,\{\theta_1+\epsilon, \theta_2\}}^{(-\beta)},
$$

where $M$ is the maximal radius of $\Omega$. 

We can now deduce the following estimates:
which leads to
\[ \|r\|^{(1/2 - 2\beta)}_{2,\alpha;[\theta_1 + \epsilon, \theta_2]} \leq C(\beta). \]
Since \( \beta \in (0, 1) \) is arbitrary, we can deduce (3.13).

3.4. Structure near the sonic arc. In this subsection, we obtain the regularity results near the sonic arc. Here we assume that \( \theta^*_w \in (\theta^*_w, \frac{\pi}{2}) \) and obtain estimates depending on \( \theta^*_w \) and initial data, valid for wedge angles \( \theta^*_w \in [\theta^*_w, \frac{\pi}{2}) \), which means that these estimates depend on the length of the sonic arc but not uniform in \( \theta_w \in [\theta^*_w, \frac{\pi}{2}) \).

3.4.1. Equations in \((x, y)\)-coordinates near \(\Gamma_{\text{sonic}}\). Let \((r, \theta)\) be the polar coordinates for \(\xi\), and set
\[ (x, y) := (c_2 - r, \theta). \] (3.14)
Then \(\Gamma_{\text{sonic}} = \{(x, y) | x = 0, \theta_w \leq y \leq \theta_1\}\). Also,
\[ \Gamma_{\text{shock}} \cap \partial \Omega_e := \{y = f(x), 0 < x < \epsilon\}, \]
\[ \Omega_e = \{0 < x < \epsilon, \theta_w < y < f(x)\}. \]
Define \(\varphi = p_2 - p\). Since \(p = p_2\) on \(\Gamma_{\text{sonic}}\), \(\varphi\) tends to zero near \(\Gamma_{\text{sonic}}\). Due to (1.7), we note that \(\varphi\) satisfies the following equation
\[ \sum_{i=1}^{2} A_{ii}(x, \varphi)\varphi_{ii} + A_1(x, \varphi)\varphi_1 - B_1(x, \varphi)\varphi_1^2 = 0, \] (3.15)
where
\[ A_{11}(x, z) = 2c_2x - z + O_1(x, z), \quad O_1(x, z) = -x^2, \]
\[ A_{22}(x, z) = 1 + O_2(x, z), \quad O_2(x, z) = \frac{2c_2x - z - x^2}{(c_2 - x)^2}, \]
\[ A_1(x, z) = c_2 + O_3(x, z), \quad O_3(x, z) = -3x + \frac{z - x^2}{c_2 - x}, \]
\[ B_1(x, z) = 1 + O_4(x, z), \quad O_4(x, z) = \frac{z - 2c_2x + x^2}{p_2 - z}. \] (3.16)

As for the pointwise estimate of \(\varphi\), we have a Lipschitz bound which follows from the degenerate ellipticity.

**Lemma 3.9.** There exist \(\epsilon_1 > 0\) and \(\delta > 0\) depending on initial data and \(\theta^*_w\) such that for any admissible solution \((p, r)\) with \(\theta_w \in [\theta^*_w, \frac{\pi}{2})\), we have
\[ 0 \leq \varphi \leq (2c_2 - \delta)x \quad \text{for} \quad x \in (0, \epsilon_1). \]

**Proof.** \(\varphi \geq 0\) follows from \(p \leq p_2\). From (3.8) we have
\[ \lambda x \leq \lambda \text{dist}(\xi, \Gamma_{\text{sonic}}) \leq p - (c_2 - x)^2 = -\varphi + 2c_2x - x^2. \]
This lemma can be proved by taking \(\epsilon_1 = \lambda/2, \delta = \lambda/2\).

Therefore, we have
\[ \frac{|O_1(x, z)|}{x^2} + \frac{|DO_1(x, z)|}{x} + \frac{O_1(x, z)}{|x| + |z|} + |DO_1(x, z)| \leq C, \]
for some constant \(C > 0\) and \(i = 2, 3, 4\).

Following these expressions we can deduce directly
Lemma 3.10. For any $\delta \in (0, c_2)$, there exists $\epsilon > 0$ such that, if $0 < x < \epsilon$ and $0 < z < (2c_2 - \delta)x$, then

$$\frac{\delta}{2} |\kappa|^2 \leq \frac{A_{11}(x, z)}{x} \kappa_1^2 + A_{22}(x, z) \kappa_2^2 \leq \frac{\delta}{2} |\kappa|^2$$

for any $\kappa = (\kappa_1, \kappa_2) \in \mathbb{R}^2$, $\kappa \neq 0$.

3.4.2. Structure of $\Gamma_{\text{shock}}$ and Rankine-Hugoniot condition near $\Gamma_{\text{sonic}}$. We rewrite the shock equation (2.4) as

$$r'(\theta) = -\frac{c_2}{\sqrt{p_2 - \overline{p}}} \sqrt{\frac{p_2 - \overline{p}}{p_2 - p_1}} + O_5(x, \varphi),$$

where $\overline{p} = \frac{p_1 + p_2 - \varphi}{2}$, and

$$\left| \frac{O_5(x, z)}{|x| + |z|} \right| + |DO_5(x, z)| \leq C.$$ 

Hence, for $\Gamma_{\text{shock}} \cap \partial \Omega_\epsilon := \{ y = f(x), 0 < x < \epsilon \}$,

$$f'(x) = -\frac{1}{r'(\theta)} = \frac{1}{c_2} \sqrt{\frac{p_2 - \overline{p}}{p_2 - p_1}} + O_6(x, \varphi) \quad (3.17)$$

with

$$\left| \frac{O_6(x, z)}{|x| + |z|} \right| + |DO_6(x, z)| \leq C.$$ 

Moreover, the boundary conditions (2.5)-(2.6) are equivalent to

$$M \varphi := \hat{\beta}_1 \varphi_x + \hat{\beta}_2 \varphi_y = 0, \quad (3.18)$$

where

$$\hat{\beta}_1 = -\frac{c_2}{\sqrt{p_2 - \overline{p}}} \left( \frac{4p_1(p_2 - \overline{p}) - (p_2 - p_1)p_2}{2p_2 \overline{p}} \right) + O_7(x, \varphi),$$

$$\hat{\beta}_2 = -\frac{8p(p_2 - \overline{p}) - (p - p_1)p_2}{2p_2 \overline{p}} + O_8(x, \varphi). \quad (3.19)$$

For $\hat{\nu} = (f'(x), -1)$,

$$\hat{\beta} \cdot \hat{\nu} = \frac{4p_1(p_2 - \overline{p}) + (p_2 - p_1)p_2 - \overline{p}p_2}{2p_2 \overline{p}} + O_9(x, \varphi),$$

Also for $i = 7, 8, 9$,

$$\left| \frac{O_i(x, z)}{|x| + |z|} \right| + |DO_i(x, z)| \leq C. \quad (3.20)$$

Considering the straight shock $S_1$ separating state (1) and (2):

$$\eta - \xi_{10} \tan \theta_w = \tan \theta_s (\xi - \xi_{10}) \quad \text{in} \; (\xi, \eta)-\text{coordinates},$$

and rewriting it in $(x, y)$-coordinates, we have

$$y = \frac{\pi}{4} \sin^{-1} \left( \frac{A}{\sqrt{2(c_2 - x)}} \right) =: \hat{f}_0(x), \quad A = \xi_{10}(\tan \theta_w - \tan \theta_s),$$

which satisfies

$$\hat{f}'_0(x) = \frac{A}{2(c_2 - x)\sqrt{2(c_2 - x)^2 - A^2}}.$$

From these expressions we obtain
where

\[ \sigma > \Gamma \]

\[ \varphi \]

3.5. Regularity estimates in the scaled H"older norms near \( \Gamma_{\text{sonic}} \). We will make a regularity estimate for \( \varphi = p_{21} - p \) and the shock curve \( \{ y = f(x) \} \) near \( \Gamma_{\text{sonic}} \). Since equation (3.15) degenerates on \( \Gamma_{\text{sonic}} \), we will use the parabolic H"older norms following [9, 27].

For \( \alpha \in (0, 1) \), denote the parabolic distance

\[ \delta^{(\text{par})}_\alpha(x, \dot{x}) := (|x_1 - \dot{x}_1|^2 + \max\{|x_1, \dot{x}_1| x_2 - \dot{x}_2|^2|^{\alpha/2}. \]

Let \( \mathcal{D} \subset \{(x_1, x_2)|x_1 > 0 \} \) be an open domain. For any nonnegative integer \( m \), real \( \sigma > 0 \), \( \alpha \in (0, 1) \), the parabolic H"older norms, weighted and scaled by the distance to \( \{x_1 = 0\} \) is defined by

\[ \|u\|^{(\sigma), (\text{par})}_{m, 0, \mathcal{D}} = \sum_{0 \leq k + l \leq m} \sup_{x \in \mathcal{D}} \left( x_1^{k + l - \sigma} \left| \partial^k_{x_1} \partial^l_{x_2} u(x) \right| \right), \]

\[ [u]^{(\sigma), (\text{par})}_{m, 0, \mathcal{D}} = \sum_{k + l = m} \sup_{x \in \mathcal{D}} \min \left( \partial^k_{x_1} \partial^l_{x_2} u(x), \partial^k_{x_1} \partial^l_{x_2} u(x) - \partial^k_{x_1} \partial^l_{x_2} u(\dot{x}) \right), \]

\[ \|u\|^{(\sigma), (\text{par})}_{m, 0, \mathcal{D}} = \|u\|^{(\sigma), (\text{par})}_{m, 0, \mathcal{D}} + [u]^{(\sigma), (\text{par})}_{m, 0, \mathcal{D}}. \]

We denote by \( C^{m, \alpha}_{\sigma, (\text{par})}(\mathcal{D}) \) the completion of the set \( \{ u \in C^\infty(\mathcal{D}) : \|u\|^{(\sigma), (\text{par})}_{m, \alpha, \mathcal{D}} < \infty \} \).

For simplicity we denote

\[ \|u\|^{(\text{par})}_{m, 0, \mathcal{D}} = \|u\|^{(\text{par})}_{m, 0, \mathcal{D}}, \]

\[ C^{m, \alpha}_{(\text{par})}(\mathcal{D}) = C^{m, \alpha}_{1, (\text{par})}(\mathcal{D}). \]

The regularity of \( \varphi \) near \( \Gamma_{\text{sonic}} \) is stated as follows.
Lemma 3.13. There exist $\sigma > 0$ and $\alpha \in (0, 1)$ depending on the initial data and $\theta_w^*$, such that the following holds: Let $l > 0$, $\epsilon = \min\{\sigma, l^2\}$. If $\theta_w \in [\theta_w^*, \frac{\pi}{2})$ satisfies

$$\theta_1 - \theta_w \geq l,$$

and $(p, r)$ is an admissible solution with the wedge angle $\theta_w$, then there exists $C < \infty$ such that $\varphi = p_2 - p$ satisfies

$$\|\varphi\|^{(par)}_{1,\alpha, \Omega_\epsilon} \leq C. \quad (3.22)$$

Proof. First of all, we rescale the domain. For any $z_0 = (x_0, y_0) \in \Omega_\epsilon$ and $r \in (0, 1)$, we define

$$\tilde{R}_{z_0, r} := \left\{(s, t) : |s - x_0| < \frac{r x_0}{2}, |t - y_0| < \frac{r \sqrt{x_0}}{2}\right\},$$

$$R_{z_0, r} = \tilde{R}_{z_0, r} \cap \Omega_{2\epsilon}.$$

It is obvious that

$$R_{z_0, r} \cap \Gamma_{\text{sonic}} = \emptyset \quad \text{for all } z_0 \in \Omega_\epsilon,$$

and

$$R_{z_0, r} \subset \Omega \cap \left\{(s, t) : 3 \frac{3}{4} x_0 \leq s \leq \frac{5}{4} x_0\right\} \subset \Omega_{2\epsilon}.$$

Next, we observe that the following three types of domains

$$\begin{cases}
R_{z_0, 1/10} & \text{for } z_0 \in \Omega_\epsilon \text{ with } \tilde{R}_{z_0, 1/10} \subset \Omega_{2\epsilon} \\
R_{z_0, 1} & \text{for } z_0 \in \Gamma_{\text{wedge}} \cap \{0 < x < \epsilon\}, \\
R_{z_0, 1} & \text{for } z_0 \in \Gamma_{\text{shock}} \cap \{0 < x < \epsilon\}
\end{cases}$$

consist an open covering of $\Omega_\epsilon$. Moreover, from (3.17), we see that

$$f(x) \geq f(0) = \theta_1 \geq l + \theta_w \quad \text{for all } x \in (0, \epsilon).$$

Hence, for small $\epsilon > 0$

$$R_{z_0, 1} \cap \Gamma_{\text{shock}} = \emptyset \quad \text{for all } z_0 \in \Gamma_{\text{wedge}} \cap \{0 < x < \epsilon\},$$

$$R_{z_0, 1} \cap \Gamma_{\text{wedge}} = \emptyset \quad \text{for all } z_0 \in \Gamma_{\text{shock}} \cap \{0 < x < \epsilon\}.$$

It means that $\Gamma_{\text{wedge}}$ and $\Gamma_{\text{shock}}$ can not close to each other. This fact fails if $\theta_w$ tends to $\theta_w^*$. We denote

$$Q_r = (-r, r)^2,$$

and for fixed $z_0 = (x_0, y_0) \in \Omega_\epsilon$, introduce the new variables $(S, T)$ defined by

$$(x, y) = (x_0 + \frac{x_0}{4} S, x_0 + \frac{\sqrt{x_0}}{4} T),$$

so that for a subset $Q_r^{(z_0)} \subset Q_r$,

$$R_{z_0, r}^{(z_0)} = \left\{(x, y) : (S, T) \in Q_r^{(z_0)}\right\}.$$

Next in order to obtain the estimates in $\Omega_\epsilon$, we consider the three kinds of domains, respectively.

Case I. Consider $R_{z_0, 1/10}$ for $z_0 \in \Omega_\epsilon$ with $\tilde{R}_{z_0, 1/10} \subset \Omega_{2\epsilon}$. In this case, $Q_r^{(z_0)} = Q_r$ for all $r \in (0, 1/10]$. We now rescale $\varphi$ on $R_{z_0, 1/10}$ by defining

$$\varphi^{(z_0)}(S, T) := \frac{1}{x_0} \varphi \left(x_0 + \frac{x_0}{4} S, y_0 + \frac{\sqrt{x_0}}{4} T\right) \quad \text{for } (S, T) \in Q_{1/10}.$$
We find \( \varphi^{(z_0)} \) satisfies
\[
A_{11}^{(z_0)}(S, \varphi^{(z_0)})\varphi_S^{(z_0)} + A_{22}^{(z_0)}(S, \varphi^{(z_0)})\varphi_T^{(z_0)} + A_1^{(z_0)}(S, \varphi^{(z_0)})\varphi_S^{(z_0)} + B_1^{(z_0)}(S, \varphi^{(z_0)})(\varphi_S^{(z_0)})^2 = 0,
\]
where
\[
A_{11}^{(z_0)}(S, z) = x_0^{-1}A_{11}(x_0(1 + \frac{S}{4}), x_0z), \quad A_{22}^{(z_0)}(S, z) = A_{22}(x_0(1 + \frac{S}{4}), x_0z),
\]
\[
A_1^{(z_0)}(S, z) = \frac{1}{4}A_1(x_0(1 + \frac{S}{4}), x_0z), \quad B_1^{(z_0)}(S, z) = B_1(x_0(1 + \frac{S}{4}), x_0z).
\]
We observe that the coefficients satisfy
\[
\|A_{11}^{(z_0)}, A_{22}^{(z_0)}, A_1^{(z_0)}, B_1^{(z_0)}\|_{L^\infty(Q_1^{(z_0)} \times R)} \leq C,
\]
\[
\|D(S, z)(A_{11}^{(z_0)}, A_{22}^{(z_0)}, A_1^{(z_0)})\|_{L^\infty(Q_1^{(z_0)} \times R)} \leq C,
\]
for some \( C < \infty \). Moreover,
\[
\frac{\lambda}{4}|\kappa|^2 \leq A_{11}^{(z_0)}(S, z)|\kappa|^2 + A_{22}^{(z_0)}(S, z)|\kappa|^2 \leq \frac{4}{\lambda}|\kappa|^2
\]
for any \( \kappa \in \mathbb{R}^2 \), \( \lambda \) depends only on the initial data and \( \theta^*_w \). Hence, by the standard arguments for strictly elliptic equations, we obtain, for any \( \alpha \in (0, 1) \),
\[
\|\varphi^{(z_0)}\|_{C^{1,\alpha}(Q_1^{(z_0)})} \leq C(\alpha).
\]

**Case II.** Consider \( R_{z_0,1} \) for \( z_0 \in \Gamma_{\text{wedge}} \cap \{0 < x < \epsilon\} \). In this case
\[
Q_r^{(z_0)} = \{(S, T) \in Q_r, T > 0\} \text{ for all } r \in (0, 1].
\]
We also consider the scaled function \( \varphi^{(z_0)} \) in \( Q_1^{(z_0)} \cap \{T > 0\} \). \( \varphi^{(z_0)} \) satisfies the same equation (3.23) and estimates (3.24). Moreover, the boundary condition on \( \Gamma_{\text{wedge}} \) is
\[
\partial_T \varphi^{(z_0)} = 0 \quad \text{on } T = 0.
\]
The slip boundary can be dealt by reflecting in \( T = 0 \). So as the same in Case I, we have
\[
\|\varphi^{(z_0)}\|_{C^{1,\alpha}(Q_1^{(z_0)})} \leq C(\alpha).
\]

**Case III.** Consider \( R_{z_0,1} \) for \( z_0 \in \Gamma_{\text{shock}} \cap \{0 < x < \epsilon\} \). In this case
\[
Q_r^{(z_0)} = \{(S, T) \in Q_r, T < F^{(z_0)}(S)\},
\]
where \( F^{(z_0)}(S) = \frac{4}{\sqrt{z_0}} \left(f(x_0 + \frac{z_0}{4} S) - f(x_0)\right) \). From (3.21) one has
\[
\|F^{(z_0)}\|_{C^{0,1}((-r, r))} \leq C \sqrt{z_0}.
\]
The Rankine-Hugoniot condition (3.18) can be rewritten as
\[
\partial_s \varphi^{(z_0)} = \sqrt{x_0}b^{(z_0)}(S, \varphi^{(z_0)})\partial_T \varphi^{(z_0)} \quad \text{on } \Gamma_{\text{shock}},
\]
where
\[
b^{(z_0)}(S, z) = -\frac{\hat{\beta}_2}{\hat{\beta}_1} \left( x_0z, x_0(1 + \frac{S}{4}) \right).
\]
From (3.19), it follows that
\[
\|b^{(z_0)}(S, z)\|_{C^1((-1, 1) \times R)} \leq C.
\]
Hence for the strictly elliptic equation with oblique derivative boundary condition, we also obtain
\[ \| \varphi^{(z_0)} \|_{C^{1,\alpha}(\Omega_{1,2})} \leq C(\alpha). \] (3.27)
Finally, combining (3.25)-(3.27), we obtain
\[ \| \varphi^{(z_0)} \|_{C^{1,\alpha}(\Omega_{1,2})} \leq C(\alpha) \quad \text{in } \Omega_c \setminus \Gamma_{\text{sonic}}, \]
which implies (3.22). The proof is completed. \(\square\)

Combining Lemma 3.8 and 3.13, we now deduce the following **global estimate** of \( \varphi \) in the subsonic domain \( \Omega \).

**Theorem 3.14.** Let \( \theta_w^* \in (\theta_w^e, \frac{\pi}{2}) \), \( \epsilon_0 > 0 \) be as in Lemma 3.11, then there exists \( \alpha \in (0,1) \) depending only on the initial data and \( \theta_w^* \), such that for any \( \beta \in (0,1) \) and \( \epsilon \in (0, \epsilon_0) \), there exists \( C < \infty \) depending on the initial data, \( \epsilon, \beta, \theta_w^* \), such that, for any admissible solution \( (p, r) \) of Problem 3 with wedge angle \( \theta_w \in [\theta_w^*, \frac{\pi}{2}] \), we have
\[
\varphi \in C^{0,1}(\overline{\Omega} \setminus \{P_2\}), \quad \| \varphi \|_{1,\alpha,\Omega_{1,10}} \leq C, \quad \| \varphi \|_{1,\alpha,\Omega_0} \leq C, \\
\| f - \hat{f}_0 \|_{1,\alpha,\Omega_0} \leq C.
\]

**3.6. Existence of admissible solutions.** We will use Leray-Schauder fixed point degree to prove the existence of admissible solutions. The relevant degree theory can be referred to [31]. The main points of the existence argument is as follows.

We first construct an invertible mapping which can map the elliptic region \( \Omega \) to the iteration set \( Q^{\text{iter}} = (0,1)^2 \). Then we define the iteration boundary problem and iteration functions \( (u, v) \) corresponding to \( (p, r) \) in \( Q^{\text{iter}} \). We construct an iteration map \( T \) and prove that this map has a fixed point using Leray-Schauder point theory. We invert the fixed point to obtain an admissible solution.

Since our problem is a free boundary problem, the elliptic region \( \Omega \) of the solution is a priori unknown. We should consider a larger domain for the iteration procedure. To this aim, for fixed wedge angle \( \theta_w \in (\theta_w^e, \frac{\pi}{2}) \), we move the straight reflected shock \( S_1 := \{ \xi \cdot \nu_{S_1} = -\sqrt{\gamma_{12}} \} \) upwards, and let
\[
S_{1,d} := \{ \xi \cdot \nu_{S_1} = -\sqrt{\gamma_{12}} - d \},
\]
where \( d > 0 \) will be defined later. Let \( \hat{P}_1 \) be the closest point of intersection of \( S_{1,d} \) with \( \partial B_{c_2}(O) \), and \( \Gamma_{\text{sonic}}^{(d)} \) be arc \( \hat{P}_1 P_4 \).

Let \( Q = Q(\theta_w) \) be the domain bounded by \( S_{1,d}, \Gamma_{\text{sonic}}^{(d)}, \Gamma_{\text{wedge}}, \Gamma_{\text{sym}} \) and \( \{ \xi = -2M \} \), where \( M = \sup_{\theta_w \in (\theta_w^e, \frac{\pi}{2})} c_2(\theta_w) \). Here we take the boundary \( \{ \xi = -2M \} \) to ensure that \( Q \) is bounded. It is obvious that \( \Omega \subset Q \).

In the \((x,y)\)-coordinates, for \( \epsilon \in (0, \epsilon_1) \), denote
\[
\mathcal{D}_{d,\epsilon} = Q \cap N_{\epsilon_1}(\Gamma_{\text{sonic}}^{(d)}) \cap \{ 0 < x < \epsilon \}.
\]
Then, for small \( d > 0 \), there exists \( f_d \in C^\infty([-\epsilon_1, \epsilon_1]) \), such that
\[
\mathcal{D}_{d,\epsilon} = \{(x,y)|x \in (0,\epsilon), \theta_w < y < f_d(x)\}, \\
S_{1,d} \cap \partial \mathcal{D}_{d,\epsilon} = \{(x,y)|x \in (\epsilon,0), y = f_d(x)\}.
\]

We want to convert the free boundary value problem to fixed boundary problem, so we first construct the coordinate map and then define the related functions in the new coordinates. The construction is the same as that in Lemma 12.2.2 [9] since domain \( \Omega \) has suitably similar structure.
Lemma 3.15. There exist $\delta > 0$ and $0 < C < \infty$ depending on the data, a one-to-one map $F_1 \equiv F_1(\theta_w) : \mathbb{R} \to [0, c_2(\theta)] \times [0, \infty)$ for each $\theta_w \in \left(\theta_w^*, \frac{\pi}{2}\right)$, and a function 
\[ \eta \equiv (\theta) : [0, c_2] \to \mathbb{R}^+ \text{ such that} \]

(I) $F_1(Q)$ is open, $F_1(\partial Q) = \partial F_1(Q)$, $F_1$ depends smoothly on $\theta_w$.

(II) $\|F_1\|_{C^3(Q)} + \|F_1^{-1}\|_{C^3(Q)} \leq C$, and $|\det(F_1^{-1})| \geq \frac{1}{C}$ in $F_1(Q)$.

(III) $F_1(Q) = \{(s, t) | s \in (0, c_2), t \in (0, \eta(s))\}$. For all $P \in D_{d, \epsilon}$, $F_1(P) = (x, y - \theta_w)$. In particular,

\[ F_1(P_1) = (0, \theta_1 - \theta_w), \quad F_1(P_4) = (0, 0), \]
\[ F_1(\Gamma_{\text{sonic}}) = \{(0, t) : 0 \leq t \leq \theta_1 - \theta_w\}, \]
\[ F_1(P_3) = (c_2(\theta_w), 0), \quad F_1(\Gamma_{\text{sym}}) = \{(s, t) | s = c_2, 0 \leq t \leq \eta(c_2)\}. \]
\[ F_1(\Gamma_{\text{wedge}}) = \{(s, 0) | 0 \leq s \leq c_2\}, \quad \text{and} \quad e_1 \cdot \nu_w \geq \delta \text{ on } \Gamma_{\text{wedge}}. \]

(IV) $\partial_t(F_1)^{-1}(s, t) \cdot e_\xi \leq -\delta$ for all $(s, t) \in F_1(Q)$.

(V) $e_1 \cdot \nu_{\text{sh}} \leq -\delta$ on $\Gamma_{\text{shock}}$ for any admissible solution $(p, r)$ with wedge angle $\theta_w \in \left(\theta_w^*, \frac{\pi}{2}\right)$.

Furthermore, there exists $g_{sh} : [0, c_2] \to \mathbb{R}^+$ such that

(VI) $\Omega = F_1(\Omega) = \{(s, t) | s \in (0, c_2), t \in (0, g_{sh}(s))\}$, $F_1(\Gamma_{\text{shock}}) = \{s \in [0, c_2], t = g_{sh}(s)\}$.

(VII) $\|g_{sh}\|_{C^1(\Omega)} \leq C$.

(VIII) If we denote $D_{\epsilon_0} = \{\xi \cdot \nu > -\sqrt{p_{12}}\} \cap N_{\epsilon_1}(\Gamma_{\text{sonic}}) \cap \{0 < x < \epsilon\}$,

\[ F_1(D_{\epsilon_0}) = \{(s, t) | s \in (0, \epsilon_0), t \in (0, g_{S_1}(s))\}, \]

then
\[ \|g_{sh} - g_{S_1}\|_{C^0(\Omega)} \leq C(\alpha). \]

Moreover,
\[ C^{-1} \leq g_{sh}(s) \leq \eta(s) - C^{-1}. \]

Next we need to construct a function $\bar{f}$ to approximate $\Gamma_{\text{shock}}$. We note that $\Gamma_{\text{shock}}$ matches $S_1$ up to the first order at $P_1$, and is vertical to $\{\eta = 0\}$ at $P_2$. Hence, define

\[ f_{S_1}(\xi) = \xi \cdot \nu_{S_1} + \sqrt{p_{12}}, \quad f_0(\xi) = \xi + \sqrt{p_{12}}, \]

and take
\[ \bar{f}(\xi) = \chi(\eta)f_{S_1}(\xi) + (1 - \chi(\eta))f_0(\xi), \]

where
\[ \chi(t) = \begin{cases} 0 & \text{for } \eta < \frac{1}{2M}, \\ 1 & \text{for } \eta > \frac{1}{M}. \end{cases} \]

It is easy to prove that

Proposition 5. For each $\theta_w \in (\theta_w^*, \frac{\pi}{2})$, $\bar{f} = \bar{f}(\theta_w)$ satisfies

\[ \begin{cases} \bar{f}(\xi) = \xi \cdot \nu_{S_1} + \sqrt{p_{12}} \text{ in } \Omega \text{ for } \theta = \frac{\pi}{2}, \\ \bar{f} = \xi \cdot \nu_{S_1} \text{ in } D_{d, \frac{\pi}{2}} \text{ for all } \theta_w \in (\theta_w^*, \frac{\pi}{2}), \\ \|\bar{f}\|_{C^3(Q)} \leq C, \quad \partial_\nu \bar{f}(\xi, 0) = 0, \\ \partial_\nu \bar{f} \circ F_1^{-1}(s, t) \leq -\delta \text{ for all } (s, t) \in F_1(Q). \end{cases} \]
Furthermore we rescale the domain by defining

\[
F_{2,gsh}(s,t) = \left(\frac{s}{c_2}, \frac{t}{g_{sh}(s)}\right) \text{ for } (s,t) \in \Omega.
\]

Then

\[
F_{2,gsh}(\Omega) = (0,1)^2 =: Q^{iter},
\]

and \(F_{2,gsh}\) is invertible given by

\[
F^{-1}_{2,gsh}(s,t) = (c_2 s, t g_{sh}(c_2 s)). \tag{3.28}
\]

Now for an admissible solution \((p,r)\), define \(u : Q^{iter} \to \mathbb{R}, v : (0,1) \to \mathbb{R}\) by

\[
u = \varphi \circ F^{-1}_{1} \circ F^{-1}_{2,gsh}(s,t),
\]

\[
v = \bar{f} \circ F^{-1}_{1} \circ F^{-1}_{2,gsh}(s,t). \tag{3.29}
\]

Then from Theorem 3.14 and the construction of the functions \(u\) and \(v\), we have

**Lemma 3.16.** Fix \(\theta_w^* \in (\theta_w^*, \frac{\pi}{2})\) and let \((p,r)\) be an admissible solution with wedge angle \(\theta_w \in [\theta_w^*, \frac{\pi}{2}]\). Then there exist \(\alpha \in (0, \frac{1}{2})\) and \(\epsilon > 0\), such that the functions \(u\) and \(v\) defined by (3.29) satisfy

\[
\|u\|_{1,\alpha, Q^{iter} \cap \{s > \epsilon/10\}} + \|u\|_{1,\alpha, Q^{iter} \cap \{s < \epsilon\}} \leq C,
\]

\[
\|v\|_{0,1,\alpha, (0,1) \cap \{s > \epsilon\}} + \|v\|_{0,1,\alpha, (0,1) \cap \{s < \epsilon\}} \leq C,
\]

where \(C < \infty\) depends on \(\theta_w^*, \alpha\) and initial data, and the parabolic norms are with respect to \(s = 0\). Besides,

\[
u(1) \geq 0, \quad v(0) = v'(0) = v'(1) = 0. \tag{3.30}
\]

We will invert the mapping \((p,r) \to (u,v)\), i.e. obtain \((p,r)\) from \((u,v)\) to be able to determine our iteration equation. Let \(\Theta = \Theta^{(\alpha,\beta)}\) be defined by

\[
\Theta = \left\{(u,v,\theta_w) \in C^{0,\alpha}(Q^{iter}) \times C^{1,\beta}([0,1]) \times [\theta_w^*, \frac{\pi}{2}] \mid (u,v) \text{ satisfy (3.30)}\right\}.
\]

We define

(i) A function \(h_{sh} : (0,1) \to \mathbb{R}\) such that \(h_{sh}(s^*) = t^*\), where \((s^*,t^*)\) is uniquely determined by \(f(F^{-1}_{1}(c_2 s^*,t^*)) = v(s^*)\). Let \(g_{sh}(s) = h_{sh}\left(\frac{s}{c_2}\right)\).

(ii) A map \(F : Q^{iter} \to \Lambda(\theta)\) given by

\[
F = F^{-1}_{1} \circ (F_{2,gsh})^{-1}.
\]

Then \(\Omega = F(Q^{iter}), \Gamma_{shock} = F((0,1) \times \{1\})\).

Moreover, we require \((u,v,\theta_w) \in \Theta\) satisfy

\[
u < p_2 - p_1, \quad a < g_{sh} < \frac{1}{a}, \quad \xi \cdot \nu \leq -M \text{ on } \Gamma_{shock} \tag{3.31}
\]

for some \(0 < a < \infty\). Let \(\Theta_{a,M}^{(\alpha,\beta)} = \{(u,v,\theta_w) \in \Theta | (3.31) \text{ hold}\}\). Then we define \(\varphi\) on \(\Omega\) by setting

\[
\varphi = u(F^{-1}(\xi)),
\]

and

\[
\Gamma_{shock} = \{(r,\theta) | r = r(\theta), \theta_1 \leq \theta \leq \theta_2\}, \quad \Omega = \{r < r(\theta), r < c_2, \theta_w < \theta < \theta_w + \pi\}.
\]
Since $\Gamma_{\text{shock}}$ is a free boundary, in the iteration process we should modify it to keep the original properties. So we give an oblique cutoff for $r(\theta)$ near $P_2$. Define

$$\bar{r}(\theta) = c_2 + \int_{\theta_1}^{\theta} \zeta_2(\pi - s)r'(s) + (1 - \zeta(\pi - s))((\pi - s)^{1/2}\zeta_1 \left( \frac{r'(s)}{(\pi - s)^{1/2}} \right)) ds,$$

where $\zeta_1, \zeta_2 \in C^\infty(\mathbb{R})$ satisfy

$$\zeta_1(t) = \begin{cases} t & \text{for } t > \delta, \\ -\frac{\delta}{2} & \text{for } t < -\frac{3}{4}\delta, \end{cases} \quad \zeta_2(t) = \begin{cases} 1 & \text{for } t > \sigma_1, \\ 0 & \text{for } t < \sigma_1/2, \end{cases}$$

$0 \leq \zeta'_1(t) \leq 2$, $\zeta_1(t) \leq t$, and $\zeta'_2(t) \geq 0$.

We can prove that $\bar{r}$ arising from this cutoff still defines a suitable shock curve $\bar{\Gamma}_{\text{shock}}$, satisfying

$$\bar{r}'(\theta_2) = 0, \quad ||\bar{r}||_{2,0,\{\theta_1,\theta_2\}} \leq C,$$

and $e_t \cdot \nu_{sh} \leq -\delta$ on $\bar{\Gamma}_{\text{shock}}$. In the $(s,t)$-coordinates,

$$\bar{\Gamma}_{\text{shock}} = \{(s,t) | t = \bar{g}_{sh}(s), s \in [0,\epsilon_2]\},$$

and we then define $\varphi$ from $u$ using $\bar{g}_{sh}(s)$ instead of $\bar{g}_{sh}(s)$.

3.6.1. Iteration set. Denote

$$\|u\|_{1,\alpha, Q_{\text{iter}}} = \|u\|_{\mathcal{D}^{\alpha}, Q_{\text{iter}} \cap \{s > 0\}/10} + \|u\|_{\mathcal{D}^{\alpha}, Q_{\text{iter}} \cap \{s < 0\}/10},$$

and $C^{1,\alpha}_{*,\sigma}(Q_{\text{iter}})$ the space of $u$ with finite norm $\|\cdot\|_{1,\alpha, Q_{\text{iter}}}$. We use the $C^{(1-\beta)}_{2,0}$ $((0,1))$ norm for $v$.

Set

$$\mathcal{B} = C^{1,\alpha}_{*,\sigma}(Q_{\text{iter}}) \times C^{(1-\beta)}_{2,0}((0,1)),$$

$$\mathcal{C} = \mathcal{B} \cap \{ (u,v) \text{ satisfy (3.30)} \}.$$

Note that $\mathcal{B}$ is a Banach space, and $\mathcal{C}$ is a closed, convex subset of $\mathcal{B}$.

Definition 3.17 (Iteration set). The iteration set $K \subset \mathcal{C} \times [\theta_w^*, \pi_2]$ is the set of all $(u,v,\theta_w)$ such that

(i) $(u,v)$ satisfy

$$\|u\|_{1,\alpha, Q_{\text{iter}}} + \|v\|_{2,0/2,0} < N_1.$$

(ii) $(u,v,\theta_w) \in \Theta$, so that $\bar{g}_{sh}, \Omega, \Gamma_{\text{shock}}$, and $\varphi$ are defined as above. Also, denoting the inner unit normal to $\Gamma_{\text{shock}}$ as $\nu$, we have

$$\xi \cdot \nu < -C,$$

so that $\Gamma_{\text{shock}}$ can be written as $\{ r = r(\theta) \}$ for $r \in C^{1,\beta}([\theta_1, \theta_2]) \cap C^{(1-\beta)-1/2}([\theta_1, \theta_2])$, and has a cutoff stated above. Moreover,

$$\frac{1}{N_2} \leq \bar{g}_{sh}(s) - \eta^{\theta_w}(s) - \frac{1}{N_2},$$

(iii) $p$ is well defined in $\Omega$, and satisfies $p_1 + \tau_1 < p \leq p_2$.

(iv) $c^2, r(\theta)$ and $\varphi = p_2 - p$ determined by $u$ and $v$ satisfy

$$|\varphi| < (2c_2 - 2\delta)\epsilon \quad \text{in $\Omega \setminus \mathcal{D}_{\epsilon_1/10}$},$$

$$|\varphi| < (2c_2 - 2\delta)\epsilon \quad \text{in $\mathcal{D}_{\epsilon_1}$},$$
(v) Uniform ellipticity in $\overline{\Omega \setminus D_{c_o/10}}$,
$$p - r^2 > \lambda_1 \text{dist}(\xi, \Gamma_{\text{sonic}}) \quad \text{for all } \xi \in \overline{\Omega \setminus D_{c_o/10}}.$$

(vi) Uniform obliqueness away from $P_2$.
$$r^2 - p > \delta_3 (\pi - \theta) \quad \text{on } \Gamma_{\text{shock}} \cap \{ \theta_2 - \theta > \sigma/10 \}.$$

3.6.2. Boundary value problem for the iteration. Consider the iteration problem

$$N^\circ \hat{\varphi} = \sum_{i,j=1}^2 A_{ij} \hat{\varphi}_{ij} + \sum_{i,j=1}^2 B_{ij} \hat{\varphi}_{i,j} + \sum_{i=1}^2 A_i \hat{\varphi}_i = 0 \quad \text{in } \Omega,$$  \hbox{(3.32)}

$$\begin{cases}
M\varphi = \beta_1 \hat{\varphi}_r + \beta_2 \hat{\varphi}_\theta = 0 & \text{on } \Gamma_{\text{shock}}, \\
\hat{\varphi} = 0 & \text{on } \Gamma_{\text{sonic}}, \\
\nabla \hat{\varphi} \cdot \nu = 0 & \text{on } \Gamma_{\text{wedge}} \cup \Gamma_{\text{sym}}, \\
\hat{\varphi} = \varphi & \text{at } P_2,
\end{cases}$$  \hbox{(3.33)}

$$\tilde{r}(\theta) = c_2 + \int_{\tilde{\theta}_1}^{\theta} \zeta_2(\pi - s) r'(s) + (1 - \zeta(\pi - s))(\pi - s)^{1/2} \zeta_1(\pi - s)^{1/2} ds. \quad \hbox{(3.34)}$$

First, for the oblique boundary condition on $\Gamma_{\text{shock}}$, we freeze the coefficients at $\varphi$ and $r$ determined by $(u, v)$ as
$$\beta_1(\varphi, r) \hat{\varphi}_r + \beta_2(\varphi, r) \hat{\varphi}_\theta = 0,$$
and now the boundary condition is linearized.

As before, we can prove
$$C(\theta_2 - \theta)^{1/2} \leq \beta \cdot \nu \leq C \sigma^{1/2},$$ \hbox{(3.35)}

hence it is uniformly oblique away from $P_2$ and non-oblique at $P_2$. Moreover,
$$\|\beta\|_{0, \alpha, \Gamma_{\text{shock}}} \leq C, \quad \|\beta_i\|_{1, \alpha/2, \Gamma_{\text{shock}} \setminus \partial \Omega_{\epsilon}} \leq C.$$

For the equation, we use (3.32) in $\Omega \setminus D_{c_{eq}/10}$ and freeze the coefficients to linearize it
$$N^\circ \hat{\varphi} := \sum_{i,j=1}^2 A_{ij}^1 \hat{\varphi}_{ij} + \sum_{i,j=1}^2 B_{ij}^1 \hat{\varphi}_{i,j} + \sum_{i=1}^2 A_i^1 \hat{\varphi}_i = 0 \quad \hbox{(3.36)}$$
where
$$\begin{pmatrix} A_{ij}^1, A_i^1, B_{ij}^1 \end{pmatrix}(\xi) = \begin{pmatrix} A_{ij}^1, A_i^1, B_{ij}^1 \end{pmatrix}(\varphi(\xi), \xi).$$

Then (3.36) is uniformly elliptic in $\Omega \setminus D_{c_{eq}/10}$ and the coefficients satisfy
$$\left\| \begin{pmatrix} A_{ij}^1, A_i^1, B_{ij}^1 \end{pmatrix} \right\|_{C^{1, \alpha/2, \partial \Omega \setminus D_{c_{eq}/10}}} \leq C. \quad \hbox{(3.37)}$$

Near $\Gamma_{\text{sonic}}$, the equation and oblique boundary conditions should be paid more attention. We will work in the $(x, y)$ coordinates. For our iteration, we only obtain $|\varphi| \leq N_0 x^{\delta_2}$, which is not sufficient to guarantee the ellipticity of (3.32). We should make a cutoff. Let $\varsigma_1(t) \in C^\infty(\mathbb{R})$ satisfy
$$\varsigma_1(t) = \begin{cases} t, & \text{if } |t| < 2c_2 - \frac{\delta_2}{10} \\
2c_2 - \frac{\delta_2}{10} \text{sign}(t), & \text{if } |t| > 2c_2 - \frac{\delta_2}{10}, \end{cases}$$
and $\varsigma_1' \geq 0$, $\varsigma_1(-t) = -\varsigma_1(t)$ on $\mathbb{R}$. 

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Then the modified equation in the \((x, y)\) coordinates is defined as follows
\[
\left(2c_2x - x\varsigma_1 \left(\frac{\tilde{\varphi}_x}{x}\right) + \tilde{O}_1\right)\tilde{\varphi}_{xx} + (c_2 + \tilde{O}_2)\tilde{\varphi}_x - (1 + \tilde{O}_3)\tilde{\varphi}_x^2 + (1 + \tilde{O}_4)\tilde{\varphi}_{yy} = 0, \tag{3.38}
\]
where \(\tilde{O}_i\) are defined as
\[
\tilde{O}_i = O_i \left( x, x\varsigma_1 \left(\frac{\tilde{\varphi}_x}{x}\right) \right), \quad i = 1, \ldots, 5.
\]
From the expressions of \(O_i\) \((i = 1, \ldots, 5)\), we have

**Lemma 3.18.** Let \(\epsilon_{eq} \in (0, \epsilon_1)\) and \((u, v, \theta_u) \in \Theta\), then
\[
|\tilde{O}_1| \leq Cx^2, \quad |\tilde{O}_i| \leq Cx, \quad \text{for } i = 2, \ldots, 5, \tag{3.39}
\]
where \(C < \infty\) depends only on the initial data.

Now we write (3.32) in \((x, y)\)-coordinates in the form
\[
N^2\tilde{\varphi} = \tilde{A}_{11}^2\tilde{\varphi}_{xx} + \tilde{A}_{22}^2\tilde{\varphi}_{yy} + \tilde{A}_1^2\tilde{\varphi}_x + \tilde{B}_1\tilde{\varphi}_x^2 = 0,
\]
and an equivalent form in the \((\xi, \eta)\) coordinates
\[
N^2\tilde{\varphi} = \sum_{ij} A_{ij}^2(\xi)\tilde{\varphi}_{ij} + \sum_{ij} B_{ij}^2(\xi)\tilde{\varphi}_i\tilde{\varphi}_j + \sum_{i} A_i(\xi)\tilde{\varphi}_i = 0. \tag{3.40}
\]
By the construction, we find that

**Lemma 3.19.** There exist \(\lambda > 0\), \(\epsilon_{eq} \in (0, \frac{\epsilon_1}{2})\), and \(N_{eq} \geq 1\) depending only on the data such that

1. For any \((x, y) \in \Omega \cap D_{eq}\), \(\kappa \in \mathbb{R}^2\),
\[
\lambda|\kappa|^2 \leq \tilde{A}_{11}^2(x, y)\kappa_1^2 + \tilde{A}_{22}^2(x, y)\kappa_2^2 \leq \frac{1}{\lambda}|\kappa|^2.
\]
2. \((\tilde{A}_{ij}^2, \tilde{A}_{ij}^2, \tilde{B}_{ij}^2) \in C^{1, \alpha/2}(\Omega \cap D_{eq} \setminus \Gamma_{sonic})\), and
\[
\|(\tilde{A}_{ij}^2, \tilde{A}_{ij}^2, \tilde{B}_{ij}^2)\|_{0, \alpha; \Gamma_{sonic}} \leq N_{eq},
\]
\[
\|(\tilde{A}_{ij}^2, \tilde{A}_{ij}^2, \tilde{B}_{ij}^2)\|_{1, \alpha/2; (\Omega \cap D_{eq}) \setminus N_\epsilon(\Gamma_{sonic})} \leq N_{eq} s^{-4}.
\]
3. For all \((x, y) \in \Omega \cap D_{eq}\),
\[
\tilde{A}_{11} \leq c_2 + N_{eq}\epsilon_{eq}, \quad \tilde{B}_{11} \leq -1 + N_{eq}\epsilon_{eq}.
\]
4. Suppose that \(\varphi = \tilde{\varphi}\) and \(\varphi\) satisfies
\[
|\varphi(x, y)| \leq (2c_2 - \frac{\delta_2}{2})x \quad \text{in } \Omega \cap D_{eq},
\]
then \(\varphi\) satisfies (3.32) in \(\Omega \cap D_{eq}\).

Next we combine the two equations above as follows. Let
\[
s_2(t) = \begin{cases} 0, & \text{if } t \leq \frac{r}{2} \\ 1, & \text{if } t \geq r, \end{cases}
\]
and \(0 \leq s_2(t) \leq \frac{10}{r}\) on \(\mathbb{R}\).

We then define, for \(\theta_u \in [\theta_u, \frac{\pi}{2})\) and \(r \in (0, \frac{\pi}{4})\), with map \(F_1 \equiv F_1^{\theta_u}\),
\[
s_2^{(r, \theta_u)} := s_2(F_1(\xi)) \quad \text{for all } \xi \in Q.
\]
We note the following properties

\[(\xi, \theta_w) \rightarrow \varsigma_2^{(\tau, \theta_w)}(\xi) \in C^3(Q^r),\]

\[\|\varsigma_2^{(\tau, \theta_w)}\|_{C^3(Q)} \leq C \quad \text{for all} \ \theta_w \in (\theta_w^*, \frac{\pi}{2}),\]

\[\varsigma_2^{(\tau, \theta_w)}(t) = \begin{cases} 0, & \text{on} \ \Omega \cap D_{\tau/2}; \\ 1, & \text{on} \ \Omega \setminus D_{\tau}. \end{cases}\]

Define

\[(A_{ij}, A_1, B_{ij})(\xi) = \varsigma_2^{(\tau, \theta_w)}(\xi)(A_{ij}^1, A_1^1, B_{ij}^1)(\xi) + (1 - \varsigma_2^{(\tau, \theta_w)}(\xi))(A_{ij}^2, A_1^2, B_{ij}^2)(\xi),\]

(3.41)

we obtain the following properties.

**Lemma 3.20.** Let the coefficients be defined as in (3.41), then the resulting equation satisfies

1. There exists $\lambda_0 > 0$ depending only on the data such that for any $\xi \in \Omega$, $\kappa \in \mathbb{R}^2$,

\[\lambda_0 \text{dist}(\xi, \Gamma_{\text{sonic}})|\kappa|^2 \leq \sum_{i,j=1}^2 A_{ij}(\xi)\kappa_i\kappa_j \leq \frac{1}{\lambda_0} |\kappa|^2.\]

2. For $s \in (0, \frac{\pi}{2})$,

\[\|A_{ij}, A_1, B_{ij}\|_{0,0,0} \leq N, \quad \|(A_{ij}, A_1, B_{ij})\|_{1,0,2}(\Omega \setminus D_\kappa) \leq Ns^{-4}.\]

3. In the region $\Omega \cap D_{\kappa/2}$, (3.32) in the $(x, y)$-coordinates is equivalent to (3.38).

Next we solve the iteration problem.

**Proposition 6.** Suppose that the parameters $\sigma_1, \epsilon$ and $\delta_\kappa$ are small depending only on the data and $\theta_w^*$. Then for each $(u, v, \theta) \in \mathcal{K}$ which satisfies

\[\|u - u^\#\|_{L^\infty(Q^{iter})} + \|v - v^\#\|_{C^2([0,1])} + \|\theta_w - \theta_w^\#\| \leq \delta_\kappa\]

for some $(u^\#, v^\#, \theta_w^\#) \in \mathcal{K}$, there exists a unique solution

\[\varphi \in C^2(\Omega) \cap C^1(\overline{\Omega} \setminus (\Gamma_{\text{sonic}} \cup \{P_2\})) \cap C(\overline{\Omega})\]

of iteration problem determined by $(u, v, \theta_w)$ such that

\[0 \leq \varphi \leq Cx \quad \text{in} \ \Omega \cap D_\kappa, \quad 0 \leq \varphi \leq \varphi \quad \text{in} \ \overline{\Omega},\]

where $C < \infty$ depending only on the data. Moreover, we can obtain the following estimates of $(\varphi, \hat{\varphi})$

\[\|\varphi\|_{1,\alpha/2,\Omega_1} \leq C \quad \text{and} \quad \|\varphi\|_{1,\alpha/2,[\theta_1, \pi]} \leq C.\]

This leads to a better regularity of $(u, v)$.

**Proposition 7.** Let $\hat{u} : \Omega^{iter} \to \mathbb{R}$ and $\hat{\varphi} : (0, 1) \to \mathbb{R}$ be determined by

\[\hat{u} = \hat{\varphi} \circ F_{1}^{-1} \circ (F_{2, g_{sh}})^{-1}(s, t),\]

where $g_{sh}$ was obtained from $v$. Also, $\hat{\Gamma}_{\text{shock}} = \{r = \hat{\varphi}(\theta)|\theta_1 \leq \theta \leq \theta_2\}$. Then $\hat{v}$ defined as (3.29) satisfies

\[\|\hat{u}\|_{1,\hat{\Gamma}_{\text{shock}}^{iter}} \leq C \quad \text{and} \quad \|\hat{v}\|_{2,\hat{\Gamma}_{\text{shock}}^{iter}(0,1)} \leq C.\]
Now we define the iteration map $I = T^0$ on $\mathcal{K}$ as

$$I(u,v,\theta_w) = (\hat{u}, \hat{v}),$$

where $\hat{u}, \hat{v}$ are defined in Proposition 7. Then $I$ is continuous, and $i$-compact.

Moreover, we have

**Lemma 3.21.** Let $(p, r)$ correspond to a fixed point $(u, v)$ of $I$ for some wedge angle $\theta_w$. Then, if $\epsilon$ in the iteration set is chosen sufficiently small depending on the data, $\varphi$ corresponding to $p$ satisfies

$$0 \leq \varphi \leq (2c_2 - \frac{\delta_2}{3})x \quad \text{in } \overline{\Omega} \cap \{0 < x < \frac{\epsilon}{10}\},$$

$$\frac{r}{\sqrt{p}} \sqrt{r^2 - p} \geq \delta_2 (\theta_2 - \theta)^{1/2} \quad \text{on } \Gamma_{\text{shock}} \cap \{0 < \theta_2 - \theta < \sigma\}.$$

Using the estimates of Lemma 3.21 we can remove the cutoff functions $\varsigma_1, \zeta_1$.

Thus we conclude that

**Theorem 3.22.** The iteration map $I$ has a fixed point $(u, v, \theta_w)$ which does not lie on the boundary of the iteration set. Meanwhile, the corresponding $(p, r)$ is an admissible solution of Problem 3.

In order to use the Leray-Schauder fixed point theory to obtain the existence of admissible solutions for all supersonic wedge angles, we need to show that the fixed point degree of the iteration map for $\theta_w = \frac{\pi}{2}$ is non-zero. We first note that when $\theta_w = \frac{\pi}{2}$, the normal reflection is the unique admissible solution. Therefore, the unique fixed point of $I^\frac{\pi}{2}$ is $(0, 0)$. From Theorem 3.4.7 in [9], it follows that

$$\text{Ind}(I^\frac{\pi}{2}, \mathcal{K}(\frac{\pi}{2})) = 1.$$

Finally, applying the Leray-Schauder theory, we conclude that $I^\theta_w$ has a fixed point for any angle $\theta_w \in [\theta^d, \frac{\pi}{2}]$. As $\theta^d_w$ is arbitrary, we have proven the existence of admissible solutions for all supersonic wedge angles.

4. **Admissible solutions up to the detachment angle.** In this section, we consider the subsonic shock reflection when $\theta_w \in (\theta^d_w, \frac{\pi}{2})$. In this case the subregion $P_0 P_1 P_4$ shrinks to one point $P_0$, and we let $P_1 = P_4 = P_0$.

4.1. **Regularity away from the reflected point.** First we show the monotonicity of $p$ on $\Gamma_{\text{shock}}$ and convexity of $\Omega$.

**Lemma 4.1.** Let $(p, r)$ be an admissible solution of problem 4. Then $p$ is monotone increasing from $P_0$ to $P_2$ on $\Gamma_{\text{shock}}$, and the elliptic region $\Omega$ is convex.

**Lemma 4.2.** Let $\nu_w$ denote the outer unit normal to $\Gamma_{\text{wedge}}$. Then, for any admissible solution to problem 4, denoting by $\nu_{sh}$ the inner unit normal to $\Gamma_{\text{shock}}$, then

$$\nu_{sh} \cdot \nu_w \leq -\delta,$$

where $\delta > 0$ depends only on the initial data.

The estimates for $p$ and the elliptic region $\Omega$ are stated as follows.

**Lemma 4.3.** Let $(p, r)$ be an admissible solution with wedge angle $\theta_w \in (\theta^d_w, \frac{\pi}{2})$. Then there exists $C < \infty$ depending only on the initial data such that

$$\Omega \subset B_{c_2}(O), \quad p_1 < p \leq C, \quad \text{Lip}(r) \leq C.$$
As for the separation of the shock curve from the wedge in the supersonic reflection, here we can ensure uniform separation away from the reflection point only.

**Lemma 4.4.** Let $\delta > 0$. Then there exists $C > 0$ depending only on the initial data and $C(\delta)$ depending on the initial data and $\delta$, such that for any admissible solution $(p,r)$ with $\theta_w \in (\theta^d_w, \theta^s_w)$, we have

$$\text{dist}(P_0, \Gamma_{\text{sym}}) \geq C \quad \text{and} \quad \text{dist}(\Gamma_{\text{shock}} \setminus B_\delta(P_0), \Gamma_{\text{wedge}}) \geq C(\delta).$$

**Lemma 4.5.** Let $\theta^*_w \in (\theta^d_w, \theta^s_w)$. There exists $\tau > 0$ depending only on the initial data, such that for any admissible solution $(p,r)$ with wedge angle $\theta_w \in [\theta^*_w, \theta^s_w]$, we have

$$p_1 + \tau < p \leq p_2 \quad \text{in} \quad \Omega, \quad r > c_1 + \tau \quad \text{on} \quad \Gamma_{\text{shock}}.$$

We can now use the global lower bound for $p - p_1$ to deduce global ellipticity and obliqueness, which extends Proposition 4.

**Proposition 8.** Let $\theta^*_w \in (\theta^d_w, \theta^s_w)$. There exist $\lambda > 0, \sigma > 0, \delta > 0$ depending only on the initial data and $\theta^*_w$, such that for any admissible solution $(p,r)$ with wedge angle $\theta_w \in [\theta^*_w, \theta^s_w]$, we have

$$p - r^2 \geq \min \{ p(P_0) - r^2(P_0) + \lambda |\xi - P_0|, \sigma \} \quad \text{in} \quad \Omega,$$

$$r^2 - \lambda > \delta (\theta^s - \theta) \quad \text{on} \quad \Gamma_{\text{shock}}.$$

Let $\Omega_\epsilon = \{ \xi \in \Omega | \text{dist}(\xi, \Gamma_{\text{sonic}}) < \epsilon \}$, where $\Gamma_{\text{sonic}} = \{ P_0 \}$ in the subsonic case. Away from the reflected point $P_0$, the domain $\Omega \setminus \Omega_\epsilon$ has the same structure as that for supersonic shock reflection, so we have the following estimates for admissible solutions $(p,r)$ as that in Lemma 3.8.

**Lemma 4.6.** Let $\theta^*_w \in (\theta^d_w, \frac{\pi}{2})$. Then there exist $\alpha \in (0,1)$ and $\epsilon_0 > 0$, depending on the initial data and $\theta^*_w$, such that for $0 < \epsilon < \epsilon_0$ and $\beta \in (0,1)$, there exists $C < \infty$ depending on $\epsilon, \beta, \theta^*_w$ and the initial data such that for any admissible solution with the wedge angle $\theta_w \in (\theta^s_w, \theta^s_w)$,

$$r^2 - \lambda \leq C(\theta^s_w - \theta) \quad \text{on} \quad \Gamma_{\text{shock}},$$

$$-C(\theta^s_w - \theta)^{1/2} \leq r^2(\theta) \leq -\frac{1}{C}(\theta^s_w - \theta)^{1/2} \quad \text{for} \quad \theta \in [\theta_1, \theta_2],$$

$$\beta \cdot \nu \geq \frac{1}{r}(\theta^s_w - \theta)^{1/2} \quad \text{on} \quad \Gamma_{\text{shock}},$$

and

$$||p||_{1,\alpha,\Omega_\epsilon} \leq C, \quad ||r||_{2,\alpha,\Omega_\epsilon} \leq C.$$

### 4.2. Estimates up to the sonic arc

In this subsection, we consider the regularity near $\Gamma_{\text{sonic}}$ up to the detachment angle. Fix $\theta^*_w \in (\theta^d_w, \frac{\pi}{2})$, we make a priori estimates up to $\Gamma_{\text{sonic}}$, uniform in $\theta_w \in [\theta^s_w, \frac{\pi}{2})$. For small $\delta^s > 0$, we shall consider the following four cases:

(i) The supersonic-away from sonic case $\theta_w \in [\theta^s_w + \delta^s, \frac{\pi}{2})$. This is precisely the case in Section 3.

(ii) The supersonic-near sonic case $\theta_w \in (\theta^s_w, \theta^s_w + \delta^s)$. In this case, the sonic arc has a positive, but very small length. We shall first prove the algebraic growth estimates when the sonic arc is sufficiently small and then use them along a different scaling to derive parabolic $C^{1,\alpha}$ estimates.

(iii) The subsonic-near sonic case $\theta_w \in (\theta^s_w - \delta^s, \theta^s_w)$. In this case the sonic arc shrinks to zero but $p(P_0) - r^2(P_0)$ is also very small.
(iv) The subsonic-away from sonic case \( \theta_{w} \in (\theta_{w}^{d}, \theta_{w}^\ast - \delta^\ast] \). In this case the sonic arc has zero length, and the ellipticity is strict. We will obtain \( C^{0, \beta} \) regularity up to the reflection point for any \( \beta \in (0, 1) \).

We note that \( S_2 \) near \( P_1 \) can be written as \( \{ y = \hat{f}_0(x) \} \). We have the following pointwise estimates for admissible solutions.

**Lemma 4.7.** Let \( \theta_{w} \in (\theta_{w}^{d}, \frac{\pi}{2}) \). Then there exist \( \epsilon_0 > 0 \), \( \omega > 0 \), and \( M \geq 1 \) such that

\[
2\omega \leq \hat{f}_0(x) \leq M \quad \text{for} \quad x \in (-\epsilon_0, \epsilon_0) \quad \text{and} \quad \| \hat{f}_0(x) \|_{C^{2}(\{\epsilon_0, \epsilon_0\})} \leq M.
\]

**Lemma 4.8.** Let \( (p, r) \) be an admissible solution and \( \varphi = p_2 - p \). Then there exists \( C \leq \infty \) depending only on the initial data such that

\[
0 \leq \varphi \leq C \quad \text{in} \quad \Omega.
\]

**Lemma 4.9.** Let \( \theta_{w} \in (\theta_{w}^{d}, \frac{\pi}{2}) \). There exist \( C > 0 \) and \( \epsilon_1 > 0 \) depending only on \( \theta_{w}^\ast \), and \( \epsilon, \varphi \in (0, \epsilon_1) \),

(i) \( \Omega_{\epsilon} \) can be written as

\[
\Omega_{\epsilon} = \{ 0 < \epsilon - \epsilon_0 < \epsilon, \theta_{w} < y < f(x) \},
\]

and

\[
\partial \Omega_{\epsilon} \cap \Gamma_{\text{shock}} = \{ 0 < \epsilon - \epsilon_0 < \epsilon, y = f(x) \}.
\]

(ii) The boundary condition on \( \Gamma_{\text{shock}} \cap \partial \Omega_{\epsilon} \) can be written as

\[
\vec{\beta} \cdot \nabla \varphi = \vec{\beta}_1 \varphi_x + \vec{\beta}_2 \varphi_y = 0.
\]

The coefficients \( (\vec{\beta}_1, \vec{\beta}_2) \) in the oblique boundary condition satisfy

\[
\vec{\beta} \cdot \nu \geq \frac{1}{C}, \quad \vec{\beta}_1 \leq -\frac{1}{C},
\]

for some \( C \leq \infty \), where \( \nu \) is the inner unit normal to \( \Omega \).

(iii) For some \( \omega, M > 0 \), \( \omega \leq f'(x) \leq M \), and

\[
f'(x) = h(x, \varphi) \quad \text{for} \quad x \in (0, \epsilon),
\]

where \( h \in C^\infty(\mathbb{R} \times \mathbb{R}) \) and \( \partial_x^{k-1}h(x, 0) = \hat{f}_0(x) \) for \( k \geq 1 \).

**Lemma 4.10.** There exist \( \epsilon_1 \in (0, \epsilon_0) \), \( \varphi > 0 \) and \( \delta^\ast > 0 \), depending only on the initial data such that for any admissible solution \( (p, r) \) with \( \theta_{w} \in (\theta_{w}^\ast - \delta^\ast, \frac{\pi}{2}] \),

\[
0 \leq \varphi \leq (2\epsilon_2 - \delta)x \quad \text{for} \quad x \in (0, \epsilon_1).
\]

**Lemma 4.11.** There exist \( \delta^\ast > 0 \) and \( C \leq \infty \) depending only on the initial data such that \( \theta_{w} \in [\theta_{w}^\ast - \delta^\ast, \frac{\pi}{2}] \), then the estimates of Lemma 3.12 hold,

\[
\left| \frac{f(x) - \hat{f}_0(x)}{x^2} + \frac{f'(x) - \hat{f}_0(x)}{x} \right| \leq C, \quad \vec{\beta}_1 \leq -\frac{1}{C}.
\]

**Lemma 4.12.** There exist \( \epsilon_0 > 0 \) and \( C \leq \infty \) depending only on the initial data and \( \theta_{w}^\ast \) such that for any admissible solution \( (p, r) \), we have

\[
0 \leq \varphi \leq Cx \quad \text{in} \quad \Omega_{\epsilon_0}.
\]

Next we make clear the estimates for the four different case respectively.

Case(1). The supersonic-away from sonic case \( \theta_{w} \in [\theta_{w}^\ast + \delta^\ast, \frac{\pi}{2}] \), which is exactly the case as in Section 3. We state
Corollary 2. Let \( \delta^* > 0, \epsilon_0 \) be as in Lemma 4.7. Then there exists \( \alpha \in (0,1) \) depending on the data and \( \delta^* \) such that for each \( \beta \in (0,1) \) and \( \epsilon \in (0, \epsilon_0] \), there exists \( C < \infty \) depending on the initial data, \( \epsilon, \delta^* \), and \( \beta \) such that, for any admissible solution \((p,r)\) with wedge angle \( \theta_w \in [\theta_w^* + \delta^*, \frac{\pi}{2}] \), then

\[
\varphi \in C^{0,1}(\Omega \setminus \{P_2\}), \quad \|\varphi\|_{1,\beta,\Omega}\|_{\Omega,10} + \|\varphi\|_{1,\alpha,\Omega} \leq C,
\]

\[
\|r\|_{2,\alpha,\Omega,\{[\theta_1,\pi]\}} + \|f - \hat{f}_0\|_{2,\alpha,\Omega,\{0,\epsilon\}} \leq C.
\]

Case (2). The supersonic-near sonic case \( \theta_w \in (\theta_w^*, \theta_w^* + \delta^*) \). As \( \theta_w \rightarrow \theta_w^* \), Lemma 3.13 no longer provides estimates in \( \Omega_\epsilon \) uniform in \( \epsilon \), so a different technique must be used.

Note that there exists \( k > 0 \) depending only on the initial data such that

\[
\Omega_\epsilon \subset \{ x \in (0, \epsilon), y - \theta_w \in (0, b_{so} + kx) \},
\]

\[
\{ x \in (0, \epsilon), y - \theta_w \in \left( 0, b_{so} + \frac{x}{k} \right) \} \subset \Omega_\epsilon,
\]

where \( b_{so} = \theta_1 - \theta_w \). We want to prove the algebraic growth bounds for \( \varphi \). Consider equation (3.23) as a quasilinear equation, obtained by freezing the coefficients at \( \varphi \),

\[
\sum_{i=1}^{2} a_{ii}(x,y)\partial_{x^i}\varphi + a_1(x,y)\partial_x\varphi - b_1(x,y)(\partial_x\varphi)^2 = 0,
\]

where the coefficients satisfy

\[
C^{-1} x \leq a_{11}(x,y) \leq C x, \quad C^{-1} \leq a_{22}(x,y) \leq C, \quad |(a_1, b_1)| \leq C, \quad b_1 \leq 0,
\]

for some \( C < \infty \). Furthermore, the boundary condition on \( \Gamma_{\text{shock}} \) can be rewritten as a linear condition

\[
\mathcal{M}\varphi = \beta(x,y) \cdot \nabla\varphi = 0 \quad \text{on} \quad \Gamma_{\text{shock}} \cap \partial \Omega_\epsilon,
\]

and the boundary condition is strictly oblique in the sense that

\[
\beta(x,y) \cdot \nu > \frac{1}{C}.
\]

Also, the other boundary conditions are

\[
\partial_{\nu}\varphi = 0 \quad \text{on} \quad \Gamma_{\text{wedge}} \cup \Gamma_{\text{sym}}, \quad \varphi = 0 \quad \text{on} \quad \Gamma_{\text{sonic}}.
\]

Basing on this, we can construct a suitable supersolution to (4.1) with boundary condition (4.2)-(4.3).

Lemma 4.13. Let \( m \geq 2 \) be integer. There exists \( \delta^* > 0, \tilde{\epsilon} > 0, \) and \( \tilde{C} > 0 \) depending only on the initial data and \( m \) such that, for any admissible solution \((p,r)\) with wedge angle \( \theta_w \in (\theta_w^*, \theta_w^* + \delta^*) \) and \( \epsilon \leq \tilde{\epsilon} \), we have

\[
\varphi(x,y) \leq \tilde{C}\max_{\Omega_\epsilon \cap \{x=\epsilon\}} \frac{\varphi}{\epsilon^m} x^m \leq C x^m \quad \text{in} \quad \Omega_\epsilon \cap \{ x < b_{so}^2 \epsilon \}.
\]

Now using the growth estimates above, we will prove a priori estimates of solutions in the weighted and scaled \( C^{1,\alpha} \) spaces.

Lemma 4.14. Let \( \theta_w \in (\theta_w^*, \frac{\pi}{2}) \), \( \epsilon^* > 10 x_{P_1} \geq 0 \) and let \( \Omega_{\epsilon^*} \) be of the structure with \( f(0) \geq \theta_w, \) i.e. including both supersonic and subsonic cases. Note that \( x = x_{P_1} \) on \( \Gamma_{\text{sonic}} \). Denote

\[
d_{so}(x) = x - x_{P_1},
\]
Assume that \( \phi \) satisfies the boundary conditions and estimates
\[
\hat{\phi} = 1 \quad \text{for all } x \in \Gamma_{\text{bot}} \cap \{x < x_{bot} + \epsilon\},
\]
and \( h = (x, y) \in \Omega_{\epsilon} \), \( |\phi(x, y)| \leq M |x - x_{bot}| \), \( |\phi(x, y)| \leq M |x - x_{bot}| \), \( |\phi(x, y)| \leq M |x - x_{bot}| \).

Let \( \epsilon \in (0, \epsilon') \) and \( \varphi \in C(\overline{\Omega_{2\epsilon}}) \cap C^1(\overline{\Omega_{2\epsilon}} \cap \{x_{P_1} < x < x_{P_1} + 2\epsilon\}) \cap C^2(\Omega_{2\epsilon}) \) satisfy the equation
\[
\sum_{i=1}^{2} A_{ii}(x, y) \partial_{ii} \varphi + A_1(x, y) \partial_{x} \varphi + B_1(x, y) (\partial_y \varphi)^2 = 0,
\]
with the degenerate elliptic structure
\[
\lambda |\nabla \varphi|^2 \leq \frac{A_{11}(x, y)}{x} \kappa_1^2 + A_{22} \kappa_2^2 \leq \lambda |\nabla \varphi|^2,
\]
and the bounds
\[
|A_{11}(\varphi(x, y), x, y)| \leq M x, \quad |A_{22}(x, y)| \leq M, \quad |x^{-\frac{1}{2}} h(x) \partial_x A_{11}(x, y)| + |x^{-1} h(x) \partial_y A_{11}| \leq M.
\]

Assume that \( x_{bot} \in [x_{P_1}, x_{P_1} + \epsilon] \), and
\[
|\phi(x, y)| \leq M g(x) \quad \text{for all } (x, y) \in \Omega_{2\epsilon} \cap \{d_{so}(x) > \frac{d_{so}(x_{bot})}{2}\}.
\]

For \( z_0 = (x_0, y_0) \in \Omega_{\epsilon} \cap \{x > x_{bot}\} \) and \( r \in (0, 1] \), consider the rectangles:
\[
\hat{R}_{z_0, r} = \{(x, y)|x - x_0| < r \sqrt{x_0 h(x_0)}, |y - y_0| < rh(x_0)\},
\]
\[
\hat{R}_{z_0, r} = \hat{R}_{z_0, r} \cap \Omega_{2\epsilon}.
\]

We assume that \( f(\cdot), h(\cdot) \) and \( \epsilon \) are such that
\[
\hat{R}_{z_{w, 1}} \cap \Gamma_{\text{shock}} = \emptyset \quad \text{for all } z_w = (x, 0) \in \Gamma_{\text{wedge}} \cap \{x_{bot} < x < x_{P_1} + \epsilon\},
\]
\[
\hat{R}_{z_{z, 1}} \cap \Gamma_{\text{shock}} = \emptyset \quad \text{for all } z_s = (x, f(x)) \in \Gamma_{\text{shock}} \cap \{x_{bot} < x < x_{P_1} + \epsilon\}.
\]

Furthermore, introduce by \((S, T)\) the invertible change of variables
\[
(x, y) = (x_0 + \sqrt{x_0 h(x_0)} S, y_0 + h(x_0) T).
\]

Define
\[
\varphi(z_0)(S, T) = \frac{1}{g(x_0)} \varphi(x_0 + \sqrt{x_0 h(x_0)} S, y_0 + h(x_0) T),
\]
we find that there exists \( Q_{r}^z \subset Q_r = (-r, r)^2 \) such that
\[
\hat{R}_{z_0, r} = \{(x_0 + \sqrt{x_0 h(x_0)} S, y_0 + h(x_0) T) | (S, T) \in Q_{r}^z\}.
\]
Then there exist $\epsilon^*_1 \in (0, \epsilon^*)$ and $\alpha \in (0, 1)$ depending only on the constant $M$, such that if $\epsilon \leq \frac{\epsilon^*_1}{2}$, $\epsilon > \frac{\epsilon^*_1}{2}$,
\[ \| \varphi^{(z_0)} \|_{C^{1, \alpha}(Q_{1/10}^{(z_0)})} \leq C(\alpha) \quad \text{for } z_0 \in \overline{\Omega} \cap \{x < x_{bot}\}. \]

Then we can prove the a priori estimate for admissible solutions.

**Lemma 4.15.** There exists $\delta^* > 0$, $\alpha \in (0, 1)$ and $\epsilon \in (0, \frac{\epsilon^*_1}{2})$ depending only on the initial data such that the following holds: for any admissible solution $(p, r)$ of Problem 4 with wedge angle $\theta_w \in (\theta_w^*, \theta_w^* + \delta^*)$, it holds
\[ \| \varphi \|^{(\text{par})}_{1, \alpha, \overline{\Omega}} \leq C. \]

**Corollary 3.** Let $\delta^* > 0$ be as in Lemma 4.15 and $\epsilon_0$ be as in Lemma 4.7. Then there exists $\alpha \in (0, 1)$ depending only on the data, such that for each $\beta \in (0, 1)$ and $\epsilon \in (0, \epsilon_0]$, there exists $C < \infty$ depending on the initial data, $\epsilon$, and $\beta$ such that if $(p, r)$ is an admissible solution with wedge angle $\theta_w \in (\theta_w^*, \theta_w^* + \delta^*)$, then
\[
\begin{cases}
\varphi \in C^{0, 1}(\overline{\Omega} \backslash \{P_2\}), \\
\| \varphi \|^{(-\beta)}_{1, \alpha, \overline{\Omega}^{(\text{par})}} + \| \varphi \|^{(\text{par})}_{1, \alpha, \Omega_x} \leq C, \\
\| r \|^{(-1-\beta/2)}_{2, \alpha, \{\theta_2, \theta_2\}} + \| f - f_0 \|^{(\text{par})}_{2, \alpha, (0, 0, \epsilon)} \leq C.
\end{cases}
\]

Case(3). The subsonic-near sonic case. Fix $\theta_w \in (\theta_w^* - \delta^*, \theta_w^*)$. We also have small elliptic structure which is the same as the supersonic-near sonic case. In the subsonic case, we also have
\[
\begin{align*}
\Omega_{\epsilon} & \subset \{x - x_{P_1} \in (0, \epsilon), y - \theta_w \in (0, kd_{so}(x))\}, \\
\{x - x_{P_1} \in (0, \epsilon), y - \theta_w \in (0, \frac{1}{k}d_{so}(x))\} & \subset \Omega_{\epsilon},
\end{align*}
\]
where $k > 1, d_{so}(x) = x - x_{P_1}$.

Similarly, we have

**Lemma 4.16.** Let $m \geq 2$ be integer. There exists $\delta^* > 0$, $\hat{\epsilon} > 0$, and $\hat{C} > 0$ depending only on the initial data and $m$ such that for any admissible solution $(p, r)$ with wedge angle $\theta_w \in (\theta_w^* - \delta^*, \theta_w^*)$ and $\epsilon \leq \hat{\epsilon}$, we have
\[ \varphi(x, y) \leq \hat{C} \frac{\varphi}{\delta_1 \cap (x = x_{P_1} + \epsilon)} \leq C(x - x_{P_1})^m \leq C(x - x_{P_1})^m \quad \text{in } \Omega_{\epsilon}. \]

Near the reflection point $P_1$, we have the following regularity result.

**Lemma 4.17.** There exists $\delta^* > 0$, $\epsilon \in (0, \frac{\epsilon^*_1}{2})$, $C < \infty$ and $\alpha \in (0, 1)$ depending only on the
\[ \| \varphi \|_{C^{1, \alpha}(\overline{\Omega_1})} \leq C, \quad (\varphi(P_1), \nabla \varphi(P_1)) = (0, 0). \]

Finally we have

**Corollary 4.** Let $\delta^* > 0$ be as in Lemma 4.17. Then there exists $\alpha \in (0, 1)$ depending on the data, such that for each $\beta \in (0, 1)$, there exists $C < \infty$ depending only on the initial data and $\beta$ such that, for any admissible solution $(p, r)$ with wedge angle $\theta_w \in [\theta_w^* - \delta^*, \theta_w^*]$, \n\[ \varphi \in C^{1, \alpha}(\overline{\Omega} \backslash \{P_2\}), \quad \| \varphi \|^{(-\beta)}_{1, \alpha, \Omega} \leq C, \quad \| r \|^{(-1-\beta/2)}_{2, \alpha, \{\theta_2, \theta_2\}} \leq C, \]
and \n\[ \varphi(P_1) = 0, \quad \nabla \varphi(P_1) = 0, \quad f^{(k)}(x_{P_1}) = f_0^{(k)}(x_{P_1}), \quad k = 0, 1, 2. \]
Case(4). The subsonic-away from sonic case. In this case we have strict ellipticity, and a one-point Dirichlet condition at \( P_1 \). We will attain \( C^{0, \beta}(P_1) \) estimates for each \( \beta \in (0, 1) \) using the same argument as that for \( P_2 \). Fix \( \theta_w^* \in (\theta_w^*, \theta_w^* - \delta^*) \) for \( \delta^* > 0 \). We first show that for any \( A > 0 \) and integer \( m \geq 2 \),

\[
A(y - \theta_w)^m - \varphi \quad \text{does not achieve a minimum on} \quad \Gamma_{\text{shock}} \cap \partial \Omega_{\epsilon},
\]

then

\[
0 \leq \varphi \leq A(y - \theta_w)^m \quad \text{on} \quad \Gamma_{\text{shock}} \cap \partial \Omega_{\epsilon}.
\]

Next, \( \Omega_{\epsilon} \) satisfies an exterior ball condition at \( P_1 \), so we may use the barrier function to deduce that for any \( \beta \in (0, 1) \), there exists \( C \) depending only on \( \beta, \theta_w^* \) such that

\[
0 \leq \varphi \leq C|x - x_{P_1}|^\beta \quad \text{in} \quad \Omega_{\epsilon}.
\]

Using the same argument used for \( P_2 \), we find that

**Lemma 4.18.** There exist \( \alpha \in (0, 1) \) depending only on the data, \( \theta_w^* \) and \( \delta^* \), such that for any \( \beta \in (0, 1) \), there exists \( C < \infty \) depending on the data, \( \theta_w^* \), \( \delta^* \) and \( \alpha \), such that for any admissible solution \((p, r)\) with wedge angle \( \theta_w \in [\theta_w^*, \theta_w^* - \delta^*] \),

\[
\|\varphi\|_{1, \alpha, \Omega} \leq C.
\]

Finally we conclude

**Corollary 5.** Let \( \delta^* > 0 \) be as in Lemma 4.17. Then there exists \( \alpha \in (0, 1) \) depending on the data and \( \theta_w^* \), such that for each \( \beta \in (0, 1) \), there exists \( C < \infty \) depending only on the initial data, \( \theta_w^* \) and \( \beta \) such that, for any admissible solution \((p, r)\) with wedge angle \( \theta_w \in [\theta_w^*, \theta_w^* - \delta^*] \), it holds

\[
\|\varphi\|_{1, \alpha, \Omega} \leq C, \quad \|r\|_{2, \alpha, [\theta, \theta^*]} \leq C.
\]

### 4.3. Weighted norm.

There are two difficulties in obtaining the existence of admissible solutions up to the detachment angle. The first one is how to combine the estimates near \( \Gamma_{\text{sonic}} \) for \( \varphi \) into a unified estimates for \( u \) in the iteration set. The second one is how to extend the iteration set to the subsonic case.

We first extend the mapping construction to all wedge angle \( \theta_w \in (\theta_w^d, \frac{\pi}{2}) \). In subsonic case, we denote \( P_0 = P_1 = P_4 \) and \( \Gamma_{\text{sonic}} = \{P_1\} \). Let

\[
\tilde{s}(\theta_w) := |OP_4|(\theta_w).
\]

For supersonic and sonic wedge angles, \( \tilde{s}(\theta_w) = c_2(\theta_w) \), and for subsonic wedge angles, \( \tilde{s}(\theta_w) < c_2(\theta_w) \). In subsonic case, we want to define a new \( \Gamma_{\text{sonic}} \) instead of \( \{P_1\} \). Let \( S_{1, \delta} \) be defined as \( \{\xi \cdot \nu = -\sqrt{P_1^2 - \delta^*}\} \), and \( \hat{P}_1 \) be the closed intersection point of \( S_{1, \delta} \) with \( \partial B_{\tilde{s}(\theta_w)}(O) \), and \( \Gamma_{\text{sonic}} = \hat{P}_1 P_4 \). Let

\[
D_{\delta, \epsilon}^{(\theta_w)} = Q \cap N_{\epsilon_1}(\Gamma_{\text{sonic}}) \cap \{x_{P_1} < x < x_{P_1} + \epsilon\},
\]

then for sufficiently small \( \delta, \epsilon \), there exists \( f_{0, \delta, \epsilon} \in C^\infty([0, \epsilon_1]) \) such that

\[
D_{\delta, \epsilon}^{(\theta_w)} = \{(x, y)|x_{P_1} < x < x_{P_1} + \epsilon, \theta_w < y < f_{0, \delta, \epsilon}(x)\},
\]

and

\[
S_{1, \delta} \cap \partial D_{\delta, \epsilon}^{(\theta_w)} = \{(x, y)|x_{P_1} < x < x_{P_1} + \epsilon, y = f_{0, \delta, \epsilon}(x)\}.
\]

We now extend Lemma 3.15 as follows. Define

\[
F_1(P) = (x - x_{P_1}, y - \theta_w) \quad \text{for all} \quad P \in D_{\delta, \epsilon}^{(\theta_w)}.
\]
In particular,
\[ F_1(P_1) = (0, \theta_1 - \theta_w), \quad F_1(P_4) = (0, 0), \]
\[ F_1(T_{\text{sonic}}) = \{(0, t) | 0 \leq t \leq \theta_1 - \theta_w\}, \]
and
\[ F_1(Q \setminus D_{\hat{s}, \hat{\epsilon}_0}) = F_1(Q) \cap \{s \geq \frac{3}{4}\epsilon_0\}. \]

The shock \( g_{sh} \) satisfies
\[ \|g_{sh}\|^{(-1-\beta),\{0,\hat{s}\}} \leq C(\beta). \]

Furthermore, the strict positivity of \( g_{sh} \) is now replaced by linear separation
\[ \min \left( g_{sh}(0) + \frac{s}{M}, \frac{1}{M} \right) \leq g_{sh}(x) \leq \min \left( g_{sh}(0) + Ms, \eta(\theta_w) - \frac{1}{M} \right) \]
for any \( s \in (0, \hat{s}) \) and \( \theta_w \in [\theta^*_w, \frac{\pi}{2}] \), \( g_{sh}(0) = \theta_1 - \theta_w \geq 0 \) and \( M \geq 1 \) depends on the initial data.

Given an admissible solution \((p, r)\), we define
\[ g_{sh} : [0, \hat{s}] \to \mathbb{R}, \quad F_{2, g_{sh}}^{-1} : Q^{iter} \to \Omega \]
as in (3.28), as well as \( u : Q^{iter} \to \mathbb{R} \) and \( \nu : (0, 1) \to \mathbb{R} \) defined by (3.29). In subsonic case \( g_{sh}(0) = 0 \), so the transformation \( F_{2, g_{sh}} \) is now singular at \( \{s = 0\} \). In order to estimate \( u \), we need a new weighted and scaled norm.

For any \( \xi \in \Omega_{\epsilon_{10}/10} \), we set \( (s, t) = F_{2, g_{sh}} \circ F_1(\xi) \). Then the \((x, y)\)-coordinates for \( \xi \) satisfy, for \( \hat{h} = \hat{s} \),
\[ (x - x_{P_1}, y - \theta_w) = (hs, g_{sh}(hs)t) \quad \text{for} \quad s \in (0, \frac{\epsilon_0}{10}), \quad t \in (0, 1), \quad (4.4) \]
where
\[ \epsilon_0 = \frac{\epsilon_0}{\max_{\theta_w \in [\theta^*_w, \frac{\pi}{2}]} s(\theta_w)}. \]

Now for \( \varphi \) corresponding to \((p, r)\), we have
\[ \varphi(x, y) = u \left( \frac{x - x_{P_1}}{h}, \frac{y - \theta_w}{g_{sh}(x - x_{P_1})} \right) \quad \text{for all} \quad (x, y) \in \Omega_{\epsilon/10}. \quad (4.5) \]
The estimates in Section 4.1 imply that \( \varphi \in C^{(-\beta),\{P_1\}}_{\lambda, \Omega_{\epsilon}} \) with \( \varphi(P_1) = 0 \) for any \( \epsilon \in (0, 1) \), \( \theta_w \in (\theta^*_w, \theta^s_w) \).

Next we investigate equivalent estimates for \( u \) using the linear separation
\[ \frac{\epsilon_0}{M} \leq g_{sh}(s) \leq M \epsilon_0 \quad \text{for} \quad s \in (0, \hat{h}). \]
We define the mapping \( \mathcal{X} : \mathbb{R}^2 \to \mathbb{R}^2 \) given by (4.4). It is easily verified that \( \mathcal{X} \) is one-to-one on \((0, \frac{\epsilon_0}{10}) \times (0, 1)\), but not one-to-one on \([0, \frac{\epsilon_0}{10}] \times (0, 1)\), more precisely, \( \mathcal{X} \) is not one-to-one on \( \{s = 0\} \times (0, 1) \).

For \( \epsilon \in (0, \hat{h}) \), let
\[ \Omega_\epsilon = \mathcal{X}(Q_{\epsilon_0}), \quad P_0 = P_1 = P_4 = \mathcal{X}(0, 0), \]
where \( \epsilon_0 = \frac{\epsilon_0}{10} \). Then \( \Omega_\epsilon = \{0 < x < \epsilon, \theta_w < y < f(x)\} \) with \( f(x) = \theta_w + g_{sh}(x - x_{P_1}) \). Now for \( k \in \{0, 1\}, \beta \in (0, 1) \) satisfying \( k + \alpha \geq \beta \), denote
\[ \mathcal{C}_{k, \alpha, \Omega_\epsilon}^{(-\beta),\{P_1\}}(\varphi) = \left\{ \varphi \in C^{(-\beta),\{P_1\}}_{k, \alpha, \Omega_\epsilon} \mid \varphi(P_1) = 0 \right\}. \]
We define a new norm as follows. For $\alpha \in (0,1)$ and $s = (s,t)$, $\bar{s} = (\bar{s}, \bar{t})$, denote
\[
\delta_{\alpha}^{(sub)}(s, \bar{s}) = (|s - \bar{s}|^2 + (\max(s, \bar{s}))^2|t - \bar{t}|^2)^{\frac{\alpha}{2}},
\]
and then for $\sigma > 0$, $\alpha \in (0,1)$ and $m \in \mathbb{N}$, define
\[
\|u\|^{(\sigma), (sub)}_{m,0,Q_{\epsilon}} = \sum_{0 \leq k + l \leq m} \sup_{s \in Q_{\epsilon}} (s^{k-\sigma} |\partial_s^k \partial_t^l u(s)|),
\]
\[
[u]^{(\sigma), (sub)}_{m,\alpha,Q_{\epsilon}} = \sum_{k+l=m, s, \bar{s} \in Q_{\epsilon}} \left( \min \left( s^{k+\alpha - \sigma} + s^{k+\alpha - \sigma} \right) \right) \frac{\partial_s^k \partial_t^l u(s) - \partial_s^k \partial_t^l u(\bar{s})}{\delta_{\alpha}^{(sub)}(s, \bar{s})},
\]
\[
\|u\|^{(\sigma), (sub)}_{m,\alpha,Q_{\epsilon}} = \|u\|^{(\sigma), (sub)}_{m,0,Q_{\epsilon}} + [u]^{(\sigma), (sub)}_{m,\alpha,Q_{\epsilon}},
\]
where $Q = \Omega^{iter}$, $Q_{\epsilon} = \Omega \cap \{s < \epsilon\}$, and $k, l \in \mathbb{N}$. We define
\[
C_{m,\alpha}^{(\sigma), (sub)}(Q_{\epsilon}) = \left\{ u \in C^m(Q_{\epsilon}) \mid \|u\|^{(\sigma), (sub)}_{m,\alpha,Q_{\epsilon}} < \infty \right\}.
\]

We now have the following properties about the space $C_{m,\alpha}^{(\sigma), (sub)}(Q_{\epsilon})$.

Lemma 4.19. Let $M, M_1 \geq 1$, $m \in \{0,1\}$, and $\alpha, \beta \in (0,1)$ such that $m + \alpha \geq \beta$. Let $\epsilon \in (\frac{1}{M_1}, M_1)$ and $\epsilon \in (0, \frac{1}{2})$. Assume that $g_{sh}$ defined on $(0, \epsilon)$ satisfies
\[
\|g_{sh}\|^{(-1), (0)}_{m,\alpha, (0, \epsilon)} \leq M.
\]

Define a function $u$ on $Q_{\epsilon}$ by
\[
u(s, t) = \varphi(x_{P_1} + hs, \theta_w + g_{sh}(hs)t).
\]

Then $\varphi \in C_{m,\alpha}^{\epsilon, (P_1)}(\Omega_{\epsilon})$ if and only if $u \in C_{\sigma, (sub)}^{m,\alpha}(Q_{\epsilon})$, and
\[
\frac{1}{C} \|\varphi\|^{(-\beta), (P_1)}_{m,\alpha, \Omega_{\epsilon}} \leq \|u\|^{(\beta), (sub)}_{m,\alpha,Q_{\epsilon}} \leq C \|\varphi\|^{(-\beta), (P_1)}_{m,\alpha, \Omega_{\epsilon}},
\]
where $C < \infty$ depends only on $(M, M_1)$.

Moreover, we have the following

Proposition 9. Let $\epsilon \in (0,1)$ and $m, m_1 \in \{0,1\}$, $\alpha, \alpha_1, \beta, \tilde{\beta} \in (0,1)$, such that $m + \alpha < m_1 + \alpha_1$ and $\beta \leq \tilde{\beta}$, $m + \alpha \geq \beta$, $m_1 + \alpha_1 \geq \tilde{\beta}$. Then if there exists $M > 0$ such that if $\|u_k\|^{(\beta), (sub)}_{m,\alpha, Q_{\epsilon}} \leq M$ for a sequence of functions $\{u_k\}_{k=1}^{\infty}$ on $Q_{\epsilon}$, then there exists a subsequence $\{u_{k_j}\}_{j=1}^{\infty}$ converging in norm $\|u_k\|^{(\beta), (sub)}_{m,\alpha, Q_{\epsilon}}$.

Next we turn to estimates that also apply in supersonic cases. We first compare the subsonic norm with the parabolic norm. Let $g_{sh}$ satisfy the linear growth conditions
\[
\min \left( g_{sh}(0) + \frac{s}{M}, \delta_{sh} \right) \leq g_{sh}(s) \leq g_{sh}(0) + M s,
\]
where $\delta_{sh} > 0$, and $0 \leq g_{sh}(0) \leq \delta_{sh}$. Now with $x_{P_1}$ fixed, we again use the map $\mathcal{X}$ to denote
\[
\Omega_{\epsilon} = \mathcal{X}(Q_{\epsilon'}) \text{ with } \epsilon' = \frac{\epsilon}{10} \text{ and } Q_{\epsilon'} = Q^{iter} \cap \{s < \epsilon'\},
\]
\[
P_1 = \mathcal{X}(0,1) = (x_{P_1}, \theta_w + g_{sh}(0)), \quad P_4 = \mathcal{X}(0,0) = (x_{P_1}, \theta_w),
\]
\[
\Gamma_{\text{sonic}} = \partial \Omega_{\epsilon} \cap \{x = x_{P_1}\} = \mathcal{X}(\{0\} \times (0,1)),
\]
and for $k \in \{0,1\}$, $\alpha' \in (0,1)$ with $k + \alpha' \geq \beta$, we define
\[
\mathcal{C}_{\alpha, \alpha', \Omega_{\epsilon}}^{(-\beta), (\Gamma_{\text{sonic}})} = \{ \varphi \in C^{(-\beta), (\Gamma_{\text{sonic}})}_{k, \alpha', \Omega_{\epsilon}} \mid \varphi = 0 \text{ on } \Gamma_{\text{sonic}} \}.
\]

Lemma 4.20. Let $M, M_1, M_2 \geq 1$, $h \in \left(\frac{1}{M_1}, M_1\right)$, $\delta_{sh} \geq \frac{1}{M_2}$ and $\epsilon \in (0, \frac{h}{2})$. Let $m \in \{0, 1\}$ and $\alpha, \beta \in (0, 1)$ with $m + \alpha \geq \beta$. Assume that $g_{sh}(s)$ on $(0, \epsilon)$ satisfies
\[ \|g_{sh}\|_{m, \alpha, (0, \epsilon)} \leq M. \]
Let $u$ be a function on $Q_{\epsilon'}$, and define $\varphi$ on $\Omega_e$ by (4.5). Then if $u \in C^m_{\beta, (\text{sub})} (Q_{\epsilon'})$, one has $\varphi \in C_{m, \alpha, \Omega_e}$ and
\[ \|\varphi\|_{m, \alpha, \Omega_e} \leq C \|u\|_{m, \alpha, Q_{\epsilon'}}, \]
where $C$ depends on $(M, M_1, M_2)$.

Thus we derive the relation between parabolic norm and subsonic norm.

Lemma 4.21. Let $\epsilon \in (0, 1)$, $\alpha \in (0, 1)$, $\sigma \geq 0$, and $m \in \mathbb{N}$. Then the subsonic norm is stronger than parabolic norm, i.e.
\[ \|u\|_{m, \alpha, Q_{\epsilon}}^{(\sigma, \text{par})} \leq \|u\|_{m, \alpha, Q_{\epsilon}}^{(\sigma, \text{sub})}. \]

Lemma 4.22. Let $\epsilon \in (0, 1)$, $\alpha \in (0, \frac{1}{3}]$. If $u \in C_{1,0}^{1,0, \text{(par)}} Q_{\epsilon')}$, then $u \in C_{0,0, \text{(sub)}}^{0,0} Q_{\epsilon'}$, and
\[ \|u\|_{0,0, Q_{\epsilon}}^{(\alpha, \text{sub})} \leq 2 \|u\|_{1,0, Q_{\epsilon}}^{(1,\text{par})}, \]
where the parabolic norm is with respect to $\{s = 0\}$.

We can now extend the estimates for $(u, v)$ in Proposition 7 to wedge angles $\theta_w \in (\theta_w^d, \frac{\pi}{2})$.

Proposition 10. Let $\theta_w^* \in (\theta_w^d, \frac{\pi}{2})$ and $\beta \in (0, \frac{1}{3})$. Then there exist $M, M(\beta) \geq 0$ and $\sigma \in (0, \frac{1}{3}]$ depending only on the data and $\theta_w^*$, such that for any admissible solution $(p, r)$ for $\theta_w \in [\theta_w^*, \frac{\pi}{2}]$, functions $(u, v)$ satisfy that for any $\alpha \in (0, \sigma]$, $u \in C_{1,0}^{1,0, \text{par}} Q_{\epsilon'}$, and $v \in C_{1,0}^{1,0, \text{par}} Q_{\epsilon'}$, $u \in C_{1,0}^{1,0} Q_{\epsilon'}$, and $v \in C_{1,0}^{1,0} Q_{\epsilon'}$.

4.4. Iteration set. We now extend our iteration set up to the detachment angle. Fix $\theta_w \in (\theta_w^d, \frac{\pi}{2})$ and we take the norms used in our iteration set as follows, for $\alpha \in (0, 1)$,
\[ \|u\|_{1,0, Q_{\epsilon'}}^{(\alpha, \text{par})} = \|u\|_{1,0, Q_{\epsilon'}}^{(\alpha, \text{par})} + \|u\|_{1,0, Q_{\epsilon'}}^{(\alpha, \text{par})} + \|u\|_{1,0, Q_{\epsilon'}}^{(\alpha, \text{par})}, \]
For $\sigma \in (0, \epsilon_0]$, denote
\[ D_\sigma = \{\theta_w : y < \tilde{f}_0(x)\} \cap N_{\epsilon_1}(\Gamma_{\text{sonic}}^*) \cap \{x_{P_1} < x < x_{P_1} + \sigma\}, \]
\[ D_{0, \sigma} = \{\theta_w : y < \tilde{f}_0(x)\} \cap N_{\epsilon_1}(\Gamma_{\text{sonic}}^*) \cap \{x_{P_1} < x < \sigma\}. \]
It follows that
\[ \begin{cases} D_{0, \sigma} \subset D_\sigma & \text{for } \theta_w \in (\theta_w^d, \frac{\pi}{2}), \\ D_{0, \sigma} = D_\sigma & \text{for } \theta_w \in [\theta_w^*, \frac{\pi}{2}], \\ D_{0, \sigma} = \emptyset & \text{for } \theta_w \in (\theta_w^d, \theta_w^*) \text{ with } x_{P_1} > \sigma, \\ \Omega_\sigma = \Omega \cap D_\sigma & \text{for any } \sigma \in (0, \epsilon_0]. \end{cases} \]
Let $\alpha, \beta \in (0, 1)$ and $1 + \alpha \leq 2\beta + \frac{1}{2}$, we define

$$B = C_{\alpha, \sigma}^{1, \alpha}(Q^{\text{iter}}) \times C_{2, \alpha/2}^{-1, \beta},$$

$$C = B \cap \{u(0, 0) = v(0) = v'(0) = v'(1) = 0, u \geq 0, v(1) \geq 0\}.$$

The iteration set $K \subset C \times \left[\theta_{w}^{*}, \frac{\pi}{2}\right]$ is redefined as follows.

(i) $(u, v)$ satisfy

$$\|u\|_{1, \alpha, Q^{\text{iter}}}^{(\alpha)} \leq N_{0}, \quad \|v\|_{2, \alpha, (0, 1)}^{-1, \beta} \leq N_{0}.$$

(ii) $(u, v)$ are defined by (3.29). The only change is that $g_{sh}(s)$ satisfies the linear growth estimate

$$\min \left(g_{sh}(0) + \frac{s}{M} \frac{1}{M}\right) \leq g_{sh}(x) \leq \min \left(g_{sh}(0) + M s, \eta(\theta_{w}) - \frac{1}{M}\right)$$

for any $s \in (0, \tilde{s}(\theta_{w}))$, and $\theta_{w} \in \left[\theta_{w}^{*}, \frac{\pi}{2}\right]$, where we note that $g_{sh}(0) = t_{P_{1}}(\theta_{w}) \geq 0$.

(iii) $p$ is well-defined, and satisfies $p_{1} + \tau \leq p \leq p_{2}$ in $\Omega$.

(iv) $c^{2}, r(\theta)$ and $\tilde{\varphi} = p_{2} - p$ determined by $(u, v)$ satisfy

$$\begin{cases}
|\varphi| \leq \eta_{2}(\theta) \tilde{\varphi} \text{ in } \Omega \setminus (D_{\epsilon} \cup D_{0, \epsilon/10}) , \\
|\varphi| \leq N_{3} \epsilon \text{ in } \Omega \cap D_{\epsilon} , \\
|r^{2} - \tilde{\varphi}| \leq N_{4} \sigma , \quad |r'\tilde{\varphi})| < N_{4} \sigma^{1/2} \text{ on } \Gamma_{\text{shock}} \cap \{\pi - \theta < \sigma\} , \\
\frac{1}{2} g(r, \theta) < r'\tilde{\varphi}) < 2g(r, \theta) \text{ on } \Gamma_{\text{shock}} \cap \{\pi - \theta > \sigma/10\} ,
\end{cases}$$

where

$$\eta_{2}(\theta) = \begin{cases}
2 - \mu_{0}, & \text{if } \theta_{w} \in \left[\theta_{w}^{*} - \frac{\delta^{*}}{2}, \frac{\pi}{2}\right], \\
N_{3}, & \text{if } \theta_{w} \in \left[\theta_{w}^{*}, \theta_{w}^{*} - \delta^{*}\right], \\
\text{linear} & \text{otherwise}.
\end{cases}$$

(v) Uniform ellipticity in $\Omega \setminus D_{0, \epsilon/10}$.

$$p - r^{2} > \min \{p(P_{1}) - r^{2}(P_{1}) - \lambda_{1} \text{dist}(\xi, \Gamma_{\text{sonic}}), \lambda_{2}\}$$

for all $\xi \in \Omega \setminus D_{0, \epsilon/10}$.

(vi) Uniform obliqueness away from $P_{2}$.

$$r^{2} - \tilde{\varphi} \geq \delta_{3}(\theta_{2} - \theta) \text{ on } \Gamma_{\text{shock}} \cap \{\theta_{2} - \theta > \sigma/10\}.$$

We next extend the iteration problem to subsonic wedge angles. We should modify the elliptic equation and the oblique boundary coefficients on $\Gamma_{\text{shock}}$. First, we consider the quasilinear iteration equation. We can choose $\epsilon$ small in the iteration set, depending only on the data to ensure that

$$x_{P_{1}} > \frac{\epsilon}{10} \quad \text{when } \theta_{w} < \theta_{w}^{*} - \frac{d^{*}}{4} ,$$

i.e. we always either have the degenerate structure, or strict ellipticity. Then the modified coefficients are defined as follows:

(1) If $\theta_{w} \in \left(\theta_{w}^{*} - \frac{d^{*}}{2}, \theta_{w}^{*}\right)$, we can define the coefficients as that in supersonic-near sonic case, and the cutoff can be removed.

(2) If $\theta_{w} \in \left(\theta_{w}^{*} - \frac{\delta^{*}}{2}, \theta_{w}^{*}\right)$, the equation does not contain a cutoff, i.e. (3.38) is strict elliptic.

Let $c_{3} \in C^{\infty}(\mathbb{R})$ satisfy

$$c_{3}(t) = \begin{cases}
0, & \text{if } t \geq \theta_{w}^{*} - \frac{\delta^{*}}{2}, \\
1, & \text{if } t \leq \theta_{w}^{*} - \frac{\delta^{*}}{2},
\end{cases}$$

$$c_{3}'(t) \geq 0.$$
Define
\[
(A_{ij}, A_i, B_i)(\xi) = \varsigma_3(\theta_w)(A^{1}_{ij}, A^1_i, B^1_i)(\xi) + (1 - \varsigma_3(\theta_w))(A^{(sup)}_{ij}, A^{(sup)}_i, B^{(sup)}_i)(\xi) = (\tilde{A}_{ij}, \tilde{A}_i, \tilde{B}_i)(x, y)
\]
where \((A^{(sup)}_{ij}, A^{(sup)}_i, B^{(sup)}_i)\) are given by (3.41) for the supersonic wedge angles.

We have the following estimates

**Lemma 4.23.** There exists \(\lambda, \lambda_0 > 0\), \(\epsilon_{eq} \in (0, \frac{\pi}{2})\) and \(N_{eq} \geq 1\) depending only on the data, such that for \(\theta_w \in [\theta^*_w - \delta^*, \frac{\pi}{2}]\),

(i) For any \((x, y) \in \Omega \cap D_{eq}, \xi \in \Omega\) and \(\kappa \in \mathbb{R}^2\),

\[
\lambda |\kappa|^2 \leq \frac{\tilde{A}_{11}(x, y)}{x} \kappa_1^2 + \tilde{A}_{22}(x, y) \kappa_2^2 \leq \frac{1}{\lambda} |\kappa|^2,
\]

and

\[
\lambda_0 \text{dist}(\xi, \Gamma_{\text{sonic}})|\kappa|^2 \leq \sum_{i,j=1}^{2} A_{ij}(\xi) \kappa_i \kappa_j \leq \frac{1}{\lambda_0} |\kappa|^2.
\]

(ii) For \(s \in (0, \frac{\pi}{2})\),

\[
\| (A_{ij}, A_i, B_i) \|_{0, \alpha, \Omega} \leq N_{eq}, \quad \| (A_{ij}, A_i, B_i) \|_{1, \alpha/2, \Omega \setminus D_{0, s}} \leq N_s.
\]

(iii) For \((x, y) \in \Omega \cap D_{eq}\),

\[
\tilde{A}_1 \leq C_2 + N_{eq} \epsilon_{eq}, \quad \tilde{B}_1 \leq -1 + N_{eq} \epsilon_{eq}.
\]

**Lemma 4.24.** Let \(\delta > 0\), there exists \(0 < \lambda(\delta) < \infty\) depending on the data, \(\theta^*_w\) and \(\delta\) such that for \(\theta_w \in [\theta^*_w, \theta^*_w - \delta]\),

(i) For any \(\xi \in \Omega, \kappa \in \mathbb{R}^2\),

\[
\lambda(\delta)|\kappa|^2 \leq \sum_{i,j=1}^{2} A_{ij}(\xi) \kappa_i \kappa_j \leq (\lambda(\delta))^{-1} |\kappa|^2.
\]

(ii) For \(s \in (0, \frac{\pi}{2})\), there exists \(N_{eq}\) depending on the data, \(\theta^*_w\), and \(\delta\), and \(N_s\) also depending on \(s\), such that

\[
\| (A_{ij}, A_i, B_i) \|_{0, \alpha, \Omega} \leq N_{eq}, \quad \| (A_{ij}, A_i, B_i) \|_{1, \alpha/2, \Omega \setminus D_{0, s}} \leq N_s.
\]

We have the following estimates for removing the elliptic cutoff.

**Lemma 4.25.** Let \(\epsilon_{eq}\) be as in Lemma 4.23. If \(\epsilon \in (0, \frac{\epsilon_{eq}}{2})\), then the following holds: Let \((u, v, \theta_w) \in K\) satisfy \(\varphi = \hat{\varphi}\) where \(\hat{\varphi}\) solves (3.32). Then, if either

(i) The wedge angle \(\theta_w\) satisfies \(\theta_w \in [\theta^*_w, \theta^*_w - \frac{\delta^*}{4}]\), or

(ii) In the \((x, y)\) coordinates,

\[
|\varphi| \leq (2c_2 - \frac{\mu_0}{5})x \quad \text{in} \ \Omega \cap D_{0, \epsilon/10},
\]

then \(\varphi\) satisfies the original nonlinear equation (3.36) in \(\Omega\).
4.5. Existence and estimates of solutions of the iteration problem. First we have the existence and basic estimates of solutions to the iteration problem.

Proposition 11. Suppose the parameters \(\sigma, \epsilon, \delta K > 0\) are small depending only on the data and \(\theta^*\). Then, for each \((u, v, \theta) \in \mathcal{K}\) which satisfies

\[
\|u^* - u\|_{L^\infty(\Omega_{iter})} + \|v - v\|_{C^1(\Omega_{iter})} + |\theta^* - \theta^*| \leq \delta K,
\]

for some \((u^*, v^*, \theta^*) \in \mathcal{K}\), there exists \(\hat{\varphi} \in C^2(\Omega) \cap C^1(\Omega \setminus (\Omega_{sonic} \cup \{P_2\})) \cap C(\Omega)\) of Problem 4, determined by \((u, v, \theta)\) such that

\[
\|\hat{\varphi}\|_{0, \Omega} \leq C, \quad 0 \leq \hat{\varphi} \leq \varphi \quad \text{in} \quad \Omega \cap \mathcal{D}_e,
\]

where \(C\) depending only on the data. Moreover, if \(\theta \in [\theta^* - \frac{3\delta}{4}, \frac{\pi}{2}]\), then

\[
0 \leq \varphi \leq Cx \quad \text{in} \quad \Omega_e.
\]

Away from the sonic arc, the estimates are as follows.

Proposition 12. Let \(\hat{\varphi}\) be the solution to the iteration problem. Then \(\hat{\varphi}\) is well-defined by (3.34). Fix \(s \in (0, \frac{\pi}{2})\), there exists \(\hat{\alpha} \in (0, 1)\) depending on \(s\), such that \(\hat{\alpha} \leq 2s^*\), where if we choose the parameters \(\alpha = \frac{s}{2}\), \(\beta > \hat{\alpha}\) in the iteration set, then the following holds:

For each \(\alpha \in (0, \hat{\alpha})\), and \(\beta \in (0, \frac{1}{2})\), there exists \(C_{s, \alpha} \geq 1\) depending only on the data, \(\theta^*\), and \(\alpha\), and \(C_{s, \alpha, \beta}\) also depending on \(\beta\) such that \(\hat{\varphi}\) and \(\hat{\varphi}\) satisfy

\[
\|\hat{\varphi}\|_{1, 2} \leq C_{s, \alpha, \beta}, \quad \|\hat{\varphi}\|_{2} \leq C_{s, \alpha, \beta}.
\]

Next we give the estimates near \(\Gamma_{sonic}\) for \((\varphi, r)\) in each of the different cases.

Lemma 4.26. (1) Supersonic-away from sonic wedge angles. Let \(\delta > 0\). Then there exists \(\epsilon_{eq} \in (0, \epsilon_{eq})\) depending only on the data and \(\delta\), such that under the same assumption in Proposition, and the wedge angle \(\theta_w \in [\theta^w + \delta, \frac{\pi}{2}]\),

\[
\|\hat{\varphi}\|_{1, 2} \leq C_{s, \alpha, \beta}, \quad \|\hat{\varphi}\|_{2} \leq C_{s, \alpha, \beta}.
\]

(2) Supersonic-near sonic wedge angles. There exists \(\epsilon_{eq} \in (0, \epsilon_{eq})\), \(\delta_{sn} > 0\) depending only on the data such that for any wedge angle \(\theta_w \in [\theta^w, \theta^w + \delta_{sn}]\),

\[
\|\hat{\varphi}\|_{1, 2} \leq C_{s, \alpha, \beta}, \quad \|\hat{\varphi}\|_{2} \leq C_{s, \alpha, \beta}.
\]

(3) Subsonic-near sonic wedge angles. There exists \(\epsilon_{eq} \in (0, \epsilon_{eq})\), \(\delta_{sn} > 0\) depending only on the data such that for any wedge angle \(\theta_w \in [\theta^w - \delta_{sn}, \theta^w]\),

\[
\|\hat{\varphi}\|_{1, 2} \leq C_{s, \alpha, \beta}, \quad \|\hat{\varphi}\|_{2} \leq C_{s, \alpha, \beta}.
\]

(4) Subsonic-away from sonic wedge angles. Let \(\delta > 0\). There exists \(\epsilon_{eq} \in (0, \epsilon_{eq})\) depending only on the data and \(\delta\), such that for \(\theta_w \in [\theta^w, \theta^w - \delta]\),

\[
\|\hat{\varphi}\|_{1, 2} \leq C_{s, \alpha, \beta}, \quad \|\hat{\varphi}\|_{2} \leq C_{s, \alpha, \beta}.
\]

Basing on the regularity estimates stated as above, we have the unified estimate for \((\hat{u}, \hat{v})\), which extends Proposition 7.

Proposition 13. Suppose the assumptions of Proposition 6 hold. Let \(\hat{u} : Q_{iter} \to \mathbb{R}\) and \(v : (0, 1) \to \mathbb{R}\) be determined as

\[
\hat{u} = \hat{\varphi} \circ F^{-1}_1 \circ (F_{2, g, h})^{-1}(s, t),
\]
where \( g_{sh} \) was obtained from \( v \). Also, we may take \( \delta_3 \) small enough in the iteration set, depending on the data, so that
\[
\hat{\Gamma}_{\text{shock}} = \{ r = \hat{r}(\theta) \mid \theta \in [\theta_1, \theta_2] \},
\]
and define \( \hat{v} \) by the formula (3.29). Then \( \hat{u} \) and \( \hat{v} \) satisfy
\[
\| \hat{u} \|_{1, \hat{\alpha}, Q_{\text{iter}}} \leq C, \quad \| \hat{v} \|_{\frac{3}{2} - \frac{\beta}{2}, 2, (0, 1)} \leq C,
\]
where \( C \) only depends on the data and \( \theta^*_w \).

Finally according to the gain-in-regularity, we conclude the proof of existence of admissible solutions up to the detachment angle in the same way as for supersonic wedge angles in Section 3.

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Received December 2019; revised December 2019.

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