Heisenberg spin chains based on $s\ell(2|1)$ symmetry.

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Abstract. We find solutions of the Yang-Baxter equation acting on tensor product of arbitrary representations of the superalgebra $s\ell(2|1)$. Based on these solutions we construct the local Hamiltonians for integrable homogeneous periodic chains and open chains.
1 Introduction

The superalgebra $sl(2|1)$ appears in various quantum systems as underlying their symmetry and dynamics. Finite-dimensional representations describe spin states. For example, a lattice site which is allowed to be empty or occupied by an electron with the two spin states $\pm \frac{1}{2}$, but not by two electrons, corresponds to the three-dimensional fundamental representation. Chains consisting of sites carrying the fundamental representation with integrable short-range interaction have been constructed. The Hamiltonian of $t-J$ model is obtained from the transfer matrix of the integrable model based on the three dimensional fundamental representation of $sl(2|1)$. The superalgebra $sl(2|1)$ has a series of four-dimensional representations parametrized by a parameter $b \neq \pm \frac{1}{2}$. Integrable models built from R-matrices defined on tensor products of two different four-dimensional representations have been considered in [5], [6], [9].

The simplest integrable chain structure is the homogeneous periodic spin chain; most applications make use of this case. There exist some important modifications. The construction of open spin chains is well known [10]. The treatment of integrable inhomogeneous spin chains is more involved. An integrable model has been constructed [11]...
in view of the relevance for systems with impurities and in particular for the Kondo effect. The representations of $sl(2|1)$ accommodate $sl(2)$ representations of spin $s$ and $s \pm \frac{1}{2}$ and allow in this way to construct chains with mobile impurities [12].

The integrable chains turn out to describe approximately the effective interaction in four-dimensional gauge theories in the Regge limit [14], [15] and in the Bjorken limit [16]. Unlike the above examples here the sites carry infinite-dimensional representations of $sl(2)$ accommodating all the momentum states of reggeons and partons. Besides of the case of homogeneous periodic chains also the case of open chains is encountered both in the Bjorken limit [14, 17] and in the Regge limit [18]. A particular $sl(2|1)$ representation of interest are the infinitesimal conformal transformations in one dimension together with their twofold supersymmetric extensions. This symmetry applies to the Bjorken limit of four-dimensional supersymmetric Yang-Mills theory at least up to one loop [19], [20]. This means, the one-loop renormalization of quasipartonic composite operators can be obtained by $sl(2|1)$ symmetric pairwise interactions of partons, the states (light-cone momenta, helicity, fermion number) of which form an infinite-dimensional lowest weight module of this algebra.

In the present paper we consider the algebra $sl(2|1)$, its lowest weight modules and construct on this basis the solutions of the Yang-Baxter equation, i.e. the R-matrix acting on the tensor product of two those modules. R-matrices acting on tensor products of two fundamental $sl(2|1)$-representations and of two four-dimensional representations have been constructed in the above mentioned papers [7], [5], [6], [9]. General integrable models based on R-matrices acting on tensor product of two arbitrary finite-dimensional atypical (chiral) representations of $sl(2|1)$ have been constructed quite recently [21] by generalizing the known approach [22] from $sl(2)$ to the case of $sl(2|1)$.

We propose the alternative approach and generalize these results. Motivated by possible applications to the Bjorken limit in QCD, we represent the lowest weight modules by polynomials in one even ($z$) and two odd ($\theta, \bar{\theta}$) variables. The general R-matrices are in fact operators acting on two-point functions, i.e. (polynomial) functions of two sets $(z_1, \theta_1, \bar{\theta}_1)$ and $(z_2, \theta_2, \bar{\theta}_2)$ representing the tensor product. We construct the R-operators acting on tensor product of two arbitrary (finite or infinite-dimensional) $sl(2|1)$-modules. This is done by calculating the two-point eigenfunctions of the lowest weight and the eigenvalues of the R-operator.

From the particular result for arbitrary but isomorphic representations we derive the integrable nearest neighbour interaction Hamiltonian for homogeneous periodic chains with sites carrying arbitrary isomorphic representations.

In the case of $sl(2)$ there exist integrable nearest neighbour interactions in open chains homogeneous inside but with arbitrary different representations corresponding to the end points [16]. We extend this result to the case of $sl(2|1)$ and construct the corresponding Hamiltonians.

The presentation is organized as follows. In Section 2 we introduce definitions and summarize the standard facts about the superalgebra $sl(2|1)$ and its representations. We represent the lowest weight modules by polynomials in one even ($z$) and two odd variables ($\theta, \bar{\theta}$) and the $sl(2|1)$-generators as first order differential operators.

In Section 3 we derive the defining relation for the general R-matrix, i.e. the solution of the Yang-Baxter equation acting on tensor products of two arbitrary representations, the elements of which are polynomial functions of $(z_1, \theta_1, \bar{\theta}_1)$ and $(z_2, \theta_2, \bar{\theta}_2)$. We solve this defining relation in the space of lowest weights.

In Section 4 we construct the local integrable Hamiltonians in the simplest case of homogeneous periodic chain carrying arbitrary isomorphic representations on the sites.
In Section 5 we construct the local integrable Hamiltonians for the open chain with arbitrary isomorphic representations inside and other arbitrary representations at the end points.

Finally, in Section 6 we summarize. Appendix A contains some technical details of calculations. In Appendix B we give the expression for the R-matrix acting in the tensor product of chiral modules. In Appendix C we discuss shortly the case of finite-dimensional representations and show that obtained general R-matrix reduces to the known ones \[9\] for the tensor product of modules with minimal dimensions.

2 Algebra $\mathfrak{sl}(2|1)$

2.1 Commutators and Casimir Operators

The superalgebra $\mathfrak{sl}(2|1)$ has eight generators: four odd $V^\pm, W^\pm$ and four even ones $S, S^\pm$ and $B$. The commutation relations have the following form \[1\]: anticommutators

\[
\{V^\pm, V^\pm\} = 0 ; \{V^\pm, V^\mp\} = 0 ; \{W^\pm, W^\pm\} = 0 ; \{W^\pm, W^\mp\} = 0
\]

commutators

\[
[S, S^\pm] = \pm S^\pm ; \{S^+, S^-\} = 2S
\]

\[
[B, S^\pm] = 0 ; \{B, S\} = 0 ; \{S^\pm, V^\pm\} = 0 ; \{S^\pm, W^\pm\} = 0
\]

\[
[B, V^\pm] = \frac{1}{2}V^\pm ; \{B, W^\pm\} = -\frac{1}{2}W^\pm ; \{S, V^\pm\} = \pm \frac{1}{2}V^\pm ; \{S, W^\pm\} = \pm \frac{1}{2}W^\pm
\]

\[
[S^\pm, V^\mp] = V^\pm ; \{S^\pm, W^\mp\} = W^\pm .
\]

These generators are linear combinations of the generators $E_{AB}$ of the superalgebra $\mathfrak{g}\ell(2|1)$ \[2\]. The commutation relations for the nine generators of $\mathfrak{g}\ell(2|1)$ can be written compactly in the form:

\[
[E_{AB}, E_{CD}] = \delta_{CB} E_{AD} - (-)^{(\bar{A}+\bar{B})(\bar{C}+\bar{D})}\delta_{AD} E_{CB}
\]

where the graded commutator is defined as:

\[
[E_{AB}, E_{CD}] = E_{AB} \cdot E_{CD} - (-)^{(\bar{A}+\bar{B})(\bar{C}+\bar{D})} E_{CD} \cdot E_{AB}
\]

The indices $A, B, C, D = 1, 2, 3$ and we choose the grading: $\bar{1} = 3 = 0 ; \bar{2} = 1$. The connection between both sets of generators is the following:

\[
S^- = E_{31} ; W^- = -E_{21} ; V^- = E_{32}
\]

\[
S^+ = E_{13} ; W^+ = E_{23} ; V^+ = E_{12}
\]

\[
S = \frac{1}{2}E_{11} - \frac{1}{2}E_{33} ; B = -\frac{1}{2}E_{11} - E_{22} - \frac{1}{2}E_{33}
\]

In the fundamental representation all generators $e_{AB}$ of $\mathfrak{g}\ell(2|1)$ are $3 \times 3$-matrices and the basis in the space of these matrices can be chosen in the standard way:

\[
(e_{AB})_{CD} = \delta_{AC} \delta_{BD} ; e_{AB}e_{CD} = \delta_{CB}e_{AD}
\]
In the fundamental representation the generators $W^\pm, V^\pm, S^\pm, S, B$ have the form:

\[
S^- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}; \quad W^- = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad V^- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}
\]

\[
S^+ = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad W^+ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}; \quad V^+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]

\[
S = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix}; \quad B = \begin{pmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix}
\]

(2.1.3)

There exists a simple construction for the central elements of the enveloping algebra of $\mathfrak{gl}(2|1)$. The first step is the construction of covariant operators: suppose we have two covariant operators $V_{CD}^{(i)}$, i.e. operators which have the following commutation relations with generators

\[
[E_{AB}, V_{CD}^{(i)}] = \delta_{CB}V_{AD}^{(i)} - (-)^{(\hat{A} + \hat{B})(\hat{C} + \hat{D})}\delta_{AD}V_{CB}^{(i)}; \quad i = 1, 2.
\]

It is easy to check that operator $V_{AB} = \sum C(-)^{\hat{C}}V_{AC}^{(1)}V_{CB}^{(2)}$ is also covariant. This simple observation allows to construct covariant operators using generators $E_{CD}$ as simplest building blocks. The second step is the construction of a central element from the covariant operator: for any covariant operator $V_{CD}$ the operator $V = \sum C V_{CC}$ belongs to the center of the algebra

\[
[E_{AB}, V] = 0.
\]

Repeating this construction we obtain central elements $K_n$, $n = 1, 2, 3...$ for the enveloping algebra of $\mathfrak{gl}(2|1)$:

\[
K_1 = \sum_A E_{AA}; \quad K_2 = \sum_{AB} (-)^{\hat{B}}E_{AB}E_{BA}; \quad K_3 = \sum_{ABC} (-)^{\hat{B}+\hat{C}}E_{AB}E_{BC}E_{CA}; \quad ...
\]

(2.1.4)

The eight generators of algebra $\mathfrak{s\ell}(2|1)$ may be introduced by defining:

\[
\mathcal{E}_{AB} \equiv E_{AB} - \delta_{AB}(-)^{\hat{B}}\sum_A E_{AA}
\]

\[
\mathcal{E}_{31} = S^-; \quad \mathcal{E}_{21} = -W^-; \quad \mathcal{E}_{32} = V^-; \quad \mathcal{E}_{13} = S^+; \quad \mathcal{E}_{23} = W^+; \quad \mathcal{E}_{12} = V^+; \quad \mathcal{E}_{11} = B - S; \quad \mathcal{E}_{22} = -2B; \quad \mathcal{E}_{33} = B + S.
\]

It can be verified that generators $\mathcal{E}_{AB}$ satisfy the same commutation relations as $E_{AB}$:

\[
[\mathcal{E}_{AB}, \mathcal{E}_{CD}] = \delta_{CB}\mathcal{E}_{AD} - (-)^{(\hat{A} + \hat{B})(\hat{C} + \hat{D})}\delta_{AD}\mathcal{E}_{CB}
\]

There exists only one restriction for the Cartan generators: $\mathcal{E}_{11} + \mathcal{E}_{22} + \mathcal{E}_{33} = 0$ and the independent eight generators can be chosen in the form (2.1.4).

The center of algebra $\mathfrak{s\ell}(2|1)$ is generated by Casimir operators $C_n$, $n = 2, 3...$ for $\mathfrak{sl}(2|1)$. We shall use only two of them:

\[
C_2 = \frac{1}{2!} \sum_{AB} (-)^{\hat{B}}\mathcal{E}_{AB}\mathcal{E}_{BA} = S^2 - B^2 + S^+S^- + V^+W^- + W^+V^-
\]

(2.1.6)
\[ C_3 = \frac{1}{3!} \sum_{ABC} (-)^{B+C} \epsilon_{AB} \epsilon_{BC} \epsilon_{CA} = B(S^2 - B^2) + BS^+S^- + \frac{3}{2} B(V^+W^- + W^+V^-) + (2.1.7) \]
\[ + \frac{1}{4}(W^+V^+ - V^+W^+)S^- + \frac{1}{4}S^+(V^-W^- - W^-V^-) + \frac{1}{2}(S - 1)(V^+W^- - W^+V^-). \]

### 2.2 Global form of superconformal transformations

We represent the generators as first order differential operators, acting on the space of polynomials \( \Phi(z, \theta, \bar{\theta}) \). Lowering (decreasing the polynomial degree) operators have the form
\[
S^- = -\partial \ ; \ V^- = \partial_\theta + \frac{1}{2} \bar{\theta} \partial \ ; \ W^- = \partial_{\bar{\theta}} + \frac{1}{2} \theta \partial
\]
and generate the following global transformations
\[
e^{\lambda S^-} \Phi(z; \theta, \bar{\theta}) = \Phi(z - \lambda; \theta, \bar{\theta}), \quad (2.2.2)
\]
\[
e^{V^-} \Phi(z; \theta, \bar{\theta}) = \Phi \left( z + \frac{\bar{\theta} \partial}{2}; \theta + \epsilon \theta, \bar{\theta} \right) \ ; \ e^{W^-} \Phi(z; \theta, \bar{\theta}) = \Phi \left( z + \frac{\theta \partial}{2}; \theta, \bar{\theta} + \epsilon \right).
\]
Rising (increasing the polynomial degree) operators
\[
V^+ = -\left[ z \partial_\theta + \frac{1}{2} \bar{\theta} z \partial + \frac{1}{2} \theta \partial \partial_\theta \right] - (\ell - b) \bar{\theta} \ ; \ W^+ = -\left[ z \partial_{\bar{\theta}} + \frac{1}{2} \theta z \partial + \frac{1}{2} \theta \partial \partial_{\bar{\theta}} \right] - (\ell + b) \theta,
\]
\[
S^+ = z^2 \partial + z \theta \partial_\theta + z \bar{\theta} \partial_{\bar{\theta}} + 2\ell z - b \theta \bar{\theta}, \quad (2.2.3)
\]
generate the global transformations
\[
e^{\lambda S^+} \Phi(z; \theta, \bar{\theta}) = \left[ 1 + \frac{\theta \partial \lambda}{(1 - \lambda z)} \right]^{-b} \frac{1}{(1 - \lambda z)^{2\ell}} \Phi \left( \frac{z}{1 - \lambda z}; \frac{\theta}{1 - \lambda z}, \frac{\bar{\theta}}{1 - \lambda z} \right),
\]
\[
e^{V^+} \Phi(z; \theta, \bar{\theta}) = \frac{1}{(1 + \epsilon \theta)^{\ell - b}} \Phi \left( \frac{z}{1 + \epsilon \theta}; \frac{\theta - \epsilon z}{1 + \epsilon \theta}, \bar{\theta} \right),
\]
\[
e^{W^+} \Phi(z; \theta, \bar{\theta}) = \frac{1}{(1 + \epsilon \theta)^{\ell + b}} \Phi \left( \frac{z}{1 + \epsilon \theta}; \frac{\bar{\theta} - \epsilon z}{1 + \epsilon \theta}, \theta \right).
\]
Two remaining elements of the Cartan subalgebra:
\[
S = z \partial + \frac{1}{2} \theta \partial_\theta + \frac{1}{2} \bar{\theta} \partial_{\bar{\theta}} + \ell \ ; \ B = \frac{1}{2} \partial_\theta - \frac{1}{2} \partial_{\bar{\theta}} + b
\]
generate the transformations:
\[
e^{\lambda S} \Phi(z; \theta, \bar{\theta}) = e^{\ell \lambda} \Phi \left( e^{\lambda} z; e^{\frac{\lambda}{2}} \theta, e^{\frac{\lambda}{2}} \bar{\theta} \right)
\]
\[
e^{\lambda B} \Phi(z; \theta, \bar{\theta}) = e^{b \lambda} \Phi \left( z; e^{-\frac{\lambda}{2}} \theta, e^{-\frac{\lambda}{2}} \bar{\theta} \right)
\]
We use a natural convention here and assign scaling dimension one and \( U(1) \)-charge zero to the even variable \( z \) and the scaling dimension \( \frac{1}{2} \) and \( U(1) \)-charge \( \mp \frac{1}{2} \) to the odd variables \( \theta \) and \( \bar{\theta} \).
2.3 $s\ell(2|1)$-lowest weight modules

The lowest weight $s\ell(2|1)$-module $V_{\ell,b} = V_{\ell}^\ell$, $\ell = (\ell, b)$ is built on the lowest weight vector $\psi$ obeying:

$$V_+ \psi = 0 ; \quad W_- \psi = 0 ; \quad S^- \psi = 0 ; \quad S^\psi = \ell \hat{\psi} ; \quad B \psi = b \psi$$

In generic situation $\ell \neq \pm b$ the module is characterised uniquely by the action of the Casimir operators on its elements:

$$C_2 \psi = (\ell^2 - b^2) \psi ; \quad C_3 \psi = b(\ell^2 - b^2) \psi ; \quad \psi \in V_{\ell,b}$$

It is useful to introduce the subspaces of functions with definite chirality. Let us define for the lowest weight $\ell$-module $V_{\ell,b}$ of polynomials $\Phi(z, \theta, \bar{\theta})$ of variables $z, \theta, \bar{\theta}$.

It is easy to obtain the expression for the coherent states $e^{\lambda S^+ \psi}$, $e^{\lambda S^+ W^+ \psi}$, $e^{\lambda S^+ V^+ \psi}$, $e^{\lambda S^+ W^+ \psi}$, $e^{\lambda S^+ W^+ \psi}$ for the lowest weight $\ell = 1$ using the formulae of the previous section. They are the generating functions, the power expansion in $\lambda$ of which produces the basis:

$$A_k = (S^+)^k \psi ; \quad B_k = (S^+)^{-1} W^+ V^+ \psi , \quad k \in \mathbb{Z}_+$$

and the odd vectors

$$V_k = (S^+)^k V^+ \psi ; \quad W_k = (S^+)^{k-1} W^+ V^+ \psi , \quad k \in \mathbb{Z}_+$$

We shall use the above realization of the $s\ell(2|1)$-generators as the differential operators of first order acting on the infinite-dimensional (for generic $\ell$) space $V_{\ell,b}$ of polynomials $\Phi(z, \theta, \bar{\theta})$.

It is useful to introduce the subspaces of functions with definite chirality. Let us define for this purpose two operators called supercovariant derivatives:

$$D^+ = -\partial_\theta + \frac{1}{2} \bar{\theta} \partial \bar{\theta} ; \quad D^- = -\partial_\bar{\theta} + \frac{1}{2} \theta \partial \theta$$

and two subspaces $V_{\ell,b}^\ell \equiv \ker D^\pm \cap V_{\ell,b}$:

$$\Phi(z, \theta, \bar{\theta}) \in V_{\ell,b}^\ell \Rightarrow \Phi(z, \theta, \bar{\theta}) = \Phi(z_+, \theta_+) ; \quad z_+ \equiv z + \frac{1}{2} \theta \bar{\theta} , \quad \theta_+ \equiv \bar{\theta}$$

$$\Phi(z, \theta, \bar{\theta}) \in V_{\ell,b}^- \Rightarrow \Phi(z, \theta, \bar{\theta}) = \Phi(z_-, \theta_-) ; \quad z_- \equiv z - \frac{1}{2} \theta \bar{\theta} , \quad \theta_- \equiv \theta$$

In the generic case the chiral subspaces $V_{\ell,b}^\pm$ are not $s\ell(2|1)$-invariant ones. Indeed, the operators $D^\pm$ have the following commutation relations with $s\ell(2|1)$-generators:

$$\{D^+, V^-\} = 0 ; \quad \{D^+, W^-\} = 0 ; \quad \{D^+, S^-\} = 0$$

$$[D^\pm, S] = \frac{1}{2} D^\pm ; \quad [D^\pm, B] = \pm \frac{1}{2} D^\pm$$

(2.3.3)

$$\{D^+, V^+\} = \bar{\theta} D^+ ; \quad \{D^+, W^+\} = \ell + b ; \quad \{D^+, S^+\} = (\ell + b) \bar{\theta} + z_+ D^+$$

(2.3.4)
There exist some special values of \( \ell \): \( \ell = \mp b \). In this case the whole module \( V_{\ell, \pm \ell} \) has definite chirality \( V_{\ell, \pm \ell} = V_{\ell, \pm \ell}^\pm \):

\[
D^- v = 0 \ , \ v \in V_{\ell, \ell} \ ; \ D^+ v = 0 \ , \ v \in V_{\ell, -\ell}.
\]

The notions of chirality and chiral representations are used here as they are common in supersymmetric field theory. In the mathematical literature about superalgebra representations generic representations are called typical and chiral representations are called atypical [1], [3]. There exist some special values of \( \ell \): \( \ell = -n \); \( n = \frac{1}{2} \mathbb{Z}_+ \), for which the module \( V_{\ell,b} \) becomes a finite-dimensional vector space [1], [3]. Indeed it is evident from (2.3.1) that all basis vectors are equal zero for \( k \geq n + 1 \). There are three cases depending on the relation between \( b \) and \( n \). The first case is for generic \( b \)-typical representations): \( b \neq \pm n \); \( \dim V_{-n,b} = 8n \)

\[
\Phi_k^\pm = z_+^k \ , \ k = 1 \ldots 2n - 1 \ , \ \Phi_0 = 1 \ , \ \Phi_{2n} = \left(z + \frac{b}{2} \theta \right)^{2n} ; \ \Psi_k^\pm = \theta_\pm z_\pm^k \ , \ k = 0 \ldots 2n - 1
\]

The second and third case appears for \( b = \pm n \) and here the representation spaces have definite chirality (atypical representations).

\( b = \pm n \); \( V_{-n,\pm n} = V_{-n,-n} \); \( \dim V_{-n,\pm n} = 4n + 1 \)

\[
\Phi_k^\pm = z_\pm^k \ , \ k = 0 \ldots 2n \ ; \ \Psi_k^\pm = \theta_\pm z_\pm^k \ , \ k = 0 \ldots 2n - 1
\]

Let us introduce the special notation for the fundamental \( s\ell(2|1) \)-module:

\( V_{-\frac{1}{2}, -\frac{1}{2}} \equiv V_f, \ \tilde{f} = (-\frac{1}{2}, -\frac{1}{2}) \). In the basis

\[
e_1 = -z_- \leftrightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} ; \ e_2 = \theta_- \leftrightarrow \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} ; \ e_3 = -1 \leftrightarrow \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
\]

the \( s\ell(2|1) \)-generators take their fundamental form (2.1.3).

### 2.4 Tensor products of two \( s\ell(2|1) \)-modules

The tensor product of two \( s\ell(2|1) \)-modules has the following direct sum decomposition [3]:

\[
V_{\ell_1,b_1} \otimes V_{\ell_2,b_2} = V_{\ell,b} + 2 \sum_{n=1}^{\infty} V_{\ell+n,b} + \sum_{n=0}^{\infty} V_{\ell+n,\frac{1}{2} b - \frac{1}{2}} + \sum_{n=0}^{\infty} V_{\ell+n,\frac{1}{2} b + \frac{1}{2}} ; \ \ell_i \neq \pm b_i \quad (2.4.1)
\]

\( \ell = \ell_1 + \ell_2 \); \( b = b_1 + b_2 \)

Note that this formula is applicable in the generic situation \( \ell_i \neq \pm b_i \). The direct sum decomposition reduces for the tensor product involving chiral modules:

\[
V_{\ell_1,\pm \ell_1} \otimes V_{\ell_2,\pm b_2} = \sum_{n=0}^{\infty} V_{\ell+n,b} + \sum_{n=0}^{\infty} V_{\ell+n,\frac{1}{2} b + \frac{1}{2}} ; \ \ell_2 \neq \pm b_2
\]

\[
V_{\ell_1,\pm \ell_1} \otimes V_{\ell_2,\pm \ell_2} = \sum_{n=0}^{\infty} V_{\ell+n,b} ; \ V_{\ell_1,\mp \ell_1} \otimes V_{\ell_2,\mp \ell_2} = \sum_{n=0}^{\infty} V_{\ell+n,\frac{1}{2} b + \frac{1}{2}}
\]
In Appendix C we discuss the modifications of \((2.4.1)\) arising for finite-dimensional representations.\(^3\)

For the proof of \((2.4.1)\) one has to determine all possible lowest weight vectors appearing in the tensor product \(V_{\ell_1,b_1} \otimes V_{\ell_2,b_2}\). In the realization on functions of \(z, \theta, \bar{\theta}\) the space \(V_{\ell_1,b_1} \otimes V_{\ell_2,b_2}\) is isomorphic to the space of polynomials on two even variables \(z_1, z_2\) and four odd variables \(\theta_1, \theta_2, \bar{\theta}_1, \bar{\theta}_2\) called for the sake of brevity two-point functions. The \(sl(2|1)\)-generators acting on the \(V_{\ell_1,b_1} \otimes V_{\ell_2,b_2}\) are just the sums of corresponding generators acting in \(V_{\ell_1,b_1}\). The lowest weight vectors of the irreducible representations in the decomposition of \(V_{\ell_1,b_1} \otimes V_{\ell_2,b_2}\) are defined as the common solutions of the equations:

\[
S^- \Phi = V^- \Phi = W^- \Phi = 0 \quad (2.4.2)
\]

which have the form

\[
\Phi(z_1, z_2; \theta_1, \theta_2; \bar{\theta}_1, \bar{\theta}_2) = \Phi(Z_{12}, \theta_{12}, \bar{\theta}_{12}) \quad (2.4.3)
\]

where

\[
Z_{12} \equiv z_1 - z_2 + \frac{1}{2} (\theta_1 \theta_2 + \theta_1 \bar{\theta}_2) ; \quad \theta_{12} \equiv \theta_1 - \theta_2 ; \quad \bar{\theta}_{12} \equiv \bar{\theta}_1 - \bar{\theta}_2.
\]

Indeed, from \((2.4.2)\) follows immediately that the function \(\Phi\) has to be invariant with respect to global transformations \((2.2.2)\). This invariance predicts the general form of \(\Phi\):

\[
e^{\beta \Phi^-} e^{\alpha V^-} e^{\beta S^-} \Phi(z_1, z_2; \theta_1, \theta_2; \bar{\theta}_1, \bar{\theta}_2) = \Phi \left( z_1 - a + \frac{\alpha(\bar{\theta}_1 + \beta)}{2} + \frac{\beta \theta_1}{2}, z_2 - a + \frac{\alpha(\bar{\theta}_2 + \beta)}{2} + \frac{\beta \theta_2}{2}; \theta_1 + \alpha, \theta_2 + \alpha; \bar{\theta}_1 + \beta, \bar{\theta}_2 + \beta \right)
\]

and choosing \(a = z_2 + \frac{1}{2} \theta_2 \bar{\theta}_2\), \(\alpha = -\theta_2\), \(\beta = -\bar{\theta}_2\) we obtain \((2.4.3)\).

There are additional restrictions of definite chirality for the lowest weights in the tensor product of the chiral modules:

\[
D_1^+ \Phi = 0 \Rightarrow \Phi = \Phi \left( Z_{12} \pm \frac{1}{2} \theta_{12} \bar{\theta}_{12}, \theta_{12}^\pm \right) ; \quad \theta_{12}^\pm \equiv \theta_1^\pm - \theta_2^\pm
\]

\[
D_2^\pm \Phi = 0 \Rightarrow \Phi = \Phi \left( Z_{12} \mp \frac{1}{2} \theta_{12} \bar{\theta}_{12}, \theta_{12}^\pm \right).
\]

Now, the lowest weight vectors in the decomposition of the tensor product are constructed from functions \((2.4.3)\) being eigenfunctions of generators \(S\) and \(B\). The eigenfunctions of the operator \(S\) are the polynomials with scaling dimension \(n\) and the eigenfunctions of the operator \(B\) are the polynomials with one of the possible \(U(1)\)-charges: \(0, \pm \frac{1}{2}\).

Finally we obtain that all lowest weights in the space \(V_{\ell_1,b_1} \otimes V_{\ell_2,b_2}\) are divided on two sets, the even lowest weights:

\[
\Phi_n^\pm \equiv \left( Z_{12} \pm \frac{1}{2} \theta_{12} \bar{\theta}_{12} \right)^n ; \quad D_1^\pm \Phi_n^\pm = 0 , \quad S \Phi_n^\pm = (n + \ell) \Phi_n^\pm , \quad B \Phi_n^\pm = b \Phi_n^\pm \quad (2.4.4)
\]

and the odd lowest weights:

\[
\Psi_n^- \equiv \theta_{12} Z_{12}^n ; \quad \Psi_n^+ \equiv \bar{\theta}_{12} Z_{12}^n ; \quad S \Psi_n^\pm = (n + \ell + \frac{1}{2}) \Psi_n^\pm , \quad B \Psi_n^\pm = (b \pm \frac{1}{2}) \Psi_n^\pm \quad (2.4.5)
\]

It is convenient to choose the chiral basis \(D_1^\pm \Phi_n^\pm = 0\) for the even lowest weights. Thus we have obtained the full set of lowest weights appearing in the expansion of \(V_{\ell_1,b_1} \otimes V_{\ell_2,b_2}\).\(^4\)
Let us consider the three operators $R$ notation:

We are going to find the general solution $R$ Yang-Baxter equation in the space $V$.

Let three steps.

Comparing operator coefficients of $R$ are $\bar{\theta}$.

Indeed, the proof is the following. The Yang-Baxter equation has the simple form in short steps.

First one considers the simplest situation: $\bar{\ell}$ for all $i = 1, 2, 3$ so that the space $V_f = V_{1, -1/2}$ has the minimal possible dimension: $\dim V_f = 3$. In the second step we fix $\bar{\ell}_i = \bar{f}$ for $i = 1, 2$ and obtain the solution $R_{\bar{f}, \bar{f}}$ for arbitrary $\bar{\ell}$. In the third step we fix $\bar{\ell}_3 = \bar{f}$ and using the result for the operator $R_{\bar{f}, \bar{f}}$ we obtain and solve the defining equation for the general $R$-matrix $R_{\bar{f}, \bar{f}}(u)$. It should be noted that the analogous approach was used for the derivation of the $\mathfrak{sl}(2)$-invariant $R$-matrix [22].

## 3 Yang-Baxter equation and general operator $R_{\bar{f}, \bar{f}}(u)$

Let $V_{\ell_i, b_i}$ : $i = 1, 2, 3$ be three lowest weight $\mathfrak{sl}(2|1)$-modules. We shall use the short-hand notation:

$\bar{\ell} = (\ell, b) ; V_{\bar{\ell}} = V_{\ell, b}$

Let us consider the three operators $R_{\bar{f}, \bar{f}}(u)$ which are acting in $V_{\bar{f}} \otimes V_{\bar{f}}$ and obey the Yang-Baxter equation in the space $V_{\bar{f}} \otimes V_{\bar{f}} \otimes V_{\bar{f}}$ [23]:

$$R_{\bar{f}, \bar{f}}(u-v)R_{\bar{f}, \bar{f}}(u)R_{\bar{f}, \bar{f}}(v) = R_{\bar{f}, \bar{f}}(v)R_{\bar{f}, \bar{f}}(u)R_{\bar{f}, \bar{f}}(u-v) \quad (3.0.1)$$

We are going to find the general solution $R_{\bar{f}, \bar{f}}(u)$ of Yang-Baxter equation by the following three steps.

First one obtains the operator $R_{\bar{f}, \bar{f}}(u)$ in the simplest situation: $\bar{\ell}_i = \bar{f}$ for all $i = 1, 2, 3$ so that the space $V_f = V_{1, -1/2}$ has the minimal possible dimension: $\dim V_f = 3$. In the second step we fix $\bar{\ell}_1 = \bar{f}$ and $\bar{\ell}_2 = \bar{f}$ for $i = 1, 2$ and obtain the solution $R_{\bar{f}, \bar{f}}$ for arbitrary $\bar{\ell}$. In the third step we fix $\bar{\ell}_3 = \bar{f}$ and using the result for the operator $R_{\bar{f}, \bar{f}}$ we obtain and solve the defining equation for the general $R$-matrix $R_{\bar{f}, \bar{f}}(u)$. It should be noted that the analogous approach was used for the derivation of the $\mathfrak{sl}(2)$-invariant $R$-matrix [22].

### 3.1 Fundamental solution $R_{\bar{f}, \bar{f}}$

First one considers the simplest situation: $\ell_i = -1/2 , b_i = -1/2 \leftrightarrow \bar{\ell}_i = \bar{f}_i$. We shall prove that the operator:

$$R_{\bar{f}_i, \bar{f}_j}(u) = u + \eta P_{ij} ; P_{ij} \equiv \sum_{AB} (-)^B e^i_A e^j_B \quad (3.1.1)$$

where $e^j_A$ are generators acting in the space $V_{\bar{f}_i}$, is the solution of the Yang-Baxter equation [23]:

$$R_{\bar{f}_i, \bar{f}_j}(u-v)R_{\bar{f}_i, \bar{f}_j}(u)R_{\bar{f}_i, \bar{f}_j}(v) = R_{\bar{f}_i, \bar{f}_j}(v)R_{\bar{f}_i, \bar{f}_j}(u)R_{\bar{f}_i, \bar{f}_j}(u-v)$$

Indeed, the proof is the following. The Yang-Baxter equation has the simple form in short notations:

$$(u-v + \eta P_{12})(u+\eta P_{13})(v + \eta P_{23}) = (v + \eta P_{23})(u + \eta P_{13})(u - v + \eta P_{12})$$

Comparing operator coefficients of $u^k$ on both sides of this equation yields:

$$u^0 : P_{12}P_{13}P_{23} = P_{23}P_{13}P_{12} \quad (3.1.2)$$

$$u^1 : P_{13}P_{23} + P_{12}P_{23} = P_{23}P_{12} + P_{23}P_{13} \quad (3.1.3)$$
Using (2.1.2) one can prove that the operator $P_{ij}$ is the permutation:

$$P_{ij} e^j_{AB} = e^j_{AB} P_{ij} \Rightarrow P_{ij} P_{jk} = P_{ik} P_{ij}$$

and this commutation relation for $P_{ij}$ allows to check that eqs. (3.1.2), (3.1.3) hold and this proves that $R_{f_f}$ (1.1.1) obeys the Yang-Baxter equation.

### 3.2 The solution for the operator $R_{\vec{f} \vec{f}}(u)$

We fix $\vec{f}_i = \vec{f}_i$ for $i = 1, 2$ and obtain the solution $R_{\vec{f} \vec{f}}$ for arbitrary $\vec{f}$. The operator:

$$R_{\vec{f} \vec{f}}(u) = u + \eta \sum_{AB} (-)^B e_{AB} E_{BA},$$

where $E_{AB}$ are generators in arbitrary representation $\vec{f}$, is the solution of the Yang-Baxter equation:

$$R_{\vec{f}_1 \vec{f}_2}(u - v)R_{\vec{f}_1 \vec{f}_3}(u)R_{\vec{f}_2 \vec{f}_3}(v) = R_{\vec{f}_2 \vec{f}_3}(v)R_{\vec{f}_1 \vec{f}_3}(u)R_{\vec{f}_1 \vec{f}_2}(u - v)$$

The proof is the following [24]. The Yang-Baxter equation has the simple form in short-hand notations:

$$(u - v + \eta P_{12})(u + \eta e \otimes 1 \otimes E)(v + \eta 1 \otimes e \otimes E) = (v + \eta 1 \otimes e \otimes E)(u + \eta e \otimes 1 \otimes E)(u - v + \eta P_{12})$$

Matching operator coefficients of $u^k$ on both sides of this equality yields:

$$u^0 : P_{12}(e \otimes 1 \otimes E)(1 \otimes e \otimes E) = (1 \otimes e \otimes E)(e \otimes 1 \otimes E)P_{12}$$

$$u^1 : (e \otimes 1 \otimes E)(1 \otimes e \otimes E) + P_{12}(1 \otimes e \otimes E) = (1 \otimes e \otimes E)P_{12} + (1 \otimes e \otimes E)(e \otimes 1 \otimes E)$$

The first equation is a simple consequence of the properties of $P_{12}$. Using the fact that the matrices $e_{AB}$ form a basis we obtain that the second equation is equivalent to the following system of equations:

$$E_{AB} \cdot E_{CD} - (-)\delta_{CB}E_{AD} - (-)\delta_{AD}E_{CB}.$$ 

This is nothing else but the commutation relations for generators $E_{AB}$.

Let us represent the operator under consideration:

$$R_{\vec{f} \vec{f}}(u) = u + \eta \sum_{AB} (-)^B e_{AB} E_{BA}$$

in the matrix form in the standard basis with the grading $\bar{1}, \bar{3} = 0$; $\bar{2} = 1$:

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}; \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}; \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

We shall use the following definition of the matrix of an operator:

$$F_{e_A} = e_B F_{BA}, \quad (3.2.1)$$

which leads to the validity of the common rule for the matrix product, i.e. without additional sign factors,

$$FG_{e_A} = e_B (FG)_{BA}; \quad (FG)_{BA} = F_{BC} G_{CA}.$$
Let us calculate the matrix of R-operator using the definitions (1.2.1) and (2.1.2):

\[
(e_{AB}E_{BA})e_C = (-)^{(\bar{B} + \bar{A})}\bar{C}e_{AB}e_C = (-)^{(\bar{B} + \bar{A})}\bar{C}e_D(e_{AB})D_C E_{BA}.
\]

Therefore the matrix element of operator \( e_{AB}E_{BA} \) has the form

\[
\sum_{AB} (-)^{B}(e_{AB}E_{BA})_{CD} = (-)^{\bar{C}}E_{DC}.
\]

Finally one obtains

\[
\mathbb{R}_{\bar{f}f}(u) = u + \eta \sum_{AB} (-)^{B}e_{AB}E_{BA} = \begin{pmatrix}
  u + \eta E_{11} & \eta E_{21} & \eta E_{31} \\
  \eta E_{12} & u - \eta E_{22} & \eta E_{32} \\
  \eta E_{13} & \eta E_{23} & u + \eta E_{33}
\end{pmatrix}.
\]

This is the expression for the \( g\ell(2|1) \)-invariant R-matrix. The \( s\ell(2|1) \)-invariant R-matrix can be derived from this result in a simple way: the operator \( K_1 = E_{11} + E_{22} + E_{33} \) belongs to the center of the algebra and therefore the R-operator \( \mathbb{R}_{\bar{f}f}(u - \eta K_1) \) is also a solution of the Yang-Baxter equation. Using the connection between \( E_{AB} \) and \( s\ell(2|1) \)-generators we obtain the \( s\ell(2|1) \)-invariant R-matrix \([2]\)

\[
\mathbb{R}_{\bar{\ell}_i\ell_i}(u - \eta K_1) = \begin{pmatrix}
  u + \eta (S + B) & -\eta W^- & \eta S^- \\
  \eta V^+ & u + 2\eta B & \eta V^- \\
  \eta S^+ & \eta W^+ & u + (B - S)
\end{pmatrix}.
\]

3.3 General R-matrix \( \mathbb{R}_{\bar{\ell}_1\ell_2}(u) \)

To obtain the defining relation for the general R-operator we consider the special case \( \bar{\ell}_3 = \bar{f} \) in (1.0.1). Then one can choose the above matrix realization in \( V_{\bar{\ell}_i} \) and the operators \( \mathbb{R}_{\bar{\ell}_1\bar{f}}, \mathbb{R}_{\bar{\ell}_2\bar{f}} \) are linear functions of spectral parameter \( u \) in this particular case

\[
\mathbb{R}_{\bar{\ell}_i\bar{f}}(u - \eta K_1) = u + \eta \mathbb{R}_i; \quad \mathbb{R}_i = \begin{pmatrix}
  S_i + B_i & -W_i^- & S_i^- \\
  V_i^+ & 2B_i & V_i^- \\
  S_i^+ & W_i^+ & B_i - S_i
\end{pmatrix}; \quad i = 1, 2
\]

Now the general R-matrix \( \mathbb{R}_{\bar{\ell}_1\bar{\ell}_2}(u) \) acting in the tensor product \( V_{\bar{\ell}_1} \otimes V_{\bar{\ell}_2} \) of arbitrary modules, is fixed by the condition

\[
\mathbb{R}_{\bar{\ell}_1\bar{\ell}_2}(u - v)\mathbb{R}_{\bar{\ell}_1\bar{f}}(u - \eta K_1)\mathbb{R}_{\bar{\ell}_2\bar{f}}(v - \eta K_1) = \mathbb{R}_{\bar{\ell}_1\bar{f}}(v - \eta K_1)\mathbb{R}_{\bar{\ell}_2\bar{f}}(u - \eta K_1)\mathbb{R}_{\bar{\ell}_1\bar{\ell}_2}(u - v) \tag{3.3.1}
\]

or equivalently:

\[
\mathbb{R}_{\bar{\ell}_1\bar{\ell}_2}(u - v) \left( \frac{uv}{\eta^2} + \frac{u + v}{2\eta} (R_1 + R_2) + \frac{u - v}{2\eta} (R_2 - R_1) + R_1 R_2 \right) =
\]

\[
= \left( \frac{uv}{\eta^2} + \frac{v + u}{2\eta} (R_2 + R_1) + \frac{v - u}{2\eta} (R_1 - R_2) + R_2 R_1 \right) \mathbb{R}_{\bar{\ell}_1\bar{\ell}_2}(u - v).
\]

After separation of \( u + v \) and \( u - v \) dependence we obtain two equations \( (u - v \to u) \):

\[
\mathbb{R}_{\bar{\ell}_1\bar{\ell}_2}(u)(R_1 + R_2) = (R_1 + R_2)\mathbb{R}_{\bar{\ell}_1\bar{\ell}_2}(u) \tag{3.3.2}
\]

\[
\mathbb{R}_{\bar{\ell}_1\bar{\ell}_2}(u) \left( \frac{u}{2\eta} (R_2 - R_1) + R_1 R_2 \right) = \left( \frac{u}{2\eta} (R_2 - R_1) + R_2 R_1 \right) \mathbb{R}_{\bar{\ell}_1\bar{\ell}_2}(u) \tag{3.3.3}
\]
The first equation \[3.3.2\] expresses the fact that \( \mathbb{R}(u) \) has to be invariant with respect to the action of \( \mathfrak{sl}(2|1) \)-algebra and the second equation is the wanted defining relation for the operator \( \mathbb{R}_{\vec{t}_1 \vec{t}_2}(u) \).

The \( \mathfrak{sl}(2|1) \)-invariance of the operator \( \mathbb{R}_{\vec{t}_1 \vec{t}_2}(u) \) allows to simplify the problem. Due to \( \mathfrak{sl}(2|1) \)-invariance any eigenspace of the operator \( \mathbb{R}_{\vec{t}_1 \vec{t}_2}(u) \) is a lowest weight \( \mathfrak{sl}(2|1) \)-module generated by some lowest weight eigenvector. Therefore without loss of generality we can solve the defining relation \(3.3.3\) in the space of lowest weights. Let us consider in more details the structure of eigenspace of the \( \mathfrak{sl}(2|1) \)-invariant operator acting on the tensor product \( V_{\ell_1,b_1} \otimes V_{\ell_2,b_2} \). As we have seen from direct sum decomposition:

\[
V_{\ell_1,b_1} \otimes V_{\ell_2,b_2} = V_{\ell,b} + 2 \sum_{n=1}^{\infty} V_{\ell+n,b} + \sum_{n=0}^{\infty} V_{\ell+n+\frac{1}{2}} \cdot b - \frac{1}{2} + \sum_{n=0}^{\infty} V_{\ell+n+\frac{1}{2},b+\frac{1}{2}}
\]

for every fixed \( n \) the space of lowest weight vectors with eigenvalue \( b \) is two-dimensional and the ones with eigenvalues \( b \pm \frac{1}{2} \) are one-dimensional. Therefore the operator \( \mathbb{R}_{\vec{t}_1 \vec{t}_2}(u) \) is diagonal on odd lowest weight vectors \( \Psi^+_n \) and \( \Psi^-_n \) but acts non-trivially on the two-dimensional subspace of even lowest weight vectors spanned on \( \Phi^+_n \) and \( \Phi^-_n \).

In matrix form we have:

\[
\mathbb{R}_{\vec{t}_1 \vec{t}_2}(u) \begin{pmatrix} \Phi^+_n \\ \Phi^-_n \\ \Psi^+_n \\ \Psi^-_n \end{pmatrix} = \begin{pmatrix} A_n(u) & B_n(u) & 0 & 0 \\ C_n(u) & D_n(u) & 0 & 0 \\ 0 & 0 & F_n(u) & 0 \\ 0 & 0 & 0 & E_n(u) \end{pmatrix} \begin{pmatrix} \Phi^+_n \\ \Phi^-_n \\ \Psi^+_n \\ \Psi^-_n \end{pmatrix}
\]

The matrix relation \(3.3.3\) leads to a set of recurrence relations for the coefficients \( A_n, ..., E_n \).

Some details of calculations can be found in Appendix and here we present the final expression for the general solution of these recurrence relations:

\[
A_n(u) = (-1)^{n+1} \frac{\Gamma(u + \ell_n + 1)}{\Gamma(-u + \ell_n)} \cdot \frac{u + b_1 - b_2}{(\ell_1 - b_1)(\ell_2 + b_2)}
\]

\[
B_n(u) = (-1)^n \frac{\Gamma(u + \ell_n + 1)}{\Gamma(-u + \ell_n + 1)} \cdot \frac{(\ell_1 + b_1)(\ell_2 - b_2)}{(\ell_1 - b_1)(\ell_2 + b_2)}
\]

\[
C_n(u) = (-1)^n \frac{\Gamma(u + \ell_n)}{\Gamma(-u + \ell_n + 1)} \cdot \frac{\ell_1 + 1}{(\ell_2 - b_2)(\ell_2 + b_2) (u - b_1 - b_2) - (u + b_1 + b_2)(u - b_2 - \ell_1)(u - b_2 + \ell_1)}
\]

\[
D_n(u) = (-1)^{n+1} \frac{\Gamma(u + \ell_n + 1)}{\Gamma(-u + \ell_n + 1)} \cdot \frac{\ell_2 + 1}{(\ell_1 - b_1)(\ell_2 + b_2)}
\]

\[
E_n(u) = (-1)^n \frac{\Gamma(u + \ell_n + 1)}{\Gamma(-u + \ell_n + 1)} \cdot \frac{u - b_1 - b_2}{(\ell_1 - b_1)(\ell_2 + b_2)}
\]

\[
F_n(u) = (-1)^n \frac{\Gamma(u + \ell_n + 1)}{\Gamma(-u + \ell_n + 1)} \cdot \frac{u - b_2 - \ell_1}{(\ell_1 - b_1)(\ell_2 + b_2)}
\]

where we used the notations:

\( \ell_n \equiv n + \ell_1 + \ell_2 \); \( u \equiv \frac{u}{\eta} + b_1 - b_2 \).

As usual the obtained general solution of the Yang-Baxter equation is fixed up to overall normalization. We choose the normalization such that the R-matrix coincides with the permutation operator for \( u = 0 \) and \( \vec{t}_1 = \vec{t}_2 \).
The obtained R-matrix (4.3.4) acts on the space of two-point functions which are polynomials in \( z_i, \theta_i, \bar{\theta}_i, \ i = 1, 2 \). This holds also if the representation parameters \( \ell_i, b_i, \ i = 1, 2 \) correspond to chiral or antichiral cases. If one or both modules in the tensor product are chiral or antichiral then the tensor product representation space is a proper subspace of the space of all two-point polynomials (compare (2.4.1)). It is important to observe that in these cases the action of R-matrix can be consistently restricted to the corresponding subspace. Indeed, after multiplying with the overall factor \((\ell_1 - b_1)(\ell_2 + b_2)\), the matrix becomes triangular in these cases in the way as expected. We list the reduced R-matrices in all special cases involving chiral representations in Appendix B.

4 Homogeneous periodic chain

4.1 Commuting transfer matrices

Let us construct the set of commuting \( \mathfrak{sl}(2|1) \)-invariant operators the generating function of which is the transfer-matrix \( T_\bar{m}(u) \). We construct \( T_\bar{m}(u) \) as the supertrace of a monodromy matrix built of the elementary \( \mathbb{R} \)-matrix blocks [24].

We introduce the \( N \) spaces \( V_{\ell_i} \) and \( N \) operators \( \mathbb{R}_{\bar{m}, \ell_i}(u) \):

\[
\mathbb{R}_{\bar{m}, \ell_i}(u) : V_{\bar{m}} \otimes V_{\ell_i} \mapsto V_{\bar{m}} \otimes V_{\ell_i}
\]

The periodicity convention \( N + 1 \equiv 1 \) is implied. The monodromy matrix

\[
\mathbb{R}_{\bar{m}}(u) \equiv \mathbb{R}_{\bar{m}, \ell_1}(u - c_1) \cdots \mathbb{R}_{\bar{m}, \ell_N}(u - c_N)
\]  

(4.1.1)

acts then on the space \( V_{\bar{m}} \otimes V_{\ell_1} \otimes \cdots \otimes V_{\ell_N} \), and \( T_\bar{m}(u) \) is obtained by taking the supertrace in the auxiliary space \( V_{\bar{m}} \):

\[
T_\bar{m}(u) = \text{str}_{V_{\bar{m}}} \mathbb{R}_{\bar{m}}(u).
\]

These monodromy matrices form the commutative family:

\[
T_{\bar{m}_1}(u) T_{\bar{m}_2}(v) = T_{\bar{m}_2}(v) T_{\bar{m}_1}(u)
\]  

(4.1.2)

The relation (1.1.2) follows from the fact that there exists the operator \( \mathbb{R}_{\bar{m}_1, \bar{m}_2} \) such that

\[
\mathbb{R}_{\bar{m}_1, \bar{m}_2}(u - v) \mathbb{R}_{\bar{m}_1, \ell_i}(u) \mathbb{R}_{\bar{m}_2, \ell_i}(v) = \mathbb{R}_{\bar{m}_2, \ell_i}(v) \mathbb{R}_{\bar{m}_1, \ell_i}(u) \mathbb{R}_{\bar{m}_1, \bar{m}_2}(u - v)
\]

The \( \mathbb{R} \)-operator is even. Therefore by using standard arguments one derives an analogous equation for the monodromy matrices:

\[
\mathbb{R}_{\bar{m}_1, \bar{m}_2}(u - v) \mathbb{R}_{\bar{m}_1}(u) \mathbb{R}_{\bar{m}_2}(v) = \mathbb{R}_{\bar{m}_2}(v) \mathbb{R}_{\bar{m}_1}(u) \mathbb{R}_{\bar{m}_1, \bar{m}_2}(u - v)
\]  

(4.1.3)

From this one can derive easily that corresponding traces and supertraces are commuting operators separately:

\[
T^{\pm}_{\bar{m}}(u) = \text{tr}_{V_{\bar{m}}} \mathbb{R}_{\bar{m}}(u) ; \ T^{\pm}_{\bar{m}_1}(u) T^{\pm}_{\bar{m}_2}(v) = T^{\pm}_{\bar{m}_2}(v) T^{\pm}_{\bar{m}_1}(u)
\]

\[
T^{-}_{\bar{m}}(u) = \text{str}_{V_{\bar{m}}} \mathbb{R}_{\bar{m}}(u) ; \ T^{-}_{\bar{m}_1}(u) T^{-}_{\bar{m}_2}(v) = T^{-}_{\bar{m}_2}(v) T^{-}_{\bar{m}_1}(u)
\]

but only \( T_\bar{m}(u) \equiv T^-_{\bar{m}}(u) \) is the generating function for the \( \mathfrak{sl}(2|1) \)-invariant operators.
Instead of giving the general proof we demonstrate all this on the example of $T_f(u)$, where the auxiliary space corresponds to the fundamental representation. Let us represent $T_f(u)$ in the form:

$$T_m(u) = e_{AB} T_{AB}(u)$$

where operators $T_{AB}(u)$ act in tensor product $V_{\vec{f}}_1 \otimes \cdots \otimes V_{\vec{f}}_n$ and we assume the summation over repeated indices. The general equation (1.1.3) has the form in this case

$$\left[u - v + \eta(-)^G e^i_{FG} e^j_{GF}\right] e^i_{AB} T_{AB}(u) e^j_{CD} T_{CD}(v) =$$

$$= e^j_{CD} T_{CD}(v) e^i_{AB} T_{AB}(u) \left[u - v + \eta(-)^G e^i_{FG} e^j_{GF}\right].$$

The traces and supertraces of generators $e_{CD}$ are calculated as follows:

$$\text{tr } e_{AB} \equiv (e_{AB})_{CC} = \delta_{AB} ; \quad \text{str } e_{AB} \equiv (-)^C (e_{AB})_{CC} = (-)^A \delta_{AB}$$

Using $e_{AB} e_{CD} = \delta_{CB} e_{AD}$ and taking (super-)traces in corresponding spaces one easily obtains:

$$\text{tr} : T_{AA}(u) T_{CC}(v) = T_{CC}(v) T_{AA}(u),$$

$$\text{str} : (-)^A T_{AA}(u)(-)^C T_{CC}(v) = (-)^C T_{CC}(v)(-)^A T_{AA}(u),$$

The $g\ell(2|1)$-invariance can be demonstrated in the simplest example $N = 2$. The generalization to arbitrary $n$ is straightforward.

$$T_f^+(u) = \text{tr} \left(u + \eta(-)^B e_{AB} E^1_{BA}\right) \left(u + \eta(-)^D e_{CD} E^2_{DC}\right) =$$

$$= 3u^2 + \eta u (-)^A \left(E^1_{AA} + E^2_{AA}\right) + \eta^2 E^1_{AB} E^2_{BA}$$

$$T_f^-(u) = \text{str} \left(u + \eta(-)^B e_{AB} E^1_{BA}\right) \left(u + \eta(-)^D e_{CD} E^2_{DC}\right) =$$

$$= u^2 + \eta u \left(E^1_{AA} + E^2_{AA}\right) + \eta^2 (-)^B E^1_{AB} E^2_{BA}.$$
where $\mathbb{P}$ is permutation operator.
Indeed, the permutation operator $\mathbb{P}$ acts as follows on the lowest weight basis:
\[
\mathbb{P} \begin{pmatrix} \Phi^+_n \\ \Phi^-_n \\ \Psi^+_n \\ \Psi^-_n \end{pmatrix} = (-1)^n \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \Phi^+_n \\ \Phi^-_n \\ \Psi^+_n \\ \Psi^-_n \end{pmatrix}
\]

The expression for matrix coefficients of $R$-operator takes the simple form in homogeneous case ($\ell_1 = \ell_2$, $b_1 = b_2$):
\[
A_n(u) = (-1)^{n+1} \frac{\Gamma(u + \ell_n + 1)}{\Gamma(-u + \ell_n)} \cdot \frac{u}{(\ell + b)(\ell - b)} \tag{4.2.2}
\]
\[
B_n(u) = (-1)^n \frac{\Gamma(u + \ell_n + 1)}{\Gamma(-u + \ell_n + 1)} ; \quad C_n(u) = (-1)^n \frac{\Gamma(u + \ell_n)}{\Gamma(-u + \ell_n)}
\]
\[
D_n(u) = (-1)^{n+1} \frac{\Gamma(u + \ell_n)}{\Gamma(-u + \ell_n + 1)} \cdot \frac{u(u^2 - 2\ell^2 - 2b^2)}{(\ell + b)(\ell - b)}
\]
\[
E_n(u) = (-1)^n \frac{\Gamma(u + \ell_n + 1)}{\Gamma(-u + \ell_n + 1)} \cdot \frac{(u + b - \ell)(u + b + \ell)}{(\ell + b)(\ell - b)}
\]
\[
F_n(u) = (-1)^n \frac{\Gamma(u + \ell_n + 1)}{\Gamma(-u + \ell_n + 1)} \cdot \frac{(u - b - \ell)(u - b + \ell)}{(\ell + b)(\ell - b)}
\]

where
\[
\ell_1 = \ell_2 = \ell ; \quad b_1 = b_2 = b ; \quad u \equiv \frac{u}{\eta} ; \quad \ell_n \equiv n + 2\ell.
\]

The equality (4.2.1) can be easily checked.
The first coefficient in the Taylor expansion of the operator $T_\ell(u)$ is proportional to the operator of cyclic shift:
\[
T_\ell(0) = \text{str}_\ell \mathbb{P}_{\ell,\ell_1} \mathbb{P}_{\ell,\ell_2} \cdots \mathbb{P}_{\ell,\ell_N} = \text{const} \cdot \mathbb{P}_{\ell_1,\ell_{N-1}} \mathbb{P}_{\ell_2,\ell_N-2} \cdots \mathbb{P}_{\ell_2,\ell_1}
\]

This is readily checked. First we move $\mathbb{P}_{\ell,\ell_1}$ to the right, then $\mathbb{P}_{\ell_1,\ell_2}$ to the right and so on and after all we use $\text{str}_\ell \mathbb{P}_{\ell,\ell_2} = \text{const}$.

The second coefficient has the form:
\[
T'_\ell(0) = \sum_i \text{str}_\ell \mathbb{P}_{\ell,\ell_1} \cdots \mathbb{P}_{\ell,\ell_i} \mathbb{R}'_{\ell,\ell_i}(0) \cdots \mathbb{P}_{\ell,\ell_N}
\]

In order to simplify this expression let us consider the $i$-th term and move the $\mathbb{R}'_{\ell,\ell_i}(0)$ to the right:
\[
\text{str}_\ell \mathbb{P}_{\ell,\ell_1} \cdots \mathbb{P}_{\ell,\ell_i-1} \mathbb{R}'_{\ell,\ell_i}(0) \mathbb{P}_{\ell,\ell_{i+1}} \cdots \mathbb{P}_{\ell,\ell_N} = \text{str}_\ell \mathbb{P}_{\ell,\ell_1} \cdots \mathbb{P}_{\ell,\ell_{i-1}} \mathbb{P}_{\ell,\ell_{i+1}} \cdots \mathbb{P}_{\ell,\ell_N} \mathbb{R}'_{\ell_{i+1},\ell_{i+1}}(0)
\]

By moving first $\mathbb{P}_{\ell,\ell_1}$ to the right, then $\mathbb{P}_{\ell_1,\ell_2}$ and so on we transform the remaining term
\[
\text{str}_\ell \mathbb{P}_{\ell,\ell_1} \cdots \mathbb{P}_{\ell,\ell_{i-1}} \mathbb{P}_{\ell,\ell_{i+1}} \cdots \mathbb{P}_{\ell,\ell_N} = \text{const} \cdot \mathbb{P}_{\ell_{N-1},\ell_N} \cdots \mathbb{P}_{\ell_{i-1},\ell_{i+1}} \cdots \mathbb{P}_{\ell,\ell_2}
\]

On the last stage we multiply the obtained expression by the operator
\[
T_\ell^{-1}(0) = (\text{const})^{-1} \mathbb{P}_{\ell,\ell_2} \cdots \mathbb{P}_{\ell_{N-1},\ell_N}
\]
from the left and obtain the following expression:

$$\mathbf{T}^{-1}_{\ell}(0) \mathbf{T}'_{\ell}(0) = \sum_i \mathbb{P}_{\ell_i, \ell_i+1} \mathbb{R}'_{\ell_i, \ell_i+1}(0) = \sum_i \mathbb{R}'_{\ell_i, \ell_i+1}(0) \cdot \mathbb{P}_{\ell_i, \ell_i+1} = \sum_i \mathcal{H}_{\ell_i, \ell_i+1}, \quad (4.2.3)$$

The resulting operator can be chosen as the Hamiltonian. It commutes with the integrals of motions generated by $\mathbf{T}_{\ell}(u)$ and is a sum of operators acting on two adjacent sites only. The two-particle Hamiltonians in the sum are

$$\mathcal{H}_{\ell_i, \ell_i+1} = \mathbb{R}'_{\ell_i, \ell_i+1}(0) \cdot \mathbb{P}_{\ell_i, \ell_i+1}$$

and have the following matrix elements:

$$\mathcal{H}_{\ell_i, \ell_i+1} = \eta^{-1} \begin{pmatrix}
  2\psi(\ell_n + 1) & -\frac{\ell_n}{(\ell-b)(\ell+b)} & 0 & 0 \\
  -\frac{2}{\ell_n} \frac{\ell^2+b^2}{(\ell-b)(\ell+b)} & 2\psi(\ell_n) & 0 & 0 \\
  0 & 0 & 2\psi(\ell_n + 1) + \frac{2b}{(\ell-b)(\ell+b)} & 0 \\
  0 & 0 & 0 & 2\psi(\ell_n + 1) - \frac{2b}{(\ell-b)(\ell+b)}
\end{pmatrix}.$$  

The eigenvalues of this matrix and corresponding eigenvalues can be easily calculated.

## 5 Inhomogeneous open chain

### 5.1 Integrals of motion

In this section we shall consider the open spin chain [10]. To start with we introduce the operator $\mathcal{R}_{\ell_1, \ell_2}(u)$:

$$\mathcal{R}_{\ell_1, \ell_2}(u) \equiv \mathbb{R}_{\ell_1, \ell_2}(u) \cdot \mathbb{R}_{\ell_1, \ell_2}^{-1}(-u).$$

It is possible to derive the following commutation relation for $\mathcal{R}(u)$:

$$\mathbb{R}_{\ell_1, \ell_2}(u-v) \cdot \mathcal{R}_{\ell_1, \ell_2}(u) \cdot \mathbb{R}_{\ell_1, \ell_2}(u+v) \cdot \mathcal{R}_{\ell_1, \ell_2}(v) = \mathcal{R}_{\ell_2, \ell_1}(v) \cdot \mathbb{R}_{\ell_2, \ell_1}(u) \cdot \mathcal{R}_{\ell_2, \ell_1}(u) \cdot \mathbb{R}_{\ell_2, \ell_1}(u-v).$$

It is evident that the operator $\mathcal{R}_{m}(u)$ constructed from the monodromy matrix $[4.1.1]$ satisfies the analogous equation:

$$\mathbb{R}_{m_1, m_2}(u-v) \cdot \mathcal{R}_{m_1}(u) \cdot \mathbb{R}_{m_1, m_2}(u+v) \cdot \mathcal{R}_{m_2}(v) = \mathcal{R}_{m_2}(v) \cdot \mathbb{R}_{m_1, m_2}(u+v) \cdot \mathcal{R}_{m_1}(u) \cdot \mathbb{R}_{m_1, m_2}(u-v).$$

Using this equation it is possible to show that corresponding supertraces are commuting operators:

$$\mathcal{T}_{\ell_1}(u) \mathcal{T}_{\ell_2}(v) = \mathcal{T}_{\ell_2}(v) \mathcal{T}_{\ell_1}(u); \mathcal{T}_{m}(u) = \text{str}_{v_m} \mathcal{R}_{m}(u) = \text{str}_{v_m} \mathbb{R}_{m}(u) \cdot \mathbb{R}_{m}^{-1}(-u).$$

If one fixes the representation $\tilde{m} = \tilde{f}$ we obtain the generating function of integrals of motions for the open chain.
5.2 Local Hamiltonian

Let us suppose that representations \( \vec{\ell}_i \) and shifts \( c_i \) in the product:

\[
\mathbb{R}_{\ell}(u) \equiv \mathbb{R}_{\ell,\vec{\ell}_1}(u + c_1)\mathbb{R}_{\ell,\vec{\ell}_2}(u + c_2)\ldots\mathbb{R}_{\ell,\vec{\ell}_N}(u + c_N)
\]

are fixed as follows:

\[
\vec{\ell}_2 = \vec{\ell}_3 = \ldots \vec{\ell}_{N-1} = \vec{\ell}; \quad c_2 = c_3 = \ldots c_{N-1} = 0 \Rightarrow \mathbb{R}_{\ell,\vec{\ell}_i}(0) = \mathbb{P}_{\ell,\vec{\ell}_i}, \quad i = 2, \ldots, n - 1,
\]

where \( \mathbb{P} \) is simply the permutation. In the case of \( \text{sl}(2) \) there exist integrable nearest neighbour interactions for this slightly inhomogeneous chain \([16]\). We extend this result to the case of \( \text{sl}(2|1) \) and construct the corresponding Hamiltonians.

The R-matrix \([3.3.4]\) obeys the equation:

\[
\mathbb{R}_{\ell,\vec{\ell}_1}(u) \cdot \mathbb{R}_{\ell,\vec{\ell}_2}(-u) = P(u) \cdot 1 \tag{5.2.1}
\]

where right side of equation is proportional of unit operator and \( P(u) \) is the function:

\[
P(u) \equiv \frac{(u + b_1 - \ell_2)(u + b_1 + \ell_2)(u - b_2 + \ell_1)(u - b_2 - \ell_1)}{(\ell_1 - b_1)(\ell_1 + b_1)(\ell_2 - b_2)(\ell_2 + b_2)},
\]

which follows from the expression for matrix elements of the R-matrix \([3.3.5]\). The explicit form of the matrix \( \mathbb{R}_{\ell,\vec{\ell}_1}(-u) \) is the following:

\[
\mathbb{R}_{\ell,\vec{\ell}_1}(-u) = \begin{pmatrix}
D_n(-u) & C_n(-u) & 0 & 0 \\
B_n(-u) & A_n(-u) & 0 & 0 \\
0 & 0 & F_n(-u) & 0 \\
0 & 0 & 0 & E_n(-u)
\end{pmatrix}
\]

where the matrix elements \( A_n, B_n, \ldots \) are obtained from the elements \( A, B, \ldots \) by formal change of variables \( \ell_1 \leftrightarrow \ell_2 \) and \( b_1 \leftrightarrow b_2 \). Due to equation \((5.2.1)\) we have

\[
\mathbb{R}_{\ell,\vec{\ell}_1}^{-1}(u) \sim \mathbb{R}_{\ell,\vec{\ell}_1}(u) \tag{5.2.2}
\]

so that we can use the operator \( \mathbb{R}_{\ell,\vec{\ell}_1}^{-1}(-u) \) instead of operator \( \mathbb{R}_{\ell,\vec{\ell}_1}^{-1}(u) \). This changes the normalization of the operator \( T_{\vec{n}}(u) \) only.

Let us calculate the first two coefficients in the Taylor expansion of the operator

\[
T_{\vec{\ell}}(u) = \text{str}_{\vec{\ell}} \mathbb{R}_{\ell,\vec{\ell}_1}(u + c_1) \ldots \mathbb{R}_{\ell,\vec{\ell}_N}(u + c_N) \mathbb{R}_{\ell,\vec{\ell}_1}(u - c_N) \ldots \mathbb{R}_{\ell,\vec{\ell}_1}(u - c_1).
\]

The first coefficient is proportional to the unit operator for arbitrary representations \( \vec{\ell}_i \) and shifts \( c_i \) due to the property \((5.2.2)\),

\[
T_{\vec{\ell}}(0) = \text{str}_{\vec{\ell}} \mathbb{R}_{\ell,\vec{\ell}_1}(c_1) \ldots \mathbb{R}_{\ell,\vec{\ell}_N}(c_N) \mathbb{R}_{\ell,\vec{\ell}_1}(-c_N) \ldots \mathbb{R}_{\ell,\vec{\ell}_1}(-c_1) \sim \text{str}_{\vec{\ell}} 1.
\]

Let us introduce the short-hand notation

\[
\mathcal{H}_{\ell,\vec{m}}(c) = \mathbb{R}_{\ell,\vec{m}}(c) \mathbb{R}_{\ell,\vec{m}}(-c) + \mathbb{R}_{\ell,\vec{m}}(c) \mathbb{R}_{\ell,\vec{m}}(-c) \tag{5.2.3}
\]

and calculate the expression for \( T_{\vec{\ell}}(0) \) which contains several terms,

\[
T_{\vec{\ell}}(0) = \text{str}_{\vec{\ell}} \mathcal{H}_{\ell,\vec{\ell}_1}(c_1) + \text{str}_{\vec{\ell}} \mathbb{R}_{\ell,\vec{\ell}_1}(c_1) \mathcal{H}_{\ell,\vec{\ell}_2}(0) \mathbb{R}_{\ell,\vec{\ell}_1}(-c_1) + \ldots
\]
\[ + \sum_{i=2}^{N} \text{str}_{\vec{\ell}} \mathcal{H}_{\vec{\ell},i}(c_i) \cdot \mathcal{P}_{\vec{\ell},i-1} \mathcal{H}_{\vec{\ell},i}(c_i) \mathcal{P}_{\vec{\ell},i-1,\vec{\ell}} \cdots \mathcal{P}_{\vec{\ell},1,\vec{\ell}},(-c_1). \]

Note that this formula is true for \( c_2 = \cdots = c_{N-1} = 0 \) only. Let us consider each term separately. The first term is constant
\[ \text{str}_{\vec{\ell}} \mathcal{H}_{\vec{\ell},i}(c_i) = \text{const} \]

The \( i \)-th term in the sum can be transformed easily to the simpler expression
\[ \text{str}_{\vec{\ell}} \mathcal{H}_{\vec{\ell},i}(c_i) \mathcal{P}_{\vec{\ell},i-1} \mathcal{H}_{\vec{\ell},i}(c_i) \mathcal{P}_{\vec{\ell},i-1,\vec{\ell}} \cdots \mathcal{P}_{\vec{\ell},1,\vec{\ell}},(-c_1) = \mathcal{H}_{\vec{\ell},i,\vec{\ell}}(c_i) \cdot \text{str}_{\vec{\ell}} 1 \]

It turns out that the second term can be transformed to the analogous form too (remember \( c_2 = 0 \))
\[ \text{str}_{\vec{\ell}} \mathcal{H}_{\vec{\ell},i}(c_i) \mathcal{H}_{\vec{\ell},i}(0) \mathcal{H}_{\vec{\ell},i}(c_i) = \mathcal{H}_{\vec{\ell},i,\vec{\ell}}(c_i) \cdot \text{str}_{\vec{\ell}} 1 + \text{const} \quad (5.2.4) \]

For the proof we start from the Yang-Baxter equation
\[ \mathcal{R}_{\vec{\ell}_2,\vec{\ell}_1}(u) \mathcal{R}_{\vec{\ell}_2,\vec{\ell}_1}(v) = \mathcal{R}_{\vec{\ell}_1,\vec{\ell}_1}(v) \mathcal{R}_{\vec{\ell}_1,\vec{\ell}_1}(u), \]

differentiate this equation with respect to \( u \) and then put \( v = -u \):
\[ \mathcal{R}_{\vec{\ell}_2,\vec{\ell}_1}(-u) \mathcal{R}_{\vec{\ell}_2,\vec{\ell}_1}(0) \mathcal{R}_{\vec{\ell}_2,\vec{\ell}_1}(-u) = \mathcal{R}_{\vec{\ell}_1,\vec{\ell}_1}(-u) \mathcal{R}_{\vec{\ell}_1,\vec{\ell}_1}(0) \mathcal{R}_{\vec{\ell}_1,\vec{\ell}_1}(-u), \]

then multiply both sides of the obtained equation by the permutation \( \mathcal{P}_{\vec{\ell}_2,\vec{\ell}_1} \) from the left \( (\vec{\ell}_2 = \vec{\ell}) \):
\[ \mathcal{R}_{\vec{\ell}_1,\vec{\ell}_1}(-u) \mathcal{R}_{\vec{\ell}_2,\vec{\ell}_1}(0) \mathcal{R}_{\vec{\ell}_2,\vec{\ell}_1}(-u) = \mathcal{R}_{\vec{\ell}_1,\vec{\ell}_1}(-u) \mathcal{R}_{\vec{\ell}_1,\vec{\ell}_1}(0) \mathcal{R}_{\vec{\ell}_1,\vec{\ell}_1}(-u), \]

and calculate \( \text{str}_{\vec{\ell}_1} \) using the equalities:
\[ \text{str}_{\vec{\ell}_1} \mathcal{R}_{\vec{\ell}_2,\vec{\ell}_1}(-u) = \text{const} ; \text{str}_{\vec{\ell}_1} \mathcal{P}_{\vec{\ell}_2,\vec{\ell}_1} \mathcal{R}_{\vec{\ell}_2,\vec{\ell}_1}(0) = \text{const}. \]

After all we obtain
\[ \text{str}_{\vec{\ell}_1} \mathcal{R}_{\vec{\ell}_2,\vec{\ell}_1}(c_1) \mathcal{R}_{\vec{\ell}_2,\vec{\ell}_1}(0) \mathcal{R}_{\vec{\ell}_1,\vec{\ell}_1}(c_1) \mathcal{R}_{\vec{\ell}_1,\vec{\ell}_1}(0) = \mathcal{H}_{\vec{\ell},\vec{\ell}}(-u) \mathcal{R}_{\vec{\ell},\vec{\ell}}(0) \mathcal{H}_{\vec{\ell},\vec{\ell}}(-u) \cdot \text{str}_{\vec{\ell}} 1 + \text{const} \]

and using the evident identity (consequence of eq. \([5.2.1]\)):
\[ \mathcal{R}_{\vec{\ell},\vec{\ell}}(-u) \mathcal{R}_{\vec{\ell},\vec{\ell}}(0) = \mathcal{R}_{\vec{\ell},\vec{\ell}}(-u) \mathcal{R}_{\vec{\ell},\vec{\ell}}(0) + \text{const} \]

we arrive at the general formula:
\[ \text{str}_{\vec{\ell}_1} \mathcal{H}_{\vec{\ell},\vec{\ell}}(c_1) \mathcal{H}_{\vec{\ell},\vec{\ell}}(0) \mathcal{H}_{\vec{\ell},\vec{\ell}}(c_1) = \mathcal{H}_{\vec{\ell},\vec{\ell}}(-u) \cdot \text{str}_{\vec{\ell}} 1 + \text{const} \]

Finally we obtain the following representation for \( T_{\vec{\ell}}(0) \):
\[ T_{\vec{\ell}}^{-1}(0) T_{\vec{\ell}}(0) = \mathcal{H}_{\vec{\ell},\vec{\ell}}(-c_1) + \sum_{i=3}^{N-1} \mathcal{H}_{\vec{\ell}_{i-1},\vec{\ell}_i}(0) + \mathcal{H}_{\vec{\ell}_N,\vec{\ell}_1}(0) \]

This operator, commuting with all integrals of motions, is a sum of two-particle operators and can be considered as the Hamiltonian. The two-particle Hamiltonians entering the sum can be easily calculated from the universal R-matrix by \([5.2.3]\). The expression for \( \mathcal{H}_{\vec{\ell}_{i-1},\vec{\ell}_i}(0) \) coincides with the two-particle Hamiltonian for the periodic chain \([4.2.4]\), up to overall coefficient two but the expression for \( \mathcal{H}_{\vec{\ell},\vec{\ell}}(c) \) is rather lengthy to be presented here.
6 Conclusions

In this paper we have obtained the general solution of the Yang-Baxter equation acting on the tensor product of arbitrary representations of the superalgebra $s\ell(2|1)$.

We have represented the lowest weight module of $s\ell(2|1)$ by polynomials in one even ($z$) and two odd ($\theta, \bar{\theta}$) variables. Therefore the general R-matrices is an operator acting on two-point functions being polynomials in the two sets of variables $(z_1, \theta_1, \bar{\theta}_1)$ and $(z_2, \theta_2, \bar{\theta}_2)$. Instead of calculating this operator explicitly we have obtained its matrix elements on the space of lowest weights. The eigenfunctions and eigenvalues can be easily calculated from the obtained matrix.

From the general R-matrix for two isomorphic representations we have calculated the nearest neighbour interaction Hamiltonians for an homogeneous closed chain. The result applies both for the finite and infinite-dimensional representations on the sites.

Using the general R-matrix an integrable open chain can be constructed with arbitrary isomorphic representations on the inner sites and other arbitrary representations at the end points. The nearest neighbour interaction Hamiltonian has been calculated.

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8 Appendix A

In this Appendix we discuss briefly the derivation of the expression (3.3.3) for the general R-matrix.

In matrix form the defining relation for the R-matrix reads as follows:

$$\mathcal{R}_{\ell_1, b_1} R_{\ell_2, b_2} K_{A,B} = \mathcal{K}_{A,B} R_{\ell_1, b_1} R_{\ell_2, b_2} ; A, B = 1, 2, 3. \quad (8.0.5)$$

where:

$$K = \begin{pmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{pmatrix} \equiv \frac{u}{2\eta} (R_2 - R_1) + R_1 R_2 ; \quad \mathcal{K} \equiv \frac{u}{2\eta} (R_2 - R_1) + R_2 R_1.$$

The matrix element $K_{A,B}$ can be obtained from $K_{A,B}$ by formal substitution $1 \leftrightarrow 2$ and $u \leftrightarrow -u$:

$$\ell_1, b_1, Z_1 \leftrightarrow \ell_2, b_2, Z_2 ; \quad u \leftrightarrow -u,$$

where $Z \equiv (z, \theta, \bar{\theta})$.

The operators $K_{A,B}$ and lowest weights transform as follows:

$$K_{A,B} \leftrightarrow \mathcal{K}_{A,B} ; \quad \Phi_n^\pm \leftrightarrow (-1)^n \Phi_n^\mp ; \quad \Psi_n^\pm \leftrightarrow (-1)^{n+1} \Psi_n^\mp.$$

There are nine equations and we start from the simplest one.
8.1 Equation $\mathbb{R}K_{13} = \tilde{K}_{13}\mathbb{R}$

The operator $K_{13}$ commutes with the lowering generators $V^-, W^-, S^-$ and the covariant derivatives $D^\pm_1$. Therefore operator $K_{13}$ maps lowest weight vectors with definite chirality to lowest weight vector with the same chirality and decrease its power by one:

$$K_{13}\Phi_n^\pm = \alpha_n^\pm(u)\Phi_{n-1}^\pm; \ K_{13}\Psi_n^\pm = \beta_n^\pm(u)\Psi_{n-1}^\pm$$

The explicit calculation gives:

$$\alpha_n^+(u) = n(u + \ell_n); \ \alpha_n^-(u) = n(u + \ell_n - 1); \ \beta_n^+(u) = n(u + \ell_n),$$

where:

$$\ell_n \equiv n + \ell_1 + \ell_2; \ u \equiv \frac{u}{\eta} + b_1 - b_2.$$  

The action of operator $\tilde{K}_{13}$ on lowest weights vectors can be obtained from formulae for $K_{13}$ by formal substitution $\ell_1, b_1, Z_1 \leftrightarrow \ell_2, b_2, Z_2$ and $u \leftrightarrow -u$:

$$\tilde{K}_{13}\Phi_n^\pm = -\alpha_n^+(u)\Phi_{n-1}^\pm; \ \tilde{K}_{13}\Psi_n^\pm = -\beta_n^+(u)\Psi_{n-1}^\pm.$$

We project the operator equation $\mathbb{R}K_{13} = \tilde{K}_{13}\mathbb{R}$ onto the lowest weight vectors $\Phi_n^\pm$:

$$\mathbb{R}K_{13}\Phi_n^+ = \tilde{K}_{13}\Phi_n^+ \Rightarrow 
\alpha_n^+(u)[A_{n-1}\Phi_{n-1}^+ + B_{n-1}\Phi_{n-1}^-] = -A_n\alpha_n^-(u)\Phi_{n-1}^+ - B_n\alpha_n^+(u)\Phi_{n-1}^- 
\mathbb{R}K_{13}\Phi_n^- = \tilde{K}_{13}\Phi_n^- \Rightarrow 
\alpha_n^-(u)[C_{n-1}\Phi_{n-1}^+ + D_{n-1}\Phi_{n-1}^-] = -C_n\alpha_n^-(u)\Phi_{n-1}^+ - D_n\alpha_n^+(u)\Phi_{n-1}^-$$

which results in the recurrent relations:

$$\alpha_n^+(u)A_{n-1} = -\alpha_n^-(u)A_n; \ \alpha_n^+(u)B_{n-1} = -\alpha_n^+(u)B_n$$

$$\alpha_n^-(u)C_{n-1} = -\alpha_n^-(u)C_n; \ \alpha_n^-(u)D_{n-1} = -\alpha_n^-(u)D_n$$

with the following general solution:

$$A_n(u) = A(-1)^n \frac{\Gamma(u + \ell_n + 1)}{\Gamma(-u + \ell_n + 1)}; \ B_n(u) = B(-1)^n \frac{\Gamma(u + \ell_n + 1)}{\Gamma(-u + \ell_n + 1)}$$

$$C_n(u) = C(-1)^n \frac{\Gamma(u + \ell_n)}{\Gamma(-u + \ell_n)}; \ D_n(u) = D(-1)^n \frac{\Gamma(u + \ell_n)}{\Gamma(-u + \ell_n + 1)}$$

The projection onto the odd lowest weight vectors $\Psi_n^\pm$ leads to analogous recurrent relations:

$$\beta_n(u)F_{n-1} = -\beta_n(u)F_n; \ \beta_n(u)E_{n-1} = -\beta_n(u)E_n$$

with general solution:

$$E_n(u) = E(-1)^n \frac{\Gamma(u + \ell_n + 1)}{\Gamma(-u + \ell_n + 1)}; \ F_n(u) = F(-1)^n \frac{\Gamma(u + \ell_n + 1)}{\Gamma(-u + \ell_n + 1)}.$$

We see that equation $\mathbb{R}K_{13} = \tilde{K}_{13}\mathbb{R}$ fixes the $n$-dependence of the matrix elements of $\mathbb{R}$-matrix. The remaining equations fix the coefficients $A, B, \ldots$ up to overall normalization.
8.2 Equation $\mathbb{R}K_{12} = K_{12}\mathbb{R}$

Due to commutativity of the operator $K_{12}$ with lowering generators $W^-, S^-$ we can write down the general formulae for the action of operator $K_{12}$ on even lowest weight vectors:

$$K_{12}\Phi^\pm_n = a^\pm W^+\Phi^\pm_{n-1} + b^\pm W^+\Phi_{n-1}^- + c^\pm\Psi^-_{n-1}.$$

The coefficients $a^\pm$ and $b^\pm$ can be calculated using the following commutation relation:

$$\{V^-, K_{12}\} = K_{13}.$$

Remembering the known results for operator $K_{13}$

$$\{V^-, K_{12}\} = K_{13} \Rightarrow V^+ K_{12} \Phi^\pm_n = K_{13} \Phi^\pm_n = \alpha_n^\pm(u) \Phi^\pm_{n-1}$$

and the simple formula ($b = b_1 + b_2$):

$$V^+ W^+ \Phi^\pm_n = - (\ell_n + b) \Phi^\pm_n \Rightarrow V^+ K_{12} \Phi^\pm_n = - a^\pm (\ell_{n-1} + b) \Phi^\pm_{n-1} - b^\pm (\ell_{n-1} + b) \Phi^-_{n-1},$$

we obtain:

$$(\ell_{n-1} + b) a^+ = - \alpha_n^+(u) , \ b^+ = 0 ; \ a^- = 0 , \ (\ell_{n-1} + b) b^- = - \alpha_n^-(u).$$

The coefficient $c^+$ can be calculated using the commutativity of $K_{12}$ and $D_1^+$:

$$\{D_1^+, K_{12}\} = 0 \Rightarrow D_1^+ K_{12} \Phi^\pm_n = a^+ D_1^+ W^+ \Phi^\pm_{n-1} + c^+ D_1^+ \Psi^-_{n-1} = 0$$

and the simple formulae:

$$D_1^+ W^+ \Phi^\pm_{n-1} = (\ell_1 + b_1) \Phi^\pm_{n-1} ; \ D_1^+ \Psi^-_{n-1} = - \Phi^\pm_{n-1}.$$ We obtain

$$(\ell_1 + b_1) a^+ = c^+.$$ The coefficient $c^-$ can be calculated using the commutation relation with the covariant derivative $D_2^+$:

$$\{D_2^+, K_{12}\} = D_2^+ W^- + (\ell_2 + b_2) S^-_1 \Rightarrow$$

$$D_2^+ K_{12} \Phi^-_n = b^- D_2^+ W^+ \Phi^-_{n-1} + c^- D_2^+ \Psi^-_{n-1} = (\ell_2 + b_2) S^-_1 \Phi^-_n$$

and the formulae:

$$D_2^+ W^+ \Phi^-_{n-1} = (\ell_2 + b_2) \Phi^-_{n-1} ; \ D_2^+ \Psi^-_{n-1} = \Phi^\pm_{n-1} ; \ S^-_1 \Phi^-_n = - n \Phi^-_{n-1}.$$ We obtain

$$(\ell_2 + b_2) b^- + c^- = - n (\ell_2 + b_2).$$ Finally we have ($b = b_1 + b_2$):

$$K_{12} \Phi^+_n = - \frac{\alpha_n^+(u)}{\ell_{n-1} + b} W^+ \Phi^+_{n-1} - \frac{\alpha_n^+(u) (\ell_1 + b_1)}{\ell_{n-1} + b} \Psi^-_{n-1}$$

$$K_{12} \Phi^-_n = - \frac{\alpha_n^-(u)}{\ell_{n-1} + b} W^+ \Phi^-_{n-1} + \frac{n (u - b_1 - b_2) (\ell_2 + b_2)}{\ell_{n-1} + b} \Psi^-_{n-1}.$$
The analogous calculations for odd lowest weight vectors give:

\[ K_{12} \Psi_n^+ = \frac{\beta_n(u)}{\ell_n + b} W^+ \Psi_{n-1}^+ + \frac{(u - b)(\ell_2 + b_2)}{\ell_n + b} \Phi_n^+ + \frac{\beta_n(u)(\ell_1 + b_1)}{n(\ell_n + b)} \Phi_n^- ; \\
K_{12} \Psi_n^- = -\frac{\beta_n(u)}{\ell_n + b} W^+ \Psi_{n-1}^- \]

The expression for the action of the operator \( \tilde{K}_{12} \) is obtained by symmetry:

\[ \tilde{K}_{12} \Phi_n^+ = \frac{\alpha_n^+(-u)}{\ell_{n-1} + b} W^+ \Phi_{n-1}^+ - \frac{\alpha_n^+(-u)(\ell_2 + b_2)}{\ell_{n-1} + b} \Psi_{n-1}^- \\
\tilde{K}_{12} \Phi_n^- = \frac{\alpha_n^+(-u)}{\ell_{n-1} + b} W^+ \Phi_{n-1}^- - \frac{n(u + b)(\ell_1 + b_1)}{\ell_{n-1} + b} \Phi_n^n \]

\[ \tilde{K}_{12} \Psi_n^+ = \frac{\beta_n(-u)}{\ell_n + b} W^+ \Psi_{n-1}^+ + \frac{(u + b)(\ell_1 + b_1)}{\ell_n + b} \Phi_n^- - \frac{\beta_n(-u)(\ell_2 + b_2)}{n(\ell_n + b)} \Phi_n^- ; \\
\tilde{K}_{12} \Psi_n^- = \frac{\beta_n(-u)}{\ell_n + b} W^+ \Psi_{n-1}^- \]

The projection of the operator equation \( \mathbb{R}K_{12} = \tilde{K}_{12}\mathbb{R} \) onto lowest weight vectors leads to the following new recurrent relations \((b = b_1 + b_2)\):

\[ \mathbb{R}K_{12} \Phi_n^+ = \tilde{K}_{12}\mathbb{R} \Phi_n^+ \implies \]

\[ (\ell_1 + b_1)\alpha_n^+(-u)E_{n-1} = n(\ell_1 + b_1)(u + b_1)A_n + (\ell_2 + b_2)\alpha_n^+(-u)B_n \]

\[ \mathbb{R}K_{12} \Phi_n^- = \tilde{K}_{12}\mathbb{R} \Phi_n^- \implies \]

\[ -n(\ell_2 + b_2)(u - b)E_{n-1} = n(\ell_1 + b_1)(u + b_1 + b_2)C_n + (\ell_2 + b_2)\alpha_n^+(-u)D_n \]

\[ \mathbb{R}K_{13} \Phi_n^+ = \tilde{K}_{13}\mathbb{R} \Phi_n^+ \implies \]

\[ n(\ell_2 + b_2)(u - b)A_n + (\ell_1 + b_1)\beta_n(u)C_n = -(\ell_2 + b_2)\beta_n(-u)F_n \]

\[ n(\ell_2 + b_2)(u - b)B_n + (\ell_1 + b_1)\beta_n(u)D_n = n(\ell_1 + b_1)(u + b_1 + b_2)F_n \]

### 8.3 Equation \( \mathbb{R}K_{23} = \tilde{K}_{23}\mathbb{R} \)

There exists an automorphism of the algebra \( s\ell(2|1) \):

\[ W^\pm \leftrightarrow V^\pm ; \ B \leftrightarrow -B \]

which has the following form in our representation:

\[ \theta \leftrightarrow \tilde{\theta} ; \ b \leftrightarrow -b. \]

Due to this automorphism the matrix \( K_{AB} \) has definite symmetry properties with respect to the transformation:

\[ \ell_1, z_1 \leftrightarrow \ell_2, z_2 ; \ \theta_1, b_1 \leftrightarrow -\tilde{\theta}_2, -b_2 \]

The operators \( K_{12}, K_{23} \) and lowest weights transform as follows:

\[ K_{12} \leftrightarrow K_{23} ; \ \Phi_n^+ \leftrightarrow (-1)^n \Phi_n^+ ; \ \Psi_n^+ \leftrightarrow (-1)^{n+1} \Psi_n^+. \]

This symmetry allows to use the results of previous section and write down the formulae for the action of the operator \( K_{23} \) on lowest weight vectors \((b = b_1 + b_2)\):

\[ K_{23} \Phi_n^+ = \frac{\alpha_n^+(u)}{\ell_{n-1} - b} V^+ \Phi_{n-1}^+ - \frac{\alpha_n^+(u)(\ell_2 - b_2)}{\ell_{n-1} - b} \Psi_{n-1}^+, \]

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$$K_{23} \Phi_n^- = \frac{\alpha_n^-(u)}{\ell_{n-1} - b} V^+ \Phi_{n-1}^- + \frac{n(u + b)(\ell_1 - b_1)}{\ell_{n-1} - b} \Psi_n^+,$$

$$K_{23} \Psi_n^- = \frac{\beta_n(u)}{\ell_{n-1} - b} V^+ \Psi_{n-1}^- \frac{(u + b)(\ell_1 - b_1)}{\ell_{n-1} - b} \Phi_n^- + \frac{\beta_n(u)(\ell_2 - b_2)}{n(\ell_{n-1} - b)} \Phi_n^- ; K_{23} \Psi_n^+ = \frac{\beta_n(u)}{\ell_{n-1} - b} V^+ \Psi_{n-1}^+,$$

$$K_{23} \Phi_n^- = \frac{\alpha_n^-(u)}{\ell_{n-1} - b} V^+ \Phi_{n-1}^- \frac{(u - b)(\ell_1 - b_1)}{\ell_{n-1} - b} \Psi_n^- + \frac{\beta_n(u)(\ell_1 - b_1)}{n(\ell_{n-1} - b)} \Phi_n^- ; K_{23} \Psi_n^+ = \frac{\beta_n(u)}{\ell_{n-1} - b} V^+ \Psi_{n-1}^+.$$

The projection of the operator equation $\mathbb{R} K_{23} = \tilde{K}_{23} \mathbb{R}$ onto lowest weight vectors leads to the following new recurrent relations $(b = b_1 + b_2)$:

$$\mathbb{R} K_{23} \Phi_n^+ = \tilde{K}_{23} \mathbb{R} \Phi_n^+ \implies$$

$$(\ell - b_2)\alpha_n^+(u)F_{n-1} = n(\ell - b_2)(u - b)A_n + (\ell_1 - b_1)\alpha_n^+(u)B_n$$

$$\mathbb{R} K_{23} \Phi_n^- = \tilde{K}_{23} \mathbb{R} \Phi_n^- \implies$$

$$-n(\ell_1 - b_1)(u + b)F_{n-1} = n(\ell_2 - b_2)(u - b)C_n + (\ell_1 - b_1)\alpha_n^+(u)D_n$$

$$\mathbb{R} K_{23} \Phi_n^+ = \tilde{K}_{23} \mathbb{R} \Phi_n^+ \implies$$

$$n(\ell_1 - b_1)(u + b)A_n + (\ell_2 - b_2)\beta_n(u)C_n = -(\ell_1 - b_1)\beta_n(u)E_n$$

$$n(\ell_1 - b_1)(u + b)B_n + (\ell_2 - b_2)\beta_n(u)D_n = n(\ell_2 - b_2)(u - b)E_n$$

The systems of equations \[[8.2.1], [8.2.2]\] fix the coefficients $A, B, C, \ldots$ up two arbitrary constants. The next operator equation $\mathbb{R} K_{11} = \tilde{K}_{11} \mathbb{R}$ fixes the remaining ambiguity but we avoid presenting the rather lengthy formulae here. Alternatively the missing equation can be obtained as follows. The even lowest weight vectors $\Phi_n^\pm$ coincide for $n = 0$. Therefore one obtains:

$$\mathbb{R} \Phi_0^+ = \Phi_0^- \implies -A(u + \ell_0) + B = \frac{C(-u + \ell_0)}{u + \ell_0} - \frac{D}{u + \ell_0}$$

Finally the systems \[[8.2.1], [8.3.1]\] and this last equation fix the solution completely:

$$A = u + b_1 - b_2 ; B = (\ell_1 + b_1)(\ell_2 - b_2) ; C = (\ell_1 - b_1)(\ell_2 + b_2)$$

$$D = (\ell_2 - b_2)(\ell_2 + b_2)(u - b_1 - b_2) - (u + b_1 + b_2)(u - b_2 - \ell_1)(u - b_2 + \ell_1)$$

$$E = (u + b_1 - \ell_2)(u + b_1 + \ell_2) ; F = (u - b_2 - \ell_1)(u - b_2 + \ell_1).$$

We have checked that obtained $\mathbb{R}$-matrix really is the solution also of the remaining seven equations but the involved formulae become rather lengthy starting from the equation $\mathbb{R} K_{11} = \tilde{K}_{11} \mathbb{R}$.  

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9 Appendix B

The R-matrix acting in the tensor product of chiral modules can be obtained by simple reduction from the general R-matrix (3.3.4). First of all we have to change the overall normalization multiplying all matrix elements by the factor \((\ell_1 - b_1)(\ell_2 + b_2)\). Let us consider all possible special cases:

1. chiral at 1, generic at 2

\[
V_{\ell_1, -\ell_1} \otimes V_{\ell_2, b_2} = \sum_{n=0}^{\infty} V_{\ell+n, b} + \sum_{n=0}^{\infty} V_{\ell+n+\frac{1}{2}, b+\frac{1}{2}} ; \; \ell_2 \neq \pm b_2 ; \; \Phi_n^+ \rightarrow V_{\ell+n, b} ; \; \Psi_n^+ \rightarrow V_{\ell+n+\frac{1}{2}, b+\frac{1}{2}}
\]

\[
A_n(u) = (-1)^{n+1} \frac{\Gamma(u + \ell_n + 1)}{\Gamma(-u + \ell_n)} ; \; F_n(u) = (-1)^n \frac{\Gamma(u + \ell_n + 1)}{\Gamma(-u + \ell_n + 1)} \cdot (u + \ell_1 - b_2)
\]

2. antichiral at 1, generic at 2

\[
V_{\ell_1, \ell_1} \otimes V_{\ell_2, b_2} = \sum_{n=0}^{\infty} V_{\ell+n, b} + \sum_{n=0}^{\infty} V_{\ell+n+\frac{1}{2}, b+\frac{1}{2}} ; \; \ell_2 \neq \pm b_2 ; \; \Phi_n^- \rightarrow V_{\ell+n, b} ; \; \Psi_n^- \rightarrow V_{\ell+n+\frac{1}{2}, b+\frac{1}{2}}
\]

\[
D_n(u) = (-1)^{n+1} \frac{\Gamma(u + \ell_n)}{\Gamma(-u + \ell_n + 1)} \cdot (u - \ell_1 - b_2) ; \; E_n(u) = (-1)^n \frac{\Gamma(u + \ell_n + 1)}{\Gamma(-u + \ell_n + 1)}
\]

3. chiral at 1, antichiral at 2

\[
V_{\ell_1, -\ell_1} \otimes V_{\ell_2, -\ell_2} = \sum_{n=0}^{\infty} V_{\ell+n, b} ; \; \Phi_n^+ \rightarrow V_{\ell+n, b}
\]

\[
\mathcal{R}_{\ell_1, \ell_2}(u) \Phi_n^+ = R \cdot A_n(u) \Phi_n^+ ; \; A_n(u) = (-1)^n \frac{\Gamma(u + \ell_n + 1)}{\Gamma(-u + \ell_n)}
\]

4. antichiral at 1, chiral at 2

\[
V_{\ell_1, \ell_1} \otimes V_{\ell_2, -\ell_2} = \sum_{n=0}^{\infty} V_{\ell+n, b} ; \; \Phi_n^- \rightarrow V_{\ell+n, b}
\]

\[
\mathcal{R}_{\ell_1, \ell_2}(u) \Phi_n^- = R \cdot D_n(u) \Phi_n^- ; \; D_n(u) = (-1)^n \frac{\Gamma(u + \ell_n)}{\Gamma(-u + \ell_n + 1)}
\]

5. antichiral at 1 and 2

\[
V_{\ell_1, \ell_1} \otimes V_{\ell_2, \ell_2} = \sum_{n=0}^{\infty} V_{\ell+n+\frac{1}{2}, b+\frac{1}{2}} ; \; \Psi_n^- \rightarrow V_{\ell+n+\frac{1}{2}, b+\frac{1}{2}}
\]

\[
\mathcal{R}_{\ell_1, \ell_2}(u) \Psi_n^- = R \cdot E_n(u) \Psi_n^- ; \; E_n(u) = (-1)^n \frac{\Gamma(u + \ell_n + 1)}{\Gamma(-u + \ell_n + 1)}
\]

6. chiral at 1 and 2

\[
V_{\ell_1, -\ell_1} \otimes V_{\ell_2, -\ell_2} = \sum_{n=0}^{\infty} V_{\ell+n+\frac{1}{2}, b-\frac{1}{2}} ; \; \Psi_n^+ \rightarrow V_{\ell+n+\frac{1}{2}, b-\frac{1}{2}}
\]

\[
\mathcal{R}_{\ell_1, \ell_2}(u) \Psi_n^+ = R \cdot F_n(u) \Psi_n^+ ; \; F_n(u) = (-1)^n \frac{\Gamma(u + \ell_n + 1)}{\Gamma(-u + \ell_n + 1)}
\]

Note that we do not fix the overall normalization of the obtained R-matrix.
10 Appendix C

In this Appendix we discuss shortly the case of finite-dimensional representations and show that the obtained general R-matrix reduces to the known ones [9] for the tensor product of modules with minimal dimensions.

The tensor product of two finite-dimensional $s\ell(2|1)$-modules has the following direct sum decomposition [3]:

$$V_{\ell_1,b_1} \otimes V_{\ell_2,b_2} = V_{\ell,b} + V_{\ell+1,b} + 2 \sum_{n=1}^{N-1} V_{\ell+n,b} + \sum_{n=0}^{N-1} V_{\ell+n+\frac{1}{2},b-\frac{1}{2}} + \sum_{n=0}^{N-1} V_{\ell+n+\frac{1}{2},b+\frac{1}{2}} ; \; \ell_i \neq \pm b_i \tag{10.0.2}$$

\[ \ell_1 = -\frac{n_1}{2} , \; n_1 \geq 2 ; \; \ell_2 = -\frac{n_2}{2} , \; n_2 \geq 2 ; \; N \equiv \min(n_1, n_2) ; \; \ell = -\frac{n_1 + n_2}{2}, \; b = b_1 + b_2 \]

Note that this formula is applicable in the generic situation $\ell_i \neq \pm b_i , \ell_i \neq -1/2$. The direct sum decomposition reduces for the module $V_{-\frac{1}{2},b}$:

$$V_{\ell_1,b_1} \otimes V_{-\frac{1}{2},b_2} = V_{\ell,b} + V_{\ell+1,b} + V_{\ell+\frac{1}{2},b-\frac{1}{2}} + V_{\ell+\frac{1}{2},b+\frac{1}{2}} ; \; \ell = -\frac{n_1 + 1}{2}, \; b = b_1 + b_2 ; \; n_1 \geq 2 , \; \ell_2 = -\frac{1}{2}$$

The origin of modifications for the tensor product involving the $V_{-\frac{1}{2},b}$-module is very simple. The module $V_{-\frac{1}{2},b}$ is the four-dimensional vector space with the following basis:

$$\Phi_0 = 1 ; \; \Phi_1 = z + b\theta \bar{\theta} ; \; \Psi^+_0 = \bar{\theta} , \; \Psi^-_0 = \theta.$$ 

It is evident that we are able to construct the following two-point lowest weight vectors only:

$$V_{\ell_1,b_1} \otimes V_{-\frac{1}{2},b_2} \leftrightarrow \Phi^+_0 = \Phi^-_0 = 1 ; \; (b_2 - \frac{1}{2})\Phi^+_0 - (b_2 + \frac{1}{2})\Phi^-_0 ; \; \Psi^+_0 = \bar{\theta}_{12} ; \; \Psi^-_0 = \theta_{12}.$$ 

Note that

$$(b_2 - \frac{1}{2})\Phi^+_0 - (b_2 + \frac{1}{2})\Phi^-_0 = -z_1 + b_2 \theta_1 \bar{\theta}_1 + z_2 + b_2 \theta_2 \bar{\theta}_2 + (b_2 - \frac{1}{2}) \bar{\theta}_1 \theta_2 - (b_2 + \frac{1}{2}) \theta_1 \bar{\theta}_2 ,$$

$$V_{-\frac{1}{2},b_1} \otimes V_{-\frac{1}{2},b_2} \leftrightarrow \Phi^+_0 = \Phi^-_0 = 1 ; \; \Psi^+_0 = \bar{\theta}_{12} ; \; \Psi^-_0 = \theta_{12}.$$ 

Let us consider the general R-matrix $[3,3,4]$ acting in the tensor product $V_{-\frac{1}{2},b_1} \otimes V_{-\frac{1}{2},b_2}.$

In this case we have:

$$\mathbb{P}_{\ell_1,\ell_2}(u) = (A_0(u) + B_0(u)) \cdot \mathbb{P}_1 + E_0(u) \cdot \mathbb{P}_2 + F_0(u) \cdot \mathbb{P}_3$$

where $\mathbb{P}_i$ are projectors on the modules in the direct sum decomposition:

$$V_{-\frac{1}{2},b_1} \otimes V_{-\frac{1}{2},b_2} = V_{-1,b} + V_{-\frac{1}{2},b-\frac{1}{2}} + V_{-\frac{1}{2},b+\frac{1}{2}} ; \; b = b_1 + b_2 , \; \ell_1 = \ell_2 = -\frac{1}{2}$$

$$\mathbb{P}_1 \rightarrow V_{-1,b} ; \; \mathbb{P}_2 \rightarrow V_{-\frac{1}{2},b-\frac{1}{2}} ; \; \mathbb{P}_3 \rightarrow V_{-\frac{1}{2},b+\frac{1}{2}}.$$ 

After a simple calculation one obtains:

$$\mathbb{P}_{\ell_1,\ell_2}(u) \sim \mathbb{P}_1 + \frac{2u - 1 + 2b_1}{2u + 1 - 2b_2} \cdot \mathbb{P}_2 + \frac{2u - 1 - 2b_2}{2u + 1 + 2b_1} \cdot \mathbb{P}_3.$$
This result coincides with the expression for the R-matrix given in [9]:

\[ R(\mu) = P_1 + \frac{4\mu - 1 + b_1 + b_2}{4\mu + 1 - b_1 - b_2} \cdot P_2 + \frac{4\mu - 1 - b_1 - b_2}{4\mu + 1 + b_1 + b_2} \cdot P_3 \]

up to the overall normalization and a redefinition of the spectral parameter:

\[ u = 2\mu - \frac{b_1 - b_2}{2}. \]

The direct sum decomposition for the tensor product of the chiral modules (atypical representations) is well known [3]. Now we need the simplest ones:

\[ V_{-\frac{1}{2},\frac{1}{2}} \otimes V_{-\frac{1}{2},b} = V_{-1,b} + V_{-\frac{1}{2},b-\frac{1}{2}} ; \quad b = b_1 + b_2 \], \quad \ell_1 = \ell_2 = -\frac{1}{2} \], \quad b_1 = \frac{1}{2} \]

Let us consider the R-matrix (3.3.4) acting in the tensor product \( V_{-\frac{1}{2},\frac{1}{2}} \otimes V_{-\frac{1}{2},b_2} \). In this case we have (see Appendix B):

\[ R_{\ell_1 \ell_2}(u) = A_0(u) \cdot P_1 + F_0(u) \cdot P_2 \]

where \( P_i \) are projectors on the modules in the direct sum decomposition:

\[ P_1 \rightarrow V_{-1,b} ; \quad P_2 \rightarrow V_{-\frac{1}{2},b-\frac{1}{2}}. \]

After a simple calculation one obtains:

\[ R_{\ell_1 \ell_2}(u) \sim P_1 + \frac{2u - 1 - 2b_2}{2u + 2} \cdot P_2. \]

This result also coincides with the expression for such a \( R \)-matrix given in [9].

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