Generation of entangled light via dynamical Casimir effect

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This paper addresses the excitation of vacuum fluctuations of the electromagnetic field through periodic modulations of a refractive index and the possibility of using entanglement as a distinctive marker of the quantum nature of the phenomenon. It introduces a lossy environment and analyses its implications on the possibility of generating such an effect and measuring entanglement, concluding that it is not entirely destructive when the produced particles share the same environment.

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Introduction. The dynamical Casimir effect (DCE) consists basically in applying time varying boundary conditions or refractive index modulations to the electromagnetic vacuum in order to excite real particles out of the zero–point fluctuations (ZPF) [1–4]. Even though this is probably the most direct method to probe and enhance ZPF, detecting it in optical cavities is not easy, albeit there being some promising attempts in that direction [5, 6] and observations in analogue experiments [7]. The obstacles have mainly been attributed to the difficulty in producing intense and fast enough optical perturbations [5, 8] and to technical aspects of the experiment (cavity losses, detector sensitivity, etc.) capable of measuring the scarce radiation produced [8, 9]. Finite temperature has two added consequences: it provides thermal photons which seed the DCE, thus masking the ZPF, and produces decoherence and losses via the thermalisation of the optical cavity by the environment. In fact, previous works have shown that a seed thermal state greatly improves the photon production [10]. However, as it will be demonstrated here, this amplification of thermal radiation is also predicted with a classical model, rendering the DCE at finite temperatures, in practice, a classical effect.

The key aspect of zero temperature DCE is that, unlike thermal fluctuations, ZPF are coherent and the parametric amplification produced by a time varying optical cavity preserves this coherence. For example, in an alternative DCE scheme, which considers a medium with a time varying refractive index in the absence of any spatial boundary, known as time refraction (TR), the parametric amplification of ZPF generates pairs of counter propagating photons which are maximally entangled and exit the medium separately [11]. For DCE in linear cavities the counter propagating photons reflect into each other, rendering them indistinguishable and thus impeding the use of entanglement as a marker of quantum character. However, this difficulty can be circumvented using different geometries.

Previous work in the problem considered the enhancement of the output radiation due to thermal seeding [10] and the impact of losses [5, 9, 12] (phase damping, dissipation, Markovian baths, etc.) in linear optical cavities. At finite temperature (T > 0) the losses in the cavity do not force the cavity to the vacuum state but to the thermal state of the cavity which is in equilibrium with the environment. This provides constant thermal reseeding of the cavity (not just at the initial moment) masking even further any mark of ZPF.

This letter introduces the idea of producing DCE in an optical ring resonator (like a Sagnac interferometer) filled with a dielectric subjected to a periodic modulation of the refractive index. This configuration possesses the following advantages: (i) the pairs of photons generated by DCE do not reflect into each other but propagate in opposite directions and can be extracted independently (see Fig. 1), remaining therefore distinguishable; (ii) the two photon beams propagate along the same optical path and experience the same decoherence and losses (they share the same bath) which allows for the existence of a decoherence free subspace (DFS) [13]; and (iii) the degree of entanglement between the two beams is a mark of ZPF even at finite temperature.

The models for DCE in linear cavities predict that the number of photons only grows exponentially if the rate of their creation is faster than they decay (weak losses regime) [9], but even then their growth rate is reduced by losses. For ring cavities, the DCE can support an exponential photon generation and entanglement, even in the strong losses regime.

Classical vs. quantum model. The classical model for an optical cavity filled with a time dependent dielectric is basically the same for both linear and ring resonators and can be derived directly from the macroscopic Maxwell equations by imposing a time modulation of the refractive index n = n(t) = n_0 + δn h(t), with h(t) ∈ [−1, 1]. The resulting dynamical eq. for the amplitude of electric field E of an harmonic mode is

\[ \ddot{E} + 4\delta n E + (\omega^2 + 2i\delta n + 2\delta^2)E = 0 \]

and can be converted into the Hill equation [14]

\[ \ddot{y} = -f^2 y, \]
resonance (PR). Including optical losses, the photon yield is bounded by
\[
\langle n \rangle = \gamma T \langle \hat{n} \rangle + \frac{\gamma}{2} \sum_{i,j} \left( \delta n_{i,j} \hat{a}_i \hat{a}_j - \hat{a}_i \hat{a}_j \right) e^{i\omega_{ij} t}.
\]

For periodic modulations \( f \) with period \( T \), the formal solution of eq. (\ref{eq:PR}) is
\[
\begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix} = S(t) \begin{bmatrix} y(t = 0) \\ \dot{y}(t = 0) \end{bmatrix}.
\]

According to Floquet’s theory \cite{14}, to characterise the evolution of the system at times \( t = mT \) (with \( m \) an integer) it suffices to consider the Lyapunov exponent \( \mu \) (i.e. the rate of growth of the amplitude of the solution) which is computed as
\[
\mu = T^{-1} \cosh^{-1} \left( \text{Tr} [2S(T)] \right).
\]

The solutions are bounded if \( |\text{Tr} [2S(T)]| < 1 \), and unbounded for \( |\text{Tr} [2S(T)]| > 1 \), as a result of a parametric resonance (PR). Including optical losses, the photon yield at a PR is
\[
N(mT) = \frac{2\epsilon_0 n_0^2}{\hbar \omega_0} \langle E_0^2 \rangle e^{2(\mu - \gamma) m T}
\]
where \( E_0 \) is the initial field amplitude and \( \gamma \) is the coupling of the cavity with the environment.

At \( T = 0 \), the amount of light emitted from the ground state of the field depends on the nature of the field considered: in the classical model \( \langle E_0^2 \rangle = 0 \) and there is no photon emission whereas in the quantum model \( \langle E_0^2 \rangle > 0 \) due to the ZPF, yielding an exponential amplification of the field. But for \( T > 0 \), the difference between the classical and quantum prediction is very small because the contribution from thermal fluctuations present in both models is relevant and masks the contribution from ZPF.

At \( T > 0 \), the quantum character of the DCE is hidden in the quantum correlations, calculated using a fully quantum dynamics. This is done replacing the \( \mathbf{E} \) field in eq. \ref{eq:Hamiltonian} by an operator and computing the effective Hamiltonian for each mode. The difference between linear and ring resonators results from the mode structure of the \( \mathbf{E} \) operator determined by the geometry of the cavity. The effects of losses and decoherence are modelled via the coupling to a Markovian bath according to the master eq. (similar to refs \cite{9,13})
\[
\dot{\rho} = -i [H, \rho] + \frac{\gamma}{2} \sum_{i,j} \left( \langle \hat{n} \rangle + 1 \right) a_i \hat{a}_j - \langle \hat{n} \rangle \hat{a}_i \hat{a}_j - \frac{\gamma}{2} \left[ a_i^\dagger a_j + a_j^\dagger a_i \right] + \frac{\gamma}{2} \left( a_i^\dagger \rho a_j - \rho \right) \left( a_j^\dagger \rho a_i - a_i \rho a_j \right)\]
\]

**DCE in linear resonators.** For linear cavities the \( \mathbf{E} \) operator for transverse modes is
\[
\mathbf{E} = \frac{i}{\sqrt{2}} \sum_j \xi_j (a_j - a_j^\dagger) \psi_j(x) e_j,
\]
where \( x \) is the coordinate along the optical axis of the resonator, \( \xi_j(t) = \sqrt{\hbar \kappa_0 j} / 2n^2(t) \), \( a_j^\dagger \) and \( a_j \) are the creation and annihilation field operators (with \( [a_j, a_k^\dagger] = \delta_{ij} \)), \( e_j \) is the polarisation unit vector and \( \psi_j(x) = \sin(k_j x) \) with \( k_j = \pi j / L \) (with \( j \) a positive integer).

Replacing \ref{eq:Hamiltonian} into \ref{eq:PR} yields the evolution eqs. for \( a_j^\dagger \) and \( a_j \), which for each mode \( \psi_j \) correspond to the Hamiltonian
\[
H(t) = f(t) a_j^\dagger a_j + ig(t) [a_j^2 - a_j^2],
\]
with \( f(t) = 1/\kappa(t) \) and \( g(t) = \kappa(t) / (2 \kappa(t) \ln \kappa(t)) \). From this point all eqs. are written in natural units of the vacuum frequency, defined by \( \hbar = 1 \) and \( c = 1 \).

For Gaussian states \cite{13}, the Wigner function has the form \( W(X) = (2\pi)^{-d} \det |\sigma|^{-1} \exp \left\{ -\frac{1}{2} (X - \bar{X})^T \sigma^{-1} (X - \bar{X}) \right\} \), where \( X = [x, p] \) and \( \sigma \) is the covariance matrix defined by \( \sigma = XX^T - \bar{X} \bar{X}^T \). For the Hamiltonian \ref{eq:Hamiltonian}, the master eq. \ref{eq:master} yields the evolution eqs.
\[
\dot{\bar{X}} = -M_2^T \bar{X} + \gamma / 2 \bar{X}, \quad \dot{\bar{M}} = -M_2^T \bar{X} - \bar{X} M_2 - \gamma \bar{X} (\sigma - \sigma_\infty),
\]
where
\[
M_2 = 2 \begin{bmatrix} -g & f \\ -f & g \end{bmatrix},
\]
and where \( \sigma_\infty = (2\bar{n} + 1) \text{Id}_2 / 4 \) and \( \text{Id}_n \) is the \( n \)-dimensional identity matrix. The state \( \sigma_\infty \) describes the cavity in equilibrium with the bath in the absence of the refractive index modulation. Notice that \( \sigma_\infty \) is de facto the ground state of the cavity at \( T > 0 \) therefore, it is simultaneously the initial state of the cavity at \( t = 0 \) (i.e. \( \sigma_0 = \sigma_\infty \)) and the asymptotic state to which the cavity decays if the modulation if turned off.
**DCE in ring resonators.** For ring cavities the \( E \) operator for transverse modes is

\[
E = i \sum_j \xi_j(t) \left[ a_j(t) \phi_j(x) - a_j^\dagger(t) \phi_j^*(x) \right] e_j,  \tag{12}
\]

where \( \phi_j(x) = \exp(ik_jx) \), with \( k_{\pm j} = \pm 2\pi j/L \) and \( j \) an integer \((+j \text{ for light propagation along the optical axis and } -j \text{ in the opposite direction})\). In this case, there is a coupling between modes with symmetric wave number \( j \) resulting in the Hamiltonian \[16\]

\[
H(t) = f(t) [a_j^\dagger a_j + a_{-j}^\dagger a_{-j}] + ig(t) [a_j^\dagger a_{-j} - a_{-j}^\dagger a_j],  \tag{13}
\]

Making \( X' = \Gamma X \), with \( X = [x_j, p_j, x_{-j}, p_{-j}]^T \) and

\[
\Gamma = \frac{1}{\sqrt{2}} \begin{bmatrix} \text{Id}_2 & \text{Id}_2 \\ \text{Id}_2 & -\text{Id}_2 \end{bmatrix},  \tag{14}
\]

the evolution eqs. for \( \vec{X} \) and \( \sigma \) become

\[
\begin{align*}
\vec{X}' &= - (M'_T + \gamma/2) \vec{X},  \\
\dot{\sigma}' &= -M_T' \sigma' - \sigma' M_4 - \gamma/2 (\alpha \sigma' + \sigma' \alpha - 2\alpha \sigma_{\infty》),  
\end{align*} \tag{15}\tag{16}
\]

where \( M'_T \), \( \alpha \) and \( \sigma'_{\infty》 \) are 4 × 4 matrices with \( \sigma'_{\infty》 = (2\pi + 1)\alpha'/4 \) and

\[
M_4 = \frac{1}{2} \begin{bmatrix} M_2 & 0 \\ 0 & -M_T' \end{bmatrix}, \quad \alpha' = \begin{bmatrix} \text{Id}_2 & 0 \\ 0 & 0 \end{bmatrix}. \tag{17}
\]

Notice that \( M_4, \alpha \) and \( \sigma'_{\infty》 \) are block diagonal. Then, eqs. \( 15 \) and \( 16 \) imply that, in terms of quadratures \( X' \), two coupled modes of ring resonator can be decomposed into two independent modes of a linear resonator.

Moreover, one these modes is isolated from the bath and constitutes a DFS. Hence, when a PR occurs, it grows exponentially with \( t \), regardless of the losses. This feature of the ring geometry does not exist for linear cavities, where all modes are coupled to the bath.

**Evolution of the state of the system.** The solutions of eqs. \( 15 \) and \( 16 \) for a ring resonator are

\[
\begin{align*}
\vec{X}'(t) &= U_{th} \vec{X}_0,  \\
\dot{\sigma}'(t) &= U_{th} \sigma'_p U_{th}^T + \sigma'_p,  \tag{18}\tag{19}
\end{align*}
\]

where \( \sigma'_0 \equiv \sigma'_{\infty》 \) and \( \vec{X}_0 \) describe the initial state of the cavity, \( U_{th} = e^{-\gamma/2 a^\dagger a} U \), \( U \) solves the eqs. in the absence of the bath and \( \sigma'_p \) is a particular solution of eq. \( 19 \). In this case, \( U \) and \( \sigma'_p \) are

\[
U = \frac{1}{2} \begin{bmatrix} U_+ & 0 \\ 0 & U_- \end{bmatrix}, \quad \sigma'_p = \frac{1}{2} \begin{bmatrix} \sigma'_+ & 0 \\ 0 & \sigma'_- \end{bmatrix} \sigma_{\infty》,  \tag{20}
\]

where \( U_\pm = e^{\pm \Gamma \Sigma} R_{\pm}^{-1} S R_\pm \) with \( G = \int_0^t g(\tau) d\tau \), \( f_0 = f(t = 0) \),

\[
R_+ = \begin{bmatrix} 0 & 1 \\ -f_0 & 0 \end{bmatrix}, R_- = \begin{bmatrix} 1 & 0 \\ 0 & f_0 \end{bmatrix}, \Sigma = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},
\]

and \( \sigma_+ = \gamma U_+ \int_0^t (U_T U_+)^{-1} d\tau U_T^T. \) The matrix \( S \) is again obtained as the solution of eq. \( 14 \). The solutions of eqs. \( 9 \) and \( 10 \) for a linear resonator are formally identical to those of eqs. \( 15 \) and \( 16 \) for the degree of freedom coupled to the bath expressed in terms of \( X' \).

**Examples of refractive index modulations.** The impact of the DFS is clearer by considering the two particular optical modulations: (i) a periodic and instantaneous change of \( n \) between two values, \( n_1 \) and \( n_2 \), which is directly related to TR; and (ii) a sinusoidal modulation, similar to more conventional setups of DCE \[8\].

For a sequence of instantaneous perturbation of the refractive index with period \( T \) given by

\[
f(t) = \begin{cases} f_1, & mT < t < mT + t_1, \\ f_2, & mT + t_1 < t < mT + t_1 + t_2 = (m + 1)T, \end{cases} \tag{21}
\]

Using eqs. \( 15 \) and \( 16 \), it results that

\[
\text{Tr}[2S(T)] = \frac{1 + f_2^2}{4f_r} \cos \theta_+ - \frac{1 - f_2^2}{4f_r} \cos \theta_- , \tag{22}
\]

with \( \theta_\pm = \theta_1 \pm \theta_2, \theta_1 = f_1 t_1, \theta_2 = f_2 t_2 \) and \( f_r = f_2/f_1 \).

For \( \gamma = 0 \), eq. \( 22 \) shows that the highest photon emission rate occurs for \( \theta_1 = \pi/2 + k\pi \) (with \( k \) integer) and \( \mu = T^{-1} |\ln f_r| \). For the same \( \theta_1 \) but with \( \gamma \neq 0 \), the asymptotic values of the number of photons emitted \( N + 1 \equiv \sum_{i=1,2} (a_i^\dagger a_i + h.c.)/2 \) and of the entanglement of the state (measured using logarithmic negativity) at
exhibits PR for eq. (2) reduces to Mathieu’s eq. $F_e$ with $\delta n/n$ refractive index, given by $\tau_e$ for $N_e$ degree of freedom which is coupled to the bath and has $2$ proportional to $\mu m_T$ result from the degree of freedom associated with $\gamma t$ that photons are proportional to $\delta n/n_0$ and for $\delta n/n_0 \ll 1$, yields that $f^2 \simeq [1 - 2(\delta n/n_0) \sin \Omega t]/n_0^2$ and eq. (2) reduces to Mathieu’s eq. $\ddot{y} + (\delta + \epsilon \sin 2t) y = 0$ (25) with $t' = \Omega t/2$, $\delta = 4/n_0^2 \Omega^2$ and $\epsilon = -8\delta n/n_0^2 \Omega^2$. Eq. (25) exhibits PR for $\delta = m^2$ (with $m$ integer) or alternatively $\Omega = 2/mn_0$, yielding $\mu = \delta n/2n_0^2$.

The asymptotic values of $N$ and $E_N$ at $t = mT \to \infty$ are

$$\langle N + 1 \rangle \to \frac{2\pi + 1}{4} e^{2\mu mT} + e^{-2\eta - mT} + + F_+^\prime (m) + F_-^\prime (m) \rangle \to \max \left[ 0, \frac{\mu \mu mT}{\ln 2} - \log_2 \left( (2\pi + 1) F_+^\prime (m)^{1/2} \right) \right]$$

(23)

(26)

with $F_+^\prime (m) \equiv [1 - e^{-2\eta T}][1 - e^{2\gamma t_1} + e^{2\gamma t_1}(1 - e^{2\gamma t_2})]f_{\pm 1}^T/[1 - e^{2\eta T}]$ and $\eta_{\pm} = (\gamma \pm \mu)$.

In the case of a small sinusoidal perturbation of the refractive index, given by $n = n_0 + \delta n \sin \omega t$, and for $\delta n/n_0 \ll 1$, yields that $f^2 \simeq [1 - 2(\delta n/n_0) \sin \Omega t]/n_0^2$ and eq. (2) reduces to Mathieu’s eq. $\ddot{y} + (\delta + \epsilon \sin 2t) y = 0$ (25) with $t' = \Omega t/2$, $\delta = 4/n_0^2 \Omega^2$ and $\epsilon = -8\delta n/n_0^2 \Omega^2$. Eq. (25) exhibits PR for $\delta = m^2$ (with $m$ integer) or alternatively $\Omega = 2/mn_0$, yielding $\mu = \delta n/2n_0^2$.

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(26)

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In both cases there are two distinct contributions for $N$. The terms in (23) and (26) proportional to $e^{2\mu mT}$ result from the degree of freedom associated with the DFS, resulting in a photon emission rate of $\tau_{DFS} = 2\mu$, independently of the bath or losses. The terms proportional to $e^{-2\eta - mT}$ and $F_-$, result from the remaining degree of freedom which is coupled to the bath and has two regimes: (i) a strong losses regime for $\eta_+ > 0$, when there is no exponential amplification of the field and (ii) a weak losses regime for $\eta_- < 0$, when photons are produced at the rate $\tau_{\phi} = 2(\mu - \gamma)$. This is the emission rate for linear resonators, in which there is no DFS.

The other relevant feature of the solution is the existence of entanglement in both regimes. In fact, since $F_+ (\infty)$ is constant, the linear growth term always dominates for large $m$. This means that a long enough modulation of the medium will always result in entanglement after an occurrence time

$$t_{occ} = \frac{\log_2 \left( (2\pi + 1) F_+^\prime (\infty)^{1/2} \right) \ln 2}{\mu T}.$$  

(28)

It is important to note that a finite temperature does not forbid the existence of entanglement but delays its occurrence, which enforces the need to work with low temperatures. These results compare well with reference [13], where the author considered two harmonic oscillators prepared in an initial two mode squeezed state, not directly coupled but sharing a thermal bath. It was shown that, if the initial state was sufficiently entangled, it would preserve some amount of entanglement after any arbitrarily large time. In the case presented here the initial state is disentangled and it is the DCE that produces correlations until they reach a threshold imposed by the thermal bath, after which entanglement is present. The common factor in both cases is the existence of a DFS which shields the entanglement from the environment.

**Conclusions.** This letter introduces the idea of using entanglement as a marker of the quantum nature of the DCE at finite temperature and proposes a new experimental setting which has a DFS and is capable of supporting and amplifying entanglement. In more detail, it was shown that for a resonant modulation of the refractive index of the dielectric medium in a ring resonator it is always possible to amplify radiation, regardless of the loss regime. Furthermore, it was verified that the existence of a decoherence free subspace allows quantum correlations to survive, making entanglement viable even at high temperatures. Entanglement is a distinctive characteristic of a quantum effect which can be used to sort out the DCE from classical parametric amplification of thermal fluctuations. Hopefully these results will pave way to the observation of the DCE and entanglement at finite temperatures.

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