Approximate Equivalence of Group Actions

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Abstract. We consider several weaker versions of the notion of conjugacy and orbit equivalence of measure preserving actions of countable groups on probability spaces, involving equivalence of the ultrapower actions and asymptotic intertwining conditions. We compare them with the other existing equivalence relations between group actions, and study the usual type of rigidity questions around these new concepts (superrigidity, calculation of invariants, etc).

1. Introduction

By a celebrated result of Ornstein and Weiss ([OW1]), any two free ergodic probability measure preserving (pmp) actions of countable amenable groups $\Gamma \curvearrowright (X, \mu)$, $\Lambda \curvearrowright (Y, \nu)$ are orbit equivalent (OE). In turn, any non-amenable group $\Gamma$ is known to have “many” non-OE free ergodic pmp actions (cf. [CW], [H], [GP], [PS], [P6], [I1, GL, E]). Moreover, certain free ergodic pmp actions of non-amenable groups $\Gamma \curvearrowright X$ exhibit various degrees of rigidity, where mere orbit equivalence with another action $\Lambda \curvearrowright Y$ may imply isomorphism of the groups $\Gamma \simeq \Lambda$, and even isomorphism (conjugacy) of the two actions (see e.g. [Z], [Fu1], [MS], [P7, P8, P9], [I2], [K1], [K2], etc). There has been constant interest in establishing such results, due to their intrinsic unexpected nature, their relevance to rigidity phenomena in von Neumann algebras and group theory, and their applications to Borel classification in descriptive set theory.

There has also been much interest in obtaining existence of “many” distinct actions and rigidity results with respect to weaker notions of equivalence of actions, such as $W^*$-equivalence (or von Neumann equivalence) considered in [P7], which requires the isomorphism of the associated group measure space von Neumann...
algebras $L^\infty(X) \rtimes \Gamma \simeq L^\infty(Y) \rtimes \Lambda$, a condition well known to be weaker than OE of actions (cf. [S], [CJ], [PV]; for more on $W^*$-equivalence see [P10], [V2], [I3] and the references therein). More recently, a weak version of equivalence between actions of groups, that we will call here \textit{weak conjugacy}, has been proposed in [Ke]. It requires that each one of the two actions can be “simulated” inside the other, in moments. Several results about weak conjugacy of actions have been obtained in [AE], [AW], [T], [IT], with many open problems still remaining.

In this paper, we consider some additional weak versions of conjugacy and orbit equivalence of group actions, give examples and address the usual type of rigidity questions.

All equivalences between actions that we study involve some form of approximation of the classical notions of conjugacy and orbit equivalence. As such, they can often be best formulated in the language of ultra products of actions and algebras, along some fixed (but arbitrary) free ultrafilter $\omega$ on $\mathbb{N}$. For instance, using the “ultraproduct framework”, weak conjugacy of pmp actions $\Gamma \rtimes^\sigma X, \Gamma \rtimes^\rho Y$ amounts to the ultrapower actions $\Gamma \rtimes^{\sigma^\omega} A^\omega, \Gamma \rtimes^{\rho^\omega} B^\omega$ (where $A = L^\infty(X), B = L^\infty(Y)$) containing $\Gamma$-invariant subalgebras $A_0 \subset A^\omega, B_0 \subset B^\omega$, such that $\Gamma \rtimes A_0$ is isomorphic to $\Gamma \rtimes^\rho B$ and $\Gamma \rtimes B_0$ is isomorphic to $\Gamma \rtimes^\sigma A$.

In this same spirit, we will say that the two actions $\sigma, \rho$ are \textit{\omega-conjugate} (respectively \textit{\omega-orbit equivalent}, abbreviated \textit{\omega-OE}), if the ultrapower actions $\Gamma \rtimes^{\sigma^\omega} A^\omega, \Lambda \rtimes^{\rho^\omega} B^\omega$ (respectively their full groups $[\sigma^\omega], [\rho^\omega]$) are conjugate. The actions $\sigma, \rho$ are \textit{approximately conjugate} (respectively \textit{approximately orbit equivalent}, abbreviated \textit{app-OE}) if they are \omega-conjugate (resp. \omega-OE) via an isomorphism $\theta : A^\omega \simeq B^\omega$ that’s of ultrapower form, $\theta = (\theta_n)_n$, with $\theta_n : A \simeq B, \forall n$. This is easily seen to be equivalent to the existence of a sequence of isomorphisms $\theta_n : (X, \mu) \simeq (Y, \nu)$ that asymptotically intertwine the two actions (respectively the two orbit equivalence relations).

The app-conjugacy of actions obviously implies \omega-conjugacy which implies weak conjugacy. We also consider an OE-version of weak conjugacy, which is weaker than \omega-OE (which in turn is implied by app-OE). We notice that the cost and the $L^2$-Betti numbers of an action $\Gamma \rtimes X$ (as defined in [Le], [G1] and respectively [G2]) are weak-OE invariant. We also show that co-rigidity ([A], [P3]) and relative Haagerup property ([Bo], [P5]) are weak-OE invariant. We prove that strong ergodicity of actions (as defined in [Sc]) is a weak-OE invariant as well, and deduce from this and [CW] that if a countable group $\Gamma$ is non-amenable and does not have property (T), then $\Gamma$ has at least two non weak-OE (thus non \omega-OE as well) free ergodic pmp actions (see Proposition 4.8 and Corollary 4.9).

We show that if a group $\Gamma$ has an infinite amenable quotient (in particular, if $\Gamma$ has a free group as a quotient, or if it is the product between an infinite amenable
group and an arbitrary group), then it has continuously many non-conjugate actions that are all approximately conjugate (see 3.7). On the other hand, we notice that for a property (T) group $\Gamma$, app-conjugacy is the same as conjugacy (see 3.11). Related to this, it would be interesting to decide whether any group that does not have property (T) admits many app-conjugate actions that are not conjugate. We also leave open the question of whether $\omega$-conjugacy is different from conjugacy for property (T) groups.

By a result in [AW], all quotients of all Bernoulli actions are weakly equivalent, once they are free (see also [T] and 5.1 in [P11] for more general results along these lines). Complementary to this, we show here that weak-conjugacy behaves well with respect to co-induction, so in particular if $\Gamma$ has an infinite amenable subgroup $H$, then the co-induction from $H$ to $\Gamma$ of any two free ergodic pmp actions of $H$ are weakly conjugate (Proposition 3.8).

Finally, we prove that if a strongly ergodic pmp action $\Gamma \curvearrowright X$ is OE-superrigid (i.e., any OE between $\Gamma$ and another free action $\Lambda \curvearrowright Y$ comes from a conjugacy), then it is app-OE superrigid (resp. $\omega$-OE superrigid), i.e., any app-OE (resp. $\omega$-OE) of this action with another free pmp action $\Lambda \curvearrowright Y$ comes from an approximate conjugacy (resp. $\omega$-conjugacy) of $\Gamma \curvearrowright X$, $\Lambda \curvearrowright Y$ (see Proposition 5.1). We deduce from this (in fact, from the proof of this result) and the OE-superrigidity results in [K1, K2], [CK] that if $\Gamma$ is either a product of mapping class groups arising from compact orientable surfaces with higher complexity [K1], or a certain type of amalgamated free product of higher rank lattices (such as $SL(3, \mathbb{Z}) *_{\Sigma} SL(3, \mathbb{Z})$, with $\Sigma$ the subgroup of matrices $(t_{ij})$ with $t_{31} = t_{32} = 0$; see [K2]), or a central quotient of a surface braid group (cf. [CK]), then any Bernoulli $\Gamma$-action (more generally, any strongly ergodic aperiodic $\Gamma$-action) is app-OE superrigid and $\omega$-OE superrigid (see Corollary 5.2). We derive in a similar way an app-OE/$\omega$-OE strong rigidity result from OE strong rigidity in [MS] (see Corollary 5.3).

We have included several comments and open problems in the last part of Section 5. For instance, given a free ergodic pmp action $\Gamma \curvearrowright X$ and a free ultrafilter $\omega$ on $\mathbb{N}$, we consider the 1-cohomology group of the Cartan inclusion $A^\omega \subset A^\omega \rtimes_{\sigma^\omega} \Gamma$, where $A = L^\infty(X)$ and $\sigma^\omega$ is the ultrapower action. This group, which we denote $H^1_\omega(\sigma)$, is an $\omega$-OE invariant for $\Gamma \curvearrowright^\sigma X$ (thus also app-OE invariant). It contains the group $(H^1(\sigma))^\omega$, where $H^1(\sigma)$ is the 1-cohomology group of $\sigma$, and we conjecture that for certain property (T) groups $\Gamma$ one should have equality, and therefore $H^1_\omega(\sigma) = H^1(\sigma)$ whenever $H^1(\sigma)$ is finite (e.g., when $\sigma$ is Bernoulli, cf. [PS]). Showing this would amount to proving vanishing results for $U(A)$-valued approximate cocycles for $\Gamma \curvearrowright^\sigma A$, i.e. for sequences of maps $w^g : \Gamma \to U(A)$ satisfying $\lim_\omega \| w^g_n \sigma_g(w^h_n) - w^g_{gh}\|_2 = 0$, $\forall g, h \in \Gamma$. But the deformation-rigidity
arguments used so far to prove such results for cocycles of Bernoulili actions break when passing to approximate cocycles, where a new idea seems to be needed.

Such calculations of the app-OE invariant $H^1_\omega(\sigma)$ for Bernoulli actions and their quotients, as in ([P6]), would show in particular that $\Gamma$ has infinitely many non $\omega$-OE actions that are all weak-OE (by [AW]). We in fact leave open the problem of whether any property (T) group has at least two non $\omega$-OE free ergodic pmp actions. This question would of course be answered if one could show that any property (T) group $\Gamma$ has at least two non weakly conjugate actions (cf. [Ke], [T]).

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2. Basics on Cartan inclusions

The most basic example of operator algebras (and historically the first to be considered) were the group measure space von Neumann algebras, introduced by Murray and von Neumann in [MvN]. These are crossed-product von Neumann algebras arising from groups acting freely on measure spaces, $\Gamma \actson (X, \mu)$, or equivalently from groups acting on function spaces, $\Gamma \actson L^\infty(X, \mu)$, and denoted $L^\infty(X) \rtimes \Gamma$. Cartan subalgebras and Cartan inclusions of von Neumann algebras are abstractions of the inclusion $L^\infty(X) \subset L^\infty(X) \rtimes \Gamma$, as developed in [D], [S], [Dy], [FM]. We will recall in this section the definition, basic properties and main invariants of these objects.

Classically, these considerations are made in a separable framework, meaning that all von Neumann algebras are assumed to act on a separable Hilbert space, corresponding to the fact that they arise from countable groups acting on standard probability measure spaces, $\Gamma \actson (X, \mu)$. This can be done by either emphasizing the countable equivalence relation $R_{\Gamma \actson X}$, given by the orbits of the action, as in [S], [FM], or by emphasizing the full group $[\Gamma]$ associated with the action, as in ([Dy], [P2], or 1.3 in [P6]). The notions of approximate equivalence of actions that we are interested in in this paper lead us to also consider ultrapowers of such group actions, $\Gamma \actson L^\infty(X)^\omega$, and the Cartan inclusion of non-separable von Neumann algebras $L^\infty(X)^\omega \subset L^\infty(X)^\omega \rtimes \Gamma$ that they entail. We thus need to consider these objects without any separability assumption, a fact that imposes the “full group formalism”. However, all proofs are the same as in the separable case and will thus be only sketched, or even omitted.

2.1. Definition. 1° Let $(M, \tau)$ be a finite von Neumann algebra with a faithful normal trace state. A Cartan subalgebra of $M$ is a maximal abelian $^\ast$-subalgebra
(MASA) \( A \subset M \) whose \textit{normalizer} in \( M \), \( \mathcal{N}_M(A) = \{ u \in \mathcal{U}(M) \mid uAu^*A \} \), generates \( M \) as a von Neumann algebra. We should note that in most cases considered, we will deal with Cartan subalgebras \( A \subset M \) for which \( \mathcal{N}_M(A) \) is countably generated over \( A \), in the sense that there exists a countable subgroup \( \mathcal{N}_0 \subset \mathcal{N}_M(A) \) such that \( \mathcal{N}_0 \vee A = M \). Note that this is equivalent to \( M \) having a countable orthornormal basis over \( A \) (in the sense of [PP]), i.e. \( \dim_A L^2(M) = \aleph_0 \). We also denote by \( q\mathcal{N}_M(A) \) the \textit{quasi-normalizer} of \( A \) in \( M \), i.e. the set of partial isometries \( v \in M \) with \( vv^*, v^*v \in A \), \( vAv^* = Avv^* \). It is easy to see that \( q\mathcal{N}_M(A) = \{ uq \mid u \in \mathcal{N}_M(A), q \in \mathcal{P}(A) \} \) and that \( q\mathcal{N}_M(A) \) with the multiplication operation inherited from \( M \) is a pseudogroup. Note that one can always find a \([PP]\)-orthonormal basis of \( M \) over \( A \) with elements in \( q\mathcal{N}_M(A) \) (see 2.1 in [P2]). Moreover, by (2.3-2.5 in [P2]), if \( M \) is a factor, then there exists an orthonormal basis of \( M \) over \( A \) with unitary elements in \( \mathcal{N}_M(A) \).

2° Let \( (A, \tau) \) be an abelian von Neumann algebra with a normal faithful trace state. If \( \Gamma \) is a subgroup of \( \text{Aut}(A, \tau) \), then we denote by \( [\Gamma] \) the \textit{full group} generated by \( \Gamma \), i.e. the group of all automorphisms \( \theta \in \text{Aut}(A, \tau) \) with the property that there exists a partition of \( 1 \) with projections \( p_n \in A \) and automorphisms \( \theta_n \in \Gamma \), such that \( \theta_n(p_n) \) are mutually disjoint and \( \theta(a) = \sum_n \theta_n(ap_n), \forall a \in A \).

We also denote by \( [[\Gamma]] \) the \textit{full pseudogroup} generated by \( \Gamma \), i.e., the set of all partial isomorphisms \( \phi : Ap \rightarrow Aq \), with \( p, q \in \mathcal{P}(A) \), for which there exists a partition of \( p \) with projections \( p_n \in A \) and automorphisms \( \theta_n \in \Gamma \) such that \( \sum_n \theta_n(p_n) = q \) and \( \theta(a) = \sum_n \theta_n(ap_n), \forall a \in Ap \). We denote by \( [[\Gamma]]_0 \) the sub-pseudogroup of identity partial isomorphisms in \( [[\Gamma]] \), which coincides with the units of \( [[\Gamma]] \), and which can be identified with \( \mathcal{P}(A) \). If \( \Gamma \curvearrowleft (A, \tau) \) is an action of \( \Gamma \) by automorphisms then we still denote by \( [[\Gamma]] \) (resp. \( [\Gamma] \)) the full pseudogroup (resp. full group) generated by its image in \( \text{Aut}(A, \tau) \).

Note that, by a standard maximality argument, we have \( [[\Gamma]] = \{ \phi \mid \exists \theta \in [\Gamma], p \in \mathcal{P}(A) \text{ such that } \theta|_{Ap} = \phi \} \) (see e.g. [Dy]).

3° If \( \phi \) is a partial isomorphism of \( A \) (for instance, from some full pseudogroup \([\Gamma]\)), then we denote by \( L(\phi), R(\phi) \in \mathcal{P}(A) \) its left and respectively right \textit{supports} and by \( p_{\phi} \in \mathcal{P}(A) \) the maximal projection under \( R(\phi) \) on which \( \phi \) acts as the identity. Note that it is equal to the maximal projection \( p \) under \( L(\phi) \) for which \( p\phi = p \) and that \( p_{\phi^{-1}} = p_{\phi} \). If \( \psi \) is another partial isomorphism on \( A \), then we define a distance between \( \phi \) and \( \psi \) by letting \( \| \phi - \psi \|_2 = (\| \tau(R(\phi)) + \tau(R(\psi)) - 2\tau(p_{\phi^{-1}}))^{1/2} \). We will see in 2.5-2.6 below that this norm corresponds to the Hilbert-norm associated with the canonical trace of a full pseudogroup, thus justifying the notation \( \| \cdot \|_2 \). It is an easy exercise to show that \( \| \phi - \psi \|_2 \) is equivalent to \( \sup\{ \| \phi(aR(\phi)) - \psi(aR(\psi)) \|_2 \mid a \in (A)_1 \} \).
Note that if $A \subset M$ is a Cartan subalgebra in a finite von Neumann algebra, then \( \{ \text{Ad}(u) \mid u \in N_M(A) \} = [N_M(A)] \) is a full group. Note further that this group is naturally isomorphic to \( N_M(A)/\mathcal{U}(A) \) and that the associated full pseudogroup \([N_M(A)]\) can be naturally identified with \( qN_M(A)/\mathcal{I}(A) \), where \( \mathcal{I}(A) \) denotes the set of partial isometries in \( A \).

We will see shortly that, conversely, if \([\Gamma]\) is a full group on \((A, \tau)\), then there exists a Cartan inclusion \((A \subset M, \tau)\) such that \([N_M(A)] = [\Gamma]\) (cf. 2.5 below).

2.2. Examples. 1° The typical example of a Cartan inclusion is provided by the Murray-von Neumann group measure space construction, as follows. Let \( \Gamma \curvearrowright (X, \mu) \) be a probability measure preserving (pmp) free action of a countable group \( \Gamma \). Let \( A = L^\infty(X) \) be endowed with the trace \( \tau(a) = \int ad\mu \) and consider the action \( \Gamma \curvearrowright (A, \tau) \) given by \( g(a)(t) = a(g^{-1}(t)), \ t \in X, \ g \in \Gamma, \ a \in A = L^\infty(X) \).

Denote by \( M_0 \) the \(*\)-algebra generated by a copy of \( A \) and a copy of the group \( \Gamma \), denoted \( \{ u_g \}_{g \in \Gamma} \) (the group of canonical unitaries), satisfying \( u_g a u_g^* = g(a) \), \( \forall g \in \Gamma, \ a \in A \). Thus, the elements in \( M_0 \) are formal finite sums \( x = \Sigma g a_g u_g \), with \( a_g \in A \), with product given by \( (a_g u_g)(a_h u_h) = a_g g(a_h) u_{gh} \) and \(*\)-operation by \( (a_g u_g)^* = g^{-1}(a^*) u_{g^{-1}} \). \( M_0 \) comes endowed with a trace state \( \tau \) extending the trace on \( A \), defined by \( \tau(\Sigma_g a_g u_g) = \tau(a_e) \), where \( e \in \Gamma \) is the neutral element. The completion of \( M_0 \) in the Hilbert norm given by \( \|x\|_2 = \tau(x^* x)^{1/2} \), \( x \in M_0 \), identifies naturally with the Hilbert space \( \oplus_g L^2(X) u_g \simeq L^2(X) \otimes \ell^2 \Gamma \), on which \( M_0 \) acts by left multiplication. The group measure space von Neumann algebra associated with \( \Gamma \curvearrowright X \), denoted \( L^\infty(X) \times \Gamma \), is by definition the weak closure of \( M_0 \) in this representation. The fact that the action \( \Gamma \curvearrowright X \) is free is equivalent to the fact that the subalgebra \( A \subset M \) is a MASA in \( M \). Since \( M \) is generated by the unitaries in \( A \) and \( \{ u_g \}_g \), which all normalize \( A \), it follows that if \( \Gamma \curvearrowright X \) is free then \( A \subset M \) is a Cartan inclusion.

2° One can generalize the group measure space construction from the case of free pmp actions to the case of pmp actions of countable groups \( \Gamma \curvearrowright (X, \mu) \) that are not necessarily free, and more generally to trace preserving actions of arbitrary groups \( \Gamma \) on abelian von Neumann algebras \((A, \tau)\). In the case \( \Gamma \) is countable and \( A = L^\infty(X) \), with \( \Gamma \curvearrowright L^\infty(X) \) arising from a pmp action \( \Gamma \curvearrowright X \), this construction can be described as follows (cf. [FM]).

Let \( \mathcal{R} = \mathcal{R}_{\Gamma \curvearrowright X} \overset{\text{def}}{=} \{(t, gt) \mid t \in X, g \in \Gamma \} \) be the orbit equivalence relation generated by the action (viewed up to null sets in the first variable). Let \( m \) be the unique measure on \( \mathcal{R} \) satisfying \( m(\{(t, gt) \mid t \in X_0\}) = \mu(X_0), \ \forall X_0 \subset X \) and \( g \in \Gamma \). Each element \( \xi \in L^2(\mathcal{R}, m) \) can be viewed as a matrix \( \xi = (\xi(t, t'))_{t \sim t'} \), with \( \langle |x(t, t')|^2 dm \rangle^{1/2} < \infty \). For each automorphism \( \theta \in \text{Aut}(X, \mu) \) implemented by some \( g \in \Gamma \), one denotes by \( u_\theta \) the matrix with \( u_\theta(t, t') \) equal to 1 if \( t' = \theta(t) \)
and 0 otherwise. Any finite sum \( x = \sum_\theta a_\theta u_\theta \) acts on \( \xi \in L^2(\mathcal{R}, m) \) by matrix multiplication, \( x_\xi(t, t') = \sum_{s \sim t} x(t, s)\xi(s, t') \), \( \forall (t, t') \in \mathcal{R} \). The set of such elements is a \( ^* \)-subalgebra \( M_0 = \mathcal{B}(L^2(\mathcal{R}, m)) \) whose weak closure is the von Neumann algebra associated with \( \mathcal{R} \). This algebra, denoted \( L(\mathcal{R}) \), has \( A \simeq L^\infty(X) \subset L(\mathcal{R}) \) as a Cartan subalgebra (when viewed as the set of matrices \( x \) supported on the diagonal \( \{(t, t) \mid t \in X \} \subset \mathcal{R} \)).

This construction can alternatively be described by using the full group \([\Gamma]\), or the full pseudogroup \([[[\Gamma]]]\) (see e.g. Section 1.3 in [P6]), this approach having the advantage of working for arbitrary \( \Gamma \) and \( A \) (not necessarily separable). We will describe a more general version of this construction (involving also a 2-cocycle on \([\Gamma]\)) in Example 2.5 and Proposition 2.6 below.

3° If \((A \subset M, \tau)\) is a Cartan inclusion (e.g., as in 2.2.1° or 2.2.2° above) and \( \omega \) is a free ultrafilter on \( \mathbb{N} \), then denote by \( A_\omega \subset M_\omega \) the corresponding ultrapower inclusion (cf. [W]; see also Section 1.2 in [P11]). Denote by \( M(\omega) \) the von Neumann subalgebra of \( M_\omega \) generated by \( N(M)(A) \) and \( A_\omega \), \( M(\omega) = N(M)(A) \lor A_\omega \). Note that if \( M = A \rtimes \Gamma \), for some free action \( \Gamma \rhd (A, \tau) \), then \( \Gamma \) also induces a free action on \( A_\omega \), by \( g((a_n)_n) = (g(a_n))_n \), where \( (a_n)_n \in A_\omega \), and \( M(\omega) \) naturally identifies with \( A_\omega \rtimes \Gamma \).

2.3. Definition. 1° A 2-cocycle for the full group \( G \) on \((A, \tau)\) is a map \( v : G \times G \to U(A) \) satisfying the following properties

\[
\begin{align*}
&v_{\theta, \phi} v_{\theta, \psi} = \theta(v_{\phi, \psi}) v_{\theta, \psi}, \forall \theta, \phi, \psi \in G; \\
&p_{\theta_1, \theta_2}^{-1} v_{\theta_1, \psi} = p_{\theta_1, \theta_2}^{-1} v_{\theta_2, \psi}, \psi(p_{\theta_1, \theta_2}^{-1}) v_{\psi, \theta_1} = \psi(p_{\theta_1, \theta_2}^{-1}) v_{\psi, \theta_2}, \forall \theta_1, \theta_2, \psi \in G.
\end{align*}
\]

Such a cocycle is normalized if the following conditions hold true:

\[
\begin{align*}
&(b) \quad p_\theta v_{\theta, \psi} = p_\theta; \theta(p_\psi)v_{\theta, \psi} = \theta(p_\psi); v_{\theta, \psi} p_\theta = p_{\theta \psi}, \forall \theta, \psi \in G.
\end{align*}
\]

Two cocycles \( v, v' \) are equivalent, \( v \sim v' \), if there exists \( w : G \to U(A) \) satisfying the condition

\[
\begin{align*}
&(c) \quad p_{\theta_1, \theta_2}^{-1} w_{\theta_1} = p_{\theta_1, \theta_2}^{-1} w_{\theta_2}, \forall \theta_1, \theta_2 \in G,
\end{align*}
\]

such that

\[
\begin{align*}
&(d) \quad v_{\theta, \psi}' = w_\theta \theta(w_\psi)v_{\theta, \psi} w_\theta^*, \forall \theta, \psi \in G.
\end{align*}
\]

It is easy to check that any 2-cocycle is equivalent to a normalized 2-cocycle. Moreover, one can check that if \( \Gamma \subset G \) is a subgroup such that \([\Gamma] = G \) and one has
a map \(v : \Gamma \times \Gamma \to \mathcal{U}(A)\) satisfying the properties \((a), (b)\) for all \(\theta_1, \theta_2, \psi \in \Gamma\), then it extends uniquely to a normalized 2-cocycle on \(G\).

Note that the set \(Z^2(G)\) of cocycles on \(G\) is an abelian group under multiplication as \(\mathcal{U}(A)\)-valued functions on \(G \times G\), and that the equivalence of 2-cocycles amounts to equivalence modulo its subgroup \(B^2(G)\) of 2-cocycles of the form \((\theta, \psi) \mapsto w_\theta \theta_1(w_\psi) w_{\theta_1 \psi}^*, \) with \(w : G \to \mathcal{U}(A)\) satisfying \(p_{\theta_1 \theta_2}^{-1} w_{\theta_1} = p_{\theta_1 \theta_2}^{-1} w_{\theta_2}, \forall \theta_1, \theta_2 \in G\).

2° A 2-cocycle for a full pseudogroup \([[G]]\) on \((A, \tau)\) is a map \(v\) from \([[G]] \times [[G]]\) into \(\mathcal{I}(A)\) (the set of partial isometries in \(A\)) which formally satisfies exactly the same properties as \((a)\) above (but with \(\theta, \theta_1, \theta_2, \phi, \psi\) in \([[G]]\)). The cocycle \(v\) is normalized if the conditions \((b)\) are formally satisfied, \(\forall \theta, \psi \in [[G]]\). Two normalized cocycles \(v, v'\) for \([[G]]\) are equivalent if there exists \(w : [[G]] \to \mathcal{I}(A)\) such that \((c)\) and \((d)\) are satisfied, \(\forall \theta, \theta_1, \theta_2, \psi \in [[G]]\).

### 2.4. Example

Let \(\Gamma\) be a countable group and \(\Gamma \acts X\) a free ergodic pmp \(\Gamma\)-action. A map \(v : \Gamma \times \Gamma \to \mathcal{U}(A)\) satisfying the conditions \(v_{g,h}v_{gh,k} = \sigma_g(v_{h,k})v_{g,hk},\) \(\forall g, h, k \in \Gamma,\) is called a 2-cocycle of \(\sigma\). We denote by \(Z^2(\sigma)\) the multiplicative group of such 2-cocycles \(v\) and call it the 2-cohomology group of \(\sigma\). We also denote by \(B^2(\sigma)\) its subgroup of 2-cocycles of the form \(v_{g,h} = w_g \sigma_g(w_h)w_{gh}^*\), for some \(w : \Gamma \to \mathcal{U}(A)\).

It is easy to check that any \(v \in Z^2(\sigma)\) extends uniquely to a 2-cocycle \(\tilde{v}\) of the full group \([\Gamma]\). If in addition \(v\) satisfies \(v_{e, h} = v_{g,e} = v_{g,q^{-1}} = 1, \forall g, h \in \Gamma,\) then \(\tilde{v}\) is a normalized 2-cocycle of \([\Gamma]\). The map \(v \mapsto \tilde{v}\) implements an isomorphism \(Z^2(\sigma) \simeq Z^2([\sigma])\) whose inverse is the restriction of \(\tilde{v}\) to \(\Gamma \times \Gamma\) and which takes \(B^2(\sigma)\) onto \(B^2([\sigma])\).

Note that if we denote \(Z^2(\Gamma, \mathbb{T}) = \{ \lambda : \Gamma \times \Gamma \to \mathbb{T} \mid \lambda_{g,h} \lambda_{gh,k} = \lambda_{h,k} \lambda_{g,hk}, \forall g, h, k \in \Gamma \}\), the 2nd cohomology group of \(\Gamma\), and by \(B^2(\Gamma, \mathbb{T})\) the subgroup of coboundaries, then given any free pmp \(\Gamma\)-action \(\Gamma \acts X\), any element \(\lambda \in Z^2(\Gamma, \mathbb{T})\) determines an element in \(Z^2(\sigma)\), thus in \(Z^2([\sigma])\). We will discuss in 5.7.4 the problem of whether some \(\lambda \in Z^2(\Gamma, \mathbb{T})\) which is non-trivial in \(H^2(\Gamma, \mathbb{T}) = Z^2(\Gamma, \mathbb{T})/B^2(\Gamma, \mathbb{T})\) is still non-trivial in \(Z^2([\Gamma])\).

### 2.5. Example

Let \(A \subset M\) be a Cartan subalgebra. We can then write \([\mathcal{N}_M(A)]\) as a well ordered set \(\{\theta_i\}_{i \in I}\), with \(i d_A\) as its first element. We choose \(u_{id} = 1\). Assume that for the “first” \(I_0 \subset I\) elements in \(I\), we have chosen elements \(\{u_{\theta_i}\}_{i \in I_0} \subset \mathcal{N}_M(A)\) such that if \(i, j \in I_0\) and \(q \in \mathcal{P}(A)\) is so that \(\theta_i, \theta_j\) agree on \(Aq\), then \(u_{\theta_i} q = u_{\theta_j} q.\) If now \(k\) is the first element of \(I \setminus I_0\), then there is a maximal projection \(p \in A\) such that the restriction of \(\theta_k\) to \(Ap\) does not agree with any \(\theta_i, i \in I_0\), on \(Aq\) for any \(q \in \mathcal{P}(Ap)\). Thus, we have \(\theta_k = \oplus_{i \in I_0} \theta_i \mid Aq \oplus \theta_k \mid Ap\), for some mutually orthogonal projections \(\{q_i\}_i\) and \(p\). We define \(u_{\theta_k} = \sum_{i \in I_0} u_{\theta_i} q_i + vp,\)
where \( v \in \mathcal{N}_M(A) \) is any unitary that implements \( \theta_k \).

In this manner, we have obtained a set of unitaries \( \{ u_\theta \mid \theta \in \mathcal{N}_M(A) \} \subset \mathcal{N}_M(A) \) such that \( u_{id} = 1 \), \( \text{Ad}(u_\theta) = \theta \), \( \forall \theta \), and \( u_\theta q = u_\psi q \) for any projection \( q \in A \) with \( \theta, \psi \) agreeing on \( Aq \).

Note that this implies \( p_{\theta_1, \theta_2} u_{\theta_1} u_{\theta_2}^{-1} = p_{\theta_1, \theta_2}, \forall \theta_1, \theta_2 \in \mathcal{N}_M(A) \). This in turn easily implies that \( v_{\theta, \psi} = u_\theta u_\psi u_{\theta \psi}^* \), for \( \theta, \psi \in \mathcal{N}_M(A) \), satisfies conditions (a) in 2.3.1°, and is thus a 2-cocycle for the full group \([\mathcal{N}_M(A)]\). The choice of \( v \) is unique up to the equivalence relation \( \sim \) defined above.

Thus, to each Cartan subalgebra \( A \subset M \) one can in fact associate a pair \((\mathcal{G}, v/\sim)\), consisting of a full group \( \mathcal{G} = [\mathcal{N}_M(A)] \) on the abelian von Neumann algebra \((A, \tau)\) and the equivalence class of a \( \mathcal{U}(A) \) valued 2-cocycle \( v \) for \( \mathcal{G} \), as defined above. This provides a functor, from the category of tracial Cartan inclusions \((A \subset M, \tau)\) with morphisms given by trace preserving isomorphisms between the ambient algebras \((M, \tau)\) carrying the Cartan subalgebras \( A \) onto each other, to the category of pairs \((\mathcal{G} \curvearrowleft (A, \tau), v/\sim)\), consisting of a full group \( \mathcal{G} \) on an abelian von Neumann algebra \((A, \tau)\) and a (equivalence class of a) 2-cocycle \( v : \mathcal{G} \times \mathcal{G} \to \mathcal{U}(A) \), with morphisms given by trace preserving isomorphisms between the algebras \((A, \tau)\) that carry the full groups (resp. class of 2-cycles) onto each other.

This functor is in fact one to one and onto, its inverse being constructed as follows. Let \((\mathcal{G}, v/\sim)\) be a pair consisting of a full group \( \mathcal{G} \) on \((A, \tau)\) and a (class of a) normalized 2-cocycle \( v : \mathcal{G} \times \mathcal{G} \to \mathcal{U}(A) \) for \( \mathcal{G} \curvearrowleft A \). Let \( M_0 \) be the vector space of finite sums \( \Sigma a_\theta u_\theta \), with \( a_\theta \in A \) and indeterminates \( u_\theta, \theta \in \mathcal{G} \) satisfying \( u_\theta u_\phi = v_{\theta, \phi} u_{\theta \phi} \). On \( M_0 \) one defines the product rule \((a_\theta u_\theta)(a_\phi u_\phi) = a_\theta \theta(a_\phi) v_{\theta, \phi} u_{\theta \phi} \) and \(*\)-operation given by \((a_\theta u_\theta)^* = \theta^{-1}(a_\theta^*)u_{\theta^{-1}}\). Define a functional \( \tau \) on \( M_0 \) by \( \tau(a_\theta u_\theta) = \tau(a_\theta p_\theta) \), which clearly factors through the expectation \( E \) of \( M_0 \) onto \( A \) given by \( E(\Sigma a_\theta u_\theta) = \Sigma a_\theta p_\theta \). Also, define the sesquilinear form on \( M_0 \) by \( \langle x, y \rangle_{\tau} = \tau(y^* x) \), which is easily seen to be positive, semi-definite. Denote by \( \mathcal{H} \) the Hilbert space completion of \( M_0/I_{\tau} \), where \( I_{\tau} = \{ x \in M_0 \mid \tau(x^* x) = 0 \} \). Note that any \( x \in M_0 \) acts as a bounded linear operator on \( \mathcal{H} \), by left multiplication.

Finally, define \( M = L(\mathcal{G}, v) \) to be the weak closure of \( M_0 \) in this representation. Then \( \tau \) defines a normal faithful trace state on \( L(\mathcal{G}, v) \) and \( A \subset L(\mathcal{G}, v) \) is a Cartan inclusion. It is straightforward to check that if a pair \((\mathcal{G}, v)\) comes from a Cartan inclusion \((A \subset M)\), as described above, then this new Cartan inclusion \( A \subset L(\mathcal{G}, v) \) naturally identifies with the initial one \((A \subset M)\); and that the pair \([\mathcal{N}_M(A)], v\) described above, for the Cartan inclusion \( A \subset L(\mathcal{G}, v) \), identifies naturally with \((\mathcal{G}, v)\).

Altogether, we have just shown the following version of a well known result of Feldman-Moore ([FM]), formulated in terms of full groups rather than countable equivalence relations on a standard measure space, a fact that allows us avoid...
separability conditions.

2.6. Proposition. The functor from Cartan inclusions to pairs consisting of a full group and a normalized 2-cocycle defined above, $(A \subset M, \tau) \mapsto ([N_M(A)], v/\sim)$, is one to one and onto, its inverse being the functor $(\mathcal{G} \rhd (A, \tau), v/\sim) \mapsto (A \subset L(\mathcal{G}, v))$. Via these functors, the isomorphisms of Cartan inclusions correspond to isomorphisms of full groups intertwining the corresponding (classes of) 2-cocycles.

We will often need to consider commuting squares of Cartan inclusions $A_0 \subset M_0$, $A \subset M$, as also considered in (Section 1.1 of [P11]), under an additional non-degeneracy condition. Recall in this respect that two inclusions of finite von Neumann algebras $B_0 \subset N_0$, $B \subset N$, with $N_0 \subset N$, $B_0 \subset B$, are in commuting square position if $B \cap N_0 = B_0$ and the trace preserving expectations of $N$ onto $B$ and $N$ onto $N_0$ commute, with their product giving the $\tau$-preserving expectation of $N$ onto $B_0$, i.e. $E_B^{N_0}E_B^{N} = E_B^{N_0}E_B^{N} = E_B^{N=0}$ (see 1.2 in [P1]). For these conditions to hold true, it is in fact sufficient that $E_B^{N_0}(N_0) \subset N_0$ (equivalently $E_B^{N_0}(B) \subset B$).

It is important to note that if $B_0 \subset N_0$, $B \subset N$ are MASAs, with $B_0 \subset B$, $N_0 \subset N$, then the commuting square condition automatically holds true. This is because $E_B^{N_0}(B) \subset B_0 \cap N_0 = B_0$ (since $B_0 \subset B$ are commutative).

2.7. Definition. Two Cartan subalgebras $A_0 \subset M_0$, $A \subset M$ are said to form a non-degenerate embedding if $A_0 \subset A$, $M_0 \subset M$ (so by the above discussion, the commuting square relation $E_A(M_0) = A_0$ is automatically satisfied), $\mathcal{N}_{M_0}(A_0) \subset \mathcal{N}_M(A)$ and $\text{sp}(M_0A) = M$ (equivalently, $\mathcal{N}_{M_0}(A_0) \lor A = M$). If this is the case, then we also say that $A_0 \subset M_0$ is sub-Cartan inclusion of $A \subset M$, or that $A \subset M$ is an extension of $A_0 \subset M_0$.

Note that for Cartan inclusions arising from full (pseudo)groups $\mathcal{G}_0$ on $(A_0, \tau_0)$ and $\mathcal{G}$ on $(A, \tau)$, with trivial 2-cocycle, such a Cartan embedding amounts to the existence of a subgroup $\mathcal{G}_0 \subset \mathcal{G}$ and a $\mathcal{G}_0$-invariant subalgebra $A_0' \subset A$, such that $\mathcal{G}_0'$ generates $\mathcal{G}$ as a full (pseudo)group on $A$ and $\mathcal{G}_0' \rhd A_0'$ is isomorphic to $\mathcal{G}_0' \rhd A_0'$.

In turn, if the Cartan inclusions involve separable von Neumann algebras, and we view them as coming from countable equivalence relations $\mathcal{R}_0$ on $(X_0, \mu_0)$ and $\mathcal{R}$ on $(X, \mu)$, then a non-degenerate Cartan embedding of $L^\infty(X_0) \subset L(\mathcal{R}_0)$ into $L^\infty(X) \subset L(\mathcal{R})$ amounts to a local OE (or local isomorphism) of $\mathcal{R}, \mathcal{R}_0$ in the sense of (Definition 1.4.2 in [P8]), i.e., an a.e. surjective measure preserving map $\Delta : (X, \mu) \rightarrow (X_0, \mu_0)$ for which there exists a measure 0 subset $S \subset X$ such that for any $t \in X \setminus S$, $\Delta$ is a bijection from the $\mathcal{R}$-orbit of $t$ onto the $\mathcal{R}_0$-orbit of $\Delta(t)$. The terminologies (class bijective) extension and orbit bijection are also used for a map $\Delta$ satisfying such a property.

Note that if $\mathcal{R} = \mathcal{R}_{\Gamma \cap X}$, $\mathcal{R}_0 = \mathcal{R}_{\Gamma \cap X_0}$ for some pmp actions of a countable group
\( \Gamma \), and \( \Delta : X \to X_0 \) is a local isomorphism between \( \mathcal{R}, \mathcal{R}_0 \), then \( \Gamma \wr X \) is free iff \( \Gamma \wr X_0 \) is free. If this is the case, then such a local-isomorphism corresponds to an actual quotient of the free action \( \Gamma \wr X \) to a free action \( \Gamma \wr X_0 \). So indeed, one can view \( \mathcal{R} \) as an “extension of \( \mathcal{R}_0 \)”, while \( \mathcal{R}_0 \) can be viewed as a (free) “quotient of \( \mathcal{R} \)” (when interpreting them as groupoids). Altogether, non-degenerate embeddings between Cartan inclusions can be seen as algebraic abstractions of local-OE (or of quotient) between equivalence relations.

2.8. Lemma. 1° If a Cartan inclusion \( A \subset M \) is such that \( \dim_AM \) is countable, then for any \( X \subset A \) countable subset and \( U_0 \subset \mathcal{N}_M(A) \) countable subgroup with \( U_0 \vee A = M \), there exists an inclusion of separable von Neumann algebras \( A_0 \subset M_0 \) with \( X \subset A_0 \subset A \), \( M_0 \subset M \), such that: \( A_0 \) is Cartan in \( M_0 \); \( U_0 \subset \mathcal{N}_{M_0}(A_0) \subset \mathcal{N}_M(A) \); \( A_0 \subset M_0 \) is a sub-Cartan inclusion of \( A \subset M \).

2° If \( [\Gamma] \) is a full group on \((A, \tau)\) with \( A \) countable and \( X \subset A \) is a countable subset, then there exists a \( \Gamma \)-invariant separable von Neumann subalgebra \( A_0 \subset A \) containing \( X \) and a subgroup \( G_0 \subset [\Gamma] \) that contains \( \Gamma \) and leaves \( A_0 \) invariant, such that the full group of \( \Gamma|_{A_0} \) on \( \text{Aut}(A_0, \tau) \) coincides with \( G_0|_{A_0} \) and such that for any \( \theta \in G_0 \), the projection \( p_\theta \) (defined for \( \theta \) viewed as an element in \([\Gamma]\)),, belongs to \( A_0 \).

Proof. The two parts 1° and 2° are clearly equivalent by Proposition 2.6. The proof of 1° can be easily adapted from the proof of (Section 1.2 Lemma in [P11], or Lemma 3.8 in [P12]), and is thus left as an exercise for the reader. □

2.9. Definition. A Cartan inclusion \( A \subset M \) is strongly ergodic if for any \( \delta_0 > 0 \), there exist a finite set of unitaries \( F \subset \mathcal{N}_M(A) \) and \( \epsilon_0 > 0 \), such that any \( x \in A \), \( \|x\| \leq 1 \), that satisfies \( \|u, x\|_2 \leq \epsilon_0 \), \( \forall u \in F \), must satisfy \( \|x - \tau(x)1\|_2 \leq \delta_0 \).

If \( M \) has countable orthonormal basis over \( A \) (equivalently, \( \mathcal{N}_M(A) \) contains a countable subgroup \( U_0 \) such that \( U_0 \vee A = M \)), then strong ergodicity is easily seen to be equivalent to \( M' \cap A^\omega = \mathbb{C} \), for some free ultrafilter \( \omega \) on \( \mathbb{N} \), and also equivalent to this being true for any free ultrafilter \( \omega \). Moreover, \( A \subset M \) is not strongly ergodic if and only if \( \forall F \subset \mathcal{N}_M(A), \epsilon > 0, \exists v \in U(A) \) such that \( \tau(v) = 0 \) and \( \|[u, v]\|_2 \leq \epsilon, \forall u \in F \).

A Cartan inclusion \( A \subset M \) has spectral gap if for any \( \delta_0 > 0 \), there exist a finite set of unitaries \( F \subset \mathcal{N}_M(A) \) and \( \epsilon_0 > 0 \), such that if \( \xi \in L^2(A) \oplus \mathbb{C} \), \( \|\xi\|_2 \leq 1 \), satisfies \( \|[u, \xi]\|_2 \leq \epsilon_0 \), \( \forall u \in F \), then \( \|\xi\|_2 \leq \delta_0 \). This condition obviously implies strong ergodicity. It has been shown by K. Schmidt in [Sc1] (inspired by arguments in [C1]) that if \( A \subset M \) are separable, then in fact the two conditions are equivalent (i.e., strong ergodicity implies spectral gap as well). His argument is easily seen to work for arbitrary (not necessarily separable) Cartan inclusions.
A full group $G$ on $(A, \tau)$ is strongly ergodic (respectively has spectral gap) if its corresponding Cartan inclusion is strongly ergodic (resp. has spectral gap).

2.10. Proposition. 1° Assume $A \subset M$ is a sub-Cartan inclusion of $B \subset N$. If $B \subset N$ is strongly ergodic then $A \subset M$ is strongly ergodic.

2° Let $A \subset M$ be a Cartan inclusion and $\omega$ a free ultrafilter. As in 2.2.3°, denote $A^\omega \subset M(\omega) = A^\omega \vee \mathcal{N}_M(A)$. Then $A \subset M$ is strongly ergodic if and only if $A^\omega \subset M(\omega)$ is strongly ergodic.

Proof. 1° If $A \subset M$ is not strongly ergodic, then for any $F \subset \mathcal{N}_M(A)$, $\varepsilon > 0$, there exists $v \in \mathcal{U}(A)$ with $\tau(v) = 0$ and $\|v, v\|_2 \leq \varepsilon$, $\forall u \in F$.

Let now $v_1, ..., v_n \subset \mathcal{N}_N(B)$ and $\delta > 0$. We will prove that there exists $v \in \mathcal{U}(B)$ such that $\tau(v) = 0$ and $\|v, v\|_2 \leq \delta$, $\forall i$.

To do this, note first that for any $\alpha > 0$, there exist $u_1, ..., u_m \in \mathcal{N}_M(A)$, partitions of 1 with projections $\{p_{ij}\}_{1 \leq j \leq m} \subset B$, $1 \leq i \leq n$, and unitary elements $w_{i,j} \in B$, $1 \leq i \leq n$, $1 \leq j \leq m$, such that $\tau(1 - \sum_{j=1}^{m}p_{ij}) \leq \alpha^2$, $\forall i$, and $p_{ij}v_i = w_{i,j}p_{ij}u_j$, $\forall 1 \leq i \leq n$, $1 \leq j \leq m$.

If we now take $v \in \mathcal{U}(A)$ to have trace 0 and to satisfy $\|[u, v]\|_2 \leq \varepsilon$, for some $\varepsilon > 0$ and all $j \leq m$, then $v$ commutes with $p_{ij}, w_{i,j}$ and we get the estimates

$$\|v_i v - vv_i\|_2 \leq \|\sum_{j=1}^{m}(p_{ij}v_i v - v p_{ij}v_i)\|_2 + 2\alpha$$

$$= \|\sum_{j=1}^{m}(w_{i,j}p_{ij}u_j v - vw_{i,j}p_{ij}u_j)\|_2 + 2\alpha$$

$$= (\sum_{j=1}^{m}\|p_{ij}w_{i,j}[v, u_j]\|_2^2)^{1/2} + 2\alpha \leq m^{1/2}\varepsilon + 2\alpha.$$

Thus, if we take $\varepsilon < \delta/2m^{1/2}$, $\alpha \leq \delta/4$, then we get $\|v_i, u\|_2 \leq \delta$, $\forall i$.

2° By the first part, if $A^\omega \subset M(\omega)$ is strongly ergodic then $A \subset M$ is strongly ergodic. If in turn $A^\omega \subset M(\omega)$ is not strongly ergodic, then noticing that $\mathcal{N}_M(A) \subset \mathcal{N}_M(A^\omega)$, it follows that for any $F \subset \mathcal{N}_M(A)$ finite and $\varepsilon > 0$, there exists $v \in \mathcal{U}(A^\omega)$ of trace 0 such that $\|[u, v]\|_2 \leq \varepsilon/2$, $\forall u \in F$. Thus, if $v = (v_n)_n$ with $v_n \in \mathcal{U}(A)$, then $\lim_{n \to \omega} \|[u, v_n]\|_2 \leq \varepsilon/2$, $\forall u \in F$, implying that for some $n$ large enough we have $\|[u, v_n]\|_2 \leq \varepsilon$, $\forall u \in F$. Thus, $A \subset M$ is not strongly ergodic.

Two important OE invariants for a pmp group action $\Gamma \curvearrowright X$ are the cost ([L], [G1]) and Gaboriau’s $L^2$-Betti numbers ([G2]). They can also be viewed as invariants for the associated full group $[\Gamma]$ or Cartan inclusion $A = L^\infty(X) \subset L^\infty(X) \rtimes \Gamma = M$, a fact that allows us to extend the definitions to Cartan inclusions $A \subset M$ where $A$ is not necessarily separable.

2.11. Definitions. 1° If $[\Gamma]$ is a full group on $(A, \tau)$ and a subset $\{\phi_i\}_i \subset [\Gamma]$ generates $[[\Gamma]]$ as a full pseudogroup, then its cost $c(\{\phi_i\})$ is defined as $\Sigma_i \tau(L(\phi_i)) \in \mathcal{U}(A)$.
The cost of $[[\Gamma]]$ is equal to the infimum over all costs $c(\{\phi\}_i)$ of generating subsets $\{\phi_i\}_i \subset [[\Gamma]]$. If $A \subset M$ is a Cartan inclusion, then its cost is by definition the cost of $[[\mathcal{N}_M(A)]]$.

2° If $A \subset M$ is a separable Cartan inclusion, then the $n$th $L^2$-Betti number of $A \subset M$, denoted $\beta_n^{(2)}(A \subset M)$, is by definition the $n$th $L^2$-Betti number (as defined in [G2]) of the countable equivalence relation $\mathcal{R}_{A \subset M}$. Notice that by [G2], if $A_0 \subset M_0$ is a sub-Cartan inclusion of $A \subset M$, then $\beta_n^{(2)}(A \subset M) = \beta_n^{(2)}(A_0 \subset M_0)$. (N.B.: this amounts to showing that if $\mathcal{R}_0$ is a free quotient of $\mathcal{R}$, then $\beta_n^{(2)}(\mathcal{R}) = \beta_n^{(2)}(\mathcal{R}_0)$; this is in fact only proved in [G2] in case $\mathcal{R}, \mathcal{R}_0$ come from free actions of groups, but the proof for free quotients of groupoids is exactly the same; see also the end of Remark 9.24 in [PSV].)

Let now $A \subset M$ be a Cartan inclusion with countable orthonormal basis but $A$ not necessarily separable. By Lemma 2.8, there exist separable sub-Cartan inclusions $A_0 \subset M_0$ of $A \subset M$ and any two separable sub-Cartan inclusions are included into a larger separable sub-Cartan inclusion of $A \subset M$. The $n$th $L^2$-Betti number of $A \subset M$, denoted $\beta_n^{(2)}(A \subset M)$, is by definition the $n$th $L^2$-Betti number of any of its separable sub-Cartan inclusions (which by the above remark does not depend on the choice of the separable sub-Cartan inclusion).

2.12. Proposition. 1° If $A_0 \subset M_0$ is a sub-Cartan inclusion of $A \subset M$, then $c(A_0 \subset M_0) \geq c(A \subset M)$.

2° Let $A \subset M$ be a Cartan inclusion and denote $A^\omega \subset M(\omega) = A^\omega \vee \mathcal{N}_M(A)$. Then $c(A \subset M) = c(A^\omega \subset M(\omega))$.

3° If $A_0 \subset M_0$ is a sub-Cartan inclusion of $A \subset M$, then they have the same $L^2$-Betti numbers. Moreover, for a Cartan inclusion coming from a free action, $A \subset A \rtimes \Gamma = M$, the $L^2$-Betti numbers of $A \subset M$ coincide with those of $\Gamma$.

Proof. Part 1° is trivial by the definitions while part 3° is implicit in ([G2]; see also the discussion at the end of 2.1.2° above).

By 1°, in order to prove 2° we only need to show that $c = c(A \subset M) \leq c(A^\omega \subset M(\omega)) = c_\omega$. If $c_\omega = \infty$ then there is nothing to prove, so we may assume $c_\omega < \infty$. Let $F \subset q\mathcal{N}_M(A)$ be an arbitrary finite set and $\varepsilon > 0$. Let now $\{v_i\}_{i \geq 1} \subset q\mathcal{N}_M(\omega)(A^\omega)$ be a set of partial isometries that generate $[[\mathcal{N}_M(\omega)(A^\omega)]]$ as a full pseudo-group and satisfies $c(\{v_i\}_i) < c_\omega + \varepsilon/2$.

Represent each $v_i$ as $(v_{i,m})_m$ with $v_{i,m} \in q\mathcal{N}_M(A)$. Since each $w \in F$ is a constant sequence when viewed in $\mathcal{M}(A^\omega)$ and since it can be approximated arbitrarily well by an appropriate finite sum of reductions by projections in $A^\omega$ of products of $\{v_i\}_i$ and their adjoints, it follows that there exists $m$ and $k$ such that $c(\{v_{i,m}\}_{i \leq k}) < c_\omega + \varepsilon/2$ while each element in $F$ is $\varepsilon/2|F|$-contained in the full
pseudogroup generated by \( \{v_{i,m}\}_{i \leq k} \). Thus, by Definition 2.1.3° we can find a set \( E \subset qN_M(A) \) that contains \( \{v_{i,m}\}_{i \leq k} \) such that \( c(E) \leq c_\omega + \varepsilon \) and such that the full pseudogroup it generates on \( A \) contains \( F \). Since \( F \) was arbitrary, this shows that \( c \leq c_\omega + \varepsilon \).

Recall now two more properties for a Cartan inclusion \( A \subset M \):

(a) co-rigidity considered in [P3] and [A] (see also Sec. 9 in [P4], or 5.6 in [P5]), an alternative terminology being “\( M \) has property (T) relative to \( A \)”;

(b) relative Haagerup property, considered in [Bo] and in Sec. 2 of [P5].

We define the same properties for an orbit equivalence relation \( \mathcal{R}_{\Gamma \times X} \), and more generally for a full group \( [\Gamma] \) on an abelian von Neumann algebra \( (A, \tau) \), by requiring that the associated Cartan inclusion \( A \subset L([\Gamma]) \) has the corresponding property.

2.13. Proposition. Assume \( A_0 \subset M_0 \) is a sub-Cartan inclusion of \( A \subset M \). Then \( A \subset M \) is co-rigid (respectively has relative Haagerup property) if and only if \( A_0 \subset M_0 \) has this same property.

Proof. First note that one can find a \( \| \|_2 \)-dense countable subset \( \mathcal{X} \) in \( N_{M_0}(A_0) \) with the property that each \( x \in \mathcal{X} \) has finite spectrum. This can be seen by first taking an arbitrary \( \| \|_2 \)-dense subset \( \mathcal{X}_0 \subset N_{M_0}(A_0) \), then approximating each \( x_0 \in \mathcal{X}_0 \) by a sequence of unitaries in \( N_{M_0}(A_0) \) that have finite spectrum (by using Rokhlin lemma) and then taking \( \mathcal{X} \) to be the set of all the resulting unitaries. Now note that for each \( u \in \mathcal{X} \), any orthonormal basis \( \{m_j^u\}_j \) of \( \{u\}' \cap A_0 \subset \{u\}' \cap A \) is an orthonormal basis for \( A_0 \subset A \) as well.

Finally, notice that if \( \Phi : M_0 \to M_0 \) is an \( A_0 \)-bimodular \( \tau \)-preserving completely positive (cp) unital map and we take \( u \in N_{M_0}(A_0) \), with \( \theta = Adu \) denoting the automorphism of \( A_0 \) that it implements, then \( \Phi(u)a = \Phi(ua) = \Phi(\theta(a)u) = \theta(a)\Phi(u), \ \forall a \in A_0 \). Thus, \( \Phi(u) = u\theta(a) \) for some \( a_u \in A_0 \). In particular, it follows that if \( u \in \mathcal{X} \) and \( \{m_j^u\}_j \subset A \) as above, then \( [\Phi(u), m_j^u] = 0, \ \forall j \).

Altogether, this shows that the proof of (Lemma 9.2 in [P4]) applies for the non-degenerate commuting square embedding of \( A_0 \subset M_0 \) into \( A \subset M \), implying that any \( A_0 \)-bimodular \( \tau \)-preserving completely positive (cp) map \( \Phi : M_0 \to M_0 \) extends (uniquely) to an \( A \)-bimodular \( \tau \)-preserving cp map \( \tilde{\Phi} \) on \( M \).

Thus, the proof of (9.3 in [P4]) applies in exactly the same way to show that if \( A \subset M \) is co-rigid then \( A_0 \subset M_0 \) is co-rigid. On the other hand, the argument in the proof of 9.4 in [P4] (in fact a much simplified version of it) shows that if \( A_0 \subset M_0 \) is co-rigid, then \( A \subset M \) is co-rigid.

For the relative Haagerup property, the proof that \( A \subset M \) will have this property once \( A_0 \subset M_0 \) has it, is quite trivial, by noticing that if \( \Phi : M_0 \to M_0 \) is \( A_0 \)-
bimodular, \(\tau\)-preserving, cp map that’s \textit{compact relative to} \(A_0\), then its extension to an \(A\)-bimodular cp map \(\Phi\) on \(M\) constructed above is compact relative to \(A\). The opposite implication is trivial. \(\square\)

\textbf{2.14. Remark.} Note that in view of the remarks we made after Definition 2.7, Proposition 2.10.1° shows in particular that if \(\mathcal{R}, \mathcal{R}_0\) are orbit equivalence relations arising from pmp actions of countable groups on standard probability spaces and \(\mathcal{R}\) is an (class bijective) extension of \(\mathcal{R}_0\), then \(\mathcal{R}\) strongly ergodic implies \(\mathcal{R}_0\) is strongly ergodic, while Proposition 2.13 shows that \(\mathcal{R}\) has Haagerup property (resp. is co-rigid) if and only if \(\mathcal{R}_0\) does.

\textbf{2.15. Definition.} We end this section by recalling the definitions of the three \textit{symmetry groups} of a Cartan inclusion \(A \subset M\) (respectively orbit equivalence relation \(\mathcal{R}_\Gamma\), resp. full group \([\Gamma]\)):

1° The \textit{automorphism group} of \(A \subset M\), denoted \(\operatorname{Aut}(A \subset M)\), is the group of automorphisms of \((M, \tau)\) that leave \(A\) invariant, and which we endow with the point-\(\|\|_2\) convergence. Note that in case \(M\) is separable, this is a Polish group. The \textit{outer automorphism group} \(\operatorname{Out}(A \subset M)\) is the quotient of \(\operatorname{Aut}(A \subset M)\) by its (normal) subgroup \(\operatorname{Int}(A \subset M) = \{\operatorname{Ad}(u) \mid u \in \mathcal{N}_M(A)\}\). Similarly, for equivalence relations \(\mathcal{R}_\Gamma\), we denote by \(\operatorname{Aut}(\mathcal{R}_\Gamma)\) the group of automorphisms of \((X, \mu)\) that normalize \(\mathcal{R}_\Gamma\) (i.e. take each \(\Gamma\)-orbit onto a \(\Gamma\)-orbit) and by \(\operatorname{Out}(\mathcal{R}_\Gamma)\) its quotient by the subgroup \(\operatorname{Int}(\mathcal{R}_\Gamma)\) of automorphisms that leave each \(\Gamma\)-orbit fixed. Similarly for \(\operatorname{Aut}([\Gamma])\), \(\operatorname{Out}([\Gamma])\).

2° We denote by \(\operatorname{Aut}_0(A \subset M)\) the subgroup of \(\operatorname{Aut}(A \subset M)\) consisting of all automorphisms of \(M\) that act trivially on \(A\), and by \(\operatorname{Out}_0(A \subset M)\) its quotient by the (normal) subgroup of inner such automorphisms, i.e., \(\operatorname{Int}_0(A \subset M) = \{\operatorname{Ad}(v) \mid v \in \mathcal{U}(A)\}\). As noticed in ([S] and Sec. 1.2-1.5 in [P6]), one has a natural identification between \(\operatorname{Aut}_0(A \subset M)\) and the group \(Z^1([\Gamma])\) of \(\mathcal{U}(A)\)-valued 1-cocycles for \([\Gamma] \actson A\), where \([\Gamma] = \mathcal{N}_M(A)/\mathcal{U}(A)\). Also, via this identification, \(\operatorname{Int}_0(A \subset M)\) corresponds to the group of coboundary 1-cocycles \(B^1([\Gamma])\), and thus \(\operatorname{Out}_0(A \subset M)\) corresponds to the first cohomology group \(H^1([\Gamma]) = \overline{Z^1([\Gamma])/B^1([\Gamma])}\), which is by definition the \textit{first cohomology group} \(H^1(A \subset M)\) of the Cartan inclusion \(A \subset M\).

As in (1.1 of [P6]), in the case \(\Gamma \actson X\) is a free pmp action and \(A = L^\infty(X) \subset L^\infty(X) \times_\sigma \Gamma = M\), then we also denote \(Z^1(\sigma) = Z^1(A \subset M), B^1(\sigma) = B^1(A \subset M), H^1(\sigma) = Z^1(\sigma)/B^1(\sigma) = H^1(A \subset M)\). Same in the case \(\Gamma \actson (A, \tau)\) is a free action on an abelian von Neumann algebra which is not necessarily separable. Note that in this case, an element \(c \in Z^1(\sigma)\) corresponds to a map \(c : \Gamma \to \mathcal{U}(A)\) satisfying \(c_g^\sigma g(c_h) = c_{gh}, \forall g, h \in \Gamma\) and the topology on \(Z^1(\sigma)\) inherited from \(\operatorname{Aut}(A \subset M)\) corresponds to point convergence in \(\|\|_2\): a net \(c_i \in Z^1(\sigma)\) converges to \(c \in Z^1(\sigma)\).
if \( \lim_i \|c_{i,g} - c_g\|_2 = 0, \forall g \in \Gamma. \)

Recall from ([Sc1], or Sec. 1.5 in [P6]) that if \( A \subset M \) is strongly ergodic (which in the case \( A \subset M = A \rtimes_{\sigma} \Gamma \) amounts to \( \sigma \) being strongly ergodic) then \( \text{Int}_0(A \subset M) \) is closed in \( \text{Aut}_0(A \subset M) \), equivalently \( B^1(A \subset M) \) is closed in \( Z^1(A \subset M) \). By [Sc1], if \( M \) is separable, then the converse is in fact true as well.

3° If \( A \subset M \) is a Cartan inclusion, with \( M \) a II\(_1\) factor, and \( t > 0 \), then its \( t\)-amplification \((A \subset M)^t\) is defined as the (isomorphism class of the) Cartan inclusion \((A \otimes D_n)p \subset p(M \otimes M_{n \times n}(\mathbb{C}))p\), for some \( n \geq t \), where \( p \in A \otimes D_n \) is a projection of (normalized) trace \( \tau(p) = t/n \), with \( D_n \subset M_{n \times n}(\mathbb{C}) \) the diagonal subalgebra. The fundamental group \( \mathcal{F}(A \subset M) \) of \( A \subset M \) is the set of all \( t > 0 \) with the property that \((A \subset M)^t \cong (A \subset M).\) Since \((A \subset M)^t)^s = (A \subset M)^{ts},\) we see that \( \mathcal{F}(A \subset M) \) is a multiplicative subgroup of \( \mathbb{R}^*_+ = (0, \infty) \).

The definition of \( t\)-amplification of a Cartan inclusion \( A \subset M \) actually makes sense whenever \( M \) is of type II\(_1\), but not necessarily a factor, by taking \((A \subset M)^t\) to be the isomorphism class of \((A \otimes D_n)p \subset p(M \otimes M_{n \times n}(\mathbb{C}))p\), for some \( n \geq t \), with \( p \in \mathcal{P}(A \otimes D_n) \) a projection of central trace equal to \((t/n)1_M\) (we leave it as an exercise to check that this doesn’t depend on \( n \geq t \), nor on the choice of \( p \), and that in this more general case we still have \(( (A \subset M)^t )^s = (A \subset M)^{st} \). Then one defines the fundamental group \( \mathcal{F}(A \subset M) \) the same way.

One should note that, with the above definition of amplification, \( \text{Out}((A \subset M)^t) \) naturally identifies with \( \text{Out}(A \subset M) \) and \( \text{Out}_0((A \subset M)^t) = H^1((A \subset M)^t) \) with \( \text{Out}_0(A \subset M) = H^1(A \subset M) \). In particular, when \( A \subset M \) comes from an ergodic pmp action of a countable group \( \Gamma \rhd X \), these invariants are stable orbit equivalence invariants of \( \Gamma \rhd X \).

At the same time, by Gaboriau’s results in [G2], the \( L^2\)-Betti numbers of \( A \subset M \) satisfy the scaling formula \( \beta_n^{(2)}((A \subset M)^t) = \beta_n^{(2)}(A \subset M)/t \) and thus, if one of the \( L^2\)-Betti numbers is non-zero and \( \neq \infty \), then no amplification by \( t \neq 1 \) of \( A \subset M \) can be isomorphic to \((A \subset M)\), i.e. \( \mathcal{F}(A \subset M) = \{1\} \).

3. Approximate Conjugacy

Recall that two pmp group actions \( \Gamma \rhd X, \Lambda \rhd Y \) are conjugate (or isomorphic) if there exist isomorphisms \( \Delta : (X, \mu) \cong (Y, \nu), \delta : \Gamma \to \Lambda, \) such that \( \Delta(\sigma_a(t)) = \rho_{\delta(g)}(\Delta(t)), \forall a \in X, \) and \( \forall g \in \Gamma. \) Note that this is equivalent to the existence of an integral preserving isomorphism \( \Delta : L^\infty(X) \cong L^\infty(Y) \) such that for any \( a \in L^\infty(X) \) we have \( \Delta(\sigma_a(t)) = \rho_{\delta(g)}(\Delta(a)) \). Let us also recall the weak version of this notion proposed by Kechris in [Ke], which merely requires that each one of the actions can be “simulated” inside the other, in moments:

3.1. Definition [Ke]. A pmp action \( \Gamma \rhd X \) is weakly embeddable into a pmp
action $\Lambda \curvearrowright^\rho Y$ with respect to some isomorphism $\delta : \Gamma \simeq \Lambda$, if for any finite $F \subset \Gamma$, any $p_1, \ldots, p_n \in \mathcal{P}(L^\infty(X))$, any $K \geq 1$ and any $\varepsilon > 0$, there exist $q_1, \ldots, q_n \in \mathcal{P}(L^\infty(Y))$ such that for any $1 \leq k \leq K$, any choice of elements $g_1, \ldots, g_k \in F$ and $1 \leq n_1, \ldots, n_k \leq n$, we have

\begin{equation}
|\tau(\Pi_{i=1}^k \sigma_{g_i}(p_{n_i})) - \tau(\Pi_{i=1}^k \rho_{\delta(g_i)}(q_{n_i}))| < \varepsilon
\end{equation}

The two actions $\Gamma \curvearrowright^\sigma X$, $\Lambda \curvearrowright^\rho Y$ are weakly conjugate if each one of them can be embedded into the other, with respect to the same group isomorphism.

Note that in fact the original definition of weak conjugacy in [Ke] only requires condition 3.1 to be satisfied for moments of degree $\leq 2$, i.e. for $K = 1, 2$, but by (Proposition 10.1 in [Ke]), this is sufficient for it to hold true for all $K$. We have chosen to formulate weak conjugacy in the form 3.1, as an “approximation in moments” (simulation) condition, because it is closer to the spirit of our paper. This formulation has also the advantage of translating into equivalent formulations in ultrapower framework, as follows (cf. [CKT]):

3.2. Proposition. Let $\Gamma \curvearrowright^\sigma X$, $\Lambda \curvearrowright^\rho Y$ be pmp actions of countable groups. Denote $A = L^\infty(X), B = L^\infty(Y)$. The following properties are equivalent:

(a) The actions $\sigma, \rho$ are weakly conjugate.

(b) There exists a free ultrafilter $\omega$ on $\mathbb{N}$, a $\Gamma$-invariant subalgebra $B_0 \subset A_\omega$, $\Lambda$-invariant subalgebra $A_0 \subset B_\omega$ (of the corresponding ultrapower of $\sigma, \rho$), such that $\sigma$ is isomorphic to $\Lambda \curvearrowright A_0$ and $\rho$ to $\Gamma \curvearrowright B_0$, with respect to the same identification $\Gamma \simeq \Lambda$.

(c) Property (b) holds true for any free ultrafilter $\omega$ on $\mathbb{N}$.

Proof. Condition (b) is clearly just a reformulation of 3.1 above, and it does not depend on $\omega$, so (a), (b), (c) are all equivalent. □

Let us point out that results from [AW], [T], combined with results in [B1], [PS], [P6], give plenty of examples of non-conjugate actions that are weakly conjugate.

3.3. Proposition ([AW], [T]). Let $\Gamma$ be a countable group.

1° Any two free ergodic quotients of Bernoulli $\Gamma$-actions are weakly conjugate. More generally, if $\{H_j^i\}_j, \{H_k^j\}_k \subset \Gamma$ are countable families of amenable subgroups and for each $i = 1, 2$, $\Gamma \curvearrowright^\sigma_i X_i$, is a free ergodic $\Gamma$-action which can be realized as the quotient of a generalized Bernoulli action $\Gamma \curvearrowright \Pi_j[0, 1]^{\Gamma/H^i_j}$, then $\sigma_1, \sigma_2$ are weakly conjugate.

2° If $\{H_j\}_j \subset \Gamma$ is a family of amenable subgroups, then given any free ergodic pmp action $\Gamma \curvearrowright^\sigma X$ and any quotient $\rho$ of a generalized Bernoulli action $\Gamma \curvearrowright \Pi_j[0, 1]^{\Gamma/H_j}$, the diagonal $\Gamma$-action $\sigma \times \rho$ is weakly conjugate to $\sigma$. 
Proof. The case of (quotients of) Bernoulli actions is a result in [AW], while the more general case of generalized Bernoulli actions and part \(2^\circ\) are results from [T]. A slightly more general result, proved directly in ultrapower framework, can be found in (5.1 of [P11])

\(\square\)

Note that if a group \(\Gamma\) is sofic, then by results of Bowen in [B1], Bernoulli \(\Gamma\)-actions with different entropy base give non-conjugate actions, while by 3.3 above, they are all weakly conjugate. Also, for certain classes of groups \(\Gamma\) (e.g. \(\Gamma\) having property \(T\), or \(\Gamma\) a product of a non-amenable and an infinite group), then by [PS], [P6] there are many free quotients of generalized Bernoulli \(\Gamma\)-actions that are not conjugate, while by 3.3 they are all weakly conjugate.

We'll now consider two additional notions of equivalence for group actions, situated between conjugacy and weak conjugacy. The first one requires straight conjugacy of the ultrapower actions.

3.4. Definition. Let \(\omega\) be a free ultrafilter on \(\mathbb{N}\). Let \(\Gamma \curvearrowright \sigma X\), \(\Lambda \curvearrowright \rho Y\) be pmp actions of countable groups and denote \(A = L^\infty(X)\), \(B = L^\infty(Y)\). The two actions are \(\omega\)-conjugate if there exists a trace preserving isomorphism \(\theta : A^\omega \simeq B^\omega\) that intertwines the ultrapower actions \(\sigma^\omega, \rho^\omega\), up to some group isomorphism \(\delta : \Gamma \to \Lambda\).

While such \(\omega\)-conjugacy is clearly an equivalence relation between group actions, it is not clear whether or not it depends on the chosen ultrafilter \(\omega\). However, if we impose the conjugacy between \(\sigma^\omega\) and \(\rho^\omega\) to be of ultraproduct form \(\theta \in (\theta_n)_n\), with \(\theta_n : A \simeq B\), \(\forall n\), then the corresponding relation no longer depends on \(\omega\) and can in fact be described “locally”, without using the ultrapowers of the actions.

From this point on, it may be useful to recall from Definition 2.1 that if \(\theta_1, \theta_2 \in \text{Aut}(X, \mu)\), then \(\|\theta_1 - \theta_2\|_2 = (2 - 2\tau(p_{\theta_1, \theta_2}^{-1}))^{1/2}\). More generally, recalling that if \(\phi\) is a local isomorphism on \((X, \mu)\) then \(p_\phi\) denotes the projection in \(L^\infty(X)\) on which it acts as the identity, then for local isomorphisms \(\phi_1, \phi_2\) we have denoted \(\|\phi_1 - \phi_2\|_2 = (\tau(p_{\phi_1, \phi_1}^{-1}) + \tau(p_{\phi_2, \phi_2}^{-1}) - 2\tau(p_{\phi_1, \phi_2}^{-1}))^{1/2}\). If \(G\) is a full pseudogroup on \((X, \mu)\) containing \(\phi_i\) (for instance \(\{\phi_i\}_i\) ), then this norm corresponds to the Hilbert norm \(\|u_{\phi_1} - u_{\phi_2}\|_2\) given by the canonical trace on \(L(G)\) that we have considered in 2.5, 2.6.

3.5. Proposition. Let \(\Gamma \curvearrowright \sigma X\), \(\Lambda \curvearrowright \rho Y\) be pmp actions of countable groups. Denote \(A = L^\infty(X), B = L^\infty(Y)\). The following properties are equivalent:

(a) There exists \(\delta : \Gamma \simeq \Lambda\) such that for any finite set \(F \subset \Gamma\) and any \(\varepsilon > 0\), there exists an isomorphism \(\theta : (X, \mu) \simeq (Y, \nu)\) satisfying \(\|\theta \sigma_g \theta^{-1} - \rho_{\delta(g)}\|_2 \leq \varepsilon, \ \forall g \in F\).

\(^1\)Note that the papers [AW], [T] are unfortunately not cited in [P11], as at the time of writing this paper the second named author was unaware of this prior work.
(b) There exists $\delta : \Gamma \simeq \Lambda$ and a sequence of isomorphisms $\theta_n : (X, \mu) \simeq (Y, \nu)$ such that $\lim_n \|\theta_n \sigma_g \theta_n^{-1} - \rho_{\delta(g)}\|_2 = 0$, $\forall g \in \Gamma$.

(c) Given any free ultrafilter $\omega$ on $\mathbb{N}$, there exist a group isomorphism $\delta : \Gamma \rightarrow \Lambda$ and a trace preserving isomorphism $\theta : A^\omega \simeq B^\omega$, of ultra product form $\theta = (\theta_n)_{n=1}^\infty$, $\forall n$, implementing an isomorphism of the ultrapower actions $\Gamma \rtimes \sigma^\omega A^\omega$, $\Gamma \rtimes \rho^\omega B^\omega$, i.e., $\theta \sigma^\omega_g \theta^{-1} = \rho^\omega_g \delta(g)$, $\forall g \in \Gamma$.

(d) Property (c) holds true for some free ultrafilter $\omega$ on $\mathbb{N}$.

Proof. Clearly $(a) \Leftrightarrow (b) \Rightarrow (c) \Rightarrow (d)$. If (d) holds and $\theta = (\theta_n)_{n=1}^\infty : A^\omega \simeq B^\omega$ is the isomorphism satisfying $\theta \sigma^\omega_g \theta^{-1} = \rho^\omega_g \delta(g)$, $\forall g$, then

\[(3.5.1) \lim_{n \to \omega} \sup \{ \| \theta_n (\sigma_g (a)) - \rho_{\delta(g)} (\theta_n (a)) \|_2 \mid a \in (A)_1 \} = 0, \forall g \in \Gamma \]

Let now $F \subset \Gamma$ be a finite set and $\varepsilon > 0$. By (3.5.1), there exists a neighborhood $V$ of $\omega \in \mathbb{N}$ such that, when we view $V$ as a subset of $\mathbb{N}$, for any $m \in V$ we have:

\[(3.5.2) \sup \{ \| \theta_m (\sigma_g (a)) - \rho_{\delta(g)} (\theta_m (a)) \|_2 \mid a \in (A)_1 \} \leq \varepsilon, \forall g \in F \]

showing that the actions $\sigma, \rho$ satisfy (a). \hfill \Box

3.6. Definition. Two pmp actions of countable groups $\Gamma \rtimes \sigma X$, $\Lambda \rtimes \rho Y$ are \textit{approximately conjugate} (app-conjugate) if any of the equivalent conditions in 3.5 holds true.

Note that we obviously have the implications “conjugacy $\Rightarrow$ approximate conjugacy $\Rightarrow$ $\omega$-conjugacy $\Rightarrow$ weak conjugacy”. It is not clear whether there exist $\omega$-conjugate actions that are not app-conjugate. It would also be interesting to decide whether the property of being $\omega$-conjugate for two group actions is independent of $\omega$.

If $\Gamma$ is a countable amenable group, then all free ergodic pmp $\Gamma$-actions are app-equivalent. This can be easily deduced from the Ornstein-Weiss’ hyperfinite approximation of actions of amenable groups (or directly from the Rohlin lemma in [OW1]). It has in fact already been pointed out in (5.2 of [P11]). Indeed, if $\Gamma \rtimes \sigma X$, $\Gamma \rtimes \rho Y$ are two such actions, and $A = L^\infty(X)$, $B = L^\infty(Y)$, then by (5.2 in [P11]) there exists $v \in \mathcal{N}_M(A^\omega)$ that intertwines the actions $\Gamma \rtimes \sigma^\omega A^\omega$, $\Gamma \rtimes \rho^\omega B^\omega = A^\omega$. But $v$ can be represented as a sequence $(v_n)_n$ with $v_n \in \mathcal{N}_M(A)$, whence the automorphism $\text{Ad}(v)$ of $A^\omega$ is of ultraproduct type $(\text{Ad}v_n)_n$. 

From the app-equivalence of $\Gamma$-actions in the case $\Gamma$ is amenable, we will now derive that if a group $\Gamma$ has an infinite amenable quotient, then it has many non-conjugate actions that are approximately conjugate. Note that the class of such groups includes the free groups $\Gamma = \mathbb{F}_n, 2 \leq n \leq \infty$, product groups $\Gamma = H \times K$, with $H$ infinite amenable and $K$ arbitrary, and groups of the form $\Gamma = \mathbb{Z}^2 \rtimes \Gamma_0$, with $\Gamma_0 \subset SL(2, \mathbb{Z})$.

3.7. Proposition. Let $\Gamma$ be a countable group which has an infinite amenable quotient $\pi : \Gamma \to H$. If $H \curvearrowright^\rho_i Y_i, i \in I$, are free ergodic pmp actions and $\Gamma \curvearrowright^\rho Y$ is a weak mixing pmp action which is free on $\ker \pi$, then the actions $\sigma_i$ of $\Gamma$ on $X_i = Y_i \times Y$ defined by $\sigma_i(g)(t, s) = (\rho_i(\pi(g))(t), \rho_i(g)(s))$ are free, ergodic and mutually app-conjugate (thus $\omega$-conjugate, $\forall \omega$, and weakly-conjugate as well). If in addition $\rho_i, i \in I$, are Bernoulli $H$-actions with distinct entropy and $\rho$ is mixing on $\ker \pi$, then $\sigma_i, i \in I$, are mutually non-conjugate.

Proof. Since obviously the (diagonal) product of actions behaves well to app-conjugacy, the first part follows from the arguments above.

If one takes two actions $\sigma_i, \sigma_j$ as above and $\delta : \Gamma \simeq \Gamma$ is an automorphism of $\Gamma$ such that $\sigma_i$ is conjugate with $\sigma_j$ with respect to $\delta$, and $\Delta : X_i \simeq X_j$ implements this conjugacy, then this is the same as $\sigma_i, \sigma_j \circ \delta$ being intertwined by $\Delta$.

Now note that if we denote $K = \ker \pi$, then for any automorphism $\delta'$ of $\Gamma$, the quotient of $\Gamma$ by $\delta'(K) \cap K$ is still amenable. In particular, $K_0 = \delta^{-1}(K) \cap K$ is infinite, with $K_0 \subset K, \delta(K_0) \subset K$. Since $\Delta \sigma_i(K_0) \Delta^{-1} = \sigma_j(\delta(K_0))$, the fixed point algebra $(L^\infty(X_i))^\sigma_i(K_0) = B_i$ is taken by $\Delta$ onto the fixed point algebra $(L^\infty(X_j))^\sigma_j(\delta(K_0)) =: B_j$.

Since $K_0$ is contained in $K$ and is infinite, it follows that $\sigma_i(K_0)$ is mixing on $L^\infty(Y) = 1 \otimes L^\infty(Y) \subset L^\infty(X_i)$ and acts trivially on $L^\infty(Y_i) = L^\infty(Y_i) \otimes 1$. Thus, $B_i = L^\infty(Y_i)$. In exactly the same way we deduce that $B_j = L^\infty(Y_j)$.

Thus, $\Delta(L^\infty(Y_i)) = L^\infty(Y_j)$. Since $\Delta$ intertwines $\sigma_i, \sigma_j \circ \delta$, this also implies that $\delta(\ker \pi) = \ker \pi$ and that $\Delta$ actually implements a conjugacy of the Bernoulli $H$-actions $\rho_i, \rho_j$. Since $\sigma_i, i \in I$, have distinct Kolmogorov-Sinai type entropies for different $i$'s (as defined for actions of amenable groups by Ornstein and Weiss in [OW2]), this shows that if $\sigma_i, \sigma_j$ are conjugate, then $i = j$. \hfill $\square$

As we mentioned in Proposition 3.3, results in [AW] show that all free quotients of Bernoulli actions of a group $\Gamma$ are weakly conjugate. The previous result derives weak conjugacy of certain actions from their app-conjugacy. We note below that the co-induction of two weak-conjugate actions gives rise to weak conjugacy as well.

To this end, let us first recall the construction of the co-induction of group actions (cf. [Lu]). Thus, let $H \subset \Gamma$ be an inclusion of groups and $H \curvearrowright^\sigma (A, \tau)$ a trace
preserving action on an abelian von Neumann algebra with a faithful normal trace \( \tau \) (but with \( A \) not necessarily separable). Let \( S \subset \Gamma \) be a set of representatives for \( \Gamma/H \) and define \( \pi: \Gamma \to H \) by \( \pi(sh) = h, \forall s \in S, h \in H \). The map \( c: \Gamma \times (\Gamma/H) \to H \) defined by \( c(g,sH) = \pi(gs)\pi(s)^{-1}, s \in S, g \in \Gamma, \) is then a 1-cocycle for the action \( \Gamma \curvearrowright \Gamma/H \), with different choices of \( \pi \) giving equivalent cocycles.

Let \( (\tilde{A}, \tau) = (A, \tau) \otimes_{\Gamma/H} \). We define on it a trace preserving \( \Gamma \)-action \( \tilde{\sigma} \) as follows.

For each \( a \in A \) and \( s \in S \) denote \( \tilde{a}_{sH} = \cdots 1 \otimes a \otimes 1 \cdots \in A^{\otimes_{\Gamma/H}} \), where \( a \) appears on the \( sH \) position. If \( g \in \Gamma \) we then let \( \tilde{\sigma}_g(\tilde{a}_{sH}) = \tilde{b}_{gsH} \), where \( b = \sigma_{c(g, sH)}(a) \). The cocycle relation implies that \( \tilde{\sigma}_g \) is a 1-cocycle for the action \( \Gamma \curvearrowright \tilde{A} \), with different choices of \( \tilde{\sigma} \) giving equivalent cocycles.

3.8. Proposition. Let \( \Gamma \) be a countable group with an infinite subgroup \( H \subset \Gamma \). Let \( H \curvearrowright^\sigma X_i \) be a free ergodic pmp action and denote \( \Gamma \curvearrowright^{\tilde{\sigma}_i} \tilde{X}_i \) its corresponding co-induction to \( \Gamma \), \( i = 1, 2 \). If \( H \curvearrowright^\sigma X_1 \) is weakly-conjugate to \( H \curvearrowright^\sigma X_2 \), then \( \Gamma \curvearrowright^{\tilde{\sigma}_1} \tilde{X}_1 \) is weakly-conjugate to \( \Gamma \curvearrowright^{\tilde{\sigma}_2} \tilde{X}_2 \).

Proof. This is trivial from the above observations, by taking into account that if \( H \curvearrowright^{\rho_0} A_0 \) is an action, then the co-induction from \( H \) to \( \Gamma \) of its ultrapower \( \rho_0^\omega \) identifies with the restriction to a \( \Gamma \)-invariant subalgebra of the ultrapower of \( \Gamma \curvearrowright^{\rho_0} A_0 \).

Note that 3.8 gives an alternative proof of the weak-equivalence of Bernoulli \( \Gamma \) actions ([AW]) for all groups \( \Gamma \) that contain an infinite amenable subgroup (so a particular case of Proposition 3.3). More generally we have:

3.9. Corollary. If a group \( \Gamma \) has an infinite amenable subgroup \( H \subset \Gamma \), then all pmp \( \Gamma \)-actions that are co-induced from free ergodic pmp \( H \)-actions, are weakly-equivalent.

Proof. We have already noticed that any two free ergodic pmp \( H \)-actions are app-conjugate. By Proposition 3.8 it follows that the co-induction from \( H \) to \( \Gamma \) of these \( H \)-actions are weakly-conjugate as well.

While Proposition 3.7 provides a rather large class of groups \( \Gamma \) that have many non-conjugate free ergodic pmp actions which are all app-conjugate, these groups \( \Gamma \) need to have infinite amenable quotients. Thus, they cannot have property (T). In fact, we notice below that for property (T) groups app-conjugacy is the same as
The proof reproduces arguments in ([P7], [P8], proof of 6.1 in [PV]; see also 14.2 in [Ke]), but we have included a full proof for completeness.

**3.10. Lemma.** Let \( \Gamma \) be a property (T) group and \( F \subset \Gamma \) a finite set of generators. For any \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that if \( \Gamma \simeq^\sigma (X, \mu) \), \( \Gamma \simeq^\rho (Y, \nu) \) are free ergodic pmp \( \Gamma \)-actions and \( \theta : (X, \mu) \simeq (Y, \nu) \) satisfies \( \|\theta \sigma - \rho\|_2 \leq \delta \), \( \forall g \in F \), then there exists \( \theta' : (X, \mu) \simeq (Y, \nu) \) such that \( \theta' \sigma_g = \rho_g \theta' \), \( \forall g \in \Gamma \), and \( \|\theta' - \theta\|_2 \leq \varepsilon \).

**Proof.** By property (T), we can choose \( \delta > 0 \) such that, if \( \pi \) is a unitary representation of \( \Gamma \) on a Hilbert space \( K \) and \( \xi \in K \), \( \|\xi\| = 1 \), satisfies \( \|\pi_g(\xi) - \xi\| \leq \varepsilon \), \( \forall g \in F \), then the projection \( \xi_0 \) of \( \xi \) onto the space of vectors fixed by \( \Gamma \) satisfies \( \|\xi - \xi_0\| \leq \varepsilon^2/4 \). We will prove that this \( \delta \) checks the required condition.

Let \( A = L^\infty(X) \) with \( \tau \) the trace on \( A \) implemented by \( \mu \). Denote by \( G \) the full group generated in \( \text{Aut}(A, \tau) \) by \( \tau^{-1} \rho \theta \) and \( \sigma \Gamma \). Denote \( A \subset L(\mathcal{G}) = L \) the Cartan inclusion of this full group and for each \( \phi \in G \), denote by \( u_\phi \) the corresponding canonical unitary.

Let \( \langle M, e_A \rangle \) denote as usual the basic construction von Neumann algebra for \( A \subset M \), generated in \( B(L^2(M)) \) by \( M \) (viewed as left multiplication operators \( L_x \) by elements \( x \in M \)) and by the orthogonal projection \( e_A \), of \( L^2(M) \) onto \( L^2(A) \). Thus, \( \langle M, e_A \rangle \) is the weak closure of sums of elements of the form \( xe_{Ay} \), with \( x, y \in M \), and it has a canonical faithful normal semifinite trace \( Tr \) defined by \( Tr(xe_{Ay}) = \tau(xy) \). Also, \( xe_{Ay} \mapsto xy \) gives an \( M \)-bimodular operator valued normal semifinite faithful weight \( \Phi \) of \( \langle M, e_A \rangle \) onto \( M \) satisfying \( \tau(\Phi(xe_{Ay})) = Tr(xe_{Ay}) \).

The algebra \( \langle M, e_A \rangle \) can also be described as the commutant of \( JAJ \) in \( B(L^2M) \), where \( J = J_M \) is the canonical involution on \( L^2(M) \). It follows that if \( u, v \) are unitaries in \( M \) that normalize \( A \), then \( \text{Ad}(u)\text{Ad}(JvJ) \) normalizes \( \langle M, e_A \rangle \), acting on elements of the form \( xe_{Ay} \) by \( \text{Ad}(u)\text{Ad}(JvJ)(xe_{Ay}) = u xv^*e_Avyu^* \) and thus preserving the trace \( Tr \).

Moreover, since both \( \text{Ad}(u) \) and \( \text{Ad}(JvJ) \) leave invariant \( A \) and \( JAJ \), their product leaves invariant \( \tilde{A} = A \vee JAJ = \vee_u u(Ae_A)u^* \). Finally, note that for any \( u, v \) as above, \( \text{Ad}(u)\text{Ad}(JvJ) \) leaves \( M \) invariant, acting on it as \( \text{Ad}(u) \), and that \( \Phi(\text{Ad}(u)\text{Ad}(JvJ)(X)) = u\Phi(X)u^* \), \( \forall X \in \text{sp}Me_AM \), \( \Phi(\tilde{A}) = A \).

Denote \( u_g = u_{g^0} \) and \( v_g = u_{g^{-1}g^0} \), \( g \in \Gamma \). Thus, we can define a unitary representation \( \pi \) of \( \Gamma \) on the Hilbert space \( \mathcal{H} = L^2(\langle M, e_A \rangle, Tr) \), by \( \pi_g = \text{Ad}(u_g)\text{Ad}(Jv_gJ) \). Since \( \pi_g(e_A) = u_gv_g^*e_Av_gu_g^* \) and \( \|\sigma_g - \theta^{-1}\rho_g\theta\|_2 = \|u_g - v_g\|_2 \), by the definition of \( Tr \) it follows that

\[
\|\pi_g(e_A) - e_A\|_{2,Tr} = \|u_g - v_g\|_2 \leq \delta
\]

Thus, the element \( \xi_0 \) of minimal norm \( \|\cdot\|_{2,Tr} \) in the weakly compact convex set obtained by taking weak closure of the convex set \( \text{co}\pi_\Gamma(e_A) = \text{co}\{u_gv_g^*e_Av_gu_g^* \mid g \in \Gamma\} \), check the required condition.
\[ \langle \text{Corollary.} \rangle \]

Let \( \Gamma \) be a property (T) group.

1° Two free ergodic pmp \( \Gamma \)-actions are app-conjugate if and only if they are conjugate.

2° Let \( \Gamma \subset Aut(X,\mu) \) be a pmp \( \Gamma \)-action. If \( (\theta_n)_n \subset Aut(X,\mu) \) are so that \( \lim_n \| [\theta_n,\sigma_g] \|_2 = 0, \forall g \in \Gamma \), then there exist \( (\theta'_n)_n \subset Aut(X,\mu) \) such that \( [\theta'_n,\sigma_g] = 0, \forall n, g \), and \( \lim_n \| \theta'_n - \theta_n \|_2 = 0 \).

\[ \text{Proof.} \]

1° Let \( F \subset \Gamma \) be a finite set of generators. Approximate conjugacy of two free ergodic pmp actions \( \Gamma \subset Aut(X,\mu) \) means there exists a sequence of measure preserving isomorphisms \( \theta_n : X \simeq Y \) such that if we denote \( \delta_n = \max_{g \in F} \| \theta_n \sigma_g \theta_n^{-1} - \rho_g \|_2 \), then \( \delta_n \to 0 \). By Lemma 3.10, there exist measure preserving isomorphisms \( \theta'_n : X \simeq Y \) such that \( \lim_n \| \theta'_n - \theta_n \|_2 = 0 \) and \( \theta'_n \sigma_g = \rho_g \theta'_n, \forall g \in \Gamma \).

2° This is just Lemma 3.10 applied in the case \( \rho_g = \sigma_g \) and \( \theta = \theta_n \in Aut(X,\mu) \), as \( n \to \infty \).

\[ \text{3.12. Remark.} \]

1° Note that the spectral behavior of \( \Gamma \subset L^2(A^\omega) \) is an \( \omega \)-conjugacy invariant for \( \Gamma \subset (A,\tau) \). In particular, strong ergodicity of \( \Gamma \subset A \) and weak mixingness of \( \Gamma \subset A^\omega \) are \( \omega \)-conjugacy invariants. Also, the property of having a \( \Gamma \)-invariant subalgebra \( A_0 \subset A^\omega \) satisfying certain properties, like relative property (T) in the sense of (Sec. 4 in [P5]), is of course \( \omega \)-conjugacy invariant.

2° Another app-conjugacy invariant for an action \( \Gamma \subset (A,\tau) \) is its app-\( \omega \)-centralizer group \( Aut_{\text{app,}\omega}(A) \), defined as the quotient between the group of sequences of automorphisms \( (\theta_n)_n \) of \( (A,\tau) \) that \( \omega \)-asymptotically commute with
\[ \sigma_g, \forall g \in \Gamma, \text{ i.e., } \lim_\omega \| [\sigma_g, \theta_n] \|_2 = 0, \forall g, \text{ by the subgroup of sequences of automorphisms } (\theta_n)_n \text{ satisfying } \lim_\omega \| \theta_n - id_A \|_2 = 0. \] This group coincides with the group of automorphisms \( \theta \) of \( A^\omega \) that are of ultraproduct form, \( \theta = (\theta_n)_n \), and commute with \( \sigma^\omega \).

Note that this group can be identified with a subgroup of the group of automorphisms of \( M(\omega) \) that leave \( A^\omega \) invariant and are of ultraproduct form when restricted to \( A^\omega \). As such, it has a topology inherited from \( \text{Aut}(A^\omega \subset M(\omega)) \).

But we also endow the group \( \text{Aut}_{\text{app},\omega}^\omega(A) \) with the topology given by the distance \( \| (\theta_n)_n - (\theta'_n)_n \|_2 := \lim_{n \to \omega} \| \theta_n - \theta'_n \|_2 \).

Note that \( \text{Aut}_{\text{app},\omega}^\omega(A) \) contains the group \( (\text{Aut}\sigma(A))^{\omega} \), of sequences of automorphisms \( (\theta_n)_n \) of \((A, \tau)\), with all \( \theta_n \) in the centralizer of \( \sigma \) in \( \text{Aut}(A) \), \( \text{Aut}\sigma(A) \).

Moreover, the restriction of the distance on \( \text{Aut}_{\text{app},\omega}^\omega(A) \) to this subgroup gives the discrete \( \{ 0, \sqrt{2} \} \)-valued distance.

An interesting result here would be to show that in certain cases we have \( \text{Aut}_{\text{app},\omega}^\omega(A) = (\text{Aut}\sigma(A))^{\omega} \). This amounts to proving that if \( \lim_\omega \| [\sigma_g, \theta_n] \|_2 = 0, \forall g \), then there exist \( \theta'_n \in \text{Aut}\sigma(A) \) such that \( \lim_\omega \| \theta_n - \theta'_n \|_2 = 0 \). Note that 3.11.2° shows that for a group \( \Gamma \) with the property (T), this is indeed the case.

3° We do not know whether \( \omega \)-conjugacy for property (T) groups is weaker than conjugacy. In fact, \( \omega \)-conjugacy of actions seems to be a rather esoteric, hard to understand concept, for which it seems unlikely that an entropy invariant can be developed and whose symmetry groups seem difficult to calculate (see also 5.6.2).

4° It is not known whether every non-amenable group \( \Gamma \) has at least two non weakly conjugate free ergodic pmp actions. But it has been noted in [AE] that strong ergodicity is a weak conjugacy invariant, and thus if \( \Gamma \) does not have property (T) then by ([Sc2], [CW]) it does have two non weakly conjugate free ergodic pmp actions (see 4.9 for a strengthening of this result). The general belief is that in fact any non-amenable group \( \Gamma \) has continuously many non weakly conjugate actions. Note that by [AE], this has been checked for many groups \( \Gamma \), including free groups \( \mathbb{F}_n, 2 \leq n \leq \infty \), and linear groups with the property (T). Since the app and \( \omega \)-conjugacies are stronger than weak-conjugacy, each one of these examples of non weak conjugate actions of a fixed group \( \Gamma \) gives examples of actions that are not app/\( \omega \)-conjugate either. Since for property (T) groups app-conjugacy is the same as conjugacy, it follows that any non-amenable group has at least two non app-conjugate actions. But we do not know how to prove the same for \( \omega \)-conjugacy, nor how to show that any non-amenable group has uncountably many non app/\( \omega \)-conjugate free ergodic pmp actions.
4. APPROXIMATE ORBIT EQUIVALENCE

The description of app-conjugacy of group actions in terms of their ultrapowers, as well as the notion of \( \omega \)-conjugacy, lead to two weak versions of the orbit equivalence of group actions (respectively of the equivalence of Cartan inclusions) that we discuss in this section. We also introduce a notion of weak orbit equivalence of group actions (engendered by weak conjugacy) and relate it with these other concepts.

4.1. Definition. Let \( \omega \) be a free ultrafilter on \( \mathbb{N} \). Two pmp actions \( \Gamma \curvearrowright X \), \( \Lambda \curvearrowright Y \) of countable groups \( \Gamma, \Lambda \) are called \( \omega \)-orbit equivalent (\( \omega \)-OE), if there exists an isomorphism \( \theta \) of Cartan inclusions \( (A^\omega \subset M(\omega)) \simeq (B^\omega \subset N(\omega)) \), where we have denoted \( A = L^\infty(X) \), \( B = L^\infty(Y) \), \( M = L(R_\sigma) \), \( N = L(R_\rho) \), \( M(\omega) = A^\omega \lor N_M(A) \subset M^\omega \), \( N(\omega) = B^\omega \lor N_N(B) \subset N^\omega \).

Note that by Proposition 2.6, if we denote by \([\sigma^\omega], [\rho^\omega]\) the full groups of ultrapower actions \( \Gamma \curvearrowright A^\omega \), \( \Lambda \curvearrowright B^\omega \) (or alternatively, of the the Cartan inclusions \( A^\omega \subset M(\omega), B^\omega \subset N(\omega) \)), then this condition is equivalent to the existence of an isomorphism \( \theta : A^\omega \simeq B^\omega \) intertwining their full groups, \( \theta([\sigma^\omega])\theta^{-1} = [\rho^\omega] \).

More generally, two Cartan inclusions \( A \subset M, B \subset N \) are \( \omega \)-equivalent if there exists an isomorphism \( \theta \) of the Cartan inclusions \( (A^\omega \subset M(\omega)) \simeq (B^\omega \subset N(\omega)) \).

Like for \( \omega \)-conjugacy, we do not know whether the \( \omega \)-OE property depends on the free ultrafilter \( \omega \). However, we will next show that the strengthening of this condition requiring that the isomorphism \( \theta : A^\omega \simeq B^\omega \) intertwining the full groups \([\sigma^\omega], [\rho^\omega]\), be of ultraproduct form \( \theta = (\theta_n)_n \), where \( \theta_n : A \simeq B, \forall n \), does not depend on \( \omega \). We will do this by showing that this condition is equivalent to a local condition which can be viewed as the OE-type equivalence relation for group actions that’s entailed by app-conjugacy, and which we will thus call approximate orbit equivalence.

4.2. Proposition. Let \( \Gamma \curvearrowright X, \Lambda \curvearrowright Y \) be pmp actions of countable groups. Denote \( L^\infty(X) = A \subset M = L(R_{\Gamma \curvearrowright X}), L^\infty(Y) = B \subset N = L(R_{\Lambda \curvearrowright Y}) \) the associated Cartan inclusions. The following properties are equivalent:

(a) There exists a free ultrafilter \( \omega \) on \( \mathbb{N} \) for which the following property holds: there exists an isomorphism \( \theta : (A^\omega \subset M(\omega)) \simeq (B^\omega \subset N(\omega)) \) whose restriction to \( A^\omega \) is of ultra product form, \( \theta = (\theta_n)_n \), with \( \theta_n : A \simeq B, \forall n \).

(b) The property (a) holds true for any free ultrafilter \( \omega \) on \( \mathbb{N} \).

(c) There exist maps \( \Gamma \ni \phi \mapsto t^\phi = \{t^\phi_\psi\}_{\psi \in \Lambda} \) and \( \Lambda \ni \psi \mapsto s^\psi = \{s^\psi_\phi\}_{\phi \in \Gamma} \), with \( \Sigma_\phi s^{\psi_\phi}_\phi = \Sigma_\psi t^{\phi_\psi}_\psi = 1, 0 \leq s^{\psi_\phi}_\phi \leq 1, \forall \phi, \psi \), such that for any \( E \subset \Gamma, F \subset \Lambda \) finite and any \( \varepsilon > 0 \), there exist an isomorphism \( \theta' : A \simeq B \) and mutually
orthogonal projections \( \{p_\phi^\psi\}_{\phi \in \Gamma} \subset A \), resp. \( \{q_\psi^\phi\}_{\psi \in \Lambda} \subset B \), so that \( \{\phi(p_\phi^\psi)\}_\phi \) (resp. \( \{\psi(q_\psi^\phi)\}_\psi \)) are mutually disjoint as well, \( \tau(p_\phi^\psi) \leq s_\phi^\psi \), \( \tau(q_\psi^\phi) \leq t_\psi^\phi \), and \( \|\theta'\phi_0\theta'^{-1} - \psi_0\|_{Bq_\psi^\phi} \leq \varepsilon \), \( \|\theta'^{-1}\psi_0\theta' - \phi_0\|_{Ap_\phi^\psi} \leq \varepsilon \), \( \forall \phi_0 \in E, \psi_0 \in F \).

(d) There exist a sequence of isomorphisms \( \theta_n : A \cong B \) and sequences of mutually orthogonal projections \( \{p_\phi^\psi\}_\phi \subset \mathcal{P}(A) \), for \( \psi \in \Lambda \), (respectively \( \{q_\phi^\psi\}_\phi \), for \( \phi \in \Gamma \)) with \( \{\phi(p_\phi^\psi)\}_\phi \) (resp. \( \{\psi(q_\phi^\psi)\}_\phi \)) mutually orthogonal, such that

\[
\lim_n \|\theta_n\phi\theta'^{-1}_n - \psi_0\|_{Bq_\psi^\phi} \leq 0, \forall \phi \in \Gamma
\]

\[
\lim_n \|\theta'^{-1}_n\psi\theta_n - \phi_0\|_{Ap_\phi^\psi} \leq 0, \forall \psi \in \Lambda
\]

\[
\Sigma_\phi \liminf_n \tau(p_\phi^\psi) = 1, \Sigma_\psi \liminf_n \tau(q_\phi^\psi) = 1, \forall \psi \in \Lambda, \phi \in \Gamma.
\]

**Proof.** Let \( \{u_\phi \mid \phi \in \Gamma\} \subset M = L([\Gamma]) = L(\mathcal{R}_\Gamma) \) (respectively \( \{v_\psi \mid \psi \in \Lambda\} \subset N = L([\Lambda]) = L(\mathcal{R}_\Lambda) \)) denote the canonical unitaries in \( M \) (resp. \( N \)), implementing the action \( \Gamma \triangleright A \) (resp. \( \Lambda \triangleright B \)).

Assume (a) holds true. Then for each \( \phi_0 \in \Gamma, \psi_0 \in \Lambda \), there exist partitions of 1 with projections \( \{Q_\phi^\psi\}_\phi \subset \mathcal{P}(B^\omega), \{P_\phi^\psi\}_\phi \subset \mathcal{P}(A^\omega) \), such that

(4.2.1) \( \theta(u_{\phi_0}) = (\Sigma_\psi v_\psi Q_\phi^\psi)b_{\phi_0}, \theta^{-1}(v_{\psi_0}) = (\Sigma_\phi u_\phi P_\phi^\psi)a_{\psi_0} \),

for some \( a_{\psi_0} \in \mathcal{U}(A^\omega), b_{\phi_0} \in \mathcal{U}(B^\omega) \).

Denote \( s_\phi^\psi = \tau(P_\phi^\psi), t_\psi^\phi = \tau(Q_\phi^\psi) \), then choose projections \( P_\phi^\psi \subset \mathcal{P}(A) \), \( Q_\psi^\phi \subset \mathcal{P}(B) \) so that \( P_\phi^\psi = (P_\phi^\psi)_m \) and \( Q_\psi^\phi = (Q_\psi^\phi)_m \). Moreover, by taking into account the properties that \( P_\phi^\psi, Q_\psi^\phi \) satisfy, it follows that we can make this choice such that for each \( m \) we have \( \tau(P_\phi^\psi) \leq s_\phi^\psi, \tau(Q_\psi^\phi) \leq t_\psi^\phi \) and \( \Sigma_\phi P_\phi^\psi \leq 1 \), \( \Sigma_\psi Q_\psi^\phi \leq 1 \), \( \Sigma_\phi u_\phi P_\phi^\psi u_\phi^* \leq 1 \), \( \Sigma_\psi v_\psi Q_\psi^\phi v_\psi^* \leq 1 \).

But then, if \( E \subset \Gamma, F \subset \Lambda \) are finite sets and \( \varepsilon > 0 \), there exists \( m \) “close to \( \omega \)” so that \( p_\phi^\psi = P_\phi^\psi \) and \( q_\psi^\phi = Q_\psi^\phi \) satisfy all the conditions in (c).

To see that (c) implies (d), choose sequences of finite subsets \( E_n \not
\Gamma, F_n \not\Lambda \) and for each \( n \) apply (c) to \( E_n, F_n, \varepsilon_n = 2^{-n} \) to get \( \{p_\phi^\psi\}_\phi \subset \mathcal{P}(A), \{q_\phi^\psi\}_\phi \subset \mathcal{P}(B) \) and \( \theta_n : A \cong B \). Then \( \theta_n : A \cong B \) clearly satisfy the conditions in (c) for all \( \phi_0 \in E_n, \psi_0 \in F_n \). Finally, assume (d) holds true and let \( \omega \) be a given (but arbitrary) free ultrafilter on \( \mathbb{N} \). Denote \( \theta' = (\theta_n)_n : A^\omega \cong B^\omega \). Let \( P_\phi^\psi = (p_\phi^\psi)_n \in \mathcal{P}(A^\omega) \),
4.3. Definition. Two pmp actions of countable groups $\Gamma \acts X$, $\Lambda \acts Y$ are approximately orbit equivalent (app-OE) if any of the equivalent conditions in 4.2 is satisfied.

4.4. Remark. We define app-equivalence of separable Cartan inclusions $A_i \subset M_i$, $i = 1, 2$, by viewing them as pairs $(G_i, v_i/\sim)$, (a full group and a 2-cocycle, cf. Proposition 2.6), then noticing that analogues of (a), (b), (d) in 4.2 above are equivalent, where (a), (b) are the same, and (d) adds the condition that $\theta_n$ asymptotically intertwine the 2-cocycles $v_1/\sim$, $v_2/\sim$, and asking that any of these equivalent conditions holds true. If $v_1 = 1 = v_2$, then the Cartan inclusions arise from pmp actions of countable groups $\Gamma_i \acts A_i$, and the app-equivalence of $A_i \subset M_i$, $i = 1, 2$, amounts to app-OE of these group actions, in the sense of 4.3.

4.5. Lemma. Let $A \subset M$, $B \subset N$ be Cartan inclusions with $M, N$ separable. Let $\mathcal{U} \subset \mathcal{N}_M(A)$ be a countable $\|\|_2$-dense subgroup, with $\mathcal{U}_0 = \mathcal{U} \cap A$ normal in $\mathcal{U}$ and dense in $\mathcal{U}(A)$, and let $v_0 = 1, v_1, \ldots \in q\mathcal{N}_N(B)$ be an orthonormal basis of $N$ over $B$. The following conditions are equivalent:

(a) There exists a non-degenerate Cartan embedding of $A \subset M$ into $B^\omega \subset N(\omega)$ for some free ultrafilter $\omega$ on $N$.

(b) There exists a non-degenerate Cartan embedding of $A \subset M$ into $B^\omega \subset N(\omega)$ for any free ultrafilter $\omega$ on $N$.

(c) The following property, which we denote $(A \subset M) \prec_w (B \subset N)$, holds true:

There exists a map $\mathcal{U} \ni u \mapsto t^u = \{t^u_l\}_{l \geq 0}$, with $\Sigma_l t^u_l = 1$, $0 \leq t^u_l \leq 1$, $\forall u, l$, such that for any $L \geq 1$, any $u_1, \ldots, u_{n} \in \mathcal{U}$, any $\varepsilon > 0$ and $K \geq 1$, there exist $u_{n+1}, \ldots, u_{m} \in \mathcal{U}$, $\{q^i_l\}_{l \geq 0} \in \mathcal{P}(B)$, with $\Sigma_l q^i_l$, $\Sigma_l v_i q^i_l v^*_i \leq 1$, $\tau(q^i_l) \leq t^u_l$, $\forall l \geq 0, 1 \leq i \leq m$, $q^0_l = 1$ whenever $u_i \in \mathcal{U}_0$, and $b_1, \ldots, b_m \in \mathcal{U}(B)$ such that if we denote $u'_i = (\Sigma_l v_i q^i_l) b_i$, $1 \leq i \leq m$, then $v_l \in _\varepsilon \Sigma_{i=1}^m u'_i B$, $\forall 0 \leq l \leq L$, and given any word $w$ of length $\leq K$ in $m$ letters (and their formal inverses), we have $|\tau(w(\{u'_i\}_i))| - \varepsilon$. 

\[
Q^\phi_\psi = (q^\phi_\psi)_{\omega} \in \mathcal{P}(B^\omega) \text{ and note that by the last condition in (d) we have }
\Sigma_\phi \tau(P^\phi_\psi) = \Sigma_\phi \lim_\omega \tau(p^\phi_\psi) = 1, \text{ and thus } \Sigma_\phi P^\phi_\psi = 1, \forall \psi_0. \text{ Similarly, } \Sigma_\psi Q^\phi_\psi = 1, \forall \phi_0. \text{ Moreover, by conditions at the beginning of (d) we also get } \Sigma_\phi \phi(P^\phi_\psi) = 1, \Sigma_\psi \psi(Q^\phi_\psi) = 1.
\]

If we now let $\theta(u_{\phi_0}) = \Sigma_\psi v_\psi Q^\phi_\psi$, $\forall \phi_0 \in \Gamma$, then it is easy to check that there exists a unique isomorphism $\theta : M(\omega) \simeq N(\omega)$ which on $A^\omega$ acts as $\theta'$ and on $\{u_\phi\}_\phi$ takes these assigned values. This shows that (d) implies (b) and thus finishes the proof. \qed
\[ \tau(w(\{u_i\}_i)) \leq \varepsilon. \]

**Proof.** (a) \( \implies \) (c) Denote by \( \theta \) the isomorphism of \( M \) into \( N(\omega) \) taking \( A \) into \( B^\omega \) and \( N_M(A) \) into \( N_{N(\omega)}(B^\omega) \).

Thus, for each \( u \in \mathcal{U} \) there exist \( \{q^u_l\}_{l \geq 0} \subset \mathcal{P}(B^\omega) \), \( b^u \in \mathcal{U}(B^\omega) \) such that:

(1) \[ \sum_l q^u_l = 1 = \sum_l v_i q^u_l v_i^*; \]

(2) \[ \theta(u) = (\sum_l v_i q^u_l) b^u, \forall u \in \mathcal{U}; \]

(3) \[ \{v_l\}_l \subset \text{sp} \theta(\mathcal{U})B^{\omega}. \]

Denote \( t^u_l = \tau(q^u_l) \) and represent each \( q^u_l \) as \( (q^u_l)_{k} \) with \( q^u_{l,k} \in \mathcal{P}(B) \), \( \tau(q^u_{l,k}) \leq t^u_l \), \( \sum_l q^u_{l,k} \leq 1 \), \( \sum_l v_i q^u_{l,k} v_i^* \leq 1 \), \( \forall u \in \mathcal{U} \), \( \forall k \geq 1 \). Also, let \( b^u = (b^u_k)_k \) with \( b^u_k \in \mathcal{U}(B) \).

Take now \( u_1, \ldots, u_n \in \mathcal{U}, L = 1 \) and \( \varepsilon > 0 \). Note first that by (3), there exists \( u_{n+1}, \ldots, u_m \in \mathcal{U} \) such that \( v_l \in \varepsilon/2 \sum_{i=1}^{m} \theta(u_i)B^\omega \), \( \forall l \leq L \). Thus, if \( K >> 1 \) is also given, then there exists some \( k \geq 1 \) “close to \( \omega^\omega \), such that if for each \( 1 \leq i \leq m \) we denote \( q^i_l = q^u_{l,k}, b_i = b^u_k \), then all conditions in 4.6.(c) are satisfied.

To see that (c) implies (b), choose first an enumeration \( \mathcal{U} = \{u_j\}_{j \geq 1} \) and for each \( n \) apply 4.6.(c) to \( u_1, \ldots, u_n, F = \{v_0, v_1, \ldots, v_n\} \) and \( \varepsilon = 2^{-n} \), to get some larger integer \( k_n \geq n \), projections \( \{q^j_{l,n} \in \mathcal{P}(B) | 1 \leq j \leq k_n, l \geq 0\} \), and \( b_{j,n} \in \mathcal{U}(B), 1 \leq j \leq k_n \), such that \( q^j_0 = 1 \) whenever \( u_j \in \mathcal{U}_0 \), \( \sum_{l \geq 0} q^j_{l,n}, \sum_l v_i q^j_{l,n} v_i^* \leq 1 \), \( \tau(q^j_{l,n}) \leq t^u_{l,n} \), and such that if we denote \( u'_{j,n} = (\sum_l v_i q^j_{l,n})b_{j,n} \) then any word \( w \) of length \( \leq 2^n \) in \( k_n \) letters satisfies the moments condition \( |\tau(w(\{u'_{j,n}\}_{1 \leq j \leq k_n})) - \tau(w(\{u_j\}_{1 \leq j \leq k_n}))| \leq 2^{-n}. \)

Denote \( q^j_{l,n} = (q^j_{l,n}) \in \mathcal{P}(B^\omega), b_j = (b_{j,n}) \in \mathcal{U}(B^\omega), j \geq 1, u'_j = (u'_{j,n})_{n} = (\sum_{l \geq 0} v_l q^j_{l,n})b_j \in N^\omega \). Note that, by the definitions, \( u'_j \) lie in the closure of \( \sum_{l \geq 0} v_l B^{\omega} \), i.e. in \( N(\omega) \). Moreover, the moments condition implies that the map \( \theta : \mathcal{U} \rightarrow N(\omega) \) defined by \( \theta(u_j) = u'_j, j \geq 1 \), extends to a trace-preserving isomorphism of \( M \) into \( N(\omega) \) with \( \theta(N_M(A)) \subset N(\omega)(B^\omega) \). Also, the condition \( q^j_0 = 1 \) implies that \( \theta(\mathcal{U}_0) \subset B^\omega \). Thus, \( \theta \) implements a commuting square embedding of \( A \subset M \) into \( B^\omega \subset N(\omega) \), which by the condition \( v_l \in \Sigma_i u'_l B^{\omega}, \forall l \geq 0 \), follows non-degenerate. 

\( \square \)
4.6. Corollary. Let $A \subset M$, $B \subset N$ be Cartan inclusions of separable von Neumann algebras. With the same notations as in 4.2, the following conditions are equivalent:

(a) There exists a free ultrafilter $\omega$ on $\mathbb{N}$ for which there exist non-degenerate Cartan embeddings of $A \subset M$ into $B^\omega \subset N(\omega)$ and of $B \subset N$ into $A^\omega \subset M(\omega)$.

(b) Property (a) holds true for any free ultrafilter $\omega$ on $\mathbb{N}$.

(c) With the notation in 4.5, we have $(A \subset M) \prec_w (B \subset N)$ and $(B \subset N) \prec_w (A \subset M)$.

Proof. This is now trivial by 4.5. \hfill \Box

4.7. Definition. Two pmp actions of countable groups $\Gamma \curvearrowright X$, $\Lambda \curvearrowright Y$ with their associated Cartan inclusions $A \subset M = L(\mathcal{R}_{\Gamma \curvearrowright X})$, $B \subset N = L(\mathcal{R}_{\Lambda \curvearrowright Y})$ (respectively two arbitrary separable Cartan inclusions $A \subset M$, $B \subset N$) are weakly orbit equivalent, abbreviated weakly-OE (respectively weakly equivalent) if any of the equivalent conditions in 4.6 above holds true.

It is clear that OE $\implies$ app-OE $\implies$ $\omega$-OE $\implies$ weak-OE and that app-OE is implied by app-conjugacy, $\omega$-OE is implied by $\omega$-conjugacy and weak-OE is implied by weak-conjugacy. Clearly $\omega$-OE and app-OE are equivalence relations. The characterization 4.5.(c) is easily seen to imply that the subordination relation $\prec_w$ is transitive, implying that weak-OE (respectively weak-equivalence) is an equivalence relation for group actions (resp. for Cartan inclusions).

4.8. Proposition. 1° Strong ergodicity is a weak-OE invariant (thus an app-OE and $\omega$-OE invariant as well).

2° Amenability, relative Haagerup property, co-rigidity are weak-OE invariants (thus app-OE and $\omega$-OE invariants as well).

3° Cost is a weak-OE invariant (thus app-OE and $\omega$-OE invariant as well).

4° $L^2$-Betti numbers are weak-OE invariants (thus app-OE and $\omega$-OE invariants as well).

Proof. Parts 1° and 3° are trivial by the characterization 4.6(a) of weak-OE and Propositions 2.10 respectively 2.12.2°.

The weak-OE invariance of Haagerup property and of co-rigidity in part 2° are consequences of 4.6(a) and of Proposition 2.13, while the weak-OE invariance of amenability is trivial, for instance by using [CFW].

Part 4° is an immediate consequence of 4.6(a) and 2.12.3°. \hfill \Box

4.9. Corollary. If $\Gamma$ is non-amenable and does not have property (T), then $\Gamma$ has two non weak-OE (thus also non $\omega$-OE and non app-OE) free ergodic pmp actions.
Proof. By [Sc2] any Bernoulli action of a non-amenable group is strongly ergodic, while by [CW], if \( \Gamma \) does not have property (T) then it has a free ergodic pmp action that is not strongly ergodic, and so the statement follows from 4.8.1. \( \square \)

4.10. Remarks. 1° Note that the property \((A \subset M) \prec_w (B \subset N)\) in 4.5(c) requires much more than just the fact that “\( A \subset M \) can be simulated inside \( B \subset N \)” Indeed, such a simulation condition would only require

\[
\forall v_1, ..., v_n \in qN_M(A), \varepsilon > 0, K \geq 1, \exists v'_1, ..., v'_n \in qN_N(B), \text{ such that given any word } w \text{ of length } \leq K \text{ in } n \text{ letters, we have } |\tau(w(v'_i)) - \tau(w(v_i))| \leq \varepsilon.
\]

In addition to this, the definition of \( \prec_w \) in 4.5(c) requires that: (1) the “simulating elements” \( v'_i \) are obtained from the initial group elements by patching them with uniform weights (independent of \( \varepsilon \)); (2) the simulating elements \( v'_i \) tend to exhaust \( N \) (as \( \varepsilon \to 0 \)). We have encountered the “uniform weights” condition also in the definition of app-OE and it will appear again when defining app-cocycles in Sec. 5.6. This corresponds to the fact that \( v'_i \) come from the normalizer of \( B^\omega \) in \( N(\omega) \) (not from the normalizer of \( B^\omega \) in \( N^\omega \)).

The above simulation condition, which we will denote \((A \subset M) \prec_{CAE} (B \subset N)\), is a version for Cartan inclusions of “Connes approximate embedding” (abbreviated CAE) subordination between \( \Pi_1 \) factors, \( M \prec_{CAE} N \), which requires that \( M \) can be “simulated” inside \( N \), in moments, with the corresponding CAE equivalence \( M \sim_{CAE} N \) requiring \( N \prec_{CAE} M \) as well. If \( D \subset R \) denotes the (unique) Cartan MAS in the hyperfinite \( \Pi_1 \) factor, then one has a (plain) Cartan embedding of \((D \subset R)\) into any Cartan inclusion \((B \subset N)\) with \( N \) a \( \Pi_1 \) factor, so in particular \((D \subset R) \prec_{CAE} (B \subset N)\). The question of whether \((B \subset N) \prec_{CAE} (D \subset R)\) for a Cartan inclusion coming from a free ergodic pmp action of a countable group \( \Lambda \), \( B \subset N = B \rtimes \Lambda \), amounts to asking whether the action is sofic. While asking this to be true for some free \( \Lambda \)-action (equivalently for a Bernoulli \( \Lambda \)-action, cf. [EL], or 5.1 in [P11]), amounts to \( \Lambda \) being a sofic group.

If one denotes by \( N(\omega) \) the normalizer of \( D(\omega) = \Pi_\omega D_n \) in \( M(\omega) = \Pi_\omega M_{n \times n}(\mathbb{C}) \), where \( D_n \) denotes the diagonal subalgebra in \( M_{n \times n}(\mathbb{C}) \), then this is easily seen to be equivalent to the existence of an embedding \( \Lambda \subset N(\omega) \) with \( \Lambda \sim D(\omega) \) free, for some (equivalently any) free ultrafilter \( \omega \) on \( \mathbb{N} \). Such an embedding of a sofic group into \( N(\omega) \) is called a sofic approximation of \( \Lambda \).

Recall that there are no known examples of groups, or actions, that are not sofic. So apriori, it may be that all Cartan inclusions \( B \subset N \) are CAE equivalent to \( D \subset R \), the same way we may have \( M \sim_{CAE} R \), for any \( \Pi_1 \) factor \( M \) (cf. CAE conjecture).

2° One can define yet another subordination property between Cartan inclusions \( A \subset M \) and \( B \subset N \), weaker than \( \prec_w \) in 4.5, but stronger than \( \prec_{CAE} \) in 1° above,
by requiring the existence of a Cartan embedding of \((A \subset M)\) into \((B^\omega \subset N(\omega))\), in the sense of 1.1 in [P11] (so an embedding of \(M\) into \(N(\omega)\), with \(A\) taken into \(B^\omega\), \(N_M(A)\) into \(N_N(\omega)(B^\omega)\), but without the non-degeneracy condition \(\text{sp} MB^\omega\) dense in \(N(\omega)\)). A description in local terms of this property in the spirit of 4.5.(c), is as follows (where \(\mathcal{U}, \{v_l\}_{l \geq 0}\) are as in 4.5):

There exists a map \(\mathcal{U} \ni u \mapsto t^u = \{t^u_l\}_{l \geq 0}\), with \(\Sigma t^u_l = 1, 0 \leq t^u_l \leq 1, \forall u \in \mathcal{U}, l \geq 0\), such that for any \(u_1, \ldots, u_n \in \mathcal{U}\), any \(\varepsilon > 0\) and \(K \geq 1\), there exist \(\{q^i_l\}_l \subset \mathcal{P}(B)\), with \(\Sigma_i q^i_l \leq 1, \Sigma_i q^i_l v^i_l \leq 1, \tau(q^i_l) = t^u_l, \forall l \geq 0, 1 \leq i \leq n\), such that if we denote \(u^i_l = \Sigma_i q^i_l v^i_l\), \(1 \leq i \leq n\), then given any word \(w\) of length \(\leq K\) in \(n\) letters, we have \(|\tau(w(\{u^i_l\}_l)) - \tau(w(\{u_l\}_l))| \leq \varepsilon\).

We will denote by \(<_{ww}\) this subordination relation. Note that hereditarity under \(<_{ww}\) does hold for co-amenable and relative Haagerup property, but not for strong ergodicity (nor for cost, \(L^2\)-invariants, or rigidity). Moreover, by [AW] and [GL], one has the following characterization of amenability in this framework, in the spirit of “von Neumann’s problem”: if \(\Gamma\) is a countable discrete group and \(\Gamma \prec (X, \mu)\) is an arbitrary free ergodic pmp \(\Gamma\)-action, then \(\Gamma\) is non-amenable if and only if \((F_2 \sim [0, 1]^{[\alpha]} \prec_{ww} (\Gamma \prec X))\), where the \(<_{ww}\) subordination for free ergodic pmp actions means that the corresponding Cartan subalgebras satisfy the \(<_{ww}\) subordination. It would be interesting to provide a direct proof of this result, that would not use [GL] and its percolation methods. Indeed, by [E] (see also [BHI]), we see that such a result is sufficient for deriving existence of many non-OE actions of a given non-amenable group \(\Gamma\), by using the co-induction technique in [E].

The \(<_{ww}\)-subordination gives rise to the equivalence relation \((A \subset M) \sim_{ww} (B \subset N)\) between Cartan inclusions \(A \subset M, B \subset N\) (and free ergodic pmp actions that entail them), by requiring that \((A \subset M) \prec_{ww} (B \subset N)\) and \((B \subset N) \prec_{ww} (A \subset M)\), and which may be interesting to study in its own right.

3° Yet another notion of “weak subordination” between free pmp actions \(\Gamma \prec^{\sigma} X, \Gamma \prec^{\rho} Y\) has been considered in [B2]. It requires that \(\sigma\) can be simulated better and better by \(\Gamma\)-actions that are orbit equivalent to \(\rho\). More in the spirit of this paper, this amounts to the existence of a sequence of free pmp \(\Gamma\)-actions \(\Gamma \prec^{\rho_n} Y_n\) that are OE to \(\rho\) and such that the ultraproduct action \((\rho_n)_n\) of \(\Gamma\) on \(B = \Pi_n \rightarrow \omega L^\infty(Y_n)\), contains a \(\Gamma\)-invariant subalgebra \(B \subset \hat{B}\) with \(\Gamma \prec B\) isomorphic to \(\sigma\). It is shown in [B2] that any two free pmp actions \(\sigma, \rho\) of a free group \(\Gamma = F_n, 1 \leq n \leq \infty\), are subordinated one to the other, in this sense.

5. \(\omega\)-OE RIGIDITY AND OPEN PROBLEMS

In this section we discuss the app-OE and \(\omega\)-OE versions of the various types of OE-rigidity paradigms for group actions: OE strong rigidity, OE-superrigidity,
cycle superrigidity, and calculation of OE symmetry groups, i.e., the automorphism group, cohomology groups, the fundamental group.

Recall that a free ergodic pmp action $\Gamma \curvearrowright X$ is called OE-superrigid if any OE between $\Gamma \curvearrowright X$ and another free pmp action $\Lambda \curvearrowright Y$ comes from a conjugacy. An OE strong rigidity type result is a weaker version of this, deriving automatic conjugacy from the orbit equivalence between $\Gamma \curvearrowright X$ and an action $\Lambda \curvearrowright Y$ belonging to a special class of group actions.

Instead, we will assume here the app-OE or $\omega$-OE between a certain group action $\Gamma \curvearrowright X$ and another action $\Lambda \curvearrowright Y$ (arbitrary, or from a specific class), trying to prove that this automatically entails a stronger equivalence between them. The natural stronger equivalence to seek for in this case is app-conjugacy and respectively $\omega$-conjugacy. We will provide below large classes of group actions $\Gamma \curvearrowright X$ for which, indeed, any $\omega$-OE (resp. app-OE) with another free pmp action comes from an $\omega$-conjugacy (resp. app-conjugacy).

We will do this by first proving that such app-OE/$\omega$-OE superrigidity for a certain free ergodic pmp action $\Gamma \curvearrowright X$ of a group $\Gamma$ follows from the fact that all free ergodic pmp $\Gamma$-actions that satisfy certain properties are OE-superrigid. Combining this with the striking OE-rigidity results in ([K1, K2], [CK], [MS]), we can then derive many examples of such phenomena.

5.1. Proposition. Assume the countable group $\Gamma$ has the property that any free, strongly ergodic pmp action $\Gamma \curvearrowright^\sigma X$ is OE-superrigid. Then any such action is $\omega$-OE superrigid (resp. app-OE superrigid) as well, i.e., any $\omega$-OE (resp. app-OE) of this action with another free pmp action $\Lambda \curvearrowright^\rho Y$ comes from an $\omega$-conjugacy (resp. app-conjugacy) of $\Gamma \curvearrowright X$, $\Lambda \curvearrowright Y$. More specifically, if $\theta : A^\omega \simeq B^\omega$ implements an $\omega$-OE (respectively app-OE) between the two group actions, then there exists $u \in N_{\mathbb{N}(\omega)}(B^\omega)$ such that $\text{Ad}(u)(\theta(\sigma^\omega(\Gamma))^{\theta^{-1}}) = \rho^\omega(\Gamma)$.

Proof. Assume $\Gamma \curvearrowright^\sigma X$ and $\Lambda \curvearrowright^\rho Y$ are $\omega$-OE. Thus, with the usual notations $A = L^\infty(X) \subset L^\infty(X) \rtimes \Gamma = M$, $B = L^\infty(Y) \subset L^\infty(Y) \rtimes \Lambda = N$, $M(\omega) = A^\omega \rtimes \Gamma$, $N(\omega) = B^\omega \rtimes \Lambda$, we have an isomorphism $\theta : (A^\omega \subset M(\omega)) \simeq (B^\omega \subset N(\omega))$. Let $\{u_g \mid g \in \Gamma\}$ (respectively $\{v_h \mid h \in \Lambda\} \subset N(\omega)$) be the canonical unitaries implementing $\Gamma \curvearrowright A^\omega$ (resp. $\Lambda \curvearrowright B^\omega$). Denote by $\mathcal{Y}_0$ the set of projections $\{q^g_h \mid g \in \Gamma, h \in \Lambda\} \subset B^\omega$ such that $\theta(u_g) = \sum_h v_h q^g_h, \forall g \in \Gamma$.

Define now $A_0 = A$, $B_0 = \vee_{h \in \Lambda} \rho^\omega_h(B \cup \mathcal{Y}_0 \cup \theta(A))$ and for each $n \geq 1$ define recursively $A_n = \vee_{g \in \Gamma} \sigma^\omega_g(A_{n-1} \cup \theta^{-1}(B_{n-1}))$, $B_n = \vee_{h \in \Lambda} \rho^\omega_h(B_{n-1} \cup \theta(A_n))$. If we then let $\tilde{A} = \bigcup_n A_n$, $\tilde{B} = \bigcup_n B_n$, then we clearly have $\theta(A) = \tilde{B}$, $\theta(\{u_g\}_g) \subset \tilde{B} \setminus \{v_h\}_h$, with $\Gamma$ (resp. $\Lambda$) leaving invariant $\tilde{A}$ (resp. $\tilde{B}$) and acting freely on it. Thus, if we denote $\tilde{M} = \tilde{A} \cup \{v_h\}_h \simeq \tilde{A} \rtimes \Gamma$, $\tilde{N} = \tilde{B} \cup \{v_h\}_h \simeq \tilde{B} \rtimes \Lambda$, then the restriction of $\theta$ to $\tilde{M}$ implements an isomorphism of the Cartan inclusions.
(\tilde{\mathcal{A}} \subset \tilde{M}) \simeq (\tilde{\mathcal{B}} \subset \tilde{\mathcal{N}}).

By Proposition 2.10, it follows that \( \Gamma \bowtie \tilde{\mathcal{A}} \) is strongly ergodic (and also free, with \( \tilde{\mathcal{A}} \) separable). The hypothesis implies that \( \Gamma \bowtie \tilde{\mathcal{A}} \) is OE-superrigid. Thus, there exists a unitary element \( u \in N_{\tilde{\mathcal{N}}} (\tilde{\mathcal{B}}) \) such that \( u\theta (\{u_g \mid g \in \Gamma \})u^* \subset \{v_h v \mid h \in \Lambda, v \in U(\mathcal{B})\} \), where \( u_g \in \mathcal{M} \subset \mathcal{M} (\omega), v_h \in \tilde{\mathcal{N}} \subset N(\omega) \) are the canonical unitaries. Since \( N_{\tilde{\mathcal{N}}} (\tilde{\mathcal{B}}) \subset N_{\mathcal{N}_\omega} (B^\omega) \), we have this way shown that \( \theta \) comes from an \( \omega \)-conjugacy. Moreover, since \( u \in N_{\mathcal{N}_\omega} (B^\omega) \) can be represented as \( (u_n)_n \) with \( u_n \in N_{\mathcal{N} (\mathcal{B})} \), it follows that \( A u \theta = (A u_n)_n \) is of ultraproduct form as well. Thus, if \( \theta = (\theta_n)_n \) implements an app-OE, then \( A u \circ \theta \) implements an app-conjugacy of 

\[ \Gamma \bowtie X, \Lambda \bowtie Y. \]

 Strictly speaking, we will not be able to apply Proposition 5.1 the way it is stated. But its proof provides the template of how to deduce app-OE/\( \omega \)-OE superrigidity from OE-superrigidity for many “special” free strongly ergodic pmp actions \( \Gamma \bowtie \sigma \) \( X \). Thus, if say we know that OE-superrigidity holds for all actions of \( \Gamma \) that satisfy a certain property \( \mathcal{P} \), then we can deduce from it app-OE superrigidity for all \( \Gamma \)-actions \( \sigma \) with the property that any “quotient of its ultrapower \( \sigma^\omega \)” still satisfies property \( \mathcal{P} \). This will force us to assume that \( \sigma \) satisfies some strengthened form of \( \mathcal{P} \), which will insure it survives to all (separable) quotients of \( \sigma^\omega \) (i.e., \( \sigma \) has good \( \omega \)-permanence).

**5.2. Corollary.** 1° If \( \Gamma = SL(3, \mathbb{Z}) * \Sigma SL(3, \mathbb{Z}) \), where \( \Sigma \subset SL(3, \mathbb{Z}) \) is the subgroup of matrices \((t_{ij})\) with \( t_{31} = t_{32} = 0 \), then any free, strongly ergodic, aperiodic pmp \( \Gamma \)-action is app-OE superrigid and \( \omega \)-superrigid.

2° If \( \Gamma \) is of the form \( \Gamma = \Gamma_1 * \mathcal{H} \Gamma_2 \), with \( \Gamma_1, \Gamma_2 \) lattices in non-compact connected simple Lie groups with trivial center and real rank \( \geq 2 \), and with the common subgroup \( \mathcal{H} \) non-amenable and satisfying \( |\Gamma_i / \mathcal{H}| = \infty \), as well as the singularity condition \( g \in \Gamma_i, [H : g H g^{-1} \cap H] < \infty \implies g \in H, i = 1, 2 \), then any \( \Gamma \)-action whose restriction to \( H \) is both strongly ergodic and aperiodic (e.g., any quotient of a Bernoulli \( \Gamma \)-action) is app-OE superrigid and \( \omega \)-superrigid.

3° The following \( \Gamma \) have the property that any free, aperiodic, strongly ergodic, pmp \( \Gamma \)-action (e.g., any quotient of a Bernoulli \( \Gamma \)-action) is app-OE superrigid and \( \omega \)-superrigid:

(a) \( \Gamma \) is a finite index subgroup of a finite product of mapping class groups \( \Gamma_1, \ldots, \Gamma_n \), of compact orientable surfaces, with the genus \( g_i \) and number of boundary components \( p_i \) of \( \Gamma_i \), satisfying \( 3g_i + p_i - 4 > 0 \) and \( (g_i, p_i) \neq (2, 0), (1.2), \forall i \);

(b) \( \Gamma \) is the pure braid group with \( k \geq 2 \) strands on a closed, orientable surface of genus \( g \geq 2 \).

**Proof.** Let us first notice that if a free ergodic pmp action \( G \bowtie^\rho \mathbb{Z} \) of a count-
able group $G$ is strongly ergodic and aperiodic, then the restriction of $\rho^\omega : G \to \text{Aut}(L^\infty(Z)^\omega)$ to any $G$-invariant von Neumann subalgebra $C \subset L^\infty(Z)^\omega$ on which it acts freely is still strongly ergodic and aperiodic (i.e., given any subgroup of finite index $G_0 \subset G$, the action $G_0 \actson C$ is free and strongly ergodic). Indeed, this can be trivially seen by the proof of (1.11(i) in [PP]; see also 3.1 in [AE]).

Now, to prove 1°, recall first that by (Theorem 1.5(b) in [K2]), any free ergodic pmp action of the group $\Gamma = \text{SL}(3,\mathbb{Z}) \ast_\Sigma \text{SL}(3,\mathbb{Z})$ is stably (or virtually) OE-superrigid. But the group $\Gamma$ has no finite normal subgroups and also, since our $\Gamma$-action is strongly ergodic and aperiodic, by the above remark its restriction to any finite index subgroup of $\Gamma$ is still strongly ergodic, so in particular it is ergodic. Thus, any free strongly ergodic aperiodic pmp $\Gamma$-action is in fact OE-superrigid, and the proof of Proposition 5.1 applies.

Let now $\Gamma = \Gamma_1 \ast_H \Gamma_2$ be as in 2°. Recall from (Theorem 1.3 in [K2]) that for such a group $\Gamma$, any free ergodic pmp $\Gamma$-action which is aperiodic when restricted to $H$ is OE-superrigid. But if $\Gamma \actson X$ is strongly ergodic and aperiodic on $H$, the above remark shows that it is strongly ergodic on any finite index subgroup of $H$. Thus, the ultrapower action $\Gamma \actson A^\omega$ is ergodic on any infinite index subgroup of $H$, where $A = L^\infty(X)$ as usual. Moreover, the same is true for the restriction of this action to any $\Gamma$-invariant separable von Neumann subalgebra $A_0 \subset A^\omega$ on which $\Gamma$ acts freely. Thus, any such $\Gamma \actson A_0$ is OE-superrigid. Hence, if $\Lambda \actson Y$ is a free ergodic pmp action that’s $\omega$-OE (respectively app-OE) with $\Gamma \actson X$, then the same argument as in the proof of 5.1 applies to get that the two actions follow $\omega$-conjugate (resp. app-conjugate).

Finally, recall from [K1] (respectively [CK]) that if a pmp action $\Gamma \actson^\sigma X$ is as in part (a) of 3° (respectively as in part (b) of 3°) then $\sigma$ is OE-superrigid. But since $\Gamma \actson A = L^\infty(X)$ is strongly ergodic, by 2.10 so is its ultrapower $\Gamma \actson^{\sigma^\omega} A^\omega$. If in addition $\sigma$ is aperiodic then its restriction to any subgroup $\Gamma_0 \subset \Gamma$ of finite index is still strongly ergodic, so $\Gamma_0 \actson A^\omega$ is strongly ergodic as well. Moreover, the same is true for the restriction of $\sigma^\omega$ to any $\Gamma$-invariant separable subalgebra $A_0 \subset A^\omega$ on which $\Gamma$ acts freely. Thus, any such $\Gamma \actson A_0$ follows OE-superrigid. The proof of 5.1 applies again to get the conclusion. □

5.3. Corollary. Let $\Gamma = \Gamma_1 \times \Gamma_2$, with $\Gamma_i$ torsion free groups in the class $C_{reg}$ of Monod-Shalom [MS], $i = 1, 2$. Let $\Gamma \actson X$ be a free pmp $\Gamma$-action whose restriction to $\Gamma_1, \Gamma_2$, is strongly ergodic (e.g., a quotient of a Bernoulli $\Gamma$-action). Then we have:

1° Any app-OE (respectively $\omega$-OE) between $\Gamma \actson X$ and an arbitrary free pmp $\Gamma$-action $\Gamma \actson Y$ comes from an app-conjugacy (resp. $\omega$-conjugacy).

2° If in addition $\Gamma_1, \Gamma_2$ have property (T) (e.g., if each $\Gamma_i$ is an arithmetic lattice
in some $Sp(n,1)$, $n \geq 2$, or a quotient of such a group), then any app-OE between
$\Gamma \curvearrowright X$ and an arbitrary free pmp $\Gamma$-action $\Gamma \curvearrowright Y$ comes from a conjugacy.

**Proof.** 1° Recall from (Theorem 1.6 in [MS]) that if $\Gamma = \Gamma_1 \times \Gamma_2$ is as in the
hypothesis, then an OE between a free pmp action $\Gamma \curvearrowright X$ whose restriction to
$\Gamma_1, \Gamma_2$ is still ergodic, and an arbitrary free pmp action $\Gamma \curvearrowright Y$, comes from a
conjugacy.

But if in addition $\sigma|_{\Gamma_i}$ are strongly ergodic, $i = 1, 2$, then by Proposition 2.10 any
restriction of the ultrapower action $\Gamma \curvearrowright^\omega X$ to a $\Gamma$-invariant subalgebra $\tilde{A} \subset A^\omega$
is still strongly ergodic on $\Gamma_1, \Gamma_2$ (so in particular ergodic).

Then the same argument as in the proof of 5.1 applies to show that any $\omega$-OE
(respectively app-OE) between a free pmp action $\Gamma \curvearrowright X$ and another free pmp $\Gamma$-action comes from
a $\omega$-conjugacy (respectively an app-conjugacy).

Part 2° is then an immediate consequence of the first part and 3.11. \qed

5.4. Definition. Let $A \subset M$ be a Cartan inclusion and $\omega$ a free ultrafilter on
$\mathbb{N}$. We will define the $\omega$ and app symmetry groups of $A \subset M$ as the corresponding
(genuine) symmetry groups of the Cartan inclusion $A^\omega \subset M(\omega)$, where $M(\omega)$
denotes, as before, the von Neumann algebra generated inside $M^\omega$ by $A^\omega$ and $M$.
Thus, if $U \subset \mathcal{N}_M(A)$ is any group with $U \cup A = M$, then $M(\omega) = U \cup A^\omega = spUA^\omega$.

It is useful to note that if $B \subset N$ denotes the Cartan inclusion $(A \subset M)^t$
then $B^\omega \subset N(\omega)$ is naturally isomorphic to $(A^\omega \subset N(\omega))^t$, with the corresponding
identification between $(A^t)^\omega$ and $(A^\omega)^t$ being of ultraproduct type.

1° We denote by $\text{Aut}_\omega(A \subset M)$ the automorphism group of $A^\omega \subset M(\omega)$, as
declared in 2.15.1°, and by $\text{Aut}_{app,\omega}(A \subset M)$ its subgroup of automorphisms which
restricted to $A^\omega$ are of ultrapower form. We denote $\text{Out}_\omega(A \subset M)$ (respectively
$\text{Out}_{app,\omega}(A \subset M)$) the quotient of $\text{Aut}_\omega(A \subset M)$ (resp. of $\text{Aut}_{app,\omega}(A \subset M)$) by
the subgroup of automorphisms implemented by unitaries in $\mathcal{N}_M(A^\omega)$.

If $\Gamma \curvearrowright^\sigma X$ is a pmp action of a countable group, then we denote $\text{Aut}_\omega(\mathcal{R}_{\Gamma})$,
$\text{Out}_\omega(\mathcal{R}_{\sigma})$, and $\text{Aut}_{app,\omega}(\mathcal{R}_{\sigma})$, $\text{Out}_{app,\omega}(\mathcal{R}_{\sigma})$ the corresponding groups of the Cartan
inclusion $L^\infty(X) \subset L(\mathcal{R}_{\sigma})$. Note that $\text{Out}_\omega(\mathcal{R}_{\sigma})$ (respectively $\text{Out}_{app,\omega}(\mathcal{R}_{\sigma})$)
is an $\omega$-OE (resp. app-OE) invariant for $\sigma$.

2° We denote $Z^1_\omega(A \subset M) = Z^1(A^\omega \subset M(\omega))$, $H^1_\omega(A \subset M) = H^1(A^\omega \subset M(\omega))$, the
first cohomology groups of $A^\omega \subset M(\omega)$, as defined in 2.15.2°. We identify these
groups, as in 2.15.2°, with $\text{Aut}_0(A^\omega \subset M(\omega))$, respectively $\text{Out}_0(A^\omega \subset M(\omega))$, of (classes of) automorphisms of $A^\omega \subset M(\omega)$ that leave $A^\omega$ fixed. Note that such
automorphisms are of ultrapower form when restricted to $A^\omega$, so they can be viewed
as subgroups of $\text{Aut}_{app,\omega}(A \subset M) \subset \text{Aut}_\omega(A \subset M)$.

If $L^\infty(X) = A \subset M = L^\infty(X) \rtimes_\sigma \Gamma$ for some free ergodic pmp action $\Gamma \curvearrowright^\sigma X$
then, with the notations in 2.15.2°, we have $Z^1_x(A \subset M) = Z^1(\sigma^w)$, $H^1_x(A \subset M) = H^1(\sigma^w)$, the first cohomology groups of $\Gamma \ltimes \sigma^w A^w$. Thus, an element in $Z^1_\sigma(A \subset M)$ is given by a map $c : \Gamma \to \mathcal{U}(A^w)$ satisfying $c_g \sigma^w_g(c_n) = c_{gh}$, $\forall g, h \in \Gamma$. As pointed out in 2.15.2°, the topology inherited from $\text{Aut}_\omega(A \subset A \rtimes^\sigma \Gamma) = \text{Aut}(A^w \subset A^{w \rtimes^\sigma \Gamma})$ (of point-$\| \|_2$ convergence) corresponds in $Z^1_\sigma(\sigma)$ to the point-$\| \|_2$-convergence of cocycles, viewed as functions $\Gamma \to \mathcal{U}(A^w)$.

By writing $c_g = (c_{g,n})_n \in \mathcal{U}(A^w)$ with $c_{g,n} \in \mathcal{U}(A)$, $\forall n$, we see that $c$ corresponds to a sequence of functions $c_n : \Gamma \to \mathcal{U}(A)$ which satisfy the approximate cocycle relation $\lim_\omega \| c_{g,n} \sigma_g(c_{h,n}) - c_{gh,n} \|_2 = 0$, $\forall g, h \in \Gamma$. Two such sequences $c = (c_{g,n})_n$, $c' = (c'_{g,n})_n$ give the same element in $Z^1_\sigma(\sigma)$ if $\lim_\omega \| c'_{g,n} - c_{g,n} \|_2 = 0$, $\forall g \in \Gamma$. This shows that any sequence of cocycles $c_n \in Z^1(\sigma)$ gives rise to a $\sigma^w$-cocycle $c = (c_{g,n})_n$. We denote by $Z^1(\sigma)^\omega$ the subgroup of such cocycles. It is easy to see that it is closed in $Z^1(\sigma^w)$. Also, we always have $(B^1(\sigma))^\omega = B^1(\sigma^w)$.

By [Sc1] we have $B^1(\sigma)$ closed in $Z^1(\sigma)$ iff $\sigma$ is strongly ergodic and this condition is easily seen to imply that $B^1(\sigma^w)$ is closed in $Z^1(\sigma^w)$. However, the converse is not necessarily true. For instance, the co-boundary group of $\sigma^w$, $B^1(\sigma^w)$ (which we saw is equal to $(B^1(\sigma))^\omega$) coincides with the closed subgroup $Z^1(\sigma)^\omega$ of $Z^1(\sigma^w)$ whenever $B^1(\sigma)$ is dense in $Z^1(\sigma)$ (e.g., when $\Gamma$ is amenable).

Note that if $H^1(\sigma)$ is discrete (e.g., if $\Gamma$ has property (T), see [Sc1]), then the embedding $(Z^1(\sigma))^\omega \subset Z^1(\sigma^w)$ induces an embedding of the ultrapower group $H^1(\sigma)$ into $H^1_\omega(\sigma)$. The interesting problem here is to show that, in certain cases, these groups coincide, $Z^1_\omega(\sigma) = Z^1(\sigma)^\omega$, $H^1_\omega(\sigma) = H^1(\sigma)^\omega$, i.e., any approximate cocycle comes from a cocycle. This would show, for instance, that if $H^1(\sigma)$ is finite, then $H^1_\omega(\sigma) = H^1(\sigma)$.

3° We denote by $\mathcal{F}_\omega(A \subset M)$ the fundamental group of $A^w \subset M(\omega)$ and by $\mathcal{F}_{app,\omega}(A \subset M)$ the subgroup of all $t > 0$ with the property that there exists an isomorphism $(A^w \subset M(\omega)) \simeq (A^w \subset M(\omega))^t$ which is of ultraproduct form when restricted to $A^w$. Note that we have natural embeddings $\mathcal{F}(A \subset M) \subset \mathcal{F}_{app}(A \subset M) \subset \mathcal{F}_\omega(A \subset M)$ and that the group $\mathcal{F}_\omega(A \subset M)$ is an $\omega$-OE invariant while $\mathcal{F}_{app}(A \subset M)$ is an app-OE invariant.

By 3.11.2°, 3.12.2°, Corollary 5.3 above and (7.13.3 in [IPP]), we thus obtain the following concrete calculations of $\text{Out}_{app,\omega}(A \subset M)$ for group measure space Cartan inclusions from some special classes of actions of property (T) groups.

5.5. Corollary. Let $\Gamma$ be an ICC hyperbolic group with the property (T) and $\Gamma \times \Gamma \ltimes \sigma^i X^1_i$ be double Bernoulli $\Gamma \times \Gamma$-actions with finite base $(X_i, \mu_i)$, $i \in I$. Two such actions $\sigma_i, \sigma_j$ are app-OE if and only if $(X_i, \mu_i) \simeq (X_j, \mu_j)$. Also, $\text{Out}_{app,\omega}(\mathcal{R}_{\sigma_i}) = \text{Aut}(X_i, \mu_i)$. 

On the other hand, by Proposition 2.12.3° and the remark at the end of 2.15.3°, we get:

5.6. Corollary. If \( \Gamma \) has one non-zero, finite \( L^2 \)-Betti number, then any free ergodic pmp \( \Gamma \)-action has trivial \( \omega \)-fundamental group, \( \mathcal{F}_\omega(\mathcal{R}_\Gamma) = \{1\} \).

5.7. Final remarks. It is tempting to re-examine existing OE rigidity results and calculations of invariants and try to use them, or try to adapt their proofs, to obtain app-OE rigidity results. But the only OE-rigidity results that we were able to use to derive app-OE/\( \omega \)-OE rigidity were the ones in [K1, K2], [CK], [MS]. This is because the OE-rigidity results in these papers have almost no restrictions on the “source action” (albeit putting much restrictions on the acting group).

In turn, the OE rigidity results that require very specific restrictions on the source action, like being Bernoulli, Gaussian, or profinite, cannot be applied directly, and the deformation-rigidity techniques used to prove them do not seem to adapt well to the “ultrapower framework” (e.g., because separability and deformation arguments fail).

So a general problem that’s of interest in this respect is to search for more examples of OE-superrigid group actions \( \Gamma \curvearrowright \sigma X \) where the only conditions that would be imposed on the action \( \sigma \) have good “\( \omega \)-permanence”, i.e., they “survive” to all separable sub-actions of the ultrapower \( \Gamma \)-action \( \sigma^\omega \). For once one has such a result, the proof of Proposition 5.1 applies and we can derive app-OE rigidity for \( \sigma \).

Such OE-rigidity results are particularly interesting to have for property (T) groups \( \Gamma \), because as we have seen in Corollary 3.11, app-conjugacy for such actions is the same as conjugacy. This can also lead to complete calculations of app-symmetry groups, like in Corollary 5.5 above.

5.7.1. Calculations of \( \text{Out}_{\text{app},\omega}(\mathcal{R}_\Gamma) \). Strong rigidity results like Corollaries 5.3, or like Corollary 5.2 applied to just the case \( \Gamma = \Lambda \), reduce the calculation of the app-Outer automorphism group \( \text{Out}_{\text{app},\omega}(\mathcal{R}_\Gamma) \) to the calculation of the app-conjugacy invariant \( \text{Aut}_{\text{app},\omega}^\sigma(X) \), defined in 3.11.2° as the group of sequences of automorphisms \( \theta_n \) of \( (X, \mu) \) that \( \omega \)-asymptotically commute with \( \sigma_n, \forall n \). To see this, we send the reader to page 445 in [P7], or to similar considerations in [F2], [MS] (we have already seen in 5.5 above a sample of how this works).

So in order to produce more classes of group actions \( \Gamma \curvearrowright \sigma X \) with calculable \( \text{Out}_{\text{app},\omega}(\mathcal{R}_\sigma) \) one needs to obtain more classes of groups \( \Gamma \) and actions \( \Gamma \curvearrowright X \) with the property that, for any \( \Gamma \)-invariant separable subalgebra \( A_0 \subset A^\omega \) that contains \( A \), any automorphism of \( A_0 \subset A_0 \times \Gamma \) comes from a self-conjugacy of \( \Gamma \curvearrowright A_0 \).

5.7.2. Untwisting app-cocycles. We leave wide open the problem of concrete cal-
culations of the app-OE invariant $H^1_{\omega}(\sigma)$ of a free ergodic pmp action $\Gamma \bowtie^\sigma X$. As explained before, the actions to consider should be strongly ergodic, a property that’s automatic when $\Gamma$ has property (T). For such group actions, the result to seek is that the natural embedding $(H^1(\sigma))^\omega \subset H^1_{\omega}(\sigma)$ is in fact an equality.

This amounts to proving that if $c_n : \Gamma \to \mathcal{U}(A)$ are maps satisfying
\[
\lim_{n \to \omega} \|c_n(g)\sigma_g(c_n(h)) - c_n(gh)\|_2 = 0, \forall g, h \in \Gamma,
\]
then there exist $c'_n \in Z^1(\sigma)$ such that $\lim_{\omega} \|c'_n(g) - c_n(g)\|_2 = 0, \forall g$. Once this is proved, if $H^1(\sigma)$ itself is calculable (e.g., when $\sigma$ belongs to calculable cohomology in [Ge], [PS], [P6]), we are done. Or we can try to prove directly that the cocycle $c = (c_n)_n$ for $\Gamma \bowtie^\sigma A^\omega$ “untwists” (like in [Ge], [PS], [P6]).

We can in fact view this problem as a particular case of a general question about untwisting app-cocycle with $\mathcal{U}_{fin}$ targets, in the spirit of ([P8]). While this is somewhat long and tedious to formulate in its full generality, we will only state it in the case of discrete targets (the other interesting case being scalar targets, which we have already stated above).

Thus, let $\Gamma \bowtie^\sigma (X, \mu)$ be a free ergodic pmp action of a countable group and $\Lambda$ another countable group. We let $\Gamma$ act on the space $\Lambda^X$ of measurable maps from $X$ to $\Lambda$ (viewed as a subgroup of unitaries in $L^\infty(X)\overline{\otimes} L(\Lambda)$) by $\sigma_g(b)(t) = b(\sigma_{g^{-1}}(t))$, $\forall t \in X, g \in \Gamma$. An app-cocycle (respectively $\omega$-cocycle) for $\sigma$ with target group $\Lambda$ is a sequence of maps $c_n : \Gamma \to \Lambda^X$ with the property that $\lim_{n \to \infty} \|c_n(g)\sigma_g(c_n(h)) - c_n(gh)\|_2 = 0$ (resp. $\lim_{n \to \omega} \|c_n(g)\sigma_g(c_n(h)) - c_n(gh)\|_2 = 0$), $\forall g, h \in \Gamma$, and such that for each $g \in \Gamma$, $h \in \Lambda$, we have $\mu(\{t \in X \mid c_n(g)(t) = h\})$ constant in $n$.

We say that $c = (c_n)_n$ can be app-untwisted (resp. $\omega$-untwisted) if there exist a group morphism $\delta : \Gamma \to \Lambda$ and a sequence of maps $w_n \in \Lambda^X$ such that $\lim_{n \to \infty} \|w_n c_n(g)\sigma_g(w_n^*) - \delta(g)\|_2 = 0$ (resp. $\lim_{n \to \omega} \|w_n c_n(g)\sigma_g(w_n^*) - \delta(g)\|_2 = 0$), $\forall g$, and such that for each $h \in \Lambda$, $\mu(\{t \in X \mid w_n(t) = h\})$ is constant in $n$.

We say that $\Gamma \bowtie^\sigma X$ is app-cocycle superrigid (resp. $\omega$-cocycle superrigid) if any app-cocycle (resp. $\omega$-cocycle) with discrete targets can be app-untwisted (resp. $\omega$-untwisted). Arguing like on (page 287 of [P8]), we see that any approximate orbit equivalence $\theta = (\theta_n)_n$ (resp. $\omega$-OE $\theta$) between two free ergodic pmp actions $\Gamma \bowtie^\sigma X, \Lambda \bowtie^\rho Y$ gives rise to an app-cocycle (resp. $\omega$ cocycle) $c = (c_n)_n$, with $c_n : \Gamma \to Y^\Lambda$, by writing each $\theta \sigma^\omega(g)\theta^{-1}$ as $(\oplus_h \rho(h)|_{\mathcal{U}_{q_n,h}B})_n$ and letting $c_n(g)$ be equal to $h$ when restricted to the subset of $Y$ corresponding to $q_{n,h}^g$. As in ([P8]), untwisting the app-cocycle $c = (c_n)_n$ by some $w = (w_n)_n$ in the above sense means that $w_n \in \Lambda^Y$ can be interpreted as elements in the normalizer of $B^\omega$ in $N(\omega)$ satisfying $w\theta \sigma^\omega(\Gamma)\theta^{-1}w^* \subset \rho^\omega(\Lambda)$, with the fact that $\tilde{\theta}(M(\omega)) = N(\omega)$ implying that we actually have equality (where $\tilde{\theta}$ is the extension of $\theta : L^\infty(X)^\omega \sim L^\infty(Y)^\omega$.
given by 4.1). In other words, $\theta$ and $w$ implement an app-conjugacy (resp. $\omega$-conjugacy) of the two actions.

Thus, if we can prove app-cocycle superrigidity (resp. $\omega$-cocycle superrigidity) for a free pmp $\Gamma$ action $\sigma$, then we also get app-OE superrigidity (resp. $\omega$-OE superrigidity) for $\sigma$.

5.7.3. On the existence of non weak-OE actions. As we mentioned before, we leave open the question of whether every property (T) group $\Gamma$ has two non weak-OE free ergodic pmp actions. And further, whether any non-amenable group $\Gamma$ has “many” non weak-OE group actions, or at least that it has “many” non app-OE actions (see [AE] for partial answers to this problem).

For the “classic” OE-case of this question, the successful final answer in ([I1], [GL], [E]) used deformation-rigidity arguments, by exploiting the existence of many free ergodic pmp $\mathbb{F}_2$-actions with relative property (T) in [GP]. But it seems difficult to make such arguments work in ultrapower framework. In the absence of that, the calculation of app-cohomology $H^1_\omega(\sigma)$ as described in 5.7.2 would at least solve the problem for existence of many non app-OE actions of property (T) groups.

For the weak-OE version of this problem, another approach is to seek properties of an action that have good $\omega$-permanence. In this respect, it would be interesting to investigate whether the Cartan inclusion $A^\omega \subset M(\omega) = A^\omega \rtimes_{\sigma^\omega} \Gamma$ associated with a Bernoulli $\Gamma$-action $\sigma$ of a non-amenable group $\Gamma$ can admit a sub-Cartan inclusion $B \subset N$ that has relative property (T) (or is rigid) in the sense of ([P5]), or at least that there exists $A_0 \subset A^\omega$ diffuse such that $A_0 \subset M(\omega)$ is rigid. Another question is whether $A^\omega \subset M(\omega)$ can admit sub-Cartan inclusions $B \subset N$ that come from profinite actions (e.g., $B \subset N$ weakly-compact, in the sense of [OP]).

5.7.4. A connection to 2nd cohomology. A well known open problem in OE rigidity is to effectively calculate the 2-cohomology group $H^2(\sigma)$ of a free ergodic pmp action $\Gamma \curvearrowright X$ (see 6.6 in [P8]). Recall from Example 2.4 that this (abelian) group is defined as the quotient between the group of (normalized) 2-cocycles for $\sigma$, $Z^2(\sigma) =$ $\{v : \Gamma \times \Gamma \to U(A) \mid v_{g,h}v_{gh,k} = \sigma_g(v_{h,k})v_{g,hk}, \forall g, h, k\}$, by the subgroup of (normalized) co-boundary cocycles $B^2(\sigma) =$ $\{v \in Z^2(\sigma) \mid \exists w : \Gamma \to U(A), v_{g,h} = w_g\sigma_g(w_h)w_{gh}^*, \forall g, h\}$,

We endow the group $Z^2(\sigma)$ with the topology of pointwise convergence in the norm $\|\|_2$. It is a Polish group with respect to this topology.

Note that $B^2(\sigma)$ is the image under the boundary map $\partial$ from the product group $U(A)^\Gamma$, defined by $\partial w(g, h) = w_g\sigma_g(w_h)w_{gh}^*$, $\forall g, h \in \Gamma$. The group morphism $\partial$ is
clearly continuous and its kernel is the (closed) subgroup $Z^1(\sigma)$ of 1-cocycles for $\sigma$. Thus, $\partial$ induces a group isomorphism $\tilde{\partial}$ from the (Polish) group $\mathcal{U}(A)^c / Z^1(\sigma)$ onto the (topological) group $B^2(\sigma)$. But while $\tilde{\partial}$ is continuous, its inverse is continuous iff $B^2(\sigma)$ is closed in $Z^2(\sigma)$, see Proposition 5.8 below.

It is a consequence of [CFW] that $H^2(\sigma) = \{1\}$ for any ergodic pmp action $\sigma$ of an amenable group $\Gamma$. But surprisingly, as of now, there has been no calculation of $H^2(\sigma)$ for a free ergodic pmp action of a non-amenable group! As already mentioned in Example 2.4, any element in the 2nd cohomology group of $\Gamma$

$$Z^2(\Gamma, T) = \{ \lambda : \Gamma \times \Gamma \to T \mid \lambda_{g,h} \lambda_{gh,k} = \lambda_{h,k} \lambda_{g,hk}, \forall g, h, k \in \Gamma \}$$

implements an element in $Z^2(\sigma)$. But it is not easy to decide when a scalar cocycle gives a non-trivial element in $H^2(\sigma)$. One situation that’s particularly desirable is for $H^2(\sigma)$ to actually coincide with $H^2(\Gamma, T)$, something we believe should be true in certain cases (e.g., when $\Gamma$ has some strong form of property (T) and $\sigma$ is Bernoulli). For this to happen (and more generally for $H^2(\sigma)$ to be calculable) a necessary condition is that $H^2(\sigma)$ be a Polish group, i.e., $B^2(\sigma)$ be closed in $Z^2(\sigma)$.

The next Proposition shows that $B^2(\sigma)$ is closed iff the condition $(Z^1(\sigma))^\omega = Z^1(\sigma^\omega)$, that we discussed in 5.7.2 and which is needed in the calculation of the app 1-cohomology group $H^1_\omega(\sigma)$, holds true. This partly explains why both the calculation of $H^1_\omega(\sigma)$ and $H^2(\sigma)$ are so difficult.

5.8. Proposition. The following conditions are equivalent:

(a) $Z^1(\sigma)^\omega = Z^1(\sigma^\omega)$;

(b) $B^2(\sigma)$ is closed in $Z^2(\sigma)$;

(c) The boundary map $\partial : \mathcal{U}(A)^c \to B^2(\sigma)$ is open. In other words, the group isomorphism $\mathcal{U}(A)^c / Z^1(\sigma) \simeq B^2(\sigma)$, implemented by the boundary map, is an isomorphism of Polish groups.

Proof. The proof follows closely arguments in [C2]. We will prove that (a) $\Rightarrow$ (c) $\Rightarrow$ (b) $\Rightarrow$ (a).

(a) $\Rightarrow$ (c): To show that $\partial$ is open, it is sufficient to prove that if a sequence of 2-coboundaries $\tilde{v}_g,h = \tilde{w}_g^n \sigma_g(\tilde{w}_h^n)^*$ converges to 1, then there is a subsequence $(\tilde{w}^{n_k})$ and $s^k \in Z^1(\sigma)$ such that $s^k_g \tilde{w}^{n_k}_g \to 1$ strongly, $\forall g \in \Gamma$. Suppose that $\tilde{w}_g^n \sigma_g(\tilde{w}_h^n)^* \to 1$ strongly, $\forall g, h \in \Gamma$. Then $g \mapsto (\tilde{w}_g^n)_n$ belongs to $Z^1(\sigma^\omega)$. Since $Z^1(\sigma)^\omega = Z^1(\sigma^\omega)$, there is a sequence $(w_g^n)_{g \in G}$ of 1-cocycles for $\sigma$ such that $\lim_{n \to \omega} \|w_g^n - w_g^n\|_2 = 0, \forall g \in \Gamma$. We claim that this implies the desired conclusion. Indeed, as $\Gamma$ is countable, we can construct an increasing map $\mathbb{N} \ni k \mapsto n_k \in \mathbb{N}$
such that $\|\tilde{w}^{n_k}_g - w^{n_k}_g\|_2 \to 0$ as $k \to \infty$, $\forall g \in \Gamma$, and then $((w^{n_k})^{-1}\tilde{w}^{n_k})$ converges to $1$ in the topology on $U(A)^\Gamma$.

(c) $\Rightarrow$ (b): View $U(A)^\Gamma$ as a subset of $N = \bigoplus_{g \in \Gamma}(A)_g = \ell^\infty(\Gamma)\otimes A$. Put $V_n = \{a \in (N)_1 : \|a\xi_j\| < 2^{-n} \text{ for } 1 \leq j \leq n\}$, where $\{\xi_j\}_{j=1}^\infty$ is an orthonormal basis in $\ell^2(\Gamma)\otimes L^2(\Gamma)$. Note that $w^n \to 0$ strongly in $N = \ell^\infty(\Gamma)\otimes A$ if and only if $(w^n_g)_{g \in \Gamma} \to 0$ in $U(A)^\Gamma$ and that the $V_n$ form a decreasing sequence of open neighborhoods of $0 \in N$ (in the strong operator topology restricted to the unit ball) that shrinks to a point and has the property that $uV_n \subset V_n$ for all $u \in U(A)^\Gamma$. Choose open sets $W_n \subseteq \partial((V_n + Z^1(\sigma)) \cap U(A)^\Gamma)$ that contain $1$, noting that $\overline{(V_n + Z^1(\sigma)) \cap U(A)^\Gamma}$ is an open set. Then

$$\partial(u) \in W_n \Rightarrow u \in V_n + Z^1(\sigma).$$

Let $v \in \overline{B^2(\sigma)}$ be given. Choose $\partial(w^n) \in B^2(\sigma)$ such that $\partial(w^n) \to v$ and $\partial((w^{n+1})^*w^n) \in W_n$ for all $n$. Then $(w^{n+1})^*w^n \in V_n + Z^1(\sigma)$, i.e., $w^n = a_n + w^{n+1}s^n$ with $a_n \in V_n$, $s^n \in Z^1(\sigma)$. So by multiplying each $w^n$ by a 1-cocycle if necessary, we may arrange that $(w^n_g)_{g \in \Gamma}$ has a strong limit in $N$ (as $(w^n\xi_j)_n$ is Cauchy for each $j$, converging to $\eta_j$, say, and we can define an operator $w$ by $w(\sum \alpha_j\xi_j) = \sum \alpha_j\eta_j$, which is certainly the pointwise limit of the $(w^n)$ on the dense linear span of the $\xi_j$, hence is the strong limit of the $(w^n)$), and therefore also that $(w^n_g)$ has a strong limit $w_g \in U(A)$, $\forall g \in \Gamma$ (defined by $w_g(\xi) = w(\delta_g \otimes \xi)$, $\forall g \in \Gamma$), which is necessarily unitary, since $A$ is abelian. It follows that $v = \partial(w) \in B^2(\sigma)$.

(b) $\Rightarrow$ (a): Suppose that $B^2(\sigma)$ is a closed subgroup of $Z^2(\sigma)$. Then $B^2(\sigma)$ is a Polish space, so the continuous bijective homomorphism

$$\tilde{\partial}: U(A)^\Gamma/Z^1(\sigma) \to B^2(\sigma)$$

automatically has a continuous inverse map (cf. e.g. a lemma in [C2]). Let $g \mapsto (\tilde{w}^n_g)$ be an element of $Z^1(\sigma^\omega)$. Then

$$\lim_{n \to \omega} \|\partial(\tilde{w}^n)_{g,h} - 1\|_2 = \lim_{n \to \omega} \|\tilde{w}^n_g\sigma_g(\tilde{w}^n_h)(\tilde{w}^n_{gh})^* - 1\|_2 = 0$$

for all $g, h \in \Gamma$. Since $U(A)^\Gamma \times \Gamma$ is equipped with the product topology, we infer that $\lim_{n \to \omega} \tilde{\partial}(\tilde{w}^n) = 1$, where $\overline{w^n}$ is the class of $\tilde{w}^n$ in $U(A)^\Gamma/Z^1(\sigma)$. By continuity of $\tilde{\partial}^{-1}$, it follows easily that $\lim_{n \to \omega} \tilde{w}^n = \overline{1}$, hence that $(\tilde{w}^n_g) \in Z^1(\sigma^\omega)$. \hfill $\square$
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