On Diagonalization in Map(M, G)

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Abstract

Motivated by some questions in the path integral approach to (topological) gauge theories, we are led to address the following question: given a smooth map from a manifold $M$ to a compact group $G$, is it possible to smoothly ‘diagonalize’ it, i.e. conjugate it into a map to a maximal torus $T$ of $G$?

We analyze the local and global obstructions and give a complete solution to the problem for regular maps. We establish that these can always be smoothly diagonalized locally and that the obstructions to doing this globally are non-trivial Weyl group and torus bundles on $M$. We show how the patching of local diagonalizing maps gives rise to non-trivial $T$-bundles, explain the relation to winding numbers of maps into $G/T$ and restrictions of the structure group and examine the behaviour of gauge fields under this diagonalization. We also discuss the complications that arise for non-regular maps and in the presence of non-trivial $G$-bundles. In particular, we establish a relation between the existence of regular sections of a non-trivial adjoint bundle and restrictions of the structure group of a principal $G$-bundle to $T$.

We use these results to justify a Weyl integral formula for functional integrals which, as a novel feature not seen in the finite-dimensional case, contains a summation over all those topological $T$-sectors which arise as restrictions of a trivial principal $G$ bundle and which was used previously to solve completely Yang-Mills theory and the $G/G$ model in two dimensions.

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1 Introduction

One of the most useful properties of a compact Lie group $G$ is that its elements can be ‘diagonalized’ or, more formally, conjugated into a fixed maximal torus $T \subset G$. In this paper we investigate to which extent this property continues to hold for spaces of smooth maps from a manifold $M$ to a compact Lie group $G$. Thus, given a smooth map $g: M \to G$, the first thing one would like to know is if it can be written as

$$g(x) = h^{-1}(x)t(x)h(x),$$

where $t: M \to T$ and $h: M \to G$ are smooth globally defined maps. It is easy to see (by examples) that this cannot be true in general, not even for loop groups ($M = S^1$), and we are thus led to ask instead the following questions:

1. Under which conditions can (1.1) be achieved locally on $M$?

2. Under which conditions will $t(x)$ be smooth (while possibly relaxing the conditions on $h$)?

3. What are the obstructions to representing $g$ as in (1.1) globally?

We will not be able to answer these questions in full generality. For those maps, however, which take values in the dense set $G_r$ of regular elements of $G$ we provide complete answers to 1-3. We establish that conjugation into $T$ can always be achieved locally and that non-trivial $T$-bundles on $M$ are the obstructions to finding smooth functions $h$ which accomplish (1.1) globally. Furthermore we prove that if either $G$ or $M$ is simply connected the diagonalized map $t$ will be smooth globally. These results confirm the intuition that (in $SU(n)$ language) obstructions to diagonalization can arise from the ambiguities in either the phase of $h$ or in the ordering of the eigenvalues of $t$.

While these questions seem to be interesting in their own right, they also arise naturally within the context of gauge fixing in non-Abelian gauge theories. In [3], ’t Hooft has argued that a ‘diagonalizing gauge’ may not only be technically useful but also essential for unravelling the physical content of these theories. For us the motivation for looking at this issue arose originally in the context of low-dimensional gauge theories. In particular, in [4, 5] we used a path integral version of the Weyl integral formula, which relates the integral of a conjugation invariant function over $G$ to an integral over $T$, to effectively abelianize non-Abelian gauge theories like 2d Yang-Mills theory and the $G/G$ gauged Wess-Zumino-Witten model. The path integrals for the partition function and correlation functions
on arbitrary two-dimensional closed surfaces $\Sigma$ could then be calculated explicitly and straightforwardly. Formally this Abelianization was achieved by using the local conjugation (gauge) invariance of the action to impose the ‘gauge condition’ $g(x) \in T$ (or its Lie algebra counterpart in the case of Yang-Mills theory). The correct results emerged when the resulting Abelian theory was summed over all topological sectors of $T$-bundles on $\Sigma$, even though the original $G$-bundle was trivial. This method has been reviewed and applied to some other models recently in [13].

In the light of the above, the occurrence of the sum over isomorphism classes of $T$-bundles can now be understood as a consequence of the fact that the chosen gauge condition cannot necessarily be achieved globally on $M = \Sigma$ by smooth gauge transformations. But while it is certainly legitimate to use a change of variables in the path integral which is not a gauge transformation, one needs to exercise more care when keeping track of the consequences of such a change of variables. Thus to the above list of questions we add (with hindsight)

4. What happens to $G$ gauge fields $A$ under the possibly non-smooth gauge transformation $A \to A^h = h^{-1}Ah + h^{-1}dh$? In particular, does this give rise to $T$ gauge fields on non-trivial $T$ bundles on $M$?

5. What is the correct version of the path integral analogue of the Weyl integral formula taking into account the global obstructions to achieving (1.1) globally? In particular, does this explain the appearance of the sum over all isomorphism classes of $T$ bundles?

It turns out that indeed connections on $T$-bundles appear in that way and that the Weyl integral formula should include a sum over those topological sectors which appear as obstructions to diagonalization.

When $M$ and $G$ are such that there are no non-trivial $G$ bundles on $M$, all isomorphism classes of torus bundles appear as obstructions (because then all torus bundles are restrictions of the trivial $G$ bundle). In particular, modulo one subtlety which we will come back to below, this takes care of the two- and three-dimensional models considered in [1,2] (as regular maps are generic in those cases and the contributions from the non-regular maps are suppressed by the zeros of the Faddeev-Popov determinant).

The close relation between restrictions of the structure group of a principal $G$ bundle and the existence of regular sections of its adjoint bundle (which hence have smooth diagonalizations in the simply-connected case) is expressed most clearly by our Proposition 6 which states that such a restriction exists if and only if the
adjoint bundle has a regular section. This is, in a sense, the fundamental result of this paper. Our proof relies on the previously established results concerning the diagonalizability of maps. If one had a different and more direct proof of this theorem (which, after all, does not refer explicitly to the issue of conjugating maps into a maximal torus), then the other existence results obtained in this paper could be obtained more or less directly as Corollaries.

The situation concerning non-regular maps is much murkier and we will not be able to say much about them. But we illustrate the difficulties which arise in that case (and in the presence of non-trivial $G$ bundles) by examples and discuss why and to which extent our present treatment fails in these cases. The subtlety mentioned above arises because the Wess-Zumino term in the $G/G$ model requires the extension of a $G$ valued map $g$ to a bounding three-manifold and there are situations where the extension is necessarily non-regular even if $g$ is regular. This problem as well as the related issue of localization in the $G/G$ model are under investigation at the moment [3].

This paper is organized as follows: In section 2 we briefly recall the basic facts we need from the theory of Lie groups: maximal tori, the Weyl group, universal and Weyl group coverings of the set of regular elements. In section 3 we discuss three prototypical examples which illustrate the possible ways in which (1.1) can fail either locally or globally. The first of these, a smooth map from $S^1$ to $SU(2)$, shows that not even $t(x)$ is necessarily smooth in general. The second, a regular map from $S^2$ to $SU(2)$, can be smoothly diagonalized locally but not globally. It provides a preliminary identification of certain obstructions in terms of winding numbers of maps from $M$ to $G/T$ and also shows quite clearly how and why connections on non-trivial $T$-bundles emerge. Finally, the third example (a map into $SO(3)$) illustrates how global smoothness of $t$ can fail even for regular $g$ when both $M$ and $G$ are not simply connected.

Sections 4-6 contain the main mathematical results of this paper. In section 4 we prove that regular maps can be smoothly conjugated into the torus over any contractible open set in $M$ and we identify the obstructions to doing this globally. The results of this section are summarized in Propositions 1 and 2. Proposition 3 contains the corresponding statements for Lie algebra valued maps.

In section 5 we investigate what happens when we try to patch together the local diagonalizing functions $h$ and rederive the previously found global obstructions from that point of view. Focussing on the case when $G$ is simply connected, we explain how finding a solution to (1.1) is related to restricting the structure group of a (trivial) principal $G$ bundle $P_G$ to $T$. We also look at what hap-
pens to gauge fields on $P_G$ and explain the relation between the Chern classes of non-trivial torus bundles on $M$ and the winding numbers of maps from $M$ to $G/T$. One of the reasons why winding numbers enter is because, in contrast with the space of maps from a two-manifold into $G$, the space of regular maps is not connected (Proposition 4). These results can be captured concisely by making use of an integral representation for a generalized winding number of such maps, depending also on a $G$-connection $A$, and are contained in Proposition 5 and the subsequent Corollaries.

In section 6 we return to those cases not covered by the previous analysis and explain the complications which arise. In particular, we establish the above-mentioned relation between the existence of regular sections of the adjoint bundle $\text{Ad}P_G$ of a non-trivial $G$ bundle $P_G$ and restrictions of the structure group to $T$ or, equivalently, sections of the associated $G/T$ bundle (Proposition 6). We then discuss some higher dimensional examples which serve to illustrate possible obstruction to restrictions of the structure group. We also address the issue of genericity of regular maps and make some (non-conclusive) comments on the problem of conjugating non-regular maps into the torus.

Finally, in section 7, we turn to an applications of the above results. We use them to justify a version of the Weyl integral formula for functional integrals over spaces of maps into a simply connected group. As a novel feature not present in the finite dimensional (or quantum mechanical path integral) version this formula includes a sum over all those topological sectors of $T$ bundles which arise as restrictions of a trivial principal $G$ bundle, justifying the method used in [1, 2] to solve exactly some low-dimensional (topological) gauge theories.

Although the entire paper has been phrased in the context of group valued maps, most of it carries over, mutatis mutandis, to the case of Lie algebra valued maps. We will point out as we go along whenever a non-obvious difference arises between the two cases.

After having completed our investigations we came across a 1984 paper by Grove and Pedersen [3] in which the local obstructions we find in section 4 are also identified, albeit using quite different techniques, see [3, Theorem 1.4]. The global issues which are our main concern in the present paper, in particular the relation between conjugation into the torus and restrictions of the structure group and the behaviour of gauge fields, are not addressed in [3], the emphasis there being on characterizing those spaces on which every continuous function taking values in normal matrices can be continuously diagonalized. These turn out to be so-called sub-Stonean spaces of dimension $\leq 2$ satisfying certain additional
criteria, \[3\] Theorem 5.6].

A final remark on terminology: we will (as above) occasionally find it convenient to use $SU(n)$ terminology even when dealing with a general compact Lie group $G$. Thus we might say ‘diagonalize’ when we should properly be saying ‘conjugate into the maximal torus’ and we may loosely refer to the action of the Weyl group as ‘a permutation of the eigenvalues’. We denote the space of maps from a manifold $M$ into a group $G$ by $\text{Map}(M, G)$. Unless specified otherwise, these maps are taken to be smooth.

2 Background from the Theory of Lie Groups

Let $G$ be a compact connected Lie group. A maximal torus $T$ of $G$ is a maximal compact connected Abelian subgroup of $G$. Its dimension is called the rank $r$ of $G$. Any two maximal tori of $G$ are conjugate to each other, i.e. if $T$ and $T'$ are maximal tori of $G$, there exists a $h \in G$ such that $T' = h^{-1}Th$ and we will henceforth choose one maximal torus arbitrarily and fix it. Since any element of $G$ lies in some maximal torus, it follows that any element of $G$ can be conjugated into $T$,

$$\forall g \in G \exists h \in G : hgh^{-1} \in T. \quad (2.1)$$

Such a $h$ is of course not unique. First of all, $h$ can be multiplied on the left by any element of $T$, $h \to th, t \in T$ as $T$ is Abelian. The residual conjugation action of $G$ on $T$ (conjugation by elements of $G$ which leave $T$ invariant) is that of a finite group, the Weyl group $W$. From the above description it follows that the Weyl group can be thought of as the quotient $W = N(T)/T$, where

$$N(T) = \{g \in G : g^{-1}tg \in T \forall t \in T\} \quad (2.2)$$

denotes the normalizer of $T$ in $G$. Thus the complete ambiguity in $h$ satisfying (2.1) for a given $g$ is $h \to nh, n \in N(T)$ and if $hgh^{-1} = t \in T$ then $(nh)g(nh)^{-1} = ntn^{-1} \in T$ is one of the finite number of images $w(t)$ of $t$ under the action of the Weyl group $W$.

For $G = SU(n)$, one has $T \sim U(1)^{n-1}$, which can be realized by diagonal matrices in the fundamental representation of $SU(n)$. $W$ is the permutation group $S_n$ on $n$ objects acting on an element of $T$ by permutation of the diagonal entries.

While it is true that any two maximal tori are conjugate to each other, it is not necessarily true that the centralizer $C(g)$ of an element $g \in G$ (i.e. the set of elements of $G$ commuting with $g$) is some maximal torus. For example, for $g$
an element of the center $Z(G)$ of $G$ one obviously has $C(g) = G$. However, the set of elements of $G$ for which $\dim C(g) = \dim T$ is open and dense in $G$ and is called the set $G_r$ of regular elements of $G$,

$$G_r = \{ g \in G : \dim C(g) = \dim T \} .$$

We also denote by $T_r$ the set of regular elements of $T$, $T_r = T \cap G_r$. The regular elements of $G$ and $T$ can alternatively be characterized by the fact that they lie in one and only one maximal torus of $G$ and this will give us some useful information on the diagonalizability of regular maps in section 5. Not only is $G_r$ open and dense in $G$ but the non-regular elements actually form a set of codimension three in $G_r$. Although this set may not be a manifold, $G_r$ and $G$ have the same fundamental group,

$$\pi_1(G_r) = \pi_1(G) .$$

Even for a regular element $g \in G_r$, the centralizer $C(g)$ need not be a maximal torus and hence conjugate to $T$ (we will see an example of that below) and it will be convenient to single out two further distinguished dense subsets of $T$ and (via conjugation) $G$. We denote by $T_n$ and $T_w$ the sets of elements $t$ of $T$ satisfying

$$T_n = \{ t \in T : C(t) = T \} ,$$

$$T_w = \{ t \in T : w(t) \neq t \ \forall w \in W, w \neq 1 \} .

While perhaps not immediately evident, it is nevertheless true that these two conditions are equivalent, $T_n = T_w$, so that, as a consequence of the obvious inclusion $T_n \subset T_r$, one also has $T_w \subset T_r$. Furthermore, if $G$ is simply connected, $\pi_1(G) = 0$, one has

$$\pi_1(G) = 0 \Rightarrow T_r = T_n = T_w .$$

It can be shown (see e.g. [4, 7]) that the conjugation map

$$q : G/T \times T_r \to G_r$$

$$([h], t) \mapsto h^{-1}th$$

is a $|W|$-fold covering onto $G_r$ and that $G/T \times T_r$ is the total space of a principal $W$ bundle over $G_r$.

If $G$ is simply connected, (2.4) implies that $G/T \times T_r$ is the total space of a trivial $W$-bundle over $G_r$ as any covering of $G_r$ is then necessarily trivial. We will see in section 4 that this simplifies the issue of diagonalizability of regular maps in that case. It follows from the above that, for $G$ simply connected, the Weyl group
acts freely on each connected component $P_r$ of $T_r = T_w$ and simply transitively on the set of components. Thus we can identify $P_r$, the image of a Weyl alcove under the exponential map, with a fundamental domain $D$ for the action of $W$ on $T_r$ and the restriction of $q$ to $P_r$ provides an isomorphism between $G/T \times P_r$ and $G_r$. In particular, as

\[
\pi_2(G/T) = \pi_1(T) = \mathbb{Z}^r, \tag{2.8}
\]

this tells us that the second homotopy group of $G_r$ is

\[
\pi_1(G) = 0 \Rightarrow \pi_2(G_r) = \mathbb{Z}^r \tag{2.9}
\]

(to be contrasted with $\pi_2(G) = 0$). As the higher homotopy groups $\pi_k(T), k > 1$, are trivial, it follows from the homotopy exact sequence associated to the fibration $G \to G/T$ that $\pi_k(G/T) = \pi_k(G)$ for $k > 2$. Thus by the same argument as above we can conclude that

\[
\pi_1(G) = 0 \Rightarrow \pi_k(G_r) = \pi_k(G) \quad \forall k > 2. \tag{2.10}
\]

In general, if one restricts $q$ to $G/T \times P_r$, it becomes a universal covering of $G_r$. Thus $P_r$ will in general contain points related by Weyl transformations as well as fixed points of $W$ as manifested by the fact that $T_w$ is not necessarily equal to $T_r$ unless $\pi_1(G) = 0$. Therefore, for general compact $G$ the above covering (2.7) is neither trivial nor connected. Nevertheless, the fact that, away from the non-regular points, the above map $q$ is a smooth fibration (with discrete fibers) will be of utmost importance in our discussion in section 4.

As an illustration of the above, let us consider the groups $SU(2)$ and $SO(3)$. The only non-regular elements of $SU(2)$ are $\pm 1$. Thus $SU(2)_r$ is isomorphic to a cylinder $S^1 \times I, I = (0, 2\pi)$ and one sees that $\pi_2(SU(2)_r) = \pi_3(SU(2)_r) = \mathbb{Z}$, in accordance with (2.9) and (2.10). The space $T_r$ of regular elements of the torus consists of two connected components, $T_r \sim (0, \pi) \cup (\pi, 2\pi)$, which explains the triviality of the fibration (2.7) in that case.

$SO(3)_r$, on the other hand, is obtained from $SO(3) \sim \mathbb{RP}^3$ by removing one point, corresponding to the identity element. The non-trivial double-covering $SU(2) \to SO(3)$ restricts to the non-trivial double covering $SU(2)_r \to SO(3)_r$ and coincides with the fibration (2.7) as $T_r(SO(3)) \sim I$. The Weyl group acts on $T_r(SO(3))$ by $\varphi \to 2\pi - \varphi$. Thus the point $\pi \in T_r(SO(3))$, corresponding to the element $t = \text{diag}(-1, -1, 1)$ in the standard embedding of $T(SO(3)) = SO(2)$ into $SO(3)$, is, while regular, a fixed point of the Weyl group, $t \notin T_w$. This is reflected in the fact that the centralizer of this element is $O(2), t \notin T_n$, which is strictly larger than than $SO(2)$ while preserving the regularity condition $\dim C(t) = \dim T$.  

8
3 Examples: Obstructions to Globally Conjugating to the Torus

We will now take a look at three examples of maps which illustrate the obstructions to achieving (1.1) globally or smoothly. The first one, which we will only deal with briefly, illustrates what can go wrong with maps which pass through non-regular points of $G$. We shall from then on (and until section 6) focus exclusively on regular maps and try to come to terms with them. The second example, a simple map from $S^2$ to $SU(2)$, allows us to detect an obstruction to globally and smoothly diagonalizing it more or less by inspection. This obstruction turns out to be a winding number associated with that map. Refining that winding number to include a gauge field contribution one can moreover read off directly that any attempt to force the map into the torus by a possibly non-smooth (discontinuous) $h$ will give rise to non-trivial torus gauge fields. The third example, a map from the circle to $SO(3)$, highlights another obstruction which can only arise when neither $G$ nor $M$ is simply connected.

Example 1: A Map from $S^1$ to $SU(2)$

Let $f$ be any smooth $\mathbb{R}$-valued function on the real line such that $f(x+2\pi) = -f(x)$. Then the map $g \in \text{Map}(S^1, SU(2))$ (the loop group of $SU(2)$) defined by

$$g(x) = \begin{pmatrix} \cos f(x) & -ie^{-ix/2} \sin f(x) \\ -ie^{ix/2} \sin f(x) & \cos f(x) \end{pmatrix}$$

(3.1)

is single-valued, $g(x+2\pi) = g(x)$, and smooth. As $f$ is necessarily zero somewhere, $g$ passes through the (non-regular) identity element. $g$ can be diagonalized by a map $h$, $hgh^{-1} = t$, but neither $h$ nor $t$ are single valued on $S^1$. For instance, $h$ can be chosen to be

$$h(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{ix/2} & -1 \\ 1 & e^{-ix/2} \end{pmatrix},$$

(3.2)

and $t$ turns out to be

$$t(x) = \begin{pmatrix} e^{if(x)} & 0 \\ 0 & e^{-if(x)} \end{pmatrix},$$

(3.3)

$$t(x+2\pi) = t^{-1}(x) \neq t(x).$$

(3.4)

What happens here is that, upon going around the circle, $t(x)$ comes back to itself only up to the action of the Weyl group, reflecting the ambiguity $h \to nh$ mentioned in section 2. Had $g$ been regular to start off with, this ambiguity could
have been consistently eliminated by giving a particular ordering prescription for the diagonal elements. Such a prescription, however, becomes ambiguous when two of the diagonal elements coincide (as at the identity element of the group).

### Example 2: A map from $S^2$ to $SU(2)$

A nice example (suggested to us by E. Witten) giving us a first idea of the possible obstructions in the case of regular maps and the role of non-trivial torus bundles is afforded by the following map from the sphere into $SU(2)$,

$$g(x) = \begin{pmatrix} ix_3 & x_1 + ix_2 \\ -x_1 + ix_2 & -ix_3 \end{pmatrix}$$

(3.5)

where the sphere is living inside $\mathbb{R}^3$ with cartesian co-ordinates $(x_1, x_2, x_3)$ and $x_1^2 + x_2^2 + x_3^2 = 1$. This map can also be written as $g(x) = \sum_k x_k \sigma_k$ which defines our conventions for the Pauli matrices $\sigma_k$.

This map is clearly regular (the only non-regular elements of $SU(2)$ being plus or minus the identity element). It is a smooth map from the two-sphere to a two-sphere in $SU(2)$ and is, in fact, the identity map when one considers $SU(2) \sim S^3$ living inside $\mathbb{R}^4$ with cartesian co-ordinates $(x_1, x_2, x_3, x_4)$, subject to $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$. We represent elements of $SU(2)$ as

$$\begin{pmatrix} x_4 + ix_3 & x_1 + ix_2 \\ -x_1 + ix_2 & x_4 - ix_3 \end{pmatrix},$$

(3.6)

so that $g$ maps the sphere to itself thought of as the equator of $S^3$ ($x_4 = 0$).

### The Obstruction: Winding Numbers

To any map $f$ from the two sphere to the two sphere we may assign an integer, the winding number $n(f)$ of that map. This is a homotopy invariant and measures the number of times that the map covers the sphere. Writing (as above) $f = \sum_k f_k \sigma_k$ with $\sum_k (f_k)^2 = 1$, an integral representation of its winding number is

$$n(f) = -\frac{1}{4\pi} \int_{S^2} Tr f [df, df],$$

(3.7)

the integral of the the pull-back of the volume form on the two sphere. This counts the covering by telling us how many times the volume one picks up. The minus sign appears in (3.7) since (in our conventions) $Tr \sigma_k \sigma_l = -2\delta_{kl}$. Clearly for (3.5) we have $n(g) = 1$, as can e.g. be seen by converting the $x_k$ to polar coordinates.
Now suppose that one can smoothly conjugate the map $g$ into a map $t : S^2 \to U(1)$ via some map $h$. As the space of maps from $S^2$ to $SU(2)$ is connected, $\pi_2(SU(2)) = 0$, $g$ is homotopic to $t$. This can be seen by choosing a homotopy $h_s$ between the identity and $h$ and defining $g_s = h_s^{-1}th_s$: then one has $g_0 = g$ and $g_1 = t$. As (3.7) is a homotopy invariant, one has $n(g) = n(t)$ but, since $g^2 = -I$, $t$ is a constant map. Actually $t$ can be either $\sigma_3 = \text{diag}(i, -i)$ or $(-\sigma_3)$. We fix on one of these throughout $S^2$ so that $t$ is smooth. This is justified in the next section. But, as $t$ is constant its winding number is zero, $n(t) = 0$, a contradiction.

More generally, if $f : S^2 \to S^2 \subset S^3$ and one is able to smoothly conjugate this map to a map into $U(1)$, then one necessarily has $n(f) = 0$. So what we have learnt is that one may not, in general, smoothly and globally conjugate into the maximal torus. We will see in the next section that we can smoothly conjugate into the maximal torus in open neighbourhoods.

Non-Trivial Torus Bundles

There is a disadvantage in simply considering the number (3.7) for it does not tell us how non-trivial $U(1)$ bundles will arise if we insist, in any case, on conjugating into $U(1)$, regardless of whether we can do so smoothly or not. There is a slight generalisation of the formula (3.7) which is not only a homotopy invariant, but for which conjugation (gauge) invariance can be established directly without any integration by parts. The advantage of such a formula is that it allows one to conjugate with arbitrary maps, not just smooth ones, and so to relate maps which are not homotopic.

Let $A$ be a connection on the $SU(2)$ product bundle over the sphere. As the bundle is trivial such an $A$ can be thought of as a Lie algebra valued one-form on $S^2$, $A \in \Omega^1(S^2, su(2))$. The number we want is

$$n(f, A) = -\frac{1}{32\pi} \int_{S^2} \text{Tr} f [df, df] - \frac{1}{2\pi} \int_{S^2} \text{Tr} [d(fA)] ,$$

(3.8)

and obviously coincides with (3.7) when both $f$ and $A$ are smooth. Furthermore $n(f, A)$ is gauge invariant, i.e. invariant under simultaneous transformation of $f$ and $A$,

$$n(h^{-1}fh, A^h) = n(f, A)$$

(3.9)

where $A^h = h^{-1}Ah + h^{-1}dh$, even for discontinuous $h$. This is seen most readily by rewriting (3.8) in manifestly gauge invariant form,

$$n(f, A) = -\frac{1}{32\pi} \int_{S^2} \text{Tr} f [dA_f, dA_f] - \frac{1}{2\pi} \int_{S^2} \text{Tr} [f F_A] ,$$

(3.10)
with \( dAf = df + [A, f] \) and \( F_A = dA + \frac{1}{2}[A, A] \).

Let us now choose \( h \) so that it conjugates our favourite map \( g \) into \( U(1) \), say \( g = h^{-1}\sigma_3h \). Using (3.9) we find

\[
n(g, A) = 1 = -\frac{1}{2\pi} \int_{S^2} \text{Tr} \sigma_3 d(A^{h^{-1}}) .
\]

(3.11)

In particular, if we introduce the Abelian gauge field \( a = -\text{Tr} \sigma_3 A^{h^{-1}} \) (we will see in section 5 that this is consistent with the gauge transformations of \( a \) and the global geometry of the problem) we obtain

\[
n(g, A) = 1 = \frac{1}{2\pi} \int_{S^2} da .
\]

(3.12)

We now see the price of conjugating into the torus. The first Chern class of the \( U(1) \) component of the gauge field \( A^{h^{-1}} \) is equal to the winding number of the original map! We have picked up the sought for non-trivial torus bundles.

If one chooses to conjugate \( g \) into \((-\sigma_3)\) instead by replacing \( h \) by \( nh \) for a suitable \( n \in \mathbb{N}(\mathbb{T}) \), the expression (3.11) remains invariant as

\[
\text{Tr}(-\sigma_3) A^{(nh)^{-1}} = \text{Tr} \sigma_3 A^{h^{-1}} .
\]

(3.13)

Thus the \( \mathbb{T} \)-bundle which emerges is independent of the choice of \( t \) and is hence canonically associated with the original map \( g \). In this case it is just the pull-back of the \( U(1) \)-bundle \( SU(2) \to SU(2)/U(1) \sim S^2 \) via \( g \) and this turns out to be more or less what happens in general. As both \( g \) and its diagonalization \( \sigma_3 \) may just as well be regarded as Lie algebra valued maps, this example establishes that obstructions to diagonalization will also arise in the (seemingly topologically trivial) case of Lie algebra valued maps.

It is possible to generalise both (3.7) and (3.10) to other manifolds and to other groups (we will do so further along). In due course we will also tie these up with some general results on the classification of torus bundles over surfaces.

Example 3: A Map from \( S^1 \) to \( SO(3) \)

While we have seen in example 1 that non-regularity is one obstruction to finding a globally well-defined smooth diagonalization \( t \), even for regular \( g \) an obstruction to finding such a \( t \) may arise. We will establish in section 4 that this can only happen when neither \( G \) nor \( M \) is simply connected. The raison d’être of this obstruction is the fact that diagonalization involves lifting a map into \( \mathcal{G}_r \) to a map into \( \mathcal{G}/\mathbb{T} \times \mathbb{T} \), which may not be possible if the fibration (2.7) is non-trivial.
Here we illustrate this obstruction by a map from $S^1$ into $SO(3)_r$ (cf. the remarks at the end of section 2).

Consider first of all the following path in $SU(2)_r$,

$$\tilde{\gamma}(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{ix/2} & ie^{-ix/2} \\ ie^{ix/2} & e^{-ix/2} \end{pmatrix}. \quad (3.14)$$

As $\tilde{\gamma}(2\pi) = -\tilde{\gamma}(0)$, $\tilde{\gamma}$ will project to a non-contractible loop $g \equiv \text{Ad}(\tilde{\gamma}) \in \text{Map}(S^1, SO(3)_r)$. Explicitly, this $g$, satisfying

$$\tilde{\gamma}^{-1}\sigma_k \tilde{\gamma} = g_{kl}\sigma_l \quad (3.15)$$

and $g(2\pi) = g(0)$, is given by

$$g(x) = \begin{pmatrix} 0 & 0 & 1 \\ \sin x & \cos x & 0 \\ -\cos x & \sin x & 0 \end{pmatrix}. \quad (3.16)$$

There is no obstruction to diagonalizing $\tilde{\gamma}$, $\tilde{\gamma} = \bar{h}^{-1}\tilde{t}\bar{h}$ and there are two solutions $\tilde{t}_\pm$ differing by a Weyl transformation (exchange of the diagonal entries). It can be checked that $\tilde{t}_\pm(2\pi)$ differs from $\tilde{t}_\pm(0)$ not only by a sign but also by a Weyl transformation,

$$\tilde{t}_\pm(0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \pm i & 0 \\ 0 & 1 \mp i \end{pmatrix} = -\tilde{t}_\pm(2\pi) \quad (3.17)$$

Hence $\tilde{t}$ will not project to a closed loop in $SO(3)$ and the diagonalization $t$ of $g$ will necessarily be discontinuous (non-periodic), as can also be checked directly.

Choosing the torus $SO(2) \subset SO(3)$ to consist of elements of the form

$$\begin{pmatrix} \cos y & -\sin y & 0 \\ \sin y & \cos y & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (3.18)$$

with the Weyl group acting as $y \to -y$, one finds that

$$t(0) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (3.19)$$

while

$$t(2\pi) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (3.20)$$
Hence the periodic regular map $g$ cannot be diagonalized to a periodic map $t$ and, regarded as map from $S^1$ into $SO(2)_r$, $t$ will only be smooth locally.

This concludes our visit to the zoo of obstructions and we now turn to establishing that at least locally regular maps can always be smoothly conjugated into the maximal torus.

4 Local Conjugation to the Maximal Torus and Global Obstructions

Let $P_G$ be a principal $G$ bundle over a smooth connected manifold $M$ and denote by $AdP_G$ the group bundle associated to $P_G$ via the adjoint action of $G$ on itself and let $g$ be a section of $AdP_G$. Locally, i.e. over a trivializing open neighbourhood $U \subset M$, $g$ can be regarded as a $G$ valued map transforming in the adjoint representation. We will from now on take $U$ to be a contractible open set. We assume that $g$ is regular, i.e. takes values in $G_r \subset G$. Note that

a) the regularity of $g$ is independent of the chosen local trivializations of $P_G$ (as $G_r$ is preserved by the adjoint action), and

b) regular $g$ always exist when $P_G$ is trivial (for more comments on the existence of regular sections see section 6).

Being able to locally conjugate smoothly into the maximal torus is the statement that we can find smooth maps $h_U \in \text{Map}(U, G)$ and $t_U \in \text{Map}(U, T)$ such that the restriction $g_U$ of $g$ to $U$ can be written as $g_U = h_U^{-1}t_Uh_U$. In other words, we are looking for a (local) lift of the map $g \in \text{Map}(M, G_r)$ to a map $(h, t) \in \text{Map}(M, G) \times \text{Map}(M, T_r)$. We will establish the existence of this lift in a two-step procedure indicated in diagram (4.1).

$$
\begin{array}{ccc}
G \times T_r & \xrightarrow{p \times 1} & G/T \times T_r \\
\downarrow{(h, t)} & & \downarrow{q} \\
M & \xrightarrow{g} & G_r
\end{array}
$$

(4.1)

In the first step we lift $g$ along the diagonal, i.e. we construct a pair $(f, t)$, where $f \in \text{Map}(M, G/T)$, which projects down to $g$ via the projection $q$ introduced in (2.7). The obstruction to doing this globally is related to the possibility of having
non-trivial $W$ bundles on $M$ (as in examples 1 and 3 of the previous section) but only arises if neither $G$ nor $M$ is simply connected.

In the second step, dealing with the upper triangle, we will lift $f$ locally to $\text{Map}(M, G)$, and the obstruction to doing this globally is given by non-trivial $T$ bundles on $M$ (as in example 2).

**The First Lifting-Problem: $W$-Bundles**

We begin by recalling that the conjugation map $q : G/T \times T_r \to G_r$, given by $([h], t) \mapsto h^{-1}th$, is a smooth $|W|$-fold covering of $G_r$ so that $G/T \times T_r$ is the total space of a principal fibre bundle over $G_r$ with fibre and structure group $W$ and projection $q$. Given the map $g$ into $G_r$, the base space of this bundle, we would like to lift this to a map into the total space, i.e. we want to find a pair $(f, t) \in \text{Map}(M, G/T) \times \text{Map}(M, T_r)$ such that diagram (4.2) commutes.

\[
\begin{array}{ccc}
G/T \times T_r & \xrightarrow{q} & G_r \\
\downarrow{(f, t)} & & \downarrow{g} \\
M & \xrightarrow{g} & G_r
\end{array}
\]

That such a map indeed exists locally is a consequence of the following fundamental result on the lifting of maps (see e.g. [9] for this and most of the other topological results used in this paper): If $P$ is a (smooth) principal fiber bundle with base space $B$ and $f$ is a (smooth) map from a manifold $X$ to $B$ then $f$ can be lifted to a (smooth) map into $P$ if and only if the pull-back bundle $f^*P$ over $X$ is trivial. It is indeed easy to see that there is a direct correspondence between lifts of $f$ and trivializing sections of $f^*P$.

The first implication of this result is that locally, i.e. over some contractible open set $U \subset M$, the desired lift can always be found as the pull-back bundle will certainly be trivializable over $U$.

However, in certain cases we can sharpen this statement to establish the existence of a global lift. Consider e.g. the case when $G$ is simply connected. As the principal $W$-bundle $G/T \times T_r \to G_r$ is then trivial, so is its pull-back to $M$ via any map $g \in \text{Map}(M, G_r)$. Hence a lift $(f, t)$ making the above diagram commute exists globally on $M$. There is an obvious $|W|$-fold ambiguity in the choice of such a lift.

Even if $G$ is not simply connected but $M$ is, the pull-back bundle is necessarily
trivial over \( M \) (otherwise it would be a non-trivial covering of \( M \)) and again a lift \((f, t)\) will exist globally.

Finally, there is a class of maps for which the \( W \)-obstruction does not arise regardless of what \( M \) and \( G \) are. This class consists of those maps \( g \) which are conjugate to a constant map \( t \) into \( T \). We will have more to say about these maps and why they are interesting in section 7.

The Second Lifting Problem: \( T \)- Bundles

It remains to lift the \( G/T \) valued map \( f \) to \( G \). Thus we are looking for a \( h \in \text{Map}(M, G) \) making the following diagram commute (with the replacement of \( M \) by \( U \) if only the local existence of \((f, t)\) could be established):

\[
\begin{array}{ccc}
M & \to & G/T \\
\downarrow{f} & & \downarrow{p} \\
G & \to & G/T \\
\uparrow{h} & \nearrow{p} & \searrow{h} \\
M & & G/T
\end{array}
\]

(4.3)

Here \( p \) is the projection of the principal fibration \( p : G \to G/T \). By construction this map will then satisfy \( g = h^{-1}th \). However, by the same result on the lifting of maps quoted above there will be an obstruction to finding such an \( h \) globally. As \( G \) can be regarded as the total space of a principal \( T \)-bundle over \( G/T \), the same reasoning as above leads us to conclude that such a lift exists iff \( f^*G \) is a trivializable \( T \) bundle over \( M \). Whether or not this is the case will depend on the interplay between the homotopy class of \( f \) and the classification of torus bundles on \( M \). However, if we restrict \( f \) to \( U \subset M \), then a lift \( h_U \) of \( f \) over \( U \) will always exist as the pull-back bundle is certainly trivializable over the contractible set \( U \). The upshot of this is that, for a regular map \( g \) we can always locally find smooth \( G \)-valued functions \( h_U \) such that \( h_U g_U h_U^{-1} \) takes values in \( T_r \).

We summarize the results about the possibility to conjugate a map locally into a maximal torus in

**Proposition 1:** Let \( G \) be a compact Lie group, \( T \) a maximal torus, \( M \) a smooth manifold, \( U \subset M \) a contractible open set in \( M \), \( P_G \) a principal \( G \) bundle over \( M \) and \( g \) a section of \( \text{Ad}P_G \). If \( g|_U \equiv g_U \) is regular, then it can be smoothly conjugated into \( T \). In other words, under these circumstances there exist smooth functions \( t_U \in \text{Map}(U, T_r) \) and \( h_U \in \text{Map}(U, G) \) such that \( g_U = h_U^{-1}t_U h_U \).

Of course, we already know a little bit more than that, for instance that under
certain conditions the diagonalized map \( t \) will exist globally. We can also be more precise about the obstruction occurring in the second lifting problem, as torus bundles are classified by \( H^2(M, \mathbb{Z}^r) \), where \( r = \dim T \) is the rank of \( G \). We have therefore established the following results concerning global obstructions to conjugating a map \( g : M \to G_r \) into the torus:

**Proposition 2:** Let \( g : M \to G_r \) be a smooth regular map. Then a smooth map \( t : M \to T_r \) satisfying \( g = h^{-1}th \) for some (not necessarily smooth) map \( h : M \to G \) exists globally if \( g^*(G/T \times T_r) \) is the total space of a trivial \( W \)-bundle over \( M \). If, furthermore, \( f^*(G) \) (where \( f \) is the \( G/T \)-part of the lift of \( g \)) is a trivial \( T \) bundle over \( M \), then \( h \) can be chosen to be smooth globally.

**Corollary 1:** If either \( M \) or \( G \) is simply connected, a smooth diagonalization \( t \in \text{Map}(M, T_r) \) of a regular \( g \) will exist globally. If, moreover, \( H^2(M, \mathbb{Z}) = 0 \), then a smooth regular map \( g \) can be smoothly conjugated into a maximal torus, i.e. there exists a smooth function \( h \in \text{Map}(M, G) \) such that \( g = h^{-1}th \).

As loop groups are a particularly interesting and well studied class of spaces of group valued maps \([10]\), we also mention separately the following immediate consequence of the above considerations:

**Corollary 2:** If \( G \) is simply connected, every regular element of the group \( LG \) of smooth loops in \( G \) can be smoothly diagonalized.

Examples 1 and 3 of section 3 show that both regularity and simple connectivity are necessary conditions. What we have shown is that they are also sufficient.

At least when \( G \) is simply connected, there is a slightly more canonical way of describing the results obtained in Proposition 2, one which does not depend on the (arbitrary) choice of a maximal torus \( T \) of \( G \). We first observe that over \( G_r \) there is a natural torus bundle \( P_C \) (the centralizer bundle) with total space

\[
P_C = \{(g_r, g) \in G_r \times G : g \in C(g_r)\}
\]  

(4.4)

and projection \((g_r, g) \mapsto g_r\). For any map \( g \in \text{Map}(M, G_r) \) this bundle can be pulled back to a torus bundle \( g^*P_C \) over \( M \) and it is the possible non-triviality of this bundle which is the obstruction to finding a globally smooth \( h \) accomplishing the diagonalization. To make contact with the previous construction, we note that under the isomorphism \( q : G/T \times P_r \to G_r \) the bundle \( P_C \) pulls back to the \( T \)-bundle \( G \times P_r \to G/T \times P_r \), while the lift \((f, t)\) in diagram (4.2) can be
written as \((f, t) = q^{-1} \circ g\). This is illustrated in the diagram below.

\[
\begin{CD}
g^*P_G @>>> \hat{G} @>>> G \times P_r \\
M @>g>> G_r @>q>> G/T \times P_r \\
\end{CD}
\]

Conjugation of \(g\)-valued Maps into the Cartan Subalgebra

The question of diagonalizability of Lie algebra valued maps (the case of interest in e.g. Yang-Mills or Chern-Simons theory) can be addressed in complete analogy with the analysis for group valued maps performed above. It will turn out that the only substantial difference between the two is that the first obstruction (non-trivial \(W\)-bundles) does not arise. That the second obstruction, related to non-trivial torus bundles, persists can already be read off from example 2 of section 3 as the map \(g = \sum_k x_k \sigma_k\) and its diagonalization \(t = \pm \sigma_3\) considered there can equally well be regarded as Lie algebra valued maps.

Let us denote by \(g\) and \(t\) (a Cartan subalgebra of \(g\)) the Lie algebras of \(G\) and \(T\) respectively and by \(g_r\) and \(t_r\) their regular elements. As in (2.7) there is a smooth \(|W|\)-fold covering

\[
q' : G/T \times t_r \to g_r ,
q'([h], \tau) = h^{-1} \tau h .
\] (4.6)

However, \(g\) is a vector space and hence simply connected. As a consequence \(g_r\) is simply connected as well. Therefore this \(W\)-bundle is necessarily trivial and the first lifting problem can always be solved globally on \(M\). This establishes the global existence of a lift \((f, \tau)\) of a smooth map \(\phi \in \text{Map}(M, g_r)\) to \(G/T \times t_r\). In particular, a smooth global diagonalization \(\tau \in \text{Map}(M, t_r)\) of \(\phi\) always exists.

The second lifting problem depends only on the \(G/T\)-part \(f\) of the lift and is identical with that for group valued maps. Therefore the situation concerning Lie algebra valued maps is the following:

**Proposition 3:** Let \(\phi \in \text{Map}(M, g_r)\) be a smooth regular map into the Lie algebra \(g\) of a compact Lie group. Then a smooth diagonalization \(\tau \in \text{Map}(M, t_r)\) exists globally. If \(f^*G\) is the total space of a trivial principal \(T\)-bundle over \(M\) then there exists a smooth functions \(h \in \text{Map}(M, G)\) such that \(\phi = h^{-1} \tau h\) globally.
Corollary 3: If $H^2(M, \mathbb{Z}) = 0$, any $\phi \in \text{Map}(M, g_r)$ can be smoothly diagonalized.

5 Global Conjugation and Non-Trivial Torus Bundles

Having demonstrated that we are able to conjugate into the maximal Torus smoothly in open contractible neighbourhoods we turn to more global questions. In particular we want to investigate what happens when we perform the diagonalizations patch-wise and then try to glue the resulting data together globally. This has a three-fold purpose. Firstly, it allows us to understand the obstructions we encountered above in a more down-to-earth way by constructing explicitly the non-liftable maps into $G/T$ and the transition functions of non-trivial $T$-bundles in terms of the local data $(h_U, t_U)$. Secondly it will allow us to establish a partial converse to the above results in that we are e.g. able to show that under certain circumstances any non-trivial $T$ bundle will appear as the obstruction for some map. And thirdly it is the most convenient language to analyse what happens to gauge fields when one insists on diagonalizing globally by a collection of locally defined $h$’s. Unless stated otherwise, $G$ will be assumed to be simply connected in this section.

Glueing the Local Data

Let again $g \in \text{Map}(M, G_r)$ be a regular smooth map and let us choose a covering of $M$ by open contractible sets $\{U_\alpha\}$. By Proposition 1 we have, on each $U_\alpha$, smooth functions $(h_\alpha, t_\alpha)$ such that $g_\alpha = h_\alpha^{-1} t_\alpha h_\alpha$ where $g_\alpha = g|_{U_\alpha}$. Actually, we already know that the $t_\alpha$’s glue together to a global map (we are assuming that $\pi_1(G) = 0$), but we will rederive this result below in a different way, one which gives more insight into why the condition of regularity is so crucial.

As $g$ is globally defined, on the overlaps $U_\alpha \cap U_\beta$ of patches we must have

$$h_\alpha^{-1} t_\alpha h_\alpha = h_\beta^{-1} t_\beta h_\beta .$$  \hspace{1cm} (5.1)

Put another way, the $t_\alpha$’s in different patches are related by

$$t_\alpha = h_{\alpha\beta} t_\beta h_{\alpha\beta}^{-1} ,$$  \hspace{1cm} (5.2)

where

$$h_{\alpha\beta} = h_\alpha h_\beta^{-1} .$$  \hspace{1cm} (5.3)

The functions $h_{\alpha\beta}$, a priori taking values in $G$, can be regarded as the transition functions of a trivial $G$ bundle on $M$ (that they satisfy the cocycle condition
\( h_{\alpha \beta} h_{\beta \gamma} = h_{\alpha \gamma} \) is clear from the definition (5.3)). The \( t_\alpha \) thus transform as sections of the (trivial) adjoint bundle \( \text{Ad}P_G \) of \( P_G \sim M \times G \).

Restrictions of the Structure Group and Torus Bundles

The first thing we will show is that regularity of the \( t_\alpha \) implies that the transition functions \( h_{\alpha \beta} \) take values in the normalizer \( N(T) \). To see that, note that pointwise (5.2) implies that \( t_\alpha \) is contained not only in \( T \) but also in the maximal torus \( h_{\alpha \beta} T h_{\alpha \beta}^{-1} \). By regularity of \( t_\alpha \) this implies that \( T = h_{\alpha \beta} T h_{\alpha \beta}^{-1} \). From this we conclude that the \( h_{\alpha \beta} \) take values in the normalizer of \( T \) in \( G \),

\[
h_{\alpha \beta} : U_\alpha \cap U_\beta \to N(T) ,
\]

so that a choice of \( h_\alpha \)'s diagonalizing \( g \) gives rise to a restriction of the structure group of the (trivial) principal \( G \)-bundle to \( N(T) \) (for the precise definition of the restriction of structure groups see section 6). Then (5.2) means that on overlaps \( U_\alpha \cap U_\beta \) the \( t_\alpha \)'s are related by Weyl transformations.

So far the condition \( \pi_1(G) = 0 \) has not entered, in agreement with the results of section 4 which allow one to read off that the ambiguity in patching together the local solutions \( t_\alpha \) can always be reduced to a \( W \) ambiguity.

If, however, \( G \) is simply connected, by choosing the \( h_\alpha \)'s appropriately, the \( h_{\alpha \beta} \)'s can be arranged to take values in \( T \) and hence the structure group will be reduced to \( T \). For \( G = SU(n) \) this can be done by adopting some particular ordering prescription for the diagonal entries, e.g. according to size. As no two eigenvalues are the same for regular elements, this eliminates the Weyl ambiguity (permutation of the diagonal entries). And in general this is achieved in the following way. Let us choose a fundamental domain \( D \) for the action of the Weyl group \( W \) on \( T_r \), i.e. some identification of \( T_r/W \) with a connected component \( D \sim P_r \) of \( T_r \) (this is where the assumption of simple connectivity of \( G \) enters in a crucial way). Let us now, using the ambiguity \( h_\alpha \to n_\alpha h_\alpha, n_\alpha \in \text{Map}(U_\alpha, N(T)) \), choose the \( h_\alpha \)'s in such a way that the \( t_\alpha \)'s take values in \( D \). In the language of the previous section this choice corresponds to a choice of lift in diagram (4.2). Then (5.2), read as \( t_\alpha = w_{\alpha \beta}(t_\beta), w_{\alpha \beta} \in W \), implies \( w_{\alpha \beta} = 1 \) as \( t_\alpha \) and \( t_\beta \) take values in \( D \) while \( D \cap w(D) = \emptyset \) unless \( w = 1 \). But this implies that

\[
t_\alpha = t_\beta \quad \text{on} \quad U_\alpha \cap U_\beta ,
\]

so that there exists a globally defined smooth \( t = \{ t_\alpha \} \) (as we had already seen in the previous section). It also implies that the \( h_{\alpha \beta} \) are \( T \)-valued functions on the overlaps \( U_\alpha \cap U_\beta \),

\[
h_{\alpha \beta} \equiv h_\alpha h_\beta^{-1} : U_\alpha \cap U_\beta \to T .
\]
Thus the $h_{\alpha \beta}$ define a (possibly non-trivial) torus bundle on $M$ which is however trivial when regarded as a principal $G$ bundle (since $h_{\alpha \beta} = h_{\alpha} h_{\beta}^{-1}$). Using the trivial identity $h_{\alpha} = h_{\alpha \beta} h_{\beta}$ we also see that we can interpret the $h_{\alpha}$’s as (local trivializing) sections of the principal $T$ bundle $P_T$ with fibre $Th_{\alpha}(x) = \{th_{\alpha}(x), t \in T\}$ above the point $x \in M$, establishing once again that the $h_{\alpha}$’s can be defined globally iff the bundle $P_T$ is trivial.

Thus, given a regular smooth map $g$ we obtain, upon choice of a fundamental domain $D$, a smooth $T_r$-valued map $t$ and a principal $T$ bundle $P_T$ characterized by the transition functions $h_{\alpha \beta}$. We now want to establish a converse to this result.

Let us assume that $M$ and $G$ are such that all $G$-bundles on $M$ are trivial. Given some $T$ bundle $P_T$ and a regular smooth map $t$, one obtains a map $g$ as follows. As every $T$ bundle is, in particular, a $G$ bundle ($T \subset G$), regarded as a $G$ bundle $P_T$ is necessarily trivial. This means that the transition functions $h_{\alpha \beta}$ on $U_\alpha \cap U_\beta$ that define $P_T$ can be expressed as $h_{\alpha \beta} = h_{\alpha} h_{\beta}^{-1}$ where the $h_{\alpha}$ are $G$-valued maps on the respective patches $U_\alpha$. Armed with this data one defines on each patch $U_\alpha$

$$g_\alpha = h_{\alpha}^{-1} t_\alpha h_{\alpha} \quad (5.7)$$

where $t_\alpha = t|_{U_\alpha}$. These $g_\alpha$ patch together to define a globally well-defined smooth $G_r$-valued map since on overlaps one has

$$g_\alpha = h_{\alpha}^{-1} t_\alpha h_{\alpha}$$
$$= h_{\beta}^{-1} h_{\alpha}^{-1} t_\alpha h_{\alpha} h_{\beta}$$
$$= h_{\beta}^{-1} t_\beta h_{\beta} = g_\beta \quad . \quad (5.8)$$

By construction, this map $g$ will give rise to the transition functions of $P_T$ upon diagonalization. In particular, therefore, in the case at hand every isomorphism class of torus bundles will appear as the obstruction to global diagonalizability for some regular map $g$. We will see below that this can also be understood directly in terms of classifying maps and universal bundles.

**Maps into $G/T$**

We have seen in section 4 that, in addition to a smooth $T_r$ valued map $t$, a regular $g$ also gives rise to a map $f \in \text{Map}(M, G/T)$ governing the obstruction to conjugating $g$ into $T$ smoothly. In terms of the local data $h_{\alpha}$ associated with $g$ and $t$, i.e. chosen to be compatible with a fixed fundamental domain $D$, these maps can be constructed in the following way.
We realize $G/T$ as the coadjoint orbit $\mathcal{O}_\mu$ through a regular element $\mu \in \mathfrak{t}^*$ ($\mathfrak{t}$ denoting the Lie algebra of $T$) so that e.g. the principal fibration $p : G \to G/T$ can be written as $p(g) = g^{-1}\mu g$ (see the discussion following (5.16) below for more on coadjoint orbits). We then define local $G/T$-valued maps $f_\alpha$ by

$$f_\alpha : U_\alpha \to \mathcal{O}_\mu,$$

$$f_\alpha = h_\alpha^{-1}\mu h_\alpha. \quad (5.9)$$

As upon a choice of $D$ the $h_\alpha$’s are unique up to left-multiplication by $T$-valued maps, the $f_\alpha$’s are well-defined and independent of which $h_\alpha$’s one chooses. On overlaps $U_\alpha \cap U_\beta$ one finds that

$$f_\beta = h_\beta^{-1}\mu h_\beta$$

$$= h_\alpha^{-1}(h_\alpha h_\beta^{-1})\mu(h_\beta^{-1}h_\alpha)h_\alpha$$

$$= h_\alpha^{-1}h_\alpha\beta\mu h_\alpha^{-1}h_\alpha$$

$$= h_\alpha^{-1}\mu h_\alpha = f_\alpha, \quad (5.10)$$

since the transition functions $h_\alpha\beta$ are $T$-valued and act trivially on $\mu$. Thus the $f_\alpha$ define a globally well-defined map $f \in \text{Map}(M, G/T)$ whose local lifts to $G$ are given by the $h_\alpha$, as in diagram (4.3). It is clear that $f$ is conjugate to the constant map $\mu$ if a diagonalizing $h$ exists globally.

We shall see below how, for simplicity in the case that $M$ is two-dimensional, the winding numbers of $f$ are related to the Chern classes of the corresponding torus bundle over $M$.

**Relation between Connections on $G$ and $T$ Bundles**

We consider again the case of regular maps $g \in \text{Map}(M, G_r)$, i.e. sections of the adjoint bundle associated to the trivial bundle $P_G \sim M \times G$. Via a choice of trivialization, connections on $P_G$ can be identified with Lie-algebra valued one-forms on $M$. Gauge transformations (vertical automorphisms of $P_G$) can be identified with sections $h$ of $\text{Ad}P_G$ and the induced action of $h$ on $g$ is given by $g \to hgh^{-1}$. Thus, when considering connections on $P_G$, such a transformation has to be accompanied by a gauge transformation on the gauge fields, $A \to hAh^{-1} + hdh^{-1}$. We now look at what happens to gauge fields when we gauge transform them patch-wise using the diagonalizing maps $h_\alpha$. Let the connection obtained in this way on an open set $U_\alpha$ be denoted by $A_\alpha$. On the overlap $U_\alpha \cap U_\beta$ one has

$$A_\alpha = h_\alpha\beta A_\beta h_\alpha^{-1} + h_\alpha\beta dh_\alpha^{-1}. \quad (5.11)$$
Decomposing the Lie-algebra \( g \) as \( g = t \oplus k \) and correspondingly the gauge field as \( A_\alpha = A^t_\alpha + A^k_\alpha \), one finds that

\[
A^t_\alpha = A^t_\beta + h_{\alpha\beta} d h^{-1}_{\alpha\beta},
\]
\[
A^k_\alpha = h_{\alpha\beta} A^k_\beta h^{-1}_{\alpha\beta},
\]
(5.12)

as the \( h_{\alpha\beta} \) take values in \( T \). Thus only the \( t \)-component of the new gauge field \( A^{h^{-1}} = \{A_\alpha\} \) transforms inhomogeneously and defines a connection on the torus bundle \( P_T \) determined by the transition functions \( h_{\alpha\beta} \). The \( k \)-component, on the other hand, transforms as a one-form with values in sections of the bundle \( P_T \times_T k \) associated to \( P_T \) via the adjoint action of \( T \) on \( k \). The very same conclusions can be reached if one starts off with a connection on a non-trivial \( G \)-bundle admitting a restriction to \( T \) and then proceeds with the local analysis as in section 6.

Classification of Torus Bundles

We would now like to bring some of the threads together which have appeared in this and the previous section. We have seen that, associated with a regular map \( g \in \text{Map}(M, G_r) \) and a connection \( A \) on \( P_G \sim M \times G \) we have the following data:

- a smooth map \( t \in \text{Map}(M, T_r) \), unique up to \( W \)-transformations;
- a corresponding collection of maps \( h_\alpha \in \text{Map}(U_\alpha, G) \), unique up to multiplication by \( T \)-valued functions on the left;
- a principal \( T \) bundle \( P_T \), determined by the transition functions \( h_{\alpha\beta} = h_\alpha h^{-1}_\beta \);
- a smooth map \( f \in \text{Map}(M, G/T) \), uniquely determined by the choice of \( t \);
- a connection \( A^t = \{A^t_\alpha\} \) on \( P_T \);
- a one-form \( \{A^k_\alpha\} \) with values in the sections of the associated bundle \( P_T \times_T k \).

On the basis of the arguments presented in section 4 one would expect there to be a close relation between the topological type (Chern classes) of \( P_T \) and the homotopy class (winding numbers) of \( f \). To make this relation as explicit as possible, we consider in the following the case of two-dimensional orientable manifolds \( M = \Sigma \) (and, as before, simply connected groups).

In order to proceed it will be helpful to make use of the notions of universal bundles and classifying spaces \([9]\). By definition, a universal \( H \) bundle, \( H \) some
(compact) group, is a principal $\mathcal{H}$ bundle with contractible total space $E\mathcal{H}$. It can be shown that isomorphism classes of principal $\mathcal{H}$ bundles on $M$ are in one-to-one correspondence with homotopy classes of based maps from $M$ to $B\mathcal{H} = E\mathcal{H}/\mathcal{H}$, the base space of the universal bundle. For this reason, $B\mathcal{H}$ is called the classifying space. The correspondence is given by pull-back, that is, every principal $\mathcal{H}$ bundle over $M$ can be realized as $f^*E\mathcal{H}$ for some $f : M \to B\mathcal{H}$ and two bundles $f_1^*E\mathcal{H}$ and $f_2^*E\mathcal{H}$ are isomorphic if and only if $f_1$ and $f_2$ are homotopic.

Usually, $E\mathcal{H}$ is infinite-dimensional (e.g. $EU(1) = S^\infty$ with $BU(1) = \mathbb{CP}^\infty$), but if one is only interested in classifying bundles in (or up to) a given dimension $n$ it is sufficient to consider a bundle $E\mathcal{H}^{(n)} \to B\mathcal{H}^{(n)}$ which is $n$-universal, i.e. for which $\pi_0(E\mathcal{H}^{(n)}) = \pi_1(E\mathcal{H}^{(n)}) = \ldots = \pi_n(E\mathcal{H}^{(n)}) = 0$. $n$-universal bundles and $n$-classifying spaces can be chosen to be finite dimensional (and are typically Stiefel-bundles over Grassmannians).

Thus, in order to classify torus bundles over a surface $\Sigma$, we need a 2-universal bundle $ET^{(2)}$. If $G$ is simply-connected, then the bundle $G \to G/T$ with $BT^{(2)} = G/T$ precisely satisfies this requirement as one has $\pi_0(G) = \pi_1(G) = \pi_2(G) = 0$ in this case. This means that isomorphism classes of torus bundles are in one to one correspondence with homotopy classes of maps from $\Sigma$ to $G/T$. As $G/T$ is simply connected, $\pi_1(G/T) = 0$, up to the two-skeleton it can be regarded as an Eilenberg - MacLane space $K(\pi, 2)$, where $\pi = \pi_2(G/T)$ and one therefore has

$$[\Sigma, BT]_* \sim H^2(\Sigma, \pi_2(G/T)) \sim \pi_2(G/T) \sim \mathbb{Z}^r .$$

Hence torus bundles on $\Sigma$ are classified by an $r$-tuple of integers measuring the winding around non-contractible two-spheres in $G/T$.

In particular, therefore, with each regular map $g \in \text{Map}(\Sigma, G_r)$ there is associated an $r$-tuple of integers. Furthermore, it is clear (and can also be read off from the integral representations of the winding numbers given below) that these integers do not change under regular homotopy, i.e. under a homotopy $g_s$, $s \in [0, 1]$ between $g_0$ and $g_1$ where $g_s$ is regular for all $s$. Now, all maps from $\Sigma$ to $G$ are homotopic as

$$\pi_0(\text{Map}(\Sigma, G)) = \pi_2(G) = 0 .$$

In particular, therefore, all maps into $G_r$ can be homotoped into each other (by possibly non-regular homotopies). But, as a consequence of (2.9), the space of regular maps consists of a $\mathbb{Z}^r$'s worth of disconnected components,

$$\pi_0(\text{Map}(\Sigma, G_r)) = \pi_2(G_r) = \mathbb{Z}^r .$$

Because the proof of (2.9) hinges on the fact that $\pi_2(G/T)$ and $\pi_2(G_r)$ are equal under the isomorphism provided by the conjugation map $q$, it also follows that two
maps \(g_0\) and \(g_1\) are regularly homotopic if and only if they give rise to homotopic maps \(f_0\) and \(f_1\) into \(G/T\) (i.e. to maps with the same winding numbers). Thus we have shown

**Proposition 4:** Let \(G\) be a simply connected compact Lie group and \(\Sigma\) a two-dimensional manifold. Then \(\pi_0(\text{Map}(\Sigma, G_r)) = \mathbb{Z}^r\) and two regular maps are regularly homotopic if and only if their lifts to \(G/T\) are homotopic in \(\text{Map}(\Sigma, G/T)\).

**Coadjoint Orbits and Symplectic Forms**

The winding numbers of the maps into \(G/T\) can be given an integral representation which we shall use to relate them to the Chern classes of torus bundles, as in the discussion in example 2 of section 3. They are best understood in terms of the Kirilov-Kostant symplectic forms on \(G/T\) thought of as a coadjoint orbit of \(G\), i.e. in terms of volume forms on the non-contractible two-spheres in \(G/T\). We thus make a short excursion into the symplectic geometry of coadjoint orbits.

For \(\mu\) a regular element of \(t \sim t^*\) let \(O_\mu\) be the coadjoint orbit through \(\mu\),

\[
O_\mu = \{ g^{-1}\mu g : g \in G \}.
\]

(5.16)

Because \(\mu\) is regular, the stabilizer of \(\mu\) is isomorphic to \(T\) and one has \(O_\mu \sim G/T\). The coadjoint orbit comes equipped with a natural symplectic form (Kirillov-Kostant form) which is defined as follows. The infinitesimal version of the \(G\)-action on \(O_\mu\) is \(\delta_X \mu = [\mu, X]\) with \(X \in \mathfrak{g}\) defined mod \(t\). As the \(G\) action is transitive, the tangent space to \(O_\mu\) at \(\mu\) is spanned by tangent vectors of this form. The symplectic form at \(\mu\) is defined by

\[
\omega_\mu(\delta_X \mu, \delta_Y \mu) = \text{Tr} \mu [X, Y]
\]

(5.17)

and extended to all of \(O_\mu\) by the \(G\)-action. It is easily verified that the right hand side of (5.17) depends on \(X\) and \(Y\) only modulo \(t\) and that it defines a closed non-degenerate two-form on \(O_\mu\), i.e. a symplectic form. Varying \(\mu\) in \(t_r\) one obtains an \(r\)-parameter family of symplectic forms on \(G/T\) and for certain values of \(\mu\) these symplectic forms are integral. Let us assume that \(\text{Tr}\) is normalized in such a way that the \(r\) symplectic forms \(\omega^k, k = 1, \ldots, r\) for \(\mu = \alpha^k\) a simple root are the generators of \(H^2(G/T, 2\pi\mathbb{Z}) \sim 2\pi\mathbb{Z}^r\).

One can then assign \(r\) homotopy invariant integers \(n^k(f)\) to any map \(f \in \text{Map}(\Sigma, G/T)\) by

\[
n^k(f) = \frac{1}{2\pi} \int_\Sigma f^*(\omega^k)
\]

(5.18)

which measure the windings of \(f\) around the non-contractible two-spheres in \(G/T\) Poincaré dual to the two-forms \(\omega^k\). In (5.18) it is useful (but not mandatory) to
think of \( f \) as a map into the coadjoint orbit \( \mathcal{O}^k \sim G/T \) through \( \alpha^k \) and we will henceforth make use of this identification. Let us therefore write \( f = h^{-1} \alpha^k h \) where \( h \) is some (not necessarily continuous) function defined by a global section of the pull-back bundle \( f^*(G) \) over \( \Sigma \) (e.g. via the local diagonalizing functions \( \{h_\alpha\} \)). Then, as a consequence of \( df = [f, h^{-1}dh] \), (5.18) can be written more explicitly as

\[
n^k(f) = \frac{1}{4\pi} \int_\Sigma \text{Tr} h^{-1} \alpha^k [h^{-1}dh, h^{-1}dh]
\]

\[
= \frac{1}{4\pi} \int_\Sigma \text{Tr} \alpha^k [dh^{-1}, dh^{-1}] .
\]

(5.19)

Note that this vanishes if \( h \) is globally defined as \( [dh^{-1}, dh^{-1}] = 2d(dh^{-1}) \) is then exact, in agreement with the fact that \( f \) should have winding number zero in that case.

The expression for the winding number in the \( SU(2) \)-case given in (3.7) is not manifestly of the above form, so let us show that the two definitions nevertheless agree in that case. Learning how to reduce (3.7) to (5.19) will also allow us to extend the generalized winding number (3.8) to groups other than \( SU(2) \) as verification of the homotopy invariance of (3.7) requires the use of identities which are special to \( SU(2) \). Thus, in (3.7) write \( f = h^{-1} \mu h \). Using the \( SU(2) \) identity \( \text{Tr}[a,b][c,d] = 4(\text{Tr}ac\text{Tr}bd - \text{Tr}ad\text{Tr}bc) \) one finds

\[
\text{Tr} f[df, df] = 4 \text{Tr} \mu^2 \text{Tr} \mu [dh^{-1}, dh^{-1}] ,
\]

(5.20)

so that indeed for \( \mu = \sigma_3 \) the two expressions (3.7) and (5.19) for the winding numbers of maps into \( S^2 \sim SU(2)/U(1) \) agree.

**Generalized Winding Numbers**

On the other hand we know that torus bundles over a surface \( \Sigma \) are also classified in terms of \( r \) first Chern classes \( c^k_1(A^t) \), where the \( A^t \) are connections on these bundles. The formula here is

\[
c^k_1(A^t) = \frac{1}{2\pi} \int_\Sigma da^k
\]

(5.21)

where \( a^k = -\text{Tr} \alpha^k A^t \) is the \( k \)’th ‘component’ of \( A^t \). As the integers \( c^k_1(A^t) \) determine the bundle completely, one would expect a relationship with the integers \( n^k(f) \). We will now show that these two descriptions of the torus bundles are bridged by the considerations involved in establishing that one may conjugate maps into the torus.
To that end we introduce a formula that interpolates between (5.18) and (5.21) and which generalises that given for the case of $SU(2)$ in (3.10). We first put the expression for the generalized $SU(2)$ winding number into a form which is amenable to generalization to other groups. Writing $f$ as $f = h^{-1} \mu h$ and using the above trace identity for $SU(2)$ and 
\[ d_A f \equiv df + [A, f] = h^{-1}[A^{h^{-1}}, \mu]h , \] (5.22)
one finds that (3.8,3.10) can be written as
\[ n(f, A) = -\frac{1}{2\pi} \int \text{Tr} \mu d(A^{h^{-1}}) , \] (5.23)
(cf. (3.11)). Thus, for $A$ a gauge field on the trivial principal $G$ bundle we are led to define
\[ n^k(f, A) = -\frac{1}{2\pi} \int_{\Sigma} \text{Tr} \alpha^{k} d(A^{h^{-1}}) \] (5.24)
as the generalized winding numbers of $f \sim h^{-1} \alpha^{k} h$. We will see in Corollary 5 below that they can be interpreted as monopole numbers. As such they should provide integral representations for the magnetic numbers introduced in \[3].

While (5.24) is a very compact way of writing the winding number, there are two alternative expressions (corresponding to (3.8) and (3.10) respectively) which make one or the other of the properties of (5.24) manifest. First of all, as in the $SU(2)$ case, this generalized winding number differs from the winding number $n^{k}(f)$ only by a total derivative,
\[ n^{k}(f, A) = n^{k}(f) - \frac{1}{2\pi} \int_{\Sigma} d(\text{Tr} f A) . \] (5.25)
Furthermore, it can also be written in terms of the curvature $F_A$ of $A$ and the covariant derivative of $f$. We write $d_A f$ as $d_A f = [D(A, f), f]$, so that $D(A, f) = A - h^{-1}dh$ modulo terms that commute with $f$. Then $n^{k}(f, A)$ can be written as
\[ n^{k}(f, A) = -\frac{1}{2\pi} \int \text{Tr} f[D(A, f), D(A, f)] - \frac{1}{2\pi} \int \text{Tr} f F_A , \] (5.26)
which makes its analogy with (3.10) manifest. This also shows that the generalized winding number makes sense for non-trivial principal bundles. The following Proposition lists the main properties of (5.24).

**Proposition 5:** Let $f \in \text{Map}(\Sigma, G/T)$ be a smooth map and denote by $h \in \text{Map}(\Sigma, G)$ a (possibly discontinuous) lift of $f$ to $G$. Let $A \in \Omega^1(\Sigma, g)$ represent a gauge field on the trivial principal $G$ bundle $P_G \sim \Sigma \times G$. Then $n^{k}(f, A)$ has the following properties:
1. $n^k(f, A)$ is independent of $A \in \Omega^1(\Sigma, g)$

2. $n^k(f, A)$ is gauge invariant, i.e.

$$n^k(g^{-1}fg, g^{-1}Ag + g^{-1}dg) = n^k(f, A)$$

for any $g \in \text{Map}(\Sigma, G)$.

3. If $h$ can be chosen to be smooth, $n^k(f, A) = 0$.

4. For $A = 0$, $n^k(f, A)$ reduces to the integral of the pull-back of the Kirilov-Kostant form $\omega$ to $\Sigma$, i.e. to the winding number $(5.18)$ of $f$,

$$n^k(f, A = 0) = n^k(f) .$$

**Proof:** Properties 1 and 4 follow immediately from $(5.25)$ and property 2 from $(5.26)$. Alternatively, one can argue as follows. The variation of $(5.24)$ with respect to $A$ is

$$\delta n^k(f, A) = -\frac{1}{2\pi} \int_{\Sigma} d(\text{Tr} \alpha^k h \delta Ah^{-1})$$

$$= -\frac{1}{2\pi} \int_{\Sigma} d(\text{Tr} f \delta A) = 0 , \quad (5.27)$$

as $f$ and $\delta A$ are globally defined. This establishes property 1. Note that this argument also goes through for $A$ a connection on a non-trivial $G$ bundle $P_G$. Property 2 follows from the observation that $f \to g^{-1}fg$ corresponds to $h \to hg$ so that $A^{h^{-1}}$ is the manifestly gauge invariant combination of $A$ and $h$. Property 3 holds because the integrand of $(5.24)$ is globally exact if $h$ is smooth, and property 4 is a consequence of

$$\text{Tr} \alpha^k d(dhh^{-1}) = \frac{1}{2} \text{Tr} \alpha^k [dh^{-1}, dhh^{-1}] . \quad (5.28)$$

This completes the proof of the proposition an immediate consequence of which is the equality of the winding numbers $(5.18)$ and the torus bundle Chern classes $(5.21)$ claimed above:

**Corollary 4:** Let $f \in \text{Map}(\Sigma, G/T)$, $h \in \text{Map}(\Sigma, G)$ and $A$ be as above. Define the torus gauge field $A^t$ to be the $t$ component of $A^{h^{-1}}$. Then the winding numbers of $f$ are equal to the Chern numbers of $A^t$,

$$n^k(f) = c_1^k(A^t) .$$

Returning to our problem of conjugating maps into the torus, we can now read off directly from the above that a smooth map $g \in \text{Map}(\Sigma, G_T)$ can be
smoothly conjugated into the torus iff the (generalized) winding number of \( f \) is zero. Furthermore, if one insists on conjugating into the torus nevertheless, albeit by a non-continuous \( h \), the resulting map \( f \) is a constant map (with winding number zero) but \( n^k(f, A) \) will remain unchanged, measuring the obstruction to doing this smoothly. This establishes

**Corollary 5:** Let \( g \in \text{Map}(\Sigma, G_r) \) be a smooth regular map, \((f, t)\) a lift of \( g \) to \( G/T \times T_r \), \( P_T \) the corresponding \( T \)-bundle. Then the generalized winding numbers \( n^k(f, A) \) are the Chern numbers of \( P_T \) and \( g \) can be smoothly conjugated into \( t \) iff the \( n^k(f, A), k = 1, \ldots, r \) are zero for some (and hence all) \( A \in \Omega^1(\Sigma, g) \).

### 6 Generalizations: Non-Regular Maps and Non-Trivial G Bundles

In this section we will take a look at some of the topics we have only touched briefly or glossed over completely so far. In particular, we will extend the analysis of section 5 from regular maps to (regular) sections of non-trivial Ad-bundles, and we will come back to the question of non-regular maps we had quickly abandoned after the first example of section 3. As it turns out, these two issues are not unrelated as there may be obstructions to finding *any* regular sections.

#### Diagonalizing Sections of Non-Trivial Ad-Bundles

We consider now the situation where the bundle \( P_G \) is possibly non-trivial and characterized by a set of transition functions \( \{g_{\alpha\beta}\} \) with respect to a contractible open covering \( \{U_\alpha\} \) of the base space \( M \). We furthermore assume the existence of a regular section \( g = \{g_\alpha\} \) of \( \text{Ad}P_G \), i.e. a section such that all the \( g_\alpha \) take values in \( G_r \). As \( G_r \) is invariant under conjugation, the notion of a regular section is independent of the choice of local trivialization and hence well defined. We will see below, however, that the assumption that a regular section exists is non-trivial and imposes some topological restrictions on \( P_G \) (which turn out to be precisely those which permit the regular sections to be diagonalized).

Since \( g \) is a section of the adjoint bundle, its local representatives are related on overlaps \( U_\alpha \cap U_\beta \) by

\[
g_\alpha = g_{\alpha\beta} g_\beta g_{\alpha\beta}^{-1}.
\]

(6.1)

Locally the situation is exactly as in section 5 and hence we can assume the existence of smooth local diagonalizing functions \( h_\alpha \in \text{Map}(U_\alpha, G) \) such that \( h_\alpha g_\alpha h_\alpha^{-1} = t_\alpha \) takes values in \( T_r \) (this has already been established in Proposition
1). It then follows from (6.1) that

\[ h^{-1}_a t_a h_a = g_{a\beta} h^{-1}_\beta t_\beta h_{\beta} g^{-1}_{a\beta} \]  

or, that on overlaps the \( t_a \) are related by

\[ t_a = (h_a g_{a\beta} h^{-1}_\beta) t_\beta (h_a g_{a\beta} h^{-1}_\beta)^{-1} \]  

We can now argue exactly as in section 5 to conclude that, as the \( t_a \) are regular, the (transition) functions \( h_a g_{a\beta} h^{-1}_\beta \) take values in \( N(T) \). Moreover, if \( G \) is simply connected one can use the ambiguity \( h_a \rightarrow n_a h_a \) with \( n_a : U_a \rightarrow N(T) \) to conjugate all the \( t_a \) into the same fundamental domain \( D \sim T_r/W \). We can then conclude from (6.3) that the \( h_a g_{a\beta} h^{-1}_\beta \) can actually be chosen to take values in \( T \),

\[ h_a g_{a\beta} h^{-1}_\beta : U_a \cap U_\beta \rightarrow T \]  

and that the locally defined diagonalized maps \( t_a \) piece together to a globally well defined \( T_r \)-valued function \( t = \{ t_a \} \),

\[ t_a = t_\beta \text{ on } U_a \cap U_\beta \]  

These results are the precise counterparts of (5.5) and (5.6) obtained in section 5 in the case of trivial bundles where we interpreted them in terms of restrictions of the structure group of a trivial principal \( G \) bundle. Here we appear to reach the conclusion that any \( G \) bundle can be restricted to a \( T \) bundle which can obviously not be correct. We will come back to this below.

In analogy with (5.9) we can also define local \( G/T \)-valued maps \( f_\alpha = h^{-1}_a \mu h_a \). However, unlike in the case considered there, these do not automatically piece together to give a globally well defined map into \( G/T \). Rather, on overlaps, they transform as

\[ f_\alpha = g_{a\beta} f_\beta g^{-1}_{a\beta} \]  

This means that the \( \{ f_\alpha \} \) define a global section of the homogenous bundle \( E_{G/T} \) with fibre \( G/T \) to be introduced below.

As one expects \( t \) to define a section of the adjoint bundle of some principal \( T \) bundle \( P_T \) which is trivial for any \( P_T \) (the adjoint action of \( T \) on itself is trivial), the result (5.5) is eminently reasonable. The crux of the matter lies in the conclusion (5.4) which cannot be fulfilled in general and which implies that some non-innocuous topological assumption has slipped into our above analysis. Alternatively, the possible non-existence of global sections of \( E_{G/T} \) constitutes a further obstruction to smoothly and globally conjugating the regular section.
Let $P_G$ be a principal $G$ bundle and $H$ a subgroup of $G$. One says that the structure group of $P_G$ can be restricted to $H$ if there exists a principal $H$ bundle $P_H$ and an embedding $j : P_H \hookrightarrow P_G$ which is a strong (i.e. fiber preserving) principal bundle morphism which induces the embedding $i : H \hookrightarrow G$ on the fibers. Alternatively, in terms of local coordinates and transition functions $g_{\alpha\beta}$ of $P_G$, $P_G$ is said to have a restriction to $H$ if there exist functions $h_{\alpha} \in \text{Map}(U_{\alpha}, G)$ such that the equivalent transition functions $h_{\alpha}g_{\alpha\beta}h_{\beta}^{-1}$ (corresponding to a change of local trivialization) take values in $H$.

Two of the more elementary results concerning restrictions of structure groups are that the structure group can always be restricted to a maximal compact subgroup and that it can be restricted to the trivial group $\{1\}$ (and hence any subgroup of $G$) iff $P_G$ is trivial. The latter already shows that in general there may be (topological) obstructions to the restriction of structure groups.

To describe the general situation when $H$ is a non-trivial subgroup of $G$ we need to introduce the quotient space $E_{G/H} = P_G/H$ which is well defined since $G$ (and hence $H$) acts freely on the right on $P_G$. $E_{G/H}$ is a fiber bundle over $M$ with typical fiber $G/T$ associated to $P_G$ via the action of $G$ on $G/T$. Then the fundamental result concerning restrictions of structure groups is that there is a bijective correspondence between

- principal $H$ bundles $P_H$ which are restrictions of $P_G$, and
- global sections of the associated bundle $E_{G/H}$.

As the proof of this result is not too hard and gives some insight into the manipulations performed in section 5 and above, we give a sketch of it here. First of all, given a section $s : M \rightarrow E_{G/H}$, one can use it to pull back the principal $H$ bundle $P_G \rightarrow E_{G/H}$ to $M$. The resulting principal $H$ bundle $P_H$ over $M$ is easily seen to satisfy all the requirements of a restriction. Conversely, given a restriction $(P_H, j)$, one defines a section of $E_{G/H}$ by composing $j : P_H \hookrightarrow P_G$ with the restriction of the projection map

$$
\chi : P_G \times G/T \rightarrow P_G \times_G G/T = E_{G/T}
$$

(6.7)

to the origin $o \in G/T$,

$$
s(x) := \chi(j(p), o) \ .
$$

(6.8)
Here \( p \) is any point in the fiber of \( P_H \) above \( x \in M \) and the right hand side does not depend on the choice of \( p \) because \( j \) is by assumption a principal bundle morphism, \( j(ph) = j(p)i(h) \) for \( h \in H \).

Let us now compare this with what we did in sections 4 and 5 under the assumption that \( P_G \) is trivial. Then \( E_{G/T} \) is trivial as well and therefore there are no obstructions to restricting the structure group of \( P_G \) to \( T \). Restrictions simply correspond to maps from \( M \) to \( G/T \), homotopic maps giving rise to isomorphic principal \( T \) bundles over \( M \). This is just what we found in a more pedestrian way in section 5; see in particular equation (5.3) which shows that the structure group of \( P_G \) \( \sim M \times G \) has been reduced to \( T \), and (5.10) which exhibits the corresponding map into \( G/T \). Furthermore, if there are no non-trivial \( G \) bundles on \( M \), every principal \( T \) bundle \( P_T \) arises as the restriction of \( P_G \) for some map \( f : M \to G/T \), as the transition functions \( t_{\alpha\beta} \) of \( P_T \) can \( a \) fortiori be regarded as the transition functions of a \( G \) bundle by composing them with \( i : T \hookrightarrow G \).

In general, however, when \( P_G \) and \( E_{G/T} \) are non-trivial, there will be topological obstructions to the existence of global sections of \( E_{G/T} \) and hence to restrictions of the structure group, the primary obstructions typically lying in \( H^k(M, \pi_{k-1}(G/T)) \) for some \( k \). What (5.4) shows, on the other hand, is that a restriction to \( T \) exists if \( \text{Ad}P_G \) has a regular section while (6.6) exhibits the corresponding section of \( E_{G/T} \). As the converse, if the bundle admits a restriction then there is a regular section, is easily established in general (by following the reasoning leading to (5.7) and (5.8)), we can summarize the consequences of the above considerations in

**Proposition 6**: Let \( P_G \) be a principal \( G \) bundle over \( M \) and \( E_{G/T} = P_G/T \) its associated homogeneous bundle. \( P_G \) admits a restriction to \( T \) (equivalently, \( E_{G/T} \) has a global section) if and only if \( \text{Ad}P_G \) has a regular section.

In a sense this is the central result of this paper. It explains the intimate relationship we found between diagonalization and restriction of the structure group and it highlights the crucial role played by the assumption of regularity.

Nevertheless this result may seem to be somewhat curious as \( a \) priori the condition of regularity is not a cohomological condition while it nevertheless implies that there are no topological obstructions to the existence of a global section of \( E_{G/T} \). However, it is not unlike the relation between the triviality of a line bundle and the existence of a nowhere vanishing section in that an algebraic condition has a topological implication.

It would be nice to have a demonstration of Proposition 6 which does not rely on diagonalization but deals directly with the obstructions instead, but we have
been unable to find such a direct proof. In four dimensions, however, necessary and sufficient conditions for the existence of restrictions of $SU(n)$ bundles can be read off more or less by inspection and this gives some insight into the nature of this problem.

We recall first that $SU(n)$ bundles $P$ on a compact oriented four-manifold are completely classified by the second Chern class $c_2(P) \in H^4(M, \mathbb{Z}) \sim \mathbb{Z}$. In terms of the curvature $F_A$ of a connection $A$ on $P$ the Chern-Weil representative of $c_2(P)$ is

$$c_2(P) = \frac{1}{8\pi^2} \int_M \text{Tr} F_A F_A$$

(with the trace normalized to $\text{Tr} \lambda^a \lambda^b = 2\delta^{ab}$, the $\lambda^a$ a basis of the Lie algebra of $SU(n)$). Torus bundles $P_T$, $T \sim U(1)^{n-1}$, on the other hand are classified by $H^2(M, \mathbb{Z}^{n-1})$. As all $T$ bundles can be regarded as $SU(n)$ bundles, they will all arise as the restriction of some $SU(n)$ bundle but not necessarily as restrictions of the trivial $SU(n)$ bundle. Moreover, some $SU(n)$ bundles may have no restrictions at all while others may admit several inequivalent restrictions. In this four-dimensional context it is straightforward to find obstructions to such an Abelianization. Let us first write $\text{Tr} F_A F_A$ locally as

$$\text{Tr} F_A F_A = d \text{Tr}(A dA + \frac{2}{3} A^3) \quad (6.10)$$

If one has been able to abelianize (with transition functions as in (6.4)), then one may as well write

$$\text{Tr} F_A F_A = d \text{Tr} A^t dA^t + d \text{Tr} A^k d_A^t A^k \quad (6.11)$$

As the second term transforms homogeneously under gauge transformations (see (5.12)) and hence under change of local trivializations, the second term is globally defined and does not contribute to the integral (6.9). Hence one finds that for a principal $SU(n)$ bundle which admits a restriction to a $T$ bundle $P_T$, its second Chern class is related to the curvature of a connection on $P_T$ by

$$c_2(P) = \frac{1}{8\pi^2} \int_M \text{Tr} dA^t dA^t \quad (6.12)$$

By looking at some concrete examples of four-manifolds we will see that this relation can impose severe constraints on $c_2(P)$.

Let us, for instance, take $M$ to be the four-sphere $M = S^4$. Then there are no non-trivial $T$ bundles on $M$ as $H^2(M, \mathbb{Z}) = 0$, and the right hand side of (6.12) is zero as the integrand is then necessarily globally exact. Hence we reach the conclusion that only the trivial $SU(n)$ bundle on $S^4$ admits a restriction to a $T$ bundle (the trivial $T$ bundle in this case). This may also be seen in a different
way by noting that, on any $n$-sphere, the bundle is characterized by the glueing (transition) function $h$ from the equator $\sim S^{n-1}$ to the group $G$. If $h$ takes values in $T$, then its winding number is zero ($\pi_{n-1}(T) = 0$ for $n > 2$) and hence

$$8\pi^2 c_2(P) = \int_{S^3} Tr(h^{-1}dh)^3 = 0.$$  \hspace{1cm} (6.13)

Thus we conclude that the adjoint bundles of non-trivial $SU(n)$ bundles over $S^4$ have no regular sections whatsoever.

This is not to mean that only trivial $SU(n)$ bundles can be reduced to $T$ bundles. As another example consider $M = \mathbb{CP}^2$ and $G = SU(2)$. In this case, $H^2(M, \mathbb{Z}) \sim H^4(M, \mathbb{Z}) \sim \mathbb{Z}$, generated by the Kähler form $\omega$. Thus there are non-trivial torus and $SU(2)$ bundles on $\mathbb{CP}^2$. The curvature of the connection on a $U(1)$ bundle is cohomologous to $k\omega$ for $k \in \mathbb{Z}$ and, as $\omega^2[\mathbb{CP}^2] = 1$, a necessary condition for an $SU(2)$ bundle $P$ to be reducible to $U(1)$ is that $c_2(P) = k^2$ for some $k \in \mathbb{Z}$. As any $U(1)$ bundle with first Chern class $k$ is the reduction of some $SU(2)$ bundle, this condition is also sufficient and for every non-trivial $SU(2)$ bundle on $\mathbb{CP}^2$ there are two inequivalent reductions to $U(1)$, characterized by the first Chern class $\pm k$.

This situation is more or less the same for all compact four-manifolds. If a torus bundle, thought of as an $SU(n)$ bundle, has second Chern class $c_2 = m$, then it can be obtained as the reduction of this $SU(n)$ bundle. Conversely, if an integer $m$ does not arise as the second Chern class of some torus bundle, the corresponding $SU(n)$ bundle with $c_2(P) = m$ cannot be Abelianized. As a consequence of Proposition 6 such bundles have no regular sections whatsoever.

By the above reasoning one can establish in general that if a principal $G$ bundle $P_G$ has a restriction to a principal $H$ bundle $P_H$, where $H$ is any subgroup of $G$ containing $T$, then the characteristic classes of these bundles will be the same. While this is more or less obvious on general grounds, the considerations involving diagonalization (or conjugation into $H$ in the more general case) permit one to be quite explicit about this.

There is one further complication that arises when $M$ admits non-trivial $G$ bundles, already implicit in the above discussion. It is certainly still true that every $T$ bundle on $M$ will arise as the restriction of some $G$ bundle. However, a given principal $G$ bundle $P_G$ will only give rise to those principal $T$ bundles after diagonalization of its regular sections which arise as restrictions of $P_G$. This will have to be reflected in the corresponding Weyl integral formula which will then include a sum over only a restricted class of isomorphism classes of principal $T$ bundles (unless, of course, the original theory is defined as a sum over all
Are Regular Maps Generic?

While we have seen above that non-trivial adjoint bundles may admit no regular sections at all, which forces us face the task of diagonalizing non-regular maps, one may have hoped that at least for trivial bundles regular maps are generic so that ‘most’ maps can indeed be conjugated into smooth torus-valued functions by the results of sections 4 and 5, at least via locally defined or discontinous diagonalizing functions $h$. This turns out to be so for Lie algebra valued maps but a simple example will show that it is not necessarily true for group valued maps.

Let us look at the Lie algebra case first. If $P_G$ is trivial, sections of the adjoint bundle $\text{ad}P_G = P_G \times_{\text{ad}} g$, a vector bundle over $M$ with fiber $g$, can be identified with maps from $M$ to $g$. Now the non-regular points in $g$ form a set of codimension at least three: the non-regular elements of $t$ form a set of codimension one (dimension $r - 1$), as they partition $t$ into its Weyl alcoves; the dimension of a coadjoint orbit through a non-regular element is strictly smaller than the dimension $\dim G - r$ of $G/T$ and in fact at most $\dim G - r - 2$ because the orbit is symplectic and hence even dimensional; hence the dimension of the set of non-regular elements is at most

$$\dim(g \setminus g_r) \leq (r - 1) + (\dim G - r - 2) = \dim G - 3$$

It follows that regular maps are indeed generic in any dimension.

Because of topological complications not present in the Lie algebra case, the corresponding statement for group valued maps is false. To see that, let us consider as a simple example the space of maps from $M = S^3$ to $G = SU(2) \sim S^3$. This space consists of an infinite number of connected components labelled by the winding number of the map in $\pi_3(SU(2)) = \mathbb{Z}$. As the only non-regular elements of $SU(2)$ are plus or minus the identity, regular maps are those which avoid the north and south poles of the target $S^3$. Clearly generic maps in the zero winding number sector have this property. As there are no non-trivial $U(1)$ bundles on $S^3$, $H^2(S^3, \mathbb{Z}) = 0$, any such map can be globally and smoothly conjugated into $U(1)$. On the other hand, any map in one of the other sectors has in particular the property that its image is the entire $SU(2)$, covered an appropriate number of times. Hence, no map with a non-trivial winding number can be regular.

The upshot of this is that neither regular $G$-valued maps nor regular sections of non-trivial $\text{Ad}P_G$ or $\text{ad}P_G$ bundles can be expected to be generic in general, the
only exception being \( g \)-valued maps.

**Diagonalization of Non-Regular Maps?**

The fact that even for trivial bundles there may be too many non-regular maps for comfort provides an additional impetus for coming to terms with the diagonalization of these maps. Unfortunately, this problem appears to be much harder than the corresponding one for regular maps and in the following we will only make a few remarks and tentative suggestions in that direction.

Let us first recall at which points in our analysis the assumption of regularity entered (we take \( G \) to be simply connected):

1. The fact that the conjugation map \( q : G/T \times T \to G, \) \( (2.7) \), is proper and, in fact, a (trivial) fibration away from the non-regular points allowed us to solve the first lifting problem in section 4 for regular maps.

2. Regularity of \( g \) (and hence of the \( t_\alpha \)) allowed us to conclude from \( (5.2) \) that the transition functions \( h_{\alpha\beta} = h_\alpha h_\beta^{-1} \) take values in \( N(T) \) (and can even be chosen to reduce the structure group to \( T \)).

If \( g \in \text{Map}(M, G) \) is not regular then clearly no such restriction will necessarily be imposed on the transition functions defined by the local diagonalizing maps. E.g. if \( g \) is the constant identity map, the \( h_\alpha \) are completely arbitrary. As this \( g \) is already diagonal, this may not be too much of a concern, but other problems arise for maps which take on both regular and non-regular values as we have already seen in example 1 of section 3. For instance, the triviality of the fibration \( (2.7) \) may alternatively be expressed by saying that the quotient of \( G_r \) by the adjoint action of \( G \) is a smooth manifold,

\[
G_r/\text{Ad}G \sim T_r/W
\]

and that topologically (and smoothly) one has

\[
G_r \sim T_r/W \times G/T , \tag{6.16}
\]

the points in \( D \sim T_r/W \) labelling the regular (maximal) coadjoint orbits in \( G_r \). If one considers maps taking values in all of \( G \), one has to come to terms with the fact that the quotient

\[
G/\text{Ad}G \sim T/W \tag{6.17}
\]

is not a smooth manifold (but the closure of a Weyl alcove or, rather, its image under the exponential map), and that the fiber of \( G \to G/\text{Ad}G \) above a singular
(non-regular) point is isomorphic to the coadjoint orbit through that non-regular point and hence strictly smaller than that at a regular point. Clearly this is a rather singular situation to consider and accounts for most of the problems associated with non-regular maps.

Looking back at example 1 of section 3 we see that the failure to be smoothly diagonalizable is due to the combined effect of having a non-regular map and a non-simply connected base space, the diagonalized map $t(x)$ being well defined on the non-trivial double cover of the circle. This and the fact that there are no non-trivial $W$ bundles on simply connected manifolds suggest that it may be possible to prove stronger statements regarding non-regular maps in that case.

7 Applications: A Weyl Integral Formula for Path Integrals

In the previous sections we have analyzed the problem of diagonalizing maps from a manifold $M$ into a compact Lie group $G$ or its Lie algebra $g$. As mentioned in the Introduction, this problem arose in a field theoretic context when we attempted to exploit the rather large local gauge symmetry present in certain low-dimensional non-Abelian gauge theories to abelianize (and hence more or less trivialize) the theories via diagonalization [1, 2, 13]. Assuming that the contributions from non-regular maps can indeed be neglected in these examples (and we have nothing to add to the arguments put forward in [4] to that effect), the analysis of the present paper can be regarded as a topological justification for the formal path integral version of the Weyl integral formula we used to solve these theories.

The Weyl integral formula expresses the integral of a smooth (real or complex valued) function over $G$ in terms of an integral over $T$ and $G/T$, using the conjugation map $q$ (2.7) to pull back the Haar measure on $G$ to $G/T \times T$ and reads

$$\int_G dg f(g) = \int_T dt \Delta(t) \int_{G/T} dg f(g^{-1}tg) \quad (7.1)$$

Here $\Delta(t)$, the Weyl determinant, is the Jacobian of $q$. Its precise form will not interest us here and we just note that it vanishes precisely at the non-regular points of $T$ (this being the mechanism by which contributions from non-regular points should be suppressed in the functional integral). For an explanation of the standard proof of (7.1) and for a derivation in the spirit of the Faddeev-Popov trick see [1, 2].

The case of interest to us is when the function $f$ is conjugation invariant (a
class function), i.e. when $f$ satisfies

$$f(h^{-1}gh) = f(g) \quad \forall \, g, h \in G. \quad (7.2)$$

In that case, since any element of $G$ is conjugate to some element of $T$, both $f$ and its integral over $G$ are determined by their restriction to $T$ and the Weyl integral formula reflects this fact,

$$\int_G dg \, f(g) = \int_T dt \, \Delta(t) f(t). \quad (7.3)$$

It is this formula which we would like to generalize to functional integrals, i.e. to a formula which relates an integral over a space of maps into $G$ to an integral over a space of maps into $T$.

For concreteness, consider a local functional $S[g; A]$ (the ‘action’) of maps $g \in \text{Map}(M, G)$ and gauge fields $A \in \Omega^1(M, g)$, i.e. of sections of $\text{Ad}P_G$ and connections on a trivial principal $G$ bundle $P_G \sim M \times G$ (a dependence on other fields could be included as well). Assume that $\exp iS[g; A]$ is gauge invariant,

$$\exp iS[g; A] = \exp iS[h^{-1}gh; A^h] \quad \forall \, h \in \text{Map}(M, G), \quad (7.4)$$

at least for smooth $h$. If e.g. a partial integration is involved in establishing the gauge invariance (as in Chern-Simons theory), this may fail for non-smooth $h$’s and more care has to be exercised when such a gauge transformation is performed. Then the functional $F[g]$ obtained by integrating $\exp iS[g; A]$ over $A$,

$$F[g] := \int D[A] \exp iS[g; A], \quad (7.5)$$

is conjugation invariant,

$$F[h^{-1}gh] = F[g]. \quad (7.6)$$

It is then tempting to use a formal analogue of (7.3) to reduce the remaining integral over $g$ to an integral over maps taking values in the Abelian group $T$. In field theory language this amounts to using the gauge invariance (7.4) to impose the ‘gauge condition’ $g(x) \in T$. The first modification of (7.3) will then be the replacement of the Weyl determinant $\Delta(t)$ by a functional determinant $\Delta[t]$ of the same form which needs to be regularized appropriately (see the Appendix of [2]).

However, the main point of this paper is that this is of course not the whole story. We already know that this ‘gauge condition’ cannot necessarily be achieved smoothly and globally. Insisting on achieving this ‘gauge’ nevertheless, albeit via non-continuous field transformations, turns the $t$-component $A^t$ of the transformed gauge field $A^{h^{-1}}$ into a gauge field on a possibly non-trivial $T$ bundle $P_T$. 
(while the $k$-components transform as sections of an associated bundle). Moreover we know that all those $T$ bundles will contribute which arise as restrictions of the (trivial) bundle $P_G$. Let us denote the set of isomorphism classes of these $T$ bundles by $[P_T; P_G]$. Hence the ‘correct’ (meaning correct modulo the analytical difficulties inherent in making any field theory functional integral rigorous) version of the Weyl integral formula, capturing the topological aspects of the situation, is one which includes a sum over the contributions from the connections on all the isomorphism classes of bundles in $[P_T; P_G]$.

Let us denote the space of connections on $P_G$ and on a principal $T$ bundle $P^l_T$ representing an element $l \in [P_T; P_G]$ by $\mathcal{A}$ and $\mathcal{A}[l]$ respectively and the space of one-forms with values in the sections of $P^l_T \times_T k$ by $\mathcal{B}[l]$. Then, with

$$Z[P_G] = \int_{\mathcal{A}} D[A] \int D[g] \exp iS[g; A] , \quad (7.7)$$

the Weyl integral formula for functional integrals reads

$$Z[P_G] = \sum_{l \in [P_T; P_G]} \int_{\mathcal{A}[l]} D[A^t] \int_{\mathcal{B}[l]} D[A^k] \int D[t] \Delta[t] \exp iS[t; A^t, A^k] \quad (7.8)$$

(modulo a normalization constant on the right hand side). The $t$-integrals carry no $l$-label as the spaces of sections of $\text{Ad} P^l_T$ are all isomorphic to the space of maps into $T$.

In the examples considered in [1, 2], Chern-Simons theory on three-manifolds of the form $\Sigma \times S^1$, 2d Yang-Mills theory and the $G/G$ gauged Wess-Zumino-Witten model, the fields $A^k$ entered purely quadratically in the reduced action $S[t; A^t, A^k]$ and could be integrated out directly, leaving behind an effective Abelian theory depending on the fields $t$ and $A^t$ with a measure determined by $\Delta[t]$ and the (inverse) functional determinant coming from the $A^k$-integration. The general structure of these terms and the ‘quantum corrections’ coming from the regularization has been determined in [13].

A further property these models were found to have is that they localize onto reducible connections and their isotropy groups (in the case of the $G/G$ model) respectively algebras (for Yang-Mills theory) so that, in practice, the necessity only ever arose to diagonalize these maps. This is possible globally even if the group is not simply connected (when, as we recall from section 4, the existence of a globally smooth diagonalized map $t$ or $\tau$ is not guaranteed \textit{a priori}). The reason for this is the following (for group valued maps - the Lie algebra case is entirely analogous):

The reducibility condition $A^g = A$ implies that $\text{Tr} g^n$ is constant for all $n$. This allows one to determine that $g$ is conjugate to a $t$ which is constant globally.
and (of course) unique up to an overall $W$-transformation. This provides the $T$, part of the lift in diagram (4.2). Furthermore, the constancy of the traces implies that $g$ can itself be regarded as a map into $G/T$ and hence furnishes the $G/T$-part $f$ of the lift. At this point the argument can then proceed as in the simply-connected case. The fact that isotropy groups of connections are indeed conjugate to subgroups of $G$ (thought of as spaces of constant maps) is well known. What seems to be less generally appreciated is the fact that the conjugation itself cannot necessarily be done globally.

We have also applied this formula to several other models like $BF$ theories in three dimensions (related to $3d$ gravity) and the supersymmetric Chern-Simons models of Rozansky and Saleur [11]. The formula can also be used to go some way towards evaluating the generating functional for Donaldson theory on Kähler manifolds with the action as in [12]. These results will be presented elsewhere.

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