Lisa Amplitude Modulation: A Study of the Angular Resolution of LISA for Monochromatic Gravitational Waves

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Abstract. We present formulae for the amplitude modulation of the X,Y and Z TDI combinations for monochromatic gravitational waves and use these to study the LISA angular resolution in the case of large SNR. The angular resolution, \( \Delta \Theta / (0.1 \times SNR) \), is found to lie between 2° and 5° and is somewhat dependent on the ecliptic colatitude angle (\( \beta \)), on the polarisation (\( h_+ \), \( h_\times \)) and much less on the ecliptic longitude angle (\( \lambda \)). Comparisons with other studies are presented. Future studies will treat the case of small SNR.

1. Introduction

The aim of this work is to provide a relatively simple method to extract the angular location and polarisation of a monochromatic gravitational wave (GW) of frequency \( f \) from the LISA TDI time series (X,Y and Z, for this study, A and E in a further development) and to estimate the precision with which this can be done. A description of the future LISA mission can be found in Danzmann et al. [1] and an explanation of the TDI (Time Delay Interferometry) laser noise supression method in Armstrong, Estabrook, and Tinto [2] and in Dhurandhar, Rajesh Nayak and Vinet [3].

The method is based on an analytical formula of the amplitude modulation produced by these TDI generators. The formula is based on two assumptions: the LISA triangle has equal arms (\( L_1 = L_2 = L_3 = L \)) and the GWs have "low frequencies" (\( f \ll L/c \)). Although it is applied here to monochromatic GW sources, it can and will be generalised to GW emitted from coalescing black holes binaries.

The results presented here show the angular resolution obtained with this method, as a function of the source direction (\( \vec{\omega} \)) in the sky (defined by \( \beta \) : ecliptic colatitude and \( \lambda \) ecliptic longitude, or equivalently by \( \theta = \pi/2 - \beta \) and \( \varphi = \pi - \lambda \)), of the polarisation (\( h_+ \), \( h_\times \)) defined in the barycentric reference frame and of the signal to noise ratio (SNR).

As it is based only on the amplitude modulation, the precision of the measured direction (\( \beta, \lambda \)) will not be as good as the one obtained from methods which also include the phase
modulation \([4, 5]\), particularly at high frequencies \(> 10^{-3} Hz\). A study of how much precision is lost will be performed in the near future.

2. The Amplitude Modulation formula

Within the two approximations mentioned above, the amplitude modulation of the TDI \(X_i\) \((i = 1\) for \(X\), \(i = 2\) for \(Y\) and \(i = 3\) for \(Z\)) is given by (see Petiteau \([6]\), page 162, see also Cornish and Rubbo \([7]\)):

\[
A_{X_i}(t; \theta, \varphi, h_+, h_\times, f) = 4\pi f L \sin(2\pi f L) \left[ h_+^2 F_{i+}^2 + h_\times^2 F_{i\times}^2 \right]^{1/2},
\]

where \(F_{i+}\) and \(F_{i\times}\) are the LISA antenna patterns defined by

\[
F_{i+} = \left[ (\vec{\theta} \cdot \vec{n}_{i+2})^2 - (\vec{\phi} \cdot \vec{n}_{i+2})^2 \right] - \left[ (\vec{\theta} \cdot \vec{n}_{i+1})^2 - (\vec{\phi} \cdot \vec{n}_{i+1})^2 \right]
\]

\[
F_{i\times} = \left[ 2 (\vec{\theta} \cdot \vec{n}_{i+2}) \cdot (\vec{\phi} \cdot \vec{n}_{i+2}) \right] - \left[ 2 (\vec{\theta} \cdot \vec{n}_{i+1}) \cdot (\vec{\phi} \cdot \vec{n}_{i+1}) \right],
\]

where \(n_i\) are the unit vectors of the arms of LISA and \(\vec{\theta}\) and \(\vec{\phi}\) are vectors defined by

\[
\vec{\theta} = -\frac{\partial \vec{\omega}}{\partial \theta} = -\frac{\partial \vec{\omega}}{\partial \beta}
\]

\[
\vec{\phi} = -\frac{1}{\sin \theta} \frac{\partial \vec{\omega}}{\partial \varphi} = -\frac{1}{\cos \beta} \frac{\partial \vec{\omega}}{\partial \lambda},
\]

and \(\vec{\omega}\) is the unit vector of the direction defined by the angles \(\beta\) and \(\lambda\).

3. Data Generation

A method to generate amplitude modulation data would be to use a LISA simulator (LISACode \([8]\) in our case) to produce a year long (TDI) time series for a given source and instrumental noise function. Using this, one would split it into many \(N_d\) time windows of duration \(\Delta T = 1\) year/\(N_d\), obtain a measure of the amplitude (from a FFT analysis) and fit these \(N_d\) data points with equation (1).

Even with a fast simulator such as LISACode, generating all the necessary cases to cover the possible angular domain, polarisation values and signal to noise ratios would take a prohibitive amount of computing time.

We have therefore simplified this method in the following way. We use LISACode to generate the time series for the various angular locations and polarisation values in the absence of instrumental noise. From this time series, we extract the value of the amplitude at the central time of each of the \(N_d\) windows. To this amplitude, we add a noise vector by randomly choosing its 2-dimensional components from Gaussian distributions of variance \(\sigma\), calculated for the requested SNR value (see eq. 11 below). The measured data corresponds then to the amplitude of the vector thus created. This method is correct as long as the power of the monochromatic signal is contained in only one frequency bin \(\Delta f = 1/\Delta T\). The typical values of \(N_d\), \(\Delta T\) and \(\Delta f\) used in this work \((31, 11.5\) days and \(10^{-6}Hz\)) guarantee that this is generally the case. Otherwise the value of the noise has to be taken into account, as well as the number of bins over which the data is spread.

Figure 1 shows an example of data generated (black dots with error bars) for a SNR of 115 (same value as treated in ref. \([4]\)) as well as the corresponding amplitude modulation (red line) as calculated from equation (1). The half size of the error bars corresponds to the value of \(\sigma\). The title of the figure gives the fit values for the parameters and, in parenthesis, the exact
values. Figure 2 shows a similar example for a SNR of 23. It is clear from this last figure that the probability distributions of the data are not Gaussian (for example, negative values are seen to be possible from the extension of the error bars). In fact, even on figure 1, some of the points that are close to zero will not have a Gaussian probability distribution. We will discuss this below.

Figure 1. Data (black dots with error bars) generated for a SNR=115. Angular location and polarisation of the source are indicated on the figure. The red line corresponds to a fit to the data.

Figure 2. Data (black dots with error bars) generated for a SNR=23. Angular location and polarisation of the source are indicated on the figure. The red line corresponds to a fit to the data.

4. Calculation of the experimental error ($\sigma$) for a given SNR

We assume here, that for frequencies close to $f$, the instrumental noise is represented by a white noise. Assuming a bandwidth limited white noise, of average value $\Gamma_0$ in the frequency domain $-B/2 < f < B/2$, and a signal $S(t)$ (or $S(t)$), the Signal to Noise ratio (SNR) is defined by

$$SNR^2 = \int_{-\infty}^{+\infty} \frac{S^2(f')\delta(f' - f)}{\Gamma(f')} df' = \frac{1}{\Gamma_0} \int_0^T S^2(t) \, dt.$$  \hspace{1cm} (6)

The frequency band $B$ and time window width $\Delta T$ are related by

$$B = \frac{1}{\Delta T} = \frac{N_d}{T},$$  \hspace{1cm} (7)

where $N_d$ is the number of time windows over a total time of $T$. The average power of the noise is defined by

$$\Gamma_0 = \frac{\sigma^2}{B}.$$  \hspace{1cm} (8)

Converting from an integral to a discrete sum, one obtains

$$SNR^2 = \frac{B}{\sigma^2} \int_0^T S^2(t) \, dt \approx \frac{B}{\sigma^2} \sum_{i=1}^{N_d} S^2(t_i) \Delta T = \frac{1}{\sigma^2} \sum_{i=1}^{N_d} S^2(t_i),$$  \hspace{1cm} (9)

so that $\sigma$ can thus be related to the SNR by
\[ \sigma = \frac{1}{SNR} \sqrt{\sum_{i=1}^{N_d} S^2(t_i)}. \] (10)

Note that, as three different signals (TDI X, Y and Z) will be analysed simultaneously, the (time independent) variance will be defined by

\[ \sigma = \frac{1}{SNR} \sqrt{N_d \sum_{i=1}^{N_d} (S^2_X(t_i) + S^2_Y(t_i) + S^2_Z(t_i))}. \] (11)

An important point to notice is that, for a fixed SNR, the value of \( \sigma \) will depend on the value and number of data points and hence on the direction of the wave, on its polarisation and on the number of TDI combinations studied.

5. Data Analysis

Using these sets of data and the amplitude modulation equation (1), a \( \chi^2 \)-minimisation on parameters \( \theta \), \( \phi \), \( h_+ \) and \( h_\times \) is performed.

This \( \chi^2 \)-minimisation produces a covariance matrix, \( C_{P_j P_k} \) for the parameters \( P_j \) and \( P_k \), from which the estimation of the angular resolution is calculated. The error on \( \theta \) and \( \phi \) are given by

\[ \Delta \theta = C_{\theta \theta}^{1/2} \quad \text{and} \quad \Delta \phi = C_{\phi \phi}^{1/2}, \] (12)

and the error on the solid angle is defined by

\[ \Delta \Omega = \sin(\theta) \Delta \theta \Delta \varphi. \] (13)

The angular error on the direction of the wave is then defined as

\[ \Delta \Theta = \sqrt{\frac{\Delta \Omega}{\pi}}. \] (14)

This error can also be estimated from the Fisher Information Matrix (FIM) [9, 10, 11]. The Fisher matrix for a signal \( S(t) \) depending on parameters \( P_j \) and \( P_k \) and with errors identical for all data points, can be approximated in the monochromatic case by

\[ J_{P_j P_k} = \frac{1}{\sigma^2} \sum_{i=1}^{N_d} \left( \partial_{P_j} S(t_i) \right) \left( \partial_{P_k} S(t_i) \right), \] (15)

where \( \partial_{P_j} S(t_i) \) is the derivative of the signal at time \( t \) with respect to parameter \( P_j \).

For \( P_j = P_k = \theta \), and using the above value of \( \sigma \), one obtains

\[ J_{\theta \theta} = SNR^2 \frac{\sum_{i=1}^{N_d} \left( \partial_{\theta} S_X(t_i) \right)^2 + \left( \partial_{\theta} S_Y(t_i) \right)^2 + \left( \partial_{\theta} S_Z(t_i) \right)^2}{N_d \sum_{i=1}^{N_d} (S^2_X(t_i) + S^2_Y(t_i) + S^2_Z(t_i))}. \] (16)

The error \( \Delta_F \theta \) on \( \theta \) obtained from the FIM is given by

\[ \Delta_F \theta = \sqrt{(J^{-1})_{\theta \theta}}, \] (17)

and the angular error on the direction of the wave is given by
\[ \Delta F \Theta = \sqrt{\frac{\sin(\theta) \Delta F \theta \Delta F \phi}{\pi}}. \] (18)

Note that for all cases treated in this work \( \sqrt{(J^{-1})_{\theta \theta}} \approx (J_{\theta \theta})^{-1/2} \), i.e. the FIM is almost diagonal.

Figure 3 shows the values of \( \Delta \Theta \) (see eq. 14) for monochromatic waves \( (f=0.1 \text{ mHz}) \), for a variety of polarisation (points with solid lines) and for a SNR of 115. The legend gives, from top to bottom, the polarisation values of the curves of decreasing \( \Delta \Theta \). The dotted lines show the predictions of the Fisher matrix (see eq. 18).

![Figure 3](image-url) **Figure 3.** Variation of the angular resolution of LISA as a function of \( \beta \) and polarisation \( (h_+, h_\times) \) for \( \lambda = 180^\circ \) and \( SNR = 115 \). Note that the amplitude of the GW \( (h_+^2 + h_\times^2)^{1/2} \) is kept fixed. The dotted lines give the predictions of the Fisher matrix (see text).

From this figure one can observe that the angular resolution of LISA, for most of the angular range, lies between 0.2\(^\circ\) and 0.5\(^\circ\) for a SNR of 115 and hence possibly between 2\(^\circ\) and 5\(^\circ\) for a SNR close to 10, a value which could apply to effective observations by LISA. A clear effect due to the polarisation is also seen.

The predictions of the Fisher matrix (dashed line on these figures) follow very closely the results obtained from the minimisation of the \( \chi^2 \) as is expected for large SNR values. The discrepancies observed for \( (h_+, h_\times) = (0.7,0.7) \) and \( \beta = \pm 90^\circ \) have been traced to numerical problems: it has been checked that, for very large SNR (i.e. 999), the resolutions follow precisely the values of the Fisher Matrix.

It has also been checked that because of the duration of the data acquisition (1 year) and of the use of different TDI combinations, the resolutions calculated depend very little of the ecliptic longitude \( \lambda \). Hence, figure 3 maps most of the possible values the LISA angular resolution can take.

Figure 4 shows, for a polarisation \( (h_+, h_\times) = (1,0) \), the evolution of the resolution as a function of the SNR for \( \lambda = 180^\circ \). Note that the polarisation is fixed at \( (h_+, h_\times) = (1,0) \).

![Figure 4](image-url) **Figure 4.** Variation of the angular resolution of LISA as a function of \( \beta \) and SNR for \( \lambda = 180^\circ \). Note that the polarisation is fixed at \( (h_+, h_\times) = (1,0) \).
6. Comparison with other studies

As stated above, other studies have been performed on the angular resolution of LISA. Peterseim et al. [4] have treated the case of SNR = 115 with, as we understand, a unique TDI (or equivalent) combination. In this case, we deduce that this corresponds to an effective value of SNR divided by $\sqrt{3}$. Their results give (for $f = 3 mHz$, $(h_+, h_\times) = (1, 0)$, $\beta = 0, \lambda = 0$) a value of $\Delta \Omega = 16 \mu str$ compared to $70/\sqrt{3} \approx 23 \mu str$ obtained in our study (equivalent to $\Delta \Theta = 0.3^\circ$). This difference is understandable because their method uses both the amplitude and the phase modulation and should therefore yield better results.

The study of Cutler [5] also makes use of the amplitude and phase modulation, but using two TDI combinations. His results, at low frequencies ($f = 10^{-4} Hz$) where the phase modulation should have little influence, give an average value of $\Delta \Omega (\times 2\pi) = 8 \times 10^{-2} str$ for a SNR of 10.

In our study, this should correspond to $79 \mustr (= 0.08 \times \sqrt{3} \times (10/115)^2/(2\pi))$ and is quite close to the values we obtain.

7. Summary and future work

Using realistic data generated from LISACode and a simple analysis method based on the study of the amplitude modulation of the LISA response for monochromatic GW, the angular resolution of LISA is shown to lie between $\Delta \Theta \times (SNR/115) = 0.1^\circ$ and $0.5^\circ$, somewhat dependent on the ecliptic colatitude angle ($\beta$), on the polarisation ($h_+, h_\times$) and much less on the ecliptic longitude angle ($\lambda$). Because this method uses only the amplitude modulation (and not the phase modulation), the results do not depend on the frequency of the GW, contrary to the cases for the other studies mentioned [5, 4]. These conclusions are valid, however, for large SNR because Gaussian data distributions are assumed. For small SNR ($\leq 50$), this is probably not the case and more exact methods have to be used. It is probable that the verification binary sources considered for LISA will fall into this class and such a study will be performed in the near future. We will also move from the X,Y and Z TDI combinations to A and E as these have zero covariance and are therefore better adapted to a standard statistical analysis.

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