Large and moderate deviations for a $\mathbb{R}^d$-valued branching random walk with a random environment in time

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Abstract

We consider a $\mathbb{R}^d$-valued branching random walk with a stationary and ergodic environment $\xi = (\xi_n)$ indexed by time $n \in \mathbb{N}$. Let $Z_n$ be the counting measure of particles of generation $n$. With help of the uniform convergence of martingale and the multifractal analysis, we establish a large deviation result for the measures $Z_n$ as well as a moderate deviation principle.

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1 Introduction

As a generalization of the classical branching random walks (see e.g. [6, 7, 9, 12, 29, 1, 23, 24]), Branching random walk with a random environment in time (BRWRE) characterises branching random walks influenced by time: the distributions of the point processes, which were indexed by particles and formulated by the number of its offspring and their displacements, vary from generations according to a time random environment. First proposed by Biggins and Kyprianou [10], BRWRE has been under wider and wider investigation recently, see e.g. [19, 20, 26, 36, 44]. Compared with the other models of branching random walks in random environments studied largely in the literature, see e.g. [21, 5, 13, 14, 15, 25, 38, 45, 30], this model does not consider the influence of space environments, and moreover, it considers particles walking in the real space $\mathbb{R}^d$ rather than the integer lattice $\mathbb{Z}^d$. Recently, for BRWRE in $\mathbb{R}$ (with dimension $d = 1$), Huang et al [20] established a large deviation principle for the counting measure of particles of generation $n$, while Wang and Huang [44] showed a corresponding moderate deviation principle. This paper purpose to generalize such results to high dimensional real space $\mathbb{R}^d$ with dimension $d \geq 1$.

Let’s describe the model in details. The random environment in time, denoted by $\xi = (\xi_n)$, is a stationary and ergodic sequence of random variables, indexed by the time $n \in \mathbb{N} = \{0, 1, 2, \cdots\}$, taking values in some measurable space $(\Theta, \mathcal{E})$. Without loss of generality we can suppose that $\xi$ is defined on the product space $(\Theta^\mathbb{N}, \mathcal{E}^\mathbb{N}, \tau)$, with $\tau$ the law of $\xi$. Each realization of $\xi_n$ corresponds to a distribution $\eta_n = \eta(\xi_n)$ on $\mathbb{N} \times \mathbb{R}^d \times \mathbb{R}^d \times \cdots$, where $d \geq 1$ is the dimension of the real space.

Given the environment $\xi = (\xi_n)$, the process can be described as follows:

- At time 0, one initial particle $\emptyset$ of generation 0 is located at $S_0 = (0, 0, \ldots, 0) \in \mathbb{R}^d$;
- At time 1, $\emptyset$ is replaced by $N = N(\emptyset)$ particles of generation 1, located at $L_1 = L_1(\emptyset) = (L_{1,1}, L_{1,2}, \ldots, L_{1,d})$, $1 \leq i \leq N$, where the random vector $X(\emptyset) = (N, L_1, L_2, \ldots) \in \mathbb{N} \times \mathbb{R}^d \times \mathbb{R}^d \times \cdots$ is of distribution $\eta_0 = \eta(\xi_0)$.
- In general, each particle $u = u_1, \ldots, u_n = (u_1, \ldots, u_n)$ of generation $n$ located at $S_n = (S_n^1, \ldots, S_n^d) \in \mathbb{R}^d$ is replaced at time $n+1$ by $N(u)$ new particles $ui$ of generation $n+1$, located at $S_{ui} = S_n + L_i(u)$ ($1 \leq i \leq N(u)$).

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where the random vector $X(u) = (N(u), L_1(u), L_2(u), \cdots)$ is of distribution $\eta_n = \eta(\xi_n)$. Note that the values $L_i(u)$ for $i > N(u)$ do not play any role for our model; we introduce them only for convenience.

All particles behave independently conditioned on the environment $\xi$.

For each realization $\xi \in \Theta^\mathbb{N}$ of the environment sequence, let $(\Gamma, \mathcal{G}, \mathbb{P}_\xi)$ be the probability space under which the process is defined. The probability $\mathbb{P}_\xi$ is usually called quenched law. The total probability space can be formulated as the product space $(\Theta^\mathbb{N} \times \Gamma, \mathcal{E}^\mathbb{N} \otimes \mathcal{G}, \mathbb{P})$, where $\mathbb{P} = \mathbb{P}(\delta_\xi \otimes \mathbb{P}_\xi)$ with $\delta_\xi$ the Dirac measure at $\xi$ and $\mathbb{E}$ the expectation with respect to the law of $\xi$, so that for all measurable and positive function $g$ defined on $\Theta^\mathbb{N} \times \Gamma$, we have

$$\int_{\Theta^\mathbb{N} \times \Gamma} g(x, y) d\mathbb{P}(x, y) = \mathbb{E} \int_{\Gamma} g(\xi, y) d\mathbb{P}_\xi(y).$$

The total probability $\mathbb{P}$ is usually called annealed law. The quenched law $\mathbb{P}_\xi$ may be considered to be the conditional probability of $\mathbb{P}$ given $\xi$. The expectation with respect to $\mathbb{P}$ will still be denoted by $\mathbb{E}$; there will be no confusion for reason of consistence. The expectation with respect to $\mathbb{P}_\xi$ will be denoted by $\mathbb{E}_\xi$.

Let

$$U = \{\emptyset\} \bigcup_{n \geq 1} \mathbb{N}^n$$

be the set of all finite sequence $u = u_1 \cdots u_n$ and $I = \mathbb{N}^{\mathbb{N}^*}$ be the set of all infinite sequences, where $\mathbb{N}^* = \{1, 2, \cdots\}$. For $u \in U$ or $I$, we write $u$ for the length of $u$, and $u|n$ for the restriction to the first $n$ terms of $u$, with the convention that $u|0 = \emptyset$. By definition, under $\mathbb{P}_\xi$, the random vectors $\{X(u)\}$, indexed by $u \in U$, are independent of each other, and each $X(u)$ has distribution $\eta_n = \eta(\xi_n)$ if $|u| = n$.

Let $T$ be the Galton-Watson tree with defining element $\{N(u)\}$. We have: (a) $\emptyset \in T$; (b) if $u \in T$, then $ui \in T$ if and only if $1 \leq i \leq N(u)$; (c) $ui \in T$ implies $u \in T$. Let $T_n = \{u \in T : |u| = n\}$ be the set of particles of generation $n$. Here the null sequence $\emptyset$ (of length 0), which represents the initial particle, can be regarded as the root of the tree $T$; $ui$ represents the $i$th child of the particle $u$; $N(u)$ represents the number of offspring of $u$. For the Galton-Watson tree $T$, the boundary of $T$ is defined as

$$\partial T = \{u \in I : u|n \in T \text{ for all } n \in \mathbb{N}\}.$$

As a subset of $I = \mathbb{N}^{\mathbb{N}^*}$, $\partial T$ is a metrical and compact topological space with

$$[u] = \{w \in \partial T : w|u| = u\}, \quad u \in T$$

its topological basis, and with the standard ultrametric distance

$$d(u, v) = e^{-\sup \{||w| : u, v \in [w]\}}.$$  

For $n \in \mathbb{N}$, let

$$Z_n(\cdot) = \sum_{u \in T_n} \delta_{S_u}(\cdot)$$  

be the counting measure of particles of generation $n$. For a measurable subset $A$ of $\mathbb{R}^d$,

$$Z_n(A) = \#\{u \in T_n \colon S_u \in A\}$$

presents the number of particles of generation $n$ located in $A$. In this paper, we are interested in large and moderate deviations associated to the sequence of measures $\{Z_n\}$.

For $n \in \mathbb{N}$ and $z = x + iy \in \mathbb{C}^d$ (later and throughout the paper we use $x$ and $y$ to represent the real and imaginary parts of $z \in \mathbb{C}^d$ respectively, i.e. $x = \text{Re} z, y = \text{Im} z \in \mathbb{R}^d$, while we use $t$ to represent a real number in $\mathbb{R}^d$), put

$$m_n(z) = \mathbb{E}_\xi \sum_{i=1}^{N(u)} e^{\langle z, L_i(u) \rangle} \quad (|u| = n),$$

where $\langle \cdot, \cdot \rangle$ is the notation of inner product that $\langle z_1, z_2 \rangle = \sum_{i=1}^{d} z_1^i \bar{z}_2^i$ if $z_1 = (z_1^1, \cdots, z_1^d)$ and $z_2 = (z_2^1, \cdots, z_2^d) \in \mathbb{C}^d$. Throughout the paper, we assume that

$$\mathbb{E} \log m_0(0) > 0 \quad \text{and} \quad N \geq 1.$$
Theorem 1.1 (Multifractal analysis and large deviations) Let \( J = \{ \nabla \Lambda(t) \in \mathbb{R}^d : t \in \Omega \} \).

(a) With probability 1, for all \( \alpha \in J \), we have
\[
\tilde{\Lambda}^*(\alpha) = \Lambda^*(\alpha) \quad \text{and} \quad \dim E(\alpha) = -\Lambda^*(\alpha).
\]
(b) With probability 1, for all measurable $A \subset \mathcal{J}$,

$$- \inf_{\alpha \in A^c} \Lambda^*(\alpha) \leq \liminf_{n \to \infty} \frac{1}{n} \log Z_n(nA) \leq \limsup_{n \to \infty} \frac{1}{n} \log Z_n(nA) \leq - \inf_{\alpha \in A} \Lambda^*(\alpha),$$

where $A^c$ denotes the interior of $A$ and $\bar{A}$ its closure.

Theorem 1.1(a) means that the multifractal formalism holds at $\alpha$, which is an extension of the result of Attia [3] for a branching random walk in a deterministic environment; Theorem 1.1(b) describes a large deviation property about the the sequence of measures $\{Z_n(nA)\}$, which generalizes the result of Huang et al [20] for one-dimensional case ($d = 1$). From Theorem 1.1 we can deduce the following corollary, under an additional condition for $L_1$.

**Corollary 1.2.** Assume that $\mathbb{E}[\mathcal{P}_{t}\{\|L_1\| \leq a\}] < \infty$ for some constant $a > 0$. Let $\tilde{\mathcal{J}} = \{\alpha \in \mathbb{R}^d : \Lambda^*(\alpha) < 0\}$.

(a) With probability 1, for all $\alpha \in \text{int}\tilde{\mathcal{J}} \cup \mathcal{J}$, we have

$$\tilde{\Lambda}^*(\alpha) = \Lambda^*(\alpha) \quad \text{and} \quad \text{dim} E(\alpha) = -\Lambda^*(\alpha).$$

(b) With probability 1, for all measurable $A \subset \tilde{\mathcal{J}}$,

$$- \inf_{\alpha \in A^c} \Lambda^*(\alpha) \leq \liminf_{n \to \infty} \frac{1}{n} \log Z_n(nA) \leq \limsup_{n \to \infty} \frac{1}{n} \log Z_n(nA) \leq - \inf_{\alpha \in A} \Lambda^*(\alpha),$$

where $A^c$ denotes the interior of $A$ and $\bar{A}$ its closure.

**Remark 1.1.** It can be seen that $\mathcal{J} \subset \tilde{\mathcal{J}}$. Under the assumptions of Corollary 1.2, it follows that with probability 1, for all $\alpha \in \text{int}\tilde{\mathcal{J}}$,

$$\lim_{\varepsilon \to 0^+} \lim_{n \to \infty} \frac{1}{n} \log Z_n(nB(\alpha, \varepsilon)) = \Lambda^*(\alpha).$$

(1.7)

For the deterministic environment case, the formula (1.7) was shown in Attia [3], Attia and Barral [4], and it also can be deduced from [8]. It is worth mentioning that Attia and Barral [4] showed $\dim \mathcal{E}(\alpha) = -\Lambda^*(\alpha)$ for all $\alpha \in \{\alpha \in \mathbb{R}^d : \Lambda^*(\alpha) < 0\}$ (namely, the boundary problem was solved) through tedious approximation and computation, and with that method, they also calculated the limit of the free energy which is not addressed in this paper. Here, in order to highlight the treatment of random environments and avoid a great length of the article, we shall not discuss such a method for our model.

Now we establish a moderate deviation principle for normalized measures $\frac{Z_n(a_n)}{Z_n(b_n)}$, where $(a_n)$ is a sequence of positive numbers satisfying

$$\frac{a_n}{A_n} \to 0 \quad \text{and} \quad \frac{a_n}{\sqrt{A_n}} \to \infty.$$  

(1.8)

**Theorem 1.3** (Moderate deviation principle). Write $\pi_0 = m_0(0)$. Assume that $\mathbb{E}[\xi \sum_{u \in T_1} S_u] = 0$ a.s. and either of the following statements is satisfied:

(i) \( \epsilon \sup_{\pi_0} \frac{1}{n} \mathbb{E}[\xi \sum_{u \in T_1} e^{\delta|S_u|}] < \infty \) for some $\delta > 0$;

(ii) \( \lim_{n \to \infty} \frac{a_n}{n} = 0 \) for some $\alpha \in (\frac{1}{2}, 1)$ and $\mathbb{E}[\frac{1}{\pi_0} \sum_{u \in T_1} e^{\delta|S_u|}] < \infty$ for some $\delta > 0$.

If $0 \in \Omega_1 \cap \Omega_2$, then with probability 1, the sequence of finite measures $\frac{Z_n(a_n)}{Z_n(b_n)}$ satisfies a moderate deviation principle: for all measurable $A \subset \mathbb{R}^d$,

$$- \inf_{x \in \mathbb{R}^d} \Gamma^*(x) \leq \liminf_{n \to \infty} \frac{1}{a_n} \log \frac{Z_n(a_n)}{Z_n(b_n)} \leq \limsup_{n \to \infty} \frac{1}{a_n} \log \frac{Z_n(a_n)}{Z_n(b_n)} \leq - \inf_{x \in A} \Gamma^*(x),$$

where $A^c$ denotes the interior of $A$ and $\bar{A}$ its closure, and the rate function $\Gamma^*(x) = \sup_{t \in \mathbb{R}^d} \{\langle t, \alpha \rangle - \Gamma(t)\}$ is the Legendre transform of $\Gamma(t) = \frac{1}{2} \langle t, Ct \rangle$ ($t \in \mathbb{R}^d$), where $C = (c_{ij})$ is the matrix with elements $c_{ij} = \mathbb{E}[\frac{1}{\pi_0} \sum_{u \in T_1} S_u^{ij} S_{u}^{ij}]$. 

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Remark 1.2. If the matrix $C$ is reversible, denoting its reverse by $C^{-1}$, then we have $\Gamma^*(x) = \frac{1}{2} (x, C^{-1}x)$.

Theorem 2.1 generalizes and improves the result of Wang and Huang \[44\] for the case $d = 1$, in which the stronger condition $(i)$ was required. Obviously, the condition $\mathbb{E} \sum_{u \in T_n} e^{\delta \|S_u\|} < \infty$ in $(ii)$ is weaker than $(i)$, but under that condition, we can not deal with the case where $\frac{a_n}{n} \to 0$ but $\frac{\log a_n}{n} \to 0$, for example $a_n = \frac{1}{\log n}$.

The rest part of the paper is organised as follows. Firstly, in Section 2, we introduce the natural martingale in BRWRE and study its uniform convergence so as to make preparations for later proofs. Then in Sections 3 and 4, we work on large deviations and prove Theorem 1.1 in Section 3 and Corollary 1.2 in Section 4 respectively. Finally, Section 5 is devoted to moderate deviations where we give the proof of Theorem 1.3.

2 Uniform convergence of martingale

We start with the introduction of the natural martingale in BRWRE. For $n \in \mathbb{N}$ and $z = x + iy \in \mathbb{C}^d$, where $x, y \in \mathbb{R}^d$, we denote the Laplace transform of $Z_n$ by

$$
\tilde{Z}_n(z) = \int e^{\langle z, \omega \rangle} Z_n(d\omega) = \sum_{u \in T_n} e^{\langle z, S_u \rangle}.
$$

Put

$$
P_0(z) = 1, \quad P_n(z) = \mathbb{E}_\xi \tilde{Z}_n(z) = \prod_{i=0}^{n-1} m_i(z),
$$

and

$$
W_n(z) = \frac{\tilde{Z}_n(z)}{P_n(z)} = \sum_{u \in T_n} \tilde{X}_u(z)
$$

Let

$$
\mathcal{F}_0 = \sigma(\xi), \quad \mathcal{F}_n = \sigma(\xi, (X(u) : |u| < n)) \quad \text{for } n \geq 1
$$

be the $\sigma$-field containing all the information concerning the first $n$ generations. It is not difficult to verify that for each $z$ fixed, $W_n(z)$ forms a complex martingale with respect to the filtration $\mathcal{F}_n$ under both laws $\mathbb{P}_\xi$ and $\mathbb{P}$. The convergence of $W_n(z)$ is always useful for studying the asymptotic properties of $Z_n$ and $\tilde{Z}_n$. In the deterministic environment case, this martingale has been studied by Kahane and Peyrière \[27\], Biggins \[6, 8\], Durrett and Liggett \[17\], Guivarc’h \[22\], Lyons \[35\] and Liu \[32, 33, 34\], etc. in different contexts. In particular, for $t \in \mathbb{R}^d$, $W_n(t)$ is nonnegative and hence converges almost surely (a.s.) to a limit random variable $W(t)$ with $\mathbb{E}_\xi W(t) \leq 1$. In order to study large and moderate deviations associated to $Z_n$, we need the uniform convergence of $W_n(z)$, especially for $z$ in a neighbourhood of $t \in \mathbb{R}^d$. For this subject, Biggins \[8\] have found the uniform convergence region of the complex martingale $W_n(z)$ for branching random walks in deterministic environments, and Wang and Huang \[44\] showed similar results for the non-negative martingale $W_n(t)$ for BRWRE in $\mathbb{R}$. The following is our conclusion.

Theorem 2.1. Let $K$ be a compact subset of $\Omega$ and $K(\delta) = \{z \in \mathbb{C}^d : |z_j - t_j| < \delta, \forall j = 1, \ldots, d, \ t \in K \}$ ($\delta > 0$) be the neighbourhood of $K$ in space $\mathbb{C}^d$. Then there exist constants $\delta > 0$ and $p_K > 1$ such that the complex martingale $W_n(z)$ converges uniformly to a limit random variable $W(z)$ almost surely (a.s.) and in $\mathbb{P}_\xi L^p$ on $K(\delta)$ for $p \in (1, p_K]$, namely,

$$
\lim_{n \to \infty} \sup_{z \in K(\delta)} |W_n(z) - W(z)| = 0 \quad \text{a.s.},
$$

$$
\lim_{n \to \infty} \sup_{z \in K(\delta)} \mathbb{E}_\xi |W_n(z) - W(z)|^p = 0 \quad \text{a.s.}.
$$

We shall prove Theorem 2.1 with the method of Biggins \[8\] (also see Attia \[3\]). The crucial technique is to use Cauchy’s formula and the inequality for martingale. The following lemma is deduced from Cauchy’s formula.
Lemma 2.2 ([8], Lemma 3). Let \( z_0 = x_0 + iy_0 \in \mathbb{C}^d \) and \( \epsilon > 0 \). If the function \( f(z) \) is analytic on \( D(z_0, 2\epsilon') \) with \( \epsilon' > \epsilon \), then

\[
\sup_{z \in D(z_0, \epsilon)} |f(z)| \leq (2\pi\epsilon)^{-d} \int_{\partial D(z_0, 2\epsilon)} |f(\zeta)| d\zeta.
\]

Lemma 2.3 ([8], Lemma 1). If \( \{X_t\} \) are independent complex random variable with \( \mathbb{E}X_t = 0 \) or more generally, martingale differences, then \( \mathbb{E} \sum X_t^p \leq 2^p \sum \mathbb{E}|X_t|^p \) for \( 1 \leq p \leq 2 \).

For our model in random environments, compared to the deterministic environment case, the main differences in the proof details are reflected in the following lemmas.

Lemma 2.4. Let \( z_0 = x_0 + iy_0 \in \mathbb{C}^d \) and \( \epsilon > 0 \). Put \( W^*(z_0, \epsilon) = \sup_{z \in D(z_0, \epsilon)} |W_1(z)| \). Then

\[ W^*(z_0, \epsilon) \leq o_0(z_0, \epsilon)^{-1} \sum_{s \in S} \tilde{Z}_1(s), \]

where \( S = \{ s = (s^1, \ldots, s^d) \in \mathbb{R}^d : s^j = s^j_1 \text{ or } s^j \neq s^j_1, j = 1, \ldots, d \} \) with \( s^j_1 := x_0^j - \epsilon \) and \( s^j_2 := x_0^j + \epsilon \).

Proof. For \( z \in D(z_0, \epsilon) \), we have \( |x_j - x_0^j| < \epsilon \), so that \( s^j_1 < x_j < s^j_2 \) (for all \( j \)). Thus for all \( z \in D(z_0, \epsilon) \),

\[
|\tilde{Z}_1(z)| \leq \sum_{u \in T_1} e^{(x,S_u)} = \sum_{u \in T_1} e^{x^j s^j_1} 1_{S_u^1} + 1_{s^j_1 \geq 0} \exp \left( \sum_{j=2}^d x^j s^j_2 \right) \exp \left( \sum_{j=2}^d x^j s^j_1 \right) \leq \sum_{u \in T_1} e^{x^j s^j_1} \exp \left( \sum_{j=2}^d x^j s^j_1 \right) \exp \left( \sum_{j=2}^d x^j s^j_2 \right).
\]

Therefore,

\[
W^*(z_0, \epsilon) = \sup_{z \in D(z_0, \epsilon)} \left| \tilde{Z}_1(z) \right| = \frac{1}{\alpha_0(z_0, \epsilon)}.
\]

Lemma 2.5. If \( t_0 \in \Omega_1 \cap \Omega_2 \), then there exist \( \epsilon_0 > 0 \) and \( p_0 > 1 \) such that for all \( 0 < \epsilon \leq \epsilon_0 \) and \( 1 < p \leq p_0 \),

\[
\mathbb{E} \log^+ \left( \mathbb{E}_z W_1(z)^p \right)^{1/p} < \infty.
\]

Proof. Since \( t_0 \in \Omega_2 \), there exists \( \delta > 0 \) such that \( \mathbb{E} \log^{-} \alpha_0(t_0, \delta) < \infty \). Notice that \( \alpha_0(t_0, \delta) \leq \alpha_0(t_0, \epsilon) \) if \( \epsilon \leq \delta \), so that \( \mathbb{E} \log^{-} \alpha_0(t_0, \epsilon) \leq \mathbb{E} \log^{-} \alpha_0(t_0, \delta) < \infty \). Put \( S_{t_0} = \{ s = (s^1, \ldots, s^d) \in \mathbb{R}^d : s^j = t_0^j + \epsilon_0 \text{ or } t_0^j - \epsilon_0, j = 1, \ldots, d \} \). Since \( t_0 \in \Omega_1 \), we can choose \( 0 < \epsilon_0 \leq \delta \) small enough such that \( S_{t_0} \subset \Omega_1 \). Thus for every \( s \in S_{t_0} \), there exists \( p_s > 1 \) such that \( \mathbb{E} \log^+ \mathbb{E}_s \tilde{Z}_1(t)^{p_s} < \infty \). Take \( p_0 = \min(p_s) \). Using Hölder’s inequality, Lemma 2.4 and Minkowski’s inequality, we obtain for all \( 0 < \epsilon \leq \epsilon_0 \) and \( 1 < p \leq p_0 \),

\[
\sup_{z \in D(t_0, \epsilon)} \left( \mathbb{E}_z W_1(z)^p \right)^{1/p} \leq \left( \sup_{z \in D(t_0, \epsilon)} \mathbb{E}_z W_1(z)^{p_0} \right)^{1/p_0} \leq \frac{1}{\alpha_0(t_0, \epsilon)} \left( \mathbb{E}_S \tilde{Z}_1(t)^{p_0} \right)^{1/p_0},
\]

which implies (2.5) immediately since \( \mathbb{E} \log^{-} \alpha_0(t_0, \epsilon_0) < \infty \) and \( \mathbb{E} \log^+ \mathbb{E}_S \tilde{Z}_1(t)^{p_0} < \infty \).

Lemma 2.6. If \( t_0 \in I \cap \Omega_2 \), then there exists \( p_0 > 1 \) such that for all \( 1 < p \leq p_0 \),

\[
\lim_{\epsilon \downarrow 0} \mathbb{E} \log \left( \sup_{z \in D(t_0, \epsilon)} \frac{m_0(p \epsilon)}{\mathbb{E}_z |m_0(z)|^p} \right) < 0.
\]

(2.6)
Proof. For $z = x + iy \in \mathbb{C}^d$ and $p \in \mathbb{R}$, set $f(z, p) = \log m_0(px) - p \log |m_0(z)|$ and $g(z, p) = \mathbb{E}f(z, p)$. While $t_0$ is fixed, taking the derivative of the function $g(t_0, p)$ with respective to $p$ gives
\[
\frac{\partial g(t_0, p)}{\partial p} = \langle t_0, \nabla \Lambda(t_0) \rangle - \Lambda(t_0) < 0,
\]
since $t_0 \in I$. Therefore, as a function varying with $p$, $g(t_0, p)$ is strictly decreasing near $p = 1$, hence there exists $p_1 > 1$ such that $g(t_0, p) < g(t_0, 1) = 0$ for $1 < p \leq p_1$.

Notice that
\[
\mathbb{E}\log \sup_{z \in D(t_0, \varepsilon)} \frac{m_0(px)}{|m_0(z)|^p} = \mathbb{E}\sup_{z \in D(t_0, \varepsilon)} \log \frac{m_0(px)}{|m_0(z)|^p} = \mathbb{E}\sup_{z \in D(t_0, \varepsilon)} f(z, p). \tag{2.7}
\]
We shall prove that there exist $\varepsilon_1 > 0$ and $p_2 > 1$ such that for $1 < p \leq p_2$,
\[
\mathbb{E}\sup_{z \in D(t_0, \varepsilon_1)} |f(z, p)| < \infty. \tag{2.8}
\]
Since $t_0 \in \Omega_2$, there exists $\delta > 0$ such that $\mathbb{E}\log^- \alpha_0(t_0, \delta) < \infty$. Take $0 < \varepsilon_1 < \delta$. Then $D(t_0, \varepsilon_1) \subset D(t_0, \delta)$, so that $|m_0(z)| \geq \alpha_0(t_0, \delta)$ for $z \in D(t_0, \varepsilon_1)$. Moreover, we can choose $p_2 > 1$ such that $px \in D(t_0, \delta)$ for $z = x + iy \in D(t_0, \varepsilon_1)$ if $1 < p \leq p_2$. In fact, for $1 < p \leq p_2$, it is clear that
\[
|px^j - t_0^j| \leq p|x^j - t_0^j| + (p - 1)|t_0^j| \leq p_2\varepsilon_1 + (p_2 - 1)|t_0^j| < \delta
\]
if we take $p_2 > 1$ small enough. On the other hand, by taking expectation $\mathbb{E}_x$ in (2.4), we get for $z = x + iy \in D(t_0, \varepsilon_1)$ and $1 < p \leq p_2$,
\[
\max\{m_0(x), m_0(px)\} \leq \sum_{s \in \mathcal{S}_d} m_0(s),
\]
where $\mathcal{S}_d = \{s = (s^1, \cdots, s^d) \in \mathbb{R}^d : \forall j \in \{1, \cdots, d\}, s^j = t_0^j + \delta \text{ or } t_0^j - \delta\}$. Thus for $1 < p \leq p_2$,
\[
\sup_{z \in D(t_0, \varepsilon_1)} |f(z, p)| \leq \sup_{z \in D(t_0, \varepsilon_1)} \left(\log^+ m_0(px) + \log^- m_0(px) + p \log^+ m_0(x) + p \log^- |m_0(z)|\right) \leq (p + 1) \left(\sum_{s \in \mathcal{S}_d} m_0(s) + \log^- \alpha_0(t_0, \delta)\right).
\]
Taking expectation $\mathbb{E}$ in the above inequality yields (2.8), since $\mathbb{E}\log^- \alpha_0(t_0, \delta) < \infty$ and $\Lambda(s)$ is a real number for every $s$.

Take $p_0 = \min\{p_1, p_2\}$. For $1 < p \leq p_0$, letting $\varepsilon_1$, $\delta$ in (2.7) and noticing (2.8), by the dominated convergence theorem, we deduce
\[
\lim_{\varepsilon_1 \downarrow 0} \mathbb{E}\log \sup_{z \in D(t_0, \varepsilon)} \frac{m_0(px)}{|m_0(z)|^p} = \lim_{\varepsilon_1 \downarrow 0} \mathbb{E}\sup_{z \in D(t_0, \varepsilon)} f(z, p) = \mathbb{E}f(t_0, p) = g(t_0, p) < 0.
\]
The proof is complete. \hfill \square

Lemma 2.7. Let $(\alpha_n, \beta_n, \gamma_n)_{n \geq 0}$ be a stationary and ergodic sequence of non-negative random variables. If $\mathbb{E}\log \alpha_n < \infty$, $\mathbb{E}\log \beta_n < \infty$ and $\mathbb{E}\log^+ \gamma_n < \infty$, then
\[
\limsup_{n \rightarrow \infty} \frac{1}{n} \log \left(\gamma_n \sum_{k=0}^{n} \alpha_k \cdots \alpha_{k-1} \beta_k \cdots \beta_{n-1}\right) \leq \max\{\mathbb{E}\log \alpha_0, \mathbb{E}\log \beta_0\} \quad \text{a.s.} \tag{2.9}
\]
Proof. Set $d_n = \alpha_n / \beta_n$ and $d = \mathbb{E}\log d_0$. By the ergodic theorem, we have a.s.
\[
\lim_{n \rightarrow \infty} \frac{1}{n} \log (\beta_0 \cdots \beta_{n-1}) = \mathbb{E}\log \beta_0, \tag{2.10}
\]
\[
\lim_{n \rightarrow \infty} \frac{1}{n} \log (d_0 \cdots d_{n-1}) = d = \mathbb{E}\log \alpha_0 - \mathbb{E}\log \beta_0. \tag{2.11}
\]
and
\[\limsup_{n \to \infty} \frac{1}{n} \log \gamma_n \leq \limsup_{n \to \infty} \frac{1}{n} \log^+ \gamma_n\]
\[= \limsup_{n \to \infty} \frac{1}{n} \left[ (\log^+ \gamma_0 + \cdots + \log^+ \gamma_n) - (\log^+ \gamma_0 + \cdots + \log^+ \gamma_{n-1}) \right]\]
\[= \mathbb{E} \log^+ \gamma_0 - \mathbb{E} \log^+ \gamma_0 = 0. \] (2.12)

By (2.11), for all \(\varepsilon > 0\), there exists an entire \(n_\varepsilon > 0\) such that for \(n \geq n_\varepsilon\),
\[d_0 \cdots d_{n-1} < e^{(d+\varepsilon)n} \text{ a.s.,}\]
so that
\[\frac{1}{n} \log \left( \sum_{k=0}^{n} \alpha_0 \cdots \alpha_{k-1} \beta_k \cdots \beta_{n-1} \right) \]
\[= \frac{1}{n} \log \gamma_n + \frac{1}{n} \log (\beta_0 \cdots \beta_{n-1}) + \frac{1}{n} \log \left( A_\varepsilon + \sum_{k=n_\varepsilon}^{n} d_0 \cdots d_{k-1} \right)\]
\[< \frac{1}{n} \log \gamma_n + \frac{1}{n} \log (\beta_0 \cdots \beta_{n-1}) + \frac{1}{n} \log \left( A_\varepsilon + \sum_{k=n_\varepsilon}^{n} e^{(d+\varepsilon)k} \right), \] (2.13)
where \(A_\varepsilon := \sum_{k=0}^{n_\varepsilon-1} d_0 \cdots d_{k-1}\). Notice that a.s.
\[\lim_{n \to \infty} \frac{1}{n} \log \left( A_\varepsilon + \sum_{k=n_\varepsilon}^{n} e^{(d+\varepsilon)k} \right) = \begin{cases} 0 & \text{if } d + \varepsilon \leq 0, \\ d + \varepsilon & \text{if } d + \varepsilon > 0. \end{cases} \] (2.14)

Let \(n \to \infty\) in (2.13). It follows from (2.11), (2.12) and (2.14) that
\[\limsup_{n \to \infty} \frac{1}{n} \log \left( \sum_{k=0}^{n} \alpha_0 \cdots \alpha_{k-1} \beta_k \cdots \beta_{n-1} \right) \leq \mathbb{E} \log \beta_0 + \max\{0, d + \varepsilon\}\]
\[= \max\{\mathbb{E} \log \beta_0, \mathbb{E} \log \alpha_0 + \varepsilon\} \text{ a.s.,}\]
which yields (2.9) by the arbitrary of \(\varepsilon\).

From Lemma 2.7 we deduce the following statements.

**Corollary 2.8.** Let \((\alpha_n, \beta_n, \gamma_n)_{n \geq 0}\) be a stationary and ergodic sequence of non-negative random variables. Assume that \(\mathbb{E} |\log \alpha_0| < \infty, \mathbb{E} |\log \beta_0| < \infty\) and \(\mathbb{E} \log^+ \gamma_0 < \infty\).

(a) If \(\max\{\mathbb{E} \log \alpha_0, \mathbb{E} \log \beta_0\} < 0\), then the series
\[\sum_{n=0}^{\infty} \gamma_n \sum_{k=0}^{n} \alpha_0 \cdots \alpha_{k-1} \beta_k \cdots \beta_{n-1} < \infty \text{ a.s..} \] (2.15)

(b) If \(\mathbb{E} \log \alpha_0 < 0\), then the series
\[\sum_{n=0}^{\infty} \alpha_0 \cdots \alpha_{n-1} \gamma_n < \infty \text{ a.s..} \] (2.16)

Now we can state the proof of Theorem 2.1.

**Proof of Theorem 2.1.** It suffices to prove that for each \(t_0 \in K\), there exists a polydic \(D(t_0, \varepsilon/2) \subset \mathbb{C}^d \) \((\varepsilon > 0\) small enough) such that
\[\lim_{n \to \infty} \sup_{z \in D(t_0, \varepsilon/2)} |W_{n+1}(z) - W_n(z)| = 0 \text{ a.s.,} \] (2.17)
\[\lim_{n \to \infty} \sup_{z \in D(t_0, \varepsilon/2)} \mathbb{E}_z |W_{n+1}(z) - W_n(z)|^p = 0 \text{ a.s.} \] (2.18)
for suitable $p \in (1, 2]$. As the function $W_n(z)$ is analytic on $\mathbb{C}^d$, applying Lemma 2.2 we get
\[
\sup_{z \in D(t, \varepsilon/2)} |W_{n+1}(z) - W_n(z)|^p \leq C \int_{\partial D(t, \varepsilon)} |W_{n+1}(\zeta) - W_n(\zeta)|^p d\zeta,
\]
where $C > 0$ is a constant, and in general it does not necessarily stand for the same constant throughout. Thus, to show (2.17) and (2.18), we can turn to show the series
\[
\sum_{n} \sup_{z \in D(t, \varepsilon)} (\mathbb{E}_\zeta |W_{n+1}(z) - W_n(z)|^p)^{1/p} < \infty \quad \text{a.s.} \quad (2.19)
\]
Observe that
\[
W_{n+1}(z) - W_n(z) = \sum_{u \in \mathcal{T}_n} \tilde{X}_u(z)(W_1(u, z) - 1),
\]
where $W_k(u, z)$ is defined in (3.2) (see next section). Under quenched law $\mathbb{P}_\zeta, \{W_k(u, z)\}_{|u|=n}$ are i.i.d. and independent of $\mathcal{F}_n$ with common distribution determined by $\mathbb{P}_\zeta(W_k(u, z) \in \cdot) = \mathbb{P}_{T^n \zeta}(W_k(z) \in \cdot)$. The notation $T$ represents the shift operator: $T^n \xi = (\xi_n, \xi_{n+1}, \cdots)$ if $\xi = (\xi_0, \xi_1, \cdots)$. Applying Lemma 2.3 to $\{\tilde{X}_u(z)(W_1(u, z) - 1)\}$, we obtain
\[
\mathbb{E}_\zeta |W_{n+1}(z) - W_n(z)|^p \leq C \sup_{u \in \mathcal{T}_n} |\tilde{X}_u(z)|^p |W_1(u, z) - 1|^p \leq C \frac{P_n(px)}{|P_n(z)|^p} \mathbb{E}_{T^n \zeta} |W_1(z) - 1|^p. \tag{2.20}
\]
Put
\[
\beta_n = \sup_{z \in D(t, \varepsilon)} \frac{m_n(px)}{|m_n(z)|^p} \quad \text{and} \quad \gamma_n = \sup_{z \in D(t, \varepsilon)} (\mathbb{E}_{T^n \zeta} |W_1(z) - 1|^p)^{1/p}.
\]
By Lemma 2.4 there exist $\varepsilon_1 > 0$ and $p_1 > 1$ such that (2.3) holds for all $0 < \varepsilon \leq \varepsilon_1$ and $1 < p \leq p_1$. Meanwhile, Lemma 2.4 implies that there exists $p_2 > 1$ such that (2.6) holds for all $1 < p \leq p_2$. Take $p \leq \min\{p_1, p_2\}$ and fix this $p$. By (2.6), there exists $\varepsilon_p > 0$ such that for all $0 < \varepsilon \leq \varepsilon_p$,
\[
\mathbb{E} \log \beta_0 = \frac{1}{p} \mathbb{E} \log \sup_{z \in D(t, \varepsilon)} \frac{m_0(px)}{|m_0(z)|^p} < 0.
\]
Take $0 < \varepsilon \leq \min\{\varepsilon_1, \varepsilon_p\}$. So we have $\mathbb{E} \log \beta_0 < 0$ and $\mathbb{E} \log^+ \gamma_0 < \infty$ (by (2.5)). Then it follows from Corollary 2.8(b) that
\[
\sum_{n} \sup_{z \in D(t, \varepsilon)} (\mathbb{E}_\zeta |W_{n+1}(z) - W_n(z)|^p)^{1/p} \leq \sum_{n} \beta_0 \cdots \beta_{n-1} \gamma_n < \infty \quad \text{a.s.},
\]
which completes the proof.

**Remark 2.1.** Theorem 2.4 says that $W_n(z)$ converges to $W(z)$ uniformly a.s. and in $\mathbb{P}_\zeta$-$L^p$ for some $p \in (1, 2)$ on set $K(\delta)$. Noticing (2.20), and by Minkowski’s inequality, we deduce that for every $z \in K(\delta)$,
\[
(\mathbb{E}_\zeta |W(z)|^p)^{1/p} = \lim_{n \to \infty} (\mathbb{E}_\zeta |W_n(z)|^p)^{1/p} \leq \lim_{n \to \infty} (\mathbb{E}_\zeta |W_n(z) - 1|^p)^{1/p} + 1 \leq \lim_{n \to \infty} \sum_{k=0}^{n-1} (\mathbb{E}_\zeta |W_{k+1}(z) - W_k(z)|^p)^{1/p} + 1 \leq C \sum_{k=0}^{\infty} \frac{P_k(px)}{|P_k(z)|} (\mathbb{E}_{T^n \zeta} |W_1(z) - 1|^p)^{1/p} + 1. \tag{2.22}
\]

Theorem 2.4 is principally concerned with the uniform convergence of $W_n(z)$ near the real number. From similar arguments to the proof of Theorem 2.4, we can obtain a general uniform convergence region for the complex martingale $W_n(z)$, see Theorem 2.9 below whose proof is omitted. Set
\[
I'_p = \{ z \in \mathbb{C}^d : \Lambda(px) - pE \log |m_0(z)| < 0 \},
\]
\[
\Omega_{1,p} = \text{int}\{ z \in \mathbb{C}^d : E \log |E \mathbb{E}_\zeta \tilde{Z}_1(x)| < \infty \},
\]
\[
\Omega' = \{ z \in \mathbb{C}^d : \exists \delta_2 > 0 \text{ such that } E \log^+ a_0(z, \delta_2) < \infty \},
\]
\[
\Omega'_p = I'_p \cap \Omega_{1,p} \text{ and } \Omega' = \left( \bigcup_{1 \leq p \leq 2} \Omega'_p \right) \cap \Omega_2.
\]
Theorem 2.9. Let $K$ be a compact subset of $\Omega$. Then there exists constant $p_K > 1$ such that the complex martingale $W_n(z)$ converges uniformly to a limit random variable $W(z)$ almost surely (a.s.) and in $P_\xi - L^p$ on $K$ for $p \in (1, p_K]$.

As the complication of Theorem 2.4, Theorem 2.9 generalizes the result of Biggins ([8], Theorem 2) for classical branching random walks. It can be seen that $\Omega$ is actually the intersection of $\Omega'$ with the real space, i.e. $\Omega = \Omega' \cap \mathbb{R}^d$, which means that Theorem 2.1 is in fact contained in Theorem 2.9.

3 Proof of Theorem 1.1

The proof of Theorem 1.1 is based on the natural approach of finding the upper and lower bounds for the Hausdorff dimensions of the level sets $E(\alpha)$, according to the steps showed in Attia [3]. Firstly, the upper bounds can be deduced from the two propositions below.

Proposition 3.1. With probability 1, $\tilde{\Lambda}(t) \leq \Lambda(t)$ for all $t \in \mathbb{R}^d$, and then $\tilde{\Lambda}^*(\alpha) \geq \Lambda^*(\alpha)$ for all $t \in \mathbb{R}^d$.

Proof. Since the functions $\tilde{\Lambda}(t)$ and $\Lambda(t)$ are convex and thus continuous, we only need to prove $\tilde{\Lambda}(t) \leq \Lambda(t)$ a.s. for each $t \in \mathbb{R}^d$. Fix $t \in \mathbb{R}^d$. For $s > \Lambda(t)$, we have $E \log(e^{-s}m_0(t)) = \Lambda(t) - s < 0$. Thus by Corollary 2.3 [b],

$$E \xi \sum_n e^{-ns} \tilde{Z}_n(t) = \sum_n e^{-ns} P_n(t) < \infty \quad \text{a.s.,}$$

which implies that $\sum_n e^{-ns} \tilde{Z}_n(t) < \infty$ a.s.. Hence, $\tilde{Z}_n(t) = O(e^{ns})$, which yields $\tilde{\Lambda}(t) \leq s$. Since $s > \Lambda(t)$ is arbitrary, we have the conclusion.

Proposition 3.2 ([3], Proposition 2.2). With probability 1, for all $\alpha \in \mathbb{R}^d$, $\text{dim } E(\alpha) \leq -\tilde{\Lambda}^*(\alpha)$, where $\text{dim } E(\alpha) < 0$ means that $E(\alpha)$ is empty.

In order to obtain the lower bounds, we need to rely on the associated Mandelbrot measure [37]. Let $T(u)$ be the Galton-Watson tree rooted at $u \in T$ and $T_n(u) = \{uv \in T : |v| = n\}$ be the set of particles in the $n$-th generation of $T(u)$. For $u \in T$ and $z \in \mathbb{C}^d$, denote

$$\tilde{Z}_n(u,z) = \sum_{uv \in T_n(u)} e^{(z,S_u - S_v)} \quad (3.1)$$

and

$$W_n(u,z) = \frac{\tilde{Z}_n(u,z)}{E_\xi \tilde{Z}_n(u,z)} = \frac{\sum_{uv \in T_n(u)} e^{(z,S_u - S_v)}}{m_{|u|}(z) \cdots m_{|u|+n-1}(z)}. \quad (3.2)$$

In particular, we have $\tilde{Z}_n(0,z) = \tilde{Z}_n(z)$ and $W_n(0,z) = W_n(z)$. For $t \in \mathbb{R}^d$, the martingale $\{W_n(u,t)\}$ is non-negative, and hence it converges a.s. to a limit

$$W(u,t) := \lim_{n \to \infty} W_n(u,t). \quad (3.3)$$

By the branching property, we can see that

$$W(u,t) = \sum_{u \in T_1(u)} \frac{e^{(t,L_i(u))}}{m_{|u|}(t)} W(ui,t).$$

Thus for each $t \in \mathbb{R}$, we can define a unique measure $\mu_t$ on $\partial T$ such that

$$\mu_t([u]) = \tilde{X}_u(t) W(u,t). \quad (3.4)$$

This measure $\mu_t$ is the so-called Mandelbrot measure for BRWRE. Clearly, $\mu_t$ is finite with $\mu_t(\partial T) = W(t)$.

Theorem 3.3. With probability 1, for all $t \in \Omega$,

$$\lim_{n \to \infty} \frac{1}{n} \log \mu_t([u^n]) = \langle t, \nabla \Lambda(t) \rangle - \Lambda(t) \quad \text{for } \mu_t\text{-a.s. } u \in \partial T.$$
Theorem 3.3 generalizes the result of Liu and Rouault [31] for the Galton-Watson processes, and that of Attia [3] for classical branching random walks. Such a result allows to further calculate the dimension of $\mu_t$, as well as the Hausdorff and Packing dimensions of the support of $\mu_t$ and those of the level sets $E(\alpha)$, just as what were done in [27, 28, 33, 3, 4].

By the definition of $\mu_t$, we see that
\[
\frac{1}{n} \log \mu_t([u|n]) = (\frac{S_n(u)}{n}) - \frac{1}{n} \log P_n(t) + \frac{1}{n} \log W(u|n, t).
\] (3.5)

The ergodic theorem gives that $\frac{1}{n} \log P_n(t) \to \Lambda(t)$ a.s. as $n \to \infty$. To prove Theorem 3.3 we need to calculate the limits of the rest two terms in the right hand side of (3.5). To this end, the following lemma is useful.

**Lemma 3.4.** Let $(\alpha_n)_{n \geq 0}$ be a stationary and ergodic sequence of non-negative random variables satisfying $\mathbb{E} \log \alpha_0 \in (-\infty, 0)$. If $t_0 \in \Omega$, then there exist $\varepsilon_0 > 0$ and $p_0 \in (1, 2]$ such that for all $0 < \varepsilon \leq \varepsilon_0$ and $1 \leq p \leq p_0$, the series
\[
\sum_{n} \alpha_0 \cdots \alpha_{n-1} \sup_{z \in D(t, \varepsilon)} (\mathbb{E}_{T=\xi} |W(z)|^p)^{1/p} < \infty \quad \text{a.s.,}
\] (3.6)
where $a > 1$ is a constant.

**Proof.** Recall $\beta_n$ and $\gamma_n$ defined in (2.21). According to the proof of Theorem 2.21 we can take $\varepsilon_0 > 0$ and $p_0 > 1$ such that $\mathbb{E} \log \beta_0 < 0$ and $\mathbb{E} \log^{+} \gamma_0 < \infty$ with $\varepsilon = \varepsilon_0$ and $p = p_0$. By (2.22), we deduce that for all $0 < \varepsilon \leq \varepsilon_0$ and $1 \leq p \leq \min\{2, p_0\}$,
\[
\sup_{z \in D(t, \varepsilon)} (\mathbb{E}_{T=\xi} |W(z)|^p)^{1/p} \leq \sup_{z \in D(t, \varepsilon_0)} (\mathbb{E}_{T=\xi} |W(z)|^{p_0})^{1/p_0} \leq C \sum_{k=0}^{\infty} \beta_0 \cdots \beta_{k-1} \gamma_k + 1.
\]

Thus
\[
\sum_{n=0}^{\infty} \alpha_0 \cdots \alpha_{n-1} \sup_{z \in D(t, \varepsilon)} (\mathbb{E}_{T=\xi} |W(z)|^p)^{1/p} \leq C \sum_{n=0}^{\infty} \alpha_0 \cdots \alpha_{n-1} \sum_{k=0}^{\infty} \beta_0 \cdots \beta_{k-1} \gamma_k + \sum_{n=0}^{\infty} \alpha_0 \cdots \alpha_{n-1}.
\]

Corollary 2.3(b) shows that the second series in the right hand side of the inequality above converges a.s. since $\mathbb{E} \log \alpha_0 < 0$. For the first series, we have
\[
\sum_{n=0}^{\infty} \alpha_0 \cdots \alpha_{n-1} \sum_{k=0}^{\infty} \beta_0 \cdots \beta_{k-1} \gamma_k = \sum_{k=0}^{\infty} \gamma_k \sum_{n=0}^{\infty} \alpha_0 \cdots \alpha_{n-1} \beta_0 \cdots \beta_{k-1} < \infty \quad \text{a.s.}
\]
from Corollary 2.3(a), since $\mathbb{E} \log^{+} \gamma_0 < \infty$ and $\max\{\mathbb{E} \log \alpha_0, \mathbb{E} \log \beta_0\} < 0$.

**Proposition 3.5.** With probability 1, for all $t \in \Omega$,
\[
\lim_{n \to \infty} \frac{1}{n} \log W(u|n, t) = 0 \quad \text{for } \mu_t, \text{a.s. } u \in \partial T.
\]

**Proof.** For $a > 1$, $t \in \mathbb{R}^d$ and $n \geq 1$, we set
\[
E_n^- = \{u \in \partial T^+ : W(u|n, t) \leq a^{-n}\},
\]
\[
E_n^+ = \{u \in \partial T^+ : W(u|n, t) \geq a^n\}.
\]

Let $K$ be a compact subset of $\Omega$. It suffices to show that for $E_n \in \{E_n^-, E_n^+\}$,
\[
\mathbb{E}_{\xi} \left( \sup_{t \in K} \sum_n \mu_t(E_n) \right) < \infty \quad \text{a.s.,}
\] (3.7)
which leads to the desired result by Borel-Cantelli lemma. As $K$ is compact, it can be covered by a finite number of sets $B(t_i, \varepsilon_i/2)$. Therefore, to show (3.7), we only need to prove that for each $t_0 \in \Omega$, there exists $\varepsilon > 0$ small enough such that
\[
\mathbb{E}_{\xi} \left( \sup_{t \in B(t_0, \varepsilon/2)} \sum_n \mu_t(E_n) \right) < \infty \quad \text{a.s.}
\] (3.8)
We first consider the case where $E_n = E_n^-$. We have

$$\mu(E_n^-) = \sum_{u \in \mathcal{T}_n} \tilde{X}_{a}(t)W(u, t)1_{(W(u, t) \leq a^{-n})} \leq a^{-n} \sum_{u \in \mathcal{T}_n} \tilde{X}_{a}(t).$$

Applying Lemma 2.2 to the analytic function $\sum_{u \in \mathcal{T}_n} \tilde{X}_{a}(z)$, we get

$$\mathbb{E}_\xi \left( \sup_{t \in B(t_0, \varepsilon/2)} \mu_t(E_n^-) \right) \leq Ca^{-n} \sup_{z \in D(t_0, \varepsilon)} \mathbb{E}_\xi \sum_{u \in \mathcal{T}_n} |\tilde{X}_{a}(z)| \leq C d_0(\varepsilon) \cdots d_{n-1}(\varepsilon),$$

where

$$d_n(\varepsilon) = a^{-1} \sup_{z \in D(t_0, \varepsilon)} \frac{m_n(x)}{|m_n(z)|}.$$

Similarly to the proof of Lemma 2.6, the dominated convergence theorem gives

$$\lim_{\varepsilon \downarrow 0} \mathbb{E} \log d_0(\varepsilon) = -\log a < 0,$$

so that by Corollary 2.8(b), the series $\sum_n d_0(\varepsilon) \cdots d_{n-1}(\varepsilon) < \infty$ a.s. for $\varepsilon > 0$ small enough. Thus $E_n = E_n^-$. Now we consider the case where $E_n = E_n^+$. The proof is similar. Notice that for $p > 1$,

$$\mu(E_n^+) = \sum_{u \in \mathcal{T}_n} \tilde{X}_{a}(t)W(u, t)1_{(W(u, t) \geq a^n)} \leq a^{-(p-1)n} \sum_{u \in \mathcal{T}_n} \tilde{X}_{a}(t)W(u, t)^p.$$

Theorem 2.1 ensures that the limit $W(u, z) := \lim_n W_n(u, z)$ exists a.s. and analytic on $D(t_0, \varepsilon)$ for some $\varepsilon > 0$. Applying Lemma 2.2 again, we obtain

$$\sum_n \mathbb{E}_\xi \left( \sup_{t \in B(t_0, \varepsilon/2)} \mu_t(E_n^+) \right) \leq C \sum_n a^{-(p-1)n} \sup_{z \in D(t_0, \varepsilon)} \mathbb{E}_\xi \sum_{u \in \mathcal{T}_n} |\tilde{X}_{a}(z)||W(u, z)|^p \leq C \sum_n a^{-(p-1)n} \sup_{z \in D(t_0, \varepsilon)} \frac{P_a(x)}{|P_a(z)|} \mathbb{E}_{T^\varepsilon} |W(z)|^p \leq C \left( \sum_n \tilde{d}_0(\varepsilon) \cdots \tilde{d}_{n-1}(\varepsilon) \left( \mathbb{E}_{T^\varepsilon} |W(z)|^p \right)^{1/p} \right)^p,$$

where

$$\tilde{d}_n(\varepsilon) = a^{-\frac{1}{p}} \left( \sup_{z \in D(t_0, \varepsilon)} \frac{m_n(x)}{|m_n(z)|} \right)^{1/p}.$$ 

By the dominated convergence theorem,

$$\lim_{\varepsilon \downarrow 0} \mathbb{E} \log \tilde{d}_0(\varepsilon) = -\frac{p-1}{p} \log a < 0.$$

Therefore, by Lemma 3.4 the series in the last line of (3.9) converges a.s. for some $p \in (1, 2]$ and $\varepsilon > 0$ small enough, so (3.8) holds for $E_n = E_n^+$. 

For $t \in \Omega$ and $n \in \mathbb{N}$, let us define a measure on $\mathbb{R}^d$ as:

$$\nu_{t, n}(\cdot) = \mu_t \left( u \in \partial T : \frac{S_{u|n}}{n} \in \cdot \right).$$

For $(t, \lambda) \in \mathbb{R}^d \times \mathbb{R}^d$, let

$$L_n(t, \lambda) = \frac{1}{n} \log \int_{\mathbb{R}^d} e^{n(\lambda, x)} d\nu_{t, n}(x)$$

and

$$L_t(\lambda) = \lim_{n \to \infty} L_n(t, \lambda).$$
Proposition 3.6. Let \( t_0 \in \Omega \). Then there exists \( \varepsilon_0 > 0 \), such that
\[
\mathbb{P}(L(t) - \Lambda(t + \varepsilon_0)) = 1. \tag{3.13}
\]

Proof. For \((z, z') \in \mathbb{C}^d \times \mathbb{C}^d\), denote
\[
V_n(z, z') = \sum_{u \in T_n} X_u(z + z') W(u, z)
\]
when \( W(u, z) := \lim_n W_n(u, z) \) exists. In particular, \( V_n(t, z') \) is always well defined for \((t, z') \in \mathbb{R}^d \times \mathbb{C}^d\), as \( W(u, t) \) is always exists for all \( t \in \mathbb{R}^d \). Calculate that for \((t, \lambda) \in \mathbb{R}^d \times \mathbb{R}^d\),
\[
L_n(t, \lambda) = \frac{1}{n} \log \int_{u \in \Theta} e^{\lambda \langle u, x \rangle} d\mu_t(u)
= \frac{1}{n} \log \sum_{u \in T_n} e^{\langle u, x \rangle} \frac{P_n(t)}{P_n(t)} W(u, t)
= \frac{1}{n} \log V_n(t, \lambda) + \frac{1}{n} \log P_n(t + \lambda) - \frac{1}{n} \log P_n(t). \tag{3.14}
\]

The ergodic theorem implies that
\[
\lim_{n \to \infty} \frac{1}{n} \log P_n(t) = \mathbb{E} \log m_0(t) \quad a.s..
\]

Hence to obtain (3.13), it remains to show that \( V_n(t, \lambda) \) converges uniformly a.s. and in \( \mathbb{P}_x \)-\( L^1 \) on \( B(t_0, \varepsilon_0) \times B(0, \varepsilon_0) \) for some \( \varepsilon_0 > 0 \), which ensures that a.s., the limit
\[
V(t, \lambda) = \lim_{n \to \infty} V_n(t, \lambda) > 0 \quad {\text{on}} \ B(t_0, \varepsilon_0) \times B(0, \varepsilon_0),
\]
and so (3.13) holds by letting \( n \to \infty \) in (3.14).

Now we work on the uniform convergence (a.s. and in \( \mathbb{P}_x \)-\( L^1 \)) of \( V_n(t, \lambda) \). By Theorem 2.1, we see that there exists \( \delta > 0 \) such that \( V_n(z, z') \) is well defined and analytic on \((z, z') \in D(t_0, \delta) \times \mathbb{C}^d\). Similarly to the proof of Theorem 2.1, we can turn to show that
\[
\sum_n \sup_{z \in D(t_0, \varepsilon)} \left( \mathbb{E}_x |V_{n+1}(z, z') - V_n(z, z')|^p \right)^{1/p} < \infty \quad a.s. \tag{3.15}
\]
for suitable \( 0 < \varepsilon \leq \delta \) and \( p \in (1, 2] \). Observe that
\[
V_{n+1}(z, z') - V_n(z, z') = \sum_{u \in T_n} \tilde{X}_u(z + z') \left( \sum_{u \in T_1(u)} \frac{e^{\langle u, x \rangle}}{m_n(z + z')} W(u, z) - W(u, z) \right).
\]

Using Lemma 2.3, we deduce
\[
\mathbb{E}_x |V_{n+1}(z, z') - V_n(z, z')|^p \leq C \frac{P_n(p(x + x'))}{P_n(z + z')^p} \mathbb{E}_{T^n_x} |V_1(z, z') - W(z)|^p. \tag{3.16}
\]

By Minkowski’s inequality, we have
\[
(\mathbb{E}_x |V_1(z, z') - W(z)|^p)^{1/p} \leq (\mathbb{E}_x |V_1(z, z')|^p)^{1/p} + (\mathbb{E}_x |W(z)|^p)^{1/p} \tag{3.17}
\]
and
\[
(\mathbb{E}_x |V_1(z, z')|^p)^{1/p} = \left( \mathbb{E}_x \left| \sum_{u \in T_1} \tilde{X}_1(z + z') (W(u, z) - \mathbb{E}_x W(u, z) + \mathbb{E}_x W(u, z)) \right|^p \right)^{1/p}
\leq \left( \mathbb{E}_x \left| \sum_{u \in T_1} \tilde{X}_1(z + z') (W(u, z) - \mathbb{E}_x W(u, z)) \right|^p \right)^{1/p}
+ (\mathbb{E}_x |W_1(z + z')|^p)^{1/p} \mathbb{E}_{T^n_x} W(z). \tag{3.18}
\]
Lemma 2.3 yields

$$\mathbb{E}_\xi \left| \sum_{z \in T_1} \tilde{X}_1(z + z') (W(u, z) - \mathbb{E}_\xi W(u, z)) \right|^p \leq C \mathbb{E}_\xi \sum_{z \in T_1} |\tilde{X}_1(z + z')|^p |W(u, z) - \mathbb{E}_\xi W(u, z)|^p$$

$$\leq C \left( \frac{m_0(p(x + z'))}{|m_0(z + z')|^p} \right) \mathbb{E}_\xi |W(z)|^p. \quad (3.19)$$

Combining (3.16)-(3.19), we obtain

$$\sup_{z \in D(t_0, 2\varepsilon)} \left( \mathbb{E}_\xi |V_{n+1}(z, z') - V_n(z, z')|^p \right)^{1/p} \leq C \left( \sum_{z \in D(t_0, 2\varepsilon)} \frac{P_n(px)^{1/p}}{P_n(z)} \sup_{z \in D(t_0, 2\varepsilon)} (\mathbb{E}_{T^n} |W(z)|^p)^{1/p} \right) + C \left( \sum_{z \in D(t_0, 2\varepsilon)} \frac{P_{n+1}(px)^{1/p}}{P_{n+1}(z)} \sup_{z \in D(t_0, 2\varepsilon)} (\mathbb{E}_{T^{n+1}} |W(z)|^p)^{1/p} \right).$$

Thus

$$\sum_{n} \sup_{z \in D(t_0, \varepsilon)} \left( \mathbb{E}_\xi |V_{n+1}(z, z') - V_n(z, z')|^p \right)^{1/p} \leq C \left( \sum_{n} \beta_0 \cdots \beta_{n-1} \sup_{z \in D(t_0, 2\varepsilon)} (\mathbb{E}_{T^n} |W(z)|^p)^{1/p} + \sum_{n} \beta_0 \cdots \beta_{n-1} \tilde{\gamma}_n \sup_{z \in D(t_0, 2\varepsilon)} \mathbb{E}_{T^{n+1}} |W(z)| \right), \quad (3.20)$$

where $\beta_n$ is defined in 2.1 with $\varepsilon$ replaced by $2\varepsilon$, and $\tilde{\gamma}_n = \sup_{z \in D(t_0, 2\varepsilon)} (\mathbb{E}_{T^n} |W(z)|^p)^{1/p}$. Lemmas 2.5 and 2.6 ensure that $\mathbb{E} \log \beta_0 < 0$ and $\mathbb{E} \log^+ \tilde{\gamma}_0 < \infty$ for suitable $0 < \varepsilon \leq \delta$ and $p \in (1, 2]$. Therefore, the first series in right hand side of the inequality (3.20) above converges a.s. by Lemma 3.4. For the second series, notice that

$$\lim_{n \to \infty} \sup_{z \in D(t_0, 2\varepsilon)} \frac{1}{n} \log (\beta_0 \cdots \beta_{n-1} \tilde{\gamma}_n) \leq \mathbb{E} \log \beta_0 < 0,$$

which implies that $\beta_0 \cdots \beta_{n-1} \tilde{\gamma}_n < a^{-n}$ a.s. for some constant $a > 1$ as $n$ large enough. It follows from Lemma 3.4 that the series $\sum_{n} \sup_{z \in D(t_0, 2\varepsilon)} \mathbb{E}_{T^{n+1}} |W(z)| < \infty$ a.s., which implies the a.s. convergence of the second series in the right hand side of (3.20).

\[
\text{Proposition 3.7. With probability 1, for all } t \in \Omega, \quad \lim_{n \to \infty} \frac{S_{u|n}}{n} = \nabla \Lambda(t) \quad \text{for } \mu_t\text{-a.s. } u \in \partial \bar{T}.
\]

\[
\text{Proof. For rational } r, \text{ denote } \mathcal{E}_r = \{ \omega \in \Theta^N \times \Gamma : L_r(\lambda) = \Lambda(t + \lambda) - \Lambda(t), \forall t \in B(r, \varepsilon_r), \forall \lambda \in B(0, \varepsilon_r) \} \\
\text{and } \hat{\mathcal{E}} = \bigcap_r \mathcal{E}_r. \text{ Then we have } \mathbb{P}(\hat{\mathcal{E}}) = 1 \text{ from Proposition 3.6. Moreover, put } \alpha_t = \nabla \Lambda(t) \text{ and } \mathcal{E}' = \{ \omega \in \Theta^N \times \Gamma : \mu_t \left( u \in \partial \bar{T} : \lim_{n \to \infty} \frac{S_{u|n}}{n} \neq \alpha_t \right) = 0, \forall t \in \Omega \}. \\
\text{We shall show } \hat{\mathcal{E}} \subset \mathcal{E}', \text{ which yields } \mathbb{P}(\mathcal{E}') = 1.
\]

Notice that for every $t \in \Omega$, there must exists a rational $r$ such that $t \in B(r, \varepsilon_r)$. Thus for $\omega \in \hat{\mathcal{E}}$, we can see that for every $t \in \Omega$, $L_r(\lambda) = \Lambda(t + \lambda) - \Lambda(t)$ is differentiable in the neighborhood (depending on
t) of \( \lambda = 0 \), so that \( L_t(0) = 0 \), and \( \nabla L_t(0) \) exists with value \( \alpha_t \). Fix \( \omega \in \tilde{\mathcal{E}} \). For this \( \omega \), we need to prove that for all \( \varepsilon > 0 \),

\[
\sum_n \mu_t \left( u \in \partial T : \left\| \frac{S_{u,n}}{n} - \alpha_t \right\| \geq \varepsilon \right) < \infty, \quad \forall t \in \Omega,
\]

(3.21)

which then implies \( \omega \in \mathcal{E} \) by Borel-Cantelli lemma. For \( t \in \Omega \) and \( \varepsilon > 0 \), write \( A_{t,\varepsilon} = \{ \alpha \in \mathbb{R}^d : \| \alpha - \alpha_t \| \geq \varepsilon \} \). It follows from the Gärtner-Ellis theorem ([16], Theorem 4.5.3) that

\[
l_{n} \mu_t \left( u \in \partial T : \left\| \frac{S_{u,n}}{n} - \alpha_t \right\| \geq \varepsilon \right) = \inf_{\alpha \in A_{t,\varepsilon}} L^*_t(\alpha),
\]

where \( L^*_t(\alpha) = \sup_{\lambda \in \mathbb{R}^d} \{ \langle \lambda, \alpha \rangle - L_t(\lambda) \} \) is the Legendre transform of \( L_t(\lambda) \). Notice that

\[
\mu_t \left( u \in \partial T : \left\| \frac{S_{u,n}}{n} - \alpha_t \right\| \geq \varepsilon \right) = \mu_{n,t}(A_{t,\varepsilon}).
\]

To obtain (3.21), we only need to show that

\[
\inf_{\alpha \in A_{t,\varepsilon}} L^*_t(\alpha) > 0.
\]

(3.22)

By the definition, we have \( L^*_t(\alpha) \geq \langle \lambda, \alpha \rangle - L_t(\lambda) \rangle_{\lambda=0} = 0 \) and \( L^*_t(\alpha_t) = \langle \alpha_t, \alpha_t \rangle - L_t(0) = 0 \), which means that the function \( L^*_t(\alpha) \) attains its minimum 0 at the point \( \alpha = \alpha_t \). Besides, for all \( \alpha \neq \alpha_t \), we can see that \( L^*_t(\alpha) > 0 \), then (3.22) immediately holds from the convexity of the function \( L^*_t \). Indeed, suppose that \( L^*_t(\alpha) = 0 \) for some \( \alpha \neq \alpha_t \). Then by the definition,

\[
L_t(0) = 0 = L^*_t(\alpha) \geq \langle \lambda, \alpha \rangle - L_t(\lambda), \quad \forall \lambda \in \mathbb{R}^d,
\]

or equivalently,

\[
L_t(\lambda) \geq L_t(0) + \langle \lambda, \alpha \rangle, \quad \forall \lambda \in \mathbb{R}^d,
\]

which implies that \( \alpha = \nabla L_t(0) = \alpha_t \), since \( L_t(\lambda) \) is convex and differentiable at 0. The proof is finished.

\[\square\]

**Proof of Theorem 3.3.** Letting \( n \) tends to infinity in (3.20) and using Propositions 3.5 and 3.4, we immediately obtain the desired result.

We will use Theorem 3.3 to calculate the lower bounds of the Hausdorff dimensions \( \dim E(\alpha) \), so as to further prove the achievement of Theorem 1.1.

**Proof of Theorem 1.1.** We first prove the assertion (a). By Propositions 3.1 and 3.2, with probability 1, we have \( \dim E(\alpha) \leq -\Lambda^*(\alpha) \leq -\Lambda^*(\alpha) \) for all \( \alpha \in \mathbb{R}^d \). On the other hand, Proposition 3.7 shows that with probability 1, for all \( \alpha \in \mathcal{J} \) (so that \( \alpha = \nabla \Lambda(t) \) for some \( t \in \Omega \), \( 0 < \mu_t(u \in \partial T : \lim_n \frac{S_{u,n}}{n} = \alpha) \) \((< \infty)\). Noticing Theorem 3.5 and using ([18], Theorem 4.2), we deduce \( \dim E(\alpha) \geq \Lambda(t) - (t, \nabla \Lambda(t)) = -\Lambda^*(\alpha) \).

Now we prove the assertion (b). For the upper bounds, by ([16], Theorem 4.5.3) and Proposition 3.1, we have with probability 1, for all measurable sets \( A \subset \mathbb{R}^d \),

\[
\limsup_{n \to \infty} \frac{1}{n} \log \inf_{\alpha \in A} \Lambda^*(\alpha) \leq - \inf_{\alpha \in A} \Lambda^*(\alpha).
\]

(3.25)

For the lower bounds, denote \( Z = \inf_n \frac{1}{n} \log \inf_{u \in E_n} \lambda u \), then \( Z \leq \mathbb{E} \log m_0(0) < \infty \) a.s.. For \( \alpha \in A^c \), take \( \varepsilon > 0 \) small enough such that \( B(\alpha, \varepsilon) \subset A \). For \( u \in E(\alpha) \), we have \( \| \frac{S_{u,n}}{n} - \alpha \| \leq \varepsilon \) for \( n \) large enough, which means that \( E(\alpha) \subset \bigcup_{u \in E_n} [u] \), where \( E_n = \{ u \in \mathbb{T}_n : S_u \in nB(\alpha, \varepsilon) \} \). Obviously, the diameter of the set \( [u] \) for \( u \in E_n \) is less than \( e^{-n} \), and so is less than \( \delta > 0 \) for \( n \) large enough. Therefore, the \( s \)-dimension Hausdorff measure of \( E(\alpha) \) satisfies

\[
\mathcal{H}^s(E(\alpha)) \leq \liminf_{n \to \infty} e^{-ns} Z_n(nA) = 0
\]

provided \( s > Z \), which implies that \( \dim E(\alpha) \leq s \) and furthermore \( \dim E(\alpha) \leq Z \) by the arbitrary of \( s \). Taking superior on \( \alpha \), we get with probability 1, \( \sup \dim E(\alpha) \leq Z \). Combing this result with the assertion (a) yields

\[
\inf_{\alpha \in A^c} \Lambda^*(\alpha) = \sup_{\alpha \in A^c} \dim E(\alpha) \leq Z \text{ if } A^c \subset \mathcal{J}.
\]

\[\square\]
4 Proof of Corollary 1.2

In order to give the proof of Corollary 1.2, we need a technique of truncation. Letting \( a > 0 \) be a rational number, we introduce the point process related to \( u \in T \) as \( X^a(u) = (N^a(u), L_1^a(u), L_2^a(u), \ldots) \), where \( N^a(u) = N(u) \land a \) with notation \( a_1 \land a_2 = \min(a_1, a_2) \), \( L_n^a(u) \) equals to \( L_n(u) \) if \( \|L_n(u)\| \leq a \) and is empty otherwise. Let us construct a new BRWRE where the point process formed by a particle \( u \) is \( X^a(u) \). Denote

\[
m_0^a(z) = \mathbb{E}_\xi \sum_{i=1}^{N_{T\land a}} e^{(z,L_i^a)} = \mathbb{E}_\xi \sum_{i=1}^{N_{T\land a}} e^{(z,L_i)} 1_{\|L_i\| \leq a} \quad (z \in \mathbb{C}^d),
\]

and the other notations can be extended similarly. If \( \mathbb{E} \left[ P_\xi(\|L_1\| \leq a_0)^{-1} \right] < \infty \) for some constant \( a_0 > 0 \), it is not difficult to verify that for all \( t \in \mathbb{R}^d \) and \( a \geq a_0 \),

\[
\mathbb{E} \log m_0^a(t) < \infty \quad \text{and} \quad \mathbb{E} \left\| \frac{\nabla m_0^a(t)}{m_0(t)} \right\| < \infty,
\]

which ensures that the function \( \Lambda_a(t) = \mathbb{E} \log m_0^a(t) \) is well defined as real number on \( \mathbb{R}^d \) and differential everywhere. Indeed, notice that \( \log^+ m_0^a(t) \leq \log^+ m_0(t) \) and for \( a \geq a_0 \),

\[
m_0^a(t) \geq \mathbb{E}_\xi e^{(t,L_i)} 1_{\|L_i\| \leq a} \geq \mathbb{E}_\xi e^{(t,L_i)} 1_{\|L_i\| \leq a_0} \geq e^{-\|\xi\|a_0 \mathbb{P}_\xi(\|L_1\| \leq a_0)},
\]

so that

\[
\log^+ m_0^a(t) \leq \|t\|a_0 - \log \mathbb{P}_\xi(\|L_1\| \leq a_0) = \|t\|a_0 + \log \mathbb{P}_\xi(\|L_1\| \leq a_0)^{-1}.
\]

The fact that \( \mathbb{E} \log^+ m_0(t) < \infty \) and (by Jensen’s inequality)

\[
\mathbb{E} \log \mathbb{P}_\xi(\|L_1\| \leq a_0)^{-1} \leq \log \mathbb{E} \mathbb{P}_\xi(\|L_1\| \leq a_0)^{-1} < \infty
\]

ensures \( \mathbb{E} \log m_0^a(t) < \infty \), and also implies that \( \Lambda_a(t) \uparrow \Lambda(t) \) as \( a \uparrow \infty \). Besides, by (4.2), we can deduce

\[
\left\| \frac{\nabla m_0^a(t)}{m_0(t)} \right\| \leq \mathbb{E}_\xi \sum_{i=1}^{N_{T\land a}} e^{(t,L_i)} 1_{\|L_i\| \leq a} \mathbb{P}_\xi(\|L_i\| \leq a_0) \leq a^2 e^{(a-a_0)\|t\|} \mathbb{P}_\xi(\|L_1\| \leq a_0)^{-1},
\]

so \( \mathbb{E} \left\| \frac{\nabla m_0^a(t)}{m_0(t)} \right\| < \infty \) since \( \mathbb{E}\mathbb{P}_\xi(\|L_1\| \leq a_0)^{-1} < \infty \).

According to the arguments in \((\text{H})\), proofs of Proposition 2.3 and Corollary 2.1), we can get the following lemma, which is a generalization of \((\text{H}), \text{Corollary 2.1}) \) for \( \mathbb{R}^d \)-valued BREWE.

**Lemma 4.1.** Assume that \( \mathbb{E} \log \mathbb{P}_\xi(\|L_1\| \leq a_0) > -\infty \) for some constant \( a_0 > 0 \). If \( \alpha \in \text{int} \{ \alpha : \Lambda^*(\alpha) < \infty \} \), then \( \Lambda_\alpha^*(\alpha) \downarrow \Lambda^*(\alpha) \) as \( a \uparrow \infty \).

**Lemma 4.2.** Assume that \( \mathbb{E} \log \mathbb{P}_\xi(\|L_1\| \leq a_0) > -\infty \) for some constant \( a_0 > 0 \). Then \( I_a = \Omega_a \) for \( a \geq a_0 \), where \( \Omega_a = I_a \cap \Omega^a_1 \cap \Omega^a_2 \).

**Proof.** We shall prove that \( \Omega^a_1 = \Omega^a_2 = \mathbb{R}^d \) for \( a \geq a_0 \), which implies \( I_a = \Omega_a \). On the one hand, for \( t \in \mathbb{R}^d \), one can see that

\[
\mathbb{E}_\xi \mathbb{Z}_1^a(t)^p = \mathbb{E}_\xi \left( \sum_{i=1}^{N_{T\land a}} e^{(t,L_i)} 1_{\|L_i\| \leq a} \right)^p \leq a^p \mathbb{E}_\xi \|t\|^p,
\]

which implies that \( \mathbb{E} \log^+ \mathbb{E}_\xi \mathbb{Z}_1^a(t)^p < \infty \), hence we have \( \Omega^a_1 = \mathbb{R}^d \) (since \( \mathbb{R}^d \) is open).

On the other hand, notice that the function \( h(\theta) = \sin \theta + \cos \theta \geq \frac{1}{2} \) on \( [-\delta_1, \delta_1] \) for some \( \delta_1 > 0 \). Take \( \delta > 0 \) small enough such that \( a \sqrt{\delta} \leq \delta_1 \). For \( t \in \mathbb{R}^d \) and \( z = x + iy \in D(t, \delta) \), we have \( \|z\| \leq \|x-t\| + \|t\| < \sqrt{\delta} \|t\| \) and \( \|y\| < \sqrt{\delta} \|t\| \). Therefore, for \( a \geq a_0 \),

\[
|m_0^a(z)|^2 = \left( \mathbb{E}_\xi \sum_{i=1}^{N_{T\land a}} e^{(z,L_i)} \cos(y,L_i) 1_{\|L_i\| \leq a} \right)^2 + \left( \mathbb{E}_\xi \sum_{i=1}^{N_{T\land a}} e^{(z,L_i)} \sin(y,L_i) 1_{\|L_i\| \leq a} \right)^2 \geq \frac{1}{2} \left( \mathbb{E}_\xi \sum_{i=1}^{N_{T\land a}} e^{(z,L_i)} h(y,L_i) 1_{\|L_i\| \leq a} \right)^2 \geq \frac{1}{8} e^{-2a_0 \sqrt{\beta \|t\|}} \mathbb{E}_\xi(\|L_1\| \leq a_0)^2,
\]
so that
\[ \mathbb{E} \log^+ \frac{\alpha_n(t, \delta)}{\alpha_n} \leq 2\sqrt{2} \mathbb{E} \log^+ \delta + \|t\| - \mathbb{E} \log \mathbb{P}_\xi(\|L_t\| \leq \alpha_0) < \infty. \]

Thus \( \Omega_n^2 = \mathbb{R}^d \). The proof is finished. \( \square \)

**Proof of Corollary 1.2.** For the assertion (a), we only need to show that with probability 1, for all \( \alpha \in \text{int}\tilde{J} \), the lower bound of the Hausdorff dimension of the set \( E(\alpha) \) satisfies \( \dim E(\alpha) \geq -\Lambda^*(\alpha) \). Firstly, it is obvious that Calton-Watson tree \( T^\infty \subset T \), thus
\[
E_\alpha^{\text{a}}(\alpha) = \{ u \in \partial T^\alpha : \lim_{n \to \infty} \frac{S_u(n)}{n} = \alpha \} \subset E(\alpha),
\]
which leads to \( \dim E(\alpha) \geq \dim E_\alpha^{\text{a}}(\alpha) \) for all \( \alpha \in \mathbb{R}^d \). Denote
\[
\tilde{E} = \bigcap_a \{ \omega \in \Theta^N \times \Gamma : \dim E_\alpha^{\text{a}}(\alpha) = \Lambda^*_\alpha(\alpha), \ \forall \alpha \in J_a \}.
\]

Then \( \mathbb{P}(\tilde{E}) = 1 \) by Theorem 1.1. Fix \( \omega \in \tilde{E} \). For \( \alpha \in \text{int}\tilde{J} \subset \text{int}\{ \alpha : \Lambda^*(\alpha) < \infty \} \), by Lemma 4.1 we have \( \Lambda^*_\alpha(\alpha) < 0 \) for a large enough. Take a large enough. Since \( \Lambda_\alpha(t) \) is differentiable, there exists \( t_a \in I_a \) such that \( \alpha = \nabla \Lambda_\alpha(t_a) \) (see [11], p227). Lemma 4.2 shows \( I_a = \Omega_a \), so that \( \alpha \in J_a \). Thus \( \dim E(\alpha) \geq \dim E_\alpha^{\text{a}}(\alpha) = -\Lambda^*_\alpha(\alpha) \) for a large enough. Letting \( a \uparrow \infty \) gives \( \dim E(\alpha) \geq -\Lambda^*(\alpha) \). Hence \( \tilde{E} \subset \{ \omega : \dim E(\alpha) \geq -\Lambda^*(\alpha), \ \forall \alpha \in \text{int}\tilde{J} \} \). The assertion (b) is from the proof of Theorem 1.1 and the assertion (a). \( \square \)

### 5 Proof of Theorem 1.3

We first establish a moderate deviation principle for the quenched means.

**Theorem 5.1 (Moderate deviation principle for \( \frac{\mathbb{E} \xi Z_n(a_n)}{\mathbb{E} \xi Z_n(\mathbb{R}^d)} \)).** Assume that \( \mathbb{E} \xi \sum_{u \in \tilde{T}_1} S_u = 0 \) a.s. and either of the following statements is established:

(i) \( \text{ess sup} \frac{1}{n} \mathbb{E}_{\xi u \in \tilde{T}_1} e^{\delta \| S_u \|} < \infty \) for some \( \delta > 0 \);

(ii) \( \lim_{n \to \infty} \frac{m_n}{n} = 0 \) for some \( \alpha \in (\frac{1}{2}, 1) \) and \( \mathbb{E} \frac{1}{n} \sum_{u \in \tilde{T}_1} e^{\delta \| S_u \|} < \infty \) for some \( \delta > 0 \).

Then the sequence of finite measures \( \frac{\mathbb{E} \xi Z_n(a_n)}{\mathbb{E} \xi Z_n(\mathbb{R}^d)} \) satisfies a moderate deviation principle: for measurable \( A \subset \mathbb{R}^d \),
\[
-\inf_{x \in A^n} \Gamma^*(x) \leq \liminf_{n \to \infty} \frac{n}{a_n^2} \log \frac{\mathbb{E} \xi Z_n(a_n A)}{\mathbb{E} \xi Z_n(\mathbb{R}^d)} \leq \limsup_{n \to \infty} \frac{n}{a_n^2} \log \frac{\mathbb{E} \xi Z_n(a_n A)}{\mathbb{E} \xi Z_n(\mathbb{R}^d)} \leq -\inf_{x \in A^n} \Gamma^*(x) \quad (5.1)
\]

for almost all \( \xi \), where the rate function \( \Gamma^*(x) \) is defined in Theorem 1.3.

**Proof.** We consider the probability measures \( q_n(\cdot) = \frac{\mathbb{E} \xi Z_n(a_n)}{\mathbb{E} \xi Z_n(\mathbb{R}^d)} \). For \( t \in \mathbb{R}^d \), put
\[
\lambda_n(t) = \log \int e^{\xi(t,x)} q_n(dx) = \log \left[ \frac{\mathbb{E} \xi \int e^{\xi(t,\xi)} Z_n(dy)}{\mathbb{E} \xi Z_n(\mathbb{R}^d)} \right] = \log \frac{P_n(a_n^2 t)}{P_n(0)} = \sum_{i=0}^{n-1} \log \left[ \frac{m_i(a_n^2 t)}{\pi_i} \right]. \quad (5.2)
\]

We need to prove that for each \( t \in \mathbb{R}^d \),
\[
\lim_{n \to \infty} \frac{n}{a_n^2} \lambda_n \left( \frac{a_n^2 t}{n} \right) = \Gamma(t) \quad \text{a.s.} \quad (5.3)
\]

Then we have
\[
\mathbb{P} \left( \lim_{n \to \infty} \frac{n}{a_n^2} \lambda_n \left( \frac{a_n^2 t}{n} \right) = \Gamma(t), \ \forall t \in \mathbb{R}^d \right) = 1,
\]

Since \( \lambda_n(t) \) and \( \Gamma(t) \) are convex and hence continuous. Applying the Gärtner-Ellis theorem (cf. [16], p52, Exercises 2.3.20), we obtain (5.3).
Put $\Delta_{n,i} = \frac{1}{\pi_i} m_i(\frac{a_n t}{n}) - 1$. We shall show that for each $t \in \mathbb{R}^d$,
\[
\sup_{0 \leq i \leq n-1} |\Delta_{n,i}| < 1 \quad \text{a.s.} \tag{5.4}
\]
for $n$ large enough. For simplicity, denote
\[
X_n(\cdot) = \sum_{i=1}^{N(u)} \delta_{L_i(u)}(\cdot) \quad (u \in \mathbb{T}_n)
\tag{5.5}
\]
the counting measure corresponding to the random vector $X(u)$. Let
\[
Q_n = \frac{1}{\pi_n} E \xi \int e^{\delta \|x\|} X_n(dx) = \frac{1}{\pi_n} E \xi \sum_{i=1}^{N(u)} e^{\delta \|L_i(u)\|} (u \in \mathbb{T}_n).
\]
If the statement (i) holds, we have $Q_n \leq C$ for some constant $C$; if the statement (ii) holds, we have $E Q_n = EQ_0 < \infty$. By the ergodic theorem, $\lim_n \frac{1}{n} \sum_{i=0}^{n-1} Q_i = EQ_0 < \infty$ a.s., hence for $n$ large enough,
\[
\sup_{0 \leq i \leq n-1} Q_i \leq \sum_{i=1}^{n-1} Q_i \leq Cn \quad \text{a.s.} \tag{5.6}
\]
Since for $n$ large enough (such that $\frac{a_n}{n} \|t\| < \delta$),
\[
\sum_{k=0}^{\infty} \frac{1}{k!} \pi_i \xi \int \left(\frac{a_n}{n} t, x\right)^k X_i(dx) = \frac{1}{\pi_i} E \xi \int e^{\delta \|t,x\|} X_i(dx) \leq \frac{1}{\pi_i} E \xi \int e^{\delta \|x\|} X_i(dx) = Q_i < \infty \quad \text{a.s.},
\]
noticing $\frac{E \xi}{\sum_{u \in \mathbb{T}_1} S_u} = 0$ a.s., we can write $\Delta_{n,i}$ as
\[
\Delta_{n,i} = \frac{1}{\pi_i} E \xi \int \sum_{k=0}^{\infty} \left(\frac{a_n}{n} t, x\right)^k X_i(dx) - 1 = \sum_{k=0}^{\infty} \frac{1}{k!} \pi_i \xi \int \left(\frac{a_n}{n} t, x\right)^k X_i(dx) - 1 = \sum_{k=2}^{\infty} \gamma_{ik}^n,
\]
with the notation
\[
\gamma_{ik}^n = \frac{1}{k! \pi_i} \xi \int \left(\frac{a_n}{n} t, x\right)^k X_i(dx)
\]
\[
= \frac{1}{k! \pi_i} \xi \int \left(\frac{a_n}{n} t, x\right)^k 1_{\{\|x\| \leq c_n\}} X_i(dx) + \frac{1}{k! \pi_i} \xi \int \left(\frac{a_n}{n} t, x\right)^k 1_{\{\|x\| > c_n\}} X_i(dx)
\]
\[
= \alpha_{ik}^n + \beta_{ik}^n,
\]
where $\{c_n\}$ is a sequence of positive constants whose values will be determined later. We can calculate that
\[
|\alpha_{ik}^n| \leq \frac{1}{k!} \left(\frac{a_n}{n} \|t\|\right)^k \frac{1}{\pi_i} E \xi \int \|x\|^k 1_{\{\|x\| \leq c_n\}} X_i(dx) \leq \frac{1}{k!} \left(\frac{a_n}{n} c_n \|t\|\right)^k \tag{5.7}
\]
and
\[
|\beta_{ik}^n| \leq \frac{1}{k!} \left(\frac{a_n}{n} \|t\|\right)^k \frac{1}{\pi_i} E \xi \int \|x\|^k 1_{\{\|x\| > c_n\}} X_i(dx) \leq \left(\frac{2 a_n}{\delta n} \|t\|\right)^k e^{-\frac{\delta}{\delta} c_n} Q_i. \tag{5.8}
\]
In the last line of the inequality above, we have used the fact that $\frac{1}{\pi_i} \left(\frac{a_n}{n} \|t\|\right)^k \leq e^{\frac{\delta}{\delta} \|x\|}$ for all $k$. If the statement (i) holds, taking $c_n = 1$, then we deduce from (5.7) and (5.8) that
\[
\sup_{i \geq 2} |\gamma_{ik}^n| \leq \sup_{i \geq 1} |\alpha_{ik}^n| + \sup_{i \geq 1} |\beta_{ik}^n| \leq \left(\frac{a_n}{n} \|t\|\right)^k \left(\frac{2 a_n}{\delta n} \|t\|\right)^k \leq C \left(\frac{\theta a_n}{n} \|t\|\right)^k,
\]
where $\theta = \max\{1, \frac{\delta}{\delta}\}$. If the statement (ii) holds, taking $c_n = \frac{\delta}{\delta} \log n$, then by (5.7), (5.8) and (5.6), we see that for $n$ large enough,
\[
\sup_{1 \leq i \leq n-1} |\gamma_{ik}^n| \leq \left(\frac{2 a_n}{\delta n} \log n \|t\|\right)^k + C \left(\frac{2 a_n}{\delta n} \|t\|\right)^k \leq C \left(\frac{2 a_n}{\delta n} \log n \|t\|\right)^k \quad \text{a.s.}
\]

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Thus, whether (i) or (ii) is satisfied, we always have for \( n \) large enough,
\[
\sup_{1 \leq i \leq n-1} |\gamma_{ik}^n| \leq C (\theta d_n \| t \|)^k \quad \text{a.s.,}
\]
(5.9)
where
\[
d_n = \begin{cases} \frac{a_n}{n}, & \text{if (i) holds,} \\ \frac{a_n}{n} \log n, & \text{if (ii) holds.} \end{cases}
\]
Therefore, for \( n \) large enough,
\[
\sup_{1 \leq i \leq n-1} |\Delta_{n,i}| \leq \sum_{k=2}^{\infty} \sup_{1 \leq i \leq n-1} |\gamma_{ik}^n| \leq C \sum_{k=2}^{\infty} (\theta d_n \| t \|)^k \leq M_1 d_n^2 \quad \text{a.s.,}
\]
(5.10)
where \( M_1 > 0 \) is a constant (related to \( t \)). It is clear that \( \lim_n d_n = 0 \), so that (5.4) holds for \( n \) sufficiently large.

Now we calculate (5.3) and (5.4), when \( n \) is large enough,
\[
\frac{n}{a_n^2} \lambda_n \left( \frac{\alpha_1}{n} \right) = \frac{n}{a_n^2} \sum_{i=0}^{n-1} \log(1 + \Delta_{n,i})
\]
\[
= \frac{n}{a_n^2} \sum_{i=0}^{n-1} (\sum_{j=1}^{n-1} (-1)^{j+1} j! (\Delta_{n,i})^j)
\]
\[
= \frac{n}{a_n^2} \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j!} \sum_{i=0}^{n-1} (\Delta_{n,i})^j
\]
\[
= \frac{n}{a_n^2} \sum_{i=0}^{n-1} \Delta_{n,i} + \frac{n}{a_n^2} \sum_{j=2}^{\infty} \frac{(-1)^{j+1}}{j!} \sum_{i=0}^{n-1} (\Delta_{n,i})^j
\]
\[
= : A_n + B_n.
\]
For \( B_n \), by (5.10), we get for \( n \) large enough,
\[
|B_n| \leq \frac{n}{a_n^2} \sum_{j=2}^{\infty} \sum_{i=0}^{n-1} |\Delta_{n,i}|^j \leq \frac{n^2}{a_n^2} \sum_{j=2}^{\infty} M_1^2 d_n^2 j! \leq \frac{n^2}{a_n^2} \sum_{j=2}^{\infty} M_1^2 d_n^2 j! \leq M_2 n^2 d_n^3 \quad \text{a.s.,}
\]
(5.11)
where \( M_2 > 0 \) is a constant. Since \( \lim_n d_n = 0 \) and
\[
\lim_{n \to \infty} \frac{n^2}{a_n^2} d_n^3 = \begin{cases} \lim_{n \to \infty} \frac{a_n}{n} = 0, & \text{if (i) holds,} \\ \lim_{n \to \infty} \frac{(\log n)^3}{n^2} = 0, & \text{if (ii) holds,} \end{cases}
\]
(5.12)
we deduce \( B_n \to 0 \) a.s. immediately from (5.11). For \( A_n \), noticing that the series \( \sum_{k=2}^{\infty} \sum_{i=0}^{n-1} |\gamma_{ik}^n| < \infty \) a.s. for \( n \) large enough, we can decompose
\[
A_n = \frac{n}{a_n^2} \sum_{i=0}^{n-1} \sum_{k=2}^{\infty} \gamma_{ik} = \frac{n}{a_n^2} \sum_{k=2}^{\infty} \sum_{i=0}^{n-1} \gamma_{ik} = \frac{n}{a_n^2} \sum_{i=0}^{n-1} \gamma_{i2} + \frac{n}{a_n^2} \sum_{k=3}^{\infty} \sum_{i=0}^{n-1} \gamma_{ik} =: C_n + D_n.
\]
For \( D_n \), by (5.10), for \( n \) large enough,
\[
|D_n| \leq \frac{n}{a_n^2} \sum_{k=3}^{\infty} \sum_{i=0}^{n-1} |\gamma_{ik}^n| \leq C \frac{n^2}{a_n^2} \sum_{k=3}^{\infty} (\theta d_n \| t \|)^k \leq M_3 n^2 a_n^2 d_n^3 \quad \text{a.s.,}
\]
where \( M_3 > 0 \) is a constant, so that \( D_n \to 0 \) a.s. by (5.12). Finally, it remains to calculate the limit of \( C_n \). By the ergodic theorem,
\[
\lim_{n \to \infty} C_n = \lim_{n \to \infty} \frac{n}{a_n^2} \sum_{i=0}^{n-1} \frac{1}{2} n_1 \frac{1}{2} \int \left( \frac{a_n}{n} t, x \right)^2 X_i (dx) = \Gamma (t) \quad \text{a.s.,}
\]
which completes the proof.
Applying (5.3) and the uniform convergence of \( W_n(t) \) near 0, we carry on the proof of Theorem 1.3.

**Proof of Theorem 1.3.** Let

\[
\Gamma_n(t) = \log \left[ \int e^{\langle t, \cdot \rangle} Z_n(dx) \right] = \log \left[ \tilde{Z}_n(a_{n}^{-1}t) \right].
\]

Notice that

\[
\frac{n}{a_n^2} \Gamma_n\left(\frac{a_n^2}{n} t \right) = n \frac{a_n^2}{n} \log W_n\left(\frac{a_n^2}{n} t \right) + n \frac{a_n^2}{n} \lambda_n\left(\frac{a_n^2}{n} t \right) - n \frac{a_n^2}{n} \log W_n(0).
\]

(5.13)

It is evident that 0 \( \in \) \( I \), since \( -\Lambda(0) = -E \log m_0(0) < 0 \). So we have 0 \( \in \) \( \Omega \). By Theorem 2.1, \( W_n(z) \) converges uniformly a.s. in a neighbourhood of 0 \( \in \mathbb{R}^d \), so that the limit \( W(z) \) is continuous at 0. Therefore,

\[
\lim_{n \to \infty} W_n\left(\frac{a_n}{n} t \right) = W(0) > 0 \quad \text{a.s.}
\]

Letting \( n \to \infty \) in (5.13) and noticing (5.3), we obtain for each \( t \in \mathbb{R}^d \),

\[
\lim_{n \to \infty} \frac{n}{a_n^2} \Gamma_n\left(\frac{a_n^2}{n} t \right) = \Gamma(t) \quad \text{a.s.}
\]

(5.14)

So (5.14) a.s. holds for all rational \( t \), and hence for all \( t \in \mathbb{R}^d \) by the convexity of \( \Gamma_n(t) \) and the continuity of \( \Gamma(t) \). Then apply the Gärtner-Ellis theorem.

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