On Local Convergence of Iterative Hard Thresholding for Matrix Completion

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Abstract—Iterative hard thresholding (IHT) has gained in popularity over the past decades in large-scale optimization. However, convergence properties of this method have only been explored recently in non-convex settings. In matrix completion, existing works often focus on the guarantee of global convergence of IHT via standard assumptions such as incoherence property and uniform sampling. While such analysis provides a global upper bound on the linear convergence rate, it does not describe the actual performance of IHT in practice. In this paper, we provide a novel insight into the local convergence of a specific variant of IHT for matrix completion. We uncover the exact linear rate of IHT in a closed-form expression and identify the region of convergence in which the algorithm is guaranteed to converge. Furthermore, we utilize random matrix theory to study the linear rate of convergence of IHTSVD for large-scale matrix completion. We find that asymptotically, the rate can be expressed in closed form in terms of the relative rank and the sampling rate. Finally, we present various numerical results to verify the aforementioned theoretical analysis.

Index Terms—Matrix completion, iterative hard thresholding, local convergence analysis, random matrix theory.

I. INTRODUCTION

MAtrix completion is a fundamental problem that arises in many areas of signal processing and machine learning such as collaborative filtering [1]–[4], system identification [5]–[7] and dimension reduction [8], [9]. The problem can be explained as follows. Let $M \in \mathbb{R}^{n_1 \times n_2}$ be the underlying matrix with rank $r$ and $\Omega$ be the set of locations corresponding to the observed entries of $M$, i.e., $(i, j) \in \Omega$ if $M_{ij}$ is observed. The goal is to recover the unknown entries of $M$, belonging to the complement set $\Omega$.

To understand the feasibility of matrix completion, let us describe $M$ as

$$M = \sum_{i=1}^{r} \sigma_i u_i v_i^T,$$

where $\sigma_i$ is the $i$-th largest singular value of $M$, $u_i$ and $v_i$ are the corresponding left and right singular vectors. Since each set of the left and right singular vectors are orthonormal, the degrees of freedom of matrix completion is given by

$$r + \sum_{i=1}^{r} (n_1 - i) + \sum_{j=1}^{r} (n_2 - j) = (n_1 + n_2 - r)r,$$

which is significantly less than the total number of entries in $M$ when $r$ is small. This implies the possibility of recovering the entire matrix even when only a few entries are observed. However, not every matrix with more than $(n_1 + n_2 - r)r$ observed entries can be completed. For instance, if an entire column (or row) of a rank-one matrix is missing, then the matrix cannot be recovered. Similarly, if a low-rank matrix contains too many zero entries, then the observed entries might end up being all zero, thereby not providing any clue about the missing entries. The aforementioned argument motivates the two standard assumptions in matrix completion: the incoherence condition and the random sampling model. Under these assumptions, Candès and Recht [10] showed that matrix completion can be solved exactly for most settings of the low-rank matrix $M$ and the sampling set $\Omega$. This breakthrough has started a long line of research on efficient methods for solving matrix completion.

In the same work, Candès and Recht [10] proposed a convex relaxation approach to matrix completion, replacing the original linearly constrained rank minimization problem by a linearly constrained nuclear norm minimization problem. Their result leads to a well-known class of proximal-type algorithms for nuclear norm minimization [11]–[14] with rigorous mathematical guarantees and extensions of classic acceleration techniques. Nonetheless, convex-relaxed methods are generally considered slow compared to their non-convex counterparts in practice. On the one hand, interior-point methods for solving the nuclear norm minimization problem are computationally expensive and even infeasible for large matrices. On the other hand, proximal-type algorithms suffer from slow convergence due to the conservative nature of the soft-thresholding operator [15], [16].

Another approach to matrix completion is known as iterative hard thresholding. To address the computational concern from the use of convex relaxation, IHT methods have been proposed to directly solve the non-convex rank minimization problem [17], [18]. Each IHT iteration takes one step in the opposite direction of the gradient and another step projecting the result onto the set of rank-$r$ matrices. Since the process resembles hard-thresholding singular values, we refer to the class of algorithms using this technique as iterative hard thresholding. When the solution is low-rank, hard-thresholding algorithms
is more efficient than their soft-thresholding counterparts in both computational complexity per iteration and convergence speed. Variants of plain IHT with faster convergence have also been developed, including normalized IHT \cite{19}, conjugate gradient IHT \cite{20}, Nesterov’s accelerated gradient IHT \cite{16}, Heavy-Ball IHT \cite{21}, just to name a few. The drawback of IHT methods, however, is the lack of mathematical guarantees on their convergence behavior. As pointed out in \cite{17}, the restricted isometry property (RIP), which is widely used in establishing the global convergence in matrix sensing, does not hold for matrix completion. Therefore, the global convergence of IHT methods for matrix completion is still an open question. Until recently, the only guarantee on the global convergence of a IHT method, to the best of our knowledge, is provided in \cite{22}. In their work, the authors considered a variant of the singular value projection (SVP) algorithm with a resampling scheme and proved the fast linear convergence of the proposed algorithm with a sample complexity that depends on the condition number and desired accuracy. Nonetheless, this result imposes some limitations at conceptual, practical, and theoretical levels due to the requirement of resampling \cite{23}. In a different perspective, local convergence of IHT methods has also been studied by Chunikhina et al. \cite{24}. In particular, by considering a special case of the SVP algorithm with unit step size, called iterative hard-thresholded singular value decomposition (IHTSVD), the authors showed that IHTSVD converges linearly to the solution $M$ as long as the algorithm is initialized close enough to $M$. Consequently, this analysis explains the superior performance of IHT methods over proximal-type methods in practice. A similar approach can be found in the unpublished work of Lai and Varghese \cite{25}. However, we remark that while the latter work proves the existence of an upper bound on the linear convergence rate of IHTSVD, the former provides an exact expression of the rate that depends directly on the structure of $M$ and $\Omega$.

The most popular approach to matrix completion is non-convex factorization. This approach stems from the Burer-Monteiro factorization \cite{20}, whereby the low-rank matrix is viewed as a product of two low-rank components. The resulting least-squares problem is unconstrained albeit non-convex. Recent progress in this approach has shown that any local minimum of the re-parameterized problem is also a local minimum \cite{23}, \cite{27}. Thus, basic optimization procedures such as gradient descent \cite{23}, \cite{28}, \cite{29} and alternating minimization \cite{30}–\cite{33} can provably find the global solution at a linear convergence rate. The exact linear convergence rate of gradient descent for matrix completion has recently been studied by Vu and Raich \cite{34}. In Table \ref{table1} we summarize the aforementioned approaches to matrix completion and the corresponding algorithms existing in the literature.

This paper is developed based on the work of Chunikhina et al. \cite{24} on the local convergence of the IHTSVD algorithm for matrix completion. Our main contribution is three-fold. First, we propose a novel analysis of the local convergence of IHTSVD for matrix completion. The proposed analysis establishes the region of convergence that is proportional to the least non-zero singular value of $M$. Moreover, we show that the convergence is asymptotically linear and the exact rate can be described in a closed-form expression of the projections onto the (left and right) null spaces of $M$ and the sampling pattern $\Omega$. Second, based on the exact linear rate, we utilize random matrix theory to study the asymptotic behavior of IHTSVD in large-scale matrix completion. As the size of $M$ grows to infinity, we uncover the linear rate of IHTSVD converges to a deterministic constant that can be expressed in closed form in terms of the relative rank and the sampling rate. Finally, we present extensive results to verify our proposed exact rate of convergence as well as the asymptotic rate of IHTSVD in large-scale settings.

### II. Preliminaries

#### A. Notations

Throughout the paper, we use the notations $\|\cdot\|_F$, $\|\cdot\|_2$ and $\|\cdot\|_{2,\infty}$ to denote the Frobenius norm, the spectral norm and the $l_2/l_\infty$ norm (i.e., the largest $l_2$ norm of the rows) of a matrix, respectively. Occasionally, $\|\cdot\|_1$ is used on a vector to denote the Euclidean norm. The notation $[n]$ refers to the set $\{1, 2, \ldots, n\}$. Boldfaced symbols are reserved for vectors and matrices. In addition, let $I_n$ denote the $n \times n$ identity matrix. $\otimes$ denotes the Kronecker product between two matrices.

For a matrix $X \in \mathbb{R}^{n_1 \times n_2}$, $X_{ij}$ refers to the $(i, j)$ element of $X$. We denote $\sigma_{\text{max}}(X)$ and $\sigma_{\text{min}}(X)$ as the largest and smallest singular values of $X$, respectively, and denote $\kappa(X) = \sigma_{\text{max}}(X)/\sigma_{\text{min}}(X)$ as the condition number of $X$. $\text{vec}(X)$ denotes the vectorization of $X$ by stacking its columns on top of one another. Let $F(X)$ be a matrix-valued function of $X$. Then, for some $k > 0$, we use $F(X) = O(\|X\|_F^k)$ to imply

$$\lim_{\delta \to 0} \sup_{\|X\|_F = \delta} \frac{\|F(X)\|_F}{\|X\|_F^k} < \infty.$$  

#### B. Background

Let us use $M$ to denote the underlying $n_1 \times n_2$ real matrix with rank

$$1 \leq r \leq m = \min\{n_1, n_2\}.$$  

The sampling set $\Omega$ is a subset of the Cartesian product $[n_1] \times [n_2]$, with cardinality of $1 \leq s < n_1n_2$. Furthermore, the orthogonal projection associated with $\Omega$ is given in the following:

**Definition 1.** The orthogonal projection onto the set of matrices supported in $\Omega$ is defined as a linear operator $P_{\Omega} : \mathbb{R}^{n_1 \times n_2} \to \mathbb{R}^{n_1 \times n_2}$ satisfying

$$[P_{\Omega}(X)]_{ij} = \begin{cases} X_{ij} & \text{if } (i, j) \in \Omega, \\ 0 & \text{if } (i, j) \in \bar{\Omega}, \end{cases}$$

where $\bar{\Omega}$ denotes the complement set of $\Omega$.

If we consider vector spaces instead of matrix spaces, the orthogonal projection $P_{\Omega}$ can also be viewed as a selection matrix corresponding to $\Omega$.\footnote{Convergence guarantees on proximal-type methods for matrix completion are often sub-linear \cite{11}, \cite{14}.}
Definition 2. The selection matrix $S_{\Omega} \in \mathbb{R}^{n_1 n_2 \times s}$ comprises a subset of $s$ columns of the identity matrix of dimension $n_1 n_2$ such that

$$
\begin{align*}
\begin{cases}
S_{\Omega}^T S_{\Omega} &= I_s, \\
\text{vec}(P_\Omega(X)) &= S_{\Omega} S_{\Omega}^T \text{vec}(X).
\end{cases}
\end{align*}
$$

Corresponding to the complement set $\bar{\Omega}$, we also define similar notations for $P_{\bar{\Omega}} : \mathbb{R}^{n_1 n_2} \rightarrow \mathbb{R}^{n_1 n_2}$ and $S_{\bar{\Omega}} \in \mathbb{R}^{n_1 n_2 \times (n_1 n_2 - s)}$.

Next, using the notation of $P_{\Omega}$, we can formulate the matrix completion problem as follows:

$$
\min_{X \in \mathbb{R}^{n_1 n_2}} \frac{1}{2} \| P_{\Omega}(X - M) \|_F^2 \text{ s.t. } \text{rank}(X) \leq r. \quad (2)
$$

One natural approach to the optimization problem (2) is projected gradient descent. Starting at some $X^{(0)}$, we iteratively update the current matrix by (i) taking a step in the opposite direction of the gradient and (ii) projecting the result back onto the set of matrices with rank less than or equal to $r$. It follows that

$$
X^{(k+1)} = P_r(X^{(k)} - \eta P_{\Omega}(X^{(k)} - M)), \quad (3)
$$

where $\eta$ is the step size and $P_r$ is the rank-$r$ projection (formally defined later in Definition 3). In the literature, PGD with step size $\eta = n_1 n_2 / s$ is also known as the Singular Value Projection (SVP) algorithm for matrix completion [17]. It is interesting to note that under certain assumptions, showed that the algorithm enjoys a fast global linear convergence with this choice of step size. On the other hand, setting the step size $\eta = 1$ yields the following update

$$
X^{(k+1)} = P_r(\hat{X}^{(k)} - \eta \hat{P}_{\Omega}(\hat{X}^{(k)} - M)) = P_r(\hat{P}_{\Omega}(X^{(k)}) + \hat{P}_r(M)).
$$

This motivates the IHTSVD algorithm [24] that alternates between two projection steps: the projection onto the manifold of rank-$r$ matrices and the projection onto the set of matrices supported in $\Omega$ (see Algorithm 1). This paper, developed based on [24], focuses on local convergence properties of IHTSVD. Compared to the existing global convergence analysis for matrix completion, our setting does not require certain assumptions such as the incoherence of $M$, the uniform randomness of $\Omega$, and the low sample complexity, e.g., $s = O(r^5 n \log n)$ in [22]. We also note that the proposed analysis can be extended to other variants of PGD with different step sizes.

Finally, we present a formal definition of the rank-$r$ projection. Consider matrix $X \in \mathbb{R}^{n_1 n_2}$ with the singular value decomposition

$$
X = \sum_{i=1}^{n_1} \sigma_i(X) u_i(X) v_i^T(X),
$$

where $\sigma_1(X) \geq \ldots \geq \sigma_m(X) \geq 0$ are the singular values of $X$ and $\{u_1(X), \ldots, u_m(X)\}$, $\{v_1(X), \ldots, v_m(X)\}$ are the sets of left and right singular vectors of $X$, respectively.

Definition 3. The rank-$r$ projection of $X$ is defined as

$$
P_r(X) = \sum_{i=1}^{r} \sigma_i(X) u_i(X) v_i^T(X).
$$

The rank-$r$ projection of $X$ is unique if and only if $\sigma_r(X) > \sigma_r(X)$ or $\sigma_r(X) = 0$ [38]. Since $P_r(X)$ zeroes out all the small singular value of $X$, it is often referred as the singular value hard-thresholding operator. Since $M$ is a rank-$r$ matrix, we have

$$
M = P_r(M) = \sum_{i=1}^{r} \sigma_i u_i v_i^T = U_r \Sigma_r V_r^T,
$$

where $\Sigma_r = \text{diag}(\sigma_1, \ldots, \sigma_r)$ contains the singular values of $M$ and $U_r = [u_1, \ldots, u_r] \in \mathbb{R}^{n_1 \times r}$, $V_r = [v_1, \ldots, v_r] \in \mathbb{R}^{n_2 \times r}$ are comprised of the first $r$ left and right singular vectors of $M$, respectively. Denote $U_{l} = [u_{r+1}, \ldots, u_{n_1}] \in \mathbb{R}^{n_1 \times (n_1 - r)}$ and $V_{l} = [v_{r+1}, \ldots, v_{n_2}] \in \mathbb{R}^{n_2 \times (n_2 - r)}$. The projections onto the left and right null spaces of $M$ are uniquely defined as $P_{U_{l}} = U_{l} U_{l}^T = I_{n_1} - \sum_{i=1}^{r} u_i u_i^T$ and $P_{V_{l}} = V_{l} V_{l}^T = I_{n_2} - \sum_{i=1}^{r} v_i v_i^T$, respectively.

C. Related Work

Traditional approaches to matrix completion often make assumptions on the incoherence of the underlying matrix $M$ and the randomness of the sampling set. First, the incoherence condition for matrix completion, introduced by Candès and Recht [10], is stated as:

Assumption 1 (Incoherence). The matrix $M = U_r \Sigma_r V_r^T$ is $\mu$-incoherent, i.e.,

$$
\|U_r\|_{2, \infty} \leq \sqrt{\frac{\mu}{n_1}} \quad \text{and} \quad \|V_r\|_{2, \infty} \leq \sqrt{\frac{\mu}{n_2}}.
$$

In the rest of this paper, we omit the parameter in the notation of the singular values and the singular vectors of $M$ for simplicity.
Intuitively, an incoherent matrix has well-spread singular vectors and is less likely in the null space of the sampling operator. A common setting that generates incoherent matrices is the random orthogonal model:

**Definition 4** (Random orthogonal model). The Haar measure provides a uniform and translation-invariant distribution over the group of orthogonal matrices $\mathcal{O}(n)$. $M$ is said to follow a random orthogonal model if $U_r$ and $V_r$ are sub-matrices of Haar-distributed matrices in $\mathcal{O}(n_1)$ and $\mathcal{O}(n_2)$, respectively.

Second, to avoid adversarial patterns in the sampling set, it is common to assume that each entry in $\Omega$ is selected randomly:

**Assumption 2** (Uniform sampling). The sampling set $\Omega$ is obtained by selecting $s$ elements uniformly at random from the Cartesian product $[n_1] \times [n_2]$.

We note that a similar but not equivalent assumption on the sampling set is the Bernoulli model in which each entry of $M$ is observed independently with probability $s/n_1n_2$.[23] Under these two standard assumptions, Candés and Recht [10] showed that symmetric matrix completion of size $n$ can be solved exactly provided that the number of observations is sufficiently large, i.e., $s = \mathcal{O}(n^{1/2} \log n)$. Later on, global convergence guarantees for various matrix-completion algorithms have been actively developed, with improved bounds on the sample complexity. Examples of these works include [22, 23, 28, 33, 39]. It is worthwhile mentioning that ideally, one would like to recover the low-rank matrix from a minimum number of observations, which is in the order of the degrees of freedom of the problem, i.e., $\mathcal{O}(nr)$.

In this paper, we study the convergence of IHT for matrix completion from a different perspective. Without any assumptions about the incoherence of $M$ and the randomness of the sampling set $\Omega$, we identify a deterministic condition on the structure of $M$ and $\Omega$ such that the local linear convergence of IHTSVD can be guaranteed. Compared to the aforementioned bounds on the global convergence rate, our result is exact and tighter thanks to the exploitation of the local structure of the problem. Our technique utilizes the recently developed error bound for the first-order Taylor expansion of the rank-$r$ projection, proposed by Vu et al. in [40]. The result is rephrased below.

**Proposition 1** (Rephrased from [40]). For any $\Delta \in \mathbb{R}^{n_1 \times n_2}$, we have

$$\mathcal{P}_r(M + \Delta) = M + \Delta - \mathcal{P}_{U_r} \Delta \mathcal{P}_{V_r} + R(\Delta),$$

where the residual $R: \mathbb{R}^{n_1 \times n_2} \to \mathbb{R}^{n_1 \times n_2}$ satisfies:

$$\|R(\Delta)\|_F \leq \frac{c_1}{\sigma_r} \|\Delta\|_F^2,$$

for some universal constant $1 + 1/\sqrt{2} \leq c_1 \leq 4(1 + \sqrt{2})$.

The rest of the paper is organized as follows. In Section III we provide the local convergence analysis of IHTSVD for matrix completion and the proof of the main result. Next, Section IV presents a summary of related results in random matrix theory, followed by our novel result on the asymptotic behavior of the convergence rate in large-scale settings. The numerical results to verify the analysis in Sections III and IV are given in Section V. Finally, we put the detailed proofs of all the main theorems and lemmas in the appendix.

### Algorithm 1: IHTSVD

**Input:** $\mathcal{P}_\Omega(M), r, K$

**Output:** $X^{(K)}$

1. Initialize $X^{(0)}$
2. for $k = 0, 1, \ldots, K - 1$
   3. $Y^{(k)} = \mathcal{P}_r(X^{(k)})$
   4. $X^{(k+1)} = \mathcal{P}_\Omega(Y^{(k)}) + \mathcal{P}_\Omega(M)$

### III. LOCAL CONVERGENCE OF IHTSVD

This section presents our analysis of local convergence of IHTSVD. First, we leverage the results in perturbation analysis to identify the Taylor series expansion of the rank-$r$ projection. Next, the approximation allows us to derive the nonlinear difference equation that describes the change in the distance to the local optimum through IHT iterations. Closed-form expressions of the asymptotic convergence rate and the region of convergence are also given as a result of our analysis.

#### A. Main Result

Our local convergence result is stated as follows:

**Theorem 1.** Let $\{X^{(k)}\}$ be the sequence of matrices generated by Algorithm 2, i.e., for $k = 0, 1, \ldots$:

$$X^{(k+1)} = \mathcal{P}_\Omega(\mathcal{P}_r(X^{(k)})) + \mathcal{P}_\Omega(M)$$

such that $X^{(0)}$ satisfies:

$$\|X^{(0)} - M\|_F < \frac{\lambda_{min}(H)}{c_1} \sigma_r,$$

where $H$ is an $(n_1n_2 - s)$ square matrix given by

$$H = S_{\Omega}^{-1}(\mathcal{P}_{U_r} \otimes \mathcal{P}_{V_r})S_{\Omega}^{-1}.$$

Then, $\|X^{(k)} - M\|_F$ converge asymptotically at a linear rate

$$\rho = 1 - \lambda_{min}(H).$$

Specifically, for any $\epsilon > 0$, $\|X^{(k)} - M\|_F \leq \epsilon \|X^{(0)} - M\|_F$ for all integer $k$ such that

$$k \geq K(\epsilon) = \frac{\log(1/\epsilon)}{\log(1/(1 - \lambda_{min}(H)))} + c,$$

where $c > 0$ is a constant depending only on $X^{(0)}$ and $M$.

Theorem provides a closed-form expression of the linear convergence rate of IHTSVD for matrix completion. As can be seen in (9), the speed of convergence depends strongly on how close the smallest eigenvalue of $H$ is to zero; as $\lambda_{min}(H)$ approaches 0, the number of iterations needed to reach a relative accuracy of $\epsilon$, i.e., $K(\epsilon)$, grows to infinity. When $\lambda_{min}(H) = 0$, the condition in (9) cannot be satisfied and hence, there is no linear convergence guarantee provided.

[3] The explicit expression of $c$ is given in Theorem 1 in [41].
by our theorem in this case. On the other hand, from \( \mathbb{1} \), one can verify that all eigenvalues of \( H \) lie between 0 and 1 since the norm of either a projection matrix or a selection matrix is less than or equal to 1. This combined with the aforementioned condition that \( \lambda_{\min}(H) > 0 \) ensures the linear convergence rate \( \rho \in [0,1) \).

**Remark 1.** Theorem \( \mathbb{1} \) does not guarantee linear convergence when \( \lambda_{\min}(H) = 0 \). Interestingly, one such situation is when \( H \) is rank-deficient. Let us represent

\[
H = S_{\Omega}^T(V_\perp \otimes U_\perp)(V_\perp \otimes U_\perp)^T S_{\Omega} = WW^T,
\]

where \( W = S_{\Omega}^T(V_\perp \otimes U_\perp) \in \mathbb{R}^{(n_1n_2-s) \times (n_1-r)(n_2-r)} \). If \( W \) is a tall matrix, i.e.,

\[
s < (n_1 + n_2 - r)r, \quad (10)
\]

then it follows that \( H \) is rank-deficient and \( \lambda_{\min}(H) = 0 \). We note that in this case the number of sampled entries is less than the degrees of freedom of the problem.

**Remark 2.** When \( s \geq (n_1 + n_2 - 2)r \), it is possible that \( \lambda_{\min}(H) = 0 \) for certain (adversarial) sampling patterns. For example, consider a \( 3 \times 2 \) rank-1 matrix

\[
M = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix}^T.
\]

One choice of the matrices \( U_\perp \) and \( V_\perp \) is

\[
U_\perp = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad V_\perp = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.
\]

If we observe \( s = 4 \) entries of the first two rows of \( M \), namely, \((1, 1), (1, 2), (2, 1), \) and \((2, 3)\), the selection matrix corresponding to the unobserved entries \((3, 1)\) and \((3, 2)\) is given by

\[
S_{\Omega}^T = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.
\]

Then, we have

\[
H = S_{\Omega}^T(V_\perp \otimes U_\perp)(V_\perp \otimes U_\perp)^T S_{\Omega} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}
\]

and \( \lambda_{\min}(H) = 0 \). While Theorem \( \mathbb{1} \) does not guarantee linear convergence of IHTSVD, one may realize that it is impossible to recover the last row of \( M \) in this case.

**Lemma 1.** Let us define the error matrix and its economy vectorized version, respectively, as

\[
E^{(k)} = X^{(k)} - M \quad \text{and} \quad e^{(k)} = S_{\Omega}^T \text{vec}(E^{(k)}).
\]

Then, we have

\[
E^{(k+1)} = \mathcal{P}_{\Omega}(E^{(k)} - P_{U_\perp}E^{(k)}P_{V_\perp} + R(E^{(k)})) \quad (11)
\]

and

\[
e^{(k+1)} = (I - S_{\Omega}^T(P_{V_\perp} \otimes P_{U_\perp})S_{\Omega})e^{(k)} + r(e^{(k)}), \quad (12)
\]

where \( R(\cdot) \) is the residual defined in Proposition \( \mathbb{1} \) and

\[
r(e) = S_{\Omega}^T \text{vec}(R(\text{vec}^{-1}(S_{\Omega}e))).
\]

Here we recall that \( \text{vec}^{-1}(\cdot) \) is the inverse vectorization operator such that \( \text{vec}^{-1} \circ \text{vec} \) is identity.

Note that \( E^{(k)} \) belongs to the set of matrices supported in \( \Omega \) and hence, \( \|E^{(k)}\|_F = \|e^{(k)}\|_2 \). Next, using the definition of the operator norm, one can obtain the following bound on the norm of the error matrix:

**Lemma 2.** The Frobenius norm of the error matrix satisfies

\[
\|E^{(k+1)}\|_F \leq (1 - \lambda_{\min}(H))\|E^{(k)}\|_F + \frac{c_1}{\sigma_r}\|E^{(k)}\|_F^2. \quad (13)
\]

The nonlinear difference equation \((13)\) has been well-studied in the stability theory of difference equations \([41]–[43]\). In fact, our theorem follows directly on applying Theorem 1 in \([41]\) to \([43]\), with \( a_0 = \|E^{(0)}\|_F, \rho = 1 - \lambda_{\min}(H), \) and \( q = c_1/\sigma_r \). The proofs of Lemmas \( \mathbb{1} \) and \( \mathbb{2} \) are given in Appendix \( \mathbb{A} \).

**IV. CONVERGENCE OF IHTSVD FOR LARGE-SCALE MATRIX COMPLETION**

In this section, we study the convergence of IHTSVD for large-scale matrix completion, a setting of practical interest in the rise of big data. Using recent results in random matrix theory, we show that, as its dimensions grow to infinity, the spectral distribution of \( H \) converges almost surely to a deterministic distribution with a bounded support. Consequently, we propose a large-scale asymptotic estimate of the linear convergence rate of IHTSVD that is a closed-form expression of the relative rank and the sampling rate.

**A. Overview**

We are interested in the asymptotic setting in which the size of \( M \) grows to infinity, i.e., \( m = \min\{n_1, n_2\} \to \infty \). Let us assume that the ratio \( n_1/n_2 \) remains to be a non-zero constant as \( m \to \infty \). In addition, we introduce two concepts that are the normalization of the degrees of freedom and the number of measurements:

**Definition 5** (Relative rank). The rank \( r \) increases as \( m \to \infty \) such that the relative rank remains to be a constant

\[
\rho_r = 1 - \sqrt{\frac{p}{n_1}}(1 - \frac{p}{n_2}) \in (0,1]. \quad (14)
\]
Definition 6 (Sampling rate). The number of observations increases as \( m \to \infty \) such that the sampling rate remains to be a constant

\[
\rho_s = \frac{s}{n_1n_2} \in (0, 1].
\]  

(15)

When \( \rho_s < 1 - (1 - \rho_r)^2 \), we recover the case in Remark 1 where the number of measurements is less than the degrees of freedom. As far as the local linear rate of IHTSVD is concerned, we only consider the case \( \rho_s \geq 1 - (1 - \rho_r)^2 \).

Remark 3. When \( r = m \), we have \( \rho_r = 1 \). Moreover, when \( n_1 = n_2 = m \), the relative rank is exactly the ratio \( r/m \). As can be seen below, the proposed definition of the relative rank incorporates both dimensions of \( M \) to enable the compact representation of \( \rho \) in terms of \( \rho_r \) and \( \rho_s \).

We are in position to state our result on the asymptotic behavior of the linear rate \( \rho \) in large-scale matrix completion:

Theorem 2 (Informal). For \( \rho_s > 1 - (1 - \rho_r)^2 \), the linear convergence rate \( \rho \) of IHTSVD approaches

\[
\rho_\infty = 1 - \left( \sqrt{(1 - \rho_r)^2\rho_s} - \sqrt{\rho_r(2 - \rho_r)(1 - \rho_s)} \right)^2,
\]

(16)

as \( m \to \infty \).

Note that \( \rho_\infty \) is independent of the structure of the solution matrix \( M \) and the sampling set \( \Omega \). Moreover, it depends only on the relative rank and the sampling rate. Figure 1 depicts the contour plot of \( \rho_\infty \) as a function of \( \rho_r \) and \( \rho_s \). It can be seen that for a fixed value of \( \rho_r \), the asymptotic rate decreases towards 0 as the number of observed entries increases. This matches with the intuition that more information leads to faster convergence. Conversely, for a fixed value of \( \rho_s \), the algorithm converges slower as the rank of the matrix increases, due to the increasing uncertainty (i.e., more degrees of freedom) in the set \( \Omega \). On the boundary where \( \rho_s = 1 - (1 - \rho_r)^2 \), there is no linear convergence predicted by our theory since \( \rho_\infty = 1 \). In this case, we recall that the number of observed entries equals the degrees of freedom of the problem.

Our technique relies on recent results in random matrix theory to exploit the special structure of \( H \). First, when \( n_1/n_2 \) remains constant, it holds that \( n = n_1n_2 \to \infty \) as \( m \to \infty \). Then, \( H \) can be viewed as an element of a sequence of matrices of form

\[
H_n = W_{pq}^n(W_{pq}^n)^T,
\]

(17)

where \( W_{pq}^n \in \mathbb{R}^{m_1n_2 \times qn_1n_2} \) is a truncation of the orthogonal matrix \( W_{pq}^n = V_{n_2}^n \otimes U_{n_1}^n \), for \( U_{n_1}^n \) and \( V_{n_2}^n \) orthogonal matrices of dimensions \( n_1 \times n_1 \) and \( n_2 \times n_2 \), respectively, and

\[
p = \frac{n_1n_2 - s}{n_1n_2} = 1 - \rho_s,
\]

\[
q = \frac{(n_1 - r)(n_2 - r)}{n_1n_2} = (1 - \rho_r)^2.
\]

As \( n \) grows to infinity, we are interested in finding the limit (or even the limiting distribution) of the smallest eigenvalue of \( H_n \), which is a random truncation of the Kronecker product of two large dimensional semi-orthogonal matrices.

Fig. 1: Contour plot of \( \rho_\infty \) as a 2-D function of \( \rho_r \) and \( \rho_s \) given by (16). The isoline at which \( \rho_\infty = 1 \) is represented by the dashed line. The yellow region below this isoline corresponds to the under-determined setting \( \rho_s < 1 - (1 - \rho_r) \).

B. Truncations of Large Dimensional Orthogonal Matrices

Random matrix theory studies the asymptotic behavior of eigenvalues of matrices with entries drawn randomly from various matrix ensembles such as Gaussian orthogonal ensemble (GOE), Wishart ensemble, MANOVA ensemble [44]. The closest random matrix ensemble to our matrix ensemble \( \{H_n\}_{n\in\mathbb{N}^+} \) is the MANOVA ensemble in which truncations of large dimensional Haar orthogonal matrices are considered. Here we recall that the Haar measure provides a uniform distribution over the set of all \( n \times n \) orthogonal matrices \( \mathcal{O}(n) \). Indeed, it is a unique translation-invariant probability measure on \( \mathcal{O}(n) \). If we assume that the matrix \( M \) follows a random orthogonal model [10], then \( U_{\perp} \) and \( V_{\perp} \) are essentially sub-matrices of Haar orthogonal matrices in \( \mathcal{O}(n_1) \) and \( \mathcal{O}(n_2) \), respectively, and \( \{H_n\}_{n\in\mathbb{N}^+} \) is a sequence of truncations of the Kronecker product of two Haar orthogonal matrices.

There have been certain theoretical works on truncations of Haar invariant matrices in the literature. In 1980, Wachter [45] established the limiting distribution of the eigenvalues in the MANOVA ensemble. Later on, the density function of the eigenvalues of such matrix has been shown to be the same as that of a Jacobi matrix [46]–[48]. Shortly afterward, Johnstone proved the Tracy-Widom behavior of the largest eigenvalue in [49]. More recently, Farrell and Nadakuditi relaxed the constraint on the uniform (Haar) distribution of the orthogonal matrix considered the Kronecker products of Haar-distributed orthogonal matrices, which is similar to our matrix completion setting in this paper. The authors showed that the limiting density of their truncations remains the same as the original case without Kronecker products. Further results on the eigenvalue distribution of truncations of Haar orthogonal matrices were also given in [50]–[52]. To the best of our knowledge, no result has been shown for the limiting behavior of the smallest eigenvalue of random MANOVA matrices.

In our context, we leverage the recent result in [53], which assumes the randomness on the truncation rather than the
Definition 7. Let $H_n$ be an $n \times n$ real symmetric matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$. The empirical spectral distribution (ESD) of $H_n$, denoted by $\mu_{H_n}$, is the probability measure which puts equal mass at each of the eigenvalues of $H_n$:

$$\mu_{H_n} \triangleq \frac{1}{n} \sum_{i=1}^{n} \delta_{\lambda_i},$$

where $\delta_{\lambda}$ is the Dirac mass at $\lambda$.

Next, we define the concepts of a sequence of row sub-sampled matrices and the concentration property:

Definition 8. For each $n \in \mathbb{N}^+$, consider the row matrix $W_q^n = [w_1^n, \ldots, w_{qn}^n]^T$, where $w_i^n \in \mathbb{R}^{qn}$ and $q$ is a constant in $(0,1)$. Let $P_n$ be a $p$-uniform permutation of $[n]$ selected uniformly at random, for $p$ is a constant in $(0,1)$, and $W_{pqn}^n \in \mathbb{R}^{pn \times qn}$ be the random matrix obtained by selecting the corresponding set of $pn$ rows from $W_q^n$. Then, the sequence $\{W_q^n\}_{n \in \mathbb{N}^+}$ is called a sequence of $q$-tall matrices, and the sequence $\{W_{pqn}^n\}_{n \in \mathbb{N}^+}$ is called a sequence of row sub-sampled matrices of $\{W_q^n\}_{n \in \mathbb{N}^+}$.

Definition 9. Given the setting in Definition 8 for each $j \in P_n$, denote $P_n^j = P_n \setminus \{j\}$. In addition, for $z \in \mathbb{C}$, define

$$R_j(z) = \left( \sum_{i \in P_n^j} w_i^n (w_i^n)^T - zI_{qn} \right)^{-1}.$$

Then, the sequence $\{W_q^n\}_{n \in \mathbb{N}^+}$ is concentrated if and only if for any $j \in P_n$ and $z \in \mathbb{C}$, we have

$$\left( w_j^n \right)^T R_j(z) w_j^n - \mathbb{E}_{j \in P_n^j} \left( \left( w_j^n \right)^T R_j(z) w_j^n \right) \xrightarrow{\text{d}} 0. \quad (18)$$

In the following, we consider examples of sequences of matrices that are concentrated, as well as an example of the sequence of incoherent matrices that are not concentrated.

Example 1. Random settings:

1. The sequence of $q$-tall matrices $\{A_q^n\}_{n \in \mathbb{N}^+}$, where the entries of $A_q^n$ are i.i.d. $\mathcal{N}(0,1/n)$, is concentrated.
2. The sequence $\{B_q^n \otimes C_q^n\}_{n \in \mathbb{N}^+}$, where $\{B_q^n\}_{n \in \mathbb{N}^+}$ and $\{C_q^n\}_{n \in \mathbb{N}^+}$ are two sequences of $q$-tall matrices whose entries are i.i.d $\mathcal{N}(0,1/n)$, is also concentrated.

Example 2. Deterministic settings:

1. The sequence of $q$-tall matrices $\{D_q^n\}_{n \in \mathbb{N}^+}$, where the entries of $D_q^n$ are all 1, is concentrated.
2. The sequence of $1/2$-tall matrices $\{E_q^n\}_{n \in \mathbb{N}^+}$ where

$$E_q^n = \begin{bmatrix} 0.6 \sqrt{\frac{2}{nH_{n/2}}} & 0 \\ 0.8 \sqrt{\frac{2}{nH_{n/2}}} & 0 \end{bmatrix}$$

for $H_{n/2}$ being a Hadamard matrix of order $n/2$ \cite{hadamard}, is not concentrated. On the other hand, one can verify that $E_q^n$ is $\mu$-incoherent, for

$$\mu = \|0.8 \sqrt{2/nH_{n/2}}\|^2 \frac{n}{n/2} = 1.28.$$

Thus, the concentration assumption in Definition 9 is stronger than the widely-used incoherence assumption.

\(^4\)The detail of this example is provided in Appendix 8.
With these definitions in place, we now state the result on the limiting ESD of a truncation of orthogonal matrices. To fit our matrix completion setting in this paper, we rephrase the result in [53] to the case of row sub-sampled semi-orthogonal matrices (as opposed to column sub-sampled semi-orthogonal matrices in the aforementioned paper).

Proposition 2 (Rephrased from [53]). Let \( \{W^n_q\}_{n \in \mathbb{N}^+} \) be a sequence of \( q \)-tall matrices that is concentrated. In addition, assume that \( W^n_q \) is semi-orthogonal for all \( n \in \mathbb{N}^+ \), i.e., \( (W^n_q)^\top W^n_q = I_{qn} \). Let \( \{W^n_q\}_{n \in \mathbb{N}^+} \) be a sequence of row sub-sampled matrices of \( \{W^n_q\}_{n \in \mathbb{N}^+} \). Then, as \( n \to \infty \), the ESD of \( H_n = W^n_{pq}(W^n_{pq})^\top \) converges almost surely to the deterministic distribution \( \mu_{pq} \) such that

\[
d\mu_{pq} = \left(1 - \frac{q}{p}\right)\delta(x)dx + \left(\frac{p + q - 1}{p}\right)\delta(x - 1)dx + \frac{\sqrt{(\lambda^+ - x)(x - \lambda^-)}}{2\pi px(1 - x)}\mathbb{I}[-\lambda^+, \lambda^+]dx,
\]

where \( \delta \) is the Dirac delta function and

\[
\lambda^\pm = \left(\sqrt{q(1 - p)} \pm \sqrt{p(1 - q)}\right)^2.
\]

The proposition asserts that the limiting ESD of \( H_n \) exists and depends only on the row ratio \( p \) and the column ratio \( q \), provided that \( \{W^n_q\}_{n \in \mathbb{N}^+} \) is concentrated. We note that the distribution \( \mu_{pq} \) is exactly the same as the limiting distribution of the MANOVA ensemble. Indeed, one can show that the MANOVA ensemble is a concentrated matrix sequence:

Lemma 3. Let \( W^n \) be a Haar-distributed orthogonal matrix in \( \bigotimes(n) \) and \( W^n_q \) be the semi-orthogonal matrices obtained from any \( q^n \) (for \( q \in (0, 1) \)) columns of \( W^n \). Then the sequence \( \{W^n_q\}_{n \in \mathbb{N}^+} \) is concentrated.

Furthermore, the Kronecker product of two Haar-distributed orthogonal matrices also possesses the concentration property:

Lemma 4. Let \( U_{n_1}^q \) and \( V_{n_2}^q \) be Haar-distributed orthogonal matrices in \( \bigotimes(n_1) \) and \( \bigotimes(n_2) \), respectively. Define \( U_{n_1}^{q_1} \otimes V_{n_2}^{q_2} \) as the semi-orthogonal matrices obtained from any \( q_1 \) and \( q_2 \) (for \( q_1, q_2 \in (0, 1) \)) columns of \( U_{n_1}^q \) and \( V_{n_2}^q \), respectively. Then the sequence \( \{W^n_q = U_{n_1}^{q_1} \otimes V_{n_2}^{q_2}\}_{n \in \mathbb{N}^+} \) with \( q = q_1q_2 \) is concentrated.

Lemmas 3 and 4 are immediate consequences of Lemma 3.1 in [53], so we omit the proof of these lemmas here.

C. Proposed Estimation of the Linear Rate \( \rho \)

In order to apply Proposition 2 to our matrix completion setting, we recall that \( W^n_q \) can be viewed as the \( n \)-th element of a sequence of row sub-sampled matrices of \( \{W^n_q\}_{n \in \mathbb{N}^+} \), where \( W^n_q = V_1^{q_1} \otimes U_1^{q_2} \). If the sequence \( \{W^n_q\}_{n \in \mathbb{N}^+} \) is concentrated, then (19) holds for \( p = 1 - \rho_s \) and \( q = (1 - \rho_s)^2 \).

Therefore, one might expect that the smallest eigenvalue of \( H_n = W^n_{pq}(W^n_{pq})^\top \) converges to

\[
\lambda^- = \left(\sqrt{q(1 - p)} - \sqrt{p(1 - q)}\right)^2.
\]

Thus, by Theorem 1, the convergence rate \( \rho \) converges to \( 1 - \lambda^- \). The following theorem is an immediate application of Proposition 2 to our large-scale matrix completion setting:

Theorem 3. As \( m \to \infty \), assume that \( M \) is generated in a way that the Kronecker product \( W^n_q = V_1^{q_1} \otimes U_1^{q_2} \) forms a sequence of semi-orthogonal matrices that is concentrated. Then, provided \( \rho_s \geq 1 - (1 - \rho_r)^2 \), the ESD \( \mu_{H_n} \) converges almost surely to the deterministic distribution \( \mu_{pr, \rho} \) such that

\[
d\mu_{pr, \rho_s} = \left(\frac{1 - \rho_s}{1 - \rho_s} - \rho_s\right)\delta(x - 1)dx + \frac{\sqrt{(\lambda^+ - x)(x - \lambda^-)}}{2\pi px(1 - x)}\mathbb{I}[-\lambda^+, \lambda^+]dx,
\]

where \( \lambda^\pm = \left(\sqrt{(1 - \rho_r)^2\rho_s} \pm \sqrt{\rho_r(2 - \rho_r)(1 - \rho_s)}\right)^2 \).

While the theorem claims the convergence of the spectral distribution of \( H \), it does not imply the convergence of its smallest eigenvalue to \( \lambda^- \). In fact, it is not trivial to prove this fact. The following conjecture is left for future work.

Conjecture 1. Assume the same setting as in Theorem 2. As \( m \to \infty \), the linear rate \( \rho \) defined in [53] converges almost surely to \( \rho_p \) defined in [16].

V. NUMERICAL RESULTS

In this section, we provide numerical results to verify the exact linear convergence rate of IHSTVD in [53] and to compare this analytical rate with the asymptotic rate in [16] in large-scale settings.

A. Analytical Rate versus Empirical Rate

In this experiment, we verify the analytical expression of the linear convergence rate of IHSTVD by comparing it with the empirical rate obtained by measuring the decrease in the norm of the error matrix. Our goal is to demonstrate that they agree in various settings of \( \rho_r \) and \( \rho_s \).

Data generation. We first set the dimensions \( n_1 = 50 \) and \( n_2 = 40 \). Next, for each \( r \) in \( \{1, 2, \ldots, 12\} \), we generate the rank-\( r \) matrix \( M \) as follows. We construct the random
Fig. 4: The analytical rate and the empirical rate of convergence of IHT-SVD as a function of the relative rank $\rho_r$ and the sampling ratio $\rho_s$, with $n_1 = 50$ and $n_2 = 40$. (a) Contour plot of the analytical rate as a function of $\rho_r$ and $\rho_s$. (b) Contour plot of the empirical rate as a function of $\rho_r$ and $\rho_s$. (c) Empirical probability of linear convergence based on the analytical rate. (d) Empirical probability of linear convergence based on the empirical rate. The black color corresponds to linear convergence, while the white color corresponds to no linear convergence. In each plot, the data is interpolated based on a $12 \times 21$ grid over $\rho_r$ and $\rho_s$, in which the value of each point is evaluated by 1000 runs. Additionally, a dashed line is included to indicate the line $1 - \rho_s = (1 - \rho_r)^2$. The similarity between the left column and the right column demonstrates the utility of our analysis.

Estimating Analytical Rate and Empirical Rate. We calculate the analytical rate for each aforementioned setting of $M$ and $\Omega$ using (8). Due to numerical errors in computing small eigenvalues, we need to set all the resulting rates that are greater than 1 to 1, indicating there is no linear convergence in such cases. For the calculation of the empirical rate, we run Algorithm 1 in the same setting with $K = 10000$ iterations. The initial point $X^{(0)}$ is obtained by adding i.i.d. normally distributed noise with standard deviation $\sigma = 10^{-4}$ to the entries of $M$. Here we note that $\sigma$ is chosen to be small for two reasons: (i) for large matrices, even small $\sigma$ for individual entry can add up to a large error on the entire matrix; and (ii) while the cost of computing $\lambda_{\text{min}}$ (and hence, the region of convergence) is prohibitively expensive for large matrices, choosing small $\sigma$ empirically guarantees the initialization is inside the region of convergence. Then, we record the error sequence $\{\|X^{(k)} - M\|_F\}_{k=1}^K$ and determine if the algorithm converges linearly to $M$ by checking whether there existsORTHOPEDIC TRACKS AND CLINICS
\( K \leq K \) such that \( \|X^{(k)} - M\|_F < \epsilon \|X^{(0)} - M\|_F \), for \( \epsilon = 10^{-8} \). If the relative error is above \( \epsilon \), we set the empirical rate to 1 to indicate that the algorithm does not converge linearly. However, it is important to note that this heuristic does not perfectly detect linear convergence since it overlooks the case in which the linear rate is extremely close to 1 and it requires more than \( K = 10000 \) iterations to reach a relative error below \( \epsilon \). As can be seen later, to compromise this computational limit, we resort to setting the analytical rate that is greater than 0.998 to 1 when making a comparison between the analytical rate and the empirical rate\(^6\). In case the relative error is less than \( \epsilon \), we terminate the algorithm at the \( K \)-th iteration (early stop) and perform a simple fitting for an exponential decrease on the error sequence \( \{\|X^{(k)} - M\|_F\}^K_{k=1} \) to obtain the empirical rate.

After obtaining the analytical rate and the empirical rate over the 2-D grid, we report the result in the contour plots of the rate as a function of \( \rho_r \) and \( \rho_s \) in Fig. 4(a) and (b). Since our original grid is non-uniform, we perform a scattered data interpolation, which uses a Delaunay triangulation of the scattered sample points to perform interpolation\(^6\), to evaluate the rate over a \( 1001 \times 1001 \) uniform grid based on \( \rho_r \) and \( \rho_s \). Due to the aforementioned limitation of estimating the empirical rate, we apply a threshold of 0.998 to both of the interpolated data for the analytical rate and the empirical rate, setting any value above the threshold to 1. In addition, we calculate the probability of linear convergence for the analytical rate and the empirical rate and visualize the result in Fig. 4(c) and (d). As mentioned, we use a threshold of 0.998 to determine the linear convergence.

**Results.** Given the values of the analytical rate and the empirical rate of 1000 matrix completion settings for each point on the \( 12 \times 21 \) grid, the mean squared difference between the two rates in our experiment is \( 2.9659 \times 10^{-5} \). Figures 4 illustrate the similarity between the analytical rate and the empirical rate evaluated under various settings of matrix completion. In both Fig. 4(a) and Fig. 4(b), we observed a matching behavior as in Fig. 1 smaller rank and more observation result in faster linear convergence of IHTSVD. However, the contour lines in Fig. 4 are not as smooth as those with asymptotic behavior in Fig. 1 due to the large variance when \( n_1 \) and \( n_2 \) are relatively small. On the other hand, it can be seen in Fig. 4(c) and Fig. 4(d) that there is a linear-convergence area (black) above the boundary line at \( 1 - \rho_s = (1 - \rho_r)^2 \) and a no-linear-convergence area (white) below the boundary line. The transition area (gray) near above the boundary line corresponds to the settings in which some sampling sets result in \( \lambda_{\min}(H) = 0 \) (no linear convergence) while some other sampling sets result in non-zero \( \lambda_{\min}(H) \) (linear convergence). Note that in order to obtain the analytical rate, we need to compute the smallest eigenvalue of a \( (n - s) \times (n - s) \) matrix, which is computationally expensive for large \( n = n_1n_2 \). In particular, when \( s = \Omega(n) \), the cost of computing the analytical rate is \( \Omega(n^2) \).

Algorithm\(^6\) whose computational complexity per iteration is \( O(nr) \).

### B. Non-asymptotic Rate versus Asymptotic Rate

In this section, we compare the asymptotic rate given in Theorem 3 with the convergence rate of IHTSVD for large-scale matrix completion. For convenience, we refer the latter as the non-asymptotic rate. As mentioned, we use the empirical rate instead of the analytical rate to estimate the non-asymptotic rate due to the computational efficiency.

**Data generation.** We consider two settings of \( (n_1, n_2) \): \( n_1 = 500, n_2 = 400 \) and \( n_1 = 1200, n_2 = 1000 \). Similar to the previous experiment, we generate \( M \) and \( \Omega \) based on a 2-D grid over \( r \) and \( s \). While the values of \( s \) are still selected from the set \( \{0.2n, 0.23n, 0.26n, \ldots, 0.8n\} \), the values of \( r \) are chosen differently for each setting of \( (n_1, n_2) \). In particular, for \( n_1 = 500, n_2 = 400 \), we select the values of \( r \) from the linearly spaced set \( \{1, 4, 7, \ldots, 118\} \). For \( n_1 = 1200, n_2 = 1000 \), we select the values of \( r \) from the linearly spaced set \( \{1, 9, 17, \ldots, 297\} \). Thus, in the former setting, the grid size is \( 40 \times 21 \), while in the latter setting, the grid size is \( 38 \times 21 \).

**Implementation.** The calculation of the empirical rate is the same as the previous experiment. For computational efficiency, we omit the points on the grid that are below the boundary line, i.e., \( s < (n_1 + n_2 - r)r \), since it is evident that there is no linear convergence at such points. No analytical rate is given in this experiment because calculating the smallest eigenvalue of a \( (n-s) \times (n-s) \) matrix is computationally expensive for large \( n_1 \) and \( n_2 \). On the other hand, the contour plot of the asymptotic rate is straightforward to obtain using (16).

**Results.** Fig. 5 compares the non-asymptotic rate and the asymptotic rate in various settings of \( \rho_r \) and \( \rho_s \). As \( n_1 \) and \( n_2 \) increase, we observe that the contour lines of the non-asymptotic rate become smoother and approach those of the asymptotic rate. Compared with Fig. 4, it can also be seen that the isoline of the same rate shifts downward towards the boundary line as \( n_1 \) and \( n_2 \) increases.

### VI. CONCLUSIONS AND FUTURE WORK

In this paper, we established a closed-form expression of the linear convergence rate of an iterative hard thresholding method for solving matrix completion. We also identified the local region around the solution that guarantees the convergence of the algorithm. Furthermore, in large-scale settings, we leveraged the result from random matrix theory to offer a simple estimation of the asymptotic convergence rate in practice. Under certain assumptions, we showed that the convergence rate of IHTSVD converges almost surely to our proposed estimate.

In future work, we would like to extend our local convergence analysis to other IHT methods with different step sizes, e.g., SVP\(^7\) and accelerated IHT\(^7\). Moreover, it would be interesting to study the non-asymptotic behavior of the convergence rate in large-scale settings. Finally, we believe

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\(^6\)Substituting \( \epsilon = 10^{-8} \) and \( K(\epsilon) = 10000 \) into (9) and assuming the constant \( c \) is negligible, we obtain \( \lambda_{\min}(H) \approx 1.8 \times 10^{-3} \).

\(^7\)The black color corresponds to linear convergence, while the white color corresponds to no linear convergence.
the technique presented in this manuscript can be applied to study the local convergence of other non-convex methods such as alternating minimization \[31\] and gradient descent \[23\].

**APPENDIX A**

**PROOF OF THEOREM** \[1\]

**A. Proof of Lemma** \[7\]

By the definition of the error matrix, we have

\[
E^{(k+1)} = X^{(k+1)} - M \\
= \left( P_{\Omega} \left( P_r(X^{(k)}) \right) + P_{\Omega}(M) \right) - \left( P_{\Omega}(M) + P_{\Omega}(M) \right) \\
= P_{\Omega} \left( P_r(M + E^{(k)}) - M \right). \tag{21}
\]

From Proposition \[1\] we can reorganize (21) to obtain

\[
P_r(M + E^{(k)}) - M = E^{(k)} - P_{U_{\perp}} E^{(k)} P_{V_{\perp}} + R(E^{(k)}).
\]

Substituting the last equation back into (21) yields the recursion on the error matrix as in (11).

Next, let us denote \( e^{(k)} = S_{\Omega}^T \text{vec}(E^{(k)}) \), for \( k = 1, 2, \ldots \). Vectorizing equation (11) and left-multiplying both sides with \( S_{\Omega} \) yield

\[
e^{(k+1)} = S_{\Omega}^T \text{vec} \left( P_{\Omega} \left( E^{(k)} - P_{U_{\perp}} E^{(k)} P_{V_{\perp}} + R(E^{(k)}) \right) \right).
\]

Using the property of selection matrices in Definition \[2\] we further have

\[
e^{(k+1)} = S_{\Omega}^T S_{\Omega} S_{\Omega}^T \text{vec} \left( E^{(k)} - P_{U_{\perp}} E^{(k)} P_{V_{\perp}} + R(E^{(k)}) \right) \\
= S_{\Omega}^T \text{vec} \left( E^{(k)} - P_{U_{\perp}} E^{(k)} P_{V_{\perp}} + R(E^{(k)}) \right).
\]

Since \( \text{vec}(P_{U_{\perp}} E^{(k)} P_{V_{\perp}}) = (P_{V_{\perp}} \otimes P_{U_{\perp}}) \text{vec}(E^{(k)}) \), the last equation can be represented as

\[
e^{(k+1)} = S_{\Omega}^T \text{vec}(E^{(k)}) - S_{\Omega}^T (P_{V_{\perp}} \otimes P_{U_{\perp}}) \text{vec}(E^{(k)}) \\
+ S_{\Omega}^T \text{vec}(R(E^{(k)})). \tag{22}
\]

On the other hand, (11) implies, for any \( k \geq 1, E^{(k)} = P_{\Omega}(E^{(k)}) \) and

\[
\text{vec}(E^{(k)}) = \text{vec}(P_{\Omega}(E^{(k)})) = S_{\Omega} S_{\Omega}^T \text{vec}(E^{(k)}) = S_{\Omega} e^{(k)}.
\]

Substituting the last equation into the RHS of (22) yields (12).

**B. Proof of Lemma** \[2\]

Applying the triangle inequality to the RHS of (12) yields

\[
\|e^{(k+1)}\|_2 \leq \| (I - H) e^{(k)} \|_2 + \| r(e^{(k)}) \|_2, \tag{23}
\]

Fig. 5: The empirical rate and the asymptotic rate of convergence of IHTSVD as a function of the relative rank \( \rho_r \) and the sampling ratio \( \rho_s \). (a) Contour plot of the empirical rate as a function of \( \rho_r \) and \( \rho_s \) for \( n_1 = 500, n_2 = 400 \). (b) Contour plot of the empirical rate as a function of \( \rho_r \) and \( \rho_s \) for \( n_1 = 1200, n_2 = 1000 \). (c) A zoom-in contour plot of the asymptotic rate as a function of \( \rho_r \) and \( \rho_s \). (d) Empirical probability of linear convergence based on the empirical rate in (a). (e) Empirical probability of linear convergence based on the empirical rate in (b). (f) A zoom-out contour plot of the asymptotic rate as a function of \( \rho_r \) and \( \rho_s \). The red solid rectangular corresponds to the zoomed-in region in (c). In each plot, the data is interpolated based on a 2-D grid over \( \rho_r \) and \( \rho_s \), in which the value of each point is evaluated by 100 runs. Additionally, a dashed line is included to indicate the line \( 1 - \rho_s = (1 - \rho_r)^2 \).
where we recall \( H = S_{\Omega}^T (P_{\mathcal{V}_\Omega} \otimes P_{\mathcal{U}_\perp}) S_{\Omega} \). By the definition of the operator norm, we have
\[
\| (I - H) e^{(k)} \|_2 \leq \| I - H \|_2 \| e^{(k)} \|_2 = \max \{ 1 - \lambda_i(H) \} \cdot \| e^{(k)} \|_2 \leq (1 - \lambda_{\min}(H)) \| e^{(k)} \|_2,
\]  
where the last equality stems from the fact that all eigenvalues of \( H \) lie between 0 and 1. From (23) and (24), we obtain
\[
\| e^{(k+1)} \|_2 \leq (1 - \lambda_{\min}(H)) \| e^{(k)} \|_2 + \| r(e^{(k)}) \|_2.
\]  
The conclusion of lemma follows from the fact that
\[
\| e^{(k)} \|_2 = \| \mathcal{P}_{\Omega}(E^{(k)}) \|_F = \| E^{(k)} \|_F
\]  
and
\[
\| r(e^{(k)}) \|_2 \leq \| R(E^{(k)}) \|_F \leq \frac{C_1}{\sigma_r} \| E^{(k)} \|_F^2.
\]

APPENDIX B
PROOF OF EXAMPLE [7]

A. The first case in Example 7

Using the same argument as in Lemma 5.3 in [57], we can replace the complex matrix in (13) by a real PSD matrix and prove the following lemma:

**Lemma 5.** Let \( a = [a_1, \ldots, a_{qn}]^T \) is a random vector with \( i.i.d \) entries, where \( a_i \sim \mathcal{N}(0, 1/n) \). Then for any sequence of \( qn \times qn \) PSD matrices \( M_{qn} \) with uniformly bounded spectral norms \( \| M_{qn} \|_2 \), we have
\[
(a^T M_{qn} a - \frac{1}{n} \text{tr}(M_{qn})) \overset{P}{\to} 0 \text{ as } n \to \infty.
\]

**Proof.** To simplify our notation, let us denote the \((i, j)\)-th entry of \( M_{qn} \) by \( M_{ij} \) and \( \delta_{ij} \) is the indicator of the event \( i = j \). Since \( a_i \) are \( i.i.d \) normally distributed, we have
\[
E[a_i] = 0, \quad E[a_i a_j] = \delta_{ij} \frac{1}{n},
\]
\[
E[a_i a_j a_k a_l] = (\delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \frac{1}{n^2},
\]  
for any indices \( 1 \leq i, j, k, l \leq n \). In order to prove \( (a^T M_{qn} a - \frac{1}{n} \text{tr}(M_{qn})) \overset{P}{\to} 0 \), it is sufficient to show that
\[
E[a^T M_{qn} a] = \frac{1}{n} \text{tr}(M_{qn}), \quad \text{Var}(a^T M_{qn} a) \to 0 \text{ as } n \to \infty.
\]
First, by the linearity of expectation, we have
\[
E[a^T M_{qn} a] = E\left[\sum_{i,j} M_{ij} a_i a_j\right]
\]
\[
= \sum_{i,j} M_{ij} E[a_i a_j]
\]
\[
= \sum_{i,j} M_{ij} \delta_{ij} \frac{1}{n}
\]
\[
= \frac{1}{n} \sum_{i,j} A_{ii}
\]
\[
= \frac{1}{n} \text{tr}(M_{qn}).
\]

Second, by rewriting the variance of the summation \( \sum_{i,j} M_{ij} a_i a_j \) in terms of the sum of covariances, we obtain
\[
\text{Var}(a^T M_{qn} a) = \text{Var}\left(\sum_{i,j} M_{ij} a_i a_j\right)
\]
\[
= \sum_{i,j,k,l} \text{Cov}(M_{ij} a_i a_j, M_{kl} a_k a_l).
\]

Using the formula
\[
\text{Cov}(X, Y) = E[XY] - E[X]E[Y],
\]
and the linearity of expectation, (28) can be represented as
\[
\text{Var}(a^T M_{qn} a)
\]
\[
= \sum_{i,j,k,l} M_{ij} M_{kl} (E[a_i a_j a_k a_l] - E[a_i a_j]E[a_k a_l])
\]
\[
= \sum_{i,j,k,l} M_{ij} M_{kl} (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \frac{1}{n^2}
\]
\[
= \frac{2}{n^2} \sum_{i,j} M_{ij}^2
\]
\[
= \frac{2}{n^2} \| M_{qn} \|_F^2.
\]  
Since \( M_{qn} \) is PSD and has bounded spectral norm, all of its eigenvalues are bounded by \( 0 \leq \lambda_i(M_{qn}) \leq C \), for some constant \( C \), and hence,
\[
\| M_{qn} \|_F^2 = \sum_{i=1}^{qn} \lambda_i^2(M_{qn}) \leq qnC^2.
\]
Thus, substituting back into (30) yields
\[
\text{Var}(a^T M_{qn} a) \leq \frac{2}{n^2} qnC^2 \to 0 \text{ as } n \to \infty.
\]
This completes our proof of the lemma.

The second case in Example 7

Similarly, we consider the following lemma:

**Lemma 6.** Let \( b = [b_1, \ldots, b_{qn}] \) and \( c = [c_1, \ldots, c_{qn}] \) are random vectors with \( i.i.d \) entries, where \( b_i, c_j \sim \mathcal{N}(0, 1/n) \). Denote \( m = n^2, k = q^2 \) and \( a = b \otimes c \). Then for any sequence of \( km \times km \) PSD matrices \( M_{km} \) with uniformly bounded spectral norms \( \| M_{km} \|_2 \), we have
\[
(a^T M_{km} a - \frac{1}{m} \text{tr}(M_{km})) \overset{P}{\to} 0 \text{ as } n \to \infty.
\]

**Proof.** Denote \( M_{(i,j)} \) is the \((i, j)\)-th \( qn \times qn \) block of \( M_{km} \). Then it is straightforward to verify that
\[
a^T M_{km} a = \sum_{i,j} b_i (c^T M_{(i,j)} c)b_j.
\]

In order to prove \( (a^T M_{km} a - \frac{1}{m} \text{tr}(M_{km})) \overset{P}{\to} 0 \), it is sufficient to show that
\[
E[a^T M_{km} a] = \frac{1}{m} \text{tr}(M_{km}), \quad \text{Var}(a^T M_{km} a) \to 0 \text{ as } n \to \infty.
\]
First, we use the linearity of expectation to obtain
\[
\mathbb{E}[a^\top M_{kn}a] = \mathbb{E}\left[\sum_{i,j} b_i (c^\top M_{ij}c)b_j\right]
= \sum_{i,j} \mathbb{E}[b_i b_j] \mathbb{E}[c^\top M_{ij}c].
\]

From (29) and Lemma 5 the last equation is equivalent to
\[
\mathbb{E}[a^\top M_{kn}a] = \sum_{i,j} \delta_{ij} \frac{1}{n} \cdot \frac{1}{n} \text{tr}(M_{ij})
= \frac{1}{m} \text{tr}(M_{km}).
\]

Second, we have
\[
\text{Var}(a^\top M_{kn}a) = \text{Var}\left(\sum_{i,j} b_i (c^\top M_{ij}c)b_j\right)
= \sum_{i,j,k,l} \text{Cov}(b_i (c^\top M_{ij}c)b_j, b_k (c^\top M_{kl}c)b_l).
\]

From (29), each covariance on the RHS of (31) can be represented as
\[
\text{Cov}(b_i (c^\top M_{ij}c)b_j, b_k (c^\top M_{kl}c)b_l) = \mathbb{E}[b_i b_j b_k b_l] - \mathbb{E}[b_i b_j] \cdot \mathbb{E}[c^\top M_{ij}c] \cdot \mathbb{E}[c^\top M_{kl}c].
\]

Lemma 7. Let \(P \) and \(Q \) be matrices in \(\mathbb{R}^{m \times q} \). Then
\[
\mathbb{E}[c^\top P c \cdot c^\top Q c] = \frac{\text{tr}(P) \text{tr}(Q) + \text{tr}(PQ^\top) + \text{tr}(PQ)}{n^2}.
\]

The proof of Lemma 7 is straightforward from (29) and is omitted in this manuscript. From Lemma 7 and (26), we can simplify (32) as
\[
\text{Cov}(b_i (c^\top M_{ij}c)b_j, b_k (c^\top M_{kl}c)b_l) = \frac{1}{n^4} \left(\text{tr}(M_{ij} M_{kl}) + \text{tr}(M_{ij} M_{kl}^\top) + \text{tr}(M_{ij}^\top M_{kl}) + \text{tr}(M_{ij}^\top M_{kl}^\top)\right).
\]

Substituting the last equation back into (31) yields
\[
\text{Var}(a^\top M_{km}a) = \frac{2}{n^4} \left(\sum_{i,j} \text{tr}(M_{ij})^2 + \sum_{i,j} \text{tr}(M_{ij} M_{jj}) + \sum_{i,j} \text{tr}(M_{ij}^2)\right).
\]

Next, we bound each term on the RHS of (33). To that end, we utilize the following lemma:

Lemma 8. For any matrices \(A, B \in \mathbb{R}^{n \times n} \), it holds that
\[
\begin{align*}
1) & \quad \|A\| \leq \sqrt{n} \|A\|_2, \\
2) & \quad \text{tr}(A)^2 \leq n \|A\|_F^2, \\
3) & \quad \text{tr}(AB) \leq \|A\|_F \|B\|_F \leq n \|A\|_2 \|B\|_2, \\
4) & \quad \text{tr}(A^2) \leq \|A\|_F^2 = \text{tr}(A^\top A).
\end{align*}
\]

The proof of Lemma 8 can be found in [58] - Chapter 5. Applying Lemma 8 with the blocks of size \(qn \times qn \), we obtain
\[
\sum_{i,j} \text{tr}^2(M_{ij}) \leq \sum_{i,j} qn \|M_{ij}\|_F^2 = qn \|M\|_F^2 
\leq (qn)^3 \|M\|_2 \leq C(qn)^3,
\]
\[
\sum_{i,j} \text{tr}(M_{ij} M_{jj}) \leq \sum_{i,j} qn \|M_{ij}\|_2 \|M_{jj}\|_2 
\leq \left(\frac{1}{n}\right)^2 qn \|M\|_2 = C(qn)^2,
\]
\[
\sum_{i,j} \text{tr}(M_{ij}^2) \leq \sum_{i,j} qn \|M_{ij}\|_2^2 = qn \|M\|_2^2 = C(qn)^2.
\]

Therefore, (33) can be bounded as
\[
\text{Var}(a^\top M_{km}a) \leq \frac{2}{n^4} (C(qn)^3 + C^2(qn)^3 + 2C(qn)^2).
\]

The conclusion of the lemma follows by the fact that the RHS of the last equation which approaches 0 as \(n \to \infty \).

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