A high-order corrector estimate for a semi-linear elliptic system in perforated domains

Vo Anh Khoa*

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Abstract

We derive in this note a high-order corrector estimate for the homogenization of a microscopic semi-linear elliptic system posed in perforated domains. The major challenges are the presence of nonlinear volume and surface reaction rates. This type of correctors justifies mathematically the convergence rate of formal asymptotic expansions for the two-scale homogenization settings. As main tool, we follow the standard approach by the energy-like method to investigate the error estimate between the micro and macro concentrations and micro and macro concentration gradients. This work aims at generalizing the results reported in [2, 7].

Keywords: Corrector estimate, Homogenization, Elliptic systems, Perforated domains

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1 Introduction

This note is devoted to the derivation of a high-order corrector for a microscopic semi-linear elliptic system posed in heterogeneous/perforated domains. In the terminology of homogenization, a corrector or corrector estimate wants to quantify the error between the approximate solution (governed by the asymptotic procedure) and the exact solution. Typically, this kind of estimates is helpful also in controlling the approximation error of numerical methods to multiscale problems (e.g. [1, 6]). The main result of this note is Theorem 1 where we report the upper bound of the corrector up to an arbitrary high order.

In [7], the investigated microscopic semi-linear system resembles a steady state-type of thermo–diffusion systems ([10, 9]). Essentially, we have analyzed the solvability of the microscopic system in [7], derived the upscaled equations as well as the corresponding effective coefficients, and proved the high-order corrector estimates for the differences of concentrations and their gradients in which the standard energy method has been used. Furthermore, we also solved a reduced similar problem where the Picard iterations-based method is applied to deal with the nonlinear auxiliary problems ([8]).

It is worth noting that the following errors have been so far obtained for the homogenization of the above-mentioned microscopic elliptic system in [7] and further in [2]:

\[ u^\varepsilon - \sum_{m=0}^{M} \varepsilon^m u_m \quad \text{and} \quad u^\varepsilon - u_0 - m^\varepsilon \sum_{m=1}^{M} \varepsilon^m u_m, \quad M \geq 2. \]

*Author for correspondence. Mathematics and Computer Science Division, Gran Sasso Science Institute, L’Aquila, Italy. (khoa.vo@gssi.infn.it, vakhoa.hcmus@gmail.com)
In this note, we prove a corrector in the form of

\[ u^\varepsilon - \sum_{k=0}^{K} \varepsilon^k u_k - \sum_{m=K+1}^{M} \varepsilon^m u_m, \]  

(1.1)
in which we fix \( K \in \mathbb{N} \) such that \( 0 \leq K \leq M - 2 \).

2 Problem settings

We consider the semi-linear elliptic boundary value problem

\[ \mathcal{A}^\varepsilon u_i^\varepsilon \equiv \nabla \cdot (-d_i^\varepsilon \nabla u_i^\varepsilon) = R_i(u_1^\varepsilon, ..., u_N^\varepsilon) \quad \text{in} \quad \Omega^\varepsilon, \]

associated with the boundary conditions

\[ d_i^\varepsilon \nabla u_i^\varepsilon \cdot n = \varepsilon (a_i^\varepsilon u_i^\varepsilon - b_i^\varepsilon F_i(u_i^\varepsilon)) \quad \text{across} \quad \Gamma^\varepsilon, \]
\[ u_i^\varepsilon = 0 \quad \text{across} \quad \Gamma^{\text{ext}}, \]

for \( i \in \{1, ..., N\} \) with \( N \geq 2 \) being the number of concentrations. For simplicity, we refer this problem as \((P^\varepsilon)\).

This problem is connected to the Smoluchowski–Soret–Dufour modeling of the evolution of temperature and colloid concentrations [3, 10]. Here, \( u^\varepsilon := (u_1^\varepsilon, ..., u_N^\varepsilon) \) denotes the vector of the concentrations, \( d_i^\varepsilon \) represents the molecular diffusion with \( R_i \) being the volume reaction rate and \( a_i^\varepsilon, b_i^\varepsilon \) are deposition coefficients, whilst \( F_i \) indicates a surface chemical reaction for the immobile species. Notice that the quantity \( \varepsilon \) is called the homogenization parameter or the scale factor. The perforated domain \( \Omega^\varepsilon \subset \mathbb{R}^d \) herein approximates a porous medium and its precise description can be found in [5, 7]. As an example, we depict in Figure 2.1 an admissible geometry of our medium and the corresponding microstructure.

![Figure 2.1: Admissible 2-D perforated domain (left) and basic geometry of the microstructure (right). (By courtesy of Mai Thanh Nhat Truong, Hankyong National University, Republic of Korea.)](image-url)
Denote by \( x \in \Omega^\varepsilon \) the macroscopic variable and by \( y = x/\varepsilon \) the microscopic variable representing high oscillations at the microscopic geometry. Henceforward, we understand throughout this paper the following convention:
\[
d_i^\varepsilon (x) = d_i \left( \frac{x}{\varepsilon} \right) = d_i (y), \quad x \in \Omega^\varepsilon, y \in Y_1,
\]
with the same meaning for all the oscillating data such as \( a_i^\varepsilon, b_i^\varepsilon \), e.g.

We introduce the function space
\[
V^\varepsilon := \{ v \in H^1 (\Omega^\varepsilon) \; | \; v = 0 \text{ on } \Gamma^\text{ext} \},
\]
which is a closed subspace of the Hilbert space \( H^1 (\Omega^\varepsilon) \) with the semi-norm
\[
\| v \|_{V^\varepsilon} = \left( \sum_{i=1}^d \int_{\Omega^\varepsilon} \left| \frac{\partial v}{\partial x_i} \right|^2 \, dx \right)^{1/2}
\]
for all \( v \in V^\varepsilon \).

Denote also \( V^\varepsilon = V^\varepsilon \times \ldots \times V^\varepsilon \) and \( W^q (\Omega^\varepsilon) = L^q (\Omega^\varepsilon) \times \ldots \times L^q (\Omega^\varepsilon) \) for \( q \in (2, \infty] \).

Unless otherwise specified, all the constants \( C \) are independent of the homogenization parameter \( \varepsilon \), but the respective values may differ from line to line and may change even within a single chain of estimates.

### 3 Corrector estimate

Consider the two-scale asymptotic expansion up to \( M \)th-level (\( M \geq 2 \)) given by
\[
u_i^\varepsilon (x) = \sum_{m=0}^M \varepsilon^m u_{i,m} (x, y) + O (\varepsilon^{M+1} ), \quad x \in \Omega^\varepsilon, \tag{3.1}
\]
where \( u_{i,m} (x, \cdot) \) is \( Y \)-periodic for \( 0 \leq m \leq M \) and \( i \in \{1, \ldots, N\} \).

Here we keep the assumptions from [2] on the coefficients. This implicitly guarantees the well-posedness of \((P^\varepsilon)\) and \( L^\infty \) bounds for all concentrations \( u_i^\varepsilon \) for \( i \in \{1, \ldots, N\} \). We also remind the crucial assumptions on the Smoluchowski production term, i.e.
\[
R_i \left( \sum_{m=0}^M \varepsilon^m u_{1,m}, \ldots, \sum_{m=0}^M \varepsilon^m u_{N,m} \right) = \sum_{m=0}^M \varepsilon^m \tilde{R}_i (u_{1,m}, \ldots, u_{N,m}) + O (\varepsilon^{M+1} ), \tag{3.2}
\]
\[
F_i \left( \sum_{m=0}^M \varepsilon^m u_{i,m} \right) = \sum_{m=0}^M \varepsilon^m \tilde{F}_i (u_{i,m}) + O (\varepsilon^{M+1} ), \tag{3.3}
\]
where \( \tilde{R}_i \) and \( \tilde{F}_i \) are global Lipschitz functions with the Lipschitz constants \( L_i \) and \( K_i \), respectively, for \( i \in \{1, \ldots, N\} \).

Let \( m^\varepsilon \in C^\infty_c (\Omega) \) be a cut-off function such that \( \varepsilon |\nabla m^\varepsilon| \leq C \) and
\[
m^\varepsilon (x) := \begin{cases} 
1, & \text{if dist} (x, \Gamma) \leq \varepsilon, \\
0, & \text{if dist} (x, \Gamma) \geq 2\varepsilon,
\end{cases}
\]
(see [2] for more properties of \( m^\varepsilon \)). We aim at proving the following result:

**Theorem 1.** Let \( u^\varepsilon \) be the vector of solutions of the elliptic system \((P^\varepsilon)\). Consider the high-order asymptotic expansion \[3.1\] up to \( M \)-level (\( M \geq 2 \)) and take \((u_0, u_m) \in W^\infty (\Omega^\varepsilon) \times W^\infty (\Omega^\varepsilon; H^1_# (Y_1) / \mathbb{R})\) for all \( 0 \leq m \leq M \). For a fixed \( K \in \mathbb{N} \) such that \( 0 \leq K \leq M - 2 \) the following corrector estimate holds:
\[
\left\| u^\varepsilon - \sum_{k=0}^K \varepsilon^k u_k - m^\varepsilon \sum_{m=K+1}^M \varepsilon^m u_m \right\|_{V^\varepsilon} \leq C \left( \varepsilon^{M-1} + \varepsilon^M + \sum_{m=K+1}^M \left( \varepsilon^{m-\frac{1}{2}} + \varepsilon^{m+\frac{1}{2}} \right) \right), \tag{3.4}
\]
where \( C > 0 \) is a generic \( \varepsilon \)-independent constant.
4 Proof of Theorem \[ \text{(1)} \]

The following useful estimates (cf. [4]) hold true:

\[
\|1 - m\|^\|L^2(\Omega^\varepsilon)\| \leq C\varepsilon^{1/2}, \quad \varepsilon \|\nabla m\|^\|L^2(\Omega^\varepsilon)\| \leq C\varepsilon^{1/2}.
\] (4.1)

To bound from above in term of \(\varepsilon\) the quantity \(\text{(1.1)}\), we define the function \(\Psi_i^\varepsilon\) by

\[
\Psi_i^\varepsilon := \varphi_i^\varepsilon + (1 - m)^{\sum_{m=K+1}^M} \varepsilon^m u_{i,m},
\]

where we denote

\[
\varphi_i^\varepsilon := u_i^\varepsilon - \sum_{m=0}^M \varepsilon^m u_{i,m} \quad \text{for} \quad i \in \{1, ..., N\}.
\]

By induction, one can easily obtain that the function \(\varphi_i^\varepsilon\) satisfies the following equation:

\[
\mathcal{A}_i^\varepsilon \varphi_i^\varepsilon = R_i^\varepsilon (u_i^\varepsilon) - \sum_{m=0}^{M-2} \varepsilon^m \bar{R}_i^\varepsilon (u_m) - \varepsilon^{M-1} (\mathcal{A}_1 u_{i,M} + \mathcal{A}_2 u_{i,M-1}) - \varepsilon^M \mathcal{A}_2 u_{i,M} \text{ in } \Omega^\varepsilon,
\] (4.2)

associated with the following boundary condition at \(\Gamma^\varepsilon\)

\[
- d_i^\varepsilon \nabla \varphi_i^\varepsilon \cdot n = \varepsilon^M d_i^\varepsilon \nabla u_{i,M} \cdot n + \varepsilon \left[ d_i^\varepsilon \left( \sum_{m=0}^{M-2} \varepsilon^m u_{i,m} - u_i^\varepsilon \right) \right] + b_i^\varepsilon \left( F_i^\varepsilon (u_i^\varepsilon) - \sum_{m=0}^{M-2} \varepsilon^m \bar{F}_i^\varepsilon (u_{i,m}) \right). \quad (4.3)
\]

In (4.3), \(\mathcal{A}_1\) and \(\mathcal{A}_2\) are defined, respectively, as follows:

\[
\mathcal{A}_1 := \nabla_y \cdot (-d_i (y) \nabla_y) + \nabla_y \cdot (-d_i (y) \nabla_x),
\]

\[
\mathcal{A}_2 := \nabla_y \cdot (-d_i (y) \nabla_x).
\]

Multiplying (4.2) by \(\varphi_i \in V^\varepsilon\) and integrating by parts with using (4.3), we arrive at

\[
\int_{\Omega^\varepsilon} d_i^\varepsilon \nabla \varphi_i^\varepsilon \nabla \varphi_i dx = \left\langle R_i^\varepsilon (u_i^\varepsilon) - \sum_{m=0}^{M-2} \varepsilon^m \bar{R}_i^\varepsilon (u_m), \varphi_i \right\rangle_{L^2(\Omega^\varepsilon)}
\]

\[
-\varepsilon^{M-1} (\mathcal{A}_1 u_{i,M} + \mathcal{A}_2 u_{i,M-1} + \varepsilon \mathcal{A}_2 u_{i,M}, \varphi_i)_{L^2(\Omega^\varepsilon)}
\]

\[
-\varepsilon \left( d_i^\varepsilon \left( \sum_{m=0}^{M-2} \varepsilon^m u_{i,m} - u_i^\varepsilon \right) \right) + b_i^\varepsilon \left( F_i^\varepsilon (u_i^\varepsilon) - \sum_{m=0}^{M-2} \varepsilon^m \bar{F}_i^\varepsilon (u_{i,m}) \right)_{L^2(\Gamma^\varepsilon)}
\]

\[
-\varepsilon^M \int_{\Gamma^\varepsilon} d_i^\varepsilon \nabla u_{i,M} \cdot n \varphi_i dS^\varepsilon. \tag{4.4}
\]

We can now gain the first part of the corrector (3.4), i.e. we shall estimate each integral on the right-hand side of (4.4) which we denote by \(I_1, I_2, I_3\) and \(I_4\), respectively.

Let \(L := \max \{ L_1, ..., L_N \}\). Using (3.2) in combination with the structural inequality \(\|R_i (u_m)\|^\|L^2(\Omega^\varepsilon)\| \leq L \|u_m\|^\|W^2(\Omega^\varepsilon)\| + \|R_i (0)\|^\|L^2(\Omega^\varepsilon)\|\) for all \(0 \leq m \leq M\), we see that

\[
\left\langle R_i^\varepsilon (u_i^\varepsilon) - \sum_{m=0}^{M-2} \varepsilon^m \bar{R}_i^\varepsilon (u_m), \varphi_i \right\rangle_{L^2(\Omega^\varepsilon)} \leq \varepsilon^{M-1} \left( L \|u_{M-1}\|^\|W^2(\Omega^\varepsilon)\| + \|R_i (0)\|^\|L^2(\Omega^\varepsilon)\| \right) \|\varphi_i\|^\|L^2(\Omega^\varepsilon)\|
\]

\[
+ \varepsilon^M \left( L \|u_m\|^\|W^2(\Omega^\varepsilon)\| + \|\bar{R}_i (0)\|^\|L^2(\Omega^\varepsilon)\| \right) \|\varphi_i\|^\|L^2(\Omega^\varepsilon)\|
\]

\[
\leq C \left( \varepsilon^{M-1} + \varepsilon^M \right) \|\varphi_i\|^\|L^2(\Omega^\varepsilon)\|. \tag{4.5}
\]
The second integral $I_2$ can be bounded above by
\[
\varepsilon^{M-1} \left| \langle A_1 u_i, M + A_2 u_{i,M-1} + \varepsilon A_2 u_{i,M}, \varphi_i \rangle_{L^2(\Omega)} \right| \leq C \varepsilon^{M-1} \| \varphi_i \|_{L^2(\Omega)}.
\] (4.6)

Let $\tilde{K} := 1 + \max \{ \bar{K}_1, ..., \bar{K}_N \}$. For the integral $I_3$, we proceed as the proof of (4.5). We thus claim that
\[
\varepsilon \left| \left\langle a^\varepsilon \left( \sum_{m=0}^{M-2} \varepsilon^m u_{i,m} - u_i^\varepsilon \right) + b^\varepsilon_i \left( F_i(u_i^\varepsilon) - \sum_{m=0}^{M-2} \varepsilon^m F_i(u_{i,m}) \right), \varphi_i \right\rangle_{L^2(\Gamma^\varepsilon)} \right| \leq C \left( \varepsilon^{M-1} + \varepsilon^M \right) \| \varphi_i \|_{L^2(\Omega)}.
\] (4.7)
in which we have used (3.1) and (3.3) together with the Hölder inequality, as well as the trace inequality (cf. [2] Lemma 2.31). On top of that, it yields for the last integral $I_4$ that
\[
\varepsilon^M \left| \int_{\Gamma^\varepsilon} d_i^\varepsilon \nabla_x u_i, M \cdot n \varphi_i dS_i \right| \leq \varepsilon^M \left| \int_{\Gamma^\varepsilon} d_i^\varepsilon \nabla_x u_i, M \cdot n \right|_{L^2(\Gamma^\varepsilon)} \left\| \varphi_i \right\|_{L^2(\Gamma^\varepsilon)}
\leq C \varepsilon^{M-1} \| \varphi_i \|_{L^2(\Omega)}.
\] (4.8)
Combining (4.4)-(4.8), we observe
\[
|\langle \varphi_i, \varphi_i \rangle_{V^\varepsilon}| \leq C \left( \varepsilon^{M-1} + \varepsilon^M \right) \| \varphi_i \|_{L^2(\Omega)} \quad \text{for } \varphi_i \in V^\varepsilon \text{ and } i \in \{ 1, ..., N \},
\] (4.9)
which then leads to $\| \varphi_i \|_{V^\varepsilon} \leq C \varepsilon^{M-1}$ by choosing $\varphi_i = \varphi_i^\varepsilon$ for $i \in \{ 1, ..., N \}$.

It remains to consider the following quantity:
\[
\left\langle \left( 1 - m^\varepsilon \right) \sum_{m=K+1}^{M} \varepsilon^m u_{i,m}, \varphi_i \right\rangle_{V^\varepsilon} \quad \text{for } \varphi_i \in V^\varepsilon \text{ and } i \in \{ 1, ..., N \}.
\]
At this stage, the following estimate is straightforward due to (4.1):
\[
\left\| \left( 1 - m^\varepsilon \right) \sum_{m=K+1}^{M} \varepsilon^m u_{i,m}, \varphi_i \right\|_{V^\varepsilon} \leq C \left\| \nabla \left( 1 - m^\varepsilon \right) \sum_{m=K+1}^{M} \varepsilon^m u_{i,m} \right\|_{L^2(\Omega)} \| \varphi_i \|_{V^\varepsilon}
+ C \left\| \left( 1 - m^\varepsilon \right) \nabla \sum_{m=K+1}^{M} \varepsilon^m u_{i,m} \right\|_{L^2(\Omega)} \| \varphi_i \|_{V^\varepsilon}
\leq C \sum_{m=K+1}^{M} \varepsilon^m \| \nabla \left( 1 - m^\varepsilon \right) \|_{L^2(\Omega)} \| \varphi_i \|_{V^\varepsilon}
+ C \sum_{m=K+1}^{M} \varepsilon^m \| 1 - m^\varepsilon \|_{L^2(\Omega)} \| \varphi_i \|_{V^\varepsilon}
\leq C \sum_{m=K+1}^{M} \left( \varepsilon^{m-\frac{1}{2}} + \varepsilon^{m+\frac{1}{2}} \right) \| \varphi_i \|_{V^\varepsilon} \quad \text{for all } \varphi_i \in V^\varepsilon.
\] (4.10)
Thanks to the triangle inequality, we combine (4.9) and (4.10) and then sum up the resulting estimates to get
\[
\sum_{i=1}^{N} |\langle \Psi_i, \varphi_i \rangle_{V^\varepsilon}| \leq C \left( \varepsilon^{M-1} + \varepsilon^M + \sum_{m=K+1}^{M} \left( \varepsilon^{m-\frac{1}{2}} + \varepsilon^{m+\frac{1}{2}} \right) \right) \| \varphi \|_{V^\varepsilon} \quad \text{for } \varphi \in V^\varepsilon.
\]
By choosing $\varphi = \Psi^\varepsilon$ and then by simplifying both sides of the resulting estimate by $\|\Psi^\varepsilon\|_{\mathcal{V}^\varepsilon}$, we obtain that
\[
\|\Psi^\varepsilon\|_{\mathcal{V}^\varepsilon} \leq C \left( \varepsilon^{M-1} + \varepsilon^M + \sum_{m=K+1}^{M} \left( \varepsilon^{m-\frac{1}{2}} + \varepsilon^{m+\frac{1}{2}} \right) \right).
\]

This completes the proof of Theorem 1.

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References

[1] C. Le Bris, F. Legoll, and A. Lozinski. An MsFEM type approach for perforated domains. *SIAM Multiscale Modeling and Simulation*, 12(3):1046–1077, 2014.

[2] D. Cioranescu and J. Saint Jean Paulin. *Homogenization of Reticulated Structures*. Springer, 1999.

[3] S.R. de Groot and P. Mazur. *Non-equilibrium Thermodynamics*. North-Holland Publishing Company, Amsterdam, 1962.

[4] C. Eck. Homogenization of a phase field model for binary mixtures. *Multiscale Modeling and Simulation*, 3:1–27, 2004.

[5] U. Hornung and W. Jäger. Diffusion, convection, adsorption, and reaction of chemicals in porous media. *Journal of Differential Equations*, 92:199–225, 1991.

[6] T. Hou, X. H. Wu, and Z. Cai. Convergence of a multiscale finite element method for elliptic problems with rapidly oscillating coefficients. *Mathematics of Computation*, 68(227):913–943, 1999.

[7] V.A. Khoa and A. Muntean. Asymptotic analysis of a semi-linear elliptic system in perforated domains: Well-posedness and corrector for the homogenization limit. *Journal of Mathematical Analysis and Applications*, 439:271–295, 2016.

[8] V.A. Khoa and A. Muntean. A note on iterations-based derivations of high-order homogenization correctors for multiscale semi-linear elliptic equations. *Applied Mathematics Letters*, 58:103–109, 2016.

[9] O. Krehel, T. Aiki, and A. Muntean. Homogenization of a thermo-diffusion system with Smoluchowski interactions. *Networks and Heterogeneous Media*, 9(4):739–762, 2014.

[10] O. Krehel, A. Muntean, and P. Knabner. Multiscale modeling of colloidal dynamics in porous media including aggregation and deposition. *Advances in Water Resources*, 86:209–216, 2015.