ON THE ISOMORPHISM PROBLEM FOR NON-MINIMAL
TRANSFORMATIONS WITH DISCRETE SPECTRUM

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Abstract. The article addresses the isomorphism problem for non-minimal
topological dynamical systems with discrete spectrum, giving a solution under
appropriate topological constraints. Moreover, it is shown that trivial systems,
group rotations and their products, up to factors, make up all systems with
discrete spectrum. These results are then translated into corresponding results
for non-ergodic measure-preserving systems with discrete spectrum.

1. Introduction. The isomorphism problem is one of the most important prob-
lems in the theory of dynamical systems, first formulated by von Neumann in [17,
pp. 592–593], his seminal work on the Koopman operator method and on dynamical
systems with “pure point spectrum” (or “discrete spectrum”). Von Neumann, in
particular, asked whether unitary equivalence of the associated Koopman opera-
tors (“spectral isomorphy”) implied the existence of a point isomorphism between
two systems (“point isomorphy”). In [17, Satz IV.5], he showed that two ergodic
measure-preserving dynamical systems with discrete spectrum on standard proba-
bility spaces are point isomorphic if and only if the point spectra of their Koopman
operators coincide, which in turn is equivalent to their spectral equivalence. These
first results on the isomorphism problem considerably shaped the ensuing devel-
opment of ergodic theory. The next step in this direction was the Halmos-von
Neumann article [10] in which the authors gave a more complete solution to the
isomorphism problem by addressing three different aspects:

• Uniqueness: For which class of dynamical systems is a given isomorphism
invariant $\Gamma$ complete, meaning that two systems of the class $(X, \phi)$ and $(Y, \psi)$
are isomorphic if and only if $\Gamma(X, \phi) = \Gamma(Y, \psi)$?

• Representation: What are canonical representatives of isomorphy classes of
dynamical systems?

• Realization: Given an isomorphism invariant $\Gamma$, what is the class of objects
that occur as $\Gamma(X, \phi)$ for some dynamical system $(X, \phi)$?

In addition to the uniqueness theorem from [17] for the isomorphism invariant
given by the point spectrum, the Halmos-von Neumann representation theorem showed
that for each isomorphy class of ergodic dynamical systems with discrete spectrum

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lem, Halmos-von Neumann.
on a standard probability space, there are canonical representatives given by ergodic rotations on compact groups. Moreover, they established that every countable subgroup of \( T \) can be realized as the point spectrum of the Koopman operator corresponding to such a system. Hence, there is, up to point isomorphy, a one-to-one correspondence between countable subgroups of \( T \) and ergodic systems with discrete spectrum on standard probability spaces.

However, these results relied heavily on the assumption of ergodicity which was justified by von Neumann in [17, p. 624] by referring to the possibility of ergodic decomposition. Later, Choksi [3] showed that the notion of spectral equivalence is not suited for the classification of non-ergodic dynamical systems with discrete spectrum and that, therefore, no Hilbert space invariant of the Koopman operator (such as its point spectrum) is complete for such systems. It was only in 1981 that Kwiatkowski [12] gave an affirmative answer to the isomorphism problem for non-ergodic systems with discrete spectrum by using, as proposed by von Neumann, the ergodic decomposition as well as measure-theoretic methods.

However, it appears that the analogue of the Halmos-von Neumann theorem for minimal topological dynamical systems with discrete spectrum has not been extended to non-minimal systems. The purpose of this article is thus twofold: In our main result, Theorem 5.6, we show that such an extension is possible via group bundles, provided a topological constraint is fulfilled. Along the way, we then show how this can be translated into a corresponding result for measure-preserving dynamical systems. These results are different in nature from Kwiatkowski’s which were formulated in the language of measure theory. We, however, will construct topological models via operator theoretic means in order to establish a relation between the respective answers to the isomorphism problem in topological dynamics and in ergodic theory. This provides, in particular, a new representation result for (non-ergodic) measure-preserving systems with discrete spectrum, see Theorem 4.10 and Remark 4.12.

The article is organized as follows: In Section 2, we fix our notation, recall basic results about systems with discrete spectrum and give key examples for the following discussion. In Section 3, we give the notion of a bundle of topological dynamical systems as well as group rotation bundles and several tools needed for later sections. The solution of the isomorphism problem is then broken down into the representation theorem in Section 4 and the uniqueness and realization results in Section 5. Our general philosophy was inspired by [9], i.e., we show how the uniqueness and realization results follow from the representation theorem.

2. Preliminaries. Our notation and terminology are generally that of [5]. We assume all compact spaces to be Hausdorff and if \( K \) is a compact space and \( \phi: K \to K \) is continuous, we call \((K, \phi)\) a topological dynamical system. For such a system, denote its Koopman operator \( T_\phi: C(K) \to C(K) \) by \( T_\phi f := f \circ \phi \) for \( f \in C(K) \). We denote the space of regular Borel measures on \( K \) by \( M(K) \) and identify it with the dual of \( C(K) \) via the Riesz-Markov-Kakutani representation theorem. We also let \( M_\phi(K) := \{ \mu \in M(K) \mid \phi^* \mu = \mu \} \) denote the subspace of \( \phi \)-invariant measures and \( M_\phi^1(K) := \{ \mu \in M_\phi(K) \mid \mu \geq 0, \langle 1, \mu \rangle = 1 \} \) denote the subspace of \( \phi \)-invariant probability measures.

We abbreviate a probability space \((X, \Sigma, \mu)\) by \( X := (X, \Sigma, \mu) \) and if \( \phi: X \to X \) is measurable and measure-preserving, we call \((X, \phi)\) a measure-preserving dynamical system. For such a system, we again define the Koopman operator \( T_\phi: L^p(X) \to L^p(X) \) for \( 1 \leq p \leq \infty \).
L^p(X), 1 ≤ p ≤ ∞, via T_φ f := f ◦ φ for f ∈ L^p(X). With this definition, T_φ is a Markov embedding, i.e., T_φ|f = |T_φ f| for all f ∈ L^p(X), T_φ 1 = 1, and T_φ^* 1 = 1 where T_φ^* denotes the adjoint operator of T_φ. Two measure-preserving dynamical systems (X, φ) and (Y, ψ) are point isomorphic if there exists an essentially invertible, measurable, measure-preserving map θ: X → Y such that θ ◦ φ = ψ ◦ θ. They are Markov isomorphic if there is an invertible Markov embedding S: L^1(Y) → L^1(X) such that T_φ S = ST_ψ. If S is merely a Markov embedding, we call (Y, ψ) a Markov factor of (X, φ). If X and Y are standard probability spaces, these notions of isomorphism and factors coincide by von Neumann’s theorem [5, Theorem 7.20].

If G is a compact topological group and a ∈ G we define φ_a: G → G, φ_a(g) := ag and call the dynamical system (G, φ_a) the group rotation with a. We may also abbreviate (G, φ_a) by writing (G, a). Since the Haar measure m on G is invariant under rotation, the rotation can also be considered as a measure-preserving dynamical system (G, m; a).

Remark 2.1. For a measure-preserving dynamical system, we usually consider the associated Koopman operator on the L^1-space instead of the L^2-space, following the philosophy advocated in [5]: When only using the Banach lattice structure of the L^p-spaces, it is natural to work on the biggest of them, the L^1-space, unless Hilbert space methods are also explicitly needed. Standard interpolation arguments show that this choice is justified. And indeed, this article will not require any Hilbert space structure.

If T is a linear operator on a vector space E, we denote by

\[ A_n[T] := \frac{1}{n} \sum_{k=0}^{n-1} T^k \]

its nth Cesàro mean and drop T from the notation if there is no room for ambiguity. Furthermore, we call fix(T) := \{x ∈ E \mid Tx = x\} the fixed space of T. If F ⊂ E is a T-invariant subspace, we set fix_F(T) := fix(T|_F). If (K, φ) is a topological dynamical system, the fixed space fix(T_φ) of its Koopman operator is a C*-subalgebra of C(K). Similarly, if (X, φ) is a measure-preserving dynamical system, fix_{L^1(X)}(T_φ) is a C*-subalgebra of L^∞(X). By the Gelfand representation theorem (cf. [16, Theorem 1.4.4]) there is a compact space L such that fix_{L^1(X)}(T_φ) ∼= C(L). The space L is necessarily extremally disconnected: Since fix_{L^1(X)}(T_φ) is a closed sublattice of L^1(X), the representation theorem for AL-spaces (see [15, Theorem II.8.5]) shows that there is a compact space M and a Borel probability measure μ_M on M such that

\[ C(L) \cong fix_{L^1(X)}(T_φ) \cong L^∞(M, μ_M). \]

But by [15, Theorem II.9.3], C(L) is isomorphic to a dual Banach lattice if and only if L is hyperstonean. In particular, L is extremally disconnected. This will be crucial for Theorem 4.10.

2.1. Operators with discrete spectrum. We start with a power-bounded operator T on a Banach space E, i.e., an operator satisfying sup_n∈N \|T^n\| < ∞, and briefly recall the definition of discrete spectrum and the Jacobs semigroup generated by T. This semigroup was first considered by Jacobs in [11, Definition III.1].
Definition 2.2. Let $E$ be a Banach space and $T \in \mathcal{L}(E)$ a power-bounded operator on $E$.

(i) The operator $T$ has discrete spectrum if its Kronecker space given by the closed linear span

$$\text{Kro}(T) := \varinjlim \bigcup_{|\lambda|=1} \ker (\lambda I - T)$$

is all of $E$.

(ii) The Jacobs semigroup generated by $T$ is

$$J(T) := \{T^n \mid n \in \mathbb{N}\}^{\text{wot}},$$

where the closure is taken with respect to the weak operator topology and the semigroup operation is the composition of operators.

The following characterization of an operator having discrete spectrum can be found in [5, Theorem 16.36].

Theorem 2.3. The following assertions are equivalent.

(i) $T$ has discrete spectrum.

(ii) $J(T)$ is a weakly/strongly compact group of invertible operators.

(iii) The orbit $\{T^n x \mid n \geq 0\}$ is relatively compact and $\inf_{n \geq 0} \|T^n x\| > 0$ for all $0 \neq x \in E$.

Remark 2.4. If $T$ has discrete spectrum, it is mean ergodic and $J(T)$ is a compact abelian group on which the weak and strong operator topology coincide. It is metrizable if $E$ is.

2.2. Systems with discrete spectrum. Next, we consider Koopman operators corresponding to dynamical systems. See [5, Chapters 4, 7] for general information.

Definition 2.5. We say that a measure-preserving dynamical system $(X, \phi)$ has discrete spectrum if its Koopman operator $T_\phi$ has discrete spectrum on $L^1(X)$. Similarly, we say that a topological dynamical system $(K, \phi)$ has discrete spectrum if $T_\phi$ has discrete spectrum as an operator on $C(K)$.

Example 2.6. If $B$ is a compact space, the trivial dynamical system $(B, \text{id}_B)$ has discrete spectrum. Also, if $G$ is a compact group and $a \in G$, the measure-preserving dynamical system $(G, m; a)$ has discrete spectrum and so does the topological dynamical system $(G, a)$. As we will see in Corollary 4.7 and Corollary 4.11, trivial systems and group rotations are, up to factors, the building blocks of all transformations with discrete spectrum.

If $(K, \phi)$ is a topological dynamical system and $T_\phi \in \mathcal{L}(C(K))$ has discrete spectrum, the Jacobs semigroup $J(T_\phi)$ is related to the Ellis semigroup $E(K, \phi) \subset K^K$ defined as $E(K, \phi) := \{\phi^n \mid n \in \mathbb{N}\}$, see [5, Section 19.3]. The following well-known result establishes this connection and gives a topological characterization of the operator theoretic notion of discrete spectrum.

Proposition 2.7. Let $(K, \phi)$ be a topological dynamical system. For the Koopman operator $T_\phi$, the following assertions are equivalent.

(i) $T_\phi$ has discrete spectrum.

(ii) $J(T_\phi)$ is an (abelian) group of Koopman operators.

(iii) $E(K, \phi)$ is an (abelian) group of equicontinuous transformations on $K$. 


(iv) \((K, \phi)\) is equicontinuous and invertible.
Moreover, if these conditions are fulfilled, the map
\[
\Phi: J(T_\theta) \to E(K, \phi), \quad T_\theta \mapsto \theta
\]
is an isomorphism of compact topological groups.

The equivalence of (i) and (ii) follows from Theorem 2.3 and [5, Theorem 4.13].
The equivalence of (ii) and (iii) follows via the canonical isomorphism \(\theta \mapsto T_\theta\),
and for the equivalence of (iii) and (iv) see [7, Proposition 2.5].

3. Bundles of dynamical systems. Bundles, e.g. in differential geometry or
algebraic topology, allow to decompose an object into smaller objects such that the
small parts fit together in a structured way. This perspective is important when
dealing with dynamical systems which are not “irreducible”, i.e., not minimal or
ergodic. We therefore start by studying bundles of topological dynamical systems.

Definition 3.1. A triple \((K, B, p)\) is called a bundle if \(K\) and \(B\) are topological
spaces, \(B\) is compact and \(p: K \to B\) is a continuous surjection. The subsets
\(K_b := p^{-1}(b)\) are called the fibers of the bundle and if \(f: K \to S\) is a function into a set \(S\),
we denote by \(f_b\) its restriction to \(K_b\). A bundle \((K, B, p)\) is called compact
if \(K\) is compact. A tuple \((K, B, p; \phi)\) is called a compact bundle of topological dynamical
systems if \((K, B, p)\) is a compact bundle and \((K, \phi)\) is a topological dynamical
system such that each fiber \(K_b\) is \(\phi\)-invariant.

Remark 3.2. For a dynamical system \((K, \phi)\) and a compact space \(B\), a tuple
\((K, B, p; \phi)\) is a bundle of dynamical systems if and only if \(p\) is a factor map from
\((K, \phi)\) to \((B, \text{id}_B)\).

Example 3.3. (1) Let \((K, \phi)\) be a topological dynamical system, \(B\) a singleton
and \(p: K \to B\) the unique map from \(K\) to \(B\). Then \((K, B, p; \phi)\) is a compact
bundle of dynamical systems. If \((K, \phi)\) is minimal, this is the only possible
choice of \(B\). However, the converse is not true as the system \(([0, 1], x \mapsto x^2)\)
demonstrates.
(2) Let \(B = [0, 1]\), \(K = \mathbb{T} \times B\), \(\alpha: B \to \mathbb{T}\) be continuous and \(\phi_\alpha: K \to K\), \((z, t) \mapsto (\alpha(t)z, t)\) be the associated rotation on the cylinder \(K\). Then \((K, B, p_B; \phi_\alpha)\) is
a compact bundle of topological dynamical systems. If \(\alpha \equiv a\) for some \(a \in \mathbb{T}\),
the system \((K, \phi)\) is just the product of the torus rotation \((\mathbb{T}, \phi_a)\) and the
trivial system \((B, \text{id}_B)\).
(3) More generally, let \((M, \psi)\) be a topological dynamical system and \(B\) be a
compact space. Then the product system \((M \times B, \psi \times \text{id}_B)\) can be viewed as
the bundle \((M \times B, B; \psi \times \text{id}_B)\). Bundles of this form are called trivial.

Definition 3.4. A bundle morphism of bundles \((K_1, B_1, p_1)\) and \((K_2, B_2, p_2)\) is a
pair \((\Theta, \theta)\) consisting of continuous functions \(\Theta: K_1 \to K_2\) and \(\theta: B_1 \to B_2\) such that the following diagram commutes:

\[
\begin{array}{ccc}
K_1 & \xrightarrow{\Theta} & K_2 \\
\downarrow{p_1} & & \downarrow{p_2} \\
B_1 & \xrightarrow{\theta} & B_2
\end{array}
\]

A morphism of compact bundles of topological dynamical systems \((K_1, B_1, p_1; \phi_1)\)
and \((K_2, B_2, p_2; \phi_2)\) is a morphism \((\Theta, \theta)\) of the corresponding bundles such that
Θ is, in addition, a morphism of topological dynamical systems. If Θ and θ are homeomorphisms, we call \((Θ, θ)\) an isomorphism.

3.1. Sections. An important tool for capturing structure in bundles relative to the base space are sections. We recall the following definition.

**Definition 3.5.** Let \((K, B, p)\) be a bundle. A function \(s: B \to K\) is called a section of \((K, B, p)\) if \(s(b) \in K_b\) for each \(b \in B\).

Although the existence of sections is guaranteed by the axiom of choice, there may not exist continuous sections in general.

**Example 3.6.** Take \(K = B = \mathbb{T}\) and define \(p: K \to B\) by \(p(z) = z^2\). Then this bundle has no continuous section because sections must be injective and an injective, continuous function \(s: \mathbb{T} \to \mathbb{T}\) is necessarily surjective and hence a homeomorphism. Clearly, \(p\) can only have a bijective section if it is itself a bijection.

Under additional topological conditions, there are positive results. The first result, due to Michael [13, Corollary 1.4], involves zero-dimensional spaces. Since every totally disconnected compact Hausdorff space is zero-dimensional (see [1, Proposition 3.1.7]) we only state the following special case.

**Proposition 3.7.** Let \((K, B, p)\) be a compact bundle such that \(K\) is metric, \(B\) is totally disconnected and \(p\) is open. Then there exists a continuous section.

For not necessarily metric \(K\), [8, Theorem 2.5] yields the following. Recall that a topological space is called extremally disconnected if the closure of every open set is open.

**Theorem 3.8.** Let \((K, B, p)\) be a compact bundle such that \(B\) is extremally disconnected. Then there exists a continuous section.

In case there is no continuous section, the so-called pullback construction allows to construct bigger bundles admitting sections. We repeat this standard construction from category theory in our setting since it will play an important role.

**Construction 3.9** (Pullback bundle). Let \((K, B, p; φ)\) be a bundle of topological dynamical systems, \(M\) a compact space and \(r: M \to B\) continuous. We then define \(r^* K := \{(m, k) \in M \times K \mid r(m) = p(k)\}\) and denote the restriction of the canonical projection \(p_M: M \times K \to M\) to \(r^* K\) by \(π_M\) and the restriction of \(id_M \times φ\) to \(r^* K\) by \(r^* φ\). Then \((r^* K, M, π_M; r^* φ)\) is a bundle of topological dynamical systems called the pullback bundle of \((K, B, p; φ)\) under \(r\) and \((K, φ)\) is a factor of \((r^* K, r^* φ)\) with respect to the projection \(π_K\) onto the second component. We obtain the following commutative diagram of dynamical systems:

\[
\begin{array}{ccc}
(r^* K, r^* φ) & \xrightarrow{π_K} & (K, φ) \\
\downarrow{π_M} & & \downarrow{p} \\
(M, id_M) & \xrightarrow{r} & (B, id_B)
\end{array}
\]

**Example 3.10.** Let \(K = B = \mathbb{T}\), define \(φ: K \to K\) by setting \(φ(z) = -z\) and \(p: K \to B\) as \(p(z) = z^2\). Set \(r: [0, 1] \to \mathbb{T}, t \mapsto e^{2πit}\). Then the pullback bundle with respect to \(r\) is isomorphic to the bundle \([0, 1] \times \{-1, 1\}; [0, 1], p; (t, x) \mapsto (t, -x)\).
Remark 3.11. Given a bundle $(K, B, p; \phi)$, the pullback bundle $(p^* K, K, \pi; p^* \phi)$ admits a continuous section: This pullback bundle is constructed by gluing to each point in $K$ its fiber and so the map $s : K \to p^* K$, $k \mapsto (k, k)$ is a canonical continuous section. In particular, every bundle of topological dynamical systems is a factor of a bundle admitting a continuous section. Moreover, properties such as minimality and unique ergodicity of each fiber as well as global properties such as equicontinuity, invertibility and mean ergodicity are preserved under forming pullback bundles.

3.2. Maximal trivial factor and mean ergodicity. The following proposition shows that, up to isomorphism, there is a one-to-one correspondence between unital $C^\ast$-subalgebras of $\text{fix}(T_\phi)$ and trivial factors $(B, \text{id}_B)$ of the system $(K, \phi)$.

Proposition 3.12. Let $(K, \phi)$ be a topological dynamical system and $A$ a unital $C^\ast$-subalgebra $A$ of $\text{fix}(T_\phi)$. Then there is an (up to isomorphism) unique bundle $(K, B, p; \phi)$ such that $T_p(C(B)) = A$ where $T_p$ denotes the Koopman operator of $p$.

Proof. Let $A$ be a unital $C^\ast$-subalgebra of $\text{fix}(T_\phi)$. By the Gelfand-Naimark theorem, there is a compact space $B$ such that $A \cong C(B)$. The induced $C^\ast$-embedding $C(B) \to C(K)$ is given by a Koopman operator $T_p$ for a continuous map $p : K \to B$. Because $T_p$ is injective, $p$ is surjective. Moreover, one obtains from the commutativity of the two diagrams

$$
\begin{array}{ccc}
C(K) & \xrightarrow{T_\phi} & C(K) \\
\downarrow T_p & & \downarrow T_p \\
C(B) & \xrightarrow{T_b} & C(B)
\end{array}
\quad
\begin{array}{ccc}
K & \xrightarrow{\phi} & K \\
\downarrow p & & \downarrow p \\
B & \xrightarrow{\text{id}_B} & B
\end{array}
$$

that $\phi(K_b) \subset K_b$, so $(K, B, p; \phi)$ is indeed a bundle of topological dynamical systems such that $T_p(C(B)) = A$.

Now take two such bundles $(K, B, p; \phi)$ and $(K, B', p'; \phi)$ of dynamical systems. Then $C(B) \cong A \cong C(B')$ and this isomorphism is again given by a Koopman operator $T_p : C(B) \to C(B')$ corresponding to a homeomorphism $\theta : B' \to B$. This yields that $(\text{id}, \theta)$ is an isomorphism between the two bundles. \qed

Remark 3.13. Proposition 3.12 allows to order the bundles corresponding to a system $(K, \phi)$ by saying that $(K, B_1, p_1; \phi)$ is finer than $(K, B_2, p_2; \phi)$ if $T_{p_1}(C(B_1)) \supset T_{p_2}(C(B_2))$. The term finer is used here because the above inclusion induces a surjective map $r : B_1 \to B_2$. In light of Proposition 3.12, there is (up to isomorphy) a maximal trivial factor of $(K, \phi)$ associated to the fixed space $\text{fix}(T_\phi)$. We denote this factor by $(L_\phi, \text{id}_{L_\phi})$ with the corresponding factor map $q_\phi : K \to L_\phi$, but omit $\phi$ from the notation if the context leaves no room for ambiguity.

The maximal trivial factor allows to decompose a topological dynamical system into closed, invariant subsets in a canonical way and will be important throughout. As a first illustration of its use, we characterize mean ergodicity, showing that the global notion of mean ergodicity is in fact equivalent to a notion of fiberwise unique ergodicity. The elegant proof of the implication $\text{(b)} \implies \text{(a)}$ was proposed by M. Haase in personal communication and is now presented here in favor of the original proof.
Theorem 3.14. Let \((K, \phi)\) be a topological dynamical system and \(q: K \rightarrow L\) the projection onto its maximal trivial factor. Then the following assertions are equivalent.

(a) The Koopman operator \(T_\phi\) is mean ergodic on \(C(K)\).
(b) Each fiber \((K_l, \phi_l)\) is uniquely ergodic.
(c) For each \(l \in L\) and \(f \in C(K)\) there is a \(c_l \in \mathbb{C}\) such that \(A_n f(x) \rightarrow c_l\) for all \(x \in K_l\).
(d) The map \(M_\phi(K) \rightarrow M(L), \mu \mapsto T_q^* \mu = q_* \mu\) is an isomorphism.

Proof. Suppose that \(T_\phi\) is mean ergodic. For each \(f \in \text{fix}(T_\phi)\) there is a function \(\tilde{f} \in C(K)\) such that \(\tilde{f}|_{K_l} = f\). If we denote the mean ergodic projection of \(T_\phi\) by \(P\), then \(P\tilde{f} \in \text{fix}(T_\phi) = T_\phi(C(L))\) and hence \(P\tilde{f}\) is constant on each fiber. Therefore, \(f = P\tilde{f}|_{K_l}\) is constant and \(\text{fix}(T_\phi)\) is one-dimensional. Thus, \(T_\phi\) is mean ergodic since the Cesàro averages converge uniformly on \(K\) and in particular on \(K_l\). Hence, each fiber \((K_l, \phi_l)\) is uniquely ergodic.

Now assume that each fiber \((K_l, \phi_l)\) is uniquely ergodic and let \(\mu_l\) denote the corresponding unique invariant probability measure. Using this and Lemma 3.15 below, we obtain that the graph of the map \(l \mapsto \mu_l\) is closed. Since this map takes values in the compact set \(M_\phi(K)\), the closed graph theorem for compact spaces (see [4, Theorem XI.2.7]) yields that the map \(l \mapsto \mu_l\) is weak*-continuous. Since each fiber is uniquely ergodic, we also have

\[
\lim_{n \to \infty} A_n f(x) = \int_K f \, \text{d}\mu_q(x) = \langle f, \mu_q(x) \rangle
\]

and this depends continuously on \(x\), showing that \(T_\phi\) is mean ergodic. The equivalence of (b) and (c) is well-known for each fiber. Assertion (d) implies that \(\text{fix}(T_\phi)\) separates \(\text{fix}(T_\phi)\) and hence that \(T_\phi\) is mean ergodic. Conversely, if \(T_\phi\) is mean ergodic, a short calculation shows that the inverse of the map in (d) is given by \(\nu \mapsto \int_L \mu_l \, \text{d}\nu\) where \(\mu_l\) is the unique \(\phi\)-invariant probability measure on \(K_l\).

Lemma 3.15. Let \((K, \phi)\) be a topological dynamical system, \(q: K \rightarrow L\) the projection onto its maximal trivial factor, and \(\mu \in M(K)\) a probability measure. Then \(\text{supp}(\mu) \subset K_1\) if and only if \(T_q^* \mu = \delta_1\).

Proof. Assume that \(\text{supp}(\mu) \subset K_1\). If \(g \in C(L)\) satisfies \(g(l) = 0\), then \(T_q g\) is zero on \(K_1\) and hence on \(\text{supp}(\mu)\), meaning that

\[
\langle g, T_q^* \mu \rangle = \langle T_q g, \mu \rangle = 0.
\]

So \(\text{supp}(T_q^* \mu) \subset \{l\}\) and since \(T_q^* \mu\) is a probability measure, we conclude that \(T_q^* \mu = \delta_1\).

Now suppose \(T_q^* \mu = \delta_1\). If \(f \in C(K)\) is positive and such that \(\|f\| \leq 1\) and \(\text{supp}(f) \cap K_l = \emptyset\), then \(l \notin q(\text{supp}(f))\) and by Urysohn’s lemma there is a function \(g \in C(L)\) equal to 1 on \(q(\text{supp}(f))\) satisfying \(g(l) = 0\). But then \(f \leq T_q g\) and hence

\[
0 \leq \langle f, \mu \rangle = \langle T_q g, \mu \rangle = \langle g, T_q^* \mu \rangle = \langle g, \delta_1 \rangle = 0.
\]

So \(\langle f, \mu \rangle = 0\) and we conclude that \(\text{supp}(\mu) \subset K_1\).

Remark 3.16. In the proof of the implication \((b) \implies (a)\), the fact that we considered fibers with respect to the maximal trivial factor \(L\) was not used. Indeed, let \(B\) be any trivial factor such that the corresponding fibers are uniquely ergodic. The existence of a continuous surjection \(r: L \rightarrow B\) from Remark 3.13 then shows
that each fiber \(K_b\) is contained in a fiber \(K_l\). But since each fiber \((K_1, \phi_l)\) is also uniquely ergodic by Theorem 3.14, it cannot contain more than one of the sets \(K_b\) and so \(r\) has to be a homeomorphism. Therefore, any bundle of topological dynamical systems \((K, B, p; \phi)\) with uniquely ergodic fibers is automatically isomorphic to the bundle \((K, L, q; \phi)\) and we may hence assume that \(B = L\) and \(p = q\).

3.3. **Group bundles.** We now introduce the main object of this article: bundles of topological dynamical systems for which each fiber is a group rotation.

**Definition 3.17.** A bundle \((G, B, p)\) is called a **group bundle** if \((G, B, p)\) is a bundle and each fiber \(G_b\) carries a group structure such that

(i) the multiplication
\[
\{(g, g') \in G \times G \mid p(g) = p(g')\} \to G, \quad (g, g') \mapsto gg'
\]
is continuous,

(ii) the inversion \(G \to G, \ g \mapsto g^{-1}\) is continuous, and

(iii) the neutral element \(e_b \in G_b\) depends continuously on \(b \in B\).

A bundle \((G, B, p; \phi)\) of topological dynamical systems is called a **group rotation bundle** if \((G, B, p)\) is a group bundle, \(\phi : G \to G\) is continuous and

(iv) there is a continuous section \(\alpha : B \to G\) with \((G_b, \phi_b) = (G_b, \phi_{\alpha(b)})\).

A morphism \((\Theta, \theta) : (G_1, B_1, p_1) \to (G_2, B_2, p_2)\) of group bundles is a **bundle morphism** such that \(\Theta\) is a group homomorphism restricted to each fiber. It is called a **morphism of group rotation bundles** if, in addition, \(\Theta\) is a morphism of the corresponding dynamical systems. A group bundle \((G, B, p)\) is called **trivial** if there is a group \(G\) such that \((G, B, p) = (G \times B, B, \pi_B)\). We call \((G, B, p)\) **trivializable** if there is an isomorphism
\[
\iota : (G, B, p) \to (G \times B, B, \pi_B).
\]
We call it **subtrivializable** and \(\iota\) a **(G-)subtrivialization** if \(\iota\) is merely an embedding.

We say that two subtrivializations
\[
\iota_1 : (G_1, B_1, p_1) \to (G \times B_1, B_1, \pi_{B_1})
\]
\[
\iota_2 : (G_2, B_2, p_2) \to (G \times B_2, B_2, \pi_{B_2})
\]
are **isomorphic** if there is an isomorphism \((\Theta, \theta) : (G_1, B_1, p_1) \to (G_2, B_2, p_2)\) such that the diagram
\[
\begin{array}{ccc}
G \times B_1 & \xrightarrow{(g, b) \mapsto (g, \theta(b))} & G \times B_2 \\
\iota_1 \downarrow & & \downarrow \iota_2 \\
G_1 & \xrightarrow{\Theta} & G_2
\end{array}
\]
commutes.

**Example 3.18.** As an example of a bundle of topological dynamical systems for which each fiber is a group rotation, yet no continuous section \(\alpha : B \to K\) exists, recall the bundle from Example 3.6 and equip it with the dynamic \(\phi : K \to K, \ z \mapsto -z\). The fibers here may be interpreted as copies of \((Z_2, n \mapsto n + 1)\) and it was seen in Example 3.6 that this bundle does not admit any continuous sections.

**Remark 3.19.** Products and pullbacks of group rotation bundles canonically are again group rotation bundles. However, when passing to factors, the existence of continuous sections may be lost, as seen in Example 3.10. If, however, such a factor
(G', B', p'; ϕ') has a continuous section s: B' → G', it is again a group rotation bundle.

**Remark 3.20.** The notion of group bundles is not new: It has been considered as a special case of locally compact groupoids in, e.g., [14, Chapter 1].

In order to decompose systems with discrete spectrum, we single out group rotation bundles for which each fiber is minimal. Recall the following characterization of minimal group rotations.

**Remark 3.21.** Let (G, B, p; ϕ) be a group rotation bundle such that each fiber is minimal. Then by [5, Theorem 10.13], every fiber is uniquely ergodic, the unique ϕ-invariant probability measure being the Haar measure m_b on the group G_b. Remark 3.16 yields that we may therefore assume that B = L and p = q where q: G → L is the projection onto the maximal trivial factor. Moreover, if m_l denotes the Haar measure on G_l, the map l → m_l is weak*-continuous. If µ is a ϕ-invariant measure on G, we define the pushforward measure ν := q_∗µ on L and disintegrate µ as in the proof of Theorem 3.14 via

$$
\mu = \int_L m_l \, d
$$

This will be important for Theorem 4.10.

The remainder of this section is dedicated to dual group bundles and their properties, which will be somewhat technical but crucial for Section 5 where we generalize Pontryagin duality to bundles.

**Construction 3.22.** Let (G, B, q) be a locally compact group bundle. Set

$$
G^* := \bigcup_{b \in B} (G_b)^*
$$

where (G_b)^* is the dual group of G_b and denote by π_B: G^* → B the canonical projection onto B. Next, let h ∈ C_c(G), F ∈ C(G) and ϵ > 0. Set

$$
N(h, F, ϵ) := \{\chi \in G^* \mid \|\chi h_{π_B(\chi)} - (Fh)_{π_B(\chi)}\|_{∞} < ϵ\}.
$$

The family of these sets forms a subbasis for a topology which we call the topology of compact convergence on G^*.

With this topology, the projection π_B is continuous as can be deduced from the continuity of the neutral element section e: B → G by invoking Urysohn’s lemma and Tietze’s extension theorem to construct appropriate functions h and F. Therefore, (G^*, B, π_B) is a bundle which we call the dual bundle of (G, B, q) and also denote by (G, B, q)^*. If (Θ, θ): (G, L, q) → (H, L, p) is a morphism of group bundles such that θ is bijective, define its dual morphism (Θ^*, θ^−1): (H^*, B', q) → (G^*, B, p) by setting Θ^*: H^* → G^*, χ → (Θ_{π_L(χ)})^∗χ.

For later reference and the convenience of the reader, we list some basic properties of dual bundles. To this end, we recall the following notions.

**Definition 3.23.** Let X and Y be topological spaces. A map F: X → P(Y) to the power set P(Y) of Y is called lower- (resp. upper-) semicontinuous in a point x ∈ X if for every open U ⊂ Y such that F(x) ∩ U ≠ ∅ (resp. F(x) ⊂ U) there exists an open neighborhood V of x such that for all x' ∈ V one has F(x') ∩ U ≠ ∅ (resp. F(x') ⊂ U). A bundle (K, B, p) is called lower- (resp. upper-) semicontinuous if the map b → K_b is lower-semicontinuous in each point b ∈ B and continuous if it is both lower- and upper-semicontinuous.
Remark 3.24. A bundle \((K, B, p)\) is lower-semicontinuous if and only if \(p\) is an open map.

Definition 3.25. Let \((X, B, p)\) and \((Y, B, q)\) be two bundles. Then their sum is defined as \((X \oplus Y, B, \pi_B)\) where
\[
X \oplus Y := \{(x, y) \in X \times Y \mid p(x) = q(y)\}
\]
and \(\pi_B(x, y) := p(x)\).

Proposition 3.26. Let \((\mathcal{G}, B, p)\) and \((\mathcal{H}, B', p')\) be locally compact abelian group bundles and \((\Theta, \theta) : (\mathcal{G}, B, p) \to (\mathcal{H}, B', p')\) a morphism of group bundles such that \(\theta\) is bijective.

(i) The evaluation map \(ev : \mathcal{G}^* \oplus \mathcal{G} \to \mathbb{C}, (\chi, g) \mapsto \chi(g)\) is continuous. In fact, a net \((\chi_i)_{i \in I}\) converges to \(\chi \in \mathcal{G}^*\) if and only if \(\pi_B(\chi_i) \to \pi_B(\chi)\) and for every convergent net \((g_i)_{i \in I}\) with \(p(g_i) = \pi_B(\chi_i)\) and limit \(g \in \mathcal{G}\) we have \(\chi_i(g_i) \to \chi(g)\).

(ii) For \(b \in B\), \(id_{(\mathcal{G}_b)}^* : (\mathcal{G}_b)^* \to (\mathcal{G}^*)_b\) is an isomorphism of locally compact groups. In particular, the notation \(\mathcal{G}_b^*\) is unambiguous.

(iii) If \(G\) is a locally compact group and \(L\) is a compact space, \((G \times L, \pi_L)^* = (G^* \times L, L, \pi_L)\).

(iv) If the bundle \((\mathcal{G}, B, p)\) is lower-semicontinuous, \(\mathcal{G}^*\) is a Hausdorff space.

(v) The dual morphism \(\Theta^*\) is continuous and
\[
(\Theta^*, \theta^{-1}) : (\mathcal{H}^*, B', \pi_{B'}) \to (\mathcal{G}^*, B, \pi_B)
\]
is a morphism of group bundles.

(vi) If \(\Theta\) is proper and surjective, \(\Theta^* : \mathcal{H}^* \to \mathcal{G}^*\) is an embedding.

(vii) If \(G\) is compact and \(\Theta\) surjective, \(\Theta^* : \mathcal{H}^* \to \mathcal{G}^*\) is an embedding.

Proof. The first part of (i) follows from the definition of the topology on \(\mathcal{G}^*\) using local compactness to invoke Urysohn’s lemma and Tietze’s extension theorem which provide appropriate functions \(h\) and \(F\). The second part of (i) is a simple proof by contradiction. For part (ii), it suffices to show that the two sets carry the same topology. This follows from (i) since it shows that the two topologies have the same convergent nets. By the same argument, (iii) follows directly from (i) and so does (v), since it suffices to show that \(\Theta^*\) is continuous. In (iv), we obtain the Hausdorff property from lower-semicontinuity and (i), showing that every convergent net in \(\mathcal{G}^*\) has a unique limit.

For part (vi) (which trivially implies (vii)), note that \(\Theta^*\) is injective because \(\Theta\) is surjective. Let \((\chi_i)_{i \in I}\) be a net in \(\mathcal{H}^*\) such that \(\Phi^*(\chi_i)\) converges to \(\eta \in \mathcal{G}_b^*\). Then \(\eta(g) = \eta(g')\) if \(\Theta(g) = \Theta(g')\) and so \(\eta = \chi \circ \Theta\) for a function \(\chi : H_b \to \mathbb{C}\). It is again multiplicative and continuous because \(H_b\) carries the final topology with respect to \(\Theta_b\), so \(\chi \in H_b^*\). Let \(N(U, h, F, \epsilon)\) be an open neighborhood of \(\chi\). Then \(N(\theta^{-1}(U), h \circ \Theta, F \circ \Theta, \epsilon)\) is an open neighborhood of \(\chi \circ \Theta\) and so \(\chi_i \circ \Theta \in N(\theta^{-1}(U), h \circ \Theta, F \circ \Theta, \epsilon)\) for \(i \geq i_0\), implying \(\chi_i \in N(U, h, F, \epsilon)\) for \(i \geq i_0\), hence \(\chi_i \to \chi\).

4. Representation. The classical examples for systems with discrete spectrum are group rotations \((G, a)\) and trivial systems \((B, id_B)\) as seen in Example 2.6. In Corollary 4.7 we show that, in fact, every system with discrete spectrum is a canonical factor of a product \((G, a) \times (B, id_B)\) and therefore arises from these two basic systems. This is an easy consequence of our Halmos-von Neumann representation theorem for not necessarily minimal or ergodic systems with discrete spectrum, see Theorem 4.6 and Theorem 4.10.
We briefly recall the Halmos-von Neumann theorem for minimal topological systems \((K,\phi)\) and, because the proof of Theorem 4.6 below is based on it, sketch a proof using the Ellis (semi)group \(E(K,\phi) := \{\phi^k \mid k \in \mathbb{N}\} \subset K^K\) introduced by Ellis as the enveloping semigroup, see [6].

**Theorem 4.1.** Let \((K,\phi)\) be a minimal topological dynamical system with discrete spectrum. Then \((K,\phi)\) is isomorphic to a minimal group rotation \((G,\phi_a)\) on an abelian compact group \(G\). More precisely, for each \(x_0 \in K\) there is a unique isomorphism \(\delta_{x_0} : (E(K,\phi),\phi) \to (K,\phi)\) such that \(\delta_{x_0}(\text{id}_K) = x_0\).

**Proof.** Pick a point \(x_0 \in K\) and consider the map

\[
\delta_{x_0} : E(K,\phi) \to K, \quad \psi \mapsto \psi(x_0).
\]

Since \(K\) is minimal, \(\delta_{x_0}\) is injective. Moreover, \(\delta_{x_0}(E(K,\phi))\) is a closed, invariant subset of \(K\) which is not empty and hence \(\delta_{x_0}(E(K,\phi)) = K\). It is not difficult to check that the system \((E(K,\phi),\phi)\) is isomorphic to \((K,\phi)\) via \(\delta_{x_0}\).

Note that the isomorphism in Theorem 4.1 depends on the (non-canonical) choice of \(x_0 \in K\). In order to extend this result to non-minimal systems, we need the following definition.

**Definition 4.2.** Let \((K,B,p;\phi)\) be a bundle of topological dynamical systems. Set

\[
E(K,B,p;\phi) := \bigcup_{b \in B} E(K_b,\phi_b),
\]

\[
\alpha : B \to E(K,B,p;\phi), \quad b \mapsto \phi_b,
\]

\[
\phi_\alpha : E(K,B,p;\phi) \to E(K,B,p;\phi), \quad \psi \mapsto \psi \circ \phi_b
\]

and let \(\pi_B : E(K,B,p;\phi) \to B\) denote the projection onto the second component. We equip \(E(K,B,p;\phi)\) with the final topology induced by the map \(\rho : E(K,\phi) \times B \to E(K,B,p;\phi), (\psi,b) \mapsto \psi_b\) and call \((E(K,B,p;\phi),B,\pi_B;\alpha)\) the Ellis semigroup bundle of \((K,B,p;\phi)\).

We abbreviate the Ellis semigroup bundle by \(E(K,B,p;\phi)\) if the context leaves no room for ambiguity. We also note that it is a group rotation bundle if it is compact and \(E(K,\phi)\) is a group, in which case we call it the Ellis group bundle. We now give a criterion for the space \(E(K,B,p;\phi)\) to be compact.

**Lemma 4.3.** Let \(X\) be a locally compact space, \(Y\) a (Hausdorff) uniform space, \(B\) a compact space and \(F : B \to \mathcal{P}(X)\) a set-valued map. Define an equivalence relation \(\sim_F\) on \(C(X,Y) \times B\) via

\[
(f,b) \sim_F (g,b') \quad \text{if} \quad b = b' \quad \text{and} \quad f|_{F(b)} = g|_{F(b)}
\]

and endow \(C(X,Y)\) with the topology of locally uniform convergence. Moreover, let \(A \subset C(X,Y)\) be a compact subset. If \(F\) is lower-semicontinuous, then the quotient \(A \times B/\sim_F\) is a compact space.

**Proof.** Since the quotient of a compact space by a closed equivalence relation is again compact (cf. [2, Proposition 10.4.8]), it suffices to show that \(\sim_F\) is closed. So let \(((f_i,b_i),(g_i,b_i))_{i \in I}\) be a net in \(\sim_F\) with limit \(((f,b),(g,b)) \in C(X,Y) \times B)^2\). Pick \(x \in F(b)\). Since \(F\) is lower-semicontinuous and \(b_i \to b\), there is a net \((x_i)_{i \in I}\) such that \(x_i \in F(b_i)\) and \(x_i \to x\). But since \((f_i)_{i \in I}\) and \((g_i)_{i \in I}\) converge locally uniformly,

\[
f(x) = \lim_{i \to \infty} f_i(x_i) = \lim_{i \to \infty} g_i(x_i) = g(x).
\]
Since $x \in F(b)$ was arbitrary, it follows that $f|_{F(b)} = g|_{F(b)}$ and so $\sim_{F}$ is closed. 

**Lemma 4.4.** Let $(K, B, p; \phi)$ be a bundle of topological dynamical systems such that $(K, \phi)$ is equicontinuous.

(i) If $p$ is open, the space $E(K, B, p; \phi)$ is compact.

(ii) If each fiber $(K_b, \phi_b)$ is minimal, then $p$ is open.

**Proof.** Part (i) is a special case of Lemma 4.3 with $A = E(K, \phi)$ and $F: B \to \mathcal{P}(K)$ with $F(b) = K_b$ since the topologies of pointwise and uniform convergence coincide on equicontinuous subsets of $C(K, K)$.

For (ii), assume that each fiber $(K_b, \phi_b)$ is minimal. If $U \subset K_b$ is open in $K_b$, then

$$K_b = \bigcup_{k=0}^{\infty} \phi_b^{-k}(U)$$

since $(K_b, \phi_b)$ is minimal (cf. [5, Proposition 3.3]). Therefore, if $U \subset K$ is open, then

$$p^{-1}(p(U)) = \bigcup_{k=0}^{\infty} \phi^{-k}(U)$$

is open in $K$ and so $p(U)$ is open.

**Lemma 4.5.** Let $(K, \phi)$ be a topological dynamical system with discrete spectrum and $q: K \to L$ the canonical projection onto the maximal trivial factor. Then each fiber $(K_l, \phi_l)$ is minimal and has discrete spectrum.

**Proof.** Each fiber $(K_l, \phi_l)$ has discrete spectrum since $E(K_l, \phi_l) = \{ \psi|_{K_l} \mid \psi \in E(K, \phi) \}$, use Proposition 2.7. Moreover, for $x, y \in K_l$ one has $\overline{\text{orb}}(x) = E(K_l, \phi_l)x$ and $\overline{\text{orb}}(y) = E(K_l, \phi_l)y$. Since $E(K_l, \phi_l)$ is a group, we conclude that either $\overline{\text{orb}}(x) = \overline{\text{orb}}(y)$ or $\overline{\text{orb}}(x) \cap \overline{\text{orb}}(y) = \emptyset$. However, by Remark 2.4 the system $(K, \phi)$ is mean ergodic and hence $(K_l, \phi_l)$ is uniquely ergodic by Theorem 3.14. We now conclude from the Krylov-Bogoljubov Theorem that $K_l$ cannot contain two disjoint closed orbits. Consequently, $\overline{\text{orb}}(x) = \overline{\text{orb}}(y)$ for all $x, y \in K_l$ and hence $(K_l, \phi_l)$ is minimal.

**Theorem 4.6.** Let $(K, \phi)$ be a topological dynamical system with discrete spectrum and assume that the canonical projection $q: K \to L$ onto the maximal trivial factor admits a continuous section. Then $(K, L, q; \phi)$ is isomorphic to its Ellis group bundle.

**Proof.** Let $s: L \to K$ be a continuous section for $q$. By Lemma 4.5, every fiber $(K_l, \phi_l)$ is minimal and has discrete spectrum. By Theorem 4.1, we obtain an isomorphism $\Phi_l: (E(K_l, \phi_l), \phi_l) \to (K_l, \phi_l)$ satisfying $\Phi_l(id_{K_l}) = s(l)$. This yields a bijection

$$\Phi: E(K, L, q; \phi) \to K, \quad \psi_l \mapsto \psi_l(s(l)).$$

Because $(K, \phi)$ has discrete spectrum, the map

$$E(K, \phi) \times L \to K, \quad (\psi, l) \mapsto \psi(s(l))$$

is continuous, hence $\Phi$ is continuous and an isomorphism of topological dynamical systems.
Example 3.18 shows that there are systems with discrete spectrum which are not isomorphic to a group rotation bundle. However, the following is still true.

**Corollary 4.7.** Let \((K, \phi)\) be a topological dynamical system with discrete spectrum. Then \((K, \phi)\) is a factor of a trivial group rotation bundle \((G, a) \times (B, \text{id}_B)\) where the group rotation \((G, a)\) is minimal and can be chosen as \((G, a) = (E(K, \phi), \phi)\).

**Proof.** Let \((K, \phi)\) be a topological dynamical system with discrete spectrum and \(q: K \to L\) the projection onto its maximal trivial factor. As noted in Remark 3.11, the associated pullback system \((q^*K, K, \pi_K, q^*\phi)\) also has discrete spectrum. Moreover, its fibers are uniquely ergodic and so Remark 3.16 shows that its maximal trivial factor is homeomorphic to \(K\). This, combined with Remark 3.11 yields that the canonical projection onto its maximal trivial factor admits a continuous section \(s: K \to q^*K\). By Theorem 4.6 we obtain that the bundle \((q^*K, K, \pi_K, q^*\phi)\) is isomorphic to its Ellis group bundle which is, by construction, a factor of the system \((E(K, \phi), \phi)\). We now consider the following maps:

\[
Q: E(K, \phi) \to E(q^*K, q^*\phi), \quad \psi \mapsto q^*\psi,
\]

\[
P: E(q^*K, q^*\phi) \to E(K, \phi), \quad \tilde{\psi} \mapsto p_2 \circ \tilde{\psi} \circ s
\]

where \(Q\) and \(P\) are continuous and satisfy \(Q(\phi^k) = (q^*\phi)^k\) and \(P((q^*\phi)^k) = \phi^k\) for all \(k \in \mathbb{N}\). Since \(\phi\) and \(q^*\phi\) generate their respective Ellis groups, \(P\) and \(Q\) are mutually inverse. Hence,

\[
(E(q^*K, q^*\phi) \times (K, \text{id}_K) \cong (E(K, \phi), \phi) \times (K, \text{id}_K).
\]

\(\square\)

**Remark 4.8.** The group rotation \((E(K, \phi), \phi)\) is the smallest group rotation that can be taken as \((G, a)\) in Corollary 4.7 in the sense that any such group rotation \((G, a)\) admits an epimorphism \(\eta: (G, a) \to (E(K, \phi), \phi)\). This is true because a factor map \(\theta: (G, a) \times (B, \text{id}_B) \to (K, \phi)\) induces a continuous, surjective group homomorphism

\[
E(\theta): E((G, a) \times (B, \text{id}_B)) \to E(K, \phi)
\]

satisfying \(E(\theta)(a \times \text{id}_B) = \phi\) and

\[
(E((G, a) \times (B, \text{id}_B)), a \times \text{id}_B) \cong (E(G, a), \phi_a) \cong (G, a).
\]

**Remark 4.9.** If \((K, \phi)\) has discrete spectrum and the canonical projection \(q: K \to L\) admits a continuous section, the system is already isomorphic to its Ellis group bundle and hence, by definition of the latter, a factor of the system \((E(K, \phi), \phi) \times (L, \text{id}_L)\). In this case, one can take \(B = L\) in Corollary 4.7.

4.1. **The measure-preserving case.** Since the problem of finding continuous sections can be solved for topological models of measure spaces as shown below, we obtain a stronger result for measure-preserving systems. This is a generalization of the Halmos-von Neumann representation theorem to the non-ergodic case. It is proved by constructing a topological model and then applying Theorem 4.6. For background information on topological models, see [5, Chapter 12].
**Theorem 4.10.** Let \((X, \phi)\) be a measure-preserving system with discrete spectrum. Then \((X, \phi)\) is Markov-isomorphic to the rotation on a compact group rotation bundle. More precisely, there is a compact group rotation bundle \((G, B, p; \phi_0)\) with minimal fibers and a \(\phi_0\)-invariant measure \(\mu_G\) on \(G\) such that \((X, \phi)\) and \((G, \mu_G; \phi_0)\) are Markov-isomorphic. Moreover, this group rotation bundle can be chosen such that the canonical map \(j: \text{Kro}_C(G)(T_{\phi_0}) \to \text{Kro}_L(G, \mu_G)(T_{\phi_0})\) of corresponding Kronecker spaces is an isomorphism.

**Proof.** We define

\[
\mathcal{A} := \text{cl}_{L^\infty} \bigcup_{|\lambda| = 1} \ker_{L^\infty}(\lambda I - T_{\phi})
\]

and note that this is a \(T_{\phi}\)-invariant, unital \(C^*\)-subalgebra of \(L^\infty(X)\) being dense in \(L^1(X)\) by [5, Lemma 17.3] since \((X, \phi)\) has discrete spectrum. The Gelfand representation theorem (cf. [16, Theorem I.4.4]) yields that there is a compact space \(K\) and a \(C^*\)-isomorphism \(S: C(K) \to \mathcal{A}\). The Riesz-Markov-Kakutani representation theorem shows that there is a unique Borel probability measure \(\mu_K\) on \(K\) such that

\[
\int_K f \, d\mu_K = \int_X Sf \, d\mu_X \quad \text{for all } f \in C(K).
\]

Moreover, \(T := S^{-1} \circ T_\phi \circ S: C(K) \to C(K)\) defines a \(C^*\)-homomorphism and so (cf. [5, Theorem 4.13]) there is a continuous map \(\psi: K \to K\) such that \(T = T_\psi\). The operator \(S\) is, by construction, an \(L^1\)-isometry and \(S|f| = |Sf|\) for all \(f \in C(K)\) by [5, Theorem 7.23]. Since \(\mathcal{A}\) is dense in \(L^1(X)\), we conclude that \(S\) extends to a Markov embedding \(S: L^1(K, \mu_K) \to L^1(X)\).

The (topological) system \((K, \psi)\) still has discrete spectrum by construction. Let \(L_\psi\) denote the maximal trivial factor of \((K, \psi)\). Then \(C(L_\psi) \cong \text{fix}(T_\psi) \cong \text{fix}_{L^\infty(X)}(T_{\phi})\) and so \(L_\psi\) is extremally disconnected as noted in Section 2. From Theorem 3.8 we therefore conclude that the canonical projection \(q: K \to L_\psi\) has a continuous section. Theorem 4.6 shows that there is an isomorphism \(\theta: (K, \psi) \to (G, \alpha)\) where \((G, \alpha)\) is the rotation on some compact group rotation bundle with minimal fibers. Equipping \((G, \alpha)\) with the push-forward measure \(\mu_G := \theta_* \mu_K\), we obtain that the system \((X, \phi)\) is isomorphic to the system \((G, \mu_G; \alpha)\).

**Corollary 4.11.** Let \((X, \phi)\) be a measure-preserving dynamical system with discrete spectrum and \((L, \nu; \text{id}_L)\) a topological model for \(\text{fix}_{L^\infty(X)}(T_{\phi})\). Then \((X, \phi)\) is a Markov factor of the trivial group rotation bundle \((J(T_{\phi}), m; T_{\phi}) \times (L, \nu; \text{id}_L)\).

**Proof.** This follows from Theorem 4.10 and Remark 4.9.

**Remark 4.12.** It is not difficult to see that if the measure space \(X\) is separable, the group rotation bundle in Theorem 4.10 can be chosen to be metrizable: Going back to the proof of Theorem 4.10, the algebra \(\mathcal{A}\) needs to be replaced by a separable subalgebra \(\mathcal{B}\) which is still dense in \(L^1(X)\). Using that \(T_{\phi}\) is mean ergodic on \(\mathcal{A}\) and that there hence is a projection \(P: \mathcal{A} \to \text{fix}_\mathcal{A}(T_{\phi})\), this can be done in such a way that \(\text{fix}_G(T_{\phi})\) is generated by its characteristic functions. Therefore, its Gelfand representation space is totally disconnected and using Proposition 3.7 instead of Theorem 3.8, one can continue the proof of Theorem 4.10 analogously. Hence, if \(X\) is a standard probability space, one obtains versions of Theorem 4.10 and Corollary 4.11 with point isomorphism and point factors. However, the group rotation bundles involved are not canonical.
Remark 4.13. We can also interpret the Halmos-von Neumann theorem in the following way: If $(X, \phi)$ is an ergodic, measure-preserving system with discrete spectrum, there is a compact, ergodic group rotation $(G, a)$ and a Markov isomorphism $S: L^1(X) \to L^1(G, m)$ such that the diagram

$$
\begin{array}{c}
L^1(X) \xrightarrow{S} L^1(G, m) \\
\downarrow T_\phi \quad \quad \quad \quad \downarrow T_{\phi a} \\
L^1(X) \xrightarrow{S} L^1(G, m)
\end{array}
$$

commutes, i.e., $T_\phi$ acts like an ergodic rotation on scalar-valued functions. If $(X, \phi)$ is not ergodic, we can interpret Corollary 4.11 similarly: There is a compact, ergodic group rotation $(G, a)$, a compact probability space $(L, \nu)$ and a Markov embedding $S: L^1(X) \to L^1(G \times L, m \times \nu)$ such that $T_{\phi a \times id_L} S = S T_\phi$. The rotation $\phi_a$ induces a Koopman operator $T_{\phi a}$ on the vector-valued functions in $L^1(G, m; L^1(L, \nu))$. Using the $\pi$-tensor product, we obtain

$$
L^1(G, m; L^1(L, \nu)) \cong L^1(G, m) \otimes L^1(L, \nu) \cong L^1(G \times L, m \times \nu).
$$

Now, the diagram

$$
\begin{array}{c}
L^1(X; \mathbb{C}) \xrightarrow{C} L^1(G \times L, m \times \nu) \xrightarrow{\cong} L^1(G, m; L^1(L, \nu)) \\
\downarrow T_\phi \quad \quad \quad \quad \downarrow T_{\phi a \times id_L} \quad \quad \downarrow T_{\phi a} \\
L^1(X; \mathbb{C}) \xrightarrow{C} L^1(G \times L, m \times \nu) \xrightarrow{\cong} L^1(G, m; L^1(L, \nu))
\end{array}
$$

also commutes, i.e., $T_\phi$ acts like an ergodic rotation on vector-valued functions. We can interpret the topological Halmos-von Neumann theorem Theorem 4.1 and Corollary 4.7 analogously.

5. Realization and uniqueness. The topological Halmos-von Neumann theorem shows that every minimal dynamical system with discrete spectrum is isomorphic to a minimal group rotation $(G, a)$. Therefore, minimal group rotations can be seen as the canonical representatives of minimal systems with discrete spectrum. Moreover, the Pontryagin duality theorem shows that $(G, a)$ and $(G^{**}, \delta_a)$ are isomorphic which has two consequences: On the one hand, $G^* \cong G^*(a)$ via $\chi \mapsto \chi(a)$ and $G^*(a) = \sigma_p(T_{\phi a})$ where $T_{\phi a}$ denotes the Koopman operator of $\phi_a$, see [5, Propositions 14.22 and 14.24]. In particular, $\sigma_p(T_{\phi a})$ is a subgroup of $\mathbb{T}$ and for the canonical inclusion $\iota: \sigma_p(T_{\phi a}) \hookrightarrow \mathbb{T}$

$$
(G, a) \cong (G^{**}, \delta_a) \cong (G^*(a)^*, \iota) = (\sigma_p(T_{\phi a})^*, \iota)
$$

if $\sigma_p(T_{\phi})$ is endowed with the discrete topology. Therefore, the point spectrum $\sigma_p(T_{\phi a})$ is a complete isomorphism invariant for the minimal group rotation $(G, a)$. Combined with the Halmos-von Neumann theorem, this shows that the point spectrum $\sigma_p(T_{\phi})$ is a complete isomorphism invariant for the entire class of minimal topological dynamical systems $(K, \phi)$ with discrete spectrum. On the other hand, the Pontryagin duality theorem also implies that every subgroup of $\mathbb{T}$ can be realized as $\sigma_p(T_{\phi a})$ for some group rotation $(G, a)$. This completes the picture, showing that minimal systems with discrete spectrum are, up to isomorphism, in one-to-one correspondence with subgroups of $\mathbb{T}$. 
In order to generalize these results to the non-minimal setting, we need to adapt the Pontryagin duality theorem to group rotation bundles using the preparations from Section 3.3. We start with the necessary terminology.

**Construction 5.1 (Dual bundles).** If \((G, L, q; \alpha)\) is a compact group rotation bundle with minimal fibers and discrete spectrum, the map

\[
\rho: E(G, \phi_\alpha) \times L \to G, \quad (\psi, l) \mapsto \psi(e_l)
\]

yields a surjective morphism \((\rho, \text{id}_L)\) of group bundles which induces, by Proposition 3.26, an embedding \(\rho^*: G^* \hookrightarrow E(G, \phi_\alpha)^* \times L\). Since \(E(G, \phi_\alpha)\) is compact, its dual group is discrete and so we also have the embedding

\[
j: E(G, \phi_\alpha)^* \times L \hookrightarrow \mathbb{T} \times L, \quad (\chi, l) \mapsto (\chi(\phi_\alpha), l)
\]

where \(\mathbb{T}\) carries the discrete topology. The composition \(\iota: G^* \to \mathbb{T} \times L\) of these two maps is hence a subtrivialization of \(G^*\) and we call \((G, L, q; \alpha)^* := (G^*, L, \pi_L; \iota)\) the dual bundle of \((G, L, q; \alpha)\). (Note that \(G^*\) is, in general, neither locally compact nor Hausdorff.) If, conversely, \((G, L, q; \iota)\) is a group bundle with a \(T\)-subtrivialization \(\iota\), we set \(\alpha: L \to G^*, l \mapsto \iota_l\) and call \((G, L, q; \iota)^* := (G^*, L, \pi_L; \alpha)\) the dual bundle of \((G, L, q; \iota)\). We say that two group bundles with \(T\)-subtrivializations \((G, L, q; \iota)\) and \((G', L', q'; \iota')\) are isomorphic if their respective subtrivializations are, i.e., if there is an isomorphism \((\Theta, \theta): (G, L, q) \to (G', L', q')\) such that the diagram

\[
\begin{array}{ccc}
\mathbb{T} \times L & \xrightarrow{(z, l) \mapsto (z, \theta(l))} & \mathbb{T} \times L' \\
\downarrow \iota & & \downarrow \iota' \\
G^* & \xrightarrow{\Theta} & G'^*
\end{array}
\]

commutes. If \(L = L' = \text{pt}\), this means that \(\iota\) and \(\iota'\) have the same image.

**Definition 5.2.** Let \((K, \phi)\) be a topological dynamical system and \(q: K \to L\) the projection onto its maximal trivial factor \(L\). Then we define

\[
\Sigma_p(K, \phi) := \bigcup_{l \in L} \sigma_p(T_{\phi_l}) \times \{l\} \subset \mathbb{C} \times L.
\]

We denote the projection onto the second component by \(\pi_L\) and equip \(\Sigma_p(K, \phi)\) with the subspace topology induced by \(\mathbb{C} \times L\) if \(\mathbb{C}\) carries the discrete topology. The bundle \((\Sigma_p(K, \phi), L, \pi_L)\) is then called the point spectrum bundle of \((K, \phi)\). We say that the point spectrum bundles of two systems are isomorphic if there is an isomorphism of their canonical subtrivializations, i.e., if there is a homeomorphism \(\eta: L_\phi \to L_\psi\) such that

\[
H: \Sigma_p(K, \phi) \to \Sigma_p(M, \psi), \quad (z, l) \mapsto (z, \eta(l))
\]

is a (well-defined) homeomorphism. We call \((H, \eta)\) an isomorphism of the point spectrum bundles.

**Remark 5.3.** Let \((G, L, q; \iota)\) be a group bundle with a \(T\)-subtrivialization \(\iota: G \to \mathbb{T} \times L\). Then \(\iota\) induces an isomorphism

\[
(G, L, q; \iota) \cong (\iota(G), L, \pi_L; \text{id}_{\iota(G)})
\]

and hence

\[
(G, L, q; \iota)^* \cong (\iota(G)^*, L, \pi_L; (\text{id}_{\iota(G)})_l l \in L).
\]
In particular, $\mathcal{G}$ and hence its dual are completely determined by $\iota(\mathcal{G})$. Now, if $(\mathcal{G}, L, \pi_L; \iota)$ is the dual of a compact group rotation bundle $(\mathcal{H}, L, p; \alpha)$ with minimal fibers and discrete spectrum, it follows from the introduction to this section that

$$\left(\iota(\mathcal{G})^*, L, \pi_L, (\text{id}_{\iota(\mathcal{G})}) \in L \right) = \left(\Sigma_p(\mathcal{H}, \phi_\alpha)^*, L, \pi_L, (\text{id}_{\Sigma_p(\mathcal{H}, \phi_\alpha)}) \in L \right).$$

So we see that the dual bundle of a group rotation bundle with discrete spectrum and minimal fibers is completely determined by the system’s point spectrum bundle.

**Lemma 5.4.** Let $(K, \phi)$ be a topological dynamical system with discrete spectrum. Then its point spectrum bundle is lower-semicontinuous.

**Proof.** Suppose $(\lambda, l) \in \Sigma_p(T)$ and let $f \in C(K)$ be a corresponding eigenfunction. Since $T\phi$ has discrete spectrum, $T\phi$ is mean ergodic. So as in the proof of Theorem 3.14, $f$ can be extended to a global fixed function $\tilde{f} \in C(K)$ of $\lambda T\phi$. Since the map $q: K \to L$ onto the maximal trivial factor is open by Lemma 4.4, $q([f])$ is open, showing that there is an open set $U \subset L$ such that for each $l \in U$, $f \neq 0$ and $T\phi_l(f) = \lambda f_l$.

**Proposition 5.5.** Let $(\mathcal{G}, L, q; \alpha)$ be a compact group rotation bundle with discrete spectrum and minimal fibers. Then it is isomorphic to its bi-dual bundle.

**Proof.** The following diagram commutes:

$$(E(\mathcal{G}, \phi_\alpha) \times L, \phi_\alpha) \xrightarrow{\delta_{\phi_\alpha}} (E(\mathcal{G}, \phi_\alpha)^* \times L, \delta_{\phi_\alpha})$$

Combining Remark 5.3 and Lemma 5.4, we see that $(\mathcal{G}, L, q)^*$ is lower-continuous. Proposition 3.26 then shows that $\mathcal{G}^*$ embeds into $E(\mathcal{G}, \phi_\alpha) \times L$, is therefore locally compact and so $\mathcal{G}^{**}$ is Hausdorff. Since $\mathcal{G}$ is a surjective, continuous map between compact spaces, $\mathcal{G}$ carries the final topology with respect to $\rho$, which shows that the map $g \mapsto \delta_g$ is continuous and bijective. Since $\mathcal{G}^{**}$ is Hausdorff, this shows that $\mathcal{G} \cong \mathcal{G}^{**}$ and the claim follows.

We can now formulate the answer to the three aspects of the isomorphism problem already discussed in the introduction.

**Theorem 5.6.** Let $(K, \phi)$ and $(M, \psi)$ be topological dynamical systems with discrete spectrum and continuous sections of the canonical projections onto their respective maximal trivial factor.

(a) (Representation) The system $(K, \phi)$ is isomorphic to a compact group rotation bundle with minimal fibers.

(b) (Uniqueness) The systems $(K, \phi)$ and $(M, \psi)$ are isomorphic if and only if their point spectrum bundles are.

(c) (Realization) The point spectrum bundle of $(K, \phi)$ is lower-semicontinuous. Conversely, if $L$ is a compact space, every lower-semicontinuous sub-group bundle of $(\mathbb{T} \times L, L, \pi_L)$ can be realized as the point spectrum bundle of a topological dynamical system with discrete spectrum.
Proof. The representation result is Theorem 4.6. Moreover, Remark 5.3 and Proposition 5.5 show that the point spectrum bundle is a complete isomorphism invariant for compact group rotation bundles with minimal fibers and discrete spectrum and the representation theorem allows to extend this to \((K, \phi)\) and \((M, \psi)\). The last part follows, analogously to the minimal case, from Proposition 3.26(iv), Proposition 5.5 and Remark 5.3.

Remark 5.7. Note that the statement of Theorem 5.6 is false if the assumption of a continuous section is removed. Indeed, one obtains a counterexample from Example 3.18.

We obtain a similar result for measure-preserving systems with discrete spectrum using topological models. This requires the following definition.

Definition 5.8. Let \((X, \phi)\) be a measure-preserving dynamical system and take \((K, \mu_K; \psi)\) to be a topological model corresponding to the algebra \(A := \text{Kro}_{L_\infty} (T_\phi) = \text{cl}_{L_\infty} \bigcup_{|\lambda| = 1} \ker_{L_\infty} (\lambda I - T_\phi) \subset L_\infty(X)\).

Let \((\Sigma_p(X, \phi), L, p)\) be the point spectrum bundle of \((K, \phi)\) and set \(\nu := p_* \mu_K\). We then call \((\Sigma_p(X, \phi), L, p, \nu)\) the point spectrum bundle of \((X, \phi)\). We say that the point spectrum bundles of two systems \((X, \phi)\) and \((Y, \psi)\) are isomorphic if there is an isomorphism \((\Theta, \theta): (\Sigma_p(X, \phi), L, p) \to (\Sigma_p(Y, \psi), L', p')\) such that \(\theta\) is measure-preserving.

Remark 5.9. Let \((K, \phi)\) be a topological dynamical system, \(\mu\) a regular Borel measure on \(K\), \(q: K \to L\) the canonical projection onto the maximal trivial factor of \((K, \phi)\) and \(\nu := q_* \mu\). If the canonical map \(j: \text{Kro}_{C(K)}(T_\phi) \to \text{Kro}_{L_\infty}(K, \mu)(T_\phi)\) is an isomorphism, then \(\Sigma_p(K, \phi) = \Sigma_p(K, \mu; \phi)\). This is in particular the case for the group rotation bundles constructed in Theorem 4.10.

Recall that a regular Borel measure \(\mu\) on a (hyper)stonean space \(K\) is called normal if all rare sets are null-sets. If \(\mu\) is a normal measure on \(K\) with full support, then the canonical embedding \(C(K) \hookrightarrow L_\infty(K, \mu)\) is an isomorphism, cf. [16, Corollary III.1.16]. After this reminder, we can state the analogue of Theorem 5.6 for measure-preserving systems.

Theorem 5.10. Let \((X, \phi)\) and \((Y, \psi)\) be measure-preserving dynamical systems with discrete spectrum.

(a) (Representation) The system \((X, \phi)\) is Markov-isomorphic to a compact group rotation bundle \((G, \mu_G; \phi_\alpha)\) with minimal fibers.

(b) (Uniqueness) The systems \((X, \phi)\) and \((Y, \psi)\) are Markov-isomorphic if and only if their point spectrum bundles are isomorphic. In that case, the systems are also point isomorphic, provided \(X\) and \(Y\) are standard probability spaces.

(c) (Realization) The point spectrum bundle of \((X, \phi)\) is continuous. Conversely, if \((L, \nu)\) is a hyperstonean compact probability space, \(\nu\) is normal, and \(\text{supp} \nu = L\) and \((\Sigma, L, p)\) is a continuous sub-group bundle of \((\mathbb{T} \times L, L, p)\), then \((\Sigma, L, p; \nu)\) can be realized as the point spectrum bundle of a measure-preserving dynamical system with discrete spectrum.

Proof. The representation result was proved in Theorem 4.10. Using it, the uniqueness can be reduced to the case of the special group rotation bundles from Theorem 4.10 and for these, it follows from Remark 5.9. Indeed, let \((G, \mu_G; \phi_\alpha)\) and
(\mathcal{H}, \mu_\mathcal{H}; \phi_\beta) \) be two such rotations and
\((\Theta, \theta): (\Sigma_p(\mathcal{G}, \mu_\mathcal{G}; \phi_\alpha), L, p, \nu) \to (\Sigma_p(\mathcal{H}, \mu_\mathcal{H}; \phi_\beta), M, q, \eta)\)
an isomorphism of their point spectrum bundles. Then Remark 5.9 shows that
\((\Theta, \theta)\) is, in particular, an isomorphism of their topological point spectrum bundles
and thus induces a (topological) isomorphism \((\Theta^*, \theta^{-1})\) of the corresponding dual bundles. By Proposition 5.5, this yields an isomorphism
\[(\Psi, \theta^{-1}): (\mathcal{H}, M, q; \phi_\beta) \to (\mathcal{G}, L, p; \phi_\alpha).\]
Using the disintegration formula from Remark 3.21, one quickly checks that \(\Psi_\ast \mu_\mathcal{H} = \mu_\mathcal{G}\) because \(\theta^{-1}\eta = \nu\).

For part (c), let \((L, \nu)\) be a hyperstonean compact probability space such that \(\nu\) is normal and \(\text{supp} \nu = L\) and let \((\Sigma, L, p)\) be a continuous sub-group bundle of \((T \times L, L, p)\). Let \((\mathcal{G}, L, \pi_L, \phi_\alpha)\) be its dual group rotation bundle endowed with the measure
\[
\mu_\mathcal{G} := \int_L m_l \, d\nu.
\]
To prove that \(\Sigma = \Sigma_p(\mathcal{G}, \mu_\mathcal{G}; \phi_\alpha)\), it suffices to show that each eigenfunction \(f \in L^\infty(\mathcal{G}, \mu_\mathcal{G})\) has a representative \(g \in C(\mathcal{G})\) since then
\[
\Sigma = \Sigma_p(\mathcal{G}, \phi_\alpha) = \Sigma_p(\mathcal{G}, \mu_\mathcal{G}, \phi_\alpha)
\]
by Proposition 5.5 and Remark 5.9. So take \([f] \in L^\infty(\mathcal{G}, \mu_\mathcal{G})\) with \(T_g[f] = \lambda[f]\). Then \(T_{\phi_\alpha}(f) = \lambda[f]\) for \(\nu\)-almost all \(l \in L\). Let \(U_\lambda \subseteq L\) be the open subset of \(l \in L\) such that \((\lambda, l) \in \Sigma\) and note that \(U_\lambda\) is also closed since \((\Sigma, L, p)\) is upper-semicontinuous.

Since \(\mathcal{G}^*\) is isomorphic to the point spectrum bundle \(\Sigma_p(\mathcal{G}, \phi_\alpha)\) via an isomorphism \(\Phi\) by Proposition 5.5, the map \(\eta: U_\lambda \to \mathcal{G}^*\), \(l \mapsto \Phi^{-1}(\lambda, l)\) is continuous. Extend \(\eta\) to all of \(L\) by setting \(\eta(l)\) to the trivial character in \(\mathcal{G}^*_l\) for \(l \in L \setminus U_\lambda\) and note that \(\eta\) is continuous since \(U\) is open and closed.

Now, for \(l \in U_\lambda\), each fiber \((\mathcal{G}_l, \phi_\lambda, l)\) of \((\mathcal{G}, \phi_\alpha)\) is a minimal group rotation and hence the eigenspace of the Koopman operator \(T_{\phi_\alpha, l}\) corresponding to \(\lambda\) is at most one-dimensional and therefore spanned by \(\eta(l) \in \mathcal{G}^*_l\). So for \(\nu\)-almost every \(l \in U_\lambda\), there is a constant \(c_l \in \mathbb{C}\) such that \(f_l = c_l \eta(l)\) \(\mu\)-almost everywhere. If we extend \(c\) to \(L\) by \(0\), \([c] \in L^\infty(L, \nu)\) since \([f] \in L^\infty(\mathcal{G}, \mu)\). But \(C(L) \cong L^\infty(L, \nu)\) via the canonical embedding and so we may assume that \(c\) is continuous. If \(q: \mathcal{G} \to L\) is the projection onto \(L\), using (i) of Proposition 3.26, we see that the function \(\hat{f} : \mathcal{G} \to \mathbb{C}\), \(x \mapsto c_q(x) \eta_q(x)(x)\) is in \(C(\mathcal{G})\), \(f = \hat{f}\) \(\mu\)-almost everywhere, and \(T_{\phi_\alpha} \hat{f} = \lambda \hat{f}\) by construction.

Now let \((X, \phi)\) be a measure-preserving dynamical system with discrete spectrum. In order to show that its point spectrum bundle is upper-semicontinuous, we may switch to its model \((\mathcal{G}, \mu_\mathcal{G}, \phi_\alpha)\) on a compact group rotation bundle \((\mathcal{G}, L, p; \phi_\alpha)\) constructed in Theorem 4.10. Take \(\lambda \in \mathbb{T}\). By Remark 5.9 and Lemma 5.4, the set
\[
U_\lambda := \{l \in L \mid (\lambda, l) \in \Sigma_p(\mathcal{G}, \mu_\mathcal{G}; \phi_\alpha)\}
\]
is open. Via the isomorphism \(\Theta: \Sigma_p(\mathcal{G}, \phi_\alpha) \cong \mathcal{G}^*\), we see that the function \(F: U_\lambda \to \mathcal{G}^*, l \mapsto \Theta(\lambda, l)\) selecting the (unique) character on \(\mathcal{G}_l\) corresponding to the eigenvalue \(\lambda\) is continuous. By (i) of Proposition 3.26, \(F\) defines a continuous function \(f: \mathbb{C}(U_\lambda) \to \mathbb{C}\) and we may extend \(f\) to a measurable function on all of \(\mathcal{G}\) by
0. Then $T_{\phi_{\alpha}}f = \lambda f$ and since the $C(G)$- and $L^\infty(G, \mu_G)$-Kronecker space for $T_{\phi_{\alpha}}$ are canonically isomorphic, we can find a continuous representative $g \in C(G)$ for the class $[f]$. Since $|f| \equiv 1$ a.e. on $p^{-1}(U_{\lambda})$, $|g| \equiv 1$ on $p^{-1}(\overline{U}_{\lambda}) = p^{-1}(U_{\lambda})$, where the last equality holds because $p$ is open by Lemma 4.4. Therefore, $\overline{U}_{\lambda} \subset U_{\lambda}$ and hence $U_{\lambda} = \overline{U}_{\lambda}$. This shows that the point spectrum bundle of $(X, \phi)$ is upper-semicontinuous.

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