QUANTUM WALK ON EXTENSION OF ASSOCIATION SCHEMES

HIROSHI MIKI
Meteorological College, Asahi-Cho, Kashiwa, 277 0852, Japan

SATOSHI TSUJIMOTO
Department of Applied Mathematics and Physics, Graduate School of Informatics, Kyoto University, Sakyo-Ku, Kyoto, 606 8501, Japan

DA ZHAO
Department of Applied Mathematics and Physics, Graduate School of Informatics, Kyoto University, Sakyo-Ku, Kyoto, 606 8501, Japan

Abstract. In this paper, we study quantum walks on the extension of association schemes. Various state transfers can be achieved on these graphs, such as multiple state transfer among extreme points of a simplex, fractional revival on subsimplexes. We also investigate the relation between perfect state transfer and the generation of maximal entanglement. Characterization of zero transfer is given as well, which has not been well-studied in the area of quantum walks.

1. Introduction
Given a graph $\Gamma = (X, E)$, which might be weighted or oriented, the continuous-time quantum walk on this graph is given by $U(t) = \exp(-\text{i}tA)$, where $A$ is a Hermitian matrix associated to the graph $\Gamma$. This notion was raised by Farhi and Guttmann (11) to develop quantum
algorithms. Bose (3) investigated quantum walk on a path for the study of quantum information transmission.

Let $|e_x⟩$ be the characteristic vector for the vertex $x ∈ X$, and let $V = ⊕_{x ∈ X} C|e_x⟩$ be the space on which $U(t)$ acts. Suppose one starts with a state, say $|e_u⟩$, after evolution by time $t$, one reaches

$$U(t)|e_u⟩ = \sum_{x ∈ S} c_x |e_x⟩,$$

where $S$ is a subset of $X$ and $\sum_{x ∈ S} |c_x|^2 = 1$. We say fractional revival (FR) occurs on $S$ at time $t$. Several special cases have been focused in the research.

- **Generation of maximal entanglement (GME).**
  If $S = \{v, w\}$ and $|c_v| = |c_w| = \frac{1}{\sqrt{2}}$ for $x ∈ S$, then the resulting state at time $t$ is a maximally entangled state of $e_v$ and $e_w$.

- **Perfect state transfer (PST).**
  If $S = \{v\}$, in other words there exists a real constant $\gamma$ such that $U(t)|e_u⟩ = e^{i\gamma}|e_v⟩$, then we say that there exists perfect state transfer from $u$ to $v$ at time $t$.

- **Pretty good state transfer (PGST).**
  If there exists a real sequence $\{t_k\}$ and a constant $\gamma ∈ \mathbb{R}$ such that $\lim_{k \to \infty} U(t_k)|e_u⟩ = e^{i\gamma}|e_v⟩$, then we say that there exists pretty good state transfer from $u$ to $v$.

- **Multiple state transfer (MST) and universal state transfer (UST).**
  Let $C$ be a subset of $X$. If for every $u, v ∈ C$, there exists perfect/pretty good state transfer from $u$ to $v$, then we say that there exists multiple perfect state transfer/multiple pretty good state transfer in $C$. In particular, if $C = X$, then we say that there exists universal state transfer in the graph $Γ$.

- **Zero transfer (ZT).**
  If there exists $v ∈ X$ such that for every $t ∈ \mathbb{R}$, $⟨e_v|U(t)|e_u⟩ = 0$, then we say that there is zero transfer between $u$ and $v$. In other words, the state on $u$ can never be seen on $v$ at any time.

After Bose, the study turns to general graphs as perfect state transfer between antipodal points of an unweighted path only occurs when the length is 2 or 3. Christandl et al. (7) showed that perfect state transfer can be achieved between antipodal points of a hypercube of arbitrary dimension, which gives the perfect state transfer between the two ends of a weighted path through projection. As perfect state transfers are hard to obtain, the notation of pretty good state transfer was introduced by Godsil (14). It was shown by Kay (17) that in an unoriented graph, if one fixes the initial state on $u$, then one can only achieve
perfect state transfer between two vertices $u$ and $v$, but not a third vertex. Recently Chaves et al. (6) discussed ‘Why and how to add direction to a quantum walk’. They also studied the phenomenon of zero transfer. The ambitious universal state transfer was constructed by Cameron et al. (4) in oriented graphs, which is impossible for unoriented graphs. Graphs that carry universal perfect state transfer and universal pretty good state transfer are partially constructed and characterized in (4; 8). However, these graphs are dense in the sense that almost every pair of vertices are adjacent. Multiple state transfer, as a relaxation of universal state transfer, was proposed by Godsil and Lato in (15).

Association schemes provide fruitful examples of graphs with good algebraic and combinatorial properties. Quantum walks on these graphs have been studied extensively, (5; 9; 16; 18; 20) to name a few. In this paper, we consider quantum walks on simplexes using the extension of association schemes. Through this approach, we can construct (multiple) perfect state transfers among extreme points of a simplex, where the underlying graph is sparse. Moreover, the distance between extreme points of a simplex can be arbitrarily large. Fractional revival on subsimplexes can be achieved as well, which explains the phenomenon in (18).

The paper is organized as follows. In Section 2, we first recall the characterizations of perfect state transfer, pretty good state transfer, and universal state transfer. Then we provide a characterization of zero transfer and show that perfect state transfer can be modified to generate maximal entanglement. In Section 3, we introduce the definition and basic properties of association schemes. The extension of association schemes is emphasized. In Section 4, we analyze quantum walks on extension of association schemes. In Section 5, we apply the analysis in Section 4 to several base association schemes, which gives various examples of state transfer, such as perfect state transfer among extreme points of a simplex, multiple perfect state transfer among extreme points of a simplex, and fractional revival on subsimplexes of a simplex.

2. State transfer on graphs

In this section, we first recall characterizations of perfect state transfer, pretty good state transfer, and universal state transfer. Emphasis will be put on the distinction between unoriented graphs and oriented graphs. The properties which rely on the rationality of entries of $A$ will be omitted. Then we give a characterization of zero transfer. At
last, a construction is given, which turns perfect state transfer into the
generation of maximal entanglement.

The following characterization of perfect state transfer was stated
for unweighted unoriented graphs in (14). In fact it holds for general
graphs. Note that there is a sign change since in some literature the
definition of quantum walk is given by
\[ U(t) = \exp(-itA) \] instead of
\[ U(t) = \exp(itA) \].

**Theorem 1** ((14, Lemma 2.1)). Let \( A \) be the Hermitian matrix as-
associated to a graph \( \Gamma \). Let \( E_1, E_2, \ldots, E_m \) be the idempotents in the
spectral decomposition \( A = \sum_{r=1}^{m} \lambda_r E_r \), where \( \lambda_1, \lambda_2, \ldots, \lambda_m \) are the
corresponding eigenvalues. Then there exists a perfect state transfer
from \( u \) to \( v \) at time \( \tau \) if and only if there exists a real constant \( \gamma \) such
that
\[ E_r |e_u\rangle = e^{i(\gamma + \tau \lambda_r)} E_r |e_v\rangle, \]
for \( r = 1, 2, \ldots, m \).

**Corollary 2** ((14, Corollary 2.2)). If the graph \( \Gamma \) in Theorem 1 is un-
oriented, in other words, the Hermitian matrix \( A \) is in fact real sym-
metric, then \( E_r |e_u\rangle = \pm E_r |e_v\rangle \) for \( r = 1, 2, \ldots, m \).

Concerning unoriented graphs, perfect state transfer implies periodic-
ity. More specifically, if there is perfect state transfer from \( u \) to \( v \) at time \( \tau \) on an unoriented graph \( \Gamma \), then perfect state transfer occurs
from \( v \) to \( u \) at time \( \tau \) as well. Consequently, there is perfect state transfer from \( u \) to itself at time \( 2\tau \). In such case, one says that the
graph \( \Gamma \) is periodic with respect to vertex \( u \). As one can see from (2)
that as long as \( E_r |e_u\rangle \neq 0 \), the phase factor \( e^{i(\gamma + \tau \lambda_r)} \) is equal to 1 if
the graph is periodic at \( u \). We define the eigenvalue support of \( u \) to be
the set of eigenvalues \( \lambda \) such that \( E_\lambda |e_u\rangle \neq 0 \).

**Theorem 3** ((13, Theorem 2.2)). Let \( A \) be the Hermitian matrix as-
associated to a graph \( \Gamma \). Let \( E_1, E_2, \ldots, E_m \) be the idempotents in the
spectral decomposition \( A = \sum_{r=1}^{m} \lambda_r E_r \), where \( \lambda_1, \lambda_2, \ldots, \lambda_m \) are the
corresponding eigenvalues. Let \( \lambda_k, \lambda_\ell, \lambda_r, \lambda_s \) be eigenvalues in the sup-
port of \( u \) and \( \lambda_r \neq \lambda_s \), then
\[ \frac{\lambda_k - \lambda_\ell}{\lambda_r - \lambda_s} \in \mathbb{Q}. \]

The above theorem was stated for unweighted unoriented graphs in
(13). Indeed it holds for general graphs as well. On the other hand,
the nearly-integral condition on eigenvalues of unweighted unoriented
graphs, namely the eigenvalues are in a field of the form \( \mathbb{Q}(\sqrt{\Delta}) \), no
longer holds for general graphs. This is due to the fact the weights in a graph are not necessarily rational numbers.

Pretty good state transfer is a relaxation of perfect state transfer. In fact, their properties do not differ by much. Before we proceed, an equivalent definition of pretty good state transfer is given, where the real constant $\gamma$ can be discarded.

**Theorem 4.** Let $A$ be the Hermitian matrix associated to a graph $\Gamma$. Let $u$ and $v$ be two vertices of $\Gamma$. Then the followings are equivalent.

1. There exists a real sequence $\{t_k\}$ and a constant $\gamma \in \mathbb{R}$ such that $\lim_{k \to \infty} U(t_k) |e_u\rangle = e^{i\gamma} |e_v\rangle$.
2. There exists a real sequence $\{t_k\}$ such that $\lim_{k \to \infty} |\langle e_u | U(t_k) | e_v\rangle| = 1$.

**Proof.** Suppose 2 holds. Consider the coefficient $c_w(t)$ in the expansion $U(t) |e_u\rangle = \sum_{w \in X} c_w(t) |e_w\rangle$. Since $U(t)$ is unitary, the norm $c_w(t)$ is at most 1, in other words, $c_w(t)$ is contained in the unit disk. Note that the unit disk is compact, hence there exists a subsequence $\{t_{k_i}\}$ such that $\lim_{k \to \infty} c_v(t_{k_i})$ converges to a point $\xi$. Since $1 = \lim_{k \to \infty} |\langle e_v | U(t_{k_i}) | e_u\rangle| = \lim_{k \to \infty} |c_v(t_{k_i})| = |\xi|$, we obtain that $\xi = e^{i\gamma}$ for some real constant $\gamma \in \mathbb{R}$. The other coefficients $c_w(t)$, $w \neq v$ vanishes since $\sum_{w \in X} |c_w(t)|^2 = 1$. Therefore 1 holds.

The proof of the other direction is trivial. \qed

The following theorem was stated for perfect state transfer in oriented graphs (15) as well as pretty good state transfer in unoriented graphs (14). In fact, it holds for pretty good state transfer in general graphs.

**Theorem 5.** Let $A$ be the Hermitian matrix associated to a graph $\Gamma$. Let $E_1, E_2, \ldots, E_m$ be the idempotents in the spectral decomposition $A = \sum_{r=1}^m \lambda_r E_r$, where $\lambda_1, \lambda_2, \ldots, \lambda_m$ are the corresponding eigenvalues. If there exists a pretty good state transfer from $u$ to $v$ at time $\tau$, then there exist real constants $\gamma, \eta_r \ (r = 1, 2, \ldots, m)$ such that

$$E_r |e_u\rangle = e^{i(\gamma + \eta_r)} E_r |e_v\rangle,$$

for $r = 1, 2, \ldots, m$.

**Proof.** Since the unit circle is compact, there exists a subsequence $\{t_{k_i}\}$ such that $\lim_{k \to \infty} \exp(-it_{k_i} \lambda_r) = \exp(-i\eta_r)$. On one hand, we have

$$\lim_{k \to \infty} E_r U(t_k) |e_u\rangle = E_r e^{i\gamma} |e_v\rangle = e^{i\gamma} E_r |e_v\rangle.$$

On the other hand, we have

$$\lim_{k \to \infty} E_r U(t_k) |e_u\rangle = \lim_{k \to \infty} e^{-it_{k_i} \lambda_r} E_r |e_u\rangle = e^{-i\eta_r} E_r |e_u\rangle.$$
Therefore \( E_r |e_u\rangle = e^{i(\gamma + \eta_r)} E_r |e_v\rangle \) for \( r = 1, 2, \ldots, m \).

**Remark 6.** It is tempting to take the limit \( \lim_{k \to \infty} \exp(-it_{k}) = \exp(-i\tau) \), resulting in \( \eta_r = \tau \lambda_r \). But the branching of complex power functions forbid it.

**Corollary 7 ((14, Lemma 13.1)).** If the graph \( \Gamma \) in Theorem 5 is unoriented, in other words, the Hermitian matrix \( A \) is in fact real symmetric, then \( E_r |e_u\rangle = \pm E_r |e_v\rangle \) for \( r = 1, 2, \ldots, m \).

**Corollary 8 ((15, Lemma 5.1)).** Under the condition of Theorem 5. Let \( |z\rangle \) be an eigenvector of \( A \) corresponding to eigenvalue \( \lambda_r \). Then there exists a phase factor \( q_r(u,v) \) such that \( \langle z | e_u \rangle = e^{i q_r(u,v)} \langle z | e_v \rangle \). Here \( q_r(u,v) \) depends on \( u, v \), and \( r \), but not on the \( \lambda_r \)-eigenvector. In particular, the magnitude of entries in \( |z\rangle \) corresponding to \( u \) and \( v \) are the same.

**Proof.** Without loss of generality, we may assume that \( |z\rangle \) is a unit vector. The conclusion follows by left multiplying \( |z\rangle \langle z| \) on both sides of (4). \( \square \)

It is known that graphs with universal pretty good state transfer have distinct eigenvalues (4). The following theorem generalizes this property to multiple pretty good state transfer.

**Theorem 9.** Let \( A \) be the Hermitian matrix associated to a graph \( \Gamma = (X,E) \). Let \( C \) be a subset of the vertex set \( X \). Suppose there exists multiple pretty good state transfer in \( C \). Then the multiplicity of each eigenvalue of \( A \) is at most \( |X| - |C| + 1 \).

**Proof.** Let \( W \) be the eigenspace of \( A \) corresponding to eigenvalue \( \lambda_r \). Take \( u \) and \( v \) to be two vertices in \( C \). By Corollary 8, we know that \( W \) is contained in the subspace \( W_{u,v} = \{ |z\rangle \in \oplus_{x \in X} \mathbb{C} |e_x\rangle : \langle z | e_u \rangle = e^{iq_r(u,v)} \langle z | e_v \rangle \} \).

Since \( u \) and \( v \) can be taken arbitrarily in \( C \), we obtain

\[
W \subseteq \bigcap_{v \in C, v \neq u} W_{u,v}.
\]

Therefore \( \dim W \leq \dim \bigcap_{v \in C, v \neq u} W_{u,v} \leq |X| - |C| + 1 \). \( \square \)

**Corollary 10 ((4, Theorem 8)).** If \( \Gamma \) carries universal pretty good state transfer, then \( A \) has distinct eigenvalues.

Next we give a characterization of zero transfer.

**Theorem 11.** Let \( A \) be the Hermitian matrix associated to a graph \( \Gamma \). Let \( \Lambda \) be the set of eigenvalues of \( A \), and let \( E_\lambda \) be the projection matrix onto the \( \lambda \)-eigenspace for every \( \lambda \in \Lambda \). Suppose \( a \) and \( b \) are two
vertices of $\Gamma$. Then there exists zero transfer between $a$ and $b$ if and only if $\langle e_b \mid E_\lambda \mid e_a \rangle = 0$ for every $\lambda \in \Lambda$.

Proof. Suppose there exists zero transfer between $a$ and $b$. By the definition of zero transfer, the following identity holds for every $t \in \mathbb{R}$:

$$0 = \langle e_b \mid e^{-itA} \mid e_a \rangle = \sum_{\lambda \in \Lambda} e^{-it\lambda} \langle e_b \mid E_\lambda \mid e_a \rangle.$$  

Taking $k$-th derivative of both sides with respect to the variable $t$ and evaluating at $t = 0$, we obtain

$$0 = \sum_{\lambda \in \Lambda} (-i\lambda)^k \langle e_b \mid E_\lambda \mid e_a \rangle.$$  

It can be written in matrix form as $Bx = 0$, where $B$ is a Vandermonde matrix and $x$ is the vector consisting of $\langle e_b \mid E_\lambda \mid e_a \rangle$. Since the Vandermonde matrix is invertible, we conclude that $\langle e_b \mid E_\lambda \mid e_a \rangle = 0$ for every $\lambda \in \Lambda$.

The proof of the other direction is trivial. \hfill $\square$

Remark 12. The characterization of zero transfer depends only on the eigenspaces but not the eigenvalues, which clearly differs from the characterization of perfect state transfer.

Corollary 13. Let $A$ be a non-negative real symmetric matrix associated to an unoriented connected graph $\Gamma$. Then there exists no zero transfer between any two vertices of $\Gamma$.

Proof. By Perron-Frobenius Theorem, there exists a Perron eigenvector $|z_\rho\rangle$ whose associated eigenvalue $\rho$ is equal to the spectral radius. Note that the Perron eigenvector is positive, hence every entry of $E_\rho = |z_\rho\rangle \langle z_\rho|$ is positive. Contradiction. \hfill $\square$

Remark 14. Corollary 13 tells us that one shall not expect zero transfer in unoriented graphs.

To finish up this section, we give a construction which turns perfect state transfer into the generation of maximal entanglement.

Let $A$ be the Hermitian matrix associated to a graph $\Gamma = (X, E)$. And let $C$ be a subset of the vertex set of $\Gamma$. We define $A \{C^2\}$ to be the Hermitian matrix of a new graph $\Gamma'$ whose vertex set is the union of $X$ and a copy of $C$, namely $(X - C) \cup C_1 \cup C_2$. For a vertex $v$ in $C_1$
or \(C_2\), we denote by \(v^*\) the corresponding vertex in \(C\).

\[
(A\{C^2\})_{u,v} = \begin{cases} 
(A)_{u,v}, & \text{if } u, v \in X - C, \\
(A)_{u^*,v^*}, & \text{if } u, v \in C_1, \text{ or } u, v \in C_2, \\
\frac{1}{\sqrt{2}}(A)_{u,v}, & \text{if } u \in X - C, v \not\in X - C \\
\frac{1}{\sqrt{2}}(A)_{u^*,v}, & \text{if } u \not\in X - C, v \in X - C, \\
0, & \text{otherwise.}
\end{cases}
\]

**Theorem 15.** Let \(A\) be the Hermitian matrix associated to a graph \(\Gamma\). Let \(a\) and \(b\) be two vertices of \(\Gamma = (X,E)\). Suppose there exists perfect state transfer from \(a\) to \(b\) at time \(\tau\). Let \(C\) be a subset of the vertex set of \(\Gamma\) such that \(a \notin C\) and \(b \in C\). The resulting state in \(A\{C^2\}\) at time \(t\) is a maximally entangled state of \(b_1\) and \(b_2\).

**Proof.** Let \(Y = (X - C) \cup C_1 \cup C_2\), where \(C_1, C_2\) are two copies of \(C\). Let \(V = \oplus_{x \in X} \mathbb{C} |e_x\rangle\) and \(W = \oplus_{y \in Y} \mathbb{C} |e_y\rangle\). Let \(V' = (\oplus_{x \in X} \mathbb{C} |e_x\rangle) \bigoplus (\oplus_{c \in C} \mathbb{C} \frac{|e_{c_1}\rangle + |e_{c_2}\rangle}{\sqrt{2}})\).

Note that the action of \(A\{C^2\}\) on \(V'\) is identical to the action of \(A\) on \(V\).

Since \(e^{-irA} |e_a\rangle = e^{i\theta} |e_b\rangle\), we have \(e^{-irA\{C^2\}} |e_a\rangle = e^{i\theta} \frac{|e_{c_1}\rangle + |e_{c_2}\rangle}{\sqrt{2}}\). \(\Box\)

### 3. Association Scheme

In this section, we recall the definition and basic properties of commutative association scheme. We will emphasize the extension of association scheme, which will be used in later construction of state transfers.

A commutative association scheme \(X = (X, \{R_i\}_{i=0}^d)\) on \(X\) of class \(d\) consists of the relations \(R_i\) (0 \(\leq i \leq d\)) whose adjacency matrices \(A_i\) (0 \(\leq i \leq d\)) satisfy the following conditions.

1. \(A_0 = I;\)
2. \(A_0 + A_1 + \cdots + A_d = J;\)
3. \(A_i = A_{i'} \text{ for some } i' \in \{0, 1, \ldots, d\};\)
4. \(A_iA_j = A_jA_i = \sum_{k=0}^d p_{ij}^k A_k.\)

The Bose-Mesner algebra \(\mathcal{A} = \langle A_0, A_1, \ldots, A_d \rangle\) of a commutative association scheme \(X = (X, \{R_i\}_{i=0}^d)\) has another basis \(\mathcal{A} = \langle E_0, E_1, \ldots, E_d \rangle\) composed of primitive idempotents. They satisfy the following properties.

1. \(E_0 = \frac{1}{|X|}J;\)
2. \(E_0 + E_1 + \cdots + E_d = I;\)
3. \(E_{i'} = E_i = E_{i}\) for some \(i' \in \{0, 1, \ldots, d\};\)
4. \(E_i \circ E_j = \frac{1}{|X|} \sum_{k=0}^d q_{ij}^k E_k\)
Example 16. Let \( A_0 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \) and \( A_1 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \). The primitive idempotents are given by \( E_0 = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \) and \( E_1 = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \). Then \( A_0 E_0 = E_0, A_0 E_1 = E_1, A_1 E_0 = E_0, \) and \( A_1 E_1 = -E_1 \). In other words, \( A_j E_\lambda = (-1)^j \lambda E_\lambda \) for \( j, \lambda \in \{0, 1\} \). This association scheme is called the trivial association scheme \( X_2 \) of size 2.

Example 17. Let \( Z = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ & 1 & \cdots & 1 \\ & & \ddots & \cdots \\ & & & 1 \end{bmatrix} \) be the circulant matrix of size \( n \). Let \( A_k = Z_k \) for \( k = 0, 1, \ldots, n-1 \) and let \( E_\ell = \left( \zeta^{\ell k} \right)_{0 \leq j, k \leq n-1} \), where \( \zeta \) is the \( n \)-th root of unity. Then \( A_k E_\ell = \zeta^{k \ell} E_\ell \) for \( k, \ell \in \{0, 1, \ldots, n-1\} \). This association scheme is called the association scheme of directed \( n \)-gon.

Let \( X = (X, \{R_i\})_{i=0}^d \) be an association scheme on \( X \) of class \( d \). Let \( A_i (0 \leq i \leq d) \) be the adjacency matrices of \( X \), and let \( E_i (0 \leq i \leq d) \) be the primitive idempotents of \( X \). The transition matrices, the first eigenmatrix \( P = (P_{i,j})_{0 \leq i,j \leq d} \) and the second eigenmatrix \( Q = (Q_{i,j})_{0 \leq i,j \leq d} \), between the two bases are given by

\[
A_i = \sum_{j=0}^d P_{j,i} E_j, \quad E_i = \frac{1}{|X|} \sum_{j=0}^d Q_{j,i} A_j
\]

In particular we have the valencies \( k_i = P_{0,i} \) and the multiplicities \( m_i = Q_{0,i} \). The two eigenmatrix are related by \( PQ = |X| I \) and \( \frac{P_{i,j}}{k_j} = \frac{Q_{i,j}}{m_j} \). We denote the cosine matrix of these values by \( U = (u_{i,j})_{0 \leq i,j \leq d} = \left( \frac{P_{i,j}}{k_j} \right)_{0 \leq i,j \leq d} \). The reader is referred to (1) for a thorough discussion of association scheme.

3.1. Extension of association schemes. In this subsection, we exhibit two operations to construct new association schemes from old ones. The first operation is based on tensor product and the second operation is based on group action. Combining these two operations, we obtain the so-called extension of association schemes, also known as the symmetric tensor product of association schemes.

Suppose \( \mathcal{X} = (X, \{R_i\})_{i=0}^d \) is an association scheme on \( X \) of class \( d \) and \( \mathcal{Y} = (Y, \{S_j\})_{j=0}^e \) is an association scheme on \( Y \) of class \( e \). Let
$A_i$ ($0 \leq i \leq d$) and $B_j$ ($0 \leq j \leq e$) be their adjacency matrices respectively. Then

$$A_i \otimes B_j \ (0 \leq i \leq d, 0 \leq j \leq e)$$

gives an association scheme on $X \times Y$ of class $(d+1)(e+1)-1$, denoted by $\mathfrak{X} \otimes \mathfrak{Y}$. In particular one can take tensor product of an association scheme $\mathfrak{X}$ with itself, which gives $\mathfrak{X}^{\otimes 2} = \mathfrak{X} \otimes \mathfrak{X}$. By composition and associativity, we can define $\mathfrak{X}^{\otimes N} = \mathfrak{X}^{\otimes (N-1)} \otimes \mathfrak{X} = \mathfrak{X} \otimes \cdots \otimes \mathfrak{X}$. The Bose-Mesner algebra of $\mathfrak{X}^{\otimes N}$ is denoted by $\mathcal{A}^{\otimes N}$.

Let $G \leq \text{Aut}(\mathcal{A})$ be a subgroup of the automorphisms of the Bose-Mesner algebra $\mathcal{A}$ of a commutative association scheme $\mathfrak{X}$. Then the $G$-invariant matrices in $\mathcal{A}$ form a Bose-Mesner algebra of a fusion scheme (also called subscheme) of $\mathfrak{X}$.

Let us consider the group action of the symmetric group $S_N$ on the Bose-Mesner algebra of $\mathfrak{X}^{\otimes N}$ given by

$$g(A_{i_1} \otimes \cdots \otimes A_{i_N}) = A_{i_{g(1)}} \otimes \cdots \otimes A_{i_{g(N)}}$$

for every $g \in S_N$. It can be easily verified that these actions give automorphisms of $\mathcal{A}^{\otimes N}$. The fusion scheme, denoted by $\text{Sym}(\mathfrak{X}, N)$, is called the extension of $\mathfrak{X}$, also known as the $N$-th symmetric tensor product of $\mathfrak{X}$.

Consider the $N$-th symmetric tensor product $\text{Sym}(\mathfrak{X}, N)$ of $\mathfrak{X}$. We denote by $\mathfrak{I} = \{ \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in \mathbb{N}_0^d : \alpha_1 + \alpha_2 + \cdots + \alpha_d \leq N \}$ the indices for $\text{Sym}(\mathfrak{X}, N)$. The value $\alpha_i$ (or $\beta_i$) represents the number of times which $E_i$ (or $A_i$) appears in the tensor product. In particular, we denote by $e_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathfrak{I}$ the sequence of length $d$ with 1 appearing at the $i$-th position. And $N e_i$ is understood as $(0, \ldots, 0, N, 0, \ldots, 0) \in \mathfrak{I}$. We will take the convention that for $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in \mathfrak{I}$, we write $\alpha_0$ for $N - \alpha_1 - \cdots - \alpha_d$. The adjacency matrices and primitive idempotents of $\text{Sym}(\mathfrak{X}, N)$ are thus denoted by $A_\beta$ and $E_\alpha$ respectively, where $\alpha, \beta \in \mathfrak{I}$. The eigenmatrices and cosine matrices of $\text{Sym}(\mathfrak{X}, N)$ can be expressed by hypergeometric functions, more specifically the Aomoto-Gelfand series.

The $(n + 1, m + 1)$-hypergeometric function is defined by

$$F(\alpha, \beta; -N; X) = \sum_{(a_{i,j}) \in \mathbb{N}_0^{n,m-n-1}} \frac{\prod_{i=1}^{n} (\alpha_i) \prod_{j=1}^{m-n-1} (\beta_j) \prod_{i,j} a_{i,j}}{(-N)^{\sum_{i,j} a_{i,j}} \prod_{i,j} x_{i,j}^{a_{i,j}} a_{i,j}!}$$

for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n$ and $\beta = (\beta_1, \ldots, \beta_{m-n-1}) \in \mathbb{C}^{m-n-1}$, where $X$ is a matrix of variables $x_{i,j}$ ($1 \leq i \leq n, 1 \leq j \leq m - n - 1$). Then $u_{\alpha, \beta} = F(-\alpha, -\beta; -N; \Omega)$ for $\alpha, \beta \in \mathfrak{I}$, where $\Omega = (1 - u_{i,j})_{1 \leq i,j \leq d}$. The value $\alpha_i, \beta_i$ represents the number of times which $E_i$ (or $A_i$) appears in the tensor product.

The (0, 1)-hypergeometric function is defined by

$$F(\alpha, \beta; 1; X) = \sum_{(a_{i,j}) \in \mathbb{N}_0^{n,m-n-1}} \frac{\prod_{i=1}^{n} (\alpha_i) \prod_{j=1}^{m-n-1} (\beta_j) \prod_{i,j} a_{i,j}}{\prod_{i,j} x_{i,j}^{a_{i,j}} a_{i,j}!}$$

for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n$ and $\beta = (\beta_1, \ldots, \beta_{m-n}) \in \mathbb{C}^{m-n}$, where $X$ is a matrix of variables $x_{i,j}$ ($1 \leq i \leq n, 1 \leq j \leq m - n$). Then $u_{\alpha, \beta} = F(-\alpha, -\beta; 1; \Omega)$ for $\alpha, \beta \in \mathfrak{I}$, where $\Omega = (1 - u_{i,j})_{1 \leq i,j \leq d}$. The value $\alpha_i, \beta_i$ represents the number of times which $E_i$ (or $A_i$) appears in the tensor product.
We abbreviate $F(-\alpha,-\beta;-N,\Omega)$ by $G(\alpha,\beta)$. In fact $G(\alpha,\beta)$ are the multivariate Krawtchouk polynomials of Griffiths type (12). The generating function for $G(\alpha,\beta)$ is given by

$$
\prod_{i=0}^{d} \left(1 + \sum_{j=1}^{d} u_{j,i}s_{j}\right)^{\beta_{i}} = \sum_{\alpha \in \mathcal{J}} \binom{N}{\alpha} G(\alpha,\beta)s_{1}^{\alpha_{1}}s_{2}^{\alpha_{2}} \cdots s_{d}^{\alpha_{d}},
$$

where $\binom{N}{\alpha} = \frac{N!}{\alpha_{0}!\alpha_{1}! \cdots \alpha_{d}!}$ is the multinomial coefficient.

Both tensor product and symmetric tensor product can be used to construct new association schemes with multivariate polynomial structure, see (2) for latest development on bivariate polynomial association schemes.

4. STATE TRANSFER ON EXTENSIONS OF ASSOCIATION SCHEME

Consider $M = \sum_{i=1}^{d} w_{i}A_{e_{i}}$ with the restriction that $w_{i} = \overline{w_{i}}$, namely $M$ is a Hermitian. Note that $e^{-itM}$ is still contained in the Bose-Mesner algebra. We can rewrite $M$ in the basis $E_{\alpha}, (\alpha \in \mathcal{J})$ by

$$
M = \sum_{i=1}^{d} w_{i} \sum_{\alpha \in \mathcal{J}} P_{\alpha,e_{i}} E_{\alpha}
$$

$$
= \sum_{\alpha \in \mathcal{J}} \left( \sum_{i=1}^{d} w_{i} P_{\alpha,e_{i}} \right) E_{\alpha}.
$$

The eigenvalues $\lambda_{\alpha}$ of $M$ for $E_{\alpha}$ can be further expressed by eigenvalues in $\mathfrak{X}$, namely

$$
\lambda_{\alpha} = \sum_{i=1}^{d} w_{i} P_{\alpha,e_{i}}
$$

$$
= \sum_{i=1}^{d} w_{i} \sum_{j=0}^{d} \alpha_{j} P_{j,i}
$$

$$
= \sum_{j=1}^{d} \alpha_{j} \left( \sum_{i=1}^{d} w_{i} P_{j,i} \right) + \alpha_{0} \left( \sum_{i=1}^{d} w_{i} P_{0,i} \right)
$$

$$
= N \left( \sum_{i=1}^{d} w_{i} P_{0,i} \right) + \sum_{j=1}^{d} \alpha_{j} \left( \sum_{i=1}^{d} w_{i} (P_{j,i} - P_{0,i}) \right).
$$
Therefore
\( e^{-itM} = \sum_{\alpha \in I} e^{-it\lambda_\alpha} E_\alpha \)
(18)
\[ = \sum_{\alpha \in I} e^{-it\lambda_\alpha} \left( \frac{1}{|X|^N} \sum_{\beta \in I} Q_{\beta,\alpha} A_\beta \right) \]
(19)
\[ = \sum_{\beta \in I} \frac{1}{|X|^N} \left( \sum_{\alpha \in I} e^{-it\lambda_\alpha} m_{\alpha} u_{\alpha,\beta} \right) A_\beta \]
(20)

Since the generating function for \( G(\alpha, \beta) \) is given by (11), the coefficient for \( A_\beta \) in \( e^{-itM} \) is equal to
\[ f_\beta(t) = \frac{1}{|X|^N} \sum_{\alpha \in I} e^{-it\lambda_\alpha} m_{\alpha} G(\alpha, \beta) \]
(21)
\[ = \frac{1}{|X|^N} e^{-itN} \sum_{\alpha \in I} w_\alpha P_{0,i} \sum_{\alpha \in I} \binom{N}{\alpha} G(\alpha, \beta) \prod_{j=1}^d m_j e^{-it\sum_{\alpha \in I} w_\alpha (P_j,i - P_{0,i})} \]
(22)
\[ = \frac{1}{|X|^N} e^{-itN} \sum_{\alpha \in I} w_\alpha P_{0,i} \prod_{k=0}^d \left( 1 + \sum_{\ell=1}^d \frac{u_{\ell,k} m_{\ell} e^{-it\sum_{\alpha \in I} w_\alpha (P_{\ell,i} - P_{0,i})}}{u_{\ell,k} m_{\ell}} \right)^{\beta_k} \]
(23)
\[ = \frac{1}{|X|^N} e^{-itN} \sum_{\alpha \in I} w_\alpha \prod_{k=0}^d \left( 1 + \sum_{\ell=1}^d u_{\ell,k} m_{\ell} e^{-it\sum_{\alpha \in I} w_\alpha (u_{\ell,i} - 1)} \right)^{\beta_k} \]
(24)

For ease of notation, we let
\[ z_\ell = z_\ell(t) = e^{-it\sum_{\alpha \in I} w_\alpha (P_{\ell,i} - P_{0,i})}, \]
(25)
and let \( p_k = 1 + \sum_{\ell=1}^d u_{\ell,k} m_{\ell} z_\ell \).

As long as \( p_k = 0 \), the coefficient \( f_\beta(t) \) vanishes on \( \beta \in I \) with \( \beta_k \neq 0 \).
In order to achieve perfect state transfer or fractional revival, we want \( f_\beta(t) \) to vanish for as many \( \beta \) as possible.

We would like to emphasize that once the values of \( z_\ell(t), \ell = 1, 2, \ldots, d, (|z_\ell| = 1) \) are given, one can always find \( w_i \in \mathbb{R}, i = 1, 2, \ldots, d \) such that \( z_\ell = e^{-it\sum_{\alpha \in I} w_\alpha (P_{\ell,i} - P_{0,i})} \). Consider the linear equation \( \tilde{P}w = b \), where \( \tilde{P}_{\ell,i} = P_{\ell,i} - P_{0,i} \) for \( 1 \leq \ell, i \leq d \), and \( w = (w_1, w_2, \ldots, w_d)^T \), \( b = -\frac{1}{it}(\arg z_1, \arg z_2, \ldots, \arg z_d)^T \). Since \( \tilde{P} \) can be obtained by operating
an elementary row transformation on \( P \), which is of full rank, the matrix \( \tilde{P} \) is also of full rank. Therefore the linear equation \( \tilde{P}w = b \) always has solutions.

Perfect state transfer on \( \text{Sym}(\mathcal{X}, N) \) can be understood in two regards. On one hand, fix a base vertex \( x_0 \in X_N \) with the indicator vector \(|x_0\rangle = \delta_{x_0}|x_0\rangle\). Then the evolution gives \( e^{itM}|x_0\rangle = \sum_{x \in X^N} \alpha_x(t)|x\rangle \). On the other hand, let \(|Y_\beta\rangle = \frac{1}{\sqrt{k_\beta}}A_\beta|x_0\rangle\). Then \( e^{itM}|Y_0\rangle = \sum_{\beta \in \mathcal{I}} f_\beta(t) \sqrt{k_\beta}|Y_\beta\rangle \).

The computation in this section can be summarized as the following theorem.

**Theorem 18.** Let \( \mathcal{X} = (X, \{R_i\}_{i=0}^d) \) be a commutative association scheme of class \( d \). Let \( A_\beta (\beta \in \mathcal{I}) \) be the adjacency matrices in \( \text{Sym}(\mathcal{X}, N) \) in the \( N \)-th symmetric tensor product of \( \mathcal{X} \). Consider the quantum walks on a Hermitian matrix \( M = \sum_{i=1}^d w_iA_{e_i} \). Then \( e^{-itM} = \sum_{\beta \in \mathcal{I}} f_\beta(t)A_\beta \), where

\[
f_\beta(t) = \frac{1}{|X|^N} e^{-itN \sum_{i=1}^d w_i k_i} \prod_{k=0}^d \left( 1 + \sum_{\ell=1}^d \frac{u_{\ell,k} m_{\ell} e^{-it \sum_{i=1}^d w_i k_i (u_{\ell,i} - 1)}}{1 + \sum_{\ell=1}^d \frac{u_{\ell,k} m_{\ell} e^{-it \sum_{i=1}^d w_i k_i (u_{\ell,i} - 1)}}} \right)^{\beta_k}.
\]

In particular, suppose the quantum walk starts at a site \(|x_0\rangle\), then it will not occur at a site \(|Y_\beta\rangle = \frac{1}{\sqrt{k_\beta}}A_\beta|x_0\rangle\) with \( \beta_k \neq 0 \) at time \( t \) as long as

\[
1 + \sum_{\ell=1}^d \frac{u_{\ell,k} m_{\ell} e^{-it \sum_{i=1}^d w_i k_i (u_{\ell,i} - 1)}}{1 + \sum_{\ell=1}^d \frac{u_{\ell,k} m_{\ell} e^{-it \sum_{i=1}^d w_i k_i (u_{\ell,i} - 1)}}} = 0.
\]

5. Examples

In this section we apply the extension of association schemes to four base association schemes, the trivial association scheme \( \mathcal{X}_2 \), its tensor product \( \mathcal{X}_2^k \), the association scheme of directed \( n \)-gon, and the ordered binary word association scheme \( \text{OW}(2, d) \). Various fractional revivals on the corresponding association schemes are constructed.

5.1. The trivial association scheme of size 2. The first eigenmatrix, the second eigenmatrix, and the cosine matrix of the trivial association scheme \( \mathcal{X}_2 \) is given by

\[
P = Q = U = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.
\]

It is in fact the Hadamard matrix \( H_2 \) of order 2 as \( U^T U = 2I \). The \( N \)-th symmetric tensor product \( \text{Sym}(\mathcal{X}_2, N) \) of \( \mathcal{X}_2 \) is the binary Hamming association scheme of length \( N \), in other words the association scheme of the \( N \)-hypercube.
If 0 = \( p_0 = 1 + z_1 \), namely \( z_1 = -1 \), then \( f_\beta(t) \) concentrates on \( (N) \in \mathcal{J} \). In other words, we have perfect state transfer from \( |Y_{(0)}\rangle \) to \( |Y_{(N)}\rangle \).

5.2. The tensor power of the trivial association scheme. The first eigenmatrix, the second eigenmatrix, and the cosine matrix of \( \mathcal{X}_2^{\otimes d} \) is given by

\[
P = Q = U = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{\otimes d}.
\]

It is in fact the Hadamard matrix \( H_{2^d} \) of order \( 2^d \). Note that the adjacency matrices in \( \mathcal{X}_2^{\otimes d} \) are of the form

\[
A_{i_1} \otimes A_{i_2} \otimes \cdots \otimes A_{i_d},
\]

where \( i_1, i_2, \ldots, i_d \in \{0, 1\} \). In other words, the indices for \( \mathcal{X}_2^{\otimes d} \) are in one-to-one correspondence with the binary expansion of integer \( 0, 1, \ldots, 2^d - 1 \). Moreover, the multiplication rule is given by

\[
(A_{i_1} \otimes A_{i_2} \otimes \cdots \otimes A_{i_d}),(A_{j_1} \otimes A_{j_2} \otimes \cdots \otimes A_{j_d}) = A_{i_1 \oplus j_1} \otimes A_{i_2 \oplus j_2} \otimes \cdots \otimes A_{i_d \oplus j_d},
\]

where \( \oplus \) is the addition without carry, namely, \( 0 \oplus 0 = 0, 0 \oplus 1 = 1 \oplus 0 = 1, \) and \( 1 \oplus 1 = 0 \). We abbreviate the above as \( A_i \cdot A_j = A_{i \oplus j} \),

where \( i, j \in \{0, 1\}^d \). Accordingly, the value \( \alpha_k \) in the indices \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{2^d - 1}) \in \mathcal{J} \) for \( \text{Sym}(\mathcal{X}_2^{\otimes d}, N) \) is understood as the number of times which \( A_k \) appears.

Note that \( H_{2^d}^{T}H_{2^d} = 2^d I \). If we set the sequence \( z_{\ell} \) \( (1 \leq \ell \leq 2^d - 1) \) as the \( k \)-th row of \( H_{2^d}^{\otimes d} \) \( (k \neq 0) \), then we have \( p_0 = p_1 = \cdots = p_{k-1} = p_{k+1} = \cdots = p_{2^d-1} = 0 \) and \( p_k \neq 0 \). In other words, \( f_\beta(t) \) concentrates on \( \beta = Ne_k \in \mathcal{J} \). We have \( e^{itM}|Y_0\rangle = e^{i\theta}|Y_{Ne_k}\rangle \) for some \( \theta \in \mathbb{R} \). Meanwhile we have

\[
e^{itM}|Y_\beta\rangle = e^{itM}A_\beta|Y_0\rangle = A_\beta e^{itM}|Y_0\rangle = A_\beta e^{i\theta}|Y_{Ne_k}\rangle = e^{i\theta}A_\beta A_{Ne_k}|Y_0\rangle = e^{i\theta}|Y_\gamma\rangle,
\]

where \( \gamma \) is determined by \( A_\beta A_{Ne_k} = A_\gamma \). That is \( \gamma_i = \beta_{i \oplus k} \). In other words, we have perfect state transfer from \( |Y_\beta\rangle \) to \( |Y_\gamma\rangle \) with \( \gamma_i = \beta_{i \oplus k} \).
5.3. The association scheme of directed $n$-gon. As shown in Example 17, the first eigenmatrix of the directed $n$-gon is given by

$$P = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \zeta & \zeta^2 & \cdots & \zeta^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \zeta^{n-1} & \zeta^{2(n-1)} & \cdots & \zeta^{(n-1)^2} \end{bmatrix},$$

where $\zeta$ is the $n$-th root of unity, and the second eigenmatrix and the cosine matrix are given by $U = P$ and $Q = \overline{P}$. In other words, $u_{\ell,k} = \zeta^{\ell k}$ for $0 \leq \ell, k \leq n-1$. The valencies $k_j$ and multiplicities $m_i$ are all equal to 1. Here the entries of the eigenmatrices can be understood as the evaluation of orthogonal polynomials $z^k, (k = 0, 1, \ldots, n-1)$ with the weight function taking point mass at $\zeta^{0}, \zeta^{1}, \ldots, \zeta^{n-1}$ on the unit circle.

Suppose $z_{\ell}(\tau_k) = e^{\frac{2\pi i \ell \tau_k}{n}} = \zeta^{\ell k}$ for $1 \leq \ell \leq n-1$, then $p_0 = p_1 = \cdots = p_{k-1} = p_{k+1} = \cdots = p_{n-1} = 0$ since $PQ = nI$. In this case, $f_\beta(\tau_k)$ concentrates on $N_{e_k}$. In other words, we have perfect state transfer from $|Y_{(0)}\rangle$ to $|Y_{Ne_k}\rangle$ at time $t = \tau_k$.

One solution is given by $w_j = \frac{1}{\zeta^{j-1}}$ and $\tau_k = \frac{2k\pi}{n}$ for $1 \leq j, k \leq n-1$. Namely we have multiple perfect state transfers among $|Y_{(0)}\rangle$, $|Y_{Ne_1}\rangle$, $|Y_{Ne_2}\rangle$, $\ldots$, $|Y_{Ne_{n-1}}\rangle$. Similar to the case in Section 5.2, we also have multiple perfect state transfer among $|Y_{\beta}\rangle$, $|Y_{\sigma(\beta)}\rangle$, $|Y_{\sigma^2(\beta)}\rangle$, $\ldots$, $|Y_{\sigma^{n-1}(\beta)}\rangle$, where $(\sigma(\beta))_i = \beta_{i-1} \mod n$ ($0 \leq i \leq n-1$).

Figure 1 illustrates the possibility of detecting the state at different sites and at various times through the size of the bubbles (based on the 3-th symmetric tensor power of the association scheme of directed 3-gon). The Hermitian matrix is given by $A$ in (29), where $w_1 = \frac{1}{\zeta^{\frac{1}{3}-1}}$ and $w_2 = \frac{1}{\zeta^{\frac{2}{3}-1}}$ with $\zeta$ being the cubic root of unity.

$$A = \begin{bmatrix} 0 & \sqrt{3}w_1 & \sqrt{3}w_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{3}w_2 & 0 & w_1 & 2w_1 & \sqrt{2}w_2 & 0 & 0 & 0 & 0 \\ \sqrt{3}w_1 & w_2 & 0 & 0 & \sqrt{2}w_1 & 2w_2 & 0 & 0 & 0 \\ 0 & 2w_2 & 0 & 0 & \sqrt{2}w_1 & 0 & \sqrt{3}w_1 & w_2 & 0 \\ 0 & \sqrt{2}w_1 & \sqrt{2}w_2 & \sqrt{2}w_2 & 0 & \sqrt{2}w_1 & 0 & \sqrt{2}w_1 & \sqrt{2}w_2 & 0 \\ 0 & 0 & 2w_1 & 0 & \sqrt{2}w_2 & 0 & 0 & 0 & w_1 & \sqrt{3}w_2 \\ 0 & 0 & 0 & \sqrt{3}w_2 & 0 & 0 & 0 & \sqrt{3}w_1 & 0 & 0 \\ 0 & 0 & 0 & w_1 & \sqrt{2}w_2 & 0 & \sqrt{3}w_2 & 0 & 2w_1 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2}w_1 & w_2 & 0 & 2w_2 & 0 & \sqrt{3}w_1 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{3}w_1 & 0 & 0 & \sqrt{3}w_2 & 0 \end{bmatrix}$$
5.4. The ordered binary word association scheme. We first construct the ordered binary word association scheme OW(2, d) of depth d. The ordered binary Hamming association scheme of depth d can be obtained by taking the symmetric tensor product of OW(2, d).

Consider the following actions $g_j (j = 2, 3, \ldots, d)$ on $X_2^{\otimes d}$:
\begin{equation}
    g_j(A_{i_1} \otimes A_{i_2} \otimes \cdots \otimes A_{i_d}) = A_{i'_1} \otimes A_{i'_2} \otimes \cdots \otimes A_{i'_d}
\end{equation}
such that
\begin{equation}
    i'_k = \begin{cases} 
    1 - i_k, & \text{if } i_j = 1, k = j - 1, \\
    i_k, & \text{otherwise.}
    \end{cases}
\end{equation}
One can verify that $g_j (j = 2, 3, \ldots, d)$ are automorphism of the Bose-Mesner algebra of $X_2^{\otimes d}$. The fusion scheme obtained by taking $G = \langle g_2, g_3, \ldots, g_d \rangle$ acting on $X_2^{\otimes d}$ is called the ordered binary word association scheme of depth d.

The first eigenmatrix of the ordered binary word association scheme OW(2, d) of depth d is given by
\begin{equation}
    P = (P_{i,j})_{0 \leq i, j \leq d} = \begin{bmatrix}
    1 & 1 & 2 & \cdots & 2^{d-1} \\
    1 & 1 & 2 & \cdots & -2^{d-1} \\
    \vdots & \vdots & \vdots & \ddots & 0 \\
    1 & 1 & -2 & \cdots & \ddots \\
    1 & -1 & 0 & \cdots & 0
    \end{bmatrix}.
\end{equation}
and the valencies $k_j$ and $m_i$ are given by $k_j = 2^{j-1}$ and $m_i = 2^{i-1}$ with $1 \leq i, j \leq d$ and $k_0 = m_0 = 1$.

The scaled eigenmatrix of $OW(2, d)$ is given by

$$U = (u_{i,j})_{0 \leq i,j \leq d} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & -1 \\ \vdots & \vdots & \vdots & \cdots & 0 \\ 1 & 1 & -1 & \cdots & \vdots \\ 1 & -1 & 0 & \cdots & 0 \end{bmatrix}$$

We consider the $N$-th symmetric tensor power of $OW(2, d)$, namely the ordered binary Hamming association scheme of length $N$ and depth $d$.

If we demand that $f_\beta(t)$ vanishes on $\beta \in \mathcal{I}$ with $\beta_0 \neq 0$, then it requires that

$$0 = p_0 = 1 + \sum_{\ell=1}^d u_{\ell,0} m_\ell z_\ell$$

$$= 1 + \sum_{\ell=1}^d 2^{\ell-1} z_\ell.$$  \hspace{1cm} (34)

Since $|z_\ell| = 1$ for every $1 \leq \ell \leq d$, it forces that $z_\ell = 1$ for $\ell < d$ and $z_d = -1$ by triangle inequality.

If we demand that $f_\beta(t)$ vanishes on $\beta \in \mathcal{I}$ with $\beta_k \neq 0$ ($1 \leq k \leq d$), then it requires that

$$0 = p_k = 1 + \sum_{\ell=1}^d u_{\ell,k} m_\ell z_\ell$$

$$= 1 + \sum_{\ell=1}^{d-k} 2^{\ell-1} z_\ell - 2^{d-k+1} z_{d-k+1}. \hspace{1cm} (36)$$

Since $|z_\ell| = 1$ for every $1 \leq \ell \leq d$, it forces that $z_\ell = 1$ for $\ell \leq d-k+1$.

Therefore if $p_0 = 0$, then $p_2 = p_3 = \cdots = p_d = 0$. In other words, $f_\beta(t)$ concentrates on $\beta = (N, 0, \ldots, 0)$. Similarly if $p_k = 0$ for some $k \geq 1$, then $p_{k+1} = p_{k+2} = \cdots = p_d = 0$. In other words, $f_\beta(t)$ concentrates on $\beta \in \mathcal{I}$ with $\beta_k = \beta_{k+1} = \cdots = \beta_d = 0$ (a $k$-simplex).

Such a phenomenon was first observed in (18).

Figure 2 illustrates the possibility of detecting the state at different sites and at various times through the size of the bubbles (based on the ordered Hamming association scheme of depth 3 and length 5).
6. Conclusion

In this paper we investigate quantum walks on extensions of association schemes. We provide examples of multiple state transfer among extreme points in a simplex of arbitrary dimension $d$ and arbitrary distance $N$. Examples of fractional revival on subsimplexes are given as well. We also exhibit a way to turn perfect state transfer into the generation of maximal entanglement. Characterizations of universal/multiple perfect/pretty good state transfer are reviewed and summarized while characterization of zero transfer is given.

Based on these results, we propose two problems for further research.

**Problem 19.** What is the smallest (regarding the number of vertices and edges) graph which carries multiple state transfer among $m$ vertices of pairwise distance at least $N$?

Note that Section 5.3 provides examples of graphs with $\Theta(N^{m-1})$ vertices and of bounded degree $\Theta(m^2)$, which is a relatively sparse graph. In (10), the problem of the smallest unweighted graph with perfect state transfer between two vertices of distance $N$ was investigated. The problem is easy if one considers perfect state transfer between two vertices on a weighted graph. But it is not obvious if $m > 2$.

**Problem 20.** Can we construct graphs carrying both perfect state transfer and zero transfer?
Note that the property of zero transfer allows one to check the decoherence in a quantum system. In other words, zero transfer can be used to verify the integrity of a quantum walk system and contribute to robustness.

ACKNOWLEDGEMENT

The research of HM and ST is supported by JSPS KAKENHI (Grant Numbers 21H04073 and 19H01792 respectively. DZ thanks Shanghai Jiao Tong University for providing scholarships. The authors would like to thank P.-A. Bernard, N. Crampé, Luc Vinet, and M. Zaimi for discussions and sharing their results.

REFERENCES

[1] Eiichi Bannai and Tatsuro Itô. *Algebraic combinatorics I: association schemes*. Number 58 in Mathematics lecture note series. Benjamin/Cummings Pub. Co, Menlo Park, Calif, 1984.

[2] Pierre-Antoine Bernard, Nicolas Crampé, Loïc Poulain d’Andey, Luc Vinet, and Meri Zaimi. Bivariate $p$-polynomial association scheme (in preparation).

[3] Sougato Bose. Quantum Communication through an Unmodulated Spin Chain. *Physical Review Letters*, 91(20):207901, November 2003. Publisher: American Physical Society.

[4] Stephen Cameron, Shannon Fehrenbach, Leah Granger, Oliver Hemmigh, Sunrose Shrestha, and Christino Tamon. Universal state transfer on graphs. *Linear Algebra and its Applications*, 455:115–142, August 2014.

[5] Ada Chan, Gabriel Coutinho, Christino Tamon, Luc Vinet, and Hammeng Zhan. Fractional revival and association schemes. *Discrete Mathematics*, 343(11):112018, November 2020.

[6] Rodrigo Chaves, Bruno Oliveira Chagas, and Gabriel Coutinho. Why and how to add direction to a quantum walk, March 2022. arXiv:2203.13857 [quant-ph].

[7] Matthias Christandl, Nilanjana Datta, Tony C. Dorlas, Artur Ekert, Alastair Kay, and Andrew J. Landahl. Perfect transfer of arbitrary states in quantum spin networks. *Physical Review A*, 71(3):032312, March 2005. Publisher: American Physical Society.

[8] Erin Connelly, Nathaniel Grammel, Michael Kraut, Luis Serazo, and Christino Tamon. Universality in perfect state transfer. *Linear Algebra and its Applications*, 531:516–532, October 2017.
[9] G. Coutinho, C. Godsil, K. Guo, and F. Vanhove. Perfect state transfer on distance-regular graphs and association schemes. *Linear Algebra Appl.*, 478:108–130, 2015.

[10] Gabriel Coutinho. Quantum walks and the size of the graph. *Discrete Mathematics*, 342(10):2765–2769, October 2019.

[11] Edward Farhi and Sam Gutmann. Quantum computation and decision trees. *Physical Review A*, 58(2):915–928, August 1998. Publisher: American Physical Society.

[12] Vincent X. Genest, Luc Vinet, and Alexei Zhedanov. The multivariate Krawtchouk polynomials as matrix elements of the rotation group representations on oscillator states. *Journal of Physics A: Mathematical and Theoretical*, 46(50):505203, 2013. Publisher: IOP Publishing.

[13] Chris Godsil. Periodic Graphs. *The Electronic Journal of Combinatorics*, pages P23–P23, January 2011.

[14] Chris Godsil. State transfer on graphs. *Discrete Mathematics*, 312(1):129–147, January 2012.

[15] Chris Godsil and Sabrina Lato. Perfect state transfer on oriented graphs. *Linear Algebra and its Applications*, 604:278–292, November 2020.

[16] M. A. Jafarizadeh and S. Salimi. Investigation of continuous-time quantum walk via modules of Bose–Mesner and Terwilliger algebras. *Journal of Physics A: Mathematical and General*, 39(42):13295–13323, October 2006. Publisher: IOP Publishing.

[17] Alastair Kay. Basics of perfect communication through quantum networks. *Physical Review A*, 84(2):022337, August 2011. Publisher: American Physical Society.

[18] Hiroshi Miki, Satoshi Tsujimoto, and Luc Vinet. Spin Chains, Graphs and State Revival. In Mama Foupouagnigni and Wolfram Koepf, editors, *Orthogonal Polynomials*, Tutorials, Schools, and Workshops in the Mathematical Sciences, pages 495–516, Cham, 2020. Springer International Publishing.

[19] Hiroshi Mizukawa and Hajime Tanaka. \((n + 1, m + 1)\)-Hypergeometric Functions Associated to Character Algebras. *Proceedings of the American Mathematical Society*, 132(9):2613–2618, 2004. Publisher: American Mathematical Society.

[20] Natalie Ellen Mullin. *Uniform Mixing of Quantum Walks and Association Schemes*. Doctoral Thesis, University of Waterloo, September 2013.