GENETICS OF ITERATIVE ROOTS FOR PM FUNCTIONS

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Abstract. It is known that the time-one mapping of a flow defines a discrete dynamical system with the same dynamical behaviors as the flow, but conversely one wants to know whether a flow embedded by a homeomorphism preserves the dynamical behaviors of the homeomorphism. In this paper we consider iterative roots, a weak version of embedded flows, for the preservation. We refer an iterative root to be genetic if it is topologically conjugate to its parent function. We prove that none of PM functions with height being > 1 has a genetic root and none of iterative roots of height being > 1 is genetic even if the height of its parent function is equal to 1. This shows that most functions do not have a genetic iterative root. Further, we obtain a necessary and sufficient conditions under which a PM function f has a genetic iterative root in the case that f and the iterative root are both of height 1.

1. Introduction. The function φ : I := [a, b] → I, regarded as an interval mapping, is called an n-th order iterative root of f : I → I if

\[ \phi^n(x) = f(x) \quad \forall x \in I, \]

where \( \phi^n \) denotes the nth iterate of \( \phi \), i.e., \( \phi^n(x) := \phi(\phi^{n-1}(x)) \) and \( \phi^0(x) = x \) for all \( x \in I \). Since Ch. Babbage [1] studied earlier in the 19-th century, the iterative root problem has attracted attentions of research and its basic theory was established (see [8, 9, 14]) for monontonic functions. More advances can be found in survey papers [2, 17]. The non-monotonic case is much more complicated because those functions, not preserving orientation, possess complicated dynamical behaviors. Earlier results were given by Jingzhong Zhang and Lu Yang ([16, 19]) and by A. Blokh, E. Coven, M. Misurewicz and Z. Nitecki ([3]) for PM functions, a class of non-monotonic functions with finitely many forts (short name of non-monotonic points). In [16] iterative roots are obtained by extending those monotone iterative roots obtained on the “characteristic interval”, the maximal monotone subinterval covering the range of \( f \). Further progresses were made in [10, 12, 13] in recent years. This problem is important because it is a weak version of the embedding

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flow problem ([4, 6, 15, 18]), and defines fractional iterates. Moreover, it is related to information engineering ([5, 7]).

It is interesting to compare the dynamics of an iterative root \( \phi \) with its parent function \( f \). Actually not all iterative roots can inherit the dynamical properties of their parent function \( f \). An iterative root \( \phi \) of \( f \) is said to be genetic if \( \phi \) is topologically conjugate to \( f \) on \( I \), i.e., for each \( \phi \) there exists a homeomorphism \( h : I \to I \) such that

\[
h \circ \phi(x) = f \circ h(x), \quad \forall x \in I.
\]

So we need to identify which iterative roots are genetic or not. This problem was considered in [11] for a special class of PM functions of height 1, each of which has a characteristic interval bounded by an endpoint of the domain and is homeomorphic onto the characteristic interval but does not reach an endpoint of the characteristic interval outside. Theorem 2 of [12] indicates that all its iterative roots are in the simplest mode of 1-extension.

In this paper we generally discuss genetics of iterative roots for PM functions. In section 2 we introduce basic concepts about PM functions and a result on genetics. Our section 3 is devoted to non-genetic cases. We prove that none of PM functions \( f \) with height \( H(f) > 1 \) has a genetic root, none of iterative roots \( \phi \) of height \( H(\phi) > 1 \) is genetic in the case that \( H(f) = 1 \), and, in the case that both \( H(f) = 1 \) and \( H(\phi) = 1 \), the root \( \phi \) is not genetic if its monotonicity on the characteristic interval of \( f \) is different from that of \( f \). In section 4 we discuss in the case that \( H(f) = H(\phi) = 1 \) for necessary and sufficient conditions under which a PM function \( f \) has a genetic iterative root \( \phi \). We give examples to demonstrate our results in section 5.

2. Preliminaries. Let \( a, b \in \mathbb{R} \) such that \( a < b \), and let \( f : I := [a, b] \to I \) be a continuous function. A point \( c \in (a, b) \) is called a nonmonotonic point (or fort for short) of \( f \) if \( f \) is strictly monotonic in no neighborhood of \( c \). The function \( f \) is said to be piecewise monotonic and called a PM functions if it has only finitely many forts. Let \( PM(I, I) \) denote the set of all piecewise monotonic self–mappings of \( I \).

For \( f \in PM(I, I) \), we easily see that the cardinality of the set \( S(f) \) of forts of \( f \), denoted by \( N(f) \), satisfies the ascending relation

\[
0 = N(f^0) \leq N(f) \leq N(f^2) \leq \ldots \leq N(f^k) \leq N(f^{k+1}) \leq \ldots,
\]

where the least positive integer \( k \) such that \( N(f^k) = N(f^{k+1}) \) is referred to as the height \( H(f) \) of \( f \) if such a \( k \) exists. Otherwise, the height is said to be \( \infty \). The height is actually the short name of the concept nonmonotonicity height indicated in [13]. It is an important number to characterize the complexity of \( f \).

The set \( S(f) \) splits the whole interval \( I \) into subintervals such that \( f \) is strictly monotone on them, called monotone subinterval. As proved in [16, 19], \( H(f) = 1 \) if and only if the PM function \( f \) has a unique monotone subinterval, called characteristic interval, such that \( F \) is a self–mapping on the monotone subinterval and the monotone subinterval covers the range of \( F \).

For convenience we introduce the following notations:

\[
CI(I, I) := \{ f : I \to I \mid f \text{ is continuous and strictly increasing} \},
\]

\[
CD(I, I) := \{ f : I \to I \mid f \text{ is continuous and strictly decreasing} \}.
\]

Let \( \text{Fix}(f) \) consist of all fixed points of \( f : I \to I \). \( \xi \in \text{Fix}(f) \) is said to be attractive (or repulsive) if there exists a neighbourhood \( O_\xi \) of \( \xi \) such that \( \lim_{n \to +\infty} f^n(x) = \xi \)
for every $x \in O_\xi$ (or if there exists a neighbourhood $O_\xi$ of $\xi$ such that the sequence $f^n(x)$ does not tend to $\xi$ for any $x \in O_\xi \setminus \{\xi\}$). As defined in [8, p.301], a reversing correspondence is a strictly increasing function $f$ mapping $I$ into itself with a fixed point $\xi \in \text{Fix}(f)$ and a strictly decreasing function $\omega$ mapping $\text{Fix}(f)$ onto itself such that $\omega(\xi) = \xi$ and the expression $f(x) - x$ has opposite signs in the intervals $(\xi_1, \xi_2)$ and $\omega(\xi_2), \omega(\xi_1)$ for every $\xi_1, \xi_2 \in \text{Fix}f$ with $\xi_1 < \xi_2$ and $(\xi_1, \xi_2) \cap \text{Fix}f = \emptyset$. Theorems 15.7, 15.8 and 15.9 of Kuczma’s book ([8]) show that

**Case (I,I):** every $f \in CI(I, I)$ has infinitely many iterative roots $\phi$ of any order $n$ in $CI(I, I)$,

**Case (I,D):** if $f \in CI(I, I)$ is a reversing correspondence then $f$ has infinitely many iterative roots $\phi$ of any even order $n$ in $CD(I, I)$, and

**Case (D,D):** if $f \in CD(I, I)$ satisfies either $a < f(b) < f(a) < b$ or $a = f(b) < f(a) = b$ then $f$ has infinitely many iterative roots $\phi$ of any odd order $n$ in $CD(I, I)$.

More precisely, [17, Theorem 2.1] asserts the following results on genetics.

**Proposition 1.** In case (I,I) all strictly increasing iterative roots $\phi$ of every $f \in CI(I, I)$ fixing both endpoints are genetic. In case (I,D) any decreasing iterative root of a reversing correspondence $f \in CI(I, I)$ is not genetic.

In fact, the first result comes from the conjugation between iterative roots and $f$. The second results holds because of their opposite monotonicity.

In what follows, we discuss on genetics of iterative roots for PM functions generally, showing what iterative roots of a PM function are topologically conjugate to the PM function or not.

3. **Non-genetic roots.** In this section, we concentrate to non-genetic roots.

**Lemma 3.1.** Let $f \in PM(I, I)$ and $\phi$ be a continuous iterative root of order $n$ of $f$. If $\phi$ is genetic, then $\phi$ and $f$ have the same forts, i.e., $S(f) = S(\phi)$.

**Proof.** As shown in [19, p.121], one can see that

$$S(f) = S(\phi^n) = S(\phi) \cup \{x \in I : \phi(x) \in S(\phi^{n-1})\},$$

implying that

$$S(\phi) \subseteq S(f). \quad (2)$$

On the other hand, there is a homeomorphism $h$ such that $h \circ \phi = f \circ h$, since $\phi$ is genetic. It follows that

$$S(h \circ \phi) = S(\phi) \cup \{x \in I : \phi(x) \in S(h)\} = S(\phi),$$

$$S(f \circ h) = S(h) \cup \{x \in I : h(x) \in S(f)\} = \{x \in I : h(x) \in S(f)\},$$

since $h$ is a homeomorphism, i.e., $S(h) = \emptyset$. Then, we have

$$S(\phi) = \{x \in I : h(x) \in S(f)\},$$

showing that

$$\#S(\phi) = \#\{x \in I : h(x) \in S(f)\} = \#S(f).$$

By (2), we get that $S(\phi) = S(f)$. Therefore, the proof is completed.\[\square\]

Lemma 3.1 enables us to obtain the following result.
Theorem 3.2. Let $f \in PM(I, I)$. The following assertions hold:

(i) If $H(f) > 1$, then none of continuous iterative roots of $f$ is genetic.

(ii) If $H(f) = 1$, then none of continuous iterative roots $\phi$ of $f$ with $H(\phi) \geq 2$ is genetic.

(iii) If $H(f) = 1$ and $\phi$ is a genetic continuous iterative root of $f$, then $H(\phi) = 1$ and $\phi$ has the same characteristic interval as $f$.

Proof. For an indirect proof to (i), assume that $\phi$ is a genetic iterative root of order $n$ of $f$. We prove that

$$N(\phi) < N(\phi^2) < \ldots < N(\phi^n) = N(f).$$

(3)

Otherwise, $N(\phi^k) = N(\phi^{k+1})$ for a certain integer $1 \leq k \leq n - 1$. By Lemma 2.5 of [19], $N(\phi^k) = \cdots = N(\phi^n) = N(\phi^{n+1}) = \cdots = N(\phi^{2n})$, implying that $N(f) = N(f^2)$, a contradiction to the fact that $H(f) > 1$. However, (3) contradicts to the fact

$$S(\phi) = S(f),$$

(4)

which is implied by Lemma 3.1. Thus, result (i) is proved.

The proof to the case of (ii) is similar. Assume that there is a continuous and genetic iterative root $\phi$ of order $n$ of $f$ with $H(\phi) \geq 2$. Then

$$N(\phi) < N(\phi^2) \leq \ldots \leq N(\phi^n) = N(f).$$

which contradicts to (4). This completes the proof of (ii).

In the case of (iii), we also have (4), i.e., $S(\phi) = S(f)$. As shown in Lemma 3 of [12], $\phi$ has a characteristic interval, denoted by $J$. Since $[\min f, \max f] \subset [\min \phi, \max \phi] \subset J$, we see that $f$ and $\phi$ have the same characteristic interval $J$. Thus, this completes the proof of (iii) and therefore the proof is completed.

In what follows, our discussion is concentrated to the case that $H(f) = H(\phi) = 1$. In the case that $H(f) = 1$, the mapping $f$ has a unique characteristic interval as indicated in [16, 19], denoted by $K(f)$, and all iterative roots $\phi$ of $f$ with $H(\phi) = 1$ are constructed in the mode of 1-extension

$$\phi(x) := \begin{cases} \phi_0(x), & \forall x \in K(f), \\ f_{K(f)}^{-1} \circ \phi_0 \circ f(x), & \forall x \in I \setminus K(f), \end{cases}$$

(5)

as shown in the proof of Theorem 1 in [12], where $\phi_0$ is a continuous and monotone iterative roots of $f$ on $K(f)$. Let $f$ be a PM function $f$ of $H(f) = 1$ and $\phi$ be its an iterative root of $H(\phi) = 1$ on the characteristic interval $K(f)$. There are three possibilities:

- **(R-II)**: Both $f$ and $\phi$ are increasing on $K(f)$;
- **(R-ID)**: $f$ is increasing on $K(f)$ but $\phi$ is decreasing on $K(f)$;
- **(R-DD)**: Both $f$ and $\phi$ are decreasing on $K(f)$.

For convenience, let $f$ have the set of forts $S(f) := \{c_1, c_2, \ldots, c_{N(f)}\}$ and assume

$$a = c_0 < c_1 < c_2 < \ldots < c_{N(f)} < c_{N(f) + 1} = b.$$

The set $S(f)$ splits the interval $I$ into $N(f) + 1$ subintervals

$$I_i := [c_i, c_{i+1}], \ i = 0, \ldots, N(f),$$

called monotone subinterval, so that $I = \bigcup_{i=0}^{N(f)} I_i$ and $f$ is monotone on each $I_i$. We say that $f$ admits the partition $I(f) := \{I_i : i = 0, 1, \ldots, N(f)\}$. Clearly, there is a unique $\ell \in \{0, 1, \ldots, N(f)\}$ such that $I_\ell = K(f)$. By (5), $f$ has the same
monotonicity as $\phi$ on each $I_i \in I(f)$ in the cases (R-II) and (R-DD), but the opposite monotonicity to $\phi$ on each $I_i \in I(f)$ in the case (R-ID).

**Theorem 3.3.** ("non-genetic roots") Let $f \in PM(I, I)$ with $H(f) = 1$ and $\phi$ be an iterative root of $f$ with $H(\phi) = 1$ such that (R-ID) holds. Then $\phi$ is not genetic.

The proof of Theorem 3.3 needs the following lemmas.

**Lemma 3.4.** Let $f \in PM(I, I)$ with $H(f) = 1$ and $\phi$ be a genetic iterative root of $f$ with $H(\phi) = 1$ such that (i) holds with a homeomorphism $h : I \rightarrow I$. Then

(i) for each $i \in \{0, 1, ..., N(f)\}$ there exists $j \in \{0, 1, ..., N(f)\}$ such that $h(I_i) \subseteq I_j$;

(ii) $h(c_i) = c_i$ for all $i = 0, 1, ..., N(f) + 1$ in the case that $h$ is an orientation-preserving homeomorphism on $I$;

(iii) $h(c_i) = c_{N(f)+1-i}$ for all $i = 0, 1, ..., N(f) + 1$ in the case that $h$ is an orientation-reserving homeomorphism on $I$.

**Proof.** For an indirect proof, assume that there is $i \in \{0, 1, ..., N(f)\}$ such that $h(I_i) \not\subseteq I_j$, $\forall j = 0, 1, ..., N(f)$.

Then, by the continuity of $h$, there are two interior points $x_1, x_2 \in I_i$ such that $h(x_1) \in I_k$ and $h(x_2) \in I_{k+1}$

where $I_k$ and $I_{k+1}$ are two consecutive sub-intervals in $I(f)$. It follows that $f \circ h$ is non-monotone on the subinterval $I_i$ because the monotonicity of $f$ on $I_k$ is different from the monotonicity of $f$ on $I_{k+1}$. However, $\phi$ is a genetic iterative root of $f$, i.e., $f \circ h = h \circ \phi$. It further implies that $h \circ \phi$ is non-monotone on the subinterval $I_i$, but it contradicts to the fact that $h \circ \phi$ is monotone on every $I_j$ in $I(f)$, which holds because $h$ is a homeomorphism on $I$ and $\phi$ is monotone on $I_j$. This proves result (i).

In order to prove (ii), let us consider $h$ to be an orientation-preserving homeomorphism on $I$ additionally. Then $h(c_0) = c_0$. We claim that

$$h(I_i) = I_i$$

for all $i = 1, 2, ..., N(f)$. In fact, by result (i) of this lemma and the continuity of $h$, we see that $h(I_0) \subseteq I_0$. Assume that $h(I_0) \subset I_0$ but $h(I_0) \neq I_0$. Then, by (6) and the continuity of $h$, $h(I_1) \subset I_0$ because $I_1$ is adjacent to $I_0$. It follows that

$$h(I_0 \cup I_1) \subset I_0.$$  

Note from result (i) that $\bigcup_{i=2}^{N(f)} I_i$ will be mapped by $h$ into at most $N(f) - 1$ sub-intervals in $I(f)$. It follows from (8) that the whole interval $I = \bigcup_{i=0}^{N(f)} I_i$ will be mapped by $h$ into at most $N(f)$ sub-intervals in $I(f)$, which shows that $h$ is not a surjection on $I$, a contradiction to the fact that $h$ is a homeomorphism. This contradiction shows that

$$h(I_0) = I_0,$$  

showing $h(c_1) = c_1$. Furthermore, we assume that (7) holds for a certain $j \in \{2, ..., N(f) - 1\}$, i.e.,

$$h(I_k) = I_k, \forall k = 0, 1, ..., j.$$  

It follows that $h$ is an orientation-preserving homeomorphism on $I \setminus \bigcup_{i=0}^{j} I_i = \bigcup_{i=j+1}^{N(f)} I_i$. By result (i) of this lemma and the continuity of $h$, we see that $h(I_{j+1}) \subseteq$
I_{j+1}$ since $h(c_{j+1}) = c_{j+1}$. Notice that $h$ is not a surjection on $I$ if $h(I_{j+1}) \subset I_{j+1}$ with $h(c_{j+2}) \neq c_{j+2}$, which contradicts to that $h$ is a homeomorphism. Thus, $h(I_{j+1}) = I_{j+1}$. This proves the claim inductively. Hence, it concludes that $h(c_i) = c_i$ for all $i = 1, 2, \ldots, N(f)$ and completes the proof of (i) since $h$ is a homeomorphism on $I$. For result (iii), knowing from the fact that $h$ is an orientation-preserving homeomorphism on $I$, we have $h(c_0) = c_{N(f)+1}$. A similar discussion to the proof of (9) gives that

$$h(I_0) = I_{N(f)}.$$  

Furthermore, similarly to (7), we can prove by induction that

$$h(I_i) = I_{N(f)-i}, \quad \forall \ i = 0, 1, \ldots, N(f),$$

implying that $h(c_i) = c_{N(f)+1-i}$ for all $i = 0, 1, \ldots, N(f) + 1$. This completes the proof of (iii) and therefore the proof is completed.

Proof of Theorem 3.3. For an indirect proof, assume that $\phi$ is genetic. Then, there is a homeomorphism $h : I \to I$ such that (1) holds.

If $h$ is an orientation-preserving homeomorphism on $I$, then $h_\ell = h|_{K(f)}$ is an orientation-preserving homeomorphism on $K(f)$ by (ii) in Lemma 3.4. It follows that $h \circ \phi|_{K(f)} = h_\ell \circ \phi|_{K(f)}$ is strictly decreasing, since $\phi(K(f)) \subset K(f)$ and $\phi$ is strictly decreasing on $K(f)$. However, $f \circ h|_{K(f)} = f|_{K(f)} \circ h_\ell$ is strictly increasing by the monotonicity of $f|_{K(f)}$ and $h_\ell$. This gives a contradiction. Thus, there are no orientation-preserving homeomorphic solutions such that (1) holds.

If $h$ is an orientation-preserving homeomorphism on $I$, then one can see from (iii) in Lemma 3.4 that

$$h \circ \phi(I_\ell) = h_\ell \circ \phi_\ell(I_\ell) \subset h(I_\ell) = I_{N(f)+1-\ell},$$

$$f \circ h(I_\ell) = f(I_{N(f)+1-\ell}) \subset I_\ell,$$

(10) (11)

since $I_\ell$ is the characteristic interval of both $f$ and $\phi$. Noting that $h \circ \phi = f \circ h$, we get that

$$N(f) + 1 - \ell = \ell, \quad \text{i.e.,} \quad N(f) \equiv 1 \pmod{2}.$$  

(12)

On the other hand, notice that both $h$ and $\phi$ are strictly increasing on $I_\ell$. Then, so is $h \circ \phi$ by (10). It follows that $f \circ h$ is strictly increasing on $I_\ell$. By (11), we get that $f$ is strictly decreasing on $I_{N(f)+1-\ell}$. Furthermore, it is assumed that $f$ is strictly increasing on $I_\ell$. Then, we get that

$$N(f) + 1 - \ell - \ell \equiv 1 \pmod{2}, \quad \text{i.e.,} \quad N(f) \equiv 0 \pmod{2},$$

which contradicts to (12). Thus, there are no orientation-preserving homeomorphic solutions such that (1) holds. Therefore, the proof is completed.

Theorem 3.3 suggests a question: Does $f \in PM(I, I)$ with $H(f) = 1$ has a genetic iterative root $\phi$ with $H(\phi) = 1$ if $\phi$ and $f$ have the same monotonicity, i.e., in cases (R-II) and (R-DD)? This question will be fully answered in next section.

4. Genetic roots. In this section, we assume that $f \in PM(I, I)$ with $H(f) = H(\phi) = 1$ and consider cases (R-II) and (R-DD), in which $f$ and its iterative root $\phi$ have the same monotonicity on the characteristic interval $K(f)$. We still use the notation $I_\ell = K(f)$, i.e., $K(f) = [c_\ell, c_{\ell+1}]$.

Lemma 4.1. Suppose that $f \in PM(I, I)$ with $H(f) = 1$ and $\phi$ is an iterative root of $f$ with $H(\phi) = 1$ such that either (R-II) or (R-DD) holds. Then conjugation equation (1) has no orientation-reversing homeomorphic solutions $h : I \to I$. 


Proof. For an indirect proof, assume that $h$ is an orientation-reversing homeomorphic solution of equation (1). Then, one can see that for each $i \in \{0, 1, ..., N(f)\}$

$$h \circ \phi(I_i) \subset h(I_i)$$

and $f \circ h(I_i) = f(I) \subset I_i$

since $\phi(I_i) \subset I_i$. It implies that $h(I_i) = I_i$ since $h \circ \phi = f \circ h$. However, by (12) and the results in (ii),

$$h(I_\ell) = h(I_\ell) = h([c_\kappa, c_{\kappa+1}]) = [c_{\kappa-1}, c_\kappa] = I_{\kappa-1} \neq I_\ell,$$

where $\kappa := (N(F) + 1)/2$. It leads to a contradiction and therefore the proof is completed. \hfill \Box

By Lemma 4.1, the conjugation between a PM function and its an iterative root cannot be orientation-reversing in the cases (R-II) and (R-DD). In what follows, we find orientation-preserving homeomorphic solutions for equation (1) in the cases (R-II) and (R-DD).

4.1. Case (R-II). For convenience, let $\text{Fix}(f)$ denote the set of all fixed points of $f$ and let $S_*(f) := S(f) \cup \{c_0, c_{N(f)+1}\}$, treating the endpoints $c_0, c_{N+1}$ as the generalized forts. Consider

$$\mathcal{A}(f) := f(S_*(f)) \setminus \text{Fix}(f),$$

called the \textit{summit set} of $f$, which consists of those values of $f$ at (generalized) forts which do not hit a fixed point of $f$. The simplest case is that $\mathcal{A}(f) = \emptyset$, i.e., $f(c_0), ..., f(c_{N(f)+1}) \in \text{Fix}(f)$. We have

$$\mathcal{A}(f) \subset K(f) \setminus \text{Fix}(f). \tag{13}$$

In fact, $\text{Fix}(f) \subset K(f)$ and

$$f(S_*(f)) = \{f(c_0), f(c_1), ..., f(c_{N(f)+1})\} \subset K(f)$$

since the height $H(f) = 1$ implies that $f(I) \subset K(f)$. Furthermore, $\mathcal{A}(f)$ admits the decomposition

$$\mathcal{A}(f) = \bigcup_{j=1}^{m} \mathcal{A}_j(f), \quad \mathcal{A}_j(f) := \mathcal{A}(f) \cap (\alpha_j, \beta_j), \tag{14}$$

where $1 \leq m \leq N(f) + 2$ is a certain integer and $\alpha_j, \beta_j \in \text{Fix}(f)$ such that

$$\alpha_1 < \beta_1 \leq \cdots \leq \alpha_m < \beta_m \quad \text{and} \quad (\alpha_j, \beta_j) \cap \text{Fix}(f) = \emptyset \quad \text{for each} \ j = 1, ..., m.$$

Clearly, $\mathcal{A}_j \cap \mathcal{A}_k = \emptyset$ for all $j \neq k$ in $\{1, ..., m\}$. Actually, (13) shows that for each $\theta \in \mathcal{A}(f)$ there exist uniquely two consecutive fixed points $\alpha$ and $\beta$ of $f$, for which we assume $\alpha < \beta$ without loss of generality, such that $\theta \in (\alpha, \beta)$. Since the cardinal $\#\mathcal{A}(f)$ is finite, we can find finitely many distinct subintervals $(\alpha_j, \beta_j)$, $j = 1, ..., m$, where $\alpha_j < \beta_j$ are a pair of consecutive fixed points of $f$ for each $j$, such that $\bigcup_{j=1}^{m}(\alpha_j, \beta_j) \supset \mathcal{A}$. This defines a decomposition of $\mathcal{A}$ as shown in (14). Furthermore, each $\mathcal{A}_j(f)$ can be grouped in different orbits, that is,

$$\mathcal{A}_j(f) = \bigcup_{i=1}^{s(j)} \mathcal{A}_{ji},$$

where $1 \leq s(j) \leq N(f) + 2$ is an integer, $\mathcal{A}_{ji} \subset \text{Orb}_f(f(c_{ji}))$ with an value $f(c_{ji}) \in \mathcal{A}_{ji}$ for each $i = 1, ..., s(j)$ and $\text{Orb}_f(f(c_{ji})) \cap \text{Orb}_f(f(c_{jk})) = \emptyset$ for $i \neq k$. Here $\text{Orb}_f(x_0)$ denotes the orbit of $f$ starting from $x_0$.

Theorem 4.2. (“genetic roots”) Let $f \in PM(I, I)$ with $H(f) = 1$ and $\phi$ be an iterative root of $f$ of order $n$ with $H(\phi) = 1$ such that (R-II) holds. Then the following assertions hold:
In case \( \mathcal{A}(f) = \emptyset \), equation (1) has infinitely many orientation-preserving homeomorphic solutions \( h : I \to I \).

(ii) In case \( \mathcal{A}(f) \neq \emptyset \), equation (1) has an orientation-preserving homeomorphic solution \( h : I \to I \) if and only if for each \( j = 1, \ldots, s \), every sub-branch \( A_{ji}(f) \), \( i = 1, \ldots, s(j) \), is a singleton, denoted by \( f(c_{ji}) \) where \( c_{ji} \in S_n(f) \), and there is an integer \( 1 \leq k \leq s(j) \) such that \( f(c_{ji}) \) lies between \( f(c_{jk}) \) and \( f^2(c_{jk}) \). Theorem 4.2 shows that root \( \phi \) is genetic if either all points \( f(c_0), \ldots, f(c_{N(f)+1}) \) are fixed points of \( f \) or every branch \( A_j(f) \) is a singleton. This theorem also indicates that the root \( \phi \) is not genetic if a branch \( A_j(f) \) either does not lie on the same orbit of \( f \) or is not a singleton in the opposite case.

In order to prove this theorem, we need the following lemmas.

**Lemma 4.3.** ("genetic roots") Let \( f \in CI(I, I) \) and \( \phi \in CI(I, I) \) be an iterative root of order \( n \) of \( f \). Then the conjugation equation (1) has infinitely many orientation-preserving homeomorphic solutions \( h : I \to I \).

Proof. We only consider that \( f \) has no fixed points in the interior of \( I \). Otherwise, \( I \) is a union of closed intervals, each of which is bounded either by two consecutive fixed points of \( f \) or by one fixed point of \( f \) and a or \( b \), one of the endpoints. Let \( U_k \) denote such a closed interval. Clearly, \( f \) has no fixed points in the interior of \( U_k \). If there is an orientation-preserving homeomorphism \( h_k : U_k \to U_k \) such that \( h_k \circ \phi|_{U_k} = f|_{U_k} \circ h_k \), then the mapping \( h : I \to I \) defined by

\[
h(x) := h_k(x), \quad \forall x \in U_k, \quad \text{for all } k,
\]

is an orientation-preserving homeomorphic solution of (1).

When \( f \) has no fixed points in the interior of \( I \), without loss of generality, we assume that \( f(x) < x \) for all \( x \in (a, b) \) since the proof is totally similar if \( f(x) > x \) for all \( x \in (a, b) \). Then, it suffices to discuss the following two cases: (I-1) \( a = f(a) < f(b) = b \); (I-2) \( a = f(a) < f(b) < b \).

In case (I-1), it is easy to check that \( \phi(x) < x \) and \( a = \phi(a) < \phi(b) = b \). Choosing \( x_0, y_0 \in (a, b) \) arbitrarily, one can define the sequences \( (x_k) \) and \( (y_k) \) such that

\[
x_k = \phi^k(x_0), \quad y_k = f^k(y_0), \quad \forall k = \pm 1, \pm 2, \ldots \quad (15)
\]

Clearly, the sequence \( (x_k) \) is strictly decreasing and

\[
a = \lim_{k \to -\infty} x_k < \lim_{k \to -\infty} x_k = b
\]

because \( \phi \) is orientation-preserving on \( I \) and \( \phi(x) < x \). Similarly, the sequence \( (y_k) \) is also strictly decreasing and

\[
a = \lim_{k \to +\infty} y_k < \lim_{k \to +\infty} y_k = b.
\]

Let \( I_i := [x_{i+1}, x_i] \) and \( J_i := [y_{i+1}, y_i] \) for each \( i \in \mathbb{Z} \). Then,

\[
I = \bigcup_{i=-\infty}^{+\infty} I_i, \quad I = \bigcup_{i=-\infty}^{+\infty} J_i.
\]
Obviously, there are infinitely many orientation-preserving homeomorphisms \( h_0 : I_0 := [x_1, x_0] \to J_0 := [y_1, y_0], \) which naturally satisfy

\[
h_0(x_0) = y_0 \text{ and } h_0(x_1) = y_1.
\]

(16)

Each \( h_0 \) defines a sequence \((h_i)\) uniquely by the recursions

\[
h_i(x) := f \circ h_{i-1} \circ \phi^{-1}(x), \quad \forall \ x \in I_i, \quad \text{for } i > 0,
\]

\[
h_i(x) := f^{-1} \circ h_{i+1} \circ \phi(x), \quad \forall \ x \in I_i, \quad \text{for } i < 0.
\]

(17)

Let

\[
h(x) := \begin{cases} 
    h_i(x), & \forall \ x \in I_i, \ i \in \mathbb{Z}, \\
    a, & x = a, \\
    b, & x = b,
\end{cases}
\]

which can be verified easily to satisfy equation (1). Further, we claim that \( h \) is continuous. In fact, by (15) and (16) we have

\[
h_1(x_1) = f \circ h_0 \circ \phi^{-1}(x_1) = f \circ h_0(x_0) = f(y_0) = y_1,
\]

\[
h_1(x_2) = f \circ h_0 \circ \phi^{-1}(x_2) = f \circ h_0(x_1) = f(y_1) = y_2.
\]

For the same reason,

\[
h_{-1}(x_0) = f^{-1} \circ h_0 \circ \phi(x_0) = f^{-1} \circ h_0(x_1) = f^{-1}(y_1) = y_0,
\]

\[
h_{-1}(x_1) = f^{-1} \circ h_0 \circ \phi(x_1) = f^{-1} \circ h_0(x_0) = f^{-1}(y_0) = y_{-1}.
\]

One can prove by induction that

\[
h_i(x_i) = y_i \text{ and } h_i(x_{i+1}) = y_{i+1},
\]

(18)

implying that \( h_i \) is an orientation-preserving homeomorphism from \( I_i \) onto \( J_i \) for all \( i \in \mathbb{Z} \). It follows from (18) that \( h \) is continuous at each point \( x_i \) for all \( i \in \mathbb{Z} \). This proves the claimed continuity of \( h \). Obviously, \( h \) is an orientation-preserving homeomorphism on \( I \).

In case \( \textbf{(I-2)} \), it is easy to check that \( \phi(x) < x \) for all \( x \in (a, b) \) and \( a = \phi(a) < \phi(b) < b \). Choose initial points \( x_0 = \phi(b) \) and \( y_0 = f(b) \), which define sequences \((x_i)\) and \((y_i)\) respectively by (15). Similarly to case \( \textbf{(I-1)} \), one can find infinitely many orientation-preserving homeomorphisms \( \tilde{h} : [a, \phi(b)] \to [a, f(b)] \) satisfying equation (1) such that

\[
\tilde{h}(\phi^i(b)) = f^i(b), \quad \forall \ k \in \mathbb{Z}^+ \setminus \{0\}.
\]

(19)

Outside \([a, \phi(b)]\), define

\[
\tilde{h}(x) := f^{-1} \circ \tilde{h} \circ \phi(x), \quad \forall \ x \in [\phi(b), b] .
\]

Finally, let

\[
h(x) := \begin{cases} 
    \tilde{h}(x), & \forall \ x \in [a, \phi(b)], \\
    \tilde{h}(x), & \forall \ x \in [\phi(b), b],
\end{cases}
\]

which satisfies equation (1). Furthermore,

\[
\tilde{h}(\phi(b)) = f^{-1} \circ \tilde{h} \circ \phi^2(b) = f^{-1} \circ f^2(b) = f(b),
\]

\[
\tilde{h}(b) = f^{-1} \circ \tilde{h} \circ \phi(b) = f^{-1} \circ f(b) = b.
\]

By (19) we can check that \( h \) is an orientation-preserving homeomorphism. Therefore, the proof is completed. \( \square \)
Lemma 4.4. Let \( f \in PM(I,I) \) with \( H(f) = 1 \) and \( \phi \) be an iterative root of \( f \) of order \( n \) with \( H(\phi) = 1 \) such that (R-II) holds. Suppose that every nonempty branch \( A_j(f) \) is orbitally dependent, i.e., all points in \( A_j(f) \) lie on the same orbit of \( f \). Then equation (1) has an orientation-preserving homeomorphic solution \( h : I \to I \) if and only if each \( A_j(f) \) is a singleton. Additionally, equation (1) has infinitely many such solutions.

Proof. In order to prove the necessity, consider a nonempty branch \( A_j(f) \), in which every point can be presented as \( f(c_i) \) for a certain \( c_i \in S(f) \cup \{c_0,c_{N(f)+1}\} \). Since \( A_j(f) \) is assumed to be contained in an orbit of \( f \), there exists \( c \in S(f) \cup \{c_0,c_{N(f)+1}\} \) such that if \( f(c_i) \in A_j(f) \) then

\[
f(c_i) = f^{s(i)}(c)
\]

for a certain \( s(i) \in \mathbb{Z} \). It follows from (5) that

\[
\phi(c_i) = f|_{K(f)}^{-1} \circ \phi_{t} \circ f(c_i) = f|_{K(f)}^{-1} \circ \phi_{t} \circ f^{s(i)}(c)
\]

\[
= f|_{K(f)}^{-1} \circ \phi_{t} \circ f^{s(i)-1} \circ f(c)
\]

\[
= \phi_{t}^{n s(i) - 2 n + 1} \circ f(c)
\]

for all \( c_i \) satisfying \( f(c_i) \in A_j(f) \) and that

\[
\phi^{s(i)}(c) = \phi_{t}^{s(i)-1} \circ f|_{K(f)}^{-1} \circ \phi_{t} \circ f(c)
\]

\[
= \phi_{t}^{s(i)-n} \circ f(c).
\]

On the other hand, we assumed that there are orientation-preserving homeomorphisms \( h : I \to I \) such that \( h \circ \phi = f \circ h \). By result (ii) of Lemma 3.4,

\[
h \circ \phi^k(c_i) = f^k \circ h(c_i) = f^k(c_i), \quad \forall k \in \mathbb{Z}, \; i = 0, 1, \ldots, N(f) + 1,
\]

which implies that

\[
h \circ \phi(c_i) = f(c_i) \quad \text{and} \quad h \circ \phi^{s(i)}(c) = f^{s(i)}(c)
\]

for all \( c_i \) satisfying that \( f(c_i) \in A_j(f) \). By (20), (21), (22) and (23) we have

\[
\phi(c_i) = \phi^{s(i)}(c), \quad \forall c_i \text{ satisfying that } f(c_i) \in A_j(f),
\]

since \( h \) is homeomorphism on \( I \). It means that

\[
ns(i) - 2n + 1 = s(i) - n,
\]

i.e., \((n - 1)(s(i) - 1) = 0\), implying that

\[
s(i) = 1
\]

because of \( n > 1 \). Thus, one can see from (20) that

\[
f(c_i) = f(c), \quad \forall f(c_i) \in A_j(f),
\]

which implies that \( \#A_j(f) = 1 \). This completes the proof of the necessity in case (ii).

In what follows, we prove the sufficiency. We only consider the situation \( j = 1 \), i.e.,

\[
A(f) = A_1(f) = \{f(c)\},
\]

a singleton, where \( c \in S(f) \cup \{c_0,c_{N(f)+1}\} \). From (24) we see that

\[
f(S_n(f)) \subset \{f(c)\} \cup \text{Fix}(f).
\]
As shown in the proof of Theorem 1 of [12], the function \( \phi \), being an iterative root of \( f \) of order \( n \) with \( H(\phi) = 1 \), can be presented as

\[
\phi(x) := \begin{cases} 
\phi_\ell(x), & \forall x \in K(f), \\
 f_{|K(f)}^{-1} \circ \phi_\ell \circ f(x), & \forall x \in I \setminus K(f), 
\end{cases}
\]

where \( \phi_\ell \) is an iterative root of \( f_{|K(f)} \). Notice from Theorem 2 of [19] that

\[
\text{Fix}(\phi) = \text{Fix}(f)
\]

because both \( f \) and \( \phi \) are increasing on \( K(f) \). It follows from (25) and (26) that

\[
\begin{cases} 
\phi(y) = f(y), & \text{if } y \in S_1, \\
\phi(y) = \phi(c), f(y) = f(c), & \text{if } y \in S_2,
\end{cases}
\]

where \( S_1 = f^{-1}(\text{Fix}(f)) \cap S_*(f) \) and \( S_2 = f^{-1}(\{f(c)\}) \cap S_*(f) \). Therefore, we are ready to find conjugations between \( f \) and \( \phi \). First, by Lemma 4.3, we can find infinitely many orientation-preserving homeomorphisms \( h_\ell : K(f) \to K(f) \) such that

\[
 h_\ell \circ \phi|_{K(f)} = f_{|K(f)} \circ h_\ell
\]

with initial points \( x_0 = \phi(c) \) and \( y_0 = f(c) \), showing that

\[
 h_\ell \circ \phi^k(c) = f^k(c), \forall k \in \mathbb{Z}_+ \setminus \{0\};
\]

then, we extend the above \( h_\ell \) from \( K(f) \) to the whole \( I \), defining

\[
 h_i(x) := \begin{cases} 
h_\ell(x), & \forall x \in K(f) = I, \\
h_i(x), & \forall x \in I_i, i = 0,1,\ldots,N(f), i \neq \ell,
\end{cases}
\]

satisfies equation (1). Additionally, from (27) and the proof of Lemma 4.3 we see that

\[
\text{Fix}(\phi) = \text{Fix}(f) = \text{Fix}(h_\ell).
\]

It follows from (28) and (29) that

\[
 h_i(c_i) = f_{|I_i}^{-1} \circ h_\ell \circ \phi(c) = \begin{cases} 
 f_{|I_i}^{-1} \circ h_\ell \circ f(c_i) = f_{|I_i}^{-1} \circ f(c_i) = c_i, & \forall c_i \in S_1, \\
 f_{|I_i}^{-1} \circ h_\ell \circ \phi(c) = f_{|I_i}^{-1} \circ f(c) = c_i, & \forall c_i \in S_2,
\end{cases}
\]

i.e., \( h(c_i) = c_i \) for all \( i = 0,1,\ldots,N(f) + 1 \). This proves that \( h \) is an orientation-preserving homeomorphism on \( I \) and completes the proof of sufficiency. Therefore, the whole proof is completed.

**Lemma 4.5.** Let \( f \in \text{PM}(I,I) \) with \( H(f) = 1 \) and \( \phi \) be an iterative root of \( f \) of order \( n \) with \( H(\phi) = 1 \) such that (R-II) holds. Suppose that every nonempty branch \( A_j(f) \) is orbitally independent, i.e., any two points in \( A_j(f) \) do not lie on the same orbit of \( f \). Then equation (1) has infinitely many orientation-preserving homeomorphic solutions \( h : I \to I \) if and only if there is an element \( f(c) \in A_j(f) \) with \( c \in S_*(f) \) such that for every \( f(c) \in A_j(f) \setminus \{f(c)\} \)

\[
\begin{cases} 
f(c) \in \{f(c_1) < f^2(c), \text{ if } x < f(x) \forall x \in (\alpha_j,\beta_j),
\end{cases}
\]

i.e., \( f(c_i) \) lies between \( f(c) \) and \( f^2(c) \), and that for iterative roots \( \phi \)

\[
\begin{cases} 
\phi(c) \in \{\phi(c_1) < \phi^2(c), \text{ if } x < f(x) \forall x \in (\alpha_j,\beta_j),
\end{cases}
\]

i.e., \( f(c_i) \) lies between \( f(c) \) and \( f^2(c) \), and that for iterative roots \( \phi \)
we need i.e., 

This implies that since both \( f(1) < f(2) \) and both \( f(3) \) and \( f(4) \) are not contained in an orbit of \( f \). Thus there are two subcases to be considered: (C-up) there is a positive integer \( k \in \mathbb{Z}^+ \) such that \( f^k(1) < f(2) < f^{k+1}(1) \) if \( x < f(x) \) for all \( x \in (\alpha_n, \beta_n) \); (C-down) there is a positive integer \( k \in \mathbb{Z}^+ \) such that \( f^{k+1}(2) < f(1) < f^k(2) \) if \( x > f(x) \) for all \( x \in (\alpha_n, \beta_n) \). We only consider subcase (C-up) since subcase (C-down) can be discussed similarly. In subcase (C-up), it can be seen from (26) that

\[
\phi_k(d_1) = \phi_k^{-1} \circ f|_{K(f)}^{-1} \circ \phi \circ f(d_1) = \phi_k^{-1} \circ \phi_n \circ \phi \circ f(d_1) = \phi_k^{-n} \circ f(d_1),
\]

\[
\phi_k(d_1) = \phi_k \circ f|_{K(f)}^{-1} \circ \phi \circ f(d_1) = \phi_k \circ \phi_n \circ \phi \circ f(d_1) = \phi_k^{-n+1} \circ f(d_1),
\]

and

\[
\phi_k(d_2) = f|_{K(f)}^{-1} \circ \phi \circ f(d_2) \leq (f|_{K(f)}^{-1} \circ \phi \circ f^k(d_1), f|_{K(f)}^{-1} \circ \phi \circ f^{k+1}(d_1)) = (\phi_k^{-n} \circ \phi \circ \phi_k^{-n-1} \circ \phi \circ f(d_1), \phi_k^{-n} \circ \phi \circ \phi_k \circ f(d_1)) = (\phi_k^{-2n+1} \circ f(d_1), \phi_k^{n-k+1} \circ f(d_1)),
\]

since both \( f|_{K(f)}^{-1} \) and \( \phi \) are increasing. On the other hand, we have

\[
\ldots < \phi_k^{-2}(x) < \phi_k^{-1}(x) < x < \phi(x) < \phi_k(x) < \ldots, \quad \forall x \in (\alpha_n, \beta_n),
\]

since \( x < f(x) \) for all \( x \in (\alpha_n, \beta_n) \). It means that \( \phi_k^{-j}(x) < \phi_k^{-j}(x) \) for any integers \( j < k \) on the subinterval \( (\alpha_n, \beta_n) \). Moreover, we assumed that there are orientation-preserving homeomorphisms \( h : I \rightarrow I \) such that \( h \circ \phi = f \circ h \). It follows from the result (ii) of Lemma 3.4 that

\[
h \circ \phi_k(d_1) = h \circ \phi \circ \phi_k^{-1}(d_1) = f \circ h \circ \phi_k^{-1}(d_1) = \ldots = f^k \circ h(d_1) = f^k(d_1)
\]

i.e.,

\[
h \circ \phi_k(d_1) = f^k(d_1) \quad \forall \quad i = 1, 2, k \in \mathbb{N}.
\]

This implies that

\[
\phi^k(d_1) < \phi(d_2) < \phi^{k+1}(d_1)
\]

since \( h \) is an orientation-preserving homeomorphism. In order to guarantee the following inequalities

\[
\begin{cases}
\phi^{n-k+1} \circ f(d_1) < \phi(d_2) < \phi^{n-k+1} \circ f(d_1), \\
\phi^k(d_1) < \phi(d_2) < \phi^{k+1}(d_1),
\end{cases}
\]

i.e.,

\[
\begin{cases}
\phi^{n-k+1} \circ f(d_1) < \phi(d_2) < \phi^{n-k+1} \circ f(d_1), \\
\phi^k \circ f(d_1) < \phi(d_2) < \phi^{k+1} \circ f(d_1),
\end{cases}
\]

we need

\[
\begin{cases}
nk - n + 1 - (k - n) \geq 0, \\
k - n + 1 - (nk - 2n + 1) \geq 0,
\end{cases}
\]

i.e.,

\[
\begin{cases}
(n - 1)k + 1 \geq 0, \\
(1 - n)k + n \geq 0.
\end{cases}
\]

Since \( n \geq 2 \) and \( k \in \mathbb{Z}^+ \), we get \( k = 1 \), implying that

\[
f(d_1) < f(d_2) < f^2(d_1).
\]
Furthermore, the above two inequalities can be changed into
\[
\begin{align*}
\phi^{-n}_t \circ f(d_1) &< \phi_t \circ f(d_2), \\
\phi^{-n}_t \circ f(d_1) &< \phi^{-2n}_t \circ f(d_1),
\end{align*}
\]
i.e., \(\phi(d_1) < \phi(d_2) < \phi^2(d_1)\). This completes the proof of the necessity.

In what follows, we prove the sufficiency. We only consider the situation \(j = 1\), i.e.,
\[
\mathcal{A}(f) = \mathcal{A}_1(f) = \{f(e), f(d)\},
\]
where \(c, d \in S(f) \cup \{c_0, c_{N(f)+1}\}\). From (30) we see that \(f(S_\ast(f)) \subset \{f(c), f(d)\} \cup \text{Fix}(f)\).

As shown in the proof of Theorem 1 of [12], the function \(\phi\), being an iterative root of \(f\) of order \(n\) with \(H(\phi) = 1\), can be presented as in (26). Notice from Theorem 2 of [19] that \(\text{Fix}(\phi) = \text{Fix}(f)\) because both \(f\) and \(\phi\) are increasing on \(K(f)\). It follows from (26) that
\[
\begin{align*}
\phi(y) &= f(y), \quad \text{if } y \in S_1, \\
\phi(y) &= \phi(c), f(y) = f(c), \quad \text{if } y \in S_2, \\
\phi(y) &= \phi(d), f(y) = f(d), \quad \text{if } y \in S_3,
\end{align*}
\]
where \(S_1 = f^{-1}(\text{Fix}(f)) \cap S_\ast(f)\), \(S_2 = f^{-1}(\{f(c)\}) \cap S_\ast(f)\) and \(S_3 = f^{-1}(\{f(d)\}) \cap S_\ast(f)\). Therefore, we are ready to find conjugations between \(f\) and \(\phi\): First, by Lemma 4.3, we can find infinitely many orientation-preserving homeomorphisms \(h_\ell : K(f) \rightarrow K(f)\) such that
\[
h_\ell \circ \phi|_{K(f)} = f|_{K(f)} \circ h_\ell
\]
with initial points \(x_0 = \phi(c)\) and \(y_0 = f(c)\), showing that
\[
h_\ell \circ \phi^k(c) = f^k(c), \quad \forall \ k \in \mathbb{Z}_+ \setminus \{0\}.
\]
Moreover, the proof of Lemma 4.3 shows that the conjugation mapping \(h_\ell\) only requires the initial one \(h_0 : [x_0, x_1] \rightarrow [y_0, y_1]\) to be an orientation-preserving homeomorphism. Note from the assumption that
\[
\begin{align*}
f(c) < f(d) < f^2(c), & \quad \text{if } x < f(x) \forall x \in (\alpha_j, \beta_j), \\
f^2(c) < f(d) < f(c), & \quad \text{if } f(x) < x \forall x \in (\alpha_j, \beta_j),
\end{align*}
\]
and that for iterative roots \(\phi\)
\[
\begin{align*}
\phi(c) < \phi(d) < \phi^2(c), & \quad \text{if } x < f(x) \forall x \in (\alpha_j, \beta_j), \\
\phi^2(c) < \phi(d) < \phi(c), & \quad \text{if } f(x) < x \forall x \in (\alpha_j, \beta_j).
\end{align*}
\]
It leads us to choose a suitable \(h_0\) which satisfies
\[
h_0(\phi(d)) = f(d),
\]
implying that
\[
h_\ell \circ \phi^k(d) = f^k(d), \quad \forall \ k \in \mathbb{Z}_+ \setminus \{0\}
\]
by (17). Then, we extend the above \(h_\ell\) from \(K(f)\) to the whole \(I\), defining
\[
h_i(x) := f|_{I_i}^{-1} \circ h_\ell \circ \phi(x), \quad \forall \ x \in I_i,
\]
for every \(i \in \{0, 1, \ldots, N(f)\} \setminus \{\ell\}\); finally, we check that the function
\[
h(x) := \begin{cases} 
   h_\ell(x), & \forall \ x \in K(f) = I_\ell, \\
   h_i(x), & \forall \ x \in I_i, \ i = 0, 1, \ldots, N(f), \ i \neq \ell,
\end{cases}
\]
for every \(i \in \{0, 1, \ldots, N(f)\} \setminus \{\ell\}\).
where $\alpha B$ satisfies equation (1). Additionally, from the proof of Lemma 4.3 and the fact $\text{Fix}(\phi) = \text{Fix}(f)$ we see that

$$\text{Fix}(\phi) = \text{Fix}(f) = \text{Fix}(h_t).$$

It follows from (31-34) that

$$h_t(c_i) = f|_{I_t}^{-1} \circ h_t \circ f(c_i) = \begin{cases} f|_{I_t}^{-1} \circ h_t \circ f(c_i) = f|_{I_t}^{-1} \circ f(c_i) = c_i, & \forall c_i \in S_1, \\
 f|_{I_t}^{-1} \circ h_t \circ \phi(c) = f|_{I_t}^{-1} \circ f(c) = c_i, & \forall c_i \in S_2, \\
 f|_{I_t}^{-1} \circ h_t \circ \phi(d) = f|_{I_t}^{-1} \circ f(d) = c_i, & \forall c_i \in S_3, \\
\end{cases}$$

i.e., $h(c_i) = c_i$ for all $i = 0, 1, \ldots, N(f) + 1$. This proves that $h$ is an orientation-preserving homeomorphism on $I$ and completes the proof of sufficiency. Therefore, the whole proof is completed.

**Proof of Theorem 4.2.** We first consider case (i), i.e., $\mathcal{A}(f) = \emptyset$. In this case,

$$f(S_*(f)) \subset \text{Fix}(f).$$

By Theorem 2 of [19], all periodic points of $f$ lie in the characteristic interval $K(f) = I_t := [c_t, c_{t+1}]$, which implies that

$$\{c_t, c_{t+1}\} \in \text{Fix}(f)$$

because $f$ is increasing on $K(f)$. Then, a similar discussion to the sufficiency proof of Lemma 4.4 gives that the roots $\phi$ satisfy the formula (26) and conjugations $h : I \to I$ between $f$ and $\phi$ satisfy the formula (29) with the equalities $\text{Fix}(\phi) = \text{Fix}(f) = \text{Fix}(h_t)$. It follows from (35) that

$$\phi(c_i) = f|_{K(f)}^{-1} \circ h_t \circ f(c_i) = f|_{K(f)}^{-1} \circ f(c_i) = f(c_i)$$

for each $i = 0, 1, \ldots, N(f) + 1$. Hence, we have

$$h_i(c_i) = f|_{I_t}^{-1} \circ h_t \circ \phi(c_i) = f|_{I_t}^{-1} \circ h_t \circ f(c_i) = f|_{I_t}^{-1} \circ f(c_i) = c_i,$$

i.e., $h(c_i) = c_i$ for all $i = 0, 1, \ldots, N(f) + 1$. This proves that $h$ is an orientation-preserving homeomorphism on $I$ and completes the proof in case (i).

Lemmas 4.4 and 4.5 give the proof of case (ii). Therefore, this completes the whole proof.

**4.2. Case (R-DD).** For convenience, let

$$B(f) := f(S(f) \cup \{S_*(f)\}) \setminus \text{Fix}(f^2),$$

i.e., the collection of values of $f$ at generalized forts but none of those values hits a 2-periodic point or a fixed point of $f$. The simplest case is that $B(f) = \emptyset$, i.e.,

$$f(c_0), ..., f(c_{N(f)+1}) \in \text{Fix}(f) \cup \mathcal{P}_2(f),$$

where $\mathcal{P}_2(f)$ denotes the set of all 2-periodic points of $f$.

Note that $\text{Fix}(f) \cup \mathcal{P}_2(f) \subset K(f)$ and $f(S_*(f)) \subset K(f)$ since the height $H(f) = 1$. It follows that $B(f) \subset K(f) \setminus \text{Fix}(f^2)$. On the other hand, the cardinal $\#\text{Fix}(f)$ is 1 since $f$ is strictly decreasing on $K(f)$. Thus, let $B(f)$ admits the decomposition

$$B(f) = \bigcup_{j=1}^{s(j)} B_j(f), \quad B_j(f) = B(f) \cap (\alpha_j, \beta_j)$$

where $s(j)$ is an integer such that $1 \leq s(j) \leq N(f) + 2$ and $\alpha_j, \beta_j \in \text{Fix}(f^2)$ satisfy $\alpha_1 < \beta_1 \leq \ldots \leq \alpha_m < \beta_m$ and $(\alpha_j, \beta_j) \cap \text{Fix}(f^2) = \emptyset$ for each $j = 1, \ldots, s(j)$. 

Clearly, $B_j \cap B_k = \emptyset$ for all $j \neq k$ in $\{1, \ldots, m\}$. Furthermore, each $B_j(f)$ can be grouped in different orbits, that is,

$$B_j(f) = \bigcup_{i=1}^{s(j)} B_{ji},$$

where $1 \leq s(j) \leq N(f) + 2$ is an integer, $B_{ji} \subset \text{Orb}_f(f(c_{ji}))$ with an value $f(c_{ji}) \in B_{ji}$ for each $i = 1, \ldots, s(j)$ and $\text{Orb}_f(f(c_{ji})) \cap \text{Orb}_f(f(c_{jk})) = \emptyset$ for $i \neq k$. Here $\text{Orb}_f(x_0)$ denotes the orbit of $f$ initiated from $x_0$.

**Theorem 4.6.** ("genetic roots") Let $f \in PM(I, I)$ with $H(f) = 1$ and $\phi$ be an iterative root of $f$ of order $n$ with $H(\phi) = 1$ such that (R-DD) holds. Then the following assertions hold:

(i) In case $\mathcal{B}(f) = \emptyset$, the conjugation equation (1) has infinitely many orientation-preserving homeomorphic solutions $h : I \to I$.

(ii) In case $\mathcal{B}(f) \neq \emptyset$, equation (1) has an orientation-preserving homeomorphic solution $h : I \to I$ if and only if for each $j = 1, \ldots, m$ every sub-branch $B_{ji}(f)$, $i = 1, \ldots, s(j)$, is a singleton, denoted by $f(c_{ji})$ where $c_{ji} \in S_n(f)$, and there is an integer $1 \leq k \leq s(j)$ such that $f(c_{ji})$ lies between $f(c_{jk})$ and $f^2(c_{jk})$ and $\phi(c_{ji})$ lies between $\phi(c_{jk})$ and $\phi^2(c_{jk})$, i.e.,

$$\begin{cases} f^3(c_{jk}) < f(c_{ji}) < f(c_{jk}), & \phi^3(c_{jk}) < \phi(c_{ji}) < \phi(c_{jk}), \text{ if } x < f^2(x) \text{ on } (\alpha_j, \beta_j), \\ f(c_{jk}) < f(c_{ji}) < f^3(c_{jk}), & \phi(c_{jk}) < \phi(c_{ji}) < \phi^3(c_{jk}), \text{ if } f^2(x) < x \text{ on } (\alpha_j, \beta_j), \end{cases}$$

for all $i \neq k$.

**Lemma 4.7.** ("genetic roots") Let $f \in CD(I, I)$ and $\phi \in CD(I, I)$ be an iterative root of odd order $n$ of $f$. Suppose that either $a < f(b) < f(a) < b$ or $a = f(b) < f(a) = b$. Then there are infinitely many orientation-preserving homeomorphisms $h : I \to I$ such that equation (1) holds.

**Proof.** We only consider the situation that $f$ has no 2-periodic points in the interior of $I$. Otherwise, $I$ is a union of closed intervals, each of which is bounded either by two consecutive 2-period points or by one 2-period point and an endpoint. Let $U_k$ denote such a closed interval. If we can find an orientation-preserving homeomorphism $h_k : U_k \cup f(U_k) \to U_k \cup f(U_k)$ for all $k$ such that $h_k \circ \phi|_{U_k} = f|_{U_k} \circ h_k$, then the mapping $h : I \to I$ defined by

$$h(x) := h_k(x), \quad \forall \ x \in I_k, \text{ for all } k,$$

is an orientation-preserving homeomorphic solution of (1).

Since we consider $f$ has no 2-periodic points in the interior of $I$, we only need to discuss in the two cases: (I-1) $a = f(b) < f(a) = b$, and (I-2) $a < f(b) < f(a) < b$.

In case (I-1), let $\eta$ be the unique fixed point of $f$. We only prove that $\eta$ is an attractive fixed point because the case that $\eta$ is a repulsive fixed point can be proved similarly. It is easy to check that $\eta$ is also an attractive fixed point of $\phi$ and $a = \phi(b) < \phi(a) = b$. Choose initial points $x_0, y_0 \in (a, b)$ arbitrarily and extend them to two infinite-sequences $(x_k)$ and $(y_k)$ respectively as

$$x_k = \phi^k(x_0), \quad y_k = f^k(y_0), \quad \forall \ k = \pm 1, \pm 2, \ldots \quad (36)$$

Clearly,

$$a = \lim_{k \to -\infty} x_{2k} < \lim_{k \to -\infty} x_{2k+1} = b, \quad \lim_{k \to +\infty} x_{2k} = \eta = \lim_{k \to +\infty} x_{2k+1},$$
because $\phi$ is orientation-reversing on $I$ and the fixed point $\eta$ is attractive. Similarly, we have

\[ a = \lim_{k \to -\infty} y_{2k} < \lim_{k \to -\infty} y_{2k+1} = b, \quad \lim_{k \to +\infty} y_{2k} = \eta = \lim_{k \to +\infty} y_{2k+1}. \]

Let $I_i$ denote the compact interval between $x_i$ and $x_{i+2}$ for each $i \in \mathbb{Z}$. Then $I = \bigcup_{i=-\infty}^{+\infty} I_i$. Moreover, Let $J_i$ denote the compact interval between $y_i$ and $y_{i+2}$ for each $i \in \mathbb{Z}$. We similarly have $I = \bigcup_{i=-\infty}^{+\infty} J_i$. Starting from $I_0 := [x_0, x_2]$ and $J_0 := [y_0, y_2]$, we easily define a homeomorphism $h_0 : I_0 \to J_0$ such that

\[ h_0(x_0) = y_0 \text{ and } h_0(x_2) = y_2. \]  \hspace{1cm} (37)

Each $h_0$ defines a sequence $(h_i)$ uniquely by the recursions

\[ h_i(x) = f \circ h_{i-1} \circ \phi^{-1}(x), \quad \forall x \in I_i, \text{ for } i > 0; \] \hspace{1cm} (38)

\[ h_i(x) = f^{-1} \circ h_{i+1} \circ \phi(x), \quad \forall x \in I_i, \text{ for } i < 0. \] \hspace{1cm} (39)

Let

\[ h(x) := \begin{cases} h_i(x), & \text{for } x \in I_i, \ i \in \mathbb{Z}, \\ a, & \text{for } x = a, \\ b, & \text{for } x = b, \end{cases} \] \hspace{1cm} (40)

which can be verified easily by equation (1). In fact, by (36) and (37) we have

\[ h_1(x_1) = f \circ h_0 \circ \phi^{-1}(x_1) = f \circ h_0(x_0) = f(y_0) = y_1, \]

\[ h_1(x_3) = f \circ h_0 \circ \phi^{-1}(x_3) = f \circ h_0(x_2) = f(y_2) = y_3. \]

For the same reason,

\[ h_{-1}(x_1) = f^{-1} \circ h_0 \circ \phi(x_1) = f^{-1} \circ h_0(x_2) = f^{-1}(y_2) = y_1, \]

\[ h_{-1}(x_{-1}) = f^{-1} \circ h_0 \circ \phi(x_{-1}) = f^{-1} \circ h_0(x_0) = f^{-1}(y_0) = y_{-1}. \]

One can prove by induction that

\[ h_i(x_i) = y_i, \quad h_i(x_{i+2}) = y_{i+2}, \quad \forall i \in \mathbb{Z} \]

implying that $h_i$ is an orientation-preserving homeomorphism from $I_i$ onto $J_i$ for all $i \in \mathbb{Z}$. It follows from (18) that $h$ is continuous at each point $x_i$ for all $i \in \mathbb{Z}$. This proves the claimed continuity of $h$. Obviously, $h$ is an orientation-preserving homeomorphism on $I$.

In case (I-2), i.e., $a < f(b) < f(a) < b$, we construct orientation-preserving homeomorphisms $h : I \to I$ by (38), (39) and (40) as done in the same manner as in the above case (I-1) with either $x_0 = f(a)$ and $y_0 = g(a)$ or $x_0 = f(b)$ and $y_0 = g(b)$ for the starting points in (36). Therefore, this completes the proof. \hfill \Box

**Proof of Theorem 4.6.** We first consider case (i), i.e., $\mathcal{B}(f) = \emptyset$. In this case

\[ f(S_n(f)) \subset \text{Fix}(f) \cup \mathcal{P}_2(f). \] \hspace{1cm} (41)

By Theorem 2 of [19], all periodic points of $f$ lie in the characteristic interval $K(f) = I_t := [c_t, c_{t+1}]$, which implies that

\[ \{ c_t, c_{t+1} \} \subset \mathcal{P}_2(f) \cup \text{Fix}(f) \] \hspace{1cm} (42)

because $f$ is decreasing on $K(f)$. Note that the function $\phi$, being an iterative root of $f$ of order $n$ with $H(\phi) = 1$, can be presented as in (26), i.e.,

\[ \phi(x) := \begin{cases} \phi_{t-1}(x), & \forall x \in K(f), \\ f|_{K(f)} \circ \phi_t \circ f(x), & \forall x \in I \setminus K(f), \end{cases} \]
where $\phi_\ell$ is an iterative root of $f|K(f)$. Further, by Theorem 2 of [19],

$$\text{Fix}(\phi) = \text{Fix}(f) = \{\xi\}, \quad P_2(\phi) = P_2(f)$$

(43)

because both $f$ and $\phi$ are decreasing on $K(f)$. Let

$$B^1(f) := f(S(f) \cup \{c_0, c_{N(f)+1}\}) \cap \text{Fix}(f),$$

$$B^2(f) := f(S(f) \cup \{c_0, c_{N(f)+1}\}) \cap P_2(f).$$

Then by (41) and (43) we have

$$\phi(c_i) = f|_{K(f)}^{-1} \circ \phi_\ell \circ f(c_i)$$

$$= \begin{cases} f|_{K(f)}^{-1} \circ f(c_i), & \forall f(c_i) \in B^1(f) \\ \phi_\ell^{-1} \circ f(c_i), & \forall f(c_i) \in B^2(f) \end{cases}$$

$$= \begin{cases} f(c_i), & \forall f(c_i) \in B^1(f) \\ f(c_i), & \forall f(c_i) \in B^2(f) \end{cases}$$

$$= f(c_i)$$

(44)

for each $i = 0, 1, \ldots, N(f) + 1$ since $n$ is odd. Therefore, we are ready to find conjugations between $f$ and $\phi$. First, by Lemma 4.7, we can find infinitely many orientation-preserving homeomorphisms $h_\ell : K(f) \to K(f)$ such that

$$h_\ell \circ \phi|_{K(f)} = f|_{K(f)} \circ h_\ell.$$

Then, we extend the above $h_\ell$ from $K(f)$ to the whole $I$, defining

$$h_\ell(x) := f|_{I_\ell}^{-1} \circ h_\ell \circ \phi(x), \quad \forall x \in I_\ell,$$

for every $i \in \{0, 1, \ldots, N(f)\} \backslash \{\ell\}$. Finally, we can check that the function

$$h(x) := \begin{cases} h_\ell(x), & \forall x \in K(f) = I_\ell, \\ h_i(x), & \forall x \in I_i, \quad i = 0, 1, \ldots, N(f), \quad i \neq \ell, \end{cases}$$

satisfies equation (1). Additionally, from (42), (43) and the proof of Lemma 4.7 we see that

$$\text{Fix}(h_\ell) = \text{Fix}(f) \cup P_2(f) = \text{Fix}(\phi) \cup P_2(\phi).$$

It follows from (41) and (44) that

$$h_i(c_i) = f|_{I_\ell}^{-1} \circ h_\ell \circ \phi(c_i) = f|_{I_\ell}^{-1} \circ h_\ell \circ f(c_i) = f|_{I_\ell}^{-1} \circ f(c_i) = c_i,$$

i.e., $h(c_i) = c_i$ for all $i = 0, 1, \ldots, N(f) + 1$. This proves that $h$ is an orientation-preserving homeomorphism on $I$ and completes the proof in case (i).

Next, we discuss in case (ii). Let $\varphi = \varphi^2$ and $g = f^2$. Then $\varphi$ is an iterative root of $g$ of order $n$ such that $H(\varphi) = H(g) = 1$. Noting from case (R-DD) holds that $\varphi$ and $g$ satisfy case (R-II). By Theorem 4.2, we see that equation

$$h \circ \varphi(x) = g \circ h(x), \quad \forall x \in I,$$

has an orientation-preserving homeomorphic solution $h : I \to I$ if and only if for each $j = 1, \ldots, m$ every sub-branch $A_{ij}(g)$, $i = 1, \ldots, s(j)$, is a singleton, denoted by $g(c_{j})$, where $c_{j} \in S_s(f)$, and there is an integer $1 \leq k \leq s(j)$ such that $g(c_{j})$ lies between $g(c_{j})$ and $g^2(c_{j})$ and $\varphi(c_{j})$ lies between $\varphi(c_{j})$ and $\varphi^2(c_{j})$, i.e.,

$$\begin{cases} g(c_{j}) < g(c_{j}) < g^2(c_{j}), & \varphi(c_{j}) < \varphi(c_{j}) < \varphi^2(c_{j}), & \text{if } x < g(x) \text{ on } (\alpha_j, \beta_j), \\ g^2(c_{j}) < g(c_{j}) < g(c_{j}), & \varphi^2(c_{j}) < \varphi(c_{j}) < \varphi(c_{j}), & \text{if } x < g(x) \text{ on } (\alpha_j, \beta_j), \end{cases}$$

for all $i \neq k$, where $\alpha_j, \beta_j \in \text{Fix}(g)$. Furthermore, by a similar discussion to the proofs of Lemma 4.5 and case (i) of Theorem 4.6, we see that equation (1) has an orientation-preserving homeomorphic solution $h : I \to I$ if and only if for each
j = 1, ..., m every sub-branch $B_{ji}(f)$, $i = 1, \ldots, s(j)$, is a singleton, denoted by $f(c_{ji})$ where $c_{ji} \in S_i(f)$, and there is an integer $1 \leq k \leq s(j)$ such that $f(c_{ji})$ lies between $f(c_{j1})$ and $f^2(c_{j1})$ and $\phi(c_{j1})$ lies between $\phi(c_{j1})$ and $\phi^2(c_{j1})$, i.e.,

$$\begin{cases}
    f^3(c_{j1}) < f(c_{ji}) < f^2(c_{j1}), & \phi^3(c_{j1}) < \phi(c_{ji}) < \phi^2(c_{j1}), \\
    f(c_{ji}) < f^2(c_{j1}) < f^3(c_{j1}), & \phi(c_{ji}) < \phi^2(c_{j1}) < \phi^3(c_{j1})
\end{cases}$$

for all $i \neq j$, where $\alpha_j, \beta_j \in \text{Fix}(f) \cup \mathcal{P}_2(f)$. Therefore, this completes the whole proof. \hfill \square

5. Examples and remarks. First, we demonstrate our theorems with the following examples.

**Example 1.** Consider the mapping $f_1 : [0, 1] \to [0, 1]$ defined by

$$f_1(x) = \begin{cases}
    -\frac{3}{2}x + \frac{5}{8}, & \forall \, x \in \left[0, \frac{1}{4}\right], \\
    \frac{1}{3}x + \frac{3}{8}, & \forall \, x \in \left(\frac{1}{4}, \frac{3}{8}\right], \\
    -\frac{1}{2}x + \frac{15}{16}, & \forall \, x \in \left(\frac{3}{8}, 1\right].
\end{cases}$$

One can check that it has an iterative root $\phi_1 : [0, 1] \to [0, 1]$ of order 2 defined by

$$\phi_1(x) = \begin{cases}
    -x + \frac{5}{8}, & \forall \, x \in \left[0, \frac{1}{4}\right], \\
    \frac{1}{2}x + \frac{3}{8}, & \forall \, x \in \left(\frac{1}{4}, \frac{3}{8}\right], \\
    -x + \frac{11}{16}, & \forall \, x \in \left(\frac{3}{8}, 1\right].
\end{cases}$$

Obviously, both $f_1$ and $\phi_1$ have the same characteristic interval $K(f_1) = \left[\frac{1}{4}, \frac{3}{8}\right]$ and they are both strictly increasing on $K(f_1)$. One can check that $f_1(0) = f_1(\frac{3}{4}) = \frac{5}{16}$ and $f_1(\frac{1}{4}) = f_1(1) = \frac{7}{16}$, i.e., assumption (i) in Theorem 4.2 is satisfied. The proof of Lemma 4.3 shows that an orientation-preserving homeomorphism $h_\ell : K(f_1) \to K(f_1)$ can be constructed as

$$h_\ell(x) = \begin{cases}
    \frac{\frac{3}{4}x - \frac{1}{8}}{\frac{3}{4}}, & \forall \, x \in \left[\frac{1}{4}, \frac{3}{8}\right], \\
    \frac{\frac{3}{4}x - \frac{1}{8}}{\frac{3}{4}}, & \forall \, x \in \left[\frac{1}{4}, \frac{3}{8}\right], \\
    \frac{\frac{3}{4}x - \frac{1}{8}}{\frac{3}{4}}, & \forall \, x \in \left[\frac{1}{4}, \frac{3}{8}\right],
\end{cases}$$

where $x_k = \frac{1}{2} - \frac{1}{2}(\frac{1}{2})^{k}$ and $\bar{x}_k = \frac{1}{2} - \frac{1}{2}(\frac{1}{2})^{k}$, so that $h_\ell \circ \phi_1|x_{K(f_1)} = f_1|x_{K(f_1)} \circ h_\ell$. Furthermore, using the construction in Theorem 4.2, one can extend $h_\ell$ to the whole interval $[0, 1]$ and obtain

$$h(x) = \begin{cases}
    -2h_\ell(-x + \frac{3}{8}) + \frac{5}{8}, & \forall \, x \in \left[0, \frac{1}{4}\right], \\
    h_\ell(x), & \forall \, x \in \left(\frac{1}{4}, \frac{3}{8}\right], \\
    -2h_\ell(-x + \frac{11}{16}) + \frac{15}{16}, & \forall \, x \in \left(\frac{3}{8}, 1\right].
\end{cases}$$

One can check that $h$ is an orientation-preserving homeomorphism on $[0, 1]$ and satisfies $h \circ \phi_1 = f_1 \circ h$, showing that $\phi_1$ is a genetic.

**Example 2.** Consider the mapping $f_2 : [0, 1] \to [0, 1]$ defined by

$$f_2(x) = \begin{cases}
    -2x + 1, & \forall \, x \in \left[0, \frac{1}{2}\right], \\
    2x, & \forall \, x \in \left(\frac{1}{2}, \frac{3}{4}\right], \\
    -x + \frac{7}{4}, & \forall \, x \in \left(\frac{3}{4}, 1\right].
\end{cases}$$

One can check that it has an iterative root $\phi_2 : [0, 1] \to [0, 1]$ of order 3 defined by

$$\phi_2(x) = \begin{cases}
    -2x + 1, & \forall \, x \in \left[0, \frac{1}{2}\right], \\
    2x, & \forall \, x \in \left(\frac{1}{2}, \frac{3}{4}\right], \\
    -x + \frac{7}{4}, & \forall \, x \in \left(\frac{3}{4}, 1\right].
\end{cases}$$
Obviously, both $f_2$ and $\phi_2$ have the same characteristic interval $K(f_2) = [\frac{1}{2}, 1]$, and they are both strictly decreasing on $K(f_2)$. Since $f_2(0) = f_2(\frac{1}{2}) = 1$ and $f_2(\frac{1}{2}) = f_2(1) = \frac{1}{2}$, assumption (i) of Theorem 4.6 holds. According to the proof of Lemma 4.7 and a discussion in the proof of Theorem 4.6, one see that the mapping $h : [0, 1] \rightarrow [0, 1]$ defined by $h(x) = x$ for all $x \in [0, 1]$ satisfies $h \circ \phi_2 = f_2 \circ h$. Thus $\phi_2$ is genetic.

Remark that for the case $H(f) > 1$ all iterative roots of such a PM function $f$ are not conjugate to $f$ and for the case $H(f) = 1$ all iterative roots $\phi$ satisfying $H(\phi) \geq 2$ are not conjugate to $f$. It shows that most PM functions have non-genetic iterative roots, which do not possess the same dynamical behaviors as the corresponding PM functions.

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