Morphology of the nonspherically decaying radiation generated by a rotating superluminal source: reply to comment

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Abstract

The fact that the formula used by Hannay in the preceding Comment is “from a standard text on electrodynamics” neither warrants that it is universally applicable, nor that it is unequivocally correct. We have explicitly shown [J. Opt. Soc. Am. A 25, 543 (2008)] that, since it does not include the boundary contribution toward the value of the field, the formula in question is not applicable when the source is extended and has a distribution pattern that rotates faster than light in vacuo. The neglected boundary term in the retarded solution to the wave equation governing the electromagnetic field forms the basis of diffraction theory. If this term were identically zero, for the reasons given by Hannay, the diffraction of electromagnetic waves through apertures on a surface enclosing a source would have been impossible.
I. INTRODUCTION

The argument presented by Hannay in Ref. 1 is based on an incorrect solution of Maxwell’s equations. We have explicitly shown [2] that the retarded solution to the wave equation

\[ \nabla^2 \mathbf{B} - \frac{1}{c^2} \frac{\partial^2 \mathbf{B}}{\partial t^2} = -\frac{4\pi}{c} \nabla \times \mathbf{j}. \]  

(1)

governing the magnetic field \( \mathbf{B} \) is not always given by

\[ \mathbf{B}(\mathbf{x}_P, t_P) = \frac{1}{c} \int d^3x \frac{\nabla \times \mathbf{j}}{|\mathbf{x}_P - \mathbf{x}|}, \]  

(2)

as assumed by Hannay [1]; an exception is the solution of Maxwell’s equations describing the emission from a polarization current density \( \mathbf{j} \) whose distribution pattern rotates superluminally (i.e., faster than light in vacuo). [Here, \((\mathbf{x}_P, t_P)\) and \((\mathbf{x}, t)\) are the space-time coordinates of the observation point and the source points, respectively, \( c \) is the speed of light in vacuo, and the square brackets denote the retarded value of \( \nabla \times \mathbf{j} \).] The emission from such a source consists of a collection of narrowing subbeams for which the absolute value of the gradient of the radiation field \( \mathbf{B} \) increases (as \( R_P^{7/2} \)) with the distance \( R_P \) from its source, rather than decreasing as predicted by Eq. (2) [2]. The inadequacy of Eq. (2), in this case, lies in the neglect of the boundary contribution toward the value of \( \mathbf{B} \).

We first describe how the boundary contribution to the retarded solution for the potential can always be made equal to zero, irrespective of the source motion. In the Lorenz gauge, the electromagnetic fields

\[ \mathbf{E} = -\nabla_\mathbf{P} A^0 - \partial A^\mu / \partial (ct_\mathbf{P}), \quad \mathbf{B} = \nabla_\mathbf{P} \times \mathbf{A}, \]  

(3)

are given by a four-potential \( A^\mu \) that satisfies the wave equation

\[ \nabla^2 A^\mu - \frac{1}{c^2} \frac{\partial^2 A^\mu}{\partial t^2} = -\frac{4\pi}{c} j^\mu, \quad \mu = 0, \cdots, 3, \]  

(4)

where \( A^0 / c \) and \( j^0 / c \) are the electric potential and charge density and \( A^\mu \) and \( j^\mu \) for \( \mu = 1, 2, 3 \) are the Cartesian components of the magnetic potential \( \mathbf{A} \) and the current density \( \mathbf{j} \). The solution to the initial-boundary value problem for Eq. (4) is given by

\[ A^\mu(\mathbf{x}_P, t_P) = \frac{1}{c} \int_0^{t_P} dt \int_V d^3x \left( \int_0^{t_P} dt \int_{\Sigma} ds \cdot (G \nabla A^\mu - A^\mu \nabla G) \right. \]  

\[ \left. - \frac{1}{4\pi c^2} \int_V d^3x \left( A^\mu \frac{\partial G}{\partial t} - G \frac{\partial A^\mu}{\partial t} \right) \right|_{t=0}, \]  

(5)
in which $G$ is the Green’s function and $\Sigma$ is the surface enclosing the volume $V$ (see, e.g., page 893 of Ref. 4).

The potential arising from a general time-dependent localized source in unbounded space decays as $R_P^{-1}$ when $R_P \equiv |x_P| \to \infty$, so that for an arbitrary free-space potential the second term in Eq. (5) would be of the same order of magnitude ($\sim R_P^{-1}$) as the first term, in the limit that the boundary $\Sigma$ tends to infinity. However, even potentials that satisfy the Lorenz condition

$$\nabla \cdot A + c^{-2} \partial A^0 / \partial t = 0$$

are arbitrary to within a solution of the homogeneous wave equation: the gauge transformation

$$A \to A + \nabla \Lambda, \quad A^0 \to A^0 - \partial \Lambda / \partial t$$

preserves the Lorenz condition if $\nabla^2 \Lambda - c^{-2} \partial^2 \Lambda / \partial t^2 = 0$ (see Ref. 3). One can always use this gauge freedom in the choice of the potential to render the boundary contribution (the second term) in Eq. (5) equal to zero, since this term, too, satisfies the homogeneous wave equation. Under the null initial conditions $A^\mu |_{t=0} = (\partial A^\mu / \partial t)_{t=0} = 0$ assumed in this note, the contribution from the third term in Eq. (5) is identically zero.

In the absence of boundaries, the retarded Green’s function has the form

$$G(x, t; x_P, t_P) = \frac{\delta(t_P - t - R/c)}{R},$$

where $\delta$ is the Dirac delta function and $R$ is the magnitude of the separation $R \equiv x_P - x$ between the observation point $x_P$ and the source point $x$. Irrespective of whether the radiation decays spherically (as in the case of a conventional source) or nonspherically (as would apply for a rotating superluminal source), therefore, the potential $A^\mu$ due to a localized source distribution that is switched on at $t = 0$ in an unbounded space, can be calculated from the first term in Eq. (5):

$$A^\mu(x_P, t_P) = c^{-1} \int d^3x dt j^\mu(x, t) \delta(t_P - t - R/c)/R,$$

i.e., from the classical expression for the retarded potential. Whatever the Green’s function for the problem may be in the presence of boundaries, it would approach that in Eq. (7) in the limit where the boundaries tend to infinity, so that one can also use this potential to calculate the field on a boundary that lies at large distances from the source.

We now turn to the case of the field and show that an analogous assumption about the boundary contribution may not be made. Consider the wave equation (1) governing the
magnetic field; Equation (1) may be obtained by simply taking the curl of the wave equation for the vector potential \[ \text{[Eq. (4) for } \mu = 1, 2, 3] \). We write the solution to the initial-boundary value problem for Eq. (1), in analogy with Eq. (5), as

\[
B_k(x_p, t_p) = \frac{1}{c} \int_0^{t_p} dt \int_V d^3 x \left( \nabla \times j \right)_k G + \frac{1}{4\pi} \int_0^{t_p} dt \int_{\Sigma} dS \cdot \left( G B_k - B_k \nabla G \right) - \frac{1}{4\pi c^2} \int_V d^3 x \left( B_k \frac{\partial G}{\partial t} - G \frac{\partial B_k}{\partial t} \right)_{t=0},
\]

where \( k = 1, 2, 3 \) designate the Cartesian components of \( \mathbf{B} \) and \( \nabla \times \mathbf{j} \).

Here, we no longer have the freedom, offered by a gauge transformation in the case of Eq. (5), to make the boundary term zero. Nor does this term always decay faster than the source term, so that it could be neglected for a boundary that tends to infinity, as is commonly assumed in textbooks (e.g., page 246 of Ref. 3) and the published literature [5, 6, 7, 8]. The boundary contribution to the retarded solution of the wave equation governing the field [the second term on the right-hand side of Eq. (9)] entails a surface integral over the boundary values of both the field and its gradient. For a rotating superluminal source, where the gradient of the field increases as \( R_p^{7/2} \) over a solid angle that decreases as \( R_p^{-4} \), this boundary contribution turns out to be proportional to \( R_p^{-1/2} \) (see Ref. 2). Not only is this not negligible relative to the contribution from the source term [the first term on the right-hand side of Eq. (9)], but the boundary term constitutes the dominant contribution toward the value of the radiation field in this case.

If one ignores the boundary term in the retarded solution of the wave equation governing the field (as Hannay does [1, 5, 6, 7, 8]), one obtains a different result, in the superluminal regime, from that obtained by calculating the field via the retarded potential [9, 10, 11]. This apparent contradiction stems solely from having ignored a term in the solution to the wave equation that is by a factor of the order of \( R_p^{1/2} \) greater than the term that is normally kept in this solution. The contradiction disappears once the neglected term is taken into account: the solutions to the wave equations governing both the potential and the field predict that the field of a rotating superluminal source decays as \( R_p^{-1/2} \) as \( R_p \) tends to infinity, a result that has now been demonstrated experimentally [12].

For a volume \( V \) in which there are no sources, Eq. (9) reduces to

\[
B_k(x_p, t_p) = \frac{1}{4\pi} \int_0^{t_p} dt \int_{\Sigma} dS \cdot \left( G B_k - B_k \nabla G \right),
\]

(10)
under the null initial conditions $B_k|_{t=0} = (\partial B_k/\partial t)|_{t=0} = 0$. As in the customary geometry for diffraction, the closed surface $\Sigma$ can consist of two disjoint closed surfaces, $\Sigma_{\text{inner}}$ and $\Sigma_{\text{outer}}$ (e.g., two concentric spheres), both of which enclose the source (see Fig. 10.7 of Ref. 3). If the observation point does not lie in the region between $\Sigma_{\text{inner}}$ and $\Sigma_{\text{outer}}$, i.e., lies outside the closed surface $\Sigma$, then the composite surface integral in Eq. (10) vanishes:

$$\int_0^{t_p} dt \int_\Sigma dS \cdot (G\nabla B_k - B_k\nabla G) = \int_0^{t_p} dt \left( \int_{\Sigma_{\text{inner}}} + \int_{\Sigma_{\text{outer}}} \right) dS \cdot (G\nabla B_k - B_k\nabla G)$$

$$= 0$$

(11)

(see Ref. 4). But under no circumstances would the integrals over $\Sigma_{\text{inner}}$ or $\Sigma_{\text{outer}}$ vanish individually if these surfaces enclose a source, i.e., if there is a nonzero field inside $\Sigma_{\text{inner}}$. Nor does the fact that the values of these integrals are unchanged by deformations of $\Sigma_{\text{inner}}$ and $\Sigma_{\text{outer}}$ have any bearing on whether they are nonvanishing or not. Equation (11) forms the basis of diffraction theory [3]. If the surface integrals over $\Sigma_{\text{inner}}$ and $\Sigma_{\text{outer}}$ were to vanish individually, as claimed by Hannay [1], the diffraction of electromagnetic waves through apertures on a surface enclosing a source would have been impossible (see Sec. 10.5 of Ref. 3).

Finally, we must stress that what one obtains by including the boundary term in the retarded solution to the wave equation governing the field is merely a mathematical identity; it is not a solution that could be used to calculate the field arising from a given source distribution in free space. Unless its boundary term happens to be small enough relative to its source term to be neglected, a condition that cannot be known a priori, the solution in question would require that one prescribe the field in the radiation zone (i.e., what one is seeking) as a boundary condition. Thus, the role played by the classical expression for the retarded potential in radiation theory is much more fundamental than that played by the corresponding retarded solution of the wave equation governing the field. The only way to calculate the free-space radiation field of an accelerated superluminal source is to calculate the retarded potential and differentiate the resulting expression to find the field (see also Ref. 13).
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[1] J. H. Hannay (2008), Morphology of the nonspherically decaying radiation generated by a rotating superluminal source: comment, submitted to J. Opt. Soc. Am. A.

[2] H. Ardavan, A. Ardavan, J. Singleton, J. Fasel, and A. Schmidt, J. Opt. Soc. Am. A 25, 543 (2008).

[3] J. D. Jackson, Classical Electrodynamics (Wiley, New York, 1999), 3rd ed.

[4] P. M. Morse and H. Feshbach, Methods of Theoretical Physics, vol. 1 (McGraw-Hill, New York, 1953).

[5] J. H. Hannay, Proc. Roy. Soc. A 452, 2351 (1996), ISSN 1364-5021.

[6] J. H. Hannay, Phys. Rev. E 62, 3008 (2000), ISSN 1063-651X.

[7] J. H. Hannay, J. Math. Phys. 42, 3973 (2001), ISSN 0022-2488.

[8] J. H. Hannay, J. Opt. Soc. Am. A 23, 1530 (2006), ISSN 1084-7529.

[9] H. Ardavan, Phys. Rev. E 58, 6659 (1998), ISSN 1063-651X.

[10] H. Ardavan, A. Ardavan, and J. Singleton, J. Opt. Soc. Am. A 21, 858 (2004), ISSN 1084-7529.

[11] H. Ardavan, A. Ardavan, J. Singleton, J. Fasel, and A. Schmidt, J. Opt. Soc. Am. A 24, 2443 (2007).

[12] A. Ardavan, W. Hayes, J. Singleton, H. Ardavan, J. Fopma, and D. Halliday, J. Appl. Phys. 96, 7760 (2004), ISSN 0021-8979, corrected version of 96(8), 4614–4631.

[13] H. Ardavan, A. Ardavan, and J. Singleton, J. Opt. Soc. Am. A 23, 1535 (2006), ISSN 1084-7529.