KRONECKER PRODUCTS OF PERRON SIMILARITIES∗

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Abstract. An invertible matrix is called a Perron similarity if one of its columns and the corresponding row of its inverse are both nonnegative or both nonpositive. Such matrices are of relevance and import in the study of the nonnegative inverse eigenvalue problem. In this work, Kronecker products of Perron similarities are examined and used to construct ideal Perron similarities all of whose rows are extremal.

Key words. Kronecker product, Perron similarity, ideal Perron similarity, nonnegative inverse eigenvalue problem

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1. Introduction. An invertible matrix is called a Perron similarity if one of its columns and the corresponding row of its inverse are both nonnegative or both nonpositive. Real Perron similarities were introduced by Johnson and Paparella [4, 5] and the case for complex matrices is forthcoming [3].

These matrices were introduced to examine the celebrated nonnegative inverse eigenvalue problem vis-à-vis the polyhedral cone

\[ C(S) := \{ x \in \mathbb{R}^n \mid SD_xS^{-1} \geq 0 \} \]
called the (Perron) spectracone of \( S \), and the set

\[ P(S) := \left\{ x \in C(S) \left\| x \right\|_\infty = 1 \right\} , \]
called the (Perron) spectratope of \( S \). The latter is not necessarily a polytope, but in some cases is finitely-generated (this is true for some complex matrices as well). Notice that the entries of of any element in \( P(S) \) form a normalized spectrum (i.e., \( x_k = 1 \) for some \( k \) and \( \max_i \{|x_i|\} \leq 1 \)) of a nonnegative matrix.

In particular, Johnson and Paparella [4] showed that if

\[
H_n := \begin{cases} 
1 & 1 \\
1 & -1 
\end{cases}, \quad n = 2
\]

\[
H_2 \otimes H_{n-1} = \begin{bmatrix} H_{n-1} & H_{n-1} \\
H_{n-1} & -H_{n-1} \end{bmatrix}, \quad n > 2,
\]

then \( C(H_n) \) and \( P(H_n) \) coincide with the conical hull and the convex hull of the rows of \( H_n \), respectively.

In this work, Kronecker products of Perron similarities are examined. In particular, it is shown that the Kronecker product of Perron similarities is a Perron similarity. An example is constructed to refute a result presented by Johnson and Paparella [4, Corollary 3.17] (see Example 13). It is also shown that

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$C(S) \otimes C(T) \subset C(S \otimes T)$ and $P(S) \otimes P(T) \subseteq P(S \otimes T)$ (strict containment in the latter occurs for some matrices). Kronecker products of ideal Perron similarities (see Section 5 below) yield Perron similarities all of whose rows are extremal.

2. Notation and Background. For $a \in \mathbb{Z}$ and $n \in \mathbb{N}$, $a \mod n$ is abbreviated to $a \% n$. For $n \in \mathbb{N}$, the set $\{1, \ldots, n\}$ is denoted by $\langle n \rangle$.

The set of $m$-by-$n$ matrices over a field $\mathbb{F}$ is denoted by $M_{m \times n}(\mathbb{F})$; when $m = n$, the set $M_{m \times n}(\mathbb{F})$ is abbreviated to $M_n(\mathbb{F})$. If $A \in M_{m \times n}(\mathbb{F})$, then the $(i, j)$-entry of $A$ is denoted by $[A]_{ij}$, $a_{ij}$, or $a_{i,j}$.

In this work, $\mathbb{F}$ stands for $\mathbb{C}$ or $\mathbb{R}$. The set of $m$-by-$n$ matrices with entries over $\mathbb{F}$ is denoted by $M_{m \times n}(\mathbb{F}) = M_{m \times n}$; when $m = n$, $M_{n \times n}(\mathbb{F})$ is abbreviated to $M_n(\mathbb{F}) = M_n$. The set of all $n$-by-1 column vectors is identified with the set of all ordered $n$-tuples with entries in $\mathbb{F}$ and thus denoted by $\mathbb{F}^n$. The set of nonsingular matrices in $M_n$ is denoted by $\text{GL}_n(\mathbb{F}) = \text{GL}_n$.

Given $x \in \mathbb{F}^n$, $[x]_i = x_i$ denotes the $i$th entry of $x$ and $\text{diag}(x) = D_x = D_x^T \in M_n(\mathbb{F})$ denotes the diagonal matrix whose $(i, i)$-entry is $x_i$. Notice that for scalars $\alpha, \beta \in \mathbb{F}$, and vectors $x, y \in \mathbb{F}^n$, $D_{\alpha x + \beta y} = \alpha D_x + \beta D_y$.

Denote by $I$, $e$, and $e_i$ the identity matrix, the all-ones vector, and the $i$th canonical basis vector, respectively. The size of these objects is determined from the context in which they appear.

If $A \in M_{m \times n}$ and $B \in M_{p \times q}$, then the the Kronecker product of $A$ and $B$, denoted by $A \otimes B$, is the $mp$-by-$nq$ matrix defined blockwise by $A \otimes B = [a_{ij}B]$. More precisely, but less intuitively,

$$(1) \quad [A \otimes B]_{ij} = a_{[i/p],[j/q]} b_{[(i-1)\%p]+1,[j-1\%q]+1}.$$ 

If $x \in \mathbb{C}^m$ and $y \in \mathbb{C}^n$, then (1) simplifies to

$$(2) \quad [x \otimes y]_i = x_{i/\%m} y_{(i-1)\%n+1}.$$ 

If $S, T \subseteq \mathbb{F}^n$, then $S \otimes T := \{s \otimes t \mid s \in S, \ t \in T \}$.

If $S \in \text{GL}_n$, then the (Perron) spectrcone of $S$, denoted by $C(S)$, is defined by $C(S) = \{x \in \mathbb{F}^n \mid SD_x S^{-1} \geq 0\}$. The (Perron) spectrotope of $S$ is the set $P(S) := \left\{ x \in C(S) \mid \|x\|_\infty = 1 \right\}$. The conical hull and convex hull of the rows of $S$ are denoted by $C_r(S)$ and $P_r(S)$, respectively. If $S \in \text{GL}_n$, then $SD_x S^{-1} = SIS^{-1} = I \geq 0$, i.e., $\emptyset \subset \text{coni}(e) \subseteq C(S)$.

If there is an $i \in \langle n \rangle$ such that $Se_i$ and $e_i^T S^{-1}$ are both nonnegative or both nonpositive for $S \in \text{GL}_n$, then $S$ is called a Perron similarity.

3. Preliminary Results.

Lemma 1. If $i \in \mathbb{Z}$ and $n \in \mathbb{N}$, then

$$i = ([i/n] - 1)n + (i - 1)\%n + 1.$$ 

Proof. By the division algorithm,

$$i = \left\lfloor \frac{i - 1}{n} \right\rfloor n + (i - 1)\%n$$

and because

$$\left\lfloor \frac{i - 1}{n} \right\rfloor = \left\lfloor \frac{i}{n} \right\rfloor - 1.$$
it follows that

\[ i - 1 = ([i/n] - 1)n + (i - 1)\%n, \]
i.e.,

\[ i = ([i/n] - 1)n + (i - 1)\%n + 1. \]

\[ \square \]

**Lemma 2.** If \( e_k \in \mathbb{F}^m \) and \( e_\ell \in \mathbb{F}^n \), then \( e_k \otimes e_\ell = e_{(k-1)n+\ell} \in \mathbb{F}^{mn} \).

**Proof.** It suffices to show that \([e_k \otimes e_\ell]_i = 1\) if and only if \(i = (k-1)n + \ell\); to this end, if \(i = (k-1)n + \ell\), then

\[ \left\lfloor \frac{i}{n} \right\rfloor = \left\lfloor \frac{(k-1)n + \ell}{n} \right\rfloor = \left\lfloor k - 1 + \frac{\ell}{n} \right\rfloor = k \]

and

\[ (i - 1)\%n + 1 = ((k - 1)n + \ell - 1)\%n + 1 \]
\[ = (\ell - 1)\%n + 1 \]
\[ = \ell - 1 + 1 = \ell. \]

Thus, according to (2),

\[ [e_k \otimes e_\ell]_i = [e_k]_{\left\lfloor \frac{i}{n} \right\rfloor}[e_\ell]_{(i-1)\%n+1} = [e_k]_k[e_\ell]_\ell = 1. \]

Conversely, if

\[ 1 = [e_k \otimes e_\ell]_i = [e_k]_{\left\lfloor \frac{i}{n} \right\rfloor}[e_\ell]_{(i-1)\%n+1}, \]

then \(k = [i/n]\) and \(\ell = (i - 1)\%n + 1\). Hence, by the division algorithm, there is a positive integer \(q\) such that \((i - 1) = qn + \ell - 1\), i.e., \(i = qn + \ell\). Thus,

\[ k = \left\lfloor \frac{qn + \ell}{n} \right\rfloor = \left\lfloor q + \frac{\ell}{n} \right\rfloor = q + 1 \]

i.e., \(q = k - 1\). Therefore, \(i = qn + \ell = (k-1)n + \ell\). \[ \square \]

**Lemma 3.** If \(e_i \in \mathbb{F}^{mn}\), then \(e_i = e_{[i/n]} \otimes e_{(i-1)\%n+1}\), where \(e_{[i/n]} \in \mathbb{F}^m\) and \(e_{(i-1)\%n+1} \in \mathbb{F}^n\).

**Proof.** If \(e_{[i/n]} \in \mathbb{F}^m\) and \(e_{(i-1)\%n+1} \in \mathbb{F}^n\), then

\[ e_{[i/n]} \otimes e_{(i-1)\%n+1} = e_{(i/n)-1)n+(i-1)\%n+1} = e_i \]

by Lemmas 1 and 2. \[ \square \]

**Lemma 4.** If \(S \in \mathbb{M}_{m \times n}\) and \(T \in \mathbb{M}_{p \times q}\), then

\[ e_i^T (S \otimes T) = e_{[\frac{i}{p}]}(S) \otimes e_{((i-1)\%p+1)}^T(T). \]

**Proof.** By Lemma 3 and properties of the Kronecker product,

\[ e_i^T (S \otimes T) = (e_{[\frac{i}{p}]} \otimes e_{((i-1)\%p+1)})^T (S \otimes T) \]
\[ = (e_{[\frac{i}{p}]}^T \otimes e_{((i-1)\%p+1)}^T)(S \otimes T) \]
\[ = e_{[\frac{i}{p}]}^T(S) \otimes e_{((i-1)\%p+1)}^T(T). \]

\[ \square \]
**Lemma 5.** If \( x \in \mathbb{F}^m \) and \( y \in \mathbb{F}^n \), then \( D_x \otimes D_y = D_{x \otimes y} \).

*Proof.* If \( i, j \in \langle mn \rangle \), then

\[
(D_x \otimes D_y)_{ij} = [D_x]_{ij} [D_y]_{(i-1)n+1,(j-1)n+1}
\]

in view of (1). Since

\[
[D_x]_{ij} = \begin{cases} x_i, & i = j \\ 0, & i \neq j, \end{cases}
\]

it follows that \( (D_x \otimes D_y)_{ij} \neq 0 \) if and only if \( \lceil i/n \rceil = \lceil j/n \rceil \) and \((i - 1)n + 1 = (j - 1)n + 1\). These equations hold, in light of Lemma 1, if and only if \( i = j \). Thus, \( D_x \otimes D_y \) is a diagonal matrix and, when \( i = j \), notice that

\[
(D_x \otimes D_y)_{ii} = (D_x \otimes y)_i = (D_x \otimes y)_{ii},
\]

as required. \( \square \)

**Lemma 6.** If \( x \in \mathbb{F}^m \) and \( y \in \mathbb{F}^n \), then \( \| x \otimes y \|_p = \| x \|_p \| y \|_p \), \( \forall p \in [1, \infty] \).

*Proof.* Notice that

\[
\| x \otimes y \|_p = \left( \sum_{k=1}^{mn} |(x \otimes y)_k|^p \right)^{1/p} = \left( \sum_{k=1}^{mn} |(x_{\lceil k/n \rceil} y_{(k-1)n+1})|^p \right)^{1/p} = \left( \sum_{k=1}^{mn} |x_{\lceil k/n \rceil}|^p |y_{(k-1)n+1}|^p \right)^{1/p}.
\]

Since \( \lceil k/n \rceil \in \langle m \rangle \) and \((k - 1)n + 1 \in \langle n \rangle \), it follows that

\[
\| x \otimes y \|_p = \left( \sum_{i=1}^m \sum_{j=1}^n |x_i|^p |y_j|^p \right)^{1/p} = \left( \sum_{i=1}^m |x_i|^p \left( \sum_{j=1}^n |y_j|^p \right) \right)^{1/p} = \left( \sum_{i=1}^m |x_i|^p \right)^{1/p} \left( \sum_{j=1}^n |y_j|^p \right)^{1/p} = \| x \|_p \| y \|_p.
\]

The case when \( p = \infty \) follows from the fact that \( \| x \|_\infty = \lim_{p \to \infty} \| x \|_p \). \( \square \)
4. Main Results.

**Theorem 7.** If \( S \in \text{GL}_m \) and \( T \in \text{GL}_n \), then \( C(S) \odot C(T) \subseteq C(S \otimes T) \) and \( P(S) \odot P(T) \subseteq P(S \otimes T) \).

**Proof.** If \( z \in C(S) \odot C(T) \), then \( z = x \odot y \), where \( x \in C(S) \) and \( y \in C(T) \). Thus, \( SD_x S^{-1} \geq 0 \) and \( TD_y T^{-1} \geq 0 \). By Lemma 5 and properties of the Kronecker product,

\[
(SD_x S^{-1}) \odot (TD_y T^{-1}) = (S \otimes T)(D_x \otimes D_y)(S^{-1} \otimes T^{-1}) = (S \otimes T)(D_x \otimes P_y)(S \otimes T)^{-1} \geq 0,
\]

since the Kronecker product of nonnegative vectors is nonnegative. Therefore, \( z \in C(S \otimes T) \) and \( C(S) \odot C(T) \subseteq C(S \otimes T) \).

If, in addition, \( z \in P(S) \odot P(T) \), then \( x \in P(S) \) and \( y \in P(T) \), i.e., \( \|x\|_\infty = \|y\|_\infty = 1 \). By Lemma 6,

\[
\|x \odot y\|_\infty = \|x\|_\infty \|y\|_\infty = 1
\]

i.e., \( z \in P(S \otimes T) \) and \( P(S) \odot P(T) \subseteq P(S \otimes T) \).

**Theorem 8.** If \( S \in \text{GL}_m \) and \( T \in \text{GL}_n \) are Perron similarities, then \( S \otimes T \) is a Perron similarity.

**Proof.** By definition, \( \exists k \in \langle m \rangle \) and \( \exists \ell \in \langle n \rangle \) such that the vectors \( S e_k, e_k^T S^{-1}, T e_\ell, e_\ell^T T^{-1} \) are nonnegative. By Lemma 2,

\[
(S \otimes T)e_{(k-1)n + \ell} = (S \otimes T)(e_k \otimes e_\ell) = S e_k \otimes T e_\ell \geq 0
\]

and

\[
e_{(k-1)n + \ell}^T (S \otimes T)^{-1} = (e_k \otimes e_\ell)^T (S \otimes T)^{-1} = (e_k^T \otimes e_\ell^T)(S^{-1} \otimes T^{-1}) = e_k^T S^{-1} \otimes e_\ell^T T^{-1} \geq 0.
\]

Therefore, \( S \otimes T \) is a Perron similarity.

**Remark 9.** If \( x, y \in C(S) \) and \( \alpha, \beta \geq 0 \), then \( \alpha x + \beta y \in C(S) \), i.e., \( C(S) \) is a convex cone.

**Remark 10.** If \( S \in \text{GL}_n \) is a Perron similarity, then

\[
SD_{e_i} S^{-1} = (S e_i) (e_i^T S^{-1}) \geq 0,
\]

i.e., \( \exists x \in C(S) \) such that \( x \neq \alpha e \) for every nonnegative \( \alpha \).

**Lemma 11.** If \( S \) is a Perron similarity, then there is a vector \( x \in C(S) \) such that \( x \) is totally nonzero and not a scalar multiple of \( e \).

**Proof.** By Remark 10, \( \exists x' \in C(S) \) such that \( x' \) is not a scalar multiple of \( e \). Select \( \alpha \in \mathbb{R} \) such that \( \alpha > \max_i |\text{Re}(x_i')| \). If \( x := x' + \alpha e \), then \( x \) is totally nonzero, not a scalar multiple of \( e \), and belongs to \( C(S) \) by Remark 9.

**Theorem 12.** If \( S \in \text{GL}_m \) and \( T \in \text{GL}_n \) are Perron similarities such that \( m > 1 \) and \( n > 1 \), then \( C(S) \odot C(T) \subset C(S \otimes T) \).

**Proof.** By Lemma 11, we may select \( x \in C(S) \) and \( y \in C(T) \) such that \( x \) and \( y \) are totally nonzero and not scalar multiples of \( e \). If \( z := x \odot y \), then \( z \in C(S \otimes T) \) by Theorem 7. As \( x \) and \( y \) are not scalar multiples
of $e$, there are integers $i, j \in \langle m \rangle$ and $k, \ell \in \langle n \rangle$ such that $x_i \neq x_j$ and $y_k \neq y_\ell$. Notice that the vector $z$ contains the blocks $x_i y$ and $x_j y$ so the vector $z$ has the entries $z_\alpha = x_i y_k$, $z_\beta = x_i y_\ell$, $z_\gamma = x_j y_k$, and $z_\delta = x_j y_\ell$. Furthermore, notice that

$$\frac{z_\alpha}{z_\beta} = \frac{y_k}{y_\ell} = \frac{z_\gamma}{z_\delta}.$$

Select $\varepsilon > 0$ such that $z' := z + \varepsilon e$ is totally nonzero. By Remark 9, $z' \in C(S \otimes T)$. For contradiction, assume that $z' = x' \otimes y'$, where $x' \in C(S)$ and $y' \in C(T)$. The vectors $x'$ and $y'$ must be totally nonzero (otherwise, $z'$ would not be totally nonzero). By a similar argument above,

$$\frac{z'_\alpha}{z'_\beta} = \frac{y'_k}{y'_\ell} = \frac{z'_\gamma}{z'_\delta}.$$

However,

$$\frac{z'_\alpha}{z'_\beta} = \frac{z'_\gamma}{z'_\delta} \iff \frac{z_\alpha + \varepsilon}{z_\beta + \varepsilon} = \frac{z_\gamma + \varepsilon}{z_\delta + \varepsilon} \iff \frac{x_i y_k + \varepsilon}{x_i y_\ell + \varepsilon} = \frac{x_j y_k + \varepsilon}{x_j y_\ell + \varepsilon} \iff (x_i y_k + \varepsilon)(x_j y_\ell + \varepsilon) = (x_i y_\ell + \varepsilon)(x_j y_k + \varepsilon) \iff x_i x_j y_k y_\ell + \varepsilon x_i y_k + \varepsilon x_j y_\ell + \varepsilon^2 = x_i x_j y_k y_\ell + \varepsilon x_i y_\ell + \varepsilon x_j y_k + \varepsilon^2 \iff \varepsilon (x_i - x_j)(y_k - y_\ell) = 0 \iff x_i - x_j = 0 \text{ or } y_k - y_\ell = 0,$$

a contradiction. Thus, $C(S) \otimes C(T) \subset C(S \otimes T)$. 

**Example 13.** Johnson and Paparella [4, Corollary 3.17] stated that $S$ is a Perron similarity if and only if $\text{coni}(e)$ is properly contained in $C(S)$. A contribution of this work is the refutation of this result with a counterexample constructed via the Kronecker product.

Indeed, the matrix

$$S := \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & -2 \\ 1 & 1 & -1 & -1 \end{bmatrix}$$

is the Kronecker product of

$$H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

and

$$T := \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$$

The inverse of $S$ is

$$\begin{bmatrix} -0.5 & 1 & -0.5 & 1 \\ 0.5 & -0.5 & 0.5 & -0.5 \\ -0.5 & 1 & 0.5 & -1 \\ 0.5 & -0.5 & -0.5 & 0.5 \end{bmatrix}$$
Notice that neither the first or second row is nonnegative. Furthermore, if \( D = \text{diag}([2, 2, -1, -1]) \), then the matrix

\[
A := SDS^{-1} = \begin{bmatrix}
0.5 & 0 & 1.5 & 0 \\
0 & 0.5 & 0 & 1.5 \\
1.5 & 0 & 0.5 & 0 \\
0 & 1.5 & 0 & 0
\end{bmatrix}
\]

is nonnegative and nonscalar. Thus, \( \text{coni}(e) \) is properly contained in \( C(S) \), but \( S \) is not a Perron similarity.

5. Ideal Perron Similarities. If \( S \in \text{GL}_n \) is a Perron similarity, then \( S \) is called ideal if \( C(S) = C_r(S) \). For real matrices, it is known that \( S \) is ideal if and only if \( \exists k \in \langle n \rangle \) such that \( e_k^T S = e^T \) and \( e_i^T S \in C(S) \) for all \( i \in \langle n \rangle \) [5, Theorem 3.8]. A careful examination of the arguments also applies to complex matrices.

**Theorem 14.** If \( S \in \text{GL}_m \) and \( T \in \text{GL}_n \) are ideal, then \( S \otimes T \) is ideal.

**Proof.** By hypothesis, there are integers \( k \in \langle m \rangle \) and \( \ell \in \langle n \rangle \) such that \( e_k^T S = e^T \) and \( e_\ell^T T = e^T \). Notice that \( (k-1)n + \ell \in \langle mn \rangle \) and by Lemma 2,

\[
e_{(k-1)n+\ell}^T (S \otimes T) = (e_k \otimes e_\ell)^T (S \otimes T) = (e_k^T \otimes e_\ell^T)(S \otimes T) = (e_k^T S) \otimes (e_\ell^T T) = e^T \otimes e^T = e^T.
\]

If \( i \in \langle mn \rangle \), then, following Lemma 3,

\[
e_i^T (S \otimes T) = (e_{i/n} \otimes e_{(i-1)\%n+1})^T (S \otimes T) = (e_{i/n}^T \otimes e_{(i-1)\%n+1})(S \otimes T) = (e_{i/n}^T S) \otimes (e_{(i-1)\%n+1}^T T) \geq 0
\]
since \( e_{i/n}^T S \in C(S) \) and \( e_{(i-1)\%n+1}^T T \in C(T) \).

**Theorem 15.** If \( U = \{u_1, \ldots, u_p\} \subseteq \mathbb{F}_m \) and \( V = \{v_1, \ldots, v_q\} \subseteq \mathbb{F}_n \), then \( \text{coni}(U) \otimes \text{coni}(V) \subseteq \text{coni}(U \otimes V) \) and \( \text{conv}(U) \otimes \text{conv}(V) \subseteq \text{conv}(U \otimes V) \).

**Proof.** If \( x \in \text{coni}(U) \otimes \text{coni}(V) \), then \( x = u \otimes v \), where \( u \in \text{coni}(U) \) and \( v \in \text{coni}(V) \). By definition,

\[
u = \sum_{i=1}^p \lambda_i u_i, \quad \lambda_i \geq 0, \quad \forall i \in \langle p \rangle
\]

and

\[
v = \sum_{j=1}^q \mu_j v_j, \quad \mu_j \geq 0, \quad \forall j \in \langle q \rangle.
\]

By properties of the Kronecker product,

\[
x = u \otimes v = \left( \sum_{i=1}^p \lambda_i u_i \right) \otimes \left( \sum_{j=1}^q \mu_j v_j \right) = \sum_{i=1}^p \left( \lambda_i u_i \otimes \sum_{j=1}^q (\mu_j v_j) \right)
\]
and strong by Theorem 14 since \( \lambda_i \mu_j \geq 0, \forall (i, j) \in \langle p \rangle \times \langle q \rangle \).

If, in addition,

\[
\sum_{i=1}^{p} \lambda_i = \sum_{j=1}^{q} \mu_j = 1,
\]

then

\[
\sum_{i=1}^{p} \sum_{j=1}^{q} \lambda_i \mu_j = \sum_{i=1}^{p} \left( \sum_{j=1}^{q} \mu_j \right) = \sum_{i=1}^{p} \lambda_i = 1,
\]

i.e., \( \text{conv}(U) \otimes \text{conv}(V) \subseteq \text{conv}(U \otimes V) \).

Recall that a matrix is **irreducible** if and only if its digraph is strongly connected (see, e.g., Brualdi and Ryser [1, Theorem 3.2.1]). The **index of imprimitivity** of an irreducible matrix is the greatest common divisor of the lengths of the closed directed walks in its digraph [1, p. 68].

An invertible matrix \( S \) is called **strong** if there is an irreducible nonnegative matrix \( A \) such that \( A = SDS^{-1} \) (in such a case, \( S \) must be a Perron similarity since the eigenspace corresponding to the Perron root is one-dimensional). If \( S \) is strong, then \( S \) is ideal if and only if \( \mathcal{P}(S) = \mathcal{P}_r(S) \) [3].

The following result is a consequence of a result stated by Harary and Trauth [2, p. 251] and follows from a result due to McAndrew [6, Theorem 2].

**Theorem 16.** If \( A \) and \( B \) are irreducible and \( k \) and \( \ell \) are the indices of imprimitivity of \( A \) and \( B \), respectively, then \( A \otimes B \) is irreducible if and only if \( \text{gcd}(k, \ell) = 1 \).

**Corollary 17.** Suppose that \( S \) and \( T \) are ideal and strong. Let \( A \) and \( B \) be irreducible nonnegative matrices with relatively prime indices of imprimitivity \( k \) and \( \ell \), respectively, and such that \( A = SDS^{-1} \) and \( B = TDT^{-1} \). Then \( S \otimes T \) is ideal, strong, and \( \mathcal{P}(S) \otimes \mathcal{P}(T) \subset \mathcal{P}(S \otimes T) \).

**Proof.** The matrix \( S \otimes T \) is ideal by Theorem 14 and strong by Theorem 16. Thus, \( \mathcal{P}_r(S) = \mathcal{P}(S) \), \( \mathcal{P}_r(T) = \mathcal{P}(T) \), and \( \mathcal{P}_r(S \otimes T) = \mathcal{P}(S \otimes T) \). The weak containment \( \mathcal{P}(S) \otimes \mathcal{P}(T) \subseteq \mathcal{P}(S \otimes T) \) follows from Theorem 15.

By Lemma 11, we may select \( \delta \in \mathcal{C}(S) \) and \( \gamma \in \mathcal{C}(T) \) such that \( \hat{x} \) and \( \hat{y} \) are totally nonzero and not scalar multiples of \( e \). Furthermore, the totally nonzero vectors \( x := \hat{x}/||x||_\infty \) and \( y := \hat{y}/||y||_\infty \) belong to \( \mathcal{P}(S) \). If \( z := x \otimes y \), then \( z \in \mathcal{P}(S \otimes T) \) by Theorem 7. As \( x \) and \( y \) are not scalar multiples of \( e \), there are integers \( i, j \in \langle m \rangle \) and \( k, \ell \in \langle n \rangle \) such that \( x_i \neq x_j \) and \( y_k \neq y_\ell \). Notice that the vector \( z \) contains the blocks \( x_iy_j \) and \( x_jy_i \) so the vector \( z \) has the entries \( z_\alpha = x_iy_k, z_\beta = x_iy_\ell, z_\gamma = x_jy_k, \) and \( z_\delta = x_jy_\ell \). Furthermore, notice that

\[
\frac{z_\alpha}{z_\beta} = \frac{y_k}{y_\ell} = \frac{z_\gamma}{z_\delta}.
\]

Since \( \mathcal{P}(S \otimes T) = \mathcal{P}_r(S \otimes T) \), it follows that \( \mathcal{P}(S \otimes T) \) is convex. As \( x \) and \( y \) are not multiples of \( e \), they are not on the ray passing through \( e \). Thus, we may select \( \varphi, \psi > 0 \) such that \( z' := \varphi z + \psi e \) is totally nonzero, \( \varphi + \psi = 1 \), and \( z' \in \mathcal{C}(S \otimes T) \).
For contradiction, assume that $z' = x' \otimes y'$, where $x' \in \mathcal{P}(S)$ and $y' \in \mathcal{P}(T)$. By a similar argument above,
\[
\frac{z'_\alpha}{z'_\beta} = \frac{y'_k}{y'_t} = \frac{z'_\gamma}{z'_\delta}
\]
However,
\[
\frac{z'_\alpha}{z'_\beta} = \frac{z'_\gamma}{z'_\delta} \iff \frac{\varphi_r \alpha + \psi}{\varphi_r \beta + \psi} = \frac{\varphi_r \gamma + \psi}{\varphi_r \delta + \psi}
\]
\[
\iff \varphi_r x_k y_k + \psi(y_k - y_t) = 0
\]
\[
\iff x_i - x_j = 0 \text{ or } y_k - y_t = 0,
\]
a contradiction. Thus, $\mathcal{P}(S) \otimes \mathcal{P}(T) \subset \mathcal{P}(S \otimes T)$.

Example 18. For $n \in \mathbb{N}$, let $F = F_n$ be the discrete Fourier transform matrix of order $n$, i.e., $F$ is the $n$-by-$n$ matrix with $(i, j)$-entry equal to $\omega^{(i-1)(j-1)}$, where $\omega := \exp(2\pi i/n)$. Notice that
\[
F = \begin{bmatrix}
1 & 1 & \cdots & 1 & \cdots & 1 \\
1 & \omega & \cdots & \omega^k & \cdots & \omega^{n-1} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
1 & \omega^k & \cdots & \omega^{k^2} & \cdots & \omega^{k(n-1)} \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
1 & \omega^{n-1} & \cdots & \omega^{k(n-1)} & \cdots & \omega^{(n-1)^2}
\end{bmatrix}
\]
and $F$ is ideal as it is a Vandermonde matrix corresponding to the polynomial $p(t) := t^n - 1$. The companion matrix $C$ corresponding to $p$ is nonnegative and the spectrum of the nonnegative matrix $C^{k-1}$ corresponds to the $k$th-row of $F$, $k \in \langle n \rangle$. Furthermore, $F$ is strong given that $C$ is the adjacency matrix of the directed cycle of length $n$ and, hence, is irreducible (it also admits positive circulant matrices).

A normalized, realizable spectrum $x$ is called extremal if $\alpha x$ is not realizable whenever $\alpha > 1$. Notice that every row of $F$ is extremal and every point in every row is extremal in the Karpelevič region.

At the 2019 Meeting of the International Linear Algebra Society in Rio de Janeiro, the second author asked whether other such matrices exist. Notice that $F_n \otimes F_m$, $F_m \otimes H_n$, and $H_n \otimes F_m$ are matrices all of whose rows and entries are extremal. 

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