sd-type Exchange Interactions in Nonhomogeneous Ferromagnets

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Abstract

Motivated by a need to understand spin-momentum transport in CPP (current perpendicular to the plane) structures, a quantum field theoretical treatment of spin-spin interactions in ferromagnets is presented. The sd interaction of the conduction electrons and the magnetic medium is treated non-perturbatively from first principles in real space. The localized magnetic moments also interact with each other through a Heisenberg exchange potential. To take into account correlation effects, a second quantized formulation is used. The semi-classical limit is taken by using a coherent-state path-integral technique which also allows us to go beyond a linear-response approach. We derive a set of coupled equations of motion for the nonuniform magnetization, the spin current and the two-point correlation functions of the magnetization. The rate of change of the magnetization is shown to obey a generalized Landau-Lifshitz equation that takes into account interaction with the conduction electrons. Within the relaxation time approximation it is shown that the polarization of the conduction electrons obeys a diffusion equation. The diffusion tensor, which has off-diagonal terms due to the sd exchange interaction, is now explicitly dependent on the magnetization of the medium. We also show that the magnetization fluctuations satisfy a diffusion-type equation. The derived equations are used in two illustrative examples.

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I. INTRODUCTION

Spin-spin interactions in ferromagnetic metals are of paramount importance in today’s GMR recording heads. There is also currently great interest in the magnetic recording industry in using spin currents, instead of magnetic fields, to switch the magnetization in a writer device. In this case a polarized electronic current is needed, such that the net spin of the polarization becomes effectively another magnetic source which induces an interaction with the magnetic moments of the medium. One widely used approximation is to separate the degrees of freedom of the current from those of the local magnetic moment. This latter separation is not justified in conducting metals but it nevertheless produced good results in some cases. This paper explores in detail the consequences for the spin accumulation problem in ferromagnets of assuming that the interaction between the conduction electrons and the local moments is of the sd exchange type. This interaction can give rise to what is now known as spin-momentum transfer (SMT) in magnetic multilayers. This latter mechanism has been predicted by Berger and Slonczewski and later verified experimentally by various groups. Other interaction mechanisms between the conduction electrons and the magnetization vector have been proposed since the Berger-Slonczewski work. In previous work, the interaction of the polarized current with the magnetization has not been treated self-consistently. In fact the equations of motion were based on those of a similar system, that of a current interacting with magnetic impurities. We believe that this approach is not suitable for transition metals.

In this work, we start from a microscopic description of the conduction electrons and the ferromagnetic medium and then take the semi-classical limit to derive equations for macroscopic quantities of physical significance to experiment and other phenomenological approaches. Although the derivations are somewhat complex, one can go to the main results (e.g., Eqs. 66 and 84 which are generalized Boltzmann-type equations) and see that the correct physics is contained in them.

Our results include those of reference and in fact our treatment should provide a basis for the more phenomenological work. We do not assume that the magnetization is uniform as in previous work and we will focus mainly on the diffusive regime. We use many-body field theoretical methods to describe the system of magnetic moments plus conduction electrons. Even though only a single particle picture is needed, the methods we use permit
us to treat the magnetic part of the problem and the conduction electrons on the same formal level. This allows us to derive transport equations for the conduction electrons and the local magnetic moments and include relaxation effects without recourse to more phenomenological modeling. Exchange effects, which are important in transition metals, are also included self-consistently. Finite-temperature properties are naturally included through the use of a path-integral formulation of the problem. Including spin-dependent interactions in a transport problem means that we have to deal with many indexes. Path-integrals are helpful with book-keeping and hence simplify the discussion considerably as compared to Ref. 9. Finally a path-integral representation helps in making consistent approximations to the derived transport equations.

The paper is organized as follows. In Sec. II, we introduce the Hamiltonian of the interacting electron-magnetization system in second quantized form. In Sec. III, a functional $Z$, which we call the universal generating functional, is defined in terms of the density matrix of the system and external sources. This functional has a path-integral representation and will generate all possible correlation functions. For the local magnetic moments we adopt a coherent state representation which is most suitable for a semi-classical treatment of the medium such as presented here. In Sec. IV, we derive equations of motion for the magnetization (a modified Landau-Lifshitz equation) and the spin accumulation. These latter equations are then used to derive equations for the correlation functions of the magnetization. We also show that non-uniform magnetization of the medium gives rise to a spin accumulation effect similar to that due to interfaces. This is one of the main results of this work. In Sec. V, we show how to solve these equations perturbatively and derive the terms that give rise to the spin momentum transfer effect for a system with non-uniform magnetization. Finally, we derive a diffusion-type equation for the magnetization fluctuations where the diffusion coefficient is dependent on the exchange integral. In section VI, we show some applications of the derived spin accumulation equations in simple cases where the magnetization is nonuniform in the direction of flow of the current. In the last section, we summarize our results.
II. THE MODEL

We start from a quantum field picture of the electronic current and the magnetic moments of the localized electrons in a thin slab of thickness comparable to the mean free path, Fig. II. In the following we do not include explicitly an electric field, but we assume that it is part of the spin-independent part of the Hamiltonian. Its inclusion has been done in Ref. 11. For the magnetic medium, a Heisenberg Hamiltonian is assumed. We explicitly take into account only exchange effects among the Heisenberg spins. Other important effects such as the demagnetization field term can be added phenomenologically. The conduction electrons will be assumed to be in equilibrium with the magnetic medium since they relax much faster than the magnetic moments. This is the case in assisted spin-momentum transfer switching. In fact the magnetic moments can be regarded as in contact with a Fermi bath with spin degrees of freedom. In this case switching may be regarded as a dissipative effect accompanied by a shift in the magnon energies. 12 Our work is a natural generalization of the model used by Langreth and Wilkins 9 to study spin resonance in dilute magnetic alloys. The conduction electron field $\Psi_s$ satisfies the usual anti-commutation relations,

$$\{\Psi_s(r), \Psi_{s'}(r')\} = \{\Psi^\dagger_s(r), \Psi^\dagger_{s'}(r')\} = 0,$$  \hspace{1cm} (1)

$$\left\{ \Psi_s(r), \Psi^\dagger_{s'}(r') \right\} = \delta_{ss'} \delta (r - r'),$$

where $s$ is a spin index ($s = \pm \frac{1}{2}$). The number density operator of the electrons is given by

$$\rho_{ss}(r) = \Psi^\dagger_s(r) \Psi_s(r).$$  \hspace{1cm} (2)

In this system, the electrons are treated as non-interacting, i.e. no Coulomb interaction, and hence the electron field can be expanded in terms of single particle wave-functions:

$$\Psi_s(r) = \sum_i \phi_i(r) a_{s,i},$$  \hspace{1cm} (3)

$$\Psi^\dagger_s(r) = \sum_i \phi^*_i(r) a^\dagger_{s,i},$$
where $a_s(a^+_s)$ is the annihilation (creation) operator for a particle of spin $s$. The total Hamiltonian of the system is composed of a conduction electron part, $H_s$, a magnetic part, $H_d$ and an interaction part, $H_{sd}$,

$$H = H_s + H_d + H_{sd}. \quad (4)$$

The conduction electron part in a ferromagnet, such as Fe, is due to 4s-type electrons. The magnetic part is however due to 3d-type electrons. For the free conduction electrons we have

$$H_s = \int \frac{\hbar^2}{2m} \nabla \Psi^\dagger (\mathbf{r}) \cdot \nabla \Psi (\mathbf{r}) \, d\mathbf{r} \quad (5)$$

where $\Psi$ is the two-component Fermi field,

$$\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}. \quad (6)$$

The magnetic medium is microscopically a lattice with a spin vector $\mathbf{S}_i$ at each lattice point $i$. Since we are interested in the continuum limit of this model, we can define a macroscopic spin vector for the medium

$$\mathbf{S} (\mathbf{r}) = \sum_{i=1}^{N} \mathbf{S}_i \delta (\mathbf{r} - \mathbf{r}_i). \quad (7)$$

The magnetization vector is then simply given by the volume average of the global spin

$$\mathbf{M} (\mathbf{x}) = g \mu_B \frac{\mathbf{S} (\mathbf{x})}{V}, \quad (8)$$

where $g$ is the spectroscopic splitting factor and $\mu_B$ is the Bohr magneton of the electron. The spin vector has an $SU(2)$ representation and consequently so does the magnetization vector, e.g.

$$[M_x (\mathbf{x}) , M_y (\mathbf{x}') ] = 2ig \mu_B M_z (\mathbf{x}) \delta (\mathbf{x} - \mathbf{x}'). \quad (9)$$

We employ units such that $g \mu_B = \hbar = 1$. Hence the magnetization will have opposite sign to that usually defined in the literature. The Hamiltonian of the spins is taken to be of the Heisenberg type,

$$H_d = -\frac{1}{2} \sum_{ij} J (\mathbf{r}_i - \mathbf{r}_j) \mathbf{S}_i \cdot \mathbf{S}_j - B \cdot \sum_i \mathbf{S}_i. \quad (10)$$
We take account of exchange only and assume the spins to be in an external field $B$. In the continuum limit, we can write the exchange term in an integral form,

$$\frac{1}{2} \sum_{i,j=1}^{N} S_i \cdot S_j J (r_i - r_j) = \frac{1}{2} \int d r d r' \sum_{\mu=1,2,3} S_\mu (r) J (r - r') S_\mu (r').$$  \hspace{1cm} (11)

Contributions from the magnetostatic field can also be included. For the typical thin structures we are interested in, the magnetization of the medium is usually held in-plane by the magnetostatic field. The energy term associated with this field is usually non-local but for a thin film it can be well approximated by a local term proportional to the normal component of the magnetization,

$$H_{\text{demag}} = \frac{1}{2} K \int d x \left[ S (x) \cdot n \right]^2,$$  \hspace{1cm} (12)

where $n$ is a normal unit vector to the surface and $K$ is a constant depending on the shape of the slab. This latter term will lower the symmetry of the problem even further. For simplicity in the following we leave out this term and discuss it elsewhere$^{12}$.

The interaction between the electrons and the localized spins is taken to be of the s-d type, of the form

$$H_{sd} = -\frac{\lambda}{2} \int d x \left( \frac{1}{2} \sum_{\alpha=1,2} \Psi^\dagger_{\alpha} (x) \vec{\sigma} \Psi_{\alpha} (x) \right) \cdot S (x)$$  \hspace{1cm} (13)

where $\lambda$ is a coupling constant of the order of $0.1 \ eV$ and $\vec{\sigma}$ is a vector whose components are the Pauli matrices,

$$[\sigma_i, \sigma_j] = 2i \epsilon^{ijk} \sigma_k.$$  \hspace{1cm} (14)

$\epsilon^{ijk}$ is the antisymmetric unit tensor. The full Hamiltonian is then the sum of all the above terms,

$$H = \sum_{\alpha=1,2} \int d r \ \Psi^\dagger_{\alpha} (r) \left\{ \frac{1}{2m} p^2 (r) + U (r) - \frac{1}{2} \sigma \cdot B \right\} \Psi_{\alpha} (r)$$  \hspace{1cm} (15)

$$- \frac{\lambda}{2} \int d r \sum_{\alpha,\beta=1}^{2} \sum_{i=1}^{3} \Psi^\dagger_{\alpha} (r) \sigma^{(i)}_{\alpha\beta} \Psi_{\beta} (r) S_i (r)$$

$$- \frac{1}{2} \int d r d r' \sum_{i=1}^{3} \left\{ S^i (r) J (r - r') S_i (r') + 2\mu_s B^i (r) \delta (r - r') S_i (r') \right\}$$
where we have included a spin-independent external potential $U(r)$ for the conduction electrons. The external magnetic field $B$ can be taken to be time-dependent and/or spatially dependent in the following treatment.

III. THE UNIVERSAL GENERATING FUNCTIONAL

In this section, we introduce what we call a universal generating functional from which we derive equations of motion for the magnetization and the spin current. This functional is defined in terms of the density matrix $\rho$ of the current-medium system and can be used to study equilibrium as well as non-equilibrium properties. We are at present interested only in the near-equilibrium case. A general overview of the method appeared in Ref. 10. Application of this method to spin systems is however almost absent. As far as we know, Refs. 9 and 16 are the only work that apply non-equilibrium methods to spin systems. One of the other advantages of this method is that it allows equal treatment of thermal effects and non-thermal effects. The extension beyond a linear-response approach is also easily achieved, at least formally. This becomes essential when we are interested in questions that involve switching of the magnetization. We will give a general outline of the method as we apply it to the particular s-d exchange system since the steps used in the derivation of the equations of motion are hard to find in one place. The importance of these methods, which are hardly used in magnetism, is potentially very great.

To motivate the structure of the functional we are about to introduce, we recall the structure of the density matrix elements. In quantum mechanics, we usually use the energy method to solve a problem. However, in transport problems, time-dependent methods are essential. Hence the evolution of the density matrix comes into play. One of the ways to calculate density matrix elements is through a path-integral representation. We have recently used such a formalism to treat the problem of fluctuations and dissipation in coherent magnetization. The density matrix elements at a given time $t$ are usually written in terms of those at an earlier time using propagators running from past to present.
and from present to past,

\[
\langle \Phi | \rho (t) | \Phi' \rangle = \langle \Phi (t) | \rho | \Phi' (t) \rangle = \int \mathcal{D} \Phi_1 \mathcal{D} \Phi_2 \langle \Phi (t) | \Phi_1 (t_0) \rangle \langle \Phi_1 (t_0) | \rho | \Phi_2 (t_0) \rangle \langle \Phi_2 (t_0) | \Phi' (t) \rangle \\
\equiv \int \mathcal{D} \Phi_1 \mathcal{D} \Phi_2 \langle \Phi_1 | \rho (t_0) | \Phi_2 \rangle \mathcal{G}^{(+)} (\Phi (t), \Phi_1 (t_0)) \mathcal{G}^{(-)} (\Phi_2 (t_0), \Phi' (t)) (17)
\]

The propagators \( \mathcal{G}^{(+)} \) and \( \mathcal{G}^{(-)} \) are then written in terms of path-integrals. This is equivalent to using the Schröedinger equation to solve for the evolution of the density matrix. In a transport problem, we instead introduce a functional of the density matrix. This functional is then made to depend on new virtual sources \( \eta_1, \eta_1^*, \eta_2, \eta_2^*, J_1 \) and \( J_2 \). These sources are coupled to the conduction electrons’ field and the magnetic moments of the medium which will enable us to generate all kinds of correlation functions and their time-evolution. The functional is then given in terms of a trace formula,

\[
Z [ \eta_1, \eta_1^*, \eta_2, \eta_2^*, J_1, J_2, \rho ] = \text{Tr} \left\{ \rho (t_0) \left( \mathcal{T}^{-1} \exp \left[ -i \int dx \left( \eta_2^* (x) \cdot \Psi^\dagger (x) + \Psi^+ (x) \cdot \eta_2 (x) + J_2 (x) \cdot S (x) \right) \right] \right) \right\} \times \left( \mathcal{T} \exp \left[ i \int dx \left( \eta_1^* (x) \cdot \Psi^\dagger (x) + \Psi^+ (x) \cdot \eta_1 (x) + J_1 (x) \cdot S (x) \right) \right] \right) (18)
\]

where \( \mathcal{T} \) is the time-ordering operator. It orders operators with the earliest time argument to the right. \( \mathcal{T}^{-1} \) is the inverse of \( \mathcal{T} \). The external sources \( \eta_1 \) and \( \eta_2 \) are two-component classical (i.e. Grassmann) sources where \( \eta_1 \) \( (\eta_2) \) and \( \eta_1^* \) \( (\eta_2^*) \) are treated as independent. The operators are all written in the Heisenberg representation. The need for both time-ordering operators is clearly seen through the Feynman-Vernon Influence formalism, Eq. \( \textbf{(16)} \). The coefficients of the expansion of the functional \( Z \) in terms of its arguments give all possible correlation functions of the system. A few of the lowest order correlations are stated below. They can be easily verified by differentiating the functional \( Z \) with respect to the virtual sources. For example, to get the expectation values of the conduction electrons’ field, we differentiate \( Z \) with respect to \( \eta_1 \) or \( \eta_2 \),

\[
\frac{1}{Z} \frac{\delta Z}{\delta \eta_{1\lambda} (x)} \bigg|_{\eta=\eta^*=J=0} = i \langle \mathcal{T} \Psi_{1\lambda} (x) \rangle \equiv i \langle \Psi_{1\lambda} (x) \rangle , (19)
\]
\[
\frac{1}{Z} \frac{\delta Z}{\delta \eta_{\lambda}^r(x)} |_{\eta = \eta^* = J = 0} = -i \langle T^{-1} \Psi_{\lambda}(x) \rangle \equiv -i \langle \Psi_{2\lambda}(x) \rangle . \tag{20}
\]

Similarly to get expectation values for the magnetization, we differentiate with respect to the external sources \( J_1 \) and \( J_2 \),

\[
\frac{1}{Z} \frac{\delta Z}{\delta J_1^i(x)} |_{\eta = \eta^* = J = 0} = i \langle T S_i(x) \rangle \equiv i \langle S_{1i}(x) \rangle \tag{21}
\]

\[
\frac{1}{Z} \frac{\delta Z}{\delta J_2^i(x)} |_{\eta = \eta^* = J = 0} = -i \langle T^{-1} S_i(x) \rangle \equiv -i \langle S_{2i}(x) \rangle . \tag{22}
\]

Higher order correlations can be obtained simply via higher order differentiations.

The Hilbert space for the conduction electrons and the local magnetic moments of the medium is the product of the corresponding Hilbert spaces,

\[
|\Phi, \Omega\rangle \equiv |\Phi\rangle \otimes |\Omega\rangle , \tag{23}
\]

where \( |\Phi\rangle \) is a many-body fermionic state representing the conduction electrons and \( |\Omega\rangle \) is a magnetic moment state. The magnetic moment states will be represented in terms of spin-coherent states (SCS)\(^{20}\) (and references therein). Since the operators are initially taken to be in the Heisenberg picture, then in the presence of the additional external sources, \( \eta_1 \), \( \eta_2 \), and \( J \), the states are no longer time independent. Now we write the functional formula in terms of a path-integral. Since we are not interested in the transient behavior of the interaction between the current and the magnetic moments, we assume that the external electric field was turned on a long time ago and we will eventually set \( t_0 = -\infty \). Reference\(^{12}\) treats the case where \( t_0 \) is kept finite in a finite size thin film. Moreover, we assume that the density matrix is separable initially, i.e.,

\[
\rho(t = -\infty) = \rho_s(t = -\infty) \otimes \rho_d(t = -\infty) , \tag{24}
\]

where \( \rho_s \) is the density matrix of the conduction s-electrons and \( \rho_d \) is that of the local magnetic moments. Now let \( |\Phi_i, \Omega_i\rangle \) be an initial over-complete set of states for the operators \( \Psi (r, t_0) \otimes 1 \) and \( 1 \otimes S (r, t_0) \). Similarly, we let \( |\Phi_c, \Omega_c\rangle \) be an over-complete set of states
for the operators $\Psi (r, t_c) \otimes 1$ and $1 \otimes S (r, t_c)$ at the time $t_c$. At each intermediate time, we define similar states. Then the functional $Z$ can be written as follows

$$Z [\eta_1, \eta_1^*, \eta_2, \eta_2^*, J_1, J_2, \rho] =$$

$$\int D\Phi_i D\Phi_c D\Omega_i \int D\Phi_i^* D\Phi_c D\Omega_i \int D\Phi_i'^* D\Phi_i' D\Omega_i' \langle \Phi_i', \Omega_i' | \rho | \Phi_i, \Omega_i \rangle$$

$$\times \langle \Phi_i, \Omega_i | T^{-1} \exp \left[ -i \int \! d\mathbf{x} \left( \eta_2^* (\mathbf{x}) \cdot \Psi (\mathbf{x}) + \Psi^+ (\mathbf{x}) \cdot \eta_2 (\mathbf{x}) + J_2 (\mathbf{x}) \cdot S (\mathbf{x}) \right) \right] | \Phi_c, \Omega_c \rangle$$

$$\times \langle \Phi_c, \Omega_c | T \exp \left[ i \int \! d\mathbf{x} \left( \eta_1^* (\mathbf{x}) \cdot \Psi (\mathbf{x}) + \Psi^+ (\mathbf{x}) \cdot \eta_1 (\mathbf{x}) + J_1 (\mathbf{x}) \cdot S (\mathbf{x}) \right) \right] | \Phi_i', \Omega_i' \rangle.$$  \hfill (25)

Hence we can now formally write the functional $Z$ as a time-ordered path-integral around a closed path in time starting at $t = t_0$, passing through $t = t_c$ and then going back to $t = t_0$ (see figure 2). This functional then has a path-integral representation similar to the equilibrium case\textsuperscript{10,21},

$$Z [\eta^*, \eta, J, \rho] = \int D\Psi^* D\Psi D\mathbf{m} \exp \left\{ i \mathcal{A} [\Psi^*, \Psi, \mathbf{m}, \eta^*, \eta, J] \right\}$$

$$\times \langle \Psi_2, \mathbf{m}_2 | \rho | \Psi_1, \mathbf{m}_1 \rangle$$

where we have used the following notation for the now classical tensor fields $\mathbf{m}$ and $\Psi$,

$$\mathbf{m} \equiv (\mathbf{m}_+, \mathbf{m}_-) \quad (27)$$

$$\Psi \equiv (\Phi_+, \Phi_-) \quad (28)$$

where + and − stand for the component that is propagating forward and backward in time, respectively. The field $\mathbf{m}$ is therefore a $2 \times 3$ tensor, while $\Psi$ is a $2 \times 2$ tensor. Similarly, we write the source terms in terms of tensors. Using the following notation for the external sources,

$$\eta = (\eta_1, \eta_2), \quad (29)$$

$$J = (J_1, J_2).$$
\( \eta \) becomes a \( 2 \times 2 \) tensor and \( \mathbf{J} \) a \( 2 \times 3 \) tensor. This notation greatly simplifies the manipulation of the path-integral.

The action \( \mathcal{A} \) is given as the difference of two actions; one due to the fields propagating forward in time and the other due to fields propagating backward in time,

\[
\mathcal{A} [\Psi^*, \Psi, \mathbf{m}, \eta^*, \eta, \mathbf{J}] = \mathcal{A} [\Psi_1^*, \Psi_1, \mathbf{m}_1, \eta_1^*, \eta_1, \mathbf{J}_1] - \mathcal{A} [\Psi_2^*, \Psi_2, \mathbf{m}_2, \eta_2^*, \eta_2, \mathbf{J}_2].
\]

Both terms on the right are obtained in the usual way. The electron contribution is standard. The magnetic moment contribution can be obtained in the same way, but it involves a geometrical part coming from the \( SU(2) \) symmetry. Hence the forward part of the action is given by

\[
\mathcal{A} [\Psi_1^*, \Psi_1, \mathbf{m}_1, \eta_1^*, \eta_1, \mathbf{J}_1] = \mathcal{A}_{WZ} [\mathbf{m}_1] - \int d\mathbf{x} \ H_d (\mathbf{m}_1 (\mathbf{x})) 
+ \int d\mathbf{x} \left\{ \frac{i}{\hbar} \Psi_1^{\dagger} (\mathbf{x}) \frac{\partial}{\partial t} \Psi_1 (\mathbf{x}) - H_{s+d} \left( \Psi_1^{\dagger}, \Psi_1, \mathbf{m}_1 \right) \right\} 
+ \int d\mathbf{x} \left\{ \eta_1^{\dagger} (\mathbf{x}) \Psi_1 (\mathbf{x}) + \Psi_1^{\dagger} (\mathbf{x}) \eta_1 (\mathbf{x}) + \mathbf{J}_1 (\mathbf{x}) \cdot \mathbf{m}_1 (\mathbf{x}) \right\},
\]

where a summation over \( \alpha \), the spin index, is implied. The \( \mathcal{A}_{WZ} \) is the Wess-Zumino part of the action \( \mathcal{A} \). Because of the boundary conditions on the spin fields at the left ends of the time path at \( t = -\infty \) (KMS-type conditions), this WZ-term has the same form as in the equilibrium case where the path of integration is along the imaginary-time branch from \( t = 0 \) to \( t = -i\beta \).

The vector map \( \mathbf{m}(t, \tau) \) is a parametrization of the surface enclosed by the trajectory of the magnetization, Fig. 2

\[
\mathbf{m}(t, 0) = \mathbf{m}_1(t), t \in C_1
= \mathbf{m}_2(t), t \in C_2,
\]

\[
\mathbf{m}(t, 1) = \mathbf{m}_0,
\]

\[
\mathbf{m}(-\infty + i0^+, \tau) = \mathbf{m}(-\infty + i0^-, \tau).
\]

\( \mathbf{m}_0 \) is a distinguished vector and is usually taken along the quantization axis.
This WZ-term is topological in origin and can be considered as a constraint condition on the configuration space of the magnetic moments and gives rise to magnetic monopole type potentials.\textsuperscript{23}

Next we expand the initial density matrix elements in terms of the initial configurations of the conduction electron field and the magnetization field. Therefore we are led to define a new functional $F$ which may describe any initial correlations between the conduction electrons and the local magnetic moments,

$$
\langle \Psi_2, m_2 \mid \rho \mid \Psi_1, m_1 \rangle = \exp \left\{ i F [\Psi^\dagger, \Psi, m] \right\}.
$$

(35)

Since we are assuming that the density matrix of the whole system is decoupled at $t = t_0$, then all cross terms in the expansion are zero. Keeping only terms up to second order, the expansion is

$$
F [m, \Psi, \Psi^\dagger] = C^{(0)} + \int dx \epsilon^{\alpha\beta} C^{(1)}_{\alpha} (x) \cdot m_\beta (x)
+ \frac{1}{2} \int dx dy \epsilon^{\alpha\gamma} \epsilon^{\beta\lambda} m_\alpha (x) \cdot C^{(2)}_{\gamma\lambda} (x, y) \cdot m_\beta (y)
+ \int dx dy \epsilon^{\alpha\gamma} \epsilon^{\beta\lambda} \Psi_\alpha^\dagger (x) \cdot Q_{\gamma\lambda} (x, y) \cdot \Psi_\beta (y).
$$

(36)

The tensor $\epsilon$ is defined such that $\epsilon_{11} = -\epsilon_{22} = 1$, and $\epsilon_{12} = \epsilon_{21} = 0$. The functional coefficients $C^{(0)}$, $C^{(1)}$, $C^{(2)}$, and $Q$ are as yet unknown. The notation used should be clear; for example the last term involves summations over the path index and the spin index,

$$
\epsilon^{\alpha\gamma} \epsilon^{\beta\lambda} \Psi_\alpha^\dagger \cdot Q_{\gamma\lambda} \cdot \Psi_\beta = \epsilon^{\alpha\gamma} \epsilon^{\beta\lambda} \Psi_\alpha^\dagger Q^{s_s'}_{s_s'} \Psi_{s_s'}
$$

(37)

where the upper indexes on $Q$ are for spin up and spin down.

Inserting this expansion back in Eq. (26), we end up with the following expression for the
action $A$,

$$
A[\Psi^*, \Psi, m, \eta^\dagger, \eta, J, Q, C] = \epsilon^{\alpha\beta} \left\{ A_{WZ}[m] - \int dx \, H_d(m_\beta(x)) \right\}
$$

$$
+ \epsilon^{\alpha\beta} \int dx \left\{ i \, \Psi^\dagger_\beta(x) \frac{\partial}{\partial t} \Psi_\beta(x) - H_{s+sd} \left( \Psi^\dagger_\beta, \Psi_\beta, m_\beta \right) \right\}
$$

$$
+ \epsilon^{\alpha\beta} \int dx \left\{ \eta^\dagger_\beta(x) \Psi_\beta(x) + \Psi^\dagger_\beta(x) \eta_\beta(x) + J_\beta(x) \cdot m_\beta(x) \right\}
$$

$$
+ \frac{1}{2} \epsilon^{\alpha\gamma} \epsilon^{\beta\lambda} \int dxdy \, m_\alpha(x) \cdot C_{\gamma\lambda}(x,y) \cdot m_\beta(y)
$$

$$
+ \epsilon^{\alpha\gamma} \epsilon^{\beta\lambda} \int dxdy \, \Psi^\dagger_\alpha(x) \cdot Q_{\gamma\lambda}(x,y) \cdot \Psi_\beta(y),
$$

where we have made an obvious redefinition of the coefficients. The functional $Z$, now becomes of the standard form \(21\)

$$
Z[\eta^\dagger, \eta, J, Q, C] = \oint \mathcal{D}\Psi^* \mathcal{D}\Psi \mathcal{D}m \exp\left\{ i \, A[\Psi^\dagger, \Psi, m, \eta^\dagger, \eta, J, Q, C] \right\}.
$$

The integral notation emphasizes that the path in time is closed, Fig. 2. Therefore we now can apply the usual field theoretical methods to extract the equations of motion from this functional. From the correlation functions, it is clear that the functional

$$
W[\eta^\dagger, \eta, J, Q, C] = -i \ln Z[\eta^\dagger, \eta, J, Q, C]
$$

(40)

is the generator that we need to derive the irreducible Green’s functions of the system. To get the average value of the conduction electron field or the magnetization, we differentiate with respect to the coefficients in the linear terms. For the conduction electrons, we have

$$
\frac{\delta W}{\delta \eta^\dagger_{\alpha s}(x)} = \epsilon^{\alpha\beta} \langle \Psi^\dagger_{\beta s}(x) \rangle,
$$

(41)

and for the magnetization, we get

$$
\frac{\delta W}{\delta J_{\alpha i}(x)} = \epsilon^{\alpha\beta} \langle m_{\beta i}(x) \rangle.
$$

(42)

The average of the conduction electrons field, a Fermi-type field, is set to zero while we set the average of the magnetization to be

$$
M_{\alpha i}(x) = \langle m_{\alpha i}(x) \rangle.
$$

(43)
Given the above definitions, Eqs. (41,42), the two-point correlation terms are easily obtained,

\[
\frac{1}{Z} \frac{\delta Z}{\delta Q_{s,s'}(x,y)} = i \frac{\delta \ln W}{\delta Q_{s,s'}(x,y)} = i \epsilon_{\alpha\gamma} \epsilon_{\beta\lambda} \langle \Psi_{\alpha s}^\dagger(x) \Psi_{\beta s'}(y) \rangle,
\]

where \(s, s'\) are for spin up and spin down and \(i, j\) are for the spin field components. The indices \(\alpha, \beta \ldots\), denote the branch of time in fig. 2. Mixed correlation functions can be obtained in the same way.

Clearly, solving for the two-point propagators is the least we can do to have a meaningful solution that includes relaxation effects. Knowing these propagators amounts to knowing the particle density, the spin density, the current density, and the scattering amplitudes, among others. Since we assume that the conduction electrons’ field has no mean value, its two-point propagator is then explicitly given by time-ordered products,

\[
\begin{align*}
G_{ss}^{22}(x,y) &= \langle T^{-1} (\Psi_s(x) \Psi_s^+(y) ) \rangle, \\
G_{ss}^{s's'}(x,y) &= \langle \Psi_s(x) \Psi_s^+(y) \rangle, \\
G_{ss}^{s's'}(x,y) &= \langle T (\Psi_s(x) \Psi_s^+(y) ) \rangle, \\
G_{ss}^{s's'}(x,y) &= -\langle \Psi_s^+(y) \Psi_s(x) \rangle.
\end{align*}
\]

From the above expressions, it is clear that the function \(G_{21}\) is the “less than” Green’s function and \(G_{12}\) is the “greater than” Green’s function. \(G_{11}\) is the Feynman propagator, while \(G_{22}\) is the Dyson propagator.\(^24\) These Green’s functions are not all independent. From their definitions, we can see that

\[
G_{11}^{s's'}(x,y) + G_{22}^{s's'}(x,y) = G_{12}^{s's'}(x,y) + G_{21}^{s's'}(x,y).
\]

The Green’s function \(G_{12}\) is of special interest since it is related to the average of the density operator of the conduction electrons. The two-point functions for the magnetization are similarly given by
$$M_{ij}^{ij}(x, y) = \langle T - 1 (S_i(x)) S_j(y) \rangle - \langle S_i(x) \rangle \langle S_j(y) \rangle,$$

$$M_{ij}^{ij}(x, y) = \langle S_j(y) S_i(x) \rangle - \langle S_i(x) \rangle \langle S_j(y) \rangle,$$

$$M_{ij}^{ij}(x, y) = \langle T (S_i(x) S_j(y)) \rangle - \langle S_i(x) \rangle \langle S_j(y) \rangle,$$

$$M_{ij}^{ij}(x, y) = \langle S_i(x) S_j(y) \rangle - \langle S_i(x) \rangle \langle S_j(y) \rangle.$$  

These Green’s functions are easily related to the retarded and advanced Green’s functions. Since we are considering a situation which is not far from equilibrium, we will follow closely the treatment in Ref. 25. Therefore, as in the equilibrium case, we relate the “less than” functions to the distribution function of electrons and spin in the semi-classical limit.

IV. THE VARIATIONAL PRINCIPLE: EFFECTIVE ACTION METHOD

Since the functions $J, Q, C$ are not bound to a simple physical interpretation, we make the following Legendre transformation,

$$\Gamma \left[ M_{\alpha i}(x), G_{\alpha \beta}^{ss'}(x, y), M_{\alpha \beta}^{ij}(x, y) \right] = W \left[ J_{\alpha i}, Q_{\alpha \beta}^{ss'}, C_{\alpha \beta}^{ij} \right] - \int dx \ J_{\alpha i}(x) M_{\alpha i}(x)$$

$$- \int dx dy \ Q_{\alpha \beta}^{ss'}(x, y) G_{\beta \alpha}^{ss'}(y, x)$$

$$- \frac{1}{2} \int dx dy \ C_{\alpha \beta}^{ij}(x, y) \left( M_{\beta \alpha}^{ij}(y, x) + M_{\beta j}(y) M_{\alpha i}(x) \right).$$

We end up with a functional $\Gamma$ that is expressed solely in terms of magnetization and correlation functions of the current and the localized spins. The equations of motion are then found by differentiating $\Gamma$ with respect to its arguments,

$$\frac{\delta \Gamma}{\delta M_{\alpha i}(x)} = -J_{\alpha i}(x) - \int dy \ C_{\alpha \beta}^{ij}(x, y) M_{\beta j}(y),$$

$$\frac{\delta \Gamma}{\delta G_{\alpha \beta}^{ss'}(x, y)} = -Q_{\beta \alpha}^{ss'}(y, x),$$

$$\frac{\delta \Gamma}{\delta M_{\alpha \beta}^{ij}(x, y)} = -\frac{1}{2} C_{\alpha \beta}^{ij}(y, x).$$
Using the standard tools of field theory, we solve for \( J, Q, C \) in terms of \( M, G, M \). A discussion of Wick’s theorem is beyond the scope of this paper. Omitting terms of order \( \lambda^4 \) and higher, we have the approximate effective action for the conduction electrons and the localized magnetic moments,

\[
\Gamma [M, G, M] = A_0 [M] + \frac{i}{2} \ln \det M^{-1}
\]

\[
+ \frac{1}{2} \int dxdy \left[ \frac{\delta^2 A_0}{\delta M_{\alpha} (x) M_{\beta} (y)} M^{ij}_{\alpha \beta} (y, x) \right] - i \ln \det G^{-1}
\]

\[
+ \int dxdy \left[ \frac{\delta^2 A_0}{\delta \Psi_{\alpha} (x) \delta \Psi_{\beta} (y)} G^{ss'}_{\alpha \beta} (y, x) \right]
\]

\[
+ \frac{\lambda^2}{2} g^{\alpha \alpha''} g^{\beta \beta''} \sigma^{i}_{s s'} \sigma^{j}_{s s'} \frac{1}{2} \int dxdy \left[ G^{ss''}_{\alpha \beta} (x, y) M^{ij}_{\alpha \beta'} (y, x) G^{ss''}_{\alpha \beta'} (y, x) \right]
\]

\[
+ O(\lambda^4).
\]

The functional \( A_0 \) is the functional \( A \) with all the source terms set to zero. The tensor \( g^{ijk} \) is equal to 1 if \( i = j = k = 1 \) and equal to -1 if \( i = j = k = 2 \) and zero, otherwise. The last term, which is clearly valid for large magnitude of \( S \), has a simple interpretation in terms of Feynman diagrams, Fig. 3.

The equations of motion for \( M, G, M \), are obtained by minimizing \( \Gamma \) and setting the external sources to zero with the appropriate initial conditions. Within the above stated approximations, the magnetization of the medium obeys the following equation of motion

\[
e^{\alpha \beta} e^{\delta \rho} M_{\delta \rho} (x) \partial_t M_{\beta} (x) + e^{\alpha \beta} \delta_{\alpha \beta} B_i (x)
\]

\[
+ \frac{1}{2} J e^{\alpha \beta} \nabla^2 M_{\beta} (x) + \frac{\lambda}{2} e^{\alpha \alpha''} \sigma^{i}_{s s'} \sigma^{j}_{s s'} M^{ij}_{\alpha \alpha'} (x, x') = 0.
\]

Here we have taken the long-wavelength limit to get the familiar exchange term through a coarse-graining procedure where each cell is taken to have a maximum spin of \( S \). The last term on the left is clearly the interaction with the conduction electrons’ magnetic moments to all orders in \( \lambda \). The equation of motion for the conduction electrons is

\[
\left( i \partial_{ty} - \epsilon (y) \right) \delta_{s s'} + \frac{\mu}{2} \sigma^{i}_{s s'} B_i + \frac{\lambda}{2} \sigma_{s s'} M_{\alpha} (y) \right] G^{ss''}_{\gamma \alpha} (y, z)
\]

\[
+ \lambda^2 g^{\alpha \alpha''} g^{\beta \beta''} \sigma^{i}_{s s'} \sigma^{j}_{s s'} \frac{1}{2} \int dx \left[ G^{ss}_{\alpha \beta} (y, x) G^{ss'}_{\gamma \alpha} (z, x) M^{ij}_{\alpha \beta'} (x, y) \right] = -i \delta_{s s''} (z - y),
\]
where $\in (y)$ is the spin-independent energy of the conduction electron. The term of first order in $\lambda$ describes the full exchange interaction between the magnetic moments of the localized electrons and those of the current. The structure of this equation is familiar from the theory of correlation functions due to Coulomb interactions. There, the propagator $M(x, y)$ is replaced by the Hartree propagator. Therefore the solution of this equation should follow by analogy with the treatment in Ref. \[27\]. The final equation is the equation of motion for the magnetic correlation functions,

\begin{align}
\epsilon^{\alpha \beta} \epsilon^{ijk} \{ \partial_i M_{\beta k}(x) - M_{\beta k}(x) \partial_i \} \mathcal{M}^{k'j}_{\alpha'\beta'}(z, x) + \epsilon^{\alpha \beta} \int dy \left[ J(x - y) \mathcal{M}^{k'i}_{\alpha'\beta'}(z, y) \right] \\
+ \lambda^2 g^{\alpha_1 \alpha_2} g^{\beta_1 \beta_2} \sigma_{s s_1 s_2 s_3}^2 \frac{\sigma_{s s_1 s_2}}{2} \int dy \left[ \mathcal{M}^{k'j}_{\alpha'\beta'}(z, x) \mathcal{G}^{s s_1' s_2'}_{\alpha_1 \beta_1}(x, y) \mathcal{G}^{s s_3' s_4'}_{\alpha_2 \beta_2}(y, x) \right] = i \delta^{k'i}_{\alpha'\alpha}(x - z). \tag{57}
\end{align}

The integrals are all four-dimensional and hence we have defined $J(x - y) = J(\vec{x} - \vec{y}) \delta(t_x - t_y)$. This set of coupled equations, Eqs. (55-57), is the main result of this work. It is important to observe that up to this point, the propagators $M$ and $G$ are the true propagators of the theory. Hence the above equations are non-perturbative in nature. In this section and the next, we study the structure of these equations and make contact with previous work. Since we are interested in how the magnetic moment of the current influences that of the medium (or vice-versa), we define

\begin{align}
\mathcal{M}^{i}_{\alpha\beta}(x, y) &= \frac{1}{2} \sigma_{s s}^i \mathcal{G}^{s s}_{\alpha\beta}(x, y), \tag{58}
\end{align}

to be the conduction electron spin propagator. The spin “charge” of the current is easily seen to follow from $\mathcal{M}^{i}_{\alpha\beta}$ by setting $\alpha = 1, \beta = 2$ and letting $y \rightarrow x^+$,

\begin{align}
\mathcal{M}^{i}(x) &= \frac{1}{2} \sigma_{s s}^i \mathcal{G}^{s s}_{12}(x, x), \tag{59}
\end{align}

However, we will find it useful to go to the center of mass and relative coordinates (Wigner coordinate system) to make contact with the classical quantities,

\begin{align}
\mathcal{X} &= \frac{1}{2} (\vec{x} + \vec{y}), \tag{60}
T &= \frac{1}{2} (t_x + t_y), \\
\mathcal{x}_\Delta &= \vec{x} - \vec{y}, \\
\mathcal{t}_\Delta &= t_x - t_y.
\end{align}
In this new coordinate system, the magnetic moment of the conduction electrons $\mathcal{M}_i(x)$ becomes a function of the macroscopic variables $X$ and $T$ only,

$$
\mathcal{M}_i(x, y) = \int \frac{d\omega}{2\pi} \int \frac{dp}{(2\pi)^3} \exp \left[ i\omega t_\Delta - ip \cdot x_\Delta \right] M_i(x, T; \omega, p).
$$

(61)

To get the equation of motion for $\mathcal{M}_i$, the spin charge of the conduction electrons, we first multiply Eq. (56) from the L.H.S. by $\sigma_{ls''}^l$ and sum over the spin degrees of freedom. We end up with an equation for the polarized current propagator,

$$
\epsilon^{\alpha\beta} \delta_{ss'} \left[ i\partial_t - \omega_\alpha \right] \mathcal{M}_{\gamma\alpha}(y, z) + \frac{i\epsilon^{\alpha\beta}}{2} \mathbf{B} \times \mathcal{M}_{\gamma\alpha}(y, z)
$$

$$
+ \lambda_2 g_\alpha'^{\alpha''} g_\beta'^{\beta''} \sigma_{ss''}^{ij} \sigma_{s's}^{ij} \mathcal{G}_{\gamma\alpha}(z, x) \mathcal{M}_{\gamma\alpha}(y, z)
$$

$$
+ \mathcal{G}_{\gamma\alpha}(z, x) \mathcal{M}_{\gamma\alpha}(y, z)
$$

(62)

The last term on the left provides for the relaxation of the spin moment of the conduction electrons.

To derive the equation for the polarization of the current, in the following we use the relaxation time approximation and replace the last term in Eq. (62), the collision integral, by a local term. The classical polarization of the current $\mathcal{M}_c$ is found by first assuming that the $l$–th component, $\mathcal{M}_i(x)$, has the following form

$$
\mathcal{M}_l(T, X, p, \omega) = \delta (\omega - p^2/2m - V(T, X)) \mathcal{M}_l(T, X, p),
$$

(63)

where we have set $\in(p) = p^2/2m + V(T, X)$. Then by averaging over the fast degrees of freedom, we have by definition

$$
\mathcal{M}_c^l(T, X) = v \int \frac{d\omega}{2\pi} \frac{dp}{(2\pi)^3} \mathcal{M}_l(T, X, p, \omega),
$$

(64)

where $v$ is the volume of the system. The spin current $\mathcal{J}$ is defined in the usual way. However here it has a tensorial character because of the vector character of the spin charge,

$$
\mathcal{J}^{kl}(T, X) = \frac{v}{m} \int \frac{dp}{(2\pi)^3} \mathcal{M}_k(T, X, p) p^l.
$$

(65)

The equation of motion for $\mathcal{M}_c$ is found by first going to the center of mass coordinates and using the quasi-particle approximation. We find for each $k$–component
\[
\left[ \partial_T + \frac{p}{m} \cdot \partial_X \right] \mathcal{M}^k (T, X, p) + \epsilon^{klp} \left[ B^l + \lambda M^l (T, X) \right] \mathcal{M}^p (T, X, p)
\]

\[
= - \frac{\mathcal{M}^k (T, X, p) - \mathcal{M}^k_{eq}}{\tau_k} - \frac{\mathcal{M}^k (T, X, p) - \mathcal{M}^k_0}{\tau_p}, \tag{66}
\]

where \(\tau_k(\tau_p)\), is the relaxation time for spin flip (momentum) scattering processes\textsuperscript{28,29}. By definition the average of the last term over the momentum is zero. This way of writing the collision term is valid only in the absence of spin-momentum coupling terms such as \(\mathbf{L} \cdot \mathbf{S}\)-coupling.

It is worthwhile to pause here and consider the content of this equation. The first term on the left hand side is the total time derivative, with independent variables \((T, X, p)\) and \(d\mathbf{p}/dt = 0\) (we consider a non-zero electric field elsewhere\textsuperscript{12}). The other term on the left hand side is the torque due to the local moments and the right hand side is an approximate expression for the collision operator.

In the absence of gradients, the polarization of the conduction electrons is along the local effective field and there will be no spin currents. This will not be the case if there is an electric field present. The spin accumulation effects studied here are solely due to non-homogeneous magnetization of the medium. It is important to observe that this effect is present even in the absence of a current as is the case for the spin accumulation effects due to interfaces put forward by Berger\textsuperscript{1}. In fact paramagnetic-ferromagnetic interfaces are naturally included in our treatment since we are dealing with nonhomogeneous magnetization. An interface simply corresponds to an abrupt change of the magnetization from a non-zero value to a zero value. These particular effects will be treated below.

Next we multiply Eq. (66) by the velocity and then average over it. To obtain the usual Fick’s law for spin diffusion, we assume that the momentum relaxation time is small and hence the R.H.S of Eq. (66) is larger than the effects due to the local magnetization \(\mathbf{M}\). In this case we have

\[
\mathbf{J}^{kl} (T, X) = -D^k \partial_{X_l} \mathcal{M}^k_c (T, X), \tag{67}
\]

where the diffusion constants are given in terms of an average Fermi velocity \(v_F\) by

\[
D^k = \frac{1}{3} v_F^2 \tau_k^{\text{eff}}, \tag{68}
\]
where $1/\tau^{'eff}_k = 1/\tau_k + 1/\tau_p$. To get this result we made use of the following approximation for the velocities of the conduction electrons

$$\partial X_l \int \frac{dp}{4\pi} v^j v^p \mathcal{M}^k (T, X, p) \simeq \frac{1}{3} v_F^2 \partial X_j \mathcal{M}^k_c (T, X) .$$

(69)

Finally using Eqs. (66, 67), we find that the classical magnetization of the conduction electrons obeys a diffusion equation for each one of its components,

$$[\partial_T - D^k \nabla^2] \mathcal{M}^k_c (T, X) = -\frac{1}{\tau_k} (\mathcal{M}^k_c (T, X) - \mathcal{M}^k_{eq} (X)) - [(B + \lambda \mathcal{M}) \times \mathcal{M}_c]^k.$$  

(70)

This equation is however rotationally invariant and does not show the reduced symmetry of the ferromagnetic state. To get a more realistic equation we improve on Fick's law by keeping all terms in Eq. (66) and treat exchange effects between the conduction electrons and the magnetization more carefully. This amounts to taking into account the sd-exchange term in the electron propagators. For slow variations in time, we have now a modified Fick's law that takes into account the variation of local magnetization in space and in direction,

$$\mathcal{J}^{kj}_c (T, X) = -\mathcal{D}^{kp}_c \partial X_j \mathcal{M}^p_c (T, X) ,$$  

(71)

where now the diffusion constant becomes a tensor. It is defined in terms of a matrix $A$

$$\mathcal{D}^{kp} (X) = D^p (A^{-1})^{kp} (X) .$$  

(72)

There is no summation over $p$ in this equation. The matrix $A$ depends locally on the effective magnetization field $H$,

$$A(X) = \begin{bmatrix}
1 & -\tau_x H_z & \tau_x H_y \\
-\tau_y H_z & 1 & -\tau_y H_x \\
-\tau_z H_y & \tau_z H_x & 1
\end{bmatrix} .$$  

(73)

In our approximation, the effective local field is simply

$$H = B + \lambda \mathcal{M} (x) .$$  

(74)

Now in the steady state, the equation satisfied by the average magnetic moment $\mathcal{M}_c$ becomes a generalized diffusion equation

$$\sum_{p,l} \partial X_l \left[ \mathcal{D}^{kp} (X) \partial X_j \mathcal{M}^p_c (X) \right] = \frac{1}{\tau_k} (\mathcal{M}^k_c - \mathcal{M}^k_{eq}) + \lambda [\mathcal{M} \times \mathcal{M}_c]^k$$  

(75)
The tensor character of the diffusion term in Eq. (75) is not due to anisotropic transport - the flux in the \( j \)-direction in Eq. (71) is due to a gradient with respect to \( X_j \). Rather, the \( p \)-component of \( \mathbf{M}_c \) is rotated into the \( k \)-th direction by the effective field \( \mathbf{H} \), while transport takes place in the direction of the gradient. The diffusion tensor, Eq. (72), has striking similarities to the diffusion tensor of charged species in a plasma.\(^{30}\) If we restrict ourselves to the case where the local effective field is constant and along the \( z \)-axis only, then the transverse diffusion coefficients are similar to those found by Hirst\(^{31}\) and Kaplan\(^{32}\) using very different methods from the one presented here. Their work showed that in the direction perpendicular to the effective field, diffusion of polarization of the electron gas is much slower than along the field. The off-diagonal terms have their origin in the strong exchange interaction among the conduction electrons which cannot be treated perturbatively for a transition metal. In a magnetic metal such as Ni, the off-diagonal terms can be two orders of magnitude larger than the diagonal ones.\(^{31}\)

V. SPIN-MOMENTUM TRANSFER: A SELF-CONSISTENT TREATMENT

To solve the above equations of motion, we retain a subset of the terms arising in the full spin propagator. This approximation is essentially similar to the random phase approximation in the calculation of the ground state energy of an interacting electron gas.\(^{27}\) The zero order propagator is taken to be that of the electrons in the external \( \mathbf{B} \)-field and the localized spins interacting through the exchange interaction in the presence of the \( \mathbf{B} \) field. Since the magnetic moments constitute a many-body problem, a full solution is not possible in general. Hence an explicit solution to the problem requires first a calculation of the background magnetization. The magnetization satisfies a generalized Landau-Lifshitz equation which follows from Eq. (55). First we observe that when the external sources are turned off, we have

\[
\mathbf{M}_1 (\mathbf{x}) = \mathbf{M}_2 (\mathbf{x}) = \mathbf{M} (\mathbf{x}),
\]

(76)

where \( \mathbf{M} \) is the average, i.e. classical, magnetization. Equation (55) is a system of two equations for \( \mathbf{M}_1 \) and \( \mathbf{M}_2 \), the magnetization vector along the paths \( C_1 \) and \( C_2 \) respectively, Fig. 2. It is the averaging of these two equations that gives the equation of motion for the
average magnetization,
\[ \partial_t \mathbf{M}(x) = \mathbf{M}(x) \times \left[ \frac{1}{2} J \nabla^2 \mathbf{M}(x) + \mathbf{B} + \frac{\lambda}{2} \sigma_{ss'}^s \frac{1}{2} \left( G_{11}^{ss'}(x,x^+) + G_{22}^{ss'}(x,x^-) \right) \right] \] (77)
where
\[ G_{11}^{ss'}(x,x^+) = G_{11}^{ss'}(x,y) \bigg|_{y \to x^+}. \] (78)

The last term is simply the spin of the current. Recalling that at equal times and equal positions, all different Green’s functions are related, the equation of motion for \( \mathbf{M} \) simply becomes
\[ \partial_t \mathbf{M}(x) = \mathbf{M}(x) \times \left[ \frac{1}{2} J \nabla^2 \mathbf{M}(x) + \mathbf{B} + \lambda \, \mathbf{M}(x) \right]. \] (79)

The last term gives rise to dissipation and a contribution to the precession for magnetic multilayers. As we will see below, this term becomes \( J \)-dependent in the non-uniform case. This latter equation, Eq. (79), is the equivalent of the Landau-Lifshitz equation (LL) for the magnetization in the presence of a current. This form is still valid even in the presence of an electric field. The solution of Eq. (56) can be represented in terms of Feynman diagrams, Eq. (56). First, we define the propagator of a non-interacting electron in an external magnetic field \( \mathbf{B} \) and zero electric field,
\[ \mathcal{G}(0)_{\alpha\beta}^{ss'}(x,y) = \left\{ \left( i \partial_t - \epsilon_\alpha \right) + \frac{1}{2} \sigma_{ss'}^{i} B^{i} \right\}^{-1} \delta_{ss'}^{\alpha\beta}(x,y). \] (80)

and expand \( \mathcal{G} \) in powers of \( \lambda \) using Eq. (56). Keeping only terms up to order \( \lambda^2 \), we have
\[ \sigma \mathcal{G} = \sigma \mathcal{G}(0) - \lambda \sigma \mathcal{G}(0) \mathbf{M} \cdot \sigma \mathcal{G}(0) + \frac{1}{2} \lambda^2 \sigma \mathcal{G}(0) \mathbf{M} \cdot \sigma \mathcal{G}(0) \mathbf{M} \cdot \sigma \mathcal{G}(0) + \lambda^2 \mathcal{G}(0) \sigma \cdot \mathbf{M} \cdot \sigma \mathcal{G}(0) \sigma \mathcal{G}(0) \ldots \] (81)

This is a matrix equation and hence integrations over time, space and spin degrees of freedom are implicit in the above notation. Recently Mills calculated the damping contribution to order \( \lambda^2 \). One of his conclusions is that this contribution is dependent on the symmetry of the system in this case. This follows from the fact that the spin propagator of the conduction electrons in a ferromagnet is not \( O(3) \)-invariant, to first order in \( \lambda \) and higher since it depends explicitly on \( \mathbf{M} \).

Equation (57) gives the dependence of correlations on the exchange interaction and on the s-d interaction between the current and the medium. Our assumption is that exchange interactions are much stronger than the spin-spin interaction. Hence to lowest order, we neglect the latter in the equation for the fluctuations. To understand the meaning of such
an equation, we study the case with strong exchange interactions, i.e., we take the average magnetization to be a constant and assume the external $B$ field to be small. In this case Eq. (57) becomes

$$\partial_t M_{\alpha\beta}^{x} (y, z) - [M \cdot \partial_y M_{\alpha\beta}^{x} (y, z)] M = i \epsilon^{\alpha\beta} (n \times M) + \int dx \left[ J (x - y) M_{\alpha\beta}^{x} (x, z) \times M \right],$$

(82)

where for each $k$, the unit vector $n$ has components $n^i = \delta^i_k$. The notation $M^{x}$ is for a vector with components $M^i$, $i = 1, 2, 3$. Now if we average over the variable $z$, we get an equation that gives the time variation of the fluctuations of the magnetization around $M$. These fluctuations will in turn cause fluctuations in the current through the last term in Eq. (81). Next we show how this latter equation gives rise to a Boltzmann-type equation for the magnetization fluctuations $M^{il}$. First we expand $M^{x} (x, z)$ around the position $-\vec{y}$,

$$M^{x} (x, z) = M^{x} (y, z) + \partial_y M^{x} (x, z) \bigg|_{x=y} \Delta x + \frac{1}{2} \partial_x \partial_x M^{x} (x, z) \bigg|_{x=y} \Delta x \Delta x + \ldots$$

(83)

where $\Delta x = \vec{x} - \vec{y}$. If we put this back in Eq. (82), we get a diffusion-type equation for all components of the magnetization fluctuations

$$\partial_t M_{\alpha\beta}^{x} (y, z) - [M \cdot \partial_y M_{\alpha\beta}^{x} (y, z)] M + \frac{1}{2} J_2 (y) M \times \nabla^2 y M_{\alpha\beta}^{x} (y, z) = i \epsilon^{\alpha\beta} n \times M$$

$$- J_0 (y) M \times M_{\alpha\beta}^{x} (y, z),$$

(84)

where $J_0 (y)$ and $J_2 (y)$ are the zeroth and second moment of the exchange coupling,

$$J_0 (y) = \int dx J (x - y),$$

$$J_2 (y) = \frac{1}{3} \int dx J (x - y) \Delta x^2.$$

(85)

(86)

These integrals converge since the exchange coupling is short ranged. The first moment vanishes since we are assuming isotropic exchange coupling. Hence treatment of the coupling of the conduction electrons to the ambient magnetization at low temperatures or temperatures close to $T_c$ must include Eq. (84) to account for the fluctuations of the magnetization.

VI. APPLICATIONS

In this section, we mainly show how this formalism can be applied to multilayers. In Ref. 11, we showed how our results extend those of Zhang, Levy and Fert by taking into account
the indirect exchange effect of the magnetization on the conduction electrons. This is an important effect in transition metals and can not be treated by a Born approximation. In the following we study two types of structures with nonhomogeneous magnetization. First we examine CPP-type structures with very thin paramagnetic spacers and no interfacial scattering. Second we consider structures which are topologically equivalent to a torus. These examples clearly illustrate the origin of spin accumulation to be directly related to inhomogeneities in the magnetization. It is also obvious from these examples that domain walls are another physical example where the results presented here can be applied. The interface will not be represented by a step function in the examples below and will instead take the shape shown in Fig. 4 which plots the mean field $a(x)$ due to the magnetization.

For a local magnetization which is a function only of distance $x$ in the direction of the current, $M = M(x)\mathbf{z}$, the spin accumulation obeys the simplified equations

\begin{align}
-2\frac{D_{xy}^2}{Da(x)}\frac{da(x)}{dx}\frac{dm_x(x)}{dx} + \left(D_{xx} - 2\frac{D_{xy}^2}{D}\right)\frac{da(x)}{dx}\frac{dm_y(x)}{dx} &= \frac{m_x(x)}{\tau_{sf}} - \frac{a(x)m_y(x)}{\tau_{sf}}, \\
-D_{xy}\frac{d^2m_x(x)}{dx^2} + D_{yy}\frac{d^2m_y(x)}{dx^2} - \left(D_{yy} - 2\frac{D_{xy}^2}{D}\right)\frac{da(x)}{dx}\frac{dm_x(x)}{dx} - 2\frac{D_{xy}^2}{Da(x)}\frac{da(x)}{dx}\frac{dm_y(x)}{dx} &= \frac{m_y(x)}{\tau_{sf}} + \frac{a(x)m_x(x)}{\tau_{sf}}, \tag{87}
\end{align}

\begin{align}
D\frac{d^2m_z(x)}{dx^2} &= \frac{m_z}{\tau_{sf}}, \tag{89}
\end{align}

where

\begin{align}
a(x) &= \tau\lambda M(x). \tag{90}
\end{align}

The coefficients $D_{xx}, D_{yy}, D_{xy}$ and $D_{yx}$ are functions of the local magnetization, the scattering rates and the exchange constant:

\begin{align}
D_p &= D_{xx} = D_{yy} = \frac{D}{1 + (\tau\lambda M(x))^2}, \tag{91}
\end{align}

\begin{align}
D_{yx} &= -D_{xy} = \frac{\tau\lambda M(x)}{1 + (\tau\lambda M(x))^2}. \tag{92}
\end{align}
These equations will be solved for different configurations of the magnetization $\mathbf{M}$. We adopt the following parameters for our calculations: The spin diffusion length $l_{sdl} = \sqrt{D\tau_{sf}} = 100\, \text{nm}$, $D = 10^{-3}\, \text{m}^2/\text{sec}$, $D_{yx} = 100\, D_{xx}$, and $\lambda = 0.1\, \text{eV}$. It should be noted that in these equations, the torque term has the opposite sign to that which appears in e.g., Zhang et al. since we have taken the electron charge to be positive in our definitions of the magnetic moments.

First we consider a configuration with in-plane magnetization. The magnetization is assumed to vary with position along the direction of the current. The spacer has practically zero thickness, which is a reasonable approximation for most GMR devices. Fig. 5 shows $a(x)$ for a typical interface. We do not explicitly include a non-magnetic spacer but we set the magnetization to zero at the center. At the ends it is parallel to the local $z$-axis, the local direction of the equilibrium magnetization. The transverse components of the spin accumulation are set to zero at the outer ends. The respective $z$-axes make a non-zero angle in order for the spin accumulation to be non-zero. First we demonstrate the effect of inhomogeneities on the spin accumulation; we keep the relative angle between the magnetizations the same but we vary the size of the 'domain wall', the transition region of the local mean field. It is seen from Fig. 6 and Fig. 7 that the larger the inhomogeneities the larger is the spin accumulation. This effect is independent of the relative orientations and was not predicted before. Therefore spin accumulation can be enhanced by having a layer with constant direction magnetization but with spatial inhomogeneities. Such a structure can be achieved by e.g. having a temperature gradient across the slab or controlled doping that changes the magnetic saturation along the direction of the current. Figures 8 and 9 show the variation of the $z$-component of the spin accumulation with respect to the relative angle of the magnetizations and size of the sample. Figures 10 and 11 are for the $x$ and $y$ components for the smaller sample. Here, the spin accumulation is largest for the case where the two magnetizations are orthogonal to each other and the size of the sample is smallest. In all these results, the equilibrium spin accumulation is normalized to $-1$ on the left hand side and normalized to $+1$ on the right hand side. The components of the spin accumulation are taken with respect to a global frame, that of the layer on the left.

As our second example, we choose a ‘ring’ structure. This can be part of a long solenoid with a square cross section. Hence we now solve our equations with periodic boundary
conditions. In each side of the square cross section, the profile of the magnetization within a period is shown in fig. 12. The relative angle of the magnetization between neighboring sides is 90 degrees. We study the spin accumulation as a function of the size of the sides. In figs. 13, 14 and 15 we plot the three components of the total spin accumulation. The equilibrium spin accumulation is taken to be normalized to one. The solutions show the expected behavior. The spin accumulation tries to reach its equilibrium value near the middle of each side. The spin accumulation is largest when the length of each side is smallest, as expected. This geometry shows how spin accumulation can be transported over large distances and also modulated by controlling the size of the cross section, similar to what happens in a regular transformer except here we are working with spin charge.

VII. CONCLUSION

We have introduced a many-body formalism based on path-integral techniques capable of handling a system of both local magnetic moments and conduction electrons in a self-consistent manner. Transport properties can be obtained through the calculation of the two-point functions of the current and the magnetization respectively. One of the important outcomes of this treatment is that we were able to derive a set of new equations that are needed when the magnetization of the medium is no longer homogeneous. First we showed that the polarization of the current is no longer homogeneous and satisfies a generalized diffusion equation where the diffusion tensor is dependent on the direction of the magnetization. We have hence shown that exchange effects are important in a ferromagnet and need to be taken into account properly. The fluctuations of the magnetization were also shown to obey a diffusion-type equation which depends on the direction of the local magnetization. This latter equation is in addition to recovering a Boltzmann equation for the current which follows from Eq. (62) and a Landau-Lifshitz equation for the average of the magnetization, Eq. (79). We have also shown how the non-uniform magnetization affects both the conduction electrons and the spin-momentum transfer term. We gave a simple physical picture for our main results. We finally showed how our results can be applied in various configurations. Our results show that spin accumulation can be enhanced by inhomogeneities at the interface.

In this work we focused on spin accumulation and we did not deal with its effect on
the dynamics of the local magnetization. We showed how to recover the Landau-Lifshitz equation but we discussed the effect of the collisions on the classical magnetization only in qualitative terms. The use of the relaxation time approximation is another shortcoming of this work. Its improvement will complicate the treatment further. We believe these issues should be addressed in the future.

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FIG. 1: Geometry of the magnetic sample.

FIG. 2: Closed time path: branch 1 corresponds to forward propagation in time while branch 2 is that for backward propagation.

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FIG. 3: Interaction terms between the conduction electrons and the localized magnetic moments. The smooth curve represents the conduction electron propagator. The curved line represents the spin-spin correlation functions.

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FIG. 4: Series expansion of the conduction electron propagator in powers of \( \lambda \).

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FIG. 5: Profile of the interface (or molecular field $a(x)$) used in the text. The current flows perpendicular to the interface. The nonmagnetic spacer is taken to have zero thickness.
FIG. 6: The x-component of the spin accumulation as a function of the interface inhomogeneities. On the left, $M = -M_0z$ and on the right $M = M_0z$. $lda = 0.6nm$ corresponds to the sharper interface.
FIG. 7: The y and z components of the spin accumulation as a function of the interface inhomogeneities for the same configuration of $\mathbf{M}$ as in Fig. 6.
FIG. 8: The z-component of the spin accumulation as a function of the angle for $L = 100 \text{ nm}$. The angles shown on the right are expressed in degrees.
FIG. 9: Spin accumulation as a function of the angle between the magnetization in both regions: $z$-component for $L = 10 \text{ nm}$. The angles are in degrees as in fig. 8.
FIG. 10: $x$-component of the spin accumulation as a function of the angle between the magnetization in both regions for $L = 10 \text{nm}$. 
FIG. 11: $y$-component of spin accumulation as a function of the angle between the magnetization in both regions for $L = 10 \, \text{nm}$. 
FIG. 12: Profile of the local magnetization along the different sides of the square ring (or torus) in one period. The magnetization is in the yz-plane whereas the current is in the x direction. The relative angle of the magnetization between neighboring sides is 90 degrees.
FIG. 13: Spin accumulation in a torus structure with equal sides intersecting at right angles. The length of each side is $L = 10$ nm.
FIG. 14: Spin accumulation in a torus structure with equal sides intersecting at right angles. The length of each side is $L = 100 \text{ nm}$
FIG. 15: Spin accumulation in a torus structure with equal sides intersecting at right angles. The length of each side is $L = 1000$ nm
$L_1 = 1000 \text{ nm}$