CONVERGENCE OF LAGRANGE FINITE ELEMENT METHODS FOR MAXWELL EIGENVALUE PROBLEM IN 3D

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ABSTRACT. We prove convergence of the Maxwell eigenvalue problem using quadratic or higher Lagrange finite elements on Worsey-Farin splits in three dimensions. To do this, we construct two Fortin-like operators to prove uniform convergence of the corresponding source problem. We present numerical experiments to illustrate the theoretical results.

1. Introduction

It is well known that, in contrast to Nédélec edge elements [27], the direct use of Lagrange finite elements fail to approximate Maxwell’s eigenvalue problem, as they lead to erroneous solutions on generic triangulations (see for example [3, 5]). However, Wong and Cendes [31] numerically show Lagrange finite element methods (FEMs) give the correct approximations on certain meshes. In particular, in two dimensions they show that the use of linear Lagrange finite element spaces defined on Powell-Sabin [28] meshes lead to accurate approximations. Likewise, in [31, Example 3] they demonstrate that the use of quadratic Lagrange finite element spaces on “consistent tetrahedral meshes” in three dimensions lead to correct approximations of Maxwell’s eigenvalue problem. Although Wong and Cendes do not explicitly define “consistent tetrahedral meshes”, it is reasonable to assume they are referring to the three-dimensional analogue of Powell-Sabin triangulations, in particular, Worsey-Farin meshes [32] which we recall below.

Recently in [7] we theoretically justified the numerical experiments of Wong and Cendes [31] in two dimensions. We proved that indeed linear (and higher) Lagrange elements on Powell-Sabin triangulations yield discrete eigenvalues that converge to the true eigenvalues as the mesh parameter tends to zero. The theory also shows the convergence of discrete eigenvalues using quadratic (and higher) Lagrange elements on Clough-Tocher splits [13], as well as quartic (and higher) Lagrange elements on general meshes without nearly singular vertices. Similar to the two families of Nédélec edge elements, the spaces mentioned above fit into a discrete de Rham sequence (see for example [20]). To prove convergence of the eigenvalue problem is suffices to prove uniform estimates of the solution operator of the corresponding source problem. One main tool in two dimensions was the construction of a Fortin-like operator [7].

The present paper can be considered a continuation of paper [7], where we consider Lagrange elements in three dimensions on Worsey-Farin triangulations on a contractible, polyhedral domain. We, again, mathematically justify the numerical experiments of Wong and Cendes [31] and prove that the use of quadratic (or higher) Lagrange elements on Worsey-Farin splits lead to accurate approximations to Maxwell’s eigenvalue problem. The present...
analysis in three dimension is more involved than the two dimensional analysis given in [7]. In particular, we need to develop two Fortin-like operators whereas in [7] only one was needed. To do this, we exploit that Lagrange elements on Worsey-Farin splits also fit into an a discrete de Rham sequence [21]. Then, using the degrees of freedom given in [21] we construct a Fortin-like operators for both curl and divergence operators. In order to prove that these operators are bounded, we use certain embeddings that hold on Lipshitz polyhedral domains; see Section 3.2.

To the best of our knowledge, this seems to be the first paper theoretically justifying convergence of Lagrange elements on simplicial meshes in three dimensions without modifying the bilinear form. In contrast, several papers prove convergence of Lagrange elements, where they add penalization or regularization terms to the bilinear form; see [4, 8, 10, 16–18]. Parallel work by Hu et al. [11, 23, 24] also develop finite elements on different splits with Lagrange or partially discontinuous elements while leaving the bilinear forms unchanged. In particular, in [23] partially discontinuous elements where applied to the Maxwell eigenvalue problem in three dimensions on Worsey-Farin splits and they show convergence numerically.

We provide numerical results confirming our theoretical findings. In particular, we show that one must use at least quadratic Lagrange elements for Worsey Farin splits to get convergence. That is, the use of linear Lagrange elements on Worsey-Farin refinements do not yield convergent approximations.

The paper is organized as follows. In the next section, we state the Maxwell eigenvalue problem and its mixed formulation. We also introduce general primal and mixed finite element methods for the eigenvalue problem and present a convergence framework. In Section 3, we give several preliminary results including trace inequalities and Sobolev embeddings. Section 4 gives the definition of Worsey-Farin triangulations, summarizes some exactness properties of finite element spaces on such meshes, and constructs a Scott-Zhang-type interpolant. In Section 5, we construct two Fortin-like operators and show stability estimates for those two operators. As a byproduct, in Section 6, we show convergence of Lagrange finite element methods for the Maxwell eigenvalue problem on Worsey-Farin triangulations provided the polynomial degree is at least two. Finally, in Section 7, we present some numerical experiments illustrating that continuous piecewise polynomials can be applied to three dimensional Maxwell eigenvalue problem after Worsey-Farin refinement.

2. THE MAXWELL’S EIGENVALUE PROBLEM, ITS DISCRETIZATION, AND CONVERGENCE FRAMEWORK

Let \( \Omega \subset \mathbb{R}^3 \) be a contractible, Lipschitz polyhedral domain and consider the eigenvalue problem: Find \( u \in H_0(\text{curl}, \Omega) \) such that

\[
(\text{curl } u, \text{curl } v) = \eta^2(v, v), \quad \forall v \in H_0(\text{curl}, \Omega),
\]

where \((\cdot, \cdot)\) is the \(L^2(\Omega)\) inner product over \(\Omega\), and

\[
H_0(\text{curl}, \Omega) := \{ v \in L^2(\Omega) : \text{curl } v \in L^2(\Omega) \text{ and } v \times n = 0 \text{ on } \partial\Omega \}.
\]

Here, \(n\) is an exterior unit normal vector on \(\partial\Omega\).

Accordingly, a canonical finite element method with respect to a given finite-dimensional space \(\mathcal{V}_h \subset H_0(\text{curl}, \Omega)\) is to find \(u_h \in \mathcal{V}_h \setminus \{0\}\) and \(\eta_h \in \mathbb{R}\) such that

\[
(\text{curl } u_h, \text{curl } v_h) = \eta_h^2(u_h, v_h), \quad \forall v_h \in \mathcal{V}_h.
\]
When \( \eta \neq 0 \), a mixed formulation given by Boffi et al. \( [6] \) is equivalent to \((2.1)\) : find \( \lambda \in \mathbb{R}\setminus\{0\} \) and \( 0 \neq p \in H_0(\text{div}^0, \Omega) \), \( \sigma \in H_0(\text{curl}, \Omega) \) such that:
\[
(\sigma, \tau) + (p, \text{curl} \tau) = 0 \quad \forall \tau \in H_0(\text{curl}, \Omega),
\]
\[
(\text{curl} \sigma, q) = -\lambda (p, q) \quad \forall q \in H_0(\text{div}^0, \Omega),
\]
where
\[
H_0(\text{div}, \Omega) := \{ v \in L^2(\Omega) : \text{div} v = 0 \text{ on } \partial \Omega \},
\]
\[
H_0(\text{div}^0, \Omega) := \{ v \in H_0(\text{div}, \Omega) : \text{div} v = 0 \}.
\]
We note that \( H_0(\text{div}^0, \Omega) = \text{curl} H_0(\text{curl}, \Omega) \), and also \( \lambda = \eta^2 \), \( \sigma = u \) and \( p = -\frac{\text{curl} u}{\lambda} \).

An equivalent mixed formulation of the discrete problem \((2.2)\) (when \( \eta_h \neq 0 \)) is: find \( \lambda_h \in \mathbb{R}\setminus\{0\} \) and \( 0 \neq p_h \in \mathcal{D}_h \), \( \sigma_h \in \mathcal{V}_h \) such that:
\[
(\sigma_h, \tau_h) + (p_h, \text{curl} \tau_h) = 0 \quad \forall \tau_h \in \mathcal{V}_h,
\]
\[
(\text{curl} \sigma_h, q_h) = -\lambda_h (p_h, q_h) \quad \forall q_h \in \mathcal{D}_h,
\]
where \( \mathcal{D}_h := \text{curl} \mathcal{V}_h \). Analogous to the continuous setting, we have \( \lambda_h = \eta_h^2 \), \( \sigma_h = u_h \) and \( p_h = -\frac{\text{curl} u_h}{\lambda_h} \).

We follow the classical theory (e.g., \( [5] \) Section 14) and analyze the mixed finite element method \((2.4)\) by considering the corresponding source problem. Define the solution operators \( A : L^2(\Omega) \to H_0(\text{curl}, \Omega) \) and \( T : L^2(\Omega) \to H_0(\text{div}^0, \Omega) \) such that for given \( f \in L^2(\Omega) \), there holds
\[
(A f, \tau) + (T f, \text{curl} \tau) = 0 \quad \forall \tau \in H_0(\text{curl}, \Omega),
\]
\[
(\text{curl} Af, q) = (f, q) \quad \forall q \in H_0(\text{div}^0, \Omega).
\]
Likewise, the discrete solution operators \( A_h : L^2(\Omega) \to \mathcal{V}_h \) and \( T_h : L^2(\Omega) \to \mathcal{D}_h \) are defined as
\[
(A_h f, \tau_h) + (T_h f, \text{curl} \tau_h) = 0 \quad \forall \tau_h \in \mathcal{V}_h,
\]
\[
(\text{curl} A_h f, q_h) = (f, q_h) \quad \forall q_h \in \mathcal{D}_h.
\]

2.1. **Convergence theory.** It is well known that the convergence of the eigenvalues to the discrete problem \((2.4)\) converge to the exact eigenvalues (given in problem \((2.3)\)) provided the source problem converges uniformly (see for example \( [5] \) Section 7)). In the next proposition, the operator norm is defined as
\[
||T|| := \sup_{f \in L^2(\Omega)\setminus\{0\}} \frac{||Tf||_{L^2(\Omega)}}{||f||_{L^2(\Omega)}}.
\]

**Proposition 2.1.** Let \( T \) and \( T_h \) be defined from \((2.5)\) and \((2.6)\), respectively, and suppose that \( ||T - T_h|| \to 0 \) as \( h \to 0 \). Consider the problem \((2.3)\) with the nonzero eigenvalues \( 0 < \lambda^{(1)} \leq \lambda^{(2)} \leq \cdots \) and the problem \((2.4)\) with the nonzero eigenvalues \( 0 < \lambda_h^{(1)} \leq \lambda_h^{(2)} \leq \cdots \).
Then, for any fixed \( i \), \( \lim_{h \to 0} \lambda_h^{(i)} = \lambda^{(i)} \).

It will suffice to verify one assumption of our discrete spaces to guarantee \( ||T - T_h|| \to 0 \). To describe this assumption, we introduce the space
\[
\mathcal{V}^T(\mathcal{D}_h) = \{ \tau \in \mathcal{V}^T : \text{curl} \tau \in \mathcal{D}_h \}.
\]
Assumption 2.2. We assume the existence of a projection \( \Pi_{\mathcal{Y}} : \mathcal{Y}^t(\mathcal{D}_h) \to \mathcal{Y}_h \) such that

\[
\text{curl} \, \Pi_{\mathcal{Y}} \, \tau = \text{curl} \, \tau \quad \forall \tau \in \mathcal{Y}^t(\mathcal{D}_h),
\]

\[
\|\Pi_{\mathcal{Y}} \tau - \tau\| \leq \omega_0(h)(\|\tau\|_{H^{1/2+\delta}(\Omega)} + \|\text{curl} \, \tau\|_{L^2(\Omega)}) \quad \forall \tau \in \mathcal{Y}^t(\mathcal{D}_h).
\]

Furthermore, we assume that the \( L^2 \)-orthogonal projection \( \mathbb{P}_Q : L^2(\Omega) \to \mathcal{D}_h \) satisfies

\[
\|\mathbb{P}_Q p - p\|_{L^2(\Omega)} \leq \omega_1(h)\|\text{curl} p\|_{L^2(\Omega)} \quad \forall p \in H(\text{curl}, \Omega) \cap H_0(\text{div}, \Omega).
\]

Here, \( \omega_i > 0 \) satisfies \( \lim_{h \to 0^+} \omega_i(h) = 0 \) for \( i = 0, 1 \).

With this assumption we have the following result. We give the proof of this result in the appendix; it is similar to the argument in [7].

Theorem 2.3. Suppose that \((\mathcal{Y}_h, \mathcal{D}_h)\) satisfy Assumption 2.2 and let \( T \) and \( T_h \) defined as in the equations (2.5) and (2.6), respectively. Then we have

\[
\|T - T_h\| \leq C(\omega_0(h) + \omega_1(h)).
\]

The following corollary is a consequence of the above theorem and Proposition 2.1.

Corollary 2.4. Consider the problem (2.3) with the nonzero eigenvalues \( 0 < \lambda_1^{(1)} \leq \lambda_2^{(2)} \leq \cdots \) and the problem (2.4) with the nonzero eigenvalues \( 0 < \lambda_1^{(1)}_h \leq \lambda_2^{(2)}_h \leq \cdots \). Suppose that \((\mathcal{Y}_h, \mathcal{D}_h)\) satisfy Assumption 2.2. Then, for any fixed \( i \), \( \lim_{h \to 0^+} \lambda_i(h) = \lambda_i \).

3. Preliminaries

3.1. Trace Theorems and Inverse inequalities. Here we state inequalities that allow us to estimate the Fortin-type projections. We start by recalling the definition of fractional-order Sobolev spaces and their accompanying (semi-)norm; see for example [19].

Definition 3.1. Let \( U \subset \mathbb{R}^d \) be an open set. For \( 0 < s < 1 \) and \( 1 < p < \infty \) define

\[
W^{s,p}(U) := \{u \in L^p(U) : \int_U \int_U \frac{|u(x) - u(y)|^p}{|x - y|^{d+sp}} \, dx \, dy < \infty\}.
\]

The accompanying semi-norm and norm on \( W^{s,p}(U) \) are given respectively by

\[
|u|_{W^{s,p}(U)} := (\int_U \int_U \frac{|u(x) - u(y)|^p}{|x - y|^{d+sp}} \, dx \, dy)^{1/p};
\]

\[
\|u\|_{W^{s,p}(U)} := (\|u\|_L^p(U)^p + |u|^p_{W^{s,p}(U)})^{1/p}.
\]

Finally, we use the notation \( H^s(U) := W^{s,2}(U) \).

We also extend the definition of fractional-order Sobolev norms in the case \( U = \partial S \), where \( S \) is a simplex. We take the same definition as above, where we view \( U \) as a \( d \)-dimensional manifold; see [19].

We will use the following basic Sobolev embedding result; see [19, Theorem 1.4.4.1]. We restrict ourselves to a simplex for simplicity.

Lemma 3.2. Let \( K \) be a \( d \)-dimensional simplex. Then,

\[
\|w\|_{H^{t,q}(K)} \leq C\|w\|_{W^{s,p}(K)}, \quad 0 \leq t \leq s \leq 1, \ 1 \leq p \leq q < \infty,
\]

with \( s - d/p = t - d/q \). The constant \( C > 0 \) depends on \( K \).

We will also need a trace inequality; see [19, Theorem 1.5.1.2].
Lemma 3.3. Let $K$ be a $d$-dimensional simplex. Let $0 < s - 1/p < 1$, $0 \leq s \leq 1$. If $w \in W^{s,p}(K)$, then
\begin{equation}
\|w\|_{W^{s-1/p,p}(\partial K)} \leq C\|w\|_{W^{s,p}(K)}.
\end{equation}
Moreover, if $v \in W^{s-1/p,p}(\partial K)$ there exists $w \in W^{s,p}(K)$ such that $w|_{\partial K} = v$ and
\begin{equation}
\|w\|_{W^{s,p}(K)} \leq C\|v\|_{W^{s-1/p,p}(\partial K)}.
\end{equation}
The constant $C > 0$ depends on $K$.

We will often require particular inverse estimates for polynomial spaces, which follow from equivalence of norms on finite dimensional spaces; for more general inverse estimates consult, for example, [9], Section 1.6 and Section 4.5.

Lemma 3.4. Let $K$ be a $d$-dimensional simplex. For any $v \in P_r(K)$ the following bounds hold
\begin{align}
(3.4a) \quad & |v|_{W^{k,p}(K)} \leq C h_K^{\ell-k+d/p-d/q} |v|_{W^{\ell,q}(K)} \quad \forall 1 \leq p, q \leq \infty, 0 \leq \ell \leq k \leq 1, \\
(3.4b) \quad & \|v\|_{L^p(\partial K)} \leq C h^{-1/p}_K \|v\|_{L^p(K)} \quad \forall 1 \leq p \leq \infty,
\end{align}
where $C > 0$ depends on the shape-regularity of $K$, $r$, and $d$, but is independent of $h_K$.

3.2. Embeddings. To prove convergence to the solution operator, we utilize certain embeddings of vector-valued functions. To describe the results we introduce the following space notation:
\begin{align*}
\mathcal{V} := & H(\text{curl}, \Omega) \cap H(\text{div}, \Omega), \\
\mathcal{V}^t := & H_0(\text{curl}, \Omega) \cap H(\text{div}, \Omega), \\
\mathcal{V}^n := & H(\text{curl}, \Omega) \cap H_0(\text{div}, \Omega).
\end{align*}

We start with an embedding result given in [2, Proposition 3.7].

Proposition 3.5. If $\Omega$ is a Lipschitz polyhedron, there exists $\delta \in (0, \frac{1}{4}]$ and $C > 0$ such that:
\begin{equation}
\|v\|_{H^{1/2+\delta}(\Omega)} \leq C(\|v\|_{L^2(\Omega)} + \|\text{curl }v\|_{L^2(\Omega)} + \|\text{div }v\|_{L^2(\Omega)}) \quad \forall v \in \mathcal{V}^t \cup \mathcal{V}^n.
\end{equation}

Here, the constant $\delta$ in (3.5) may differ for $v \in \mathcal{V}^t$ and $v \in \mathcal{V}^n$. Thus, we choose the smaller constant $\delta$ of these two embeddings. Next, we use a result given in [3, Theorem 2.2] where we use that $\Omega$ is a contractible, Lipschitz polyhedron.

Lemma 3.6. There exists a positive constant $C$ such that for all $v \in \mathcal{V}^t \cup \mathcal{V}^n$,
\begin{equation}
\|v\|_{L^2(\Omega)} \leq C(\|\text{curl }v\|_{L^2(\Omega)} + \|\text{div }v\|_{L^2(\Omega)}).
\end{equation}

4. Finite element spaces on Worsey-Farin splits

4.1. Definitions and Notations. For a set of simplices $S$, we use $\Delta_s(S)$ to denote the set of $s$-dimensional simplices ($s$-simplices for short) in $S$. If $S$ is a simplicial triangulation of a domain $D$ with boundary, then $\Delta'_s(S)$ denotes the subset of $\Delta_s(S)$ that does not belong to the boundary of the domain. If $S$ is a simplex, then we use the convention $\Delta_s(S) = \Delta_s(\{S\})$. 
For a non-negative integer \( r \), we use \( \mathcal{P}_r(S) \) to denote the space of piecewise polynomials of degree \( \leq r \) on \( S \), and we define
\[
\mathcal{P}_r(S) = \prod_{S \in \mathcal{S}} \mathcal{P}_r(S), \quad \mathcal{P}_r^c(S) = \mathcal{P}_r(S) \cap C^0(D), \quad \text{with } D = \bigcup_{S \in \tilde{\mathcal{S}}} S,
\]
\[
\mathcal{P}_r(T) = \mathcal{P}_r(S) \cap L^2_0(D), \quad \mathcal{P}_r^c(T) = \{ v \in \mathcal{P}_r(S) : v|_{\partial D} = 0 \},
\]
where \( L^2_0(D) \) is the space of square integrable functions with vanishing mean. Analogous vector-valued spaces are denoted in boldface, e.g., \( \mathcal{P}_r^c(S) = [\mathcal{P}_r^c(S)]^3 \).

Given a family of shape-regular, simplicial triangulations \( \{ T_h \} \) of \( \Omega \), let \( h_T = \text{diam}(T) \), \( \forall T \in \mathcal{T}_h \) and \( h = \max_{T \in \mathcal{T}_h} h_T \). Since the meshes are shape regular, there exists a constant \( c_0 > 0 \) such that
\[
h_T \leq c_0 \rho_T, \quad \rho_T = \text{diameter of largest sphere contained in } T,
\]
for all \( T \in \mathcal{T}_h \). We now describe the construction of a Worsey-Farin triangulation \cite{32} from the original triangulation \( \mathcal{T}_h \).

For an arbitrary tetrahedra \( T \), we first show how to obtain a local Worsey-Farin triangulation of \( T \), denoted by \( T_{wf} \). This is done via the following two steps (cf. \cite{21} Section 2)):

1. Connect the incenter \( z_T \) of \( T \) to its (four) vertices.
2. For each face \( F \) of \( T \) choose \( m_F \in \text{int}(F) \). We then connect \( m_F \) to the three vertices of \( F \) and to the incenter \( z_T \).

Here, \( \text{int}(F) \) denotes the interior of \( F \). This procedure divides \( T \) into the twelve tetrahedra which define \( T_{wf} \).

To obtain a Worsey-Farin refinement \( T_{wf} \) of a triangulation \( \mathcal{T}_h \) we split each \( T \in \mathcal{T}_h \) by the above procedure. However, special care is needed in the choice of the point \( m_F \). For each interior face \( F = T_1 \cap T_2 \) with \( T_1, T_2 \in \mathcal{T}_h \), let \( m_F = L \cap F \) where \( L = [z_{T_1}, z_{T_2}] \), the line segment connecting the incenters of \( T_1 \) and \( T_2 \). The fact that such a \( m_F \) exists is established in \cite{25} Lemma 16.24. For a boundary face \( F \) with \( F = T \cap \partial \Omega \) with \( T \in \mathcal{T}_h \), let \( m_F \) be the barycenter of \( F \). In \cite{25} it is conjectured that the resulting triangulation is shape regular.

We will assume throughout that \( \{ T_{wf} \} \) is shape regular with the regularity constant related to the shape regularity constant of \( \{ T_h \} \).

For any \( F \in \Delta_2(T) \), we see that the refinement \( T_{wf} \) induces a Clough-Tocher triangulation of \( F \), i.e., a triangulation consisting of three triangles, each having the common vertex \( m_F \); we denote this set by \( \mathcal{E}^{ct}(F) \). Let \( e_F \in \Delta_1(\mathcal{E}^{ct}(F)) \) be an arbitrary, but fixed internal edge of \( F^{ct} \). Then further define
\[
\mathcal{E}(T_{wf}^h) = \{ e \in \Delta_1(\mathcal{E}^{ct}(F)) : \forall F \in \Delta_2(T_h) \}.
\]

Let \( T_1 \) and \( T_2 \) be adjacent tetrahedra in \( \mathcal{T}_h \) that share a face \( F \). Write \( F = [x_0, x_1, x_2] \) where \( \{ x_i \}_{i=0}^2 \) are the vertices of \( F \), and similarly write \( T_i = [a_i, x_0, x_1, x_2] \) for \( i = 1, 2 \), where \( a_i \) is the vertex of \( T_i \) not shared by \( F \). Let \( z_{T_i} \) be the incenter of \( T_i \) and set \( K_i = [z_{T_i}, x_0, x_1, x_2] \), \( i = 1, 2 \). We denote by \( K_{wf}^i \) the triangulation \( T_{wf} \) restricted to \( K_i \), that is, \( K_{wf}^i \) consists of three tetrahedra which are each in \( T_i \) and have a single face that lie in \( F \).

Let \( e \in \Delta_1(\mathcal{E}^{ct}(F)) \) be one of the three internal edges in the Clough-Tocher refinement of \( F \), and let \( \{ K^j_i : j = 1, 2 \} \subset K_{wf}^i \) be the two tetrahedra in \( K_{wf}^i \) that have \( e \) as an edge \( (i = 1, 2) \). We assume that \( K^j_i \) is labeled such that \( K^1_i \) and \( K^2_i \) share a common face. We then define
\[
\theta_e(p) = p|_{K^1_i} - p|_{K^2_i} + p|_{K^3_i} - p|_{K^4_i}, \quad \text{on } e.
\]
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(a) A representation of $F^{ct}$ and $\Delta^{I}_1(F^{ct})$ (indicated in blue).

(b) The original triangulation

(c) Worsey-Farin refinement

Figure 1. The Worsey-Farin Splits

Let $T \in \mathcal{T}_h$, $F \in \Delta_2(\mathcal{T}_h)$ with $F \subset \partial T$, and $e \in \Delta^I_1(F^{ct})$. Denote by $t_e$ be the unit vector tangent to $e$ pointing away from $m_F$, and let $n_F$ be the outward unit normal of $T$ restricted to $F$. Then there exist triangles $Q_1, Q_2 \subset F^{ct}$ such that $e = \partial Q_1 \cap \partial Q_2$. The jump of a piecewise smooth function $p$ across $e$ is defined as

$$[p]_e = (p|_{Q_1} - p|_{Q_2})s,$$

where $s = n_F \times t_e$ is a unit vector orthogonal to $t_e$ and $n_F$.

Let $p$ and $u$ be a smooth vector and scalar valued functions, respectively, on $T \in \mathcal{T}_h$, and let $F \in \Delta_2(T)$. Then the tangential part of $p$ and $u$ are given by $p_F := n \times p \times n$ and $u_F = u|_F$, respectively, where $n$ is the unit outward normal of $\partial T$. We have the following identities:

$$\text{curl}_F p_F = \text{curl} p \cdot n, \quad \text{grad}_F u_F = n \times (\text{grad} u \times n), \quad \text{on } F.$$

We define the function spaces with respect to a Clough-Tocher triangulation of a face $F$ (cf. [21, Section 3]),

$$S_r(F^{ct}) := \{v \in P^r_c(F^{ct}) : \text{grad}_F v \in P^r_c(F^{ct})\},$$

$$\mathcal{R}_r(F^{ct}) := S_r(F^{ct}) \cap P^r_c(F^{ct}), \quad \hat{S}_r(F^{ct}) := \{v \in \mathcal{R}_r(F^{ct}) : \text{grad}_F v|_{\partial F} = 0\},$$

and define function spaces with respect to the Worsey-Farin triangulation of a tetrahedron $T \in \mathcal{T}_h$,

$$S_r(T^{wf}) := \mathcal{P}_r(T^{wf}) \cap H^2(T), \quad \hat{S}_r(T^{wf}) := S_r(T^{wf}) \cap H^2_0(T),$$

$$\mathcal{N}_r(T^{wf}) := \mathcal{P}_r(T^{wf}) \cap H(\text{div}, T).$$

4.2. Global spaces and degrees of freedom. We define the global spaces (cf. [21, Section 6])

$$\mathcal{G}_r = \{v \in C^1(\Omega) : q|_T \in S_r(T^{wf}), \forall T \in \mathcal{T}_h\},$$

$$\mathcal{E}_{r-1} = \mathcal{P}^c_{r-1}(\mathcal{T}^{wf}_h),$$

$$\mathcal{M}_{r-2} = \{v \in H(\text{div}, \Omega) : v|_T \in \mathcal{N}_{r-2}(T^{wf}), \forall T \in \mathcal{T}_h, \theta_e(v \cdot t_e) = 0 \forall e \in \mathcal{E}(\mathcal{T}^{wf}_h)\},$$

$$\mathcal{W}_{r-3} = \mathcal{P}_{r-3}(\mathcal{T}^{wf}_h).$$
The definitions of the finite element spaces show the following sequence forms a complex (cf. [21])

\[ \mathbb{R} \supseteq \mathcal{S}_r \xrightarrow{\text{grad}} \mathcal{L}_{r-1} \xrightarrow{\text{curl}} \mathcal{N}_{r-2} \xrightarrow{\text{div}} \mathcal{M}_{r-3} \rightarrow 0. \]

We also consider the complex with boundary conditions:

\[ 0 \rightarrow \mathcal{S}^0_r \xrightarrow{\text{grad}} \mathcal{L}_{r-1}^f \xrightarrow{\text{curl}} \mathcal{N}_{r-2}^n \xrightarrow{\text{div}} \mathcal{M}_{r-3}^0 \rightarrow 0, \]

where

\[ \mathcal{S}^0_r = \mathcal{S}_r \cap H^1_0(\Omega), \quad \mathcal{L}_{r-1}^f = \mathcal{L}_{r-1} \cap H_0(\text{curl}, \Omega), \]
\[ \mathcal{N}_{r-2}^n = \mathcal{N}_{r-2} \cap H^1_0(\text{div}, \Omega), \quad \mathcal{M}_{r-3}^0 = \mathcal{M}_{r-3} \cap L^2_0(\Omega). \]

The following lemma summarizes the degrees of freedom (dofs) for \( \mathcal{L}_{r-1}, \mathcal{N}_{r-2}, \) and \( \mathcal{M}_{r-3} \) given in [21] Lemmas 5.4–5.6.

**Lemma 4.1.** Let \( r \geq 3. \)

1. A function \( \boldsymbol{\tau} \in \mathcal{P}^{r-1}_{r-1}(T_{wf}) \) is uniquely defined by the following conditions:

   \[ \begin{align*}
   (4.5a) \quad & \boldsymbol{\tau}(a) \quad \forall a \in \Delta_0(T) \\
   (4.5b) \quad & \int_e \boldsymbol{\tau} \cdot \boldsymbol{\kappa} \, ds \quad \forall \boldsymbol{\kappa} \in \mathcal{P}_{r-3}(e), \; \forall e \in \Delta_1(T), \\
   (4.5c) \quad & \int_e [\text{curl} \; \boldsymbol{\tau} \cdot t_e]_e \kappa \, ds \quad \forall \boldsymbol{\kappa} \in \mathcal{P}_{r-3}(e), \; \forall e \in \Delta_1^f(F^c) \setminus \{e_F\}, \; \forall F \in \Delta_2(T), \\
   (4.5d) \quad & \int_{e_F} [\text{curl} \; \boldsymbol{\tau} \cdot t_{e_F}]_{e_F} \kappa \, ds \quad \forall \boldsymbol{\kappa} \in \mathcal{P}_{r-2}(e_F), \; \forall F \in \Delta_2(T), \\
   (4.5e) \quad & \int_F (\boldsymbol{\tau} \cdot \boldsymbol{n}_F) \kappa \, dA \quad \forall \boldsymbol{\kappa} \in \mathcal{R}_{r-1}(F^c), \; \forall F \in \Delta_2(T), \\
   (4.5f) \quad & \int_F (\text{curl}_F \boldsymbol{\tau}_F) \kappa \, dA \quad \forall \boldsymbol{\kappa} \in \tilde{\mathcal{P}}_{r-2}(F^c), \; \forall F \in \Delta_2(T), \\
   (4.5g) \quad & \int_F \boldsymbol{\tau}_F \cdot \boldsymbol{\kappa} \, dA \quad \forall \boldsymbol{\kappa} \in \text{grad}_F \mathcal{S}_r(F^c), \; \forall F \in \Delta_2(T), \\
   (4.5h) \quad & \int_T \text{curl} \; \boldsymbol{\kappa} \, d \tau \quad \forall \boldsymbol{\kappa} \in \text{curl} \; \mathcal{P}^c_{r-1}(T_{wf}), \\
   (4.5i) \quad & \int_T \boldsymbol{\kappa} \, d \tau \quad \forall \boldsymbol{\kappa} \in \text{grad}_T \mathcal{S}_r(T_{wf}).
   \end{align*} \]

2. A function \( \boldsymbol{p} \in \mathcal{N}_{r-2}(T_{wf}) \) is uniquely defined by the following conditions:

   \[ \begin{align*}
   (4.6a) \quad & \int_e [\boldsymbol{p} \cdot t_e]_e \kappa \, ds \quad \forall \boldsymbol{\kappa} \in \mathcal{P}_{r-3}(e), \; \forall e \in \Delta_1^f(F^c) \setminus \{e_F\}, \; \forall F \in \Delta_2(T), \\
   (4.6b) \quad & \int_{e_F} [\boldsymbol{p} \cdot t_{e_F}]_{e_F} \kappa \, ds \quad \forall \boldsymbol{\kappa} \in \mathcal{P}_{r-2}(e_F), \; \forall F \in \Delta_2(T), \\
   (4.6c) \quad & \int_F (\boldsymbol{p} \cdot \boldsymbol{n}_F) \kappa \, dA \quad \forall \boldsymbol{\kappa} \in \mathcal{P}_{r-2}(F^c), \; \forall F \in \Delta_2(T), \\
   (4.6d) \quad & \int_T (\text{div} \; \boldsymbol{p}) \kappa \, d \tau \quad \forall \boldsymbol{\kappa} \in \tilde{\mathcal{P}}_{r-3}(T_{wf}).
   \end{align*} \]
\[ \text{(4.6e)} \quad \int_T p \cdot \kappa \, dx \quad \forall \kappa \in \text{curl} \hat{\mathcal{P}}_{r-1}^{\text{c}}(T^{\text{wf}}). \]

(3) A function \( v \in \mathcal{P}_{r-3}(T^{\text{wf}}) \) is uniquely defined by the following conditions:

\[ \text{(4.7a)} \quad \int_T v \, dx \]

\[ \text{(4.7b)} \quad \int_T v \cdot \kappa \, dx \quad \forall \kappa \in \hat{\mathcal{P}}_{r-3}(T^{\text{wf}}). \]

4.3. Equivalence norms assumptions. We will use the above dofs to build Fortin-like projections. Unfortunately, the dofs are not preserved with a Piola transform, as is the case for the Nédélec elements. Rather, our strategy to prove estimates of the projections is to map a tetrahedra to a unit size tetrahedra via dilation. To this end, we define scaled tetrahedra.

Definition 4.2. For a tetrahedron \( T \in \mathcal{T}_h \), define its dilation and induced Worsey-Farin split

\[ \text{(4.8)} \quad \hat{T} = \frac{1}{h_T} T := \{ \frac{x}{h_T} : x \in T \}, \quad \hat{T}^{\text{wf}} = \{ \frac{1}{h_T} K : K \in T^{\text{wf}} \}. \]

Note the scaled tetrahedra \( \hat{T} \) are of unit size, and \( \hat{T}^{\text{wf}} \) inherit the shape-regular properties of \( T^{\text{wf}} \). From now on, unless otherwise specified, we denote \( \hat{v}(\hat{x}) = v(h_T x) \ (\hat{x} \in \hat{T}) \) where \( v \) is either a scalar function or vector-valued function.

As a first step, we note that Lemma 4.1 and equivalence of norms in finite dimensional spaces immediately give the following lemma.

Lemma 4.3. Consider \( T \in \mathcal{T}_h \).

1. Let \( G^L_i, i = 1, \ldots, m^L \) be the functional given by dofs (4.5) on \( \hat{T} \), where \( m^L \) is the dimension of \( \mathcal{P}_{r-1}^{\text{c}}(T^{\text{wf}}) \). Then, there exists a constant \( \beta(T^{\text{wf}}) > 0 \) such that

\[ \| \hat{v} \|_{L^2(\hat{T})} \leq \beta(T^{\text{wf}}) \sum_{i=1}^{m^L} |G^L_i(\hat{v})|, \quad \forall \hat{v} \in \mathcal{P}_{r-1}^{\text{c}}(T^{\text{wf}}). \]

2. Let \( G^N_i, i = 1, \ldots, m^N \) be the functional given by dofs (4.6) on \( \hat{T} \), where \( m^N \) is the dimension of \( \mathcal{N}_{r-1}(T^{\text{wf}}) \). Then, there exists a constant \( \theta(T^{\text{wf}}) > 0 \) such that

\[ \| \hat{v} \|_{L^2(\hat{T})} \leq \theta(T^{\text{wf}}) \sum_{i=1}^{m^L} |G^N_i(\hat{v})|, \quad \forall \hat{v} \in \mathcal{N}_{r-1}(T^{\text{wf}}). \]

In the following, we make an assumption concerning the uniform bound of the constants appearing in Lemma 4.3.

Assumption 4.4. There exists constants \( \theta, \beta > 0 \) that depend only on the shape regularity of \( \{T_h\} \) such that

\[ \theta(T^{\text{wf}}) \leq \theta, \quad \forall T \in \mathcal{T}_h, \]

\[ \beta(T^{\text{wf}}) \leq \beta, \quad \forall T \in \mathcal{T}_h. \]

Remark 4.5. One common approach (see for example [30]) to verify this assumption is to argue that the constants \( \theta(T^{\text{wf}}) \) and \( \beta(T^{\text{wf}}) \) are continuous functions of the vertices of \( \hat{T}^{\text{wf}} \). If so, since the vertices live on a compact set, such a result implies the uniform bounds stated in Assumption 4.4.
4.4. The modified Scott-Zhang interpolants. The Fortin-like projections utilize Scott-Zhang-type interpolants onto piecewise linear polynomials that preserve zero tangential or normal boundary conditions. The proof of the following lemma can be found in the appendix. To state the lemma we denote the patch around $T$:

$$\omega(T) = \bigcup_{T' \in \mathcal{T}_h \atop T \cap T' \neq \emptyset} T'.$$

**Lemma 4.6.** Let $0 < \delta \leq \frac{1}{2}$. There exists a projection $I_h^{\text{curl}} : H^{1/2+\delta}(\Omega) \rightarrow \mathcal{P}_1^c(\mathcal{T}_h)$ with the following bounds:

$$h_T^{-1/2-\delta} \|\tau - I_h^{\text{curl}} \tau\|_{L^2(T)} + \|I_h^{\text{curl}} \tau\|_{H^{1/2+\delta}(T)} \leq C \|\tau\|_{H^{1/2+\delta}(\omega(T))} \quad \forall \tau \in H^{1/2+\delta}(\Omega)$$

for all $T \in \mathcal{T}_h$. Moreover, if $\tau \in H^{1/2+\delta}(\Omega) \cap H_0(\text{curl}, \Omega)$ then $I_h^{\text{curl}} \tau \in \mathcal{P}_1^c(\mathcal{T}_h) \cap H_0(\text{curl}, \Omega)$.

There also exists a projection $I_h^{\text{div}} : H^{1/2+\delta}(\Omega) \rightarrow \mathcal{P}_1^c(\mathcal{T}_h)$ with the following bounds:

$$h_T^{-1/2-\delta} \|\tau - I_h^{\text{div}} \tau\|_{L^2(T)} + \|I_h^{\text{div}} \tau\|_{H^{1/2+\delta}(T)} \leq C \|\tau\|_{H^{1/2+\delta}(\omega(T))} \quad \forall \tau \in H^{1/2+\delta}(\Omega)$$

for all $T \in \mathcal{T}_h$. Moreover, if $\tau \in H^{1/2+\delta}(\Omega) \cap H_0(\text{div}, \Omega)$ then $I_h^{\text{div}} \tau \in \mathcal{P}_1^c(\mathcal{T}_h) \cap H_0(\text{div}, \Omega)$.

5. Construction of Two Fortin-like Projections

In this section we define projections that conform to the framework given in Section 2. Let

$$H^D(\text{curl}, \Omega) := \{v \in L^2(\Omega) : \text{curl } v \in \mathfrak{N}_{r-2}\} \subset H(\text{curl}, \Omega).$$

5.1. Definition of Fortin-like operators. The following operators are defined through the use of Lemma 4.1. We separate those degrees of freedom into two parts with one part affecting commuting properties and the second one not.

**Definition 5.1.** Define the operator $\Pi_L : H^D(\text{curl}, \Omega) \cap H^{1/2+\delta}(\Omega) \to \mathfrak{L}_{r-1}$ such that on each $T \in \mathcal{T}_h$,

\begin{align*}
(5.1a) \quad & \int_e (\Pi_L v \cdot t_e) \kappa ds = \int_e (v \cdot t_e) \kappa ds \quad \forall \kappa \in \mathcal{P}_{r-3}(e), \forall e \in \Delta_1(T), \\
(5.1b) \quad & \int_e [\text{curl } \Pi_L v \cdot t_e] \kappa ds = \int_e [\text{curl } v \cdot t_e] \kappa ds \quad \forall \kappa \in \mathcal{P}_{r-3}(e), \forall e \in \Delta_1(F_{\text{ct}} \setminus \{e_F\}), \forall F \in \Delta_2(T), \\
(5.1c) \quad & \int_{e_F} [\text{curl } \Pi_L v \cdot t_{e_F}] \kappa ds = \int_{e_F} [\text{curl } v \cdot t_{e_F}] \kappa ds \quad \forall \kappa \in \mathcal{P}_{r-2}(e_F), \forall F \in \Delta_2(T), \\
(5.1d) \quad & \int_F (\text{curl}_F (\Pi_L v)_F) \kappa dA = \int_F (\text{curl}_F v_F) \kappa dA \quad \forall \kappa \in \mathcal{P}_{r-2}(F_{\text{ct}}), \forall F \in \Delta_2(T), \\
(5.1e) \quad & \int_T (\Pi_L v \cdot \kappa) dx = \int_T \text{curl } v \cdot \kappa dx \quad \forall \kappa \in \text{curl } \mathcal{P}_{r-1}(T_{\text{ct}}^w), \\
(5.1f) \quad & (\Pi_L v)(a) = 0 \quad \forall a \in \Delta_0(T), \\
(5.1g) \quad & \int_e (\Pi_L v \times t_e) \cdot \kappa ds = 0 \quad \forall \kappa \in \mathcal{P}_{r-3}(e), \forall e \in \Delta_1(T), \\
(5.1h) \quad & \int_F (\Pi_L v \cdot n_F) \kappa dA = 0 \quad \forall \kappa \in \mathcal{P}_{r-2}(F_{\text{ct}}), \forall F \in \Delta_2(T), \\
(5.1i) \quad & \int_F \Pi_L v_F \cdot \kappa dA = 0 \quad \forall \kappa \in \text{grad}_F \mathfrak{S}_r(F_{\text{ct}}), \forall F \in \Delta_2(T).
\end{align*}
\[\int_T \Pi_L v \cdot \kappa \, dx = 0 \quad \forall \kappa \in \text{grad} \hat{S}_r(T^{wf}).\]

**Definition 5.2.** The operator \( \Pi_N : H^{1/2+\delta}(\Omega) \cap H(\text{div}, \Omega) \to \mathcal{N}_{r-2} \) is defined such that, on each \( T \in \mathcal{T}_h \),

\[
\begin{aligned}
(5.2a) \quad &\int_{T} [\Pi_N p \cdot t_e] e_p \kappa \, ds = 0 \quad \forall \kappa \in \mathcal{P}_{r-3}(e), \forall e \in \Delta_1^t(F^{ct}) \setminus \{e_F\}, \forall F \in \Delta_2(T), \\
(5.2b) \quad &\int_{e_p} [\Pi_N p \cdot t_{e_p}] e_p \kappa \, ds = 0 \quad \forall \kappa \in \mathcal{P}_{r-2}(e_F), \forall F \in \Delta_2(T), \\
(5.2c) \quad &\int_{F} (\Pi_N p \cdot n_F) \kappa \, dA = \int_{F} (p \cdot n_F) \kappa \, dA \quad \forall \kappa \in \mathcal{P}_{r-2}(F^{ct}), \forall F \in \Delta_2(T), \\
(5.2d) \quad &\int_{T} (\text{div} \Pi_N p) \kappa \, dx = \int_{T} (\text{div} p) \kappa \, dx \quad \forall \kappa \in \hat{\mathcal{P}}_{r-3}(T^{wf}), \\
(5.2e) \quad &\int_{T} \Pi_N p \cdot \kappa \, dx = \int_{T} p \cdot \kappa \, dx \quad \forall \kappa \in \text{curl} \hat{\mathcal{P}}_{r-1}(T^{wf}).
\end{aligned}
\]

**Definition 5.3.** The operator \( \Pi_W : L^2(\Omega) \to \mathcal{W}_{r-3} \) is defined such that, on each \( T \in \mathcal{T}_h \),

\[
\begin{aligned}
(5.3a) \quad &\int_{T} \Pi_W v \, dx = \int_{T} v \, dx, \\
(5.3b) \quad &\int_{T} (\Pi_W v) \kappa \, dx = \int_{T} v \kappa \, dx \quad \forall \kappa \in \hat{\mathcal{P}}_{r-3}(T^{wf}).
\end{aligned}
\]

We modify the operator \( \Pi_L \) to obtain an Fortin-like projection that inherits the commuting properties of \( \Pi_L \) and approximation properties of the Scott-Zhang interpolant \( I_h^{\text{curl}} \).

**Definition 5.4.** The operator \( \Pi_V : H^D(\text{curl}, \Omega) \cap H^{1/2+\delta}(\Omega) \to \mathcal{L}_{r-1} \) is defined as

\[
\Pi_V = I_h^{\text{curl}} + \Pi_L (1 - I_h^{\text{curl}}),
\]

where \( I \) is the identity operator and \( \Pi_L \) is defined in Definition 5.1.

The next two theorems state the commuting properties of \( \Pi_V \) and \( \Pi_N \) and their approximation properties in the \( L^2 \)-norm. The proof of these theorems are postponed to Section 5.4.

**Theorem 5.5.** The operator \( \Pi_V : H^D(\text{curl}, \Omega) \cap H^{1/2+\delta}(\Omega) \to \mathcal{L}_{r-1} \) defined in Definition 5.4 satisfies

\[\text{curl} \Pi_V \tau = \text{curl} \tau, \quad \forall \tau \in H^D(\text{curl}, \Omega) \cap H^{1/2+\delta}(\Omega).\]

Moreover, under Assumption 4.4, the following bound holds for any \( \tau \in H^D(\text{curl}, \Omega) \cap H^{1/2+\delta}(\Omega) \):

\[
\|\Pi_V \tau - \tau\|_{L^2(\Omega)} \leq C(h^{1/2+\delta}\|\tau\|_{H^{1/2+\delta}(\Omega)} + h\|\text{curl} \tau\|_{L^2(\Omega)}).
\]

Finally, if \( \tau \in H^D(\text{curl}, \Omega) \cap H^{1/2+\delta}(\Omega) \cap H_0(\text{curl}, \Omega) \) then \( \Pi_V \tau \in \mathcal{L}_{r-1}^t \).

We also state the analogous result for \( \Pi_N \).

**Theorem 5.6.** The operator \( \Pi_N : H^{1/2+\delta}(\Omega) \cap H(\text{div}, \Omega) \to \mathcal{N}_{r-2} \) defined in Definition 5.1 satisfies

\[\text{div} \Pi_N p = \Pi_W \text{div} p \quad \forall p \in H^{1/2+\delta}(\Omega) \cap H(\text{div}, \Omega).\]
Moreover, under Assumption \[4.4\] the following bound holds for any \( p \in H^{1/2+\delta}(\Omega) \cap \mathbf{H}(\text{div}, \Omega) \):

\[
\| \Pi_N p - p \|_{L^2(\Omega)} \leq C \left( h^{1/2+\delta} \| p \|_{H^{1/2+\delta}(\Omega)} + h \| \text{div} p \|_{L^2(\Omega)} \right).
\]

Finally, if \( p \in H^{1/2+\delta}(\Omega) \cap \mathbf{H}_0(\text{div}, \Omega) \) then \( \Pi_N p \in \mathbf{N}_{p-2} \).

5.2. Bounds after Dilation.

5.2.1. Some inequalities on scaled tetrahedra. We first give several inequalities on the scaled tetrahedra \( \hat{T} \) from Definition \[4.2\]. The proofs of the next three results are shown in the appendix.

**Proposition 5.7.** Under Definition \[4.2\] we have the following results:

(i) There holds \( \text{diam}(\hat{T}) = 1 \leq c_0 \rho_{\hat{T}} \), where \( c_0 \) is the shape-regularity constant given in Definition \[4.1\].

(ii) \( \text{div} \hat{v} = h_T \text{div} v \) and \( \text{curl} \hat{v} = h_T \text{curl} v \).

(iii) There holds for \( 0 \leq s \leq 1 \)

\[
c_0^{-3/2-s} h_T^{-3/2+\delta} |v|_{H^s(\hat{T})} \leq |\hat{v}|_{H^s(\hat{T})} \leq h_T^{-3/2+\delta} |v|_{H^s(\hat{T})} \quad \forall v \in \mathbf{H}^s(\hat{T}).
\]

The next result establishes a trace inequality on \( \hat{T} \).

**Lemma 5.8.** For any \( \hat{v} \in \mathbf{H}^{1/2+\delta}(\hat{T}) (\delta \in (0, \frac{1}{2})) \), we have \( \hat{v} |_{\partial \hat{T}} \in L^p(\partial \hat{T}) \) for \( 2 \leq p < \frac{2}{1-\delta} \).

In particular,

\[
\| \hat{v} \|_{L^p(\partial \hat{T})} \leq C \| \hat{v} \|_{H^{1/2+\delta}(\hat{T})} \quad \forall \hat{v} \in \mathbf{H}^{1/2+\delta}(\hat{T}),
\]

where \( C \) is a uniform constant for all \( \hat{T} \).

The following lemma gives an inverse trace operator with estimate. Its proof is given in \[2\] Lemma 4.7; however, for completeness, we provide the proof with additional details in the appendix.

**Lemma 5.9.** Let \( \hat{e} \in \Delta_1(\hat{T}) \), and let \( \hat{F} \in \Delta_2(\hat{T}) \) be a face that has \( \hat{e} \) as an edge. Then, there exists an extension operator \( E : \mathcal{P}_{r-3}^{\hat{e}} \rightarrow W^{1,q}(\hat{T}) \) with \( 1 < q < 2 \) such that \( (E \hat{k}) |_{\hat{e}} = \hat{k} \), \( E\hat{k}|_{\partial \hat{F} \setminus \hat{e}} = 0 \) and \( E\hat{k}|_{\partial \hat{T} \setminus \hat{F}} = 0 \). Moreover, the following estimates hold:

\[
\begin{align*}
(5.8a) \quad & \| E\hat{k} \|_{W^{1,q}(\hat{F})} \leq C_1 \| \hat{k} \|_{W^{1-1/q,q}(\hat{e})}, \\
(5.8b) \quad & \| E\hat{k} \|_{W^{1,q}(\hat{T})} \leq C_2 \| \hat{k} \|_{W^{1-1/q,q}(\hat{e})},
\end{align*}
\]

where \( C_1, C_2 > 0 \) are two constants uniform for all \( \hat{T} \).

5.2.2. The estimates. The following two lemmas not only show that the operators \( \Pi_L \) and \( \Pi_N \) are well-defined on their respective domains but also give local estimates on the tetrahedra after dilating.

**Lemma 5.10.** Let \( \hat{\Pi}_L \) be operator given in Definition \[5.1\] defined on \( \hat{T} \) \[4.8\]. Under Assumption \[4.4\] there holds

\[
\| \hat{\Pi}_L \hat{v} \|_{L^2(\hat{T})} \leq C(\| \text{curl} \hat{v} \|_{L^2(\hat{T})}^2 + \| \hat{v} \|_{H^{1/2+\delta}(\hat{T})}^2),
\]

for all \( \hat{v} \in \mathbf{H}^{1/2+\delta}(\hat{T}) \) with \( \text{curl} \hat{v} \in \mathcal{P}_{r-2}(\hat{T}^{w/f}) \cap \mathbf{H}(\text{div}, \hat{T}) \).
Proof. We bound the corresponding non-zero functionals appearing on the right-hand side of Definition 5.1. We start with the following estimates, which follow from Hölder’s inequality and inverse estimates (3.4b) that hold since \( \nabla \mathbf{v} \) is a piecewise polynomial.

\[
| \int_{\tilde{F}} \left( \nabla \mathbf{v} \cdot \mathbf{t} \right) e \kappa | ds \leq C \| \nabla \mathbf{v} \|_{L^2(\tilde{F})} \| \kappa \|_{L^2(\tilde{F})} \quad \forall \kappa \in L^2(\tilde{F}), \forall \mathbf{v} \in \Delta_1(\tilde{T}^{u,F}), \mathbf{t} \subset \tilde{F}
\]

\[
| \int_{\tilde{F}} \left( \nabla \mathbf{v} \cdot \mathbf{n} \right) e \kappa | dA \leq C \| \nabla \mathbf{v} \|_{L^2(\tilde{T})} \| \kappa \|_{L^2(\tilde{T})} \quad \forall \kappa \in L^2(\tilde{T}), \forall \tilde{F} \in \Delta_2(\tilde{T}),
\]

\[
| \int_{\tilde{T}} \nabla \mathbf{v} \cdot \kappa | d\mathbf{x} \leq \| \nabla \mathbf{v} \|_{L^2(\tilde{T})} \| \kappa \|_{L^2(\tilde{T})} \quad \forall \kappa \in L^2(\tilde{T}).
\]

We now bound the remaining functionals coming from the right-hand side of (5.1a) by adopting the technique developed in the proof of [2, Lemma 4.7]. Let \( \mathbf{e} \in \Delta_1(\tilde{T}) \) and let \( \tilde{F} \in \Delta_2(\tilde{T}) \) have \( \mathbf{e} \) as an edge. We choose \( p \) such that \( 2 < p < \frac{2}{2-\alpha} \) in order to apply Lemma 5.8. Let \( \kappa \in \mathcal{P}_{\alpha-3}(\mathbf{e}) \), and let \( E \kappa \in W^{1,p} (\tilde{T}) \) be as in Lemma 5.9 (with \( \alpha = p' < 2 \), the Hölder conjugate of \( p \)). Integration by parts and using \( E \kappa|_{\tilde{F}} = \kappa \) and \( E \kappa|_{\partial \tilde{F} \setminus \mathbf{e}} = 0 \) gives

\[
\int_{\tilde{F}} (\mathbf{v} \cdot \mathbf{t}) \kappa | ds = \int_{\tilde{F}} \left( \nabla \mathbf{v} \right) \cdot \left( \mathbf{n} \right) \left( E \kappa \right) d\mathbf{A} + \int_{\tilde{F}} \left( \mathbf{v} \times \mathbf{n} \right) \cdot \left( \nabla E \kappa \right) d\mathbf{A}.
\]

An additional integration by parts, using \( E \kappa|_{\partial \tilde{T} \setminus \tilde{F}} = 0 \), Hölder’s inequality, estimate (5.8b), an inverse estimate (3.4a) on \( \tilde{T}^{u,F} \) and the shape regularity of \( \tilde{T}^{u,F} \) gives

\[
| \int_{\tilde{F}} \left( \nabla \mathbf{v} \right) \cdot \left( \mathbf{n} \right) \left( E \kappa \right) d\mathbf{A} | = | \int_{\tilde{T}} \nabla \mathbf{v} \cdot \nabla E \kappa d\mathbf{x} | \\
\leq \| \nabla \mathbf{v} \|_{L^p(\tilde{T})} \| E \kappa \|_{W^{1,1/p'} (\tilde{T})} \leq C \| \nabla \mathbf{v} \|_{L^2(\tilde{T})} \| E \kappa \|_{W^{1,1/p'} (\tilde{T})}.
\]

Using Hölder’s inequality, Lemma 5.8 and (5.8b) we obtain

\[
\int_{\tilde{F}} \left( \mathbf{v} \times \mathbf{n} \right) \cdot \left( \nabla E \kappa \right) d\mathbf{A} = \| \mathbf{v} \times \mathbf{n} \|_{L^p(\tilde{F})} \| E \kappa \|_{W^{1,1/p'} (\tilde{F})} \leq \| \mathbf{v} \|_{H^{1/2+\delta}(\tilde{T})} \| E \kappa \|_{W^{1,1/p'} (\tilde{T})}.
\]

Combining the above estimates and using Definition 5.1 with Assumption 4.4 yields the result (5.9). \( \square \)

We now derive a similar estimate for \( \tilde{\Pi}_N \).

Lemma 5.11. Let \( \tilde{\Pi}_N \) be the operator given in Definition 5.2 defined on \( \tilde{T} \) (4.8). Under Assumption 4.4, there holds with a uniform constant \( C \) for all \( \tilde{T} \),

\[
\| \tilde{\Pi}_N \tilde{p} \|_{L^2(\tilde{T})} \leq C \left( \| \nabla \tilde{p} \|_{L^2(\tilde{T})} + \| \tilde{p} \|_{H^{1/2+\delta}(\tilde{T})} \right) \quad \forall \tilde{p} \in \tilde{H}^{1/2+\delta}(\tilde{T}) \cap H(\nabla, \tilde{T}).
\]

Proof. Using the estimate in Lemma 5.8, the Cauchy-Schwarz inequality, along with \( \nabla \tilde{p} \in L^2(\tilde{T}) \), we obtain

\[
| \int_{\tilde{F}} (\tilde{p} \cdot \mathbf{n}_{F}) \kappa | d\mathbf{A} | \leq \| \tilde{p} \|_{H^{1/2+\delta}(\tilde{T})} \| \kappa \|_{L^2(\tilde{F})} \quad \forall \tilde{p} \in \tilde{H}^{1/2+\delta}(\tilde{T}), \forall \tilde{F} \in \Delta_2(\tilde{T}),
\]

\[
| \int_{\tilde{T}} \left( \nabla \tilde{p} \right) \kappa | d\mathbf{x} | \leq \| \nabla \tilde{p} \|_{L^2(\tilde{T})} \| \kappa \|_{L^2(\tilde{T})} \quad \forall \tilde{p} \in \tilde{L}^2(\tilde{T}),
\]

\[
| \int_{\tilde{T}} \tilde{p} \cdot \kappa | d\mathbf{x} | \leq \| \tilde{p} \|_{H^{1/2+\delta}(\tilde{T})} \| \kappa \|_{L^2(\tilde{T})} \quad \forall \tilde{p} \in \tilde{L}^2(\tilde{T}).
\]
These estimates, combined with Lemma 4.1, Definition 5.2, and Assumption 4.4, yield
\[ \| \hat{\Pi}_N \hat{p} \|_{L^2(\hat{T})} \leq C(\| \text{div} \hat{p} \|_{L^2(\hat{T})} + \| \hat{p} \|_{H^{1/2+\delta}(\hat{T})}), \]
where \( C \) is a uniform constant for all \( \hat{T} \).

\[ \square \]

5.3. Proofs of Theorems 5.5, 5.6. In order to prove these theorems, we transfer the results for \( \hat{T} \) back to \( T \). We start with the proof of Theorem 5.6.

Proof of Theorem 5.6

We first prove (5.6). Let \( p \in H^{1/2+\delta}(\Omega) \cap H(\text{div}, \Omega) \) and set \( \rho = (\text{div} \Pi_N p - \Pi_W \text{div} p) \in \mathcal{W}_{r-3} \). First, by (5.2c), (5.3a) and Stokes theorem, we have on each \( T \in \mathcal{T}_h \),
\[ \int_T \rho dx = \int_T \text{div}(\Pi_N p - p) dx = \int_{\partial T}(\Pi_N p - p) \cdot n dA = 0, \]
where we use that the constant functions are in \( \mathcal{P}_{r-2}(F^e) \). Next, for any \( \kappa \in \mathcal{P}_{r-3}(T^w) \), we use (5.2d) and (5.3b) to obtain
\[ \int_T \rho \kappa dx = \int_T \text{div}(\Pi_N p - p) \kappa dx = 0. \]
Thus \( \rho = 0 \) by (4.7), and so (5.6) holds.

Next we prove the bound (5.7). For any \( T \in \mathcal{T}_h \), with \( \hat{T} \) defined in (4.8) and \( \hat{p}(\hat{x}) = p(h_T \hat{x}) \), \( \forall \hat{x} \in \hat{T} \), it is easy to check that \( \hat{\Pi}_N \hat{p} = \Pi_N p \) by Definition 5.2 and Lemma 4.1. With Lemma 5.11 and Proposition 5.7 we have:
\[ \| \Pi_N p \|_{L^2(T)} = h_T^r \| \hat{\Pi}_N \hat{p} \|_{L^2(\hat{T})} \]
\[ \leq C h_T^r (\| \hat{\Pi}_N \hat{p} \|_{L^2(\hat{T})} + \| \hat{p} \|_{H^{1/2+\delta}(\hat{T})}) \]
\[ = C h_T^r (\| \text{div} \hat{p} \|_{L^2(\hat{T})} + \| \hat{p} \|_{H^{1/2+\delta}(\hat{T})} + \| \hat{\Pi}_N \hat{p} \|_{L^2(\hat{T})}) \]
\[ \leq C (h_T^2 \| \text{div} p \|_{L^2(T)} + h_T^{1+2\delta} \| p \|_{H^{1/2+\delta}(T)}^2 + \| \Pi_N p \|_{L^2(T)}^2), \]
where the constant \( C \) is independent of \( h_T \).

Then by Lemma 4.6, (5.11) and the inverse estimate (3.4a), we have
\[ \| \Pi_N p - I_h^\text{div} p \|_{L^2(T)}^2 \leq 2 \| \Pi_N (I_h^\text{div} p) - I_h^\text{div} p \|_{L^2(T)}^2 + 2 \| I_h^\text{div} p - p \|_{L^2(T)}^2 \]
\[ = 2 \| \Pi_N (I_h^\text{div} p - p) \|_{L^2(T)}^2 + 2 \| I_h^\text{div} p - p \|_{L^2(T)}^2 \]
\[ \leq C (h_T^2 \| \text{div} I_h^\text{div} p \|_{L^2(T)}^2 + h_T^2 \| \text{div} p \|_{L^2(T)}^2 \]
\[ + h_T^{1+2\delta} \| I_h^\text{div} p - p \|_{H^{1/2+\delta}(T)}^2 + \| I_h^\text{div} p - p \|_{L^2(T)}^2 \]
\[ \leq C (h_T^2 \| \nabla I_h^\text{div} p \|_{L^2(T)}^2 + h_T^2 \| \text{div} p \|_{L^2(T)}^2 + h_T^{1+2\delta} \| p \|_{H^{1/2+\delta}(\omega(T))}^2 \]
\[ \leq C (h_T^{1+2\delta} \| I_h^\text{div} p \|_{H^{1/2+\delta}(T)}^2 + h_T^{1+2\delta} \| p \|_{H^{1/2+\delta}(\omega(T))}^2 + h_T^2 \| \text{div} p \|_{L^2(T)}^2) \]
\[ \leq C (h_T^{1+2\delta} \| p \|_{H^{1/2+\delta}(\omega(T))}^2 + h_T^2 \| \text{div} p \|_{L^2(T)}^2). \]

Summing this result over all tetrahedra \( T \in \mathcal{T}_h \) gives (5.7).

Finally, if \( p \in H^{1/2+\delta}(\Omega) \cap H_0(\text{div}, \Omega) \), then it easily follows from (5.2c) that \( \Pi_N p \cdot n = 0 \) on \( \partial \Omega \), which implies \( \Pi_N p \in \mathcal{W}_{r-2} \). \( \square \)
Next we turn our attention to Theorem 5.3. To this end, we first state and prove an intermediate result for the operator $\Pi_L$.

**Lemma 5.12.** Under Assumption 4.4, the operator $\Pi_L : H^D(\text{curl}, \Omega) \cap H^{1/2+\delta}(\Omega) \to L_{r-1}$ defined in Definition 5.7 satisfies
\begin{equation}
\text{curl} \Pi_L \tau = \text{curl} \tau \quad \forall \tau \in H^D(\text{curl}, \Omega) \cap H^{1/2+\delta}(\Omega).
\end{equation}

Moreover, the following bound holds for any $\tau \in H^D(\text{curl}, \Omega) \cap H^{1/2+\delta}(\Omega),$
\begin{equation}
\|\Pi_L \tau\|_{L^2(\Omega)} \leq C(\|\tau\|_{L^2(\Omega)} + h^{1/2+\delta} |\tau|_{H^{1/2+\delta}(\Omega)} + h\|\text{curl} \tau\|_{L^2(\Omega)}).
\end{equation}

Finally, if $\tau \in H^D(\text{curl}, \Omega) \cap H^{1/2+\delta}(\Omega) \cap H_0(\text{curl}, \Omega)$ then $\Pi_L \tau \in L_{r-1}$.

**Proof.** We first prove (5.12). Set $\rho = \text{curl} \Pi_L \tau - \text{curl} \tau \in \mathcal{P}_{r-2}$. Then by directly using the definition of $\Pi_L$, we see that $\rho$ vanishes on the DOFs (4.6a)–(4.6b), (4.6d)–(4.6e), and
\[
\int_F (\rho \cdot n_F) \kappa \, dA = 0 \quad \forall \kappa \in \hat{\mathcal{P}}_{r-2}(F^{ct}),
\]
where we used the identity $\text{curl}_F \tau_F = \text{curl} \tau \cdot n_F$. Next, by Stokes Theorem
\[
\int_F \rho \cdot n_F \, dA = \int_F \text{curl}_F (\Pi_L \tau - \tau) \cdot n_F \, dA = \sum_{e \in \Delta_1(F)} \int_e (\Pi_L \tau - \tau) \cdot t_e \, ds = 0.
\]

Since the difference between $\mathcal{P}_{r-2}(F^{ct})$ and $\hat{\mathcal{P}}_{r-2}(F^{ct})$ is the space of the constant functions on $F$, we have
\[
\int_F (\rho \cdot n_F) \kappa \, dA = 0 \quad \forall \kappa \in \mathcal{P}_{r-2}(F^{ct}),
\]
and so $\rho$ vanishes on all the DOFs (4.6). We thus conclude $\rho \equiv 0$ by Lemma 4.1 and therefore (5.12) is satisfied.

We now prove (5.13). For any $T \in \mathcal{T}_h$, with $\hat{T}$ defined in (4.8) and $\hat{\tau}(\hat{x}) = \tau(h_T \hat{x})$, $\forall \hat{x} \in \hat{T}$, it is easy to check that $\Pi_L \tau = \Pi_L \hat{\tau}$ by Definition 5.1 and Lemma 5.1. With Lemma 5.10 and Proposition 5.7, for any $T \in \mathcal{T}_h$, we have:
\begin{equation}
\|\Pi_L \tau\|_{L^2(T)}^2 = h_T^3 \|\hat{\Pi}_L \hat{\tau}\|_{L^2(\hat{T})}^2 \leq Ch_T^3 \left( \|\text{curl} \hat{\tau}\|_{L^2(\hat{T})}^2 + \|\hat{\tau}\|_{H^{1/2+\delta}(\hat{T})}^2 \right)
\end{equation}
\begin{equation}
\quad \quad = Ch_T^3 \left( \|\hat{\text{curl}} \hat{\tau}\|_{L^2(\hat{T})}^2 + \|\hat{\tau}\|_{L^2(\hat{T})}^2 + \|\hat{\tau}\|_{H^{1/2+\delta}(\hat{T})}^2 \right)
\end{equation}
\begin{equation}
\quad \quad \quad \quad \leq C(h_T^2 \|\text{curl} \tau\|_{L^2(T)}^2 + \|\tau\|_{L^2(T)}^2 + h_T^{1+2\delta} |\tau|_{H^{1/2+\delta}(T)}^2),
\end{equation}
where the constant $C$ is independent of $h_T$. Then summing up all the tetrahedra gives the bound (5.13).

Finally, we will show that if $\tau \in H^D(\text{curl}, \Omega) \cap H^{1/2+\delta}(\Omega) \cap H_0(\text{curl}, \Omega)$, then $\Pi_L \tau \in L_{r-1}$. Since $\Pi_L \tau \in L_{r-1}$, we only need to show $\Pi_L \tau \times n = 0$ on $\partial \Omega$. Note that $\tau \times n_F = 0$ and $\text{curl} \tau \cdot n_F = 0$ on $F$ for all $F \in \Delta_2(\mathcal{T}_h)$ with $F \subset \partial \Omega$. Let $F$ be a boundary face and let $e \in \Delta_1(F)$. Then for all $\kappa \in \mathcal{P}_{r-3}(e)$, recalling $E\kappa$ in Lemma 5.9 we use integration by parts to get
\[
\int_e (\Pi_L \tau \cdot t_e) \kappa \, ds = \int_e (\tau \cdot t_e) \kappa \, ds = \int_F (\text{curl} \tau) \cdot n_F \, E\kappa \, dA + \int_F (\tau \times n_F) \cdot \text{grad} \, E\kappa \, dA = 0.
\]
Therefore, because $\Pi_L \tau$ vanishes at the vertices of $F$, $(\Pi_L \tau \cdot t_e)|_e = 0$ and hence $\text{curl}_F (\Pi_L \tau)_F \in P_{r-2}(F^e)$. By [5.1d] we get $\text{curl}_F (\Pi_L \tau)_F = 0$. Using the exactness on Clough-Tocher splits [21] (3.3e), we know $(\Pi_L \tau)_F \in \text{grad}_F S^0(F^e)$ which after applying (5.1i) shows $(\Pi_L \tau)_F = 0$ on $F$. Since $F \subset \partial \Omega$ was arbitrary, we conclude that $\Pi_L \tau \cdot n = 0$ on $\partial \Omega$ and thus $\Pi_L \tau \in \mathcal{L}^t_{r-1}$.

We finish with the proof of Theorem 5.5.

**Proof of Theorem 5.5.** The commuting property (5.4) easily follows from (5.12). Indeed,

$$\text{curl} \Pi_V \tau = \text{curl} (I_h^\text{curl} \tau) + \text{curl} (\Pi_L (\tau - I_h^\text{curl} \tau))$$

$$= \text{curl} (I_h^\text{curl} \tau) + \text{curl} (\tau - I_h^\text{curl} \tau)$$

$$= \text{curl} \tau.$$

Since $I_h^\text{curl} \tau \in \mathcal{L}^t_{r-1}$ if $\tau \in H^D(\text{curl}, \Omega) \cap H^{1/2+\delta}(\Omega) \cap H_0(\text{curl}, \Omega)$, with Lemma 5.12 we have $\Pi_V \tau \in \mathcal{L}^t_{r-1}$.

To prove (5.5) we use Lemma 4.6 and Lemma 5.12 to obtain

$$||\Pi_V \tau - \tau||_{L^2(\Omega)} \leq ||I_h^\text{curl} \tau - \tau||_{L^2(\Omega)} + ||\Pi_L (I_h^\text{curl} \tau - \tau)||_{L^2(\Omega)}$$

$$\leq C (||I_h^\text{curl} \tau - \tau||_{L^2(\Omega)} + h^{1/2+\delta} ||I_h^\text{curl} \tau - \tau||_{H^{1/2+\delta}(\Omega)})$$

$$+ h||\text{curl} (I_h^\text{curl} \tau)||_{L^2(\Omega)} + h||\text{curl} \tau||_{L^2(\Omega)}$$

(5.15)

$$\leq C (h^{1/2+\delta} ||\tau||_{H^{1/2+\delta}(\Omega)} + h||\nabla I_h^\text{curl} \tau||_{L^2(\Omega)} + h||\text{curl} \tau||_{L^2(\Omega)})$$

$$\leq C (h^{1/2+\delta} ||\tau||_{H^{1/2+\delta}(\Omega)} + h||\text{curl} \tau||_{L^2(\Omega)}),$$

where we used the inverse inequality (3.4a).

Finally, we will show that if $\tau \in H^D(\text{curl}, \Omega) \cap H^{1/2+\delta}(\Omega) \cap H_0(\text{curl}, \Omega)$, then $\Pi_V \tau \in \mathcal{L}^t_{r-1}$. Similar to the proof for $\Pi_L$, we only need to check the boundary conditions. Since $\tau \times n = 0$ on $\partial \Omega$, by Lemma 4.6 $I_h^\text{curl} \tau \times n = 0$ on $\partial \Omega$ and by Lemma 5.12 $\Pi_L \tau \times n = 0$ on $\partial \Omega$. Therefore, $\Pi_L I_h^\text{curl} \tau \times n = 0$ on $\partial \Omega$ and thus,

$$\Pi_V \tau \times n = I_h^\text{curl} \tau \times n + \Pi_L \tau \times n - \Pi_L I_h^\text{curl} \tau \times n = 0 \text{ on } \partial \Omega.$$

This gives $\Pi_V \tau \in \mathcal{L}^t_{r-1}$. \hfill $\Box$

6. Application to the Maxwell eigenvalue problem

In this section, we apply the convergence theory established in Section 2.1 and the properties of the two Fortin-like projections to show that quadratic (or higher) Lagrange finite element on Worsey-Farin meshes lead to convergent approximations of the Maxwell eigenvalue problem (2.1). First, we require the following proposition.

**Proposition 6.1.** Recall the domain $\Omega$ is contractible. Then the complex (4.2) and (4.3) are exact sequences. In particular,

$$\text{curl} \mathcal{L}^t_{r-1} = \Psi_{r-2} := \ker (\mathfrak{M}_r^{m n}(\Omega_{r-2}, \text{div})).$$

This result follows from the Bogovskii operator in [14, Theorem 4.9] and the projections $\Pi_L$, $\Pi_N$. We omit the details.

In Assumption 2.2 set $\mathcal{V}_h = \mathcal{L}^t_{r-1}$, $\mathcal{D}_h = \Psi_{r-2}$ and

$$\mathcal{V}^t(\Psi_{r-2}) := \{ \tau \in \mathcal{V}^t : \text{curl} \tau \in \Psi_{r-2} \}.$$
Then Theorems 5.5 [6] lead to the following results:

**Corollary 6.2.** The projection $\Pi_V : \mathcal{V}^t(\Psi_{r-2}) \to \mathcal{V}^{t-1}_{r-1}$ satisfies

\[
\text{curl} \Pi_V \tau = \text{curl} \tau \quad \forall \tau \in \mathcal{V}^t(\Psi_{r-2}),
\]

\[
\|\Pi_V \tau - \tau\|_{L^2(\Omega)} \leq C(h^{1/2+\delta}\|\tau\|_{H^{1/2+\delta}(\Omega)} + h\|\text{curl} \tau\|_{L^2(\Omega)}) \quad \forall \tau \in \mathcal{V}^t(\Psi_{r-2}).
\]

Furthermore, the $L^2$-orthogonal projection $\mathbb{P}_\Psi : L^2(\Omega) \to \Psi_{r-2}$ satisfies

\[
\|\mathbb{P}_\Psi p - p\|_{L^2(\Omega)} \leq Ch^{1/2+\delta}\|\text{curl} p\|_{L^2(\Omega)} \quad \forall p \in H(\text{curl},\Omega) \cap H_0(\text{div},\Omega).
\]

**Proof.** Let $\delta \in (0, \frac{1}{2}]$ be the constant in the embedding result of Proposition 3.5. The results for the operator $\Pi_V$ is then a direct consequence of Theorem 5.5 since $\mathcal{V}^t(\Psi_{r-2}) \subset H^D(\text{curl},\Omega) \cap H^{1/2+\delta}(\Omega)$.

Next, we prove the estimate for $\mathbb{P}_\Psi$. Since $H(\text{curl},\Omega) \cap H_0(\text{div},\Omega) \subset H^{1/2+\delta}(\Omega) \cap H_0(\text{div},\Omega)$, we use Theorem 5.6 to obtain $\text{div} \Pi_N p = \Pi_D \text{div} p = 0$. Thus, $\Pi_N p \in \Psi_{r-2}$. Consequently, we have the estimate for $\mathbb{P}_\Psi$:

\[
\|\mathbb{P}_\Psi p - p\|_{L^2(\Omega)} \leq \|\Pi_N p - p\|_{L^2(\Omega)} \leq C h^{1/2+\delta}\|p\|_{H^{1/2+\delta}(\Omega)}
\]

\[
\leq C h^{1/2+\delta}(\|\text{curl} p\|_{L^2(\Omega)} + \|\text{div} p\|_{L^2(\Omega)}) = C h^{1/2+\delta}\|\text{curl} p\|_{L^2(\Omega)},
\]

where we used Proposition 3.5 and Lemma 3.6. \hfill \Box

Corollary 6.2 tells us ($\mathcal{V}^{t-1}_{r-1}, \Psi_{r-2}$) satisfy Assumption 2.2 and now with Theorem 2.3 we have the final result.

**Corollary 6.3.** Let $\mathcal{V}_h = \mathcal{V}^{t}_{r-1}$ and $\Psi_h = \Psi_{r-2}$. Consider the problem $\text{(2.3)}$ with the nonzero eigenvalues $0 < \lambda^{(1)} \leq \lambda^{(2)} \leq \ldots$ and the problem $\text{(2.4)}$ with the nonzero eigenvalues $0 < \lambda^{(1)}_h \leq \lambda^{(2)}_h \leq \ldots$. Then, for any fixed $i$, \( \lim_{h \to 0} \lambda^{(i)}_h = \lambda^{(i)} \).

7. Numerical Experiments

In this section we provide numerical experiments which support our theoretical work. All the computations were carried out using Fenics [1]. We consider the domain $\Omega = (0, \pi)^3$, so that the exact eigenvectors of problem $\text{(2.1)}$ (with non-zero eigenvalues) are of the following form:

\[
\begin{pmatrix}
\cos(k_1x) \\
\sin(k_1x) \\
k_1x
\end{pmatrix}
\begin{pmatrix}
\cos(k_2y) \\
\sin(k_2y) \\
k_2y
\end{pmatrix}
\begin{pmatrix}
\cos(k_3z) \\
\sin(k_3z) \\
k_3z
\end{pmatrix}, \quad k_j \in \mathbb{N} \cup \{0\}, \quad k_1 + k_2 + k_3 \geq 2,
\]

where the coefficient $a_1, a_2, a_3$ are determined by the divergence-free equation: $a_1k_1 + a_2k_2 + a_3k_3 = 0$ and the non-zero eigenvalues of the form $\eta^2 = \lambda = k_1^2 + k_2^2 + k_3^2$. We can compute the first few eigenvalues explicitly: 2 (with multiplicity 3), 3 (with multiplicity 2), 5 (with multiplicity 6), 6 (with multiplicity 6).

We compute problem $\text{(2.2)}$ using three different choices of $\mathcal{V}_h$: (i) linear and quadratic Lagrange finite element on mesh without Worsey-Farin refinement, (ii) linear Lagrange finite element space on Worsey-Farin meshes and (iii) quadratic Lagrange finite element space on Worsey-Farin meshes.

Table 1 states the first 13 computed eigenvalues using linear and quadratic elements on a mesh without Worsey-Farin refinement with $h = \pi/8$. The numerics clearly indicate that
the discrete eigenvalues are poor approximations with $O(1)$ errors. Likewise, numerical experiments using the linear Lagrange finite element space on Worsey-Farin meshes lead to computed eigenvalues that are far from the exact solution (cf. Table 2).

Finally, we report the computed eigenvalues of method (2.2) using quadratic Lagrange elements on Worsey-Farin meshes and report the first 13 non-zero eigenvalues with $h = 1/6$ in Table 3. As expected from the theoretical results, this scenario leads to accurate approximate eigenvalues. In addition the right column in Table 3 lists the errors of the first computed (non-zero) eigenvalue and indicates converges with at least cubic rate.

| $i$ | $\lambda_h$ | $|\lambda - \lambda_h|$ |
|-----|-------------|------------------|
| 1   | 2.0610      | 6.10 $\times 10^{-2}$ |
| 2   | 2.0610      | 6.10 $\times 10^{-2}$ |
| 3   | 2.0774      | 7.74 $\times 10^{-2}$ |
| 4   | 2.0900      | 9.10 $\times 10^{-1}$ |
| 5   | 2.0900      | 9.10 $\times 10^{-1}$ |
| 6   | 2.1506      | 2.85             |
| 7   | 2.1506      | 2.85             |
| 8   | 2.2698      | 2.73             |
| 9   | 2.2698      | 2.73             |
| 10  | 2.2910      | 2.71             |
| 11  | 2.3304      | 2.67             |
| 12  | 2.3514      | 3.65             |
| 13  | 2.3514      | 3.65             |

Table 1. The case of linear (left) and quadratic (right) Lagrange finite element space on a mesh without Worsey-Farin refinement, $h = \pi/8$.

| $i$ | $\lambda_h$ | $|\lambda - \lambda_h|$ |
|-----|-------------|------------------|
| 1   | 2.6699      | 6.67 $\times 10^{-1}$ |
| 2   | 2.6766      | 6.77 $\times 10^{-1}$ |
| 3   | 2.7369      | 7.34 $\times 10^{-1}$ |
| 4   | 2.7510      | 2.49 $\times 10^{-1}$ |
| 5   | 2.7510      | 2.49 $\times 10^{-1}$ |
| 6   | 2.7615      | 2.24             |
| 7   | 2.7615      | 2.24             |
| 8   | 2.7782      | 2.22             |
| 9   | 2.7782      | 2.22             |
| 10  | 2.7941      | 2.21             |
| 11  | 2.8244      | 2.18             |
| 12  | 2.8244      | 3.17             |
| 13  | 2.8855      | 3.11             |

Table 2. The case of linear Lagrange finite element space on a Worsey-Farin mesh, $h = \pi/10$. 
Table 3. The case of $P_2$ finite element space with Worsey-Farin meshes

8. Conclusion

In this paper, we studied and justified the convergence theory of the three-dimensional Maxwell eigenvalue problem using Lagrange finite element spaces on Worsey-Farin splits. Although we only focus on Worsey-Farin splits in this paper, we provide a framework of proof which may apply to other refinements if we could fit the spaces into a de Rham complex.

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[12] P. Ciarlet, Analysis of the Scott Zhang interpolation in the fractional order Sobolev spaces, Journal of Numerical Mathematics, 21 (2013), pp. 173–180.
Before we prove Theorem 2.3 we will need a few lemmas.

**Lemma A.1.** With Assumption 2.2 there exists a positive constant $C$ such that

$$\|A\mathbb{P}_Q f - Af\|_{L^2(\Omega)} + \|T\mathbb{P}_Q f - Tf\|_{L^2(\Omega)} \leq C\omega_1(h)\|f\|_{L^2(\Omega)} \quad \forall f \in L^2(\Omega).$$

**Proof.** Let $f \in L^2(\Omega)$ and set $\sigma = Af$, $u = Tf$, $\psi = A\mathbb{P}_Q f$ and $w = T\mathbb{P}_Q f$. With $f$ and $\mathbb{P}_Q f$ in (2.5), we see that

$$(\sigma - \psi, \tau) + (u - w, \text{curl} \tau) = 0 \quad \forall \tau \in H_0(\text{curl}, \Omega),$$

$$(\text{curl} (\sigma - \psi), q) = (f - \mathbb{P}_Q f, q) \quad \forall q \in H_0(\text{div}^0, \Omega).$$

Setting $q = w - u$ and $\tau = \sigma - \psi$ in the above equations gives $\|\sigma - \psi\|_{L^2(\Omega)} = \|f - \mathbb{P}_Q f, w - u\|$, and the above equations also tell us that $\text{curl}(u - w) = \sigma - \psi$. Furthermore, $u - w \in$
$H(\text{curl}, \Omega) \cap H_0^0(\text{div}^0, \Omega)$ and there holds
\[ \|\sigma - \psi\|_{L^2(\Omega)} \leq \sup_{\phi \in H(\text{curl}, \Omega) \cap H_0^0(\text{div}^0, \Omega)} \|f - P_Q f, \phi\|_{L^2(\Omega)} \cdot \|\text{curl} \phi\|_{L^2(\Omega)}. \]

Moreover, Assumption 2.2 gives us
\[ \sup_{\phi \in H(\text{curl}, \Omega) \cap H_0^0(\text{div}^0, \Omega)} \|f - P_Q f, \phi\|_{L^2(\Omega)} = \sup_{\phi \in H(\text{curl}, \Omega) \cap H_0^0(\text{div}^0, \Omega)} \|f, \phi - P_Q \phi\|_{L^2(\Omega)} \leq \omega_1(h) \|f\|_{L^2(\Omega)}. \]

Thus, we have shown
\[ \|A P_Q f - A f\|_{L^2(\Omega)} \leq \omega_1(h) \|f\|_{L^2(\Omega)}. \]

On the other hand, since $T(P_Q f - f) \in H(\text{curl}, \Omega) \cap H_0^0(\text{div}^0, \Omega)$, we have by Lemma 3.5
\[ \|T P_Q f - T f\|_{L^2(\Omega)} \leq C \|\text{curl} (T P_Q f - T f)\|_{L^2(\Omega)} \]
\[ = C \|A P_Q f - A f\|_{L^2(\Omega)} \leq C \omega_1(h) \|f\|_{L^2(\Omega)}. \]

Since the domain $\Omega$ is contractible, then the set of harmonic forms is trivial and we have (cf. [14] Theorem 4.9):

**Lemma A.2.** Let $\Omega$ be a bounded, contractible, Lipschitz domain in $\mathbb{R}^3$. Then for all $u \in H_0^0(\text{div}^0, \Omega)$, there exists $v \in H_1^0(\Omega)$ such that $u = \text{curl} v$ and $\|v\|_{H^1(\Omega)} \leq C \|u\|_{L^2(\Omega)}$.

**Lemma A.3.** With Assumption 2.2 there exists a positive constant $C$ such that for every $p_h \in \mathcal{P}_h$, there exists $\tau_h \in \mathcal{H}_h$ such that $\text{curl} \tau_h = p_h$ and $\|\tau_h\|_{L^2(\Omega)} \leq C \|p_h\|_{L^2(\Omega)}$.

**Proof.** By Lemma A.2 for every $p_h \in \mathcal{P}_h \subset H_0^0(\text{div}^0, \Omega)$, there exists $\tau \in H_0^0(\Omega)$ such that $\text{curl} \tau = p_h$ and $\|\tau\|_{H^1(\Omega)} \leq C \|p_h\|_{L^2(\Omega)}$. Note that since $\tau \in \mathcal{V}_h(\mathcal{P}_h)$, we can set $\tau_h = \Pi_{\mathcal{V}_h} \tau$. By Assumption 2.2 we have $\text{curl} \tau_h = \text{curl} \tau = p_h$. Additionally,
\[ \|\tau_h\|_{L^2(\Omega)} \leq C (\|\tau\|_{H^{1/2+\epsilon}(\Omega)} + \|\text{curl} \tau\|_{L^2(\Omega)}) \leq C \|\tau\|_{H^1(\Omega)} \leq C \|p_h\|_{L^2(\Omega)}. \]

We are now in position to prove Theorem 2.3

**Proof of Theorem 2.3.** Let $f \in L^2(\Omega)$ and set $\sigma = A f$, $u = T f, \sigma_h = A_h f$, $u_h = T_h f$, $\psi = A P_Q f$ and $w = T P_Q f$. The first step is to estimate $\psi - \sigma_h$. We note that $\text{curl} \psi = P_Q f$ and $\text{curl} w = \psi$. From this we have that $\psi \in \mathcal{V}_h(\mathcal{P}_h)$, $\text{div} \psi = 0$ and $w \in H(\text{curl}, \Omega) \cap H_0^0(\text{div}^0, \Omega)$. Hence, by Assumption 1 we get $\text{curl} (\Pi_{\mathcal{V}_h} \psi) = \text{curl} \psi = P_Q f$. We can write the error equations
\[ (\Pi_{\mathcal{V}_h} \psi - \sigma_h, \tau_h) + (P_Q w - u_h, \text{curl} \tau_h) = (\Pi_{\mathcal{V}_h} \psi - \psi, \tau_h) \quad \forall \tau_h \in \mathcal{H}_h, \]
\[ (\text{curl} (\Pi_{\mathcal{V}_h} \psi - \sigma_h), q_h) = 0 \quad \forall q_h \in \mathcal{P}_h. \]

Setting $\tau_h = \Pi_{\mathcal{V}_h} \psi - \sigma_h$ and using the Cauchy-Schwarz inequality provides:
\[ \|\Pi_{\mathcal{V}_h} \psi - \sigma_h\|_{L^2(\Omega)} \leq \|\Pi_{\mathcal{V}_h} \psi - \psi\|_{L^2(\Omega)} \leq \omega_0(h)(\|\psi\|_{H^{1/2+\epsilon}(\Omega)} + \|\text{curl} \psi\|_{L^2(\Omega)}). \]

Next, with the embedding result in Proposition 3.3 and noting that $\text{div} \psi = 0$, we obtain
\[ \|\psi\|_{H^{1/2+\epsilon}(\Omega)} \leq C (\|\text{curl} \psi\|_{L^2(\Omega)} + \|\text{div} \psi\|_{L^2(\Omega)}) \leq C \|\text{curl} \psi\|_{L^2(\Omega)} = C \|P_Q f\|_{L^2(\Omega)} \leq C \|f\|_{L^2(\Omega)} \text{ and } \|\psi\|_{L^2(\Omega)} \leq C \|f\|_{L^2(\Omega)}. \]
Thus, we have \[ \| \Pi_r \psi - \sigma_h \|_{L^2(\Omega)} \leq C \omega(h) \| f \|_{L^2(\Omega)} \] (A.5). Lemma A.3 shows that
\[ \| \sigma - \psi \|_{L^2(\Omega)} + \| w - u \|_{L^2(\Omega)} \leq C \omega(h) \| f \|_{L^2(\Omega)}, \]
and therefore, combining (A.3), (A.4), and (A.5) we obtain
\[ \| (A - A_h)f \|_{L^2(\Omega)} = \| \sigma - \sigma_h \|_{L^2(\Omega)} \leq \| \sigma - \psi \|_{L^2(\Omega)} + \| \sigma_h - \Pi_r \psi \|_{L^2(\Omega)} + \| \Pi_r \psi - \psi \|_{L^2(\Omega)} \leq C(\omega(h) + \omega(h)) \| f \|_{L^2(\Omega)}. \]

By Lemma A.3, we know that there exists \( \tau_h \in \mathcal{V}_h \) such that \( \text{curl} \, \tau_h = \Pi_Q w - u_h \) and \( \| \tau_h \|_{L^2(\Omega)} \leq C \| \Pi_Q w - u_h \|_{L^2(\Omega)}. \) Then using (A.2a) and the Cauchy-Schwarz inequality, we have
\[ \| \Pi_Q w - u_h \|_{L^2(\Omega)}^2 = (\Pi_Q w - u_h, \text{curl} \, \tau_h) = - (\psi - \sigma_h, \tau_h) \leq C \| \Pi_Q w - u_h \|_{L^2(\Omega)} \| \psi - \sigma_h \|_{L^2(\Omega)}. \]
Furthermore, using the triangle inequality, and Assumption 1, we have
\[
\| w - u_h \|_{L^2(\Omega)} \leq \| \Pi_Q w - u_h \|_{L^2(\Omega)} + \| w - \Pi_Q w \|_{L^2(\Omega)} \leq C \| \psi - \sigma_h \|_{L^2(\Omega)} + \omega(h) \| \text{curl} \, w \|_{L^2(\Omega)} \leq C(\omega(h) + \omega(h)) \| f \|_{L^2(\Omega)}.
\]
We also have using Lemma 3.6
\[ \| \text{curl} \, w \|_{L^2(\Omega)} = \| \psi \|_{L^2(\Omega)} \leq C \| \text{curl} \, \psi \|_{L^2(\Omega)} = C \| \Pi_Q f \|_{L^2(\Omega)} \leq C \| f \|_{L^2(\Omega)}, \]
and therefore using this inequality with (A.6) and (A.5) we arrive at
\[
\| (T - T_h)f \|_{L^2(\Omega)} = \| u - u_h \|_{L^2(\Omega)} \leq \| u \|_{L^2(\Omega)} + \| w - u_h \|_{L^2(\Omega)} \leq C(\omega(h) + \omega(h)) \| f \|_{L^2(\Omega)}.
\]

□

APPENDIX B. PROOF OF LEMMA 4.6

Before proving Lemma 4.6, let us develop some notations. Let \( \{ E_1, \ldots, E_M \} \) and \( \{ v_1, \ldots, v_m \} \) be the sets of edges and (corner) vertices, respectively, of the polyhedral domain \( \Omega. \) For each \( E_j \) there exists two faces of \( \partial \Omega, \) \( f_j^1 \) and \( f_j^2 \) that share \( E_j, \) and we denote their unit-normal vectors by \( n_j^i \) for \( i = 1, 2. \) We let \( T_j^3 \) be tangent to \( E_j \) and set \( T_j^i = n_j^i \times T_j^3 \) for \( i = 1, 2. \) Note that \( T_j^i \) for \( i = 1, 2, 3 \) are linearly independent.

Let \( \mathcal{V}_h \) denote all the vertices of \( T_h^{w_f} \) that are in the interior of \( E_j. \) Recall that \( \mathcal{V}_h \) is the set of vertices of \( T_h^{w_f}. \) We decompose them in the following form
\[ \mathcal{V}_h = \mathcal{V}_h^C \cup \mathcal{V}_h^E \cup \mathcal{V}_h^0, \]
where \( \mathcal{V}_h^C = \{ v_1, \ldots, v_m \} \) are the corner points and \( \mathcal{V}_h^E = \cup_{1 \leq j \leq M} \mathcal{V}_j^{w} \) are the vertices lying on edges of \( \partial \Omega. \) Finally, \( \mathcal{V}_h^0 = \mathcal{V}_h \setminus (\mathcal{V}_h^C \cup \mathcal{V}_h^E). \)

For any \( z \in \mathcal{V}_h^{w} \) we choose \( F^j_z \in \Delta_2(T_h^{w_f}) \) such that \( z \) is a vertex of \( F^j_z \) and \( F^j_z \subset f_j^i \) for \( i = 1, 2. \) We then set \( F^3_z = F^2_z. \) We also set \( T_z^i = T_j^i \) for \( i = 1, 2, 3. \) If \( z \in \mathcal{V}_h^C \) then \( z \) is an end point of some \( E_j \) and we define \( F^1_z \) and \( T_z^i \) for \( i = 1, 2, 3 \) in the same way.
Proof of Lemma 4.6. First, we summarize the construction of the Scott-Zhang interpolant in [29]. Let \( V_h \) be the set of all the vertices of \( T_h \). For each \( z \in V_h \), let \( \phi_z \) be corresponding nodal basis of \( P_1(T_h) \), i.e., \( \phi_z \in P_1(T_h) \) satisfies \( \phi_z(y) = \delta_{yz} \) for all \( y \in V_h \). For every \( z \in V_h \), we identify an arbitrary face \( F_z \) of the mesh that contains \( z \) with the only constraint that \( F_z \) is a boundary face if \( z \) is a boundary vertex. Then there exists function \( \psi_z \in L^\infty(F_z) \) such that

\[
\int_{F_z} \psi_z \phi_y = \delta_{yz}, \quad \forall y \in V_h.
\]

Furthermore, the function \( \psi \) satisfies the estimate:

\[
\|\psi_z\|_{L^\infty(F_z)} \leq \frac{C}{|F_z|}.
\]

The Scott-Zhang interpolant \( I_h \) is given by:

\[
I_h \tau(x) = \sum_{z \in V_h} (\int_{F_z} \psi_z \tau) \phi_z(x).
\]

Construction of \( I_h^{\text{curl}} \): Similar to the construction in [7] for the two-dimensional case, we modify the Scott-Zhang interpolant \( I_h \) on edges and corner vertices of \( \Omega \) to preserve the vanishing tangential trace. We also let \( \psi^i_z \) (for \( i = 1, 2, 3 \)) satisfy:

\[
\int_{F_z^i} \psi^i_z \phi_y = \delta_{yz}, \quad \forall y \in V_h, \quad i = 1, 2, 3
\]

\[
\|\psi^i_z\|_{L^\infty(F_z^i)} \leq \frac{C}{|F_z^i|}.
\]

The modified Scott-Zhang interpolant \( I_h^{\text{curl}} \) is given as

\[
I_h^{\text{curl}} \tau(x) = \sum_{z \in V_h} (\int_{F_z} \psi_z \tau) \phi_z(x) + \sum_{z \in V_h \cup V_E} \beta^i_z(\tau) \phi_z(x)
\]

\[
\beta^i_z(\tau) := \frac{3}{3} \sum_{i=1}^3 \frac{C_i}{(T^2_z \times T^3_z) \cdot T^i_z} \int_{F_z^i} (\tau \cdot T^i_z) \psi^i_z,
\]

with \( C_1 = T^2_z \times T^3_z, C_2 = -T^1_z \times T^3_z \) and \( C_3 = T^1_z \times T^2_z \). Note that \( C_i \cdot T^i_z = \delta_{id} \).

Construction of \( I_h^{\text{div}} \): If \( z \in V_h^E \) we define \( F_z^i \) for \( i = 1, 2, 3 \) as above. Moreover, we set \( n_z^i = n^i_z \) for \( i = 1, 2 \) and \( n^3_z = T^3_z \). On the other hand, if \( z \in V_h^C \) we let \( F_z^i \in \Delta_2(T_h^{uf}) \) for \( i = 1, 2, 3 \) be such that \( F_z^i \subset \partial \Omega \) with each \( z \) being a vertex of \( F_z^i \), and such that \( F_z^i \) lies on a distinct plane. We then let \( n^i_z \) be unit normal vectors to \( F_z^i \). We then define

\[
I_h^{\text{div}} \tau(x) = \sum_{z \in V_h^0} (\int_{F_z} \psi_z \tau) \phi_z(x) + \sum_{z \in V_h^E \cup V_h^C} \beta^i_z(\tau) \phi_z(x),
\]

\[
\beta^i_z(\tau) := \frac{3}{3} \sum_{i=1}^3 \frac{D_i}{(n^1_z \times n^2_z) \cdot n^i_z} \int_{F_z^i} (\tau \cdot n^i_z) \psi^i_z,
\]

with \( D_1 = n^2_z \times n^3_z, D_2 = -n^1_z \times n^3_z \) and \( D_3 = n^1_z \times n^2_z \). Note that \( D_i \cdot n^i_z = \delta_{id} \).

Proof of estimates (4.9)–(4.10): We prove the estimates in four steps:
(ia) $I_{h}^{\text{curl}} : H^{1/2+\delta}(\Omega) \cap H_{0}(\text{curl},\Omega) \to \mathcal{P}_{1}(T_{h}) \cap H_{0}(\text{curl},\Omega)$: if $\tau \in H^{1/2+\delta}(\Omega) \cap H_{0}(\text{curl},\Omega)$, then $\tau \times n_{|\partial\Omega} = 0$, and so $(\tau \cdot T)(z) = 0$ for any tangential vector $T$ at $z \in \partial\Omega$. Therefore, for every $z \in V_{h}^{0} \cup V_{h}^{C}$, $I_{h}^{\text{curl}} \tau(z) = \beta_{1}^{T}(\tau) = 0$. On the other hand, for every $z \in V_{h}^{0} \cap \partial\Omega$, we have $I_{h}^{\text{curl}} \tau(z) \times n_{F_{z}} = \int_{F_{z}} \psi_{z}(\tau \cdot n_{F_{z}}) = 0$, where $n_{F_{z}}$ is the outward normal vector of $F_{z} \subset \partial\Omega$. These two identities yield $(I_{h}^{\text{curl}} \tau \times n)|_{\partial\Omega} = 0$.

(bia) $I_{h}^{\text{div}} : H^{1/2+\delta}(\Omega) \cap H_{0}(\text{div},\Omega) \to \mathcal{P}_{1}(T_{h}) \cap H_{0}(\text{div},\Omega)$: if $\tau \in H^{1/2+\delta}(\Omega) \cap H_{0}(\text{div},\Omega)$, then $\tau \cdot n_{|\partial\Omega} = 0$. Suppose $z \in V_{h}^{E}$, then the definition of $I_{h}^{\text{div}}$ shows $I_{h}^{\text{div}} \tau(z) \cdot n_{1}^{z} = \beta_{1}^{e}(\tau) = 0$ for $i = 1, 2$. On the other hand, if $z \in V_{h}^{C}$ then $I_{h}^{\text{div}} \tau(z) \cdot n_{1}^{z} = \beta_{2}^{e}(\tau) = 0$, and so in this case $I_{h}^{\text{div}} \tau(z) = 0$. Finally, if $z \in V_{h}^{0} \cap \partial\Omega$, we have $I_{h}^{\text{div}} \tau(z) \cdot n_{F_{z}} = \int_{F_{z}} \psi_{z}(\tau \cdot n_{F_{z}}) = 0$, where $n_{F_{z}}$ is the outward normal vector of $F_{z} \subset \partial\Omega$. We conclude that $(I_{h}^{\text{curl}} \tau \cdot n)|_{\partial\Omega} = 0$.

(ii) $I_{h}^{\text{curl}}$ and $I_{h}^{\text{div}}$ are projections. We show that if $\tau_{1} \in \mathcal{P}_{1}(T_{h}) \cap H(\text{curl},\Omega)$ and $\tau_{2} \in \mathcal{P}_{1}(T_{h}) \cap H(\text{div},\Omega)$, then $I_{h}^{\text{curl}} \tau_{1} = \tau_{2}$ and $I_{h}^{\text{div}} \tau_{2} = \tau_{2}$. Since $\tau_{1} \in \mathcal{P}_{1}(T_{h})$ ($i = 1, 2$), we can write

$$\tau_{1}(x) = \sum_{y \in V_{h}} \tau_{1}(y) \phi_{y}(x).$$

If $z \in V_{h}^{0}$, $I_{h}^{\text{curl}} \tau_{1}(z) = \int_{F_{z}} \psi_{z} \tau_{1} = \sum_{y \in V_{h}} \tau_{1}(y) \int_{F_{z}} \psi_{z} \phi_{y} = \tau_{1}(z)$ by (B.1). Similarly, $I_{h}^{\text{div}} \tau_{2}(z) = \tau_{2}(z)$. However, if $z \in V_{h}^{E} \cup V_{h}^{C}$ then $I_{h}^{\text{curl}} \tau_{1}(z) = \beta_{1}^{e} \tau_{1}$ and $I_{h}^{\text{div}} \tau_{2}(z) = \beta_{2}^{e} \tau_{2}$. We have $\beta_{i}^{e}(\tau_{1}) \cdot T_{z}^{i} = \int_{F_{z}} \psi_{z} \tau_{1} \cdot T_{z}^{i}$ for $i = 1, 2, 3$. Recalling these three tangential vectors are linearly independent, we conclude $I_{h}^{\text{curl}} \tau_{1}(z) = \tau_{1}(z)$.

Similarly, since $\beta_{i}^{e}(\tau_{2}) \cdot n_{1}^{z} = \int_{F_{z}} \psi_{z} \tau_{2} \cdot n_{1}^{z} = \tau_{2}(z) \cdot n_{1}^{z}$, $i = 1, 2, 3$, we have $I_{h}^{\text{div}} \tau_{2}(z) = \tau_{2}(z)$.

(iii) Stability estimate. By an inverse estimate (3.4) we have

$$|I_{h}^{\text{curl}} \tau|_{H^{1/2+\delta}(T)} \leq Ch_{T}^{-1/2-\delta} \|I_{h}^{\text{curl}} \tau\|_{L^{2}(T)}.$$  

We will use the following trace inequality (see [12, Proposition 3.1]; also follows from (3.2), (3.4) and a scaling argument),

$$\|\tau\|_{L^{1}(F_{z})} \leq C(h_{T}^{1/2} \|\tau\|_{L^{2}(T)} + h_{T}^{1+\delta} \|\tau\|_{H^{1/2+\delta}(T)}), \quad T \in T_{h}, F_{z} \in \Delta_{2}(T).$$

Since the number of edges and vertices of $\partial\Omega$ is finite we have $M_{t} := \max_{z \in V_{h}^{E} \cup V_{h}^{C}} \frac{1}{(T_{z}^{1} \cdots T_{z}^{3})^{3/2}}$ is finite. Using the $L^{\infty}$ estimates of $\psi_{z}$, (B.2), (B.5), (B.8) and the estimate $\|\phi_{z}\|_{L^{2}(T)} \leq C h_{T}^{3/2}$ for $z \in T$, we have

$$\|I_{h}^{\text{curl}} \tau\|_{L^{2}(T)} \leq \sum_{z \in V_{h}^{0}} \|\phi_{z}\|_{L^{2}(T)} \|\psi_{z}\|_{L^{\infty}(F_{z})} \|\tau\|_{L^{1}(F_{z})}$$

$$+ M_{t} \sum_{z \in V_{h}^{E} \cup V_{h}^{C}} \|\phi_{z}\|_{L^{2}(T)} \sum_{i=1}^{3} \|\psi_{z}^{(i)}\|_{L^{\infty}(F_{z})} \|\tau\|_{L^{1}(F_{z})}$$

$$\leq C(1 + M_{t}) \left(\|\tau\|_{L^{2}(\omega(T))} + h_{T}^{1/2+\delta} \|\tau\|_{H^{1/2+\delta}(\omega(T))}\right).$$
Therefore, we conclude
\[(B.9)\]
\[h_T^{1/2+\delta}|I_h^{\text{curl}} \mathbf{\tau}|_{H^{1/2+\delta}(T)} + \|I_h^{\text{curl}} \mathbf{\tau}\|_{L^2(T)} \leq C(1 + M_t)(\|\mathbf{\tau}\|_{L^2(\omega(T))} + h_T^{1/2+\delta}|\mathbf{\tau}|_{H^{1/2+\delta}(\omega(T))}).\]

On the other hand, by following the same process, we obtain
\[(B.10)\]
\[h_T^{1/2+\delta}|I_h^{\text{div}} \mathbf{\tau}|_{H^{1/2+\delta}(T)} + \|I_h^{\text{div}} \mathbf{\tau}\|_{L^2(T)} \leq C(1 + M_n)(\|\mathbf{\tau}\|_{L^2(\omega(T))} + h_T^{1/2+\delta}|\mathbf{\tau}|_{H^{1/2+\delta}(\omega(T))}),\]

where \(M_n := \max \frac{1}{\mathcal{V}_h^C} \bigcup_{\mathcal{V}_h^C} (n_1 \times n_2 \times n_3).\)

(iv) Estimate (4.9): Let \(\mathbf{\omega} = 1_{\omega(T)} \int_{\omega(T)} \mathbf{\tau} \, dx,\) so that by the Poincare inequality (cf. [29, Section 4] and [15, Proposition 2.1])
\[(B.11)\]
\[\|\mathbf{\tau} - \mathbf{\omega}\|_{L^2(\omega(T))} \leq C h_T^{1/2+\delta}|\mathbf{\tau}|_{H^{1/2+\delta}(\omega(T))}.\]

Since \(\mathbf{\omega}\) is a constant, there holds \(I_h^{\text{curl}} \mathbf{\omega}|_T = \mathbf{\omega}.\) Using the estimate (B.11) and the stability result (B.9), we obtain
\[\|I_h^{\text{curl}} \mathbf{\tau} - \mathbf{\omega}\|_{L^2(T)} = \|I_h^{\text{curl}}(\mathbf{\tau} - \mathbf{\omega}) - (\mathbf{\tau} - \mathbf{\omega})\|_{L^2(T)} \leq C(1 + M_t)(\|\mathbf{\tau} - \mathbf{\omega}\|_{L^2(\omega(T))} + h_T^{1/2+\delta}|\mathbf{\tau}|_{H^{1/2+\delta}(\omega(T))}) \leq C(1 + M_t) h_T^{1/2+\delta}|\mathbf{\tau}|_{H^{1/2+\delta}(\omega(T))}.

Similarly, we have
\[|I_h^{\text{curl}} \mathbf{\tau}|_{H^{1/2+\delta}(T)} = |I_h^{\text{curl}}(\mathbf{\tau} - \mathbf{\omega})|_{H^{1/2+\delta}(T)} \leq C(1 + M_t)(h_T^{-1/2-\delta}|\mathbf{\tau} - \mathbf{\omega}|_{L^2(\omega(T))} + |\mathbf{\tau}|_{H^{1/2+\delta}(\omega(T))}) \leq C(1 + M_t)|\mathbf{\tau}|_{H^{1/2+\delta}(\omega(T))}.

These last two estimates complete the proof of (4.9). By the same process, (but replacing the stability estimate (B.9) with (B.10)) we obtain the estimate for \(I_h^{\text{div}}\) (4.10).

\[\square\]

**APPENDIX C. SCALING PROPERTIES**

In this section, we need to prove Proposition 5.7, Lemma 5.8 and Lemma 5.9.

**C.1. Proof of Proposition 5.7.**

**Proof.** The proof of (i) follows from the definition of \(\hat{T}\) and Definition 4.1. The identities given in (ii) follow from the chain rule, and the scaling result in (iii) is a direct application of [22, Lemma 2.9].

To prove those two lemmas, we transform \(\hat{T}\) to the standard reference tetrahedron \(\tilde{T}\) with unit size.

**Definition C.1.** Let \(\tilde{T}\) be the tetrahedron with vertices \((0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1)\). For any tetrahedra \(\hat{T}\) set \(\phi_{\hat{T}} : \hat{T} \to \tilde{T}\) to be an affine diffeomorphism with
\[\phi_{\hat{T}}(\tilde{x}) = B_{\hat{T}} \tilde{x} + b_{\hat{T}}, \quad \tilde{x} \in \tilde{T}

for some \(B_{\hat{T}} \in \mathbb{R}^{3 \times 3}\) and \(b_{\hat{T}} \in \mathbb{R}^3\).
Remark C.2. There holds (cf. [22, Page 80 and Lemma 5.10])

\[ |\det(B_T)| = \frac{|\tilde{T}|}{|T|} = 6|\tilde{T}|, \quad \|B_T\| \leq h_T = 1, \quad \|B_T^{-1}\| \leq \rho_T^{-1} \leq c_0. \]

C.2. Proof of Lemma 5.8. We require an intermediate result to prove Lemma 5.8.

Proposition C.3. (Equivalence norms of Sobolev space)

(i) \( \forall \tilde{v} \in H^s(\tilde{T}) \) with \( 0 < s < 1 \), we have

\[
C_1|\tilde{v}|_{H^s(\tilde{T})} \leq |\tilde{v} \circ \phi_T|_{H^s(\tilde{T})} \leq C_2|\tilde{v}|_{H^s(\tilde{T})},
\]

where \( C_1 \) and \( C_2 \) depends on \( c_0 \) (the shape regularity of \( T_h \)) and \( s \).

(ii) For all \( \hat{\kappa} \in W^{s,q}(\hat{e}) \) with \( sq < 1 \) and \( 0 < s \leq 1/2 \), we have

\[
C_3\|\hat{\kappa} \circ \phi_T\|_{W^{s,q}(\hat{e})} \leq \|\hat{\kappa}\|_{W^{s,q}(\hat{e})},
\]

where \( \hat{e} = \phi_T^{-1}(\hat{e}) \), and \( C_3 \) depends on \( c_0 \), \( s \) and \( q \).

(iii) For all \( \tilde{v} \in W^{1,p}(\tilde{T}) \), we have

\[
\|\tilde{\kappa}\|_{W^{1,p}(\tilde{T})} \leq C_4\|\kappa \circ \phi_T\|_{W^{1,p}(\tilde{T})}.
\]

Moreover, for any \( \hat{F} \in \Delta_2(\hat{T}) \),

\[
\|\hat{\kappa}\|_{W^{1,p}(\hat{F})} \leq C_5\|\hat{\kappa} \circ \phi_T\|_{W^{1,p}(\hat{F})} \quad \forall \hat{\kappa} \in W^{1,p}(\hat{F}),
\]

where \( \hat{F} = \phi_T^{-1}(\hat{F}) \), and \( C_4 \) and \( C_5 \) depends on \( c_0 \) and \( p \).

Proof. (i) This estimate follows from [22, Lemma 2.9] and Remark C.2.

(ii) Let \( \tilde{v} \in W^{s,q}(\tilde{e}) \) and to ease notation, set \( \tilde{\kappa} = \tilde{v} \circ \phi_T \). Recalling the definition of \( W^{s,q}(\tilde{e}) \) and applying a change of variables, we have

\[
|\tilde{\kappa}|^q_{W^{s,q}(\tilde{e})} = \int_{\tilde{e}} \int_{\tilde{e}} \left| \tilde{\kappa}(\tilde{x}) - \tilde{\kappa}(\tilde{y}) \right|^q d\tilde{x} d\tilde{y} = \left| \frac{\tilde{e}}{e} \right|^2 \int_{\tilde{e}} \int_{\tilde{e}} \left| \tilde{\kappa}(\tilde{x}) - \tilde{\kappa}(\tilde{y}) \right|^q d\tilde{x} d\tilde{y} \\
\geq C\|\tilde{\kappa}\|^q_{W^{s,q}(\tilde{e})}.
\]

Because

\[
\|\tilde{\kappa}\|^q_{L^q(\tilde{e})} = \left| \frac{\tilde{e}}{e} \right|^q \|\tilde{\kappa}\|^q_{L^q(\tilde{e})} \geq C\|\tilde{\kappa}\|^q_{L^q(\tilde{e})},
\]

we conclude \( \|\tilde{\kappa}\|_{W^{s,q}(\tilde{e})} \geq C_3\|\tilde{\kappa}\|_{W^{s,q}(\tilde{e})} \).

(iii) The proof of (C.2) and (C.3) follows the same arguments as (ii); we omit the details.

□

Now we are ready to prove Lemma 5.8.

Proof of Lemma 5.8. Let \( \tilde{v} \in H^{1/2+\delta}(\tilde{T}) \) and set \( \tilde{v} = \tilde{v} \circ \phi_T \). Then \( \tilde{v} \in H^{1/2+\delta}(\tilde{T}) \) by Proposition C.3. Setting \( s = \delta - 1 + 3/p \) then we see that since \( p < \frac{2}{1+\delta} \) we have \( s > 1/p \). Hence by the trace inequality (3.2) we have

\[
\|\tilde{v}\|_{L^p(\partial\tilde{T})} \leq \|\tilde{v}\|_{W^{s-1/p,p}(\partial\tilde{T})} \leq C\|\tilde{v}\|_{W^{s,p}(\tilde{T})}.
\]

By our choice we have \( s - 3/p = 1/2 + \delta - 3/2 \) and hence by the Sobolev inequality (3.1)\n
\[
\|\tilde{v}\|_{W^{s,p}(\tilde{T})} \leq C\|\tilde{v}\|_{H^{1/2+\delta}(\tilde{T})}.
\]
A change of variables along with (C.1) then shows
\[ \| \hat{\nu} \|_{L^p(\partial \hat{T})} \leq C \| \hat{\nu} \|_{L^p(\partial \hat{T})} \leq C \| \hat{\nu} \|_{H^{1/2+\delta}(\hat{T})} \leq C \| \hat{\nu} \|_{H^{1/2+\delta}(\hat{T})}, \]

\[ \tag*{\Box} \]

C.3. Proof of Lemma 5.9. We start with the same result for the reference element \( \hat{T} \).

Lemma C.4. Let \( \hat{e} \in \Delta_1(\hat{T}) \) and \( \hat{F} \in \Delta_2(\hat{T}) \) be an edge and face of the reference tetrahedron, respectively, such that \( \hat{e} \in \Delta_1(\hat{F}) \). Then for \( 1 < q < 2 \), there exists \( \hat{E} : \mathcal{P}_{r-3}(\hat{e}) \to W^{1,q}(\hat{T}) \) such that \( (\hat{E} \hat{\kappa})|_{\hat{e}} = \hat{\kappa} \), \( (\hat{E} \hat{\kappa})|_{\partial F \setminus \hat{e}} = 0 \), \( (\hat{E} \hat{\kappa})|_{\partial T \setminus \hat{e}} = 0 \) for all \( \hat{\kappa} \in \mathcal{P}_{r-3}(\hat{e}) \), and the following estimates hold:
\[ \| \hat{E} \hat{\kappa} \|_{W^{1,q}(\hat{F})} \leq C \| \hat{\kappa} \|_{W^{1-1/q,q}(\hat{e})}, \]
\[ \| \hat{E} \hat{\kappa} \|_{W^{1,q}(\hat{T})} \leq C \| \hat{\kappa} \|_{W^{1-1/q,q}(\hat{e})}. \]

Proof. We first extend \( \hat{\kappa} \in \mathcal{P}_{r-3}(\hat{e}) \subset W^{1-1/q,q}(\hat{e}) \) by zero to \( \partial \hat{F} \), denoted by \( \hat{\kappa}_1 \). With \( q < 2 \) and the definition of \( W^{1-1/q,q}(\hat{e}) \), we have
\[ \| \hat{\kappa}_1 \|_{W^{1-1/q,q}(\partial \hat{F})} = \| \hat{\kappa}_1 \|_{W^{1-1/q,q}(\partial \hat{e})} + 2 \int_{\hat{e}} \int_{\hat{F} \setminus \hat{e}} \frac{|\hat{\kappa}(\hat{x})|}{|\hat{x} - \hat{y}|^q} \hat{y} d\hat{x} \]
\[ \leq \| \hat{\kappa}_1 \|_{W^{1-1/q,q}(\partial \hat{e})} + 2 \| \hat{\kappa}_1 \|_{L^{\infty}(\hat{e})} \int_{\hat{e}} \int_{\hat{F} \setminus \hat{e}} \frac{1}{|\hat{x} - \hat{y}|^q} d\hat{y} d\hat{x} \]
\[ \leq \| \hat{\kappa}_1 \|_{W^{1-1/q,q}(\partial \hat{e})} + C \| \hat{\kappa}_1 \|_{L^{\infty}(\hat{e})} \leq C \| \hat{\kappa}_1 \|_{W^{1-1/q,q}(\hat{e})}, \]
where we used that \( \int_{\hat{e}} \int_{\hat{F} \setminus \hat{e}} \frac{1}{|\hat{x} - \hat{y}|^q} d\hat{y} d\hat{x} \) is finite for \( q < 2 \) and (3.4a).

We extend \( \hat{\kappa}_1 \) to \( \hat{F} \) using (3.3) and denote the extension by \( \hat{\kappa}_2 \in W^{1,q}(\hat{F}) \) with the estimate:
\[ \| \hat{\kappa}_2 \|_{W^{1,q}(\hat{F})} \leq C \| \hat{\kappa}_1 \|_{W^{1-1/q,q}(\partial \hat{F})} \leq C \| \hat{\kappa}_2 \|_{W^{1-1/q,q}(\hat{e})}. \]
Similarly, we extend \( \hat{\kappa}_2 \in W^{1,q}(\hat{F}) \) by zero to \( \partial \hat{T} \), which we denote by \( \hat{\kappa}_3 \). Set \( s = 2/(2-q) \) then we see that using a Sobolev inequality \( \| \hat{\kappa}_2 \|_{L^{q^s}(\hat{F})} \leq C \| \hat{\kappa}_2 \|_{W^{1,q}(\hat{F})} \). Using definition of \( W^{1-1/q,q}(\partial \hat{T}) \) and Hölder’s inequality, we have:
\[ \| \hat{\kappa}_2 \|_{W^{1-1/q,q}(\partial \hat{T})} = \| \hat{\kappa}_2 \|_{W^{1-1/q,q}(\hat{F})} + \int_{\hat{F}} \int_{\hat{\partial T} \setminus \hat{F}} \frac{|\hat{\kappa}_2(\hat{x})|}{|\hat{x} - \hat{y}|^{q+1}} dA(y) dA(x) \]
\[ \leq \| \hat{\kappa}_2 \|_{W^{1-1/q,q}(\hat{F})} + \| \hat{\kappa}_2 \|_{L^{q^s}(\hat{F})} \left( \int_{\hat{F}} \int_{\hat{\partial T} \setminus \hat{F}} \frac{1}{|\hat{x} - \hat{y}|^{(q+1)s'}} dA(y) dA(x) \right)^{1/s'} \]
\[ \leq \| \hat{\kappa}_2 \|_{W^{1-1/q,q}(\hat{F})} + C \| \hat{\kappa}_2 \|_{L^{q^s}(\hat{F})} \leq C \| \hat{\kappa}_2 \|_{W^{1,q}(\hat{F})}, \]
where we used \( s' = 2/q \) which implies \( (q+1)s' = (q+1)\frac{2}{q} < 3 < 4 \) and hence the double integral is finite.

Again we use (3.3) to lift \( \hat{\kappa}_3 \) to \( \hat{T} \) where we denote the lifting by \( \hat{E} \hat{\kappa} \in W^{1,q}(\hat{T}) \) and it has the estimate
\[ \| \hat{E} \hat{\kappa} \|_{W^{1,q}(\hat{T})} \leq C \| \hat{\kappa}_3 \|_{W^{1-1/q,q}(\partial \hat{T})} \leq C \| \hat{\kappa}_2 \|_{W^{1,q}(\hat{F})} \leq C \| \hat{\kappa}_2 \|_{W^{1-1/q,q}(\hat{e})}. \]

Furthermore, we have
\[ \| \hat{E} \hat{\kappa} \|_{W^{1,q}(\hat{T})} = \| \hat{\kappa}_2 \|_{W^{1,q}(\hat{F})} \leq C \| \hat{\kappa}_2 \|_{W^{1-1/q,q}(\hat{e})}. \]
\[ \tag*{\Box} \]
Using Lemma C.4 and Proposition C.3, we now prove Lemma 5.9.

**Proof of Lemma 5.9.** Let $\tilde{\kappa} \in P_{r-3}(\hat{e})$, and let $\hat{\kappa} \in P_{r-3}(\hat{e})$ (with $\hat{e} = \phi^{-1}_T(\hat{e})$) be given as $\tilde{\kappa} = \hat{\kappa} \circ \phi_T$. By (ii) of Proposition C.3 there holds
\[
\|\hat{\kappa}\|_{W^{1,1/q,q}(\hat{e})} \geq C_3 \|\tilde{\kappa}\|_{W^{1,1/q,q}(\hat{e})}.
\]
By Lemma C.4 there exists $\tilde{E} : P_{r-3}(\hat{e}) \to W^{1,q}(\hat{T})$ such that $(\tilde{E}\hat{\kappa})|_{\hat{e}} = \hat{\kappa}$ and $(\tilde{E}\hat{\kappa})|_{\partial\hat{T}\setminus\hat{e}} = 0$, $(\tilde{E}\hat{\kappa})|_{\partial\hat{T}\setminus\hat{e}} = 0$. Let $E : P_{r-3}(e) \to W^{1,q}(\hat{T})$ be defined by $(E\tilde{\kappa}) = (\tilde{E}\hat{\kappa}) \circ \phi_T$. Then $E\tilde{\kappa}|_{\hat{e}} = \hat{\kappa}$ $(E\tilde{\kappa})|_{\partial\hat{T}\setminus\hat{e}} = 0$, $(E\tilde{\kappa})|_{\partial\hat{T}\setminus\hat{e}} = 0$ and with (iii) of Proposition C.3, we have
\[
\|E\tilde{\kappa}\|_{W^{1,q}(\hat{T})} \leq C\|\tilde{E}\hat{\kappa}\|_{W^{1,q}(\hat{T})} \leq C\|\hat{\kappa}\|_{W^{1,1/q,q}(\hat{e})} \leq C\|\tilde{\kappa}\|_{W^{1,1/q,q}(\hat{e})},
\]
and
\[
\|E\tilde{\kappa}\|_{W^{1,q}(\hat{T})} \leq C\|\tilde{E}\hat{\kappa}\|_{W^{1,q}(\hat{T})} \leq C\|\hat{\kappa}\|_{W^{1,1/q,q}(\hat{T})} \leq C\|\tilde{\kappa}\|_{W^{1,1/q,q}(\hat{T})}.
\]
\[\square\]