Characteristic polynomials, $\eta$-complexes and freeness of tame arrangements

Takuro Abe

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Abstract
We compare each coefficient of the reduced characteristic polynomial of a simple arrangement and that of its Ziegler restriction. As a consequence we can show that the former is not less than the latter in the category of tame arrangements. This is a generalization of Yoshinaga’s freeness criterion for 3-arrangements and also the recent result by the author and Yoshinaga. As a corollary, we can prove that a free arrangement is a minimal chamber arrangement, and we can give a freeness criterion in terms of chambers in the category of tame arrangements.

1 Introduction
Let $A$ be a central $\ell$-arrangement over an arbitrary field $\mathbb{K}$. Fix $H_0 \in A$ and $(A'', m)$ the Ziegler restriction of $A$ onto $H_0$. Let $dA$ be the deconing of $A$ with respect to $H_0$. For details of a notation in this section, see the next section.

Let us put a reduced characteristic polynomial of $A$, which is combinatorial, as follows:

$$\chi_0(A, t) = \chi(dA, t) = \sum_{i=0}^{\ell-1} (-1)^{\ell-1-i}b_{\ell-1-i}t^i.$$ 

Also, let us put a characteristic polynomial of $(A'', m)$, which is algebraic, as follows:

$$\chi(A'', m, t) = \sum_{i=0}^{\ell-1} (-1)^{\ell-1-i}\sigma_{\ell-1-i}t^i.$$
It is known that $b_0 = \sigma_0 = 1$ and $b_1 = \sigma_1 = |\mathcal{A}| - 1 = |m|$. Also, it is proved in \cite{3} that $b_2 \geq \sigma_2$. Moreover, in \cite{3}, the equality of $b_2$ and $\sigma_2$ is closely related to the freeness of $\mathcal{A}$. This is a generalization of Yoshinaga’s freeness criterion for 3-arrangements in \cite{13}. After introducing a characteristic polynomial of multiarrangements in \cite{2}, Yoshinaga’s criterion can be also understood in terms of the comparison of coefficients of characteristic polynomials, or minimality of chambers. Then a natural question is, what about $b_i$ and $\sigma_i$ for $i \geq 3$? The special case of this question is the relation between free arrangements and minimal chamber arrangements introduced in \cite{1}. To these problems, we can give an answer in the category of tame arrangements as follows:

**Theorem 1.1**

Let $\mathcal{A}$ be a central $\ell$-arrangement. Fix $H_0 \in \mathcal{A}$ and let $(\mathcal{A}'', m)$ be the Ziegler restriction of $\mathcal{A}$ with respect to $H_0$. If $\mathcal{A}$ and $(\mathcal{A}'', m)$ are both tame, then $b_i \geq \sigma_i \geq 0$ ($i = 0, 1, \ldots, \ell - 1$) in the notation above.

Hence in the category of tame arrangements, we say that $\mathcal{A}$ is a **minimal chamber arrangement** (MCA for short) if $(-1)^{\ell-1}\chi_0(\mathcal{A}, -1) = (-1)^{\ell-1}\chi(\mathcal{A}'', m, -1)$. When $\ell = 3$, by Yoshinaga’s criterion in \cite{13}, we can define MCA, and it holds that free arrangements and MCA are equivalent. As a corollary of Theorem 1.1, we can generalize this criterion and the relation between MCA and free arrangements as follows:

**Corollary 1.2**

With the notation in Theorem 1.1, assume again that $\mathcal{A}$ and $(\mathcal{A}'', m)$ are tame. Then it holds that $(-1)^{\ell-1}\chi_0(\mathcal{A}, -1) \geq (-1)^{\ell-1}\chi(\mathcal{A}'', m, -1) \geq 0$. Moreover, $\mathcal{A}$ is free if and only if $(\mathcal{A}'', m)$ is free and $\chi_0(\mathcal{A}, -1) = \chi(\mathcal{A}'', m, -1)$.

Corollary 1.2 is also a generalization of Yoshinaga’s criterion. In other words, if we fix a free multiarrangement $(\mathcal{A}'', m)$ and consider a family of arrangements $\mathcal{A}$ the Ziegler restriction of which are all $(\mathcal{A}'', m)$, then the freeness in this family is nothing but MCA in the tame category. Also, in the same category, a characteristic polynomial of the Ziegler restriction gives a lower bound of the value $(-1)^{\ell-1}\chi_0(\mathcal{A}, -1)$, or equivalently the cardinality of chambers over the real number field.

The main tool for the proofs is the multi-version of the $\eta$-complex, originally introduced in \cite{11}, developed in \cite{8} and \cite{12} for simple arrangements. In the proof, we also investigate several properties of this complex.

The organization of this article is as follows. In section two we introduce several definitions and results used in the rest of this article. In section three
we develop several results for the proof. Mainly, we study several variants of the $\eta$-complexes. In section four we prove Theorem 1.1 and Corollary 1.2.

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2 Preliminaries

For the rest of this article everything is considered over an arbitrary field $K$ and $V = K^\ell$. For a general reference, see [7].

Let $A$ be an affine arrangement, i.e., a finite set of affine hyperplanes in $V$. An arrangement is called to be an $\ell$-arrangement if it is in $K^\ell$. The intersection lattice $L(A)$ is a set of subspaces of the form $\cap_{H \in B} H$ with $B \subseteq A$. $L(A)$ is a poset with the reverse inclusion order and the unique minimum element $V$. Define $L_i(A) = \{ X \in L(A) \mid \operatorname{codim}_V X = i \}$. The Möbius function $\mu : L(A) \to \mathbb{Z}$ is defined by, $\mu(V) = 1$, and by $\mu(X) = -\sum_{V \supseteq Y \supseteq X} \mu(Y)$ ($X \neq V$). Then a characteristic polynomial $\chi(A, t)$ is defined as follows:

$$\chi(A, t) = \sum_{X \in L(A)} \mu(X) t^{\dim X}.$$ 

$A$ is called to be central if $0 \in H$ ($\forall H \in A$). Let $\alpha_{H} \in V^*$ be the defining form of $H \in A$. If $A$ is central, then $\chi(A, t)$ has $(t - 1)$ as a divisor. So define a reduced characteristic polynomial $\chi_0(A, t)$ by

$$\chi_0(A, t) := \frac{\chi(A, t)}{(t - 1)}.$$ 

Let $A$ be a central $\ell$-arrangement. $A$ is called essential if $\cap_{H \in A} H = \{0\}$. When $A$ is a direct product of an essential arrangement $B$ and an empty arrangement $\Phi$ (i.e., $A \cong B \times \Phi$), then $B$ is called an essentialization of $A$.

Now let us fix $H_0 \in A$. Then the deconing $dA$ of $A$ is defined as $A \cap \{ \alpha_{H_0} = 1 \}$, which is an $(\ell - 1)$-affine arrangement. Note that $\chi_0(A, t) = \chi(dA, t)$.

When the base field is $\mathbb{R}$, the set of connected components of $V \setminus \cup_{H \in A} H$ is said to be chambers, and denoted by $C(A)$.

Remark 2.1

It is well-known that $\pi(A, t) := (-t)^\ell \chi(A, -t^{-1})$ is equal to the topological Poincaré polynomial of $V \setminus \cup_{H \in A} H$ when the base field is $\mathbb{C}$. Also, when the base field is $\mathbb{R}$, $(-1)^\ell \chi(A, -1)$ is the number of chambers of the complement of hyperplanes, and $|\chi(A, 1)|$ the number of bounded chambers of that. Also, $|C(dA)| = (-1)^{\ell-1} |\chi_0(A, -1)|$.
For the rest of this article we assume that $A$ is a central $\ell$-arrangement. Let $S := \text{Sym}^\ast(V^\ast) = \mathbb{K}[x_1, \ldots, x_\ell]$ be a coordinate ring of $V$. For the module of $S$-derivations $\text{Der}S$, a module of logarithmic vector fields of $A$ is defined by

$$D(A) := \{\theta \in \text{Der}S \mid \alpha_H | \theta(\alpha_H) \ (\forall H \in A)\}.$$ 

In general $D(A)$ is a reflexive module. When $D(A)$ is a free $S$-module with homogeneous basis $\theta_1, \ldots, \theta_\ell$ of degrees $d_1, \ldots, d_\ell$, we say that $A$ is free with exponents $\exp(A) = (d_1, \ldots, d_\ell)$.

A multiplicity is a map $m: A \to \mathbb{Z}_{\geq 0}$ and a pair $(A, m)$ is a multiarrangement. A module of logarithmic vector fields of $(A, m)$ is defined by

$$D(A, m) := \{\theta \in \text{Der}S \mid \alpha^m_H | \theta(\alpha_H) \ (\forall H \in A)\}.$$ 

The freeness and exponents of a multiarrangement can be defined in the same manner. For $X \in L(A)$, let $(A_X, m_X)$ denote the localization of $(A, m)$ defined by

$$A_X := \{H \in A \mid X \subset H\},$$

$$m_X := m|_{A_X}.$$ 

Multiarrangements appear naturally when we consider the restriction operation of a central arrangement. For a central arrangement $A$ and $H_0 \in A$, the Ziegler restriction $(A'', m)$ with respect to $H_0$ is defined by

$$A'' := \{H \cap H_0 \mid H \in A \setminus \{H_0\}\},$$

$$m(H \cap H_0) := |\{K \in A \setminus \{H_0\} \mid K \cap H_0 = H \cap H_0\}|.$$ 

For the set of regular $p$-forms $\Omega^p_V$, a module of logarithmic differential $p$-forms of $(A, m)$ is defined as follows:

$$\Omega^p(A, m) := \{\omega \in Q(A, m) \Omega^p_V \mid (Q(A, m)/\alpha^m_H)\text{d}\alpha_H \wedge \omega \in \Omega^{p+1}_V \ (\forall H \in A)\},$$

where $Q(A, m) := \prod_{H \in A} \alpha^m_H$. See [13] for details of multiarrangements. By using this algebraic object, following [2], we can define a characteristic polynomial of a multiarrangement as follows:

$$\chi(A, m, t) := \lim_{x \to 1} \sum_{p=0}^t t \ Poin(\Omega^p(A, m), x)(t(1 - x) - 1)^p,$$

where $Poin(M, x) := \sum_{k \in \mathbb{Z}} \dim_k M_k x^k$ is a Poincaré series of the $S$-graded module $M = \bigoplus_{k \in \mathbb{Z}} M_k$. 


Remark 2.2
Precisely, the definition of $\chi(A, m, t)$ in the above is different from the original one in [2]. In other words, the original definition was

$$
\chi(A, m, t) := (-1)^{\ell} \lim_{x \to 1} \sum_{p=0}^{\ell} \text{Poin}(D^p(A, m), x)(t(x - 1) - 1)^p,
$$

where $D^p(A, m)$ is a dual module of $\Omega^p(A, m)$. The equality of these two definitions was proved in Remark 2.3 of [3]. So we use the definition by differential forms in this article.

Related to these characteristic polynomials, the following local-to-global formula is useful to compute each coefficient.

Theorem 2.3 ([2], Theorem 3.3)
Put

$$
\chi(A, m, t) = \sum_{i=0}^{\ell} (-1)^{\ell-i} \sigma_{\ell-i} t^i,
$$

$$
\chi(A_X, m_X, t) = t^{\ell-k} \sum_{i=0}^{k} (-1)^{k-i} \sigma_{k-i} X^i \ (X \in L_k(A)).
$$

Then $\sigma_k = \sum_{X \in L_k(A)} \sigma_k^X$.

For a fixed $(A'', m)$ where $A''$ is a central $(\ell - 1)$-arrangement, define $F(A'', m)$ to be the set of central $\ell$-arrangements the Ziegler restriction of which are all $(A'', m)$. When it holds that

$$
(-1)^{\ell-1} \chi_0(A, -1) \geq (-1)^{\ell-1} \chi(A'', m, -1) \geq 0
$$

for all $A \in F(A'', m)$, we say that $A \in F(A'', m)$ is a minimal chamber arrangement (MCA for short) if

$$
(-1)^{\ell-1} \chi_0(A, -1) = \min_{B \in F(A'', m)} (-1)^{\ell-1} \chi_0(B, -1) = (-1)^{\ell-1} \chi(A'', m, -1).
$$

We say that a multiarrangement $(A, m)$ is tame if for a projective dimension $\text{pd}_S \Omega^p(A, m)$ of the $S$-module $\Omega^p(A, m)$, it holds that

$$
\text{pd}_S \Omega^p(A, m) \leq p \ (p = 0, 1, \ldots, \ell).
$$

For example, generic arrangements and free arrangements are tame, see [9] for details. Tame arrangements were introduced, first in [8] without names,
and named in [12]. Recently, tame arrangements play important roles in several research areas of arrangements, see [4], [5] and [10] for example.

For $D(A, m) \ni \theta$ and $\Omega^p(A, m) \ni \omega = \sum g_{ij...ip}dx_i \wedge \cdots \wedge dx_p$, define a contraction

$$\langle \theta, \omega \rangle := \sum (-1)^{j-1} \theta(x_{ij}) g_{ij...ip} dx_i \wedge \cdots \wedge dx_{ij-1} \wedge dx_{ij+1} \wedge \cdots \wedge dx_p.$$ 

If $\eta$ is a homogeneous $p$-form, then it holds that

$$\langle \theta, \eta \wedge \omega \rangle = \langle \theta, \eta \rangle \wedge \omega + (-1)^p \eta \wedge \langle \theta, \omega \rangle.$$ 

The following is a generalized Yoshinaga’s freeness criterion:

**Theorem 2.4 ([3], Theorem 5.1)**

*Let $A$ be an arrangement and $(A'', m)$ the Ziegler restriction. Then $A$ is free if and only if $(A'', m)$ is free and $b_2 = \sigma_2$ in the notation of the section one.*

The following map, which is introduced in [3], is important to prove Theorem 1.1.

**Proposition 2.5 ([3])**

*Let $A$ be a central $\ell$-arrangement, $H_0 \in A$ and $(A'', m)$ the Ziegler restriction. Then there is a well-defined map $\rho : L(dA) \to L(A'')$ which keeps inclusion orders and codimensions of each flat. Also, $\rho$ is compatible with localization operations.*

### 3 Several complexes and their properties

Put $\alpha := \alpha_{H_0} = x_\ell \in S = \mathbb{K}[x_1, \ldots, x_\ell]$ and $S' = S/\alpha S = \mathbb{K}[x_1, \ldots, x_{\ell-1}]$ the coordinate ring of $H_0$. To prove Theorem 1.1 we need some lemmas and propositions, mainly on $\eta$-complexes.

**Remark 3.1**

*In this section we do not use the tameness assumption.*

**Lemma 3.2**

*The $S$-morphism

$$\Omega^p(A) \to \Omega^p(A) \wedge \frac{d\alpha}{\alpha} \to 0.$$ 

is a splitting surjection. In particular, $\text{pd}_S \Omega^p(A) \geq \text{pd}_S (\Omega^p(A) \wedge \frac{d\alpha}{\alpha}).$*
Proof. It suffices to show that the morphism has a section. Recall Proposition 4.86 in [7]. Then the section is given by

\[ \omega \wedge \frac{d\alpha}{\alpha} \mapsto (-1)^p \langle \theta_E, \omega \wedge \frac{d\alpha}{\alpha} \rangle, \]

where \( \langle , \rangle \) is a contraction. The inequality of projective dimensions follows from the long exact sequence of Ext’s. \( \square \)

Remark 3.3
Since the complex \( (\Omega^*(A), \wedge \frac{d\alpha}{\alpha}) \) is exact (see [7] for example), Lemma 3.2 shows that

\[ \Omega^p(A) \simeq (\Omega^{p-1}(A) \wedge \frac{d\alpha}{\alpha}) \oplus (\Omega^p(A) \wedge \frac{d\alpha}{\alpha}). \]

Let \( \text{res} : \Omega^p(A) \to \Omega^p(A'', m) \) be the residue map defined by

\[ \sigma \wedge \frac{d\alpha}{\alpha} + \delta \mapsto \delta|_{H_0}, \]

where \( \sigma \) and \( \delta \) are generated by \( dx_1, \ldots, dx_{\ell-1} \). Note that the residue map factors through \( \Omega^p(A) \wedge \frac{d\alpha}{\alpha} \to \Omega(A'', m) \). Let \( M^p \subset \Omega^p(A'', m) \) denote the image of the residue map and \( C^p \) its cokernel:

\[ 0 \to M^p \to \Omega^p(A'', m) \to C^p \to 0. \]

Lemma 3.4
The sequence

\[ 0 \to \Omega^p(A) \wedge \frac{d\alpha}{\alpha} \to \Omega^p(A) \wedge \frac{d\alpha}{\alpha} \to M^p \to 0 \]

is exact, where the second arrow is the product of \( \alpha \) and the third arrow is the residue map. In particular, \( \text{pd}_S M^p \leq \text{pd}_S \Omega^p(A) \).

Proof.

\[ \text{res}(\delta \wedge \frac{d\alpha}{\alpha}) = \delta|_{H_0} = 0 \iff \alpha \mid \delta. \]

Hence the exactness follows immediately. Let us prove the inequality. Since the action of \( S \) to \( M^p \) factors through \( S' = S/\alpha S \), it follows that \( \text{depth}_S M^p = \text{depth}_{S'} M^p \). Hence Auslander-Buchsbaum formula shows that \( \text{pd}_S M^p + 1 = \text{pd}_S M^p \). Also, the long exact sequence shows that \( \text{pd}_S (\Omega^p(A) \wedge \frac{d\alpha}{\alpha}) + 1 \geq \text{pd}_S M^p \). Combining this with Lemma 3.2 gives \( \text{pd}_{S'} M^p \leq \text{pd}_S \Omega^p(A) \). \( \square \)
Next let us consider the $\eta$-complex, see [7] for details. It is the complex $(\Omega^*(A), \wedge, \eta)$, where $\eta$ is some generic regular 1-form and the boundary map is given by $\wedge, \eta$. This is of course a complex, and we can define the cohomology group $H^p(\Omega^*(A))$. Let $\overline{\eta} := \eta|_{H_0}$. Since the wedge product is commutative with the inclusion $M^p \to \Omega^p(A'', m)$ and $\eta$ is regular, the wedge product of $\overline{\eta}$ is closed in $\Omega^p(A'', m)$. In other words, the boundary map $\wedge, \overline{\eta} : \Omega^p(A'', m) \to \Omega^{p+1}(A'', m)$ is induced for $p = 0, 1, \ldots, \ell - 1$. Since $M^p$ and $C^p$ are the surjective images from these two differential modules, we can define not only the complexes $(M^*, \wedge, \overline{\eta})$, $(\Omega^*(A'', m), \wedge, \overline{\eta})$ and $(C^*, \wedge, \overline{\eta})$ but also the cohomology groups $H^p(M^*), H^p(\Omega^*(A'', m))$ and $H^p(C^*)$.

**Proposition 3.5**

For an integer $d \geq 0$, there exists a regular generic 1-form $\eta$ of homogeneous degree $d$ such that all cohomology groups of both complexes $(\Omega^*(A), \wedge, \eta)$ and $(\Omega^*(A'', m), \wedge, \overline{\eta})$ are finite dimensional.

**Proof.** The proof is similar to that in [7] with a slight modification for multiarrangements. Let $S^X$ be the coordinate ring of $X \in L(A)$. Let $r_{Y,X} : S^Y \to S^X$ be the quotient map for flats $X \subset Y$ in $L(A)$, and $\Omega^1[X]^0$ the set of regular 1-forms of degree $d$ over $X$ which vanish only at the origin. It is well-known that such forms are generic in each vector spaces. Also, we can canonically extend $r_{Y,X}$ to that from the set of differential forms over $Y$ to those over $X$. Now put $N^X_d \equiv r_{V,X}^{-1}(\Omega^1[X]^0)$ and

$$N_d := \bigcap_{X \in L(A), \dim X > 0} N^X_d.$$ 

Since $r_{Y,X}$ is continuous, $N_d$ is a non-empty open set. Take an arbitrary $\eta \in N_d$. Then Proposition 4.91 in [7] shows $\dim_k H^p(\Omega^*(A)) < \infty$. Next, let us prove the multi-case. First, let us prove that, for the ideal

$$I(\overline{\eta}) := \{\{\overline{\theta}, \overline{\eta}\} \mid \overline{\theta} \in D(A'', m)\} \subset S',$$

the radical of $I(\overline{\eta})$ contains the irrelevant ideal of $S'$. It suffices to show that the zero locus $Z(I(\overline{\eta}))$ of the ideal $I(\overline{\eta})$ is contained in the origin. Take $v \in H_0 \setminus \{0\}$ and put $X_0 := \cap_{v \in H'} A^v H'$. Assume that $v \in Z(I(\overline{\eta}))$. Choose a basis $x_1, \ldots, x_{\ell - 1}$ for $H_0^*$ in such a way that $X_0 = \{x_{k+1} = \cdots = x_{\ell - 1} = 0\}$. Put $A'' := \{H' \in A'' \mid X_0 \not\subset H'\}$ and $Q'_1 := Q(A''_1, m|_{A''})$. By definition $Q'_1 \partial z_i \in D(A'', m)$ for $i = 1, \ldots, k$. Write $\overline{\eta} = f_1 dx_1 + \cdots + f_{\ell - 1} dx_{\ell - 1}$ with $f_i \in S'$. Then $\langle Q'_1 \partial z_i, \overline{\eta} \rangle = \overline{Q'_1} f_i \in I(\overline{\eta})$. By definition $Q'_1(v) \neq 0$. Hence $f_1(v) = \cdots = f_k(v) = 0$. Recalling that $r_{H_0, X_0}(\overline{\eta}) = \overline{f}_1 dx_1 + \cdots + \overline{f}_k dx_k$ vanishes only at the origin, there exists some $i$, $1 \leq i \leq k$ such that $f_i(v) \neq 0$, which is a contradiction.
Second, let \( \eta \in N_d \) and \( \overline{H}^p \) denote the \( p \)-th cohomology of the \( \overline{\eta} \)-complex \( (\Omega^*(\mathcal{A}', m), \wedge \overline{\eta}) \) where \( \overline{\eta} = r_{V,H_0}(\eta) \). First, note that \( \overline{\eta} \neq 0 \) since \( \overline{\eta} \) is chosen in such a way that it only vanishes at the origin. Before the proof, let us show the following easy but important lemma.

**Lemma 3.6**

For \( \theta \in D(\mathcal{A}', m) \) and \( \omega \in \Omega^p(\mathcal{A}', m) \), it holds that \( \langle \theta, \omega \rangle \in \Omega^{p-1}(\mathcal{A}', m) \).

**Proof.** For \( H \in \mathcal{A}' \), it holds that

\[
\langle \theta, d\alpha_H \wedge \omega \rangle = \langle \theta, d\alpha_H \rangle \omega - d\alpha_H \wedge \langle \theta, \omega \rangle = \theta(\alpha_H) \omega - d\alpha_H \wedge \langle \theta, \omega \rangle.
\]

Since \( \alpha_m(H) \mid \theta(\alpha_H) \) and \( d\alpha_H \wedge \omega \) is regular along \( H \), \( d\alpha_H \wedge \langle \theta, \omega \rangle \) is regular along \( H \). Since \( Q(\mathcal{A}', m) \theta, \omega \rangle \) is regular, \( \langle \theta, \omega \rangle \in \Omega^{p-1}(\mathcal{A}', m) \).

**Proof of Proposition 3.5, continued.** Now let \( \overline{\omega} \in \Omega^p(\mathcal{A}', m) \) be a cocycle of this complex and take \( \overline{\theta} \in D(\mathcal{A}, m) \). Then

\[
0 = \langle \overline{\theta}, \overline{\eta} \wedge \overline{\omega} \rangle = \langle \overline{\theta}, \overline{\eta} \rangle \overline{\omega} - \overline{\eta} \wedge \langle \overline{\theta}, \overline{\omega} \rangle.
\]

Hence \( I(\overline{\eta}) \) annihilates \( \overline{H}^p \), which makes the cohomology group finite dimensional.

**Remark 3.7**

Proposition 3.5 shows that we can choose the regular 1-form \( \eta \) in the proposition such that the dimensions of all cohomologies of the \( \eta \)-complex are finite for all \( m : \mathcal{A} \to \mathbb{Z}_{\geq 0} \). In other words, such a 1-form \( \eta \) depends only on \( \mathcal{A} \), independent of \( m \).

**Corollary 3.8**

In the same notation, \( H^p(M^*) \) is also finite dimensional.

**Proof.** First, by the exact sequence in Lemma 3.2, Proposition 3.5 and Remark 3.3 it holds that \( H^p(\Omega^*(\mathcal{A}) \wedge d\alpha/\alpha) \) is of finite dimensional. So the exact sequence in Lemma 3.4 shows that \( H^p(M^*) \) is all finite dimensional.

**Corollary 3.9**

In the same notation, \( H^p(C^*) \) is also finite dimensional.

**Proof.** Apply Proposition 3.5 and Corollary 3.8 to the cohomology long exact sequence of

\[
0 \to M^p \to \Omega^p(\mathcal{A}', m) \to C^p \to 0
\]

which commutes with \( \wedge \overline{\eta} \).
Before the next proposition, let us recall the fact that $C^0 = C^{ℓ-1} = 0$. We follow the proof in [10]. Since $Ω^0(A) = S$ and $Ω^0(A'', m) = S'$, it follows that $C^0 = 0$. Also, since the complex $(Ω^*(A), \frac{dα}{α})$ is exact, it follows that

$$Ω^{ℓ-1}(A) \land \frac{dα}{α} = Ω^ℓ(A) = S/Q(A)dx_1 \land \cdots \land dx_ℓ.$$

So

$$Ω^{ℓ-1}(A'', m) = S'/Q(A'', m)dx_1 \land \cdots \land dx_ℓ-1$$

implies that $C^{ℓ-1} = 0$.

**Proposition 3.10**

$$\sum_{p=0}^{ℓ-1} Poin(C^p, x)\left(t(1 - x) - 1\right)^p \in \mathbb{R}[x, x^{-1}, t] \text{ and } \sum_{p=0}^{ℓ-1} Poin(M^p, x)\left(t(1 - x) - 1\right)^p \in \mathbb{R}[x, x^{-1}, t], \text{ i.e., there are no poles along } x = 1.$$

**Proof.** Apply the same proof as Proposition 4.133 in [7] combined with Propositions 3.5, Corollaries 3.8 and 3.9. □

The following is useful to prove Theorem 1.1.

**Theorem 3.11 (Theorem 5.8, [8])**

Let $S = K[x_1, \ldots, x_ℓ]$ and $F^* = (0 \to F^0 \to F^1 \to \cdots \to F^ℓ \to 0)$ be a complex of finite $S$-modules such that every morphism is $S$-linear and that every cohomology group is finite dimensional. If a nonnegative integer $q$ satisfies

$$pd_S F^p < ℓ + p - q$$

for all $p$, then $H^q(F^*) = 0.$

**4 Proof of Theorem 1.1 and Corollary 1.2**

In this section we prove the main results of this article. Recall that we have not yet used the tameness assumption in this article. In this section we apply it.

**Proof of Theorem 1.1** Let us prove Theorem 1.1 by induction on the dimension $ℓ$ and $i$ in the setup of Theorem 1.1. For $i = 0$, $b_0 = σ_0 = 1$. For $i = 1$, $b_1 = σ_1 = |A| - 1 = |m|$. So Theorem 1.1 holds when $ℓ \leq 1$. For $ℓ = 2$, as we see in the section one, the statement is nothing but Yoshinaga’s criterion (see also [3]). Assume that $ℓ ≥ 3$ and $i < ℓ - 1$. Recall the map $ρ : L(dA) \to L(A'')$ in Proposition 2.5 and put

$$b_i = \sum_{X ∈ L_i(A'')} b_i^X,$$
where \( b_i^X \) is the sum of absolute values of \( \mu(Y) \) with \( Y \in L_i(dA) \) and \( \rho(Y) = X \). Also, let \( \sigma_i^X \) be the absolute value of the constant term of \( \chi_{\text{red}}(A'_{X}, m, X, t) := \chi(A'_{X}, m, X, t)/t^{\ell-1-i} \) for \( X \in L_i(A'/m) \). Then Theorem 2.3 and Proposition 2.5 imply that

\[
S = S_{\ell} \quad \text{are both \( d \)-limited.}
\]

Let \( \Omega_A \) be used. By definition of the tameness and the fact that the localization is exact, Proof of Theorem 1.1, continued. Lemma 4.1 allows us to apply the induction hypothesis on dimensions to the essentialization of \( A'_{X}, mX \). Hence to apply the induction hypothesis, we need the following lemma (see also [5]):

**Lemma 4.1**

Let \( A = A_1 \times A_2 \) be an \( \ell \)-arrangement which decomposes into the product of a \( d \)-arrangement \( A_1 \) in \( V_1 \) and an \((\ell-d)\)-arrangement \( A_2 \) in \( V_2 \). For \( m : A \to \mathbb{Z}_{\geq 0} \), let \( m_i \) denote the restriction of \( m \) onto \( A_i \). Let \( S_i \) denote the coordinate ring of \( A_i \). Hence \( \Omega_A \) is a projective \( S \)-module if \( P \) is a direct summand of \( S \)-module. Using the flatness of \( S \)-module, it holds that \( S \)-module. Hence \( P \) is projective. Also, it is known that

\[
\text{Hom}_A(M, S_1) \otimes \mathbb{K} S_2 \simeq \text{Hom}_S(S \cdot M, S)
\]

for any finitely generated \( S \)-module \( M \) (see [6] for example). Hence

\[
\text{Ext}^q_{S_1}(\Omega^p(A_1, m_1), S_1) \otimes \mathbb{K} S_2 \simeq \text{Ext}^q_S(S \cdot \Omega^p(A_1, m_1), S).
\]

So it holds that \( \text{pd}_S S \cdot \Omega^p(A_1, m_1) \geq \text{pd}_{S_1} \Omega^p(A_1, m_1) \). \( \square \)

**Proof of Theorem 1.1, continued.** Lemma 4.1 allows us to apply the induction hypothesis on dimensions to the essentialization of \( A_X \) and \( A'_{X}, mX \)
since they are also tame. Also, note that \( \chi_{\text{red}}(\mathcal{A}_X^n, m_X, t) \) is nothing but the characteristic polynomial of the essentialization of \((\mathcal{A}_X^n, m_X)\). Hence
\[
b_i^X \geq |\chi_{\text{red}}(\mathcal{A}_X^n, m_X, 0)| = \sigma_i^X.
\]
Then local-to-global formula above shows that \( b_i \geq \sigma_i \) \((i = 0, 1, \ldots, \ell - 2)\).

Next show that \( b_{\ell - 1} \geq \sigma_{\ell - 1} \). By the assumption and Lemma 3.4, it holds that \( \text{pd}_S M^p \leq p \) \((p = 0, 1, \ldots, \ell - 1)\). Hence Theorem 3.11 combined with Proposition 3.5 shows that \( H^p(M^*) = 0 \) \((p \leq \ell - 2)\). Also, Theorem 3.11 combined with Proposition 3.5 and the assumption that \((\mathcal{A}^\prime, m)\) is tame show that \( H^p(\Omega^*(\mathcal{A}^\prime, m)) = 0 \) \((p \leq \ell - 2)\) for every generic \( \eta_{d^*}\)-complex. Hence the long exact sequence of cohomology of the sequence
\[
0 \to M^p \to \Omega^p(\mathcal{A}^\prime, m) \to C^p \to 0
\]
shows that \( H^p(C^*) = 0 \) \((0 \leq p \leq \ell - 3)\) for every generic \( \eta_{d^*}\)-complex. By the arguments in [10],
\[
\chi_0(\mathcal{A}, t) = \sum_{p=0}^{\ell-1} \text{Poin}(M^p, x)(t(1-x) - 1)^p|_{x=1},
\]
\[
\chi(\mathcal{A}^\prime, m, t) = \sum_{p=0}^{\ell-1} \text{Poin}(\Omega^p(\mathcal{A}^\prime, m), x)(t(1-x) - 1)^p|_{x=1}.
\]
Hence
\[
\chi_0(\mathcal{A}, t) - \chi(\mathcal{A}^\prime, m, t) = -\sum_{p=0}^{\ell-1} \text{Poin}(C^p, x)(t(1-x) - 1)^p|_{x=1}.
\]
Now consider a generic \( \eta_{d^*}\)-complex combined with the cohomology vanishing in the above. Then
\[
\chi_0(\mathcal{A}, 0) - \chi(\mathcal{A}^\prime, m, 0) = -\sum_{p=0}^{\ell-1} \text{Poin}(C^p, x)(-1)^p|_{x=1}
\]
\[
= -\sum_{p=1}^{\ell-2} \text{Poin} H^p(C^p)(-1)^p|_{x=1}
\]
\[
= (-1)^{\ell-1} \dim_K H^{\ell-2}(C^*).
\]
Hence \( b_{\ell - 1} - \sigma_{\ell - 1} = \dim_K H^{\ell-2}(C^*) \geq 0 \). To complete the proof, it suffices to show the following proposition. 

□
Proposition 4.2
If \((\mathcal{A}'', m)\) is tame, then \(\sigma_i \geq 0\).

Proof. Use the similar argument to the proof of Theorem 1.1. Then it suffices to show that, by using localizations and Theorem 2.3 the Euler characteristic \(\sum_{p=0}^{\ell-1} (-1)^p \text{Poin}(\Omega^p(\mathcal{A}'', m), x)|_{x=1}\) is not negative. Then the tameness condition completes the proof. \(\square\)

Corollary 4.3
In the notation of Theorem 1.1 \((-1)^{\ell-1} \chi(\mathcal{A}'', m, -1) \geq 0\).

Proof. Proposition 4.3 and
\[(-1)^{\ell-1} \chi_0(\mathcal{A}'', m, -1) = \sum_{i=0}^{\ell-1} \sigma_i\]
completes the proof. \(\square\)

Proof of Corollary 1.2 The first statement follows immediately from Theorem 1.1 and Corollary 4.3. Let us prove the second statement. By [14] and the factorization in [2], it is easy to see that the freeness implies MCA. Assume that \(\mathcal{A}\) is MCA. Then Theorem 1.1 implies that \(\chi_0(\mathcal{A}, t) = \chi(\mathcal{A}'', m, t)\). Then Theorem 2.4 completes the proof. \(\square\)

Corollary 4.4
Assume that \(\mathcal{A}\) is a 4-arrangement. Then the statement in Theorem 1.1 holds true if \(\mathcal{A}\) is tame.

Proof. Since \(\Omega^p(\mathcal{A}'', m)\) is reflexive, Auslander-Buchsbaum formula combined with depth \(\Omega^p(\mathcal{A}'', m) \geq 2\) completes the proof. \(\square\)

Corollary 4.5 ([13], Theorem 3.2)
A 3-arrangement \(\mathcal{A}\) is free if and only if it is a minimal chamber arrangement.

Proof. Since 2 and 3-(multi)arrangements are tame, we can use Theorem 1.1 and Corollary 1.2. Assume that \(\mathcal{A}\) is a minimal chamber arrangement. Since \(C^0 = C^2 = 0\), the complex is \(0 \to C^1 \to 0\). The minimality of chambers implies that \(H^1(C^*) = 0\), which is nothing but \(C^1 = 0\). Then the result in [14] implies that \(\mathcal{A}\) is free. \(\square\)

Theorem 1.1 and Corollary 1.2 give us a direction of the research on the relation between free arrangements and MCA as follows:
Problem 4.6

Do Theorem 1.1 and Corollary 1.2 hold true without the assumption of tameness?

Problem 4.6 is a very natural one. When we construct a free arrangement by the addition theorem, we usually, or empirically, add hyperplanes in such a way that the new arrangements have the smallest chambers among all the other choices (though the addition of this type does not always work well!). This choice of the additions is justified when $\ell = 3$ by [13]. If Problem 4.6 is true, then we can obtain a better generalization of Yoshinaga’s criterion. If that is not true, then the tameness condition becomes more important, and essential condition which connects algebra and geometry of hyperplane arrangements.

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Takuro Abe
Department of Mechanical Engineering and Science
Kyoto University
Yoshida Honmachi, Sakyo-Ku, Kyoto 6068501, Japan
abe.takuro.4c@kyoto-u.ac.jp