Progress in lattice QCD with dynamical fermions

L. Del Debbio
SUPA, School of Physics and Astronomy, University of Edinburgh, Edinburgh, EH9 3JZ, Scotland
E-mail: luigi.del.debbio@ed.ac.uk

Abstract. Recent progress in lattice QCD has been characterised by significant improvements in the algorithms for simulating dynamical fermions, leading to numerical simulations that run now closer to the physical quark masses and with a variety of fermionic discretizations. In this paper we focus on two aspects that have been studied recently. The latest results obtained for the low–energy constants extract from fitting lattice data to chiral perturbation theory. Sources of systematic errors are emphasized, and results from different collaborations are compared by plotting adimensional ratios that should fall on a universal curve as the continuum limit is approached. We also report on recent results for the parameter $B_K$ which encodes the nonperturbative contribution to indirect CP violation in kaon mixing.

1. Towards the chiral limit

Important algorithmic improvements have been achieved over the last few years in simulating lattice QCD with dynamical fermions [1, 2, 3, 4, 5]. As a consequence, numerical simulations can now access unprecedented light quark masses, on large volumes, and at fine lattice spacings; last but not least, it is now viable to generate gauge configurations in full QCD using several different discretizations for the fermionic action. The improvement that has been achieved is best quantified by comparing the cost formulae for producing 100 independent gauge configurations for $N_f = 2$ flavours of clover fermions on a $2L \times L^3$ lattice. The estimate from the Lattice 2001 conference [6]:

$$5 \left( \frac{20 \text{ MeV}}{m} \right)^3 \left( \frac{L}{5 \text{ fm}} \right)^5 \left( \frac{0.1 \text{ fm}}{a} \right)^7 \text{TFlops} \times \text{yrs},$$

has been updated in 2006, after extensive experience with the DD-HMC algorithm [7, 8, 9]; the current cost of simulations close to the chiral limit is well described by the formula [10]:

$$0.05 \left( \frac{20 \text{ MeV}}{m} \right)^1 \left( \frac{L}{5 \text{ fm}} \right)^5 \left( \frac{0.1 \text{ fm}}{a} \right)^6 \text{TFlops} \times \text{yrs},$$

where $m$ is the running sea–quark mass in the $\overline{\text{MS}}$ scheme at the renormalization scale of 2 GeV. The most striking feature in comparing the two equations is the improved scaling exponent with the quark mass, which enables to simulate at much lighter quark masses with the existing computing facilities. The cost for a $64 \times 32^3$ lattice with a lattice spacing $a \approx 0.09\text{fm}$ is reported in Fig. 1. Current computing facilities yield sustained performances of the order of the TFlop (and above), and therefore the combination of improved algorithms and new machines is readily expanding the scope of lattice simulations towards the chiral regime of QCD. In practice this
Figure 1. Cost for producing 100 configurations for $N_f = 2$ flavours of clover fermions on a $64 \times 32^3$ lattice at $a \approx 0.09 \text{fm}$, expressed in TFlops × yrs as a function of the quark mass. The algorithmic improvement of the last five years is clearly visible by comparing the two curves.

translates into the possibility of first-principle calculations in QCD with robust control of the systematic errors that arise in lattice computations. Several collaborations are now generating gauge configurations with dynamical quarks. CERN [7, 8, 9] and QCDSF (see e.g. Ref. [11] and references therein) have been using two flavours of $O(a)$–improved Wilson (clover) fermions, PACS-CS simulate three flavours of clover fermions [12], and ETMC use the twisted mass formulation of QCD (tmQCD) [13]. Other collaborations put more emphasis on preserving the chiral structure of lattice discretization: RBC-UKQCD simulate $N_f = 2 + 1$ Domain Wall Fermions (DWF) [14, 15], while JLQCD use overlap fermions at fixed topology [16]. New results are also being produced by MILC using staggered fermions [17]. The latest results from each collaboration have been presented this summer at the Lattice conference in Regensburg.

In a nutshell, accurate results are being obtained for physical observables for pseudoscalar meson masses down to 300 MeV, and precision studies of the chiral regime of QCD have begun; an extensive review of current activities would be beyond the scope of this work, the interested reader should refer to the proceedings of recent Lattice conferences for an exhaustive overview of the subject. For this talk, we choose to focus on two topics which have been under intense scrutiny in the last year, namely the progress in extracting the low-energy constants (LECs) of chiral perturbation theory (ChPT), and the computation of the kaon $B$–parameter, $B_K$. The motivations behind this choice can be summarized as follows. On one hand, a precise determination from first principles of the parameters of chiral perturbation theory can be directly compared with phenomenological estimates based on experimental data, and thus provides an important test of QCD as the theory of strong interaction at the hadronic energy scales. The latest results from fits to ChPT are reported in Sect. 2. In Sect. 2 we also present a universal
scaling plot that is useful for presenting a critical comparison of different lattice data that are
obtained by the different collaborations, and that are currently used for fits to ChPT. On the
other hand the parameter $B_K$ is an important ingredient for constraining the vertex of the
Unitarity Triangle in current CKM fits [18, 19]. Sect. 3 is devoted to a discussion of the status
of two recent $B_K$ computations. One of them is performed by the RBC-UKQCD collaboration
with dynamical DWF fermions, while the other, by the ALPHA collaboration, still relies on the
quenched approximation, but sets the framework for a method that is well–suited for use with
dynamical fermions. Other determinations have been reviewed recently in Ref. [20].

Both for the determination of the LECs and for $B_K$, the main emphasis is on the precision
required for lattice measurements to have a phenomenological impact, and on the systematic
errors that may hinder the accuracy of lattice results. The different systematics that arise for
different fermionic actions are discussed.

2. Chiral logs

The ability to simulate light quark masses allows lattice QCD to approach the regime where
results can be compared with NLO prediction from chiral perturbation theory. For two flavours
of light degenerate quarks, ChPT predicts for the pseudoscalar mass and decay constant:

$$m_\pi^2 = M^2 \left[ 1 + \frac{M^2}{32\pi^2 F^2} \log \left( \frac{M^2}{\Lambda_3^2} \right) + \ldots \right], \quad M^2 = 2Bm, \quad (1)$$

$$F_\pi = F - \frac{M^2}{16\pi^2 F^2} \log \left( \frac{M^2}{\Lambda_4^2} \right) + \ldots, \quad (2)$$

where the coefficients of the logarithms are determined by the pseudoscalar decay constant in
the chiral limit, $F$, while the scales of the logarithms are further free parameters of the theory,
denoted $\Lambda_3, \Lambda_4$. These LECs can be determined from phenomenological studies, with recent
estimates yielding [21]:

$$\bar{l}_3 = \log \left. \frac{\Lambda_3^2}{m_\pi^2} \right|_{m_\pi=139\text{MeV}} = 2.9 \pm 2.4, \quad (3)$$

$$\bar{l}_4 = \log \left. \frac{\Lambda_4^2}{m_\pi^2} \right|_{m_\pi=139\text{MeV}} = 4.4 \pm 0.2. \quad (4)$$

Using the central value for $\bar{l}_3$, the size of the chiral logarithm is depicted in Fig. 2, where the red
points denote hypothetical simulation points for a realistic range of quark masses. More details
on the perspectives for simulating lighter masses will be discussed below. The relative size of
the correction due to the chiral log is given by the deviations of the ratio $m_\pi^2/(2Bm)$ from unity.

With the current values of the LECs, deviations from the LO linear behaviour appear to be of
the order of 5% when the mass of the pseudoscalar meson ranges in between 300 and 400 MeV.
Hence any lattice study aiming at a determination of $\bar{l}_3, \bar{l}_4$ needs to simulate at sufficiently light
quark masses, and needs to keep the systematic errors at the few percent level.

The most recent results from dynamical simulations with light quark masses have just reached
this regime, and have prompted novel determinations of the LECs. The mass and decay constant
of the pseudoscalar meson are usually extracted from the exponential behaviour of suitable lattice
two–point functions at large Euclidean times:

$$f_{PP}(x_0) = a^3 \sum_{x_1, x_2, x_3} \langle P_P(x) P_P(0) \rangle, \quad (5)$$

$$f_{AP}(x_0) = a^3 \sum_{x_1, x_2, x_3} \langle A_P^r(x) P_P(0) \rangle, \quad (6)$$
Figure 2. The ratio $m_{\pi}^2/(2Bm)$ as a function of the pseudoscalar mass $m_{\pi}$, obtained from NLO ChPT, using the central value of the LEC $\Lambda_3$ as obtained from phenomenological studies.

where the pseudoscalar density and the axial current are built from the quark fields, labelled here as $r$ and $s$:

$$P_{rs} = \bar{r}\gamma_5s, \quad A_{\mu rs} = \bar{r}\gamma_\mu\gamma_5s.$$  \hspace{1cm} (7)  

Different types of smearing of the mesonic quark bilinears are performed in order to increase the overlap of the operator with the lowest state in the spectrum. Large Euclidean separations are needed to guarantee that the behaviour of the correlation function is dominated by the lightest state in the spectrum. Current lattices allow for a precise identification of the pseudoscalar state, see e.g. effective mass plots in Ref. [9]. Studies of vector resonances, or baryonic states may require larger lattices to reach a similar precision. We refer the interested reader to specific papers for more details on the lattice observables, see e.g. Refs. [9, 17, 14, 13] and references therein.

As discussed above, precise results at small quark masses are needed in order to extract the mass and the pseudoscalar decay constant, and to match them to the functional behaviour predicted by chiral symmetry. The main sources of systematic errors are finite-size effects, lattice artefacts, and the determination of renormalization (and improvement) constants. Any given choice of the fermionic action has its own advantages and disadvantages in taming these systematic errors. If all systematics error were under control, the results could be extrapolated to the infinite volume limit, the lattice regulator would be removed and universal continuum results should be recovered. In the absence of these robust extrapolations, one has to define some prescriptions to compare results from different lattices; there is some degree of arbitrariness in choosing these prescriptions, which often introduce some new systematic error in the process. Here we shall try to define a procedure which enables to compare the chiral fits for $\bar{l}_3,\bar{l}_4$. The typical lattice ensembles for which there are currently available data are summarized in Tab. 1. In comparing the entries in the table, note that different collaborations use different methods to determine the lattice spacing. At present this introduces some ambiguities in the determination of the physical scale. Ambiguities could be resolved by simulating at lighter masses and setting the scale using only properties of stable hadrons, e.g. the pion and the nucleon.

Finite–volume effects can be described by ChPT or a resummed Lüscher formula [22, 23, 24];
Table 1. Data from recent simulations with dynamical fermions. The number of flavours is indicated together with the choice of the fermion formulation. The last column reports the lightest pion mass that is currently reached. QCDSF, PACS-CS, and MILC have data for several values of the lattice spacing. The value reported here

these analytical calculations suggest that corrections can become fairly large for the pseudoscalar decay constant for values of $m_\pi L \approx 3$ [25]. The range of validity of the analytical formulae is not precisely known, and finite–size corrections are not implemented uniformly by all collaborations; further numerical work is needed, in order to quantify the size of finite–volume effects.

Nonperturbative renormalization techniques, which are necessary for precision measurements, have become a common tool in lattice QCD. For the chiral fits that are discussed here, nonperturbative determinations of the axial current renormalization $Z_A$ are available for the fermionic formulations that are commonly used for numerical simulations [26, 27, 14]. For clover fermions, the axial current needs to be improved in order to remove $O(a)$ artefacts, the improvement coefficient $c_A$ has also been computed nonperturbatively [28, 29].

Results for the LECs $\bar{l}_3, \bar{l}_4$, which are obtained by fitting the chiral behaviour of $m_\pi$ and $f_\pi$, were recently summarized by S. Necco at the Lattice 07 conference [30]. Note that the radius of convergence of ChPT is unknown, and hence there is some arbitrariness in the range of masses that should be included in the fits. Moreover precise results at the lightest masses are needed for the fit to be driven by the data in the region where the expansion is more reliable. The results for $N_f = 2$ are reported in Tab. 2. Despite the different fitting procedures, results are consistent with each other at the current level of accuracy. Similar results have been obtained

Table 2. Summary of LECs for $SU(2) \times SU(2)$ ChPT; values obtained from fits to the lattice data, and presented at the Lattice conference [30].

from fitting the lattice data obtained with $N_f = 2 + 1$ flavours to $SU(3) \times SU(3)$ ChPT, and are reported in Tab. 3. Recent results from extensive runs by RBC-UKQCD collaboration have cast some doubts on the applicability of the three–flavour effective theory to describe data up to meson masses of the order of the kaon mass [31]. In the light of these observations, results in Tab. 3 should be taken with a grain of salt. Further work is in progress to clarify this issue.

In order to assess the universality of the current results, we propose a global analysis of lattice results, which treats all data in a uniform fashion; here a preliminary analysis is applied to a subset of the available results, using published data only. The analysis follows closely the
Table 3. Summary of LECs for $SU(3) \times SU(3)$ ChPT computed at the scale 770 MeV; values obtained from fits to the lattice data, and presented at the Lattice conference [30].

| Collaboration  | $2L_8 - L_5$ | $2L_6 - L_4$ | $L_4$ | $L_5$ |
|----------------|-------------|-------------|-------|-------|
| MILC          | 0.3(1)(1)   | 0.3(1)(2)   | 0.1(3)(3) | 1.4(2)(2) |
| RBC-UKQCD     | 0.243(45)   | -0.01(42)   | 0.139(80)  | 0.872(99)  |
| PACS-CS        | -0.23(5)    | 0.10(4)     | -0.02(11)  | 1.47(15)   |

one presented in Ref. [8], but only uses data at unitary points, i.e. no information coming from partially quenched data is included at this stage. The strange quark mass is fixed by requiring the ratio $M_K/M_{K^*}$ to be equal to its physical value. Here $M_K$ (respectively $M_{K^*}$) indicates the mass of the pseudoscalar (respectively vector) state made of two degenerate quarks, with masses equal to the mass of the sea quarks. Note that this prescription is different from the one in Ref. [8], since partially quenched valence quarks are not used to define the pseudoscalar meson which sets the scale; the lattice spacing could be defined by setting the reference mass $M_{K, ref}$ obtained above to the physical value of the $K$ meson mass. Instead, in order to compare data from different simulations, all dimensional quantities are expressed in units of dimensionful quantities defined at the reference point. In order to avoid the uncertainties related to the renormalization of the lattice quantities, data are compared by plotting $(f_\pi/f_{K, ref})$ as a function of $(m_\pi/m_{K, ref})^2$. In the ratios the renormalization of the axial current $Z_A$ cancels, while scaling violations could be reduced. In this way, data from different lattices can be directly compared, since all points should fall on a universal curve as simulations get sufficiently close to the continuum limit. Indeed the data reported in Fig. 3 are nicely consistent with each other, showing small scaling violations in the range of couplings that are currently explored.

![Figure 3. Universal curve for the ratio $f_\pi/f_{K, ref}$ as a function of $(m_\pi/m_{K, ref})^2$. The lightest pion mass corresponds approximately to 309 MeV](image-url)
3. Kaon mixing

The measure of indirect CP asymmetry in the neutral kaon system, $\epsilon_K$, can be expressed in terms of the sides of the Unitarity Triangle, $\bar{\rho}, \bar{\eta}$, as:

$$\epsilon_K = C\epsilon B_K f_p (1 - \bar{\rho}),$$

where $C, f_p$ are known factors [32], while the nonperturbative QCD effects are encoded in the parameter $\hat{B}_K$:

$$\hat{B}_K = \frac{\langle K^0|\hat{O}_{VV + AA}\rangle}{\frac{8}{3}\bar{m}_K^2},$$

which parametrizes the matrix element of the renormalization group invariant (RGI) four-quark operator: $\hat{O}_{VV + AA} = (\bar{s}\gamma_\mu d)(\bar{s}\gamma_\mu d) + (\bar{s}\gamma_5\gamma_\mu d)(\bar{s}\gamma_5\gamma_\mu d)$.

Lattice calculations provide a first-principle determination of $\hat{B}_K$, and thus valuable information for constraining the vertex of the Unitarity Triangle. Another possibility is to extract the nonperturbative parameters directly from fitting the abundant number of constraints that are currently available and assuming the Standard Model scenario [33]. The current bound from the latter analysis is:

$$\hat{B}_K = 0.75(9).$$

Lattice computations need to achieve a comparable accuracy, in order to provide a useful input in the quest for New Physics in the flavour sector. For the determination of $\hat{B}_K$ the main source of uncertainty in lattice calculations is due to the renormalization of the four-quark operator. Regularizations that do not preserve chiral symmetry induce a mixing between $O_{VV + AA}$ and four other operators [34, 35, 36, 37] with opposite chirality:

$$O^R_{VV + AA} = Z_{VV + AA}(g_0, a_\mu) \left[ O_{VV + AA} + \sum_{i=1}^4 c_i(g_0)O_i \right],$$

and therefore the physical result is obtained from cancellations than can induce large fluctuations. The main progresses in this area have been obtained by two methods that reformulate the lattice calculation in ways which avoid operator mixing.

The first method consists in using a fermionic regularised action which posses an exact chiral symmetry [38, 39]. The latest calculation of this kind has been performed this year by the RBC-UKQCD Collaboration, using 2 + 1 flavours of domain wall fermions. The (approximate) chiral symmetry of the action ensures that the mixing is sufficiently small and can be safely neglected. Moreover, it also guarantees that the results are automatically free of $O(a)$ lattice artefacts. The renormalization constant is computed using a nonperturbative RI-MOM technique. Simulations at fixed lattice spacing $a^{-1} = 1.73(3)\text{ GeV}$, and with two volumes $16^3 \times 32 \times 16$, and $24^3 \times 64 \times 16$ allow for a controlled extrapolation to the chiral limit. Their final result is [40]:

$$\hat{B}_K = 0.720(13)(37),$$

where the second error is the estimated systematic error. The scaling violations are difficult to assess since results are only available at one value of the lattice spacing. They are currently estimated at 4%. The error on $\hat{B}_K$ is already substantially smaller than in previous computations. Simulations on finer lattices are in progress, and will allow a better control of this remaining source of uncertainty. Perspectives for a precise determination of $\hat{B}_K$ form DWF simulations are very good.

Twisted mass QCD provides another interesting approach to compute $B_K$: it enables to extract the matrix element $\langle K^0|O_{VV + AA}|\bar{K}^0\rangle$ from the matrix element of the operator $O_{VA + AV}$. 

---

The 2007 Europhysics Conference on High Energy Physics IOP Publishing
Journal of Physics: Conference Series 110 (2008) 102004
doi:10.1088/1742-6596/110/0/102004
as originally suggested in Ref. [41]. The latter operator is protected from mixing, and therefore only renormalizes multiplicatively. So far this method has been fully implemented only in the quenched approximation. The renormalization has been studied with nonperturbative techniques in Refs. [42, 43]. Results for the matrix elements and the actual computation of \( \hat{B}_K \) have been presented in Ref. [44]. When tmQCD is regularised using Wilson fermions, the twist angle needs to be tuned by adjusting ratios of renormalized mass parameters. This procedure is particularly sensitive to the critical value of the hopping parameter, \( \kappa_c \), which requires to be tuned to high precision. Recent results [45] demonstrated the impact of the tuning of the twist angle on the determination of \( B_K \). At fixed value of the lattice spacing systematic error in the determination of the critical mass translates into a shift of the \( B_K \) value, which in turn affects the extrapolation to the continuum limit. The final number quoted in Ref. [45] is:

\[
\hat{B}_K = 0.735(71).
\] (13)

The tmQCD formulation enables to fully control the systematics related to renormalization, lattice artefacts and finite–volume effects. The only source of systematics left in the result in Eq. 13 is the quenched approximation. This approach is particularly promising in the light of the recent effort by the ETMC in generating gauge configurations with dynamical fermions [13].

A global summary of available results was presented in Ref. [20]. The new data discussed in this Section yield the updated summary plot reported in Fig. 4. It is worthwhile to emphasize the reduced error on \( B_K \) obtained for three flavours of DWF, thanks to the correct flavour content, good chiral behaviour, and \( O(a) \) improvement. By comparison, the staggered computation – HPQCD-UKQCD 06 in Fig. 4 – is plagued by large taste mixing and lack of nonperturbative renormalization, which make the error bar substantially larger.

![Figure 4. Summary of results for the determination of \( \hat{B}_K \), the plot is an update of the one presented in Ref. [20]. The points corresponding to ALPHA06, and RBC-UKQCD07 have been added to the previous plot.](image-url)

4. Perspectives
A great abundance of new results with dynamical fermions is expected in the near future, which will deliver sufficient statistics at lighter masses, larger volumes, and possibly smaller lattice spacings.

In the light of the recent progresses discussed in the two examples above, and of further results of similar quality for other observables, the perspectives for high-accuracy results from lattice simulations are very good. Precise results will be available for different discretizations, therefore testing the universality of the continuum limit. Universal scaling plots, like the one
proposed here, are a natural tool to compare results obtained with different discretizations. These results will confirm QCD as the theory of strong interactions at low–energy scales, and will be sufficiently accurate to have an impact on Standard Model phenomenology.

Acknowledgments
My contributions to simulations with dynamical fermions, and fits to the chiral regime, have been obtained in collaboration with L. Giusti, M. Lüscher, R. Petronzio, and N. Tantalo. I would like to thank them for a very pleasant collaboration. I would like to thank the organisers of the EPS conference, and the conveners for setting up an interesting parallel session. Many thanks to P. Boyle and G. Schierholz for correspondence on the data produced by RBC-UKQCD and QCDSF collaborations.

References
[1] Hasenbusch M 2001 Phys. Lett. B519 177–182 (Preprint hep-lat/0107019)
[2] Hasenbusch M and Jansen K 2003 Nucl. Phys. B659 299–320 (Preprint hep-lat/0211042)
[3] Luscher M 2005 Comput. Phys. Commun. 165 199–220 (Preprint hep-lat/0409106)
[4] Urbach C, Jansen K, Shindler A and Wenger U 2006 Comput. Phys. Commun. 174 87–98 (Preprint hep-lat/0506011)
[5] Clark M A and Kennedy A D 2007 Phys. Rev. Lett. 98 051601 (Preprint hep-lat/0608015)
[6] Ukawa A (CP-PACS and JLQCD) 2002 Nucl. Phys. Proc. Suppl. 106 195–196
[7] Del Debbio L, Giusti L, Luscher M, Petronzio R and Tantalo N 2006 JHEP 02 011 (Preprint hep-lat/0512021)
[8] Del Debbio L, Giusti L, Luscher M, Petronzio R and Tantalo N 2007 JHEP 02 056 (Preprint hep-lat/0610059)
[9] Del Debbio L, Giusti L, Luscher M, Petronzio R and Tantalo N 2007 JHEP 02 082 (Preprint hep-lat/0701009)
[10] Giusti L 2007 PoS. LAT2006 (Preprint hep-lat/0702014)
[11] Gockeler M et al. 2006 PoS LAT2006 169 (Preprint hep-lat/0610071)
[12] Kuramashi Y et al. (PACS-CS) 2006 PoS LAT2006 029 (Preprint hep-lat/0610063)
[13] Boucaud P et al. (ETM) 2007 Phys. Lett. B650 304–311 (Preprint hep-lat/0701012)
[14] Allton C et al. (RBC and UKQCD) 2007 Phys. Rev. D76 014504 (Preprint hep-lat/0701013)
[15] Antonio D J et al. (RBC and UKQCD) 2007 (Preprint arXiv:0705.2340 [hep-lat])
[16] Kaneko T et al. (JLQCD) 2006 PoS LAT2006 054 (Preprint hep-lat/0610038)
[17] Bernard C et al. (MiLCS) 2006 PoS LAT2006 163 (Preprint hep-lat/0609053)
[18] Bona M et al. (UTfit) 2005 JHEP 07 028 (Preprint hep-ph/0501199)
[19] Charles J et al. (CKMfitter Group) 2005 Eur. Phys. J. C41 1–131 (Preprint hep-ph/0406184)
[20] Tantalo N 2007 (Preprint hep-ph/0703241)
[21] Colangelo G, Gasser J and Leutwyler H 2001 Nucl. Phys. B603 125–179 (Preprint hep-ph/0103088)
[22] Gasser J and Leutwyler H 1987 Phys. Lett. B184 83
[23] Becirevic D and Villadoro G 2004 Phys. Rev. D69 054010 (Preprint hep-lat/0311028)
[24] Colangelo G, Durr S and Hafelj C 2005 Nucl. Phys. B721 136–174 (Preprint hep-lat/0503014)
[25] Urbach C Plenary talk at the Lattice 2007 conference, to appear in the proceedings
[26] Becirevic D et al. 2006 Nucl. Phys. B734 138–155 (Preprint hep-lat/0510014)
[27] Della Morte M, Hoffmann R, Knechtli F, Sommer R and Wolff U 2005 JHEP 07 007 (Preprint hep-lat/0505026)
[28] Della Morte M, Hoffmann R and Sommer R 2005 JHEP 03 029 (Preprint hep-lat/0503003)
[29] Kaneko T et al. 2007 JHEP 04 092 (Preprint hep-lat/0703006)
[30] Necco S Plenary talk at the Lattice 2007 conference, to appear in the proceedings
[31] Lin M and Scholz E E
[32] Buras A J 1998 (Preprint hep-ph/9806471)
[33] Bona M et al. (UTfit) 2006 JHEP 10 081 (Preprint hep-ph/0606167)
[34] Martinielli G 1984 Phys. Lett. B141 395
[35] Bernard C W, Soni A and Draper T 1987 Phys. Rev. D36 3224
[36] Gupta R, Daniel D, Kilcup G W, Patal A and Sharpe S R 1993 Phys. Rev. D47 5113–5127 (Preprint hep-lat/9210018)
[37] Donini A, Gimenez V, Martinelli G, Talevi M and Vladikas A 1999 Eur. Phys. J. C10 121–142 (Preprint hep-lat/9902030)
[38] Ginsparg P H and Wilson K G 1982 *Phys. Rev.* **D25** 2649
[39] Luscher M 1998 *Phys. Lett.* **B428** 342–345 (Preprint hep-lat/9802011)
[40] Antonio D J *et al.* (RBC and UKQCD) 2007 (Preprint hep-ph/0702042)
[41] Frezzotti R, Grassi P A, Sint S and Weisz P (Alpha) 2001 *JHEP* **08** 058 (Preprint hep-lat/0101001)
[42] Guagnelli M, Heitger J, Pena C, Sint S and Vladikas A (ALPHA) 2006 *JHEP* **03** 088 (Preprint hep-lat/0505002)
[43] Palombi F, Pena C and Sint S 2006 *JHEP* **03** 089 (Preprint hep-lat/0505003)
[44] Dimopoulos P *et al.* (ALPHA) 2006 *Nucl. Phys.* **B749** 69–108 (Preprint hep-ph/0601002)
[45] Dimopoulos P *et al.* 2007 *Nucl. Phys.* **B776** 258–285 (Preprint hep-lat/0702017)