Generalized Duality Symmetry of Non-Abelian Theories

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Abstract

The quantum Yang–Mills theory describing dual ($\hat{g}$) and non–dual ($g$) charges and revealing the generalized duality symmetry was developed by analogy with the Zwanziger formalism in QED.
1. Introduction

In the last years gauge theories essentially operate with the fundamental idea of duality (see, for example, reviews [1] and references there). Duality is a symmetry appearing in pure electrodynamics as invariance of the free Maxwell equations:

\[ \nabla \cdot \mathbf{\tilde{B}} = 0, \quad \nabla \times \mathbf{\tilde{E}} = -\partial_0 \mathbf{\tilde{B}}, \]  

(1)

\[ \nabla \cdot \mathbf{\tilde{E}} = 0, \quad \nabla \times \mathbf{\tilde{B}} = \partial_0 \mathbf{\tilde{E}}, \]  

(2)

under the interchange of electric and magnetic fields:

\[ \mathbf{\tilde{E}} \rightarrow \mathbf{\tilde{B}}, \quad \mathbf{\tilde{B}} \rightarrow -\mathbf{\tilde{E}}. \]  

(3)

Letting

\[ F = \partial \wedge A = -(\partial \wedge B)^*, \]  

(4)

\[ F^* = \partial \wedge B = (\partial \wedge A)^*, \]  

(5)

it is easy to see that the equations of motion:

\[ \partial_\lambda F_{\lambda \mu} = 0, \]  

(6)

which together with the Bianchi identity:

\[ \partial_\lambda F^*_{\lambda \mu} = 0 \]  

(7)

are equivalent to Eqs. (1) and (2), show the invariance under the Hodge star operation on the field tensor:

\[ F^*_{\mu \nu} = \frac{1}{2} \epsilon_{\mu \nu \rho \sigma} F_{\rho \sigma}. \]  

(8)

This Hodge star duality, having a long history [2]- [7], does not hold in general for non–Abelian theories. In Abelian theory Maxwell’s equation (6) is
equivalent to the Bianchi identity for the dual field $F^*_{\mu \nu}$, which guarantees the existence of a ”dual potential” (see $B_\mu$ in Eq.(3)).

In the non–Abelian Yang–Mills theory, one usually starts with a gauge field $F_{\mu \nu}(x)$ derivable from a potential $A_\mu(x)$:

$$F_{\mu \nu} = \partial_\nu A_\mu(x) - \partial_\mu A_\nu(x) + ig[A_\mu(x), A_\nu(x)].$$  \hspace{1cm} (9)

Considering (for simplicity of presentation) only gauge group with Lie algebra SU(N), we have:

$$A_\mu(x) = t^j A^j_\mu(x), \quad j = 1, \ldots, N^2 - 1,$$ \hspace{1cm} (10)

where $t^j$ are the generators of SU(N) group. Equations of motion obtained by extremizing the corresponding action with respect to $A_\mu(x)$ gives:

$$D_\nu F^\mu\nu(x) = 0,$$ \hspace{1cm} (11)

where $D_\nu$ is the covariant derivative defined as

$$D_\mu = \partial_\mu - ig[A_\mu(x), \cdot].$$ \hspace{1cm} (12)

The analogy to electromagnetism is still rather close. But Yang–Mills equation (11) does not imply in general the existence of a potential for the corresponding dual field $F^*_{\mu \nu}$. This Yang–Mills equation itself can no longer be interpreted as the Bianchi identity for $F^*_{\mu \nu}(x)$, nor does it imply the existence of a ”dual potential” $\tilde{A}_\mu(x)$ satisfying

$$F^*_{\mu \nu}(x) \equiv \partial_\nu \tilde{A}_\mu(x) - \partial_\mu \tilde{A}_\nu(x) + i\tilde{g}[\tilde{A}_\mu(x), \tilde{A}_\nu(x)],$$ \hspace{1cm} (13)

in parallel to (9). This result means that the dual symmetry of the Yang–Mills theory under the Hodge star operation does not hold. So one has to seek a more general form of duality for non–Abelian theories than the Hodge star operation on the field tensor.

2. The Generalized Polyakov Variables

It was shown in Refs. [3], [6] that the classical Yang–Mills theory is symmetric under a generalized dual transform which reduces to the well–known
electric–magnetic duality in the Abelian case. This dual transform is formulated in Refs. [3]- [7] in terms of loop variables similar to those introduced by A.M.Polyakov [8].

Starting with the Dirac phase factor [9]:

$$\Phi[\xi] = P_s \exp[i g \int_0^{2\pi} ds A_\mu(\xi(s)) \dot{\xi}^\mu(s)]$$

(14)

for a parametrized closed loop $\xi(s)$, $s = 0 \rightarrow 2\pi$, one can define the Polyakov variables [8]:

$$F_\mu[\xi|s] = i g \Phi^{-1}[\xi] \frac{\delta \Phi[\xi]}{\delta \xi^\mu(s)}.$$  

(15)

The duality transformation proposed in Refs. [5], [6] operates on the following variables:

$$E_\mu[\xi|s] = \Phi_{\xi}(s, 0) F_\mu[\xi|s] \Phi^{-1}_{\xi}(s, 0),$$

(16)

where

$$\Phi_{\xi}(s_1, s_2) = P_s \exp[i g \int_{s_1}^{s_2} ds A_\mu(\xi(s)) \dot{\xi}^\mu(s)].$$

(17)

Therefore $\Phi[\xi] \equiv \Phi_{\xi}(0, 2\pi)$.

With aim to understand the difference between the quantitites $F_\mu[\xi|s]$ and $E_\mu[\xi|s]$, it is convinient to give some explanations.

The loop derivative in Eq.(15) is defined as

$$\frac{\delta \Phi[\xi]}{\delta \xi^\mu(s)} = \lim_{\Delta \to 0} \frac{\Phi[\xi'] - \Phi[\xi]}{\Delta},$$

(18)

where

$$\xi'^\lambda = \xi'^\lambda(s') + \Delta \delta^\lambda_\mu (s - s').$$

(19)

The $\delta$–function $\delta(s - s')$ is a bump function centred at $s$ with width $\epsilon = s_+ - s_-$ (see [3]).

In contrast to $F_\mu[\xi|s]$, the quantity $E_\mu[\xi|s]$ depends only on a ”segment” of the loop $\xi^\mu(s)$ from $s_-$ to $s_+$. The regularization of $\delta$–function is necessary for the definition of loop derivatives used in this theory.
The quantities $E_\mu[\xi|s]$ constrained by the condition:

$$\frac{\delta E_\mu[\xi|s]}{\delta \xi^\nu} - \frac{\delta E_\nu[\xi|s]}{\delta \xi^\mu} = 0 \quad (20)$$

constitute a set of the curl–free variables valid for the description of Yang–Mills theories revealing properties of the generalized dual symmetry. Here it is necessary to note that, in contrast to the Polyakov variables $F_\mu[\xi|s]$, the variables $E_\mu[\xi|s]$ are gauge dependent quantities. But in spite of this unconvinient property, the variables $E_\mu[\xi|s]$ are more useful for studying the generalized duality.

The authors of Refs. [5], [6] consider also the dual variables $\tilde{E}_\mu[\eta|t]$ defined by the following relation:

$$\omega^{-1}(\eta(t))\tilde{E}_\mu[\eta|t]\omega(\eta(t)) =$$

$$-\frac{2}{K}\epsilon_{\mu\nu\rho\sigma}\eta^\nu(t) \int \delta \xi ds E^\rho[\xi|s] \dot{\xi}^\sigma(s) \dot{\xi}^{-2}(s) \delta(\xi(s) - \eta(t)) \quad (21)$$

where $K$ is an (infinite) normalization constant:

$$K = \int_0^{2\pi} ds \Pi_{s', s} d^4\xi(s'). \quad (22)$$

The integral in Eq.(21) is over all loops and over all points of each loop, and the factor $\omega(x)$ is just a rotational matrix allowing for the change of local frames between the two sets of variables.

As it was shown in Refs. [5], [6], the expression (21) reduces to the Hodge star operation in the Abelian case, but for a non–Abelian theory they are in general different.

The usual Yang–Mills action

$$S_0 = -\frac{1}{16\pi} \int d^4x Tr(F_{\mu\nu}F^{\mu\nu}) \quad (23)$$

can be expressed in terms of the new variables:

$$S_0 = -\frac{1}{4\pi K} \int \delta \xi ds Tr(E_\mu E^\mu) \dot{\xi}^{-2}, \quad (24)$$
or dually:
\[ \tilde{S}_0 = S_0 = -\frac{1}{4\pi K} \int \delta \eta dt Tr(\tilde{E}_\mu \tilde{E}^\mu) \tilde{\eta}^{-2}. \] (25)

The quantities \( \tilde{E}_\mu[\xi|s] \) are also described by Eqs. (14)-(17) with the following replacements:
\[ g \to \tilde{g}, \quad s \to t, \quad \xi^\mu(t) \to \eta^\mu(t), \quad \Phi[\xi] \to \tilde{\Phi}[\eta], \quad \Phi_\xi(s_1,s_2) \to \tilde{\Phi}_\eta(\eta_1,\eta_2), \]
\[ F_\mu[\xi|s] \to \tilde{F}_\mu[\xi|s], \quad E_\mu[\xi|s] \to \tilde{E}_\mu[\xi|s]. \] (26)

Summarizing the results of Refs. [5], [6], we can formulate such items:
1. \( E_\mu[\xi|s] \) is derivable from a local potential \( A_\mu(x) \) if and only if the (loop) curl of \( E_\mu[\xi|s] \) vanishes (see Eq. (20)).
2. The constrained action contains the Lagrange multipliers \( W^{\mu\nu}[\xi|s] \):
\[ S = S_0 + \int \delta \xi ds Tr\{W^{\mu\nu}[\xi|s](\delta E_\mu[\xi|s]/\delta \xi^\nu(s) - \delta E_\nu[\xi|s]/\delta \xi^\mu(s))\} \] (27)
and implies
\[ \frac{\delta E_\mu}{\delta \xi^\mu(s)} = 0, \] (28)
which is equivalent to the Yang–Mills equation (11).
3. The Eq. (28) is equivalent to the dual variables \( \tilde{E}_\mu[\eta|t] \) being curl–free, which (according to the point 1) is equivalent to the existence of a local potential \( \tilde{A}_\mu(x) \):
\[ \tilde{F}_{\mu\nu}(x) = \partial_\nu \tilde{A}_\mu(x) - \partial_\mu \tilde{A}_\nu(x) + i \tilde{g}[\tilde{A}_\mu(x), \tilde{A}_\nu(x)], \] (29)
obeying the ”dual Yang–Mills equation”:
\[ \tilde{D}^\nu \tilde{F}_{\mu\nu}(x) = 0, \] (30)
where
\[ \tilde{D}_\nu = \partial_\nu - i \tilde{g}[\tilde{A}_\nu(x), \ ] \] (31)
is the ”dual covariant derivative”.

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4. The local dual potential \( \tilde{A}_\mu(x) \) can be expressed in terms of the Lagrange multipliers \( W^{\mu\nu} \) (see [3], [4]) as:

\[
\tilde{A}_\mu(x) = \int \delta \xi ds \epsilon_{\mu\nu\rho\sigma} \omega(\xi(s)) W^{\rho\sigma}[\xi|s] \omega^{-1}(\xi(s)) \dot{\xi}^\nu(s) \dot{\xi}^{-2}(s) \delta(\xi(s) - x).
\]  

(32)

5. The following charge quantization condition exists in theory [5]:

\[
g \tilde{g} = 4\pi n, \quad n \in \mathbb{Z}.
\]

(33)

3. The Zwanziger–type Action for non–Abelian Theories

Following the idea of Zwanziger [10], [11] (see also [12], [13]) to describe symmetrically dual and non-dual Abelian fields covariantly interacting with magnetic and electric currents (respectively), we suggest to consider the generalized Zwanziger formalism for non-Abelian theories. The action of such theories is based on the Chan–Tsou generalized dual symmetry and has the following form:

\[
S = -\frac{2}{K} \int \delta \xi ds \left\{ \text{Tr} \left( E^\mu[\xi|s] E_\mu[\xi|s] \right) + \text{Tr} \left( \tilde{E}^\mu[\xi|s] \tilde{E}_\mu[\xi|s] \right) \right. \\
+ i \text{Tr} \left( E^\mu[\xi|s] \tilde{E}^{(d)}_\mu[\xi|s] \right) + i \text{Tr} \left( \tilde{E}^\mu[\xi|s] E^{(d)}_\mu[\xi|s] \right) \left. \right\} \dot{\xi}^{-2}(s) + S_{gf},
\]

(34)

where \( S_{gf} \) is the gauge–fixing action. The choice

\[
S_{gf} = \frac{2}{K} \int \delta \xi ds \left[ M_A^2 (\dot{\xi} \cdot A)^2 + M_B^2 (\dot{\theta} \cdot A)^2 \right] \dot{\xi}^{-2}
\]

(35)

excludes ghosts in the theory.

In Eq.(34) we have used the generalized dual operations:

\[
E^{(d)}_\mu[\xi|s] =
\]

\[
-\frac{2}{K} \epsilon_{\mu\nu\rho\sigma} \dot{\xi}^\nu \int \delta \eta dt \omega(\eta(t)) E^\rho[\eta|t] \omega^{-1}(\eta(t)) \dot{\eta}^\sigma(t) \dot{\eta}^{-2} \delta(\eta(t) - \xi(s)),
\]

(36)
\[ E_{\mu}^{(d)}[\xi|s] = \]

\[ \frac{2}{K} \epsilon_{\mu\nu\rho\sigma} \dot{\xi}^\nu \int \delta \eta dt \omega^{-1} (\eta(t)) \tilde{E}^\rho[\eta|t] \omega(\eta(t)) \dot{\eta}^\sigma(t) \dot{\eta}^{-2} \delta (\eta(t) - \xi(s)). \]  (37)

From action (34) we have the following equations of motions:

\[ \frac{\delta E_\mu[\xi|s]}{\delta \xi^\mu(s)} = 0, \]  (38)

\[ \frac{\delta \tilde{E}_\mu[\xi|s]}{\delta \xi^\mu(s)} = 0. \]  (39)

In the Abelian case, the following relations are easily obtained:

\[ E_\mu[\xi|s] = F_{\mu\nu}(\xi(s)) \dot{\xi}^\nu(s), \]

\[ \tilde{E}_\mu[\xi|s] = F^*_{\mu\nu}(\xi(s)) \dot{\xi}^\nu(s), \]

\[ E_\mu^{(d)}[\xi|s] = -\frac{1}{2} \epsilon_{\mu\nu\rho\sigma} \dot{\xi}_\nu F_{\rho\sigma} = -\dot{\xi} \cdot (\partial \wedge A)^* \]  (42)

and

\[ \tilde{E}_\mu^{(d)}[\xi|s] = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} \dot{\xi}^\nu \tilde{F}_{\rho\sigma} = \dot{\xi} \cdot (\partial \wedge \tilde{A})^*. \]  (43)

Introducing a unit vector:

\[ n^\mu_s \equiv n^\mu(s) = \dot{\xi}^\mu / \sqrt{\dot{\xi}^2}, \]  (44)

tangential to the loop \( \xi^\mu = \xi^\mu(s) \), we obtain the following action for Abelian fields:

\[ S = -\frac{2}{K} \int \delta \xi ds \left\{ (n_s \cdot [\partial \wedge A])^2 + (n_s \cdot [\partial \wedge \tilde{A}])^2 + \right. \]

\[ i(n_s \cdot [\partial \wedge A])(n_s \cdot [\partial \wedge \tilde{A}]^*) - i(n_s \cdot [\partial \wedge \tilde{A}]) (n_s \cdot [\partial \wedge A]^*) \right\}, \]  (45)

where

\[ (n \cdot [A \wedge B]) = n_\nu (A_\mu B_\nu - A_\nu B_\mu). \]  (46)
The Zwanziger action for Abelian fields \[10\] follows immediately from the action (45) if we choose \(n_\mu\) as a direction of a "frozen string" imagined as an unclosed loop of the fixed direction in the 4–space:

\[
S = -\frac{1}{2} \int d^4 x \{ (n \cdot [\partial \wedge A])^2 + (n \cdot [\partial \wedge \tilde{A}])^2 +
\]

\[
i(n \cdot [\partial \wedge A])(n \cdot [\partial \wedge \tilde{A}]) - i(n \cdot [\partial \wedge \tilde{A}]) (n \cdot [\partial \wedge A])\}.
\]

(47)

Let us return to the non-Abelian theories. In the light of the regularization procedure considered in this paper, we have only the following relations:

\[
\lim_{\epsilon \to 0} E_\mu[\xi|s] = F_{\mu\nu} \dot{\xi}^\nu(s),
\]

(48)

\[
\lim_{\epsilon \to 0} \tilde{E}_\mu[\xi|s] = \tilde{F}_{\mu\nu} \dot{\xi}^\nu(s).
\]

(49)

Now

\[
\lim_{\epsilon \to 0} E_\mu^{(d)}[\xi|s] \neq -\frac{1}{2} \epsilon_{\mu\rho\sigma} \dot{\xi}_\nu F_{\rho\sigma},
\]

\[
\lim_{\epsilon \to 0} \tilde{E}_\mu^{(d)}[\xi|s] \neq \frac{1}{2} \epsilon_{\mu\rho\sigma} \dot{\xi}_\nu \tilde{F}_{\rho\sigma},
\]

(50)

what means that for non-Abelian theories the reduction to the Hodge star relation does not go through.

It is obvious now that there exists the second local gauge symmetry for non–Abelian theories, which can be denoted as \(\widehat{SU(N)}\) to distinguish it from the initial local symmetry of \(A_\mu\). As a result, we deal with a doubling of the gauge symmetry from \(SU(N)\) to

\[
SU(N) \times \widehat{SU(N)}
\]

(51)

without extra degrees of freedom.

4. Conclusions

It was shown in the present paper that the Zwanziger–type action can be constructed for non–Abelian theories revealing the generalized duality
symmetry. In the Abelian limit this action corresponds to the Zwanziger formalism for quantum electro–magneto dynamics (QEMD). It was emphasized that although the generalized duality transformation is rather complicated, it is explicit in terms of the Polyakov loop space variables type.

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