A Simplified Yield Criterion of the Orthotropic Material Considering the Crystallographic Texture

Y Erisov 1,2, F Grechnikov 2 S Surudin1 and V Razhivin 1

1Metal Forming Department, Samara National Research University, Moskovskoye shosse 34, Samara, 443086, Russia, yaroslav.erisov@mail.ru (corresponding author)
2Samara Federal Research Center of Russian Academy of Sciences, Studencheskii perelok 3A, Samara, 443001, Russia

Abstract. The proposed linearized yield function considers the crystal lattice constants and the parameters of crystallographic orientation of material and has simplified form, which makes it suitable for technological calculations of sheet metal forming processes. It is shown that the Lode coefficient is determined not only by the stress state, as in the case of an isotropic material, but also depends on anisotropy parameters, which reflects the elastic constants of the crystal lattice and the crystallographic orientation of the structure.

1. Introduction
The current development stage of the technologies for aircraft production, automobile construction, shipbuilding and other industries is characterized by the continuous improvement of the existing constructional materials and technological processes of their forming, as well as by the research and development of new ones. The development of a rational, science-based technology in the metal forming processes is primarily concerned with the need of a detailed study of the structure (texture) formation, the material properties and the most extensive application of these properties in engineering analysis.

One of the specific characteristics inherent in the majority of materials is the anisotropy of their properties, which is caused by the crystallographic structure and texture formation under high plastic strains [1]. However, the assumption of the material isotropy is still being used as one of the main hypotheses in the analysis and calculations of metal forming processes. Although this hypothesis facilitates the solution of numerous metal forming problems, it does not actually meet the real deformation conditions.

The abandonment of the assumption of the medium isotropy allows to generalize the metal forming theory and to use the anisotropy effectively in technological processes [2-3]. Despite the fact that over the recent years greater attention is paid to the theoretical and experimental research in the field of anisotropic plastic deformations, there is still a number of unsolved problems. These issues are related primarily to the further development of the anisotropic plasticity theory in a form suitable for the engineering and technological analysis.

Till nowadays the most widely used yield criterion was proposed by R. Hill [4]. Verification of these yield function proved its applicability only for steels [5], so for other materials, R. Hill proposed a non-quadratic yield criterion [6]. However, it does not take into account the shear stresses and,
therefore, it is only applicable in the case when the direction of the principal stresses coincides with the anisotropy axes. This disadvantage was eliminated by F. Barlat [7]. For the general stress state, F. Barlat developed the yield function, in which the stress tensor is expressed in terms of the coordinates parallel to the anisotropy axes [8]. Using two linear transformations of stress tensor F. Barlat obtained the anisotropic criterion for plane and general stress state for the materials with the equal yield strengths of tension and compression [9-10].

Another approach is the replacement of initial isotropic invariants of the stress deviator by the similar generalized anisotropic invariants, derived on the basis of the theory of representations of tensor functions [11-12]. Thus, the modern yield functions in contrast to the earlier ones, which were derived to describe the behavior of any metal, allow taking into account the peculiarities of the specific materials. It should be noted that the high accuracy of the recently proposed criterions is achieved by a large amount of the anisotropy coefficients (up to 18), determination of which involves numerous mechanical tests at different stress states [13].

Though the applied anisotropy coefficients characterize the anisotropy of plastic deformations, they do not take into account the reason of anisotropy, i.e. the crystallographic texture. Thus, the mentioned yield functions, on the one hand, allow describing the plastic flow of anisotropic materials. On the other hand, they do not allow carrying out technological analysis of metal forming processes considering the crystallographic texture. As a result, it is impossible to determine the composition of crystallographic texture in terms of the requirements of certain metal forming operations. As shown in [14], to embed the parameters of crystallographic texture into the yield function it can be used the hypothesis of proportionality of the elastic and plastic material deviators. Alternative approach is to use the specific distortion strain energy to derive the yield criterion and the constitutive equations, which consider the crystal lattice constants and parameters of the crystallographic texture [15-19]. The main practical significance of this yield criterion is the possibility to predict the effect of crystallographic texture on the metal flow during plastic deformation. Consequently, it allows designing the composition of crystallographic texture depending on the requirements of the metal forming processes.

However, in most cases, for the analysis of metal forming processes it is convenient to use a simplified form of the yield criterion, since, on the one hand, technological calculations do not require high accuracy, and, on the other hand, it is necessary to simplify the derivation of analytical dependencies. In this regard, in this paper it is proposed the linearized yield function for the orthotropic material with consideration of the crystal lattice constants and parameters of the crystallographic texture.

2. General anisotropic yield criterion
Let us use the yield criterion, in the basic equations of which the parameters of the material’s structure are introduced [15]:

\[
\sigma_i = \frac{1}{\sqrt{2}} \left\{ \eta_{12} (\sigma_{11} - \sigma_{22})^2 + \eta_{23} (\sigma_{22} - \sigma_{33})^2 + \eta_{31} (\sigma_{33} - \sigma_{11})^2 \right. \\
+ 4 \left[ \left( \frac{5}{2} - \eta_{12} \right) \sigma_{12}^2 + \left( \frac{5}{2} - \eta_{23} \right) \sigma_{23}^2 + \left( \frac{5}{2} - \eta_{31} \right) \sigma_{31}^2 \right] \right\}^{1/2},
\]

where \( \sigma_i \) – equivalent stress; \( \sigma_{ij} \) – the components of the stress tensor; (i, j = 1, 2, 3; 1 – the rolling direction, 2 – transverse direction; 3 – the normal direction); \( \eta_{ij} \) – generalized anisotropy parameters:

\[
\eta_{ij} = 1 - \frac{15(A'-1)}{3+2A'} \left( \Delta_i + \Delta_j \right) \left( \frac{1}{\Delta_k} - \frac{1}{5} \right).
\]
A' – the parameter of anisotropy of the crystal lattice:

\[ A' = \frac{S_{1111}^r - S_{1122}^r}{2S_{2323}^r}; \]  

(3)

\[ S_{ijkl}^r \] – elastic constants of the crystal lattice; \( \Delta_i \) – parameters of crystallographic orientation of the structure:

\[ \Delta_i = \sum_{\{hkl\}\{uvw\}} p^{\{hkl\}\{uvw\}} \Delta_i^{\{hkl\}\{uvw\}}; \]  

(4)

\[ p^{\{hkl\}\{uvw\}} \] – weight fraction of \( i \)-th component \( \{hkl\}\{uvw\} \); \( \Delta_i^{\{hkl\}\{uvw\}} \) – orientation factor of ideal crystallographic orientation \( \{hkl\}\{uvw\} \):

\[ \Delta_i^{\{hkl\}\{uvw\}} = \frac{h_i^2 k_i^2 + k_i^2 l_i^2 + l_i^3 h_i^2}{(h_i^2 + k_i^2 + l_i^2)^2}; \]  

(5)

\( h_i, k_i, l_i \) – Miller indices determining the \( i \)-th direction in the crystal relative to the coordinate system associated with the sample.

**Linearized anisotropic yield criterion**

In solving specific problems of metal forming, a simplified linearized form of the yield criterion is widely used:

\[ \sigma_{\text{max}} - \sigma_{\text{min}} = \beta \sigma_i, \]  

(6)

where \( \sigma_{\text{max}} \) and \( \sigma_{\text{min}} \) are the major and minor principal stresses (\( \sigma_{\text{max}} > \sigma_{\text{av}} > \sigma_{\text{min}} \)). \( \beta \) is Lode coefficient, which characterizes the influence of the average stress \( \sigma_{\text{av}} \) and is determined by the magnitude of the stress state factor \( \beta = f(\nu_{\sigma}) \):

\[ \nu_{\sigma} = \frac{\sigma_{\text{av}} - \sigma_{\text{max}} + \sigma_{\text{min}}}{\sigma_{\text{max}} - \sigma_{\text{min}}} \cdot \frac{2}{2}. \]  

(7)

If an isotropic material is considered, then \( \beta \) does not depend on the choice of the directions of the principal axes and is determined only by the stress state.

In case of an orthotropic medium Lode coefficient is dependent on the direction of the stresses relative to the principal anisotropy axes. To derive a simplified expression of the yield criterion of an orthotropic medium in the form (6), it is necessary to consider possible options for the mutual arrangement of the principal stresses relative to the anisotropy axes. For the case \( \sigma_1 > \sigma_2 > \sigma_3 \) Eq. (7) can be rewritten in the following form:

\[ \sigma_2 = \nu_{\sigma} \left( \frac{\sigma_1 - \sigma_3}{2} + \frac{\sigma_1 + \sigma_3}{2} \right). \]  

(8)

Substituting Eq. (8) into the yield criterion (1), we obtain:

\[ \sigma_1 - \sigma_3 = \frac{2\sqrt{2}\sigma_i}{\sqrt{\eta_2 (\nu_{\sigma} - 1)^2 + \eta_3 (\nu_{\sigma} + 1)^2 + 4\eta_3^2}}. \]  

(9)
Then the Lode coefficient is determined by the dependence:

\[
\beta = \frac{2\sqrt{2}}{\sqrt{\eta_{12} (v_{\sigma} - 1)^2 + \eta_{23} (v_{\sigma} + 1)^2 + 4\eta_{31}}}.
\]

(10)

Thus, the value of the Lode coefficient is determined not only by the magnitude of the stress state factor \(v_{\sigma}\), as in the case of an isotropic material, but also depends on parameters \(\eta_{ij}\), which reflects the elastic constants of the crystal lattice and the crystallographic orientation of the structure.

Similarly, considering other combinations of arrangement of stresses \(\sigma_{\text{max}}, \sigma_{\text{min}}\) and \(\sigma_{\text{av}}\) with respect to the principal anisotropy axes 1, 2, and 3, we obtain the remaining expressions for calculating the Lode coefficient \(\beta\) (Table 1). The six cases of expressions for the Lode coefficient, which are indicated in Table 1, can be represented as a general expression:

\[
\beta = \frac{2\sqrt{2}}{\sqrt{\eta_{\text{max},\text{av}} (v_{\sigma} - 1)^2 + \eta_{\text{av},\text{min}} (v_{\sigma} + 1)^2 + 4\eta_{\text{min},\text{max}}}}.
\]

(11)

Here \(\eta_{ij}\) are the generalized anisotropy parameters, where the indices \(i, j\) are determined by the location of the stresses relative to the principal anisotropy axes. For example, when \(\sigma_2 > \sigma_3 > \sigma_1\) the indices are following: \(\text{max} = 2\), \(\text{min} = 1\) and \(\text{av} = 3\). The order of the indices does not matter: \(\eta_{ij} = \eta_{ji}\). In the case of isotropic material \((\eta_{ij} = 1)\), Eq. (11) takes the traditional form:

\[
\beta = \frac{2}{\sqrt{3 + v_{\sigma}^2}}.
\]

(12)

**Table 1. Expressions for calculating the Lode coefficient**

| Stress state          | \(\beta = f(\eta_{ij}, v_{\sigma})\) |
|-----------------------|--------------------------------------|
| \(\sigma_2 > \sigma_1 > \sigma_3\) | \[
\beta = \frac{2\sqrt{2}}{\sqrt{\eta_{12} (v_{\sigma} - 1)^2 + \eta_{31} (v_{\sigma} + 1)^2 + 4\eta_{23}}}.
\]
| \(\sigma_3 > \sigma_1 > \sigma_2\) | \[
\beta = \frac{2\sqrt{2}}{\sqrt{\eta_{31} (v_{\sigma} - 1)^2 + \eta_{23} (v_{\sigma} + 1)^2 + 4\eta_{12}}}.
\]
| \(\sigma_1 > \sigma_2 > \sigma_3\) | \[
\beta = \frac{2\sqrt{2}}{\sqrt{\eta_{12} (v_{\sigma} - 1)^2 + \eta_{23} (v_{\sigma} + 1)^2 + 4\eta_{31}}}.
\]
| \(\sigma_3 > \sigma_2 > \sigma_1\) | \[
\beta = \frac{2\sqrt{2}}{\sqrt{\eta_{23} (v_{\sigma} - 1)^2 + \eta_{12} (v_{\sigma} + 1)^2 + 4\eta_{31}}}.
\]
| \(\sigma_1 > \sigma_3 > \sigma_2\) | \[
\beta = \frac{2\sqrt{2}}{\sqrt{\eta_{31} (v_{\sigma} - 1)^2 + \eta_{23} (v_{\sigma} + 1)^2 + 4\eta_{12}}}.
\]
| \(\sigma_2 > \sigma_3 > \sigma_1\) | \[
\beta = \frac{2\sqrt{2}}{\sqrt{\eta_{12} (v_{\sigma} - 1)^2 + \eta_{31} (v_{\sigma} + 1)^2 + 4\eta_{23}}}.
\]
3. Results and discussion

Let us determine the value of the Lode coefficient for the most common cases of stress state (uniaxial compression $\nu_{\sigma} = 1$; pure shear $\nu_{\sigma} = 0$; uniaxial tension $\nu_{\sigma} = -1$):

$$
\nu_{\sigma} = 1, \beta = \frac{\sqrt{2}}{\sqrt{\eta_{\text{av}, \text{min}} + \eta_{\text{min}, \text{max}}}};
$$

$$
\nu_{\sigma} = 0, \beta = \frac{2\sqrt{2}}{\sqrt{\eta_{\text{max}, \text{av}} + \eta_{\text{av}, \text{min}} + 4\eta_{\text{min}, \text{max}}}};
$$

$$
\nu_{\sigma} = -1, \beta = \frac{\sqrt{2}}{\eta_{\text{max}, \text{av}} + \eta_{\text{min}, \text{max}}}.
$$

It follows from Eq. (13) that in the case of an orthotropic material, the value of the Lode coefficient $\beta$ on a interval $\nu_{\sigma} \in [-1; 1]$ can vary within other limits than for an isotropic medium, when $\beta \in \left[ 1; \frac{2}{\sqrt{3}} \right]$, and depends on the elastic constants of the crystal lattice and texture parameters.

When analyzing the sheet metal forming processes, in case of relatively small average stress, the $\beta$ value is taken equal to the mean value of the Lode coefficient $\beta = f \left( \eta_{ij}, \nu_{\sigma} \right)$ on the interval $\nu_{\sigma} \in [-1; 1]$:

$$
\bar{\beta} = \frac{1}{2} \int_{-1}^{1} f \left( \eta_{ij}, \nu_{\sigma} \right) d\nu_{\sigma}.
$$

Substituting Eq. (11) in (14), we obtain:

$$
\bar{\beta} = \frac{\sqrt{2}}{\sqrt{\eta_{\text{max}, cp} + \eta_{cp, \text{min}}}} \ln \frac{\eta_{12}\eta_{23}\eta_{31}N + \eta_{cp, \text{min}}^2}{\eta_{12}\eta_{23}\eta_{31}N + \eta_{max, cp}^2 - \eta_{cp, \text{min}}},
$$

where $N = \frac{1}{\eta_{12}} + \frac{1}{\eta_{23}} + \frac{1}{\eta_{31}}$.

It is easy to show that the $\bar{\beta}$ value depends only on the location of the average stress $\sigma_{av}$ relative to the principal anisotropy axes, the location of $\sigma_{\text{max}}$ and $\sigma_{\text{min}}$ can be arbitrary. Consider two cases:

1) $\sigma_1 > \sigma_2 > \sigma_3$, then Eq. (15) can be written as follows

$$
\bar{\beta}_1 = \frac{\sqrt{2}}{\eta_{12} + \eta_{23}} \ln \frac{\eta_{12}\eta_{23}\eta_{31}N + \eta_{23}^2 + \eta_{23}}{\eta_{12}\eta_{23}\eta_{31}N + \eta_{12}^2 - \eta_{12}};
$$

2) $\sigma_3 > \sigma_2 > \sigma_1$, then Eq. (15) can be written as follows

$$
\bar{\beta}_2 = \frac{\sqrt{2}}{\eta_{12} + \eta_{23}} \ln \frac{\eta_{12}\eta_{23}\eta_{31}N + \eta_{12}^2 + \eta_{12}}{\eta_{12}\eta_{23}\eta_{31}N + \eta_{23}^2 - \eta_{23}}.
$$

Equating the right-hand sides of Eq. (17) and (18), we obtain that $\bar{\beta}_1 = \bar{\beta}_2$.

In sheet metals, crystallographic orientations are not random but show preferred orientations around one or several components. After rolling, for FCC alloy sheets, these components are mainly $\{112\}<111>$ (copper), $\{110\}<112>$ (brass), $\{123\}<634>$ (S) and $\{100\}<011>$ (rotated cube) orientations, with the relative amounts dependent on rolling conditions. After annealing, depending on
the process conditions, the resulting texture mainly composed of the {100}<001> (cube) and
{110}<001> (Goss) recrystallization components.
Consider the sheet from copper for which components of compliance tensor $S'_{ijkl}$ are

$S'_{1111} = 15.0$ TPa$^{-1}$; $S'_{1222} = -6.30$ TPa$^{-1}$ and $S'_{2323} = 3.33$ TPa$^{-1}$ [16], i.e. $A' = 1.207$ (Eq. (3)). The parameters of crystallographic texture and the generalized anisotropy parameters calculated using Eq. (5) and (2)
for stated components are listed in Table 2.

Table 2 The parameters of crystallographic texture and the generalized anisotropy parameters of single ideal components

| Component       | Orientation  | $\Delta_1$ | $\Delta_2$ | $\Delta_3$ | $\eta_{12}$ | $\eta_{23}$ | $\eta_{31}$ |
|-----------------|--------------|------------|------------|------------|-------------|-------------|-------------|
| Copper          | {112}<111>   | 0.333      | 0.250      | 0.250      | 0.924       | 1.019       | 0.924       |
| Brass           | {110}<112>   | 0.250      | 0.333      | 0.250      | 0.924       | 0.924       | 1.019       |
| S               | {123}<634>   | 0.281      | 0.278      | 0.250      | 0.938       | 0.973       | 0.970       |
| Rotated cube    | {100}<011>   | 0.250      | 0.250      | 0.0        | 0.828       | 1.115       | 1.115       |
| Cube            | {100}<001>   | 0.0        | 0.0        | 0.0        | 1.115       | 1.115       | 1.115       |
| Goss            | {110}<001>   | 0.0        | 0.250      | 0.250      | 1.115       | 0.828       | 1.115       |
| Isotropy        | -            | 0.20       | 0.20       | 0.20       | 1.0         | 1.0         | 1.0         |

Table 3 shows the results of calculating the mean Lode coefficient $\bar{\beta}$ for typical crystallographic orientations.

Table 3. The value of the mean Lode coefficient $\bar{\beta}$ for typical crystallographic orientations

| Component       | Average stress $\sigma_{av}$ |
|-----------------|-----------------------------|
|                 | $\sigma_1$ | $\sigma_2$ | $\sigma_3$ |
| Copper          | 1.167     | 1.376      | 1.376      |
| Brass           | 1.376     | 1.167      | 1.376      |
| S               | 1.238     | 1.245      | 1.315      |
| Rotated cube    | 0.960     | 0.960      | 1.387      |
| Cube            | 0.842     | 0.842      | 0.842      |
| Goss            | 1.387     | 0.960      | 0.960      |
| Isotropy        | 1.099     | 1.099      | 1.099      |

From the point of view of the analysis of sheet metal forming processes, the case where the average stress is directed along the normal to the sheet surface is of most interest. As can be seen from Table 3, in this case, the crystallographic orientations of the deformation type provide higher values of the mean Lode coefficient, and hence – higher flow stresses, in comparison with the isotropic case, and the orientations of recrystallization type – lower.

4. Conclusions
The phenomenological yield criterions applied in technological analysis of the metal forming processes do not take into account the crystallographic texture of materials. The proposed linearized
yield function considers the crystal lattice constants and the parameters of crystallographic orientation of material and has simplified form, which makes it suitable for technological calculations of sheet metal forming processes. The linearized yield criterion is applicable to orthotropic materials with cubic crystal lattice in case when texture does not change, for example, under small elastic-plastic strains. The proposed yield criterion allows determining the characteristics of the metal forming processes considering the texture of sheet metal. Consequently, it allows designing the composition of crystallographic texture depending on the requirements of the metal forming processes or products performance.

The study was carried out as part of a state assignment (FSSS-2020-0016).

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