Secure list decoding
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Abstract
In this paper, we propose a new concept of secure list decoding. While the conventional list decoding requires that the list contains the transmitted message, secure list decoding requires the following additional security conditions. The first additional security condition is the impossibility of the correct decoding. This condition can be trivially satisfied when the transmission rate is larger than the channel capacity. The other additional security condition is the impossibility for the sender to estimate another element of decoded list except for the transmitted message. This protocol can be used for anonymous auction, which realizes the anonymity for bidding.

Index Terms
list decoding; anonymous auction; security condition; capacity region

I. INTRODUCTION

Independently by Elias [1] and Wozencraft [2] as relaxation of the notion of the decoding process, list decoding was introduced as the method to allow more than one element as candidates of the message sent by the encoder, in the decoder. When one of these elements coincides with the true message, the decoding is regarded as successful. In this formulation, Nishimura [3] obtained the channel capacity by showing its strong converse part. That is, he showed that the transmission rate is less than the conventional capacity plus the rate of number of list. Then, the reliable transmission rate does not increase even when list decode is allowed if the number of list does not increase exponentially. In the non-exponential case, these results was generalized by Ahlswede [4]. Further, the paper [5] showed that the upper bound of capacity by Nishimura can be attained even if the number of list increases exponentially. However, the merit of the increase of the number of list was not discussed sufficiently.

In this paper, we propose a new concept of secure list decoding. To explain this protocol, we consider the following anonymous auction scenario, which realizes the anonymity for bidding. M players participate auction for an item dealt by Bob, and they have their distinct ID from 1 to M.

(i) (Bidding) A player, Alice bids with her ID M. Here, she sends M via noisy channel. Then, Bob receives L ID numbers M1, . . . , ML as the list. The list is required to contain her ID M.

(ii) (Purchasing) Assume that Alice’s bidding price is highest. She purchases the item from Bob by showing her ID M.

This scenario has the following requirements.

(a) Bob wants to identify whether the person to purchase the item is the same as the person to bid the highest price. That is, M needs to be one of M1, . . . , ML.
(b) Alice wants to hide her ID M at the bidding step (i). Hence, she will not be identified by Bob when she loses this auction.
(c) Bob wants to avoid the situation that two players show Bob correct ID at purchasing Step (ii). That is, Alice cannot find another element among M1, . . . , ML except for M.

The requirement (a) is the condition for the requirement for the conventional list decoding while the requirements (b) and (c) are not considered in the conventional list decoding. In this paper, as a new concept to satisfy these conditions, we propose secure list decoding by imposing the following two

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The strong converse part is the argument that the average error goes to 1 if the code has a transmission rate over the capacity.
additional conditions to the list decoding. The first additional security condition is the impossibility of the correct decoding. This condition can be trivially satisfied when the transmission rate is larger than the channel capacity due to the strong converse property. The other additional security condition is the impossibility for the sender to estimate another element of decoded list except for the transmitted message. In fact, we might use an authentication protocol to identify Alice [6]. In this case, if Alice gives the key for the authentication to the third party, the third party can claim to Bob that he is also the winner of this auction. To avoid this type of spoofing, we need to use ID number. That is, the above anonymous auction scenario realizes a kind of authentication which satisfies the anonymity and forbids spoofing even when Alice colludes the third party.

In this paper, we formulate secure list decoding, and define various types of capacity regions for secure list decoding. Then, we calculate these capacity regions under some condition.

This paper is structured as follows. Section II-A gives the formulation of secure list decoding. Section II-B explains the relation with bit commitment. Section IV prepares several information quantities. Section V states the main result by deriving the capacity regions. Section VI proves the direct part.

II. PROBLEM SETTING

A. Our setting

To realize the requirements (a), (b), and (c) mentioned in Section I, given a channel \( W \) from the discrete system \( \mathcal{X} \) to the other system \( \mathcal{Y} \), we consider the following protocol with integers \( L < M \) and security parameters \( \epsilon_A, \delta_B, \delta_C \). For \( x \in \mathcal{X} \) and a distribution on \( \mathcal{X} \), we define the distribution \( W_x \) and \( W_P \) on \( \mathcal{Y} \) as:

\[
W_x(y) := W(y|x) \quad \text{and} \quad W_P(y) := \sum_{x \in \mathcal{X}} P(x)W(y|x).
\]

Alice sends her ID \( M \in \mathcal{M} := \{1, \ldots, M\} \) via noisy channel \( W \) with a code \( \phi \), which is a map from \( \mathcal{M} \) to \( \mathcal{X} \). Bob recovers the \( L \) messages \( M_1, \ldots, M_L \). The decoder is given by disjoint subsets \( D = \{D_{m_1, \ldots, m_L}\}_{(m_1, \ldots, m_L) \in \mathcal{M}} \) such that \( \bigcup_{(m_1, \ldots, m_L) \in \mathcal{M}} D_{m_1, \ldots, m_L} = \mathcal{Y} \). Then, we impose the following conditions.

(A) Verifiable condition.

\[
\epsilon_A(\phi, D) := \max_{m \in \mathcal{M}} \epsilon_{A,m}(\phi(m), D) \leq \epsilon_A \tag{1}
\]

\[
\epsilon_{A,m}(x, D) := 1 - \sum_{m_1, \ldots, m_L \setminus \{m_1, \ldots, m_L\} \ni m} W_x(D_{m_1, \ldots, m_L}) \tag{2}
\]

(B) Non-decodable condition. There is no decoder \( \{\hat{D}_m\}_{m \in \mathcal{M}} \) such that

\[
\delta_B(\phi) := \min_{\hat{D}} \sum_{m=1}^M \frac{1}{M} \delta_{B,m}(\phi(m), \hat{D}) \leq \delta_B \tag{3}
\]

\[
\delta_{B,m}(\phi(m), \hat{D}) := W_{\phi(m)}(\hat{D}_m) \tag{4}
\]

In this paper, when a decoder has only one outcome as an element of \( \mathcal{M} \) like \( \{\hat{D}_m\}_{m \in \mathcal{M}} \), it is called a single-element decoder.

(C) Non-cheating condition for honest Alice.

\[
\delta_C(\phi, D) := \max_{m \in \mathcal{M}} \delta_{C,m}(\phi(m), D) \leq \delta_C \tag{5}
\]

\[
\delta_{C,m}(x, D) := \max_{m' \setminus \{m\} \in \mathcal{M}} \sum_{m_1, \ldots, m_L \setminus \{m_1, \ldots, m_L\} \ni m'} W_x(D_{m_1, \ldots, m_L}) \tag{6}
\]

Now, we discuss how the code \( (\phi, D) \) can be used for the task explained in Section I. Assume that Alice sends her ID \( M \) to Bob by using the encoder \( \phi \) via noisy channel \( W \) and Bob gets the list \( M_1, \ldots, M_L \) by applying the decoder \( D \) as Step (i). At Step (ii), Alice shows her ID \( M \) to Bob. Verifiable condition (A) guarantees that \( M \) belongs to Bob’s list. Hence, requirement (a) is satisfied. Non-decodable condition (B)
forbids Bob to identify Alice’s ID at Step (i), hence it guarantees requirement (b). In fact, if \( m \) is Alice’s ID and \( x_0 \neq m \) is an element such that \( \delta_{C,m}(x_0, D) \) is close to 1, Alice can make the following cheating. Since Alice knows that \( x_0 \) belongs to Bob’s decoded list, she finds the third person whose ID is \( x_0 \). Then, she tells the third person this fact. At Step (ii), the third person can make spoofing by showing Bob his/her ID. Since Non-cheating condition (C) forbids Alice such a cheating, it guarantees requirement (c). Further, Bob is allowed to decode messages less than \( L \). That is, \( L \) is the maximum number of Bob can list as the candidates of the original message.

However, Condition (C) is the security evaluation for honest Alice who use the correct encoder \( \phi \). Dishonest Alice might send her message by using a different encoder. To cover such a case, we impose the following condition instead of Condition (C).

(D) Non-cheating condition for dishonest Alice.

\[
\delta_D(D) := \max_{m \in \mathcal{M}} \delta_{D,m}(D) \leq \delta_C
\]

\[
\delta_{D,m}(D) := \max_{x \in \mathcal{X}} \left\{ \delta_{C,m}(x, D) \left| \epsilon_{A,m}(x, D) \leq \frac{1}{2} \right. \right\}
\]

In the following, when a code \((\phi, D)\) satisfies conditions (A), (B) and (D), it is called a \((\epsilon_A, \delta_B, \delta_C)\) code. Also, for a code \((\phi, D)\), we denote \( \mathcal{M} \) and \( L \) by \(|(\phi, D)|_1 \) and \(|(\phi, D)|_2 \). Also, we allow stochastic encoder, in which \( \phi(m) \) is a distribution on \( \mathcal{X} \). In this case, for a function \( f \) from \( \mathcal{X} \) to \( \mathbb{R} \), \( f(\phi(m)) \) expresses \( \sum_x f(x)\phi(m)(x) \).

**B. Relation to bit commitment**

If our task is realized and \( \mathcal{M} = \mathbb{F}_2^l \), we can approximately realize bit commitment as follows while it is known that bit commitment can be realized by using noisy channel [7, 8, 9].

Assume that we have a \((\epsilon_A, \delta_B, \delta_C)\) code \((\phi, D)\) with sufficiently small security parameters \( \epsilon_A, \delta_B, \delta_C \). Then, \( X, \mathcal{M}, \) and \( Y \) are variables given in Section II-A with the code \((\phi, D)\). Also, we assume that \( \mathcal{M} \) is subject to the uniform distribution on \( \mathcal{M} = \mathbb{F}_2^l \). Since Bob cannot identify \( M \), we have \( H(M|Y) \geq -\log \delta_B \) [10, Theorem 1][11, Lemma 5.9]. First, to choose a function \( f \) from \( \mathcal{M} \) to \( \mathbb{F}_2 \). We choose surjective homomorphic universal hash function \( F \) from \( M \) to \( \mathbb{F}_2 \) [13]. Then, the universal2 hash lemma [12, 14, 15] guarantees that \( \epsilon' := I(F(M); Y|F) \) is close to zero because \( H(M|Y) \) is sufficiently large.

Now, we choose a surjective homomorphic hash function \( f \) from \( M \) to \( \mathbb{F}_2 \) such that \( I(f(M); Y) \leq \epsilon' \).

Alice has bit \( X = 0 \) or 1. Then, Alice randomly generates messages \( M \in f^{-1}(X) \). Then, Alice sends \( M \) to Bob via the above protocol in the binding phase. In the opening phase, Alice shows \( M \) to Bob. Bob calculates \( f(M) \) and finds the value of \( X \). If \( M \) does not belong to Bob’s decoded message list, Bob consider that Alice makes cheating.

In this scenario, if Alice wants to make cheating, in the opening phase, she has to find another message \( M' \in f^{-1}(X+1) \) such that \( M' \) belongs to Bob’s decoder’s list. However, it is impossible due to Condition (D). Hence, the bit commitment is realized from the code for secure list decoding.

**III. INFORMATION QUANTITIES**

Consider the channel written as the transition matrix \( W \) from \( \mathcal{X} \) to \( \mathcal{Y} \). For \( x \in \mathcal{X} \) and a distribution on \( \mathcal{X} \), we define the distribution \( W_x \) and \( W_P \) on \( \mathcal{Y} \) as \( X_x(y) := W(y|x) \) and \( W_P(y) := \sum_{x \in \mathcal{X}} P(x)W(y|x) \).

\( \mathbb{E}_x \) expresses the average with respect to a variable over the system \( \mathcal{Y} \) under the distribution \( W_x \) and \( \nabla_x \) expresses the variance with respect to a variable over the system \( \mathcal{Y} \) under the distribution \( W_x \). This notation is also applied to the \( n \)-fold extended setting.
We define
\[ C(W) := \max_P I(P, W), \]
\[ I(P, W) := \sum_x P(x) \sum_{y \in \mathcal{Y}} W_x(y)(\log W_x(y) - \log W_P(y)), \]
\[ H(P) := -\sum_x P(x) \log P(x), \]
where the base of logarithm is 2.

For \( x, x' \in \mathcal{X} \), we define
\[ F(x, x'|P) := \sum_y \mathbb{E}_x(\log W_{x'}(Y) - \log W_P(Y)) = D(W_x||W_P) - D(W_x||W_{x'}). \]

Then, we define
\[ \zeta_2(P) := \max_{x \neq x'} \max_{x''} F(x, x''|P) - F(x', x''|P) \]
\[ \zeta_1(P) := \min_{x \neq x'} F(x, x|P) - F(x', x|P) = \min_{x \neq x'} D(W_{x'}||W_x) + D(W_x||W_P) - D(W_{x'}||W_P). \]

In the following, we assume that
\[ V := \max_{x, x' \in \mathcal{X}} \mathbb{E}_x(\log W_{x'}(Y) - \log W_P(Y)) < \infty. \]

For \( s > 0 \), we define
\[ G(s, x|P) := \log \sum_{x' \in \mathcal{X}} P(x') 2^{sF(x, x'|P)} \]
\[ G(s|P) := \sum_{x \in \mathcal{X}} P(x)G(s, x|P). \]

Since the function \( s \mapsto G(s, x|P) \) is strictly convex, the function \( s \mapsto G(s|P) \) is strictly convex. Hence, we have the following lemma.

**Lemma 1:** When \( W_x \neq W_{x'} \) for \( x \neq x' \), The value \( sI(P, W) - G(s|P) - H(P) \) is negative and continuous for \( s \). It converges to zero as \( s \) goes to infinity. Also, \( \sup_{s>0} -R_1 + sR - G(s|P) > 0 \) for \( R_1 < H(P) \).

**Proof:** Since \( F(x, x'|P) > F(x, x'|P) \) for \( x \neq x' \), we have
\[ G(s|P) = \sum_x P(x) \left( \log P(x) + sF(x, x|P) + \log \left( 1 + \sum_{x' \neq x} \frac{P(x')}{P(x)} e^{-s(F(x, x'|P) - F(x, x'|P))} \right) \right). \]

Hence, since \( \sum_x P(x)F(x, x|P) = I(P, W) \),
\[ sI(P, W) - G(s|P) - H(P) = -\sum_x P(x) \log \left( 1 + \sum_{x' \neq x} \frac{P(x')}{P(x)} e^{-s(F(x, x'|P) - F(x, x'|P))} \right) < 0. \]

When \( s \to \infty \), the above value goes to zero.

Since \( -R_1 > -H(P) \), we have \( \lim_{s \to 0} -R_1 + sR - G(s|P) > \lim_{s \to 0} -H(P) + sI(P, W) - G(s|P) = 0 \).

Hence, due to the continuity for \( s \), there exists \( s > 0 \) such that \( -R_1 + sR - G(s|P) > 0 \). Then, the proof is completed. \( \blacksquare \)
IV. Main results

A. Capacity regions

To give the capacity region, we consider \( n \)-fold discrete memoryless extension \( W^n \) of the channel \( W \). A sequence of codes \( \{ (\phi_n, D_n) \} \) is called strongly secure when \( \epsilon_A(\phi_n, D_n) \to 0 \), \( \delta_B(\phi_n) \to 0 \). A sequence of codes \( \{ (\phi_n, D_n) \} \) is called weakly secure when \( \epsilon_A(\phi_n, D_n) \to 0 \), \( \delta_B(\phi_n) \to 0 \), \( \delta_C(\phi_n, D_n) \to 0 \). A rate pair \((R_1, R_2)\) is strongly deterministically (stochastically) achievable when there exists a strongly secure sequence of deterministic (stochastic) codes \( \{ (\phi_n, D_n) \} \) such that \( \frac{1}{n} \log |(\phi_n, D_n)|_1 \to R_1 \) and \( \frac{1}{n} \log |(\phi_n, D_n)|_2 \to R_2 \). A rate pair \((R_1, R_2)\) is weakly deterministically (stochastically) achievable when there exists a weakly secure sequence of deterministic (stochastic) codes \( \{ (\phi_n, D_n) \} \) such that \( \frac{1}{n} \log |(\phi_n, D_n)|_1 \to R_1 \) and \( \frac{1}{n} \log |(\phi_n, D_n)|_2 \to R_2 \). Then, we denote the set of strongly deterministically (stochastically) achievable rate pair \((R_1, R_2)\) by \( \mathcal{R}_{s,d} \) (\( \mathcal{R}_{s,s} \)). In the same way, we denote the set of weakly deterministically (stochastically) achievable rate pair \((R_1, R_2)\) by \( \mathcal{R}_{w,d} \) (\( \mathcal{R}_{w,s} \)).

Theorem 1:
\[
\mathcal{R}_{w,d} \subset \bigcup_{P_{X^U}} \{(R_1, R_2) | 0 \leq R_1 - R_2 \leq I(X; Y | U), R_1 \leq H(X | U), 0 \leq R_1, 0 \leq R_2 \}, \tag{20}
\]
\[
\mathcal{R}_{s,s} \subset \bigcup_{P_{X^U}} \{(R_1, R_2) | 0 \leq R_1 - R_2 \leq I(X; Y | U), R_1 \leq H(X | U), 0 \leq R_1, 0 \leq R_2 \}. \tag{21}
\]

Theorem 2: A rate pair \((R_1, R_2)\) is achievable when there exists a distribution \( P \) on \( \mathcal{X} \) such that \( \zeta_1(P) > 0 \) and
\[
0 < R_1 - R_2 < H(X) < R_1 < H(X). \tag{22}
\]

In fact, the condition \( R_1 - R_2 < H(P, W) \) corresponds to Verifiable condition (A), the condition \( I(P; W) < R_1 \) does to Non-decodable condition (B), and the conditions \( R_1 < H(P) \) and \( \zeta_1(P) > 0 \) do to Non-cheating condition for dishonest Alice (D). Theorems 1 and 2 are shown in Sections V and VI respectively. We have the following corollaries from Theorem 2 whose detailed derivations are given in Section IV-B.

Corollary 1: When any distribution \( P \) with support \( \mathcal{X} \) satisfies the condition \( \zeta_1(P) > 0 \), we have
\[
\mathcal{R}_{s,d} \supset \bigcup_{P_{X^U}} \{(R_1, R_2) | 0 < R_1 - R_2 < I(X; Y | U), R_1 < H(X | U), 0 < R_1, 0 < R_2 \}. \tag{23}
\]

Corollary 2: Assume that there is no distinct pair \((x, x')\) in \( \mathcal{X} \) such that \( W_x = W_{x'} \). Also, we assume that there exists \( P_0 \) such that \( C(W) = I(P_0, W) \) and \( \text{supp}(P_0) = \mathcal{X} \). Then, we have
\[
\mathcal{R}_{s,d} \supset \{(R_1, R_2) | 0 < R_1 - R_2 < C(W) < R_1 < H(P_0), 0 < R_1, 0 < R_2 \}. \tag{24}
\]

Combining Corollary 1 with Theorem 1 (Converse part), we have the capacity regions as follows.

Corollary 3: When any distribution \( P \) with support \( \mathcal{X} \) satisfies the condition \( \zeta_1(P) > 0 \),
\[
\mathcal{R}_{s,s} = \overline{\mathcal{R}_{s,d}} = \overline{\mathcal{R}_{w,d}} = \bigcup_{P_{X^U}} \{(R_1, R_2) | 0 \leq R_1 - R_2 \leq I(X; Y | U), R_1 \leq H(X | U), 0 \leq R_1, 0 \leq R_2 \}. \tag{25}
\]

Notice that when \( |\mathcal{X}| = 2 \), any distribution \( P \) satisfies the condition \( \zeta_1(P) > 0 \), i.e., \( D(W_x || W_{x'}) + D(W_{x'} || W_P) > D(W_x || W_P) \). Hence, we have the capacity region.
B. Derivations of corollaries

First, we prepare the following lemma.

Lemma 2: Given a joint distribution \( P_{XU} \), we have the Markov chain \( U - X - Y \), which gives the information quantities \( I(X;Y|U) \) and \( H(X|U) \). Then, we have

\[
\bigcup_{P_{XU}} \{ (R_1, R_2) | 0 < R_1 - R_2 < I(X;Y|U) < R_1 < H(X|U), \; 0 < R_1, \; 0 < R_2 \} = \bigcup_{P_{XU}} \{ (R_1, R_2) | 0 < R_1 - R_2 < I(X;Y|U), \; R_1 < H(X|U), \; 0 < R_1, \; 0 < R_2 \}. \tag{26}
\]

Proof: Since the relation \( \subseteq \) is trivial, it is sufficient to the relation \( \supseteq \). That is, given a distribution \( P_{XU} \) and the pair \( (R_1, R_2) \) such that \( 0 < R_1 - R_2 < R_1 < I(X;Y|U) \), it is sufficient to show the existence of a distribution \( P'_{XU} \) such that \( 0 < R_1 - R_2 < I(X;Y|U) < R_1 < H(X|U) \). There exists a distribution \( P''_{XU} \) such that \( I(X;Y|U) = 0 \). There exists a convex combination \( P''_{XU} \) of \( P_{XU} \) and \( P''_{XU} \) such that \( I(X;Y|U) < R_1 < H(X|U) \). Thus, the desired statement is shown. \( \blacksquare \)

Proof of Corollary 2: Assume that two sequences \( \{(\phi_n, D_n)\} \) and \( \{(\phi'_n, D'_n)\} \) of deterministic codes are strongly secure. Then, we define the concatenation \( \{(\phi''_n, D''_n)\} \) as follows. When \( \phi_n(\phi'_n) \) is given as a map from \( M(M') \) to \( X^n \), the encoder \( \phi''_n \) is given as a map from \( (m, m') \in M \times M' \) to \( (\phi_n(m), \phi'_n(m')) \in X^{2n} \). The decoder \( D''_n \) is given as a map from \( (y_1, \ldots, y_{2n}) \in Y^{2n} \) to \( (D_n(y_1, \ldots, y_n), D'_n(y_{n+1}, \ldots, y_{2n})) \in M \times M' \). We have \( \epsilon_A(\phi''_n, D''_n) \leq \epsilon_A(\phi_n, D_n) + \epsilon_A(\phi'_n, D'_n) \) because the code \( (\phi''_n, D''_n) \) is correctly decoded when both codes \( (\phi_n, D_n) \) and \( (\phi'_n, D'_n) \) are correctly decoded. Since the message encoded by \( \phi''_n \) is correctly decoded only when both messages encoded by encoders \( \phi_n \) and \( \phi'_n \) are correctly decoded, we have \( \delta_B(\phi''_n) \leq \min(\delta_B(\phi_n), \delta_B(\phi'_n)) \). Alice can make cheating for the decoder \( D''_n \) only when Alice makes cheating for one of the decoders \( D_n \) and \( D'_n \). Hence, \( \delta_D(D''_n) \leq \min(\delta_D(D_n), \delta_D(D'_n)) \). Therefore, the concatenation \( \{(\phi''_n, D''_n)\} \) is also strongly secure.

Since any distribution \( P \) satisfies the condition of Theorem 2 using the concatenated code given in Theorem 2 we find that

\[
\mathcal{R}_{s,d} \supseteq \bigcup_{P_{XU}} \{ (R_1, R_2) | 0 \leq R_1 - R_2 \leq I(X;Y|U) \leq R_1 \leq H(X|U), 0 \leq R_1, 0 \leq R_2 \}. \tag{27}
\]

The combination of Eq. (27) and Lemma 2 implies Theorem 1. \( \blacksquare \)

Proof of Corollary 2: Since \( D(W_x||W_P) = D(W_{x'}||W_P) \), we have

\[
\zeta_1(P_0) = \min_{x \neq x'} D(W_{x'}||W_x) > 0. \tag{28}
\]

Hence, Theorem 2 implies Eq. (24). \( \blacksquare \)

V. PROOF OF CONVERSE THEOREM

We prepare the following lemma.

Lemma 3: For \( X^n = (X_1, \ldots, X_n) \), we choose the joint distribution \( P_{X^n} \). Let \( Y^n = (Y_1, \ldots, Y_n) \) be the channel output variables of the inputs \( X^n \) via the channel \( W \). Then, we have

\[
I(X^n;Y^n) \leq \sum_{j=1}^{n} I(X_j;Y_j), \tag{29}
\]

\[
H(X^n) \leq \sum_{j=1}^{n} H(X_j). \tag{30}
\]

Proof: The relation (30) follows from

\[
H(X^n) = \sum_{j=1}^{n} H(X_j|X^{j-1}) \leq \sum_{j=1}^{n} H(X_j). \tag{31}
\]
To show Eq. (29), we define \( X_{j,c} := (X_1, \ldots, X_{j-1}, X_{j+1}, \ldots, X_n) \). Using \( Y^j := Y_1 \ldots Y_j \), we have

\[
I(\mathbf{X}^n; Y^n) = \sum_{j=1}^{n} I(\mathbf{X}^n; Y^j|Y^{j-1}) = \sum_{j=1}^{n} I(X_{j,c}X_j; Y^j|Y^{j-1}) = \sum_{j=1}^{n} I(X_j; Y^j|Y^{j-1}) + \sum_{j=1}^{n} I(X_{j,c}; Y_j|X_jY^{j-1}).
\]

(32)

In this case, we have the Markovian chain \( Y^{j-1} - X_{j,c} - X_j - Y_j \). Then, we have \( I(X_{j,c}; Y_j|Y^{j-1}) = 0 \). Also, the Markovian chain \( Y^{j-1} - X_j - Y_j \) implies that \( I(X_j; Y^j|Y^{j-1}) \lesssim I(X_j; Y_j) \). Hence, we have Eq. (29).

Proof of Theorem 7: Assume that a sequence of deterministic codes \( \{(\phi_n, D_n)\} \) is weakly secure. We assume that \( R_i := \lim_{n \rightarrow \infty} -\frac{1}{n} \log |(\phi_n, D_n)|_i \) converges for \( i = 1, 2 \). Letting \( M \) be the random variable of the message, we define the variables \( X^n = (X_1, \ldots, X_n) := \phi_n(M) \). The random variables \( Y^n = (Y_1, \ldots, Y_n) \) are defined as the output of the channel \( W^n \), which is the \( n \) times use of the channel \( W \). We define the joint distribution \( P_{X^nU^n} \) by \( P_{X^nU^n}(x, i) := \frac{1}{n} P_X(x) \). Under the distribution \( P_{X^nU^n} \), we denote the channel output by \( Y \). Then, we have \( I(X; Y|U) \) and \( H(X|U) \). When we need to describe the dependence of \( n \), we denote them by \( I_n(X; Y|U) \) and \( H_n(X|U) \). In this proof, we use the notations \( M_n := |(\phi_n, D_n)|_1 \) and \( L_n := |(\phi_n, D_n)|_2 \). Also, instead of \( \epsilon_A(\phi_n, D_n) \), we employ \( \epsilon'_A(\phi_n, D_n) := \sum_{m=1}^{M_n} \frac{1}{M_n} \epsilon_A(m, \phi_n(m), D_n) \), which goes to zero.

For a code \( (\phi_n, D_n) \), we have

\[
\log |(\phi_n, D_n)|_1 \leq H(X^n) + \epsilon_A(\phi_n, D_n) \log |(\phi_n, D_n)|_1 + \log 2
\]

(33)

(34)

where \( (b) \) follows from Lemma 3. Dividing the above by \( n \) and taking the limit, we have

To show \( (a) \) in (33), we consider the following protocol. After converting the message \( M \) to \( X^n \) by the encoder \( \phi_n(M) \), Alice sends the \( X^n \) to Bob \( K \) times. Here, we choose \( K \) to be an arbitrary large integer. Applying the decoder \( D_n \), Bob obtains \( K \) lists that contain up to \( K \) messages. Among these messages, Bob chooses \( \hat{M} \) as the element that most frequently appears in the \( K \) lists. When \( \delta_C(\phi_n, D_n) < 1 - \epsilon_A(\phi_n(M), D_n) \) and \( K \) is sufficiently large, Bob can correctly decode \( M \) by this method because \( 1 - \epsilon_A(\phi_n(M), D_n) \) is the probability that the list contains \( M \) and \( \delta_C(\phi_n, D_n) \) is the maximum of the probability that the list contains \( m' \neq M \), i.e., the element \( M \) has the highest probability to be contained in the list. Therefore, the failure of decoding is limited to the case when \( 1 - \delta_C(\phi_n, D_n) \leq \epsilon_A(\phi_n(M), D_n) \). Since the average of \( \epsilon_A(\phi_n(M), D_n) \) is \( \epsilon'_A(\phi_n, D_n) \), Markov inequality guarantees that the error probability of this protocol is bounded by \( \epsilon := \frac{\epsilon'_A(\phi_n, D_n)}{1 - \delta_C(\phi_n, D_n)} \). Fano inequality shows that

\[
H(\hat{M}|\hat{M}) \leq \epsilon' \log |(\phi_n, D_n)|_1 + \log 2.
\]

Then, we have

\[
\log |(\phi_n, D_n)|_1 - \epsilon' \log |(\phi_n, D_n)|_1 + \log 2 \leq \log |(\phi_n, D_n)|_1 - H(\hat{M}|\hat{M}) = I(M; \hat{M}) \leq H(X^n),
\]

(35)

which implies \( (a) \) in (33).

Now, we consider the hypothesis testing with two distributions \( P(m, y^n) := \frac{1}{M_n} W^n(y^n|\phi_n(m)) \) and \( Q(m, y^n) := \frac{1}{\sum_{m=1}^{M_n}} W^n(y^n|\phi_n(m)) \) on \( \mathcal{M}_n \times \mathcal{Y}^n \), where \( \mathcal{M}_n := \{1, \ldots, M_n\} \). Then, we define the region \( \mathcal{D}^*_n \subset \mathcal{M}_n \times \mathcal{Y}^n \) as \( \cup_{m_1, \ldots, m_n} \{m_1, \ldots, m_n\} \times \mathcal{D}_{m_1, \ldots, m_n} \). Using the region \( \mathcal{D}^*_n \) as our test, we define \( \epsilon_Q \) as the error probability to incorrectly support \( P \) while the true is \( Q \). Also, we define \( \epsilon_P \) as the
error probability to incorrectly support $Q$ while the true is $P$. When we apply the monotonicity for the KL divergence between $P$ and $Q$, dropping the term $\epsilon P \log(1 - \epsilon Q)$, we have
\[
- \log \epsilon_Q \leq \frac{D(P||Q) + h(1 - \epsilon_p)}{1 - \epsilon_p}.
\]
(36)
The meta converse for list decoding \[5, Section III-A\] shows that $\epsilon_Q \leq \frac{|(\phi_n, D_n)|_2}{|\phi_n, D_n|_1}$ and $\epsilon_P = \epsilon_A(\phi_n, D_n)$. Since Lemma 3 guarantees that $D(P||Q) = I(X^n; Y^n) \leq nI_n(X; Y|U)$, the relation (36) is converted to
\[
\log \frac{|(\phi_n, D_n)|_1}{|\phi_n, D_n|_2} \leq \frac{I(X^n; Y^n) + h(1 - \epsilon_A(\phi_n, D_n))}{1 - \epsilon_A(\phi_n, D_n)} \leq \frac{nI_n(X; Y|U) + h(1 - \epsilon_A(\phi_n, D_n))}{1 - \epsilon_A(\phi_n, D_n)}.
\]
(37)
Dividing the above by $n$ and taking the limit, we have
\[
\limsup_{n \to \infty} R_1 - R_2 - I_n(X; Y|U) \leq 0.
\]
(38)
Therefore, combining Eqs. (34) and (38), we obtain Eq. (20).

Assume that a sequence of stochastic codes $\{\beta (\phi_n, D_n)\}$ is strongly secure. Then, there exists a sequence of deterministic encoders $\{\phi_n\}$ such that $\epsilon_A(\phi_n, D_n) \leq \epsilon_A(\phi_n, D_n)$ and $\delta_C(\phi_n', D_n) \leq \delta_D(D_n)$. Since $\epsilon_A(\phi_n, D_n) \to 0$ and $\delta_C(\phi_n', D_n) \to 0$, combining Eq. (20), we have Eq. (21).

VI. PROOF OF DIRECT THEOREM

Here, we prove the direct theorem (Theorem 2).

A. Preparation

To show Theorem 2, we prepare notations and basic facts. Assume that $R_1$ and $R_2$ satisfies the condition 22 First, given a real number $R_3 < R_1$, we fix the size of message $M_n := 2^{nR_1}$, the size of list $X_n := 2^{nR_2}$, and a number $M'_n := 2^{nR_3}$, which is smaller than the size of message $M_n$. Then, we prepare the decoder used in this proof as follows.

Definition 1 (Decoder $D_{\phi_n}$): Given a distribution $P$ on $\mathcal{X}$, we define the decoder $D_{\phi_n}$ for a given encoder $\phi_n$ (from $\{1, \ldots, M_n\}$ to $\mathcal{X}^n$) in the following way. We define the subset $D_{x^n} := \{y^n|W_{x^n}(y^n) \geq M'_n\}$. Then, for $y^n \in D_{x^n}$, we choose up to $M_n$ elements $i_1, \ldots, i_{M_n}$ as the decoded messages such that $y^n \in D_{\phi_n(i)}$ for $i = 1, \ldots, M_n$.

For $x^n, x'^n \in \mathcal{X}^n$, we define
\[
F^n(x^n, x'^n|P) := \sum_{i=1}^{M_n} F(x^n_i, x'^n_i|P)
\]
(39)
and define $d(x^n, x'^n)$ to be the number of $k$ such that $x_k \neq x'_k$.

Define the functions $\eta^A_{\phi_n, \epsilon_1}$ and $\eta^C_{\phi_n, \epsilon_2, R_3}$ from $M_n$ to $\{0, 1\}$ as
\[
\eta^A_{\phi_n, \epsilon_1}(m) := \begin{cases} 1 & \text{when } F^n(\phi_n(i), \phi_n(i)|P) \geq n(I(P, W) + \epsilon_1) \\ 0 & \text{otherwise} \end{cases}
\]
(41)
\[
\eta^C_{\phi_n, \epsilon_2, R_3}(m) := \begin{cases} 1 & \text{when } \exists j \neq i, F^n(\phi_n(i), \phi_n(j)|P) \geq n(R_3 - \epsilon_2) \\ 0 & \text{otherwise} \end{cases}
\]
(42)

Since the condition in the subset $D_{x^n}$ can be written by the function $F^n(x^n, x'^n|P)$, Chebychev inequality implies that
\[
W_{x^n}(D_{x^n}) \leq \frac{nV}{[nR_3 - F^n(x^n, x'^n|P)]_+^2}.
\]
(43)
where \( [x]_+ := \max(x, 0) \). As shown in Section VI-C, we have the following lemma.

**Lemma 4:** For arbitrary real numbers \( \epsilon_1 > 0 \) and \( R_3 < I(P,W) \), we choose \( \epsilon_2 := \epsilon_1 + \frac{\zeta_1}{\zeta_1} (I(P,W) - R_3 + \epsilon_1) \). When a code \( \tilde{\phi} \) defined in the message set \( \tilde{M}_n \) satisfies

\[
\max_{m' \neq m \in \mathcal{M}_n} F^n(\tilde{\phi}_n(m), \tilde{\phi}_n(m')|P) < n(R_3 - \epsilon_2) \tag{44}
\]

\[
F^n(\tilde{\phi}_n(m), \tilde{\phi}_n(m)|P) < n(I(P,W) + \epsilon_1) \tag{45}
\]

for an element \( m \in \tilde{M}_n \), we have

\[
\delta_{D,m}(D_{\tilde{\phi}_n}) \leq \frac{V}{n(\epsilon_1 - \frac{\zeta_1 \sqrt{2V}}{\zeta_1})^2} \tag{46}
\]

B. Proof of Theorem 2

To show Theorem 2, we assume that the variable \( \Phi_n(m) \) for \( m \in \mathcal{M}_n \) is subject to the distribution \( P^n \) independently. Then, we have the following two lemmas, which are shown later. In this proof, we treat the code \( \Phi_n \) as a random variable. Hence, the expectation and the probability for this variable are denoted by \( \mathbb{E}_{\Phi_n} \) and \( \text{Pr}_{\Phi_n} \), respectively.

**Lemma 5:** When

\[
I(P,W) > R_3, \quad R_3 \geq R_1 - R_2, \tag{47}
\]

we have the average version of Verifiable condition (A), i.e.,

\[
\lim_{n \to \infty} \mathbb{E}_{\Phi_n} \sum_{m=1}^{M_n} \frac{1}{M_n} \epsilon_{A,m}(\Phi_n, D_{\Phi_n}) = 0. \tag{48}
\]

**Lemma 6:** When

\[
H(P) > R_1, \tag{49}
\]

for any real number \( \epsilon_2 > 0 \), we have

\[
\lim_{n \to \infty} \mathbb{E}_{\Phi_n} \sum_{m=1}^{M_n} \frac{1}{M_n} \eta_{\Phi_n,\epsilon_2,R_3}(m) = 0. \tag{50}
\]

In this proof, we set the parameters \( \epsilon_0, \epsilon_1, \epsilon_2, \epsilon_3, \) and \( R_3 \) in the following way so that the conditions in Lemmas 4, 5, and 6 are satisfied.

\[
\epsilon_0 := H(P) - R_1 \tag{51}
\]

\[
\epsilon_1 := \frac{I(P,W) - R_1 + R_2}{2} \tag{52}
\]

\[
R_3 := I(P,W) - \epsilon_1 \tag{53}
\]

\[
\epsilon_2 := (1 + \frac{2\zeta_2}{\zeta_1})\epsilon_1 = \epsilon_1 + \frac{\zeta_2}{\zeta_1} (I(P,W) - R_3 + \epsilon_1) \tag{54}
\]

\[
\epsilon_3 := \min(\frac{1}{6}, \frac{R_1 - I(P,W)}{3}). \tag{55}
\]
The law of large number guarantees that
\[
\lim_{n \to \infty} E_{\Phi_n} \sum_{m=1}^{M_n} \frac{1}{M_n} \eta_{\Phi_n,\epsilon_1}^A (m) = 0 \tag{56}
\]
\[
\lim_{n \to \infty} \Pr_{\Phi_n(i)}(D(W_{\Phi_n(i)}^n||W_{P_n}^n) < n(I(P,W) + \epsilon_3)) = 0, \tag{57}
\]
where \( \Pr_X(A) \) expresses the probability that the condition \( A \) with respect to the variable \( X \) holds. Eq. (57) implies
\[
\lim_{n \to \infty} \Pr_{\Phi_n}(\#\{m|D(W_{\Phi_n(m)}^n||W_{P_n}^n) < n(I(P,W) + \epsilon_3)\} \geq (1 - \epsilon_3)M_n) = 0. \tag{58}
\]
Due to Eqs. (56) and (58), and Lemmas 5 and 6 there exist a sequence of codes \( \phi_n \) and a sequence of real numbers \( \epsilon_{4,n} > 0 \) such that \( \epsilon_{4,n} \to 0 \) and
\[
\sum_{m=1}^{M_n} \left( \frac{1}{M_n} (\epsilon_{A,m}(\phi_n, D_{\phi_n}) + \eta_{\phi_n,\epsilon_1}^A (m) + \eta_{\phi_n,\epsilon_2,R_3}^C (m)) \right) \leq \frac{\epsilon_{4,n}}{3} \tag{59}
\]
\[
\#\{m|D(W_{\Phi_n(m)}^n||W_{P_n}^n) < n(I(P,W) + \epsilon_3)\} \geq (1 - \epsilon_3)M_n. \tag{60}
\]
Due to Eq. (59), Markov inequality guarantees that there exist \( 2M_n/3 \) elements \( \tilde{M}_n' := \{m_1, \ldots, m_{2M_n/3}\} \) such that any element \( m \in \tilde{M}_n' \) satisfies
\[
\epsilon_{A,m}(\phi_n, D_{\phi_n}) + \eta_{\phi_n,\epsilon_1}^A (m) + \delta_B(\phi_n) \leq \epsilon_{4,n}, \tag{61}
\]
which implies that
\[
\epsilon_{A,m}(\phi_n, D_{\phi_n}) \leq \epsilon_{4,n}; \quad \delta_B(\phi_n) \leq \epsilon_{4,n} \tag{62}
\]
\[
\eta_{\phi_n,\epsilon_1}^A (m) = \eta_{\phi_n,\epsilon_2,R_3}^C (m) = 0 \tag{63}
\]
because \( \eta_{\phi_n,\epsilon_1}^A \) and \( \eta_{\phi_n,\epsilon_2,R_3}^C \) take value 0 or 1. Since \( 1/6 \geq \epsilon_3 \), Eq. (60) guarantees that the number of \( m \) that does not satisfy the following condition (64) is at most \( M_n/6 \);
\[
D(W_{\phi_n(m)}^n||W_{P_n}^n) < n(I(P,W) + \epsilon_3). \tag{64}
\]
Hence, we can choose \( M_n/2 \) elements \( \tilde{M}_n := \{m_1, \ldots, m_{M_n/2}\} \) from \( \tilde{M}_n' \) to satisfy the condition (64) because \( |	ilde{M}_n'| - |\tilde{M}_n| = M_n/6 \). Now, we define a code \( \tilde{\phi}_n \) on \( \tilde{M}_n \) as \( \tilde{\phi}_n(m) := \phi_n(m) \) for \( m \in \tilde{M}_n \). Thus, for \( m \neq m' \in \tilde{M}_n \), we have
\[
\epsilon_{A,m}(\tilde{\phi}_n, D_{\tilde{\phi}_n}) \leq \epsilon_{A,m}(\phi_n, D_{\phi_n}) \leq \epsilon_{4,n} \tag{65}
\]
\[
\delta_B(\tilde{\phi}_n) \leq \epsilon_{4,n} \tag{66}
\]
\[
F^n(\tilde{\phi}_n(m), \tilde{\phi}_n(m')) < n(R_3 - \epsilon_2) \tag{67}
\]
\[
F^n(\tilde{\phi}_n(m), \tilde{\phi}_n(m)||P) < n(I(P,W) + \epsilon_1). \tag{68}
\]
Therefore, Lemma 4 guarantees Non-cheating condition for dishonest Alice (D), i.e.,
\[
\delta_{D,m}(D_{\tilde{\phi}_n}) \leq \frac{V}{n(\epsilon_1 - \xi_1 \sqrt{2\epsilon})} \tag{69}
\]
In the code \( \tilde{\phi}_n \), Eq. (64) implies that
\[
I(X^n; Y^n) \leq n(I(P,W) + \epsilon_3). \tag{70}
\]
Using the formula given in [16, Theorem 4][17, Lemma 4], we have
\[ \delta_B(\tilde{\phi}_n) \leq \left( \sum_{m \in \mathcal{M}_n} \frac{2}{M_n} W^n_{\tilde{\phi}_n(m)} \left( \left\{ y^n \mid \log W^n_{\tilde{\phi}_n(m)}(y^n) - \log W^n_{\tilde{\phi}_n(m)}(y^n) \geq n(I(P, W) + 2\epsilon_3) \right\} \right) \right) + \frac{2 \cdot 2^{n(I(P, W) + 2\epsilon_3)}}{M_n}. \] (71)

Also, we have
\[ W^n_{\tilde{\phi}_n(m)} \left( \left\{ y^n \mid \log W^n_{\tilde{\phi}_n(m)}(y^n) - \log W^n_{\tilde{\phi}_n(m)}(y^n) \geq n(I(P, W) + 2\epsilon_3) \right\} \right) \leq W^n_{\tilde{\phi}_n(m)} \left( \left\{ y^n \mid \log W^n_{\tilde{\phi}_n(m)}(y^n) - \log W^n_{\tilde{\phi}_n(m)}(y^n) - D(W^n_{\tilde{\phi}_n(m)} || W^n_{\tilde{\phi}_n(m)}) \geq n\epsilon_3 \right\} \right) \leq \frac{V}{n\epsilon_3^2}, \] (72)

where (a) follows from Eq. (64) and (b) follows from Chebychev inequality. Since \( \frac{2^{2n(I(P, W) + 2\epsilon_3)}}{M_n} \rightarrow 0 \), the combination of (71) and (72) implies that \( \delta_B(\tilde{\phi}_n) \rightarrow 0 \). Since RHSs of (65) and (69) go to zero, we obtain the desired statement.

C. Proof of Lemma 2

Now, we assume that an element \( x^n \in \mathcal{X}^n \) satisfies
\[ \epsilon_{A,m}(x^n, D_{\tilde{\phi}_n}) \leq \frac{1}{2}. \] (73)

Then,
\[ \frac{1}{2} \leq (1 - \epsilon_{A,m}(x^n, D_{\tilde{\phi}_n})) \leq W^n_{x^n}(D_{\tilde{\phi}_n(m)}) \leq \frac{nV}{[nR_3 - F^n(x^n, \tilde{\phi}_n(m)|P)]^2}, \] (74)

where (a) follows from the definition of \( D_{\tilde{\phi}_n} \) (Definition 1) and (b) follows from Eq. (43).

Hence, we have
\[ nR_3 - F^n(x^n, \tilde{\phi}_n(m)|P) \leq \sqrt{2nV}. \] (75)

Thus,
\[ nR_3 - \sqrt{2nV} \leq F^n(x^n, \tilde{\phi}_n(m)|P) \leq F^n(\tilde{\phi}_n(m), \tilde{\phi}_n(m)|P) - d(x^n, \tilde{\phi}_n(m))\zeta_1 \leq n(I(P, W) + \epsilon_1) - d(x^n, \tilde{\phi}_n(m))\zeta_1, \] (76)

where (a), (b), and (c) follow from Eq. (75), the combination of the definitions of \( \zeta_1 \) and \( d(x^n, \tilde{\phi}_n(m)) \), and Eq. (45), respectively.

Hence,
\[ d(x^n, \tilde{\phi}_n(m)) \leq \frac{n(I(P, W) - R_3 + \epsilon_1) + \sqrt{2nV}}{\zeta_1}. \] (77)

Thus, for \( m' \in \mathcal{M}_n \), we have
\[ nR_3 - F^n(x^n, \tilde{\phi}_n(m')|P) \geq nR_3 - F^n(\tilde{\phi}_n(m), \tilde{\phi}_n(m')|P) - \zeta_2d(x^n, \tilde{\phi}_n(m)) \geq n\epsilon_2 - \zeta_2d(x^n, \tilde{\phi}_n(m)) \geq n\epsilon_2 - \frac{\zeta_2}{\zeta_1}(n(I(P, W) - R_3 + \epsilon_1) + \sqrt{2nV}) \geq n\epsilon_1 - \frac{\zeta_2}{\zeta_1}\sqrt{2nV}, \] (78)
where \((a), (b), (c)\) and \((d)\) follow from the combination of the definitions of \(\zeta_1\) and \(d(x^n, \tilde{\phi}_n(m))\), Eq. (44), Eq. (77), the choice of \(\epsilon_2\) respectively.

Thus, we have

\[
\delta_{C,m}(x^n, D_{\tilde{\phi}_n}) \leq \max_{m' \neq m} \frac{nV}{(nR_3 - F_n(x^n, \tilde{\phi}_n(m'))|^2) + (\epsilon)} \leq \frac{nV}{(n\epsilon - \frac{\epsilon}{\zeta_1\sqrt{2nV}})^2} = \frac{V}{n(\epsilon_1 - \frac{\epsilon}{\zeta_1\sqrt{2n}})^2},
\]

where \((a), (b), (c)\) follow from the definition of \(\delta_{C,m}\) (Eq. (6)), Eq. (43), and Eq. (78), respectively.

\[\square\]

\section*{D. Proof of Lemma 5}

To show (48), we employ an idea similar to [16], [17].

\textbf{Lemma 7:} We have the following inequality:

\[
\epsilon_A(\Phi_n, D_{\Phi_n}) \leq \frac{1}{M_n} \sum_{i=1}^{M_n} \left( W_{\Phi_n(i)}(D_{\Phi_n(i)}) + \frac{1}{L_n} W_{\Phi_n(i)}(D_{\Phi_n(j)}) \right).
\]

\[\square\]

\textbf{Proof:} When \(i\) is sent, there are two cases for incorrectly decoded. The first case is the case that the received element \(y\) does not belong to \(D_{\Phi_n(i)}\). The second case is the case that there are more than \(L_n\) elements \(i'\) to satisfy \(y \in D_{\Phi_n(i')}\). The error probability of the first case is given in the first term of Eq. (80). The error probability of the second case is given in the second term of Eq. (80).

\textbf{Using Lemma 7} we have the following lemma.

\textbf{Lemma 8:} We have the following inequality:

\[
E_{\Phi_n} \epsilon_A(\Phi_n, D_{\Phi_n}) \leq \sum_{x^n \in X^n} P^n(x^n) \left( W_{x^n}(D_{x^n}) + \frac{M_n - 1}{L_n} W_{x^n}(D_{x^n}) \right).
\]

\[\square\]

Applying Lemma 8 we have

\[
E_{\Phi_n} \epsilon_A(\Phi_n, D_{\Phi_n}) \leq E_{X^n} W_{X^n} \left\{ \left\{ y^n \right\} | \log W_{x^n} y^n - \log W_{x^n} y^n nR_3 \right\}
+ E_{X^n} 2^{n(R_1 - R_2)} W_{x^n} \left\{ \left\{ y^n \right\} | \log W_{x^n} y^n - \log W_{x^n} y^n nR_3 \right\}.
\]

Taking the limit, we obtain Eq. (48).

\[\square\]

\section*{E. Proof of Lemma 6}

Due to Lemma 1 and Eq. (49), we can choose \(s\) such that \(-R_1 + s(R_3 - \epsilon_2) > G(s|P)\). We set \(\epsilon_5 := \frac{1}{2}(-R_1 + s(R_3 - \epsilon_2) - G(s|P) + 0). We define two conditions \(A_{n, i}\) and \(B_{n, i}\) for the encoder \(\Phi_n\) as

\[
A_{n, i} = G_n(s, \Phi_n(i)|P) < n(-R_1 + s(R_3 - \epsilon_2) - \epsilon_5).
\]

\[
B_{n, i} = \exists j \neq i, F_n(\Phi_n(i), \Phi_n(j)|P) \geq n(R_3 - \epsilon_2).
\]

When \(A_{n, i}\) holds, for \(j \neq i\), Markov inequality implies that \(Pr_{\Phi_n(j)|\Phi_n(i)} \left( F_n(\Phi_n(i), \Phi_n(j)|P) \geq n(R_3 - \epsilon_2) \right) \leq 2^{G_n(s, \Phi_n(i)|P) - sn(R_3 - \epsilon_2)}\), where \(Pr_{\Phi_n(j)|\Phi_n(i)}\) is the probability for the random variable \(\Phi_n(j)\) with the fixed variable \(\Phi_n(i)\). Hence, under this condition, we have

\[
Pr_{\Phi_n,i,j|\Phi_n(i)}(B_{n, i}) \leq 2^{nR_1 - 2^{G_n(s, \Phi_n(i)|P) - sn(R_3 - \epsilon_2)} - 2^{G_n(s, \Phi_n(i)|P) - sn(R_3 - \epsilon_2) + nR_1} \leq 2^{-n\epsilon_5},
\]

(83)
where $\Phi_{n,i,c}$ expresses the random variables $\{\Phi_n(j)\}_{j \neq i}$. Now, $\Pr_{\Phi_n}(A_{n,i}|B_{n,i})$ expresses the probability that the condition $A_{n,i}$ holds under the condition that the condition $B_{n,i}$ holds. Then, we have

$$\Pr_{\Phi_n}(A_{n,i}|B_{n,i}) \leq 2^{-n\epsilon_5}. \quad (84)$$

Thus, we have

$$\mathbb{E}_{\Phi_n} \left( \sum_{m=1}^{M_n} \frac{1}{M_n} \eta_{\Phi_n,\epsilon_2, R_3}(m) \right) = \frac{1}{M_n} \mathbb{E}_{\Phi_n} \left( \{i|B_{n,i} \text{ holds.}\} \right) = \sum_{i=1}^{M_n} \frac{1}{M_n} \Pr_{\Phi_n}(B_{n,i})$$

$$\leq \sum_{i=1}^{M_n} \frac{1}{M_n} \left( \Pr_{\Phi_n}(A_{n,i}) \Pr_{\Phi_n}(B_{n,i}|A_{n,i}) + (1 - \Pr_{\Phi_n}(A_{n,i})) \right) \leq 2^{-n\epsilon_5} + \frac{1}{M_n} \left( 1 - \Pr(A_{n,i}) \right), \quad (85)$$

where (a) follows from Eq. (84).

The random variable $G^n(s, \Phi_n(i)|P)$ can be regarded as the $n$-fold i.i.d. extension of the variable $G(s, X|P)$ whose expectation is $G(s, P)$. Since the choice of $\epsilon_5$ guarantees that

$$G(s, P) < -R_1 + s(R_3 - \epsilon_2) - \epsilon_5, \quad (86)$$

we have

$$1 - \Pr_{\Phi_n}(A_{n,i}) = \Pr_{\Phi_n} \left( G^n(s, \Phi_n(i)|P) \geq n(-R_1 + s(R_3 - \epsilon_2) - \epsilon_5) \right) \to 0. \quad (87)$$

Hence, the combination of Eqs. (85) and (87) implies the desired statement.

VII. CONCLUSION

We have proposed a new concept, secure list decoding, which has additional requirements for the conventional list decoding. This scheme has three requirements. Verifiable condition, Non-decodable condition, and Non-cheating condition. Non-cheating condition has two versions. One is the condition for honest Alice (sender). The other is the condition for dishonest Alice. Since there is a possibility that Alice use a different code, we need to guarantee the impossibility of cheating even for such a dishonest Alice. In this paper, we have shown the existence of a code to satisfy these three conditions. Also, we have defined the capacity region as the possible rate pair of the rates of the message and the list, and have derived the capacity region under a proper condition. Also, we have constructed a protocol for bit commitment from the secure list decoding. However, it is not trivial to construct secure list decoding from bit commitment. This direction is an interesting open problem.

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