A Ramsey Theorem for Multiposets

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April 9, 2019

Abstract

In the parlance of relational structures, the Finite Ramsey Theorem states that the class of all finite chains has the Ramsey property. A classical result from the 1980’s claims that the class of all finite posets with a linear extension has the Ramsey property. In 2010 Šokčić proved that the class of all finite structures consisting of several linear orders has the Ramsey property. This was followed by a 2017 result of Solecki and Zhao that the class of all finite posets with several linear extensions has the Ramsey property.

Using the categorical reinterpretation of the Ramsey property in this paper we prove a common generalization of all these results. We consider multiposets to be structures consisting of several partial orders and several linear orders. We allow partial orders to extend each other in an arbitrary but fixed way, and require that every partial order is extended by at least one of the linear orders. We then show that the class of all finite multiposets conforming to a fixed template has the Ramsey property.

Key Words: Ramsey property, finite posets
AMS Subj. Classification (2010): 05C55, 18A99
1 Introduction

Generalizing the classical results of F. P. Ramsey from the late 1920’s, the structural Ramsey theory originated at the beginning of 1970’s in a series of papers (see [16] for references). We say that a class $\mathbf{K}$ of finite structures has the Ramsey property if the following holds: for any number $k \geq 2$ of colors and all $A, B \in \mathbf{K}$ such that $A$ embeds into $B$ there is a $C \in \mathbf{K}$ such that no matter how we color the copies of $A$ in $C$ with $k$ colors, there is a monochromatic copy $B'$ of $B$ in $C$ (that is, all the copies of $A$ that fall within $B'$ are colored by the same color). In this parlance, the Finite Ramsey Theorem [22] takes the following form:

(Finite Ramsey Theorem) The class of all finite chains has the Ramsey property.

As it turns out many classes of finite linearly ordered structures have the Ramsey property. For example, the class of all finite linearly ordered graphs has the Ramsey property [1, 18]. Interestingly, this is not the case with the class of finite partial orders accompanied with arbitrary linear orders — this class is not Ramsey [9, 23]. However, Paoli, Trotter and Walker show in [21] that the class of all infinite posets with a linear extension has the Ramsey property. The same result was reproved recently using different strategies by Sokić [23, 24], Solecki and Zhao [27], Nešetřil and Rödl [20], and the second author [13]. Sokić in [23, 24] derived the result as a consequence of a result of Fouché [9], while the proof of Solecki and Zhao [27] relies on Solecki’s abstract approach to finite Ramsey theory [26]. The proof given in [20] starts from the fact that the class of all finite acyclic digraphs endowed with a linear extension is a Ramsey class (and this, in turn, follows from the Nešetřil-Rödl Theorem) and then uses the partite construction to “improve acyclic digraphs to posets”. This is the first proof where the partite construction was used to establish the Ramsey property for this class of structures. The proof given in [13], on the other hand, establishes a particular relationship (called a pre-adjunction) between the category of all finite posets with a linear extension and the Graham-Rothschild category (which is nothing but a categorical rendering of the setup of the Graham-Rothschild Theorem) and then uses the pre-adjunction between the categories to “transport the

\footnote{Let us briefly note that the proof of the ordering property presented in [21, Theorem 16, p. 362] relies on Theorem 2 (p. 354), the statement of which is not true. Fortunately, as the referee of this paper have pointed out to the second author, the finitary version of Theorem 2 (the well known Finite Product Ramsey Theorem) suffices for the proof.}
Ramsey property” from the the Graham-Rothschild category to the other one. Interestingly, the proofs presented in [27] and [13] do not require the ordering property in order to establish the Ramsey property for the class.

The fact that the class of all finite posets with a linear extension has the Ramsey property was generalized in several directions. In his paper [9] Fouché explicitly calculated Ramsey degrees of finite posets: as it turns out, the Ramsey degree of a finite poset is the number of finite linear extensions of the poset ordering. Generalizing the result in other direction, Sokić proved in his PhD thesis [23, Theorem 78, p. 96] that the class of all finite structures consisting of several linear orders has the Ramsey property. (The case for \( n = 2 \) was independently proved by Böttcher and Foniok in [3].) Sokić also proved that the class of all finite posets with a linear extension and an independent linear order has the Ramsey property [23, Theorem 80, p. 98]. This example is interesting and motivating, because it seems that as soon as we have at least one linear extension of the base partial order we can pretty much do whatever we want.

Another point of view on the same phenomenon is taken by Sokić in [23] and Bodirsky in [3, 4]. They proved that we can always “put two Ramsey classes together” to get a new one in the following sense: free interposition of two Ramsey classes with strong amalgamation is again a Ramsey class.

The next important step in understanding the Ramsey property for classes of finite partial orders endowed with additional linear orders was a result of Solecki and Zhao that the class of all finite posets with several linear extensions has the Ramsey property [27]. An alternative, shorter, proof of the Solecki-Zhao result was given by Arman and Rödl in [2]. The proof of Arman and Rödl starts from a large product of classes of finite linear orders with a linear extension and then reduces it to the class of finite posets with several linear extensions. In this paper we take a similar approach, but in order to prove our more general result we have to refine it. As in the case of the proof of Arman and Rödl our main “building tool” is the structural version of the Product Ramsey Theorem from [25], but as the “refinement tool” we use a theorem about transferring the Ramsey property from a category onto its subcategory closed in a particular way.

A problem closely related to identifying Ramsey classes is the classification of amalgamation classes of finite structures (a connection between the two notions was established by Nešetřil in [17]). Amalgamation classes of permutations (understood as finite structures with two independent linear orders) were classified by Cameron in [8]. In that paper Cameron also posed the problem of generalizing his result to classes of finite structures with three or more independent linear orders, which turned out to be quite a challenge.
A very recent result of Braunfeld and Simon [7] gives a catalog of amalgamation classes of such structures which, interestingly, contains examples of classes of structures that do not fall into the framework of this paper. The relationship of the classes from their catalog to the Ramsey property (in the context of Ramsey expansions) was further discussed by Braunfeld in [6].

Going back to the examples that motivate the main result of this paper let us collect some of the above considerations into a single statement.

Theorem 1.1 (a) (Ramsey [22]) The class of all finite chains \((A, \leq)\) has the Ramsey property.

(b) (Paoli, Trotter and Walker [21]) The class of all finite posets with a linear extension (that is, structures \((A, \leq_1, \leq_2)\) where \(\leq_1\) is a partial order on \(A\) and \(\leq_2\) a linear order on \(A\) such that \((\leq_1) \subseteq (\leq_2))\) has the Ramsey property.

(c) (Sokić [23]) The class of all finite posets with a linear extension and another linear order (that is, structures \((A, \leq_1, \leq_2, \leq_3)\) where \(\leq_1\) is a partial order, \(\leq_2\) a linear extension of \(\leq_1\) and \(\leq_3\) is an arbitrary linear order) has the Ramsey property.

(d) (Sokić [23, 24], Bodirsky [3, 4], Böttcher and Foniok [5]) For every \(n \geq 2\), the class of all finite structures of the form \((A, \leq_1, \ldots, \leq_n)\) where each \(\leq_i\) is a linear order on \(A\) has the Ramsey property.

(e) (Solecki and Zhao [27], Arman and Rödl [2]) For every \(n \geq 1\), the class of all finite structures of the form \((A, \leq_0, \leq_1, \ldots, \leq_n)\) where \(\leq_0\) is a partial order on \(A\) and each \(\leq_i, i \in \{1, 2, \ldots, n\}\), is a linear order on \(A\) extending \(\leq_0\) has the Ramsey property.

For reasons that will become apparent soon, let us depict these five situations as in Fig. 1. For example, by Fig. 1(c) we indicate that the structures we are interested in have three ordering relations, the first one is always contained in the second, while the third ordering relation is independent of the first two.

In each of these cases we have a class of structures with several ordering relations, the relations are required to form a fixed partially ordered set under set inclusion, and the maximal elements in this poset of relations are required to be linear orders. A straightforward generalization now leads to the following concept.

Let \(\mathcal{T} = (\{1, 2, \ldots, t\}, \leq_t)\), \(t \geq 1\), be a poset which we refer to as the template. A \(\mathcal{T}\)-multiposet is a structure \((A, \leq_1, \ldots, \leq_t)\) where
\[
\begin{align*}
\leq & \quad \leq_2 \quad \leq_3 \quad \cdots \quad \leq_n \\
\leq_1 & \quad \leq_1 \quad \leq_1 \quad \cdots \quad \leq_{n-1}
\end{align*}
\]

(a) (b) (c) (d) (e)

Figure 1: Relationships between ordering relations in five situations listed in Theorem 1.1

- \(\leq_1, \ldots, \leq_t\) are partial orders on \(A\),
- if \(i\) is a maximal element of \(\mathcal{T}\) then \(\leq_i\) is a linear order on \(A\), and
- if \(i \preceq j\) in \(\mathcal{T}\) then \(\langle \leq_i \rangle \subseteq \langle \leq_j \rangle\).

Let \(\mathbf{K}(\mathcal{T})\) be the class of all finite \(\mathcal{T}\)-multiposets. The purpose of this paper is to show the following result which clearly generalizes each of the results listed in Theorem 1.1:

**Theorem 1.2** For every template \(\mathcal{T}\) the class \(\mathbf{K}(\mathcal{T})\) has the Ramsey property.

It was Leeb who pointed out in 1970 [11] that the use of category theory can be quite helpful both in the formulation and in the proofs of results pertaining to structural Ramsey theory. We pursued this line of thought in several papers [12, 13, 14] and demonstrated that reinterpreting the Ramsey property in the context of category theory and using the machinery of category theory can lead to essentially new proving strategies. The proof of Theorem 1.2 will represent another demonstration of these new strategies.

In Section 2 we give a brief overview of standard notions referring to finite structures and category theory, and conclude with the reinterpretation of the Ramsey property in the language of category theory. In Section 3 we consider two ways of transferring the Ramsey property from a category to another category. We first recall a result of M. Sokić from [25] which enables us to combine Ramsey classes of finite structures over disjoint relational signatures in a particular way, and then recall a result from [14] which enables us to transfer the Ramsey property from a category to its (not necessarily full) subcategory. Using these two transfer principles, starting from Theorem 1.1 (a) and (b), in Section 4 we prove Theorem 1.2.
2 Preliminaries

In this section we give a brief overview of standard notions referring to first order structures and conclude with a reinterpretation of the Ramsey property in the language of category theory.

2.1 Structures

Let \( \Theta \) be a set of relational symbols. A \( \Theta \)-structure \( \mathcal{A} = (A, \Theta^A) \) is a set \( A \) together with a set \( \Theta^A \) of relations on \( A \) which are interpretations of the corresponding symbols in \( \Theta \). A structure \( \mathcal{A} = (A, \Theta^A) \) is finite if \( A \) is a finite set. For \( \Theta \)-structures \( \mathcal{A} \) and \( \mathcal{B} \), an embedding \( f : \mathcal{A} \hookrightarrow \mathcal{B} \) is an injection \( f : A \rightarrow B \) such that \( (a_1, \ldots, a_r) \in \rho^A \iff (f(a_1), \ldots, f(a_r)) \in \rho^B \), for every relational symbol \( \rho \in \Theta \) and all \( a_1, \ldots, a_r \in A \) where \( r = \text{ar}(\rho) \).

A structure \( \mathcal{A} \) is a substructure of a structure \( \mathcal{B} \) (\( \mathcal{A} \leq \mathcal{B} \)) if the identity map \( a \mapsto a \) is an embedding of \( \mathcal{A} \) into \( \mathcal{B} \). Let \( \mathcal{A} \) be a structure and \( \emptyset \neq B \subseteq A \). Then \( \mathcal{A}|_B = (B, \Theta^A|_B) \) denotes the substructure of \( \mathcal{A} \) induced by \( B \), where \( \Theta^A|_B \) denotes the restriction of each relation in \( \Theta^A \) to \( B \). If \( \mathcal{A} \) is a \( \Theta \)-structure and \( \Sigma \subseteq \Theta \) then by \( \mathcal{A}|_{\Sigma} \) we denote the \( \Sigma \)-reduct of \( \mathcal{A} \): \( \mathcal{A}|_{\Sigma} = (A, \{\theta^A : \theta \in \Sigma\}) \).

Let \( \Theta \) be a set of relational symbols. Let \( C \) be a class of \( \Theta \)-structures and \( K \) a subclass of \( C \). Then:

- **K** has the hereditary property (HP) with respect to \( C \) if for all \( \mathcal{C} \in K \) and \( \mathcal{B} \in C \) such that \( \mathcal{B} \hookrightarrow \mathcal{C} \) we have that \( \mathcal{B} \in K \);
- **K** has the joint embedding property (JEP) if for all \( \mathcal{A}, \mathcal{B} \in K \) there is a \( \mathcal{C} \in \mathcal{K} \) such that \( \mathcal{A} \hookrightarrow \mathcal{C} \) and \( \mathcal{B} \hookrightarrow \mathcal{C} \);
- **K** has the strong amalgamation property (SAP) if for all \( \mathcal{A}, \mathcal{B}, \mathcal{C} \in K \) and embeddings \( f_1 : \mathcal{A} \hookrightarrow \mathcal{B} \) and \( f_2 : \mathcal{A} \hookrightarrow \mathcal{C} \) there is a \( \mathcal{D} \in K \) and embeddings \( g_1 : \mathcal{B} \hookrightarrow \mathcal{D} \) and \( g_2 : \mathcal{C} \hookrightarrow \mathcal{D} \) such that \( g_1 \circ f_1 = g_2 \circ f_2 \) and \( g_1(B) \cap g_2(C) = g_1(f_1(A)) = g_2(f_2(A)) \).

**Example 2.1** Let \( \text{Ch} \) denote the class of all finite chains, and let \( \text{EPos} \) denote the class of all finite posets with a linear extension. Both \( \text{Ch} \) and \( \text{EPos} \) have each of the properties listed above.

2.2 Categories, functors and the Ramsey property

In order to specify a category \( C \) one has to specify a class of objects \( \text{Ob}(C) \), a set of morphisms \( \text{hom}_C(\mathcal{A}, \mathcal{B}) \) for all \( \mathcal{A}, \mathcal{B} \in \text{Ob}(C) \), the identity morphism...
id_A for all A ∈ Ob(C), and the composition of morphisms so that id_B · f = f · id_A for all f ∈ hom_C(A, B), and (f · g) · h = f · (g · h). A morphism f ∈ hom_C(B, C) is monic or left cancellable if f · g = f · h implies g = h for all g, h ∈ hom_C(A, B) where A ∈ Ob(C) is arbitrary.

**Example 2.2** Every class of structures of the same relational type forms a category where morphisms are embeddings. Given a class K of structures (of the same relational type), whenever we refer to K as a category, we have in mind the category whose objects are structures from K and whose morphisms are embeddings.

In particular, Ch and EPos are categories where objects are the corresponding structures and morphisms are embeddings.

A category D is a subcategory of a category C if Ob(D) ⊆ Ob(C) and hom_D(A, B) ⊆ hom_C(A, B) for all A, B ∈ Ob(D). A category D is a full subcategory of a category C if Ob(D) ⊆ Ob(C) and hom_D(A, B) = hom_C(A, B) for all A, B ∈ Ob(D).

A functor F : C → D from a category C to a category D maps Ob(C) to Ob(D) and maps morphisms of C to morphisms of D so that F(f) ∈ hom_D(F(A), F(B)) whenever f ∈ hom_C(A, B), F(f · g) = F(f) · F(g) whenever f · g is defined, and F(id_A) = id_{F(A)}.

Categories C and D are isomorphic if there exist functors F : C → D and G : D → C which are inverses of one another both on objects and on morphisms.

A diagram in a category C is a functor F : Δ → C where the category Δ is referred to as the shape of the diagram. A diagram F : Δ → C is consistent in C if there exists a C ∈ Ob(C) and a family of morphisms (e_δ : F(δ) → C)_{δ ∈ Ob(Δ)} such that for every morphism g : δ → γ in Δ we have e_γ · F(g) = e_δ:

(see Fig. 2). We say that C together with the family of morphisms (e_δ)_{δ ∈ Ob(Δ)} forms a compatible cone in C over the diagram F.

Let C be a category and S a set. We say that S = X_1 ∪ ... ∪ X_k is a k-coloring of S if X_i ∩ X_j = ∅ whenever i ≠ j. For an integer k ≥ 2 and A, B, C ∈ Ob(C) we write C → (B)_k^A to denote that for every k-coloring hom_C(A, C) = X_1 ∪ ... ∪ X_k there is an i ∈ {1, ..., k} and a morphism w ∈ hom_C(B, C) such that w · hom_C(A, B) ⊆ X_i.
Definition 2.1 A category \( \mathcal{C} \) has the Ramsey property if for every integer \( k \geq 2 \) and all \( \mathcal{A}, \mathcal{B} \in \text{Ob}(\mathcal{C}) \) such that \( \text{hom}_\mathcal{C}(\mathcal{A}, \mathcal{B}) \neq \emptyset \) there is a \( \mathcal{C} \in \text{Ob}(\mathcal{C}) \) such that \( \mathcal{C} \rightarrow (\mathcal{B})^\mathcal{A}_k \).

Clearly, if \( \mathcal{C} \) and \( \mathcal{D} \) are isomorphic categories, then one of them has the Ramsey property if and only if the other one does.

Example 2.3 As we have seen, both \( \text{Ch} \) and \( \text{EPos} \) have the Ramsey property (Theorem 1.1 (a) and (b)).

3 Transferring the Ramsey property between categories

In this section we give a brief overview of two strategies of transferring the Ramsey property from a category to another category. We first recall a result of M. Sokič from [25] which enables us to combine Ramsey classes of structures over disjoint signatures in a particular way.

Let \( \Sigma_1, \Sigma_2, \ldots, \Sigma_n, n \geq 2 \), be pairwise disjoint sets of relational symbols, and for each \( i \in \{1, 2, \ldots, n\} \) let \( \mathbf{K}_i \) be a class of \( \Sigma_i \)-structures. Let \( \Theta = \bigcup_{i=1}^{n} \Sigma_i \). Then with a slight abuse of set-theoretic notation we define the class \( \bigotimes_{i=1}^{n} \mathbf{K}_i \) of \( \Theta \)-structures as follows:

\[
\bigotimes_{i=1}^{n} \mathbf{K}_i = \{ \mathcal{A} : \mathcal{A} \text{ is a } \Theta \text{-structure such that } \mathcal{A}_{|\Sigma_i} \in \mathbf{K}_i \text{ for all } i \in \{1, \ldots, n\} \}.
\]

Theorem 3.1 [25, Corollary 2] Let \( \Sigma_1, \Sigma_2, \ldots, \Sigma_n, n \geq 2 \), be pairwise disjoint sets of relational symbols, and for each \( i \in \{1, 2, \ldots, n\} \) let \( \mathbf{K}_i \) be a
class of $\Sigma_i$-structures having (HP), (JEP), (SAP) and the Ramsey property. Then $\bigotimes_{i=1}^n K_i$ has the Ramsey property.

In [14] we devised a technique to transfer the Ramsey property from a category to its (not necessarily full) subcategory, as follows. Consider a finite, acyclic, bipartite digraph with loops where all the arrows go from one class of vertices into the other and the out-degree of all the vertices in the first class is 2 (modulo loops), see Fig. 3. Such a digraph can be thought of as a category (where the loops represent the identity morphisms), and will be referred to as a binary category.

A binary diagram in a category $\mathbf{C}$ is a functor $F : \Delta \to \mathbf{C}$ where $\Delta$ is a binary category, $F$ takes the top row of $\Delta$ onto the same object, and takes the bottom row of $\Delta$ onto the same object, Fig. 4. A subcategory $\mathbf{D}$ of a category $\mathbf{C}$ is closed for binary diagrams if every binary diagram $F : \Delta \to \mathbf{D}$ which is consistent in $\mathbf{C}$ is also consistent in $\mathbf{D}$.

**Theorem 3.2** [14] Let $\mathbf{C}$ be a category such that $\text{hom}_\mathbf{C}(A, B)$ is finite for all $A, B \in \text{Ob}(\mathbf{C})$ and such that every morphism in $\mathbf{C}$ is monic. Let $\mathbf{D}$ be a
(not necessarily full) subcategory of $C$. If $C$ has the Ramsey property and $D$ is closed for binary diagrams, then $D$ has the Ramsey property.

We will use the following special case of the previous theorem.

**Corollary 3.3** Let $C$ be a class of finite structures and $K$ a subclass of $C$. If $C$ has the Ramsey property and $K$ is closed for binary diagrams (of structures and embeddings), then $K$ has the Ramsey property.

### 4 The proof

We are now ready to prove Theorem 1.2.

**Proof.** Let $T = \{(1, 2, \ldots, t), \preceq\}$, $t \in \mathbb{N}$, be a template. Let $k_1, k_2, \ldots, k_m$ be all the isolated points in $T$, let $T' = T \setminus \{k_1, k_2, \ldots, k_m\}$ and let $T'' = T|_{T'}$. Clearly, here are no isolated points in $T''$, so every maximal element of $T'$ is above an element of $T''$. Let us fix a list $(i_1, j_1), (i_2, j_2), \ldots, (i_s, j_s)$ of all the pairs $(i_\alpha, j_\alpha)$ of elements of $T'$ such that $i_\alpha \prec j_\alpha$ in $T'$ and $j_\alpha$ is a maximal element in $T'$.

Let $\mathbf{K}(T)$ be the class of structures defined as follows. For each $(A, \preceq_1, \ldots, \preceq_t) \in \mathbf{K}(T)$ the class $\mathbf{K}(T)$ contains

$$(A, \preceq_{i_1}, \preceq_{i_2}, \ldots, \preceq_{i_s}, \preceq_{j_1}, \preceq_{j_2}, \ldots, \preceq_{j_s}, \preceq_{k_1}, \preceq_{k_2}, \ldots, \preceq_{k_m})$$

and these are the only structures in $\mathbf{K}(T)$. Clearly, $\mathbf{K}(T)$ and $\mathbf{K}(T)$ are isomorphic as categories (where morphisms are embeddings), so it suffices to show that $\mathbf{K}(T)$ has the Ramsey property.

It is easy to see that a structure $(A, \preceq_1, \preceq_2, \ldots, \preceq_{2s+m})$ with $2s + m$ binary relations on $A$ belongs to $\mathbf{K}(T)$ if and only if

- (MP1) $\preceq_\alpha$ are partial and $\preceq_{s+\beta}$ linear orders on $A$ for all $\alpha \in \{1, \ldots, s\}$ and $\beta \in \{1, \ldots, s + m\}$;
- (MP2) $(\preceq_\alpha) \subseteq (\preceq_\beta)$ whenever $i_\alpha \preceq i_\beta$, for all $\alpha, \beta \in \{1, \ldots, s\}$;
- (MP3) $(\preceq_\alpha) \subseteq (\preceq_{s+\beta})$ whenever $i_\alpha \preceq j_\beta$, for all $\alpha, \beta \in \{1, \ldots, s\}$.
easy to see that by Theorem 3.1 and Examples 2.1 and 2.3.

∈ \{α \in \{1, \ldots, s\} \} but fixed linear extension of \{C \}

\{\beta \} \leq \{\alpha \} \in \{1, \ldots, s\}

\{\alpha \} \in \{1, \ldots, s\}

Let us show that \(C(s, m)\) has the Ramsey property. Let \(Σ_α = \{≤_α, ≤_{s+α}\}\) for \(α \in \{1, \ldots, s\}\) and \(Σ_{s+β} = \{≤_{2s+β}\}\) for \(β \in \{1, \ldots, m\}\). Now, for \(α \in \{1, \ldots, s\}\) let \(K_α\) be the class EPos but over the signature \(Σ_α\), and for \(β \in \{1, \ldots, m\}\) let \(K_{s+β}\) be the class Ch but over the signature \(Σ_{s+β}\). It is easy to see that \(C(s, m) = \bigotimes_{α=1}^{s+m} K_α\), whence follows that \(C(s, m)\) has the Ramsey property by Theorem 3.1 and Examples 2.1 and 2.3.

As we have seen, \(\mathcal{K}(T)\) is a subclass of a Ramsey class \(C(s, m)\). By Corollary 3.3, in order to show that \(\mathcal{K}(T)\) has the Ramsey property it suffices to show that \(\mathcal{K}(T)\) is closed for binary diagrams (of structures and embeddings) in \(C(s, m)\).

Take any \(A, B ∈ \mathcal{K}(T)\) and let \(F : Δ → \mathcal{K}(T)\) be a binary diagram that takes the top row of \(Δ\) onto \(B\) and the bottom row of \(Δ\) onto \(A\). Assume that \(F\) is consistent in \(C(s, m)\) and let \(C = (C, \leq_1, \leq_2, \ldots, \leq_{2s+m})\) together with the embeddings \(e_1, e_2, \ldots, e_n : B \hookrightarrow C\) be a compatible cone in \(C(s, m)\) over \(F\). Define \(D = (D, \leq^D_1, \leq^D_2, \ldots, \leq^D_{2s+m})\) as follows. Let \(D = e_1(B) ∪ e_2(B) ∪ \ldots ∪ e_n(B)\). For every partial order \(⊆\) on \(D\) there are many ways to choose a linear extension of \(⊆\). Let \(⊆_{lin}\) denote an arbitrary but fixed linear extension of \(⊆\) on \(D\). Now, for each \(α \in \{1, \ldots, s\}\) and \(β \in \{1, \ldots, s + m\}\) let

\[≤^α_{D} = (≤^C_{α e_1(B)} ∪ ≤^C_{α e_2(B)} ∪ \ldots ∪ ≤^C_{α e_n(B)})^+\]

and

\[≤^D_{s+β} = ((≤^C_{s+β e_1(B)} ∪ ≤^C_{s+β e_2(B)} ∪ \ldots ∪ ≤^C_{s+β e_n(B)})^+_{lin})\]

where \(^+\) denotes the transitive closure of a binary relation. Let us show that \(D ∈ \mathcal{K}(T)\), or equivalently, that \(D\) satisfies (MP1–4).

(MP1) is obvious and follows directly from the definition of \(≤^α_{D}\) and \(≤^D_{s+β}\).

(MP2) is also easy to confirm. Assume that \(i_α ≤ i_β\) for some \(α, β \in \{1, 2, \ldots, s\}\). Then \((≤^D_α) \subseteq (≤^D_β)\) because \(B ∈ \mathcal{K}(T)\). Since all the \(e_i\)’s are embeddings, \((≤^C_{α e_1(B)}) \subseteq (≤^C_{β e_1(B)})\) whence \((≤^D_α) \subseteq (≤^D_β)\).

(MP3) and (MP4) follow by analogous arguments, having in mind that \(⊆_{lin}\) is an arbitrary but fixed linear extension of \(⊆\).
Finally, define $f_1, f_2, \ldots, f_n : B \rightarrow D$ by $f_i(b) = e_i(b)$ for each $b \in B$ and $i \in \{1, \ldots, n\}$ and let us show that $f_i$'s are embeddings $B \hookrightarrow D$. Fix an $i \in \{1, \ldots, n\}$. Clearly, it suffices to prove that by taking transitive closures to obtain the partial and linear orders $\leq^D_{\alpha}$ and $\leq^D_{s+\beta}$ we do not change the orders restricted to $e_i(B)$. This is obvious for the linear orders, since the restriction to $e_i(B)$ of $\leq^C_{s+\beta}$ was already a linear order. As for the partial orders, notice that $(\leq^D_{\alpha}) \subseteq (\leq^C_{s+\beta})$ for each $\alpha$, since $\leq^D_{\alpha}$ is a transitive closure of a subrelation of $\leq^C_{\alpha}$ and hence the restriction to $e_i(B)$ remains the same.

Therefore, $D$ together with the embeddings $f_1, f_2, \ldots, f_n : B \hookrightarrow D$ is a compatible cone in $\mathbf{K}(T)$ over $F$, which completes the proof. □

5 Acknowledgements

The authors would like to thank the two anonymous referees who shared their insights into the historical development of identifying various Ramsey classes based on partial orders and (often intertwined) paths to their solutions.

The second author gratefully acknowledges the support of the Grant No. 174019 of the Ministry of Education, Science and Technological Development of the Republic of Serbia.

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