On Axially Symmetric Solutions of Fully Nonlinear Elliptic Equations

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1 Introduction

In this paper we study a class of fully nonlinear second-order elliptic equations of the form

\[ F(D^2 u) = 0 \]

defined in a domain of \( \mathbb{R}^n \). Here \( D^2 u \) denotes the Hessian of the function \( u \). We assume that \( F \) is a Lipschitz function defined on \( S^2(\mathbb{R}^n) \) of the space of \( n \times n \) symmetric matrices. Recall that (1) is called uniformly elliptic if there exists a constant \( C = C(F) \geq 1 \) (called an ellipticity constant) such that

\[ C^{-1} \| N \| \leq F(M + N) - F(M) \leq C \| N \| \]

for any non-negative definite symmetric matrix \( N \); if \( F \in C^1(D) \) then this condition is equivalent to

\[ \frac{1}{C'} |\xi|^2 \leq F_{u_{ij}} \xi_i \xi_j \leq C' |\xi|^2, \forall \xi \in \mathbb{R}^n. \]

Here, \( u_{ij} \) denotes the partial derivative \( \partial^2 u / \partial x_i \partial x_j \). A function \( u \) is called a classical solution of (1) if \( u \in C^2(\Omega) \) and \( u \) satisfies (1). Actually, any classical solution of (1) is a smooth \( (C^{\alpha+3}) \) solution, provided that \( F \) is a smooth \( (C^\alpha) \) function of its arguments.

For a matrix \( S \in S^2(\mathbb{R}^n) \) we denote by \( \lambda(S) = \{ \lambda_i : \lambda_1 \leq ... \leq \lambda_n \} \in \mathbb{R}^n \) the (ordered) set of eigenvalues of the matrix \( S \). Equation (1) is called a Hessian equation ([T1],[T2] cf. [CNS]) if the function \( F(S) \) depends only on the eigenvalues \( \lambda(S) \) of the matrix \( S \), i.e., if

\[ F(S) = f(\lambda(S)), \]

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for some function $f$ on $\mathbb{R}^n$ invariant under permutations of the coordinates.

In other words the equation (1) is called Hessian if it is invariant under the action of the group $O(n)$ on $S^2(\mathbb{R}^n)$:

$$\forall O \in O(n), \ F(O \cdot S \cdot O) = F(S).$$

Consider the Dirichlet problem

$$\begin{cases}
F(D^2u) = 0 \quad \text{in } \Omega \\
u = \varphi \quad \text{on } \partial \Omega,
\end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary $\partial \Omega$ and $\varphi$ is a continuous function on $\partial \Omega$.

The main goal of this paper is to show that the axially symmetric solutions of the Dirichlet problem are classical for Hessian elliptic equations. Recall that without the symmetricity assumption this can be false in higher dimensions [NV1, NV2].

Let $\Omega \subset \mathbb{R}^3$ be a smooth bounded axially symmetric domain. We consider the Dirichlet problem (3) in $\Omega$.

**Theorem 1.** Let $F \in C^1$ be a uniformly elliptic operator. Let $\varphi \in C^{1,\epsilon}(\partial \Omega)$ be an axially symmetric function, $0 < \epsilon < \epsilon_0$, where $\epsilon_0 > 0$ depends on the ellipticity constant of $F$. Then the Dirichlet problem (3) has a unique classical solution $u \in C^2(\Omega) \cap C^{1,\epsilon}(\bar\Omega)$.

**Remark.** The same results hold for the solutions of the $n$-dimensional axially symmetric problems (i.e., for the solutions of the form $u(x) = u(x_1, x_2^2 + \ldots + x_n^2)$).

The axially symmetric problems are essentially 2-dimensional. Outside the axis of symmetry one can rewrite the equations as two-dimensional fully non-linear equations with lower order terms. However on the axis of symmetry the equations became singular and that limits the application of the strong methods known for the dimension 2.

## 2 Proof of Theorem 1

Let $\Omega \subset \mathbb{R}^n$. Let

$$(2.1) \quad Lw = \sum a_{ij}(x) \frac{\partial^2 w}{\partial x_i \partial x_j},$$

be a linear uniformly elliptic operator defined in a domain $\Omega \subset \mathbb{R}^n$,

$$C^{-1} |\xi|^2 \leq \sum a_{ij} \xi_i \xi_j \leq C |\xi|^2.$$
We will need the following propositions, see [GT], [K].

**Proposition 1.** Let $G \subset \mathbb{R}^n$ be a bounded domain with a smooth boundary. Let $u \in C^2(\bar{G})$ be a solution of the equation
\[ Lu = 0 \quad \text{in} \quad G, \]
$u|_{\partial G} = \phi$. Then
\[ ||Du||_{C^\alpha(\partial G)} \leq C||\varphi||_{C^{1,\alpha}(\partial G)}, \]
where positive constants $\alpha$ and $C$ depend on $G$ and the ellipticity constant of the operator $L$.

**Proposition 2.** Assume that $F \in C^1$, $F(0) = 0$, $\partial \Omega \in C^2$ and the uniform ellipticity condition (2') holds. Let $u \in C^2(\bar{\Omega})$ be a solution of the Dirichlet problem (3). Then
\[ ||Du||_{C^\alpha(\partial \Omega)} \leq C||\varphi||_{C^{1,\alpha}(\partial \Omega)}, \]
where positive constants $\alpha$ and $C$ depend on $\Omega$ and on the ellipticity constants of $F$.

Two following propositions are essentially two-dimensional, see [BJS], [GT].

**Proposition 3.** Let $u \in C^2(D_1)$, where $D_r \subset \mathbb{R}^2$ be the disk $|x| < r$, and let $u$ be a solution in $D_1$ of the equation
\[ Lu = 0, \]
where $L$ is the elliptic operator (2.1). Then
\[ \text{osc}_{D_1} u_{x_1} \geq (1 + \xi)\text{osc}_{D_{1/2}} u_{x_1}, \]
where $\xi > 0$ be a constant depending only on the ellipticity constant of operator $L$.

**Proposition 4.** Let $u \in C^2(D_1)$ be a solution of a fully nonlinear elliptic equation
\[ H(D^2 u, Du, x) = 0 \]
in $D_1$, and $H(0, 0, x) = 0$. Let $|u| < M$. Then
\[ ||u||_{C^{2,\alpha}(D_{1/2})} < CM, \]
where $\alpha, C > 0$ are constants depending on the ellipticity constant of $H$ and $C^1$-norm of the function $H$.

As a corollary of Proposition 3 we have

**Lemma 1.** Let $u \in C^2(D_1)$ be a solution of the equation
\[ Lu = 0, \]
in $D_1$ and $l$ an affine linear function in $D_1$. Let $|l - u| < M$. Then for any $\epsilon > 0$ there are $\alpha, r > 0$ depending only on $\epsilon$ and the ellipticity constant of $L$ such that

$$||u - l||_{C^{1,\alpha}(D,r)} < \epsilon M.$$ 

Applying Lemma 1 to the derivative of the solutions of fully nonlinear elliptic equation we get

**Lemma 2.** Let $u \in C^2(D_1)$ be a solution of the fully nonlinear equation

$$F(D^2u) = 0,$$

in $D_1$ and $F(0) = 0$. Let $q$ be a quadratic polynomial in $D_1$ such that $|q - u| < M$. Then for any $\epsilon > 0$ there are $\alpha, \rho > 0$ depending only on $\epsilon$ and the ellipticity constant of $F$ such that

$$||u - q||_{C^{2,\alpha}(D,\rho)} < \epsilon M.$$ 

Proving Theorem 1 we may assume without loss that $F(0) = 0$.

Let $x_1, x_2, x_3$ be an orthonormal coordinate system in $\mathbb{R}^3$ and $x_1$ be an axis of symmetry of the domain $\Omega$. Denote

$$\omega = \{x \in \Omega, x_3 = 0\}.$$ 

Let $u$ be a classical axially symmetric solution of the Dirichlet problem (3). Denote

$$||u||_{C(\Omega)} = A.$$ 

Since $u_{x_3}$ is a solution of linear uniformly elliptic equation $Lu_{x_3} = 0$ and $u_{x_3} = 0$ on $\omega$ then by Proposition 1

(2.1) 

$$||u_{x_3,x_3}||_{C^{\alpha}(\omega')} \leq C||\varphi||_{C^{1,\alpha}(\partial G)},$$

where $\omega' \subset \subset \omega$, positive constants $\alpha$ and $C$ depend on $G, \omega'$ and the ellipticity constant of the operator $F$.

We define two-dimensional Hessian elliptic operators $f_a, a \in \mathbb{R}$,

$$f_a(\lambda_1, \lambda_2) = f(\lambda_1, \lambda_2, a).$$ 

Let $y \in \Omega$ be a point on the axis $x_1$. Denote

$$h = dist(y, \partial \Omega).$$

Define for $0 < r < h$ the function $u_r$ on the unit disk $D_1 \subset \mathbb{R}^2$ by

$$u_r(x) = u_r(x_1, x_2) = (u(r(x_1 - y_1, x_2)) - u(y))/r^2.$$
Set \( a = u_{x_2x_2}(y) \). Let \( v_r \) be a solution of the Dirichlet problem

\[
(2.2) \quad \begin{cases}
  f_a(\lambda(D^2v_r)) = 0 \quad \text{in } D_1 \\
  v_r = u_r \quad \text{on } \partial D_1,
\end{cases}
\]

The classical solution of two-dimensional Dirichlet problem (2.2) is known to exist, e.g. [GT].

Since our equation \( F(D^2u) = 0 \) is homogeneous we can assume without loss that the inequalities

\[ 1 < |\nabla F| < C \]

hold for a positive constant \( C \).

From (2.1) and the last inequalities it follows easily that the functions \( u_r - C_0r^\alpha(1 - |x|^2) \) and \( u_r + C_0r^\alpha(1 - |x|^2) \) are, for a sufficiently large constant \( C_0 \), sub- and supersolutions of the Dirichlet problem (2.2). Hence

\[ |u_r - v_r| \leq C_0r^\alpha. \]

Denote, \( w_r = u_r - v_r \).

Let \( \rho \) be the constant of Lemma 2 for the elliptic operator \( f \) and \( \epsilon = 1/2 \).

Define a sequence of functions \( u_n \) in \( D_1, n = 1, 2, \ldots \), by

\[ u_n = u_{h\rho^n}. \]

Correspondingly we define \( v_n = v_{h\rho^n}, w_n = u_n - v_n \).

From Lemma 2 we get the following recurrence inequalities: there are quadratic polynomials \( q_n, n = 1, 2, \ldots \), such that \( f(q_n) = 0 \) and

\[
||v_{n+1} - q_{n+1}||_{C(D_1)} \leq \frac{1}{2}||v_n - q_n||_{C(D_1)} + C_0\rho^n.
\]

Since \( |u_1| < A/h^2 \), we get

\[
||v_n - q_n||_{C(D_1)} < 2AC_0\rho^n/h^2,
\]

\[
||u_n - q_n||_{C(D_1)} < 2AC_0\rho^n/h^2,
\]

\[
||w_n||_{C(D_1)} < C_0\rho^n.
\]

for all \( n = 1, 2, \ldots \).

Hence, since the functions \( u_n \) are obtained as dilations of \( u \), it follows that

\[
(2.3) \quad ||q_{n+1} - q_n||_{C(D_1)} < 2AC_0\rho^{n-2}/h^2.
\]

Therefore

\[
(2.4) \quad ||u_n|| < AC_1/h^2
\]

for a constant \( C_1 > 0 \) depending only on the ellipticity constant of \( F \), \( n = 1, 2, \ldots \).
Denote

\[ E = \{ z = x + y : |x| < h/2, x_2/x_1 > 1/4 \}, \]
\[ G = \{ x \in D_1 : x_2 > 1/4, \text{dist}(x, \partial D_1) > 1/4 \}, \]
\[ G_n = \{ x : x/h\rho^n \in G \}, \]
\[ n = 1, 2, \ldots \]

Set \( g_n = u_n - q_n \). Then from (2.3), (2.4) and Proposition 4 we have

\[ \| g_n \|_{C^{1,\alpha}(G)} < AC_2/h^2, \]

where \( C_2 > 0 \) depends only on the ellipticity constant of \( F \). Since

\[ \| g_n \|_{C(D_1)} < 2AC_0\rho^{\alpha n}/h^2 \]

then by interpolation between the last two inequalities we get

\[ \| g_n \|_{C^{1,\alpha/2}(G_n)} < AC_3\rho^{\alpha n/2}/h^2, \]

where \( C_3 > 0 \) depends on the ellipticity constant of the equation. Thus

\[ \| u \|_{C^{1,\alpha/2}(G_n)} < AC_3/h^2, \]

for all \( n = 1, 2, \ldots \). Together with (2.3) the last inequality gives

(2.5) \[ \| u \|_{C^{1,\alpha/2}(E)} < AC_4/h^2, \]

where \( C_4 > 0 \) depends only on the ellipticity constant of the equation.

By (2.1) on the axis \( x_1 \) the second derivatives \( u_{x_2x_2} = u_{x_3x_3} \) satisfy the Hölder estimates. Since on the axis the mixed derivatives \( u_{x_ix_j} = 0 \) for \( i \neq j \) we conclude from the equation that the second derivative \( u_{x_1x_1} \) satisfies the Hölder estimates as well. These estimates together with (2.5) give the following inequality

\[ \| u \|_{C^{1,\alpha/2}(\omega')} < AC_5, \]

where \( C_5 > 0 \) depends on the ellipticity constant of the equation and the distance of \( \omega' \) to the boundary \( \partial \omega \).

Combining the last inequality with Proposition 2 we get the following apriori estimate for the axially symmetric solutions of fully nonlinear uniformly elliptic equations:

**Lemma 3.** Let \( u \in C^2(\Omega) \) be an axially symmetric solution of (3) and let \( \Omega' \) be a compact subdomain of \( \Omega \). Then the following inequalities hold:

\[ \| u \|_{C^{1,\alpha}(\Omega)} \leq C\| \varphi \|_{C^{1,\alpha}(\partial \Omega)}, \]
\[ \| u \|_{C^2(\Omega')} \leq C'\| \varphi \|_{C^{1,\alpha}(\partial \Omega)}, \]
where positive constants $C, C'$ and $\alpha$ depend on $\Omega$ and on the ellipticity constant of $F$, $C'$ depending also on the distance of $\Omega'$ to the boundary $\partial\Omega$.

The apriori estimate of Lemma 3 and the standard method of continuation by parameter, see, e.g., [GT], gives the classical solvability of the Dirichlet problem (3) for a uniformly elliptic equation.

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