The Statistical Significance of the Low CMB Multipoles

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ABSTRACT
The Wilkinson Microwave Anisotropy Probe (WMAP) has measured lower amplitudes for the temperature quadrupole and octopole anisotropies than expected in the best fitting (concordance) Λ-dominated cold dark matter (ΛCDM) cosmology. Some authors have argued that this discrepancy may require new physics. Yet the statistical significance of this result is not clear. Some authors have applied frequentist arguments and claim that the discrepancy would occur by chance about 1 time in 700 if the concordance model is correct. Other authors have used Bayesian arguments to claim that the data show marginal evidence for new physics. I investigate these confusing and apparently conflicting claims in this paper using a frequentist analysis and a simplified Bayesian analysis. On either analysis, I conclude that the WMAP results are consistent with the concordance ΛCDM model.

Key words: cosmic microwave background, cosmology.

1 INTRODUCTION
The WMAP satellite (Bennett et al. 2003a; Hinshaw et al. 2003; Spergel et al. 2003) has led to a precise measurement of the temperature anisotropy power spectrum, $C_\ell$, from multipoles $\ell = 2$ to $\ell \sim 600$. The observed temperature power spectrum is in striking agreement with the predictions of the ‘concordance’ inflationary ΛCDM cosmology with parameters consistent with those inferred from observations made prior to WMAP (compare, for example, Wang et al. 2002 and Spergel et al. 2003).

However, as pointed out by the WMAP team there may be a discrepancy between the predictions of the ΛCDM models and the observations at low multipoles. A low amplitude of the CMB quadrupole was first found by COBE (Hinshaw et al. 1996), but the new WMAP observations have led to a more accurate measurement and to tighter control of systematic errors caused by residual foreground emission from the Galaxy. The amplitude of the octopole measured by WMAP is also low compared to the best fitting ΛCDM model and the temperature autocorrelation function $C(\theta)$ shows an almost complete lack of signal on angular scales ≳ 60 degrees. Spergel et al. (2003, hereafter S03), quantify the latter discrepancy by computing the statistic

$$S = \int_{-1}^{1/2} [C(\theta)]^2 \, d\cos \theta$$

(1)

for a large number of simulated skies generated from the posterior distribution of the ΛCDM cosmology. They conclude that the probability of finding a value of $S$ smaller than that observed is about $1.5 \times 10^{-3}$. This low probability, if correct, suggests a discrepancy between the ΛCDM cosmology and the observed low CMB multipoles, indicating a need for new physics. Indeed, a number of authors have explored various models that might reproduce the low quadrupole and octopole. For example, S03 and Tegmark, de Oliveira Costa and Hamilton (2003) suggest that the effect might be associated with the small size of a finite universe, while Efstathiou (2003a) and Contaldi et al. (2003) have proposed a cut-off in the primordial power spectrum associated with spatial curvature. Cline, Crotty and Lesgourgues (2003) and Feng and Zhang (2003) consider multi-field inflation models, while DeDeo, Caldwell and Steinhardt (2003) consider quintessence models with an equation of state that leads to a partial cancellation of the usual integrated Sachs-Wolfe effect. Evidently, theorists are not short of ideas that might account for the observations.

But is new physics necessary? Is the probability of $1.5 \times 10^{-3}$ derived in S03 correct, or has the significance of the discrepancy been overestimated? Do modified models provide statistically significantly better fits to the data than the concordance ΛCDM model? Some of the recent literature on these points is confusing. For example, Bridle et al. (2003), Cline et al. (2003) and Contaldi et al. (2003) perform Bayesian analyses of the WMAP data to test whether the low multipoles require a sharp break in the primordial spectrum. Although the data favour a break at a wavenumber $k_c \sim 3 \times 10^{-3}$ Mpc$^{-1}$, the concordance model with $k_c = 0$ is not strongly excluded. Is this conclusion compatible with the SO3 analysis of the $S$ statistic? The questions raised in this paragraph are addressed in this paper. Tegmark et al. (2003) comment on a possible alignment between the quadrupole and octopole. This effect is ignored in this paper, which focuses exclusively on the statistical significance of the amplitudes of the quadrupole and octopole. For an
2 OBSERVATIONS AND FIDUCIAL CONCORDANCE MODEL

It has become common to use Monte-Carlo Markov Chains (MCMC) to evaluate the posterior distributions of cosmological parameters given observations of the CMB power spectra and their covariance matrices (see Christensen et al. 2001; Lewis and Bridle 2002; Verde et al. 2003). As a fiducial model, we follow the MCMC analysis of Bridle et al. (2003) and adopt a spatially flat $\Lambda$CDM cosmology specified by 6 parameters, a constant scalar spectral index $n_s$, spectral amplitude $A_s$, Hubble constant $h = H_0/(100\text{km s}^{-1}\text{Mpc}^{-1})$, baryon density $\omega_b \equiv \Omega_b h^2$, CDM density $\omega_c \equiv \Omega_c h^2$ and redshift of reionization $z_{\text{eff}}$. Tensor modes are ignored in this analysis. The input CMB data consists of the WMAP temperature and polarization cross-correlation power spectra (and associated programmes to compute the likelihood function, see Verde 2003). The power spectra and their covariance matrices (see Christensen et al. 2003) are used to generate Figure 1, but some have been adjusted slightly so that they are consistent with other data, e.g. the HST key project measurement of the Hubble constant (Freedman et al. 2001). The quadrupole and octopole amplitudes for this fiducial model are $\Delta T^2_2 = 1140 \, \mu K^2$ and $\Delta T^3_3 = 1060 \, \mu K^2$. To illustrate the sensitivity to the parameters of the fiducial model, we will show how various results change if the quadrupole and octopole amplitudes are lowered to $\Delta T^2_2 = 1000 \, \mu K^2$ and $\Delta T^3_3 = 930 \, \mu K^2$, i.e. towards the lower end of the allowed range according to Figure 1.

The WMAP quadrupole and octopole amplitudes in the publicly available data release are given as $\Delta T^2_2 = 123 \, \mu K^2$ and $\Delta T^3_3 = 611 \, \mu K^2$. The quadrupole amplitude in particular (shown by the dashed line in Figure 1) is much lower than the amplitude of the fiducial $\Lambda$CDM model. The Bennett et al. (2003a) WMAP summary paper lists the quadrupole amplitude at 154 $\pm$ 70 $\mu K^2$, slightly higher than the value in the public data release. The error on this number is a 95% confidence limit on the uncertainty associated mainly from modelling foreground Galactic emission. (For comparison, the quadrupole amplitude measured by COBE is $\Delta T^2_2 = 240 \pm 140 \, \mu K^2$, Hinshaw et al. 1996). Full details of how the WMAP team arrive at this error estimate have not yet been published, but it would seem to be reasonable since Tegmark et al. (2003) find a quadrupole amplitude of 202 $\mu K^2$ (including the small $\sim 4\mu K^2$ contribution from the kinematic quadrupole) from an analysis of their all-sky foreground subtracted WMAP map. Tegmark et al. (2003) find an octopole amplitude of 866 $\mu K^2$ from their all-sky map. We adopt these numbers (which for the quadrupole is at the upper end of the Bennett et al. error range) to illustrate the effects of systematic uncertainties associated with contamination by Galactic emission. Tegmark et al. find similar numbers for an all-sky analysis of the WMAP internal linear combination foreground subtracted CMB map (Bennett et al. 2003b).

3 FREQUENTIST STATISTICS

In the absence of a sky cut and instrumental noise, the distribution of $C_\ell$ estimates in a theory with Gaussian amplitudes $a_{\ell m}$ follows a $\chi^2$ distribution,

$$dP(C_\ell) \propto \left( \frac{C_\ell}{C_\ell^T} \right)^\frac{2\ell+1}{2} \exp \left( -\frac{(2\ell+1)C_\ell}{2C_\ell^T} \right) \frac{dC_\ell}{C_\ell^T},$$

where $C_\ell^T$ is the expectation value of $C_\ell$. Integrating equation (2), the probability of observing a value $\leq C_\ell$ is given by

$$P(\leq C_\ell) = \frac{\gamma \left( \frac{2\ell+1}{2}, \frac{2\ell+1}{2} \frac{C_\ell}{C_\ell^T} \right)}{\Gamma \left( \frac{2\ell+1}{2} \right)},$$

where $\gamma$ is the incomplete Gamma function.

In practice, the actual distribution depends on the estimator of $C_\ell$, the shape of any Galactic cut and, of course, instrumental noise and other sources of error. Figure 2 shows a histogram of quadrupole amplitudes determined by applying a pseudo-$C_\ell$ estimator (see e.g. Hivon et al., 2002) to a large number of simulated noise-free maps generated using the power spectrum of the fiducial $\Lambda$CDM model discussed.

* The chains have been made available by Antony Lewis at the following web site http://cosmologist.info/cosmomc/.

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in the previous section. The Kp2 Galactic cut imposed by Hinshaw et al. (2003) was used in the simulations. Figure 2 shows the resulting distribution of quadrupole amplitudes, together with a $\chi^2$ distribution (see also Wandelt, Hivon, Gorski, 2001). The effects of Galactic cuts on pseudo-$C_\ell$ estimators is discussed in detail by Efstathiou (2003b, c), however, for the modest Galactic cuts used in the analysis of WMAP, the quadrupole amplitude is weakly correlated with higher multipoles and its distribution follows a $\chi^2$ distribution quite accurately with a variance that is only marginally greater than the cosmic variance.

The corresponding frequentist statistics are given in Table 1(a) and agree with equation (2) to within about 50%. From this Table, we see that the probability of observing a quadrupole lower than $123\mu K^2$ is about 1.3% and that the joint probability of finding quadrupole and octopole amplitudes smaller than those observed is about 0.32% which is about twice the value inferred by SO3 from their analysis of the $S$ statistic. In fact, SO3 compare the $S$ statistic with simulating generated from their MCMC chains. If we rescale the quadrupole and octopole amplitudes of our simulations so that the amplitudes follow the MCMC distribution plotted in Figure 1, the joint probability for the quadrupole and octopole amplitudes drops from 0.32% to 0.21% only slightly larger than the value of 0.15% deduced by SO3 from the $S$ statistic. Given that the integration range of the $S$ statistic was chosen a posteriori, it is not suprising that SO3 find a slightly more significant discrepancy. The main conclusion to draw from this analysis is that the significance level deduced by SO3 from the $S$ statistic is understandable; a similar significance level is deduced from the quadrupole and octopole amplitudes alone. Most of the weight in the $S$ statistic is coming from the quadrupole and octopole amplitudes and any ‘a posteriori bias’ in the statistic is small.†

Table 1a: Frequentist estimates of the Quadrupole and Octopole Discrepancy

| $(\Delta T^2_q)^T$ | $(\Delta T^2_o)^T$ | $P(\Delta T^2_q < 123)$ | $P(\Delta T^2_o < 123)$ | $P(\Delta T^2_q < 123, \Delta T^2_o < 611)$ | $P(\Delta T^2_q < 202)$ | $P(\Delta T^2_o < 870)$ | $P(\Delta T^2_q < 202, \Delta T^2_o < 870)$ |
|-------------------|-------------------|-------------------|-------------------|-------------------|-------------------|-------------------|-------------------|
| 1140              | 1060              | 0.013             | 0.24              | 0.0032            | 0.036             | 0.44              | 0.016             |
| 1000              | 930               | 0.017             | 0.31              | 0.0054            | 0.046             | 0.53              | 0.025             |

Table 1b: Bayesian estimates of the Quadrupole and Octopole Discrepancy

| $\Delta T^2_q$ | $\Delta T^2_o$ | $P((\Delta T^2_q)^T > 1140)$ | $P((\Delta T^2_q)^T > 1060)$ | $P((\Delta T^2_q)^T > 1000)$ | $P((\Delta T^2_q)^T > 930)$ |
|----------------|----------------|-----------------------------|-----------------------------|-------------------------------|-----------------------------|
| 123            | 611            | 0.087                       | 0.45                        | 0.10                          | 0.53                        |
| 202            | 870            | 0.16                        | 0.66                        | 0.19                          | 0.73                        |

Note: Table 1a gives the frequencies that the observed amplitudes $\Delta T^2_q$ and $\Delta T^2_o$ will be less than the specified values (expressed in $\mu K^2$) if the true amplitudes are $(\Delta T^2_q)^T$ and $(\Delta T^2_o)^T$ (see Section 3). Table 1b gives the Bayesian frequencies that the observed values of quadrupole and octopole amplitudes are drawn for a model with true quadrupole and octopole amplitudes greater than $(\Delta T^2_q)^T$ and $(\Delta T^2_o)^T$ (see Section 4).

† We note here that Gaztañaga et al., 2003, find much higher probabilities by applying a frequentist statistic to estimates of $C(\theta)$ from WMAP using a conservative Galactic cut. Their results are puzzling, however, because: (i) they find that their statistic is extremely sensitive to the size and shape of the Galactic cut; (ii) their results are inconsistent with the pseudo-$C_\ell$ estimator used by the WMAP team and the quadratic maximum likelihood estimator used by Efstathiou (2003c).
Since the systematic errors in $C^{TE}$ from Galactic emission have not yet been quantified fully, and since the random errors are large, we ignore the WMAP $C^{TE}$ measurements in the rest of this paper.) The joint probability of finding quadrupole and octopole amplitudes below these values given the fiducial ΛCDM model is about 5 times greater than the value deduced for quadrupole and octopole amplitudes of 123μK$^2$ and 611μK$^2$, an order of magnitude greater than deduced by SO3 from the S statistic. If the parameters of the ΛCDM model are adjusted to give values for the quadrupole and octopole amplitudes that lie towards the lower end of the allowed range, the joint probability for the quadrupole and octopole rises to 2.5%. We conclude that it is premature to rule out the ΛCDM cosmology because of the low quadrupole and octopole amplitudes. The true values of the quadrupole and octopole for our CMB sky have not yet been determined with a sufficiently high accuracy, and the theoretical expectations of the Λ cosmology are not constrained tightly enough, to exclude the model at a high significance level. The SO3 analysis of the S statistic exaggerates the statistical significance of any discrepancy with the ΛCDM model because they did not consider errors in the autocorrelation function arising from inaccurate subtraction of Galactic emission.

4 BAYESIAN STATISTICS

As mentioned in the Introduction, a number of papers (e.g. Bridle et al. 2003, Contaldi et al. 2003, Cline et al. 2003) have applied Bayesian methods to analyse more complex models, for example, ΛCDM models with a sharp break or truncation in the initial power spectrum on large scales. These papers ignore the possible systematic errors in the WMAP power spectra discussed above, yet even so they report no strong evidence for the introduction of any additional parameters. How is this conclusion compatible with the SO3 analysis of the S statistic or the simple frequentist tests described in the previous Section? In this Section we provide an answer by applying Bayes’ theorem to an intentionally simple model.

According to Bayes’ theorem, the posterior probability of hypothesis $H$ given the data $D$ is

$$P(H|D) \propto P(D|H)P(H),$$

(4)

where $P(D|H)$ is the probability of the data $D$ given $H$ and $P(H)$ is the prior probability of $H$. Let us adopt the hypothesis that the true amplitude $C^T$ lies in the range $C^T_{\text{fid}} \pm 2C^T_{\text{fid}}$, then applying equation (4) and assuming a uniform prior for $C^T_{\text{fid}}$, the posterior probability distribution for $C^T_{\text{fid}}$ is

$$dP(C^T_{\text{fid}}) \propto \frac{1}{(C^T_{\text{fid}})^{2\ell+1}} \exp \left( \frac{-2\ell+1}{2C^T_{\text{fid}}} \right) dC^T_{\text{fid}},$$

(5)

where $C^T_{\text{fid}}$ is the observed amplitude. Equation (5) is proportional to the likelihood function, which has its maximum value at $C^T_{\text{fid}} = C^T_{\text{fid}}$. The distributions (5) are plotted for the quadrupole and octopole in Figure 3.

The distributions plotted in Figure 3 give the posterior probabilities that the true amplitudes $C^T_{\text{fid}}$ take on any particular value and so we can use these figures to test how ‘disconnected’ the observed values of $C^T_{2}$ and $C^T_{3}$ (corresponding to the peaks of the probability distributions) are from those of the fiducial model (indicated by the vertical dashed lines). The ratio of these probabilities is $p(C^T_{2})/p((C^T_{2})_{\text{fid}} = 28$ and $p(C^T_{3})/p((C^T_{3})_{\text{fid}} = 1.6$; neither of these ratios is high and so we conclude that the observed amplitudes do not provide strong evidence against the fiducial ΛCDM model.

Why do these numbers indicate a weaker rejection of the model than the frequentist statistics of Table 1a? Let us recast the Bayesian analysis in frequentist language. Imagine that we draw values of $(C^T_{\ell})_{\text{fid}}$ from a uniform distribution between 0 and an upper limit $(C^T_{\ell})_{\text{max}}$ (the exact value of this maximum limit is unimportant as long as it extends well into the tail of the distribution (2)). For each draw, generate a random value of $C^T_{\ell}$ from the $\chi^2$ distribution (2) and for
those values that lie within a narrow interval around the observed value of \(C_\ell\) compute the frequency with which \(C_\ell^2\) exceeds a critical value \((C_\ell^2)_{\text{crit}}\). This frequency is just the integral over the probability distribution (5)

\[
P(C_\ell^2 > (C_\ell^2)_{\text{crit}}) = \int_{(C_\ell^2)_{\text{crit}}}^{\infty} dP(C_\ell^2),
\]

(where the upper limit \((C_\ell^2)_{\text{max}}\) has been replaced by infinity). Numerical values for these frequencies for the octopole and quadrupole (neglecting minor effects from a cut sky) are given in Table 1b for values of \((C_\ell^2)_{\text{crit}}\) equal to those of the fiducial model and for values at the low end of the range found from the MCMC chains. The latter numbers are the more useful because if these frequencies turn out to be low, then there is little overlap between the posterior distributions of Figure 3 and the distributions of quadrupole and octopole amplitudes from the MCMC chains. This would force us to reject the concordance ΛCDM model.

However, we find that the frequency with which \((\Delta^2)_{\text{fid}} > 10^5\mu K^2\), given the observed WMAP quadrupole of 123\(\mu\)K2 is only 0.10, and so again we conclude that the evidence against the ΛCDM model is marginal. Of course, this test is different to the frequentist test discussed in Section 3 (Table 1a), but it is easy to understand why the two tests give different impressions of a discrepancy. We can see from Figure 2 that the probability of finding a quadrupole amplitude as low as that observed, given the fiducial model, is improbably small and so if we assume a uniform prior for \(C_2^2\), low values of \(C_2^2\) simply do not have enough weight to exclude the quadrupole amplitude of the fiducial model at high significance.

Since \(C_2^2\) varies from zero to infinity, should we not have used Jeffreys’ prior (Jeffreys 1939, Jaynes 2003), \(dC_2^2/C_2^4\), thus giving extra weight to low values of \(C_2^2\)? No, because there is a natural scale for \(C_2^2\), namely the amplitude of the fiducial model \((C_2^2)_{\text{fid}}\). In any reasonable physical model, it is impossible to get a perturbation amplitude that is very much smaller than that of the fiducial model because \(C_2\) involves an integral of the perturbation spectrum over wavenumber \(k\). Since there is strong evidence in favour of the fiducial model for wavenumbers \(k > 10^{-3}\) Mpc\(^{-1}\) (Bridle et al. 2003) we should strongly disfavour models with very low values of \(C_2^2\). The assumption of a uniform prior over the range, say, \(\sim 10^{-1}(C_2^2)_{\text{fid}}\) to a few times \((C_2^2)_{\text{fid}}\) is physically reasonable and relatively benign, although (as with all of the Bayesian analyses referred to in this paper) we must recognise that there is some dependence of the posterior probabilities on the form of the prior.

In conclusion, the Bayesian frequencies given in Table 1b provide a meaningful comparison of the fiducial ΛCDM model to the WMAP data and they indicate marginal evidence for any discrepancy. In the opinion of this author, the Bayesian analysis is preferable to the frequentist analysis of Section 3 which is, in any case, inconclusive because of systematic errors in the quadrupole and octopole amplitudes. The Bayesian frequencies listed in Table 1b could be misleading only if there is persuasive evidence that the priors on \(C_2^2\) and \(C_4^2\) should be strongly skewed towards much smaller values than those of the fiducial model, in which case the low amplitudes observed by WMAP add little new information.

5 CONCLUSIONS

(i) Do the WMAP measurements of the quadrupole and octopole amplitudes conflict with the ΛCDM cosmology? Based on the quadrupole and octopole amplitudes, the answer is unambiguously no. The frequentist tests discussed in Section 3 are inconclusive because there are significant systematic errors in the WMAP quadrupole and octopole amplitudes. These errors were neglected in SO3’s analysis of the S statistic and hence their estimate of a 1 in 700 chance of reproducing the observations according to the concordance ΛCDM model is an overestimate of the true odds. The Bayesian analysis of Section 4 suggests that a more reasonable estimate of the odds is more like 1 in 10 or 1 in 20. Whatever your statistical orientation, there is no convincing evidence for a discrepancy with the concordance ΛCDM model.

(ii) Do the WMAP measurements of the quadrupole and octopole amplitudes require new physics? Despite point (i) above, the likelihood functions plotted in Figure 3 peak at lower values than those of the fiducial ΛCDM model and so will favour models which predict low quadrupole and octopole amplitudes, provided that the number of extra parameters required to describe the models is not too large. As an example, consider the analysis of Bridle et al. (2003) of a ΛCDM model with an initial spectrum truncated sharply below a wavenumber \(k = k_c\) (see their Figure 2). The WMAP data favour a truncation at \(k_c \sim 3 \times 10^{-4}\) Mpc\(^{-1}\), thus favouring new physics, but (consistent with the results of this paper) a model with \(k_c = 0\) is not strongly excluded. We conclude that the WMAP data certainly warrant exploration of models incorporating new physics, but these models had better make other testable predictions if they are ever to be strongly preferred over the concordance ΛCDM model.

(iii) Can measurements of the low CMB multipoles be improved? As mentioned in Section 2, Bennett et al. (2003a) quote an error on the quadrupole amplitude of \(\pm 70\mu K^2\) and state that this is caused largely by errors in subtracting foreground emission. This error estimate is consistent with the difference in the quadrupole amplitude measured by Tegmark et al. (2003), who use a different method to subtract Galactic emission. It may, therefore, be possible to improve on the accuracy of the quadrupole, and other low CMB multipoles, by applying better methods of foreground subtraction.

More accurate estimates of the low multipoles can be obtained by applying an optimal estimator (see e.g. Tegmark 1997) rather than the pseudo-C\(_{\ell}\) estimator used by the WMAP team. In the noise-free limit (a good approximation for WMAP on large angular scales), an optimal estimator will return almost the exact values of low multipoles on the cut sky, provided that the sky-cut is not too large. An analysis of this sort might establish whether \(\Delta T^2\) is closer to \(100\mu K^2\) or \(200\mu K^2\), which would be useful, though as explained in Section 4, the Bayesian analysis is not particularly sensitive to variations of this magnitude.}

‡ This condition can be quantified using Bayesian methods, e.g. by computing Occam factors (Jaynes 2003, Chapter 20) or Bayesian evidence (see e.g. Saini, Weller and Bridle 2003).

§ Such an analysis has been completed since this paper was sub-
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