Compact spacelike surfaces in four-dimensional Lorentz-Minkowski spacetime with a non-degenerate lightlike normal direction

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Abstract

A spacelike surface in four-dimensional Lorentz-Minkowski spacetime through the light-cone has a meaningful lightlike normal vector field $\eta$. Several sufficient assumptions on such a surface with non-degenerate $\eta$-second fundamental form are established to prove that it must be a totally umbilical round sphere. With this aim, a new formula which relates the Gauss curvatures of the induced metric and of the $\eta$-second fundamental form is developed. Then, totally umbilical round spheres are characterized as the only compact spacelike surfaces through the lightcone such that its $\eta$-second fundamental form is non-degenerate and has constant Gauss curvature two. Another characterizations of totally umbilical round spheres in terms of the Gauss-Kronecker curvature of $\eta$ and the area of the $\eta$-second fundamental form are also given.

1 Introduction

The geometry of spacelike surfaces in 4-dimensional Lorentz-Minkowski spacetime $\mathbb{L}^4$ through the future lightcone $\Lambda$ is very rich and appealing. In fact, any 2-dimensional simply-connected Riemannian manifold may be isometrically immersed into $\Lambda$, [10], [12]. In particular, any Riemannian metric on the sphere $\mathbb{S}^2$ may be realized as the induced metric of a spacelike immersion of $\mathbb{S}^2$ in $\mathbb{L}^4$ through $\Lambda$. This situation is quite different when $\Lambda$ is replaced by a (non-degenerate) hypersurface of $\mathbb{L}^4$. For instance, there is no spacelike immersion of $\mathbb{S}^2$ in the unit De Sitter spacetime $\mathbb{S}_1^4 \subset \mathbb{L}^4$ such that the Gauss curvature of the induced metric satisfies $K > 1$, [3, Cor. 10]. On the other hand, the existence of an isometric immersion of an $n(\geq 3)$-dimensional Riemannian manifold in $\mathbb{L}^{n+2}$ through the corresponding future lightcone has a clear geometric meaning; namely, an $n(\geq 3)$-dimensional simply-connected Riemannian manifold $M^n$ admits an isometric immersion in $\mathbb{L}^{n+2}$ through the lightcone if and only if $M^n$ is conformally flat [4], which is a nice characterization of conformally flatness in terms of Lorentzian geometry.

Motivated in part by these results, spacelike surfaces through the lightcone in $\mathbb{L}^4$ have been studied from different viewpoints, [8], [9], [10]. Focusing our approach here, if $\psi : M^2 \rightarrow \mathbb{L}^4$
is a spacelike immersion such that \( \psi(M^2) \subset \Lambda \), the position vector field \( \psi \) is clearly normal and lightlike. The corresponding Weingarten operator satisfies \( A_\psi = -I \), where \( I \) denotes the identity transformation, and, therefore, it provides no information on the extrinsic geometry of \( M^2 \). However, there is another lightlike normal vector field \( \eta \), uniquely defined by \( \langle \eta, \eta \rangle = 0 \) and \( \langle \psi, \eta \rangle = 1 \). The Weingarten operator \( A_\eta \) is closely related to both intrinsic and extrinsic geometry of \( M^2 \) by the equations

\[
\tr(A_\eta) = -K = -\langle H, H \rangle,
\]

where \( K \) is the Gauss curvature of the induce metric and \( H \) the mean curvature vector field of \( \psi \) (Section 2). Moreover, the lightlike normal vector fields \( \psi \) and \( \eta \) are connected to the so-called \( S^2 \)-valued Gauss maps \( G^F \) and \( G^P \), introduced in a more general context in [7], by

\[
G^F = \frac{1}{\psi_0} \psi \quad \text{and} \quad G^P = -\frac{1}{\eta_0} \eta,
\]

where \( \psi_0 \) and \( \eta_0 \) are the time coordinates of \( \psi \) and \( \eta \), respectively.

In that follows, given a spacelike surface \( M^2 \) in \( \mathbb{L}^4 \) through \( \Lambda \), we will say that \( \eta \) is non-degenerate if the \( \eta \)-second fundamental form, \( \Pi_\eta \), is a non-degenerate metric on \( M^2 \). The assumption \( \eta \) is non-degenerate has the following geometric meanings. On the one hand, \( \eta \) is non-degenerate if and only if the Gauss map \( G^P \) is a local diffeomorphism from \( M^2 \) to \( S^2 \). On the other hand, \( \eta \) is non-degenerate if and only if \( \tilde{\psi} := -\eta \) is also a spacelike immersion (Lemma 2.5). In this case, \( \tilde{\psi} \) is said to be the conjugate immersion to \( \psi \). It is remarkable that given a non-degenerate spacelike surface through \( \Lambda \), it is totally umbilical if and only if its conjugated surface is also totally umbilical (Corollary 2.8).

A compact spacelike surface \( M^2 \) in \( \Lambda \) must be topologically a sphere \( S^2 \), [12]. It is then natural to wonder for some additional assumption in order to conclude that \( M^2 \) is a totally umbilical round sphere. Recall that all the totally umbilical compact spacelike immersions of \( S^2 \) in \( \mathbb{L}^4 \) through \( \Lambda \) were explicitly constructed in [12] as follows. If \( \psi \) is such an immersion, there exist \( u \in \mathbb{L}^4 \), \( \langle u, u \rangle = -1 \), with \( u_0 < 0 \) and \( r > 0 \), such that,

\[
\psi(S^2) = S^2(u, r) := \{ x \in \mathbb{L}^4 : \langle x, x \rangle = 0, \langle u, x \rangle = r \}.
\]

In this case, \( A_\eta = -(1/2r^2)I \) and the Riemannian metric on \( M^2 \) given by \( \Pi_\eta(X, Y) := -\langle A_\eta(X), Y \rangle \) has constant Gauss curvature \( K^\eta = 2 \), [12]. As shown in [12] Theor. 5.4] a compact spacelike surface \( M^2 \) in \( \Lambda \) with constant Gauss curvature is a totally umbilical round sphere in \( \mathbb{L}^4 \). This result gives an answer to the previous question from an intrinsic point of view. In the same philosophy of [1], [2] and [3], the main aim of this paper is to obtain several extrinsic characterizations of the totally umbilical spacelike spheres in \( \mathbb{L}^4 \) among all the compact spacelike surfaces in \( \mathbb{L}^4 \) which factors through \( \Lambda \).

When a spacelike surface \( M^2 \) in \( \Lambda \) is compact, the non-degeneracy of \( \eta \) implies that \( \Pi_\eta \) is in fact Riemannian (Proposition 2.3). Our main goal here is (Theorem 4.1),

For a compact spacelike surface \( M^2 \) of \( \mathbb{L}^4 \) through \( \Lambda \) with \( \eta \) non-degenerate, the following assertions are equivalent:

1. \( M^2 \) is a totally umbilical round sphere,
2. The Gauss-Kronecker curvature \( \mathfrak{d} = \det(A_\eta) \) is constant,
3. The Gauss curvature of the Riemannian metric \( \Pi_\eta \) satisfies \( K^\eta = 2 \).
Moreover, each one of these assumptions is equivalent to the constancy of the Gauss curvature of the induced metric on $M^2$ [12, Theor. 5.4].

In order to prove this result, our main tool will be a new formula which, for any spacelike surface of $L^4$ through $\Lambda$ with non-degenerate $\eta$, relates the Gauss curvature $K$ of the induced metric, the Gauss curvature $K^\eta$ of $\Pi_\eta$ and the Gauss-Kronecker curvature $d$ (Theorem 3.1). Note that this extrinsic quantity is closely related to the notion of quartic curvature $H$ of the spacelike surface [7]. In fact, it is not difficult to see that $H = 2d$.

The paper ends with the statement of two equivalent conditions each one equivalent to each of the three previously stated (Propositions 4.3, 4.4):

Each of the three equivalent assertions above is equivalent to,

4. The $\Pi_\eta$-area of $M^2$ satisfies, $\text{area}(M^2, \Pi_\eta) = 2\pi$, or

5. The first non-trivial eigenvalue, $\lambda_1$, of the Laplacian of the induced metric on $M^2$ satisfies,

$$
\lambda_1 = 2 \frac{\int_{M^2} \langle H, H \rangle \, dA}{\text{area}(M^2, \langle, \rangle)}.
$$

2 Preliminaries

Let $L^4$ be the Lorentz-Minkowski spacetime, that is, $\mathbb{R}^4$ endowed with the Lorentzian metric,

$$
\langle \, , \rangle = -(dx_0)^2 + (dx_1)^2 + (dx_2)^2 + (dx_3)^2,
$$

where $(x_0, x_1, x_2, x_3)$ are the canonical coordinates of $\mathbb{R}^4$. A smooth immersion $\psi : M^2 \to L^4$ of a 2-dimensional (connected) manifold $M^2$ is said to be a spacelike if the induced metric via $\psi$ (denoted also by $\langle, \rangle$) is a Riemannian metric on $M^2$.

Let $\nabla$ and $\nabla^\perp$ be the Levi-Civita connections of $M^2$ and $L^4$, respectively, and let $\nabla^\perp$ be the normal connection. The Gauss and Weingarten formulas are,

$$
\nabla_X Y = \psi_*(\nabla_X Y) + \Pi(X, Y) \quad \text{and} \quad \nabla_X N = -\psi_*(A_N X) + \nabla^\perp_X N,
$$

for any $X, Y \in \mathcal{X}(M^2)$ and $N \in \mathcal{X}^\perp(M^2)$, where $\Pi$ denotes the second fundamental form of $\psi$. The shape (or Weingarten) operator $A_N$ corresponding to $N$ is related to $\Pi$ by,

$$
\langle A_N X, Y \rangle = \langle \Pi(X, Y), N \rangle,
$$

for all $X, Y \in \mathcal{X}(M^2)$. The mean curvature vector field of $\psi$ is given by $H = \frac{1}{2} \text{tr}(\langle , \rangle) \Pi$. For each $N \in \mathcal{X}^\perp(M^2)$, the Codazzi equation gives,

$$
(\nabla_X A_N) Y - (\nabla_Y A_N) X = A_{\nabla^\perp_X N} Y - A_{\nabla^\perp_Y N} X.
$$

(1)

We denote by $\Pi_N$ the symmetric tensor field on $M^2$ defined by,

$$
\Pi_N(X, Y) = -\langle A_N X, Y \rangle.
$$

We will call $\text{det}(A_N)$ the Gauss-Kronecker curvature of $M^2$ with respect to the normal vector field $N$. The normal vector field $N$ is said to be non-degenerate whenever $\text{det}(A_N) \neq 0$ at every point $p \in M^2$, [5]. When $N$ is non-degenerate, $\Pi_N$ is a semi-Riemannian metric on $M^2$. 

We write,
\[ \Lambda = \{ v \in \mathbb{L}^4 : \langle v, v \rangle = 0, v_0 > 0 \}, \]
for the future lightcone of $\mathbb{L}^4$. A spacelike surface $\psi : M^2 \to \mathbb{L}^4$ factors through the lightcone if $\psi(M^2) \subset \Lambda$. Every spacelike surface in $\mathbb{L}^4$ which factors through the lightcone must be orientable [12, Lemma 3.2]. Therefore, we can globally take a lightlike vector field $\eta \in X^+(M^2)$ with $\langle \psi, \eta \rangle = 1$.

From now on, unless otherwise was stated, we will assume $\psi : M^2 \to \mathbb{L}^4$ is a spacelike surface which factors through the lightcone. Recall briefly several local geometric properties of such a surface. Proofs for these features can be found in [12]. First, the lightlike normal vector fields $\psi$ and $\eta$ are parallel with respect to the normal connection. The corresponding Weingarten operators are given by,
\[ A_\psi = -I, \quad A_\eta = - \frac{1 + \|\nabla \psi_0\|^2}{2\psi_0^2} I + \frac{1}{\psi_0} \nabla^2 \psi_0, \tag{2} \]
where $\nabla^2 \psi_0(v) = \nabla_v(\nabla \psi_0)$ for every $v \in T_pM^2$, $p \in M^2$ and we have written $\psi_0$ for $x_0 \circ \psi$. Recall that the Gauss curvature for the induced metric on $M^2$ satisfies,
\[ K = -\text{tr}(A_\eta) = \langle \mathbf{H}, \mathbf{H} \rangle, \]
and therefore, the second fundamental form satisfies,
\[ \langle \mathbf{II}, \mathbf{II} \rangle(p) = \sum_{i,j=1}^2 \langle \mathbf{II}(e_i, e_j), \mathbf{II}(e_i, e_j) \rangle = 2K(p), \]
where $\{e_1, e_2\}$ is an orthonormal basis of $T_pM^2$, $p \in M^2$.

We write $\mathfrak{d} = \text{det}(A_\eta)$ for the Gauss-Kronecker curvature with respect to $\eta$. From formula (3), we arrive to the following technical result which will be useful along this paper.

**Lemma 2.1.** Let $\psi : M^2 \to \mathbb{L}^4$ be a spacelike immersion which factors through the lightcone $\Lambda$. Then,
\[ 4\mathfrak{d} \leq K^2 \leq 2 \text{tr}(A_\eta^2), \tag{4} \]
and one equality holds (if and only if the other one also holds) if and only if $M^2$ is totally umbilical.

**Remark 2.2.** Note that if $M^2$ is assumed to be compact, there exists $p_0 \in M^2$ such that equalities (4) hold at $p_0$. In fact, otherwise we can define two smooth functions $f_1$ and $f_2$ on $M^2$ with $f_1 < f_2$ and $\{f_1(p), f_2(p)\}$ are the eigenvalues of $A_\eta$ at every point $p \in M^2$. Each one of the eigendirections provides a 1-dimensional distribution on $M^2$. Since $M^2$ must be a topological sphere $S^2$, this is not possible.

On the contrary, the situation for noncompact complete spacelike surfaces is completely different. Consider the following isometric immersion $\psi$ of the Euclidean plane $\mathbb{E}^2$ in $\mathbb{L}^4$ through the lightcone,
\[ \psi(x, y) = (\cosh x, \sinh x, \cos y, \sin y), \]
$(x, y) \in \mathbb{E}^2$. The lightlike normal vector field $\eta$ is given by $\eta(x, y) = \frac{1}{2}(-\cosh x, -\sinh x, \cos y, \sin y)$. A direct computation shows $A_\eta(\partial_x) = -(1/2) \partial_x$ and $A_\eta(\partial_y) = (1/2) \partial_y$. Therefore, $\mathfrak{d} = -1/4$, $K = 0$ and $2 \text{tr}(A_\eta^2) = 1$.

As a direct consequence of (2) we get.
**Proposition 2.3.** Let \( \psi : M^2 \rightarrow \mathbb{L}^4 \) be a spacelike immersion which factors through the lightcone \( \Lambda \). If the function \( \psi_0 \) attains a maximum value at \( p_0 \in M^2 \), then \( \Pi_\eta \) is positive definite in a neighborhood of \( p_0 \). In particular, if \( M^2 \) is compact and \( \eta \) is non-degenerate, \( \Pi_\eta \) is a Riemannian metric.

**Remark 2.4.** (a) If \( \Pi_\eta \) is a Riemannian metric on \( M^2 \), we get from (3) that \( K > 0 \) on all \( M^2 \).

(b) For a noncompact complete spacelike surface we can have even \( A_\eta \equiv 0 \). In fact, consider the isometric immersion \( \varphi \) of the Euclidean plane \( \mathbb{E}^2 \) in \( \mathbb{L}^4 \), given by,

\[
\varphi(x, y) = \left( \frac{x^2 + y^2 + 1}{2}, \frac{x^2 + y^2 - 1}{2}, x, y \right).
\]

Clearly \( \varphi(\mathbb{E}^2) \subset \Lambda \), \( \eta(x, y) = (-1, -1, 0, 0) \) and therefore \( A_\eta = 0 \) at every point \((x, y) \in \mathbb{E}^2\).

For every spacelike immersion \( \psi : M^2 \rightarrow \mathbb{L}^4 \) which factors through the lightcone \( \Lambda \), we consider the smooth map \( \tilde{\psi} : M^2 \rightarrow \Lambda \) given by \( \tilde{\psi} = -\eta \). In general, \( \tilde{\psi} \) fails to be an immersion (see previous Remark). In fact, for every \( v \in T_pM^2 \), we have that \( \tilde{\psi}_*(v) = -\nabla_v \eta = \psi_*(A_\eta(v)) \).

Hence we get the following result.

**Lemma 2.5.** Let \( \psi : M^2 \rightarrow \mathbb{L}^4 \) be a spacelike immersion which factors through the lightcone \( \Lambda \). Then, \( \tilde{\psi} \) is an immersion if and only if \( \eta \) is non-degenerate. In this case, the induced metric from \( \tilde{\psi} \) is Riemannian and agrees with the third fundamental form corresponding to \( \eta \), that is,

\[
\tilde{\psi}^*(u, v) = \langle A_\eta^2(u), v \rangle,
\]

for every \( u, v \in T_pM^2 \), \( p \in M^2 \).

If \( \eta \) is assumed to be non-degenerate, we will write \( \tilde{\psi}^*(u, v) = \Pi_{\tilde{\eta}} \) and \( \tilde{\eta} \) will denote the lightlike normal vector field corresponding to \( \tilde{\psi} \). Observe that \( \tilde{\eta} = -\psi \), in particular \( \tilde{\psi} = \psi \).

The Weingarten operators and the second fundamental form for \( \tilde{\psi} \) will be represented by \( \tilde{A} \) and \( \Pi_{\tilde{\eta}} \), respectively. Note that, in general, \( \tilde{A}_{\tilde{\eta}} = \tilde{A} - \psi \neq I \). The spacelike surface \( \tilde{\psi} : M^2 \rightarrow \Lambda \) is called the conjugated surface to \( \psi \).

**Proposition 2.6.** Let \( \psi : M^2 \rightarrow \mathbb{L}^4 \) be a spacelike immersion which factors through the lightcone \( \Lambda \). Assume \( \eta \) is non-degenerate. Then we have,

1. \( \tilde{A}_{\tilde{\eta}} = A_\eta^{-1} \).
2. \( \tilde{\Pi}_{\tilde{\eta}} = \Pi_\eta \).

**Proof.** The Weingarten equation for \( \tilde{\psi} \) can be written as follows,

\[
\nabla_v \tilde{\eta} = -\tilde{\psi}_*(\tilde{A}_{\tilde{\eta}}(v)) = -\psi_*(A_\eta(\tilde{A}_{\tilde{\eta}}(v)) \),
\]

for every \( v \in T_pM^2 \), \( p \in M^2 \). On the other hand, \( \nabla_v \tilde{\eta} = -\psi_*(v) \) and we deduce the first assertion. Now, for \( u, v \in T_pM^2 \),

\[
\tilde{\Pi}_{\tilde{\eta}}(u, v) = -\langle A_\eta^2(\tilde{A}_{\tilde{\eta}}(u)), v \rangle = \Pi_\eta(u, v).
\]
Corollary 2.7. Let $\psi : M^2 \to \mathbb{L}^4$ be a spacelike immersion which factors through the lightcone $\Lambda$. If $\eta$ is non-degenerate, then,

$$K^{\Pi_\eta} = \frac{K}{\circ}.$$  

(5)

Proof. This easily follows from previous result taking into account [3] for $\tilde{\psi}$. □

Now, Lemma 2.1 and Proposition 2.6 give.

Corollary 2.8. Let $\psi : M^2 \to \mathbb{L}^4$ be a spacelike immersion which factors through the lightcone $\Lambda$. Assume $\eta$ is non-degenerate. Then $\psi$ is totally umbilical if and only if $\tilde{\psi}$ is totally umbilical.

Remark 2.9. An interesting question on spacelike surfaces which factor through the lightcone is the behavior under the effect of expansions. That is, let $\psi : M^2 \to \mathbb{L}^4$ be a spacelike immersion which factors through the lightcone $\Lambda$. For every $\sigma \in C^\infty(M^2)$, consider the immersion $\psi_\sigma := e^{\sigma} \psi$. Clearly, $\psi_\sigma$ factors through the lightcone and $g_\sigma = \psi_\sigma^* (, ) = e^{2\sigma} (, )$, where as usual we have written $\psi^* (, ) = (, )$. Therefore, $\psi_\sigma$ is spacelike and the Gauss curvature $K_\sigma$ of $g_\sigma$ satisfies,

$$K_\sigma = K - \triangle \sigma e^{2\sigma}.$$  

(6)

Let $\eta_\sigma = e^{-\sigma} \eta$ be the lightlike normal vector field such that $\langle \psi_\sigma, \eta_\sigma \rangle = 1$. A straightforward computation from (2) gives that for every $X \in \mathfrak{X}(M^2)$,

$$A^\sigma_{\eta_\sigma}(X) = \frac{1}{e^{2\sigma}} \left( A_\eta(X) + \nabla_X \nabla \sigma + \frac{||\nabla \sigma||^2}{2} X - X \sigma \cdot \nabla \sigma \right),$$

where $A^\sigma_{\eta_\sigma}$ denotes the Weingarten operator corresponding to $\eta_\sigma$ with respect to the spacelike immersion $\psi_\sigma$. Observe that (6) also achieves from $K_\sigma = -\tr(A^\sigma_{\eta_\sigma})$. Assume now $A^\sigma_{\eta_\sigma}$ is non-degenerate, then

$$\Pi^\sigma_{\eta_\sigma} = \Pi_\eta + d\sigma \otimes d\sigma - \frac{||\nabla \sigma||^2}{2} (, ) - \text{Hess}^\sigma.$$

In particular, if $\sigma$ is a constant $\Pi^\sigma_{\eta_\sigma} = \Pi_\eta$. The converse holds in the compact case. In fact, from $\Pi^\sigma_{\eta_\sigma} = \Pi_\eta$ we get $\triangle \sigma = 0$.

3 The Gauss curvature of $\Pi_\eta$

Assume now $\Pi_\eta$ is a Riemannian metric on $M^2$. This section is devoted to obtain a formula which relates the Gauss curvature $K$ of the induced Riemannian metric $(, )$ and the Gauss curvature $K^\eta$ of the metric $\Pi_\eta$.

Let $\nabla^{\Pi_\eta}$ be the Levi-Civita connection of the metric $\Pi_\eta$. The difference tensor $L$ between the Levi-Civita connections $\nabla^{\Pi_\eta}$ and $\nabla$ is the symmetric tensor given by,

$$L(X, Y) = \nabla^{\Pi_\eta} X Y - \nabla_X Y,$$

for all $X, Y \in \mathfrak{X}(M^2)$. The Koszul formula [11, p. 61] for $\Pi_\eta$, the Codazzi equation (1) and $\nabla^{\perp_\eta} = 0$ show,

$$L(X, Y) = \frac{1}{2} A^{-1}_\eta \left[ (\nabla_X A_\eta) Y \right].$$  

(7)

The Riemannian curvature tensor $R^\eta$ of $\Pi_\eta$ is obtained as,

$$R^\eta = R + Q_1 + Q_2,$$
where $R$ is the Riemannian curvature tensor of the induced metric and
\[ Q_1(X, Y)Z = (D_X L)(Y, Z) - (D_Y L)(X, Z), \]
\[ Q_2(X, Y)Z = L(Y, L(X, Z)) - L(X, L(Y, Z)), \]
for all $X, Y, Z \in \mathfrak{X}(M^2)$. Therefore the Gauss curvature $K_\eta$ satisfies,
\[ 2K_\eta = \text{tr}_{\Pi_\eta} (\text{Ric}) + \text{tr}_{\Pi_\eta} (\hat{Q}_1) + \text{tr}_{\Pi_\eta} (\hat{Q}_2), \tag{8} \]
where $\hat{Q}_i(X, Y) = \text{tr} \{ Z \mapsto Q_i(Z, X)Y \}$ and $\text{tr}_{\Pi_\eta}$ stands for the trace of the $(1,1)$-tensor $\hat{T}$ defined by $\Pi_\eta(\hat{T}(X), Y) = T(X, Y)$.

**Theorem 3.1.** Let $\psi : M^2 \to \mathbb{L}^4$ be a spacelike immersion which factors through the lightcone $\Lambda$ such that $\Pi_\eta$ is a Riemannian metric. Then,
\[ 2K_\eta = \frac{K^2}{\delta} + \Pi_\eta(L, L) - \frac{1}{4\delta^2} \Pi_\eta(\nabla^{\Pi_\eta} \delta, \nabla^{\Pi_\eta} \delta). \tag{9} \]

**Proof.** Fix $p \in M^2$ and let $\{e_1, e_2\}$ be an orthonormal basis of $T_p M^2$ for $\langle , \rangle$ which satisfies $A_\eta(e_i) = -\lambda_i e_i$ with $\lambda_i > 0$ for $i = 1, 2$. Then $\{w_1, w_2\}$, where $w_i = (\lambda_i)^{-1/2} e_i$, is an orthonormal basis for $\Pi_\eta$. Taking into account (3), a direct computation shows that,
\[ \text{tr}_{\Pi_\eta} (\text{Ric}) = \frac{K^2}{\delta}. \]

From (7), we obtain that $\Pi_\eta(L(X, Y), Z)$ is symmetric in all three variables and therefore,
\[ \Pi_\eta(Q_1(X, Y)Y, X) = \Pi_\eta(Q_1(X, Y)X, Y). \]

Now, it is easily deduced that,
\[ \text{tr}_{\Pi_\eta} (\hat{Q}_1) = 0. \]

Taking into account (7), we obtain,
\[ \Pi_\eta(L(X, Y), Z) = \Pi_\eta(L(X, Z), Y), \]
for every $X, Y, Z \in \mathfrak{X}(M^2)$. A straightforward computation shows,
\[ \text{tr}_{\Pi_\eta} (\hat{Q}_2) = 2 \left( \Pi_\eta(L(w_1, w_2), L(w_1, w_2)) - \Pi_\eta(L(w_1, w_1), L(w_2, w_2)) \right) \]
\[ = \Pi_\eta(L, L) - \Pi_\eta(\text{tr}_{\Pi_\eta}(L), \text{tr}_{\Pi_\eta}(L)), \]
where
\[ \Pi_\eta(L, L) = \sum_{i,j} \Pi_\eta(L(w_i, w_j), L(w_i, w_j)), \]
and
\[ \text{tr}_{\Pi_\eta}(L) = -L(w_1, w_1) - L(w_2, w_2), \]
denotes the vector field obtained from the $\Pi_\eta$-contraction of $L$.

We end the proof with an explicit expression of $\text{tr}_{\Pi_\eta}(L)$. Let $\{E_1, E_2\}$ be a local orthonormal frame for $\langle , \rangle$ which satisfies $A_\eta(E_i) = -f_i E_i$ for smooth functions $f_i > 0$, $i = 1, 2$ (see comment...
in [I p. 1815]). Then, \{W_1, W_2\}, where \(W_i = (f_i)^{-1/2}E_i\), is a local orthonormal frame for \(\Pi_\eta\). Recall that \(\nabla^\perp \eta = 0\). Now a direct computation shows that,

\[\langle \nabla X A_\eta E_i, E_i \rangle = X(f_i)\]

for any \(X \in \mathfrak{X}(M^2)\), and the Codazzi equation implies,

\[X(f_i) = \langle \nabla E_i A_\eta E_i, X \rangle. \quad (10)\]

Finally, from (10) and (7) we obtain,

\[X(\log \vartheta) = \langle \nabla W_1 A_\eta W_1, X \rangle + \langle \nabla W_2 A_\eta W_2, X \rangle = -2\Pi_\eta (\text{tr}_{\Pi_\eta}(L), X).\]

Therefore,

\[\text{tr}_{\Pi_\eta}(L) = \frac{\nabla \Pi_\eta \vartheta}{2\vartheta}, \quad (11)\]

which completes the proof.

Remark 3.2. An alternative proof of this formula can be achieved, using a local computation, from [6, Exercise I.18]; compare with [3, Propssion 3.4 ]. On the other hand, a key fact in order to get formula (9) has been \(\nabla^\perp \eta = 0\). For an arbitrary spacelike surface in \(L^4\), every lightlike normal vector field \(\eta\) must be recurrent. That is, we have \(\nabla^\perp \eta = \omega \otimes \eta\) where \(\omega\) is a one form on \(M^2\). If in addition, \(\eta\) is assumed to be non-degenerate, a formula relating the Gauss curvatures \(K\) and \(K_\eta\) can be also obtained as a wide extension of (9).

Proposition 3.3. Let \(\psi : M^2 \to L^4\) be a spacelike immersion which factors through the lightcone \(\Lambda\) and assume \(\Pi_\eta\) is a Riemannian metric. Then, \(M^2\) is totally umbilical in \(L^4\) if and only if the Gauss-Kronecker curvature with respect to \(\eta\) is a constant and \(K_\eta = 2\).

Proof. Assume \(A_\eta = \lambda I\), where \(\lambda \in \mathbb{R}\). Then (9) reduces to,

\[2K_\eta = \frac{K^2}{\lambda^2}.\]

Now from (11) we get that \(K_\eta = 2\). For the converse, note that (9) implies that \(K^2 \leq 4 \vartheta\), and Lemma 2.1 applies to end the proof.

Remark 3.4. From Proposition 3.3 and Remark 2.4 every complete spacelike surface \(M^2\) in the lightcone \(\Lambda\) with \(\Pi_\eta\) a Riemannian metric and totally umbilical satisfies \(K = 2|\lambda| > 0\). As a consequence of the classical Myers theorem, if we assume \(M^2\) geodesically complete, \(M^2\) must be compact, and hence a round sphere.

4 Main results

For compact surfaces, Proposition 3.3 can be improved as the following result states.

Theorem 4.1. Let \(\psi : M^2 \to L^4\) be a compact spacelike immersion which factors through the lightcone \(\Lambda\). Assume \(\eta\) is non-degenerate. Then the following conditions are equivalent:
1. $M^2$ is a totally umbilical round sphere,

2. The Gauss-Kronecker curvature $\det(A_\eta)$ is constant,

3. The Gauss curvature of the Riemannian metric $\Pi_\eta$ satisfies $K^\eta = 2$.

**Proof.** From Proposition 2.3, the metric $\Pi_\eta$ is Riemannian. If we assume $M^2$ is totally umbilical, Proposition 3.3 gives that $\partial$ is constant and $K^\eta = 2$. Assume now the Gauss-Kronecker curvature $\partial$ is constant. Since $M^2$ is a topological 2-sphere \[12\], we have $\partial > 0$. Therefore, Lemma 2.1 assures that $K^\eta \geq 2\sqrt{\partial}$, with equality if and only if $M^2$ is totally umbilical. Now from Theorem 3.1, 

$$2K^\eta \geq \frac{K^2}{\partial} \geq \frac{2K}{\sqrt{\partial}}. \tag{12}$$

The area elements corresponding to $\langle , \rangle$ and $\Pi_\eta$ are related by $dA_{\Pi_\eta} = \sqrt{\partial} dA_{\langle , \rangle}$. Hence the Gauss-Bonnet formula and (12) imply,

$$8\pi = \int_{M^2} 2K^\eta dA_{\Pi_\eta} \geq 2 \int_{M^2} \frac{K}{\sqrt{\partial}} dA_{\Pi_\eta} = 2 \int_{M^2} K dA_{\langle , \rangle} = 8\pi.$$ 

We get the equality in (12) and so $K = 2\sqrt{\partial}$ and $K^\eta = 2$. Finally, under the assumption $K^\eta = 2$, Lemma 2.1 can be rewritten as follows: $K^\eta \sqrt{\partial} \leq K$, again equality holds if and only if $M^2$ is totally umbilical. From the Gauss-Bonnet formula,

$$4\pi = \int_{M^2} K^\eta dA_{\Pi_\eta} = \int_{M^2} K^\eta \sqrt{\partial} dA_{\langle , \rangle} \leq \int_{M^2} K dA_{\langle , \rangle} = 4\pi,$$

and $K^\eta \sqrt{\partial} = K$. \qed

**Remark 4.2.** A compact spacelike immersion $\psi$ which factors through the lightcone $\Lambda$ with constant Gauss curvature must be totally umbilical \[12\, Theorem 5.4\]. From (5) and Corollary 2.8 it follows that $\psi$ is totally umbilical if and only if $K/\partial$ is a constant.

We end the paper with the statement of two results which complement Theorem 4.1 from points of view.

**Proposition 4.3.** Let $\psi : M^2 \to \mathbb{L}^4$ be a compact spacelike immersion which factors through the lightcone $\Lambda$. Assume $\eta$ is non-degenerate. Then,

$$\text{area}(M^2, \Pi_\eta) \leq 2\pi,$$

and equality holds if and only if $M^2$ is totally umbilical.

**Proof.** In Theorem 4.1 we have pointed out that $dA_{\Pi_\eta} = \sqrt{\partial} dA_{\langle , \rangle}$ and $2\sqrt{\partial} \leq K$. Therefore, the result follows as a consequence of the Gauss-Bonnet formula and Lemma 2.1. \qed

For a compact submanifold $M^n$ of an Euclidean space $\mathbb{E}^{n+p}$ there is a well-known upper bound of the first non-trivial eigenvalue $\lambda_1$ of the Laplacian of $M^n$ called the classical Reilly formula \[13\]. This upper bound depends on the integral of the square length of the mean curvature vector field and the $n-$dimensional area of $M^n$. It was shown in \[12\] that the same formula does not work for any compact spacelike surface in $\mathbb{L}^4$. However,
Proposition 4.4. Let $\psi : M^2 \to \mathbb{L}^4$ be a compact spacelike immersion which factors through the lightcone $\Lambda$. We have the following inequality,

$$\lambda_1 \leq 2 \frac{\int_{M^2} \langle H, H \rangle \, dA}{\text{area}(M^2, \langle \cdot, \cdot \rangle)},$$

and equality holds if and only if $M^2$ is totally umbilical.

Proof. The inequality was obtained in [12] as a consequence of the Hersch inequality [?], taking into account (3). The equality holds in (13) if and only if $M^2$ has constant Gauss curvature. Now, [12, Theorem 5.4] ends the proof.

A compact spacelike surface $M^2$ in the 3-dimensional de Sitter space $\mathbb{S}^3_1$ with non-degenerate second fundamental form is totally umbilical if and only if the Gauss curvature $K^{II}$ of its second fundamental form is constant, [3]. A key tool in order to get this result is the Gauss formula $K = 1 - \det(A)$ where $K$ and $A$ are the Gauss curvature and the Weingarten operator of $M^2$, respectively. This relationship permits to obtain a formula which relates $K$ and $K^{II}$ and involves different ingredients of (9). This makes that the technique in [3] does not work in order to show that a compact spacelike surface in $\Lambda$, with $K^\eta$ a constant, must be totally umbilical. Note that from Theorem 4.4, this assertion is in fact equivalent to the following one: if $K^\eta$ is a constant for such a spacelike surface, then $K^\eta = 2$. At the moment the authors have no argument to support this assertion, although we think that it holds true.

Note that $M^2$ compact, $\eta$ non-degenerate and $K^\eta$ constant imply $K^\eta \geq 2$. To prove this fact, take a point $q_0 \in M^2$ where the function $\partial$ attains its maximum value. From Lemma 2.1 and Theorem 3.1 we deduce that, $2K^\eta \geq K^2(q_0)/\partial(q_0) \geq 4$.

In view of the previous discussion, we state the following

Conjecture. Every compact spacelike surface in $\mathbb{L}^4$ which factors through the lightcone $\Lambda$ such that $\eta$ is non-degenerate and $K^\eta =$ constant must be totally umbilical (that is, $K^\eta = 2$).

References

[1] J.A. Aledo, L.J. Alías and A. Romero, A new proof of Liebmann classical rigidity theorem for surfaces in space forms, Rocky Mt. J. Math., 35 (2005), 1811–1824.

[2] J.A. Aledo, S. Haesen and A. Romero, Spacelike surfaces with positive definite second fundamental form in 3D spacetimes, J. Geom. Physics, 57 (2007), 913–923.

[3] J.A. Aledo and A. Romero, Compact spacelike surfaces in the 3-dimensional de Sitter space with non-degenerate second fundamental form, Differ. Geom. Appl., 19 (2003), 97–111.

[4] A. Asperti and M. Dajczer, Conformally flat Riemannian manifolds as hypersurfaces of the light cone, Can. Math. Bull., 32 (1989), 281–285.

[5] B.Y. Chen, Geometry of Submanifolds, Marcel Dekker, New York, 1973.

[6] L.P. Eisenhart, Riemannian Geometry, 6th Edition, Princeton Univ. Press, 1996.

[7] M. Kossowski, The $S^2$-valued maps and split total curvature of a spacelike codimension-2 surface in Minkowski space, J. London Math. Soc., 40 (1989), 179–192.

[8] H.L. Liu, Surfaces in lightlike cone, J. Math. Anal. Appl., 325 (2007), 1171–1181.
[9] H.L. Liu and S. D. Jung, Hypersurfaces in lightlike cone, *J. Geom. Phys.*, **58** (2008), 913–922.

[10] H. Liu, M. Umehara and K. Yamada, The duality of conformally flat manifolds, *Bull. Braz. Math. Soc.*, **42** (2011), 131–152.

[11] B. O’Neill, *Semi-Riemannian Geometry with Applications to Relativity*, Academic Press, New York, 1983.

[12] F.J. Palomo and A. Romero, On spacelike surfaces in 4-dimensional Lorentz-Minkowski spacetime through a lightcone, to appear in *P. Roy. Soc. Edinb. A Mat.*

[13] R.C. Reilly, On the first eigenvalue of the Laplacian for compact submanifolds of Euclidean space, *Comment. Mat. Helvetici*, **52** (1977), 525–533.