A DIRECT COMPUTATION OF THE COHOMOLOGY OF THE BRACES OPERAD

VASILY DOLGUSHEV AND THOMAS WILLWACHER

ABSTRACT. We give a self-contained and purely combinatorial proof of the well known fact that the cohomology of the braces operad is the operad $\mathcal{G}e r$ governing Gerstenhaber algebras.

1. Introduction

It is a well known fact [6] that the Hochschild cohomology of an associative (or $A_\infty$) algebra $A$ carries the structure of a Gerstenhaber algebra. In 1993, P. Deligne [3] asked whether this Gerstenhaber algebra structure is induced by an action of some version of the chains operad of the little disks operad on the Hochschild cochain complex $C^*(A, A)$ of $A$. This question became known as the Deligne conjecture and was answered affirmatively by various authors including C. Berger and B. Fresse [1], M. Kontsevich and Y. Soibelman [8], J. E. McClure and J. H. Smith [11], and D. Tamarkin [15], [16].

A key role in the proof of the Deligne conjecture is played by the braces operad $\mathcal{B}r$, which encodes a set of natural operations on the Hochschild cochain complex $C^*(A, A)$ of any $A_\infty$ algebra $A$. In the form used here this differential graded (dg) operad was introduced by Kontsevich and Soibelman [8], where it is called the “minimal operad”, and by McClure and Smith [12], where it is denoted by $\mathcal{H}$.

Concretely, a braces algebra is a dg vector space $V$ together with two sets of operations

\begin{align}
    m_{1,n-1}: V \otimes n &\to V \\
    m_n: V \otimes n &\to V
\end{align}

\[
    (v_1, v_2, \ldots, v_n) \mapsto v_1 \{v_2, \ldots, v_n\} \quad (v_1, \ldots, v_n) \mapsto m_n(v_1, \ldots, v_n)
\]

for $n = 2, 3, \ldots$, satisfying certain natural compatibility relations.

As shown in [8], the braces operad comes equipped with a natural operad map into (a suitable version of) the operad of chains of the little disks operad. The proof of Deligne’s conjecture is hence completed by showing the following result.

**Theorem 1.1.** The cohomology of the braces operad is the Gerstenhaber operad.

\[ H^\bullet(\mathcal{B}r) \cong \mathcal{G}e r. \]

The goal of this short note is to provide a self-contained combinatorial proof of this result. The result itself is not new. A proof is sketched, for example, in [8, Theorem 4], and another proof may be extracted from [12], together with a small computation. Our proof applies with minor changes also to the higher versions of the braces operads $\mathcal{B}r_{n+1}$ acting naturally on the deformation complexes of $n$-algebras, cf. [2, Section 4].

Theorem 1.1 is a shadow of the very deep statement which says that the dg operad $\mathcal{B}r$ is weakly equivalent to the operad $\mathcal{G}e r$. The proof of the latter statement involves a solution of the Deligne conjecture and the formality of the dg operad $C_\bullet(E_2, K)$ where $E_2$ denotes the topological operad of little discs [14]. One possible proof [14] of the formality of $C_\bullet(E_2, K)$ involves the use of Drinfeld’s associator [5] and another possible proof [7, Section 3.3], [9] involves the use of a configuration space integral. Although Theorem 1.1 does not imply the formality of the operad $\mathcal{B}r$, it is amazing that it

---

1. In this note, the ground field $K$ is any field of characteristic zero.
can be proved in a purely combinatorial way which bypasses the use of compactified configuration spaces.

**Remark 1.2.** There is an amazing combinatorial similarity between the dg operad \( \text{Br} \) and the dg operad \( \text{Graphs} \) [7, Section 3.3], [13, Section 3]. The latter dg operad is “assembled from” graphs of certain kind with some additional data and the former dg operad is “assembled from” rooted planar trees (also with some additional data). Both dg operads are formal. In fact, both dg operads are weakly equivalent to the same operad \( \text{Ger} \). However, while the proof of formality for \( \text{Graphs} \) involves only elementary homological algebra [7, Section 3.3.4], the proof of formality for \( \text{Br} \) requires a “very heavy hammer”.

**Acknowledgements:** V.D. has been partially supported by NSF grant DMS-1161867 and T.W. has been partially supported by the Swiss National Science foundation, grant 200021_150012, and the SwissMAP NCCR funded by the Swiss National Science foundation.

2. Notation

We work over a ground field \( \mathbb{K} \) of characteristic 0. For a set \( X \) we denote by \( \mathbb{K}(X) \) the \( \mathbb{K} \)-vector space of finite linear combinations of elements in \( X \). We denote by \( s \) (resp. \( s^{-1} \)) the operation of suspension (resp. desuspension) for graded or differential graded (dg for short) \( \mathbb{K} \) vector spaces.

By a *collection* we mean a sequence \( \{P(n)\}_{n \geq 0} \) of dg vector spaces with a right action of the symmetric group \( S_n \). The category of collections carries a natural monoidal structure, the plethysm operation \( \odot \), see, e.g., [4, eqn. (5.1)].

We will freely use the language of operads. A good introduction is provided in textbook [10]. The notation \( \text{Lie} \) (resp. \( \text{As}, \text{Com}, \text{Ger} \)) is used for the operad governing Lie algebras (resp. associative, commutative or Gerstenhaber algebras without unit). Dually, the notation \( \text{coLie} \) (resp. \( \text{coAs}, \text{coCom} \)) is reserved for the cooperad governing Lie coalgebras (resp. coassociative coalgebras without counit, cocommutative (and coassociative) coalgebras without counit).

For an operad \( \mathcal{O} \) (resp. a cooperad \( \mathcal{C} \)) and a cochain complex \( V \), we denote by \( \mathcal{O}(V) \) (resp. \( \mathcal{C}(V) \)) the free \( \mathcal{O} \)-algebra (resp. cofree \( \mathcal{C} \)-coalgebra).

For an operad (resp. a cooperad) \( P \) we denote by \( \Lambda P \) the operad (resp. the cooperad) with the spaces of \( n \)-ary operations:

\[
\Lambda P(n) = s^{1-n} P(n) \otimes \text{sgn}_n,
\]

where \( \text{sgn}_n \) denotes the sign representation of \( S_n \).

For an operad \( \mathcal{O} \) and degree 0 auxiliary variables \( a_1, a_2, \ldots, a_n \), \( \mathcal{O}(n) \) is naturally identified with the subspace of the free \( \mathcal{O} \)-algebra

\[
\mathcal{O}(\mathbb{K}\langle a_1, a_2, \ldots, a_n \rangle)
\]

spanned by \( \mathcal{O} \)-monomials in which each variable from the set \( \{a_1, a_2, \ldots, a_n\} \) appears exactly once.

We often use this identification in this paper. For example, the vector space \( \text{Ger}(2) \) of the operad \( \text{Ger} \) is spanned by the degree zero vector \( a_1a_2 \) and the degree \(-1\) vector \( \{a_1, a_2\} \). The commutative (and associative) multiplication on a Gerstenhaber algebra \( V \) comes from the vector \( a_1a_2 \in \text{Ger}(2) \) and the odd Lie bracket \( \{ , \} \) on \( V \) comes from the vector \( \{a_1, a_2\} \in \text{Ger}(2) \). Similarly, the space \( \Delta\text{Lie}(n) \) of the suboperad \( \Delta\text{Lie} \subset \text{Ger} \) is spanned by \( \Delta\text{Lie} \)-monomials in \( a_1, a_2, \ldots, a_n \) in which each variable from the set \( \{a_1, a_2, \ldots, a_n\} \) appears exactly once. For example, \( \Delta\text{Lie}(2) \) is spanned by the vector \( \{a_1, a_2\} \) and \( \Delta\text{Lie}(3) \) is spanned by the vectors \( \{a_1, a_2, a_3\} \) and \( \{a_1, a_3, a_2\} \).

3. Trees, and a Graphical Description of \( \text{Br} \)

The braces operad \( \text{Br} \) may be defined combinatorially as an operad of planar trees. More precisely, \( \text{Br}(n) \) is the linear span of rooted planar trees having two kinds of non-root vertices:

- \( n \) labeled vertices, numbered \( \{1, \ldots, n\} \),
an arbitrary number of unlabeled neutral vertices.
In addition, one requires that each neutral vertex has at least two children. We will call such a tree a brace tree. For example, figure 3.1 shows a brace tree $T$ with 5 labeled vertices. So $T$ is a vector in $\text{Br}(5)$. In pictures, white circles with inscribed numbers denote labeled vertices, black circles denote neutral vertices, and the small black node (at the bottom) denotes the root.

![Fig. 3.1. An element in Br(5)](image)

The $\mathbb{Z}$-grading on $\text{Br}(n)$ is given by declaring that each brace tree has the degree

$$2 \times \# \text{ of neutral vertices} - \# \text{ of non-root edges}.$$ 

For example, the brace tree $T$ shown in figure 3.1 has degree $-2$.

The differential is graphically defined by the following formulas:

\[
\delta = \sum \pm \\
\delta = \sum \pm
\]

For more details, in particular regarding the signs and the definition of the operadic composition, we refer the reader to [4, Sections 7-9]. Note that the brace trees shown in figures 3.2 and 3.3 correspond to the operations $m_{1,n-1}$ and $m_n$ in (1.1), respectively.

![Fig. 3.2. The brace tree corresponding to $m_{1,n-1}$](image)  ![Fig. 3.3. The brace tree corresponding to $m_n$](image)

3.1. **The outline of the proof of Theorem 1.1** Theorem 1.1 is proved below by induction on the arity $n$.

Assuming that $H^j(\text{Br}(j)) \cong \text{Ger}(j)$ for all $1 \leq j \leq n - 1$, we split the graded vector space into the direct sum

$$\text{Br}(n) = V_\circ(n) \oplus V_\bullet(n),$$

where $V_\circ(n)$ is the subspace of $\text{Br}(n)$ spanned by brace trees whose lowest non-root vertex is neutral and $V_\bullet(n)$ is the subspace of $\text{Br}(n)$ spanned by brace trees whose lowest non-root vertex is labeled. The arrows in the above formula indicate the non-zero components of the differential.

It is clear that

- both $V_\circ(n)$ and $V_\bullet(n)$ may be considered as cochain complexes with the differential $\delta_0$;
The cohomology of the operad

**Theorem 4.1.** The cohomology of the operad \( \text{Br} \) is the Gerstenhaber operad:

\[
H^\bullet(\text{Br}) \cong \text{Ger}.
\]

The cohomology class corresponding to the vector \( \{a_1, a_2\} \in \text{Ger}(2) \) is represented by the sum

\[
T_{\{a_1, a_2\}} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \end{pmatrix}
\]

and the cohomology class corresponding to the vector \( a_1a_2 \in \text{Ger}(2) \) is represented by

\[
T_{a_1a_2} = \frac{1}{2} \left( \begin{pmatrix} 1 \\ 2 \end{pmatrix} + 2 \begin{pmatrix} 1 \end{pmatrix} \right).
\]

**Remark 4.2.** In fact, sending the generator \( \{a_1, a_2\} \) of \( \Lambda \text{Lie} \) to the element \( T_{\{a_1, a_2\}} \) introduced in Theorem 4.1, we obtain a map of operads

\[
j : \Lambda \text{Lie} \to \text{Br}.
\]

To see this, one has to check that the Jacobi identity holds in \( \text{Br} \). Theorem 4.1 implies that this map is injective and that the image is given by the closed elements of \( \text{Br} \) which do not contain neutral vertices. Note, however, that \( j \) does not extend to an operad map \( \text{Ger} \to \text{Br} \).

We show that \( H^\bullet(\text{Br}(n)) = \text{Ger}(n) \) by induction on \( n \). For \( n = 1 \) there is nothing to show. So suppose we know that \( H^\bullet(\text{Br}(j)) = \text{Ger}(j) \) for \( j = 1, 2, \ldots, n - 1 \) and let us tackle the statement for \( j = n \). We split

\[
\delta_0 \bigwedge \delta_1 \delta_0
\]

\[
\text{Br}(n) = V_0(n) \bigoplus V_\bullet(n)
\]
Here $V_\bullet(n)$ is the subspace of $\text{Br}(n)$ spanned by brace trees whose lowest non-root vertex is neutral, while $V_0(n)$ is the subspace of $\text{Br}(n)$ spanned by brace trees whose lowest non-root vertex is labeled. The arrows indicate the several components of the differential. For example, $V_\bullet(n)$ is a subcomplex of $\text{Br}(n)$. Clearly $\delta_1$ induces a map, called $H^\bullet(\delta_1)$, on $\delta_0$-cohomologies:

$$H^\bullet(\delta_1) : H^\bullet(V_0(n),\delta_0) \rightarrow H^\bullet(V_\bullet(n),\delta_0).$$

Furthermore, it is not hard to see that

$$H^\bullet(\text{Br}(n)) = (\ker H^\bullet(\delta_1)) \oplus (\coker H^\bullet(\delta_1)).$$

4.1. Computing $H^\bullet(V_\bullet(n),\delta_0)$. Let us observe that the linear dual $\text{Br}(n)^*$ can be canonically identified with $\text{Br}(n)$ as vector space and the differential on $\text{Br}(n)^*$ is given by edge contractions instead of vertex splitting. Using this observation we will freely switch back and forth between the cochain complexes under consideration and their linear duals.

Let us begin by computing $H^\bullet(V_\bullet(n),\delta_0^*)$.

**Claim 4.3.** We claim that

$$H^\bullet(V_\bullet^*(n),\delta_0^*) \cong s^{n-1}\mathbb{K}[n] \cong s^{n-1}\mathbb{K}[S_n]$$

as $S_n$-modules. Moreover, the class corresponding to a permutation $\lambda \in S_n$ is represented by the brace tree $T_\lambda$ shown in figure 4.3.

**Proof.** We proceed by induction on $n$. For $n = 1$ the statement is clear. Otherwise split:

$$V_0^* = W_1 \oplus W_2$$

Here $W_1$ is spanned by brace trees in which the lowest non-root vertex has exactly one child and $W_2$ is spanned by brace trees in which the lowest non-root vertex has at least two children. It is easy to see that $\delta_1^0$ is surjective and that its kernel is spanned by brace trees whose lowest non-root vertex has a labeled vertex as a child. The complex $(\ker \delta_1^0, \delta_0^0)$ is isomorphic to $(V_0^*(n - 1), \delta_0^0)$.

Thus the induction hypothesis implies that $H^\bullet(V_\bullet^*(n),\delta_0^*) \cong s^{n-1}\mathbb{K}[S_n]$ as graded vector spaces. The compatibility of the resulting isomorphism with the $S_n$-action is obvious. □

**Remark 4.4.** Note that every brace tree $T \in \text{Br}(n)^*$ without neutral vertices is automatically closed and in particular $\delta_0^0$-closed. Hence, by Claim 4.3, there exists a vector $T' \in V_0^*(n)$, such that $T - \delta_0^0 T'$ is a linear combination of string-like brace trees, i.e. brace trees of the form $T_\lambda$ (see figure 4.3).

4.2. Computing $H^\bullet(\text{Br}(n),\delta_0)$. Let us prove the following claim:

**Claim 4.5.** For the complex $(\text{Br}(n),\delta_0)$ we have

$$H^\bullet(\text{Br}(n),\delta_0) \cong \text{Com} \circ \Lambda \text{Lie}(n)/\Lambda \text{Lie}(n) \oplus s(\Lambda \text{Com} \circ \Lambda \text{Lie}(n)/\Lambda \text{Lie}(n)),$$

where $\circ$ denotes the plethysm of collections.

**Proof.** Let us filter $V_\bullet(n)$ by the number of children of the lowest non-root vertex.

$$0 = \mathcal{F}^1 V_\bullet(n) \subset \mathcal{F}^2 V_\bullet(n) \subset \mathcal{F}^3 V_\bullet(n) \subset \cdots \subset \mathcal{F}^n V_\bullet(n) = V_\bullet(n).$$

Here $\mathcal{F}^p V_\bullet(n)$ is spanned by brace trees whose lowest non-root vertex has $\leq p$ children. Let us consider the spectral sequence associated to this filtration. The first differential, say $d_0$, splits vertices except for the lowest non-root vertex. Hence,

$$\text{Gr } V_\bullet(n) \cong (s \Lambda \text{coAs}_0 \circ \text{Br})(n),$$

The only difference is that the degree of a brace tree in $\text{Br}(n)^*$ equals $\#$ of non-root edges $- 2 \times \#$ of neutral vertices.
where \( s\Lambda\coAs_0 \) is the collection with

\[
s\Lambda\coAs_0(m) = \begin{cases} s^{2-m}K[S_m] \otimes \text{sgn}_m & \text{if } m \geq 2, \\ 0 & \text{otherwise}. \end{cases}
\]

Therefore, by inductive hypothesis, we conclude that

\[
E^1V_\bullet(n) := H^\bullet(\Gr V_\bullet(n), d_0) \cong (s\Lambda\coAs_0 \odot \Ger)(n).
\]

The differential \( d_1 \) on \( E^1V_\bullet(n) \) splits the lowest non-root vertex producing a neutral child node with two children. To describe this cochain complex, we consider the free Gerstenhaber algebra \( \Ger_n \) in \( n \) auxiliary variables \( a_1, a_2, \ldots, a_n \) of degree zero. Forgetting the bracket \( \{,\} \) on \( \Ger_n \) we can view it merely as the free commutative algebra (without unit)

\[
\Ger_n = \Com(\Lambda\Lie_n)
\]

generated by the free \( \Lambda\Lie \)-algebra \( \Lambda\Lie_n \) in the auxiliary variables \( a_1, a_2, \ldots, a_n \).

Next, we introduce the cofree coassociative coalgebra

\[
\coAs(s^{-1}\Ger_n) = \bigoplus_{q \geq 1} (s^{-1}\Ger_n)^\otimes q
\]

and equip it with the coderivation \( \delta \) defined by the equation\(^3\)

\[
p \circ \delta(s^{-1}v_1 \otimes \cdots \otimes s^{-1}v_q) = \begin{cases} (-1)^{\lambda_1 + 1}s^{-1}v_1v_2 & \text{if } q = 2, \\ 0 & \text{otherwise}, \end{cases}
\]

where \( v_i \in \Ger_n \) and \( p \) is the canonical projection;

\[
p : \coAs(s^{-1}\Ger_n) \to s^{-1}\Ger_n.
\]

It is easy to see that the coderivation \( \delta \) has degree 1. Moreover, due to associativity of the multiplication on \( \Ger_n \), we have

\[
\delta^2 = 0.
\]

In other words, \( \delta \) is a differential on the coalgebra \((4.10)\).

For our purposes we need the following truncation of the cochain complex \( s^2\coAs(s^{-1}\Ger_n) \)

\[
s^2T'(s^{-1}\Ger_n) = \bigoplus_{q \geq 2} s^2(s^{-1}\Ger_n)^\otimes q
\]

with the differential \( \delta' \) given by the formula:

\[
\delta'(s^2(s^{-1}v_1 \otimes s^{-1}v_2 \otimes \cdots \otimes s^{-1}v_q)) = \begin{cases} s^2\delta(s^{-1}v_1 \otimes s^{-1}v_2 \otimes \cdots \otimes s^{-1}v_q) & \text{if } q > 2, \\ 0 & \text{if } q = 2, \end{cases} \quad v_i \in \Ger_n.
\]

It is not hard to see that \( E^1V_\bullet(n) \) \((4.9)\) is isomorphic to the subspace of \( s^2T'(s^{-1}\Ger_n) \) which is spanned by tensor monomials

\[
s^2(s^{-1}v_1 \otimes s^{-1}v_2 \otimes \cdots \otimes s^{-1}v_q), \quad v_i \in \Ger_n, \quad 2 \leq q \leq n
\]

in which each variable from the set \( \{a_1, a_2, \ldots, a_n\} \) appears exactly once. It is easy to see that this subspace is a subcomplex with respect to \( \delta' \) and, moreover, the differential \( d_1 \) coincides with the restriction of \( \delta' \) up to a total sign.

Since the augmentation

\[
\mathbf{\cdots} \xrightarrow{\delta'} (s^{-1}\Ger_n)^\otimes 2 \xrightarrow{\delta} s^{-1}\Ger_n \xrightarrow{0} K
\]

\(^3\)Note that, since the coalgebra \((4.10)\) is cofree, any coderivation \( \delta \) is uniquely determined by its composition with the projection \((4.12)\).
of the cochain complex (4.10) computes the Hochschild homology
\[(4.16) \quad HH_\bullet (S(\Lambda\text{Lie}_n), \mathbb{K})\]
of the free commutative algebra \(S(\Lambda\text{Lie}_n)\) (with unit) with the trivial coefficients, we conclude that
\[(4.17) \quad H^\bullet (\text{coAs}(s^{-1} \text{Ger}_n), \mathcal{O}) = \bigoplus_{q \geq 1} S^q(s^{-1} \Lambda\text{Lie}_n), \]
and the cohomology class of the symmetric word \((s^{-1} v_1, s^{-1} v_2, \ldots, s^{-1} v_q) \in S^q(s^{-1} \Lambda\text{Lie}_n)\) is represented by the cocycle:
\[\sum_{\sigma \in S_q} (-1)^{\varepsilon(\sigma, v_1, \ldots, v_q)} (s^{-1} v_{\sigma(1)}, s^{-1} v_{\sigma(2)}, \ldots, s^{-1} v_{\sigma(q)}) \in (s^{-1} \text{Ger}_n)^{\otimes q},\]
where the sign factors \((-1)^{\varepsilon(\sigma, v_1, \ldots, v_q)}\) are determined by the Koszul rule.

Using this observation, it is not hard to see that
\[(4.18) \quad H^\bullet (s^2 T'(s^{-1} \text{Ger}_n), \mathcal{O}') \cong \bigoplus_{q \geq 2} S^q(\Lambda\text{Lie}_n) \oplus \bigoplus_{q \geq 2} s^2 S^q(s^{-1} \Lambda\text{Lie}_n),\]
where the second summand comes from the Hochschild homology (4.16) and the first summand appears as the result of the truncation of the Hochschild chain complex (4.15).

On the other hand, \(E^1 V_\bullet(n)\) is isomorphic to the direct summand of the cochain complex \((s^2 T'(s^{-1} \text{Ger}_n), \mathcal{O}')\).

Thus we conclude that the following three facts hold:

- First,
\[(4.19) \quad E^2 V_\bullet(n) \cong \text{Com} \circ \Lambda\text{Lie}(n)/\Lambda\text{Lie}(n) \oplus s(\Lambda\text{Com} \circ \Lambda\text{Lie}(n)/\Lambda\text{Lie}(n)).\]

- Second, the class corresponding to the vector

\[s^2(s^{-1} v_1, \ldots, s^{-1} v_q) \in s\text{Com} \circ \Lambda\text{Lie}(n), \quad q \geq 2\]
is represented in the associated graded \(\text{Gr} V_\bullet\) by the cochain
\[(4.20) \quad \mu \left( \sum_{\sigma \in S_q} (-1)^{|\sigma|} \sigma(T_q^\bullet) \otimes j(v_1) \otimes j(v_2) \otimes \ldots j(v_q) \right),\]
where \(\mu\) is the operadic composition
\[\mu : \text{Br}(q) \otimes \text{Br}(m_1) \otimes \cdots \otimes \text{Br}(m_q) \to \text{Br}(m_1 + \cdots + m_q),\]
\(j\) is the operad map in (4.1), and \(T_q^\bullet\) is the brace tree shown in figure 4.1. Note that the element (4.20) is not closed in \(V_\bullet\), only in the associated graded. To obtain a cocycle in \(V_\bullet\), a correction term in \(F^{q-1}\) has to be added. We will see below that such a term may always be found.

- Finally, classes corresponding to vectors in \(\text{Com} \circ \Lambda\text{Lie}(n)/\Lambda\text{Lie}(n)\) are represented by cocycles which are obtained by composing the vectors \(T_{\{a_1, a_2\}}\) and \(T_{\{a_1 a_2\}}\) from the statement of Theorem 4.1. For example, the vector \(\{a_1, a_3\}a_2\) is represented in \(E^2 V_\bullet(n)\) by the linear combination shown in figure 4.2.

![Fig. 4.1. The brace tree \(T^\bullet_q \in T^{tw}(q)\)](image)
We claim that the spectral sequence abuts at this point. In other words,
\begin{equation}
E^\infty V_\bullet (n) = E^2 V_\bullet (n).
\end{equation}

Indeed, closed (in \( Br(n) \)) representatives for elements of \( \text{Ger}(n)/\Lambda \text{Lie}(n) \) may be given, and hence all higher differentials are zero on \( \text{Ger}(n)/\Lambda \text{Lie}(n) \). Note that \( V_\bullet (n) \) lives in (cohomological) degrees \( 0, -1, -2, \ldots , 2 - n \), the space
\begin{equation}
\left( \Lambda \Com \circ \Lambda \text{Lie}(n)/\Lambda \text{Lie}(n) \right)
\end{equation}
is concentrated in degree \( 2 - n \), and the space
\begin{equation}
\text{Ger}(n)/\Lambda \text{Lie}(n)
\end{equation}
lives in degrees \( 0, -1, -2, \ldots , 2 - n \). Hence higher differentials in the spectral sequence must map (some subspace of) \((4.22)\) to (some quotient of) \((4.23)\).

Let us also observe that there is an action of the symmetric group \( S_p \) on \( \text{Gr}^p V_\bullet (n) \) by permuting the subtrees of the lowest vertex. The space \((4.22)\) is a subspace of the symmetric pieces \((\text{Gr}^p V_\bullet (n))^{S_p}\) while the space \((4.23)\) lives in the anti-symmetric part of \( \text{Gr}^2 V_\bullet (n) \). Hence the following Lemma is almost evident.

**Lemma 4.6.** If \( x \in (\text{Gr}^p V_\bullet (n))^{S_p} \) is a cocycle of cohomological degree \( > 2 - n \), then \( x \) is exact, i.e., there is some \( x' \in \text{Gr}^p V_\bullet (n) \) whose coboundary is \( x \). Moreover, \( x' \) may be chosen to be symmetric, i.e., \( x' \in (\text{Gr}^p V_\bullet (n))^{S_p} \).

Proof of Lemma 4.6. \( \text{Gr}^p V_\bullet (n) \) splits into a direct sum of subcomplexes, one for each isotypical component with respect to the \( S_p \)-action. By the above computation, only the symmetric piece and the antisymmetric piece contains any cohomology: the former only in the cohomological degree \( 2 - n \) and the latter only for \( p = 2 \), and in the cohomological degrees \( 0, -1, -2, \ldots , 2 - n \). The lemma immediately follows. \( \diamond \)

Let us consider a non-trivial cocycle
\[
u \in s\left( \Lambda \Com \circ \Lambda \text{Lie}(n)/\Lambda \text{Lie}(n) \right) \cap (\text{Gr}^p V_\bullet (n))^{S_p}
\]
and examine how the higher differentials in the spectral sequence act on \( u \). We claim that all these higher differentials vanish. Equivalently, this means that a closed representative for \( u \) in \( V_\bullet (n) \) may be found. The class \( u \) has a unique representative, say \( u_p \in V_\bullet (n) \), that is a linear combination of trees whose lowest non-root vertex has valence \( p \). Now \(-\delta_0 u_p \in \mathcal{F}^{p-1} V_\bullet (n) \) represents some cocycle \( u_{p-1}' \in \text{Gr}^{p-1} V_\bullet (n) \) of cohomological degree \( 3 - n \). By symmetry of \( u \) it is also symmetric and hence the coboundary of some symmetric element \( u_{p-1} \) by the preceding lemma. \( u_{p-1} \) has a unique representative we also call \( u_{p-1} \in V_\bullet (n) \) that is a linear combination of trees whose lowest non-root vertex has valence \( p - 1 \). Next, \(-\delta_0 (u_p + u_{p-1}) \in \mathcal{F}^{p-2} V_\bullet (n) \) represents some symmetric cocycle \( u_{p-2}' \in \text{Gr}^{p-2} V_\bullet (n) \) which is again the coboundary of some symmetric element \( u_{p-2} \) by the preceding lemma. Continuing in this manner, we obtain some closed element
\[
u_p + u_{p-1} + \cdots + u_2 \in V_\bullet (n)
\]
representing \( u \). Hence Claim 4.5 is shown.
Remark 4.7. One may, of course, dualize the statement of Claim 4.5. The dual statement says that
\begin{equation}
H^\bullet(V^\bullet_2(n), \delta_0^\bullet) \cong X \oplus U^\bullet,
\end{equation}
where $X \subset \text{Ger}(n)^*$ is the kernel of $\text{Ger}(n)^* \to \Lambda \text{Lie}(n)^*$ and $U^\bullet$ is the linear dual of $(1.22)$.

Let us observe that degree $n-2$ vectors in $V^\bullet_2(n)$ (those are dual to degree $2-n$ vectors in $V^\bullet_4(n)$) are precisely linear combinations of brace trees with exactly one neutral vertex (and $n$ labeled vertices). For example, to every tuple $(r_1, r_2, \ldots, r_k)$ of positive integers satisfying the condition $r_1 + r_2 + \cdots + r_k = n$ and a permutation
\[
\sigma = \left( \begin{array}{cccccc}
1 & \ldots & r_1 & r_1 + 1 & \ldots & r_1 + r_2 & \ldots & n - r_k + 1 & \ldots & n \\
1 & \ldots & i_1^1 & i_1^2 & \ldots & i_1^2 & \ldots & i_k & \ldots & i_k \end{array} \right) \in S_n
\]
we assign the brace tree $T_{r_1, \ldots, r_k}$ depicted in figure 4.4. Each such brace tree is $\delta_0^\bullet$-closed and has degree $n-2$ in the dual space $V^\bullet_2(n)$.

We claim that

**Lemma 4.8.** The $\delta_0^\bullet$-cohomology classes represented by the brace trees $T_{r_1, \ldots, r_k}$ span the space $H^{n-2}(V^\bullet_2(n), \delta_0^\bullet)$.

**Proof.** An equivalent form of the statement is that if $x$ is a cohomology class in $H^\bullet(V^\bullet_2(n), \delta_0^\bullet)$ representable by a linear combination of brace trees with exactly one neutral vertex, and if furthermore all $T_{r_1, \ldots, r_k}$ evaluate to zero on $x$, then $x$ is trivial. To show this equivalent statement let us trace through the proof of Claim 4.5. We may assume that $x$ is contained either in the first or the second summand of $(4.5)$. Assume that $x$ is contained in the second summand and nontrivial. Then, by the proof of Claim 4.5, we see that it is represented by a linear combination containing brace trees of the form $T_{r_1, \ldots, r_k}$ for some $\sigma$ and $r_1, \ldots, r_k$. Hence the evaluation of $T_{r_1, \ldots, r_k}$ on $x$ is non-zero.

Next, if $x$ lives in the first summand of $(4.5)$, an explicit representative for $x$ may be written down, which is a linear combination of elements of the form
\[
\sigma' \cdot \mu(T_{a_1 a_2} \otimes L_1 \otimes L_2)
\]
where $\mu$ denotes the operadic composition, $\sigma' \in S_n$ and $L_1$ and $L_2$ are elements of $\Lambda \text{Lie} \subset \text{Br}$. Since non-zero elements of $\Lambda \text{Lie} \subset \text{Br}$ always contain string-like brace trees, the representative for $x$ necessarily contains brace trees as in figure 4.4 and hence the evaluation of $T_{r_1, \ldots, r_k}$ on $x$ is necessarily non-zero for some $\sigma$ and $r_1, \ldots, r_k$, unless $x = 0$. \hfill $\square$

4.3. A technical claim. To proceed further we need the following technical claim:

**Claim 4.9.** Let $\{T^n_\lambda\}_{\lambda \in S_n}$ be the family of brace trees shown in figure 4.3. Furthermore, let $T_{r_1, \ldots, r_k}$ be the brace tree shown in figure 4.4, as introduced above in Lemma 4.8. Then the vector
\begin{equation}
\delta_1^\bullet(T_{r_1, \ldots, r_k})
\end{equation}
is cohomologous in $(V^\bullet_2, \delta_0^\bullet)$ to
\begin{equation}
\sum_{\tau \in \text{Sh}_{r_1, \ldots, r_k}} (-1)^{|\tau|} T^n_{\sigma_\tau r_1, \ldots, r_k},
\end{equation}
where $(-1)^{|\tau|}$ is the sign of the permutation $\tau$.

**Example 4.10.** Before giving the proof, we illustrate the statement of Claim 4.9 with an example shown in figure 4.5. In the first step in figure 4.5, we apply the operator $\delta_1^\bullet$. In the second step we add a coboundary term and compute the sum. In the third step, we add another coboundary term to the result and obtain an expression of the form $(1.26)$. 
Proof. Since the operators $\delta_1^*$ and $\delta_0^*$ commute with the action of the symmetric group, it suffices to prove the desired statement for $\sigma = \text{id} \in S_n$.

The case $n = 2$ is very easy, so we leave it to the reader and take it as the base of our induction on $n$.

To perform the inductive step, we observe that the set of permutations

$$\left\{ \tau^{-1} \right\}_{\tau \in \text{Sh}_{r_1, \ldots, r_k}}$$

is the disjoint union of the subsets $\Sigma_q$, $q \geq 1$, where $\Sigma_q$ consists of permutations of the form

$$(4.28) \quad \left( \begin{array}{cccc} 1 & 2 & \ldots & n \\ t_q & \bullet & \ldots & \bullet \end{array} \right)$$

where

$$(4.29) \quad t_q = \begin{cases} 1 & \text{if } q = 1, \\ r_1 + \cdots + r_{q-1} + 1 & \text{otherwise} \end{cases}$$

and $\bullet \ldots \bullet$ is any unshuffle of the groups of indices:

$$(1, \ldots, r_1), (r_1 + 1, \ldots, r_1 + r_2), \ldots, (t_q + 1, \ldots, r_q), \ldots, (n - r_k + 1, \ldots, n).$$

Next, we denote by $X_{r_1, \ldots, r_k}^q$, $Y_{r_1, \ldots, r_k}^q$, and $T_{\text{opp}}^{\sigma}$ the brace trees shown in figures 4.6, 4.7, 4.8, respectively.
We remark that

\[(4.30) \quad \delta^*_1(T^{\text{id}_{r_1,\ldots,r_k}}) = \sum_{q=1}^{k} (-1)^{t_q-1} X^q_{r_1,\ldots,r_k} \]

and for every $1 \leq q \leq k$

\[(4.31) \quad X^q_{r_1,\ldots,r_k} - \delta_0^k(Y^q_{r_1,\ldots,r_k}) = (t_q, t_q + 1, \ldots, n) (T^{\text{opp}} \circ_1 \delta_1^* (T^{\text{id}_{r_1,\ldots,r_{q-1},r_k}})) .\]
By inductive assumption, the vector \( \delta^*_1(\tau^{{\text{id}}}_{r_1, \ldots, r_q - 1, \ldots, r_k}) \) is cohomologous in \((V^*_0, \delta^*_0)\) to

\[
\sum_{\tau' \in \text{Sh}_{r_1, \ldots, r_q - 1, \ldots, r_k}} (-1)^{|\tau'|} T_{{(\tau')-1}}^{n-1}.
\]

Thus Claim 4.9 is proved. \hfill \square

4.4. Analyzing the map (4.2). The goal of this subsection is to describe the kernel and the cokernel of the map (4.2). This way we will get the desired isomorphism \( H^*(\text{Br}(n)) \cong \text{Ger}(n) \).

More precisely,
Claim 4.11. For the map (4.2) we have

$$\ker \left( H^\bullet(\delta_1) : H^\bullet(V_n(n), \delta_0) \to H^\bullet(V_n(n), \delta_0) \right) \cong \Lambda \text{Lie}(n)$$

and

$$\operatorname{coker} \left( H^\bullet(\delta_1) : H^\bullet(V_n(n), \delta_0) \to H^\bullet(V_n(n), \delta_0) \right) \cong \text{Ger}(n)/\Lambda \text{Lie}(n).$$

Proof. We will prove Claim 4.11 by analyzing the dual map

$$H^\bullet(\delta_1^*) : H^\bullet(V_n^*(n), \delta_0^*) \to H^\bullet(V_n^*(n), \delta_0^*).$$

Recall that the right hand side has dimension $n!$ and classes are represented by string-like trees corresponding to permutations (see figure (4.3)).

Due to Claim 4.9 the image of $H^\bullet(\delta_1^*)$ in

$$H^\bullet(V_n^*(n), \delta_0^*) \cong S^{n-1} \mathbb{K}[S_n]$$

is spanned by classes which correspond to shuffle products in $\mathbb{K}[S_n]$.

Hence,

$$\operatorname{coker} (H^\bullet(\delta_1^*)) \cong \Lambda \text{Lie}(n)^*$$

or equivalently

$$\ker (H^\bullet(\delta_1)) \cong \Lambda \text{Lie}(n).$$

Next let us examine the kernel of the map $H_\bullet(\delta_1^*) : H^\bullet(V_n^*(n), \delta_0^*) \to H^\bullet(V_n^*(n), \delta_0^*)$. Since classes in $H^{\leq n-2}(V_n^*(n), \delta_0^*)$ are represented by linear combinations of brace trees with at least two neutral vertices, Claim 4.3 implies that

(4.33) 

$$H_\bullet(\delta_1^*) \big|_{H^{\leq n-2}(V_n^*(n), \delta_0^*]} = 0.$$ 

By Remark 4.7 above, $H^{n-2}(V_n^*(n), \delta_0^*)$ consists of two summands, namely the linear dual of (4.22) and a subspace which we call $X_1$ of $\text{Ger}(n)^*$. Representatives of classes in $X_1$ are given by antisymmetric linear combinations of a tree with a neutral vertex with two child strings, and the same tree, but with the order of the strings flipped. Since the shuffle product is commutative, such a class is mapped to zero by $H_\bullet(\delta_1^*)$. Combining this observation with (4.33), we conclude that

(4.34) 

$$X \subset \ker (H_\bullet(\delta_1^*)),$$

where $X$ is the component of $H^\bullet(V_n^*(n), \delta_0^*)$ isomorphic to the kernel of $\text{Ger}(n)^* \to \Lambda \text{Lie}(n)^*$.

Now we are ready to show that

(4.35) 

$$\ker (H^\bullet(\delta_1^*)) \cong X$$

or equivalently,

(4.36) 

$$\operatorname{coker} (H^\bullet(\delta_1)) \cong \text{Ger}(n)/\Lambda \text{Lie}(n).$$

Indeed, by counting dimensions we get

$$\dim \operatorname{im}(\delta_1^*) = \dim (\text{As}(n)) - \dim (\text{Lie}(n)) = n! - (n-1)!. $$

Of course $\dim \operatorname{im}(\delta_1^*) = \operatorname{codim} \ker(\delta_1^*)$. Finally

$$\operatorname{codim} X = \dim U = \dim (\text{Ger}(n)/\Lambda \text{Lie}(n)) = n! - (n-1)! = \operatorname{codim} \ker(\delta_1^*),$$

where $U$ is the space (4.22).

Thus, it follows that $\ker(\delta_1^*) = X$, and Claim 4.11 is proved. 

\[ \square \]
Now let us pull together all strings and finish the proof of Theorem 4.1. The theorem states that the map

$$\text{Ger} \to H^*(\text{Br})$$

defined on generators as stated in the theorem is a map of operads and an isomorphism. Showing that it is a map of operads amounts to checking that the Gerstenhaber relations hold between the images of the generators. This is a relatively trivial verification. Let us turn to the statement that $\text{Ger} \to H^*(\text{Br})$ is an isomorphism, i.e., that $\text{Ger}(j) \to H^*(\text{Br}(j))$ is an isomorphism for all $j$. Assume inductively the statement holds for $j < n$ and let us show it for $j = n$. By Claim 4.11 and the arguments above it we know that

$$H^*(\text{Br}(n)) \cong \Lambda \text{Lie}(n) \oplus \text{Ger}(n)/\Lambda \text{Lie}(n) \cong \text{Ger}(n),$$

but we still have to verify that the morphisms $\text{Ger}(n)/\Lambda \text{Lie}(n) \to H^*(\text{Br}(n))$ and $\Lambda \text{Lie}(n) \to H^*(\text{Br}(n))$ constructed in the proof above agree with the ones proposed in Theorem 4.1. Let us begin with the morphism $\text{Ger}(n)/\Lambda \text{Lie}(n) \to H^*(\text{Br}(n))$. Above, we constructed it recursively by sending an element $A \cdot B$ to the (class represented by the) insertion of the images of $A$ and $B$ in into the element $T_{a_1 a_2}$ given in the Theorem. Hence, under the induction hypotheses our two maps $\text{Ger}(n)/\Lambda \text{Lie}(n) \to H^*(\text{Br}(n))$ agree. Let us consider the two maps $\Lambda \text{Lie}(n) \to H^*(\text{Br}(n))$. In the proof of Claim 4.11 above we identified $\Lambda \text{Lie}(n)$ as the $\delta$-closed linear combinations of trees without neutral vertices. Moreover the latter space was identified with $\Lambda \text{Lie}(n)$ by (trivially) identifying the dual space (string-like trees modulo shuffles) with $\Lambda \text{Lie}(n)^\ast$, i.e., with linear combinations of permutations modulo shuffles. Hence we are done if we can verify that the two pairings

$$\Lambda \text{Lie}(n)^\ast \otimes \Lambda \text{Lie}(n) \to \mathbb{K}$$

that one can construct from the data given agree. One verifies this (for example) by choosing an explicit basis of $\Lambda \text{Lie}(n)$ given by Lie words of the form $\{?, ?, \ldots, ?, a_1 \} \cdots$ and tracing the maps.

Theorem 4.1 is proven. □

References

[1] C. Berger and B. Fresse, Combinatorial operad actions on cochains, Math. Proc. Cambridge Philos. Soc., 137, 1 (2004) 135–174.
[2] D. Calaque and T. Willwacher, Triviality of the higher Formality Theorem, arXiv:1310.4605.
[3] P. Deligne, Lettre to V. Drinfeld, M. Gerstenhaber, J.P. May, V. Schechtman and J. Stasheff, unpublished, 1993.
[4] V.A. Dolgushev and T. Willwacher, Operadic Twisting – with an application to Deligne’s conjecture, arXiv:1207.2180, to appear in J. Pure Appl. Alg.
[5] V.G. Drinfeld, On quasitriangular quasi-Hopf algebras and on a group that is closely connected with $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. (Russian) Algebra i Analiz 2, 4 (1990) 149–181.
[6] M. Gerstenhaber, The cohomology structure of an associative ring, Annals of Math., 78 (1963) 267–288.
[7] M. Kontsevich, Operads and motives in deformation quantization, Lett. Math. Phys., 48, 1 (1999) 35–72.
[8] M. Kontsevich and Y. Soibelman, Deformations of algebras over operads and the Deligne conjecture, Proceedings of the Moshe Flato Conference Math. Phys. Stud. 21, 255–307, Kluwer Acad. Publ., Dordrecht, 2000.
[9] P. Lambrechts and I. Voli, Formality of the little $N$-disks operad, Mem. Amer. Math. Soc. 230 (2014) no. 1079, viii+116 pp.
[10] J.-L. Loday and B. Vallette, Algebraic Operads, Grundlehren der mathematischen Wissenschaften, 346, Springer-Verlag 2012.
[11] J. E. McClure and J. H. Smith, A solution of Deligne’s Hochschild cohomology conjecture, Contemp. Math. 293 (2002) 153–193, Amer. Math. Soc., Providence, RI; math.QA/9910126.
[12] J. E. McClure and J. H. Smith, Multivariable cochain operations and little $n$-cubes, J. Amer. Math. Soc. 16, 3 (2003) 681–704.
[13] P. Severa and T. Willwacher, Equivalence of formalities of the little discs operad, Duke Math. J. 160, 1 (2011) 175–206.
[14] D. Tamarkin, Formality of chain operad of little discs, Lett. Math. Phys. 66, 1-2 (2003) 65–72.
[15] D. Tamarkin, Another proof of M. Kontsevich formality theorem, arXiv:math/9803025
[16] D. Tamarkin, What do DG categories form? Compos. Math. 143, 5 (2007) 1335–1358; math.CT/0606553.
Department of Mathematics, Temple University,
Wachman Hall Rm. 638
1805 N. Broad St.,
Philadelphia, PA, 19122 USA
E-mail address: vald@temple.edu

University of Zürich,
Institute of Mathematics,
Winterthurerstrasse 190,
8057 Zürich, Switzerland
E-mail address: thomas.willwacher@math.uzh.ch