Relations Between Hyperelliptic Integrals

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Abstract

A simple property of the integrals over the hyperelliptic surfaces of arbitrary genus is observed. Namely, the derivatives of these integrals with respect to the branching points are given by the linear combination of the same integrals. We check that this property is responsible for the solution to the level zero Knizhnik-Zamolodchikov equation given in terms of hyperelliptic integrals.

1 Introduction

The starting point of our investigation is the observation due to F.A. Smirnov that the integral representation for the solutions to the Knizhnik-Zamolodchikov equation restricted to the level zero and associated with affine \( \widehat{\mathfrak{sl}}_2 \) algebra can be rewritten as the determinant of the matrix having second kind hyperelliptic integrals as its elements. The KZ equation in this case is the system of first order differential equations for the multicomponent function \( f_{\varepsilon_1,\ldots,\varepsilon_n}(\lambda_1,\ldots,\lambda_n) \in \mathbb{C}^2 \otimes \ldots \otimes \mathbb{C}^2; \varepsilon_i = \pm, \lambda_i \in \mathbb{C} \)

\[
\left( \frac{\partial}{\partial \lambda_j} - \frac{1}{4} \sum_{i \neq j} \sigma_i^a \otimes \sigma_j^a \frac{\lambda_j - \lambda_i}{\lambda_j - \lambda_i} \right) f(\lambda_1, \ldots, \lambda_n) = 0,
\]

where the operators \( \sigma_i^a = 1 \otimes \ldots \otimes 1 \otimes \sigma_i^a \otimes 1 \otimes \ldots 1 \) act as the Pauli matrices in the \( i \)th two-dimensional vector space \( \mathbb{C}^2 \). System (1) is obviously split to the subsystems for the functions \( f_{\varepsilon_1,\ldots,\varepsilon_n}(\lambda_1,\ldots,\lambda_n) \) with \( \# \{ \varepsilon_i = + \} = l \) and \( \# \{ \varepsilon_i = - \} = m \) being fixed. These subsystems correspond to the fixed total spin \( (l - m)/2 \).

The question that we address in this letter is why the determinant combination of the full hyperelliptic integrals, which are complicated transcendental functions of the branching points \( \lambda_i \), satisfy a simple differential equation with rational coefficients.
We found the explanation to this phenomena, namely to the fact that the derivatives of the full hyperelliptic integral with respect to the branching points can be written as a linear combination of the same integrals. The proof is just the application of the integration by part rule to some third kind integral \([12]\).

Since the hyperelliptic integrals appear in many problems of the mathematical physics we hope that the formulas of this note will be useful in application to such problems. Let us formulate the main result of this letter.

Consider the Riemann hyperelliptic surface of genus \(g\) given in \(\mathbb{C}^2\) by algebraic relation

\[ y^2 = \prod_{k=1}^{n} (x - \lambda_k), \quad (2) \]

where \(n = 2g + 1\) or \(n = 2g + 2\) depending on the case whether the infinity is the branching point or not. Using rational transformation we can always place all the branching points in the finite part of the complex plane, so without losing the generality we will restrict ourselves to the case when \(n = 2g + 2\). Define the integrals on this Riemann surface

\[ K_j(\lambda) = \int_{\gamma} \frac{x^j dx}{\sqrt{P_{2g+2}(x)}}, \quad j \in \mathbb{Z}_+, \quad (3) \]

\[ P_{2g+2}(x) = \prod_{i=1}^{2g+2} (x - \lambda_i) = \sum_{i=0}^{2g+2} (-)^i \sigma_i(\lambda) x^{2g+2-i}, \quad (4) \]

where \(\sigma_i(\lambda)\) are order \(i\) homogeneous and symmetric functions of the branching points \(\lambda_j\), and \(\gamma\) is an arbitrary closed contour on the surface that in the case under consideration can be reduced to the sum of the integrals between points \(\lambda_j\). We state a

**Proposition.** The integrals \([3]\) with \(j = 0, 1, \ldots, 2g\), as functions of the branching points satisfy the “minimal” system of the differential equations of the first order

\[ \frac{\partial K_j(\lambda)}{\partial \lambda_m} = \frac{1}{2} \left( \sum_{i=0}^{j-1} \lambda_i^{-i-1} K_i(\lambda) + \frac{\lambda_i^{m}}{P_{2g+1}(\lambda_m)} \sum_{i=0}^{2g} a_i(\lambda) K_i(\lambda) \right) \]

\[ a_i(\lambda) = (-)^i \left( \sum_{k \neq m}^{2g-i} \sigma^{(m,k)}_{2g-i} - \sum_{m=0}^{2g} (-)^k \lambda_i^k \sigma^{(m)}_{2g-i-m} \right), \quad (5) \]

where \(m = 1, \ldots, 2g + 2\),

\[ \hat{P}_{2g+1}^{(m)}(x) = \prod_{l \neq m}^{2g+2} (x - \lambda_l) = \sum_{i=0}^{2g+1} (-)^i \hat{\sigma}^{(m)}_i x^{2g+1-i}, \]

and \(\hat{\sigma}^{(m)}_i\) are symmetric function of the order \(i\) of the variables \(\lambda_l\), \(l = 1, \ldots, m - 1, m + 1, \ldots, 2g + 2\). The symmetric functions \(\hat{\sigma}^{(m,k)}_i\) are defined analogously

\[ \hat{P}_{2n-2}^{(m,k)}(x) = \prod_{l = 1 \atop l \neq m,k}^{2g} (x - \lambda_l) = \sum_{i=0}^{2g} (-)^i \hat{\sigma}^{(m,k)}_i x^{2g-i}. \]

\(^1\)This word will be explained in Sect. 3.
Using the obvious identity
\[
\sum_{k=1, k\neq m}^{M} \hat{\sigma}^{(k,m)}_j = (M - 1 - j) \hat{\sigma}^{(m)}_j, \quad j = 0, 1, \ldots, M - 2
\]  
(6)
we can express the coefficients \(a_i\) only in terms of symmetric functions \(\hat{\sigma}^{(m)}_i\)
\[
a_i(\lambda) = (-i)^i \left( i\hat{\sigma}^{(m)}_{2g-i} - \sum_{k=1}^{2g-i} (-)^k \lambda^k \hat{\sigma}^{(m)}_{2g-i-k} \right).
\]  
(7)

2 Proof

To prove the above statement we observe first that the derivative
\[
\frac{\partial K_j(\lambda)}{\partial \lambda_m} = \frac{1}{2} \int_\gamma \frac{x^j dx}{\sqrt{P_{2g+2}(x)(x - \lambda_m)}}
\]  
(8)
is independent on the contour \(\gamma\). Ended, using the homology symmetry on the Riemann surface we can always remove the dependence on the variable \(\lambda_m\) in the integration limits. In what follows we will always assume that the contour \(\gamma\) is chosen in such a way that it does not cross the branching point \(\lambda_m\). As result we can conclude that derivative of \(K_j\) with respect to any branching point is given by integral (8). Dividing \(x^j/(x - \lambda_m)\), we obtain the first term in (6), and calculation of (8) is reduced to the calculation of the integral
\[
\int_\gamma \frac{dx}{\sqrt{P_{2g+2}(x)(x - \lambda_m)}}.
\]  
(9)
To calculate integral (9) we use the following trick. Let us again use the fact that the integral over any closed contour on the Riemann surface \(y^2 = P_{2g+2}(x)\) can be reduced to the integral between the branchings points. In this case the integral
\[
\int_\gamma dx \frac{dP_{2g+2}(x)}{dx (x - \lambda_m)} = 0
\]  
(10)
is identically zero if we again adjust properly the contour \(\gamma\) in (10).

Calculating the derivative under the integral in (10) we arrive to the relation
\[
\int_\gamma \frac{\sum_{k=1, k\neq m}^{2g+2} \hat{P}^{(m,k)}_{2g}(x)}{\sqrt{P_{2g+2}(x)}} = \int_\gamma \frac{\hat{P}^{(m)}_{2g+1}(x)}{\sqrt{P_{2g+2}(x)(x - \lambda_m)}}.
\]  
(11)
Dividing $\tilde{P}_{2g+1}^{(m)}(x)/(x - \lambda_m)$, we obtain

$$
\int_{\gamma} \frac{dx}{\sqrt{P_{2g+2}(x)(x - \lambda_m)}} = \frac{1}{\tilde{P}_{2g+1}^{(m)}(\lambda_m)} \sum_{i=0}^{2g} a_i(\lambda)K_i(\lambda),
$$

(12)

where coefficients $a_i$ are given by (3). This finishes the proof of the proposition.

Let us point out that for the elliptic surface the formulas (3) are well known and can be found in most books on special functions (see for example [6], formulas 8.123.1-4).

### 3 Discussion

Let us explain the word “minimal” used in the formulation of the proposition. To obtain formulas like (3), we can start from the integral $K_j(\lambda)$ for any $j \geq 0$ and calculate the derivative of this integral with respect to any branching point. Then, as we have seen in the previous section, the minimal number of integrals $K_i$ contained in the formula for the derivative is equal to $2g + 1$, $0 \leq i \leq 2g$. It explains why we choose this minimal number of integrals to write down the formulas (3). This minimal set of integrals consists of $g$ integrals of the first kind differentials $(x^i/\sqrt{P_{2g+2}(x)}$, $0 \leq i \leq g)$ with no singularities at the complex plain, $g$ integrals of the second kind differentials $(x^i/\sqrt{P_{2g+2}(x)}$, $g + 1 \leq i \leq 2g)$ with pole singularity at infinity points $\infty^{\pm}$ of the order $i - g + 1$ and one third kind integral of $x^g/\sqrt{P_{2g+2}(x)}$ with logarithmic singularity at the infinity.

Because of the uniqueness of the third kind integral we can reduce the system (3). Let us introduce the integrals of the second kind differentials which has zero residues at infinity,

$$
E_j = K_j - c_j K_g, \quad j = 0, \ldots, 2g,
$$

(13)

where

$$
c_j = \text{res}_{x=\infty} \frac{x^j}{\sqrt{P_{2g+2}(x)}}.
$$

(14)

There is a simple recurrent relation for the symmetric functions $c_j(\lambda)$ that follows from (3)

$$
2 \frac{\partial c_j(\lambda)}{\partial \lambda_m} = \sum_{i=g}^{j-1} \lambda_m^{j-i-1} c_i(\lambda), \quad j = g + 1, \ldots, 2g.
$$

(15)

Multiplying the left and right sides of (15) by $\lambda_m$ and summing over $m$, we obtain

$$
c_j(\lambda) = \frac{1}{2(j - g)} \sum_{i=g}^{j-1} s_{j-i}(\lambda)c_i(\lambda),
$$

(16)

\footnote{Equation (15) can be easily obtained by taking the derivative of (14).}
where we have used the property of the order \((j - g)\) homogeneous functions \(c_j(\lambda)\) and introduced new symmetric functions
\[
s_i(\lambda) = \sum_{m=1}^{2g+2} \lambda_m^i.
\]
(17)

Using the recurrent relation (16) with a boundary condition \(c_g = 1\), one can easily calculate the functions \(c_j\). For example,
\[
c_{g+1} = \frac{1}{2} s_1 = \frac{1}{2} \sigma_1,
\]
\[
c_{g+2} = \frac{1}{4} s_2 + \frac{1}{8} s_1^2 = \frac{3}{8} \sigma_1^2 - \frac{1}{2} \sigma_2,
\]
\[
c_{g+3} = \frac{1}{6} s_3 + \frac{1}{8} s_1 s_2 + \frac{1}{48} s_1^3 = \frac{5}{16} \sigma_1^3 - \frac{3}{4} \sigma_1 \sigma_2 + \frac{1}{2} \sigma_3.
\]

Let us point out that the set of functions \(c_j\) forms a new basis in the space of homogeneous and symmetric functions of many variables different from the bases generated by \(s_j\) and \(\sigma_j\) with a good property (15). On the other hand, the symmetric functions \(c_j(\lambda)\) are the coefficients of the expansion of the function \(1/\sqrt{P_{2g+2}(x)}\) in the vicinity of the infinity point. The recurrent relation (16) is much more convenient for calculation these coefficients than the direct expansion of this function.

Now we are in the position to reduce \(2g+1\)-dimensional system (5) for the integrals \(K_j\) to \(2g\) dimensional system for the integrals \(E_j\).
\[
\frac{\partial E_j(\lambda)}{\partial \lambda_m} = \frac{1}{2} \left( \sum_{i=0}^{g-1} \lambda_m^{j-i-1} E_i(\lambda) - c_j(\lambda) \sum_{i=0}^{g-1} \lambda_m^{g-i-1} E_i(\lambda) \right) + \frac{\lambda_j^m - c_j(\lambda) \lambda_m^g}{2 P^{(m)}_{2g+1}(\lambda_m)} \sum_{i=0}^{2g} a_i(\lambda) E_i(\lambda).
\]
(18)

This reduction follows from the nontrivial identity between the functions \(c_j(\lambda)\) and \(a_i(\lambda)\)
\[
\sum_{i=g}^{2g} a_i(\lambda) c_i(\lambda) = 0.
\]
(19)

It was demonstrated in [1] that the linear combination of the determinants composed from the integrals \(E_i, i = g + 1, \ldots, 2g,\) satisfies the level zero KZ equation (1). The papers [1, 2] were devoted to the investigation of the classical limit of quantum KZ equation [7]. The quantum KZ equation is a system of difference equations which historically originates in the bootstrap approach in quantum field theory. F.A. Smirnov developed the systematic approach to the solutions of this kind difference systems that was summarized in the book [3]. The quantum or deformed KZ equation at level zero coincide essentially with the form factors equations in the completely integrable models of quantum field theory. The solutions to the form factor equations for the models which are associated with Yangian and \(U_q(\widehat{sl}_2), |q| = 1\) symmetries
and restricted to the total spin zero case were considered in that book. Recently the same approach was successfully applied to the construction of the integral solution to the level 0 deformed KZ equation associated with $U_q(\mathfrak{sl}_2)$, $|q| < 1$ symmetry and for arbitrary values of the total spin $\frac{1}{2}$.

One of the main ingredients of the Smirnov’s approach is the identities between integrals of some meromorphic functions which are deformed analogues of the holomorphic differentials on the hyperelliptic Riemann surface. Using these identities and quasi-periodicity properties of the kernel of the integral solution, one can solve the deformed KZ equation.

The integral solutions to the classical and quantum KZ equation have quite different properties due to the problem of braiding. In the deformed case the integrands for the solutions are the meromorphic functions which have infinite number of simple poles and essential singularities at infinity that makes the braiding trivial. In the classical limit these infinite sequences of the poles are concentrating into the cuts that leads to complicated braiding which is served by a finite-dimensional quantum groups. This makes the relation between the solutions to the deformed and the ordinary KZ equations quite complicated and only asymptotical (see Ref. [1] for the precise treatment). But nevertheless, as we have seen above, it is possible to find quite simple identities between hyperelliptic integrals and then to show that the complicated transcendental solutions to the level zero KZ equation are the consequences of these relations. The integral formulas for the solution to this equation in the subsector of the total spin zero were written in [1], while those for the non-zero subsectors can be obtained after the classical limit from the formulas presented in [3].

4 Conclusion

In this note we have addressed to the question why the simple dynamical systems with rational coefficients like level zero KZ equation possess the complicated transcendental solution [1] defined on the hyperelliptic surface. We have found the simple relation between the hyperelliptic integrals which is responsible for this phenomena.

To conclude we would like to point out the questions which are very interesting from our point of view and deserve further investigation.

• Whether is it possible to obtain by the pure analytical tools the integral formula for the solution of KZ equation at the arbitrary level. We suppose that the key step to solve this problem is to use the equivalence between KZ and Calogero systems [10] and a starting point there should be the ground wave function for the Calogero system containing the different sort of particles.

• It is an interesting problem to investigate the operator content of the formulas (18) in the sense of the field theory on algebraic curves. An exhaustive investigation in this direction was done recently in preprint [11] for for the field theories on the $\mathbb{Z}_n$-symmetric algebraic curves and in [12] for the $\mathbb{D}_n$-symmetric ones.
• As it is known [13], the hyperelliptic integrals can be expressed in terms of the theta constants. It is an interesting question what relations between the theta constants correspond to formulas [9].

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