The short pulse equation
is integrable

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Abstract

We prove that the Schäfer–Wayne short pulse equation (SPE) which describes the propagation of ultra-short optical pulses in nonlinear media is integrable. First, we discover a Lax pair of the SPE which turns out to be of the Wadati–Konno–Ichikawa type. Second, we construct a chain of transformations which relates the SPE with the sine-Gordon equation.

1 Introduction

Recently, Schäfer and Wayne \textsuperscript{1} derived the short pulse equation (SPE) as a model equation, alternative to the nonlinear Schrödinger equation (NLSE), to approximate the evolution of very short optical pulses in nonlinear media. More recently, Chung, Jones, Schäfer and Wayne \textsuperscript{2} proved numerically that, as the pulse length shortens, the NLSE approximation becomes less and less accurate while the SPE provides a better and better approximation to the solution of Maxwell’s equation.

In the present paper, we study this interesting new nonlinear equation, the SPE, from the standpoint of its integrability. The NLSE is well known as one of the numerous nonlinear equations integrable by the inverse scattering
transform technique \[3\]. Therefore it is natural to ask whether the SPE, as an ultra-short pulse alternative to the NLSE, is integrable as well, or one has to rely mainly on numerical techniques when studying the SPE. We prove that the SPE is integrable. In Section 2 we find that the SPE possesses a Lax pair of the Wadati–Konno–Ichikawa (WKI) type \[4\]. Then, in Section 3 we show how the SPE can be transformed into the well-known integrable sine-Gordon equation \[3\]. Section 4 summarizes the results.

We study the SPE in the form

\[ u_{xt} = u + \frac{1}{6} (u^3)_{xx} \] (1)

which is related to the original form used in \[1, 2\] by scale transformations of the dependent and independent variables.

## 2 Zero-curvature representation

Let us try to find a zero-curvature representation (ZCR) of the SPE (1),

\[ D_t X - D_x T + [X, T] = 0, \] (2)

which is the compatibility condition of the overdetermined linear system

\[ \Psi_x = X \Psi, \] \hspace{1cm} (3)
\[ \Psi_t = T \Psi, \] \hspace{1cm} (4)

where \( X \) and \( T \) are \( n \times n \) matrix functions of \( u \) and its derivatives, \( \Psi(x, t) \) is an \( n \)-component column, \( D_x \) and \( D_t \) denote the total derivatives, and the square brackets denote the matrix commutator.

Taking for simplicity

\[ X = u_x A + B, \quad T = T(u, u_x), \] (5)

with constant matrices \( A \) and \( B \), and replacing \( u_{xt} \) by the right-hand side of (1), we find from (2) that

\[ T = \frac{1}{2} u^2 u_x A + S(u), \] (6)

where the matrix function \( S(u) \) satisfies the equations

\[ \frac{dS}{du} = [A, S - \frac{1}{2} u^2 B], \quad u A + [B, S] = 0. \] (7)
Setting $S(u)$ to be the simplest nontrivial polynomial,

$$ S = \frac{1}{2}u^2B + uP + Q \tag{8} $$

with constant matrices $P$ and $Q$, we have from (7)

$$ A = -[B, P] \tag{9} $$

and

$$ [P, [B, P]] = B, \quad [B, Q] = 0, \quad [Q, [B, P]] = P. \tag{10} $$

A nontrivial solution of the commutator equations (10) can be easily found in traceless $2 \times 2$ matrices $B$, $P$ and $Q$. Using this solution, together with (5), (6), (8) and (9), and simplifying the expressions for $X$ and $T$ by $X \mapsto GXG^{-1}$ and $T \mapsto GTG^{-1}$ with a constant matrix $G$, we obtain the following:

$$ X = \begin{pmatrix} \lambda & \lambda u_x \\ \lambda u_x & -\lambda \end{pmatrix} \tag{11} $$

and

$$ T = \begin{pmatrix} \frac{\lambda}{2}u^2 + \frac{1}{4\lambda} & \frac{\lambda}{2}u^2u_x - \frac{1}{2}u \\ \frac{\lambda}{2}u^2u_x + \frac{1}{2}u & -\frac{\lambda}{2}u^2 - \frac{1}{4\lambda} \end{pmatrix}, \tag{12} $$

where $\lambda$ is an arbitrary nonzero constant.

Consequently, the SPE (1) possesses the ZCR (2), or the Lax pair (3) and (4), with the matrices $X$ and $T$ determined by (11) and (12). The linear problem (3) with the matrix $X$ (11) is a spectral problem of the WKI type [4] with respect to $u_x$. Note, however, that the matrix $T$ (12) is nonlocal with respect to $u_x$ and contains $\lambda^{-1}$. The WKI-type ZCRs with matrices $T$ containing negative powers of the spectral parameter were studied in [5]. Nevertheless, the nonlinear equation we study, the SPE (1), did not appear in the literature prior to [1, 2], as far as we know.

We have already pointed out that the form (1) of the SPE is related to the original form used in [1, 2] by scale transformations of variables. Note, however, that some complex-valued scalings of variables are required in order to change the sign at the nonlinear term in (1). Therefore, if one does not
accept complex-valued transformations (say, as nonphysical) but is interested in the form

\[ u_{xt} = u - \frac{1}{6} (u^3)_{xx} \]  

(13)

of the SPE, one needs to use the ZCR (2) with the matrices

\[ X_{(-)} = \begin{pmatrix} \lambda & \lambda u_x \\ -\lambda u_x & -\lambda \end{pmatrix}, \]
\[ T_{(-)} = \begin{pmatrix} -\frac{\lambda}{2} u^2 + \frac{1}{4\lambda} & -\frac{\lambda}{2} u^2 u_x - \frac{1}{2} u \\ \frac{\lambda}{2} u^2 u_x - \frac{1}{2} u & \frac{\lambda}{2} u^2 - \frac{1}{4\lambda} \end{pmatrix} \]  

(14)

for the ‘minus’ form (13) of the SPE. All this is completely analogous to the situation with the ‘plus’ and ‘minus’ forms of the NLSE [3].

### 3 Equivalence transformation

Quite often an interesting newly-found nonlinear equation turns out to be equivalent to a well-studied old one through some chain of transformations. Let us see that the SPE (1) is not an exception in this respect. Of course, since the SPE possesses a WKI-type Lax pair, it is possible to use the known interrelation [6, 7] between spectral problems of the WKI type and the Ablowitz–Kaup–Newell–Segur (AKNS) type, in order to find which equation possessing an AKNS-type Lax pair corresponds to the SPE. However, we follow a different way when trying to transform the SPE into some old equation: we use a generalized symmetry of the SPE as an indicator of required transformations. This heuristic method is based on the facts that any generalized symmetry is formally represented by an evolution equation [8] and that the transformations of evolution equations are studied much better than transformations of other types of equations.

It is easy to find directly that the SPE (1) admits a third-order generalized symmetry, which is written in the evolutionary form as

\[ u_{s} = \left( u_x \left( u_x^2 + 1 \right)^{-1/2} \right)_{xx}, \]

(15)

where the subscript \(s\) denotes the derivative with respect to the group parameter \(s\). This symmetry can also be obtained using the recursion operator
\[ R = D_x^2 \cdot u_{xx}^{-1} D_x \cdot (u_x^2 + 1)^{1/2} D_x^{-1} \cdot (u_x^2 + 1)^{-3/2} u_{xx} \]

(16)

easily derivable from the matrix \( X \) (11) by the cyclic basis technique [9, 10]. This operator generates the symmetry (15) from the trivial symmetry \( u_s = 0 \), not from the translational symmetry \( u_s = u_x \), because \( Ru_x = 0 \). In what follows, however, we do not study the recursion operator (16).

We consider the generalized symmetry (15) as a third-order evolution equation in \( u(x, t, s) \), and try to transform it into a simple known evolution equation, following the way used in [10, 11].

First of all, we find the separant \( \partial f / \partial u_{xxx} \) of the equation (15), where \( f \) denotes the equation’s right-hand side, introduce the new dependent variable \( v(x, t, s) = (\partial f / \partial u_{xxx})^{1/3} \), and thus obtain the transformation

\[ v(x, t, s) = (u_x^2 + 1)^{-1/2} \]

(17)

which relates the equation (15) with the equation

\[ v_s = v^3 v_{xxx} + \frac{3v^2 v_x v_{xx}}{1 - v^2} + \frac{3v^3 v_x^3}{(1 - v^2)^2} \]

(18)

possessing the separant \( v^3 \).

Then, we introduce the new independent and dependent variables, \( y \) and \( w(y, t, s) \), such that

\[ x = w(y, t, s), \quad v(x, t, s) = w_y(y, t, s). \]

(19)

This transformation (19) relates the equation (18) with the constant-separant equation

\[ w_s = w_{yyy} + \frac{3w_y w_{yy}^2}{2 (1 - w_y^2)} + \phi(t, s) w_y, \]

(20)

where the arbitrary function \( \phi(t, s) \) appeared as a ‘constant’ of the integration over \( y \). This inessential function \( \phi(t, s) \) can be turned into zero by the redefinition of \( y \), \( y \mapsto y + \int \phi(t, s) \, ds \), which does not change \( v(x, t, s) \).

At last, trying to eliminate the second-order derivative terms from the transformed equation, we find that the new dependent variable

\[ z(y, t, s) = \arccos(w_y) \]

(21)
transforms the equation (20) (where we have already set $\phi(t, s) = 0$) into the potential modified Korteweg–de Vries equation
\begin{equation}
z_s = z_{yyy} + \frac{1}{2} z_y^3.
\end{equation}

We have transformed the generalized symmetry (15) of the SPE into the simple known evolution equation (22). The equation (22) represents a generalized symmetry of the sine-Gordon equation $z_{yt} = \sin z$ [8], but not only of it: for example, the equations $z_{yt} = \cos z$, $z_{yt} = \exp(\pm iz)$ and $z_{yt} = 0$ also admit the generalized symmetry (22). In order to see which equation possessing the generalized symmetry (22) corresponds to the SPE, we apply the chain of transformations (17), (19) and (21) directly to the SPE (1), and actually obtain the sine-Gordon equation $z_{yt} = \sin z$. Note that, when making the transformation (19), we neglect the ‘constant’ of the integration over $y$, like we did this in the equation (20) and for the same reason.

Consequently, the SPE (1) and the sine-Gordon equation
\begin{equation}
z_{yt} = \sin z
\end{equation}
are equivalent to each other through the chain of transformations
\begin{align}
v(x, t) &= \left(u_x^2 + 1\right)^{-1/2}; \quad (24) \\
x &= w(y, t), \quad v(x, t) = w_y(y, t); \quad (25) \\
z(y, t) &= \arccos(w_y). \quad (26)
\end{align}

4 Conclusion

In this paper, we studied the integrability of the short pulse equation (SPE) which describes the propagation of ultra-short optical pulses in nonlinear media. We found a Lax pair of the SPE, of the Wadati–Konno–Ichikawa type. This makes possible to use the powerful inverse scattering transform technique for obtaining and analyzing solutions of the SPE. Also we found how to transform the SPE into the well-studied sine-Gordon equation. This makes possible to derive solutions and properties of the SPE from known solutions and properties of the sine-Gordon equation. We believe that these two possibilities to study the SPE analytically, discovered in this paper, will be valuable for soliton theory and nonlinear optics.
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