GRAND-CANONICAL ENSEMBLE OF RANDOM SURFACES WITH FOUR SPECIES OF ISING SPINS

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Abstract

The grand-canonical ensemble of dynamically triangulated surfaces coupled to four species of Ising spins (c=2) is simulated on a computer. The effective string susceptibility exponent for lattices with up to 1000 vertices is found to be $\gamma = -0.195(58)$. A specific scenario for $c > 1$ models is conjectured.

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1 INTRODUCTION

The purpose of this paper is to present a calculation of the string susceptibility exponent for a statistical system of dynamically triangulated random surfaces coupled to four species of Ising spins. The method employed consists in simulating on a computer the so-called grand-canonical ensemble of surfaces with the topology of a sphere. The motivation for this study is to contribute to a better understanding of the physics of random surfaces beyond the "barrier" at $c = 1$ ($c$ denotes the central charge of matter fields interacting with the geometry).

As is well known, the spectacular progress in our understanding of two-dimensional quantum gravity, achieved during the last years, concerns models where $c \leq 1$. The string susceptibility exponent $\gamma$ is then given, for spherical topology, by the formula

$$\gamma = \frac{1}{12} [c - 1 - \sqrt{(c - 1)(c - 25)}],$$

(1)

derived first, in the continuum framework, by Knizhnik, Polyakov and Zamołodchikov [1]. The right-hand side of (1) becomes complex for $c > 1$, which is clearly unphysical and indicates that something rather dramatic happens when one attempts to move $c$ beyond unity. It is generally believed that this "decease" reflects the tachyonic instability of string models. A suggestive heuristic picture imputes the break-down of the Euclidean theory for $c > 1$ to a condensation of gravitational singularities [2].

Eq. (1) also holds in solvable models of discrete quantum gravity, when the "dynamical triangulation" recipe [3] is adopted to discretize the two-dimensional manifold. Nevertheless, the discrete models of Euclidean quantum gravity are well defined for any value of $c$. Triangulated surfaces collapse or degenerate into branched polymers when $c \to \infty$ [4, 5]. The indications of the phenomenon have been observed numerically, for $c \gtrsim 10$, many years ago [5, 6]. Also, there exist theorems establishing the absence of a non-trivial continuum limit for classes of discrete models (for a lucid discussion of these issues we refer the reader to an excellent review by Ambjørn [4]). However, in numerical simulations, one does not observe anything special to happen as one crosses the famous "barrier". Moreover, an analysis of a series expansion of the partition function, rewritten using the matrix model formalism, has been carried out by Brézin and Hikami [8]. They also did not find any sign...
of pathology at \( c > 1 \). This situation has motivated a series of numerical simulations, with the aim to achieve a better understanding of the quantum gravity coupled to matter with central charge larger than, but close to unity. As these studies were restricted to the microcanonical ensemble \([9, 10, 11, 12]\), we have decided to use the grand-canonical ensemble to determine directly the particularly interesting susceptibility exponent, for the model with four species of Ising spins \((c=2)\). In parallel, a novel method of measuring this exponent has been developed in Copenhagen \([13]\). When completing this work, we received the very recent ref. \([14]\), where \( \gamma \) is also measured for the model we have been working with. Our calculation constitutes an independent check of these results, obtained with a more traditional method. In the last section of this paper we also conjecture a specific scenario for \( c > 1 \) models.

## 2 THE NUMERICAL EXPERIMENT

### 2.1 The method

The model is defined by the partition function

\[
Z(g, \beta) = \sum_{T,N} W(T) e^{-gN} \left( \sum_{[\sigma]} \exp[ \beta \sum_{<ij>} \sigma_i \sigma_j ] \right)^n, \tag{2}
\]

where \( T \) refers to a specific triangulation and \( W(T) \) is the corresponding symmetry factor. Further, \( N \) is the number of vertices and \( n \) is the number of species of Ising spins \( \sigma \). The cosmological constant \( g \) and the spin coupling \( \beta \) are the only parameters of the model. Only nearest neighbour spins are coupled. As in the early ref. \([15]\) the spins live on the vertices of the triangulated surface (and not the dual one).

The partition function can be rewritten as

\[
Z = \sum_{N} e^{-gN} Z(N) \tag{3}
\]

For large \( N \), one expects

\[
N^3 Z(N) \sim N^\gamma e^{g_{cr}N}, \tag{4}
\]
where $\gamma$ is the susceptibility exponent and $g_{cr}$ is the (non-universal) critical cosmological constant.

A single step of the grand-canonical algorithm \[5, 6\] changes $N$ by $\pm 1$. The code employed in this work to update the geometry is essentially identical to the one used and described in detail in ref. \[6\]. The only modification is that the popular method due originally to Baumann \[16\] is adopted: the partition function is modified into

$$
\tilde{Z} = \sum_N c(N) e^{-gN} N^3 Z(N),
$$

where $c(N) = c_l e^{\Delta N}$ for $N \leq N_0$ and $c(N) = c_u e^{-\Delta N}$ for $N \geq N_0$, while $\Delta$ is an adjustable parameter. Of course, one has $c_l / c_u = \exp (-2\Delta N_0)$. Notice, that the parameters $c_l, u$ need not be specified individually since only their ratio enters the detailed balance equations.

Furthermore, $N$ is restricted to $N_0 \pm \delta N$, with $\delta N \ll N_0$. Denote by $E(N)$ the experimentally measured density. Since the algorithm satisfies the detailed balance equations, within the finite interval under consideration $E(N)$ tends to the expression under the sum in \[5\] for infinite statistics, up to global normalization. A sweep of geometry is defined as $N_0$ successive steps of the grand-canonical algorithm.

Another difference, compared to ref. \[6\], is that we consider Ising spins instead of Gaussian fields. When a new vertex is created the new spin(s) are generated using the heat bath method. A sweep of geometry is always followed by a sweep of spins. The latter is performed using Wolff’s \[17\] cluster algorithm. This algorithm is particularly appropriate in the vicinity of the critical spin coupling $\beta_c$. Sufficiently above $\beta_c$ the clusters become large and the algorithm loses its efficiency. For such values of $\beta$ the heat bath algorithm becomes competitive. We have used it occasionally as an independent check of our results.

Define

$$
g_{eff}(n, N_0) = \frac{1}{2n} \ln \left( \frac{E(N_0 + n)}{E(N_0 - n)} \right) + g
$$

Taking the asymptotic form \[4\], one gets:

$$
\frac{E(N_0 + n)}{E(N_0 - n)} = e^{-2n(g - g_{cr})} \left( \frac{N_0 + n}{N_0 - n} \right)^\gamma
$$

4
and consequently

\[ g_{\text{eff}}(n, N_0) = g_{\text{cr}} + \frac{\gamma}{N_0} \]  (8)

to leading order in \( N_0 \). The quantity \( g_{\text{eff}}(n, N_0) \) is measured for a set of values of \( N_0 \). Since the right-hand side of (8) is independent of \( n \), one averages over \( n \) to get an improved estimate of \( g_{\text{eff}}(N_0) \). The procedure is repeated many times and the error on \( g_{\text{eff}}(N_0) \) is found with the binning method. The parameters \( \gamma \) and \( g_{\text{cr}} \) are then determined from the linear fit to \( g_{\text{eff}} \) versus \( N_0^{-1} \).

The choice of \( \Delta \) is a pure matter of convenience. The experimental histogram \( E(N) \) should fall exponentially as one moves off the point \( N = N_0 \), but one wants to have good statistics for \( n \leq 5 \), say, and therefore this fall should not be too fast. The choice of the input value of \( g \) involves a subtlety. For large \( N_0 \) one has roughly

\[ \frac{E(N_0 + n)}{E(N_0 - n)} \sim e^{-2n(g - g_{\text{cr}})} \]  (9)

Thus, the experimental histogram is asymmetric. This asymmetry, together with the statistical fluctuations of the histogram, generates a systematic error in the measurement of \( g_{\text{eff}}(n, N_0) \). For example, if for a given \( n \) one has \( Z_{\text{exp}}(N_0 + n) < Z_{\text{exp}}(N_0 - n) \), then what is measured is in fact \( g_{\text{eff}}(n, N_0) - \delta \), where \( \delta \sim (1/4n)(\epsilon_n^2 - \epsilon_{-n}^2) \) and \( \epsilon_{\pm n} \) is the statistical error on \( E(N_0 \pm n) \).

In practice, one has to start with some exploratory runs in order to get a first estimate of \( g_{\text{cr}} \), and then use this estimate as the input \( g \) in the full scale simulation. Proceeding in this way one makes the systematic error insignificant compared to the statistical one.

### 2.2 Simulating solvable models

In order to check and gauge our "tools" two solvable models are simulated first, viz. pure gravity and the model with 1 spin/site.

**Pure gravity (c = 0):** The string susceptibility exponent \( \gamma \) is known to be \(-1/2\). The expression for the partition function is exactly known and the critical value of the fugacity \( z \equiv \exp(-g) \) is \( z_{\text{cr}} = 27/256 \approx 0.1055 \). To check the code, it is verified that \( E(N) \), once normalized, follows very closely the exact formula. Then, \( z \) is set to value 0.1044, on purpose slightly off.
the critical one, and we perform experiments at \( N_0 = 100, 120, 150, 200, 300, 500 \) and 1000. The Baumann parameter is given the value \( \Delta = \ln 2 \), forcing \( E(N) \) to decrease by about a factor 5 between \( N_0 \) and \( N_0 \pm 10 \). The number of sweeps per experiment is about 1 to \( 4 \times 10^5 \) sweeps (following 4000 heating sweeps) and the observed acceptance of the grand-canonical algorithm is found to be about 72% independently of \( N_0 \).

This exercise is instructive, as it helps appreciating the influence of finite size corrections on the results of an experiment of this type. Using all data points the estimate \( \gamma = -0.563(33) \) is produced, significantly different from the expected \(-0.5\). This is not just a fluctuation: the effective exponent is below \(-0.5\) precisely because our data follow closely the exact formula. However, without the points at \( N < 200 \) one obtains \( \gamma = -0.509(56) \), with a bigger error (there are less data) but closer to the theoretical value. Of course, in this particular model one can determine analytically the threshold for the applicability of (4) and the form of correction terms. This is not the case for more complicated models. We believe that, while interpreting data obtained in simulating such models, guessing the corrections to (4) is a waste of time. It is preferable to stick to the simple expression (4) and to accept the fact of life that one can only get an effective exponent. Anyway, this is the philosophy we have adopted.

1 spin/site \((c = 1/2)\): The string susceptibility is now \( \gamma = -1/3 \). The critical spin coupling is also known exactly: \( \beta_{cr} = -\frac{1}{2} \ln \tanh \frac{1}{2} \ln (108/23) \approx 0.2163 \) \( [18] \). The code is tested comparing \( E(N) \) to the first few terms of the expansion in powers of \( z \) calculated analytically; the expected values are reproduced with an error \(< 0.1\% \). We set \( \Delta = \ln 2 \), \( \delta N = 5 \) and \( \beta = \beta_{cr} \). The value of fugacity \( z = 0.05 \) is found to be acceptable as the input one. One sweep is defined as \( N_0 \) steps of the grand-canonical algorithm followed by \( n_w = N/\langle \text{cluster size} \rangle \) calls to the Wolff algorithm. The average size of clusters produced by the Wolff algorithm is determined prior to each production run and it is checked that is remains stable during the run. The efficiency of the grand-canonical algorithm is about 57\%. It is checked, using either a ”cold” or a ”hot” start, that the system thermalizes after about 2000 sweeps (cf. ref [15]). This is therefore the number of sweeps to be performed initially in order to heat the system. Simulations are carried out for \( N_0 = 200, 300, 500 \) and 1000 with about \( 1.3 \times 10^5 \) sweeps for each \( N_0 \). The resulting exponent is \( \gamma = -0.317(102) \). This is unprecise but correct and, without further ado, we go over to the truly interesting case of 4 spins/site.
2.3 Surfaces with 4 spins/site

Now, neither the string susceptibility exponent, nor the critical spin coupling $\beta_c$ are known theoretically. Of course, it is expected that $\gamma(\beta)$ has a maximum at $\beta = \beta_c$ and falls off to $-1/2$ as $\beta$ moves away from its critical value. The exponent $\gamma$ is estimated for several values of $\beta$ in order to observe the maximum in question and the corresponding value of $\gamma$. We proceed as in the 1 spin/site case, making the same tests (with similar results) and choosing the same values of $\Delta$ and $\delta N$. The fugacity is always chosen so that $E(N)$ is almost symmetric in the neighborhood of $N_0$. The observed efficiency of the grand-canonical algorithm is about 56%. The experiment is carried out for $N_0=200, 300, 500$ and 1000, with 2000 heating sweeps followed by $1.3 \times 10^6$ production sweeps, for each $N_0$.

The results are shown in fig. 1. The exponent $\gamma$ takes its maximum value near $\beta \approx 0.196$. This is reassuring, since this is the critical coupling found from microcanonical simulation in ref. [9] (in comparing with their value remember that they have been working on the dual lattice).

The relatively slow fall of $\gamma$ on the right of the critical point is associated with the worsening performances of the Wolff method (size of clusters increasing too much). We have also made some runs with the heat bath algorithm. With the latter, in order to get results close to those with Wolff, one has (for the system with 4 spins/site) carry out 3 sweeps of spins after each sweep of the geometry. And, as expected, the autocorrelation time is longer (however not larger than $\sim 100$ sweeps, this low number is presumably due to the fact that the grand-canonical algorithm permanently destroys and creates spins). But at $\beta = 0.24$ the value obtained using the heat bath algorithm is $-0.487(83)$, which is closer to the expected $-1/2$ than the rightmost point in fig. 1.

3 DISCUSSION AND SPECULATIONS

Using the two top values from fig. 1 we get the estimate $\gamma = -0.195(58)$. It is compatible with the value found in ref. [14], which is $-0.167(8)$. The respective errors should not be directly compared, since they have different significance. Our error estimate is very conservative. The error given in ref. [14] is the error obtained in fitting the baby universe distribution in a
single experiment. Contemplating the dispersion of $\gamma$’s shown in fig. 1 of ref. [14] and corresponding to nearby values of $\beta$, one suspects that this error should be multiplied by at least a factor of 2. Nevertheless, our experiment is admittedly less precise, simply due to our lower statistics.

The size of our lattices is the same as the size of baby universes observed in ref. [14]. However, with their method one can easier extend simulations to larger systems, and it is this which makes it superior to the grand-canonical technique employed in our work. Anyhow, it is certainly a good thing to have two independent and compatible estimates of the string susceptibility exponent at $c = 2$, obtained with two completely different methods.

We wish to end this paper with a speculation concerning $c > 1$ models. We conjecture that the continuum $c = 1$ theory acts as an ”attractor” within a finite interval $c_0 > c \geq 1$. More precisely, for these central charges, we suspect the long distance physics of the different models to be the same, modulo a renormalization of the length scale. In particular, this means that the microcanonical partition function behaves as follows:

$$N^3 Z(N, c) \sim \frac{e^{9c- N}}{\ln^2 (\lambda N)} [1 + ...] ,$$

where $\lambda = \lambda(c)$ is some constant and the dots represent subleading non-universal contributions. One can check that the normalized distribution of ”baby universes” given in ref. [14] for $c = 2$ is compatible with (10), even without sub-leading corrections, provided one sets $\lambda \approx 20$. With such a value of $\lambda$ the effective $\gamma$ calculated from (10) for $N \in [200, 1000]$ is also close to the one measured in this work. Such a simple rescaling does not reproduce the data [13, 14] for $c \geq 3$. Of course, one could reconcile (10) with these data playing with the sub-leading terms. This would be a futile exercise, however, in view of the lack of any theoretical prediction about the latter.

Our conjecture is simple and refutable. We obviously expect that the effective susceptibility exponent tends to zero when data with larger $N$ are used, provided $1 < c < c_0$. Actually, the authors of ref. [14] report that for $c = 1.6$, an extension by the factor of 10 of the size of ”baby universes” produces a shift of the effective $\gamma$ from $-0.212(15)$ to $-0.175(24)$. Because of the errors, the effect is not very significant. However, it goes in the right direction and the magnitude of the shift, if taken at the face value, is just what we expect with $\lambda = 10$ to 20. This is perhaps not an accident. If true,
our conjecture also explains why the critical exponents (fractal dimension and magnetic exponents) found in microcanonical simulations [9, 10, 11, 12] show so little dependence on $c$ just beyond the ”barrier”.

The general arguments reviewed in ref. [7] indicate that $\gamma > 0$ implies that this exponent has a finite value, which for spin models is most likely $\frac{1}{3}$ (see also ref. [19]). We thus expect that, in the continuum limit, the exponent $\gamma$ jumps from the value 0 it takes in the interval $[1, c_0)$ to a moderate value, say $\frac{1}{3}$, for $c \geq c_0$. We imagine that the jump occurs when the density of baby universes increases above some critical value. With this perspective, we are tempted to consider the small positive values of $\gamma$ (e.g. $\gamma \approx 0.04$ to 0.06 for $c = 2$) extracted from series expansion in ref. [8] as supporting our conjecture.

It has been suggested long time ago (first by David in ref. [3]; see also [20] and references therein) that the effective $\gamma$ becomes positive somewhere in the neighborhood of $c = 4$. The most recent data presented in ref. [14] indicate that this happens for $3 \lesssim c < 4$ This result suggests the crude estimate $3 \lesssim c_0 < 4$. It is perhaps not unreasonable to think that such a value of $c_0$ is large enough to warrant the approximate validity of the mean field calculations for $c > c_0$.

Although our conjecture differs from that put forward in ref. [14] as a plausible ”$c > 1$ hypothesis”, it is likewise pessimistic: in both scenarios the physics of a $c > 1$ model is reducible to the known theories with $c \leq 1$.

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**Figure caption**

**Fig. 1** - The string susceptibility exponent $\gamma$ versus the spin coupling $\beta$. The arrow indicates the critical coupling calculated in ref. [9].
This figure "fig1-1.png" is available in "png" format from:

http://arxiv.org/ps/hep-th/9401143v1