A NOTE ON LOWER DIAMETER BOUNDS FOR CLOSED DOMAIN MANIFOLDS OF SHRINKING RICCI-HARMONIC SOLITONS

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Abstract. In this short note, we remark that the arguments by Futaki-Sano (Asian J. Math. 17, 17-31, 2013) and Futaki et al (Ann. Global Anal. Geom. 44, 105-114, 2013) also work well for closed domain manifolds of shrinking Ricci-harmonic solitons and the arguments also give lower diameter bounds for the domain manifolds.

1. Introduction

Let \((M, g)\) be an \(n\)-dimensional Riemannian manifold with a Riemannian metric \(g = g(t)\) evolving by the following coupled system:

\[
\begin{align*}
\frac{\partial}{\partial t} g(x, t) &= -2 \text{Ric}_g(x, t) + 2\alpha_n \nabla \phi(x, t) \otimes \nabla \phi(x, t), \\
\frac{\partial}{\partial t} \phi(x, t) &= \Delta_{g(t)} \phi(x, t),
\end{align*}
\]

(1.1)

where \(\alpha_n > 0\) is a positive constant depending only on \(n\), \(\phi = \phi(t) : (M, g(t)) \rightarrow \mathbb{R}\) is a family of smooth function on \(M\), and \(\Delta_{g(t)}\) is the Laplace-Beltrami operator given by the evolving metric \(g(t)\). The flow (1.1) is called a Bernhard List’s flow and was introduced by List \([11, 12]\). The short time existence is proved. A typical example would be the Ricci flow \([8]\) plays an important role on Perelman’s work \([17]\) in which case \(\phi\) is a constant function. The motivation of studying the Bernhard List’s flow stems from its connection to general relativity. The stationary points of the flow correspond to the static Einstein vacuum equations \([11, 12]\).

After List introduced the Bernhard List’s flow, a new geometric flow was introduced. Let \((M, g)\) be an \(n\)-dimensional Riemannian manifold with a Riemannian metric \(g = g(t)\) evolving by the following coupled system:

\[
\begin{align*}
\frac{\partial}{\partial t} g(x, t) &= -2 \text{Ric}_g(x, t) + 2\alpha_n(t) \nabla \phi(x, t) \otimes \nabla \phi(x, t), \\
\frac{\partial}{\partial t} \phi(x, t) &= \tau_{g(t)} \phi(x, t),
\end{align*}
\]

(1.2)

where \(\alpha_n(t) > 0\) is a positive constant depending on \(n\) and \(t\), \(\phi = \phi(t) : (M, g(t)) \rightarrow (N, h)\) is a family of smooth map between \((M, g(t))\) and a fixed Riemannian manifold \((N, h)\), and \(\tau_{g(t)} = \text{trace} \nabla d\phi\) denotes the tension field given by the evolving metric \(g(t)\). The flow (1.2) is called a Ricci-harmonic flow and was introduced by Müller \([14, 16]\). The Bernhard List’s flow is an example of the Ricci-harmonic flow in which case \((N, h) = (\mathbb{R}, dr^2)\). More examples can be found in \([14, 16]\). Denoting as \(S_{ij} := R_{ij} - \alpha_n \phi_i \phi_j\) in local coordinates, the first equations in (1.1) and (1.2) become

\[
\frac{\partial}{\partial t} g_{ij} = -2 S_{ij}.
\]

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As with the Ricci flow, under (1.1) and (1.2), Harnack inequalities for several heat type equations for the Bernhard List’s flow and Ricci-harmonic flow are obtained, respectively [1, 20, 3, 21]. Note that the papers [2, 9, 6, 7] study these Harnack inequalities under more general settings, that is, under the flows (1.3) for general symmetric two tensors $S_{ij}$ and some technical assumptions on evolving quantities.

**Definition 1.1 (Müller [14, 16]).** A 7-tuple $(M, g, N, h, f, \phi, \rho)$, where $(M, g)$ and $(N, h)$ are Riemannian manifolds, $f : M \to \mathbb{R}$ is a smooth function on $M$, $\phi : (M, g) \to (N, h)$ is a smooth map between a domain manifold $(M, g)$ and a target manifold $(N, h)$, and $\rho \in \mathbb{R}$, is called a *Ricci-harmonic soliton* if it satisfies the coupled elliptic system

$$
\begin{align*}
\text{Ric}_g - \alpha_n |\nabla \phi|^2 + \text{Hess} f &= \rho g, \\
\tau_g \phi &= \langle \nabla \phi, \nabla f \rangle,
\end{align*}
$$

where $\alpha_n > 0$ is a positive constant depending on $n$, and $\tau_g$ denotes the tension field given by $g$. We say that the soliton $(M, g, N, h, f, \phi, \rho)$ is *shrinking*, *steady*, and *expanding* described as $\rho > 0$, $\rho = 0$, and $\rho < 0$, respectively.

The solitons defined in (1.4) are special solutions for the coupled systems [14, 16]. Note that if $(N, h) = (\mathbb{R}, dr^2)$ and $\phi : M \to \mathbb{R}$ is a constant function in (1.4), then the soliton is exactly the gradient Ricci soliton. Since the Ricci-harmonic flows are natural generalizations of Ricci flows, it is a natural question whether the same theorems in Ricci flows hold for Ricci-harmonic flows. In this direction, many fundamental theorems in Ricci flows are extended to Ricci-harmonic flows, for example, no breather theorems, non-collapsing theorems [14, 16], Perelman’s entropy formulas [10], Monotone volume formulas [15], and volume growth estimates [19]. See also [18] for more related results.

In this short note, we give lower diameter bounds for closed domain manifolds of shrinking Ricci-harmonic solitons, which generalize the works by Futaki-Sano [4] and Futaki et al [5] for closed gradient shrinking Ricci solitons. Our result is the following:

**Theorem 1.2.** Let $(M, g, N, h, f, \phi, \rho)$ be a shrinking Ricci-harmonic soliton satisfying (1.4). Suppose that the domain manifold $M$ is closed. Then the diameter of the domain manifold $(M, g)$ has the universal lower bound

$$
diam(M, g) \geq \frac{2\sqrt{2} - 1}{\sqrt{\rho}} \pi.
$$

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2. **Proof of Theorem 1.2**

Our proof is almost the same as [4] and [5]. Then, we just give an outline of the proof.

**Proof.** We first show that $2\rho$ is an eigenvalue of the Witten-Laplacian $\Delta_f$ defined by

$$
\Delta_f := \Delta - \nabla f \cdot \nabla.
$$

Here $\Delta = g^{ij} \nabla_i \nabla_j$. By [14 (4.5)], there exists a constant $C$ such that

$$
R - \alpha_n |\nabla \phi|^2 + |\nabla f|^2 - 2\rho f = C,
$$

where $R$ denotes the scalar curvature with respect to $(M, g)$. See also [13 (1.5)]. On the other hand, by taking the trace of the first equation in (1.4), we have

$$
R - \alpha_n |\nabla \phi|^2 + \Delta f = n\rho.
$$
By combining the above two equalities, we obtain

\[(2.1) \quad \Delta f = \Delta f - |\nabla f|^2 = -2\rho f + C',\]

where \(C' := n\rho - C\). By adding some constants on \(f\), we may normalize \(f\) such that

\[\int_M f e^{-f} d\text{vol}_g = 0.\]

We make this normalization throughout this paper. By this normalization and \((2.1)\), we see that \(C'\) in \((2.1)\) must be zero. Thus, \(2\rho\) is an eigenvalue of \(\Delta f\).

Next, by the first equation in \((1.4)\), we see that

\[\text{Ric}_g + \text{Hess} f = \rho g + \alpha_n \nabla \phi \otimes \nabla \phi \geq \rho g.\]

This says that Theorem 1.1 in [5] also works on our Ricci-harmonic soliton \((1.4)\). Hence, by Theorem 1.1 in [5] and the above argument, we have

\[2\rho \geq 4s(1-s)\pi^2 d^2 + s\rho\]

for all \(0 < s < 1\). By the same argument as Theorem 1.2 in [5], we obtain \((1.5)\). □

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