SPECTRAL PREORDER AND PERTURBATIONS OF DISCRETE WEIGHTED GRAPHS

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Abstract. In this article, we introduce a geometric and a spectral preorder relation on the class of weighted graphs with a magnetic potential. The first preorder is expressed through the existence of a graph homomorphism respecting the magnetic potential and fulfilling certain inequalities for the weights. The second preorder refers to the spectrum of the associated Laplacian of the magnetic weighted graph. These relations give a quantitative control of the effect of elementary and composite perturbations of the graph (deleting edges, contracting vertices, etc.) on the spectrum of the corresponding Laplacians, generalising interlacing of eigenvalues.

We give several applications of the preorders: we show how to classify graphs according to these preorders and we prove the stability of certain eigenvalues in graphs with a maximal d-clique. Moreover, we show the monotonicity of the eigenvalues when passing to spanning subgraphs and the monotonicity of magnetic Cheeger constants with respect to the geometric preorder. Finally, we prove a refined procedure to detect spectral gaps in the spectrum of an infinite covering graph.

To Hagen Neidhardt in memoriam.

1. Introduction

Analysis on graphs is an active area of research that combines several fields in mathematics including combinatorics, analysis, geometry or topology. Problems in this field range from discrete version of results in differential geometry to the study of several combinatorial aspects of the graph in terms of spectral properties of operators on graphs (typically discrete versions of continuous Laplacians), see e.g. [Moh01, CDS95, Chu97, CdV98, Hog05, Sun08, Sun12, BH12]. The interplay between discrete and continuous structures are very apparent for the class of metric graphs together with their natural Laplacians (see e.g. [EKK08, P12] and references therein).

The spectrum of a finite graph mostly refers to the spectrum of the adjacency matrix $A$ (e.g. in Cvetković, Doob and Sachs book [CDS95] or in Brouwer and Haemers’ book [BH12]), while the latter book also contains many results on the Laplacian $L = D - A$ and its signless version $Q = D + A$. Here, $D$ is the matrix with the degrees of the (numbered) vertices on its diagonal. In Chung’s book [Chu97, Section 1.2] the spectrum of a graph refers to the spectrum of its standard Laplacian $L = D^{-1/2}LD^{-1/2} = I - D^{-1/2}AD^{-1/2}$, where $I$ is the identity matrix of order $|G|$ (the standard Laplacian is sometimes also called normalised, e.g. in [Chu97], or sometimes also geometric). Colin de Verdière [CdV08] considers wider classes of discrete operators, namely discrete weighted Laplacians with electric (but without magnetic) potential. A survey considering all the above-mentioned matrices associated with a graph can be found in [Hog05]. Note that the spectra of the combinatorial, standard Laplacian and the adjacency operator are only related if the underlying graph is regular (i.e. all vertices have the same degree).

In this article, we consider general weights on the edges and vertices, in order to include the combinatorial and standard Laplacians at the same time. Moreover, we allow magnetic potentials, which can be considered also as complex-valued edge weights (of absolute value 1). Magnetic Laplacians or Schrödinger operators on graphs have also attracted much interest (see, e.g. [Moh01, HS01, LLPP15, KSI17, BGK20]); they are defined via a phase $e^{i\alpha}$ for each oriented edge $e$ in the discrete Laplacian; $\alpha$ is called the magnetic potential. The concept of balanced or signed graphs is related (as pointed out only recently in [LLPP15], see also the detailed reference list therein), and it can be seen as a special case of a magnetic Laplacian with magnetic phases $1 = e^{0}$ and $-1 = e^{i\pi}$ only. A prominent example of a magnetic Laplacian already treated in some spectral graph theory articles or books (e.g. [BH12]) is the signless (combinatorial) Laplacian $Q = D + A$ mentioned above; it can be seen as a magnetic combinatorial Laplacian with phase $-1 = e^{i\pi}$ (i.e. vector potential $\alpha = \pi$ on all edges).

We will base our analysis in a rather general setting. In particular, we allow multigraphs $G$ (i.e., graphs with multiple edges and loops) which we simply call graphs here. Moreover, we allow arbitrary weights on vertices and edges (denoted by the same symbol $w$) in order to cover the combinatorial and the standard Laplacian (and all other weighted versions). Finally, we allow a discrete vector potential $\alpha$ describing a magnetic flux on each cycle of the graph; in particular, our analysis allows to include also signed graphs or signless versions of the Laplacian. We call such graphs magnetic weighted...
graphs (or MW-graphs for short) and denote the class by \( \mathcal{G} \). The graphs in this class may have finite or infinite order. A generic element in this class is written as \( G = (G, w, \alpha) \). If we restrict to MW-graphs with combinatorial or standard weights, we use the symbols \( \mathcal{G}_c \) and \( \mathcal{G}_{\text{deg}} \), respectively.

In this article, we present two preorders on the class of MW-graphs: the first one denoted by \( G \preceq G' \) is geometric in nature and basically assumes that there is a graph homomorphism from \( G \) to \( G' \) respecting the magnetic potential and fulfilling certain inequalities on the weights, called magnetic graph homomorphisms (MW-homomorphisms for short, see Definition 2.15 for details). The existence of an MW-homomorphism is rather restrictive, e.g. for standard weights (degree on the vertices, and 1 on the edges), an MW-homomorphism is a quotient map (cf. Proposition 2.18).

The inequalities on the weights are made in such a way that

\[
G \preceq G' \implies \lambda_k(G) \leq \lambda_k(G')
\]

hold for all \( k \) (assuming that the number of vertices fulfils \(|V(G)| \geq |V(G')|\)). Here we write the spectrum of the magnetic weighted Laplacian in increasing order and counting multiplicities. This monotonicity is our first main result, see Theorem 3.14. In particular, the inequalities on the weights imply a similar inequality on the Rayleigh quotients. We state the above eigenvalue inequality as \( G \preceq G' \), our second preorder on the set of (finite) MW-graphs \( \mathcal{G} \).

Similarly, the weight inequalities characterising MW-homomorphisms are compatible with a certain isoperimetric ratio. In fact, given \( G \in \mathcal{G} \) denote by \( h_k(G) \) the \( k \)-th (magnetic weighted) Cheeger constant where we incorporate into the analysis the magnetic field via the frustration index of the graph (see Subsection 5.2 and [LLPP15]). Then, for any \( G, G' \in \mathcal{G} \) we show in Theorem 5.12 the implication

\[
G \preceq G' \implies h_k(G) \leq h_k(G')
\]

for all \( k \).

The relation \( \preceq \) can be extended by a shift \( r \in \mathbb{N}_0 \) in the list of eigenvalues in which case we use the symbol \( \preceq_r \) (cf. Definition 3.7). From the point of view of linear algebra, the spectral preorder is a very flexible generalisation of eigenvalue interlacing known for matrices (see e.g. [HJ13 Theorem 4.3.28]). Interlacing applied to graphs is also treated in [BHT12 Sections 2.5, and 3.2]. Some of our elementary operations on graphs can hence be also seen as a geometric interpretation of eigenvalue interlacing. In particular, we have already mentioned above that the geometric preorder is stronger than the spectral preorder (cf. Theorem 3.14), i.e., if \( G, G' \in \mathcal{G} \) then \( G \preceq G' \) implies \( G \preceq_r G' \).

We can also compare in a natural way the same graphs with different weights. In particular, in Corollary 3.16 we show that the \( k \)-th eigenvalue of the standard magnetic Laplacian is always bounded above by the \( k \)-th eigenvalue of the combinatorial magnetic Laplacian for every possible vector potential \( \alpha \).

In Section 4 we use the preorders \( \preceq \) and \( \preceq_r \) (with appropriate shifts) to give a quantitative estimate of the spectral effect that elementary perturbations have on the spectrum of the corresponding Laplacians (see Theorems 4.4 and 4.9 in the case of general weights). We also analyse in Subsection 4.2 composite perturbations like edge contraction or vertex deletion. In the special cases of combinatorial and standard weights, we have the following situations (cf., Corollaries 4.2 and 4.7).

- **Edge deletion**: Let \( e_0 \) be an edge and \( G, G' \in \mathcal{G} \), where \( G' = G - e_0 \) (i.e., \( e_0 \) has been removed from \( G \)).
  - If \( G, G' \in \mathcal{G}_c \), then \( G \preceq G' \) and \( G' \subseteq G \), hence \( G \preceq G' \preceq G \).
  - If \( G, G' \in \mathcal{G}_{\text{deg}} \), then \( G \preceq G' \preceq G \).

- **Vertex contraction**: Let \( v_1, v_2 \) be vertices and \( G, \tilde{G} \in \mathcal{G} \) with \( \tilde{G} = G/\{v_1, v_2\} \) (i.e., the vertices have been identified in \( \tilde{G} \) keeping all the edges, i.e. loops or multiple edges may occur).
  - If \( G, \tilde{G} \in \mathcal{G}_c \), then \( G \subseteq G \) and \( \tilde{G} \preceq G \), hence \( G \preceq \tilde{G} \preceq G \), where \( r = \min\{\deg^G(v_1), \deg^G(v_2)\} \).
  - If \( G, \tilde{G} \in \mathcal{G}_{\text{deg}} \), then \( G \subseteq \tilde{G} \) and \( \tilde{G} \preceq G \), hence \( G \preceq \tilde{G} \preceq G \).

These results are sharp in the sense that, in general, one cannot lower the value of the spectral shift. We discuss some related results in the literature: Van den Heuvel [Hef05 Lemma 2] proves the result on edge deletion for the combinatorial Laplacian and its signless version, see also [Moh91 Theorem 3.2] and [Pie73 Corollary 3.2]; the result is also used to spectrally exclude the existence of a Hamiltonian cycle e.g. in the Peterson graph (see [Moh92 Theorem 3.3]) and [Hef05 Theorem 1]).

In [CDH04 Theorem 2.3], the authors consider the specific case of the standard Laplacian and edge deletion; this result was generalised to signed graphs in [AT14 Theorem 8]. Similarly, [CDH04 Theorem 2.7] (and again generalised to the case of signed graphs in [AT14 Theorem 10]) prove a weaker version of our vertex contraction for the standard Laplacian, namely \( G \preceq \tilde{G} \preceq G \) in our notation, under the additional assumption that the vertices \( v_1, v_2 \) have combinatorial distance at least 3. The latter restriction is mainly due to the fact that both papers avoid the use of multigraphs, namely multiple edges and loops.

There are related results for so-called quantum graphs in [BKKM19] (for the notion of quantum graphs, see the references therein or e.g. [P12]): Let \( M \) be a compact metric graph, and let \( \tilde{M} \) be the metric graph obtained from \( M \) by contracting...
two vertices. The spectrum of $M$ is the ordered list of eigenvalues of its standard (also called Kirchhoff) Laplacian (repeated with respect to their multiplicity). Then Theorem 3.4 of \cite{BKKM19} states $M \preceq \tilde{M} \preceq M$ in the sense that $\lambda_k(M) \leq \lambda_k(\tilde{M}) \leq \lambda_{k+1}(M)$ for all $k$ (see also the references therein). This result shows again that the standard (also called “geometric”) Laplacian is closer to the continuous case than the combinatorial Laplacian on a graph.

We present a wide range of applications of the preorders and relations studied before: one can use the preorders to give a geometrical and spectral ordering of graphs (see Subsection 5.1.1). We also show that the eigenvalues of a magnetic weighted Laplacian of a spanning subgraph and the original graph are monotonous (i.e., the spectral ordering holds, see Corollary 5.1). In addition, we show the stability of certain eigenvalues after “small” perturbations and taking certain minors (see Subsection 5.1.3). We also prove the above mentioned monotonicity of Cheeger’s constant with respect to the geometric preorder (see Theorem 5.12).

Finally, the spectral preorder can be used also to refine the bracketing technique (known for continuous spaces under the name “Dirichlet-Neumann-bracketing”) for discrete graphs. We can apply the results on vertex contraction at the level of the finite fundamental domain to detect in some new examples spectral gaps and to almost determine completely the spectrum of the discrete Laplacian on the covering space (see Subsection 5.3). Let us conclude mentioning that spectral gaps of Schrödinger operators play an important role in spectral analysis and mathematical physics (see \cite{HN09, FCLP18, KS15, KS19} and references therein.)

Structure of the article. In Section 2 we introduce the main discrete structures needed in this article. In particular the class $\mathcal{G}$ of magnetic weighted graphs (MW-graphs for short) and the subclasses of MW-graphs with combinatorial and standard weights denoted by $\mathcal{G}_1$ and $\mathcal{G}_{deg}$, respectively. We introduce in Definition 2.20 the geometric preorder $\subseteq$ on $\mathcal{G}$ which is based on the notion of a magnetic weighted graph homomorphism. We show that it is a partial order on the class of finite MW-graphs with combinatorial or standard weights. In Section 3 we introduce the discrete magnetic Laplacian $\Delta_\alpha = d^*e^*d_\alpha$, where $d_\alpha$ is a discrete exterior derivative twisted by the magnetic potential $\alpha$. We also introduce in Definition 3.10 the spectral preorder $\preceq$ on $\mathcal{G}$ and consider also the possibility to compare shifted lists of eigenvalues of the corresponding Laplacians. In Section 4 we use the preorders $\subseteq$ and $\preceq$ to give a quantitative estimate of the spectral effect that elementary perturbations have on the spectrum of the corresponding Laplacians (see Theorems 4.1 and 4.9 in \cite{FCLP18} and 4.9 in the case of general weights). We also analyse in Subsection 4.2 composite perturbations like edge contraction or vertex deletion. In the final section we present our applications on spectral ordering of combinatorial graphs, graph minors, cliques, multiple eigenvalues, magnetic Cheeger constants and existence of spectral gaps on covering graphs.

2. Magnetic weighted graphs and their homomorphisms

In this section, we introduce the discrete structures needed and mention some basic properties and examples. We will consider discrete locally finite graphs with arbitrary weights on vertices and edges as well as an $R$-valued function on the edges which correspond to a discrete analogue of the magnetic potential. Here, $R$ is a subgroup of the Abelian group $\mathbb{R}/2\pi\mathbb{Z}$ written additively. In this section, graphs may be finite or infinite, and we will not assume that the graphs are necessarily connected.

2.1. Discrete graphs. A discrete graph (or, simply, a graph) $G = (V,E,\partial)$ consists of two disjoint (and at most countable) sets $V = V(G)$ and $E = E(G)$, the set of vertices and edges, respectively, and a connection map $\partial = \partial^G : E \to V \times V$, where $\partial_e = (\partial_e,e,\partial_e)$ denotes the pair of the initial and terminal vertex, respectively. We also say that $e$ starts at $\partial_e$ and ends at $\partial_e$. We assume that each edge $e$ (also called arrow) comes with its oppositely oriented edge $\bar{e}$, i.e. that there is an involution $:\mathcal{E} = \mathcal{E}^\partial : E \to E$ such that $e \neq \bar{e}$ and $\partial_{\bar{e}} \bar{e} = \partial_e e$ for all $e \in \mathcal{E}$. We allow multiple edges (i.e. $\partial$ is not necessarily injective, hence edges cannot be represented as pairs $(v_1,v_2)$ of vertices in general) and also loops (i.e. edges $e$ with $\partial_e = \partial_e e$). Note that also loops come in pairs $e \neq \bar{e}$. If $V(G)$ has infinitely many vertices, we say that the graph $G$ is infinite. If $V(G)$ has $n \in \mathbb{N}$ vertices, we say that $G$ is a finite graph of order $n$ and we write $|G| = |V(G)| = n$.

A path $p = (e_1,\ldots,e_r)$ of length $r$ in a graph $G$ is a finite sequence of $r$ edges $e_1,\ldots,e_r \in E$ such that $\partial_{e_{k-1}} = \partial_{e_k}$ for all $k = 1,\ldots,r$. We say that $p$ joins the vertices $\partial_{e_1}$ and $\partial_{e_r}$. The combinatorial distance of two vertices is the length of the shortest path joining these two vertices. A graph is called connected if for any vertices $x,y \in V$ there is a path $p$ joining $x$ and $y$. For two subsets $V_0 \subset V$ we denote by $E(V_-,V_+) := \{ e \in E \mid \partial_e \in V_-,\partial_e e \in V_+ \}$ the set of all edges starting in $V_-$ and ending in $V_+$. Note that $e \in E(V_-,V_+)$ if and only if $\bar{e} \in E(V_+,V_-)$. If we need to stress the graph $G$ to which $E(V_-,V_+)$ refers, we write $E^G(V_-,V_+)$. As a shortcut, we also set $E(V_0) := E(V_0,V_0)$, $E(v,V_0) := E(\{v\},V_0)$ and $E(v,x) = E(\{v\},\{x\})$ etc. for $v,x \in V$ and $V_0 \subset V$. Moreover, we denote by $E_v := E(v,V) := \{ e \in E \mid \partial_e = e \}$.

Note that in this article we switched to the more standard notation that an edge $e$ (also called an arrow) always has its oppositely oriented counterpart $\bar{e}$ in $E$; in our older papers (e.g. in \cite{FCLP18}) we used the convention that $E$ contains only one arrow (not its inverted arrow), hence in our older notation $E$ together with $\partial e = (\partial_e,e,\partial_e)$ determines already an orientation of the graph.
deleting edges

Note that a loop at a vertex \( v \) increases the degree by 2. We assume that the graph is locally finite, i.e. \( \deg(v) < \infty \) for all \( v \in V \). We call a graph simple if it has no loops and no multiple edges, i.e. if \( E(v,v) = \emptyset \) and \( |E(v,x)| \leq 1 \) for all \( v,x \in V, v \neq x \). For a simple graph, the connection map \( \partial \) is injective, hence an edge \( e \) can be identified with its pair \((\partial_+, \partial_-)\) of its initial and terminal vertex.

We first consider two elementary operations on a graph, contracting or identifying vertices while keeping the edges and deleting edges while keeping the vertices:

**Definition 2.1 (Contracting and splitting vertices).** Let \( G = (V,E,\partial) \) be a graph and let \( \sim \) be an equivalence relation on \( V \).

(i) The quotient graph \( \tilde{G} = G/\sim \) is defined by \( \tilde{G} = (\tilde{V}, \tilde{E}, \tilde{\partial}) \), where \( \tilde{V} = V/\sim, \tilde{E} = E \) and \( \tilde{\partial} e = ([\partial_+ e], [\partial_- e]) \) for all \( e \in E \). We also say that \( \tilde{G} \) is obtained from \( G \) by contracting or contracting vertices according to \( \sim \).

(ii) If the relation \( \sim \) identifies only the vertices \( v_1, \ldots, v_r \in V(G) \) to one vertex in \( \tilde{G} \), we also say that \( \tilde{G} \) is obtained from \( G \) by contracting or identifying the vertices \( v_1, \ldots, v_r \). We write \( \tilde{G} = G/(v_1, \ldots, v_r) \) for short (see Figure 3 for the case \( r = 2 \)).

(iii) The reverse operation is called splitting: we say that \( G \) is a vertex splitting of \( \tilde{G} \) if there is an equivalence relation \( \sim \) on \( V(G) \) such that \( \tilde{G} = G/\sim \).

**Remark 2.2 (Loops and multiple edges after vertex contraction).** Let us stress that in contrast to many combinatorial graph theory books we use a “topological” contraction of vertices as in [BM08 Section 2.3]: if we contract two adjacent vertices \( v_1 \) and \( v_2 \), all edges joining \( v_1 \) and \( v_2 \) become loops in \( G/(v_1, v_2) \). Moreover, contracting two vertices of combinatorial distance 2, leads to a double (or multiple) edge.

**Definition 2.3 (Deleting and adding edges).** Let \( G = (V,E,\partial) \) be a graph and let \( E_0 \subset E \).

(i) We denote by \( G - E_0 \) the graph given by \( (V,E \setminus E_0, \partial|_{E \setminus E_0}) \). We call \( G - E_0 \) the graph obtained from \( G \) by deleting the edges \( E_0 \). If \( E_0 = \{e_0\} \) we simply write \( G - e_0 \) instead of \( G - \{e_0\} \) (see, e.g. Figures 2a and 2b).

(ii) The reverse operation is called adding edges: We say that \( G \) is obtained from a graph \( G' \) by adding the edges \( E_0 \subset G' \) if \( G' = G \cup E_0 \); for short we write \( G = G' \cup E_0 \) and also \( G = G' + e_0 \) if \( E_0 = \{e_0\} \).

The operation of contracting two adjacent vertices and deleting the edges joining them is called edge contraction, and it is a combination of contracting \( v_1 \) and \( v_2 \) and deleting the adjacent edges \( E(v_1, v_2) \). Note that the order of the operations does not matter: first delete \( E(v_1, v_2) \) and then contracting \( v_1, v_2 \) or first contracting \( v_1, v_2 \) and then delete the loops obtained from \( E(v_1, v_2) \) gives the same graph.

**Definition 2.4 (Isolated and pendant vertices, pendant and bridge edges).** Let \( G = (V,E,\partial) \) be a discrete graph.

(i) A vertex \( v_0 \in V \) is called isolated if \( \deg(v_0) = 0 \).

(ii) A vertex \( v_0 \in V \) is called pendant if \( \deg(v_0) = 1 \).

(iii) An edge \( e_0 \in E \) is pendant if at least one of its vertices \( \partial_+ e_0 \) is a pendant vertex.

(iv) An edge \( e_0 \in E \) is a bridge (edge) if \( G - e_0 \) has one more connected component than \( G \).

Another way of producing graphs from given ones are (induced) subgraphs:

**Definition 2.5 ((Induced) subgraphs).** Let \( G = (V,E,\partial) \) be a graph. A subgraph \( G_0 = (V_0, E_0, \partial|_{E_0}) \) of \( G \) is given by subsets \( V_0 \subset V \) and \( E_0 \subset E(V_0) := E(V_0, V_0) \). An induced subgraph is a subgraph such that \( E_0 = E(V_0) \). The latter graph is also called subgraph induced by \( V_0 \) and is denoted by \( G[V_0] \).

Note that \( \partial(E(V_0)) \) indeed maps into \( V_0 \times V_0 \): We have \( e \in E(V_0) := E(V_0, V_0) \) if and only if \( \partial_+ e, \partial_- e \in V_0 \), hence \( \partial e \in V_0 \times V_0 \).

We introduce next another standard notation from graph theory:

**Definition 2.6 (Graph homomorphisms).** Let \( G = (V,E,\partial) \) and \( G' = (V',E',\partial') \) be two graphs. We say that \( \pi : G \rightarrow G' \) is a graph homomorphism, if \( \pi \) is a map on the vertices \( \pi[V] : V \rightarrow V' \) and on the edges \( \pi[E] : E \rightarrow E' \) (denoted by the same symbol \( \pi \)) such that

\[
\pi(\partial_+ e) = \partial'_+ (\pi e) \quad \text{and} \quad \pi(\partial_- e) = \partial'_- (\pi e)
\]

for all \( e \in E \). If \( \pi[V] \) and \( \pi[E] \) are both bijective then \( \pi \) is called an isomorphism. If there exists an isomorphism between \( G \) and \( G' \), then the graphs are called (graph-)isomorphic, for short \( G \cong G' \).

**Example 2.7.** Let \( G = (V,E,\partial) \) and \( G' = (V',E',\partial') \) be two graphs, then some basic examples of homomorphisms are given as follows:

(i) Let \( \sim \) be an equivalence relation on \( V \), then the quotient map \( \kappa := \kappa_\sim : G \rightarrow G/\sim \) given by \( \kappa(v) = [v] \) and \( \kappa(e) = e \) is a graph homomorphism.
(ii) Let $E_0 \subset E$ then the inclusion $\iota := \iota_{E_0} : G - E_0 \rightarrow G$ is a graph homomorphism.

(iii) Let $\pi : G \rightarrow G'$ be a graph homomorphism. Then the image graph is defined by $\pi(G) := (\pi(V), \pi(E), \partial' |_{\pi(E)})$.

Note that this indeed defines a graph as $\partial'(\pi(E)) \subset \pi(V) \times \pi(V)$ by (2.1), and $\pi(G)$ is a subgraph of $G'$.

Moreover, the inclusion $\iota : \pi(G) \rightarrow G'$ is a graph homomorphism, injective on the vertex and edge set.

The next lemma is an immediate consequence of Eq. (2.1):

**Lemma 2.8.** Let $\pi : G \rightarrow G'$ be a graph homomorphism, then $\pi^{-1}(E^G(\pi'(V'), V')) = E^G(\pi^{-1}(V'), \pi^{-1}(V'))$ for $V' \subset V'$.

We finish this subsection of graph theory with the following observation that will be useful in the following.

**Lemma 2.9.** Let $G = (V, E, \partial)$ and $G' = (V', E', \partial')$ be two graphs and let $\pi : G \rightarrow G'$ be a graph homomorphism.

(i) If $\pi$ is injective on the edges, then
\[
\sum_{v \in V, \pi(v) = v'} \deg^G(v) \leq \deg^{G'}(v') \quad \text{for every } v' \in V'.
\]

(ii) If $\pi$ is injective, but not surjective on the edges, then there is a vertex $v'_0 \in V'$ such that
\[
\sum_{v \in V, \pi(v) = v'_0} \deg^G(v) < \deg^{G'}(v'_0).
\]

Note that the sum over $v \in V$ with $\pi(v) = v'_0$ could be $0$, namely, if $v'_0$ is not in the range of $\pi$.

**Proof.** For $v' \in V'$ let
\[
A_{v'} := \bigcup_{v \in V, \pi(v) = v'} E^G_v = \{ e \in E | \pi(\partial_e) = v' \}
\]
be the set of edges in $G$ that are starting at $v$ for all preimages $v$ of $v'$ under $\pi$. Then the map $\pi : A_{v'} \rightarrow E^G_{v'}$ is well-defined, and injective because $\pi$ is. Moreover, note that $(E^G_v)_{v \in V}$ is a disjoint family of sets (as each edge has only one initial vertex), hence we have
\[
\sum_{v \in V, \pi(v) = v'} |E^G_v| \leq |E^G_{v'}|.
\]

The desired inequality follows from $\deg^G(v) = |E^G_v|$ and $\deg^{G'}(v') = |E^G_{v'}|$.

(ii) Choose an edge $e'_0 \in E' \setminus \pi(E)$ and $v'_0 = \partial' e'_0$. Then $\pi : A_{v'_0} \rightarrow E_{v'_0}(G') \setminus \{e'_0\}$ is injective and well-defined as a map, hence we conclude
\[
\sum_{v \in V, \pi(v) = v'_0} |E^G_v| \leq |E^G_{v'_0}| - 1 < |E^G_{v'_0}|
\]
proving the strict inequality claimed. □

### 2.2. Weights on graphs

Given a graph $G = (V, E, \partial)$ we consider a **weight** on it, i.e. a vertex $w : V \rightarrow (0, \infty)$ and $w : E \rightarrow (0, \infty)$ associating to a vertex $v$ its weight $w(v)$ and to an edge $e$ its weight $w_e$. For subsets $V_0 \subset V$ and $E_0 \subset E$, we may interpret $w$ as a **discrete measure** on the corresponding sets, and we use the natural notation
\[
\omega(V_0) = \sum_{v \in V_0} w(v) \quad \text{and} \quad \omega(E_0) = \sum_{e \in E_0} w_e.
\]

**Example 2.10** (Standard and combinatorial weights). Given a graph $G = (V, E, \partial)$ one can define two important intrinsic weights on it. The **standard** weight in $V$ given by $w(v) := \deg(v)$, $v \in V$, and $w_e := 1$, $e \in E$, and we denote it simply by $\deg$. Note that $\deg(v) > 0$ implies that a weighted graph with standard weights has no isolated vertices, i.e. vertices of degree 0 (nevertheless see Remark 3.6 for the convention with standard weights for the associated Laplacian in case that the graph has isolated vertices). The **combinatorial** weight is given by $\omega(v) := 1$, $v \in V$, and $w_e := 1$, $e \in E$ and we denote it by $\mathbb{1}$.

An edge weight on a graph determines a so-called **weighted degree** of a vertex defined by
\[
\deg^w(v) := w(E_v) = \sum_{e \in E_v} w_e.
\]

Recall that a loop counts twice in $E_v$. In particular, the combinatorial degree $\deg(v)$ agrees with the weighted degree $\deg^w v$ iff the edge weight equals 1 for all edges. We call the weight **normalized** if
\[
\deg^w(v) = w(v), \quad \text{or, equivalently,} \quad \sum_{e \in E_v} w_e = w(v), \quad \text{for all} \quad v \in V.
\]

We define the **relative weight** $\varrho := \varrho_w : V \rightarrow (0, \infty)$ of a weighted graph $(G, w)$ by
\[
\varrho_w(v) := \frac{\deg^w(v)}{w(v)} = \frac{1}{w(v)} \sum_{e \in E_v} w_e.
\]
We assume that the relative weight is bounded, i.e. the maximal \( w\)-degree of \((G, w)\) is (uniformly) bounded:
\[
\varrho_{\infty} := \sup_{v \in V} \varrho_w(v) < \infty. \tag{2.5}
\]
Note that for the standard weight, or, more generally, for a normalised weight, the relative weight is just \( \varrho_w = 1 \). In particular, the relative weight for any normalised weight is bounded.

For the combinatorial weight, the relative weight is just the usual degree, hence the relative weight is bounded if and only if the degree of the graph is bounded.

2.3. Magnetic potentials. Let \( G = (V, E, \partial) \) be a graph and let \( R \) be a subgroup of \( \mathbb{R}/2\pi\mathbb{Z} \) which we write additively. We consider the \( R \)-valued magnetic potentials.

Let \( G = (V, E, \partial) \) be a graph and let \( R \) be a subgroup of \( \mathbb{R}/2\pi\mathbb{Z} \). We consider the coboundary operator \( \delta \) and the so-called cohomological operator
\[
d: C^0(G, R) \rightarrow C^1(G, R), \quad (d\xi)_e = \xi(\partial_+ e) - \xi(\partial_- e).
\]

**Definition 2.11.** Let \( G = (V, E, \partial) \) be a graph and \( R \) be a subgroup of \( \mathbb{R}/2\pi\mathbb{Z} \).

(i) An \( R \)-valued magnetic potential \( \alpha \) is an element of \( C^1(G, R) \).

(ii) We say that \( \alpha, \tilde{\alpha} \in C^1(G, R) \) are cohomologous or gauge-equivalent and denote this as \( \tilde{\alpha} \sim \alpha \) if \( \tilde{\alpha} - \alpha \) is exact, i.e. if there is \( \xi \in C^0(G, R) \) such that \( d\xi = \tilde{\alpha} - \alpha \), and \( \xi \) is called the gauge. We denote the equivalence class or cohomology class by \([\alpha] = \{ \tilde{\alpha} \in C^1(G, R) | \tilde{\alpha} \sim \alpha \}\). We say that \( \alpha \) is trivial, if it is cohomologous to 0.

In the sequel, we will omit the Abelian group \( R \) for simplicity of notation, e.g. we will write \( C^1(G) \) instead of \( C^1(G, R) \) for the group of magnetic potential etc.

The next result says that if the vector potential is supported on a bridge, then it is trivial.

**Lemma 2.12.** Let \( G = (V, E, \partial) \) be a graph such that \( e_0 \in E(G) \) is a bridge edge. If \( \alpha \) and \( \tilde{\alpha} \) are two vector potentials having different values only on \( e_0 \) (i.e. \( \alpha_e = \tilde{\alpha}_e \) for all \( e \in E(G - e_0) = E(G \setminus \{ e_0 \}) \)), then \( \alpha \sim \tilde{\alpha} \).

**Proof.** Denote by \( C_+ \) (respectively, \( C_- \)) the two connected components of \( G - e_0 \) with \( \partial_+ e_0 \subset C_+ \) (respectively, \( \partial_- e_0 \subset C_- \)). Define a function \( \xi: V(G - e_0) \rightarrow R \) by \( \xi(v) = \alpha_{e_0} - \tilde{\alpha}_{e_0} \) for all \( v \in C_+ \) and 0 for all \( v \in C_- \). It follows that
\[
(d\xi)_e = \alpha_e - \tilde{\alpha}_e \quad \text{hence} \quad \alpha \sim \tilde{\alpha}.
\]

2.4. MW-graphs and geometric preorder. In the following definition we collect all relevant structure needed: a discrete weighted graph with vector potential.

**Definition 2.13** (Magnetic weighted graph, MW-graph). We call \( G = (G, \alpha, w) \) a magnetic weighted graph (MW-graph for short) if \( G = (V, E, \partial) \) is a discrete graph, \( w \) is a weight on the graph and \( \alpha \in C^1(G) \) is an \( R \)-valued magnetic potential, i.e. a map \( \alpha: E \rightarrow R \) such that \( \alpha_e = -\alpha_e \) for all \( e \in \partial G \), where \( R \) is a subgroup of \( \mathbb{R}/2\pi\mathbb{Z} \).

Note that \( R \) can be chosen a priori. If we choose \( R = \{0\} \), then the corresponding Laplacian defined in Section 3.1 is the usual Laplacian (without magnetic potential). If we choose \( R = \{0, \pi\} \), then the magnetic potential is also called signature, and \( G \) is called a signed graph (see, e.g. [LLPP15] and references therein for details). This setting includes the so-called signless Laplacian (see Example 3.2 (iii)) by choosing \( \alpha_e = \pi \) for all \( e \in E \).

**Definition 2.14** (Classes of MW-graphs). We denote by \( \mathcal{G} \) the class of all MW-graphs. We denote the subclasses of MW-graphs with combinatorial weight simply by \( \mathcal{G}_1 \) and with standard weights just by \( \mathcal{G}_{\text{std}} \). Moreover, for a symmetric subset \( R_0 \) of \( R \) (not necessarily a subgroup but being invariant under reflections, i.e. if \( t \in R_0 \) then \( -t \in R_0 \)) we write
\[
\mathcal{G}^{R_0} := \{ G = (G, \alpha, w) \in \mathcal{G} | \alpha_e \in R_0, \ e \in E \} \quad \text{and} \quad \mathcal{G}^t := \mathcal{G}^{\{t, -t\}}
\]
for the subclass of MW-graphs having magnetic potential with values in \( R_0 \) respectively with constant value \( t \). Similarly, we denote by \( \mathcal{G}^{R_0}_{\text{deg}} \) resp. \( \mathcal{G}^{R_0}_{\text{std}} \) and \( \mathcal{G}^t_{\text{deg}} \) resp. \( \mathcal{G}^t_{\text{std}} \) the MW-graphs with combinatorial and standard with vector potential with values in \( R_0 \) respectively with constant value \( t \).

We now introduce an important notion for this article:

**Definition 2.15** (MW-homomorphism). Let \( G = (G, \alpha, w) \) and \( G' = (G', \alpha', w') \) be two MW-graphs. We say that
\[
\pi: G \rightarrow G'
\]
is an MW-homomorphism if the map satisfies the following two conditions:

(i) \( \pi: G \rightarrow G' \) is a graph homomorphism (Definition 2.6).

(ii) The magnetic potential is invariant: \( \alpha = \alpha' \circ \pi \), i.e. \( \alpha_e = \alpha'_{\pi e} \) for all \( e \in E(G) \).

\[\text{Note that in some references the standard weight is also called normalised.}\]
Moreover, we say that

\[ \pi: G \to G' \] is an **MW-homomorphism**

if it is an M-homomorphism and satisfies the following two vertex and edge weight inequalities:

(iii) The following vertex weight inequality holds:

\[ (\pi*w)(v') := \sum_{v \in V, \pi(v) = v'} w(v) \geq w'(v') \text{ for all } v' \in V', \]

i.e. the push-forward vertex measure \( \pi*w \) fulfils the inequality \( \pi*w \geq w' \) pointwise.

(iv) The following edge weight inequality holds:

\[ (\pi*w)_{e'} := \sum_{e \in E, \pi(e) = e'} w_e \leq w'_{e'} \text{ for all } e' \in E', \]

i.e. the push-forward edge measure \( \pi*w \) fulfils the inequality \( \pi*w \leq w' \) pointwise.

(v) We say that \( \pi \) is vertex or edge measure preserving if equality holds in (iii) or (iv), respectively, i.e. \( \pi*w = w' \) for the vertex or the edge measure. We simply say that \( \pi \) is measure preserving if \( \pi \) is vertex and edge measure preserving.

(vi) We say that \( \pi \) is an **MW-isomorphism**, if \( \pi \) is bijective, and if \( \pi \) and \( \pi^{-1} \) are both MW-homomorphisms; in other words, if \( \pi \) is a graph isomorphism, if \( \alpha = \alpha' \circ \pi \) and if \( w = w' \circ \pi \). We say that the two MW-graphs are **(MW-)isomorphic** (denoted by \( G \simeq G' \)) if there exists an MW-isomorphism between \( G \) and \( G' \).

Some examples of M-homomorphism and MW-homomorphism are the following.

**Example 2.16 (MW-homomorphisms).** Let \( G = (G, \alpha, w), G' = (G', \alpha', w') \) and \( \tilde{G} = (\tilde{G}, \tilde{\alpha}, \tilde{w}) \) be MW-graphs.

(i) If \( G = G' \) and \( \alpha = \alpha' \), the identity \( \text{id}_G: G \to G \) is an M-homomorphism. In order that \( \text{id}_G \) is an MW-homomorphism, the weights must fulfill

\[ w(v) \geq w'(v) \text{ for all } v \in V(G) \text{ and } w_e \leq w'_e \text{ for all } e \in E(G). \]

In particular, this is true if \( w = \text{deg} \) and \( w' = 1 \), i.e. \( \text{id}_G: (G, \alpha, \text{deg}) \to (G, \alpha, 1) \) is an MW-homomorphism.

(ii) If \( \tilde{G} = G/\sim \) for some equivalence relation \( \sim \) on \( V(G) \), then the quotient map \( \kappa: G \to \tilde{G} \) of Example 2.7 is an M-homomorphism of the graphs with magnetic potentials \((G, \alpha)\) and \((\tilde{G}, \alpha)\). In order that \( \kappa \) is an MW-homomorphism, the weights must fulfill

\[ \sum_{v \in v_0} w(v) \geq \tilde{w}(v_0) \text{ for all } v_0 \in V(\tilde{G}) \text{ and } w_e \leq \tilde{w}_e \text{ for all } e \in E(\tilde{G}). \]

This condition is automatically true when both MW-graphs \( G \) and \( G' \) have combinatorial or standard weights. Moreover, \( \kappa \) is even measure preserving for standard weights.

(iii) If \( G' = G - e_0 \) and \( \alpha' = \alpha|_{E(G')} \), let \( \iota: G' \to G \) be the inclusion map of Example 2.7, then \( \iota \) is an M-homomorphism. In order that \( \iota \) is an MW-homomorphism, the weights must fulfill

\[ w'(v) \geq w(v) \text{ for all } v \in V(G) \text{ and } w'_e \leq w_e \text{ for all } e \in E(G). \]

The condition on the edges is true when both MW-graphs \( G \) and \( G' \) have combinatorial or standard weights. However, the condition on the vertex weights is true only for the combinatorial case (\( \iota \) is even vertex measure preserving here). Note that \( \iota \) is not edge measure preserving both for the combinatorial and standard weight.

(iv) Let \( G = (G, \alpha, w), G' = (G', \alpha', w') \) and \( G'' = (G'', \alpha'', w'') \) be three MW-graphs. If \( \pi: G \to G' \) and \( \tau: G' \to G'' \) are MW-homomorphisms, then it is easy to see that \( \tau \circ \pi: G \to G'' \) is also an MW-homomorphism.

We state below some basic consequences of MW-homomorphism:

**Proposition 2.17 (Basic properties of MW-homomorphism).** Let \( G = (G, \alpha, w) \) and \( G' = (G', \alpha', w') \) be two MW-graphs and \( \pi: G \to G' \) an MW-homomorphism, then:

(i) The map \( \pi: V(G) \to V(G') \) is surjective.

(ii) If there exists \( c > 0 \) such that \( w_e = w'_e = c \) for all \( e \in E(G) \) and \( e' \in E(G') \) (e.g. if \( G \) has standard or combinatorial weights), then the map \( \pi|_E: E(G) \to E(G') \) is injective.

**Proof.**

(i) Let \( v' \in V(G') \), then \( 0 < w'(v') \leq \sum_{v \in \pi^{-1}(v')} w(v) \), and this implies that the sum is not empty, i.e. there is \( v \in \pi^{-1}(v') \) with \( \pi(v) = v' \).

(ii) If the edge weights on \( E \) and on \( E' \) have constant value \( c > 0 \), then \( \sum_{e \in E, \pi(e) = e'} w_e \leq w'_e \) is equivalent with the fact that \( \{ e \in E \mid \pi(e) = e' \} \) has at most one element, i.e. \( \pi|_E \) is injective.

For the combinatorial and for the standard weights, the MW-homomorphism are characterised by geometrical conditions in the following lemmas.
Proposition 2.18 (Characterisations of MW-homomorphisms for standard weights). Consider two MW-graphs with standard weights $G = (G, \alpha, \deg)$ and $G' = (G', \alpha', \deg)$. Then the following conditions are equivalent:

(i) There exists an MW-homomorphism $\pi : G \rightarrow G'$.

(ii) There exists a measure-preserving MW-homomorphism $\pi : G \rightarrow G'$.

(iii) There is an equivalence relation $\sim$ on $V(G)$ such that $G' \cong G/\sim$ with $\alpha = \alpha'$.

Proof. (i)$\Rightarrow$(ii). By Proposition 2.17, the map $\pi$ is injective on edges, hence we have equality in (iv) of Definition 2.15. Moreover, from Lemma 2.9 (i) we conclude also equality in Definition 2.15 (iii).

(ii)$\Rightarrow$(iii). Let $\pi : G \rightarrow G'$ be an MW-homomorphism. Define a relation by $v_1 \sim v_2$ if $\pi(v_1) = \pi(v_2)$. As $\pi$ is surjective by Proposition 2.17, this defines an equivalence relation on $V(G)$. Let $\Phi : G/\sim \rightarrow G'$ be given by $\Phi([v]) = \pi(v)$. It is easy to see that $\Phi$ is a graph homomorphism, bijective on vertices and injective on edges. Moreover, the magnetic potentials are preserved. If $\Phi$ was not surjective on the edges, then $\pi$ would not be surjective on the edges, hence Lemma 2.9 (ii) contradicts the fact that $\pi$ is measure-preserving on the vertices. In particular, $\Phi$ is an MW-isomorphism, hence $G' \cong G/\sim$.

(iii)$\Rightarrow$(i). Let $f : G' \rightarrow G/\sim$ be an isomorphism and let $\kappa : G \rightarrow G/\sim$ be the quotient map. It is straightforward to show that the composition $\pi = f \circ \kappa$ is an MW-homomorphism.

The next lemma states a characterisation of MW-homomorphisms for the combinatorial weight; its proof is similar to the previous lemma.

Proposition 2.19 (Characterisations of MW-homomorphisms for combinatorial weight). Let $G = (G, \alpha, 1)$ and $G' = (G', \alpha', 1)$, the following are equivalent:

(i) There exists an MW-homomorphism $\pi : G \rightarrow G'$.

(ii) There is an equivalence relation $\sim$ on $V(G)$ and a subset $E_0$ such that $G' - E_0 \cong (G/\sim)$ with $\alpha|_{E(G)} = \alpha'$.

Definition 2.20 (Geometric (pre)order of MW-graphs). Let $G$ and $G'$ be two MW-graph. If there exists $\pi : G \rightarrow G'$ an MW-homomorphism, we write $G \sqsubset G'$.

Note that, by definition, $\sqsubset$ is invariant under MW-isomorphism.

Proposition 2.21. The relation $\sqsubset$ is a preorder on $\mathcal{G}$, i.e. it is reflexive and transitive. Moreover, for finite MW-graphs $\sqsubset$ is a partial order on the equivalence classes of MW-isomorphic graphs in $\mathcal{G}_{\deg}$ and in $\mathcal{G}_1$ (or on any subclass of weighted graphs with constant edge weight).

Proof. For the reflexivity of $\sqsubset$ use $\pi = \mathrm{id}_G$ as MW-homomorphism. The transitivity follows from Example 2.16 (v), hence $\sqsubset$ is a preorder on $\mathcal{G}$.

Consider two finite MW-graphs $G, G' \in \mathcal{G}_1$ with $\pi : G \rightarrow G'$ and $\pi' : G' \rightarrow G$, then by Proposition 2.17 (i) that both $\pi$ and $\pi'$ are surjective on the vertex sets. Since both sets are finite, it follows that $\pi$ and $\pi'$ are bijective on the vertex sets. Similarly, from Proposition 2.17 (ii) it follows that $\pi$ and $\pi'$ are bijective on the edge sets: In particular, $G$ and $G'$ are MW-isomorphic, and hence $\sqsubset$ is antisymmetric (i.e. a partial order) on the equivalence classes of MW-isomorphic graphs from $\mathcal{G}_{\deg}$ or $\mathcal{G}_1$.

The preceding result remains true on any subclass $\mathcal{G}'$ of $\mathcal{G}$ with edge weight given by a common constant.

Remark 2.22. On infinite MW-graphs, $\sqsubset$ is in general not a partial order on $\mathcal{G}_{\deg}$ or $\mathcal{G}_1$, as for infinite MW-graphs, the antisymmetry may fail: Consider e.g. $G = (G, 0, \deg)$ and $G' = (G', 0, \deg)$, where $G$ and $G'$ are given in Figure 1. It is easy to see that $G' \cong G/\{u, v\}$ and $G \cong G' / \{u', v'\}$, therefore $G \sqsubset G'$ and $G' \sqsubset G$ by Proposition 2.18 (iii).

Nevertheless, $G \not\sqsubset G'$ because $G$ and $G'$ are not isomorphic as graphs. In particular, $\sqsubset$ is not antisymmetric for infinite graphs for standard weights. A similar argument works for combinatorial weights.

![Figure 1](image.png)

3. Magnetic Laplacians and spectral preorder

In this section, we will introduce the discrete magnetic Laplacian associated to an MW-graph and present a new spectral relation between the magnetic Laplacian associated to different graphs.
3.1. Discrete magnetic exterior derivatives and Laplacians. We define some standard spaces related with a weighted graph \((G, w)\), namely for \(p \in [1, \infty)\) we set are

\[
\ell_p(V, w) := \left\{ \varphi : V \to \mathbb{C} \mid \|\varphi\|_{V,w}^p := \sum_{v \in V} |\varphi(v)|^p w(v) < \infty \right\},
\]

\[
\ell_p(E, w) := \left\{ \eta : E \to \mathbb{C} \mid \forall e \in E : \eta_e = -\eta_e, \|\eta\|_{E,w}^p := \frac{1}{2} \sum_{e \in E} |\eta_e|^p w_e < \infty \right\},
\]

For \(p = 2\), both spaces are Hilbert spaces. Note that functions on \(V\) can be interpreted as 0-forms and functions on \(E\) are considered as 1-forms. We will denote the corresponding canonical orthonormal basis by \(\{\delta_e \mid e \in V\}\) (respectively, \(\{\delta_e \mid e \in E\}\)) with \(\delta_e(u) = w(e)^{-1/2}\) if \(u = e\) and \(\delta_e(u) = 0\) otherwise (respectively, \(\delta_e(e') = w_e^{-1/2}\) if \(e = e'\) and \(\delta_e(e') = 0\) otherwise). If \(w = 1\), we sometimes simply write \(\ell_2(V, 1) = \ell_2(V)\) and \(\ell_2(E, 1) = \ell_2(E)\).

Let \(G = (G, \alpha, w)\) be an MW-graph. The (discrete) magnetic exterior derivative \(d_\alpha\) is defind as

\[
d_\alpha : \ell_2(V, w) \to \ell_2(E, w), \quad (d_\alpha \varphi)_e = e^{i\alpha_e} \varphi(\partial_+ e) - e^{-i\alpha_e} \varphi(\partial_- e).
\]

It is not hard to see that \(d_\alpha\) is a bounded operator if the relative weight \(\varphi_\infty\) is bounded (actually, the boundedness of \(d_\alpha\) is equivalent with the boundedness of \(\varphi_\infty\)). In [MY02], \(d_\alpha\) is called a codifferential operator for a twisted complex. In particular, if \(\alpha = 0\), then \(d_\alpha\) is the usual coboundary operator.

The adjoint \(d_\alpha : \ell_2(E, w) \to \ell_2(V, w)\) is given by

\[
(d_\alpha^* \eta)(v) = -\frac{1}{w(v)} \sum_{e \in E_v} w_e e^{i\alpha_e} \eta_e.
\]

**Definition 3.1** (Discrete magnetic weighted Laplacian). Let \(G\) be an MW-graph, the (discrete) magnetic (weighted) Laplacian is defined as

\[
\Delta_\alpha := d_\alpha^* d_\alpha : \ell_2(V, w) \to \ell_2(V, w).
\]

The discrete magnetic weighted Laplacian acts as

\[
(\Delta_\alpha \varphi)(v) = \frac{1}{w(v)} \sum_{e \in E_v} w_e (\varphi(v) - e^{i\alpha_e} \varphi(\partial_+ e)) = \varphi_w(v) \varphi(v) - \frac{1}{w(v)} \sum_{e \in E_v} w_e e^{i\alpha_e} \varphi(\partial_+ e),
\]

where \(\varphi_w\) is the relative weight defined in Eq. (2.4).

**Example 3.2** (Special cases of magnetic weighted Laplacians).

(i) Let \(G = (G, \alpha, w) \in \mathcal{G}\) be an MW-graph (respectively, \(G \in \mathcal{G}_{\deg}\), \(G \in \mathcal{G}_1\)), then \(\Delta_\alpha\) is the magnetic weighted (respectively, standard, combinatorial) Laplacian.

(ii) Let \(G = (G, 0, w) \in \mathcal{G}^0\) be an MW-graph (respectively, \(G \in \mathcal{G}^0_{\deg}\), \(G \in \mathcal{G}^0_1\)), then \(\Delta_0\) is the weighted (respectively, standard, combinatorial) Laplacian.

(iii) Let \(G = (G, \pi, w) \in \mathcal{G}^\pi\) be an MW-graph (respectively, \(G \in \mathcal{G}^\pi_{\deg}\), \(G \in \mathcal{G}^\pi_1\)), then \(\Delta_\pi\) is the weighted (respectively, standard, combinatorial) signless Laplacian.

By construction, the magnetic Laplacian is a bounded, non-negative (hence self-adjoint) operator. Moreover, its spectrum is contained in the interval \(\sigma(\Delta_\alpha) \subset [0, 2\varphi_\infty]\). Let \(\tilde{\alpha}\) and \(\alpha\) be two cohomologous (gauge-equivalent) magnetic potentials for some gauge \(\xi \in C^0(G)\) (i.e. \(\xi : V \to \mathbb{R} / 2\pi\mathbb{Z}\) with \(\tilde{\alpha} = \alpha + d\xi\)). Then the gauge \(\xi\) induces two unitary (multiplication) operators \(\Xi_0\) and \(\Xi_1\) on \(\ell_2(V, w)\) and \(\ell_2(E, w)\), respectively, defined by

\[
(\Xi_0 \varphi)(v) := e^{i\xi(v)} \varphi(v) \quad \text{and} \quad (\Xi_1 \eta)_e := e^{i\xi(\partial_+ e) + i\xi(\partial_- e)}/2 \eta_e.
\]

Here, \(\xi \mapsto \Xi_0\) and \(\xi \mapsto \Xi_1\) are unitary representations of \(\xi \in C^0(G)\) seen as an additive group on \(\ell_2(V, w)\) and \(\ell_2(E, w)\).

We will now show that magnetic Laplacians with gauge-equivalent magnetic potentials are unitarily equivalent:

**Proposition 3.3.** If \(\alpha \sim \tilde{\alpha}\) (or, more precisely, if \(\tilde{\alpha} = \alpha + d\xi\)), then

\[
d_\alpha \Xi_0 = \Xi_1 d_{\tilde{\alpha}} \quad \text{and} \quad \Delta_\alpha \Xi_0 = \Xi_1 \Delta_{\tilde{\alpha}}.
\]

In particular, \(\Delta_\alpha\) and \(\Delta_{\tilde{\alpha}}\) are unitarily equivalent, and the spectral properties of \(\Delta_\alpha\) depend only on the MW-graph class \([\alpha]\).

**Proof.** The first equation follows by a straightforward calculation, namely

\[
(d_\alpha \Xi_0 \varphi)_e = e^{i\alpha_e/2 + i\xi(\partial_+ e)} \varphi(\partial_+ e) - e^{-i\alpha_e/2 + i\xi(\partial_- e)} \varphi(\partial_- e) = e^{i\xi(\partial_+ e) + i\xi(\partial_- e)/2} (e^{i\alpha_e/2} \varphi(\partial_+ e) - e^{-i\alpha_e/2} \varphi(\partial_- e)) = (\Xi_1 d_{\tilde{\alpha}} \varphi)_e.
\]

The second intertwining relation follows from the first one and the fact that \(\Xi_0\) and \(\Xi_1\) are unitary. \(\square\)

Now, we will prove some results related with the spectrum of the magnetic Laplacian that will be useful in the next sections. The first result says that a magnetic potential increases the smallest eigenvalue.
Lemma 3.4. Let $G = (G, \alpha, w)$ be an MW-graph, such that the underlying discrete graph $G$ is connected and finite. Then $0 \in \sigma(\Delta_\alpha)$ if and only if $\alpha$ is trivial, i.e. cohomologous to 0.

Proof. “$\Rightarrow$”: Suppose that $0 \in \sigma(\Delta_\alpha)$, then there exists a non-zero $\varphi \in \ell_2(V, w)$ such that $0 = \langle \Delta_\alpha \varphi, \varphi \rangle = \|d_\alpha \varphi\|^2$. In particular, $d_\alpha \varphi = 0$, i.e. $\varphi(\partial_e) = e^{\alpha_e} \varphi(\partial_e)$ for all $e \in E$. As the graph $G$ is connected we can define a function $\xi: V \to A$ adjusting the phases in such a way that $\varphi(v)e^{\xi(v)}$ is constant on $V$. We then have $\alpha_e = (d\xi)_e (e \in E)$, hence $\alpha \sim 0$.

“$\Leftarrow$”: If $\alpha \sim 0$, then by Proposition [3,3], $\Delta_\alpha$ is unitarily equivalent with the Laplacian $\Delta_0$ without magnetic potential. For the latter, a constant function is an eigenfunction with eigenvalue 0, hence $0 \in \sigma(\Delta_\alpha)$. □

We conclude this section recalling some variations of the well-known variational characterisation of the eigenvalues (min-max principle):

Theorem 3.5 (Courant-Fischer). Let $\mathcal{H}$ be an $n$-dimensional (complex) Hilbert space and $A: \mathcal{H} \to \mathcal{H}$ linear and $A^* = A$. Moreover, denote by $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ be the eigenvalues of $A$ written in ascending order and repeated according to their multiplicities. Let $k \in \{1, 2, \ldots, n\}$, then

$$\lambda_k = \min_{S \in \mathcal{A}_{k-1}} \max_{\varphi \in S \setminus \{0\}} \frac{\langle A\varphi, \varphi \rangle}{\langle \varphi, \varphi \rangle} \quad \text{and} \quad \lambda_k = \max_{S \in \mathcal{A}_{k-1}} \min_{\varphi \in S \setminus \{0\}} \frac{\langle A\varphi, \varphi \rangle}{\langle \varphi, \varphi \rangle},$$

(3.6)

and

$$\lambda_k = \min_{S \in \mathcal{A}_k} \max_{\varphi \in S \setminus \{0\}} \frac{\langle A\varphi, \varphi \rangle}{\langle \varphi, \varphi \rangle} \quad \text{and} \quad \lambda_k = \max_{S \in \mathcal{A}_k} \min_{\varphi \in S \setminus \{0\}} \frac{\langle A\varphi, \varphi \rangle}{\langle \varphi, \varphi \rangle},$$

(3.7)

where $\mathcal{A}_k$ denotes the set of $k$-dimensional subspaces of $\mathcal{H}$.

Let $G = (G, \alpha, w) \in \mathcal{G}$ be a finite MW-graph of order $n$. The eigenvalues of its discrete magnetic Laplacian $\Delta_\alpha$ can be written as

$$\sigma(G) = \sigma(\Delta_\alpha) = (\lambda_1(G), \lambda_2(G), \ldots, \lambda_n(G)),$$

(3.8)

where the eigenvalues are written in ascending order and repeated according to their multiplicities, as in the above theorem.

Remark 3.6. For a graph with only one vertex $v$ with weight $w(v) > 0$ and no edge, the corresponding Laplacian is 0, as the sum over $e \in E_v$ is empty; the spectrum in this case is $(0)$. More generally, any isolated vertex in a graph contributes with an extra 0 in the list of eigenvalues.

For the standard weight, we have $w(v) = \deg(v) = 0$. In this case, it is convenient to associate to the graph with only one vertex and no edges again the eigenvalue 0. Hence any isolated vertex of the standard Laplacian contributes with one extra eigenvalue 0 in the list of eigenvalues (see e.g. [Chu97] bottom of p. 2); in this way, the case $r = 0$ also applies in Theorem [3.6] for the standard weight).

3.2. Order on sets of finite sequences. We next relate spectra of different MW-graphs. To do so, we first introduce an order on the set of finite increasing sequences of real numbers:

Definition 3.7. Let $\Lambda$ and $\Lambda'$ two sequences of real numbers written in increasing order with lengths $n$ and $n'$ respectively, i.e.

$$\Lambda := \{(\lambda_1, \lambda_2, \ldots, \lambda_n) | \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{n-1} \leq \lambda_n \};$$

$$\Lambda' := \{(\lambda_1', \lambda_2', \ldots, \lambda_{n'}') | \lambda_1' \leq \lambda_2' \leq \cdots \lambda_{n'-1}' \leq \lambda_{n'}' \}.$$ 

Given $r \in \mathbb{N}_0$, we say that $\Lambda$ is smaller than (or equal to) $\Lambda'$ with shift $r$ (and denote this by $\Lambda \preceq^r \Lambda'$) if $n \geq n' - r$ and $\lambda_k \leq \lambda_{k+r}'$ for $1 \leq k \leq n' - r$.

The length of the sequence $\Lambda$ is defined by $|\Lambda| := n$.

Remark 3.8. Let $\Lambda, \Lambda'$ and $\Lambda''$ be increasing sequences of real numbers as above.

(i) We denote $\Lambda \preceq^0 \Lambda'$ simply by $\Lambda \preceq \Lambda'$.

(ii) If $\Lambda \preceq \Lambda'$ and $\Lambda' \preceq \Lambda''$ we will write $\Lambda \preceq^s \Lambda' \preceq^t \Lambda''$.

(iii) The case $\Lambda \preceq^r \Lambda'$ implies that $n = |\Lambda| \geq |\Lambda'| \geq n - r$. If $|\Lambda'| = n - r$, then $\Lambda \preceq^1 \Lambda'$ is equivalent with the interlacing of $\Lambda$ and $\Lambda'$ similarly as in [BH12 Section 2.5], namely

$$\lambda_1 \leq \lambda_1' \leq \lambda_{1+r}, \quad \lambda_2 \leq \lambda_2' \leq \lambda_{2+r}, \quad \ldots, \quad \lambda_{n-r} \leq \lambda_{n-r}' \leq \lambda_n.$$

Especially if $r = 1$ it becomes the usual interlacing (explaining also the name).

$$\lambda_1 \leq \lambda_1' \leq \lambda_2 \leq \lambda_2' \leq \cdots \leq \lambda_{n-1} \leq \lambda_{n-1}' \leq \lambda_n.$$

We state in the next lemma some direct consequences of the preceding definition.
Lemma 3.9. Let $\Lambda$, $\Lambda'$ and $\Lambda''$ be three increasing sequences of real numbers and consider $r, s \in \mathbb{N}_0$.

(i) If $r \in \mathbb{N}_0$, then $\Lambda \overset{r}{\preceq} \Lambda$ (reflexivity).
(ii) If $\Lambda \not\preceq \Lambda'$, then $\Lambda = \Lambda'$ (antisymmetry).
(iii) If $\Lambda \overset{r}{\preceq} \Lambda'$ and $\Lambda' \overset{s}{\preceq} \Lambda''$, then $\Lambda \overset{r+s}{\preceq} \Lambda''$ (transitivity).
(iv) If $\Lambda \not\preceq \Lambda'$ and $r \leq s$, then $\Lambda \overset{r}{\preceq} \Lambda'$.
(v) If $\Lambda \not\preceq \Lambda'$ with $n = |\Lambda| = |\Lambda'|$ and if $\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} \lambda_i'$, then $\Lambda = \Lambda'$, i.e., $\lambda_i = \lambda_i'$, $i = 1, \ldots, n$ (equal sums).

In case $|\Lambda| = |\Lambda'|$ then $\Lambda \preceq \Lambda'$ implies (in the usual sense of majorisation of vectors in $\mathbb{R}^d$, cf., [MOA11]) that $\Lambda'$ majorises $\Lambda$. Nevertheless the fact that, in addition, we allow the shift $r$ as a parameter is particularly convenient to study and relate spectra of Laplacians.

Proof. The reflexivity property and part (i) follow from the fact that the sequence $\Lambda$ is written in increasing order. The antisymmetry property (ii) is also a direct consequence of Definition 3.7. To show (iii), note that $|\Lambda| + r \geq |\Lambda'|$ and $\lambda_k \leq \lambda_{k+r}^{\prime}$ for $1 \leq k \leq |\Lambda| - r$, as well as $|\Lambda'| + s \geq |\Lambda''|$ and $\lambda_l \leq \lambda_{l+s}^{\prime}$ for $1 \leq l \leq |\Lambda''| - s$. This implies $|\Lambda| + r + s \geq |\Lambda''|$ and $\lambda_k \leq \lambda_{k+r+s}^{\prime}$ for $1 \leq k \leq |\Lambda'| - r - s$. To show (iv) note first that the statement is trivial for $n = 1$. Assume that it is true for $n \in \mathbb{N}$. We then conclude from $\Lambda \not\preceq \Lambda'$ for two sequences of length $n + 1$, that

$$\sum_{k=1}^{n} \lambda_k - \sum_{k=1}^{n} \lambda_k' \leq 0 \quad \text{and} \quad \lambda_k' - \lambda_k \geq 0.$$ 

By assumption (equality of the traces for $n + 1$) both above left hand sides are equal, hence equal to 0. It follows by the induction hypothesis that $\lambda_k = \lambda_k'$ for $k = 1, \ldots, n$, and hence also $\lambda_{n+1} = \lambda_{n+1}'$, again by the equality of the traces for $n + 1$. \hfill $\square$

3.3. Spectral preorder. We will now apply the relation $\preceq$ described before to relate the spectrum of the magnetic Laplacian on different MW-graphs.

Definition 3.10 (Spectral preorder of magnetic weighted graphs and isospectrality). Let $G, G' \in \mathcal{G}$ be two finite MW-graphs. We say that $G$ is (spectrally) smaller than $G'$ with shift $r$ (denoted by $G \overset{r}{\preceq} G'$) if $\sigma(G) \overset{r}{\preceq} \sigma(G')$, where $\sigma(G)$ and $\sigma(G')$ are the spectra of the corresponding discrete magnetic weighted Laplacians as in Eq. (3.8), i.e., if $|G| + r \geq |G'|$ and if $\lambda_k(G) \leq \lambda_{k+r}(G')$ for all $1 \leq k \leq |G'| - r$.

If $r = 0$ we write again simply $G \preceq G'$.

We say that $G$ and $G'$ are isospectral, if $G \preceq G'$ and $G' \preceq G$, i.e., if the (magnetic weighted) Laplace spectrum of $G$ and $G'$ agrees; in other words if both graphs have the same number $n$ of vertices and $\lambda_k(G) = \lambda_k(G')$ for all $k = 1, \ldots, n$.

Proposition 3.11. The relation $\preceq$ is a preorder on $\mathcal{G}$.

Proof. Lemma 3.9 (ii) and (iii) show the reflexivity and transitivity of $\preceq$ using shifts $r = s = 0$. \hfill $\square$

Note that $\preceq$ is invariant under MW-isomorphisms, as MW-isomorphisms have isospectral magnetic Laplacians. Since there are non-isomorphic isospectral graphs it follows that $\preceq$ is not antisymmetric, i.e. equality of spectra does not imply that the graphs are (MW)-isomorphic. In particular, $\preceq$ is not a partial order.

Remark 3.12. (i) The second smallest eigenvalue of the usual Laplacian gives a measure of the connectivity of the graph (see [Fie72] and Subsection 5.2). Defining $a(G) := \lambda_2(G)$ we obtain directly from the definition of the spectral preorder that

$$G \preceq G' \implies a(G) \leq a(G').$$

Corollaries 4.2 and 4.7 give a quantitative measure of the fact that deleting edges reduces the connectivity and contracting vertices increases the connectivity of the graph.

(ii) The name spectral order has been introduced by Olson [Ols74] for two self-adjoint (bounded) operators $T_1$ and $T_2$ in a Hilbert space $\mathcal{H}$ with spectral resolutions $E_j(t) := \mathbb{I}_{(-\infty,t]}(T_j)$ ($j = 1, 2$). Then $T_1 \preceq T_2$ if and only if $E_1(t) \leq E_2(t)$ for all $t \in \mathbb{R}$ (i.e. if $E_1(t)\varphi, \varphi) \leq (E_2(t)\varphi, \varphi)$ for all $\varphi \in \mathcal{H}$). If $T_1 \geq 0$ and $T_2 \geq 0$, then $T_1 \preceq T_2$ is equivalent with $T_1^p \leq T_2^p$ for all $p \in \mathbb{N}$. If both operators have purely discrete spectrum $\lambda_k(T_j)$ (written in increasing order and repeated according to multiplicity) then we have the implications

$$T_1 \preceq T_2 \implies T_1 \leq T_2 \implies T_1 \preceq T_2,$$

where the latter means that $\lambda_k(T_1) \leq \lambda_k(T_2)$ for all $k$ (the latter implication follows from the min-max principle as in Theorem 3.5).

Next we lift MW-homomorphism to spaces of functions on vertices and edges:
Lemma 3.13. Let \( \pi : G \rightarrow G' \) be an MW-homomorphism where \( G = (G, \alpha, w) \), \( G' = (G', \alpha', w') \) and \( G = (V, E, \partial) \), \( G' = (V', E', \partial') \) denote the underlying graphs. Define the natural the identification operators on functions over vertices \( J^0 : \ell_2(V', w') \rightarrow \ell_2(V, w) \) and edges \( J^1 : \ell_2(E', w') \rightarrow \ell_2(E, w) \) by \( J^0 \phi = \varphi \circ \pi \) and \( J^1 \eta = \eta \circ \pi \), respectively. Then the following holds:

(i) We have \( \|J^0 \varphi \|_{\ell_2(V, w)} \geq \|\varphi\|_{\ell_2(V', w')} \) for all \( \varphi \in \ell_2(V', w') \). In particular, \( J^0 \) is injective. If \( \pi \) is vertex measure preserving, then \( J^0 \) is an isometry.

(ii) We have \( \|J^1 \eta\|_{\ell_2(E, w)} \leq \|\eta\|_{\ell_2(E', w')} \) for all \( \eta \in \ell_2(E', w') \). If \( \pi \) is edge measure preserving, then \( J^1 \) is an isometry.

(iii) We have \( d_\alpha J^0 = J^1 d_\alpha' \).

Proof. From Definition 2.15 (iii) we have:

\[
\|J^0 \varphi\|_{\ell_2(V, w)}^2 = \sum_{v \in V} w(v)(\varphi \circ \pi)(v)^2 = \sum_{v' \in V', \pi^{-1}(v)} w(v')|\varphi(v')|^2 = \sum_{v' \in V'} (\pi_* w)(v')|\varphi(v')|^2 \geq \sum_{v' \in V'} w'(v')|\varphi(v')|^2 = \|\varphi\|_{\ell_2(V', w')}^2.
\]

Clearly, if \( \pi \) is vertex measure preserving, then \( \pi_* w = w' \) on \( V' \), and equality in the above estimate holds.

(ii) The assertion of the identification map \( J^0 \) on the edges follows similarly from Definition 2.15 (iv).

(iii) This intertwining equation follows immediately from the properties of MW-homomorphism given in Definition 2.15 (i). □

The following result showing that the geometric preorder is stronger than the spectral preorder follows from the min-max principle mentioned in Theorem 3.5. Recall that \( G \sqsubseteq G' \) means that there is an MW-homomorphism \( \pi : G \rightarrow G' \), see Definitions 2.13 and 2.20.

Theorem 3.14. Let \( G, G' \in \mathcal{G} \), then

\[
G \sqsubseteq G' \quad \text{implies} \quad G \sqsubsetneq G'.
\]

Moreover, if the MW-homomorphism \( \pi : G \rightarrow G' \) is (vertex and edge) measure preserving, then we additionally have

\[
G' \sqsubsetneq G, \quad r = |G| - |G'| \geq 0.
\]

Proof. First note that \( \pi \) is surjective on the set of vertices by Proposition 2.17 (i), hence \( |G| \geq |G'| \) and therefore \( r \geq 0 \). From Lemma 3.13 we conclude

\[
\frac{\|d_\alpha J^0 \varphi'\|_{\ell_2(E, w)}^2}{\|J^0 \varphi'\|_{\ell_2(V, w)}^2} = \frac{\|J^1 d_\alpha' \varphi'\|_{\ell_2(E', w')}^2}{\|J^0 \varphi'\|_{\ell_2(V', w')}^2} \leq \frac{\|d_\alpha' \varphi'\|_{\ell_2(E', w')}^2}{\|\varphi'\|_{\ell_2(V', w')}^2}.
\]

Denote by \( S_k' \) the \( k \)-dimensional subspace of \( \ell_2(V', w') \) spanned by the first \( k \) eigenfunctions of \( \Delta_{G'} \). From the min-max characterisation of the \( k \)-th eigenvalue (first equality in Eq. (3.7)), we then have by the preceding estimate:

\[
\lambda_k(G) = \min_{S \in \mathcal{S}_k} \max_{\varphi \neq 0} \frac{\|d_\alpha \varphi\|_{\ell_2(E, w)}^2}{\|\varphi\|_{\ell_2(V, w)}^2} \leq \frac{\|d_\alpha J^0 \varphi'\|_{\ell_2(E, w)}^2}{\|J^0 \varphi'\|_{\ell_2(V, w)}^2} \leq \frac{\|d_\alpha' \varphi'\|_{\ell_2(E', w')}^2}{\|\varphi'\|_{\ell_2(V', w')}^2} = \lambda_k(G'),
\]

for all \( 1 \leq k \leq |G'| \), where \( \mathcal{S}_k \) is the set of all \( k \)-dimensional subspaces of \( \ell_2(V, w) \). Moreover, as \( J^0 \) is injective, \( S = J^0(S_k') \) is also \( k \)-dimensional, i.e. \( J^0(S_k') \in \mathcal{S}_k \). This shows \( G \sqsubsetneq G' \).

If \( \pi \) is measure preserving, then \( J^0 \) and \( J^1 \) are isometries, hence we have equality in Eq. (3.9). Moreover, let \( n = |G| \), \( n' = |G'| \) and denote by \( T_k' \) the space generated by the \( n-k+1 \) eigenfunctions \( \varphi'_{n-k+1}, \ldots, \varphi'_{n'} \) of the Laplacian on \( G' \), then we have similarly as before (second equality in Eq. (3.7)):

\[
\lambda_k(G) = \min_{S \in \mathcal{S}_{n-k+1}} \max_{\varphi \neq 0} \frac{\|d_\alpha' \varphi'\|_{\ell_2(E', w')}^2}{\|\varphi'\|_{\ell_2(V', w')}^2} \geq \min_{\varphi' \in T_k'} \frac{\|d_\alpha' J^0 \varphi'\|_{\ell_2(E', w')}^2}{\|J^0 \varphi'\|_{\ell_2(V, w)}^2} = \lambda_{n-(k-1)+1}(G') = \lambda_{k-r}(G'),
\]

where \( S = J^0(T_k') \) is \( (n-k+1) \)-dimensional since \( J^0 \) is injective. From Definition 3.7 and 3.10 it follows that \( G' \sqsubset G \). □
Remark 3.15. Note that the converse statement of Theorem [3.14] is wrong in general, i.e. there are MW-graphs such that $G \not\preceq G'$ but not $G \succeq G'$. As an example consider the preorder relations between $G_9$ and $G_{10}$ in Figure [6].

We have the following simple consequence of the previous theorem and Example [2.16] [1]:

**Corollary 3.16.** Let $G = (G, \alpha, \deg)$ and $G' = (G, \alpha, 1)$, then $G \succeq G'$. In particular $G \not\succeq G'$, i.e. the (magnetic) eigenvalues of the standard Laplacian are always lower or equal than the (magnetic) eigenvalues of the combinatorial Laplacian.

4. Geometric perturbations and preorders

In this section, we present several elementary and composite perturbations of finite MW-graphs (deleting edges, contracting vertices, etc.) and study systematically their effect on the spectrum of the magnetic Laplacian. We will apply the geometric and spectral preorders to quantify the effect of the perturbations. The results are stated for general weights. We treat the important special cases of combinatorial and standard weights as corollaries.

4.1. Elementary perturbations. We consider first two elementary perturbations on graphs: deleting an edge and contracting two vertices.

4.1.1. Deleting an edge. Let $G = (G, \alpha, w)$ be an MW-graph with $G = (V, E, \partial)$. Deleting an edge $e_0 \in E$ of $G$ gives the MW-graph $G' = (G', \alpha', w')$ where $G' = G - e_0$ and $\alpha' = \alpha|_{E \setminus \{e_0\}}$; we write $G' = G - e_0$ for the MW-graph obtained in this way and will specify the weight $w'$ later on.

Recall that $G' = (V', E', \partial')$ is obtained from $G = (V, E, \partial)$ by deleting $e_0 \in E$, i.e. $V' = V \setminus \{e_0\}$ and $\partial' = \partial|_{E' \setminus E'}$ (see Definition [2.3] and Figure [2a, 2b]). In particular, the inclusion $\iota: G' \hookrightarrow G$ is a graph homomorphism (see Example [2.7] [ii]).

The next result is a generalisation of the previous remarks to arbitrary vector potentials. Our results also applies to the case when a loop or a multiple edge is deleted (see e.g. Remark [5.3] [iii]).

**Theorem 4.1** (General weights). Let $G, G' \in \mathcal{G}$ with $G' = G - e_0$ for some $e_0 \in E = E(G)$.

(i) If $w' \leq w$ for all edges $e \in E \setminus \{e_0\}$ and $w(v) \leq w'(v)$ for all $v \in V \setminus \{\partial - e_0, \partial_+ e_0\}$, then

(a) $w(v) \leq w'(v)$ for $v \in \{\partial - e_0, \partial_+ e_0\}$ implies $G' \preceq G$ (and hence $G' \succeq G$).

(b) $w(v) - w(e_0) \leq w'(v)$ for $v \in \{\partial - e_0, \partial_+ e_0\}$ and $d_\infty \leq 1$ (maximal relative weight, see (2.5)) implies $G' \succeq G$.

Moreover, if $e_0$ is a loop and $\alpha_{e_0} = \pi$ then $G' \succeq G$.

(ii) If $w_\leq w' \leq w$ for all edges $e \in E \setminus \{e_0\}$ and $w'(v) \leq w(v)$ for all $v \in V$, then $G' \succeq G$.

Moreover, if $e_0$ is a loop with $\alpha_{e_0} = 0$ then $G \succeq G'$.

(iii) If $w_e = w'_e$ for all edges $e \in E \setminus \{e_0\}$, $w'(v) = w(v)$ for all $v \in V$ and if $e_0$ is a loop with $\alpha_{e_0} = 0$ then $G' \succeq G \succeq G'$, i.e. $G$ and $G'$ are isospectral.

**Proof.** Let $G = (G, \alpha, w)$ and $G = (G', \alpha', m')$ be two MW-graph with $G' = G - e_0$, $\alpha' = \alpha|_{E(G')}$ and note that $|G| = |G'|$.

(a) To show $G' \succeq G$ just observe that the inclusion $\iota: G' \hookrightarrow G$ is an MW-homomorphism, hence $G' \preceq G$ and therefore $G' \succeq G$ by Theorem [3.14].

(b) For the relation $G' \succeq G$, we have (using Theorem [3.5] twice)

$$
\lambda_k(G') = \max_{S \in \mathcal{S}_{-k-1}} \min_{\varphi \perp S} \frac{\|d_\alpha \varphi\|_{\mathcal{L}^2(E,w')}}{\|\varphi\|_{\mathcal{L}^2(V,w')}}
\leq \max_{S \in \mathcal{S}_{-k-1}} \min_{\varphi \perp S} \frac{\|d_\alpha \varphi\|_{\mathcal{L}^2(E,w)} - |(d_\alpha \varphi)_{e_0}|^2 w_{e_0}}{\|\varphi\|_{\mathcal{L}^2(V,w)} - |(\varphi(\partial - e_0)|^2 + |\varphi(\partial_+ e_0)|^2)^2 w_{e_0}}
\leq \max_{S \in \mathcal{S}_{-k-1}} \min_{\varphi \perp S} \frac{\|d_\alpha \varphi\|_{\mathcal{L}^2(E,w)} - |(d_\alpha \varphi)_{e_0}|^2 w_{e_0}}{\|\varphi\|_{\mathcal{L}^2(V,w)} - |\varphi(\partial - e_0)|^2 + |\varphi(\partial_+ e_0)|^2)^2 w_{e_0}}
= \max_{S \in \mathcal{S}_{-k-1}} \min_{\varphi \perp S} \frac{\|d_\alpha \varphi\|_{\mathcal{L}^2(E,w)} - 4|\varphi(\partial - e_0)|^2 w_{e_0}}{\|\varphi\|_{\mathcal{L}^2(V,w)} - 2|\varphi(\partial_+ e_0)|^2 w_{e_0}}
\leq \max_{S \in \mathcal{S}_{-k-1}} \min_{\varphi \perp S} \frac{\|d_\alpha \varphi\|_{\mathcal{L}^2(E,w)}}{\|\varphi\|_{\mathcal{L}^2(V,w)}} = \lambda_{k+1}(G),
$$
for $k = 1, \ldots, n - 1$, where $L'(e_0) = \mathbb{C} \psi'$ denotes the linear space generated by

$$
\psi' = \frac{1}{(w(\partial_+ e_0))^{1/2}} \delta_{\partial_+ e_0} + \frac{1}{(w(\partial_- e_0))^{1/2}} \delta_{\partial_- e_0}.
$$

We use the vertex weight inequality $w'(v) \geq w(v) - w(e_0) in the second line; we use the fact that $\varphi \perp L'(e_0)$ implies $| \langle d_\alpha \varphi \rangle_{e_0} |^2 = 4 | \varphi(\partial_+ e_0) |^2$ and $| \varphi(\partial_- e_0) |^2$ in the fourth line; and for the fifth line, we use the following inequality between real numbers $a, b$ and $\gamma$ (see [CDH04, Lemma 2.9]), namely

$$
a^2 - 2\gamma^2 \geq 0, \quad b^2 - \gamma^2 > 0 \quad \text{and} \quad \frac{a^2}{b^2} \leq 2 \implies \frac{a^2 - 2\gamma^2}{b^2 - \gamma^2} \leq \frac{a^2}{b^2}.
$$

We also used the fact that $\rho_\infty \leq 1$ as $a^2/b^2$ is the Rayleigh quotient for the graph $G$ and hence $a^2/b^2 \leq 2$.

The proof of the second part of (i) is similar to the previous. We observe that $(d_\alpha \varphi)_{e_0} = 0$ if $\alpha_{e_0} = \pi$ for a loop $e_0$, and $\psi' = 0$ in Eq. (4.1). In particular, we do not have to introduce the function $\psi'$, hence $\lambda_k(G') \leq \lambda_k(G)$.

Using Theorem 4.3 twice we obtain

$$
\lambda_{k+1}(G') = \min_{S \in \mathcal{F}_{n-1}(k+1)} \max_{\varphi \in \mathcal{S} \setminus \{0\}} \frac{\|d_\alpha \varphi \|^2_{L^2(E',w')}}{\|\varphi\|^2_{L^2(V',w')}}
\geq \min_{S \in \mathcal{F}_{n-1}(k+1)} \max_{\varphi \in \mathcal{S} \setminus \{0\}} \frac{\|d_\alpha \varphi \|^2_{L^2(E,w)}}{\|\varphi\|^2_{L^2(V,w)}}
= \min_{S \in \mathcal{F}_{n-1}(k+1)} \max_{\varphi \in \mathcal{S} \setminus \{0\}} \frac{\|d_\alpha \varphi \|^2_{L^2(E,w)}}{\|\varphi\|^2_{L^2(V,w)}}
\geq \min_{S \in \mathcal{F}_{n-1}(k+1)} \max_{\varphi \in \mathcal{S} \setminus \{0\}} \frac{\|d_\alpha \varphi \|^2_{L^2(E,w)}}{\|\varphi\|^2_{L^2(V,w)}} = \lambda_k(G),
$$

for $k = 1, \ldots, n - 1$, where $L(e_0) = \mathbb{C} \psi$ denotes the linear space generated by

$$
\psi = \frac{1}{(w(\partial_+ e_0))^{1/2}} \delta_{\partial_+ e_0} - \frac{1}{(w(\partial_- e_0))^{1/2}} \delta_{\partial_- e_0}
$$

for the canonical orthonormal basis $(\delta_v)_v$ of $L^2(V,w)$; and we used the fact that $(d_\alpha \varphi)_{e_0} = 0$ if $\varphi \perp L(e_0)$, hence we can just take the norm over $E'$ instead of $E$ for the second equality.

The proof of the second part (i) is similar to the previous, we observe that $(d_\alpha \varphi)_{e_0} = 0$ if $\alpha_{e_0} = \pi$ for a loop $e_0$, and $\psi = 0$ in Eq. (4.2). In particular, we do not have to introduce the function $\psi$, hence $\lambda_k(G') \leq \lambda_k(G)$.

(iii) From part (i) we conclude that $G' \preceq G$ and from part (ii) $G \preceq G'$ follows; finally, observe that $G$ and $G'$ have the same number of vertices; hence $G$ and $G'$ are isospectral. \hfill \Box

The above theorem generalises some known interlacing results, namely [Hen95, Lemma 2] (combinatorial Laplacian and its signless version, see also [Moh91, Theorem 3.2] and [Fie73, Corollary 3.2]) and [CDH04, Theorem 2.3] (standard Laplacian) and [AT14, Theorem 8] (signed standard Laplacians).

We state these cases now for standard and combinatorial weights as a corollary:

**Corollary 4.2.** Let $G, G' \in \mathcal{G}$ where $G' = G - e_0$ for some $e_0 \in E(G)$.

(i) If $G, G' \in \mathcal{G}_1$, then $G \preceq G'$ and $G' \subseteq G$, hence $G \preceq G' \preceq G$. Furthermore, if $e_0$ is not a loop, then there exists $1 \leq k \leq |G|$ such that $\lambda_k(G') < \lambda_k(G)$.

(ii) If $G, G' \in \mathcal{G}_{\text{deg}}$, then $G \preceq G' \preceq G$.

**Proof.** (i) By Theorem 4.1 (i) we have $G' \subseteq G$ (and hence $G' \preceq G$) and by Theorem 4.1 (ii) we conclude $G \preceq G'$.

For the second part, note that

$$
\sum_{k=1}^n \lambda_k(G) = \text{tr}(\Delta) = \sum_{v \in V} \deg_G(v) > \sum_{v \in V'} \deg_{G'}(v) = \text{tr}(\Delta_{G'}) = \sum_{k=1}^n \lambda_k(G'),
$$

hence there exists an index $k \in \{1, \ldots, n\}$ such that $\lambda_k(G') < \lambda_k(G)$.

(ii) For $G \preceq G'$ we use Theorem 4.1 (iii). Moreover, $\deg_{G'}(v) = \deg_G(v) - 1$ for $v = \partial v_0$ and $\deg_{G'}(v) = \deg_G(v)$ for all other vertices, hence by Theorem 4.1 (iii) it follows that $G' \preceq G$. \hfill \Box

**Remark 4.3.**
The magnetic potential (see Example 2.16 (ii)). We also write while keeping all edges and vector potentials. Then the quotient map \( \tilde{t} \). Recall that \( \sigma \)

Let \( G \) be an MW-graph, a spanning subgraph is given in Corollary 5.1.

If we delete a loop, we can slightly improve the previous corollary:

**Corollary 4.4.** Let \( G, G' \in \mathcal{G} \) with \( G' = G - e_0 \) for a loop \( e_0 \in E(G) \).

(i) If \( G, G' \in \mathcal{G}_1 \) and \( \alpha_{e_0} = 0 \) then \( G \nless G' \), i.e. \( G \) and \( G' \) are isospectral.

(ii) If \( G, G' \in \mathcal{G}_{\text{deg}} \) and \( \alpha_{e_0} = 0 \) then \( G \nless G' \).

**Proof.** (i) follows from Theorem 4.1 (iii). (ii) If \( \alpha_{e_0} = 0 \) it follows from Theorem 4.1 (ii) that \( G \nless G' \); and if \( \alpha_{e_0} = \pi \) it follows from Theorem 4.1 (iii) that \( G \nless G' \).

**Example 4.5.** For \( t \in [0, 2\pi] \) we consider the MW-graph \( G_t \in \mathcal{G}_1 \) defined by \( G \) in Figure 2a. We orient the edges along the closed path such that the flux through it adds up to \( t \). The spectrum \( \sigma(G_t) \) consists of five eigenvalues plotted as a solid line in Figure 2c (the spectrum depends of the value \( t \)). Let \( G'_t = G - e_0 \) with combinatorial weights, i.e. \( G'_t \in \mathcal{G}_1 \) (see Figure 2b). Since \( G' \) is a tree, we have \( \sigma(G'_t) = \sigma(G'_0) \) for all \( t \).

In particular, \( \sigma(G'_0) \) consists of five eigenvalues (dotted lines in Figure 2c). From Corollary 4.2 we conclude \( G_t \nless G'_0 \nless G_t \). In particular

\[
\sigma(G'_0) = \left( 0, \frac{1}{2} \left( 3 - \sqrt{5} \right), \frac{1}{2} \left( 5 - \sqrt{5} \right), \frac{1}{2} \left( \sqrt{5} + 3 \right), \frac{1}{2} \left( \sqrt{5} + 5 \right) \right) \approx (0, 0.381966, 1.38197, 2.61803, 3.61803),
\]

hence we can localise the spectrum of \( \sigma(G'_t) \) for any \( t \in [0, 2\pi] \), i.e. \( \lambda_i(G_t) \in [\lambda_i(G'), \lambda_{i+1}(G')] \) for \( i = 1, 2, 3 \) and 4.

![Figure 2](image)

**Figure 2.** If we delete the edge \( e_0 \) from the graph \( G \) in Figure 2a we obtain the graph \( G' = G - e_0 \) in Figure 2b. Let \( G_t \) (respectively \( G'_t \)) be in \( \mathcal{G}_1 \) with underlying graphs \( G \) (respectively, \( G' \)). In Figure 2c we plot \( \sigma(G_t) \) (respectively, \( \sigma(G'_t) \)) as a solid (respectively, dashed) line for all \( t \in [0, 2\pi] \). Note that \( G_t \nless G'_t \nless G_t \), i.e., the eigenvalues interlace.

### 4.1.2. Contracting vertices

Let \( G = (G, \alpha, w) \) be an MW-graph, a **vertex contraction** of \( G \) is the MW-graph \( \widetilde{G} = (\widetilde{G}, \widetilde{\alpha}, \widetilde{w}) \) where \( \widetilde{G} = G/\{v_1, v_2\} \) for two different vertices \( v_1, v_2 \in V(G) \) (see Definition 2.1 and Figure 2); we specify the weight \( \widetilde{w} \) later. Recall that \( \widetilde{G} \) is obtained from \( G \) by contracting the vertices \( v_1 \) and \( v_2 \) to one vertex \( \widetilde{v}_0 = [v_1] = [v_2] = \{v_1, v_2\} \) while keeping all edges and vector potentials. Then the quotient map \( \kappa: G \to \widetilde{G} \) is a graph homomorphism and preserves the magnetic potential (see Example 2.16 (iii)). We also write \( \widetilde{G} = G/\{v_1, v_2\} \). We would like to stress that contracting two adjacent vertices \( v_1, v_2 \) turns any edge in \( E(v_1, v_2) \) into a loop in \( G/\{v_1, v_2\} \) (see also Remark 2.2 for further cases).
Theorem 4.6. Let $G, \tilde{G} \in \mathcal{G}$ with $\tilde{G} = G / \{v_1, v_2\}$.

(i) If $w_e \leq \bar{w}_e$ for all $e \in E(G)$ and $\bar{w}(v) \leq w(v)$ for all $v \in V(G) \setminus \{v_1, v_2\}$ and $\bar{w}(v_1) \leq w(v_1) + w(v_2)$, then $G \not\subseteq \tilde{G}$ and $\tilde{G} \not\subseteq G$ (and hence $G \not\ll \tilde{G} \not\ll G$) where

$$r = \min\{\deg^G(v_1), \deg^G(v_2)\} \quad \text{and} \quad s = |\{e \in E^G(v_1,v_2) | \alpha_e = 0\}| \quad (4.3)$$

is the minimal degree and $s$ the number of (unoriented) edges joining $v_1$ and $v_2$ having no magnetic potential. In particular, if $v_1, v_2$ are not adjacent, then $s = 0$.

(ii) If $w_e = \bar{w}_e$ for all $e \in E(G)$, and $\bar{w}(v) = w(v)$ for all $v \in V(G) \setminus \{v_1, v_2\}$ and $\bar{w}(v_1) = w(v_1) + w(v_2)$, then $G \not\subseteq \tilde{G}$ and $\tilde{G} \not\subseteq G$ (and hence $G \not\ll \tilde{G} \not\ll G$).

Proof. By our assumption and by definition of an MW-homomorphism, $\kappa : G \rightarrow \tilde{G}$ is an MW-homomorphism, i.e., $G \not\subseteq \tilde{G}$, and $G \not\ll \tilde{G}$ follows from Theorem 3.14.

Suppose $r = \deg(v_1)$ is the minimal degree of $v_1$ and $v_2$ and that $v_1, v_2$ are not adjacent ($E(v_1,v_2) = \emptyset$). We now prove $\tilde{G} \not\ll G$ by consecutively deleting the $r$ edges of $E^G_{v_1}$. Let $G' = (G', \alpha', w')$ with $G' = G - E^G_{v_1}$ and let $\alpha'$ and $w'$ be the restrictions of the corresponding magnetic potential and weights on $G$ onto $G'$. Similarly, we define $\tilde{G}'$ with underlying graph $\tilde{G}' = \tilde{G} - E^G_{v_1}$ (recall that the edge sets of $G$ and $\tilde{G}$ are the same). From Theorem 4.1 (iii) applied $r$ times and the transitivity in Lemma 3.9 (ii) we conclude $\tilde{G} \ll \tilde{G}'$. Moreover, from Theorem 4.1 (i) applied $r$ times and the transitivity in Lemma 3.9 (ii) we conclude $G' \ll \tilde{G'}$. Since the order of vertex contraction and edge deletion does not matter, we have $G - E^G_{v_1} = (G' - E^{G'}_{v_1}) / \{v_1,v_2\}$, so that $G'$ is $G'$ together with $v_1$ as an isolated vertex. In particular, the spectrum of $G'$ is just the one of $G'$ with an extra $0$, and therefore $\tilde{G} \ll G$. The result then follows again by transitivity.

If $v_1, v_2$ are adjacent, then each edge $e \in E(v_1,v_2)$ with $\alpha_e = 0$ turns into a loop in $G$, hence the spectral shift is $0$ for each such edge (Theorem 4.1 (iii)).

Here, the MW-homomorphism $\kappa : G \rightarrow \tilde{G}$ is measure preserving, hence we conclude $G \ll \tilde{G}$ and $\tilde{G} \ll G$ both from Theorem 3.14 with $r = |V(G)| - |V(\tilde{G})| = 1$.

Similarly, [CDH04, Theorem 2.7] (and again generalised to the case of signed graphs in [ATT14, Theorem 10]) prove a weaker version of our vertex contraction for the standard Laplacian, namely $G \ll \tilde{G} \ll G$ in our notation, under the additional assumption that the vertices $v_1, v_2$ have combinatorial distance at least 3. The latter restriction is mainly due to the fact that both papers avoid the use of multigraphs, namely multiple edges and loops.

As corollary we restrict the theorem to the case of combinatorial and standard weights. Note that our result improves in particular [CDH04, Theorem 2.7] (standard Laplacian) and [ATT14, Theorem 10] (signed standard Laplacians: in both articles, only $G \ll \tilde{G} \ll G$ is proven (in our notation) for vertices $v_1, v_2$ with combinatorial distance at least 3. Our corollary does not need this restriction and gives a better shift (in the standard case):

Corollary 4.7. Let $G, \tilde{G} \in \mathcal{G}$ with $\tilde{G} = G / \{v_1, v_2\}$.

(i) If $G, \tilde{G} \in \mathcal{H}$, then $G \not\subseteq \tilde{G}$ and $\tilde{G} \not\subseteq G$ (respectively, $G \not\ll \tilde{G} \not\ll G$) where $r$ and $s$ are defined in (4.3).
(ii) If $G, \tilde{G} \in \mathcal{G}_{\text{deg}}$, then $G \preceq \tilde{G}$ and $\tilde{G} \preceq G$, hence $G \preceq \tilde{G} \preceq G$.

Proof. The claim follows as the combinatorial weights fulfil the condition in Theorem 4.6 (i).

(ii) The standard weights fulfil the condition in Theorem 4.6 (ii) as $\deg^t([v_1]) = \deg^t(v_1) + \deg^t(v_2)$, in particular we have $G \preceq \tilde{G}$ and $G \preceq \tilde{G} \preceq G$.

Example 4.8. For $t \in [0, 2\pi]$, consider the MW-graph $G_t \in \mathcal{G}_{\text{deg}}$ with underlying graph $G$ as in Figure 3a. Since $G$ is a tree, we have $\sigma(G_t) = \sigma(G_0)$ and $\sigma(G_0)$ consists of six eigenvalues (dashed lines in Figure 3b). Let now $G_t = G_t/\{v_1, v_2\}$, see Figure 3b, we orient the edges in the cycle such that the flux adds up to $t$. Figure 3c shows the five eigenvalues of $\sigma(W_t^r)$ changing $t$ from 0 to $2\pi$. By Corollary 4.7 (ii) we have $G_t \preceq \tilde{G}_t \preceq G_t$ for any $t$, then $G_0 \preceq \tilde{G}_t \preceq G_0$. In particular, we can use the spectrum of the tree $G$ to localised the spectrum of $\tilde{G}$ for any vector potential, i.e.

$$
\lambda_1(\tilde{G}_t) \in \left[0, 1 - \frac{1}{\sqrt{2}}\right], \quad \lambda_2(\tilde{G}_t) \in \left[1 - \frac{1}{\sqrt{2}}, 1 - \frac{1}{\sqrt{6}}\right], \quad \lambda_3(\tilde{G}_t) \in \left[1 - \frac{1}{\sqrt{6}}, 1 + \frac{1}{\sqrt{6}}\right],
$$

$$
\lambda_4(\tilde{G}_t) \in \left[1 + \frac{1}{\sqrt{6}}, 1 + \frac{1}{\sqrt{2}}\right] \quad \text{and} \quad \lambda_5(\tilde{G}_t) \in \left[1 + \frac{1}{\sqrt{2}}, 2\right].
$$

In this example, the previous localisation of the spectrum in the bracketing intervals remains the same if we identify any other pair of distinct vertices, i.e. $G_0 \preceq \tilde{G}_t \preceq G_0$ where $\tilde{G}_t = G/\{u, v\}$ for any distinct vertices $u, v \in V(G)$.

4.2. Composite perturbations. The following perturbation of graphs are composite and can be obtained by the operations introduced in the preceding subsection.

4.2.1. Contracting an edge. Contracting an edge (not being a loop) is just the composition of the two operations: deleting an edge $e_0$ and contracting the adjacent vertices (note that the order of the perturbations does not matter). Formally, let $G$ be an MW-graph, an edge identification of $G$ is the MW-graph $G'$ where $G' = (G - e_0)/\{\partial_+ e_0, \partial_- e_0\} = G/\{\partial_+ e_0, \partial_- e_0\} - e_0$ for some edge $e_0 \in E(G)$ (see Example 2.16 (i) and Figure 4). We write this operation simply as $G' = G/\{e_0\}$ (again, we specify the weight later).

![Figure 4](image.png)

If we contract the edge $e_0$ of the graph $G$ in Fig. 4a, we obtain the graph $G' = G/\{e_0\}$ as in Fig. 4b. Let $t \in [0, 2\pi]$ and let $G_t, G'_t \in \mathcal{G}_{\text{deg}}$ be the corresponding magnetic weighted graphs, then $\sigma(G_t)$ (respectively, $\sigma(G'_t)$) are plotted in Fig. 4c as solid (respectively, dashed) lines for $t \in [0, 2\pi]$. Here, we have $G_t \preceq G'_t \preceq G_t$, and one can see this classical interlacing by the fact that the solid and dashed lines do not intersect, and solid and dashed lines alternate. Note that the horizontal eigenvalue (independent of $t$) is an eigenvalue for both graphs.

Theorem 4.9. Let $G, G' \in \mathcal{G}$ with $G' = G/\{e_0\}$, where $e_0 \in E(G)$ is not a loop and simple (i.e., $|E(\partial_- e_0, \partial_+ e_0)| = 1$).

(i) If $G, G' \in \mathcal{G}_t$, then $G \preceq G' \preceq G$ where $r = \min\{\deg(\partial_+ e_0), \deg(\partial_- e_0)\}$. If $\alpha_{e_0} = 0$ or if $e_0$ is a bridge edge, then $G \preceq G'$.

(ii) If $G, G' \in \mathcal{G}_{\text{deg}}$, then $G \preceq G' \preceq G$. If $\alpha_{e_0} = 0$ then $G \preceq G'$.

If $\alpha_{e_0} = \pi$ then $G \preceq G'$.

If $e_0$ is a bridge edge, then $G \preceq G'$. 

\end{document}
Proof. [4.4] Suppose that $G, G' \in \mathcal{G}_1$. Let $\tilde{G} = G/\{\partial_-, \partial_0, \partial_e\} \in \mathcal{G}_1$ be the graph with combinatorial weights obtained from $G$ by contracting the two vertices of $e_0$. Recall that $e_0$ becomes a loop (for simplicity also denoted by $e_0$) in $\tilde{G}$.

From Corollary 4.7 we obtain $G \preceq \tilde{G} \preceq G$. Next we delete the loop in $\tilde{G}$, so that $G' = \tilde{G} - e_0$; from Corollary 4.2 we obtain $\tilde{G} \preceq G' \preceq G$. Combining both arguments, we have

$$G \preceq \tilde{G} \preceq G' \preceq \tilde{G} \preceq G$$

and the transitivity in Lemma 3.9 gives us the result.

If $\alpha_{e_0} = 0$, then we use Corollary 4.4 and obtain $G \preceq G' \preceq G$ (and $G'$ are trivially isospectral); the same argument as above (now with $s = 1$ in Corollary 4.7) then gives $G \preceq G' \preceq G$. If $e_0$ is a bridge edge, then we can find an equivalent vector potential $\tilde{\alpha}$ with $\tilde{\alpha}_{e_0} = 0$ by Lemma 2.12. From Proposition 3.3 we conclude that $G$ and $\tilde{G} = (G, \tilde{\alpha}, w)$ are isospectral, hence the above argument with $G$ replaced by $\tilde{G}$ yields the result.

If $\alpha_{e_0} = \pi$, then Corollary 4.4 gives us $G' \preceq G$. If $\alpha_{e_0} = \pi$, we have $\tilde{G} \preceq G'$. If $e_0$ is a bridge edge, then we can find equivalent vectors potentials with value 0 or $\pi$ on $e_0$, and use the same argument as in the combinatorial case to conclude the better estimate $G \preceq G' \preceq G$.

Example 4.10. For any $t \in [0, 2\pi]$, consider the MW-graphs $G_t \in \mathcal{G}_1$ defined by the graph $G$ in Figure 4a, again, we orient the edges along the closed path such that the flux through it adds up to $t$. Then $\sigma(G_t)$ consists of six eigenvalues that depend on the value of $t$. The spectrum $\sigma(G_t)$ is plotted as a solid line in Figure 4c for all $t \in [0, 2\pi]$. If we consider $G' \in \mathcal{G}_1$ given by $G'_t = G/\{e_0\}$, i.e. $G'$ is defined by the graph $G' = G/\{e_0\}$ in Figure 5. The spectrum $\sigma(G'_t)$ consists of five eigenvalues (dashed lines in Figure 4c). Since $e_0$ is a bridge edge, we see graphically the interlacing given by Theorem 4.9 (the solid and dashed line alternate); i.e. $G_t \preceq G'_t \preceq G_t$ for any $t \in [0, 2\pi]$.

4.2.2. Contracting a pendant edge. We now consider a special case of contraction a pedant edge (see Figure 5). In particular, a pendant edge is always a bridge edge. In particular, Theorem 4.9 gives us (now with $r = 1$):

**Corollary 4.11.** Let $G, G' \in \mathcal{G}$ where $G' = G/\{e_0\}$ and $e_0$ is a pendant edge.

(i) If $G, G' \in \mathcal{G}_1$, then $G \preceq G' \preceq G$.

(ii) If $G, G' \in \mathcal{G}_\text{deg}$, then $G \preceq G' \preceq G$.

**Example 4.12.** For all $t \in [0, 2\pi]$, consider the MW-graph $G_t \in \mathcal{G}_1$ defined by the graph $G$ in Figure 5a and choose the orientation of the edges along the cycle such that the flux through it adds up to $t$. Define $G'_t \in \mathcal{G}_1$ where $G'_t = G/\{e_0\}$, i.e. $G'_t$ is defined by the graph in Figure 5b. For each $t$, we have $\sigma(G_t)$ (respectively, $\sigma(G'_t)$) consists of five (respectively, four)

![Figure 5](image-url)

**Figure 5.** The graph $G$ with $e_0$ a pendant edge (Fig. 5a). If we make the edge contraction of $e_0$, we obtain the graph $G' = G/\{e_0\}$ in Fig. 5b. For any $t \in [0, 2\pi]$, consider $G_t, G'_t \in \mathcal{G}_1$ defined by $G$ (respectively, $G'$). In Fig. 5c we plot $\sigma(G_t)$ (respectively, $\sigma(G'_t)$) in solid (respectively, dashed) line for all $t \in [0, 2\pi]$. 
eigenvalues plotted as solid (respectively, dashed) lines for all $t \in [0, 2\pi]$ in Figure 5. From Corollary 4.11 we conclude $G_t \preceq G_{t+1} \preceq G_t$. This interlacing property for the combinatorial weights is only true for pendant edges and not for general bridge edges.

4.2.3. **Deleting a vertex.** Let $G$ be a graph and $v_0 \in V(G)$. For simplicity, we assume that there is no loop at $v_0$. The graph $G' = G - v_0$ is obtained from $G$ by deleting the vertex $v_0$ and all its adjacent edges $e \in E_{v_0}$, i.e. $V(G') = V(G) \setminus \{v_0\}$, $E' := E(G') = E(G) \setminus E_{v_0}$ and $\partial \alpha' = \partial |_{E'}$ (see [BM08, Section 2.1]). We say that the MW-graph $G' = (G', \alpha', w')$ is obtained from $G = (G, \alpha, w)$ by deleting the vertex $v_0$ (denoted by $G' = G - v_0$) if $G' = G - v_0$ and if $\alpha' = \alpha |_{E'}$; we specify the weight $w'$ in the following cases:

**Corollary 4.13.** Let $G, G' \in \mathcal{G}$ where $G' = G - v_0$. If $r = \deg(v_0)$, then

(i) Let $G, G' \in \mathcal{G}_1$, then $G \overset{r-1}{\preceq} G' \overset{1}{\preceq} G$.

(ii) Let $G, G' \in \mathcal{G}_{\deg}$, then $G \overset{r-1}{\preceq} G' \overset{1}{\preceq} G$.

**Proof.** First delete $r - 1$ edges adjacent to $v_0$ and apply Corollary 4.2 in each case, i.e. for combinatorial and standard weights. Finally, delete the pendant edge and apply the corresponding parts of Corollary 4.11. \qed

5. **Applications**

In this final section, we present a large variety of applications of the preorder relations of MW-graphs studied before. In particular, we apply our results to study certain combinatorial aspects of graphs, to prove how Cheeger’s constant change under MW-homomorphisms and to study the stability of eigenvalues under perturbation of graphs with high multiplicity; we will also identify spectral gaps in the spectrum of Laplacians on infinite covering graphs. Our results are also useful in order to show the monotonicity of certain combinatorial numbers like, e.g. the algebraic connectivity of a graph under elementary perturbations.

5.1. **Spectral graph theory and combinatorics.** In this subsection, we assume that all graphs are finite.

5.1.1. **Spectral order of graphs.** We begin mentioning some natural interaction between the preorder relations $\preceq$ and $\subseteq$ mentioned before and combinatorics. Recall that the spectral preorder $G \preceq G'$ means that the increasingly ordered list of eigenvalues $\lambda_k$ and $\lambda'_k$ of $G$ and $G'$, respectively (repeated according to their multiplicity) fulfill $\lambda_k \leq \lambda'_k$ for all indices $k$; see Definition 2.10. Moreover, the geometric (pre)order $G \subseteq G'$ means that there is an MW-homomorphism $\tau: G \to G'$, i.e., a graph homomorphism respecting the magnetic potential and fulfilling certain inequalities on the vertex and edge weights, see Definition 2.15.

First, we apply the geometric perturbation and elementary operations on graphs established in Section 4 to present a new spectral order of MW-graphs. We illustrate the method for some simple graphs up to order 6 with combinatorial weights: We have seen in Proposition 2.21 that for any fixed value $t \in [0, 2\pi]$ the family $\mathcal{G}_t$ (see Definition 2.14) is partially ordered with respect to $\subseteq$. In particular, the spectral relations below include the cases of the combinatorial Laplacian (if $t = 0$) and the signless Laplacian (if $t = \pi$). In Figure 6 we specify the spectral relations of a chain of simple graphs up to order 6. Note first that $G_i \subseteq G_{i+1}$ for $1 \leq i \leq 7$ is a consequence of Corollary 4.2(ii) and the fact any two consecutive graphs from $G_1, \ldots, G_8$ differ by an edge. Moreover, $G_8 \subseteq G_9$ follows from Corollary 4.7(ii) since the graph $G_9$ is obtained from $G_8$ by contracting the upper right vertex with the lower right vertex. Recall also that by Theorem 3.14 we directly obtain also the relation $G_i \preceq G_{i+1}$, $1 \leq i \leq 8$. Finally, note that $G_9 \nsubseteq G_{10}$ is false by Proposition 2.17 since an MW-homomorphism $G_9$ and $G_{10}$ is injective on the edges. Corollary 4.11 gives the relation $G_9 \preceq G_{10}$ because both graphs differ by a pendant edge.

![Figure 6. Example of the preorders relations in $\mathcal{G}_t$ for simple graphs up to order $n = 6$. One has $G_i \preceq G_{i+1}$, $1 \leq i \leq 9$ hence $G_i \preceq G_{i+1}$, $1 \leq i \leq 8$. Moreover, $G_9 \preceq G_{10}$ but $G_9 \nsubseteq G_{10}$ showing that the geometric preorder is stronger than the spectral preorder.](image-url)

A **spanning** subgraph of a graph $G$ is a subgraph $G - E_0$ obtained from $G$ by deleting all edges in $E_0$ where $E_0 \subset E(G)$. Note that a spanning subgraph has the same set of vertices of the original graph. A **spanning tree** of $G$ is a spanning subgraph which is a tree. For example $G_6$ is a spanning tree for $G_8$. We have the following simple consequence of Corollary 4.2(ii) and our definition of spectral preorder $\preceq$ which generalises a result of Fiedler [Pic73 Corollary 3.2], see also [Moh91, Section 3]:

- **Case 1:** $G_i \preceq G_{i+1}$, $1 \leq i \leq 8$.
- **Case 2:** $G_9 \preceq G_{10}$.
- **Case 3:** $G_9 \nsubseteq G_{10}$.
Corollary 5.1. Let \( G' = G - E_0 \in \mathcal{F}_d \) be a spanning subgraph of \( G \in \mathcal{F}_d \), then \( \lambda_k(G') \leq \lambda_k(G) \) for all \( k \in \{1, \ldots, |V(G)|\} \).

5.1.2. Cliques and stability of eigenvalues. Let \( G = (V, E, \theta) \) be a graph and \( d \in \mathbb{N} \). A \( d \)-clique of \( G \) is an induced subgraph \( G[V_0] \) (\( V_0 \subset V \)) isomorphic to the complete graph \( K_d \) of order \( d = |V_0| \) (see Definition 2.5). A \( d \)-clique is maximal if it is not a subgraph of a \((d+1)\)-clique of \( G \). The clique number of \( G \) is the maximal \( d \) such that \( G \) has a \( d \)-clique. The notion of a clique can be naturally extended to MW-graphs by restricting the weights and vector potential to the corresponding substructures. For simplicity, we will denote the \( d \)-clique by \( K_d \). All graphs here are assumed to have the combinatorial weight.

In the next theorem, we will apply the geometric and spectral preorder relations given in Definitions 2.20 and 3.10 to identify the eigenvalue \( d \) in the spectrum of the Laplacian of the graph with a \( d \)-clique and to give a lower bound of its multiplicity. Roughly speaking, the clique number \( d \) of a graph can be seen in its (combinatorial) spectrum for graphs with number of edges in a certain range depending on \( d \), see Eq. (5.1).

Theorem 5.2. Let \( G \) be a connected graph with combinatorial weights having \( m \) edges and a maximal \( d \)-clique. Assume that \( m < (d-1)(d+2)/2 \) then \( d \) is in the spectrum of \( G \) with multiplicity at least

\[
\frac{(d-1)(d+2)}{2} - m = d - r - 1,
\]

where \( r = m - d(d-1)/2 \) is the number of edges of \( G \) not in the clique.

Remark 5.3.

(i) Theorem 5.2 applies to graphs with a maximal \( d \)-clique and number of edges \( m \) fulfilling

\[
m \in \left\{ \frac{d(d-1)}{2}, \ldots, \frac{(d+2)(d-1)}{2} - 1 \right\}.
\]

Let us call such numbers of edges \( d \)-admissible. For a given \( d \), the above list of \( d \)-admissible numbers of edges has \( d-1 \) entries; for each one the multiplicity of the eigenvalue \( d \) is fixed, independently of the number \( n \) of vertices of \( G \). Nevertheless since \( G \) has a \( d \)-clique, we have \( n \geq d \) and since \( G \) is connected, we have \( n \leq d+r = m - d(d-3)/2 \).

(ii) Note that we also allow multiple edges here. For example if \( G \) is the complete graph with three vertices and one double edge (hence \( m = 4 \)), then its spectrum is \((0, 3, 5)\). This graph has a 3-clique by deleting one of the double edges, and \( d = 3 \) has multiplicity (here exactly) 1, as predicted by the theorem. Similarly, if \( G = K_4 \), i.e. \( d = 1 \), the extra double edge, then the spectrum is \((0, 4, 4, 6)\) and \( d = 4 \) is a double eigenvalue, again as predicted.

(iii) In the proof of Theorem 5.2, we do not need that the \( d \)-clique is maximal, but considering \( G \) as a graph with a \((d-1)\)-clique then the range of \((d-1)\)-admissible numbers of edges is disjoint from the range of \( d \)-admissible numbers of edges. In particular, the theorem only makes sense for maximal \( d \)-cliques.

In [Moh92], Mohar excludes certain cycles as subgraphs by just looking at the spectrum of the graph. Here, we can exclude \( d \)-cliques spectrally:

Corollary 5.4. Assume that \( G \) is a connected graph. If \( d \in \mathbb{N} \) is not in the spectrum of \( G \), and if \( G \) has less than \((d-1)(d+2)/2 \) edges, then \( G \) has no \( d \)-clique.

Remark.

(i) If the number of edges \( m \) is below \( d(d-1)/2 \), then obviously \( K_d \) cannot be a subgraph, the other values of \( m \) are admissible.

(ii) The converse of the above corollary is false (or the conclusion of Theorem 5.2 can be true also for graphs without a \( d \)-clique): the Petersen graph has (combinatorial Laplace) spectrum \((0, 2_5, 5_4)\) (the subscript indicating the multiplicity), hence \( d = 5 \) is in its spectrum with multiplicity 4 as said in Theorem 5.2. Also, the number of edges \((m = 10)\) is 5-admissible, see Eq. (5.1). But the clique number of the Petersen graph is 2 (and not 5).

Proof of Theorem 5.2. The strategy of the proof is to delete suitable edges on the complement of the maximal clique and control the spectral shifts \( s, t \) so that we finally obtain relations \( G \preccurlyeq K_d \preccurlyeq G \). Then we exploit the fact that for combinatorial weights the eigenvalue \( d \in \sigma(K_d) \) has high multiplicity, namely multiplicity \( d - 1 \).

Let \( G = (G, 0, 1) \) and set \( n = |V(G)| \). The first Betti numbers of \( G \) and \( K_d \) are \( b_1(G) := m - n + 1 \) and \( b_1(K_d) := d(d-1)/2 - d + 1 \) (see e.g. [Sim12] Section 4)). Denote its difference by

\[
p := b_1(G) - b_1(K_d) = m - n - \frac{d(d-3)}{2}.
\]

Let \( \widetilde{E} := E(G) \setminus E(K_d) \). We delete \( p \) edges \( \widetilde{E} \) from \( \widetilde{E} \) in such a way that no cycles are present in the complement of the clique. Construct first a subgraph \( G_1 = G - \widetilde{E} = (V_1, E_1, \theta) \) with \( V_1 = V(G) \) and \( E_1 = E(K_d) \cup (\widetilde{E} \setminus \widetilde{E}) \). Applying iteratively Corollary 4.2 [4], we obtain

\[
G \preccurlyeq G_1 \preccurlyeq G.
\]
Note that the graph $G_1 = E(K_d)$ is a forest. Next, we delete all $n - d$ edges of this forest starting from the leaves and proceeding towards the $d$-clique $K_d$. (Note that it is important to delete only leaves in this process, as otherwise the spectral shift is not optimal.) Then, applying iteratively Corollary 4.11, we obtain

\[ G_1 \preceq K_d \preceq G_1. \]

Using the transitivity of the relation $\preceq$ as well as Eq. (5.2) we get finally the relations

\[ G \preceq K_d \preceq G \]  

(5.3)

Let $\lambda_1, \ldots, \lambda_n$ resp. $\mu_1, \ldots, \mu_d$ be the spectrum of (the combinatorial Laplacians of) $G$ resp. $K_d$; as usual written in ascending order and repeated according to their multiplicities. For the complete graph of order $d$ we have $\mu_1 = 0$ and $\mu_k = d$ ($k = 2, \ldots, d$). The relations in Eq. (5.3) imply

\[ \lambda_k \leq \mu_{k+p} \quad \text{for} \quad 1 \leq k \leq d-p \quad \text{and} \quad \mu_k \leq \lambda_{k+(n-d)} \quad \text{for} \quad 1 \leq k \leq d. \]

Note that the relations between the orders of the graphs and the spectral shift needed in Definition 3.10 are automatically satisfied in our case since $|E(K_d)| = d(d-1)/2 \leq m$. Combining the preceding inequalities, we obtain

\[ d = \mu_k \leq \lambda_{k+n-d} \leq \lambda_{k+(n-d)+p} = d \quad \text{for} \quad 2 \leq k \leq 2d-n-p, \]

hence $d$ is an eigenvalue of $G$ with multiplicity given by

\[ 2d - n - p - 1 = \frac{d(d+1)}{2} - m - 1 = \frac{(d-1)(d+2)}{2} - m \]

which completes the proof.

\[ \square \]

Example 5.5. We illustrate the preceding theorem with some examples having combinatorial weights, no magnetic potential and a maximal 6-clique as shown in Figure 7. Concretely, the graphs $G_1$, $G_2$ and $G_3$ all have $m = 17$ edges and a maximal 6-clique $K_6$; moreover $G_1$ and $G_2$ have 8 vertices, while $G_3$ has 7 vertices. All three graphs have $d = 6$ as eigenvalue in its spectrum with multiplicity at least 3. The range of admissible number of edges for $d = 6$ here is $m \in \{15, 16, 17, 18, 19\}$. The (minimal) multiplicity of the eigenvalue $d = 6$ is then 5, 4, 3, 2, 1. In Figure 7 we have $m = 17$ and minimal multiplicity 3.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{example_graphs}
\caption{An illustration of the Theorem 5.2 for 6-clique. Let $G_1$, $G_2$ and $G_3$ be the MW-graphs (with combinatorial weight and no magnetic potential) defined by the graphs $G_1$, $G_2$ and $G_3$ respectively. All graphs have $d = 6$ in its spectrum with multiplicity at least 3.}
\end{figure}

5.1.3. Minors. A fundamental notion in combinatorics is that of a graph minor. Several fundamental results in this field are presented in terms of minors (e.g. in the Robertson-Seymour theory [Die00, Chapter 12]). A graph $H$ is called a minor of a given graph $G$ if $H$ is obtained from $G$ by applying certain elementary operations. We can generalise this construction to MW-graphs and apply the results of the previous sections to give a spectral relation between a graph and its minor. We consider the following three elementary operations:

- Deleting an edge (Subsection 4.1.1),
- Contracting an edge (Subsection 4.2.1),
- Deleting pendant vertex (the same as contracting a pendant edge, Subsection 4.2.2).

If $G' \in \mathcal{G}$ is obtained from $G$ by successive application of the previous operations, then we say that $G'$ is a minor of $G$ (see, e.g. [BM08]).

Proposition 5.6. Let $G \in \mathcal{G}$ be a simple graph without magnetic potential and let $G'$ be a minor of $G$ obtained by deleting $p$ edges, contracting $q$ edges and deleting $s$ pendant vertices.

(i) If $G, G' \in \mathcal{G}_1$, then $G^p \preceq G' \preceq G$ where $r = \sum_{e \in E_0} \min\{\deg(\partial_+ e_0), \deg(\partial_- e_0)\} (\geq q)$ and $E_0$ is the set of $q$ edges, that are contracted.

(ii) If $G, G' \in \mathcal{G}_{\text{deg}}$, then $G^{p+q-q'} \preceq G' \preceq G$ where $q'$ is the number of bridge edges that are contracted.
A simple consequence of the spectral preorder is the following:

**Corollary 5.7.** Let \( G \in \mathcal{G}_1 \). If \( G' \in \mathcal{G}_1 \) is a minor of \( G \) obtained as in Proposition 5.6, then an eigenvalue of \( G \) of multiplicity \( m \geq p + r + s + 1 \) remains and eigenvalue of \( G' \) of multiplicity \( m - p - r - s \).

Similarly, for \( G, G' \in \mathcal{G}_{\text{deg}} \), an eigenvalue of \( G \) of multiplicity \( m \geq 2p + 2q + s - q' + 1 \) is an eigenvalue of \( G' \) of multiplicity \( m - 2p - 2q - s + q' \).

### 5.2. Cheeger constants and frustration index under MW-homomorphisms.

The Cheeger constant is a quantitative measure of the connectedness of a graph. It can be used in a lower bound on the second (first non-zero) eigenvalue of a graph, usually called Cheeger inequality (and also in an upper bound). Probably the first occurrence of a lower bound on the second eigenvalue in terms of geometric quantities is in [Fe73, Section 4.3]; Dodziuk [Dod84, Theorem 2.3] proves a Cheeger inequality for the combinatorial Laplacian, see also Mohar [Moh91, Section 6] for further improvements and references, as well as Colin de Verdière [CdV98, Theorem 3.1]. A Cheeger inequality for the standard Laplacian can be found in Chung’s book [Chu97, Chapter 2]. An extension of the Cheeger constant to magnetic potentials need the so-called frustration index. For a more detailed overview on the literature concerning Cheeger constants and the frustration index on graphs and manifolds we refer to [LLPP15] and references therein.

We now show that the concept of MW-homomorphisms also gives simple inequalities for Cheeger constants. Here, we always denote the underlying graphs of \( G \) and \( G' \) by \( G = (V, E, \partial) \) and \( G' = (V', E', \partial') \).

We first define an ingredient necessary in the presence of a magnetic potential:

**Definition 5.8 (Frustration index).** Let \( G = (G, \alpha, w) \) be an MW-graph and consider a function \( \tau : V \to R \), where \( R \) is a subgroup of \( \mathbb{R}/2\pi\mathbb{Z} \). We set

\[
\iota(G, \tau) := \|d_{\alpha}(e^{i\tau})\|_{\ell_1(E, w)} = \sum_{e \in E} w_e|e^{i\arg(\partial_+ e)} - e^{-i\arg(\partial_- e)}|.
\]

The frustration index of \( G \) is defined as

\[
\iota(G) := \inf_{\tau \in R^V} \iota(G, \tau),
\]  

(5.4)

where \( R^V \) denotes the set of all maps \( \tau : V \to R \).

Note that the infimum is actually a minimum. It is not hard to see that \( \iota(G) = 0 \) if and only if \( \alpha \sim 0 \), i.e. if the magnetic potential is cohomologous to 0. An MW-homomorphism gives a natural inequality for the frustration indices:

**Lemma 5.9.** Let \( \pi : G \to G' \) be an MW-homomorphism and let \( \tau' : V' \to R \) a map, then

\[
\iota(G, \tau' \circ \pi) \leq \iota(G', \tau') \quad \text{and} \quad \iota(G) \leq \iota(G').
\]

**Proof.** Note first that \( d_{\alpha}(e^{i\tau' \circ \pi}) = (d_{\alpha'}(e^{i\tau'})) \circ \pi \) as \( \pi \) is a graph homomorphism and \( \alpha' \circ \pi = \alpha \). Moreover, we have

\[
\iota(G, \tau' \circ \pi) = \sum_{e \in E} w_e|d_{\alpha}(e^{i\tau' \circ \pi})_e| = \sum_{e' \in E'} \sum_{e \in E, \pi(e) = e'} w_{e'}|d_{\alpha'}(e^{i\tau'})_e'| = \sum_{e' \in E'} (\pi_* w)_{e'} \cdot |(d_{\alpha'}(e^{i\tau'}))_e'|
\]

\[
\leq \sum_{e' \in E'} w_{e'}|d_{\alpha'}(e^{i\tau'})_e'| = \iota(G', \tau'),
\]

as \( \pi \) is an MW-homomorphism (and in particular, \( (\pi_* w)_{e'} \leq w_{e'} \) for all \( e' \in E' \)). For the last inequality in the lemma, note that the set \( R^V \) of maps \( \tau : V \to R \) is larger than the subset \( \{ \tau' \circ \pi \mid \tau' \in R^{V'} \} \subset R^V \), hence we have

\[
\iota(G) \leq \inf_{\tau' \in R^{V'}} \iota(G, \tau' \circ \pi) \leq \inf_{\tau' \in R^{V'}} \iota(G', \tau') = \iota(G').
\]

We denote by \( G[V_0] \) the induced subgraph \( G[V_0] \) with vertex set \( V_0 \subset V \) and edge set \( E(V_0) \) (see Definition 2.5) together with the natural restrictions of \( w \) and \( \alpha \) to \( V_0 \) respectively \( E(V_0) \). A \( k \)-subpartition of \( V \) is given by \( k \) pairwise disjoint non-empty subsets \( V_1, \ldots, V_k \) of \( V \); the set of all \( k \)-subpartitions \( \Pi = \{ V_1, \ldots, V_k \} \) of \( V \) is denoted by \( \Pi_k(V) \).

**Definition 5.10.** Let \( G \) be an MW-graph. For a subset \( V_0 \subset V \) we set

\[
h(G, V_0) := \frac{\iota(G[V_0]) + w(E(V_0, V_0^c))}{w(V_0)}.
\]

The \( k \)-th (also called \( k \)-way) (magnetic weighted) Cheeger constant \( h_k(G) \) is defined as

\[
h_k(G) := \inf_{\Pi \in \Pi_k(V)} \sup_{V_0 \in \Pi} h(G, V_0).
\]

(5.5)

Note that the infimum and supremum are actually minimum and maximum. It is not hard to see that \( h_k(G) \leq h_{k+1}(G) \). Moreover, for \( k = 1 \) resp. \( k = 2 \) we have

\[
h_1(G) = \min_{V_0 \subset V, V_0 \neq \emptyset} h(G, V_0) \quad \text{resp.} \quad h_2(G) = \min_{V_0 \subset V, V_0 \neq \emptyset, V_0 \neq \emptyset} \max\{h(G, V_0), h(G, V_0^c)\}
\]
for the first and second Cheeger constant (the latter is usually called the Cheeger constant). If \( \alpha \sim 0 \) then the second (usual) Cheeger constant equals:

\[
h_2(G) = \min_{V_0 \subset V, V_0 \neq \emptyset, V_0 \neq V} \frac{w(E(V_0, V_0^c))}{\min\{w(V_0), w(V_0^c)\}}.
\]

**Lemma 5.11.** Let \( \pi: G \rightarrow G' \) be an MW-homomorphism, then

\[
h(G, \pi^{-1}(V_0')) \leq h(G', V_0')
\]

for all \( V_0' \subset V' \), \( V_0' \neq \emptyset \).

**Proof.** If \( \pi: G \rightarrow G' \) is an MW-homomorphism, then the restriction \( \pi: G[\pi^{-1}(V_0')] \rightarrow G'[V_0'] \) is defined as map (as \( \pi(\pi^{-1}(V_0')) \subset V_0' \) and \( \pi(E(G(\pi^{-1}(V_0')))) \subset E(G')(V_0') \)) by Lemma 2.8, and again an MW-homomorphism. In particular, from Lemma 5.9 we conclude that

\[
\iota(G[\pi^{-1}(V_0')]) \leq \iota(G'[V_0']).
\]

Next, we have \( E(G(\pi^{-1}(V_0'), (\pi^{-1}(V_0'))^c) = \pi^{-1}(E(G'(V_0', (V_0')^c)) \) again by Lemma 2.8 and as \( (\pi^{-1}(V_0'))^c = \pi^{-1}((V_0')^c) \). In particular, we conclude

\[
w(E(G(\pi^{-1}(V_0'), (\pi^{-1}(V_0'))^c)) = (\pi_* w)(E(G'(V_0', (V_0')^c)) \leq w'(E(G'(V_0', (V_0')^c))
\]

as \( \pi \) is an MW-homomorphism. Similarly, we have

\[
w(\pi^{-1}(V_0')) = (\pi_* w)(V_0') \geq w'(V_0'),
\]

and the desired inequality follows. \( \square \)

We are now able to prove the main result of this section:

**Theorem 5.12.** Let \( G \) and \( G' \) be two MW-graphs such that \( G \subset G' \), i.e. there is an MW-homomorphism \( \pi: G \rightarrow G' \) (see Definition 2.15), then we have

\[
h_k(G) \leq h_k(G')
\]

for all \( k \in \mathbb{N} \).

**Proof.** Let the minimum in \( h_k(G') \) be achieved at \( \Pi' = \{V_1', \ldots, V_k'\} \in \Pi_k(V') \), and the maximum at \( V_j' \), i.e. assume that \( h_k(G') = h(G', V_j') \). As \( \pi \) is surjective on the vertices (see Proposition 2.17 (i)), \( \Pi := \{\pi^{-1}(V_1'), \ldots, \pi^{-1}(V_k')\} \) is again a \( k \)-subpartition (all sets are pairwise disjoint and non-empty by the surjectivity). Now we have

\[
h_k(G) \leq \sup_{j=1,\ldots,k} h(G, \pi^{-1}(V_j')) \leq \sup_{j=1,\ldots,k} h(G', (V_j')) = h(G', V_j') = h_k(G')
\]

as \( h_k(G) \) is the infimum over all \( k \)-subpartitions, and \( \Pi \) is such a \( k \)-partition of \( V \) (first inequality). The second inequality follows from Lemma 5.11 and the last equality from the choice of the partition \( \Pi' \) and \( V_j' \). \( \square \)

**Remark 5.13.** Note that we have proven in Theorem 3.14 the inequality \( \lambda_k(G) \leq \lambda_k(G') \) if there is an MW-homomorphism \( \pi: G \rightarrow G' \). We have just proven in Theorem 5.12 that an MW-homomorphism also increases the \( k \)-th Cheeger constant, hence Theorem 5.12 is in accordance with the (magnetic weighted) Cheeger inequalities

\[
\frac{1}{2} \lambda_k(G) \leq h_k(G) \leq C k^3 \sqrt{\rho_\infty \lambda_k(G)}
\]

for all \( k \in \{1, \ldots, |G|\} \) proven in [LLPP15], where \( C > 0 \) is a universal constant (recall that \( \rho_\infty \) is the supremum of the relative weight, see (2.5)). If \( k = 1 \), then \( C = 1 \), and if \( k = 2 \) and if \( \alpha \sim 0 \), then one can choose \( C = \sqrt{2}/4 \).

**Example 5.14.** Let \( G = (G, \alpha, w) \) and \( G' = (G', w', \alpha') \) be two MW-graphs.

- **Combinatorial weight and removing edges:** If \( E_0 \subset E(G) \) and \( G, G' = G - E_0 \in G_1 \), then \( h_k(G - E_0) \leq h_k(G) \). Heuristically, this means that removing edges decreases the connectivity.
- **Standard and combinatorial weight compared:** If \( G = G' \), \( w = \deg \) and \( w' = 1 \), then \( h_k(G) \leq h_k(G') \) (the combinatorial weight has higher Cheeger constants).
- **Standard weight and contracting vertices:** If \( \sim \) is an equivalence relation on \( V(G) \), and if \( G, G' = G/\sim \in G_{\deg} \), then \( h_k(G) \leq h_k(G/\sim) \). Heuristically, this means that contracting vertices increases the connectivity.
5.3. Covering graphs and spectral gaps. In \cite{FCLP18} and \cite{LP08} we study the spectrum of the discrete Laplacian of infinite (regular) covering graphs $G \to G$ with finite quotient. For the case with a periodic vector potential see \cite{FCL19}. In Section 5 of \cite{FCLP18} we consider only Abelian covering and interpret the magnetic potential as a Floquet parameter to decompose the Laplacian $\Delta^G$ as a direct integral of discrete magnetic weighted Laplacians $\Delta^G_\alpha$ on the finite quotient. We developed a technique of virtualising specific edges and vertices on the quotient $G$ for the new graph (cf. Definition 3.4 in \cite{FCLP18}). The virtualisation of vertices $V$ to decompose the Laplacian $\Delta^G$ gives a new partial MW-graph $G, \alpha, w$ defined as $V^+ = G \setminus V_0$, $E^+ = E \setminus E(V_0)$, $w^+_v = w_v$ for all $v \in V^+$, $w^+_e = w_e$ and $\alpha^+_e = \alpha_e$ for all $e \in E^+$ (see Definition 3.9 in \cite{FCLP18} for details and additional motivation).

Denote the spectrum of $G^\pm$ by $\{\lambda_k(G^\pm)\}_k$ (written as usual in ascending order and counting multiplicities). Our techniques allow to localise the spectrum of the covering graph by

\[
\sigma(\Delta^G) \subset \bigcup_{k=1}^{\lfloor G \rfloor} [\lambda_k(G^-), \lambda_k(G^+)]
\]

where we denote by $J_k := [\lambda_k(G^-), \lambda_k(G^+)]$ the bracketing intervals. The elementary operations described in Section 4 can now be applied to refine the spectral localisation given in Eq. (5.7) and, in particular, one can discover new spectral gaps that can not be found with the method described in \cite{FCLP18, FCL19}.

We illustrate this in one specific example of covering graph but, there are many ways how to refine the spectral localisation using the methods described in Section 4. Consider the infinite $Z$-covering graph $\tilde{G}$ given in Figure 8c with standard weights.

\[J_1 \approx [0, 0.121], \quad J_2 \approx [0.116, 0.358], \quad J_3 \approx [0.5, 0.744], \quad J_4 \approx [0.713, 1.256], \]
\[J_5 \approx [1.145, 1.642], \quad J_6 \approx [1.638, 1.879], \quad \text{and} \quad J_7 \approx [1.889, 2].\]
using also the symmetry of the spectrum from bipartiteness. Note that $J_2 \cap J_3 = J_6 \cap J_7 = \emptyset$, so that we have localised two spectral gaps in $\sigma(\Delta \tilde{G})$. We can now refine the localisation of the spectrum as follows. Consider the graphs $G$ and $G'$ as in Figures 8a and 8b, where $G' = G/\{v_1, v_8\}$. Applying Theorem 4.6 we obtain the following spectral relations

$$G_0' = G_i' \preceq G_t \preceq G_i' = G_0'.$$

where for $G_i'$ we can take $t = 0$ since $G'$ is a tree. Therefore we have the alternative localisation:

$$\sigma(\Delta \tilde{G}) \subset \bigcup_{k=1}^{7} J_i'',$$

where

$$J_i' \approx [0, 0.108), \quad J_2' \approx [0.108, 0.463), \quad J_3' \approx [0.463, 1), \quad J_4' = \{1\},$$

$$J_5' \approx [1, 1.536), \quad J_6' \approx [1.536, 1.891], \quad \text{and} \quad J_7' \approx [1.891, 2].$$

Intersection both localisations $J$ and $J'$ we obtain a finer bracketing:

$$\sigma(\Delta \tilde{G}) \subset \bigcup_{k=1}^{7} J \cap J' = \bigcup_{k=1}^{7} J''',$$

where

$$J_i'' \approx [0, 0.108), \quad J_2'' \approx [0.116, 0.358], \quad J_3'' \approx [0.5, 0.744], \quad J_4'' = \{1\},$$

$$J_5'' \approx [1.145, 1.536], \quad J_6'' \approx [1.638, 1.879], \quad \text{and} \quad J_7'' \approx [1.91, 2].$$

Note that $J_k''$ are better than $J_k$ for $k \in \{1, 4, 5, 7\}$. Using this refinement obtained by applying a vertex splitting we are able to determine that $1 \in \sigma(G)$ with spectral gaps around it. We also found new spectral gaps between all bands, while in [FCLP18], we only found two bands.

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