AN OPTIMAL SUPPORT FUNCTION RELATED TO THE 
STRONG OPENNESS CONJECTURE 

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Abstract. In the present article, we obtain an optimal support function of weighted $L^2$ integrations on superlevel sets of psh weights, which implies the strong openness property of multiplier ideal sheaves.

1. Introduction

The notion of multiplier ideal sheaves plays an important role in complex geometry and algebraic geometry, which has been widely discussed (see e.g. [31, 23, 26, 3, 4, 8, 5, 22, 28, 29, 6]). Let $D$ be a domain in $\mathbb{C}^n$, and let $\psi$ be a negative plurisubharmonic function (see [7, 24, 25]). It is known that $I(\psi)$ is a coherent analytic sheaf (see [23] and [4]). In [6] (see also [8]), Demailly posed the strong openness conjecture for multiplier ideal sheaves: $I(\psi) = I_+(\psi) := \cup_{\epsilon > 0} I((1 + \epsilon)\psi)$. When $I(\psi) = \mathcal{O}$, the strong openness conjecture was called the openness conjecture. In [2], Berndtsson proved the openness conjecture (Favre-Jonsson proved 2-dim in [9]). In [16] (see also [21] and [19]), Guan-Zhou proved the strong openness conjecture, which is called the strong openness property (Jonsson-Mustata proved 2-dim in [20]). After that, Guan-Zhou [15] gave a characterization of the multiplier ideal sheaves with weights of Lelong number one by using the strong openness property of multiplier ideal sheaves. Recently, Xu [32] completed the algebraic approach to the openness conjecture, which was conjectured by Jonsson-Mustata [20].

Based on some modifications of the method in [17] (see also [13] and [14]), a support function of weighted $L^2$ integrations on the superlevel sets of the weight was obtained in [18], which implies the strong openness property. Then it is natural to ask:

Question 1.1. Can one obtain an optimal version of the above support function?

Let $z_0$ be a point in a pseudoconvex domain $D \subset \mathbb{C}^n$, and let $F$ be a holomorphic function on $D$. Let $D_t = \{ z \in D : \psi \geq -t \}$ the superlevel set of weight $\psi$, and let $C_{F,\psi,t}(z_0)$ be the infimum of $\int_{D_t} |F|^2$ for all $F \in \mathcal{O}(D)$ satisfying that $(F - F, z_0) \in I(\psi)_{z_0}$. When $C_{F,\psi,t}(z_0) = 0$ or $\infty$, we set $\frac{\int_{D_t} |F|^2 e^{-\psi}}{C_{F,\psi,t}(z_0)} = +\infty$. 

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In the present article, by doing some modifications of the method in [11], we obtain the following optimal uniform support function of $\int_D |F|^2 e^{-\psi} C_{F,\psi,t}(z_0)$, which gives a positive answer to Question 1.1.

**Theorem 1.2.** Let $D$ be a pseudoconvex domain in $\mathbb{C}^n$. Let $F$ be a holomorphic function, and let $\psi$ be a negative plurisubharmonic function on $D$. Assume that $\int_{\{\psi < -t\}} |F|^2 < +\infty$ holds for any $t > 0$. Then the inequality

$$\int_D |F|^2 e^{-\psi} C_{F,\psi,t}(z_0) \geq \frac{t}{1 - e^{-t}}$$

holds for any $t \in (0, +\infty)$, where $\frac{t}{1 - e^{-t}}$ is the optimal support function.

The following remark illustrates the optimality of the support function in inequality (1.1).

**Remark 1.3.** Take $D = \Delta \subset \mathbb{C}$, $z_0 = 0$ the origin of $\mathbb{C}$, $F \equiv 1$ and $\psi = 2 \log |z|$. It is clear that $\int_D |F|^2 < +\infty$. Note that $C_{F,\psi,t}(z_0) = \pi (1 - e^{-t})$ and $\int_D |F|^2 e^{-\psi} = t\pi$. Then $\frac{\int_D |F|^2 e^{-\psi}}{C_{F,\psi,t}(z_0)} = \frac{t}{1 - e^{-t}}$, which shows the optimality of the support function $\frac{t}{1 - e^{-t}}$.

Theorem 1.2 implies the following strong openness property. We present some details in Section 4.

**Corollary 1.4.** (see [16]) $\mathcal{I}(\psi) = \mathcal{I}_+(\psi)$.

## 2. Preparations

In this section, we will do some preparations.

### 2.1. $\bar{\partial}$-equation with $L^2$ estimates.

We call a positive smooth function $c$ on $(0, +\infty)$ in class $P$, if the following three statements hold

1. $\int_0^{+\infty} c(t) e^{-t} dt < +\infty$;
2. $c(t) e^{-t}$ is decreasing with respect to $t$;
3. for any $a > 0$, $c(t)$ has a positive lower bound on $(a, +\infty)$.

We shall prove Lemma 2.7 by using the following Lemma, whose various forms already appear in [13, 14, 11] etc.:

**Lemma 2.1.** Let $B \in (0, +\infty)$ and $t_0 > 0$ be arbitrarily given. Let $D$ be a pseudoconvex domain in $\mathbb{C}^n$. Let $\psi$ be a negative plurisubharmonic function on $D$, and $\varphi$ be a plurisubharmonic function on $D$. Let $F$ be a holomorphic function on $\{\psi < -t_0\}$ such that

$$\int_{K \cap \{\psi < -t_0\}} |F|^2 < +\infty$$

for any compact subset $K$ of $D$ and

$$\int_D \frac{1}{B} \int_{\{t_0 - B < \psi < -t_0\}} |F|^2 e^{-\varphi} \leq C < +\infty,$$

where $C \in \mathbb{R}$ is a positive constant. Then there exists a holomorphic function $\tilde{F}$ on $D$ such that

$$\int_D |\tilde{F} - (1 - b_{t_0,B}(\psi)) F|^2 e^{-\varphi + \nu_{t_0,B}(\psi) c(-\nu_{t_0,B}(\psi))} \leq C \int_0^{t_0 + B} c(t) e^{-t} dt,$$
where \( b_{t_0,B}(t) = \int_{-\infty}^t \frac{1}{B} \mathbb{1}_{\{t_0-B<s<t_0\}} ds \), \( v_{t_0,B}(t) = \int_0^t b_{t_0,B}(s) ds \), and \( c(t) \in P \).

For the sake of the convenience of the reader, we give the details of the proof of Lemma 2.1 in Section 5.2.

**Remark 2.2.** Note that \( v_{t_0,B}(t) \geq \max\{t, -(t_0 + B)\} \), \( c(t) e^{-t} \) is decreasing with respect to \( t \), and \( b_{t_0,B}(t) = 0 \) for any \( t \leq -(t_0 + B) \). Replacing \( \varphi \) by \( \psi \) and assuming \( \psi(z_0) = -\infty \), it follows from inequality (2.2) that

\[
\int_{\{\psi < -t_0 - B\}} |\bar{F} - F|^2 e^{-\psi - (t_0 + B)} c(t_0 + B) \\
\leq \int_{\{\psi < -t_0 - B\}} |\bar{F} - (1 - b_{t_0,B}(\psi)) F|^2 e^{-\varphi + v_{t_0,B}(\psi)} c(-v_{t_0,B}(\psi)) \\
\leq \int_{D} |\bar{F} - (1 - b_{t_0,B}(\psi)) F|^2 e^{-\varphi + v_{t_0,B}(\psi)} c(-v_{t_0,B}(\psi)) \\
\leq C \int_0^{t_0 + B} c(t) e^{-t} dt \\
< +\infty,
\]

which implies that \( (\bar{F} - F, z_0) \in \mathcal{I}(\psi)_{z_0} \).

**2.2. Some properties of \( G_{F,\psi,c}(t) \).** Let \( \psi \) be a negative plurisubharmonic function defined on a pseudoconvex domain \( D \subset \mathbb{C}^n \), and let \( F \) be a holomorphic function on \( \{ z \in D : \psi < -t \} \), and \( z_0 \in \{ z \in D : \psi < -t \} \). Denote that

\[
G_{F,\psi,c}(t) := \inf \left\{ \int_{\{\psi < -t\}} |F_1|^2 c(-\psi) : (F_1 - F, z_0) \in \mathcal{I}(\psi)_{z_0} \& F_1 \in \mathcal{O}(\{ \psi < -t \}) \right\},
\]

where \( c \in P \).

We recall the closedness of submodules, which can be referred to [10] (Chapter 2, §3, 3. Closedness of Submodules, page 45).

**Lemma 2.3.** (see [10]) Let \( N \) a submodule of \( \mathcal{O}^q_{\mathbb{C}^n,o} \), \( 1 \leq q < +\infty \), let \( f_j \in \mathcal{O}^q_{\mathbb{C}^n,o}(U)^q \) be a sequence of \( q \)-tuples holomorphic in an open neighborhood \( U \) of the origin. Assume that the \( f_j \) converge uniformly in \( U \) towards to a \( q \)-tuples \( f \in \mathcal{O}^q_{\mathbb{C}^n,o} \), assume furthermore that all germs \( (f_j,o) \) belong to \( N \). Then \( (f,o) \in N \).

The closedness of submodules will be used in the following discussion.

**Lemma 2.4.** Let \( F \) be a holomorphic function on \( D \), then \( (F, z_0) \in \mathcal{I}(\psi)_{z_0} \) if and only if \( G_{F,\psi,c}(0) = 0 \), where \( (F, z_0) \) denotes the germ of \( F \) at \( z_0 \).

**Proof.** It is clear that \( (F, z_0) \in \mathcal{I}(\psi)_{z_0} \) implies that \( G_{F,\psi,c}(0) = 0 \).

If \( G_{F,\psi,c}(0) = 0 \), then there exist holomorphic functions \( \{ f_j \}_{j \in \mathbb{N}} \) on \( D \) such that \( \lim_{j \to +\infty} \int_D |f_j|^2 c(-\psi) = 0 \) and \( (f_j - F, z_0) \in \mathcal{I}(\psi)_{z_0} \) for any \( j \). As \( c(-\psi) \) has positive lower bound on any compact subset of \( D \), then there exists a subsequence of \( \{ f_j \}_{j \in \mathbb{N}} \) denoted by \( \{ f_{j_k} \}_{k \in \mathbb{N}} \) compactly convergent to 0. Then we know \( f_{j_k} - F \) is compactly convergent to \( F \). It follows from Lemma 2.3 that \( (F, z_0) \in \mathcal{I}(\psi)_{z_0} \).

This proves Lemma 2.4. \( \square \)

**Lemma 2.5.** Assume that \( G_{F,\psi,c}(t) < +\infty \), then there exists a unique holomorphic function \( F_t \) on \( \{ \psi < -t \} \) satisfying \( (F_t - F, z_0) \in \mathcal{I}(\psi)_{z_0} \) and \( \int_{\{\psi < -t\}} |F_t|^2 c(-\psi) = \)
Choosing $f$. Furthermore, for any holomorphic function $\hat{F}$ on $\{\psi < -t\}$ satisfying $(\hat{F} - F, z_0) \in \mathcal{I}(\psi)_{z_0}$ and $\int_{\{\psi < -t\}} |\hat{F}|^2 c(-\psi) < +\infty$, we have the following equality

$$
(2.3) \int_{\{\psi < -t\}} |F_1|^2 c(-\psi) + \int_{\{\psi < -t\}} |\hat{F} - F_1|^2 c(-\psi) = \int_{\{\psi < -t\}} |\hat{F}|^2 c(-\psi).
$$

Proof. Firstly, we prove the existence of $F_t$. As $G_{F,\psi,c}(t) < +\infty$, then there exist holomorphic functions $\{f_j\}_{j \in \mathbb{N}}$ on $\{\psi < -t\}$ such that $\lim_{j \to +\infty} \int_{\{\psi < -t\}} |f_j|^2 c(-\psi) = G_{F,\psi,c}(t)$ and $(f_j - F, z_0) \in \mathcal{I}(\psi)_{z_0}$. Then there exists a subsequence of $\{f_j\}_{j \in \mathbb{N}}$ denoted by $\{f_{jk}\}_{k \in \mathbb{N}}$ compactly convergent to a holomorphic function $f$ on $\{\psi < -t\}$ satisfying $\int_K |f|^2 c(-\psi) \leq G_{F,\psi,c}(t)$ for any compact subset $K \subset \{\psi < -t\}$, which implies that $\int_{\{\psi < -t\}} |f|^2 c(-\psi) \leq G_{F,\psi,c}(t)$ by the monotone convergence theorem. Note that Lemma 2.3 implies that $(f - F, z_0) \in \mathcal{I}(\psi)_{z_0}$. Then we obtain the existence of $F_t(= f)$.

Secondly, we prove the uniqueness of $F_t$ by contradiction: if not, there exist two different holomorphic functions $f_1$ and $f_2$ on $\{\psi < -t\}$ satisfying $\int_{\{\psi < -t\}} |f_1|^2 c(-\psi) = \int_{\{\psi < -t\}} |f_2|^2 c(-\psi) = G_{F,\psi,c}(t)$, $(f_1 - F, z_0) \in \mathcal{I}(\psi)_{z_0}$, and $(f_2 - F, z_0) \in \mathcal{I}(\psi)_{z_0}$. Note that

$$
(2.4) \int_{\{\psi < -t\}} \left| \frac{f_1 + f_2}{2} \right|^2 c(-\psi) + \int_{\{\psi < -t\}} \left| \frac{f_1 - f_2}{2} \right|^2 c(-\psi) = G_{F,\psi,c}(t),
$$

then we obtain that

$$
\int_{\{\psi < -t\}} \left| \frac{f_1 + f_2}{2} \right|^2 c(-\psi) < G_{F,\psi,c}(t),
$$

and $(\frac{f_1 + f_2}{2} - F, z_0) \in \mathcal{I}(\psi)_{z_0}$, which contradicts the definition of $G_{F,\psi,c}(t)$.

Finally, we prove the equality (2.3). For any holomorphic function $f$ on $\{\psi < -t\}$ satisfying $(f, z_0) \in \mathcal{I}(\psi)_{z_0}$ and $\int_{\{\psi < -t\}} |f|^2 c(-\psi) < +\infty$, it is clear that for any complex number $\alpha$, $F_t + \alpha f$ satisfies

$$
((F_t + \alpha f) - F, z_0) = ((F_t - F) + \alpha f, z_0) \in \mathcal{I}(\psi)_{z_0}
$$

and the definition of $G_{F,\psi,c}(t)$ implies that

$$
\int_{\{\psi < -t\}} |F_t|^2 c(-\psi) \leq \int_{\{\psi < -t\}} |F_t + \alpha f|^2 c(-\psi) < +\infty.
$$

Note that

$$
\int_{\{\psi < -t\}} |F_t + \alpha f|^2 c(-\psi) - \int_{\{\psi < -t\}} |F_t|^2 c(-\psi) \geq 0
$$

implies

$$
\text{Re} \int_{\{\psi < -t\}} F_t \overline{f} c(-\psi) = 0,
$$

then we obtain that

$$
\int_{\{\psi < -t\}} |F_t + f|^2 c(-\psi) = \int_{\{\psi < -t\}} |F_t|^2 c(-\psi) + \int_{\{\psi < -t\}} |f|^2 c(-\psi).
$$

Choosing $f = \hat{F} - F_t$, we obtain equality (2.3). \qed
Lemma 2.6. Let $F$ be a holomorphic function on $D$ and $\psi(z_0) = -\infty$. Assume that $G_{F, \psi, c}(0) < +\infty$. Then $G_{F, \psi, c}(t)$ is decreasing with respect to $t \in (0, +\infty)$ such that $\lim_{t \to 0^+} G_{F, \psi, c}(t) = G_{F, \psi, c}(t_0)$ for any $t_0 \in (0, +\infty)$, $\lim_{t \to t_0^-} G_{F, \psi, c}(t) \geq G_{F, \psi, c}(t_0)$ for any $t_0 \in (0, +\infty)$ and $\lim_{t \to +\infty} G_{F, \psi, c}(t) = 0$. Especially, $G_{F, \psi, c}(t)$ is lower semi-continuous on $[0, +\infty)$.

Proof. By the definition of $G_{F, \psi, c}(t)$, it is clear that $G_{F, \psi, c}(t)$ is decreasing on $[0, +\infty)$, $\lim_{t \to t_0^-} G_{F, \psi, c}(t) \geq G_{F, \psi, c}(t_0)$ for any $t \in (0, +\infty)$ and $\lim_{t \to +\infty} G_{F, \psi, c}(t) = 0$. It suffices to prove $\lim_{t \to t_0^+} G_{F, \psi, c}(t) = G_{F, \psi, c}(t_0)$. We prove it by contradiction: if not, then $\lim_{t \to t_0^+} G_{F, \psi, c}(t) < G_{F, \psi, c}(t_0)$ for some $t_0 \in [0, +\infty)$.

By Lemma 2.3, there exists a unique holomorphic function $F_t$ on $\{ \psi < -t \}$ satisfying that $(F_t - F, z_0) \in I(\psi)_z$ and $\int_{\{ \psi < -t \}} |F_t|^2 c(\psi) = G_{F, \psi, c}(t)$. Note that $G_{F, \psi, c}(t)$ is decreasing implies that $\int_{\{ \psi < -t \}} |F_t|^2 c(\psi) \leq \lim_{t \to t_0^+} G_{F, \psi, c}(t)$ for any $t > t_0$. As $c(\psi)$ has positive lower bound on any compact subset of $\{ \psi < -t_0 \}$, then there exists a sequence of $\{ F_{t_j} \}$ $(t_j \to t_0)$, as $j \to +\infty$ uniformly convergent on any compact subset of $\{ \psi < t_0 \}$. Let $\hat{F}_{t_0} := \lim_{j \to +\infty} F_{t_j}$, which is a holomorphic function on $\{ \psi < -t_0 \}$ and $(\hat{F}_{t_0} - F, z_0) \in I(\psi)_z$. Then it follows the decreasing property of $G_{F, \psi, c}(t)$ that

$$\int_K |\hat{F}_{t_0}|^2 c(\psi) = \lim_{j \to +\infty} \int_K |F_{t_j}|^2 c(\psi) \leq \lim_{j \to +\infty} G_{F, \psi, c}(t_j) < G_{F, \psi, c}(t_0) - \epsilon$$

for any compact subset of $\{ \psi < -t_0 \}$ and $\epsilon > 0$ is independent of $K$. It follows from the monotone convergence theorem that $\int_{\{ \psi < -t_0 \}} |\hat{F}_{t_0}|^2 c(\psi) < G_{F, \psi, c}(t_0)$, which contradicts the definition of $G_{F, \psi, c}(t_0)$. □

The following Lemma will be used to prove Proposition 3.1.

Lemma 2.7. Let $F$ be a holomorphic function on $D$ and $\psi(z_0) = -\infty$. Assume that $G_{F, \psi, c}(0) < +\infty$. Then for any $t_0 \in (0, +\infty)$, we have

$$G_{F, \psi, c}(0) - G_{F, \psi, c}(t_0) \leq \int_0^{t_0} c(t)e^{-tdt} \liminf_{B \to 0^+} \frac{G_{F, \psi, c}(t_0) - G_{F, \psi, c}(t_0 + B)}{B}.$$  

Proof. By Lemma 2.3, there exists a holomorphic function $F_{t_0}$ on $\{ \psi < -t_0 \}$, such that $(F_{t_0} - F, z_0) \in I(\psi)_z$ and $\int_{\{ \psi < -t_0 \}} |F_{t_0}|^2 c(\psi) = G_{F, \psi, c}(t_0)$.

It suffices to consider that $\liminf_{B \to 0^+} \frac{G_{F, \psi, c}(t_0) - G_{F, \psi, c}(t_0 + B)}{B} \in [0, +\infty)$, because of the decreasing property of $G_{F, \psi, c}(t)$. Then there exists a sequence $\{ B_j \}$ $(B_j \to 0^+, \text{as } j \to +\infty)\text{ such that}$

$$\lim_{j \to +\infty} \frac{G_{F, \psi, c}(t_0) - G_{F, \psi, c}(t_0 + B_j)}{B_j} = \liminf_{B \to 0^+} \frac{G_{F, \psi, c}(t_0) - G_{F, \psi, c}(t_0 + B)}{B}$$

and $\left\{ \frac{G_{F, \psi, c}(t_0) - G_{F, \psi, c}(t_0 + B_j)}{B_j} \right\}_{j \in \mathbb{N}}$ is bounded.

Lemma 2.3 and Remark 2.2 $(\phi \sim \psi$ and $F \sim F_{t_0}, \text{where } \sim \text{ means the former replaced by the latter})$ show that for any $B_j$, there exists holomorphic function $\hat{F}_j$
on $D$ such that $(\tilde{F}_j - F_{t_0}, z_0) \in \mathcal{I}(\psi)_{z_0}$ and
\[
\int_D |\tilde{F}_j - (1 - b_{t_0,B_j}(\psi))F_{t_0}|^2 c(-v_{t_0,B_j}(\psi))e^{v_{t_0,B_j}(\psi)} \leq \int_0^{t_0+B_j} c(t)e^{-t} dt \cdot \int_D \frac{1}{B_j} \mathbb{1}_{\{t_0-B_j < \psi < -t_0\}} |F_{t_0}|^2 e^{-\psi}.
\]
(2.7)
\[
\leq \frac{e^{t_0+B_j} \int_0^{t_0+B_j} c(t)e^{-t} dt}{\inf_{t \in (t_0,t_0+B_j)} c(t)} \cdot \frac{G_{F,\psi,c}(t_0) - G_{F,\psi,c}(t_0 + B_j)}{B_j}.
\]

As $-t \leq v_{t_0,B_j}(-t)$ for any $t \geq 0$, the decreasing property of $c(t)e^{-t}$ shows that $c(t)e^{-t} \leq c(-v_{t_0,B_j}(-t))e^{v_{t_0,B_j}(-t)}$ for any $t \in (0, +\infty)$, which implies that
(2.8)
\[
c(-\psi) \leq c(-v_{t_0,B_j}(\psi))e^{v_{t_0,B_j}(\psi)}.
\]

Then combining inequality (2.7) and inequality (2.8), we obtain that
(2.9)
\[
\int_D |\tilde{F}_j - (1 - b_{t_0,B_j}(\psi))F_{t_0}|^2 c(-\psi)
\leq \frac{e^{t_0+B_j} \int_0^{t_0+B_j} c(t)e^{-t} dt}{\inf_{t \in (t_0,t_0+B_j)} c(t)} \cdot \frac{G_{F,\psi,c}(t_0) - G_{F,\psi,c}(t_0 + B_j)}{B_j}.
\]

Firstly, we will prove that $\int_D |\tilde{F}_j|^2 c(-\psi)$ is bounded with respect to $j$.

Note that
\[
\left( \int_D |\tilde{F}_j|^2 c(-\psi) \right)^\frac{1}{2} - \left( \int_D |(1 - b_{t_0,B_j}(\psi))F_{t_0}|^2 c(-\psi) \right)^\frac{1}{2}
\leq \left( \int_D |\tilde{F}_j - (1 - b_{t_0,B_j}(\psi))F_{t_0}|^2 c(-\psi) \right)^\frac{1}{2},
\]
then we obtain that
\[
\left( \int_D |\tilde{F}_j|^2 c(-\psi) \right)^\frac{1}{2} \leq \left( \int_D |(1 - b_{t_0,B_j}(\psi))F_{t_0}|^2 c(-\psi) \right)^\frac{1}{2} + \left( \frac{e^{t_0+B_j} \int_0^{t_0+B_j} c(t)e^{-t} dt}{\inf_{t \in (t_0,t_0+B_j)} c(t)} \cdot \frac{G_{F,\psi,c}(t_0) - G_{F,\psi,c}(t_0 + B_j)}{B_j} \right)^\frac{1}{2}.
\]

Since $\left\{ \frac{G_{F,\psi,c}(t_0) - G_{F,\psi,c}(t_0 + B_j)}{B_j} \right\}_{j \in \mathbb{N}}$ is bounded and
\[
\int_D |(1 - b_{t_0,B_j}(\psi))F_{t_0}|^2 c(-\psi) \leq \int_{\{\psi < -t_0\}} |F_{t_0}|^2 c(-\psi) < +\infty,
\]
then we obtain that $\int_D |\tilde{F}_j|^2 c(-\psi)$ is bounded with respect to $j$.

Secondly, we will prove the main result.

It follows from $b_{t_0,B_j}(\psi) = 1$ on $\{\psi \geq -t_0\}$ that
\[
\int_D |\tilde{F}_j - (1 - b_{t_0,B_j}(\psi))F_{t_0}|^2 c(-\psi) \leq \int_{\{\psi \geq -t_0\}} |\tilde{F}_j|^2 c(-\psi) + \int_{\{\psi < -t_0\}} |\tilde{F}_j - (1 - b_{t_0,B_j}(\psi))F_{t_0}|^2 c(-\psi).
\]
(2.10)
Denote that \( || \cdot ||_2 := \left( \int_{\{ \psi < -t_0 \}} \left| \cdot \right|^2 c(-\psi) \right)^{\frac{1}{2}} \). It is clear that
\[
|| \tilde{F}_j - (1 - b_{t_0,B_j}(\psi))F_{t_0} ||_2^2 \\
\geq \left( || \tilde{F}_j - F_{t_0} ||_2 - || b_{t_0,B_j}(\psi)F_{t_0} ||_2 \right)^2 \\
\geq || \tilde{F}_j - F_{t_0} ||_2^2 - 2|| \tilde{F}_j - F_{t_0} ||_2 || b_{t_0,B_j}(\psi)F_{t_0} ||_2 \\
\geq || \tilde{F}_j - F_{t_0} ||_2^2 - 2|| \tilde{F}_j - F_{t_0} ||_2 \left( \int_{\{ -t_0 - B_j < \psi < -t_0 \}} |F_{t_0}|^2 c(-\psi) \right)^{\frac{1}{2}},
\]
where the last inequality follows from \( 0 \leq b_{t_0,B_j}(\psi) \leq 1 \) and \( b_{t_0,B_j}(\psi) = 0 \) on \( \{ \psi \leq -t_0 - B_j \} \).

Combining equality (2.10), inequality (2.11) and equality (2.3), we obtain that
\[
\tilde{F}_j \leq -\int_{D} \tilde{F}_j^2 c(-\psi) + || \tilde{F}_j - (1 - b_{t_0,B_j}(\psi))F_{t_0} ||_2^2 \\
\geq || \tilde{F}_j - F_{t_0} ||_2^2 - 2|| \tilde{F}_j - F_{t_0} ||_2 || b_{t_0,B_j}(\psi)F_{t_0} ||_2 \\
\geq || \tilde{F}_j - F_{t_0} ||_2^2 - 2|| \tilde{F}_j - F_{t_0} ||_2 \left( \int_{\{ -t_0 - B_j < \psi < -t_0 \}} |F_{t_0}|^2 c(-\psi) \right)^{\frac{1}{2}} \\
- 2|| \tilde{F}_j - F_{t_0} ||_2 \left( \int_{\{ -t_0 - B_j < \psi < -t_0 \}} |F_{t_0}|^2 c(-\psi) \right)^{\frac{1}{2}} \\
= \int_{D} || \tilde{F}_j ||_2^2 c(-\psi) - || F_{t_0} ||_2^2 \\
- 2|| \tilde{F}_j - F_{t_0} ||_2 \left( \int_{\{ -t_0 - B_j < \psi < -t_0 \}} |F_{t_0}|^2 c(-\psi) \right)^{\frac{1}{2}}.
\]

It follows from equality (2.3) that
\[
|| \tilde{F}_j - F_{t_0} ||_2 = \left( || \tilde{F}_j ||_2^2 - || F_{t_0} ||_2^2 \right)^{\frac{1}{2}} \leq || \tilde{F}_j ||_2 \leq \left( \int_{D} || \tilde{F}_j ||_2^2 c(-\psi) \right)^{\frac{1}{2}}. \tag{2.13}
\]
Since \( \int_{D} || \tilde{F}_j ||_2^2 c(-\psi) \) is bounded with respect to \( j \), inequality (2.13) implies that
\( \left( \int_{\{ \psi < -t_0 \}} || \tilde{F}_j - F_{t_0} ||_2^2 c(-\psi) \right)^{\frac{1}{2}} \) is bounded with respect to \( j \). Using the dominated convergence theorem and \( \int_{\{ \psi < -t_0 \}} |F_{t_0}|^2 c(-\psi) < +\infty \), we obtain that
\[
\lim_{j \to +\infty} || \tilde{F}_j - F_{t_0} ||_2 \left( \int_{\{ -t_0 - B_j < \psi < -t_0 \}} |F_{t_0}|^2 c(-\psi) \right)^{\frac{1}{2}} = 0.
\]
Combining with inequality (2.12), we obtain
\[
\lim_{j \to +\infty} \int_{D} || \tilde{F}_j - (1 - b_{t_0,B_j}(\psi))F_{t_0} ||_2^2 c(-\psi) \\
\geq \lim_{j \to +\infty} \int_{D} || \tilde{F}_j ||_2^2 c(-\psi) - || F_{t_0} ||_2^2. \tag{2.14}
\]
Using inequality (2.9) and inequality (2.14), we obtain

\[
\int_{t_0}^{t} c(t) e^{-t} dt \quad \lim_{j \to +\infty} \frac{G_{F,\psi,c}(t_0) - G_{F,\psi,c}(t_0 + B_j)}{B_j} \\
= \lim_{j \to +\infty} e^{t_0 + B_j} \int_{t_0 + B_j}^{t} c(t) e^{-t} dt \quad \frac{G_{F,\psi,c}(t_0) - G_{F,\psi,c}(t_0 + B_j)}{B_j} \\
\geq \lim_{j \to +\infty} \inf_{t \in (t_0, t_0 + B_j)} c(t) (t - (1 - b_{t_0,B_j}(\psi)) F_{t_0})^2 c(-\psi) \\
\geq \lim_{j \to +\infty} \int_{D} |\tilde{F}_j|^2 c(-\psi) - ||F_{t_0}||_2^2 \\
\geq G_{F,\psi,c}(t_0) - G_{F,\psi,c}(t_0).
\]

This proves Lemma 2.7. □

The following Lemma will be used to prove Theorem 1.2.

**Lemma 2.8.** Let \( F \) be a holomorphic function on pseudoconvex domain \( D \), and let \( \psi \) be a negative plurisubharmonic function on \( D \). Assume that \( \int_{\{\psi \geq -t\}} |F|^2 e^{-\psi} < +\infty \) for a given positive number \( t \). Then

\[
\int_{\{\psi \geq -t\}} |F|^2 e^{-\psi} = \int_{-\infty}^{t} \left( \int_{\{l \leq \psi < -l\}} |F|^2 \right) e^l dl.
\]

**Proof.** It is clear that the lemma directly follows from the basic formula

\[
\int_{X} f d\mu = \int_{0}^{+\infty} \mu(\{x \in X : f(x) > l\}) dl
\]

for a measurable function \( f : X \to [0, +\infty) \) with \( X = \{\psi > -t\} \), \( f = e^{-\psi} \), and \( d\mu = |F|^2 dV_{2n} \), where \( dV_{2n} \) is the Lebesgue measure on \( \mathbb{C}^n \).

Next we will prove the equality (2.17).

\[
\int_{X} f d\mu = \int_{0}^{+\infty} \left( \int_{X} \left( \int_{0}^{+\infty} \theta_f(l,x) dl \right) d\mu \right) dl
\]

where \( \theta_f(l,x) = \begin{cases} 1 & \text{if } l < f(x) \\ 0 & \text{if } l \geq f(x) \end{cases} \) is a function defined on \( \mathbb{R} \times X \). Then the equality (2.17) has thus been proved. □

3. **Proof of Theorem 1.2**

Firstly, we prove the following proposition.
Proposition 3.1. Let $F$ be a holomorphic function on $D$, $z_0 \in D$, and $\psi$ be a negative plurisubharmonic function on $D$. Then the inequality

\begin{equation}
\int_{\{\psi < -t\}} |F|^2 c(-\psi) \geq \frac{\int_{t_0}^{+\infty} c(l)e^{-l}dl}{\int_{0}^{+\infty} c(l)e^{-l}dl} G_{F,\psi,c}(0)
\end{equation}

holds for any $t \geq 0$, which is sharp.

Especially, if $G_{F,\psi,c}(0) = +\infty$, then $\int_{\{\psi < -t\}} |F|^2 c(-\psi) = +\infty$ for any $t \geq 0$.

Remark 3.2. Let $D = \Delta \subset \mathbb{C}$ be the unit disc, $z_0 = o$ the origin of $\mathbb{C}$, $\psi = 2 \log |z|$, $F \equiv 1$, and $c \equiv 1$. It is clear that

\begin{equation}
e^{-t}\pi = \int_{\{\psi < -t\}} |F|^2 c(-\psi) \geq \frac{\int_{t_0}^{+\infty} c(l)e^{-l}dl}{\int_{0}^{+\infty} c(l)e^{-l}dl} G_{F,\psi,c}(0) = e^{-t}\pi,
\end{equation}

which is sharp.

Proof of Proposition 3.1. When $\psi(z_0) > -\infty$, it is clear that $G_{F,\psi,c}(0) = 0$. Then the inequality (3.1) holds. Next, we will only consider the case $\psi(z_0) = -\infty$.

We prove Proposition 3.1 in two steps, i.e. the case $G_{F,\psi,c}(0) < +\infty$ and the case $G_{F,\psi,c}(0) = +\infty$.

Step 1. We prove the case $G_{F,\psi,c}(0) < +\infty$. As $\int_{\{\psi < -t\}} |F|^2 c(-\psi) \geq G_{F,\psi,c}(t)$ for any $t \in [0, +\infty)$, then it suffices to prove that $G_{F,\psi,c}(t) \geq \frac{\int_{t_0}^{+\infty} c(l)e^{-l}dl}{\int_{0}^{+\infty} c(l)e^{-l}dl} G_{F,\psi,c}(0)$ for any $t \in [0, +\infty)$.

Let $H(t) := G_{F,\psi,c}(t) - \int_{t_0}^{+\infty} c(l)e^{-l}dl G_{F,\psi,c}(0)$. We prove $H(t) \geq 0$ by contradiction: if not, then there exists $t$ such that $H(t) < 0$.

Note that $G_{F,\psi,c}(t) \in [0, G_{F,\psi,c}(0)]$ is bounded on $[0, +\infty)$, then $H(t)$ is also bounded on $[0, +\infty)$, which implies that $\inf_{[0, +\infty)} H(t)$ is finite.

By Lemma 2.2, it is clear that $\lim_{t \to 0+} H(t) = H(0) = 0$ and $\lim_{t \to +\infty} H(t) = 0$. Then it follows from $\inf_{[0, +\infty)} H(t) < 0$ that there exists a closed interval $[a, b] \subset (0, +\infty)$ such that $\inf_{[a, b]} H(t) = \inf_{[0, +\infty)} H(t)$. Since $G_{F,\psi,c}$ is lower semi-continuous, then $H(t)$ is also lower semi-continuous, which implies that there exists $t_0 \in [a, b]$ such that $H(t_0) = \inf_{[0, +\infty)} H(t) < 0$.

As $H(t_0) = \inf_{[0, +\infty)} H(t)$, then it follows that $\liminf_{B \to 0+} \frac{H(t_0) - H(t_0 + B)}{B} \leq 0$. Combining with $H(t_0) < 0$, then we obtain that

\begin{equation}
H(t_0) + \int_{t_0}^{t_0 + B} \frac{c(t)e^{-t}}{c(t_0)e^{-t_0}} \liminf_{B \to 0+} \frac{H(t_0) - H(t_0 + B)}{B} < 0.
\end{equation}
Note that
\[ H(t_0) + \int_{t_0}^{t_0 + B} c(t)e^{-t} dt \liminf_{B \to 0^+} \frac{H(t_0) - H(t_0 + B)}{B} = G_{F,\psi,c}(t_0) - G_{F,\psi,c}(0) \]
\[ + \int_{t_0}^{t_0 + B} c(t)e^{-t} dt \liminf_{B \to 0^+} \frac{G_{F,\psi,c}(t_0) - G_{F,\psi,c}(t_0 + B)}{B} \]
\[ = G_{F,\psi,c}(t_0) + \int_{0}^{t_0} c(t)e^{-t} dt \liminf_{B \to 0^+} \frac{G_{F,\psi,c}(t_0) - G_{F,\psi,c}(t_0 + B)}{B} \]
\[ - G_{F,\psi,c}(0) \left( \int_{0}^{t_0} c(t)e^{-t} dt \right) \]
\[ = G_{F,\psi,c}(t_0) - G_{F,\psi,c}(0) + \int_{0}^{t_0} c(t)e^{-t} dt \liminf_{B \to 0^+} \frac{G_{F,\psi,c}(t_0) - G_{F,\psi,c}(t_0 + B)}{B} \]
then it follows from Lemma 2.7 that
\[ H(t_0) + \int_{0}^{t_0} c(t)e^{-t} dt \liminf_{B \to 0^+} \frac{H(t_0) - H(t_0 + B)}{B} \geq 0, \]
which contradicts inequality (5.2). Then the case $G_{F,\psi,c}(0) < +\infty$ has thus been proved.

Step 2. We prove the case $G_{F,\psi,c}(0) = +\infty$ by contradiction: if not, then integral
\[ \int_{t_0}^{t_0 + B} |F|^2 e(-\psi) \] is finite for some $t_0 > 0$ (when $t_0 = 0$, $\int_{t_0}^{t_0 + B} |F|^2 e(-\psi) \geq G_{F,\psi,c}(0) = +\infty$). It follows from Lemma 2.1 and Remark 2.2 that there exists a holomorphic function $F$ on $D$ satisfying
\[ (\bar{F} - F, z_0) \in \mathcal{T}(\psi)_{z_0} \]
and
\[ \int_{D} |\bar{F} - (1 - b_{t_0,B}(\psi))F|^2 c(-\psi) \]
\[ \leq \int_{D} |\bar{F} - (1 - b_{t_0,B}(\psi))F|^2 c(-\psi) e^{\psi_{t_0,B}(\psi)} e^{-\psi} \]
\[ \leq \int_{0}^{t_0 + B} \frac{2}{B} c(t)e^{-t} \int_{D} \frac{2}{B} |(1 - b_{t_0,B}(\psi))F|^2 e^{-\psi}. \]

Note that
\[ \left( \int_{D} |\bar{F}|^2 c(-\psi) \right)^{1/2} - \left( \int_{D} |(1 - b_{t_0,B}(\psi))F|^2 c(-\psi) \right)^{1/2} \]
\[ \leq \left( \int_{D} (\bar{F} - (1 - b_{t_0,B}(\psi))F)^2 c(-\psi) \right)^{1/2}, \]
then it follows from inequality (3.3) that

\[
(3.4) \quad \left( \int_D |\tilde{F}|^2 c(\psi) \right)^{\frac{1}{2}} \leq \left( \int_D |(1 - b_{t_0,B}(\psi))F|^2 c(\psi) \right)^{\frac{1}{2}} + \left( \int_0^{t_0+B} c(t) e^{-t} dt \int_D \frac{1}{B} \chi_{(-t_0-B<\psi<-t_0)} |F|^2 e^{-\psi} \right)^{\frac{1}{2}}.
\]

Since \(b_{t_0,B}(\psi) = 1\) on \(\{\psi \geq t_0\}\), \(0 \leq b_{t_0,B}(\psi) \leq 1\), \(\int_{\{\psi < -t_0\}} |F|^2 c(\psi) < +\infty\), and \(c(-\psi)\) has positive lower bound on \(\{\psi < -t_0\}\), then we obtain that

\[
\left( \int_D |(1 - b_{t_0,B}(\psi))F|^2 c(\psi) \right)^{\frac{1}{2}} < +\infty
\]

and

\[
\left( \int_0^{t_0+B} c(t) e^{-t} dt \int_D \frac{1}{B} \chi_{(-t_0-B<\psi<-t_0)} |F|^2 e^{-\psi} \right)^{\frac{1}{2}} < +\infty,
\]

which implies that

\[
\int_D |\tilde{F}|^2 e^{-\psi} < +\infty.
\]

Then we obtain \(G_{F,\psi,c}(0) \leq \int_D |\tilde{F}|^2 c(\psi) < +\infty\), which contradicts \(G_{F,\psi,c}(0) = +\infty\). The case \(G_{F,\psi,c}(0) = +\infty\) has been proved. \(\Box\)

In the following part, we will prove Theorem 1.2 by using Proposition 3.1 and Lemma 2.8.

Firstly, we will construct a family of functions \(\{c^n_t(x)\}_{n \in \mathbb{N}} \subset P\), where \(t\) is the given positive number in Theorem 1.2.

Let \(f(x) = \begin{cases} e^{-\frac{1}{1-(x-1)^2}}, & \text{if } |x-1| < 1 \\ 0, & \text{if } |x-1| \geq 1 \end{cases}\) be a real function defined on \(\mathbb{R}\). It is clear that \(f(x) \in C_0^\infty(\mathbb{R})\) and \(f(x) \geq 0\) for any \(x \in \mathbb{R}\). Then let \(g_n(x) = \frac{1}{n(n+1)} \int_0^n f(s)ds\), where \(d = \int_\mathbb{R} f(s)ds\). It follows that \(g_n(x)\) is increasing with respect to \(x\), \(g_n(x) \leq g_{n+1}(x)\) for any \(n \in \mathbb{N}\) and \(x \in \mathbb{R}\), and \(\lim_{n \to +\infty} g_n(x) = I_{\{x \in \mathbb{R} : x > 0\}}(x)\) for all \(x \in \mathbb{R}\).

Now we construct \(\{c^n_t(x)\}_{n \in \mathbb{N}}\) by setting \(c^n_t(x) = 1 - g_n(x-t)\). It follows from the properties of \(\{g_n(x)\}_{n \in \mathbb{N}}\) that \(c^n_t(x)\) is decreasing with respect to \(x\), \(c^n_t(x) \geq c^{n+1}_t(x)\) for any \(n \in \mathbb{N}\) and \(x \in \mathbb{R}\), and \(\lim_{n \to +\infty} c^n_t(x) = I_{\{x \in \mathbb{R} : x \leq t\}}(x)\) for all \(x \in \mathbb{R}\). Note that \(c^n_t(x) \in [\frac{1}{n+1}, 1]\) on \(x \in (0, +\infty)\), then \(c^n_t(x) \in P\) for any \(n \in \mathbb{N}\).

Secondly, we will prove

\[
(3.5) \quad \int_{-l \leq \psi < -l} |F|^2 \geq \frac{e^{-t} - e^{-t}}{1 - e^{-t}} C_{F,\psi,t}(z_0)
\]

for any \(l \in [0, t)\).

For the case \(l = 0\), inequality (3.5) holds by the definition of \(C_{F,\psi,t}(z_0)\), then we only consider the case \(l > 0\).

It follows from Proposition 3.1 that

\[
(3.6) \quad \int_{\psi < -l} |F|^2 c^n_t(-\psi) \geq \frac{1}{\int_{0}^{+\infty} c^n_t(s)e^{-s}ds} \int_{-l}^{+\infty} c^n_t(s)e^{-s}ds\int \int_{-l}^{+\infty} c^n_t(s)e^{-s}ds C_{F,\psi,t}(z_0).
\]

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By \( \int_{\{\psi < -t\}} |F|^2 < +\infty \) and the properties of \( \{c^n_t\}_{n \in \mathbb{N}} \), we obtain that

\[
\lim_{n \to +\infty} \int_{\{\psi < -t\}} |F|^2 c^n_t (-\psi) = \int_{\{-t \leq \psi < -l\}} |F|^2.
\]

As \( c^n_t(x) \geq \mathbb{I}_{\{x \in \mathbb{R}, x \leq t\}}(x) \) for any \( x > 0 \) and \( n \in \mathbb{N} \), then it follows from the definition of \( G_{F, \psi, c} \) and \( C_{F, \psi, t}(z_0) \) that

\[
(3.8) \quad G_{F, \psi, c^n_t}(0) \geq C_{F, \psi, t}(z_0).
\]

Combining inequality (3.6), equality (3.7), and inequality (3.8), we obtain that

\[
\int_{\{-t \leq \psi < -l\}} |F|^2 = \lim_{n \to +\infty} \int_{\{\psi < -t\}} |F|^2 c^n_t (-\psi)
\]

\[
\geq \lim_{n \to +\infty} \int_0^{+\infty} c^n_t(s) e^{-s} ds C_{F, \psi, t}(z_0)
\]

\[
= \frac{e^{-l} - e^{-t}}{1 - e^{-t}} C_{F, \psi, t}(z_0).
\]

Then inequality (3.5) has been proved.

Finally, we will finish the proof of Theorem 1.2.

By Lemma 2.8 and inequality (3.5), we obtain that

\[
\int_{D_t} |F|^2 e^{-\psi} = \int_{-\infty}^{t} \left( \int_{\{-t \leq \psi < -l\}} |F|^2 \right) e^l dl
\]

\[
= \int_{-\infty}^{0} \left( \int_{\{-t \leq \psi < -l\}} |F|^2 \right) e^l dl + \int_{0}^{t} \left( \int_{\{-t \leq \psi < -l\}} |F|^2 \right) e^l dl
\]

\[
\geq C_{F, \psi, t}(z_0) \left( \int_{-\infty}^{0} e^l dl + \int_{0}^{t} \frac{1 - e^{-t}}{1 - e^{-l}} dl \right)
\]

\[
= \frac{t}{1 - e^{-t}} C_{F, \psi, t}(z_0).
\]

Then Theorem 1.2 has thus been proved.

4. Theorem 1.2 implies Corollary 1.4

It suffices to prove that \( I(\psi)_{z_0} = I_+ (\psi)_{z_0} \) for any \( z_0 \in D \). When \( \psi(z_0) > -\infty \), it is clear that \( I(\psi)_{z_0} = I_+ (\psi)_{z_0} = \mathcal{O}_{z_0} \). In the following part, we will only consider the case that \( \psi(z_0) = -\infty \) and \( \psi \) is not identically \( -\infty \) in any of the neighborhoods of \( z_0 \).

Firstly, we will prove that there is a plurisubharmonic function \( \tilde{\psi} \) whose polar set is included in a (closed) analytic set, satisfying \( I(\psi)_{z_0} = I(\tilde{\psi})_{z_0} \) and \( I_+ (\psi)_{z_0} = I_+ (\tilde{\psi})_{z_0} \). Then it suffices to prove \( I(\tilde{\psi})_{z_0} = I_+ (\tilde{\psi})_{z_0} \).

Choosing \( \epsilon > 0 \), let \( J_\epsilon := I(3(1 + \epsilon)\psi)_{z_0} = (f_1, f_2, ..., f_m)_{z_0} \) and \( |J_\epsilon|^2 := \sum_{j=1}^{m} |f_j|^2 \). By the definition of \( I(3(1 + \epsilon)\psi)_{z_0} \), we have \( \int_{\nu} |J_\epsilon|^2 e^{-3(1+\epsilon)\psi} < +\infty \),
where \( V \ni z_0 \) is a neighborhood of \( z_0 \) in \( D \). Then we obtain that
\[
\int_V e^{-(1+\epsilon)\psi} - e^{-\max\{(1+\epsilon)\psi, \log |J_z|\}} \leq \int_{\{\log |J_z| > (1+\epsilon)\psi\} \cap V} e^{-(1+\epsilon)\psi} |J_z|^2 e^{-2(1+\epsilon)\psi} < +\infty.
\]
\[
(4.1)
\]

Let \( \tilde{\psi} = \max\{\psi, \frac{\log |f_1|}{1+\epsilon}\} \), it is clear that
\[
(4.2) \quad \int_V e^{-(1+\epsilon)\psi} - e^{-(1+\epsilon)\tilde{\psi}} \leq \int_V e^{-(1+\epsilon)\psi} - e^{-\max\{(1+\epsilon)\psi, \log |J_z|\}}.
\]

We may of course assume that \( |f_1| < 1 \) on \( V \), then we obtain that \( \tilde{\psi} < 0 \) on \( V \). For any \( s \in [0, \epsilon] \), it follows from \( e^{-(1+\epsilon)\psi} - e^{-(1+\epsilon)\tilde{\psi}} \leq e^{-(1+\epsilon)\psi} - e^{-(1+\epsilon)\tilde{\psi}} \), inequality (4.1) and inequality (4.2) that
\[
\int_V e^{-(1+\epsilon)\psi} - e^{-(1+\epsilon)\tilde{\psi}} \leq \int_V e^{-(1+\epsilon)\psi} - e^{-(1+\epsilon)\tilde{\psi}} < +\infty.
\]

Combining inequality (4.3) and \( \psi \leq \tilde{\psi} \), we obtain \( \mathcal{I}((1+s)\tilde{\psi})_{z_0} = \mathcal{I}((1+s)\tilde{\psi})_{z_0} \) for any \( s \in [0, \epsilon] \).

Next, we will prove \( \mathcal{I}(\tilde{\psi})_{z_0} = \mathcal{I}(\tilde{\psi})_{z_0} \) by contradiction. If not, then there exists a holomorphic function \( F \) near \( z_0 \) such that \((F, z_0) \in \mathcal{I}(\tilde{\psi})_{z_0} \) and \((F, z_0) \notin \mathcal{I}((1+s)\tilde{\psi})_{z_0} = \mathcal{I}(\tilde{\psi})_{z_0} \) for some \( s \in (0, \epsilon) \).

Choose a small enough neighborhood of \( z_0 \) denoted also by \( D \), which satisfies that \( D \) is a bounded pseudoconvex domain, \( D \subset V \) and
\[
\int_D |F|^2 e^{-\tilde{\psi}} < +\infty.
\]

Then we claim that there exist \( C_0 > 0 \) and \( T > 0 \) such that \( C_F(1+s)\tilde{\psi}(z_0) > C_0 \) for any \( t > T \), which implies that \( C_F(1+s')\tilde{\psi}(z_0) > C_0 \) for any \( t > T \) and \( s' \in (0, s] \).

In fact, as \( C_F(1+s)\tilde{\psi}(z_0) \) is increasing with respect to \( t \), if the claim is wrong, then we have \( C_F(1+s)\tilde{\psi}(z_0) = 0 \) for any \( t > 0 \).

As \( f_1 \) is a holomorphic function near \( z_0 = (z_{0,1}, z_{0,2}, \ldots, z_{0,n}) \) and \( f_1 \) is not identically 0, for any \( v \in \mathbb{C}^n \) the Taylor series of \( f_1 \) at \( z_0 \) can be written as
\[
f_1(z_0 + tv) = \sum_{k=0}^{+\infty} \frac{1}{k!} f_k(v),
\]
where \( f_k \) is a homogeneous polynomial of degree \( k \) on \( \mathbb{C}^n \) and \( f^{k_0} \) is not identically 0 for some \( k_0 \). Thus we may assume that \( f^{k_0}(v) \neq 0 \), where \( v = (1, 0, \ldots, 0) \). Then there exists \( r_1 \) such that \( f_1(w_1, z_{0,2}, z_{0,3}, \ldots, z_{0,n}) \neq 0 \) and \((w_1, z_{0,2}, z_{0,3}, \ldots, z_{0,n}) \in D \) when \( 0 < |w_1 - z_{0,1}| \leq r_1 \), which follows that there exist \( r > 0 \) and \( a > 0 \) such that \( D(z_{0,1}, r_1) \times \prod_{i=2}^n D(z_{0,i}, r) \subset D \) where \( D(y, r) := \{ z \in \mathbb{C} : |z - y| < r \} \) and \( |f_1(w)| > a \) for any \( w \in \{ w_1 \in \mathbb{C} : \frac{r}{2} < |w_1 - z_{0,1}| < r_1 \} \times \prod_{i=2}^n D(z_{0,i}, r) \) denoted by \( W \). Choosing \( t_0 > -\log a \), we have
\[
W \subset \{ \log |f_1| \geq -t_0 \} \cap D \subset \{(1+s)\tilde{\psi} \geq -t_0 \} \cap D.
\]
\[
(4.4)
\]
It follows from $C_{F,(1+s)\psi,t_0}(z_0) = 0$ that for any positive integer $j$ there exists $F_j \in \mathcal{O}(D)$ such that $(F_j - F, z_0) \in I((1 + s)\tilde{\psi})_{z_0}$ and

\[(4.5) \quad \int_{\omega} |F_j|^2 \leq \int_{\{(1+s)\tilde{\psi} \geq -t_0\} \cap D} |F_j|^2 < \frac{1}{j}.\]

Combining inequality (4.5) and Maximum principle, we obtain that $\{F_j\}$ compactly convergent to 0 on $D(z_{01}, r_1) \times \prod_{l=2}^n D(z_{0l}, r_l)$. Note that Lemma 2.3 implies that $(F, z_0) \in I((1+s)\tilde{\psi})_{z_0}$, which contradicts $(F, z_0) \notin I((1+s)\tilde{\psi})_{z_0}$. Then the claim holds.

As $\lim_{t \to +\infty} \int_{D_t} |F|^2 e^{-\tilde{\psi}} = \int_D |F|^2 e^{-\tilde{\psi}} < +\infty$, let $D_t = \{z \in D : \tilde{\psi} \geq -t\}$, then $\lim_{t \to +\infty} \int_{D_t} |F|^2 e^{-\tilde{\psi}} / C_0 = \int_D |F|^2 e^{-\tilde{\psi}} / C_0 < +\infty$.

But using Theorem 1.2 we obtain that

\[
\int_{D_t} |F|^2 e^{-\tilde{\psi}} / C_{F,(1+s')\tilde{\psi},t}(z_0) \geq \frac{\int_{\{(1+s')\tilde{\psi} \geq -t\} \cap D} |F|^2 e^{-\tilde{\psi}}}{\int_{D_t} |F|^2 e^{-\tilde{\psi}} / C_{F,(1+s')\tilde{\psi},t}(z_0)} \geq \frac{t}{1-e^{-t}},
\]

which contradicts (4.6). Then Corollary 1.4 has been proved.

5. Appendix

In this section, we present a concavity property of $G_{F,\psi,c}$ and the proof of Lemma 2.1

5.1. A concavity property of $G_{F,\psi,c}$. Let $h(t) = \int_t^{+\infty} c(l) e^{-l} dl$ defined on $t \in [0, +\infty)$. It is clear that $h(t) \in (0, \int_0^{+\infty} c(l) e^{-l} dl]$, and $h(t)$ is strictly decreasing with respect to $t$.

Proposition 5.1 shows that

\[G_{F,\psi,c}(t) \geq \int_t^{+\infty} c(l) e^{-l} dl G_{F,\psi,c}(0),\]

when $G_{F,\psi,c}(0) < +\infty$. Let $r = h(t)$, then we obtain that

\[G_{F,\psi,c}(h^{-1}(r)) \geq \frac{h(h^{-1}(r))}{\int_0^{+\infty} c(l) e^{-l} dl} G_{F,\psi,c}(0) \geq \frac{r}{\int_0^{+\infty} c(l) e^{-l} dl} G_{F,\psi,c} \left( h^{-1} \left( \int_0^{+\infty} c(l) e^{-l} dl \right) \right).\]

So it is natural to consider the concavity of $G_{F,\psi,c}(h^{-1}(r))$.

**Proposition 5.1.** Let $\psi$ be a negative plurisubharmonic function defined on a pseudoconvex domain $D \subset \mathbb{C}^n$, $F$ be a holomorphic function on $D$, $c \in \mathcal{P}$, and $\psi(z_0) = -\infty$. If $G_{F,\psi,c}(0) < +\infty$, then $G_{F,\psi,c}(h^{-1}(r))$ is concave with respect to $r \in (0, \int_0^{+\infty} c(l) e^{-l} dl]$.

We complete the proof of the Proposition 5.1 by the following two lemmas.
Lemma 5.2. Under the assumption of Proposition 5.1, for any \( t_1 \in [0, +\infty) \) and \( t_0 \in (0, +\infty) \), we have

\[
G_{F, \psi, c}(t_1) - G_{F, \psi, c}(t_0 + t_1) \leq \frac{\int_{t_0}^{t_0+t_1} c(l)e^{-l}dl}{c(t_0 + t_1)e^{-(t_0 + t_1)}}
\times \liminf_{B \to 0^+} \frac{G_{F, \psi, c}(t_0 + t_1) - G_{F, \psi, c}(t_0 + t_1 + B)}{B},
\]
i.e.

\[
\frac{G_{F, \psi, c}(t_1) - G_{F, \psi, c}(t_0 + t_1)}{\int_{t_0}^{t_0+t_1} c(l)e^{-l}dl - \int_{t_0 + t_1}^{\infty} c(l)e^{-l}dl} \leq \liminf_{B \to 0^+} \frac{G_{F, \psi, c}(t_0 + t_1) - G_{F, \psi, c}(t_0 + t_1 + B)}{\int_{t_0 + t_1}^{\infty} c(l)e^{-l}dl - \int_{t_0 + t_1 + B}^{\infty} c(l)e^{-l}dl}.
\]

Proof. By replacing \( D \) with \( \{ \psi < -t_1 \} \), replacing \( \psi \) with \( \psi + t_1 \) and replacing \( c(t) \) by \( c(t + t_1) \), it follows from Lemma 2.7 that this lemma holds. \( \square \)

As \( G_{F, \psi, c}(h^{-1}(t)) \) is lower semi-continuous (Lemma 5.3), then it follows from the following well-known property of concave functions (Lemma 5.3) that Lemma 5.2 implies Proposition 5.1.

Lemma 5.3. (see [11]) Let \( a(r) \) be a lower semicontinuous function on \((0, R]\). Then \( a(r) \) is concave if and only if

\[
\frac{a(r_1) - a(r_2)}{r_1 - r_2} \leq \liminf_{r_3 \to r_2} \frac{a(r_3) - a(r_2)}{r_3 - r_2}
\]
holds for any \( 0 < r_2 < r_1 \leq R \).

5.2. Proof of Lemma 5.1. The following remark shows that it suffices to consider the case of Lemma 5.1 that \( D \) is a strongly pseudoconvex domain, \( \varphi \) and \( \psi \) are plurisubharmonic functions on an open set \( U \) containing \( D \) such that \( \psi \leq 0 \) on \( U \), and \( F \) is a holomorphic function on \( U \cap \{ \psi < -t_0 \} \) such that \( \int_{D,t_0+t} \int_{\{\psi < -t_0\}} |F|^2 < +\infty \).

In the following remark, we recall some standard steps (see e.g. [26] [13] [14]) to illustrate the above statement.

Remark 5.4. It is well-known that there exist strongly pseudoconvex domains \( D_1 \in \cdots \in D_j \in D_{j+1} \in \cdots \) such that \( \bigcup_{j=1}^{+\infty} D_j = D \).

If inequality (2.1) holds on \( D \), then we obtain a sequence of holomorphic functions \( F_j \) on \( D_j \) such that

\[
\left. \begin{array}{l}
\int_{D_j} \left| \tilde{F}_j - (1 - b_{t_0,B}(\psi))F \right|^2 e^{-\varphi + v_{t_0,B}(\psi)} c(-v_{t_0,B}(\psi)) \\
\int_{D_j} c(t)e^{-t}dt \int_{D_j} \frac{1}{B} \int_{\{t_0-B<\psi<-t_0\}} |F|^2 e^{-\varphi} \\
\leq C \int_{0}^{t_0+B} c(t)e^{-t}dt
\end{array} \right\}
\]

is bounded with respect to \( j \). Note that for any given \( j \), \( e^{-\varphi + v_{t_0,B}(\psi)} c(-v_{t_0,B}(\psi)) \) has a positive lower bound on \( D_j \), then it follows that for any given \( j \), \( \int_{D_j} \left| \tilde{F}_j - (1 - b_{t_0,B}(\psi))F \right|^2 < \int_{D_j \cap \{\psi < -t_0\}} |F|^2 < +\infty \).
then we obtain that \( \int_{D_j} |\tilde{F}|^2 \) is bounded with respect to \( j' > j \).

By diagonal method, there exists a sequence of \( \tilde{F}_{j_k} \) uniformly convergent on any \( \tilde{D}_j \) to a holomorphic function on \( D \) denoted by \( \tilde{F} \). Then it follows from inequality \([5.1]\) and Fatou’s Lemma that
\[
\int_{D_j} |\tilde{F} - (1 - b_{0,B}(\psi))F|^2 e^{-\varphi + v_{0,B}(\psi)} c(-v_{0,B}(\psi)) \leq C \int_0^{t_0 + B} c(t)e^{-t}dt,
\]
then when \( j \) goes to \( +\infty \) we obtain Lemma \([2.7]\).

For the sake of completeness, we recall some lemmas on \( L^2 \) estimates for \( \bar{\partial} \)-equations, and \( \bar{\partial}^* \) means the Hilbert adjoint operator of \( \bar{\partial} \).

**Lemma 5.5.** (see [26], see also [1]) Let \( \Omega \subset \mathbb{C}^n \) be a domain with \( C^\infty \) boundary \( b\Omega, \Phi \in C^\infty(\bar{\Omega}) \). Let \( p \) be a \( C^\infty \) defining function for \( \Omega \) such that \(|dp| = 1 \) on \( b\Omega \). Let \( \eta \) be a smooth function on \( \Omega \). For any \((0,1)\)-form \( \alpha = \sum_{j=1}^n \alpha_jd\bar{z}^j \in Dom_{\Omega}(\bar{\partial}^*) \cap C^\infty(\bar{\Omega}, 1) \),
\[
\int_{\Omega} \eta|\bar{\partial}_\phi \alpha|^2 e^{-\Phi} + \int_{\Omega} |\bar{\partial}_\alpha|^2 e^{-\Phi} = \sum_{i,j=1}^n \int_{\Omega} \eta \bar{\partial}_i \bar{\partial}_j |\alpha|^2 e^{-\Phi} + \sum_{i,j=1}^n \int_{\Omega} \eta(\bar{\partial}_i \bar{\partial}_j \Phi) \alpha_i \alpha_j e^{-\Phi} + \sum_{i,j=1}^n \int_{\Omega} (\bar{\partial}_i \bar{\partial}_j \eta) \alpha_i \alpha_j e^{-\Phi} + 2\text{Re}(\bar{\partial}_\phi \alpha, \alpha_\Phi (\bar{\partial}_\eta)^2)_{\Omega, \Phi},
\]
where \( \alpha_\Phi (\bar{\partial}_\eta)^2 = \sum_{i,j} \alpha_j \bar{\partial}_j \eta \).

The symbols and notations can be referred to [13]. See also [26], [27], or [30].

**Lemma 5.6.** (see [1], see also [13]) Let \( \Omega \subset \mathbb{C}^n \) be a strictly pseudoconvex domain with \( C^\infty \) boundary \( b\Omega \) and \( \Phi \in C(\bar{\Omega}) \). Let \( \lambda \) be a \( \bar{\partial} \) closed smooth form of bidegree \((n,1)\) on \( \bar{\Omega} \). Assume the inequality
\[
|\langle \lambda, \alpha \rangle_{\Omega, \Phi}|^2 \leq C \int_{\Omega} |\bar{\partial}_\phi \alpha|^2 e^{-\Phi} < +\infty,
\]
where \( \frac{1}{\mu} \) is an integrable positive function on \( \Omega \) and \( C \) is a constant, holds for all \((n,1)\)-form \( \alpha \in Dom_{\Omega}(\bar{\partial}^*) \cap \text{Ker}(\bar{\partial}) \cap C^\infty(\bar{\Omega}, 1) \). Then there is a solution \( u \) to the equation \( \bar{\partial} u = \lambda \) such that
\[
\int_{\Omega} |u|^2 e^{-\Phi} \leq C.
\]

In the following part, we will prove Lemma \([2.1]\).

For the sake of completeness, let us recall some steps in the proof in [13] (see also [13], [17], [11]) with some slight modifications.

By Remark \([7]\), we can assume that \( D \subset \mathbb{C}^n \) is a strongly pseudoconvex domain (with smooth boundary), \( \varphi \) and \( \psi \) are plurisubharmonic functions on an open set \( U \) containing \( D \) such that \( \psi < 0 \) on \( U \), and \( F \) is a holomorphic function on \( U \cap \{ \psi < -t_0 \} \) such that
\[
\int_{D \cap \{ \psi < -t_0 \}} |F|^2 < +\infty.
\]
Then it follows from method of convolution (see e.g. [7]) that there exist smooth plurisubharmonic functions \( \psi_m \) and \( \varphi_m \) on an open set \( U \supset \bar{D} \) decreasing convergent to \( \psi \) and \( \varphi \) respectively, such that \( \sup_m \sup_D \psi_m < 0 \) and \( \sup_m \sup_D \varphi_m < +\infty \).

**Step 1: recall some Notations**

Let \( \epsilon \in (0, \frac{1}{4}B) \). Let \( \{v_\epsilon\}_{\epsilon \in (0, \frac{1}{4}B)} \) be a family of smooth increasing convex functions on \( \mathbb{R} \), which are continuous functions on \( \mathbb{R} \cup \{-\infty\} \), such that:

1. \( v_\epsilon(t) = t \) for \( t \geq -t_0 - \epsilon \), \( v_\epsilon(t) = \text{constant} \) for \( t < -t_0 - B + \epsilon \) and are pointwise convergent to \( v_{t_0,B} \), when \( \epsilon \to 0 \);

2. \( v_\epsilon'(t) \) are pointwise convergent to \( \frac{1}{B} \mathbb{I}_{(-t_0-B,-t_0)} \), when \( \epsilon \to 0 \), and \( 0 \leq v_\epsilon''(t) \leq \frac{2}{B} \mathbb{I}_{(-t_0-B+\epsilon,-t_0-\epsilon)} \) for any \( t \in \mathbb{R} \);

3. \( v_\epsilon'(t) \) are pointwise convergent to \( b_{t_0,B}(t) \) which is a continuous function on \( \mathbb{R} \cup \{-\infty\} \), when \( \epsilon \to 0 \), and \( 0 \leq v_\epsilon'(t) \leq 1 \) for any \( t \in \mathbb{R} \).

One can construct the family \( \{v_\epsilon\}_{\epsilon \in (0, \frac{1}{4}B)} \) by the setting

\[
(5.4) \quad v_\epsilon(t) := \int_0^t \left( \int_{-\infty}^{t_1} \left( \frac{1}{B - 4\epsilon} \mathbb{I}_{(-t_0-B+2\epsilon,-t_0-2\epsilon)} * \rho_{\frac{1}{2}\epsilon} \right)(s)ds \right)dt_1,
\]

where \( \rho_{\frac{1}{2}\epsilon} \) is the kernel of convolution satisfying \( \text{supp}(\rho_{\frac{1}{2}\epsilon}) \subset (-\frac{1}{4}\epsilon, \frac{1}{4}\epsilon) \). Then it follows that

\[
v_\epsilon''(t) = \frac{1}{B - 4\epsilon} \mathbb{I}_{(-t_0-B+2\epsilon,-t_0-2\epsilon)} * \rho_{\frac{1}{2}\epsilon}(t),
\]

and

\[
v_\epsilon'(t) = \int_{-\infty}^{t} \left( \frac{1}{B - 4\epsilon} \mathbb{I}_{(-t_0-B+2\epsilon,-t_0-2\epsilon)} * \rho_{\frac{1}{2}\epsilon} \right)(s)ds.
\]

It is clear that \( \lim_{\epsilon \to 0} v_\epsilon(t) = v_{t_0,B}(t) \), and \( \lim_{\epsilon \to 0} v_\epsilon'(t) = b_{t_0,B}(t) \).

Let \( \eta = s(-v_\epsilon'(\psi_m)) \) and \( \phi = u(-v_\epsilon'(\psi_m)) \), where \( s \in C^\infty((0, +\infty)) \) satisfies \( s \geq 0 \) and \( s' > 0 \), and \( u \in C^\infty((0, +\infty)) \), satisfies \( \lim_{t \to +\infty} u(t) \) exists, such that \( u''s - s'' > 0 \), and \( s' = u's = 1 \). It follows from \( \sup_m \sup_D \psi_m < 0 \) that \( \phi = u(-v_\epsilon'(\psi_m)) \) are uniformly bounded on \( D \) with respect to \( m \) and \( \epsilon \), and \( u(-v_\epsilon'(\psi)) \) are uniformly bounded on \( D \) with respect to \( \epsilon \). Let \( \Phi = \phi + \varphi_m \).

**Step 2: Solving \( \bar{\partial} \)-equation with smooth polar function and smooth weight**

Now let \( \alpha = \sum_{j=1}^n \alpha_j d\bar{z}^j \in \text{Dom}_D(\bar{\partial}^* \cap \text{Ker}(\bar{\partial}) \cap C^\infty((0,1))(\bar{D}) \). It follows from Cauchy-Schwarz inequality that

\[
2\text{Re}(\bar{\partial}_k \alpha, \alpha_m(\bar{\partial} \eta)^n)_{D,\Phi} \geq - \int_D g^{-1} |\bar{\partial}_k \alpha|^2 e^{-\Phi} + \sum_{j,k=1}^n \int_D -g(\bar{\partial}_j \eta)(\bar{\partial}_k \eta)\alpha_j \bar{\alpha}_k e^{-\Phi},
\]

where \( g \) is a positive continuous function on \( D \).
Using Lemma 5.5 and inequality (5.5), since \( s \geq 0 \), \( \varphi_m \) is a plurisubharmonic function on \( \overline{D} \) and \( D \) is a strongly pseudoconvex domain, we get

\[
\int_D (\eta + g^{-1}) |\overline{\partial} \eta^s |^2 e^{-\Phi} 
\]

\[\geq \sum_{j,k=1}^{n} \int_D (-\partial_j \partial_k \eta + \eta \partial_j \partial_k \Phi - g(\partial_j \eta)(\partial_k \eta)) \alpha_j \alpha_k e^{-\Phi} \]

\[\geq \sum_{j,k=1}^{n} \int_D (-\partial_j \tilde{\partial}_k \eta + \eta \partial_j \tilde{\partial}_k \Phi - g(\partial_j \eta)(\partial_k \eta)) \alpha_j \alpha_k e^{-\Phi}. \]

We need some calculations to determine \( g \).

We have

\[
\partial_j \partial_k \eta = -s'(-v_e(\psi_m)) \partial_j \partial_k (v_e(\psi_m)) + s''(-v_e(\psi_m)) \partial_j v_e(\psi_m) \partial_k v_e(\psi_m),
\]

and

\[
\partial_j \tilde{\partial}_k \phi = -u'(-v_e(\psi_m)) \partial_j \tilde{\partial}_k (v_e(\psi_m)) + u''(-v_e(\psi_m)) \partial_j v_e(\psi_m) \tilde{\partial}_k v_e(\psi_m)
\]

for any \( j, k(1 \leq j, k \leq n) \).

Then we have

\[
\sum_{j,k=1}^{n} (-\partial_j \partial_k \eta + \eta \partial_j \partial_k \Phi - g(\partial_j \eta)(\partial_k \eta)) \alpha_j \alpha_k = (s' - su') \sum_{j,k=1}^{n} \partial_j \tilde{\partial}_k v_e(\psi_m) \alpha_j \alpha_k
\]

\[
+ ((u'' s - s'') - gs'^2) \sum_{j,k=1}^{n} \partial_j (-v_e(\psi_m)) \tilde{\partial}_k (-v_e(\psi_m)) \alpha_j \alpha_k
\]

\[= (s' - su') \sum_{j,k=1}^{n} (v'_e(\psi_m) \partial_j \tilde{\partial}_k \psi_m + v''_e(\psi_m) \partial_j (\psi_m) \tilde{\partial}_k (\psi_m)) \alpha_j \alpha_k
\]

\[+ ((u'' s - s'') - gs'^2) \sum_{j,k=1}^{n} \partial_j (-v_e(\psi_m)) \tilde{\partial}_k (-v_e(\psi_m)) \alpha_j \alpha_k.
\]

We omit composite item \(-v_e(\psi_m)\) after \( s' - su' \) and \((u'' s - s'') - gs'^2\) in the above equalities.

Let \( g = \frac{u'' - s''}{s'^2}(-v_e(\psi_m)) \). It follows that \( \eta + g^{-1} = \left( s + \frac{s'^2}{u'' s - s''} \right)(-v_e(\psi_m)). \)

As \( v'_e \geq 0 \) and \( s' - su' = 1 \), using inequality (5.6), we obtain that

\[
\int_D (\eta + g^{-1}) |\overline{\partial} \eta^s |^2 e^{-\Phi} \geq \int_D v''_e(\psi_m)|\alpha_e(\partial \psi_m)|^2 e^{-\Phi}.
\]

As \( F \) is holomorphic on \( \{ \psi < -t_0 \} \) and \( \text{supp}(v''_e(\psi_m)) \subset \{ \psi < -t_0 \} \), then \( \lambda := \partial((1 - v'_e(\psi_m))F) \) is well-defined and smooth on \( D \). By the definition of
contraction, Cauchy-Schwarz inequality and inequality (5.10), it follows that
\[
|\lambda(\alpha)_{D,\phi}|^2 = |(\nu''(\psi_m)\bar{\partial}\psi_m F, \alpha)_{D,\phi}|^2 \\
= |(\nu''(\psi_m)F, \alpha_{L}(\bar{\partial}\psi_m)^2)_{D,\phi}|^2 \\
\leq \left( \int_D |v''(\psi_m)|^2 e^{-\Phi} \right) \left( \int_D |v''(\psi_m)(\bar{\partial}\psi_m)^2| e^{-\Phi} \right) \\
\leq \left( \int_D |v''(\psi_m)|^2 e^{-\Phi} \right) \left( \int_D (\eta + \varphi g^{-1})|\bar{\partial}\psi\alpha|^2 e^{-\Phi} \right).
\]
(5.11)

Let \( \mu := (\eta + g^{-1})^{-1} \). By Lemma 5.6 then we have locally \( L^1 \) function \( u_{m,m',\varepsilon} \) on \( D \) such that \( \partial u_{m,m',\varepsilon} = \lambda \), and
\[
\int_D |u_{m,m',\varepsilon}|^2 (\eta + g^{-1})^{-1} e^{-\Phi} \leq \int_D v''(\psi_m)|F|^2 e^{-\Phi}.
\]
(5.12)

Assume that we can choose \( \eta \) and \( \phi \) such that \( e^{\nu(\psi_m)}e^{\phi c(-\nu(\psi_m))} = (\eta + g^{-1})^{-1} \). Then inequality (5.12) becomes
\[
\int_D |u_{m,m',\varepsilon}|^2 e^{\nu(\psi_m) - \varphi m'}c(-\nu(\psi_m)) \leq \int_D v''(\psi_m)|F|^2 e^{-\phi - \varphi m'}.
\]
(5.13)

Let \( F_{m,m',\varepsilon} := -u_{m,m',\varepsilon} + (1 - v(\psi_m))F \). It is clear that \( F_{m,m',\varepsilon} \) is holomorphic on \( D \). Then inequality (5.13) becomes
\[
\int_D |F_{m,m',\varepsilon} - (1 - v(\psi_m))F|^2 e^{\nu(\psi_m) - \varphi m'}c(-\nu(\psi_m))
\]
\[
\leq \int_D (v''(\psi_m))|F|^2 e^{-\phi - \varphi m'}.
\]
(5.14)

**Step 3: Singular polar function and smooth weight**

As \( \sup_{m,\varepsilon} |\nu| = \sup_{m,\varepsilon} |u(-\nu(\psi_m))| < +\infty \) and \( \varphi_{m'} \) is continuous on \( \bar{D} \), then \( \sup_{m,\varepsilon} e^{-\phi - \varphi m'} < +\infty \). Note that
\[
v''(\psi_m)|F|^2 e^{-\phi - \varphi m'} \leq \frac{2}{B} I(\nu < -\lambda_0)|F|^2 \sup_{m,\varepsilon} e^{-\phi - \varphi m'}
\]
on D, then it follows from inequality (5.3) and the dominated convergence theorem that
\[
\lim_{m \to +\infty} \int_D v''(\psi_m)|F|^2 e^{-\phi - \varphi m'} = \int_D v''(\psi)|F|^2 e^{-u(\nu(\psi)) - \varphi m'}.
\]
(5.15)

Note that \( \inf_m \inf_D e^{\nu(\psi_m) - \varphi m'}c(-\nu(\psi_m)) > 0 \), then it follows from inequality (5.14) that \( \sup_m \int_D |F_{m,m',\varepsilon} - (1 - v(\psi_m))F|^2 < +\infty \). Note that
\[
|(1 - v(\psi_m))F| \leq |I|_{\nu < -\lambda_0}|F|,
\]
then it follows from inequality (5.3) that \( \sup_m \int_D |F_{m,m',\varepsilon}|^2 < +\infty \), which implies that there exists a subsequence of \( \{F_{m,m',\varepsilon}\}_{m \in \mathbb{N}} \) (also denoted by \( F_{m,m',\varepsilon} \)) compactly convergent to a holomorphic function \( F_{m',\varepsilon} \) on \( D \).

Note that \( e^{\nu(\psi_m) - \varphi m'}c(-\nu(\psi_m)) \) are uniformly bounded on \( D \) with respect to \( m \), then it follows from \( |F_{m,m',\varepsilon} - (1 - v(\psi_m))F|^2 \leq 2(|F_{m,m',\varepsilon}|^2 + |(1 -}
\[ v'(\psi_m)F^2 \leq 2(|F_{m,m',c}|^2 + \|F_{\psi<-\tau_0}\|^2) \] and the dominated convergence theorem that

\[
\lim_{m \to +\infty} \int_K |F_{m,m',c} - (1 - v'_e(\psi_m))F|^2 e^{v_e(\psi_m) - \varphi_{m'}} c(-v_e(\psi_m))\]

\[
= \int_K |F_{m',c} - (1 - v'_e(\psi))F|^2 e^{v_e(\psi) - \varphi_{m'}} c(-v_e(\psi))
\]

holds for any compact subset \( K \) on \( D \). Combining with inequality (5.14), equality (5.17) and equality (5.19), one can obtain that

\[
\int_K |F_{m',c} - (1 - v'_e(\psi))F|^2 e^{v_e(\psi) - \varphi_{m'}} c(-v_e(\psi)) \leq \int_D v''(\psi)|F|^2 e^{-u(-v_e(\psi)) - \varphi_{m'}}
\]

which implies

\[
\int_D |F_{m',c} - (1 - v'_e(\psi))F|^2 e^{v_e(\psi) - \varphi_{m'}} c(-v_e(\psi)) \leq \int_D v''(\psi)|F|^2 e^{-u(-v_e(\psi)) - \varphi_{m'}}.
\]

**Step 4: Nonsmooth cut-off function**

Note that \( \sup_e \sup_D e^{-u(-v_e(\psi)) - \varphi_{m'}} < +\infty \), and

\[ v''(\psi)|F|^2 e^{-u(-v_e(\psi)) - \varphi_{m'}} \leq \frac{2}{\beta} \sup_{-t_0-B<\psi<-t_0} |F|^2 e^{-u(-v_{0,B}(\psi)) - \varphi_{m'}}. \]

then it follows from inequality (5.20) and the dominated convergence theorem that

\[
\lim_{\epsilon \to 0} \int_D v''(\psi)|F|^2 e^{-u(-v_e(\psi)) - \varphi_{m'}}
\]

\[
= \int_D \frac{1}{\beta} \sup_{-t_0-B<\psi<-t_0} |F|^2 e^{-u(-v_{0,B}(\psi)) - \varphi_{m'}} - \frac{1}{\beta} \sup_{-t_0-B<\psi<-t_0} |F|^2 e^{-\varphi_{m'}} < +\infty.
\]

Note that \( \inf_e \inf_D e^{v_e(\psi) - \varphi_{m'}} c(-v_e(\psi)) > 0 \), then it follows from inequality (5.19) and (5.20) that \( \sup_e \int_D |F_{m',c} - (1 - v'_e(\psi))F|^2 < +\infty \). Combining with

\[
\sup_e \int_D |(1 - v'_e(\psi))F|^2 \leq \int_D \|F_{\psi<-\tau_0}\|^2 < +\infty,
\]

one can obtain that \( \sup_e \int_D |F_{m',c}|^2 < +\infty \), which implies that there exists a subsequence of \( \{F_{m',c}\}_{c>0} \) (also denoted by \( \{F_{m',c}\}_{c>0} \)) compactly convergent to a holomorphic function \( F_{m'} \) on \( D \).

Note that \( \sup_e \sup_D e^{v_e(\psi) - \varphi_{m'}} c(-v_e(\psi)) \leq +\infty \) and \( |F_{m',c} - (1 - v'_e(\psi))F|^2 \leq 2(|F_{m',c}|^2 + \|F_{\psi<-\tau_0}\|^2) \), then it follows from inequality (5.21) and the dominated convergence theorem on any given \( K \subset D \) that

\[
\lim_{\epsilon \to 0} \int_K |F_{m',c} - (1 - v'_e(\psi))F|^2 e^{v_e(\psi) - \varphi_{m'}} c(-v_e(\psi))
\]

\[
= \int_K |F_{m'} - (1 - b_{t_0,B}(\psi))F|^2 e^{v_{0,B}(\psi) - \varphi_{m'}} c(-v_{0,B}(\psi)).
\]
Combining with inequality (5.19), inequality (5.20) and equality (5.22), one can obtain that
\[
\int_K |F_{m'} - (1 - b_{t_0,B}(\psi))F|^2 e^{v_{t_0,B}(\psi)} - \varphi_{m'} c(-v_{t_0,B}(\psi)) \leq \left( \sup_{D} e^{-u(-v_{t_0,B}(\psi))} \right) \int_D \frac{1}{B} \int_{-t_0 - B < \psi < -t_0} |F|^2 e^{-\varphi_{m'}}
\]
which implies
\[
\int_D |F_{m'} - (1 - b_{t_0,B}(\psi))F|^2 e^{v_{t_0,B}(\psi)} - \varphi_{m'} c(-v_{t_0,B}(\psi)) \leq \left( \sup_{D} e^{-u(-v_{t_0,B}(\psi))} \right) \int_D \frac{1}{B} \int_{-t_0 - B < \psi < -t_0} |F|^2 e^{-\varphi_{m'}}. \tag{5.24}
\]

**Step 5: Singular weight**

Note that
\[
\int_D \frac{1}{B} \int_{-t_0 - B < \psi < -t_0} |F|^2 e^{-\varphi_{m'}} \leq \int_D \frac{1}{B} \int_{-t_0 - B < \psi < -t_0} |F|^2 e^{-\varphi} < +\infty,
\]
and \(\sup_{D} e^{-u(-v_{t_0,B}(\psi))} < +\infty\), then it follows from (5.24) that
\[
\sup_{m'} \int_D |F_{m'} - (1 - b_{t_0,B}(\psi))F|^2 e^{v_{t_0,B}(\psi)} - \varphi_{m'} c(-v_{t_0,B}(\psi)) < +\infty.
\]
Combining with \(\inf_{m'} \inf_{D} e^{v_{t_0,B}(\psi)} - \varphi_{m'} c(-v_{t_0,B}(\psi)) > 0\), one can obtain that
\[
\sup_{m'} \int_D |F_{m'} - (1 - b_{t_0,B}(\psi))F|^2 < +\infty.
\]
Note that
\[
\int_D |(1 - b_{t_0,B}(\psi))F|^2 \leq \int_D \|\psi < -t_0\| F|^2 < +\infty. \tag{5.26}
\]

Then \(\sup_{m'} \int_D |F_{m'}|^2 < +\infty\), which implies that there exists a compactly convergent subsequence of \(\{F_{m'}\}\) (also denoted by \(\{F_{m'}\}\)), which is convergent a holomorphic function \(\tilde{F}\) on \(D\).

Note that \(\sup_{D} e^{v_{t_0,B}(\psi)} - \varphi_{m'} c(-v_{t_0,B}(\psi)) < +\infty\), then it follows from inequality (5.26) and the dominated convergence theorem on any given compact subset \(K\) of \(D\) that
\[
\lim_{m' \to +\infty} \int_K |F_{m'} - (1 - b_{t_0,B}(\psi))F|^2 e^{v_{t_0,B}(\psi)} - \varphi_{m'} c(-v_{t_0,B}(\psi)) = \int_K |\tilde{F} - (1 - b_{t_0,B}(\psi))F|^2 e^{v_{t_0,B}(\psi)} - \varphi_{m'} c(-v_{t_0,B}(\psi)). \tag{5.27}
\]
Note that for any \( m'' \geq m' \), \( \varphi_{m''} \geq \varphi_{m'} \) holds, then it follows from inequality (5.24) and inequality (5.25) that

\[
\lim_{m'' \to +\infty} \int_K |F_{m''} - (1 - b_{t_0, B}(\psi)) F|^{2} e^{\varphi_m c(-v_{t_0, B}(\psi))} (5.31)
\]

\[
\leq \limsup_{m'' \to +\infty} \int_K |F_{m''} - (1 - b_{t_0, B}(\psi)) F|^{2} e^{\varphi_m c(-v_{t_0, B}(\psi))} (5.32)
\]

\[
\leq \limsup_{m'' \to +\infty} \left( \sup_{D} e^{u(-v_{t_0, B}(\psi))} \right) \int_D \frac{1}{B} \left\{ -t_0 < \psi < -t_0 \right\} |F|^{2} e^{-\varphi_{m''}} (5.33)
\]

\[
\leq \left( \sup_{D} e^{u(-v_{t_0, B}(\psi))} \right) C < +\infty. (5.34)
\]

Combining with equality (5.27), one can obtain that

\[
\int_D \left| F - (1 - b_{t_0, B}(\psi)) F \right|^{2} e^{\varphi_m c(-v_{t_0, B}(\psi))} (5.29)
\]

for any compact subset \( K \) of \( D \), which implies

\[
\int_D \left| F - (1 - b_{t_0, B}(\psi)) F \right|^{2} e^{\varphi_m c(-v_{t_0, B}(\psi))} (5.29)
\]

When \( m' \to +\infty \), it follows from the monotone convergence theorem that

\[
\int_D \left| F - (1 - b_{t_0, B}(\psi)) F \right|^{2} e^{\varphi_m c(-v_{t_0, B}(\psi))} (5.29)
\]

Step 6: ODE system

It suffices to find \( \eta \) and \( \phi \) such that \( \eta + g^{-1} = e^{-v_{\psi_m}} e^{-\phi} \frac{1}{c(-v_{\psi_m})} \) on \( D \) and \( s' - u's = 1 \). As \( \eta = s(-v_{\psi_m}) \) and \( \phi = u(-v_{\psi_m}) \), we have \( (\eta + g^{-1}) e^{v_{\psi_m}} c^\phi = \left( s + \frac{s'^2}{u's - s''} \right) e^{-u'} \circ (-v_{\psi_m}) \).

Summarizing the above discussion about \( s \) and \( u \), we are naturally led to a system of ODEs (see [12, 13, 14, 17]):

\[
(5.30)
\]

1. \( \left( s + \frac{s'^2}{u's - s''} \right) e^{-u'} = \frac{1}{c(t)}, \)

2. \( s' - su' = 1, \)

where \( t \in (0, +\infty) \).

It is not hard to solve the ODE system (5.30) and get \( u(t) = -\log \left( \int_0^t c(t_1) e^{-t_1} dt_1 \right) \) and \( s(t) = \frac{\int_0^t (\int_0^t c(t_1) e^{-t_1} dt_1) dt_2}{\int_0^t c(t_1) e^{-t_1} dt_1} \) (see [14]). It follows that \( s \in C^\infty((0, +\infty)) \) satisfies \( s > 0 \) and \( s' > 0 \), \( \lim_{t \to +\infty} u(t) = -\log \left( \int_0^{+\infty} c(t_1) e^{-t_1} dt_1 \right) \) and \( u \in C^\infty((0, +\infty)) \) satisfies \( u''s - s'' > 0 \).

As \( u(t) = -\log \left( \int_0^t c(t_1) e^{-t_1} dt_1 \right) \) is decreasing with respect to \( t \), then it follows from \( 0 \geq v(t) \geq \max\{t, -t_0 - B_0\} \geq -t_0 - B_0 \) for any \( t \leq 0 \) that

\[
(5.31) \sup_{D} e^{-u(-v_{t_0, B}(\psi))} \leq \sup_{t \in [0, t_0 + B]} e^{-u(t)} = \int_0^{t_0 + B} c(t_1) e^{-t_1} dt_1,
\]
therefore we are done. Thus we have proved Lemma 2.1.

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