We investigate the effects of low-lying fermion modes on the QCD partition function in the $\epsilon$-regime. With the overlap Dirac operator we calculate several tens of low-lying fermion eigenvalues on the quenched lattice. By partially incorporating the fermion determinant through the truncated determinant approximation, we calculate the partition function and other related quantities for $N_f = 1$ and compare them with the theoretical predictions obtained by Leutwyler and Smilga.

1. Introduction

The low energy behavior of QCD, and therefore lattice QCD, is effectively described by the chiral Lagrangian. Approaching the massless limit, the pion’s Compton wave length becomes longer than the spatial extent of the lattice, and the zero momentum modes dominate the partition function. This is the so-called $\epsilon$-regime of QCD, where various properties of the QCD partition function is known analytically [1]. For instance, for one-flavor QCD the partition function is given by

$$Z = e^{m \Sigma V \cos \theta}$$

for finite $\theta$ at the leading order. Here, $m$ is the quark mass, $\Sigma$ is the chiral condensate, and $V$ is the space-time volume. It can be divided into topological sectors as

$$Z_\nu(m) = \int d\theta e^{i\nu \theta} Z = I_\nu(m \Sigma V),$$

(1)

where $I_\nu(x)$ denotes the modified Bessel function, and $\nu$ is the topological charge.

For lattice QCD to study the small quark mass region, it is therefore the first step to confirm such analytic results. Eventually one can determine the low energy constants of chiral perturbation theory by measuring correlation functions in the $\epsilon$-regime, as discussed for example in [2]. In this exploratory study we investigate the mass dependence of the partition function, the sum rules for eigenvalues of the Dirac operator, as well as the topological susceptibility. The effect of low-lying fermion eigenmodes is incorporated using the reweighting technique, and thus we study the above relations in the one-flavor $N_f = 1$ case.

Since the chiral symmetry is essential in deriving the partition function in the $\epsilon$-regime, we employ the overlap Dirac operator $D = 1 + s \gamma_5 \text{sgn}(H_W)$. To compute the sign function $\text{sgn}(H_W)$, 14 lowest eigenmodes of $H_W$ are treated exactly and we used the Chebyshev polynomial of degree 100-200 to approximate the rest of the eigenmodes. On a $10^4$ lattice at $\beta = 5.85$ we calculated 50 lowest eigenvalues of $D$ using the implicit restarted Arnoldi method (ARPACK). The topological charge is obtained from the number of zero modes for each gauge configuration, and we accumulated 168, 290, and 149 configurations for $|\nu| = 0, 1,$ and 2, respectively.

2. Truncated determinant approximation

In order to study the effect of fermion determinant, we use an approximation to include only the low-lying fermion eigenmodes. Namely, the fermion determinant is replaced by a product

$$\prod_{i=1}^{N_{\max}} (|\lambda_i|^2 + m^2),$$

where $\lambda_i$ is the $i$-th lowest eigenvalue of the overlap Dirac operator $D$ for a given configuration. If $N_{\max}$ is large enough to cover physically relevant eigenmodes, this approximation is expected to reproduce the right physics. For our choice of parameters $\lambda_{10}$ lies...
around 700 MeV, and therefore should already be large enough to be treated as the ultraviolet mode. The neglected eigenmodes are the ultraviolet modes, which should be irrelevant to the low energy physics. Such idea was implemented previously as an algorithm for the Wilson fermion \[3\], in which the ultraviolet modes are also incorporated using either the effective gauge action or the multi-boson method. See also \[4\].

In order to demonstrate how the truncation works, in Fig. 1 we plot a correlation between the truncated determinants with $N_{\text{max}} = 10$ and 50 for $m = 0$. Each dot in the plot represents a gauge configuration, and the strong correlation among the different truncation indicates that the product $\prod_{i=11}^{50} |\lambda_i|^2$ is mostly independent of gauge configuration. In Fig. 2 we show how the truncated determinant varies as a function of $N_{\text{max}}$. We can observe a rapid change below $N_{\text{max}} \sim 10$, with which the order of magnitude is determined. The values still changes slowly below $N_{\text{max}} \sim 50$, and stays almost flat above it.

3. Partition function

In Fig. 3 we show the mass dependence of the partition function $Z_\nu(m)/Z(m)_{|\theta=0}$ for the topological sectors $|\nu| = 0, 1, $ and 2 at $N_f = 1$. The truncated determinant with $N_{\text{max}} = 50$ is used. We can see that the data agree with the analytic expectation $I_\nu(m\Sigma V)/\exp(m\Sigma V)$ \[I\] shown by the curves. From this fit we obtain $\Sigma = (243 \text{ MeV})^3$.

By differentiating the partition function with respect to the quark mass, Leutwyler and Smilga obtained a sum rule for the eigenvalues \[I\]

$$\left\langle \sum_n \frac{1}{(\lambda_n \Sigma V)^2} \right\rangle_\nu = \frac{1}{4(\nu + 1)}.$$ (2)

Left hand side of this equation is quadratically divergent in the ultraviolet region, since the eigenmode distribution behaves as $\lambda^3$ in the bulk.
Therefore the sum rule (2) makes sense only after the ultraviolet divergence is regulated and the particular volume dependence is extracted. Here, instead, we consider a difference of (2) between different topological sectors. Because the bulk distribution is expected to be independent of topology, the ultraviolet divergence cancels in such differences. Fig. 4 shows these differences as a function of the ultraviolet cutoff. We find that the results are completely independent of the cutoff and agree with the analytical predictions.

4. Topological susceptibility

Topological susceptibility $\chi$ is obtained from the partition function by differentiating with respect to $\theta^2$. On the lattice one can simply calculate the expectation value of the topological charge squared $\langle \nu^2 \rangle = \chi V$. In Fig. 5 we plot $\langle \nu^2 \rangle$ as a function of quark mass for both $N_f = 0$ and 1. Near the massless limit the topological susceptibility grows linearly for $N_f = 1$ as expected from the form of the partition function $e^{m \Sigma V} \cos \theta$. For larger quark masses it saturates toward the quenched value corresponding to $\chi = (205 \text{ MeV})^4$. This behavior was previously observed in [5]. By fitting the slope near the massless limit we obtain $\Sigma = (238 \text{ MeV})^3$, which agrees well with the determination from the partition function itself.

5. Conclusions

Using the truncated determinant approximation of the overlap Dirac operator, we confirmed that the Leutwyler-Smilga’s analytic predictions for the QCD partition function is satisfied for $N_f = 1$. Results are stable against the number of eigenmodes included, if it is greater than 10 ($\lambda_{10} \sim 700 \text{ MeV}$). It suggests that the dynamical fermion algorithms designed to separate low and high eigenmodes, such as [3], work effectively. For larger volume or smaller lattice spacing the number of eigenvalues to cover the equivalent physical region becomes much larger (proportional to the lattice volume), and their calculation could be numerically prohibitive. Still it would be helpful to isolate the low-lying modes, as they account for the dominant part of the fermion determinant.

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