THE GENERIC BEHAVIOR OF SOLUTIONS TO SOME EVOLUTION EQUATIONS: ASYMPTOTICS AND SOBOLEV NORMS

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Abstract. In this paper, we study the generic behavior of the solutions to a large class of evolution equations. The Schrödinger evolution is considered as an application.

In this paper, we develop methods to control the generic behavior of solutions to various evolution equations. The word “generic” will refer to the “coupling constant” that appears in front of the diagonal operator (a differential operator in most applications) which is perturbed by a potential. We were motivated by recent results where the behavior of Sobolev norms was studied for various evolution equations \cite{3, 9, 11}. The general situation of fixed coupling constant was studied in many papers (e.g., \cite{8} and references therein).

The structure of the paper is as follows. In the first section, we prove simple results for the “typical” behavior of solutions to $2 \times 2$ system of differential equations which preserve $\ell^2$ norm of the initial value in Cauchy problem. The section 2 deals with “integrable” case when the transport equation is considered on the circle. In section 3, we prove results similar to those in section 1 but for the general $N \times N$ systems. The section 4 contains the applications of these results to evolution in Hilbert spaces, e.g., the non-stationary Schrödinger equation. In the last section, we consider the most difficult case when the so-called “gap condition” deteriorates in time. We handle only the short-range case, the general situation is far more difficult and will be considered elsewhere. The paper is concluded with Appendix which contains some well-known results we use in the main text. The first two sections and Appendix have mostly pedagogical value, the main results are in sections 3 and 5.

In the text, we will use the following notations: for $f_{1(2)} \geq 0$, $f_1 \lesssim f_2$ means there is a constant $C$ so that $f_1 \leq Cf_2$. The symbols $e_j$ will denote the standard basis vectors in $\mathbb{C}^N$.

1. The model case of $2 \times 2$ system

We start this section with the following model system of two ODEs.

$$X_t = i(A + V(t))X, \quad X(0) = I$$

(1)

where

$$A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \quad \lambda_{1(2)} \in \mathbb{R}$$

(2)
and Hermitian $V(t)$

$$V(t) = \begin{pmatrix} v_{11}(t) & v_{12}(t) \\ v_{12}(t) & v_{22}(t) \end{pmatrix} \to 0, \quad t \to \infty$$

in some way to be specified below. Notice that $X(t)$ is unitary in this setting. The standard problem to address is, of course, the long-time asymptotics of $X(t)$. In the case of strong decay, i.e. when $V(t) \in L^1(\mathbb{R}^+)$, $X(t)$ has a limit as can be easily seen from iteration of the corresponding integral equations. Take

$$X = UY$$

where

$$U = \begin{pmatrix} \exp \left[ i\lambda_1 t + i \int_0^t v_{11}(\tau) d\tau \right] & 0 \\ 0 & \exp \left[ i\lambda_2 t + i \int_0^t v_{22}(\tau) d\tau \right] \end{pmatrix}$$

That reduces the problem to

$$Y_t = i \begin{pmatrix} 0 & q(t) \\ \bar{q}(t) & 0 \end{pmatrix} Y, \quad Y(0) = I \tag{3}$$

with

$$q(t) = v_{12}(t) \exp \left[ i(\lambda_2 - \lambda_1) t + i \int_0^t (v_{22}(\tau) - v_{11}(\tau)) d\tau \right]$$

and it decay fast as $v_{12}$. Does $Y(t)$ have a limit as $t \to \infty$? Clearly, for $q(t) \in L^1(\mathbb{R}^+)$ the answer is yes but already for $q(t) \in L^p(\mathbb{R}^+), \quad p > 1$ this is not the case in general. Indeed, take $q(t) = ie$ on $t \in [0, \varepsilon^{-1}]$. Then, on that interval,

$$Y(t) = \begin{pmatrix} \cos(\varepsilon t) & -\sin(\varepsilon t) \\ \sin(\varepsilon t) & \cos(\varepsilon t) \end{pmatrix}$$

a rotation by $\varepsilon t$. Clearly, $\|q\|_{L^1([0, \varepsilon^{-1}]} = 1$ but $\|q\|_{L^p([0, \varepsilon^{-1}]}) = \varepsilon^{1-p^{-1}}$. Therefore, by taking $q(t) = \varepsilon_n, \varepsilon_n \to 0$ on consecutive intervals, we can arrange for $q$ to be in $L^p(\mathbb{R}^+)$ but the solution has no limit. In the meantime, we will see that it makes sense to talk about the following problem: let $\lambda_1 = 0, \lambda_2 = k, v_{11} = v_{22} = 0$ and $v_{12} = q(t)$. Thus,

$$X_t = i \begin{pmatrix} 0 & q(t) \\ \bar{q}(t) & 0 \end{pmatrix} X, \quad X(0) = I \tag{4}$$

and we will study the asymptotics for generic frequency $k$. As the discrete version of (1) one might consider the following recursion

$$X_{n+1}(z) = \rho_n^{-1/2} \begin{pmatrix} 1 & \gamma_n \\ -\bar{\gamma}_n & 1 \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} X_n(z)$$

$$X_0(z) = I_{2 \times 2}, \quad |z| = 1, \quad \rho_n = 1 + |\gamma_n|^2$$

This is different from the recursion for polynomials orthogonal on the circle only by sign “−” in front of $\gamma_n$ and by slightly different formula for $\rho_n$.

**Remark 1.** The simple calculation shows that

$$X(t, k) = \begin{pmatrix} x_{11}(t, k) & x_{12}(t, k) \\ x_{21}(t, k) & x_{22}(t, k) \end{pmatrix} = \begin{pmatrix} x_{11}(t, k) & x_{12}(t, k) \\ -e^{ikt}x_{12}(t, k) & e^{ikt}x_{11}(t, k) \end{pmatrix}$$

for real $k$. 
Proof. Is it true that for \( q(t) \in L^2(\mathbb{R}^+) \) and Lebesgue a.e. \( k \in \mathbb{R} \), the limit \( \lim_{t \to \infty} X(t, k) \) exists? If so, how do the points of convergence correspond to frequencies \( k \) for which \( |Mq|(k) < \infty \), where \( Mq \) is the Carleson-Hunt maximal function?

The similar problem is known for the Krein systems \([4]\)

\[
Z_t = \begin{bmatrix} ik & \bar{q}(t) \\ q(t) & 0 \end{bmatrix}, \quad Z(0) = I, \quad k \in \mathbb{R}
\]

In this case, though, the solution \( Z \) is \( J \)-unitary, with

\[
J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}
\]

We will consider \([4]\) and will prove a simple result which has an analog in the theory of Krein systems (the so-called Szegő case) which gives a somewhat weaker type of convergence. That will be a warm-up for a later discussion of the general case. Consider the first column \((x_{11}(t, k), x_{21}(t, k))^t \) of \( X(t, k) \). The functions \( x_{11(21)}(t, k) \) are holomorphic in \( k \) and have the following properties for any fixed \( t \).

**Lemma 1.1.** For any locally integrable \( q(t) \), we have

(a) \( |x_{11}(t, k)|^2 + |x_{21}(t, k)|^2 = 1 - 2 \text{Im} k \int_0^t |x_{21}(\tau, k)|^2 d\tau, \quad k \in \mathbb{C} \)

(b) \( |x_{11(21)}(t, k)| \leq 1, \quad k \in \mathbb{C}^+, \text{ and } \|x_{21}(t, k)\|_2 \leq (2 \text{Im } k)^{-1/2} \text{ for } k \in \mathbb{C}^+ \).

**Proof.** This is a direct corollary of the differential equations. \( \square \)

Assuming, in addition, that \( q(t) \) is square summable, we get

**Lemma 1.2.** If \( q(t) \in L^2(\mathbb{R}^+) \), then

(a) \( x_{11}(t, k) \to x_{11}(\infty, k), \quad x_{21}(t, k) \to 0 \) uniformly on compacts in \( \mathbb{C}^+ \)

(b) \( |x_{11}(t, k)| \geq 1 - (\text{Im } k)^{-1}\|q\|_2^2, \quad \|x_{21}(t, k)\|_2 \leq \|q\|_2(\text{Im } k)^{-1}, \quad k \in \mathbb{C}^+ \)

(c) The function \( x_{11}(\infty, k) \) is nonzero function in the unit ball in \( H^\infty(\mathbb{C}^+) \) and \( x_{11}(k, t) \to x_{11}(\infty, k) \) in the weak-* sense on \( \mathbb{R} \)

(d) The following estimates hold

\[
\int_\mathbb{R} \ln |x_{11}(t, k)| dk \geq -\pi \int_0^t |q(\tau)|^2 d\tau
\]

\[
0 \geq \int_\mathbb{R} (|x_{11}(\infty, k)| - 1) dk \geq \int_\mathbb{R} \ln |x_{11}(\infty, k)| dk \geq -\pi \int_0^\infty |q(\tau)|^2 d\tau
\]

\[
[1 - x_{11}(\infty, k)]_{L^1(\mathbb{R})} \leq \|q\|_2^2, \quad \|1 - x_{11}(\infty, k)\|_{L^p(\mathbb{R})} \leq C(p)\|q\|_2^{2/p}, \quad 1 < p < \infty
\]

**Proof.** If \( x_{21} = e^{ikt}y \), then we have the following integral equations

\[
x_{11}(t, k) = 1 + i \int_0^t q(\tau)e^{ikt}y(\tau, k)d\tau, \quad y(t, k) = i \int_0^t \bar{q}(\tau)e^{-ikt}x_{11}(\tau, k)d\tau
\]
Therefore,
\[
x_{11}(t, k) = 1 - \int_0^t q(\tau_1)e^{ikt_1} \int_0^{\tau_1} \bar{q}(\tau_2)e^{-ikt_2} x_{11}(\tau_2, k) d\tau_2 d\tau_1
\] (8)

Due to the estimate \(|x_{11}(t, k)| \leq 1\) and Young's inequality, the \(L^1\) norm of the integrand in \(\tau_1\) is smaller than \(\|q\|^2(\text{Im } k)^{-1}\). That implies the uniform convergence on compacts of \(\mathbb{C}^+\) and the bound from below for \(x_{11}\). The representation
\[
x_{21}(t, k) = ie^{ikt} \int_0^t \bar{q}(\tau)e^{-ikt} x_{11}(\tau, k) d\tau
\]
gives uniform convergence to zero for \(x_{21}(t, k)\) and the estimate on its \(L^2\) norm. The function \(x_{11}(\infty, k)\) is obviously in the unit ball of \(H^\infty(\mathbb{C}^+)\) and is nonzero due to the estimate from below. Since \(x_{11}(t, k)(k + i)^{-1}, x_{11}(\infty, k)(k + i)^{-1}\) are both uniformly bounded in \(H^2(\mathbb{C}^+)\) and
\[
x_{11}(t, k)(k + i)^{-1} \rightarrow x_{11}(k + i)^{-1}
\]
as \(t \to \infty\) uniformly on the compacts in \(\mathbb{C}^+\), we have
\[
x_{11}(t, k)(k + i)^{-1} \rightarrow x_{11}(\infty, k)(k + i)^{-1}
\]
weakly in \(H^2(\mathbb{C}^+)\). Thus, we have the weak-* convergence of \(x_{11}(t, k)\) to \(x_{11}(\infty, k)\) on \(\mathbb{R}\).

Let us prove estimates in (d) now. Iterate (8) once and take the first term (the second allows stronger estimate)
\[
\int_0^t q(\tau_1)e^{ikt_1} \int_0^{\tau_1} \bar{q}(\tau_2)e^{-ikt_2} d\tau_2 d\tau_1
\]
by Plancherel, where \(\bar{q}(\omega)\) is the Fourier transform of \(q(\tau) \cdot \chi_{[0, t]}(\tau)\). Therefore, we have
\[
x_{11}(t, k) = 1 - ik^{-1} \int_0^t |q(\tau)|^2 d\tau + \pi(\text{Im } k)^{-1}
\]
as \(\text{Im } k \to +\infty\). The function \(\ln |x_{11}(t, k)|\) is subharmonic in \(\mathbb{C}^+\) so
\[
\pi^{-1} \int y \ln |x_{11}(t, s)| \frac{ds}{(s-x)^2 + y^2} \geq \ln |x_{11}(t, k)|, \quad k = x + iy
\]
Since \(\ln |x_{11}(t, s)| \leq 0\), we can take \(k = iy, y \to \infty\) and compare the first terms in asymptotics of l.h.s. and r.h.s. to get (3). The estimates for \(x_{11}(\infty, k)\) are deduced similarly.

Let us prove (e). Writing \(x_{11}(\infty, k) = 1 + h(k)\) yields \(2 \text{Re } h \leq -|h|^2\) since \(|x_{11}| \leq 1\) and
\[
\text{Re } h(iy) = \frac{y}{\pi} \int \frac{\text{Re } h(t)}{t^2 + y^2} dt
\]
Taking \(y \to +\infty\), we get
\[
\int_{\mathbb{R}} \text{Re } h(t) dt = -\pi \int_0^\infty |q(\tau)|^2 d\tau, \quad \text{Re } h(t) \leq 0, \quad |h(t)| \leq 2
\]
(this we might call the “trace formula” for our evolution). Therefore, we have
\[ [h]_{L^1(w(R))} \lesssim \|q\|_2^2 \]
and
\[ \|h\|_{L^p(R)} \leq C(p)\|q\|_2^{2/p}, \quad 1 < p < \infty \]
by interpolation.

In a similar way, one can show that \( x_{12}(t,k) \) has a limit in \( C^+ \) which we will denote by \( x_{12}(\infty,k) \). This, of course, implies that \( x_{12}(t,k) \to x_{12}(\infty,k) \) in the weak-* sense on the real line. In the next lemma, we will improve the convergence result.

**Lemma 1.3.** If \( q(t) \in L^2(R^+) \), then for any \( j = 1,2 \),
\[ \int_R |x_{1j}(t,k) - x_{1j}(\infty,k)|^2 dk \to 0, \quad t \to \infty \]

**Proof.** Take arbitrary \( t_1 < t_2 \) and notice that we have the following semigroup property
\[ X(t_1,t_2,k)X(t_1,k) = X(t_2,k) \]
Hence,
\[ x_{1j}(t_2,k) - x_{1j}(t_1,k) = (x_{11}(t_1,t_2,k) - 1)x_{1j}(t_1,k) + x_{12}(t_1,t_2,k)x_{2j}(t_1,k) \]
By (7),
\[ \|x_{11}(t_1,t_2,k) - 1\|_2 \to 0 \]
as \( t_1(2) \to \infty \) and
\[ |x_{12}(t_1,t_2,k)|^2 = 1 - |x_{11}(t_1,t_2,k)|^2 \leq -2 \ln |x_{11}(t_1,t_2,k)| \]
So, \( \|x_{12}(t_1,t_2,k)\|_2 \to 0 \) by (6). Thus, \( x_{1j}(t,k) \) is Cauchy in \( L^2(R) \) and must have the limit which will coincide with weak-* limit \( x_{1j}(\infty,k) \).

**Remark 2.** By Remark 1, \( e^{-ikt}x_{21}(t,k) \) and \( e^{-ikt}x_{22}(t,k) - 1 \) have limits in \( L^2(R) \). They are in fact related to boundary values of \( H^\infty(C^-) \) functions. Notice that
\[ |x_{11}(\infty,k)|^2 + |x_{21}(\infty,k)|^2 = 1, \quad \text{for a.e. } k \in R \]
as follows from the a.e. convergence over some subsequence.

2. **The Transport Equation on the Circle**

Another model very important for us is the transport equation. Consider
\[ u_t = ku_x + q(t,x)u \quad (9) \]
with the simplest initial condition \( u(0,x) = 1 \). We can either say that \( x \in T \) or \( x \in R \) but all functions are \( 2\pi \)-periodic in \( x \).

Notice that on the Fourier side, this equation is
\[ \hat{u}_t = -ikD\hat{u} + \hat{q}(t) * \hat{u}, \quad \hat{u}(0,n) = \delta_0 \]
where $D$ is diagonal: $(Dg)_n = ng(n)$ in $\ell^2(\mathbb{Z})$. We will work under the assumption that

$$\hat{q}(t,0) = \int_T q(t,x)dx = 0 \quad (10)$$

There is no any loss of generality since we can always satisfy this condition by subtracting $\hat{q}(t,0)I$ which corresponds to unimodular factor for $u(t,x)$.

**Remark 3.** Let $P\{0,1\}$ be the Fourier projection onto the zeroth and the first modes, $\hat{P}\{0,1\} = I - P\{0,1\}$. Then,

$$(P_{\{0,1\}}u)_t = k(P_{\{0,1\}}u)_x + (P_{\{0,1\}}qP_{\{0,1\}})(P_{\{0,1\}}u) + P_{\{0,1\}}qP_{\{0,1\}}u$$

Notice that if one drops the third term in the r.h.s., then the equation becomes equivalent to the model considered in the previous section.

Of course, the solution to (9) can be written explicitly. After periodization, we have

$$u(t,x,k) = \exp \left[ \int_0^t q(\tau, x+k(t-\tau))d\tau \right]$$

If, e.g., $q, q_x \in C([\mathbb{R}^+, \mathbb{T}])$, then this is the classical solution. The meaningful question is to study the asymptotics of $u(t, x-kt)$ or, rather,

$$\phi(t, x, k) = \int_0^t q(\tau, x-kt)d\tau$$

On the Fourier side (with respect to $x$ coordinate),

$$\hat{\phi}(t, n, k) = \int_0^t e^{ink}\hat{q}(\tau, n)d\tau$$

and

$$\hat{\phi}(t, 0, k) = 0, \quad t > 0$$

due to assumption (10). We have the following

**Lemma 2.1.** If $q(t, x) \in L^2(\mathbb{R}^+, \mathbb{T})$ and $\int_T q(t,x)dx = 0$, then

1. There is $\phi(\infty, x, k)$ such that

$$\int_{\mathbb{R}} \| \phi(t, x, k) - \phi(\infty, x, k) \|^2_{H^{1/2}(\mathbb{T})}dk \to 0$$

as $t \to \infty$.

2. For a.e. $k$,

$$\| \phi(t, x, k) - \phi(\infty, x, k) \|^2_{H^{1/2}(\mathbb{T})} \to 0 \quad t \to \infty$$

**Proof.** The proof is a trivial calculation. We have

$$\int_{\mathbb{R}} \| \phi(t, x, k) \|^2_{H^{1/2}(\mathbb{T})}dk =$$

$$= \sum_{n \neq 0} |n| \int_0^t \left| \int_0^t e^{ink\tau} \hat{q}(\tau, n)d\tau \right|^2 d\tau = (2\pi)^2 \int_T dx \int_0^t |q(\tau, x)|^2d\tau$$
by Plancherel. If \( \phi(\infty, x, k) \) is defined as function on \( \mathbb{T} \times \mathbb{R} \) having Fourier coefficients

\[
\int_{0}^{\infty} e^{inr^k \hat{q}(\tau, n)} d\tau
\]

then we easily have the first statement of the lemma.

For the second part, it is sufficient to show that for a.e. \( k \) we have

\[
h(t_1, t_2, k) = \sum_{n \in \mathbb{Z}} |n| \left( \sup_{\tau_1, \tau_2 > t} \left| \int_{\tau_1}^{\tau_2} e^{inr^k \hat{q}(\tau, n)} d\tau \right|^2 \right)
\]

We have

\[0 \leq h(t_1, t_2, k) \leq h_m(t, k), \quad t < t_1, t_2\]

and \( h_m(t, k) \) is decreasing in \( t \) (if it exists). Moreover,

\[
\int_{\mathbb{R}} h_m(t, k) dk = \sum_{n \in \mathbb{Z}} |n| \left( \sup_{\tau_1, \tau_2 > t} \left| \int_{\tau_1}^{\tau_2} e^{inr^k \hat{q}(\tau, n)} d\tau \right|^2 \right) dk
\]

\[
= \sum_{n \neq 0} \int_{-\infty}^{\infty} \left( \sup_{\tau_1, \tau_2 > t} \left| \int_{\tau_1}^{\tau_2} e^{inr^k \hat{q}(\tau, n)} d\tau \right|^2 \right) d\xi \leq \sum_{n \neq 0} \int_{-\infty}^{\infty} |\hat{q}(\tau, n)|^2 d\tau = 2\pi \|q\|_{L_2^{1}(t, \infty) \times \mathbb{T}}^2 \rightarrow 0
\]

Here we used the standard Carleson estimate for the maximal function \([6]\). Since \( h_m(t, k) \) is monotonic, we have \( h_m(t, k) \rightarrow 0 \) for a.e. \( k \).

We get the following simple corollary

**Corollary 2.1.** If \( q(t, x) \in L^2(\mathbb{R}^+, \mathbb{T}) \) and \( \int q(t, x) dx = 0 \), then for a.e. \( k \) we have \( u(t, x - kt, k) \rightarrow \nu(x, k) \) in the following sense

\[\|u(t, x - kt, k) - \nu(x, k)\|_{L_2(\mathbb{T})}^2 \rightarrow 0, \quad t \rightarrow \infty\]

If we also have \( q(t, x) \in i\mathbb{R} \), then for a.e. \( k \) there is \( \nu(x, k) \in H^{1/2}(\mathbb{T}) \) such that

\[\|u(t, x - kt, k) - \nu(x, k)\|_{H^{1/2}(\mathbb{T})}^2 \rightarrow 0, \quad t \rightarrow \infty\]

**Proof.** We know for a.e. \( k \) the limit \( \phi(\infty, x, k) \) exists as a function in \( H^{1/2}(\mathbb{T}) \). All other statements follow from Lemmas\([6.1, 6.2]\) in Appendix.

**Remark 4.** Similarly, one can show

\[
\int_{\mathbb{R}} \|u(t, x - tk, k) - \nu(x, k)\|_{H^{1/2}(\mathbb{T})}^2 dk \rightarrow 0,
\]
provided that \( q \in i\mathbb{R} \). It follows from Lemma 6.2, estimates on the maximal function, and dominated convergence theorem. Moreover, for a.e. \( k \), all Fourier coefficients of \( u(t, x - kt, k) \) converge as \( t \to \infty \) regardless of whether \( q \) is purely imaginary or not.

**Remark 5.** We considered the simplest case of initial data, i.e. \( u(0, x, k) = 1 \). The general case \( u(0, x, k) = f(x) \) is almost identical due to multiplicative structure of the problem. If the potential is square summable and purely imaginary, then we have the full measure set of \( k \) (that depends only on \( q \)) for which the equation is globally well-posed for \( f \) in the Krein algebra \( L^\infty(T) \cap H^{1/2}(T) \) \([2]\). We also have the stability and the asymptotics at infinity.

There is an instructive case \( q(t, x) = 2q(t) \cos x \) with \( q(t) \) – purely imaginary square summable on \( \mathbb{R} \). In this situation,

\[
\nu(x, k) = \exp \left[ \left( e^{-iz\hat{q}(k)} - e^{iz\overline{\hat{q}(k)}} \right) \right],
\]

where

\[
\hat{q}(k) = \int_0^\infty e^{ik\tau} q(\tau) d\tau.
\]

Notice that for a.e. \( k \) the function \( \nu(x, k) \) is infinitely smooth. Moreover, expanding into the Taylor series,

\[
\int_T \nu(x, k) dx = H(|\hat{q}(k)|)
\]

with

\[
H(z) = \sum_{n=0}^\infty \frac{(-z^2)^n}{(n!)^2} = J_0(2z)
\]

Notice that since the Bessel function \( J_0(z) \) has positive zeroes ([1], Chapter 9), it is possible to choose \( q \) such that

\[
\hat{\nu}(0, k) = \int_T \nu(x, k) dx = 0
\]

on arbitrary interval \( k \in I \) which means there is no hope to get

\[
\int_I \ln |\hat{\nu}(0, k)| dk > -\infty
\]

Consider the case when the transport equation is given on the cylinder of large size \( 2\pi h \)

\[
u_t = ku_x + q(t, x)u, \quad u(0, x) = 1,
\]

and \( u \) is \( h \)--periodic in \( x \), \( q \) is purely imaginary. Scaling in \( x \) gives

\[
\psi_t = k h^{-1} \psi_\theta + \hat{q}(t, \theta)\psi, \quad \psi(0, \theta) = 1
\]

where \( \psi(t, \theta, k) = u(t, h\theta, k), \hat{q}(t, \theta) = q(t, h\theta) \). For the new differential operator, \( ih^{-1}\partial_x \), the gaps in the spectrum are of the size \( h^{-1} \) but, nevertheless, we have
by scaling. The r.h.s. measures the $L^2$ norm in time of the space averages of $q$. If it is bounded, then $H^{1/2}$ norm of $u(T, x, k)$, when averaged over $(0, h)$, is bounded for most $k$. We expect this phenomenon for general situation when the gap condition deteriorates.

The calculations presented in this section can be easily carried out for the case when $q$ is more regular, e.g. $q \in L^2(\mathbb{R}^+, H^{1/2}(\mathbb{T}))$. That will lead to better regularity of the solution.

3. THE MODEL CASE OF $N \times N$ SYSTEM

In this section, we consider the following evolution

$$X_t = i(k\Lambda + V(t))X, \quad X(0) = I, \quad V^* = V$$

and

$$\Lambda = \begin{bmatrix}
\lambda_1 & 0 & \ldots & 0 \\
0 & \lambda_2 & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots \\
0 & 0 & \ldots & \lambda_N
\end{bmatrix}$$

with $0 = \lambda_1 < \lambda_2 < \ldots < \lambda_N$. Sometimes we will allow the eigenvalues to degenerate, that will require more careful analysis. Denote $\delta_j = \lambda_{j+1} - \lambda_j, \quad j = 1, \ldots, N - 1$. We will also assume that $V(t)$ is locally integrable on $\mathbb{R}^+$ and that $V_{jj}(t) = 0$ for all $j$. The last assumption can be made without loss of generalization. It is obvious that $X(t, k) = \{x_{mn}(t, k), 1 \leq m, n \leq N\}$ is unitary for real $k$. For general $k$, the following lemma holds true.

**Lemma 3.1.** For any $V \in L^1_{\text{loc}}(\mathbb{R}^+)$, we have

$$|X(t, k)|^2 + 2 \text{Im} \, k \int_0^t X^*(\tau, k)\Lambda X(\tau, k)d\tau = I, \quad k \in \mathbb{C}$$

and

$$0 \leq \int_0^\infty X^*(\tau, k)\Lambda X(\tau, k)d\tau \leq (2 \text{Im} \, k)^{-1}, \quad |X(t, k)| \leq I, \quad \text{Im} \, k > 0 \quad (15)$$

Moreover, $|X(t, k)|$ and $|X(t, k)|^2$ decay monotonically in $t$ for $k \in \mathbb{C}^+$.  

**Proof.** The proof is a trivial corollary from the differential equation itself and monotonicity of the square root. \hfill \square

**Lemma 3.2.** Assume that $q(t) = \|V(t)e_1\|$ belongs to $L^2(\mathbb{R}^+)$. Fix any $f \in \mathbb{C}^N$ with $\|f\| = 1$.

(a) We have $X(t, k)f \to \pi f(k)e_1$, as $t \to \infty$ uniformly on compacts in $\mathbb{C}^+$.

(b) For $k \in \mathbb{C}^+$, $|\langle X(t, k)f, e_1 \rangle| \geq |\langle f, e_1 \rangle| - (2\Lambda \text{Im} \, k)^{-1/2} \cdot \|q\|_2$ and

$$\left(\int_0^\infty \|P_f^* X(t, k)f\|^2 dt \right)^{1/2} \lesssim (\Lambda \text{Im} \, k)^{-1} \|q\|_2 + (\Lambda \text{Im} \, k)^{-1/2} \|P_f^* f\|$$
(c) \( (X(t, k)f, e_1) \) converges to \( \pi_f(k) \) on \( \mathbb{R} \) in the weak-* sense. If \( (f, e_1) \neq 0 \), then \( \pi_f(k) \) is nonzero function in the unit ball in \( H^\infty(\mathbb{C}^+) \) and so
\[
\int_{-\infty}^{\infty} \ln \frac{|\pi_f(k)|}{k^2 + 1} dk > -\infty
\]

(d) For \( x_{11}(t, k) \), we have \( x_{11}(t, k) \to x_{11}(\infty, k) \) uniformly over compacts in \( \mathbb{C}^+ \). Moreover,
\[
\int_{\mathbb{R}} \ln |x_{11}(\infty, k)| dk \gtrsim -\lambda_1^{-1} \|q\|_2^2
\]
and
\[
[1 - x_{11}(\infty, k)]_{L^1(\mathbb{R})} \lesssim \|q\|_2^2, \quad \|1 - x_{11}(\infty, k)\|_{L^p(\mathbb{R})} \leq C(p) \|q\|_2^{2/p}, \quad 1 < p < \infty
\]

Proof. Denote \( u(t, k) = X(t, k)f \). It is entire in \( k \). We have
\[
\frac{d}{dt} \langle u(t, k), e_1 \rangle = \langle f, e_1 \rangle + i \int_0^t \langle u(\tau, k), V(\tau) e_1 \rangle d\tau
\]
Since \( V_{11}(t) = 0 \), we have \( \langle u(\tau, k), V(\tau) e_1 \rangle = \langle P^c_1 u(\tau, k), V(\tau) e_1 \rangle \). From [15], we have
\[
\int_0^\infty \|P^c_1 u(\tau, k)\|_2^2 d\tau \leq (2\lambda_1 \text{Im } k)^{-1}
\]
Thus \( \langle u(\tau, k), V(\tau) e_1 \rangle \in L^1(\mathbb{R}^+) \) by Cauchy-Schwarz and that proves convergence of \( \langle u(t, k), e_1 \rangle \) to some \( \pi_f(k) \) and
\[
|\pi_f(k) - \langle f, e_1 \rangle| \leq (2\lambda_1 \text{Im } k)^{-1/2} \|q\|_2
\]

Consider \( \psi(t) = P^c_1 u \). We have
\[
\psi_t = i k\Delta P^c_1 \psi + i P^c_1 V P^c_1 \psi + i P^c_1 V P_1 u, \quad \psi(0) = P^c_1 f
\]
If
\[
l(t, k) = i P^c_1 V P_1 u = i \langle u, e_1 \rangle P^c_1 V e_1
\]
then
\[
\int_0^\infty \|l(\tau, k)\|_2^2 d\tau \leq \|q\|_2^2
\]
since \( \|\langle u, e_1 \rangle\| \leq 1 \). Thus,
\[
\frac{d}{dt} \left( \|\psi\|_2^2 \right) \leq -2\lambda_1 \text{Im } k \|\psi\|_2^2 + 2\|l\| \|\psi\|, \quad \|\psi(0, k)\| \leq 1
\]
and we have
\[
\|\psi(t, k)\| \lesssim e^{-\alpha t} \|P^c_1 f\| + \int_0^t e^{-\alpha (t-\tau)} \|l(\tau)\| d\tau, \quad \alpha = \lambda_1 \text{Im } k
\]
So,
\[
\left( \int_0^\infty \| \psi(\tau, k) \|^2 d\tau \right)^{1/2} \lesssim (\lambda_1 \text{Im } k)^{-1} \| q \|_2 + (\lambda_1 \text{Im } k)^{-1/2} \| P_1^c f \| \tag{20}
\]
and \( \| \psi(t, k) \| \to 0 \) uniformly on compacts in \( \mathbb{C}^+ \). That proves (a) through (b).
The properties of \( \pi_f(k) \) stated in (c) follow from the mean-value inequality for subharmonic function \( \ln |\pi_f(k)| \) and \[ \text{(19)}. \]

If \( f = e_1 \), then \[ \text{(18)} \] and \[ \text{(20)} \] yield
\[ |x_{11}(t, iy) - 1| \lesssim \lambda_1^{-1} y^{-1} \| q \|_2^2 \]
The proof of (d) repeats the arguments in lemma \[ \text{(19)}. \]

As a simple corollary of (c), we get existence of the weak-* limits for \( x_{1j}(t, k) \) on the real line \( (j = 2, \ldots, N) \). Denote them by \( x_{1j}(\infty, k) \). The next lemma gives a stronger convergence result and is an analog of lemma \[ \text{(13)}. \]

**Lemma 3.3.** Assume that \( q(t) = \| V(t)e_1 \| \) belongs to \( L^2(\mathbb{R}^+) \). Then,
\[
\int_{-\infty}^{\infty} |x_{1j}(t, k) - x_{1j}(\infty, k)|^2 dk \to 0, \quad j = 1, \ldots, N
\]
as \( t \to \infty \).

**Proof.** For any \( t_1 < t_2 \) the semigroup property yields
\[
x_{1j}(t_2, k) = \sum_{m=1}^N x_{1m}(t_1, t_2, k)x_{mj}(t_1, k)
\]
where \( X(t_1, t_2, k) \) has matrix elements \( \{x_{ij}(t_1, t_2, k)\} \). Therefore,
\[
x_{1j}(t_2, k) - x_{1j}(t_1, k) = \sum_{m>1} x_{1m}(t_1, t_2, k)x_{mj}(t_1, k) + (x_{11}(t_1, t_2, k) - 1)x_{1j}(t_1, k)
\]
\[
= I_1 + I_2
\]
By \[ \text{(17)}. \] we have
\[
\| I_2(k) \|_{L^2(\mathbb{R})} \to 0
\]
as \( t_1(2) \to \infty \). The Cauchy-Schwarz and unitarity of \( X \) give
\[
|I_1(k)|^2 \leq 1 - |x_{11}(t_1, t_2, k)|^2 \leq -2 \ln |x_{11}(t_1, t_2, k)|
\]
and thus
\[
\| I_1(k) \|_{L^2(\mathbb{R})} \to 0
\]
by \[ \text{(10)}. \] Therefore, \( x_{1j}(t, k) \) is Cauchy in \( L^2(\mathbb{R}) \) and must have a limit equal to the weak-* limit \( x_{1j}(\infty, k) \).

For fixed \( f \), we have \( \pi_f = x_{11}(\infty, k)f_1 + \ldots + x_{1N}(\infty, k)f_N \) and so

**Corollary 3.1.** If \( q(t) = \| V(t)e_1 \| \in L^2(\mathbb{R}^+) \), then
\[
\int_{\mathbb{R}} |P_1 X(t, k) f - \pi_f(k)|^2 dk \to 0
\]
for any \( f \in \mathbb{C}^N \).
This result is somewhat surprising since we do not assume anything about $P_1^c V P_1^c$ except local integrability.

Now, we are going to prove results on convergence of all elements of the matrix $X$ and need to assume more on $V$. Let $\|V\| \in L^2_{\text{loc}}(\mathbb{R}^+)$ and $T$ is a fixed positive constant. We start with the following simple observation. Fix $1 < j < N$ and consider vector $u(t)$ (it will be different for different $j$ but we suppress this dependence for shorthand) which solves

$$\frac{d}{dt} u(t) = i(k \Lambda + V)u, \quad 0 < t < T$$

and satisfies the following boundary conditions. Let $a(t)$ denote the vector containing the first $j - 1$ components of $u$, $b(t)$ is the $j$-th component of $u$, and $c(t)$ contains the $j + 1, \ldots, N$ components of $u$. Then, we require that $c(0) = 0$, $b(0) = 1$, and $a(T) = 0$. This solution does not have to exist, but for $\text{Im} \, k$ large enough or small $V$ it does, it is unique, and it allows two different representations. One of them is through $X$. Let $X_j = P_{1 \leq k \leq j}^P X P_{1 \leq k \leq j}$, where $P_{1 \leq k \leq j}$ is the projection onto the first $j$ coordinates. Then, assuming that $u$ exists, $X_j(T) = b_j(T) \cdot 0, \ldots, 0, 1)^t$ (22)

By the Laplace theorem for determinants, we have

$$\Delta_{j-1} b(T) = \Delta_j$$

where $\Delta_j = \det X_j$. Provided that $u_l$ exists for any $l = 1, \ldots, j$, iteration yields

$$\Delta_j(T, k) = b_1(T, k) \cdot \ldots \cdot b_j(T, k)$$

The $b_1(t, k)$ can be identified with $x_{11}(t, k)$.

The existence of $u(t, k)$ for large $\text{Im} \, k$ and its analytical properties follow from the standard asymptotical method for systems of ODEs close to diagonal. We can write (21) as

$$\begin{align*}
a' &= (ik \Lambda_a + Q_{11})a + Q_{12}b + Q_{13}c \\
b' &= Q_{21}a + i k \lambda_j b + Q_{23}c \\
c' &= Q_{31}a + Q_{32}b + (ik \Lambda_c + Q_{33})c
\end{align*}$$

where $Q_{nl}$ are the corresponding blocks of $iV$ and

$$\Lambda_a = P_{1 \leq n \leq j-1} \Lambda P_{1 \leq n \leq j-1}, \quad \Lambda_c = P_{j+1 \leq n \leq N} \Lambda P_{j+1 \leq n \leq N}$$

Let $U_1$ and $U_2$ be solutions to the following Cauchy problems

$$\begin{align*}
\frac{d}{dt} U_1(\tau, t, k) &= (ik \Lambda_a + Q_{11})U_1(\tau, t, k), \quad U_1(\tau, \tau, k) = I \\
\frac{d}{dt} U_2(\tau, t, k) &= (ik \Lambda_c + Q_{33})U_2(\tau, t, k), \quad U_2(\tau, \tau, k) = I
\end{align*}$$

Since $Q_{11}$ and $Q_{33}$ are antisymmetric, we have the following obvious estimates

$$\|U_1(\tau, t, k)\| \leq \exp(\lambda_{j-1} \cdot \text{Im} \, k \cdot (\tau - t)), \quad t < \tau$$

$$\|U_2(\tau, t, k)\| \leq \exp(-\lambda_{j+1} \cdot \text{Im} \, k \cdot (t - \tau)), \quad \tau < t$$

(25) (26)
Consider \( f \) where and Young inequalities, then \( \text{Im} \) allows to define \( b \). Thus, there is a unique solution which belongs to \( B \) and \( L \)

\[
\begin{align*}
    \mathcal{B} &= \mathcal{L} \times \ldots \times \mathcal{L} \times L^\infty(0,T) \times \mathcal{L} \times \ldots \times \mathcal{L} \\
    \text{and } \mathcal{L} \text{ is the space with the norm } \| \cdot \|_\infty + \| \cdot \|_2.
\end{align*}
\]

In fact, by (25), (26), Cauchy-Schwarz and Young inequalities,

\[
\begin{align*}
    \|(D)_{1(3)}\|_\infty &\lesssim \|Q\|_2 y^{-1/2}, \|(D)_{1(3)}\|_2 \lesssim \|Q\|_2 y^{-1}, \|(D)_{2}\|_2 \lesssim \|Q\|_2 \\
    \|(D^2)_{1(3)}\|_\infty &\lesssim \|Q\|_2^2 y^{-1/2}, \|(D^2)_{1(3)}\|_2 \lesssim \|Q\|_2^2 y^{-1}, \|(D^2)_{2}\|_2 \lesssim \|Q\|_2^2 y^{-1}
\end{align*}
\]

Thus, there is a unique solution which belongs to \( \mathcal{B} \). Assuming \( T = \infty \) and \( \|V\| \in L^2(\mathbb{R}^+) \), we have \( u(t) = \exp(i\lambda_j kt)(b(\infty, k)e_j + o(1)) \). This is a well-know result in the asymptotical theory of ODE but it is valid for either large positive \( \text{Im } k \) or fixed \( \text{Im } k > 0 \) and small \( \|V\|_2 \). It does not require any information on \( Q_{11} \) and \( Q_{33} \). Notice that \( b(T, k) = \exp(i\lambda_j kT)(1 + O((\text{Im } k)^{-1})) \). Then, (24) yields invertibility of each \( X_j \) for large \( \text{Im } k \). Also, since \( \Delta_j \) is entire in \( k \), the formula

\[
    b(T, k) = \Delta_j \Delta_j^{-1}
\]

allows to define \( b \) for any \( k \) as a meromorphic function.

For \( k = iy, \ y > 1 \) we have the following asymptotical expansion

\[
\hat{u} = f + Df + D^2f + \ldots
\]

\[
Df = 
\begin{bmatrix}
    - \int_t^T U_1(\tau, t, iy)e^{-y\lambda_j(\tau-t)}Q_{12}(\tau)d\tau \\
    0 \\
    \int_0^t U_2(\tau, t, iy)e^{y\lambda_j(t-\tau)}Q_{32}(\tau)d\tau
\end{bmatrix}
\]

and

\[
\begin{align*}
    (D^2f)_2(T) &= \int_0^T [Q_{21}(\tau)(Df)_1(\tau) + Q_{23}(\tau)(Df)_3(\tau)] d\tau
\end{align*}
\]
Denote $\Psi_2(\tau, t, y) = U_2(\tau, t, iy)e^{y\lambda_j(t-\tau)}$, $\Psi_1(\tau, t, y) = U_1(\tau, t, iy)e^{y\lambda_j(t-\tau)}$. Substituting the Duhamel expansions for $\Psi_1(2)$ into the formula above, one gets

$$(D^2f)_2(T) = g^{-1} \left( \int_0^T Q_{23}(t)(\Lambda_e - \lambda_j)^{-1}Q_{32}(t)dt 
- \int_0^T Q_{21}(t)(\lambda_j - \Lambda_a)^{-1}Q_{12}(t)dt \right) + o(y^{-1})$$

Since $Q_{ij} = iV_{ij}$ and $|(D^3f)_2(T, iy)| = o(y^{-1})$, we have

$$b(T, iy) = \exp(-yT\lambda_j) \left( 1 + y^{-1} \sum_{k=1}^{j-1} (\lambda_j - \lambda_k)^{-1} \int_0^T |V_{kj}(\tau)|^2 d\tau
- \sum_{k=j+1}^{N} (\lambda_k - \lambda_j)^{-1} \int_0^T |V_{kj}(\tau)|^2 d\tau \right) + o(y^{-1})$$

From [24], we have

$$\ln \Delta_j(T, iy) = -(\lambda_1 + \ldots + \lambda_j)yT
- y^{-1} \sum_{k=1}^{j} \sum_{l=j+1}^{N} |\lambda_l - \lambda_k|^{-1} \int_0^T |V_{lk}(\tau)|^2 d\tau + o(y^{-1}) \quad (28)$$

The similar calculation can be done in the general case when $\text{Im} \ k \to +\infty$.

We are ready to prove the following

**Theorem 3.1.** If

$$I(V) = \sum_{k=1}^{j} \sum_{l=j+1}^{N} |\lambda_l - \lambda_k|^{-1} \int_0^\infty |V_{lk}(\tau)|^2 d\tau < \infty$$

then $g(t, k) = \Delta_j(t, k)\exp(-ikt(\lambda_1 + \ldots + \lambda_j))$ converges in $\mathbb{C}^+$ to a function $g(\infty, k)$ which is in the unit ball in $H^\infty(\mathbb{C}^+)$. Moreover,

$$g(t, k) - g(\infty, k) \in H^2(\mathbb{C}^+), \quad ||g(t, k) - g(\infty, k)||_2 \to \infty$$

and

$$0 \geq \int_{\mathbb{R}} \ln|g(\infty, k)|dk \geq -\pi I(V), \quad \int_{\mathbb{R}} |1 - g(\infty, k)|^p dk \leq C(p)I(V), \quad 1 < p < \infty \quad (29)$$

**Proof.** Consider $g(t, k)$ for any $t > 0$. It is entire in $k$ and $|g(t, k)| \leq 1$ for real $k$ since $X_j$ is a contraction for $\text{Im} \ k \geq 0$. Moreover, we know its asymptotics for large $\text{Im} \ k$ which implies that $g(t, k)$ is in the unit ball in $H^\infty(\mathbb{C}^+)$ and

$$0 \geq \int_{\mathbb{R}} \ln|g(t, k)|dk \geq -\pi I(V)$$

Arguing like in the proof of lemma 1.2 we write $g(t, k) = 1 + h(t, k)$. Then $\text{Re} \ h(t, k) \leq 0$ and
\[
\int_{\mathbb{R}} \Re h(t, k) \, dk = -\pi I(V)
\]
which implies
\[
\|1 - g(t, k)\|_{L^1(\mathbb{R})} \lesssim I(V), \quad \|1 - g(t, k)\|_{L^p(\mathbb{R})} \leq C(p) [I(V)]^{1/p}, \quad 1 < p < \infty
\]

Let \(0 \leq s_1(k, t) \leq s_2(k, t) \leq \ldots \leq s_j(k, t) \leq 1\) be singular numbers of \(X_j(t, k)\). Then,
\[
0 \geq \int_{\mathbb{R}} \left( \ln s_1^2(k, t) + \ldots + \ln s_j^2(k, t) \right) \, dk \geq -2\pi I(V)
\]
and therefore
\[
\int_{\mathbb{R}} \text{tr}(I_j \times_j - |X_j(t, k)|^2) \, dk \leq 2\pi I(V) \quad (30)
\]
Write \(X(t, k)\) in the block form
\[
X(t, k) = \begin{bmatrix} X_j & Y_j \\ Z_j & W_j \end{bmatrix}
\]
Since \(X\) is unitary, (30) can be rewritten
\[
\int_{\mathbb{R}} |\text{tr}Y_j(t, k)|^2 \, dk \leq 2\pi I(V), \quad \int_{\mathbb{R}} |\text{tr}Z_j(t, k)|^2 \, dk \leq 2\pi I(V) \quad (31)
\]
Now, that all necessary uniform bounds are obtained, we can prove the convergence result. For any \(t_1 < t_2\), the semigroup property in the block form yields the identity
\[
X_j(t_2, k) = X_j(t_1, t_2, k)X_j(t_1, k) + Y_j(t_1, t_2, k)Z_j(t_1, k)
\]
As \(t_{1(2)} \to \infty\), we have
\[
\int_{\mathbb{R}} |\text{tr}Y_j(t_1, t_2, k)|^2 \, dk \to 0
\]
So,
\[
\int_{\mathbb{R}} |\det X_j(t_2, k) - \det X_j(t_1, t_2, k) \cdot \det X_j(t_3, k)|^2 \, dk \to 0, \quad t_{3(2)} \to \infty
\]
On the other hand,
\[
\int_{\mathbb{R}} |\det X_j(t_1, t_2, k) - \exp(ik(t_2 - t_1)(\lambda_1 + \ldots + \lambda_j))|^2 \, dk \to 0
\]
as \(t_{1(2)} \to \infty\). So, \(1 - g(t, k)\) is Cauchy in \(H^2(\mathbb{C}^+)\) and we have convergence \(g(t, k) \to g(\infty, k)\) uniformly over the compacts in \(\mathbb{C}^+\). Since each \(g(t, k)\) is analytic contraction, the limit \(g(\infty, k)\) as an analytic contraction as well. The bounds (29) can be obtained through the argument identical to the one used to handle \(g(t, k)\).
Remark 6. Notice that the theorem was proved under the assumption that all eigenvalues of $\Lambda$ are non-degenerate. That was used in the proof of the asymptotics for $b$. In the meantime, due to cancelation in (28), the statement of theorem 3.1 as well as (31) holds under the assumption that $\lambda_j < \lambda_{j+1}$ and the other eigenvalues can degenerate.

The estimates for determinants and (31) can be obtained for $W_{j-1}$ as well and that yields the following important result.

**Theorem 3.2.** If

$$I'(V) = \sum_{k=1}^{j} \sum_{l=j}^{N} |\lambda_l - \lambda_k|^{-1} \int_0^\infty |V_{kl}(\tau)|^2 d\tau < \infty,$$

(remember that $V_{jj}(t) = 0$)

then

$$\int_{\mathbb{R}} |x_{jj}(t,k) - \exp(i\lambda_j tk)|^2 dk \lesssim I'(V)$$

Moreover, there is $\tilde{x}_{jj}(\infty, k)$ such that

$$\int_{\mathbb{R}} |x_{jj}(t,k) \exp(-i\lambda_j tk) - \tilde{x}_{jj}(\infty, k)|^2 dk \to 0$$

and

$$\int_{\mathbb{R}} |\tilde{x}_{jj}(\infty, k) - 1|^2 dk \lesssim I'(V)$$

**Proof.** The estimate (31), applied to $X_j$ and $W_{j-1}$, yields

$$\int_{\mathbb{R}} \sum_{l\neq j} (|x_{lj}(t,k)|^2 + |x_{jl}(t,k)|^2) dk \lesssim I'(V)$$

Expanding in the last raw, we have

$$\Delta_j = x_{jj} \Delta_{j-1} + r, \quad r = x_{j1} A_{j1} + \ldots + x_{j,j-1} A_{j,j-1}$$

where $\{A_{lm}\}$ are cofactors of $X_j$.

**Lemma 3.4.** If $Z$ is a $j \times j$ contraction then the adjoint $C = \text{adj}Z$ is contraction as well. In particular,

$$|A_{11}|^2 + \ldots + |A_{1j}|^2 \leq 1$$

(33)

where $A_{ik}$ is $(i,k)$-cofactor of $Z$.

**Proof.** Take any $\alpha = (\alpha_1, \ldots, \alpha_j)$ with $\|\alpha\|_2 = 1$. Replace the first raw of $Z$ by $\alpha$ and denote the resulting matrix by $Z_\alpha$. By Laplace theorem,

$$\det Z_\alpha = \alpha_1 A_{11} + \ldots + \alpha_j A_{1j}$$

On the other hand, Hadamard’s estimate gives

$$|\det Z_\alpha| \leq h_1 \ldots h_j \leq 1$$

where $h_l$ is the $l^2$-length of the $l$-th raw of $Z_\alpha$. Since $\alpha$ is arbitrary, we get (33).

This implies $\|C e_1\| \leq 1$. Take any unitary $U$. We have $UCU^{-1} = \text{adj}(UZU^{-1})$. Therefore,

$$\|CU^{-1}e_1\| \leq 1$$

Since $U$ is arbitrary, $\|Cx\| \leq \|x\|$ for any $x$. □
By lemma, we have

\[ \int_{\mathbb{R}} |r(k)|^2 dk \lesssim I'(V) \]

Since

\[ \int_{\mathbb{R}} |\Delta_j - \exp(ikt(\lambda_1 + \ldots + \lambda_j))|^2 dk \lesssim I'(V) \]

and

\[ \int_{\mathbb{R}} |\Delta_{j-1} - \exp(ikt(\lambda_1 + \ldots + \lambda_{j-1}))|^2 dk \lesssim I'(V) \]

the formula (32) yields

\[ \int_{\mathbb{R}} |x_{jj}(t,k) - \exp(ikt\lambda_j)|^2 dk \lesssim I'(V) \]

Using the semigroup property one can show that \( x_{jj}(t,k) \exp(-ikt\lambda_j) - 1 \) is Cauchy in \( L^2(\mathbb{R}) \) which implies existence of the limit. \( \square \)

**Corollary 3.2.** Let

\[ \sum_{k \neq l} \int_0^\infty |\lambda_k - \lambda_l|^{-1} |V_{kl}(t)|^2 dt < \infty \]

Consider \( \tilde{X}(t,k) = \exp(-ikt\Lambda)X(t,k) \). Then \( \tilde{X}(t,k) \to \tilde{X}(\infty,k) \) in the strong sense, i.e.,

\[ \int_{\mathbb{R}} \| \tilde{X}(t,k)f - \tilde{X}(\infty,k)f \|^2 dk \to 0 \]

for any \( f \).

**Proof.** The proof is a standard application of the semigroup property and the previous results. \( \square \)

We are going to consider now a somewhat special case when the frequencies degenerate in different ways. The first situation is a model for Schrödinger evolution on 1–d torus.

Assume that \( \lambda_{j-1} < \lambda_j = \lambda_{j+1} < \lambda_{j+2} \) for some \( j : 1 < j < N \). We will try to understand how the \( P_{\{j,j+1\}}X(t,k)P_{\{j,j+1\}} \) part of \( X(t,k) \) behaves for large \( t \). Consider the following evolutions

\[ \Psi'(\tau,t,k) = i \left( k\lambda_j I_{2 \times 2} + \begin{bmatrix} 0 & V_{j,j+1}(t) \\ V_{j+1,j}(t) & 0 \end{bmatrix} \right) \Psi(\tau,t,k), \quad \Psi(\tau,\tau,k) = I_{2 \times 2} \]

\[ W'(\tau,t) = i \begin{bmatrix} 0 & V_{j,j+1}(t) \\ V_{j+1,j}(t) & 0 \end{bmatrix} W(\tau,t), \quad W(\tau,\tau) = I_{2 \times 2} \]

Obviously, \( W \) is \( k \)-independent and is unitary since \( V_{j,j+1} = \tilde{V}_{j+1,j} \). For \( \Psi \), we have \( \Psi = \exp(ikt\lambda_j)W \).

Notice that for real or purely imaginary \( V_{j,j+1} \) the matrix that diagonalizes the perturbation is \( t \)-independent and we can assume that \( V_{j,j+1}(t) = 0 \) without loss of generality. We will consider the general case. The proof of the following result repeats the argument given above with minor changes which we will explain.
Theorem 3.3. Assume that \( \lambda_{j-1} < \lambda_j = \lambda_{j+1} < \lambda_{j+2} \) and
\[
I''(V) = \sum_{k=1}^{j-1} \sum_{l=j}^{j+1} |\lambda_l - \lambda_k|^{-1} \int_0^\infty |V_{kl}(\tau)|^2 d\tau + \sum_{k=1}^{j+1} \sum_{l=j+2}^N |\lambda_l - \lambda_k|^{-1} \int_0^\infty |V_{kl}(\tau)|^2 d\tau < \infty,
\]
Consider \( Y(t,k) = P_{(j,j+1)}X(t,k)P_{(j,j+1)} \). Then,
\[
\int \|Y(t,k) - \Psi(0,t,k)\|^2 dk \lesssim I''(V) \tag{34}
\]
Moreover,
\[
\|\Psi^{-1}(0,t,k)Y(t,k) - \tilde{Y}(\infty,k)\| \rightarrow 0 \text{ in } L^2(\mathbb{R}) \text{ and }
\int \|\tilde{Y}(\infty,k) - I\|^2 dk \lesssim I''(V)
\]
Proof. We repeat the proofs of theorems 3.1 and 3.2 with the following modifications. Instead of a single vector \( u \) satisfying \( b(0,k) = 1 \), we consider its \( N \times 2 \) matrix version. Let us denote the matrix containing first \( j-1 \) rows of \( u \) by \( a \), \( b \) is formed by \( j,j+1 \) rows and is therefore \( 2 \times 2 \) matrix, and \( c \) is built of \( j+2,\ldots,N \) rows of \( u \). Then, the boundary conditions would be
\[
a(T,k) = 0, \quad b(0,k) = I_{2 \times 2}, \quad c(0,k) = 0
\]
The analogs of (22) and (23) are
\[
X_{j+1}(T,k) \begin{bmatrix} a(0,k) \\ I_{2 \times 2} \end{bmatrix} = \begin{bmatrix} 0 \\ b(T,k) \end{bmatrix}
\]
Multiplying from the left with the adjoint of \( X_{j+1}(T,k) \) and taking the \( 2 \times 2 \) blocks in the “southeastern corner”, we have
\[
\Delta_{j+1}(T,k) \cdot I_{2 \times 2} = A(T,k) \cdot b(T,k),
\]
\[
A(T,k) = \begin{bmatrix} A_{jj}(T,k) & A_{j,j+1}(T,k) \\ A_{j+1,j}(T,k) & A_{j+1,j+1}(T,k) \end{bmatrix}
\]
(35)
By Remark 6, we already know that
\[
\int \|\Delta_{j+1}(T,k) - \exp(ikT(\lambda_1 + \ldots + \lambda_{j-1} + 2\lambda_j))\|^2 dk \lesssim I''(V)
\]
\[
\int \|\Delta_{j-1}(T,k) - \exp(ikT(\lambda_1 + \ldots + \lambda_{j-1}))\|^2 dk \lesssim I''(V)
\]
and
\[
\int \sum_{i \neq j,j+1} \left[ |x_{i,j}(T,k)|^2 + |x_{i,j+1}(T,k)|^2 ight. \\
\left. + |x_{j,i}(T,k)|^2 + |x_{j+1,i}(T,k)|^2 \right] dk \lesssim I''(V) \tag{36}
\]
Combining these estimates and using the Laplace theorem for determinants, we have
\[
\int \| \det Y(T,k) - \exp(2ikT\lambda_j) \|^2 dk \lesssim I''(V) \tag{37}
\]
The analog of (27) is
\[
\begin{align*}
\begin{cases}
  a(t) &= -\int_t^T U_1(\tau, t, k) [Q_{12}(\tau)b(\tau) + Q_{13}(\tau)c(\tau)] d\tau \\
  b(t) &= \Psi(0, t, k) + \int_0^t \Psi(\tau, t, k) [Q_{21}(\tau)a(\tau) + Q_{23}(\tau)c(\tau)] d\tau \\
  c(t) &= \int_0^t U_2(\tau, t, k) [Q_{31}(\tau)a(\tau) + Q_{32}(\tau)b(\tau)] d\tau
\end{cases}
\end{align*}
\]

(38)

The similar perturbation argument gives
\[
b(T, iy) = \Psi(0, T, iy) \left( I_{2 \times 2} + y^{-1} \Gamma + \bar{o}(y^{-1}) \right), \quad y \to +\infty
\]

where
\[
\Gamma = \int_0^T W(t, 0)Q_{23}(t)(\Lambda_e - \lambda_j)^{-1}Q_{32}(t)W(0, t)dt
\]
\[
- \int_0^T W(t, 0)Q_{21}(t)(\lambda_j - \Lambda_a)^{-1}Q_{12}(t)W(0, t)dt
\]

Consider
\[
C(T, k) = A(T, k)\Psi(0, T, k)\exp(-ikT(\lambda_1 + \ldots + \lambda_{j-1} + 2\lambda_j))
\]
(39)

Since we know asymptotical expansion for $\Delta_{j+1}$ as $\text{Im } k \to +\infty$, (35) yields
\[
C(T, iy) = I_{2 \times 2} - y^{-1}(\Gamma + I_{j+1}(V) \cdot I_{2 \times 2}) + \bar{o}(y^{-1})
\]

Notice that
\[
0 \leq \Gamma + I_{j+1}(V) \cdot I_{2 \times 2} \lesssim I''(V)
\]

By lemma 3.3, $A(T, k)$ is contraction for $\text{Im } k \geq 0$ and so is $C(T, k)$ for real $k$. $C(T, k)$ is also entire in $k$ and we know its asymptotics for large $\text{Im } k$ which implies that $C(T, k)$ is contraction for $k \in \mathbb{C}^+$. Write $C(T, k) = I_{2 \times 2} + H(T, k)$. Then,
\[
2\Re H(T, k) + |H(T, k)|^2 \leq 0
\]
(40)

Since $(\Re H(T, k)\xi, \xi)$ is harmonic for any $\xi \in \mathbb{C}^2$, comparison of asymptotics for $y \to \infty$ gives
\[
- \int_{\mathbb{R}} \Re H(T, k) dk \lesssim I''(V)
\]
and then (40) yields
\[
\int_{\mathbb{R}} \|H(T, k)\|^2 dk \lesssim I''(V)
\]
(41)

Next, notice that
\[
A(T, k) = \Delta_{j-1}(T, k) \begin{bmatrix} x_{j+1,j+1}(T, k) & -x_{j,j+1}(T, k) \\ -x_{j+1,j}(T, k) & x_{j,j}(T, k) \end{bmatrix} + r(T, k)
\]
and
\[
\int_{\mathbb{R}} \|r(T, k)\|^2 dk \lesssim I''(V)
\]
due to \((36)\). On the other hand, we know the asymptotics of \(\Delta_{j-1}(T, k)\) which together with \((39)\) and \((41)\) give

\[
\int_{\mathbb{R}} \left\| \exp(2ikT\lambda_j) - (\text{adj} Y(T, k)) \cdot \Psi(0, T, k) \right\|^2 dk \lesssim I''(V) \tag{42}
\]

since

\[
\text{adj} Y(T, k) = \begin{bmatrix}
x_{j+1,j+1}(T, k) & -x_{j,j+1}(T, k) \\
-x_{j+1,j}(T, k) & x_{j,j}(T, k)
\end{bmatrix}
\]

Denote the matrix under the norm in \((42)\) by \(\mu\). We have \(\|Y\mu\| \leq \|\mu\|\) since \(Y\) is a contraction and therefore

\[
\int_{\mathbb{R}} \left\| \exp(2ikT\lambda_j)Y(T, k) - \det Y(T, k) \cdot \Psi(0, T, k) \right\|^2 dk \lesssim I''(V)
\]

By \((37)\), we have \((34)\). The rest is standard. \(\square\)

The method can be carried over to the case when the multiplicity of frequencies are higher than 2. Using these results we can obtain the “asymptotics” of solution for any \(\lambda_1 \leq \ldots \leq \lambda_N\) provided that \(\|V\| \in L^2(\mathbb{R}^+)\). However, the constants in our estimates will blow up when some \(\delta_j = \lambda_{j+1} - \lambda_j \sim 0\).

Assume we are in the situation when \(\lambda_1 = \lambda_2 = \ldots = \lambda_m < \lambda_{m+1}\). The simple matrix version of lemma \(3.2\) gives

**Proposition 3.1.** If \(\lambda_1 = \lambda_2 = \ldots = \lambda_m < \lambda_{m+1}\) and

\[
\tilde{v}(t) = \| P_{\{1,\ldots,m\}} V(t) P_{\{m+1,\ldots\}} \| \in L^2(\mathbb{R}^+)
\]

then

\[
\int_{\mathbb{R}} \| P_{\{m+1,\ldots\}} X(T, k) e_n \|^2 dk \lesssim \|\tilde{v}\|^2 \lambda_m^{-1}, \quad n = 1, \ldots, m
\]

If \(\lambda_{m+1} >>> 1\), this means rather strong localization of the solution. Therefore, we pose the following

**Open problem.** Is it possible to improve the estimate on

\[
\int_{\mathbb{R}} \| P_{\{m,\ldots\}} X(T, k) e_1 \|^2 dk
\]

for large \(m\) assuming only \(\lambda_1 < \lambda_2 < \ldots\) and some off-diagonal decay for \(V\)? The conjecture might be that

\[
\sup_{T>0} \int_{\mathbb{R}} \|\sqrt{A} X(T, k) e_1 \|^2 dk < \infty \tag{43}
\]

for suitable \(L^2\) condition on \(V\). That could lead to better understanding of Schrödinger evolution on the circle.

The calculations below will be extensively used in later sections to handle special evolutions equations. We will emphasize the dependence of \(I'(V)\) on \(j\) by writing \(I'_j(V)\).
(a) Let $V_{ij}(t) = q_{i-j}(t)$ where $q_{i}(t) = q_{-j}(t)$ and
\[
\|q\|_2^2 = \int_0^{\infty} \sum_{l=1}^{N} |q_l(t)|^2 dt < \infty
\]
Assume also that $|\lambda_l - \lambda_m| \gtrsim |l - m|$. Then for any $j$ we have (remember that $q_0(t) = 0$)
\[
I_j'(V) \lesssim \sum_{k \leq 0} \sum_{l \geq 0} \frac{1}{|k-l|} \int_0^{\infty} |q_{k-l}(t)|^2 dt \lesssim \|q\|_2^2\]

(b) Take the same $V$ but assume that $\lambda_j \sim j^m$. Then,
\[
I_j'(V) \lesssim \sum_{k=1}^{j} \sum_{l=j}^{\infty} \frac{1}{l^{m-k^m}} \int_0^{\infty} |q_{k-l}(t)|^2 dt \lesssim j^{-(m-1)} \|q\|_2^2 \tag{44}
\]

In the second case, the condition on $V$ can be relaxed. If $\lambda_{2j} = \lambda_{2j+1} \sim j^m$, the estimate for $I_j''(V)$ is similar.

4. The case of Hilbert spaces and applications to Schrödinger evolution on the circle

In this section, some results from the previous section are generalized to the case $N = \infty$. We will also give various applications to the Schrödinger evolution on the circle.

Consider the selfadjoint operator $\Lambda$ on $\mathcal{H} = \ell^2(\mathbb{Z})$ with discrete spectrum $\{\lambda_n\}$ where $\lambda_n$ is nondecreasing sequence. Let $Q(t)$ be operator-valued function with norm $\|Q(t)\|$ bounded for a.e. $t$ and
\[
\|Q(t)\| \in L^1_{loc}(\mathbb{R}^+). \tag{45}
\]
The weak solution to
\[
u_t(t, k) = ik\Lambda u(t, k) + Q(t)u(t, k), \quad u(0, k) = \psi \tag{46}
\]
is the solution to
\[
u(t, k) = X_0(0, t, k)\psi + \int_0^{t} X_0(\tau, t, k)Q(\tau)u(\tau, k)d\tau
\]
which follows from the Duhamel formula. Here $X_0(\tau, t, k)$ denotes solution to the unperturbed evolution. We will write $\nu(t, k) = X(0, t, k)\psi$.

There are some general results that prove the weak solution is in fact a “strong solution” provided that the initial value and potential $Q$ are “regular enough”. We will study the behavior of weak solution. The condition (45) is sufficient for the iterations of (46) to converge in the space $L^\infty([0, T], \mathcal{H})$ for any $T > 0$ with the obvious estimate
\[
\|\nu(t, k)\| \leq \exp \left( \int_0^{t} \|Q(\tau)\| d\tau \right) \|\psi\|
\]
The following stability result will allow us to use the standard approximation technique. Let $\Pi_n = P_{(-n, \ldots, -n)}$, a projection in $\ell^2(\mathbb{Z})$.

**Lemma 4.1. (Approximation lemma).** If $\|Q(t)\| \in L^1(0, T)$ and $Q_n(t) = \Pi_n Q(t)\Pi_n$, then $\|u_n(T, k) - u(T, k)\| \to 0$ as $n \to \infty$. 
Proof. Each term in the corresponding series is a multilinear operator

\[
G_m(Q, \ldots, Q) = \int_0^T \int_0^{\tau_1} \cdots \int_0^{\tau_{m-1}} X_0(\tau_1, T, k)Q(\tau_1)X_0(\tau_2, \tau_1, k) \cdots Q(\tau_m)X_0(0, \tau_m, k)\psi d\tau_1 \cdots d\tau_m
\]

or

\[
G_m(Q_n, \ldots, Q_n) = \int_0^T \int_0^{\tau_1} \cdots \int_0^{\tau_{m-1}} X_0(\tau_1, T, k)Q_n(\tau_1)X_0(\tau_2, \tau_1, k) \cdots Q_n(\tau_m)X_0(0, \tau_m, k)\psi d\tau_1 \cdots d\tau_m
\]

Write

\[
G_m(Q_n, \ldots, Q_n) = G_m(Q_n, \ldots, Q_n, Q) + G_m(Q_n, \ldots, Q_n, Q_n - Q)
\]

and use linearity for the second term, etc. Then,

\[
G_m(Q_n, \ldots, Q_n) - G_m(Q, \ldots, Q)
\]

can be written as a sum of \( m \) terms and each of them converges to zero. Indeed,

\[
\int_0^T \int_0^{\tau_{m-1}} \|Q_n(\tau_1)\| \cdots \|Q_n(\tau_{j-1})\| \cdots \|Q_n(\tau_j)\| \cdots \|Q_n(\tau_{j+1})\|X_0(\tau_{j+1}, \tau_j, k)Q(\tau_{j+1}) \cdots Q(\tau_m)X_0(0, \tau_m, k)\psi d\tau_1 \cdots d\tau_m \to 0
\]

by dominated convergence theorem. Therefore

\[
\sum_{m \geq 1} G_m(Q_n, \ldots, Q_n) \to \sum_{m \geq 1} G_m(Q, \ldots, Q), \quad n \to \infty
\]

since we also have a bound

\[
|G_m(V, \ldots, V)| \leq \frac{1}{m!} \left( \int_0^T \|V(t)\| dt \right)^m \|\psi\|
\]

which takes care of the tails in the series. \( \square \)

It is now easy to prove

**Lemma 4.2.** If \( Q(t) = iV(t) \), where \( V(t) \) is selfadjoint and \( \|V(t)\| \in L^1_{loc}(\mathbb{R}^+) \), then \( X(\tau, t, k) \) is unitary and it satisfies the semigroup property

\[
X(t_1, t_2, k) \cdot X(t_0, t_1, k) = X(t_0, t_2, k)
\]

Proof. The semigroup property and preservation of the norm follow from the Approximation lemma and the corresponding results for finite systems of ODE’s. We also have

\[
X(0, t, k) \cdot X(t, 0, k) = X(t, 0, k) \cdot X(0, t, k) = I
\]

which implies that \( X \) is unitary. \( \square \)

Next, we prove an analog of theorem 3.2. Denote the matrix elements of \( V(t) \) and \( X(t, k) \) by \( V_{mn}(t) \) and \( x_{mn}(t, k) \), respectively. For simplicity, we again make an assumption that \( V_{nn}(t) = 0 \) for any \( n \).
**Theorem 4.1.** Assume that \( \|V\| \in L^{1}_{\text{loc}}(\mathbb{R}^+) \), \( \lambda_{-1} < \lambda_0 = 0 < \lambda_1 \), and

\[
I'(V) = \sum_{k \leq 0} \sum_{l \geq 0} |\lambda_l - \lambda_k|^{-1} \int_{0}^{\infty} |V_{kl}(\tau)|^2 d\tau < \infty, \quad (V_{00}(t) = 0)
\]

Then,

\[
\int_{\mathbb{R}} |x_{00}(t, k) - x_{00}(\infty, k)|^2 dk \to 0
\]

\[
\int_{\mathbb{R}} |x_{00}(\infty, k) - 1|^2 dk \lesssim I'(V)
\]

**Proof.** For truncated potentials \( V^{(n)} = \Pi_n V \Pi_n \), the theorem \[3.2\] is applicable and the resulting estimates are uniform in \( n \). Since for each fixed \( k \) we have convergence

\[
x^{(n)}_{jl}(t, k) \to x_{jl}(t, k), \quad n \to \infty
\]

and uniform estimates

\[
\int_{\mathbb{R}} |x^{(n)}_{00}(t, k) - 1|^2 dk \lesssim I'(V)
\]

\[
\int_{\mathbb{R}} \sum_{l \neq 0} (|x^{(n)}_{0l}(t, k)|^2 + |x^{(n)}_{l0}(t, k)|^2) dk \lesssim I'(V)
\]

we can go to the limit as \( n \to \infty \) to get

\[
\int_{\mathbb{R}} |x_{00}(t, k) - 1|^2 dk \lesssim I'(V), \quad \int_{\mathbb{R}} \sum_{l \neq 0} (|x_{0l}(t, k)|^2 + |x_{l0}(t, k)|^2) dk \lesssim I'(V)
\]

The rest is the standard application of semigroup property and unitarity of \( X \). \( \square \)

Most results from the previous section can be adjusted similarly including the case when the frequencies have multiplicity. In particular, we can consider Schrödinger evolution on, say, one-dimensional circle

\[
u_t = -iku_{\theta \theta} + iV(t, \theta)u, \quad u(0, k) = \psi(\theta) \in L^2(\mathbb{T})
\]

We are interested in the weak solution and assume that \( V \) is real-valued and \( \|V(t, \theta)\|_{\infty} \in L^{1}_{\text{loc}}(\mathbb{R}^+) \). In this case, on the Fourier side, equation takes form

\[
\hat{u}_t = ik\Lambda \hat{u} + i\hat{V} \ast \hat{u}, \quad \hat{u}(0, k) = \hat{\psi} \in \ell^2
\]

where \( \Lambda \) is diagonal \( \lambda_0 = 0, \lambda_n = n^2 \) and all eigenvalues but the principal one (i.e., \( \lambda_0 = 0 \)) have multiplicity 2. Write \( \Lambda \) in the basis \( \{1, e^{i\theta}, e^{-i\theta}, \ldots, e^{in\theta}, e^{-in\theta}, \ldots\} \)

\[
\Lambda = \begin{bmatrix}
0 & 0 & 0 & 0 & \cdots \\
0 & 1 & 0 & 0 & \cdots \\
0 & 0 & 1 & 0 & \cdots \\
0 & 0 & 0 & 4 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots 
\end{bmatrix}
\]

We also always assume without loss of generality that

\[
\int_{\mathbb{T}} V(t, \theta) d\theta = 0, \quad t > 0
\]
Consider the following $k$–independent evolution

$$
\Psi_n(t) = i \begin{bmatrix} \frac{1}{V(2n,t)} & \hat{V}(2n,t) \\ \hat{V}(2n,t) & 0 \end{bmatrix} \Psi_n(t), \quad \Psi_n(0) = I_{2 \times 2}, \quad n \geq 1
$$

where

$$
\hat{V}(2n,t) = \int T(t,\theta)e^{2in\theta}d\theta
$$

Notice that if $\hat{V}(2n,t) = 0$ for all $t$, then $\Psi_n(t)$ are trivial. Also, if $\hat{V}(2n,t)$ is real or purely imaginary, we can write the explicit formula for $\Psi_n(t)$. That can be satisfied, e.g., if $V$ is even or odd.

Take

$$
W(0, t, k) = X_0(0, t, k) \cdot \begin{bmatrix} 1 & 0 & 0 & \ldots \\ 0 & \Psi_1(t) & 0 & \ldots \\ 0 & 0 & \Psi_2(t) & \ldots \end{bmatrix}
$$

where $X_0(0, t, k)$ is the free Schrödinger evolution.

**Theorem 4.2.** Assume that

$$
\|V(t, \theta)\|_\infty \in L^1_{\text{loc}}(\mathbb{R}^+)
$$

and $V(t, \theta) \in L^2(\mathbb{R}^+ \times \mathbb{T})$. Then for any $\psi \in L^2(\mathbb{T})$ the weak solution $u(t, k)$ satisfies

$$
\int_I \|W^{-1}(0, t, k)\hat{u}(t, k) - \hat{H}\psi\|^2 dk \to 0, \quad \text{as} \quad t \to \infty
$$

for any $I \subset \mathbb{R}, |I| < \infty$. The operator $H$ is defined as bounded operator from $L^2(\mathbb{T})$ to the space of functions $h(\theta, k)$ satisfying $\|h(\theta, k)\|_2^2 \in L^1_{\text{loc}}(\mathbb{R})$.

**Proof.** Denote the matrix elements of $X(t, k) = W^{-1}(0, t, k)X(t, k)$ by $\hat{x}_{\alpha\alpha}(t, k)$. Fix $n$ and take, say, $\psi(\theta) = e^{in\theta}$. Let $\hat{u}$ be the corresponding solution, i.e. the $\alpha(n)$–th column of $X$. The choice $\psi(\theta) = e^{-in\theta}$ gives the $\alpha + 1$–th column. From the Approximation lemma, theorem 4.1 (adapted by the theorem 3.3), and (44) we know

$$
\sum_{m \in \mathbb{Z}^+} |\hat{x}_{\alpha\alpha}(t, k)|^2 = 1, \quad t > 0, k \in \mathbb{R}
$$

$$
\int_{\mathbb{R}} \|1 - \hat{x}_{\alpha\alpha}(t, k)\|^2 dk \lesssim \|V\|^2, \quad \int_{\mathbb{R}} \|\hat{x}_{\alpha\alpha}(t, k) - \hat{x}_{\alpha\alpha}(\infty, k)\|^2 dk \to 0
$$

and

$$
\int_{\mathbb{R}} \sum_{m \neq \alpha} |\hat{x}_{\alpha\alpha}(t, k)|^2 dk \lesssim \|V\|^2, \quad \int_{\mathbb{R}} \|\hat{x}_{\alpha\alpha}(t, k) - \hat{x}_{\alpha\alpha}(\infty, k)\|^2 dk \to 0, \quad m \neq \alpha
$$

Moreover, (31) gives the following uniform estimates

$$
\int_{\mathbb{R}} \sum_{|m| > t} |\hat{x}_{\alpha\alpha}(t, k)|^2 dk \lesssim C(n)t^{-1}\|V\|^2, \quad t >> n
$$

which implies

$$
\int_{\mathbb{R}} \sum_{m \neq \alpha} |m|^{\gamma} |\hat{x}_{\alpha\alpha}(t, k)|^2 dk \lesssim C(n)\|V\|^2
$$
for any $\gamma < 1$. Therefore,
\[ \int_{\mathbb{R}} \sum_{m>0} |\tilde{x}_{m\alpha}(t,k) - \tilde{x}_{m\alpha}(\infty,k)|^2 dk \to 0 \]

By linearity, we can prove existence of the limit for any trigonometric polynomial $\psi = T(\theta)$. Denote the corresponding limit by $[HT](\theta,k)$. That gives a linear operator $H$ defined on the set dense in $L^2(\mathbb{T})$. We have
\[ \|\tilde{X}(t,k)\tilde{T}\| = \|T\| \]
for any $k$ and $t$ and so for any $I \subset \mathbb{R}$
\[ \int_{I} \int_{\mathbb{T}} |[HT](\theta,k)|^2 d\theta dk = |I| \cdot \|T\|^2 \]

Therefore, $H$ can be extended to a bounded operator on $L^2(\mathbb{T})$ such that
\[ \int_{I} \int_{\mathbb{T}} |[H\psi](\theta,k)|^2 d\theta dk = |I| \cdot \|\psi\|^2 \]

If $\psi$ is fixed, the last identity implies that $\|H\psi\| = \|\psi\|$ for a.e. $k$. \qed

Notice that the condition (47) was used only to guarantee the global existence of the weak solution for any $k$ and can probably be dropped. The solution corresponding to the initial value $\psi = 1$ is special in a way that we always have
\[ \int_{\mathbb{R}} \ln |x_{11}(t,k)| dk \gtrsim -\|V\|_2^2 \]
and that means $|x_{11}(t,k)| > 0$ for a.e. $k$ (compare with (13)).

The following proposition is the direct corollary from (49)

**Proposition 4.1.** Take $\psi = 1$ and assume that $\|V(t,\theta)\|_\infty \in L^1_{\text{loc}}(\mathbb{R}^+)\text{ and } V(t,\theta) \in L^2([0,T] \times \mathbb{T})$ for any $T > 0$. Then,
\[ \int_{\mathbb{R}} \|u(T,\theta,k)\|_{H^\gamma(\mathbb{T})}^2 dk \lesssim \int_{0}^{T} \int_{\mathbb{T}} |V(t,\theta)|^2 d\theta dt \]
for any $\gamma < 1/2$.

The analogous bound can be proved for any sufficiently smooth function $\psi$. Assuming that $V$ is only bounded on the strip $\mathbb{R}^+ \times \mathbb{T}$ this estimate shows that $k$–averaged $H^\gamma$ norm is finite and grows not faster than $\sqrt{T}$.

**Remark 7.** Now, assume that
\[ \int_{0}^{T} \int_{\mathbb{T}} |V(t,\theta)|^2 d\theta dt < C \]
Consider the first $T$ columns in $X(0,T,k)$ and denote by $M$ the set of those for which
\[ \int_{\mathbb{R}} \sum_{t=T+1}^{\infty} |x_{1m}(0,T,k)|^2 dk > \sigma T^{-2} \]
Due to (31), we have $|M| < C\sigma^{-1}T$ and by taking $\sigma$ large we have that at least a half of the first $T$ columns are strongly localized for many $k$. In this argument, $M$ can depend on $T$, in principle.

In the case of transport equation, our method allows us to reproduce that the solutions are in $H^{1/2}$. Indeed, we have

$$
\int_{\mathbb{R}} \sum_{m \leq 0, n \geq 0, m \neq n} |x_{mn}(t, k)|^2 dk \lesssim \|V\|_2^2
$$

where $x_{mn}(t, k)$ are the matrix elements of the evolution operator in the Fourier representation. In the meantime, we have $x_{nn}(t, k) = \exp(ikmt)x_{n-1,0}(t, k)$ which yields

$$
\int_{\mathbb{R}} \|X(t, k) \cdot 1\|_{H^{1/2}(T)}^2 dk = \int_{\mathbb{R}} \sum_{n \neq 0} |n| \cdot |x_{n0}(t, k)|^2 dk \lesssim \|V\|_2^2
$$

5. Evolution with deteriorating gap condition: the short-range interactions.

This section contains the main results of the paper. Unfortunately, they handle only the short-range potentials and even in this case are far from optimal.

Consider, e.g., the following model

$$
u_t = -ikt^{-2}u_{\theta\theta} + iV(t, \theta)u, \quad t > 1
$$

(50)

$V$ is real and

$$
u(1, \theta) = 1
$$

Similar evolution equation appears as the WKB correction in the three-dimensional Schrödinger dynamics [5]. We assume that $V$ is real-valued and satisfies

$$
\|V(t, \theta)\|_{L^\infty([0, \infty) \times T)} \lesssim t^{-\gamma},
$$

where $0 \leq \gamma \leq 1$ is to be specified later.

On the Fourier side, the equation can be written as

$$
\hat{u}' = ikt^{-2}\Lambda \hat{u} + i\hat{V} \hat{u}, \quad \hat{u}(1) = \delta_0
$$

and $\Lambda$ is diagonal with elements $n^2$, $n \geq 0$. The multiplicity of each eigenvalue is two as long as $n > 0$, the principal eigenvalue is non-degenerate. Clearly, this case can not be handled by the methods considered in the previous section since the distance between eigenvalues decays like $t^{-2}$ which might lead to significant growth of the Sobolev norms even for “typical” $k$. Instead, as results of the previous section suggest, we should introduce the scaled Sobolev norms

$$
\|u\|_{s,T} = t^{-s}\|u\|_{H^{s}(\mathbb{R})}
$$

Open problem. Assume $V(t, \theta) \in L^2([0, \infty) \times T)$. Is it true that for a.e. $k$ we have

$$
\|u\|_{1,T} \to 0
$$

as $t \to \infty$?
This conjecture is supported, e.g., by calculations (14) made for transport equation or by the Remark 7. If true, it implies
\[ \sum_{|n|>Ct} |\hat{u}(t,n)|^2 \to 0 \]
for any \( C \) and since \( \|u\|_2 = 1 \), we have
\[ \sum_{|n|<Ct} |\hat{u}(t,n)|^2 \to 1 \]
so the most of \( L^2(\mathbb{T}) \) norm is concentrated on, roughly, \( t \) first harmonics. We will call this phenomenon the concentration of \( L^2 \) norm. It does not seem to be possible to obtain any asymptotical result similar to the case when the gap condition does not deteriorate and, perhaps, the “scattering” for this model should be defined in terms of the boundedness of scaled Sobolev norms.

The simple substitution \( \tau = t^{-1} \) reduces the problem to equation
\[ \psi_{\tau} = ik\psi_{\theta\theta} - i\tau^{-2}V(\tau^{-1},\theta)\psi, \quad \psi(1) = 1, \quad 0 < \tau < 1 \]
and for \( q(\tau,\theta) = -\tau^{-2}V(\tau,\theta) \) we have the following bound as \( \tau \to +0 \)
\[ \|q(\tau,\theta)\|_{L^\infty(\mathbb{T})} \lesssim \tau^{\gamma-2} \]
Thus, (50) can be reduced to studying the standard problem on the circle
\[ \psi_t = ik\psi_{\theta\theta} + iq(t,\theta)\psi, \quad \psi(t,\theta) = 1 \] (51)
where the potential grows in the controlled way. We will study the growth of the standard Sobolev norm. Assume for a second that we could prove (which we can not! but compare to (43))
\[ \int_{\mathbb{R}} \|\psi(t,\theta,k)\|_{L^2(\mathbb{T})}^2 dk \lesssim \int_0^t \|q(\tau,\theta)\|_{L^2(\mathbb{T})}^2 d\tau \] (52)
Then, for the original problem that would mean
\[ \int_{\mathbb{R}} \|u_\theta(T,\theta,k)\|_{L^2(\mathbb{T})}^2 dk \lesssim \int_0^T t^2 \|V(t,\theta)\|_{L^2(\mathbb{T})}^2 dt \] (53)
and so
\[ \int_{\mathbb{R}} \|u(T,\theta,k)\|_{L^2(\mathbb{T})}^2 dk = T^{-2} \int_0^T t^2 \|V(t,\theta)\|_{L^2(\mathbb{T})}^2 dt \to 0 \]
provided that \( V \in L^2([0,\infty) \times \mathbb{T}) \). Notice also that by the standard time–scaling it would be sufficient to prove (52) only for \( t = 1 \).

We will start with rather simple apriori estimates. Consider the simplified version of (50)
\[ u_t = ikt^{-2}u_{\theta\theta} + iV(t,\theta)u, \quad 0 < t < T, \quad |V(t,\theta)| \lesssim T^{-\gamma}, \quad u(0,\theta) = 1 \] (54)
We start with well-known estimate
Lemma 5.1. Assume that $V(t, \theta)$ is real trigonometric polynomial of degree smaller than $T^\alpha$ for any $t \in [0, T]$ and $|V(t, \theta)| \lesssim T^{-\gamma}$. Then, for any $k \in \mathbb{R}$

$$T^{-1} \| u(T) \| _{\dot{H}^1(T)} \lesssim T^{\alpha - \gamma}$$

Therefore,

$$\sum_{|n| > CT} |u_n(T)|^2 \lesssim T^{\alpha - \gamma}$$

Proof. The proof is elementary. Differentiating (54) in angle, multiplying by $\bar{u}_\theta$ and integrating, we get

$$\| u_\theta(t) \| _2^2 \leq 2 \int_0^t \| V_\theta(\tau) \| _{L^\infty(T)} \| u_\theta(\tau) \| _2 d\tau$$

If $\max_{t \in [0, T]} \| u_\theta(t) \| _2 = \| u_\theta(t_m) \| _2$, then

$$\| u_\theta(t_m) \| _2 \leq 2 \int_0^T \| V_\theta(t) \| _{\infty} dt \leq 2T^{1+\alpha - \gamma}$$

by Bernstein.

Clearly, we have concentration of $L^2$ norm for all $k$ as long as $\alpha < \gamma$. This argument holds for transport equation as well and can be easily modified to control the higher Sobolev norms. On the other hand, for the transport equation, the $L^2$ norm can really smear over first $T^1$ harmonics as can be easily seen from van der Corput lemma applied to (12).

If one writes

$$u(t, \theta) = \exp \left( i \int_0^t V(\tau, \theta)d\tau \right) \psi(t, \theta)$$

in the previous lemma, then the equation for $\psi$ reads

$$\psi_t = ikT^{-2} \psi_\theta + I,$$

where

$$I = -2kT^{-2} \psi_\theta \int_0^t V_\theta(\tau, \theta)d\tau + ik \psi T^{-2} \left( i \int_0^t V_\theta(\tau, \theta)d\tau - \left( \int_0^t V_\theta(\tau, \theta)d\tau \right)^2 \right)$$

For $I$, we have

$$\| I(t) \| _2 \lesssim T^{2(\alpha - \gamma)} + T^{2\alpha - \gamma - 1}$$

by the previous lemma.

Thus, if $1 + 2\alpha < 2\gamma$, then $\| \psi(T, \theta) - 1 \| _2 \lesssim T^{1+2\alpha - 2\gamma} \to 0$ by Duhamel formula which proves the standard WKB asymptotics of solution for the range $\alpha < \gamma - 1/2$.

In the case just considered, the potential had an extra smoothness in $\theta$. The other extreme case is when $V$ oscillates.
Lemma 5.2. Assume that $V(t, \theta)$ is real trigonometric polynomial, $|V(t, \theta)| \lesssim T^{-\gamma}$, and $\hat{V}(n, t) = 0$ for $|n| < T^\alpha$ and $|n| > CT$. Then,

$$\int_{\mathbb{R}} |\hat{u}_0(T, k)|^2 dk \lesssim T^{5-2\alpha-4\gamma}$$

Proof. On the Fourier side, apply the Duhamel formula to $\hat{u}(t, k) = \exp(ikT^{-2}A\tau)\psi(t, k)$ to get

$$\psi(t, k) = \delta_0 + i \int_0^t e^{-ikT^{-2}A\tau} \hat{V}(\tau)e^{ikT^{-2}A\tau}\psi(\tau, k)d\tau$$

Taking the scalar product with $\delta_0$ and integrating by parts

$$\langle \psi(T, k), \delta_0 \rangle = 1 + I,$$

where

$$I = i \int_0^T \langle \psi'(t, k), \int_t^T e^{-ikT^{-2}A\tau} \hat{V}(\tau)e^{ikT^{-2}A\tau}\delta_0 d\tau \rangle dt$$

and

$$\|I\|_{L^2(\mathbb{R}, dk)} \lesssim T^{-\gamma} \int_0^T \left\| \int_t^T e^{-ikT^{-2}A\tau} \hat{V}(\tau)e^{ikT^{-2}A\tau}\delta_0 d\tau \right\| dt$$

By Plancherel,

$$\left(\int_{\mathbb{R}} \|I\|^2_{L^2} dk \right)^{1/2} \lesssim T^{5/2-\alpha-2\gamma}$$

due to the limitations on the support of $\hat{V}$. □

Clearly, by taking $\alpha + 2\gamma > 5/2$, we have localization of almost all of the $L^2$-norm on the first harmonic for most $k$ but this argument does not say much about the Sobolev norms.

In the rest of this section, we will focus on \[51\] with short range potential, e.g. $V(t, \theta) = \cos(\theta)q(t)$. For simplicity, we start with the following problem where all eigenvalues are non-degenerate

$$x_t = i\Lambda x + iQx, \quad x(0, k) = \delta_0 \quad (55)$$

where $\Lambda$ is diagonal with eigenvalues $\lambda_n = n^2$, $n = 0, 1, \ldots$ and $Q$ is symmetric Toeplitz operator: $Q_{mn}(t) = q_{m-n}(t), q_0(t) = 0, q_{-m}(t) = \tilde{q}_m(t)$, $m, n \geq 0$.

We will use the following notations: given a function $v(t)$, let $\sigma_\alpha(t) = \langle t \rangle^{-1-\alpha} + \langle t \rangle^{-\alpha}|v(t)|$ where $\langle t \rangle = (1 + t^2)^{1/2}$ and $\alpha \geq 0$ is to be specified later.

Theorem 5.1. Assume that $q_n(t) = v(t)(\delta_{-1} + \delta_1)$ and $|v(t)| \lesssim t^{-\gamma}, \gamma > 3/4$. Then, for a.e. $k$, we have

$$\sup_{t>0} \sum_{n \geq 0} n^s |x(t, n, k)|^2 < \infty, \quad \forall s \in \mathbb{N}$$

Fix any $(a, b)$ not containing 0. Then, for any $s \geq 1$

$$\left\| \sup_{t \leq T} \sum_{n=1}^\infty n^s |x_n(t, k)|^2 \right\|_{L^2(\mathbb{R},(a,b))} \lesssim C^s_1(T)C^s_2(T) + 1 \quad (56)$$
Here,
\[ C_1(T) = \left( \int_0^T (\tau)^{2\alpha} v^2(\tau) d\tau \right)^{1/2}, \quad C_2(T) = \int_0^T \sigma(\tau) d\tau, \quad 1 - \gamma < \alpha < \gamma - 1/2. \]

**Proof.** We have
\[ x_n'(t, k) = iv(t)x_{n-1}(t, k) + i kn^2 x_n(t, k) + iv(t)x_{n+1}(t, k), \quad n > 0 \]
and
\[ x_0'(t, k) = iv(t)x_1(t, k), \quad x_n(0, k) = \delta_0. \]
Thus, we have
\[ S_N(T) = \sum_{n=N}^{\infty} |x_n(T, k)|^2 = -2 \text{Im} \left[ \int_0^T v(t)x_{N-1}(t, k)\overline{x}_N(t, k) dt \right], \quad N > 1 \]
Writing \( x_n = \exp(ikn^2t)\psi_n \), we have
\[ \psi_n' = iv(t)\exp(-ikn^2t)(x_{n-1} + x_{n+1}) \]
and so
\[ S_N(T) \lesssim \left| \int_0^T v_N(t, k)(t)^{-\alpha}\psi_{N-1}\overline{\psi}_N dt \right| \]
where
\[ v_N(t, k) = -\int_t^T (\tau)^\alpha v(\tau) \exp(-ik(2N-1)\tau) d\tau \]
Taking \( N > 2 \) and integrating by parts,
\[ S_N(T) \lesssim \int_0^T |v_N(t, k)| \cdot \left( (t)^{-1-\alpha} + (t)^{-\alpha}|v(t)| \right) \cdot \]
\[ \left( |x_{N-2}x_N| + |x_N|^2 + |x_{N-1}|^2 + |x_{N-1}x_{N+1}| \right) dt \quad (57) \]
Notice that for any \( t \), we have
\[ |v_N(t, k)| \lesssim M(k(2N-1)) \]
where \( M(k) \) is Carleson-Hunt maximal function for \( (t)^\alpha v(t) \) and \( M(k) \in L^2(\mathbb{R}) \).
Let
\[ \mu(k) = \left( \sum_{n=1}^{\infty} |M(kn)|^2 \right)^{1/2} \]
For \( 0 < a < b \), we have
\[ \int_a^b \mu^2(k) dk \lesssim \sum_{m=1}^{\infty} \sum_{n=2^m}^{2^{m+1}} n^{-1} \int_{\alpha_n}^{\beta_n} |M(\xi)|^2 d\xi \lesssim \sum_{m=1}^{\infty} \int_{\alpha 2^m}^{\beta 2^m} |M(\xi)|^2 d\xi \lesssim \|M\|^2 \lesssim C_1^2(T) \]
Thus, by Fubini, we have \( M(kn) \in \ell^2(Z^+) \subset \ell^\infty(Z^+) \) for a.e. \( k \).
\[ S_N(T) \leq \mu(k) \int_0^T \left( (t)^{-1-\alpha} + (t)^{-\alpha}|v(t)| \right) \left( |x_{N-2}x_N| + |x_N|^2 + |x_{N-1}|^2 + |x_{N-1}x_{N+1}| \right) dt \quad (58) \]
Sum these inequalities over $N$ using $\|x(t,k)\|_2 = 1$ for any $t$
\[
\sup_{t \leq T} \sum_{n=1}^{\infty} n|x_n(t,k)|^2 \lesssim C_2(T)\mu(k) + 1
\]
By induction,
\[
\sup_{t \leq T} \sum_{n=1}^{\infty} n^s|x_n(t,k)|^2 \lesssim C_2^s(T)\mu^s(k) + 1 \tag{59}
\]
for any $s$. Thus, there is a full measure set such that
\[
\sup_{t > 0} \sum_{n=1}^{\infty} n^s|x_n(t,k)|^2 < \infty
\]
for any $s$. Integration of (59) gives (56). □

We also can improve this result to get real analyticity for a.e. $k$.

**Proposition 5.1.** Under the conditions of the theorem 5.1, there is a full measure set in $k$ for which the solution is real analytic.

**Proof.** We will work on the Fourier side. Summing (58) from $N=2$ to $\infty$
\[
\sum_{N=2}^{\infty} |x_N(T,k)|^2(N-1) \leq C\mu(k) \int_0^T \sigma_\alpha(t) dt \tag{60}
\]
Multiply (58) by $N-3$ and sum from $N=4$ to $\infty$. (60) gives
\[
\sum_{N=4}^{\infty} |x_N(T,k)|^2(N-3)^2 \leq C^2 \cdot 2\mu^2(k) \int_0^t \int_0^{t_1} \sigma_\alpha(t_1) \sigma_\alpha(t_2) dt_2 dt_1
\]
By induction
\[
\sum_{N=2l}^{\infty} |x_N(T,k)|^2(N-2l-1)^l \leq (C\mu(k))^l \left( \int_0^T \sigma_\alpha(t) dt \right)^l
\]
Taking, say, $N \sim 4l$, we have
\[
\sup_{t \geq 0} |x_N(t,k)|^2 \leq \left( \frac{C\mu(k)\|\sigma_\alpha\|_1}{l} \right)^l
\]
which shows that the solution is real analytic for a.e. $k$. □

In theorem 5.1 the integration is restricted to an interval $(a,b)$ which must be finite, not containing $0$. Below we show that this condition can be dropped.

**Theorem 5.2.** Under the conditions of theorem 5.1, we have
\[
\sup_{t > 0} \sum_{n=1}^{\infty} n^2|x_n(t,k)|^2 \in L^1_{loc}(\mathbb{R}) \tag{61}
\]

**Proof.** Notice that the function $(t)^{\alpha}v(t) \in L^\nu(\mathbb{R}^+)$ for some $\nu(\gamma) < 2$ and therefore $M(k) \in L^\nu(\mathbb{R})$ with $\zeta$ dual to $\nu$. Multiply (57) by $N$ and sum from $N=2$ to infinity. We have
\[
I(T,k) = \sum_{n=1}^{\infty} n^2|x_n(T,k)|^2 \lesssim 1+
\]
\[
\int_0^T \sigma_\alpha(t) \left( \sum_{n \geq 2} n^{-\epsilon} |M((2n-1)k)| \cdot n^{1+\epsilon} \left( |x_{n-2}(t,k)x_n(t,k)| + |x_n(t,k)|^2 + |x_{n-1}(t,k)x_{n+1}(t,k)| \right) dt \right. \\
where \epsilon > 0.\]

By Young's inequality, we have

\[
I(T, k) \lesssim 1 + \int_0^T \sigma_\alpha(t) \sum_{n \geq 2} \left( \frac{n^{-\zeta \epsilon} |M((2n-1)k)|^{\zeta}}{\zeta} + \frac{n^{\nu(1+\epsilon)} |x_{n-2}(t,k)|^{2\nu}}{\nu} \right) dt
\]

Taking \( \epsilon = (2 - \nu)/\nu \), we get

\[
I(T, k) \lesssim 1 + A(k) + \int_0^T \sigma_\alpha(t) I(t, k) dt
\]

where

\[
A(k) = \left( \int_0^T \sigma_\alpha(t) dt \right) \cdot \left( \sum_{n \geq 2} n^{-\zeta \epsilon} |M((2n-1)k)|^{\zeta} \right) \in L^1(\mathbb{R})
\]

The Gronwall lemma yields

\[
I(T, k) \lesssim (1 + A(k)) \exp(C_2(T))
\]

which implies (61).

The similar argument can handle the higher Sobolev norms.

The next theorem studies the \( L^p(\mathbb{R}, dk) \) norms of

\[
S_N(T, k) = \sum_{n=N}^{\infty} |x_n(T,k)|^2
\]

**Theorem 5.3.** Assume that conditions of the theorem 5.1 hold. Then, for any \( 2 \leq p \leq \infty, N > 1 \), we have

\[
\|S_N(T, k)\|_p \lesssim N^{-2+2p^{-1}} \left( \int_0^T |v(t)| dt \right)^{2-2/p} \left( \int_0^T v^2(\tau) d\tau \right)^{1/p}
\]  
(62)

**Proof.** We have

\[
S_m(T, k) \lesssim \int_0^T |v(t)| \left| \int_t^T v(\tau) e^{i(2m-1)k\tau} d\tau \right| |x_{m-2}x_m| + |x_m|^2 + |x_{m-1}|^2 + |x_{m-1}x_{m+1}| dt
\]

Sum these inequalities in \( m \) from \( N/2 \) to \( N \). We get

\[
NS_N(T, k) \lesssim \int_0^T |v(t)| \max_{m=N/2, \ldots, N} \left| \int_t^T v(\tau) e^{i(2m-1)k\tau} d\tau \right| dt
\]
Taking the $L^2(\mathbb{R}, dk)$ norm of both sides, we have by Minkowski
\[
\| S_N \|_2 \lesssim N^{-1} \int_0^T |v(t)| \left( \int \sum_{n=N/2}^N \left( \int v(\tau)e^{i(2m-1)k\tau}d\tau \right)^2 dk \right)^{1/2} dt
\]
so
\[
\| S_N \|_2 \lesssim N^{-1} \int_0^T |v(t)| \left( \int |v(\tau)|^2 d\tau \right)^{1/2} dt,
\]
(63) The argument similar to the one employed in the proof of lemma 5.1 gives
\[
\sum_{n \geq 0} n^2|y_n|^2 \lesssim \left( \int_0^T |v(t)| dt \right)^2
\]
uniformly in $k$. Thus,
\[
\| S_N \|_\infty \lesssim N^{-2} \left( \int_0^T |v(t)| dt \right)^2
\]
(64) Interpolation between (63) and (64) gives the statement of the theorem. \hfill \Box

Repeating the same arguments for the case when the eigenvalues $\{\lambda_j\}, j > 0$ have multiplicity two, one has

**Theorem 5.4.** Let $\psi(t, \theta, k)$ be the solution to (51) and $q(t, \theta) = \cos(\theta)q(t)$ where $|q(t)| \lesssim t^{-\gamma}, \gamma > 3/4$. Then

1. For a.e. $k$ we have
   \[
   \sup_{t > 0} \| \psi(t, \theta, k) \|_{H^s(T)} < \infty, \quad s \in \mathbb{Z}^+
   \]
   and $\psi(t, \theta, k)$ is real analytic in $\theta$ for any $t$.

2. For any finite interval $(a, b)$ not containing zero,
   \[
   \sup_{t > 0} \| \psi(t, \theta, k) \|_{H^{1/2}(T)}^2 \in L^{2/s}(a, b)
   \]

3. If $S_N(T, k) = \| P_{|n| \geq N} \psi(T, \theta, k) \|^2$, then
   \[
   \| S_N(T, k) \|_p \lesssim N^{-2+2p^{-1}} \left( \int_0^T |q(t)| dt \right)^{2-2/p} \left( \int_0^T q^2(\tau) d\tau \right)^{1/p}, \quad 2 \leq p \leq \infty
   \]

Consider the model (54) with potential $V(t, \theta) = q(t) \cos(\mu \theta)$ where $\mu$ is integer and $\mu \sim T^\beta, \beta \in [0, 1]$. Then, obviously, $u(T, \theta, k) = \phi(T, \mu \theta, k\mu^2 T^{-2})$ and
\[
i \phi_t = ik\phi_{\theta\theta} + iq(t)\cos(\theta)\phi, \quad \phi(0, \theta) = 1
\]
We have
\[
\int_{\mathbb{R}} \left( \sum_{|n| > N} |\phi_n(T, k)|^2 \right)^2 dk \lesssim N^{-2}T^{3-4\gamma}
\]
Taking \( N \sim T^{-\mu} \), we have
\[
\int_{\mathbb{R}} \left( \sum_{|n| > T} |u_n(T, k)|^2 \right)^2 dk \to 0
\]
for the original solution (as long as \( \gamma > 3/4 \)).

The methods developed in this section can handle the case of transport equation or equation with the symbol \(|n|\). Some of them are applicable to the general short-range potentials \( V \) as well. The perturbation arguments at some places are taken from [7].

In conclusion, we will mention the case for which rather satisfactory results can be obtained. Consider the following short range evolution
\[
u_t = kT^{-\alpha} u_\theta + 2iq(t) \cos(\theta) u, \quad u(0) = f(\theta), \quad 0 < t < T
\]
(65)
where \( 0 < \alpha < 1 \). Notice that \( q \) is not necessarily real-valued but we require \( |q| \lesssim T^{-\gamma} \). We will be interested in the case \( \gamma < (1 + \alpha)/2 \).

The scaled solution is
\[
u(t, \theta - kT^{-\alpha} t, k) = f(\theta) \exp \left( i z Q(k, t) + i \bar{z} Q(-k, t) \right)
\]
where
\[
z = e^{i\theta}, \quad Q(k, t) = \int_0^t q(t) \exp(ikT^{-\alpha} t) dt
\]
We have
\[
\left\| \max_{t \in [0, T]} |Q(k, t)| \right\|_2 \lesssim T^{(1+\alpha)/2-\gamma}
\]
so for most \( k \),
\[
\max_{t \in [0, T]} |Q(k, t)| \lesssim T^{(1+\alpha)/2-\gamma}
\]
Take \( k \) such that
\[
\max_{t \in [0, T]} |Q(\pm k, t)| \leq Q = CT^{(1+\alpha)/2-\gamma}
\]
and expand into Taylor series to get
\[
\exp \left( i z Q(k, t) + i \bar{z} Q(-k, t) \right) = \sum_{l \in \mathbb{Z}} z^l \alpha_l
\]
and
\[
|\alpha_l| \lesssim \sum_{j=0}^{\infty} \frac{Q^{l+j} Q_j}{(l+j)!}! \quad l > 0
\]
Notice that
\[
\frac{Q^{l+j}}{(l+j)!}
\]
decays in \( j \) as long as \( l > Q \). So,
\[
|\alpha_l| \lesssim \frac{Q^l}{l} e^Q < (d/e)^{-l} e^Q,
\]
where \( l > dQ \). Then,
\[
\sum_{l > dQ} |\alpha_l|^2 < 2^{-Q}
\]
for $d$ large enough. This means exponential localization to the range $|l| < dQ$ (since $Q$ is large) for $f = 1$ and for any other column of the monodromy matrix. Such a strong localization result allows us to run a simple perturbation argument. Take initial value $f(\theta) = e^{ij\theta}$ with large positive $j$. Fix $k$ such that the localization property holds and act on $(65)$ with Riesz projection

$$(Pu)_t = kT^{-\alpha}(Pu)_0 + 2iPq(t)\cos(\theta)Pu + \psi$$

where

$$\psi = 2iPq(t)\cos(\theta)P^\perp u$$

Due to strong localization, we have

$$\|\psi\| \lesssim T^{1-\gamma-\frac{Q}{2}}$$

for each $t \in [0,T]$ provided that $j > dQ$ so $Pu$ is an approximate solution to the problem

$$y_t = ikT^{-\alpha}\partial_y y + 2iq(t)P\cos \theta Py, \quad y(0) = ek_j$$

(66)

Assuming that $q$ is real-valued and using Duhamel formula, we get

$$\max_{t \in [0,T]} \|Pu - y\| \lesssim T^{1-\gamma-\frac{Q}{2}}$$

In particular, that means $\max_{t \in [0,T]} |x(t,k)| < T^{1-\gamma-\frac{Q}{2}}$ for each $l : dQ < l < T$ where $x_{ij}$ are elements of the monodromy matrix for the problem (66). Due to symmetry of the monodromy matrix (or time reversal), we have

$$\max_{t \in [0,T]} \sum_{l = dQ}^T |x_{i0}(t,k)|^2 \lesssim T^{3-2\gamma-\frac{CT\epsilon}{2}}, \quad \epsilon = (1 + \alpha)/2 - \gamma > 0$$

Since the first column is always localized to the range $(0,T^{1-\gamma})$, we obtain its localization to the range $(0,T^{1-\gamma}(dT^{(\alpha-1)/2}))$ for most $k$. By simple scaling, one proves that the solution to

$$y_t = ikT^{-1}\partial_y y + iq(t)P\cos(\mu\theta)Py, \quad y(0) = 1, \quad \mu \leq T$$

(67)

is localized to $[0,T^{1-\gamma}\sqrt{\mu}]$ for most $k$ versus $[0,T^{1-\gamma}\mu]$ for all $k$. If $\gamma \geq 1/2$, then $[0,T^{1-\gamma}\sqrt{\mu}] \subseteq [0,T]$, as expected.

6. Appendix

In this section, we collect rather standard results that we used in the main text. The following lemma is well-known

**Lemma 6.1.** If $f \in H^{1/2}(\mathbb{T})$, then

$$\|e^f\|_2 < C_1 e^{C_2\|f\|_{H^{1/2}(\mathbb{T})}}$$

Also, this map is continuous on $H^{1/2}(\mathbb{T})$.

**Proof.** We have

$$e^f = \sum_{n=0}^{\infty} \frac{f^n}{n!}$$

$$\|f^n\|_2 = \|\hat{f} \ast \ldots \ast \hat{f}\|_2$$

By Hölder,

$$\|\hat{f}\|_p \lesssim \left( \frac{C}{p-1} \right)^{(2-p)/(2p)} \|f\|_{H^{1/2}(\mathbb{T})}$$
and by Young’s inequality
\[ \| f \|^2_{H^{1/2}(\mathbb{T})} \sim |\hat{f}(0)|^2 + \int_{\mathbb{T}} \int_{\mathbb{T}} \frac{|f(x) - f(y)|^2}{|x - y|^2} \, dx \, dy \]  
(68)

Since
\[ \left| \int_{\mathbb{T}} \left( e^{if(x)} - 1 \right) dx \right|^2 \leq \left| \int_{\mathbb{T}} |f(x)| dx \right|^2 \lesssim \| f \|^2 \leq \| f \|^2_{H^{1/2}(\mathbb{T})} \]

and
\[ |e^{if(x)} - e^{if(y)}| = \left| \int_{f(x)}^{f(y)} e^{it} dt \right| \leq |f(x) - f(y)| \]

we have the first statement of the lemma. The continuity of exponential at zero is elementary. Now, assume that \( \| f_n - f \|_{H^{1/2}} \to 0 \). Clearly,
\[ \int_{\mathbb{T}} e^{if_n} dx \to \int_{\mathbb{T}} e^{if} dx \]

For the second term in (68), we have
\[ e^{if_n(x)} - e^{if(x)} = (e^{if_n(x)} - e^{if(y)}) = (e^{i(f_n(x) - f(x))} - e^{i(f_n(y) - f(y))})e^{if(x)} + (e^{i(f_n(y) - f(y))} - 1)(e^{if(x)} - e^{if(y)}) \]

and we just need to show that
\[ \int_{\mathbb{T}} \int_{\mathbb{T}} \frac{|(e^{i(f_n(y) - f(y))} - 1)(e^{if(x)} - e^{if(y)})|^2}{|x - y|^2} \, dx \, dy \to 0 \]

The function \( F_n(y) = e^{i(f_n(y) - f(y))} - 1 \) satisfies \( |F_n| \leq 2 \) and \( \| F_n \|_1 \to 0 \). So \( F_n = F_n^1 + F_n^2 \) such that \( F_n^1 = F_n \cdot \chi_{|F_n| < \epsilon} \), \( |F_n^1| < \epsilon \) and
\[ |F_n^2| \leq 2, \quad |\text{supp}(F_n^2)| \lesssim \epsilon^{-1} \int_{\mathbb{T}} |F_n| dx \to 0 \]
Since $\epsilon$ is arbitrary positive number,
\[
\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{|F_n(y)(e^{i f(x)} - e^{i f(y)})|^2}{|x - y|^2} \, dx \, dy \to 0, \quad n \to \infty
\]

\[\square\]

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