Dynamical engineering of squeezed thermal states

Léonce Dupays\textsuperscript{1,2} Aurélie Chenu\textsuperscript{1,3,4}

\textsuperscript{1} Donostia International Physics Center, E-20018 San Sebastián, Spain
\textsuperscript{2} University of the Basque Country UPV/EHU, E-48080 Bilbao, Spain
\textsuperscript{3} Ikerbasque, Basque Foundation for Science, E-48013 Bilbao, Spain
\textsuperscript{4} Massachusetts Institute of Technology, Cambridge, MA 02139, USA

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Control protocols known as shortcuts to adiabaticity allow to drive a quantum system from an initial to a final state arbitrarily fast. These techniques have recently been proposed for open quantum systems, thus extending their application to allow for fast thermalization. Here, we engineer dynamical schemes for the fast preparation of squeezed thermal states at controlled temperature. We derive the equations of motion of squeezed thermal states in harmonic oscillators under unitary and open dynamics, allowing for temperature and entropy variations between the initial and final states. The counter-diabatic Hamiltonians and associated dissipators are provided, and whenever possible, given in a form relevant to experimental applications. The technique is detailed in the setting of trapped-ion experiments with two-photon Raman interaction, where the desired open dynamics is obtained from stochastically shaking the trapping potential, or driving the system with a laser of stochastic amplitude. In this context, we find solutions for the control parameters—namely laser amplitude, phase, and dephasing strength—that allow creating a squeezed thermal state at controlled temperature in arbitrary time.

1 Introduction

Squeezing is a paradigmatic quantum effect that allows reducing fluctuations of one variable beneath the standard quantum limit. This is achieved at the expenses of increasing the variance of the conjugated variable, such that Heisenberg uncertainty principle still holds true for the product of the variances. Squeezed states have kept their promise in improving measurement accuracy beyond quantum noise \cite{1,2} and have become central in quantum optics \cite{3} through demonstrated application in quantum metrology and sensing \cite{4,5}. Advanced techniques to generated squeezed light \cite{6–8} facilitated the detection of gravitational wave \cite{9–11}.

Squeezing is not restricted to optics, nor to the generation of minimal uncertainty states. A mechanical oscillator can be prepared in a squeezed thermal state \cite{12–14} by periodically modulating the spring constant \cite{15}. The resulting state has reduced thermal fluctuations in one quadrature (e.g. position) and increased fluctuations in the conjugate quadrature (e.g. momentum). Squeezed thermal states \cite{16,17} are thus also characterized by an asymmetric phase-space density, and can be viewed as the classical counterpart of squeezed coherent states \cite{18}. Theoretical works have proposed applications in quantum information, where coupling a qubit to a squeezed reservoir allows erasing information below the Landauer's limit \cite{19}. In the context of quantum thermodynamics, the proposed theories of coupling the working medium of a nanoscale heat engine to a squeezed reservoir to generate work beyond the Carnot’s limit \cite{20–25} have been experimentally demonstrated using a vibrating nano-beam driven by squeezed electronic noise \cite{26}.

Diverse theoretical proposals have been put forward to prepare stationary squeezed reservoirs, such as using a mechanical resonator in a quadratically coupled optomechanical system \cite{27}, or from reservoir engineering techniques \cite{21,28}. Continuous quantum measurements, where measuring one variable, e.g. position, precisely reduces its associated variance, present an intuitive technique that has been recently used to design control protocols \cite{29–31}. However, those schemes do not permit an easy control of the temperature.

Figure 1: Schematic representation of the control processes studied in this work. Starting from an initial thermal state with isotropic density in phase-space (top), we design dynamical protocols to generate a squeezed thermal state (bottom) at controlled temperature in arbitrary time. We start with the case $\phi = 0$ (left), that corresponds to a harmonic oscillator with time-dependent frequency $\omega_t$, and generalize to allow for squeezing at an arbitrary phase (right).

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Here, we propose various schemes to dynamically engineer squeezed thermal states in arbitrary time and with controlled temperature, starting from an initial Gaussian state of a quantum harmonic oscillator. This works falls under the umbrella of techniques known as Shortcuts to Adiabaticity (STA) that aim at controlling states without relying on adiabatic protocols. Those techniques have been recently extended to open quantum systems \[32–36\]. We here provide techniques and experimental set-ups to control both the squeezing and thermalization processes. The control of temperature is all the more relevant in squeezed reservoirs, since their variance not only depends on the squeezing amplitude but also on the average thermal phonon number \[37\].

The manuscript is organized as follow: we introduce below the squeezed thermal state to set the focus of the paper and the notation. We first look at the particular case of squeezing along the \(\hat{x}\) or \(\hat{p}\) quadratures in Section 3 and show how a time-dependent harmonic oscillator can be viewed as the initial-time squeezed or dilated oscillator (Fig. 1, left scenario). We provide the equations of motion to create a thermal squeezed state in unitary or non-unitary dynamics, and detail the set of parameters for which the open dynamics can be recast in an experimentally-friendly form, as recently proposed for thermal state \[36\]. Section 4 extends the results to allow for squeezing along any angle in phase space (Fig. 1, right scenario). Thereby, we provide the equations of motion for the closed dynamics in the most general case. Open dynamics is presented, in view of experimental implementation, in a trapped ion setting (Section 5). We thus find the control parameters to generate a squeezed thermal state via two-photon Raman interaction and propose to engineer the dissipative part via (i) a stochastically-shaken trap, and (ii) two additional laser beam, one of which having a stochastic amplitude. Concluding remarks are presented in Section 6.

2 Squeezed thermal state: definition and notation

Let us first introduce the notation and goal of the present work. Squeezing is obtained through non-linear effects described by the operator

\[ S_{r,\phi} = \exp \left( \frac{r_t}{2} (e^{-i\phi_t} a_t^2 - e^{i\phi_t} a_t^{\dagger 2}) \right) \]  

where the time-dependent annihilation operator reads

\[ a_t = \sqrt{\frac{m\omega_t}{2\hbar}} \hat{x} + i\sqrt{\frac{1}{2\hbar m\omega_t}} \hat{p}. \]

The complex-valued squeezing parameter \( z_t = \frac{r_t}{2} e^{-i\phi_t} \) is given by the squeezing amplitude \( r_t \) and phase \( \phi_t \). These are taken time-dependent and will reach a desired value at the end of the control protocol.

We denote \( \sigma_t = e^{-\beta_t H_t} / \text{Tr}(e^{-\beta_t H_t}) \) the instantaneous thermal state of the time-dependent harmonic oscillator \( H_t = \hbar \omega_t (a_t^\dagger a_t + 1/2) \) with frequency \( \omega_t \) and constant mass \( m \), and consider the ‘squeezed thermal state’,

\[ \rho_t = S_{r,\phi} \sigma_t S_{r,\phi}^\dagger. \]

Its static properties have been thoroughly studied and described in e.g. \[38\]. We provide its dynamical properties, starting first with squeezing at a null phase and then generalizing to an arbitrary phase. In both cases, we consider unitary and open dynamics, allowing for variations in the state entropy in the latter case.

3 Time-dependent Harmonic Oscillator: a squeezed oscillator

We first recall some known results about the time-dependent harmonic oscillator (HO), specifically, how its Hamiltonian and thermal state can be obtained from squeezing their initial counterparts. By considering the evolution of the instantaneous thermal state under unitary dynamics, we recover the known invariant of motion \[39\] and counter-diabatic results \[40, 41\]. We extend these results to the case of a system with time-dependent temperature, for which we characterize the open dynamics.

3.1 Time-dependent HO from dilatation of the time-independent HO

We show how the instantaneous thermal state \( \sigma_t \) of a HO with time-dependent frequency \( \omega_t \) can be seen as a squeezed state, at \( \phi = 0 \), of a HO with initial-time frequency \( \omega_0 = \omega_{t=0} \). This becomes explicit considering the evolution of the annihilation operator (2) that reads \( \dot{a}_t = \frac{i}{\hbar \omega_t} a_t^\dagger \). By taking \( r_t = \ln \sqrt{\omega_t/\omega_0} \), this evolution can be written as \( \dot{a}_t = r_t a_t^\dagger \). It is easy to verify\(^1\) that \( a_t = a_0 \cosh r_t + a_0^\dagger \sinh r_t \) is a solution, which can also be written as \( a_t = S_{r,0} a_0 S_{r,0}^\dagger \), with

\[^1\text{Note that } a_0^2 - a_0^{\dagger 2} = a^2_t - a_t^{\dagger 2} \text{ because } \hat{x} \text{ and } \hat{p} \text{ are time independent, so } (a_0 + a_0^\dagger) / \sqrt{\omega_0} = (a_t + a_t^\dagger) / \sqrt{\omega_t} \text{ and } \sqrt{\omega_0}(a_0 - a_0^\dagger) = \sqrt{\omega_t}(a_t - a_t^\dagger).\]
3.2 Unitary dynamics: counter-diabatic Hamiltonian and its control

The evolution of the time-dependent thermal state can be obtained from (i) the dilatation of the original Hamiltonian, \( \sigma_t = S_{r,0} \sigma_0 S_{r,0}^\dagger \), whose time derivation, using \( \partial(S_{r,0} \sigma_0 S_{r,0}^\dagger)/\partial t = 0 \), yields \( \dot{\sigma}_t = [\dot{S}_{r,0} S_{r,0}^\dagger, \sigma_t] = -\frac{i}{2} [a_t^2 - a_t^{12}, \sigma_t] \); or through an alternative road that will prove useful, that is, (ii) from direct derivation of the thermal operator (5). From

\[
\frac{d(a_t^i a_t)^k}{dt} = \frac{\omega_t}{4\omega_t} [a_t^2 - a_t^{12}, (a_t^i a_t)^k],
\]

which in turn gives the evolution of the operator (5) as

\[
\dot{\sigma}_t = \frac{\omega_t}{4\omega_t} [a_t^2 - a_t^{12}, \sigma_t].
\]

Since the instantaneous Hamiltonian commutes with the instantaneous thermal state, it can be added into the dynamics, which becomes

\[
\dot{\sigma}_t = -\frac{i}{\hbar} [H_{\text{cd}}^{(0)}, \sigma_t].
\]

The Hamiltonian

\[
H_{\text{cd}}^{(0)} = H_t + i\hbar \dot{S}_{r,0} S_{r,0}^\dagger
\]

\[
= \hbar \omega_t (a_t^i a_t + \frac{1}{2}) + i\hbar \frac{\omega_t}{4\omega_t} (a_t^2 - a_t^{12})
\]

is known as the counter-diabatic Hamiltonian \([40, 43, 44]\). The term \( H_t \equiv i\hbar \dot{S}_{r,0} S_{r,0}^\dagger \) ensures that each eigenstate remains instantaneous eigenstate of the time-dependent Hamiltonian, i.e., each Fock state evolves as \( i\hbar |n_t\rangle = H_{\text{cd}}^{(0)} |n_t\rangle \). Note that, since

\[
-ih(a_t^2 - a_t^{12}) = \langle \dot{x}, \dot{\rho}\rangle,
\]

this recovers the known result \([40]\), written in first quantization as

\[
H_{\text{cd}}^{(0)} = \frac{\vec{p}^2}{2m} + \frac{1}{2} \hbar \omega_t \sqrt{2} \dot{z} - \frac{\omega_t}{4\omega_t} \langle \dot{x}, \dot{\rho}\rangle.
\]

The non-local, quadratic term \( H_t \) can be transformed into a local form by applying yet another spatial transformation (also see Fig. 2). Consider the time-dependent operator \( \hat{\sigma}_0 = U_{\Omega_0} \sigma_0 U_{\Omega_0}^\dagger \), with the unitary \( U_{\Omega_0} = \exp(i\frac{\Omega_0}{2\hbar} \hat{z}^2) = \exp(i\frac{\Omega_0}{2\hbar} (a_t + a_t^i)^2) \), where \( \Omega_0 \) so far is an arbitrary, time-dependent frequency. The dynamics of this state matrix is governed by the effective Hamiltonian \( \hat{H}_{\theta_0} \equiv U_{\Omega_0}(H_t + H_t)U_{\Omega_0}^\dagger + ih\dot{U}_{\Omega_0} U_{\Omega_0}^\dagger \) and reads

\[
\dot{\hat{\sigma}}_0 = -\frac{i}{\hbar} [H_{\theta_0}, \hat{\sigma}_0].
\]

Using the fact that \( U_{\Omega_0} a_t U_{\Omega_0}^\dagger = a_t - i\frac{\Omega_0}{2\omega_t} (a_t + a_t^i) \), this Hamiltonian takes the explicit form

\[
S_{r,0} = e^{\frac{\omega_t}{2} (a_t^2 - a_t^{12})} \text{ defined from Eq. (1). The time-dependent annihilation operator thus corresponds to the squeezed initial operator, where squeezing acts only on the amplitude } r_t, \text{ the phase being kept to zero. This operation is also known as a dilatation [39], and often written in first quantization as } T_w = \exp\left(-\frac{i\log(w)}{2\hbar}(\dot{x}\dot{p} + \dot{\rho}\dot{\sigma})\right) \text{ with the scaling factor } w \equiv \sqrt{\omega_0/\omega_t}. \text{ The transformation of position and momentum then follows as } S_{r,0} f(x) S_{r,0}^\dagger = f(x/w) \text{ and } S_{r,0} f(p) S_{r,0}^\dagger = f(wp), \text{ respectively. So the time-dependent HO itself is equivalently a squeezing or dilatation from the initial time HO, specifically (also see Fig. 2)}

\[
H_t = \frac{\omega_t}{\omega_0} S_{r,0} H_0 S_{r,0}^\dagger = T_w H_0 T_w^\dagger.
\]

This has already been used to generate squeezing in trapped ion through the sudden switch of the trap frequency [42].

Let us now look at the instantaneous thermal state of the time-dependent HO,

\[
\sigma_t = \frac{e^{-\beta_t H_t}}{\text{Tr}(e^{-\beta_t H_t})} = e^{-\beta_t \hbar \omega_t a_t a_t^i} (1 - e^{-\beta_t \hbar \omega_t}).
\]

This operator is diagonal in the basis of instantaneous Fock states, \( |n_t\rangle = \frac{1}{\sqrt{n_t!}} |0\rangle \), and a projection in this basis reads \( \sigma_t = \sum_{n_t} p_{n_t,t} |n_t\rangle \langle n_t| \). The probabilities are given by the well-known Boltzmann distribution, \( p_{n_t,t} = e^{-\varepsilon_t n} (1 - e^{-\varepsilon_t}) \), where we have defined \( \varepsilon_t \equiv \beta_t \hbar \omega_t \). The von Neumann entropy of this state is \( -\text{Tr}(\sigma_t \ln \sigma_t) = \varepsilon_t / (e^{\varepsilon_t} - 1) - \ln(1 - e^{-\varepsilon_t}) \). For a closed system, the entropy is constant, and \( \beta_t \omega_t = \beta_0 \omega_0 \). So the instantaneous thermal state \( \sigma_t \) can be seen as a dilatation of the initial thermal state \( \sigma_0 \),

\[
\sigma_t = S_{r,0} \sigma_0 S_{r,0}^\dagger.
\]

We next characterize the evolution of the thermal state and distinguish between (a) a unitary dynamics, where \( \varepsilon \) and the probabilities \( p_n \) are time independent, and (b) open dynamics, where the probabilities are time dependent and the von Neumann entropy of the state is allowed to change.
Thus, taking \( \Omega_0 = -\dot{\omega}_t/(2\omega_t) = -\dot{r}_t \) removes the correlations in position and momentum, and yields a time-dependent HO with an effective frequency

\[
\omega_{\text{eff}}^2 \equiv \omega_t^2 + \Omega_0^2 + \frac{\dot{\omega}_t}{\omega_t} - \dot{\Omega}_0
\]

The equation for the effective frequency (10a) provides the effective frequency of a HO for an arbitrary rotation \( U_{\Omega_0} \). The case \( \Omega_0 = -\dot{r}_t \) in (10b) recovers the time-dependent frequency used for the non-adiabatic control of a harmonic trap associated with local counterdiabatic driving [43, 45].

Since \( i\hbar \dot{\omega}_0 - [H_{\text{eo}}, \omega_0] = 0 \), \( \omega_0 \) is an invariant of \( H_{\text{eo}} \). This is consistent with the results for the invariant of motion of a HO. Indeed, the operator \( I_t = T_t H_{\text{eo}} T_t^{\dagger} \) is known to be an invariant of motion for the system with Hamiltonian \( H_{\text{inv}} = i\hbar \dot{T}_t T_t^{\dagger} + F_t(I_t) \), where the function \( F_t(I) = T_t F_t(H_0) T_t^{\dagger} \). Note that for a closed dynamics, \( \beta \omega_t = \beta_0 \omega_0 \), so \( e^{-\beta_0 \hbar t} = U_{\Omega_0} e^{-\beta_0 \frac{\hbar}{\omega_0} \dot{H}_0(t_0)} U_{\Omega_0}^{\dagger} = \omega_0 \). Taking the transformation as the product of two spatial unitary transformations, \( T_t = U_{\Omega_0} S_{\omega_t, \alpha} \), and the function as \( F_t(H_0) = H_0/w_2 \), the Hamiltonian \( H_{\text{inv}} \) becomes exactly equal to \( H_{\text{eo}} \). Indeed \( H_{\text{inv}} = i\hbar \dot{T}_t T_t^{\dagger} + T_t H_{\text{eo}} T_t^{\dagger} = i\hbar \dot{U}_{\Omega_0} U_{\Omega_0}^{\dagger} + U_{\Omega_0} \dot{S}_{\omega_t, \alpha} U_{\Omega_0}^{\dagger} + U_{\Omega_0} H_0 U_{\Omega_0}^{\dagger} = H_{\text{eo}} \). For the particular choice of parameter \( \dot{\Omega}_0 = \dot{\omega}_t/w \), this Hamiltonian becomes a time-dependent HO provided that \( w \) satisfies the nonlinear equation introduced by Ermakov [46], \( \dot{w} + \omega_{\text{eff}}^2 w = \omega_0^2 / w^3 \). This solution of Ermakov equation also directly recovers the effective frequency in Eq. (10b).

### 3.3 Open dynamics: dissipators and master equations

We now allow for temperature and entropy changes during the dynamics by taking \( \varepsilon_t = \beta_t \hbar \omega_t \) time-dependent. The equation of motion of the instantaneous thermal state (5), using (7), reads

\[
\dot{\sigma}_t = -\frac{i}{\hbar} [H_{\text{eo}}^{(0)}, \sigma_t] - \varepsilon_t \sigma_t \left( a_t^\dagger a_t + \frac{1}{1 - e^{\varepsilon_t}} \right).
\]

(11)

In the Fock state basis, the instantaneous thermal state \( \sigma_t = \sum_n p_{n,t} |n_t\rangle \langle n_t| \) now has time-dependent populations. We show below how this impacts the dynamics of the control Hamiltonian, and propose a solution for implementation in the lab.

To do so, we consider the state matrix \( \varrho_t = U_{\Omega} \sigma_t U_{\Omega}^{\dagger} \) rotated by the unitary \( U_{\Omega} = \exp \left( i \frac{\Omega_t}{\omega_t} (a_t + a_t^\dagger)^2 \right) \), and provide a possible choice for the frequency \( \Omega_t \). This matrix evolves according to

\[
\dot{\varrho}_t = -\frac{i}{\hbar} [H_{\text{eo}}, \varrho_t] + \mathcal{D}_c(\varrho_t).
\]

The counter-diabatic dissipator defined here takes into account all terms coming from the variation of the population and in the Fock state basis reads

\[
\mathcal{D}_c(\varrho_t) = \sum_n \hat{p}_{n,t} U_{\Omega} |n_t\rangle \langle n_t| U_{\Omega}^{\dagger}.
\]

Alternatively, part of the counter-diabatic Hamiltonian can be written as a ‘control’ harmonic oscillator, \( H_c \), with a ‘control’ frequency chosen of the form of the closed results (10b), i.e. \( \omega_c^2 = \omega_t^2 - \Omega_t^2 - \dot{r}_t \). Explicitly, this yields

\[
H_{\text{cd}} = H_c + \frac{\hbar}{4\omega_t} \left( 2 \Omega_t^2 + \dot{\Omega}_t \right) (a_t + a_t^\dagger)^2 + i\hbar \left( \frac{\Omega_t^2}{2} + \dot{\Omega}_t \right) (a_t^2 - a_t^\dagger)^2
\]

(13)

with \( H_c = \frac{\dot{\rho}^2}{2m} + \frac{1}{2} m \omega_c^2 \dot{x}^2 + \hbar \omega_t (a_t^\dagger a_t + \frac{1}{2}) + \frac{\hbar}{4\omega_t} \omega_c^2 (a_t^\dagger + a_t)^2 \). Then, the frequency \( \Omega_t \equiv \Omega_0 + \Omega_1 \) is taken to be composed of \( \Omega_0 = -\dot{\omega}/(2\omega_t) \) to cancel the term in \( a_t^2 - a_t^\dagger^2 \) if the dynamics were unitary (cf. Eq. 10a), and an additional frequency \( \Omega_1 \) to account for changes due to the open dynamics. The master equation (12) thus becomes

\[
\dot{\varrho}_t = -\frac{i}{\hbar} [H_c, \varrho_t] + \mathcal{D}_c(\varrho_t).
\]

(14)

The ‘control’ dissipator can be written in a compact form by defining the annihilation operator \( b_t \equiv U_{\Omega} a_t U_{\Omega}^{\dagger} = a_t - i \frac{\Omega_t}{2\omega_t} (a_t + a_t^\dagger) \), which gives (see App. A for details)

\[
\mathcal{D}_c(\varrho_t) = \frac{\Omega_1}{2} [b_t^2 - b_t^\dagger^2, \varrho_t] - \varepsilon_t \varrho_t \left( b_t^\dagger b_t + \frac{1}{1 - e^{\varepsilon_t}} \right).
\]

(15)

Note that \( a_t + a_t^\dagger = b_t + b_t^\dagger \), so the position operator \( \hat{x} \) is equivalently written in one basis or the other.
This dissipator can be further written in a more ‘experimentally-friendly’ form. For this, we note that the relations $\alpha_i \sigma_i = e^{-\epsilon (\alpha_i \alpha_i^\dagger + 1/2)} a_i Z_i^{-1} = \sigma_i a_i e^{-\epsilon \hat{z}}$ and $\alpha_i^\dagger \sigma_i = \sigma_i a_i^\dagger e^{\epsilon \hat{z}}$ translate to $b_i \dot{q}_i = \dot{q}_i b_i e^{-\epsilon \hat{z}}$ and $b_i^\dagger \dot{q}_i = \dot{q}_i b_i^\dagger e^{\epsilon \hat{z}}$. By setting $\Omega_1 \equiv (1-e^{-\epsilon \gamma})/(1-e^{-\epsilon \gamma})$, the dissipator (15) can be recast as
\[
\mathcal{D}_c (\dot{q}_i) = -\Gamma_t \left[ (b_i + b_i^\dagger), [\dot{x}_i, \dot{q}_i] \right],
\]
where $\Gamma_t \equiv \frac{\dot{q}_i}{2(1-e^{-\epsilon \gamma})(1-e^{-\epsilon \gamma})} = \frac{n}{2m\omega} \gamma_i t$. The control dissipator can thus be recognized as the well-known form of localization in the position eigenbasis, often referred to as Joos-Zeh term [47,48], and is easily implementable in current experimental platforms. In turn, the modulation of $\gamma_i$ can be engineered, e.g., by post-selection measurement of the position or via stochastic parametric driving, as proposed in [36] and also used in Section 4.

To summarize the main results of this section, a time-dependent HO is equivalent to squeezing the initial HO along the $\hat{x}$ or $\hat{p}$ quadratures (left scheme in Fig. 1). Under unitary dynamics, the instantaneous thermal state evolves according to a non-local Hamiltonian with correlated position and momentum. In a rotated frame, the state can be recast as the thermal state of a HO with an effective frequency $\omega_{\text{eff}}$ given in Eq. (10b). Allowing for changes in the temperature equivalently yields to the master equations (12) and (14). The difference in the Hamiltonians controlling the unitary parts is accounted for in the different dissipators. Eq. (12) follows from using the original instantaneous thermal state to define the unitary part. In turn, using the results of the closed dynamics to defines the unitary part yields to Eq. (14), that has an obvious benefit for experimental implementation. Indeed, a particular choice of the frequency controlling the open part, $\Omega_1$, allows to recast the master equation into

\[
\rho_t = S_{r,\phi} \sigma_t S_{r,\phi}^\dagger
\]

\[
= \frac{1}{Z_t} e^{-\beta_t H_{\text{zemo}}} S_{r,\phi} S_{r,\phi}^\dagger = \frac{1}{Z_t} e^{-\epsilon \hat{z}} (A_t^\dagger A_t + 1/2).
\]
Here, we have inserted the identity in \( H \) to write
\[
S_{r,\phi}a_t^{\dagger}a_tS_{r,\phi}^\dagger = (S_{r,\phi}a_t^{\dagger}S_{r,\phi})(S_{r,\phi}a_tS_{r,\phi}^\dagger) \equiv A_t^\dagger A_t.
\]
The new creation and annihilation operators, denoted \( A_t^\dagger \) and \( A_t \) respectively, can be explicitly found by using the Baker-Campbell-Hausdorff (BCH) formula [49], where the odd and even terms in the infinite sum can be recognized as hyperbolic functions. Explicitly, this yields the known Bogoliubov transformation
\[
A_t \equiv S_{r,\phi}a_t S_{r,\phi}^\dagger = f_+ a_t + f_- a_t^{\dagger}, \tag{18a}
\]
\[
a_t = S_{r,\phi}^{\dagger}A_t S_{r,\phi} = f_+ A_t - f_- A_t^{\dagger}, \tag{18b}
\]
with the time-dependent dimensionless prefactors \( f_+ = \cosh r_t \) and \( f_- = e^{i\phi_t} \sinh r_t \), fulfilling \( |f_+|^2 - |f_-|^2 = 1 \). These \( A_t \) operators correspond to bosonic operators and fulfill the commutation relation \([A_t, A_t^\dagger] = 1\). For each \( A_t^\dagger \) boson created, there is both creation and annihilation of some ‘a’ bosons.²

In this basis, the squeezed harmonic oscillator simply reads
\[
H_{\text{gho}} \equiv S_{r,\phi}H_t S_{r,\phi}^{\dagger} = \hbar\omega_t(A_t^\dagger A_t + 1/2). \tag{19}
\]
It corresponds to the well-known generalized harmonic oscillator (see e.g. [52–55]), and in first quantization takes the form
\[
H_{\text{gho}} = \frac{\vec{p}^2}{2M_t} + \frac{1}{2}M_t \Omega_t^2 \hat{x}^2 + \frac{1}{2} \omega_t \sinh(2r_t)\{\hat{x}, \hat{p}\}, \tag{20}
\]
with the time-dependent mass and frequency defined from \( m/M_t = \cosh(2r_t) - \sinh(2r_t) \cos \phi_t \), and \( \Omega_t^2/\omega_t^2 = \cosh(2r_t) - \cos^2 \phi_t \sinh^2(2r_t) \). This Hamiltonian is quadratic in the basis of the a operators, as easily seen from inserting Eq. (18a) into Eq. (19), yielding
\[
H_{\text{gho}} = \hbar\omega_t (\cosh(2r_t) (a_t^{\dagger}a_t + 1/2) + \frac{1}{2} \sinh(2r_t) (e^{-i\phi_t}a_t^2 + e^{i\phi_t}a_t^{\dagger 2})).
\]
We proceed to describe the evolution of its thermal state.

### 4.2 Unitary dynamics: the counter-diabatic Hamiltonian

We are interested in describing the dynamics of the squeezed thermal state defined in Eqs. (3;17), and look for an expression of the Hamiltonian governing the unitary dynamics as \( \dot{\rho}_t = -\frac{i}{\hbar}[H_{\text{cd}}, \rho_t] \), with
\[
H_{\text{cd}} \equiv S_{r,\phi}(H_t + H_1)S_{r,\phi}^{\dagger} + \frac{1}{2} S_{r,\phi}^{\dagger}S_{r,\phi}^{\dagger}. \tag{21}
\]
The transformed Hamiltonians in the first term directly follow from the definition of the ‘A’ operators (18a), and are given by Eq. (19) and \( S_{r,\phi}H_t S_{r,\phi}^{\dagger} = i\hbar \frac{\omega_t}{2\omega_t^2} (A_t^2 - A_t^{\dagger 2}) \).

The challenge in getting the full \( H_{\text{cd}} \) for an arbitrary phase is to find an expression for \( S_{r,\phi}S_{r,\phi}^{\dagger} \).

App. B shows how to obtain it from the \( k \)th time-derivatives of \( (e^{-i\phi_t}a_t^2 - e^{i\phi_t}a_t^{\dagger 2}) \). This allows for an explicit expression of the counter-diabatic Hamiltonian into the form
\[
H_{\text{cd}} = \hbar\omega_t (A_t^\dagger A_t + 1/2) + \hbar \frac{\dot{\phi}_t}{2} (A_t^\dagger A_t - A_t A_t^\dagger) + i\hbar \left( \frac{\dot{\omega}_t}{4\omega_t} (a_t^2 - a_t^{\dagger 2}) + \dot{r}_t (a_t^2 e^{-i\phi_t} - a_t^{\dagger 2} e^{i\phi_t}) \right) \equiv H_{\text{gho}} + H_{\text{cd}1}^{\text{gho}}. \tag{20}
\]
In this counter-diabatic Hamiltonian, we recognize the free Hamiltonian of the time-dependent generalized HO, \( H_{\text{gho}} \) in the first term. The remaining terms, gathered in \( H_{\text{cd}1}^{\text{gho}} \), prevent the creation of excitation such that each eigenstate remains eigenstate of the instantaneous squeezed Hamiltonian. We verify that for \( \phi_t = 0 \), the later contribution exactly recovers the form in Eq. (8) when taking \( r_t \rightarrow \frac{r_t}{2} = \frac{1}{2} \ln \sqrt{\omega_t/\omega_0} \). This is to be expected because, in this case, the operators are identical. Specifically, the annihilation operator becomes \( A_t(r_t) = a_0 \cosh(2r_t) + a_0^\dagger \sinh(2r_t) = a_0(2r_t) \), and the thermal squeezed state reads \( \rho_t(\phi = 0) = S_{r,0} \sigma_t S_{r,0}^{\dagger} \).

Equation (20) shows how squeezing at an arbitrary phase non-trivially affects the counter-diabatic Hamiltonian. The variation of the phase modifies the harmonic part proportionally to the difference between the generalized and original HO. It also adds a phase shift in the quadratic term, that can be captured through a modified operator \( \tilde{a}_t \equiv e^{-i\phi_t/4} \).

Doing this and setting \( r_t = \ln \sqrt{\omega_t/\omega_0} \), we get a form (not shown here) similar to that presented in Section 4.3, in which the quadratic terms could be removed by applying a rotation. However, for a non-zero phase, there will always be terms that correlate the position and momentum, so we believe that Eq. (20) remains one of the simplest possible expressions.

### 4.3 Open dynamics: general master equation

With the set of operators defined above, it is straightforward to derive a formal expression for the master equation dictating the dynamics of the squeezed thermal state when entropy is allowed to change (\( \varepsilon_t \) is now time-dependent). Indeed, direct derivation of the state matrix \( \rho_t \) and the master equation of the thermal state \( \sigma_t \) given in Eq. (11)

.multipart

²Note that squeezing then appears similar to the physical setup of the independent-boson model [50], which is best dealt with using two different basis for the bosons [51].
directly yield to
\[ \dot{\rho}_t = -\frac{i}{\hbar} [H_{cd}, \rho_t] - \dot{\epsilon}_t \rho_t \left( A_t^\dagger A_t + \frac{1}{1 - e^{\epsilon_t}} \right), \]
where \( H_{cd} \) is given in Eq. (20). In the next Section, we propose an experimental setup for the experimental implementation of the dynamical scheme generating a squeezed thermal state.

5 Proposed setup to generate a squeezed thermal state

5.1 Equations of motion for the squeezed thermal state

With the former general results at hand, it is easy to derive the equations of motion for a harmonic oscillator squeezed at arbitrary phase. Its thermal state reads
\[ \rho_t = S_{r,\phi} \sigma S_{r,\phi}^\dagger, \]
where \( \sigma \equiv e^{-\epsilon_t (a^\dagger a + 1/2)} \) denotes the thermal state of the oscillator \( \hbar \nu (a^\dagger a + 1/2) \) depending only on time through its temperature, \( \epsilon_t \equiv \beta_t \hbar \nu \). The bosonic operator is time-independent and denoted as
\[ a = \sqrt{\frac{m \nu}{2 \hbar}} \dot{x} + i \sqrt{\frac{1}{2 m \nu}} \dot{p}. \]
We denote the relevant operator \( A_0 \equiv \cosh \beta_t a + \sinh \beta_t e^{i \phi_0} a^\dagger \), which corresponds to \( A_t \) defined in Eq. (18a) for \( \omega_t \to \nu \).

The counter-diabatic Hamiltonian directly follows from Eq. (20) as
\[ H_{cd} = \hbar \nu (A_0^\dagger A_0 + \frac{1}{2}) + \hbar \frac{\dot{\phi}}{2} (A_0^\dagger A_0 - a^\dagger a) \]
\[ + \hbar \frac{\dot{\gamma}_t}{2} (i e^{-i \phi_t} a^2 - i e^{i \phi_t} a^\dagger a^\dagger). \]
The master equation for the thermal squeezed state simplifies from Eq. (21) as
\[ \dot{\rho}_t = - i \left[ - \frac{\dot{\phi}}{2} (a^\dagger a + \frac{1}{2}) + \frac{\dot{\gamma}_t}{2} (i e^{-i \phi_t} a^2 - i e^{i \phi_t} a^\dagger a^\dagger), \rho_t \right] \]
\[ - \dot{\beta}_t \hbar \nu \rho_t \left( A_0^\dagger A_0 + \frac{1}{1 - e^{\beta_t \hbar \nu}} \right). \]

In the case of unitary dynamics (\( \dot{\beta}_t = 0 \)), the squeezing Hamiltonian, first line in (25), can be easily generated using a two-photon Raman interaction \[56\]. The amplitude of interaction \( \dot{\gamma}_t/2 \) is then controlled experimentally through the laser characteristics, that we provide in details in Sec. 5.2.

Such a Hamiltonian has been extensively used for the squeezing of external fields in cavity quantum electrodynamics \[57, 58\]. However, having the variation of the squeezing parameter directly related to the laser parameters restricts the accessible squeezing variance, since it is then related to the process time (see illustration in Fig. 3). Such restrictions can be lifted with the use of STA techniques and reverse engineering of the dynamics that allow to choose the time independently of the squeezing parameter and give freedom on the final variance of the squeezed state.

**Figure 3:** (a) Evolution of the squeezing amplitude \( r_t \) as function of the process time with linear variation (blue curves) or through a controlled dynamics (red curve). (b) Corresponding evolution of the squeezed thermal state normalized variance in position, computed using the definition given in [37]. Increasing the squeezing parameter linearly in time (\( \dot{r}_t \) constant, see subfigure) yields to a variance that decreases exponentially in time. STA techniques, through reverse-engineering of the dynamics, allow to reach a target squeezing amplitude in an controlled time. For example, taking a fifth-order polynomial Ansatz for the squeezing parameter, \( r_\tau = r_0 + (r_f - r_0)(10\tau^3 - 15\tau^4 + 6\tau^5) \) with \( \tau = t/t_f \), allows to reach a desired squeezing in arbitrary time (\( r_f = 4 \) here), as also seen in the variance (b).

The generality of the open dynamics considered in Eq. (25) makes it challenging to find an experimentally friendly expression. So we decide to take a reverse approach and in the following engineer a particular master equation that does generate the desired squeezed thermal state at controlled temperature in arbitrary time. The next subsections propose a scheme for implementation in trapped-ion platforms.

We present how to engineer an effective master equation where the dissipator is obtained from a stochastic system and provide the control parameters for which the squeezed thermal state is a solution.
5.2 Engineering the master equation in a stochastically-shaken trapped ion

Consider an ion trapped in an electric potential of quadrupolar spatial shape and interacting with two mono-chromatic laser beams as illustrated in Fig. 1. In the experimental situation of interest, the internal electronic structure of the ion is reduced to a two-level system, with levels $|g\rangle$ and $|e\rangle$ of energy difference $\hbar \omega$, the atomic Hamiltonian reading $H_a = \hbar \omega \sigma_z$, with $\sigma_z = |e\rangle\langle e| - |g\rangle\langle g|$. The motion of the trapped atom can be considered harmonic in all three dimensions, as obtained either from a classical or quantum-mechanical treatment \cite{59,60}. The motional Hamiltonian along one of the axis of the trap is thus taken as $H_m = \hbar \nu (a^\dagger a + 1/2)$. With suitable electromagnetic fields, the internal levels can be coupled to each other and to the external motional degrees. The interaction Hamiltonian resulting from the applied two laser fields can be described as \cite{60}

\[
H_{\text{int}}(t) = \sum_{l=1,2} \frac{\hbar}{2} \Omega_l \sigma_x \left( e^{i(k_l x - \omega_l t - \Phi_l)} + \text{h.c.} \right),
\]

with $\sigma_x = |g\rangle\langle e| + |e\rangle\langle g|$, each of the electromagnetic field being treated as a classical plane wave of the form, in the direction $\mathbf{x}$ of interest, $E_l(\tilde{x}, t) \cdot \mathbf{x} = A_l(t) (e^{i(k_l \tilde{x} - \omega_l t - \Phi_l)} + \text{c.c.})/2$ with time-dependent amplitude $A_l(t)$, wave vector $k_l = k_l \mathbf{x}$, and detuning $\delta_l$ from the atomic transition, $\omega_l - \delta_l = \omega$. The Rabi frequency describing dipole coupling to a single charge $q$ is given by $\hbar \Omega_l/2 = q(\hbar \tilde{x}) A_l(t)$.

We aim at preparing a squeezed thermal state on the vibrational levels of the system with total Hamiltonian

\[
H_{\text{st}} = -\frac{\hbar \Delta}{2} \sigma_z + U_{t,\text{tot}} H_{\text{int}}(t) U_{t,\text{tot}}^\dagger + \hbar \sqrt{2 \gamma_l \xi_l \hat{\tau}_t} \otimes |g\rangle\langle g|,
\]

starting from an initial vibrational state that is thermal or has a (possibly generalized) Gaussian representation in coordinate space. As discussed above, reaching a desired temperature can require changes in entropy, that we achieve through engineering the master equation for the density matrix. While Section 4.3 shows a general form for the open dynamics of a squeezed thermal state, we here adapt the experimentally friendly form that demonstrated useful at null phase and look for a dissipator in the position coordinate, similar to the dissipator in Eq. (16).

We show below that a squeezed thermal state can be engineered through implementation of the stochastic Hamiltonian

\[
h_{\text{st}} = H_{\text{tot}}(t) + \hbar \sqrt{2 \gamma_l \xi_l \hat{\tau}_t} \otimes |g\rangle\langle g|,
\]

that is characterized by the Wiener process $W_t = W_0 + \int_0^t \xi_t \, dt'$ defined in terms of the normally distributed random variable $\xi_t$ with zero mean and vanishing correlation, $\langle \xi_t \xi_{t'} \rangle = \delta(t - t')$. The stochastic term allows to create the control dissipator, and has the advantage of being readily implementable via continuous quantum measurement or in a stochastically-shaken trap \cite{61}. Note that the stochastic term in (28) is taken as acting only on the electronic ground state, which allow for a rigorous derivation, but can be experimentally changing. To overcome this difficulty, Sec. E presents an alternative scheme that uses, instead of shaking the trap, two additional laser beams, one having a stochastic amplitude.

Let $|\psi_t\rangle$ denote the solution of the Schrödinger equation, $|\psi_{t+\Delta t}\rangle = e^{-\frac{i}{\hbar} H_{\text{tot}} \Delta t} |\psi_t\rangle$. It is useful to change the energy scale \cite{62} and look at the evolution of the state vector $|\Psi_t\rangle = U_{t,\text{tot}} |\psi_t\rangle$ rotated by a unitary transformation $U_{t,\text{tot}} \equiv e^{i \hat{H}_{\text{tot}} \Delta t}$. The rescaling Hamiltonian $H_{t} = H_a + H_m + \frac{\hbar}{2} \Delta \sigma_z$ effectively shifts the electronic energy of the gap $\hbar \omega$ into an energy defined by the average of the laser detuning $\hbar \Delta = \hbar (\delta_1 + \delta_2)/2$, and yields to an interaction picture. The rotated state evolves as $i \hbar |\Psi_t\rangle = H_{st} |\Psi_t\rangle$, with $H_{st} \equiv U_{t,\text{tot}} H_{\text{tot}} U_{t,\text{tot}}^\dagger + i \hbar U_{t,\text{tot}} U_{t,\text{tot}}^\dagger$ taking the explicit form (see App. D for details)
5.3 Solving the dynamics

We consider the system initialized in $|g\rangle\langle g| \otimes \sigma_{t=0}$, where $\sigma_t \equiv e^{-\beta_E t}/Z_t$ denotes the thermal state of the vibrational manifold at initial inverse temperature $\beta(t=0) = \beta_0$. We next solve the dynamics to find the dynamical control parameters $\{\alpha_t, \kappa_t\}$ for which the squeezed thermal state

$$
\rho_t = |g\rangle\langle g| \otimes S_{r_{t}, \phi_t} \sigma_0 S^\dagger_{r_{t}, \phi_t}
$$

is a solution of the master equation (33). Note that the parameter $\lambda_t \equiv -\beta_t \hbar \nu$ is taken time-dependent to allow for changes in the temperature. For this, it is useful to work with the factorized form that we derive in normal ordering following McCoy [66] as (see derivation in App. C)

$$
e^{\lambda_t S_{r_{t}, \phi_t} a^\dagger a S^\dagger_{r_{t}, \phi_t}} = e^{\lambda_t (\cosh(2\nu t) a^\dagger a + \cosh \nu t \sinh \nu t (e^{i\phi_{t}} a^2 + e^{-i\phi_{t}} a^2) + \sinh^2 \nu t)}
$$

$$= K_t e^{J_t a^\dagger a} e^{-B_t a^2},$$

(35)

where the parameters are defined as $J_t \equiv f(r_t, \lambda_t) e^{i\phi_{t}}$ with $f(r_t, \lambda_t) \equiv \frac{1}{2} \left( \frac{\sinh(2\nu t)}{\sinh^2(\nu t) - \sinh^2(\nu t) e^{2\lambda_t}} \right) B_t = \lambda_t$.
5.3 Solving the dynamics

\[ g(r_t, \lambda_t) = -\ln \left( \frac{1 + (e^{i \lambda_t} - 1)(\cosh^2(r_t) + \sinh^2(r_t))e^{i \lambda_t}}{\cosh^2(r_t) + \sinh^2(r_t)e^{i \lambda_t}} \right), \]

and \( K_t \) a normalizing constant given explicitly in Eq. (S22).

The master equation (33) gives

\[ \frac{d\rho_t}{dt} \rho_t^{-1} = -i\alpha_t a^2 - i\alpha_t a^2 - 2\kappa_t - 2\kappa_t a^a a + \rho_t \left( i\alpha_t a^2 + i\alpha_t a^2 - 2\kappa_t a^a a \right) \rho_t^{-1} \]
\[ + 2\kappa_t a^a a^a a^a a^a a^{-1} + 2\kappa_t a^a a^{-1} a^{-1}. \] (36)

Using the adjoint representation \( A_A \) of an operator \( A \), we compute the terms \( \rho A \rho^{-1} \) for each operator \( A \) in the basis \( B \equiv \{ a^2, a^2, 1, a^a a, a^a a \} \) and express them in a matrix form. Details are provided in App. D. This eventually yields to the simple system (34) that can be expressed in the matrix form

\[ \begin{pmatrix} \kappa \\ \alpha_R \\ \alpha_I \end{pmatrix} = M_t^{-1} \begin{pmatrix} \dot{J}_R \\ \dot{J}_I \\ \dot{B} \end{pmatrix}, \]

where the transfer matrix reads

\[ \begin{pmatrix} 4(e^{-B} - 1)J_R & 4(e^{-B} - 1)J_I & -8J_I J_R \\ 4(J_R^2 - J_I^2) + (1 - e^{-B}) & 8J_R J_I & 4J_R J_I + (e^{-2B} - 1) \\ -8J_R^2 & 8J_R J_I & -8J_R \end{pmatrix}. \] (37)

This is the main result of this section. It provides the control parameters \( \alpha = \alpha_R + i\alpha_I \) and \( \kappa_t = \kappa_0 t \) to engineer the squeezed state characterized by \( J = f(r_t, \lambda_t) = J_R + iJ_I \) and \( B = g(r_t, \lambda_t) \) at the desired temperature through \( \lambda_t \equiv -\hbar \nu \beta_t \).

![Figure 5: Experimental control parameters](image)

- **Cooling**
- **Isothermal**
- **Heating**

**Figure 5:** **Experimental control parameters:** (top) laser relative amplitude and (bottom) dephasing strength for (a) cooling \( \lambda_t = -2 \), (b) isothermal \( \lambda_t = \lambda_t \), and (c) heating \( \lambda_t = -0.5 \) processes. The initial state is isotropic \( r_t = 0 = \phi_t = 0 \) at \( \lambda_t = -1 \). The final state is a thermal state with (plain lines) no squeezing \( r_f = \phi_f = 0 \); (dash-dotted lines) squeezing at \( r_f = 1, \phi_f = 0 \) or (grey dashed lines) squeezing at \( r_f = 1 \) and angle \( \phi_f = \pi \). Note that the control parameters are here normalized, and see Figs. 6-7 for the influence of temperature and squeezing on their maxima.

We show a numerical application for a system initialized in a (possibly squeezed) thermal state characterized by \( \{ r_0, \phi_0, \lambda_0 \} \) and reaching a target final state \( \{ r_f, \phi_f, \lambda_f \} \). The state parameters \( J_R, J_I \) and \( B \) are assumed to follow a smooth evolution taken as a fifth-order polynomial, with boundary conditions \( J_R(t_0/f) = J_I(t_0/f) = B(t_0/f) = 0 \), first, and second derivative equal to zero. The relative detuning between the lasers is fixed to \( \omega_2 - \omega_1 = 2\nu \), as required to generate the squeezing Hamiltonian (31). The control parameters are obtained by solving Eqs. (37, 38). The dynamics can thus be implemented through the controlled dephasing strength \( \kappa_t = \gamma_0 t^2 \), the controlled laser amplitudes and their Rabi frequencies. The latter are directly related to the control parameters \( \alpha = |\alpha| e^{i(\Phi_1 - \Phi_2)} \) that gives the relative laser phases \( \Phi_1 - \Phi_2 = \arctan (\alpha_1 / \alpha_2) \) and Rabi frequencies through \( |\alpha| = (\eta_2 - \eta_1) e^{2\Omega_1 \Omega_2} \).

![Figure 6: Influence of temperature on the control maxima](image)

- (a) Various quantum processes
- (b) Influence of temperature

**Figure 6:** **Influence of temperature on the control maxima:** Maximum (a) laser amplitude and (b) dephasing strength as function of changes in the temperature \( |\lambda_f| = \beta_0 \nu \beta_0 \) for heating (orange background) and cooling (blue background) processes. Results are shown for states with constant squeezing amplitude, starting with \( |\lambda_0| = 1 \) and \( \phi_0 = \phi_f = 0 \).

Figure 5 shows the control parameters for squeezing with different temperature conditions, namely isothermal, heating, and cooling. The normalized laser amplitude appears to be quite similar for all squeezing processes, which can be expected as it mainly controls the squeezing amplitude. The maximum is influenced by the variation of squeezing, as shown in Fig. 7. In the case that the state is kept thermal, no squeezing term is needed, as intuitively expected. Figures 6 and 7 show the influence of changing the temperature and squeezing amplitude, respectively. We verify that the dephasing strength is symmetric in the sense that squeezing by a variation \( |\Delta r| \) only changes the sign of the dephasing,
not its strength.

Figure 7: Influence of squeezing on the control maxima: Maximum (a-b) dephasing strength and (c) laser amplitude as function of the initial \( r_i \), or final \( r_f \) squeezing amplitude for different variation of squeezing \( \Delta r = r_f - r_i \).

As mentioned above, implementation of the stochastic Hamiltonian (28) assumes a spin-dependent term on the position of the trap, that could be developed following the techniques proposed in e.g. [67]. This allowed for a rigorous derivation of the effective Hamiltonian through the adiabatic elimination. Shaking the full trap (ground and excited electronic states) would require further approximations of the excited state populations, although the adiabatic elimination might still hold at large detunings. Further work could be done using the recently developed adiabatic elimination for open bipartite systems [68–70]. An experimental alternative is to install a feedback loop that enforces the qubit to remain in its ground state [29]. Should the proposed model still be limiting experimentally, we provide in App. E an alternative scheme where the dissipator is engineered with two additional laser field instead of shaking the trap.

Note that we have here focused on the trapped-ion setup for the sake of proposing a scheme that can be directly implemented experimentally. However, the dissipator need not be of the form \([\hat{x}, \hat{x}, \rho]\).

For example, using a dissipator of the type \( \mathcal{D}_{\text{sq}} = \gamma(n_{th} + 1)\mathcal{L}(a)\rho + \gamma n_{th}\mathcal{L}(a^\dagger)\rho \) in Eq. (25), with \( \mathcal{L}(a) = a\rho a^\dagger - \frac{1}{2} (a^\dagger a\rho + \rho a^\dagger a) \), could also be used to generate a squeezed thermal state in a photonic platform, where the squeezing Hamiltonian could be obtained with, e.g., parametric downconversion.

6 Conclusion

We have derived the master equations describing the evolution of the squeezed thermal state for a (i) harmonic oscillator with (ii) time-dependent frequency, (iii) and a generalized harmonic oscillator. Results include open dynamics in order to allow for changes in the state temperature.

Through reverse engineering of the dynamics, we have found the control parameters that allow generating a thermal state that is squeezed at a target amplitude, phase, and temperature, in a controlled time. Dissipation has been engineered through the use of stochastic fields. This has been detailed in the setup of a trapped-ion platform, for which we provided the control laser amplitude, relative phase and strength of dephasing to generate the dynamics. The general formalism is however not restricted to any particular platform, and could capture e.g. photonic thermal states squeezed by parametric downconversion in a lossy cavity [71]. The presented formalism as been kept as general as possible to be adaptable accordingly to the experimental platform of interest.

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A Control dissipator \( \mathcal{D}_c \) (15)

The ‘control’ dissipator is defined using Eqs. (11-14) as

\[
\mathcal{D}_c(\rho) = -\frac{i}{\hbar} \left[ \frac{\hbar}{4\omega} \left( 2\Omega_l^2 + \Omega_t^2 \right) (a_t + a_t^\dagger)^2 + i\hbar \frac{\Omega_t}{2} (a_t^2 - a_t^\dagger)^2, \rho \right] - \tilde{\epsilon}_t \rho_t \left( U_{\Omega} a_t^\dagger a_t U_{\Omega}^\dagger + \frac{1}{1 - e^{\tilde{\epsilon}_t}} \right) \\
= \frac{\Omega_l}{2} \left[ a_t^2 - a_t^\dagger a_t - i\frac{\Omega_t}{\omega_l} (a_t + a_t^\dagger)^2, \rho \right] - \tilde{\epsilon}_t \rho_t \left( U_{\Omega} a_t^\dagger a_t U_{\Omega}^\dagger + \frac{1}{1 - e^{\tilde{\epsilon}_t}} \right) 
\]

In order to find a compact form, it is useful to define the operator

\[
b_t \equiv U_{\Omega} a_t^\dagger U_{\Omega}^\dagger = a_t - i\frac{\Omega_t}{2\omega_l} (a_t + a_t^\dagger).
\]
First note that since \(a_t + a_t^\dagger = b_t + b_t^\dagger\), the position operator is equally represented in both operator basis, namely \(\hat{x} = \sqrt{\frac{\hbar}{2m\omega}} (a_t + a_t^\dagger) = \sqrt{\frac{\hbar}{2m\omega}} (b_t + b_t^\dagger)\). Then, noticing that \(b_t^2 - b_t^{12} = a_t^2 - a_t^{12} - i\frac{\Omega_t}{\omega_t} (a_t + a_t^\dagger)^2\), we can recast the control dissipator (S1) into the compact form given in Eq. (15) of the main text.

B Finding an expression for \(S_{r,\phi} \hat{S}_{r,\phi}^\dagger\)

To find an explicit expression of \(S_{r,\phi} \hat{S}_{r,\phi}^\dagger\) for any squeezing angle \(\phi_t\), it is useful to define the operator \(\hat{a}_t = e^{-i\phi_t/2} a_t\). It fulfills the bosonic commutator relation \([\hat{a}_t, \hat{a}_t^\dagger]\), and gives \(S_{r,\phi} = e^{\frac{\Omega_t}{2} (\hat{a}_t^2 - \hat{a}_t^{12})}\). We can expand the exponential in Taylor series and look for \(S_{r,\phi} \hat{S}_{r,\phi}^\dagger\) in terms of the \(k\)th derivatives of \(\hat{a}_t^2 - \hat{a}_t^{12}\). From the time derivative \(\hat{a}_t = -i\frac{\hbar}{2}\hat{a}_t + \frac{\hbar}{2\omega_t} e^{-i\phi_t} \hat{a}_t^\dagger\), it follows that \(\frac{d(\hat{a}_t^2 - \hat{a}_t^{12})}{dt} = -i\frac{\hbar}{2} \hat{a}_t \hat{a}_t^\dagger + \frac{\hbar}{2\omega_t} (e^{-i\phi_t} - e^{-i\phi_t}) (\hat{a}_t^\dagger \hat{a}_t + \hat{a}_t \hat{a}_t^\dagger)\). We use the fact that \(\hat{a}_t^\dagger \hat{a}_t + \hat{a}_t \hat{a}_t^\dagger = -\frac{1}{2} [\hat{a}_t^2 + \hat{a}_t^{12}, \hat{a}_t^2 - \hat{a}_t^{12}]\) and \(\hat{a}_t^2 + \hat{a}_t^{12} = \frac{1}{2} [\hat{a}_t^2 - \hat{a}_t^{12}, \hat{a}_t^\dagger \hat{a}_t]\), to obtain the expression for \(k = 1\) in the form of commutators, explicitly,

\[
\frac{d(\hat{a}_t^2 - \hat{a}_t^{12})}{dt} = i \frac{\hbar}{2} [\hat{a}_t^\dagger \hat{a}_t, \hat{a}_t^2 - \hat{a}_t^{12}] + \frac{\hbar}{2\omega_t} [\hat{a}_t^\dagger + \hat{a}_t, \hat{a}_t^2 - \hat{a}_t^{12}].
\] (S3)

This form allows to generalize the results and obtain

\[
\frac{d(\hat{a}_t^2 - \hat{a}_t^{12})^k}{dt} = i \left[ \frac{\hbar}{2} \frac{\hbar}{2\omega_t} \sin \phi_t (\hat{a}_t^2 + \hat{a}_t^{12}), (\hat{a}_t^2 - \hat{a}_t^{12})^k \right].
\] (S4)

Eventually, we get

\[
S_{r,\phi} \hat{S}_{r,\phi}^\dagger = S_{r,\phi} \sum_{k=0}^{\infty} \frac{1}{k!} \left( \frac{\hbar}{2} \frac{\hbar}{2\omega_t} \sin \phi_t (\hat{a}_t^2 + \hat{a}_t^{12}) \right)^k
\]

\[
= -\frac{\hat{r}_t}{2} (\hat{a}_t^2 - \hat{a}_t^{12}) + i S_{r,\phi} \left[ \frac{\hbar}{2} \frac{\hbar}{2\omega_t} \sin \phi_t (\hat{a}_t^2 + \hat{a}_t^{12}), S_{r,\phi}^\dagger \right]
\]

\[
= -\frac{\hat{r}_t}{2} (e^{-i\phi_t} \hat{a}_t^2 - e^{-i\phi_t} \hat{a}_t^{12}) + i \frac{\hbar}{2} (A_1^\dagger A_t - a_1^\dagger a_t) + i \frac{\hbar}{2\omega_t} \sin \phi_t \left( e^{-i\phi_t} (A_t^2 - a_t^2) + e^{i\phi_t} (A_t^2 - a_t^{12}) \right)
\] (S5)

The counter-diabatic Hamiltonian \(H_{CD} = S_{r,\phi} (H_t + H_1) S_{r,\phi}^\dagger + \frac{\hbar}{\i} S_{r,\phi} \hat{S}_{r,\phi}^\dagger\) follows as

\[
H_{CD} = \hbar \omega_t (A_1^\dagger A_t + \frac{1}{2}) + \hbar \frac{\hbar}{2} (A_1^\dagger A_t - \hat{a}_t^\dagger \hat{a}_t)
\]

\[
+ i \hbar \left( \frac{\hbar}{2\omega_t} (A_t^2 - A_t^{12}) + \frac{\hbar}{2} (\hat{a}_t^2 - \hat{a}_t^{12}) \right) + \hbar \frac{\hbar}{2\omega_t} \sin \phi_t \left( S_{r,\phi} (\hat{a}_t^2 + \hat{a}_t^{12}), S_{r,\phi}^\dagger - (\hat{a}_t^2 + \hat{a}_t^{12}) \right).
\] (S6)

Since,

\[
A_t^2 - A_t^{12} = a_t^2 - a_t^{12} + 2i \sin \phi_t \sinh^2(r_t) (e^{i\phi_t} a_t^{12} + e^{-i\phi_t} a_t^2) + i \sin \phi_t \sinh(2r_t) (a_t^\dagger a_t + a_t^\dagger a_t^\dagger),
\] (S7)

we note that

\[
S_{r,\phi} (\hat{a}_t^2 + \hat{a}_t^{12}) S_{r,\phi}^\dagger - (\hat{a}_t^2 + \hat{a}_t^{12}) = e^{-i\phi_t} (c_t^2 - a_t^2) + e^{i\phi_t} (c_t^{12} - a_t^{12})
\]

\[
= 2 \sinh^2 r_t (e^{i\phi_t} a_t^{12} + e^{-i\phi_t} a_t^2) + \sinh(2r_t) (a_t^\dagger a_t + a_t^\dagger a_t^\dagger)
\]

\[
= \frac{1}{i \sin(\phi_t)} \left( A_t^2 - A_t^{12} - a_t^2 + a_t^{12} \right),
\] (S8)

which allows simplifying (S6) into Eq. (20) given in the main text.
C Factorization of the squeezed thermal state

It is useful to write the squeezed thermal state in a product form in order to solve its dynamics. We show how to obtain
\[ e^{S_{\phi} + a^{\dagger}a} e^{i\phi} = e^{\lambda A^{\dagger}A} = Ke^{\frac{\lambda}{2}a^{\dagger}a} e^{\frac{\lambda}{2}a^{\dagger}a}. \] (S9)

The full analytical demonstration we propose here is alternative to the one provided in [72].

The factorized form of a function can be obtained following the use of differential equations, as first proposed by McCoy [66]. For any function \( g(a^{\dagger}, a) \) of the non-commuting operators \([a, a^{\dagger}] = c\), the partial derivative can be defined as [73]
\[ c \cdot \frac{\partial g}{\partial a^{\dagger}} = [a, g], \quad \text{and} \quad c \cdot \frac{\partial g}{\partial a} = -[a^{\dagger}, g]. \] (S10)

Note that \( c \) is a constant that will be taken equal to unity at the end, but is useful in the general derivation for the purpose of normalization. The expression of the function \( g \) in normal ordering (annihilation operators \( a \) to the right, creation \( a^{\dagger} \) to the left) is obtained by integration of a system of partial derivatives. We consider the particular function \( \rho = e^{\lambda A^{\dagger}A} = \sum_{n} \frac{\lambda^{n}}{n!} (A^{\dagger}A)^{n} \), which is quadratic in \( a \) and \( a^{\dagger} \) since, for \( A = f_+ a + f_- a^{\dagger} \), we have
\[ A^{\dagger}A = a^{\dagger}a + f_+ f_- a^{\dagger 2} + f_+ f_- a^{2} + c |f_-|^2 \mathbb{1}. \] (S11)

To obtain differential equations, we start from the obvious observation that \((A^{\dagger}A)^{n} A^{\dagger} = A^{\dagger}(AA^{\dagger})^{n}\), which gives
\[ e^{\lambda A^{\dagger}A} A^{\dagger} = A^{\dagger} e^{\lambda A^{\dagger}A} = e^{c^{\lambda} A^{\dagger}A} e^{\lambda A^{\dagger}A}. \] (S12)

Using the relations between the ‘\( A \)’ and ‘\( a \)’ operators (18a) this readily gives
\[ f_{+}^{*} [\rho, a^{\dagger}] + f_{-}^{*} [\rho, a] = (f_{+}^{*} a^{\dagger} + f_{+}^{*} a)(e^{c^{\lambda}} - 1)\rho. \] (S13)

With the future integration in mind, we write the \( a \rho \) term on the r.h.s as \( \rho a + \frac{\partial \rho}{\partial a^{\dagger}} \) and obtain the first differential equation
\[ cf_{+}^{*} \frac{\partial \rho}{\partial a} - cf_{-}^{*} e^{c^{\lambda}} \frac{\partial \rho}{\partial a^{\dagger}} = f_{+}^{*} (e^{c^{\lambda}} - 1) \rho a + f_{+}^{*} (e^{c^{\lambda}} - 1) a^{\dagger} \rho. \] (S14)

A similar equation can be obtained starting from the observation that \((A^{\dagger}A)^{n} = (AA^{\dagger})^{n} A\). This gives \([A, \rho] = (e^{c^{\lambda}} - 1) \rho A\) and yields to the differential equation
\[- e^{c^{\lambda}} f_{-} \frac{\partial \rho}{\partial a} + f_{+} \frac{\partial \rho}{\partial a^{\dagger}} = (e^{c^{\lambda}} - 1) f_{+} \rho a + f_{-} (e^{c^{\lambda}} - 1) a^{\dagger} \rho \] (S15)

where again, we have chosen to have terms on the r.h.s in the ordering \( \rho a \) and \( a^{\dagger} \rho \). So we now have the system of differential equation
\[ \begin{cases} \frac{\partial \rho}{\partial a} = X \rho a + (e^{Y} - 1) a^{\dagger} \rho \\ \frac{\partial \rho}{\partial a^{\dagger}} = (e^{Y} - 1) \rho a + X a^{\dagger} \rho \end{cases} \] (S16)

with the constants
\[ X = \frac{f_{-} f_{+}^{*} (e^{2c^{\lambda}} - 1)}{c |f_{+}|^{2} - |f_{-}|^{2} e^{2c^{\lambda}}} \] (S17)
\[ e^{Y} - 1 = \frac{(e^{c^{\lambda}} - 1)(|f_{+}|^{2} + |f_{-}|^{2} e^{c^{\lambda}})}{c |f_{+}|^{2} - |f_{-}|^{2} e^{2c^{\lambda}}} \equiv \gamma \] (S18)

It is now easy to verify that the factorized form (S9) is solution of the system (S16). So we obtain the following factorized form, in normal ordering
\[ e^{\lambda A^{\dagger}A} = e^{\lambda (a^{\dagger}a + f_{+}^{*} f_{-} a^{\dagger 2} + f_{+} f_{-} a^{2} + c |f_{-}|^{2} \mathbb{1})} = Ke^{\frac{\lambda}{2}a^{\dagger}a} e^{\ln(1+y) a^{\dagger}a} e^{\frac{\lambda}{2}a^{\dagger}a}. \] (S19)
For the squeezed thermal state, \( f_+ = \cosh(r_t) \) and \( f_- = \sinh(r_t) e^{i\phi} \), which allows to relate directly the squeezing parameters with the factorized form as given in \((35)\).

In order to compute the constant \( K \), we further follow the derivation proposed by McCoy [66]. During the factorization, only the commutation relation is important. So we choose to replace \( a \to \hat{x} \) and \( a \to \hat{c} \). We look at how \( \lambda A^d \) acts on \( \mathbb{1} \), and denote this action \( \lambda A^d \{\mathbb{1}\} \). With the change of operators, \( \lambda A^d = \lambda (f_+^2 + f_-^2) f_+ f_- \), so \( \lambda A^d \{\mathbb{1}\} = \lambda (|f_+|^2 c + f_+^* f_-^2) \). The constant term in \((\lambda A^d)^n\) can be found in applying \( n \) times the operator \((\lambda A^d)^n\) on the identity, and is of the form \( a_n c^n \). Let us denote the constant in the first term of the serie \( P_1(c) \) such that \( P_1(c) = \sum_{n=0}^{\infty} a_n c^n / n! \). We denote \( P_2(c) \) the constant when acting twice \((\lambda A^d)^2\), namely the constant term in \( e^{\frac{\lambda}{A^d}} \{\mathbb{1}\} \). This yields to \( P_2(c) = K(c) \lambda (|f_+|^2 + c^2 f_+^* f_-) \), which lead to \( \frac{\partial K(c)}{\partial c} = K(c) \lambda (|f_+|^2 + c^2 f_+^* f_-) \). At the limit for which the operators commute, \( K(c \to 0) \) tends to unity. Hence,

\[
K(c) = \exp \left( \lambda \left( |f_+|^2 c + f_+^* f_- \int_0^c \zeta X(\zeta) d\zeta \right) \right). \tag{S20}
\]

One can compute the integral

\[
\int_0^c \zeta f_+^* f_- (e^{2\lambda \zeta} - 1) |f_+|^2 - |f_-|^2 e^{2\lambda \zeta} \zeta d\zeta = f_+^* f_- \frac{1}{|f_+|^2 - |f_-|^2} \int_0^c \left( -1 + \frac{|f_+|^2}{|f_+|^2 - |f_-|^2} \right) \left( \frac{|f_+|^2 e^{-2\lambda} - |f_-|^2}{|f_+|^2 - |f_-|^2} \right) d\zeta
\]

Inserting in \((S20)\) and using the constant \( c = 1 \), we obtain the normalization constant in the factorized state \((35, S9, S19)\) we get

\[
K = \exp \left( \lambda \left( |f_+|^2 - f_+^* f_- \right) \left( \frac{|f_+|^2 e^{-2\lambda} - |f_-|^2}{|f_+|^2 - |f_-|^2} \right) \right)^{f_+^* f_-}. \tag{S22}
\]

### D Dynamics for an ion in a stochastically shaken trap and driven with two-photon Raman interaction

We are interested in the evolution of the state \(|\Psi_t\rangle\) characterized by the stochastic Hamiltonian

\[
H_{st} = -\frac{\hbar \Delta}{2} \sigma_z + U_{r,t} H_{\text{int}}(t) U_{r,t}^\dagger + \hbar \sqrt{2 \gamma t} \xi t\hat{x} U_{r,t}^\dagger \otimes |g\rangle \langle g|. \tag{S23}
\]

Let us first give the explicit form of the interaction Hamiltonian in the rotated frame. The atomic part evolves as \( e^{\frac{1}{2}(\omega + \Delta)t} \sigma_x e^{-\frac{1}{2}(\omega + \Delta)t} \sigma_x = e^{-i(\omega + \Delta)t} |g\rangle \langle e| + \text{h.c.} \). The bosonic part is obtained from \( e^{i \nu t a \dagger a} = a^\dagger e^{i \nu (a + a^\dagger - 1)} \) that gives \( e^{i \nu t a \dagger a} e^{\text{i} \nu (a^\dagger + a)} e^{-i \nu t a} = e^{\text{i} \nu t (a e^{i \nu t} + a^\dagger e^{-i \nu t})} \). Keeping only the terms with the lowest frequency (RWA), we thus have

\[
U_{r,t} H_{\text{int}}(t) U_{r,t}^\dagger = \frac{\hbar}{2} \sum_{l=1,2} \Omega_l(t) \left( e^{\frac{1}{2}(\omega + \Delta)t} \sigma_x e^{-\frac{1}{2}(\omega + \Delta)t} \sigma_x \right) \left( e^{i \nu t a \dagger a} e^{\text{i} \nu (a^\dagger + a)} e^{-i \nu t a} - i \Phi_t + \text{h.c.} \right)
\]

\[
\approx \frac{\hbar}{2} \sum_{l=1,2} \Omega_l(t) \left( e^{i (\omega_l - \omega - \Delta) t} e^{\text{i} \nu t} e^{-i \nu (a e^{i \nu t} + a^\dagger e^{-i \nu t})} |g\rangle \langle e| + \text{h.c.} \right). \tag{S24}
\]

We are looking for a solution of the wave function as a linear combination of the dressed basis

\[
|\Psi_t\rangle = \sum_n (e_n(t) |e, n\rangle + g_n(t) |g, n\rangle). \tag{S25}
\]
The evolution of this state over a small increment of time $dt$, given in Eq. (30), gives the excited and ground state populations evolving as

$$\dot{e}_n(t) = i \frac{\Delta}{2} e_n(t) - i \sum_{l=1,2} \Omega_l e^{i(\omega_l - \omega) t} e^{-i \Phi_l} \sum_{n'} \langle n | e^{i \eta_l (a_l^+ + a_l)} | n' \rangle g_{n'}(t),$$  
(S26a)

$$\dot{g}_n(t) = -i \frac{\Delta}{2} g_n(t) - i \sum_{n', l=1,2} \Omega_l e^{i(\omega_l - \omega - \Delta) t} e^{i \Phi_l} \langle n | e^{-i \eta_l (a_l^+ + a_l)} | n' \rangle e_{n'}(t) - \sum_{n'} \langle n | \left( i \sqrt{2 \gamma} \dot{x}_l \frac{dW_l}{dt} + \gamma_l \dot{x}_l^2 \right) | n' \rangle g_{n'}(t).$$  
(S26b)

For large detuning, $|\Delta| \gg |\Omega|, \nu$, a state initially in the electronic ground state mainly remains in this electronic level. The small population of the electronic excited state can be eliminated abscastically. We thus set $\dot{e}_n(t) = 0$, and the evolution follows as

$$i \hbar \frac{d|\Psi_i\rangle}{dt} = i \hbar \sum_n \dot{g}_n(t) |g, n\rangle$$

$$= \frac{\hbar}{2} \left( \Delta + \frac{\Omega_1^2 + \Omega_2^2}{\Delta} + \frac{\Omega_1 \Omega_2}{\Delta} (e^{i(\omega_1 - \omega) t} e^{i(\Phi_1 - \Phi_2)} e^{i(\eta_2 - \eta_1) (a_1^+ + a_1)} + \text{h.c.}) \right) |g\rangle \langle g |\Psi_i\rangle$$

$$- i \hbar \left( i \sqrt{2 \gamma} \dot{x}_l \frac{dW_l}{dt} + \gamma_l \dot{x}_l^2 \right) \sum_{n'} |g, n'\rangle \langle g, n'| \Psi_i\rangle,$$

$$= \tilde{H}_{\text{eff}} |\Psi_i\rangle - i \hbar \left( i \sqrt{2 \gamma} \dot{x}_l \frac{dW_l}{dt} + \gamma_l \dot{x}_l^2 \right) |g\rangle \langle g |\Psi_i\rangle,$$

where we have defined the effective Hamiltonian

$$\tilde{H}_{\text{eff}} = \frac{\hbar}{2} \left( \Delta + \frac{\Omega_1^2 + \Omega_2^2}{\Delta} \right) |g\rangle \langle g| + \hbar \frac{\Omega_1 \Omega_2}{2 \Delta} (e^{i(\omega_1 - \omega) t} e^{i(\Phi_1 - \Phi_2)} e^{i(\eta_2 - \eta_1) (a_1^+ + a_1)} + \text{h.c.}) |g\rangle \langle g|.$$  
(S27)

We then use Glauber formula to write $e^{i(\eta_2 - \eta_1) (a_1^+ + a_1)} = e^{i(\eta_2 - \eta_1) a_1^+} e^{i(\eta_2 - \eta_1) a_1} e^{-(\eta_2 - \eta_1)^2/2}$, and expand the exponentials in series to keep only the first resonant term. For $\omega_1 - \omega_2 = 2 \nu$, this leads, in leading order of $(\eta_2 - \eta_1)$, to

$$\tilde{H}_{\text{eff}} \approx \frac{\hbar}{2} \left( \Delta + \frac{\Omega_1^2 + \Omega_2^2}{\Delta} \right) |g\rangle \langle g| + \frac{\hbar}{4} (\eta_2 - \eta_1)^2 \frac{\Omega_1 \Omega_2}{\Delta} (e^{i(\Phi_1 - \Phi_2)} a_1^2 + \text{h.c.}) |g\rangle \langle g|.$$  
(S29)

This corresponds, up to the Lamb-shift term that we neglect, to the effective Hamiltonian given in Eq. (31) of the main text.

The evolution of the wave function (S27) yields to the master equation (S44a) for the noise-average density matrix. We solve this equation and find the control parameters for which the squeezed thermal state (34) is a solution. To do so, we look at $\frac{d\rho}{dt} = \rho^{-1}$ and use the factorized form of the squeezed thermal state that allows recasting all needed terms of the master equation (36) in the form $e^A B e^{-A}$ for all elements $\{A, B\} \in \mathcal{B} \equiv \{a_1^2, a_2^2, 1, a_1^+ a_1, a_1, a_2\}$ basis. We denote these terms with the adjoint operator $A$ of an operator $A$, defined by recurrence from $A^n B = [A, A^{n-1}]$ with $A^n B = [A, B]$ and $A^0 = 1$. The BCH formula then becomes

$$e^A B e^{-A} = [A, B] + \frac{1}{2!} [A, [A, B]] + \frac{1}{3!} [A, [A, [A, B]]] + \ldots$$

$$= \sum_{n=0}^{\infty} \frac{A^n}{n!} B = e^{A_n} B.$$  
(S30)

We then explicit the transformation for each element of the basis, e.g. $e^{A_j a^2} a^2 = a^2 + 4J^2 a^2 - 2J \mathbb{1} - 4Ja a^1$. The action of the adjoint is thus a linear transformation that can be represented in matrix form in
the basis $\mathcal{B}$, the needed terms being explicitly

$$
e^{A_{J^*a}^2} = \begin{pmatrix} 1 & 4J^2 & -2J^* & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -2J^* & 1 & 0 & 0 \\ 0 & -4J^* & 4J & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad e^{A_{Ja}^2} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2J & 0 & 0 \\ 2J & 0 & 1 & 0 & 0 \\ 4J & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

and

$$e^{A_{-Ba}^2}_{|\mathcal{B}} = \text{diag}(e^{-2B}, e^{2B}, 1, 1, e^{-B}, e^B). \quad (S31)$$

Equation (36) then follows, in matrix representation in the $\mathcal{B}$ basis, as

$$
d\rho/dt = \begin{pmatrix} \dot{J}^* + 2\dot{B}J^* + 4e^{2B}\dot{J}(J^*)^2 \\ e^{2B}\dot{J} \\ -B - 4e^{2B}\dot{J}J^* \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -i\alpha_R - \alpha_I \\ -i\alpha_R + \alpha_I \\ -2\kappa \\ 0 \end{pmatrix} \rho^{-1} + 2\kappa \alpha^\dagger \rho \alpha \rho^{-1} + \begin{pmatrix} (i\alpha_R + \alpha_I) \\ (i\alpha_R - \alpha_I) \\ 0 \\ 0 \end{pmatrix} \rho^{-1} + 2\kappa \alpha^\dagger \rho \alpha \rho^{-1}.
$$

Finally,

$$
d\rho/dt = \begin{pmatrix} (i\alpha_R (-1 + 4e^{2B}J^2 + e^{-2B}(1 - 4e^{2B}J^2)^2) + \alpha_I (-1 - 4e^{2B}J^2 + e^{-2B}(1 - 4e^{2B}J^2)^2)) + \kappa (-4e^B J^2 - 4J^2 (-1 + 4e^{2B}J^2)) \\ 2i\alpha_R (J^2 + e^{2B}(1 + 4J^2)^2) + \alpha_I (J^2 + e^{-2B}(1 + 4J^2)^2) + \kappa (2(-1 + e^{-B}) - 8e^{2B}J^2 + 8e^{2B}J^2) \\ 4i\alpha_R (J^2 - e^{2B}(1 + 4J^2)^2) + 4\alpha_I (J^2 + e^{-2B}(1 - 4J^2)^2) + \kappa (2(-1 + \cosh(B)) - 8e^B|J|^2 + 16e^{2B}|J|^2) \end{pmatrix}. \quad (S33)
$$

By linear combination of equations of the system (S33), we can identify the evolution parameters of the squeezed thermal state and obtain the coupled differential equations

$$
\dot{J}_t = -4e^{-B}(-1 + e^B)J_t \kappa + i(-e^{-2B} + (1 + 4J_t^2)) \alpha_R + (e^{-2B} + (1 + 4J_t^2)) \alpha_I, \quad (S34a)
\dot{B} = -4 \left( \kappa \left(-1 + \cosh(B) + 2e^B|J|^2\right) + i\alpha_R (J_t - J_t^*) + \alpha_I (J_t + J_t^*) \right), \quad (S34b)
$$

that provide the control parameters as function of the state characteristics.

### E Two-photon Raman interaction and stochastically driven Jaynes-Cummings Hamiltonian

We now consider to generate the dissipator through a stochastic laser field rather than through shaking of the trap. This leads to a Jaynes-Cummings (JC) Hamiltonian [57,58] in its stochastic form [74,75]. Note that the effect of dissipation in the JC model has been considered [76–78], mainly focusing on the influence over the populations.

The set-up is similar to the one presented Sec. 5, but with two additional beams used to engineer the dissipator (see Fig. 8 for an illustration). The interaction Hamiltonian resulting from the applied laser fields now reads [60]

$$H_{\text{int}}(t) = \frac{\hbar}{2} \sum_{l=0,\ldots,3} \Omega_l \sigma_x (e^{i(k_i z - \omega_l t - \Phi_l)} + \text{h.c.}), \quad (S35)$$
where the Rabi frequency $\Omega_0$ will be taken as stochastic $\Omega_0^s$. We aim at preparing a squeezed thermal state on the vibrational levels of the system with total Hamiltonian

$$h_{\text{tot}}(t) = H_\text{a} + H_\text{m} + H_{\text{int}}(t).$$  \hfill (S36)

starting from an initial vibrational state that is thermal. As discussed above, this will be done by reverse engineering of the master equation to allow both squeezing and thermalization. We proceed as before and look at the evolution of the rotated vector $|\Psi_t\rangle \equiv U_{r,t} |\Psi_t\rangle$. The unitary $U_{r,t} \equiv e^{\frac{\i}{\hbar} H_{r,t} t}$ is defined from the rotation Hamiltonian $H_r = H_\text{a} + H_\text{m} + \frac{\hbar}{2} \sigma_z$, the average detuning now being $\hat{\Delta} = \sum_{t=0}^{t=3} \delta_t / 4$. The rotated state evolves as $|\Psi_{t+dt}\rangle = e^{-\frac{\i}{\hbar} H_{\text{rot}} dt} |\Psi_t\rangle$ with $H_{\text{rot}} \equiv U_{r,t} H_{\text{tot}} U_{r,t}^\dagger + i \hbar U_{r,t} U_{r,t}^\dagger = -\frac{\hbar}{2} \sigma_z + U_{r,t} H_{\text{int}}(t) U_{r,t}^\dagger$. The interaction Hamiltonian in the rotated frame, after the RWA, reads

$$U_{r,t} H_{\text{int}}(t) U_{r,t}^\dagger = \frac{\hbar}{2} \sum_l \Omega_l(t) \left( e^{\frac{\i}{2}(\omega+\Delta)\sigma_z} e^{-\frac{\i}{2}(\omega+\Delta)\sigma_z} \right) \left( e^{i\nu l a^\dagger} e^{i\nu l (a^\dagger+a)} e^{-i\nu l a^\dagger} e^{-i(\Phi_l+\omega l t)} + \text{h.c.} \right).$$  \hfill (S37)

where we have defined $\hat{\Psi} t \equiv e^{-i(\omega-\Delta-\omega_l) t} e^{i\Phi_l} e^{-i\nu l (a^\dagger+a)}$.

![Stochastic field](image)

Figure 8: **Experimental setup:** 2-photon Raman interaction is generated by the (blue) laser pair with $\omega_2 - \omega_3 = 2\nu$, while dephasing is generated with the (red) laser pair, $\omega_1 - \omega_0 = \nu$, one amplitude being taken as stochastic.

The open dynamics is generated using a white noise on top of the ‘$0$’ laser’s amplitude, namely taking $\Omega^s_0 \rightarrow \sqrt{\Omega_0} \xi_t$. It is then convenient to split the total Hamiltonian into its deterministic and stochastic contributions $H_{\text{tot}} = H_{\text{det}} + \xi_t H_0$, defined as

$$H_{\text{det}} = -\frac{\hbar}{2} \hat{\Delta} \sigma_z + \frac{\hbar}{2} \sum_{t=1,2,3} \Omega_t(\hat{\Psi} t |g\rangle \langle e| + \hat{\Psi}^\dagger t |e\rangle \langle g|)$$  \hfill (S38a)

$$\xi_t H_0 = \xi_t \frac{\hbar}{2} \sqrt{\Omega_0} (\hat{\Psi} t |g\rangle \langle e| + \hat{\Psi}^\dagger t |e\rangle \langle g|).$$  \hfill (S38b)

We look for a solution of the wave function as $|\Psi_t\rangle = \sum_{n=0}^{\infty} (e_n(t) |e, n\rangle + g_n(t) |g, n\rangle)$. The evolution of this state over a small increment of time $dt$ reads $d|\Psi_t\rangle = -\frac{i}{\hbar} (H_{\text{det}} dt + H_0 dW_t) - \frac{1}{2\hbar^2} H^2_0 dt$. This yields the coefficients evolving as

$$\dot{e}_n(t) = \frac{i}{2} \omega e_n(t) - \frac{i}{2} \sum_{n'} \left( \sum_{l \neq 0} \Omega_l(n |\hat{\Psi}^\dagger l |n', \sqrt{\Omega_0} \xi_t |n| \hat{\Psi}^\dagger t |n\rangle \right) \right) g_{n'}(t) - \frac{1}{8} \Omega_0(n |\hat{\Psi}^\dagger l \hat{\Psi} t |n\rangle |e_n(t) \right)$$  \hfill (S39a)

$$\dot{g}_n(t) = -\frac{i}{2} \frac{\Delta}{2} g_n(t) - \frac{i}{2} \sum_{n'} \left( \sum_{l \neq 0} \Omega_l(n |\hat{\Psi}^\dagger l |n', \sqrt{\Omega_0} \xi_t |n| \hat{\Psi}^\dagger t |n\rangle \right) \right) e_{n'}(t) - \frac{1}{8} \Omega_0(n |\hat{\Psi}^\dagger l \hat{\Psi} t |n\rangle g_{n'}(t)$$  \hfill (S39b)
For large detuning, \(|\tilde{\Delta}| \gg |\Omega_l|, \nu\), a state initially in the electronic ground state mainly remains in this electronic level. The small population of the electronic excited state can be eliminated abiatically. We thus set \(\dot{\rho}_{e}(t) = 0\), and the evolution follows as (assuming \(\frac{\Omega_0}{\tilde{\Delta}} \ll 1\))

\[
\frac{i\hbar}{\hbar} \frac{d}{dt} |\Psi_t\rangle = \frac{\hbar}{2} \left( \tilde{\Delta} + \frac{\sqrt{\Omega_0}}{\tilde{\Delta}} + i \frac{\Omega_0}{4} + \sum_{l \neq 0} \frac{\Omega_l}{\tilde{\Delta}} \left( \sum_{l' \neq 0} \Omega_{l'} \hat{h}_{l'} + \Omega_0 \xi_l (\hat{h}_{l} \hat{h}_{l}^\dagger + \hat{h}_{l}^\dagger \hat{h}_{l}) \right) \right) |g\rangle \langle g| \Psi_t \rangle.
\]

(S40)

We then split the term \(e^{-i(\eta_l - \eta_{l'})|a_l^\dagger + a_l|}\) and expand the exponentials in series to keep only the first resonant term. Choosing \(\omega_2 - \omega_3 = 2\nu\), the first resonant term brings a quadratic contribution of the form \(a^2 e^{i(\Phi_{2} - \Phi_{3})}\); and \(\omega_1 - \omega_0 = \nu\) gives the slowest oscillating term as linear, \(ae^{i(\Phi_{1} - \Phi_{0})}\). Thus, the resonant contributions are between the pairs of lasers, and read, in leading order of \((\eta_l - \eta_{l'})\),

\[
\hat{h}_l \hat{h}_{l'}^\dagger = e^{i(\omega_l - \omega_{l'})t} e^{i(\Phi_{1} - \Phi_{0})} e^{-i(\eta_l - \eta_{l'})|a_l^\dagger + a_l|} = e^{i(\omega_l - \omega_{l'})t} e^{i(\Phi_{1} - \Phi_{0})} \sum_{j,j'} (-i)^{j+j'} \frac{\xi_l}{j!j'} a^j \hat{a}^j e^{i\nu t(j-j')} e^{-(\eta_{l'} - \eta_l)^2/2} \approx \delta_{l,l'} (\frac{-i}{2}) \frac{\eta_l}{2} (\eta_l + \eta_{l'}) (a^2 e^{i(\Phi_{2} - \Phi_{3})} + h.c.) - i\delta_{l,l'} \xi_l (\eta_l - \eta_{l'}) (ae^{i(\Phi_{1} - \Phi_{0})} - h.c.).
\]

(S41)

The evolution of the wave function then becomes

\[
\frac{i\hbar}{\hbar} \frac{d}{dt} |\Psi_t\rangle = \frac{\hbar}{2} \left( \tilde{\Delta} + \sum_{l \neq 0} \frac{\Omega_l^2 + \xi_l \Omega_l}{\tilde{\Delta}} - \frac{\Omega_0 \Omega_3}{2\tilde{\Delta}} (\eta_l - \eta_{l'}) \xi_l (\eta_l - \eta_{l'}) \right) |g\rangle \langle g| \Psi_t \rangle
\]

\[
- i\frac{\hbar}{2} \left( \frac{\Omega_0}{4} + \frac{\Omega_1 \sqrt{\Omega_0}}{\tilde{\Delta}} \xi_l (\eta_l - \eta_{l'}) (ae^{i(\Phi_{1} - \Phi_{0})} - h.c.) \right) |g\rangle \langle g| \Psi_t \rangle.
\]

(S42)

We can thus define an effective Hamiltonian \(H_{\text{se}} \equiv (\alpha_t a^2 + h.c.) |g\rangle \langle g|\), where \(\alpha_t = -\frac{\Omega_0 \Omega_3}{4\tilde{\Delta}} (\eta_l - \eta_{l'})^2 e^{i(\Phi_{2} - \Phi_{3})}\), and a dissipator \(D_a = \hbar \frac{\Omega_0 \sqrt{\Omega_0}}{2\tilde{\Delta}} \xi_l (\eta_l - \eta_{l'}) (ae^{i(\Phi_{1} - \Phi_{0})} - h.c.) |g\rangle \langle g|\) and obtain the compact expression (neglecting the Lamb shift)

\[
\frac{i\hbar}{\hbar} \frac{d}{dt} |\Psi_t\rangle = H_{\text{se}} |\Psi_t\rangle - i\left( \frac{\Omega_0}{8} |g\rangle \langle g| + \xi_l D_a \right) |\Psi_t\rangle.
\]

(S43)

Using the previously defined Leibnitz chain rule, we obtain the master equation for the noise-average density matrix

\[
\frac{d\rho_t}{dt} = -i \left[ H_{\text{se}}, \rho_t \right] - \hbar \frac{\Omega_0}{8} (|g\rangle \langle g|, \rho_t) + \frac{1}{4} D_a \rho_t D_a^\dagger,
\]

(S44a)

\[
= -i[\alpha_t a^2 + \alpha^* a^2, \rho_t] - \hbar \frac{\Omega_0}{8} (|g\rangle \langle g| \rho_t + \rho_t |g\rangle \langle g|) + \frac{\kappa_t}{4} (a + a^\dagger) \rho_t (a + a^\dagger).
\]

(S44b)

In the second line, we have applied the RWA, set \(\Phi_1 - \Phi_0 = \pi/2\), and defined \(\kappa_t = (\hbar \frac{\Omega_0 \sqrt{\Omega_0}}{2\tilde{\Delta}} (\eta_l - \eta_{l'})^2\) to express the dissipator as \(D_a = (i\sqrt{\kappa_t} |g\rangle \langle g| - h.c.)\).

We next solve the dynamics to find the dynamical control parameters \(\{\alpha_t, \kappa_t\}\) for which the squeezed thermal state \(|g\rangle \otimes \frac{\kappa_t}{\sqrt{2\pi}} e^{i\tau a^2} e^{-B a^\dagger a} e^{J a^2}\) is solution of (S44b). Proceeding similarly to the other setup, the master equation (S44b) is rewritten in the basis \(B\) and now reads

\[
\frac{d\rho_t}{dt} \rho_t^{-1} = \begin{pmatrix}
-i\alpha_R - \alpha_I & 0 \\
-i\alpha_R + \alpha_I & 0 \\
0 & 0 \\
0 & 0
\end{pmatrix} + e^{A_{j} a^2} e^{A_{-B a^\dagger a} e^{A_{j} a^2}} \begin{pmatrix}
(i\alpha_R + \alpha_I) \\
(i\alpha_R - \alpha_I) \\
0 & 0 \\
0 & 0
\end{pmatrix} + \frac{1}{4} \kappa_t (a + a^\dagger) e^{A_{j} a^2} e^{A_{-B a^\dagger a} e^{A_{j} a^2}} (a + a^\dagger) - \hbar \frac{\Omega_0}{4} \mathbb{1}.
\]

(S45)
By linear combination of the equations in the system (S45), the control parameters are found as solutions of

\[
\dot{J}_t = \frac{1}{4} e^{-B}(1 + 2J_t) \kappa_t + i \alpha_R (-e^{-2B} + 1 + 4J_t^2) + \alpha_I (e^{-2B} - 1 + 4J_t^2) \quad (S46a)
\]

\[
\dot{B} = -\frac{1}{4} (e^{-B} + e^{B}(1 + 2J_t)(1 + 2J_t^*)) \kappa_t - 4i \alpha_R (J_t - J_t^*) - 4 \alpha_I (J_t + J_t^*). \quad (S46b)
\]

So this dynamics creates the squeezed thermal state (34) provided that the control parameters fulfill

\[
\begin{pmatrix}
\kappa \\
\alpha_R \\
\alpha_I
\end{pmatrix} = M_t^{-1} \begin{pmatrix}
\dot{J}_R \\
\dot{J}_I \\
\dot{B}
\end{pmatrix},
\]

with the matrix now reading

\[
M_t = \begin{pmatrix}
-\frac{1}{4} e^{-B}(1+2J_t) & \frac{1}{4} e^{-B} J_t & -8J_t J_I \\
-\frac{1}{2} e^{-B}(J_R^2+J_I^2+J_R) & 4(J_R^2-J_I^2)+(1-e^{-2B}) & 4(J_R^2-J_I^2)+(e^{-2B}-1) \\
-(\frac{1}{2} \cosh B + e^B(J_R^2+J_I^2+J_R)) & 8J_I & -8J_R
\end{pmatrix}. \quad (S48)
\]

Figure 9 presents the control parameters for implementation of the dynamics for cooling, isothermal and heating processes. Interestingly, in the case of simple cooling and heating (with no squeezing), the squeezing hamiltonian is not zero anymore, which is different from the former setup (cf. Fig. 5). Adding squeezing (dashed curves) leads to similar results. In turn, the parameter controlling the dephasing, \(\kappa_t\), is positive for heating and a negative for cooling, which matches with intuition. The influence of temperature and squeezing variations on the maxima of control parameters are presented in Figures 10 and 11.

Figure 9: Control parameters: relative laser amplitude (top) and dephasing strength (bottom) for (a) cooling (\(\lambda_f = -2\)), (b) isothermal (\(\lambda_f = \lambda_i\)), and (c) heating (\(\lambda_f = -0.5\)) processes. The initial state is isotropic \(r_i = \phi_i = 0\) at \(\lambda_i = -1\). The final state is a thermal state with (plain lines) no squeezing \(r_f = \phi_f = 0\); (dash-dotted lines) squeezing at \(r_f = 1, \phi_f = 0\) or (dashed lines) squeezing at \(r_f = 1\) and angle \(\phi_f = \frac{\pi}{4}\).

Figure 10: Influence of temperature on the control maxima: Maximum (a) laser amplitude and (b) dephasing strength as function of changes in the temperature \(|\lambda_f| = \beta_f \hbar \nu\) for heating (orange background) and cooling (blue background) processes. Results are shown for states with constant squeezing amplitude, starting with \(|\lambda_i| = 1\) and \(\phi_i = \phi_f = 0\).
Figure 11: Maximum of the control parameters $\kappa_{\text{max}}$ and $|\alpha|_{\text{max}}$ as function of the initial or final squeezing parameter, for different variation of squeezing $\Delta r = r_f - r_i$. The two control parameters are symmetric and their amplitude is the same for a given variation in squeezing. In other words, a unique value of $|\kappa_{\text{max}}|$ is associated to a given couple of values $(r_i, r_f)$. We also note that all the points corresponding to a final $r_f = 0$ are higher than those to go back from a squeezed state to a less squeezed state. In other words, thermalization costs more than the only change of the squeezing parameter. We note that a high variation of the squeezing parameter is hard to engineer, however it doesn’t seem to be a problem since the variance is evolving exponentially with the squeezing parameter, for instance $\Delta r = 2$ corresponds to reduce the variance by seven times.

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