Free energy distribution of the stationary O’Connell–Yor directed random polymer model

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Abstract
We study the semi-discrete directed polymer model introduced by O’Connell–Yor in its stationary regime, based on our previous work on the stationary q-totally asymmetric simple exclusion process (q-TASEP) using a two-sided q-Whittaker process. We give a formula for the free energy distribution of the polymer model in terms of Fredholm determinant and show that the universal KPZ stationary distribution appears in the long time limit. We also consider the limit to the stationary KPZ equation and discuss the connections with previously found formulas.

Keywords: integrable probability, KPZ universality, directed random polymer, q-TASEP

(Some figures may appear in colour only in the online journal)

1. Introduction
The O’Connell–Yor (OY) polymer model introduced in [21] is a finite temperature directed polymer model in a Brownian motion environment. At zero temperature, this is related to the GUE random matrix [3, 13, 32] and can be studied by the techniques of random matrix theory. The finite temperature version is more difficult to treat, but still has nice algebraic properties. In particular the connection to the quantum Toda lattice was discovered in [20], which was further generalized to the Macdonald process [5]. A few other algebraic properties have been discussed in [7, 19].
The original OY model is defined for the case where the polymer starts and ends at specified positions (point-to-point geometry). One can also consider other geometries such as the point-to-line geometry. In this paper we consider the model in the stationary situation [21, 26].

In our previous paper [16] we studied the stationary $q$-TASEP. The $q$-TASEP, introduced in [5], is a version of the totally asymmetric simple exclusion process (TASEP), in which particles on $\mathbb{Z}$ hop to the right neighboring site under the exclusion condition, i.e. more than one particle cannot be on the same site. While the hopping rate of each particle is unity in the standard TASEP, it is generalized to $1 - q^{gap}$ in $q$-TASEP, where the ‘gap’ means the number of empty sites between the particle and the right neighboring particle. The special case in which $q = 0$ corresponds to the standard TASEP. In [16], we first showed that the $q$-TASEP with a random initial condition can be encoded as a marginal of a two-sided version of the $q$-Whittaker process. Then by rewriting the Cauchy identity for the ordinary $q$-Whittaker function and applying the Ramanujan’s summation formula and the Cauchy determinant identity for the theta function, we were able to find a Fredholm determinant formula for the $q$-Laplace transform for a position of the $N$th particle. We also showed that the limiting distribution is given by the Baik–Rains distribution $F_{BR}$ [2].

In this paper, we discuss the stationary OY polymer model by taking a scaling limit with $q \to 1$ of the $q$-TASEP and the two-sided $q$-Whittaker process. We first show that the OY polymer model with boundary sources appears as a marginal of a limiting case of the two-sided $q$-Whittaker process with two set of parameters. We can obtain the stationary OY model by modifying this model with an appropriate control of the sources as with the case of the two-sided $q$-Whittaker process [16]. The same OY model with parameters already appeared in [6] but the real stationary case was not covered there. We will give a formula for the free energy distribution for both the two parameter model and for the stationary case. We also show that the Baik–Rains distribution $F_{BR}$ appears in the long time limit. It is well known that $F_{BR}$ describes the universal height distribution in the stationary regime of models in the KPZ universality class [6, 17, 18, 22]. Although the distribution has not been clearly observed in an experiment, unlike the GUE/GOE Tracy–Widom distributions [29, 30], the crossover from the flat to the stationary regimes can be observed in an experiment using the turbulent liquid crystals [28], and some universal properties of this crossover have been discussed recently in [11, 14, 28].

The stationary OY model goes to the stationary KPZ equation under appropriate scalings of the model parameters. The latter was already studied in [6, 18] but we will discuss the relations among a few representations.

The paper is organized as follows. In section 2, we discuss some basic properties of the OY model especially focusing on the stationary situation. In section 3, we introduce the OY model with boundary sources and state its relation to the stationary OY model. In section 4, we show that the OY model with boundary sources appears as a limit of a marginal of the two-sided $q$-Whittaker process studied in [16]. In section 5, we present formulas for the distribution of the free energy for the stationary OY model in terms of Fredholm determinant. We also study the long time limit and show that the Baik–Rains distribution appears as the limiting distribution. In section 6, we take a scaling limit to the stationary KPZ equation and discuss the connections to previous representations [6, 18]. In appendix A, we summarize basic definitions and properties of the two-sided $q$-Whittaker function which are relevant in this paper. In appendix B, we discuss the inverse Laplace transform. Appendix C contains the details of the asymptotic analysis from section 6.
2. The stationary O’Connell–Yor polymer model

The partition function of the OY polymer model is defined by

\[ Z_j(\tau) = \int_{0 < s_0 < \cdots < s_{j-1} \leq \tau} ds_j \cdot e^{\sum_{k=1}^{j-1}(B_i(s_k) - B_i(s_{k-1}))}, \quad (2.1) \]

where \( s_0 = 0, s_j = \tau, j \in \mathbb{Z}_+ \) and \( B_i(\tau), i \in \mathbb{Z}_+ \) are the independent standard Brownian motions without drift [21]. (For \( j = 1, (2.1) \) should be understood to represent \( Z_i(\tau) = e^{R(\tau, \tau)} \).)

\( \tau \) is described by independent Brownian motions, which starts at the site \( \tau = j \) at \( \tau = 0 \) and ends at the site \( \tau \) at time \( \tau \) (point-to-point geometry). By using Itô’s formula, we find that it satisfies the discrete stochastic heat equations,

\[ dZ_j(\tau) = Z_{j-1}(\tau)d\tau + Z_j(\tau)dB_j(\tau), \quad (2.2) \]

where \( Z_0(\tau) = 0 \) and we interpret the second term as \( \text{Itô type. One can extend the values of the index } j \text{ to the whole } j \in \mathbb{Z} \text{ and consider the process for } r_j(\tau) := \log Z_{j+1}(\tau) - \log Z_j(\tau). \)

This process has a stationary measure labeled by a parameter \( \alpha \epsilon \mathbb{R} \) in which all \( r_j \)'s are independent random variables and each \( e^{-r_j} \) obeys the Gamma distribution with parameter \( \alpha(>0) \).

\[ \mathbb{P}[e^{-r_j} \in dx] = \frac{\alpha^{\alpha-1}e^{-x}}{\Gamma(\alpha)}dx, \quad (2.3) \]

see [27]. We sometimes write \( (2.3) \) as \( r_j \sim -\log \Gamma(\alpha). \)

Using a version of Burke’s theorem [8, 21, 26], one can replace the effects of the whole \( Z_j(\tau), j \leq 0 \) by \( Z_i(\tau) \) driven by the Brownian motion with drift \( \alpha \). This situation with the normalization condition \( Z_i(0) = 1 \) is described by the SDEs (2.2) with \( j \geq 0 \) and \( B_i(\tau) \) replaced by a standard Brownian motion with drift \( \alpha \). Let us denote the partition function as \( Z_j(\tau, \alpha) \) specifying the dependence on \( \alpha \). In [21, 26], it has been shown that it can be represented as

\[ Z_j(\tau, \alpha) = \int_{-\infty < s_0 < \cdots < s_{j-1} \leq \tau} e^{\sum_{k=1}^{j-1}(\tilde{B}_i(s_k) - \tilde{B}_i(s_{k-1}))} \prod_{k=1}^{j-1} ds_k, \quad (2.4) \]

where \( s_0 = 0, s_j = \tau, \) and \( \tilde{B}_i(s), j = 1, \cdots, N \) are independent two-sided Brownian motions among which \( \tilde{B}_i(s) \) has drift \( \alpha \) while \( \tilde{B}_i(s), j = 2, \cdots, N \) have no drifts. Here the two-sided Brownian motion \( \tilde{B}(x) \) with drift \( v \) is defined as

\[ \tilde{B}(x) = \begin{cases} B_+(x) + vx, & x \geq 0, \\ B_-(x) + vx, & x < 0 \end{cases} \quad (2.5) \]

with \( B_k(x) \) are the independent standard Brownian motions. Hereafter we call \( Z_j(\tau, \alpha) (2.4) \) the partition function of the stationary OY model.

Note that, by the conditioning on the smallest positive \( s_k \), the rhs of (2.4) with \( j = N \) is rewritten as

\[ Z_N(\tau, \alpha) = \sum_{k=1}^{N} \int_{-\infty < s_0 < \cdots < s_{k-1} \leq 0} e^{\sum_{k=1}^{N-1}(\tilde{B}_i(s_k) - \tilde{B}_i(s_{k-1}))} \prod_{j=1}^{k-1} ds_j \\
\times \int_{0 < s_k < \cdots < s_{N-1} \leq \tau} e^{\sum_{k=1}^{N-1}(\tilde{B}_i(s_k) - \tilde{B}_i(s_{k-1}))} \prod_{j=k}^{N-1} ds_j, \quad (2.6) \]
Since the first factor corresponds to the case \( \tau = 0 \) in (2.4), it is equal to \( e^{\sum_{i=1}^{n-1} y_i} \) in distribution where \( y_j, j = 1, 2, \ldots \) are i.i.d. random variables with \( y_j \sim -\log \Gamma(\alpha) \)'s while the second one is equal to the partition function \( Z_{N-k+1}(\tau) \) for the point-to-point polymer in (2.1). To summarize, we have seen that the partition function of the stationary OY polymer with parameter \( \alpha \) can be written as

\[
Z_N(\tau, \alpha) = \sum_{k=1}^{N} e^{\sum_{i=1}^{l} y_i} Z_{N-k+1}(\tau) \tag{2.7}
\]

in distribution, where the random variables \( y_j, j = 1, \ldots, N \) are independent and identically distributed as \( -\log \Gamma(\alpha) \).

### 3. The O’Connell–Yor polymer model with boundary sources

Here we introduce a directed random polymer model related to (2.7), which has a direct connection to the Whittaker process. This model is defined as a composition of the OY model with point-to-point geometry (with drifts) and the log-Gamma discrete random polymer model [25]. Let us consider a slight modification of (2.1), in which the polymer starts at site \( j = n \) and ends at \( j = N \) and the Brownian motions \( B_j(t), j = 1, 2, \ldots, N \) are the independent standard Brownian motions with drift \( \alpha j \in \mathbb{R} \) starting at the origin. The partition function of the OY model for this situation is given by

\[
Z_{n,N}^{\text{OY}}(\tau, a) = \int_{0 < t_1 < \ldots < t_{n-1} \leq \tau} \prod_{j=1}^{N-n} ds_j \cdot e^{\sum_{i=1}^{N-n}(B_{t_i}(s_{i+n-1}) - B_{t_i}(s_{i-1}))}, \tag{3.1}
\]

where \( s_0 = 0 \) (i.e. \( B_{t_i-1}(s_0) = 0 \)), \( s_{N-n+1} = \tau \) and \( a = (a_1, \ldots, a_N) \in \mathbb{R}^N \). Note that for the case \( a_i = \cdots = a_N = 0 \), \( Z_{n,N}^{\text{OY}}(\tau, 0) = Z_N(\tau) \) in (2.1).

To introduce the log-Gamma discrete random polymer model, let us consider the two dimensional lattice \((i, j), \ i = 1, \ldots, N, j = 1, \ldots, n\). Let the discrete up/right path from \((1, 1)\) to \((N, n)\) be an ordered set \((i_1, j_1), (i_2, j_2), \ldots, (i_{N-n+1}, j_{N-n+1})\) with \((i_1, j_1) = (1, 1)\) and \((i_{N-n+1}, j_{N-n+1}) = (N, n)\) such that \((i_k, j_k) \in \mathbb{Z}^2\) and \((i_{k+1} - i_k, j_{k+1} - j_k) \in \{(1, 0), (0, 1)\}\). The partition function of the log-Gamma polymer model is defined as

\[
Z_{n,N}^{\text{LGM}}(\alpha, a) = \sum_{(i_1, j_1), \ldots, (i_{N-n+1}, j_{N-n+1}) \in \Omega_{n,N}} e^{\sum_{i=1}^{N-n+1} \omega_{i,j}}, \tag{3.2}
\]

where \( \Omega_{n,N} \) represents a set of the discrete up/right paths from \((1, 1)\) to \((N, n)\) and \( \alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{R}^N, \ a = (a_1, \ldots, a_N) \in \mathbb{R}^N \) in the lhs are parameters such that \( \alpha_i - a_i > 0 \) for \( i, j = 1, \ldots, N \). \( \omega_{i,j}, \ i, j = 1, \ldots, N \) in the rhs are i.i.d. random variables with \( \omega_{i,j} \sim -\log \Gamma(\alpha_i - a_i) \).

In terms of the two polymers above, a semi-discrete polymer model is defined as follows.

**Definition 3.1 ([6]).** The partition function of the OY polymer model with boundary sources is defined as

\[
Z(\tau, \alpha, a) = \sum_{n=1}^{N} Z_{n,N}^{\text{LGM}}(\alpha, a) Z_{n,N}^{\text{OY}}(\tau, a). \tag{3.3}
\]

The first part \( Z_{n,N}^{\text{LGM}}(\alpha, a) \) can be regarded as representing boundary sources. See figure 1. When we set \( \alpha_1, \ldots, \alpha_N \to \infty \) and \( a_1 = \cdots = a_N = 0 \), we see from (2.3) that the whole weights \( e^{\omega_{i,j}} \)'s in the log-Gamma polymer model (3.2) vanish. Noting that in (3.2), the number
of lattice points, to which we assign the weights is $N + n - 1$, which increases with $n$, we find that in (3.3), the contribution of $n = 1$ becomes dominant. (In other words, in figure 1, the path crossing the bottom points $(1, 1), (2, 1), \ldots, (5, 1)$ in the left plane becomes dominant.) Thus in this limit $Z(\tau, \alpha, a)$ reduces to the OY model without sources, $Z_{OY}^{(N)}(\tau, 0)$.

This model is related to the stationary OY model (2.4) in the following way. To describe the stationary situation we need to specialize the parameters of the OY model with boundaries as

$$\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_N \to \infty, \quad a_1 = a, a_2 = \cdots = a_N = 0$$

(3.4)

and take the limit $a \to \alpha$. However note that in this limit, the model is not well-defined since $\omega_{1,1} \sim -\log \Gamma(\alpha - a)$ becomes singular in the limit. Thus we introduce the modified model which is defined in a same way as the original one (3.3) except that $\omega_{1,1} = 0$. We write the partition function of the modified model as $Z^{(0)}(\tau, \alpha, a)$. Note that it is related to $Z(\tau, \alpha, a)$ as

$$Z(\tau, \alpha, a) = e^{\omega_{1,1}} Z^{(0)}(\tau, \alpha, a).$$

(3.3)

Considering (2.7), we find under (3.4)

$$\lim_{{a \to \alpha}} Z^{(0)}(\tau, \alpha, a) = Z_0(\tau, \alpha)$$

(3.5)

in distribution. In this way we can study the stationary OY model by considering a limiting case of the OY model with boundary sources.

4. The O’Connell–Yor model with boundary sources and the Whittaker process

In [16], we studied the stationary $q$-TASEP using a two-sided version of the $q$-Whittaker process. In this section we will see that the $q$-Whittaker functions with signatures (see definitions 7.1 and 7.2) go to the Whittaker functions with two sets of parameters, which is previously shown to be related to the OY polymer with the boundary sources in [6]. This opens the way to study the stationary OY model by considering a limit of the analysis in [16] using the two-sided $q$-Whittaker process. In appendix A, we give a brief summary of the definitions and properties of the two-sided $q$-Whittaker functions and process, which are used in this paper.
The Whittaker process is defined as follows. Let $Y \in \mathbb{R}^{N(N+1)/2}$ be a triangular array $Y = (y^{(1)}, \ldots, y^{(N)})$ where $y^{(i)} = (y^{(1)}_i, \ldots, y^{(N)}_i)$ with $y^{(j)}_j \in \mathbb{R}$ for $1 \leq j \leq k \leq N$. The Whittaker function $\Psi_\nu(y^{(N)})$ with parameter $\nu = (\nu_1, \ldots, \nu_N) \in \mathbb{R}^N$ has the following integral representation [12],

$$
\Psi_\nu(y^{(N)}) = \int_{\mathbb{R}^{N(N-1)/2}} e^{\mathcal{F}_\nu(Y)} \prod_{1 \leq j \leq k \leq N} dy^{(k)}_j,
$$

(4.1)

where $\mathcal{F}_\nu(Y)$ is defined by

$$
\mathcal{F}_\nu(Y) = i \sum_{k=1}^N y^k \left( \sum_{j=1}^{k-1} y^{(k-1)}_j - \sum_{j=1}^{k-1} y^{(k-1)}_j \right) - \sum_{k=1}^{N-1} \sum_{j=1}^k \left( e^{y^{(k)}_j - y^{(k+1)}_j} + e^{y^{(k+1)}_j - y^{(k)}_j} \right).
$$

(4.2)

By definition, one sees that $\Psi_\nu(y^{(N)})$ is symmetric in $\nu = (\nu_1, \ldots, \nu_N)$.

We also define the function $\theta_{\mu, \tau}(y^{(N)})$ with parameters $\mu = (\mu_1, \ldots, \mu_N) \in \mathbb{R}^N$ and $\tau > 0$,

$$
\theta_{\mu, \tau}(y^{(N)}) = \int_{\mathbb{R}^N} \prod_{j=1}^N d\nu_j \cdot \Psi_\nu(y^{(N)}) e^{-\tau \sum_{j=1}^N \nu_j^2} \cdot \prod_{m,n=1}^N \Gamma(\mu_m + i\nu_n) \cdot m_N(\nu),
$$

(4.3)

where the Sklyanin measure $m_N(\nu)$ is defined by

$$
m_N(\nu) = \frac{1}{(2\pi)^N} \prod_{i \neq j} \frac{1}{\Gamma(i\nu_i - i\nu_j)}.
$$

(4.4)

As will be shown below in the proof of proposition 4.3, the functions $\Psi_\nu(y^{(N)})$ (4.1) and $\theta_{\mu, \tau}(y^{(N)})$ (4.3) can be regarded as the $q \to 1$ scaling limit of $P_\lambda(a)$ (A.5) and $Q_\lambda(\alpha, t)$ (A.6) respectively (see (4.12) and (4.13) below.)

The Whittaker process with parameters $a, \alpha \in \mathbb{R}^N$ and $\tau > 0$ is defined in terms of (4.1) and (4.3) as follows [6]:

**Definition 4.1.** For $a, \alpha \in \mathbb{R}^N$ such that $a_i + a_j > 0$ for $1 \leq i, j \leq N$ and $\tau > 0$, the Whittaker process is defined as a probability measure on $\mathbb{R}^{N(N+1)/2}$ with the pdf

$$
W_{a, \alpha, \tau}(Y) = e^{\mathcal{F}_a(Y)} \theta_{\alpha, \tau}(y^{(N)}) e^{-\tau \sum_{j=1}^N \nu_j^2} \prod_{m,n=1}^N \frac{1}{\Gamma(\alpha_m + a_n)},
$$

(4.5)

In the limit as $\alpha_j \to \infty$, $j = 1, \ldots, N$, the density function (4.5) reduces to the one in [5, 20],

$$
e^{\mathcal{F}_a(Y)} \int_{\mathbb{R}^N} \prod_{j=1}^N d\nu_j \cdot \Psi_\nu(y^{(N)}) e^{-\tau \sum_{j=1}^N \nu_j^2} m_N(\nu) \cdot e^{-\tau \sum_{j=1}^N \nu_j^2}.
$$

(4.6)

Furthermore from (4.1) and (4.5), we immediately have the following

**Proposition 4.2.** The pdf of the marginal density of $W_{a, \alpha, \tau}(Y)$ (4.5) on $y^{(N)}$ is expressed as

$$
\Psi_a(y^{(N)}) \theta_{\alpha, \tau}(y^{(N)}) e^{-\tau \sum_{j=1}^N \nu_j^2} \prod_{m,n=1}^N \frac{1}{\Gamma(\alpha_m + a_n)}.
$$

(4.7)

We call (4.2) the Whittaker measure.
We will show that the Whittaker process $W_{-\nu,\tau}(Y)$ appears as a limit of our two-sided $q$-Whittaker process (A.9) and one can study the OY model with boundary sources (3.3) by considering the limit of our results for the $q$-TASEP in [16]. In this section, we rewrite the parameters $a_j$ and $\alpha_j$, $j = 1, \cdots, N$ in (A.9) as $\tilde{a}_j$ and $\tilde{\alpha}_j$ to distinguish them from $a_j$, $\alpha_j$ in $W_{-\nu,\tau}(Y)$. We scale each variable and parameter of (A.9) as

$$ q = e^{-\epsilon}, \ t = \tau e^{-2}, \ \tilde{a}_j = e^{-\tilde{\epsilon}a_j}, \ \tilde{\alpha}_j = e^{-\tilde{\epsilon}\alpha_j}, $$

(4.8)

for $j = 1, 2, \cdots, N$ and taking the limit $\epsilon \to 0$. Here we assumed $0 < \tilde{\epsilon} < \infty$, $1 \leq j \leq N$. When we study the case where only $M$ of them are finite and $\alpha_j \to \infty, N - M + 1 \leq j \leq N$, we should set $\tilde{\alpha}_j = 0, N - M + 1 \leq j \leq N$ and replace $-N\epsilon^{-1}\log\epsilon^{-1}$ by $-M\epsilon^{-1}\log\epsilon^{-1}$. This is an aspect which is different from the previously studied scaling limit from q-Whittaker processes to the Whittaker process [5, 6]. There the term $-M\epsilon^{-1}\log\epsilon^{-1}$ is replaced by $+M\epsilon^{-1}\log\epsilon^{-1}$. The minus sign of $-M\epsilon^{-1}\log\epsilon^{-1}$ results from the ‘two-sided’ nature of (A.9): our process is defined on the signature $s_\nu$ (A.1) of which each element can take negative value. Due to this property, the scaling changes to the minus direction.

We obtain the following

**Proposition 4.3.** Under the scaling (4.8), the $\epsilon \to 0$ limit of $P_{\nu}(\tilde{\lambda}_N)$ (A.9) becomes

$$ \lim_{\epsilon \to 0} e^{\epsilon N(a_i-1)} P_{\nu}(\tilde{\lambda}_N) = e^{\epsilon N(a_i-1)} P_{\nu}(\tilde{\lambda}_N) $$

(4.9)

where $-Y = (-y(1), \cdots, -y(N))$ with $-y(j) = (-y(j), \cdots, -y(1))$, $j = 1, \cdots, N$.

**Proof.** For comparing (4.1) with (A.5), we write $e^{F_e(Y)}$ in (4.1) as

$$ e^{F_e(Y)} = \prod_{j=1}^{N} \Psi_{\nu}(y(j-1), y(j)), $$

(10.1)

where

$$ \Psi_{\nu}(y(j-1), y(j)) = \exp \left( \beta_j \left( \sum_{i=1}^{j} y_i(y(j) - y(i)) - \sum_{i=1}^{j-1} \left( e^{y(j-1)-y(i)} + e^{y(j-1)-y(i)} \right) \right) \right). $$

(10.11)

We will show that the skew $q$-Whittaker function $P_{\nu}(\tilde{\lambda}_N)(\tilde{a}_j)$ with some factors goes to this function (10.11), i.e.

$$ \lim_{\epsilon \to 0} e^{\epsilon N(a_i-1)} C(\nu) e^{\epsilon N(a_i-1)} P_{\nu}(\tilde{\lambda}_N)(\tilde{a}_j) = \Psi_{\nu}(y(j-1), y(j)) = \Psi_{\nu}(y(j-1), y(j)) $$

(10.12)

where $C(\nu) = -\frac{\pi^2}{6} e^{-1} - \frac{1}{2} \log \frac{e}{2\pi}$. Furthermore we will also show that

$$ \lim_{\epsilon \to 0} e^{\epsilon N(a_i-1)} C(\nu) e^{\epsilon N(a_i-1)} \prod_{j=1}^{N} \tilde{a}_j \log \epsilon \cdot Q_{\nu}(\tilde{\lambda}_N)(\tilde{a}_j, t) = \theta_{\nu}(\tilde{\lambda}_N)(\tilde{a}_j, t) $$

(10.13)

$$ \lim_{\epsilon \to 0} e^{\epsilon N(a_i-1)} C(\nu) e^{\epsilon N(a_i-1)} \prod_{j=1}^{N} \tilde{a}_j \log \epsilon \cdot \prod_{k=1}^{N} \tilde{a}_k \log \epsilon \cdot \prod_{k=1}^{N} \Gamma(\alpha_k - a_j). $$

(10.14)

Then (4.9) immediately follows from (10.12)–(10.14).
Hereafter we give proofs of (4.12)–(4.14). Limiting behaviors of various factors can be taken from [5].

**Proof of (4.12).** Here we show the first equality since the second equality follows immediately by definitions of $\Psi_{\nu}(y(t))$ (4.1) and $-y(t)$ written below (4.9). Substituting (4.8) into (A.4) with $\lambda = \lambda(t)$, $\mu = \lambda(t-1)$, and $a = \tilde{a}_j$, we have

\[ P_{\lambda(j)/\lambda(j-1)}(\tilde{a}_j) = a_j^{2N-1} \prod_{i=1}^{j-1} \left( \frac{q; q}{q - e^{\epsilon} (y_{i-1} - y_i)} \right)^{\epsilon - \log e} \times \exp \left( -a_j \left( \sum_{i=1}^{j} y_i^{(j)} - \sum_{i=1}^{j-1} y_i^{(j-1)} \right) \right). \]  \hfill (4.15)

Here we see, for $c > 0$ and $y \in \mathbb{R}$

\[ (q; q)_{-c}^{-\epsilon \log e - c - y} = e^{c(\epsilon)+c^{-1}e^{-y}} \]  \hfill (4.16)

by corollary 4.10 in [5]. Applying this to the second factor in (4.15), we obtain (4.12). □

**Proofs of (4.13) and (4.14).** For showing (4.13), we consider the limiting behavior of each factor in the definition of $Q_{\lambda(t)}(\tilde{\alpha}; t)$ (A.6). Hereafter we change integration variables $z_j$ in (A.6) to $z_j = e^{i\omega}$, $j = 1, \ldots, N$. First, one sees that under the scaling (4.8),

\[ \lim_{\epsilon \to 0} \prod_{j=1}^{N-1} (q_j; q_{j+1}^{-}\lambda(t); q) = 1 \]  \hfill (4.17)

from lemma 4.25 in [5]. Next for $P_{\lambda(t)}(1/z)$ we use (4.12) and have

\[ \lim_{\epsilon \to 0} e^{2N(\epsilon)-2N(\epsilon)\epsilon} \prod_{j=1}^{N} z_j^{-2N+1} P_{\lambda(t)}(1/z) = \Psi_{-\nu}(y(t)) = \Psi_{\nu}(-y(t)). \]  \hfill (4.18)

For $\Pi(z; \tilde{\alpha}, t)$ (A.8), we have

\[ \Pi(z; \tilde{\alpha}, t) = \prod_{i,j=1}^{N} \left( e^{-\epsilon(\alpha_i+\omega_j)}; e^{-\epsilon} \right)^{-1} \prod_{j=1}^{N} e^{i\epsilon\omega_j \epsilon^{-2} \tau_j}. \]  \hfill (4.19)

Using the relations

\[ \lim_{\epsilon \to 0} e^{\epsilon(x)} e^{x} = \Gamma(x), \quad \lim_{\epsilon \to 0} e^{i\omega \epsilon^{-2} \tau - \epsilon^{-1} \omega \epsilon \tau} = -\tau \omega^2/2, \]  \hfill (4.20)

where the first one is given in (4.55) in [5], we have

\[ \lim_{\epsilon \to 0} e^{N\epsilon(x)} e^{N\epsilon^2} \prod_{i,j=1}^{N} \left( \frac{\tilde{a}_i}{\tilde{a}_j} \right)^{\epsilon^{-1} \log e} \prod_{j=1}^{N} \frac{1}{\tilde{a}_j} \Pi(z; \tilde{\alpha}, t) = \prod_{i,j=1}^{N} \Gamma(\alpha_i + \omega_j) \prod_{j=1}^{N} e^{-\tau \omega_j^2/2}. \]  \hfill (4.21)

At last for $m(z) \prod_{\nu=1}^{N} d\nu_{\nu}/z_{\nu}$, we find
\[
\lim_{\epsilon \to 0} e^{-N^2} e^{-N(N-1)\mathcal{L}(v)} m_N^a(z) \prod_{j=1}^N \frac{dz_j}{z_j} = m_N(w) \prod_{j=1}^N dw_j
\]  

(4.22)

by using (4.36) in [5]. Combining (A.8) with the scaling limits (4.17), (4.18), (4.20), and (4.22), we arrive at (4.13). Then (4.14) can be obtained by (4.21) with \(z\) and \(iw_j\) replaced by \(1/\alpha\) and \(-a_j\), respectively.

Thus we have shown that the Whittaker process with two parameters appears as a limit of our two-sided \(q\)-Whittaker process. In addition the relationships between (3.3) and (4.5) is also known:

**Proposition 4.4.** \( \log Z(\tau, \alpha, a) \) has the same distribution as \( Y_1^{(N)} \) in the Whittaker process \( W_{-\alpha, \alpha, \tau}(Y) \) (4.5).

This relation was obtained by introducing a version of \(q\)-Whittaker process, which is different from ours (A.9) and by taking \(q \to 1\) scaling limit [6].

From propositions 4.3 and 4.4, we find a relation between \( \lambda_N^{(N)} \) in (A.11) and the OY polymer with the boundary sources \( Z(\tau, \alpha, a) \) (3.3). From the definition of \(-Y\) (see below (4.9)), one sees that the marginal density of \( W_{-\alpha, \alpha, \tau}(Y) \) on \( Y_1^{(N)} \) is equal to that of \( W_{-\alpha, \alpha, \tau}(-Y) \) on \(-Y_1^{(N)}\). Combining this with proposition 4.3, we conclude that the marginal density function of \( W_{-\alpha, \alpha, \tau}(Y) \) on \(-Y_1^{(N)}\) is given by the scaling limit (4.8) of the marginal density of \( P_{\lambda}(\lambda_N) \) on \( \lambda_N^{(N)} \), which describes the CM particle of the \(q\)-TASEP. Furthermore combining this with the proposition 4.4 we see that the marginal density of \( P_{\lambda}(\lambda_N) \) on \( \lambda_N^{(N)} \) (or equivalently the marginal density of the two-sided \(q\)-Whittaker measure (A.10) on \( \lambda_N \)) goes to the density function of \(-\log Z(\tau, \alpha, a)\) under the scaling limit (4.8) with \(\epsilon \to 0\). Thus in this limit, theorem A.5 becomes a relation on \( Z(\tau, \alpha, a) \). We have the following:

**Proposition 4.5.** The Laplace transform of \( Z(\tau, \alpha, a) \) (3.3) is written as the Fredholm determinant,

\[
\left\langle e^{-uZ(\tau, \alpha, a)} \right\rangle = \det \left(1 - f_u K \right)_{L^2(\mathbb{R})}.
\]

(4.23)

Here in the lhs, \( u \in \mathbb{C} \) with \( \text{Re } u > 0 \), \( \left\langle \cdot \right\rangle \) represents the average over the random variables \( \omega_{ij}, 1 \leq i, j \leq N \) and \( B_i, i = 1, \cdots, N \) in (3.3) and on the rhs the kernel \( f_u K \) is given by

\[
f_u(x) = \frac{1}{1 + e^{-\tau u}}.
\]

(4.24)

\[
K(x_1, x_2) = \sum_{j=0}^{N-1} \phi_j(x_1) \psi_j(x_2),
\]

(4.25)

where the functions \( \Phi_j(x) \) and \( \Psi_j(x) \) are given as

\[
\phi_j(x) = \frac{1}{2\pi i} \int \frac{e^{px - \frac{v^2}{2} \tau^2}}{v - \alpha_j} \prod_{i=1}^J \frac{v - \alpha_i}{v - a_i} \prod_{i=1}^N \frac{\Gamma(1 + v - a_j)}{\Gamma(1 + \alpha_j - v)}
\]

(4.26)

\[
\psi_j(x) = \frac{\alpha_j - a_j}{2\pi} \int_{-\infty}^\infty \frac{e^{-ixv \cdot \tau^2}}{\alpha_j - iv} \prod_{k=1}^J \frac{iw - \alpha_k}{iw - a_k} \prod_{i=1}^N \frac{\Gamma(1 + \alpha_j - iw)}{\Gamma(1 + i + a_j - a_j)},
\]

(4.27)

where in (4.26), the contour encloses \( a_j, j = 1, \cdots, N \) positively.
In the proof below, we take the $q \to 1$ scaling limit of (A.11). As with $\tilde{a}_i$ and $\tilde{a}_j$ in (4.8), we rewrite $\phi_i(n)$ (A.14) and $\psi_i(n)$ (A.15) as $\tilde{\phi}_i(n)$ and $\tilde{\psi}_i(n)$ to distinguish them from (4.26) and (4.27) respectively.

Proof. We consider the $\epsilon \to 0$ limit of (A.11) under the scaling (4.8) and

$$\zeta = -e^{2N\epsilon \epsilon^{-1}}. \tag{4.28}$$

First substituting (4.8) and (4.28) into the lhs of (A.11) we have

$$\lim_{\epsilon \to 0} \frac{1}{\zeta q^{\infty}} = \lim_{\epsilon \to 0} e_q(x_q) |_{q=e^{-\epsilon}} = e^{-e^{-\epsilon}^{(N)}}, \tag{4.29}$$

where $e_q(x_q) = 1 / ((1 - q) x_q; q)_\infty$ is the $q$-exponential function with $x_q = -e^{-\epsilon}^{(N)} / (1 - q)$ and we used the fact $\lim_{q \to 1} e_q(x) = e^x$ uniformly on $x \in (-\infty, 0)$. Thus from the remark below proposition 4.4, we have

$$\lim_{\epsilon \to 0} \left\langle \frac{1}{\zeta q^{\infty}} \right\rangle = e^{-e^{-\epsilon}} \tag{4.30}$$

under the scalings (4.8) and (4.28).

Next we consider the rhs of (A.11). We begin with the function $f(n)$ (A.12). Associated with (4.28), we scale $n$ as

$$n = -\tau e^{-2} - 2N \epsilon^{-1} \log \epsilon + x \epsilon^{-1}. \tag{4.31}$$

Substituting $q = e^{-\epsilon}$, (4.28) and (4.31) into (A.12), one immediately sees $\lim_{\epsilon \to 0} f(n) = f_0(x)$.

Next we show that under (4.8) and (4.31)

$$\lim_{\epsilon \to 0} e^{\tau \epsilon^{-2}} \prod_{j=1}^N e^{\epsilon_q + \epsilon_j} \cdot \tilde{\phi}_i(n) = \phi_i(x), \quad \lim_{\epsilon \to 0} e^{-\epsilon \epsilon^{-2}} \prod_{j=1}^N \frac{1}{e^{\epsilon_q + \epsilon_j}} \cdot \tilde{\psi}_i(n) = \psi_i(x) \tag{4.32}$$

by simple saddle point analyses. Here we consider only the case of $\tilde{\phi}_i(n)$ since that of $\tilde{\psi}_i(n)$ can be obtained in a similar way. Substituting the scalings (4.8) and (4.31) into the definition of $\tilde{\phi}_i(n)$ (A.14), we have

$$\tilde{\phi}_i(n) = \int_D \frac{dv}{2\pi i} e^{-\frac{\tilde{v}}{\epsilon} \tilde{g}(\tilde{v})} \frac{1}{v - q^{\alpha_j}} \prod_{j=1}^N \frac{v - q^{\alpha_j}}{v - q^{\alpha_j}} \prod_{j=1}^N \Gamma_q(1 - \alpha_j + \log \tilde{w}) \Gamma_q(1 - \alpha_j - \log \tilde{w}) (1 - q)^{2v - \alpha_j - a_j}, \tag{4.33}$$

where $g(\tilde{v}) = \tilde{v} - \log \tilde{v}$ and we used the $q$-Gamma function $\Gamma_q(x) = (1 - q)^{1 - \frac{1}{q}(q^\tilde{w})}\Gamma_q(1 - q^\tilde{w})$. Noting the saddle point $v_c$ such that $g'(v_c) = 0$ is 1, we scale $v$ around the saddle point,

$$v = q^w = e^{-cw}. \tag{4.34}$$

Thus we find
\[ \lim_{\epsilon \to 0} \frac{-\tau}{\epsilon^2} (g(\epsilon) - 1) = \lim_{\epsilon \to 0} \frac{-\tau g''(1)}{2\epsilon^2} (\epsilon - 1)^2 + O(\epsilon) = -\frac{\tau w^2}{2}, \quad (4.35) \]

\[ \lim_{\epsilon \to 0} \frac{1}{\epsilon^2 - 4\epsilon} = e^{w}, \quad (4.36) \]

\[ \lim_{\epsilon \to 0} \frac{1}{\epsilon^2 - q^{\alpha}} = \frac{1}{\epsilon^2 - q^\alpha}, \quad (4.37) \]

\[ \lim_{\epsilon \to 0} \frac{1}{\epsilon^2 - 2\epsilon} \log \epsilon + N \prod_{j=1}^N (1 - q) = 1. \quad (4.38) \]

Furthermore noting \( \lim_{\epsilon \to 1} \Gamma_q(x) = \Gamma(x) \) and \( \log_q \epsilon = w \), we have

\[ \lim_{\epsilon \to 0} \prod_{j=1}^N \Gamma_q(1 - a_j + \log_q \epsilon) = \prod_{j=1}^N \Gamma(1 - a_j) \quad (4.39) \]

From (4.35)–(4.39), we arrive at the first relation in (4.32).

The relation (4.23) is a generalization of proposition 12 in [19] to the case with two sets of boundary parameters \( \alpha = (a_1, \ldots, a_N), \ a = (a_1, \ldots, a_N) \). In the limiting case \( \alpha_1, \ldots, \alpha_N \to \infty, \ a_1 = \cdots = a_N = 0 \), one finds that (4.25) reduces to \( K(x; y; \tau) \) (4.10) in [19].

5. Distributions in the stationary O’Connell–Yor model

By proposition 4.4, we have found that the law of \( \log Z(\tau, \alpha, a) \) is the same as the marginal low on \( y^N \) of the \( q \)-Whittaker process (4.5) with \( a \) replaced by \( -a \) or equivalently of the \( q \)-Whittaker measure (4.7) with \( a \) replaced by \( -a \). Note that (4.7) is symmetric in \( a = (a_1, \ldots, a_N) \) and \( \alpha = (\alpha_1, \ldots, \alpha_N) \). Due to the symmetry the specialization (3.4) is equivalent to

\[ \alpha_1, \ldots, \alpha_{N-1} \to \infty, \ \alpha_N = \alpha, \ a_1 = \cdots = a_{N-1} = 0, \ a_N = a. \quad (5.1) \]

Hereafter we adopt (5.1) and write \( Z(\tau, \alpha, a) \) (3.3) and \( Z^{(0)}(\tau, \alpha, a) \) defined below (3.4) under (5.1) as \( Z(\tau, \alpha, a) \) and \( Z^{(0)}(\tau, \alpha, a) \) respectively.

We define

\[ F(y) = \mathbb{P}[\log Z(\tau, \alpha, a) \leq y], \ F_0(y) = \mathbb{P}[\log Z^{(0)}(\tau, \alpha, a) \leq y], \quad (5.2) \]

\[ G(u) = \langle e^{-uZ(\tau, \alpha, a)} \rangle, \ G_0(u) = \langle e^{-uZ^{(0)}(\tau, \alpha, a)} \rangle. \quad (5.3) \]

Note that the averages on the rhs in (5.3) are different for \( G(u) \) and \( G_0(u) \) and they are with respect to the unmodified and the modified model respectively. To consider the stationary limit (5.1) with \( a \to \alpha \), we need to have the relations which connect \( Z(\tau, \alpha, a) \) and the modified one \( Z^{(0)}(\tau, \alpha, a) \). We use the results from appendix B.2. For the OY model, the random variable \( \chi \) is distributed according to \( -\log \Gamma(\nu) \) with parameter \( \nu = \alpha - a \). (For the definition of \( -\log \Gamma(\nu) \), see (2.3).) Its Laplace (or Fourier for \( \xi \in \mathbb{R} \)) transform is

\[ g(\xi) = \langle e^{-\xi\nu} \rangle = \int_{\mathbb{R}} \frac{\xi e^{-\xi\nu}}{\Gamma(\nu)} \ dx = \frac{\Gamma(\nu + \xi)}{\Gamma(\nu)}. \quad (5.4) \]
One can find an expression for $F_0(y)$ (5.2) in terms of $G(u)$ (5.3).

**Proposition 5.1.** Let $Z(\tau, \alpha, a)$ (resp. $Z^{(0)}(\tau, \alpha, a)$) be the partition function of the OY polymer model (3.3) when the parameters are given by (3.4), $\alpha > a$ (resp. and $\omega_1$ in (3.2) is set to be zero). The distribution function $F_0(y)$ for $\log Z^{(0)}(5.2)$ is recovered from $G(u)$ (5.3) by the following formula.

$$F_0(y) = \int_{iR} \frac{d\xi}{2\pi i} \frac{\Gamma(\alpha - a)e^{\xi y}}{\Gamma(\alpha - a + \xi)\Gamma(1 + \xi)} \int_0^\infty u^{\xi-1}G(u)du. \quad (5.5)$$

**Remark.** A similar formula was obtained in [6] for the stationary KPZ equation using a property of the 0th Bessel function which appears for this special case. Here we show that the formula is a consequence of a combination of a few basic facts.

**Proof.** Let us set the distribution function $F(y)$, $y \in \mathbb{R}$ in the argument below (B.9) to be the one in (5.2). (In this case $\varphi(x)$ in appendix B becomes $\varphi(x) := F(\log x) = \mathbb{P}(Z(\tau, \alpha, a) \leq x), x > 0$, which is the distribution function of $Z(\tau, \alpha, a)$.) By (B.13), one has

$$F^\sharp(\xi) = \frac{1}{\Gamma(\xi + 1)} \int_0^\infty u^{\xi-1}G(u)du. \quad (5.6)$$

where $F^\sharp(\xi)$ is the Fourier transform of $F(y)$ (see (B.13) for more detailed argument.) On the other hand, due to $Z(\tau, \alpha, a) = Z^{(0)}(\tau, \alpha, a)e^{\chi}$, we find

$$F^\sharp_0(\xi) = \frac{\Gamma(\alpha - a)}{\Gamma(\alpha - a + \xi)}F^\sharp(\xi). \quad (5.7)$$

Combining (5.6),(5.7) and applying the inverse Fourier transform, we arrive at (5.5). \qed

For the case of the OY polymer model, the distribution function is infinitely differentiable since it is expressed as the marginal distribution of the Whittaker measure (4.7) with $a$ replaced by $-a$ on $y_1^{(N)}$ and the Whittaker function (4.1) appearing in (4.7) is infinitely differentiable with respect to each $y_j^{(N)}$, $j = 1, \cdots, N$. We have

**Corollary 5.2.**

$$F_0(y) = \Gamma(\nu) \sum_{n=0}^\infty \frac{1}{n!} \frac{d^n}{d\nu^n} \left( \frac{1}{\Gamma(\nu)} \right) F^{(n)}(y), \quad (5.8)$$

where $F^{(n)}$ means the $n$th derivative of $F(y)$.

**Remark.** Note that on the rhs one can use any representation of $F(y)$. For example, if one employs the formula (B.9), there is no complex integral and hence (5.8) with this formula is useful for numerical evaluation. Formally (5.8) can be written as

$$F_0(y) = \frac{\Gamma(\alpha - a)}{\Gamma(\alpha - a + d/dy)}F(y). \quad (5.9)$$

This type of formula was obtained for the stationary KPZ equation in [18]. Though it may look awkward with the derivative in the denominator, it has a solid meaning and is practically useful as explained above.
Proof. Using the integral representation of $1/\Gamma(z)$,
\[
\frac{1}{\Gamma(z)} = \frac{i}{2\pi} \int_{\gamma} (-w)^{-z} e^{-w} dw,
\]
(5.10)
where $\gamma$ is the contour in figure 2 and (B.11), we see

\[
\text{rhs of (5.5)} =\sum_{n=0}^{\infty} \frac{\Gamma(n+\nu)}{n!} \frac{1}{\Gamma(\nu)} \int_{\delta+iR}^{\delta+i\infty} d\xi \frac{\xi^n}{\Gamma(1+\nu)} \int_{0}^{\infty} u^{\nu-1} G(u) du
\]

which is the rhs of (5.5).

There is also a relation at the level of the Laplace transform. From (B.23) with (5.4), we have
\[
G_0(\epsilon^v) = \Gamma(\alpha - \frac{\theta}{\epsilon}) \int_{\delta+i\infty}^{\delta+i\infty} d\xi \frac{\epsilon^n}{\Gamma(1+\nu)} \int_{0}^{\infty} u^{\nu-1} G(u) du
\]
(5.12)
Expanding formally the rhs of the equation above around $d/dv$, we obtain the relation.
\[
G_0(\epsilon^v) = \Gamma(\alpha - \frac{\theta}{\epsilon}) \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{d^n}{d\epsilon^n} \left( \frac{1}{\Gamma(\nu)} \right)_{\nu \to \alpha - \theta} \int_{0}^{\infty} u^{\nu-1} G(u) du.
\]
(5.13)
where $G_0(\nu)$ and $G(\nu)$ are given by (5.3). The representation (5.12) or (5.13) can be used to establish the $F_0$ asymptotics for the stationary OY model just as in the case of the stationary $q$-TASEP. (See section 4 in [16])

Hereafter we explain how we can obtain an integral representation for the free energy distribution of the stationary OY model $\log Z_N(\tau, \alpha)$ where $Z_N(\tau, \alpha)$ is introduced above (2.4).
Recall that as discussed in (3.5), it has the same distribution as $\lim_{a \to \alpha} Z^{(a)}(\tau, \alpha, a)$. From (5.5), we have
\[
\mathbb{P}(\log Z_N(\tau, \alpha) \leq y) = \lim_{a \to \alpha} F_0(u) = \int_{\delta+i\infty}^{\delta+i\infty} d\xi \frac{\epsilon^\xi}{\Gamma(1+\nu)} \int_{0}^{\infty} u^{\nu-1} \hat{G}(u),
\]
(5.14)
where
\[
\hat{G}(u) = \lim_{a \to \alpha} \Gamma(\alpha - \frac{\theta}{\epsilon}) G(u).
\]
(5.15)
On the other hand, if one takes (5.9), we find
\[ P(\log Z_N(\tau, \alpha) \leq y) = \sum_{n=0}^{\infty} \frac{1}{n!} \left. \frac{d^n}{dx^n} \left( \frac{1}{\Gamma(\nu)} \right) \right|_{x=1} \hat{F}^{(n+1)}(y), \]  

(5.16)

where \( \hat{F}^{(n)}(y) \) is the \( n \)th derivative of \( \hat{F}(y) \) and

\[ \hat{F}(y) = \lim_{a \to \alpha} \Gamma(\alpha - a) F(y). \]  

(5.17)

Note that whilst \( F(y) \) is a distribution function, \( \hat{F}(y) \) is not expected to be so. One can use any expression of \( \hat{F}(y) \), for instance (B.10) and (B.11) with \( F(y) \) (resp \( G(u) \)) replaced by \( \hat{F}(y) \) (resp \( \hat{G}(u) \)). We can use also (B.14), through it is more suitable to write down a formula for the first derivative. One has

\[ \hat{F}^{(1)}(y) = \int_{\mathbb{R}} dw e^{-\nu y} \lim_{\epsilon \downarrow 0} \frac{1}{\pi} \text{Im} \hat{G}(-e^{-\nu} + i\epsilon). \]  

(5.18)

Note that although the functions \( \hat{G}(u) \) and \( \hat{F}(y) \) are defined through \( a \to \alpha \) limit of \( G(u) \) and \( F(y) \), they have a connection directly with the partition function of the stationary OY model \( Z_N(\tau, \alpha) \) introduced in section 2: since \( Z(\tau, \alpha, a) = Z^{(0)}(\tau, \alpha, a) e^y \) where \( \chi \sim -\log \Gamma(\nu) \) with \( \nu = \alpha - a \) (see (2.3)), we find \( G(u) \) and \( G_0(u) \) are related as

\[ G(u) = \int_{\mathbb{R}} dw \frac{(ue^y)^\nu e^{-ue^y}}{\Gamma(\nu)} G_0(e^{-y}). \]  

(5.19)

Thus noting (3.5) and (5.2), we have

\[ \hat{G}(u) := \lim_{\nu \to 0} \Gamma(\nu) G(u) = \int_{\mathbb{R}} dw e^{-ue^y} \langle e^{-e^y Z_N(\tau, \alpha)} \rangle. \]  

(5.20)

We can also have the relation about \( \hat{F}(u) \) in a similar way:

\[ \hat{F}(y) = \int_{\mathbb{R}} dw e^{-\nu y} \lim_{\epsilon \downarrow 0} \frac{1}{\pi} \text{Im} \hat{G}(-e^{-\nu} + i\epsilon). \]  

(5.21)

In both expressions (5.14) and (5.16) with (5.18), the remaining problem is to estimate \( \hat{G}(u) \). Note that in our approach, we first take the \( q \to 1 \) scaling limit in section 3 and then take the stationary limit \( a \to \alpha \). As another approach, it would be possible to exchange these two limits, i.e. (5.15) can also be obtained by taking \( q \to 1 \) scaling limit for the proposition 5.6 in [16].

Under the specialization (5.1), the kernel \( K(x_1, x_2) \) (4.25) can be written as

\[ K(x_1, x_2) = \sum_{l=0}^{N-2} \phi_l(x_1) \psi_l(x_2) + (\alpha - a) B_1(x_1) B_2(x_2), \]  

(5.22)

where

\[ \phi_l(x) = \frac{1}{2\pi i} \int_{C_l} dv e^{\nu v - v^2/2} \Gamma(1 + v)^{N-1} \Gamma(1 + v - a) \frac{\Gamma(1 + \alpha - v)}{\Gamma(1 + \alpha - v)} \]  

(5.23)

\[ \psi_l(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dw e^{-iwx - w^2/2} (iw)^l \frac{1}{\Gamma(1 + iw)^{N-1}} \Gamma(1 + \alpha - iw) \frac{\Gamma(1 + \alpha - iw)}{\Gamma(1 + \alpha - v)} \]  

(5.24)

\[ B_1(x) = \frac{1}{2\pi i} \int_{C_1} dv e^{\nu v - v^2/2} \frac{\Gamma(1 + v - a)\Gamma(1 + v)^{N-1}}{\Gamma(1 + \alpha - v)} \]  

(5.25)
\[ B_2(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dw \frac{e^{-iwx-w^2/2}}{iw-\alpha} \frac{(iw)^{N-1}}{\Gamma(1+\alpha-ia)} \frac{\Gamma(1+\alpha-ia)}{\Gamma(1+iw)^{N-1}}. \] (5.26)

for \( l = 0, 1, \ldots, N - 2 \). Note that although under (5.1), \( \phi_l(x) \) (4.26) vanishes and \( \psi_l(x) \) (4.27) diverges due to the factors with \( \alpha_j \), \( j = 1, \ldots, N - 1 \), their product \( \phi_l(x)\psi_l(x) \) converges to that of (5.23) and (5.24) for \( l \leq N - 2 \) and the second factor in (5.22) for \( l = N - 1 \). The fact that each (4.26) and (4.27) do not go to (5.23) and (5.24) respectively reflects from the scaling of (4.8). Since in the limit (5.1), the number of finite \( \alpha_j \) is 1, we should replace \( N \) in the scaling of \( \lambda^{(l)}_j \) in (4.8) by \( M = 1 \) as stated in the argument above (4.9).

We further rewrite the relation (4.23) under the specialization (5.1), (see (5.27) in [16] for the \( q \)-TASEP) as

\[ G(u) = \det(1 - A_{a,\alpha} - (\alpha - a)f_a) \chi_1 \otimes B_2, \]

\[ = (\alpha - a) \det(1 - A_{a,\alpha}) \left( L_{\alpha,a} - \int_{\mathbb{R}} dx (A_{a,\alpha} \rho A_{a,\alpha} f_a) B_2(x) \right), \] (5.27)

where \( A_{a,\alpha}(x_1, x_2) = f_a(x_1) \sum_{N_{\alpha,\alpha} - 2} \phi_k(x_1) \psi_k(x_2), f_a \) is defined by (4.24), \( \rho_{\alpha,\alpha} = (1 - A_{a,\alpha})^{-1} \) and

\[ L_{\alpha,a} = \frac{1}{\alpha - a} - \int_{\mathbb{R}} dx f_a B_1(x) B_2(x). \] (5.28)

Furthermore we decompose \( B_1(x) \) and \( B_2(x) \) as (corresponding to (5.30) in [16] for \( q \)-TASEP),

\[ B_1(x) = B_1^{(1)}(x) + B_1^{(2)}(x), \]

\[ B_2(x) = B_2^{(1)}(x) + B_2^{(2)}(x), \] (5.29)

where \( B_1^{(1)}(x) \) (resp. \( B_2^{(1)}(x) \)) is the residue at \( x = a \) (resp. \( x = -ia \)), while \( B_1^{(2)}(x), i = 1, 2 \) come from the remaining contributions,

\[ B_1^{(1)}(x) = \frac{e^{x/a} - e^{x-a}}{a^{N-1}} \frac{\Gamma(1+a)^{N-1}}{\Gamma(1+a-a)}, \] (5.30)

\[ B_1^{(2)}(x) = \frac{e^{-x/a} - e^{-x+a}}{a^{N-1}} \frac{\alpha^{N-1}}{\Gamma(1+a+a)}, \] (5.31)

\[ B_2^{(2)}(x) = \frac{1}{2\pi i} \int_{|\sigma| < a} d\sigma \frac{e^{i\sigma x - \sigma^2/2}}{\sigma^{N-1}(\sigma - v)} \frac{\Gamma(1+\alpha-\alpha/v)}{\Gamma(1+\alpha-v)}, \] (5.32)

\[ B_2^{(2)}(x) = \frac{1}{2\pi i} \int_{|\sigma| < a} d\sigma \frac{e^{-i\sigma x - \sigma^2/2}}{\sigma^{N-1}(\sigma - \alpha)} \frac{\Gamma(1+\alpha-\alpha - i\sigma)}{\Gamma(1+\alpha+i\sigma-\alpha)}, \] (5.33)

where \( \alpha \) in (5.33) satisfies \( \alpha < c < \alpha + 1 \). Using these, we write (5.28) as

\[ L_{\alpha,a} = \frac{1}{\alpha - a} - \int_{\mathbb{R}} dx f_a(x) B_1^{(1)}(x) B_2^{(1)}(x) \sum_{(i,j) \neq (1,1)} \int_{\mathbb{R}} dx f_a(x) B_1^{(i)}(x) B_2^{(j)}(x). \] (5.34)

Here we take the stationary limit \( a \to \alpha \) in (5.14) and (5.15). As with lemma 5.5 in [16] for the stationary \( q \)-TASEP, the following lemma is important.
Lemma 5.3. Let $L_{\alpha} = \lim_{a \to \alpha} L_{\alpha,a}$. We have

$$L_{\alpha} = (N - 1) \left( \frac{\Gamma'(1 + \alpha)}{\Gamma(1 + \alpha)} - \frac{1}{\alpha} \right) - 2\gamma_E - \alpha \tau - \log \alpha - \sum_{(i,j) \neq (1,1)}^{2} \int_{\mathbb{R}} \text{d}x f_{\alpha}(x) B_{i}^{(j)}(x; \alpha) B_{2}^{(j)}(x; \alpha),$$

(5.35)

where $\gamma_E$ is the Euler–Mascheroni constant and $B_{i}^{(j)}(x; \alpha)$, $i, j = 1, 2$ are $a \to \alpha$ limit of (5.30)–(5.33).

Proof. It is easy to see that the last term in (5.34) goes to the one in (5.35). The remaining part is to establish

$$\lim_{a \to \alpha} \frac{1}{\alpha - a} - \int_{\mathbb{R}} \text{d}x f_{\alpha}(x) B_{1}^{(1)}(x) B_{2}^{(1)}(x) = (N - 1) \left( \frac{\Gamma'(1 + \alpha)}{\Gamma(1 + \alpha)} - \frac{1}{\alpha} \right) - 2\gamma_E - \alpha \tau - \log \alpha.$$

(5.36)

For this purpose, we calculate

$$\int_{\mathbb{R}} f_{\alpha}(x) B_{1}^{(1)}(x) B_{2}^{(1)}(x) = \left( \frac{\alpha}{\alpha} \frac{\Gamma(1 + a)}{\Gamma(1 + \alpha)} \right)^{N-1} \frac{e^{(a^2 - a^2)\tau/2}}{\Gamma(1 + a - a)^2} \int_{\mathbb{R}} \text{d}x \frac{e^{(a^2 - a^2)\tau/2}}{\Gamma(1 + a - a)^2} \frac{\pi a - a}{\sin \pi(a - a)}.$$

(5.37)

where in the second equality we use the formula

$$\int_{\mathbb{R}} \text{d}x \frac{e^{ax}}{1 + e^{x}} = \frac{\pi}{\sin \pi c}$$

for $0 < \text{Re} c < 1$. Noting

$$\left( \frac{\alpha}{\alpha} \frac{\Gamma(1 + a)}{\Gamma(1 + \alpha)} \right)^{N-1} = 1 - (N - 1) \left( \frac{1}{\alpha} - \frac{\Gamma'(1 + \alpha)}{\Gamma(1 + \alpha)} \right) (a - \alpha) + O((a - \alpha)^2),$$

(5.39)

$$\frac{1}{\Gamma(1 + a - a)^2} = 1 - 2\gamma_E (a - \alpha) + O((a - \alpha)^2),$$

(5.40)

$$e^{(a^2 - a^2)\tau/2} a - a = 1 - (a \tau + \log \alpha)(a - \alpha) + O((a - \alpha)^2),$$

(5.41)

$$\frac{\pi}{\sin \pi(a - a)} = \frac{1}{a - \alpha} + O(a - \alpha),$$

(5.42)

where in (5.40), we used the fact $\Gamma'(1) = -\gamma_E$, we arrive at (5.36).

Combining (5.27) with lemma 5.3, we find a representation for $\tilde{G}(u)$ (5.15). Thus we obtain two representations of the free energy distribution for the stationary OY model substituting the representation of $\tilde{G}(u)$ into (5.14) and (5.16):

**Theorem 5.4.** The free energy distribution of the stationary OY model with parameter $\alpha$ is given by

$$\mathbb{P}(\log Z_{N}(\tau, \alpha) \leq y) = \int_{\mathbb{R}} \frac{\text{d}x}{2\pi i} \frac{e^{ix \xi}}{\Gamma(1 + \xi)} \int_{0}^{\infty} \text{d}u e^{iu - 1} \tilde{G}(u),$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\text{d}^{n}}{\text{d}\alpha^{n}} \left( \frac{1}{\Gamma(n + 1)} \right)_{\alpha=1} \tilde{F}^{(n+1)}(y).$$

(5.43)
Here \( \hat{G}(u) \) is given by

\[
\hat{G}(u) = \det(1 - A_\alpha) \left( L_\alpha - \int_\mathbb{R} dx(A_\alpha \rho A_\alpha f_\alpha B_1)(x; \alpha) B_2(x; \alpha) \right),
\]

where \( A_\alpha = \lim_{a \to \alpha} A_{\alpha,a} \) with \( A_{\alpha,a} \) defined below (5.27), \( \rho A_\alpha = (1 - A_\alpha)^{-1} \), \( B_j(x; \alpha) = \lim_{a \to \alpha} B_j(x) \) for \( j = 1, 2 \) and \( L_\alpha \) is given by (5.35). In the second expression we can choose several representations, for instance (B.10) and (B.11) with \( F(y) \) (resp. \( G(u) \)) replaced by \( \hat{F}(y) \) (resp. \( \hat{G}(u) \)), or (5.18) for \( \hat{F}^{(1)}(y) \).

**Proof.** We immediately obtain (5.44) substituting (5.27) into the rhs of (5.15) and using lemma 5.3.

Using the relation

\[
\det(1 - A_\alpha) \left( 1 - \int_\mathbb{R} dx(A_\alpha \rho A_\alpha f_\alpha B_1)(x; \alpha) B_2(x; \alpha) \right) = \det(1 - A_\alpha - (A_\alpha f_\alpha B_1(\alpha) \otimes B_2),
\]

we find (5.44) can also be written as

\[
\hat{G}(u) = \det(1 - A_\alpha) (L_\alpha - 1) + \det(1 - A_\alpha - (A_\alpha f_\alpha B_1 \otimes B_2).
\]

Combining (5.18) with (5.44), we obtain the following representation of \( \hat{F}(y) \):

\[
\hat{F}(y) = \int_\mathbb{R} dw e^{-w-y} \left( \hat{G}(e^{-w}) - \hat{G}^{(\delta)}(e^{-w}) \right).
\]

Here \( \hat{G}^{(\delta)}(e^{-w}) \) is defined by (5.44) with \( f_{-e^{-x}}(x) = 1/(1 - e^{w-x}) \) (see (4.24)) replaced by \( 1/(1 - e^{w-x}) - \delta(x - w) \). For showing (5.47), it is sufficient to show that the part \(-\frac{1}{\pi} \lim_{w \to 0} \text{Im} \hat{G}(e^{-w} + i\epsilon) \) in (5.18) is written as

\[
-\frac{1}{\pi} \lim_{\epsilon \to 0} \text{Im} \hat{G}(e^{-w} + i\epsilon) = \hat{G}(e^{-w}) - \hat{G}^{(\delta)}(e^{-w}).
\]

This is obtained by using the fact \( 1/(x + i\epsilon) = \mathcal{P}(1/x) - i\pi \delta(x) \), where \( \mathcal{P} \) represents the Cauchy principal value and basic properties of determinant.

Although \( \hat{G}^{(\delta)}(e^{-w}) \) includes the delta function terms, we find that it is finite: as with the representation (5.46) of \( \hat{G}(u) \), we can also express \( \hat{G}^{(\delta)}(e^{-w}) \) as

\[
\hat{G}^{(\delta)}(e^{-w}) = \det \left( 1 - A^{(\delta)}_\alpha \right) (L^{(\delta)} - 1) + \det \left( 1 - A^{(\delta)}_\alpha - (A^{(\delta)}_\alpha f^{(\delta)}_{-e^{-x}} B_1 \otimes B_2 \right),
\]

where \( L^{(\delta)}_\alpha = L_\alpha + B_1(w)B_2(w) \), \( A^{(\delta)}_\alpha(x,y) = (f_{-e^{-x}}(x) - \delta(x - w))K(x,y) \) and \( f^{(\delta)}_{-e^{-x}}(x) = f_{-e^{-x}}(x) - \delta(x - w) \). Note that each integral in the expansion of the above two Fredholm determinants is finite even if it has delta function contributions.

Applying the same arguments and calculations as for the long-time limit of the \( q \)-TASEP (see section 4.3 in [16]), we finally obtain the limiting distribution.

**Corollary 5.5.**

\[
\lim_{N \to \infty} \mathbb{P} \left( \frac{\log Z_N(sN, \alpha) - \eta N}{\gamma N^{1/3}} \leq s \right) = F_{BR}(s; \omega),
\]

where \( F_{BR}(s; \omega) \) denotes the Baik–Rains distribution and
\[\alpha = \theta - \frac{\omega}{\gamma^{1/3}}, \quad \kappa = \Phi'(\theta), \quad \eta = \Phi'(\theta)\theta - \Phi(\theta), \quad \gamma = \left(\frac{-\Phi''(\theta)}{2}\right)^{1/3}, \tag{5.51}\]

where \(\theta > 0\) and \(\Phi(z) = \Gamma'(z)/\Gamma(z)\) is the digamma function. (Note that \(\Phi''(z) < 0\) for \(z > 0\).)

**Remark.** The Baik–Rains distribution has a few different representations. Here it is convenient to choose the one in [16, 18]

\[F_{\text{BR}}(s; \omega) = \frac{\partial}{\partial s} \nu_{\omega}(s),\]

\[\nu_{\omega}(s) = F_2(s) \left( s - \omega^2 - \sum_{j,l=1, j \neq l}^{2} \int_{\gamma}^{\infty} d\xi \left( B_{\omega}^{(j)}(\xi) B_{\omega}^{(l)}(\xi) - \int_{\gamma}^{\infty} d\xi \left( \rho_A B_{\omega}(\xi) \right) B_{\omega}(\xi) \right), \right) \tag{5.52}\]

where \(F_2(s) = \det(1 - A)_{(1,2)}\) with the Airy kernel \(A(x, y) = 1_{>\epsilon}(x) \int_{0}^{\infty} d\lambda \text{Ai}(x + \lambda) \text{Ai}(y + \lambda)\) is the GUE Tracy–Widom distribution [31] and

\[\rho_A(x, y) = (1 - A)^{-1}(x, y), \quad B_{\omega}^{(1)}(\xi) = e^{-\omega \sqrt{1-\omega \xi}}, \quad B_{\omega}^{(2)}(\xi) = -\int_{0}^{\infty} d\xi e^{\omega \xi} \text{Ai}(\xi + z),\]

\[B_{\omega}(\xi) = B_{\omega}^{(1)}(\xi) + B_{\omega}^{(2)}(\xi). \tag{5.53}\]

**Proof.** For showing (5.50), it is convenient to use (5.16). Scaling \(\tau\) and \(y\) as \(\tau = \kappa N, \quad y = \eta N + \gamma N^{1/3}\) in both hand sides, we have

\[\lim_{N \to \infty} \frac{1}{\gamma N^{1/3}} \mathbb{P}(\log \mathcal{Z}_N(\tau, \alpha) \leq y) = \frac{d}{ds} \lim_{N \to \infty} \frac{\hat{F}(y)}{\gamma N^{1/3}}. \tag{5.54}\]

In order to consider the limit in the rhs, we first focus on the relation (5.21). Associated with the scaling of \(y\) stated above (5.54), we scale \(w\) in (5.21) also as \(w = \eta N + \gamma N^{1/3}\) then take the limit \(N \to \infty\). We have

\[\lim_{N \to \infty} \frac{\hat{F}(y)}{\gamma N^{1/3}} = \lim_{N \to \infty} \int_{\gamma}^{\infty} dx e^{-\gamma x^{1/3}(x-\alpha)} \mathbb{P}(\log \mathcal{Z}_N(\tau, \alpha) \leq w) \]

\[= \int_{-\infty}^{\infty} dx \lim_{N \to \infty} \mathbb{P}(\log \mathcal{Z}_N(\tau, \alpha) \leq w), \tag{5.55}\]

where we used the fact \(\lim_{N \to \infty} e^{-\gamma N} = 1_{>0}(x)\). Noting the relation

\[\lim_{N \to \infty} \mathbb{P}(\log \mathcal{Z}_N(\tau, \alpha) \leq w) = \lim_{N \to \infty} \left( e^{-\gamma w} \mathcal{Z}_N(\tau, \alpha) \right) \tag{5.56}\]

and (5.20), we eventually obtain

\[\lim_{N \to \infty} \frac{\hat{F}(y)}{\gamma N^{1/3}} = \int_{-\infty}^{\infty} dx \lim_{N \to \infty} \left( e^{-\gamma w} \mathcal{Z}_N(\tau, \alpha) \right) = \lim_{N \to \infty} \frac{\hat{G}(e^{-\gamma})}{\gamma N^{1/3}}. \tag{5.57}\]

Thus for establishing (5.50), it is sufficient to estimate \(\lim_{N \to \infty} \frac{\hat{G}(e^{-\gamma})}{\gamma N^{1/3}}\). As with lemma 5.10 and 5.11 in [16] for the stationary \(q\)-TASEP, We show that under the scaling
\[ x = \eta N + \gamma N^{1/3} \xi, \quad I = N - \gamma N^2 \theta \lambda, \quad (5.58) \]

we have

\[
\lim_{N \to \infty} C_{N,\theta,\xi} = \lim_{N \to \infty} C_{N,\theta,\xi}^{-1} = \mathcal{A} (\xi + \lambda),
\]

\[
\lim_{N \to \infty} C_{N,\theta,\xi} B_i^{(i)}(x) = B_i^{\omega}(\xi), \quad \lim_{N \to \infty} C_{N,\theta,\xi} B_i^{(i)}(x) = B_i^{(i)}(\xi), \quad \text{for } i = 1, 2,
\]

\[
\lim_{N \to \infty} \frac{1}{\gamma N^{1/3}} \left[ (N - 1) \left( \frac{\Gamma'(1 + \alpha)}{\Gamma(1 + \alpha)} - \frac{1}{\alpha} \right) - 2\gamma - \alpha \tau + \gamma \right] = s - \omega^2,
\]

where \( C_{N,\theta,\xi} \) is

\[
C_{N,\theta,\xi} = \frac{e^{N(\kappa\theta^2/2 - \eta \xi)} \phi}{\Gamma(1 + \theta) N^{-1}}.
\]

Here we give a sketch of proofs of these relations. For (5.59) and (5.60), we only focus on the second one in each relation since the first one can be obtained in a parallel way. They can be obtained by the saddle point analyses. First we focus on the result on \( \Psi_i(x; \alpha) \) in (5.59). Setting \( z = i \omega \), one has

\[
\psi_i(x; \alpha) = \frac{1}{2\pi i} \int_{\mathbb{R}} dz e^{N(z)} e^{-\gamma N^{1/3} \xi z} \Gamma(1 + z) \Gamma(1 + \alpha - z) \Gamma(1 + \alpha - z)
\]

where \( f(z) = \kappa z^2/2 - \eta z - \log \Gamma(z) \). One easily finds \( z = \theta \) is a double saddle point i.e. \( f'(\theta) = f''(\theta) = 0 \). Thus scaling \( z \) around this double saddle point as \( z = z(\sigma) = \theta - i\sigma / \gamma N^{1/3} \), we obtain

\[
N f(z) = N f(\theta) + \frac{i}{3} \sigma^3 + O \left( N^{-1/3} \right).
\]

Combining this with the relations

\[
\lim_{N \to \infty} e^{-\gamma N^{1/3} \xi(z - \theta)} = e^{2\sigma}, \quad \lim_{N \to \infty} \theta^\gamma N^{1/3} \xi z = e^{\lambda \sigma}, \quad \lim_{N \to \infty} \frac{\Gamma(1 + z) \Gamma(1 + \alpha - z)}{\Gamma(1 + \theta) \Gamma(1 + \alpha - z)} = 1,
\]

we obtain the second relation in (5.59).

Next we consider the limit of \( B_i^{(1)}(x; \alpha) \) and \( B_i^{(2)}(x; \alpha) \). The former one can be written as

\[
B_2^{(1)}(x; \alpha) = e^{N(\alpha)} e^{-\alpha \gamma N^{1/3} \xi} \Gamma(\alpha).
\]

Here the function \( f(z) \) is defined below (5.63). As with (5.64) and (5.65) noting \( N f(\alpha) = N f(\theta) - \omega^2 z + O(1) \), \( e^{-\gamma N^{1/3} \xi} = e^{-\theta \gamma N^{1/3} \xi + \xi \omega} \), we arrive at the second relation in (5.60) with \( i = 1 \). Also we rewrite \( B_2^{(2)}(x; \alpha) \) as

\[
B_2^{(2)}(x; \alpha) = \frac{1}{2\pi i} \int_{\mathbb{R} - c} dz e^{N(\alpha)} e^{-\gamma N^{1/3} \xi z} \Gamma(\alpha) \frac{\Gamma(1 + \alpha - z)}{\Gamma(1 + z - \alpha)}
\]

with \( \alpha < c < \alpha + 1 \). Changing \( z \) to \( \sigma \) defined above (5.64), and using (5.64) and (5.65) with the relation
\[ \frac{1}{\gamma N^{1/3}(\alpha - z)} = \frac{1}{i \sigma - \omega} = - \int_0^\infty d\omega e^{(i \sigma - \omega)z}, \]  
where we used the fact that the contour of \( \sigma \) become \( \mathbb{R} - ic \) with \( c < \omega \), we obtain the second relation in (5.60) with \( i = 2 \).

For the last one (5.61), noting \( \Gamma'(1 + \alpha)/\Gamma(1 + \alpha) - 1/\alpha = \Phi(\alpha) \) where \( \Phi(\alpha) := \Gamma'(\alpha)/\Gamma(\alpha) \) is the digamma function, and expand each term in the lhs up to \( O(N^{1/3}) \), we obtain the rhs.

These relations with \( \lim_{N \to \infty} f e^{-y}(x) = 1 \) under the scaling of \( x \) and \( y \) stated (5.58) and above (5.54) lead to

\[ \lim_{N \to \infty} \tilde{G}(e^{-\gamma}) = \nu_\omega(x). \]  
Thus from this equation with (5.54) and (5.57), we obtain (5.50).

### 6. The stationary KPZ equation

In this section we consider the limit to the KPZ equation,

\[ \partial_t h = \frac{1}{2} \partial_x^2 h + \frac{1}{2} (\partial_x h)^2 + \eta, \]  
where \( h = h(x, t) \) represents the height at position \( x \in \mathbb{R} \) and at time \( t \geq 0 \) and \( \eta = \eta(x, t) \) is the space-time Gaussian white noise with mean zero and covariance \( \langle \eta(x, t) \eta(x', t') \rangle = \delta(x-x')\delta(t-t') \), particularly for the stationary situation. It has been known that in the stationary KPZ equation, the height difference \( h(x, t) - h(0, t) \) is given by the two-sided Brownian motion. Thus we prepare the initial condition as

\[ h(x, 0) = h(x, 0) - h(0, 0) = \tilde{B}(x), \]  
where in the lhs, we set \( h(0, 0) = 0 \) due to the translational invariance and \( \tilde{B}(x) \) is the two-sided Brownian motion with drift \( v \) (2.5). The KPZ equation is (formally) transformed to the stochastic heat equation (SHE). By applying the Cole-Hopf transformation \( Z(x, t) = e^{h(x, t)} \), \( Z(x, t) \) solves the stochastic heat equation (SHE)

\[ \frac{\partial}{\partial t} Z(x, t) = \frac{1}{2} \frac{\partial^2}{\partial x^2} Z(x, t) + \eta(x, t) Z(x, t), \]

\[ Z(x, 0) = e^{\tilde{h}(x)}. \]  
A precise meaning of the KPZ equation (6.1) for the case of the initial data with the two-sided Brownian motion is given for example using the Cole-Hopf transformation [4].

Now let us consider the scaling limit of the stationary OY model \( Z_N(\tau, \alpha) \) in (2.4) to \( Z(x, t) \) in (6.3). The scaling limit of the OY model or more generalized discrete models to the SHE has been studied in [10]. In our case, we scale \( \tau \) and \( \alpha \) as

\[ \tau = \sqrt{4N + x}, \quad \alpha = \sqrt{N \frac{2}{4}} + \frac{1}{2} v, \]  
and set
\[ C(N, x, t) := \exp \left( N + \frac{\sqrt{N}}{2} + x \sqrt{\frac{N}{\tau}} \right) \left( \frac{t}{N} \right)^{\frac{\alpha-1}{2}}. \]  

(6.5)

Then \( Z_N(\tau, \alpha) \) goes to the solution for the stationary SHE (6.3),

\[ \lim_{N \to \infty} \frac{Z_N(\tau, \alpha)}{C(N, x, t)} = Z(x, t). \]

(6.6)

Here we give a derivation of (6.6) based on the discussion in section 4.1 in [19].

Let \( Z_{j, \beta}(\tau), j \in \{1, 2, \cdots, N\} \) be

\[ Z_{j, \beta}(\tau) = e^{-\tau - \beta^2 \tau/2} Z_i(\beta^2 \tau, \alpha). \]

(6.7)

Noting \( Z_j(\tau, \alpha) \) (2.4) solves (2.2) where \( B_1(\tau) \) is the standard Brownian motion with drift \( \alpha \) while the other ones \( B_2(\tau), \cdots, B_N(\tau) \) are the independent standard normal Brownian motions without drift, we see that the deformed one (6.7) satisfies the stochastic differential equation

\[ d\tilde{Z}_{j, \beta}(\tau) = (\tilde{Z}_{j-1, \beta}(\tau) - \tilde{Z}_{j, \beta}(\tau)) d\tau + \beta \tilde{Z}_{j, \beta}(\tau) dB(\tau), \]

(6.8)

where we set \( \tilde{Z}_{0, \beta}(t) = 0 \). Now let us take the diffusion scaling for (6.8): we set

\[ \tau = tM, \ j = tM - x \sqrt{M} \]

(6.9)

with \( M > 0 \). Note that at this stage the scaling is different from (6.4). At the same time we scale \( \beta \) as

\[ \beta = M^{-1/4}, \]

(6.10)

then take the large \( M \) limit. Under this scaling limit, we see that (6.8) goes to SHE i.e. the first equation in (6.3). An explanation of this property was given in section 4.1 in [19].

Next we consider the initial condition. Considering \( \tau = 0 \) in (6.9), we notice that only the negative region \( x < 0 \) appears. This comes from the replacement of the whole \( Z_j(\tau, \alpha) \) with the negative index \( j \leq 0 \) by a single \( Z_1(\tau, \alpha) \) with the Brownian motion with drift \( \alpha \) (see the explanation above (2.4)). Here in order to take the region \( x > 0 \) into account, we consider \( Z_j(0, \alpha) \) with \( j \in \mathbb{Z} \) rather than \( j \in \{1, \cdots, N\} \). According to theorem 3.3 in [26], one has

\[ \tilde{Z}_{j, \beta}(0) = \frac{Z_j(0, \alpha)}{\beta^{2(j-1)}} \sim \begin{cases} \sum_{k=1}^{j} \log(\alpha + 1), & j \geq 0, \\ \sum_{k=1}^{-j} \log|\alpha - 1|, & j \leq -1, \end{cases} \]

(6.11)

where \( r_k(0), k \in \mathbb{Z} \) are i.i.d. random variables with \( r_k(0) \sim -\log \Gamma(\alpha) \) (see below (2.2)). Here we scale \( j \) as (6.9) with \( t = 0 \) and

\[ \alpha = \beta^{-2} + v + \frac{1}{2}. \]

(6.12)

From the properties of the distribution \( -\log \Gamma(\alpha) \), we see that

\[ \mathbb{E}(r_k(0)) = -\log \Gamma(\alpha)' = -\log \alpha + \frac{1}{2\alpha} + o\left(\frac{1}{\alpha}\right) = \log \beta^2 - \frac{v}{M^{1/2}} + o\left(\frac{1}{M^{1/2}}\right), \]

(6.13)

\[ \text{Var}(r_k(0)) = \log \Gamma(\alpha)'' = \frac{1}{\alpha} + o\left(\frac{1}{\alpha}\right) = \frac{1}{M^{1/2}} + o\left(\frac{1}{M^{1/2}}\right). \]

(6.14)
Using them and Donsker’s theorem [15], we find that

\[ \lim_{M \to \infty} \sum_{k=1}^{j-1} \left( r_k(0) - \log \beta^2 \right) = B_\pm(x) + \nu x, \quad \text{for } j \geq 0 \text{ and } x \leq 0, \]

\[ \lim_{M \to \infty} \sum_{k=1}^{j-1} \left( r_{-k}(0) - \log \beta^2 \right) = B_\pm(x) + \nu x, \quad \text{for } j < 0 \text{ and } x > 0, \]

(6.15)

where \( B_\pm(x) \) are independent standard Brownian motion without drift.

Therefore under the scaling (6.9), (6.10) and (6.12), the following limiting property is established.

\[ \lim_{M \to \infty} \tilde{Z}_{\beta}(\tau) = Z(x, t). \]

(6.16)

At last we show that the relation (6.6) with (6.4) and (6.5) is equivalent to (6.16) with (6.9), (6.10), and (6.12). We note that even if we slightly change the scaling (6.10) to \( \beta = (M - x \sqrt{M}/t)^{-1/4} \), we have the same result (6.16). Setting \( N = tM - x \sqrt{M} \), one sees that (6.9) and (6.12) with the modification above is equivalent to

\[ \tau = N + x \sqrt{N/T}, \quad j = N, \quad \beta = \left( \frac{N}{T} \right)^{-1/4}, \quad \alpha = \left( \frac{N}{T} \right)^{1/2} + v + \frac{1}{2}. \]

(6.17)

Applying the above scaling to the rhs of (6.7) and noticing

\[ \beta^2 \tau = \sqrt{TN} + x, \quad \beta^{2(N-1)} e^\tau + \beta \tau / 2 = C(N, x, t), \]

(6.18)

where \( C(N, x, t) \) is defined in (6.5), we find that (6.16) is equivalent to (6.6).

The goal of this section is to obtain the height distribution function of the stationary KPZ equation (6.1) and (6.2) by considering the scaling limit of theorem 5.4 in the stationary OY model. Hereafter we put a tilde \((\sim)\) on each \( G, F, A_\alpha \) and \( B_{\alpha, i}(x) \), \( i = 1, 2 \) in (5.43) and (5.44), for the quantities for the OY model while those without tildes represent the corresponding quantities for the KPZ equation (6.1).

We define \( \tilde{G}(u) \) and \( \tilde{F}(s) \) as

\[ \tilde{G}(u) = \int_R d\omega e^{-\omega s} \left\langle \exp \left( -e^{\beta (2\gamma_2 y \tau) + \frac{\gamma_3}{12} + \gamma_3 \omega^2} \right) \right\rangle, \]

(6.19)

\[ \tilde{F}(s) = \int_R d\omega e^{-\omega s} \left\langle \exp \left( h(2\gamma_2 y \tau, t) + \frac{\gamma_3}{12} + \gamma_3 \omega^2 \right) \right\rangle, \]

(6.20)

where we set

\[ x = 2\gamma_2 y, \quad v = \omega/\gamma_2, \quad \gamma_2 = (t/2)^{1/3}. \]

(6.21)

Note that these can be obtained as the KPZ equation limit of \( \tilde{G}(\tilde{u}) \) (5.20) and \( \tilde{F}(\tilde{y}) \) (5.21) respectively for the stationary OY model, i.e. in addition to (6.4) with (6.21), we scale \( \tilde{u} \) in \( \tilde{G}(\tilde{u}) \) and \( \tilde{y} \) in \( \tilde{F}(\tilde{y}) \) as

\[ \tilde{u} = uc^{3/2} + \gamma_3 \gamma_2^2 / C(N, t, 2\gamma_2^2) \]

(6.22)

\[ \tilde{y} = s - \frac{\gamma_3}{12} - \gamma_3 y^2 + \log C(N, t, 2\gamma_2^2) \]

respectively. Then under the above scaling we have
\( \hat{G}(u) = \lim_{N \to \infty} \hat{G}(\tilde{u}), \ F(s) = \lim_{N \to \infty} \hat{F}(\tilde{y}). \)  

(6.23)

Before stating the result, we further define some functions:

\[
A \Gamma_1(\xi, \omega) = \frac{1}{2\pi} \int_{\lambda} \omega \xi + i \zeta \Gamma(1 - (\omega - i\zeta)/\gamma_r) \Gamma(1 + (\omega - i\zeta)/\gamma_l).
\]

(6.24)

\[ B_0^1(\xi) = e^{-\omega^2/3 + \omega^2}, \ B_1^2(\xi) = -\int_0^\infty \omega e^{-\omega^2} A \Gamma_1(\xi + \lambda, \omega). \]

(6.25)

\[ B_{\omega}(\xi) = B_0^1(\xi) + B_1^2(\xi). \]

(6.26)

In (6.24), \( \Gamma_\omega \) represents the contour from \(-\infty\) to \(\infty\) passing below the pole \(i(\gamma_r - \omega)\). Then we have the following.

**Proposition 6.1.** We have the following representation for \( \hat{G}(u) \) (6.19).

\[
\hat{G}(u) = \gamma \lambda \det(1 - \Lambda_{\omega + y, y}) \left( L_{\omega + y, y} - \int_{\mathbb{R}} \omega \xi (A_{\omega + y, y} \rho_{\Lambda_{\omega + y}} f_{\lambda, \gamma} B_{\omega + y})(\xi) B_{-(\omega + y)}(\xi) \right).
\]

(6.27)

where \( f_{\lambda, \gamma} (\xi) = f_{\lambda}(\gamma \xi) \) and

\[
L_{\omega, y} = - \frac{2\gamma E + \log u}{\gamma_r} - \omega^2 - \sum_{l \neq 1} \int_{\partial \Omega} \omega \xi A_{\omega, y} \xi B^l_{\omega}(\xi) B^l_{-\omega}(\xi).
\]

(6.28)

\[
A_{\omega, y}(\xi_1, \xi_2) = f_{\lambda}(\gamma \xi_1) \int_0^\infty \omega \xi_2 A \Gamma_1(\xi_1 + \lambda, \omega) A \Gamma_1(\xi_2 + \lambda, -\omega).
\]

(6.29)

For the proof this proposition, the following lemma plays a crucial role. The proof will be given in appendix C.

**Lemma 6.2.** In addition to (6.4) with (6.21), we scale \( \tilde{u} \) as the first one in (6.22) and \( \tilde{x} \), the argument of the functions \( \Phi(\tilde{x}), \Psi(\tilde{x}), \ B^l_j(\tilde{x}), i, j = 1, 2, \) and \( l \) as

\[
\tilde{x} = \gamma \xi - \frac{3}{12} - \gamma y^2 + \log C(N, t, 2\gamma y), \ l = N - \sqrt{N/2\gamma}.
\]

(6.30)

Then under (6.4) and (6.30), we have

\[
\lim_{N \to \infty} \frac{\sqrt{N} e^{j(\tau, \tilde{x}, \tilde{y})}}{2\gamma_r^{1/2}} \Phi_l(\tilde{x}) = A \Gamma_1(\xi + \lambda, \omega + y),
\]

(6.31)

\[
\lim_{N \to \infty} \frac{1}{e^{j(\tau, \tilde{x}, \tilde{y})}} \Psi_l(\tilde{x}) = A \Gamma_1(\xi + \lambda, -\omega - y),
\]

\[
\lim_{N \to \infty} \frac{1}{e^{j(\tau, \tilde{x}, \tilde{y})}} \frac{\Gamma(1 + \sqrt{N}/l)}{\Gamma(1 + \sqrt{N}/l - 1)} \frac{B^l(\tilde{x})}{B^l_{\omega - y}(\tilde{x})} = B^l_{\omega - y}(\tilde{x}).
\]

(6.32)
\[
\lim_{N \to \infty} \frac{1}{\Gamma(1 + \sqrt{N/t})} B_1^{(2)}(\tilde{x}) = \frac{1}{\omega + y} B_1^{(2)}(\xi),
\]
\[
\lim_{N \to \infty} \frac{1}{\Gamma(1 + \sqrt{N/t})} B_2^{(2)}(\tilde{x}) = \frac{1}{\omega - y} B_2^{(2)}(\xi), \tag{6.33}
\]
\[
\lim_{N \to \infty} \frac{(N - 1)}{\Gamma(1 + \alpha)} - 2\gamma E - \alpha \tau - \log \tilde{u} = -2\gamma E - \log u - \gamma(\omega + y)^2, \tag{6.34}
\]

where \( f_0(\tilde{z}, \tilde{x}, l) \) is defined by (C.2) and \( \tilde{z} \) is given as (C.13) with \( \sigma = 0 \).

**Proof of proposition 6.1.** Combining the first relation in (6.23) with lemma 6.2, we readily obtain (6.27)

Thus we arrive at an expression of the height distribution for the stationary KPZ equation:

**Theorem 6.3.** Set \( \gamma_1 = (1/2)^{1/3} \). For \( y \in \mathbb{R}, s \in \mathbb{R} \), we have

\[
P \left( \frac{h(2\gamma_1^2 y, t) + \tilde{\gamma} \bar{y} + \gamma_1 s}{\gamma_1} \leq s \right) = \frac{\gamma_1}{2\pi i} \int_{\mathbb{C}} \frac{e^{\gamma \alpha t}}{\Gamma(1 + \xi)\Gamma(1 + \xi)} \int_{\mathbb{R}} d\omega e^{-\gamma \omega \xi} \tilde{G}(e^{-\gamma \omega})
\]
\[
= \sum_{n=0}^{\infty} \frac{1}{\gamma_1^{n+1}} \frac{(\xi)}{\Gamma(\nu)} \int_{\mathbb{R}} d\omega \left( \frac{1}{\Gamma(\nu)} \right) \frac{d^{n+1}}{d\omega d^{n+1}} \tilde{F}(\gamma \omega). \tag{6.35}
\]

Here \( \tilde{G}(\omega) \) is given by (6.27). For \( \tilde{F}(s) \) in the second expression, which is defined in (6.20), one can use the inversion formulas in appendix B, for example, (B.10) or (B.11) with \( F(y) \) and \( G(u) \) replaced by \( \tilde{F}(y) \) and \( \tilde{G}(u) \). One can also use (5.18) in which \( \tilde{F}(y) \) and \( \tilde{G}(u) \) are replaced by those for the KPZ equations (6.20) and (6.19).

**Proof of theorem 6.3.** We take the KPZ equation limit (the limit \( N \to \infty \) under the scaling (6.4) with (6.21) and (6.22) with \( u \) and \( s \) replaced by \( e^{-\gamma \omega} \) and \( \gamma_1 s \) for (5.43). (Recall that we put tilde on \( u \) and \( y \) in (5.43).) We have

\[
P \left( \frac{h(2\gamma_1^2 y, t) + \tilde{\gamma} \bar{y} + \gamma_1 s}{\gamma_1} \leq s \right) = \frac{\gamma_1}{2\pi i} \int_{\mathbb{C}} \frac{e^{\gamma \alpha t}}{\Gamma(1 + \xi)\Gamma(1 + \xi)} \int_{\mathbb{R}} d\omega e^{-\gamma \omega \xi} \lim_{N \to \infty} \tilde{G}(\tilde{\omega})
\]
\[
= \sum_{n=0}^{\infty} \frac{1}{\gamma_1^{n+1}} \frac{(\xi)}{\Gamma(\nu)} \int_{\mathbb{R}} d\omega \left( \frac{1}{\Gamma(\nu)} \right) \frac{d^{n+1}}{d\omega d^{n+1}} \lim_{N \to \infty} \tilde{F}(\tilde{\gamma} \omega). \tag{6.36}
\]

Combining this with (6.23), we immediately obtain (6.35). \( \square \)

The height distribution for the stationary KPZ equation has already been studied in [17, 18] using the replica method and by [6] using the Macdonald process technique. The first expression in (6.35) is close to that in [6]. The only difference is that \( \gamma_1^{-1} \tilde{G}(e^{-\gamma \omega}) \) (6.27) is replaced by \( \Xi(S, h, \sigma) \) of (2.11) in [6] with \( S = e^{-h\omega}, b = (\omega + y)/\gamma_1 \), and \( \sigma = 1/\gamma_1 \). This comes from the choice of the Fredholm determinant representations. In this paper, we used the formula (4.23) where under the specialization (5.1), the kernel \( f_0(x_1)K(x_1, x_2) \) is expressed as a product of (4.24) and (5.22). On the other hand in [6], the authors take the KPZ equation limit before taking the stationary limit \( \tilde{a} \to \tilde{a} \), i.e. they consider the KPZ equation with the initial
condition $h(x,0) = 1_{x>0}(B_+ + ax) + 1_{x<0}(B_- + cx)$ in place of (6.2), where $B_{\pm}(x)$ are the independent standard Brownian motions and obtained a Fredholm determinant with the kernel (see (2.5) and (2.6) in [6].)

$$K_{\alpha,\beta}(x,y) = \frac{\gamma_i}{(2\pi i)^{2n}} \int dw \int dz \frac{\pi e^{\gamma_i(z-w)}}{\sin \pi(z-w)} e^{(\gamma_i)^2/\pi \gamma_x} \Gamma(\alpha-z) \Gamma(w-a)$$

(6.37)

where $x, y \in \mathbb{R}$ and $z$ and $w$ satisfy $0 < \text{Re}(z-w) < 1/2$ (for more precise information of the contours see theorem 2.7 in [6]). One feature of our kernel $f_d(x_1)K(x_1,x_2)$ is that the function $f_d(x_1)$ is completely separated from $K(x_1,x_2)$ (5.22). On the other hand in (6.37), the information of $f_d(x)$ is included in the factor $\pi e^{\gamma_i(z-w)}/\sin \pi(z-w)$ as one sees from (see (5.38))

$$\frac{\pi e^{\gamma_i(z-w)}}{\sin \pi(z-w)} = \int_\mathbb{R} df e^{-\gamma_i(z-w)}.$$  

(6.38)

In the form of (6.37), it does not seem clear how one can find a simple rank 1 perturbation of the kernel, but one can still calculate a rank three perturbation of the kernel (see section 6 in [18]), from which proposition 2.14 in [6] follows.

On the other hand the other expression in [18] is obtained from the second expression in (6.35) where

$$\hat{F}(z) = \int dw e^{-\gamma_i(z-w)/2} \left( \hat{G}(-e^{-\gamma_i}) - \hat{G}^{(\delta)}(-e^{-\gamma_i}) \right).$$

(6.39)

and $\hat{G}^{(\delta)}(-e^{-\gamma_i})$ is defined in the same way as $\hat{G}(-e^{-\gamma_i})$ (6.27) with $f_{-e^{-\gamma_i}}(\gamma_i \xi) = 1/(1 - e^{\gamma_i(z-w)})$ replaced by $1/(1 - e^{\gamma_i(z-w)}) - \delta(\xi - w)$. As discussed in (5.46) below, we find that $\hat{G}^{(\delta)}(-e^{-\gamma_i})$ is finite even if it includes the delta function term.

Finally, in the large $t$ limit, our formula (6.35) goes to the distribution $F_{\text{BKR}}(z;\omega)$ which was introduced in [2]. (See remark below corollary 5.5.) This can be easily seen by taking the $t \to \infty$ limit of the second relation of (6.35), which has been done in section 6 in [18].

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Appendix A. Two-sided $q$-Whittaker processes

In this appendix, we summarize basic definitions and properties of the two-sided $q$-Whittaker function. For more details, see [16].

A.1. Definitions

The set of $n$-tuples of non-increasing integers, each of which can take both positive and negative value is denoted by

$$S_n := \{ \lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{Z}^n | \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \}.$$  

(A.1)

An element $\lambda \in S_n$ is called a signature. The set of $N(N + 1)/2$-tuples of integers with interlacing conditions,
\[ G_N := \{ (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(n)}), \lambda^{(n)} \in S_n, 1 \leq n \leq N | \lambda^{(m+1)}_{\ell+1} \leq \lambda^{(m)}_{\ell+1} \leq \lambda^{(m)}_{\ell}, 1 \leq \ell \leq m \leq N - 1 \} \]  

is called the Gelfand–Tsetlin cone for signatures. See figure A1. An element of \( \lambda_N \in G_N \) can also be regarded as a point in \( \mathbb{Z}^N \) with the above interlacing conditions.

Next we explain the (skew) \( q \)-Whittaker function labeled by signatures. Hereafter we use the following notations of the \( q \)-Pochhammer symbols. For \( a \in \mathbb{C}, |q| < 1 \) and \( n \in \mathbb{N} \),

\[ (a; q)_n = \frac{1 - q^a}{1 - q}, \quad (a; q)_\infty = \prod_{j=0}^{\infty} (1 - aq^j). \]  

**Definition 7.1.** Let \( \lambda \in S_n, \mu \in S_{n-1} \) be two signatures of length \( n \) and \( n-1 \) respectively and \( a \) be an indeterminate. The skew \( q \)-Whittaker function (with one variable) is defined as

\[ P_{\lambda/\mu}(a) = \sum_{\lambda^{(1)}_{m+1} \leq \cdots \leq \lambda^{(1)}} P_{\lambda^{(0)}/\lambda^{(1)}}(a_j). \]  

Using this, for a signature \( \lambda \in S_N \) and \( N \) indeterminates \( a = (a_1, \cdots, a_N) \), we define the \( q \)-Whittaker function with \( N \) variables as

\[ P_{\lambda}(a) = \prod_{j=1}^{N} P_{\lambda^{(j-1)}/\lambda^{(j)}}(a_j). \]  

Here the sum is over the Gelfand–Tsetlin cone \( G_N \) with the condition \( \lambda^{(N)} = \lambda \).

We also define another function labeled by a signature.

**Definition 7.2.** For a signature \( \lambda \) of length \( N \) (A.1), \( t > 0 \) and \( \alpha = (\alpha_1, \cdots, \alpha_N) \in [0, 1)^N \), we define

\[ Q_{\lambda}(\alpha, t) = \prod_{i=1}^{N} (q^{\lambda_i - \lambda_{i+1} + 1}; q)_{\infty} \int_0^t \prod_{i=1}^{N} \frac{dz_i}{z_i} \cdot P_{\lambda}(1/z) \Pi(z; \alpha, t) m^{(i)}_{\lambda}(z), \]  

![Figure A1. The Gelfand–Tsetlin cone as a triangular array.](image_url)
where $z = (z_1, \cdots, z_N)$ and $1/z = (1/z_1, \cdots, 1/z_N)$ are shorthand notations,

$$m_q(z) = \frac{1}{(2\pi i)^N N!} \prod_{1 \leq i < j \leq N} (z_i/z_j; q)_{\infty} (z_j/z_i; q)_{\infty}$$  \hspace{1cm} (A.7)

is the $q$-Sklyanin measure,

$$\Pi(z; \alpha, t) = \prod_{i=1}^{N} \frac{1}{(\alpha_i/z_i; q)_{\infty}} \prod_{j=1}^{N} e^{z_j t}.$$  \hspace{1cm} (A.8)

Using definitions (A.4) and (A.6), we introduce a measure on $G_N$.

**Definition 7.3.** For $\lambda^N \in G_N$, we define

$$P_t(\lambda^N) = \prod_{j=1}^{N} P_{\lambda_j} \left( \frac{a_j}{\lambda_j} \right) / \lambda_j \left( \frac{a_j}{\lambda_j} - 1 \right).$$  \hspace{1cm} (A.9)

We call this the two-sided $q$-Whittaker process. Furthermore using (A.5), the marginal distribution of $P_t(\lambda^N)$ on $\lambda^N \in S_N$ can be written as

**Proposition 7.4.** For $\lambda \in S_N$, we have

$$P(\lambda^N = \lambda) = \frac{P_{\lambda}(\alpha) Q_{\lambda}(\alpha, t)}{\Pi(a; \alpha, t)}.$$  \hspace{1cm} (A.10)

We call this the two-sided $q$-Whittaker measure.

One of the main results in [16] was that the $q$-Laplace transform for $\lambda^N$ is written as a Fredholm determinant formula.

**Theorem A.5.** For the two-sided $q$-Whittaker measure (A.10) with $0 \leq \alpha_i < a_j \leq 1, 1 \leq i, j \leq N$ and with $\zeta \neq q^n, n \in \mathbb{Z}$,

$$\left\langle \frac{1}{(\zeta q^{\lambda^N}; q)_{\infty}} \right\rangle = \text{det} (1 - fK)_{L^2(Z)},$$  \hspace{1cm} (A.11)

where $\langle \cdots \rangle$ means the average and

$$f(n) = \frac{1}{1 - q^n/\zeta},$$  \hspace{1cm} (A.12)

$$K(m, n) = \sum_{l=0}^{N-1} \phi_l(m) \psi_l(n),$$  \hspace{1cm} (A.13)

$$\phi_l(n) = \int_D dv \frac{e^{-\alpha_i v}}{\alpha_i v + N} \frac{1}{v - a_l + 1} \prod_{j=1}^{I} \frac{v - \alpha_j}{v - a_j} \prod_{k=1}^{N} \frac{(qz_k/v; q)_{\infty}}{(qz_k/a_k; q)_{\infty}},$$  \hspace{1cm} (A.14)

$$\psi_l(n) = (a_{l+1} - \alpha_{l+1}) \int_C dz e^{z/N} \frac{e^{z}}{z - a_{l+1}} \prod_{j=1}^{I} \frac{z - \alpha_j}{z - \alpha_j} \prod_{k=1}^{N} \frac{(qz_k/z_k; q)_{\infty}}{(qz_k/a_k; q)_{\infty}}.$$  \hspace{1cm} (A.15)

Here the contour $D$ is around $\{a_i, 1 \leq i \leq N\}$ and the contour $C_r$ is around $\{0, q^{\alpha_j}, i = 0, 1, 2, \cdots, 1 \leq j \leq N\}$.
Appendix B. Inverse Laplace transforms

B.1. Three versions

For a real function $\varphi(x)$, defined for $x > 0$, the Laplace transform is defined as [33]

$$\tilde{\varphi}(u) = \int_0^\infty e^{-ux} \varphi(x) \, dx, \quad u \in \mathbb{C}. \quad (B.1)$$

The region of $u$ in which the integral converges depends on $\varphi(x)$. A formula to recover the original $\varphi$ from its Laplace transform $\tilde{\varphi}$ is well known,

$$\varphi(x) = \frac{1}{2\pi i} \int_{\mathcal{C}} du e^{ux} \tilde{\varphi}(u), \quad x > 0, \quad (B.2)$$

where $\delta$ should be taken so that the singularities of $\tilde{\varphi}$ are to the left of the integration contour.

In this appendix, we mainly consider the case where $\varphi(x)$ is a (probability) distribution function on $(0, \infty)$. If a random variable having this distribution function $\varphi(x)$ is denoted by $X$, its generating function $G(u) = \langle e^{-uX} \rangle$ is written as

$$G(u) = \int_0^\infty e^{-ux} \varphi(x) \, dx = u \tilde{\varphi}(u). \quad (B.3)$$

When $\varphi(x)$ is a distribution function on $(0, \infty)$, $\tilde{\varphi}(u)$ and hence also $G(u)$ are analytic for $\Re u > 0$. Hence in the inversion formula (B.2), the condition on $\delta$ is simply taken to be $\delta > 0$.

Here we discuss another inversion formula.

**Proposition 8.1.** Let $\varphi(x)$ be a distribution function on $(0, \infty)$, decaying as $\varphi(x) = O(x^a)$, $a > 0$ as $x$ approaches zero. Then we have

$$\varphi(x) = \frac{1}{2\pi i} \int_{\mathcal{C}} du \frac{x^\xi}{\Gamma(\xi + 1)} \int_0^\infty u^\xi \tilde{\varphi}(u) \, du, \quad x > 0, \quad (B.4)$$

where $\delta > a$.

**Remark.** Note that in (B.4) one needs only $\tilde{\varphi}(u)$ for real $u > 0$ to recover the original $\varphi(x)$ whereas in (B.2) the information of $\tilde{\varphi}(u)$ for $\Re u > 0$ is necessary.

**Proof.** First we check the rhs is finite. Since $\varphi(x) = O(x^a)$ as $x \downarrow 0$, $\lim_{x \to \infty} \varphi(x) = 1$ and $\Re \xi > a$, the integral $\int_0^\infty dx \frac{\varphi(x)}{x^{\xi+1}}$ converges. (Note the integral does not converge when $\varphi(x) = O(1/\log x), x \downarrow 0$.) One then writes (for $\xi$ s.t. $\Re \xi > 0$,

$$\Gamma(\xi + 1) \int_0^\infty dx \frac{\varphi(x)}{x^{\xi+1}} = \int_0^\infty dx \varphi(x) \int_0^\infty du u^\xi e^{-ux}$$

$$= \int_0^\infty du u^\xi \int_0^\infty dx e^{-ux} \varphi(x) = \int_0^\infty u^\xi \tilde{\varphi}(u) \, du, \quad (B.5)$$

where for the second equality we used Fubini’s theorem. Hence

$$\text{rhs of (B.4)} = \frac{1}{2\pi i} \int_{\mathcal{C}} du \frac{x^\xi}{\Gamma(\xi + 1)} \int_0^\infty dx_1 \frac{\varphi(x_1)}{x_1^{\xi+1}} = \int_0^\infty dx_1 \frac{\varphi(x_1)}{x_1^{\xi+1}} \frac{1}{2\pi i} \int_{\mathcal{C}} du \frac{x}{x_1} \xi$$

$$= \int_{\mathbb{R}} dy_1 \varphi(e^{y_1}) \frac{1}{2\pi i} \int_{\mathcal{C}} \xi e^{(y_1-\gamma_1)\xi} = \varphi(e^\gamma) = \varphi(x). \quad (B.6)$$

\[\square\]
The Laplace transform $\tilde{\varphi}(u)$ is often analytically continued to the region $\mathbb{C} \setminus \mathbb{R}_-$. One may find an analytic continuation directly from an expression for $\varphi$. There is also a rather general lemma.

**Lemma 8.2 ([23] appendix B).** Suppose $\varphi(x)$ is analytic for $\text{Re} \ x > 0$ and at $x = 0$, and satisfies

$$|\varphi(re^{i\theta})| \leq C(\chi), \ \forall (r, \theta) \in [0, \infty) \times [-\chi, \chi], \quad (B.7)$$

where $\chi \in [0, \pi/2]$ and $C(\chi)$ is a positive non-decreasing function on $[0, \pi/2]$. Then $\tilde{\varphi}(u)$ can be analytically continued to the region $\mathbb{C} \setminus \mathbb{R}_-$. In such a case, we have the following third inversion formula.

**Proposition 8.3 ([1]).** Suppose $\tilde{\varphi}(u)$ satisfies the following.

(i) $\tilde{\varphi}(u)$ is analytic on $\mathbb{C} \setminus \mathbb{R}_-$.

(ii) $\tilde{\varphi}(u) = \tilde{\varphi}(u)$.

(iii) The limiting value $\tilde{\varphi}(\pm t) := \tilde{\varphi}(-t \pm i0), t > 0$ exist and $\tilde{\varphi}(t) = \overline{\tilde{\varphi}(t)}$ holds.

(iv) $\tilde{\varphi}(u) = o(1)$ for $|u| \to \infty$ and $\tilde{\varphi}(u) = o(1/|u|)$ for $|u| \to 0$, uniformly in any sector $|\arg u| < \pi - \eta, \pi > \eta > 0$.

(v) There exists $\epsilon > 0$ s.t. for every $\pi - \epsilon < \phi < \pi$,

$$\frac{\tilde{\varphi}(re^{\pm i\phi})}{1 + r} \in L^1(\mathbb{R}_+), \ \|\tilde{\varphi}(re^{\pm i\phi})\| < a(r), \quad (B.8)$$

where $a(r)$ does not depend on $\phi$ and $a(r)e^{-\delta r} \in L^1(\mathbb{R}_+)$ for any $\delta > 0$.

Then

$$\varphi(x) = \frac{1}{\pi} \int_0^\infty e^{-uy} \text{Im} \tilde{\varphi}_- (t) \, dt, \quad (B.9)$$

Basically the formula is obtained by changing the contour in (B.2) to the one around $\mathbb{R}_-$ and then take the limit to $\mathbb{R}_-$ from both above and below.

Suppose $F(y)$ is a distribution function on $\mathbb{R}$ associated with a random variable $Y$. Then $\varphi(x) = F(\log x), \ x > 0$ is a distribution function on $(0, \infty)$ and the above formulas can be applied. First, combining (B.2) and (B.3) with $x = e^u$, we have

$$F(y) = \frac{1}{2\pi i} \int_{\delta + i\mathbb{R}} \frac{du}{u} e^{uy} G(u), \quad (B.10)$$

where $G(u)$ is written in terms of $Y$ as $G(u) = \langle e^{-uy} \rangle$. Next (B.4) is rewritten as follows. **Corollary 8.4.** For a random variable $Y$, set $G(u) = \langle e^{-uy} \rangle$. The distribution function of $Y$ is recovered from $G(u)$ as

$$F(y) = \frac{1}{2\pi i} \int_{\delta + i\mathbb{R}} d\xi \frac{e^{\xi y}}{\Gamma(\xi + 1)} \int_0^\infty u^{\xi - 1} G(u) \, du, \quad (B.11)$$

The corresponding density function $f(y) = F'(y)$, if it exists, is given by

$$f(y) = \frac{1}{2\pi i} \int_{\delta + i\mathbb{R}} d\xi \frac{e^{\xi y}}{\Gamma(\xi)} \int_0^\infty u^{\xi - 1} G(u) \, du. \quad (B.12)$$
There is an analogous formula for general $n$th derivative, when they exist. Note (B.11) is equivalent to

$$F^\sharp(\xi) := \int_\mathbb{R} dy e^{-y\xi} F(y) = \frac{1}{\Gamma(\xi+1)} \int_0^\infty u^{\xi-1} G(u) du,$$  \hspace{1cm} (B.13)

where $\xi \in i \mathbb{R} + \delta$, $\delta > 0$. For $\delta \to 0$, this is the Fourier transform.

The third Laplace inversion formula (B.9) reads

$$F(y) = 1 - \int_\mathbb{R} dw e^{-e^{-y-w}} \frac{1}{\pi} \lim_{\epsilon \downarrow 0} \Im G(-e^{-w} + i\epsilon).$$  \hspace{1cm} (B.14)

One notices that this is written in a form of convolution including the Gumbel distribution. In the context of the KPZ equation, the distribution function in this form appeared in [24] and an explanation of the appearance of the Gumbel distribution was given in [9]. An advantage of this inversion formula as compared to (B.11) is that whereas the latter still contains the complex integral over $\xi$, in the former all quantities are real. This is a useful property, for example when evaluating the numerical value of the distribution function.

## B.2. Sum of two independent random variables

Let us consider the case in which a random variable $Y$ can be written in a form,

$$Y = Y_0 + \chi,$$  \hspace{1cm} (B.15)

where $Y_0$ and $\chi$ are independent random variables. We discuss a few formulas which give the distribution $F_0(y) := P[Y_0 \leq y]$ in terms of the information on $Y$ and $\chi$. The discussions in this subsection are rather formal. We simply assume that all the quantities have appropriate analytic properties. A most standard approach would be to consider the Fourier transforms. By the independence, $F^\sharp(\xi)$ and $F^\sharp_0(\xi)$, the Fourier transforms of the distribution functions of $Y$ and $Y_0$ respectively (see (B.13)) and $g(\xi) := \langle e^{-\xi \chi} \rangle$, the Fourier transform of the pdf of $\chi$ are related as

$$F^\sharp(\xi) = g(\xi) F^\sharp_0(\xi).$$  \hspace{1cm} (B.16)

Hence the distribution function $F_0$ can be written as

$$F_0(y) = \int d\xi e^{\xi y} F^\sharp_0(\xi) = \int d\xi \frac{e^{\xi y} F^\sharp(\xi)}{g(\xi)}.$$  \hspace{1cm} (B.17)

The formula (5.5) is a result of the combination of this and (B.11). For the distributions themselves, the independence implies

$$F(y) = g \left( \frac{d}{dy} \right) F_0(y).$$  \hspace{1cm} (B.18)

Note that (B.16) is the Fourier transform of this relation. By inverting this, we find

$$F_0(y) = \frac{1}{g} \left( \frac{d}{dy} \right) F(y).$$  \hspace{1cm} (B.19)

This may look a formal expression, but when $1/g(\xi)$ can be Taylor expanded and the resulting series

$$\sum_{n=0}^\infty \frac{1}{n!} \left. \frac{d^n}{dy^n} \left( \frac{1}{g(y)} \right) \right|_{y=0} \cdot F^{(n)}(y)$$  \hspace{1cm} (B.20)
converges, this makes sense. The formula (2.22) in [18] is a result of the combination of this and (B.14).

We can also discuss similar relations for the generating functions. Let us set $Z = e^Y, Z_0 = e^{Y_0}$ and

$$G(u) = \langle e^{-uz} \rangle, \quad G_0(u) = \langle e^{-uz_0} \rangle. \quad \text{(B.21)}$$

By the independence, we have

$$G(e^{w}) = g \left( -\frac{d}{dw} \right) G_0(e^{w}). \quad \text{(B.22)}$$

By inverting this, we find

$$G_0(e^{w}) = \frac{1}{g \left( -\frac{d}{dw} \right) G(e^{w})}. \quad \text{(B.23)}$$

**Appendix C. Proof of lemma 6.2**

First we consider (6.31) by the saddle point analysis, in particular $\psi(\tilde{x})$ (5.24) since we can deal with $\phi(\tilde{x})$ (5.23) in a parallel way. Changing $w = -i\sqrt{N}z$ in (5.24), one has

$$\psi(\tilde{x}) = \frac{\sqrt{N}}{2\pi i} \int_{-i\infty}^{i\infty} dz e^{\xi(z; \tau, \tilde{x})} \frac{\Gamma(1 + \sqrt{N}z) \Gamma(1 + \alpha - \sqrt{N}z)}{\Gamma(1 - \alpha + \sqrt{N}z)}, \quad \text{(C.1)}$$

where

$$f_N(z; \tau, \tilde{x}, l) = -\sqrt{N}z\tilde{x} + N\frac{z^2}{2} + (l - N) \log(\sqrt{N}z) - N \log \Gamma(\sqrt{N}z). \quad \text{(C.2)}$$

Substituting (6.4) with (6.21) and (6.30) into (C.2), we find that the first three terms of (C.2) becomes

$$\begin{align*}
&\quad -\sqrt{N}z\tilde{x} + N\frac{z^2}{2} + (l - N) \log(\sqrt{N}z) \\
&= N^{3/2} \log N \left( z^2 + z \left( -\frac{1/2}{2} - \frac{z}{2} \log t \right) + N \left( \frac{\gamma_2 z^2}{2} - 2\gamma_2 \sqrt{2\gamma_1} \gamma_1^{-1/2} \frac{t^{1/2}}{2} \right) \\
&- N^{1/2} \log N \cdot \frac{\lambda}{2(2\gamma_1)^{1/2}} + N^{1/2} \left( \frac{\gamma_2}{12} z - \gamma_2 \gamma_3 z \gamma_3 - \frac{\lambda}{(2\gamma_1)^{1/2}} \log z \right). \quad \text{(C.3)}
\end{align*}$$

For the last term we use the Stirling formula,

$$\log \Gamma(n) = n \log n - n - \frac{\log 2\pi n}{2} + \frac{1}{12n} + O(n^{-3}) \quad \text{(C.4)}$$

and have

$$\begin{align*}
-N \log \Gamma(\sqrt{N}z) &= -N^{3/2} \log N \cdot \frac{z}{2} + N^{3/2} (z - z \log z) + N^2 \log N \left( N \log \frac{2\pi z}{2} \\
&- N^{1/2} \frac{1}{12z} + O(N^{-1/2}). \quad \text{(C.5)}
\end{align*}$$

Using (C.3) and (C.5), we write $f_N(z; \tau, \tilde{x}, l)$ (C.2) as
\(f_\delta(z; \tau, \tilde{x}, l) = N^{3/2}f_1(z) + Nf_2(z) + N^{1/2}f_3(z) + C_1 + O(N^{-1/2}), \quad (6.6)\)

\(f_1(z) = \frac{t^{1/2}z^2}{2} - z\log z - \frac{z}{2}\log t, \quad (6.7)\)

\(f_2(z) = -\frac{t^{1/2}z}{2} + \frac{\log z}{2} + \gamma_2^2z^2 - 2\gamma_2^2y, \quad (6.8)\)

\(f_3(z) = -\frac{1}{12z} + \left(\frac{\gamma_1^3}{12} - \gamma_2^2y - \gamma_2(\xi - y^2)\right)z - \frac{\lambda}{(2\gamma_1)^{1/2}}\log z. \quad (6.9)\)

Here \(C_1(N)\) is a constant, which does not depend on \(z\), \(C_1 = -N^{1/2}\log N\cdot\frac{\lambda}{2(2\gamma_1)^{1/2}} + \frac{\lambda}{2}\log 2\pi\sqrt{N}\). We note that \(f_1(z)\) above has a double saddle point \(z_c = t^{-1/2}\) such that \(f''_1(z_c) = f''_2(z_c) = 0\). We expand \(f_1(z), f_2(z), f_3(z)\) around \(z_c\). Noting \(f''''(z_c) = 2\gamma_1^3\), \(f''_2(z_c) = 0\), \(f''_2(z_c) = 2\gamma_1^2y - \gamma_2\), \(f_1(z_c) = \gamma_1^2/4 - \gamma_2^2y - \gamma_2(\lambda + \xi - y^2)\), we obtain

\[N^{3/2}f_1(z) = N^{3/2}f_1(z_c) + N^{3/2}\gamma_1^3\left(z - z_c\right)^3 + O(N^{3/2}(z - z_c)^4), \quad (6.10)\]

\[Nf_2(z) = Nf_2(z_c) - N\left(\frac{\gamma_1^3}{2} - \gamma_2^2y\right)\left(z - z_c\right)^2 + O(N(z - z_c)^3), \quad (6.11)\]

\[N^{1/2}f_3(z) = N^{1/2}f_3(z_c) + N^{1/2}\left(\frac{\gamma_1^3}{4} - \gamma_2^2y - \gamma_2(\lambda + \xi - y^2)\right)\left(z - z_c\right) + O(N^{1/2}(z - z_c)^2). \quad (6.12)\]

Here we scale \(z\) around the critical point \(z_c = 1/\sqrt{t}\)

\[z = z_c + \frac{1}{\sqrt{N}}\left(\frac{1}{2} - \frac{1}{\gamma_1}(y + i\sigma)\right). \quad (6.13)\]

Using (6.6)–(6.12) with the fact

\[f_\delta(\tilde{z}; \tau, \tilde{x}, l) = C_1 + C_2 + N^{3/2}f_1(z_c) + Nf_2(z_c) + N^{1/2}f_3(z_c) + O(N^{-1/2}), \quad (6.14)\]

where \(\tilde{z}\) is defined as (6.13) with \(\sigma = 0\), \(C_1\) is defined below (6.9) and

\[C_2 = -\frac{1}{3}\left(y - \frac{\gamma_1^3}{2}\right)^3 + (\xi - \lambda)\left(y - \frac{\gamma_1^3}{2}\right), \quad (6.15)\]

we have

\[f_\delta(z; \tau, \tilde{x}, l) = \frac{i}{3}\sigma^3 + i(\xi + \lambda)\sigma + f_\delta(\tilde{z}; \tau, \tilde{x}, l) + O\left(N^{-1/2}\right). \quad (6.16)\]

At last we evaluate the remaining factor with the Gamma functions in (6.1). Substituting

\[\frac{\Gamma(1 + \sqrt{N})\Gamma(1 + \alpha - \sqrt{N})}{\Gamma(1 - \alpha + \sqrt{N})} = \Gamma\left(\frac{3}{2} - \frac{y + i\sigma}{\gamma_1} + \sqrt{\frac{N}{t}}\right)\Gamma\left(\frac{1 + (i\sigma + y + \omega)/\gamma_1}{1 - (i\sigma + y + \omega)/\gamma_1}\right). \quad (6.17)\]

Thus from (6.1) and (6.16), we find that the limiting form of \(\psi(\tilde{x})\) becomes
\[
\lim_{N \to \infty} \frac{\mathcal{H}(z; \tau, \tilde{x})}{\Gamma(1 + \sqrt{Nz})} \psi(\tilde{x}) = \frac{1}{2\pi i} \frac{\Gamma(1 - \alpha + \sqrt{N})}{\Gamma(1 + \sqrt{Nz})} \Gamma(1 + \alpha - \sqrt{Nz}) \int_{-\infty}^{\infty} \frac{d\sigma}{\zeta - \xi} \left( \frac{1}{1 + \sqrt{Nz}} \right) \Gamma(1 - \alpha - \sqrt{Nz})
\]

which is the second relation in (6.31).

We can show the result of \( \phi(\tilde{x}) \) in (6.31) in a parallel way. By changing \( \nu = \sqrt{Nz} \), \( \phi(\tilde{x}; t) \) (5.23) is rewritten as

\[
\phi(\tilde{x}) = \frac{1}{2\pi i} \int \frac{dz}{\eta - \xi} \frac{\Gamma(1 - \alpha + \sqrt{N})}{\Gamma(1 + \sqrt{Nz})} \Gamma(1 + \alpha - \sqrt{Nz})
\]

where \( f_N(\zeta; \tau, x, l) \) is given in (C.2). Applying the same techniques as the case of \( \psi(\tilde{x}) \), we obtain the first relation in (6.31).

Next we derive (6.32). As in the case of (6.31), we mainly consider \( B_2^{(1)}(\tilde{x}) \) (5.31). From the definition of \( f_N(\zeta; \tau, x, k) \) (C.2), it can be written as

\[
B_2^{(1)}(\tilde{x}) = e^{\mathcal{H}(\alpha/\sqrt{N}, \tau, \tilde{x}, N - 1)} \Gamma(1 + \alpha).
\]

Note that from (6.4), \( \alpha/\sqrt{N} \) is scaled as \( \frac{\alpha}{\sqrt{N}} = \frac{z_c}{\sqrt{N}} \). Comparing this with (C.13), \( f_N(\alpha/\sqrt{N}, \tau, \tilde{x}, N - 1) \) in (C.20) can be estimated by (C.16) with \( \lambda = 0 \) and \( \sigma = i^{(\nu + y)} \) leading to

\[
f_N(\alpha/\sqrt{N}, \tau, \tilde{x}, N - 1) = \frac{1}{3} (\omega + y)^3 - x(\omega + y) + f_N \left( z_c + \frac{1}{2\sqrt{N}} \xi; \tau, \tilde{x}, N - 1 \right) + O(N^{-\frac{1}{2}}).
\]

The second part of (6.32) follows immediately from (C.20) and (C.21). We can also obtain the first part in a similar way.

Third we derive (6.33). We mainly consider the second relation. As with the case (6.31), by the change of the variable \( w = -i\sqrt{Nz} \), \( B_2^{(2)}(\tilde{x}) \) (5.33) can be expressed as

\[
B_2^{(2)}(\tilde{x}) = \frac{\sqrt{N}}{2\pi i} \int_{-i\pi + c}^{i\pi - c} \frac{d\zeta}{\alpha - \sqrt{Nz}} \Gamma(1 + \sqrt{Nz}) \Gamma(1 + \alpha - \sqrt{Nz})
\]

with \( \alpha/\sqrt{N} < c < (\alpha + 1)/\sqrt{N} \). Thus we obtain

\[
\lim_{N \to \infty} \frac{\mathcal{H}(\tau, \tilde{x}; N - 1)}{\Gamma(1 + \sqrt{Nz})} B_2^{(2)}(\tilde{x}) = \frac{1}{2\pi i} \int_{-\pi + ic}^{\pi - ic} \frac{d\sigma}{y + \omega + i\sigma} \left( \frac{1}{1 + \sqrt{Nz}} \right) \Gamma(1 - \alpha - \sqrt{Nz})
\]

with \( y + \omega < c < y + \omega + \gamma \). Using the relation

\[
\frac{1}{y + \omega + i\sigma} = - \int_{0}^{\infty} d\lambda e^{\lambda(y + \omega + i\sigma)},
\]

which is confirmed by \( \text{Re}(y + \omega + i\sigma) < 0 \), we arrive at the second relation of (6.33). The first relation can also be shown in the same way.

At last we consider (6.34). Taking the scalings (6.4) with (6.21) and (6.22) into account, we see that each term in the lhs of (6.34) becomes
\[
(N - 1) \frac{\Gamma'(1 + \alpha)}{\Gamma(1 + \alpha)} = \frac{1}{2} N - 1 \log \frac{N}{t} + \sqrt{N t} \left(1 + \frac{\omega}{\gamma_t} - t \left(\frac{11}{24} + \frac{\omega^2}{2 \gamma_t^2} + \frac{\omega}{\gamma_t} \right) + O(N^{-\frac{1}{2}}),
\]
(C.25)

\[
- \frac{N - 1}{\alpha} = - \sqrt{N t} + t \left(\frac{1}{2} + \frac{\omega}{\gamma_t} \right) + O(N^{-\frac{1}{2}}),
\]
(C.26)

\[
- \alpha \tau = - N + \sqrt{N t} \left(\frac{\sqrt{t} \omega}{\gamma_t} - \frac{2 \gamma_t^2 y}{\sqrt{t}} \right) - 2 \gamma_t^2 y \left(\frac{1}{2} + \frac{\omega}{\gamma_t} \right) + O(N^{-\frac{1}{2}}),
\]
(C.27)

\[
- \log \bar{u} = - \frac{N - 1}{2} \log \frac{N}{t} + N + \sqrt{N t} \left(\frac{\sqrt{t}}{2} + \frac{2 \gamma_t^2 y}{\sqrt{t}} \right) - \gamma_t^2 \frac{t}{12} + \gamma_t^2 y + \gamma_t (s - y^2) - \log u + O(N^{-\frac{1}{2}}),
\]
(C.28)

where in (C.25), we used (C.4). (6.34) follows immediately from (C.25)–(C.28).

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