Solvability of a boundary value problem with a Nagumo Condition

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ABSTRACT
The aim of this paper is to discuss the existence and localization of solutions for a generalized Emden-Fowler equation involving a conformable derivative and with a Dirichlet boundary condition. Our approach is based on the lower and upper solutions method and Schauder fixed-point theorem under a Nagumo condition.

1. Introduction
Recently, Khalil et al. introduced a conformable derivative based on a limit form as in the case of the classical derivative [1]. Since this derivative has properties similar to those of the classical one, it has attracted more attention.

The conformable derivative of order 0 < α < 1, of a function h : [a, ∞) → ℝ is defined by

\[ T_\alpha^uh(t) = \lim_{\epsilon \to 0} \frac{h(t + \epsilon (t - a)^{1-\alpha}) - h(t)}{\epsilon}, \quad t > a. \]

If \( T_\alpha^uh(t) \) exists on \((a, b), b > a\) and \( \lim_{\epsilon \to 0} T_\alpha^uh(t) \) exists, then we define \( T_\alpha^uh(a) = \lim_{\epsilon \to 0} T_\alpha^uh(t) \).

Moreover, if \( h^{(n)} \) exists, the conformable derivative of order \( n < \alpha < n + 1 \) of \( h \), is defined by

\[ T_\alpha^uh(t) = T_\beta^nh^{(n)}(t), \]

where \( \beta = \alpha - n \in (0, 1) \). Later, many papers consider boundary value problems with conformable derivatives are presented [2, 3].

Our interest will be focused upon the problem to prove the existence of solution for the following boundary value problem:

\[ T_\alpha^uh(t) + f(t, u(t), u'(t)) = 0, \quad a < t < b, \quad u(a) = u(b) = 0, \quad (1) \]

where \( T_\alpha^uh \) denotes the conformable derivative of order \( 1 < \alpha < 2 \), \( u \) is the unknown function and \( f : [a, b] \times \mathbb{R}^2 \to \mathbb{R} \) is a given function. To solve this problem, we use the method of upper and lower solutions together with Schauder’s fixed-point theorem. Since the nonlinear term \( f \) depends on the derivative of the function \( u \), we have to find a priori bound for the derivative of the solution. The method of upper and lower solutions is efficient since it gives both the existence and localization of the solution. For some problems applying this method for the study of the existence of solutions, we refer to [4–10].

Some particular cases of problem (P) may represent important problems such as the well-known Emden–Fowler equation with a Dirichlet condition of the form

\[ u'' + a(t)u'(t) = 0, \quad t \in (0, 1), \quad u(0) = u(1) = 0. \]

that appear in many branches of physics and engineering, for example in fluid dynamics, stellar dynamics, quantum mechanics . . . , see [10–12].

Let us recall some essential definitions on conformable derivative.

Let \( n < \alpha < n + 1 \), and set \( \beta = \alpha - n \), for a function \( h : [a, \infty) \to \mathbb{R} \), we denote by

\[ T_\alpha^n h(t) = \int_a^t (s - a)^{n-1} h(s) \, ds, \quad n = 0, \]

and

\[ T_\alpha^n h(t) = \frac{1}{n!} \int_a^t (s - a)^{n-1} h(s) \, ds, \quad n \geq 1. \]

For the properties of the conformable derivative, we mention the following:

Let \( n < \alpha < n + 1 \) and \( h \) be an \((n + 1)\)-differentiable at \( t > a \), then we have

\[ T_\alpha^n h(t) = (t - a)^{n+1-\alpha} h^{(n+1)}(t). \]
and

\[ \rho_a^n T^a_\sigma h(t) = h(t) - \sum_{k=0}^{n} h^{(k)}(a)(t-a)^k. \]

Next, we give a property on the extremum of a function that its conformable derivative exists:

**Proposition 1.1 ([13]):** Let \( 1 < \alpha < 2 \), if a function \( f \in C^1[a,b] \) attains a global maximum (respectively, minimum) at some point \( \xi \in (a,b) \), then \( T^a_\sigma f(\xi) \leq 0 \) (respectively \( T^a_\sigma f(\xi) \geq 0 \)).

### 2. Main results

Let \( AC[a,b] \) be the space of absolutely continuous functions on \( [a,b] \). Denote \( AC^2[a,b] = \{ u \in C^1[a,b], u' \in AC[a,b] \} \) and \( L^1((a,b), \rho(s) ds) \) the Banach space of Lebesgue integrable functions on \( [a,b] \) with respect to the positive weight function \( \rho(s) = (s-a)^{\alpha-2} \in L^1[a,b], 1 < \alpha < 2 \).

The lower and upper solutions for problem (P) are defined as follows.

**Definition 2.1:** The functions \( \underline{\sigma} \) and \( \overline{\sigma} \in AC^2[a,b] \) are called lower and upper solutions of problem (P) respectively, if

(a) \( T^a_\sigma \overline{\sigma}(t) + f(t, \overline{\sigma}(t), \overline{\sigma}'(t)) \geq 0 \),
for all \( t \in [a,b], \overline{\sigma}(a) \leq 0, \overline{\sigma}(b) \leq 0 \).

(b) \( T^a_\sigma \underline{\sigma}(t) + f(t, \underline{\sigma}(t), \underline{\sigma}'(t)) \leq 0 \),
for all \( t \in [a,b], \underline{\sigma}(a) \geq 0, \underline{\sigma}(b) \geq 0 \).

Next, we solve the corresponding linear problem.

**Lemma 2.2:** Assume that \( y \in C[a,b] \), then the linear problem

\[ T^a_\sigma u(t) + y(t) = 0, \quad a < t < b \]

\[ u(a) = u(b) = 0, \]

has a unique solution given by

\[ u(t) = \int_a^b G(t,s) y(s) \rho(s) ds, \]

where

\[ G(t,s) = \frac{1}{(b-a)} \begin{cases} \frac{-(b-a)(t-s)+(b-s)(t-a)}{(b-s)\left(a-t\right)}, & a \leq s \leq t < b, \\ \frac{(t-a)}{(b-s)\left(a-t\right)}, & 0 \leq s \leq t \leq b. \end{cases} \]

**Proof:** Applying the integral \( \rho_a^n \), to both sides of the differential equation (3), it yields

\[ u(t) - u(a) - c(t-a) + \rho_a^n y(t) = 0. \]

Thanks to the boundary condition \( u(a) = 0 \), we get

\[ u(t) = -\rho_a^n y(t) + c(t-a). \]

Since \( u(b) = 0 \), then

\[ c = \frac{1}{(b-a)} \rho_a^n y(t) \bigg|_{t=b} \]

Substituting \( c \) by its value in Equation (6) gives

\[ u(t) = -\rho_a^n y(t) + \frac{(t-a)}{(b-a)} \rho_a^n y(t) \bigg|_{t=b} \]

\[ = -\int_a^b (s-a)\left((s-a)^{\alpha-2} - 1\right) y(s) ds + \]

\[ \frac{(t-a)}{(b-a)} \int_a^b (s-a)^{\alpha-2} y(s) ds \]

\[ = \int_a^b G(t,s) y(s)\rho(s) ds, \]

where the Green function \( G \) is given by Equation (5).

**Lemma 2.3:** The function \( G \) is nonnegative, continuous and satisfies

(1) \( \max_{t \in [a,b]} G(t,s) = (b-a)/4 \),

(2) \( |G'(t,s)| \leq 1 \), for all \( s, t \in [a,b] \).

**Proof:** Putting

\[ g_1(t,s) = -(b-a)(t-s) + (b-s)(t-a), \quad a \leq s \leq t \leq b, \]

\[ g_2(t,s) = (b-s)(t-a), \quad a \leq t \leq s \leq b, \]

we can easily see that the function \( g_1 \) is decreasing and the function \( g_2 \) is increasing with respect to \( t \), consequently

\[ 0 \leq g_1(t,s) \leq (b-s)(s-a) = (b-a)G(s,s), \]

\[ a \leq s \leq t \leq b, \]

\[ 0 \leq g_2(t,s) \leq (b-s)(s-a) = (b-a)G(s,s), \]

\[ a \leq t \leq s \leq b, \]

from which yields

\[ G(t,s) \geq 0, \quad s, t \in [a,b], \]

\[ \max_{t \in [a,b]} G(t,s) = \max_{s \in [a,b]} G(s,s) = \frac{b-a}{4}. \]

Since the nonlinear term \( f \) depends on the derivative of the function \( u \), we have to find a priori bound for the derivative of the solution, for this reason, the function \( f \) should satisfy the following Nagumo condition.
**Definition 2.4:** (Nagumo condition) Let \( \sigma \) and \( \overline{\sigma} \) be the lower and upper solutions of problem (P) such that \( \sigma(t) \leq \overline{\sigma}(t) \), \( \forall t \in [0, 1] \). Consider the set

\[
D = \left\{ (t, x, y) \in [a, b] \times \mathbb{R}^2 / \sigma(t) \leq x \leq \overline{\sigma}(t) \right\}.
\]

A function \( f \in C([a, b] \times \mathbb{R}^2, \mathbb{R}) \) is said to satisfy Nagumo condition on \( D \) if there exists a function \( H \in C(\mathbb{R}^+, (0, +\infty)) \) such that

\[
|f(t, x, y)| \leq (t - a)^{2-\alpha} H(|y|), \quad \forall (t, x, y) \in D, |y| \geq r
\]
and

\[
\int_{t-r}^{t+r} \frac{sds}{H(s)} = \max_{t \in [a, b]} \frac{\overline{\sigma}(t) - \min_{t \in [a, b]} \sigma(t)}{r},
\]
where \( r = \max(\overline{\sigma}(b) - \sigma(a))/(b - a), (\overline{\sigma}(a) - \sigma(b))/(b - a) \geq 0 \).

Now we give the existence result for the nonlinear problem (P).

**Theorem 2.5:** Let \( \sigma \) and \( \overline{\sigma} \) be the lower and upper solutions of problem (P) such that \( \sigma \leq \overline{\sigma} \) and assume that \( f(t, x, y) \) is continuous on \( D \). Then the problem (P) has at least one solution \( u \in AC^2([a, b]) \) such that

\[
\sigma(t) \leq u(t) \leq \overline{\sigma}(t), \quad a < t < b.
\]

**Proof:** Define the modified problem

\[
(MP) \quad \begin{cases}
T_a u(t) + F(t, u(t), u'(t)) = 0, & a < t < b \\
u(a) = u(b) = 0,
\end{cases}
\]
where the function \( F(t, x, y) \) is a modification of \( f(t, x, y) \) associated with the coupled of lower and upper solutions \( \sigma \) and \( \overline{\sigma} \):

\[
F(t, x, y) = \begin{cases}
f^x(t, \overline{\sigma}(t), x) + \frac{\overline{\sigma}(t) - x}{x - \overline{\sigma}(t) + 1}, & \text{for } x > \overline{\sigma}(t), \\
f^x(t, x, y), & \text{for } \sigma(t) \leq x \leq \overline{\sigma}(t), \\
f^x(t, \sigma(t), y) + \frac{\sigma(t) - x}{\overline{\sigma}(t) - x + 1}, & \text{for } x < \sigma(t),
\end{cases}
\]

\[
F^*(t, x, y) = \begin{cases}
f^*(t, x, -R), & \text{if } y < -R, \\
f(t, x, y), & \text{if } -R \leq y \leq R, \\
f(t, x, R), & \text{if } y > R,
\end{cases}
\]

where \( R \) is a positive constant and large enough such that

\[
R \geq \max \left\{ |\overline{\sigma}'(t)|, |\sigma'(t)|, \quad t \in [a, b] \right\}
\]
and

\[
\int_{t}^{R} \frac{sds}{H(s)} > \max_{t \in [a, b]} \sigma(t) - \min_{t \in [a, b]} \sigma(t),
\]
that exists from Equation (8). The proof of Theorem 2.5 will be done in three steps.

**Step 1:** Existence of solution of the modified problem (MP). From the definition of \( F \), we see that it is continuous and bounded, i.e., \(|F(t, x, y)| \leq M\) on \([a, b] \times \mathbb{R}^2\), with

\[
M = \max \{|f(t, x, y)|, (t, x, y) \in D, |y| \leq R\}.
\]
Set \( \Omega = \{u \in C^1[\Omega_1, b], \|u\|_{C^1[\Omega_1, b]} \leq M((b - a)^{\alpha-1}/(\alpha - 1)) \max((b - a)/4, 1)\} \) and define the operator \( A \) on \( C^1[\Omega, b] \) by

\[
Au(t) = \int_{a}^{b} G(t, s) (s - a)^{\alpha-2} F(s, u(s), u'(s)) \ ds,
\]
where \( t \leq b \).

Let us prove that \( A(\Omega) \) is uniformly bounded. For \( u \in \Omega \) and using Lemma 2.3, we get

\[
|Au(t)| \leq \int_{a}^{b} G(t, s) (s - a)^{\alpha-2} |F(s, u(s), u'(s))| \ ds
\]

\[
\leq M \frac{(b - a)^{\alpha}}{4(\alpha - 1)},
\]

\[
|Au'(t)| \leq \int_{a}^{b} G'(t, s) (s - a)^{\alpha-2} |F(s, u(s), u'(s))| \ ds
\]

\[
\leq M \frac{(b - a)^{\alpha-1}}{(\alpha - 1)},
\]

thus \( A(\Omega) \) is uniformly bounded and \( A(\Omega) \subset \Omega \).

Now we prove that \( A(\Omega) \) is equicontinuous. For \( a \leq t_1 < t_2 \leq b \), it yields

\[
|Au(t_1) - Au(t_2)|
\]

\[
\leq (t_2 - t_1) M \int_{t_1}^{t_2} (s - a)^{\alpha-2} \ ds
\]

\[
+ M \int_{t_1}^{t_2} \frac{(t_2 - s)}{(b - a)} (s - a)^{\alpha-2} \ ds
\]

\[
\leq 2 (t_2 - t_1) M \int_{a}^{b} (s - a)^{\alpha-2} \ ds
\]

\[
+ M (b - a) \int_{t_1}^{t_2} (s - a)^{\alpha-2} \ ds \to 0, t_1 \to t_2.
\]

On the other hand, we have

\[
|Au'(t_1) - Au'(t_2)|
\]

\[
= \left| \int_{t_1}^{t_2} (s - a)^{\alpha-2} F(s, u(s), u'(s)) \ ds \right|
\]

\[
\leq M \int_{t_1}^{t_2} (s - a)^{\alpha-2} \ ds \to 0, t_1 \to t_2.
\]

Hence, \( A(\Omega) \) is equicontinuous. Thanks to Arzela-Ascoli's theorem we get that \( A \) is completely continuous. Moreover, by Schauder fixed-point theorem, we conclude that \( A \) has a fixed point \( u \in \Omega \) which is a solution of the modified problem (MP).
Step 2: Localization of the solutions. Let us prove that if \( u \) is a solution of the modified problem (MP), it satisfies

\[
\wbar{\sigma} (t) \leq u (t) \leq \overline{\sigma} (t).
\]

Set \( w = u - \sigma \). Suppose that there exists \( t_0 [a, b] \) such that

\[
\max_{t \in [a, b]} w(t) = w(t_0) > 0,
\]

therefore, if \( t_0 \in (a, b) \), then Proposition 1.1 implies \( T^\alpha a w(t_0) \leq 0 \). Since \( \sigma \) is an upper solution for problem (P), then

\[
T^\alpha a w (t_0) = T^\alpha a u (t_0) - T^\alpha a \sigma (t_0)
\]

\[
= -F(t, u (t_0), u' (t_0)) - T^\alpha a \sigma (t_0)
\]

\[
= -f(t, \sigma (t_0), \sigma' (t_0)) - \frac{\sigma (t_0) - u (t_0)}{u (t_0) - \sigma (t_0) + 1} - T^\alpha a \sigma (t_0) > 0,
\]

that leads to a contradiction. Now, if \( t_0 = a \), we get

\[
w(a) = u (a) - \sigma (a) > 0,
\]

taking into account that \( u(a) = 0 \) then \( \sigma(a) < 0 \), which contradicts the fact that \( \sigma \) is an upper solution of problem (P). By the same way, we get a contradiction if \( t_0 = b \).

Applying similar reasoning, we prove that \( \sigma (t) \leq u (t) \), \( \forall \ t \in [a, b] \).

Step 3: Priorly bound for the derivative of the solution.

Now let us prove that if \( u \) is a solution of the modified problem (MP), then we have

\[
-R \leq u' (t) \leq R, \ \forall \ t \in [a, b].
\]

Let \( u \) be a solution of the modified problem (MP), then it satisfies from step 2 the inequalities \( \sigma (t) \leq u (t) \leq \overline{\sigma} (t) \), \( \forall \ t \in [a, b] \). Suppose that there exists \( t_0 \in [a, b] \) such that \( u'(t_0) > R \). On the other side, by the Mean value theorem and conditions (2), there exists \( \theta \in (a, b) \) such that \( u' (\theta) = (u(b) - u(a))/(b - a) = 0 \). Since the function \( u' \) is continuous then there exist two points \( t_1, t_2 \) between \( t_0 \) and \( \theta \) such that \( u'(t_1) = r, u'(t_2) = R \) and \( u' (t) \geq r, \forall \ t \in [t_1, t_2] \) (or \( [t_2, t_1] \). By the change of variable \( s = u' (t) \) and in view of Equation (7), it yields

\[
\int_{t_1}^{t_2} \frac{sds}{H(s)} = \int_{t_1}^{t_2} \frac{u'(t) u''(t)}{H(u'(t))} dt
\]

\[
= \int_{t_1}^{t_2} \frac{(t - a)^{\alpha-2} u'(t) T^\alpha a u (t)}{H(u'(t))} dt
\]

\[
= -\int_{t_1}^{t_2} \frac{u'(t) F(t, u(t), u'(t))}{(t - a)^{\alpha-2} H(u'(t))} dt
\]

\[
= -\int_{t_1}^{t_2} \frac{u'(t) F(t, u(t), u'(t))}{(t - a)^{\alpha-2} H(u'(t))} dt
\]

\[
\leq \int_{t_1}^{t_2} u'(t) dt
\]

\[
= u(t_2) - u(t_1) \leq \max_{t \in [a, b]} \overline{\sigma} (t) - \min_{t \in [a, b]} \sigma (t)
\]

\[
< \int_{t_1}^{t_2} \sigma (t) dt.
\]

that gives a contradiction, thus \( u'(t) \leq R, \forall \ t \in [a, b] \). Similarly, we show that \( u'(t) \geq -R, \forall \ t \in [a, b] \).

Finally, since the solutions of the modified problem (MP) lie in a region where \( f \) is unmodified i.e. \( \sigma (t) \leq u (t) \leq \overline{\sigma} (t) \) and \( -R \leq u'(t) \leq R \), hence \( u \) is a solution of the problem (P). The proof is completed.

3. Conclusion

We discussed the existence of solutions for differential equations involving a new type of derivative that is the conformable one. An important particular case is the Emden–Fowler equation that appears in various areas of knowledge. We adapted the lower and upper solutions method to this new type of derivative. A far as we know, this approach is original and will provide new insights to further research on the subject.

Acknowledgments

The authors are grateful to an anonymous referee for valuable comments and suggestions, which helped to improve the quality of the paper.

Disclosure statement

No potential conflict of interest was reported by the authors.

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