Spinorial Snyder and Yang Models
From Superalgebras And
Noncommutative Quantum Superspaces

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Abstract

The relativistic Lorentz-covariant quantum space-times obtained by
Snyder can be described by the coset generators of (anti) de-Sitter alge-
bras. Similarly, the Lorentz-covariant quantum phase spaces introduced
by Yang, which contain additionally quantum curved fourmomenta and
quantum-deformed relativistic Heisenberg algebra, can be defined by suit-
ably chosen coset generators of conformal algebras. We extend such alge-
braic construction to the respective superalgebras, which provide quantum
Lorentz-covariant superspaces (SUSY Snyder model) and indicate also
how to obtain the quantum relativistic phase superspaces (SUSY Yang
model). In last Section we recall briefly other ways of deriving quantum
phase (super)spaces and we compare the spinorial Snyder type models
defining bosonic or fermionic quantum-deformed spinors.

1 Introduction

In order to obtain quantum gravity models which reconcile two basic theories
in physics, namely general relativity (GR) and quantum mechanics (QM), it
appears desirable to introduce noncommutative (NC) quantum space-times (see
e.g. [1]-[5]) and look for also the quantum deformations of quantum-mechanical
relativistic phase space algebra with quantum NC fourmomenta (see e.g. [6]-
[9]). Further, because the spinorial variables are even more fundamental that
the vectorial ones (see e.g. quark model of hadrons or Penrose twistor theory)
it is interesting to look for the algebraic systems providing quantum-deformed
spinorial variables.

In this short paper we consider firstly various NC Lorentz-covariant models
of quantum space-times and quantum-deformed relativistic phase spaces, which
originate from the ones proposed firstly in 1947 by Snyder [1] and Yang [6].
These historically first NC structures of quantum space-times and quantum relativistic phase spaces are directly associated with the Lie algebras describing D=4 space-time symmetries which contain $D = 4$ Lorentz algebra as its subalgebra. Further, these models will be naturally extended to supersymmetric ones, with supercharges promoted to fundamental quantum-deformed fermionic spinorial variables.

In standard $D = 4$ Snyder models one introduces NC space-time as described by the coset generators $O(4,1)/O(3,1)$ (dS Snyder model) or $O(3,2)/O(3,1)$ (AdS Snyder model). Each of these two models can be extended in two-fold way into Yang type models describing quantum-deformed relativistic phases

i) dS Snyder model can be embedded into $O(5,1)$ or $O(4,2)$ Yang models, with quantum relativistic phase space variables $(\hat{x}_\mu, \hat{p}_\mu)$ as described by the generators of cosets $O(5,1)/O(3,1)\otimes O(2)$ or $O(4,2)/O(3,1)\otimes O(1,1)$;

ii) AdS Snyder model can be embedded into $O(3,3)$ Yang models with quantum relativistic phase space sector described by the cosets $O(4,2)/O(3,1)\otimes O(2)$ or $O(3,3)/O(3,1)\otimes O(1,1)$.

The aim of this paper is to consider the supersymmetric extensions of four Lie algebras $\hat{o}(4,1), \hat{o}(3,2), \hat{o}(5,1), \hat{o}(4,2)$ and construct respective supersymmetric extensions of Snyder quantum space-times and Yang quantum relativistic phase spaces. For that purpose we will study the superalgebras containing as bosonic subalgebras the spinorial coverings of Lie groups describing $D = 4$ space-time symmetries which we mentioned above:

$$O(4,1) = U(1,1|H) \simeq USp(2,2), \quad O(3,2) = Sp(4|\mathbb{R})$$

$$O(5,1) = SL(2|H) \simeq SU^{\star}(4), \quad O(4,2) = SU(2,2)$$

The way of constructing NC spaces and NC superspaces from Lie-algebraic and Lie-superalgebraic generators firstly proposed with the use of linear formulae by Snyder [1] and Yang [9], was further generalized to nonlinear relations by Madore, Gazeau and Buric (see e.g. [18]-[20]). Promoting the symmetry generators to quantum space-times or quantum phase space variables we will call "Snyderization procedure". For that purpose let us split the (super)Lie algebra $\hat{g}$ into the (super)Riemannian symmetric pair

$$\hat{g} = \hat{k} \oplus \hat{h}$$

1We assume that the NC space-time coordinates and Lorentz $\hat{O}(3,1)$ generators are algebraically independent. Recently such models were studied under the name of extended Snyder models [10],[11]. We add that dS Snyder model is also called "Snyder model" and AdS Snyder model as "anti-Snyder model" (see e.g. [12]).

2$\hat{o}(4,2)$ provides $D = 4$ conformal algebra and $\hat{o}(5,1)$ can be considered as $D = 4$ Euclidean conformal algebra. $\hat{o}(3,3)$ algebra is physically more exotic, represents the conformal extension of $\hat{o}(2,2)$ algebra ($\hat{o}(2,2) = \hat{o}(2,1) \oplus \hat{o}(2,1)$), which appears as a counterpart of Lorentz algebra in models with two times (see e.g. [13],[14]). Further we consider only $\hat{o}(5,1)$ and $\hat{o}(4,2)$ Yang models.

3The N-component quaternionic real spinors can be described as the pair of N-component complex spinors satisfying symplectic SU(2)-Majorana subsidiary conditions [15]-[17].
where $\hat{g} = \hat{k} \oplus \hat{h}$ and

$$[\hat{k}, \hat{k}] \subset \hat{h} \quad [\hat{h}, \hat{k}] \subset \hat{k} \quad [\hat{k}, \hat{h}] \subset \hat{h}. \quad (1.5)$$

The sub(super)algebra $\hat{h}$ enters as the covariance (super)algebra, and $\hat{g}(3,1) \subset \hat{g}$ the generators $\hat{k}$ via Snyderization procedure are defining quantum (super) space coordinates, and to quantum phase (super)space if we are able to embed in $\hat{k}$ some quantum-deformation of relativistic Heisenberg algebra.

We observe that in our Snyderization procedure of superalgebra $\hat{g}$ all super-charges belonging to $\hat{k}$ will be promoted to quantum-deformed spinors, which will transform under the spinorial covering groups (see e.g. (1.1)-(1.2)). In the case of semisimple superalgebras $\hat{g}$ the fermionic odd spinorial generators form an algebraic basis of $\hat{g}$ because for such Lie superalgebras all the bosonic generators can be described as bilinear products of supercharges. Subsequently, in our quantum-deformed covariant superspaces the spinorial quantum coordinates are primary, i.e. they describe the algebraic basis of $U(\hat{g})$. That property inclined us to use the names “spinorial Snyder” or “spinorial Yang” models for those obtained by the Snyderization of supercharges in semisimple Lie superalgebras.

We add that the quantum (super) spaces obtained via described above Snyderization procedure have two important properties:

i) The algebraic construction of quantum (super)spaces inherits from Lie (super)algebras the validity of Jacobi identities i.e. the respective quantum (super)spaces are described by the associative (super)algebras.

ii) Because underlying classical Lie (super)algebras are endowed with Hopf-algebraic structure, the quantum (super)spaces inherit as well the coalgebraic structure, described for classical Lie (super)algebra generators by primitive co-products.

The plan of this paper is the following. Firstly in Sect. 2 we recall briefly the algebraic construction of quantum D=4 Snyder space-time and quantum D=4 Yang relativistic phase spaces. In Sect. 3 we consider two $D = 4$ supersymmetric Snyder models of quantum dS and quantum AdS superspaces. Further, in Sect. 4 we describe three types of quantum Yang phase superspaces (one in $D = 3$, two in $D = 4$), which can be linked with quantum-deformed supersymmetric Heisenberg algebras. In last Section we provide outlook and final remarks.

It should be mentioned that the idea of using Lie (super)algebra relations for description of NC quantum superspaces was already considered earlier in the literature (see e.g. [23]-[24]), but these examples studied in the literature were not providing physically the most interesting physically cases of $D = 4$ Lorentz-covariant quantum superspaces with AdS and dS quantum fermionic spinors.

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4Because superalgebra $\hat{g}$ is split into bosonic and fermionic sector $\hat{g} = \hat{g}(0) \oplus \hat{g}(1)$, by $[\cdot, \cdot]$ we denote graded commutator ($i, j = 0, 1 \text{ mod } 2$)

$$[g(i):g(j)] = g(i)g(j) - (-1)^{ij}g(j)g(i). \quad (1.4)$$

5We will consider here only $D = 4$ relativistic-covariant Lorentz Snyder models; if $\hat{g}(4) \subset \hat{h}$ one gets the class of $D = 4$ Euclidean Snyder models (see e.g. [21]).
describing their odd sectors. One can add however that interest in Snyder and Yang type models is increasing; in particular recently Zoupanos et al [9], [25] applied the ideas based on Yang type models in order to describe the dynamics of $D = 4$ fundamental interactions, with included quantum gravity sector.

2 Snyder quantum space-times and Yang quantum phase spaces

2.1 Snyder quantum space-times

D=4 dS and AdS algebras are described by the following five-dimensional orthogonal algebras ($A=0,1,2,3,4$)

$$[M_{AB}, M_{CD}] = i(\eta_{AD}M_{BC} + \eta_{BC}M_{AD} - \eta_{AC}M_{BD} - \eta_{BD}M_{AC})$$  \hspace{1cm} (2.1)

with signature $\eta_{AB} = \text{diag}(-1, 1, 1, 1, \epsilon)$ and $\epsilon = \eta_{44} = \pm 1$, ($\epsilon = 1$ for dS algebra and $\epsilon = -1$ for AdS algebra). If following the Snyderization procedure we postulate that $M_{\mu 4} = \frac{1}{\lambda} \hat{x}_\mu$, where $\lambda$ is an elementary length in quantum physics given usually by the Planck length $\lambda_p = \sqrt{\frac{\hbar G}{c^3}} \simeq 1.6 \cdot 10^{-33}$ cm, we obtain besides the Lorentz algebra generators $M_{\mu \nu}$ ($\mu = 0, 1, 2, 3$) the relations describing NC Snyder space-times:

$$[M_{\mu \nu}, \hat{x}_\rho] = i(\eta_{\nu \rho} \hat{x}_\mu - \eta_{\mu \rho} \hat{x}_\nu)$$  \hspace{1cm} (2.2)

$$[\hat{x}_\mu, \hat{x}_\nu] = i\epsilon \beta M_{\mu \nu} \hspace{0.5cm} \beta = \lambda^2 > 0$$  \hspace{1cm} (2.3)

The difference between dS and AdS Snyder space-times consists only in difference of sign on rhs of relation (2.3). The algebra with the basis $(M_{\mu \nu}, \hat{x}_\rho)$ introduces algebraically an elementary relativistic quantum system, with D=4 quantum (A)dS space-times $\hat{x}_\mu$ as Lorentz algebra module and Lorentz transformations providing the D=4 relativistic covariance of Snyder equations (2.2, 2.3).

Snyder models (see (2.1, 2.4)) are Born-dual $(\hat{x}_\mu \leftrightarrow \hat{p}_\mu, M_{\mu \nu} \text{ unchanged, } \lambda \rightarrow \frac{1}{\lambda})$ to the momentum space realizations $(M_{\mu \nu}, \hat{p}_\mu)$ of $\hat{o}(4, 1)$ or $\hat{o}(3, 2)$ algebras with generators describing the automorphisms of five-dimensional pseudospheres

$$x^A \eta_{AB} x^B = -x_0^2 + x_1^2 + x_2^2 + x_3^2 + \epsilon x_4^2 = R^2$$  \hspace{1cm} (2.4)

where $M_{\mu A} = R \hat{p}_\mu$ describe the generators of curved translations on the pseudospheres $\frac{O(4,1)}{O(3,1)}$ or $\frac{O(3,2)}{O(3,1)}$ described by (2.4).

In fact Snyder constructed his model as aimed at the description of NC geometry at ultra short (Planckian) distances, in order to regularize geometrically the ultraviolet divergencies in renormalization procedure of quantized
fields. Born duality formalizes a physical as well as some philosophical concept that one can relate the micro and macro world phenomena - the first ones of quantum nature described by NC geometry, and the second linked with classical de-Sitter dynamics of general relativity at very large cosmological distances.

2.2 Yang $D = 4$ quantum phase spaces

Already in 1947 C.N. Yang observed that by considering $D=6$ rotations algebras (i.e. putting in $A = 0, 1, 2, 3, 4, 5$) one can interpret the presence of rotation generators $M_{5\mu}$ as adding to Snyder model the NC fourmomenta $\hat{p}_\mu$. The sixth dimension can be added to dS or AdS Snyder model in two-fold way, by postulating that $\eta_{55} = \epsilon' = \pm 1$. Assuming that $M_{\mu5} = R\hat{p}_\mu$ one gets the following extension of Snyder equations (2.2)-(2.3):

$$[M_{\mu\nu}, \hat{p}_\rho] = i(\eta_{\rho\nu}\hat{p}_\mu - \eta_{\mu\rho}\hat{p}_\nu)$$  \hspace{1cm} (2.5)

$$[\hat{p}_\mu, \hat{p}_\nu] = i\epsilon' \gamma M_{\mu\nu}, \quad \gamma = \frac{1}{R^2}$$  \hspace{1cm} (2.6)

Additionally besides (2.3) and (2.6), one gets the quantum-deformed canonical phase space commutator

$$[\hat{x}_\mu, \hat{p}_\nu] = i\lambda R \eta_{\mu\nu} M_{45} = i\eta_{\mu\nu} \hat{d}$$  \hspace{1cm} (2.7)

with operator-valued substitution of Planck constant $\hbar$ by rescaled generator $\hat{d} = \lambda M_{45}$, which is a $D = 4$ Lorentz scalar ($[M_{\mu\nu}, \hat{d}] = 0$). The generator $M_{45}$ commutes with $\hat{x}_\mu$ and $\hat{p}_\mu$ as follows

$$[\hat{d}, \hat{x}_\mu] = i\epsilon R \hat{p}_\mu$$  \hspace{1cm} (2.8)

$$[\hat{d}, \hat{p}_\mu] = i\epsilon' \lambda R \hat{x}_\mu$$  \hspace{1cm} (2.9)

and describes in $D = 4$ o(2) (if $\epsilon = -\epsilon'$) or o(1, 1) (if $\epsilon = \epsilon'$) internal symmetries.

As we mentioned in Introduction we can obtain in such a way four types of Yang models, with two dychotomic parameters $\epsilon = \pm 1$ (see (2.3)) and $\epsilon' = \pm 1$ (see (2.6)), which could be also called dSdS, dSAdS, AdSdS, AdSAdS Yang models.

3 From D=4 SUSY Snyder model to quantum superspaces

3.1 Quantum $D = 4$ AdS superspace from $D = 4$ SUSY AdS Snyder model

In $o\hat{sp}(1|4)$ superalgebra the generators $M_{\mu\nu}$ and $\hat{x}_\mu = \lambda M_{\mu4}$ form the $D = 4$ AdS $o(2, 3) \simeq Sp(4)$ subalgebra, with the generators $M_{\mu\nu}$ describing its $D = 4$
Lorentz subalgebra. The osp(1|4) superalgebra is obtained by adding to \( D = 4 \) AdS algebra the four additional real Majorana spinor supercharges \( Q_\alpha \) which due to "Snyderization procedure" will be interpreted as quantum NC fermionic real components of \( D=4 \) AdS spinors \( \hat{\xi}_\alpha \). The osp(1|4) superalgebra is described by the following supersymmetric extension of relations (2.1)-(2.3) with \( \beta \to -\beta \) \((\beta = L^2)\)

\[
[M_{\mu\nu}, M_{\rho\sigma}] = i(\eta_{\mu\sigma}M_{\nu\rho} - \eta_{\mu\rho}M_{\nu\sigma} - \eta_{\nu\rho}M_{\mu\sigma} - \eta_{\nu\sigma}M_{\mu\rho}) \\
[M_{\mu\nu}, \hat{x}_\rho] = i(\eta_{\mu\rho}\hat{x}_\nu - \eta_{\nu\rho}\hat{x}_\mu) \\
[\hat{x}_\mu, \hat{x}_\nu] = -i\beta M_{\mu\nu}, \\
\{\hat{\xi}_\alpha, \hat{\xi}_\beta\} = -2(C\gamma^\mu)_{\alpha\beta}\hat{x}_\mu + \beta\hat{\gamma}^\mu (C\gamma^\rho)_{\alpha\beta} M_{\mu\rho} \\
[M_{\mu\nu}, \hat{\xi}_\alpha] = -\frac{i}{2}\hat{\xi}_\beta (\gamma_{\mu\nu})^\beta_\alpha \\
[\hat{x}_\mu, \hat{\xi}_\alpha] = -\frac{i}{2}\beta\hat{\gamma}^\mu (\gamma_\mu)_{\beta\alpha}.
\]

where the quantum spinors \( \hat{\xi}_\alpha \) appearing in the place of supercharges have (see (3.4)) the length dimensionality \([\hat{\xi}_\alpha] = L^{\frac{1}{2}}\). The parameter \( \beta \) is the AdS\(_4\) Planckian length square and \( \gamma_{\mu\nu} \) are \( D=4 \) Dirac \( O(3,1) \) matrices in real Majorana representation; \( \gamma_{\mu\nu} = \frac{1}{2}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu) \). Further, by \( C_{\alpha\beta} = (\gamma_0)_{\alpha\beta} \) we denote the charge conjugation matrix with the properties \( C^T = -C \), \( (\gamma^\mu C^{-1})^T = -\gamma^\mu C^{-1} \). \( C^2 = -1 \)

The superalgebra (3.1)-(3.6) describes graded associative quantum superspace \((\hat{x}^{AdS}_\mu; \hat{\xi}_\alpha)\)

\[
X^{(4,4)}_{AdS} = (\hat{x}^{AdS}_\mu; \hat{\xi}_\alpha|M_{\mu\nu})
\]

where the generators \( M_{\mu\nu} \) describe the Lorentz covariance algebra (see (3.2) and (3.5)). Alternatively, one can Snyderize in osp(1|4) only the supercharges, and obtain in such a way purely spinorial model of Snyder type \((\hat{\xi}_\alpha|M_{AB})\), with anticommuting \( D = 4 \) AdS quantized spinors, covariant under the \( D = 4 \) AdS transformations.

3.2 Quantum \( D = 4 \) dS superspace from \( D = 4 \) SUSY dS Snyder model

For \( D = 4 \) dS algebra \( \hat{o}(4,1) \) the supercharges transform as fundamental spinor realizations of the quaternionic spinorial covering \( O(4,1) = U(1,1; H) \equiv osp(1,2; H) = usp(2,2) \) [10]-[13]. The supersymmetrization of \( D = 4 \) dS algebra requires a pair of quaternionic supercharges, which can be equivalently represented by the pair of four-component complex spinors \( \hat{Q}_\lambda^A(i = 1, 2; A = 1 \ldots 4) \) with their quaternionic structure represented by symplectic SU(2) Majorana condition [15]-[17],[35].
The simple \((N = 1)\) \(D = 4\) dS quaternionic superalgebra is described as the intersection of two complex superalgebras \[u_u(1, 1; |H) = \hat{su}(2, 2) \cap \hat{osp}(4; 2|C)\] (3.8)

with bosonic sector \(\hat{u}(1, 1; |H) \oplus \hat{u}_\alpha(1; |H) \equiv u\hat{sp}(2, 2) \oplus \hat{o}(2)\). Using complex spinors notation the superalgebra (3.8) is described by the following set of (anti)commutators \((A, B = 0, 1, 2, 3, 4; \alpha, \beta = 1, 2, 3, 4; i, j = 1, 2)\)

\[
[M_{AB}, M_{CD}] = i(\eta_{AD}M_{BC} + \eta_{BC}M_{AD} - \eta_{AC}M_{BD} - \eta_{BD}M_{AC})
\]

(3.9)

\[
\{\hat{Q}_\alpha^i, \hat{Q}_\beta^j\} = \delta^{ij}(\Sigma_{AB}C)_{\alpha\beta}M_{AB} + \epsilon^{ij}C_{\alpha\beta}T
\]

(3.10)

\[
[M_{AB}, \hat{Q}_\alpha^i] = -\eta_{AB}(\Sigma_{CD})_{\alpha\beta}\hat{Q}_\beta
\]

(3.11)

\[
[M_{AB}, T] = 0
\]

(3.12)

\[
[T, \hat{Q}_\alpha^i] = -\epsilon^{ij}\hat{Q}_\alpha^j
\]

(3.13)

where \(\eta_{AB} = diag(1, 1, 1, -1), M_{AB}\) are \(\hat{o}(4; 1)\) generators, \(T\) is a scalar internal \(\hat{o}(2)\) symmetry generator and \(\Sigma_{AB} = \frac{1}{2}[\gamma_A, \gamma_B]\) represents the \(4 \times 4\) complex matrix realization of \(\hat{o}(4, 1)\) algebra \([19, 20]\). The complex \(\hat{o}(4, 1)\) Dirac matrices can be chosen for \(A = 0, 1, 2, 3\) as real \((\gamma^\mu_dS = \gamma_\mu)\) and the choice for \(A = 4\) \(\gamma^4_dS = i\gamma_5\) is purely imaginary. The fermionic supercharger \(\hat{Q}_\alpha^i\) satisfy the following quaternionic \(SU(2)\)-symplectic Majorana condition

\[
\hat{Q}_\alpha^i = \epsilon^{ij}(\gamma_5\hat{Q}_j)_\alpha, \quad \hat{Q} = \hat{Q}^\dagger C.
\]

(3.14)

In the Snyderization procedure we replace the generators in the coset \(UU_u(1, 1; |H) \subset \hat{sl}(2; |C)\) by quantum \(D = 4\) dS superspace coordinates as follows

\[
\hat{\psi}_\alpha = \sqrt{1 - \beta^4}\hat{Q}_\alpha^1, \quad \hat{\psi}_\alpha^\dagger = -\sqrt{1 - \beta^4}(\gamma_5\hat{Q}_\alpha^2)_\alpha, \quad \hat{x}_\mu = \beta^4M_\mu,
\]

(3.15)

and we obtain the following superalgebra defining quantum \(D = 4\) dS superspace

\[
[M_{\mu\nu}, M_{\rho\sigma}] = i(\eta_{\mu\rho}M_{\nu\sigma} + \eta_{\nu\rho}M_{\mu\sigma} - \eta_{\nu\sigma}M_{\mu\rho} - \eta_{\mu\sigma}M_{\nu\rho})
\]

(3.16)

\[
[M_{\mu\nu}, \hat{x}_\rho] = i(\eta_{\rho\sigma}\hat{x}_\mu - \eta_{\mu\sigma}\hat{x}_\rho)
\]

(3.17)

\[
[\hat{x}_\mu, \hat{x}_\nu] = i\beta M_{\mu\nu}
\]

(3.18)

\[
\{\hat{\psi}_\alpha, \hat{\psi}_\beta\} = -i(\gamma^\mu C)_{\alpha\beta}\hat{x}_\mu + i\beta(\gamma^{\mu\nu} C)_{\phi\beta}M_{\mu\nu}
\]

(3.19)

\[\text{We use the notation, where } N \text{ denotes the number of 2-component quaternionic } D = 4 \text{ dS supercharges (see e.g. [16, 17]).}\]
\[ \left\{ \hat{\psi}_\alpha^*, \hat{\psi}_\beta \right\} = -i(\gamma^\mu C)_{\alpha\beta} \hat{x}_\mu - i\beta (\gamma^{\mu\nu} C)_{\alpha\beta} M_{\mu\nu} \]  
\[ \left\{ \hat{\psi}_\alpha, \hat{\psi}_\beta^* \right\} = -(\gamma_5)_{\alpha\beta} T \]  
\[ [M_{\mu\nu}, \hat{\psi}_\alpha] = - (\gamma_{\mu\nu})^\beta_\alpha \hat{\psi}_\beta \]  
\[ [\hat{x}_\mu, \hat{\psi}_\alpha] = i\beta \gamma^\mu (\gamma_\mu)^\beta_\alpha \hat{\psi}_\beta \]  
\[ [T, \hat{\psi}_\alpha] = \gamma_5 \hat{\psi}_\alpha \]  

with the length dimensionalities

\[ [M_{\mu\nu}] = 0, \quad [\hat{x}_\mu] = 1, \quad [\hat{\psi}_\alpha] = [\hat{\psi}_\alpha^*] = \frac{1}{2}, \quad [T] = 1. \]  

The superalgebraic relations define the quantum \( D = 4 \) dS superspace

\[ X_{dS}^{(5;4+4)} = (\hat{x}_\mu^{dS}; \hat{\psi}_\alpha, \hat{\psi}_\alpha^* | M_{\mu\nu}, T) \]  

where \( M_{\mu\nu} \) are the Lorentz generators and \( T \) describes the internal \( O(2) \) symmetries, generator \( T \).

It follows from relation (4.31) and traceless \( \gamma_5 \) matrix that by putting \( \alpha = \beta \) in (4.31) one gets \( \sum_{\alpha=1}^4 |\hat{\psi}_\alpha|^2 = 0 \). The nonvanishing quantum spinors can be therefore only realized in Hilbert-Krein space of states with indefinite metric (see e.g. [29],[30]) and local gauging of quaternionic superalgebra [38] leads to \( D = 4 \) dS supergravity with appearing necessarily gauge ghost fields [36].

### 4 From \( D = 4 \) \( N = 2 \) SUSY Yang models to quantum phase superspaces

#### 4.1 Quantum \( D = 3 \) SUSY AdS phase space from \( D = 3 \) AdS Yang model

In order to obtain \( D = 3 \) SUSY AdS Yang model to the \( \hat{o}(3,2) \) generators \( M_{AB} \) (see [27], \( \epsilon = -1 \)) describing \( D = 3 \) AdS Snyder model we add the pair of real \( O(3,2) \) spinorial supercharges \( Q_i^\alpha \) \((i = 1, 2; \alpha = 1 \ldots 4)\) and \( \hat{\omega}(2) \) internal symmetry generator \( T \). The underlying \( \hat{o}\hat{sp}(2;4) \) superalgebra looks as follows (see e.g. [38])

\[ \left\{ Q_\alpha^i, Q_\beta^j \right\} = \delta^{ij} (\gamma^{AB} C)_{\alpha\beta} M_{AB} - \epsilon^{ij} C_{\alpha\beta} T \]  
\[ [M_{AB}, Q_\alpha^i] = - (\gamma_{AB})^\beta_\alpha Q_\beta^i \]  
\[ [T, Q_\alpha^i] = - \epsilon^{ij} Q_\alpha^j \]  

where \( \gamma_{AB} = \frac{1}{2}[\gamma_A, \gamma_B] \) and \( \gamma_A \) denotes the real \( O(3,2) \) Dirac-Majorana matrices. Part of the generators in (4.1)-(4.3) describe the \( D = 3 \) covariance
generators

\[ M_{rs} \oplus M_{34} \oplus T \simeq \hat{o}(2,1) \oplus \hat{o}(1,1) \oplus \hat{o}(2) \]  
\quad (4.4)

and the remaining ones are Snydered as follows

\[ M_{3r} \oplus M_{4r} \oplus \tilde{Q}^1_\alpha \oplus \tilde{Q}^2_\alpha \simeq \frac{1}{\lambda} \hat{x}_r \oplus R\hat{p}_r \oplus \hat{\xi}_\alpha \oplus \hat{\pi}_\alpha \]  
\quad (4.5)

where \((\hat{x}_r, \hat{p}_r)\) and \((\hat{\xi}_\alpha, \hat{\pi}_\alpha)\) describe even and odd canonical pairs of vectorial and spinorial positions and momenta. Denoting \(O(1,1)\) generator \(M_{45} = \frac{\hat{R} \hat{d}}{\lambda}\) we obtain the following SUSY-extended quantum-deformed Heisenberg algebra

i) odd-odd relations

\[ \{\hat{\xi}_\alpha, \hat{\xi}_\beta\} = \{\hat{\pi}_\alpha, \hat{\pi}_\beta\} = \left(\gamma^{rs}\right)_{\alpha\beta} M_{rs} + \frac{1}{\lambda} \left(\gamma^{3r}\right)_{\alpha\beta} \hat{x}_r \] 
\quad + \frac{1}{\lambda} \left(\gamma^{4r}\right)_{\alpha\beta} \hat{p}_r + \left(\gamma^{34}\right)_{\alpha\beta} \hat{d} \]  
\quad (4.6)

\[ \{\hat{\pi}_\alpha, \hat{\xi}_\beta\} = C_{\alpha\beta} T \]  
\quad (4.7)

ii) even-even relations

\[ [\hat{x}_r, \hat{x}_s] = \lambda^2 M_{rs}, \quad [\hat{p}_r, \hat{p}_s] = \frac{1}{R^2} M_{rs} \] 
\quad [\hat{x}_r, \hat{x}_s] = i\eta_{rs} \hat{d} \]  
\quad (4.8)

iii) crossed even-odd relations

\[ [\hat{x}_r, \hat{\xi}_\alpha] = \left(\gamma^{3r}\right)_\alpha \beta \hat{\xi}_\beta \quad [\hat{p}_r, \hat{\xi}_\alpha] = \left(\gamma^{4r}\right)_\alpha \beta \hat{\pi}_\beta \] 
\quad [\hat{x}_r, \hat{\pi}_\alpha] = -\left(\gamma^{3r}\right)_\alpha \beta \hat{\pi}_\beta \quad [\hat{p}_r, \hat{\pi}_\alpha] = -\left(\gamma^{4r}\right)_\alpha \beta \hat{\xi}_\beta. \]  
\quad (4.9)

We see that the relations (4.6)-(4.9) depend on the quantum superspace coordinates (4.5) as well as the covariance symmetry generators (4.4). In order to solve eq. (4.7)-(4.9) one can introduce in place of generators (4.4) their irreducible algebraic or fundamental matrix realizations.

#### 4.2 \( D = 4 \) SUSY AdS phase space from \( N = 2 \) \( D = 4 \) SUSY AdS Yang model

Using \( N = 2 \) superextension of \( D = 4 \) Minkowskian conformal algebra

\[ \hat{o}(4,2) \simeq \hat{s}u(2,2) \xrightarrow{\text{SUSY}} \hat{s}u(2,2) \]  
\quad (4.10)

by Snyderization of the coset \( SU(2,2)/SO(2,2)\times\hat{o}(2)\) generators one can define the spinorial coordinates and momenta which span the odd sector of quantum-deformed supersymmetric Heisenberg algebra. The covariance algebra is described by the following extension of \( D = 4 \) Lorentz algebra \((\mu, \nu = 0, 1, 2, 3)\)

\[ M_{\mu\nu} \rightarrow (M_{\mu\nu}, M_{45}, I, I_r) \simeq \hat{o}(3,1) \oplus \hat{o}(1,1) \oplus \hat{o}(2) \oplus \hat{o}(3) \]  
\quad (4.11)
where \((I, I_r)\) \((r = 1, 2)\) describe the internal \(u(2)\) symmetries commuting with conformal \(o(4, 2)\) algebra and the Snyderization procedure should be applied to the remaining generators in order to introduce the quantum-deformed superspace coordinates:

\[
(M_{3\mu}, M_{4\mu}, \tilde{Q}^a_{\alpha}, \tilde{S}_a^\alpha) \overset{S}{\to} \left(\frac{1}{\lambda}\tilde{x}_\mu, R\tilde{p}_\mu, \tilde{\psi}_\alpha^a, \tilde{\hat{s}}_\alpha^a\right). \tag{4.12}
\]

The fermionic sector of \(\hat{su}(2, 2; 2)\) can be conveniently described by two pairs of four component real Majorana supercharges \(Q^a_\alpha, S^\alpha_a\) \((a = 1, 2; \alpha, \beta = 1 \ldots 4)\), satisfying the following basic superalgebraic relations (we use standard basis \((M_{\mu\nu}, P_\mu, K_\mu, D)\) for the \(D = 4\) conformal algebra generators; see [32]-[33])

\[
\begin{align*}
\{Q^a_\alpha, Q^b_\beta\} &= 2\delta^{ab}(\gamma^\mu C)_{\alpha\beta} P_\mu \\
\{S^a_\alpha, S^b_\beta\} &= -2\delta^{ab}(\gamma^\mu C)_{\alpha\beta} K_\mu \\
\{Q^a_\alpha, S^b_\beta\} &= \delta^{ab}[(\gamma^{\mu\nu} C)_{\alpha\beta} M_{\mu\nu} + 2i(C_{\alpha\beta} D) \\
&\quad + \epsilon^{ab}C_{\alpha\beta}I_2 + i\tau_i^{(ab)}(\gamma_5 C)_{\alpha\beta}I_k]
\end{align*}
\]  \tag{4.13}

where \(k = 0, 1, 2, 3\) and \(2 \times 2\) symmetric matrices \(\tau_k^{(ab)} = (1_2, \sigma_1, \sigma_3)\) and generators \((I = I_0, I_1, I_2, I_3)\) describe internal symmetry algebra \(u(2) \simeq o(2) \oplus o(3)\). In order to reexpress \(P_\mu, K_\mu\) by generators \(M_{3\mu}, M_{4\mu}\) we use before employment of the Snyderization procedure the following formulae

\[
\begin{align*}
M_{4\mu} &= \frac{1}{\sqrt{2}}(RP_\mu + \frac{1}{\lambda}K_\mu) S \frac{1}{\sqrt{2}\lambda} \tilde{x}_\mu \\
M_{5\mu} &= \frac{1}{\sqrt{2}}(RP_\mu - \frac{1}{\lambda}K_\mu) S \frac{R}{\sqrt{2}} \tilde{p}_\mu.
\end{align*}
\]  \tag{4.14}

The supercharges \(\tilde{Q}^a_\alpha, \tilde{S}_a^\alpha\) employed in the Snyderization procedure (see [4.12]) are defined in terms of supercharges \(Q^a_\alpha, S^\alpha_a\) (see [4.13]) as follows \([\tilde{Q}^a_\alpha] = [\tilde{S}_a^\alpha] = \frac{1}{2}\) \(\frac{1}{2}\)

\[
\begin{align*}
\tilde{Q}^a_\alpha &= \frac{1}{\sqrt{2}}(Q^a_\alpha + (\lambda R)^{-\frac{1}{2}} S_a^\alpha) \\
\tilde{S}_a^\alpha &= \frac{1}{\sqrt{2}}(S_a^\alpha - (\lambda R)^{\frac{1}{2}} Q^a_\alpha).
\end{align*}
\]  \tag{4.16}

The algebra [4.13] after using relations [4.14]-[4.16] can be rewritten in term of the supercharger [4.10] in the following way

\[
\begin{align*}
\{\tilde{Q}^a_\alpha, \tilde{Q}^b_\beta\} &= \delta^{ab}[(\gamma^\mu C)_{\alpha\beta} M_{3\mu} + (\gamma^{\mu\nu} C)_{\alpha\beta} M_{\mu\nu}] + \epsilon^{ab}C_{\alpha\beta}I_2 \\
\{\tilde{S}_a^\alpha, \tilde{S}_b^\beta\} &= -\delta^{ab}[(\gamma^\mu C)_{\alpha\beta} \lambda M_{4\mu} + (\gamma^{\mu\nu} C)_{\alpha\beta} M_{\mu\nu}] - \epsilon^{ab}C_{\alpha\beta}I_2 \\
\{\tilde{Q}^a_\alpha, \tilde{S}_b^\beta\} &= -\delta^{ab}C_{\alpha\beta}D + i\tau_i^{(ab)}(\gamma_5 C)_{\alpha\beta}I_k.
\end{align*}
\]  \tag{4.17}

After Snyderization (see [4.12]) we obtain the following fermionic odd-odd sector of algebra [4.17]-[4.19]

\[
\begin{align*}
\{\psi^a_\alpha, \psi^b_\beta\} &= \delta^{ab}[(\gamma^\mu C)_{\alpha\beta} \hat{p}_\mu + (\gamma^{\mu\nu} C)_{\alpha\beta} M_{\mu\nu}] + \epsilon^{ab}C_{\alpha\beta}I_2 \\
\{\pi^a_\alpha, \pi^b_\beta\} &= -\delta^{ab}[(\gamma^\mu C)_{\alpha\beta} \hat{p}_\mu + (\gamma^{\mu\nu} C)_{\alpha\beta} M_{\mu\nu}] - \epsilon^{ab}C_{\alpha\beta}I_2 \\
\{\pi^a_\alpha, \psi^b_\beta\} &= \delta^{ab}C_{\alpha\beta}D + i\tau_i^{(ab)}(\gamma_5 C)_{\alpha\beta}I_k.
\end{align*}
\]  \tag{4.20}
We see that in the above relations besides the "bosonic" and "fermionic" phase space coordinates \((\hat{x}_\mu, \hat{p}_\mu), (\psi^a_\alpha, \pi^a_\alpha)\) enter all the generators of covariance algebra (4.11) (we recall that \(D = M_{45}\)).

4.3 \(D = 4\) SUSY dS phase space from \(D = 4\) SUSY dS Yang model

Following the supersymmetrization of \(\hat{o}(5,1) \simeq sl(2;H)\) algebra, in dS Yang model one should consider the following \(N = 1\) quaternionic (\(N = 2\) complex) superalgebras

\[
sl(2; H) \simeq su^*(4) \xrightarrow{\text{SUSY}} sl(2; 1|H) \simeq su^*(4, 2) \tag{4.23}
\]

with bosonic sector

\[
sl(2; H) \otimes gl(1|H) \simeq su^*(4) \otimes u^*(2) \tag{4.24}
\]

where \(su^*(4) \simeq \hat{o}(5,1)\) and \(u^*(2) = \hat{o}(2) \oplus \hat{o}(2,1)\). We see that due to quaternionic structure in complex notation we will employ for our \(D = 4\) Yang models with complex superalgebra \(su^*(4,2)\).

The superalgebras \(su^*(4;2N)\) can be obtained by so called Weyl trick (see e.g. [31]) from the \(SU(4;2N)\) superalgebra which supersymetrizes \(SU(4) \simeq \hat{o}(6)\). For \(SU(4;2N)\) algebra one can introduce the following \(Z_4\)-grading:

\[
\begin{array}{c}
L_0 \\
USp(4) \oplus USp(2N) \\
U^+ \\
L_1 \\
L_2 \\
L_3 \\
Q^+ \\
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
USp(4) \\
USp(2N)
\end{array} \\
\oplus \\
\oplus \\
\oplus \\
USp(2N)
\end{array}
\]

where (see footnote 4)

\[
[L_r, L_s] = L_{r+s} \mod 4. \tag{4.26}
\]

One passes from \(SU(4;2N)\) to \(SU^*(4;2N)\) by multiplication of the generators from sectors (4.25) according to the following compact formula

\[
SU(4;2N) \rightsquigarrow SU^*(4;2N) \leftrightarrow L_r \rightsquigarrow \exp \left( \frac{ir\pi}{2} \right) L_r. \tag{4.27}
\]

We choose for \(D = 4\) SUSY dS phase space the following covariance algebra (compare with (4.11))

\[
(M_{rs} \oplus M_{45} \oplus I \oplus \hat{I}_r) \simeq \hat{o}(3,1) \oplus \hat{o}(2) \oplus \hat{o}(2) \oplus \hat{o}(2,1). \tag{4.28}
\]

The remaining generators, which include all supercharges, define via Snyderization procedure the \(D = 4\) SUSY dS phase superspace

\[
(M_{34}, M_{44}, z^a_\alpha, u^b_\beta) \xrightarrow{S} \left( \frac{1}{\lambda} \hat{x}_\mu, \hat{R}_\mu, \chi^a_\alpha, \rho^a_\alpha \right) \tag{4.29}
\]

where due to quaternionic \(SU(2)\)-symplectic Majorana condition (see e.g. [3.14]) it follows that independent degrees of freedom are described by \(z_\alpha \equiv z^1_\alpha, \bar{z}_\alpha \sim z^2_\alpha\).
and \( u_\alpha \equiv u_1^\alpha, \bar{u}_\alpha \sim u_2^\alpha \) and after Snyderization by \( \chi_\alpha \equiv \chi_1^\alpha, \bar{\chi}_\alpha \sim \chi_2^\alpha \) and \( \rho_\alpha \equiv \rho_1^\alpha, \bar{\rho}_\alpha \sim \rho_2^\alpha \).

The \( D = 5 \) dS superalgebra is the same as \( D = 4 \) Euclidean conformal superalgebra, which has been studied in explicit form (see e.g. [15], [26], [33]). The fermionic odd-odd sector \( (A = B = 1, 2, \ldots, 5; \Sigma_{AB} = \frac{1}{2}[\gamma_A, \gamma_B]C, \text{where } C \text{ is a charge conjugation matrix}) \)

\[
\{ z_\alpha, \bar{u}_\beta \} = 2[(\Sigma_{AB} \gamma_5)_{\alpha\beta} M^{AB} + C_{\alpha\beta}(I_1 + i I_2)] \tag{4.30}
\]

\[
\{ z_\alpha, \bar{z}_\beta \} = \{ u_\alpha, \bar{u}_\beta \} = 0 \tag{4.31}
\]

\[
\{ z_\alpha, u_\beta \} = i(\gamma_5 C)_{\alpha\beta}(I + \bar{I}_3) \tag{4.32}
\]

where \( I \oplus I_r \) \((r = 1, 2, 3)\) describe the internal symmetry \( O(2) \oplus O(2, 1) \).

After Snyderization given by formulae (4.29) we get the relations for the complex spinors \( \chi_\alpha, \rho_\alpha \) and \( \bar{\chi}_\alpha, \bar{\rho}_\alpha \) depending on the covariance algebra generators (4.28) and bosonic phase space coordinates \( \hat{x}_\mu, \hat{p}_\mu \).

It should be mentioned that putting \( \alpha = \beta \) in relations (4.31) one gets that (see [37]) \( \sum_\alpha |z_\alpha|^2 = \sum_\alpha |u_\alpha|^2 = 0 \) or \( \sum_\alpha |\chi_\alpha|^2 = \sum_\alpha |\rho_\alpha|^2 = 0 \) and one can conclude that after quantization, similarly as in Sect. 3.2, the local gauging of superalgebra \( su^*(4; 2) \) leads to \( D = 5 \) dS supergravity with ghost gauge fields.

5 Outlook

The Snyder dS and AdS models of NC Lorentz-covariant quantum space-time coordinates \( \hat{x}_\mu \) are described by the algebras

\[
O_x(4, 1) \xrightarrow{S} \frac{1}{\lambda} \hat{x}_\mu^{dS} \oplus M_{\mu\nu} \quad O_x(3, 2) \xrightarrow{S} \frac{1}{\lambda} \hat{x}_\mu^{AdS} \oplus M_{\mu\nu} \tag{5.1}
\]

Performing semi-dual Born mapp \( \hat{x}_\mu \leftrightarrow \hat{p}_\mu, \lambda \leftrightarrow \frac{1}{\lambda} \) one gets analogous algebraic structure with Lorentz-covariant quantum NC four-momenta \( \hat{p}_\mu \)

\[
O_p(4, 1) \xrightarrow{S} R\hat{p}_\mu^{dS} \oplus M_{\mu\nu} \quad O_p(3, 2) \xrightarrow{S} R\hat{p}_\mu^{AdS} \oplus M_{\mu\nu} \tag{5.2}
\]

Subsequently in dS and AdS Yang models the pairs of algebras (5.1), (5.2) are embedded in \( D=6 \) pseudo-orthogonal quantum deformed Yang algebras \( \hat{\mathcal{O}}_H(5, 1) \) and \( \hat{\mathcal{O}}_H(4, 2) \), where subindex \( H \) denotes that they contain basic generators \( \hat{x}_\mu, \hat{p}_\mu \) defining quantum-deformed Heisenberg algebras. We have the following two diagrams denoting the chains of subalgebras

\[
\bigcup \hat{\mathcal{O}}_H(5, 1) \bigcup \hat{\mathcal{O}}_H(4, 2) \tag{5.3}
\]

In this paper we introduced in Sect.4 the \( N = 2 \) supersymmetric extensions of \( \hat{\mathcal{O}}_H(5, 1) \) and \( \hat{\mathcal{O}}_H(4, 2) \) algebras, which contain pairs of \( N = 1 \) SUSY Snyder
models, with the same (self-dual) Lorentz algebra sectors and dual choices of Lorentz algebra modules.

In the outlook we would like to comment on some possible directions of future studies, namely:

i) Our method of passing from Snyder models describing quantum space-times to Yang models providing quantum-deformed Heisenberg algebra relations is quite general, what also permits to extend supersymmetrically Yang models in order to obtain quantum-deformed supersymmetric extension of Heisenberg algebra. It should be recalled however the old Snyder idea of adding “by hand” to quantum space-time coordinates \( \hat{x}_\mu \) the commuting fourmomenta \( p_\mu \). In such an axiomatic method we postulate further the general covariant formula for the quantum-deformed basic commutator (for \( \beta \) see (2.3)) [22]

\[
[\hat{x}_\mu, p_\nu] = i\eta_{\mu\nu} F(\beta p^2) + \beta p_\mu p_\nu G(\beta p^2)
\]

and impose all required Jacobi identities. Such a problem posed for \( D = 4 \) AdS Snyder models has a general solution with one of the functions \( F, G \) remaining arbitrary. Such axiomatic approach can be proposed also for supersymmetric extension of Snyder model in order to specify the anticommutator relations for fermionic counterparts \( (\hat{\xi}_\alpha, \hat{\pi}_\alpha) \) of the quantum-deformed Heisenberg algebra basis \( (\hat{x}_\mu, \hat{p}_\mu) \).

ii) If one uses superalgebras for obtaining via Snyderization the spinorial degrees of freedom, the spinors will appear necessarily as Grassmannian, what is desirable in the framework of QFT. One can however employ the ”bosonic” coset of matrix groups with employment of spinorial covering groups - in such a case one gets curved bosonic spinors. A good example is provided by the case of conformal Penrose twistors \((t_A \in \mathbb{C}; A = 1 \cdots 4)\), which are the fundamental representations of \( D = 4 \) conformal ”bosonic” group. Describing twistors by bosonic cosets or as \( N = 1 \) superconformal odd cosets one gets the following two different choices (see e.g. [34])

\[
\begin{align*}
\text{Penrose twistors } t_A^{(B)} & = \frac{SU(2,3)}{SU(2,2) \otimes U(1)} \leftrightarrow \text{Fermionic twistors } t_A^{(F)} = \frac{SU(2,2;1)}{SU(2,2) \otimes U(1)}.
\end{align*}
\]

iii) It is known since eighties [36], [37] that local gauging of \( D = 4 \) and \( D = 5 \) quaternionic dS superalgebras leads to the appearance of gauge ghost fields. Recently however it appeared proposals (see e.g. [39]) that \( D = 4 \) dS supergravity without ghost fields can be obtained by suitably spontaneously broken \( D = 4 \) superconformal gravity. Further algebraic understanding of this idea is still desired.

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9One of the authors (JL) would like to acknowledge that in 2019 using ”axiomatic approach” there were also undertaken with E.A.Ivanov the attempts to construct the spinorial version of Snyder model, however without the use of supersymmetry.
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References

[1] H.S. Snyder, Phys.Rev **71**, 38 (1947)
[2] S. Majid, H. Ruegg, Phys.Lett. **B314**, 348 (1994)
[3] L.J. Garay, Int.J.Mod.Phys. **A10**, 145(1995)
[4] S. Doplicher, F. Fredenhagen, J.E. Roberts, Comm.Math.Phys. **172**, 187(1995); [hep-th/0303037]
[5] S.A. Kempf, Proc. of 36th School od Subnuclear Physics, Erice, Sicily, Sept. 1998 [hep-th/9801021]
[6] C.N. Yang, Phys.Rev. **72**, 874 (1947)
[7] H.-Y. Guo, C.-G. Huang, H.-T. Wu, Phys.Lett. **B663**, 270 (2008); [arXiv:0801.1146 [hep-th]]
[8] J.J. Heckman, H. Verlinde, Nucl.Phys. **B894**, 58(2015); [arXiv:1401.1810 [hep-th]]
[9] G. Manolakos, P. Manousselis, G. Zoupanos, JHEP **08**, 001 (2020); [arXiv:2104.13746 [hep-th]]
[10] S. Meljanac, S. Mignemi, Phys.Rev. **D102**, 12011 (2020)
[11] S. Meljanac, S. Pachol, Symmetry **13**(6), 1055 (2021); [arXiv:2101.02512 [hep-th]]
[12] S. Mignemi, Phys. Rev. **D84**, 025021 (2021); [arXiv:1104.0490 [hep-th]]
[13] Y. Bars, Class.Quant.Grav. **18**, 3113(2001); [hep-th/0008164]
[14] J.M. Romero, A. Zamora, Phys.Rev. **70**, 105006(2004)
[15] T. Kugo, P. Townsend, Nucl. Phys. **B221**, 357 (1983)
[16] J. Lukierski, A. Nowicki, Phys. Lett. **B127**, 40 (1983)
[17] J. Lukierski, L. Rytel, Phys. Rev. **D27**, 2354 (1983)
[18] J. Madore, An Introduction to Noncommutative Differential Geometry and its Physical Applications, CUP (1995)
[19] J. P. Gazeau, J. Mourad, J. Queva, quant-ph/0610222
[20] M. Buric, J. Madore, Eur. Phys. J. C75, 502 (2015)
[21] A. Ballesteros, G. Gubitosi, F.J. Herranz, Class. Quant. Grav. 37 195021 (2020); arXiv:1912.12878 [hep-th]
[22] M.V. Battisti, S. Meljanac, Phys. Rev. D79, 067505(2009); D82 024028 (2010)
[23] M. Hatsuda, S. Iso, H. Umetsu, Nucl.Phys. B671 217 (2003); hep-th/0306251
[24] M. Hatsuda, W. Siegel, Nucl.Phys. B681 152 (2004); hep-th/0311002
[25] G. Manolakos, P. Manousselis, G. Zoupanos, PoS Corfu 2019, 236 (2020)
[26] V.G. Kadyshevsky, JETP14, 1310 (1962)
[27] B.G. Kadyshevski, R.M. Kasimov, N.B. Skachkov, Nuovo Cim. A55, 233 (1968)
[28] J. Lukierski, A. Nowicki, Annals of Physics 166, 164(1986)
[29] K.L. Nagy, State Vector Spaces with Indefinite Metric in QFT, Akademiai K., Budapest (1966)
[30] W. Heisenberg, Indefinite Metric in State Space in W. Blum, H.P. Durr, H. Rechenberg, Collected Works, vol. B p. 904, ed. Springer (1984)
[31] R. Gilmore, Lie Groups, Lie Algebras and Some of Their Applications, John Willey and Sons, Sect. (1974)
[32] S. Ferrara, M. Kaku, P.K. Townsend, P.van Nieuwenhuizen, Nucl. Phys. B129 125 (1977)
[33] J.A. de Azcarraga, J. Lukierski, Phys. Lett. B678, 411 (2009)
[34] J. Lukierski, J.Math.Phys. 21. 561 (1980)
[35] J. Lukierski, A. Nowicki, Fortsch.d.Physik 30, 75 (1982)
[36] K. Pilch, P. Sohnius, P. van Nieuwenhuizen, Comm.Math.Phys. 98, 105 (1985)
[37] J. Lukierski, A. Nowicki, Phys. Lett. B151, 382 (1985)
[38] P. van. Nieuwenhuizen, P. Townsend, Phys.Lett. B67, 439(1977)
[39] E.A. Bergshoeff, D.Z. Freedman, R. Kallosh, A. van Proeyen, Phys. Rev. D92, 085040 (2015); arXiv:1507.08264 [hep-th]