Noncommutative $SU(N)$ and Gauge Invariant Baryon Operator

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Abstract

We propose a constraint on the noncommutative gauge theory with $U(N)$ gauge group which gives rise to a noncommutative version of the $SU(N)$ gauge group. The baryon operator is also constructed.
1 Introduction

Noncommutative geometry of the form

\[ [x^\mu, x^\nu] = i\theta^{\mu\nu}, \]  

has got a lot of interest recently. (See [1, 2, 3] for a comprehensive introduction and an extensive list of references.) Part of the reason is because it appears in a certain corner of moduli space of string [4, 5, 6] and M theory [7, 8] and so cannot be ignored.

Noncommutative gauge theory with gauge group \( U(N) \) has been constructed and analysed quite extensively in the literature. It was first pointed out in [1] that there is an obstacle in the naive way to construct noncommutative gauge theory with gauge group other than \( U(N) \). Since then there had been a number of proposals [9, 10, 11, 12] to construct noncommutative gauge theory with gauge group different from \( U(N) \).

In this letter we propose a construction of noncommutative \( SU(N) \) gauge theory. The construction follows similar ideas as in [9, 11] by imposing a constraint on the noncommutative \( U(N) \) gauge configurations. The constraint selects out the corresponding gauge configurations that we propose to be identified as noncommutative \( SU(N) \) configurations. We also construct a gauge invariant baryon operator.

2 Noncommutative \( SU(N) \)

Consider a noncommutative gauge theory with gauge group \( U(N) \). The action is

\[ S = \frac{1}{4} \int dx \; \text{Tr} F_{\mu\nu} \ast F_{\mu\nu}, \]  

where

\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig(A_\mu \ast A_\nu - A_\nu \ast A_\mu) \]  

and \( g \) is the gauge coupling. The gauge transformation is given by

\[ A_\mu(x) \rightarrow U(x) \ast A_\mu(x) \ast U(x)^\dagger - \frac{i}{g} U(x) \ast \partial_\mu U(x)^\dagger, \]  

where \( U(x) \in U(N) \) with \( U(x) \ast U(x)^\dagger = U(x)^\dagger \ast U(x) = 1 \).

The Wilson line in noncommutative gauge theories is defined by

\[ W(x, C) = P_\ast \exp \left( ig \int_0^1 d\sigma \frac{d\zeta_\mu}{d\sigma} A_\mu(x + \zeta(\sigma)) \right), \]  

where \( \zeta(\sigma) \) is a path in the moduli space.
where $C$ is the curve

$$C = \{ \zeta^\mu(\sigma), 0 \leq \sigma \leq 1 \mid \zeta(0) = 0, \zeta(1) = l \},$$  \hspace{1cm} (6)

and $P_*$ is the path ordering with respect to the star product

$$W(x, C) = \sum_{n=0}^{\infty} (ig)^n \int_0^1 d\sigma_1 \int_1^1 d\sigma_2 \ldots \int_{\sigma_{n-1}}^1 \zeta'_{\mu_1}(\sigma_1) \ldots \zeta'_{\mu_n}(\sigma_n) A_{\mu_1}(x + \zeta(\sigma_1)) \ast \ldots \ast A_{\mu_n}(x + \zeta(\sigma_n)).$$  \hspace{1cm} (7)

It is easy to verify that $W(x, C)$ transforms under a gauge transformation as

$$W(x, C) \rightarrow U(x) \ast W(x, C) \ast U^\dagger(x + l).$$  \hspace{1cm} (8)

The open Wilson line is an important building block for constructing gauge invariant operators \cite{13, 14}. The crucial observation is that in noncommutative geometry \cite{11}, the plane wave $e^{ikx}$ is a translational operator

$$e^{ikx} \ast f(x) = f(x + \theta k) \ast e^{ikx},$$  \hspace{1cm} (9)

therefore one can construct gauge invariant operators with the help of the open Wilson line. Let $\mathcal{O}(x)$ be an operator transforming in the adjoint (e.g. $\text{Tr} F^n$), i.e.

$$\mathcal{O}(x) \rightarrow U(x) \ast \mathcal{O}(x) \ast U^\dagger(x),$$  \hspace{1cm} (10)

then one can introduce

$$\tilde{\mathcal{O}}(k) := \int dx \, \mathcal{O}(x) \ast W(x, C_k) \ast e^{ikx},$$  \hspace{1cm} (11)

where the subscript $k$ of $C_k$ denotes the possible $k$ dependence of the contour. $\tilde{\mathcal{O}}(k)$ is a generalization of the Fourier transform of the operator $\mathcal{O}$ to the noncommutative case. It reduces to the usual Fourier transform in the commutative limit. The tilde reminds us that $\tilde{\mathcal{O}}(k)$ is not exactly the usual Fourier transform of $\mathcal{O}(x)$. It is easy to show that the following momentum space operator

$$\text{Tr} \, \tilde{\mathcal{O}}(k),$$  \hspace{1cm} (12)

is gauge invariant if $C_k$ satisfies the condition

$$l^\mu = \theta^{\mu\nu} k_\nu.$$  \hspace{1cm} (13)

Although (13) is sufficient to guarantee gauge invariance, straight contours play a special role. Then the insertion point for $\mathcal{O}$ on $C_k$ is arbitrary \cite{14} and in addition one has the remarkable identity \cite{15, 16}

$$e^{ik(x - g\theta A(x))} = W(x, C_k) \ast e^{ikx}.$$  \hspace{1cm} (14)

The subscript $\ast$ on the l.h.s. indicates that the exponential is understood as a power series of the star multiplied exponent. The combination $x - g\theta A(x)$ is just the covariant coordinate in the sense of \cite{17}. From now on we always choose straight contours.

To construct noncommutative gauge theory with gauge group $SU(N)$, we can try to follow the approach of \cite{4, 11} by imposing constraints on the gauge configurations $A$ and gauge transformation parameters of the noncommutative $U(N)$. For finding a suitable constraint
to fix a noncommutative version of $SU(N)$ (we denote it from now by $ncSU(N)$ ) it is helpful to recall the reason why simple tensoring $SU(N)$ with the star product fails. If one imposes $\text{Tr}\lambda(x) = 0$ to single out the modified $ncSU(N)$ Lie algebra, it turns out that this condition due to the non-commutativity of the star product does not close, i.e. $\text{Tr}[\lambda(x), \mu(x)]_{*} \neq 0$. A formulation of the same problem in terms of $U(N)$ group elements would impose the vanishing trace condition on the Maurer-Cartan form, i.e. $\text{Tr}(U^\dagger * dU) = 0$. Since $\text{Tr}(U * V)^\dagger * d(U * V) = \text{Tr}(V^\dagger * U^\dagger * dU * V) + \text{Tr}(U^\dagger * dU)$, we see again that the lack of cyclic invariance under the matrix trace prevents $U * V$ to fulfil the constraint if $U$ and $V$ do separately. Cyclic invariance is restored if the total trace with respect to internal indices and the spacetime points is taken.

Now certainly $\int dx \text{Tr}(U^\dagger * dU) = 0$ is too weak and one should look for a localized version, i.e. in the language of Fourier transforms for an extension from momentum $k = 0$ to generic $k \neq 0$. This problem is similar to the one described above in connection with the construction of gauge invariant quantities.

Motivated by these remarks we impose the following constraint on the allowed gauge field configurations:

$$\text{Tr} \tilde{A}(k) := \int dx \text{Tr}(A(x) * W(x, C_k)) * e^{ikx} = 0, \quad \forall k.$$  \hfill (15)

The constraint (15) is a condition on the allowed gauge configurations $A$ of $U(N)$ that can be identified as $ncSU(N)$ configurations. It is the generalization of the traceless condition for the commutative $SU(N)$ gauge fields. Under a gauge transformation, it transforms as

$$\text{Tr} \tilde{A}(k) \rightarrow \text{Tr} \tilde{A}(k) + i \int dx \text{Tr}(U^\dagger(x) * dU(x) * W(x, C_k)) * e^{ikx}. \hfill (16)$$

So in order for (15) to be gauge invariant, we need to impose the condition

$$\int dx \text{Tr}(U^\dagger(x) * dU(x) * W(x, C_k)) * e^{ikx} = 0, \quad \forall k, \quad \text{for } x\text{-dependent } U,$$  \hfill (17)

on the allowed gauge transformations. Note that the allowed gauge transformation $U(x)$ is generally a gauge field dependent gauge transformation. Strictly speaking we should write $U^A$. In the following we will drop the superscript and simply write $U$. This is to be distinguished from the case of noncommutative $U(N)$. In the commutative limit, (17) reduces to the usual traceless condition $\text{Tr}(d\lambda) = 0$ where $U = \exp i\lambda \in SU(N)$. For $x$-independent gauge transformations, the condition (17) gives no extra information and it is natural to consider gauge transformations that are traceless

$$\text{Tr} \lambda = 0, \quad \text{for } x\text{-independent } U = e^{i\lambda}.$$  \hfill (18)

Thus we propose that (13), (17) and (18) together provide a characterization of noncommutative $ncSU(N)$ gauge configurations.

Furthermore, we note the following composition law for our $ncSU(N)$. Denote the constraint (17) as $f(U, A) = 0$ and as $A^U$ the gauge transform of $A$ according to (1), then one has the composition law

$$f(V * U, A) = f(V, A^U) + f(U, A)$$  \hfill (19)
for $ncSU(N)$. This ensures the consistency of imposing (15) and (17) under successive gauge transformations.

Before closing this section we want to comment on the issue of nontrivial solutions for our constraints. Both constraints are understood to be imposed for any $k$. Therefore, an equivalent form of (15) and (17) which no longer contains $k$ would be highly welcome. For the commutative case $\theta = 0$ the Wilson line $W$ is absent and we have the situation of standard Fourier transformation. Vanishing of the Fourier transform of $\text{Tr} A$ for all $k$ is equivalent to $\text{Tr} A = 0$ for all $x$. Our aim is to get for $\theta \neq 0$ a similar pure coordinate space constraint. To this goal let us expand $W$ and further Taylor expand the appearing $A(x+\sigma\theta k)$. All arising factors of $k$ can be thought as generated by differentiations of $e^{ikx}$. Then these differentiations, by partial integrations, will be moved to the remaining factors in the $x$-integral. Under this integral the $\ast$ in front of $e^{ikx}$ can be dropped and we arrive at the standard Fourier transformation of an infinite power series in $A, \theta, \partial$. Performing these manipulations explicitly we find up to $O(\theta^3)$

\[
\text{Tr} \left( A_\nu - g(\theta \partial)\mu (A_\nu \ast A_\mu) - \frac{i}{2} g(\theta \partial)^{\mu_1}(\theta \partial)^{\mu_2}(A_\nu \ast \partial_{\mu_1} A_{\mu_2}) + \frac{1}{2} g^2(\theta \partial)^{\mu_1}(\theta \partial)^{\mu_2}(A_\nu \ast A_{\mu_1} \ast A_{\mu_2}) \right) + O(\theta^3) = 0 .
\]

The corresponding equivalent to (17) is obtained by replacing $A_\nu$ by $U^\dagger \partial_\nu U$ and keeping the $A_\mu$. While obviously all finite order approximations have nontrivial solutions, at this level of discussions it is far from obvious whether the infinite power series allows for solutions.

However one can nevertheless find an argument for the existence of nontrivial solutions. Let us define (A similar construction for quantities transforming in the adjoint has been used in [16].)

\[
\hat{A}(y) = \frac{1}{(2\pi)^D} \int dk \ e^{-iky} \hat{A}(k)
\]

and similar for $U^\dagger \partial_\nu U$. Then $\text{Tr} \hat{A}(k) = 0$ for all $k$ is equivalent to $\text{Tr} \hat{A}(y) = 0$ for all $y$. The map $A \rightarrow \hat{A}$ is a map of coordinate space functions. For $\theta = 0$ it is the identity map hence invertible. Assuming that continuity ensures invertibility also for $\theta \neq 0$ we have nontrivial solutions of our constraints for free.

The vacuum configuration $A = 0$ has a distinguished position. Then $W = 1$ and (17) says the admissible $ncSU(N)$ gauge transformations satisfy

\[
\int dx \ \text{Tr} \ (U^\dagger(x) \ast dU(x)) \ast e^{ikx} = 0 , \ \forall k .
\]

This is equivalent to $\text{Tr} \ (U^\dagger(x) \ast dU(x)) = 0$ and on substituting to (15) implies that then $A = \text{id} U \ast U^\dagger / g$, is also a $ncSU(N)$ configuration.
3 Gauge invariant baryon operator

We start with a commutative gauge theory with a color gauge group $SU(N)$. Let $q^i$ be a set of fermionic fields in the fundamental representations of $SU(N)$. They transform under $SU(N)$ as

$$q^i \rightarrow U^i_j q^j, \quad U \in SU(N).$$

One can introduce the operator

$$M^i_j = q^i \bar{q}^j.$$ 

It transforms as

$$M \rightarrow U M U^\dagger.$$ 

A set of $SU(N)$ gauge invariant operators can be constructed from powers of $M$ as

$$\text{Tr} M^n, \quad n = 1, 2, \ldots.$$ 

In addition, the determinant $\text{Det} M$ can be related to traces of powers of $M$ using the formula

$$\text{Det} M = \sum_{n_1+2n_2+\ldots+Nn_N=N} \epsilon^{(N)}_{n_1 n_2 \ldots n_N} (\text{Tr} M)^{n_1} (\text{Tr} M^2)^{n_2} \ldots (\text{Tr} M^N)^{n_N},$$

which e.g. for $N = 2, 3$ means

$$\text{Det} M = \begin{cases} -\frac{1}{2} \left( \text{Tr} M^2 - (\text{Tr} M)^2 \right), & N = 2 \\ \frac{1}{3} \text{Tr} M^3 - \frac{1}{2} \text{Tr} M \text{Tr} M^2 + \frac{1}{6} (\text{Tr} M)^3, & N = 3. \end{cases}$$

In the commutative case, the standard gauge invariant baryon operator is given by

$$B(x) = \frac{1}{N!} \epsilon^{i_1 i_2 \ldots i_N} q^{i_1}(x) \ldots q^{i_N}(x).$$

The magnitude of this baryon operator $B$ is related to $\text{Det} M$ through the formula

$$\text{Det} M = N! BB^\dagger.$$ 

Up to now we have related the absolute value of the baryon operator to traces of powers of $M$, which transform in the adjoint.

This can be used as a starting point for the definition of the square of the absolute value of a baryon operator in the noncommutative case. We know how to form gauge invariant quantities out of operators transforming in the adjoint. Therefore we define

$$N!BB^\dagger(y) := \sum_{n_1+2n_2+\ldots+Nn_N=N} \epsilon^{(N)}_{n_1 n_2 \ldots n_N} (\text{Tr} \hat{M}(y))^{n_1} (\text{Tr} \hat{M}^2(y))^{n_2} \ldots (\text{Tr} \hat{M}^N(y))^{n_N},$$

We suppress flavor and spin indices.
with
\[ \tilde{M}^j(y) = \frac{1}{(2\pi)^D} \int dk \ e^{-iky} \tilde{M}^j(k). \] (32)

The quantity defined by (26) is gauge invariant under noncommutative $U(N)$ and reproduces $N!BB^\dagger(y)$ with $B(y)$ given by (24) in the commutative limit. The phase of $B$ remains undetermined in the construction just presented. No use has been made of the $ncSU(N)$ constraint.

To proceed with a different construction, we consider Wilson lines for contours $C$ running to infinity, in particular $W(x, C_{\infty})$ with
\[ C_{\infty} = \{ \zeta^\mu(\sigma), 0 \leq \sigma \leq 1 \mid \zeta(0) = \infty, \zeta(1) = 0 \}. \] (33)

They transform under the gauge transformation (4) as
\[ W(x, C_{\infty}) \rightarrow U(\infty)^* W(x, C_{\infty}) U^\dagger(x). \] (34)

For gauge transformation that are trivial at infinity, i.e. $U(\infty) = 1$, this becomes
\[ W(x, C_{\infty}) \rightarrow W(x, C_{\infty}) U^\dagger(x), \] (35)

which effectively is a transformation in the anti-fundamental representation. Using $W(x, C_{\infty})$, one can construct the manifestly gauge invariant combination $W * q$ and use it as the building block for a gauge invariant baryon operator. We define in the $x$-space the following operator
\[ B(x) = \frac{1}{N!} \varepsilon_{i_1 \cdots i_N} (W^{i_1 j_1} * q^{j_1}) * \cdots * (W^{i_N j_N} * q^{j_N}). \] (36)

Note that $B$ is manifestly gauge invariant for all noncommutative $U(N)$ transformations approaching the identity at infinity, and not just for $ncSU(N)$ ones. However, one should nevertheless restrict oneself to noncommutative $SU(N)$ configuration in the definition (36) of $B$. Indeed in the commutative limit,
\[ B(x) = \text{Det} \ W(x, C_{\infty}) \cdot B(x) \] (37)

where $B$ given by (24) is the usual baryon operator. Thus $B = B$ only if $A$ is in commutative $SU(N)$, since then
\[ \text{Det} \ W(x, C_{\infty}) = 1, \quad \text{in the commutative limit.} \] (38)

Just this limiting property is realised if the $ncSU(N)$ constraint (15) is imposed.

We also note that our baryon $B$ is invariant with respect to the rigid, i.e. $x$-independent, $SU(N)$ gauge transformations since constants can be factored out of the star products.

Finally we remark that the construction in this section by parallel transporting the quarks from infinity works so long as the Wilson loop in anti-fundamental representation (34) can be constructed. The correct commutative limit is guaranteed by (38). For example, our construction here can be applied equally well to [10, 12].
4 Discussions

In this paper, we have proposed a definition of noncommutative $SU(N)$. We would like to comment on the relation of our work to other approaches to the construction of noncommutative gauge theories beyond $U(N)$. Working with enveloping algebra valued gauge fields whose components are functions of standard Lie algebra valued fields, noncommutative gauge theories have been constructed for arbitrary Lie algebras [10, 12]. The construction is based on the use of the Seiberg-Witten map. For practical calculations this map is treatable as a power expansion in $\theta$ only. Since our main motivation was the construction of physical relevant gauge invariant quantities in all orders of $\theta$, we followed another approach, namely the use of suitable constraints within noncommutative $U(N)$. Our constraint for $ncSU(N)$ differs qualitatively from those used for the noncommutative versions of $O(N)$ and $Usp(2N)$ [4, 11]. The new feature is the dependence of the gauge transformation constraint on the gauge field. Thinking in terms of covariant coordinates [17], this seems to be a quite generic feature in noncommutative geometry.

Just this property reminds one a little bit on the dependence of the noncommutative gauge transformation both on the commutative one and the commutative gauge field within the Seiberg-Witten map. Nevertheless a sketchy look at the $U(N)$ SW map indicates that, to lowest order in $\theta$, the images of commutative $SU(N)$ fields do not necessarily obey our constraint. This would imply that our noncommutative version of $SU(N)$ is different from the one constructed along the lines of [10, 12].

The consideration in this paper is mostly a classical one. Our constraint is motivated purely from the field theory side. It is possible that the noncommutative $SU(N)$ gauge theory is an effective description of a certain string construction. It would also be interesting to see if there is a dual description in terms of gravity. An interesting related issue is to understand the existence of the baryon vertex operator from the AdS/CFT point of view [14, 18, 19]. However these are completely open for the moment.

On the other hand, one may try to think of the noncommutative $SU(N)$ gauge theory as a quantum theory and try to analyse its quantum properties. As a first step, one needs a correct implementation of the constraint at the quantum level, for example using the Dirac quantization. It would be interesting to perform an one loop analysis similar to those in [20, 21] and clarify its relation to that for noncommutative $U(N)$.

On the more phenomenological level, it may be interesting to adopt our construction of the noncommutative $SU(N)$ and the baryon operator in studying standard model with noncommutative $SU(3) \times SU(2) \times U(1)$ gauge symmetry.

It would also be interesting to generalize our construction to include matters transforming in other nontrivial representations other than the fundamental representation of the noncommutative $SU(N)$ by imposing an appropriate set of constraints.
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