SU(N) polynomial integrals and some applications

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Abstract

We use the method of the Weingarten functions to evaluate SU(N) integrals of the polynomial type. As an application we calculate various one-link integrals for lattice gauge and spin SU(N) theories.

1 Introduction

Analytical methods of integral calculations in the lattice gauge theories (LGTs) had been an important ingredient since the early days of LGT. Many integrals appearing in the strong coupling expansion of the LGT can be found in [1,2]. Important progress relevant for this paper was made in Refs. [3–9] for U(N) integrals. Many U(N) integrals considered here have been studied in Refs. [10–15]. Some integrals for SU(N) group can be found in [1,16,17]. Last decade has seen a thorough development of the theory of the integrals over U(N) group. These integrals are of polynomial type

\[ I_N(r,s) = \int_{U(N)} dU \prod_{k=1}^{r} U_{i_kj_k} \prod_{n=1}^{s} U_{m_n}^* \]  

and can be expressed through summation over permutations of the group indices. The weight of the permutation is given by the Weingarten function whose theory has been developed in a series of papers [18–25]. Precisely, the integral is given by

\[ I_N(r,s) = \delta_{r,s} I_N(r) , \]

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\[ I_N(r) = \sum_{\tau,\sigma \in S_r} Wg^N(\tau^{-1}\sigma) \prod_{k=1}^{r} \delta_{i_k,m_{\sigma(k)}} \delta_{j_k,l_{\tau(k)}}. \]  

(3)

\( S_r \) is a group of permutations of \( r \) elements and \( Wg^N(\sigma) \) is the Weingarten function which depends only on the length of the cycles of a permutation \( \sigma \). Its explicit form is given by

\[ Wg^N(\sigma) = \frac{1}{(r!)^2} \sum_{\lambda \vdash r} \frac{d^2(\lambda)}{s_\lambda(1^N)} \chi_\lambda(\sigma), \]  

(4)

where \( d(\lambda), \chi_\lambda(\sigma) \) are the dimension and the character of the irreducible representation \( \lambda \) of \( S_r \). The irreducible representations \( \lambda \) are enumerated by partitions \( \lambda = (\lambda_1, \lambda_2, \cdots, \lambda_{l(\lambda)}) \) of \( r \), i.e. \( \sum_{i=1}^{l(\lambda)} \lambda_i = |\lambda| = r \), where \( l(\lambda) \) is the length of the partition and \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{l(\lambda)} > 0 \). The sum in (4) is taken over all \( \lambda \) such that \( l(\lambda) \leq N \). \( s_\lambda(X), X = (x_1, x_2, \cdots, x_N) \), is the Schur function \( (s_\lambda(1^N) \) is the dimension of \( U(N) \) representation). To the best of our knowledge, the character expansion of the form (4), valid for \( r \leq N \), had been derived for the first time in [6], where the recurrence relations for \( Wg^N(\sigma) \) were also presented. This result was rederived in [18], again for the case \( r \leq N \). In Ref. [19] it was shown that the result (3) and the expression (4) remains valid also when \( r > N \). In this case the restriction on the length of the partition \( l(\lambda) \) can be omitted. If so, and for \( r > N \) the function \( Wg^N(\sigma) \), as a function of \( N \) will have poles at integers \(-r+1, \cdots, r-1\). These poles will, however cancel after summation over all permutations in (3).

For many applications in the lattice Quantum Chromodynamics (QCD) an extension of the integral \( I(r,s) \) to \( SU(N) \) group is very important. Such integrals appear in the strong coupling expansion of the pure gauge action and are useful in the evaluation of the baryon spectrum. Integrals of this type appear during calculations of the dual representations of some spin models describing effective interaction between Polyakov loops [26, 27] and in principal chiral models [28]. Many one-link integrals in the strong-coupled LGT can be reduced to the computation of (1). These are one-link integrals with the staggered and the Wilson fermions as well as the one-link integrals of the scalar QCD. Most importantly, the knowledge of these integrals is necessary for the construction of the dual formulations of the full lattice QCD [29, 31].

In the paper [31] we outlined a certain approach to the duality transformations based on the Taylor expansion of the Boltzmann weight both for \( U(N) \) spin models and for \( U(N) \) LGTs. Before presenting the details of our calculations and their development we think it would be useful to extend the integration methods to the special unitary group \( SU(N) \). It will help in generalizing the approach of Ref. [31].
to lattice QCD. Actually, the generating functional for SU($N$) integrals $I_N(r,s)$ has been calculated long ago (see the Appendix of the review [2]). Recently, the case $s - r = N$ has been examined in some details in [25]. In this paper we present our results for SU($N$) group integrals of the type (1) for arbitrary $r, s$. In deriving them we use the approach of Ref. [22]. In Section 2 we obtain and describe in details our results for SU($N$) integrals. As the simplest but important applications we compute various one-link and one-site integrals relevant for LGT in Section 3. The results and perspectives are discussed in Section 4. In Appendix we collected some formula used in computation of group integrals.

2 SU($N$) polynomial integrals

In this section we calculate the following integral over SU($N$) group

$$I_N(r, s) = \int_{SU(N)} dU \prod_{k=1}^{r} U_{ikj_k} \prod_{n=1}^{s} U_{mnl_n}^* .$$

(5)

Similar to U($N$) case this integral can be re-written as a sum of the invariant integrals. The product of the matrix elements in the integrand is presented in the form

$$\prod_{k=1}^{r} U_{ikj_k} = \frac{1}{r!} \prod_{k=1}^{r} \frac{\partial}{\partial A_{ikj_k}^*} P_{1r}(A^{\dagger} U) ,$$

(6)

where $P_{1r}(X)$ is the power sum symmetric function [82]. The integral becomes

$$I_N(r, s) = \frac{1}{r!s!} \prod_{k=1}^{r} \frac{\partial}{\partial A_{ikj_k}^*} \prod_{n=1}^{s} \frac{\partial}{\partial B_{mnl_n}} \mathcal{F} ,$$

(7)

where

$$\mathcal{F} = \int_{SU(N)} dU P_{1r}(A^{\dagger} U) P_{1s}(B U^{\dagger}) .$$

(8)

Using Eq.(83) from Appendix the last integral can be brought to

$$\mathcal{F} = \sum_{\lambda \vdash r} \chi_{\lambda}(1^r) \sum_{\mu \vdash s} \chi_{\mu}(1^s) \int_{SU(N)} dU s_{\lambda}(A^{\dagger} U) s_{\mu}(B U^{\dagger}) .$$

(9)

We assume here the partitions $\lambda$ and $\mu$ are those described after Eq.(4). Consider first the following integral

$$\mathcal{F}_0 = \int_{SU(N)} dU s_{\lambda}(U) s_{\mu}(U^{\dagger}) .$$

(10)
Let $s_\lambda(U) = s_\lambda(u_1, \cdots, u_N)$, where $u_i$ are eigenvalues of $U$. Then

$$s_\lambda(U) = \frac{\det u_i^{\lambda_j + N-j}}{\det u_i^{N-j}}. \quad (11)$$

Last two integrals can be written as integrals over $U(N)$ measure with additional constraint

$$\det U = \prod_{i=1}^{N} u_i = 1. \quad (12)$$

Then, it is straightforward to prove

$$F_0 = \sum_{q=-\infty}^{\infty} \prod_{i=1}^{N} \delta_{\lambda_i - \mu_i + q, 0}. \quad (13)$$

Summation over $q$ enforces the constraint $s_\lambda(U) = 1$. We thus have

$$\sum_{i=1}^{N} (\mu_i - \lambda_i) = s - r = Nq, \quad (14)$$

i.e., the difference $s - r$ must be multiple of $N$. This is, of course, a consequence of $Z(N)$ symmetry. Clearly, the same holds for the integral in (9), therefore we can write

$$s_\mu(BU^\dagger) = s_{\lambda+q^N}(BU^\dagger) = s_{q^N}(BU^\dagger) s_\lambda(BU^\dagger) = \left(\det BU^\dagger\right)^q s_\lambda(BU^\dagger), \quad (15)$$

where $\lambda + q^N = (\lambda_1 + q, \cdots, \lambda_N + q)$ and $s_{q^N}(X) = s_{(q, \cdots, q)}(X)$. Last two equalities follow from the representation (11). Suppose that $s \geq r$. Substituting last expression into (9) and performing integration we find

$$\mathcal{F} = \sum_{q=0}^{\infty} \delta_{s-r,Nq} (\det B)^q \sum_{\lambda-\nu} d(\lambda) d(\lambda + q^N) \frac{s_\lambda(A^\dagger B)}{s_\lambda(1^N)}. \quad (16)$$

which results in the following intermediate result

$$\sum_{q=0}^{\infty} \delta_{s-r,Nq} \frac{1}{r!(r + Nq)!} \prod_{\lambda-\nu} \frac{d(\lambda) d(\lambda + q^N)}{s_\lambda(1^N)}$$

$$\times \prod_{k=1}^{r} \frac{\partial}{\partial A_{ik,jk}} \prod_{n=1}^{r+Nq} \frac{\partial}{\partial B_{mnl}} s_{q^N}(B) s_\lambda(A^\dagger B). \quad (17)$$
The equality \((\det B)^q = s_q N(B)\) was used here. To take all the derivatives one employs the relation between the Schur functions and the power sum symmetric functions \(\text{(81)}\). This gives for the second line of \((17)\)

\[
\frac{1}{r!(Nq)!} \sum_{\sigma \in S_r} \sum_{\tau \in S_{Nq}} \chi_\lambda(\sigma) \chi_{q^N}(\tau) \sum_{a_1, \ldots, a_r=1}^N \sum_{b_1, \ldots, b_r=1}^N \sum_{c_1, \ldots, c_{Nq}=1}^N \prod_{k=1}^r \frac{\partial}{\partial A^*_{k,jk}} \prod_{n=1}^{r+qN} \frac{\partial}{\partial \text{B}_{mn} a_n} \prod_{k=1}^r A^*_{bk,a_k} B_{bk,a_{\sigma(k)}} \prod_{n=1}^{Nq} B_{c_n,c_{r(n)}}. \tag{18}
\]

Taking all derivatives and performing summations over matrix indices \(a_i, b_i, c_i\) we finally obtain for \(s \geq r\)

\[
\mathcal{I}_N(r,s) = \sum_{q=0}^\infty \delta_{s-r,Nq} \sum_{\sigma \in S_r} Wg^{N,q}(\sigma) \sum_{\rho \in S_{r+qN}} \prod_{k=1}^r \delta_{i_k,m_{\rho(k)}} \delta_{j_{\rho(k)},l_{\rho(k)}} \times \frac{1}{(Nq)!} \sum_{\tau \in S_{Nq}} \chi_{q^N}(\tau) \prod_{k=1}^{Nq} \delta_{m_{\rho(k)+r},l_{\rho(k)+r}}. \tag{19}
\]

where we defined the \(SU(N)\) Weingarten function as

\[
Wg^{N,q}(\sigma) = \frac{1}{r!(r+qN)!} \sum_{\lambda \vdash r} \frac{d(\lambda) d(\lambda+qN)}{s_{\lambda}(1^N)} \chi_\lambda(\sigma). \tag{20}
\]

When \(r = s\) one restores the result \((2)\). The extension to \(r > s\) is trivial. The second line in \((19)\) can be re-written in an equivalent form through completely anti-symmetric tensors with the help of identity

\[
\sum_{\tau \in S_{Nq}} \chi_{q^N}(\tau) \prod_{k=1}^{Nq} \delta_{m_{\rho(k)+r},l_{\rho(k)+r}} = \frac{1}{(Nq)!} \sum_{\sigma \in S_{Nq}} \epsilon_{m_{\sigma(1)} \cdots m_{\sigma(N)}} \cdots \\
\epsilon_{m_{\sigma((q-1)N+1)} \cdots m_{\sigma(qN)}} \epsilon_{l_{\sigma(1)} \cdots l_{\sigma(N)}} \cdots \epsilon_{l_{\sigma((q-1)N+1)} \cdots l_{\sigma(qN)}}. \tag{21}
\]

While not so compact, this representation is more useful for practical calculations in LGT. The expressions \((19), (20)\) and \((21)\) are main result of this paper. They allow to evaluate many \(SU(N)\) integrals and present the result in a compact form.

Similar result for the generating function

\[
\int_{SU(N)} dU \ (\text{Tr} JU)^r \ (\text{Tr} KU^\dagger)^s \tag{22}
\]
has recently been derived in [32]. In this paper the authors define the $SU(N)$ Weingarten function as

$$\tilde{W}_g^{N,q}(\sigma) = \frac{1}{(r!)^2} \sum_{\lambda \vdash r} \frac{d^2(\lambda)}{s_\lambda(1^{N+q})} \chi_\lambda(\sigma).$$

(23)

The relation between two definitions is given by

$$\prod_{k=0}^{N-1} \frac{k!}{(k+q)!} \tilde{W}_g^{N,q}(\sigma) = W_g^{N,q}(\sigma).$$

(24)

With this relation taken into account our result for the generating functional [22] agree with that of [32].

We finish this section with a conjecture about large-$N$ asymptotic behaviour of the $SU(N)$ Weingarten function. In case of the $U(N)$ group it is well-known and the leading term of the asymptotic expansion reads [19]

$$W_g^{N}(\sigma) = \frac{1}{N^r} \left( \prod_k \delta_{\sigma(k),k} + \mathcal{O}(N^{-1}) \right),$$

(25)

if $\sigma \in S_r$. The simple way to obtain the leading term for the $SU(N)$ group is to consider the integral [29] (see next section). Then, one can readily find

$$(r + Nq)! \sum_{\sigma \in S_r} W_g^{N,q}(\sigma)N^{N\sigma} = Q_N(r,s).$$

(26)

Here, $|\sigma|$ is a number of cycles in the permutation $\sigma$ and $Q_N(r,s)$ is given in Eq.(31). Taking asymptotics on the right-hand side of the last expression leads to the equality

$$(r + Nq)! \sum_{\sigma \in S_r} W_g^{N,q}(\sigma)N^{N|\sigma|} = \frac{(r + Nq)!}{(N!)^q} \left( 1 + \mathcal{O}(N^{-1}) \right).$$

(27)

If we suppose that, similar to $U(N)$ group, the leading contribution to the large-$N$ expansion comes from the identity permutation $\sigma(k) = k$ we obtain

$$W_g^{N,q}(\sigma) = \frac{1}{N^r(N!)^q} \left( \prod_k \delta_{\sigma(k),k} + \mathcal{O}(N^{-1}) \right).$$

(28)

Note, that $N^r(N!)^q \sim N^{r+Nq} = N^s$. 

6
3 Applications

In this section we apply the formula (19) for the evaluation of various $SU(N)$ integrals which often appear in some spin models and LGTs.

3.1 Invariant integrals

The simplest integrals which can be calculated with the help of (19) are those whose integrands depend only on the traces of the group elements

$$Q_N(r, s) = \int_G dU \ (\text{Tr}U)^r \ (\text{Tr}U^*)^s ,$$

(29)

where $G = U(N), SU(N)$. Such integrals appear in the effective spin models describing the interaction of the Polyakov loops in the finite-temperature LGTs. Writing the traces as

$$\text{Tr}U^s = \sum_{i_1=1}^{N} \cdots \sum_{i_s=1}^{N} \prod_{k=1}^{s} U_{i_k i_k}$$

(30)

one obtains the integral of the type (1) or (5). Using (3) and (19) one finds after summation over group indices

$$Q_N(r, s) = \begin{cases} 
\delta_{r,s} \sum_{\lambda \vdash \min(r,s)} d^2(\lambda) , & G = U(N) , \\
\sum_{q=-\infty}^{\infty} \delta_{s-r,qN} \sum_{\lambda \vdash \min(r,s)} d(\lambda) \ d(\lambda + |q|^N) , & G = SU(N) .
\end{cases}$$

(31)

Of course, the simpler way to compute (29) is to use the expansion $\text{Tr}U^s = (u_1 + u_2 + \cdots + u_N)^s = \sum_{\lambda \vdash s} d(\lambda)s(\lambda(U))$. Then, the result (31) follows from the orthogonality of the Schur functions.

The next example is provided by the coefficients of the character expansion of an invariant function $H(x, y)$

$$C_\lambda(a, b) = \int_G dU \ s_\lambda(U) \ H\left(\frac{a}{2} \text{Tr}U, \frac{b}{2} \text{Tr}U^*\right) .$$

(32)

For the most common case $H(x, y) = e^{x+y}$ the result is well known for many decades and reads for $G = SU(N)$

$$C_\lambda(a, b) = \left(\frac{b}{a}\right)^{\frac{1}{2} |\lambda|} \sum_{m=-\infty}^{\infty} \left(\frac{b}{a}\right)^{\frac{1}{2} Nm} \det I_{\lambda, -i+j+m}(\sqrt{ab})_{1 \leq i,j \leq N} ,$$

(33)
where $|\lambda| = \sum_i \lambda_i$. Only term with $m = 0$ contributes for $U(N)$ group. An alternative expression for the coefficients $C_\lambda$ can be obtained with the help of (19). First, using the Taylor expansion of the function $H(x, y)$, the Eq. (32) is written as

$$C_\lambda(a, b) = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} h(r, s) \frac{(a/2)^r}{r!} \frac{(b/2)^s}{s!} \int_G dU \ s_\lambda(U) (\text{Tr}U)^r (\text{Tr}U^\dagger)^s,$$  \hspace{1cm} (34)

where $h(r, s) = \left. \frac{\partial^{r+s} H(a, b)}{\partial a^r \partial b^s} \right|_{a=b=0}$. The last integral can again be presented as an integral over the power sum symmetric functions following Eqs. (6)-(8) and using the relations (81), (82). This leads to the computation of the integral of the form

$$\int_G dU \ P_\sigma(U) P_1(A U) P_1(B U^\dagger), \sigma \in S_{|\lambda|}, |\lambda| = \sum_i \lambda_i.$$  \hspace{1cm} (35)

Repeating all the calculations from the previous section for the last integral we end up with the following result

$$C_\lambda(a, b) = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} h(r, s) \frac{(a/2)^r}{r!} \frac{(b/2)^s}{s!} \sum_{\mu \vdash r} \sum_{\nu \vdash s} C_{\nu \lambda \mu}(G) d(\mu) d(\nu).$$  \hspace{1cm} (36)

The Littlewood-Richardson coefficients $C_{\nu \lambda \mu}(G)$ appear in the final answer in the form

$$C_{\nu \lambda \mu}(U(N)) = \frac{1}{r!s!} \sum_{\sigma \in S_r} \sum_{\tau \in S_s} \chi_\lambda(\sigma) \chi_\mu(\tau) \chi_\nu([\sigma : \tau]),$$  \hspace{1cm} (37)

for $G = U(N)$ and

$$C_{\nu \lambda \mu}(SU(N)) = \sum_{q=0}^{\infty} (Nq)! \frac{d(\nu + q|N)}{s_\nu(1^N)} \sum_{\rho \vdash s+Nq} \frac{s_\rho(1^N)}{d(\rho)} C_{\nu \lambda \mu}(U(N)) C_{\rho \mu}(U(N)).$$  \hspace{1cm} (38)

for $G = SU(N)$. The permutation $[\sigma : \tau] \in S_{r+s}$ is defined as

$$[\sigma : \tau](k) = \begin{cases} \sigma(k), & k \leq r, \\ \tau(k-r) + r, & r < k \leq r+s. \end{cases}$$  \hspace{1cm} (39)

The constraints on partitions $r + |\lambda| = s$ for $U(N)$ and $r + |\lambda| = s + Nq$ for $SU(N)$ are evident from the Littlewood-Richardson coefficients. The expression (36) can be further simplified using the summation formula

$$\sum_{\mu \vdash r} C_{\nu \lambda \mu}(U(N)) d(\mu) = d(\nu/\lambda).$$  \hspace{1cm} (40)

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Here, \( d(\nu/\lambda) \) is the dimension of a skew representation defined by a corresponding skew Young diagram

\[
d(\nu/\lambda) = \frac{1}{|\lambda|!} \sum_{\sigma \in S_{|\lambda|}} \chi_\nu([I : \sigma]) \chi_\lambda(\sigma) .
\] (41)

Then, e.g. for \( U(N) \) we get a simple expression

\[
C_\lambda(a,b) = \sum_{r=0}^{\infty} h(r, r + |\lambda|) \frac{(ab/4)^r}{r!} \frac{(b/2)^{|\lambda|}}{(r + |\lambda|)!} \sum_{\nu \vdash r + |\lambda|} d(\nu/\lambda) \ d(\nu)
\] (42)

and similar formula can be written down for \( SU(N) \) if one uses Eqs. (36), (38).

### 3.2 One-link integrals

Consider the following integral over \( G = U(N), SU(N) \) with arbitrary \( N \times N \) matrices \( A, B \)

\[
Z_G(A, B) = \int_G dU \ \exp \left[ \frac{\beta}{2} (\text{Tr} AU + \text{Tr} BU^\dagger) \right] .
\] (43)

Expanding the integrand in the Taylor series one gets

\[
Z_G(A, B) = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(\beta/2)^{r+s}}{r! s!} \int_G \text{d}U \ (\text{Tr} AU)^r \ (\text{Tr} BU^\dagger)^s .
\] (44)

This can be presented as

\[
Z_G(A, B) = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(\beta/2)^{r+s}}{r! s!} \sum_{i_1 \cdots i_r} \sum_{j_1 \cdots j_r} \sum_{m_1 \cdots m_s} \sum_{l_1 \cdots l_s} \prod_{k=1}^{r} A_{i_k j_k} \prod_{n=1}^{s} B_{l_n m_n} \ I_N(r, s) .
\] (45)

Using the result of the previous section (19) for the group integral we can calculate all summations over group indices and over all permutations. All necessary formulae to do this are given in the Appendix. We find for \( G = SU(N) \)

\[
Z_{SU(N)}(A, B) = \sum_{s=0}^{\infty} \sum_{q=-\infty}^{\infty} \frac{1}{s!} \left( \frac{\beta}{2} \right)^{2s+N|q|} (\text{det} C)^{|q|} \sum_{\sigma \in S_s} W g_{\sigma}^{N,4}(\sigma) P_{\sigma}(AB) .
\] (46)

With the help of (30) and (33) this can be re-written as

\[
Z_{SU(N)}(A, B) = \sum_{s=0}^{\infty} \sum_{q=-\infty}^{\infty} \left( \frac{\beta}{2} \right)^{2s+N|q|} \sum_{\lambda \vdash s} \frac{d(\lambda) d(\lambda + |q| N)}{s! (s + N|q|)!} \frac{s_\lambda(AB)}{s_\lambda(1^N)} (\text{det} C)^{|q|} .
\] (47)
where $C = A$ if $q > 0$ and $C = B$ if $q < 0$. Equivalent form of (47) reads

$$Z_{SU(N)}(A, B) = \sum_{q=-\infty}^{\infty} \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N \geq 0 \frac{(\beta^2 \sum \lambda_i + N|q|)}{(\sum \lambda_i)! (\sum \lambda_i + N|q|)!} d(\lambda) d(\lambda + |q|N)$$

$$\times \frac{s\lambda(AB)}{s\lambda(1N)} (\det C)^{|q|} . \quad (48)$$

When $B = 0$ the last formula reduces to

$$Z_{SU(N)}(A, 0) = \sum_{q=0}^{\infty} \left( \frac{\beta}{2} \right)^N q^N (\det A)^q \prod_{k=0}^{N-1} \frac{k!}{(q+k)!} . \quad (49)$$

This result coincides with [16] for $A = aI$, where $I$ is a unit matrix. For $U(N)$ group [18] simplifies to

$$Z_{U(N)}(A, B) = \sum_{\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N \geq 0} \left( \frac{\beta}{2} \right) 2^{\sum \lambda_i} \frac{d^2(\lambda)}{((\sum \lambda_i)!)^2} \frac{s\lambda(AB)}{s\lambda(1N)} . \quad (50)$$

Integrals of the more general form

$$Z_G(A, B, C_k, D_l) = \int_G dU \prod_k \text{Tr} C_k U \prod_l \text{Tr} D_l U^\dagger \exp \left[ \frac{\beta}{2} (\text{Tr} A U + \text{Tr} B U^\dagger) \right] \quad (51)$$

can be evaluated by differentiating the basic integral (43), for example

$$Z_G(A, B, C, D) = \frac{\partial^2}{\partial h_1 \partial h_2} Z_G(A + h_1 C, B + h_2 D) \bigg|_{h_1 = h_2 = 0} . \quad (52)$$

The final result is quite cumbersome and will not be given here.

### 3.3 Staggered fermions

Formulae (46)-(50) can be further specified for many particular cases. As an important example, let us consider the partition function with staggered fermions. For $N_f$ flavours of the staggered fermions the matrices $A$ and $B$ from (43) read

$$A^{ij} = \eta_{\nu}(x) y_{\nu} \sum_{f=1}^{N_f} \bar{\psi}^i_f(x) \psi^j_f(x + e_\nu) , \quad (53)$$
\[ B^{ij} = -\eta_\nu(x) y_\nu^{-1} \sum_{f=1}^{N_f} \tilde{\psi}_f^i(x + e_\nu) \psi_f^j(x) . \] (54)

\( \eta_\nu(x) \) is given by
\[ \eta_0(x) = 1 ; \quad \eta_\nu(x) = \xi (-1)^{x_0 + x_1 + \cdots + x_{\nu-1}} , \quad \nu = 1, \cdots, d . \] (55)

\( \xi = a_t / a_s \) is the lattice anisotropy and the chemical potential \( \mu \) is introduced via \( y_\nu \)
\[ y_\nu = \begin{cases} \exp(\mu), & \nu = 0 \ , \ \mu = a_t \mu_q \\ 1, & \nu = 1, \cdots, d . \end{cases} \] (56)

We define the composite meson fields as
\[ \sigma_{ff'}(x) = \sum_{i=1}^N \tilde{\psi}_f^i(x) \psi_{f'}^i(x) \] (57)

and the composite (anti-)baryon fields as
\[ B_{f_1 \cdots f_N}(x) = \frac{1}{N!} \sum_{i_1 \cdots i_N=1}^N \epsilon_{i_1 \cdots i_N} \psi_{f_1}^{i_1}(x) \cdots \psi_{f_N}^{i_N}(x) , \] (58)
\[ \bar{B}_{f_1 \cdots f_N}(x) = \frac{1}{N!} \sum_{i_1 \cdots i_N=1}^N \epsilon_{i_1 \cdots i_N} \bar{\psi}_{f_1}^{i_1}(x) \cdots \bar{\psi}_{f_N}^{i_N}(x) . \] (59)

Consider first the \( U(N) \) model (50). Using representation (89) for the Schur functions and definitions (53), (54) it is easy to prove that
\[ \text{Tr}(AB)^j = (\eta_\nu(x))^{2j} (-1)^{j+1} \text{Tr} \Sigma^j , \] (60)

where we introduced matrix
\[ \Sigma_{f_1 f_2} = \sum_{f=1}^{N_f} \sigma_{f_1 f}(x) \sigma_{f f_2}(x + e_\nu) . \] (61)

Substituting this result into (50) and using \( \sum i\tau_i = s \), we obtain \( (\beta = 1) \)
\[ Z_{U(N)}(A, B) = \sum_{s=0}^{NN_f} \left( \frac{\eta_\nu(x)}{2} \right)^{2s} \sum_{\lambda \vdash s} \frac{d^2(\lambda)}{(s!)^2} \frac{s_{\lambda}(\Sigma)}{s_\lambda(1^N)} . \] (62)
Here, $\lambda'$ is a representation dual to $\lambda$. The dual representation is defined by exchanging rows and columns in the corresponding Young diagram, i.e. $\lambda' = \sum_j 1_{\lambda_j \geq i}$. In order to extend this result to $SU(N)$ we need to calculate determinants of $A$ and $B$ matrices. They are

$$\det A = (-1)^{N(N-1)/2} N! (\eta_\nu(x) y_\nu)^N \sum_{f_i=1}^{N_f} \bar{B}_{f_1 \ldots f_N}(x) B_{f_1 \ldots f_N}(x + e_\nu) ,$$

$$\det B = (-1)^{N(N-1)/2} N! (-\eta_\nu(x) y_\nu^{-1})^N \sum_{f_i=1}^{N_f} \bar{B}_{f_1 \ldots f_N}(x + e_\nu) B_{f_1 \ldots f_N}(x) .$$

The factor $(-1)^{N(N-1)/2}$ appears due to $\sum_{i=0}^{N-1} i = N(N-1)/2$ commutations of the fermion fields to gather them into baryons. Combining these expressions with (62) and (47) we can write down the partition function with an arbitrary number of staggered fermion flavours as

$$Z_{SU(N)}(A, B) = Z_{U(N)}(A, B) + \sum_{q=1}^{N_f} (-1)^{N(N-1)/2} q!(N!)^q \sum_{s=0}^{N(N_f-q)} \left( \eta_\nu(x) \right)^{2s+Nq} \times \sum_{\lambda' \prec \lambda} \frac{d(\lambda)d(\lambda + q\eta^N)}{s!(s + Nq)!} s_{\lambda'}(\Sigma) \left[ y_\nu^{Nq} Q^q + (-1)^{Nq} y_\nu^{-Nq} \bar{Q}^q \right] ,$$

where notations have been used

$$Q = \sum_{f_i=1}^{N_f} \bar{B}_{f_1 \ldots f_N}(x) B_{f_1 \ldots f_N}(x + e_\nu) ,$$

$$\bar{Q} = \sum_{f_i=1}^{N_f} \bar{B}_{f_1 \ldots f_N}(x + e_\nu) B_{f_1 \ldots f_N}(x) .$$

As the simplest case, consider one-flavour system. Then, the matrix $\Sigma$ becomes a scalar $\Sigma = \sigma(x)\sigma(x + e_\nu)$ and the summation over $\lambda$ consists of one term, namely $\lambda' = (s, 0, \ldots, 0)$ (or $\lambda_i = 1, i \in [1, s]$ and zero otherwise). This, together with (88), yields

$$Z_{SU(N)}(A, B) = \sum_{s=0}^{N} \left( \frac{\eta_\nu(x)}{2} \right)^{2s} \frac{(N-s)!}{s!N!} (\sigma(x)\sigma(x + e_\nu))^{s} + (-1)^{N(N-1)/2} \left( \frac{\eta_\nu(x)}{2} \right)^N [y_\nu^N \bar{B}(x) B(x + e_\nu) + (-1)^N y_\nu^{-N} \bar{B}(x + e_\nu) B(x)] .$$
This agrees with the well-known result of Ref. [33]. Similar method of computing one-link integral with one flavour of the staggered fermions has been used in [34]. Slightly more complicated is the case of two staggered flavours, \(N_f = 2\). Parameterizing \(\lambda' = (u_1, u_2)\), \(u_1 + u_2 = s\), \(N \geq u_1 \geq u_2 \geq 0\) and utilizing the property \(\text{(88)}\) of the Schur functions we present result for the two-flavour partition function as

\[
Z_{SU(N)(A,B)} = Z_0 + Z_1 + Z_2 ,
\]

\[
Z_0 = \sum_{u_1 \geq u_2 \geq 0} \left( \frac{\eta_{\nu}(x)}{2} \right)^{2(u_1+u_2)} \frac{(u_1 - u_2 + 1)(N - u_1)!(N - u_2 + 1)!}{(u_1 + 1)!u_2!(N + 1)!}
\]

\[
\times \sum_{i=0}^{[u_1-u_2]} C_{u_1-u_2-i}^i (-1)^i (\text{det} \Sigma)^{u_2+i}(\text{Tr} \Sigma)^{u_1-u_2-2i} ,
\]

\[
Z_1 = \sum_{u_1 \geq u_2 \geq 0} \left( \frac{\eta_{\nu}(x)}{2} \right)^{2(u_1+u_2)+N} \frac{(u_1 - u_2 + 1)(N - u_1 + 1)!(N - u_2 + 2)!}{(u_1 + 1)!u_2!(N + 1)!(N + 2)!}
\]

\[
(-1)^{\frac{N(N-1)}{2}} [y^N_{\nu} Q + (-1)^N y_{\nu}^{-N} \bar{Q}] \sum_{i=0}^{[u_1-u_2]} C_{u_1-u_2-i}^i (-1)^i (\text{det} \Sigma)^{u_2+i}(\text{Tr} \Sigma)^{u_1-u_2-2i} ,
\]

\[
Z_2 = \left( \frac{\eta_{\nu}(x)}{2} \right)^{2N} \frac{1}{(N + 1)} \left[ y_{\nu}^{2N} Q^2 + y_{\nu}^{-2N} \bar{Q}^2 \right] .
\]

Here, \(C_k^j\) are the binomial coefficients. To calculate the Schur functions we have used their relations with the complete symmetric functions \(h_k\) and the elementary symmetric functions \(e_k\), \([86]-[87]\).

### 3.4 Reduced principal chiral model

Our last example is the so-called reduced principal chiral model whose partition function reads

\[
Z = \int_G dU \exp \left[ \beta \sum_{\mu=1}^d \text{Re} \text{Tr} U \Gamma_\mu U^\dagger \Gamma_\mu^\dagger \right] .
\]

The matrices \(\Gamma_\mu \in SU(N)\) and satisfy certain commutation relations exact form of which is not important here. It is expected that the reduced model gives an exact
solution for the principal chiral model in the large-\(N\) limit (see, for instance (33) and refs. therein). Here we discuss the group integration in (72) using the method of the Weingarten functions. For simplicity, we restrict ourselves to the two-dimensional case. Expanding the integrand into two character series and using the relation (81) again one finds

\[
Z = \sum_{\lambda_1} \sum_{\lambda_2} \frac{C_{\lambda_1}(\beta) C_{\lambda_2}(\beta)}{|\lambda_1|! |\lambda_2|!} \sum_{\sigma_1 \in S_{|\lambda_1|}} \sum_{\sigma_2 \in S_{|\lambda_2|}} \chi_{\lambda_1}(\sigma_1) \chi_{\lambda_2}(\sigma_2) J_{\sigma_1,\sigma_2}(\{\Gamma_{\mu}\}) ,
\]

where \(C_{\lambda_i}(\beta)\) are given by either (33) or (42) and the resulting group integral takes the form

\[
J_{\sigma_1,\sigma_2}(\{\Gamma_{\mu}\}) = \int_G dU P_{\sigma_1}(U \Gamma_1 U^\dagger) P_{\sigma_2}(U \Gamma_2 U^\dagger) .
\]

Applying Eq. (82) one sees the last integral is of a type (5), therefore the equation (19) can be used (it is sufficient to take only the term with \(q = 0\) in the limit of large \(N\)) and gives after partial summation over group indices

\[
J_{\sigma_1,\sigma_2}(\{\Gamma_{\mu}\}) = \sum_{\tau,\rho \in S_{r_1+r_2}} W g^N(\tau^{-1}\rho) P_{\tau,\tau}(\{\Gamma_{\mu}\}) P_{\rho,\sigma_1,\rho,\sigma_2}(\{\Gamma_{\mu}^\dagger\}) .
\]

We introduced here the following invariant function

\[
P_{\tau,\rho}(\{\Gamma_{\mu}\}) = \sum_{i_1, \cdots, i_{r_1+r_2}} \prod_{k=1}^{r_1} \Gamma_1^{i_{\tau(k)}-i_k} \prod_{k=1}^{r_2} \Gamma_2^{i_{\tau(r_1+k)}-i_{r_1+k}} .
\]

The right-hand side of the last equation can be expressed in terms of traces of products of matrices \(\Gamma_{\mu}\). Detailed investigation will be published elsewhere. Here we would like to emphasize that we expect certain simplifications in the large-\(N\) limit which is the only one relevant here. Indeed, taking the asymptotics of the Weingarten function (25) in Eq. (75) we obtain

\[
J_{\sigma_1,\sigma_2}(\{\Gamma_{\mu}\}) = \sum_{\rho \in S_{r_1+r_2}} \frac{1}{N^{r_1+r_2}} P_{\rho,\rho}(\{\Gamma_{\mu}\}) P_{\rho,\sigma_1,\rho,\sigma_2}(\{\Gamma_{\mu}^\dagger\}) .
\]

4 Summary and perspectives

In this article we have calculated certain integrals over \(SU(N)\) group. The basic integral (5) is of polynomial type whose integrand includes product of an arbitrary
number of the group matrices and their conjugates in the fundamental representation. The result of the integration is expressed through the summation over permutations of the group indices which are contracted via products of the Kronecker deltas and the totally anti-symmetric tensors, Eqs. (19) and (21). The weight is given by the $SU(N)$ Weingarten function (20). We have considered several applications of this general result. In particular, we have calculated the integrals of powers of traces of the group elements, the coefficients of the character expansion of the invariant function of an arbitrary form and the general one-link integral appearing in LGT. In the latter case, the result was used to evaluate one-link integrals with arbitrary number of staggered fermion flavours. Also, we applied our approach for the computation of the integrals in the expression for the partition function of the reduced principal chiral model.

The method of the Weingarten functions is a powerful and very general tool which can be used both for the derivation of alternative representations of known integrals and for calculation of new and more complex ones appearing in lattice QCD and spin models. Some further applications of this method may include one-link integrals with $N_f$ flavours of the Wilson fermions, the scalar lattice QCD in the strong coupling region, $SU(N)$ principal chiral model at finite $N$. Interesting is an extension of 3.3 to the Eguchi-Kawai model and its twisted version.

The main goal of this article was to give a mathematical background to our computations of the dual representations of 1) the effective Polyakov loop spin models and 2) pure gauge LGT and lattice QCD with the staggered fermions, as outlined in [31]. Details of these calculations will be reported elsewhere. Also, an interesting and important direction is to explore the large-$N$ expansion of $SU(N)$ models in frameworks of the present method. This can be done by using the asymptotic expansion of the Weingarten function (28) and is planned for future investigations.

Appendix

Let $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_N)$ be a partition $\lambda \vdash r$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \geq 0$ and $\sum_{i=1}^{N} \lambda_i = r$. $\chi_{\lambda}(\sigma)$ denotes a character of $\sigma \in S_r$ in representation $\lambda$. $d(\lambda) = \chi_{\lambda}(1)$ is the dimension of the representation $\lambda$. The Schur function $s_{\lambda}(X)$ is a character of the unitary group $G$, thus $s_{\lambda}(1^N)$ is the dimension of the irreducible representation $\lambda$ of $G$. If $l(\lambda)$ is the length of the partition $\lambda$, i.e., the number of non-vanishing parts $\lambda_i$, then

$$d(\lambda) = r! \frac{\prod_{1 \leq i < j \leq l(\lambda)} (\lambda_i - \lambda_j + j - i)}{\prod_{i=1}^{l(\lambda)} (\lambda_i + l(\lambda) - i)!},$$

(78)
Below we list some formulae used in evaluating the group integrals. Orthogonality relation for the characters of the permutation group is written as

$$\sum_{\tau \in S_r} \chi_\mu(\tau)\chi_\lambda(\tau\sigma) = r! \delta_{\mu,\lambda} \frac{\chi_\lambda(\sigma)}{d(\lambda)}. \quad (80)$$

Relation between Schur functions and power sum symmetric functions of matrix argument is given by

$$s_\lambda(X) = \frac{1}{r!} \sum_{\tau \in S_r} \chi_\lambda(\tau)P_\tau(X), \quad (81)$$

where $X$ has the dimension $N$ and

$$P_\tau(X) = \sum_{i_1 \cdots i_r} \prod_{k=1}^{r} X_{i_k \tau(k)}. \quad (82)$$

Inverse relation reads

$$P_\tau(X) = \sum_{\mu \vdash r} \chi_\mu(\tau)s_\mu(X). \quad (83)$$

The power sum symmetric function of the unit matrix equals

$$P_\tau(I) = N^{\tau}, \quad (84)$$

where $|\tau|$ is the number of cycles in the permutation $\tau$. We mention the following formula ($1 \leq i \leq N$)

$$\sum_{j_1 \cdots j_r} \sum_{m_1 \cdots m_r} \prod_{k=1}^{r} \delta_{j_k,m_{\tau(k)}} \delta_{j_k,m_{\sigma(k)}} y_{j_k} y_{m_k} = P_{\tau^{-1}\sigma}(x), \quad x_i = y_i^2. \quad (85)$$

Given the complete symmetric functions $h_k$ and the elementary symmetric functions $e_k$ in $m$ variables $x_1, \ldots, x_m$

$$h_k = \sum_{1 \leq n_1 \leq \cdots \leq n_k \leq m} x_{n_1} \cdots x_{n_k}, \quad e_k = \sum_{1 \leq n_1 < \cdots < n_k \leq m} x_{n_1} \cdots x_{n_k}, \quad \text{(86)}$$
the Schur functions can be computed with the help of identities

\[
s_\lambda(X) = s(\lambda_1, \ldots, \lambda_m)(x_1, \ldots, x_m) = \det (h_{\lambda_i - i + j})_{1 \leq i, j \leq m},
\]
\[
s_\lambda(X) = s(\lambda_1, \ldots, \lambda_m)(x_1, \ldots, x_m) = \det (e_{\lambda'_i - i + j})_{1 \leq i, j \leq m},
\] (87)

where \( \lambda' \) is a partition dual to \( \lambda \) and the following rule is understood

\[
s(\lambda_1, \ldots, \lambda_n)(x_1, \ldots, x_{n-1}, 0) = \begin{cases} 
0, & \text{if } \lambda_n \neq 0, \\
\lambda_1, \ldots, \lambda_{n-1})(x_1, \ldots, x_{n-1}), & \text{if } \lambda_n = 0.
\end{cases}
\] (88)

Another expression for the Schur function used in the text reads

\[
s_\lambda(X) = \sum_{\tau_1, \ldots, \tau_s} \chi_\lambda(\tau) \prod_{j=1}^s \frac{1}{\tau_j!} \left[ \text{Tr}(X)^j \right]^{\tau_j}
\] (89)

with \( \tau \) being a permutation such that the number of cycles of length \( j \) in \( \tau \) is \( \tau_j \).

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