ACCELERATING FRONTS IN SEMILINEAR WAVE EQUATIONS

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Abstract. We study dynamics of interfaces in solutions of the equation
\[ \varepsilon \Box u + \frac{1}{\varepsilon} f_\varepsilon(u) = 0, \]
for \( f_\varepsilon \) of the form \( f_\varepsilon(u) = (u^2 - 1)(2u - \varepsilon \kappa) \), for \( \kappa \in \mathbb{R} \), as well as more general, but qualitatively similar, nonlinearities. We prove that for suitable initial data, solutions exhibit interfaces that sweep out timelike hypersurfaces of mean curvature proportional to \( \kappa \). In particular, in 1 dimension these interfaces behave like a relativistic point particle subject to constant acceleration.

1. INTRODUCTION

In this paper we consider the dynamics of interfaces in semilinear hyperbolic equations. The simplest example that we study is the equation
\[ \varepsilon (u_{tt} - u_{xx}) + \frac{1}{\varepsilon} (u^2 - 1)(2u - \varepsilon \kappa) = 0, \]
where we assume for concreteness that \( \kappa > 0 \). Here the nonlinearity \( f_\varepsilon(u) = (u^2 - 1)(2u - \varepsilon \kappa) \) has the form \( f_\varepsilon = F'_\varepsilon \), where \( F_\varepsilon \) has local minima at \( u = \pm 1 \), with \( F_\varepsilon(\pm 1) = \pm \frac{2}{3} \varepsilon \kappa \). Thus the state \( u = -1 \) has slightly lower energy than the state \( u = 1 \), and one might expect that there exist solutions in which the low-energy phase \( u = -1 \) grows at the expense of the higher-energy phase. This is what we prove. In fact we show that, for suitable initial data, solutions exhibit an interface that behaves like a relativistic mass subject to constant acceleration proportional to the parameter \( \kappa \). Equivalently, the interface sweeps out a timelike curve of constant Minkowskian curvature, proportional to \( \kappa \), in the \((t,x)\)-plane.

It turns out that our analysis extends with rather few changes to wave equations on suitable Lorentzian manifolds \((N,h)\). Thus we will also consider the equation
\[ \varepsilon \Box_h u + \frac{1}{\varepsilon} f_0(u) + \kappa f_1(u) = 0 \]
where \( \Box_h \) is the Laplace-Beltrami (wave) operator on \((N,h)\), \( \kappa \) is a smooth function, and \( \frac{1}{\varepsilon} f_0(u) + \kappa f_1(u) \) generalizes the nonlinearity in \((1.1)\) in a natural way. In this situation, analogous to \((1.1)\), we show that for well-prepared data, interfaces sweep out timelike hypersurfaces of prescribed mean curvature \( \kappa \), with respect to the Lorentzian metric \( h \).

In the case when \((N,h)\) is just \(1+n\)-dimensional Minkowski space and \( \kappa \equiv 0 \), corresponding to the situation when the two potential wells have equal depth, similar results were proved by the first author in [9], following partial results of [3]. Thus, the present paper consists of a number of improvements of the basic argument developed in [9]: we extend the results to the case \( \kappa \neq 0 \), we show that they remain valid on Lorentzian manifolds more general than Minkowski space, and we drop some convenient but artificial restrictions imposed in [9] on the topological type of the hypersurfaces considered. A key point in our analysis is that if \( \kappa \) is a nonzero constant, then in certain weighted energy estimates it is much more useful to use, not the canonical conserved energy associated to the actual equation \((1.2)\) under study, but rather the conserved energy associated to the \( \kappa = 0 \) equation. (See Remark 2.1.)

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observation plays a crucial role in our arguments and makes the extension of techniques
developed in [9] to the more general situation considered here surprisingly straightforward.

Equations such as (1.2), with \( \kappa \neq 0 \), have been studied in the cosmological literature as
models for what is called the decay of a false vacuum. This arises from models in which the
universe is described by a quantum field theory for which an equation like (1.2) (or a more
complicated but in some ways similar equation) is a low-energy limit, and whose state is
initially given by a constant function \( u \equiv v_f \), where \( v_f \) is a “false vacuum”: a local, but not
global, minimum of some underlying potential function. In the example (1.1), if \( \kappa > 0 \) then
\( v_f = 1 \), and the “true vacuum”, or global minimizer of the potential function \( F_{\varepsilon} \), is \( v_t = -1 \).
In this situation, a quantum tunnelling event could in principle lead to the nucleation of
region in which \( u = v_t \). This scenario was investigated in a series of papers by Coleman
and coworkers; see for example [4], which estimates via a formal semiclassical approximation
the probability per unit time per unit volume of such a tunneling event. Our results have
nothing to say about this, but describe the dynamics of a fully-formed interface between
false and true vacuums in a universe governed by (1.2), showing that if the interface has an
energetically optimal structure, then it behaves like a hypersurface of constant Lorentzian
mean curvature proportional to the difference in energy between the true and false vacuums.

Earlier work on dynamics of energy concentration sets in hyperbolic equations includes
[8, 11, 6, 16], all of which consider situations in which energy concentrates around points
rather than submanifolds, as in [3, 9] and the present paper. The dynamics of interfaces in
equations such as (1.2) is studied from a formal point of view in [14].

A lengthy discussion of related elliptic and parabolic results, with a heavy bias toward
the \( \kappa = 0 \) case, is contained in [9]. For \( \kappa \neq 0 \), there is a rather strong analogy between the
phenomena we study and propagating fronts in semilinear parabolic equations, a subject that
has attracted a great deal of study, dating back to the 1930s [10]. In particular, the problem
that formally determines the profile and (relativistic) acceleration of interfaces (see (2.5) or
(3.17)) is exactly the same one that determines the profile and velocity of propagating fronts
in certain parabolic problems, see for example [1, 5, 2]. There is also an analogy between our
work and results that establish an asymptotic connection between elliptic analogs of (1.2)
and surfaces of prescribed Euclidean (or more generally Riemannian) mean curvature, see for
example [13, 7].

This paper is organized as follows: In order to highlight some main ideas with as few
preliminaries as possible, we consider in Section 2 the case (1.1) of an equation in one space
dimension associated with two potential wells of unequal depth. This discussion is included
just to illustrate our arguments in a simple setting, and is not needed in later pars of the
paper.

We therefore defer until Section 3 both the statement of our main result, and the intro-
duction of some notation that is used throughout the rest of the paper. In Section 4 we
introduce a coordinate system in which many of our main estimates will take place, stating
the properties that we will need and deferring most proofs to Section 7. The heart of our
argument consists of weighted energy estimates in this adapted coordinate system. These
are carried out in Section 5. In Section 6 these estimates are combined with rather standard
energy estimates away from the interface in an iterative argument that completes the proof
of our main theorem.

2. THE SIMPLEST NONTRIVIAL EQUATION

In this section we consider the 1-dimensional equation (1.1). All the results in this section
are essentially subsumed in Theorem 3.2 and most of the main ideas in Theorem 3.2 appear
here, in somewhat simpler form.
It is convenient to consider initial data such that at $t = 0$,
\begin{equation}
(u, u_t) = (-1, 0) \text{ for all } x \text{ near } 0, \quad (u, u_t) = (1, 0) \text{ for all } x \geq R
\end{equation}
for some $R$. (More conditions on the data will be imposed later.) Noting that the constant functions $\pm 1$ are both solutions of (1.1), standard facts about finite propagation speed for solutions of (1.1) imply that
\begin{equation}
u(t, x) = -1 \text{ for } (t, x) \text{ near } (0, 0), \quad u(t, x) = 1 \text{ for } x \geq R + \lvert t \rvert.
\end{equation}

2.1. change of variables. As suggested above, one might guess that for suitable initial data, solutions will exhibit an interface that sweeps out a timelike curve of constant (nonzero) Minkowskian curvature proportional to the parameter $\kappa$ that controls to the difference in depth of the two energy wells. Such curves have the form \{$(x, t) : x^2 - t^2 = c, x > 0$\} modulo translations and reflections. We thus start by changing variables in such a way as to “straighten out” a 1-parameter family of such curves. Thus, we introduce new coordinates $(\theta, r)$ defined (for $\theta \in \mathbb{R}, r > 0$) by
\begin{equation}
(t, x) = (r \sinh \theta, r \cosh \theta) = \psi(\theta, r) \in \{ (t, x) : |t| < x \}.
\end{equation}
These are just Minkowskian polar coordinates, with $\theta$ being the angular and $r$ the radial coordinate. Note that every coordinate line $r = r_0$ is a timelike curve of constant curvature $\frac{1}{r_0}$ with respect to the Minkowski metric, which in these coordinates takes the form $ds^2 = -r^2 dt^2 + dr^2$. We will treat $r$ as a timelike coordinate, and $\theta$ as spacelike.

If $u$ solves (1.1) and we write $v = u \circ \psi$, then we find that $v$ satisfies
\begin{equation}
\varepsilon \left( \frac{1}{r^2} v_{\theta \theta} - v_{rr} - \frac{1}{r} v_r \right) + \frac{1}{\varepsilon} (v^2 - 1)(2v - \varepsilon \kappa) = 0, \quad \theta \in \mathbb{R}, r > 0.
\end{equation}
If we imagine that $v_{\theta \theta} \approx 0$ and that $\frac{1}{r} \approx c$ constant, then this looks like the equation
\begin{equation}
\varepsilon (-q'' - cq') + \frac{1}{\varepsilon} (q^2 - 1)(2q - \varepsilon \kappa) = 0
\end{equation}
for the profile $q$ and wave speed $c$ of traveling wave solutions of the parabolic counterpart of (1.1). This is known to have the 1-parameter family of solutions
\begin{equation}
c = \kappa, \quad q = \tanh(\frac{r - r_0}{\varepsilon}), \quad r_0 \in \mathbb{R}.
\end{equation}
(We have set things up so that the profile $q$ is independent of the parameter $\kappa$.) Note that if we choose $r_0 = \frac{1}{\kappa}$, then all the nontrivial behavior of $q_\varepsilon(r) := \tanh(\frac{1}{\varepsilon}(r - \frac{1}{\kappa}))$ is concentrated in an $\varepsilon$-neighborhood of $r = \frac{1}{\kappa}$, which is consistent with the heuristic $\frac{1}{r} \approx \kappa = c$.

Thus, we will study (2.4) with initial data such that
\begin{equation}
v(0, r) \approx \tanh(\frac{1}{\varepsilon}(r - r_0)), \quad v_\theta(0, r) \approx 0 \quad \text{for } r > 0
\end{equation}
where henceforth we set
\begin{equation}
r_0 := \frac{1}{\kappa}.
\end{equation}
Indeed, we will show that for data of this form, solutions are approximately independent of $\theta$, and hence remain concentrated about the curve $r = \frac{1}{\kappa}$. Recall also that we are assuming \begin{equation}, which in the new variables implies that for $\theta = 0$,
\begin{equation}
(v, v_\theta) = (-1, 0) \text{ for all } r \text{ close to } 0, \quad (v, v_\theta) = (1, 0) \text{ for } r \geq R,
\end{equation}
and this implies (2.2), which translates to
\begin{equation}
v(\theta, r) = -1 \text{ for } r \text{ near } 0, \quad v(\theta, r) = 1 \text{ for } r \geq R e^{\lvert \theta \rvert}.
\end{equation}
for every $\theta \in \mathbb{R}$.
2.2. differential energy inequality. We next define
\[ e_\varepsilon(v) := \varepsilon \left( \frac{v^2}{2} + v_r^2 + \frac{1}{2\varepsilon}(v^2 - 1)^2 \right). \]

A short computation shows that if \( v \) is a sufficiently smooth solution of (2.4), then
\[ (2.11) \quad \frac{d}{d\theta} e_\varepsilon(v) = \varepsilon(v_\theta v_r)_r + \text{Term 1} + \text{Term 2} \]
where
\[ \text{Term 1} = \varepsilon \frac{v_\theta}{r} v_r (1 - kr), \quad \text{Term 2} = \kappa \varepsilon v_\theta (v_r - \frac{1}{\varepsilon}(1 - v^2)). \]

Term 1 should be small in \( L^1 \) if \( v_\theta \) is small and \( v_r \) is concentrated near \( r = \frac{1}{\kappa} \). Also, the profile \( q_\varepsilon(r) = \tanh(\frac{1}{\varepsilon}(r - \frac{1}{\kappa})) \) satisfies \( \partial_r q_\varepsilon = \frac{1}{\varepsilon}(1 - q_\varepsilon^2) \), so that if \( v \approx q_\varepsilon \) in a sufficiently strong sense, then Term 2 should be small.

**Remark 2.1.** Note that (2.4) has an exactly conserved energy: a computation shows that
\[ \frac{d}{d\theta} \left( \frac{\varepsilon}{2} v_\theta^2 + \frac{\varepsilon}{2} v_r^2 + \frac{1}{\varepsilon} F_\varepsilon(v) \right) = (rv_r v_\theta)_r, \quad \text{where } F_\varepsilon'(s) = (s^2 - 1)(2s - \kappa \varepsilon). \]

It turns out that it is much more useful to work with the approximately conserved energy \( e_\varepsilon(v) \) defined above. This observation, although very simple, is a key point in our analysis.

2.3. lower energy bound. Note that if \( v \) satisfies (2.10),
\[ \int_0^\infty \frac{\varepsilon}{2} v_\theta^2 + \frac{1}{2\varepsilon}(v^2 - 1)^2 \, dr \geq \int_0^\infty |(1 - v^2)v_r| \, dr \]
\[ \geq \left| \int_0^\infty (v - \frac{1}{3} v^3)_r \, dr \right| \]
\[ = \frac{4}{3} =: c_0. \]

(2.12)

2.4. weighted energy estimates in new variables. Next, given a solution \( v \) of (2.4), we will write
\[ (2.13) \quad \zeta_1(\theta) = \int_0^\infty [1 + (r - r_0)^2] e_\varepsilon(v) \, dr \bigg|_{\theta} - c_0. \]

For the initial data we consider, (2.10) holds, and then (2.12) implies that
\[ (2.14) \quad \zeta_1(\theta) \geq \zeta_2(\theta) := \int_0^\infty \frac{\varepsilon}{2} \frac{v_\theta^2}{r^2} + (r - r_0)^2 \left( \frac{\varepsilon}{2} v_r^2 + \frac{1}{2\varepsilon}(v^2 - 1)^2 \right) \, dr \bigg|_{\theta}. \]

Using (2.11), we compute
\[ \zeta_1'(\theta) = \int_0^\infty [1 + (r - r_0)^2] [\varepsilon(v_\theta v_r)_r + \text{Term 1} + \text{Term 2}] \, dr. \]

Every term in the integrand contains a factor of \( v_\theta \), which due to (2.10) has compact support in \((0, \infty)\), so the integral clearly exists, and we can integrate by parts without problems. It also follows from (2.10) that \( 1 \leq Re^{v_\theta} \frac{1}{r} \) on the support of \( v_\theta \). Thus
\[ \int_0^\infty [1 + (r - r_0)^2] \varepsilon(v_\theta v_r)_r = - \int_0^\infty 2\varepsilon v_\theta (r - r_0)v_r \]
\[ \leq Re^{v_\theta} \int_0^\infty 2\varepsilon \frac{|v_\theta|}{r} \, |r - r_0| \, |v_r| \leq 2Re^{v_\theta} \zeta_2(\theta). \]

Recalling that \( \kappa = r_0^{-1} \), elementary estimates yield
\[ |\text{Term 1}| \leq \frac{\varepsilon \kappa}{2} \left( \frac{v_\theta^2}{r^2} + (r - r_0)^2 v_r^2 \right), \]
and 
\[ |\text{Term 2}| \leq \frac{r \epsilon^{\frac{1}{2}}}{2} \left[ \left( \frac{v_{\theta}}{r} \right)^2 + \left( v_r - \frac{1}{\epsilon} (1 - v^2) \right)^2 \right]. \]
Repeatedly using (2.10) to bound \( r \), and recalling that \( \zeta_2 \leq \zeta_1 \), we deduce that 
\[ \zeta'_1(\theta) \leq C R e^{2|\theta|} \zeta_1(\theta) + C R^3 e^{3|\theta|} \int_0^\infty \frac{\epsilon}{2} (v_r - \frac{1}{\epsilon} (1 - v^2))^2. \]
However, arguing as in (2.12), 
\[
\int_0^\infty \frac{\epsilon}{2} (v_r - \frac{1}{\epsilon} (1 - v^2))^2 = \int_0^\infty \frac{\epsilon}{2} v_r^2 + \frac{1}{\epsilon} (1 - v^2)^2 - \int_0^\infty (1 - v^2) v_r \\
\overset{(2.10)}{=} \int_0^\infty \frac{\epsilon}{2} v_r^2 + \frac{1}{\epsilon} (1 - v^2)^2 - c_0 \\
\leq \zeta_1(\theta).
\]
We conclude that 
\[ \zeta'_1(\theta) \leq C(R + R^3) e^{3|\theta|} \zeta_1(\theta). \]

2.5. conclusions about \( v \). At this point, we have proved most of the following proposition.

**Proposition 2.2.** Let \( v \) solve (2.4) with initial data satisfying (2.9). Let \( r_0 = \kappa^{-1} \), and define \( \zeta_1, \zeta_2 \) as in (2.13), (2.14). Then there exists a constant \( C \), depending on the parameter \( R \) in (2.9), such that for every \( \theta \in \mathbb{R} \), 
\[ \zeta_2(\theta) \leq \zeta_1(\theta) \leq e^{C(\epsilon^{3|\theta|}) - 1} \zeta_1(0) \quad \text{for every } \theta, \]
As a result, 
\[ \int_0^\infty |v(\theta, r) - v(0, r)|^2 \frac{dr}{r^2} \leq C(\theta, R) \frac{\zeta_1(0)}{\epsilon}, \quad C(\theta, R) := C|\theta| \int_0^\theta e^{C(R)(\epsilon^{3|\theta|}) - 1} ds. \]
In particular, there exists initial data for which the solution \( v \) satisfies 
\[ \int_0^\infty |v(\theta, r) - \text{sign}(r - r_0)|^2 \frac{dr}{r^2} \leq C(\theta, R) \epsilon \quad \text{for all } \theta. \]

**Remark 2.3.** For any \( \delta > 0 \), there exists \( C = C(\delta, r_0) \) such that 
\[ \int_0^\infty |\text{tanh}(\frac{r - r_0}{\epsilon}) - \text{sign}(r - r_0)|^2 \frac{dr}{\max(r, \delta)^2} \leq C \epsilon. \]
So (2.19) implies that 
\[ \int_0^\infty |v(\theta, r) - \text{tanh}(\frac{r - r_0}{\epsilon})|^2 \frac{dr}{\max(r, \delta)^2} \leq C \epsilon. \]
However, (2.20) says precisely that (2.19) is not a sharp enough estimate to determine the profile of \( v \), so it would arguably be a little misleading to insist on \( \text{tanh} \) rather than \( \text{sign} \) in estimates such as (2.21), (2.19).

On the other hand, standard spectral estimates imply that for every \( \theta \), there exists some \( r_\epsilon(\theta) \) such that the solution \( v \) in (2.19) satisfies 
\[ \int_0^\infty |v(\theta, r) - \text{tanh}(\frac{r - r_\epsilon(\theta)}{\epsilon})|^2 \frac{dr}{\epsilon} \leq C \zeta_1(\theta) \leq C(\theta) \epsilon^2, \]
and then it follows from (2.19) that \( |r_\epsilon(\theta) - r_0| \leq C \epsilon \). So although it is not captured in (2.19), our estimates do in fact show that \( v \) is close to a scaled, translated hyperbolic tangent.
Proof. To complete the proof, notice first that (2.17) follows from (2.16) and (2.14), via a form of Grönwall’s inequality. Next, for every \( r \),
\[
\frac{1}{r^2} \left( v(\theta, r) - v(0, r) \right)^2 \leq |\theta| \int_0^\theta \frac{1}{r^2} v_\theta^2(s, r) ds.
\]
We deduce (2.18) by integrating this inequality with respect to \( r \) then using Fubini’s Theorem and (2.17). Finally, to prove (2.19), it now suffices to exhibit initial data \( (v, v_\theta)|_{\theta=0} \) satisfying (2.9), and such that
\[
(2.22) \quad \zeta_1(0) \leq C \varepsilon^2, \quad \int_0^\infty |v(0, r) - \text{sign}(r - r_0)|^2 \frac{dr}{r^2} \leq C \varepsilon^2.
\]
To do this, let \( v\theta|_{\theta=0} = 0 \), and let
\[
v(0, r) := \bar{q}_{\varepsilon, r_0}(r - r_0),
\]
where
\[
(2.23) \quad \bar{q}_{\varepsilon, r_0}(s) = \chi_{r_0}(s) \tanh(\frac{s}{\varepsilon}) + (1 - \chi_{r_0}(s)) \text{sign}(s),
\]
and for \( r > 0 \) we define \( \chi_r \in C^\infty_c(\mathbb{R}) \) to be a function such that
\[
(2.24) \quad \chi_r(s) = 1 \text{ if } |s| \leq \frac{r}{3} \text{ and } \chi_r(s) = 0 \text{ if } |s| \geq \frac{2r}{3}, \quad |\chi_r'| \leq C/r.
\]
It is straightforward to verify that this initial data satisfies (2.22). \( \square \)

2.6. Conclusions about \( u \). Proposition 2.2 yields uniform estimates for \( v \) on sets of the form \( \{ (r, \theta) : r > 0, |\theta| < \Theta \} \), corresponding to uniform estimates of the original solution \( u \) in a sector \( \{ (t, x) : x > 0, |t| < x \tanh \Theta \} \). (Recall that \( u = v \circ \psi \) for \( \psi \) defined in (2.3).) We next show uniform estimates of \( u \) in a spacetime slab \( (-T, T) \times \mathbb{R} \).

Proposition 2.4. Fix \( \varepsilon \in (0, 1] \) and let \( u \) solve \( \text{(1.1)} \) with initial data \( u(0, x) = \bar{q}_{\varepsilon, r_0}(x - r_0) \), \( u_t(0, x) = 0 \), where \( \bar{q}_{\varepsilon, r_0} \) is defined in (2.23) above and \( r_0 = \kappa^{-1} \).

For every \( T > 0 \), there is a constant \( C(T) \), independent of \( \varepsilon \), such that
\[
(2.25) \quad \int_{-T}^T |u(t, x) - \text{sign}(x - \gamma(t))|^2 \, dx \, dt \leq C(T) \varepsilon.
\]
where \( \gamma(t) = (r_0^2 + t^2)^{1/2} \).

Remark 2.5. Although we have stated here only an analog of (2.19), our arguments also establish an analog of (2.17), i.e., energy estimates showing that, for a large class of initial data, energy concentrates near the curve \( (t, \gamma(t)) \). We already know this, modulo a change of variables, in the sector \( \{ (t, x) : x > 0, |t| \leq x \tanh \Theta \} \) so the new point is energy estimates outside this sector, which are established in (2.28) below.

Proof. Fix \( T > 0 \) and let \( \Theta \) be such that \( T = \frac{3}{3} \sinh \Theta \), i.e., \( \Theta = \sinh^{-1} \frac{3T}{r_0} \). We will start by considering a solution \( u \) for initial data satisfying
\[
(2.26) \quad (u, u_t)(0, x) = (-1, 0) \text{ for all } x < \delta, \quad (u, u_t)(0, x) = (1, 0) \text{ for all } x > R,
\]
for some \( 0 < \delta < r_0 < R \). We will only specialize later to the initial data in the statement of the Proposition. Until we do so, all constants in our argument may depend on \( r_0 = \frac{1}{\kappa} \), \( \Theta \) (hence on \( T \)), and the parameter \( R \) above.

The choice (2.26) of initial data implies that Proposition 2.2 applies to \( v = u \circ \psi^{-1} \). We will write
\[
x_0(t) := \sqrt{t^2 + \frac{1}{9} r_0^2}, \quad x_1(t) := \sqrt{t^2 + \frac{4}{9} r_0^2}.
\]
The idea is simply that results from Proposition 2.2 imply that \( u \approx -1 \) (with respect to the \( H^1 \) norm) in the set \( \{ (t, x) : x_0(t) < x < x_1(t) \} \), see the shaded region in Figure 1. We
combine this with the fact that \((u, u_t)|_{t=0} \approx (-1, 0)\) to argue that \(u \approx -1\) in \(H^1\) in the entire set \\(\{(t, x) : |t| < T, x < x_1(t)\}\).

**Figure 1.**

**Step 1.** Indeed, let \(\chi\) be a smooth function such that \(0 \leq \chi \leq 1\) and
\[
\chi(x, t) = 1 \text{ if } x \leq x_0(t), \quad \chi(x, t) = 0 \text{ if } x \geq x_1(t).
\]

We will write \(e^\varepsilon_\varepsilon(u)\) for an energy density in the original coordinates defined by
\[
e^\varepsilon_\varepsilon(u) = \varepsilon^2 (u_t^2 + u_x^2) + \frac{1}{2\varepsilon} (1 - u^2)^2.
\]

Then rather standard energy arguments, which we recall in Step 3 below, imply that
\[
\int \chi e^\varepsilon_\varepsilon(u) \bigg|_{t} \leq e^{\kappa t} \int \chi e^\varepsilon_\varepsilon(u) \bigg|_{0} + C_1 \int_{t_0}^{x_1(t)} e^{\kappa(t-s)} e^\varepsilon_\varepsilon(u) \, dx \, ds.
\]

(2.27)

It is straightforward to check that there exists some \(C\) such that \(e^\varepsilon_\varepsilon(u) \circ \psi \leq C e^\varepsilon_\varepsilon(v)\) in \([-\Theta, \Theta] \times [\frac{1}{2} r_0, \frac{2}{3} r_0]\). On the same set, the Jacobian determinant \(\det(D\psi)\) is bounded and
\[
\frac{1}{C} e^\varepsilon_\varepsilon(v) \leq \frac{\varepsilon}{2} \left(\frac{r_0}{r}\right)^2 + (r - r_0)^2 \left(\frac{\varepsilon}{2} v_r^2 + \frac{1}{2\varepsilon} (v^2 - 1)^2\right).
\]

Hence by a change of variables,
\[
\int_{t_0}^{t} \int_{x_0(t)}^{x_1(t)} e^\varepsilon_\varepsilon(u) \, dx \, ds \leq C \int_{\Theta}^{2r_0/3} e^\varepsilon_\varepsilon(v) \, dr \, d\theta \leq C \int_{0}^{\Theta} \zeta_2(\theta) d\theta.
\]

We combine this with (2.27) and using (2.17), and note that \(\int \chi(t, x) e^\varepsilon_\varepsilon(u)(0, x) \, dx \leq C \zeta_1(0)\), as a result of (2.1). These computations lead to the inequality
\[
(2.28) \quad \int_{\{(t, x) : |t| \leq T, x < x_0(t)\}} e^\varepsilon_\varepsilon(u) \, dx \, dt \leq C \zeta_1(0).
\]

**Step 2.** We now use the above estimates to prove (2.25) for \(u\) solving (1.1) with initial data satisfying (2.26) together with
\[
(2.29) \quad \zeta_1(0) \leq C \varepsilon^2 \quad \text{for } v = u \circ \psi^{-1}, \quad \int_{R} |u(0, x) - \text{sign}(x - r_0)|^2 \leq C \varepsilon.
\]
In particular, these conditions are satisfied by the specific data described in the statement of the proposition, exactly as in the proof of Proposition 2.2. We will write $Z(t, y) := (t, y + x_0(t))$. Then for every $y,$

$$u(Z(t, y)) - u(Z(0, y)) = \int_0^t u_t(Z(s, y)) + x'_0(s) u_x(Z(s, y)) \, ds \leq C \left( t \int_0^t \frac{e^0_u(u(Z(s, y)))}{\varepsilon} \, ds \right)^{1/2}.$$  

Also, $\{(t, x) : |t| \leq T, x < x_0(t)\} = \{Z(t, y) : |t| \leq T, y \leq 0\}$, so by squaring the above inequality, integrating from $y = -\infty$ to $y = 0$, integrating in $t$, changing variables, and using (2.28) and (2.29), we find that

$$\int_{-T}^T \int_{x_0(t)}^{x_0(t)} |u(t, x) + 1|^2 \, dx \, dt \leq \frac{C}{\varepsilon} \zeta_1(0) \leq C \varepsilon.  \tag{2.30}$$

Next, recall that $u \circ \psi = v$, so by a change of variables (see (2.10) and (2.19),

$$\int_{-T}^T \int_{x_0(t)}^{x_0(t)} |u - \text{sign}(x - \gamma(t))|^2 \leq C \int \int |v(\theta, r) - \text{sign}(r - r_0)|^2 \, dr \, d\theta \leq C \varepsilon.$$  

Recalling that $u(t, x) \equiv 1$ for $x \geq R + |t|$, we deduce that if (2.26), (2.29) hold then

$$\int_{-T}^T \int_R \left| u(t, x) - \text{sign}(x - \gamma(t)) \right|^2 \leq C \varepsilon.  \tag{2.31}$$

**Step 3: proof of (2.27):** A short calculation shows that

$$\frac{d}{dt} e^0_{\varepsilon}(u) = \varepsilon(u_t u_x)_x + \kappa u_t (1 - u^2).  \tag{2.32}$$

Then it follows from (2.32) that

$$\frac{d}{dt} \int \chi e^0_{\varepsilon}(u) = \int \chi_t e^0_{\varepsilon} + \varepsilon \int \chi (u_t u_x)_x + \kappa \int \chi u_t (1 - u^2).$$

We integrate by parts and use some elementary inequalities to find

$$\frac{d}{dt} \int \chi e^0_{\varepsilon}(u) \leq \int (|\chi_t| + |\chi_x|) e^0_{\varepsilon} + \kappa \int \chi e^0_{\varepsilon}.$$  

This implies that

$$\frac{d}{dt} \left( e^{-\kappa t} \int \chi e^0_{\varepsilon}(u) \right) \leq e^{-\kappa t} \int (|\chi_t| + |\chi_x|) e^0_{\varepsilon}(u) \leq C_1 e^{-\kappa t} \int_{\text{supp} \chi_x} e^0_{\varepsilon}(u),$$

for $|t| \leq T$, where we may take $C_1 = \frac{C(1 + T)}{r_0}$. We arrive at (2.27) by integrating this expression from 0 to $t$.

□

### 3. Statement of Main Theorem

In this section we state our main theorem, which relates a semilinear wave equation to a hypersurface of prescribed mean curvature on a Lorentzian manifold. We first introduce these ingredients.

We will always use greek letters such as $\alpha, \beta, \ldots$ to denote indices that run between 0 and $n$, and we implicitly sum repeated greek indices from 0 to $n$. We will explicitly indicate sums that run over different ranges, we will generally not use greek letters for indices that belong to some proper subset of $\{0, \ldots, n\}$. 

[89x398]
3.1. the Lorentzian manifold. We consider a manifold $N$ that we assume to be homeomorphic to $\mathbb{R}^{1+n}$, $n \geq 2$, and we fix global coordinates $(x^0, \ldots, x^n)$. We will write $h$ to denote a Lorentzian inner product on $N$, and we let $(h_{\alpha\beta})$ denote the components of the metric tensor with respect to the given coordinates. We also write
dd(h_{\alpha\beta}^{\alpha\beta}) = (h_{\alpha\beta})^{-1}, \quad h = \det(h_{\alpha\beta}).$
We will assume that $(h_{\alpha\beta})$ is smooth and that there exists some constant $c_1 > 0$ such that
everywhere in $N$. Thus $x^0$ may be thought of as a time coordinate, and we will sometimes write $t$ for $x^0$. We will further assume that
\begin{equation}
\label{eq:00}
h_{0i} = 0, \quad i = 1, \ldots, n.
\end{equation}
Then it is clear that $0 < c \leq -h \leq C$ everywhere in $N$. For $0 \leq t \leq T < \infty$, we will use the notation
\[ \Sigma_t := \{ x \in N : x^0 = t \}, \quad N_T := \{ x \in N : |x^0| < T \}. \]
In view of \eqref{eq:00}, we can obtain a Riemannian metric from $h$ by changing the sign of $h_{00}$. Since this metric is uniformly equivalent to the Euclidean metric, and the associated volume element is uniformly comparable to the Euclidean volume element on $\mathbb{R}^{1+n}$, we will for simplicity use the Euclidean structure on $\mathbb{R}^{1+n}$ to define $L^p$ and Sobolev norms on $N$.

3.2. the semilinear wave equation. Let $\kappa : N \to \mathbb{R}$ be a fixed smooth function. We will consider the equation
\begin{equation}
\label{eq:semilinear}
\varepsilon \Box_h u + \frac{1}{\varepsilon} f_0(u) + \kappa f_1(u) = 0, \quad u : N \to \mathbb{R}
\end{equation}
where
\[ \Box_h u := \frac{-1}{\sqrt{-h}} \partial_{x^0} \left( \sqrt{-h} h^{\alpha\beta} \partial_{x^0} u \right). \]
It follows from \eqref{eq:00} that \eqref{eq:semilinear} is a hyperbolic equation. We always assume that the nonlinearities $f_0, f_1$ in \eqref{eq:semilinear} have the form
\begin{equation}
\label{eq:nonlinearities}
f_0 = F', \quad f_1 = \begin{cases} 
\sqrt{2F} & \text{in } [-1,1] \\
-\sqrt{2F} & \text{elsewhere}.
\end{cases}
\end{equation}
for $F : \mathbb{R} \to \mathbb{R}$ a smooth function such that
\begin{equation}
\label{eq:potential}
F(x) > 0 \text{ if } |x| \neq 1, \quad c(1 - |x|)^2 \leq F(x) \leq C(1 - |x|)^2 \text{ if } |x| \leq 2.
\end{equation}
Note that $f_0, f_1$ are smooth as a consequence of \eqref{eq:potential} and the smoothness of $F$.

We do not address questions about well-posedness of \eqref{eq:semilinear}. When $(N, h)$ is flat Minkowski space $\mathbb{R}^{1+n}$ then global well-posedness in the energy space can be guaranteed by imposing suitable growth conditions on $F$. In particular, if $n \leq 4$ then \eqref{eq:semilinear} on Minkowski space is globally well-posed for $f_0(u) = 2(u^2 - 1)u$ and $f_1(u) = 1 - u^2$, associated to the potential $F(u) = \frac{1}{2}(1 - u^2)^2$.
The form of the nonlinearity is further discussed in Section 3.6, where we show that the assumptions \eqref{eq:nonlinearities} are not actually restrictive if $\kappa$ is constant. Note also that one can easily generate nonlinearities satisfying the above conditions by starting from $f_1$ such that $\text{sign } f_1(s) = \text{sign}(1 - s^2), |f_1'(\pm 1)| \neq 0$, then defining $F = \frac{1}{2} f_1^2$ and $f_0 = F' = f_1 f_1'$.\end{document}
3.3. the hypersurface of prescribed mean curvature. We assume that $I$ is a bounded open subset of $N_{T^*}$, for some $T^* > 0$, such that $\Gamma := \partial I \cap N_{T^*}$ is a smooth embedded timelike hypersurface satisfying the prescribed mean curvature equation
\begin{equation}
- \kappa(x) = \text{mean curvature in } (N, h) \text{ of } \Gamma \text{ at } x
\end{equation}
and that
\begin{equation}
\Gamma \text{ is orthogonal to the initial hypersurface } \Sigma_0.
\end{equation}
This means that $\Gamma$ has zero initial velocity with respect to the initial hypersurface. See (4.2), (4.3) below for a precise formulation.

For smooth data and smooth $\kappa$, such as we consider here, local existence of smooth embedded submanifolds $\Gamma \subset N$ satisfying (3.6), (3.7) follows from arguments in Milbredt [12].

Remark 3.1. In fact Milbredt [12] studies a rather general Cauchy problem for the $\kappa = 0$ case of (3.6) on a larger class of Lorentzian manifolds than we consider here. Modifying his arguments to extend to the case of smooth $\kappa$ presents no difficulty. His basic existence results are ultimately proved by solving (3.6) in coordinate charts (with a suitable choice of gauge) and then piecing together these local solutions, using finite propagation speed for the equation. Solvability in coordinate charts depends on local solvability results for the Cauchy problem for a general quasilinear hyperbolic equation, of the form
\begin{equation}
g^{\alpha\beta}(\Psi, D\Psi) \partial_\alpha \partial_\beta \Psi = f(\Psi, D\Psi),
\end{equation}
where $g^{00} \leq -\lambda$ and $g^{ij} \geq \mu \delta^{ij}$. Changing the equation to allow nonzero $\kappa$ simply adds some additional smooth lower-order terms on the right-hand side, and the existence results used in [12] apply with no change to the equation once it is modified in this way. Other aspects of the argument, such as piecing together local solutions, are similarly unaffected.

3.4. main theorem. The main result of this paper is the following. The statement uses terminology and notation introduced in Sections 3.1 - 3.3, and for any set $U$, we will write
\begin{equation}
\text{sign}_U(x) := \begin{cases} 
1 & \text{if } x \in U, \\
-1 & \text{if not.}
\end{cases}
\end{equation}

Theorem 3.2. Assume that $T_0 < T^*$.

Then there exists a neighborhood $N'$ of $\Gamma \cap N_{T_0}$ and a unique smooth function $d_\Gamma : N' \to \mathbb{R}$ such that
\begin{equation}
d_\Gamma(x) = 0 \text{ on } \Gamma, \quad h^{\alpha\beta} \partial_\alpha d_\Gamma \partial_\beta d_\Gamma = 1, \quad d_\Gamma > 0 \text{ in } I \cap N'.
\end{equation}
Moreover, $d_\Gamma$ is bounded away from 0 outside of every neighbourhood of $\Gamma$.

In addition, for every $\varepsilon \in (0, 1]$, there exists smooth initial data $(u_0, u_1) \in \dot{H}^1 \times L^2(\Sigma_0)$ such that, if $u$ is a smooth solution of (3.3) with $(u, \partial_\nu u) = (u_0, u_1)$ on $\Sigma_0$, then
\begin{equation}
\int_{N_{T_0}} |u - \text{sign}_I|^2 \leq C\varepsilon,
\end{equation}
and
\begin{equation}
\int_{N_{T_0}} \left[ d_\Gamma^2 T_{N'} \right] \left( \varepsilon |Du|^2 + \frac{1}{\varepsilon} F(u) \right) \leq C\varepsilon^2.
\end{equation}

Here $C$ is a constant that depends on $h, F, \Gamma$ but is independent of $\varepsilon$.

\footnote{Our sign conventions for the unit normal, and hence the mean curvature, are described in Section 7.3 where we also review some basic properties of mean curvature. These sign conventions are such that the curve around which the solution in Section 2 concentrates, with the orientation we have implicitly chosen there, in fact has “mean curvature” equal to $-\kappa$ rather than $\kappa$.}
The function $d_\Gamma$ from the theorem is the signed distance to $\Gamma$ with respect to the $h$ metric. In fact under our hypotheses it is uniformly comparable to the signed Euclidean distance to $\Gamma$, so we could replace $d_\Gamma$ in (3.10) by the Euclidean squared distance with changes only to constants (depending on $\Gamma$ and the choice of the neighborhood $N'$).

Our proof yields additional information that we have not recorded in the statement of the theorem, including the following:

- We show that $\int \varepsilon |Du|^2 + \frac{1}{2} F(u) \ge C > 0$ for all small $\varepsilon$, so (3.10) implies that the energy is strongly concentrated near $\Gamma$.

- We find certain vector fields $X$, depending only on the geometry of $\Gamma$, such that $\|X \cdot Du\|_{L^2(N_{r_0})} \le C\varepsilon$.

- Our arguments in fact establish not just estimates of some specific solutions, but also more general stability estimates, see Proposition 5.1 for example.

3.5. Discussion. As in [9] and Section 2, the heart of the proof of Theorem 3.2 consists of weighted energy estimates in well-chosen coordinates near $\Gamma$. In particular, we will introduce coordinates $(y^0, \ldots, y^n)$ such that $\Gamma = \{ (y^0, \ldots, y^n) : y^n = 0 \}$, and in addition $y^n \mapsto c(y^n) = (y^0, \ldots, y^{n-1})$ is (approximately) a geodesic with respect to the Lorentzian metric for every $(y^0, \ldots, y^{n-1})$, with $c(0) \in \Gamma$ and $c'(0)$ normal to $\Gamma$. A key point is that the geometry of $\Gamma$ is exactly such that, when the equation (3.3) is written in the these coordinates, some cancellations occur that make very strong energy estimates possible.

Writing $v$ to denote the solution of (3.3) in the $y$ coordinates, the initial data we consider will have the form

$$v(0, y^1, \ldots, y^n) \approx q(y^n) =: q_\varepsilon(n), \quad \partial_{y^n} v(0, y^1, \ldots, y^n) = 0$$

near $\Gamma = \{ y^n = 0 \}$, where $q$ solves

$$-q'' + f_0(q) = 0, \quad q(0) = 0, \quad q(s) \to \pm 1 \text{ as } s \to \pm \infty.$$  

Existence of a profile $q$ solving (3.12) is standard. Indeed, multiplying by $q'$, integrating, and using (3.4), one finds that the unique solution of

$$q' - f_1(q) = 0, \quad q(0) = 0$$

also satisfies (3.12). We remark for future reference that standard ODE arguments, using the fact that $f_1'(\pm 1) > 0$ (see (3.5)), imply that

$$|q(s) - \text{sign}(s)| \le C e^{-c|s|} \quad \text{as } s \to \pm \infty.$$  

As is well-known, the profile $q_\varepsilon$ is characterized by an optimality property. Indeed, for any $\tilde{q} : \mathbb{R} \to \mathbb{R}$,

$$f_1(\tilde{q}) \tilde{q}' \le \frac{\varepsilon}{2} \varepsilon^2 + \frac{1}{2\varepsilon} f_1^2(\tilde{q}) = \frac{\varepsilon}{2} \tilde{q}'^2 + \frac{1}{\varepsilon} F(\tilde{q})$$

by (3.4). Thus if $\tilde{q}(s) \to \pm 1$ as $s \to \pm \infty$, then

$$c_0 := \int_{-1}^{1} f_1(s) \, ds = \int_{-\infty}^{\infty} f_1(\tilde{q}(s)) \tilde{q}'(s) \, ds \le \int_{-\infty}^{\infty} \frac{\varepsilon}{2} \tilde{q}'^2 + \frac{1}{\varepsilon} F(\tilde{q}) \, ds.$$  

Moreover, equality holds if and only if $\varepsilon \tilde{q}' = f_1(\tilde{q})$, which occurs exactly when $\tilde{q}$ is a translate of the profile $q_\varepsilon$ above.

Formal arguments suggest that the solution $v$ with initial data (3.11) should satisfy

$$v(y^0, \ldots, y^n) \approx q_\varepsilon(y^n).$$

As in Remark 2.3, our basic estimate (3.9) in fact implies that

$$\int [v(y^0, \ldots, y^n) - q_\varepsilon(y^n)]^2 \le C\varepsilon,$$

but is not sharp enough to distinguish the shape of the profile; that is, it does not allow us to say whether $\text{sign}(y^n)$ or $q_\varepsilon(y^n)$ is closer to $v$. But,
again as in Remark 2.3, estimates established in the course of the proof in fact imply that for most \((y^0, \ldots, y^{n-1})\),
\[
\int \left| v(y^1, \ldots, y^{n-1}, y^n) - q\left(\frac{y^n - y_0^n}{\varepsilon}\right)\right|^2 \frac{dy^n}{\varepsilon} \leq C\varepsilon^2
\]
for some translation \(y_0^n = y_0^n(y^0, \ldots, y^{n-1})\) such that \(|y_0^n| \leq C\varepsilon\). Indeed, this follows from estimates we establish in Proposition 5.1 of \(\zeta_1\), defined in (5.4) below, together with spectral estimates like those discussed in Remark 2.3.

3.6. **About the nonlinearity.** In equation (3.3), we have assumed a nonlinearity \(f_\varepsilon(x, u) := f_0(u) + \varepsilon\kappa(x)f_1(u)\), for \(f_0, f_1\) satisfying (3.4), that appears to have a very special form.

This is not actually the case if \(f_\varepsilon\) depends only on \(u\); in this case a nonlinearity \(f_\varepsilon\) associated to a general double-well potential can be written in this form. Indeed, assume that \(f\) has a choice of properties of \(f\) such that \(f(\varepsilon x) = f(x)\) and in particular vanishing exactly at \(y_0^n\), for some translation \(y_0^n = y_0^n(y^0, \ldots, y^{n-1})\) for most \((y^0, \ldots, y^{n-1})\), such that \(|y_0^n| \leq C\varepsilon\). Indeed, this follows from estimates we establish in Proposition 5.1 of \(\zeta_1\), defined in (5.4) below, together with spectral estimates like those discussed in Remark 2.3.

We will always use **standard local coordinates** on \(M_T\), by which we mean coordinates of the form \((y^0, y')\), where \(y^0 \in (-T, T)\) and \(y' = (y^1, \ldots, y^{n-1})\) are local coordinates on \(M\). We

4. **An adapted coordinate system**

We now introduce the coordinate system near \(\Gamma = \partial I \cap N_{-1}\) in which our main estimates will take place.

4.1. **A good parametrization of \(\Gamma\).** Fix a smooth \((n-1)\)-dimensional manifold \(M\) diffeomorphic to \(\Gamma_0 := \Gamma \cap \Sigma_0\). Our assumptions imply that \(M\) is compact. The example that arises most naturally in cosmological settings is \(M = S^{n-1}\). We will write \((y^1, \ldots, y^{n-1})\), or simply \(y'\), to denote local coordinates on \(M\).

For \(T > 0\), we will write
\[
M_T := (-T, T) \times M,
\]
We will always use **standard local coordinates** on \(M_T\), by which we mean coordinates of the form \((y^0, y')\), where \(y^0 \in (-T, T)\) and \(y' = (y^1, \ldots, y^{n-1})\) are local coordinates on \(M\). We

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will often write \( y^\tau = (y^0, y') \) to denote a point in \( M_T \), where the superscript \( \tau \) stands for “tangential”.

We will parametrize \( \Gamma \subset N_T^\tau \) by a smooth map \( \Psi : M_T^\tau \to N_T^\tau \) of the form

\[
\Psi(y^\tau) = (y^0, \psi(y^\tau)) \quad \text{for some } \psi : M_T^\tau \to \mathbb{R}^n.
\]

(Here and below, we use the fixed coordinate system on \( N \) to identify it with \( \mathbb{R}^{1+n} \).) Note that with this convention and condition \( (3.2) \) on the metric \( (h_{\alpha\beta}) \), assumption \( (3.7) \) becomes

\[
\frac{\partial \psi}{\partial y^0}(0, y') = 0 \quad \text{for all } y' \in M.
\]

We also impose the condition

\[
\gamma_{0a} = \gamma_{a0} = 0 \quad \text{for } a = 1, \ldots, n - 1,
\]

where here and in what follows, we use the notation

\[
\gamma_{ab} := h\left( \frac{\partial \Psi}{\partial y^a}, \frac{\partial \Psi}{\partial y^b} \right) \quad \text{for } a, b = 0, \ldots, n - 1.
\]

The existence of such a parametrization \( \Psi \) is rather standard. Indeed, suppose we are given local coordinates \( y' = (y', \ldots, y^{n-1}) \) on a subset of \( M \). By definition \( M \) is diffeomorphic to \( \Gamma_0 \), so we may fix a diffeomorphism \( y' \mapsto \psi(y') \in \Gamma_0 \). Then for every \( y' \) and sufficiently small \( \delta > 0 \) there exists a unique curve \( p = p(\cdot; y') : (-\delta, \delta) \to N \), with components \( (p^0, \ldots, p^n) \), such that

\[
p^0(t) = t, \quad p(0; y') = \psi_0(y'), \quad h(p'(t), \frac{\partial \Psi}{\partial y^a}) = 0 \quad \text{for } a = 0, \ldots, n - 1 \text{ and } |t| < \delta.
\]

Indeed, in coordinates this is just a first-order ODE for \( p(\cdot) \) to which standard theorems apply. It is then easy to see that \( \delta = \delta(y') \) is bounded away from zero on small enough subsets, and on those sets can define \( \Psi(y^0, y') := p(y^0, y') \).

### 4.2. almost-normal coordinates near the hypersurface.

For \( r \) positive, we will write

\[
M^\tau := M \times (-r, r). \quad M_T^\tau := M_T \times (-r, r).
\]

As above, we will always use standard local coordinates on these spaces, that is, coordinate systems that respect the product structure. Thus, in these coordinates, points in \( M_T^\tau \) have the form \( (y^\tau, y^n) = (y^0, y', y^n) \), where \( y' \) are local coordinates on \( M, |y^0| < T \) and \( |y^n| < r \).

The next proposition introduces a map \( \phi : M_T^{2\rho} \to N \) that parametrizes a neighbourhood of \( \Gamma \) and such that certain good properties are enjoyed by the pullback metric

\[
g_{\alpha\beta} := h\left( \frac{\partial \phi}{\partial y^\alpha}, \frac{\partial \phi}{\partial y^\beta} \right), \quad (g^{\alpha\beta}) := (g_{\alpha\beta})^{-1}, \quad g := \det(g_{\alpha\beta})
\]

where \( \alpha, \beta \in \{0, \ldots, n\} \). The proof will be deferred to Section \ref{sec:proof}. We remark however that \( \phi \) essentially defines a Gaussian normal coordinate system near \( \Gamma \), modified slightly to arrange that condition \( (4.10) \) below holds; this condition implies that changing variables using \( \phi \) maps Cauchy problems for \( (3.3) \), with data given for \( x^0 = 0 \), to Cauchy problems with data given on the hypersurface \( \{y^n = 0\} \). This will be useful.

**Proposition 4.1.** For every \( T_0 < T^\star \), there exists \( \phi : M_T^{2\rho} \to N \), for some \( T \in (T_0, T^\star) \) and \( \rho > 0 \), such that such that \( \phi \) is a diffeomorphism onto its image, and the following hold. First, \( \phi(y^\tau, 0) = \Psi(y^\tau) \), which implies that

\[
\Gamma \cap N_{T_0} = \phi(M_{T_0} \times \{0\}),
\]

and hence that

\[
N' := \phi(M_T^{2\rho}) \text{ is an open neighbourhood of } \Gamma \cap N_{T_0}.
\]
In addition,

\[(4.9) \quad \text{if } y^0 \in M^T \text{ and } |y^0| = T, \text{ then } |x^0| > T_0 \text{ for } x = \phi(y).\]

Second,

\[(4.10) \quad \phi(\{0\} \times M^{2\rho}) \subset \Sigma_0 = \{ x \in N \cong \mathbb{R}^{1+n} : x^0 = 0 \}.\]

Third, the metric satisfies

\[(4.11) \quad (g_{\alpha\beta})(y^\gamma, y^n) = \begin{pmatrix} (\gamma_{ab})(y^\gamma) & 0 \\ 0 & 1 \end{pmatrix} + O \begin{pmatrix} |y^n| \\ |y^n| \end{pmatrix} \quad (in \ block \ form), \quad \text{where } (\gamma_{ab}) \text{ was introduced in (4.5). Hence}\]

\[(4.12) \quad (g^\alpha{}_{\beta})(y^\gamma, y^n) = \begin{pmatrix} (\gamma^a{}_{b})(y^\gamma) & 0 \\ 0 & 1 \end{pmatrix} + O \begin{pmatrix} |y^n| \\ |y^n| \end{pmatrix} \quad (in \ block \ form).\]

In addition,

\[(4.13) \quad (\partial_0 g^{\alpha\beta})(y^\gamma, y^n) = O \begin{pmatrix} 1 \\ |y^n| \end{pmatrix} \quad (in \ block \ form).\]

Next, the eikonal equation (4.8) has a unique smooth solution \(d_{\Gamma} \) on \(N'\), and if we define \(\pi^n(y^0, \ldots, y^n) = y^n\), then

\[(4.14) \quad \pi^n \circ \phi^{-1} = d_{\Gamma} + O(d_{\Gamma}^2).\]

Finally,

\[(4.15) \quad -\frac{1}{\sqrt{-g}} g^{\alpha\xi} \partial_{y^\alpha} \sqrt{-g} = -\frac{1}{\sqrt{-g}} \partial_{y^n} \sqrt{-g} + O(|y^n|) = -\kappa(y^n) + O(|y^n|).\]

**Remark 4.2.** The implied constants in the above estimates could in principle depend on the choice of local coordinates for \(M\). However, our assumptions imply that \(M\) is compact, and so we may once and for all fix a cover of \(M\) by a finite collection of coordinate neighborhoods \(U_1, \ldots, U_k\), such that all the above estimates are uniform on \((-T, T) \times U_i \times (-2\rho, 2\rho)\) for every \(i\). (This last point will be evident from the proof of the proposition.) We can then require that all subsequent computations in local coordinates are carried out in one of these fixed coordinate systems. Having done this, all the above constants are uniform on \(M^{2\rho}\). The same remark applies below as well.

Our later energy estimates will contain a symmetric tensor \((a_{\alpha\beta})\), defined by

\[(4.16) \quad \frac{1}{2} a_{\alpha\beta} \xi_\alpha \xi_\beta := -g^{\alpha\xi} \xi_\xi + \frac{1}{2} g^{\alpha\beta} \xi_\alpha \xi_\beta = -\frac{1}{2} g^{\alpha\alpha} \xi_\xi + \frac{1}{2} \sum_{i,j=1}^n g^{ij} \xi_i \xi_j.\]

It follows from (4.12) that if \(\rho\) is taken to be small enough (which we henceforth assume to be the case) then there exists some positive constants \(c_2, c_3, c_4, c_5\) such that

\[(4.17) \quad \frac{1}{2} \sum_{a,b=0}^{n-1} a_{ab} \xi_a \xi_b + (1 + (y^n)^2) \xi_n^2 \leq (1 + c_2(y^n)^2) a_{\alpha\beta} \xi_\alpha \xi_\beta \leq 2 \sum_{a,b=0}^{n-1} a_{ab} \xi_a \xi_b + (1 + c_3(y^n)^2) \xi_n^2,\]

\[(4.18) \quad \sum_{a,b=0}^{n-1} \delta^{ab} \xi_a \xi_b \leq c_4 \sum_{a,b=0}^{n-1} a_{ab} \xi_a \xi_b,\]

and

\[(4.19) \quad |g^{\alpha\xi} \xi_\xi| \leq \frac{c_5}{2} a_{\alpha\beta} \xi_\alpha \xi_\beta\]

everywhere in \(M^{2\rho}\), for all \(\xi \in \mathbb{R}^{1+n}\).
5. Weighted Energy Estimates in Normal Coordinates

In this section we study the wave equation

\begin{equation}
\varepsilon \square_g v + \frac{1}{\varepsilon} f_0(v) + \kappa f_1(v) = 0, \quad v : M_T^{2\rho} \to \mathbb{R}
\end{equation}

Here \( g \) is a metric on \( M_T^{2\rho} \) satisfying the conclusions of Proposition 4.1. \( \kappa \) is a smooth, bounded function on \( M_T^{2\rho} \), and \( f_0, f_1 \) satisfy (3.4). In particular, if \( u \) solves (3.3) on \( N \) then \( v := u \circ \phi \) solves (5.1) on \( M_T^{2\rho} \).

To state our main estimates, we need some notation. We start by fixing a smooth volume form \( d(vol)_0 \) on \( M \), and we extend it to a volume form \( d(vol) \) on \( M_T^{2\rho} \) by requiring that

\begin{equation}
(5.2) \quad d(vol) = dy^0 \wedge d(vol)_0 \wedge dy^n
\end{equation}

in standard local coordinates. We emphasize that \( d(vol) \) in general does not coincide with the volume form associated to the Lorentzian metric \( g \).

We will similarly extend \( d(vol)_0 \) to \( M_T \) and \( M^{2\rho} \), writing \( d(vol) \) in every case; the meaning should always be clear from the context.

Thus, in standard local coordinates these are represented by expressions of the form

\[
d(vol)_0 = \omega_0(y') dy^1 \wedge \cdots \wedge dy^n, \\
d(vol) = \omega(y) dy^0 \wedge dy^1 \wedge \cdots \wedge dy^n \wedge dy^n \quad \text{on } M_T^{2\rho}
\]

where \( \omega(y', y^n) = \omega_0(y') \) in \( M_T^{2\rho} \). Here \( \omega_0 \) is a smooth positive function that depends on our choice of local coordinates for \( M \). Similar expressions hold for \( d(vol) \) on \( M_T \) and \( M^{2\rho} \).

Next, we define a natural energy density associated to (5.1). For \( v \in H^1(M_T^{2\rho}) \), let

\begin{equation}
(5.3) \quad e_\varepsilon(v; g) := \frac{\varepsilon}{2} a^{\alpha\beta} \partial_{y^\alpha} v \partial_{y^\beta} v + \frac{1}{\varepsilon} F(v),
\end{equation}

for \( F \) defined in (3.5), and \( a^{\alpha\beta} \) defined in (4.16). We will write simply \( e_\varepsilon(v) \) when there is no ambiguity, which will be the case throughout this section. Finally, recall that we have defined

\[
c_0 := \int_{-1}^1 f_1(u) \, du,
\]

and that, as noted in (3.15), \( c_0 \) is a lower bound for the energy of a 1-d interface connecting the equilibrium states \( \{ \pm \} \), and this lower bound is attained by the profile \( q \).

The following estimates are the heart of the proof of Theorem 3.2

**Proposition 5.1.** Let \( v \) be a smooth solution of (5.1) on \( M_T^{2\rho} \), and assume that \( g \) satisfies the conclusions of Proposition 4.1. Define \( \rho(s) = \rho - c_5 s \), for \( c_5 \) defined in (4.19), and

\begin{align}
(5.4) \quad \zeta_1(s) &:= \int_{M^{\rho(s)}} (1 + c_2(y^n)^2) e_\varepsilon(v) d(vol) \bigg|_{y^\rho=s} - c_0 vol_0(M) \\
(5.5) \quad \zeta_2(s) &:= \int_{M^{\rho/2}} |y^n| |v - \text{sign}(y^n)|^2 d(vol) \bigg|_{y^\rho=s} \\
(5.6) \quad \zeta_3(s) &:= \int_{M^{\rho(s)}} \frac{\varepsilon}{2} \sum_{a,b=0}^{n-1} a^{ab} v_{y^a} v_{y^b} + (y^n)^2 \left[ \frac{1}{2} \varepsilon |\partial_{y^n} v|^2 + \frac{1}{\varepsilon} F(v) \right] d(vol) \bigg|_{y^\rho=s}
\end{align}

Then there exists a constant \( C \), independent of \( v \) and of \( \varepsilon \in (0, 1) \), such that

\begin{equation}
(5.7) \quad \zeta_i(s) \leq C \max(\zeta_1(0), \zeta_2(0)) \quad \text{for } i = 1, 2, 3 \text{ and } 0 < s < s_1 := \min(T, \rho/(3c_5))
\end{equation}
5.1. differential energy inequality.

**Lemma 5.2.** Assume the hypotheses of Proposition 5.1. Then

\[
\frac{\partial}{\partial y^0} \epsilon \varphi (v) \leq \epsilon C \left( \sum_{a,b=0}^{n-1} \alpha^\beta \partial_{y^a} v \partial_{y^b} v + |g|^2 |\partial_{y^a} v|^2 \right) + \epsilon \operatorname{div}_{M^2\varphi} \varphi + \kappa \left[ \epsilon v^{\alpha} - f_1 (v) \right] \cdot v_{y^\varphi}
\]

where

\[
\varphi : = (\varphi^1, \ldots, \varphi^n), \quad \varphi^i : = g^{i\alpha} v_{y^\alpha} \cdot v_{y^\varphi}
\]

and \( \operatorname{div}_{M^2\varphi} \) denotes the divergence on \( M^2\varphi \) with respect to the fixed volume form \( d(\text{vol}) \), so that

\[
\operatorname{div}_{M^2\varphi} \varphi := \frac{1}{\omega} \sum_{i=1}^{\omega} \partial_{y^i} (\omega \varphi^i).
\]

Note that \( \epsilon \varphi \in \{ \varphi \in \mathbb{R}^n : \epsilon \varphi \leq 0 \} \), and the constants in Proposition 5.1 may depend for example on \( \| \kappa \|_{\infty} \), but they are independent of \( \epsilon \in (0, 1] \).

**Proof.** In standard local coordinates on \( M^2\varphi \), our equation (5.1) takes the form

\[
-\frac{\epsilon}{\omega} \partial_{y^\alpha} \left( \sqrt{-g} g^{\alpha\beta} \partial_{y^\beta} v \right) + \frac{1}{\epsilon} f_0 (v) + \kappa f_1 (v) = 0.
\]

We rewrite the leading term as a divergence with respect to \( d(\text{vol}) \) on \( M^2\varphi \), leading to

\[
-\frac{\epsilon}{\omega} \partial_{y^\alpha} \left( \omega g^{\alpha\beta} \partial_{y^\beta} v \right) - \epsilon b^\alpha \partial_{y^\alpha} v + \frac{1}{\epsilon} f_0 (v) + \kappa f_1 (v) = 0
\]

where

\[
b^\beta := \frac{\omega}{\sqrt{-g}} \left( \sqrt{-g} g^{\alpha\beta} \partial_{y^\alpha} \left( \sqrt{-g} \right) \right).
\]

Multiply this by \( v_{y^\varphi} \) and rewrite, recalling (3.4), to find that

\[
-\frac{\epsilon}{\omega} \partial_{y^\alpha} (\omega g^{\alpha\beta} v_{y^\beta} v_{y^\varphi}) + \epsilon g^{\alpha\beta} v_{y^\alpha} v_{y^\varphi} v_{y^\varphi} + \frac{1}{\epsilon} F (v) v_{y^\varphi} = \epsilon b^\alpha v_{y^\alpha} v_{y^\varphi} - \kappa f_1 (v) v_{y^\varphi}
\]

Note that

\[
g^{\alpha\beta} v_{y^\alpha} v_{y^\varphi} v_{y^\varphi} = \frac{1}{2} \partial_{y^\alpha} (g^{\alpha\beta} v_{y^\beta} v_{y^\varphi}) - \frac{1}{2} \partial_{y^\alpha} (g^{\alpha\beta} v_{y^\beta} v_{y^\varphi})
\]

and that \( \partial_{y^\varphi} \omega = \partial_{y^\alpha} \omega = 0 \) on \( M^2\varphi \). We use these facts and collect all the terms of the form \( \partial_{y^\alpha} [ \cdots ] \) to the left-hand side to obtain

\[
\partial_{y^\varphi} \left[ -\epsilon g^{\alpha\beta} v_{y^\beta} \cdot v_{y^\varphi} + \frac{\epsilon}{2} g^{\alpha\beta} v_{y^\alpha} v_{y^\varphi} + \frac{1}{\epsilon} F (v) \right]
\]

\[
= \frac{\epsilon}{\omega} \partial_{y^\alpha} (\omega g^{\alpha\beta} v_{y^\beta} v_{y^\varphi}) + \frac{\epsilon}{2} g^{\alpha\beta} v_{y^\alpha} v_{y^\varphi} v_{y^\varphi} + \epsilon b^\alpha v_{y^\alpha} v_{y^\varphi} - \kappa f_1 (v) v_{y^\varphi}.
\]

By definition, the left-hand side is just \( \partial_{y^\varphi} \epsilon \varphi (v) \), and the first term on the right-hand side is exactly \( \epsilon \operatorname{div}_{M^2\varphi} \varphi \). It follows from (4.13) and (4.18) that

\[
g^{\alpha\beta} v_{y^\alpha} v_{y^\varphi} v_{y^\varphi} \leq C \sum_{a,b=0}^{n-1} a^{ab} v_{y^a} v_{y^b}.
\]
To estimate the remaining terms on the right-hand side of (5.12) first note that
\[ \varepsilon b^n v y^n v y^\rho - \kappa f_1(v) v y_0 = \varepsilon \sum_{a=0}^{n-1} b^a v y^n v y^\rho + \varepsilon (b^n - \kappa) v y^n v y^\rho + \kappa [\varepsilon v y_n - f_1(v)] v y^\rho. \]

Also, since \( \partial_\rho \omega = 0 \), we see that (4.15) states exactly that \( |b^n - \kappa| = O(|y^n|) \). Thus (4.18) implies that
\[ |(b^n - \kappa) v y^n v y^\rho| \leq C |y^n v y^n v y^\rho| \leq C \sum_{a,b=0}^{n-1} a^b v y^n v y^\rho + (y^n)^2 |v y^n|^2 \]
and that
\[ \sum_{a=0}^{n-1} b^a v y^n v y^\rho \leq C \sum_{a,b=0}^{n-1} a^b v y^n v y^\rho. \]

Thus the Lemma follows by combining these facts with (5.12). \( \square \)

5.2. stability of the profile. In this section we collect a couple of lemmas that encode some stability properties of initial data for which \( \zeta_i(0) \) is small, \( i = 1, 2 \). These concern functions of a single variable, which we will denote \( y^n \), since later we will apply these results to the functions \( y^n \mapsto v(y^r, y^n) \) for \( y^r \in M_T \) fixed.

Lemma 5.3. There exists a constant \( c_6 = c_6(\rho) \) such that if \( v \in H^1(-\rho, \rho) \) and if
\[ (5.13) \quad \int_{-\rho}^{\rho} |y^n| |v - \text{sign}(y^n)|^2 dy^n \leq c_6 \]
then
\[ (5.14) \quad \int_{-\rho}^{\rho} e_{\varepsilon, \nu}(v) dy^n \geq c_0 - C e^{-c/\varepsilon}, \quad \text{for } e_{\varepsilon, \nu}(v) := \frac{\varepsilon}{2} |\partial_{y^n} v|^2 + \frac{1}{\varepsilon} F(v). \]
Moreover, there exists a constant \( c_7 = c_7(\rho) > 0 \) such that if (5.13) holds and if
\[ (5.15) \quad \int_{-\rho}^{\rho} e_{\varepsilon, \nu}(v) dy^n - c_0 \leq c_7 \]
then for every \( \bar{\rho} \geq \rho \),
\[ (5.16) \quad \int_{-\rho}^{\bar{\rho}} \frac{1}{2} \left( \sqrt{\varepsilon} v y^n - \frac{f_1(v)}{\sqrt{\varepsilon}} \right)^2 dy^n \leq C \left( \int_{-\rho}^{\bar{\rho}} e_{\varepsilon, \nu}(v) dy^n - c_0 \right) + C e^{-c/\varepsilon} \]
as long as \( v \) is defined on \(( -\bar{\rho}, \bar{\rho} ) \), with \( C \) independent of \( \bar{\rho} \).

These are largely proved in Lemma of [9], but since we have modified the statement here in some ways, we present some details.

Proof. Steps 3 and 4 of the proof of Lemma 11 of [9] show that if \( c_6 \) and \( c_7 \) are chosen to be suitably small and (5.13), (5.15) hold, then there exists a function \( v_1 \) and points \( s_- < s_+ \) in \(( -\rho, \rho ) \) such that
\[ (5.17) \quad |v_1(s_\pm) + \pm 1| \leq C e^{-c/\varepsilon}, \quad v_1(s) = v(s) \quad \text{if } |s| \geq \rho, \]
and
\[ (5.18) \quad \int_{-\rho}^{\rho} e_{\varepsilon, \nu}(v_1) \leq \int_{-\rho}^{\rho} e_{\varepsilon, \nu}(v). \]

In fact, \( v_1 \) is found by minimizing \( w \mapsto \int e_{\varepsilon, \nu}(w) \) among the space of functions that agree with \( v \) outside of the intervals \(( -\rho, -\frac{3}{2} \rho ) \cup ( \frac{3}{2} \rho, \rho ) \). Then (5.18) is clear, and a maximum principle argument, together with (5.13), (5.15) and the choices of \( c_6, c_7 \), can be used to show
that $\pm v_1(\pm \frac{7}{8}\rho) \geq 1 - Ce^{-c/\varepsilon}$. If $v_1(\frac{7}{8}\rho) \leq 1$ we can take $s_+ = \frac{7}{8}\rho$, and otherwise we can find some $s_+$ near $\frac{7}{8}\rho$ where $v_1(s_+) = 1$. The choice of $s_-$ is similar.

Let $Q(s) := \int_0^s f_1(t) \, dt$, and note from the definition (3.4) of $f_1$ that

$$
(5.19) \quad e_{\varepsilon,\nu}(w) = \sqrt{2F(w)}|\partial_y w| = |f_1(w) \partial_y w| = |\partial_y Q(w(y^n))| \geq \partial_y Q(w(y^n))
$$

for every $w$. Thus

$$
(5.20) \quad \int_a^b e_{\varepsilon,\nu}(w) \, dy^n \geq |Q(w(b)) - Q(w(b))|
$$

for every $w \in H^1$ and every $a < b$. Applying this inequality with $w = v_1$ and $a = s_-, b = s_+$ rather easily yields (5.14); see [9] for a little more detail.

To prove (5.16), assume that $v \in H^1((-\bar{\rho}, \bar{\rho}))$ for $\bar{\rho} \geq \rho$, and note that since $F = \frac{1}{2}f_1^2$,

$$
\int_{-\bar{\rho}}^{\bar{\rho}} \left( \frac{\sqrt{\varepsilon} v_{y^n} - \frac{f_1(v)}{\sqrt{\varepsilon}}} \right)^2 \, dy^n = \left( \int_{-\bar{\rho}}^{\bar{\rho}} e_{\varepsilon,\nu}(v) \, dy^n - c_0 \right) + (c_0 - Q(v(\bar{\rho})) + Q(v(-\bar{\rho})))
$$

Since $v = v_1$ at $\pm \bar{\rho}$ and $c_0 = Q(1) - Q(-1) = Q(v(s_+)) - Q(v(s_-)) + O(e^{-c/\varepsilon})$, we again use (5.20) to find that

$$
c_0 - Q(v(\bar{\rho})) + Q(v(-\bar{\rho})) \leq \left[ Q(v_1(s_+)) - Q(v_1(\bar{\rho})) \right] + \left[ Q(v_1(-\bar{\rho})) - Q(v_1(s_-)) \right] + Ce^{-c/\varepsilon}
$$

$$
\leq \int_{-\bar{\rho}}^{s_-} e_{\varepsilon,\nu}(v_1) + \int_{s_+}^{\bar{\rho}} e_{\varepsilon,\nu}(v_1) + Ce^{-c/\varepsilon}
$$

$$
= \left( \int_{-\bar{\rho}}^{s_-} e_{\varepsilon,\nu}(v_1) + \int_{s_+}^{\bar{\rho}} e_{\varepsilon,\nu}(v_1) + c_0 \right) - c_0 + Ce^{-c/\varepsilon}.
$$

And again using the choice of $s_\pm$ and (5.20), we have

$$
c_0 \leq Q(v_1(s_+)) - Q(v_1(s_-)) + Ce^{-c/\varepsilon} \leq \int_{s_-}^{s_+} e_{\varepsilon,\nu}(v_1) + Ce^{-c/\varepsilon}.
$$

The proof of (5.16) is completed by combining the last three estimates and recalling (5.18). 

Our next result is exactly Lemma 12 in [9]. Here, in view of future applications, we write $v$ as a function of two variables, $y^0 \in (0, \tau)$ and $y^n \in (-\rho, \rho)$.

**Lemma 5.4.** Let $v \in H^1((0, \tau) \times (-\rho, \rho))$ for some $\tau > 0$. Then there exists a constant $C$, depending on $\rho$ but independent of $\tau$ and of $\varepsilon \in (0, 1)$, such that

$$
\int_{(-\rho,\rho)} |y^n| \, |v(0, y^n) - v(\tau, y^n)|^2 \, dy^n \leq C \int_{(0,\tau) \times (-\rho,\rho)} \frac{\varepsilon}{2} v_0^2 + \frac{(y^n)^2}{\varepsilon} F(v) \, dy^n \, dy^0.
$$

**5.3. Weighted energy estimate.** Now we give the

**Proof of Proposition 5.1.** We will write $\zeta_0 := \max \left( \zeta_1(0), \zeta_2(0) \right)$.

**Step 1.** Since

$$
|v(s, y, y^n) - \text{sign}(y^n)|^2 \leq 2 \left( |v(s, y, y^n) - v(0, y, y^n)|^2 + |v(0, y, y^n) - \text{sign}(y^n)|^2 \right),
$$
we find from Lemma 5.4 that
\[ \zeta_2(s) \leq 2\zeta_2(0) + 2 \int_M \int_{-\rho/2}^{\rho/2} |y^n| |v(s', y^n) - v(0, y^n)|^2 dy^n \ d(vol)_0 \]
\[ \leq 2\zeta_0 + C \int_M \left( \int_0^s \int_{-\rho/2}^{\rho/2} \frac{\varepsilon}{2} v_0^2 + \frac{(y^n)^2}{\varepsilon} F(y) \ dy^n \ dy^0 \right) \ d(vol)_0 \]
\[ \leq 2\zeta_0 + C \int_0^s \zeta_3(\sigma) \ d\sigma. \tag{5.21} \]

**Step 2.** For the next few steps of the proof, we regard \( s \) as fixed, and we write \( v(\cdot) \) rather than \( v(s, \cdot) \).

We will say that a point \( y' \in M \) is **good** if
\[ \int_{-\rho/2}^{\rho/2} |y^n| |v(y', y^n) - \text{sign}(y^n)|^2 dy^n \leq c_6(\rho/2) \tag{5.22} \]
and in addition
\[ \int_{-\rho/2}^{\rho/2} e_{\varepsilon, \nu}(y', y^n) dy^n - c_0 \leq c_7(\rho/2). \tag{5.23} \]
where \( c_6, c_7 \) were fixed in Lemma 5.3. We will say that a point is **bad** if it is not **good**.

We claim that
\[ \text{vol}_0(\{y' \in M : y' \text{ is bad}\}) \leq C (\zeta_1(s) + \zeta_2(s)) + Ce^{-c/\varepsilon}. \tag{5.24} \]
To prove this, first note that by Chebyshev’s inequality,
\[ \text{vol}_0(\{y' \in M : (5.22) \text{ fails}\}) \leq \frac{1}{c_6} \int_M \int_{-\rho/2}^{\rho/2} |y^n| |v(y', y^n) - \text{sign}(y^n)|^2 dy^n \ d(vol)_0 \]
\[ = C\zeta_2(s). \tag{5.25} \]

Next, note that
\[ \int_{-\rho(\nu)}^{\rho(\nu)} e_{\varepsilon, \nu}(s, y', y^n) dy^n - c_0 \geq \begin{cases} \ -Ce^{-c/\varepsilon} & \text{if } y' \text{ is good} \\ \ c_7 & \text{if } y' \text{ is bad and } (5.22) \text{ holds} \\ \ -c_0 & \text{always, and in particular if } (5.22) \text{ fails at } y' \end{cases} \]
where we have used Lemma 5.3 for the first case. We integrate to find
\[ \zeta_1(s) \geq \int_M \left( \int_{-\rho(\nu)}^{\rho(\nu)} e_{\varepsilon, \nu}(s, y', y^n) dy^n - c_0 \right) \ d(vol)_0 \]
\[ \geq -c_0 \text{vol}_0(\{y' \in M : (5.22) \text{ fails}\}) + c_7 \text{vol}_0(\{y' \in M : y' \text{ is bad but } (5.22) \text{ holds}\}) - Ce^{-c/\varepsilon}. \tag{5.26} \]

Using (5.25), we deduce that
\[ (\text{vol}_0(\{y' \in M : y' \text{ is bad but } (5.22) \text{ holds}\}) \leq C (\zeta_1(s) + \zeta_2(s)) + Ce^{-c/\varepsilon}, \]
and this together with (5.25) implies (5.24).

**Step 3.** Next we estimate \( \zeta_3(s) \). We claim that
\[ \zeta_3(s) \leq \zeta_1(s) + C\zeta_2(s) + Ce^{-c/\varepsilon} \tag{5.27} \]
for every \( s \). The choice (4.17) of \( c_2 \) implies that
\[ (1 + c_2(y^n)^2) e_{\varepsilon}(v) \geq \frac{\varepsilon}{4} \sum_{a, b = 0}^{n-1} a^{ab} \partial_{y^a} v \partial_{y^b} v + (1 + (y^n)^2) e_{\varepsilon, \nu}(v). \]
By the definitions of $\zeta_1$ and $\zeta_3$, it follows that

$$\zeta_1(s) \geq \frac{1}{2} \zeta_3(s) + \int_{M^{\rho(s)}} e_{\epsilon, \nu}(v) \, d(vol) - c_0 \, vol_0(M).$$

So to complete the proof of (5.27), it suffices to show that

$$c_0 \, vol_0(M) - \int_{M^{\rho(s)}} e_{\epsilon, \nu}(v) \, d(vol) \leq C \zeta_2(s) + C e^{-c/\epsilon},$$

and this follows directly from (5.26) and (5.25).

**Step 4.** We now claim that

$$\zeta_1'(s) \leq C(\zeta_1(s) + \zeta_2(s) + \zeta_3(s)) + C e^{-c/\epsilon}.$$  

Recalling the definition $\rho(s) := \rho - c_5 s$, we compute $\zeta_1'(s) = I_1 - c_5 I_2$, where

$$I_1 := \int_{\{s\} \times M^{\rho(s)}} (1 + (y^n)^2) \partial_y e_{\epsilon}(v) \, d(vol)$$

$$I_2 := \int_{\{s\} \times M} (1 + (y^n)^2) e_{\epsilon}(v) \, d(vol) \bigg|_{y^n = \rho(s)}^{y^n = -\rho(s)}$$

It follows directly from the differential energy inequality of Lemma 5.2 that

$$I_1 \leq C \zeta_3(s) + I_{1a} + I_{1b}$$

where

$$I_{1a} := \int_{\{s\} \times M^{\rho(s)}} \epsilon (1 + c_2(y^n)^2) \text{div}_{M^\rho} \varphi \, d(vol),$$

$$I_{1b} := \int_{\{s\} \times M^{\rho(s)}} \epsilon (1 + c_2(y^n)^2) \kappa \left( v y^n - \frac{1}{\epsilon} f_1(v) \right) \, d(vol).$$

**Step 5.** To bound $I_{1a}$, note that

$$(1 + (y^n)^2) \text{div}_{M^\rho} \varphi = \text{div}_{M^\rho} \left((1 + (y^n)^2) \varphi \right) - 2y^n \varphi^n$$

by (5.10), since $\omega$ is independent of $y^n$. It is easy to see from the definition (5.9) of $\varphi$ that

$$|y^n| \, |\varphi^n| \leq C \left| \sum_{a,b=0}^{n-1} a^{ab} \partial_y a \, v \, \partial_y v + \epsilon^{-1} |y^n|^2 e_{\epsilon, \nu}(v) \right|$$

and it follows that

$$\epsilon \int_{\{s\} \times M^{\rho(s)}} |y^n| \, |\varphi^n| \, d(vol) \leq C \zeta_3(s).$$

For the other term, note that $\partial(\{s\} \times M^{\rho(s)} = \{s\} \times M \times \{-\rho(s), \rho(s)\}$ (appropriately oriented), and that the induced volume form on $\partial M$ is just $(vol)_0$. This yields

$$\epsilon \int_{\{s\} \times M^{\rho(s)}} \text{div}_{M^\rho} (1 + (y^n)^2) \, d(vol) = \epsilon \int_{\{s\} \times M} (1 + (y^n)^2) \, |\varphi^n| \, d(vol) \bigg|_{y^n = -\rho(s)}^{y^n = \rho(s)}$$

Next, the choice (4.19) of $c_5$ was arranged exactly so that $\epsilon |\varphi^n| \leq c_5 e_{\epsilon}(v)$, so it follows from the above that

$$I_{1a} - c_5 I_2 \leq C \zeta_3(s).$$
Step 6. We estimate $I_{1b}$ as follows. First,

$$I_{1b} \leq \frac{\varepsilon \lVert \kappa \rVert_{\infty}}{2} \int_{\{s\} \times M^p(s)} \left[ |v_y| \right]^2 + \left( v'_y - \frac{1}{\varepsilon} f_1(v) \right)^2 \, d(\nu) \leq C \zeta_3(s) + C \int_{\{s\} \times M^p(s)} \left( \sqrt{\varepsilon v'_y} - \frac{f_1(v)}{\sqrt{\varepsilon}} \right)^2 \, d(\nu).$$

Note that Lemma 5.3 implies that if $y'$ is good in the sense of (5.22), (5.23), then

$$\int_{-\rho(s)}^{\rho(s)} \left( \sqrt{\varepsilon v'_y} - \frac{f_1(v)}{\sqrt{\varepsilon}} \right)^2 \, dy \leq C \left( \int_{-\rho(s)}^{\rho(s)} e_{\varepsilon, \nu}(v) \, dy - c_0 \right) + Ce^{-c/\varepsilon}$$

at $y'$. Integrating this, we find that

$$\int_{\{y' \in M : y' \text{ is good} \}} \int_{-\rho(s)}^{\rho(s)} \left( \sqrt{\varepsilon v'_y} - \frac{f_1(v)}{\sqrt{\varepsilon}} \right)^2 \, dy \, d(\nu)_0 \leq C \left( \int_{\{y' \in M : y' \text{ is good} \}} \int_{-\rho(s)}^{\rho(s)} e_{\varepsilon, \nu}(v) \, dy - c_0 \right) \, d(\nu)_0 + Ce^{-c/\varepsilon} \leq C \left( \zeta_1(s) + c_0 \, vol_0(\{y' \in M : y' \text{ is bad} \}) \right) + Ce^{-c/\varepsilon}.$$  

Combining these estimates with (5.24), we conclude that $I_{1b} \leq C(\zeta_1(s) + \zeta_2(s) + \zeta_3(s) + e^{-c/\varepsilon})$, and this together with (5.30) and (5.31) implies (5.29).

Step 7. Having proved (5.29), (5.21), and (5.27), the conclusions of the Proposition follow by a Grönwall inequality argument, exactly as in the proof of Proposition 10 in [9].

6. Proof of Theorem 3.2

In this section we prove our main theorem.

6.1. construction of initial data. We will prove that the conclusions of the theorem are satisfied by a smooth solution $u : N_{T_0} \to \mathbb{R}$ of (3.3) with initial data

$$(u, \partial_x^\alpha u)_{x^\rho=0} = (u_0, u_1) \text{ constructed below}.$$  

To define $u_0, u_1$, we first define $\phi_0 : M^{2\rho} \to \Sigma_0$ by requiring that

$$\phi = (0, y', y^n) = (0, \phi_0(y', y^n)).$$

This definition makes sense in view of (4.10). Next, we set

$$(6.1) \quad u_0 = \text{sign} I_0, \quad u_1 = 0 \quad \text{ in } \Sigma_0 \setminus \text{Image}(\phi_0),$$

where $I_0 := \{ x \in \Sigma_0 \cong \mathbb{R}^n : (0, x) \in I \}$. In Image($\phi_0$), it is convenient to specify initial data in term of the $y$ coordinates introduced in Proposition 4.1. We would like $v = u \circ \phi$ to satisfy

$$(6.2) \quad v = v_0, \quad \partial_y^\rho v = 0 \quad \text{in } M^{2\rho},$$

when $y^0 = 0$, where

$$(6.3) \quad v_0(y', y^n) = \bar{q}_\varepsilon(y^n) := \chi_\rho(y^n) q(y^n) + (1 - \chi_\rho(y^n)) \text{sign}(y^n)$$

and $\chi_\rho \in C^\infty(\mathbb{R})$ satisfies (2.24). Thus, we complete the definition of $u_0$ by

$$(6.4) \quad u_0 = v_0 \circ \phi_0^{-1} \text{ in } \text{Image}(\phi_0).$$

We specify $u_1$ in Image($\phi_0$) by requiring that

$$0 = \partial_y^\rho v = (\partial_x^\alpha u \circ \phi) \partial_y^\rho \phi^\alpha,$$
when \( y^0 = 0 \). Thus we define \( u_1 = \partial_x^0 u|_{x^0=0} \) in \( \text{Image} (\phi_0) \) by the identity
\[
(6.5) \quad (u_1 \circ \phi) \partial_y^0 \phi^0 = - \sum_{i=1}^{n} (\partial_x^i u_0 \circ \phi) \partial_y^i \phi^0 \quad \text{in } \{ 0 \} \times M_\rho.
\]
The construction of \( \phi \) implies that \( \partial_y^0 \phi^0 \) does not vanish in \( \{ 0 \} \times M_\rho \), so \( u_1 \) is well-defined, and \( (6.2) \) holds as desired.

Observe that the definitions imply that \( u_0 \) and \( u_1 \) are constant, hence smooth, near \( \partial(\text{Image}(\phi_0)) \). As a result they are smooth everywhere.

6.2. **first estimates of** \( v \). As remarked above, \( v = u \circ \phi \) solves \( (5.1) \) (where we are abusing notation and writing \( \kappa \) in place of \( \kappa \circ \phi \)) with initial data satisfying \( (6.2) \) and coefficients of the metric tensor satisfying the conclusions of Proposition 4.1. Thus Proposition 5.1 in the rightmost inequality in \( (4.17) \), shows that for this choice of initial data, the quantity \( \kappa \) notation and writing \( (6.6) \)
\[
(6.7) \quad e \phi (x) := \varepsilon \left( -h^{00}(\partial_x^0 u)^2 + \sum_{i,j=1}^{n} h^{ij} \partial_x^i u \partial_x^j u \right) + \frac{1}{\varepsilon} F(u).
\]
Note that our assumptions \( (3.1) \) on the metric \( h \) imply that
\[
-h^{00}(\partial_x^0 u)^2 + \sum_{i,j=1}^{n} h^{ij} \partial_x^i u \partial_x^j u \approx |D_0 u|^2 := \sum_{\alpha=0}^{n} (\partial_x^\alpha u)^2
\]
in \( N \), where \( A \approx B \) means that there exists some constant \( C \) such that \( C^{-1} B \leq A \leq CB \) pointwise. One can also check that
\[
(6.8) \quad e \phi (u; h) \circ \phi \approx e \phi (v; g) \quad \text{in } M_{T}^{2\rho}.
\]
We next define a smooth cutoff function \( \chi^u : N_T \to \mathbb{R} \) such that \( \chi^u = 1 \) outside of \( N' = \text{Image}(\phi) \), and in \( N' \), we require that
\[
\chi^u \circ \phi (y) = 1 - \chi_\rho (y^n), \quad \text{where } \chi_\rho : \mathbb{R} \to \mathbb{R} \text{ is defined in } (2.24).
\]
Then \( \chi^u \) is smooth, and \( \chi^u = 0 \) near \( \Gamma \). In particular, \( \chi^u \circ \phi (y) = 0 \) if \( |y^n| \leq \frac{1}{3} \rho \), and \( \nabla \chi^u \circ \phi (y) = 0 \) unless \( \frac{1}{3} \rho \leq |y^n| \leq \frac{2}{3} \rho \). Now fix \( t_1 > 0 \) such that
\[
(6.9) \quad \{ x \in N : 0 \leq x^0 \leq t_1, \nabla \chi^u (x) \neq 0 \} \subset \left\{ \phi (y) : y \in M_{T}^{2\rho}, 0 \leq y^0 \leq s_1, \frac{1}{3} \rho \leq |y^n| \leq \frac{2}{3} \rho \right\}.
\]
Using \( (6.7) \) and \( (6.6) \), we deduce that
\[
\int_{\{ x \in N : 0 < x^0 < t_1, D_1 \chi^u \neq 0 \}} e \phi (u; h) \, dx \leq C \int_{\{ y \in M_{T}^{2\rho} : 0 \leq y^0 \leq s_1, \frac{1}{3} \rho \leq |y^n| \leq \frac{2}{3} \rho \}} e \phi (v; g) \, d(\text{vol}) \leq C \int_{0}^{s_1} \zeta_3 (s) \, ds \leq C \varepsilon^2.
\]
Here and below, we use Remark 4.2 to obtain uniform bounds on the Jacobian arising in the change of variables.

6.4. energy estimates for $u$. Computations very similar to those in the proof of Lemma 5.2 show that

$$
\frac{\partial}{\partial x^0} e_\varepsilon(u; h) - \sum_{i,j=1}^n \partial_{x^i} (\varepsilon h^{ij} \partial_{x^j} u \partial_{x^0} u) = \frac{\varepsilon}{2} (\partial_{x^0} h^{\alpha\beta}) \partial_{x^\alpha} u \partial_{x^\beta} u + \varepsilon b^\beta \partial_{x^\beta} u \partial_{x^0} u - \kappa f_1(u) \partial_{x^0} u
$$

where

$$
b^\beta := \frac{1}{\sqrt{-h}} h^{\alpha\beta} \partial_{x^\alpha} \sqrt{-h}.
$$

Since $f^2_1(u) = 2F(u)$ by definition, it easily follows that

$$
(6.10) \quad \frac{\partial}{\partial x^0} e_\varepsilon(u; h) \leq \sum_{i,j=1}^n \partial_{x^i} (h^{ij} \partial_{x^j} u \partial_{x^0} u) + C e_\varepsilon(u; h).
$$

For $0 < \tau < t_1$, we deduce that

$$
\zeta(\tau) := \int_{\Sigma_\tau} \chi^u e_\varepsilon(u; h)
$$

$$
\leq \zeta(0) + \int_{\Sigma_\tau} \partial_{x^0} \chi^u e_\varepsilon(u; h) + \chi^u \left[ \sum_{i,j=1}^n \partial_{x^i} (h^{ij} \partial_{x^j} u \partial_{x^0} u) + C e_\varepsilon(u; h) \right]
$$

$$
\leq \zeta(0) + C \int_{\Sigma_\tau} \chi^u e_\varepsilon(u; h) + \|D\chi^u\|_{\infty} \int_{\{x:0<x_0<t, |D\chi^u(x)| \neq 0\}} e_\varepsilon(u; h)
$$

$$
\leq C \varepsilon^2 + C \int_0^\tau \zeta(t) \, dt.
$$

In the last line we have used (6.9) and an estimate of $\zeta(0)$ that follows easily from our choice of initial data. The integration by parts is justified since $u(t, \cdot) = -1$ outside a compact set. Hence we can conclude by Grönwall’s inequality that

$$
(6.11) \quad \zeta(t) := \int_{\Sigma_t} \chi^u e_\varepsilon(u; h) \leq C \varepsilon^2 \quad \text{for all } 0 \leq t \leq t_1.
$$

6.5. iterate. We now define $\chi^v : M^{2\rho} \to \mathbb{R}$ of the form $\chi^v(y^0, y', y^n) = \tilde{\chi}(|y^n|)$, where

$$
\tilde{\chi} \in C^\infty(\mathbb{R}), \quad 0 \leq \tilde{\chi} \leq 1, \quad \tilde{\chi} = 1 \text{ in } \left(\frac{1}{2}\rho, \rho\right), \quad \text{supp}(\tilde{\chi}) \subset \left(\frac{\rho}{3}, 2\rho\right).
$$

By arguing as in the proof of (6.9), we deduce from (6.11) find that there exists some $s_0 \in (0, s_1)$ such that

$$
\int_{\{y \in M^{2\rho} : 0 < y^0 < s_0, \rho < |y^n| \leq 2\rho\}} e_\varepsilon(v; g) \, d(vol) \leq C \varepsilon^2.
$$
By combining this with \((6.6)\), we find that
\[
\int_{\{y \in M^2_{\rho,0 < y^0 < s_0, D\chi \neq 0}\}} e_\varepsilon(v; g) d(vol) \leq C\varepsilon^2.
\]
Then by using a differential energy inequality satisfied by \(v\), see \((5.12)\), and arguing as in the proof of \((6.11)\), we find that
\[
\int_{\{s_0\} \times M \times \{y^n : \frac{\rho}{2} \leq |y^n| \leq \rho\}} e_\varepsilon(v; g) d(vol) \leq C\varepsilon^2.
\]
Recalling from Section 6.2 that \(\zeta_1(s_0) \leq C\varepsilon^2\) we deduce that \(\zeta_1(0; s_0) \leq C\varepsilon^2\), for
\[
(6.12) \quad \zeta_1(s; s_0) := \int_{M^{\rho}(s)} (1 + c_2(y^n)^2) e_\varepsilon(v) d(vol) \bigg|_{y^0 = (s + s_0)} - c_0 vol_0(M)
\]
We also know from Section 6.2 that
\[
(6.13) \quad \zeta_2(0; s_0) \leq C\varepsilon^2, \quad \text{for} \quad \zeta_2(s; s_0) := \zeta_2(s - s_0).
\]
Now we have shown that \(v|_{y^0 = s_0}\) satisfies estimates of the same form (though with larger constants) as \(v|_{y^0 = 0}\). We can thus iterate the above argument to extend estimates first of \(v\), then of \(u\), to somewhat longer time intervals. We claim that after piecing together finitely many iterations, we can obtain the estimates
\[
(6.14) \quad \int_{N_T \setminus N'} e_\varepsilon(u; h) \, dx \leq C\varepsilon^2,
\]
\[
(6.15) \quad \int_0^T \left( \int_{M^{2\rho}} (1 + c_2(y^n)^2) e_\varepsilon(v; g) d(vol) - c_0 vol_0(M) \right) dy^0 \leq C\varepsilon^2,
\]
\[
(6.16) \quad \int_0^T \int_{M^{2\rho}} \frac{\varepsilon}{2} \sum_{a,b=0}^{n-1} a^{ab} v_{y^a} v_{y^b} + (y^n)^2 \left[ \frac{\varepsilon}{2} |\partial y^n v|^2 + \frac{1}{\varepsilon} F(v) \right] d(vol) \leq C\varepsilon^2.
\]
A proof that finitely many iterations suffice is given in [9, proof of Theorem 22] for \(\kappa = 0\) and in flat Minkowski space, but exactly the same proof is valid here. The point is that the proof only involves piecing together estimates in the standard and normal coordinate systems, and the algorithm for doing so applies equally in this situation, since the \((h_{\alpha\beta})\) is uniformly comparable to the Minkowski metric, see \((3.1)\).

6.6. conclusion of proof. Now \((3.10)\) follows from \((6.14)\) and \((6.16)\), together with \((4.14)\), \((6.7)\) and a change of variables.

The other conclusion of Theorem 3.2, that is the estimate \((2.25)\) of \(\|u - \text{sign}_I\|_{L^2}\), follows from \((6.14)\), \((6.16)\) and a Poincaré inequality. We omit the details, which are just a slightly more complicated version of the argument used to deduce \((2.25)\) from \((2.17)\) (via \((2.19)\)) and \((2.28)\) in the simple model problem in Section 2.

7. Proof of Proposition 4.1

In this section we construct a map \(\phi : M^2_T \to N\) with the properties summarized in Proposition 4.1.
7.1. construction of \( \phi \). To start, for \( y^\tau \in M_T \) we define \( \nu(y^\tau) \in T_{\Psi(y^\tau)} \Gamma \subset T_{\Psi(y^\tau)} N \) to be the unit normal to \( \Gamma \), so that

\[
(7.1) \quad h(\nu, \nu) = 1, \quad h(\nu, \tau) = 0 \quad \text{for all} \quad \tau \in T_{\Psi(y^\tau)} \Gamma.
\]

We will fix a sign by requiring that \( \nu \) point “into \( \Gamma \), see below. These conditions uniquely determine \( \nu \), and our assumptions imply that \( y^\tau \mapsto \nu(y^\tau) \) is smooth. We next define \( \tilde{\phi} : M_{2^\rho}^0 \to N \), for \( \rho > 0 \), by

\[
(7.2) \quad \tilde{\phi}(y^\tau, y^n) := \exp_{\Psi(y^\tau)}(y^n \nu(y^\tau)).
\]

The condition that \( \nu \) point into \( I \) means that \( \tilde{\phi}(y^\tau, y^n) \in I \) for all sufficiently small \( y^n > 0 \). Thus \( \tilde{\phi} \) exactly determine Gaussian normal coordinates for \( N \) near \( \Gamma \).

We will sometimes write \( (\tilde{\phi}_0, \ldots, \tilde{\phi}^n) \) to denote the components of \( \tilde{\phi} \) in the fixed coordinate system on \( N \). Definition \( (7.2) \) states that the components satisfy the system of differential equations

\[
(7.3) \quad \left( \frac{\partial}{\partial y^n} \right)^2 \tilde{\phi}^\alpha + \Gamma^\alpha_{\mu\nu} \frac{\partial \tilde{\phi}^\mu}{\partial y^n} \frac{\partial \tilde{\phi}^\nu}{\partial y^n} = 0, \quad \tilde{\phi}^\alpha(y^\tau, 0) = \psi^\alpha(y^\tau), \quad \frac{\partial \tilde{\phi}^\alpha}{\partial y^n}(y^\tau, 0) = \nu^\alpha(y^\tau),
\]

where \( \nu^\alpha, \alpha = 0, \ldots, n \) are the components of \( \nu \) and

\[
(7.4) \quad \Gamma^\alpha_{\mu\nu} = \frac{1}{2} h^{\alpha\beta} \left( \frac{\partial}{\partial y^n} h_{\beta\nu} + \frac{\partial}{\partial y^\mu} h_{\beta\nu} - \frac{\partial}{\partial y^n} h_{\mu\nu} \right)
\]

are the usual Christoffel symbols.

Finally, we define

\[
\phi(y^0, \ldots, y^n) := \tilde{\phi}(y^0 - \sigma(y^0, y^n), y^1, \ldots, y^n)
\]

where \( \sigma : M_{2^\rho} \to \mathbb{R} \) is chosen exactly so that \( (4.10) \) holds. Thus, we require that \( \sigma \) satisfy

\[
(7.5) \quad \tilde{\phi}^0(-\sigma(y^0, y^n), y^0, y^n) = 0 \quad \text{in} \quad M_{2^\rho}^0.
\]

The next lemma implies that the definition of \( \phi \) makes sense.

**Lemma 7.1.** For \( \rho \) sufficiently small, there exists \( \sigma : M_{\rho} \to \mathbb{R} \) satisfying \( (7.3) \) and in addition

\[
(7.6) \quad |\sigma(y^0, y^n)| \leq C(y^n)^2.
\]

**Proof.** The definitions \( (7.2) \) and \( (4.2) \) of \( \tilde{\phi} \) and \( \Psi \) implies that \( \tilde{\phi}^0(y^\tau, 0) = y^0 \). Thus it is clear that \( \partial_{\psi^0} \tilde{\phi}^0(y^\tau, 0) = 1 \). Since \( M \) is compact, the implicit function theorem thus implies that, taking \( \rho \) smaller if necessary, there exists a function \( \sigma : M_{2^\rho} \to \mathbb{R} \) such that \( (7.5) \) holds. To prove \( (7.6) \), it suffices (again using the compactness of \( M \)) to show that

\[
(7.7) \quad \sigma(y^0, 0) = 0, \quad \frac{\partial \sigma}{\partial y^n}(y^0, 0) = 0 \quad \text{for all} \quad y^0 \in M.
\]

The first of these assertions is clear. For the second, we differentiate \( (7.5) \) with respect to \( y^n \) and evaluate at a point \( (y^0, 0) \), to find that

\[
\frac{\partial \tilde{\phi}^0}{\partial y^n}(0, y^0, 0) = \frac{\partial \tilde{\phi}^0}{\partial y^0}(0, y^0, 0) \frac{\partial \sigma}{\partial y^n}(y^0, 0) = \frac{\partial \sigma}{\partial y^n}(y^0, 0).
\]

So in view of \( (7.3) \), to complete the proof of \( (7.7) \) it suffices to prove that \( \nu^0(0, y^0) = 0 \) for all \( y^0 \in M \). But this follows from noting that at points of the form \( (0, y^0) \),

\[
0 = h(\nu, \frac{\partial \psi}{\partial y^0}) = h_{\alpha\beta} \nu^\alpha \frac{\partial \psi^\beta}{\partial y^0} = h_{00} \nu^0 + h_{0\alpha} \nu^\alpha + h_{\alpha\beta} \nu^\alpha \nu^\beta.
\]

\( \square \)
Our arguments will imply that \( \phi \) is locally invertible in a neighborhood of every point of \( M_T \), and it follows from this that for every \( T \in (T_0, T^*) \) there exists \( \rho > 0 \) such that \( \phi \) is a diffeomorphism of \( M^T \) onto its image. We henceforth assume that this holds. We will also feel free to decrease the size of \( \rho \) throughout our argument.

7.2. estimates of components of the metric tensor. We next prove (4.11), (4.12), and (4.13). We will use the notation

\[
\tilde{g}_{\alpha\beta} := h(\frac{\partial \tilde{\phi}}{\partial y^\alpha}, \frac{\partial \tilde{\phi}}{\partial y^\beta}), \quad (\tilde{g}^{\alpha\beta}) := (\tilde{g}_{\alpha\beta})^{-1}, \quad \tilde{g} := \det(\tilde{g}_{\alpha\beta})
\]

for \( \alpha, \beta = 0, \ldots, n \). We first remark that

\[
(\tilde{g}_{\alpha\beta}) (y^\tau, y^n) = \left( \begin{array}{cc} (\gamma_{ab}) (y^\tau) & 0 \\ 0 & 1 \end{array} \right) + \left( \begin{array}{cc} O(|y^n|^2) & 0 \\ 0 & 0 \end{array} \right)
\]

(in block form), where \( (\gamma_{ab}) \) was introduced in (4.5). The estimate \( \tilde{g}_{ab}(y^\tau, y^n) = \gamma_{ab}(y^\tau) + O(y^n) \) for \( a, b < n \) is immediate when \( y^n = 0 \), since \( \tilde{\phi}(y^\tau, 0) = \Psi(y^\tau) \), and then follows by the smoothness of \( \tilde{\phi} \). The claim above that \( \tilde{g}_{an} = \tilde{g}_{n,\alpha} = \delta_{n\alpha} \) for all \( (y^\tau, y^n) \) is standard; see for example Wald [17] Section 3.3.

It is convenient to define \( \tilde{\phi}(y^0, \ldots, y^n) := (y^0 - \sigma(y', y^n), y^1, \ldots, y^n) \), so that \( \phi = \tilde{\phi} \circ \phi \). Then the definitions imply that

\[
g_{\alpha\beta}(y) = \tilde{g}_{\mu\nu}(\tilde{\phi}(y)) \frac{\partial \tilde{\phi}^\mu}{\partial y^\alpha}(y) \frac{\partial \tilde{\phi}^\nu}{\partial y^\beta}(y).
\]

It follows from (7.6) that \( g_{\mu\nu}(\tilde{\phi}(y)) = g_{\mu\nu}(y) + O((y^n)^2) \) and that \( \frac{\partial g_{\mu\nu}}{\partial y^\gamma} = O(|y^n|) \) for \( i \geq 1 \) and from (1.4) and (7.8) that \( \tilde{g}_{0i} = \tilde{g}_{i0} = O(|y^n|) \) for \( i \geq 1 \). From these one can check that

\[
\tilde{g}_{\alpha\beta}(y) = g_{\alpha\beta}(y) + O((y^n)^2).
\]

It follows from this and (7.8) that

\[
(\tilde{g}_{\alpha\beta})(y^\tau, y^n) = \left( \begin{array}{cc} (\gamma_{ab}) (y^\tau) & 0 \\ 0 & 1 \end{array} \right) + \left( \begin{array}{cc} O(|y^n|) & O(|y^n|) \\ O(|y^n|) & O((y^n)^2) \end{array} \right)
\]

which is (4.11). Then (4.12) follows by elementary linear algebra considerations; see for example Lemma 26 in [9].

We must also estimate \( \partial_0 g^{\alpha\beta} \). To do this, we follow [9] and differentiate the identity \( g^{\alpha\beta} g_{\beta\gamma} = \delta^\alpha_\gamma \) and rearrange to find that

\[
\partial_0 g^{\alpha\beta} = -g^{\alpha\mu} \partial_0 g_{\mu\nu} g^{\nu\beta}.
\]

Next, after differentiating (7.9) and using (7.8), some calculations show that

\[
(\partial_0 g_{\mu\nu}) = \left( \begin{array}{cc} O(1) & O(|y^n|) \\ O(|y^n|) & O((y^n)^2) \end{array} \right),
\]

and then (4.13) can be deduced from this and the estimates (4.12) for \( (g^{\alpha\beta}) \) found above.

7.3. mean curvature in almost-normal coordinates. We next prove (4.15), which states that

\[
\frac{1}{\sqrt{-\tilde{g}}} g^{\alpha\mu} \partial_\gamma g^\mu_\alpha \sqrt{-\tilde{g}} = -\kappa(y^\tau) + O(|y^n|).
\]

where the left-hand side is evaluated at \( (y^\tau, y^n) \) and \(-\kappa(y^n)\) is the mean curvature of \( \Gamma \) at \( \phi(y^n) \), since \( \Gamma \) is assumed to satisfy (3.6). Since the left-hand side is smooth, it suffices to check this when \( y^n = 0 \).
follows from the inverse function theorem that $\phi_\Gamma$. This implies that $\phi_\Gamma$ and expressing $(\dot{g}^\alpha)^n_{\alpha\beta} = \frac{1}{2}g^{\alpha\beta} \partial_y^\alpha g_{\alpha\beta}$. Thus the fact that $\tilde{\phi}$ and by (4.12), this agrees with the right-hand side of (7.12) when $y^n = 0$.  

**Remark 7.2.** The above proof shows that, in addition to (7.8), we have 

$$(\tilde{g}^\alpha)^n_{\alpha\beta}(y^\tau, y^n) = \begin{pmatrix} (\gamma^{ab}(y^\tau) & 0 \\ 0 & I \end{pmatrix} + \begin{pmatrix} O(|y^n|) & 0 \\ 0 & 0 \end{pmatrix},$$

and 

$$-\frac{1}{1 + g^{-1/n}} \tilde{g}^{\alpha\beta} \partial_{y^n} \gamma^{-1} = -\frac{1}{\sqrt{-g}} \partial_{y^n} \gamma^{-1} = -\kappa(y^\tau) + O(|y^n|).$$

In particular the first two of these facts, together with (7.8), are somewhat better than the corresponding properties of $(g_{\alpha\beta})$.  

### 7.4. Solution of the eikonal equation.

Finally, it is well-known that if we define $d_\Gamma := \pi^n \circ \tilde{\phi}^{-1}$, where $\pi^n(y^0, \ldots, y^n) = y^n$, then $d_\Gamma$ satisfies (3.8) in Image($\phi$). Indeed, the definitions imply that $d_\Gamma = 0$ on $\Gamma$, and we have fixed signs such that $d_\Gamma > 0$ in Image($\phi$) $\cap I$. Finally, by inverting the definition 

$$\tilde{g}^{\alpha\beta} = h_{\mu\nu} \partial_{y^\mu} \tilde{\phi}^\nu \partial_{y^\beta} \tilde{\phi}^\nu$$

and expressing $(D\phi)^{-1}$ in terms of $D(\tilde{\phi}^{-1}) \circ \phi$, we find that 

$$\tilde{g}^{\alpha\beta} \circ \phi^{-1} = h^{\mu\nu} \partial_{x^\mu} (\tilde{\phi}^{-1})^{\alpha} \partial_{x^\nu} (\tilde{\phi}^{-1})^{\beta}.$$ 

Thus the fact that $\tilde{g}^{nn} = 1$ states exactly that $h^{\alpha\beta} \partial_{x^\mu} d_\Gamma, \partial_{x^\beta} d_\Gamma = 1$. Finally, we deduce from (7.6) that $\phi^{-1} = \tilde{\phi}^{-1}$ and $D\phi = D\tilde{\phi}^{-1}$ everywhere on $\{y^n = 0\}$, and it follows from the inverse function theorem that $\phi^{-1} = \tilde{\phi}^{-1}, D(\phi^{-1}) = D(\tilde{\phi}^{-1})$ everywhere on $\Gamma$. This implies that $\phi^{-1} - \tilde{\phi}^{-1} = O(d_\Gamma^2)$, and since $d_\Gamma = (\tilde{\phi}^{-1})^n$, we arrive at (4.14).
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