Still better nonlinear codes from modular curves

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Abstract. We give a new construction of nonlinear error-correcting codes over suitable finite fields \( k \) from the geometry of modular curves with many rational points over \( k \), combining two recent improvements on Goppa’s construction. The resulting codes are asymptotically the best currently known.

1. Introduction.

1.1 Review of Goppa’s construction. Fix a finite field \( k \) of \( q = p^\alpha \) elements. Let \( C \) be a (projective, smooth, irreducible) algebraic curve of genus \( g \) defined over \( k \), with \( N \) rational points. It is known that

\[
N < (q^{1/2} - 1 + o(1))g
\]

as \( g \to \infty \) (Drinfeld-Vlăduţ \[11\]). We say a curve of genus \( g \to \infty \) is “asymptotically optimal” if it has at least \( (q^{1/2} - 1 - o(1))g \) rational points over \( k \). If \( \alpha \) is even (that is, if \( q_0 := \sqrt{q} \) is an integer), then modular curves of various flavors — classical (elliptic), Shimura, or Drinfeld — attain

\[
N \geq (q_0 - 1)(g - 1) = (q^{1/2} - 1 - o(1))g
\]

\[10, 13\], and are thus asymptotically optimal.

Let \( D \) be a divisor on \( C \) of degree \( < N \). Goppa \([8]\), see also \[12\]) regards the space \( L(D) \) of sections of \( D \) as a linear code in \( k^N \), whose dimension \( r \) and minimal distance \( d \) satisfy

\[
r \geq \deg(D) - g + 1 \quad \text{(by the Riemann-Roch theorem)}
\]

and

\[
d \geq N - \deg(D) \quad \text{(because a nonzero section of \( D \) has at most \( \deg(D) \) zeros)}
\]

Thus the transmission rate \( R = r/N \) and the error-detection rate \( \delta = d/N \) of Goppa’s codes are related by

\[
R + \delta > 1 - \frac{g}{N}.
\]

This lower bound improves as \( N/g \) increases. For \( q = q_0^2 \), we may take \( C \) asymptotically optimal, and find

\[
R + \delta > 1 - \frac{1}{q_0 - 1} - o(1)
\]

for an infinite family of linear codes over \( k \), which is the best that can be obtained from \([8]\).
Let us say that \((R_0, \delta_0)\) is asymptotically feasible if \(R_0, \delta_0\) are positive and there exist arbitrarily long codes over \(k\), linear or not, with \(R > R_0\) and \(\delta > \delta_0\). Then Goppa’s construction yields the asymptotic feasibility of \((R_0, \delta_0)\) for any positive \(R_0, \delta_0\) such that \(R_0 + \delta_0 < 1 - (1/\q0)\). This is true because \(\deg(D)\) is an arbitrary integer in \((0, N)\), and nontrivial (in the sense that such \((R_0, \delta_0)\) exist) once \(\q0 > 2\).

By comparison, a random code or random linear code of length \(N \to \infty\) and rate \(R\) has error-detection rate at least \(\delta - o(1)\) with probability \(1 - o(1)\) provided \(\delta < (q - 1)/q\) and \(R + H_q(\delta) < 1\), where \(H_q\) is the normalized entropy function

\[
H_q(\delta) := \delta \log_q(q - 1) - \delta \log_q\delta - (1 - \delta) \log_q(1 - \delta) \tag{5}
\]

\[
= \delta \log_q\left((q - 1)\frac{1 - \delta}{\delta}\right) - \log_q(1 - \delta) \tag{6}
\]

\[
= \lim_{N \to \infty} N^{-1} \log_q\left(\frac{N}{N\delta N}\right).
\]

Therefore if \(0 < \delta_0 < (q - 1)/q\) and \(0 < R_0 < 1 - H_q(\delta_0)\) then \((R_0, \delta_0)\) is asymptotically feasible. This is the Gilbert-Varshamov bound.

Once \(q_0 \geq 7\), Goppa’s construction yields asymptotically feasible \((R_0, \delta_0)\) beyond the Gilbert-Varshamov bound. This was the first construction to improve on Gilbert-Varshamov in this sense, and it remains the only such construction that yields linear codes.

1.2 Beyond Goppa. For about 20 years Goppa’s technique remained the best construction of codes over an alphabet of size \(q_0^2 \geq 49\) beyond the Gilbert-Varshamov bound. Refinements concerned only algorithmic improvements, for exhibiting suitable curves \(C\) (see [2] for classical and Shimura curves, [3] for further Shimura curves, and [6, 7, 4, 12] for Drinfeld modular curves) and using the resulting codes for error-resistant communication (polynomial-time encoding and decoding, see [9, 11]). In [5], we used rational functions on \(C\) to construct algebraic-geometry codes over the \((q + 1)\)-letter alphabet \(P^1(k) = k \cup \{\infty\}\), and gave asymptotic estimates on their parameters \(R, \delta\). We argued in [5] that these codes improve on Goppa’s in a range of parameters that includes all the Goppa codes that improve on Gilbert-Varshamov; but our comparison was necessarily indirect due to the different alphabet sizes.

Even more recently, Xing [14] gave a new construction of nonlinear algebraic-geometry codes over \(k\). Like Goppa, Xing uses sections of line bundles, but he cleverly exploits the sections’ derivatives to find codes with better asymptotic parameters than Goppa’s. The Xing codes have

\[
R + \delta > 1 - \frac{1}{q_0 - 1} + \sum_{i=2}^{\infty} \log_q\left(1 + \frac{q - 1}{q^{2i}}\right) - o(1), \tag{7}
\]

\(^1\)As usual, the rate of a nonlinear code \(C \subset k^n\) is defined by \(R = N^{-1} \log_q(\#C)\), which equals \(r/N\) when \(C\) is linear.

\(^2\)While [5] was the first publication, we obtained these results in the mid-1990’s and included them in several conference and seminar talks starting in 1996.
which improves on \( [4] \) by
\[
c_q := \sum_{i=2}^{\infty} \log_q \left( 1 + \frac{q-1}{q^{2i}} \right) = \frac{1}{\log q} (q^{-3} - q^{-4} + O(q^{-5})). \tag{8}
\]

In particular, this is the first construction of algebraic-geometry codes over a 4-letter alphabet with \( R, \delta \) both bounded away from zero. (Our codes over \( \mathbf{P}^1(k) \) attained this for an alphabet of 5 letters.) Xing’s construction does not require that \( q = q_0^2 \), and yields an improvement of \( c_q \) over the Goppa bound \( [3] \) for all \( q \).

In this paper we obtain a further improvement, at least for \( q = q_0^2 \), by applying Xing’s technique to our codes of \([5]\). While those codes used an alphabet of \( q+1 \) letters, our new codes \( \mathcal{C}_D(h) \) use the \( q \)-letter alphabet \( k \), and can thus be compared directly with Goppa’s and Xing’s codes. We find that when \( q = q_0^2 \) our \( \mathcal{C}_D(h) \) have parameters that improve on Xing’s, replacing the sum \( [8] \) by
\[
\log_q \left( 1 + \frac{1}{q^3} \right) = \frac{1}{\log q} (q^{-3} - O(q^{-6})). \tag{9}
\]

We conclude that \( (R_0, \delta_0) \) is asymptotically feasible for any positive \( R_0, \delta_0 \) such that
\[
R_0 + \delta_0 < 1 - \frac{1}{q_0 - 1} + \log_q \left( 1 + \frac{1}{q^3} \right). \tag{10}
\]

This also gives further support to our claim that the codes of \([5]\) improve on Goppa’s. Xing’s refinement applies to both constructions and yields nonlinear codes over the same alphabet, making possible a direct comparison in which the codes of \([5]\) come out ahead.

Like the codes of \([5]\) and (probably) \([14]\), and unlike Goppa codes, our new codes \( \mathcal{C}_D(h) \) are still very far from any practical use: it is not even clear that one can efficiently encode integers in \([1, (\# \mathcal{C}_D(h))^{1-o(1)}]\) or recognize whether a given word in \( k^N \) is in that code, let alone solve the error-correcting problem.

The rest of this paper is organized as follows. We first review the nonlinear algebraic-geometry codes \( C_m, C_D(h) \) of \([14, 5]\). We then combine these two constructions, and show that our new codes \( \mathcal{C}_D(h) \) attain the claimed improvement over Xing’s codes.

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2. Two variations on a theme of Goppa

2.1 Xing: nonlinear codes from derivatives of sections. Let \( C \) be a (projective, smooth, irreducible) algebraic curve of genus \( g \) defined over \( k \), with \( N \) rational points \( P_1, \ldots, P_N \). For each \( j = 1, \ldots, N \), choose a local uniformizing parameter \( t_j \) at \( P_j \), that is, a rational function on \( C \) vanishing to order 1 at \( P_j \). (Different choices of \( t_j \) will yield isomorphic codes.) For any rational function
Let $D$ be a divisor on $C$. As in [14], we simplify the exposition by assuming that the support of $D$ is disjoint from $C(k) = \{P_1, \ldots, P_N\}$. (Little is lost by this assumption, because any divisor on $C$ is linearly equivalent to one satisfying the disjointness condition, and linearly equivalent divisors yield equivalent codes.) Note that we do not assume that $\deg(D) < N$, and indeed we shall use divisors of degree considerably larger than $N$. We denote by $\mathcal{L}(D)$ the vector space of global sections of $D$. For any distinct $f, f' \in \mathcal{L}(D)$, the difference $f - f'$ has a total of $\deg(D)$ zeros in $C(K)$, counted with multiplicity. Goppa’s construction does not exploit multiplicity, using only the corollary that $f(P_j) = f'(P_j)$ holds for at most $\deg(D)$ values of $j$. Multiple zeros are the key to Xing’s improvement.

For $r \geq 0$ define $\phi_r : \mathcal{L}(D) \to k^N$ by

$$\phi_r(f) = (f_1^{(r)}, f_2^{(r)}, \ldots, f_N^{(r)}).$$

Recall that the Hamming distance $d(\cdot, \cdot)$ on $k^N$ is defined by

$$d(c, c') := |\{ j \in \{1, 2, \ldots, N\} | c_j \neq c'_j\}|.$$

For each positive integer $m$, Xing defines a code $C_m$ as follows. For $r = 0, 1, \ldots, m - 1$, fix positive real numbers $\sigma_r < (q - 1)/q$, which will be specified later to optimize $C_m$. Choose $c_r \in k^N$ to maximize the size of

$$\mathcal{M}_m := \{ f \in \mathcal{L}(D) | d(c_r, \phi_r(f)) \leq \sigma_r N \text{ for each } r = 0, 1, \ldots, m - 1\},$$

and define

$$C_m := \phi_m(\mathcal{M}_m).$$

As $(c_0, \ldots, c_{m-1})$ varies over all $q^{mN}$ possible choices, the average size of $\mathcal{M}_m$ equals $q^{-mN} \#(\mathcal{L}(D))$ times the product of the sizes of closed Hamming balls of radius $\sigma_0 N, \sigma_1 N, \ldots, \sigma_{m-1} N$. The maximal $\mathcal{M}_m$ is therefore at least as large, so

$$\#\mathcal{M}_m \geq q^{(H - o(1))N} \#(\mathcal{L}(D)),$$

where the negative number $H$ is given in terms of the entropy function [19] by

$$H = \sum_{r=0}^{m-1} (H_q(\sigma_r) - 1).$$

Now let $f, f'$ be distinct functions in $\mathcal{M}_m$, and for each $r \geq 0$ let $I_r$ be the set of $i \in \{1, \ldots, N\}$ such that $f_i^{(r)} = f'_i^{(r)}$. If $r < m$ then $\#I_r \geq (1 - 2\sigma_r)N$, since $N - \#I_r$ is the distance between the words $\phi_r(f), \phi_r(f')$ in the same Hamming ball of radius $\sigma_r N$. On the other hand, the total number of zeros of $f - f'$ at $P_1, \ldots, P_n$, counted with multiplicity, is
$$\sum_{s=0}^{\infty} \#(\cap_{r=0}^{s} I_r) \geq \sum_{s=0}^{m} \# \left( \bigcap_{r=0}^{s} I_r \right) \geq (m+1)N - \sum_{s=0}^{m} \left( \sum_{r=0}^{s} (N - \# I_r) \right) \tag{17}$$

$$= (m+1)N - \sum_{r=0}^{m} (m+1-r)(N - \# I_r) \geq \left( m - 2 \sum_{r=0}^{m-1} (m+1-r)\sigma_r \right)N + \# I_m.$$  

Since this number may not exceed the degree of $D$, we deduce that

$$\# I_m \leq \deg(D) - \left( m - 2 \sum_{r=0}^{m-1} (m+1-r)\sigma_r \right)N. \tag{18}$$

Define $d_0$, then, by

$$d_0 := \left( m + 1 - 2 \sum_{r=0}^{m-1} (m+1-r)\sigma_r \right)N - \deg(D). \tag{19}$$

Then $\# I_m \leq N - d_0$. That is, $d(\phi_m(f), \phi_m(f')) \geq d_0$. Therefore $C_m$ has minimum distance at least $d_0$. Assume that $\deg(D)$ is small enough that $d_0 > 0$:

$$\deg(D) < \left( m + 1 - 2 \sum_{r=0}^{m-1} (m+1-r)\sigma_r \right)N. \tag{20}$$

Then $\# I_m < N$, that is, $\phi_m(f) \neq \phi_m(f')$. Therefore $\phi_m$ is injective and $\# C_m = \# M_m$.

It remains to optimize $\deg(D)$ and $\sigma_r$ given $\delta = d_0/N$. By Riemann-Roch, $L(D)$ is a $k$-vector space of dimension at least $\deg(D) - g + 1$, with equality if $D > 2g - 2$ (which will be the case for all $D$ that we use). Combining this with (19) and the estimate (15) on $\# M_m$, we find

$$\log_q \frac{\# C_m}{N} + \frac{d_0}{N} > 1 - \frac{g}{N} + \sum_{r=0}^{m-1} \left( H_q(\sigma_r) - 2(m+1-r)\sigma_r \right) - o(1). \tag{21}$$

The left-hand side is a lower bound on $R + \delta$ for the code $C_m$. If we take all $\sigma_r = 0$, the right-hand side reduces to $1 - (g/N)$, and we recover the Goppa bound (23) — which is to be expected because in this case $C_m$ is equivalent to the Goppa code $L(D - \sum_{i=1}^{N} (P_i))$ (this is most easily seen if we also choose each $c_r = 0$). Since the derivative of $H_q(\sigma) - 2(m+1-r)$ at $\sigma = 0$ is $+\infty$ for any $m, r$, there must exist positive $\sigma_r$ that improve on Goppa. By differentiating each term of the sum in (21), Xing computed that the optimal choice is

$$\sigma_r = (q-1)/(q^{2(m+1-r)} + q - 1), \tag{22}$$
corresponding to \( \sigma_r/(1-\sigma_r) = q^{-2(m+1-r)}(q-1) \). Using the equivalent formula for \( H_q \) and taking \( i = m+1-r \), the \( m \)-th term of \( (21) \) is seen to equal 
\[ \log_q(1+q^{-2}(q-1)) \]
\[ < N/2 \]
\[ \text{if } \sigma_i \text{ is the multiplicity of } \sigma_i \]
\[ \text{at the optimal } \sigma_i, \text{ whence the codes } C_m \text{ attain} \]
\[ R + \delta > 1 - \frac{1}{q_0-1} + \sum_{i=2}^{m+1} \log_q \left( 1 + \frac{q-1}{q^{2i}} \right) - o(1). \]  
(23)

Taking \( m \to \infty \) and \( q = q_0^2 \) recovers \( (3) \); without the hypothesis \( q = q_0^2 \), Xing’s construction still improves on \( (3) \) by adding \( c_q \) to the lower bound on \( R + \delta \).

2.2 Codes over \( \mathbb{P}^1(k) \) from rational functions and sections.

The codes we introduced in \( [5] \) use rational functions on \( C \) instead of the global sections that comprise Goppa’s codes. Let \( D \) be a divisor of degree zero on \( C \). For a nonnegative integer \( h \), we define \( M_D(h) \) to be the set of rational sections of degree \( \leq h \) of the line bundle \( L_D \) associated to \( D \). That is, \( M_D(h) \subset k(C) \) consists of the zero function together with the rational functions \( f \) on \( C \) whose divisor \( (f) \) is of the form \( E - D \), where \( E \) is a divisor whose positive and negative parts each have degree at most \( h \). For instance, \( M_0(h) \) is the set of rational functions of degree at most \( h \). Here, as opposed to \( [5] \), we do not assume that \( h < N/2 \), and indeed we shall use considerably larger \( h \).

To use these \( M_D(h) \) for coding, we need an upper bound on the number of solutions, with multiplicity, of \( f(P) = f'(P) \) for distinct rational sections \( f, f' \) of \( L_D \), given the degrees of \( f \) and \( f' \). We define this multiplicity as follows. For each \( P \in C(k) \) choose a rational function \( \varphi_P \) whose divisor has the same order at \( P \) as \( D \). (The definition will be clearly independent of the choice of \( \varphi_P \).) Then \( P \) is a solution of \( f = f' \) if the rational functions \( \varphi_P f \) and \( \varphi_P f' \) have the same value, finite or infinite, at \( P \). In the former case, the multiplicity is the valuation of \( (\varphi_P f) - (\varphi_P f') \) at \( P \). In the latter case, the multiplicity is the valuation of \( (\varphi_P f)^{-1} - (\varphi_P f')^{-1} \) at \( P \). Of course if \( P \) is not a solution of \( f = f' \) then its multiplicity is zero.

**Proposition.** Suppose \( f, f' \) are distinct rational sections of \( L_D \), with degrees \( h, h' \). For \( P \in C(k) \) let \( m(P) \) be the multiplicity of \( P \) as a solution of \( f = f' \). Then 
\[ \sum_{P \in \mathcal{C}(k)} m(P) = h + h'. \]

**Proof.** For \( P \in C(k) \), if \( P \) is a pole of \( \varphi_P f \), let \( \mu(P) \) be the multiplicity of this pole, and otherwise set \( \mu(P) = 0 \); define \( \mu'(P) \) likewise using \( \varphi_P f' \). Then 
\[ h + h' = \sum_{P \in \mathcal{C}(k)} (\mu(P) + \mu'(P)). \]
But we claim that \( m(P) - (\mu(P) + \mu'(P)) \) is the valuation at \( P \) of \( \varphi_P f - \varphi_P f' \), considered also as a rational section of \( L_D \). This claim is immediate if neither \( \varphi_P f \) nor \( \varphi_P f' \) has a pole at \( P \); if just one of them has a pole there then \( \varphi_P f - \varphi_P f' \) has a pole of the same order, which equals \( \mu(P) + \mu'(P) \), while \( m(P) = 0 \); finally, if both \( \varphi_P f, \varphi_P f' \) have poles at \( P \), then \( m(P) > 0 \), and the claim follows from the identity 
\[ (\varphi_P f)^{-1} - (\varphi_P f')^{-1} = \frac{(\varphi_P f') - (\varphi_P f)}{(\varphi_P f')(\varphi_P f)} \]
by taking valuations at \( P \). This establishes our claim in all cases. But the sum
over $P \in C(k)$ of $v_P(\varphi_P f - \varphi_P f')$ vanishes, since $\sum_{P \in C(k)} v_P(f - f') = 0$ while $\sum_{P \in C(k)} v_P(\varphi_P) = \deg(D)$ was also assumed to equal zero. Since
\[
\sum_{P} v_P(\varphi_P f - \varphi_P f') = \sum_{P} (m(P) - (\mu(P) + \mu'(P))) = \sum_{P} m(P) - (h + h'),
\]
this proves that $m(P) = h + h'$. □

**Remark.** This result is the analogue for rational sections of the fact that a nonzero element of $L(D)$ has $\deg(D)$ zeros counted with multiplicity. It refines Proposition 1 of [5], where we showed only that $f = f'$ has at most $h + h'$ solutions not counting multiplicity.

In particular, if $2h < N$ then the evaluation map $\phi_0 : M_D(h) \to (\mathbb{P}^1(k))^N$ taking $f$ to $((\varphi_P f)(P_1), \ldots, (\varphi_P f)(P_N))$ is injective, and its image is a code of length $N$ over $\mathbb{P}^1(k)$ with minimal distance at least $N - 2h$. We call this code $C_D(h)$. Note that it is $2h$, rather than $h$, that plays the role analogous to the degree of the divisor on Goppa’s construction; the notation $h$ should suggest both the $h$eight of a rational section of $L_D$ and half of the degree of Goppa’s divisor.

We also need the size of this code. It turns out to be easier, though still far from trivial, to estimate not individual $\#M_D(h)$ but the average of $\#M_D(h)$ as $D$ ranges over (representatives of) the Jacobian $J_C$, which is the group of equivalence classes of degree-zero divisors on $C$. We quote the following from [5, Thm.1]:

\[\left(\frac{q + 1}{q}\right)^{N-o(N)} q^{2h-g}.\]  

Hence there exist $D$ for which $M_D(h)$ has size at least \[24\], which exceeds by a factor of $(q + 1)/q^{N-o(N)}$ the Riemann-Roch estimate on the size of $\mathcal{L}(D)$ when $\deg(D) = 2h$. Note that we do not require that $2h < N$. We shall use the result also for some $h \geq N/2$, in which case we cannot deduce a lower bound on $C_D(h)$ (and anyway we have no nontrivial lower bound on the minimal distance of $C_D(h)$), but will be able to construct another code $\mathcal{C}_D(h)$ by adapting Xing’s technique.

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3. New nonlinear codes over $k$

We again regard the evaluation map $\phi_0$ on $M_D(h)$ as the first of an infinite series $\phi_0, \phi_1, \phi_2, \ldots$ that records not just the values but also the derivatives of rational sections of $D$ at $P_1, \ldots, P_N$. While $\phi_0$ takes values in $(\mathbf{P}^1(k))^N$, the $\phi_r$ for $r > 0$ take values in $k^N$; this is why we construct codes with alphabet $k$ rather than $\mathbf{P}^1(k)$. We define $\phi_1, \phi_2, \ldots$ as follows. If $(\varphi_{P_j}, f)(P_j) = \infty$ then the $j$-th coordinate of $\phi_r(f)$ is the $t_j^r$ coefficient of the expansion of $(\varphi_{P_j} f)^{-1}$ in powers of $t_j$. Otherwise that coordinate is the $t_j^r$ coefficient of the expansion of $\varphi_{P_j} f$.

In either case, $P_j$ is a solution of $f = f'$ of multiplicity $m$ if and only if the $j$-th coordinates of $\phi_r(f), \phi_r(f')$ coincide for $0 \leq r < m$ but not for $r = m$. As in [14], we could have dispensed with the $\varphi_{P_j}$ by choosing a linearly equivalent $D$ whose support is disjoint from $\{P_1, \ldots, P_N\}$.

Fix positive $\sigma_0 < q/(q + 1)$. Choose $c_0 \in (\mathbf{P}^1(k))^N$ to maximize the size of

$$\mathcal{M}_D(h) := \{f \in M_D(h) \mid d(c_0, \phi_0(f)) \leq \sigma_0 N\},$$

and define

$$\mathcal{C}_D(h) := \phi_1(\mathcal{M}_D(h)).$$

As $c_0$ varies over all $(q + 1)^N$ possible choices, the average size of $\mathcal{M}_D(h)$ equals $(q + 1)^{-N} \#(\mathcal{M}_D(h))$ times size of the closed Hamming ball of radius $\sigma_0 N$ in $(\mathbf{P}^1(k))^N$. The maximal $\mathcal{M}_D(h)$ is therefore at least as large, so

$$\#(\mathcal{M}_D(h)) > (q + 1)^{(H_{q+1}(\sigma_0) - 1 - o(1))N} \#(M_D(h)).$$

(27)

If moreover $C$ is asymptotically optimal and $h \geq \rho N$ for some $\rho > q/(q^2 - 1)$, then there exists $D$ such that $\#(M_D(h))$ is bounded below by [24]. We then have

$$\#(\mathcal{M}_D(h)) > q^{2h-q} \exp[N(\log(q + 1)H_{q+1}(\sigma_0) - \log q - o_q(1))].$$

(28)

We next fix $d_0 > 0$ and show that if

$$2h \leq (2 - 4\sigma_0)N - d_0$$

(29)

then $\phi_1$ is injective on $\mathcal{M}_D(h)$ and its image $\mathcal{C}_D(h)$ is a code of minimum distance at least $d_0$. We have seen that for any distinct $f, f' \in M_D(h)$ the total multiplicity of solutions of $f = f'$ is at most $2h$. If $f, f' \in \mathcal{M}_D(h)$ then $f(P_j) = f'(P_j)$ for at least $N - 2\sigma_0 N$ values of $j$. For at least

$$N - 2\sigma_0 N - d(\phi_1(f), \phi_1(f'))$$

of these, the $j$-th coordinates of $\phi_1(f)$ and $\phi_1(f')$ also coincide, so $P_j$ is a solution of $f = f'$ of multiplicity at least 2. Hence

$$(2 - 4\sigma_0)N - d(\phi_1(f), \phi_1(f')) \leq 2h.$$
Therefore if (29) holds then \(d(\phi_1(f), \phi_1(f')) \geq d_0\), which proves are claim that \(\phi_1\) maps \(\mathcal{M}_D(h)\) injectively to a code of minimum distance \(\geq d_0\). (This is of course a direct adaptation of the case \(m = 1\) of Xing’s argument, which is Theorem 1.2 and §II of his paper [13].)

Finally we take \(d_0 = \delta N\), let \(2h = (2 - 4\sigma_0 - \delta - o(1))N\), and optimize \(\sigma_0\). Combining our results thus far, we have

\[
\log_q \frac{\#C_m}{N} + \delta > 1 - \frac{q}{N} + \log_q (q + 1) \cdot H_{q+1}(\sigma_0) - 4\sigma_0 - o(1). \tag{30}
\]

If we took \(\sigma_0 = 0\), we would again recover the Goppa bound [4]. Since the derivative with respect to \(\sigma_0\) of the bound (30) is \(+\infty\), the optimal \(\sigma_0\) must improve on (41) as was the case for Xing’s construction. The resulting bound improves on Xing’s because (30) involves the entropy function for an alphabet of \(q + 1\) letters rather than \(q\). Here we calculate that the optimal \(\sigma_0\) is \(1/(q^3 + 1)\), corresponding to \(\sigma_0/(1 - \sigma_0) = q^{-3}\). Substituting this into (30) we obtain

\[
\log_q \frac{\#C_m}{N} + \delta > 1 - \frac{q}{N} + \log_q \left(1 + \frac{1}{q^3}\right) - o(1). \tag{31}
\]

With this choice of \(\sigma_0\), the ratio \(2h/N\) easily exceeds the threshold of \(2q/(q^2 - 1)\) even for \(\delta = 1\) as long as \(q_0 \geq 4\). (We must in any event exclude \(q = 2\) or \(3\), because for those \(q\) none of the methods described here could yield codes with positive \(R, \delta\) even if asymptotically optimal curves were known.)

We have therefore proved:

**Theorem.** Let \(q_0\) be a prime power, and \(k\) a finite field of \(q^2\) elements. For all positive \(R_0, \delta_0\) satisfying (11), and any \(N_0\), there exist a curve \(C/k\), a degree-zero divisor \(D\) on \(C\), and a positive integer \(h\), such that \(\mathcal{C}_D(h)\) is a code whose length \(N\), transmission rate \(R\), and error-detection rate \(\delta = d/N\) satisfy \(N > N_0\), \(R > R_0\), and \(\delta > \delta_0\). In particular, all positive \((R_0, \delta_0)\) satisfying (11) are asymptotically feasible.

As noted already, this construction corresponds to the case \(m = 1\) of Xing’s codes. One can formulate such a construction for any \(m\), but with worse results, because only \(\sigma_0\) increases, and the resulting improvement falls off as \(q^{-2m-2}/\log(q)\) when \(m\) grows. Can multiplicities of order \(> 2\) be exploited to yield even better codes?

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4In [5] we already noted, in a different way, that we could use \(\mathcal{C}_d(h)\) to construct nonlinear codes that are asymptotically as good as Goppa’s, and cited this observation in one of our indirect comparisons between \(\mathcal{C}_d(h)\) and Goppa codes.
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