Stochastic multiscale flux basis for Stokes-Darcy flows

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Abstract

Three algorithms are developed for uncertainty quantification in modeling coupled Stokes and Darcy flows. The porous media may consist of multiple regions with different properties. The permeability is modeled as a non-stationary stochastic variable, with its log represented as a sum of local Karhunen-Loève (KL) expansions. The problem is approximated by stochastic collocation on either tensor-product or sparse grids, coupled with a multiscale mortar mixed finite element method for the spatial discretization. A non-overlapping domain decomposition algorithm reduces the global problem to a coarse scale mortar interface problem, which is solved by an iterative solver, for each stochastic realization. In the traditional implementation, each subdomain solves a local Dirichlet or Neumann problem in every interface iteration. To reduce this cost, two additional algorithms based on deterministic or stochastic multiscale flux basis are introduced. The basis consists of the local flux (or velocity trace) responses from each mortar degree of freedom. It is computed by each subdomain independently before the interface iteration begins. The use of the multiscale flux basis avoids the need for subdomain solves on each iteration. The deterministic basis is computed at each stochastic collocation and used only at this realization. The stochastic basis is formed by further looping over all local realizations of a subdomain’s KL region before the stochastic collocation begins. It is reused over multiple realizations. Numerical tests are presented to illustrate the performance of the three algorithms, with the stochastic multiscale flux basis showing significant savings in computational cost.

Keywords. stochastic collocation, domain decomposition, multiscale basis, mortar finite element, mixed finite element, Stokes-Darcy flows

1 Introduction

Coupled Stokes-Darcy flows arise in numerous applications, including interaction between surface and groundwater flows, cardiovascular flows, industrial filtration, fuel cells, and flows in fractured or vuggy reservoirs. The Stokes equations describe the motion of incompressible fluids and the Darcy model describes the infiltration process. In this work we consider mixed velocity-pressure formulations in both regions. The Beavers-Joseph-Saffman slip with friction condition [9,41] is applied on the Stokes-Darcy interface. Existence and uniqueness

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of a weak solution has been proved in [34], see also [15] for analysis of the system with a pressure Darcy formulation. The finite element approximation of the coupled problem has been studied extensively, see, e.g., [15, 21, 34, 40] for some of the early works. To the best of our knowledge, all previous studies have considered the deterministic model. Often, due to incomplete knowledge of the physical parameters, quantification of the model uncertainty needs to be incorporated. In this paper, we study the interaction of a free fluid with a porous media with stochastic permeability. Even though the stochasticity comes only from the uncertain nature of the permeability in the porous region, the resulting solution is stochastic over the entire domain, due to the coupling conditions. We propose an efficient algorithm to compute statistical moments of the solution, such as mean, variance, and higher moments.

In this work the permeability function is parametrized using truncated Karhunen-Loéve (KL) expansion with independent identically distributed random variables as coefficients. Given a covariance relationship with empirically determined statistics, one can compute the eigenvalues and corresponding eigenfunctions that form the KL series. This approach is commonly used for stochastic permeability as shown in [35, 47, 50, 51] and in particular can be used in the framework for stochastic collocation and mixed finite elements [24]. Following [27, 35], we consider non-stationary porous media with different covariance functions in different parts of the domain, which allows to model heterogeneous media. For instance, the arrangement of sedimentary rocks in distinct layers motivate the use of such statistically independent regions, each region corresponding to a particular rock type. In the paper such regions are referred to as KL regions. The covariance between two points that lie in different KL regions is zero, while otherwise it depends on the distance between those points.

To compute the statistical moments of the solution, we employ the stochastic collocation method [8, 24, 37, 48], which samples the stochastic space at specifically chosen points. In this work we use zeros of orthogonal polynomials with either tensor product or sparse grid approximations. The method is non-intrusive, since it requires solving a sequence of deterministic problems. In this regard it resembles Monte Carlo Simulation (MCS) [20]. However, MCS may exhibit a high computational cost due to the need to generate valid representative statistics from a large number of realizations at random points in the stochastic event space. In comparison, stochastic collocation provides improved approximation of polynomial interpolation type and thus may result in better accuracy than MCS with fewer realizations. In terms of its approximation properties, stochastic collocation is comparable to polynomial chaos expansions [49] or stochastic finite element methods [14, 28]. However, the intrusive character of these methods may complicate their implementation. In addition, they result in high dimensional algebraic problems with fully coupled physical and stochastic dimensions.

In each stochastic realization we solve the coupled Stokes-Darcy problem using the multiscale mortar mixed finite element method (MMMFEM) introduced in [5, 29, 45]. The MMMFEM decomposes the physical domain into a union of non-overlapping subdomains of Stokes or Darcy type. Any Darcy subdomain is assumed to be contained in only one KL region. Each subdomain is discretized on a fine scale using stable and conforming mixed finite element spaces of Stokes or Darcy type. The grids are allowed to be non-matching along subdomain interfaces. A mortar space is introduced on the interfaces and discretized on a coarse scale. A coarse scale mortar Lagrange multiplier \( \lambda_H \) is used to impose weakly
continuity of flux. Since we allow for multiple subdomains, we must account for interfaces of Stokes-Darcy, Darcy-Darcy and Stokes-Stokes types. In particular, on Stokes-Darcy or Darcy-Darcy interfaces $\lambda_H$ is the normal stress or pressure, respectively - a scalar quantity, and it is used to impose continuity of the normal velocity. On Stokes-Stokes interfaces $\lambda_H$ is the normal stress vector, and it is used to impose continuity of the entire velocity vector. Following the algorithm in [30,45], the global fine scale problem is reduced to a coarse scale interface problem, which can be solved by Krylov iterative solvers in parallel. In the traditional implementation, the action of the interface operator requires solving Neumann problems in Stokes subdomains and Dirichlet problems in Darcy subdomains. The finite element tearing and interconnecting (FETI) method [19,44] is employed to deal with the possibly singular Stokes subdomain problems. We refer the reader to [16,17,22,31] for previous work on domain decomposition for Stokes-Darcy flows in the two-subdomain case.

The multiscale approximation of the MMMFEM is motivated by the fact that in porous media problems, resolving fine scale accuracy is often computationally infeasible. The method is an alternative to other multiscale methods, such as the variational multiscale method [4,33] and multiscale finite elements [2,12,32]. Both have been applied to stochastic problems in [7,23] and [1,13] respectively. The MMMFEM is more flexible than the aforementioned methods, since it allows for a posteriori error estimation and adaptive refinement of the coarse scale mortar interface mesh [5].

In the traditional implementation of the MMMFEM, the dominant computational cost is the solution of Stokes or Darcy subdomain problems at each interface iteration. Even though the dimension of the interface problem is reduced due to the coarse scale mortar space, the cost can be significant and the number of iterations grow with the condition number of the interface operator. To alleviate this cost, we utilize a multiscale flux basis. We follow the approach introduced in [26] for the Darcy problem, extended to the Stokes-Darcy problem in [25] and the stochastic Darcy problem in [27]. In particular, we compute a multiscale basis consisting of the local fine scale flux (or velocity trace) responses from each coarse scale mortar degree of freedom. It is computed by each subdomain independently before the interface iteration begins. The use of the multiscale flux basis avoids the need for subdomain solves on each Krylov iteration, since the action of the interface operator can be computed by a simple linear combination of the basis functions. We develop and compare two multiscale flux basis algorithms for the stochastic Stokes-Darcy problem. In the first method, referred to as deterministic multiscale flux basis, the basis is computed at each stochastic collocation and used only during the interface iteration at this realization. In this approach the number of subdomain solves is proportional to the number of mortar degrees of freedom per subdomain and the total number of stochastic collocation points. In the second method we follow the approach from [27] and explore the fact that the stochastic parameter is represented as a sum of local KL expansions. Unlike the first method, we do not recompute the basis at the beginning of every stochastic collocation. Instead, a full multiscale basis pre-computation is carried out on each subdomain for all local stochastic realizations before the global collocation loop begins. This stochastic basis can be reused multiple times during the collocation loop, since the same local subdomain stochastic structure occurs over multiple realizations. In this method, the number of subdomain solves is proportional to the number of mortar degrees of freedom per subdomain and the number of local stochastic collocation points, resulting in additional significant computational savings.
The remainder of the paper is organized as follows. The model problem is introduced in section 2. The MMMFEM is introduced in section 3. Tensor product and sparse grid stochastic collocation algorithms are presented in section 4. Three algorithms, including traditional MMMFEM, deterministic and stochastic multiscale flux basis implementations are presented in section 5. In section 6, the algorithms are employed for several numerical tests and compared in terms of computational efficiency.

2 Model problem

In this paper, a stochastic space with probability measure \( P \) is denoted by \( \Omega \). For any random variable \( \xi(\omega): \Omega \to \mathbb{R} \) with probability density function (PDF) \( \rho(z) \), its mean or expectation is defined by

\[
E[\xi] = \int_{\Omega} \xi(\omega)dP(\omega) = \int_{\mathbb{R}} z\rho(z)dz, \tag{1}
\]

and its variance is given by

\[
\text{var}[\xi] = E[\xi^2] - (E[\xi])^2.
\]

We denote the fluid region and the porous media region by \( D_s \subset \mathbb{R}^d \) and \( D_d \subset \mathbb{R}^d \), respectively, where \( d = 2, 3 \). Let \( \Gamma_s \) be the outside boundary of \( D_s \) with outward unit normal vector \( n_s \), and let \( \Gamma_d \) be the outside boundary of \( D_d \) with outward unit normal \( n_d \). The entire physical domain is defined as \( D = D_s \cup D_d \), with the Stokes-Darcy interface denoted by \( \Gamma_{SD} = D_s \cap D_d \). Let \((u_s, p_s)\) and \((u_d, p_d)\) be the velocity and pressure unknowns in the Stokes and Darcy regions, respectively. In the Stokes region, let \( \nu_s \) be the viscosity coefficient and define the deformation rate tensor \( D \) and stress tensor \( T \) by

\[
D(u_s) := \frac{1}{2}(\nabla u_s + (\nabla u_s)^T), \quad T(u_s, p_s) := -p_s I + 2\nu_s D(u_s).
\]

In the Darcy region, let \( \nu_d \) be the viscosity coefficient and \( K(x, \omega) \) be a stochastic function defined on \( D \times \Omega \) representing the non-stationary permeability of porous media. We assume \( K \) to be uniformly positive definite for \( P \)-almost every \( \omega \in \Omega \) with components in \( L^\infty(D_d) \). The coupled Stokes-Darcy flow satisfies the following equations (2)–(10): For \( P \)-almost every \( \omega \in \Omega \), in the Stokes region, \((u_s, p_s)\) satisfy

\[
-\nabla \cdot T(u_s, p_s) = -2\nu_s \nabla \cdot D(u_s) + \nabla p_s = f_s \quad \text{in } D_s, \tag{2}
\]

\[
\nabla \cdot u_s = 0 \quad \text{in } D_s, \tag{3}
\]

\[
u_s u_s = 0 \quad \text{on } \Gamma_s. \tag{4}
\]

where \( f_s \) represents the body force. In the Darcy region, \((u_d, p_d)\) satisfy

\[
\nu_d K(x, \omega)^{-1} u_d + \nabla p_d = f_d \quad \text{in } D_d, \tag{5}
\]

\[
\nabla \cdot u_d = q_d \quad \text{in } D_d, \tag{6}
\]

\[
u_d u_d \cdot n_d = 0 \quad \text{on } \Gamma_d. \tag{7}
\]
where $f_d$ is the gravity force and $q_d$ is an external source or sink term satisfying the solvability condition
\[
\int_{D_d} q_d \, dx = 0.
\]
The two regions are coupled on $\Gamma_{SD}$ through the following interface conditions:
\[
\begin{align*}
  \mathbf{u}_s \cdot \mathbf{n}_s + \mathbf{u}_d \cdot \mathbf{n}_d &= 0 \quad \text{on } \Gamma_{SD}, \\
  -\mathbf{T}(\mathbf{u}_s, p_s) \cdot \mathbf{n}_s &\equiv p_s - 2\nu_s (D(\mathbf{u}_s) \mathbf{n}_s) \cdot \mathbf{n}_s = p_d \quad \text{on } \Gamma_{SD}, \\
  \mathbf{T}(\mathbf{u}_s, p_s) \cdot \tau_j &\equiv -2\nu_s (D(\mathbf{u}_s) \mathbf{n}_s) \cdot \tau_j = \frac{\nu_s \alpha \sqrt{K_j}}{\sqrt{K_j}} \mathbf{u}_s \cdot \tau_j, \\
  &\quad \text{on } \Gamma_{SD},
\end{align*}
\]
Conditions (8) and (9) denote the continuity of flux and normal stress through $\Gamma_{SD}$, respectively. Condition (10) is the well-known Beaver-Joseph-Saffman law [9, 41], where $\{\tau_j\}_{j=1}^{d-1}$ is an orthogonal system of unit tangent vectors on $\Gamma_{SD}$ and $K_j = (K(x, \omega) \tau_j) \cdot \tau_j$ is the friction coefficient. The constant $\alpha > 0$ is a slip coefficient determined experimentally.

Let $Y(x, \omega) = \ln(K(x, \omega))$ be the log permeability and define
\[
Y'(x, \omega) := Y(x, \omega) - E[Y](x).
\]
To characterize $Y'$, we divide the Darcy region $D_d$ into several non-overlapping KL regions $D_d = \bigcup_{i=1}^{N_\Omega} D_{KL}^{(i)}$, where the stochastic structure of every region is independent from the others. In other words, the covariance between any pair of points from different KL regions is zero. The stochastic space is then divided correspondingly by
\[
\Omega = \bigotimes_{i=1}^{N_\Omega} \Omega^{(i)}.
\]
Therefore, for each event $\omega \in \Omega$, we can write $\omega = (\omega^{(1)}, \ldots, \omega^{(N_\Omega)})$ and
\[
Y'(x, \omega) = \sum_{i=1}^{N_\Omega} Y^{(i)}(x, \omega^{(i)}).
\]

2.1 Karhunen-Loève (KL) expansion

Each $Y^{(i)}$ has a physical support in $D_{KL}^{(i)}$ and is given a covariance function:
\[
C_{Y^{(i)}}(x, \bar{x}) = E[Y^{(i)}(x, \omega^{(i)})Y^{(i)}(\bar{x}, \omega^{(i)})].
\]
Since it is symmetric and positive definite, it can be decomposed into the following series expansion:
\[
C_{Y^{(i)}}(x, \bar{x}) = \sum_{j=1}^{\infty} \lambda_j^{(i)} f_j^{(i)}(x) f_j^{(i)}(\bar{x}),
\]
where the eigenvalues and eigenfunctions $\lambda^{(i)}_j$, $f^{(i)}_j$ respectively, are computed by using $C_Y^{(i)}$ as the kernel of Fredholm integral equation:

$$\int_{D_{KL}^{(i)}} C_Y^{(i)}(x, \bar{x}) f^{(i)}_j(x) dx = \lambda^{(i)}_j f^{(i)}_j(\bar{x}). \tag{11}$$

The eigenfunctions of $C_Y^{(i)}$ are mutually orthogonal and form a complete spanning set since each $C_Y^{(i)}$ is symmetric and positive definite, therefore the Karhunen-Loève expansion for the log permeability can be expressed as

$$Y'(x, \omega) = \sum_{i=1}^{N_\Omega} \sum_{j=1}^{N_{\text{term}}(i)} \xi^{(i)}_j(\omega(i)) \sqrt{\lambda^{(i)}_j} f^{(i)}_j(x), \tag{12}$$

where the eigenfunctions $f^{(i)}_j(x)$ computed in [11] have been extended by zero outside of $D_{KL}^{(i)}$ and the $\xi^{(i)}_j : \Omega_i \to \mathbb{R}$ are independent identically distributed random variables [28]. We would assume for this work that $Y^{(i)}$ are given by Gaussian process, so each $\xi^{(i)}_j$ is a random variable with zero mean and unit variance, with the following probability density function: $\rho^{(i)}_j(y) = 1/\sqrt{2\pi} \exp[-y^2/2]$.

It is reasonable to truncate the KL-expansion after $N$ terms as the eigenvalues $\lambda^{(i)}_j$ show rapid decay when $N$ is large [51]. If the expansion is truncated prematurely, the permeability may appear too smooth in a particular KL region. In our case, for any KL region $i$, we truncate the expansion after its first $N_{\text{term}}(i)$ terms. Increasing $N_{\text{term}}(i)$ introduces more heterogeneity into the permeability realizations for a chosen region. It is beyond the scope of this paper to address the modeling error associated with truncating the KL expansion. Some work has been done to quantify the modeling error [37] and it can be reduced a posteriori [11]. The truncation after $n_T$ terms allows us to approximate [12] by

$$Y'(x, \omega) \approx \sum_{i=1}^{N_\Omega} \sum_{j=1}^{N_{\text{term}}(i)} \xi^{(i)}_j(\omega(i)) \sqrt{\lambda^{(i)}_j} f^{(i)}_j(x). \tag{13}$$

The above also shows that globally we have $N_{\text{term}} := \sum_{i=1}^{N_\Omega} N_{\text{term}}(i)$ terms in $Y'$.

The images of the random variables $S^{(i)}_j = \xi^{(i)}_j(\Omega(i))$ make up finite dimensional spaces which are local to each KL region: $S^{(i)} = \prod_{j=1}^{N_{\text{term}}(i)} S^{(i)}_j \subseteq \mathbb{R}^{N_{\text{term}}(i)}$ and also vector space that is global: $S = \prod_{i=1}^{N_\Omega} S^{(i)} \subseteq \mathbb{R}^{N_{\text{term}}}$.

Let us introduce a function $\kappa$ that provides a natural ordering for the global number of stochastic dimensions. Then the $j$-th stochastic parameter of the $i$-th KL region have a global index in $\{1, \ldots, N_{\text{term}}\}$ given by the function

$$\kappa(i, j) = \begin{cases} j, & \text{if } i = 1 \\ j + \sum_{k=1}^{i-1} N_{\text{term}}(k), & \text{if } i > 1. \end{cases}$$

Since if $\rho^{(i)}_j$ is the PDF of each $\xi^{(i)}_j$, then joint PDF for $\xi$ is defined to be $\rho = \prod_i \prod_j \rho^{(i)}_j$. These allows us to write $Y(x, \omega) \approx Y(x, y)$, where $y = \left(\xi^{(i)}_j(\omega(i))\right)_\kappa$. 

6
Note that for the remainder of this paper, we abuse notation by replacing $K(x, \omega)$ with its finite dimensional spectral approximation $\hat{K}(x, y)$ given by equation (13). We also identify each stochastic subspace $\Omega^{(i)}$ with its parametrization $\mathbb{S}^{(i)}$. Therefore the modeling error between the true stochastic solution and its finite dimensional approximation is neglected.

When the exact eigenvalues and eigenfunctions of the covariance function $C$ can be found, the KL expansion is the most efficient method for representing a random field. However, in most cases, closed-form eigenfunctions and eigenvalues are not readily available and numerical procedures need be performed for solving the integral equation (11).

3 Multiscale mortar mixed finite element method

3.1 Domain decomposition

The coupled Stokes-Darcy flow is solved using domain decomposition method following the approach described in [45]. The Stokes and Darcy domains are partitioned into $N_s$ and $N_d$ non-overlapping subdomains, respectively. Let $N = N_s + N_d$, with $D_s = \bigcup_{i=1}^{N_s} D_i$, $D_d = \bigcup_{i=N_s+1}^N D_i$, and $D_i \cap D_j = \emptyset$ for $i \neq j$. Let the interface between adjacent subdomains be $\Gamma_{i,j} = \partial D_i \cap \partial D_j$. Depending on the models of adjacent domains, we group all interfaces into three different types: Stokes-Stokes type, Darcy-Darcy type and Stokes-Darcy type, denoted by $\Gamma_{SS}$, $\Gamma_{DD}$ and $\Gamma_{SD}$, respectively. The union of all interfaces is then defined as $\Gamma = \Gamma_{SS} \cup \Gamma_{DD} \cup \Gamma_{SD}$. In addition, it is assumed that in $D_d$ each KL region is an exact union of subdomains.

Several interface conditions are also applied, we impose continuity of velocity and stress on $\Gamma_{SS}$, and continuity of normal velocity and pressure on $\Gamma_{DD}$. Let $(u_i, p_i) = (u_s|_{D_i}, p_s|_{D_i})$ if $D_i$ is a Stokes subdomain and $(u_i, p_i) = (u_d|_{D_i}, p_d|_{D_i})$ if it is a Darcy subdomain. Then for $P$-almost every $y \in \mathbb{S}$, we seek $(u_i, p_i)$ for $i = 1, \ldots, N$ that satisfy the equations (2)–(10) in each subdomain $D_i$ with the following interface conditions:

\begin{align}
[T(u, p)n] &= 0 \quad \text{on } \Gamma_{SS}, \quad [p] = 0 \quad \text{on } \Gamma_{DD}, \quad (14) \\
[u] &= 0 \quad \text{on } \Gamma_{SS}, \quad [u \cdot n] = 0 \quad \text{on } \Gamma_{DD}, \quad (15)
\end{align}

where $[ \cdot ]$ represents the jump on the interface. For example, on $\Gamma_{ij}$, $[u] = (u_i - u_j)|_{\Gamma_{ij}}$, $[u \cdot n] = u_i \cdot n_i + u_j \cdot n_j$, using the notation $u_i = u|_{D_i}$, and $n_i$ is the outer unit normal to $\partial D_i$.

3.2 Variational formulation

For clarity, we rename $D_i$ to $D_{s,i}$ if $1 \leq i \leq N_s$, and to $D_{d,i-N_s}$ if $N_s + 1 \leq i \leq N$. In the deterministic setting, the velocity spaces in each domain are defined as

\begin{align*}
\tilde{V} &= \{ v \in L^2(D)^d; \ v|_{D_{s,i}} \in \tilde{V}_D, \ v|_{D_{d,i}} \in \tilde{V}_S \}, \\
\tilde{V}_D &= \{ v \in L^2(D)^d; \ v_{d,i} := v|_{D_{d,i}} \in H(\text{div}; D_{d,i}), \ 1 \leq i \leq N_d, \ v \cdot n = 0 \text{ on } \Gamma_d \}, \\
\tilde{V}_S &= \{ v \in L^2(D)^d; \ v_{s,i} := v|_{D_{s,i}} \in H^1(D_{s,i})^d, \ 1 \leq i \leq N_s, \ v = 0 \text{ on } \Gamma_s \},
\end{align*}
where
\[ H(\text{div}; D_{d,i}) = \{ \mathbf{v}_{d,i} \in (L^2(D_{d,i}))^d \mid \nabla \cdot \mathbf{v}_{d,i} \in L^2(D_{d,i}) \}, \]
equipped with the norm
\[ \| \mathbf{v} \|_{H(\text{div}; D_{d,i})} = (\| \mathbf{v} \|^2_{L^2(D_{d,i})} + \| \nabla \cdot \mathbf{v} \|^2_{L^2(D_{d,i})})^{1/2}. \]

The pressure space is defined as \( \tilde{W} = L_0^2(D) \), and we also need a space for the Lagrange multiplier to impose continuity on the interfaces with
\[ \tilde{\Lambda} = \tilde{\Lambda}_{SD} \times \tilde{\Lambda}_{DD} \times \tilde{\Lambda}_{SS}, \]
where on \( \Gamma_{SS} \) the Lagrange multiplier has the physical meaning of stress and on \( \Gamma_{DD} \cup \Gamma_{SD} \) it has the meaning of pressure.

Now we define the following \( L^2 \) space on \( S \):
\[ L^2_{\rho}(S) = \left\{ \mathbf{v} : S \to \mathbb{R}^d \mid \left( \int_S \| \mathbf{v}(y) \|^2 \rho(y) dy \right)^{1/2} < \infty \right\}, \]
and form the stochastic spaces by taking its tensor product with all the deterministic spaces above:
\[ V := V(S) = \tilde{V} \otimes L^2_{\rho}(S), \quad V_S := V_S(S) = \tilde{V}_S \otimes L^2_{\rho}(S), \quad V_D := V_D(S) = \tilde{V}_D \otimes L^2_{\rho}(S), \]
\[ W := W(S) = \tilde{W} \otimes L^2_{\rho}(S), \quad \Lambda := \Lambda(S) = \tilde{\Lambda} \otimes L^2_{\rho}(S). \]

The weak formulation for the coupled Stokes-Darcy problem (2)–(10) and (14)–(15) is given by: Find \( (\mathbf{u}, p, \lambda) \in V \times W \times \Lambda \), such that
\[ a(\mathbf{u}, \mathbf{v}, S) + b(\mathbf{v}, p, S) + b_\Lambda(\mathbf{v}, \lambda, S) = \int_S \left( \int_D \mathbf{f} \cdot \mathbf{v} \right) \rho(y) dy, \quad \forall \mathbf{v} \in V, \] (16)
\[ b(\mathbf{u}, w, S) = -\int_S \left( \int_{D_d} w q_d \right) \rho(y) dy, \quad \forall w \in W, \] (17)
\[ b_\Lambda(\mathbf{u}, \mu, S) = 0, \quad \forall \mu \in \Lambda. \] (18)
where

\[
\tilde{a}_{s,i}(u_{s,i}, v_{s,i}) = 2 \nu_s \int_{D_{s,i}} D(u_{s,i}) : D(v_{s,i}) + \sum_{j=1}^{d-1} \int_{\partial D_{s,i} \cap \Gamma_{sd}} \frac{\nu_s \alpha}{k_j} (u_{s,i} \cdot \tau_j)(v_{s,i} \cdot \tau_j),
\]

\[
1 \leq i \leq N_s, \quad \forall (u_{s,i}, v_{s,i}) \in V_S \times V_S,
\]

\[
\tilde{a}_{d,i}(u_{d,i}, v_{d,i}) = \nu_d \int_{D_{d,i}} K^{-1} u_{d,i} \cdot v_{d,i}, \quad 1 \leq i \leq N_d, \quad \forall (u_{d,i}, v_{d,i}) \in V_D \times V_D,
\]

\[
\tilde{b}_i(v_i, w_i) = - \int_{D_i} w_i \nabla \cdot v_i, \quad 1 \leq i \leq N, \quad \forall v_i \in V, \quad \forall w_i \in W;
\]

\[
\tilde{a}(u, v) = \sum_{i=1}^{N_s} \tilde{a}_{s,i}(u, v) + \sum_{i=1}^{N_d} \tilde{a}_{d,i}(u, v), \quad \tilde{b}(v, w) = \sum_{i=1}^{N} \tilde{b}_i(v, w),
\]

\[
\tilde{b}_\Lambda(v, \mu) = \int_{\Gamma_{SS}} [v] \mu + \int_{\Gamma_{DD}} [v \cdot n] \mu + \int_{\Gamma_{SD}} [v] \mu, \quad \forall (v, \mu) \in V \times \Lambda,
\]

\[
a(u, v, S) = \int_S \tilde{a}(u, v) \rho(y) dy, \quad b(u, v, S) = \int_S \tilde{b}(u, v) \rho(y) dy, \quad \forall (u, v) \in V \times V,
\]

\[
b_\Lambda(v, \mu, S) = \int_S \tilde{b}_\Lambda(v, \mu) \rho(y) dy, \quad \forall (v, \mu) \in V \times \Lambda.
\]

### 3.3 Finite element discretization

Following the idea in [27], a semidiscrete approximation to the weak solution of the stochastic variational form [16] – [18] is applied. To do this, we implement the multiscale mortar mixed finite element method (MMMFEM) in the physical dimensions, and then employ the stochastic collocation method, such as tensor product or sparse grid collocation using Gauss-Hermite quadrature rule for the additional stochastic dimensions. Therefore, solving (16) – (18) is becoming solving a sequence of independent deterministic problem where the original domain decomposition algorithm for Stokes-Darcy coupled problem applies. The resulting solutions are realizations in stochastic space and function values in the quadrature rule.

The MMMFEM we are using allows non-conforming meshes along the subdomain interfaces. Each subdomain \( D_i \) is partitioned into a \( d \)-dimensional shape regular finite element discretization \( T_{h_i} \) with \( h_i \) being the maximal element diameter. Let \( T_h = \bigcup_{i=1}^{N} T_{h_i} \) be the global fine mesh and \( h = \max_{i=1}^{N} h_i \). On Stokes domains, i.e. \( 1 \leq i \leq N_s \), let \( \tilde{V}_{h,i}(D_i) \times \tilde{W}_{h,i}(D_i) \subset \tilde{V}_i(D_i) \times \tilde{W}_i(D_i) \) be any finite element spaces satisfying the inf-sup condition

\[
\inf_{0 \neq w_{h,i} \in \tilde{W}_{h,i}} \sup_{0 \neq v_{h,i} \in \tilde{V}_{h,i}} \frac{(w_{h,i}, \nabla \cdot v_{h,i})_{D_i}}{\|v_{h,i}\|_{H^1(D_i)} \|w_{h,i}\|_{L^2(D_i)}} \geq \beta_s > 0. \quad (19)
\]

where \( \tilde{V}_i(D_i) = \tilde{V}_S|_{D_i} \) and \( \tilde{W}_i(D_i) = \tilde{W}_D|_{D_i} \). Examples of working spaces include the Taylor-Hood elements [43], the MINI elements [6] and the conforming Crouzeix-Raviart elements [13]. Similarly, on Darcy domains, i.e. \( N_s + 1 \leq i \leq N \), we let \( \tilde{W}_{h,i}(D_i) \subset \tilde{V}_i(D_i) \times \tilde{W}_i(D_i) \) be a mixed finite element space on \( T_{h,i} \). Any of the well-known pairs would
the boundary types of ker \( \tilde{\Lambda} \), where \( SS \) and tangential stress on \( \Gamma \) with specified velocity and pressure spaces are given as \( \tilde{V}_h = \bigoplus_{i=1}^N \tilde{V}_{h,i} \) and \( \tilde{W}_h = \bigoplus_{i=1}^N \tilde{W}_{h,i} \), respectively.

Next, each subdomain interface \( \Gamma_{i,j} \) is partitioned into a coarse \((d-1)\)-dimensional quasi-uniform affine mesh \( T_{H_{i,j}} \) with maximal element diameter \( H_{i,j} \). A mortar space \( \tilde{\Lambda}_{H_{i,j}}(\Gamma_{i,j}) \subset L^2(\Gamma_{i,j}) \) is defined to weakly impose the continuity of normal fluxes for the discrete velocity across the non-matching grids. It contains continuous or discontinuous piecewise polynomials. Let \( H = \max_{i,j} H_{i,j} \) and \( \tilde{\Lambda}_H = \bigoplus_{i,j} \tilde{\Lambda}_{H_{i,j}} \). Then the semidiscrete approximation to \((16) - (18)\) with stochastic MMMFEM is to find \( \tilde{u}_h : S \to \tilde{V}_h \), \( \tilde{p}_h : S \to \tilde{W}_h \), \( \lambda_H : S \to \tilde{\Lambda}_H \)

such that for \( \rho \)-almost every \( y \in S \), the following deterministic problem holds:

\[
\tilde{a}(\tilde{u}_h, v_h) + \tilde{b}(\tilde{v}_h, \tilde{p}_h) + \tilde{b}_\lambda(v_h, \lambda_H) = \int_D f \cdot v_h, \quad \forall v_h \in \tilde{V}_h, \quad (20)
\]

\[
\tilde{b}(\tilde{u}_h, w_h) = -\int_{D_{d'}} w_h q_d, \quad \forall w_h \in \tilde{W}_h, \quad (21)
\]

\[
\tilde{b}_\lambda(\tilde{u}_h, \mu) = 0, \quad \forall \mu \in \tilde{\Lambda}_H. \quad (22)
\]

The function in mortar space represents the stress on \( \Gamma_{SS} \) and pressure on \( \Gamma_{DD} \). Therefore the interface condition \((14)\) is satisfied on the mortar mesh. On the other hand, condition \((15)\), which represents the continuity of velocity on \( \Gamma_{SS} \) and normal velocity on \( \Gamma_{DD} \), is fulfilled weakly by \((22)\).

For well-posedness of the above deterministic problem, refer to \([34]\).

### 3.4 Reduction to an interface problem

We use the algorithm introduced in \([45]\) to solve \((20) - (22)\) by reducing them to a symmetric and positive definite interface problem. To do so, on each subdomain we split \((20) - (22)\) into two subproblems. For Stokes domains, \( D_{s,i} \), \( 1 \leq i \leq N_s \), one of the subproblems is to find

\[
\tilde{u}_{h,i}^s(\lambda) : S \to \tilde{V}_{h,i}, \quad \tilde{p}_{h,i}^s(\lambda) : S \to \tilde{W}_{h,i},
\]

with specified \( \lambda = (\lambda_n, \lambda_r) \), where \( \lambda_n \) and \( \lambda_r = (\lambda^1_r, \ldots, \lambda^{d-1}_r) \) represent the normal stress and tangential stress on \( \Gamma_{SS} \), respectively, such that for \( \rho \)-almost every \( y \in S \),

\[
\tilde{a}_{s,i}(\tilde{u}_{h,i}^s(\lambda), v_i) + \tilde{b}_i(v_i, \tilde{p}_{h,i}^s(\lambda)) = -\langle \lambda_n, v_i \cdot n_i \rangle_{\partial D_{s,i} \setminus \partial D} - \sum_{l=1}^{d-1} \langle \lambda^l_r, v_i \cdot \tau^l_i \rangle_{\partial D_{s,i} \cap \Gamma_{SS}},
\]

\[
\forall v_i \in \tilde{V}_{h,i} / \ker \tilde{a}_i, \quad (23)
\]

\[
\tilde{b}_i(\tilde{u}_{h,i}^s(\lambda), w_i) = 0, \quad \forall w_i \in \tilde{W}_{h,i}, \quad (24)
\]

where \( \{\tau^1_i\}_{i=1}^{d-1} \) is an orthogonal set of unit vectors tangential to \( \partial D_{s,i} \) and the kernel space \( \ker \tilde{a}_i := \{ v \in \tilde{V}_i : \tilde{a}_i(v, v) = 0 \} \) consists of a subset of all rigid body motions depending on the boundary types of \( D_{s,i} \). For more detail we refer the reader to \([45]\).
The complementary subproblem is to find
\[ \tilde{u}_{h,i} : S \rightarrow \tilde{V}_{h,i}, \quad \tilde{p}_{h,i} : S \rightarrow \tilde{W}_{h,i}, \]
such that for \( \rho \)-almost every \( y \in S \),
\[ \tilde{a}_{s,i}(\tilde{u}_{h,i}, \mathbf{v}_i) + \tilde{b}_i(\mathbf{v}_i, \tilde{p}_{h,i}) = \int_{D_{s,i}} \mathbf{f}_i \cdot \mathbf{v}_i, \quad \forall \mathbf{v}_i \in \tilde{V}_{h,i}/\text{ker} \ a_i, \]  \( \text{(25)} \)
\[ \tilde{b}_i(\tilde{u}_{h,i}, w_i) = 0, \quad \forall w_i \in \tilde{W}_{h,i}. \]  \( \text{(26)} \)

Notice the first problem \( (23) - (24) \) has specified stress on the interface with zero boundary condition and source, while the second problem \( (25) - (26) \) has specified boundary condition and source but with zero stress on the interface. Similarly, we can do the same splitting on Darcy domains \( D_{d,i}, 1 \leq i \leq N_d \), one subproblem is to find
\[ u_{h,i}^*(\lambda) : S \rightarrow \tilde{V}_{h,i}, \quad p_{h,i}^*(\lambda) : S \rightarrow \tilde{W}_{h,i}, \]
with specified pressure \( \lambda \) such that for \( \rho \)-almost every \( y \in S \),
\[ \tilde{a}_{d,i}(u_{h,i}^*(\lambda), \mathbf{v}_i) + \tilde{b}_i(\mathbf{v}_i, p_{h,i}^*(\lambda)) = -\langle \lambda, \mathbf{v}_i \cdot \mathbf{n}_i \rangle_{\partial D_{d,i} \setminus \partial D}, \quad \forall \mathbf{v}_i \in \tilde{V}_{h,i}, \]  \( \text{(27)} \)
\[ \tilde{b}_i(u_{h,i}^*(\lambda), w_i) = 0, \quad \forall w_i \in \tilde{W}_{h,i}. \]  \( \text{(28)} \)

The complementary problem is to find
\[ \tilde{u}_{h,i} : S \rightarrow \tilde{V}_{h,i}, \quad \tilde{p}_{h,i} : S \rightarrow \tilde{W}_{h,i}, \]
such that for \( \rho \)-almost every \( y \in S \),
\[ \tilde{a}_{d,i}(\tilde{u}_{h,i}, \mathbf{v}_i) + \tilde{b}_i(\mathbf{v}_i, \tilde{p}_{h,i}) = \int_{D_{d,i}} \mathbf{f}_i \cdot \mathbf{v}_i, \quad \forall \mathbf{v}_i \in \tilde{V}_{h,i}, \]  \( \text{(29)} \)
\[ \tilde{b}_i(\tilde{u}_i, w_i) = -\int_{D_{d,i}} w_i q_d, \quad \forall w_i \in \tilde{W}_{h,i}. \]  \( \text{(30)} \)

Therefore by equating the solutions via condition \( (22) \), the original problem \( (20) - (22) \) is equivalent to this interface problem: finding \( \lambda_H : S \rightarrow \bar{\Lambda}_H \) such that for \( \rho \)-almost every \( y \in S \),
\[ s_H(\lambda_H, \mu) := -\tilde{b}_H(u_{h,i}^*(\lambda_H), \mu) = \tilde{b}_H(\tilde{u}_i, \mu), \quad \mu \in \bar{\Lambda}_H. \]  \( \text{(31)} \)

Problem \( (31) \) can be solved by a Krylov space solver. In this paper, we use the Conjugate Gradient (CG) method in all numerical examples. After solving \( \lambda_H \), the global velocity \( u_h \) and pressure \( p_h \) can be recovered by
\[ u_h = u_{h,i}^*(\lambda_H) + \tilde{u}_i, \quad p_h = p_{h,i}^*(\lambda_H) + \tilde{p}_i. \]
4 Stochastic collocation

The stochastic collocation method is to approximate the semidiscrete solution \((u_h, p_h, \lambda_H)\) by an interpolant \(\mathcal{I}_m\), where \(m\) (or \(m\)) is a multi-index indicating the desired polynomial degree of accuracy in stochastic dimensions. It is uniquely formed on a set of \(N_{real}\) stochastic points \(y_k\), where \(N_{real}\) is a function of \(m\) and the fully discrete solution is given by

\[ u_{h,m}(x, y) = \mathcal{I}_m u_h(x, y), \quad p_{h,m}(x, y) = \mathcal{I}_m p_h(x, y), \quad \lambda_{H,m}(x, y) = \mathcal{I}_m \lambda_H(x, y). \]

Choose the Lagrange basis \(\{L_m^{(k)}(y)\}\) such that \(\{L_m^{(k)}(y_j)\} = \delta_{kj}\). Then the Lagrange representation for the fully discrete solution is

\[ (u_{h,m}, p_{h,m}, \lambda_{H,m})(x, y) = \sum_{k=1}^{N_{real}} (u_h^{(k)}, p_h^{(k)}, \lambda_H^{(k)})(x)L_m^{(k)}(y), \]

where \((u_h^{(k)}, p_h^{(k)}, \lambda_H^{(k)})\) is the evaluation of semidiscrete solution \((u_h, p_h, \lambda_H)\) at the point in stochastic space \(y_k\). In other words, for each permeability realization \(K^{(k)}(x) = K(x, y_k)\), \(k = 1, \ldots, N_{real}\), we solve the deterministic problem: find \(u_h^{(k)} \in \tilde{V}_h\), \(p_h^{(k)} \in \tilde{W}_h\) and \(\lambda_H^{(k)} \in \tilde{\Lambda}_H\), such that,

\[
\begin{align*}
\tilde{a}(u_h^{(k)}, v_h) + \tilde{b}(v_h, p_h^{(k)}) + \tilde{b}_\lambda(v_h, \lambda_H^{(k)}) &= \int_D f \cdot v_h, \quad \forall v_h \in \tilde{V}_h, \quad (32) \\
\tilde{b}(u_h^{(k)}, w_h) &= -\int_{D_d} w_h q_d, \quad \forall w_h \in \tilde{W}_h, \quad (33) \\
\tilde{b}_\lambda(u_h^{(k)}, \mu) &= 0, \quad \forall \mu \in \tilde{\Lambda}_H. \quad (34)
\end{align*}
\]

Similarly, equations \((32)-(34)\) can be reduced to the interface problem: find \(\lambda_H^{(k)} \in \tilde{\Lambda}_H\) such that

\[ s_H(\lambda_H^{(k)}, \mu) := -\tilde{b}_\lambda(u_h^{(k)}(\lambda_H^{(k)}), \mu) = \tilde{b}_\lambda(u_h^{(k)}, \mu), \quad \mu \in \tilde{\Lambda}_H. \quad (35) \]

using the same technique in section \(3.4\) to split the solutions into “bar” and “star” components.

To get the quadrature rule, one may plug the Lagrange representation of the fully discreet solution into the expectation integral \([1]\). Different collocation methods could be obtained by choosing different collocation points \(y_k\). In this paper we considered tensor product and sparse grids, both of them are constructed using the one-dimensional rules, where the points in dimension \(S_j^{(i)}\) are the zeros of orthogonal polynomials with respect to the \(L_p^2(S_j^{(i)})\)-inner-product. Since the random variables used in the paper are Gaussian random variables, it was natural to choose the zeros of the “probabilist” \(N(0,1)\) Hermite polynomials

\[ H_m(y) = m! \sum_{k=0}^{[m/2]} (-1)^k \frac{(2y)^{m-2k}}{k!(m-2k)!}. \]

The weights and abscissae can be computed with a symbolic manipulation software package or, alternatively, by using a table of rules for the “physicist” \(N(0,1/2)\) Hermite polynomials listed in \([3]\) with proper coefficients.
4.1 Collocation on tensor product grids

In tensor product collocation, the polynomial accuracy is prescribed in terms of *component* degree. This makes anisotropic rules to be very easy constructed, but the number of points in tensor product rule grows exponentially as number of dimensions or polynomial accuracy are increased. Thus, they are usually used in problems with relatively low number of stochastic dimensions.

Let \( N_{\text{term}}(i) \) be the stochastic dimension in KL region \( i \) and \( N_{\text{term}} \) be the total stochastic dimension. We choose \( N_{\text{coll}}(i,j) \) many collocation points in stochastic dimension \( j \) of KL region \( i \), and define \( m = (N_{\text{coll}}(i,j))_\kappa \), which is an \( N_{\text{term}} \)-dimensional multi-index, as the required component degree of the interpolant in the stochastic space \( \mathbb{S} \). The corresponding anisotropic tensor product Gauss-Hermite interpolant in \( N_{\text{term}} \)-dimensions is given by

\[
I_{m}^{\text{TG}} f(y) = (I_{m(1)}^{1} \otimes \cdots \otimes I_{m(N_{\text{term}})}^{N_{\text{term}}}) f(y) = \sum_{k_{1}=1}^{N_{\text{term}}} \cdots \sum_{k_{N_{\text{term}}}=1}^{N_{\text{term}}} f(h_{m(1)}^{k_{1}}, \ldots, h_{m(N_{\text{term}})}^{k_{N_{\text{term}}}}) I_{m(1)}^{k_{1}}(y_{1}) \cdots I_{m(N_{\text{term}})}^{k_{N_{\text{term}}}}(y_{N_{\text{term}}}).
\]

The set of abscissae for this rule is

\[
T(m) = \bigotimes_{k=1}^{N_{\text{term}}} \mathcal{H}(m(k)) = \bigotimes_{i=1}^{N_{\text{term}}(i)} \mathcal{H}(N_{\text{coll}}(i,j)), \tag{36}
\]

which interpolates the semi-discrete solution into the polynomial space \( P_{m} = \prod_{k} P_{m(k)} \) in the stochastic dimensions. The tensor product weight for the point \( (h_{m(1)}^{k_{1}}, \ldots, h_{m(N_{\text{term}})}^{k_{N_{\text{term}}}}) \) is given by

\[
w(k) = \prod_{i=1}^{N_{\text{term}}} w_{m(i)}^{k_{i}}.
\]

In a fixed stochastic dimension, the one dimensional Gauss-Hermite quadrature rules are accurate to degree \( 2m - 1 \).

4.2 Collocation on sparse grids

In sparse grid collocation, the polynomial accuracy is prescribed in terms of *total* degree. Sparse grids rules require much fewer points than tensor product rules as the dimension increases, but have the same asymptotic accuracy. This makes them applicable for problems with high number of stochastic dimensions. A picture of comparable sparse grid and tensor grid rules is shown in Figure [1].

Sparse grid rules are linear combination of tensor products on a family of nested one-dimensional rules. By the construction sparse grid rules have two main properties: only products with relatively small number of nodes are used and the total polynomial degree is independent of dimension. The characterization of a sparse grid rule is a level \( \ell_{\max} \), where the \( N_{\text{term}} \)-dimensional sparse grid quadrature rule of level \( \ell_{\max} \) is accurate to degree \( (2 \cdot \ell_{\max} + 1) \).
Each level between $\ell_{\text{max}}$ and $\ell_{\text{min}} = \max\{0, \ell_{\text{max}} - N_{\text{term}} + 1\}$ is an integer split into $N_{\text{term}}$ non-negative parts. These partitions define multi-indices $\mathbf{p} = (p_1, \ldots, p_{N_{\text{term}}})$, where $|\mathbf{p}| = \sum p_i$ denotes the levels of one dimensional rules, which are used for each stochastic dimension. We consider the Gauss-Hermite points $\mathcal{H}(2^{p_i+1} - 1)$ as the one dimensional abscissae of level $p$. Level 0 consists of a single point, and for every subsequent level the number of points doubles plus one.

For each partition $\mathbf{p}$ consider the multi-index $m = 2^{p_i+1} - 1$. The corresponding isotropic sparse grid Gauss-Hermite interpolant in $N_{\text{term}}$-dimensions is given by

$$I_{\ell_{\text{max}}}^{SG} f(y) = \sum_{\ell_{\text{min}} \leq |\mathbf{p}| \leq \ell_{\text{max}}} (-1)^{\ell_{\text{max}} - |\mathbf{p}|} \cdot \left(\frac{N_{\text{term}} - 1}{\ell_{\text{max}} - |\mathbf{p}|}\right) \cdot I_{m}^{TG} f(y).$$

The set of abscissae for this rule is

$$S(\ell_{\text{min}}, \ell_{\text{max}}, N_{\text{term}}) = \bigcup_{\ell_{\text{min}} \leq |\mathbf{p}| \leq \ell_{\text{max}}} N_{\text{term}} \bigotimes_{i=1}^{N_{\text{term}}} \mathcal{H}(2^{p_i+1} - 1).$$

Note that the origin is a value which is repeated in each one dimensional rule. Thus, the points in (37) are nested weakly.

5 Collocation-MMMFEM algorithms for Stokes-Darcy

In this section, we will introduce three different algorithms on the fully discrete stochastic problem (32)–(34). Recall that on each stochastic realization we solve a deterministic interface problem using CG method which is discretized by MMMFEM. That is, we solve for $\lambda^{(k)}_{H} \in \Lambda_{H}$ such that for any $k = 1, \ldots, N_{\text{real}}$,

$$s^{(k)}_{H}(\lambda^{(k)}_{H}, \mu) = \tilde{b}_{H}(u^{(k)}_{h}, \mu), \quad \forall \mu \in \Lambda_{H}. \quad (38)$$
For simplicity we introduce the Steklov–Poincaré type operator $S^{(k)}_H : \tilde{\Lambda}_H \to \tilde{\Lambda}_H$ with 
\[
\left(S^{(k)}_H \lambda, \mu \right) = s^{(k)}_H(\lambda, \mu), \quad \forall \lambda, \mu \in \tilde{\Lambda}_H.
\]
Then we can rewrite (38) in the operator form
\[
S^{(k)}_H \lambda^{(k)}_H = g^{(k)}_H,
\]
where $g^{(k)}_H$ is defined by $(g^{(k)}_H, \mu)_\Gamma = \tilde{b}_\Lambda(u^{(k)}_h, \mu), \quad \forall \mu \in \tilde{\Lambda}_H$.

It is important to note that the major cost in solving (39) comes from the operator action of $S^{(k)}_H$ in every CG iteration. This action involves the solving of a Neumann-to-Dirichlet problem (23)−(24) in Stokes domains, and a Dirichlet-to-Neumann problem (27)−(28) in Darcy domains.

Three different collocation algorithms are presented below: the first is a traditional MMMFEM collocation algorithm, the second and the third are implemented with deterministic and stochastic multiscale flux basis, respectively. The second and the third algorithms are dedicated to reduce the number of solves in traditional implementation and achieve a better computational efficiency.

5.1 Method S1: Collocation with traditional MMMFEM

**Method S1** (without multiscale flux basis)

For $k = 1, \ldots, N_{real}$, do

- **Step 1:** Generate permeability realization $K^{(k)}$ corresponding to the global collocation index $k$.
- **Step 2:** Solve (39) using traditional MMMFEM for $\lambda^{(k)}_H$.
- **Step 3:** Add the solution to the statistical moments with collocation weight applied: $\lambda_H = \lambda_H + \lambda^{(k)}_H \cdot w^{(k)}_m$.

End do

As mentioned before, in any subdomain, step 2 costs one subdomain solve from applying $S^{(k)}_H$ in every CG iteration. Given mortar function $\lambda_H$, the action of $S^{(k)}_H \lambda_H$ in subdomain $D_i$ include:

1. Project $\lambda_H$ onto subdomain boundaries: $\lambda_h = L_{h,i} \lambda_H$, where $L_{h,i} : L^2(\Gamma_i) \to \tilde{V}_{h,i} \cdot n_{i}|_{\Gamma_i}$ is the $L^2$-projection operator onto the normal trace of the velocity space on $D_i$.

2. If $D_i$ is a Darcy domain, solve the subdomain problem (27)−(28) with Dirichlet data $\lambda_h$; If $D_i$ is a Stokes domain, solve the subdomain problem (23)−(24) with Neumann data $\lambda_h$.

3. Project the resulting flux in Darcy or velocity in Stokes back to mortar space and compute the jump across the interfaces.

Let $N_{iter}(k)$ be the number of CG iterations for $k$th realization. Then in any subdomain $D_i$ the leading term in the number of solves for method S1 is 
\[
N_{S1}(i) := \sum_{k=1}^{N_{real}} N_{iter}(k),
\]
omitting the extra solves in the right hand side and solution recovery.

In this traditional MMMFEM implementation, since the condition number of our deterministic problem increases with the number of subdomains and magnitude of the permeability, the number $N_{\text{iter}}$ in CG can sometimes become large and thus method S1 will be very costly.

### 5.2 Method S2: Collocation with deterministic multiscale flux basis

One way to reduce the cost in MMMFEM for Stokes-Darcy flow is to implement multiscale flux basis following the method introduced in [24] and [25]. When solving the deterministic problem (39), we compute the subdomain solution before CG starts for every degree of freedom on mortar space and keep all results as a basis of flux responses in Darcy (or velocity responses in Stokes). Then to evaluate the action of $S^{(k)}_H$ in every interface iteration, we simply use the linear combination of the basis instead of costing an extra subdomain solve. The algorithm of this method is presented as follow:

| Method S2 (with deterministic multiscale flux basis) |
|------------------------------------------------------|
| For $k = 1, \ldots, N_{\text{real}}$, do             |
|   Step 1: Generate permeability realization $K^{(k)}$ corresponding to the global collocation index $k$. |
|   Step 2: Compute and save the multiscale flux basis $\{\phi_{H,i}^{(k)}\}_{i=1}^{N_{\text{dof}}}$ |
|   Step 3: Solve (39) for $\lambda_H^{(k)}$ using MMMFEM with the computed basis. |
|   Step 4: Add the solution to the statistical moments with collocation weight applied: $\lambda_H = \lambda_H + \lambda_H^{(k)} \cdot w^{(k)}$. |
| End do                                               |

In step 2, we follow the routines from traditional MMMFEM and apply the action of operator $S^k_H$ on each mortar basis function. The mortar basis functions $\{\xi_{H,i}^{(k)}\}_{i=1}^{N_{\text{dof}}} \subset \widetilde{\Lambda}_H$ are defined such that any $\lambda_H^{(k)} \in \widetilde{\Lambda}_H$ can be expressed uniquely by their linear combination:

$$
\lambda_H^{(k)} = \sum_{i=1}^{N_{\text{dof}}} c_i^{(k)} \xi_{H,i}^{(k)}.
$$

Then for any $i = 1 \ldots N_{\text{dof}}$, the multiscale flux basis is computed by

$$
\phi_{H,i}^{(k)} = S^k_H \xi_{H,i}^{(k)}.
$$

To evaluate the result of $S^k_H \lambda_H^{(k)}$ in step 3, we simply compute it by

$$
S^k_H \lambda_H^{(k)} = \sum_{i=1}^{N_{\text{dof}}} c_i^{(k)} \phi_{H,i}^{(k)}.
$$

Since there is no solve needed inside the CG loop, total solves for method S2 would not depend on the number of CG iterations. Therefore, in every stochastic realization, method
S2 costs less than method S1 if the number of CG iterations $N_{\text{iter}}(k)$ is bigger than the maximum number of mortar degrees of freedom among all subdomains. In any subdomain $D_i$, the number of solves for method S2 has the leading term

$$N_{S2}(i) := N_{\text{dof}}(i) \cdot N_{\text{real}}.$$  

5.3 Method S3: Collocation with stochastic multiscale flux basis

The third algorithm (method S3) was first presented in [27] with an implementation of stochastic multiscale flux basis. This method can achieve greater computational savings than method S2 due to the fact that in the same KL region, both tensor product and sparse grid collocations have a repeated local structure. The main idea is to form a pre-computation loop before the collocation loop, where multiscale flux basis are computed and stored for all local realizations of a subdomain’s KL region. Below we present the algorithm of method S3:

**Method S3** (with stochastic multiscale flux basis)

For any subdomain $D_i$, if it is Darcy, then $N_{\text{real}}(j)$ is the number of local realization in the KL region $j$ where $D_i$ belongs to; if it is Stokes, then $N_{\text{real}}(j)$ is set to 1.

**Pre-computation loop**

For $k = 1, \ldots, N_{\text{real}}(j)$, do

1. **Step 1:** Generate permeability realization $K^{(k)}$ corresponding to the local collocation index $k$.
2. **Step 2:** Compute and save the multiscale flux basis under current local index.

End do

**Main loop**

For $l = 1, \ldots, N_{\text{real}}$, do

1. **Step 3:** Generate permeability realization $K^{(l)}$ corresponding to the global collocation index $l$.
2. **Step 4:** Convert the global index to the subdomain’s local index $k$.
3. **Step 5:** Solve (39) for $\lambda_H^{(k)}$ using MMMFEM with the computed basis for local index $k$ in step 2.
4. **Step 6:** Add the solution to the statistical moments with collocation weight applied: $\lambda_H = \lambda_H + \lambda_H^{(k)} \cdot w_m^{(k)}$.

End do

The algorithms for index conversion for both tensor product grid and sparse grid in step 4 are given in [27], numbered as algorithm 5 and 6, respectively. Notice that all subdomain solves (except the extra two for RHS computation and solution recovery) in method S3 are done in the pre-computation loop only. Therefore in subdomain $D_i$, the total number of solves has the lead term

$$N_{S3}(i) := N_{\text{dof}}(i) \cdot N_{\text{real}}(j).$$

Compared to the leading term in $N_{S2}$ for method S2, $N_{S3}$ is proportional to $N_{\text{real}}(j)$ instead of $N_{\text{real}}$. When there exists more than one KL region in the problem setting, in
any KL region j the local realization \( N_{\text{real}}(j) \) will be less than the global realization \( N_{\text{real}} \), which makes method S3 cost less in subdomain solves than method S2.

6 Numerical tests

In this section, three numerical examples are presented to test both tensor product and sparse grid collocations and on all methods (S1, S2 and S3). For all tests, we use Taylor-Hood triangular finite element in Stokes and lowest order Raviart-Thomas rectangular finite element in the Darcy for space discretization. All interfaces are discretized via discontinuous piecewise linear mortar finite element.

6.1 Test case 1

The first test case has a global domain \([0,1] \times [0,1.2]\), where \( D_d = [0,1] \times [0,0.8] \) and \( D_s = [0,1] \times [0.8,1.2] \). The problem is divided into 6 equal-size subdomains with two Stokes and four Darcy domains. The outside boundary conditions are given as the following: in Darcy, zero pressure is specified on the bottom edge, and no-flow condition on the left and the right; in Stokes, velocity is specified on the left and top edges, while normal and tangential stresses are specified on the right.

There are two rectangular KL regions in Darcy, defined as \([0,1] \times [0,0.4]\) and \([0,1] \times [0.4,0.8]\), with \( N_{\text{term}}(1) = 2 \times 1 \) and \( N_{\text{term}}(2) = 3 \times 3 \) for tensor product grid, \( N_{\text{term}}(1) = N_{\text{term}}(2) = 5 \times 5 \) for sparse grid. The mean values of both KL regions are read-in from a file with the variances equal to 1.0 and the correlation lengths equal to 0.1. In tensor product collocation, the grid is isotropic and \( N_{\text{coll}} = 2 \); in sparse grid collocation, \( \ell_{\text{max}} = 1 \), so both ways have a third degree of accuracy in stochastic dimension. The number of global realization is \( N_{\text{real}} = 2048 \) for tensor product and \( N_{\text{real}} = 201 \) for sparse grid.

For space discretization, we have a local mesh of \( 64 \times 64 \) in every Darcy domain and \( 32 \times 32 \) in every Stokes domain. A mortar mesh of \( 8 \times 1 \) is applied to every interface on \( \Gamma_{DD} \) and \( 4 \times 1 \) to \( \Gamma_{SD} \cup \Gamma_{SS} \).

Table 1: Number of solves and runtime in seconds with the three algorithms for Case 1.

| | Tensor Product Collocation, \( N_{\text{term}} = 11 \), \( N_{\text{coll}} = 2 \) (2048 realizations); | | Sparse Grid Collocation, \( N_{\text{term}} = 50 \), \( \ell_{\text{max}} = 1 \) (101 realizations); |
|---|---|---|---|
| | Method S1 | Method S2 | Method S3 (Pre-comp) | Method S1 | Method S2 | Method S3 (Pre-comp) |
| Max. number of solves | 1,499,257 | 167,936 | 45,056 (40960) | 71,237 | 8,282 | 4,282 (4,080) |
| Runtime in seconds | 33457.4 | 4464.9 | 3357.1 (23.7) | 1566.1 | 220.5 | 156.8 (2.5) |

Test results for all three methods on tensor product and sparse grid collocations are shown in Table 1. As seen from Figure 3, the maximum permeability in this case is of
size $10^3$, which leads to a large condition number and the number of iterations $N_{\text{iter}}$ in each realization. With tensor product collocation, method S1 requires more than 1 million number of solves which is very expensive due to the high value of $N_{\text{real}}$ as well as $N_{\text{iter}}$. Method S2 and S3, on the other hand, only cost about $1/10$ or less the number of solves and runtime as in method S1, which illustrates some great computational efficiency with multiscale flux basis implemented. Similar results held in sparse grid collocation as well.

It can be easily concluded from Table 1 that method S3 has the best performance among all three algorithms. Also, from the values in parenthesis which indicate the cost of the pre-computation, we note that most subdomain solves are completed in the pre-computation loop as expected, while the vast majority of runtime was spent in the main loop. This is because the adding-up cost of the subdomain communications in every CG iteration is much more significant than the cost of multiscale flux basis computation.

Moreover, comparing between method S2 and S3 on both collocation grids, we can see that stochastic multiscale flux basis implementation saves more solves in tensor product (nearly 75%) than sparse grid (nearly 50%). This is due to the fact that for any KL region $j$, the global to local ratio $N_{\text{real}} : N_{\text{real}}(j)$ is higher in tensor product than in sparse grid.

### 6.2 Test case 2

In this test, the global domain $[0, 1]^2$ is divided in half with one Stokes region on the top and one Darcy region on the bottom, and is partitioned into 32 subdomains. Outside boundary condition is given as the following: In Darcy, zero pressure is specified on the bottom edge, with no-flow condition on the left and right; in Stokes, velocity is specified on the left edge,
horizontal velocity and normal stress are specified on the top, normal and tangential stresses are specified on the right.

As shown in Figure 4, an L-shape KL region $S_1$ (red part) for stochastic permeability is inscribed in the Darcy region, with its complement $S_2$ defined to be the second KL region (gray part). In $S_1$, the mean value is $e$, variance is 1.0 and correlation lengths are 0.01. In $S_2$, the mean value is $e^{-1}$, variance is 1.0 and correlation lengths are 0.1. For tensor product collocation, the grid is isotropic with $N_{\text{coll}} = 2$ and $N_{\text{term}} = 11$, where $N_{\text{term}}(1) = 3 \times 3$ and $N_{\text{term}}(2) = 2 \times 1$; for sparse grid, we have $\ell_{\text{max}} = 1$ and $N_{\text{term}} = 200$, where $N_{\text{term}}(1) = 14 \times 14$ and $N_{\text{term}}(2) = 2 \times 2$. The number of global realization is $N_{\text{real}} = 2048$ for tensor product and $N_{\text{real}} = 401$ for sparse grid.

The space discretization is set as follows: for tensor product, we have a local mesh of $25 \times 25$ in every Darcy domain and $8 \times 8$ in every Stokes domain. A mortar mesh of $10 \times 1$ is applied to every interface on $\Gamma_{DD}$ and $4 \times 1$ to $\Gamma_{SD} \cup \Gamma_{SS}$; for sparse grid, we have a local mesh of $20 \times 20$ in every domain in $S_1$ and $4 \times 4$ elsewhere. A mortar mesh of $10 \times 1$ is applied to every interior interface of $S_1$, $4 \times 1$ to every outside boundaries of $S_1$ and $2 \times 1$ elsewhere.

![Figure 4: Case 2, illustration of different regions.](image)

The results of this test are shown in Table 2 and Figures 5-6. When multiscale flux basis implemented, the subdomain that having the maximum number of solves is the one at the corner of L-shape who has the highest mortar degrees of freedom on its interfaces. Comparing method S2 with S1, we note that it again saves more than 85% in number of solves, but saves only 50% in time for tensor product and 20% for sparse grid. It is because this test has a much heavier interprocessor communication involving 32 cores. The gain in method S3 is moderate for tensor product collocation but a bit limited in sparse grid. Specifically, in tensor product, we reduce the number of solves by 70% and only by 10% in time compared with method S2; in sparse grid, the save is 2% in solves and 10% in time. Significant communication cost explains the limited gain in time for S3 since in both cases it took very little time to pre-compute all the stochastic multiscale basis.
Figure 5: Case 2, tensor product, realization of permeability (top-left), realization of solution (top-right), mean value of pressure (middle-left), mean value of vertical solution (middle-right), variance of pressure (bottom-left), variance of vertical solution (bottom-right)
Figure 6: Case 2 sparse grid, realization of permeability (top-left), realization of solution (top-right), mean value of pressure (middle-left), mean value of vertical solution (middle-right), variance of pressure (bottom-left), variance of vertical solution (bottom-right)

Another reason that we are not gaining much in method S3, especially in sparse grid, is a very low global to local ratio $N_{\text{real}} : N_{\text{real}(j)}$. In KL region $S_1$, where $N_{\text{term}}(1) = 196$,
Table 2: Number of solves and runtime in seconds with the three algorithms for Case 2.

| Tensor Product | Method S1 | Method S2 | Method S3 (Pre-comp) |
|----------------|-----------|-----------|----------------------|
| Max. number of solves | 3,033,494 | 331,776 | 86,016 (81,920) |
| Runtime in seconds | 5893.3 | 3155.9 | 2707.6 (16.5) |

| Sparse Grid | Method S1 | Method S2 | Method S3 (Pre-comp) |
|-------------|-----------|-----------|----------------------|
| Max. number of solves | 346,709 | 45,714 | 44,818 (44,016) |
| Runtime in seconds | 525.5 | 435.1 | 391.2 (2.9) |

the local realization number is $N_{\text{real}}(1) = 393$. Therefore, $N_{\text{real}}: N_{\text{real}}(1)$ equals only about 1.02 in $S_1$, making the difference very little between $N_{S3}(i)$ and $N_{S2}(i)$ for any $D_i \subset S_1$.

6.3 Test case 3

The third test is a coupled surface water and groundwater flow with realistic geometry. The outside boundary conditions for the Darcy region are the same as in test case 1; for the Stokes region, the velocity is specified on the left and right, normal and tangential stresses are specified on the top. There are $4 \times 2 = 8$ domains on the global irregular region with Stokes on the top half and Darcy on the bottom. The permeability field in one example tensor product realization is presented in Figure 7. As shown in Figure 8, it is generated from four KL regions, represented by each one of the four Darcy subdomains, respectively. In all KL regions, the mean values of the permeability are read from a file, the variance equals to 1 and correlation lengths are 0.1. For tensor product, the grid is isotropic with $N_{\text{coll}} = 2$, the stochastic dimensions are set by $N_{\text{term}}(j) = 2 \times 1$ for $j = 1, 2, 3, 4$ and $N_{\text{term}} = 8$ in total. For sparse grid, we have $\ell_{\text{max}} = 1$ and $N_{\text{term}} = 100$ with $N_{\text{term}}(j) = 5 \times 5$ for $j = 1, 2, 3, 4$. The physical grids are alternating $18 \times 15$ and $15 \times 12$ in Darcy domains, and $12 \times 15$ and $9 \times 12$ in Stokes. A mortar mesh of $4 \times 1$ are used on all interfaces.

To handle irregular geometry, we employ the multipoint flux mixed finite element method in the Darcy region [46], which can reduce to a cell-centered finite difference for the pressure. In the Stokes region, we impose the mortar conditions on curved interfaces by mapping the physical grids to reference grids with flat interfaces. For more details about this implementation, please refer to [42].

Figures [7][11] show the permeability and solution realization in sparse grid, with solution means and variance in both grids. There is little difference between the computed means in tensor product and sparse grid since both cases have the same mean permeability sourced from a file. The computed variance is of larger size in sparse grid due to the more collocation points picked for every KL region.

In this problem, a smaller size in stochastic dimension leads to less global realizations in the loop, with $N_{\text{real}} = 256$ in tensor product and $N_{\text{real}} = 201$ in sparse grid. In general, a smaller collocation loop means a fewer saves in aggregate for methods S2 and S3. The computation costs of this test case are presented in Table 3. Compared with traditional...
implementation, Method S2 results in a 50% ~ 60% decrease in solves and a 30% ~ 40% decrease in runtime.

With 4 independent KL regions, we have a larger global to local ratio such that for any \( j \), \( \frac{N_{\text{real}}}{N_{\text{real}(j)}} = 256 : 4 = 64 \) for tensor product and \( \frac{N_{\text{real}}}{N_{\text{real}(j)}} = 201 : 51 \approx 3.94 \) for sparse grid. That is why Methods S3 becomes significantly better than method S2 with a large gain in numbers of solves and a moderate gain in time, especially for tensor product.
Figure 10: *Case 3, mean of vertical velocity for tensor product (left) and sparse grid (right).*

Figure 11: *Case 3, solution variance for tensor product (left) and sparse grid (right)*

Also we note that this is a very cheap improvement providing the low cost of runtime in pre-computation.

7 Conclusions

We have presented three algorithms for solving the coupled stochastic Stokes-Darcy problem, one of which does not use a multiscale flux basis and the other two use deterministic
and stochastic multiscale flux basis, respectively. The discretization include stochastic collocation methods in the stochastic dimension and MMMFEM in the physical dimension. Numerical tests in Section 6 demonstrate a significant saving in number of solves and run-time with multiscale flux basis implemented on the reduced coarse-scale mortar interface problem. In the cases with large global to local realizations ratio, such as using tensor product collocation or having more KL regions, the stochastic multiscale flux basis gains significantly from the extra pre-computation and it is the most computationally efficient. In all algorithms, the interprocessor communication cost in each CG iteration becomes a factor when the local problems are relatively cheap. A balancing preconditioner for the Darcy region [36,38] could be used to reduce the number of iterations. A possible extension of this work is to incorporate stochasticity in the source term of Stokes equation, e.g., in modeling rainfall in surface-subsurface flow simulations. In that case, one would expect even larger computational savings from the use of the stochastic multiscale flux basis.

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