A posteriori error analysis for finite element solution of one-dimensional elliptic differential equations using equidistributing meshes

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The paper is concerned with the adaptive linear finite element solution of linear one-dimensional elliptic differential equations using equidistributing meshes. A strategy is developed for defining such meshes based on a residual-based a posteriori error estimate. The mesh and the finite element solution are determined by the coupled system of the finite element equation and the equidistribution relation (i.e., the mesh equation). An iterative algorithm is proposed for solving this system for the mesh and the finite element solution. The existence of an equidistributing mesh is proven for a given sufficiently large number of the points with help of a result on the continuous dependence of the finite element solution on the mesh, which is also established in the current work. Error bounds for the finite element solution are obtained for the equidistributing and quasi-equidistributing meshes. They show that adaptive meshes can lead to more accurate solutions than a uniform mesh and it is unnecessary to compute the equidistribution relation accurately for the equidistributing meshes. The departure from the equidistributing meshes has only a mild effect on the finite element error. Numerical examples are given to illustrate the convergence of the iterative algorithm and the theoretical findings.

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1 Introduction

We are concerned with the convergence of the linear finite element (FE) solution of elliptic differential equations using equidistributing meshes. An equidistributing mesh of $N$ elements for $\Omega \equiv (0, 1)$ is a mesh $x_0 = 0 < x_1 < \cdots < x_N = 1$ satisfying the so-called equidistribution principle \cite{12, 19}

\begin{equation}
\int_{x_{i-1}}^{x_i} \rho(x) dx = \frac{1}{N} \int_{0}^{1} \rho(x) dx, \quad i = 1, \ldots, N
\end{equation}

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where $\rho = \rho(x)$ is a user-prescribed, strictly positive function. Function $\rho(x)$, referred to as a monitor function, can be interpreted as an “error” density function, with $\int_0^1 \rho(x) \, dx$ being the total “error”. Equation (1) implies that $\rho(x)$ is evenly distributed among the mesh elements.

Equidistributing meshes are known to produce optimal error bounds and have been widely used for adaptive numerical solution of differential equations. Their theoretical studies have also attracted considerable attention from researchers; e.g., see [12, 19, 20, 22, 30, 35] for best approximations with variable nodes, [39, 40, 41] for regression problems in statistics, [2, 34, 44] for adaptive numerical solution of differential equations, and [6, 7, 8, 16, 25, 26, 27, 28, 29, 36, 37] for more recent works. A focus of these studies has been on error analysis, i.e., to understand how accurate an approximation or a numerical solution can be on an equidistributing mesh. Unfortunately, this has proven to be a difficult task due to the highly nonlinear coupling between the mesh and the solution. The analysis can be significantly simplified by taking a priori meshes defined using the exact solution or some information of the exact solution. Interestingly, almost all of the existing analyses have been done in this way. For example, Pereyra and Sewell [34] choose a mesh to equidistribute a form of the truncation error and obtain an asymptotical bound for it for the finite difference solution of two-point boundary value problems. Qiu et al. [36, 37] and Beckett and Mackenzie [6, 7, 8, 28] investigate the uniform convergence of finite difference and FE approximations for singularly perturbed problems for meshes determined using the equidistribution principle and the singular part of the exact solution. Chen and Xu [17] show that a standard FE method and a new streamline diffusion FE method produce stable and accurate approximations for a singularly perturbed convection-diffusion problem provided that the mesh properly adapts to the singularity of the solution. Huang et al. [25, 26, 27] and Chen et al. [16] study multi-dimensional interpolation problems using equidistributing meshes which depend on the function under consideration.

The noticeable exceptions are the work by Babuška and Rheinboldt [2] and Kopteva and Stynes [28] where a posteriori equidistributing meshes, or equidistributing meshes determined by the computed solution, are considered. More specifically, Babuška and Rheinboldt consider the linear FE solution of a one-dimensional elliptic problem and develop a functional from a residual-based a posteriori error estimate in lieu of asymptotic approximation and coordinate transformation. Using the optimal coordinate transformation obtained by minimizing the functional, they show that a mesh is asymptotically optimal if the residual-based error estimate is evenly distributed among the mesh elements. Kopteva and Stynes [28] study an upwind finite difference discretization of one-dimensional quasi-linear convection-diffusion problems without turning points and develop a convergence analysis for the discretization where the mesh is determined by the computed solution through the equidistribution principle and the arc-length monitor function.

In this paper we are concerned with convergence analysis for the FE solution using a posteriori equidistributing meshes. The goal is to develop a systematic approach for defining these meshes such that both their error analysis and computation can be done in an a posteriori manner. In the meantime, we would like the approach to be general enough so that it can apply to other FE methods and have no essential limitations for multi-dimensional generalizations. Furthermore, the approach should be mathematically rigorous. Particularly, it should not rely on asymptotic approximation or continuous coordinate transformations as in [2, 27]. Several other issues, such as the existence and computation of equidistributing meshes and the continuous dependence of the linear FE solution on...
mesh, are also studied in the paper.

Since Dörfler’s seminal work [21] significant progress has been made on the convergence analysis of adaptive FE methods based on a posteriori error estimates; e.g. see [9, 13, 14, 15, 31, 32, 42]. However, it should be pointed out that there are essential differences between those works and the current one. The former ones are dealt with adaptive mesh refinement using specially designed marking strategies and their convergence results are typically measured in terms of refinement levels, whereas the current work is concerned with equidistributing meshes (including their existence, generation, optimality, and error analysis) and our results are measured in terms of the number of mesh elements (cf. Theorems 3.2 and 3.3). The existing convergence analysis for mesh refinement cannot apply directly to equidistributing meshes. Neither can the results established in this work be covered by the existing ones. On the other hand, adaptive mesh refinement and equidistribution do share some common ground. For example, an equidistributing mesh can be generated through mesh refinement (e.g., see [10, 26]) (and other strategies (e.g. see [21] for a variational approach)), and the concept of mesh equidistribution is often used in mesh refinement algorithms and computer codes for maximizing the efficiency of computation (e.g., see [33]). Relations between convergence results for adaptive mesh refinement and equidistribution may thus deserve further investigations.

The paper is organized as follows. The mathematical problem and its FE discretization are described in §2. In §3 an a posteriori error estimate is developed. How this estimate is used to define the monitor function and thus the corresponding equidistributing meshes as well as the FE error on the equidistributing meshes are also discussed in the section. An iterative algorithm for computing the meshes and FE solution is proposed and numerical results are presented in §4. The continuous dependence of the FE solution on mesh and the existence of equidistributing meshes are studied in §5 and §6, respectively. Finally, §7 contains the conclusions.

2 Finite element discretization

We consider the boundary value problem of a linear elliptic differential equation

\[-(au')' + bu' + cu = f, \quad \text{in } \Omega \equiv (0,1)\]

\[u(0) = u(1) = 0,\]

where \(a(x), b(x), c(x),\) and \(f(x)\) are given functions satisfying

\[a, b \in W^{1,\infty}(\Omega), \quad c \in L^\infty(\Omega), \quad f \in L^2(\Omega),\]

\[a(x) \geq a_0 > 0, \quad c(x) - \frac{1}{2} b'(x) \geq 0, \quad \text{a.e. in } \Omega\]

for some constant \(a_0\.\) Here, \(W^{1,\infty}(\Omega)\) denotes the Sobolev space of functions whose derivatives are in \(L^\infty(\Omega)\). The variational form of problem (2) and (3) is to find \(u \in V \equiv H^1_0(\Omega)\) such that

\[B(u, v) = (f, v), \quad \forall v \in V\]

where

\[B(u, v) = \int_{\Omega} (au'v' + bu'v + cuv)dx, \quad (f, v) = \int_{\Omega} fx dx.\]
For a given mesh

\[ \pi_h : \ x_0 = 0 < x_1 < \cdots < x_N = 1 \]

with \( h = \max_i(x_i - x_{i-1}) \), the linear FE approximation to (4) is to find \( u_h \in V_h \) such that

\[ B(u_h, v_h) = (f, v_h), \quad \forall v_h \in V_h \]  

(6)

where the linear FE space is chosen as \( V_h = \text{span}\{\phi_1, ..., \phi_{N-1}\} \) with \( \phi_i \)'s being linear basis functions associated with mesh points \( x_i \)'s.

We are concerned with adaptive FE solution of (2) and (3) using equidistributing meshes. For this purpose, we choose the mesh according to the equidistribution principle (1), i.e.,

\[ \rho_i h_i = \sigma h, \quad i = 1, ..., N \]  

(7)

where \( h_i = x_i - x_{i-1} \), \( \sigma h = \sum_i \rho_i h_i \), and \( \rho = \rho(x) \) is the monitor function used for controlling the mesh concentration. In the current work the monitor function, which will be defined in the next section based on an a posteriori error estimate, is always considered as a piecewise constant function, i.e.,

\[ \rho(x) = \rho_i, \quad \text{for} \ x \in (x_{i-1}, x_i), \ i = 1, ..., N. \]

It is easy to see that

\[ \int_{x_{i-1}}^{x_i} \rho(x)dx = \rho_i h_i. \]

In the remainder of the paper we will estimate the error of the finite element solution defined in (6) for these equidistributing meshes and study their existence for a properly chosen monitor function (depending on the computed solution). We will also investigate similar issues for quasi-equidistributing meshes which are defined as meshes satisfying

\[ \frac{N \rho_i h_i}{\sigma h} \leq \kappa, \quad i = 1, ..., N \]  

(8)

for some positive, modest constant \( \kappa \) independent of \( i \) and \( N \). Quasi-equidistributing meshes are more realistic to compute in practical computation than exact equidistributing meshes.

Hereafter, we denote the \( L^2 \)-norm on \( \Omega \) by \( \| \cdot \| \) and other \( L^p \)-norm by \( \| \cdot \|_{L^p(\Omega)} \), with the latter being extended to the situation \( 0 < p < 1 \). For any \( v \in L^\infty(\Omega) \), we denote \( \tilde{v} = \| v \|_{L^\infty(\Omega)} \). We use \( C \) as a generic constant which may have different values at different appearances. In most part of this paper, constants are considered as numbers that may depend on the domain and coefficients \( a(x), b(x), \) and \( c(x) \) of differential equation (2) but not on the solution \( u \), the right-hand side \( f \), and the mesh employed in the FE solution. The exceptions are Theorems 5.1 and 6.1 and §5 and §6 where constants may further depend on \( u \) and \( f \).

### 3 Error analysis for FE solution using equidistributing meshes

In this section we present an error analysis for equidistributing and quasi-equidistributing meshes satisfying (7) and (8), respectively. The approach we use consists of three major steps, deriving a residual-based a posteriori error estimate, defining the monitor function (see (15) below) based on
the estimate, and developing the error analysis for the corresponding equidistributing mesh. This approach shares some similarity with that used in [25, 26, 27] for analyzing interpolation error in multi-dimensions. The main difference lies in that the current analysis is based on an a posteriori error estimate and is mathematically rigorous, whereas the analysis in [25, 26, 27] is based on interpolation error bounds (depending on the exact solution) and valid only in an asymptotic sense.

3.1 Preliminary results

For completeness and for easy reference we list here some preliminary results without giving their proofs. These results can be found in most FE textbooks, e.g., [11, 18].

Lemma 3.1. The bilinear form $B(\cdot, \cdot)$ defined in (5) has the properties

$$a_0 \|v\|^2 \leq B(v, v) \leq C\|v\|^2, \quad \forall v \in V$$

$$B(u, v) \leq C\|u\|\|v\|, \quad \forall u, v \in V.$$ More completely, the solution of the problem (4) satisfies

$$\|u\| \leq C\|f\|.$$ 

Lemma 3.2. Given a mesh $\pi_h$, denote by $\Pi_h$ the operator for piecewise linear interpolation, i.e.,

$$\Pi_h v(x) = \sum_{i=0}^{N} v(x_i)\phi_i(x), \quad \forall v \in H^1(\Omega)$$

where $\phi_i$'s are the linear basis functions associated with mesh points $x_i$'s. Then, for any $K_i = (x_{i-1}, x_i)$,

$$\|v - \Pi_h v\|_{K_i} \leq Ch_i\|v - \Pi_h v\|_{K_i}, \quad \forall v \in H^1(K_i)$$

$$\|(v - \Pi_h v)'\|_{K_i} \leq Ch_i\|v''\|_{K_i}, \quad \forall v \in H^2(K_i)$$

$$\|(v - \Pi_h v)'\|_{K_i} \leq C\|v'\|_{K_i}, \quad \forall v \in H^1(K_i)$$

where $\|\cdot\|_{K_i}$ denotes the $L^2$-norm on $K_i$.

The error for the FE solution $u_h$, $e_h = u - u_h$, satisfies the orthogonality property and the error equation, viz.,

$$B(e_h, v_h) = 0, \quad \forall v_h \in V_h$$

$$B(e_h, v) = (f, v) - B(u_h, v), \quad \forall v \in V.$$ 

Lemma 3.3. The FE solution $u_h$ defined in (6) satisfies

$$\|u_h\| \leq C_1\|f\|$$

for some constant $C_1$. Moreover, if the solution of the continuous problem (4) satisfies $u \in H^2(\Omega)$ and the mesh has the property

$$h \equiv \max_i h_i \leq \frac{C_2}{N}$$
for some positive constant $C_2$, the error is bounded by

$$
\|(u - u_h)'\| \leq \frac{C\|r\|}{N},
$$

(11)

where $C$ is a constant and the continuous “residual” function $r$ is defined as

$$
r = f + a'u' - bu' - cu.
$$

It is remarked that the bound (11) can be obtained by combining the conventional bound

$$
\|(u - u_h)'\| \leq Ch\|u''\|
$$

and the equivalence $\|u''\| \sim \|r\|$. The latter is a consequence of the assumption $u \in H^2(\Omega)$ and the fact that $u$ satisfies (4). The main reason the residual function $r$ is used in the bound is that, as we will see later, the FE error with equidistributing meshes can also be bounded by some norm of $r$.

Obviously, a uniform mesh satisfies the condition (10). The FE error for a uniform mesh can thus be bounded as in (11).

### 3.2 An a posteriori error estimate

We now derive a residual-based a posteriori error estimate for the FE solution. The general procedure for this type of error estimation can be seen, e.g., in [1, 3, 4, 5, 43].

**Lemma 3.4.** The error $e_h = u - u_h$ is bounded by

$$
\|(u - u_h)'\|^2 \leq C \sum_i h_i^2 \|r_h\|^2_{K_i},
$$

(12)

where the residual $r_h$ is defined as

$$
r_h = f + a'u_h' - bu_h' - cu.
$$

**Proof.** Using orthogonality property (6), error equation (9), integration by parts, Lemma 3.2, and Schwarz’ inequality, we have, for any $v \in V$,

\[
B(e_h, v) = B(e_h, v - \Pi_h v) = (f, v - \Pi_h v) - B(u_h, v - \Pi_h v) = \sum_i \int_{K_i} r_h (v - \Pi_h v) dx \\
\leq \sum_i \|r_h\|_{K_i} \|v - \Pi_h v\|_{K_i} \\
\leq C \sum_i h_i \|r_h\|_{K_i} \|v'\|_{K_i} \\
\leq C \left( \sum_i h_i^2 \|r_h\|^2_{K_i} \right)^{1/2} \|v'\|.
\]

Then (12) follows by taking $v = e_h$ in the above inequality and using Lemma 3.1. \qed
3.3 Determination of optimal monitor function

Up to this point the mesh has been assumed to be arbitrary. From now on we focus on equidistributing meshes determined according to the a posteriori error estimate (12).

As we can see from (1), the key for the determination of equidistributing meshes is to define an appropriate monitor function

\[ \rho = \rho(x) > 0. \]

To this end, we regularize the bound in (12) with a positive constant \( \alpha_h > 0 \) (to be determined), i.e.,

\[
\sum_i h_i^2 \|r_h\|^2_{K_i} = \sum_i h_i^3 \langle r_h \rangle_i^2 \leq \sum_i h_i^3 \left( \alpha_h + \langle r_h \rangle_i^2 \right) = \alpha_h \sum_i h_i^3 \left( 1 + \frac{1}{\alpha_h} \langle r_h \rangle_i^2 \right), \tag{13}
\]

where we have defined the \( L^2 \) average of \( r_h \) over \( K_i \) as

\[
\langle r_h \rangle_i = \left( \frac{1}{h_i} \int_{K_i} |r_h|^2 dx \right)^{\frac{1}{2}} = \frac{1}{\sqrt{h_i}} \|r_h\|_{K_i}.
\]

From Hölder’s inequality, we have

\[
\left( \frac{1}{N} \sum_i \left[ h_i \left( 1 + \frac{1}{\alpha_h} \langle r_h \rangle_i^2 \right)^{\frac{1}{3}} \right] \right)^3 \geq \frac{1}{N} \sum_i h_i \left( 1 + \frac{1}{\alpha_h} \langle r_h \rangle_i^2 \right)^{\frac{1}{3}},
\]

and thus,

\[
\sum_i h_i^3 \left( 1 + \frac{1}{\alpha_h} \langle r_h \rangle_i^2 \right)^{\frac{1}{3}} = \sum_i \left[ h_i \left( 1 + \frac{1}{\alpha_h} \langle r_h \rangle_i^2 \right)^{\frac{1}{3}} \right]^3 \geq \frac{1}{N^2} \left[ \sum_i h_i \left( 1 + \frac{1}{\alpha_h} \langle r_h \rangle_i^2 \right)^{\frac{1}{3}} \right]^3, \tag{14}
\]

with equality if and only if

\[
h_i \left( 1 + \frac{1}{\alpha_h} \langle r_h \rangle_i^2 \right)^{\frac{1}{3}} = \frac{1}{N} \sum_j h_j \left( 1 + \frac{1}{\alpha_h} \langle r_h \rangle_j^2 \right)^{\frac{1}{3}}, \quad i = 1, \ldots, N.
\]

Comparing this with the equidistribution principle (7) suggests that we choose the monitor function as

\[
\rho_i = \left( 1 + \frac{1}{\alpha_h} \langle r_h \rangle_i^2 \right)^{\frac{1}{3}}, \quad i = 1, \ldots, N. \tag{15}
\]

For any equidistributing mesh associated with this monitor function, from Lemma (3.4) and inequalities (13) and (14) we have

\[
\|u - u_h\|^2 \leq C \alpha_h \sigma_h^3, \tag{16}
\]

where

\[
\sigma_h \equiv \sum_i h_i \rho_i = \sum_i h_i \left( 1 + \frac{1}{\alpha_h} \langle r_h \rangle_i^2 \right)^{\frac{1}{3}}. \tag{17}
\]

Notice that the bound in (16) is not the lowest one since both \( \alpha_h \) (to be determined) and \( \sigma_h \) vary with the mesh. Nevertheless, we may expect

\[
\alpha_h \left[ \sum_i h_i \left( 1 + \frac{1}{\alpha_h} \langle r_h \rangle_i^2 \right)^{\frac{1}{3}} \right]^3 \to \alpha \left[ \int_{\Omega} \left( 1 + \frac{1}{\alpha} r^2 \right)^{\frac{1}{3}} dx \right]^3 \text{ as } N \to \infty,
\]
where $\alpha = \lim_{N \to \infty} \alpha_h$. When this is the case, (16) is an asymptotically lowest bound for the error, and in this sense the choice (15) is asymptotically optimal.

To complete the definition, we need to determine the parameter $\alpha_h$. We follow [23] to choose it such that

$$\sigma_h \equiv \sum_i h_i \rho_i \leq 2.$$  (18)

In this way, roughly fifty percents of the mesh points are placed in the region where $\rho \gg 1$ [23]. From Jensen’s inequality and (15),

$$\sigma_h = \sum_i h_i \left( 1 + \frac{1}{\alpha_h} \langle r_h i \rangle^2 \right)^{\frac{3}{2}} \leq \sum_i h_i \left( 1 + \alpha_h^{-\frac{1}{2}} \langle r_h i \rangle^2 \right) = 1 + \alpha_h^{-\frac{1}{2}} \sum_i h_i \langle r_h i \rangle^2.$$  Then (18) holds when $\alpha_h$ is chosen as

$$\alpha_h = \left( \sum_i h_i \langle r_h i \rangle^2 \right)^{\frac{3}{2}}.$$  (19)

Combining this with (16) and (18), we obtain

$$\| (u - u_h) \| \leq C \sqrt{\alpha_h} \sqrt{N}.$$  (20)

The boundedness of $\alpha_h$ as $N \to \infty$ is investigated in the next subsection.

It is emphasized that the choice (15) for the monitor function is based on an a posteriori error estimate for the linear FE solution and depends on the computed solution. Unfortunately, this also means that the mesh and the computed solution are coupled with each other. The system for $u_h$ and $\pi_h$ consists of algebraic equations (6) and (7) and the boundary conditions $x_0 = 0$ and $x_N = 1$. An iterative algorithm for solving the system is given in §4.

3.4 Convergence for equidistributing and quasi-equidistributing meshes

We notice that the monitor function defined in (15) satisfies $\rho_i \geq 1$, $i = 1, \ldots, N$. As a consequence, (18) implies that the equidistributing mesh (7) has the property (10) with $C_1 = 2$. Combining this with Lemma 3.3 we have the following theorem.

**Theorem 3.1.** Assume that $\rho$ and $\alpha_h$ are defined as in (15) and (19), respectively. If $u \in H^2(\Omega)$, then for any mesh equidistributing $\rho$ the error in the FE solution to problem (4) is bounded by (17), i.e.,

$$\| (u - u_h) \| \leq C \sqrt{\rho} \sqrt{N}.$$  (21)

As mentioned before, the error bound for a uniform mesh also has the same form given by (21). Although a bound like (21) for an equidistributing mesh is useful in some situations such as in proving Lemma 3.8, it does not show advantages of using an adaptive mesh over a uniform one. In the following we shall derive a sharper bound based on the a posteriori error bound (20). The key is to estimate $\alpha_h$, and that is done in a series of lemmas.
Lemma 3.5. (Power Inequalities)

(i) Given a real number $0 < q \leq 1$, for any $x, y \in \mathbb{R}$,

$$|x + y|^q \leq |x|^q + |y|^q,$$

$$||x|^q - |y|^q| \leq |x - y|^q.$$

(ii) Given a real number $0 < q \leq 1$, for any two functions $v$ and $w$ in a function space equipped with a norm $\| \cdot \|$,

$$\|v + w\|^q \leq \|v\|^q + \|w\|^q,$$

$$\|\|v\| - \|w\|\| \leq \|v - w\|^q.$$

Proof. These inequalities are obtained using the triangle and Jensen’s inequalities. \qed

Lemma 3.6. For any real number $0 < q \leq 1$ and any mesh $\pi_h$ for $\Omega$,

$$\|v\|^{2q}_{L^2(\Omega)} \leq \sum_i h_i^{1-q}\|v\|^{2q}_{K_i} = \sum_i h_i \langle v \rangle_i^{2q} \leq |\Omega|^{1-q}\|v\|^{2q}, \quad \forall v \in L^2(\Omega).$$

Proof. The estimates follow from

$$\sum_i h_i \langle v \rangle_i^{2q} = \sum_i h_i \left( \frac{1}{h_i} \int_{K_i} |v|^2 dx \right)^q \geq \sum_i h_i \left( \frac{1}{h_i} \int_{K_i} |v|^{2q} dx \right) = \|v\|^{2q}_{L^{2q}(\Omega)},$$

$$\left( \frac{1}{|\Omega|} \sum_i h_i \left( \frac{1}{h_i} \|v\|^2_{K_i} \right)^q \right)^{\frac{1}{q}} \leq \frac{1}{|\Omega|} \sum_i h_i \left( \frac{1}{h_i} \|v\|^2_{K_i} \right) = \frac{1}{|\Omega|}\|v\|^2.$$ \qed

Lemma 3.7. For any real number $0 < q \leq \frac{1}{2}$ and any mesh $\pi_h$ for $\Omega$,

$$\|v\|^{2q}_{L^{2q}(\Omega)} \leq \sum_i h_i^{1-q}\|v\|^{2q}_{K_i} \leq \|v\|^{2q}_{L^{2q}(\Omega)} + 2h^{2q}|\Omega|^{1-2q}\|v'\|^{2q}_{L^1(\Omega)}, \quad \forall v \in L^2(\Omega), \; v' \in L^1(\Omega)$$

where $h = \max_i h_i$.

Proof. The left inequality is a consequence of Lemma 3.6.

To prove the right inequality, define the element-wise average of $v$ as

$$v_{K_i} = \frac{1}{h_i} \int_{K_i} v dx.$$

Then, from Lemma 3.5

$$\sum_i h_i^{1-q}\|v\|^{2q}_{K_i} - \|v\|^{2q}_{L^{2q}(\Omega)}$$

$$= \sum_i h_i^{1-q}\|v - v_{K_i}\|^{2q}_{K_i} - \sum_i \int_{K_i} |v|^{2q} dx$$
\[ \sum_{i} h_{i}^{-q} \| v - v_{K_{i}} \|_{2q}^{2} + \sum_{i} h_{i}^{-q} \| v_{K_{i}} \|_{K_{i}}^{2q} - \sum_{i} \int_{K_{i}} |v|^{2q} \, dx \]
\[ = \sum_{i} h_{i}^{-q} \| v - v_{K_{i}} \|_{K_{i}}^{2q} + \sum_{i} \int_{K_{i}} (|v_{K_{i}}|^{2q} - |v|^{2q}) \, dx \]
\[ \leq \sum_{i} h_{i}^{-q} \| v - v_{K_{i}} \|_{K_{i}}^{2q} + \sum_{i} \int_{K_{i}} |v_{K_{i}} - v|^{2q} \, dx \]
\[ \leq \sum_{i} h_{i}^{-q} \| v - v_{K_{i}} \|_{K_{i}}^{2q} + \sum_{i} h_{i} \left( \frac{1}{h_{i}} \int_{K_{i}} |v_{K_{i}} - v|^{2} \, dx \right)^{q} \]
\[ \leq 2 \sum_{i} h_{i}^{-q} \| v - v_{K_{i}} \|_{K_{i}}^{2q}. \]  

(23)

From the assumption \( v' \in L^{1}(\Omega) \), we have
\[
\| v - v_{K_{i}} \|_{K_{i}}^{2} = \int_{K_{i}} \left| v(x) - \frac{1}{h_{i}} \int_{K_{i}} v(t) \, dt \right|^{2} \, dx
\]
\[ = \frac{1}{h_{i}^{2}} \int_{K_{i}} \int_{K_{i}} (v(x) - v(t))^{2} \, dt \, dx
\]
\[ = \frac{1}{h_{i}^{2}} \int_{K_{i}} \int_{K_{i}} \int_{t}^{x} v'(s) \, ds \, dt \, dx
\]
\[ \leq h_{i} \| v' \|_{L^{1}(K_{i})}^{2}. \]  

(24)

Combining (24) with (23) and using Hölder’s inequality we get
\[
\sum_{i} h_{i}^{-q} \| v \|_{K_{i}}^{2q} - \| v \|_{K_{i}}^{2q} \leq 2 \sum_{i} h_{i} \| v' \|_{L^{1}(K_{i})}^{2q}
\]
\[ \leq 2h^{2q} \sum_{i} h_{i}^{-2q} \| v' \|_{L^{1}(K_{i})}^{2q}
\]
\[ \leq 2h^{2q} |\Omega|^{1-2q} \| v' \|_{L^{1}(\Omega)}^{2q}, \]

which gives the right inequality of (22). \( \square \)

**Theorem 3.2.** (Convergence for equidistributing meshes) Define \( \rho \) and \( \alpha_{h} \) as in (13) and (19), respectively. For any equidistributing mesh satisfying (7), the error for the linear FE solution (6) is bounded by
\[
\| (u - u_{h})' \| \leq C \sqrt{\alpha_{h}} N^{-\frac{3}{2}}, \]  

(25)

where \( \alpha_{h} \) has the property
\[
\lim_{N \to \infty} \sqrt{\alpha_{h}} = \| r \|_{L^{2}(\Omega)} \]  

(26)

If further \( r' \in L^{1}(\Omega) \), then there exists a positive constant \( c \) such that for \( N > c \),
\[
\left( 1 + \left( \frac{c}{N} \right)^{\frac{2}{3}} \right)^{-\frac{3}{2}} \| r \|_{L^{2}(\Omega)} \leq \sqrt{\alpha_{h}} \leq \left( 1 - \left( \frac{c}{N} \right)^{\frac{2}{3}} \right)^{-\frac{3}{2}} \left[ \| r \|_{L^{2}(\Omega)}^{\frac{2}{3}} + \left( \frac{\| r \|_{L^{2}(\Omega)}^{\frac{2}{3}}}{N} \right)^{\frac{2}{3}} \right]^{\frac{3}{2}}. \]  

(27)
Proof. The bound (25) is given by (20). We first prove (27) and then (26) in the following. For (27), from (25) and Lemmas 3.5 and 3.6 we have
\[ \sum_i h_i^\frac{2}{3} \| r - r_h \|_{K_i}^\frac{2}{3} \leq \| r - r_h \|^{\frac{2}{3}} \leq C ((u - u_h)' + \| u - u_h \|)^{\frac{2}{3}} \]
\[ \leq C (u - u_h)' \|^{\frac{2}{3}} \]
\[ \leq CN^{-\frac{2}{3}} \alpha_h^{\frac{1}{3}} \ldots \tag{28} \]
Then from (19), (29), and Lemmas 3.5 and 3.6 it follows that
\[ \alpha_h^{\frac{1}{3}} = \sum_i h_i^\frac{2}{3} \| r_h \|_{K_i}^\frac{2}{3} \geq \sum_i h_i^\frac{2}{3} \| r \|_{K_i}^\frac{2}{3} - \sum_i h_i^\frac{2}{3} \| r - r_h \|_{K_i}^\frac{2}{3} \geq \| r \|_{L^\frac{2}{3} (\Omega)}^\frac{2}{3} - CN^{-\frac{2}{3}} \alpha_h^{\frac{1}{3}}, \ldots \tag{30} \]
which leads to the left inequality of (27) (with \( c = C^\frac{2}{3} \)). From (10), (29), Lemmas 3.5 and 3.7 we have
\[ \alpha_h^{\frac{1}{3}} = \sum_i h_i^\frac{2}{3} \| r_h \|_{K_i}^\frac{2}{3} \leq \sum_i h_i^\frac{2}{3} \| r \|_{K_i}^\frac{2}{3} + \sum_i h_i^\frac{2}{3} \| r - r_h \|_{K_i}^\frac{2}{3} \leq \sum_i h_i^\frac{2}{3} \| r \|_{K_i}^\frac{2}{3} + CN^{-\frac{2}{3}} \alpha_h^{\frac{1}{3}} \ldots \tag{31} \]
which yields the right inequality of (27).
We now prove (26). In this situation, \( r \in L^2 (\Omega) \). From (30) and (31) we can see that, for \( N > C^\frac{3}{2} \),
\[ \left( 1 + CN^{-\frac{2}{3}} \right)^{-1} \| r \|_{L^\frac{2}{3} (\Omega)}^{\frac{2}{3}} \leq \alpha_h^{\frac{1}{3}} \leq \left( 1 - CN^{-\frac{2}{3}} \right)^{-1} \sum_i h_i^\frac{2}{3} \| r \|_{K_i}^\frac{2}{3} \ldots \tag{32} \]
Since functions having derivatives in \( L^1 (\Omega) \) are dense in \( L^2 (\Omega) \), given any \( \epsilon > 0 \) there exists a function \( \tilde{r} \) such that
\[ \tilde{r}' \in L^1 (\Omega) \quad \text{and} \quad \| r - \tilde{r} \| \leq \epsilon. \]
Then, from Lemmas 3.5, 3.6, and 3.7 we have
\[ \sum_i h_i^\frac{2}{3} \| r \|_{K_i}^\frac{2}{3} - \int_{\Omega} |r|^{\frac{2}{3}} dx \]
\[ = \sum_i h_i^\frac{2}{3} \| r - \tilde{r} \|_{K_i}^\frac{2}{3} - \int_{\Omega} |\tilde{r}|^{\frac{2}{3}} dx + \left( \int_{\Omega} |\tilde{r}|^{\frac{2}{3}} dx - \int_{\Omega} |r|^{\frac{2}{3}} dx \right) \]
\[ \leq \sum_i h_i^\frac{2}{3} \left( \| r - \tilde{r} \|_{K_i}^\frac{2}{3} + \| \tilde{r} \|_{K_i}^\frac{2}{3} \right) - \int_{\Omega} |\tilde{r}|^{\frac{2}{3}} dx + \int_{\Omega} |\tilde{r} - r|^{\frac{2}{3}} dx \]
\[ = \left( \sum_i h_i^\frac{2}{3} \| \tilde{r} \|_{K_i}^\frac{2}{3} - \int_{\Omega} |\tilde{r}|^{\frac{2}{3}} dx \right) + \sum_i h_i \left( \frac{1}{h_i} \| r - \tilde{r} \|_{K_i}^\frac{2}{3} \right) + \int_{\Omega} |\tilde{r} - r|^{\frac{2}{3}} dx \]
\[ CN^{-\frac{2}{3}} \| r' \|_{L^2(\Omega)}^2 + 2 \| r - \tilde{r} \|_{L^2(\Omega)}^2 \leq CN^{-\frac{2}{3}} \| \tilde{r}' \|_{L^2(\Omega)}^2 \]

Inserting this into (32) gives

\[
(1 + CN^{-\frac{2}{3}})^{-1} \| r \|_{L^3(\Omega)}^2 \leq \alpha_h^\frac{1}{3} \leq \left( 1 - CN^{-\frac{2}{3}} \right)^{-1} \left[ \| r \|_{L^3(\Omega)}^2 + CN^{-\frac{2}{3}} \| r' \|_{L^1(\Omega)}^2 + 2\epsilon^\frac{2}{3} \right].
\]

Taking limit as \( N \to \infty \) in the above inequality yields

\[
\| r \|_{L^2(\Omega)}^2 \leq \lim_{N \to \infty} \alpha_h^\frac{1}{3} \leq \| r \|_{L^2(\Omega)}^2 + 2\epsilon^\frac{2}{3}.
\]

Finally, taking limit as \( \epsilon \to 0 \) in the above inequality gives (26).

The above theorem shows that the FE error has the asymptotic bound

\[
\lim_{N \to \infty} N \|(u - u_h)'\| \leq C \| r \|_{L^2(\Omega)}.
\]

This is compared with the error bound for a uniform mesh (cf. Lemma 3.3)

\[
\|(u - u_h)'\| \leq C_1 \| r \| / N.
\]

Since

\[
\| r \|_{L^2(\Omega)} \leq \| r \|
\]

and particularly, the left-hand side is much smaller than the right-hand side when \( r (\sim au'') \) is non-smooth, the theorem implies that the error bound for an equidistributing mesh can be much smaller than that for a uniform mesh. This explains why an adaptive mesh often produces a more accurate solution than a uniform one when the solution is non-smooth.

We also note that we can avoid using (25) and thus the assumption that the mesh is an equidistributing mesh when proving (26) and (27). The idea is to use the a priori error bound (11) to estimate \( \|(u - u_h)'\| \) in (28). The result is stated in the following lemma without proof. Such bounds for \( \alpha_h \) are needed when we prove the existence of equidistributing meshes (cf. Lemma 6.1).

**Lemma 3.8.** Assume that the solution to problem (4) satisfies \( u \in H^2(\Omega) \) and the mesh (not necessarily an equidistributing mesh) satisfies the property (19). If \( r \) is only \( L^2 \) integrable, then \( \alpha_h \) satisfies the property (26). If further \( r' \in L^1(\Omega) \), then \( \alpha_h \) is bounded by

\[
\| r \|_{L^2(\Omega)}^2 - \left( \frac{C \| r \|}{N} \right)^{\frac{2}{3}} \leq \alpha_h^{\frac{2}{3}} \leq \| r \|_{L^2(\Omega)}^2 + \left( \frac{C \| r' \|_{L^1(\Omega)}}{N} \right)^{\frac{2}{3}} + \left( \frac{C \| r \|}{N} \right)^{\frac{2}{3}}.
\]

To conclude this section, we show that similar results also hold for quasi-equidistributing meshes.

**Theorem 3.3.** (Convergence for quasi-equidistributing meshes) Define \( \rho \) and \( \alpha_h \) as in (15) and (19), respectively. Then for any quasi-equidistributing mesh satisfying (8), the error for the linear FE solution (6) is bounded by

\[
\|(u - u_h)'\| \leq \frac{C \sqrt{\alpha_h \kappa^3}}{N},
\]
Figure 1: Illustration of the iterative solution procedure for the FE solution using equidistributing meshes.

where $\alpha_h$ satisfies (26). If further $r' \in L^1(\Omega)$, then there exists a positive constant $c$ such that for $N > \kappa c$,

$$
\left(1 + \left(\frac{KC}{N}\right)^{\frac{3}{2}}\right)^{-\frac{3}{2}} \|r\|_{L^2(\Omega)} \leq \sqrt{\alpha_h} \leq \left(1 - \left(\frac{KC}{N}\right)^{\frac{3}{2}}\right)^{-\frac{3}{2}} \left[\|r\|_{L^2(\Omega)}^2 + \left(\frac{KC\|r\|_{L^1(\Omega)}^2}{N}\right)^{\frac{2}{3}}\right]^{\frac{3}{2}}.
$$

Proof. For $\rho$ and $\alpha_h$ defined as in (15) and (19), respectively, a quasi-equidistributing mesh (8) has the property

$$h_i \leq \frac{2\kappa}{N}, \quad i = 1, \ldots, N. \quad (36)$$

Moreover, from (8), (13), (18), and Lemma 3.4 we have

$$\|u - u_h\|^2 \leq C\alpha_h \sum_i h_i^3 \left(1 + \frac{1}{\alpha_h} \langle r_h \rangle_i^2\right) = \alpha_h \sum_i (h_i \rho_i)^3 \leq \alpha_h \sum_i \left(\frac{KC\rho_i}{N}\right)^3 \leq \frac{8\alpha_h\kappa^3 N^2}{N^2}.
$$

The remaining of the theorem can be proven similarly as for Theorem 3.2.

This theorem shows that we do not have to compute equidistributing meshes exactly. Indeed, the FE error is only affected by a factor of $\kappa^{1.5}$, where $\kappa$ is a constant in (8) which measures how closely the equidistribution principle is satisfied by the mesh. As long as $\kappa$ is not very large (or the mesh is not very far from being equidistributing), the error bound will not be affected significantly.

4 An iterative algorithm for computing equidistributing mesh and FE solution; numerical examples

In this section we consider the issue of computing the equidistributing mesh and FE solution. We recall that the FE equation (6), the equidistribution relation (7) (with the monitor function and regularity parameter defined in (15) and (19)), and the boundary conditions $x_0 = 0$ and $x_N = 1$ form a nonlinear algebraic system for the physical solution $u_h$ and the mesh $\pi_h$. This system can be solved using any nonlinear equation solvers such as Newton’s method. We describe an iterative algorithm below. Starting from an initial mesh $\pi_h^{(0)}$, it produces a sequence of meshes and solutions, $\{\pi_h^{(k)}, u_h^{(k)}\}$; see Fig. 1

Algorithm for computing equidistributing mesh and FE solution. Given an integer $N > 0$ and an initial mesh $\pi_h^{(0)}$, for $k = 0, 1, \ldots$ do
(i) Solve the boundary value problem using mesh $\pi_h^{(k)}$. This step is to find $u_h^{(k)} \in V_h^{(k)}$ such that

$$B(u_h^{(k)}, v_h) = (f, v_h), \quad \forall v_h \in V_h^{(k)}.$$ 

(ii) Generate the equidistributing mesh. This step is to compute the new equidistributing mesh using the equidistribution relation (7), i.e.,

$$\int_{x_i^{(k+1)}}^{x_i^{(k+1)}} \rho^{(k)}(x)dx = \frac{i}{N} \sigma_h^{(k)}, \quad i = 1, ..., N - 1$$

where

$$\rho^{(k)}(x) = \rho_i^{(k)} \equiv \left(1 + \frac{1}{\alpha_h^{(k)}} \left< \frac{r^{(k)}}{r_h^{(k)}} \right> \right)^{\frac{1}{3}}, \quad \forall x \in K_i^{(k)}, \quad i = 1, ..., N$$

$$\alpha_h^{(k)} = \left[ \sum_i h_i^{(k)} \left< \frac{r_i^{(k)}}{r_h^{(k)}} \right> \right]^{\frac{2}{3}} K_i^{(k)}$$

$$\sigma_h^{(k)} = \sum_{i=1}^{N} h_i^{(k)} \rho_i^{(k)}.$$

Note that the left-hand side of (37) is a monotone and piecewise linear function of $x_i^{(k+1)}$ and an explicit formula for $x_i^{(k+1)}$ can be found as

$$x_i^{(k+1)} = x_i^{(k)} + \left( \frac{i}{N} \sigma_h^{(k)} - \sum_{l=1}^{j-1} h_l^{(k)} \rho_l^{(k)} \right) \rho_j^{(k)},$$

where $j$ is the index satisfying

$$\sum_{l=1}^{j-1} h_l^{(k)} \rho_l^{(k)} < \frac{i}{N} \sigma_h^{(k)} \leq \sum_{l=1}^{j} h_l^{(k)} \rho_l^{(k)}.$$

Moreover, Steps (i) and (ii) define a map

$$G_N : \mathbb{R}^{N-1} \to \mathbb{R}^{N-1} : \quad X^{(k+1)} = G_N X^{(k)},$$

where $X^{(k)}$ and $X^{(k+1)}$ are the $(N - 1)$-component vectors corresponding to the meshes $\pi_h^{(k)}$ and $\pi_h^{(k+1)}$, respectively. In this definition, the FE solution $u_h^{(k)}$ is considered as an intermediate variable. It is not difficult to see that a fixed point of this map satisfies (\Pi) and is an equidistributing mesh. Furthermore, the computation can be stopped when

$$\|\pi_h^{(k+1)} - \pi_h^{(k)}\|_\infty \equiv \max_i |x_i^{(k+1)} - x_i^{(k)}| \leq \epsilon$$

or

$$Q_{eq,i}^{(k)} \equiv \frac{N \rho_i^{(k)} h_i^{(k)}}{\sigma_h^{(k)}} \leq \kappa, \quad i = 1, ..., N$$

(39)
where $\epsilon > 0$ is a prescribed tolerance, $\kappa$ is a number chosen to be close to and greater than one, and $Q^{(k)}_{eq,i}$ is the so-called quality measure of equidistribution [25]. The second stopping criterion needs some explanation. It is not difficult to see that $Q^{(k)}_{eq,i}$ has the properties

$$\frac{1}{N} \sum_{i} Q^{(k)}_{eq,i} = 1, \quad \max_{i} Q^{(k)}_{eq,i} \geq 1. \quad (41)$$

In addition, $\max_{i} Q^{(k)}_{eq,i} = 1$ if and only if the mesh is an equidistributing mesh satisfying (7). Thus, if the mesh sequence $\pi_{h}(k)$ converges to an equidistributing mesh we will have $\max_{i} Q^{(k)}_{eq,i} \to 1$ as $k \to \infty$; and vice versa. This implies that (40) is an effective stopping criterion.

It is interesting to point out that $\max_{i} Q^{(k)}_{eq,i}$ actually measures how closely the equidistribution relation (7) is satisfied by the mesh; see [25] for detailed discussion. Moreover, by the definition (8) one can see that any mesh satisfying (40) is a quasi-equidistributing mesh. Finally, from (40) and (41) we have

$$-N(\kappa - 1) + \kappa \leq Q^{(k)}_{eq,i} \leq \kappa, \quad i = 1, \ldots, N$$

where $-N(\kappa - 1) + \kappa > 0$ when $\kappa$ is sufficiently close to one.

We now present numerical results to demonstrate the convergence of the algorithm.

**Example 4.1.** This example is a reaction-diffusion equation

$$- \epsilon u'' + u = -2\epsilon - x(1 - x) - 1$$

subject to the boundary condition (3). The exact solution is given by

$$u = \frac{1}{1 - e^{-\frac{2}{\sqrt{\epsilon}}}} \left( e^{-\frac{1}{\sqrt{\epsilon}}} - e^{-\frac{1 + x}{\sqrt{\epsilon}}} + e^{-\frac{x}{\sqrt{\epsilon}}} - e^{-\frac{2 - x}{\sqrt{\epsilon}}}, \quad x \right) - x(1 - x) - 1.$$

It exhibits boundary layers at both ends of interval $(0, 1)$ when $\epsilon$ is small. The parameter is taken as $\epsilon = 10^{-5}$.

A typical adaptive mesh and the corresponding computed solution are shown in Fig. 2(a). In Fig. 2(b), $\|\pi_{h}(k+1) - \pi_{h}(k)\|_{\infty}$, $\max_{i}(Q_{eq,i} - 1)$, and $\|u_{h} - u\|$ are plotted as functions of the number of iteration. It can be seen that both (39) and (40) are effective stopping criteria and $\|\pi_{h}(k+1) - \pi_{h}(k)\|_{\infty}$ and $\max_{i}(Q_{eq,i} - 1)$ converge in a similar manner. Moreover, the solution error quickly reaches its lowest level (in one or two iterations for the current case).

The number of iterations required to reach the stopping criterion $\max_{i} Q_{eq,i} \leq 1.01$ and the solution error and the modified a posteriori estimator on the final mesh of each run are listed in Table 1. The results show that the underlying iterative algorithm may fail for small $N$ but is convergent for sufficiently large $N$. Moreover, the algorithm converges faster for larger $N$. These results are consistent with the observations made in Pryce [35] and Xu et al. [45] for the convergence of de Boor’s algorithm for generating equidistributing meshes for a given analytical function. It can also be seen that $\|u_{h} - u\|$ is smaller than the error estimator $\tilde{\eta}_{h}$ and both $\|u_{h} - u\|$ and $\tilde{\eta}_{h}$ converge in the same order $O(1/N)$ as $N \to \infty$. These results conform the theoretical predictions in Theorems 3.2 and 3.3.
Table 1: Example 4.1. \(\text{Iter}\) is the number of iterations required to reach the stopping criterion \(\max_i Q_{eq, i} \leq 1.01\) or the maximum allowed number (1000 is used in the computation). \(\| (u_h - u)' \|\) and \(\tilde{\eta}_h = \sqrt{\sum_i h_i^3 (\alpha_h + \langle r_h \rangle_i^2)}\) are the error and estimate obtained for the final mesh of each computation.

| \(N\)  | 21   | 41   | 81   | 161  | 321  | 641  |
|-------|------|------|------|------|------|------|
| \(\text{Iter}\) | 1000 | 1000 | 39   | 4    | 3    | 2    |
| \(\| (u_h - u)' \|\) | 3.07 | 1.41 | 6.86e-1 | 3.39e-1 | 1.69e-1 | 8.46e-2 |
| \(\tilde{\eta}_h\) | 8.10 | 3.65 | 1.72 | 8.35e-1 | 4.15e-1 | 2.07e-1 |

Example 4.2. Our second example is a convection-dominated differential equation

\[-\epsilon u'' + \left(1 - \frac{1}{2} \epsilon\right) u' + \frac{1}{4} \left(1 - \frac{1}{4} \epsilon\right) u = e^{-\frac{x}{4}},\]

where \(\epsilon = 2 \times 10^{-3}\). The exact solution is given by

\[u = e^{-\frac{x}{4}} \left(x - \frac{e^{-\frac{1 - x}{4}} - e^{-\frac{1}{4}}}{1 - e^{-\frac{x}{4}}}\right),\]

which has the boundary layer at \(x = 1\) when \(\epsilon\) is small. The numerical results are showed in Fig. 3 and Table 2. These results confirm the observations made from the previous example. Particularly, the algorithm converges for sufficiently large \(N\) and faster for larger \(N\). Moreover, the \(H^1\) semi-norm of the error converges in the first order \(O(1/N)\) as \(N \to \infty\).

Example 4.3. This example has been used by Babuška and Rheinboldt [2]. It takes the form

\[-((x + \alpha)^p u')' + (x + \alpha)^q u = f,\]
Table 2: Example 4.2. Iter is the number of iterations required to reach the stopping criterion $\max_i Q_{eq,i} \leq 1.01$ or the maximum allowed number (1000 is used in the computation). $\| (u_h - u)' \|$ and $\tilde{\eta}_h = \sqrt{\sum_i h_i^3 (\alpha_h + \langle r_h \rangle_i)}$ are the error and estimate obtained for the final mesh of each computation.

| $N$   | 21  | 41  | 81  | 161 | 321 | 641 |
|-------|-----|-----|-----|-----|-----|-----|
| Iter | 1000 | 327 | 83  | 9   | 5   | 3   |
| $\| (u_h - u)' \|$ | 1.20 | 5.07e-1 | 2.54e-1 | 1.20e-1 | 5.95e-2 | 2.96e-2 |
| $\tilde{\eta}_h$ | 8.08 | 1.30 | 6.44e-1 | 2.97e-1 | 1.46e-1 | 7.26e-2 |

Figure 3: Example 4.2. (a) An adaptive mesh of $N = 161$ points is plotted on the curve of the computed solution. (b) The difference between consecutive meshes ($\| \pi_h^{(k+1)} - \pi_h^{(k)} \|_\infty$), the equidistribution quality measure ($\max_i (Q_{eq,i} - 1)$), and the solution error ($\| (u_h - u)' \|$) are plotted against the number of iteration, $k$.

where $f$ is chosen such that the exact solution of the boundary value problem (with boundary condition $u(x) = (x + \alpha)^r - ((1 + \alpha)^r x)$ is

$$u = (x + \alpha)^r - ((1 + \alpha)^r x).$$

In our computation, the parameters are taken as $p = 2$, $q = 1$, $r = -1$, and $\alpha = 1/100$. The numerical results are shown in Fig. 4 and Table 3. Once again, these results confirm the observations made from the previous examples.

5 Continuous dependence of FE solution on mesh

The main purpose of this and next sections is to prove that there exists an equidistributing mesh satisfying the relation $\rho$ with $\alpha_h$ defined in (15) and (19) for a given, sufficiently large $N$. The procedure is to consider the mapping $G_N$ defined through the iterative algorithm described in the previous section and show that $G_N$ is continuous from a convex subset of $\mathbb{R}^{N-1}$ into itself. To this end, in this section we show that the FE solution depends on the mesh continuously. The existence
Table 3: Example 4.3  
Iter is the number of iterations required to reach the stopping criterion $\max_i Q_{eq,i} \leq 1.01$ or the maximum allowed number (1000 is used in the computation). $\| (u_h - u) \|$ and $\tilde{\eta}_h = \sqrt{\sum_i h_i^3 (\alpha_h + \langle r_h \rangle_i^2)}$ are the error and estimate obtained for the final mesh of each computation.

| N   | 21 | 41 | 81 | 161 | 321 | 641 |
|-----|----|----|----|-----|-----|-----|
| Iter | 4  | 3  | 3  | 2   | 2   | 2   |
| $\| (u_h - u) \|$ | 2.15e2 | 1.13e2 | 5.73e1 | 2.88e1 | 1.44e1 | 7.20 |
| $\tilde{\eta}_h$ | 3.22e3 | 1.51e3 | 7.41e2 | 3.69e2 | 1.84e2 | 9.21e1 |

Figure 4: Example 4.3  
An adaptive mesh of $N = 41$ points is plotted on the curve of the computed solution.

of the equidistributing mesh is proven in the next section.

To show the continuous dependence of the FE solution on the mesh, we need to establish some error bounds in the $L^\infty$ norm, which are also needed in the next section in obtaining the upper and lower bounds for the monitor function. It should be pointed out that we do not require the mesh to be an equidistributing mesh in this section. Instead, all results hold for any mesh in

$$ S_N = \left\{ X \in \mathbb{R}^{N-1} : x_0 = 0 < x_1 < \cdots < x_{N-1} < x_N = 1, \frac{1}{\rho_0 N} \leq h_i \leq \frac{2}{N}, \ i = 1, \ldots, N \right\}, $$

where

$$ \rho_0 = \left[ 1 + \gamma \left( \frac{\| r \|_{L^\infty(\Omega)}}{\| r \|_{L^3(\Omega)}} \right)^\frac{2}{3} \right]^2, \quad (42) $$

and $\gamma$ is a positive constant dependent on the domain and coefficients of equation (2). The definition of $\gamma$ is given in the proof of Lemma 6.1, which shows that $\rho_0$ is an upper bound of the monitor function defined in (15). It is easy to verify that $S_N$ is a closed, convex subset of $\mathbb{R}^{N-1}$. The set will
be equipped with the maximum norm, viz.,

\[ \|X\|_\infty = \max_i |x_i|, \quad \forall X \in S_N. \]

We use two different meshes, \( \pi_h \) and \( \pi_\tilde{h} \) or \( X \) and \( \tilde{X} \), in this and next sections. To distinguish the dependence we denote any quantity or function (say \( v \)) associated with mesh \( \tilde{X} \) by \( \tilde{v} \). Moreover, in these two sections constants are considered as numbers that may further depend on the solution \( u \) and the right-hand side function \( f \) (but not on the mesh).

We start with establishing two inequalities in Lemma 5.1 and error bounds in the \( L^\infty \) norm in Lemma 5.2.

**Lemma 5.1.**

\[ |v(x)|^2 \leq \int_{a_1}^{b_1} |v| |v'| dt, \quad \forall x \in [a_1, b_1], \quad \forall v \in H^1_0(a_1, b_1) \]  

(43)

\[ |v(x)|^2 \leq 2 \int_{a_1}^{b_1} |v| |v'| dt + \frac{1}{b_1 - a_1} \int_{a_1}^{b_1} |v|^2 dt, \quad \forall x \in [a_1, b_1], \quad \forall v \in H^1(a_1, b_1). \]  

(44)

**Proof.** For \( v \in H^1_0(a_1, b_1) \) we have

\[ |v(x)|^2 = 2 \int_{a_1}^{x} v v' dt \leq 2 \int_{a_1}^{x} |v| \cdot |v'| dt, \]

\[ |v(x)|^2 = -2 \int_{x}^{b_1} v v' dt \leq 2 \int_{x}^{b_1} |v| \cdot |v'| dt. \]

Summing these inequalities yields (43).

Moreover, for \( v \in H^1(a_1, b_1) \), using integration by parts we have

\[ \int_{a_1}^{x} |v|^2 dt = (x - a_1)|v(x)|^2 - 2 \int_{a_1}^{x} (t - a_1)v v' dt \geq (x - a_1)|v(x)|^2 - 2(b_1 - a_1) \int_{a_1}^{x} |v| \cdot |v'| dt, \]

\[ \int_{x}^{b_1} |v|^2 dt = (b_1 - x)|v(x)|^2 - 2 \int_{x}^{b_1} (t - b_1)v v' dt \geq (b_1 - x)|v(x)|^2 - 2(b_1 - a_1) \int_{x}^{b_1} |v| \cdot |v'| dt. \]

Summing these two inequalities gives (44). \( \square \)

**Lemma 5.2.** Assume that \( X \in S_N \) and \( u \in H^2(\Omega) \). Then the FE error can be bounded in \( L^\infty \) norm as

\[ \|e_h\|_{L^\infty(\Omega)} \leq \frac{C\|r\|}{N}, \]  

(45)

\[ \|e'_h\|_{L^\infty(\Omega)} \leq \frac{\sqrt{\rho_0 + 1} C\|r\|}{\sqrt{N}}. \]  

(46)

**Proof.** From Poincare’s inequality and Lemma 3.3 we know that the error can be bounded in \( L^2 \) norm as

\[ \|e_h\| \leq \frac{C\|r\|}{N}, \quad \|e'_h\| \leq \frac{C\|r\|}{N}. \]  

(47)
Then from Lemma 5.1 and Schwarz’ inequality we get
\[ \|e_h\|_{L^\infty(\Omega)}^2 \leq \|e_h\| \|e'_h\| \leq \frac{C\|r\|^2}{N^2}, \]
which leads to (45).

To prove (46), taking \( v = e'_h \) and \((a_1, b_1) = K_i \) \((i = 1, \ldots, N)\) in (44) we have
\[ \|e'_h\|^2_{L^\infty(K_i)} \leq \frac{1}{h_i} \|e'_h\|^2_{K_i} + 2\|e'_h\|_{K_i} \|u''\|_{K_i}, \tag{48} \]
Noticing that \( 1/(\rho_0 N) \leq h_i \), we get from (47) and (48) that
\[ \|e'_h\|^2_{L^\infty(K_i)} \leq \rho_0 N \|e'_h\|^2_{\Omega} + 2\|e'_h\|_{\Omega} \|u''\|_{\Omega} \leq \frac{(\rho_0 + 1) C\|r\|^2}{N}, \]
which gives (46).

It is remarked that the dependence of the bound on the constant \( \rho_0 \) is spelled out explicitly in (46). This is needed for the definition of \( \rho_0 \); see the proof of Lemma 6.1. Moreover, in the above lemma the convergence order for \( \|e_h\|_{L^\infty(\Omega)} \) is not optimal. Tighter bounds are obtained in the following lemma.

**Lemma 5.3.** Assume that \( X \in S_N \) and \( u \in H^2(\Omega) \). Then the FE error \( e_h \) is bounded by
\[ \|e_h\| \leq \frac{C}{N^2}, \quad \|e'_h\| \leq \frac{C}{N^2}, \quad \|e_h\|_{L^\infty(\Omega)} \leq \frac{C}{\sqrt{N}}, \quad \|e'_h\|_{L^\infty(\Omega)} \leq \frac{C}{\sqrt{N}}, \tag{50} \]
\[ \|u_h\|_{L^\infty(\Omega)} \leq C, \quad \|u'_h\|_{L^\infty(\Omega)} \leq C, \tag{51} \]
where the generic constant \( C \) may further depend on the solution \( u \) and the right-hand side function \( f \).

**Proof.** The second inequality of (49) is a consequence of Lemma 3.3. The first inequality can readily be proven by making use of the Nitsche trick. The proof for (50) is similar to that for Lemma 5.2 by using (49). The inequalities in (51) follow from (50), the triangle inequality, and the boundedness of \( u \) and \( u' \) in \( L^\infty \) norm.

We now consider the continuous dependence of the FE solution on the mesh.

**Lemma 5.4.** Assume that \( X, \tilde{X} \in S_N, f \in L^\infty(\Omega), \) and \( u \in H^2(\Omega) \). Then the FE solutions \( u_h \) related to mesh \( X \) and \( u_{\tilde{h}} \) related to \( \tilde{X} \) satisfy
\[ \|(u_h - \hat{u}_h)'\|^2_{\Omega} + \|u_h - \hat{u}_h\|^2_{\Omega} + \sum_{i=1}^{N} \frac{1}{h_i} \|u_h(x_i) - u_h(\tilde{x}_i) - (u_h(x_{i-1}) - u_h(\tilde{x}_{i-1}))\|^2 + \sum_{i=1}^{N} h_i |u_h(x_i) - u_h(\tilde{x}_i)|^2 \leq CN^2 \|X - \tilde{X}\|^2, \tag{52} \]
where
\[ \hat{u}_h(x) = \sum_i u_h(\tilde{x}_i) \phi_i(x). \]
Proof. We notice that the FE solutions can be expressed as
\[ u_h = \sum_{i=1}^{N-1} u_h(x_i)\phi_i(x), \quad \tilde{u}_h = \sum_{i=1}^{N-1} \tilde{u}_h(\tilde{x}_i)\tilde{\phi}_i(x). \]

Moreover, (6) can be rewritten into matrix form as
\[ AU = F, \quad \tilde{A}\tilde{U} = \tilde{F}, \]  
where
\[ A = (a_{ij}), \quad \tilde{A} = (\tilde{a}_{ij}), \quad a_{ij} = B(\phi_j, \phi_i), \quad \tilde{a}_{ij} = B(\tilde{\phi}_j, \tilde{\phi}_i), \]
\[ U = (u_h(x_1), \ldots, u_h(x_{N-1}))^\top, \quad \tilde{U} = (u_{\tilde{h}}(\tilde{x}_1), \ldots, u_{\tilde{h}}(\tilde{x}_{N-1}))^\top, \]
\[ F = (F_1, \ldots, F_{N-1})^\top, \quad \tilde{F} = (\tilde{F}_1, \ldots, \tilde{F}_{N-1})^\top, \quad F_i = (f, \phi_i), \quad \tilde{F}_i = (f, \tilde{\phi}_i). \]

Let
\[ V = (V_1, \ldots, V_{N-1})^\top, \quad V_i = u_h(x_i) - \hat{u}_h(x_i) = u_h(x_i) - u_{\tilde{h}}(\tilde{x}_i), \]
\[ v = u_h - \hat{u}_h = \sum_{i=1}^{N-1} V_i\phi_i(x) \in V_h. \]

By subtracting the second equation from the first one in (53), re-grouping the terms, and taking the inner product of the resulting equation with \( V \), we obtain
\[ V^\top AV + V^\top (A - \tilde{A})\tilde{U} = V^\top (F - \tilde{F}). \]  
(54)

We now estimate the terms in (54) separately. First, from Lemma 3.1 we have
\[ V^\top AV = B(v, v) \geq a_0\|v'\|^2. \]  
(55)

It is not difficult to verify that
\[ \|v'\|^2 = \sum_{i=1}^N \frac{1}{h_i} |V_i - V_{i-1}|^2. \]

Moreover, from Poincare’s inequality we have
\[ \|v'\|^2 \geq 8\|v\|^2 = 8\sum_i |V_i\phi_i|^2 \]
\[ = 8 \sum_{i=1}^{N-1} V_i \left[ \frac{(h_i + h_{i+1})}{3} V_i + \frac{h_i}{6} V_{i-1} + \frac{h_{i+1}}{6} V_{i+1} \right] \]
\[ \geq \frac{8}{6} \sum_{i=1}^{N-1} (h_i + h_{i+1})V_i^2. \]  
(56)

Combining (55)–(56) we get
\[ V^\top AV \]
\[
\begin{align*}
\geq & \quad \frac{a_0}{4} \|v'\|^2 + \frac{a_0}{8} \|v'\|^2 + \frac{a_0}{4} \|v'\|^2 + \frac{3a_0}{8} \|v'\|^2 \\
\geq & \quad \frac{a_0}{4} \|u_h - \hat{u}_h\|^2 + a_0 \|u_h - \hat{u}_h\|^2 + \frac{a_0}{4} \sum_{i=1}^{N} \frac{1}{h_i} |V_i - V_{i-1}|^2 + \frac{a_0}{2} \sum_{i=1}^{N} (h_i + h_{i+1}) V_i^2. \quad (57)
\end{align*}
\]

Next, we estimate the term \( V^\top (F - \tilde{F}) \). Noticing that \( \phi_i + \phi_{i+1} = 1 \) on \((x_i, x_{i+1})\) and \( \tilde{\phi}_i + \tilde{\phi}_{i+1} = 1 \) on \((\tilde{x}_i, \tilde{x}_{i+1})\), we have

\[
F_i - \tilde{F}_i = \int_{x_{i-1}}^{x_i} f \phi_i dx - \int_{\tilde{x}_{i-1}}^{\tilde{x}_i} f \tilde{\phi}_i dx \\
= \left( \int_{x_{i-1}}^{x_i} f \phi_i dx - \int_{\tilde{x}_{i-1}}^{\tilde{x}_i} f \tilde{\phi}_i dx \right) + \left( \int_{x_{i-1}}^{x_i} f \phi_i dx - \int_{\tilde{x}_{i-1}}^{\tilde{x}_i} f \tilde{\phi}_i dx \right) \\
= \left( \int_{x_{i-1}}^{x_i} f \phi_i dx - \int_{\tilde{x}_{i-1}}^{\tilde{x}_i} f \tilde{\phi}_i dx \right) - \left( \int_{x_{i-1}}^{x_i} f \phi_i dx - \int_{\tilde{x}_{i-1}}^{\tilde{x}_i} f \tilde{\phi}_i dx \right) \\
+ \left( \int_{x_{i-1}}^{x_i} f dx - \int_{\tilde{x}_{i-1}}^{\tilde{x}_i} f dx \right).
\]

Thus,

\[
V^\top (F - \tilde{F}) = \sum_{i=1}^{N} \left( \int_{x_{i-1}}^{x_i} f \phi_i dx - \int_{\tilde{x}_{i-1}}^{\tilde{x}_i} f \tilde{\phi}_i dx \right) (V_i - V_{i-1}) + \sum_{i=1}^{N} (V_{i-1} - V_i) \int_{\tilde{x}_i}^{x_i} f dx. \quad (58)
\]

Denote

\[
x_i^- = \min\{x_i, \tilde{x}_i\}, \quad x_i^+ = \max\{x_i, \tilde{x}_i\}. \quad (59)
\]

When \((x_{i-1}, x_i)\) and \((\tilde{x}_{i-1}, \tilde{x}_i)\) overlap, we have

\[
\begin{align*}
& \quad \left| \int_{x_{i-1}}^{x_i} f \phi_i dx - \int_{\tilde{x}_{i-1}}^{\tilde{x}_i} f \tilde{\phi}_i dx \right| \\
& = \left| \int_{x_{i-1}}^{x_i} f \phi_i dx + \int_{x_i}^{x_i} f \phi_i dx + \int_{x_i}^{x_i} f (\phi_i - \tilde{\phi}_i) dx - \int_{x_i}^{\tilde{x}_i} f \tilde{\phi}_i dx - \int_{x_i}^{\tilde{x}_i} f \phi_i dx \right| \\
& \leq \|f\|_{L^\infty} (\Omega) \left( |x_{i-1} - \tilde{x}_{i-1}| + |x_i - \tilde{x}_i| \right) \\
& + \|f\|_{L^\infty} (\Omega) \int_{x_i}^{x_i} \left| \frac{(\tilde{h}_i - h_i)}{h_i} (x - x_{i-1}) + \frac{(\tilde{h}_i - h_i)}{h_i} (x - x_{i-1}) \right| dx \\
& \leq \|f\|_{L^\infty} (\Omega) \left( 2|x_{i-1} - \tilde{x}_{i-1}| + |x_i - \tilde{x}_i| + |\tilde{h}_i - h_i| \right) \\
& \leq 3\|f\|_{L^\infty} (\Omega) \left( |x_{i-1} - \tilde{x}_{i-1}| + |x_i - \tilde{x}_i| \right).
\end{align*}
\]
On the other hand, if \((x_{i-1}, x_i)\) and \((\tilde{x}_{i-1}, \tilde{x}_i)\) do not overlap, we have

\[
\left| \int_{x_{i-1}}^{x_i} f \phi_i \, dx - \int_{\tilde{x}_{i-1}}^{\tilde{x}_i} f \tilde{\phi}_i \, dx \right| = \int_{x_{i-1}}^{x_i} |f \phi_i| \, dx + \int_{x_i}^{\tilde{x}_i} |f \tilde{\phi}_i| \, dx \\
\leq \|f\|_{L^\infty(\Omega)} (|x_{i-1} - \tilde{x}_{i-1}| + |x_i - \tilde{x}_i|).
\]

For both cases we thus have

\[
\left| \int_{x_{i-1}}^{x_i} f \phi_i \, dx - \int_{\tilde{x}_{i-1}}^{\tilde{x}_i} f \tilde{\phi}_i \, dx \right| \leq 3\|f\|_{L^\infty(\Omega)} (|x_{i-1} - \tilde{x}_{i-1}| + |x_i - \tilde{x}_i|). \tag{60}
\]

Inserting (60) into (58) and using Young’s inequality and \(h_i \leq 2/N\), we get

\[
|V^T (F - \tilde{F})| \leq \frac{a_0}{20} \sum_{i=1}^{N-1} h_i V_i^2 + C\|X - \tilde{X}\|_{\infty}^2. \tag{61}
\]

We now proceed to estimate \(V^T (A - \tilde{A})\tilde{U}\). We start with computing the non-zero entries of \(A = (a_{ij})\). Noticing that

\[
\phi_i \phi'_{i-1} = -\phi'_i \phi_i = -\frac{1}{2} (\phi_i^2)', \quad \phi_i + \phi_{i-1} = 1, \quad \text{on } (x_{i-1}, x_i) \]
\[
\phi_i \phi'_{i+1} = \phi'_{i+1} - \phi_{i+1} \phi_{i+1} = \phi'_{i+1} - \frac{1}{2} (\phi_{i+1}^2)', \quad \phi_i + \phi_{i+1} = 1, \quad \text{on } (x_i, x_{i+1})
\]

by direct calculation we have

\[
a_{i,i-1} = B(\phi_{i-1}, \phi_i) = \int_{x_{i-1}}^{x_i} \left[ a \phi'_i \phi'_{i-1} + b \phi_i \phi'_{i-1} + c \phi_i \phi_{i-1} \right] dx \\
= -\frac{a_i}{h_i} \frac{1}{2} b(x_i) + \frac{1}{2} \int_{x_{i-1}}^{x_i} b' \phi_i \, dx + \int_{x_{i-1}}^{x_i} (c - \frac{1}{2} b') \phi_i \phi_{i-1} \, dx,
\]
\[
a_{i,i+1} = B(\phi_{i+1}, \phi_i) = \int_{x_i}^{x_{i+1}} \left[ a \phi'_i \phi'_{i+1} + b \phi_i \phi'_{i+1} + c \phi_i \phi_{i+1} \right] dx \\
= -\frac{a_{i+1}}{h_{i+1}} \frac{1}{2} b(x_{i+1}) - \frac{1}{2} \int_{x_i}^{x_{i+1}} b' \phi_{i+1} \, dx + \int_{x_i}^{x_{i+1}} (c - \frac{1}{2} b') \phi_i \phi_{i+1} \, dx,
\]
\[
a_{i,i} = B(\phi_i, \phi_i) = \int_{x_{i-1}}^{x_{i+1}} \left[ a \phi'_i \phi'_i + (c - \frac{1}{2} b') \phi_i^2 \right] dx \\
= \frac{a_i}{h_i} + \frac{a_{i+1}}{h_{i+1}} + \int_{x_{i-1}}^{x_{i+1}} (c - \frac{1}{2} b') \phi_i^2 \, dx,
\]

where

\[
a_i = \frac{1}{h_i} \int_{x_{i-1}}^{x_i} a(x) \, dx.
\]

Matrix \(\tilde{A} = (\tilde{a}_{ij})\) has similar expressions.
Using the above expressions, we have

\[ V^T(A - \tilde{A})\tilde{U} \]

\[ = \sum_{i=1}^{N-1} V_i \left[ \left( \frac{a_i}{h_i} - \frac{\tilde{a}_i}{h_i} + \frac{a_{i+1}}{h_{i+1}} - \frac{\tilde{a}_{i+1}}{h_{i+1}} \right) \tilde{U}_i - \left( \frac{a_i}{h_i} - \frac{\tilde{a}_i}{h_i} \right) \tilde{U}_{i-1} - \left( \frac{a_{i+1}}{h_{i+1}} - \frac{\tilde{a}_{i+1}}{h_{i+1}} \right) \tilde{U}_{i+1} \right] \]

\[ - \sum_{i=1}^{N-1} \frac{1}{2} V_i \left[ (b(x_i) - b(\tilde{x}_i)) \tilde{U}_{i-1} - (b(x_{i+1}) - b(\tilde{x}_{i+1})) \tilde{U}_{i+1} \right] \]

\[ + \sum_{i=1}^{N-1} \frac{1}{2} V_i \left[ (B_i - \tilde{B}_i) \tilde{U}_{i-1} + (B_{i+1} - \tilde{B}_{i+1}) \tilde{U}_{i+1} \right] \]

\[ + \sum_{i=1}^{N-1} V_i \left[ (C_i - \tilde{C}_i) \tilde{U}_{i-1} + (C_{i+1} - \tilde{C}_{i+1}) \tilde{U}_{i+1} \right] \]

\[ = I_1 + I_2 + I_3 + I_4, \quad (62) \]

where \[ B_i = \int_{x_{i-1}}^{x_i} b'(\phi_i) dx, \quad C_i = \int_{x_{i-1}}^{x_i} (c - \frac{1}{2} b') \phi_i \phi_{i-1} dx, \quad C_{2,i} = \int_{x_{i-1}}^{x_{i+1}} (c - \frac{1}{2} b') \phi_i^2 dx. \]

Noticing that Lemma 5.3 implies \[ ||u_i^*||_{L^\infty(\Omega)} \leq C, \] we have

\[ |I_1| = \left| \sum_{i=1}^{N-1} V_i \left[ \left( \frac{a_i}{h_i} - \frac{\tilde{a}_i}{h_i} + \frac{a_{i+1}}{h_{i+1}} - \frac{\tilde{a}_{i+1}}{h_{i+1}} \right) \tilde{U}_i - \left( \frac{a_i}{h_i} - \frac{\tilde{a}_i}{h_i} \right) \tilde{U}_{i-1} \right] \right| \]

\[ = \sum_{i=1}^{N} |V_i - V_{i-1}| \left( \frac{a_i}{h_i} - \frac{\tilde{a}_i}{h_i} \right) \left| \tilde{U}_i - \tilde{U}_{i-1} \right| \]

\[ \leq C \sum_{i=1}^{N} |V_i - V_{i-1}| \left( \frac{a_i}{h_i} - \frac{\tilde{a}_i}{h_i} \right) \]

\[ \leq \frac{a_0}{40} \sum_{i=1}^{N} \frac{1}{h_i} |V_i - V_{i-1}|^2 + C \sum_{i=1}^{N} |a_i h_i - \tilde{a}_i h_i|^2 \frac{1}{h_i}. \]

From the definitions of \( a_i \) and \( \tilde{a}_i \),

\[ a_i \tilde{h}_i - \tilde{a}_i h_i = a_i (\tilde{h}_i - h_i) + h_i (a_i - \tilde{a}_i) \]

\[ = a_i (\tilde{h}_i - h_i) + \left( \frac{\tilde{h}_i - h_i}{h_i} \right) h_i \left\{ \int_{x_{i-1}}^{x_i} a(x) dx + \frac{h_i}{\tilde{h}_i} \left( \int_{x_{i-1}}^{x_i} a(x) dx - \int_{x_{i-1}}^{\tilde{x}_i} a(x) dx \right) \right\}. \]

From this it is not difficult to obtain

\[ |a_i \tilde{h}_i - \tilde{a}_i h_i|^2 \frac{1}{h_i} \leq \frac{C h^2}{h^3} (|x_i - \tilde{x}_i| + |x_{i-1} - \tilde{x}_{i-1}|). \]

Thus, from the fact that \[ 1/(\rho_0 N) \leq \tilde{h} \leq h_i \leq 2/N \] it follows

\[ |I_1| \leq \frac{a_0}{40} \sum_{i=1}^{N} \frac{1}{h_i} |V_i - V_{i-1}|^2 + C N^2 \|X - \tilde{X}\|_{\infty}^2. \quad (63) \]
Similarly, we have

\[ |I_2| = \frac{1}{2} \left| \sum_{i=1}^{N-1} (b(x_i) - b(\bar{x}_i))(V_i \bar{U}_{i-1} - V_{i-1} \bar{U}_i) \right| \]

\[ \leq \frac{1}{2} \sum_{i=1}^{N-1} h_i |V_i| \left( |V_i| |\bar{U}_i - \bar{U}_{i-1}| + |V_i - V_{i-1}| |\bar{U}_i| \right) \]

\[ \leq \frac{a_0}{20} \sum_{i=1}^{N-1} h_i |V_i|^2 + C \|X - \bar{X}\|_\infty^2 + \frac{a_0}{40} \sum_{i=1}^{N} \frac{1}{h_i} |V_i - V_{i-1}|^2. \]  

(64)

The difference \( B_i - \tilde{B}_i \) involved in \( I_3 \) can be estimated in the same manner as for \( (\int_{x_{i-1}}^{x_i} f \phi dx - \int_{\tilde{x}_{i-1}}^{\tilde{x}_i} \tilde{f} \phi dx) \) in \( (F_i - \tilde{F}_i) \). We have

\[ |B_i - \tilde{B}_i| \leq 3 \|V_i - \tilde{V}_i\|_\infty \left( |x_{i-1} - \tilde{x}_{i-1}| + |x_i - \tilde{x}_i| \right). \]  

(65)

Using these estimates we obtain

\[ |I_3| = \frac{1}{2} \left| \sum_{i=1}^{N-1} (B_i - \tilde{B}_i)(V_i \bar{U}_{i-1} - V_{i-1} \bar{U}_i) \right| \]

\[ \leq \frac{1}{2} \sum_{i=1}^{N-1} h_i |V_i| \left( |V_i| |\bar{U}_i - \bar{U}_{i-1}| + |V_i - V_{i-1}| |\bar{U}_i| \right) \]

\[ \leq \frac{a_0}{20} \sum_{i=1}^{N-1} h_i |V_i|^2 + C \|X - \bar{X}\|_\infty^2 + \frac{a_0}{40} \sum_{i=1}^{N} \frac{1}{h_i} |V_i - V_{i-1}|^2. \]  

(66)

To estimate \( I_4 \), we denote

\[ C_{0,i} = \int_{x_{i-1}}^{x_i} (c - \frac{1}{2} b') dx, \quad C_{1,i} = \int_{x_{i-1}}^{x_i} (c - \frac{1}{2} b') \phi_i dx. \]

Like for \( B_i - \tilde{B}_i \), we have the estimates

\[ |C_i - \tilde{C}_i| \leq \frac{(c - b'/2)(3|x_{i-1} - \tilde{x}_{i-1}| + 3|x_i - \tilde{x}_i|)}{3}, \]

\[ |C_{1,i} - \tilde{C}_{1,i}| \leq \frac{(c - b'/2)(3|x_{i-1} - \tilde{x}_{i-1}| + 3|x_i - \tilde{x}_i|)}{3}. \]

Moreover,

\[ C_{2,i} = \int_{x_{i-1}}^{x_i} (c - \frac{1}{2} b') \phi_i^2 dx + \int_{x_i}^{x_{i+1}} (c - \frac{1}{2} b') \phi_i^2 dx \]

\[ = \int_{x_{i-1}}^{x_i} (c - \frac{1}{2} b') \phi_i(1 - \phi_{i-1}) dx + \int_{x_i}^{x_{i+1}} (c - \frac{1}{2} b') \phi_i(1 - \phi_{i+1}) dx \]

\[ = C_{1,i} - C_i + \int_{x_i}^{x_{i+1}} (c - \frac{1}{2} b') \phi_i dx - C_i + \]

\[ = C_{1,i} - C_i + \int_{x_i}^{x_{i+1}} (c - \frac{1}{2} b')(1 - \phi_{i+1}) dx - C_i + \]

\[ = C_{1,i} - C_i + \int_{x_i}^{x_{i+1}} (c - \frac{1}{2} b')(1 - \phi_{i+1}) dx - C_i + \]

\[ + \quad \]

\[ \]
\[ = C_{0,i+1} + C_{1,i} - C_{1,i+1} - C_i - C_{i+1}. \]

Then,

\[
I_4 = \sum_{i=1}^{N-1} V_i \left[ (C_i - \tilde{C}_i)(\tilde{U}_{i-1} - \tilde{U}_i) - (C_{i+1} - \tilde{C}_{i+1})(\tilde{U}_i - \tilde{U}_{i+1}) \right. \\
+ \left. ((C_{1,i} - \tilde{C}_{1,i}) - (C_{1,i+1} - \tilde{C}_{1,i+1}))\tilde{U}_i + (C_{0,i+1} - \tilde{C}_{0,i+1})\tilde{U}_i \right]
\]

\[
= \sum_{i=1}^{N-1} (V_i - V_{i-1})(C_i - \tilde{C}_i)(\tilde{U}_{i-1} - \tilde{U}_i) + \sum_{i=1}^{N-1} (C_{1,i} - \tilde{C}_{1,i})(V_i\tilde{U}_i - V_{i-1}\tilde{U}_{i-1}) \\
+ \sum_{i=1}^{N-1} (V_i\tilde{U}_i - V_{i-1}\tilde{U}_{i-1}) \int_{\bar{x}_i}^{x_i} (c - \frac{1}{2}b')dx.
\]

From this we get

\[
|I_4| \leq \frac{a_0}{20} \sum_{i=1}^{N-1} (h_i + h_{i+1})V_i^2 + C\|X - \tilde{X}\|_\infty^2 + \frac{a_0}{40} \sum_{i=1}^{N} \frac{1}{h_i} |V_i - V_{i-1}|^2.
\] (67)

Combining (62), (63), (64), (66), and (67), we get

\[
V^T(A - \tilde{A})\tilde{U} \leq \frac{a_0}{8} \sum_{i=1}^{N} \frac{1}{V_i - V_{i-1}}^2 + \frac{a_0}{4} \sum_{i=1}^{N} (h_i + h_{i+1})V_i^2 + CN^2\|X - \tilde{X}\|_\infty^2.
\] (68)

Finally, (52) follows from (67), (61), and (68).

**Theorem 5.1.** (Continuous dependence of the FE solution on the mesh) Assume that \( f \in L^\infty(\Omega) \) and \( u \in H^2(\Omega) \). Then for any meshes \( X, \tilde{X} \in S_N \) satisfying

\[ \|X - \tilde{X}\|_\infty < \frac{1}{\rho_0 N}, \] (69)

the corresponding linear FE solutions, \( u_h \) and \( \tilde{u}_h \), satisfy

\[
\|(u_h - \tilde{u}_h)'\|_{L^1(\Omega)} \leq CN\|X - \tilde{X}\|_\infty,
\] (70)

\[
\|u_h - \tilde{u}_h\| \leq CN^{-\frac{1}{2}}\|X - \tilde{X}\|_2^\frac{1}{2},
\] (71)

\[
\|u_h\|_{\tilde{E}}^2 - \|u_h\|_{\tilde{E}}^2 \leq CN^{-\frac{1}{2}}\|X - \tilde{X}\|_2^\frac{1}{2},
\] (72)

where \( \| \cdot \|_E \) denotes the energy norm associated with the bilinear form \( B(\cdot, \cdot) \), viz.,

\[
\|v\|_E^2 = \int_{\Omega} \left( av'^2 + (c - \frac{1}{2}b')v^2 \right) dx.
\]

**Proof.** From Lemma 5.4 we can see that the key to the proof of this theorem is to estimate \( \|\tilde{u}'_h - u'_h\|_{L^1(\Omega)} \) and \( \|\tilde{u}_h - u_h\|_{L^1(\Omega)} \). For this purpose, we notice from assumption (69) that \( \|X - \tilde{X}\|_\infty < \min_i \{h_i, \tilde{h}_i\} \) and \( (x_{i-1}, x_i) \cap (\tilde{x}_{i-1}, \tilde{x}_i) \neq \emptyset, \quad i = 1, \ldots, N. \)
As a consequence, we can divide \([x_{i-1}, x_i]\) into subintervals \([x_{i-1}^-, x_i^-]\), \([x_{i-1}^+, x_i^+]\), and \([x_i^-, x_i^+]\). On these intervals \(\hat{u}_h - u_h\) can be expressed as

\[
(\hat{u}_h - u_h)_{|[x_{i-1}, x_i^+]} = (u_h(\hat{x}_i)\phi_i(x) + u_h(\check{x}_{i-1})\phi_{i-1}(x)) - \left( u_h(\check{x}_{i-1})\tilde{\phi}_{i-1}(x) + u_h(\check{x}_{i-2})\tilde{\phi}_{i-2}(x) \right) \\
= (u_h(\hat{x}_i) - u_h(\check{x}_{i-1}))\phi_i(x) + (u_h(\check{x}_{i-1}) - u_h(\check{x}_{i-2}))\phi_{i-1}(x) \\
+ (u_h(\check{x}_{i-1}) - u_h(\check{x}_i))\left( \phi_i(x) - \tilde{\phi}_{i-1}(x) \right),
\]

(73)

\[
(\hat{u}_h - u_h)_{|[x_{i-1}^-, x_i^-]} = (u_h(\check{x}_i) - u_h(\check{x}_{i-1}))\left( \phi_i(x) - \tilde{\phi}_{i}(x) \right),
\]

(74)

Integrating \(|\hat{u}_h - u_h|\) over the subintervals and using the above expressions and Lemma 5.3 we get

\[
\|\hat{u}_h - u_h\|_{L^1(x_{i-1}, x_i^+)} \leq C(\tilde{h}_i + \check{h}_i) |x_{i-1} - \check{x}_{i-1}|, \\
\|\hat{u}_h - u_h\|_{L^1(x_{i-1}^-, x_i^-)} \leq C h_i (|x_i - \check{x}_i| + |x_{i-1} - \check{x}_{i-1}|), \\
\|\hat{u}_h - u_h\|_{L^1(x_i^-, x_i)} \leq C(\tilde{h}_i + \check{h}_i) |x_i - \check{x}_i|.
\]

Summing these estimates from \(i = 1\) to \(i = N\) yields

\[
\|\hat{u}_h - u_h\|_{L^1(\Omega)} \leq C\|X - \hat{X}\|_{\infty}.
\]

(75)

Moreover, differentiating (73)-(74) leads to

\[
(\hat{u}_h' - u_h')_{|[x_{i-1}, x_i^+]}) = u_h'(\check{x}_i)(\tilde{h}_i - \check{h}_i), \\
(\hat{u}_h' - u_h')_{|[x_{i-1}^-, x_i^-]} = u_h'(\check{x}_i)(\tilde{h}_i - \check{h}_i), \\
(\hat{u}_h' - u_h')_{|[x_i^-, x_i^+]} = u_h'(\check{x}_i)(\tilde{h}_i - \check{h}_i).
\]

Integrating \(|\hat{u}_h' - u_h'|\) over the subintervals and using Lemma 5.3, we obtain

\[
\|\hat{u}_h' - u_h'\|_{L^1(x_{i-1}, x_i^+)} \leq C(1 + \frac{\tilde{h}_i}{\check{h}_i}) |x_{i-1} - \check{x}_{i-1}|, \\
\|\hat{u}_h' - u_h'\|_{L^1(x_{i-1}^-, x_i^-)} \leq C |\tilde{h}_i - \check{h}_i|, \\
\|\hat{u}_h' - u_h'\|_{L^1(x_i^-, x_i^+)} \leq C(1 + \frac{\tilde{h}_i}{\check{h}_i}) |x_i - \check{x}_i|.
\]

Thus, combining these estimates gives

\[
\|\hat{u}_h' - u_h'\|_{L^1(\Omega)} \leq C N \|X - \hat{X}\|_{\infty}.
\]

Inequality (70) follows from the above estimate, Lemma 5.4 and the triangle inequality.
Next, recalling from Lemma 5.4 and (75) that

\[ \| u_h - \hat{u}_h \| \leq C \| X - \tilde{X} \|, \quad \| \hat{u}_h \|_{L^1(\Omega)} \leq C \| X - \tilde{X} \|, \]

by Schwarz’ inequality and Lemma 5.3 we have

\[ \| u_h - u_{\tilde{h}} \|^2 = \int_{\Omega} | u_h - u_{\tilde{h}} | \cdot | u_h - u_{\hat{h}} | dx \]
\[ \leq \int_{\Omega} | u_h - u_{\hat{h}} | \cdot | u_h - u_{\hat{h}} | dx + \int_{\Omega} | u_h - u_{\hat{h}} | \cdot | \hat{u}_h - u_{\hat{h}} | dx \]
\[ \leq \| u_h - u_{\hat{h}} \| \| u_h - u_{\hat{h}} \| + \| u_h - u_{\hat{h}} \|_L^\infty(\Omega) \| \hat{u}_h - u_{\hat{h}} \|_{L^1(\Omega)} \]
\[ \leq C N^{-2} \cdot N \| X - \tilde{X} \|_{\infty} + C N^{-\frac{3}{2}} \cdot \| X - \tilde{X} \|_{\infty} \]
\[ \leq C N^{-1} \| X - \tilde{X} \|_{\infty}, \]

which gives (71).

Finally, the FE equation (6) implies that

\[ \| u_h \|^2_E - \| u_{\tilde{h}} \|^2_E = (f, u_h - u_{\tilde{h}}). \]

From (70) we have

\[ \| u_h \|^2_E - \| u_{\hat{h}} \|^2_E = \| f \| \| u_h - u_{\hat{h}} \| \leq \alpha_1 \frac{1}{h} \leq 3 \frac{1}{2} \| r \|^2_{L^2(\Omega)}, \]

which gives (72).

6 Existence of equidistributing meshes

We prove the existence of equidistributing meshes in this section. Notice that this is equivalent to the existence of fixed points of the mapping \( G_N \) defined by the iterative algorithm in §4. The key is to show that \( G_N \) maps \( S_N \) into \( S_N \) and is continuous.

Recall from (37) that the mesh \( Y = G_N X \) satisfies

\[ \int_{y_{i-1}}^{y_i} \rho(x) dx = \frac{\sigma_h}{N}, \quad i = 1, \ldots, N \]  

(76)

where \( \rho(x) \) and \( \sigma_h \) are defined in (15) and (17) based on the solution \( u_h \) obtained on mesh \( X \) (i.e., \( \pi_h \)).

**Lemma 6.1.** Assume that \( X \in S_N \) and \( u \in H^2(\Omega) \). Then there exists a positive integer \( N_0 \), independent of the FE approximation and the mesh, such that, for any \( N \geq N_0 \),

\[ \frac{1}{2} \| r \|^2_{L^2(\Omega)} \leq \alpha_h^\frac{1}{4} \leq \frac{3}{2} \| r \|^2_{L^2(\Omega)}. \]

Moreover, for any \( N \geq N_0 \),

\[ 1 \leq \rho \leq \rho_0, \quad 1 \leq \sigma_h \leq 2, \]

where \( \rho_0 \) is a constant defined in (42).
Proof. The existence of \( N_0 \) is guaranteed by Lemma 3.8 (which does not assume the mesh to be equidistributing). Its independence of the FE approximation and the mesh is clear from (34) for the situation \( r' \in L^1(\Omega) \). For the situation \( r \in L^2(\Omega) \), we can choose \( \epsilon = \theta \|r\|_{L^2(\Omega)}^2 \) for some value of \( \theta \) (cf. the proof of Theorem 3.2). Then, a smoother function \( \tilde{r} \) which is independent of the approximation and the mesh can be chosen and an inequality similar to (33) can be obtained. Thus, an \( N_0 \) independent of the FE approximation and the mesh also exists for the situation \( r \in L^2(\Omega) \).

The inequality \( \rho \geq 1 \) follows immediately from the definition of \( \rho \). For the upper bound of \( \rho \), from (15), (77), Lemma 5.2, Young’s inequality, and the inequalities \( \|r\|_{L^2(\Omega)} \leq \|r\|_{L^\infty(\Omega)} \) and \( N \geq 1 \) we have

\[
\rho_i &= (1 + \alpha_h^{-1}(r_h)_i^2)^{\frac{1}{3}} \\
&\leq 1 + \alpha_h^{-\frac{1}{3}}\|r_h\|_{L^\infty(\Omega)}^\frac{2}{3} \\
&= 1 + \alpha_h^{-\frac{1}{3}}\|r - a'e_h + be_h + ce_h\|_{L^\infty(\Omega)}^\frac{2}{3} \\
&\leq 1 + 2\|r\|_{L^\infty(\Omega)}^\frac{2}{3} \left(\|r\|_{L^\infty(\Omega)}^\frac{2}{3} + (b-a')\|e_h\|_{L^\infty(\Omega)} + c\|e_h\|_{L^\infty(\Omega)}\right)^\frac{2}{3} \\
&\leq 1 + 2\|r\|_{L^\infty(\Omega)}^\frac{2}{3} \left(1 + \frac{\sqrt{\rho_0 + TC_1}}{\sqrt{N}} + \frac{C_2}{N}\right)^\frac{2}{3} \\
&\leq 1 + 2\|r\|_{L^\infty(\Omega)}^\frac{2}{3} \|r\|_{L^\infty(\Omega)}^\frac{2}{3} (1 + C_1 + C_2 + C_1\sqrt{\rho_0})^\frac{2}{3} \\
&\leq 1 + 2\|r\|_{L^\infty(\Omega)}^\frac{2}{3} \|r\|_{L^\infty(\Omega)}^\frac{2}{3} (1 + C_1 + C_2)^\frac{2}{3} + 2\|r\|_{L^\infty(\Omega)}^\frac{2}{3} \|r\|_{L^\infty(\Omega)}^\frac{2}{3} C_1^\frac{2}{3} \rho_0^\frac{1}{3} \\
&\leq 1 + 2\|r\|_{L^\infty(\Omega)}^\frac{2}{3} \|r\|_{L^\infty(\Omega)}^\frac{2}{3} (1 + C_1 + C_2)^\frac{2}{3} + 2\|r\|_{L^\infty(\Omega)}^\frac{2}{3} \|r\|_{L^\infty(\Omega)}^\frac{2}{3} (1 + C_1 + C_2)^\frac{2}{3} \rho_0^\frac{1}{3},
\]

where \( C_1 \) and \( C_2 \) denote the constants in (45) and (46), respectively. Notice that \( \rho_0 \geq 1 \) and thus \( \rho_0^\frac{1}{3} \leq \rho_0^\frac{1}{3} \). Letting \( \gamma = 2(1 + C_1 + C_2)^\frac{2}{3} \), from the definition of \( \rho_0 \), (42), we thus have

\[
\rho_i \leq \left[1 + \gamma\|r\|_{L^\infty(\Omega)}^\frac{2}{3} \|r\|_{L^\infty(\Omega)}^\frac{2}{3} + \gamma\|r\|_{L^\infty(\Omega)}^\frac{2}{3} \|r\|_{L^\infty(\Omega)}^\frac{2}{3}\right]^2 \rho_0^\frac{1}{3} = \rho_0\rho_0.
\]

The bounds for \( \sigma_h \) follow from the bounds for \( \rho \) and the definitions of \( \sigma_h \) and \( \alpha_h \). \( \square \)

Lemma 6.2. Assume that \( u \in H^2(\Omega) \). For any \( N \geq N_0 \) where \( N_0 \) is defined in Lemma 6.1, then \( G_N(S_N) \subset S_N \) or \( G_N : S_N \to S_N \).

Proof. Lemma 6.1 implies that for any given \( X \in S_N \), \( 1 \leq \rho \leq \rho_0 \) and \( 1 \leq \sigma_h \leq 2 \). From (76) we then have

\[
(y_i - y_i-1) \leq \int_{y_{i-1}}^{y_i} \rho(x)dx = \frac{\sigma_h}{N} \leq \frac{2}{N}
\]

and

\[
(y_i - y_i-1)\rho_0 \geq \int_{y_{i-1}}^{y_i} \rho(x)dx = \frac{\sigma_h}{N} \geq \frac{1}{N}.
\]

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Thus, \( Y = G_N X \in S_N \) for any \( X \in S_N \).

**Lemma 6.3.** Assume that \( f \in L^\infty(\Omega) \), \( u \in H^2(\Omega) \), and \( X, \bar{X} \in S_N \). Then,

\[
\tilde{h}_i \left| \langle r_h \rangle^2_i - \langle r_{\tilde{h}} \rangle^2_i \right| \leq C \left( \| X - \bar{X} \|_\infty + \| u_h' - u_{\tilde{h}}' \|_{L^1(K_i)} + \| u_h - u_{\tilde{h}} \|_{L^1(K_i)} \right).
\]

(78)

**Proof.** Using the notation (59) we have

\[
\langle r_h \rangle^2_i - \langle r_{\tilde{h}} \rangle^2_i = \tilde{h}_i - h_i \langle r_h \rangle^2_i + \frac{1}{\tilde{h}_i} \int_{x_{i-1}^-}^{x_i^-} (|r_h|^2 - |r_{\tilde{h}}|^2) dx
\]

\[
+ \frac{1}{\tilde{h}_i} \left( \int_{x_i^-}^{x_i^+} + \int_{x_{i+1}^-}^{x_{i+1}^+} \right) |r_h|^2 dx - \frac{1}{\tilde{h}_i} \left( \int_{x_{i-1}^-}^{x_{i-1}^+} + \int_{x_{i+1}^-}^{x_{i+1}^+} \right) |r_{\tilde{h}}|^2 dx.
\]

From the assumption \( f \in L^\infty(\Omega) \) and Lemma 6.3, we have \( \| r_h \|_{L^\infty(\Omega)} \leq C \) and \( \| r_{\tilde{h}} \|_{L^\infty(\Omega)} \leq C \). It follows that

\[
\left| \langle r_h \rangle^2_i - \langle r_{\tilde{h}} \rangle^2_i \right| \leq \frac{1}{\tilde{h}_i} \left( \| X - \bar{X} \|_\infty + \| u_h - u_{\tilde{h}} \|_{L^1(K_i)} \right) \left( \| r_h \|_{L^\infty(\Omega)} + \| r_{\tilde{h}} \|_{L^\infty(\Omega)} \right)
\]

which gives (78).

**Lemma 6.4.** Assume that \( f \in L^\infty(\Omega) \), \( u \in H^2(\Omega) \), and \( X, \bar{X} \in S_N \) satisfying \( \| X - \bar{X} \|_\infty < 1/(\rho_0 N) \). We also assume that \( N \geq N_0 \) where \( N_0 \) is defined in Lemma 6.1. Then,

\[
\int_0^1 \left| \rho - \bar{\rho} \right| dx \leq CN \| X - \bar{X} \|_\infty + C(N \| X - \bar{X} \|_\infty)^{\frac{1}{4}}.
\]

(79)

**Proof.** Using Lemmas 6.1, 5.1, and 6.3 and the inequality \( \| u_h - u_{\tilde{h}} \|_{L^1(\Omega)} \leq \|(u_h - u_{\tilde{h}})'\|_{L^1(\Omega)} \), we have

\[
|\alpha_h^2 - \alpha_{\tilde{h}}^2 | \leq \sum_{i=1}^N \left[ \tilde{h}_i - h_i \langle r_h \rangle^2_i + \tilde{h}_i \langle r_h \rangle^2_i - \langle r_{\tilde{h}} \rangle^2_i \right]
\]

\[
\leq C N \| X - \bar{X} \|_\infty + \sum_{i=1}^N (h_i \langle r_h \rangle^2_i - \langle r_{\tilde{h}} \rangle^2_i)^{\frac{1}{4}}
\]

\[
\leq C N \| X - \bar{X} \|_\infty + \left( \sum_{i=1}^N h_i \langle r_h \rangle^2_i - \langle r_{\tilde{h}} \rangle^2_i \right)^{\frac{1}{4}}
\]

\[
\leq C N \| X - \bar{X} \|_\infty^4 + C \left( \|(u_h - u_{\tilde{h}})'\|_{L^1(\Omega)} + \| u_h - u_{\tilde{h}} \|_{L^1(\Omega)} \right)^{\frac{1}{4}}
\]
\[
\frac{|\rho_i - \tilde{\rho}_i|}{\rho_i^2 + \rho_i \tilde{\rho}_i + \tilde{\rho}_i^2} \leq \frac{\frac{1}{3} \alpha_h \alpha_{\tilde{h}}}{\alpha_h \alpha_{\tilde{h}}} \left( \alpha_h^2 + \frac{1}{3} \alpha_{\tilde{h}} \right) \left| \langle r_h \rangle_i^2 - \langle \tilde{r}_h \rangle_i^2 \right| 
\]

It follows from (81), Lemma 6.3, and Theorem 5.1 that

\[
\int_0^1 |\rho - \tilde{\rho}| dx = \sum_{i=1}^N \left( \int_{x_i}^{x_{i-1}} |\rho - \tilde{\rho}| dx + \sum_{i=1}^N (x_i - x_{i-1}) |\rho_i - \tilde{\rho}_i| \right) 
\]

\[
\leq \sum_{i=1}^N \left( \left( x_i - x_{i-1} \right) + (x_{i-1}^+ - x_{i-1}) \right) |\rho_0| + \sum_{i=1}^N \tilde{\rho}_i |\rho_i - \tilde{\rho}_i| 
\]

\[
\leq C |\alpha_h^{\frac{1}{2}} - \alpha_{\tilde{h}}^{\frac{1}{2}}| + C N \|X - \tilde{X}\|_\infty + C (\|u_h - u_{\tilde{h}}\|_{L^1(\Omega)} + \|u_h - u_{\tilde{h}}\|_{L^1(\Omega)}) 
\]

\[
\leq C |\alpha_h^{\frac{1}{2}} - \alpha_{\tilde{h}}^{\frac{1}{2}}| + C N \|X - \tilde{X}\|_\infty, 
\]

which, together with (80), gives (79).

**Lemma 6.5.** Under the assumptions of Lemma 6.3, \( G_N \) is a continuous map from \( S_N \) into \( S_N \):

\[
\|G_N X - G_N \tilde{X}\|_\infty \leq C N \|X - \tilde{X}\|_\infty + C (N \|X - \tilde{X}\|_\infty)^{\frac{1}{2}}. 
\]

**Proof.** Let \( Y = G_N X \) and \( \tilde{Y} = G_N \tilde{X} \). From the equidistribution relation (76), we obtain

\[
\int_0^{y_i} \rho dx - \int_0^{\tilde{y}_i} \tilde{\rho} dx = \frac{i}{N} (\sigma_h - \sigma_{\tilde{h}}) 
\]

or

\[
\int_0^{y_i} \rho dx = \frac{i}{N} \int_0^1 (\rho - \tilde{\rho}) dx + \int_0^{\tilde{y}_i} (\tilde{\rho} - \rho) dx. 
\]

It follows from Lemmas 6.1 and 6.4 that

\[
\left| y_i - \tilde{y}_i \right| \leq \int_{\tilde{y}_i}^{y_i} \left| \rho - \tilde{\rho} \right| dx 
\]

\[
\leq 2 \int_0^1 \left| \rho - \tilde{\rho} \right| dx 
\]
\[ \leq CN\|X - \tilde{X}\|_\infty + C(N\|X - \tilde{X}\|_\infty)^{\frac{1}{3}}, \]

which gives (83).

The term involving \((N\|X - \tilde{X}\|_\infty)^{\frac{1}{3}}\) in the above lemma can be dropped for a situation shown in the following lemma.

**Lemma 6.6.** Assume that the assumptions of Lemma 6.4 hold. If further \(r_0 = \min_{x \in \Omega} |r(x)| > 0\) and \(N\) is sufficiently large, then \(G_N : S_N \to S_N\) is a continuous map satisfying

\[ \|G_N X - G_N \tilde{X}\|_\infty \leq C N\|X - \tilde{X}\|_\infty. \]  

**Proof.** Recall from Lemma 5.3 that

\[ \|e_h\|_{L^\infty(\Omega)} \leq C \frac{N^{3/2}}{N}, \quad \|e'_h\|_{L^\infty(\Omega)} \leq C \sqrt{N}. \]

There exists a sufficiently large \(N\) such that

\[ |r_h(x)| = |f(x) - (b - a')w_h(x) - cu_h(x)| = |r(x) + (b - a')e_h(x) + ce_h(x)| \geq r_0 - (b - a')\|e_h\|_{L^\infty(\Omega)} - \bar{c}\|e_h\|_{L^\infty(\Omega)} \geq \frac{1}{2}r_0, \quad \forall x \in \Omega. \]

From this and Lemma 6.3, we can estimate \(|\alpha^{1/3}_h - \alpha^{1/3}_{\tilde{h}}|\) as

\[ |\alpha^{1/3}_h - \alpha^{1/3}_{\tilde{h}}| \leq \sum_{i=1}^{N} \left| \tilde{h}_i - \tilde{h}_i \right| \left| \langle r_h \rangle_i - \langle r_{\tilde{h}} \rangle_i \right| \leq C N\|X - \tilde{X}\|_\infty + \sum_{i=1}^{N} \tilde{h}_i \frac{|\langle r_h \rangle_i^2 - \langle r_{\tilde{h}} \rangle_i^2|}{r_h^2(\xi_i) + r_{\tilde{h}}^2(\xi_i) + \frac{2}{3}r_{\tilde{h}}^3(\xi_i)} \leq C N\|X - \tilde{X}\|_\infty, \]

where \(\xi_i \in K_i, \quad \tilde{\xi}_i \in \tilde{K}_i\). Combining (86) with (82) gives

\[ \int_0^1 |\rho - \tilde{\rho}| dx \leq C|\alpha^{1/3}_h - \alpha^{1/3}_{\tilde{h}}| + C N\|X - \tilde{X}\|_\infty \leq C N\|X - \tilde{X}\|_\infty. \]  

Finally, by (87) and (84) we obtain

\[ \|Y - \tilde{Y}\|_\infty \leq 2 \int_0^1 |\rho - \tilde{\rho}| dx \leq CN\|X - \tilde{X}\|_\infty, \]

which gives (85).
Theorem 6.1. (Existence of equidistributing meshes) Assume that $f \in L^\infty(\Omega)$ and $u \in H^2(\Omega)$. For sufficiently large $N$ (i.e., $N \geq N_0$ where $N_0$ is defined in Lemma 6.1), there exists at least an equidistributing mesh satisfying (7).

Proof. From Lemmas 6.2 and 6.5 we see that $G_N$ is a continuous map from $S_N$ to $S_N$. Recall that $S_N$ is a closed, convex set. By Brouwer’s theorem, $G_N$ has at least a fixed point in $S_N$. Since any fixed point of $G_N$ is an equidistributing mesh, we have proven that an equidistributing mesh exists and is in $S_N$. □

7 Conclusions

In the previous sections we have studied the linear FE solution of elliptic differential equations in the form (2) using an equidistributing mesh. The mesh is determined using the equidistribution principle (7) and the monitor function (15), with the latter being defined in terms of the residual of the FE solution. The procedure is completely a posteriori and uses no prior knowledge about the solution of the continuous problem. As a consequence, a coupled system of the FE equation (4) and the mesh equation (7) has to be solved for the FE solution and the mesh.

A major effort of this work has been to prove the existence of the equidistributing mesh for a sufficiently large $N$. The result is stated in Theorem 6.1. One of the keys in the proof is to show that the FE solution depends on the mesh continuously; see Theorem 5.1. The error bounds for the FE solution obtained with an equidistributing mesh or a quasi-equidistributing mesh are given in Theorems 3.2 and 3.3. Theorem 3.3 also shows that it is unnecessary to compute the equidistribution relation exactly for the equidistributing mesh. A quasi-equidistributing mesh with a modest value of $\kappa$ (cf. (8)) can still give an accurate FE solution.

An iterative algorithm was given in §4 for solving the coupled system of the FE and mesh equations for the mesh and the FE solution. Numerical examples show that it converges when $N$ is large enough and converges faster for larger $N$. We were unable to show the convergence of the iterative algorithm theoretically, which could be an interesting topic to pursue in the future.

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References

[1] M. Ainsworth and J. T. Oden. A posteriori error estimation in finite element analysis. Pure and Applied Mathematics (New York). Wiley-Interscience [John Wiley & Sons], New York, 2000.

[2] I. Babuška and W. C. Rheinboldt. Analysis of optimal finite-element meshes in $R^1$. Math. Comput., 33:435–463, 1979.
[3] I. Babuška and T. Strouboulis. *The Finite Element Method and Its Reliability*. Oxford Science Publication, New York, 2001. Numer. Math. Sci. Comput.

[4] R. E. Bank and A. Weiser. Some a posteriori error estimators for elliptic differential equations. *Math. Comput.*, 44:283–301, 1985.

[5] R. Becker and R. Rannacher. An optimal control approach to a posteriori error estimation in finite element methods. *Acta Numer.*, 10:1–102, 2001.

[6] G. Beckett and J. A. Mackenzie. Convergence analysis of finite-difference approximations on equidistributed grids to a singularly perturbed boundary value problems. *J. Comput. Appl. Math.*, 35:109–131, 2000.

[7] G. Beckett and J. A. Mackenzie. On a uniformly accurate finite difference approximation of a singularly perturbed reaction-diffusion problem using grid equidistribution. *J. Comput. Appl. Math.*, 131:381–405, 2001.

[8] G. Beckett and J. A. Mackenzie. Uniformly convergent high order finite element solutions of a singularly perturbed reaction-diffusion equation using mesh equidistribution. *Appl. Numer. Math.*, 39:31–45, 2001.

[9] P. Binev, W. Dahmen, and R. DeVore. Adaptive finite element methods with convergence rates. *Numer. Math.*, 97:219–268, 2004.

[10] H. Borouchaki, P. L. George, P. Hecht, P. Laug, and E. Saletl. Delaunay mesh generation governed by metric specification: Part I. algorithms. *Fin. Elem. Anal. Des.*, 25:61–83, 1997.

[11] S. C. Brenner and L. R. Scott. *The Mathematical Theory of Finite Element Methods*. Springer-Verlag, New York, 1994.

[12] H. G. Burchard. Splines (with optimal knots) are better. *Appl. Anal.*, 3:309–319, 1974.

[13] C. Carstensen and H. W. Hoppe. Error reduction and convergence for an adaptive mixed finite element methods. *Math. Comput.*, 75:1033–1042, 2006.

[14] J. M. Cascon, C. Kreuzer, R. H. Nochetto, and K. G. Siebert. Quasi-optimal convergence rate for an adaptive finite element method. *SIAM J. Numer. Anal.*, 46:2524–2550, 2008.

[15] L. Chen, M. Holst, and J. Xu. Convergence and optimality of adaptive mixed finite element methods. *Math. Comput.*, 78:35–53, 2009.

[16] L. Chen, P. Sun, and J. C. Xu. Optimal anisotropic meshes for minimizing interpolation errors in $L^p$-norm. *Math. Comput.*, 76:179–204, 2007.

[17] L. Chen and J. C. Xu. Stability and accuracy of adapted finite element methods for singularly perturbed problems. *Numer. Math.*, 109:167–191, 2008.

[18] P. G. Ciarlet. *The Finite Element Method for Elliptic Problems*. North-Holland, Amsterdam, 1978.
[19] C. de Boor. Good approximation by splines with variable knots. In A. Meir and A. Sharma, editors, *Spline Functions and Approximation Theory*, pages 57–73, Basel und Stuttgart, 1973. Birkhäuser Verlag.

[20] C. de Boor. Good approximation by splines with variables knots II. In G. A. Watson, editor, *Lecture Notes in Mathematics 363*, pages 12–20, Berlin, 1974. Springer-Verlag. Conference on the Numerical Solution of Differential Equations, Dundee, Scotland, 1973.

[21] W. Dörfler. A convergent adaptive algorithm for Poisson’s equation. *SIAM J. Numer. Anal.*, 33:1106–1124, 1996.

[22] D. S. Dodson. Optimal order approximation by polynomial spline functions. Technical report, Purdue University, 1972. Ph.D. thesis.

[23] W. Huang. Practical aspects of formulation and solution of moving mesh partial differential equations. *J. Comput. Phys.*, 171:753–775, 2001.

[24] W. Huang. Variational mesh adaptation: isotropy and equidistribution. *J. Comput. Phys.*, 174:903–924, 2001.

[25] W. Huang. Measuring mesh qualities and application to variational mesh adaptation. *SIAM J. Sci. Comput.*, 26:1643–1666, 2005.

[26] W. Huang. Metric tensors for anisotropic mesh generation. *J. Comput. Phys.*, 204:633–665, 2005.

[27] W. Huang and W. Sun. Variational mesh adaptation II: error estimates and monitor functions. *J. Comput. Phys.*, 184:619–648, 2003.

[28] N. Kopteva and M. Stynes. A robust adaptive method for a quasi-linear one-dimensional convection-diffusion problem. *SIAM J. Numer. Anal.*, 39:1446–1467, 2001.

[29] J. Mackenzie. Uniform convergence analysis of an upwind finite-difference approximation of a convection-diffusion boundary value problem on an adaptive grid. *IMA J. Numer. Anal.*, 19:233–249, 1999.

[30] D. E. McClure. Convergence of segmented approximations of smooth functions on a bounded interval. *AMS Notices*, 17:252, abstract 672–584, 1970.

[31] P. Morin, R. H. Nochetto, and K. G. Siebert. Data oscillation and convergence of adaptive FEM. *SIAM J. Numer. Anal.*, 38:466–488, 2000.

[32] P. Morin, R. H. Nochetto, and K. G. Siebert. Convergence of adaptive finite element methods. *SIAM J. Numer. Anal.*, 44:631–658, 2002.

[33] F. Moukalled and S. Acharya. A local adaptive grid procedure for incompressible flows with multigrid and equidistribution concepts. *Int. J. Numer. Meth. Fluids*, 13:1085–1111, 1991.

[34] V. Pereyra and E. G. Sewell. Mesh selection for discrete solution of boundary problems in ordinary differential equations. *Numer. Math.*, 23:261–268, 1975.
[35] J. D. Pryce. On the convergence of iterated remeshing. *IMA J. Numer. Anal.*, 9:315–335, 1989.

[36] Y. Qiu and D. M. Sloan. Analysis of difference approximations to a singularly perturbed two-point boundary value problem on an adaptively generated grid. *J. Comput. Appl. Math.*, 101:1–25, 1999.

[37] Y. Qiu, D. M. Sloan, and T. Tang. Numerical solution of a singularly perturbed two-point boundary value problem using equidistribution: analysis of convergence. *J. Comput. Appl. Math.*, 116:121–143, 2000.

[38] J. R. Rice. On the degree of convergence of nonlinear spline approximation. In I. J. Schoenberg, editor, *Approximations with Special Emphasis on Spline Functions*, pages 349–365, New York, London, 1969. Academic Press.

[39] J. Sacks and D. Ylvisaker. Designs for regression problems with corrected errors. *Ann. Math. Stat.*, 37:66–89, 1966.

[40] J. Sacks and D. Ylvisaker. Designs for regression problems with corrected errors; many parameters. *Ann. Math. Stat.*, 39:49–69, 1968.

[41] J. Sacks and D. Ylvisaker. Designs for regression problems with corrected errors III. *Ann. Math. Stat.*, 41:2057–2074, 1970.

[42] R. Stevenson. An optimal adaptive finite element method. *SIAM J. Numer. Anal.*, 42:2188–2217, 2005.

[43] R. Verfürth. *A Review of A-Posteriori Error Estimation and Adaptive Mesh Refinement Techniques*. John Wiley and Teubner, Germany, 1996. Advances in Numerical Mathematics.

[44] A. B. White Jr. On selection of equidistributing meshes for two-point boundary-value problems. *SIAM J. Numer. Anal.*, 16:472–502, 1979.

[45] X. Xu, W. Huang, R. D. Russell, and J. F. Williams. Convergence of de Boor’s algorithm for generation of equidistributing meshes. *IMA J. Numer. Anal.* 31, 558–596, 2011.

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