Anisotropic Spherical Solutions through Extended Gravitational Decoupling Approach

M. Sharif *and Qanitah Ama-Tul-Mughani †
Department of Mathematics, University of the Punjab, Quaid-e-Azam Campus, Lahore-54590, Pakistan.

Abstract

This paper is devoted to evaluating exact anisotropic spherical solutions for static self-gravitating systems through extended geometric deformation decoupling technique. For this purpose, we consider an isotropic Tolman IV solution and extend it to anisotropic domain by transforming both temporal as well as radial metric potentials. To examine the physical viability and stability of interior anisotropic solutions, we plot energy bounds, TOV equation, causality condition and adiabatic index for the stars Her X-I and PSR J 1416-2230. It is found that both obtained models show realistic behavior as they fulfill all physical constraints as well as stability criterion. We conclude that the extended gravitational decoupling approach provides more proficient results to discuss the interior configuration of stellar structures.

Keywords: Extended geometric deformation; Anisotropy; Exact solutions.
PACS: 04.20.-q; 04.20.Jb; 04.40.Dg.

*msharif.math@pu.edu.pk
†qanitah94@gmail.com
1 Introduction

General relativity as the geometric theory of gravitation provides an elementary insight to the salient features of self-gravitating objects. In astrophysics, the formulation of exact solutions of Einstein field equations describe the interior distribution of stellar structures. Schwarzschild [1] found the spherical vacuum solution which specifies the exterior region of perfect matter distribution. Tolman [2] computed several solutions for perfect fluid in the presence of cosmological constant and investigated smooth matching conditions of interior spacetime with the exterior one. Lemaitre [3] proposed that anisotropy in the interior of celestial objects may arise due to phase transition, rotational motion, presence of magnetic field or mixture of two fluids, etc. To study the prominent features of anisotropic fluid distribution, many physically acceptable solutions have been evaluated in relativistic theories of gravity [4]-[10].

The study of exact interior anisotropic solutions is a difficult task due to non-linear characteristics of the evolution equations. Over the years, several techniques have been proposed to obtain physically acceptable models of celestial objects. In this context, the gravitational decoupling through minimal geometric deformation (MGD) approach has been used to formulate such analytical solutions in cosmology as well as astrophysics. This approach was first proposed by Ovalle [11] to derive new consistent solutions for static spherically symmetric spacetime in the background of astrophysics braneworld. The MGD technique applies deformation on the radial metric functions and splits the field equations in two sets of differential equations that are convenient to solve as compared to the original system. To attain the solution of the complete model, the results of the decoupled equations are combined using the principle of superposition. Following this procedure, Ovalle and Linares [12] computed an exact anisotropic solution for compact spherical distribution and concluded that the model corresponds to braneworld version of Tolman IV solution. Later, Contreras and Bargueño [13] applied this method on 1+2 static circularly symmetric spacetime and evaluated anisotropic solutions from the static BTZ model.

Ovalle [14] used the systematic approach of MGD to decouple gravitational sources and obtained anisotropic spherical solutions from the perfect fluid configuration. Ovalle et al. [15] included the effects of anisotropy in corresponding isotropic interior solution and derived three exact anisotropic models from Tolman IV solution by means of MGD technique. Sharif and
Sadiq [16] extended the singularity-free Krori-Barua solution to anisotropic domain through this technique in the presence of the electromagnetic field. In the same perspective, Gabbanelli et al. [17] constructed new anisotropic solutions by taking Durgapal-Fluria isotropic fluid as an interior of stellar system. Graterol [18] applied MGD decoupling phenomenon to generate anisotropic analytic solutions from Buchdahl perfect fluid model for static spherical self-gravitating system. Sharif and collaborators [19, 20] obtained viable anisotropic solutions through this procedure in modified gravity. Sharif and Ama-Tul-Mughani [21] computed exact charged isotropic as well as anisotropic solutions in a cloud of strings. Recently, Casadio et al. [22] used this method to continuously isotropize the anisotropic solution with vanishing complexity factor for the static sphere.

The MGD technique assists to study the essential characteristics of stellar structures but it works under some limitations, e.g., the geometric deformation can only be performed as long as the interaction between the matter sources is purely gravitational. Probably, its main drawback is that the transformation endures by the metric component is minimal, i.e., it only modifies the radial coordinate by leaving temporal metric potential as an invariant quantity which may lead to certain shortcomings in the decoupling phenomenon. To overcome this issue, Casadio et al. [23] introduced an extended version of MGD approach by employing the deformation on both temporal as well as radial metric functions and obtained a new solution for spherically symmetric spacetime. However, this extension can only study the vacuum solutions as the conservation law no longer holds in the presence of matter. Consequently, the interior structure as well as intrinsic properties of self-gravitating objects cannot be discussed through this extended approach. Ovalle [24] proposed a novel idea of extended geometric deformation (EGD) by modifying both metric potentials ($g_{tt}, g_{rr}$) which remains valid for entire spacetime without depending upon the choice of matter distribution. Contreras and Bargueño [25] successfully decoupled the field equations in 1+2-dimensional gravity through EGD technique and implemented it to obtain exterior charged BTZ model from the corresponding vacuum solution.

The aim of this paper is to study the most general extension of MGD decoupling approach in the context of perfect spherical geometry. We derive two exact anisotropic models from the well-known isotropic Tolman IV solution by employing this extended decoupling phenomenon. The paper is organized as follows. The next section provides the basics of EGD approach and deals with the decoupling of the field equations. In section 3, we com-
pute anisotropic solutions by implying some physical constraints on the new gravitational source. Section 4 is devoted to discussing the stability of the resulting models and the final comments are summarized in the last section.

2 Gravitational Decoupled Field Equations

The interior of spherically symmetric object in Schwarzschild coordinates \((x^\alpha) = (t, r, \theta, \phi)\) can be expressed as

\[
ds^2 = -e^{\eta} dt^2 + e^{\chi} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2),
\]

where \(\eta\) and \(\chi\) have dependence on radial coordinate \(r\) which varies from center to the boundary of star, i.e., \((0 \leq r \leq R)\). The energy-momentum tensor describing the internal configuration of the stellar structure is taken as

\[
\tilde{T}^\beta_\alpha = T^\beta_\alpha + \sigma \vartheta^\beta_\alpha,
\]

such that

\[
T^\beta_\alpha = (\rho + p) u_\alpha u^\beta + p \delta^\beta_\alpha,
\]

where \(\rho\) stands for energy density, \(p\) for pressure and \(u_\alpha\) corresponds to four velocity of the fluid. The factor \(\vartheta^\beta_\alpha\) describes an additional source which is gravitationally coupled to the perfect fluid by a free parameter \(\sigma\). This source term generally produces anisotropy in the stellar objects by incorporating new scalar, vector or tensor fields in the respective model. The Einstein field equations for non-generic gravitational sources explicitly read

\[
T^0_0 + \sigma \vartheta^0_0 = -\frac{1}{r^2} - e^{-\chi} \left( \frac{\chi'}{r} - \frac{1}{r^2} \right),
\]

\[
T^1_1 + \sigma \vartheta^1_1 = -\frac{1}{r^2} + e^{-\chi} \left( \frac{\eta'}{r} + \frac{1}{r^2} \right),
\]

\[
T^2_2 + \sigma \vartheta^2_2 = e^{-\chi} \left( 2\eta'' + 2\eta'^2 - \eta' \chi' + \frac{2}{r} (\eta' - \chi') \right).
\]

where prime means derivative with respect to \(r\).

The conservation equation corresponding to metric (1) turns out to be

\[
p' + \frac{\eta'}{2} (\rho + p) + \sigma \left( \vartheta^1_1 + \frac{\eta'}{2} (\vartheta^1_1 - \vartheta^0_0) + \frac{2}{r} (\vartheta^1_1 - \vartheta^2_2) \right) = 0.
\]
Through direct analysis, the matter components can be defined as

\[ \hat{\rho} = \rho - \sigma \vartheta_0^0, \quad \hat{\rho}_r = p + \sigma \vartheta_1^1, \quad \hat{\rho}_t = p + \sigma \vartheta_2^2, \]

where \( \hat{\rho} \) is the effective density, \( \hat{\rho}_r \) and \( \hat{\rho}_t \) denote the effective radial and tangential pressures, respectively. The anisotropy in the interior of celestial objects takes the form

\[ \hat{\Delta} = \sigma (\vartheta_2^2 - \vartheta_1^1), \]

which indicates that the insertion of new source term \( \vartheta_\alpha^\beta \) produces anisotropy in self-gravitating objects. Now, we have a system of three equations (4)-(6) with seven unknowns, i.e., two metric components \( \chi, \eta \) and five matter variables \( \rho, p, \vartheta_0^0, \vartheta_1^1, \vartheta_2^2 \). To compute the exact solution of the prescribed model, we use a novel approach named as EGD and evaluate these unknown functions.

### 2.1 Extended Geometric Decoupling Approach

To solve the system of non-linear differential equations (4)-(6), we implement the EGD technique which transforms the field equations associated with source \( \vartheta_\alpha^\beta \) into an “effective quasi-Einstein system”. We start by considering a known perfect fluid solution (\( \vartheta_\alpha^\beta = 0 \)) for the metric

\[ ds^2 = -e^{\xi(r)} dt^2 + e^{\mu(r)} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \]

where \( e^{-\mu(r)} = 1 - \frac{2m(r)}{r} \) with the Misner-Sharp mass \( m(r) \). To analyze the impact of \( \vartheta_\alpha^\beta \) on \( T_{\alpha \beta}^\alpha \), we consider the non-zero values of free parameter and observe the following geometric deformations on the metric functions, namely,

\[ e^{-\mu} \rightarrow e^{-\chi} = e^{-\mu} + \sigma g(r), \quad \xi \rightarrow \eta = \xi + \sigma h(r), \]

where \( g(r) \) and \( h(r) \) denote the deformation functions subject to the radial and temporal coordinates, respectively. We substitute the above decompositions in Eqs.(4)-(6) to split them into two set of differential equations. The first set contains the standard field equations for \( T_{\alpha \beta}^\alpha \) given by

\[ \rho = \frac{1}{r^2} + e^{-\mu} \left( \frac{\mu'}{r} - \frac{1}{r^2} \right), \]
\( p = -\frac{1}{r^2} + e^{-\mu} \left( \frac{\xi'}{r} + \frac{1}{r^2} \right), \)  
(13)

\( p = e^{-\mu} \left( \frac{\xi''}{2} + \frac{\xi'^2}{4} - \frac{\xi' \mu'}{4} + \frac{1}{2r}(\xi' - \mu') \right), \)  
(14)

while the second set consists of evolution equations for \( \vartheta^\beta_{\alpha} \)

\( \vartheta^0_{0} = \frac{g}{r^2} + \frac{g'}{r}, \)  
(15)

\( \vartheta^1_{1} = g \left( \frac{1}{r^2} + \frac{\eta'}{r} \right) + e^{-\mu} \frac{h'}{r}, \)  
(16)

\( \vartheta^2_{2} = \frac{g}{4} \left( 2\eta'' + \eta'^2 + \frac{2\eta'}{r} \right) + \frac{g'}{4} \left( \eta' + \frac{2}{r} \right) + e^{-\mu} \frac{1}{4} \left( 2h'' + \sigma h'^2 + \frac{2h'}{r} \right) + 2(\xi' - \mu' \h'). \)  
(17)

The continuity equation, \( \tilde{T}^\beta_{\alpha,\beta} = 0, \) for the line-element (10) yields

\[ p' + \frac{\xi'}{2}(\rho + p) + \frac{\sigma h'}{2}(\rho + p) + \sigma \left( \vartheta^1_{1} + \frac{\eta'}{2}(\vartheta^1_{1} - \vartheta^0_{0}) + \frac{2}{r}(\vartheta^1_{1} - \vartheta^2_{2}) \right) = 0. \]  
(18)

Setting \( h'(r) = 0 \) in the above identity leads to the specific case of MGD technique and the system of equations (15)-(17) reduce to the quasi-Einstein system [14, 26].

Now, we study the critical behavior of conservation equation to obtain a successful decoupling approach. Since the Bianchi identity for the perfect fluid configuration remains preserved for the metric \((\xi, \mu)\), so that

\[ \nabla_{(\xi,\mu)} T^\alpha_{\beta} = 0, \]  
(19)

whereas the conservation of \( T^\alpha_{\beta} \) corresponding to metric (11) takes the form

\[ \nabla_{\alpha} T^\alpha_{\beta} = \nabla_{(\xi,\mu)} T^\alpha_{\beta} + \frac{\sigma h'}{2}(\rho + p)\delta^1_{\beta}. \]  
(20)

Using Eq. (19) in the above expression gives

\[ \nabla_{\alpha} T^\alpha_{\beta} = \frac{\sigma h'}{2}(\rho + p)\delta^1_{\beta}, \]  
(21)
while the divergence of the gravitational source $\vartheta^\alpha_\beta$ reads
\[
\nabla_\alpha \vartheta^\alpha_\beta = -\frac{h'}{2}(\rho + p)\delta^1_\beta,
\]
which can be expressed as the linear combination of Eqs. (15)-(17), so we have
\[
\vartheta_1' + \frac{\eta^1}{2}(\vartheta_1^1 - \vartheta_0^0) + \frac{2}{r}(\vartheta_1^1 - \vartheta_2^2) = -\frac{h'}{2}(\rho + p).
\]
(23)

From the above identities (21) and (22), we can conclude that matter sources $T^{\alpha}_{\beta}$ and $\vartheta^\alpha_\beta$ can be decoupled successfully until the energy is able to transform from one source to another. However, the MGD approach allows a purely gravitational interaction between matter sources by restricting the exchange of energy between them. It is worthwhile to mention here that EGD approach can also work without exchange of energy in merely two scenarios: either the fluid is barotropic, i.e., $T^0_0 = T^1_1$ or the isotropic sector represents the vacuum solution, i.e., $T^\alpha_\beta = 0$.

The equations of motion for anisotropic source (15)-(17) can be identified as the field equations for anisotropic conserved tensor $\vartheta^*_{\alpha \beta}$ defined by
\[
\vartheta^*_{\alpha \beta} = \vartheta^\alpha_\beta - \frac{1}{r^2}\delta^0_\beta\delta^\alpha_0 - \left(X_1 + \frac{1}{r^2}\right)\delta^1_\beta\delta^0_1 - X_2\delta^2_\beta\delta^0_2,
\]
(24)
which is explicitly written as
\[
\begin{align*}
\vartheta^{*0} &= \vartheta^0 - \frac{1}{r^2}, \\
\vartheta^{*1} &= \vartheta^1 - \left(X_1 + \frac{1}{r^2}\right), \\
\vartheta^{*2} &= \vartheta^2 - X_2,
\end{align*}
\]
(25) (26) (27)
with
\[
X_1 = \frac{e^{-\mu}h'}{r}, \quad X_2 = \frac{e^{-\mu}}{4} \left(2h'' + \sigma h'^2 + \frac{2h'}{r} + 2\xi' h' - \mu' h'\right).
\]
In this scenario, the continuity equation turns out to be
\[
(\vartheta^{*1})' + \frac{\eta^1}{2}(\vartheta^{*1} - \vartheta^{*0}) + \frac{2}{r}(\vartheta^{*1} - \vartheta^{*2}) = 0,
\]
(28)
with the metric
\[
ds^2 = -e^\eta dt^2 + \frac{dr^2}{g(r)} + r^2(d\theta^2 + \sin^2 \theta d\phi^2).
\]
(29)
3 Anisotropic Solutions

To derive the anisotropic solutions for the interior of self-gravitating system, we solve Eqs. (15)-(17) through EGD approach. In this regard, we first turn off the free parameter and consider a known solution for perfect matter distribution in the background of spherically symmetric spacetime. We choose a well-known Tolman IV model for perfect fluid given by \[27\]

\[e^\xi = B^2(1 + \frac{r^2}{A^2}), \quad (30)\]

\[e^\mu = \frac{1 + 2r^2}{(1 - \frac{r^2}{A^2})(1 + \frac{r^2}{A^2})}, \quad (31)\]

\[\rho = \frac{3A^4 + A^2(3C^2 + 7r^2) + 2r^2(C^2 + 3r^2)}{C^2(A^2 + 2r^2)^2}, \quad (32)\]

\[p = \frac{C^2 - A^2 - 3r^2}{C^2(A^2 + 2r^2)}. \quad (33)\]

The integration constants \(A, B\) and \(C\) are evaluated by the smooth matching of interior and exterior spacetimes. Here, we choose Schwarzschild as an exterior metric which yields

\[A^2 = \frac{R^2(R - 3M)}{M}, \quad B^2 = 1 - \frac{3M}{R}, \quad C^2 = \frac{R^3}{M}, \quad (34)\]

with the compactness factor \(\frac{M}{2R} = \frac{2}{9}\). The above expressions ensure the continuity of isotropic solution with the exterior geometry at the boundary \((r = R)\). However, the inclusion of gravitational source \(\vartheta_{\alpha \beta}\) in the perfect fluid distribution will modify the respective quantities, accordingly.

To obtain the anisotropic solutions, we take \(\sigma \neq 0\) in the interior of spherical object and solve the equations of motion (15)-(17). These equations interlink the five unknown functions, namely, geometric deformations \((h(r), g(r))\) and source term \(\vartheta_{\alpha \beta}\). Hence, we need some additional constraints to minimize the number of unknowns. For this purpose, we impose an equation of state and two physical constraints on the components of \(\vartheta_{\alpha \beta}\) to obtain exact as well as viable models in the following subsections.
3.1 Solution I

In order to compute exact anisotropic solution, we implement a linear equation of state on \( \vartheta^{\alpha} \)

\[
\vartheta^{0}_0 = \delta \vartheta^1_1 + \gamma \vartheta^2_2,
\]

and apply a constraint on \( \vartheta^1_1 \) to close the system. From the junction conditions of Schwarzschild spacetime with the interior geometry, we get \( p(R) \sim -\sigma(\vartheta^1_1(R))_\sim \). Thus the suitable choice is taken to be

\[
\vartheta^1_1 = -p.
\]

Here, we set \( \delta = 1 \) and \( \gamma = 0 \) which lead to the relation \( \vartheta^{0}_0 = \vartheta^1_1 \). Using the field Eqs. (15) and (16), the deformation functions are evaluated as

\[
g = \frac{-2r (A^2 + 2C^2 - 2r^2) + \sqrt{2}A (A^2 + 2C^2) \tan^{-1} \left( \frac{\sqrt{2}r}{A} \right)}{8C^2 r},
\]

\[
h = \int \frac{1}{\varepsilon} \left( 2r (A^6 + 8C^2 r^4 + 6A^2 r^2 (C^2 + 2r^2) + A^4 (2C^2 + 7r^2) \right) - \sqrt{2}A \\
\times (A^2 + 2C^2) (A^4 + 5A^2 r^2 + 6r^4) \tan^{-1} \left( \frac{\sqrt{2}r}{A} \right) \right) dr,
\]

where

\[
\varepsilon = r \left( A^2 + r^2 \right) (8(C - r)r(C + r) (A^2 + r^2) + (A^2 + 2r^2) \sigma(4r^3) \\
- 2 (A^2 + 2C^2) r + \sqrt{2}A (A^2 + 2C^2) \tan^{-1} \left( \frac{\sqrt{2}r}{A} \right) )
\]

which yield the temporal as well as radial metric components as

\[
\eta = \ln \left( B^2 (1 + \frac{r^2}{A^2}) \right) + \sigma \int \frac{1}{\varepsilon} \left( 2r (A^6 + 8C^2 r^4 + 6A^2 r^2 (C^2 + 2r^2) \\
+ A^4 (2C^2 + 7r^2) \right) - \sqrt{2}A (A^2 + 2C^2) (A^4 + 5A^2 r^2 + 6r^4) \\
\times \tan^{-1} \left( \frac{\sqrt{2}r}{A} \right) \right) dr,
\]

\[
e^{-\chi} = \frac{(1 - \frac{r^2}{C^2})(1 + \frac{r^2}{A^2})}{1 + \frac{2r^2}{A^2}} + \frac{\sigma}{8C^2 r} \left( \sqrt{2}A (A^2 + 2C^2) \tan^{-1} \left( \frac{\sqrt{2}r}{A} \right) \right)
\]
\[ -2r (A^2 + 2C^2 - 2r^2). \]

Notice that for \( \sigma = 0 \), the above equations reduce to the standard Tolman IV spherical solution for perfect matter distribution.

To measure the effects of anisotropy on the constants \((A, B, C)\), we evaluate their expressions through matching conditions. The continuity of the first fundamental form leads to the following identities

\[
\ln \left( 1 - \frac{2M}{R} \right) = \ln \left( B^2 (1 + \frac{R^2}{A^2}) \right) + \sigma \left( \int \frac{1}{\varepsilon} \left( 2r (A^6 + 8C^2r^4 + 6A^2r^2 \right.ight.
\times (C^2 + 2r^2) + A^4 (2C^2 + 7r^2) - \sqrt{2A} (A^2 + 2C^2)
\times (A^4 + 5A^2r^2 + 6r^4) \tan^{-1} \left( \frac{\sqrt{2r}}{A} \right) \right) \right) \bigg|_{r=R} dr,
\]

\[
1 - \frac{2M}{R} = \left( 1 - \frac{\frac{R^2}{A^2}}{1 + \frac{2R^2}{Ax^2}} \right) + \frac{\sigma}{8C^2 R} (\sqrt{2A} (A^2 + 2C^2) \tan^{-1} \left( \frac{\sqrt{2R}}{A} \right)
- 2R (A^2 + 2C^2 - 2R^2)),
\]

whereas the continuity of second fundamental form \((p(R) + \sigma (\psi_1'(R)) = 0)\) yields
\[
p(R) = 0 \quad \Rightarrow \quad C^2 = 3R^2 + A^2.
\]

These equations \((41)-(43)\) are the necessary and sufficient conditions for the smooth matching of exterior and interior spacetimes. In the case of pressure-like constraint, the matter variables are evaluated as

\[
\dot{\rho} = \frac{3A^4 + A^2 (3C^2 + 7r^2) + 2r^2 (C^2 + 3r^2)}{C^2 (A^2 + 2r^2)} + \sigma \left( \frac{C^2 - A^2 - 3r^2}{C^2 (A^2 + 2r^2)} \right),
\]
\[
\dot{p} = \frac{C^2 - A^2 - 3r^2}{C^2 (A^2 + 2r^2)} (1 - \sigma),
\]
\[
\dot{p} = \frac{C^2 - A^2 - 3r^2}{C^2 (A^2 + 2r^2)} + \sigma (2r (16r^4 (C^4 - 3C^2r^2 (-1 + \sigma) + 3r^4 (-1 + \sigma) + A^8 (1 - 2\sigma) + 4A^4 (C^2 r^2 (21 - 8\sigma) + C^4 (-1 + \sigma) - r^4 (11 + \sigma) + 4A^2 r^2 (17 - 10\sigma) + 12r^4 (-2 + \sigma) + C^4 (1 + 2\sigma)) - A^6 (2C^2 (-6 + \sigma) + r^2 (1 + 12\sigma)) + \sqrt{2A} (A^2 + 2C^2) (A^2 + 2r^2) \times (- (A^2 + 2C^2) (A^2 + 3r^2) + 2 (A^4 + 6r^4 - A^2 (C^2 - 6r^2)) \sigma) \tan^{-1} (\frac{\sqrt{2r}}{A}))
\]

\[10\]
\[ \times \left( 2C^2 \left( A^2 + 2r^2 \right)^2 \left[ 8(C - r)r(C + r) (A^2 + r^2) + (A^2 + 2r^2) \right] \right) \times \sigma \left( -2 \left( A^2 + 2C^2 \right) r + 4r^3 + \sqrt{2}A \left( A^2 + 2C^2 \right) \tan^{-1} \left( \frac{\sqrt{2}r}{A} \right) \right)^{-1} \], (46)

\[ \hat{\Delta} = \left( A^2 + 2C^2 \right) \sigma \left( 2r\left( A^6 + 2A^2r^2 \left( 2r^2 + C^2(7 - 2\sigma) \right) \right) + A^4(2C^2 + r^2(7 - 2\sigma)) - 8r^4(C^2(-2 + \sigma) - r^2(-1 + \sigma)) \right) - \sqrt{2}A \left( A^2 + 2C^2 \right) \left( A^2 + 2r^2 \right) \left( A^2 + r^2(3 - 2\sigma) \right) \tan^{-1} \left( \frac{\sqrt{2}r}{A} \right) \times \left( 2C^2(A^2 + 2r^2)^2(8(C - r)r(C + r)(A^2 + r^2) + (A^2 + 2r^2) \right) \times \sigma \left( -2 \left( A^2 + 2C^2 \right) r + 4r^3 + \sqrt{2}A \left( A^2 + 2C^2 \right) \tan^{-1} \left( \frac{\sqrt{2}r}{A} \right) \right)^{-1} \]. (47)

The corresponding mass function takes the form

\[ m = 4\pi \int r^2 \hat{\rho} dr, \]

\[ = \pi (\sigma - 1) \left( \sqrt{2}A \left( A^2 + 2C^2 \right) \tan^{-1} \left( \frac{\sqrt{2}r}{A} \right) - 2r \left( A^2 + 2C^2 - 2r^2 \right) \right) \]

while the central density and pressure are given by

\[ \hat{\rho}_c = \frac{(1 - \sigma) \left( 3A^4 + 3A^2C^2 \right)}{A^4C^2}, \]

\[ \hat{p}_{rc} = \frac{(1 - \sigma) \left( C^2 - A^2 \right)}{A^2C^2}, \]

which have positive as well as finite characteristics for \( \sigma < 1 \).

We plot the above solution to analyze the physical characteristic of matter variables for the stars Her X-I (\( M = 1.25375 \text{ km}, \ R = 8.10 \text{ km} \)) and PSR J 1416-2230 (\( M = 2.90575 \text{ km}, \ R = 13 \text{ km} \)). Here, we choose \( A \) as a free parameter whose value is presented in isotropic solution (34) while the expression of \( C \) is taken from Eq. (43). For the feasible behavior of stellar structure, the energy density and pressures (\( \hat{\rho}_r, \hat{p}_t \)) must be positive, finite as well as maximum at the center of star. The profiles of energy density and anisotropic pressures are displayed in Figure 1 which indicate that density has the maximum value in the interior of compact star (\( r = 0 \)) with monotonically decreasing behavior towards the surface. The sketch of \( \hat{\rho}_r \) and \( \hat{p}_t \)
Figure 1: Plots of $\hat{\rho}$, $\hat{p}_r$, $\hat{p}_t$ and $\hat{\Delta}$ for Her X-I (green) and PSR J 1416-2230 (blue) for solution I.
observe the same physical trend as that of energy density and ultimately become zero at the boundary of the star. From the plot of anisotropic parameter (right plot, second row of Figure 1), one can easily investigate that the anisotropy vanishes at the center of self-gravitating object and attains a maximum towards its surface. Moreover, it is noted that the decoupling parameter $\sigma$ decreases the ranges of anisotropic pressures ($\hat{p}_r, \hat{p}_t$) whereas the density as well as anisotropy in the system slightly increases under its effects. Figure 2 indicates that the mass of stellar structure has direct relation with its radius whereas the decoupling parameter depicts the inverse scenario.

To measure the viability of developed solution, we examine the physical behavior of energy bounds. These bounds are some restrictions that are enforced on the energy-momentum tensor to identify the realistic fluid distribution. For anisotropic configuration, these conditions turn out to be

$$\hat{\rho} \geq 0, \quad \hat{\rho} + \hat{p}_r \geq 0, \quad \hat{\rho} + \hat{p}_t \geq 0, \quad \hat{\rho} - \hat{p}_r \geq 0, \quad \hat{\rho} - \hat{p}_t \geq 0, \quad \hat{\rho} + \hat{p}_r + 2\hat{p}_t \geq 0.$$ 

It is found that anisotropic model satisfies all energy conditions which ensure the viability of our constructed solution as shown in Figure 3.
Figure 3: Plots of energy bounds for Her X-I (green) and PSR J 1416-2230 (blue) for solution I.
3.2 Solution II

In this section, we attain an alternative form of anisotropic solution by employing a physically acceptable constraint on $\vartheta_0$. The particular form of density-like constraint is taken as

$$\vartheta_0 = \rho.$$  \hspace{1cm} (48)

With the help of Eqs. (12), (15) and (16), the deformation functions take the form

$$g = \frac{r^2 (A^2 + C^2 + r^2)}{C^2 (A^2 + 2r^2)}, \hspace{1cm} (49)$$

$$h = \frac{1}{2} \left( -\frac{2 \ln (A^2 + r^2)}{\sigma} - \frac{2 \ln (A^2 + 2r^2)}{-1 + \sigma} + (A^2 (-1 + \sigma) + C^2 (-1 + 3\sigma)) (-1 + 2\sigma) \ln (C^2 (1 + \sigma) - \rho + (A^2 + 2r^2) (-1 + \sigma)) (\rho \sigma (-1 + \sigma))^{-1} + C^2 (1 + \sigma) + \rho) ((-1 + \sigma) \sigma \rho) \right)^{-1}, \hspace{1cm} (50)$$

where

$$\sigma = \sqrt{A^2 (-1 + \sigma)^2 + 2A^2 C^2 (-1 + \sigma)^2 + C^4 (1 + \sigma)^2}.$$  

By using the same strategy as applied in solution I, the matching conditions are computed as

$$\ln \left( 1 - \frac{2M}{R} \right) = \ln \left( B^2 (1 + \frac{R^2}{A^2}) \right) + \frac{\sigma}{2} \left( -\frac{2 \ln (A^2 + r^2)}{\sigma} - \frac{2 \ln (A^2 + 2r^2)}{-1 + \sigma} \right) + (A^2 (-1 + \sigma) + C^2 (-1 + 3\sigma)) + (-1 + 2\sigma) \ln (C^2 (1 + \sigma) - \rho + (A^2 + 2r^2) (-1 + \sigma)) (\rho \sigma (-1 + \sigma))^{-1} + C^2 (1 - 3\sigma) - A^2 (-1 + \sigma)) \ln ((A^2 + 2r^2) (-1 + \sigma) + C^2 (1 + \sigma) + \rho) ((-1 + \sigma) \sigma \rho)^{-1}, \hspace{1cm} (51)$$

$$A = \frac{\sqrt{-4C^2 M R^2 + C^2 R^4 + R^6 - C^2 R^4 \sigma - R^6 \sigma}}{\sqrt{2C^2 M - R^3 + R^3 \sigma}}. \hspace{1cm} (52)$$

The expressions of fluid parameters $$(\hat{\rho}, \hat{p}_r, \hat{p}_t)$$ for anisotropic solution are evaluated as

$$\hat{\rho} = \frac{3A^4 + A^2 (3C^2 + 7r^2) + 2r^2 (C^2 + 3r^2)}{C^2 (A^2 + 2r^2)^2} (1 - \sigma), \hspace{1cm} (53)$$
\[ \hat{p}_r = \frac{C^2 - A^2 - 3r^2}{C^2(A^2 + 2r^2)} + \sigma \left( \frac{3A^4 + A^2(3C^2 + 7r^2) + 2r^2(C^2 + 3r^2)}{C^2(A^2 + 2r^2)^2} \right), \]  
(54)

\[ \hat{p}_t = \frac{C^2 - A^2 - 3r^2}{C^2(A^2 + 2r^2)} + \sigma \left( 3A^6 + 3A^4C^2 + 8A^4r^2 - 2A^2C^2r^2 + 18A^2r^4 + 12r^6 \right. 
+ \left. \frac{(A^2 + 2C^2)r^2 (A^6 + A^2r^4 - 2C^2r^4 + A^4(C^2 + 2r^2))}{(C - r)(C + r)(A^2 + r^2) + r^2(A^2 + C^2 + r^2)\sigma} \right) (C^2(A^2 + 2r^2)^3)^{-1}. \]  
(55)

In this scenario, the anisotropic factor becomes

\[ \hat{\Delta} = \left( (-A^2 + C^2 - 3r^2) (A^2 + 2r^2)^2 + \sigma \left( 3A^6 + 3A^4C^2 + 8A^4r^2 - 2A^2C^2r^2 \right. 
+ \left. 18A^2r^4 + 12r^6 + \frac{(A^2 + 2C^2)r^2 (A^6 + A^2r^4 - 2C^2r^4 + A^4(C^2 + 2r^2))}{(C - r)(C + r)(A^2 + r^2) + r^2(A^2 + C^2 + r^2)\sigma} \right) \right) \] 
\times \left( C^2(A^2 + 2r^2)^3 \right)^{-1}. \]  
(56)

We note that for \( \sigma = 0 \), the anisotropic factor vanishes and our model reduces to isotropic solution. The mass function, central density and pressure take the form

\[ m = -\frac{4\pi r^3(\sigma - 1) (A^2 + C^2 + r^2)}{C^2(A^2 + 2r^2)}, \]

\[ \hat{\rho}_c = \frac{(1 - \sigma) (3A^4 + 3A^2C^2)}{A^2C^2}, \]

\[ \hat{p}_{rc} = \frac{3\sigma + 1}{A^2} + \frac{3\sigma - 1}{C^2}. \]

To determine the physical characteristics of solution II, the expressions of \( C \) and \( A \) are taken from Eqs. (34) and (52), respectively. The graphical behavior of matter variables \( (\hat{\rho}, \hat{p}_r, \hat{p}_t, \hat{\Delta}) \) for the stars Her X-I \( (M = 1.25375\text{km}, R = 8.10\text{km}) \) and PSR J 1416-2230 \( (M = 2.90575\text{km}, R = 13\text{km}) \) are shown in Figure 4. It is found that the density as well as anisotropic pressures follow the pattern of solution I, i.e., they are finite, monotonically decreasing as well as regular. However, the impact of anisotropic factor is found to be negative which indicates that the anisotropic force leads the system towards more compact matter distribution. It is noted that this solution satisfies the viability criterion for smaller choices of \( \sigma \) as compared to solution I. For larger values of decoupling parameter, the plot
Figure 4: Plots of $\dot{\rho}$, $\dot{p}_r$, $\dot{p}_t$ and $\Delta$ for Her X-I (green) and PSR J 1416-2230 (blue) for solution II.
of tangential pressure becomes negative which exhibits the non-realistic behavior of the stellar structure. The consistency of mass function and energy bounds are displayed in Figures 5 and 6, respectively, which represent that the model fulfills all criteria of being physically acceptable solution.

4 Dynamical Equilibrium and Stability Constraints

This section investigates to determine the equilibrium state through Tolman-Oppenheimer-Volkoff (TOV) equation as well as to perform the stability criteria via speed of sound constraint and adiabatic index for the developed anisotropic models.

4.1 TOV Equation for Gravitationally Decoupled Models

Tolman [27], Oppenheimer and Volkoff [28] proposed that the sum of all physical forces, viz. hydrostatic force \( f_h \), gravitational force \( f_g \) and anisotropic force \( f_a \) must be zero to maintain the system into an equilibrium state, i.e.,

\[
\begin{align*}
  f_h + f_g + f_a &= 0. 
\end{align*}
\]  

(57)
Figure 6: Plots of energy bounds for Her X-I (green) and PSR J 1416-2230 (blue) for solution II.
Figure 7: Plots of fundamental forces ($f_g$ (green), $f_a$ (red) and $f_h$ (blue)) for Her X-I (left) and PSR J 1416-2230 (right) corresponding to solution $I$.

For our spacetime, the TOV equation takes the form

$$-\left[ p' + \sigma \vartheta_1'' \right] - \frac{\xi'}{2}(\rho + p) + \frac{\sigma h'}{2}(\rho + p) + \frac{\sigma \eta'}{2}(\vartheta_1^1 - \vartheta_0^0) + \frac{2\sigma}{r}(\vartheta_2^2 - \vartheta_1^1) = 0. \quad (58)$$

The factors in square brackets denote the forces $f_h$, $f_g$ and $f_a$, respectively which explicitly can be written as

$$f_h = - (p' + \sigma \vartheta_1''), \quad (59)$$

$$f_g = - \left( \frac{\xi'}{2}(\rho + p) + \frac{\sigma h'}{2}(\rho + p) + \frac{\sigma \eta'}{2}(\vartheta_1^1 - \vartheta_0^0) \right), \quad (60)$$

$$f_a = - \frac{2\sigma}{r}(\vartheta_2^2 - \vartheta_1^1). \quad (61)$$

The profile of these forces corresponding to the solutions $I$ and $II$ are plotted in Figures 7 and 8, respectively. The figures show the dominating behavior of the gravitational force which is counter-balanced by the effects of anisotropic as well as hydrostatic forces. Hence, our proposed models are in equilibrium state under the combine influence of these fundamental forces.
Figure 8: Plots of fundamental forces ($f_q$ (green), $f_a$ (red) and $f_h$ (blue)) for Her X-I (left) and PSR J 1416-2230 (right) corresponding to solution II.

### 4.2 Causality Condition

Once the system has achieved its equilibrium stage then the next question arises whether it is stable or not. To do so, we will study the stability of the anisotropic models through Abreu et al. [29] technique based on Herrera’s cracking concept [30] which demands

\[
\begin{align*}
-1 &\leq v_{st}^2 - v_{sr}^2 \leq 0 \Rightarrow \text{Potentially stable,} \\
0 &< v_{st}^2 - v_{sr}^2 \leq 1 \Rightarrow \text{Potentially unstable,}
\end{align*}
\]

where

\[
v_{st}^2 = \frac{d\hat{p}_t}{d\hat{\rho}}, \quad v_{sr}^2 = \frac{d\hat{p}_r}{d\hat{\rho}}.
\]

The above conditions can be unified as $|v_{st}^2 - v_{sr}^2| \leq 1$ [31] which proposed that “no cracking” concept is necessary for the potentially stable regions. This procedure requires that for an physically acceptable model the transverse as well as radial sound velocities should be less than 1, i.e., $v_{st}^2$, $v_{sr}^2 < 1$ which are also known as causality conditions. It is clear from Figures 9 and 10 that both anisotropic models meet the causality constraints along with the no cracking condition. Thus, the stability criteria are satisfied for both constructed solutions.
Figure 9: Plots of $v_{sr}^2$, $v_{st}^2$ and $|v_{st}^2 - v_{sr}^2|$ for Her X-I (green) and PSR J 1416-2230 (blue) corresponding to solution I.
Figure 10: Plots of $v_{sr}^2$, $v_{st}^2$, and $|v_{st}^2 - v_{sr}^2|$ for Her X-I (green) and PSR J 1416-2230 (blue) corresponding to solution II.
Figure 11: Plots of adiabatic index for Her X-I (green) and PSR J 1416-2230 (blue) corresponding to solutions I (left) and II (right).

4.3 Adiabatic Index

The adiabatic index $\Gamma$ as a stiffness parameter has significant importance to study the stable behavior of relativistic stellar structure. Chandrasekhar [32] and many researchers [33]-[35] discussed the stability of gaseous stars against radial adiabatic perturbation. It is found that $\Gamma$ should be greater than $4/3$ in the interior of stable isotropic celestial objects. For anisotropic sphere, the adiabatic index is defined by

$$\Gamma_r = \frac{\hat{\rho} + \hat{p}_r}{\hat{\rho}_r} \frac{\nu_{sr}^2}{r^2}.$$  (63)

The graphical representation of $\Gamma$ is displayed in Figure 11 which shows that our constructed anisotropic models indicate dynamical stable behavior as the value of $\Gamma > \frac{4}{3}$ throughout the domain. Moreover, it is noted that the adiabatic index becomes undefined at the boundary due to the vanishing of radial pressure at $r = R$.

5 Concluding Remarks

The formulation of analytic solutions describing the interior constituents of self-gravitating objects has captured the attention of many researchers. In
this context, gravitational decoupling by MGD technique has effectively been used to explore anisotropic solutions for matter sources. Here, matter sources interact gravitationally with no exchange of energy. To resolve this issue, an extended version of the MGD was presented \cite{24} which enables the transformation of energy between matter contents. In this paper, we have derived exact spherical anisotropic solutions from a known isotropic model through the EGD technique. To include the effects of anisotropy, we have deformed both temporal as well as radial metric potentials and successfully decoupled the field equations. The Bianchi identities for both non-generic matter sources have been studied which provide a particular relation between them as displayed in Eqs. (21) and (22). Moreover, the unknown constants of the models are computed through the matching of interior anisotropic solutions with the Schwarzschild spacetime.

To examine the consistency of the EGD approach, we have first considered the isotropic Tolman IV solution and incorporated the effects of anisotropy by adding an additional matter source in the perfect fluid configuration. In order to evaluate the exact anisotropic solutions, we have used a barotropic equation of state for $\rho^0_0$ and imposed the conditions on pressure and energy density which yield the solutions I and II, respectively. We have investigated physical properties of the constructed models through the graphical portray of matter variables, anisotropic factor and energy conditions. It is observed that both solutions are physically acceptable as they satisfy all the essential viability conditions for the star candidates “Her X-I” and “PSR J 1416-2230”. Moreover, the stability criteria provided by the speed of sound constraint and adiabatic index are fulfilled for proposed models that depict the potentially stable structure of compact stars. It is worthwhile to mention here that the both derived anisotropic solutions given in Eqs.(37)-(40) and (49)-(50) satisfy the field equations (4)-(6) as shown in Appendix A.

Ovalle et al. \cite{15} derived anisotropic spherical solutions from perfect Tolman IV model using the MGD approach but the energy bounds as well as stability conditions were not investigated for their solutions. Sharif and Sadiq \cite{16} extended Krori-Barua solution for charged spherical system to anisotropic domain and deduced that the first solution corresponding to pressure constraint exhibits the stable behavior whereas the second solution violates the viability criteria. In a recent paper, Sharif and Saba \cite{19} evaluated anisotropic spherical solutions using MGD in the background of $f(G)$ gravity and found only one viable as well as stable solution. It is worth mentioning here that our both anisotropic models indicate the consistent behavior and
satisfy the stability criteria. We would like to re-iterate that EGD technique is more powerful as this can provide transformation of energy between matter contents and helps to study the physical characteristics of self-gravitating objects.

Appendix A

Here we show that the anisotropic solutions given in Eqs. (37)-(40) and (49)-(50) satisfy the field equations (4)-(6). Moreover, the conservation equations (18) and (23) are verified. For this purpose, we proceed as follows. Differentiating Eqs. (39) and (40) with respect to \( r \), it follows that

\[
\eta' = \frac{2r}{A^2 \left( \frac{A^2}{r^2} + 1 \right)} + \sigma \frac{1}{\varepsilon} \left( 2r \left( A^6 + 8C^2r^4 + 6A^2r^2 (C^2 + 2r^2) + A^4 \left( 2C^2 + 77r^2 \right) \right) \right) \left( A^2 + 2C^2 \right) \left( A^4 + 5A^2r^2 + 6r^4 \right) \left( \frac{\sqrt{2r}}{A} \right) \left( A^2 + 2C^2 \right) \left( A^4 + 5A^2r^2 + 6r^4 \right) \left( \frac{\sqrt{2r}}{A} \right) \tag{A1}
\]

\[
\eta'' = -((2\sigma (A^2 + r^2) \left( \sqrt{2A} \left( A^2 + 2C^2 \right) \left( 5A^2 + 12r^2 \right) \tan^{-1} \left( \frac{\sqrt{2r}}{A} \right) \right) + 8r \left( A^2 + r^2 \right) \left( C^2 - r^2 \right)) - 2r \left( 3A^2 + 2C^2 \right) \left( 3A^2 + 10r^2 \right) \left( A^2 + 2C^2 \right) \left( A^4 + 5A^2r^2 + 6r^4 \right) \left( \frac{\sqrt{2r}}{A} \right) \left( A^2 + 2C^2 \right) \left( A^4 + 5A^2r^2 + 6r^4 \right) \left( \frac{\sqrt{2r}}{A} \right) \right) \times \left( \sqrt{2A} \left( A^2 + 2C^2 \right) \tan^{-1} \left( \frac{\sqrt{2r}}{A} \right) + 4r^3 \right) + 8r \left( A^2 + r^2 \right) \left( C^2 - r^2 \right) \right) - (4\sigma (A^2 + r^2) \left( 3A^2 + 2C^2 \right) \left( 3A^2 + 10r^2 \right) \left( A^2 + 2C^2 \right) \left( A^4 + 5A^2r^2 + 6r^4 \right) \left( \frac{\sqrt{2r}}{A} \right) \left( A^2 + 2C^2 \right) \left( A^4 + 5A^2r^2 + 6r^4 \right) \left( \frac{\sqrt{2r}}{A} \right) \right)
\]

26
The derivatives of Eqs. (30), (33) and (38) lead to

\[ \times \ r^2 (5r^2 - 3C^2) + \sqrt{2} A r (A^2 + 2C^2) \tan^{-1} \left( \frac{\sqrt{2}r}{A} \right) \]

\[ \times \ (\sigma (A^2 + 2r^2) (-2r (A^2 + 2C^2) + \sqrt{2} A (A^2 + 2C^2) \tan^{-1} \left( \frac{\sqrt{2}r}{A} \right) + 4r^3) + 8r (A^2 + r^2) (C^2 - r^2))^{-1} + (\sigma (A^2 + r^2) (2r (A^6 + A^4 (2C^2 + 7r^2) + 6A^2 r^2 (C^2 + 2r^2) + 8C^2 r^4) - \sqrt{2} A (A^2 + 2C^2)) \times \ (A^4 + 5A^2 r^2 + 6r^4) \tan^{-1} \left( \frac{\sqrt{2}r}{A} \right) (r^2 (\sigma (A^2 + 2r^2) (-2r \times \ (A^2 + 2C^2) + \sqrt{2} A (A^2 + 2C^2) \tan^{-1} \left( \frac{\sqrt{2}r}{A} \right) + 4r^3) + 8r \times \ (A^2 + r^2) (C^2 - r^2))^{-1} + \right) (A^4 + 5A^2 r^2 + 6r^4) \tan^{-1} \left( \frac{\sqrt{2}r}{A} \right), (A2) \]

\[ \chi' \ e^{-\chi} = (\sqrt{2} A \sigma (A^2 + 2C^2) (A^2 + 2r^2) \tan^{-1} \left( \frac{\sqrt{2}r}{A} \right) - 2r \sigma (A^2 + 2r^2) \times \ (A^4 + 2A^2 (C^2 + 2r^2) + 8r^4) + 16r^3 (A^4 + A^2 (C^2 + 2r^2) + 2r^4) \times \ (8C^2 r^2 (A^2 + 2r^2)^2)^{-1}. (A3) \]

The derivatives of Eqs. (30), (33) and (38) lead to

\[ \xi' = \frac{2r}{a^2 \left( \frac{r^2}{a^2} + 1 \right)}, (A4) \]

\[ p' = - \frac{2r (a^2 + 2C^2)}{c^2 (a^2 + 2r^2)^2} = -\varphi_1', (A5) \]

\[ h' = \frac{1}{\varepsilon} \left( 2r (A^6 + 8C^2 r^4 + 6A^2 r^2 (C^2 + 2r^2) + A^4 (2C^2 + 7r^2)) - \sqrt{2} A \times \ (A^2 + 2C^2) (A^4 + 5A^2 r^2 + 6r^4) \tan^{-1} \left( \frac{\sqrt{2}r}{A} \right) \right), (A6) \]

The field equation (1) can be written as

\[ \hat{\rho} = \frac{1}{r^2} + e^{-\chi} \left( \frac{\chi'}{r} - \frac{1}{r^2} \right). (A7) \]

Using Eq. (A3), the right side of the above equation takes the form

\[ \frac{3A^4 + A^2 (3C^2 + 7r^2) + 2r^2 (C^2 + 3r^2)}{C^2 (A^2 + 2r^2)^2} + \sigma \left( \frac{C^2 - A^2 - 3r^2}{C^2 (A^2 + 2r^2)} \right), \]

27
which is just equal to \( \hat{\rho} \) given in Eq. \( \text{(44)} \). Consider the right side of Eq. \( \text{(5)} \) as

\[-\frac{1}{r^2} + e^{-\chi} \left( \frac{\eta'}{r} + \frac{1}{r^2} \right).\]

Using Eqs. \( \text{(A1)} \) and \( \text{(A3)} \), we have

\[\frac{C^2 - A^2 - 3r^2}{C^2(A^2 + 2r^2)} (1 - \sigma),\]

which is equal to \( \hat{p}_r = p + \sigma \vartheta^1 \) given in Eq. \( \text{(45)} \). Similarly, the right side of Eq. \( \text{(6)} \) is expressed as

\[e^{-\chi} \left( 2\eta'' + \eta' \chi' + \frac{2}{r}(\eta' - \chi') \right),\]

which, through Eqs. \( \text{(A1)} \) and \( \text{(A3)} \) gives rise to

\[
\frac{C^2 - A^2 - 3r^2}{C^2(A^2 + 2r^2)} + \sigma \left( 2r(16r^4(C^4 - 3C^2r^2(-1 + \sigma)) + 3r^4(-1 + \sigma)) \right) \\
+ A^8(1 - 2\sigma) + 4A^4(C^2r^2(21 - 8\sigma) + C^4(-1 + \sigma) - r^4(11 + \sigma)) \\
+ 4A^3r^2(2C^2r^2(17 - 10\sigma) + 12r^4(-2 + \sigma) + C^4(1 + 2\sigma)) \\
- A^6(2C^2(-6 + \sigma) + r^2(1 + 12\sigma))) + \sqrt{2}A(A^2 + 2C^2)(A^2 + 2r^2) \\
\times \left( -\left( A^2 + 2C^2 \right)A^2 + 3r^2 \right) + 2(A^4 + 6r^4 - A^2(C^2 - 6r^2)) \sigma \tan^{-1} \left( \frac{\sqrt{2}r}{A} \right) \\
\times \left( 2C^2 \left( A^2 + 2r^2 \right)^2 \right) [8(C - r)r(C + r) \left( A^2 + r^2 \right) + \left( A^2 + 2r^2 \right)] \\
\times \sigma(-2 \left( A^2 + 2C^2 \right) r + 4r^3 + \sqrt{2}A \left( A^2 + 2C^2 \right) \tan^{-1} \left( \frac{\sqrt{2}r}{A} \right))^{-1}.
\]

This is exactly the same as \( \hat{p}_r = p + \sigma \vartheta^1 \) given in Eq. \( \text{(47)} \). Hence, the solution \( I \) satisfies the field equations. Using Eqs. \( \text{(13)} \) and \( \text{(14)} \), we obtain the following constraint

\[-\frac{1}{r^2} + e^{-\mu} \left( \frac{\xi'}{r} + \frac{1}{r^2} \right) = e^{-\mu} \left( \frac{\xi''}{2} + \frac{\xi'^2}{4} - \frac{\xi' \mu'}{4} + \frac{1}{2r}(\xi' - \mu') \right), \quad \text{(A8)}\]

where its left side reads

\[\frac{1}{2} \left( -\frac{A^2 + 2C^2}{C^2(A^2 + 2r^2)} + \frac{3}{C^2} - \frac{4}{r^2} \right), \quad \text{(A9)}\]
which turns out to be the same to the right side of Eq. (A8) after inserting the values. This indicates that solutions of the field equations satisfy the constraints derived from them. The conservation equation (18) can be written as

\[ p' + \frac{\xi'}{2}(\rho + p) + \frac{\sigma h'}{2}(\rho + p) = -\sigma \left( \phi_1' - \frac{\eta'}{2}(\phi_1^0 - \phi_0^0) + \frac{2}{r}(\phi_1^1 - \phi_2^1) \right). \quad (A10) \]

Using Eqs. (A4)-(A6) and (32)-(33), its left side is evaluated as

\[
- (\sigma (A^2 + 2C^2) (\sqrt{2} A (A^2 + 2C^2) (A^4 + 5A^2 r^2 + 6r^4) \tan^{-1} \left( \frac{\sqrt{2} r}{A} \right) - 2r \\
\times \left( A^6 + A^4 (2C^2 + 7r^2) + 6A^2 r^2 (C^2 + 2r^2) + 8C^2 r^2) \right) (C^2 r (A^2 + 2r^2)^2 \\
\times (\sigma (A^2 + 2r^2) \left( -2r (A^2 + 2C^2) + \sqrt{2} A (A^2 + 2C^2) \tan^{-1} \left( \frac{\sqrt{2} r}{A} \right) + 4r^3 \right) \\
+ 8r (a^2 + r^2) (C - r)(C + r))^{-1},
\]

which is equal to the right side of Eq. (A10). Similarly, the second conservation equation can be satisfied. Hence, the anisotropic solution I satisfies the conservation equations.

Now, we repeat the same procedure for the second anisotropic solution. The metric potentials take the form

\[ e^{-\chi} = \frac{(1 - \frac{r^2}{2A^2})(1 + \frac{r^2}{2A^2})}{1 + \frac{2r^2}{A^2}} + \sigma \left( \frac{r^2 (A^2 + C^2 + r^2)}{C^2 (A^2 + 2r^2)} \right), \quad (A11) \]

\[ \eta = \sigma \left( \frac{1}{2} \left( -\frac{2 \ln(A^2 + r^2)}{r} - \frac{2 \ln(A^2 + 2r^2)}{r} \right) - \frac{1 + \sigma}{1 + \sigma} + \left( \frac{A^2(-1 + \sigma) + C^2(-1 + 3\sigma)}{C^2(1 + \sigma) - \rho + (A^2 + 2r^2)(-1 + \sigma)) (\rho (A^2 + 2r^2)(-1 + \sigma) \right)^{-1} + \left( (\rho (-1 + 2\sigma) + C^2(1 - 3\sigma) - A^2(-1 + \sigma)) \ln \left( \frac{(A^2 + 2r^2)}{(-1 + \sigma)} \right) \right) \right) \left( (-1 + \sigma) \rho \right)^{-1} + \ln \left( B^2(1 + \frac{r^2}{A^2}) \right), \quad (A12) \]

The derivatives of the above equations lead to

\[ \chi' e^{-\chi} = -2r (\sigma - 1) \left( A^4 + A^2 (C^2 + 2r^2) + 2r^4) \right) \left( C^2 (A^2 + 2r^2)^2 \right)^{-1}, \quad (A13) \]

\[ \eta' = 2r \left( (A^2 + 2r^2) (C^2 - r^2) + \sigma \left( A^4 + A^2 (C^2 + 2r^2) + 2r^4) \right) \right) \]
Using Eq. (A13), the right side of Eq. (A7) is evaluated as

$$ (A10) $$

is computed as

$$ (A11) $$

side of Eq. (5) reads

$$ (A12) $$

is satisfied. Hence, both anisotropic solutions satisfy the field equations as

$$ (A13) $$

which is equal to its right side. Similarly, the second conservation equation

$$ (A14) $$

is similar to $\hat{\rho}$ given in (53). Through Eqs. (A13) and (A14), the right side of Eq. (A7) takes

$$ (A15) $$

which is similar to $\hat{\rho}_r$ given in (54). Similarly, the right side of Eq. (6) takes

$$ (A16) $$

which is similar to $\hat{p}_r$ given in (55). The left side of conservation equation

$$ (A17) $$

is computed as

$$ (A18) $$

which is equal to its right side. Similarly, the second conservation equation

$$ (A19) $$

is satisfied. Hence, both anisotropic solutions satisfy the field equations as

$$ (A20) $$

as well as conservation equations.
Acknowledgement

One of us (QM) would like to thank the Higher Education Commission, Islamabad, Pakistan for its financial support through the Indigenous Ph.D. Fellowship, Phase-II, Batch-III.

References

[1] Schwarzschild, K.: Kl. Math. Phys. 24(1916)424.
[2] Tolman, R.C.: Phys. Rev. 55(1939)364.
[3] Lemaitre, G.: Ann. Soc. Sci. Bruxells A53(1933)51.
[4] Ruderman, R.: Ann. Rev. Astron. Astrophys. 10(1972)427.
[5] Bowers, R.L. and Liang, E.P.T.: Astrophys. J. 188(1974)657.
[6] Abbas, G. et al.: Astrophys. Space Sci. 357(2015)158.
[7] Tripathy, S.K. and Mishra, B.: Eur. Phys. J. Plus 131(2016)273.
[8] Murad, M.H.: Astrophys. Space Sci. 20(2016)361.
[9] Maurya, S.K. and Maharaj, S.D.: Eur. Phys. J. C 77(2017)328.
[10] Matondo, D.K., Maharaj, S.D. and Ray, S.: Eur. Phys. J. C 78(2018)437.
[11] Ovalle, J.: Mod. Phys. Lett. A 23(2008)3247.
[12] Ovalle, J. and Linares, F.: Phys. Rev. D 88(2013)104026.
[13] Contreras, E. and Bargueño, P.: Eur. Phys. J. C 78(2018)558.
[14] Ovalle, J.: Phys. Rev. D 95(2017)104019.
[15] Ovalle, J. et al.: Eur. Phys. J. C 78(2018)122.
[16] Sharif, M. and Sadiq, S.: Eur. Phys. J. C 78(2018)410.
[17] Gabbanelli, L., Rincon, A. and Rubio, C.: Eur. Phys. J. C 78(2018)370.
[18] Graterol, R.P.: Eur. Phys. J. Plus 133(2018)244.
[19] Sharif, M. and Saba, S.: Eur. Phys. J. C 78(2018)921.
[20] Sharif, M. and Waseem, A.: Ann. Phys. 405(2019)14.
[21] Sharif, M. and Ama-Tul-Mughani, Q.: Int. J. Geom. Methods Mod. Phys. 16(2019)1950187; Mod. Phys. Lett. A (to appear, 2020).
[22] Casadio, R. et al.: Eur. Phys. J. C 79(2019)826.
[23] Casadio, R., Ovalle, J. and da Rocha, R.: Class. Quantum Grav. 32(2015)215020.
[24] Ovalle, J.: Phys. Lett. B 788(2019)213.
[25] Contreras, E. and Bargueño, P.: Class. Quantum Grav. 36(2019)215009.
[26] Ovalle, J. et al.: Eur. Phys. J. C 78(2018)960.
[27] Tolman, R. C.: Phys. Rev. 55(1939)364.
[28] Oppenheimer, J.R. and Volkoff, G.M.: Phys. Rev. 55(1939)374.
[29] Abreu, H., Hernandez, H. and Nunez, L.A.: Class. Quantum Gravit. 24(2007)4631.
[30] Herrera, L.: Phys. Lett. A 165(1992)206.
[31] Andreasson, H.: Commun. Math. Phys. 288(2009)715.
[32] Chandrasekhar, S.: Astrophys. J. 140(1964)417.
[33] Heintzmann, H.: Hillebrandt, W.: Astron. Astrophys. 38(1975)51.
[34] Hillebrandt, W. and Steinmetz, K.O.: Astron. Astrophys. 53(1976)283.
[35] Bombaci, I.: Astron. Astrophys. 305(1996)871.