Conformal n-point functions in momentum space

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We present a Feynman integral representation for the general momentum-space scalar n-point function in any conformal field theory. This representation solves the conformal Ward identities and features an arbitrary function of n(n − 3)/2 variables which play the role of momentum-space conformal cross-ratios. It involves (n − 1)(n − 2)/2 integrations over momenta, with the momenta running over the edges of an (n − 1)-simplex. We provide the details in the simplest non-trivial case (4-point functions), and for this case we identify values of the operator and spacetime dimensions for which singularities arise leading to anomalies and beta functions, and discuss several illustrative examples from perturbative quantum field theory and holography.

I. MOTIVATION

The structure of correlation functions in a conformal field theory (CFT) is highly constrained by conformal symmetry. It has been known since the work of Polyakov [1, 2] that the most general 4-point function of scalar primary operators OΔi, each of dimension Δi, takes the form

$$\langle O_{Δ_1}(x_1)O_{Δ_2}(x_2)O_{Δ_3}(x_3)O_{Δ_4}(x_4) \rangle = f(u,v) \prod_{1 \leq i < j \leq 4} x_{ij}^{2δ_{ij}},$$

(1)

where $x_{ij} = |x_i - x_j|$ are the coordinate separations and

$$2δ_{ij} = Δ_i - Δ_j - Δ_j, \quad Δ_i = \sum_{i=1}^{4} Δ_i. \quad (2)$$

The 4-point function is thus determined up to an arbitrary (theory-specific) function $f$ of the two conformal cross-ratios,

$$u = \frac{x_{13}^2 x_{34}^2}{x_{14}^2 x_{23}^2}, \quad v = \frac{x_{12}^2 x_{34}^2}{x_{14}^2 x_{23}^2}. \quad (3)$$

This result straightforwardly generalizes to n-point functions, which now involve an arbitrary function of $n(n − 3)/2$ cross ratios.

These results are easy to derive in position space where the conformal group acts naturally. Yet for many modern applications, including cosmology [3–19], condensed matter [20–25], anomalies [26–28] and the bootstrap programme [29–33], it would be highly desirable to know the analogue of this result – and, indeed, the analogue of the conformal cross-ratios themselves – in momentum space.

Despite the lapse of nearly five decades, such an understanding has yet to be achieved. Nevertheless, through recent efforts, all the necessary prerequisites are now in place. Firstly, the momentum-space 3-point functions of general scalar and tensorial operators are known, including the cases where anomalies and beta functions arise as a result of renormalization [34–46]. Secondly, momentum-space studies of the 4-point function have yielded special classes of solutions to the conformal Ward identities [15, 32, 47–51]. Here, our aim is now to provide the general solution for the momentum-space n-point function. We start by providing a complete discussion of the 4-point function and an exploration of its properties, and we then present the result for the n-point function.

II. MOMENTUM-SPACE REPRESENTATION

For scalar 4-point functions, our main result is the general momentum-space representation:

$$\langle\langle O_{Δ_1}(p_1)O_{Δ_2}(p_2)O_{Δ_3}(p_3)O_{Δ_4}(p_4)\rangle\rangle = \int \frac{d^d q_1}{(2\pi)^d} \frac{d^d q_2}{(2\pi)^d} \frac{d^d q_3}{(2\pi)^d} \frac{d^d q_4}{(2\pi)^d} \frac{f(\hat{u}, \hat{v})}{\text{Den}_3(q_j, p_k)}. \quad (4)$$

Here, ⟨⟨...⟩⟩ = ⟨⟨...⟩⟩(2π)dδ(∑j p_j), d is the spacetime dimension, and the denominator is

$$\text{Den}_3(q_j, p_k) = q_3^{2δ_{31}+d} q_2^{2δ_{32}+d} q_1^{2δ_{31}+d} |p_1 + q_2 - q_3|^{2δ_{31}+d} \times |p_2 + q_3 - q_1|^{2δ_{32}+d} |p_3 + q_1 - q_2|^{2δ_{31}+d}. \quad (5)$$

where the $δ_{ij}$ are given in (2). We work in Euclidean signature throughout. As expected from (1), this 4-point function depends on an arbitrary function $f(\hat{u}, \hat{v})$ of two variables:

$$\hat{u} = \frac{q_1^2 |p_1 + q_2 - q_3|^2}{q_2^2 |p_2 + q_3 - q_1|^2}, \quad \hat{v} = \frac{q_2^2 |p_2 + q_3 - q_1|^2}{q_3^2 |p_3 + q_1 - q_2|^2}. \quad (6)$$
The role of $\tilde{u}$ and $\tilde{v}$ is analogous to that of the position-space cross-ratios $u$ and $v$ defined in (3). These variables are thus the desired momentum-space cross-ratios, though notice they depend on the momenta $q_j$ that are subject to integration in (4).

A. Proof of conformal invariance

The conformal invariance of (4) can be verified by direct substitution into the conformal Ward identities (CWIs). Its Poincaré invariance is manifest, and its scaling dimension is given by the sum of powers in (5) minus $3d$ from the three integrals. This gives $-2\delta_3 - 3d = \Delta_3 - 3d$, the correct result in momentum space.

The remaining CWIs associated with special conformal transformations are implemented by the second-order differential operator $K^\kappa = \sum_{j=1}^3 K_j^\kappa$, where [36]

$$K_j^\kappa = p_j^\alpha \frac{\partial}{\partial p_j^\alpha} \frac{\partial}{\partial p_j^3} - 2p_j^\alpha \frac{\partial}{\partial p_j^3} \frac{\partial}{\partial p_j^\alpha} + 2(\Delta_j - d) \frac{\partial}{\partial p_j^3}. \tag{7}$$

By acting with $K^\kappa$ on the integrand of (4), one can show

$$K^\kappa \left( \frac{\hat{f}(\tilde{u}, \tilde{v})}{\text{Den}_3(q_j, p_k)} \right) = \sum_{n=1}^3 \frac{\partial}{\partial q_n^\beta} \left[ \frac{\langle q_n \rangle_n}{\text{Den}_3(q_j, p_k)} \right] \times \left( A^{\alpha \mu \kappa}_{(n)} \frac{\partial}{\partial \tilde{u}} + B^{\alpha \mu \kappa}_{(n)} \frac{\partial}{\partial \tilde{v}} + C^{\alpha \mu \kappa}_{(n)} \hat{f} \right). \tag{8}$$

In order to write these coefficients explicitly, we define

$$A^{\alpha \mu \kappa}_{(n)} = 2k_n^\beta \left( \delta^{\alpha \mu} \delta^\beta_3 - \delta^{\mu \alpha} \delta^\beta_3 - \delta^{\mu \beta} \delta^\alpha_3 \right), \tag{9}$$

where the $k_n$ are the vectors featuring in (5), i.e., $k_1 = p_1 + q_2 - q_3$ along with cyclic permutations. The coefficients in equation (8) are then

$$C_{(1)}^{\alpha \mu \kappa} = (\frac{1}{2} + \delta_{34}) A^{\alpha \mu \kappa}_{(2)} + (\frac{1}{2} + \delta_{23}) A^{\alpha \mu \kappa}_{(3)}, \tag{10}$$

with $C_{(2)}^{\alpha \mu \kappa}$ and $C_{(3)}^{\alpha \mu \kappa}$ following by cyclic permutation of the indices 1, 2, 3, while

$$A^{\alpha \mu \kappa}_{(1)} = A^{\alpha \mu \kappa}_{(2)}, \quad B^{\alpha \mu \kappa}_{(1)} = A^{\alpha \mu \kappa}_{(3)} - A^{\alpha \mu \kappa}_{(2)}, \quad C^{\alpha \mu \kappa}_{(1)} = - A^{\alpha \mu \kappa}_{(3)};$$

$$A^{\alpha \mu \kappa}_{(2)} = - A^{\alpha \mu \kappa}_{(1)}, \quad B^{\alpha \mu \kappa}_{(2)} = A^{\alpha \mu \kappa}_{(3)}, \quad C^{\alpha \mu \kappa}_{(2)} = - A^{\alpha \mu \kappa}_{(3)};$$

$$A^{\alpha \mu \kappa}_{(3)} = A^{\alpha \mu \kappa}_{(2)} - A^{\alpha \mu \kappa}_{(1)}, \quad B^{\alpha \mu \kappa}_{(3)} = - A^{\alpha \mu \kappa}_{(2)}, \quad C^{\alpha \mu \kappa}_{(3)} = - A^{\alpha \mu \kappa}_{(2)}. \tag{11}$$

As the action of $K^\kappa$ on the integrand of (4) is a total derivative, the integral itself is then invariant. This proves the conformal invariance of the representation (4).

B. The tetrahedron

The momentum-space expression (4) is not the direct Fourier transform of the position-space expression (1).

![FIG. 1. The 3-loop tetrahedral integral (15), where each internal line corresponds to a generalized propagator in (16).](image)

Rather, for $f(u, v) = u^\alpha v^\beta$, the Fourier transform is given by (4) with

$$\hat{f}(\tilde{u}, \tilde{v}) = C_{\beta}^{\alpha \phi \delta 4} C_{\alpha - \beta}^{\gamma \delta 3} C_{\alpha - \gamma}^{\delta \gamma 2} \tilde{u}^\alpha \tilde{v}^\beta, \tag{12}$$

where

$$C_{\alpha \beta}^{\delta + \delta' + 2 \sigma + d} d \Gamma \left( \frac{\sigma + \delta + \sigma}{\Gamma(-\delta - \sigma) \Gamma(-\delta' - \sigma)}. \tag{13}$$

This follows since the Fourier transform of a product is a convolution of Fourier transforms, and so we can write

$$\mathcal{F} \left[ \frac{2(\beta + \delta_{23})}{x_{12}} \frac{2(\beta + \delta_{34})}{x_{34}} \frac{2(\alpha - \beta + \delta_{13})}{x_{13}} \frac{2(\alpha - \beta + \delta_{24})}{x_{24}} \right] \times \frac{2(\alpha - \beta + \delta_{14})}{x_{14}} \frac{2(\alpha - \beta + \delta_{23})}{x_{23}} \frac{2(\alpha - \beta + \delta_{34})}{x_{34}} \times \frac{2(\alpha - \beta + \delta_{23})}{x_{23}}$$

where * denotes the convolution in all variables, namely $(f * g)(p_k) = \int \prod_{j=1}^4 \frac{d^4q_j}{(2\pi)^4} f(q_j) g(p_j - q_j)$. With the $\hat{f}$ in (12), the momentum-space integral in (4) becomes

$$W_{\alpha, \beta} = \int \frac{d^4q_1}{(2\pi)^4} \frac{d^4q_2}{(2\pi)^4} \frac{d^4q_3}{(2\pi)^4} \frac{d^4q_4}{(2\pi)^4} \frac{1}{\text{Den}_3^{(\alpha \beta)}(q_j, p_k)}, \tag{15}$$

where

$$\text{Den}_3^{(\alpha \beta)}(q_j, p_k) = q_2^{\gamma_{12}} q_3^{\gamma_{13}} q_1^{\gamma_{34}} |p_1 + q_2 - q_3|^{2\gamma_{14}}$$

and

$$\gamma_{12} = \delta_{12} + \beta + d/2, \quad \gamma_{13} = \delta_{13} + \alpha - \beta + d/2;$$

$$\gamma_{23} = \delta_{23} - \alpha + d/2, \quad \gamma_{14} = 2\delta_{14} - \alpha + d/2;$$

$$\gamma_{24} = \delta_{24} + \alpha - \beta + d/2, \quad \gamma_{34} = \delta_{34} + \beta + d/2. \tag{17}$$
This is a 3-loop Feynman integral with the topology of a tetrahedron as presented in Fig. 1. The four momenta entering the vertices are those of the external operators, while the six internal lines describe generalized propagators in which the momenta are raised to the specific powers given in (17).

C. Spectral representation

Where convergence permits, the function \( f(\hat{u}, \hat{v}) \) can be expressed as a double inverse Mellin transform over the monomial \( \hat{u}^\alpha \hat{v}^\beta \). The 4-point function (4) then admits the spectral representation

\[
\langle \mathcal{O}_{\Delta_1}(p_1) \mathcal{O}_{\Delta_2}(p_2) \mathcal{O}_{\Delta_3}(p_3) \mathcal{O}_{\Delta_4}(p_4) \rangle = \frac{1}{(2\pi i)^2} \int_{b_1-i\infty}^{b_1+i\infty} \int_{b_2-i\infty}^{b_2+i\infty} d\alpha d\beta \rho(\alpha, \beta) W_{\alpha,\beta}
\]

for an appropriate choice of integration contour specified by \( b_1 \) and \( b_2 \). Here, \( W_{\alpha,\beta} \) is a universal kernel corresponding to the tetrahedron integral (15) and \( \rho(\alpha, \beta) \) is a theory-specific spectral function derived from the Mellin transform of \( f(\hat{u}, \hat{v}) \). Where the position-space Mellin representation of a 4-point function is known – as is often the case for holographic CFTs [38–40] – the corresponding \( \rho(\alpha, \beta) \) in momentum space can be read off immediately using equations (12) and (13).

To evaluate the spectral integral, we close the contour and sum the residues. For certain \( \alpha \) and \( \beta \), these residues are simple to evaluate due to reductions in the loop order of \( W_{\alpha,\beta} \). Such reductions arise whenever a propagator in the denominator (16) appears with a power \( \gamma_{ij} = d + 2n \), for some non-negative integer \( n \). This can be seen by noting that, in a distributional sense as \( q \to 0 \),

\[
\lim_{\epsilon \to 0} \epsilon^{d/2} q^{d+2n-2} = \pi^{d/2} \frac{n!\Gamma(d/2+n)}{4^n n!\Gamma(d/2)} \delta^{(d)}(q).
\]

We then obtain a pole in \( \alpha, \beta \) whose residue is given by a 2-loop integral as shown in Fig. 2a. Where the external dimensions permit, such poles can also coincide. In Fig. 2b, we illustrate the case where \( \alpha - \beta = \delta_{14} = \delta_{23} \), creating a pair of delta functions \( \delta(q_1) \delta(p_2 + q_3 - q_1) \) for which the residue is a 1-loop box.

Simplifications of a different kind occur whenever a propagator in (16) appears with a vanishing power, or more generally for \( \gamma_{ij} = -2n \). This results in a contraction of the corresponding leg of the tetrahedron, producing a 1-loop triangle for which two of the legs are bubbles as shown in Fig. 2c. Evaluating the bubbles, one obtains a pure 1-loop triangle whose propagators are raised to new powers. This integral is equivalent to a general CFT 3-point function [36]. The locality can be understood by noting that a propagator \( q^{-2n} \) corresponds to a factor \( x_{ij}^{-(d+2n)} \) in position space, which is equivalent to a delta function via the position-space analogue of (19).

D. Singularities and renormalization

For special values of the spacetime and operator dimensions, momentum-space CFT correlators exhibit divergences requiring regularization and renormalization. All divergences are local can be removed through the addition of covariant counterterms giving rise to conformal anomalies and beta functions for composite operators. The renormalization of 3-point functions was studied in [38–40]. For 4-point functions, a similar analysis holds as we now discuss.

Firstly, renormalizability requires that all UV divergences should be either ultralocal, with support only when all four position-space insertions are coincident, or else semilocal, meaning they are supported only in the cases where either (i) \( x_1 = x_2 = x_3 \neq x_4 \), (ii) \( x_1 = x_2 \neq x_3 = x_4 \), or (iii) \( x_1 = x_2 \neq x_3 \neq x_4 \), along with all related cases obtained by permutation. In momentum space, ultralocal divergences are thus analytic in all the squared momenta, while semilocal divergences are analytic in at least one squared momentum. (Cases (i) and (ii) have a momentum-dependence matching that of a 2-point function, while that of case (iii) corresponds to a 3-point function.)

These divergences constitute local solutions of the CWI. Their form, as well as the \( d \) and \( \Delta \), for which they appear, can be predicted from an analysis of local counterterms. Such counterterms exist only in cases where

\[
d + \sum_{j=1}^{4} \sigma_j (\Delta_j - d/2) = -2n
\]

for some \( n \) non-negative integer, with signs \( \sigma_j \) whose values are either all minus, or else three minus and one plus. Ultralocal divergences are removed by counterterms that are quartic in the sources \( \varphi_j \) for the operators \( \mathcal{O}_{\Delta_j} \). These feature \( 2n \) fully-contracted derivatives whose action is distributed over the sources, and exist whenever (20) is satisfied with all minus signs. Since the scaling dimension of \( \varphi_j \) is \( d - \Delta_j \), this ensures the counterterm has overall dimension \( d \). The appearance of ultralocal divergences when this condition is satisfied can be seen by examining the region of integration where all three loop momenta in the kernel \( W_{\alpha,\beta} \) become large simultaneously.
Re-parametrizing $q_j = \lambda \hat{q}_j$ where $q_1^2 = 1$, as $\lambda \to \infty$ the denominator in (15) scales as $\lambda^{6d-6\Delta}(1 + O(\lambda^{-2}))$, while the numerator contributes a Jacobian factor $\int d\lambda \lambda^{2d-4}$. The $\lambda$ integral is then logarithmically divergent precisely when (20) is satisfied with all minus signs. (For nonzero $n$, the divergence derives from expanding the denominator to subleading order in powers of $\lambda^{-2}$.) After the divergence is subtracted and the regulator removed, the renormalized correlator has the expected nonlocal momentum dependence and obeys anomalous CWIs due to the RG scale introduced by the counterterm, see [38].

Semilocal divergences are removed by counterterms featuring one operator and multiple sources. For quartic counterterms, we have $2n$ fully-contracted derivatives whose action is distributed over $\varphi_1 \varphi_2 \varphi_3 \mathcal{O}_{\Delta_4}$. Such counterterms exist whenever (20) is satisfied with signs $(- - +)$ (or some permutation thereof), ensuring the counterterm has dimension $d$. The resulting 4-point contribution then has the momentum-dependence of a 2-point function and corresponds to case (i) above. This counterterm effectively reparametrizes the source for $\mathcal{O}_{\Delta_4}$ and we obtain a beta function in the renormalized theory.

The appearance of a semilocal divergence in $W_{\alpha,\beta}$ when the $(- - +)$ condition is satisfied can be seen by re-parametrizing the loop momenta in (15) as $q_1 = \lambda \hat{q}_1$ with $q_1^2 = 1$, $q_2 = \lambda \hat{q}_1 + p_3 + \ell_2$ and $q_3 = \lambda \hat{q}_1 - p_3 + \ell_3$. The $\lambda$ integral is then logarithmically divergent when this condition is satisfied, and has a semilocal momentum dependence that is non-analytic in $p_2^2$ only. For the permuted cases featuring $\mathcal{O}_{\Delta_4}$ with $j = 1, 2, 3$ in place of $\mathcal{O}_{\Delta_4}$, the corresponding reparametrization is simply $q_j = \lambda \hat{q}_j$ leaving the other loop momenta fixed. This difference reflects our use of momentum conservation to eliminate $p_3$ in (15). After renormalization, the correlator is again fully nonlocal and obeys anomalous CWIs reflecting the presence of the beta function, see [38].

Besides the quartic counterterms discussed above, which contribute solely to 4- and higher-point functions, we may also have cubic and quadratic counterterms. Their form is already fixed from the renormalization of 2- and 3-point functions [38–40, 55], but they nevertheless contribute to 4-point functions as well [56]. In particular, cubic counterterms with two sources and one operator remove semilocal divergences of types (ii) and (iii).

### III. FREE FIELDS

Consider a free spin-0 massless field $\phi$ and connected 4-point functions of the operators of the form $\phi^n$. In all cases, the function $f$ in position space is a sum of monomials of the form $u^a v^b$. For example, for the connected 4-point function $\langle \phi^2 :\phi^2 :\phi^2 :\phi^2 :\text{conn} \rangle$, one has

$$f(u, v) \sim \left( \frac{u}{v} \right)^{\frac{1}{2} \Delta_{\phi^2} - \frac{\epsilon}{2}} + \text{2 cycl. perms.,}$$

where $\Delta_{\phi^2} = d - 2$ is the dimension of $\phi^2$, and to write $f$ and $\tilde{f}$ succinctly we introduce the additional conformal ratios $w$ and $\tilde{w}$ defined by $ww = 1$ and $\hat{w} = 1$. Equation (12) now yields the momentum space $\tilde{f}$. In this case, however, the prefactor in (12) vanishes as two out of the six gamma functions in the denominator of (13) diverge. This means we have to consider the regulated expression with regulated $\tilde{f}$, namely

$$\tilde{f}(\hat{u}, \hat{v}) = 16\epsilon^2 \left( \frac{\hat{u}}{\hat{v}} \right)^{\frac{1}{2} \Delta_{\phi^2} - \frac{\epsilon}{2}} + 2 \text{ cycl. perms.,}$$

where $\epsilon = (4\pi)^{d/2}/\Gamma(d/2)$ and 2 cycl. perms. denotes two remaining terms with cyclic permutations of the ratios, $\hat{u} \leftrightarrow \hat{v} \leftrightarrow \hat{w} \leftrightarrow \hat{\epsilon}$. After this is substituted into (4) and the momentum space integrals carried out, the limit $\epsilon \to 0$ should be taken.

The appearance of the double zero in (22) reflects the fact that the only Feynman diagram contributing to this correlator has the topology of a box. If instead we consider the 4-point function of $\phi^4$; the contributing Feynman diagram topologies are as presented in Fig. 3. Up to an overall symmetry factor, the regulated $\tilde{f}$ reads

$$\tilde{f}(\hat{u}, \hat{v}) \sim \left( \frac{\hat{v}}{\hat{u}} \right)^{\frac{1}{2} \Delta_{\phi^4} - \frac{\epsilon}{2}} + \text{2 cycl. perms.,}$$

where $\Delta_{\phi^4} = 2(d - 2)$. The constants $c_n$ are defined recursively through

$$c_{n+1} = c_n \frac{\Gamma(\Delta_{\phi})\Gamma(n\Delta_{\phi})\Gamma(1 - n\Delta_{\phi})}{(4\pi)^d/2\Gamma((n + 1)\Delta_{\phi})\Gamma(1 - (n - 1)\Delta_{\phi})}$$
with \( c_1 = 1 \) and \( \Delta_\phi = d/2 - 1 \). These coefficients arise from the evaluation of effective propagators. Denoting the standard massless propagator as \( D_1(p) = 1/p^2 \), the effective propagator \( D_n(p) \) with \( n \) lines in Fig. 3 is

\[
D_n(p) = \frac{c_n}{p^{2(n-1)\Delta_\phi}} = \int \frac{d^d q}{(2\pi)^d} \frac{D_{n-1}(q)}{|p-q|^2}.
\]  

(25)

Finally, the disconnected part of any correlator can also be represented by the function \( f \). As an example, consider a generalized free field \( O \) of dimension \( \Delta_O \), for which the position-space 4-point function has

\[
f(u, v) \sim \left( \frac{u}{v} \right)^{\Delta_O} + 2 \text{ cycl. perms}.
\]  

(26)

The momentum-space expression is then proportional to \( p_1^{2\Delta_O-4} p_3^{2\Delta_O-4} \delta(p_1 + p_2)\delta(p_3 + p_4) \) plus permutations, and can be represented by a regulated \( \hat{f} \) with quadruple zero,

\[
\hat{f}(\hat{u}, \hat{v}) = \varepsilon^4 \left[ \left( \frac{\hat{u}}{\hat{v}} \right)^{\Delta_O - \gamma} + 2 \text{ cycl. perms} \right].
\]  

(27)

### IV. HOLOGRAPHIC CFTS

Holographic 4-point functions are obtained by evaluating Witten diagrams in anti-de Sitter space. These yield compact scalar integral representations for the momentum-space 4-point functions. Such expressions must again be special cases of the general solution (4) for some appropriate \( \hat{f} \). This function can be found in several ways as we now discuss. Since exchange Witten diagrams can be reduced to a sum of contact diagrams [57, 58], we focus here on the latter deferring a complete discussion to [59]. In the simplest case of a quartic bulk interaction without derivatives, we find

\[
\Phi = \langle O_{\Delta_1}(p_1)O_{\Delta_2}(p_2)O_{\Delta_3}(p_3)O_{\Delta_4}(p_4) \rangle
\]

\[
= c_W \int_0^\infty dz z^{-d/2-1} \prod_{j=1}^4 \Gamma(z_j - d/2) K_{z_j - d/2}(p_j z),
\]  

(28)

where \( c_W = 2^{2d+4-\Delta_i}/\prod_{j=1}^4 \Gamma(\Delta_j - d/2) \) and the four modified Bessel-K functions represent bulk-boundary propagators.

This integral can now be mapped to a tetrahedral topology via the star-mesh transformation from electrical circuit theory. Schwinger parametrizing the Bessel functions in (28) and evaluating the integral, we find

\[
\Phi = c_W' \prod_{j=1}^4 \int_0^\infty dZ_j Z_j^{-d/2-1} Z_j^{(d-\Delta_j)/2} e^{-p_j^2/2Z_j},
\]  

(29)

for \( c_W' = 2^{(\Delta_i - d)/2 - 5} \Gamma(\frac{1}{2} (\Delta_i - d)) c_W \) and \( Z_j = \sum_{j=1}^4 Z_j \). The exponent describes the power dissipated in a network of four impedances \( Z_j \) arranged in a star configuration. Such a network is equivalent, however, to a tetrahedral network where the impedance connecting the vertices \((i, j)\) is \( z_{ij} = Z_iZ_j/Z_\ell \) (see Fig. 4). Since all products of the impedances on opposite edges are equal, \( z^2 = z_{12}z_{34} = z_{13}z_{24} = z_{14}z_{23} \), we can re-parametrize the tetrahedron in terms of \( z \) and the three variables \( s_i = z_{1i} \) for \( i = 1, 2, 3 \). With this change of variables, the contact diagram (28) can be mapped to the form (4), with

\[
\hat{f}(\hat{u}, \hat{v}) = 16 c'_W (2\pi)^{3d/2} \left( \frac{\hat{u}}{\hat{v}} \right)^{-\Delta_i/12+d/2}
\]

\[
\times \int_0^\infty dz z^{-\Delta_i/2+3d-1} K_{\delta_{12} - \delta_{23}}(z)
\]

\[
\times K_{\delta_{23} - \delta_{14}}(z\sqrt{\hat{u}}) K_{\delta_{12} - \delta_{34}}(z/\sqrt{\hat{v}}).
\]  

(30)

This can be directly verified by Schwinger parametrizing the three Bessel functions in terms of the \( s_i \) and performing the Gaussian integrations over the momenta \( q_i \) in (4). (For full details, see [59]). Remarkably, this \( f \) features precisely the same integral (the ‘triple-\( K \)’) that describes the momentum-space 3-point function [36].

An alternative derivation of (30) starts from the position-space Mellin representation for the contact Witten diagram [53]. Applying (13), one immediately obtains a spectral representation of the form (18) with

\[
\rho(\alpha, \beta) = c'_W 2^{-\Delta_i/2+3d} (2\pi)^{3d/2} \prod_{i<j} \Gamma(\gamma_{ij})
\]  

(31)

and the \( \gamma_{ij} \) defined in (17). The equivalence of this result with (30) is seen by writing the latter as a double inverse Mellin transform. The poles of this spectral function now give residues of \( W_{\alpha, \beta} \) for which the propagators in (16) have powers \( \gamma_{ij} = -2n \). The ensuing reduction to 3-point functions shown in Fig. 2c then accounts for the appearance of the triple-\( K \) integral in (30). It would be interesting to understand if this simplification of residues is a general feature of holographic 4-point functions.

![Diagram](image-url)
V. N-POINT FUNCTION

Generalizing our discussion above, the conformal n-point function takes the form [59]

\[
\langle O_1(p_1) \ldots O_n(p_n) \rangle = \prod_{1 \leq i < j \leq n} \int \frac{d^d q_{ij}}{(2\pi)^d} \frac{\hat{f}(\{\hat{u}\})}{q_{ij}^{2\Delta + d}} \prod_{k=1}^n \int \frac{(2\pi)^d}{(2\pi)^d} \delta \left( p_k - \sum_{l=1}^n q_{kl} \right),
\]

where \( \sum_{1 \leq i < j \leq n} 2\delta_{ij} = -\Delta \), and \( \hat{f} \) is an arbitrary function of \( n(n - 3)/2 \) ‘conformal ratios’ which we denote collectively as \( \{\hat{u}\} = q_{ij}^2 q_{kl}^2 / q_{ij} q_{kl} \). The tetrahedron thus generalizes to an \((n - 1)\)-simplex where \( q_{ij} \) is the momentum running from vertex \( i \) to \( j \). We then have \( n(n - 1)/2 \) integrals and \( n - 1 \) delta functions (setting one aside for overall momentum conservation), leaving \((n - 1)(n - 2)/2\) integrals to perform. If \( n = 4 \), integrating out the delta functions and using \( q_{ij} = \epsilon_{abc} q_{bc} \), where \( a, b, c = 1, 2, 3 \) and \( \epsilon_{abc} \) is the Levi-Civita symbol, we recover (4).

VI. CONCLUSIONS

We have presented a general momentum-space representation for the scalar n-point function of any CFT. This features an arbitrary function of \( n(n - 3)/2 \) variables which play the role of momentum-space conformal ratios, and is a solution of the conformal Ward identities. It would be interesting to generalize this to tensorial correlators.

Following the success of the conformal bootstrap program in position space [60, 61], it may prove useful to develop a version in momentum space, see [29–33]. This requires understanding the expansion of the 4-point function in conformal partial waves [29, 62–65]. One then seeks to impose consistency with the operator product expansion (OPE). To correctly implement the OPE in momentum space requires a careful treatment of the short-distance singularities [66]. To understand these better, and for practical calculational purposes, it would be useful to find a compact scalar parametric representation of the general solutions (4) and (32). For 3-point functions this is provided by the triple-\( K \) integral, while for holographic n-point functions we have Witten diagrams. This suggests the existence of a similarly compact scalar representation for the general CFT n-point function. We hope to report on these questions in the near future.

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References:

[1] Alexander M. Polyakov, “Conformal symmetry of critical fluctuations,” JETP Lett. 12, 381–383 (1970).
[2] P. Di Francesco, P. Mathieu, and D. Senechal, Conformal Field Theory (Springer, New York, 1997).
[3] Ignatios Antoniadis, Pawel O. Mazur, and Emil Motlola, “Conformal invariance, dark energy, and CMB non-Gaussianity,” JCAP 09, 024 (2012), arXiv:1103.4164 [gr-qc].
[4] Juan M. Maldacena and Guilherme L. Pimentel, “On graviton non-Gaussianities during inflation,” JHEP 09, 045 (2011), arXiv:1104.2846 [hep-th].
[5] Adam Bzowski, Paul McFadden, and Kostas Skenderis, “Holographic predictions for cosmological 3-point functions,” JHEP 03, 091 (2012), arXiv:1112.1967 [hep-th].
[6] A. Kehagias and A. Riotto, “The Four-Point Correlator in Multifield Inflation, the Operator Product Expansion and the Symmetries of de Sitter,” Nucl. Phys. B868, 577–595 (2013), arXiv:1210.1918 [hep-th].
[7] Adam Bzowski, Paul McFadden, and Kostas Skenderis, “Holography for inflation using conformal perturbation theory,” JHEP 04, 047 (2013), arXiv:1211.4550 [hep-th].
[8] Ishan Mata, Suvarat Raju, and Sandip Trivedi, “CMB from CFT,” JHEP 07, 015 (2013), arXiv:1211.5482 [hep-th].
[9] Paul McFadden, “On the power spectrum of inflationary cosmologies dual to a deformed CFT,” JHEP 10, 071 (2013), arXiv:1308.0331 [hep-th].
[10] Archisman Ghosh, Nilay Kundu, Suvarat Raju, and Sandip P. Trivedi, “Conformal Invariance and the Four Point Scalar Correlator in Slow-Roll Inflation,” JHEP 07, 011 (2014), arXiv:1401.3626 [hep-th].
[11] Nilay Kundu, Ashish Shukla, and Sandip P. Trivedi, “Constraints from Conformal Symmetry on the Three Point Scalar Correlator in Inflation,” (2014), arXiv:1410.2606 [hep-th].
[12] Dionyssios Anninos, Tarek Anous, Daniel Z. Freedman, and George Konstantinidis, “Late-time Structure of the Bunch-Davies De Sitter Wavefunction,” JCAP 1511, 048 (2015), arXiv:1406.5490 [hep-th].
[13] Nima Arkani-Hamed and Juan Maldacena, “Cosmological Collider Physics,” (2015), arXiv:1503.08043 [hep-th].
[14] Hiroshi Isono, Toshifumi Nouni, Gary Shiu, Sam C. Wong, and Siyi Zhou, “Holographic non-Gaussianities in general single-field inflation,” JHEP 12, 028 (2016), arXiv:1610.01258 [hep-th].
[15] Nima Arkani-Hamed, Daniel Baumann, Hayden Lee, and Guilherme L. Pimentel, “The Cosmological Bootstrap: Inflationary Correlators from Symmetries and Singularities,” (2018), arXiv:1811.00024 [hep-th].
[16] Nima Arkani-Hamed, Paolo Benincasa, and Alexander Postnikov, “Cosmological Polytopes and the Wavefunction of the Universe,” (2017), arXiv:1709.02813 [hep-th].
[17] D. Anninos, V. De Luca, G. Franciolini, A. Kehagias, and A. Riotto, “Cosmological Shapes of Higher-Spin Gravity,” JCAP 1904, 045 (2019), arXiv:1902.01251 [hep-th].
[18] Charlotte Sleight, “A Mellin Space Approach to Cosmological Correlators,” (2019), arXiv:1906.12302 [hep-th].
[19] Charlotte Sleight and Massimo Tarannor, “Bootstrap Inflationary Correlators in Mellin Space,” (2019), arXiv:1907.01143 [hep-th].

[20] Debanjan Chowdhury, Suverat Raju, Subir Sachdev, Ajay Singh, and Philipp Strack, “Multipoint correlators of conformal field theories: implications for quantum critical transport,” Phys. Rev. B87, 051306 (2013), arXiv:1210.5247 [cond-mat.str-el].

[21] Yejin Huh, Philipp Strack, and Subir Sachdev, “Conserved current correlators of conformal field theories in 2+1 dimensions,” Phys. Rev. B88, 155109 (2013), [Erratum: Phys. Rev. B90, no.19, 199902 (2014)], arXiv:1307.6863 [cond-mat.str-el].

[22] V. P. J. Jacobs, Panagiotis Betzios, Umut Gursoy, and H. T. C. Stoof, “Electromagnetic response of interacting Weyl semimetals,” Phys. Rev. B93, 195104 (2016), arXiv:1512.04883 [hep-th].

[23] Andrew Lucas, Snir Gazit, Daniel Podolsky, and William Witczak-Krempa, “Dynamical response near quantum critical points,” Phys. Rev. Lett. 118, 056601 (2017), arXiv:1608.09286 [cond-mat.str-el].

[24] Robert C. Myers, Todd Sierens, and William Witczak-Krempa, “A Holographic Model for Quantum Critical Responses,” JHEP 05, 073 (2016), [Addendum: JHEP 09(2016)066], arXiv:1602.05599 [hep-th].

[25] Andrew Lucas, Todd Sierens, and William Witczak-Krempa, “Quantum critical response: from conformal perturbation theory to holography,” JHEP 07, 149 (2017), arXiv:1704.05461 [hep-th].

[26] Claudio Coriano, Matteo Maria Maglio, and Emil Mottola, “TTT in CFT: Trace Identities and the Conformal Anomaly Effective Action,” (2017), arXiv:1703.08860 [hep-th].

[27] Marc Gillioz, Xiaochuan Lu, and Markus A. Luty, “Graviton Scattering and a Sum Rule for the c Anomaly in 4D CFT,” JHEP 09, 025 (2018), arXiv:1801.05807 [hep-th].

[28] Claudio Coriano and Matteo Maria Maglio, “Renormalization, Conformal Ward Identities and the Origin of a Conformal Anomaly Pole,” Phys. Lett. B781, 283–289 (2018), arXiv:1802.01501 [hep-th].

[29] A. M. Polyakov, “Nonhamiltonian approach to conformal quantum field theory,” Zh. Eksp. Teor. Fiz. 66, 23–42 (1974), [Sov. Phys. JETP39,9(1974)].

[30] Rajesh Gopakumar, Apratim Kaviraj, Kaloll Sen, and Aninda Sinha, “Conformal Bootstrap in Mellin Space,” Phys. Rev. Lett. 118, 081601 (2017), arXiv:1609.00572 [hep-th].

[31] Rajesh Gopakumar, Apratim Kaviraj, Kaloll Sen, and Aninda Sinha, “A Mellin space approach to the conformal bootstrap,” JHEP 05, 027 (2017), arXiv:1611.08407 [hep-th].

[32] Hiroshi Isono, Toshihumi Noumi, and Gary Shiu, “Momentum space approach to crossing symmetric CFT correlators,” JHEP 07, 136 (2018), arXiv:1805.11107 [hep-th].

[33] Hiroshi Isono, Toshihumi Noumi, and Gary Shiu, “Momentum space approach to crossing symmetric CFT correlators II: General spacetime dimension,” (2019), arXiv:1908.04572 [hep-th].

[34] Roberta Armillis, Claudio Coriano, and Luigi Delle Rose, “Conformal Anomalies and the Gravitational Effective Action: The TJJ Correlator for a Dirac Fermion,” Phys. Rev. D81, 085001 (2010), arXiv:0910.3381 [hep-ph].

[35] Claudio Coriano, Luigi Delle Rose, Emil Mottola, and Mirko Serino, “Graviton Vertices and the Mapping of Anomalous Correlators to Momentum Space for a General Conformal Field Theory,” JHEP 08, 147 (2012), arXiv:1203.1339 [hep-th].

[36] Adam Bzowski, Paul McFadden, and Kostas Skenderis, “Implications of conformal invariance in momentum space,” JHEP 03, 111 (2014), arXiv:1304.7760 [hep-th].

[37] Claudio Coriano, Luigi Delle Rose, Emil Mottola, and Mirko Serino, “Solving the Conformal Constraints for Scalar Operators in Momentum Space and the Evaluation of Feynman’s Master Integrals,” JHEP 07, 011 (2013), arXiv:1304.6944 [hep-th].

[38] Adam Bzowski, Paul McFadden, and Kostas Skenderis, “Scalar 3-point functions in CFT; renormalisation, beta functions and anomalies,” JHEP 03, 066 (2016), arXiv:1510.08442 [hep-th].

[39] Adam Bzowski, Paul McFadden, and Kostas Skenderis, “Renormalised 3-point functions of stress tensors and conserved currents in CFT,” JHEP 11, 153 (2018), arXiv:1711.09105 [hep-th].

[40] Adam Bzowski, Paul McFadden, and Kostas Skenderis, “Renormalised CFT 3-point functions of scalars, currents and stress tensors,” JHEP 11, 159 (2018), arXiv:1805.12100 [hep-th].

[41] Claudio Coriano and Matteo Maria Maglio, “Exact Correlators from Conformal Ward Identities in Momentum Space and the Perturbative TJJ Vertex,” (2018), arXiv:1802.07675 [hep-th].

[42] Marc Gillioz, “Momentum-space conformal blocks on the light cone,” JHEP 10, 125 (2018), arXiv:1807.07003 [hep-th].

[43] Joseph A. Farrow, Arthur E. Lipstein, and Paul McFadden, “Double copy structure of CFT correlators,” JHEP 02, 130 (2019), arXiv:1812.11129 [hep-th].

[44] Hiroshi Isono, Toshihumi Noumi, and Toshiaki Takeuchi, “Momentum space conformal three-point functions of conserved currents and a general spinning operator,” JHEP 05, 057 (2019), arXiv:1903.01110 [hep-th].

[45] Teresa Bautista and Hadi Godazgar, “Lorentzian CFT 3-point functions in momentum space,” (2019), arXiv:1908.04733 [hep-th].

[46] Marc Gillioz, “Conformal 3-point functions and the Lorentzian OPE in momentum space,” (2019), arXiv:1909.00878 [hep-th].

[47] Suvrat Raju, “Four point functions of the stress tensor and conserved currents in AdS4/CFT3,” Phys. Rev. D85, 126008 (2012), arXiv:1201.6452 [hep-th].

[48] Soner Albayrak and Savan Kharel, “Towards the higher point holographic momentum space amplitudes,” JHEP 02, 040 (2019), arXiv:1810.12459 [hep-th].

[49] Shing Yan Li, Yi Wang, and Siyi Zhou, “KLT-Like Behaviour of Inflationary Graviton Correlators,” JCAP 1812, 023 (2018), arXiv:1806.06242 [hep-th].

[50] Soner Albayrak, Chandramouli Chowdhiury, and Savan Kharel, “New relation for AdS amplitudes,” (2019), arXiv:1904.10043 [hep-th].

[51] Claudio Coriano and Matteo Maria Maglio, “On Some Hypergeometric Solutions of the Conformal Ward Identities of Scalar 4-point Functions in Momentum Space,” (2019), arXiv:1903.05047 [hep-th].

[52] Gerhard Mack, “D-independent representation of conformal field theories in D dimensions via transformation
to auxiliary dual resonance models. Scalar amplitudes,” (2009), arXiv:0907.2407 [hep-th].

[53] Joao Penedones, “Writing CFT correlation functions as AdS scattering amplitudes,” JHEP 03, 025 (2011), arXiv:1011.1485 [hep-th].

[54] A. Liam Fitzpatrick, Jared Kaplan, Joao Penedones, Su- vrat Raju, and Balt C. van Rees, “A Natural Language for AdS/CFT Correlators,” JHEP 11, 095 (2011), arXiv:1107.1499 [hep-th].

[55] Anastasios Petkou and Kostas Skenderis, “A Nonrenormalization theorem for conformal anomalies,” Nucl. Phys. B561, 100–116 (1999), arXiv:hep-th/9906030 [hep-th].

[56] Adam Bzowski, “Dimensional renormalization in AdS/CFT,” (2016), arXiv:1612.03915 [hep-th].

[57] Eric D’Hoker, Daniel Z. Freedman, and Leonardo Rastelli, “AdS / CFT four point functions: How to succeed at z integrals without really trying,” Nucl. Phys. B562, 395–411 (1999), arXiv:hep-th/9905049 [hep-th].

[58] F. A. Dolan and H. Osborn, “Conformal four point functions and the operator product expansion,” Nucl. Phys. B599, 459–496 (2001), arXiv:hep-th/0011040 [hep-th].

[59] A. Bzowski, P. McFadden and K. Skenderis, to appear.

[60] David Simmons-Duffin, “The Conformal Bootstrap,” in Proceedings, Theoretical Advanced Study Institute in Elementary Particle Physics: New Frontiers in Fields and Strings (TASI 2015): Boulder, CO, USA, June 1-26, 2015 (2017) pp. 1–74, arXiv:1602.07982 [hep-th].

[61] David Poland, Slava Rychkov, and Alessandro Vichi, “The Conformal Bootstrap: Theory, Numerical Techniques, and Applications,” Rev. Mod. Phys. 91, 015002 (2019), arXiv:1805.04405 [hep-th].

[62] S. Ferrara, A. F. Grillo, G. Parisi, and Raoul Gatto, “Covariant expansion of the conformal four-point function,” Nucl. Phys. B49, 77–98 (1972), [Erratum: Nucl. Phys.B53,643(1973)].

[63] S. Ferrara, A. F. Grillo, and R. Gatto, “Tensor representations of conformal algebra and conformally covariant operator product expansion,” Annals Phys. 76, 161–188 (1973).

[64] F. A. Dolan and H. Osborn, “Conformal partial waves and the operator product expansion,” Nucl. Phys. B678, 491–507 (2004), arXiv:hep-th/0309180 [hep-th].

[65] F. A. Dolan and H. Osborn, “Conformal Partial Waves: Further Mathematical Results,” (2011), arXiv:1108.6194 [hep-th].

[66] Adam Bzowski and Kostas Skenderis, “Comments on scale and conformal invariance,” JHEP 08, 027 (2014), arXiv:1402.3208 [hep-th].