Sequences of point blow-ups from a combinatorial point of view

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Abstract

We define equivalence relations for both sequences of point blow-ups of smooth $d$–dimensional projective varieties and for associated sequential morphisms and we prove that there exists a bijection between the respective equivalence classes of sequences of point blow-ups and the associated sequential morphisms. Moreover we show some differences when dealing with the more general case of sequences of blow-ups of smooth 3–dimensional projective varieties, where all possible dimensions for the centers are allowed.

1 Introduction

Sequences of blow-ups of smooth varieties along smooth centers are useful for general algebraic geometric purposes, in particular, for resolution and classification of singularities. We will assume, additionally, that the center of each blow up is transversal to the already created exceptional divisors by the precedent blow ups, i.e. that at every point there are suitable regular systems of parameters such that both the center and the divisor components are locally defined by ideals generated by some of those parameters.

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The composition of the successive blow ups of such sequences is a projective (and therefore proper) birational morphisms $Z \to X$, where $Z$ and $X$ are smooth algebraic varieties which are respectively called sky and ground. A morphism obtained in that way will be called sequential morphism. Each sequential morphism has an exceptional divisor which is the reduced divisor of $Z$ whose support consists of those points of $Z$ at which the morphism is not a local isomorphism. By the choice of the successive centers, the exceptional divisor of a sequential morphism has only normal crossing. In fact, in this paper, we restrict to the case of point blow-ups, as it is involved in many geometric contexts, in particular, the study of algebraic curves and surfaces.

Campillo, González-Sprinberg and Lejeune-Jalabert in [1] studied sequences of point blow-ups using several combinatorial objects associated to the sequential morphism such as the $d$-ary intersection form on the abelian group of divisors with exceptional support, and Campillo and Reguera in [3] also studied other equivalent objects as the weighted dual polyhedron or the weighted tree. These results are based on the proximity of infinitely near points. More details and other applications of proximity can be found on the survey [2]. More recently, Stepanov has studied in [12] another combinatorial object, the dual complex associated to a log-resolution, and has proved that the dual complexes of two log-resolutions of an isolated singularity are homotopy equivalent. Moreover, in [4] de Fernex, Kollár and Xu have proved that in many cases, for instance in the case of isolated singularities, we can do even better by selecting a minimal representative such that it is well defined up-to piecewise linear homeomorphism.

Throughout this paper we revisit some of the results in [1], [3] focusing in the $d$–ary intersection form. Our approach is geometrical in the sense that it makes an intensive use of intersection theory, looking forward it could become useful for the more general case of varieties of dimension $d \geq 3$, where all possible dimensions for the centers are allowed. More precisely, we will focus on birational contractions (see [11]) starting from the final components. Moreover we consider the case of
any base field $K$ which is perfect but non necessarily algebraically closed.

For sequences of point blow-ups, a combinatorial equivalence notion can be defined for them, see definition 2.25. For their associated sequential morphism another combinatorial equivalence notion of equivalence can be defined in terms of the intersection forms of their divisor with exceptional support, see definition 2.20. Naturally, sequences of blow-ups and sequential morphisms which are algebraically isomorphic are equivalent for their corresponding notions.

Our first main result, theorem 3.1, states that two sequences of point blow ups are combinatorially equivalent if and only if their associated sequential morphisms are combinatorially equivalent too. In particular, the equivalence of sequences of blow ups whose composition is a sequential morphism becomes also another well defined equivalence notion for those morphisms. To prove all this, we show a rather strong result, which is interesting for its own (see theorem 5.14).

Namely, in our second main result, theorem 3.2, we show that the intersection form on the sky $Z$ determines those exceptional components which are final, i.e. divisors obtained by one of the point blow ups of the sequence. Also, we show that the intersection form on the blow down of the final components can be explicitly computed in terms of the intersection form of $Z$. This allows to recover from the sequential morphism, not only the blow up sequence up to combinatorial equivalence, but to recover, up to algebraic equivalence, the blow up sequence itself.

Finally, we illustrate with some examples that the information contained in the intersection form is minimal in some sense for our classification purpose and as well as highlight some of the main differences with the more general case in dimension $d = 3$. 
2 Sequences of blow ups and sequential morphisms

Fix a perfect field $k$ and chose an algebraic closure $\overline{k}$. Throughout this paper, unless otherwise stated, a variety will mean a reduced projective scheme over a perfect field $K$, with $K$ an algebraic extension of $k$, so it is also perfect, such that $K \subset \overline{k}$, and a point will mean a closed point.

**Definition 2.1.** A sequence of blow-ups over $K$ is defined as a sequence of blow-ups at smooth closed subvarieties $C_i$ of smooth $d-$dimensional projective varieties $Z_i$

$$Z = Z_s \xrightarrow{\pi_s} Z_{s-1} \xrightarrow{\pi_{s-1}} \cdots \xrightarrow{\pi_2} Z_1 \xrightarrow{\pi_1} Z_0,$$

such that for $i \in \{0, 1, ..., s-1\}$:

i. if we denote by $C_{i+1}$ to the center of $\pi_{i+1}$, then $C_{i+1}$ is a smooth subvariety of $Z_i$ defined over $K$ that could be reducible, that is $C_{i+1} = \sqcup C_{i+1,j}$ with $C_{i+1,j}$ irreducible over $K$

ii. $\text{codim}(C_{i+1}) \geq 2$

iii. if we denote by $E_j^i$ the exceptional divisor of $\pi_j$, and for $k > j$ we denote by $E_j^k$ the strict transform of $E_j^i$ in $Z_k$, then $C_{i+1}$ has simple normal crossings with $\{E_1^i, E_2^i, ..., E_s^i\}$

We denote by $\pi$ the composition $\pi_s \circ \pi_{s-1} \circ ... \circ \pi_1$. A morphism $\pi : Z \to Z_0$ which can be expressed, in at least one way, as a composition of blow ups with the conditions in definition [2.1] will be called a sequential morphism over $K$.

**Remark 2.2.** Given a sequence of blow-ups $\pi$, we denote by $\pi_{s,i} : Z \to Z_i$ where $\pi_{s,i} = \pi_{i+1} \circ \pi_{i+2} \circ ... \circ \pi_{s-1} \circ \pi_s$.

**Remark 2.3.** We will refer to $X = Z_0$ and $Z$ as the ground and the sky of the sequential morphism $\pi$ respectively.
Definition 2.4. The length $m$ over $K$ of a sequence of blow ups is defined as $\sum_{i=1}^{s} \#C_i$, where $\#C_i$ denotes the number of irreducible components of $C_i$ over $K$. Notice that it coincides with the number of irreducible components of the exceptional divisor $E$ (over $K$ too). Therefore, the length depends on the sequential morphism $\pi$ and it can be also called the length of $\pi$ over $K$, and it will be denoted by $m = \text{length}_K(\pi)$. Notice that $s \leq m$, and $s = m$ exactly when all the blow up centers are irreducible over $K$.

Remark 2.5. Note that in the case of sequences of point blow-ups if $K = \overline{k}$, with $\overline{k}$ the algebraic closure of $k$, then $m = \text{length}_K(\pi) = \sum_{i=1}^{s} \left[ K(C_i) : K \right]$.

Remark 2.6. Moreover we will denote by $H_\beta$ the irreducible components over $K$ of the exceptional divisor $E$ of $\pi$, that is we have $E = \bigcup_{\beta} H_\beta$.

Definition 2.7. We denote by $E_{Z,\pi,K}$ (or simply $E_Z$ if there is no confusion) to be the abelian group of divisors of $Z$ of the form $\sum_{\beta=1}^{m} n_\beta H_\beta$ where $n_\beta$ are integers, that is, the free group generated by the irreducible components of $E$ over $K$. The set $\{H_\beta\}_{\beta=1}^{m}$ is a basis of the $\mathbb{Z}$−module $E_Z$.

Definition 2.8. Given a sequential morphism $\pi : Z \to Z_0$, we consider the $d$-ary multilinear intersection form

$$I_{Z,E_Z} : E_Z \times E_Z \times \cdots \times E_Z \to \mathbb{Z},$$

defined by intersecting cycles in the sky $Z$ and taking degrees, that is

$$I_{Z,E_Z}(H_{i_1}, H_{i_2}, ..., H_{i_d}) = \deg(H_{i_1} \cdot H_{i_2} \cdot H_{i_3} \cdot \cdots \cdot H_{i_d}),$$

where $H_{i_1} \cdot H_{i_2} \cdot H_{i_3} \cdots H_{i_d}$ is an intersection class of $0$-cycles in the abelian group $A_0(Z)$, and $\deg$ stands for the degree.

Remark 2.9. For the sake of simplicity we will denote $H_{i_1} \cdot H_{i_2} \cdots H_{i_d} = I_{Z,E_Z}(H_{i_1}, H_{i_2}, ..., H_{i_d})$.

Definition 2.10. Given a sequential morphism $\pi$ as in definition 2.1, it induces a natural isomorphism $E_Z \cong \mathbb{Z}^m$, where the standard basis of $\mathbb{Z}^m$ is the image of
the $\mathbb{Z}$-basis $\{H_i\}_{i=1}^m$. In this way, the multilinear form of intersection give rise to a multilinear form

$$\Phi_{Z,E}: \mathbb{Z}^m \times \cdots \times \mathbb{Z}^m \to \mathbb{Z}$$

We say that $\Phi_{Z,E}$ is the multilinear form associated to $\pi$. The permutation group $S_m$ acts on the set of multilinear forms $\mathbb{Z}^m \times \cdots \times \mathbb{Z}^m \to \mathbb{Z}$ by interchanging the elements of the standard basis of $\mathbb{Z}^m$. It is clear that if we denote by $\Psi_{Z,E}$ to the orbit of $\Phi_{Z,E}$, then $\Psi_{Z,E}$ does not depend on the labeling of the elements of the basis $\{H_i\}$.

In order to consider different fields $K$, with $k \subset K \subset \bar{k}$, we define the notion of compatible partition of the exceptional divisor $E$. Combinatorial compatibility with a sequential morphism 2.11 will mean compatibility of the $d$–ary multilinear intersection form. Compatibility with a sequence of point blow-ups 2.12 will mean compatibility of proximity relations and degrees of the residue field extensions.

Also we will define the notion of algebraic compatibility, stronger than combinatorial, where the partition comes, by fiber product, from a sequential morphism 2.14 (resp. a sequence of blow-ups 2.15) defined over a smaller field $\bar{K}$, with $k \subset \bar{K} \subset K$.

**Definition 2.11.** Given a sequential morphism $\pi : Z_s \to Z_0$ as in definition 2.7 and a partition $E = \bigsqcup_{i=1}^l F_i$, we will say that the partition is combinatorially compatible with $\pi$ if for each $i = 1, \ldots, l$, and $H_{j_1}, H_{j_2} \in F_i$ there exists $\sigma \in S_m$ such that

i. $\sigma(j_1) = j_2$,

ii. $I_{Z,E}(H_{i_1}, H_{i_2}, \ldots, H_{i_d}) = I_{Z,E}(H_{\sigma(i_1)}, H_{\sigma(i_2)}, \ldots, H_{\sigma(i_d)})$ \(\forall i_1, \ldots, i_d\)

Let $(Z_s, \ldots, Z_0, \pi)$ be a sequence of blow ups of length $m$, and $H_1, \ldots, H_m$ the irreducible components of the exceptional divisor over $K$ of the associated sequential
morphism. For each $i$, with $i = 1, 2, \ldots, m$, let $r(i)$ be the integer such that the image of $H_i$ at $Z_{r(i)}$ is a component of the center (codimension at least 2) whose blow-up creates $H_i$. If $j$ is different from $i$, and the image of $H_j$ at $Z_{r(i)}$ has codimension 1 and contains the image of $H_i$ at $Z_{r(i)}$, then $H_i$ is said to be proximate to $j$ and we denote it by $H_i \to H_j$. It is clear that one has $r(i) > r(j)$ when $H_i$ is proximate to $H_j$.

For sequences of point blow-ups we denote $\deg(H_i) = \lceil K(P_i) : K \rceil$, where $P_i$ is the point in the center of $\pi_{r(i)}$ such that the image of $H_i$ in $Z_{r(i)}$ is $P_i$.

**Definition 2.12.** Given a sequence of point blow-ups $(Z_0, \ldots, Z_s, \pi)$ and a partition $E = \bigsqcup_{i=1}^l F_i$, we will say that the partition is combinatorially compatible with the sequence $(Z_0, \ldots, Z_s, \pi)$ if for each $i = 1, \ldots, l$ and $H_{j_1}, H_{j_2} \in F_i$ there exists $\sigma \in S_m$ such that

i. $\sigma(j_1) = j_2$,

ii. $\deg(H_{j_1}) = [K(P_{j_1}) : K] = [K(P_{\sigma(j_1)}) : K] = \deg(H_{\sigma(j_1)})$,

iii. if $H_{j_1} \in F_{i_1}$, $H_{j_k} \in F_{i_k}$ and $H_{j_k} \to H_{j_1}$, then $H_{\sigma(j_k)} \to H_{\sigma(j_1)}$.

**Remark 2.13.** Note that it makes sense to define $F_i \to F_j$ if $\exists H_i \in F_i, H_j \in F_j$ with $H_i \to H_j$.

**Definition 2.14.** Given a sequential morphism $\pi : Z_s \to Z_0$ as in definition 2.7 and a partition $E = \bigsqcup_{i=1}^l F_i$, we will say that the partition is algebraically compatible with the morphism $\pi$ if there exist a smaller field $\tilde{K} \subset K$ with $k \subset \tilde{K}$, there are $\tilde{K}$-varieties $\tilde{Z}_0$ and $\tilde{Z}$ and a $\tilde{K}$-morphism $\tilde{\pi} : \tilde{Z} \to \tilde{Z}_0$

\[
\begin{array}{c}
Z \cong \tilde{Z} \times_{\text{Spec}(\tilde{K})} \text{Spec}(K) \xrightarrow{\tilde{\pi}} Z_0 \cong \tilde{Z}_0 \times_{\text{Spec}(\tilde{K})} \text{Spec}(K) \\
\beta \\
\tilde{Z} \xrightarrow{\tilde{\pi}} \tilde{Z}_0
\end{array}
\]

such that the exceptional divisor of $\tilde{\pi}$, $\tilde{E}$, has irreducible components $\tilde{H}_1, \ldots, \tilde{H}_l$ and for each $i = 1, \ldots, l$ then $\forall H \in F_i$ $\beta(H) = \tilde{H}_i$. 

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Definition 2.15. Given a sequence of point blow-ups \((Z_0, \ldots, Z_s, \pi)\) and a partition 
\(E = \sqcup_{i=1}^l F_i\), we will say that the partition is algebraically compatible with the 
sequence \((Z_0, \ldots, Z_s, \pi)\) if there exist a smaller field \(\widetilde{K} \subset K\) with \(k \subset \widetilde{K}\) and there 
are \(\widetilde{K}\)-varieties \(\widetilde{Z}_i\) and \(\widetilde{K}\)-morphisms \(\widetilde{Z}_{i+1} \to \widetilde{Z}_i\)

\[
\begin{array}{c}
Z_s \\ \pi_s \\
\downarrow \\
Z_{s-1} \\ \pi_{s-1} \\
\downarrow \\
Z_1 \\ \pi_1 \\
\downarrow \\
Z_0 \\
\end{array}
\]

\[
\begin{array}{c}
\beta \\
\uparrow \\
Z_s \\ \pi_s \\
\downarrow \\
Z_{s-1} \\ \pi_{s-1} \\
\downarrow \\
Z_1 \\ \pi_1 \\
\downarrow \\
Z_0 \\
\end{array}
\]

where \(Z_i \cong \widetilde{Z}_i \times_{\text{Spec}(\widetilde{K})} \text{Spec}(K)\) \(\forall i = 1, \ldots, s\), such that the exceptional divisor of 
\((\widetilde{Z}_0, \ldots, \widetilde{Z}_l, \widetilde{\pi})\) has irreducible components \(\widetilde{H}_1, \ldots, \widetilde{H}_l\) and for each \(i = 1, \ldots, l\) then 
\(\forall H \in F_i \beta(H) = \widetilde{H}_i\).

Remark 2.16. Note that since \(k\) is perfect then \(\widetilde{K} \subset K\) is a separable algebraic 
extension, so \(\widetilde{K}\) and \(K\) are both perfect fields.

A combinatorially (resp. algebraically) marked sequential morphism is denoted 
\((\pi : Z_s \to Z_0, \sqcup_{i=1}^l F_i)_{\text{comb}}\) (resp. \((\pi : Z_s \to Z_0, \sqcup_{i=1}^l F_i)_{\text{alg}}\)) where \(\sqcup_{i=1}^l F_i\) is a partition 
combinatorially (resp. algebraically) compatible with \(\pi\). The same notation 
holds for sequences.

Note also that if a partition is algebraically compatible with a sequential morphism (resp. a sequence) then the partition is combinatorially compatible with the sequential morphism (resp. the sequence).

Now we define our notions of equivalence (algebraic and combinatorial) for marked sequential morphisms (definitions 2.17 and 2.20).

Definition 2.17. We say that two algebraically marked sequential morphisms 
\((\pi : Z \to Z_0, \sqcup_{i=1}^l F_i)_{\text{alg}}\) and \((\pi' : Z' \to Z'_0, \sqcup_{i=1}^l F'_i)_{\text{alg}}\) over \(K\) are algebraically 
equivalent, and we denote it by \(\pi \sim_K \pi'\), if and only if there exist smaller fields
\( \widetilde{K}, \widetilde{K}' \subset K \) with \( \widetilde{K} \cong_k \widetilde{K}' \) satisfying the conditions of definition 2.14.

\[
\begin{array}{c}
Z_s \cong \widetilde{Z}_s \times \text{Spec}(\widetilde{K}) \xrightarrow{\pi} Z_0 \cong \widetilde{Z}_0 \times \text{Spec}(\widetilde{K}) \xrightarrow{\text{Spec}(K)} \\
\beta \downarrow \quad \pi \downarrow \\
\widetilde{Z}_s \quad \widetilde{Z}_0
\end{array}
\]

\[
\begin{array}{c}
Z'_s \cong \widetilde{Z}'_s \times \text{Spec}(\widetilde{K}') \xrightarrow{\pi'} Z'_0 \cong \widetilde{Z}'_0 \times \text{Spec}(\widetilde{K}') \xrightarrow{\text{Spec}(K)} \\
\beta' \downarrow \quad \pi' \downarrow \\
\widetilde{Z}'_s \quad \widetilde{Z}'_0
\end{array}
\]

and there exist isomorphisms \( a \) and \( b \) such that the following diagram is commutative.

\[
\begin{array}{c}
\widetilde{Z} \xrightarrow{b} \widetilde{Z}' \\
\beta \downarrow \quad \beta' \downarrow \\
\widetilde{Z}_0 \xrightarrow{a} \widetilde{Z}'_0
\end{array}
\]

**Remark 2.18.** Notice that two different sequences of point blow ups can have isomorphic algebraically marked associated sequential morphisms. For instance, if one blow ups with a center having two or more components (over \( K \)) of a ground variety \( X = Z_0 \), then one gets the same sequential morphism, up to algebraic isomorphism, than if one blow ups a proper subset of components followed by successive subsets of the strict transforms of the others. Such a situation can happen at every step of a sequence of blow ups. However, it is clear that those examples do not provide essential changes on the geometry of the sequences.

**Definition 2.19.** Given a combinatorially marked sequential morphism \( (\pi : Z_s \to Z_0, \sqcup_{i=1}^l F_i)_{\text{comb}} \), we can also consider the \( d \)-ary multilinear intersection form associated to the partition

\[
\mathcal{I}_{Z, \sqcup_{i=1}^l F_i} : F_Z \times F_Z \times \cdots \times F_Z \to \mathbb{Z},
\]

where \( F_Z \) is the free abelian group generated by \( \{F_i\} \) and by an abuse of notation \( F_i = \sum_{H \in F_i} H \). The intersection form is defined by intersecting cycles in the sky.
and taking degrees, that is
\[ I_{Z, \sqcup_{i=1}^d F_i}(F_{i_1}, F_{i_2}, \ldots, F_{i_d}) = \deg(F_{i_1} \cdot F_{i_2} \cdot F_{i_3} \cdot \ldots \cdot F_{i_d}), \]
where \( F_{i_1} \cdot F_{i_2} \cdot F_{i_3} \cdot \ldots \cdot F_{i_d} \) is an intersection class of 0–cycles in the abelian group \( A_0(Z) \), and \( \deg \) stands for the degree.

**Definition 2.20.** Given two combinatorially marked sequential morphisms \((\pi : Z_s \to Z_0, \sqcup_{i=1}^d F_i)_{\text{comb}}\) and \((\pi' : Z'_s \to Z'_0, \sqcup_{i=1}^d F'_i)_{\text{comb}}\) we say that the associated multilinear forms \( \Phi_{Z, \sqcup_{i=1}^d F_i} \) and \( \Phi_{Z', \sqcup_{i=1}^d F'_i} \) are equivalent, and we denote it by \( \Phi_{Z, \sqcup_{i=1}^d F_i} \sim \Phi_{Z', \sqcup_{i=1}^d F'_i} \), if there exists \( \tau \in S_l \) such that
\[ \tau(\Phi_{Z, \sqcup_{i=1}^d F_i}) = \Phi_{Z', \sqcup_{i=1}^d F'_i}. \]
Moreover, the combinatorially marked sequential morphisms \((\pi : Z_s \to Z_0, \sqcup_{i=1}^d F_i)_{\text{comb}}\) and \((\pi' : Z'_s \to Z'_0, \sqcup_{i=1}^d F'_i)_{\text{comb}}\) are said to be combinatorially equivalent, when their associated multilinear maps \( \Phi_{Z, \sqcup_{i=1}^d F_i} \) and \( \Phi_{Z', \sqcup_{i=1}^d F'_i} \) are equivalent.

**Remark 2.21.** If \( \pi : Z_s \to Z_0 \) and \( \pi' : Z'_s \to Z'_0 \) are algebraically equivalent, then \( b(H_i) = H'_{\sigma(i)} \) for some permutation \( \sigma \in S_m \), so the multilinear intersection forms are equivalent as in definition 2.20. However, the converse is not true. For instance, for \( d = 2 \) and we consider sequences of five blow ups, the first on a rational point of a smooth surface and the other at four different rational points of the exceptional divisor created by the blow up of the original point. Then the 5–multilinear form, up to a permutation of \( S_5 \), is independent on the choice of the four exceptional points; however, two choices with a different cross-ratio provide sequential morphism which are not algebraically isomorphic.

**Definition 2.22.** Given a variety \( X \) we will call a brick blow-up with ground \( X \) to a sequential morphism obtained as a composition of point blow-ups with disjoint centers \( \sqcup_{j=1}^l C_j \subset X \), \( X' = X_0 \to X_{l-1} \to \ldots \to X_1 \to X \). Note that \( Z_i \to Z_{i-1} \) is the brick blow-up at \( C_i \), where \( C_i \) need not to be irreducible.

**Definition 2.23.** We say that two algebraically marked sequences of point blow ups, \((Z_s, \ldots, Z_0, \pi, \sqcup_{i=1}^d F_i)_{\text{alg}}\), and \((Z'_s, \ldots, Z'_0, \pi', \sqcup_{i=1}^d F'_i)_{\text{alg}}\), are algebraically
 equivalent over $K$, and we denote it by $(Z_s, ..., Z_0, \pi, \sqcup_{i=1}^t F_i)_{alg} \sim_K (Z_{s'}, ..., Z_{0'}, \pi', \sqcup_{i=1}^{t'} F'_i)_{alg}$, if and only if $l = l'$ and there exist smaller fields $\bar{K}, \bar{K'} \subset K$ with $\bar{K} \cong \bar{K'}$.

\[
\begin{array}{c}
Z_s \xrightarrow{\pi_s} Z_{s-1} \xrightarrow{\pi_{s-1}} \cdots \xrightarrow{\pi_1} Z_1 \xrightarrow{\pi_0} Z_0 \\
\downarrow \beta \quad \downarrow \pi_s \quad \downarrow \pi_{s-1} \quad \cdots \quad \downarrow \pi_1 \quad \downarrow \pi_0 \quad \downarrow \beta \\
\bar{Z}_s \xrightarrow{\bar{\pi}_s} \bar{Z}_{s-1} \xrightarrow{\bar{\pi}_{s-1}} \cdots \xrightarrow{\bar{\pi}_1} \bar{Z}_1 \xrightarrow{\bar{\pi}_0} \bar{Z}_0
\end{array}
\]

with $Z_i \cong \bar{Z}_i \times_{\text{Spec}(K)} \text{Spec}(K)$ (resp. $Z_i' \cong \bar{Z}_i' \times_{\text{Spec}(K')} \text{Spec}(K')$) and algebraic isomorphisms $a, b = b_1, b_{t-1}, ..., b_1$, with $t \leq s$, such that there are indexes $r_1, ..., r_t = s \in \{1, ..., t\}$ and $r'_{1}, ..., r'_{t'} = s' \in \{1, ..., s'\}$, where $Z_{r_i} \to Z_{r_i-1} \to \cdots \to Z_{r_{t-1}}$ (resp. $Z_{r_i'} \to Z_{r_i'-1} \to \cdots \to Z_{r_{t'-1}'}$), with $r_i > r_i-1$ (resp. $r_i' > r_i'-1$), is a brick blow-up $\forall i = 1, ..., t$ as in definition 2.22 and the diagram

\[
\begin{array}{c}
\bar{Z}_s \xrightarrow{b} \bar{Z}_{r_{t-1}} \xrightarrow{b_{t-1}} \bar{Z}_{r_{t-2}} \cdots \xrightarrow{b_1} \bar{Z}_{r_1} \xrightarrow{a} \bar{Z}_0 \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
\bar{Z}_s' \xrightarrow{b'} \bar{Z}_{r_{t'-1}}' \xrightarrow{b'_{t-1}} \bar{Z}_{r_{t'-2}}' \cdots \xrightarrow{b'_1} \bar{Z}_{r_1}' \xrightarrow{a'} \bar{Z}_0'
\end{array}
\]

is commutative.

**Remark 2.24.** If two algebraically marked sequences of blow ups, $(Z_s, ..., Z_0, \sqcup_{i=1}^t F_i, \pi)_{alg}$ and $(Z'_s, ..., Z'_0, \sqcup_{i=1}^{t'} F'_i, \pi')_{alg}$, are algebraically equivalent, then their associated sequential morphisms are also algebraically equivalent. Therefore, in particular, one has $b(H_i) = H'_{\sigma(i)}$ where $\sigma \in S_m$ is a permutation. Moreover, for two different indexes $i, j$, one has that $H_i$ is proximate to $H_j$ if and only if $H'_{\sigma(i)}$ is proximate to $H'_{\sigma(j)}$.

**Definition 2.25.** We say that two combinatorially marked sequences of point blow ups, $(Z_s, ..., Z_0, \sqcup_{i=1}^t F_i, \pi)_{comb}$ and $(Z'_s, ..., Z'_0, \sqcup_{i=1}^{t'} F'_i, \pi')_{comb}$ as before with respective partitions $E = \sqcup_{i=1}^t F_i$ and $E' = \sqcup_{i=1}^{t'} F'_i$ and irreducible components of
the exceptional divisor $H_1, \ldots, H_m; H'_1, \ldots, H'_m$, with $l = l'$, are combinatorially equivalent if and only there is a permutation $\tau$ in $S_l$ such that for every two different indexes $i, j$ one has

i. $F_i$ is proximate to $F_j$ if and only if $F'_{\tau(i)}$ is proximate to $F'_{\tau(j)}$,

ii. $\deg(F_i) = \sum_{H \in F_i} \deg(H) = \sum_{H' \in F'_i} \deg(H') = \deg(F'_{\tau(i)})$

3 Statements of the main results

Throughout this paper we will restrict ourselves to the case of sequences of point blow-ups, that is sequences of blow-up in which all centers are closed points. Notice that in this case, for each exceptional component $E_i$, the value of $r(i)$ is the greatest integer such that the image of $E_i$ at $Z_{r(i)}$ is 0−dimensional.

Our main theorem states the following result.

**Theorem 3.1.** Two combinatorially marked sequences of point blow-ups $(Z_s, \ldots, Z_0, \sqcup_{i=1}^l F_i, \pi)_{\text{comb}}$ and $(Z'_s, \ldots, Z'_0, \sqcup_{i=1}^{l'} F'_i, \pi')_{\text{comb}}$, with $l = l'$, are combinatorially equivalent over $K$ as in definition 2.23 if and only if their associated combinatorially marked sequential morphisms $(\pi : Z_s \to Z_0, \sqcup_{i=1}^l F_i)_{\text{comb}}$ and $(\pi' : Z'_s \to Z'_0, \sqcup_{i=1}^{l'} F'_i)_{\text{comb}}$ are combinatorially equivalent over $K$ as in definition 2.20, and both statements are true if and only if the associated multilinear forms $\Phi_{Z, \sqcup_{i=1}^l F_i, \pi, K}$ and $\Phi_{Z', \sqcup_{i=1}^{l'} F'_i, \pi', K}$ are equivalent as in definition 2.20.

Furthermore, using the necessary condition in remark 2.21 that is $\pi \sim_K \pi' \Rightarrow \Phi_{Z, \sqcup_{i=1}^l F_i, \pi, K} \sim \Phi_{Z', \sqcup_{i=1}^{l'} F'_i, \pi', K}^g$, it follows a stronger result which is the core result of this article.

**Theorem 3.2.** Given two algebraically marked sequential morphisms $(\pi : Z_s \to Z_0, \sqcup_{i=1}^l F_i)_{\text{alg}}$ and $(\pi' : Z'_s \to Z'_0, \sqcup_{i=1}^{l'} F'_i)_{\text{alg}}$, then they are algebraically equivalent over $K$ as in definition 2.17 if and only if there exist algebraically marked
sequences of point blow-ups \((Z_s, ..., Z_0, \bigcup_{i=1}^l F_i, \pi)_{\text{alg}}\) and \((Z'_s, ..., Z'_0, \bigcup_{i=1}^{l'} F'_i, \pi')_{\text{alg}}\) associated to \((\pi : Z_s \to Z_0, \bigcup_{i=1}^l F_i)_{\text{alg}}\) and \((\pi' : Z'_s \to Z'_0, \bigcup_{i=1}^{l'} F'_i)_{\text{alg}}\) respectively such that they are algebraically equivalent over \(K\) as in definition 2.23.

4 Some useful intersection theory

In this section we will recall some known results on intersection theory which we will use in the proof of our results. We do not present an exhaustive list, but the aim of this section is to list some results which either are critical in our development, or are very specific in intersection theory, or we did not find any proof in the literature.

Along this section, \(X\) and \(Z\) will stand for smooth algebraic varieties over \(K\), the considered maps \(Z \to X\) for proper morphisms, and \(d\) for the dimension of \(X\).

Remark 4.1. Let \(V\) be a \(k\)-dimensional irreducible subvariety of \(X\). Although it is common to denote by \([V]\) to the equivalence class of \(V\) in the Chow ring \(A(X)\), for simplicity we will denote also by \(V\) to the equivalence class whenever there is not possible confusion.

Definition 4.2. [7, Section 2.6] Let \(D\) be an effective Cartier divisor on a variety \(X\), and let \(i : D \to X\) be the inclusion. There are Gysin homomorphisms

\[ i^* : Z_k X \to A_{k-1} D \]

for \(k = 1, ..., \dim(X)\) defined by the formula

\[ i^*(\alpha) = D \cdot \alpha \]

Proposition 4.3. [7, Proposition 2.6.] There are therefore induced homomorphisms:

\[ i^* : A_k X \to A_{k-1} D \]

for \(k = 1, ..., \dim(X)\) such that one has
a If $\alpha$ is a $k$-cycle on $X$, then

$$i_\ast i^\ast (\alpha) = c_1(\mathcal{O}_X(D)) \cap \alpha$$

b If $\alpha$ is a $k$-cycle on $D$, then

$$i^\ast i_\ast (\alpha) = c_1(N_{D/X}) \cap \alpha$$

Notice that if $f : Y \to X$ is a proper morphism, $V$ is an irreducible subvariety of $Y$ and $W = f(V)$ its image by $f$, then one defines the push-forward of the cycle associated to $V$ as follows

$$f_\ast (V) = \begin{cases} 0 & \text{if } \dim V > \dim W \\ [K(V) : K(W)]W & \text{if } \dim V = \dim W \end{cases}$$

where $[K(V) : K(W)]$ denotes the degree of the extension of function fields $K(V)/K(W)$. This definition allows also to define, by linearity, the push-forward $f_\ast (\alpha)$ for any cycle $\alpha$.

The next result known as the projection formula is specially useful and it will play an essential role in some of our intersection computations.

**Proposition 4.4.** [7, Proposition 2.3.] Let $D$ be a divisor on $X$, $f : Z \to X$ a proper morphism, $\alpha$ a $k$-cycle on $Z$, and $g$ the morphism from $f^{-1}(|D|) \cap |\alpha|$ to $|D| \cap f(|\alpha|)$ induced by $f$. Then

$$g_\ast (f^\ast D \cdot \alpha) = D \cdot f_\ast (\alpha)$$

in $A_{k-1}(|D| \cap f(|\alpha|))$.

The following proposition is obtained as a particular case of the one above.

**Proposition 4.5.** [5, Proposition 1.10] Let $f : Z \to X$ be a proper surjective morphism. Let $D_1, D_2, \ldots, D_r$ be Cartier divisors on $X$ with $r = d = \dim(X)$. Then, one has

$$f^\ast D_1 \cdot f^\ast D_2 \cdots f^\ast D_r = \deg(f)D_1 \cdot D_2 \cdots D_r$$

where $\deg(f) = [K(Z) : K(X)]$, if $\deg(f)$ is finite.
Let $P$ be a closed point of a variety $X$. Let $\tilde{X}$ be the blow-up of $X$ at $P$, and let $E$ be the exceptional divisor. We have then a fibre square

$$
\begin{array}{ccc}
E & \xrightarrow{j} & \tilde{X} \\
\downarrow{g} & & \downarrow{f} \\
P & \xrightarrow{i} & X \\
\end{array}
$$

In order to apply the projection formula (either proposition 4.4 or proposition 4.5) then we need previously the following result

**Theorem 4.6.** [7, Corollary 6.7.1.] Let $V$ be a $k$-dimensional subvariety of $X$, and let $\tilde{V} \subset \tilde{X}$ be the proper transform of $V$. Then

$$
f^*(V) = \tilde{V} + (e_P V) j^*(L),
$$

in $A_k(\tilde{X})$, where $L$ is a $k$-dimensional linear subspace of $E = \mathbb{P}^{d-1}_{K(P)}$, $K(P)$ is the residue field of $\mathcal{O}_{P,X}$, and $e_P V$ is the multiplicity of $P$ on $V$.

The next two results refers to the computation of the normal bundle of certain varieties we are interested in. More concretely, varieties resulting from the intersection of two Cartier divisors meeting regularly (i.e. transversally at smooth points of both) and varieties obtained as the strict transforms of other ones by a blow-up.

**Lemma 4.7.** Let $D$ and $F$ be two irreducible components of a simple normal crossing divisor $E$ that is regularly embedded in $X$. If we denote by $G = D \cap F$ then

$$
N_{F/X}|_G \cong N_{G/D}
$$

**Proof.** Let $i_{G,D} : G \rightarrow D$ and $i_{D,X} : D \rightarrow X$ be regular embeddings. Then the composite $i_{D,X} \circ i_{G,D}$ is a regular imbedding, and there is an exact sequence of vector bundles on $G$ (see [10, Proposition 19.1.5])

$$
0 \rightarrow N_{G/D} \rightarrow N_{G/X} \rightarrow N_{D/X}|_G \rightarrow 0
$$
Since $D$ and $F$ meet regularly in $X$, then (see [8 Proposition 3.6.]):

$$N_{G/X} \cong N_{D/X}|_G \oplus N_{F/X}|_G$$

So, we have the following exact sequences of vector bundles on $G$

$$0 \to N_{G/D} \to N_{D/X}|_G \oplus N_{F/X}|_G \to N_{D/X}|_G \to 0$$

Then it follows that $N_{G/D} \cong N_{F/X}|_G$.

**Proposition 4.8.** [7 Appendix B.6.10.] If $C \subset Y$ and $Y \subset X$ are regular embeddings, let $\tilde{X} = Bl_C X$, $E$ the exceptional divisor in $\tilde{X}$, $\rho$ the projection from $\tilde{X}$ to $X$. Let $\tilde{Y} = Bl_C Y$. Then $\tilde{Y} \subset \rho^{-1}(Y)$, $E \subset \rho^{-1}(Y)$, and $\tilde{Y}$ is the residual scheme to $E$ in $\rho^{-1}(Y)$, i.e., the ideal sheaves of $\tilde{Y}$, $E$ and $\rho^{-1}(Y)$ in $\tilde{X}$ are related by

$$\mathcal{I}(\tilde{Y}) \cdot \mathcal{I}(E) = \mathcal{I}(\rho^{-1}(Y)).$$

In addition, the canonical embedding of $\tilde{Y}$ in $\tilde{X}$ is a regular imbedding, with normal bundle

$$N_{\tilde{Y}/\tilde{X}} \cong \pi^* N_{Y/X} \otimes O(-F)$$

where $\pi$ is the projection from $\tilde{Y}$ to $Y$, and $F$ is the exceptional divisor on $\tilde{Y}$ of such projection.

**Proposition 4.9.** [7 Appendix B.6.3] With the same notations as above

$$N_{E/\tilde{Z}} \cong O_E(-1)$$

The last three results of this section deals with the structure of the Chow ring of a projective bundle over a smooth projective variety and the computation of class corresponding to a projective subbundle.

**Theorem 4.10.** [6 Theorem 9.6.] Let $E$ be a vector bundle of rank $r + 1$ on a smooth projective variety $X$, and let $\varsigma = c_1(O_{\mathcal{P}(E)}(1)) \in A^1(\mathcal{P}(E))$, and $p_{\mathcal{P}(E)} : \mathcal{P}(E) \to X$ the projection of the induced projective bundle. The map $p_{\mathcal{P}(E)}^* : A(X) \to A(\mathcal{P}(E))$ is an injective ring homomorphism, and via this map one has the isomorphism of $A(X)$-algebras given by

$$A(\mathcal{P}(E)) \cong A(X)[c]/(c^{r+1} + c_1(E)c^r + \cdots + c_{r+1}(E))$$
Corollary 4.11. [6, Theorem 2.1.] The Chow ring of $\mathbb{P}^n$ is

$$A(\mathbb{P}^n) \cong \mathbb{Z}[\varsigma]/(\varsigma^{n+1})$$

where $\varsigma \in A^1(\mathbb{P}^n)$ is the rational equivalence class of all the hyperplane; more generally, the class of all closed subschemes of codimension $k$ and degree $d$ is $d\varsigma^k$.

Proposition 4.12. [6, Proposition 9.13] If $X$ is a smooth projective variety and $\mathcal{F} \subset \mathcal{E}$ are vector bundles on $X$ of ranks $s$ and $r$ respectively, then

$$[\mathbb{P}(\mathcal{F})] = \varsigma^{r-s} + \gamma_1\varsigma^{r-s-1} + ... + \gamma_{r-s} \in A^{r-s}(\mathbb{P}(\mathcal{E})),$$

where $\varsigma = c_1(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))$ and $\gamma_k = c_k(\mathcal{E}/\mathcal{F})$. Moreover, the normal bundle of $\mathbb{P}\mathcal{F}$ in $\mathbb{P}\mathcal{E}$ is $\mathcal{O}_{\mathbb{P}\mathcal{F}}(1) \otimes p^*\mathcal{F}(\mathcal{E}/\mathcal{F})$.

5 Proof of the main theorems

Now we will prove our main theorems 3.1 and 3.2. First we need to define and characterize the notion of final component of $\mathcal{E}$. The naive idea is that a component $H_i$ will be final if there is some other sequence of point blow-ups $(Z_s', ..., Z_0', \pi')$ associated to the sequential morphism $\pi: Z_s \to Z_0$ such that $H_i$ will be the exceptional divisor of the last blowup of $(Z_s', ..., Z_0', \pi')$.

Definition 5.1. Let $(Z_s, ..., Z_0, \pi)$ be a sequence of blow-ups. The exceptional divisor $E$ in $Z$ is $\{E_1, ..., E_s\}$ (see definition 2.7). Assume $H_i \in E_i$ be an irreducible component over $K$. Set $H_i^1 \subset E_i^1$ to be the image of $H_i$ in $Z_i$. We say that $H_i$ is final with respect to $(Z_s, ..., Z_0, \pi)$ if there exists an open set $U_i$ on $Z_i$ such that $H_i^1 \subset U_i$, $V_i = \pi_{s,i}^{-1}(U_i) \subset Z$, and $\pi_{s,i}|_{V_i} : V_i \to U_i$ is an isomorphism (see remark 2.7 for $\pi_{s,i}$).

Definition 5.2. Let $\pi: Z_s \to Z_0$ be a sequential morphism. We say that an irreducible component $H$ of $E$ is final with respect to $K$ if it exists a sequence of blow-ups $(Z_s, ..., Z_0, \pi)$ associated to $\pi: Z_s \to Z_0$ such that $H$ is final with respect to this sequence.
Lemma 5.3. In the case of sequences of point blow-ups, if \( H_i \in E_i^s \) and \( H_j \in E_j^s \) are both final then \( H_i \cap H_j = \emptyset \), that is \( H_i \) and \( H_j \) have not geometric points in common over \( K \).

Proof. Set \( P_i \in Z_{i-1}, P_j \in Z_{j-1} \), to be the points such that \( H_i \) maps to \( P_i \) and \( H_j \) maps to \( P_j \). If \( H_\beta \) is final with \( \beta \in \{i, j\} \), then \( H_\beta \cong \mathbb{P}^{d-1}_{K(P_\beta)} \) and \( N_{H_\beta/Z} \cong \mathcal{O}_{H_\beta}(-1) \).

Let us suppose that \( H_i \cap H_j \neq \emptyset \). Then either \( P_i \) is proximate to \( P_j \) or \( P_j \) is proximate to \( P_i \). In the first case \( H_j \not\cong \mathbb{P}^{d-1}_{K(P_j)} \) and \( N_{H_j/Z} \not\cong \mathcal{O}_{H_j}(-1) \) so \( H_j \) is not final, whereas in the second case \( H_i \not\cong \mathbb{P}^{d-1}_{K(P_i)} \) and \( N_{H_i/Z} \not\cong \mathcal{O}_{H_i}(-1) \) so \( H_i \) is not final. Through any of them we get to a contradiction.

The result above makes a huge difference with respect to the more general case, where two final divisors may not have an empty intersection (see example 6.4).

Remark 5.4. Assume that we have a sequential morphism associated to a sequence of point blow-ups. If an irreducible component \( H_\alpha \) of \( E \) is final with respect to one representative of the sequences associated to this sequential morphism then it is final with respect to all. This fact drastically changes when more general centers are allowed (see example 6.4).

Before characterizing numerically final divisors, we need a numerical characterization of empty intersections \( H_i \cap H_j = \emptyset \).

Lemma 5.5. In case of sequences of point blow-ups \( H_i \cap H_j = \emptyset \) if and only if \( (H_i)^s(H_j)^r = 0 \) for all \( r \neq 0 \) and \( s \neq 0 \) with \( r + s = d \).

Proof. If \( H_i \cap H_j = \emptyset \) then \( (H_i)^s(H_j)^r = 0 \) follows directly.

In order to prove the necessary condition, we will prove that \( H_i \cap H_j \neq \emptyset \Rightarrow \exists r \neq 0, s \neq 0 \) such that \( (H_i)^s(H_j)^r \neq 0 \), it is enough to prove it in the case of a sequence of point blow-ups of length \( m = 3 \) since the general result follows by induction.

First let \( \pi_1 : Z_1 \rightarrow Z_0 \) be the blow-up with center \( P_1 \). Now we blow-up \( Z_1 \) with
center $P_2$ such that $P_2 \in E_1^1$, that is such that $P_2$ is proximate to $P_1$. Thus we have the following diagram

\[
\begin{array}{ccc}
H_1^2 \cap H_2^2 & \xrightarrow{i_{H_2^2 \cap H_2^2}} & H_1^2 \\
\downarrow & & \downarrow \\
H_1^2 & \xrightarrow{i_{P_2, H_1^2}} & H_1^2
\end{array}
\]

Then it follows by propositions 4.3 and 4.12 that

\[
H_1^2 \cdot H_2^2 \cdot H_2^2 = i_{H_2^2 \cdot Z_2 \cdot H_2^2}(c_1(N_{H_2^2 / Z_2}) \cap (H_1^2 \cap H_2^2))
\]

By proposition 4.8

\[
N_{H_2^2 / Z_2} \cong \pi_2 \mid_{H_1^2}(N_{H_1^2 / Z_2}) \otimes \mathcal{O}(-H_1^2 \cap H_2^2)
\]

Furthermore we have the following commutative diagram where all the morphisms are regular imbeddings

\[
\begin{array}{ccc}
H_1^2 \cap H_2^2 & \xrightarrow{i_{H_2^2 \cap H_2^2, Z_2}} & H_2^2 \\
\downarrow & & \downarrow \\
H_1^2 \cap H_2^2 & \xrightarrow{i_{H_2^2 \cap H_2^2, Z_2}} & Z_2 \\
\downarrow & & \downarrow \\
H_2^2 & \xrightarrow{j_{H_2^2, Z_2}} & H_2^2
\end{array}
\]

Now due to the commutativity of the diagram above and the result of lemma 4.7 then

\[
H_1^2 \cdot H_2^2 \cdot H_2^2 = i_{H_2^2 \cdot Z_2 \cdot H_2^2}(c_1(N_{H_2^2 \cap H_2^2 / H_1^2}) \cap (H_1^2 \cap H_2^2))
\]

so

\[
H_1^2 \cdot H_2^2 \cdot H_2^2 = j_{H_2^2, Z_2}(i_{H_2^2 \cap H_2^2, H_1^2 \cdot H_2^2}(i_{H_1^2 \cap H_2^2})(H_1^2 \cap H_2^2)^2)
\]

\[
H_1^2 \cdot H_2^2 \cdot H_1^2 = j_{H_1^2, Z_2}((-i_{H_1^2 \cap H_2^2, H_2^2 \cdot H_2^2}(i_{H_1^2 \cap H_2^2})(H_1^2 \cap H_2^2)^2))
\]
By induction on \( r \) and \( s \) respectively it follows

\[
H_1^2(H_2^2)^r = j_{H_1^2,z_2*}((j_{H_1^2,H_2^2,1*}(H_1^2 \cap H_2^2))^r)
\]

\[
(H_1^3)^sH_2^3 = (-1)^{s-1}j_{H_1^3,z_2*}((j_{H_1^2,H_2^3,1*}(H_1^2 \cap H_2^3))^s)
\]

Finally we can conclude that

\[
(H_1^3)^s(H_2^3)^r = (-1)^{s-1}deg(j_{H_1^3,z_2*}((i_{H_1^2,H_2^3,1*}(H_1^2 \cap H_2^3))^{r+s-1}))
\]

Note that \((H_1^3)^s(H_2^3)^r \neq 0\) and furthermore if we denote by \(\Delta_{1,2}\) to \(deg(j_{H_1^2,H_2^3,1*}(H_1^2 \cap H_2^3))^{r+s-1})\) then \((H_1^3)^s(H_2^3)^r = (-1)^{s-1}\Delta_{1,2}\) if \(r + s = d\).

Let \(\pi : Z_3 \rightarrow Z_2\) be the blow-up of \(Z_2\) with center \(P_3\), such that \(P_3 \in H_1^2 \cap H_2^3\),

that is \(P_3\) is proximate to \(P_1\) and to \(P_2\). Then it follows that by theorem \(3.2\)

\[
(H_1^3)^s(H_2^3)^r = (\pi_3^*(H_1^2) - j_{H_1^3,z_2*}(H_3^3)^s)(\pi_3^*(H_2^3) - j_{H_1^3,z_2*}(H_3^3))^r
\]

and due to the projection formula, propositions \(4.4\) and \(4.5\) then

\[
(H_1^3)^s(H_2^3)^r = (\pi_3^*(H_1^2))^s(\pi_3^*(H_2^3))^r + (-1)^d(j_{H_1^3,z_2*}(H_3^3))^d = (H_1^3)^s(H_2^3)^r + (-1)^d(j_{H_1^3,z_2*}(H_3^3))^d
\]

(1)

Since \((H_1^3)^s(H_2^3)^r \neq 0\) and furthermore that it is of the form \((H_1^3)^s(H_2^3)^r = (-1)^{s-1}\Delta_{1,2}\), then it must exist \(r, s\) with \(r \neq 0\) and \(s \neq 0\) such that \((H_1^3)^s(H_2^3)^r \neq 0\).

For the more general case, let us suppose that \(\{P_{\alpha_1}, P_{\alpha_2}, ..., P_{\alpha_k}\}\) are proximate to both \(P_1\) and \(P_2\). Then by iterating equation (1)

\[
(H_1^3)^s(H_2^3)^r = (H_1^3)^s(H_2^3)^r + (-1)^d \sum_{j=1}^{k} (j_{H_1^3,z_2*}(H_{\alpha_j}^3))^d,
\]

so it must exist \(r, s\) with \(r \neq 0\) and \(s \neq 0\) such that \((H_1^3)^s(H_2^3)^r \neq 0\).

Now we are ready to characterize numerically when an irreducible component \(H_i\) of the exceptional divisor \(E\) is final.

**Proposition 5.6.** \(H_i\) is final if and only if

\[
(H_i)^d = (-1)^r (H_i)^s \cdot (H_j)^r \text{ and } (H_i) \cdot (H_j)^d-1 > 0
\]
for every \( j \) such that \( H_i \cap H_j \neq \emptyset \) (see lemma 5.3 for a numerical characterization) and for all natural numbers \( r \) and \( s \) with \( r + s = d \).

**Proof.** We have the following commutative diagram where we denote by \( D_{i,j} \) to the scheme theoretic intersection \( H_i \cap H_j \) and all the morphism are regular imbeddings

\[
\begin{array}{ccc}
H_i & \xrightarrow{i_{D_{i,j}, H_i}} & j_{H_i, Z} \\
\downarrow{i_{D_{i,j}, H_j}} & & \downarrow{j_{H_j, Z}} \\
D_{i,j} & \xrightarrow{i_{D_{i,j}, Z}} & Z \\
\end{array}
\]

First let us suppose that \( H_i \) is final. Then \( H_i \cong \mathbb{P}^{d-1} \) and by proposition 4.9

\[ N_{H_i/Z} = O_{H_i}(-1), \]

so it follows by proposition 4.3 that

\[ (H_i)^d = (-1)^{d-1} \deg(j_{H_i, Z*}(\zeta^{d-1})), \]

where \( \zeta = c_1(O_{H_i}(1)) \).

Moreover as we have seen in proposition 4.3

\[ H_i \cdot H_j = j_{H_i, Z*}j_{H_i, Z}(H_j), \]

and \( D_{i,j} \) determines a projective subbundle on \( H_i \), so by proposition 4.12

\[ D_{i,j} = \zeta, \]

in \( A(H_i) \). Since \( E \) is simple normal crossing divisor, then by lemma 4.7

\[ N_{H_j/Z}|_{D_{i,j}} \cong N_{D_{i,j}/H_i}. \]

It follows by propositions 4.3 and 4.12 that

\[ H_i \cdot H_j \cdot H_j = j_{H_j, Z*}(c_1(N_{H_j/Z}) \cap D_{i,j}) \]

\[ H_i \cdot H_j \cdot H_j = j_{H_i, Z*}(c_1(N_{H_i/Z} \cap D_{i,j}) = (-1)^{j_{H_i, Z*}(\zeta^2)} \]

Now due to the commutativity of the diagram above and the result of lemma 4.7 then

\[ H_i \cdot H_j \cdot H_j = j_{H_i, Z*}(c_1(N_{D_{i,j}/H_i}) \cap D_{i,j}) = j_{H_i, Z*}(\zeta^2) \]
By induction on $r$ and $s$ respectively it follows

$$H_i(H_j)^r = j_{H_i,Z^\ast}((s^r)$$

$$(H_i)^s H_j = (-1)^{s-1} j_{H_i,Z^\ast}((s^s))$$

So we can conclude that

$$(H_i)^s \cdot (H_j)^r = (-1)^{s-1}\deg(j_{H_i,Z^\ast}((s^{r+s-1})))$$

and more concretely

$$(-1)^r(H_i)^s \cdot (H_j)^r = (-1)^{r+s-1}\deg(j_{H_i,Z^\ast}((s^{r+s-1})))$$

Then $(H_i)^d = (-1)^r(H_i)^s \cdot (H_j)^r$. Moreover $(H_i) \cdot (H_j)^{d-1} = \deg(j_{H_i,Z^\ast}((s^{d-1}))) > 0$.

Now let us suppose that $H_i$ is not final. If $P_\alpha$ is proximate to $P_i$, then we have the following commutative diagram

$$
\begin{array}{c}
H_i^\alpha \cap H_\alpha^\alpha \\
\pi_\alpha|_\mu^\alpha \cap H_i^\alpha \\
P_\alpha \\
\end{array}
\xrightarrow{\iota_{\mu^\alpha \cap H_i^\alpha}} H_i^\alpha \\
\pi_\alpha|_\mu^\alpha \cap H_\alpha^\alpha \\
\iota_{\mu^\alpha \cap H_\alpha^\alpha} \\
$$

Among all the index satisfying $\alpha \to i$ there must exist an index $j$ such that $j \to i$ but that there not exists $k$ with $k \to i$ and $k \to j$. Let $j$ be such index. Since $H_i$ is not final then by proposition 4.8 its normal bundle satisfies

$$N_{H_i/Z} = \pi^\ast_{n,i}|_{H_i}(N_{H_i/Z_i}) \otimes \bigotimes_{\alpha \to i} \pi^\ast_{n,\alpha}|_{H_i^\alpha}(O(-H_i^\alpha \cap H_\alpha^\alpha))$$

Now, by the projection formula (see proposition 4.3)

$$deg(j_{H_i,Z^\ast}((\pi^\ast_{n,i}|_{H_i}(c_1(N_{H_i/Z_i})^{n_i}))) \prod_{\alpha \to i} (\pi^\ast_{n,\alpha}|_{H_i^\alpha}((-1)^{d-1}(i_{H_i^\alpha \cap H_\alpha^\alpha}^\ast(H_i^\alpha \cap H_\alpha^\alpha)^{d-1})))) = 0$$

with $n_i + \sum_{\alpha \to i} n_\alpha = d$, and then

$$(H_i)^d = deg(j_{H_i,Z^\ast}((\pi^\ast_{n,i}|_{H_i}(c_1(N_{H_i/Z_i})^{d-1}))) + \sum_{\alpha \to i} deg(j_{H_i,Z^\ast}((\pi^\ast_{n,\alpha}|_{H_i^\alpha}((-1)^{d-1}(i_{H_i^\alpha \cap H_\alpha^\alpha}^\ast(H_i^\alpha \cap H_\alpha^\alpha)^{d-1}))))$$
Furthermore, by an analogous reasoning to the case when $H_i$ is final then

$$H_i \cdot H_j = j_{H_i, Z}((c_1(N_{D_{i,j}}/H_i) \cap D_{i,j})) = j_{H_i, Z}((i_{H_i \cap H_j, H_i^\ast}(D_{i,j}))^2)$$

and

$$H_i \cdot H_j = j_{H_i, Z}((i_{H_i \cap H_j, H_i^\ast}(H_j^i \cap H_j^j)) = j_{H_i, Z}((i_{H_i \cap H_j, H_i^\ast}(D_{i,j}))^2)$$

By induction on $r$ and $s$ respectively it follows

$$H_i(H_j)^r = j_{H_i, Z}((i_{H_i \cap H_j, H_i^\ast}(D_{i,j}))^r)$$

so it follows that

$$(H_i)^s(H_j)^r = (-1)^{s-1}j_{H_i, Z}((i_{H_i \cap H_j, H_i^\ast}(D_{i,j}))^{r+s-1})$$

If $d$ is even then

$$(-1)^r(H_i)^s(H_j)^r = (-1)^{r+s-1}j_{H_i, Z}((i_{H_i \cap H_j, H_i^\ast}(D_{i,j}))^{r+s-1}) \neq (H_i)^d$$

since

$$\deg(j_{H_i, Z}((\pi^\ast_{n,1} |_{H_i}(c_1(N_{H_j^i/Z_j})^{d-1})))) + \sum_{\alpha \neq j} \deg(j_{H_i, Z}((\pi^\ast_{n,\alpha} |_{H_i^\ast}(i_{H_i \cap H_j, H_i^\ast}(H_j^i \cap H_j^\alpha)^{d-1})))) < 0$$

If $d$ is odd then

$$(H_i) \cdot (H_j)^{d-1} = \deg(j_{H_i, Z}((i_{H_i \cap H_j, H_i^\ast}(D_{i,j}))^{r+s-1})), $$

so $(H_i)(H_j)^{d-1} < 0$. \hfill \Box

**Remark 5.7.** Note that whereas in [3] the authors show two different numerical characterizations for final divisors, distinguishing when the dimension $d$ is even or odd, our numerical characterization works for both cases.

**Proposition 5.8.** Given an algebraically marked sequence $(Z_s, ..., Z_0, \cup_{i=1}^j F_i, \pi)_{alg}$ with $H, H' \in F_i$, then $H$ is final if and only if $H'$ is final too.

**Proof.** If $H, H' \in F_i$ then there exist a sequence $(\tilde{Z}_s, ..., \tilde{Z}_0, \tilde{E}, \tilde{\pi})$ over $\tilde{K}$ such that $\beta(H) = \beta(H')$, where $\beta : Z_s \times_{\text{Spec}(\tilde{K})} \text{Spec}(K) \to \tilde{Z}_s$, so it follows that if $H$ satisfies the numerical condition of proposition [5.6] $H'$ will satisfy it too. \hfill \Box
Proposition 5.9. Given an algebraically marked sequence $(Z_s, \ldots, Z_0, \sqcup_{i=1}^{l} F_i, \pi)_{alg}$ then $F_i$ is final if and only if

$$(F_i)^d = (-1)^r (F_i)^s \cdot (F_j)^r \text{ and } (F_i) \cdot (F_j)^{d-1} > 0$$

for every $j$ such that $F_i \cap F_j \neq \emptyset$ and for all natural numbers $r$ and $s$ with $r+s = d$.

Proof. This is a consequence of proposition 5.6 since if $H, H' \in F_i$ and $H \neq H'$ then $H \cap H' = \emptyset$. \qed

Corollary 5.10. Let $(\pi : Z_s \rightarrow Z_0, \sqcup_{i=1}^{l} F_i)_{alg}$ and $(\pi' : Z'_s \rightarrow Z'_0, \sqcup_{i=1}^{l'} F'_i)_{alg}$ algebraically marked sequential morphisms that are algebraically equivalent. If we denote by $b'$ to be the isomorphism $b' : Z_s = \tilde{Z}_s \times_{Spec(\tilde{K})} Spec(K) \rightarrow Z'_s = \tilde{Z}'_s \times_{Spec(\tilde{K}')} Spec(K)$, that is the extension of that in definition 2.17, then $F_i$ is final if and only if $b'(F_i)$ is final.

Now we will define a key tool for our proof of theorems 3.1 and 3.2, that of a regular projective contraction.

Definition 5.11. Let $X'$ be a $d$-dimensional variety, $L$ be an effective Cartier divisor on $X'$ and $Y$ an $r$-dimensional variety with $r < d-1$. Then we say that $L$ is contractible to $Y$ within $X'$ if there exist a variety $X$ and a proper birational morphism $f : X' \rightarrow X$ such that

i. $f(L) = Y$

ii. $L$ is a closed subset of $X'$ which consists of the points where $f$ is not an isomorphism

We call this triple $(X', f, X)$ a contraction of $L$ to $Y$, or simply a contraction. We shall say that $L$ is regularly and projectively contractible to $Y$ within $X'$, when moreover

iii. $X$ is a non-singular projective variety

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Before continuing, we need to prove the following technical lemma that is crucial for the uniqueness of the regular projective contractions.

**Lemma 5.12.** Let $X$ and $Y$ be two affine normal varieties such that $X = \text{Spec}(A)$ and $Y = \text{Spec}(B)$. Let $\pi : Z \rightarrow X$ and $\pi' : Z' \rightarrow Y$ be proper morphisms. If $Z$ is isomorphic to $Z'$ then $A \cong B$.

**Proof.** By [9, Theorem 3.2.1], since $\pi$ and $\pi'$ are proper morphisms then $\pi_*(\mathcal{O}_Z)$ and $\pi'_*(\mathcal{O}_{Z'})$ are a coherent sheaves on $X$ and $Y$ respectively. Since $X$ and $Y$ are both normal, then $\mathcal{O}_X \cong \pi_*(\mathcal{O}_Z)$ and $\mathcal{O}_Y \cong \pi'_*(\mathcal{O}_{Z'})$, so

\[
A \cong H^0(X, \mathcal{O}_X) \cong H^0(Z, \mathcal{O}_Z),
\]

\[
B \cong H^0(Y, \mathcal{O}_Y) \cong H^0(Z', \mathcal{O}_{Z'}).
\]

Since $H^0(Z, \mathcal{O}_Z) \cong H^0(Z', \mathcal{O}_{Z'})$, then it follows that $A \cong B$. \hfill \Box

**Proposition 5.13.** Let $(Z_s, ..., Z_0, \pi)$ be a sequence of point blow-ups (as in definition 2.1) of length $m$ and let $H_i \in E_i^s$ be an irreducible component of the exceptional divisor of $\pi$. If $H_i$ is final, then there exists a regular projective contraction $(Z, f_m, X_{m-1})$ of $H_i$ to a point such that $f_m(E)$ is a simple normal crossing divisor and $X_{m-1}$ is the sky of a sequence of point blow-ups with ground $Z_0$.

**Proof.** Since $H_i$ is final there must exist an isomorphism between the two opens sets $U_i \subset Z_i$ and $V_i \subset Z_s$ via $\pi_{s,i}$. After shrinking $U_i$ if necessary, we may assume that $U_i \setminus H_i^s$ is isomorphic via $\pi_i$ to an open set of $Z_{i-1} \setminus \{P_i\}$ where $P_i = \pi_s(H_i^s)$. Note that $W_i = \pi_i(U_i)$ is an open set in $Z_{i-1}$. In fact $\pi_i|_{U_i}$ is the blow-up of $W_i$ at $P_i$. 

\[
\begin{array}{c}
V_i \\
\pi_{s,i|U_i} \downarrow \quad \downarrow \pi_i|_{U_i} \\
U_i \\
\pi_i|_{W_i} \downarrow \\
W_i
\end{array}
\]
Set $\phi = (\pi_i \circ \pi_{s,i})|_{V_i}$ the composition morphism from $V_i$ to $W_i$

\[
\begin{array}{c}
V_i \\
\pi_{s,i|V_i} \downarrow \phi \downarrow \pi_{|U_i}
\end{array}
\]

where $\phi := \pi_i \circ \pi_{s,i}$.

Set $\overline{W_i} = Z \setminus H_i$. We construct $X_{m-1}$ by gluing $W_i$ and $\overline{W_i}$ along the open isomorphic sets $W_i \setminus \{P_i\} \subset W_i$ and $V_i \setminus H_i \subset \overline{W_i}$. Note that $W_i \setminus \{P_i\} \cong U_i \setminus H_i \cong V_i \setminus H_i$.

Now we define $f_m : Z \to X_{m-1}$, $f_m|_{\overline{W_i}} = Id_{\overline{W_i}}$, $f_m|_{V_i} = \phi$, which is well defined by the isomorphisms.

Finally, it is clear from the construction that if we denote by $D_{X_{m-1}}$ to the image $f_m(E)$, then $D_{X_{m-1}}$ is a simple normal crossing divisor.

An alternative construction of the contraction if $K = \mathbb{R}$.

Since $H_i$ is final, then $H_i \cong \mathbb{P}^d_{K(P_i)}$, where $P_i = \pi_{s,i}(H_i)$, and moreover by proposition 1.9 its normal bundle $N_{H_i/Z} \cong \mathcal{O}(H_i)(-1)$. Let $\mathcal{F}$ be a very ample line bundle on $Z$. Then $\mathcal{F} \otimes \mathcal{O}_{H_i} = \mathcal{L} \otimes \mathcal{O}_{H_i}(u)$. If we consider the line bundle $\mathcal{L} := \mathcal{F} \otimes \mathcal{O}(H_i) \otimes u$, then by 11 Corollary 2 there exists a regular projective contraction $(Z, \varphi, X'_{m-1})$ of $H_i$ to a closed point, that we will denote by $P'_i$, with $\varphi|_{H_i} = P'_i$, such that $\varphi$ is defined by the complete linear system $|\mathcal{L}|$. To see that $D_{X'_{m-1}} := \varphi(E)$ is still a simple normal crossing divisor we prove that the contraction is unique up to isomorphism. By 11 Theorem 3 $(Z, \varphi)$ is the blowing up of $X'_{m-1}$ at a point $P'_i$. Let $Y_i$ be an affine open neighborhood of $P_i$ in $X_{m-1}$. If we denote by $Y'_i := \varphi(f^{-1}_m(Y_i))$, then $Y'_i$ is an affine neighborhood of $P'_i$ in $X'_{m-1}$ and there exist two proper morphisms

\[
\begin{array}{c}
f_m|_{f^{-1}_m(Y_i)} \\
Y_i \downarrow \varphi|_{f^{-1}_m(Y_i)} \downarrow Y'_i
\end{array}
\]
By lemma 5.12 this implies that $Y_i \cong Y'_i$, so it's then clear that $D_{X_{m-1}}$ is a simple normal crossing divisor.

So we have proved that there exists a regular projective contraction $(Z, f_m, X_{m-1})$ of $H_i$ to a point $P_i \in X_{m-1}$.

Following the notations of definition 5.1, let $W_i = \pi_i(U_i)$. Then by definition 5.11 $f_m|_{Z \setminus V_i} : Z \setminus V_i \to X_{m-1} \setminus f_m(V_i)$ is an isomorphism. Now we define $g : X_{m-1} \to Z_{i-1}$ as follows: $g|_{W_i} = \pi_{s,i-1}|_{W_i}$ and $g|_{W_i} = Id_{W_i}$. By our construction of $X_{m-1}$ $g$ is well defined, and by the definition $g : X_{m-1} \to Z_{i-1}$ is a sequence of point blow-ups.

Hence the composition $X_{m-1} \to Z_{i-1} \to Z_0$ is a sequence of blow-ups.
Theorem 5.14. Let \((\pi : Z_s \to Z_0, \sqcup_{i=1}^l F_i)_{\text{alg}}\) be an algebraically marked sequential morphism. Given the \(d\)-ary multilinear intersection form associated to the partition \(I_{Z, \sqcup_{i=1}^l F_i}\) (see definition 2.19) we can recover all the algebraically marked sequences of point blow-ups that are associated to algebraically marked sequential morphisms in the same algebraic equivalence class of \((\pi : Z_s \to Z_0, \sqcup_{i=1}^l F_i)_{\text{alg}}\).

Proof. Since \(\sqcup_{i=1}^l F_i\) is a partition algebraically compatible with \(\pi\) then \(\exists \bar{K} \subset K\) as in definition 2.13. If \(H \in F_i\) is final then \(\bar{H} = \beta(H)\) is final for \(\bar{\pi} : \bar{Z}_s \to \bar{Z}_0\).

We will prove this result first by contracting one irreducible component of the exceptional divisor \(\bar{E}\) each time.

Since the set formed by final divisors is not empty, let us suppose that \(\bar{H}_i\) is final,
then by proposition 5.6 there exists a regular projective contraction \((\tilde{Z}_s, \tilde{f}_l, \tilde{X}_{l-1})\) of \(\tilde{H}_i\) to a point such that \(\tilde{X}_{l-1}\) is the sky of a sequence of point blow-ups with ground \(\tilde{Z}_0\).

\[
\begin{array}{c}
\tilde{Z}_s \\
\downarrow \tilde{\pi}_s \\
\vdots \\
\tilde{Z}_i \\
\downarrow \tilde{\pi}_i \\
\vdots \\
\tilde{Z}_{l-1} \\
\downarrow \tilde{\pi}_{l-1} \\
\tilde{X}_{l-1} \\
\end{array}
\]

The next step in our proof refers to how to obtain the intersection form in \(\tilde{X}_{l-1}\) associated to the simple normal crossing divisor \(\tilde{D}_{\tilde{X}_{l-1}}\).

If we denote by \(\tilde{H}_{\tilde{X}_{l-1}, i_1}\) to \(\tilde{f}_i(\tilde{H}_i)\), then by the projection formula (proposition 4.5)

\[
\tilde{H}_{\tilde{X}_{l-1}, i_1} \cdot \tilde{H}_{\tilde{X}_{l-1}, i_2} \cdots \tilde{H}_{\tilde{X}_{l-1}, i_d} = \tilde{f}_l^* (\tilde{H}_{\tilde{X}_{l-1}, i_1}) \cdot \tilde{f}_l^* (\tilde{H}_{\tilde{X}_{l-1}, i_2}) \cdots \tilde{f}_l^* (\tilde{H}_{\tilde{X}_{l-1}, i_d}),
\]

Applying the result of theorem 4.6 then

\[
\tilde{H}_{\tilde{X}_{l-1}, i_1} \cdot \tilde{H}_{\tilde{X}_{l-1}, i_2} \cdots \tilde{H}_{\tilde{X}_{l-1}, i_d} = (\tilde{H}_{i_1} + \delta_{i_1, i} \tilde{H}_i) \cdot (\tilde{H}_{i_2} + \delta_{i_2, i} \tilde{H}_i) \cdots (\tilde{H}_{i_d} + \delta_{i_d, i} \tilde{H}_i), \quad (2)
\]
where $\delta_{i,i} = 1$ if $\tilde{H}_i \cap \tilde{H}_j \neq \emptyset$ (see numerical characterization in lemma 5.5) and $\delta_{i,i} = 0$ otherwise.

**Remark 5.15.** It follows then that by iterating the above process, that is by contracting a final divisor at each step, we will obtain a sequence of point blow-ups of length $l$. The algebraically marked sequence obtained depends on the choice of final components. Below we will prove that all the algebraically marked sequential morphisms associated to the sequences constructed in this way are algebraically equivalent.

**Proposition 5.16.** Any of the algebraically marked sequences obtained as in 5.15, that is as composition of regular projective contractions from a fixed sky $Z$ and a fixed simple normal crossing divisor $E$, are associated to algebraically marked sequential morphisms in the same algebraic equivalence class (see definition 2.17).

Before proving this, we need the following lemma

**Lemma 5.17.** Given a fixed sky $Z$ and a fixed simple normal crossing divisor $E$, let us suppose that $H_i$ and $H_j$ are both finals. Then there is an isomorphism $X_{m-2} \cong X'_{m-2}$ making the following diagram commutative

\[
\begin{array}{c}
\text{Z} \\
f_m & \searrow & f'_m \\
\swarrow & X_{m-1} & X'_{m-1} \\
X_{m-1} & \searrow & X'_{m-1} \\
& \searrow^{f_m} & \nearrow^{f'_m} \\
X_{m-2} & \cong & X'_{m-2} \\
& \searrow^{f_{m-1}} & \nearrow^{f'_{m-1}} \\
& \searrow & \nearrow \\
& & \end{array}
\]

where $f_m$ is the contraction of $H_i$ and $f_{m-1}$ is the contraction of $H_{X_{m-1},j}$, whereas $f'_m$ is the contraction of $H_j$ and $f'_{m-1}$ is the contraction of $H'_{X'_{m-1},i}$.

**Proof.** To begin with, if we denote by $O_{i,m-2} = f_{m-1} \circ f_m(H_i)$, $O_{j,m-2} = f_{m-1}(H_{X_{m-1},j})$, $O'_{i,m-2} = f'_{m-1} \circ f'_m(H_i)$ and $O'_{j,m-2} = f'_{m-1}(H'_{X'_{m-1},j})$, then it follows that

\[
X_{m-2} \setminus \{O_{i,m-2}, O_{j,m-2}\} \cong Z \setminus H_i \cup H_j \cong X'_{m-2} \setminus \{O'_{i,m-2}, O'_{j,m-2}\}
\]
Let $W_j$ be an open affine open neighborhood of $O_{j,m-2}$. If we denote by $V_j$ to the inverse image $f_m^{-1} \circ f_{m-1}^{-1}(W_j)$, then the image $W'_j = f'_m \circ f'_{m-1}(V_j)$ will be an affine open neighborhood of $O'_{j,m-2}$. Then since $f_{m-1} \circ f_m|_{V_j}$ and $f'_{m-1} \circ f'_m|_{V_j}$ are both proper morphisms, it follows by lemma [5.12] $W_j \cong W'_j$.

If we denote by $W_i$ to an open affine neighborhood of $O_{i,m-2}$ and $W'_i = f'_m \circ f'_{m-1}(V_i)$, where $V_i$ is the inverse image $f_m^{-1} \circ f_{m-1}^{-1}(W_i)$, then in a similar way we can prove that $W_i \cong W'_i$, so it follows $X_{m-2} \cong X'_m$ since all isomorphisms are given by global sections.

Consequently, we have the following corollary, which means that proposition 5.16 holds for length 2.

**Corollary 5.18.** If $Z$ is the sky of a sequence of point blow-ups of length 2, then any of the two sequences of point blow-ups obtained following the procedure in [5.15] are associated to algebraically marked sequential morphisms in the same algebraic equivalence class.

In order to prove proposition 5.16 we need the following definition.

**Definition 5.19.** We say that two sequences of point blow-ups obtained as in remark [5.16] that is through the composition of regular projective contractions from a fixed sky $Z$ and a fixed simple normal crossing divisor $E$,

\[
\begin{array}{cccccccccc}
Z & \xrightarrow{f_s} & X_{s-1} & \xrightarrow{f_{s-1}} & X_{s-2} & \xrightarrow{f_{s-2}} & \cdots & \xrightarrow{f_2} & X_1 & \xrightarrow{f_1} & X_0 \\
\uparrow & & \uparrow & & \uparrow & & \ddots & & \uparrow & & \uparrow \\
Z & \xrightarrow{f'_s} & X'_{s-1} & \xrightarrow{f'_{s-1}} & X'_{s-2} & \xrightarrow{f'_{s-2}} & \cdots & \xrightarrow{f'_2} & X'_1 & \xrightarrow{f'_1} & X'_0 \\
\end{array}
\]

have the same end if at least the first contraction is common to both. i.e. one has $f_s = f'_s$.

**Proof of Proposition 5.16** Let us suppose then that $Z$ is the sky of a sequence of $n+1$ point blow ups and that proposition 5.16 is true for sequences of length lower or equal than $n$. If two sequences obtained as above $\rho := f_1 \circ f_2 \circ \cdots \circ f_n \circ f_{n+1}$
and \( \rho' := f'_1 \circ f'_2 \circ \ldots \circ f'_n \circ f'_{n+1} \) have the same end, then it is clear that both are associated to algebraically marked sequential morphism in the same algebraic equivalence class. It is a direct consequence of the fact that by hypothesis the assertion is true for sequences of length lower or equal than \( n \).

If two sequences \( \rho := f_1 \circ f_2 \circ \ldots \circ f_n \circ f_{n+1} \) and \( \sigma := g_1 \circ g_2 \circ \ldots \circ g_n \circ g_{n+1} \) have not the same end, then let us suppose that \( f_{n+1} \) and \( g_{n+1} \) correspond to the contraction of \( H_i \) and \( H_j \) respectively. Consider all the sequences that belong to the tree contracting \( H_j \) first, there must exist a sequence \( \rho' := f'_1 \circ f'_2 \circ \ldots \circ f'_n \circ f'_{n+1} \) contracting \( H_{X_n,i} \) secondly. Analogously, if we consider all sequences contracting \( H_i \) first, there must exist a sequence \( \sigma' := g'_1 \circ g'_2 \circ \ldots \circ g'_n \circ g'_{n+1} \) contracting \( H_{Y_n,j} \) secondly.

By corollary \[5.18\] the sequences \( f'_n \circ f'_{n+1} \) and \( g'_n \circ g'_{n+1} \) of length 2 are associated to sequential morphism in the same equivalence class, so it just remain to proof that \( f'_1 \circ f'_2 \circ \ldots \circ f'_{n-2} \circ f'_{n-1} \) belong to the same equivalence class that \( g'_1 \circ g'_2 \circ \ldots \circ g'_{n-2} \circ g'_{n-1} \).
But this equivalence follows directly from the hypothesis, so $\rho \sim \sigma$.

**Proof of Theorem 5.14.** We can apply proposition 5.16 to the morphism $\tilde{Z}_s \to \tilde{Z}_0$ and by scalar extension $\times_{\text{Spec}(\tilde{K})} \text{Spec}(K)$ the algebraically marked sequences of point blow-ups constructed as above

$$
\begin{array}{cccc}
Z & \xrightarrow{X_{i-1}} & \cdots & \xrightarrow{X_1} & \xrightarrow{X_0} \\
\downarrow & & & & \\
\tilde{Z} & \xrightarrow{\tilde{X}_{i-1}} & \cdots & \xrightarrow{\tilde{X}_1} & \xrightarrow{\tilde{X}_0} \\
\end{array}
$$

where $X_i \cong \tilde{X}_i \times_{\text{Spec}(\tilde{K})} \text{Spec}(K)$, so the theorem is proved. 

Now we are able to prove our main theorems 3.1 and 3.2.

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Theorem 3.1. Two combinatorially marked sequences of point blow-ups \((Z_s, ..., Z_0, \bigsqcup_{i=1}^{l} F_i, \pi)_{\text{comb}}\) and \((Z'_s, ..., Z'_0, \bigsqcup_{i=1}^{l'} F'_i, \pi')_{\text{comb}}\), with \(l = l'\), are combinatorially equivalent over \(K\) as in definition 2.25 if and only if their associated combinatorially marked sequential morphisms \((\pi : Z_s \to Z_0, \bigsqcup_{i=1}^{l} F_i)_{\text{comb}}\) and \((\pi' : Z'_s \to Z'_0, \bigsqcup_{i=1}^{l'} F'_i)_{\text{comb}}\) are combinatorially equivalent over \(K\) as in definition 2.20, and both statements are true if and only if the associated multilinear maps \(\Phi_{Z, \bigsqcup_{i=1}^{l} F_i}\) and \(\Phi_{Z', \bigsqcup_{i=1}^{l'} F'_i}\) are equivalent too as in definition 2.20.

First we will prove that if two combinatorially marked sequential morphisms \((\pi : Z_s \to Z_0, \bigsqcup_{i=1}^{l} F_i)_{\text{comb}}\) and \((\pi' : Z'_s \to Z'_0, \bigsqcup_{i=1}^{l'} F'_i)_{\text{comb}}\) are combinatorially equivalent then the associated combinatorially marked sequences of points blow-ups are combinatorially equivalent too. To begin with, we need a numerical characterization of proximity.

Lemma 5.21. Let \((Z_s, ..., Z_0, \bigsqcup_{i=1}^{l} F_i, \pi)_{\text{comb}}\) be a combinatorially marked sequence. Then \(P_i \to P_j\) if and only if

\begin{enumerate}
  \item \(\exists \alpha \in \{2, 3, ..., m-1, m\}\) such that \(H_{X\alpha, i} \cap H_{X\alpha, j} \neq \emptyset\) (see numerical characterization of lemma 5.9).
  \item \((H_{X\alpha, i})^{d} = (-1)^{r}(H_{X\alpha, i})^{s}(H_{X\alpha, k})^{r} (H_{X\alpha, j})^{d-1} > 0 \forall k, H_{X\alpha, i} \cap H_{X\alpha, k} \neq \emptyset\).
\end{enumerate}

where \(Z_s = X_m \to X_{m-1} \to \cdots \to X_\alpha \to \cdots \to X_0 = Z_0\) is any sequence of contractions obtained as in remark 5.15.

Proof. If \(P_i\) is proximate to \(P_j\) then \(H^i_{X\alpha, i} \cap H^j_{X\alpha, j} \neq \emptyset\). Moreover, if \(P_i \in Z_r\), then \(H^i_{X_{\alpha-1}}\) is final for the sequence \(\pi_{r+1} \circ \pi_{r} \circ \cdots \circ \pi_1\), for some \(r \geq i\) and we have \(i\).

Conversely, if \(H_{X\alpha, i}\) is final for the sequence \(\pi_{r+1} \circ \pi_{r} \circ \cdots \circ \pi_1\) for some \(r \geq i\) then by proposition 5.13 there exist a regular projective contraction \(f_\alpha : X_\alpha \to X_{\alpha-1}\) of \(H_{X\alpha, i}\) such that \(f_\alpha(H_{X\alpha, i}) = O_{\alpha-1} \subset H_{X_{\alpha-1}}\) .

\(\Box\)
Remark 5.22. The result of the previous lemma also holds for characterizing numerically the proximity between elements of the combinatorially compatible partition $F_i \to F_j$.

Proof of Theorem 3.1. Assume that the combinatorially marked sequential morphisms $(\pi : Z_s \to Z_0, \sqcup_{i=1}^{l} F_i)_{comb}$ and $(\pi' : Z'_s \to Z'_0, \sqcup_{i=1}^{l'} F'_i)_{comb}$ are combinatorially equivalent. If $F_i$ is final, then there exists $\tau \in S_l$ such that

i. $F'_{\tau(i)}$ is final,

ii. $F_i \cap F_\beta \neq \emptyset$ if and only if $F'_{\tau(i)} \cap F'_{\tau(\beta)} \neq \emptyset$,

iii. $F_{\beta_1} \cdot F_{\beta_2} \cdots F_{\beta_d} = F'_{\tau(\beta_1)} \cdot F'_{\tau(\beta_2)} \cdots F'_{\tau(\beta_d)}$

Furthermore, by equation (2)

$$H_{X_{m-1,\beta_1}} \cdot H_{X_{m-1,\beta_2}} \cdots H_{X_{m-1,\beta_d}} = (H_{\beta_1} + \delta_{\beta_1,i}H_i)(H_{\beta_2} + \delta_{\beta_2,i}H_i) \cdots (H_{\beta_d} + \delta_{\beta_d,i}H_i),$$

so it follows then that there exists $\bar{\tau} \in S_{l-1}$ such that

$$F_{X_{l-1,\beta_1}} \cdot F_{X_{l-1,\beta_2}} \cdots F_{X_{l-1,\beta_d}} = F'_{X'_{l-1,\bar{\tau}(\beta_1)}} \cdot F'_{X'_{l-1,\bar{\tau}(\beta_2)}} \cdots F'_{X'_{l-1,\bar{\tau}(\beta_d)}}$$

Consequently we have that $\Phi_{Z,i \sqcup_{i=1}^{l} F_i} \sim \Phi_{Z',i \sqcup_{i=1}^{l'} F'_i}$. Furthermore, by iterating the above process, then $\Phi_{X\alpha,i \sqcup_{i=1}^{m} F_i} \sim \Phi_{X'\alpha,i \sqcup_{i=1}^{m} F'_i}$ for $\alpha = 1, \ldots, l$. So as a consequence of lemma 5.21 any two combinatorially marked sequential morphisms combinatorially equivalent preserve the proximity relations. Moreover, $deg(F_i) = deg(F'_{\tau(i)})$ so combinatorially equivalent sequential morphism also preserve degrees.

Conversely assume now that the combinatorially marked sequences of point blow-ups with $l = l'$ are combinatorially equivalent. We want to prove that their associated combinatorially marked sequential morphisms $(\pi : Z_s \to Z_0, \sqcup_{i=1}^{l} F_i)_{comb}$ and $(\pi' : Z'_s \to Z'_0, \sqcup_{i=1}^{l'} F'_i)_{comb}$ are combinatorially equivalent. First, there exists
σ ∈ S_m such that by applying iteratively theorem 4.6 we get

\[ H_i = H_i^* - \sum_{\beta \rightarrow i} H_\beta^* \]

\[ H_{\sigma(i)}' = H_{\sigma(i)}'^* - \sum_{\sigma(\beta) \rightarrow \sigma(i)} H_{\sigma(\beta)}'^* \]

Moreover, as a consequence proposition 4.4

\[ H_\beta_1^* H_\beta_2^* \cdots H_\beta_d^* \neq 0 \text{ if and only if } \beta_1 = \beta_2 = \cdots = \beta_d \]

and if \( H_i \) is final then \( H_i = H_i^* \), so it follows that there exists \( \tau \in S_i \) such that

\[ \text{deg}(F_i^*) = \text{deg}(F_{\tau(i)}'^*) \quad \forall i = 1, \ldots, l \]

Finally, and as a consequence of theorem 4.6

\[ H_{\beta_1} \cdot H_{\beta_2} \cdots H_{\beta_d} = (H_{\beta_1}^* - \sum_{\delta \rightarrow \beta_1} H_\delta^*) \cdot (H_{\beta_2}^* - \sum_{\delta \rightarrow \beta_2} H_\delta^*) \cdots (H_{\beta_d}^* - \sum_{\delta \rightarrow \beta_d} H_\delta^*) \]

so

\[ F_{\beta_1} \cdot F_{\beta_2} \cdots F_{\beta_d} = F_{\tau(\beta_1)}' \cdot F_{\tau(\beta_2)}' \cdots F_{\tau(\beta_d)}' \]

\[ \square \]

**Theorem 3.2.** Given two algebraically marked sequential morphisms \((\pi : Z_s \rightarrow Z_0, \sqcup_{i=1}^l F_i)_{alg}\) and \((\pi' : Z'_s \rightarrow Z'_0, \sqcup_{i=1}^l F'_i)_{alg}\), then they are algebraically equivalent over \( K \) as in definition 2.23 if and only if there exist algebraically marked sequences of point blow-ups \((Z_s, \ldots, Z_0, \sqcup_{i=1}^l F_i, \pi)_{alg}\) and \((Z'_s, \ldots, Z'_0, \sqcup_{i=1}^l F'_i, \pi')_{alg}\) associated to \((\pi : Z_s \rightarrow Z_0, \sqcup_{i=1}^l F_i)_{alg}\) and \((\pi : Z'_s \rightarrow Z'_0, \sqcup_{i=1}^l F'_i)_{alg}\) respectively such that they are algebraically equivalent over \( K \) as in definition 2.23.

**Proof.** If two algebraically marked sequences of point blow-ups are algebraically equivalent, then it follows directly by definition 2.23 that the associated algebraically marked sequential morphisms are algebraically equivalent too.

Now we will prove that if two algebraically marked sequential morphism \((\pi : Z_s \rightarrow Z_0, \sqcup_{i=1}^l F_i)_{alg}\) and \((\pi' : Z'_s \rightarrow Z'_0, \sqcup_{i=1}^l F'_i)_{alg}\) are algebraically equivalent, then there exist algebraically marked sequences of point blow-ups associated to them.
that are algebraically equivalent too. By theorem 5.14, given a certain sky $Z$ associated to an algebraically marked sequential morphism $(\pi : Z_s \to Z_0, \sqcup_{i=1}^d F_i)_{\text{alg}}$, all the algebraically marked sequences of point blow-ups obtained by regular projective contractions are associated to algebraically marked sequential morphisms in the same algebraic equivalence class. Since $(\pi : Z_s \to Z_0, \sqcup_{i=1}^d F_i)_{\text{alg}}$ and $(\pi' : Z'_s \to Z'_0, \sqcup_{i=1}^d F'_i)_{\text{alg}}$ are algebraically equivalent, then $\exists K \subset K$ such that there exist an isomorphism $\tilde{b} : \tilde{Z} \to \tilde{Z}'$. By applying proposition 5.13 and proposition 5.16, we conclude the result by scalar extension $\times_{\text{Spec}(K)} \text{Spec}(K)$.

\[ \square \]

6 Examples and comments

In the following example we will show that the numerical information that appears in the $d$–ary multilinear intersection form $I_{Z,E}$ is minimal in some sense for our classification purpose.

**Example 6.1.** Consider a $d$–dimensional smooth algebraic variety $X$ over $k$, where $k$ is a perfect field not algebraically closed, and $d$ is an even natural number. Let $P = P_1$ be a $k$–rational point on $X$ and let $\pi_1 : Z_1 \to X$ be the blowing up with center $P$. Take a point $P_2$ on $Z_1$ such that the degree of the field extension $k \subset k(P_2)$ is 1, and a point $Q_2$ in $Z_1$ such that $[k(Q_2) : k] = 2$. Now, let $\pi_2 : Z_2 \to Z_2$ (respectively $\pi'_2 : Z'_2 \to Z_1$) be the blowing up of $P_2$ (of $Q_2$, respectively) and take a point $P_3$ in $Z_2$ such that $P_3$ is not proximate to $P_1$ and such that the degree of the field extension $k(P_2) \subset k(P_3)$ is 3 (respectively, a point $Q_3$ in $Z'_2$ no proximate to $P_1$ such that $[k(Q_3) : k(Q_2)] = 1$). Consider now the blow-up $\pi_3 : Z_3 \to Z_2$.
\((\pi'_3 : Z'_3 \to Z'_2)\) of \(Z_2 (Z'_2)\) with center \(P_3 (Q_3\) respectively).

Let \(\Delta : E \to \overline{E \times E \times \cdots \times E}\) be the diagonal map, that is \(\Delta(E_i) = (E_i, E_i, ..., E_i)\).

Instead of the \(d\)-ary multilinear intersection form \(I_{Z,E}\), let us consider the map \(I_{Z,E} \circ \Delta\) we will refer to as the diagonal of the \(d\)-ary intersection form. Then the numerical information contained in this map for \(\pi = \pi_1 \circ \pi_2 \circ \pi_3\) and \(\pi' = \pi'_1 \circ \pi'_2 \circ \pi'_3\) is the following one respectively

\[
\begin{align*}
(E_1)^d &= -2 \\
(E_2)^d &= -4 \\
(E_3)^d &= -3 \\
(E'_1)^d &= -3 \\
(E'_2)^d &= -4 \\
(E'_3)^d &= -2
\end{align*}
\]

So if we had just considered the diagonal of the \(d\)-ary intersection form, then it is clear that by naturally extending our combinatorial equivalence relation (see definition 2.20) both diagonals would be equivalent. However, it is clear that the combinatorially marked sequences of point blow-ups \(\pi\) and \(\pi'\) are not. This is due to the fact that although diagonals are equivalent, the associated \(d\)-ary intersection forms are not.

\[
\begin{align*}
(E_1)^r \cdot (E_2)^s &= (-1)^{r+1} \\
(E_2)^r \cdot (E_3)^s &= (-1)^r \cdot (-3)
\end{align*}
\]
\[
\begin{align*}
(E_1')^r \cdot (E_2')^s &= (-1)^{r+1} \cdot 2 \\
(E_2')^r \cdot (E_3')^s &= (-1)^r \cdot (-2)
\end{align*}
\]

The next two examples pretend to highlight some of the differences that appear in the more general case, even when we restrict to the case of varieties of dimension \(d = 3\).

**Example 6.2.** Let \(C_1, C_2 \subset \mathbb{P}^2 \subset \mathbb{P}^3\) be two smooth plane curves intersecting transversally at a point \(P\), and let \(\pi_1 : Z_1 \rightarrow Z_0 = \mathbb{P}^3\) be the blow-up of \(\mathbb{P}^3\) with center \(C_1 = C_1\). If we denote by \(\tilde{C}_2\) to the strict transform of \(C_2\), let \(\pi_2 : Z_2 \rightarrow Z_1\) be the blow-up of \(Z_1\) with center \(C_2 = \tilde{C}_2\).

\[
Z_2 \xrightarrow{\pi_2} Z_1 \xrightarrow{\pi_1} Z_0
\]

Note that although \(C_2\) is not proximate to \(C_1\) in the usual sense, the blow-up with center \(C_2\) modifies \(E_1\) by creating an exceptional curve and increasing its Picard number by one. Consequently, \(E_1^2\) is not a projective bundle any more, so it can not be regularly contractible from \(Z_2\) by \([11, \text{Theorem 3}]\) and thus it can not be final.

This fact lead us to introduce a new proximity relation between the centers of a sequence of blow-ups

**Definition 6.3.** Let us suppose that \(C_j \cap E_j^{j-1} \neq \emptyset\). Then we say that \(C_j\) is proximate to \(C_i\), and we write \(C_j \rightarrow C_i\) if and only if \(C_j \subset E_i^{j-1}\). Otherwise we say that \(C_j\) is \(t\)-proximate to \(C_i\), and we write \(C_j \xrightarrow{t} C_i\).

The unique final divisor of the sequence above in example 6.2 is \(E_2\), and following the above notations \(C_2\) is \(t\)-proximate to \(C_1\), and we would denote it by \(C_2 \xrightarrow{t} C_1\).

**Example 6.4.** Let \(C \subset \mathbb{P}^3\) be a smooth curve of degree \(d\) and genus \(g\), and let \(\pi_1 : Z_1 \rightarrow Z_0 = \mathbb{P}^3\) be the blow-up of \(\mathbb{P}^3\) with center \(C_1 = C\). If \(P \in C\) is a closed point, and we denote by \(F\) to the inverse image \(\pi_1^{-1}(P)\), let \(\pi_2 : Z_2 \rightarrow Z_1\) be the blow-up of \(Z_1\) with center \(C_2 = F\).
Let $\pi'_1 : Z'_1 \to Z_0$ be the blow-up of $\mathbb{P}^3$ with center $C'_1 = P$. If we denote by $\tilde{C}$ the strict transform of $C$, let $\pi'_2 : Z'_2 \to Z'_1$ be the blow-up of $Z'_1$ with center $C'_2 = \tilde{C}$.

It can be proved by the universal property of blow-ups that there exist two morphisms $p_1 : Z_2 \to Z'_1$ and $p_2 : Z_2 \to Z'_2$ such that $\pi'_1 \circ p'_1 = \pi_1 \circ \pi_2$ and $p_2 \circ \pi'_2 = p_1$. In particular, $p_2$ is an isomorphism, so $Z_2 \cong Z'_2$. Then if we denote by $\pi$ to the sequence $\pi_1 \circ \pi_2$ and $\pi'$ to the sequence $\pi'_1 \circ \pi'_2$ it follows that $\pi \sim \pi'$. Following the notations of definition 2.17 there exists and isomorphism $b : Z_2 \to Z'_2$. Note that $b(E_2) = E'_1$ and $b(E_1) = E'_2$. Furthermore, $E_2$ and $E'_2$ are both final (see definition 5.2), but whereas $E_2 \cong E'_1$ is final with respect to $\pi$ but not with respect to $\pi'$ ($C'_2$ is $t$-proximate to $C'_1$), $E'_2 \cong E_1$ is final with respect to $\pi'$ but not with respect to $\pi$ ($C_2$ is proximate to $C_1$). Since both are finals, then by [11, Theorem 3] there exist two regular projective contraction from $Z_2 \cong Z'_2$, one contracting $E_2$ to $F$ and the other one contracting $E'_2$ to $\tilde{C}$.

Note that just allowing one dimensional centers the situation is much more rich than just in the case of point blow-ups. In fact there could exist final divisors with non empty intersection, as shown in this example.

References

[1] Antonio Campillo, Gérard Gonzalez-Sprinberg, and Monique Lejeune-Jalabert. Clusters of infinitely near points. *Math. Ann.*, 306(1):169–194, 1996.
[2] Antonio Campillo, Gérard Gonzalez-Sprinberg, and Francisco Monserrat. Configurations of infinitely near points. *São Paulo J. Math. Sci.*, 3(1):115–160, 2009.

[3] Antonio Campillo and Ana José Reguera. Combinatorial aspects of sequences of point blowing ups. *Manuscripta Math.*, 84(1):29–46, 1994.

[4] Tommaso de Fernex, János Kollár, and Chenyang Xu. The dual complex of singularities. In *Higher dimensional algebraic geometry—in honour of Professor Yujiro Kawamata’s sixtieth birthday*, volume 74 of *Adv. Stud. Pure Math.*, pages 103–129. Math. Soc. Japan, Tokyo, 2017.

[5] Olivier Debarre. *Higher-dimensional algebraic geometry*. Universitext. Springer-Verlag, New York, 2001.

[6] David Eisenbud and Joe Harris. *3264 and all that—a second course in algebraic geometry*. Cambridge University Press, Cambridge, 2016.

[7] William Fulton. *Intersection theory*, volume 2 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, second edition, 1998.

[8] William Fulton and Serge Lang. *Riemann-Roch algebra*, volume 277 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, New York, 1985.

[9] Alexandre Grothendieck. Éléments de géométrie algébrique. III. Étude cohomologique des faisceaux cohérents. I. *Inst. Hautes Études Sci. Publ. Math.*, (11):167, 1961.

[10] Alexandre Grothendieck. Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas IV. *Inst. Hautes Études Sci. Publ. Math.*, (32):361, 1967.

[11] Shihoko Ishii. Some projective contraction theorems. *Manuscripta Math.*, 22(4):343–358, 1977.
[12] Dmitry Anatol’evich Stepanov. A remark on the dual complex of a resolution of singularities. *Uspekhi Mat. Nauk*, 61(1(367)):185–186, 2006.