Delay dependent criteria for the consensus of second-order multi-agent systems subject to communication delay

Amir Nejadvali1 | Reza Mahboobi Esfanjani1 | Arash Farnam2,3 | Guillaume Crevecoeur2,3

1 Department of Electrical Engineering, Sahand University of Technology, Tabriz, Iran
2 Department of Electrical Energy, Metals, Mechanical Constructions and Systems, Ghent University, Ghent, Belgium
3 EEDT Decision and Control, Flanders Make, Belgium

Correspondence
Reza Mahboobi Esfanjani, Department of Electrical Engineering, Sahand University of Technology, Tabriz, Iran.
Email: mahboobi@sut.ac.ir

Abstract
In this paper, a distributed controller is designed for the consensus of multi-agent systems in which each agent has a general second-order linear dynamic and the information is exchanged over a data-delaying communication network. Using the sensitivity of system poles to the parameters of the control protocol, graphical delay-dependent synthesis conditions are derived to tune the controller gains. A systematic procedure is developed to attain maximum tolerable transmission delay in the system. Moreover, simpler condition is provided for the special case where the second-order model is reduced to a double integrator. Simulation results are presented to illustrate the merits of the proposed scheme compared to some recent rival ones in the literature.

1 | INTRODUCTION

A canonical concept in the field of cooperative systems is called consensus, in which the agents of the system agree on some physical or virtual quantities to work with together. Consensus applications have been expanded in the different aspects of the multi-agent coordination such as rendezvous [1], flocking [2–4] and formation [5, 6]. Also, the notion of consensus is used in smart electricity networks [7] and behaviour analysis of biological groups [8].

Various control strategies were designed to force the agents achieve a common interest. In the primitive works like [9], consensus protocols were developed for multi-agent systems (MASs), considering the single integrator dynamic for each agent with frequency-domain methods. Also, in [5, 10–13], control methods were presented for the consensus of first order networked system. However, in most of the above-mentioned results communication delay in data exchange between the agents of the system is neglected; while transmission imperfections are unavoidable in real-world applications because of limited channel bandwidth and data congestion. Since the existence of delay deteriorates the performance of the networked systems; it should be taken into account in the design procedure. The presence of delay makes the design and the practical implementation of cooperative controllers a challenging problem. The monographs [14, 15] present a wide range of analytical methods to stability analysis and the stabilization of single-loop time-delay dynamical systems using unified eigenvalue-based approaches. Especially, the sensitivity of the system behavior with respect to delay value was discussed based on frequency-domain tools like Routh theorem and its variants. Also, the asymptotic behavior of critical poles’ location with respect to system parameters was studied in [16] by solving a simple eigenvalue-based problem. There are a few articles in the literature that investigate the consensus of MASs with communication delay. In [17–20], the first order agents were considered. More realistic second-order dynamic with communication delay was considered in [3, 21, 22] in which controllers were provided for systems with double integrator agents.

The second-order consensus in the presence of delay was studied in [23], where a necessary and sufficient condition was presented for MASs with the directed spanning tree, provided that the time delay is less than a certain value. In [24], sufficient condition was given in terms of linear matrix inequalities for the consensus of second-order multi-agent systems with jointly connected topologies and time-delay. In [25], a proportional-integral (PI)-type distributed controller was proposed using the location of the characteristic roots of the quasi-polynomial of
the delayed system. The delay margin was obtained to guarantee that the system reach consensus provided that the delay is less than the critical value. The influence of communication delay on the second-order consensus was investigated in [26]. It was shown that time-varying communication delay has positive effects on the consensus of MASs with second-order dynamics if a certain condition on the delay bound is satisfied. Then, by utilizing Lyapunov–Krasovskii functional, sufficient conditions for the consensus of MASs with second-order dynamics were derived. In [27], consensus conditions were suggested for the MASs with general second-order agents in the presence of communication delay; wherein, based on the frequency domain analysis, synthesis criteria were first extracted for choosing controller parameters in the delay-free case. Then, for the fixed gains which are obtained from the delay-free criterion, the maximum tolerable communication delay was computed by analysing the characteristic equation. The main drawback of the mentioned approach is that the method is very conservative with respect to delay value.

In this paper, along the lines of [14, 15], a novel control strategy is developed for the consensus of MASs with general second-order linear agents considering communication delay between agents. Motivated by the ideas of [27, 28], a common control protocol is employed for each agent; then, a graphical technique is developed to tune the parameters of the control law. The main contribution of this paper is to develop a novel delay-dependent synthesis condition for the consensus protocol. The key idea is to determine a stable region for the roots of the characteristic quasi-polynomial of the system using the frequency domain analysis approach which is not employed before in the consensus problem of second-order agents. Specifically, a systematic procedure is proposed to determine a region in the plane of controller gains which ensures consensus of agents. The area of this admissible set decreases when the value of time delay increases. The main feature of the considered problem is that the connection topology of the agents is described by a directed graph which means that the data is not transmitted bi-directionally in the system. Therefore, the challenge of handling complex eigenvalues of the Laplacian matrix of the graph appears in the calculations.

The most relevant work in the literature to ours is [27] in which the gains of the consensus controller are computed based on a delay-free condition; afterwards, the tolerable delay is obtained by trial and error using the characteristic equation of the system for the specified controller parameters; our method incorporates explicitly the delay value in the criterion which is used for tuning the controller, so the maximum admissible delay is increased noticeably. Comparative simulation results are presented to illustrate the merits of the suggested method compared to [25] and [27].

It is worth noting that in the time-domain design methods which are mostly based on Lyapunov–Krasovskii argument, the controller parameters are determined explicitly by computationally demanding solution of LMIs without any insight to the properties of the system’s response; while in the proposed frequency-domain scheme, a region is obtained for the admissible gains which can be selected based on desired specifications in this set. Namely, choosing the gains from the central part of the region lead to more stable but slow response.

In the rest of this paper, the considered consensus problem is formulated in Section 2 and some preliminary facts are recalled. In Section 3, the admissible set for controller gains are derived. In Section 4, numerical examples are provided to illustrate the merits of the proposed method. Finally, the conclusion and future work are given in Section 5.

Notation 1. In this paper, \( \mathbf{1}_m \) denote the m-dimensional column vector with all component 1. \( \mathbf{I}_m \) denote the m-dimensional identity matrix, \( \mathbf{0}_m \) denote the zero matrices with an appropriate dimension. \( \{ \mathbf{R}_m \text{ or } \mathbb{H}_m \} \) and \( \{ \mathbf{L}_m \text{ or } \mathbb{I}_m \} \) are the real and imaginary parts of the complex number \( \theta \), respectively.

## 2 PROBLEM STATEMENT

Consider a multi-agent system composed of \( N \) agents with communication topology described by the graph \( \mathcal{G} = \{ \mathcal{N}, \mathcal{E}, \mathcal{A} \} \), where \( \mathcal{N} = 1, 2, \ldots, N \) is the set of nodes, \( \mathcal{E} \subset \{(i, j) : i, j \in \mathcal{N} \} \) denotes the set of edges and \( \mathcal{A} = [a_{ij}] \in \mathbb{R}^{N \times N} \) is the Adjacency matrix. Each agent \( i \in \mathcal{N} \) has the following general second-order linear dynamic:

\[
\begin{align*}
\dot{x}_i(t) &= y_i(t) \\
\dot{y}_i(t) &= \alpha x_i(t) + \beta y_i(t) + u_i(t)
\end{align*}
\]

(1)

The following commonly used control protocol, \( u_i(t) \) is employed to achieve consensus between agents:

\[
\begin{align*}
u_i(t) = k_x \sum_{j=1}^{N} a_{ij}(x_j(t - \tau) - x_i(t - \tau)) + \\
k_y \sum_{j=1}^{N} a_{ij}(y_j(t - \tau) - y_i(t - \tau)),
\end{align*}
\]

(2)

where \( k_x \) and \( k_y \) are proportional gains and \( \tau \) is the known communication delay between agents. The most recent available information from agents are compared in (2) to compute the error signal in feedback law. In what follows, a systematic procedure is developed to determine the parameters of (2) such that the tolerable delay is increased in the system.

Definition 1. An MAS composed of N agents as (1) achieves consensus if for any initial condition and \( i \neq j, i, j = 1, 2, \ldots , N \), the errors between the states of agent \( i \) and agent \( j \) converge to zero:

\[
\lim_{t \to +\infty} \|x_i(t) - x_j(t)\| = 0,
\]

\[
\lim_{t \to +\infty} \|y_i(t) - y_j(t)\| = 0.
\]

(3)

So, the aim is to determine the unknown gains \( k_x \) and \( k_y \) in (2) such that (3) holds for the closed-loop system (1)-(2).
Before proceeding further, some useful facts from the graph theory are recalled. A graph $G$ is represented by the weighted adjacency matrix, $A = [a_{ij}] \in \mathbb{R}^{N \times N}$ in which the entry $(i,j)$ is non-zero if the agent $j$ can exchange information with agent $i$. The allowed set $N_i$ of the agent $i$, includes the agents that exchange information with agent $i$. The elements of matrix $A = [a_{ij}]$ are non-negative and non-zero if and only if $j \in N_i$. Specifically, $a_{ij} > 0$ if $(i,j) \in E$, or $a_{ij} = 0$. The indegree matrix is defined as $D = \text{diag}(d_1, d_2, \ldots, d_N)$, where $d_i = \sum_{j=1}^{N} a_{ij}$ is the membership degree of agent $i$. The Laplacian matrix of graph $G$ is $\mathcal{L} = D - A$.

**Lemma 1.** If $G$ is a directed graph with directed spanning tree, that is, $a_{ij} \neq a_{ji}$ for each $(i,j) \in \mathcal{N}$, eigenvalues of Laplacian matrix $\mathcal{L}$ are complex or real numbers. Namely, $\mathcal{L}$ has one simple zero eigenvalue, $\lambda_1 = 0$, and all others have positive real parts.

$$0 = \lambda_1 < R_{32} \leq \ldots \leq R_{1N}. \quad (4)$$

### 3 MAIN RESULTS

In this section, an approach is derived to tune the gains of the control protocol (2). First, the generalized Routh criterion [29] is applied to extract a preliminary condition for the delay-free case; then, some conditions are obtained for the delayed case by computing the sensitivity of the system poles with respect to variation of controller parameters. These criteria are combined to determine an admissible region in the plane of controller gains such that the considered MAS achieves consensus.

Without loss of generality, agent number 1 is assumed to be the group leader. The error dynamical equation is defined with two states $e_i(t)$ and $\hat{e}_i(t)$, which stands for the error between states of agents $i = 2, 3, \ldots, N$ with agent 1.

$$\begin{cases} e_i(t) = x_i(t) - x_1(t) \\ \hat{e}_i(t) = y_i(t) - y_1(t) \end{cases} \quad i = 2, 3, \ldots, N. \quad (5)$$

By augmenting the error vectors of agents in

$$E(t) = [e_2(t), e_3(t), \ldots, e_N(t), \hat{e}_2(t), \ldots, \hat{e}_N(t)]^T,$$

the overall error dynamic for the MAS is obtained as follows:

$$\dot{E}(t) = \begin{bmatrix} 0 & I_{N-1} \\ \alpha I_{N-1} & \beta I_{N-1} \end{bmatrix} E(t) - \begin{bmatrix} 0 & 0 \\ k_e \hat{\mathcal{L}} & k_j \mathcal{L} \end{bmatrix} E(t - \tau),$$

where $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 \mathcal{L}_2^T$ and $\mathcal{L}_1 = \begin{bmatrix} d_2 & -a_{23} & \cdots & -a_{2N} \\ -a_{32} & d_3 & \cdots & -a_{3N} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{N2} & -a_{N3} & \cdots & d_N \end{bmatrix}$, $\mathcal{L}_2 = \begin{bmatrix} a_{12} \\ \vdots \\ a_{1N} \end{bmatrix}$.

If condition (3) holds, it implies that error dynamic (5) is asymptotically stable. So, the problem of interest can be restated as follows: determine gains $k_e$ and $k_j$ to render system (5) asymptotically stable. Let $\mathcal{L}_3 = [a_{21}, a_{31}, \ldots, a_{N1}]^T$, then the Laplacian matrix can be rewritten as:

$$\mathcal{L} = \begin{bmatrix} d_2 & \mathcal{L}_3 \\ -\mathcal{L}_3^T & \mathcal{L}_1 \end{bmatrix}.$$  

Considering the transformation matrix,

$$P = \begin{bmatrix} 1 & 0^T \ldots \mathcal{L}_{N-1} \\ 0_{N-1} & \mathcal{L}_{N-1} \end{bmatrix},$$

the following similar form for matrix $\mathcal{L}$ is obtained:

$$\hat{\mathcal{L}} = P^{-1} \mathcal{L} P = \begin{bmatrix} 0 & -\mathcal{L}_3^T \\ 0_{N-1} & \mathcal{L}_2 \end{bmatrix}. \quad (6)$$

Regarding (6), the eigenvalues of $\mathcal{L}$ are the same as the ones of $\hat{\mathcal{L}}$, so:

$$\text{eig}(\mathcal{L}) = \text{eig}(\hat{\mathcal{L}}) = \{\lambda_1 = 0, \text{eig}(\hat{\mathcal{L}})\}.$$

Then, Lemma 1 implies that:

$$\text{eig}(\hat{\mathcal{L}}) = \{\lambda_2, \lambda_3, \ldots, \lambda_N\}.$$

Now, the characteristic equation for the system (5) is computed as the following:

$$\det(sI_{2(N-1)}) - \begin{bmatrix} 0 & I_{N-1} \\ \alpha I_{N-1} & \beta I_{N-1} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ k_e \hat{\mathcal{L}} & k_j \mathcal{L} \end{bmatrix} e^{-\tau s} = \prod_{i=2}^{N} \Delta_i(s, \tau),$$

where, $\Delta_i(s, \tau)$ are quasi polynomials as:

$$\Delta_i(s, \tau) = s^2 - \beta s - \alpha + e^{-\tau s} (k_j s + k_e), \quad (7)$$

It is clear that achieving consensus in the MAS which is equivalent to the asymptotic stability of (5) depends on the roots of the characteristic equation (7). Note that $\Delta_i(s, \tau)$ has infinite number of roots, but the stability is checked by the sign of the rightmost of them [14]. Let $\sigma$ be the abscissa of $\Delta_i(s, \tau)$ defined by:

$$\sigma = \{R_k \Delta_k, \Delta_i(s, \tau) > 0\}.$$

If all the roots are simple, then the rightmost root(s) is (are) the real number $\sigma$, or conjugate complex numbers with $\sigma$ as their real part.

**Lemma 2.** Control protocol (2) forces the MAS achieves consensus if and only if characteristic equation (7) is stable for all $i = 2, 3, \ldots, N$. 

TABLE 1  Generalized Roth criterion

| \(s\) | \(r^2\) | \(k_j I_{h_j}\) | \(k_c R_{h_i} - \alpha\) |
|---|---|---|---|
| 1 | \(k_j R_{h_i} - \beta\) | \(k_c I_{h_i}\) | 0 |
| 1 | \((k_j R_{h_i} - \beta)k_j I_{h_j} - k_c I_{h_i}\) | \((k_j R_{h_i} - \beta)\) | 0 |
| 1 | \((k_j R_{h_i} - \beta)^2\) \((k_c R_{h_i} - \beta)k_j I_{h_j} + k_c I_{h_i}\) | 0 | 0 |
| 0 | \((k_j R_{h_i} - \beta)^2\) \((k_c R_{h_i} - \beta)\) | 0 | 0 |

First, for the delay-free case, \(\tau = 0\) some conditions are obtained in Proposition 1 to assure that the roots of (7) are in LHP; afterwards, in Proposition 2 the delayed case, \(\tau \neq 0\) is analysed. Finally, merging these results, the stabilizing set on the \(k_c - k_j\) plane is determined in Theorem 1. It is worth noting that in the proposed method the topology graph of the network is directed. Let \(\lambda_i = R\lambda_i + \iota I\lambda_i\) be the eigenvalues of the Laplacian matrix of the graph, where \(\iota\) is the complex unit and \(i = 2, 3, \ldots, N\). Since \(G\) is directed, then the imaginary part of \(\lambda_i\) can be either \(I\lambda_i = 0\) or \(I\lambda_i \neq 0\) and \(R\lambda_i > 0\).

Proposition 1. **MAS (1) achieves consensus without communication delay by control protocol (2) if and only if the proportional gains \(k_c\) and \(k_j\) satisfy (8):**

\[
\begin{align*}
    k_j R_{h_i} - \beta &> 0 \\
    (k_j R_{h_i} - \beta)^2(k_c R_{h_i} - \alpha) + ((k_j R_{h_i} - \beta)k_j - k_c)k_j I_{h_i}^2 &> 0 \\
\end{align*}
\]

(8)

**Proof.** Taking \(\tau = 0\) converts Equation (7) to:

\[
    s^2 + s(k_j(R_{h_i} + \iota I_{h_i}) - \beta) + (k_c(R_{h_i} + \iota I_{h_i}) - \alpha).
\]

(9)

The obtained characteristic equation in (9) is a second-order polynomial with complex coefficients:

\[
\{1, (k_j R_{h_i} - \beta) + \iota(k_j I_{h_j}), (k_c R_{h_i} - \alpha) + \iota(k_c I_{h_i})\}.
\]

To analyse the stability of the above-mentioned complex polynomial, the generalized Routh criterion is used from \([29]\) as follows:According to Table 1, polynomial (9) is Hurwitz stable provided that condition (8) holds for \(i = 2, 3, \ldots, N\). So, the error dynamic (5) is asymptotically stable and consequently, MAS (1) achieves consensus. This concludes the proof. \(\square\)

Condition (8) is portrayed on \(k_j - k_c\) plane in Figure 1. The region \(Q_0\) in Figure 1 indicates the stabilizing set for proportional gains \(k_c\) and \(k_j\).

**Remark 1.** For the case that \(\lambda_i\) is a real number and \(\tau = 0\), polynomial (9) is simplified to a second-order polynomial which has real coefficients \(\{1, k_j\lambda_i - \beta, k_c\lambda_i - \alpha\}\). Clearly, condition (8) is transformed to:

\[
\begin{align*}
    k_j\lambda_i - \beta &> 0 \\
    k_c\lambda_i - \alpha &> 0 \\
\end{align*}
\]

(10)

If condition (10) holds for \(i = 2, 3, \ldots, N\), the error dynamic in (5) is asymptotically stable and MAS (1) achieves consensus.

Figure 2 represents the stable region \(Q_0\) which is obtained by condition (10). In the reminder of this section, it is assumed that...
the gains $k_x$ and $k_y$ satisfy:

$$
\begin{align*}
  k_x &= \max\{\text{condition}(8), 0\} \quad \text{or} \quad \min\{\text{condition}(10), 0\} \\
  k_y &= \max\{\text{condition}(8), 0\} \quad \text{or} \quad \min\{\text{condition}(10), 0\}
\end{align*}
$$

**Proposition 2.** If the values of the gains $k_x$ and $k_y$ satisfy (11), then the rightmost roots of characteristic equation (7) with $\tau > 0$ lie on the imaginary axis.

$$
H_i : \begin{cases}
  k_x = \frac{\mu_1 \cos(\tau \omega_i) + \mu_2 \sin(\tau \omega_i)}{|\lambda_i|^2} \\
  k_y = \frac{\mu_1 \sin(\tau \omega_i) - \mu_2 \cos(\tau \omega_i)}{\omega_i |\lambda_i|^2}
\end{cases}, \quad (11)
$$

where:

$$
\mu_1 = R_3(\omega_j^2 + \alpha) + l_3 \beta \omega_i, \quad \text{and} \quad \mu_2 = l_3(\omega_f^2 + \alpha) - R_3 \beta \omega_i.
$$

**Proof.** Let $s_i = \tau \omega_i$, $\omega_i \in (0, \infty)$ be the imaginary root of characteristic equation (7), then $s_i = \tau \omega_i$ satisfies $\Delta_i(\omega_i, \tau) = 0$. So, both of its real and imaginary parts are equal to zero as (12) and (13):

$$
\begin{align*}
  \cos(\tau \omega_i)(k_x R_{3j} - k_y l_3 \omega_i) - \omega_j^2 - \alpha^2 + \\
  \sin(\tau \omega_i)(k_y R_3 \omega_i + k_x l_3) &= 0, \\
  \sin(\tau \omega_i)(k_y l_3 \omega_i - k_x R_3) - \beta \omega_i + \\
  \cos(\tau \omega_i)(k_x l_3 + k_y R_3 \omega_i) &= 0.
\end{align*}
$$

Solving (12) and (13) with respect to $k_x$ and $k_y$ leads to explicit relations (11) in terms of $\omega_i$. This concludes the proof. \(\square\)

Figure 3 shows a typical $H_i$ in the $k_x - k_y$ plane. The curve $H_i$ in Figure 3 divides the $k_x - k_y$ plane into three regions: top of the curve, on the curve and bottom of the curve. From Proposition 2, it is clear that for $k_x, k_y \in H_i$, the rightmost roots of (7) are on the imaginary axis. Proposition 3 investigates the position of these roots with respect to the variation of these gains (i.e. if $k_x$ and $k_y$ belong to the top or bottom region of $H_i$, the roots of (7) move to the left or right of the critical axis).

**Proposition 3.** If the gains $k_x$ and $k_y$ are selected from the bottom region of $H_i$ with $\omega_i \in \{\omega_i^-, \omega_i^+\}$ where $\omega_i^-, \omega_i^+ \in \{0, +\infty\}$, then the roots of (7) are in the LHP.

**Proof.** In characteristic equation (7), consider the proportional gain $k_x$ as a variable and $k_y$, $\tau$, $\lambda_i$ as the parameters, also let $s_i(k_x)$ be the root of (7) that depends on $k_x$ and satisfies (14):

$$
R\{s_i(k_x)\} = 0, \quad I\{s_i(k_x)\} = \omega_i, \quad (14)
$$

where $i = 2, 3, \ldots, N$. Differentiating Equation (7) with respect to $k_x$ yields (15)

$$
(2s_i(k_x) - \beta) \frac{ds_i(k_x)}{dk_x} + e^{-\tau s_i(k_x)} \lambda_i s_i(k_x) \frac{ds_i(k_x)}{dk_x} + 1
$$

$$
- e^{-\tau s_i(k_x)} \lambda_i \frac{ds_i(k_x)}{dk_x} = 0, \quad (15)
$$

in which, $\frac{ds_i(k_x)}{dk_x}$ denotes the sensitivity of the root of (7) to $k_x$. Rearranging (15), we have:

$$
\left\{\frac{ds_i(k_x)}{dk_x}\right\}^{-1} = \frac{\beta - 2s_i(k_x)}{\lambda_i} e^{-\tau s_i(k_x)} + \frac{\tau k_x + \tau t_i s_i(k_x) - k_y}{\lambda_i},
$$

by substituting $s_i(k_x) = \omega_i$, $\lambda_i = R_3 + t_i l_3$ and using (11), we have:

$$
\left\{\frac{ds_i(k_x)}{dk_x}\right\}^{-1} = \frac{(\omega_i^2 - \alpha)(R_3 \sin(\tau \omega_i) - l_3 \cos(\tau \omega_i))}{\omega_i |\lambda_i|^2} + \tau k_x.
$$

It is known that:

$$
a \sin(x) + b \cos(x) = \sqrt{a^2 + b^2} \sin(x + \tan^{-1}\left(\frac{b}{a}\right)). \quad (17)
$$

Utilizing (17), relation (16) is rewritten as:

$$
R\left\{\frac{ds_i(k_x)}{dk_x}\right\}^{-1} = \frac{(\omega_i^2 - \alpha) \sin(\tau \omega_i) - \gamma_i}{\omega_i |\lambda_i|^2} + \tau k_x, \quad (18)
$$

FIGURE 3 The curve $H_i$ for (37), $\lambda_i = 2$ and different time delay.
where \( \gamma_i = \tan^{-1} \left( \frac{I_y}{I_x} \right) \). The term \( \tau k_x \) in (18) is always positive, so equation (18) remains positive if \( (\omega_i^2 - \alpha) \sin(\tau \omega_i - \gamma_i) > 0 \). This fact implies that:

\[
\begin{cases}
  \text{if } \alpha > 0 & \omega_i > \sqrt{\alpha} \\
  \text{if } \alpha \leq 0 & \omega_i > 0 \text{ and } \frac{n\pi + \gamma_i}{\tau} < \omega_i < \frac{m\pi + \gamma_i}{\tau},
\end{cases}
\]

(19)

where \( n \) and \( m \) are the even and odd numbers, respectively. Common range for \( \omega_i \) in conditions (19) is obtained as follows:

\[
\omega_i^i < \omega_i < \omega_i^+, \]

where:

\[
\begin{cases}
  \text{if } \alpha > 0 & \omega_i^i = \max \left\{ \sqrt{\alpha}, \frac{\gamma_i}{\tau} \right\} \\
  \text{if } \alpha \leq 0 & \omega_i^i = 0, \text{ and } \omega_i^+ = \frac{\pi + \gamma_i}{\tau}.
\end{cases}
\]

(20)

Since (20) holds:

\[
R \left\{ \frac{d_s(k_y)}{dk_x} \right\}_{i(k_x)} > 0,
\]

(21)

which means that the sensitivity of critical roots to variation of the parameter \( k_x \) is positive. Namely, the imaginary roots move in the same direction of \( k_x \).

Regarding Proposition 2, if \( k_x, k_y \in H_i \), the rightmost poles of (7) lie on the imaginary axis and (21) is met for fixed \( k_y \) on the curve \( H_i \); if the gain \( k_x \) decreases, then the critical poles start to move into the LHP. Also, for fixed \( k_x \) on the curve \( H_i \), if the gain \( k_y \) increases, then the poles on the imaginary axis start to enter the RHP. Therefore, characteristic equation (7) has at least one root in \( R \) if \( k_x \) and \( k_y \) belong to the upper region of the curve \( H_i \) and all of the roots are in the LHP, if \( k_x \) and \( k_y \) are chosen from the bottom region of the curve \( H_i \) in interval (20). This concludes the proof.

According to Proposition 1, there exist more restrictions on the \( k_x \) and \( k_y \) to be combined with the bottom area of \( H_i \) to define the stable closed region \( Q_j \) for the characteristic equation \( \Delta_j(\tau, \lambda) \). Specifically, characteristic equation (7) is asymptotically stable for all \( i = 2, 3, \ldots, N \), or equivalently, MAS (1) achieve consensus if the gains are selected from non-empty region \( Q = \bigcap_{i=2}^{N} Q_j \) \( i \neq j \) and \( i = 2, 3, \ldots, N \). The above discussion is summarized in Theorem 1 as follows:

**Theorem 1.** The control protocol (2) makes the MAS (1) achieve consensus if the proportional gains \( k_x \) and \( k_y \) belong to stabilizing set \( Q \) which is given by:

\[
Q = \bigcap_{i=2}^{N} Q_j \quad \bigcap_{i \neq j} Q \quad i = 2, 3, \ldots, N, \]

(22)

where \( Q_i \) is the stable region for the characteristic equation (7) and enclosed by the following four conditions:

1. curve \( \hat{H} \in \omega \in (\omega_i, \omega_i^+) \)
2. condition (8) if \( \hat{a}_i \neq 0 \) or condition (10) if \( \hat{a}_i = 0 \)
3. vertical line \( k_x^+ = k_x \in (\omega_i, \omega_i^-) \), obtained from (11.2)
4. vertical line \( k_y^+ = k_y \in (\omega_i, \omega_i^+) \), obtained from (11.2)

Note that boundaries of stable region in \( k_x - k_y \) plane are constructed by two main conditions: delay free one (which is independent from delay value) and the delay dependent one. Figure 3 demonstrates that for delay values, \( \tau_2 < \tau_1 \) the stable region corresponding to \( \tau_2 \) contains the one of \( \tau_1 \); therefore, roughly speaking, delay margin can be obtained by increasing delay value.

**Remark 2.** By replacing \( \alpha = \beta = 0 \) in dynamic equation (1), the dynamical equation of agents is converted to double integrator as follows:

\[
\begin{cases}
  \dot{y}_i(t) = y_j(t) \\
  \dot{y}_j(t) = u_i(t)
\end{cases}
\]

(24)

Thus, the corresponding characteristic equation for error dynamic, \( E(t) \) is given by (25):

\[
\begin{align*}
\begin{bmatrix}
  \frac{d}{dt} (U_{2(N-1)}) - & \begin{bmatrix}
  0 & I_{N-1} \\
  0 & k_x \mathcal{L}
\end{bmatrix} + \begin{bmatrix}
  0 & 0 \\
  k_x \mathcal{L} & k_y \mathcal{L}
\end{bmatrix} e^{-\tau t}
\end{bmatrix}
\end{align*}
\]

(25)

where \( \Delta_j(\tau, \lambda) \) are quasi polynomials as:

\[
\Delta_j(\tau, \lambda) = \lambda^2 - \tau \lambda + \tau^2 (k_x + k_x \lambda),
\]

Consequently, condition (8) obtained in Proposition 1 is relaxed as follows:

\[
\begin{cases}
  k_y > 0 \\
  k_x^2 k_x \lambda^2 x^{3} + k_y^2 \lambda x^{3} > 0
\end{cases}
\]

(26)

For the case that the graph \( G \) is undirected, \( \lambda \) is a real number and polynomial (25) is simplified to a second-order polynomial with real coefficients \( \{1, k_x \lambda, i, k_x \lambda_i \} \). Hence, the condition (26) is simplified to:

\[
\begin{cases}
  k_y > 0 \\
  k_x^2 > 0
\end{cases}
\]

(27)
Moreover, \( H_i \) in (11) is rewritten as follows:

\[
\begin{align*}
    k_x & = \omega_j \frac{\sum_{i=1}^{N} R_{ij} \cos(\tau \omega_i) + I_{ij} \sin(\tau \omega_i)}{\left| \lambda_i \right|^2}, \\
    k_y & = \omega_j \frac{\sum_{i=1}^{N} R_{ij} \sin(\tau \omega_i) - I_{ij} \cos(\tau \omega_i)}{\left| \lambda_i \right|^2}.
\end{align*}
\]

The sufficient synthesis condition which is derived in Theorem 1 (Algorithm 3.1) for the general second-order agents is converted to necessary and sufficient one for the second-order agents with double integrator dynamic in Corollary 1 (Algorithm 3.2).

**Corollary 1.** Control protocol (2) makes MAS (24) achieve consensus if and only if proportional gains \( k_x \) and \( k_y \) belong to stabilizing set \( q \) which is given by:

\[
q = \left\{ q_i \mid i \neq j \text{ and } i, j = 2, 3, \ldots, N \right\},
\]

where \( q_i \) is the stable region for double integrator characteristic equation (25) and enclosed by following two conditions:

\[
\begin{align*}
1. \; & \left. h_i \right|_{\omega_j \in (0, \omega_j^*)} \\
2. \; & \text{condition (26) if } I_{i} \neq 0 \text{ or condition (27) if } I_{i} = 0,
\end{align*}
\]

where \( \omega_j^* = \frac{(\pi - \delta_j)}{\tau} \) and \( \delta_j = \tan^{-1} \left( \frac{R_{ij}}{I_{ij}} \right) \).

**Proof.** It is shown that if the gains \( k_x \) and \( k_y \) are selected from the bottom region of (28), then the roots of (25) are in the LHP. In characteristic equation (25), consider the proportional gain \( k_x \) as a variable and \( \{ \tau, \; k_y, \; \lambda_i \} \) as the parameters, also let \( s_i(k_x) \) be the root of (25) that depends on \( k_x \) and satisfies (14). Differentiating equation (25) with respect to \( k_x \) yields:

\[
-\frac{d}{dk_x} s_i(k_x) - k_x \frac{d}{dk_x} s_i(k_x) - \left( k_y \frac{d}{dk_x} s_i(k_x) + 1 \right)
\]

\[
+ 2s_i(k_x) \frac{d}{dk_x} s_i(k_x) = 0.
\]

Rearranging the obtained equation leads to:

\[
\left\{ \frac{d}{dk_x} s_i(k_x) \right\}^{-1} = \frac{-2s_i(k_x)}{\lambda_i} \frac{d}{dk_x} s_i(k_x) - k_y = \frac{d}{dk_x} s_i(k_x) - k_y.
\]

Substituting \( s_i(k_x) = \omega_j, \lambda_i = R_{ij} + I_{ij} \) and using (28) yields:

\[
R \left\{ \frac{d}{dk_x} s_i(k_x) \right\}^{-1} = k_y + \tau k_x.
\]

In which \( k_x, \; k_y \) and \( \tau \) are positive; so, (31) always remain positive, then:

\[
R \left\{ \frac{d}{dk_x} s_i(k_x) \right\} > 0.
\]

Which means that the sensitivity of imaginary roots of (25) to variation of parameter \( k_x \) is positive for all \( \omega_j \in (0, \infty) \); namely, the imaginary roots move in the same direction of \( k_x \). Similarly to the general case, the bottom area of curve (28) is restricted with the obtained delay-free conditions (26) and (27) which organizes the stable region \( q_i \) for \( i \) th characteristic equation (25) as in (30). Regarding Theorem 1, stabilizing set for gains \( k_x \) and \( k_y \) is specified with (29). Unlike the general case, there is no restriction on \( \omega_j \) to assure the positivity of (32), so curve (28) can be plotted for \( \omega_j \in (0, \infty) \) as in Figure 4.

**FIGURE 4** The curve \( h_i \) for (39), \( \tau = 0.5, \lambda_i = 2 \) and \( \omega_j \in (0, \infty) \).

From Remark 2, the stable region for delay-free case lies in the first quarter of \( k_x - k_y \) plane, as depicted in Figure 4. According to (32), if the gains \( k_x \) and \( k_y \) belong to the top region of each admissible areas on the Figure 4, the characteristic equation (25) has at least one pole in the RHP and if the gains \( k_x \) and \( k_y \) belong to the bottom region of these regions, all roots are in the LHP. Namely, for a fixed parameter \( k_x \) and variable parameter \( k_x \), crossing from bottom to the upper area of the curve \( h_i \), the rightmost pole that was in the LHP, crosses from the imaginary axis and enters the RHP. Thus, the only stable region \( q_i \) is the area enclosed by domain 1 and delay-free conditions (26) and (27). It is obvious that the \( i \)th characteristic equation in (25) is asymptotically stable if and only if the gains \( k_x \) and \( k_y \) are chosen from \( q_i \). Similarly to Theorem 1, MAS with double integrator agent (24) achieves consensus if and only if the characteristic equation (25) be asymptotically stable for \( i = 2, 3, \ldots, N \). To ensure the asymptotic stability of (25) for all \( i = 2, 3, \ldots, N \), existence of a stable region \( q \in \{ q_j \cap q_i \mid i \neq j \text{ and } i, j = 2, 3, \ldots, N \} \) is necessary. As it is clear, domain 1 is obtained from interval \( \omega_j \in (0, \omega_j^*) \), where
for the values of $\omega_i = 0$ and $\omega_i = \omega_j^*$ the equation (28.1) must be equal to zero.

$$k_x = \omega_j^2 \frac{R_j \cos(\tau \omega_j) + I_j \sin(\tau \omega_j)}{|\lambda_i|^2} = 0. \quad (33)$$

Relation (33) is satisfied, if $\omega_i = 0$ or:

$$R_j \cos(\tau \omega_j) + I_j \sin(\tau \omega_j) = 0. \quad (34)$$

(34) with (17) implies that:

$$\sin(\tau \omega_j + \delta_j) = 0. \quad (35)$$

Equation (35) is satisfied if $\omega_i = \frac{p\pi - \delta_j}{\tau}$, where $p = 0, 1, 2 \ldots$.

Thus, the admissible range for $\omega_i$ is as follows:

$$\omega_i \in (0, \omega_j^*) = \left(0, \frac{\pi - \delta_j}{\tau}\right). \quad (36)$$

Briefly, an algorithm is introduced to determine the stabilizing set on the $k_x - k_j$ plane and corresponding tolerable delay $\tau$ for MAS with a general second-order dynamic (1).

### 3.1 Algorithm for MAS’s with general second-order dynamic

1. Compute $\lambda_i$ from $det(\lambda_i I_N - \mathcal{L}) = 0$.
2. Generate characteristic equation (7) having parameters $\alpha$, $\beta$, $\lambda_i$ and set $\tau = 0$.
3. If $I_{\lambda_i} \neq 0$, plot condition (8) on $k_x - k_j$ plane or if $I_{\lambda_i} = 0$, plot condition (10) on $k_x - k_j$ plane.
4. Take a small initial time delay $\tau$.
5. Determine the stable region $Q_i$ and $Q$ from (22)-(23) as depicted in Figure 5.
6. If the region $Q$ is non-empty, then increase the value of $\tau$ and go to step (5).
7. End when the region $Q$ disappears.

Figure 5 shows a typical procedure done with the above algorithm. Specifically, following algorithm can be used to determine the exact stabilizing set for the case that MAS has double integrator agents as (24).

### 3.2 Algorithm for MAS’s with double integrator dynamic

1. Compute $\lambda_i$ from $det(\lambda_i I_N - \mathcal{L}) = 0$.
2. Generate characteristic equation (25) substituting $\lambda_i$ and set $\tau = 0$.
3. If $I_{\lambda_i} \neq 0$, plot condition (26) on $k_x - k_j$ plane or if $I_{\lambda_i} = 0$, plot condition (27) on $k_x - k_j$ plane.
4. Take an small initial time delay $\tau$.

5. Determine the stable region $q_i$ and $q$ from (30) and (29).
6. If the region $q$ is non-empty, then increase the value of $\tau$ and go to step (5).
7. End when the region $q$ disappears.

*Remark 3.* Differently from [27], the above-mentioned procedures lead to delay-dependent regions for choosing controller gains. Moreover, compared to [25] the employed control law and aforementioned tuning algorithms are more amenable for implementation. Simulation examples in the next section demonstrate that the suggested control strategy yields to greater tolerable communication delay, compared to rival methods, [25] and [27].

### 4 COMPARATIVE NUMERICAL EXAMPLE

In this section, the advantages and applicability of the proposed scheme are demonstrated by simulations. The obtained results are compared to the ones of [25] and [27] to demonstrate that the proposed controller can tolerate more delay than this rival ones. It is worth noting that unlike [27], our derived theory is not applicable for the varying time-delay; so, the comparison is reasonable only in the case that the method of [27] is implemented for the fixed and known delay value which is a special case of their problem.

*Example 1.* The following MAS is adopted from [27] in which five agents are described by second-order dynamical equations as

$$\begin{cases}
\dot{x}_i(t) = y_i(t), \\
\dot{y}_i(t) = 2x_i(t) + y_i(t) + n_i(t),
\end{cases} \quad (37)$$
where Laplacian and adjacency matrices are as:

\[
\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 0 & 2 \\ 3 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{L} = \begin{bmatrix} 2 & 0 & 0 & 0 & -2 \\ -3 & 3 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 2 & 0 \\ 0 & -4 & 0 & 0 & 4 \end{bmatrix},
\]

The communication delay between agents is assumed to be \( \tau = 0.07 \) which leads to an unstable system by the method of [27]. Algorithm 3.1 is used to obtain the stabilizing values for \( k_x \) and \( k_y \) in control protocol (2).

\[\text{Step 1.}\] The Laplacian matrix has the following eigenvalues:

\[\lambda_1 = 0, \quad \lambda_2 = 1, \quad \lambda_3 = 2, \quad \lambda_4 = \frac{9}{2} + t\sqrt{\frac{23}{2}}, \quad \lambda_5 = \frac{9}{2} - t\sqrt{\frac{23}{2}}.\]

\[\text{Step 2.}\] Error dynamic (5) has the following characteristic equation:

\[\prod_{i=2}^{5} \Delta_i(s, \tau),\]

where \( \Delta_i(s, \tau) \) is given by:

\[\Delta_2(s, \tau) = s^2 - 2 + e^{-\tau}(k_x s + k_y)(1),\]
\[\Delta_3(s, \tau) = s^2 - 2 + e^{-\tau}(k_x s + k_y)(2),\]
\[\Delta_4(s, \tau) = s^2 - 2 + e^{-\tau}(k_y s + k_x) \left( \frac{9}{2} + t\sqrt{\frac{23}{2}} \right),\]
\[\Delta_5(s, \tau) = s^2 - 2 + e^{-\tau}(k_y s + k_x) \left( \frac{9}{2} - t\sqrt{\frac{23}{2}} \right).\]

\[\text{Step 3.}\] Employing conditions (8) and (10) on above mentioned characteristic equations leads to the following stability criteria for the delay-free case:

\[\Delta_2(s, \tau) : \text{is stable if } k_x > 2, \quad k_y > 1,\]
\[\Delta_3(s, \tau) : \text{is stable if } k_x > 1, \quad k_y > \frac{1}{2},\]
\[\Delta_4(s, \tau) \text{ and } \Delta_5(s, \tau) : \text{are stable if }\]

\[\begin{cases} 
 k_y > \frac{2}{9} \\
 \frac{(9k_y - 2)^2}{2} + \frac{(9k_x - 4)^2}{2} + \frac{(9k_y - 2)(k_xk_y - k_x^2)}{2} > 0.
\end{cases}\]

The aforementioned conditions are summarized as follows:

\[\begin{cases} 
 k_x > 2 \\
 \frac{(9k_x - 2)^2}{2} + \frac{(9k_y - 4)^2}{2} + \frac{(9k_x - 2)(k_xk_y - k_x^2)}{2} > 0.
\end{cases}\]

(38)

FIGURE 6 The stable region \( Q \) for (37) and \( \tau = 0.07 \)

\[\text{Step 4.}\] Regarding \( \tau = 0.07 \), the admissible \( \omega_i \in (\omega_i^-, \omega_i^+) \) for \( i \)th curve \( H_i \) is obtained from (20) as follows:

\[\begin{cases} 
 \omega_x^- = \omega_x^+ = \omega_x^- = 2, \omega_y^- = 6.99 \\
 \omega_x^+ = 44.87, \omega_y^+ = 51.87, \omega_y^+ = 37.88
\end{cases}\]

and the corresponding limiter vertical lines in the \( k_x - k_y \) plane is computed by (23.3) and (23.4).

As shown in Figure 6, \( Q \) is enclosed by curves \( H_i \) in the interval \( \omega_i \in (\omega_i^-, \omega_i^+) \) together with obtained vertical lines in step 4 and the conditions extracted in step 3. Finally, the stabilizing region \( Q \) is obtained from (22) which is enclosed by \( \{k_x^1 = 1.27, \quad \text{curve } H_2, \quad \text{curve } H_3, \quad k_y = 2\} \). From Theorem 1, if the gains \( k_x \) and \( k_y \) are chosen from \( Q \) then the MAS achieves consensus.

Figure 7 demonstrates the trajectories of errors for the gain values \( \{k_x, k_y\} = [4.5, 2.2] \in Q \) and the communication delay \( \tau = 0.07 \). As seen, differently from the control method of [27] which cannot stabilize this scenario (see Figure 2 of [27]), the proposed scheme leads to desirable behaviour.

The maximum tolerable communication delay is determined by algorithm 3.1 as \( \tau_{\text{max}} = 0.1064 \). Furthermore, consensus of the agents with the controller gains, \( \{k_x, k_y\} = [2.003, 1.414] \in Q \) and communication delay \( \tau = 0.105 \) are shown on Figure 8.

**Example 2.** The double integrator dynamic is studied under the communication topology of Example 1. Substituting \( \alpha = \beta = 0 \) in the (37) leads to:

\[\begin{cases} 
 \ddot{y}_i(t) = y_i(t) \\
 \dot{y}_i(t) = u_i(t)
\end{cases}\]

(39)

Execution of algorithm 3.2 is described in the following steps:
Step 1. The eigenvalues of the Laplacian matrix are computed as before.

Step 2. Characteristic equation (25) for the (39) and $i = 2, 3, 4, 5$ are given by:

$$
\Delta_2(s, \tau) = s^2 + e^{-\tau}(k_x s + k_y),
$$

$$
\Delta_3(s, \tau) = s^2 + e^{-\tau}(k_x s + k_y),
$$

$$
\Delta_4(s, \tau) = s^2 + e^{-\tau}(k_x s + k_y) \left( \frac{9}{2} + t \frac{\sqrt{23}}{2} \right),
$$

$$
\Delta_5(s, \tau) = s^2 + e^{-\tau}(k_x s + k_y) \left( \frac{9}{2} - t \frac{\sqrt{23}}{2} \right).
$$

Step 3. Utilizing conditions (26) and (27) leads to the following stability criteria for the delay-free case:

$$
\Delta_2(s, \tau) : \text{is stable if } k_x > 0, k_y > 0,
$$

$$
\Delta_3(s, \tau) : \text{is stable if } k_x > 0, k_y > 0,
$$

$$
\Delta_4(s, \tau) \text{ and } \Delta_5(s, \tau) : \text{are stable if}
$$

$$
\begin{cases}
  k_y > 0 \\
  \left( \frac{9k_y}{2} \right)^2 \left( \frac{9k_x}{2} \right) + \left( \frac{9k_x}{2} \right) k_x k_y - k_x^2 \frac{23}{4} > 0.
\end{cases}
$$

These conditions are summarized as follows:

$$
\begin{cases}
  k_x > 0 \\
  \left( \frac{9k_y}{2} \right)^2 \left( \frac{9k_x}{2} \right) + \left( \frac{9k_x}{2} \right) k_x k_y - k_x^2 \frac{23}{4} > 0.
\end{cases}
$$

Step 4. Assuming $\tau = 0.07$, as shown in Figure 9, the stable regions $q_i$ are enclosed by curves $h_i$ in the interval $\omega_i \in (0, \omega_i^*)$, where:

$$
\omega_2^* = 22.439, \omega_3^* = 22.439, \omega_4^* = 29.434, \omega_5^* = 60.325.
$$

Together with the conditions extracted in step 3, the stable region $q$ is obtained finally from

$$
q \in \{ q_j \setminus q_i \mid i \neq j \text{ and } i, j = 2, 3, 4, 5 \},
$$

which is enclosed by the horizontal axis $k_y$ and curve $h_5$. The above procedure is depicted in Figure 9.
Figure 10 displays the consensus of the double integrator MAS with the gain values $[k_x, k_y] = [6, 2] \in q$ and the communication delay $\tau = 0.07$. It is worth noting that for this value of delay, the method of [27] does not stabilize this MAS (see Figure 4 of [27]).

In Figure 11, the states of error dynamic (5) are shown for MAS with double integrator agents and communication delay $\tau = 1$, which still are asymptotically stable.

Note that to demonstrate the robustness of the system to the large delay values, the simulation results in Figures 7, 8, 10, and 11 are reported for the marginal values of delay, where the area of the admissible region is very small and consequently the fluctuations in time responses are considerable. Furthermore, as seen from Figures 7 and 10, for the same delay value, double integrator agents reach consensus faster than general second-order agents because of simpler dynamics.

Example 3. Consider the MAS with double-integrator agents and following Adjacency-Laplacian matrix from [25]:

$$
\mathbf{A} = \begin{bmatrix}
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix}, \quad \mathbf{L} = \begin{bmatrix}
1 & 0 & -1 & 0 \\
-1 & 1 & 0 & 0 \\
0 & -1 & 1 & 0 \\
-1 & 0 & 0 & 1
\end{bmatrix}.
$$

In [25], the consensus was achieved with distributed PI controller if and only if $\tau < 0.2563$. By the proposed method the agents reach consensus even for $\tau = 3$ with the controller parameters $[k_x, k_y] = [0.006, 0.15]$ which belong to the stable region shown on Figure 12. Figure 13 depicts the consensus of double-integrator MAS with above-mentioned communication topology.
5 | CONCLUSION

In this paper, the stabilizing set has been obtained in the plane of control protocol parameters for MASs with general second-order linear dynamic and communication delay between agents. Furthermore, the consensus of MASs with double integrator agents has been studied in the proposed framework. The main feature of the suggested scheme is that the maximum tolerable communication delay between agents is increased remarkably compared to the rival approaches in the literature. Future studies can be directed to MASs with high order and non-linear dynamics for agents.

ORCID

Arash Farnam https://orcid.org/0000-0002-9139-234X

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How to cite this article: Nejadvali A, Esfanjani RM, Farnam A, Crevecoeur G. Delay dependent criteria for the consensus of second-order multi-agent systems subject to communication delay. IET Control Theory Appl. 2021;1–12. https://doi.org/10.1049/cth2.12154