SOME OBSERVATIONS ON THE INDEX OF $C_p$-SPACES

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Abstract. In this paper, we consider numerical indices associated to spaces with free $C_p$-action. We prove that the Stiefel manifolds provide an example of non-tidy spaces for $p = 2$, which are those whose co-index and index disagree. In the case of odd primes, we construct examples of co-index 3 whose index may be arbitrarily large.

1. Introduction

In topological combinatorics, one often has to rule out equivariant maps between $G$-spaces. In such cases an important family of invariants comes from various indices associated to $G$-spaces [9]. One such index is the Fadell-Husseini index [6] which is an ideal in the cohomology of $BG$.

In the case of a cyclic group of prime order the universal space $EG$ may be filtered using spheres. For the group $C_2$, the spheres with antipodal action form the skeleton of $EC_2$. For $p$ odd, the odd dimensional spheres form the odd skeleton of $EC_p$. This leads to the definition of co-index and index of $C_p$-spaces depending of which spheres map to $X$ or which spheres possess a map from $X$. A crucial role here is played by the Borsuk-Ulam theorem which says that there are no $C_p$-maps from spheres of higher dimension to those of lower dimension.

This paper investigates examples of spaces $X$ whose co-index is $<$ the index. These are called “non-tidy” spaces by Matousek [9]. For the spheres described, these two are equal. We show using Steenrod operations that Stiefel manifolds provide examples of “non-tidy” spaces in many cases (cf. Theorem 3.1). The index computations of Stiefel manifolds have been earlier applied to combinatorial problems in [3].

Matsushita [10] has constructed $C_2$-spaces of co-index 1 whose index is large. We analogously consider co-index and index for $C_p$-spaces. These have also earlier been studied using Bredon cohomology [1], [2]. We provide examples of $C_p$-spaces of co-index 3 and index arbitrarily large (cf. Theorem 4.2).

1.1. Organisation. In section 2 we recall some basic definitions, useful results regarding index and co-index and some basic notations used in rest of the paper. In section 3 we construct certain Stiefel manifolds as examples of ”non-tidy” space. In section 4 we construct inductively free $C_p$-spaces whose co-index remain constant but index becomes large.

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2. Preliminaries

In this section we will define some basic notations, definitions which will be used frequently in the rest of the paper. The category of the $G$-spaces has objects topological spaces with $G$-action and morphism as $G$-equivariant maps. We recall various indices associated to the free $G$-spaces in the case where $G$ is a cyclic group of prime order.

Notation 2.1. We use $S(V)$ to denote the sphere inside a $G$-representation $V$ and $\sigma$ to denote sign representation of $C_2$. For an odd prime $p$, $\lambda$ denotes the one dimensional

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complex representation of $C_p$ where the chosen generator $\tau$ acts by multiplying a complex number with $e^{2\pi \tau}$. We use the model for the universal space $EC_p$ as the $C_p$-CW-complex whose odd and even dimensional skeletons are defined by

$$E^{(2n-1)}C_p = S(n\lambda)$$

$$E^{(2n)}C_p = S(n\lambda) \cup_a C_p \times D^{2n}$$

where $a : C_p \times S^{2n-1} \to S^{2n-1}$ is the action map. If we consider $C_2$, the $k$-th skeleton of $EC_2$ becomes $S((k+1)\sigma)$ which is $S^k$ with antipodal action.

Note that for any topological group $G$ there always exists an universal bundle $p : EG \to EG/G = BG$ \footnote{which is unique upto homotopy. Our construction of universal bundle is slightly specialised which will be useful in proving Theorem 4.1} which is homotopically equivalent to $X/G$. Consider the fibration

$$X \to X_{hG} \xrightarrow{p_X} BG.$$

**Definition 2.2.** Let $X$ be a free $C_2$-space. Then one may define

$$\text{Ind}_{C_2}(X) = \min\{n \geq 0 \mid \exists a \text{ $C_2$-map } X \to S((n+1)\sigma)\},$$

$$\text{Coind}_{C_2}(X) = \max\{n \geq 0 \mid \exists a \text{ $C_2$-map } S((n+1)\sigma) \to X\}.$$

To rule out the equivariant maps between certain $G$-spaces one useful invariant is Fadell-Husseini index. For a free $G$-space $X$ recall that the homotopy orbit space $X_{hG} := EG \times_G X$ is homotopically equivalent to $X/G$. Consider the fibration

$$X \to X_{hG} \xrightarrow{p_X} BG.$$

**Definition 2.3.** \footnote{The Fadell-Husseini index $\text{Index}_G(X)$ of a $G$-space $X$ is defined as $\text{Ker}(p^*)$ where $p^* : BG \to X_{hG}$.} Some basic properties of Fadell-Husseini index are \footnote{\begin{itemize}
  \item **Monotonicity:** If $X \to Y$ is a $G$-equivariant map, then $\text{Index}_G(Y) \subseteq \text{Index}_G(X)$.
  \item **Additivity:** If $(X_1 \cup X_2, X_1, X_2)$ is an excisive triple of $G$-spaces, then
    $$\text{Index}_G(X_1) \text{Index}_G(X_2) \subseteq \text{Index}_G(X_1 \cup X_2).$$
  \item **Join:** Let $X$ and $Y$ be $G$-spaces, then $\text{Index}_G(X) \text{Index}_G(Y) \subseteq \text{Index}_G(X * Y)$.
\end{itemize} Fadell-Husseini index inspires one more similar numerical invariant as defined below.} \footnote{The cohomological index, denoted by $C$-index$_G(X)$ equals the maximum $n$ such that the ideal $\text{Index}_G(X) = 0$ upto degree $n$.}

**Definition 2.4.** We call a $C_2$-space "tidy" if $\text{Ind}_{C_2}(X) = \text{Coind}_{C_2}(X)$. The existence of $C_2$-maps between "tidy" spaces are completely determined by the $C_2$-index. A $C_2$-space is "non-tidy" if $\text{Ind}_{C_2}(X) \neq \text{Coind}_{C_2}(X)$. Examples of "non-tidy" spaces are not so trivial. We will construct certain Stieffel manifolds as examples of "non-tidy" spaces in section 3. The $C_2$-indices are related by the following inequality,

**Proposition 2.5.** \footnote{For any topological space $X$, $\text{Coind}_{C_2}(X) \leq \text{C-index}_{C_2}(X) \leq \text{Ind}_{C_2}(X)$.} We can generalize the definitions of $C_2$-indices for an arbitrary $G$-space $X$ by replacing the representation sphere to universal spaces $E^{(n)}G$. But we will restrict our attention to $C_p$-spaces in the rest of the section.

**Definition 2.6.** Let $X$ be a free $C_p$-space. Then one may define

$$\text{Ind}_{C_p}(X) = \min\{n \geq 0 \mid \exists a \text{ $C_p$-map } X \to E^{(n)}C_p\},$$

$$\text{Coind}_{C_p}(X) = \max\{n \geq 0 \mid \exists a \text{ $C_p$-map } E^{(n)}C_p \to X\}.$$
Since there exists $C_p$-map between any two $E^{(n)}C_p$ spaces the definitions are well defined. We can observe similar inequality like free $C_2$-spaces in free $C_p$-space too which is given by

$$\text{Coind}_{C_p}(X) \leq \text{C-index}_{C_p}(X) \leq \text{Ind}_{C_p}(X).$$

In the case of $C_p$-spaces, the cohomological index is often computed using the height of a cohomology class defined below.

**Definition 2.7.** The height of a cohomology class is defined by

$$ht(v) = \min\{n : v^n = 0\}.$$

Let $V_{l,k}$ denote the Stiefel manifold of $k$ orthonormal vectors in $\mathbb{R}^l$. The group $C_2$ acts on $V_{l,k}$ by sending $(v_1, \ldots, v_k)$ to $(\pm v_1, \ldots, \pm v_k)$. The projective Stiefel manifold $X_{l,k}$ is the orbit space $V_{l,k}/C_2$. The mod 2 cohomology of Stiefel manifold is given by

$$H^*(V_{l,k}) = \Lambda\mathbb{Z}_2(x_{l-k}, \ldots, x_{l-1}).$$

From the Serre spectral sequence of the fibration $V_{l,k} \xrightarrow{i} X_{l,k} \xrightarrow{p} BC_2$ we obtain the mod 2 cohomology of $X_{l,k}$ additively (cf. Theorem 1.6 [7]).

**Theorem 2.8.** $H^*(X_{l,k}; \mathbb{Z}_2) = \mathbb{Z}_2[z]/\langle z^N \rangle \otimes \Lambda\mathbb{Z}_2(z_{l-k}, \ldots, z_{l-1})$, where degree of $z$ is 1 and $N = \min \{j : l - k + 1 \leq j \leq l\}$ and $\binom{\ell}{j} \neq 0$ (mod 2). Moreover $p^*(u) = z$ where $u$ is the generator of $H^1(BC_2; \mathbb{Z}_2)$ and $i^*(z_1) = x_1$.

**Theorem 2.9.** Index$_{C_2} V_{l,k}$ is the ideal $\langle u^N \rangle$ in the cohomology of $BC_2$, where $N$ is as described in (2.8).

**Proof.** Considering the fibration

$$V_{l,k} \rightarrow X_{l,k} \rightarrow BC_2$$

the proof will directly follow from the Theorem 2.8. \qed

3. **Stiefel manifolds as an example of non-tidy spaces**

In this section we will try to construct certain Stiefel manifolds as an example of ”non-tidy” spaces. Note that as $V_{l,k}$ is $l - k - 1$ connected, by equivariant obstruction theory there is a $C_2$-map from $S((l - k)\sigma) \rightarrow V_{l,k}$. Therefore we can say that the co-index of $V_{l,k}$ is at least $l - k - 1$. By monotonicity of Fadell-Husseini index we can rule out the existence of $C_2$-equivariant map from $V_{l,k} \rightarrow S(r\sigma)$ if $r < N$, where $N = \min \{j : l - k + 1 \leq j \leq l\}$ and $\binom{\ell}{j} \equiv 1$ (mod 2). The next result addresses the following question. Does there exists a $C_2$-map

$$f : V_{l,k} \rightarrow S(N\sigma)$$

for suitable $l$ and $k$?

**Theorem 3.1.** For $l = k - 1 + \alpha 2^s$, $k < 2^s$ we have C-index$_{C_2}(V_{l,k}) = \alpha 2^s - 1$. Further if $s$ is the least positive integer such that $k < 2^s$ then Ind$_{C_2}(V_{k-1+\alpha 2^s k}) > \alpha 2^s - 1$.

**Proof.** From Theorem (2.9) it follows that C-index$_{C_2}(V_{l,k})$ is $N - 1$ where $N$ is as described in (2.8). We have

$$\binom{l}{l - k + 1} = \frac{(k - 1 + 2^s\alpha)(k - 2 + 2^s\alpha) \cdots (1 + 2^s\alpha)}{(k - 1) \cdots 1} = \frac{(k - 1 + 2^s\alpha)}{(k - 1)} \frac{(k - 2 + 2^s\alpha)}{(k - 2)} \cdots \frac{(1 + 2^s\alpha)}{1}.$$
Now if $k - i$ is odd the expression \( \frac{(k-i+2^s \alpha)}{(k-i)} \) is odd. If $k - i$ is even we can factor out $2^m$ part ($m < s$) from both the numerator and denominator and the expression becomes odd. Therefore we can conclude

\[
\left( \frac{l}{l - k + 1} \right) \equiv 1 \pmod{2}.
\]

Hence $N = l - k + 1 = \alpha 2^s$. This completes the first part of the proof.

For computing the topological index suppose there exists a $C_2$-map $f : \mathcal{V}_{l,k} \to S(N\sigma)$, then it will induce the following commutative diagram between fibrations

\[
\begin{array}{ccc}
\mathcal{V}_{l,k} & \xrightarrow{f} & S(N\sigma) \\
\downarrow & & \downarrow \\
\mathcal{X}_{l,k} & \xrightarrow{f_{\cdot}C_2} & \mathbb{R}P^{N-1} \\
\downarrow & & \downarrow \\
BC_2 & \xrightarrow{hC_2} & BC_2.
\end{array}
\]

As both $\text{Index}_{C_2} \mathcal{V}_{l,k}$ and $\text{Index}_{C_2} S(N\sigma)$ is \( \langle u \rangle \)

\[
f^*(\epsilon_{N-1}) = \begin{cases} 
  x_{N-1} \quad \text{(mod } I^2) \quad \text{if } 2k > l \\
  x_{N-1} \quad \text{if } 2k < l
\end{cases}
\]

where $\epsilon_{N-1}$ is the generator of top cohomology of $S(N\sigma)$ and $I$ is the ideal $\langle x_{l-k}, \cdots, x_{l-1} \rangle$.

Observe that for $k > 0$

\[
Sq^k(x_{N-1}) = Sq^k f^*(\epsilon_{N-1}) = f^*Sq^k \epsilon_{N-1} = 0.
\]

Now if we can show that $Sq^k(x_{N-1}) \neq 0$ for some $k > 0$, we will obtain a contradiction.

There is a map $i : \mathbb{R}P^{l-1} \to SO(l)/SO(l-k) \cong \mathcal{V}_{l,k}$, (Ch.(5)[12]). Consider the diagram

\[
\begin{array}{ccc}
H^{N-1}(\mathcal{V}_{l,k}) & \xrightarrow{i^*} & H^{N-1}(\mathbb{R}P^{l-1}) \\
\downarrow Sq^{2^s-1} & & \downarrow Sq^{2^s-1} \\
H^{N-1+2^s-1}(\mathcal{V}_{l,k}) & \xrightarrow{i^*} & H^{N-1+2^s-1}(\mathbb{R}P^{l-1}).
\end{array}
\]

We know from the property of Steenrod squares that (Ch.4,[8])

\[
Sq^{2^s-1}(u_{N-1}) = \left( \frac{N-1}{2^s-1} \right) u^{N-1+2^s-1}.
\]

The expression would be non zero for following two conditions

\[
\left( \frac{N-1}{2^s-1} \right) \equiv 1 \pmod{2} \quad \text{(2)}
\]

and

\[
N - 1 + 2^s-1 \leq l - 1,
\]

i.e.

\[
N + 2^s-1 \leq l. \quad \text{(3)}
\]

We have

\[
l = k - 1 + \alpha 2^s, \\
N = \alpha 2^s.
\]
Expanding (2) we get
\[
\frac{(N - 1)(N - 2) \cdots (N - 2^s - 1)}{2^s \cdots 1} = \frac{(2^s \alpha - 1)}{1} \cdots \frac{(2^s \alpha - 2^{s-1} + 1)}{(2^{s-1} - 1)} \cdot \frac{2^s \alpha - 2^{s-1}}{2^s - 1}
\]
which is odd by a similar argument as in (1).

So for \( \alpha \geq 1 \) and \( l = k - 1 + \alpha 2^s \) both the conditions (2) and (3) are satisfied implying, \( Sq^{2^{s-1}}(u^{N-1}) \neq 0 \) and we obtain a contradiction. This implies \( \text{Ind}_{C_2}(V_{k-1+\alpha 2^s,k}) > \alpha 2^s - 1 \). □

From Proposition 2.6 we obtain the following result.

**Corollary 3.2.** With \( k, s \) and \( \alpha \) as above, \( V_{k-1+\alpha 2^s,k} \) are examples of "non-tidy" spaces.

\[\text{4. } C_p\text{-space of high index}\]

In this section we provide examples of spaces with small co-index and high C-index. We first prove a lemma which identifies the space of co-index 1.

Let \( X \) be a free \( C_p \) path connected space and \( \bar{X} \) be its orbit space. From covering space theory we have \( \pi_1(\bar{X})/\pi_1(X) \cong C_p \). Let \( f : X \to Y \) be a \( C_p \)-equivariant map between two free path connected \( C_p \)-spaces. This map will induce \( f_* : \pi_1(\bar{X})/\pi_1(X) \to \pi_1(Y)/\pi_1(Y) \).

We have a commutative diagram
\[
\begin{array}{ccc}
\pi_1(\bar{X}) & \xrightarrow{f_*} & \pi_1(Y) \\
\Downarrow & & \Downarrow \\
C_p & = & C_p.
\end{array}
\]

We call an element \( \alpha \in \pi_1(\bar{X}) \) prime to \( p \) if it does not belong to \( \pi_1(X) \).

**Theorem 4.1.** For an 1-connected free \( C_p \)-space \( X \), there exists a map \( g : E^{(2)}C_p \to X \) iff \( \pi_1(\bar{X}) \) has an element prime to \( p \) whose order is \( p \).

**Proof.** The proof is similar to Theorem 2.2 described in [10]. Let \( \tau \in \pi_1(E^{(2)}C_p) \cong C_p \) be the generator of \( C_p \). By the commutative diagram (1), existence of \( g \) implies \( \bar{g}_*(\tau) \) is an element prime to \( p \) of order \( p \).

Let \( \beta \in \pi_1(\bar{X}) \) be an element prime to \( p \) such that \( \beta^p = 1 \). Let \( \bar{\gamma} \) be a representative of \( \beta \) and \( \gamma \) be its lift in \( X \). We can choose \( \gamma \) as
\[
\gamma : [0, 2\pi/p] \to X.
\]

Since \( \beta \) is prime to \( p \) we have a non-trivial generator \( \tau \) of \( C_p \), such that \( \tau \gamma(0) = \gamma(\frac{2\pi}{p}) \).

Define \( \phi = \gamma \cdot (\tau \gamma) \cdots (\tau^{p-1} \gamma) \). Then \( \phi : S^1 = E^{(1)}C_p \to X \) is a \( C_p \)-equivariant map and
\[
p_*[\phi] = \beta^p = 1
\]
where \( p : X \to \bar{X} \) is the covering projection. Since \( p_* \) is injective in \( \pi_1 \) we can conclude \( \phi \) is null-homotopic and it can be extended to a map \( g : E^{(2)}C_p \to X \) by equivariant obstruction theory [9]. □

We want to construct a sequence of spaces \( X(k) \) whose C-index \( C_p(X(k)) \) becomes large and \( \text{Coin}(X(k)) \) is small. Start with \( X(0) = S^3 \) where \( C_p \) acts freely and the left action is generated by \( g : (z_0, z_1) = (e^{2\pi i/p} z_0, z_1) \) and the right action is generated by \( (z_0, z_1) \cdot g = (e^{2\pi i/p} z_0, e^{2\pi i/p} z_1) \). Define \( X(k+1) = X(k) \times_{C_p} S^3 \). We see
\[
S^3 \to X(k+1) \to X(k)/C_p
\]
is a $S^3$ bundle over $X(k)/C_p$. More specifically, it is the bundle $S(L \oplus \epsilon C)$ over $X(k)/C_p$, where the line bundle $L$ is classified by

$$X(k)/C_p \xrightarrow{\psi_k} BC_p \xrightarrow{\phi} BS^1.$$  

Recall that

$$H^*(BC_p; \mathbb{Z}/p) = \mathbb{Z}/p[y]/(y^2)$$

where $|y| = 2$, $|\epsilon| = 1$ and the two generators are related by Bockstein, $\beta(\epsilon) = y$. Also recall,

$$H^*(BS^1; \mathbb{Z}/p) = H^*(\mathbb{C}P^\infty; \mathbb{Z}/p) = \mathbb{Z}/p[x]$$

where $\phi^*(x) = y$.

From the fiber bundle (5) we have a $C_p$-equivariant inclusion $S^3 \to X(k+1)$. This implies $\text{Coind}_{C_p}(X(k+1)) \geq 3$.

**Theorem 4.2.** $C$-index $C_p(X(k+1)) \geq 2k + 2$ and $\text{Coind}_{C_p}(X(k+1)) = 3$.

**Proof.** Consider the fibration

(6)  

$$S^3/C_p = L_p(3) \twoheadrightarrow X(k+1)/C_p \to X(k)/C_p$$

and the commutative diagram of fibrations

$$S^3 \xrightarrow{i} X(k+1) \xrightarrow{\psi_{k+1}} BC_p$$

As $\text{Coind}_{C_p}(X(k+1)) \geq 3$, $C$-index $C_p(X(k+1)) \geq 3$. This implies $\text{Index}_{C_p}(X(k+1))$ does not have any element in degree 3 and

$$\psi_{k+1}^*(y) \neq 0, \psi_{k+1}^*(\epsilon) \neq 0, \psi_{k+1}^*(\epsilon y) \neq 0$$

all of which get pulled back to the respective generators of the cohomology of $L_p(3)$. The spectral sequence for (6) collapses at $E_2$ page.

Consider the commutative diagram of fibrations

(7)  

$$X(k+1) \xrightarrow{i} X(k+1)/C_p \xrightarrow{\Lambda_{k+1}} P(L \oplus \epsilon) \xrightarrow{\Lambda_{k+1}} X(k)/C_p \xrightarrow{\Lambda_{k+1}} X(k)/C_p.$$  

From the construction of Chern classes [5]

$$H^*(P(L \oplus \epsilon); \mathbb{Z}/p) = H^*(X(k)/C_p)[x]/(x^2 + z_k x)$$

where $z_k$ is the first Chern class of the line bundle $L$ over $X(k)/C_p$ as described above, that is

$$c_1(L \oplus \epsilon) = c_1(L) = \psi_k^*(y) = z_k.$$  

Let

$$\psi_{k+1}^*(\epsilon) = e_{k+1}, \quad \psi_{k+1}^*(y) = z_{k+1}.$$  

As cohomology of $L_p(3)$ is freely generated by $i^*(e_{k+1}), i^*(z_{k+1}), i^*(\epsilon_{k+1} z_{k+1})$, Lerray-Hirsch theorem for the bundle

$$L_p(3) \twoheadrightarrow X(k+1)/C_p \to X(k)/C_p$$

implies $H^*(X(k+1)/C_p; \mathbb{Z}/p)$ is the $H^*(X(k)/C_p; \mathbb{Z}/p)$-module generated by $e_{k+1}, z_{k+1}, \epsilon_{k+1} z_{k+1}$. To find the ring structure of $H^*(X(k+1)/C_p; \mathbb{Z}/p)$ first observe that $\Lambda^*$ is a ring map. From the right hand square of the diagram (7) we have

$$\Lambda^*(x) = \psi_{k+1}^*(y) = z_{k+1},$$

$$\Lambda^*(x^2 + z_k x) = 0.$$
This completes the proof of the first part of the theorem.

And $e^2 = 0$ implies $e_{k+1}^2 = 0$. Thus

$$H^*(X(k + 1)/C_p; \mathbb{Z}_p) = H^*(X(k)/C_p; \mathbb{Z}_p)[e_{k+1}, z_k]/(e_{k+1}^2, z_{k+1}^2)$$

Therefore in this ring,

$$z_{k+1}^n = \pm z_{k+1}^{n-1} z_k$$

This gives $ht(z_{k+1}) = ht(z_k) + 1$. As $\beta(e_{k+1}) = z_{k+1}$, we conclude that

$C$-index$_{C_p}(X(k + 1)) = 2(ht(z_{k+1}) - 1)$

or

$$= 2ht(z_{k+1}) - 1.$$ 

So we have $C$-index$_{C_p}(X(k)) = 2k + 2$ or $2k + 3$, which is arbitrary large as $k$ increases. This completes the proof of the first part of the theorem.

For the second part consider any arbitrary $C_p$-map $f : S^3 \to X(k + 1)$. We will show this map is not null-homotopic. This implies that the map does not extend to $E(4)C_p$ and thus $\text{Coind}_{C_p}(X(k + 1)) = 3$. By construction of $X(k)$ consider the commutative diagram

$$\begin{array}{ccc}
S^3 & \xrightarrow{f} & X(k + 1) \\
\downarrow{q} & & \downarrow{g} \\
S^3/C_p & \xrightarrow{g \circ q} & X(k)/C_p
\end{array}$$

If for every $r$, image of $\pi_1(S^3/C_p)$ is non-zero under the composite map

$$S^3/C_p \to X(k + 1)/C_p \to \cdots \to X(r)/C_p$$

we reach $X(0)/C_p$ and have $g_0 : S^3/C_p \to X(0)/C_p = S^3/C_p$ for which $g_0$ is not 0. That means $g_0$ induces isomorphism in $\pi_1$. This implies $g_0$ lifts to an $C_p$-equivariant map $\tilde{g}_0 : S^3 \to S^3$ with

$$\tilde{g}_0 : H_3(S^3; \mathbb{Q}) \to H_3(S^3; \mathbb{Q})$$

is an isomorphism. The quotient map $q : S^3 \to S^3/C_p$ induces an isomorphism

$$q_* : H_3(S^3; \mathbb{Q}) \cong H_3(S^3/C_p; \mathbb{Q}).$$

Applying $H_3(\cdot; \mathbb{Q})$ to the diagram (8) gives

$$\begin{array}{ccc}
H_3(S^3; \mathbb{Q}) & \xrightarrow{f_*} & H_3(X(k + 1); \mathbb{Q}) \\
\downarrow{q_*} & & \downarrow{g_{0*}} \\
H_3(S^3/C_p; \mathbb{Q}) & \xrightarrow{g_{0*}} & H_3(X(0)/C_p; \mathbb{Q})
\end{array}$$

In above diagram as the composite $g_*g_* \neq 0$ we must have

$$f : S^3 \to X(k + 1)$$

not homotopic to zero as it induces non-trivial map on rational homology.
Now we may assume for $k = r$

$$\begin{xy}
 0<0,0> = \text{Diagram}
 0a1{S^3/C_p} = \ar[r]^{g} & \ar[r] X(r + 1)/C_p \\
 0a0{X(r)/C_p} = \ar[r] & \ar[r] \text{and hence the map } g_∗ \text{ is 0 in } \pi_1. \text{ From the covering space theory we have a lift of } g \text{ in the universal cover of } X(k)/C_p \text{ by the following diagram}

\begin{xy}
 0<0,0> = \text{Diagram}
 0a1{S^3 × S^3 × \cdots × S^3} = \ar[r]^{\lambda} \ar[r] X(k) \\
 0a0{S^3/C_p} = \ar[r]^{g} \ar[r] X(k)/C_p.
\end{xy}

The 3-equivalence $S^3 \to K(\mathbb{Z}/3)$ implies the following isomorphism between the based homotopy classes of maps

$$[S^3/C_p, S^3 × S^3 × \cdots × S^3] \cong [S^3/C_p, K(\mathbb{Z}/3) × \cdots × K(\mathbb{Z}/3)]
\cong \bigoplus_{k+1} H^3(S^3/C_p; \mathbb{Z})
\cong \bigoplus_{k+1} \mathbb{Z}.$$

If $\lambda$ is not homotopic to $\ast$, $\lambda^*: H^3((S^3)^{k+1}; \mathbb{Z}) \to H^3(S^3/C_p; \mathbb{Z})$ is non-trivial. This implies

$$\lambda^*: H^3((S^3)^{k+1}; \mathbb{Q}) \to H^3(S^3; \mathbb{Q})$$

is non-zero. From the long exact sequence of fibration $S^3 × \cdots × S^3 \to X(k)/C_p$ we have $H^3((X(k)/C_p); \mathbb{Q}) \cong H^3((S^3)^{k+1}; \mathbb{Q})$. This implies that the map $gq : S^3 \to X(k)/C_p$ in the commutative diagram is not homotopic to zero. Thus $f : S^3 \to X(k + 1)$ is not null-homotopic as it induces an isomorphism on $H^*(-; \mathbb{Q})$.

If $\lambda$ is homotopic to $\ast$ then $g$ is null-homotopic. From the covering lifting property of fibration we have

$$\begin{xy}
 0<0,0> = \text{Diagram}
 0a1{S^3/C_p} = \ar[r]^{i} \ar[r] X(k + 1)/C_p \\
 0a0{S^3/C_p × I} = \ar[r] \ar[r] \ar[r] S^3/C_p × \{1\} \ar[r] \ar[r] \ar[r] X(k)/C_p.
\end{xy}

The map $S^3/C_p \to X(k + 1)/C_p$ on orbit spaces induced from $f$ is non-trivial, so we deduce that $\mu$ is an isomorphism in $\pi_1$ and in $\mathbb{Z}/p$-cohomology. Now consider the commutative
We have $\mu^*$ non-zero in upper row. We will show $i^*$ is non-zero in upper row. It suffices to prove this rationally by the commutative diagram

$$
\begin{array}{ccc}
H^3(X(k+1)/C_p; \mathbb{Z}) & \stackrel{i^*}{\longrightarrow} & H^3(S^3/C_p; \mathbb{Z}) \\
\downarrow & & \downarrow \\
H^3(X(k+1)/C_p; \mathbb{Z}/p) & \stackrel{\mu^*}{\longrightarrow} & H^3(S^3/C_p; \mathbb{Z}/p).
\end{array}
$$

We will show $i^*: H^3(X(k+1); \mathbb{Q}) \to H^3(S^3; \mathbb{Q})$ is non-zero. For this we will analyze Serre spectral sequence associated to the principal fibration

$$S^3 \to X(k+1)/C_p \to X(k)/C_p.$$ 

As mentioned earlier in this section this is exactly the $S(L \oplus \epsilon_C)$ bundle over $X(k)/C_p$. Observe that $\pi_1(X(k)/C_p) = \oplus_{k+1} \mathbb{Z}/p$ does not act non-trivially on $\mathrm{Aut}(\mathbb{Z})$. The $\mathbb{Z}$-cohomology spectral sequence associated to the bundle collapses at $E_2$ page. Denote by $\beta$ the generator of top cohomology of $S^3$. Then the differential is given by

$$d(\beta) = e(L \oplus \epsilon_C) = 0,$$

where $e$ denotes the Euler class of the bundle. This implies $i^*: H^3(X(k+1); \mathbb{Z}) \to H^3(S^3; \mathbb{Z})$ is non-trivial and so with rational coefficient. This implies that the homomorphism $\mu^* \circ i^*$ in the top row of the diagram (9) is non-zero. Therefore the top horizontal map is non-trivial in the following commutative diagram as the down horizontal map is non-zero.

$$
\begin{array}{ccc}
H^3(X(k+1); \mathbb{Q}) & \stackrel{i^*}{\longrightarrow} & H^3(S^3; \mathbb{Q}) \\
\downarrow & & \downarrow \\
H^3(X(k+1)/C_p; \mathbb{Q}) & \stackrel{\mu^* \circ i^*}{\longrightarrow} & H^3(S^3/C_p; \mathbb{Q}).
\end{array}
$$

This implies $f: S^3 \to X(k+1)$ is not null-homotopic and completes the proof.

\[\square\]

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