Finding Cliques of a Graph using Prime Numbers

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Abstract

This paper proposes a new algorithm for solving maximal cliques for simple undirected graphs using the theory of prime numbers. A novel approach using prime numbers is used to find cliques and ends with a discussion of the algorithm.

1 Introduction:

Graph-theoretic clustering techniques find their application in myriad of problems in information science. One such technique is finding all the cliques (maximal complete subgraphs) of a given graph. A first general algorithm which enumerates all cliques of the graph was given by Bierstone\cite{1}. The Bierstone algorithm attempts to find the cliques of the current node and its neighboring nodes which can be merged with the subgraphs already generated to give the maximal sub graph of the graph. A correction of the Bierstone’s algorithm was proposed by Gordon D. Mulligan and D.G. Corneil in 1972\cite{2}. In this paper we propose a new notation for graphs using prime numbers\cite{3}. This is followed by an algorithm to enumerate all the cliques of a general graph. The paper is commenced by a discussion.

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2 Notation Used:

In general we consider a graph given by $G(V, E)$. Each vertex $u \in V$ is identified by a unique prime number denoted by $v_u$. Every vertex $u$, has a weight, denoted by $w_u = \prod_{i \in N[u]} v_i$ where $N[i]$ is the closed set of all vertices adjacent to $i$.

**Theorem 2.1 (Fundamental Theorem of Arithmetic)** Every positive number greater than 1 can be written as a product of prime numbers in only one way.

**Corollary 2.1.1** If $p, q_1, q_2, \ldots, q_n$ are all primes and $p|q_1q_2\cdots q_n$ then $p = q_k$ for some $k$ where $1 \leq k \leq n$.

**Proposition 2.2** If $v_i|w_j$, there exists an edge from $j$ to $i$.

**Proof** $w_j = \prod_{u \in N[v]} v_u$. From [2.1.1] and the fact that $v_i$ is prime, $v_i = v_u$ for some $u \Rightarrow i \in N[j]$.

Consider $g = \gcd(w_i, w_j)$. Using [2.1] we can factorise $g$ in unique primes common to both $w_i$ and $w_j$. These are values of vertexes common to both $i$ and $j$.

**Proposition 2.3** A clique can be uniquely identified by the product of the values of its participating vertices.

**Proof** A clique is identified by the vertices participating in it. The value of each vertex is a unique prime. Thus the product is also unique.

3 Algorithm:

3.1 Theme:

A graph is represented by a list $Q$ of tuples $\{v_u, w_u\}$ where each tuple represents a vertex. A vertex participates in all the cliques of its induced subgraph. Hence any arbitrary vertex is chosen as pivot - $p$ and two graphs (lists) are generated. Left$Q$ represents the induced
subgraph of pivot. RightQ represents the subgraph in which pivot doesnot participate in any clique.
Consider any arbitrary vertex $u$. If $u \in N[p] \Rightarrow v_p|w_u$:

**Case 1:** For every clique in which $u$ participates, pivot is one of the vertex. This implies that $N[u] \subseteq N[p] \Rightarrow w_u|w_p$. Thus $u \in LeftQ$. In the induced subgraph, the neighborhood of $u$ is $N[u] - p$. Hence $w_u = w_u/v_p$.

**Case 2:** For some cliques in which $u$ participates, pivot is one of the vertex. This implies that $N[u] \nsubseteq N[p]$ but $N[u] \cap N[p] \neq \phi$. Here, $u \in LeftQ \land u \in RightQ$. Two vertices $u_L$ and $u_R$ are created for $LeftQ$ and $RightQ$ respectively where $v_{u_L} = v_{u_R} = v_u$. $N[u_L] = N[u] \cap N[p] - p \Rightarrow w_{u_L} = \gcd(w_u, w_p)/v_p$. $N[u_R] = N[u] - p$, all vertices that fall in case 1.

Else, $u \in RightQ$. The cliques of the graphs thus generated are found out by recursion. The terminating condition is a single vertex or a null graph. The pivot is added to all the cliques of the induced subgraph ($LeftQ$).

### 3.2 Merging of vertices:

If two vertices have the same neighborhood, they participate in the same cliques. Hence, such vertices can be logically considered as a single vertex. Thus, if there exist $u_1, u_2 \ldots u_n$ such that $w_{u_1} = w_{u_2} = \ldots = w_{u_n}$, they can be merged in one vertex $u_m$ such that $v_{u_m} = v_{u_1} \times v_{u_2} \times \ldots \times v_{u_n}$ and $w_{u_m} = w_{u_1}$. Logically speaking, $u_m$ represents a clique $\{u_1, u_2 \ldots u_n\}$. A normal case would demand $n$ recursions (one for every vertex) against a single recursion after merging. Thus, if given graph is a clique, the vertices will coalesce into a single vertex reducing the amount of computation drastically.
3.3 Algorithm is given below:

**Find-Clique**($Q$)

1. $CliqueQ \leftarrow NIL$
2. Sort-By-Weight($Q$)
3. if $|Q| = 0$
4. then return $CliqueQ$
5. for $i \leftarrow 1$ to $|Q|$
6. do
7. while $w_{Q[i]} = w_{Q[i+1]}$
8. do $v_{Q[i]} \leftarrow v_{Q[i]} \times v_{Q[i+1]}$
9. Remove($Q, Q[i+1]$)
10. if $|Q| = 1$
11. then Insert($CliqueQ, v_{Q[1]}$)
12. return $CliqueQ$
13. $p \leftarrow Q[1]$
14. Left$Q$, Right$Q$, Pivot$Q \leftarrow NIL$
15. for $j \leftarrow 2$ to $|Q|$
16. do if $v_p | w_{Q[j]}$
17. then $w_{Q[j]} \leftarrow w_{Q[j]} / v_p$
18. if $w_{Q[j]} | w_p$
19. then Insert($LeftQ, Q[j]$)
20. Insert($NewQ, Q[j]$)
21. else $n_L, n_R \leftarrow NIL$
22. $v_{n_L}, v_{n_R} \leftarrow v_{Q[j]}$
23. $w_{n_L} \leftarrow \text{GCD}(w_p, w_{Q[j]})$
24. $w_{n_R} \leftarrow w_{Q[j]}$
25. Insert($LeftQ, n_L$), Insert($RightQ, n_R$)
26. else Insert($RightQ, Q[j]$)
27. for $i \leftarrow 1$ to $|RightQ|$
28. do for $j \leftarrow 1$ to $|PivotQ|$
do if $v_{PivotQ[j]} \mid w_{RightQ[i]}$
then $w_{RightQ[i]} \leftarrow w_{RightQ[i]} / v_{PivotQ[j]}
$LeftCliqueQ, RightCliqueQ $\leftarrow$ NIL
LeftCliqueQ $\leftarrow$ FIND-CLIQUE(LeftQ)
RightCliqueQ $\leftarrow$ FIND-CLIQUE(RightQ)
for $i \leftarrow 1$ to $|LeftCliqueQ|
do LeftCliqueQ[i] $\leftarrow$ LeftCliqueQ[i] $\times$ $v_p$
CliqueQ $=$ LeftCliqueQ + RightCliqueQ
return CliqueQ

3.4 Explanation of the algorithm:
Initially all the tuples are sorted by weight to facilitate identification of vertices having same neighborhood if so then they are merged. If the input graph was a single vertex or a clique, the degree of graph reduces to one after merging. The algorithm terminates here reporting the value of the vertex as the clique in the graph.

If the given graph is not a clique, then the first element is chosen as the pivot. Based on this pivot, the graphs LeftQ and RightQ are generated. All the vertices belonging to case 1 are also stored in PivotQ. These are then eliminated one-by-one from the neighborhood of the vertices in RightQ. The cliques of subgraphs LeftQ and RightQ are found out using recursion. The pivot is then added to every clique of LeftQ. The two lists of cliques are merged and the algorithm terminates.

3.5 Analysis:
The best case is when the given graph $G(V, E)$ is a clique $C_{|V|}$. Then all the vertices merge into a single vertex. The time required to find the clique then depends only upon the time required to sort the vertices on the basis of their weight. Using quick sort, best case complexity converges to $O(n \log(n))$.
The choice of the pivot governs the two subgraphs generated. Hence if the pivot is chosen such that that we have the induced subgraph of order $|V| - 2$ and the other subgraph of
order $|V| - 1$. Such a case will occur only when the pivot participates in a clique with every other vertex except one, say $y$ and $y$ participates with every other vertex except pivot. This gives,

$$T(n) = T(n - 2) + T(n - 1) = O(2^n).$$

Thus the worst case complexity converges to $O(n)$.

4 Conclusion:

The theory of prime numbers can be used to perform simple set operations like union and intersection using arithmetic functions like gcd (greatest common divisor) and lcm (least common multiple). The major drawback of this method is its storage space, while it scores on simplicity.

Applying this principle to the theory of graphs, we can compute all the maximal subsets of the graph efficiently by removing the unwanted vertices by the process of merging, thus reducing the complexity for some special type of graphs to $O(n \log(n))$.

More work however remains to be done. This spans the complexity in the average case, and also possibility of a heuristic on choosing the pivot such that the generated subgraphs are more or less mutually exclusive.

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5 References:

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