DIMENSIONAL REDUCTION AND CATALYSIS OF DYNAMICAL SYMMETRY BREAKING BY A MAGNETIC FIELD

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ABSTRACT. It is shown that a constant magnetic field in 3+1 and 2+1 dimensions is a strong catalyst of dynamical chiral symmetry breaking, leading to the generation of a fermion dynamical mass even at the weakest attractive interaction between fermions. The essence of this effect is the dimensional reduction $D \to D - 2$ in the dynamics of fermion pairing in a magnetic field. The effect is illustrated in the Nambu–Jona–Lasinio (NJL) model and QED. In the NJL model in a magnetic field, the low–energy effective action and the spectrum of long wavelength collective excitations are derived. In QED (in ladder and improved ladder approximations) the dynamical mass of fermions (energy gap in the fermion spectrum) is determined. Possible applications of this effect and its extension to inhomogeneous field configurations are discussed.
1. INTRODUCTION

At present there are only a few firmly established non-perturbative phenomena in 2+1 and, especially, (3 + 1)–dimensional field theories. In this paper, we will establish and describe one more such phenomenon: dynamical chiral symmetry breaking by a magnetic field.

The problem of fermions in a constant magnetic field had been considered by Schwinger long ago [1]. In that classical work, while the interaction with the external magnetic field was considered in all orders in the coupling constant, quantum dynamics was treated perturbatively. There is no spontaneous chiral symmetry breaking in this approximation. In this paper we will reconsider this problem, treating quantum dynamics non-perturbatively. We will show that in 3+1 and 2+1 dimensions, a constant magnetic field is a strong catalyst of dynamical chiral symmetry breaking, leading to generating a fermion mass even at the weakest attractive interaction between fermions. We stress that this effect is universal, i.e. model independent.

The essence of this effect is the dimensional reduction $D \to D - 2$ in the dynamics of fermion pairing in a magnetic field: while at $D = 2 + 1$ the reduction is $2 + 1 \to 0 + 1$, at $D = 3 + 1$ it is $3 + 1 \to 1 + 1$. The physical reason of this reduction is the fact that the motion of charged particles is restricted in directions perpendicular to the magnetic field.

Since the case of 2+1 dimensions has been already considered in detail in Ref. [2], the emphasis in this paper will be on (3 + 1)–dimensional field theories. However, it will be instructive to compare the dynamics in 2+1 and 3+1 dimensions.

As concrete models for the quantum dynamics, we consider the Nambu-Jona-Lasinio (NJL) model [3] and QED. We will show that the dynamics of the lowest Landau level (LLL) plays the crucial role in catalyzing spontaneous chiral symmetry breaking. Actually, we will see that the LLL plays here the role similar to that of the Fermi surface in the BCS theory of superconductivity [4].

As we shall show in this paper, the dimensional reduction $D \to D - 2$ is reflected in the structure of the equation describing the Nambu–Goldstone (NG) modes in a magnetic field. In Euclidean space, for weakly interacting fermions, it has the form of a two-dimensional (one–dimensional) Schrödinger equation at $D = 3 + 1$ ($D = 2 + 1$):

$$(-\Delta + m^2_{\text{dyn}} + V(\mathbf{r}))\Psi(\mathbf{r}) = 0.$$  \hspace{1cm} (1)

Here $\Psi(\mathbf{r})$ is expressed through the Bethe–Salpeter (BS) function of NG bosons,

$$\Delta = \frac{\partial^2}{\partial x_3^2} + \frac{\partial^2}{\partial x_4^2}.$$
(the magnetic field is in the $+x_3$ direction, $x_4 = it$) for $D = 3 + 1$, and $\Delta = \frac{\partial^2}{\partial x_3^2}$, $x_3 = it$, for $D = 2 + 1$. The attractive potential $V(\mathbf{r})$ is model dependent. In the NJL model (both at $D = 2 + 1$ and $D = 3 + 1$), $V(\mathbf{r})$ is a $\delta$–like potential. In $(3 + 1)$–dimensional ladder QED, the potential $V(\mathbf{r})$ is

$$V(\mathbf{r}) = \frac{\alpha}{\pi \ell^2} \exp\left(\frac{r^2}{2\ell^2}\right) Ei\left(-\frac{r^2}{2\ell^2}\right), \quad r^2 = x_3^2 + x_4^2,$$

where $Ei(x) = - \int_{-x}^{\infty} dt \exp(-t)/t$ is the integral exponential function [5], $\alpha = \frac{e^2}{4\pi}$ is the renormalized coupling constant and $\ell \equiv |eB|^{-1/2}$ is the magnetic length.

Since $-m_{\text{dyn}}^2$ plays the role of energy $E$ in this equation and $V(\mathbf{r})$ is an attractive potential, the problem is reduced to finding the spectrum of bound states (with $E = -m_{\text{dyn}}^2 < 0$) of the Schrödinger equation with such a potential. More precisely, since only the largest possible value of $m_{\text{dyn}}^2$ defines the stable vacuum [6], we need to find the lowest eigenvalue of $E$. For this purpose, we can use results proved in the literature for the one–dimensional ($d = 1$) and two–dimensional ($d = 2$) Schrödinger equation [7]. These results ensure that there is at least one bound state for an attractive potential for $d = 1$ and $d = 2$. The energy of the lowest level $E$ has the form:

$$E(\lambda) = -m_{\text{dyn}}^2(\lambda) = -|eB| f(\lambda),$$

where $\lambda$ is a coupling constant ($\lambda = \lambda = G$ in the NJL model and $\lambda = \alpha$ in QED). While for $d = 1$, $f(\lambda)$ is an analytic function of $\lambda$ at $\lambda = 0$, for $d = 2$, it is non–analytic at $\lambda = 0$. Actually we find that, as $G \to 0$,

$$m_{\text{dyn}}^2 = |eB| \frac{N_c^2 G^2 |eB|}{4\pi^2},$$

where $N_c$ is the number of fermion colors, in $(2 + 1)$–dimensional NJL model [2], and

$$m_{\text{dyn}}^2 = \frac{|eB|}{\pi} \exp\left(-\frac{4\pi^2(1 - g)}{|eB| N_c G}\right),$$

where $g \equiv N_c G \Lambda^2/(4\pi^2)$, in $(3+1)$–dimensional NJL model. In $(3+1)$–dimensional ladder QED, $m_{\text{dyn}}$ is

$$m_{\text{dyn}} = C \sqrt{|eB|} \exp \left[-\frac{\pi}{2} \left(\frac{\pi}{2\alpha}\right)^{1/2}\right],$$

where the constant $C$ is of order one and $\alpha$ is the renormalized coupling constant. As we will show below, this expression for $m_{\text{dyn}}$ in QED is gauge invariant.

It is important that, as we shall show below, an infrared dynamics, with a weak QED coupling, is responsible for spontaneous symmetry breaking in QED in a
magnetic field. This suggests that the ladder approximation can be reliable in this problem. However, because of the (1 + 1)–dimensional form of the fermion propagator in the infrared region, there may be also relevant higher order contributions. As we shall show in this paper, there are indeed relevant one–loop contributions in the photon propagator. Taking into account these contributions (that corresponds to the so called improved ladder approximation), we get the expression for \( m_{\text{dyn}} \) of the form (3) with \( \alpha \rightarrow \alpha/2 \). We shall discuss the physics underlying this effect and the general status of the validity of the expansion in \( \alpha \) in the infrared region in QED in a magnetic field in Sec. 9.

As we will discuss in this paper, there may exist interesting applications of the phenomenon of chiral symmetry breaking by a magnetic field: in planar condensed matter systems, in cosmology, in the interpretation of the heavy-ion scattering experiments, and for understanding of the structure of the QCD vacuum. We will also discuss an extension of these results to inhomogeneous field configurations.

The paper is organized as follows. In Section 2 we consider the problem of a free relativistic fermion in a magnetic field in 3+1 and 2+1 dimensions. We show that the roots of the fact that a magnetic field is a strong catalyst of dynamical chiral symmetry breaking are actually in this problem. In Section 3 we show that the dimensional reduction \( 3 + 1 \rightarrow 1 + 1 \) (\( 2 + 1 \rightarrow 0 + 1 \)) in the dynamics of fermion pairing in a magnetic field is consistent with spontaneous chiral symmetry breaking and does not contradict the Mermin-Wagner-Coleman theorem forbidding the spontaneous breakdown of continuous symmetries in less than 2+1 dimensions. In Sections 4-8 we study the NJL model in a magnetic field in 3+1 dimensions. We derive the low-energy effective action and determine the spectrum of long wavelength collective excitations in this model. We also compare these results with those in the (2 + 1)–dimensional NJL model [2]. In Section 9 we study dynamical chiral symmetry breaking in QED in a magnetic field. In Section 10 we summarize the main results of the paper and discuss possible applications of this effect. In Appendix A some useful formulas and relations are derived. In Appendix B the reliability of the \( 1/N_c \) expansion in the NJL model in a magnetic field is discussed. In Appendix C we analyze the Bethe-Salpeter equation for Nambu-Goldstone bosons in QED in a magnetic field.
2. FERMIONS IN A CONSTANT MAGNETIC FIELD

In this section we will discuss the problem of relativistic fermions in a magnetic field in 3+1 dimensions and compare it with the same problem in 2+1 dimensions. We will show that the roots of the fact that a magnetic field is a strong catalyst of chiral symmetry breaking are actually in this dynamics.

The Lagrangian density in the problem of a relativistic fermion in a constant magnetic field $B$ takes the form

$$\mathcal{L} = \frac{1}{2} [\bar{\psi} \ (i \gamma^\mu D_\mu - m) \psi], \quad \mu = 0, 1, 2, 3,$$

where the covariant derivative is

$$D_\mu = \partial_\mu - i e A^\text{ext}_\mu.$$  \hfill (8)

We will use two gauges in this paper:

$$A^\text{ext}_\mu = - \delta_{\mu 1} B x_2$$  \hfill (9)

(the Landau gauge) and

$$A^\text{ext}_\mu = - \frac{1}{2} \delta_{\mu 1} B x_2 + \frac{1}{2} \delta_{\mu 2} B x_1$$  \hfill (10)

(the symmetric gauge). The magnetic field is in the $+x_3$ direction.

The energy spectrum of fermions is \[8\]:

$$E_n(k_3) = \pm \sqrt{m^2 + 2 |eB| n + k_3^2}, \quad n = 0, 1, 2, \ldots$$  \hfill (11)

(the Landau levels). Each Landau level is degenerate: at each value of the momentum $k_3$, the number of states is

$$dN_0 = S_{12} L_3 \frac{|eB| dk_3}{2\pi}$$

at $n=0$, and

$$dN_n = S_{12} L_3 \frac{|eB| dk_3}{\pi}$$

at $n \geq 1$ (where $L_3$ is the size in the $x_3$-direction and $S_{12}$ is the square in the $x_1 x_2$-plane). In the Landau gauge \[8\], the degeneracy is connected with the momentum $k_1$ which at the same time coincides with the $x_2$ coordinate of the center of a fermion orbit in the $x_1 x_2$-plane \[8\]. In the symmetric gauge \[10\], the
degeneracy is connected with the angular momentum \( J_{12} \) in the \( x_1x_2 \)-plane (for a recent review see Ref. [9]).

As the fermion mass \( m \) goes to zero, there is no energy gap between the vacuum and the lowest Landau level (LLL) with \( n = 0 \). The density of the number of states of fermions on the energy surface with \( E_0 = 0 \) is

\[
\nu_0 = V^{-1} \left. \frac{dN_0}{dE_0} \right|_{E_0=0} = S_{12}^{-1} L_3^{-1} \left. \frac{dN_0}{dE_0} \right|_{E_0=0} = \frac{|eB|}{4\pi^2},
\]

where \( E_0 = |k_3| \) and \( dN_0 = V \frac{|eB|}{2\pi} \frac{dk_3}{2\pi} \) (here \( V = S_{12} L_3 \) is the volume of the system). We will see that the dynamics of the LLL plays the crucial role in catalyzing spontaneous chiral symmetry breaking. In particular, the density \( \nu_0 \) plays the same role here as the density of states on the Fermi surface \( \nu_F \) in the theory of superconductivity [4].

The important point is that the dynamics of the LLL is essentially (1+1)-dimensional. In order to see this, let us consider the fermion propagator in a magnetic field. It was calculated by Schwinger [1] and has the following form both in the gauge (9) and the gauge (10):

\[
S(x, y) = \exp \left[ \frac{i\epsilon}{2} (x - y) \gamma^\mu A^\mu_{\text{ext}} (x + y) \right] \tilde{S}(x - y),
\]

where the Fourier transform of \( \tilde{S} \) is

\[
\tilde{S}(k) = \int_0^\infty ds \exp \left[ is(k_0^2 - k_3^2 - |eB|^2 \tan(eBs) - m^2) \right] \times \left[(k_0 \gamma^0 - k_3 \gamma^3 + m)(1 + \gamma^1 \gamma^2 \tan(eBs)) - k_\perp \gamma_\perp (1 + \tan^2(eBs))\right]
\]

(here \( k_\perp = (k_1, k_2), \gamma_\perp = (\gamma_1, \gamma_2) \)). Then by using the identity \( i\tan(x) = 1 - 2 \exp(-2ix)/(1 + \exp(-2ix)) \) and the relation \[\int_0^\infty z^{-\alpha-1} \exp(\frac{zx}{z-1}) = \sum_{n=0}^{\infty} L_n^\alpha(x) z^n,\]

where \( L_n^\alpha(x) \) are the generalized Laguerre polynomials, the propagator \( \tilde{S}(k) \) can be decomposed over the Landau poles [10]:

\[
\tilde{S}(k) = i \exp \left( -\frac{k_\perp^2}{|eB|} \right) \sum_{n=0}^{\infty} (-1)^n \frac{D_n(eB, k)}{k_0^2 - k_3^2 - m^2 - 2|eB|n} \]

with

\[
D_n(eB, k) = (k_0 \gamma^0 - k_3 \gamma^3 + m) \left[(1 - i\gamma^1 \gamma^2 \text{sign}(eB))L_n \left(2 \frac{k_\perp^2}{|eB|}\right)ight]

- (1 + i\gamma^1 \gamma^2 \text{sign}(eB))L_{n-1} \left(2 \frac{k_\perp^2}{|eB|}\right) + 4(k_1 \gamma^1 + k_2 \gamma^2)L_{n-1}^1 \left(2 \frac{k_\perp^2}{|eB|}\right),
\]

6
where $L_n \equiv L_n^0$ and $L_{\alpha - 1} = 0$ by definition. The LLL pole is

$$\tilde{S}^{(0)}(k) = i \exp \left( -\frac{k_2^2}{|eB|} \right) \left( k_0^0 \gamma_0 - k_3^3 \gamma_3 + m \right) \frac{k_2^2 - k_3^3 - m^2}{k_2^2 - k_3^3 - m^2} (1 - i\gamma^1 \gamma^2 \text{sign}(eB)) .$$

This equation clearly demonstrates the $(1 + 1)$–dimensional character of the LLL dynamics in the infrared region, with $k_2^2 \ll |eB|$. Since at $m^2, k_0^0, k_3^3, k_2^2 \ll |eB|$ the LLL pole dominates in the fermion propagator, one concludes that the dimensional reduction $3 + 1 \rightarrow 1 + 1$ takes place for the infrared dynamics in a strong (with $|eB| \gg m^2$) magnetic field. It is clear that such a dimensional reduction reflects the fact that the motion of charged particles is restricted in directions perpendicular to the magnetic field.

The LLL dominance can, in particular, be seen in the calculation of the chiral condensate:

$$\langle 0 | \bar{\psi} \psi | 0 \rangle = - \lim_{x \rightarrow y} \text{tr} S(x, y) = - \frac{i}{(2\pi)^4} \text{tr} \int d^4k \tilde{S}_E(k)$$

$$= - \frac{4m}{(2\pi)^4} \int d^4k \int_{1/\Lambda^2}^\infty ds \exp \left[ -s \left( m^2 + k_4^4 + k_3^3 + k_2^2 \tanh(eBs) \right) \right]$$

$$= - |eB| m \frac{1}{4\pi^2} \int_{1/\Lambda^2}^\infty \frac{ds}{s} e^{-sm^2} \coth(|eBs|) \rightarrow - |eB| \frac{m}{4\pi^2} \left( \ln \frac{\Lambda^2}{m^2} + O(m^0) \right),$$

where $\tilde{S}_E(k)$ is the image of $\tilde{S}(k)$ in Euclidean space and $\Lambda$ is an ultraviolet cutoff. As it is clear from Eqs. (16) and (18), the logarithmic singularity in the condensate appears due to the LLL dynamics.

The above consideration suggests that there is a universal mechanism for enhancing the generation of fermion masses by a strong magnetic field in 3+1 dimensions: the fermion pairing takes place essentially for fermions at the LLL and this pairing dynamics is $(1 + 1)$–dimensional (and therefore strong) in the infrared region. This in turn suggests that in a magnetic field, spontaneous chiral symmetry breaking takes place even at the weakest attractive interaction between fermions in 3+1 dimensions. In this paper we will show the existence of this effect in the NJL model and QED.

In conclusion, let us compare the dynamics in a magnetic field in 3+1 dimensions with that in 2+1 dimensions [2]. In 2+1 dimensions, we consider the four–component fermions $\gamma^a$, connected with a four–dimensional (reducible) representation of Dirac’s matrices

$$\gamma^0 = \left( \begin{array}{cc} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{array} \right), \quad \gamma^1 = \left( \begin{array}{cc} i\sigma_1 & 0 \\ 0 & -i\sigma_1 \end{array} \right), \quad \gamma^2 = \left( \begin{array}{cc} i\sigma_2 & 0 \\ 0 & -i\sigma_2 \end{array} \right).$$

The Lagrangian density is:

$$\mathcal{L} = \frac{1}{2} [\bar{\psi}, \gamma^a \gamma^b D_\mu - m] \psi],$$
where $D_\mu = \partial_\mu - ieA^\text{ext}_\mu$ with $A^\text{ext}_\mu = (0, -Bx_2, 0)$ or $A^\text{ext}_\mu = (0, -\frac{B}{2}x_2, \frac{B}{2}x_1)$. At $m = 0$, this Lagrangian density is invariant under the flavor (chiral) $U(2)$ transformations with the generators

$$T_0 = I, \ T_1 = \gamma_5, \ T_2 = \frac{1}{i}\gamma^3, \ T_3 = \gamma^3\gamma^5$$

(22)

where $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$. The mass term breaks this symmetry down to the $U(1) \times U(1)$ with the generators $T_0$ and $T_3$. The fermion propagator in 2+1-dimensions is [2]:

$$S(x, y) = \exp\left(\frac{i\epsilon}{2}(x - y)^\mu A^\text{ext}_\mu(x + y)\right)\tilde{S}(x - y),$$

(23)

where the Fourier transform $\tilde{S}(k)$ of $\tilde{S}(x)$ is

$$\tilde{S}(k) = \int_{1/A^2}^\infty ds \exp\left[-ism^2 + isk_0^2 - isk^2\tan(eBs)\right] \cdot [(k^\mu\gamma_\mu + m + (k^2\gamma^1 - k^1\gamma^2)\tan(eBs))(1 + \gamma^1\gamma^2\tan(eBs))] .$$

(24)

The decomposition $\tilde{S}(k)$ over the Landau level poles now takes the following form:

$$\tilde{S}(k) = i \exp\left(-\frac{k^2}{|eB|}\right) \sum_{n=0}^\infty (-1)^n \frac{D_n(eB, k)}{k_0^2 - m^2 - 2|eB|n}$$

(25)

with

$$D_n(eB, k) = (k^0\gamma^0 + m)\left[(1 - i\gamma^1\gamma^2\text{sign}(eB))L_n\left(2\frac{k^2}{|eB|}\right)\right]$$

$$- (1 + i\gamma^1\gamma^2\text{sign}(eB))L_{n-1}\left(2\frac{k^2}{|eB|}\right) + 4(k^1\gamma^1 + k^2\gamma^2)L_{n-1}^1\left(2\frac{k^2}{|eB|}\right).$$

(26)

Then Eq. (25) implies that as $m \to 0$, the condensate $\langle 0|\bar{\psi}\psi|0 \rangle$ remains non-zero due to the LLL:

$$\langle 0|\bar{\psi}\psi|0 \rangle = -\lim_{m \to 0} \frac{m}{(2\pi)^3} \int d^3k \exp\left(-\frac{k^2}{|eB|}\right) \frac{1}{k_0^2 + m^2} = -\frac{|eB|}{2\pi}$$

(27)

(for concreteness, we consider $m \geq 0$). The appearance of the condensate in the flavor (chiral) limit, $m \to 0$, signals the spontaneous breakdown of $U(2)$ to $U(1) \times U(1)$ [2]. As we will discuss in Section 5, this in turn provides the analyticity of the dynamical mass $m_{\text{dyn}}$ as a function of the coupling constant $G$ at $G = 0$ in the (2+1)-dimensional NJL model.
3. IS THE DIMENSIONAL REDUCTION 3+1 → 1+1 (2+1 → 0+1) CONSISTENT WITH SPONTANEOUS CHIRAL SYMMETRY BREAKING?

In this section we consider the question whether the dimensional reduction 3+1 → 1+1 (2+1 → 0+1) in the dynamics of the fermion pairing in a magnetic field is consistent with spontaneous chiral symmetry breaking. This question occurs naturally since, due to the Mermin-Wagner-Coleman (MWC) theorem [12], there cannot be spontaneous breakdown of continuous symmetries at \( D = 1 + 1 \) and \( D = 0 + 1 \). The MWC theorem is based on the fact that gapless Nambu-Goldstone (NG) bosons cannot exist in dimensions less than 2+1. This is in particular reflected in that the (1 + 1)–dimensional propagator of would be NG bosons would lead to infrared divergences in perturbation theory (as indeed happens in the \( 1/N_c \) expansion in the (1 + 1)–dimensional Gross–Neveu model with a continuous symmetry [13]).

However, the MWC theorem is not applicable to the present problem. The central point is that the condensate \( \langle 0 | \bar{\psi} \psi | 0 \rangle \) and the NG modes are neutral in this problem and the dimensional reduction in a magnetic field does not affect the dynamics of the center of mass of neutral excitations. Indeed, the dimensional reduction \( D \rightarrow D - 2 \) in the fermion propagator, in the infrared region, reflects the fact that the motion of charged particles is restricted in the directions perpendicular to the magnetic field. Since there is no such restriction for the motion of the center of mass of neutral excitations, their propagators have \( D \)–dimensional form in the infrared region (since the structure of excitations is irrelevant at long distances, this is correct both for elementary and composite neutral excitations). This fact will be shown for neutral bound states in the NJL model in a magnetic field, in the \( 1/N_c \) expansion, in Section 6 and Appendix B. Since, besides that, the propagator of massive fermions is, though \( (D-2) \)–dimensional, nonsingular at small momenta, the infrared dynamics is soft in a magnetic field, and spontaneous chiral symmetry breaking is not washed out by the interactions, as happens, for example, in the (1 + 1)–dimensional Gross–Neveu model [13].

This point is intimately connected with the status of the space-translation symmetry in a constant magnetic field. In the gauge (9), the translation symmetry along the \( x_2 \) direction is broken; in the gauge (10), the translation symmetry along both the \( x_1 \) and \( x_2 \) directions is broken. However, for neutral states, all the components of the momentum of their center of mass are conserved quantum numbers (this property is gauge invariant) [14]. In order to show this, let us in-

\[ D(P) \sim \left( P_0^2 - C_\perp P_\perp^2 - C_3 P_3^2 \right)^{-1} \]

\[ C_\perp, C_3 \neq 0. \]

\[ ^1 \text{The Lorentz invariance is broken by a magnetic field in this problem. By the } D \text{-dimensional form, we understand that the denominator of the propagator depends on energy and all the components of momentum. That is, for } D = 3 + 1, \ D(P) \sim (P_0^2 - C_\perp P_\perp^2 - C_3 P_3^2)^{-1} \text{ with } C_\perp, C_3 \neq 0. \]
Introduce the following operators (generators of the group of magnetic translations) describing the translations in first quantized theory:

\[
\hat{P}_{x_1} = \frac{1}{i} \frac{\partial}{\partial x_1}, \quad \hat{P}_{x_2} = \frac{1}{i} \frac{\partial}{\partial x_2} + \hat{Q} B x_1, \quad \hat{P}_{x_3} = \frac{1}{i} \frac{\partial}{\partial x_3}
\]  

(28) in gauge (9), and

\[
\hat{P}_{x_1} = \frac{1}{i} \frac{\partial}{\partial x_1} - \frac{\hat{Q}}{2} B x_2, \quad \hat{P}_{x_2} = \frac{1}{i} \frac{\partial}{\partial x_2} + \frac{\hat{Q}}{2} B x_1, \quad \hat{P}_{x_3} = \frac{1}{i} \frac{\partial}{\partial x_3}
\]  

(29) in gauge (10) (\(\hat{Q}\) is the charge operator). One can easily check that these operators commute with the Hamiltonian of the Dirac equation in a constant magnetic field. Also, we get:

\[
[\hat{P}_{x_1}, \hat{P}_{x_2}] = \frac{1}{i} \hat{Q} B, \quad [\hat{P}_{x_1}, \hat{P}_{x_3}] = [\hat{P}_{x_2}, \hat{P}_{x_3}] = 0.
\]  

(30)

Therefore all the commutators equal zero for neutral states, and the momentum \(\mathbf{P} = (P_1, P_2, P_3)\) can be used to describe the dynamics of the center of mass of neutral states.
4. THE NAMBU-JONA-LASINIO MODEL IN A MAGNETIC FIELD: GENERAL CONSIDERATION

In this and the next four sections we shall consider the NJL model in a magnetic field. This model gives a clear illustration of the general fact that a constant magnetic field is a strong catalyst for the generation of a fermion dynamical mass.

Let us consider the $(3 + 1)$–dimensional NJL model with the $U_L(1) \times U_R(1)$ chiral symmetry:

$$
L = \frac{1}{2}[\bar{\psi}, (i \gamma^\mu D_\mu)\psi] + \frac{G}{2}[(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma^5\psi)^2],
$$

(31)

where $D_\mu$ is the covariant derivative and fermion fields carry an additional “color” index $\alpha = 1, 2, \ldots, N_c$. The theory is equivalent to the theory with the Lagrangian density

$$
L = \frac{1}{2}[\bar{\psi}, (i \gamma^\mu D_\mu)\psi] - \bar{\psi}(\sigma + i\gamma^5\pi)\psi - \frac{1}{2G}(\sigma^2 + \pi^2).
$$

(32)

The Euler-Lagrange equations for the auxiliary fields $\sigma$ and $\pi$ take the form of constraints:

$$
\sigma = -G(\bar{\psi}\psi), \quad \pi = -G(\bar{\psi}i\gamma^5\psi),
$$

(33)

and the Lagrangian density (32) reproduces Eq. (31) upon application of the constraints (33).

The effective action for the composite fields is expressed through the path integral over fermions:

$$
\Gamma(\sigma, \pi) = \tilde{\Gamma}(\sigma, \pi) - \frac{1}{2G} \int d^4x(\sigma^2 + \pi^2),
$$

(34)

$$
\exp(i\tilde{\Gamma}) = \int[d\bar{\psi}][d\psi] \exp \left\{ \frac{i}{2} \int d^4x[\bar{\psi}, \{i\gamma^\mu D_\mu - (\sigma + i\gamma^5\pi)\}\psi] \right\}
$$

$$
= \exp \left\{ \text{Tr} \ln \left[ i\gamma^\mu D_\mu - (\sigma + i\gamma^5\pi) \right] \right\},
$$

(35)

i.e.

$$
\tilde{\Gamma}(\sigma, \pi) = -i\text{Tr} \ln[i\gamma^\mu D_\mu - (\sigma + i\gamma^5\pi)].
$$

(36)

As $N_c \to \infty$, the path integral over the composite fields $\sigma$ and $\pi$ is dominated by the stationary points of the action: $\delta \Gamma/\delta \sigma = \delta \Gamma/\delta \pi = 0$. We will analyze
the dynamics in this limit by using the expansion of the action $\Gamma$ in powers of derivatives of the composite fields.

Is the $1/N_c$ expansion reliable in this problem? The answer to this question is “yes”. It is connected with the fact, already discussed in the previous section, that the dimensional reduction $3+1 \to 1+1$ by a magnetic field does not affect the dynamics of the center of mass of the NG bosons. If the reduction affected it, the $1/N_c$ perturbative expansion would be unreliable. In particular, the contribution of the NG modes in the gap equation, in next-to-leading order in $1/N_c$, would lead to infrared divergences (as happens in the $1+1$-dimensional Gross–Neveu model with a continuous chiral symmetry [13]). This is not the case here. Actually, as we will show in Appendix B, the next-to-leading order in $1/N_c$ yields small corrections to the whole dynamics at sufficiently large values of $N_c$. 
5. THE NJL MODEL IN A MAGNETIC FIELD: THE EFFECTIVE POTENTIAL

We begin the calculation of the effective action $\Gamma$ by calculating the effective potential $V$. Since $V$ depends only on the $U_L(1) \times U_R(1)$-invariant $\rho^2 = \sigma^2 + \pi^2$, it is sufficient to consider a configuration with $\pi = 0$ and $\sigma$ independent of $x$. So now $\tilde{\Gamma}(\sigma)$ in Eq. (36) is:

$$\tilde{\Gamma} = -i \Tr \ln(i \hat{D} - \sigma) = -i \ln \Det(i \hat{D} - \sigma),$$

(37)

where $\hat{D} \equiv \gamma^\mu D_\mu$. Since

$$\Det(i \hat{D} - \sigma) = \Det(\gamma^5 (i \hat{D} - \sigma) \gamma^5) = \Det(-i \hat{D} - \sigma),$$

(38)

we find that

$$\tilde{\Gamma}(\sigma) = -\frac{i}{2} \Tr [\ln(i \hat{D} - \sigma) + \ln(-i \hat{D} - \sigma)] = -\frac{i}{2} \Tr \ln(\hat{D}^2 + \sigma^2).$$

(39)

Therefore $\tilde{\Gamma}(\sigma)$ can be expressed through the following integral over the proper time $s$:

$$\tilde{\Gamma}(\sigma) = -\frac{i}{2} \Tr \ln(\hat{D}^2 + \sigma^2) = \frac{i}{2} \int d^4x \int_0^\infty \frac{ds}{s} \Tr(x | e^{-is(\hat{D}^2 + \sigma^2)} | x)$$

(40)

where

$$\hat{D}^2 = D_\mu D^\mu = \frac{i e}{2} \gamma^\mu \gamma^\nu F^{ext}_\mu\nu = D_\mu D^\mu + i e \gamma^1 \gamma^2 B.$$

(41)

The matrix element $\langle x | e^{i s(\hat{D}^2 + \sigma^2)} | y \rangle$ was calculated by Schwinger [1]:

$$\langle x | e^{i s(\hat{D}^2 + \sigma^2)} | y \rangle = e^{-i s \sigma^2} \langle x | e^{-i s D_\mu D^\mu} | y \rangle [\cos(eBs) + \gamma^1 \gamma^2 \sin(eBs)]$$

$$= \frac{-i}{(4\pi s)^2} e^{-i(s^2 - S_{cl})} [eBs \cot(eBs) + \gamma^1 \gamma^2 eBs],$$

(42)

where

$$S_{cl} = e \int_y^x A^{ext}_\lambda dz^\lambda - \frac{1}{4s} (x - y)_\nu (g^{\mu\nu} + \frac{(F^{ext}_{\mu\nu})^{\nu\mu}}{B^2}) (1 - eBs \cot(eBs))(x - y)_\mu.$$

(43)

Here the integral $\int_y^x A^{ext}_\lambda dz^\lambda$ is taken along a straight line. Substituting expression (42) into Eq. (40), we find

$$\tilde{\Gamma}(\sigma) = \frac{N_c}{8\pi^2} \int d^4x \int_0^\infty \frac{ds}{s^2} e^{-i s^2} eB \cot(eBs).$$

(44)
Therefore the effective potential is

\[ V(\rho) = \frac{\rho^2}{2G} + \tilde{V}(\rho) = \frac{\rho^2}{2G} + \frac{N_c}{8\pi^2} \int_{1/\Lambda^2}^{\infty} \frac{ds}{s^2} e^{-s\rho^2} eB \coth(eBs) , \]  

(45)

where now the ultraviolet cutoff has been explicitly introduced.

By using the integral representation for the generalized Riemann zeta function \( \zeta \),

\[ \int_0^\infty ds s^{\mu-1} e^{-\beta s} \coth s = \Gamma(\mu) \left[ 2^{1-\mu} \zeta \left( \mu, \frac{\beta}{2} \right) - \beta^{-\mu} \right] , \]

(46)

which is valid at \( \mu > 1 \), and analytically continuing this representation to negative values of \( \mu \), we can rewrite Eq. (45) as

\[ V(\rho) = \frac{\rho^2}{2G} + \frac{N_c}{8\pi^2} \left[ \frac{\Lambda^4}{2} + \frac{1}{3\ell^4} \ln(\Lambda\ell)^2 + \frac{1-\gamma-\ln 2}{3\ell^4} \right. \\
- (\rho\Lambda)^2 + \frac{\rho^4}{2} \ln(\Lambda\ell)^2 + \frac{\rho^4}{2}(1-\gamma-\ln 2) + \frac{\rho^2\ell^2}{\ell^2} \ln \frac{\rho^2\ell^2}{2} \\
- \frac{4}{\ell^4} \zeta'(-1, \frac{\rho^2\ell^2}{2} + 1) \bigg] + O(1/\Lambda) , \]

(47)

where the magnetic length \( \ell \equiv |eB|^{-1/2} \), \( \zeta'(-1, x) = \frac{d}{dx} \zeta(x, x)|_{x=-1} \), and \( \gamma \approx 0.577 \) is the Euler constant.

The gap equation \( dV/d\rho = 0 \) is

\[ \rho\Lambda^2 \left( \frac{1}{g} - 1 \right) = -\rho^3 \ln \left( \frac{(\Lambda\ell)^2}{2} \right) + \gamma \rho^3 + \ell^{-2} \rho \ln \left( \frac{(\rho\ell)^2}{4\pi} \right) \\
+ 2\ell^{-2} \rho \ln \Gamma \left( \frac{\rho^2\ell^2}{2} \right) + O(1/\Lambda) , \]

(48)

where the dimensionless coupling constant \( g = N_c G \Lambda^2 / (4\pi^2) \). In the derivation of this equation, we used the relations [5]:

\[ \frac{d}{dx} \zeta(x, x) = -\nu \zeta(\nu + 1, x) \]  

(49)

\[ \frac{d}{d\nu} \zeta(\nu, x)|_{\nu=0} = \ln(\Gamma(x) - \frac{1}{2} \ln 2\pi) , \quad \zeta(0, x) = \frac{1}{2} - x . \]

(50)

---

2In this paper, for simplicity, we consider the case of a large ultraviolet cutoff: \( \Lambda^2 \gg \tilde{\rho}^2, |eB| \), where \( \tilde{\rho} \) is a minimum of the potential \( V \). Notice, however, that in some cases (as in the application of the NJL model to hadron dynamics [3]), \( \Lambda^2 \) and \( \tilde{\rho}^2 \) may be comparable in magnitude.
As $B \to 0 \ (\ell \to \infty)$, we recover the known gap equation in the NJL model [3, 6]:

$$
\rho \Lambda^2 \left( \frac{1}{g} - 1 \right) = -\rho^3 \ln \frac{\Lambda}{\rho^2}.
$$

(51)

This equation admits a nontrivial solution only if $g$ is supercritical, $g > g_c = 1$ (as Eq. (32) implies, a solution to Eq. (18), $\rho = \bar{\sigma}$, coincides with the fermion dynamical mass, $\bar{\sigma} = m_{\text{dyn}}$, and the dispersion relation for fermions is Eq. (11) with $m$ replaced by $\bar{\sigma}$). We will show that the magnetic field changes the situation dramatically: at $B \neq 0$, a nontrivial solution exists for all $g > 0$.

We shall first consider the case of subcritical $g$, $g < g_c = 1$, which in turn can be divided into two subcases: a) $g \ll g_c$ and b) $g \to g_c - 0$ (nearcritical $g$). Since at $g < g_c = 1$, the left-hand side in Eq. (18) is positive and the first term on the right-hand side in this equation is negative, we conclude that a nontrivial solution to this equation may exist only at $\rho^2 \ln(\Lambda \ell)^2 \ll \ell^{-2} \ln(\rho \ell)^{-2}$. Then we find the solution at $g \ll 1$:

$$
m_{\text{dyn}}^2 \equiv \bar{\sigma}^2 = \frac{|eB|}{\pi} \exp\left( -\frac{4\pi^2 (1-g)}{|eB|\sqrt{N_c G}} \right).
$$

(52)

One can check that the dynamical mass is mostly generated in the infrared region, with $k \lesssim \ell^{-1} = |eB|^{1/2}$, where the contribution of the LLL dominates. In order to show this, one should check that essentially the same result for $m_{\text{dyn}}$ is obtained if the full fermion propagator in the gap equation is replaced by the LLL contribution and the cutoff $\Lambda$ is replaced by $\sqrt{|eB|}$ (see Section 8 below).

It is instructive to compare the relation (52) with the relations for the dynamical mass in the $(1+1)$–dimensional Gross–Neveu model [13], in the BCS theory of superconductivity [4], and in the $(2+1)$–dimensional NJL model in a magnetic field [2].

The relation for $m_{\text{dyn}}^2$ in the Gross-Neveu model is

$$
m_{\text{dyn}}^2 = \Lambda^2 \exp\left( -\frac{2\pi}{N_c G^{(0)}} \right).
$$

(53)

where $G^{(0)}$ is the bare coupling, which is dimensionless at $D = 1 + 1$. The similarity between relations (52) and (53) is evident: $|eB|$ and $|eB|G$ in Eq. (52) play the role of an ultraviolet cutoff and the dimensionless coupling constant in Eq. (53), respectively. This of course reflects the point that the dynamics of the fermion pairing in the $(3+1)$–dimensional NJL model in a magnetic field is essentially $(1+1)$–dimensional. We shall return to the discussion of the connection between this dynamics and that in the Gross-Neveu model in Sec. 7.

We recall that, because of the Fermi surface, the dynamics of the electron in superconductivity is also $(1+1)$–dimensional. This analogy is rather deep. In
particular, the expression (52) for $m_{\text{dyn}}$ can be rewritten in a form similar to that for the energy gap $\Delta$ in the BCS theory: while $\Delta \sim \omega_D \exp(-\text{const.}/\nu_F G_S)$, where $\omega_D$ is the Debye frequency, $G_S$ is a coupling constant and $\nu_F$ is the density of states on the Fermi surface, the mass $m_{\text{dyn}}$ is $m_{\text{dyn}} \sim \sqrt{|eB|} \exp(-1/2G\nu_0)$, where the density of states $\nu_0$ on the energy surface $E = 0$ of the LLL is now given by expression (12) multiplied by the factor $N_c$. Thus the energy surface $E = 0$ plays here the role of the Fermi surface.

Let us now compare the relation (52) with that in the $(2 + 1)$–dimensional NJL model in a magnetic field, in a weak coupling regime. It is [4]:

$$m_{\text{dyn}}^2 = |eB|^2 \frac{N_c^2 G^2}{4\pi^2}$$

(54)

While the expression (52) for $m_{\text{dyn}}^2$ has an essential singularity at $G = 0$, $m_{\text{dyn}}^2$ in the $(2 + 1)$–dimensional NJL model is analytic at $G = 0$. The latter is connected with the fact that in 2+1 dimensions the condensate $\langle 0 | \bar{\psi} \psi | 0 \rangle$ is non-zero even for free fermions in a magnetic field (see Eq. (27)). Indeed, Eq. (54) implies that $m_{\text{dyn}} = \langle 0 | \sigma | 0 \rangle = -G \langle 0 | \bar{\psi} \psi | 0 \rangle$. From this fact, and Eq. (27), we get the relation (54), to leading order in $G$. Therefore the dynamical mass $m_{\text{dyn}}$ is essentially perturbative in $G$ in this case. As we will see in Section 8, this is in turn intimately connected with the fact that, for D=2+1, the dynamics of fermion pairing in a magnetic field is $(0 + 1)$–dimensional.

Let us now consider near-critical values of $g$, with $\Lambda^2(1 - g)/g \rho^2 \ll \ln(\Lambda^2 \rho)^2$. Looking for a solution to the gap equation (48) with $\rho^2 \ell^2 \ll 1$, we arrive at the following equation:

$$\frac{1}{\rho^2 \ell^2} \ln \frac{1}{\pi \rho^2 \ell^2} \simeq \ln \Lambda^2 \ell^2,$$

(55)
i.e.

$$m_{\text{dyn}}^2 = \tilde{\sigma}^2 \simeq |eB| \frac{\ln[(\ln \Lambda^2 \ell^2)/\pi]}{\ln \Lambda^2 \ell^2}.$$  

(56)

What is the physical meaning of this relation? Let us recall that at $g = g_c$, the NJL model is equivalent to the (renormalizable) Yukawa model [3]. In leading order in $1/N_c$, the renormalized Yukawa coupling $\alpha_Y^{(\ell^{-1})} = (g_Y^{(\ell^{-1})})^2/(4\pi)$, corresponding to the scale $\mu = \ell^{-1}$, is $\alpha_Y^{(\ell^{-1})} = \pi/\ln \Lambda^2 \ell^2$ [3]. Therefore the expression (56) for the dynamical mass can be rewritten as

$$m_{\text{dyn}}^2 \simeq |eB| \frac{\alpha_Y^{(\ell^{-1})}}{\pi} \ln \frac{1}{\alpha_Y^{(\ell^{-1})}}.$$  

(57)
Thus, as has to be the case in a renormalizable theory, the cutoff \( \Lambda \) is removed, through the renormalization of parameters (the coupling constant, in this case), from the observable \( m_{\text{dyn}} \).

Let us now consider the case of supercritical values of \( g \) (\( g > g_c \)). In this case an analytic expression for \( m_{\text{dyn}} \) can be obtained for a weak magnetic field, satisfying the condition \( |eB|^{1/2}/m_{\text{dyn}}^{(0)} \ll 1 \), where \( m_{\text{dyn}}^{(0)} \) is the solution to the gap equation (51) with \( B = 0 \). Then we find from Eq. (48):

\[
m_{\text{dyn}}^2 \simeq (m_{\text{dyn}}^{(0)})^2 \left[ 1 + \frac{|eB|^2}{3(m_{\text{dyn}}^{(0)})^4 \ln(\Lambda/m_{\text{dyn}}^{(0)})^2} \right],
\]

i.e. \( m_{\text{dyn}} \) increases with \( |B| \). Notice that in the near-critical region, with \( g - g_c \ll 1 \), this expression for \( m_{\text{dyn}}^2 \) can be rewritten as

\[
m_{\text{dyn}}^2 \simeq (m_{\text{dyn}}^{(0)})^2 \left[ 1 + \frac{1}{3\pi} \alpha_{Y}^{(m_{\text{dyn}})} \frac{|eB|^2}{(m_{\text{dyn}}^{(0)})^4} \right],
\]

where, to leading order in \( 1/N_c \), \( \alpha_{Y}^{(m_{\text{dyn}})} = \frac{\pi}{\ln(\Lambda/m_{\text{dyn}}^{(0)})^2} \equiv \frac{\pi}{\ln(\Lambda/m_{\text{dyn}})^2} \) is the renormalized Yukawa coupling related to the scale \( \mu = m_{\text{dyn}} \).

In conclusion, let us discuss the validity of the LLL dominance approximation in more detail. As Eq. (33) implies, the dynamical mass is

\[
m_{\text{dyn}} = \langle 0|\sigma|0 \rangle = -G \langle 0|\bar{\psi}\psi|0 \rangle.
\]

Calculating the condensate \( \langle 0|\bar{\psi}\psi|0 \rangle \) in the LLL dominance approximation, we find (see Eqs. (18) and (19)):

\[
m_{\text{dyn}} = N_c G \frac{m_{\text{dyn}} |eB|}{4\pi^2} \ln \frac{\Lambda^2}{m_{\text{dyn}}^2},
\]

i.e.

\[
m_{\text{dyn}}^2 = \Lambda^2 \exp\left( -\frac{4\pi^2}{|eB| N_c G} \right).
\]

Comparing this relation with Eq. (52), corresponding to dynamics with \( g \) outside the near–critical region, we conclude that this approximation reproduces correctly the essential singularity in \( m_{\text{dyn}}^2 \) at \( G = 0 \). On the other hand, the coefficient of the exponent in Eq. (52) gets a contribution from all the Landau levels.

\[^{3}\text{The fact that in the supercritical phase of the NJL model, } m_{\text{dyn}} \text{ increases with } |B| \text{ has been already pointed out by several authors (for a review, see Ref. [13]).}\]
In the near-critical region, with \( 1 - g \ll \frac{m^2_{\text{dyn}}}{\Lambda^2} \ln \Lambda^2 \ell^2 \), all the Landau levels become equally important (see Eq. (56)), and the LLL dominance approximation ceases to be valid. This also happens (in a weak magnetic field) at supercritical values of \( g, g > g_c \), when the dynamical mass is generated even without a magnetic field.

Thus we conclude that, like the BCS approximation in the theory of superconductivity, the LLL dominance approximation is appropriate for the description of the dynamics of weakly interacting fermions. This point will be especially important in QED, considered in Section 9.
6. THE NJL MODEL IN A MAGNETIC FIELD: THE KINETIC TERM

Let us now consider the kinetic term $L_k$ in the effective action (34). The chiral $U_L(1) \times U_R(1)$ symmetry implies that the general form of the kinetic term is

$$L_k = F_{\mu\nu}^1 (\partial_\mu \rho_j \partial_\nu \rho_j) + F_{\mu\nu}^2 (\rho_j \partial_\mu \rho_j)(\rho_i \partial_\nu \rho_i)$$  \hspace{1cm} (63)$$

where $\rho = (\sigma, \pi)$ and $F_{\mu\nu}^1, F_{\mu\nu}^2$ are functions of $\rho^2$. We found these functions by using the method of Ref. [16]. The derivation of $L_k$ is considered in Appendix A. Here we shall present the final results.

The functions $F_{\mu\nu}^1, F_{\mu\nu}^2$ are $F_{\mu\nu}^1 = g_{\mu\nu} F_{\mu\nu}^1$, $F_{\mu\nu}^2 = g_{\mu\nu} F_{\mu\nu}^2$ with

$$F_{00}^0 = F_{33}^3 = \frac{N_c}{8 \pi^2} \left[ \ln \frac{A^2 \ell^2}{2} - \psi \left( \frac{\rho^2 \ell^2}{2} + 1 \right) + \frac{1}{\rho^2 \ell^2} - \gamma + \frac{1}{3} \right],$$

$$F_{11}^1 = F_{22}^2 = \frac{N_c}{8 \pi^2} \left[ \ln \frac{2}{\rho^2} - \gamma + \frac{1}{3} \right],$$

$$F_{00}^2 = F_{33}^2 = -\frac{N_c}{24 \pi^2} \left[ \frac{\rho^2 \ell^2}{2} \zeta \left( 2, \frac{\rho^2 \ell^2}{2} + 1 \right) + \frac{1}{\rho^2 \ell^2} \right],$$

$$F_{11}^2 = F_{22}^2 = \frac{N_c}{8 \pi^2} \left[ \rho^4 \ell^4 \psi \left( \frac{\rho^2 \ell^2}{2} + 1 \right) - 2 \rho^2 \ell^2 \ln \Gamma \left( \frac{\rho^2 \ell^2}{2} \right) \right. - \left. \rho^2 \ell^2 \ln \frac{\rho^2 \ell^2}{4 \pi} - \rho^4 \ell^4 - \rho^2 \ell^2 + 1 \right]$$  \hspace{1cm} (64)$$

where $\psi(x) = d(\ln \Gamma(x))/dx$. (We recall that the magnetic length is $\ell \equiv |eB|^{-1/2}$.)

As follows from Eqs. (33) and (34), the propagators of $\sigma$ and $\pi$ in leading order in $1/N_c$ have a genuine $(3 + 1)$–dimensional form. This agrees with the general arguments in support of the $(3 + 1)$–dimensional form of propagators of neutral particles in a magnetic field considered in Section 3. We shall see in Appendix B that this point is crucial for providing the reliability of the $1/N_c$ expansion in this problem.

Now, knowing the effective potential and the kinetic term, we can find the dispersion law for the collective excitations $\sigma$ and $\pi$. 

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7. THE NJL MODEL IN A MAGNETIC FIELD: THE SPECTRUM OF THE COLLECTIVE EXCITATIONS

The derivative expansion for the effective action is reliable in the infrared region, with \( k \lesssim F_\pi \), where \( F_\pi \sim m_{\text{dyn}} \) is the decay constant of \( \pi \). Therefore it is appropriate for the description of the low-energy dynamics of the gapless NG mode \( \pi \), but may, at most, aspire to describe only qualitatively the dynamics of the \( \sigma \) mode. Nevertheless, for completeness, we will derive the dispersion law both for \( \pi \) and \( \sigma \).

We begin by considering the spectrum of the collective excitations in the subcritical, \( g < g_c \), region. At \( g \ll g_c = 1 \) we find the following dispersion laws from Eqs. (47), (63) and (64):

\[
E_\pi \simeq \left[ \frac{m_{\text{dyn}}}{|eB|} \ln \left( \frac{|eB|}{\pi m_{\text{dyn}}^2} \right) k_\perp^2 + k_3^2 \right]^{1/2}. \tag{65}
\]

\[
E_\sigma \simeq \left[ 12 m_{\text{dyn}}^2 + \frac{3m_{\text{dyn}}^2}{|eB|} \ln \left( \frac{|eB|}{\pi m_{\text{dyn}}^2} \right) k_\perp^2 + k_3^2 \right]^{1/2}. \tag{66}
\]

with \( m_{\text{dyn}} \) defined in Eq. (52). Thus the \( \pi \) is a gapless NG mode. Taking into account Eq. (52), we find that the transverse velocity \( |v_\perp| = |\partial E_\pi / \partial k_\perp| \) of \( \pi \) is less than 1.

In the near–critical region, with \( g - g_c \ll 1 \) (where the NJL model is equivalent to the Yukawa model), we find from Eqs. (47), (57), (63) and (64):

\[
E_\pi \simeq \left[ \left( 1 - \frac{1}{\ln \pi/\alpha_Y^{(e-1)}} \right) k_\perp^2 + k_3^2 \right]^{1/2}, \tag{67}
\]

\[
E_\sigma = \left[ 4m_{\text{dyn}}^2 \left( 1 + \frac{2}{3 \ln \pi/\alpha_Y^{(e-1)}} \right) + \left( 1 - \frac{1}{3 \ln \pi/\alpha_Y^{(e-1)}} \right) k_\perp^2 + k_3^2 \right]^{1/2}. \tag{68}
\]

Thus, as has to be the case in a renormalizable model, the cutoff \( \Lambda \) disappears from the observables \( E_\pi \) and \( E_\sigma \). Note also that the transverse velocity \( v_\perp \) of \( \pi \) is again less than 1.

Since the derivative expansion is valid only at small values of \( k_\perp, k_3 \), satisfying \( k_\perp, k_3 \lesssim F_\pi \sim m_{\text{dyn}} \), the relations (63)-(68) are valid only in the infrared region (as we already indicated above, the relations (66) and (68) for \( \sigma \) are only estimates). In particular, since, as \( B \to 0 \), the dynamical mass \( m_{\text{dyn}} \) and the decay constant \( F_\pi \sim m_{\text{dyn}} \) go to zero in the subcritical region, these relations cease to be valid, even in the infrared region, at \( B = 0 \).

Let us now turn to the supercritical region, with \( g > g_c \). The analytic expressions for \( E_\pi \) and \( E_\sigma \) can be obtained for weak magnetic fields, satisfying \( |eB|^{1/2} \ll
where $m_{\text{dyn}}^{(0)}$ is the dynamical mass of fermions at $B = 0$. We find from Eqs. (47), (63) and (64):

\begin{align}
E_\pi & \simeq \left[ \left( 1 - \frac{(eB)^2}{3m_{\text{dyn}}^4 \ln(\Lambda^2/m_{\text{dyn}}^2)^2} \right) k_\perp^2 + k_3^2 \right]^{1/2}, \\
E_\sigma & \simeq \left[ 4m_{\text{dyn}}^2 \left( 1 + \frac{1}{3 \ln(\Lambda/m_{\text{dyn}})^2} + \frac{4(eB)^2}{9m_{\text{dyn}}^4 \ln(\Lambda/m_{\text{dyn}})^2} \right) + \left( 1 - \frac{11(eB)^2}{45m_{\text{dyn}}^4 \ln(\Lambda/m_{\text{dyn}})^2} \right) k_\perp^2 + k_3^2 \right]^{1/2},
\end{align}

with $m_{\text{dyn}}$ given in Eq. (58). One can see that the magnetic field reduces the transverse velocity $v_\perp$ of $\pi$ in this case as well.

Since in the supercritical region, with $g - g_c \ll 1$, the renormalized Yukawa coupling is $\alpha_{Y}^{(m_{\text{dyn}})} = \pi / \ln(\Lambda/m_{\text{dyn}})^2$ to leading order in $1/N_c$, Eqs. (69) and (70) can be rewritten in that region as

\begin{align}
E_\pi & \simeq \left[ \left( 1 - \alpha_{Y}^{(m_{\text{dyn}})} \frac{(eB)^2}{3\pi m_{\text{dyn}}^4} \right) k_\perp^2 + k_3^2 \right]^{1/2}, \\
E_\sigma & \simeq \left[ 4m_{\text{dyn}}^2 \left( 1 + \alpha_{Y}^{(m_{\text{dyn}})} \left( \frac{1}{3\pi} + \frac{4(eB)^2}{9\pi m_{\text{dyn}}^4} \right) \right) + \left( 1 - \alpha_{Y}^{(m_{\text{dy}}) n} \frac{11(eB)^2}{45\pi m_{\text{dy}}^4} \right) k_\perp^2 + k_3^2 \right]^{1/2}.
\end{align}

Let us now return to the discussion of the connection between the dynamics in the $(3 + 1)$-dimensional NJL model in a magnetic field (with a weak coupling $g \ll 1$) and that in the Gross-Neveu (GN) model. Comparison of Eqs. (52) and (53) suggests that the infrared dynamics in these two models may be similar. The GN model is asymptotically free, with the bare coupling constant $G^{(0)} = 2\pi/N_c \ln(\Lambda^2/m_{\text{dy}}^2) \to 0$ as $\Lambda \to \infty$. Therefore there is the dimensional transmutation in the model: in the scaling region, at $G^{(0)} \ll 1$, the infrared dynamics with $(\ln(\Lambda^2/k^2))^{-1} \ll 1$ is essentially independent either of the coupling constant $G^{(0)}$ or the cutoff $\Lambda$; the only relevant parameter is the dynamical mass $m_{\text{dy}}$ (which is an analogue of the parameter $\Lambda_{QCD}$ in QCD). One might expect that a similar dimensional transmutation should take place in the $(3 + 1)$-dimensional NJL model in a magnetic field: at $|eB|G N_c \ll 1$ (see Eq. (52)), the infrared dynamics should be essentially independent of the magnetic field and the coupling constant $G$.

The situation in this model is however more subtle. As follows from Eq. (18) (with $m$ replaced by $m_{\text{dy}}$), the fermion propagator in the infrared region is indeed independent (up to power corrections $\sim (k_\perp^2/|eB|)^n$ for $n \geq 1$) of the magnetic
field. However, the dependence of the propagator of the NG mode $\pi$ on $|eB|$ is essential. Though, as follows from Eqs. (52) and (65), the maximum transverse velocity $|v_\perp| = \sqrt{(m_{\text{dyn}}^2 / |eB|) \ln(|eB| / \pi m_{\text{dyn}}^2)}$ of $\pi$ is small at $|eB|/GN_c \ll 1$, it is crucial that it is nonzero. The latter provides the $(3 + 1)$–dimensional character of the NG propagator and, as was already explained in Sec. 3, this in turn is crucial for the realization of spontaneous chiral symmetry breaking in the model.

We recall that in the $(1+1)$–dimensional GN model with the chiral $U_L(1) \times U_R(1)$ symmetry, there is no spontaneous chiral symmetry breaking, despite the fact that the fermion dynamical mass is nonzero: in accordance with the MWC theorem [12], there are no NG bosons in this model. As was suggested by Witten (see the second paper in Ref. [13]), in the GN model, the Berezinsky-Kosterlitz-Thouless (BKT) phase is presumably realized, with the NG bosons being replaced by the BKT vortex collective excitations.

Therefore, in $(3 + 1)$–dimensional NJL model, the magnetic length $\ell \equiv |eB|^{-1/2}$ acts as a (physical) regulator: at finite $\ell$, there is spontaneous chiral symmetry breaking, the propagator of NG mode $\pi$ has a $(3 + 1)$–dimensional form and, as a result, the $1/N_c$ expansion is reliable (see Appendix B). In a sense, the magnetic length $\ell$ plays here the same role as the $\epsilon$ parameter in the $(2 + \epsilon)$–dimensional GN model [13].

It is noticeable that the observables $E_\pi$ and $E_\sigma$ in Eqs. (55) and (66) do not depend explicitly on cutoff $\Lambda$. It happens because, due to the relation (52) for $m_{\text{dyn}}$, the term $1/\rho^2\ell^2 |v_{\rho^2}\approx \ell^2 - m_{\text{dyn}}^2$ dominates in the functions $F_{10}^{00} = F_{10}^{33}$ and $F_{20}^{00} = F_{20}^{33}$ in the kinetic term (see Eq. (64)). In this dynamical regime, there is a strong hierarchy of two scales: $m_{\text{dyn}}^2 \ll |eB|$. In a sense, with respect to the infrared dynamics with $k^2 \ll |eB|$, the scale $|eB|$ plays the role of a (physical) ultraviolet cutoff.

Let us consider the renormalization group properties of this dynamics.

The reliability of the $1/N_c$ expansion implies that the infrared dynamics in this model is (at least at sufficiently large $N_c$) under control: the effects connected with nonleading orders in $1/N_c$ are small (see Appendix B). Let us consider the running Yukawa coupling in this model. In leading order in $1/N_c$, the bare Yukawa coupling and both the renormalization constant of the fermion field and that of the Yukawa vertex equal one (see Appendix B). Therefore the behavior of the running Yukawa coupling is determined by the renormalization constant

\footnote{Comparing the effective actions in the $(3+1)$–dimensional NJL model in a magnetic field and $(1+1)$–dimensional GN model, one can show (at least formally) that, as $|eB| \rightarrow \infty$, the NJL model, with $N_c$ colors becomes equivalent to the GN model with the number of colors $\tilde{N}_c = N_c N$, where $N = |eB|S_{12}/2\pi \rightarrow \infty$ is the number of states at the LLL ($S_{12}$ is the square in the $x_1x_2$-plane). This interesting limit will be considered in more detail elsewhere.}
of the chiral field $\rho$: $g_Y^{(\mu)} \sim \sqrt{Z_\rho^{(\mu)}}$. Eq.(64) implies that $Z_\rho^{(\mu)}^{-1} \sim N_c|eB|/8\pi^2\mu^2$ at $\mu \sim m_{\text{dyn}}$. Therefore the running Yukawa coupling is weak in the infrared region: $g_Y^{(\mu)} \sim \sqrt{8\pi^2\mu^2/N_c|eB|}$ and $g_Y^{(\mu)} \to 0$ as $|eB|$ (and therefore also $\Lambda$) goes to infinity, i.e. there is a Gaussian infrared fixed point.

Examining the effective potential and the kinetic term in the effective action, one can show that a similar behavior occurs for the rest of running couplings, describing self–interactions of the chiral field $\rho$.

We emphasize that this discussion relates only to the NJL model with a weak coupling constant. In the case of the NJL model with a near–critical $g$, the situation is different. In that case, one finds from (64) that the renormalization constant $Z_\rho^{(\mu)}^{-1}$ is $Z_\rho^{(\mu)}^{-1} \sim (N_c/4\pi^2) \ln(\Lambda/\mu)$. Therefore the behavior of the running coupling is similar to that in the NJL model without magnetic field (and with the same near–critical $g$) [3]: $g_Y^{(\mu)} \sim 2\pi/\sqrt{N_c \ln \Lambda/\mu}$, i.e. there is the usual Gaussian infrared fixed point in this case.

As was already indicated in the previous section, the difference between these two dynamical regimes is connected with the point that, while at $g \ll 1$ the LLL dominates, at a near–critical $g$, all Landau levels become relevant.

In the next section, we will derive the Bethe–Salpeter equation for the NG mode $\pi$ in the LLL dominance approximation. This will yield a further insight into the physics of the dimensional reduction $D \to D - 2$ in the dynamics of fermion pairing in a magnetic field.
8. THE NJL MODEL IN A MAGNETIC FIELD: 
THE BETHE-SALPETER EQUATION
FOR THE NG MODE

The homogeneous Bethe-Salpeter (BS) equation for the NG bound state \( \pi \) takes the form (for a review, see Ref. [3]):

\[
\chi_{AB}(x, y; P) = -i \int d^4x_1 d^4y_1 d^4x_2 d^4y_2 G_{AA_1}(x, x_1) K_{A_1 B_1; A_2 B_2}(x_1 y_1, x_2 y_2) \\
\times \chi_{A_2 B_2}(x_2, y_2; P) G_{B_1 B}(y_1, y),
\]

(73)

where the BS wave function \( \chi_{AB}(x, y; P) = \langle 0|T \psi_A(x) \bar{\psi}_B(y)|P; \pi \rangle \) and the fermion propagator \( G_{AB}(x, y) = \langle 0|T \bar{\psi}_A(x) \psi_B(y)|0 \rangle \); the indices \( A = (n \alpha) \) and \( B = (m \beta) \) include both Dirac \((n, m)\) and color \((\alpha, \beta)\) indices. Notice that though the external field \( A_\mu^{\text{ext}} \) breaks the conventional translation invariance, the total momentum \( P \) is a good, conserved, quantum number for neutral bound states, in particular for the \( \pi \) (see Section 3). Henceforth we will use the symmetric gauge [10].

In leading order in \( 1/N_c \), the BS kernel in the NJL model is [3]:

\[
K_{A_1 B_1; A_2 B_2}(x_1 y_1, x_2 y_2) = G[\delta_{A_1 B_1} \delta_{A_2 B_2} + \delta_{A_1 B_2} \delta_{A_2 B_1} - \delta_{A_1 B_1} \delta_{A_2 B_2} + \delta_{A_1 B_2} \delta_{A_2 B_1}]
\times \delta^4(x_1 - y_1) \delta^4(x_2 - y_2).
\]

(74)

Also, as was already indicated in Section 5, in this approximation, the full-fermion propagator coincides with the propagator \( S \) (13) of a free fermion (with \( m = m_{\text{dyn}} \)) in a magnetic field.

Let us now factorize (as in Eq. (13)) the Schwinger phase factor in the BS wave function:

\[
\chi_{AB}(R, r; P) = \delta_{\alpha \beta} \exp(i er^\mu A_\mu^{\text{ext}}(R)) e^{-iPR} \chi_{nm}(R, r; P),
\]

(75)

where we introduced the relative coordinate, \( r = x - y \), and the center of mass coordinate, \( R = (x + y)/2 \). Then Eq. (73) can be rewritten as

\[
\tilde{\chi}_{nm}(R, r; P) = -i N_c G \int d^4R_1 \tilde{S}_{nm} \left( \frac{r}{2} + R - R_1 \right) \left[ \delta_{n_1 m_1} \text{tr}(\tilde{\chi}(R_1, 0; P)) - \frac{1}{N_c} \tilde{\chi}_{n_1 m_1}(R_1, 0; P) \left( \frac{r}{2} + R - R_1 \right) \right]
\times \int \left( \frac{r}{2} - R + R_1 \right) \exp[-ier^\mu A_\mu^{\text{ext}}(R - R_1)] \exp[iP(R - R_1)].
\]

(76)
The important fact is that the effect of translation symmetry breaking by the magnetic field is factorized in the Schwinger phase factor in Eq. (75), and Eq. (76) admits a translation invariant solution: \( \tilde{\chi}_{nm}(R, r; P) = \tilde{\chi}_{nm}(r, P) \). Then, transforming this equation into momentum space, we get:

\[
\tilde{\chi}_{nm}(p; P) = -i N_c G \int \frac{d^2 q_\perp d^2 R_\perp d^2 k_\perp d^2 k_\parallel}{(2\pi)^6} \times \exp\left[i(p_\parallel - q_\parallel)R_\parallel + \frac{P_\parallel}{2}\right] \tilde{S}\nabla_1 \left( \frac{p_\parallel}{2}, p_\perp + eA^{\text{ext}}(R_\perp) + \frac{q_\perp}{2} \right) \quad (77)
\]

where \( p_\parallel \equiv (p^0, p^3) \), \( p_\perp \equiv (p^1, p^2) \). Henceforth we will consider the equation with the total momentum \( P_\mu \to 0 \).

We shall consider the case of weakly interacting fermions, when the LLL pole approximation for the fermion propagator is justified. Henceforth, for concreteness, we will consider the case \( eB > 0 \). Then

\[
\tilde{S}(p) \approx i \exp(-\ell^2 p_\perp^2) \hat{p}_\parallel + m_{\text{dyn}} \left( 1 - i\gamma^1 \gamma^2 \right) \quad (78)
\]

(see Eq. (18)), where \( \hat{p}_\parallel = p^0 \gamma^0 - p^3 \gamma^3 \), and Eq. (77) transforms into the following one:

\[
\rho(p_\parallel, p_\perp) = i N_c G \ell^2 (2\pi)^5 e^{-\ell^2 p_\perp^2} \int d^2 A_\perp d^2 k_\perp d^2 k_\parallel e^{-\ell^2 A_\perp^2} \left( 1 - i\gamma^1 \gamma^2 \right) \hat{F}[\rho(k_\parallel, k_\perp)] (1 - i\gamma^1 \gamma^2) , \quad (79)
\]

where

\[
\rho(p_\parallel, p_\perp) = (\hat{p}_\parallel - m_{\text{dyn}}) \tilde{\chi}(p_\parallel, p_\perp) (\hat{p}_\perp - m_{\text{dyn}}) \quad (80)
\]

with \( \chi(p_\parallel, p_\perp) \equiv \chi(p_\parallel, p_\perp; P)|_{P=0} \), and the operator symbol \( \hat{F}[\rho] \) means:

\[
\hat{F}[\rho(k_\parallel, k_\perp)] = \text{tr} \left( \frac{k_\parallel + m_{\text{dyn}}}{k_\parallel^2 - m_{\text{dyn}}^2} \rho(k_\parallel, k_\perp) \frac{k_\parallel + m_{\text{dyn}}}{k_\parallel^2 - m_{\text{dyn}}^2} \right)
\]

This point is intimately connected with the fact that these bound states are neutral: the Schwinger phase factor is universal for neutral fermion-antifermion bound states.
\( - \gamma_5 \text{tr} \left( \frac{k_\parallel + m_{\text{dyn}}}{k_\parallel^2 - m_{\text{dyn}}} \rho(k_\parallel, k_\perp) \frac{k_\parallel + m_{\text{dyn}}}{k_\parallel^2 - m_{\text{dyn}}} \right) \)
\( - \frac{1}{N_c} \frac{k_\parallel + m_{\text{dyn}}}{k_\parallel^2 - m_{\text{dyn}}} \rho(k_\parallel, k_\perp) \frac{k_\parallel + m_{\text{dyn}}}{k_\parallel^2 - m_{\text{dyn}}} \)
\( + \frac{1}{N_c} \gamma_5 \frac{k_\parallel + m_{\text{dyn}}}{k_\parallel^2 - m_{\text{dyn}}} \rho(k_\parallel, k_\perp) \frac{k_\parallel + m_{\text{dyn}}}{k_\parallel^2 - m_{\text{dyn}}} \gamma_5 \).  

Eq. (73) implies that
\( \rho(p_\parallel, p_\perp) = \exp(-\ell^2 p_\perp^2) \varphi(p_\parallel) \), where \( \varphi(p_\parallel) \) satisfies the equation:
\[ \varphi(p_\parallel) = iN_c G_{4\pi^3} \int d^2 k (1 - i\gamma_1 \gamma_2) F[k_\parallel] \varphi(k_\parallel) \varphi(1 - i\gamma_1 \gamma_2) \).  

Thus the BS equation has been reduced to a two–dimensional integral equation. Of course, this fact reflects the two–dimensional character of the dynamics of the LLL, that can be explicitly read off from Eq. (78).

Henceforth we will use Euclidean space with \( k_4 = -ik^0 \). In order to define the matrix structure of the wavefunction \( \varphi(p_\parallel) \) of the \( \pi \), note that in a magnetic field, in the symmetric gauge (10), there is the symmetry \( SO(2) \times SO(2) \times \mathcal{P} \), where the \( SO(2) \times SO(2) \) is connected with rotations in the \( x_1 - x_2 \) and \( x_3 - x_4 \) planes and \( \mathcal{P} \) is the inversion transformation \( x_3 \rightarrow -x_3 \) (under which a fermion field transforms as \( \psi \rightarrow i\gamma_5 \gamma_3 \psi \)). This symmetry implies that the function \( \varphi(p_\parallel) \) takes the form:
\[ \varphi(p_\parallel) = \gamma_5 (A + i\gamma_1 \gamma_2 B + \hat{p}_\parallel C + i\gamma_1 \gamma_2 \hat{p}_\parallel D) \]  

where \( \hat{p}_\parallel = p_3 \gamma_3 + p_4 \gamma_4 \) and \( A, B, C \) and \( D \) are functions of \( p_\parallel^2 \) (\( \gamma_\mu \) are antihermitian in Euclidean space). Substituting expansion \( (83) \) into Eq. \( (82) \), we find that \( B = -A, C = D = 0 \), i.e. \( \varphi(p_\parallel) = A\gamma_5 (1 - i\gamma_1 \gamma_2) \). The function \( A \) satisfies the equation:
\[ A(p) = \frac{N_c G_{4\pi^3}}{4\pi^3 \ell^2} \int d^2 k A(k) \frac{A(k)}{k^2 + m_{\text{dyn}}^2} \]  

The solution to this equation is \( A(p) = \) constant, and introducing the ultraviolet cutoff \( \Lambda \), we get the gap equation for \( m_{\text{dyn}}^2 \):
\[ 1 = \frac{N_c G_{4\pi^3}}{4\pi^3 \ell^2} \int_0^\Lambda \frac{dk^2}{k^2 + m_{\text{dyn}}^2} \]  

\( ^6 \)The occurrence of the projection operator \( (1 - i\gamma_1 \gamma_2)/2 \) in the wave function \( \varphi \) reflects the fact that the spin of fermions at the LLL is polarized along the magnetic field.
It leads to the expression
\[ m_{\text{dyn}}^2 = \Lambda^2 \exp \left( -\frac{4\pi^2\ell^2}{N_c G} \right) \]
for \( m_{\text{dyn}}^2 \) which coincides with expression (72) derived in Section 5.

The integral equation (84) can be rewritten in the form of a two–dimensional Schrödinger equation with a \( \delta \)–like potential. Indeed, introducing the wave function
\[ \Psi(r) = \int \frac{d^2k}{(2\pi)^2} \frac{e^{-ikr}}{k^2 + m_{\text{dyn}}^2} A(k), \]
we find
\[ \left( -\Delta + m_{\text{dyn}}^2 - \frac{N_c G}{\pi \ell^2} \delta^2(r) \right) \Psi(r) = 0 \]
where
\[ \Delta = \frac{\partial^2}{\partial r_1^2} + \frac{\partial^2}{\partial r_2^2} \]
and
\[ \delta^2(r) = \int \frac{d^2k}{(2\pi)^2} e^{-ikr} \]
(88)

Notice that in the same way, one can show that in the (2 + 1)–dimensional NJL model [2], the analog of Eq. (87) has the form of a one–dimensional Schrödinger equation:
\[ \left( -\frac{\partial^2}{\partial r^2} + m_{\text{dyn}}^2 - \frac{N_c G}{\pi^2} \delta(r) \right) \Psi(r) = 0 \]
with \( \delta(r) = \int_{-\Lambda}^{\Lambda} \frac{dk}{2\pi} e^{-ikr} \). It leads to the following gap equation for \( m_{\text{dyn}}^2 \):
\[ 1 = \frac{N_c G}{2\pi^2 \ell^2} \int_{-\Lambda}^{\Lambda} \frac{dk}{k^2 + m_{\text{dyn}}^2}, \]
which yields expression (4) for \( m_{\text{dyn}}^2 \) in the limit \( \Lambda \to \infty \): \( m_{\text{dyn}}^2 = (|eB|N_c G/2\pi)^2 \).

Thus, since \( (-m_{\text{dyn}}^2) \) plays the role of the energy \( E \) in the Schrödinger equation with a negative, i.e. attractive, potential, the problem has been reduced to finding the spectrum of bound states (with \( E = -m_{\text{dyn}}^2 < 0 \)) of the two–dimensional and one–dimensional Schrödinger equation with a short–range, \( \delta \)–like, potential. (Actually, since only the largest value of \( m_{\text{dyn}}^2 \) defines the stable vacuum [6], one
has to find the value of the energy for the lowest stationary level.) This allows us to use some general results proved in the literature that will in turn yield some additional insight into the dynamics of chiral symmetry breaking by a magnetic field.

First, there is at least one bound state for the one–dimensional \((d = 1)\) and two–dimensional \((d = 2)\) Schrödinger equation with an attractive potential [7]. Second, while the energy of the lowest level \(E(G)\) is an analytic function of the coupling constant, around \(G = 0\), at \(d = 1\), it is non–analytic at \(G = 0\) at \(d = 2\) [4]. Moreover, at \(d = 2\) for short–range potentials, the energy \(E(G) = -m^2_{\text{dyn}}(G)\) takes the form \(E(G) \sim -\exp[1/(aG)]\) (with \(a\) being a positive constant) as \(G \to 0\) [7].

Thus the results obtained in the NJL model at \(D = 2 + 1\) and \(D = 3 + 1\) agree with these general results.

The fact that the effect of spontaneous chiral symmetry breaking by a magnetic field is based on the dimensional reduction \(D \to D - 2\) in a magnetic field suggests that this effect is general, and not restricted to the NJL model. In the next section, we shall consider this effect in \((3 + 1)\)–dimensional QED.
9. SPONTANEOUS CHIRAL SYMMETRY BREAKING
BY A MAGNETIC FIELD IN QED

The dynamics of fermions in a constant magnetic field in QED was first considered
by Schwinger [1]. In that classical work, while the interaction with the external
magnetic field was considered in all orders in the coupling constant, the quantum
dynamics was treated perturbatively. There is no dynamical chiral symmetry
breaking in this approximation [17]. In this section we reconsider this problem,
treating the QED dynamics non-perturbatively, and show that, in ladder and
improved ladder approximations, a constant magnetic field leads to spontaneous
chiral symmetry breaking.

We will use the same strategy as in the previous section and derive the BS equa-
tion for the NG modes.

The Lagrangian density of massless QED in a magnetic field is:

\[ L = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} [\bar{\psi}, (i\gamma^\mu D_\mu) \psi] , \]  \hspace{1cm} (91)

where the covariant derivative \( D_\mu \) is

\[ D_\mu = \partial_\mu - ie (A^\text{ext}_\mu + A_\mu) , \quad A^\text{ext}_\mu = (0, -\frac{B}{2} x_2, \frac{B}{2} x_1, 0) , \]  \hspace{1cm} (92)

i.e. we use the symmetric gauge [10]. Besides the Dirac index \( n \), the fermion
field carries an additional flavor index \( a = 1, 2, \ldots, N_f \). Then the Lagrangian
density (91) is invariant under the chiral \( SU_L(N_f) \times SU_R(N_f) \times U_{L+R}(1) \) (we
will not discuss the dynamics related to the anomalous, singlet, axial current \( J_5 \)
in this paper). Since we consider the weak coupling phase of QED, there is no
spontaneous chiral symmetry breaking at \( B = 0 \) [3, 18]. We will show that the
magnetic field changes the situation dramatically: at \( B \neq 0 \) the chiral symmetry
\( SU_L(N_f) \times SU_R(N_f) \) breaks down to \( SU_V(N_f) \equiv SU_{L+R}(N_f) \). As a result, the
dynamical mass \( m_{\text{dyn}} \) is generated, and \( N_f^2 - 1 \) gapless bosons, composed of
fermions and antifermions, appear.

The homogeneous BS equation for the \( N_f^2 - 1 \) NG bound states takes the form

\[ \chi_{AB}^\beta(x, y; P) = -i \int d^4 x_1 d^4 y_1 d^4 x_2 d^4 y_2 G_{A_1 A_2; B_1 B_2}(x_1 y_1, x_2 y_2)
\times \chi_{A_2 B_2}^\beta(x_2, y_2; P) G_{B_1 B}(y_2, y) , \]  \hspace{1cm} (93)

where the BS wave function \( \chi_{AB}^\beta = \langle 0| T \psi_A(x) \bar{\psi}_B(y)| P; \beta \rangle , \beta = 1, \ldots, N_f^2 - 1 \),
and the fermion propagator \( G_{AB}(x, y) = \langle 0| T \psi_A(x) \bar{\psi}_B(y)|0 \rangle \); the indices \( A = (na) \)
and \( B = (mb) \) include both Dirac \( n, m \) and flavor \( a, b \) indices.
As will be shown below, the NG bosons are formed in the infrared region, where the QED coupling is weak. This seems suggest that the BS kernel in leading order in $\alpha$ is a reliable approximation. However, because of the $(1 + 1)$–dimensional form of the fermion propagator in the infrared region, there may be also relevant higher order contributions. We shall return to this problem below, but first, we shall analyze the BS equation with the kernel in leading order in $\alpha$.

The BS kernel in leading order in $\alpha$ is [3]:

$$
K_{A_1B_1;A_2B_2}(x_1y_1, x_2, y_2) = -4\pi i\alpha \delta_{a_1a_2}\delta_{b_1b_2}\gamma^\mu_{n_1n_2}\gamma^\nu_{m_2m_1}D_{\mu\nu}(y_2 - x_2)
\times \delta(x_1 - x_2)\delta(y_1 - y_2) + 4\pi i\alpha \delta_{a_1b_1}\delta_{b_2a_2}\gamma^\mu_{n_1m_1}\gamma^\nu_{m_2n_2}D_{\mu\nu}(x_1 - x_2)
\times \delta(x_1 - y_1)\delta(x_2 - y_2),
$$

(94)

where the photon propagator

$$
D_{\mu\nu}(x) = \frac{-i}{(2\pi)^4} \int d^4k e^{ikx} (g_{\mu\nu} - \frac{\lambda k_\mu k_\nu}{k^2}) \frac{1}{k^2}
$$

(95)

($\lambda$ is a gauge parameter). The first term on the right–hand side of Eq. (14) corresponds to the ladder approximation. The second (annihilation) term does not contribute to the BS equation for NG bosons (this follows from the fact that, due to the Ward identities for axial currents, the BS equation for NG bosons can be reduced to the Schwinger-Dyson equation for the fermion propagator where there is no contribution of the annihilation term [3]). For this reason we shall omit this term in the following. Then the BS equation takes the form:

$$
\chi^\beta_{AB}(x, y; P) = -4\pi \alpha \int d^4x_1 d^4y_1 S_{AA_1}(x, x_1)\delta_{a_1a_2}\gamma^\mu_{n_1n_2}\chi^\beta_{A_2B_2}(x_1, y_1; P)
\times \gamma^\nu_{m_2m_1}S_{B_1B}(y_1, y)D_{\mu\nu}(y_1 - x_1),
$$

(96)

where, since the lowest in $\alpha$ (ladder) approximation is used, the full fermion propagator $G_{AB}(x, y)$ is replaced by the propagator $S$ of a free fermion (with the mass $m = m_{\text{dyn}}$) in a magnetic field (see Eqs. (13), (14)).

Using the new variables, the center of mass coordinate $R = \frac{x_1 + x_2}{2}$, and the relative coordinate $r = x - y$, Eq. (96) can be rewritten as

$$
\tilde{\chi}_{nm}(R, r; P) = -4\pi \alpha \int d^4R_1 d^4r_1 \tilde{S}_{mn_1}(R - R_1 + \frac{r - r_1}{2})
\times \gamma^\mu_{n_1n_2}\tilde{\chi}_{n_2m_2}(R_1, r_1; P)\gamma^\nu_{m_2m_1}\tilde{S}_{m_1m}(\frac{r - r_1}{2} - R + R_1)D_{\mu\nu}(-r_1)
\times \exp[-ie(r + r_1)\mu A^\text{ext}_\mu(R - R_1)] \times \exp[iP(R - R_1)].
$$

(97)

Here $\tilde{S}$ is defined in Eqs. (13) and (14), and the function $\tilde{\chi}_{nm}(R, r; P)$ is defined from the equation

$$
\chi^\beta_{AB}(x, y; P) \equiv \langle 0 | T\psi_A(x)\bar{\psi}_B(y) | P, \beta \rangle
= \lambda^\beta_{ab} e^{-iPR} \exp[i(e^\mu A^\text{ext}_\mu(R))] \tilde{\chi}_{nm}(R, r; P)
$$

(98)
where $\lambda^\beta$ are $N_f^2 - 1$ flavor matrices (tr$(\lambda^\beta \lambda^\gamma) = 2\delta_{\beta\gamma}$; $\beta, \gamma \equiv 1, \ldots, N_f^2 - 1$).

The important fact is that, like in the case of the BS equation for the $\pi$ in the NJL model considered in the previous section, the effect of translation symmetry breaking by the magnetic field is factorized in the Schwinger phase factor in Eq. (78), and Eq. (19) admits a translation invariant solution, $\tilde{\chi}_{nm}(R, r; P) = \tilde{\chi}(r; P)$. Then, transforming this equation into momentum space, we get

$$\tilde{\chi}_{nm}(p; P) = -4\pi\alpha \int \frac{d^2q_\perp d^2R_\perp d^2k_\perp d^2k_\parallel}{(2\pi)^6} \times \exp\left[i(P_\parallel - q_\parallel)R_\parallel\right] \tilde{S}_{nm1}(p_\parallel + \frac{P_\parallel}{2}, p_\perp + eA^{ext}(R_\parallel) + \frac{q_\perp}{2}) \times \gamma^\mu_{n_1 n_2} \tilde{\chi}_{nm}(p_\parallel - \frac{P_\parallel}{2}, p_\perp + eA^{ext}(R_\parallel) - \frac{q_\perp}{2}) \times D_{\mu\nu}(k_\parallel - p_\parallel, k_\perp - p_\perp - 2eA^{ext}(R_\parallel)) \ (99)$$

(we recall that $p_\parallel \equiv (p^0, p^4)$, $p_\perp \equiv (p^1, p^2)$). Henceforth we will consider the equation with the total momentum $P_\mu \to 0$.

The crucial point for further analysis will be the assumption that $m_{dyn} << \sqrt{|eB|}$ and that the region mostly responsible for generating the mass is the infrared region with $k \lesssim m_{dyn} \ll \sqrt{|eB|}$. As we shall see, this assumption is self-consistent (see Eq. (3)). The assumption allows us to replace the propagator $\tilde{S}_{nm}$ in Eq. (99) by the pole contribution of the LLL (see Eq. (78)), and Eq. (99) transforms into the following one:

$$\rho(p_\parallel, p_\perp) = \frac{2\alpha\ell^2}{(2\pi)^4} e^{-\ell^2p_\perp^2} \int d^2k_\perp d^2k_\parallel e^{-\ell^2A^2_\perp} (1 - i\gamma^1\gamma^2)\gamma^\mu \left(\frac{k_\parallel + m_{dyn}}{k_\parallel^2 - m_{dyn}^2}\rho(k_\parallel, k_\perp) \frac{k_\parallel + m_{dyn}}{k_\parallel^2 - m_{dyn}^2}\gamma^\nu (1 - i\gamma^1\gamma^2)D_{\mu\nu}(k_\parallel - p_\parallel, k_\perp - A_\perp) \right) \ (100)$$

where $\rho(p_\parallel, p_\perp) = (\hat{\rho}_\parallel - m_{dyn})\tilde{\chi}(p)(\hat{\rho}_\parallel - m_{dyn})$. Eq. (100) implies that $\rho(p_\parallel, p_\perp) = \exp(-\ell^2p_\perp^2)\varphi(p_\parallel)$, where $\varphi(p_\parallel)$ satisfies the equation

$$\varphi(p_\parallel) = \frac{\pi\alpha}{(2\pi)^4} \int d^2k_\parallel (1 - i\gamma^1\gamma^2)\gamma^\mu \left(\frac{k_\parallel + m_{dyn}}{k_\parallel^2 - m_{dyn}^2}\varphi(k_\parallel) \right) \times \frac{k_\parallel + m_{dyn}}{k_\parallel^2 - m_{dyn}^2}\gamma^\nu (1 - i\gamma^1\gamma^2)D_{\mu\nu}(k_\parallel - p_\parallel) \ . \ (101)$$

Here

$$D_{\mu\nu}(k_\parallel - p_\parallel) = \int d^2k_\perp \exp\left(-\frac{\ell^2k_\perp^2}{2}\right)D_{\mu\nu}(k_\parallel - p_\parallel, k_\perp) \ . \ (102)$$
Thus, as in the NJL model in the previous section, the BS equation has been reduced to a two-dimensional integral equation.

Henceforth we will use Euclidean space with $k_4 = -ik_0$. Then, because of the symmetry $SO(2) \times SO(2) \times \mathcal{P}$ in a magnetic field, we arrive at the matrix structure \[ \text{(103)} \] for $\varphi(p_\|)$:

$$\varphi(p_\|) = \gamma_5(A + i\gamma_1\gamma_2B + \hat{p}_\|C + i\gamma_1\gamma_2\hat{p}_\|D)$$

where $A, B, C$ and $D$ are functions of $p_\|$.

We begin the analysis of Eq. \[ \text{(101)} \] by choosing the Feynman gauge (the general covariant gauge will be considered below). Then

$$D_{\mu\nu}(k_\| - p_\|) = i\delta_{\mu\nu}\pi\int_0^\infty dx \exp(-s^2/2)\left(\frac{k_\| - p_\|}{(k_\| - p_\|)^2 + x}\right),$$

and, substituting the expression \[ \text{(103)} \] for $\varphi(p_\|)$ into Eq. \[ \text{(101)} \], we find that $B = -A$, $C = D = 0$, i.e.

$$\varphi(p_\|) = A\gamma_5(1 - i\gamma_1\gamma_2)$$

and the function $A$ satisfies the equation:

$$A(p) = \frac{\alpha}{2\pi^2} \int \frac{d^2kA(k)}{k^2 + m^2_{\text{dyn}}} \int_0^\infty dx \exp(-s^2/2)\left(\frac{1}{(k - p)^2 + x}\right)$$

(henceforth we will omit the symbol $\parallel$; for the explanation of physical reasons of the appearance of the projection operator $(1 - i\gamma_1\gamma_2)/2$ in $\varphi(p)$, see footnote \[ \text{[3]} \].

Now introducing the function

$$\Psi(r) = \int \frac{d^2k}{(2\pi)^2} A(k) e^{ikr},$$

we get the two-dimensional Schrödinger equation from Eq. \[ \text{(100)} \]

$$(-\Delta + m^2_{\text{dyn}} + V(r))\Psi(r) = 0,$$

where the potential $V(r)$ is

$$V(r) = -\frac{\alpha}{2\pi^2} \int d^2pr e^{ipr} \int_0^\infty dx \exp(-s^2/2)\left(\frac{1}{(r^2 + x^2)^2 + x}\right) = \frac{\alpha}{\pi\ell^2} \int_0^\infty dx e^{-x/2}K_0\left(\frac{r}{\ell}\sqrt{x}\right)$$

$$= \frac{\alpha}{\pi\ell^2} \exp\left(\frac{r^2}{2\ell^2}\right) Ei\left(-\frac{r^2}{2\ell^2}\right)$$

\[ \text{(109)} \]
(\(K_0\) is the Bessel function). The essential difference of the potential (109) with respect to the \(\delta\)-like potential (88) in the NJL model is that it is long range. Indeed, using the asymptotic relations for \(E_i(x)\) \([5]\), we get:

\[
\begin{align*}
V(r) & \approx -\frac{2\alpha}{\pi} \frac{1}{r^2}, & r \to \infty, \\
V(r) & \approx -\frac{\alpha}{\pi \ell^2} \left( \gamma + \ln \frac{2\ell^2}{r^2} \right), & r \to 0,
\end{align*}
\]

where \(\gamma \approx 0.577\) is the Euler constant. Therefore, the theorem of Ref.\([7]\) (asserting that, for short-range potentials, \(-E(\alpha) \equiv m_{\text{dyn}}^2(\alpha) \sim \exp(-1/\alpha a)\), with \(a > 0\), as \(\alpha \to 0\)) cannot be applied to this case. In order to find \(m_{\text{dyn}}^2(\alpha)\), we shall use the integral equation (106). This equation is analyzed in Appendix C by using both analytical and numerical methods. The result is:

\[
m_{\text{dyn}} = C \sqrt{|eB|} \exp \left[ -\frac{\pi}{2} \left( \frac{\pi}{2\alpha} \right)^{1/2} \right],
\]

where the constant \(C = O(1)\). Note that this result agrees with the analysis of Ref.\([19]\) where the analytic properties of \(E(\alpha)\) were studied for the Schrödinger equation with potentials having the asymptotics \(V(r) \to \text{const}/r^2\) as \(r \to \infty\).

Since

\[
\lim_{\alpha \to 0} \frac{\exp[-1/a\sqrt{\alpha}]}{\exp[-1/a'\alpha]} = \infty,
\]

at \(a, a' > 0\), we see that the long-range character of the potential leads to the essential enhancement of the dynamical mass.

Let us now turn to considering the general covariant gauge (95). As is known, the ladder approximation is not gauge invariant. However, let us show that because the present effect is due to the infrared dynamics in QED, where the coupling constant is small, the leading term in \(\ln(m_{\text{dyn}}^2 \ell^2)\), \(\ln(m_{\text{dyn}}^2 \ell^2) \approx -\pi \sqrt{\pi/2\alpha}\), is the same in all covariant gauges.

Acting in the same way as before, we find that the wave function \(\varphi(p)\) now takes the form

\[
\varphi(p) = \gamma_5 (1 - i\gamma^1 \gamma^2) (A(p^2) + \hat{p} C(p^2))
\]

where the functions \(A(p^2)\) and \(C(p^2)\) satisfy the equations:

\[
\begin{align*}
A(p^2) &= \frac{\alpha}{2\pi^2} \int_0^\infty \frac{d^2 k A(k^2)}{k^2 + m_{\text{dyn}}^2} \int_0^\infty dx \frac{(1 - \lambda x \ell^2/4) \exp(-x \ell^2/2)}{(k - p)^2 + x}, \\
C(p^2) &= \frac{\alpha \lambda}{4\pi^2} \int_0^\infty \frac{d^2 k C(k^2)}{k^2 + m_{\text{dyn}}^2} \left[ 2k^2 - (kp) \right. \left. - \frac{k^2(kp)}{p^2} \right] \\
&\quad \times \int_0^\infty dx \exp(-x \ell^2/2) \left[ \frac{1}{((k - p)^2 + x)^2} \right].
\end{align*}
\]
One can see that the dominant contribution on the right-hand side of Eq. (114) (proportional to \[ \ln m_d^2 \ell^2 \]^2 and formed at small \( k^2 \)) is independent of the gauge parameter \( \lambda \). Thus the leading contribution in \( \ln(m_d^2 \ell^2) \), \( \ln(m_d^2 \ell^2) \approx -\pi \sqrt{\pi/2\alpha} \), is indeed gauge–invariant (for more details see Appendix C).

This concludes the derivation of Eqs. (3), (108) and (109) describing spontaneous chiral symmetry breaking by a magnetic field in ladder QED.

Note the following important point. Because of the exponent \( \exp(-\ell^2 x/2) \) in integral equations (106), (114) and (115), the present effect is connected with the infrared dynamics in QED: the natural cutoff in this problem is \(|eB|\). The fact that the non–perturbative infrared dynamics in this problem decouples from the ultraviolet dynamics is reflected also in the asymptotic behavior of the functions \( A(p^2) \) and \( C(p^2) \): as follows from Eqs. (106), (114) and (115), these functions rapidly \((A(p^2) \sim 1/p^2, C(p^2) \sim \lambda/p^2)\) decrease as \( p^2 \to \infty \). Therefore, since at \( p^2 \gg |eB| \) the magnetic field essentially does not affect the behavior of the running coupling in QED [24], the coupling constant \( \alpha \) in equations (106), (114) and (115) has to be interpreted as the value of the running coupling related to the scale \( \mu^2 \sim |eB| \).

Up to now we have considered the ladder approximation in QED in a magnetic field. As was already mentioned above, because of the \((1+1)\)–dimensional form of the fermion propagator in the infrared region, there may be also relevant higher order contributions. In particular, the dimensional reduction may essentially affect the vacuum polarization, thus changing the behavior of the running coupling in the infrared region. We will show that this is indeed the case. Actually, in this problem, because a magnetic field breaks Lorentz and \( SO(3) \) rotational symmetries, it is more appropriate to speak not about a single running coupling but about a running coupling tensor, \( i.e. \) about the full photon propagator \( D_{\mu \nu} \) in the magnetic field.

Let us consider the photon propagator in a strong magnetic field in the infrared region \(|eB| \gg m_d^2, |k_\parallel^2|, |k_\perp^2|\), with the polarization operator calculated in one–loop approximation [20, 21]. One can rewrite it in the following form:

\[
D_{\mu \nu} = -i \left( \frac{1}{k^2} g_{\mu \nu} + \frac{k_\parallel k_\parallel}{k^2 k^2} \right) + \frac{1}{k^2 + k_\parallel^2 \Pi(k_\parallel^2)} \left( g_{\mu \nu} - \frac{k_\parallel k_\parallel}{k^2} \right) - \frac{\lambda}{k^2} \frac{k_\mu k_\nu}{k^2}, \quad (116)
\]

\[
\Pi(k_\parallel^2) = -\frac{\alpha}{2\pi} \frac{|eB|}{m_d^2} \left[ \frac{4m_d^2}{k^2} - \frac{8m_d^4}{k^2 \sqrt{(k_\parallel^2)^2 - 4m_d^2 k_\parallel^2}} \right] \times
\]
\[ x \ln \frac{\sqrt{(k^2)^2 - 4m^2_{\text{dyn}}k^2 - k^2 \parallel}}{\sqrt{(k^2)^2 - 4m^2_{\text{dyn}}k^2 + k^2 \parallel}}, \quad (117) \]

where the symbols \( \perp \) and \( \parallel \) in \( g_{\mu\nu} \) are related to the \((1, 2)\) and \((0, 3)\) components, respectively. The asymptotic behavior of \( \Pi(k^2 \parallel) \) is:

\[
\Pi(k^2 \parallel) \rightarrow \frac{\alpha |eB|}{3\pi m^2_{\text{dyn}}} \quad \text{at} \quad |k^2 \parallel| \ll m^2_{\text{dyn}}, \quad (118)
\]

\[
\Pi(k^2 \parallel) \rightarrow -\frac{2\alpha |eB|}{\pi k^2 \parallel} \quad \text{at} \quad |k^2 \parallel| \gg m^2_{\text{dyn}}. \quad (119)
\]

Note the following characteristic points:

a) The expressions (116), (117) correspond to the one–loop contribution with the fermions from the LLL. Actually, expression (117) is the leading term in the \( 1/|eB| \) expansion of the one–loop polarization operator.

b) The polarization effects are absent in the transverse components of \( D_{\mu\nu} \). This is because the spin of fermions from the LLL is polarized along the magnetic field (see footnote 3). Indeed, this implies that in the QED–vertex, with two fermions from the LLL, the photon spin equals zero along the magnetic field, \( i.e. \) only the longitudinal, \((0, 3)\), components, are relevant in this case.

c) There is a strong screening effect in the \((g^\parallel_{\mu\nu} - k^\parallel_{\mu}k^\parallel_{\nu}/k^2 \parallel)\)–component of the photon propagator. In particular there is a pole at \( k^2 \parallel = 0 \) in \( \Pi(k^2 \parallel) \) as \( m^2_{\text{dyn}} \rightarrow 0 \): this is a reminiscence of the Higgs effect in the \((1+1)\)–dimensional massless QED (Schwinger model) \[22\].

d) Is the dynamics in QED in a magnetic field similar to that in the Schwinger model, as \( |eB| \rightarrow \infty? \) The answer to this question is “no”. The crucial point is that, unlike the NJL model, there is a neutral fundamental field in the QED Lagrangian: the photon field. As one can see from Eq. (113), the photon propagator does not have a \((1+1)\)–dimensional form even as \( |eB| \rightarrow \infty \): there is the quantity \( k^2 = k^2_0 - k^2_\perp - k^2_3 \) in the denominator of the photon propagator. This leads to the interaction which is very different from that in the Schwinger model. Notice that the interaction at long distances in this problem is much weaker than that in the \((1+1)\)–dimensional Schwinger model: while here, in Euclidean space, the potential \( V(r) \) is \( V(r) \sim -1/r^2 \) as \( r \rightarrow \infty \) (see Eq. (110)), it is \( V(r) \sim \ln r \) in the Schwinger model. Thus the infrared dynamics in QED in a magnetic field is a complex mixture of \((3+1)\) and \((1+1)\)–dimensional dynamics.

Let us now consider the BS equation in the so called improved ladder approximation: in this approximation, the free photon propagator is replaced by the one–loop propagator (116). As was already indicated above, the transverse com-
ponent of the photon propagator decouple from the fermions at the LLL. The longitudinal components lead to the following equation (in the Feynman gauge):

\[ A(p_\parallel^2) = \frac{\alpha}{4\pi^2} \int \frac{d^2k_\parallel}{k_\parallel^2 + m_{\text{dyn}}^2} \int_0^\infty dx e^{-\ell x/2} \left[ \frac{1}{x + (k_\parallel - p_\parallel)^2} + \frac{1}{x + (k_\parallel - p_\parallel)^2 + (k_\parallel - p_\parallel)^2 \Pi_E(k_\parallel - p_\parallel)} \right], \]  

(120)

where \( \Pi_E \) is the polarization operator (117) in Euclidean space (compare with Eq.(106)). The first term in the square brackets on the right-hand side of Eq.(120) comes from the (unscreened) \( k_\mu^\parallel k_\nu^\parallel/k_\parallel^2 \)-component; the second term comes from the (screened) \( (g_\mu^\parallel \mu^\nu - k_\mu^\parallel k_\nu^\parallel/k_\parallel^2) \)-component. It is not difficult to show that at \( p^2 = 0 \) the latter does not contribute to the dominant term (proportional to \( \ln m_{\text{dyn}}^2/\ell^2 \)) on the right-hand side of this equation. Indeed, one finds

\[ A(0) \sim \frac{\alpha}{4\pi^2} A(0) \int \frac{d^2k_\parallel}{k_\parallel^2 + m_{\text{dyn}}^2} \int_0^\infty dx e^{-\ell x/2} \left[ \frac{1}{x + k_\parallel^2} + \frac{1}{x + k_\parallel^2 + k_\parallel^2 \Pi_E(k_\parallel)} \right], \]  

(121)

and, matching asymptotics (118), (119) at \( k_\parallel^2 = 6m_{\text{dyn}}^2 \) in Euclidean space, we get the following estimate for the second term:

\[
\begin{align*}
\frac{\alpha}{4\pi^2} A(0) & \int \frac{d^2k_\parallel}{k_\parallel^2 + m_{\text{dyn}}^2} \int_0^\infty dx e^{-\ell x/2} \left[ \frac{1}{x + k_\parallel^2 + k_\parallel^2 \Pi_E(k_\parallel)} \right] \\
& \sim \frac{\alpha}{4\pi^2} A(0) \int_0^{6m_{\text{dyn}}^2} dy e^{-\ell y/2} \left[ \int_0^{y + m_{\text{dyn}}^2} dy \right] \\
& \sim \frac{\alpha}{4\pi} A(0) \ln \left( \frac{2 \ell m_{\text{dyn}}^2}{\alpha} \right) \ln \frac{\pi}{\alpha}. \quad \text{(122)}
\end{align*}
\]

Therefore the dominant contribution in this equation is connected with the unscreened \( k_\mu^\parallel k_\nu^\parallel/k_\mu^\parallel k_\nu^\parallel \)-component (since \( k_\parallel^\mu \) is different from \( k_\mu \), this term is not a gauge artifact which can be removed by a gauge transformation). As a result, we get the same equation as in the ladder approximation but with \( \alpha \) replaced by \( \alpha/2 \). The expression for \( m_{\text{dyn}} \) is now given by Eq. (111) with \( \alpha \to \alpha/2 \). It is gauge invariant. Also, since \( \alpha \) is the value of the running coupling at a definite scale, \( \mu^2 \sim |eB| \), this expression is renormalization group invariant.

The present consideration shows that, despite the smallness of \( \alpha \), the expansion in \( \alpha \) is broken in the infrared region in this model. There are two reasons for
that: the (1 + 1)–dimensional character of the fermion propagator in the infrared region and the smallness of the dynamical mass $m_{\text{dyn}}$ as compared to $|eB|^{1/2}$ 

$$\ln(|eB|/m_{\text{dyn}}^2) \sim 1/\sqrt{\alpha}.$$ 

It is well known that in massive QED, infrared singularities (on mass shell) in Green’s functions can be completely factorized if a certain set of diagrams is summed up [8]. The situation in massless QED is much more complicated: the set of relevant diagrams becomes essentially larger and, up to now, the problem of the complete description of the infrared singularities in that model remains unsolved. The smallness of $m_{\text{dyn}}$ in QED in a magnetic field makes this model closer to massless QED than to massive one. This implies that, despite the smallness of the coupling constant $\alpha$, the infrared dynamics in this model is quite complicated.

It is a challenge to define the class of all those diagrams in QED in a magnetic field which give a relevant contribution in this problem. Since the QED coupling constant is weak in the infrared region, this problem, though hard, seems not to be hopeless.
10. CONCLUSION

In this paper we showed that a constant magnetic field is a strong catalyst of spontaneous chiral symmetry breaking in 3+1 and 2+1 dimensions, leading to the generation of a fermion dynamical mass even at the weakest attractive interaction between fermions. The essence of this effect is the dimensional reduction $D \rightarrow D - 2$ in the dynamics of fermion pairing in a magnetic field. In particular, the dynamics of NG modes, connected with spontaneous chiral symmetry breaking by a magnetic field is described by the two (one)-dimensional Schrödinger equation at $D = 3 + 1 \ (D = 2 + 1)$:

$$\left( -\Delta + m^2_{\text{dyn}} + V(\textbf{r}) \right) \Psi(\textbf{r}) = 0 ,$$

where the attractive potential is model dependent and it defines the form of the dynamical mass as a function of the coupling constant. Since the general theorem of Ref. [7] assures the existence of at least one bound state for the two- and one-dimensional Schrödinger equations with an attractive potential, the generation of the dynamical mass $m_{\text{dyn}}$ takes place even at the weakest attractive interaction between fermions at $D = 3 + 1$ and $D = 2 + 1$. This general effect was illustrated in the NJL model and QED.

In this paper, we considered the dynamics in the presence of a constant magnetic field only. It would be interesting to extend this analysis to the case of inhomogeneous electromagnetic fields. In connection with this, note that in 2+1 dimensions, the present effect is intimately connected with the fact that the massless Dirac equation in a constant magnetic field admits an infinite number of normalized solutions with $E = 0$ (zero modes) [2]. More precisely, the density of the zero modes

$$\bar{\nu}_0 = \lim_{S \rightarrow \infty} S^{-1} N(E) \bigg|_{E=0}$$

(where $S$ is a two-dimensional volume of the system) is finite. As has been already pointed out (see the second paper in Ref. [2] and Ref. [23]), spontaneous flavor (chiral) symmetry breaking in 2+1 dimensions should be catalysed by all stationary (i.e. independent of time) field configurations with $\bar{\nu}_0$ being finite. On the other hand, as we saw in this paper, the density

$$\nu_0 = \lim_{V \rightarrow \infty} V^{-1} \frac{dN(E)}{dE} \bigg|_{E=0}$$

of the states with $E = 0$ (from a continuous spectrum) plays the crucial role in the catalysis of chiral symmetry breaking in 3+1 dimensions. One may expect that the density $\nu_0$ should play an important role also in the case of (stationary) inhomogeneous configurations in 3+1 dimensions.
As a first step in studying this problem, it would be important to extend the Schwinger results [1] to inhomogeneous field configurations. Interesting results in this direction have been recently obtained in Ref. [24].

In conclusion, let us discuss possible applications of this effect.

Since (2 + 1)–dimensional relativistic field theories may serve as effective theories for the description of long wavelength excitations in planar condensed matter systems [25], this effect may be relevant for such systems. It would be also interesting to take into account this effect in studying the possibility of the generation of a magnetic field in the vacuum, i.e. spontaneous breakdown of the Lorentz symmetry, in (2 + 1)–dimensional QED [26].

In 3+1 dimensions, one potential application of the effect can be connected with the possibility of the existence of very strong magnetic fields \( B \sim 10^{24} G \) during the electroweak phase transition in the early universe [27]. As the results obtained in this paper suggest, such fields might essentially change the character of the electroweak phase transition.

Another application of this effect can be connected with the role of chromomagnetic backgrounds as models for the QCD vacuum (the Copenhagen vacuum [28]).

Yet another potentially interesting application is the interpretation of the results of the GSI heavy-ion scattering experiments in which narrow peaks are seen in the energy spectra of emitted \( e^+e^- \) pairs [29]. One proposed explanation [30] is that a strong electromagnetic field, created by the heavy ions, induces a phase transition in QED to a phase with spontaneous chiral symmetry breaking and the observed peaks are due to the decay of positronium-like states in the phase. The catalysis of chiral symmetry breaking by a magnetic field in QED, studied in this paper, can serve as a toy example of such a phenomenon. In order to get a more realistic model, it would be interesting to extend this analysis to non-constant background fields [31].

We believe that the effect of the dimensional reduction by external field configurations may be quite general and relevant for different non-perturbative phenomena. It deserves further study.
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APPENDIX A

In this Appendix, we derive the kinetic term $L_k$ in the effective action $\Gamma$ (34) in the NJL model.

The structure of the kinetic term (63) is:

$$L = \frac{\rho}{2} \left( \partial^\mu \rho_j \partial_{\rho_j} \rho \right) + \rho \left( \partial^\mu \rho_j \partial_{\rho_j} \rho \right), \tag{124}$$

where $\rho = (\sigma, \pi)$ and $F_{1}^{\mu \nu}$, $F_{2}^{\mu \nu}$ depend on the $U_L(1) \times U_R(1)$-invariant $\rho^2 = \sigma^2 + \pi^2$.

The definition $\Gamma = \int d^4x L$ and Eq. (124) imply that the functions $F_{1}^{\mu \nu}$, $F_{2}^{\mu \nu}$ are determined from the equations:

$$\frac{\delta^2 \Gamma}{\delta \sigma(x) \delta \sigma(0)} \bigg|_{\sigma=\text{const.}} = -(F_{1}^{\mu \nu} + 2F_{2}^{\mu \nu}) \bigg|_{\sigma=\text{const.}} \partial^\mu \partial_{\nu} \delta^4(x), \tag{125}$$

$$\frac{\delta^2 \Gamma}{\delta \pi(x) \delta \pi(0)} \bigg|_{\sigma=\text{const.}} = -F_{1}^{\mu \nu} \bigg|_{\sigma=\text{const.}} \partial^\mu \partial_{\nu} \delta^4(x). \tag{126}$$

Here $\Gamma_k$ is the part of the effective action containing terms with two derivatives. Eq. (34) implies that $\Gamma_k = \tilde{\Gamma}_k$. Therefore we find from Eq. (126) that

$$F_{1}^{\mu \nu} = -\frac{1}{2} \int d^4x x^\mu x^\nu \frac{\delta^2 \tilde{\Gamma}}{\delta \pi(x) \delta \pi(0)} = -\frac{1}{2} \int d^4x x^\mu x^\nu \frac{\delta^2 \tilde{\Gamma}}{\delta \pi(x) \delta \pi(0)} \tag{127}$$

(henceforth we shall not write explicitly the condition $\sigma=\text{const.}, \pi = 0$). Taking into account the definition of the fermion propagator,

$$iS^{-1} = i\tilde{D} - \sigma, \tag{128}$$

we find from Eq. (36):

$$\frac{\delta^2 \tilde{\Gamma}}{\delta \pi(x) \delta \pi(0)} = -i \text{ tr}(S(x,0)i\gamma^5 S(0,x)i\gamma^5)$$

$$= -i \text{ tr}(\tilde{S}(x)i\gamma^5 \tilde{S}(-x)i\gamma^5)$$

$$= -i \int \frac{d^4q d^4k}{(2\pi)^8} e^{iqx} \text{ tr}(\tilde{S}(k)i\gamma^5 \tilde{S}(k+q)i\gamma^5) \tag{129}$$

(the functions $\tilde{S}(x)$ and $\tilde{S}(k)$ are given in Eqs. (13) and (14) with $m = \sigma$). Therefore

$$F_{1}^{\mu \nu} = -\frac{i}{2} \int \frac{d^4k}{(2\pi)^4} \text{ tr} \left( \tilde{S}(k)i\gamma^5 \frac{\partial^2 \tilde{S}(k)}{\partial k_\mu \partial k_\nu} i\gamma^5 \right) \tag{130}$$
In the same way, we find that

\[ F_2^{\mu\nu} = -\frac{i}{4} \int \frac{d^4k}{(2\pi)^4} \text{tr} \left( \hat{S}(k) \frac{\partial^2 \hat{S}(k)}{\partial k_{\mu} \partial k_{\nu}} - \hat{S}(k)i\gamma^5 \frac{\partial^2 \hat{S}(k)}{\partial k_{\mu} \partial k_{\nu}} i\gamma^5 \right) . \]  

Taking into account the expression for \( \hat{S}(k) \) in Eq. (14) (with \( m = \sigma \)), we get:

\[ \frac{\partial^2 \hat{S}(k)}{\partial k_0 \partial k_0} = 2iN_c \ell^4 \int_0^\infty dt \cdot te^{-R(t)} [(2i\ell(\ell k_0)^2)(k^0\gamma^0 - k^3\gamma^3 + \sigma)] + 3k^0\gamma^0 - k^3\gamma^3 + \sigma)(1 + \eta^1\gamma^2T) - k_\perp \gamma_\perp (1 + 2i(\ell k_0)^2t) \times (1 + T^2) , \]  

(132)

\[ \frac{\delta^2 \hat{S}(k)}{\partial k_3 \partial k_3} = -2iN_c \ell^4 \int_0^\infty dt \cdot te^{-R(t)} [(-2i\ell(\ell k_3)^2)(k^0\gamma^0 - k^3\gamma^3 + \sigma)] + k^0\gamma^0 - 3k^3\gamma^3 + \sigma)(1 + \eta^1\gamma^2T) - k_\perp \gamma_\perp (1 - 2i(\ell k_3)^2t) \times (1 + T^2) , \]  

(133)

\[ \frac{\partial^2 \hat{S}(k)}{\partial k_j \partial k_j} = -2iN_c \ell^4 \int_0^\infty dt \cdot T e^{-R(t)} [(-2i\ell(\ell k_j)^2)(k^0\gamma^0 - k^3\gamma^3 + \sigma)] + k^0\gamma^0 - k^3\gamma^3 + \sigma)(1 + \eta^1\gamma^2T) - k_\perp \gamma_\perp (1 - 2i(\ell k_j)^2t)(1 + T^2) - 2k^3\gamma^j(1 + T^2) , \]  

(134)

where \( T = \tan t, \eta = \text{sign}(eB), R(t) = i\ell^2t((k^0)^2 - (k^3)^2 - \sigma^2) - i\ell^2\vec{k}_\perp T, \) and \( \ell = 1, 2 \) (there is no summation over \( j \)).

Eqs. (14), (130), and (131) imply that non-diagonal terms \( F_1^{\mu\nu} \) and \( F_2^{\mu\nu} \) are equal to zero. Below, we will consider in detail the calculation of the function \( F_1^{00} \); other functions \( F_1^{0\mu}, F_2^{\nu\nu} \) can similarly be found.

Substituting expressions (14) and (132) into Eq. (130) for \( F_1^{00} \), we get

\[ F_1^{00} = \frac{N_c \ell^4}{4\pi^4} \int d^4k \int_0^\infty \int_0^\infty d\tau dt e^{(R(t)+R(\tau))} t A [2i(\ell k_0)^2 t ((k^0)^2 - (k^3)^2 - \sigma^2)] + 3(k^0)^2 - (k^3)^2 - \sigma^2 - B\vec{k}_\perp^2 (1 + 2i(\ell k_0)^2t) , \]  

(135)

where \( A = 1 - T \tau, B = (1 + T^2)(1 + T^2), \tau = \tan \tau. \) After integrating over \( k \), we find:

\[ F_1^{00} = \frac{N_c}{4\pi^2} \int_0^\infty \int_0^\infty \frac{d\tau dt}{(\tau + \ell t)^2} \tau e^{-i\ell^2\sigma^2(t + \tau)} \times \left[ \frac{A}{T + \tau + i\ell^2\sigma^2} + \frac{B}{(T + \tau)^2} \right] . \]  

(136)

Changing \( t \) and \( \tau \) to new variables,

\[ t = \frac{s}{2}(1 + u), \quad \tau = \frac{s}{2}(1 - u) , \]  

(137)
and then introducing the imaginary (dimensionless) proper time, $s \rightarrow -is$, we arrive at the expression:

$$F_{1}^{00} = \frac{N_{c}}{24\pi^{2}} \int_{0}^{\infty} \frac{ds}{s} e^{-(\ell^{2}\sigma^{2}s)} \left[ (2s \coth s - 2) + \left( \frac{s^{2}}{\sinh^{2} s} - 1 \right) + \ell^{2}\sigma^{2}s^{2} \coth s \right]$$

$$+ \frac{1}{8\pi^{2}} \int_{\frac{1}{\Lambda^{2}}}^{\infty} \frac{ds}{s} e^{-(\ell^{2}\sigma^{2}s)}

= \frac{N_{c}}{8\pi^{2}} \left[ \ln \frac{\Lambda^{2}\ell^{2}}{2} - \psi\left( \frac{\sigma^{2}\ell^{2}}{2} + 1 \right) + \frac{1}{\sigma^{2}\ell^{2}} - \gamma + \frac{1}{3} \right]. \tag{138}$$

Here we used relation (136) and the relations [5]:

$$\int_{0}^{\infty} e^{-\beta x} \left( \frac{1}{x} - \coth x \right) dx = \psi\left( 1 + \frac{1}{\beta} \right) - \ln \frac{1}{\beta} - \frac{1}{\beta}; \quad \text{Re} \ \beta > 0, \tag{139}$$

$$\int_{0}^{\infty} x^{\mu-1} e^{-\beta \tau} d\tau = 2^{1-\mu} \Gamma(\mu) \left[ 2\zeta\left( \frac{\mu - 1}{2}, \frac{\beta}{2} \right) - \beta \zeta\left( \mu, \frac{\beta}{2} \right) \right]; \quad \mu > 2 \tag{140}$$

All the other functions $F_{1}^{\mu\mu}, F_{2}^{\mu\mu}$ can similarly be found.
APPENDIX B

In this Appendix we analyze the next-to-leading order in $1/N_c$ expansion in the $(3 + 1)$-dimensional NJL model (the analysis of the $(2 + 1)$-dimensional NJL model is similar). Our main goal is to show that the propagator of the neutral NG boson $\pi$ has a $(3+1)$-dimensional form in this approximation and that (unlike the $(1 + 1)$-dimensional Gross-Neveu model [13]), the $1/N_c$ expansion is reliable in this model.

For an excellent review of the $1/N_c$ expansion see Ref. [32]. For our purposes, it is sufficient to know that the perturbative expansion is given by Feynman diagrams with the vertices and the propagators of fermions and composite particles $\sigma$ and $\pi$ calculated in leading order in $1/N_c$. In the leading order, the fermion propagator is given in Eqs. (13) and (14) (with $m$ replaced by $m_{\text{dyn}}$). As follows from Eq. (32), the bare Yukawa coupling of fermions with $\sigma$ and $\pi$ is $g_Y(0) = 1$ in this approximation. The inverse propagators of $\sigma$ and $\pi$ are [14]:

$$D_{\rho}^{-1}(x) = N_c \left( \frac{\Lambda^2}{4\pi^2 g} \delta^4(x) + i \text{tr}[S(x,0)T_{\rho}S(0,x)T_{\rho}] \right),$$

(141)

where $\rho = (\sigma, \pi)$ and $T_\sigma = 1$, $T_\pi = i\gamma^5$. Here $S(x,0)$ is the fermion propagator (13) with the mass $m_{\text{dyn}} = \bar{\sigma}$ defined from the gap equation (48). Actually, for our purposes, we need to know the form of the propagator of $\pi$ at small momenta only. We find from Eqs. (63) and (64):

$$D_\pi(k) = -\frac{8\pi^2}{N_c} f_1(\Lambda\ell, \bar{\sigma}\ell)[k_0^2 - k_3^2 - k_2^2 f_2(\Lambda\ell, \bar{\sigma}\ell)]^{-1},$$

(142)

where

$$f_1 = \left[ \ln \frac{\Lambda^2 \ell^2}{2} - \psi \left( \frac{\bar{\sigma}^2 \ell^2}{2} + 1 \right) + \frac{1}{\bar{\sigma}^2 \ell^2} - \gamma + \frac{1}{3} \right]^{-1},$$

(143)

$$f_2 = \left( \ln \frac{\Lambda^2}{\bar{\sigma}^2} - \gamma + \frac{1}{3} \right) f_1.$$

(144)

The crucial point for us is that, because of the dynamical mass $m_{\text{dyn}}$, the fermion propagator (despite its $(1+1)$-dimensional form) is soft in the infrared region and that the propagator of $\pi$ has a $(3+1)$-dimensional form in the infrared region (as follows from Eqs. (63) and (64), the propagator of $\sigma$ has of course also a $(3+1)$-dimensional form).

Let us begin the analysis by considering the next-to-leading order corrections in the effective potential. The diagram which contributes to the effective potential in this order is shown in Fig. 1a. Because of the structure of the propagators pointed
out above, there are no infrared divergences in this contribution to the potential. (Note that this is in contrast to the Gross–Neveu model: the contribution of this diagram is logarithmically divergent in the infrared region in that model, i.e. the $1/N_c$ expansion is unreliable in that case). Therefore the diagram in Fig. 1a leads to a finite, $O(1)$, correction to the potential $V$ (we recall that the leading contribution in $V$ is of order $N_c$). As a result, at sufficiently large values of $N_c$ the gap equation in this model admits a non-trivial solution $\bar{\sigma} \neq 0$ in next-to-leading order in $1/N_c$, i.e. there is spontaneous chiral symmetry breaking in this approximation.

Let us now consider the next-to-leading order corrections to the propagator of the NG mode $\pi$. First of all, note that in a constant magnetic field, the propagator of a neutral local field $\varphi(x)$, $D_\varphi(x,y)$, is translation invariant, i.e. it depends on $(x-y)$. This immediately follows from the fact that the operators of space translations in Eqs. (28) and (29) take the canonical form for neutral fields (the operator of time translations is $i\partial/\partial t$ both for neutral and charged fields in a constant magnetic field). The diagram contributing to the propagators of the NG mode in this order is shown in Fig. 1b. Because of the dynamical mass $m_{\text{dyn}}$ in the fermion propagator, this contribution is analytic at $k_\mu = 0$. Since at large $N_c$ the gap equation has a non-trivial solution in this approximation, there is no contribution of $O(k^0) \sim \text{const.}$ in the inverse propagator of $\pi$. Therefore the first term in the momentum expansion in its inverse propagator has the form $C_1(k_0^2-k_3^2)-C_2k_\perp^2$, where $C_1$ and $C_2$ are functions of $\bar{\sigma} \ell$ and $\Lambda \ell$ and the propagator takes the following form in this approximation:

$$D_\pi(k) = \lim_{k \to 0} \frac{8\pi^2}{N_c} f_1(\Lambda \ell, \bar{\sigma} \ell) \left[ 1 - \frac{1}{N} \tilde{C}_1(\Lambda \ell, \bar{\sigma} \ell) \right] (k_0^2 - k_3^2)$$

$$- \left( f_2(\Lambda \ell, \bar{\sigma} \ell) - \frac{1}{N} \tilde{C}_2(\Lambda \ell, \bar{\sigma} \ell) \right) k_\perp^2 \right]^{-1}$$

(compare with Eq. (142)).

For the same reasons, there are also no infrared divergences in the fermion propagator (see Fig.1c) nor in the Yukawa vertices (see Fig.1d) in this order. Therefore, at sufficiently large values of $N_c$, the results retain essentially the same as in leading order in $1/N_c$.

We believe that there should be no essential obstacles to extend this analysis for all orders in $1/N_c$. 

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APPENDIX C

In this Appendix we analyze integral equations (106) and (114) for the function $A(p)$. We will show that these equations lead to the expression (111) for $m_{\text{dyn}}$.

The integral equations were analyzed by using both analytical and numerical methods. The numerical plot of the dynamical mass as a function of $\alpha$ is shown in Fig.2. Below we consider the analytical analysis of these equations.

We begin by considering equation (106) in the Feynman gauge. After the angular integration, this equation becomes

$$A(p^2) = \frac{\alpha}{2\pi} \int_0^\infty \frac{dk^2 A(k^2)}{k^2 + m_{\text{dyn}}^2} K(p^2, k^2) \int_0^\infty \frac{dz \exp(-z\ell^2/2)}{p^2 + z}.$$

(146)

with the kernel

$$K(p^2, k^2) = \int_0^\infty \frac{dz \exp(-z\ell^2/2)}{\sqrt{(k^2 + p^2 + z)^2 - 4k^2 p^2}}.$$

(147)

To study Eq.(146) analytically it is convenient to break the momentum integration into two regions and expand the kernel appropriately for each region (compare with Ref.[11]):

$$A(p^2) = \frac{\alpha}{2\pi} \left[ \int_0^{p^2} \frac{dk^2 A(k^2)}{k^2 + m_{\text{dyn}}^2} \int_0^\infty \frac{dz \exp(-z\ell^2/2)}{p^2 + z} \right] + \frac{\alpha}{2\pi} \left[ \int_0^\infty \frac{dk^2 A(k^2)}{k^2 + m_{\text{dyn}}^2} \int_0^\infty \frac{dz \exp(-z\ell^2/2)}{k^2 + z} \right].$$

(148)

Introducing dimensionless variables $x = p^2\ell^2/2$, $y = k^2\ell^2/2$, $a = m_{\text{dyn}}^2\ell^2/2$, we rewrite Eq.(148) in the form

$$A(x) = \frac{\alpha}{2\pi} \left[ g(x) \int_0^x \frac{dy A(y)}{y + a^2} + \int_x^\infty \frac{dy A(y) g(y)}{y + a^2} \right].$$

(149)

where

$$g(x) = \int_0^\infty \frac{dz e^{-z}}{z + x} = -e^x Ei(-x),$$

(150)

We thank Anthony Hams and Manuel Reenders for their help in numerical solving these equations.
and $Ei(x)$ is the integral exponential function. The solutions of integral equation (149) satisfy the second order differential equation

$$ A'' - \frac{g''}{g'} A' - \frac{\alpha}{2\pi} \frac{A}{x + a^2} = 0, \quad (151) $$

where $(t)$ denotes derivative with respect to $x$. The boundary conditions to this equation follow from the integral equation (149):

$$ \frac{A'}{g'} \bigg|_{x=0}, \quad (A - \frac{gA'}{g'}) \bigg|_{x=\infty} = 0. \quad (152) $$

For the derivatives of the function $g(x)$ we have relations

$$ g' = -\frac{1}{x} + g(z), \quad g'' = \frac{1}{x^2} - \frac{1}{x} + g(x), \quad (154) $$

and $g(x)$ has asymptotics

$$ g(x) \sim \ln \frac{e^{-\gamma}}{x}, \quad x \to 0, $$

$$ g(x) \sim \frac{1}{x} - \frac{1}{x^2} + \frac{2}{x^3}, \quad x \to \infty. \quad (155) $$

Using Eqs. (154),(155) we find that Eq. (151) has two independent solutions which behave as $A(x) \sim \text{const}$, $A(x) \sim \ln 1/x$ and $A(x) \sim \text{const}$, $A(x) \sim \frac{1}{x^2}$, near $x = 0$ and $x = \infty$, respectively. The infrared boundary condition (BC) (152) leaves only the solution with regular behaviour, $A(x) \sim \text{const}$, while the ultraviolet BC gives an equation to determine $a = a(\alpha)$. To find analytically $a(\alpha)$ we will solve approximate equations in regions $x << 1$ and $x >> 1$ and then match the solutions at the point $x = 1$. This provides insight into the critical behaviour of the solution at $\alpha \to 0$. A numerical study of the full Eq.(146) reveals the same approach to criticality (see the plot in Fig.2).

In the region $x << 1$ Eq.(151) is reduced to a hypergeometric type equation:

$$ A'' + \frac{1}{x} A' + \frac{\alpha}{2\pi x(x + a^2)} A = 0. \quad (156) $$

The regular at $x = 0$ solution has the form

$$ A_1(x) = C_1 F(i\nu, -i\nu, 1; -\frac{x}{a^2}), \quad \nu = \sqrt{\frac{\alpha}{2\pi}}, \quad (157) $$
where $F$ is a hypergeometric function \[5\]. In the region $x >> 1$ Eq. (151) takes the form

$$A'' + \frac{2}{x} A' + \frac{\alpha}{2\pi x^2(x + a^2)} A = 0. \quad (158)$$

The solution satisfying ultraviolet BC (153) is

$$A_2(x) = C_2 \frac{1}{x} F\left(\frac{1 + i\mu}{2}, \frac{1 - i\mu}{2}; 2; -\frac{a^2}{x}\right), \quad \mu = \sqrt{\frac{2\alpha}{\pi a^2} - 1}. \quad (159)$$

Equating now logarithmic derivatives of $A_1$ and $A_2$ at $x = 1$ we arrive at the equation determining the quantity $a(\alpha)$:

$$\left. \frac{d}{dx} \left\{ \ln \frac{x F(\nu, -\nu; 1; -\frac{x}{a^2})}{F\left(\frac{1 + i\mu}{2}, \frac{1 - i\mu}{2}; 2; -\frac{a^2}{x}\right)} \right\} \right|_{x=1} = 0. \quad (160)$$

Note that up to now we have not made any assumptions on the value of the parameter $a$. Let us seek now for solutions of Eq. (160) with $a << 1$ (which corresponds to the assumption of the LLL dominance). Then the hypergeometric function in denominator of Eq. (160) can be replaced by 1 and we are left with equation

$$-\frac{1}{a^2} \nu^2 F(1 + i\nu, 1 - i\nu; 2; -\frac{1}{a^2}) + F(i\nu, -i\nu; 1; -\frac{1}{a^2}) = 0, \quad (161)$$

where we used the formula for differentiating the hypergeometric function \[5\]

$$\frac{d}{dz} F(a, b, c; z) = \frac{ab}{c} F(a + 1, b + 1, c + 1; z). \quad (162)$$

Now, because of $a << 1$, we can use the formula of asymptotic behavior of hypergeometric function at large values of its argument $z$ \[5\]:

$$F(a, b, c; z) \sim \frac{\Gamma(c)\Gamma(b - a)}{\Gamma(b)\Gamma(c - a)}(-z)^{-a} + \frac{\Gamma(c)\Gamma(a - b)}{\Gamma(a)\Gamma(c - b)}(-z)^{-b}. \quad (163)$$

Then Eq. (161) is reduced to the following one:

$$\cos \left[ \nu \ln \frac{1}{a^2} + \arg \Sigma(\nu) \right] = 0,$$

$$\Sigma(\nu) = \frac{1 + i\nu \Gamma(1 + 2i\nu)}{2 \Gamma^2(1 + i\nu)} \quad (164)$$

and we get

$$m_{\text{dyn}}^2 = 2|eB| \exp \left[ - \frac{\pi(2n + 1)/2 - \arg \Sigma(\nu)}{\nu} \right], \quad (165)$$
where $n$ is zero or positive integer. The arg $\Sigma(\nu)$ can be rewritten as
\[
\text{arg } \Sigma(\nu) = \arctan \nu + \arg \Gamma(1 + 2i\nu) - 2 \arg \Gamma(1 + i\nu)
\] (166)
and in the limit $\nu \to 0$ Eq. (160) takes the form
\[
m_{\text{dyn}}^2 = 2|eB|e \exp \left[ -\frac{\pi}{2\nu}(2n + 1) \right] = 2|eB|e \exp \left[ -\pi \sqrt{\frac{\pi}{2\alpha}}(2n + 1) \right]
\] (167)
(the second factor $e$ here is $e \approx 2.718$ and not the electric charge!). The stable vacuum corresponds to the largest value of $m_{\text{dyn}}^2$ with $n = 0$.

Let us now turn to equation (114) in an arbitrary covariant gauge. The function $g(x)$ is now replaced by
\[
\tilde{g}(x) = \int_0^\infty \frac{dz e^{-z} (1 - \lambda z/2)}{z + x} = g(x) + \frac{1}{2} \lambda x g'(x).
\] (168)
As it is easy to verify, this does not change equation (156) in the region $x \ll 1$. At $x \gg 1$ we have
\[
\frac{\tilde{g}'(x)}{\tilde{g}(x)} \sim -\frac{2 - \lambda}{2x^2} + \frac{1 - \lambda}{x^3},
\]
\[
\frac{\tilde{g}''(x)}{\tilde{g}'(x)} = -\frac{2}{x} \frac{2 - \lambda - 6 \frac{1 - \lambda}{x}}{2 - \lambda - 4 \frac{1 - \lambda}{x}}.
\] (169)
Therefore in any gauge, except $\lambda = 2$, the differentional equation at $x \gg 1$ takes the form
\[
A'' + \frac{2}{x} A' + \frac{\alpha(2 - \lambda)}{4\pi} \frac{A}{x^2(x + a^2)} = 0
\] (170)
with asymptotic solution $A(x) \sim \frac{1}{x}$. In the gauge $\lambda = 2$, instead of Eq. (170), we have
\[
A'' + \frac{3}{x} A' + \frac{\alpha}{\pi} \frac{A}{x^3(x + a^2)} = 0,
\] (171)
which gives more rapidly decreasing behavior $A(x) \sim \frac{1}{x^2}$ when $x \to \infty$. Repeating the previous analysis, we are led to expression (111) for $m_{\text{dyn}}$. 

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References

[1] J. Schwinger, Phys. Rev. 82 (1951) 664.

[2] V. P. Gusynin, V. A. Miransky, and I. A. Shovkovy, Phys. Rev. Lett. 73 (1994) 3499; Phys. Rev. D 52 (1995) 4718.

[3] Y. Nambu and G. Jona-Lasinio, Phys. Rev. 122 (1961) 345.

[4] J. Bardeen, L. N. Cooper, and J. R. Schrieffer, Phys. Rev. 108 (1957) 1175.

[5] I. S. Gradshtein and I. M. Ryzhik, *Table of Integrals, Series and Products* (Academic Press, Orlando, 1980).

[6] V. A. Miransky, *Dynamical Symmetry Breaking in Quantum Field Theories* (World Scientific Co., Singapore, 1993).

[7] B. Simon, Ann. Phys. 97 (1976) 279.

[8] A. I. Akhiezer and V. B. Berestetsky, *Quantum Electrodynamics* (Interscience, NY, 1965).

[9] A. Cappelli, C. A. Trugenberger, and G. R. Zemba, Nucl. Phys. B396 (1993) 465.

[10] A. Chodos, I. Everding, and D. A. Owen, Phys. Rev. D42 (1990) 2881.

[11] T. Appelquist, M. Bowick, D. Karabali, and L. C. R. Wijewardhana, Phys. Rev. D33 (1986) 3704.

[12] N. D. Mermin and H. Wagner, Phys. Rev. Lett. 17 (1966) 1133; S. Coleman, Commun. Math. Phys. 31 (1973) 259.

[13] D. Gross and A. Neveu, Phys. Rev. D10 (1974) 3235; E. Witten, Nucl. Phys. B145 (1978) 110.

[14] J. Zak, Phys. Rev. 134 (1964) A1602; J. E. Avron, I. W. Herbst, and B. Simon, Ann. Phys. 114 (1978) 431.

[15] S. P. Klevansky, Rev. Mod. Phys. 64 (1992) 649.

[16] V. P. Gusynin and V. A. Miransky, Mod. Phys. Lett. A6 (1991) 2443; Sov. Phys. JETP 74 (1992) 216; V. A. Miransky, Int. J. Mod. Phys. A8 (1993) 135.

[17] D. Caldi, A. Chodos, K. Everding, D. Owen, and S. Vafaeisefat, Phys. Rev. D39 (1989) 1432.
[18] P. I. Fomin, V. P. Gusynin, V. A. Miransky, and Yu. A. Sitenko, Riv. del Nuovo Cimento 6, N5 (1983) 1.

[19] A. M. Perelomov and V. S. Popov. Theor. Math. Phys. 4 (1970) 664.

[20] W. Dittrich and M. Reuter, Effective Lagrangians in Quantum Electrodynamics (Springer–Verlag, Berlin, 1985).

[21] G. Calucci and R. Ragazzon, J. Phys. A 27 (1994) 2161.

[22] J. Schwinger, Phys. Rev. 125 (1962) 397; J.H. Lowenstein and J.A. Swieca, Ann. Phys. (N.Y.) 68 (1971) 172.

[23] R. Parwani, Phys. Lett. B 358 (1995) 101.

[24] D. Cangemi, E. D’Hoker, and G. Dunne, Phys. Rev. D 51 (1995) R2513; Phys. Rev. D 52 (1995) R3163.

[25] R. Jackiw, Phys. Rev. D 29 (1984) 2375; I. V. Krive and A. S. Rozhavsky, Sov. Phys. Usp. 30 (1987) 370; A. Kovner and B. Rosenstein, Phys. Rev. B 42 (1990) 4748; G. W. Semenoff and L. C. R. Wijewardhana, Phys. Rev. D 45 (1992) 1342; N. Dorey and N. E. Mavromatos, Nucl. Phys. B 368 (1992) 614.

[26] Y. Hosotani, Phys. Lett. B 319 (1993) 332.

[27] T. Vachaspati, Phys. Lett. B 265 (1991) 258; K. Enqvist and P. Olesen, Phys. Lett. B 319 (1993) 178; J. M. Cornwall, in Unified Symmetry: In the Small and Large, eds. B. N. Kursunoglu et al. (Plenum Press, New York, 1995).

[28] N. K. Nielsen and P. Olesen, Nucl. Phys. B 144 (1978) 376.

[29] P. Salabura et al., Phys. Lett. B 245 (1990) 153; I. Koeing et al., Z. Phys. A 346 (1993) 153.

[30] L. S. Celenza, V. K. Mishra, C. M. Shakin, and K. F. Lin, Phys. Rev. Lett. 57 (1986) 55; D. G. Caldi and A. Chodos, Phys. Rev. D 36 (1987) 2876; Y. J. Ng and Y. Kikuchi, Phys. Rev. D 36 (1987) 2880.

[31] D. G. Caldi and S. Vafaeisefat, Phys. Lett. B 287 (1992) 185.

[32] S. Coleman, Aspects of Symmetry (Cambridge University Press, Cambridge, 1985).
FIGURE CAPTIONS

1. Fig. 1. Diagrams in next-to-leading order in $1/N_c$. A solid line denotes the fermion propagator and a dashed line denotes the propagators of $\sigma$ and $\pi$ in leading order in $1/N_c$.

2. Fig. 2. A numerical fit of the dynamical mass as a function of the coupling constant $\alpha$: $m_{\text{dyn}}/\sqrt{2eB} = \exp[\pi \frac{\sqrt{2}}{2\alpha} a + b]$. The fitting parameters are: $a = 1.000059$, $b = -0.283847$.
Figure 1: Diagrams in next-to-leading order in $1/N_c$. A solid line denotes the fermion propagator and a dashed line denotes the propagators of $\sigma$ and $\pi$ in leading order in $1/N_c$. 
Figure 2: A numerical fit of the dynamical mass as a function of the coupling constant $\alpha$: $m_{\text{dyn}}/\sqrt{2eB} = \exp[-\frac{\pi}{2}\sqrt{2a} + b]$. The fitting parameters are:

$a = 1.000059$, $b = -0.283847$. 

