Viscosity Solutions for McKean-Vlasov Control I:
one-dimensional torus

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Abstract

An optimal control problem in the space of probability measures, and the viscosity solutions of the corresponding dynamic programming equations defined using the intrinsic linear derivative are studied. The value function is shown to be Lipschitz continuous with respect to a novel smooth Fourier-Wasserstein metric. A comparison result between the Lipschitz viscosity sub and super solutions of the dynamic programming equation is proved using this metric, characterizing the value function as the unique Lipschitz viscosity solution.

Key words: Mean Field Games, Wasserstein metric, Viscosity Solutions, McKean-Vlasov.

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1 Introduction

McKean–Vlasov optimal control is a part of the overarching program of Lasry & Lions [22, 23, 24] as articulated by Lions through his College de France lectures [25], and independently initiated by Huang, Malhamé, & Caines [21]. We refer the reader to the classical book of Carmona & Delarue [7] and to the lecture notes of Cardaliaguet [5] for detailed information and more references on this exciting scientific activity.

Main feature of the McKean-Vlasov type optimization is the dependence of its evolution and cost not only on the position of the state but also on its distribution, making the set of probability measures as its state space. Thus, the dynamic programming approach results in nonlinear partial differential equations in this space. Without common noise, they are first order Hamilton-Jacobi-Bellman equations with an unbounded Hamiltonian, as it is almost always the case for optimal control problems set in infinite dimensional spaces [18]. They are analogous to the coupled Hamilton-Jacobi and Fokker-Planck-Kolmogorov systems that characterize the solutions of the mean-field games for which deep regularity results are proved in [6] under some structural conditions. However, in general the dynamic programming equations for the McKean-Vlasov control problems are not expected to admit classical solutions (see subsection 4.1 below), and a weak formulation is needed.

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As in the finite-dimensional setting, maximum principle is still the salient feature of these settings as well and therefore, the viscosity solutions of Crandall & Lions [14, 15, 16, 19] is the appropriate choice of a weak solution. However, due to the unboundedness of the Hamiltonian, original definition must be modified. In fact, such modifications of viscosity solutions in infinite dimensional spaces have already been studied extensively, and the book [18] provides an exhaustive account of these results. Still, it is believed that more can be achieved in the context of McKean–Vlasov due to the special structure of the set of probability measures. Indeed, an approach developed by Lions lifts the problems from the Wasserstein space to a regular $L^2$ space, and then exploits the Hilbert structure to obtain new comparison results. This procedure also delivers the novel Lions derivative which has many useful properties, and we refer to [7] for its definition and more information. This method is further developed in several papers including [1, 2, 11, 27, 28]. The choice of the appropriate notion of a derivative is also explored in the recent paper [20], which then utilizes the deep connections to geometry to prove uniqueness results for bounded Hamiltonians.

Our main goals are to develop a viscosity theory directly on the space of probability measures using the linear derivative, provide a comparison result, and obtain a characterization of the value function as the unique viscosity solution in a certain class of functions. A natural approach towards this goal is to project the problem onto finite-dimensional spaces to leverage the already developed theory on these structures. A second-order problem studied in [13] provides a clear example of this approach as its projections exactly solve the projected finite dimensional equations. However, in general these projections are only approximate solutions, and [12] uses the Ekeland variational principle together with Gaussian smoothed Wasserstein metrics as gauge functions to control the approximation errors. A different technical tool is developed in [3], and [20] studies the pure projection problem. Other approaches include the path-dependent equations used in [30], and gradient flows in [10]. Recent paper [9] exploits the semi-convexity, and also provides an extensive survey.

We on the other hand employ the classical viscosity technique of doubling the variables as done in [4] in lieu of projection. The central difficulty of this approach is to appropriately replace the "distance-square" term $|x - y|^2$ used in the finite dimensional comparison proofs with the square of a metric on the space of measures. Thus, the crucial ingredient of our method is a novel smooth metric $\rho_3$ defined by a Fourier based modification of the Wasserstein metrics. In addition to several intriguing properties of $\rho_3$ proved in Section 5, our other main results are a comparison between Lipschitz continuous sub and super viscosity solutions, Theorem 4.3 and the Lipschitz continuity of the value function with respect to another metric $\hat{\rho}_3 \leq c_3 \rho_3$, Theorem 4.1. Although the Lipschitz property of the value function is rather elementary for the Wasserstein metric $W_1$, it requires detailed analysis for $\hat{\rho}_3$. Indeed, a technical estimate, Proposition 7.1, on the dependence of the solutions of the McKean–Vlasov stochastic differential equation on the initial distribution is needed for this property.

As our approach contains several new steps, we study the simplest problem that allows us to showcase its details and power concisely. In particular, to ease the notation we omit the dependence of all functions on the time variable which can be added directly. Additionally, dynamics with jumps can be included as done in [4]. The one-dimensional compact structure of the torus is clearly a simplifying feature as well. In our accompanying paper [29] we remove most of these restrictions and study the extension of our method in higher dimensions.

The paper is organized as follow. General structure and notations are given in the next section, in Section 3 we define the problem and state the assumptions. The main results are stated in Section 4. We construct a family of Fourier-Wasserstein metrics in Section 5. The comparison result is proved in Section 6, and the Lipschitz property in Section 7. Standard results of dynamic programming and viscosity property are proved in Section 8 and respectively in Section 9.
2 Notations

In this section, we summarize the notations and known results used in the sequel. \( T > 0 \) is the finite horizon, \( Z \) is the set of all integers, and \( \mathbb{Z}_0 = \mathbb{Z} \setminus \{0\} \). \( \mathbb{T} = \mathbb{R} / (2\pi \mathbb{Z}) \) is the one dimensional torus with the metric given by \( d(x,y) := \inf_{k \in \mathbb{Z}} |x - y - 2k\pi| \). All periodic functions on \( \mathbb{R} \) are uniquely related to functions on \( \mathbb{T} \), and we do not distinguish them in our notation. We use a filtered probability space \( (\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P}) \) that supports Brownian motions. We also assume that initial filtration \( \mathcal{F}_0 \) is rich enough so that for any probability measure on \( \mathbb{T} \), there exists a random variable on \( \Omega \) whose distribution is equal to this measure.

For a metric space \((E,d)\), \( \mathcal{M}(E) \) is the set of all Radon measures on \( E \), and \( \mathcal{P}(E) \) denotes the set of all probability measures on \( E \). Let \( L^0(E) \) be the set of all \( E \)-valued random variables. For \( X \in L^0(E) \), \( \mathcal{L}(X) \in \mathcal{P}(E) \) is the law of \( X \). For \( \mu \in \mathcal{P}(\mathbb{T}) \), \( t \in [0,T] \), we set

\[
\mathbb{L}_t := \{ \xi \in L^0(\mathbb{T}) : \xi \text{ is } \mathcal{F}_t \text{ - measurable} \}, \quad \mathbb{L}_t^\mu := \{ \xi \in \mathbb{L}_t : \mathcal{L}(\xi) = \mu \}.
\]

\( \mathcal{C}(E) \) is the set of all continuous real-valued functions on \( E \), and \( \mathcal{C}_b(E) \subset \mathcal{C}(E) \) is the set of all bounded ones. We write \( \mathcal{C}(E,d) \) when the dependence on the metric is relevant, and \( \mathcal{C}(E \to Y) \) if the range \( Y \) is not the real numbers. For a positive integer \( k \), \( \mathcal{C}^k(E) \) is the set of all \( k \)-times continuously differentiable, real-valued functions with bounded derivatives. For \( \mu \in \mathcal{M}(E) \), \( f \in \mathcal{C}_b(E) \), \( \mu(f) := \int_E f(x) \mu(dx) \). We endow the space of probability measures \( \mathcal{M}(E) \) with the weak* topology \( \sigma(\mathcal{P}(E),\mathcal{C}_b(E)) \), and write \( \mu_n \rightharpoonup \mu \) when \( \mu_n \) converges to \( \mu \) in this topology, i.e. \( \lim_{n \to \infty} \mu_n(f) = \mu(f) \) for every \( f \in \mathcal{C}_b(E) \). We set \( \mathcal{O} := (0,T) \times \mathcal{P}(\mathbb{T}) \).

We use the standard (linear) derivative on the convex set \( \mathcal{P}(\mathbb{T}) \), i.e. \( f \in \mathcal{C}(\mathcal{P}(\mathbb{T})) \) is continuously differentiable if there exists \( \partial_\nu f \in c(\mathcal{P}(\mathbb{T}) \to \mathcal{C}(\mathbb{T})) \) satisfying,

\[
f(\nu) = f(\mu) + \int_0^1 \int_T \partial_\nu f(\mu + \tau(\nu - \mu))(x)(\nu - \mu)(dx) \, d\tau, \quad \forall f, \nu \in \mathcal{P}(\mathbb{T}).
\]

For \( \psi \in \mathcal{C}(\mathcal{O}) \) and \( (t,\mu) \in \mathcal{O} \), \( \partial_t \psi(t,\mu) \) is the time derivative evaluated at \( (t,\mu) \), and \( \partial_\mu \psi(t,\mu) \) is the derivative in the \( \mu \)-variable. We again note that \( \partial_\mu \psi(t,\mu) \in \mathcal{C}(\mathbb{T}) \) by its definition.

We use the following orthonormal basis of \( L^2(\mathbb{T}) \),

\[
e_k(x) := \frac{1}{\sqrt{2\pi}} e^{ikx}, \quad x \in \mathbb{T}, \quad k \in \mathbb{Z}, \quad (2.1)
\]

where \( i = \sqrt{-1} \) and \( z^* \) be the complex conjugate of \( z \). In particular, for any \( f \in L^2(\mathbb{T}) \),

\[
f = \sum_{k \in \mathbb{Z}} F_k(f) e_k, \quad \text{where} \quad F_k(f) := \int_\mathbb{T} f(x)e^*_k(x) \, dx, \quad k \in \mathbb{Z}.
\]

We use the following distances on \( \mathcal{P}(\mathbb{T}) \) given by their dual representations,

\[
W_1(\mu,\nu) := \sup \{ (\mu - \nu)(\psi) : |\psi(x) - \psi(y)| \leq |x - y|, \forall x,y \in \mathbb{T} \}, \quad \rho_\lambda(\mu,\nu) := \sup \{ (\mu - \nu)(\psi) : \psi \in \mathcal{H}_\lambda(\mathbb{T}), \|\psi\|_\lambda \leq 1 \}, \quad \lambda \geq 1,
\]

\[
\hat{\rho}_k(\mu,\nu) := \sup \{ (\mu - \nu)(\psi) : \psi \in \mathcal{C}^k(\mathbb{T}), \|\psi\|_{C^k} \leq 1 \}, \quad k = 1,2,\ldots
\]

where in view of Kantorovich duality, \( W_1 \) is the Wasserstein-one distance, and for \( \lambda \geq 1 \),

\[
\mathcal{H}_\lambda(\mathbb{T}) := \{ f \in L^0(\mathbb{T}) : \|f\|_\lambda < \infty \}, \quad \|f\|_\lambda := \sum_{k \in \mathbb{Z}} (1 + k^2)^\lambda |F_k(f)|^2 \hat{\psi}.
\]
3 McKean-Vlasov control

In this section, we define the McKean-Vlasov optimal control problem, and for an introduction to it we refer the reader to Chapter 6 in [7]. Formally, starting from $t \in [0, T]$, the goal is to choose a control process $(\alpha_u)_{u \in [t, T]}$ so as to minimize

$$\int_t^T \mathbb{E}[\ell(X_u, \mathcal{L}(X_u), \alpha_u)] \, du + \varphi(X_T),$$

where $\ell$ is the running cost, $\varphi$ is the terminal cost, $b, \sigma$ are given functions, and with a Brownian motion $B$, $dX_u = b(X_u, \mathcal{L}(X_u), \alpha_u) \, du + \sigma(X_u, \mathcal{L}(X_u), \alpha_u) \, dB_u$.

We continue by defining this problem properly.

3.1 Controlled processes

Suppose that $A$ is a closed Euclidean space and let the control set $\mathcal{C}_a$ be a subset of $C(T \rightarrow A)$ containing all constant functions, and the admissible controls $A$ be the set of (deterministic) measurable functions $\alpha : [0, T] \rightarrow \mathcal{C}_a$. We denote the value of any $\alpha \in A$ at time $u \in [0, T]$ by $\alpha_u \in \mathcal{C}_a$. We continue by stating our standing regularity assumptions on the given functions

$$b, \sigma, \ell : T \times \mathcal{P}(\mathbb{T}) \times A \rightarrow \mathbb{R}, \quad \varphi : \mathcal{P}(\mathbb{T}) \rightarrow \mathbb{R}.$$

For $\alpha \in \mathcal{C}_a$, $x \in T$, and $\mu \in \mathcal{P}(T)$, set

$$b^\alpha(x, \mu) := b(x, \mu, \alpha(x)), \quad \sigma^\alpha(x, \mu) := \sigma(x, \mu, \alpha(x)), \quad \ell^\alpha(x, \mu) := \ell(x, \mu, \alpha(x)).$$

**Assumption 3.1** (Regularity). There exists $c_a < \infty$ such that for all $\alpha \in \mathcal{C}_a$ and $\mu \in \mathcal{P}(T)$,

$$\|b^\alpha(\cdot, \mu)\|_{C^3(T)} + \|\sigma^\alpha(\cdot, \mu)\|_{C^3(T)} + \|\ell^\alpha(\cdot, \mu)\|_{C^3(T)} \leq c_a,$$

and for $h = b, \sigma, \ell, \varphi,$

$$|h(x, \mu, a) - h(x, \nu, a)| \leq c_a \rho_h(\mu, \nu), \quad \forall x \in T, \mu, \nu \in \mathcal{P}(T), a \in A.$$

Under this regularity condition, for any $\alpha \in A$, $t \in [0, T]$, and $\xi \in \mathbb{L}_t^\alpha$, there is a unique $\mathbb{F}$-adapted solution $X_t^{t, \alpha, \alpha}$ of the following McKean-Vlasov stochastic differential equation,

$$X_t^{t, \alpha, \alpha} = \xi + \int_t^s b^{\alpha}(X_u^{t, \mu, \alpha}, \mathcal{L}_u^{t, \mu, \alpha}) \, du + \int_t^s \sigma^{\alpha}(X_u^{t, \mu, \alpha}, \mathcal{L}_u^{t, \mu, \alpha}) \, dB_u, \quad s \in [t, T], \tag{3.1}$$

where $B$ is a Brownian motion.

Although the solution $X_t^{t, \alpha, \alpha}$ depends on the choice of the initial condition $\xi$ and the Brownian increments $(B_u - B_t)_{u \in [t, T]}$, as the Brownian increments are independent of $\mathcal{F}_t$ and we consider feedback controls, the flow $(\mathcal{L}_u^{t, \mu, \alpha})_{u \in [t, T]}$ depends only on the law of the initial condition $\xi \in \mathbb{L}_t^\alpha$ instead of $\xi$ itself.

Clearly, the existence and uniqueness of solutions of (3.1) can be obtained under weaker assumptions. However, the stronger condition with three derivatives is needed for the comparison and the Lipschitz continuity results. We also emphasize that the regularity Assumption 3.1 puts implicit restrictions of the control set $\mathcal{C}_a$, see Remark 3.2 below.
3.2 Problem

For a control process $\alpha \in \mathcal{A}$, we define the pay-off functional $J$ at $(t, \mu) \in \overline{\Omega}$ by,

$$J(t, \mu, \alpha) := \int_0^T \mathbb{E}[\ell^\mu(X_u^t, \Lambda^\mu_t, \alpha)] \, du \, + \, \phi(L^\mu_t, \alpha), \quad \alpha \in \mathcal{A}, \ (t, \mu) \in \overline{\Omega}. \quad (3.2)$$

Since $\mathbb{E}[\ell^\mu(X_u^t, \Lambda^\mu_t, \alpha)] = \Lambda^\mu_t(\ell(\cdot, \Lambda^\mu_t, \alpha(\cdot))), J(t, \mu, \alpha)$ is a function of $\mu$ independent of the choice of the initial random variable $\xi \in \mathcal{L}_0^\mu$. Although, this property, called law-invariance, holds directly in our setting, in general structures it is quite subtle. We refer to Proposition 2.4 of [17] and Theorem 3.5 in [11] for its general proof, and to Section 6.5 and Definition 6.27 of [7] for a discussion.

Then the McKean-Vlasov optimal control problem is to minimize the pay-off functional $J$ over $\alpha \in \mathcal{A}$, and the value function is given by,

$$v(t, \mu) := \inf_{\alpha \in \mathcal{A}} J(t, \mu, \alpha), \quad (t, \mu) \in \overline{\Omega}.$$  

Remark 3.2. Suppose that $\mathcal{C}_\alpha = \{ \alpha \in \mathcal{C}(T \to A) : \| \alpha \|_{C_3} \leq c_3 \}$ for some constant $c_3 \geq 0$. Consider class of functions of the form $h(x, \mu(f), a)$ for some $f \in \mathcal{C}^3$, and $h : T \times \mathbb{R} \times A \to \mathbb{R}$, satisfying $\|h(\cdot, y, \cdot)\|_{C_3} + \|h(x, \cdot, a)\|_{C_3} \leq c_3$ for every $x \in \mathcal{T}$, $y \in \mathbb{R}$, and $a \in \mathcal{A}$. Then,

$$|h(x, \mu(f), a) - h(x, \nu(f), a)| \leq c_3 \| (\mu - \nu)(f) \| \leq c_3 \| \mu - \nu \|_{C_3} \hat{\rho}_a(\mu, \nu).$$

Hence, this class of functions with the control set $\mathcal{C}_\alpha$ as above satisfy the regularity assumption. More generally, functions $h(x, \mu(f_1), \ldots, \mu(f_k), a)$ for some integer $k$, with $f_1, \ldots, f_k \in \mathcal{C}^3(T)$, and $h : T \times \mathbb{R}^k \times A \to \mathbb{R}$ uniformly Lipschitz and uniformly three-times differentiable in the $x$ and $a$ variables, satisfy the regularity assumptions.

Assumptions made above hold in a large class of examples studied in the mean-field games. In particular, for the Kuramoto problem studied in [8], for some constants $\kappa, \sigma > 0$,

$$\ell(\mu, a) = \frac{1}{2} \sigma^2 + \kappa [1 - (\mu(\cos))^2 - (\nu(\sin))^2], \quad b(x, \mu, a) = a, \quad \sigma(a) = \sigma.$$  

3.3 Dynamic programming principle

We next state the dynamic programming principle which is central to the viscosity approach to optimal control. A general proof in a different setting is given in [17]. The specific structure of the problem studied here allows for a simpler proof that we provide in Section 8. Our proof is based on the Lipschitz continuity of the value function proved in Section 7 and the standard techniques outlined in [19].

Theorem 3.3 (Dynamic programming). For every $\mu \in \mathcal{P}(\mathbb{T})$ and $0 \leq t \leq \tau \leq T$,

$$v(t, \mu) = \inf_{\alpha \in \mathcal{A}_\mu} \int_t^\tau \mathbb{E}[\ell^\mu(X_u^t, \Lambda^\mu_t, \alpha)] \, du \, + \, \phi(L^\mu_t, \alpha), \quad \mu \in \mathcal{P}(T). \quad (3.3)$$

It is well known that the dynamic programming can be used directly to show that the value function is a viscosity solution of the dynamic programming equation

$$- \Delta v(t, \mu) = H(\mu, \partial_\mu v(t, \mu)), \quad t \in [0, T), \mu \in \mathcal{P}(T),$$

where for $\phi \in \mathcal{C}_2(\mathbb{T})$, $\mu \in \mathcal{P}(\mathbb{T})$,

$$H(\mu, \phi) := \inf_{\alpha \in \mathcal{C}_\alpha} \left\{ \int_\mathbb{T} (\ell(x, \mu, \alpha(x)) + \mathcal{M}_\alpha^\mu[\phi](x)) \, \mu(dx) \right\},$$

$$\mathcal{M}_\alpha^\mu[\phi](x) := b(x, \mu, \alpha(x)) \partial_x \phi(x) + \frac{1}{2} \sigma^2(x, \mu, \alpha(x)) \partial_{xx} \phi(x), \quad x \in \mathbb{T}, \, \alpha \in \mathcal{C}_\alpha.$$
The value function also trivially satisfies the following terminal condition,

\[ v(T, \mu) = \varphi(\mu), \quad \forall \mu \in \mathcal{P}(\mathbb{T}). \]  

(3.5)

As the value function is not necessarily differentiable, a weak formulation is needed, and we use the notion of viscosity solutions. The definition that we use is exactly the classical one in which the auxiliary test functions are continuously differentiable functions on \( \mathcal{O} = [0, T] \times \mathcal{P}(\mathbb{T}) \), with the linear derivative in \( \mathcal{P}(\mathbb{T}) \) recalled in Section 2. We continue by specifying the auxiliary functions used in the definition of viscosity solutions.

**Definition 3.4.** We say that \( \psi \in \mathcal{C}(\mathcal{O}) \) is a test function, if \( \psi \) is continuously differentiable with \( \partial_t \psi(t, \mu) \in \mathcal{C}^2(\mathcal{T}) \) for every \( (t, \mu) \in \mathcal{O} \), and the map \( (t, \mu) \in \mathcal{O} \mapsto H(\mu, \partial_t \psi(t, \mu)) \) is continuous. We let \( \mathcal{C}(\mathcal{O}) \) be the set of all test functions.

**Definition 3.5.** A continuous function \( u \in \mathcal{C}(\mathcal{O}) \) is a viscosity subsolution of (3.4), if every \( \psi \in \mathcal{C}(\mathcal{O}), \ (t_0, \mu_0) \in [0, T) \times \mathcal{P}(\mathbb{T}), \) satisfying \( (u - \psi)(t_0, \mu_0) = \max_{\mathcal{O}}(u - \psi) \), also satisfies

\[ -\partial_t \psi(t_0, \mu_0) \leq H(\mu_0, \partial_t \psi(t_0, \mu_0)). \]

A continuous function \( w \in \mathcal{C}(\mathcal{O}) \) is a viscosity supersolution of (3.4), if every \( \psi \in \mathcal{C}(\mathcal{O}), \ (t_0, \mu_0) \in [0, T) \times \mathcal{P}(\mathbb{T}), \) satisfying \( (w - \psi)(t_0, \mu_0) = \min_{\mathcal{O}}(w - \psi) \), also satisfies

\[ -\partial_t \psi(t_0, \mu_0) \geq H(\mu_0, \partial_t \psi(t_0, \mu_0)). \]

Finally, \( v \in \mathcal{C}(\mathcal{O}) \) is a viscosity solution of (3.4), if it is both a sub and a super solution.

## 4 Main results

Our main result is the characterization of the value function as the unique Lipschitz (in the \( \mu \)-variable) viscosity solution of the dynamic programming equation (3.4) and the terminal condition (3.5). Recall the metrics \( \rho_3, \tilde{\rho}_3 \) defined in Section 2.

The regularity Assumption 3.1 is assumed in all results.

**Theorem 4.1 (Continuity).** There exists a constant \( L_\alpha > 0 \) depending only on the horizon \( T \) and the constant \( c_\alpha \) of Assumption 3.1, so that

\[ |v(t, \mu) - v(s, \nu)| \leq L_\alpha \left[ \tilde{\rho}_3(\mu, \nu) + |t - s|^{\frac{1}{2}} \right], \quad \forall \mu, \nu \in \mathcal{P}(\mathbb{T}), \ t, s \in [0, T]. \]

(4.1)

The above theorem is proved in Section 7. As \( \tilde{\rho}_3 \leq c_3 \rho_3 \) for some constant \( c_3 \), above also implies Lipschitz continuity with respect to metric \( \rho_3 \). The following result follows directly from the standard viscosity theory [19], and its proof is given in Section 9.

**Theorem 4.2 (Viscosity property).** The value function is a viscosity solution of (3.4) in \( \mathcal{O} \), satisfying the terminal condition (3.5).

Following comparison result is proved in Section 6 below.

**Theorem 4.3 (Comparison).** Suppose that \( u \in \mathcal{C}(\mathcal{O}) \) is a viscosity subsolution, and respectively \( w \in \mathcal{C}(\mathcal{O}) \) is a viscosity supersolution of (3.4) and (3.5). If \( u \) or \( w \) is Lipschitz continuous in the \( \mu \)-variable with respect to the metric \( \rho_3 \), then \( u \leq v \) on \( \mathcal{O} \).

In particular, any continuous viscosity subsolution is less than or equal to the value function \( v \), and any continuous viscosity supersolution is greater than or equal to \( v \).

**Remark 4.4.** In the comparison result, we could use any metric \( \rho_\lambda \) with \( \lambda > \frac{1}{2} \). However, our proof for Lipschitz continuity requires us to employ the smaller metric \( \tilde{\rho}_k \) and only for integer values of \( k \). This combination of the results dictates the global choice \( \lambda = 3 \).
4.1 An example

In this subsection, we provide a simple example to illustrate the notation and also the need for viscosity solutions. We take $T = 1$, $A = \mathbb{R}$, $b(x, \mu, a) = a$, $\sigma \equiv 1$, $\varphi \equiv 0$, and $\ell(\mu, a) := \frac{1}{2}a^2 + L(m(\mu))$, where $m(\mu) := \int_x \mu(dx)$, and $L : [-\pi, \pi] \rightarrow \mathbb{R}$ is a given Lipschitz function. It can be shown that the value function of the above problem is independent of the control set $\mathcal{C}_\alpha$, and is given by,

$$v(t, \mu) = w(t, \mu), \quad (t, \mu) \in \overline{O},$$

where $w(t, \mu) := \inf_{\alpha \in \mathcal{A}} \int_t^T \frac{1}{2}(\hat{\alpha}_s)^2 + L(Y_u^{t, \mu, \hat{\alpha}}) \, du$, $(t, \mu) \in [0, 1] \times \mathbb{T}$, and $Y_u^{t, \mu, \hat{\alpha}} = y + \int_t^t \hat{\alpha}_s \, ds$. It is well known that $w$ is the unique viscosity solution of the Eikonal equation,

$$-\partial_t w(t, y) = -\frac{1}{2}(\partial_y w(t, y))^2 + L(y), \quad y \in \mathbb{T},$$

called Eikonal equation (4.2), and $w(1, \cdot) \equiv 0$. Since $w$ is not always differentiable, we conclude that $v$ is not either, and therefore a weak theory is needed. On the other hand, when $w$ is differentiable, we have

$$\partial_v v(t, \mu)(x) = \partial_v w(t, \mu)(x) \Rightarrow \partial_v (\partial_v v(t, \mu)(x)) = \partial_v w(t, \mu).$$

Hence, by Jensen’s inequality,

$$H(\mu, \partial_v v(t, \mu)) = \inf_{\alpha \in \mathcal{C}_\alpha} \int_T \frac{1}{2} \alpha(x)^2 + \alpha(x) \partial_v w(t, \mu) \mu(dx) + L(m(\mu))$$

$$\geq \inf_{\alpha \in \mathcal{A}} \left\{ \frac{1}{2} \left( \int_T \alpha(x) \mu(dx) \right)^2 + \left( \int_T \alpha(x) \mu(dx) \right) \partial_v w(t, \mu) \right\} + L(m(\mu))$$

$$\geq \inf_{\alpha \in \mathcal{A}} \left\{ \frac{1}{2} a^2 + a \partial_v w(t, \mu) \right\} + L(m(\mu))$$

$$= -\frac{1}{2}(\partial_v w(t, \mu))^2 + L(m(\mu)).$$

As constant functions $\alpha \equiv a$ are always in $\mathcal{C}_\alpha$, we also have the opposite inequality. Therefore,

$$H(\mu, \partial_v v(t, \mu)) = -\frac{1}{2}(\partial_v w(t, \mu))^2 + L(m(\mu)).$$

Since $\partial_v v(t, \mu) = \partial_v w(t, \mu)$, the Eikonal equation (4.2) implies that when $w$ is differentiable, $v$ is a classical solution of the dynamic programming equation for every $\mathcal{C}_\alpha$,

$$-\partial_v v(t, \mu) = H(\mu, \partial_v v(t, \mu)), \quad (t, \mu) \in \mathcal{O}.$$

5 Fourier-Wasserstein metrics

In this section, we study the properties of the norms and the metric $\rho_\lambda$ defined in Section 2. Similar metrics are also defined in [26] using a dual representation with Sobolev functions.

Recall that $z^*$ is the complex conjugate of $z$, and the orthonormal basis $\{e_k\}_{k \in \mathbb{Z}}$, Fourier coefficients $F_k(f)$ are defined in Section 2. For $\mu \in \mathcal{M}(\mathbb{T})$, $k \in \mathbb{Z}$, we also set $F_k(\mu) := \mu(e_k)$. As $\mathbb{T}$ is compact, $F_k(\mu)$ is finite for every $k$, and $F_0(\mu) = 1$ for all $\mu \in \mathcal{P}(\mathbb{T})$. 7
For $\lambda \geq 1$, we define a norm on $\mathcal{M}(\mathbb{T})$, dual to $\| \cdot \|_\lambda$ by,

$$|\eta|_\lambda := \sup\{ \eta(\psi) : \psi \in \mathbb{H}_\lambda(\mathbb{T}), \|\psi\|_\lambda \leq 1 \}, \quad \eta \in \mathcal{M}(\mathbb{T}),$$

so that $\rho_\lambda(\mu, \nu) = |\mu - \nu|_\lambda$.

**Lemma 5.1.** For $\lambda \geq 1$, $\eta \in \mathcal{M}(\mathbb{T})$, $|\eta|_\lambda < \infty$ and has the following dual representation,

$$|\eta|_\lambda = \left( \sum_{k \in \mathbb{Z}} \frac{1}{(1 + k^2)^\lambda} |F_k(\eta)|^2 \right)^{\frac{1}{2}}. \quad (5.1)$$

**Proof.** Let $d(\eta)$ be the expression in the right hand side of (5.1) and $TV(\eta)$ be the total variation of the measure $\eta$. Then, $|F_k(\eta)| \leq TV(\eta)$ and therefore, $d(\eta) \leq c\lambda TV(\eta)$, for some constant $c\lambda$. For $\psi \in C(\mathbb{T})$, the Fourier representation $\psi = \sum_{k \in \mathbb{Z}} F_k(\psi) e_k$ implies that,

$$\eta(\psi) = \sum_{k \in \mathbb{Z}} F_k(\psi) \eta(e_k) = \sum_{k \in \mathbb{Z}} F_k(\psi) F_k^*(\eta)$$

$$= \sum_{k \in \mathbb{Z}} [(1 + k^2)^{\frac{\lambda}{2}} F_k(\psi)] \left( \frac{1}{(1 + k^2)^{\frac{\lambda}{2}}} F_k^*(\eta) \right)$$

$$\leq \left( \sum_{k \in \mathbb{Z}} (1 + k^2)^{\lambda} |F_k(\psi)|^2 \right)^{\frac{1}{2}} \left( \sum_{k \in \mathbb{Z}} \frac{1}{(1 + k^2)^{\lambda}} |F_k^*(\eta)|^2 \right)^{\frac{1}{2}} = \|\psi\|_\lambda d(\eta). \quad (5.2)$$

In view of the definition of $| \cdot |_\lambda$, $|\eta|_\lambda \leq d(\eta)$, for any $\eta \in \mathcal{M}(\mathbb{T})$.

To prove the opposite inequality, fix $\eta \in \mathcal{M}(\mathbb{T})$ and define a function $\tilde{\psi}$ by,

$$\tilde{\psi}(x) := \sum_{k \in \mathbb{Z}_0} \frac{1}{(1 + k^2)^{\lambda}} F_k(\eta) e_k(x), \quad \Rightarrow \quad F_k(\tilde{\psi}) = \frac{F_k(\eta)}{(1 + k^2)^{\lambda}}, \quad k \in \mathbb{Z}_0,$$

and $F_0(\tilde{\psi}) = 0$. Moreover,

$$\|\tilde{\psi}\|^2_\lambda = \sum_{k \in \mathbb{Z}_0} (1 + k^2)^{\lambda} |F_k(\tilde{\psi})|^2 = \sum_{k \in \mathbb{Z}_0} \frac{|F_k(\eta)|^2}{(1 + k^2)^{\lambda}} = d^2(\eta) < \infty.$$

Hence $\tilde{\psi} \in \mathbb{H}_\lambda(\mathbb{T})$, and by (5.2),

$$\eta(\tilde{\psi}) = \sum_{k \in \mathbb{Z}_0} F_k(\tilde{\psi}) F_k^*(\eta) = \sum_{k \in \mathbb{Z}_0} \frac{|F_k(\eta)|^2}{(1 + k^2)^{\lambda}} = d^2(\eta) = \|\tilde{\psi}\|_\lambda d(\eta).$$

Then from the definition of $| \cdot |_\lambda$, we have

$$|\eta|_\lambda \geq \eta(\tilde{\psi} \frac{\tilde{\psi}}{\|\tilde{\psi}\|_\lambda}) = d(\eta) \quad \square$$

An immediate corollary is the following.

**Corollary 5.2.** For any $\lambda \geq 1$, $\rho_\lambda$ is a metric on $\mathcal{P}(\mathbb{T})$ with dual representations given by,

$$\rho_\lambda(\mu, \nu) = \max\{ |\mu - \nu|(\psi) : \|\psi\|_\lambda \leq 1 \} = \left( \sum_{k \in \mathbb{Z}} \frac{1}{(1 + k^2)^{\lambda}} |F_k(\mu - \nu)|^2 \right)^{\frac{1}{2}}.$$
Proof. The dual representation follows directly from the previous lemma. Suppose that $\rho_3(\mu, \nu) = 0$, then $F_k(\mu) = F_k(\nu)$ for every $k \in \mathbb{Z}$. As $\mu, \nu$ have the same Fourier series, we conclude that $\mu = \nu$. The fact that $\rho_3$ is a metric now follows from the dual representation. □

Lemma 5.3. For every $\mu, \nu \in \mathcal{P}(\mathbb{T})$, $W_1(\mu, \nu) \leq \sqrt{\pi^3 + 2\pi} \rho_1(\mu, \nu)$.

Proof. Recall that $W_1(\mu, \nu) = \text{sup}\{\mu - \nu(\psi) : \psi \in \text{Lip}_1(\mathbb{T})\}$, where Lip$_1(\mathbb{T})$ is the set of all Lipschitz continuous functions on $\mathbb{T}$ with a Lipschitz constant less than one. Also, for any $\psi \in \text{Lip}_1(\mathbb{T})$, $\|\psi\|^2_1 = \|\psi\|^2_{L^2(\mathbb{T})} + \|\psi'\|^2_{L^2(\mathbb{T})} \leq \pi^3 + 2\pi$. Then, by the previous lemma,

$$\rho_1(\mu, \nu) \leq \sqrt{\pi^3 + 2\pi} \rho_1(\mu, \nu),$$

for every $\psi \in \text{Lip}_1(\mathbb{T})$. By Kantorovich duality, $W_1(\mu, \nu) \leq \sqrt{\pi^3 + 2\pi} \rho_1(\mu, \nu)$. □

Lemma 5.4. Fix $\lambda \geq 1$, $\nu \in \mathcal{P}(\mathbb{T})$ and set $h(\mu) := \frac{1}{\pi} \rho_3(\mu, \nu)$. Then,

$$\partial_{\mu} h(\mu)(x) = \sum_{k=1}^{\infty} \frac{2}{\sqrt{2\pi}} \frac{1}{(1 + k^2)\lambda} F_k(\mu) \cos(kx).$$

Moreover, $\|\partial_{\mu} h(\mu)\|_{L_{\lambda}} = \rho_3(\mu, \nu)$, and $\partial_{\mu} h(\mu) \in C^2(\mathbb{T})$ for every $\lambda > \frac{1}{2}$.

Proof. Fix $\mu, \nu \in \mathcal{P}(\mathbb{T})$. For each $k \in \mathbb{Z}_0$, set $a_k(\mu) := \frac{1}{\pi} |F_k(\mu - \nu)|^2$. Then, we directly calculate that $\partial_{\mu} a_k(\mu)(x) = F_k(\mu - \nu)e_k(x)$. Then,

$$\partial_{\mu} h(\mu)(x) = \sum_{k \in \mathbb{Z}_0} \frac{1}{(1 + k^2)\lambda} \partial_{\mu} a_k(\mu)(x) = \sum_{k \in \mathbb{Z}_0} \frac{1}{(1 + k^2)\lambda} F_k(\mu - \nu)e_k(x) = \sum_{k=1}^{\infty} \frac{2}{\sqrt{2\pi}} \frac{1}{(1 + k^2)\lambda} F_k(\mu - \nu) \cos(kx).$$

The above formula implies that $F_0(\partial_{\mu} h(\mu)) = 0$, and

$$F_k(\partial_{\mu} h(\mu)) = \frac{1}{(1 + k^2)\lambda} F_k(\mu - \nu), \quad k \in \mathbb{Z}_0.$$

Hence,

$$\|\partial_{\mu} h(\mu)\|_{L_{\lambda}}^2 = \sum_{k \in \mathbb{Z}_0} \frac{1}{(1 + k^2)\lambda} |F_k(\partial_{\mu} h(\mu))|^2 = \sum_{k \in \mathbb{Z}_0} \frac{1}{(1 + k^2)\lambda} |F_k(\mu - \nu)|^2 = \rho_3(\mu, \nu).$$

The final statement follows from the classical Sobolev embedding of $\mathbb{H}_{\lambda}(\mathbb{T})$ into $C^k(\mathbb{T})$ for every $\lambda > k + \frac{1}{2}$. □

6 Comparison

In this section we prove Theorem 4.3 in several steps. Recall the test functions $C_c(\mathbb{T})$ of Definition 3.4, and $\mathbb{H}_{\lambda}, \rho_3$ defined in Section 2.

Step 1 (Set-up). Let $u, w$ be as in the statement of the theorem. Towards a contraposition suppose that $\text{sup}_{(t, \omega) \in \mathbb{S}} (u - w) > 0$. We fix a sufficiently small $\delta > 0$ satisfying

$$l := \max_{(t, \omega) \in \mathbb{S}} \{(u - w)(t, \mu) - \delta(T - t)\} > 0.$$
Set \( \bar{u}(t, \mu) := u(t, \mu) - \delta(T - t) \). Then, \( \bar{u} \) is a continuous viscosity subsolution of
\[
- \partial_t \bar{u}(t, \mu) = H(\mu, \partial_t \bar{u}(t, \mu)) - \delta.
\] (6.1)

**Step 2** (Doubling the variables). For \( \epsilon > 0 \), set
\[
\Phi_\epsilon(t, \mu, s, \nu) := \bar{u}(t, \mu) - w(s, \nu) - \frac{1}{2\epsilon} \left( \rho_3^2(\mu, \nu) + (t - s)^2 \right).
\]
As \( \overline{\mathcal{O}} \) is compact and \( \bar{u}, w \) are continuous, there exists \( (t_*, s_*, \mu_*, \nu_*) \in \overline{\mathcal{O}} \times \overline{\mathcal{O}} \) satisfying
\[
\Phi_\epsilon(t_*, \mu_*, s_*, \nu_*) = \max_{\overline{\mathcal{O}} \times \overline{\mathcal{O}}} \Phi_\epsilon \geq \lambda > 0.
\]
Set \( M := \max \bar{u}, m := \min \nu, \zeta := \rho_3^2(\mu_*, \nu_*) + (t_* - s_*)^2 \), so that
\[
0 \leq \zeta \leq 2 \epsilon (M + m - \lambda).
\] (6.2)

**Step 3** (Letting \( \epsilon \) to zero). Since \( \overline{\mathcal{O}} \) is compact, there is a subsequence \( \{ (t_\epsilon, s_\epsilon, \mu_\epsilon, \nu_\epsilon) \} \subset \overline{\mathcal{O}} \times \overline{\mathcal{O}} \), denoted by \( \epsilon \) again, and \( (t^*, \mu^*, s^*, \nu^*) \in \overline{\mathcal{O}} \times \overline{\mathcal{O}} \), such that
\[
\mu_\epsilon \rightharpoonup \mu^*, \quad \nu_\epsilon \rightharpoonup \nu^*, \quad t_\epsilon \to t^*, \quad s_\epsilon \to s^*, \quad \text{as} \ \epsilon \downarrow 0.
\]
By (6.2) it is clear that \( t^* = s^* \), and \( \rho_3(\mu^*, \nu^*) = 0 \). Then, by Lemma 5.3, \( \mu^* = \nu^* \).

If \( t_* \) were to be equal to \( T \), by the terminal condition (3.5), we would have
\[
0 < l \leq \liminf_{\epsilon \downarrow 0} \Phi_\epsilon(t_\epsilon, \mu_\epsilon, s_\epsilon, \nu_\epsilon) \leq \liminf_{\epsilon \downarrow 0} [\bar{u}(t_\epsilon, \mu_\epsilon) - w(s_\epsilon, \nu_\epsilon)] = \bar{u}(T, \mu^*) - w(T, \mu^*) \leq 0.
\]
Hence, \( t_* < T \) and \( t_\epsilon, s_\epsilon < T \) for all sufficiently small \( \epsilon > 0 \).

**Step 4** (Distance estimate). Without loss of generality, suppose that \( w \) is Lipschitz, i.e,
\[
|w(t, \mu) - w(t, \nu)| \leq \frac{1}{2} L_w \rho_3(\mu, \nu), \quad \mu, \nu \in P(T), \ t \in [0, T].
\]
Then, for each \( \epsilon > 0 \),
\[
\bar{u}(t_\epsilon, \mu_\epsilon) - w(s_\epsilon, \nu_\epsilon) - \frac{1}{2\epsilon} \zeta = \Phi_\epsilon(t_\epsilon, \mu_\epsilon, s_\epsilon, \nu_\epsilon) \geq \Phi_\epsilon(t_*, \mu_*, s_*, \nu_*)
= \bar{u}(t_*, \mu_*) - w(s_*, \nu_*) - \frac{1}{2\epsilon} (t_* - s_*)^2.
\]
Therefore, \( \rho_3^2(\mu_*, \nu_*) = \zeta - (t_* - s_*)^2 \leq 2\epsilon |w(s_*, \mu_*) - w(s_*, \nu_*)| \leq 2\epsilon L_w \rho_3(\mu_*, \nu_*) \). Hence,
\[
\rho_3(\mu_*, \nu_*) \leq \epsilon L_w \quad \forall \epsilon > 0.
\] (6.3)

**Step 5** (Viscosity property). Set
\[
\psi_\epsilon(t, \mu) := \frac{1}{2\epsilon} [\rho_3^2(\mu, \nu_*) + (t - s_*)^2], \quad \phi_\epsilon(s, \nu) := -\frac{1}{2\epsilon} [\rho_3^2(\mu_*, \nu) + (t_* - s_*)^2].
\]
By Lemma 5.4 with \( \lambda = 3 \), both \( \partial_\mu \psi_\epsilon(t, \mu), \partial_\nu \phi_\epsilon(t, \mu) \in C^2(T) \). Moreover, by the regularity Assumption 3.1, maps \( (t, \mu) \mapsto H(\mu, \partial_\mu \psi_\epsilon(t, \mu)) \), and \( (t, \nu) \mapsto H(\nu, \partial_\nu \phi_\epsilon(t, \nu)) \) are continuous. Hence, \( \psi_\epsilon \) and \( \phi_\epsilon \) are smooth test functions. Set
\[
\kappa_\epsilon(x) := \partial_\nu \psi_\epsilon(t_\epsilon, \mu_\epsilon)(x) = \partial_\nu \phi_\epsilon(s_\epsilon, \nu_\epsilon)(x)
= \frac{1}{\epsilon} \sum_{k=1}^{\infty} \frac{2}{\sqrt{2\pi} (1 + k^2)^3} F_k(\mu_\epsilon - \nu_\epsilon) \cos(kx), \quad x \in T.
\]
Also, \( \bar{u}(t, \mu) - \psi_r(t, \mu) \) is maximized at \( t_r, \mu_r \). Since \( t_r < T, \psi_r \in \mathcal{C}_\epsilon(\overline{\mathcal{T}}) \) and \( \bar{u} \) is a viscosity subsolution of (6.1), then
\[
-\frac{t_r - s_*}{\epsilon} \leq H(\mu_r, \kappa_r) - \delta.
\]
By the viscosity property of \( w \), a similar argument implies that
\[
-\frac{t_r - s_*}{\epsilon} \geq H(\nu_r, \kappa_r).
\]
We subtract the above inequalities to arrive at
\[
0 < \delta \leq H(\mu_r, \kappa_r) - H(\nu_r, \kappa_r).
\] (6.4)

Step 6 (Estimation). Since \( H(\mu, \kappa_r) = \inf_{\alpha \in \mathcal{C}_\alpha} \{ \mu(\ell^\alpha(\cdot, \mu) + \mathcal{M}^{\alpha, \mu}[\kappa_r](\cdot)) \} \),
\[
|H(\mu_r, \kappa_r) - H(\nu_r, \kappa_r)| \leq \sup_{\alpha \in \mathcal{C}_\alpha} T^\alpha_r + \sup_{\alpha \in \mathcal{C}_\alpha} J^\alpha_r,
\]
where
\[
T^\alpha_r := |\mu(\ell^\alpha(\cdot, \mu_r)) - \nu_r(\ell^\alpha(\cdot, \nu_r))|, \\
J^\alpha_r := |\mu_r - \nu_r(\mathcal{M}^{\alpha, \mu_r}[\kappa_r](\cdot))|.
\]

Step 7 (Estimating \( T^\alpha_r \)). By the regularity Assumption 3.1 and the estimate (6.3),
\[
|\mu(\ell^\alpha(\cdot, \mu_r)) - \nu_r(\ell^\alpha(\cdot, \nu_r))| \leq |(\mu_r - \nu_r)(\ell^\alpha(\cdot, \mu_r))| + |\nu_r(\ell^\alpha(\cdot, \mu_r) - \ell^\alpha(\cdot, \nu_r))| \\
\leq \rho_3(\mu_r, \nu_r) \|\ell^\alpha(\cdot, \mu_r)\|_{\mathcal{C}_3} + \sup_{x \in \mathcal{T}} |\ell^\alpha(x, \mu_r) - \ell^\alpha(x, \nu_r)| \\
\leq 2\epsilon \rho_3(\mu_r, \nu_r) \leq 2\epsilon \rho_1(\mu_r, \nu_r).
\]

Hence, we have \( \lim_{\epsilon \downarrow 0} \sup_{\alpha \in \mathcal{C}_\alpha} T^\alpha_r = 0 \).

Step 8 (Estimating \( \mathcal{I} \)). For \( x \in \mathbb{T}, \mu, \nu \in \mathcal{P}(\mathbb{T}) \), and \( \alpha \in \mathcal{C}_\alpha \), set
\[
\beta^\alpha_r(x, \mu) := \mathcal{M}^{\alpha, \mu}[\cos(k \cdot)](x) = -[kk^\alpha(x, \mu) \sin(kx) + \frac{k^2}{2}(\sigma^\alpha(x, \mu))^2 \cos(kx)].
\]

Then,
\[
\mathcal{M}^{\alpha, \mu_r}[\kappa_r](x) = \frac{2}{\epsilon \sqrt{2\pi}} \sum_{k=1}^{\infty} \frac{1}{(1 + k^2)^{3/2}} F_k(\mu_r - \nu_r) \beta^\alpha_k(x, \mu_r).
\]

This in turn implies that
\[
\mathcal{I}^\alpha_r \leq \frac{2}{\epsilon \sqrt{2\pi}} \sum_{k=1}^{\infty} \frac{1}{(1 + k^2)^{3/2}} |F_k(\mu_r - \nu_r)|||\mu_r - \nu_r)\beta^\alpha_k(\cdot, \mu_r)|| \\
\leq \frac{2}{\epsilon \sqrt{2\pi}} \frac{1}{\rho_3(\mu_r, \nu_r)} \sum_{k=1}^{\infty} \frac{2}{(1 + k^2)^{3/2}} \left( \sum_{k=1}^{\infty} \frac{((\mu_r - \nu_r)\beta^\alpha_k(\cdot, \mu_r))}{(1 + k^2)^{3/2}} \right)^{1/2} \\
\leq \frac{2}{\epsilon \sqrt{2\pi}} \frac{\rho_3(\mu_r, \nu_r)}{\rho_3(\mu_r, \nu_r)} \sum_{k=1}^{\infty} \frac{2}{(1 + k^2)^{3/2}} \left( \sum_{k=1}^{\infty} \frac{\beta^\alpha_k(\cdot, \mu_r)}{1 + k^2} \right)^{1/2},
\]
where
\[
\beta^\alpha_{k, \epsilon} := (1 + k^2)^{-1} \sup_{\alpha \in \mathcal{C}_\alpha} |(\mu_r - \nu_r)(\beta^\alpha_k(\cdot, \mu_r))|, \quad k = 1, 2, \ldots
\]
Again by Assumption 3.1, $|\beta_{k,\epsilon}| \leq c_\epsilon + c_\epsilon^2$, and $\beta_{k,\epsilon}$ is Lipschitz continuous with a Lipschitz constant $c_\epsilon$. Hence, by Kantorovich duality $\beta_{k,\epsilon} \leq c_k W_1(\mu_\epsilon, \nu_\epsilon)$. As $\mu_\epsilon - \nu_\epsilon$ converges weakly to zero, we conclude that $\beta_{k,\epsilon}$ also converges to zero for every $k$. Since $\sum_{k=1}^{\infty} (1+k^2)^{-1}$ is finite and $|\beta_{k,\epsilon}|$ is uniformly bounded, we conclude by dominated convergence that the sequence $\sum_{k=1}^{\infty} (1+k^2)^{-1} \beta_{k,\epsilon}^2$ converges to zero as $\epsilon \downarrow 0$. Hence, by (6.3),

$$\lim_{\epsilon \downarrow 0} \sup_{\alpha \in \mathcal{C}_\alpha} T_\alpha^\epsilon \leq \lim_{\epsilon \downarrow 0} \frac{2L_\epsilon}{\sqrt{2\pi}} \left( \sum_{k=1}^{\infty} \frac{\beta_{k,\epsilon}^2}{1+k^2} \right)^\frac{1}{2} = 0.$$

**Step 9 (Estimating $J_\epsilon^\alpha$).** The definition of $J_\epsilon^\alpha$ and the regularity Assumption 3.1 imply that

$$J_\epsilon^\alpha \leq \sup_{\alpha \in \mathcal{A}} \{ |M^{\alpha,\mu_\epsilon}[\kappa_\epsilon](x) - M^{\alpha,\nu_\epsilon}[\kappa_\epsilon](x)| \}.$$

For $\alpha \in \mathcal{C}_\alpha$, $x \in \mathbb{T}$, and $k \in \mathbb{Z}_0$, set

$$\gamma_{k,\epsilon}^\alpha(x) := M^{\alpha,\mu_\epsilon}[\cos(k \cdot)](x) - M^{\alpha,\nu_\epsilon}[\cos(k \cdot)](x)$$

$$= -k(\hat{b}^\alpha(x,\mu_\epsilon) - \hat{b}^\alpha(x,\nu_\epsilon)) \sin(kx) - \frac{k^2}{2}(\sigma^\alpha(x,\mu_\epsilon))^2 - (\sigma^\alpha(x,\nu_\epsilon))^2 \cos(kx)$$

$$\leq 2(k + k^2)(\|\hat{b}^\alpha\|_\infty + \|\sigma^\alpha\|_\infty).$$

By the regularity Assumption 3.1, there exists $c_2$ such that

$$\sup_{x \in \mathbb{T}} |\gamma_{k,\epsilon}^\alpha(x)| \leq \frac{1}{2} c_2 (k + k^2) \hat{\rho}_3(\mu_\epsilon, \nu_\epsilon) \leq c_2 (1 + k^2) \hat{\rho}_3(\mu_\epsilon, \nu_\epsilon), \quad \forall \alpha \in \mathcal{C}_\alpha, \ k \in \mathbb{Z}_0.$$

Hence, for every $\alpha \in \mathcal{A}$,

$$J_\epsilon^\alpha \leq \sup_{\alpha \in \mathcal{A}} \{ |M^{\alpha,\mu_\epsilon}[\kappa_\epsilon](x) - M^{\alpha,\nu_\epsilon}[\kappa_\epsilon](x)| \}$$

$$\leq \frac{2}{\epsilon \sqrt{2\pi}} \sum_{k=1}^{\infty} \frac{|F_k(\mu_\epsilon - \nu_\epsilon)|}{(1+k^2)^{1/2}} \sup_{\alpha \in \mathcal{A}} |\gamma_{k,\epsilon}^\alpha(x)|$$

$$\leq \frac{2c_2}{\epsilon \sqrt{2\pi}} \frac{1}{\sqrt[1/2]{\epsilon}} \left( \sum_{k=1}^{\infty} \frac{|F_k(\mu_\epsilon - \nu_\epsilon)|^2}{(1+k^2)^{3/2}} \right)^{1/2} \left( \sum_{k=1}^{\infty} \frac{1}{1+k^2} \right)^{1/2} \hat{\rho}_3(\mu_\epsilon, \nu_\epsilon)$$

$$= \frac{2c_2}{\epsilon \sqrt{2\pi}} \frac{\rho_3(\mu_\epsilon, \nu_\epsilon)}{\epsilon} \left( \sum_{k=1}^{\infty} \frac{1}{1+k^2} \right)^{1/2} \hat{\rho}_3(\mu_\epsilon, \nu_\epsilon)$$

$$\leq \frac{2c_2 L_\epsilon}{\epsilon \sqrt{2\pi}} \left( \sum_{k=1}^{\infty} \frac{1}{1+k^2} \right)^{1/2} \hat{\rho}_3(\mu_\epsilon, \nu_\epsilon)$$

$$= \hat{c} \hat{\rho}_3(\mu_\epsilon, \nu_\epsilon).$$

Therefore, $\sup_{\alpha \in \mathcal{C}_\alpha} J_\epsilon^\alpha \leq \hat{c} \hat{\rho}_3(\mu_\epsilon, \nu_\epsilon)$. As $\hat{\rho}_3(\mu_\epsilon, \nu_\epsilon) \leq \rho_3(\mu_\epsilon, \nu_\epsilon)$, we have

$$\lim_{\epsilon \downarrow 0} \sup_{\alpha \in \mathcal{C}_\alpha} J_\epsilon^\alpha = 0.$$

**Step 10 (Conclusion).** By (6.4) and Steps 7, 8 and 9,

$$0 < \delta \leq \lim_{\epsilon \downarrow 0} \sup_{\alpha \in \mathcal{C}_\alpha} |H(\mu_\epsilon, \kappa_\epsilon) - H(\nu_\epsilon, \kappa_\epsilon)| \leq 0.$$

This clear contradiction implies that $\max_{\mathbb{T}} (u - v) \leq 0$. 

\[ \square \]
7 Lipschitz continuity

In this section, we prove Theorem 4.1.

7.1 Regularity in space

We first prove the continuous dependence of the solutions of the McKean-Vlasov stochastic differential equation (3.1) on its initial data.

**Proposition 7.1.** Suppose that the regularity Assumption 3.1 holds. Then, there exists \( c > 0 \) depending on \( T \) and the constant \( c_0 \) in Assumption 3.1, such that

\[
\hat{\rho}_3(L^e_{u,\mu}, L^e_{u,\nu}) \leq c \hat{\rho}_3(\mu, \nu), \quad \forall 0 \leq t \leq T, \; \mu, \nu \in \mathcal{P}(\mathbb{T}), \; \alpha \in \mathcal{A}.
\]

**Proof.** We complete the proof in several steps.

**Step 1 (Setting).** We fix \( t \in [0, T] \), \( \mu, \nu \in \mathcal{P}(\mathbb{T}) \), \( \alpha \in \mathcal{A} \), and set

\[
Y_u := X^e_{t,\mu,\alpha}, \quad \mu_u := L^e_{t,\mu,\alpha}, \quad Z_u := X^e_{t,\nu,\alpha}, \quad \nu_u := L^e_{t,\nu,\alpha}, \quad u \in [t, T].
\]

By the definition of \( \hat{\rho}_3 \), we need to prove the following estimate for every \( u \in [t, T] \),

\[
(\mu_u - \nu_u)(\psi) \leq c \hat{\rho}_3(\mu, \nu) \|\psi\|_{C^3}, \quad \forall \psi \in C^3(\mathbb{T}).
\]

**Step 2 (SDEs).** For \( x \in \mathbb{T} \), let \( Y^x \), \( Z^x \) be the solutions of the stochastic differential equations,

\[
Y_u^x = x + \int_t^u [b^{\alpha}(Y_u^x, \mu_u) ds + \sigma^{\alpha}(Y_u^x, \mu_u) dB_s],
\]

\[
Z_u^x = x + \int_t^u [b^{\alpha}(Z_u^x, \nu_u) ds + \sigma^{\alpha}(Z_u^x, \nu_u) dB_s].
\]

Set \( L_u^x(\cdot) := \mathbb{E}[\psi(Y_u^x)] \), and \( L_u^x(\cdot) := \mathbb{E}[\psi(Z_u^x)] \). Then, by conditioning we have

\[
\mu_u(\psi) = \mathbb{E}[\psi(Y_u^x)] = \mu(L_u^x), \quad \nu_u(\psi) = \mathbb{E}[\psi(Z_u^x)] = \nu(L_u^x).
\]

Therefore,

\[
(\mu_u - \nu_u)(\psi) \leq (\mu - \nu)(L_u^x) + \nu(L_u^x - L_u^x) := I_u(\psi) + J_u(\psi).
\]

**Step 3 (I_u estimate).** By the regularity Assumption 3.1, there exists a constant \( c_1 \) satisfying

\[
\|b^{\alpha}(\cdot, \mu_u)\|_{C^3} + \|\sigma^{\alpha}(\cdot, \mu_u)\|_{C^3} \leq c_1, \quad \forall u \in [t, T].
\]

Hence, the map \( x \in \mathbb{T} \to Y_u^x \) is three times differentiable. Therefore, \( L_u^x \in C^3(\mathbb{T}) \) and there exists a constant \( c_2 > 0 \) depending only on \( c_0 \) of Assumption 3.1, satisfying

\[
\|L_u^x\|_{C^3} \leq c_2 \|\psi\|_{C^3}, \quad \forall u \in [t, T], \mu \in \mathcal{P}(\mathbb{T}).
\]

This implies that

\[
I_u(\psi) = (\mu - \nu)(L_u^x) \leq c_2 \hat{\rho}_3(\mu, \nu) \|\psi\|_{C^3}.
\]

**Step 4 (J_u estimate).** Fix \( x \in \mathbb{T} \), and set

\[
m_s := \mathbb{E}[(Y_u^x - Z_u^x)^2], \quad n_s := \hat{\rho}_3(\mu_s, \nu_s)^2, \quad s \in [t, T].
\]
We directly estimate that

\[ m_u \leq 2 \left( \int_t^u E[|b^{\alpha_u}(Y^u_x, \mu_u) - b^{\alpha_u}(Z^u_x, \nu_u)|^2] \, ds \right)^{1/2} + 2 \int_t^u E[(\sigma^{\alpha_u}(Y^u_x, \mu_u) - \sigma^{\alpha_u}(Z^u_x, \nu_u))^2] \, ds. \]

By the regularity Assumption 3.1,

\[ |b^{\alpha_u}(Y^u_x, \mu_u) - b^{\alpha_u}(Z^u_x, \nu_u)| \leq c_n \left( |Y^u_x - Z^u_x| + \hat{\rho}_3(\mu_u, \nu_u) \right) \]

Same estimate also holds for \( |\sigma^{\alpha_u}(Y^u_x, \mu_u) - \sigma^{\alpha_u}(Z^u_x, \nu_u)| \). Hence there exists \( \hat{c}_4 > 0 \) satisfying,

\[ m_u \leq \hat{c}_3 \int_t^u (m_s + n_s) \, ds, \quad \forall u \in [t, T]. \]

By Grönwall’s inequality, there exists \( \hat{c}_4 > 0 \) satisfying \( m_u \leq \hat{c}_4 \int_t^u n_s \, ds \) for all \( u \in [t, T] \).

Since \( E[|Y^u_x - Z^u_x|^2] \leq \sqrt{m_u} \),

\[ \mathcal{J}_u \leq \|\psi\|_{\mathcal{C}^1} E[|Y^u_x - Z^u_x|] \leq \hat{c}_4 \left( \int_t^u n_s \, ds \right)^{1/2} \|\psi\|_{\mathcal{C}^1}, \quad \forall u \in [t, T]. \]

**Step 5 (Conclusion).** By the previous steps,

\[ (\mu_u - \nu_u)(\psi) \leq \left( \hat{c}_2 \hat{\rho}_3(\mu, \nu) + \hat{c}_4 \left( \int_t^u n_s \, ds \right)^{1/2} \right) \|\psi\|_{\mathcal{C}^1}, \quad \forall \psi \in \mathcal{C}^1([T]). \]

The definitions of \( \hat{\rho}_3 \) and \( n_u \) imply that

\[ \sqrt{n_u} = \hat{\rho}_3(\mu_u, \nu_u) \leq \hat{c}_2 \hat{\rho}_3(\mu, \nu) + \hat{c}_4 \left( \int_t^u n_s \, ds \right)^{1/2}, \quad \forall u \in [t, T]. \]

Hence,

\[ n_u \leq 2\hat{c}_2^2 \hat{\rho}_3(\mu, \nu)^2 + 2\hat{c}_4^2 \int_t^u n_s \, ds, \quad \forall u \in [t, T]. \]

Again by Grönwall, there exists \( \hat{c} > 0 \) such that

\[ (\hat{\rho}_3(\mu_u, \nu_u))^2 = n_u \leq \hat{c}^2 (\hat{\rho}_3(\mu, \nu))^2, \quad \forall u \in [t, T]. \]

The following is an immediate consequence of the above estimate.

**Lemma 7.2.** Under the regularity Assumption 3.1, there exists \( L_1 > 0 \) such that

\[ |J(t, \mu, \alpha) - J(t, \nu, \alpha)| \leq L_1 \hat{\rho}_3(\mu, \nu), \quad \forall \alpha \in A, \mu, \nu \in \mathcal{P}(\mathbb{T}), t \in [0, T]. \]

Consequently,

\[ |v(t, \mu) - v(t, \nu)| \leq L_1 \hat{\rho}_3(\mu, \nu), \quad \forall \mu, \nu \in \mathcal{P}(\mathbb{T}), t \in [0, T]. \]

**Proof.** We fix \( \alpha \in A, \mu, \nu \in \mathcal{P}(\mathbb{T}), t \in [t, T] \), and use the same notation as in Proposition 7.1. For \( u \in [t, T] \), the regularity Assumption 3.1 implies that

\[ |E[\ell^{\alpha_u}(Y_u, \mu_u) - \ell^{\alpha_u}(Z_u, \nu_u)]| \]

\[ \leq |E[\ell^{\alpha_u}(Y_u, \mu_u) - \ell^{\alpha_u}(Z_u, \mu_u)]| + |E[\ell^{\alpha_u}(Z_u, \mu_u) - \ell^{\alpha_u}(Z_u, \nu_u)]| \]

\[ = |(\mu_u - \nu_u)(\ell^{\alpha_u}(\cdot, \mu_u))| + c_n \hat{\rho}_3(\mu_u, \nu_u) \]

\[ \leq \hat{\rho}_3(\mu_u, \nu_u)|\ell^{\alpha_u}(\cdot, \mu_u)|_{C^1} + c_n \hat{\rho}_3(\mu_u, \nu_u) \]

\[ \leq 2c_n \hat{c}_3 \hat{\rho}_3(\mu, \nu). \]
We now directly estimate using the above estimate to obtain the following inequalities,

\[ |J(t, \mu, \alpha) - J(t, \nu, \alpha)| \leq \int_t^T |E[\ell^{\alpha u}(Y_u, \mu_u) - \ell^{\alpha u}(Z_u, \nu_u)]| \, du + |E[\varphi(\mu_T) - \varphi(\nu_T)]| \]

\[ \leq 2c_n \hat{c}(T - t) \tilde{p}_3(\mu, \nu) + c_n \tilde{p}_3(\mu, \nu_T) \]

\[ \leq c_n \hat{c}(2(T - t) + 1)\tilde{p}_3(\mu, \nu). \]

As \(|v(t, \mu) - v(t, \nu)| \leq \sup_{\alpha \in A} |J(t, \mu, \alpha) - J(t, \nu, \alpha)|\), the proof of the lemma is complete.

\[ \square \]

**7.2 Time Regularity**

**Proposition 7.3.** Suppose that the regularity Assumption 3.1 holds. Then there exists \(L_2 > 0\) depending on \(T\) and the constant \(c_n\) in Assumption 3.1, such that

\[ |v(t, \mu) - v(\tau, \mu)| \leq L_2 |t - \tau|^\frac{3}{2}, \quad \forall \ t, \tau \in [0, T], \ \mu \in \mathcal{P}(\mathbb{T}). \]

**Proof.** Fix \(0 \leq t \leq \tau \leq T, \ \mu \in \mathcal{P}(\mathbb{T}), \ \alpha \in A,\) and set \(h := \tau - t.\) With an arbitrary constant \(\alpha_* \in A,\) we define

\[ \tilde{\alpha}_u(\cdot) := \begin{cases} \alpha_{u+h}(\cdot) & \text{if } u \in [t, T-h], \\ \alpha_* & \text{if } u \in [T-h, T]. \end{cases} \]

It is clear that \(\tilde{\alpha} \in A.\) Set

\[ \tilde{\mu}_u := \mathcal{L}^{\tilde{\alpha}}_{u,T-h}, \quad u \in [t, T-h], \quad \text{and} \quad \mu_\tau := \mathcal{L}^{\tau,\alpha}_u, \quad u \in [\tau, T]. \]

Then \(\tilde{\mu}_u = \mu_{u+h}\) for every \(u \in [t, T-h].\) In particular,

\[ E[\ell^{\tilde{\alpha}}(X_u^{\tilde{\alpha},\mu,\tilde{\alpha}})] = E[\ell^{\alpha}(X_{u+h}^{\alpha,\mu,\alpha})], \quad \forall u \in [t, T-h]. \]

Since \(\mu_T = \tilde{\mu}_{T-h} = \mathcal{L}(X_{T-h}^{\tilde{\alpha},\mu,\tilde{\alpha}}),\) and \(\mu_T = \mathcal{L}(X_T^{\alpha,\mu,\alpha}),\)

\[ W_1(\tilde{\mu}_T, \mu_T) \leq \mathcal{E}[\|X_T^{\tilde{\alpha},\mu,\tilde{\alpha}} - X_{T-h}^{\alpha,\mu,\alpha}\|] \leq \mathcal{E}[(X_T^{\tilde{\alpha},\mu,\tilde{\alpha}} - X_{T-h}^{\alpha,\mu,\alpha})^2]^{\frac{1}{2}}. \]

As \(\tilde{\beta}, \sigma\) are bounded by \(c_n,\) there is \(\tilde{c}_1 > 0\) satisfying, \(W_1(\tilde{\mu}_T, \mu_T) \leq \tilde{c}_1 \sqrt{h}.\) Therefore,

\[ |\varphi(\tilde{\mu}_T) - \varphi(\mu_T)| \leq c_n \tilde{p}_3(\tilde{\mu}_T, \mu_T) \leq c_n W_1(\tilde{\mu}_T, \mu_T) \leq \tilde{c}_1 c_n \sqrt{h}. \]

Above estimate imply that for any \(\alpha \in A,\)

\[ v(t, \mu) - J(\tau, \mu, \alpha) \leq J(t, \mu, \tilde{\alpha}) - J(\tau, \mu, \alpha) \]

\[ = \int_{T-h}^T \mathcal{E}[\ell^{\tilde{\alpha}}(X_u^{\tilde{\alpha},\mu,\tilde{\alpha}})] \, du + \varphi(\tilde{\mu}_T) - \varphi(\mu_T) \leq c_n h + \tilde{c}_1 c_n \sqrt{h}. \]

Hence,

\[ v(t, \mu) - v(\tau, \mu) = \sup_{\alpha \in A} (v(t, \mu) - J(t, \mu, \alpha)) \leq c_n h + \tilde{c}_1 c_n \sqrt{h}. \]

We prove the opposite inequality by using the control

\[ \tilde{\alpha}_u(\cdot) := \begin{cases} \alpha_{u-h}(\cdot) & \text{if } u \in [h, T], \\ \alpha_* & \text{if } u \in [0, h]. \end{cases} \]
Again $\alpha \in A$, and we set
\[
\hat{\mu}_u := \mathcal{L}^{\infty} u, \quad u \in [\tau, T], \quad \text{and} \quad \mu_u := \mathcal{L}^L u, \quad u \in [t, T].
\]
Then $\hat{\mu}_u = \mu_{u-h}$ for every $u \in [\tau, T]$ and $\hat{\mu}_T = \mu_{T-h}$. Following the above steps mutatis
mutandis, we obtain the following inequality for any $\alpha \in A$,
\[
v(\tau, \mu) - J(t, \mu, \alpha) \leq J(\tau, \mu, \hat{\alpha}) - J(t, \mu, \alpha)
\]
\[
= -\int_t^\tau \mathbb{E}[\hat{\alpha}_u (X_u^{\infty}, L_u^L)] \, du + \varphi(\hat{\mu}_T) - \varphi(\mu_T) \leq c_a h + \frac{\bar{c}_1}{16} c_a \sqrt{h}.
\]
Hence,
\[
v(\tau, \mu) - v(t, \mu) = \sup_{\alpha \in A} (v(\tau, \mu) - J(t, \mu, \alpha)) \leq c_a h + \frac{\bar{c}_1}{16} c_a \sqrt{h}.
\]

8 Dynamic Programming

In this section we prove Theorem 3.3. For a general result but in a different setting, we refer
the reader to [17].

Proof of Theorem 3.3. We fix $(t, \mu) \in \mathcal{O}$, $\tau \in [t, T]$, and set
\[
Q(\alpha) := \int_t^\tau \mathbb{E}[\hat{\alpha}_u (X_u^{\infty}, L_u^L)] \, du + v(\tau, L_u^L), \quad \alpha \in A.
\]
Then, the dynamic programming principle can be stated as $v(t, \mu) = \inf_{\alpha \in A} Q(\alpha)$. Recall that $v(t, \mu) = \inf_{\alpha \in A} J(t, \mu, \alpha)$. For any $\alpha \in A$, and $s \in [\tau, T]$, Markov property implies
that $X_s^{\tau, \mu, \alpha} = X_{s-}^{\tau, \mu, \alpha}$, and consequently $L_s^L = L_{s-}^L$. Hence,
\[
\int_\tau^T \mathbb{E}[\hat{\alpha}(X_{s-}^{\tau, \mu, \alpha}, L_{s-}^L)] \, ds + \varphi(L_T^L)
\]
\[
= \int_\tau^T \mathbb{E}[\hat{\alpha}(X_{s-}^{\tau, \mu, \alpha}, L_{s-}^L)] \, ds + \varphi(L_T^L)
\]
\[
= J(\tau, L_T^L), \quad \alpha \geq v(\tau, L_T^L).
\]
This implies that
\[
J(t, \mu, \alpha) = \int_t^\tau \mathbb{E}[\hat{\alpha}(X_{s-}^{\tau, \mu, \alpha}, L_{s-}^L)] \, ds + \left(\int_\tau^T \mathbb{E}[\hat{\alpha}(X_{s-}^{\tau, \mu, \alpha}, L_{s-}^L)] \, ds + \varphi(L_T^L)\right)
\]
\[
\geq \int_t^\tau \mathbb{E}[\hat{\alpha}(X_{s-}^{\tau, \mu, \alpha}, L_{s-}^L)] \, ds + v(\tau, L_{s-}^L) = Q(\alpha).
\]
Therefore, $v(t, \mu) = \inf_{\alpha \in A} J(t, \mu, \alpha) \geq \inf_{\alpha \in A} Q(\alpha)$.

To prove the opposite inequality, we fix $\varepsilon > 0$, and set $\delta := \varepsilon/(4 L_1)$. By Lemma 7.2,
whenever $\hat{\rho}_3(\nu, \eta) \leq \delta$, we have $|J(\tau, \nu, \alpha) - J(\tau, \nu, \alpha)| \leq \varepsilon/4$, for every $\alpha \in A$, and also $|v(\tau, \nu) - v(\tau, \eta)| \leq \varepsilon/4$. Consider a covering of $\mathcal{P}(\mathcal{O})$ given by
\[
\mathcal{B}(\nu) := \{ \eta \in \mathcal{P}(\mathcal{O}) : \hat{\rho}_3(\nu, \eta) < \delta \}, \quad \nu \in \mathcal{P}(\mathcal{O}).
\]
It is clear that each $\mathcal{B}(\nu)$ is an open set as $\tilde{\rho}_1$ is continuous with respect to the weak* topology. Then, since $\mathcal{P}(\mathbb{T})$ is weak* compact, there exits $\{\nu_j\}_{j=1,\ldots,n} \subset \mathcal{P}(\mathbb{T})$ such that $\mathcal{P}(\mathbb{T}) = \bigcup_{j=1}^n \mathcal{B}(\nu_j)$. Set $\mathcal{B}_1 := \mathcal{B}(\nu_1)$, and and recursively define

$$
\mathcal{B}_{j+1} := \mathcal{B}(\nu_{j+1}) \setminus \bigcup_{i=1}^j \mathcal{B}_i,
$$

so that $\{\mathcal{B}_j\}_{j=1,\ldots,n}$ forms a disjoint covering of $\mathcal{P}(\mathbb{T})$. Moreover, for any $\nu \in \mathcal{B}_j \subset \mathcal{B}(\nu_j)$, $\tilde{\rho}_1(\nu, \nu_j) \leq \delta$, and therefore,

$$
|v(\tau, \nu) - v(\tau, \nu_j)| \leq \frac{\varepsilon}{4},
$$

and $|J(\tau, \nu, \alpha) - J(\tau, \nu_j, \alpha)| \leq \frac{\varepsilon}{4}$ for all $\alpha \in \mathcal{A}$.

For each $j$, choose $\alpha^j$ so that $J(\tau, \nu_j, \alpha^j) \leq v(\tau, \nu_j) + \frac{\varepsilon}{4}$. Then,

$$
J(\tau, \nu, \alpha^j) \leq J(\tau, \nu_j, \alpha^j) + \frac{\varepsilon}{4} \leq v(\tau, \nu_j) + \frac{\varepsilon}{2} \leq v(\tau, \nu) + \frac{3\varepsilon}{4},
$$

for all $\nu \in \mathcal{B}_j$. (8.1)

We choose $\alpha^* \in \mathcal{A}$ satisfying $Q(\alpha^*) \leq \inf_{\alpha \in \mathcal{A}} Q(\alpha) + \frac{\varepsilon}{4}$, and define a control process $\alpha^*$ by,

$$
\alpha^*_n(x) = \begin{cases} 
\alpha^*_n(x), & \text{if } u \in [t, \tau), \\
\sum_{j=1}^n \alpha^*_n(x) \chi_{B_j}(L^\mu_{t,\alpha^*}), & \text{if } u \in [\tau, T], 
\end{cases}
$$

As $\alpha^*$ and $\alpha^j$ agree on $[t, \tau)$, we have $L^\mu_{t,\alpha^*} = L^\mu_{t,\alpha^j}$ for all $u \in [t, \tau]$. Hence,

$$
\inf_{\alpha \in \mathcal{A}} Q(\alpha) + \frac{\varepsilon}{4} \geq Q(\alpha^*) = \int_t^\tau E[\rho^\mu(\gamma_{t,\alpha^*}(\gamma_x, L^\mu_{x,\alpha^*}))] ds + v(\tau, L^\mu_{t,\alpha^*}) = Q(\alpha^*).
$$

Moreover, by the definition of $\alpha^*$ and (8.1),

$$
v(\tau, L^\mu_{t,\alpha^*}) = \sum_{j=1}^n v(\tau, L^\mu_{t,\alpha^*}) \chi_{B_j}(L^\mu_{t,\alpha^*}) \geq \sum_{j=1}^n J(\tau, L^\mu_{t,\alpha^*}, \alpha^j) \chi_{B_j}(L^\mu_{t,\alpha^*}) - \frac{3\varepsilon}{4} = J(\tau, L^\mu_{t,\alpha^*}, \alpha^*) - \frac{3\varepsilon}{4}.
$$

Hence,

$$
\inf_{\alpha \in \mathcal{A}} Q(\alpha) + \varepsilon \geq Q(\alpha^*) + \frac{3\varepsilon}{4} = \int_t^\tau E[\rho^\mu(\gamma_{t,\alpha^*}(\gamma_x, L^\mu_{x,\alpha^*}))] ds + \left(v(\tau, L^\mu_{t,\alpha^*}) + \frac{3\varepsilon}{4}\right) 
$$

$$
\geq \int_t^\tau E[\rho^\mu(\gamma_{t,\alpha^*}(\gamma_x, L^\mu_{x,\alpha^*}))] ds + J(\tau, L^\mu_{t,\alpha^*}, \alpha^*) 
$$

$$
= J(t, \mu, \alpha^*) \geq v(t, \mu).
$$

\[\square\]

9 Viscosity property

In this section, we prove the viscosity property of the value function. Although the below proof follows the standard one very closely, we provide it for completeness.

The following version of the Itô’s formula along flows of measures follows from Proposition 5.102 of [7]. Recall that $X^t,\mu,\alpha$ is the solution of (3.1), $L^\mu_{t,\alpha} = L(X^\mu_{t,\alpha})$, and the operator $\mathcal{M}^\mu,\alpha$ is defined in subsection 3.3.

Lemma 9.1. For every $\psi \in \mathcal{C}(\mathbb{T})$, $(t, \mu) \in \mathbb{T}$, $u \in [t, T]$, and $\alpha \in \mathcal{A}$,

$$
\psi(u, L^\mu_{t,\alpha}) = \psi(t, \mu) + \int_t^u \left(\partial_t \psi(s, L^\mu_{s,\alpha}) + \mathbb{E}[\partial_x \psi(s, L^\mu_{s,\alpha})(X^\mu_{s,\alpha})]ight) ds.
$$
9.1 Subsolution

Suppose that for \((t_0, \mu_0) \in [0, T) \times \mathcal{P}(T)\) and test function \(\psi \in \mathcal{C}_a(O)\),
\[
0 = (v - \psi)(t_0, \mu_0) = \max_{\mathcal{O}}(v - \psi).
\]

For \(\alpha \in \mathcal{C}_a\), set
\[
k^\alpha(t, x, \mu) := \ell(x, \mu, \alpha(x)) + \mathcal{M}^{\alpha, \mu}[\partial_\mu \psi(t, \mu)](x), \quad t \in [0, T], \ x \in \mathbb{R}, \ \mu \in \mathcal{P}(\mathbb{R}).
\]

As \(H(\mu_0, \partial_\mu \psi(t_0, \mu_0)) = \inf_{\alpha \in \mathcal{C}_a} \mu_0(k^\alpha(t_0, \cdot, \mu_0))\), for any \(\epsilon > 0\) there is \(\alpha^* \in \mathcal{C}_a\) satisfying,
\[
\mu_0(k^\alpha(t_0, \cdot, \mu_0)) \leq H(\mu_0, \partial_\mu \psi(t_0, \mu_0)) + \epsilon.
\]

Set \(\alpha^*_0 \equiv \alpha^*\) and let \(X^*_0 := X^*_{t_0, \mu_0, \alpha^*}\) and \(\mu^*_0 := \mathcal{L}^0_{\mu_0, \alpha^*}\) for \(u \in [t_0, T]\). Since \(v \leq \psi\),
dynamic programming principle. Theorem 3.3 with \(\tau = t_0 + h \leq T\) implies that
\[
v(t_0, \mu_0) \leq \int_{t_0}^{t_0+h} \mathbb{E}[\ell(X^*_0, \mu^*_0, \alpha^*(X^*_0))] ds + \psi(t_0 + h, \mu^*_0).
\]

By Lemma 9.1,
\[
\psi(t_0 + h, \mu^*_0) = \psi(t_0, \mu_0) + \int_{t_0}^{t_0+h} \left( \partial_\mu \psi(s, \mu^*_0) + \mathbb{E}[\mathcal{M}^{\alpha^*, \mu^*_0}[\partial_\mu \psi(s, \mu^*_0)](X^*_0)] \right) ds.
\]

Since \(\psi(t_0, \mu_0) = v(t_0, \mu_0)\), above inequalities imply that
\[
0 \leq \frac{1}{h} \int_{t_0}^{t_0+h} \left( \partial_\mu \psi(s, \mu^*_0) + \mathbb{E}[k^\alpha(s, X^*_0, \mu^*_0)] \right) ds. \tag{9.1}
\]

We now let \(h\) tend to zero to arrive at the following inequality,
\[
-\partial_\mu \psi(t_0, \mu_0) \leq \mathbb{E}[k^\alpha(t_0, X^*_0, \mu_0)] = \mu_0(k^\alpha(t_0, \cdot, \mu_0)) \leq H(\mu_0, \partial_\mu \psi(t_0, \mu_0)) + \epsilon.
\]

9.2 Supersolution

Suppose that for \((t_0, \mu_0) \in [0, T) \times \mathcal{P}(T)\) and a test function \(\psi \in \mathcal{C}_a(O)\),
\[
0 = (v - \psi)(t_0, \mu_0) = \min_{\mathcal{O}}(v - \psi).
\]

We may assume that the minimum is strict. Towards a counterposition, suppose that
\[
-\partial_\mu \psi(t_0, \mu_0) < H(\mu_0, \partial_\mu \psi(t_0, \mu_0)) = \inf_{\alpha \in \mathcal{C}_a} \{ \mu_0(k^\alpha(t_0, \cdot, \mu_0)) \},
\]

where \(k^\alpha(t, x, \mu) = \ell^\alpha(x, \mu) + \mathcal{M}^{\alpha, \mu}[\partial_\mu \psi(t, \mu)](x)\) is as in the previous subsection. By the definition of test functions \(\mathcal{C}_a(O)\), \(H\) is continuous. Therefore, there exists \(\delta > 0\) and a neighborhood \(B \subseteq \mathcal{O}\) of \((t_0, \mu_0)\) such that
\[
-\partial_\mu \psi(s, \mu) + \delta \leq H(\mu, \partial_\mu \psi(t, \mu)) = \inf_{\alpha \in \mathcal{C}_a} \{ \mu(k^\alpha(t, \cdot, \mu)) \}, \quad \forall (t, \mu) \in B.
\]

For \(\alpha \in \mathcal{A}\), set \(X^*_0 := X^*_{t_0, \mu_0, \alpha}\), \(\mu^*_0 := \mathcal{L}^0_{\mu_0, \alpha}\), and consider the (deterministic) time
\[
\tau^\alpha := \inf \{ s \in [t_0, T] : (s, \mu^*_0) \notin B \},
\]
so that for every $s \in [t_0, \tau^\alpha)$, $(s, \mu^\alpha_s) \in \mathcal{B}$, and consequently
\[
\mu^\alpha_s(k^\alpha(s, \cdot, \mu^\alpha_s)) \geq H(\mu^\alpha_s, \partial_v \psi(s, \mu^\alpha_s)) \geq -\partial_v \psi(s, \mu^\alpha_s) + \delta.
\]
As $\mathbb{E}[k^\alpha(s, X^\alpha_s, \mu^\alpha_s)] = \mu^\alpha_s(k^\alpha(s, \cdot, \mu^\alpha_s))$,
\[
\int_{t_0}^{\tau^\alpha} \left( \mathbb{E}[k^\alpha(X^\alpha_s, \mu^\alpha_s)] + \partial_v \psi(s, \mu^\alpha_s) \right) ds \geq \delta(\tau^\alpha - t_0).
\]
Then, by Lemma 9.1, we obtain the following inequality,
\[
\psi(\tau^\alpha, \mu^\alpha_{\tau^\alpha}) = \psi(t_0, \mu_0) + \int_{t_0}^{\tau^\alpha} (\partial_v \psi(s, \mu^\alpha_s) + \mathbb{E}[M^{\alpha,s} \partial_v \psi(s, \mu^\alpha_s)](X^\alpha_s) ) ds
\]
\[
= \psi(t_0, \mu_0) + \int_{t_0}^{\tau^\alpha} (\partial_v \psi(s, \mu^\alpha_s) + \mathbb{E}[k^\alpha(s, X^\alpha_s, \mu^\alpha_s)] - \mathbb{E}[k^\alpha(s, X^\alpha_s, \mu^\alpha_s)]) ds
\]
\[
\geq \psi(t_0, \mu_0) - \int_{t_0}^{\tau^\alpha} \mathbb{E}[k^\alpha(X^\alpha_s, \mu^\alpha_s)] ds + \delta(\tau^\alpha - t_0).
\]
Since $v \geq \psi$ and $\psi(t_0, \mu_0) = v(t_0, \mu_0)$, above implies that
\[
v(t_0, \mu_0) \leq \int_{t_0}^{\tau^\alpha} \mathbb{E}[k^\alpha(X^\alpha_s, \mu^\alpha_s)] ds + v(\tau^\alpha, \mu^\alpha_{\tau^\alpha}) - g(\alpha), \quad \forall \alpha \in \mathcal{A}.
\]
where $g(\alpha) := \delta(\tau^\alpha - t_0) + (v(\tau^\alpha, \mu^\alpha_{\tau^\alpha}) - \psi(\tau^\alpha, \mu^\alpha_{\tau^\alpha}))$. We now claim that
\[
\delta_0 := \inf_{\alpha \in \mathcal{A}} g(\alpha) > 0.
\]
Indeed, since $v \geq \psi$, if $\tau^\alpha = T$, then $g(\alpha) \geq \delta(T - t_0)$. On the other hand if $\tau^\alpha < T$, then $(\tau^\alpha, \mu^\alpha_{\tau^\alpha}) \in \partial \mathcal{B}$. As $\mathcal{B}$ is compact and $(t_0, \mu_0) \notin \partial \mathcal{B}$ is the strict minimizer of $v - \psi$, we have
\[
(v - \psi)(\tau^\alpha, \mu^\alpha_{\tau^\alpha}) \geq \inf_{(t, \mu) \in \partial \mathcal{B}} (v - \psi)(t, \mu) > 0.
\]
Hence, $\delta_0 > 0$ and the above inequalities imply that for every $\alpha \in \mathcal{A}$,
\[
v(t_0, \mu_0) \leq \int_{t_0}^{\tau^\alpha} \mathbb{E}[k^\alpha(X^\alpha_s, \mu^\alpha_s)] ds + v(\tau^\alpha, \mu^\alpha_{\tau^\alpha}) - \delta_0.
\]
This contradiction to dynamic programming implies that $-\psi(t_0, \mu_0) \geq H(\mu_0, \partial_v \psi(t_0, \mu_0))$.

\[\square\]

\textbf{References}

[1] E. Bandini, A. Cosso, M. Fuhrman, and H. Pham. Randomized filtering and Bellman equation in Wasserstein space for partial observation control problem. \textit{Stochastic Processes and their Applications}, 129(2):674–711, 2019.

[2] E. Bayraktar, A. Cosso, and H. Pham. Randomized dynamic programming principle and Feynman-Kac representation for optimal control of McKean–Vlasov dynamics. \textit{Transactions of the American Mathematical Society}, 370(3):2115–2160, 2018.

[3] E. Bayraktar, I. Ekren, and X. Zhang. A smooth variational principle on Wasserstein space. \textit{arXiv:2209.15028}, 2022.
[4] M. Burzoni, V. Ignazio, M. Reppen, and H. M. Soner. Viscosity solutions for controlled McKean–Vlasov jump-diffusions. *SIAM Journal on Control and Optimization*, 58(3):1676–1699, 2020.

[5] P. Cardaliaguet. Notes on mean field games. Technical report, Technical report, 2010.

[6] P. Cardaliaguet, F. Delarue, J.-M. Lasry, and P.-L. Lions. *The master equation and the convergence problem in mean field games*: (AMS-201). Princeton University Press, 2019.

[7] R. Carmona and F. Delarue. *Probabilistic theory of mean field games with applications I–II*. Springer, 2018.

[8] R. Carmona, Q. Cormier, and H. M. Soner. Synchronization in a Kuramoto mean field game. arXiv:2210.12912, 2022.

[9] A. Cecchin and F. Delarue. Weak solutions to the master equation of potential mean field games. arXiv:2204.04315, 2022.

[10] G. Conforti, R. Kraaij, and D. Tonon. Hamilton–Jacobi equations for controlled gradient flows: the comparison principle. arXiv:2111.13258, 2021.

[11] A. Cosso, F. Gozzi, I. Kharroubi, H. Pham, and M. Rosestolato. Optimal control of path-dependent McKean-Vlasov SDGs in infinite dimension. arXiv:2012.14772, 2020.

[12] A. Cosso, F. Gozzi, I. Kharroubi, H. Pham, and M. Rosestolato. Master Bellman equation in the Wasserstein space: Uniqueness of viscosity solutions. arXiv:2107.10535, 2021.

[13] A. M. Cox, S. Källblad, M. Larsson, and S. Svaluto-Ferro. Controlled measure-valued martingales: a viscosity solution approach. arXiv:2109.00064, 2021.

[14] M. G. Crandall and P.-L. Lions. Viscosity solutions of Hamilton–Jacobi equations. *Transactions of the American Mathematical Society*, 277(1):1–42, 1983.

[15] M. G. Crandall, L. C. Evans, and P.-L. Lions. Some properties of viscosity solutions of Hamilton–Jacobi equations. *Transactions of the American Mathematical Society*, 282(2):487–502, 1984.

[16] M. G. Crandall, H. Ishii, and P.-L. Lions. User’s guide to viscosity solutions of second order partial differential equations. *Bulletin of the American Mathematical Society*, 27(1):1–67, 1992.

[17] M. F. Djete, D. Possamaï, and X. Tan. McKean–Vlasov optimal control: the dynamic programming principle. *The Annals of Probability*, 50(2):791–833, 2022.

[18] G. Fabbri, F. Gozzi, and A. Swiech. *Stochastic optimal control in infinite dimension*. Springer, 2017.

[19] W. H. Fleming and H. M. Soner. *Controlled Markov processes and viscosity solutions*, volume 25. Springer Science & Business Media, 2006.

[20] W. Gangbo, S. Mayorga, and A. Swiech. Finite dimensional approximations of Hamilton–Jacobi–Bellman equations in spaces of probability measures. *SIAM Journal on Mathematical Analysis*, 53(2):1320–1356, 2021.
[21] M. Huang, R. P. Malhamé, and P. E. Caines. Large population stochastic dynamic games: closed-loop McKean-Vlasov systems and the Nash certainty equivalence principle. *Communications in Information & Systems*, 6(3):221–252, 2006.

[22] J.-M. Lasry and P.-L. Lions. Jeux à champ moyen. I–Le cas stationnaire. *Comptes Rendus Mathématique*, 343(9):619–625, 2006.

[23] J.-M. Lasry and P.-L. Lions. Jeux à champ moyen. II–Horizon fini et contrôle optimal. *Comptes Rendus Mathématique*, 343(10):679–684, 2006.

[24] J.-M. Lasry and P.-L. Lions. Mean field games. *Japanese Journal of Mathematics*, 2(1):229–260, 2007.

[25] P.-L. Lions. Cours au collège de France. Available at [www.college-de-france.fr](http://www.college-de-france.fr), 2007.

[26] Y. Mroueh, C.-L. Li, T. Sercu, A. Raj, and Y. Cheng. Sobolev GAN. *arXiv:1711.04894*, 2017.

[27] H. Pham and X. Wei. Dynamic programming for optimal control of stochastic McKean–Vlasov dynamics. *SIAM Journal on Control and Optimization*, 55(2):1069–1101, 2017.

[28] H. Pham and X. Wei. Bellman equation and viscosity solutions for mean-field stochastic control problem. *ESAIM: Control, Optimisation and Calculus of Variations*, 24(1):437–461, 2018.

[29] H. M. Soner and Q. Yan. Viscosity solutions for McKean-Vlasov control II: d-dimensional torus. *in preparation*, 2022.

[30] C. Wu and J. Zhang. Viscosity solutions to parabolic master equations and McKean–Vlasov SDEs with closed-loop controls. *The Annals of Applied Probability*, 30(2):936–986, 2020.