Some Results on the Extended Hypergeometric Matrix Functions and Related Functions

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Received 21 August 2021; Accepted 3 September 2021; Published 22 September 2021

In this article, we discuss certain properties for generalized gamma and Euler’s beta matrix functions and the generalized hypergeometric matrix functions. The current results for these functions include integral representations, transformation formula, recurrence relations, and integral transforms.

1. Introduction

Matrix generalizations of some known classical special functions are important both from the theoretical and applied point of view (see, for example, [1–10]). These new extensions have proved to be very useful in various fields such as physics, engineering, statistics, actuarial sciences, life testing, and telecommunications.

In particular, various results of gamma, Euler’s beta, and hypergeometric matrix functions have been presented and investigated (see, e.g., [11–17]). Motivated by investigations of the extended gamma, beta, and Gauss hypergeometric matrix functions given in [14–16, 18, 19], we aim to derive certain properties of matrix generalizations of gamma, Euler’s beta, Gauss, and confluent hypergeometric functions. The results given by many authors [12, 20–24] follow as special cases in this work.

The plan of this work is described below:

In Section 2, we propose to derive some integral representations and recurrence relations for generalizations of gamma and Euler’s beta matrix functions. The various properties for the generalizations of Gauss and confluent hypergeometric matrix functions are investigated in Section 3. Finally, we end up with the conclusion in Section 4.

Throughout this paper, let I and 0 denote the identity matrix and null matrix in \( C^{r \times r} \), respectively. A matrix A in \( C^{r \times r} \) is a positive stable matrix if \( \Re(\lambda) > 0 \) for all \( \lambda \in \sigma(A) \) where \( \sigma(A) \) denotes the set of all eigenvalues of A. In [12], if \( f(z) \) and \( g(z) \) are holomorphic functions in an open set \( \Omega \) of the complex plane and if A is a matrix in \( C^{r \times r} \) for which \( \sigma(A) \subset \Omega \), then

\[
f(A)g(A) = g(A)f(A).
\]

Let A be a positive stable matrix in \( C^{r \times r} \); then, the gamma matrix function in [11, 12] is defined by

\[
\Gamma(A) = \int_0^\infty e^{-t} t^{A-1} dt, \quad t^{A-1} = \exp[(A - I)\ln t].
\]

For positive stable matrices \( A \) and \( B \in C^{r \times r} \), the beta matrix function in [11, 12] is defined by

\[
\beta(A, B) = \int_0^1 t^{A-1} (1 - t)^{B-1} dt.
\]
Also, if $A$, $B$, and $A + B$ are positive stable matrices in $\mathbb{C}^{m \times r}$ and $AB = BA$, then
\[
\mathbb{B} (A, B) = \Gamma (A) \mathbb{I} (B) \Gamma^{-1} (A + B).
\] (4)

From [12], for a matrix $A \in \mathbb{C}^{n \times r}$, the matrix version of Pochhammer symbol is defined as
\[
(A)_n = A (A + I) (A + 2I), \ldots, (A + (n - 1) I); n \geq 1; (A)_0 = I.
\] (5)

The Gauss hypergeometric matrix function $\mathbf{2F1}(A, B; C; z)$ is given in [12] as follows:
\[
\mathbf{2F1}(A, B; C; z) = \sum_{n=0}^{\infty} \frac{(A)_n (B)_n}{n!} \frac{\Gamma (C)_n}{\Gamma (C + n)} z^n,
\] (6)
where $|z| < 1$ and $A, B, C \in \mathbb{C}^{m \times r}$ such that $C + n I$ is invertible for all integer $n \geq 0$.

Also, the confluent hypergeometric matrix function is defined by
\[
\mathbf{Y} (z) = \mathbf{1F1}(A; B; z) = \sum_{n=0}^{\infty} \frac{(A)_n}{n!} \frac{\Gamma (B)_n}{\Gamma (B + n)} z^n,
\] (7)
and it satisfies the following matrix differential equation:
\[
z \mathbf{Y}'' + (B - zI) \mathbf{Y} = 0,
\] (8)
where $A$ and $B \in \mathbb{C}^{m \times r}$ and $B + n I$ is invertible for every integer $n \geq 0$. Furthermore, we have
\[
\mathbf{Y} (z) = \Theta^{-1} (A, B - A) \int_{0}^{1} e^{tz} t^{A - I} (1 - t)^{B - A - I} dt,
\] (9)
where $A, B$, and $B - A$ are positive stable matrices and $AB = BA$.

In the recent paper [14], for any arbitrary parameter $p$ with $\text{Re} (p) > 0$, the matrix generalizations of gamma and Euler’s beta functions are given as follows:
\[
\Gamma (\mathcal{X}; p) = \int_{0}^{\infty} \exp \left( -t - \frac{p}{t} \right) t^{\mathcal{X} - 1} dt,
\] (10)
\[
\mathcal{B} (\mathcal{X}, \mathcal{Y}; p) = \int_{0}^{1} \exp \left( -\frac{p}{t(1-t)} \right) t^{\mathcal{X} - 1} (1 - t)^{\mathcal{Y} - 1} dt,
\] (11)
for $p = 0$, which gives gamma and Euler’s beta matrix functions given by (2) and (3), respectively:
\[
\Gamma^{(A, B)} (\mathcal{X}; p) = \int_{0}^{\infty} i \Gamma_{1} \left( A; B; -t - \frac{p}{t} \right) t^{\mathcal{X} - 1} dt,
\] (12)
and
\[
\mathcal{B}^{(A, B)} (\mathcal{X}, \mathcal{Y}; p) = \int_{0}^{1} t^{\mathcal{X} - 1} (1 - t)^{\mathcal{Y} - 1} i \Gamma_{1} \left( A; B; -\frac{p}{t(1-t)} \right) dt
\] (13)
respectively, where $A, B, \mathcal{X}$, and $\mathcal{Y}$ are positive stable matrices in $\mathbb{C}^{r \times r}$ and $p$ is any arbitrary parameter with $\text{Re} (p) > 0$. Also, these are matrix versions of gamma and beta functions [23]. The case of $A = B$ in (12) and (13) gives us generalizations of gamma and Euler’s beta matrix functions defined by (10) and (11), respectively.

Moreover, author in [14] defined the generalized Gauss and confluent hypergeometric matrix functions as follows:

\textbf{Definition 1.} Let $A, B, A^*, B^*$, and $C^* \in \mathbb{C}^{m \times r}$ satisfying conditions that $A, B, B^*, C^* - B^*$, and $C^*$ are the positive stable matrix, and
\[
B^* C^* = C^* B^*,
\] (14)
and $p$ be a number with $\text{Re} (p) > 0$. Then, the generalized Gauss hypergeometric matrix function (GGHMF) is defined in [14] by
\[
\mathbf{F}^{(A, B)} (A^*, B^*; C^*; z; p) = \sum_{n=0}^{\infty} (A^*)_n \mathbf{B}^{(A, B)} (B^* + n I, C^* - B^*; p) \mathbf{B}^{-1} (B^* + n I, C^* - B^*; p) \mathbf{z}^n
\] (15)
and the generalized confluent hypergeometric matrix function (GCHMF) in the form
\[
\mathbf{F}^{(A, B)} (B^*; C^*; z; p) = \sum_{n=0}^{\infty} \mathbf{B}^{(A, B)} (B^* + n I, C^* - B^*; p) \mathbf{B}^{-1} (B^* + n I, C^* - B^*; p) \mathbf{z}^n
\] (16)

\textbf{Remark 1.} For $A = B$ in (15) and (16), we have
In this section, we derive some properties of generalized gamma and Euler's beta functions, which are defined by (12) and (13), as follows:

\[ F(A^*, B^*; C^*; z; p) = \sum_{n=0}^{\infty} (A^*)_n \mathcal{B} (B^* + nI, C^* - B^*; p) \mathcal{B}^{-1} (B^*, C^* - B^*) \frac{z^n}{n!} \]  

(17)

where \( p = 0 \) in (15), it reduces

\[ F^{(A, B)} (A^*, B^*; C^*; z) = 2 F_1 (A^*, B^*; C^*; z), \]  

(18)

and also, at \( p = 0 \) in (16), we get

\[ F_1^{(A, B)} (B^*, C^*; z) = F_1 (B^*, C^*; z). \]  

(19)

Now, let us give the following definition as a generalization of the functions in (15) and (16).

\[ \Gamma^{(A, B)} (z; p) \]

where \( r \) and \( s \in \mathbb{N} \).

**2. Properties of Generalizations of Gamma and Beta Matrix Functions**

In this section, we derive some properties of generalized gamma and Euler's beta matrix functions, which are defined by (12) and (13), as follows:

\[ \Gamma^{(A, B)} (X; p) = \int_0^1 \eta^{-X} \Gamma (X; p\eta^2) \mathcal{B}^{-1} (A, B - A) \eta^{A-1} (1 - \eta)^{B-A-1} d\eta, \]  

(20)

where \( B - A \) is a positive stable matrix in \( C^{r \times r} \) and \( AB = BA \).

**Proof.** Using (9) in (12), we have

\[ \Gamma^{(A, B)} (X; p) = \int_0^{\infty} u^{-X-1} \left[ \mathcal{B}^{-1} (A, B - A) \right] \int_0^1 e^{-ut-(pt/u)} t^{A-1} (1 - t)^{B-A-1} dt du. \]  

(23)

Taking \( \nu = ut \) and \( \eta = t \) in the above equation, we can write

\[ \Gamma^{(A, B)} (X; p) = \int_0^1 \eta^{-X} \left[ \int_0^{\infty} \nu^{A-1} e^{-(\nu^2/p)} d\nu \right] \left[ \mathcal{B}^{-1} (A, B - A) \right] \eta^{A-1} (1 - \eta)^{B-A-1} d\eta. \]  

(24)

Then, by (10), we complete the proof of the theorem. \( \square \)
Theorem 2. The following integral representation for generalized Euler’s beta matrix function in (13) holds well:

\[
\mathcal{B}^{(A,B)}(\mathcal{X}, Y; p) = \int_0^1 \mathcal{B}(\mathcal{X}, Y, pt) t^{A-I} (1-t)^{B-A-I} \left[ \mathbb{B}^{-1}(A,B-A) \right] dt,
\]

where \( B - A \) is a positive stable matrix in \( \mathbb{C}^{r \times r} \) and \( AB = BA \).

Proof. The proof of the theorem is completed similar to Theorem 1. \( \square \)

Theorem 3. For generalized Euler’s beta matrix function \( \mathcal{B}^{(A,B)}(\mathcal{X}, Y; p) \), we have the next property:

\[
\int_0^1 p^{S-I} \mathcal{B}^{(A,B)}(\mathcal{X}, Y; p) dp = \mathbb{B}(S+\mathcal{X}, Y+S)^{(-A,B)}(S),
\]

(26)

where \( \mathbb{AY} = YA \) and \( \mathbb{BY} = YB \).

Proof. It follows straightforwardly from (12). \( \square \)

Remark 2. If we take \( A = B = p = 0 \) in Theorem 4, we obtain

\[
\mathbb{B}(\mathcal{X}, Y) = \Gamma(\mathcal{X}) \Gamma(Y)^{-1} (X + Y),
\]

(28)

where \( X,Y, \) and \( X+Y \) are positive stable matrices and \( \mathcal{XY} = Y \mathcal{X} \).

Theorem 4. For generalized gamma matrix function \( \Gamma^{(A,B)}(\mathcal{X}; p) \), we get

\[
\Gamma^{(A,B)}(\mathcal{X}; p) \Gamma^{(A,B)}(Y; p) = 4 \int_0^{\pi/2} \int_0^\infty (r \cos \theta)^{2\mathcal{X}-1} (r \sin \theta)^{2Y-1}

\times \frac{1}{4} F_1 \left( A; B; -r^2 \cos^2 \theta - \frac{p}{r^2 \cos^2 \theta} \right) \frac{1}{4} F_1 \left( A; B; -r^2 \sin^2 \theta - \frac{p}{r^2 \sin^2 \theta} \right) r dr d\theta,
\]

(27)

where \( \mathbb{AY} = YA \) and \( \mathbb{BY} = YB \).

Proof. It is enough to use (13), interchange the order of integration, take transformations \( \nu = pt(1-t) \) and \( \eta = t \) in (13), and then use (12) in the left-hand side of the above equation, respectively. \( \square \)

On the contrary, if we consider Taylor expansion of \( (1-t)^{-Y} \) at \( t = 0 \), we can write

\[
(1-t)^{-Y} = \sum_{n=0}^{\infty} \frac{(Y)_n}{n!} t^n.
\]

(31)

Then, using (13), the theorem can be proved. \( \square \)

Remark 3. For \( A = B \) in Theorem 5, we get

\[
\mathcal{B}(\mathcal{X}, I - Y; p) = \sum_{n=0}^{\infty} \frac{(Y)_n}{n!} \mathcal{B}(\mathcal{X} + nI, I; p),
\]

(32)

and for \( p = 0 \) in (32), we have

\[
\mathbb{B}(\mathcal{X}, I - Y) = \sum_{n=0}^{\infty} \frac{(Y)_n}{n!} \mathbb{B}(\mathcal{X} + nI, I).
\]

(33)
**Theorem 6.** The following recurrence relation for the generalized gamma matrix function holds well:

\[
\frac{d^2}{dp^2} [\Gamma^{(A,B)}(\mathcal{X} + 5I; p)] + p^2 \frac{d^2}{dp^2} [\Gamma^{(A,B)}(\mathcal{X} + 3I; p)] - \frac{d}{dp} [\Gamma^{(A,B)}(\mathcal{X} + 2I; p)] B
\]

\[
- \frac{d}{dp} [\Gamma^{(A,B)}(\mathcal{X} + 3I; p)] - p \frac{d}{dp} [\Gamma^{(A,B)}(\mathcal{X} + I; p)] + A \Gamma^{(A,B)}(\mathcal{X}; p)
\]

\[= 0.\]

**Proof.** From (12), by using the Leibnitz rule, it is easily seen that the left-hand side of the above equation can be written as

\[
\int_0^\infty e^{x-t} \left[ (t^3 + pt) \frac{d^2 Z}{dp^2} + \frac{dZ}{dp} \left[ Bt + (t^3 + p) I \right] + AZ \right] dt,
\]

where \( Z = \psi F_1(A; B; (-t - (p/t))) \). On the one hand, due to (8), we have

\[
(t^3 + pt) \frac{d^2 Z}{dp^2} + \frac{dZ}{dp} \left[ Bt + (t^3 + p) I \right] + AZ = 0.
\]

Thus, the proof is completed. \( \square \)

**Remark 4.** For \( A = B \) in Theorem 6, we get

\[
\frac{d^2}{dp^2} [\Gamma^{(A,B)}(\mathcal{X} + 5I; p)] + p^2 \frac{d^2}{dp^2} [\Gamma^{(A,B)}(\mathcal{X} + 3I; p)] - \frac{d}{dp} [\Gamma^{(A,B)}(\mathcal{X} + 2I; p)] A
\]

\[
- \frac{d}{dp} [\Gamma^{(A,B)}(\mathcal{X} + 3I; p)] - p \frac{d}{dp} [\Gamma^{(A,B)}(\mathcal{X} + I; p)] + A \Gamma^{(A,B)}(\mathcal{X}; p)
\]

\[= 0.\]

**Theorem 7.** Generalized Euler’s beta matrix function verifies the next recurrence relation:

\[
p \frac{d^2}{dp^2} \left[ \beta^{(A,B)}(\mathcal{X} + Y + 3I; p) \right] + \frac{d}{dp} \left[ \beta^{(A,B)}(\mathcal{X} + Y + 2I; p) \right] B
\]

\[
+ p \frac{d}{dp} \left[ \beta^{(A,B)}(\mathcal{X} + Y + I; p) \right] + A \beta^{(A,B)}(\mathcal{X}, Y; p)
\]

\[= 0.\]

**Proof.** In view of (13), the left-hand side of above equation is equal to

\[
\int_0^\infty t^{x-t} (1-t)^{y-t} \left[ pt (1-t) \frac{d^2 Z}{dp^2} + \frac{dZ}{dp} \left[ Bt (1-t) + p I \right] + AZ \right] dt,
\]

where \( Z = \psi F_1(A; B; (-p/t (1-t))) \). On the other hand, by using equation (8), we obtain
\[ \frac{d^2 Z}{dp^2} pt (1-t) + \frac{dZ}{dp} [(1-t)Bt + pI] + ZA = 0, \quad (40) \]

which yields the desired result of Theorem 7. \[ \square \]

**Remark 5.** The case of \( A = B \) in Theorem 7 gives

\[ p \frac{d^2}{dp^2} [\mathcal{R}(X + 3I, Y + 3I; p)] + \frac{d}{dp} [\mathcal{R}(X + 2I, Y + 2I; p)]A \]

\[ + p \frac{d}{dp} [\mathcal{R}(X + I, Y + I; p)] + A\mathcal{R}(X, Y; p) \]

\[ = 0. \quad (41) \]

### 3. Properties of the GGHMF and GCHMF

In this section, we give some of the main results of the GGHMF and GCHMF as follows.

**Theorem 8.** For the GGHMF \( F^{(A,B)}(A,B^*;C^*;z;p) \), the following integral form holds true:

\[ F^{(A,B)}(A,B^*;C^*;z;p) = \int_0^1 (1-tz)^{A^*} t^{B^* - I} (1-t)^{C^* - B^* - I} \frac{B}{t} \frac{1}{(1-t)} F_1 \left( A; B; -\frac{-p}{t} \right) dt, \quad (42) \]

where \( |\arg (1-z)| < \pi. \)

**Proof.** If we take

\[ \mathcal{R}_p^{(A,B)}(B^* + nI, C^* - B^*) = \int_0^1 t^{B^* + nI - 1} (1-t)^{C^* - B^* - I} F_1 \left( A; B; -\frac{-p}{t} \right) dt, \quad (43) \]

in (15) and use the following relation:

\[ \sum_{n=0}^{\infty} \frac{(A^*)^n}{n!} (tz)^n = (1-tz)^{-A^*}, \quad (44) \]

then we arrive at the required result. \[ \square \]

**Corollary 1.** If we apply the substitution \( t = \sin^2 \nu \) in (42), we derive

\[ F^{(A,B)}(A^*, B^*; C^*; z; p) = 2 \int_0^{\pi/2} \left( 1 - z \sin^2 \nu \right)^{-A^*} \sin^{2B^* - I} \nu \cos^{2C^* - 2B^* - I} \nu \times B^{-1} (B^*, C^* - B^*)_1 F_1 \left( A; B; \frac{-p}{\sin^2 \nu \cos^2 \nu} \right) d\nu, \quad (45) \]

and applying the substitution \( t = u/1 + u \) in (42), we obtain
\[ F_{p}^{(A,B)}(A^*, B^*; C^*; z) \]
\[ = \int_{0}^{\infty} (1 + u(1 - z))^{-A^*} (1 + u)^{-A^* - C^*} u^{B^* - 1} \]
\[ \times F_{1}(A; B; -2p - pu - pu^{-1}) B^{-1}(B^*, C^* - B^*) du, \]
\[ (46) \]
\[ F_{p}^{(A,B)}(A^*, B^*; C^*; z; p) = \int_{0}^{1} F(A^*, B^*; C^*; z; pt) t^{A^* - 1} (1 - t)^{B^* - 1} B^{-1}(A, B - A) dt, \]
\[ (47) \]
where \( B - A \) is a positive stable matrix in \( \mathbb{C}^{mxr} \) and \( AB = BA \).

**Proof.** From (15) and Theorem 2, we have the desired relation. \( \square \)

\[ F_{1}^{(A,B)}(B^*; C^*; z; p) \]
\[ = \int_{0}^{1} t^{B^* - 1} (1 - t)^{C^* - B^* - 1} B^{-1}(B^*, C^* - B^*) F_{1}(A; B; -\frac{p}{(1 - t)}) t^{B^* - 1} \]
\[ (48) \]
\[ e^{\frac{z}{u}} \int_{0}^{1} (1 - u)^{B^* - 1} u^{C^* - B^* - 1} B^{-1}(B^*, C^* - B^*) F_{1}(A; B; -\frac{p}{u(1 - u)}) e^{-zu} du. \]
\[ (49) \]

**Theorem 9.** The following integral form for GGHMF holds true:

**Theorem 10.** The GCHMF \( F_{1}^{(A,B;p)}(B^*; C^*; z) \) has the next representation:

**Corollary 2.** If we take \( t = 1 - u \) in (48), we derive

\[ F_{1}^{(A,B;p)}(B^*; C^*; z) \]
\[ = e^{\frac{z}{u}} \int_{0}^{1} (1 - u)^{B^* - 1} u^{C^* - B^* - 1} B^{-1}(B^*, C^* - B^*) F_{1}(A; B; -\frac{p}{u(1 - u)}) e^{-zu} du. \]
\[ (49) \]

**Theorem 11.** For the GGHMF with \( |\arg(1 - z)| < \pi \), then the following transformation formula holds true:

\[ F^{(A,B)}(A^*, B^*; C^*; z; p) = (1 - z)^{-A^*} F^{(A,B)}(A^*, C^* - B^*; C^*; \frac{z}{z - 1}; p). \]
\[ (50) \]

**Proof.** In (42), by writing \((1 - t)\) instead of \( t \) and using the following equation
\[ (1 - z(1 - t))^{-A^*} = (1 - z)^{-A^*} \left( 1 + \frac{z}{z - 1} \right)^{-A^*}, \]
\[ (51) \]
we obtain

\[ F^{(A,B)}(A^*, B^*; C^*; z; p) = (1 - z)^{-A^*} F^{(A,B)}(A^*, C^* - B^*; C^*; \frac{z}{z - 1}; p). \]
\[ (52) \]

**Remark 6.** For \( A = B \) in Theorem 11, one can easily obtain
\[
F(A^*, B^*; C^*; z; p) = (1 - z)^{-A^*} F\left(A^*, C^* - B^*; C^*; \frac{z}{z - 1}; p\right),
\]
and also, for \( p = 0 \),
\[
_{2} F_{1}(A^*, B^*; C^*; z) = (1 - z)^{-A^*} _{2} F_{1}\left(A^*, C^* - B^*; C^*; \frac{z}{z - 1}\right),
\]
(54)
and it is satisfied which is given in [25].

**Corollary 3.** From Theorem 11, the GGHMF \( F^{(A,B)}(A^*, B^*; C^*; z; p) \) satisfies the transformation formula:
\[
_{2} F_{1}(A^*, B^*; C^*; 1 - \frac{1}{z}; p) = z^{A^*} F^{(A,B)}(A^*, C^* - B^*; C^*; 1 - z; p),
\]
(55)

where \(|\arg z| < \pi\).

**Remark 7.** In the case of \( A = B \) in Corollary 3, we have
\[
F\left(A^*, B^*; C^*; 1 - \frac{1}{z}; p\right) = z^{A^*} F\left(A^*, C^* - B^*; C^*; 1 - z; p\right),
\]
(56)
and also, if we get \( p = 0 \), we find
\[
F^{(A,B)}\left(A^*, B^*; C^*; \frac{z}{z + 1}; p\right) = (1 + z)^{A^*} F^{(A,B)}\left(A^*, C^* - B^*; C^*; -z; p\right),
\]
(58)
where \(|\arg (1 + z)| < \pi\).

**Theorem 12.** The GGHMF \( F^{(A,B)}(A^*, B^*; C^*; z; p) \) verifies the recurrence relation:
\[
\begin{align*}
&\mathcal{B}(B^* + 3I; C^* - B^* + 3I) p \frac{d^2}{dp^2} \left[ F^{(A,B)}\left(A^*, B^* + 3I; C^* + 6I; z; p\right)\right] \\
&- B(B^* + 2I; C^* - B^* + 2I) B \frac{d}{dp} \left[ F^{(A,B)}\left(A^*, B^* + 2I; C^* + 4I; z; p\right)\right] \\
&- B(B^* + I; C^* - B^* + I) p \frac{d}{dp} \left[ F^{(A,B)}\left(A^*, B^* + I; C^* + 2I; z; p\right)\right] \\
&+ AF^{(A,B)}\left(A^*, B^*; C^*; z; p\right) \\
&= 0.
\end{align*}
\]
(59)

**Proof.** By using relation (42), we can write the left-hand side of (59) in the following form:
\[
\int_{0}^{1} t^{B^* - I} (1 - t)^{C^* - B^* - I} \left[ pt(1 - t) \frac{d^2 Z}{dp^2} + \frac{d Z}{dp} \left[ Bt(1 - t) + pI + AZ \right] \right] (1 - tz)^{-A^*} dt,
\]
(60)
where \( Z = F_1(A; B; -(p/t(1-t))) \). On the other hand, it follows from (8):
\[
\frac{d^2Z}{dp^2}pt(1-t) + \frac{dZ}{dp}[(1-t)Bt + pI] + ZA = 0, \quad (61)
\]

which proves the theorem. \( \square \)

**Theorem 13.** For the GCHMF \( F_1^{(A,B)}(B^*; C^*; z; p) \), the following recurrence relation holds true:

\[
\mathcal{B}(B^* + 3I; C^* - B^* + 3I) p \frac{d^2}{dp^2} \left[ F_1^{(A,B)}(B^* + 3I; C^* + 6I; z; p) \right] \\
- \mathcal{B}(B^* + 2I; C^* - B^* + 2I) p \frac{d}{dp} \left[ F_1^{(A,B)}(B^* + 2I; C^* + 4I; z; p) \right] \\
- \mathcal{B}(B^* + I; C^* - B^* + I) p \frac{d}{dp} \left[ F_1^{(A,B)}(B^* + I; C^* + 2I; z; p) \right]
+ A_1 F_1^{(A,B)}(B^*; C^*; z; p)
= 0.
\]

**Proof.** It is enough to make similar calculations as in Theorem 11. \( \square \)

Next, the integral transforms for the GGHMF \( F^{(A,B)}(A^*, B^*; C^*; z; p) \) are given as follows:

**Theorem 14.** The following beta matrix transform formula holds true:
\[
\mathcal{B}\left\{F^{(A,B)}(P + Q, B^*; C^*; yz; p) : P, Q\right\} = \mathcal{B}(P, Q) F^{(A,B)}(P, B^*; C^*; x; p), \quad (63)
\]

where \( A, B, B^*, C^*, P, Q, \) and \( P + Q \) are positive stable matrices and commutative in \( C^ {\infty} \) with \( \text{Re}(p) \geq 0, |x| < 1 \), and the beta matrix transform of \( f(z) \) is defined as follows [25]:
\[
\mathcal{B}[f(z) : P, Q] = \int_0^1 z^{P-1}(1-z)^{Q-1} f(z)dz,
\]
where \( P \) and \( Q \) are positive stable matrices in \( C^ {\infty} \).

**Proof.** Using relation (64) and applying (15) to the beta matrix transform of (63), we have
\[
\int_0^1 z^{P-1}(1-z)^{Q-1} F^{(A,B)}(P + Q, B^*, C^*; xz)dz

= \int_0^1 z^{P-1}(1-z)^{Q-1} \sum_{n=0}^{\infty} (P + Q)_n \rho^{(A,B)}(B^* + nl; C^* - B^*; p) \mathcal{B}^{-1}(B^*, C^* - B^*) \frac{(xz)^n}{n!}dz.
\]

By interchanging the order of integration and summation with (4), we obtain
\[
\int_0^1 z^{P-1}(1-z)^{Q-1} F^{(A,B)}(P + Q, B^*; C^*; xz; p)dz

= \sum_{n=0}^{\infty} (P + Q)_n \rho^{(A,B)}(B^* + nl; C^* - B^*; p) \mathcal{B}^{-1}(B^*, C^* - B^*) \frac{(xz)^n}{n!}

= \mathcal{B}(P, Q) \sum_{n=0}^{\infty} (P)_n \rho^{(A,B)}(B^* + nl, C^* - B^*; p) \mathcal{B}^{-1}(B^*, C^* - B^*) \frac{x^n}{n!}.
\]

(65)
which, according to (15), yields our desired result (63). This completes the proof of Theorem 13. □

Theorem 15. If $\Re(s) > 0, \Re(p) \geq 0, M \in \mathcal{C}^{\infty}$, and $|x/s| < 1$, then the Laplace transform holds true:

$$\mathcal{L}\left\{ z^{M-1} F^{(A,B)}(A^*, B^*, C^*; xz; \rho) \right\} = s^{-M} \Gamma(M) F^{(A,B)}(A^*, M, B^*; C^*; \frac{x}{s}; \rho),$$

(67)

where the Laplace transform of $f(z)$ is defined as follows [26]:

$$\mathcal{L}\{ f(z) \} = \int_{0}^{\infty} e^{-sz} f(z) \, dz.$$ (68)

Proof. From definition (68) and (15), we find that

$$\int_{0}^{\infty} z^{M-1} e^{-sz} F^{(A,B)}(A^*, B^*, C^*; xz; \rho) \, dz$$

$$= \int_{0}^{\infty} z^{M-1} e^{-sz} \sum_{n=0}^{\infty} (A)_n \mathcal{B}^{(A,B)}(B^* + \ln, C^* - B^*; \rho) \mathcal{B}^{-1}(B^*, C^* - B^*) \frac{(xz)^n}{n!} \, dz.$$ (69)

By interchanging the order of integration and summation and using Laplace transform, we have

$$\int_{0}^{\infty} z^{M-1} e^{-sz} F^{(A,B)}(A^*, B^*, C^*; xz; \rho) \, dz$$

$$= \sum_{n=0}^{\infty} (A)_n \mathcal{B}^{(A,B)}(B^*, +n\ln, C^* - B^*; \rho) \mathcal{B}^{-1}(B^*, C^* - B^*) \frac{s^{M+n}}{n!} \Gamma(M + n\ln).$$ (70)

which, upon using (21), yields our desired result (67). □

Theorem 16. If $\rho$ and $\delta \in \mathbb{C}$, $\Re(p) \geq 0$, and $|\omega/\delta| < 1$, then the following Whittaker transform formula hold true:

$$\mathcal{F} = \int_{0}^{\infty} \left( \frac{\delta}{\delta \omega} \right)^{n-1} e^{-\omega/\delta} W_{\lambda,\mu}(\delta t) F^{(A,B)}(A^*, B^*; C^*; \omega t; \rho) \, d\omega$$

$$= \delta^{-n} \Gamma(1/2 + \mu + \rho) \Gamma(1/2 - \mu + \rho)$$

$$\times t^{\rho-1} F_{1}^{(A,B)}\left( A^*, \left( \frac{1}{2} + \mu + \rho \right), I\left( \frac{1}{2} - \mu + \rho \right), I, B^*; C^*; (1 - \lambda + \rho)I; \frac{\omega}{\delta}; \rho \right).$$ (71)

Proof. Starting with $\delta t = \nu$ in L.H.S of (71), we obtain

$$\mathcal{F} = \int_{0}^{\infty} \left( \frac{\delta}{\delta \nu} \right)^{n-1} e^{-\nu/\delta} W_{\lambda,\mu}(\nu) \sum_{n=0}^{\infty} (A^n)_{n} \mathcal{B}^{-1}(B^*, C^* - B^*) \mathcal{B}^{(A,B)}(B^* + n\ln, C^* - B^*; \rho) \frac{(\omega\nu)^n}{\delta^n n!} \, d\nu.$$ (72)

By interchanging the order of integration and summation, we obtain
\[
\mathcal{T} = \delta^\rho \sum_{n=0}^{\infty} (A^*)_n B^{-1} (B^* + nI, C^* - B^*; p) \frac{w^n}{\delta^n n!} \\
n \times \int_0^\infty \rho^{p+n-1} e^{-\rho t} W_{\lambda, \mu}(\nu) d\nu.
\]

Using the following integral form (cf. [26]),

\[
\int_0^\infty t^{\nu-1} e^{-t} W_{\lambda, \mu}(t) dt = \frac{\Gamma(1/2 + \mu + \nu) \Gamma(1/2 - \mu + \nu)}{\Gamma(1/2 - \lambda + \nu)}, \quad \left( \Re(\nu + \mu) > -\frac{1}{2} \right),
\]

and (73) becomes the following equation:

\[
\mathcal{T} = \delta^\rho \sum_{n=0}^{\infty} (A^*)_n B^{-1} (B^* + nI, C^* - B^*; p) \frac{w^n}{\delta^n n!} \\
\times \frac{\Gamma(1/2 + \mu + \nu + n) \Gamma(1/2 - \mu + \nu + n)}{\Gamma(1 - \lambda + \nu + n)},
\]

and relation (15) evidently leads us to the required result. \(\square\)

4. Conclusion

This manuscript is a continuation of the recent papers [14–16, 18, 19]. In the current paper, we introduced new properties of extension of the gamma and beta matrix function. We also introduced new extensions of the Gauss hypergeometric matrix function and confluent hypergeometric matrix function. Then, we discussed certain properties of these extended matrix functions such as the integral representations, transformation formulae, recurrence relations, and integral transforms. In addition, some interesting special cases of our main results are archived.

Data Availability

No data were used to support the study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

The second authors extend their appreciation to the Deanship of Scientific Research at King Khalid University for funding this work through Research Group Project under Grant no. (R.G.P-2/53/42).

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