Fractional Dynamics of Systems with Long-Range Space Interaction and Temporal Memory

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Abstract

Field equations with time and coordinates derivatives of noninteger order are derived from stationary action principle for the cases of power-law memory function and long-range interaction in systems. The method is applied to obtain a fractional generalization of the Ginzburg-Landau and nonlinear Schrödinger equations. As another example, dynamical equations for particles chain with power-law interaction and memory are considered in the continuous limit. The obtained fractional equations can be applied to complex media with/without random parameters or processes.

1 Introduction

From the contemporary vision of complex media, where microscopic processes take place and where many important applications are utilized, it is too far from considering the media as uniform gases, liquids, or solids. The most typical features of the new physical objects and/or processes are fractality of their structure and of intrinsic dynamics or kinetics. Observation of fractality of the basic processes began fairly long ago (see for review [1, 2]). Typically the complexity of systems is linked to the long term memory, long-range interactions, non-markovianity of the kinetics, and particularly with the Levy-type processes (Levy flights) [3]. The literature on this subject is vast. Let us mention some of the most related references,
where the indication of the complexity can lead, in one or another way, to the fractional
description of the dynamic and/or kinetic processes with fractional time [4, 5, 6]; systems
of many coupled elements [7, 8]; colloidal aggregates and chemical reaction medium [9, 10];
wave processes [11, 12, 13]; porous media [14]; quantum mechanics and quantum field theory
[15, 16, 17]; plasma physics [18, 19, 20]; magnetosphere [21]; random processes and random
walks [22, 23, 24, 25, 26]; fractional diffusion and Brownian motion [27, 28, 6]; weak and strong
turbulence [12, 29, 30]; fractional kinetics and chaos theory [31] (see for review [32, 33]).

It seems that the basic formal tool to be applied to is the description of the processes by
fractional equations, i.e., by the ones that contain fractional derivatives or integrals [34, 35, 36,
37]. The theory of derivatives of non-integer order goes back to Leibnitz, Liouville, Riemann,
Grunwald, and Letnikov [34, 35]. Derivatives and integrals of fractional order have found
many applications in recent studies in physics because of their continually growing numerous
applications.

Usually, onset of fractional derivatives (integral) is linked to different power type asymptotic
interactions or time memories. Depending on what kind of specific features characterize the
physical object, the fractional derivative (integral) can be with respect to time or space coor-
dinate. In the description of particles transport, when the dynamics is chaotic, the fractional
derivatives emerge in space and time simultaneously as a natural reflection of scaling proper-
ties of the phase space dynamics [31, 32]. The diffusion described by the fractional equations
is called the anomalous one. The occurrence of such derivatives could also be related to the
space-time decay [38, 39], i.e., to pure dynamical processes without kinetics or diffusion. Par-
ticularly it was shown in [40, 41, 42, 43, 44] how the long-range interaction between different
oscillators can be described by the fractional differential equations in the continuous medium
limit. Another way to connect the fractional equations with specific dispersion laws of the
media was considered in [12, 11, 45].

The goal of this paper is to provide a systematic approach to the onset of fractional equations
as a result of existence of long-range interaction in a corresponding space and long-range time
memory in the system of fields or particles depending on what kind of physical objects are
considered. The notions of long terms memory or interaction can be exactly specified by power
laws in time for a memory function and power law interaction between different elements of
the medium. It is of importance to understand the conditions when the fractional derivatives (integrals) occur since it allows us to involve into the consideration power tools of fractional calculus.

In Sec. 2, we consider the variation of action functional that describes field with memory and long-range interaction. The long-time memory and long-range interaction can be introduced through power-like kernels of the action functional. The corresponding powers are defined by the exponent $\alpha$ (for space) and $\beta$ (for time), which in general can be fractional. The Euler-Lagrange equations lead to the equation with fractional ($\alpha, \beta$)-derivatives. In Sec. 3, the obtained results are used for derivation of ($\alpha, \beta$)-generalization of the Ginzburg-Landau and nonlinear Schrödinger equations. In Sec. 4, we consider chains of particles with long-range interaction and memory function. Applying the results of Sec. 2, we derive the continuous limit of the particle dynamics equations. In two Appendices, we provide a brief information on the Riemann-Liouville, Caputo and Riesz fractional derivatives used in paper, and $n$-dimensional generalization of the final fractional equations.

## 2 Action functional and its variation

### 2.1 Action functional

Let us define the action functional as

\[
S[u] = \int_{R} d^2x \int_{R} d^2y \left( \frac{1}{2} \partial_t u(x) g_0(x,y) \partial_t u(y) + \frac{1}{2} \partial_r u(x) g_1(x,y) \partial_r u(y) - V(u(x), u(y)) \right). \tag{1}
\]

Here $x = (t,r)$, $t$ is time, $r$ is coordinate, and $y = (t',r')$. The integration is carried out over a region $R$ of the 2-dimensional space $\mathbb{R}^2$ to which $x$ belong. The field $u(x)$ is defined in a 2-dimensional region $R$ of $\mathbb{R}^2$. We assume that $u(x)$ has partial derivatives

\[
\partial_t u(x) = \frac{\partial u(t,r)}{\partial t}, \quad \partial_r u(x) = \frac{\partial u(t,r)}{\partial r},
\]

which are smooth functions with respect to time and coordinate.

Here are three examples of this action.

(a) If

\[
g_0(x,y) = -g_1(x,y) = \delta(x - y),
\]
\[
V(u(x), u(y)) = V(u(x))\delta(x - y),
\]
then we get the usual action
\[
S[u] = \int_{\mathbb{R}} d^2x \left( \frac{1}{2} [\partial_x u(x)]^2 - \frac{1}{2} [\partial_r u(x)]^2 - V(u(x)) \right).
\]

(b) If
\[
g_0(x, y) = -g_1(x, y) = \delta(x - y)C_1(D, r),
\]
\[
V(u(x), u(y)) = V(u(x))\delta(x - y)C_1(D, r),
\]
where
\[
C_1(D, r) = \frac{|r|^{D-1}}{\Gamma(D)}, \quad (0 < D < 1),
\]
then we obtain
\[
S[u] = \int dt \int dl_D \left( \frac{1}{2} [\partial_t u(x)]^2 - \frac{1}{2} [\partial_r u(x)]^2 - V(u(x)) \right),
\]
where
\[
dl_D = C_1(D, r)dr.
\]
This action defines the field \( u(x) \) in a medium with the fractional Hausdorff dimension \( D \).

(c) If
\[
V(u(x), u(y)) = V(u(x))\delta(x - y),
\]
\[
g_0(x, y) = g_0\delta(r - r')K_0(t, t'),
\]
\[
g_1(x, y) = g_1\delta(t - t')K_1(r, r'),
\]
then it follows from (1) and (3),
\[
S[u] = \frac{1}{2} g_0 \int_{\mathbb{R}} dr \int_{\mathbb{R}} dt \int_{\mathbb{R}} dt' \partial_t u(t, r)K_0(t, t')\partial_{r'} u(t', r) +
\]
\[
+ \frac{1}{2} g_1 \int_{\mathbb{R}} dt \int_{\mathbb{R}} dr \int_{\mathbb{R}} dr' \partial_r u(t, r)K_1(r, r')\partial_{r'} u(t, r') - \int dt \int dr V(u(t, r)),
\]
and the time and space dependent kernels are separated in the terms with derivatives.

We will be interested in a homogeneous case
\[
K_1(r, r') = K_1(r - r'),
\]
and an algebraically decaying kernel $K_1$ with a power tail, i.e.,

$$K_1(\lambda r) = (\lambda)^{1-\alpha}K_1(r), \quad (1 < \alpha < 2). \tag{5}$$

Similarly, we can consider

$$K_0(t, t') = K_0(t - t')$$

for $0 < t' < t$ as a homogeneous function of order $1 - \beta$:

$$K_0(\lambda t') = \lambda^{1-\beta}K_0(t'), \quad (0 < \beta < 2, \quad 0 < t' < t). \tag{6}$$

The relation (5) means that we have power-law long-range interaction in the system. Equation (6) indicates the memory effects with power-law memory function, which can be regarded as the influence of the environment. Just this case of the power-law dependences of $K_0(t)$ and $K_1(r)$, (5) and (6), will be considered to derive the field equations with fractional derivatives.

### 2.2 Gateaux differential and variation of action

The field equations will be derived by using the Gateaux differential \cite{46, 47, 48} of $S[u]$ at the point $u(x)$, which is defined as the limit

$$\delta S[u,h] = \left( \frac{d}{d\varepsilon} S[u + \varepsilon h] \right)_{\varepsilon=0} = \lim_{\varepsilon \to 0} \frac{S[u + \varepsilon h] - S[u]}{\varepsilon}, \tag{7}$$

and which exists for fairly smooth integrable functions $h(x) = \delta u(x)$. The Gateaux derivative is slightly different from the Frechet derivative $\delta_F S[u, h]$, where

$$\lim_{\|h\| \to 0} \frac{\|S[u + h] - S[u] - \delta_F S[u, h]\|}{\|h\|} = 0. \tag{8}$$

The Gateaux derivative is more general concept than Frechet derivative. If a function is Frechet differentiable, it is also Gateaux differentiable, and $\delta S[u, h]$ is a linear operator. However, not every Gateaux differentiable function is Frechet differentiable. In general, unlike other forms of derivatives, the Gateaux derivative is not linear with respect to $h(x)$.

The action (1) for $u + \varepsilon h$ is

$$S[u + \varepsilon h] = \int_R d^2x \int_R d^2y \left( \frac{1}{2} \partial_t (u(x) + \varepsilon h(x)) g_0(x, y) \partial_t (u(y) + \varepsilon h(y)) + \right.$$
This expression up to the order \( \varepsilon \) has the form

\[
S[u + \varepsilon h] = S[u] + \varepsilon \int_R d^2x \int_R d^2y \left( \frac{1}{2} \partial_r h(x) g_0(x, y) \partial_r u(y) + \frac{1}{2} \partial_r u(x) g_0(x, y) \partial_r h(y) + \frac{1}{2} \partial_r h(x) g_1(x, y) \partial_r u(y) + \frac{1}{2} \partial_r u(x) g_1(x, y) \partial_r h(y) - \frac{\partial V(u(x), u(y))}{\partial u(x)} h(x) - \frac{\partial V(u(x), u(y))}{\partial u(y)} h(y) \right) + \ldots
\]

In the second, fourth and 6th terms of the right hand side, we change the variables \( x \leftrightarrow y \).

Then

\[
S[u + \varepsilon h] = S[u] + \varepsilon \int_R d^2x \int_R d^2y \left( \frac{1}{2} \partial_r h(x) g_0(x, y) + g_0(y, x) \right) \partial_r u(y) + \frac{1}{2} \partial_r h(x) [g_1(x, y) + g_1(y, x)] \partial_r u(y) - \frac{\partial [V(u(x), u(y)) + V(u(y), u(x))]}{\partial u(x)} h(x) \right) + \ldots
\]

It is convenient to introduce the functions

\[
K_0(x, y) = \frac{1}{2} [g_0(x, y) + g_0(y, x)],
\]

\[
K_1(x, y) = \frac{1}{2} [g_1(x, y) + g_1(y, x)],
\]

\[(9) \quad U(u(x), u(y)) = V(u(x), u(y)) + V(u(y), u(x)).\]

Then the variation of action is

\[
\delta S[u, h] = \lim_{\varepsilon \to 0} \frac{S[u + \varepsilon h] - S[u]}{\varepsilon} = \int_R d^2x \int_R d^2y \left( \partial_r h(x) K_0(x, y) \partial_r u(y) + \partial_r h(x) K_1(x, y) \partial_r u(y) - \frac{\partial U(u(x), u(y))}{\partial u(x)} h(x) \right).
\]

Using the relations

\[
\partial_r h(x) K_0(x, y) \partial_r u(y) = \partial_r [h(x) K_0(x, y) \partial_r u(y)] - \partial_r [K_0(x, y) \partial_r u(y)] h(x),
\]

\[
\partial_r h(x) K_1(x, y) \partial_r u(y) = \partial_r [h(x) K_1(x, y) \partial_r u(y)] - \partial_r [K_1(x, y) \partial_r u(y)] h(x),
\]

\[
\partial_r \partial_r u(y) = \partial_r \partial_r u(y) = 0.
\]
and the boundary condition

$$[h(y)]_{\partial R} = 0,$$

we get

$$\delta S[u, h] = \int_R d^2 x \, h(x) \int_R d^2 y \left( -\partial_t [K_0(x, y)] \partial_r u(y) - \partial_r [K_1(x, y)] \partial_r u(y) - \frac{\partial U(u(x), u(y))}{\partial u(x)} \right).$$  \hspace{1cm} (10)

For the symmetric potential

$$U(u(x), u(y)) = U(u(x))\delta(x - y),$$

equation (10) transforms into

$$\delta S[u, h] = -\int_R d^2 x h(x) \left( \int_R d^2 y \partial_t K_0(x, y) \partial_r u(y) + \int_R d^2 y \partial_r K_1(x, y) \partial_r u(y) + \frac{\partial U(u(x))}{\partial u(x)} \right).$$  \hspace{1cm} (11)

The dynamical equation follows from the stationary action principle

$$\delta S[u, h] = 0$$

for any $h$. The field $u = u(x)$, which leads to a minimum or saddle values of $S[u]$, describes the space-time evolution. For the action (11), the stationary principle gives

$$\int_R d^2 y \partial_t K_0(x, y) \partial_r u(y) + \int_R d^2 y \partial_r K_1(x, y) \partial_r u(y) + \frac{\partial U(u(x))}{\partial u(x)} = 0.$$  \hspace{1cm} (12)

It is an integro-differential equation, which allows us to derive field equations for different cases of the kernels $K_0(x, y)$ and $K_1(x, y)$.

### 2.3 Special cases

Let us consider here two special cases: (a) system without memory and with local interaction in space, (b) field with power-law memory and long-range interaction.

(a) In absence of memory and for local interaction the kernels (9) are defined at the only instant $t$ and point $r$, i.e.,

$$K_0(x, y) = g_0\delta(x - y), \quad K_1(x, y) = g_1\delta(x - y)$$
with some constants $g_0$ and $g_1$. Then equation (12) gives
\[ g_0 \partial_t^2 u(t, r) + g_1 \partial_r^2 u(t, r) + \frac{\partial U(u(t, r))}{\partial u(t, r)} = 0. \]

For $g_0 = 1$, $g_1 = -1$, and
\[ U(u(t, r)) = -\cos u(t, r), \]
we get the sine-Gordon equation
\[ \partial_t^2 u(t, r) - \partial_r^2 u(t, r) + \sin u(t, r) = 0. \] (13)

(b) In this example, we show how time and space variables can be separated leaving a possibility to consider the system with power-law memory and long-range interaction. Let $K_0(x, y)$ and $K_1(x, y)$ have the form
\[ K_0(x, y) = \delta(r - r')K_0(t, t'), \] (14)
\[ K_1(x, y) = \delta(t - t')K_1(r, r'), \] (15)
where $x = (t, r)$, and $y = (t', r')$. Then field equation (12) can be presented as
\[ Z_t(t, r) + Z_r(t, r) + \frac{\partial U(u(t, r))}{\partial u(t, r)} = 0, \] (16)
where
\[ Z_t(t, r) = \int_{-\infty}^{+\infty} dt' \partial_t K_0(t, t') \partial_r u(t', r), \] (17)
\[ Z_r(t, r) = \int_{-\infty}^{+\infty} dr' \partial_r K_1(r, r') \frac{\partial u(t, r')}{\partial r'} \] (18)
with separated spatial and temporal kernels. Till now, the kernels $K_0(t, t')$ and $K_1(r, r')$ were not defined. Their specific choice to present a long-term memory and long-range interaction will be in the next two subsections.

2.4 Power-law memory

Consider the kernel $\partial_t K_0(t, t')$ of integral (17) in the interval $t' \in (0, t)$ such that
\[ \partial_t K_0(t, t') = \begin{cases} \mathcal{M}(t - t'), & 0 < t' < t; \\ 0, & t' > t, \quad t' < 0. \end{cases} \] (19)
Then

$$Z_t(t, r) = \int_0^t dt' \mathcal{M}(t - t') \partial_{t'} u(t', r) = \mathcal{M}(t) * \partial_t u(t, r).$$

(20)

As the result, we have the evolution field equation in which the quantity $Z_t(t, r)$ is related to another quantity $\partial_{t'} u(t', r)$ through a memory function $\mathcal{M}(t)$. Equation (20) is a typical non-Markovian equation obtained in studying of systems coupled to an environment, where environmental degrees of freedom being averaged. For a system without memory, we have

$$\mathcal{M}(t - t') = \delta(t - t'),$$

(21)

and

$$Z_t(t, r) = \int_0^t \delta(t - t') \partial_{t'} u(t', r) dt' = \partial_t u(t, r),$$

(22)

i.e., the function $Z_t(t, r)$ is defined by $\partial_t u(t, r)$ at the only current instant $t$.

Consider now the power-like memory function

$$\mathcal{M}(t - t') = \frac{g_0}{\Gamma(1 - \beta)} \frac{1}{(t - t')^\beta}, \quad (0 < \beta < 1),$$

(23)

where $g_0$ is a constant that can be presented as a strength of perturbation induced by the environment, and $\Gamma(1 - \beta)$ is the Gamma function.

Substitution of (23) into (20) gives

$$Z_t(t, r) = \frac{g_0}{\Gamma(1 - \beta)} \int_0^t (t - t')^{-\beta} \partial_{t'} u(t', r) dt' = g_0 C_0^\beta D_t^\beta u(t, r), \quad (0 < \beta < 1),$$

(24)

where $C_0^\beta D_t^\beta$ is the left fractional Caputo derivative [36, 37].

For the kernel $\partial_t \mathcal{K}_0(t, t')$ in the integral (17) such that

$$\partial_t \mathcal{K}_0(t, t') = \begin{cases} \mathcal{M}'(t' - t), & t < t' < 0; \\ 0, & t' > 0, \ t' < t, \end{cases}$$

(25)

where

$$\mathcal{M}'(t' - t) = \frac{(-1)g_0'}{\Gamma(1 - \beta)} \frac{1}{(t' - t)\theta}, \quad (0 < \beta < 1),$$

(26)

we get

$$Z_t(t, r) = \frac{(-1)g_0'}{\Gamma(1 - \beta)} \int_t^0 \frac{\partial_{t'} u(t', r')}{(t' - t)\theta} dt' = g_0' C_0^\beta D_t^\beta u, \quad (0 < \beta < 1),$$

(27)
which is the right fractional Caputo derivative \[37, 36\].

In general, the kernel \( K_0(t, t') \) can include positive and negative intervals of time. Then

\[
\partial_t K_0(t, t') = \begin{cases} 
\mathcal{M}(t - t'), & 0 < t' < t; \\
\mathcal{M}'(t' - t), & t < t' < 0; \\
0, & 0 < t < t', \quad t' < t < 0,
\end{cases}
\]  

(28)

where \( \mathcal{M}(t - t') \) and \( \mathcal{M}'(t' - t) \) are defined by (23) and (26). Then, we get a linear combination of left and right Caputo derivatives

\[
Z_t(t, r) = g_0 \, C_0 \, D_t^{\beta+1} u(t, r) + g_0' \, C_t \, D_0^{\beta+1} u(t, r), \quad (0 < \beta < 1).
\]  

(29)

As a result, field equation (12) consists of fractional time derivatives, and it will be written in Sec. 2.6.

We also will be interested in the case when

\[
K_0(t, t') = \begin{cases} 
\mathcal{M}(t - t'), & 0 < t' < t; \\
0, & t > t', \quad t' < 0,
\end{cases}
\]  

(30)

or

\[
K_0(t, t') = \begin{cases} 
\mathcal{M}(t - t'), & 0 < t' < t; \\
\mathcal{M}'(t' - t), & t < t' < 0; \\
0, & 0 < t < t', \quad t' < t < 0,
\end{cases}
\]  

(31)

(compare to (19) and (25)), with the functions \( \mathcal{M}, \mathcal{M}' \) as in (23) and (26). Substitution of (19) and (25) into (17), and integration by parts gives, similarly to (24) and (29),

\[
Z_t(t, r) = g_0 \, C_0 \, D_t^{\beta+1} u(t, r), \quad (0 < \beta < 1),
\]  

(32)

or

\[
Z_t(t, r) = g_0 \, C_0 \, D_t^{\beta+1} u(t, r) + g_0' \, C_t \, D_0^{\beta+1} u(t, r), \quad (0 < \beta < 1)
\]  

(33)

with the same field equation (16). Depending on different kernels (19), (25) and (30), (31), we obtain field equations with different order of time derivatives (see in Sec. 2.6).
The Caputo fractional derivatives can be linked to fractional powers of variable $s$ for the corresponding Laplace-transformed equation. It is known \[50, 51, 36\], that the Laplace transform of the Caputo fractional derivative is

\[
\int_0^\infty e^{-st} \left[ C_0 D_t^\beta u(t, r) \right] dt = s^\beta v(s, r) - \sum_{s=0}^{m-1} s^{\beta - q - 1} u^{(q)}(0, r), \tag{34}
\]

where $m - 1 < \beta \leq m$,

\[
u^{(q)}(t, r) = \frac{\partial^q u(t, r)}{\partial t^q},
\]

and $v(s, r)$ is the Laplace transform of $u(t, r)$:

\[
v(s, r) = \int_0^\infty e^{-st} u(t, r) dt. \tag{35}
\]

Note that formula (34) involves the initial conditions $u^{(q)}(0, r)$ as integer derivatives $u^{(q)}(t, r)$ with respect to time. Therefore we can put the initial conditions in a usual way. The functions $u(t, r)$ satisfy the condition

\[
\int_0^\infty e^{-st} |u(t, r)| < \infty. \tag{36}
\]

For $0 < \beta \leq 1$, Eq. (34) has the form

\[
\int_0^\infty e^{-st} \left[ C_0 D_t^\beta u(t, r) \right] dt = s^\beta v(s, r) - s^{\beta - 1} u(0, r). \tag{37}
\]

Inversion of (37) gives

\[
C_0 D_t^\beta u(t, r) = \frac{1}{2\pi i} \int_{Br} e^{st} \left[ s^\beta v(s, r) - s^{\beta - 1} u(0, r) \right] ds, \tag{38}
\]

where $Br$ denotes the Bromwich contour.

The final equation of $u(t, r)$ will be written in Sec. 2.6.

### 2.5 Non-local interaction

Consider the kernel $K_1(r, r')$ of the integral (18) as

\[
K_1(r, r') = C(|r - r'|) = \frac{-g_1}{\cos(\pi\alpha/2)\Gamma(2 - \alpha)} \frac{1}{|r - r'|^{\alpha - 1}}, \quad (1 < \alpha < 2) \tag{39}
\]

that describes the power law interaction. Then, we obtain

\[
Z_r(t, r) = \int_{-\infty}^{+\infty} dr' C(|r - r'|) \frac{\partial^2 u(t, r')}{\partial r'^2} = g_1 \frac{\partial^\alpha}{\partial |r'|^\alpha} u(t, r), \tag{40}
\]
where the fractional Riesz derivative with respect to coordinates is introduced [34, 37] (see also Appendix 1).

It is known [34] the connection between the Riesz fractional derivative and its Fourier transform

$$\mathcal{F} : \frac{\partial^\alpha}{\partial |r|^\alpha} \to -|k|^\alpha,$$  \hspace{1cm} (41)

where $\mathcal{F}$ is defined by

$$\tilde{f}(k) = (\mathcal{F} f)(k) = \int_{-\infty}^{+\infty} f(r) e^{-i k r} dr,$$  \hspace{1cm} (42)

and $\mathcal{F}^{-1}$ is an inverse Fourier transform

$$f(r) = (\mathcal{F}^{-1} \tilde{f})(r) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \tilde{f}(k) e^{i k r} dk.$$  \hspace{1cm} (43)

The fractional Riesz derivatives describes properties of fractal media or complex media with fractional dispersion law (see for example in [45]).

### 2.6 Field equations with fractional derivatives

Substitution of (29) and (39) into (16) gives the fractional field equation

$$g_0 C^\beta_0 D_t^\beta u(t, r) + g_0' C^\beta_0 D_t^\beta u(t, r) + g_1 \frac{\partial^\alpha}{\partial |r|^\alpha} u(t, r) + \frac{\partial U(u(t, r))}{\partial u(t, r)} = 0, \hspace{1cm} (1 < \alpha < 2, \hspace{0.5cm} 0 < \beta < 1).$$  \hspace{1cm} (44)

This equation describes the field of the system with power-law memory and long-range interaction. Depending on the situation, $g_0$ or $g_0'$ could be zero or not. For example, the potential

$$U(u(t, r)) = \frac{a}{2} u^2(t, r) + \frac{b}{4} u^4(t, r)$$

in Eq. (44) gives the fractional time-dependent generalization of the Ginzburg-Landau equation

$$g_0 C^\beta_0 D_t^\beta u(t, r) + g_0' C^\beta_0 D_t^\beta u(t, r) + g_1 \frac{\partial^\alpha}{\partial |r|^\alpha} u(t, r) + au(t, r) + bu^3(t, r) = 0, \hspace{1cm} (1 < \alpha < 2, \hspace{0.5cm} 0 < \beta < 1).$$  \hspace{1cm} (45)

In the case of the time kernel [31], Eq. (44) is replaced by

$$g_0 C^\beta_1 D_t^{\beta + 1} u(t, r) + g_0' C^\beta_1 D_t^{\beta + 1} u(t, r) + g_1 \frac{\partial^\alpha}{\partial |r|^\alpha} u(t, r) + \frac{\partial U(u(t, r))}{\partial u(t, r)} = 0, \hspace{1cm} (1 < \alpha < 2, \hspace{0.5cm} 0 < \beta < 1).$$  \hspace{1cm} (46)
This equation has increased by one the order of time derivative and can be applied to the wave propagation in media with fractional dispersion law. Particularly in the case when only right time derivative should be used (one-directional wave propagation) and the potential is

\[ U(u(t, r)) = -\cos u(t, r), \]

Eq. (46) gives the fractional sine-Gordon equation

\[ \frac{C}{\partial} \partial_{t}^{\beta+1} u(t, r) - \frac{\partial^\alpha}{\partial |r|^\alpha} u(t, r) + \sin u(t, r) = 0, \quad (1 < \alpha < 2, \quad 0 < \beta < 1), \]

where we put \( g_0 = 1, \ g'_0 = 0 \) and \( g_1 = -1 \), and which is a generalization of (13) for noninteger derivatives with respect to time and coordinate.

Finally, let us simplify the notation and write down Eqs. (44) or (46) as

\[ g \frac{\partial^\beta}{\partial t^\beta} u(t, r) + g_1 \frac{\partial^\alpha}{\partial |r|^\alpha} u(t, r) + U'(u(t, r)) = 0, \]

where \( \partial^\beta/\partial t^\beta \) stays for left, right, or both Caputo derivatives (in the latter case, the constant \( g \) can be different for different derivatives), and \( 0 < \beta < 2, \ 0 < \alpha < 2, \) and \( U'(u) = \partial U/\partial u \). Let us comment that the choice of the derivative \( \partial^\beta/\partial t^\beta \) depends on the type of initial conditions and the processes, and other than Caputo derivative can appear.

In Appendix 2, we present a generalization of Eqs. (43), (46) for the \( n \)-dimensional coordinate case.

**3 Fractional Ginzburg-Landau equation**

Since the variable \( x \) in \((1)\) can be not specified, one can apply a similar technique to other problems, defined by the extremum of a functional with long-range interaction. As an example, consider a free energy functional for a model of Ginzburg-Landau equation (GLE) that consists of long-range interaction.

The fractional generalization of the Ginzburg-Landau equation (FGLE) was suggested in Ref. [45]. This equation can be used to describe the processes in complex media [52, 53]. Some properties of FGLE are discussed in [41, 54, 55].
It is known [56] that the stationary GLE
\[ g\Delta u - au - bu^3 = 0 \]
can be derived as the variational Euler-Lagrange equation
\[ \frac{\delta F[u]}{\delta u(r)} = 0 \] (49)
for the free energy functional
\[ F[u] = F_0 + \frac{1}{2} \int_R [g(\partial u)^2 + au^2 + \frac{b}{2}u^4]dr, \] (50)
where \( \partial u = \partial u(r)/\partial r \), and the integration is over a region \( R \). Here \( F_0 \) is a free energy of the normal state, i.e. \( F[u] \) for \( u = 0 \).

Consider the thermodynamic potential (free energy functional) \( F[u] \) for the non-equilibrium state of a medium with power-law non-local interaction. The generalized free energy functional has the form
\[ F[u] = F_0 + \int_R dr \int_R dr' F(u(r), u(r'), \partial u(r), \partial u(r')), \] (51)
where the generalized density of free energy
\[ F(u(r), u(r'), \partial u(r), \partial u(r')) = \]
\[ = \frac{1}{2}gK_1(r, r')\partial^\alpha u(r')\partial^\alpha u(r') + \left( \frac{a}{2}u^2(r) + \frac{b}{4}u^4(r) \right) \delta(r - r') \] (52)
has the kernel \( K_1(r, r') \) defined as in (39). The variational equation (49) gives
\[ g\partial^\alpha [\partial |r|^\alpha u(r) + au(r) + bu^3(r)] = 0, \quad (1 < \alpha < 2), \] (53)
which can be called the \( \alpha \)-FGLE. This equation can be easily generalized for the 3-dimensional variable \( r \) (see Appendix 2).

In the non-stationary case, Eq. (49) should be replaced by
\[ \frac{\partial u(t, r)}{\partial t} = \frac{\delta F[u]}{\delta u(t, r)}, \] (54)
(see [57]), where there is no explicit time memory effects. To put such memory into (54), we can write
\[ \int_0^t dt'\mathcal{M}(t-t')\frac{\partial u(t', r)}{\partial t'} = \frac{\delta F[u(t, r)]}{\delta u(t, r)}, \] (55)
and to assume for $\mathcal{M}(t-t')$ the power law \[^{23}\]. Then we arrive to a nonstationary generalization of $(\alpha, \beta)$-FGLE

$$\frac{\partial^\beta u(t, r)}{\partial t^\beta} = g \frac{\partial^\alpha}{\partial |r|^\alpha} u(t, r) + au(t, r) + bu^3(t, r), \quad (0 < \beta < 1, \quad 1 < \alpha < 2), \quad (56)$$

where $\partial^\beta /\partial t^\beta$ is used for Caputo derivative while any other fractional derivative can be applied by modifying the memory kernel $\mathcal{M}(t)$, and initial conditions.

It is worthwhile to compare Eq. (56) to its counterpart nonlinear Schrödinger equation (NSE)

$$i \frac{\partial u}{\partial t} = g \Delta u + au + b|u|^2u \quad (57)$$

with $u = u(t, r)$, and complex $a, b$. In the case of $\Delta_\perp$ instead of $\Delta$, Eq. (57) also known as parabolic equation for wave propagation. Generalization of (57) for the case of fractional space derivative and non-local interaction (\(\alpha\)-NLS) was considered in \[^{45}\] \[^{55}\]:

$$i \frac{\partial u}{\partial t} = -g(-\Delta)^{\alpha/2} u + au + b|u|^2u, \quad (1 < \alpha < 2), \quad (58)$$

where the fractional Laplacian is defined through the Fourier transform and Riesz derivatives \[^{34}\]:

$$\mathcal{F} : (-\Delta)^{\alpha/2} \longrightarrow (k^2)^{\alpha/2}. \quad (59)$$

Similar to (56) generalized $(\alpha, \beta)$-NLS equation has the form

$$\frac{\partial^\beta u}{\partial t^\beta} = -g(-\Delta)^{\alpha/2} u + au + b|u|^2u, \quad (60)$$

where $\partial^\beta /\partial t^\beta$ is now Riemann-Liouville derivative with the Fourier transform

$$\mathcal{F} : \frac{\partial^\beta}{\partial t^\beta} \longrightarrow (i\omega)^\beta, \quad (61)$$

and the memory function is working through the parameter $\beta$.

It is convenient also to interpret (60) through the nonlinear dispersion law by applying to (60) Fourier transform in both time and space. Then it gives with the help of (59) and (61)

$$(i\omega)^\beta = -g(k^2)^{\alpha/2} + a + b|u|, \quad (62)$$

which was derived for $\beta = 1$ in \[^{45}\] \[^{55}\]. Onset of fractional time derivative in (60) can stop self-focusing of waves, steepening of the solution, developing of a singularity. These phenomena need a special analysis.
Eq. (60) can be easily generalized for the anisotropic case

\[
\frac{\partial^3 u}{\partial t^3} = -g_\perp (-\Delta_\perp)^{\alpha_\perp/2} u - g_\parallel (-\Delta_\parallel)^{\alpha_\parallel/2} u + au + b|u|^2 u
\]

(63)

with a corresponding anisotropic dispersion equation instead of (62) (see also [45, 55] for \(\beta = 1\)).

4 Discrete system with memory and long-range interaction

4.1 Equation for discrete chains

In this section, we show how the obtained results of Sec. 2 can be applied to discrete systems, for example chains of interacting particles.

Long-range interaction is a subject of a great interest since a long time. Thermodynamics of a model of classical spins with long-range interactions has been considered in [58, 59, 60, 61, 62]. The long-range interactions have been widely studied in discrete systems of lattices as well as in their continuous analogues: solitons in one-dimensional lattice with the Lennard-Jones-type interaction [63]; kinks in the Frenkel-Kontorova model [64]; time periodic spatially localized solutions (breathers) [65, 66]; energy and decay properties of discrete breathers in the framework of the Klein-Gordon equation [68], and discrete nonlinear Schrödinger equations [19]. A remarkable property of the dynamics described by the equation with fractional space derivatives is that the solutions have power-like tails. Similar features were observed in the lattice models with power-like long-range interactions [69, 65, 66, 70, 71, 42]. Long-range interaction can be relevant to the systems such as neuron populations [72] and Josephson junctions [73]. The synchronization of chaotic systems with power-law long-range interactions were considered in [73, 41, 74]. A model of coupled map lattices with coupling that decays in a power-law, was considered in [75, 76, 77, 74]. Note that the fractional power-law dependence can be linked to fractal properties of heterogeneous surfaces [9] and power-law decay of structure factor for geometry of colloids aggregates [10]. It will be shown how long-range coupling of particles and memory function with power tails can reveal a new type of particle equations with fractional derivatives and the connection of these equations to their continuous media.
counterpart.

Consider an one-dimensional chain of interacting oscillators that can be described by the action,

$$ S[u] = \int_{-\infty}^{+\infty} dt \int_{-\infty}^{+\infty} dt' \sum_{n=-\infty}^{+\infty} L(u_n(t), u_n(t'), \dot{u}_n(t), \dot{u}_n(t')) $$

where $u_n$ are displacements of the oscillators from the equilibrium and $L$ is a Lagrangian. If

$$ L(u_n(t), u_n(t'), \dot{u}_n(t), \dot{u}_n(t')) = L(u_n(t), \dot{u}_n(t)) \delta(t - t'), $$

then we have the chain without memory.

Let us introduce a generalization of (64) with the action

$$ S[u_n] = \int_{-\infty}^{+\infty} dt \int_{-\infty}^{+\infty} dt' \left( \sum_{n=-\infty}^{+\infty} \left[ \frac{1}{2} K_0(t, t') \dot{u}_n(t) \dot{u}_n(t') - V(u_n(t), u_n(t')) \right] - \sum_{n,m=-\infty}^{+\infty} U(u_n(t), u_m(t')) \right). $$

(65)

In the same way as in Sec. 2, let us separate the kinetic energy from the long-range interaction and potential parts:

$$ U(u_n(t), u_m(t')) = \frac{1}{4} g_0 J_\alpha(|n - m|) (u_n(t) - u_m(t))^2 \delta(t - t'), $$

(66)

$$ V(u_n(t), u_n(t')) = V(u_n(t)) \delta(t - t'). $$

(67)

Note that (67), (66) in (64) are equivalent to

$$ U(u_n(t), u_m(t')) = -\frac{1}{2} g_0 J_\alpha(|n - m|) u_n(t) u_m(t) \delta(t - t'), $$

(68)

$$ V(u_n(t), u_n(t')) = \left( V(u_n(t)) + \frac{1}{2} \tilde{g} u_n^2(t) \right) \delta(t - t'), $$

(69)

where

$$ \tilde{g} = g_0 \sum_{m \neq 0} J_\alpha(|m|). $$

The second term in the right hand side of (69) removes the infinity of the interaction (68) in the continuous medium limit. The interparticle interaction $J_\alpha(|n - m|)$ in (66) is defined by

$$ J_\alpha(|n - m|) = \frac{1}{|n - m|^{\alpha+1}}, \quad (\alpha > 0). $$

(70)
Some other examples of functions $J_\alpha(n)$ can be found in [44].

Using (19) and (23), for the kernel $K_0(t, t')$ in (65), i.e.

$$
\partial_t K_0(t, t') = \begin{cases} 
\frac{\mathbb{g}_0}{t^{(1-\beta)}} (t-t')^{-\beta}, & 0 < t' < t \ (0 < \beta < 1); \\
0, & t' > t, \ t' < 0,
\end{cases} \quad (71)
$$

we obtain the corresponding Euler-Lagrange equations

$$
C_0 D_\alpha^\beta u_n(t) + g_0 \sum_{m=-\infty}^{+\infty} \sum_{m \neq n} J_\alpha(|n-m|) [u_m(t) - u_n(t)] + F(u_n(t)) = 0, \quad (72)
$$

where $F(u) = \partial V(u)/\partial u$.

A continuous limit of equation (72) can be defined by a transform operation from $u_n(t)$ to $u(x, t)$ [40, 41, 42, 43, 44]. First, define $u_n(t)$ as Fourier coefficients of some function $\hat{u}(k, t)$, $k \in [-K/2, K/2]$, i.e.

$$
\hat{u}(t, k) = \sum_{n=-\infty}^{+\infty} u_n(t) e^{-ikx_n} = \mathcal{F}\{u_n(t)\}, \quad (73)
$$

where $x_n = n\Delta x$, and $\Delta x = 2\pi/K$ is a distance between nearest particles in the chain, and

$$
u_n(t) = \frac{1}{K} \int_{-K/2}^{+K/2} dk \ \hat{u}(t, k) e^{ikx_n} = \mathcal{F}_{\Delta}^{-1}\{\hat{u}(t, k)\}. \quad (74)
$$

Secondly, in the limit $\Delta x \to 0 \ (K \to \infty)$ replace $u_n(t) = (2\pi/K)u(x_n, t) \to u(x, t)dx$, and $x_n = n\Delta x = 2\pi n/K \to x$. In this limit, Eqs. (73), (74) are transformed into the integrals

$$
\hat{u}(t, k) = \int_{-\infty}^{+\infty} dx \ e^{-ikx} u(t, x) = \mathcal{F}\{u(t, x)\} = \lim_{\Delta x \to 0} \mathcal{F}_{\Delta}\{u_n(t)\}, \quad (75)
$$

$$
u(t, x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dk \ e^{ikx} \hat{u}(t, k) = \mathcal{F}^{-1}\{\hat{u}(t, k)\} = \lim_{\Delta x \to 0} \mathcal{F}_{\Delta}^{-1}\{\hat{u}(t, k)\}. \quad (76)
$$

Applying (73) to (72) and performing the limit (75), we obtain

$$
\frac{\partial^\beta u(t, x)}{\partial t^\beta} + g_\alpha \frac{\partial^\alpha u(t, x)}{\partial |x|^\alpha} + F(u(t, x)) = 0, \quad (0 < \beta < 2, \ 1 < \alpha < 2), \quad (77)
$$

where

$$
g_\alpha = 2g_0(\Delta x)^\alpha \Gamma(-\alpha) \cos\left(\frac{\pi\alpha}{2}\right) \quad (78)
$$
is the renormalized constant. The Caputo time derivative is written in a simplified form $\partial^{\beta}/\partial t^{\beta}$, and the value of $\beta$ depends on the choice of memory function. The equation (77) can be generalized to a nonlinear long-range interaction. Consider, instead of (72),

$$C_{0} D_{t}^{\beta} u_{n} + g_{0} \sum_{m=-\infty}^{+\infty} J_{\alpha}(|n - m|) [f(u_{m}) - f(u_{n})] + F(u_{n}) = 0,$$

where $f(u)$ is a function of $u$. For example, $f(u) = u^{2}$ or $f(u) = u - gu^{2}$. Then the corresponding continuous limit for the same $J_{\alpha}(|n - m|)$ as (70) leads to the time-space fractional equation

$$\frac{\partial^{\beta} u(t, x)}{\partial t^{\beta}} + g_{s} \frac{\partial^{\alpha}}{\partial |x|^{\alpha}} f(u(t, x)) + F(u(t, x)) = 0, \quad (0 < \beta < 2, \ 1 < \alpha < 2).$$

Equations (77) and (80) for $\beta = 1$ were considered in [40, 41, 42, 43, 44]. Generalization to $0 < \beta < 2$ significantly extends the area of their applications. A physical motivation is that a dynamical process typically reveals fractional features simultaneously in space and time. Such situation just was considered in chaotic dynamics [31, 32]. Now we have such a possibility far beyond the fractional kinetics. An evident generalization of (80) is for the inter-particle interactions with two or more different kernels. For example one can consider regular terms without long memory together with a term with long memory:

$$\frac{\partial^{\beta} u(t, x)}{\partial t^{\beta}} + g_{s} \frac{\partial^{\alpha}}{\partial |x|^{\alpha}} u(t, x) + g_{\alpha} \frac{\partial^{\alpha} u(t, x)}{\partial |x|^{\alpha}} + F(u(t, x)) = 0, \quad (0 < \beta < 2, \ 1 < \alpha < 2)$$

with some $g_{s}$ and $g_{\alpha}$ and integer $s$.

5 Conclusion

Starting from a variation of the action functional, we consider different type of kernels that define the character of particle interaction and the influence of an environment on the memory function. The main stress is on the long-range interaction and memory that occur in complex media. The case when the interaction or memory function have power-law structure the system can be described by the equation of motion with fractional derivatives $\partial^{\beta}/\partial t^{\beta}$ and $\partial^{\alpha}/\partial |x|^{\alpha}$ depending on the power of interaction and memory function. We have discussed how different types of the derivatives and possible values $(\alpha, \beta)$ may occur with respect to the type of memory.
and interaction. The final equations of motions can be considered as a new kind of tool to study dynamics with space-time distributed interactions. Number of examples of such kind of systems can be found in the reviews [1][32] related to random or chaotic processes. The study of this paper shows that the list of possible applications of fractional equations can be naturally expanded to include non-chaotic and non-random dynamics as well.

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"The nature of electronic states in a disordered chain with long-ranged hopping amplitudes" Physica A 256 (1998) 18-29; R.P.A. Lima, M.L. Lyra, J.C. Cressoni, "Multifractality of one electron eigen states in 1D disordered long-range models" Physica A 295 (2001) 154-157.

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Appendix 1: Fractional derivatives

The fractional derivative has different definitions \[34, 35\], and exploiting any of them depends on the kind of the problems, initial (boundary) conditions, and the specifics of the considered physical processes. The classical definition is the so-called Riemann-Liouville derivative \[34, 35\]. The left and right Riemann-Liouville derivatives for an interval \([a, b]\) are defined by

\[
a D^\alpha_t u(x) = \frac{1}{\Gamma(n - \alpha)} \frac{\partial^n}{\partial x^n} \int_a^x \frac{u(z)dz}{(x - z)^{\alpha-n+1}},
\]

\[
\iota D^\alpha_t u(x) = \frac{(-1)^n}{\Gamma(n - \alpha)} \frac{\partial^n}{\partial x^n} \int_x^b \frac{u(z)dz}{(z - x)^{\alpha-n+1}}, \tag{82}
\]

where \(n - 1 < \alpha < n\). End-points \(a, b\) can be extended to \(-\infty, \infty\) if the integral exists.

Due to reasons, concerning the initial conditions, it is more convenient to use the Caputo fractional derivatives \[36\]. Its main advantage is that the initial conditions take the same form as for integer-order differential equations. The Caputo fractional derivatives are

\[
C_a D^\alpha_x u(x) = \frac{1}{\Gamma(n - \alpha)} \int_a^x \frac{u^{(n)}(z)dz}{(x - z)^{\alpha-n+1}},
\]

\[
C_x D^\alpha_b u(x) = \frac{(-1)^n}{\Gamma(n - \alpha)} \int_x^b \frac{u^{(n)}(z)dz}{(z - x)^{\alpha-n+1}}, \tag{83}
\]

where \(u^{(n)}(z) = d^n u(z)/dz^n\), and \(n - 1 < \alpha < n\). The Caputo fractional derivatives can be defined through the Riemann-Liouville derivatives \[37\] by

\[
C_a D^\alpha_x u(x) = a D^\alpha_x \left( u(x) - \sum_{k=0}^{n-1} \frac{(x - a)^k}{k!} u^{(k)}(a) \right),
\]

\[
C_x D^\alpha_b u(x) = x D^\alpha_b \left( u(x) - \sum_{k=0}^{n-1} \frac{(b - x)^k}{k!} u^{(k)}(b) \right), \tag{84}
\]

where \(n - 1 < \alpha < n\). These equations give

\[
a D^\alpha_x u(x) = C_a D^\alpha_x u(x) + \sum_{k=0}^{n-1} \frac{(x - a)^{k-\alpha}}{\Gamma(k - \alpha + 1)} u^{(k)}(a), \tag{85}
\]

\[
x D^\alpha_b u(x) = C_x D^\alpha_b u(x) + \sum_{k=0}^{n-1} \frac{(b - x)^{k-\alpha}}{\Gamma(k - \alpha + 1)} u^{(k)}(b), \tag{86}
\]
The Riesz fractional derivative of order $\alpha$ are

$$\frac{\partial^{\alpha}}{\partial |x|^{\alpha}} u(x) = -\frac{1}{2 \cos(\pi \alpha/2)} \left( \mathcal{D}_+^{\alpha} u(x) + \mathcal{D}_-^{\alpha} u(x) \right),$$

where $\alpha \neq 1, 3, 5, \ldots$, and $\mathcal{D}_\pm^{\alpha}$ are Riemann-Liouville fractional derivatives with infinite limits:

$$\mathcal{D}_+^{\alpha} u(x) = \frac{1}{\Gamma(n - \alpha)} \frac{\partial^n}{\partial x^n} \int_{-\infty}^{x} \frac{u(z)dz}{(x - z)^{\alpha - n + 1}},$$

$$\mathcal{D}_-^{\alpha} u(x) = \frac{(-1)^n}{\Gamma(n - \alpha)} \frac{\partial^n}{\partial x^n} \int_{x}^{\infty} \frac{u(z)dz}{(z - x)^{\alpha - n + 1}}. \quad (88)$$

Substitution of Eqs. (88) into Eq. (87) gives

$$\frac{\partial^{\alpha}}{\partial |x|^{\alpha}} u(x) = -\frac{1}{2 \cos(\pi \alpha/2)} \frac{\partial^n}{\partial x^n} \left( \int_{-\infty}^{x} \frac{u(z)dz}{(x - z)^{\alpha - n + 1}} + \int_{x}^{\infty} \frac{(-1)^n u(z)dz}{(z - x)^{\alpha - n + 1}} \right).$$

(89)

The Fourier transform of the fractional derivatives \cite{34, 37} are

$$\mathcal{F} \left( \frac{\partial^{\alpha}}{\partial |x|^{\alpha}} u(x) \right)(k) = -|k|^\alpha \tilde{u}(k),$$

$$\mathcal{F} \left( \mathcal{D}_+^{\alpha} u(x) \right)(k) = (\pm ik)^\alpha \tilde{u}(k),$$

where $\mathcal{F}$ is defined by

$$\tilde{u}(k) = (\mathcal{F} u)(k) = \int_{-\infty}^{+\infty} u(x) e^{-ikx} dx.$$  \quad (92)

The inverse Fourier transform is

$$u(x) = (\mathcal{F}^{-1} \tilde{u})(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \tilde{u}(k) e^{ikx} dk.$$  \quad (93)

**Appendix 2: n-dimensional case**

The generalization the action (1) for the case $r \in \mathbb{R}^n$, where $x = (t, r)$, and $r = (x^1, \ldots, x^n)$, gives the field equation

$$g_0 \frac{C_{\alpha t}}{C_{\alpha t}^2} u(t, r) + g_0 \frac{C_{\alpha r}}{C_{\alpha r}^2} u(t, r) + \sum_{k=1}^{n} g_k \frac{\partial^{\alpha}}{\partial |x|^\alpha} u(t, r) + \frac{\partial U(u(t, r))}{\partial u(t, r)} = 0. \quad (94)$$

For the case $r \in \mathbb{R}^n$, there exists the other possibility to define the kernels $M_t(x, y)$ and $M_r(x, y) = \{M_k(x, y), k = 1, \ldots, n\}$. We can consider

$$Z_r(t, r) = \int_{\mathbb{R}^n} d^nr' \sum_{k=1}^{n} M_k(r, r') \frac{\partial u(t, r')}{\partial x^k}. \quad (95)$$
where \( M_k(r, r') \) is Riesz kernel \([34]\):

\[
M_k(r, r') = K_{\alpha_k} (r - r') = \frac{1}{\gamma_n(\alpha_k)} \left\{ \begin{array}{ll}
|r - r'|^{\alpha_k - n} & \alpha_k - n \neq 0, 2, 4, \ldots \\
-|r - r'|^{\alpha_k - n} \ln |r - r'| & \alpha_k - n = 0, 2, 4, \ldots 
\end{array} \right.
\]

Here \( \alpha_k > 0 \), \( (\alpha_k \neq n, n + 2, n + 4, \ldots) \), and

\[
\gamma_n(\alpha) = \begin{cases} 
2^{\alpha \pi n/2} \Gamma(\alpha/2)/\Gamma(\frac{n-\alpha}{2}) & \alpha \neq n + 2k, \; n \neq -2k, \\
1 & n = -2k, \\
(-1)^{(n-\alpha)/2} 2^{\alpha - 1} \pi^{n/2} \Gamma(\alpha/2) \left[ \frac{\alpha-n}{2} \right]! & \alpha \neq n + 2k.
\end{cases}
\] (96)

Note that the multivariable Riesz integral

\[
(I^\alpha u)(t, r) = \frac{1}{\gamma_n(\alpha)} \int_{\mathbb{R}^n} \frac{u(t, r') dr'}{|r - r'|^{n-\alpha}},
\] (97)

where \( \alpha > 0 \), can be presented as convolution:

\[
(I^\alpha u)(t, r) = \int_{\mathbb{R}^n} K_\alpha (r - r') u(t, r') d^n r',
\] (98)

with the Riesz kernel \( K_\alpha (r) \). It allows us to write \([95]\) as

\[
Z_r(t, r) = \sum_{k=1}^{n} I^{\alpha_k} \frac{\partial u(t, r)}{\partial x^k}.
\] (99)

The fractional Riesz integrals of orders \( \alpha_k \) \( (k = 1, \ldots, n) \) in the field equations describe the fractal media. Then the field equation is

\[
g_0 \overset{C}{D}_t^\beta u(t, r) + g_0' \overset{C}{D}_t^\beta u(t, r) + \sum_{k=1}^{n} I^{\alpha_k} \frac{\partial u(t, r)}{\partial x^k} + \frac{\partial U(u(t, r))}{\partial u(t, r)} = 0.
\] (100)

If \( M_k(r, r') \) in \([95]\) is an operator such that

\[
M_k(r, r') u(t, r') = g_k \frac{1}{d_{n,l}(\alpha_k)} \frac{(\Delta^l u)(t, r)}{|r' |^{n+\alpha_k}},
\] (101)

where

\[
d_{n,l}(\alpha) = \frac{2^{-\alpha \pi^{1+n/2}}}{\Gamma(1+\alpha/2) \Gamma((n+\alpha)/2) \sin(\alpha \pi/2)} \sum_{k=0}^{l} \frac{(-1)^{k-l}l!}{(l-k)! k^\alpha}.
\]

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is normalized multiplier \[37\], and

\[
(\Delta_{r'}^l u)(t, r) = \sum_{k=0}^{l} (-1)^{k-1} \frac{l!}{(l-k)!k!} u(t, r - kr')
\]

is symmetrized difference \[37\], then

\[
Z_r(t, r) = \sum_{k=1}^{n} g_k \frac{\partial^{\alpha_k}}{\partial |r|^{\alpha_k}} \frac{\partial u(t, r)}{\partial x^k}.
\]

As a result, we have

\[
g_0 D_0^\beta u(t, r) + g' D_0^\beta u(t, r) + g_k \sum_{k=1}^{n} \frac{\partial^{\alpha_k}}{\partial |r|^{\alpha_k}} \frac{\partial u(t, r)}{\partial x^k} + \frac{\partial U(u(t, r))}{\partial u(t, r)} = 0,
\]

which is the field equations with \( n \) fractional Riesz multivariable derivatives.