Immirzi parameter in the Barrett-Crane model?

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We study the generalised constrained BF theory described in [1] in order to introduce the Immirzi parameter in spin foam models. We show that the resulting spin foam model is still based on simple representations and that the generalised BF action is simply a deformation of the Barrett-Crane model. The Immirzi parameter doesn’t change the representations used in the spin foam model, so it doesn’t affect the geometry of the model. However we show how it may still appear as a factor in the area spectrum.

I. INTRODUCTION

Spin foam models are an attempt to quantize general relativity in a covariant way. They can be seen as a theory of evolving spin networks [5]. The most successful model is the Barrett-Crane model [7] in Euclidian general relativity, which has shown to have some very interesting finiteness properties [13]. There is also a similar model in Lorentzian general relativity [7,14]. They are based on the simple representations of the symmetry algebra $so(4)$ and of $so(3,1)$.

There are two ways of deriving the Barrett-Crane model as a theory of gravity (or quantum geometry). First, there is a geometrical method, which relies on a simplicial triangulation of the space-time 4-manifold and uses the isomorphism between the set of bivectors and the rotation Lie algebra ($so(4)$ or $so(3,1)$) to associate to each triangle of the decomposition a (simple) representation [7,15]. Then, there is the approach by discretisation of a constrained BF theory [9,16]. The constraint restricts us to the use of simple representations again.

Recently, a generalised constrained action has been proposed by Capovilla, Montesinos, Prieto and Rojas [1] in order to introduce an Immirzi parameter as in the canonical approach (loop quantum gravity) [3,4]. It seemed, as suggested in [1], that discretising that new BF action would lead us to introduce non-simple representations in spin foam models. Then, checking if those models be consistent or not, it could lead to some constraints on the possible values of the Immirzi parameter, or else we would have an arbitrary parameter in spin foam models as in loop quantum gravity.

Here, I will show that one might, instead, reconsider the way the bivector $B$ field is associated to generators of the symmetry algebra. Instead of associating it as usual with the canonical generators of $so(4)$ (or $so(3,1)$), we are lead to make a change of basis which allow us to keep the simple representations. I then argue that this could have been seen directly by a change of variables on the initial action. The Immirzi parameter then doesn’t have any effect on the representations used in the spin foam model, so that we are still studying the same Barrett-Crane model. I conclude with a calculation of the area spectrum, by trying to apply canonical methods to the BF action, and show how the Immirzi parameter can still introduce a factor between the area and the non-zero Casimir operator of $so(4)$ in the Euclidian case or $so(3,1)$ in the Lorentzian case.

II. EUCLIDIAN BF THEORY WITH IMMIRZI PARAMETER

I start by recalling the action proposed in [1] for the constrained BF theory for Euclidian general relativity:

$$S = \int B^{IJ} \wedge F_{IJ} - \frac{1}{2} \phi_{IJKL} B^{IJ} \wedge B^{KL} + \mu H$$

(1)

where $H = a_1 \phi_{IJKL} + a_2 \phi_{IJKL}^\epsilon$. $\phi$ and $\mu$ are Lagrange multipliers and $\phi$ has the symmetries $\phi_{IJKL} = -\phi_{IJLK} = -\phi_{IJKL} = \phi_{KLIJ}$; $I, J, K, L$ are internal indices. The $\wedge$ product is defined on the space-time indices and the $*$ operator acts on internal indices: $*B_{IJ} = 1/2 \epsilon_{IJKL} B^{KL}$ and $*^2 = 1$.

The same action can also be used for Lorentzian general relativity with the same definitions. The difference is that $*^2 = -1$ in the Lorentzian case, so that all the equations will be the same up to a sign.

The constraints on the field $B$ are then:
\[ B^{IJ} \wedge B^{KL} = \frac{1}{6}(B^{MN} \wedge B_{MN})\eta^{[I[K}\eta^{J]L} + \frac{1}{12}(B^{MN} \wedge *B_{MN})\epsilon^{IJKL} \]  

\[ 2a_2B^{IJ} \wedge B_{IJ} - a_1B^{IJ} \wedge *B_{IJ} = 0 \]  

Solving these constraints for the field \( B \), we get:

\[ B^{IJ} = \alpha * (e^I \wedge e^J) + \beta e^I \wedge e^J \]  

where

\[ \frac{a_2}{a_1} = \frac{\alpha^2 + \beta^2}{4\alpha\beta} \]  

Replacing this expression in the original action, we get an action where the general relativity action is coupled to the “trivial” sector:

\[ S = \alpha \int * (e^I \wedge e^J) \wedge F_{IJ} + \beta \int e^I \wedge e^J \wedge F_{IJ} \]  

Let’s note that the equations of motion ignore the “trivial” part and still gives the Einstein equations.

We can have a look at the particular cases for which we take only one of the two terms in the constraint \( H \). For \( a_2 = 0 \), we get \( \alpha = \beta = 0 \), it is a “empty” theory. In fact, only degenerate tetrads are going to contribute to the action. For \( a_1 = 0 \), we have \( \alpha = 0 \) or \( \beta = 0 \), which means that we have two distinct sectors of solutions: the general relativity one \( * (e \wedge e) \) and the “trivial” one \( e \wedge e \). More precisely, we have four sectors: \( \alpha = \pm 1, \beta = 0 \) and \( \alpha = 0, \beta = \pm 1 \).

Following \( 8 \), the Immirzi parameter is \( \gamma = \alpha/\beta \). The equation relating \( a_1, a_2 \) and \( \gamma \) is:

\[ \frac{a_2}{a_1} = \frac{1}{4} \left( \frac{\gamma + 1}{\gamma} \right) \]  

There are two solutions to this equation, inverse of each other. We have two sectors in our theory with Immirzi parameter \( \gamma \) or \( 1/\gamma \): there is a symmetry exchanging \( \alpha \) and \( \beta \). I will call that symmetry the \( * \)-symmetry since the resulting \( B \) field get changed into its image by the Hodge operator \( * \) under the exchange of \( \alpha \) and \( \beta \).

To introduce these modifications into the spin foam, we rewrite \( 3 \) in an equivalent form:

\[ 2a_2 \frac{1}{2} \epsilon_{IJKL}B^{IJ}_{ab}B^{KL}_{ab} - a_1B^{IJ}_{ab}B^{ab}_{IJ} = 0 \]  

Indeed, we can get to \( 3 \) from \( 8 \) by changing the role of the internal indices and space-time indices. What makes the two equations equivalent is that they have the same set of solutions, given by \( 8 \) (see in appendix for more details).

We could also deal directly with the condition \( 3 \) as shown by Reisenberger \( [10,11] \) and by De Pietri and Freidel \( [12] \) in the case \( a_1 = 0 \), and it would lead to (a modification of) the Reisenberger spin foam model, which is a different model than the Barrett-Crane one.

Then, we can translate the above constraint on the field \( B \) into a constraint on the two Casimirs of \( so(4) \), using the equivalence between this field and the generators of the \( so(4) \approx su(2)_L \oplus su(2)_R \) algebra discussed in the framework of spin foam models \( 7 \). More precisely, replacing \( B^{IJ}_{ab} \) by the canonical generators \( J^{IJ} \) of \( so(4) \), we get:

\[ \frac{B^{IJ}_{ab}B^{ab}_{IJ}}{2\epsilon_{IJKL}B^{IJ}_{ab}B^{KL}_{ab}} \rightarrow J^{IJ}J_{IJ} = 2((J^{01})^2 + \ldots) = 2C_1 \]  

\[ \frac{1}{2}\epsilon_{IJKL}J^{IJ}J^{KL} = 4(J^{01}J^{23} + \ldots) = 2C_2 \]  

Using the above correspondence, we get to a modified simplicity constraint:

\[ 2a_2C_2 - a_1C_1 = 0 \]  

or equivalently:

\[ 2a_1C_1 = (\alpha^2 + \beta^2)C_2 \]
Then we could conclude as stated in [4] that we should use non-simple representations in our spin foam model. This conclusion is problematic in the Euclidean case where \(C_1\) and \(C_2\) are discrete. More precisely, in the Euclidean case we are studying, an equation \(xC_1 = yC_2\) has an infinite number of solutions only in 3 cases. If \(x = y\), we get the representations of \(su(2)_L\) \((j^- = 0)\). If \(x = -y\), we use the representations of \(su(2)_R\). And finally \(x = 0\) gives us the simple representations of \(so(4)\). In the other cases, we have in general no solutions, or sometimes only one solution (one representation \((j^+, j^-)\)), which would give quite an ill-defined spin foam model.

Apart from those problems, it is hard to understand why we should give up the “simple representations” which are understood to be the simplest and most natural representations of \(so(N)\) [8].

In the next section, I describe an alternative: we can identify the \(B\) field to some generators of the \(so(4)\) algebras in such a way to that we keep the simple representations. This correspondence is at the heart of the “spin foam quantization” because it is through it that we translate the simplicity (and intersection) constraint to the quantum level.

### III. MODIFYING THE CORRESPONDENCE \(B \rightarrow J\)

We could directly replace \(B^{IJ}\) by the canonical generator \(J^{IJ}\) of \(so(4)\). This would give us the mixed simplicity condition [9]. On the other hand, looking at [9], we can associate the following generators to the field \(B\):

\[
\tilde{J}^{IJ} = \alpha J^{IJ} + \beta \ast J^{IJ}
\]

(11)

This is simply a change of basis. It is still the same algebra \(so(4)\) we are dealing with, in the same chosen representation we would be studying. I will call a choice of basis “consistent” (with the Barrett-Crane simplicity constraint) if it is a change of basis which leads to the simple representations after translating the simplicity constraint [8] to the Lie algebra generators. [11] is a consistent choice as I show below. It corresponds to associating \(J^{IJ}\) to \(\ast(e^I \wedge e^J)\) (or to \(e^I \wedge e^J\)) instead of associating it directly to the bivector field \(B^{IJ}\).

There are other consistent choices of \(J^{IJ}\). First, we can exchange \(\alpha\) and \(\beta\) in the above formula. This doesn’t seem to affect anything in the theory. We could also rotate (by an \(so(4)\) rotation) \(J^{IJ}\). This wouldn’t change the Casimir, so it wouldn’t affect the discussion below. And also, we could rescale \(J^{IJ}\) by a factor \(\lambda\), this corresponds to a rescaling of the frame field \(e^I_a\). It rescales the Casimirs by a factor \(\lambda^2\) and doesn’t seem to affect any result, except maybe the spectrum of the area as shown in section [11]. There are also other changes of basis which don’t affect any of the following result like changing the signs of the generators \(J^{0i}\), but, we aren’t going to study these here.

Taking a non-consistent choice of basis would lead to a mixed simplicity condition \(xC_1 = yC_2\) which doesn’t give any self-consistent spin foam model in general. Moreover, the choice [11] is natural as we replace the geometrical meaningful \(\ast(e^I \wedge e^J)\) (general relativity sector) by the canonical generators \(J^{IJ}\).

The Casimir corresponding to \(B^{IJ} \wedge B_{IJ}\) is then

\[
\tilde{C}_2 = \frac{1}{4} \epsilon_{IJKL} \tilde{J}^{IJ} \tilde{J}^{KL} = 2\alpha \beta C_1 + (\alpha^2 + \beta^2) \tilde{C}_2
\]

where \(C_1\) and \(C_2\) are the usual Casimirs associated to the generators \(J^{IJ}\):

\[
C_1 = \frac{1}{2} (J_i^+)^2 (J_i^-)^2 = (j^+ (j^+ + 1) + j^- (j^- + 1))
\]

\[
C_2 = \frac{1}{2} ((J_i^+)^2 - (J_i^-)^2) = (j^+ (j^+ + 1) - j^- (j^- + 1))
\]

Also the Casimir associated to \(B^{IJ} \wedge \ast B_{IJ}\) becomes

\[
\tilde{C}_1 = \tilde{J}^{IJ} \tilde{J}_{IJ} = (\alpha^2 + \beta^2) C_1 + 2\alpha \beta C_2
\]

(13)

Now writing the simplicity condition [3], we get the equation:

\[
2\alpha \beta \tilde{C}_1 = (\alpha^2 + \beta^2) \tilde{C}_2 \Rightarrow (\alpha^2 - \beta^2)^2 C_2 = 0
\]

(14)

And in the case \(\alpha^2 \neq \beta^2\) \((\gamma^2 \neq 1)\), we find back the usual simplicity condition \(C_2 = 0\) and the simple representations given by \(j^+ = j^-\). And the whole modification of the initial action is absorbed by a suitable redefinition of the
correspondence between the field $B$ and the generators of the Lie algebra. In the degenerate case $\gamma = \pm 1$ (Barbero’s choice for canonical gravity), we have to see whether we want to impose directly by hand the condition $C_2 = 0$ or maybe impose something else as discussed in section [V]. The degeneracy comes from the fact that the $*$-symmetry $(\alpha, \beta) \rightarrow (\beta, \alpha)$ is exactly realised when $\alpha = \beta$.

We can interpret the Casimir $C_1$ of a representation as the squared area associated to it. A more rigorous proof of this is done in section [VII] through a canonical treatment. But here, in our covariant framework, there is no reason (maybe impose something else as discussed in section V). The degeneracy comes from the fact that the bivector $\mathbf{B}$ to each triangle (2-simplex) a bivector $\mathbf{B}$. The geometrical picture consists of a simplicial triangulation of the 4-manifold (into 4-simplices). We then associate the bivector $\mathbf{B}$ and is given by the equation (4). Discretising it, we can translate the constraint (3) by a constraint on the bivector $\mathbf{B}$.

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\[
\mathcal{A}^2 = C_1 = \frac{1}{(\alpha^2 - \beta^2)^2} \left( (\alpha^2 + \beta^2)\tilde{C}_1 - 2\alpha\beta\tilde{C}_2 \right) \tag{15}\]

This definition allows us to relate the area with the correlation of the $B$ field that we could calculate using the spin foam model. It could be interesting to check whether calculating those correlations directly (as in [8]) would yield the same result as above (the Casimir). In fact, the two results are likely to be different. Already in two dimensions and three dimensions (Euclidian), calculating the length associated to an edge labelled by $a$ and three dimensions (Euclidian), calculating the length associated to an edge labelled by $a$ and three dimensions (Euclidian), calculating the length associated to an edge labelled by $a$

\[
\text{IV. LINK WITH THE GEOMETRICAL PICTURE}\]

The geometrical picture consists of a simplicial triangulation of the 4-manifold (into 4-simplices). We then associate to each triangle (2-simplex) a bivector $B$. The Barrett-Crane model [7] associates to a triangle defined by its edges $e, f, g$ the bivector $B = e \wedge f = f \wedge g = g \wedge e$. This is a simple bivector. This can be linked up to a discretisation of the constrained BF action for $q_1 = 0$. In the mixed case, the $B$ field is not simple anymore, it satisfies the relation (3) and is given by the equation (4). Discretising it, we can translate the constraint (3) by a constraint on the bivector $B$.

\[
2\alpha\beta \alpha B \wedge B = (\alpha^2 + \beta^2)B \wedge *B \tag{16}\]

This leads to the solution

\[
B = \alpha E + \beta * E \tag{17}\]

where $E$ is a simple bivector ($*E$ is then also a simple bivector). This solution corresponds to a discretisation of the classical solution (4).

For two triangles which share an edge, we must impose an intersection constraint stating that $B = B_1 + B_2$ should still be of the mixed simple form (17). It is easy to check that the corresponding condition can be written:

\[
2\alpha\beta B_1 \wedge B_2 = (\alpha^2 + \beta^2)B_1 \wedge *B_2 \tag{18}\]

and that in that case, the simple bivector $E$ associated to $B$ is simply $E_1 + E_2$. This shows us that although we have modified the field (or bivector) $B$, everything is still working as if we were using simple bivectors (the field or bivector $E = (\alpha B - \beta * B)/(\alpha^2 - \beta^2)$). The simple bivectors (and therefore simple representations) give the whole structure of the theory. We can also see that directly on the action (4) as shown in the following section.
V. FROM MIXED VARIABLES TO PURE SIMPLE VARIABLES

Starting from the initial action \([1]\), there is a change of variables which makes it obvious that the constraint on the field \(B\) is still the pure simplicity constraint. We start by finding \(\alpha\) and \(\beta\) verifying \([1]\) for some coefficient \(\lambda\):

\[
\begin{align*}
 a_1 &= \lambda(4\alpha\beta) \\
 a_2 &= \lambda(\alpha^2 + \beta^2)
\end{align*}
\]

Then, we can do the change of variables \((B, \phi) \rightarrow (E, \tilde{\phi})\):

\[
\begin{align*}
 B^{IJ} &= \alpha E^{IJ} + \beta \ast E^{IJ} \\
 \phi_{IJKL} &= (\alpha + \beta \frac{1}{2} \epsilon_{IJK}^{\quad AB}) \phi_{ABCD}(\alpha + \beta \frac{1}{2} \epsilon_{KL}^{\quad CD})
\end{align*}
\] (19)

This change of variables is well-defined and invertible only if \(\alpha^2 \neq \beta^2\). This excludes three possibilities: \(a_2 = 0\) and \(2a_2 = \pm a_1\) which we will discuss below. When the change is well-defined, the action \([1]\) becomes:

\[
S = \frac{1}{\alpha^2 - \beta^2} \int (\alpha E^{IJ} + \beta \ast E^{IJ}) \wedge F_{IJ} - \frac{1}{2} \tilde{\phi}_{IJKL} E^{IJ} \wedge E^{KL} + \mu \lambda IJKL \bar{\phi}_{IJKL} (20)
\]

Looking at \([20]\), we notice that the simplicity constraint on the bivector field \(E^{IJ}\) is the pure simplicity constraint. This shows us how to get to the correspondence \([11]\) which lead from the mixed simplicity condition to the pure simplicity condition.

The case \(a_2 = 0\) leads to \(\alpha = \beta = 0\) and the set of solutions for the field \(B\) contains only degenerate tetrads. The cases \(2a_2 = \pm a_1\) are more interesting. They correspond to the self-dual and the anti-self dual cases. As we have seen earlier, they do not lead directly to the simplicity constraint \(C_2 = 0\) after the change of basis of the generators \(J^{IJ}\) \([11]\). So we have many possibilities. First, we could impose by hand the pure simplicity constraint and say the model is equivalent once more to the Barrett-Crane model (at an infinite coefficient). Or we could forget the change of basis. In fact, as it corresponds to the change of variables \([19]\), it might be ill-defined. Keeping \(B^{IJ} \rightarrow J^{IJ}\) where \(J^{IJ}\) is the canonical basis, we get modified simplicity constraints:

\[
\begin{align*}
 C_1 &= C_2 \quad \text{for} \quad 2a_2 = a_1 \\
 \phi_{IJKL} &= (\alpha + \beta \frac{1}{2} \epsilon_{IJK}^{\quad AB}) \phi_{ABCD}(\alpha + \beta \frac{1}{2} \epsilon_{KL}^{\quad CD})
\end{align*}
\]

The first case leads to \(j^{-} = 0\) and the second to \(j^{+} = 0\). So we are reduced from \(so(4)\) to \(su(2)_L\) or to \(su(2)_R\). What theory do we then get? If we want to build a spin foam model similar to the Barrett-Crane model in 4 dimensions but based on \(SU(2)_L\) only (or \(SU(2)_R\) only), we would naturally be lead to the \([15]\) and to a topological theory corresponding to \(SU(2)\) BF theory. However, it can not be the topological \(SU(2)\) BF theory since although we are reduced to a \(su(2)\) gauge symmetry, we still have the constraint that the \(B\) field should be derived from a tetrad field, which is not implemented in the topological theory. More precisely, in the case we are reduced to the representations \(j^{+}\) and therefore to the gauge group \(SU(2)_L\), the \(SU(2)_R\) degrees of freedom of the connection decouple and we can forget them. We then get the left-handed (self-dual) action for Euclidian gravity. There exist a corresponding spin foam model: it is the Reisenberger model for left-handed euclidian gravity \([1]\). We derive it discretising the constraint \([6]\) and not its equivalent form \([8]\). But this equivalence works only when \(2a_2 \neq \pm a_1\) (see appendix). So, perhaps, although the Reisenberger model is well-defined in both the full case \([11, 12]\) and the self-dual case \([10, 9]\), the Barrett-Crane approach is ill-defined in the self-dual case and need to be regularised by a choice \(\alpha\) slightly different than \(\pm \beta\). It is important to understand what happens precisely for these degenerate cases correspond classically to the self-dual (and anti-self-dual) connection action which is usually used in the canonical treatment of Euclidian general relativity.

VI. LORENTZIAN CASE

In the Lorentzian case, starting from the same action \([1]\), we can write the same equations as in the Euclidian case. We are lead to the same conclusion that the representations we should use are the simple ones. More precisely, there is a difference of sign in some equations due to \(\ast^2 = -1\) instead of \(\ast^2 = 1\). First, the equation defining \(\alpha\) and \(\beta\) is:

\[
\frac{a_2}{a_1} = \frac{\alpha^2 - \beta^2}{4\alpha\beta} (21)
\]

\(a_2 = 0\) now corresonds to the case \(\alpha^2 = \beta^2\) while \(a_1 = 0\) still corresponds to \(\alpha = 0\) or \(\beta = 0\). The \(\ast\)-symmetry is now \((\alpha, \beta) \rightarrow (\beta, -\alpha)\) and \(\gamma \rightarrow -1/\gamma\).
The simplicity constraint becomes:

\[ 2a_2B^{IJ} \wedge B_{IJ} + a_1B^{IJ} \wedge *B_{IJ} = 0 \]  

(22)

It is equivalent (same set of solutions) to the equation:

\[ 2a_2 \frac{1}{2} c^{IJKL} B_{ab}^I B_{ab}^J - a_1 B_{ab}^I B_{ab}^J = 0 \]  

(23)

which can be translated to the Lie algebras generators:

\[ 2a_2 \tilde{C}_2 - a_1 \tilde{C}_1 = 0 \Rightarrow (\alpha^2 - \beta^2) \tilde{C}_2 = 2\alpha\beta \tilde{C}_1 \]  

(24)

It is the same modified simplicity condition as in the Euclidian case expressed in terms of \( a_1, a_2 \). We modify a sign in the change of basis:

\[ B^{IJ} \rightarrow \tilde{J}^{IJ} = \alpha J^{IJ} - \beta *J^{IJ} \]

The minus sign is normal since if we associate \( J^{IJ} \) to \( \ast (e^I \wedge e^J) \) then we must associate \( -\ast J^{IJ} \) to \( e^I \wedge e^J = -\ast (e^I \wedge e^J) \).

Then the modified Casimirs are:

\[ \tilde{C}_1 = (\alpha^2 - \beta^2)C_1 - 2\alpha\beta C_2 \]

\[ \tilde{C}_2 = +2\alpha\beta C_1 + (\alpha^2 - \beta^2)C_2 \]

(25)

The two Casimirs are \([7,14]\), where \( j \) is a positive half-integer bigger than 1 and \( \rho \) a positive real number:

\[ C_1 = j^2 - \rho^2 - 1 \]

\[ C_2 = 2j\rho \]

The modified simplicity condition then reads \((\alpha^2 + \beta^2)C_2 = 0\). Since \( \alpha^2 + \beta^2 \neq 0 \) in all real cases, we get to the pure simplicity condition \( C_2 = 0 \). We have two series of simple representations: a discrete series given by \( \rho = 0 \) and a continuous series defined by \( j = 0 \). We can note that the degenerate case \( \alpha^2 + \beta^2 = 0 \) would be the Ashtekar choice of variables with \( \alpha = 1, \beta = i \) and \( \gamma = \pm i \). This choice of \( \gamma \) makes the \( \ast \)-symmetry obvious in the Lorentzin case. The Casimirs are then:

\[ \tilde{C}_1 = 2(C_1 + iC_2) \]

\[ \tilde{C}_2 = -2i(C_1 + iC_2) \]

We can do the same change of variables as in the Euclidian case. In contrast with the Euclidian case, everything is always well-defined. More precisely, we need \( \alpha \) and \( \beta \) such that

\[ a_1 = \lambda(4\alpha\beta) \]

\[ a_2 = \lambda(\alpha^2 - \beta^2) \]

Then the change of variable \([19]\) is invertible if \( \alpha^2 + \beta^2 \neq 0 \), which is the case for any real \( a_1, a_2 \) coefficients. So we don’t have any ambiguous cases as in the Euclidian case. All the cases lead to the Lorentzin Barrett-Crane model.

VII. CALCULATION OF THE AREA SPECTRUM FOR \( SO(4) \) AND \( SO(3,1) \) SPIN NETWORKS

It is interesting to calculate the area spectrum of the theory to see if we find a factor corresponding to the Immirzi parameter as in the case of loop quantum gravity \([1]\). We could calculate the the area expectation values from the spin foam action as in \([8]\). However, I choose a more direct approach to compute the area spectrum for this constrained BF theory. I will use the same tools as when calculating the area spectrum in the canonical theory \([17]\) and try to adapt it to use it on \( so(4) \) and \( so(3,1) \) spin networks. More precisely, I will follow the canonical approach as described in \([18]\). Nevertheless, I will assume a natural quantification for the connections and their conjugate momenta, check
whether it gives us consistent results and derive from it a Poisson bracket that one should be able to find directly from the action.

We study the action:

\[
S(A, e) = \alpha \int (e^I \wedge e^J) \wedge F_{IJ} + \beta \int e^I \wedge e^J \wedge F_{IJ}
\]

The metric is given by \( g_{\mu \nu} = e^I \epsilon_{\mu \nu} \).

We are working with a real \( so(4) \) connection \( A \), its strength field \( F \) and a real tetrad \( e \). The conjugate momentum of the variable \( A^I_J \) is:

\[
\Pi^c_{IJ} = \epsilon^{abc} (\alpha \epsilon_{IJKL} e^K_a e^L_b + \beta e_a [I e_b J])
\]

In a first calculation, we are going to ignore the (simplicity) constraints linking those conjugate momenta.

We are doing a hamiltonian treatment choosing \( a = 0 \) as our time direction and \( M \) parametrised by \( x_i, i = 1, 2, 3 \) as our initial hypersurface. As we are interested by spin foams whose surfaces are labelled by \( so(4) \) representations, we are going to study \( so(4) \) spin networks. More precisely, the “observables” we consider are the holonomies of the connection \( A^I_J \) so that our spin networks are graphs imbedded in \( M \) and labelled by principal unitary irreducible representations of \( so(4) \). The possible representations are restricted by the (modified) simplicity condition discussed earlier.

We are considering infinitesimal 2-surface \( \Sigma \) with the coordinate system \( \vec{\sigma} = (\sigma_1, \sigma_2) \) and we want to compute its area. We will only consider spin networks which intersect that surface just once and never at a node. \( \Pi \) is the conjugate momentum of the connection \( A \). If we cheat and ignore the constraints, the corresponding operator acting on the spin networks is given by:

\[
\Pi^a_{IJ} \rightarrow -i \hbar \delta \delta A^a_{IJ} \approx \hbar J_{IJ}
\]

To get the area, we must integrate the induced 2-metric on \( \Sigma \). We introduce the following operator:

\[
\mathcal{B}(\Sigma) = \sqrt{\mathcal{B}_{IJ}(\Sigma) \mathcal{B}_{IJ}(\Sigma)}
\]

where

\[
\mathcal{B}_{IJ}(\Sigma) = \int_\Sigma d\sigma^1 d\sigma^2 \epsilon_{abc} \frac{\partial x^a(\vec{\sigma})}{\partial \sigma^1} \frac{\partial x^b(\vec{\sigma})}{\partial \sigma^2} \Pi^c_{IJ}
\]

This operator is closely related to the area. To show this, let’s choose a special parametrisation of \( \Sigma \). Let’s choose \( x^3 = 0 \) on \( \Sigma \), \( x^{1,2} = \sigma^{1,2} \). Then the quantity \( \mathcal{B} \) becomes:

\[
\mathcal{B}(\Sigma) = \int_\Sigma d\sigma^1 d\sigma^2 \sqrt{\epsilon_{abc} \frac{\partial x^a(\vec{\sigma})}{\partial \sigma^1} \frac{\partial x^b(\vec{\sigma})}{\partial \sigma^2} \Pi^c_{IJ} \epsilon_{def} \frac{\partial x^d(\vec{\sigma})}{\partial \sigma^1} \frac{\partial x^e(\vec{\sigma})}{\partial \sigma^2} \Pi^f_{IJ}}
\]

Looking up at the definition of \( \Pi \) and of \( g_{\mu \nu} \), we get:

\[
\mathcal{B}(\Sigma) = \int_\Sigma d\sigma^1 d\sigma^2 \sqrt{(\beta^2 + \alpha^2)(g_{11}g_{22} - g_{12}g_{12})} = \int_\Sigma d\sigma^1 d\sigma^2 \sqrt{(\beta^2 + \alpha^2) \det^2 g(\Sigma)}
\]

Promoting \( \Pi \) to an operator and we get an operator \( \mathcal{B} \) acting on \( so(4) \) spin networks and whose values are the area to a factor \( \sqrt{\beta^2 + \alpha^2} \). Replacing the variable \( \Pi \) by the generators of the \( so(4) \) algebra as in [18], we find that the
The eigenvalues of the operator $B$ are given by the square root of the Casimir $C_1 = J^{IJ} J_{IJ}$ (without any extra sign) times $\hbar$.

To sum up the results, we get that the squared area is given by:

$$A^2 = \frac{1}{\beta^2 + \alpha^2} C_1$$  \hspace{1cm} (28)

We could also decide to use the operator $C$ defined as follows:

$$C(\Sigma) = \sqrt{B^{IJ}(\Sigma) * B_{IJ}(\Sigma)}$$

Following the same procedure as in the case of the $B$ operator, we find:

$$C(\Sigma) = \int_{\Sigma} d\sigma^1 d\sigma^2 \sqrt{(2\alpha \beta) \det^2 g(\Sigma)}$$  \hspace{1cm} (29)

$$A^2 = \frac{1}{2\alpha \beta} C_2$$  \hspace{1cm} (30)

So if we want the results to be consistent ($A^2 = A^2$), we must impose:

$$2a_2 C_2 = a_1 C_1$$

We see that assuming the canonical bracket between the field $B$ and the connection $A$, we find the simplicity condition. But we have showed earlier that this wasn’t the “right” condition: we need to use a new correspondence between $B$ and the Lie algebra, changing the Casimirs $C_1, C_2$ into the new Casimirs $\tilde{C}_1, \tilde{C}_2$. This means that we must change the assumed Poisson bracket between $B$ and $A$ so that:

$$\Pi^a_{IJ} \rightarrow -i\hbar \left( \alpha \frac{\delta}{\delta A^I_a} + \beta \frac{\delta}{\delta A^J_a} \right) = \hbar \tilde{J}^{IJ}$$  \hspace{1cm} (31)

We then get the right simplicity condition

$$2a_2 \tilde{C}_2 = a_1 \tilde{C}_1$$

and, as soon as this condition is fulfilled, the area is

$$A^2 = \frac{1}{\beta^2 + \alpha^2} \tilde{C}_1 = \frac{1}{2\alpha \beta} \tilde{C}_2 = C_1$$  \hspace{1cm} (32)

And we find no Immirzi parameter. Still, there were different consistent choices for $\tilde{J}^{IJ}$. First, we could exchange $\alpha$ and $\beta$:

$$\Pi^a_{IJ} \rightarrow -i\hbar \left( \beta \frac{\delta}{\delta A^I_a} + \alpha \frac{\delta}{\delta A^J_a} \right)$$  \hspace{1cm} (33)

That doesn’t change anything. On another hand, we could rescale the relations (31) and (33) by a factor $\lambda$ and that would rescale the area (square) by a factor $\lambda$ (square).

More precisely, to make the link with the canonical formalism [6], we can chose $\alpha = 1$, and then the Immirzi parameter is $\gamma = 1/\beta$. Normalising (31) to have a first term $\gamma \delta / \delta A^I_a$, we need a factor $\lambda = 1/\beta = \gamma$. Then in that case, we find the “usual” area spectrum:

$$\Pi^a_{IJ} \rightarrow -i\hbar \left( \frac{1}{\beta} \frac{\delta}{\delta A^I_a} + \frac{\delta}{\delta * A^J_a} \right)$$  \hspace{1cm} (34)
The Lorentzian case works mostly the same way as the Euclidian case. We choose:

\[ A^2 = \frac{1}{\beta^2} C_1 = \gamma^2 C_1 \quad A = \gamma \sqrt{C_1} = \gamma \sqrt{2j^+ (j^+ + 1)} \]  

(35)

First, the choice (34) is consistent with the (classical) simplicity constraint (3) (replacing B by Π) as soon as we choose the simple representations \( C_2 = 0 \) in the holonomies which are the state space of our quantum theory. It works because the simple representations automatically verify \( 2\alpha \beta \tilde{C}_1 = (\alpha^2 + \beta^2)C_2 \) which is the direct translation of the constraint (3) once we have chosen the quantization (34).

Then, we have

\[ [A^i_a, \Pi^a_{KL}] = i\hbar \delta^b_a \left( \gamma \delta^i_j K^j_{KL} + \frac{1}{2} \epsilon^i_{KLM} \right) \]  

(36)

Let’s introduce another field \( E \):

\[ E^a_{iJ} = \epsilon^{abc} \frac{1}{2} \epsilon_{KLM} \bar{e}^a_a e^b_{iJ} \]  

(37)

Then we have the following commutation relation

\[ [A^i_a, E^a_{iJ}] = i\hbar \delta^b_a \delta^i_j K^j_{KL} \]  

(38)

This commutator corresponds to a classical Poisson bracket:

\[ \{A^i_a(x), E^a_{iJ}(y)\} = \gamma \delta^a_a \delta^i_j K^j_{KL} \delta(x - y) \]  

(39)

This is to be compared with the Poisson bracket of the canonical approach (loop quantum gravity) which leads directly to a factor \( \gamma \) in the spectrum of the area [18]:

\[ \{A^i_a(x) = \Gamma^i_{a} + \gamma K^i_{a}, E^a_{iJ}(y)\} = \gamma \delta^a_a \delta^i_j K^j_{KL} \delta(x - y) \]  

(40)

The two expressions share the same factor \( \gamma \). What we have done is transfer the transformation done on the spin-connection \( \Gamma \) to a transformation on the tetrads. This is why, in a way, [24] seems to be a natural choice of operator for \( \Pi^i_a \). Nevertheless, the Immirzi parameter is here introduced in the area spectrum in an artificial way by a normalisation. Thus, it would be interesting to derive the Poisson bracket (39) from a direct treatment of the operator for \( \Pi^i_a \) as soon as we have chosen the quantization (34).

To conclude, the Immirzi parameter doesn’t change the representations used in the spin foam model, so it doesn’t affect the geometry of the space-time. There is also no reason to introduce it in a area spectrum in the covariant framework since the simplicity constraint is independent of rescaling. Nevertheless, in a canonical framework, the choice of quantization (34) leads to the same factor equal to the Immirzi parameter in the spectrum of the area as in loop quantum gravity. We should check if the choice (34) and the resulting Poisson bracket (39) can be recovered by performing a hamiltonian treatment directly on the classical BF action. To understand what happens, we should also investigate in more details the link between the spin foam quantization (through the Barrett-Crane procedure or the discretisation of BF theory) and the canonical approach (loop quantum gravity). We could then possibly apply the same procedure to higher-dimensional constrained BF theory as proposed in [8] in order to quantize \( D \geq 4 \) theories of gravity without breaking explicitly the Lorentz invariance.
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APPENDIX: EQUIVALENCE OF THE SIMPLICITY CONDITIONS

We would like to prove the equivalence of the two simplicity conditions (3) and (8). We are going to follow the proof given in [12] for the case $a_1 = 0$ and we will restrict ourselves to the Euclidian case. This proof can be adapted to the Lorentzian case provided we change some signs (due to $s^2 = -1$).

First, we define:

$$ e = \frac{1}{4!} \epsilon_{IJKL} B^{IJ} \wedge B^{KL} \quad (A1) $$

Then supposing $a_2 = 0$, the constraints (3) and (8) on the $B$ field can be written as:

$$ \epsilon^{abcd} B_{ab}^{IJ} B_{cd}^{KL} = e \left( \epsilon^{IJKL} + \frac{a_1}{a_2} \eta_{[IK} \eta_{J]L} \right) = e \Omega^{IJKL} \quad (A2) $$

By raising and lowering indices, we can rewrite (A2) under a more convenient form in which we consider all the variables as $6 \times 6$ matrices (on the antisymmetric couples $[IJ]$):

$$ B_{IJ}^{ab} \epsilon_{ab}^{cd} B_{cd}^{KL} = e \Omega_{IJ}^{KL} \quad (A3) $$

Then $\Omega_{IJ}^{KL}$ is invertible when

$$ \left( \frac{a_1}{2a_2} \right)^2 \neq 1 $$

In that case, assuming that $e \neq 0$ (the $B$ field is non-degenerate), we can define

$$ \Sigma_{IJ}^{cd} = \frac{1}{e} \epsilon^{abcd} \frac{2}{\left( \frac{a_1}{a_2} \right)^2 - 4} \left( -\epsilon_{IJKL} + \frac{a_1}{2a_2} \eta_{[IK} \eta_{J]L} \right) B_{cd}^{KL} \quad (A4) $$

and rewrite (A3) under the simple form:

$$ \Sigma_{IJ}^{cd} B_{cd}^{KL} = \delta_{IJ}^{KL} \quad (A5) $$

(A5) means that $\Sigma_{IJ}^{cd}$ and $B_{cd}^{KL}$ are invertible and inverse of each other. This means that (A5) is equivalent to:

$$ B_{IJ}^{ab} \Sigma_{IJ}^{cd} = \epsilon_{cd}^{ab} \quad (A6) $$

After writing explicitly this new equation and passing the tensor $\epsilon^{abcd}$ from the LHS to the RHS, we find:

$$ 2 \left( -\epsilon_{IJKM} + \frac{a_1}{2a_2} \eta_{[IK} \eta_{J]M} \right) B_{cd}^{MN} B_{ab}^{IJ} = \epsilon_{abcd} e \left( \frac{a_1}{a_2} \right)^2 - 4 \right) \quad (A7) $$

and we recover the simplicity condition (8) in the case $(ab) = (cd)$.

Let’s note that in the Lorentzian case, the condition on $a_1, a_2$ reads

$$ \left( \frac{a_1}{2a_2} \right)^2 \neq 1 $$

and is automatically satisfied if we keep real variables.
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