A NOTE ON LOWER BOUNDS ESTIMATES FOR THE NEUMANN EIGENVALUES OF MANIFOLDS WITH POSITIVE RICCI CURVATURE

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Abstract. We study new heat kernel estimates for the Neumann heat kernel on a compact manifold with positive Ricci curvature and convex boundary. As a consequence, we obtain lower bounds for the Neumann eigenvalues which are consistent with Weyl’s asymptotics.

Contents

1. Introduction 1
2. Li-Yau type estimates on manifolds with positive Ricci curvature and convex boundary 2
2.1. The Neumann semigroup 2
2.2. Li-Yau gradient type estimate and heat kernel bounds 4
2.3. Harnack inequality 4
2.4. On diagonal heat kernel estimates 6
3. Lower bounds for the eigenvalues 8
References 9

1. Introduction

Eigenvalues of compact Riemannian manifolds have been extensively studied (see for instance Chavel [6], Cheng [7], Li-Yau [9], [10], and the references therein). In particular, it has been proved by Li and Yau [10] that for the Neumann eigenvalues of a compact Riemannian manifold with non negative Ricci curvature and convex boundary

$$\lambda_k \geq C(n) \frac{k^{2/n}}{D(\mathbb{M})^2},$$

where $C(n)$ is a constant that only depends on the dimension $n$ of the manifold and where $D(\mathbb{M})$ is the diameter of $\mathbb{M}$. These lower bound estimates are obtained by proving an on-diagonal upper bound for the Neumann heat kernel. In this note, we follow the approach of Li and Yau, but use the tools introduced in Bakry-Qian [4], Bakry-Ledoux [3] and Baudoin-Garofalo [5] to prove new upper bounds for the Neumann heat kernel in the case where the Ricci curvature is bounded from below by a positive constant $\rho$. These new heat kernel upper bounds lead to lower bounds of the form

$$\lambda_k \geq C_1(n, \rho, k),$$

and

$$\lambda_k \geq C_2(n, \rho, D(\mathbb{M}), k)$$

where $C_1(n, \rho, k)$ and $C_2(n, \rho, D(\mathbb{M}), k)$ have order $k^{2/n}$ when $k \to \infty$, which is consistent with Weyl’s asymptotics.

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2. Li-Yau type estimates on manifolds with positive Ricci curvature and convex boundary

2.1. The Neumann semigroup. Let $M$ be a $n$-dimensional smooth, compact and connected Riemannian manifold with boundary $\partial M$. Let us denote by $N$ the outward unit vector field on $\partial M$. The second fundamental form of $\partial M$ is defined on vector fields tangent to $\partial M$ by

$$\Pi(X, Y) = \langle \nabla_N X, Y \rangle.$$

The boundary is then said to be convex if $\Pi \geq 0$ as a symmetric bilinear form. Throughout this paper, we will assume that the boundary $\partial M$ is convex. We shall moreover assume that the Ricci curvature tensor of $M$ satisfies $\text{Ric} \geq \rho$ for some $\rho > 0$.

Let $\Delta$ be the Laplace-Beltrami operator of $M$, with the sign convention that makes $\Delta$ a non-positive symmetric operator on $C^\infty(M)$. It is well-known that $\Delta$ is essentially self-adjoint on $D = \{ f \in C^\infty(M), Nf = 0 \text{ on } \partial M \}$. The Friedrichs extension of $\Delta$ is then the generator of strongly continuous Markov semigroup which is called the Neumann semigroup. We shall denote this semigroup by $(P_t)_{t \geq 0}$.

By ellipticity of $\Delta$, for every $f \in L^p(M)$, $1 \leq p \leq +\infty$, $P_t f \in D$, $t > 0$ and

$$\frac{\partial P_t f}{\partial t} = \Delta P_t f.$$

Also (see for instance [12]), if $f \in C^2(M)$ is such that $Nf \leq 0$ on $\partial M$, then

(2.1) \[ \frac{\partial P_t f}{\partial t} \geq P_t (\Delta f). \]

Moreover $(P_t)_{t \geq 0}$ has a smooth heat kernel, that is there exists a smooth function $p : (0, +\infty) \times M \times M \to (0, +\infty)$ such that for every $f \in L^\infty(M)$:

$$P_t f(x) = \int_M p(t, x, y)f(y)d\mu(y).$$

2.2. Li-Yau gradient type estimate and heat kernel bounds.

Theorem 2.1. Let $f \in C^2(M)$, $f > 0$. For $t > 0$, and $x \in M$,

$$\|\nabla \ln P_t f(x)\|^2 \leq e^{-\frac{2\rho t}{3}} \frac{\Delta P_t f(x)}{P_t f(x)} + \frac{n\rho}{3} \frac{e^{-\frac{4\rho t}{3}}}{1 - e^{-\frac{4\rho t}{3}}}.$$

Proof. Consider the functional

$$\Phi(t, x) = a(t)(P_t f)(x)\|\nabla \ln P_t f(x)\|^2, \quad t > 0, x \in M,$$

where

$$a(t) = e^{\frac{2\rho t}{3}} \left( e^{\frac{2\rho t}{3}} - 1 \right)^2.$$

Since $f \in C^2(M)$, let us first observe that according to Qian [11], $\|\nabla P_t f\|^2(x) \leq e^{-2\rho t} P_t \|\nabla f\|^2(x)$, so that we have, uniformly on $M$,

(2.2) \[ \lim_{t \to 0} \Phi(t, x) = 0. \]

We now compute

(2.3) \[ \frac{\partial \Phi}{\partial t} = L\Phi + \frac{a'(t)}{a(t)} \Phi - a(t)(P_t f)(x) \left( \Delta \|\nabla \ln P_t f\|^2 - 2\langle \nabla \ln P_t f, \nabla \Delta \ln P_t f \rangle \right). \]
This computation made be performed by using the so-called $\Gamma$-calculus developed in [2]. More precisely, denote for functions $u$ and $v$,
\[
\Gamma(u, v) = \frac{1}{2} (\Delta(u) - u\Delta v - v\Delta u) = \langle \nabla u, \nabla v \rangle,
\]
and
\[
\Gamma^2(u, v) = \frac{1}{2} (\Delta\Gamma(u, v) - \Gamma(u, \Delta v) - \Gamma(v, \Delta u)).
\]
Using then the change of variable formula (see [1] or [3]),
\[
\Gamma^2(\ln u, \ln u) = \frac{1}{u^2} \Gamma^2(u, u) - \frac{1}{u^3} \Gamma(u, \Gamma(u, u)) + \frac{1}{u^4} \Gamma(u, u)^2.
\]
Using Bochner’s formula, we have
\[
\Delta \|\nabla \ln P_t f\|^2 - 2\langle \nabla \ln P_t f, \nabla \Delta \ln P_t f \rangle = 2\|\nabla^2 \ln P_t f\|^2 + 2\text{Ric}(\nabla \ln P_t f, \nabla \ln P_t f),
\]
and Cauchy-Schwarz inequality implies,
\[
\|\nabla^2 \ln P_t f\|^2 \geq \frac{1}{n} (\Delta \ln P_t f)^2.
\]
Since by assumption
\[
\text{Ric}(\nabla \ln P_t f, \nabla \ln P_t f) \geq \rho \|\nabla \ln P_t f\|^2,
\]
we obtain therefore
\[
\frac{\partial \Phi}{\partial t} \leq \Delta \Phi + L\Phi + \frac{a'(t)}{a(t)} \Phi - a(t)(P_t f(x)) \left( \frac{2}{n} (\Delta \ln P_t f)^2 + 2\rho \|\nabla \ln P_t f\|^2 \right).
\]
Now, observe that for every $\gamma \in \mathbb{R}$,
\[
(\Delta \ln P_t f)^2 \geq 2\gamma \Delta \ln P_t f - \gamma^2 = 2\gamma \frac{\Delta P_t f}{P_t f} - 2\gamma \|\nabla \ln P_t f\|^2 - \gamma^2.
\]
In particular, by chosing
\[
\gamma(t) = -\frac{np}{3} \frac{1}{e^{\frac{2\gamma}{3}} - 1},
\]
so that
\[
\frac{a'(t)}{a(t)} + \frac{4}{n} \gamma(t) - 2\rho = 0
\]
and then coming back to (2.3), we infer
\[
\frac{\partial \Phi}{\partial t} \leq \Delta \Phi - \frac{4a(t)\gamma(t)}{n} \Delta P_t f + \frac{2a(t)\gamma(t)^2}{n} P_t f.
\]
We now make the crucial observation that on $\partial M$,
\[
N (\langle P_t f \|\nabla \ln P_t f\|^2 \rangle) = N \left( \frac{\|\nabla P_t f\|^2}{P_t f} \right) = -\frac{N P_t f}{(P_t f)^2} \|\nabla P_t f\|^2 + \frac{N \|\nabla P_t f\|^2}{P_t f} = N \|\nabla P_t f\|^2, \]
and, that by the convexity assumption,
\[
N \|\nabla P_t f\|^2 = -2\Pi(\nabla P_t f, \nabla P_t f) \leq 0.
\]
As a conclusion, on $\partial M$, we have
\[
(2.5) \quad N \Phi \leq 0.
\]
We fix now $T > 0$, $x \in M$ and consider
\[
\Psi(t) = (P_{T-t}\Phi)(x).
\]
As a consequence of (2.1) and (2.5), we thus get
\[
\Psi'(t) \leq P_{T-t} \left( \frac{\partial \Phi}{\partial t} - \Delta \Phi \right)(x)
\]
\[
\leq P_{T-t} \left( -\frac{4a(t)\gamma(t)}{n} \Delta P_t f + 2\frac{a(t)\gamma(t)^2}{n} P_t f \right)(x)
\]
\[
\leq -\frac{4a(t)\gamma(t)}{n} \Delta P_T f(x) + 2\frac{a(t)\gamma(t)^2}{n} P_T f(x)
\]
We now integrate the previous inequality from 0 to \( T \), use (2.2), and end up with
\[
\Phi(T, x) \leq -\int_0^T \frac{4\gamma(t)}{n} a(t) \gamma(t) dt \Delta P_T f(x) + 2\frac{\gamma(t)^2}{n} a(t) dt P_T f(x).
\]
Since
\[
a(t) = e^{2\rho t} \left( e^{\frac{2\rho t}{3}} - 1 \right)^2,
\]
the conclusion is reached by computing
\[
\int_0^T a(t) \gamma(t) dt = -\frac{n}{4} \left( e^{\frac{2\rho T}{3}} - 1 \right)^2
\]
and
\[
\int_0^T a(t) \gamma(t)^2 dt = \frac{n^2 \rho}{6} \left( e^{\frac{2\rho T}{3}} - 1 \right)
\]
□

Remark 2.2.
- In [5], in the case where the manifold has no boundary, the same inequality was obtained as a by product of a class of more general Li-Yau type inequalities.
- In the case \( \rho = 0 \), considering the functional
  \[
  \Phi(t, x) = t^2 (P_t f)(x) \| \nabla \ln P_t f(x) \|^2, \quad t \geq 0, x \in M,
  \]
would lead to the celebrated Li-Yau inequality for the Neumann semigroup on manifolds with convex boundaries (see [10], [4]):
  \[
  \| \nabla \ln P_t f(x) \|^2 \leq \frac{\Delta P_t f(x)}{P_t f(x)} + \frac{n}{2t}.
  \]
- In the case \( \rho = 0 \), a Li-Yau type inequality is obtained in [13] without the assumption that the boundary is convex.

2.3. Harnack inequality. As is well-known since Li-Yau [10], gradients bounds like Theorem 2.1 imply by integrating along geodesics a Harnack inequality for the heat semigroup:

**Theorem 2.3.** Let \( f \in L^\infty(M) \), \( f > 0 \). For \( 0 \leq s < t \) and \( x, y \in M \),
\[
P_s f(x) \leq \left( \frac{1 - e^{-\frac{2\rho s}{3}}}{1 - e^{-\frac{2\rho t}{3}}} \right)^{n/2} e^{\frac{d(x, y)^2}{4\rho}} e^{-\frac{d(x, y)^2}{4\rho} + \frac{n}{4\rho}} P_t f(y).
\]

**Proof.** We first assume that \( f \in C^2(M) \). Let \( x, y \in M \) and let \( \gamma : [s, t] \to M \), \( s < t \) be an absolutely continuous path such that \( \gamma(s) = x, \gamma(t) = y \). We write Theorem 2.1 in the form
\[
\| \nabla \ln P_u f(x) \|^2 \leq a(u) \frac{\Delta P_u f(x)}{P_u f(x)} + b(u),
\]
where
\[
a(u) = e^{-\frac{2\rho u}{3}},
\]
and
\[
b(u) = e^{-\frac{2\rho u}{3}}.
\]

\[\square\]
and
\[ b(u) = \frac{n\rho}{3} \frac{e^{-\frac{4au}{3}}}{1 - e^{-\frac{2a}{3}}}. \]

Let us now consider
\[ \phi(u) = \ln P_u f(\gamma(u)). \]

We compute
\[ \phi(u) = (\partial_u \ln P_u f)(\gamma(u)) + \langle \nabla \ln P_u f(\gamma(u)), \gamma'(u) \rangle. \]

Now, for every \( \lambda > 0 \), we have
\[ \langle \nabla \ln P_u f(\gamma(u)), \gamma'(u) \rangle \geq -\frac{1}{2\lambda^2} \| \nabla \ln P_u f(x) \|^2 - \frac{\lambda^2}{2} \| \gamma'(u) \|^2. \]

Choosing \( \lambda = \sqrt{\frac{a(u)}{2}} \) and using then (2.6) yields
\[ \phi'(u) \geq -\frac{a(u)}{b(u)} - \frac{1}{4} a(u) \| \gamma'(u) \|^2. \]

By integrating this inequality from \( s \) to \( t \) we get as a result.
\[ \ln P_t f(y) - \ln P_s f(x) \geq -\int_s^t \frac{a(u)}{b(u)} du - \frac{1}{4} \int_s^t a(u) \| \gamma'(u) \|^2 du. \]

We now minimize the quantity \( \int_s^t a(u) \| \gamma'(u) \|^2 du \) over the set of absolutely continuous paths such that \( \gamma(s) = x, \gamma(t) = y \). By using reparametrization of paths, it is seen that
\[ \int_s^t a(u) \| \gamma'(u) \|^2 du \leq \frac{d^2(x, y)}{\int_s^t d\tilde{u}}. \]

with equality achieved for \( \gamma(u) = \sigma \left( \frac{\int_s^t a(u)}{\int_s^t d\tilde{u}} \right) \) where \( \sigma : [0, 1] \to \mathbb{M} \) is a unit geodesic joining \( x \) and \( y \). As a conclusion,
\[ P_s f(x) \leq \exp \left( \int_s^t \frac{a(u)}{b(u)} du + \frac{d^2(x, y)}{4 \int_s^t d\tilde{u}} \right) P_t f(y). \]

Using finally the expressions of \( a \) and \( b \) leads to
\[ P_s f(x) \leq \left( \frac{1 - e^{-\frac{2a}{3}}}{1 - e^{-\frac{2a}{3}}} \right)^{n/2} e^{-\frac{d(x, y)^2}{3} - \frac{d^2(x, y)}{6}} P_t f(y). \]

If \( f \in L^\infty(\mathbb{M}) \) but \( f \notin C^2(\mathbb{M}) \), in the previous argument we replace \( f \) by \( P_\tau f, \tau > 0 \) and, at the end, let \( \tau \to 0 \).

As a straightforward corollary, we get a Harnack inequality for the Neumann heat kernel:

**Corollary 2.4.** Let \( p(t, x, y) \) be the Neumann heat kernel of \( \mathbb{M} \). For \( 0 < s < t \) and \( x, y, z \in \mathbb{M} \),
\[ p(s, x, y) \leq \left( \frac{1 - e^{-\frac{2a}{3}}}{1 - e^{-\frac{2a}{3}}} \right)^{n/2} e^{-\frac{d(y, z)^2}{3} - \frac{d^2(y, z)}{6}} p(t, x, z). \]
2.4. On diagonal heat kernel estimates. We now prove on-diagonal heat kernel estimates for the Neumann heat kernel that stem from the previous Harnack inequalities. We shall essentially focus on two types of estimates: Estimates that only depend on the curvature parameter \( \rho \) or estimates that depend on \( \rho \) and the diameter of \( M \).

**Proposition 2.5.** Let \( p(t, x, y) \) be the Neumann heat kernel of \( M \). For \( t > 0, x \in M \),

\[
\left( \frac{\rho}{6\pi} \right)^{n/2} \frac{1}{\left( 1 - e^{-\frac{2\rho t}{3}} \right)^{n/2}} \leq p(t, x, x) \leq \frac{1}{\mu(M)} \left( \frac{1}{1 - e^{-\frac{2\rho t}{3}}} \right)^{n/2}.
\]

**Proof.** From Corollary 2.4, for \( 0 \leq s < t \) and \( x, y \in M \),

\[
p(s, x, x) \leq \left( \frac{1 - e^{-\frac{2\rho t}{3}}}{1 - e^{-\frac{2\rho s}{3}}} \right)^{n/2} p(t, x, x).
\]

We have \( \lim_{t \to +\infty} p(t, x, x) = \frac{1}{\mu(M)} \). Thus by letting \( t \to +\infty \) in (2.7), we get

\[
p(s, x, x) \leq \frac{1}{\mu(M)} \left( \frac{1}{1 - e^{-\frac{2\rho s}{3}}} \right)^{n/2}.
\]

On the other hand, \( \lim_{s \to 0} p(s, x, x)(4\pi s)^{n/2} = 1 \), so by letting \( s \to 0 \) in (2.7), we deduce

\[
p(t, x, x) \geq \left( \frac{\rho}{6\pi} \right)^{n/2} \frac{1}{\left( 1 - e^{-\frac{2\rho t}{3}} \right)^{n/2}}.
\]

\( \square \)

**Remark 2.6.** Interestingly, Proposition 2.5 contains the geometric bound

\[
\mu(M) \leq \left( \frac{6\pi}{\rho} \right)^{n/2}.
\]

This bound is not sharp since from the Bishop’s volume comparison theorem the volume of \( M \) is less than the volume of the \( n \)-dimensional sphere with radius \( \sqrt{\frac{\mu(M)}{2\pi}} \) which is \( \frac{2\pi^{\frac{n+1}{2}}}{\Gamma\left( \frac{n+1}{2} \right)} \left( \frac{n-1}{\rho} \right)^{n/2} \).

However, by using Stirling’s equivalent we observe that, when \( n \to \infty \), the ratio between \( \left( \frac{6\pi}{\rho} \right)^{n/2} \) and \( \frac{2\pi^{\frac{n+1}{2}}}{\Gamma\left( \frac{n+1}{2} \right)} \left( \frac{n-1}{\rho} \right)^{n/2} \) only has order \( (\frac{3}{e})^n \).

Since

\[
\int_M p(t, x, x) d\mu(x) = \sum_{k=0}^{+\infty} e^{-\lambda_k t},
\]

where \( 0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \lambda_n \leq \cdots \) are the Neumann eigenvalues of \( M \), we deduce from the previous estimates

\[
\left( \frac{\rho}{6\pi} \right)^{n/2} \frac{\mu(M)}{\left( 1 - e^{-\frac{2\rho t}{3}} \right)^{n/2}} \leq \sum_{k=0}^{+\infty} e^{-\lambda_k t} \leq \frac{1}{\left( 1 - e^{-\frac{2\rho t}{3}} \right)^{n/2}}.
\]

**Proposition 2.7.** Let \( p(t, x, y) \) be the Neumann heat kernel of \( M \). For \( t > 0, x \in M \),

\[
p(t, x, x) \leq \frac{\left( 1 + e^{-\frac{2\rho t}{3}} \right)^{n/2} e^{n/2}}{\mu \left( B(x, \sqrt{r(t)}) \right)}
\]

with \( r(t) = \frac{3n}{\rho} \left( e^{\frac{4\rho t}{3}} - e^{\frac{2\rho t}{3}} \right) \).
Proof. From Corollary 2.4,

\[ p(t, x, x) \leq \left( \frac{1 - e^{-\frac{4\rho t}{3}}}{1 - e^{-\frac{2\rho t}{3}}} \right)^{n/2} e^{\frac{\rho D(x, y)^2}{4 \rho} - \frac{4\rho t}{3}} p(2t, x, y). \]

Thus, for \( y \in B \left( x, \sqrt{\frac{3n}{\rho} \left( e^{-\frac{4\rho t}{3}} - e^{-\frac{2\rho t}{3}} \right)} \right) \),

\[ p(t, x, x) \leq \left( 1 + e^{-\frac{2\rho t}{3}} \right)^{n/2} \mu \left( B \left( x, \sqrt{\frac{3n}{\rho} \left( e^{-\frac{4\rho t}{3}} - e^{-\frac{2\rho t}{3}} \right)} \right) \right) \int_{B \left( x, \sqrt{\frac{3n}{\rho} \left( e^{-\frac{4\rho t}{3}} - e^{-\frac{2\rho t}{3}} \right)} \right)} p(2t, x, y) \mu(dy) \]

Integrating with respect to \( y \) over the ball \( B \left( x, \sqrt{\frac{3n}{\rho} \left( e^{-\frac{4\rho t}{3}} - e^{-\frac{2\rho t}{3}} \right)} \right) \) therefore yields

\[ p(t, x, x) \leq \left( 1 + e^{-\frac{2\rho t}{3}} \right)^{n/2} \frac{\mu \left( B \left( x, \sqrt{\frac{3n}{\rho} \left( e^{-\frac{4\rho t}{3}} - e^{-\frac{2\rho t}{3}} \right)} \right) \right)}{\mu \left( B \left( x, \sqrt{\frac{3n}{\rho} \left( e^{-\frac{4\rho t}{3}} - e^{-\frac{2\rho t}{3}} \right)} \right) \right)} \]

\[ \int_{B \left( x, \sqrt{\frac{3n}{\rho} \left( e^{-\frac{4\rho t}{3}} - e^{-\frac{2\rho t}{3}} \right)} \right)} p(2t, x, y) \mu(dy) \]

\[ \leq \left( 1 + e^{-\frac{2\rho t}{3}} \right)^{n/2} \frac{\mu \left( B \left( x, \sqrt{\frac{3n}{\rho} \left( e^{-\frac{4\rho t}{3}} - e^{-\frac{2\rho t}{3}} \right)} \right) \right)}{\mu \left( B \left( x, \sqrt{\frac{3n}{\rho} \left( e^{-\frac{4\rho t}{3}} - e^{-\frac{2\rho t}{3}} \right)} \right) \right)} \]

\[ \mu \left( B \left( x, \sqrt{\frac{3n}{\rho} \left( e^{-\frac{4\rho t}{3}} - e^{-\frac{2\rho t}{3}} \right)} \right) \right) \]

\[ \square \]

Combining the previous estimate with the Bishop-Gromov comparison theorem yields the following upper bound estimate for the heat kernel, which in small times, may be better than the upper bound of Proposition 2.5.

**Corollary 2.8.** Let \( p(t, x, y) \) be the Neumann heat kernel of \( M \). Denote \( D(M) \) the diameter of \( M \) and consider

\[ \tau = \frac{3}{2\rho} \ln \left( 1 + \sqrt{1 + \frac{4\rho D(M)^2}{3n}} \right) \]

For \( x \in M \):

- If \( 0 < t \leq \tau \),

\[ p(t, x, x) \leq \left( 1 + e^{-\frac{2\rho t}{3}} \right)^{n/2} e^{\rho D(M)} \frac{\mu(D(M))}{\mu(\sqrt{r(t)})}, \]

with \( r(t) = \frac{3n}{\rho} \left( e^{-\frac{4\rho t}{3}} - e^{-\frac{2\rho t}{3}} \right) \).

- If \( t \geq \tau \),

\[ p(t, x, x) \leq \left( 1 + e^{-\frac{2\rho t}{3}} \right)^{n/2} \frac{\mu(D(M))}{\mu(M)}, \]

where

\[ V_\rho(s) = \int_0^s \sin^{n-1} \left( \sqrt{\frac{\rho}{n-1}} u \right) du. \]
3. Lower bounds for the eigenvalues

Heat kernel upper bounds are a well-known device to prove lower bounds on the spectrum (see [8], [10]). We therefore apply the results of the previous Section and obtain:

**Theorem 3.1.** Let $0 = \lambda_0 < \lambda_1 \leq \cdots \leq \lambda_k \leq \cdots$ be the Neumann eigenvalues of $\mathcal{M}$.

- For every $k \in \mathbb{N}$,
  \[ \lambda_k \geq -\frac{n \rho}{3 \ln \left( 1 - \frac{1}{(1+e^{-n/2k})^{2/n}} \right)} \]

- For every $k \in \mathbb{N}$, $k > 2^{n/2}e^n - e^{n/2}$,
  \[ \lambda_k \geq \frac{n \rho}{3 \ln \left( 1 + \sqrt{1 + \frac{4\rho}{3n}V^{(2/3)} - \frac{1}{\rho} \left( \frac{(2e)^{n/2}V^e(D(\mathcal{M}))}{V^e(D(M))} \right)^2} \right)} \]

**Proof.**

- Thanks to (2.8), we have for every $t > 0$,
  \[ 1 + ke^{-\lambda_k t} \leq \sum_{k=0}^{+\infty} e^{-\lambda_k t} \leq \frac{1}{(1 - e^{-2\tau})^{n/2}}. \]

Choosing then $t = \frac{n}{2\lambda_k}$ yields the lower bound
  \[ \lambda_k \geq -\frac{n \rho}{3 \ln \left( 1 - \frac{1}{(1+e^{-n/2k})^{2/n}} \right)} \]

- Thanks to Corollary 2.8, if $t \leq \tau = \frac{3}{2\rho} \ln \left( 1 + \sqrt{1 + \frac{4\rho}{3n}D^2 \rho(D(\mathcal{M}))} \right)$, then
  \[ 1 + ke^{-\lambda_k t} \leq \frac{(2e)^{n/2}V^e(D(\mathcal{M}))}{V^e(D(\mathcal{M}))} \]

And, if $t \geq \tau$, then
  \[ 1 + ke^{-\lambda_k t} \leq (2e)^{n/2}. \]

For $k > 2^{n/2}e^n - e^{n/2}$, we have
  \[ 1 + ke^{-n/2} > (2e)^{n/2} \]

and thus $\frac{n}{2\lambda_k} \leq \tau$. This implies
  \[ 1 + ke^{-n/2} \leq \frac{(2e)^{n/2}V^e(D(\mathcal{M}))}{V^e(D(\mathcal{M}))} \frac{\sqrt{n}}{(2\lambda_k)} \]

and the result follows by direct computations.

**Remark 3.2.** When $k \to \infty$, we have
  \[ -\frac{n \rho}{3 \ln \left( 1 - \frac{1}{(1+e^{-n/2k})^{2/n}} \right)} \sim_{k \to \infty} \frac{n \rho}{3e} k^{2/n} \]
and

\[
\frac{n\rho}{3\ln \left( 1 + \sqrt{1 + \frac{4e^{n\rho}}{3n}\left( \frac{(2e)^{n/2}}{V_\rho(D(M))} \right)^2} \right)} \sim_{k \to +\infty} n \left( \frac{n\rho}{2e^2} \right)^{1 - \frac{1}{n}} \left( \frac{k}{V_\rho(D(M))} \right)^{2/n}
\]

which is consistent with the Weyl asymptotics

\[
\lambda_k^{n/2} \sim_{k \to \infty} \frac{(4\pi)^{n/2}}{\mu(M)} k.
\]

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