Nuclear dimension and $\mathcal{Z}$-stability of pure C*-algebras

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In Erinnerung an meinen Vater

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Abstract In this article I study a number of topological and algebraic dimension type properties of simple C*-algebras and their interplay. In particular, a simple C*-algebra is defined to be (tracially) $(m, \bar{m})$-pure, if it has (strong tracial) $m$-comparison and is (tracially) $\bar{m}$-almost divisible. These notions are related to each other, and to nuclear dimension.

The main result says that if a separable, simple, nonelementary, unital C*-algebra with locally finite nuclear dimension is $(m, \bar{m})$-pure, then it absorbs the Jiang–Su algebra $\mathcal{Z}$ tensorially. It follows that a separable, simple, nonelementary, unital C*-algebra with locally finite nuclear dimension is $\mathcal{Z}$-stable if and only if it has the Cuntz semigroup of a $\mathcal{Z}$-stable C*-algebra. The result may be regarded as a version of Kirchberg’s celebrated theorem that separable, simple, nuclear, purely infinite C*-algebras absorb the Cuntz algebra $O_\infty$ tensorially.

As a corollary we obtain that finite nuclear dimension implies $\mathcal{Z}$-stability for separable, simple, nonelementary, unital C*-algebras; this settles an important case of a conjecture by Toms and the author.

The main result also has a number of consequences for Elliott’s program to classify nuclear C*-algebras by their K-theory data. In particular, it completes the classification of simple, unital, approximately homogeneous algebras with slow dimension growth by their Elliott invariants, a question left open in the Elliott–Gong–Li classification of simple AH algebras.

Another consequence is that for simple, unital, approximately subhomogeneous algebras, slow dimension growth and $\mathcal{Z}$-stability are equivalent. In
the case where projections separate traces, this completes the classification of
simple, unital, approximately subhomogeneous algebras with slow dimension
growth by their ordered K-groups.

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1 Introduction

In the structure and classification theory of nuclear C\(^*\)-algebras, the inter-
play between algebraic and topological regularity properties is becoming in-
creasingly important. Algebraic regularity occurs in many different ways; it
might be related to the number of generators of certain modules, to tenso-
rial absorption of suitable touchstone algebras, or to structural properties of
homological invariants. Topological regularity properties usually stem from
the interpretation of C\(^*\)-algebras as noncommutative topological spaces. They
might manifest themselves as properties of the primitive ideal space (e.g. sep-
oration properties), but there is also a rich supply of noncommutative versions
of topological dimension, which carry nontrivial information even when the
spectrum doesn’t. Examples of such concepts are stable and real rank [4, 24],
or (in increasing order of generality) approximately homogeneous dimension,
approximately subhomogeneous dimension, decomposition rank, and nuclear
dimension (cf. [19, 27, 43]). Of these, the former are linked by design to cer-
tain algebraic regularity properties; the latter concepts behave much more like
topological invariants, and their connections to algebraic regularity are subtle.

In [35, 40, 43], some links between such regularity properties have been
proven, and some others have been formally established as conjectures (see
also [12]). In particular, A.S. Toms and the author have conjectured that fi-
nite decomposition rank, Z-stability and strict comparison are equivalent, at
least for separable, simple, nonelementary, unital, nuclear and stably finite
C\(^*\)-algebras. The main result of [40] establishes one implication of this con-
jecture: a separable, simple, nonelementary, unital C\(^*\)-algebra with finite de-
composition rank is Z-stable, i.e., absorbs the Jiang–Su algebra Z tensorially.
By [28], this also implies strict comparison of positive elements (i.e., the or-
der structure of the Cuntz semigroup is determined by lower semicontinuous
dimension functions, which in turn are given by tracial states). The result has
been particularly useful for Elliott’s classification program for nuclear C\(^*\)-
algebras; for example it has led to the complete classification of C\(^*\)-algebras
associated to uniquely ergodic, minimal, finite-dimensional dynamical sys-
tems, in [33, 34].

The conjecture of Toms and the author was formulated in the not neces-
sarily finite case in [43], replacing decomposition rank by nuclear dimension.
This generalization is interesting since on the one hand it allows a unified
view on the purely infinite and the purely finite branches of the classification program; on the other hand, nuclear dimension is highly accessible to applications, in particular in dynamical systems, but also in coarse geometry.

In the present paper some more regularity properties for simple C*-algebras are studied, \( m \)-comparison and \( m \)-almost divisibility. These have an algebraic flavour and can be formulated at the level of the Cuntz semigroup, but they might as well be interpreted as dimension type conditions. I also introduce and study tracial versions of these conditions, (strong) tracial \( m \)-comparison and tracial \( m \)-almost divisibility. I then call a simple C*-algebra (tracially) \((m, \tilde{m})\)-pure, if it has (strong tracial) \( m \)-comparison and (tracial) \( \tilde{m} \)-almost divisibility; it is shown that tracial \((m, \tilde{m})\)-purity implies \((\tilde{m}, \tilde{m})\)-purity (for a suitable \( \tilde{m} \) depending on \( m \) and \( \tilde{m} \)).

L. Robert has shown in [25] that if a C*-algebra has nuclear dimension at most \( m \), then it has \( m \)-comparison in the sense of [25] (in the simple case, Robert’s notion of \( m \)-comparison agrees with the definition used here). I will use a result of [43] on Kirchberg’s covering number to show that finite nuclear dimension (at least in the simple case) implies \( \tilde{m} \)-almost divisibility for some \( \tilde{m} \in \mathbb{N} \); tracial \((m, \tilde{m})\)-purity will also be derived in this situation. This generalizes results of [40, Sect. 3] to the case of nuclear dimension; [40, Lemma 3.10 and Corollary 3.12] essentially yield tracial \((m, \tilde{m})\)-purity for simple, unital C*-algebras with finite decomposition rank, and provide crucial ingredients for the main result of [40].

While the subtle interplay between nuclear dimension and (tracial) \((m, \tilde{m})\)-purity is interesting by itself, these relations become particularly useful when combined with the main technical result, Theorem 7.1. This says that a simple, unital, nonelementary C*-algebra with locally finite nuclear dimension is \( \mathcal{Z} \)-stable, if it is tracially \((m, \tilde{m})\)-pure. The proof of Theorem 7.1 is quite technical; it will to some extent follow ideas from [40], isolating and generalizing some of the abstract machinery of that paper.

As a consequence of these results we obtain that a separable, simple, unital, nonelementary C*-algebra with finite nuclear dimension is \( \mathcal{Z} \)-stable, Corollary 7.3, thus establishing one of the implications of [43, Conjecture 9.3]. As a second consequence, the statement that a pure, separable, simple and unital C*-algebra with locally finite nuclear dimension is \( \mathcal{Z} \)-stable (Corollary 7.2) partially verifies another implication of [43, Conjecture 9.3]. (The reverse implication was established by M. Rørdam, who has shown that pureness is a necessary condition for \( \mathcal{Z} \)-stability.) Remarkably, then, if \( A \) is simple and unital with locally finite nuclear dimension, \( \mathcal{Z} \)-stability can be read off from the Cuntz semigroup, and, in fact, \( A \) is \( \mathcal{Z} \)-stable if and only if the Cuntz semigroups of \( A \) and \( A \otimes \mathcal{Z} \) agree, see Corollary 7.4.

Corollaries 7.2 and 7.3 are both nontrivial, and both are useful in the structure theory and in the classification program for nuclear C*-algebras. For example, the result on finite nuclear dimension provides an alternative (to some
extent more direct, but not necessarily simpler) access to [33, Theorem 0.2] (still using [33, Theorem 0.3]). It will also be a crucial ingredient of one of the main results of [14], which says that if \( A \) is a separable, simple, unital C*-algebra with finite nuclear dimension, then there is a dense \( G_\delta \) subset of the space of automorphisms of \( A \) such that the respective crossed product C*-algebras are again simple with finite nuclear dimension.

The result on \( \mathcal{Z} \)-stability of pure C*-algebras in combination with a theorem of A.S. Toms [30] can be used to make progress on the classification and structure theory for simple, unital, approximately (sub)homogeneous C*-algebras. Corollary 7.5 says that for such algebras slow dimension growth is equivalent to \( \mathcal{Z} \)-stability. In the approximately subhomogeneous case with slow dimension growth, this confirms the Elliott conjecture provided projections separate traces—Corollary 7.6. In the approximately homogeneous case with slow dimension growth, we obtain classification without any additional trace space condition—see Corollary 7.7; this answers an important technical question left open in [11], where classification was proven under the more restrictive (and less natural) condition of very slow dimension growth. (It should be noted, however, that the result of [11] is still a necessary step.)

In the absence of traces, a pure, separable, simple, unital C*-algebra is purely infinite in the sense of [8]. Therefore, the statement that pureness implies \( \mathcal{Z} \)-stability (at least under the additional structural hypothesis of locally finite nuclear dimension) may be regarded as a partial generalization of E. Kirchberg’s celebrated result that purely infinite, separable, simple, nuclear C*-algebras absorb the Cuntz algebra \( \mathcal{O}_\infty \); see [16]. (Kirchberg’s theorem later became one of the cornerstones of the Kirchberg–Phillips classification of purely infinite, simple, nuclear C*-algebras.)

It should be mentioned that it remains an open question how restrictive the hypothesis of locally finite nuclear dimension really is. In fact, at this point we do not know of any nuclear C*-algebra which does not have locally finite nuclear dimension.

I also wish to point out that the methods of this article do not in any way involve a dichotomy between the purely finite (i.e., finite and \( \mathcal{Z} \)-stable) and the purely infinite (i.e., \( \mathcal{O}_\infty \)-stable or, equivalently, infinite and \( \mathcal{Z} \)-stable) situation. This may be interpreted as promising evidence towards an approach to classification which would unify the purely finite and the purely infinite case.

The paper is organized as follows. Section 2 introduces some notation, recalls the notions of order zero maps, decomposition rank and nuclear dimension, as well as a useful criterion for \( \mathcal{Z} \)-stability. In Sect. 3, the interplay of several versions of \( m \)-comparison and \( m \)-almost divisibility for C*-algebras is studied and the notion of (tracially) \((m, \bar{m})\)-pure C*-algebras is introduced. In Sect. 4 it is shown that finite nuclear dimension implies tracial \((m, \bar{m})\)-pureness, at least in the simple and unital case. Section 5 generalizes

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[40, Sect. 3] (it provides tracially large almost central order zero maps into tracially \(m\)-almost divisible C\(^*\)-algebras with locally finite nuclear dimension); in a somewhat similar manner, Sect. 6 generalizes [40, Sect. 4]. The main result and its corollaries are established in Sect. 7.

2 Preliminaries

Below we recall the definition of and some facts about order zero maps, the Cuntz semigroup and \(\mathcal{Z}\)-stability; we also fix some notation which will be used frequently throughout the paper.

2.1

Recall from [42] that a completely positive (c.p.) map \(\varphi: A \to B\) between C\(^*\)-algebras \(A\) and \(B\) is said to have order zero if it preserves orthogonality. We collect below some facts about order zero maps from [42] (see [36, Proposition 3.2(a)] and [37, 1.2] for the case of finite-dimensional domains).

**Proposition** Let \(A\) and \(B\) be C\(^*\)-algebras, and let \(\varphi: A \to B\) be a completely positive contractive (c.p.c.) order zero map.

(i) There is a unique \(*\)-homomorphism

\[
\tilde{\varphi}: C_0((0, 1]) \otimes A \to B
\]

such that

\[
\varphi(x) = \tilde{\varphi}(\iota \otimes x)
\]

for all \(x \in A\), where \(\iota(t) = t\).

(ii) There are a \(*\)-homomorphism

\[
\pi: A \to \mathcal{M}(C^*(\varphi(A))) \subseteq B''
\]

and a positive element

\[
0 \leq h \in \mathcal{M}(C^*(\varphi(A)))
\]

(both uniquely determined) such that

\[
\varphi(x) = \pi(x)h = h\pi(x)
\]

for all \(x \in A\). If \(A\) is unital, then \(h = \varphi(1_A)\).
(iii) If, for some \( u \in B \) with \( \|u\| \leq 1 \), the element \( u^*u \) commutes with \( \pi(A) \), then the map
\[
\varphi_u : A \to B
\]
given by
\[
\varphi_u(x) = u\pi(x)u^*
\]
for \( x \in A \), is a c.p.c. order zero map.

**Notation 2.2** The map \( \pi \) in 2.1(ii) above will be called the *canonical supporting \(*-homomorphism* of \( \varphi \). Following [42] (also cf. [40, 1.3]), whenever \( \varphi : A \to B \) is a c.p.c. order zero map and \( f \in C^0((0, 1]) \) is a positive function of norm at most one, we define a c.p.c. order zero map
\[
f(\varphi) : A \to B
\]
by setting
\[
f(\varphi)(x) := \pi(x)f(h).
\]

**Remark 2.3** Let \( F \) be finite-dimensional, \( \pi : A \to B \) a surjective \(*-homomorphism* and \( \varphi : F \to B \) a c.p.c. order zero map. Then, \( \varphi \) lifts to a c.p.c. order zero map \( \tilde{\varphi} : F \to A \), see [36].

2.4

It will be convenient to introduce the following notation.

**Notation** For positive numbers \( 0 \leq \eta < \epsilon \leq 1 \) define continuous functions
\[
f_{\eta, \epsilon}, g_{\eta, \epsilon} : [0, 1] \to [0, 1]
\]
by
\[
g_{\eta, \epsilon}(t) = \begin{cases} 
0 & \text{if } t \leq \eta, \\
1 & \text{if } \epsilon \leq t \leq 1, \\
\text{linear} & \text{else},
\end{cases}
\]
and
\[
f_{\eta, \epsilon}(t) = \begin{cases} 
0 & \text{if } t \leq \eta, \\
t & \text{if } \epsilon \leq t \leq 1, \\
\text{linear} & \text{else}.
\end{cases}
\]
Notation 2.5 For a given $L \in \mathbb{N}$, by a standard partition of unity for the unit interval $[0, 1]$ we mean a set of sawtooth functions

$$h_0^{(0)}, \ldots, h_L^{(0)}, h_1^{(1)}, \ldots, h_L^{(1)} \in \mathcal{C}([0, 1])$$

defined by

$$h_i^{(j)}(t) = \begin{cases} 
0 & \text{if } t \leq \frac{2l-i-1}{2L}, \\
1 & \text{if } t = \frac{2l-i}{2L}, \\
0 & \text{if } t \geq \frac{2l-i+1}{2L}, \\
\text{linear} & \text{else.}
\end{cases}$$

Note that the functions $h_1^{(0)}, \ldots, h_L^{(0)}, h_1^{(1)}, \ldots, h_L^{(1)}$ are in $\mathcal{C}_0((0, 1))$; we will slightly abuse notation and call these a standard partition of unity for the interval $(0, 1]$.

Notation 2.6 If $A$ is a unital C*-algebra, we denote by $\mathcal{T}(A)$ the space of tracial states and by $\mathcal{QT}(A)$ the space of normalized quasitraces, see [1]. $\mathcal{T}(A)$ and $\mathcal{QT}(A)$ are compact (and metrizable if $A$ is separable). If $A$ is simple, then any $0 \neq a \in A_+$ is strictly positive and continuous on $\mathcal{QT}(A)$, and there is $\eta > 0$ such that $\tau(a) > \eta$ for all $\tau \in \mathcal{QT}(A)$.

Remark 2.7 We associate to any $\tau \in \mathcal{QT}(A)$ a lower semicontinuous dimension function $d_\tau$ on $A_+$ by

$$d_\tau(a) := \lim_{n \to \infty} \tau(a^n);$$

it was shown in [2] that every lower semicontinuous dimension function on $A$ is induced by a quasitrace. One checks

$$d_\tau(g_{\epsilon,0}(a)) = \lim_{\epsilon \downarrow 0} \tau(g_{\epsilon,2\epsilon}(a))$$

and

$$d_\tau((a - \beta)_+) \leq \tau(g_{\beta/2,\beta}(a)) \quad (1)$$

whenever $\beta > 0$.

If $f$ is a positive continuous function on $\mathcal{QT}(A)$ and $a \in A_+$ satisfies

$$d_\tau(a) > f(\tau)$$

for all $\tau \in \mathcal{QT}(A)$, then there are $n \in \mathbb{N}$ and $\eta > 0$ such that

$$\tau(a^{\frac{1}{n}}) > f(\tau)$$
and

$$\tau(g_{\eta,2\eta}(a)) > f(\tau)$$

for all \(\tau \in QT(A)\) (we still assume \(A\) to be unital, so that \(QT(A)\) is compact). The argument is essentially the same as for Dini’s Theorem, the details are omitted.

**Remark 2.8** It was shown in [42, Corollary 3.4] that the composition of a trace with a c.p. order zero map is again a trace; the respective statement holds for quasitraces and for (lower semicontinuous) dimension functions.

**Notation 2.9** If \(A\) is a C*-algebra, we write

$$A_\infty := \prod_N A \big/ \bigoplus_N A;$$

we regard \(A\) as being embedded in \(A_\infty\) as constant sequences.

We denote by \(T_\infty(A)\) those tracial states on \(A_\infty\) which are of the form

$$[(a_l)_{l \in \mathbb{N}}] \mapsto \lim_{\omega} \tau_l(a_l)$$

for a free ultrafilter \(\omega\) on \(\mathbb{N}\) and a sequence \((\tau_l)_{l \in \mathbb{N}} \subset T(A)\). Note that any \(\tau \in T_\infty(A)\) restricts to a tracial state on \(A\).

**Notation 2.10** If \(A\) is a C*-algebra we denote by \(W(A)\) its Cuntz semigroup; this consists of equivalence classes of elements in

$$M_\infty(A)_+ = \bigcup_{k=1}^\infty M_k(A)_+.$$

Two elements \(a, b \in M_\infty(A)_+\) are equivalent, \(a \sim b\), if they are Cuntz subequivalent to each other, \(a \preceq b\) and \(b \preceq a\). The elements \(a\) and \(b\) are Murray–von Neumann equivalent, \(a \approx b\), if there is \(x \in M_\infty(A)\) such that \(x^*x = a\) and \(xx^* = b\).

Any dimension function \(d_\tau\) on \(A\) as in 2.7 induces a positive real-valued character (also denoted by \(d_\tau\)) on the Cuntz semigroup \(W(A)\) with its natural order (cf. [9]); if \(A\) is unital and \(\tau\) is a normalized quasitrace, then \(d_\tau : W(A) \to \mathbb{R}_+\) is a state.

The reader is referred to [28] and [26] for more detailed information and for background material on Cuntz comparison of positive elements and on the Cuntz semigroup.
Proposition 2.11 Let $A$ be a C*-algebra, $a, d \in A$ positive contractions and $0 < \beta < 1$ a number. If $d \preceq a$, then there is $x \in A$ such that

$$x^*x = g_{\beta/2, \beta}(b) \quad \text{and} \quad xx^* \in \text{her}(a) \subset A.$$  

Proof Since $d \preceq a$, there is $y \in A$ such that

$$\|d - y^* ay\| < \frac{\beta}{4}.$$  

By [18, Lemma 2.2], there is $z \in A$ with

$$(d - \beta/4)_+ = z^* y^* ay z.$$  

By functional calculus there is $t \in A$ such that

$$t^* (d - \beta/4)_+ t = g_{\beta/2, \beta}(d).$$  

Now

$$x := a^{1/2} y z t$$

will have the desired properties. $\square$

Proposition 2.12 Let $A$ be a C*-algebra, $a, b \in A$ positive contractions, $k \in \mathbb{N}$, $0 < \beta < 1$, and $f \in C_0((\beta, 1])$ a positive function of norm at most one. If

$$k \cdot \langle b \rangle \leq \langle a \rangle$$  

in $W(A)$, then there is a c.p.c. order zero map

$$\varphi : M_k \rightarrow \text{her}(a) \subset A$$

such that

$$\varphi(e_{11}) \approx f(b).$$

Proof Set

$$d := 1_k \otimes b \in M_k \otimes A,$$

then

$$\langle d \rangle = k \cdot \langle b \rangle \leq \langle a \rangle$$

in $W(A)$. By Proposition 2.11, there is $x \in M_k \otimes A$ such that

$$x^* x = g_{\beta/2, \beta}(d) \quad \text{and} \quad xx^* \in \text{her}(a).$$
2.13 Define a c.p.c. map
\[ \varphi : M_k \to \text{her}(a) \]
by
\[ \varphi(y) := x(y \otimes f(b))x^* \quad \text{for} \ y \in M_k. \]
Using Proposition 2.1, it is straightforward to check that \( \varphi \) has the desired properties. \( \square \)

Remark 2.13 Recall from [15] that the Jiang–Su algebra \( \mathcal{Z} \) is the uniquely determined simple and monotracial inductive limit of so-called prime dimension drop \( C^* \)-algebras,

\[ \mathcal{Z} = \lim_{\to} Z_{p_\nu,q_\nu}, \]

where
\[ Z_{p_\nu,q_\nu} = \{ f \in C([0, 1], M_{p_\nu} \otimes M_{q_\nu}) \mid f(0) \in M_{p_\nu} \otimes 1_{M_{q_\nu}} \text{ and } f(1) \in 1_{M_{p_\nu}} \otimes M_{q_\nu} \} \]

for \( \nu \in \mathbb{N} \), and \( p_\nu \) and \( q_\nu \) are relatively prime.

It was shown in [15] that \( \mathcal{Z} \) is strongly self-absorbing in the sense of [31], and that it is KK-equivalent to the complex numbers. In [29], several alternative characterizations of the Jiang–Su algebra were given; see also [10, 41].

2.14

In order to show that a separable and unital \( C^* \)-algebra \( A \) is \( \mathcal{Z} \)-stable, i.e., \( A \cong A \otimes \mathcal{Z} \), it will suffice to construct approximately central unital \( * \)-homomorphisms from (arbitrarily large) prime dimension drop intervals to \( A \); cf. [32, Proposition 2.2]. Using [29, Proposition 5.1], to this end it will be enough to realize the generators and relations of [29, Proposition 5.1(iii)] approximately—and in an approximately central way. This was formalized in terms of the relations \( \mathcal{R}(n,F,\eta) \) in [40, Sect. 2]. For our purposes the following combination of [40, Propositions 2.3 and 2.4] will be most useful.

**Proposition** Let \( A \) be a separable and unital \( C^* \)-algebra. Suppose that, for any \( n \in \mathbb{N} \), any finite subset \( F \subset A \), and for any numbers \( 0 < \delta < 1 \) and \( 0 < \zeta < 1 \), there are a c.p.c. order zero map
\[ \varphi' : M_n \to A \]
and
\[ v' \in A \]
satisfying

(i) \( \| (v')^* v' - (1_A - \varphi'(1_{M_n})) \| < \delta \),

(ii) \( v'(v')^* \in (\varphi'(e_{11}) - \zeta)_+ + A(\varphi'(e_{11}) - \zeta)_+ \),

(iii) \( \|[\varphi'(x), a]\| \leq \delta \|x\| \) for all \( x \in M_n, a \in \mathcal{F} \),

(iv) \( \|[v', a]\| < \delta \) for all \( a \in \mathcal{F} \).

Then, \( A \) is \( \mathcal{Z} \)-stable.

3 Pure C*-algebras

In this section we define and relate several versions of comparison and almost divisibility of a C*-algebra (or its Cuntz semigroup, respectively). The notion of a pure C*-algebra is based on these concepts.

**Definition 3.1** Let \( A \) be a separable, simple, unital C*-algebra and \( m \in \mathbb{N} \).

(i) We say \( A \) has \( m \)-comparison of positive elements, if for any nonzero positive contractions \( a, b_0, \ldots, b_m \in M_\infty(A) \) we have

\[
a \precsim b_0 \oplus \cdots \oplus b_m
\]

whenever

\[
d_\tau(a) < d_\tau(b_i)
\]

for every \( \tau \in QT(A) \) and \( i = 0, \ldots, m \).

(ii) \( A \) has tracial \( m \)-comparison of positive elements, if for any nonzero positive contractions \( a, b_0, \ldots, b_m \in M_\infty(A) \) we have

\[
a \precsim b_0 \oplus \cdots \oplus b_m
\]

whenever

\[
d_\tau(a) < \tau(b_i)
\]

for every \( \tau \in QT(A) \) and \( i = 0, \ldots, m \).

(iii) \( A \) has strong tracial \( m \)-comparison of positive elements, if for any nonzero positive contractions \( a, b \in M_\infty(A) \) we have

\[
a \precsim b
\]

whenever

\[
d_\tau(a) < \frac{1}{m+1} \cdot \tau(b)
\]

for every \( \tau \in QT(A) \).
Remarks 3.2 Let $A$ and $m$ be as above.

(i) Note that 0-comparison is the same as strict comparison as defined in [28]; we will occasionally just write ‘comparison’ for brevity. Note also that tracial comparison and strong tracial comparison (i.e., in the case $m = 0$) agree, and that strong tracial $m$-comparison implies tracial $m$-comparison. (We will derive a partial converse below.)

(ii) The definition of $m$-comparison above differs slightly from that of [25] since here we restrict ourselves to the case of simple $C^*$-algebras. In this situation, it is not hard to show that Definition 3.1(i) and [25, Definition 2] are in fact equivalent (using [25, Lemma 1] and an argument similar to that of Proposition 3.4 below).

(iii) I regard $m$-comparison as an algebraic interpretation of dimension. This is the reason why I write ‘$m$-comparison’ when the definition involves $m + 1$ positive elements $b_i$, as this notation is more coherent with what is common in dimension theory. The same comment applies to (strong) tracial $m$-comparison and to $m$-almost divisibility, see Definition 3.5 below.

Proposition 3.3 If $A$ is separable, simple and unital, then $A$ has $m$-comparison if and only if $A$ has tracial $m$-comparison.

Proof We have $\tau(b) \leq d_\tau(b)$ for every $\tau \in QT(A)$ and whenever $b$ is a positive contraction, so clearly $m$-comparison implies tracial $m$-comparison.

For the converse, let $a, b_0, \ldots, b_m \in M_\infty(A)$ be positive contractions such that

$$d_\tau(a) < d_\tau(b_i)$$

for every $\tau \in QT(A)$ and $i = 0, \ldots, m$. By passing to a matrix algebra, we may assume that $a, b_i \in A$.

Let $\epsilon > 0$, then

$$d_\tau(f_{2\epsilon,3\epsilon}(a)) \leq \tau(f_{\epsilon,2\epsilon}(a))$$
$$\leq d_\tau(a)$$
$$< d_\tau(b_i)$$
$$= \lim_{\eta \downarrow 0} \tau(g_{\eta,2\eta}(b_i))$$

for every $\tau \in QT(A)$ and $i = 0, \ldots, m$. By Dini’s Theorem (cf. 2.7) there is $\eta_\epsilon > 0$ such that

$$\tau(f_{\epsilon,2\epsilon}(a)) < \tau(g_{\eta_\epsilon,2\eta_\epsilon}(b_i))$$
for every $\tau \in QT(A)$ and $i = 0, \ldots, m$. We obtain

$$d_\tau(f_{2\varepsilon,3\varepsilon}(a)) < \tau(g_{\eta\varepsilon,2\eta\varepsilon}(b_i))$$

for every $\tau \in QT(A)$ and $i = 0, \ldots, m$, whence

$$f_{2\varepsilon,3\varepsilon}(a) \lesssim g_{\eta\varepsilon,2\eta\varepsilon}(b_0) \oplus \cdots \oplus g_{\eta\varepsilon,2\eta\varepsilon}(b_m) \lesssim b_0 \oplus \cdots \oplus b_m$$

by tracial $m$-comparison. Since $\varepsilon > 0$ was arbitrary, this yields

$$a \lesssim b_0 \oplus \cdots \oplus b_m,$$

as desired. $\square$

**Proposition 3.4**  Let $A$ be a separable, simple, unital C*-algebra and $m \in \mathbb{N}$. Suppose that, whenever $a, b \in M_\infty(A)$ are positive contractions and $\eta > 0$ is a number such that

$$d_\tau(a) < \frac{1}{m+1} \cdot \tau(b) - \eta$$

for every $\tau \in QT(A)$, then $a \lesssim b$.

Then, $A$ has strong tracial $m$-comparison.

**Proof**  Let $a, b \in M_\infty(A)$ be positive contractions such that

$$d_\tau(a) < \frac{1}{m+1} \cdot \tau(b)$$

for every $\tau \in QT(A)$.

If $a$ is Cuntz equivalent to a projection, we may as well assume that $a$ is a projection itself. In this case,

$$\tau(a) = d_\tau(a) < \frac{1}{m+1} \cdot \tau(b)$$

for every $\tau \in QT(A)$. Since $A$ is simple and unital, the assignment

$$\tau \mapsto \frac{1}{m+1} \cdot \tau(b) - \tau(a)$$

is a strictly positive continuous function on the compact space $QT(A)$, whence there is $\eta > 0$ such that

$$\frac{1}{m+1} \cdot \tau(b) - d_\tau(a) = \frac{1}{m+1} \cdot \tau(b) - \tau(a) > \eta$$

for all $\tau \in QT(A)$ (cf. 2.6); we now have $a \lesssim b$ by our hypothesis.
If $a$ is not equivalent to a projection, then 0 is an accumulation point of its spectrum and $a - f_{\epsilon, 2\epsilon}(a)$ is positive and nonzero for any $\epsilon > 0$. But then

$$\tau(a - f_{\epsilon, 2\epsilon}(a)) > \eta_{\epsilon}$$

for some $\eta_{\epsilon} > 0$ and for all $\tau \in QT(A)$ (again cf. 2.6).

We now estimate

$$d_\tau(f_{2\epsilon, 3\epsilon}(a)) + \eta_{\epsilon} < d_\tau(f_{2\epsilon, 3\epsilon}(a)) + \tau(a - f_{\epsilon, 2\epsilon}(a))$$

$$\leq d_\tau(f_{2\epsilon, 3\epsilon}(a)) + d_\tau(a - f_{\epsilon, 2\epsilon}(a))$$

$$\leq d_\tau(a)$$

$$< \frac{1}{m + 1} \cdot \tau(b),$$

using the fact that $f_{2\epsilon, 3\epsilon}(a)$ and $a - f_{\epsilon, 2\epsilon}(a)$ are orthogonal, with sum dominated by $a$. By our hypothesis we then have $f_{2\epsilon, 3\epsilon}(a) \precsim b$; since $\epsilon > 0$ was arbitrary, this implies $a \precsim b$. □

**Definition 3.5** Let $A$ be a unital $\mathbb{C}^*$-algebra and $m \in \mathbb{N}$.

(i) $A$ has $m$-almost divisible Cuntz semigroup, if for any positive contraction $a \in M_\infty(A)$ and $0 \neq k \in \mathbb{N}$ there is $x \in W(A)$ such that

$$k \cdot x \leq \langle a \rangle \leq (k + 1)(m + 1) \cdot x.$$  

(ii) We say $A$ is tracially $m$-almost divisible, if for any positive contraction $a \in M_\infty(A)$, $\epsilon > 0$ and $0 \neq k \in \mathbb{N}$ there is a c.p.c. order zero map

$$\psi : M_k \to \text{her}(a) \subset M_\infty(A)$$

such that

$$\tau(\psi(1_k)) \geq \frac{1}{m + 1} \cdot \tau(a) - \epsilon$$

for all $\tau \in QT(A)$.

**Definition 3.6** Let $A$ be a separable, simple, unital $\mathbb{C}^*$-algebra and $m, \bar{m} \in \mathbb{N}$.

(i) We say $A$ is $(m, \bar{m})$-pure, if it has $m$-comparison of positive elements and if it is $\bar{m}$-almost divisible. We say $A$ is pure, if it is $(0, 0)$-pure.

(ii) We say $A$ is tracially $(m, \bar{m})$-pure, if it has strong tracial $m$-comparison of positive elements and if it is tracially $\bar{m}$-almost divisible. We say $A$ is tracially pure, if it is tracially $(0, 0)$-pure.
For convenience, let us note the following combination of results by Rørdam explicitly.

**Proposition** A separable, simple, unital and \( \mathcal{Z} \)-stable \( C^* \)-algebra is pure.

**Proof** By [28, Theorem 4.5], a \( \mathcal{Z} \)-stable \( C^* \)-algebra \( A \) has almost unperforated Cuntz semigroup, hence comparison (see, for example, [23, Proposition 2.1]).

To show almost divisibility, let \( 0 \neq k \in \mathbb{N} \) and let \( a \in M_\infty(A) \) be a positive contraction. By [28, Lemma 4.2], there is a positive contraction \( e \in \mathcal{Z} \) such that

\[
k \cdot \langle e \rangle \leq 1 \mathcal{Z} \leq (k + 1) \cdot \langle e \rangle.
\]

Now \( a \otimes e \in M_\infty(A \otimes \mathcal{Z}) \) will satisfy

\[
k \cdot \langle a \otimes e \rangle \leq a \otimes 1 \mathcal{Z} \leq (k + 1) \cdot \langle a \otimes e \rangle
\]

in \( W(A \otimes \mathcal{Z}) \).

By [31] there is an isomorphism

\[
\varphi : A \to A \otimes \mathcal{Z}
\]

which is approximately unitarily equivalent to the embedding

\[
id_A \otimes 1 \mathcal{Z} : A \to A \otimes \mathcal{Z}.
\]

This in turn implies that \( \text{id}_A \otimes 1 \mathcal{Z} \) and \( \varphi \) both induce the same isomorphism between Cuntz semigroups, say

\[
\psi : W(A) \overset{\cong}{\to} W(A \otimes \mathcal{Z}),
\]

whence

\[
\psi^{-1}(\langle a \otimes 1 \mathcal{Z} \rangle) = \langle a \rangle.
\]

It follows from (2) and (3) that

\[
x := \psi^{-1}(\langle a \otimes e \rangle) \in W(A)
\]

satisfies

\[
k \cdot x \leq \langle a \rangle \leq (k + 1) \cdot x,
\]

as desired. \( \square \)
Proposition 3.8 Let $A$ be a separable, simple, unital C*-algebra and $m \in \mathbb{N}$. If $A$ has $m$-almost divisible Cuntz semigroup, then $A$ is tracially $m$-almost divisible.

Proof Let $a \in M_\infty(A)$ a positive contraction, $\epsilon > 0$ and $0 \neq k \in \mathbb{N}$ be given. By passing to a matrix algebra over $A$, we may as well assume that $a \in A$.

Choose $\bar{k} \in \mathbb{N}$ such that $k$ divides $\bar{k}$ and such that
\[
\frac{1}{\bar{k}} \leq \frac{\epsilon}{2}.
\]

By $m$-almost divisibility, there is $x \in W(A)$ such that
\[
\bar{k} \cdot x \leq \langle a \rangle \leq (\bar{k} + 1)(m + 1) \cdot x. \tag{4}
\]

Suppose that $x = \langle b \rangle$ for some positive contraction $b \in M_r(A)$ and $r \in \mathbb{N}$. We then have
\[
\frac{1}{m + 1} \cdot \tau(a) \leq \frac{1}{m + 1} \cdot d_\tau(a) \leq (\bar{k} + 1) \cdot d_\tau(b)
\]
for every $\tau \in QT(A)$.

Since
\[
d_\tau(b) = \lim_{\eta \searrow 0} \tau(g_{\eta,2\eta}(b))
\]
for every $\tau$, by Dini’s Theorem (cf. 2.7) there is $\eta > 0$ such that
\[
\frac{1}{m + 1} \cdot \tau(a) \leq (\bar{k} + 1) \cdot \tau(g_{\eta,2\eta}(b)) + \frac{\epsilon}{2} \tag{5}
\]
for every $\tau \in QT(A)$. We also have
\[
\tau(g_{\eta,2\eta}(b)) \leq d_\tau(b) \leq \frac{1}{\bar{k}} \cdot d_\tau(a) \leq \frac{1}{\bar{k}} < \frac{\epsilon}{2}, \tag{6}
\]
whence
\[
\frac{1}{m + 1} \cdot \tau(a) \leq (\bar{k} + 1) \cdot \tau(g_{\eta,2\eta}(b)) + \epsilon \tag{7}
\]
for every $\tau \in QT(A)$.

Since
\[
\bar{k} \cdot x \leq \langle a \rangle,
\]
by Proposition 2.12 there is a c.p.c. order zero map
\[
\tilde{\psi} : M_{\bar{k}} \to \text{her}(a) \subset A
\]
such that
\[ \bar{\psi}(e_{11}) \approx g_{\eta, 2\eta}(b), \]
which in turn implies that
\[ \tau(\bar{\psi}(1_{k})) = \bar{k} \cdot \tau(\bar{\psi}(e_{11})) = \bar{k} \cdot \tau(g_{\eta, 2\eta}(b)) \geq \frac{1}{m+1} \cdot \tau(a) - \epsilon \]
for every \( \tau \in QT(A) \).

Upon composing \( \bar{\psi} \) with a unital embedding of \( M_k \) into \( M_{\bar{k}} \) (\( k \) divides \( \bar{k} \)), we obtain \( \psi \) as desired. \( \square \)

**Proposition 3.9** Given \( m, \bar{m} \in \mathbb{N} \), there is \( \tilde{m} \in \mathbb{N} \) such that the following holds:

If \( A \) is a separable, simple, unital \( C^* \)-algebra which has tracial \( m \)-comparison and tracial \( \bar{m} \)-almost divisibility, then \( A \) has strong tracial \( \tilde{m} \)-comparison, i.e., \( A \) is \((\tilde{m}, \bar{m})\)-pure.

**Proof** Take
\[ \tilde{m} := (m+1)(\bar{m}+1) - 1. \]

Now suppose we have positive contractions \( a, b \in A \) and some \( \eta > 0 \) with
\[ d_\tau(a) < \frac{1}{m+1} \cdot \tau(b) - \eta \]
for all \( \tau \in QT(A) \). By Proposition 3.4 it will suffice to show that \( a \preceq b \). (We do not have to consider matrix algebras, since any matrix algebra over \( A \) satisfies the hypotheses of the proposition just as \( A \) does.)

By tracial \( \bar{m} \)-almost divisibility, there is a c.p.c. order zero map
\[ \psi : M_{m+1} \to \text{her}(b) \subset A \]
such that
\[ \tau(\psi(1_{m+1})) \geq \frac{1}{\bar{m}+1} \cdot \tau(b) - \frac{\bar{m}+1}{\tilde{m}+1} \cdot \eta \]
for all \( \tau \in QT(A) \). Then (9) and (10) together give
\[ d_\tau(a) < \frac{\tilde{m}+1}{\tilde{m}+1} \cdot \tau(\psi(1_{m+1})) \]
\[ \overset{2.8}{=} \frac{(\bar{m}+1)(m+1)}{(\tilde{m}+1)} \cdot \tau(\psi(e_{11})) \]
\[ = \tau(\psi(e_{11})) \]
for all $\tau \in QT(A)$; by tracial $m$-comparison and since $\psi$ has order zero, we obtain

$$ a \preceq \psi(1_{m+1}) \preceq b, $$

as desired. \hfill \Box

3.10

Let us explicitly note the following corollary, in which $\tilde{m}$ comes from Proposition 3.9.

**Corollary** If $A$ is separable, simple, unital and $(m, \tilde{m})$-pure, then $A$ is tracially $(\tilde{m}, \tilde{m})$-pure.

**Proof** This combines Propositions 3.3, 3.8 and 3.9. \hfill \Box

3.11

The final result of this section says that pureness and tracial pureness are in fact equivalent. At this point I do not know whether Corollary 3.10 admits a more general converse.

**Theorem** Let $A$ be a separable, simple and unital $C^*$-algebra.

Then, $A$ is pure if and only if $A$ is tracially pure.

**Proof** The forward implication is a special case of Corollary 3.10, observing that Proposition 3.9 yields $\tilde{m} = 0$ if $m = \tilde{m} = 0$ by (8).

So let us assume that $A$ is tracially pure, i.e., $A$ has strong tracial comparison and is tracially almost divisible.

But strong tracial comparison is just tracial comparison, which implies comparison by Proposition 3.3. It therefore remains to show that $W(A)$ is almost divisible.

To this end, let $a \in M_\infty(A)$ be a (nonzero) positive contraction and $0 \neq k \in \mathbb{N}$ a number; we need to find $x \in W(A)$ such that

$$ k \cdot x \leq \langle a \rangle \leq (k + 1) \cdot x. $$

By passing to a matrix algebra over $A$, we may clearly assume that $a \in A$.

We first consider the case when $a$ is Cuntz equivalent to a projection, so that we may as well assume that $a$ itself is a projection. But then

$$ \tau(a) = d_\tau(a) $$

for all $\tau \in QT(A)$; by 2.6 there is $\delta > 0$ such that

$$ d_\tau(a) = \tau(a) > \delta $$
for every $\tau \in QT(A)$.

By tracial almost divisibility there is a c.p.c. order zero map
\[ \psi : M_k \to \text{her}(a) \subset A \]
such that
\[ \tau(\psi(1_k)) \geq \tau(a) - \frac{\delta}{k+1} = d_\tau(a) - \frac{\delta}{k+1} \quad (13) \]
for every $\tau \in QT(A)$. But then we have
\[ (k+1) \cdot d_\tau(\psi(e_{11})) \geq (k+1) \cdot \tau(\psi(e_{11})) \]
\[ \geq \frac{(k+1)}{k} \cdot \tau(\psi(1_k)) \]
\[ \geq \frac{1}{k}((k+1) \cdot \tau(a) - \delta) \]
\[ \geq d_\tau(a) + \frac{1}{k} \cdot (d_\tau(a) - \delta) \]
\[ > d_\tau(a) \quad (14) \]
for every $\tau \in QT(A)$. Setting
\[ x := \langle \psi(e_{11}) \rangle, \]
we obtain
\[ d_\tau((k+1) \cdot x) = (k+1) \cdot d_\tau(x) \quad (14) \]
for all $\tau$, whence
\[ \langle a \rangle \leq (k+1) \cdot x \]
by comparison. We have
\[ k \cdot x \leq \langle a \rangle \]
since $\psi(1_k) \in \text{her}(a)$.

We now consider the case when $a$ is not Cuntz equivalent to a projection. But then 0 is an accumulation point of the spectrum, and (using 2.6) it is straightforward to construct a strictly decreasing sequence $(\eta_l)_{l \in \mathbb{N}}$ of strictly positive numbers such that the elements
\[ a_l := g_{\eta_l, 2\eta_l}(a) \in A \]
form a strictly increasing sequence of positive contractions, satisfying
\[ \tau(a_l) > d_\tau(a_{l-1}) > \tau(a_{l-1}) \quad (15) \]
for all $l \geq 1$ and $\tau \in QT(A)$; note that

$$d_\tau(a_l) \xrightarrow{l \to \infty} d_\tau(a)$$

(16)

for all $\tau$ by lower semicontinuity.

For $1 \leq l \in \mathbb{N}$, set

$$\epsilon_l := \frac{1}{2} \cdot (\tau(a_l) - d_\tau(a_l-1)) > 0.$$  

(17)

By tracial almost divisibility, for each $l \in \mathbb{N}$ there is a c.p.c. order zero map

$$\psi_l : M_k \to \text{her}(a_l)$$

(18)

such that

$$\tau(\psi_l(1_k)) \geq \tau(a_l) - \epsilon_l$$

(19)

for every $\tau \in QT(A)$. We then have

$$d_\tau(a) \overset{\text{(16), (18)}}{\geq} d_\tau(\psi_l(1_k))$$

$$\quad \geq \tau(\psi_l(1_k))$$

$$\quad \overset{\text{(19)}}{\geq} \tau(a_l) - \epsilon_l$$

$$\quad \overset{\text{(17)}}{>} d_\tau(a_l-1)$$

$$\quad \overset{\text{(18)}}{\geq} d_\tau(\psi_{l-1}(1_k))$$

(20)

for all $l \geq 1$ and $\tau \in QT(A)$.

Set

$$x_l := \langle \psi_l(e_{11}) \rangle \in W(A),$$

then

$$d_\tau(k \cdot x_l) = k \cdot d_\tau(x_l) \overset{\text{2.8}}{=} d_\tau(\psi_l(1_k))$$

(21)

for all $l$ and $\tau$; by (20) we have

$$d_\tau(x_l) < d_\tau(x_{l+1})$$

for all $l$ and $\tau$, so comparison yields

$$x_l \leq x_{l+1} \leq \cdots$$

and

$$k \cdot x_l \leq k \cdot x_{l+1} \leq \cdots \leq \langle a \rangle.$$
Let

\[ x := \sup \{ x_l \} \in W(A \otimes K) \]

be the supremum of the \( x_l \), cf. [7] or [5]; it follows from [3] that in fact

\[ x \in W(A) \subseteq W(A \otimes K). \]

By [7] we also have

\[ k \cdot x = \sup \{ k \cdot x_l \} \leq \langle a \rangle. \]

Moreover, we have

\[ d_\tau (x) = \frac{1}{k} \cdot d_\tau (k \cdot x) \geq \frac{1}{k} \cdot d_\tau (\psi I (1_k)) \xrightarrow{l \rightarrow \infty} \frac{1}{k} \cdot d_\tau (a) > 0, \]

whence

\[ d_\tau (a) < (k + 1) \cdot d_\tau (x) = d_\tau ((k + 1) \cdot x) \]

for all \( \tau \in QT(A) \). Comparison yields

\[ \langle a \rangle \leq (k + 1) \cdot x. \]

We have now found \( x \in W(A) \) such that

\[ k \cdot x \leq \langle a \rangle \leq (k + 1) \cdot x, \]

regardless of whether \( a \) is equivalent to a projection or not. \( \square \)

4 Pureness and nuclear dimension

Corollary 7.3 below says that separable, simple, nonelementary, unital C\(^*\)-algebras with finite nuclear dimension are \( \mathcal{Z} \)-stable; together with Proposition 3.7 this entails pureness.

As an intermediate step towards \( \mathcal{Z} \)-stability we prove in this section that, for separable, simple, unital C\(^*\)-algebras, finite nuclear dimension implies tracial \((m, \bar{m})\)-pureness (for suitable \( m, \bar{m} \in \mathbb{N} \)). The key features here are \( m \)-comparison as established in [25], and finite covering number of the central sequence algebra (see [17]) of a C\(^*\)-algebra of nuclear dimension at most \( m \).

4.1

The following definitions are collected from [19, 39, 43]. Note that in (iii), we may additionally ask for the composition \( \varphi \circ \psi \) to be contractive by [43, Remark 2.2(iv)].
Definition Let $A$ be a C*-algebra, $m \in \mathbb{N}$.

(i) A c.p. map $\varphi : F \to A$ is $m$-decomposable (where $F$ is a finite-dimensional C*-algebra), if there is a decomposition

$$F = F^{(0)} \oplus \ldots \oplus F^{(m)}$$

such that the restriction $\varphi^{(i)}$ of $\varphi$ to $F^{(i)}$ has order zero for each $i \in \{0, \ldots, m\}$; we say $\varphi$ is $m$-decomposable with respect to $F = F^{(0)} \oplus \ldots \oplus F^{(m)}$.

(ii) $A$ has decomposition rank $m$, $\text{dr} A = m$, if $m$ is the least integer such that the following holds: For any finite subset $\mathcal{G} \subset A$ and $\varepsilon > 0$, there is a finite-dimensional c.p.c. approximation $(F, \psi, \varphi)$ for $\mathcal{G}$ to within $\varepsilon$ (i.e., $F$ is finite-dimensional $\psi : A \to F$ and $\varphi : F \to A$ are c.p.c. and $\|\varphi \psi (b) - b\| < \varepsilon \ \forall b \in \mathcal{G}$) such that $\varphi$ is $m$-decomposable. If no such $m$ exists, we write $\text{dr} A = \infty$.

(iii) $A$ has nuclear dimension $m$, $\dim_{\text{nuc}} A = m$, if $m$ is the least integer such that the following holds: For any finite subset $\mathcal{G} \subset A$ and $\varepsilon > 0$, there is a finite-dimensional c.p.c. approximation $(F, \psi, \varphi)$ for $\mathcal{G}$ to within $\varepsilon$ (i.e., $F$ is finite-dimensional, $\psi : A \to F$ and $\varphi : F \to A$ are c.p. and $\|\varphi \psi (b) - b\| < \varepsilon \ \forall b \in \mathcal{G}$) such that $\psi$ is c.p.c., and $\varphi$ is $m$-decomposable with c.p.c. order zero components $\varphi^{(i)}$. If no such $m$ exists, we write $\dim_{\text{nuc}} A = \infty$.

(iv) $A$ has locally finite decomposition rank (or locally finite nuclear dimension, respectively), if for any finite subset $\mathcal{G} \subset A$ and $\varepsilon > 0$, there is a C*-subalgebra $B \subset A$ such that $\text{dr} B < \infty$ (or $\dim_{\text{nuc}} B < \infty$, respectively) and $\mathcal{G} \subset \epsilon B$.

4.2

The following is contained in the proof of [43, Proposition 4.3]. An explicit proof is extracted here for convenience of the reader.

Proposition Let $A$ be a unital and separable C*-algebra with $\dim_{\text{nuc}} A \leq m < \infty$.

Then, there is a system of finite-dimensional $m$-decomposable c.p. approximations for $A$

$$(F_p = F_p^{(0)} \oplus \ldots \oplus F_p^{(m)}, \psi_p, \varphi_p)_{p \in \mathbb{N}}$$

such that the

$$\psi_p : A \to F_p$$

are c.p.c. maps, the

$$\varphi_p^{(i)} : F_p^{(i)} \to A$$
are c.p.c. order zero maps, and such that

\[ \| \varphi_p \psi_p (b) - b \| \overset{p \to \infty}{\longrightarrow} 0, \]  
(22)

\[ \| [\varphi_p \psi_p (b), \varphi_p^{(i)} (\psi_p^{(i)} (1_A))] \| \overset{p \to \infty}{\longrightarrow} 0 \]  
(23)

and

\[ \| b \varphi_p^{(i)} (\psi_p^{(i)} (1_A)) - \varphi_p^{(i)} \psi_p^{(i)} (b) \| \overset{p \to \infty}{\longrightarrow} 0 \]  
(24)

for all \( b \in A \), \( i \in \{0, \ldots, m\} \).

**Proof** Choose a system

\[ (F_p = F_p^{(0)} \oplus \cdots \oplus F_p^{(m)}, \psi_p, \varphi_p = \sum_{i=0}^{m} \varphi_p^{(i)})_{p \in \mathbb{N}} \]

of \( m \)-decomposable c.p. approximations for \( A \) such that (22) holds and the maps

\[ \psi_p, \varphi_p^{(i)}, \varphi_p \psi_p \]

are all c.p. contractions.

For each \( p \in \mathbb{N} \), we define

\[ \hat{\psi}_p (\cdot) := \psi_p (1_A)^{-\frac{1}{2}} \psi_p (\cdot) \psi_p (1_A)^{-\frac{1}{2}}, \]

where the inverses are taken in the hereditary subalgebras

\[ \tilde{F}_p := \text{her}(\psi_p (1_A)) \subset F_p \]

and

\[ \hat{\varphi}_p (\cdot) := \varphi_p (\psi_p (1_A)^{\frac{1}{2}} \psi_p (1_A)^{\frac{1}{2}}), \]

then

\[ \hat{\varphi}_p \hat{\psi}_p = \varphi_p \psi_p; \]

moreover, the

\[ \hat{\psi}_p : A \to \tilde{F}_p \]

are unital c.p. and the

\[ \hat{\varphi}_p : \tilde{F}_p \to A \]

are c.p.c. maps.
From [19, Lemma 3.6], we see that for \( i \in \{0, \ldots, m\} \), \( p \in \mathbb{N} \) and any projection \( q_p \in \tilde{F}_p \) we have
\[
\|\hat{\psi}_p(q_p \hat{\psi}_p(a)) - \hat{\psi}_p(q_p)\hat{\psi}_p(a)\| \\
\leq 3 \cdot \max\{\|\hat{\psi}_p\hat{\psi}_p(a) - a\|, \|\hat{\psi}_p\hat{\psi}_p(a^2) - a^2\|\}^{1/2},
\]
from which follows that
\[
\|\varphi(i)\psi(i)(a) - \varphi(i)\psi(i)(1_A)\varphi_p\psi_p(a)\| \xrightarrow{p \to \infty} 0
\]
for any \( a \in A \), i.e., (24) holds. From this we obtain (23) by taking adjoints. □

**Remark 4.3** A celebrated result of U. Haagerup says that every quasitrace on an exact C*-algebra is a trace, see [13]. In [6], N. Brown and the author give a short proof of this fact in the case of C*-algebras with locally finite nuclear dimension.

### 4.4

In [43, Proposition 4.3] it was shown that if \( A \) is separable, simple, nonelementary and unital with \( \dim_{\text{nuc}} A \leq m \), then \( A_\omega \cap A' \) (where \( \omega \) is a free ultrafilter on \( \mathbb{N} \)) has Kirchberg covering number (in the sense of [17]) at most \((m + 1)^2\). If \( A \) is unital, then the proof shows that in fact \( A_\infty \cap A' \) has covering number at most \((m + 1)^2\). We note here a straightforward generalization of that result. The proof involves a standard diagonal sequence argument, which is spelled out explicitly since it will appear repeatedly in the sequel.

**Proposition** Let \( A \) be a separable, simple, nonelementary and unital C*-algebra with \( \dim_{\text{nuc}} A \leq m < \infty \); let \( X \subset A_\infty \) be a separable subspace.

Then, for any \( k \in \mathbb{N} \) there are c.p.c. order zero maps
\[
\psi^{(1)}, \ldots, \psi^{(m+1)^2} : M_k \oplus M_{k+1} \to A_\infty \cap A' \cap X'
\]
satisfying
\[
\sum_{j=1}^{(m+1)^2} \psi^{(j)}(1_k \oplus 1_{k+1}) \geq 1_{A_\infty}.
\]

**Proof** By (the proof of) [43, Proposition 4.3], there are c.p.c. order zero maps
\[
\varphi^{(1)}, \ldots, \varphi^{(m+1)^2} : M_k \oplus M_{k+1} \to A_\infty \cap A'
\]
satisfying
\[
\sum_{j=1}^{(m+1)^2} \varphi^{(j)}(1_k \oplus 1_{k+1}) \geq 1_{A_\infty}.
\] (28)

For each \( j \), let
\[
\tilde{\varphi}^{(j)} : M_k \oplus M_{k+1} \to \prod_{\mathbb{N}} A
\]
be a c.p.c. order zero lift of \( \varphi^{(j)} \) (cf. [19, Remark 2.4]) with c.p.c. order zero components
\[
\tilde{\varphi}_l^{(j)} : M_k \oplus M_{k+1} \to A, \ l \in \mathbb{N}.
\]
Let
\[
\{x_i \mid i \in \mathbb{N}\} \subset X
\]
be a countable dense subset; for each \( i \in \mathbb{N} \) let
\[
\tilde{x}_i = (\tilde{x}_{i,r})_{r \in \mathbb{N}} \in \prod_{\mathbb{N}} A
\]
be an isometric lift of \( x_i \).

Since \( \tilde{\varphi}^{(j)} \) lifts \( \varphi^{(j)} \), using (27) it is straightforward to find a strictly increasing sequence \( (l_r)_{r \in \mathbb{N}} \subset \mathbb{N} \) such that for each \( r \in \mathbb{N}, \ i \leq r \) and \( y \in M_k \oplus M_{k+1} \) we have
\[
\|[\tilde{\varphi}_r^{(j)}(y), \tilde{x}_{i,r}]\| \leq \frac{1}{r+1} \|y\|.
\] (29)

We obtain c.p.c. order zero maps
\[
\tilde{\psi}^{(j)} := (\tilde{\varphi}_r^{(j)})_{r \in \mathbb{N}} : M_k \oplus M_{k+1} \to \prod_{\mathbb{N}} A
\]
which induce c.p.c. order zero maps
\[
\psi^{(j)} : M_k \oplus M_{k+1} \to A_\infty, \ j = 1, \ldots, (m + 1)^2.
\]
The images of the \( \psi^{(j)} \) commute with \( X \) by (29), they commute with \( A \) by (27); (26) follows directly from (28).

4.5

It was shown in [35, Lemma 6.1] that decomposition rank at most \( m \) implies \( m \)-comparison (in the simple and unital case; it was observed in [23] that essentially the same proof holds also in the general situation). Let us recall a
result of L. Robert [25, Theorem 1], which establishes the respective state-
ment for nuclear dimension. Note that the theorem indeed applies since our
definition of $m$-comparison agrees with that of [25] in the simple situation,
 cf. 3.2(ii).

**Theorem (L. Robert)** If $A$ is a separable C*-algebra with $\dim_{\text{nuc}} A \leq m$, then $A$ has $m$-comparison.

4.6

The following proposition nicely illustrates a technique which will be used
frequently in the sequel, in particular in Sect. 5.

**Proposition** Given $m, k \in \mathbb{N}$ (with $k \geq 1$), there is $1 \leq L_{m,k} \in \mathbb{N}$ such that the following holds:

If $A$ is a separable, simple, nonelementary and unital C*-algebra with $\dim_{\text{nuc}} A \leq m$, and if $X \subset (A_\infty)_+$ is a separable subspace, then there are 
pairwise orthogonal positive contractions

$$d^{(1)}, \ldots, d^{(k)} \in A_\infty \cap A' \cap X'$$

such that

$$\tau(d^{(i)} b) \geq \frac{1}{L_{m,k}} \cdot \tau(b) \quad (30)$$

for all $b \in \mathcal{C}^*(A, X)_+ \subset A_\infty$ and $\tau \in T_\infty(A)$.

**Proof** For $k = 1$ we may take $L_{m,1} := 1$ and $d^{(1)} := 1_{A_\infty}$, so let us prove the statement for $k = 2$.

Take

$$L_{m,2} := 2(m+1)^2 + 1; \quad (31)$$

for notational convenience set

$$\bar{k} := 2(m+1)^2. \quad (32)$$

By Proposition 4.4 there are c.p.c. order zero maps

$$\psi^{(1)}, \ldots, \psi^{(m+1)^2} : M_{\bar{k}} \oplus M_{\bar{k}+1} \to A_\infty \cap A' \cap X'$$

satisfying

$$\sum_{j=1}^{(m+1)^2} \psi^{(j)}(1_{\bar{k}} \oplus 1_{\bar{k}+1}) \geq 1_{A_\infty}. \quad (30)$$
We then have

$$\tau \left( \sum_{j=1}^{(m+1)^2} \psi^{(j)}(e_{11} \oplus e_{11})b \right) \geq \frac{1}{k+1} \cdot \tau(b) \quad (33)$$

for all $\tau \in T_\infty(A)$ and $b \in C^*(A, X)_+$ (using the fact that $\psi^{(j)}(\cdot)b$ is a c.p.c. order zero map, and that the composition of an order zero map with a trace is again a trace by 2.8).

For $\eta > 0$, define

$$d^{(0)}_\eta := g_{\eta, 2\eta} \left( \sum_{j=1}^{(m+1)^2} \psi^{(j)}(e_{11} \oplus e_{11}) \right) \quad (34)$$

and

$$d^{(1)}_\eta := g_{0, \eta} \left( \sum_{j=1}^{(m+1)^2} \psi^{(j)}(e_{11} \oplus e_{11}) \right) \quad (35)$$

We have

$$d^{(0)}_\eta \perp d^{(1)}_\eta,$$

$$d^{(0)}_\eta, d^{(1)}_\eta \in A_\infty \cap A' \cap X'$$

and

$$\tau(d^{(0)}_\eta b) \geq \frac{1}{k+1} \cdot \tau(b) - \eta \quad (36)$$

for all $\tau \in T_\infty(A)$ and all $b \in C^*(A, X)_+$ of norm at most one. Moreover, for all $\tau \in T_\infty(A)$ and all $b \in C^*(A, X)_+$ of norm at most one we estimate

$$\tau((1_{A_\infty} - d^{(1)}_\eta)b) \geq \lim_{l \to \infty} \tau \left( \left( \sum_{j=1}^{(m+1)^2} \psi^{(j)}(e_{11} \oplus e_{11}) \right)^{1/l} b \right) \leq \sum_{j=1}^{(m+1)^2} \lim_{l \to \infty} \tau((\psi^{(j)}(e_{11} \oplus e_{11}))^{1/l} b)$$
\[
\sum_{j=1}^{(m+1)^2} \lim_{l \to \infty} \tau((\psi(j))^{1/l}(e_{11} \oplus e_{11})b) = \sum_{j=1}^{(m+1)^2} \lim_{l \to \infty} \frac{1}{k} \cdot \tau((\psi(j))^{1/l}(1_k \oplus 1_{k+1})b) \leq \frac{(m+1)^2}{2(m+1)^2} \cdot \tau(b) = \frac{1}{2} \cdot \tau(b),
\]
whence
\[
\tau(d_{\eta}^{(1)} b) \geq \left(1 - \frac{1}{2}\right) \cdot \tau(b) = \frac{1}{2} \cdot \tau(b).
\]
Here, for the second estimate we have used that the map
\[
a \mapsto \lim_{l \to \infty} \tau(a^{1/l} b)
\]
is a dimension function on
\[
\mathcal{C}^*(\psi(j)(e_{11} \oplus e_{11}) | j = 1, \ldots, (m+1)^2) \subset A_\infty \cap A' \cap X',
\]
and as such respects the order structure of the Cuntz semigroup, cf. [2]; for the third inequality we have used that
\[
\tau((\psi(j))^{1/l} (\cdot) b) : M_k \oplus M_{k+1} \to \mathbb{C}
\]
is a trace by 2.8 (cf. the argument for (33)). Taking \(\frac{1}{l+1}\) for \(\eta\), we then obtain sequences
\[
(d_{\frac{1}{l+1}}^{(0)})_{l \in \mathbb{N}}, (d_{\frac{1}{l+1}}^{(1)})_{l \in \mathbb{N}} \subset A_\infty \cap A' \cap X'
\]
of positive contractions satisfying
\[
d_{\frac{1}{l+1}}^{(0)} \perp d_{\frac{1}{l+1}}^{(1)}
\]
and (by (36) and (37))
\[
\tau(d_{\frac{1}{l+1}}^{(i)} b) \geq \frac{1}{k+1} \cdot \tau(b) - \frac{1}{l+1}, \quad i = 0, 1,
\]
for each \(l \in \mathbb{N}\), \(\tau \in T_\infty(A)\) and \(b \in \mathcal{C}^*(A, X)_+\) of norm at most one.
For each $l \in \mathbb{N}$ and $i = 0, 1$, let

$$(d_{l, r}^{(i)})_{r \in \mathbb{N}} \in \prod_{\mathbb{N}} A$$

be a sequence of positive contractions lifting $d_{l, r}^{(i)}$. Let

$$(G_l)_{l \in \mathbb{N}}$$

be a nested sequence of finite subsets in the unit ball of $(\prod_{\mathbb{N}} A)^+$ such that the image of $\bigcup_{l \in \mathbb{N}} G_l \subset \prod A$ in $A_\infty$ is a dense subset of the unit ball of $C^*(A, X)^+$.

We claim that for each $l \in \mathbb{N}$ there is $K_l \in \mathbb{N}$ such that

$$\tau(\bar{d}_{l, r}^{(i)} f_r) \geq \frac{1}{k + 1} \cdot \tau(f_r) - \frac{2}{l + 1}$$

for $i = 0, 1$, for each $\tau \in T(A)$, each $(f_r)_{r \in \mathbb{N}} \in G_l$ and for each $r \geq K_l$.

Indeed, if this was not true, then for some $l \in \mathbb{N}$ there were $i \in \{0, 1\}$, $(f_r)_{r \in \mathbb{N}} \in G_l$, an increasing sequence $(r_s)_{s \in \mathbb{N}} \subset \mathbb{N}$ and a sequence $(\tau_s)_{s \in \mathbb{N}} \subset T(A)$ with

$$\tau_s(\bar{d}_{l, r}^{(i)} f_{r_s}) < \frac{1}{k + 1} \cdot \tau_s(f_{r_s}) - \frac{2}{l + 1}.$$  

(42)

Let $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$ be a free ultrafilter, then

$$[(a_r)_{r \in \mathbb{N}}] \mapsto \lim_{\omega} \tau_s(a_{r_s})$$

is a trace in $T_\infty(A)$. Taking the limit along $\omega$ on both sides of (42), we obtain a contradiction to (40), hence the claim holds.

On increasing the $K_l$, if necessary, we may also assume that they form a strictly increasing sequence, that

$$\|\bar{d}_{l, r}^{(0)} \bar{d}_{l, r}^{(1)}\| < \frac{1}{l + 1}$$

and that

$$\|\bar{d}_{l, r}^{(i)} f_r\| < \frac{1}{l + 1}, \quad i = 0, 1,$$

(44)

for any $r \geq K_l$ and any $(f_r)_{r \in \mathbb{N}} \in G_l$ (using (39) and (38)).

We now define sequences

$$(\tilde{d}_r^{(i)})_{r \in \mathbb{N}} \in \prod_{\mathbb{N}} A, \quad i = 0, 1,$$
of positive contractions by
\[ \tilde{d}_r^{(i)} := \begin{cases} \tilde{d}_{l,K_l} & \text{if } K_l \leq r < K_{l+1} \\ 0 & \text{if } r < K_0. \end{cases} \] (45)

By combining (45) with (43), (44), (41) and the facts that the \( G_l \) are nested and the \( K_l \) are strictly increasing, it is straightforward to check the following properties:

(a) \( \| \tilde{d}_r^{(0)} \tilde{d}_r^{(1)} \| \xrightarrow{r \to \infty} 0 \)

(b) \( \| [\tilde{d}_r^{(i)}, f_r] \| \xrightarrow{r \to \infty} 0, \ i = 0, 1, \) for every \((f_r)_{r \in \mathbb{N}} \in \bigcup_{l \in \mathbb{N}} G_l\)

(c) for any \( \delta > 0 \) and any \((f_r)_{r \in \mathbb{N}} \in \bigcup_{l \in \mathbb{N}} G_l\) there is \( \tilde{r} \in \mathbb{N} \) such that

\[ \tau(\tilde{d}_r^{(i)} f_r) \geq \frac{1}{k+1} \cdot \tau(f_r) - \delta, \quad i = 0, 1, \]

for any \( r \geq \tilde{r} \) and any \( \tau \in T(A) \).

Then,
\[ d^{(i)} := [(\tilde{d}_r^{(i)})_{r \in \mathbb{N}}] \in A_\infty, \quad i = 0, 1, \]
will have the desired properties: They are positive contractions because all the \( \tilde{d}_r^{(i)} \) are; they are orthogonal because of (a) and they commute with \( C^*(A, X) \) because of (b). Finally, (c) shows that for any \( \tau \in T_\infty(A) \) and any \( b \in C^*(A, X)_+ \) we have
\[ \tau(\tilde{d}_r^{(i)} b) \geq \frac{1}{k+1} \cdot \tau(b) \overset{(31)}{=} \frac{1}{L_{m,2}} \cdot \tau(b), \quad i = 0, 1. \]

This proves the proposition for \( k = 2 \).

Let us now use induction to handle the cases when \( k = 2^l \) for \( 1 \leq l \in \mathbb{N} \). The anchor step \( l = 1 \) has already been taken care of, so let us assume that the statement of the proposition holds for \( k = 2^l \) for some \( l \geq 1 \), with constant \( L_{m,2^l} \) and positive elements
\[ \hat{d}^{(1)}, \ldots, \hat{d}^{(2^l)} \subset A_\infty \cap A' \cap X'. \]

Set
\[ L_{m,2^{l+1}} := L_{m,2} \cdot L_{m,2^l} \]
and apply the first part of the proof (with
\[ \tilde{X} := X \cup \{ \hat{d}^{(1)}, \ldots, \hat{d}^{(2^l)} \} \]
in place of $X$) to obtain positive elements

$$\tilde{d}^{(0)}, \tilde{d}^{(1)} \in A_\infty \cap A' \cap \tilde{X}'.$$ 

Set

$$d^{(j)} := \begin{cases} 
\tilde{d}^{(0)} \tilde{d}^{(j)} & \text{if } 1 \leq j \leq 2^l \\
\tilde{d}^{(1)} \tilde{d}^{(j-2^l)} & \text{if } 2^l < j \leq 2^{l+1};
\end{cases}$$

it is straightforward to check that these indeed satisfy the assertion of the proposition.

This settles the case $k = 2^l$ for $1 \leq l \in \mathbb{N}$. The case of general $k \in \mathbb{N}$ is an immediate consequence. \hfill \Box

**Proposition 4.7** Given $m \in \mathbb{N}$, there is $\tilde{m} \in \mathbb{N}$ such that the following holds:

If $A$ is a separable, simple, nonelementary and unital $C^*$-algebra with $\dim_{\text{nuc}} A \leq m$, then $A$ has tracial $\tilde{m}$-almost divisibility.

**Proof** Invoke Proposition 4.6 to define

$$\tilde{m} := L_{m,(m+1)^2} - 1. \quad (46)$$

Let $b \in A$ a positive contraction, $k \in \mathbb{N}$ and $\epsilon > 0$ be given. (Since any matrix algebra over $A$ will satisfy the hypotheses of the proposition just as $A$ does, to test tracial $\tilde{m}$-almost divisibility, it will suffice to consider a positive contraction $b$ in $A$, as opposed to $M_\infty(A)$.)

Choose $\bar{k} \in \mathbb{N}$ such that

$$\frac{1}{\bar{k} + 1} < \frac{\epsilon}{4} \quad (47)$$

and such that $k$ divides $\bar{k}$. Use Proposition 4.4 to find c.p.c. order zero maps

$$\varphi^{(1)}, \ldots, \varphi^{((m+1)^2)} : M_{\bar{k}} \oplus M_{\bar{k}+1} \to A_\infty \cap A'$$

such that

$$\sum_{r=1}^{(m+1)^2} \varphi^{(r)}(1_{\bar{k}} \oplus 1_{\bar{k}+1}) \geq 1_{A_\infty}. \quad (48)$$

Take pairwise orthogonal positive contractions

$$d^{(1)}, \ldots, d^{((m+1)^2)} \in A_\infty \cap A' \cap \left( \bigcup_{r=1}^{(m+1)^2} \varphi^{(r)}(M_{\bar{k}} \oplus M_{\bar{k}+1}) \right)'$$

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as in Proposition 4.6 (with \((m + 1)^2\) in place of \(k\) and \(\bigcup_{r=1}^{(m+1)^2} \varphi^{(r)}(M_{\tilde{k}} \oplus M_{\tilde{k}+1})\) in place of \(X\)).

We then obtain a c.p. map

\[
\bar{\varphi} : M_{\tilde{k}} \oplus M_{\tilde{k}+1} \to A_\infty \cap A'
\]

by setting

\[
\bar{\varphi}(x) := \sum_{r=1}^{(m+1)^2} \varphi^{(r)}(x)d^{(r)};
\]

since the \(d^{(j)}\) are pairwise orthogonal and commute with the images of the c.p.c. order zero maps \(\varphi^{(j)}\), the map \(\bar{\varphi}\) is itself c.p.c. order zero. Note that we have

\[
\tau(b\bar{\varphi}(e_{jj} \oplus e_{jj})) = \tau(b\bar{\varphi}(e_{11} \oplus e_{11}))
\]

\[
\geq \tau \left( \sum_{r=1}^{(m+1)^2} b\varphi^{(r)}(e_{11} \oplus e_{11})d^{(r)} \right)
\]

\[
\geq \frac{1}{L_{m,(m+1)^2}(\tilde{k}+1)} \cdot \tau \left( \sum_{r=1}^{(m+1)^2} b\varphi^{(r)}(1_{\tilde{k}} \oplus 1_{\tilde{k}+1}) \right)
\]

\[
\geq \frac{1}{L_{m,(m+1)^2}(\tilde{k}+1)} \cdot \tau (b)
\]

for \(j = 1, \ldots, \tilde{k}\) and for all \(\tau \in T_\infty(A)\) (for the first equality and for the second inequality we use Remark 2.8, cf. the argument for (33)). From this we obtain

\[
\tau \left( f_{\epsilon/4,\epsilon/2}(b) \bar{\varphi} \left( \sum_{l=1}^{\tilde{k}} e_{ll} \oplus e_{ll} \right) \right)
\]

\[
\geq \tau \left( b\bar{\varphi} \left( \sum_{l=1}^{\tilde{k}} e_{ll} \oplus e_{ll} \right) \right) - \frac{\epsilon}{4}
\]

\[
\geq \frac{\tilde{k}}{L_{m,(m+1)^2}(\tilde{k}+1)} \cdot \tau (b) - \frac{\epsilon}{4}
\]
\[
\frac{1}{L_{m,(m+1)^2}} \cdot \tau(b) - \frac{1}{k+1} - \frac{\epsilon}{4} \geq 1
\]

for all \( \tau \in T_\infty(A) \).

Let \( \tilde{\phi} = (\tilde{\phi}_l)_{l \in \mathbb{N}} : M_{\tilde{k}} \oplus M_{\tilde{k}+1} \to b\left(\prod_{\mathbb{N}} A\right)b \subset \prod_{\mathbb{N}} A \) (52)

be a c.p.c. order zero lift (cf. [19, Remark 2.4]) of the c.p.c. order zero map

\[ x \mapsto f_{\epsilon/4, \epsilon/2}(b)\tilde{\phi}(x). \]

We claim that there is \( \tilde{l} \in \mathbb{N} \) such that

\[
\tau\left(\tilde{\phi}_l\left(\sum_{j=1}^{\tilde{k}} e_{jj} \oplus e_{jj}\right)\right) \geq \frac{1}{L_{m,(m+1)^2}} \cdot \tau(b) - \epsilon
\] (53)

for all \( \tau \in T(A) \).

If not, then for every \( l \in \mathbb{N} \) there is some \( \tau_l \in T(A) \) such that

\[
\tau_l\left(\tilde{\phi}_l\left(\sum_{j=1}^{\tilde{k}} e_{jj} \oplus e_{jj}\right)\right) < \frac{1}{L_{m,(m+1)^2}} \cdot \tau_l(b) - \epsilon;
\]

let \( \omega \in \beta\mathbb{N} \setminus \mathbb{N} \) be a free ultrafilter, then

\[
\lim_{\omega} \tau_l\left(\tilde{\phi}_l\left(\sum_{j=1}^{\tilde{k}} e_{jj} \oplus e_{jj}\right)\right) \leq \frac{1}{L_{m,(m+1)^2}} \cdot \lim_{\omega} \tau_l(b) - \epsilon,
\]

a contradiction to (51), thus proving the claim.

Let

\[ \iota_1 : M_k \to M_{\tilde{k}} \]

be a unital embedding (\( k \) divides \( \tilde{k} \)) and let

\[ \iota_2 : M_{\tilde{k}} \to M_{\tilde{k}+1} \]

be the upper left corner embedding; set

\[ \iota := (\text{id}_{M_{\tilde{k}}} \oplus \iota_2) \circ \iota_1 : M_k \to M_{\tilde{k}} \oplus M_{\tilde{k}+1} \]
and

$$\psi := \tilde{\varphi} \circ \iota,$$

(54)

then

$$\psi : M_k \to \text{her}(b) \subseteq A$$

is c.p.c. order zero (cf. (52)) and

$$\tau(\psi(1_k)) \geq \frac{1}{L_m, (m+1)^2} \cdot \tau(b) - \epsilon = \frac{1}{\tilde{m} + 1} \cdot \tau(b) - \epsilon$$

for all $$\tau \in T(A).$$ \qed

Proposition 4.8  Given $$m \in \mathbb{N},$$ there is $$\tilde{m} \in \mathbb{N}$$ such that the following holds:

If $$A$$ is a separable, simple, unital $$\text{C}^*$$-algebra with $$\text{dim}_{\text{nuc}} A \leq m,$$ then $$A$$ has strong tracial $$\tilde{m}$$-comparison.

Proof  Take

$$\tilde{m} := L_m, (m+1)^2 \cdot (m + 2) - 1,$$

(55)

where $$L_m, (m+1)^2$$ comes from Proposition 4.6.

Since any matrix algebra over $$A$$ will satisfy the hypotheses of the proposition just as $$A$$ does, to test strong tracial $$\tilde{m}$$-comparison it will suffice to consider positive contractions $$a$$ and $$b$$ in $$A$$ (as opposed to $$M_\infty(A)).$$ In view of Proposition 3.4 it will suffice to show that, if

$$d_\tau(a) < \frac{1}{\tilde{m} + 1} \cdot \tau(b) - \eta$$

(56)

for some $$\eta > 0$$ and for all $$\tau \in T(A),$$ then

$$f_{\epsilon, 2\epsilon}(a) \preceq b$$

for any given $$\epsilon > 0.$$ (We may use traces as opposed to quasitraces by Remark 4.3.)

So, let $$a, b, \eta$$ and $$\epsilon$$ as above be given. We then have

$$\tau(g_{\epsilon/2, \epsilon}(a)) \leq d_\tau(a) < \frac{1}{\tilde{m} + 1} \cdot \tau(b) - \eta \leq \frac{1}{\tilde{m} + 1} \cdot \tau(g_{\eta/2, \eta}(b)) - \frac{\eta}{2}$$

(57)

for all $$\tau \in T(A).$$ Use Proposition 4.4 to find c.p.c. order zero maps

$$\varphi^{(1)}, \ldots, \varphi^{((m+1)^2)} : M_{m+1} \oplus M_{m+2} \to A_\infty \cap A'$$

(58)
such that
\[(m+1)^2 \sum_{j=1}^{(m+1)^2} \varphi(j)(1_{m+1} \oplus 1_{m+2}) \geq 1_{A_\infty} \cdot (59)\]

Take pairwise orthogonal positive contractions
\[d^{(1)}, \ldots, d^{(m+1)^2} \in A_\infty \cap A' \cap \left( \bigcup_{j=1}^{(m+1)^2} \varphi(j)(M_{m+1} \oplus M_{m+2}) \right)' \cdot (60)\]
as in Proposition 4.6 (with \((m + 1)^2\) in place of \(k\)). We then have
\[\tau(c\varphi(j)(e_{11} \oplus e_{11})d^{(j)}) \geq \frac{1}{L_{m,(m+1)^2}} \cdot \tau(c\varphi(j)(e_{11} \oplus e_{11})) \cdot (61)\]
for all \(j = 1, \ldots, (m + 1)^2, c \in A_+\) and \(\tau \in T_\infty(A)\). Let
\[\bar{\varphi} : M_{m+1} \oplus M_{m+2} \to bA_\infty b\]
be the c.p. map given by
\[\bar{\varphi}(\cdot) := \sum_{j=1}^{(m+1)^2} g_{\eta/2,\eta}(b)\varphi(j)(\cdot)d^{(j)}; \cdot (62)\]
using (58) and (60) and mutual orthogonality of the \(d^{(j)}\) one checks that \(\bar{\varphi}\) in fact is a c.p.c. order zero map. We compute
\[\tau(\bar{\varphi}(e_{11} \oplus e_{11})) \cdot (62)\]
\[\equiv \tau \left( \sum_{j=1}^{(m+1)^2} g_{\eta/2,\eta}(b)\varphi(j)(e_{11} \oplus e_{11})d^{(j)} \right) \cdot (62)\]
\[\geq \frac{1}{L_{m,(m+1)^2}} \cdot \tau \left( \sum_{j=1}^{(m+1)^2} g_{\eta/2,\eta}(b)\varphi(j)(e_{11} \oplus e_{11}) \right) \cdot (61)\]
\[\geq \frac{1}{L_{m,(m+1)^2} \cdot (m+2)} \cdot \tau \left( \sum_{j=1}^{(m+1)^2} g_{\eta/2,\eta}(b)\varphi(j)(1_{m+1} \oplus 1_{m+2}) \right) \cdot (59)\]
\[\geq \frac{1}{L_{m,(m+1)^2} \cdot (m+2)} \cdot \tau(g_{\eta/2,\eta}(b)) \cdot \tau(c\varphi(j)(e_{11} \oplus e_{11})) \cdot \tau(c\varphi(j)(e_{11} \oplus e_{11})) \cdot (61)\]
for all $\tau \in T_\infty(A)$. For the second inequality we have once more used (2.8) as in the argument for (33); for the last inequality we restrict $\tau$ to $A$, so that (57) indeed applies.

Take a c.p.c. order zero lift (cf. [19, Remark 2.4])

$$\Phi = (\Phi_l)_{l \in \mathbb{N}} : M_{m+1} \oplus M_{m+2} \to b(\prod_{\mathbb{N}} A) b \subset \prod_{\mathbb{N}} A$$

(64)

of the c.p.c. order zero map $\tilde{\varphi}$.

We claim that there is $\bar{l} \in \mathbb{N}$ such that

$$\tau(\Phi_{\bar{l}}(e_{11} \oplus e_{11})) \geq \tau(g_{\epsilon/2,\epsilon}(a)) + \frac{\eta}{4}$$

(65)

for all $\tau \in T(A)$. If not, for every $l \in \mathbb{N}$ there is $\tau_l \in T(A)$ such that

$$\tau_l(\Phi_l(e_{11} \oplus e_{11})) < \tau_l(g_{\epsilon/2,\epsilon}(a)) + \frac{\eta}{4};$$

for a free ultrafilter $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$ we obtain

$$\lim_{\omega} \tau_l(\Phi_l(e_{11} \oplus e_{11})) \leq \lim_{\omega} \tau_l(g_{\epsilon/2,\epsilon}(a)) + \frac{\eta}{4},$$

a contradiction to (63).

We obtain

$$d_\tau(\Phi_l(e_{11} \oplus e_{11})) \geq \tau(\Phi_l(e_{11} \oplus e_{11})) \geq \tau(g_{\epsilon/2,\epsilon}(a)) + \frac{\eta}{4} > d_\tau(f_{\epsilon,2\epsilon}(a))$$

for all $\tau \in T(A)$. Now by $m$-comparison (see Theorem 4.5) and since $\Phi_l$ is c.p.c. of order zero, we have

$$f_{\epsilon,2\epsilon}(a) \precsim \sum_{j=1}^{m+1} \Phi_l(e_{jj} \oplus e_{jj}) \precsim b,$$

(64)

as desired. \qed

4.9

We note the following direct consequence of Propositions 4.7 and 4.8 explicitly.
Corollary If $A$ is a separable, simple, nonelementary and unital C*-algebra with finite nuclear dimension, then $A$ is tracially $(m, \bar{m})$-pure for suitable $m, \bar{m} \in \mathbb{N}$.

5 Tracial matrix cone absorption

We now set out to prove a refined version of [40, Lemma 3.10], Lemma 5.11 below. The strategy is to isolate some of the abstract machinery in [40, Sect. 3] so that we do not have to ask for finite decomposition rank (or finite nuclear dimension) beforehand.

Proposition 5.1 Let $A$ be a separable, simple, unital C*-algebra with tracial $m$-almost divisibility, let $0 \neq k \in \mathbb{N}$ and $d \in A_\infty$ be a positive contraction.

Then, for any $\eta > 0$ there is a c.p.c. order zero map

$$\varphi : M_k \to dA_\infty d$$

such that

$$\tau(\varphi(1_k)) \geq \frac{1}{m+1} \cdot \tau(d) - \eta$$

for all $\tau \in T_\infty(A)$.

Proof Let

$$\tilde{d} = (\tilde{d}_n)_{n \in \mathbb{N}} \in \prod_{\mathbb{N}} A$$

be a positive contraction lifting $d$. Set

$$\tilde{d} := g_{\eta/2, \eta}(\tilde{d}) = (g_{\eta/2, \eta}(\tilde{d}_n))_{n \in \mathbb{N}}.$$

For each $n \in \mathbb{N}$, by tracial $m$-almost divisibility there is a c.p.c. order zero map

$$\varphi_n : M_k \to \text{her}(g_{\eta/2, \eta}(\tilde{d}_n)) \subset A \quad (66)$$

such that

$$\tau(\varphi_n(1_k)) \geq \frac{1}{m+1} \cdot \tau(g_{\eta/2, \eta}(\tilde{d}_n)) - \frac{\eta}{2}$$

$$\geq \frac{1}{m+1} \cdot \tau(\tilde{d}_n) - \eta \quad (67)$$

for all $\tau \in T(A)$. Let

$$\tilde{\varphi} : M_k \to \prod_{\mathbb{N}} A$$
and

$$\varphi : M_k \to A_\infty$$

be the induced c.p.c. maps; it is clear (from (66)) that

$$g_{0, \eta/2}(\bar{d})\bar{\varphi}(1_k) = \bar{\varphi}(1_k),$$

whence

$$g_{0, \eta/2}(d)\varphi(1_k) = \varphi(1_k)$$

and

$$\varphi(M_k) \subset \overline{dA_\infty d} \subset A_\infty.$$ Moreover, by (67) we have

$$\tau(\varphi(1_k)) \geq \frac{1}{m+1} \cdot \tau(d) - \eta$$

for all $\tau \in T_\infty(A)$. \qed

**Proposition 5.2** Let $A$ be a separable, simple, unital C*-algebra with tracial $m$-almost divisibility, let $0 \neq k \in \mathbb{N}$ and $d \in A_\infty$ be a positive contraction.

Then, there are c.p.c. order zero maps

$$\psi^{(0)}, \psi^{(1)} : M_k \to A_\infty \cap \{d\}'$$

satisfying

$$\tau((\psi^{(0)}(1_k) + \psi^{(1)}(1_k)) f(d)) \geq \frac{1}{m+1} \cdot \tau(f(d))$$

for all $\tau \in T_\infty(A)$ and all $f \in \mathcal{C}_0((0, 1])_\ast$.

**Proof** Let

$$\tilde{d} = (\tilde{d}_q)_{q \in \mathbb{N}} \in \prod_{\mathbb{N}} A$$

be a positive contraction lifting $d$. Let $(\mathcal{G}_q)_{q \in \mathbb{N}}$ be a nested sequence of finite subsets of the unit ball of $\mathcal{C}_0((0, 1])_\ast$ with dense union. For each $q \in \mathbb{N}$, it is straightforward to choose $0 \neq L_q \in \mathbb{N}$ such that the sawtooth functions
$f^{(i)}_{q,j} \in \mathcal{C}_0((0, 1])_+$, given by

$$
f^{(1)}_{q,j}(t) = \begin{cases} 
0 & \text{if } 0 \leq t \leq \frac{j-1}{L_q} \\
1 & \text{if } t = \frac{j-1/2}{L_q} \\
0 & \text{if } \frac{j}{L_q} \leq t \leq 1 \\
\text{linear} & \text{else}
\end{cases}
$$

and

$$
f^{(0)}_{q,j}(t) = \begin{cases} 
0 & \text{if } 0 \leq t \leq \frac{1}{2L_q} \\
f^{(1)}_{q,j}(t - \frac{1}{2L_q}) & \text{else}
\end{cases}
$$

for $j = 1, \ldots, L_q$, satisfy

$$
\| f \cdot g(f^{(i)}_{q,j}) - f(j/L_q) \cdot g(f^{(i)}_{q,j}) \| \leq \frac{1}{q+1} \tag{68}
$$

for $j = 1, \ldots, L_q$, $i = 0, 1$, $f \in \mathcal{G}_q$ and $g$ in the unit ball of $\mathcal{C}_0((0, 1])_+$, and

$$
\left\| \sum_{i,j} f(j/L_q) \cdot f^{(i)}_{q,j} - f \right\| \leq \frac{1}{q+1} \tag{69}
$$

for $f \in \mathcal{G}_q$.

By tracial $m$-almost divisibility, for each $q \in \mathbb{N}$, $i \in \{0, 1\}$, $j \in \{1, \ldots, L_q\}$ there exists a c.p.c. order zero map

$$
\psi^{(i)}_{q,j} : M_k \to \text{her}(f^{(i)}_{q,j}(\bar{d}_q)) \subset A \tag{70}
$$

such that

$$
\tau(\psi^{(i)}_{q,j}(1_k)) \geq \frac{1}{m+1} \cdot \tau(f^{(i)}_{q,j}(\bar{d}_q)) - \frac{1}{L_q \cdot (q+1)} \tag{71}
$$

for all $\tau \in T(A)$. Since for fixed $q \in \mathbb{N}$ and $i \in \{0, 1\}$ the functions $f^{(i)}_{q,j}$, $j = 1, \ldots, L_q$, are pairwise orthogonal, the maps

$$
\psi^{(i)}_q := \bigoplus_{j=1}^{L_q} \psi^{(i)}_{q,j} : M_k \to A
$$

are c.p.c. order zero (by (70)).
Using (68) and (70) it is straightforward to estimate

\[
\left\| f(\tilde{a}_q)\psi_q^{(i)}(x) - \sum_{j=1}^{L_q} f(j/L_q) \cdot \psi_q^{(i)}(x) \right\| \leq \frac{1}{q + 1} \tag{72}
\]

for \(q \in \mathbb{N}, i \in \{0, 1\}, x \) in the unit ball of \((M_k)_+\) and \(f \in \mathcal{G}_q\), whence

\[
\| [f(\tilde{a}_q), \psi_q^{(i)}(x)] \| \leq \frac{2}{q + 1}. \tag{73}
\]

Moreover, we have

\[
\tau\left( \sum_{i=0,1} f(\tilde{a}_q)\psi_q^{(i)}(1_k) \right) \geq \frac{1}{m + 1} \cdot \tau\left( \sum_{i=0,1; j=1}^{L_q} f(j/L_q) \cdot \psi_q^{(i)}(1_k) \right) - \frac{2}{q + 1} \tag{72}
\]

\[
\geq \frac{1}{m + 1} \cdot \tau\left( \sum_{i=0,1; j=1}^{L_q} f(j/L_q) \cdot f_q^{(i)}(\tilde{d}_q) \right) - \frac{4}{q + 1} \tag{71}
\]

\[
\geq \frac{1}{m + 1} \cdot \tau(f(\tilde{d}_q)) - \frac{5}{q + 1} \tag{69}
\]

for all \(q \in \mathbb{N}, f \in \mathcal{G}_q\) and \(\tau \in T(A)\).

For \(i = 0, 1\) let

\[
\psi^{(i)} : M_k \to A_\infty
\]

be the maps induced by the \(\psi_q^{(i)}\); it follows directly from (73) and (74) that these have the desired properties. \(\square\)

5.3

The following will be a crucial ingredient for Propositions 5.7 and 5.8, which in turn are variations of Proposition 4.6.

**Proposition** Let \(A\) be a separable, simple, unital C*-algebra with tracial \(m\)-almost divisibility, and let \(d \in A_\infty\) be a positive contraction.

Then, there are orthogonal positive contractions

\[
d_0, d_1 \in A_\infty \cap \{d\}'
\]
satisfying
\[ \tau(d_i f(d)) \geq \frac{1}{4(m+1)} \cdot \tau(f(d)) \] (75)
for all \( \tau \in T_\infty(A) \), all \( f \in C_0((0,1])_+ \) and \( i = 0, 1 \).

Proof By Proposition 5.2 there are c.p.c. order zero maps
\[ \psi_1, \psi_2 : M_4 \to A_\infty \cap \{d\}' \]
satisfying
\[ \tau((\psi_1(e_{11}) + \psi_2(e_{11})) f(d)) \geq \frac{1}{4(m+1)} \cdot \tau(f(d)) \]
for all \( \tau \in T_\infty(A) \) and \( f \in C_0((0,1])_+ \) (note that \( \tau((\psi_1 + \psi_2)(\cdot)f(d)) \) is a trace on \( M_4 \)).

For \( \eta > 0 \), define positive contractions
\[ d^{(0)}_\eta := g_{\eta,2\eta}(\psi_1(e_{11}) + \psi_2(e_{11})) \]
and
\[ d^{(1)}_\eta := 1_{A_\infty} - g_{0,\eta}(\psi_1(e_{11}) + \psi_2(e_{11})). \] (76)

We have
\[ d^{(0)}_\eta \perp d^{(1)}_\eta, \]
\[ d^{(0)}_\eta, d^{(1)}_\eta \in A_\infty \cap \{d\}' \]
and
\[ \tau(d^{(0)}_\eta f(d)) \geq \frac{1}{4(m+1)} \cdot \tau(f(d)) - \eta \] (77)
for all \( \tau \in T_\infty(A) \) and all \( f \in C_0((0,1])_+ \) of norm at most one. Moreover, for all \( \tau \in T_\infty(A) \) and all \( f \in C_0((0,1])_+ \) we estimate
\[
\tau((1_{A_\infty} - d^{(1)}_\eta) f(d)) \overset{(76)}{=} \tau(g_{0,\eta}(\psi_1(e_{11}) + \psi_2(e_{11})) f(d)) \\
\leq \lim_{l \to \infty} \tau((\psi_1(e_{11}) + \psi_2(e_{11}))^{1/l} f(d)) \\
\leq \lim_{l \to \infty} \tau((\psi_1(e_{11}))^{1/l} f(d)) + \lim_{l \to \infty} \tau((\psi_2(e_{11}))^{1/l} f(d)) \\
\overset{\text{2.2}}{=} \lim_{l \to \infty} \tau(\psi_1^{1/l}(e_{11}) f(d)) + \lim_{l \to \infty} \tau(\psi_2^{1/l}(e_{11}) f(d))
\]
\[\begin{align*}
&= \lim_{l \to \infty} \frac{1}{4} \cdot \tau(\psi_1^{1/l}(1_f) f(d)) \\
&\quad + \lim_{l \to \infty} \frac{1}{4} \cdot \tau(\psi_2^{1/l}(1_f) f(d)) \\
&\leq \frac{1}{4} \cdot \tau(f(d)) + \frac{1}{4} \cdot \tau(f(d)) \\
&= \frac{1}{2} \cdot \tau(f(d)),
\end{align*}\]
whence
\[\tau(d_{\eta}^{(1)} f(d)) \geq \frac{1}{2} \cdot \tau(f(d)). \quad (78)\]

Here, for the second estimate we have used that the map \(a \mapsto \lim_{l \to \infty} \tau(a^{1/l} \cdot f(d))\) is a dimension function on \(C^*(\psi_1(e_{11}), \psi_2(e_{11})) \subset A_\infty \cap \{d\}'\), and as such respects the order structure of the Cuntz semigroup; for the third equality we have used that \(\psi_i^{1/l}(\cdot) f(d) : M_k \to A_\infty\) is a c.p.c. order zero map, and that the composition of an order zero map with a trace is again a trace (cf. 2.8).

Taking \(\frac{1}{l+1}\) for \(\eta\), we then obtain sequences
\[(d^{(0)}_{l,1})_{l \in \mathbb{N}}, (d^{(1)}_{l,1})_{l \in \mathbb{N}} \subset A_\infty \cap \{d\}'\]
(79)
of positive contractions satisfying
\[d^{(0)}_{l,1} \perp d^{(1)}_{l,1} \quad (80)\]
and
\[\tau(d^{(i)}_{l,1} f(d)) \geq \frac{1}{4(m+1)} \cdot \tau(f(d)) - \frac{1}{l+1}, \quad i = 0, 1, \quad (81)\]
for each \(l \in \mathbb{N}, \tau \in T_\infty(A)\) and \(f \in C_0((0, 1])^+\) of norm at most one.

For each \(l \in \mathbb{N}\) and \(i = 0, 1\), let
\[(d^{(i)}_{l,r})_{r \in \mathbb{N}} \in \prod_{\mathbb{N}} A\]
be a sequence of positive contractions lifting \(d^{(i)}_{l,1}\), and let
\[(d_r)_{r \in \mathbb{N}} \in \prod_{\mathbb{N}} A\]
be a sequence of positive contractions lifting $d$. Let $(G_l)_{l \in \mathbb{N}}$ be a nested sequence of finite subsets in the unit ball of $C_0((0, 1])_+$ with dense union.

We claim that for each $l \in \mathbb{N}$ there is $K_l \in \mathbb{N}$ such that
\[
\tau(\bar{d}^{(i)}_{l, r} f(dr)) \geq \frac{1}{4(m+1)} \cdot \tau(f(dr)) - \frac{2}{l+1}
\] (82)
for $i = 0, 1$, for each $\tau \in T(A)$, each $f \in G_l$ and for each $r \geq K_l$.

Indeed, if this was not true, then for some $l \in \mathbb{N}$ there were $i \in \{0, 1\}$, $f \in G_l$, an increasing sequence $(r_s)_{s \in \mathbb{N}} \subset \mathbb{N}$ and a sequence $(\tau_s)_{s \in \mathbb{N}} \subset T(A)$ with
\[
\tau_s(\bar{d}^{(i)}_{l, r_s} f(dr_s)) < \frac{1}{4(m+1)} \cdot \tau_s(f(dr_s)) - \frac{2}{l+1}.
\] (83)
If $\omega \in \beta\mathbb{N}\setminus\mathbb{N}$ is a free ultrafilter, then
\[
[(a_r)_{r \in \mathbb{N}}] \mapsto \lim_{\omega} \tau_s(a_{r_s})
\]
is a trace in $T_\infty(A)$. Taking the limit along $\omega$ on both sides of (83), this yields a contradiction to (81), hence the claim holds.

On increasing the $K_l$, if necessary, by (79) and (80) we may also assume that they form a strictly increasing sequence, that
\[
\|\bar{d}^{(0)}_{l, r} \bar{d}^{(1)}_{l, r}\| < \frac{1}{l+1}
\] (84)
and that
\[
\|\bar{d}^{(i)}_{l, r}, d_r\| < \frac{1}{l+1}, \quad i = 0, 1,
\] (85)
for any $r \geq K_l$.

We now define sequences
\[
(\tilde{d}^{(i)}_r)_{r \in \mathbb{N}} \in \prod_{\mathbb{N}} A, \quad i = 0, 1,
\]
of positive contractions by
\[
\tilde{d}^{(i)}_r := \begin{cases} 
  d^{(i)}_{l, r} & \text{if } K_l \leq r < K_{l+1} \\
  0 & \text{if } r < K_0.
\end{cases}
\]
The following properties are straightforward to check, using (84), (85) and (82):
(a) \( \| d_i^{(0)} - d_i^{(1)} \| \rightarrow 0 \)
(b) \( \| [\tilde{d}_r^{(i)}, d_r] \| \rightarrow 0, \ i = 0, 1 \)
(c) for any \( \delta > 0 \) and any \( f \in \bigcup_{l \in \mathbb{N}} G_l \) there is \( \tilde{r} \in \mathbb{N} \) such that

\[
\tau(\tilde{d}_r^{(i)} f( d_r )) \geq \frac{1}{4(m+1)} \cdot \tau(f( d_r )) - \delta, \quad i = 0, 1,
\]

for any \( r \geq \tilde{r} \) and any \( \tau \in T(A) \).

Then,

\[
d_i := [(\tilde{d}_r^{(i)})_{r \in \mathbb{N}}] \in A_\infty, \quad i = 0, 1,
\]

will have the desired properties: They are positive contractions because all the \( \tilde{d}_r^{(i)} \) are; they are orthogonal because of (a) and they commute with \( d \) because of (b). Finally, (c) shows that for any \( \tau \in T_\infty(A) \) and any \( f \in C_0((0, 1]) \) we have

\[
\tau(d_i f( d )) \geq \frac{1}{4(m+1)} \cdot \tau(f( d )), \quad i = 0, 1. \quad \square
\]

5.4

The following will play a similar role for the proof of Lemma 5.11 as [40, Proposition 3.7] does for [40, Lemma 3.10]

**Proposition** Let \( A \) be a separable, simple, unital \( C^* \)-algebra with tracial \( m \)-almost divisibility, let \( 0 \neq k \in \mathbb{N} \) and \( d \in A_\infty \) be a positive contraction.

Then, there is a c.p.c. order zero map

\[
\Phi : M_k \rightarrow A_\infty \cap \{d\}'
\]

satisfying

\[
\tau(\Phi(1_k)d) \geq \frac{1}{4(m+1)^2} \cdot \tau(d) \quad (86)
\]

for all \( \tau \in T_\infty(A) \).

**Proof** For a given \( \eta > 0 \), choose \( L \in \mathbb{N} \) with

\[
1/L < \eta \quad (87)
\]

and a standard partition of unity of the interval \( (0, 1] \)

\[
h_1^{(i)}, \ldots, h_L^{(i)} \in C_0((0, 1]), \quad i = 0, 1, \quad (88)
\]

as in 2.5.
Choose orthogonal positive contractions
\[ d_0, d_1 \in A_{\infty} \cap \{d\}' \]  
(89)

as in Proposition 5.3. By Proposition 5.1, there are c.p.c. order zero maps
\[ \varphi_{\eta,l}^{(i)} : M_k \to d_i h_l^{(i)}(d) A_{\infty} d_i h_l^{(i)}(d) \]  
(90)

for \( i = 0, 1 \) and \( l = 1, \ldots, L \) satisfying
\[ \tau(\varphi_{\eta,l}^{(i)}(1_k)) \geq \frac{1}{m + 1} \cdot \tau(d_i h_l^{(i)}(d)) - \frac{\eta}{L} \]  
(91)

for all \( \tau \in T_{\infty}(A) \).

The elements \( d_i h_l^{(i)}(d) \) are pairwise orthogonal (using (89), and since \( d_0 \perp d_1 \), and since the \( h_l^{(i)} \) are pairwise orthogonal for fixed \( i \)), and so the images of the \( \varphi_{\eta,l}^{(i)} \) are pairwise orthogonal by (90), whence the \( \varphi_{\eta,l}^{(i)} \) add up to a c.p.c. order zero map
\[ \varphi_{\eta} := \bigoplus_{i=0,1; l=1,\ldots,L} \varphi_{\eta,l}^{(i)} : M_k \to A_{\infty}. \]  
(92)

One checks that
\[ \left\| \sum_{i,l} f(h_l^{(i)}(d)) d - \sum_{i,l} \frac{l}{L} \cdot f(h_l^{(i)}(d)) \right\| \leq \frac{1}{L} \]  
(87)

for any positive \( f \in C((0, 1]) \) of norm at most one, from which follows that
\[ \left\| \bigoplus_{i,l} \varphi_{\eta,l}^{(i)}(x) d - \bigoplus_{i,l} \frac{l}{L} \cdot \varphi_{\eta,l}^{(i)}(x) \right\| \leq \eta \|x\| \]  
(94)

for all \( x \in M_k \). We obtain
\[ \|\varphi_{\eta}(x), d\| \leq 2\eta \|x\| \]  
(95)

for all \( x \in M_k \).

Moreover, we have
\[ \tau(\varphi_{\eta}(1_k)d) \overset{(92)}{=} \tau\left(\sum_{i,l} \varphi_{\eta,l}^{(i)}(1_k)d\right) \overset{(94)}{\geq} \sum_{i=0,1} \tau\left(\sum_{l=1,\ldots,L} \frac{l}{L} \cdot \varphi_{\eta,l}^{(i)}(1_k)\right) - \eta \]
\[
\begin{align*}
(91) & \quad \sum_{i=0,1} \frac{1}{m+1} \cdot \tau \left( \sum_{l=1,\ldots,L} \frac{l}{L} \cdot d_i h_i^{(i)}(d) \right) - 3\eta \\
(75) & \quad \sum_{i=0,1} \frac{1}{4(m+1)^2} \cdot \tau \left( \sum_{l=1,\ldots,L} \frac{l}{L} \cdot h_i^{(i)}(d) \right) - 3\eta \\
(93),(88) & \quad \geq \frac{1}{4(m+1)^2} \cdot \tau(d) - 4\eta
\end{align*}
\]

for any \( \tau \in T_\infty(A) \).

For \( l \in \mathbb{N} \), we may now take \( \frac{1}{l+1} \) in place of \( \eta \), and apply the preceding construction to obtain c.p.c. order zero maps

\[
\varphi_{\frac{1}{l+1}} : M_k \rightarrow A_\infty
\]
such that

\[
\tau(\varphi_{\frac{1}{l+1}}(1_k)d) \geq \frac{1}{4(m+1)^2} \cdot \tau(d) - \frac{4}{l+1}
\]

for all \( \tau \in T_\infty(A) \) and

\[
\|\varphi_{\frac{1}{l+1}}(x), d\| \leq \frac{2}{l+1} \|x\|
\]

for all \( x \in M_k \).

Let

\[
(d_r)_{r \in \mathbb{N}} \in \prod_{\mathbb{N}} A
\]

be a sequence of positive contractions lifting \( d \), and let

\[
(\tilde{\varphi}_{l,r})_{r \in \mathbb{N}} : M_k \rightarrow \prod_{\mathbb{N}} A
\]

be a c.p.c. order zero map lifting \( \varphi_{\frac{1}{l+1}} \).

We claim that, for any \( l \in \mathbb{N} \), there is \( K_l \in \mathbb{N} \) such that

\[
\tau(\tilde{\varphi}_{l,r}(1_k)d_r) \geq \frac{1}{4(m+1)^2} \cdot \tau(d_r) - \frac{5}{l+1}
\]

for each \( \tau \in T(A) \) and each \( r \geq K_l \).

If not, for some \( l \in \mathbb{N} \) there were an increasing sequence \( (r_s)_{s \in \mathbb{N}} \subset \mathbb{N} \) and a sequence \( (\tau_s)_{s \in \mathbb{N}} \subset T(A) \) satisfying

\[
\tau_s(\tilde{\varphi}_{l,r_s}(1_k)d_{r_s}) < \frac{1}{4(m+1)^2} \cdot \tau(d_{r_s}) - \frac{5}{l+1},
\]
a contradiction to (97), hence proving the claim (cf. the argument for (82)).

Upon increasing the $K_l$, if necessary, we may also assume that they form a strictly increasing sequence and that

$$\|[[\bar{\phi}_{l,r}(x),d_r]]\|_{l+1} \leq \frac{3}{l+1} \|x\|$$

(98)

(100)

for all $x \in M_k$ and $r \geq K_l$.

We now define a sequence

$$(\tilde{\phi}_r : M_k \to A)_{r \in \mathbb{N}}$$

of c.p.c. order zero maps by

$$\tilde{\phi}_r := \begin{cases} \bar{\phi}_{l,r} & \text{if } K_l \leq r < K_{l+1} \\ 0 & \text{if } r < K_0. \end{cases}$$

From (100) and (99) we obtain

(a) $\|[[\tilde{\phi}_r(x),d_r]]\|_{r \to \infty} \to 0$ for all $x \in M_k$

(b) for any $\delta > 0$ there is $\tilde{r} \in \mathbb{N}$ such that

$$\tau(\tilde{\phi}_r(1_k)d_r) \geq \frac{1}{4(m+1)^2} \cdot \tau(d_r) - \delta$$

for any $r \geq \tilde{r}$ and $\tau \in T(A)$.

Define

$$\Phi : M_k \to A_\infty$$

by

$$\Phi(x) := [[(\tilde{\phi}_r(x))_{r \in \mathbb{N}}]],$$

then $\Phi$ is a c.p.c. order zero map (since all the $\tilde{\phi}_r$ are) satisfying

$$\|[[\Phi(M_k),d]]\| = 0$$

(by a above) and

$$\tau(\Phi(1_k)d) \geq \frac{1}{4(m+1)^2} \cdot \tau(d)$$

for any $\tau \in T_\infty(A)$ (by b above).
The next proposition generalizes [40, 3.8].

**Proposition** Let \( A \) be a separable, simple, unital \( C^\ast \)-algebra with tracial \( m \)-almost divisibility, and let \( 0 \neq k \in \mathbb{N} \). Suppose further that \( F \) is a finite-dimensional \( C^\ast \)-algebra and let
\[
\varphi : F \to A_\infty
\]
be a c.p.c. order zero map.

Then, there is a c.p.c. order zero map
\[
\Phi : M_k \to A_\infty \cap \varphi(F)'
\]
satisfying
\[
\tau(\Phi(1_k)\varphi(x)) \geq \frac{1}{4(m+1)^2} \cdot \tau(\varphi(x))
\]
(101)
for all \( x \in F_+ \) and \( \tau \in T_\infty(A) \).

**Proof** Write
\[
F = M_{r_1} \oplus \cdots \oplus M_{r_s}
\]
and let
\[
\varphi^{(i)} : M_{r_i} \to A_\infty
\]
denote the (pairwise orthogonal) components of \( \varphi \); let \( \pi^{(i)} \) denote the respective supporting \( \ast \)-homomorphisms.

For \( i = 1, \ldots, s \), apply Proposition 5.4 to obtain c.p.c. order zero maps
\[
\Phi^{(i)} : M_k \to A_\infty \cap \varphi^{(i)}(e^{(i)}_{11})'
\]
(102)
satisfying
\[
\tau(\Phi^{(i)}(1_k)\varphi^{(i)}(e^{(i)}_{11})) \geq \frac{1}{4(m+1)^2} \cdot \tau(\varphi^{(i)}(e^{(i)}_{11}))
\]
(86)
for all \( \tau \in T_\infty(A) \), where \( \{e^{(i)}_{pq}\} \) denotes a set of matrix units for \( M_{r_i} \).

For \( \eta > 0 \), we may define pairwise orthogonal maps
\[
\hat{\Phi}^{(i)}_{\eta} : M_k \to (g_0,\eta(\varphi^{(i)})(1_k))A_\infty (g_0,\eta(\varphi^{(i)})(1_k))
\]
(103)
by setting
\[ \tilde{\Phi}_\eta^{(i)}(y) := \sum_{p=1}^{r_i} \pi^{(i)}(e^{(i)}_{p1})(g_0, \eta(\varphi^{(i)}))(e^{(i)}_{11})(\Phi^{(i)}(y))\pi^{(i)}(e^{(i)}_{1p}) \]
(104)
for \( y \in M_k \). One checks that
\[ \hat{\Phi}_\eta := \bigoplus_{i=1}^{s} \hat{\Phi}^{(i)}_\eta \]
is c.p.c. order zero, that
\[ [\hat{\Phi}_\eta(M_k), \varphi(F)] = 0, \]
(105)
and that
\[ \tau(\hat{\Phi}_\eta(1_k)\varphi(x)) \geq \frac{1}{4(m+1)^2} \cdot \tau(\varphi(x)) - \eta \]
(106)
for all \( x \in F_+ \) of norm at most one and for all \( \tau \in T_\infty(A) \) (using (102), (103), (104)).

For \( l \in \mathbb{N} \) we may now take \( \frac{1}{l+1} \) in place of \( \eta \) and choose c.p.c. order zero maps
\[ \bar{\Phi}_{l,r} : M_k \to A, \quad r \in \mathbb{N}, \]
such that
\[ [(\bar{\Phi}_{l,r}(y))_{r \in \mathbb{N}}] = \hat{\Phi}_{\frac{1}{l+1}}(y) \]
for \( y \in M_k \).

Let
\[ (\bar{\varphi}_r : F \to A)_{r \in \mathbb{N}} \]
be a sequence of c.p.c. order zero maps lifting \( \varphi \). Just as in the proof of 5.4 (see the argument for (99)), one checks by contradiction to (106) that for each \( l \in \mathbb{N} \) there is \( K_l \in \mathbb{N} \) such that
\[ \tau(\bar{\Phi}_{l,r}(1_k)\bar{\varphi}_r(x)) \geq \frac{1}{4(m+1)^2} \cdot \tau(\bar{\varphi}_r(x)) - \frac{2}{l+1} \]
(107)
for all \( x \in F_+ \) of norm at most one and \( r \geq K_l \). Upon increasing the \( K_l \) if necessary, we may assume the sequence \( (K_l)_{l \in \mathbb{N}} \) to be strictly increasing, and (by (105)) that
\[ \|[\tilde{\Phi}_{l,r}(y), \bar{\varphi}_r(x)]\| \leq \frac{1}{l+1} \]
(108)
for normalized elements $y \in M_k$ and $x \in F$, if $r \geq K_l$.

Again as in the proof of 5.4, we may define a sequence

$$(\tilde{\Phi}_r : M_k \to A)_{r \in \mathbb{N}}$$

of c.p.c. order zero maps by

$$\tilde{\Phi}_r := \begin{cases} \tilde{\Phi}_{l,r} & \text{if } K_l \leq r < K_{l+1} \\ 0 & \text{if } r < K_0; \end{cases}$$

using (108) and (107) one checks that

(a) $\|[[\tilde{\Phi}_r(y), \varphi_r(x)]]\| \xrightarrow{r \to \infty} 0$ for all $x \in F$, $y \in M_k$

(b) for any $\delta > 0$ there is $\bar{r} \in \mathbb{N}$ such that

$$\tau(\tilde{\Phi}_r(1_k)\tilde{\varphi}_r(x)) \geq \frac{1}{4(m+1)^2} \cdot \tau(\tilde{\varphi}_r(x)) - \delta$$

for any $r \geq \bar{r}$, $x \in F_+$ of norm at most one, and $\tau \in T(A)$.

Define

$$\Phi : M_k \to A_\infty$$

by

$$\Phi(x) := [(\tilde{\Phi}_r(x))_{r \in \mathbb{N}}],$$

then $\Phi$ is a c.p.c. order zero map since the $\tilde{\Phi}_r$ are; $\Phi(M_k)$ commutes with $\varphi(F)$ by (a) above and

$$\tau(\Phi(1_k)\varphi(x)) \geq \frac{1}{4(m+1)^2} \cdot \tau(\varphi(x))$$

for all $x \in F_+$ and $\tau \in T_\infty(A)$ by (b) above. $\square$

5.6

The next proposition will play a role for 5.11 similar to that of [40, Proposition 3.9] for the proof of [40, Lemma 3.10]. The subsequent Propositions 5.7 through 5.10 will get this analogy to work.

**Proposition** Let $A$ be a separable, simple, unital C*-algebra with tracial $m$-almost divisibility, and let $0 \neq k \in \mathbb{N}$. Suppose further that $F$ is a finite-dimensional C*-algebra and let

$$\varphi : F \to A_\infty$$
be a c.p.c. order zero map. Let
\[ d \in A_\infty \cap \varphi(F)' \] (109)
be a positive contraction.

Then, there are c.p.c. order zero maps
\[ \Phi^{(0)}, \ldots, \Phi^{(3)} : M_k \to A_\infty \cap \varphi(F)' \cap \{d\}' \]
satisfying
\[ \tau((\Phi^{(0)}(1_k) + \cdots + \Phi^{(3)}(1_k)) f(d)\varphi(x)) \geq \frac{1}{4(m+1)^2} \cdot \tau(f(d)\varphi(x)) \]
for all \( x \in F_+, f \in C_0((0, 1])_+ \) and \( \tau \in T_\infty(A) \).

**Proof** Let
\[(G_q)_{q \in \mathbb{N}}\]
be a nested sequence of finite subsets of the unit ball of \( C_0((0, 1])_+ \) with dense union. For each \( q \in \mathbb{N} \), find \( L_q \in \mathbb{N} \) such that the standard partition of unity
\[ f_q^{(i)}, \ldots, f_q^{(i)} \in C_0((0, 1]), \quad i = 0, 1, \] (110)
of the interval \( (0, 1] \) as in 2.5 satisfies
\[ \left\| \bigoplus_{r,s=1}^{L_q} \left( f_q^{(i)}(d) f_q^{(j)}(\varphi(1_F)) f(d)\varphi(x) - f \left( \frac{r}{L_q} \right) \cdot f_q^{(i)}(d) f_q^{(j)}(\varphi)(x) \right) \right\| \leq \frac{1}{q + 1} \] (111)
for \( q \in \mathbb{N}, i, j \in \{0, 1\}, f \in G_q \) and \( x \) in the unit ball of \( F_+ \).

For each \( q \in \mathbb{N}, r, s \in \{1, \ldots, L_q\}, i, j \in \{0, 1\}, \), the map
\[ x \mapsto f_q^{(i)}(d) f_q^{(j)}(\varphi)(x) \] (112)
is c.p.c. order zero, so that we may apply Proposition 5.5 to obtain c.p.c. order zero maps
\[ \Phi^{(i,j)}_{q,r,s} : M_k \to A_\infty \cap (f_q^{(i)}(d)(f_q^{(j)}(\varphi)(F)))' \] (113)
For fixed \( q, i \) and \( j \), the \( f_q^{(i)}(d) f_q^{(j)}(\varphi)(1_F) \) are pairwise orthogonal positive contractions by (109) and (110).
We therefore have c.p.c. order zero maps
\[ \Phi_{q}^{(i,j)} : M_k \to A_\infty \cap \left( \bigoplus_{r,s} f_{q,\tau}^{(i)}(d) f_{q,s}^{(j)}(\varphi)(F) \right) \]
given by
\[ \Phi_{q}^{(i,j)}(\cdot) := \bigoplus_{r,s} \Phi_{q,r,s}^{(i,j)}(\cdot) f_{q,r}^{(i)}(d) f_{q,s}^{(j)}(\varphi(1_F)). \] (114)

For fixed \( q, i, j \), one checks that
\[ \left\| \Phi_{q}^{(i,j)}(y) f(d) - \bigoplus_{r,s} \Phi_{q,r,s}^{(i,j)}(y) f_{r,L_q}^{(i)}(d) f_{q,s}^{(j)}(\varphi)(1_F) \right\| \leq \frac{1}{q + 1} \cdot \|y\| \] (115)
for \( y \in (M_k)_+ \) and \( f \in \mathcal{G}_q \), from which (in connection with (113)) it follows that
\[ \|[\Phi_{q}^{(i,j)}(y), f(d)]\| \leq \frac{2}{q + 1} \cdot \|y\| \] (116)
for \( y \in (M_k)_+, f \in \mathcal{G}_q, q \in \mathbb{N} \) and \( i, j \in \{0, 1\} \).

Similarly, we have
\[ \left\| \Phi_{q}^{(i,j)}(y) \varphi(x) - \bigoplus_{r,s} \Phi_{q,r,s}^{(i,j)}(y) f_{q,s}^{(j)}(\varphi)(x) \right\| \leq \frac{1}{q + 1} \cdot \|x\| \|y\| \] (117)
for \( x \in F_+ \) and \( y \in (M_k)_+ \), from which we see that
\[ \|[\Phi_{q}^{(i,j)}(y), \varphi(x)]\| \leq \frac{2}{q + 1} \cdot \|x\| \|y\| \] (118)
for \( x \in F_+, y \in (M_k)_+, q \in \mathbb{N} \) and \( i, j \in \{0, 1\} \).

Next, we check that for \( q \in \mathbb{N} \), \( x \) in the unit ball of \( F_+ \), \( f \in \mathcal{G}_q \) and \( \tau \in T_\infty(A) \)
\[ \tau \left( \sum_{i,j \in \{0,1\}} \Phi_{q}^{(i,j)}(1_k) f(d) \varphi(x) \right) \]
\[ \geq \tau \left( \sum_{i,j \in \{0,1\}} \bigoplus_{r,s=1}^{L_q} \Phi_{q,r,s}^{(i,j)}(1_k) f_{r,L_q}^{(i)}(d) f_{q,s}^{(j)}(\varphi)(x) \right) \] (114),(111)
\[
\begin{align*}
(122), (101) & \geq \frac{1}{4(m+1)^2} \cdot \tau \left( \sum_{i,j} \bigoplus_{r,s} f \left( \frac{r}{L_q} \right) \cdot \frac{s}{L_q} \cdot f_{q,r}^{(i)}(d) f_{q,s}^{(j)}(\varphi)(x) \right) \\
& \geq \frac{1}{4(m+1)^2} \cdot \tau(\sum_{i,j} \Phi_{1}^{q}(i,j) M_k) \to \prod_{\mathbb{N}} A \\
& \geq \frac{1}{4(m+1)^2} \cdot \tau(f(d)\varphi(x)) - \frac{7}{q+1}.
\end{align*}
\]

For each \( q, i, j \) let

\[ \Phi_{1}^{q}(i,j) : M_k \to \prod_{\mathbb{N}} A \]

be a c.p.c. order zero lift for \( \Phi_{q}^{i,j} \) (cf. [19, Remark 2.4]); let

\[ \Phi_{q,l}^{i,j} : M_k \to A, \quad l \in \mathbb{N} \]

denote the components. Let

\[ (d_l)_{l \in \mathbb{N}} \in \prod_{\mathbb{N}} A \]

be a positive contraction lifting \( d \), and let

\[ (\varphi_l)_{l \in \mathbb{N}} : F \to \prod_{\mathbb{N}} A \]

be a c.p.c. order zero lift for \( \varphi \).

We claim that for each \( q \in \mathbb{N} \) there is \( R_q \in \mathbb{N} \) such that

\[ \tau \left( \sum_{i,j} \Phi_{q,l}^{i,j}(1_k) f(d_l)\varphi_l(x) \right) \geq \frac{1}{4(m+1)^2} \cdot \tau(f(d_l)\varphi_l(x)) - \frac{7}{q+1} \]

for each \( l \geq R_q, \tau \in T(A), f \in G_q \) and \( x \) in the unit ball of \( F_+ \).

To prove the claim, note that otherwise for some \( q \in \mathbb{N} \) there were \( f \in G_q \), an increasing sequence \((l_t)_{t \in \mathbb{N}} \subset \mathbb{N} \), a sequence \((x_{l_t})_{t \in \mathbb{N}} \) in the unit ball of \( F_+ \), and a sequence \((\tau_{l_t})_{t \in \mathbb{N}} \subset T(A) \) such that

\[ \tau_{l_t} \left( \sum_{i,j} \Phi_{q,l_t}^{i,j}(1_k) f(d_{l_t})\varphi_{l_t}(x_{l_t}) \right) < \frac{1}{4(m+1)^2} \cdot \tau_{l_t}(f(d_{l_t})\varphi_{l_t}(x_{l_t})) - \frac{7}{q+1} \]
for each $t \in \mathbb{N}$. By compactness and passing to a subsequence of $(l_t)_{t \in \mathbb{N}}$ we may assume that $(x_{l_t})_{t \in \mathbb{N}}$ converges to some $\bar{x}$ in the unit ball of $F_+$. Let $\omega \in \beta \mathbb{N} \setminus \mathbb{N}$ be a free ultrafilter along the sequence $(l_t)_{t \in \mathbb{N}}$, then

$$
\lim_{\omega} \tau_l \left( \sum_{i,j} \tilde{\Phi}_{q,l}^{(i,j)}(1_k) f(\tilde{d}_l) \tilde{\varphi}_l(\bar{x}) \right) \leq \frac{1}{4(m+1)^2} \lim_{\omega} \tau_l( f(\tilde{d}_l) \tilde{\varphi}_l(\bar{x}))) - \frac{7}{q+1},
$$

a contradiction to (119), since the map $[(a_l)_{l \in \mathbb{N}}] \mapsto \lim_{\omega} \tau_l(\cdot)$ is in $T_\infty(A)$. This establishes the claim (120).

We may also assume that the sequence $(R_q)_{q \in \mathbb{N}}$ is strictly increasing and that, for each $q \in \mathbb{N}$, $R_q$ is so large that

$$
\| [\tilde{\Phi}_{q,l}^{(i,j)}(y), f(\tilde{d}_l)] \| \leq \frac{3}{q+1} \| y \|
$$

and

$$
\| [\tilde{\Phi}_{q,l}^{(i,j)}(y), \bar{\varphi}_l(x)] \| \leq \frac{3}{q+1} \| x \| \| y \|
$$

for all $i, j \in \{0, 1\}, x \in F_+, y \in (M_k)_+, f \in G_q$ and for all $l \geq R_q$ (using (116) and (118)).

Now for $l \in \mathbb{N}$ we may define c.p.c. order zero maps

$$\tilde{\Phi}_{l}^{(i,j)} : M_k \to A$$

by

$$
\tilde{\Phi}_{l}^{(i,j)} := \begin{cases} 
\tilde{\Phi}_{q,l}^{(i,j)} & \text{if } R_q \leq l < R_q+1 \\
0 & \text{if } l < R_0,
\end{cases}
$$

and let

$$\Phi^{(i,j)} : M_k \to A_\infty$$

be the c.p.c. order zero map induced by the $\tilde{\Phi}_{l}^{(i,j)}, i, j \in \{0, 1\}$. After relabeling the $\Phi^{(i,j)}$, it is easy to check that we indeed have maps $\Phi^{(0)}, \ldots, \Phi^{(3)}$ with the desired properties.

5.7

Some of the ideas of the next two propositions we have already encountered in the proof of Proposition 4.6, which was used to show that finite nuclear dimension implies tracial $m$-almost divisibility (Proposition 4.7). In the present context we assume tracial $m$-almost divisibility to arrive at a statement similar to that of 4.6.
**Proposition**  For any \( m, \bar{m} \in \mathbb{N} \), there is \( \beta_{m, \bar{m}} > 0 \) such that the following holds:

Let \( A \) be a separable, simple, unital \( \mathrm{C}^\ast \)-algebra with tracial \( \bar{m} \)-almost divisibility, and let \( B \subset A_\infty \) be a separable, unital \( \mathrm{C}^\ast \)-subalgebra with \( \dim_{\text{nuc}} B \leq m \) and

\[
d \in A_\infty \cap B'\tag{121}
\]
a positive contraction.

Then, there are orthogonal positive contractions

\[
d^{(0)}, d^{(1)} \in A_\infty \cap B' \cap \{d\}'
\]
satisfying

\[
\tau(d^{(j)} f(d)b) \geq \beta_{m, \bar{m}} \cdot \tau(f(d)b) \tag{122}
\]
for all \( b \in B_+, f \in C_0((0, 1])_+, \tau \in T_\infty(A) \) and \( j = 0, 1 \).

**Proof** Take

\[
\beta_{m, \bar{m}} := \frac{1}{32(m + 1)(\bar{m} + 1)^2}. \tag{123}
\]

Let \( A, B, d \) be as in the proposition. Let \( (\mathcal{G}_p)_{p \in \mathbb{N}} \) be a nested sequence of finite subsets (each containing \( 1_B \)) of the unit ball of \( B_+ \) with dense union.

By Proposition 4.2, there is a system of \( m \)-decomposable c.p. approximations for \( B \)

\[
(F_p = F_p^{(0)} \oplus \cdots \oplus F_p^{(m)}, \psi_p, \varphi_p = \varphi_p^{(0)} + \cdots + \varphi_p^{(m)})_{p \in \mathbb{N}}
\]
such that the

\[
\psi_p : B \to F_p
\]
are c.p.c. maps, the

\[
\varphi_p^{(i)} : F_p^{(i)} \to B
\]
are c.p.c. order zero maps, and such that

\[
\|\varphi_p \psi_p(b) - b\| < \frac{1}{p + 1} \tag{124}
\]

\[
\|[\varphi_p \psi_p(b), \varphi_p^{(i)}(\psi_p^{(i)}(1_B))]\| < \frac{1}{p + 1} \tag{125}
\]

and

\[
\|b \varphi_p^{(i)}(\psi_p^{(i)}(1_B)) - \varphi_p^{(i)} \psi_p^{(i)}(b)\| < \frac{1}{p + 1} \tag{126}
\]
for all \( b \in \mathcal{G}_p, i \in \{0, \ldots, m\} \) and \( p \in \mathbb{N} \).

For each \( i \in \{0, \ldots, m\} \) and \( p \in \mathbb{N} \) apply Proposition 5.6 (with \( m \) in place of \( m, \varphi^{(i)}_p \) in place of \( \varphi \) and \( 8(m + 1) \) in place of \( k \)) to obtain c.p.c. order zero maps

\[
\Phi_p^{(i,j)} : M_{8(m+1)} \to A_\infty \cap \varphi_p^{(i)}(F_p)^{\prime} \cap \{d\}^{\prime}
\]

for \( j = 0, 1, 2, 3 \) satisfying

\[
\tau \left( \sum_{j=0}^{3} \Phi_p^{(i,j)}(1_{8(m+1)}) f(d)\varphi_p^{(i)}(x) \right) \geq \frac{1}{4(m+1)^2} \cdot \tau(f(d)\varphi_p^{(i)}(x))
\]

for all \( x \in (F_p^{(i)})_+, f \in C_0((0, 1])_+ \) and \( \tau \in T_\infty(A) \).

But then one checks that, for all \( i \in \{0, \ldots, m\} \), \( j \in \{0, \ldots, 3\} \) and \( p \in \mathbb{N} \),

\[
\Phi_p^{(i,j)}(\cdot) := \Phi_p^{(i,j)}(\cdot)\varphi_p^{(i)}(\psi_p^{(i)}(b)): M_{8(m+1)} \to A_\infty \cap \{d\}^{\prime}
\]

define c.p.c. order zero maps (using (127) and (121)).

Furthermore, we have

\[
\|\tilde{\Phi}_p^{(i,j)}(y)b - b\tilde{\Phi}_p^{(i,j)}(y)\|
\leq \|\Phi_p^{(i,j)}(y)\varphi_p^{(i)}\psi_p^{(i)}(b) - \varphi_p^{(i)}\psi_p^{(i)}(b)\Phi_p^{(i,j)}(y)\| + \frac{2}{p+1}\|y\|
\]

for \( b \in \mathcal{G}_p \) and \( y \in M_{8(m+1)} \), whence

\[
\|[\tilde{\Phi}_p^{(i,j)}(y), b]\| \xrightarrow{p \to \infty} 0
\]

for \( b \in B \) and \( y \in M_{8(m+1)} \), and

\[
\tau \left( \sum_{j=0}^{3} \sum_{i=0}^{m} \Phi_p^{(i,j)}(1_{8(m+1)}) f(d)b \right)
\geq \tau \left( \sum_{j=0}^{3} \sum_{i=0}^{m} \Phi_p^{(i,j)}(1_{8(m+1)}) f(d)\varphi_p^{(i)}\psi_p^{(i)}(b) \right)
\geq \frac{1}{4(m+1)^2} \cdot \tau(f(d)b) - \frac{4(m+1)+1}{p+1}
\]
for all $b \in \mathcal{G}_p$, $f$ in the unit ball of $C_0((0, 1])_+$ and $\tau \in T_\infty(A)$.

After choosing a c.p.c. order zero lift
\[
\hat{\Phi}_p^{(i, j)} : M_{8(m+1)} \to \prod_{N} A
\]
for each $\hat{\Phi}_p^{(i, j)}$ (cf. [19, Remark 2.4]), a standard diagonal sequence argument yields c.p.c. order zero maps
\[
\tilde{\Phi}^{(i, j)} : M_{8(m+1)} \to A_\infty \cap B' \cap \{d\}'
\]
such that
\[
\tau \left( \sum_{j=0}^{3} \sum_{i=0}^{m} \tilde{\Phi}^{(i, j)} (1_{8(m+1)}) f(d)b \right) \geq \frac{1}{4(m+1)^2} \cdot \tau(f(d)b) \quad (130)
\]
for all $b \in B$, $f \in C_0((0, 1])_+$ and $\tau \in T_\infty(A)$.

Now for each $q \in \mathbb{N}$ we may define orthogonal positive contractions
\[
d_q^{(0)} := f_{\frac{1}{q+1}}, \frac{2}{q+1} \left( \sum_{i,j} \tilde{\Phi}^{(i, j)} (e_{11}) \right) \in A_\infty \cap B' \cap \{d\}' \quad (131)
\]
and
\[
d_q^{(1)} := 1_{A_\infty} - g_{\frac{1}{q+1}} \left( \sum_{i,j} \tilde{\Phi}^{(i, j)} (e_{11}) \right) \in A_\infty \cap B' \cap \{d\}'. \quad (132)
\]

We estimate
\[
\tau(d_q^{(0)} f(d)b) \quad (131) \geq \tau \left( \sum_{i,j} \tilde{\Phi}^{(i, j)} (e_{11}) f(d)b \right) - \frac{1}{q+1} \\
\geq \frac{1}{4(m+1)^2} \cdot \tau(f(d)b) - \frac{1}{q+1} \quad (133)
\]
for all $b \in B_+$ and $f \in C_0((0, 1])_+$ of norm at most one and for all $\tau \in T_\infty(A)$. Moreover,
\[
\tau((1_{A_\infty} - d_q^{(1)}) f(d)b) \quad (132) \leq \tau \left( g_{\frac{1}{q+1}} \left( \sum_{i,j} \tilde{\Phi}^{(i, j)} (e_{11}) \right) f(d)b \right) \\
\leq \lim_{l \to \infty} \tau \left( \left( \sum_{i,j} \tilde{\Phi}^{(i, j)} (e_{11}) \right)^{\frac{1}{l}} f(d)b \right)
\]
\[ \leq \sum_{i=0,\ldots,m; j=0,\ldots,3} \lim_{l \to \infty} \tau((\Phi(i,j)(e_{11}))^{\frac{1}{l}} f(d)b) \]
\[ \leq \frac{4(m + 1)}{8(m + 1)} \cdot \tau(f(d)b) \]
\[ = \frac{1}{2} \cdot \tau(f(d)b) \quad (134) \]

for \( \tau \in T_{\infty}(A) \) and \( b \in B_+ \), where for both the second and third estimate we have used that \( \tau(\cdot f(d)b) \) is a trace on \( A_{\infty} \cap B' \cap \{d\}' \), whence \( \lim_{l \to \infty} \tau((\cdot)^{\frac{1}{l}} f(d)b) \) is a dimension function on \( A_{\infty} \cap B' \cap \{d\}' \) for any \( \tau \in T_{\infty}(A) \).

As a consequence of (134),

\[ \tau(d_q^{(1)} f(d)b) \geq \frac{1}{2} \cdot \tau(f(d)b) \]

for \( b \in B_+ \), \( f \in C_0((0, 1])_+ \) and \( \tau \in T_{\infty}(A) \).

Applying a diagonal sequence argument once more, from (133) and (134) we obtain orthogonal positive contractions

\[ d^{(0)}, d^{(1)} \in A_{\infty} \cap B' \cap \{d\}' \]

such that

\[ \tau(d^{(j)} f(d)b) \geq \frac{1}{32(m + 1)(\bar{m} + 1)^2} \cdot \tau(f(d)b) \]

for \( b \in B_+ \), \( f \in C_0((0, 1])_+ \) and \( \tau \in T_{\infty}(A) \).

\[ \square \]

**Proposition 5.8**  For any \( m, \bar{m}, k \in \mathbb{N} \), there is \( \gamma_{m,\bar{m},k} > 0 \) such that the following holds:

Let \( A \) be a separable, simple, unital \( \text{C}^* \)-algebra with tracial \( \bar{m} \)-almost divisibility, and let \( B \subset A_{\infty} \) be a separable, unital \( \text{C}^* \)-subalgebra with \( \dim_{\text{nuc}} B \leq m \). Let \( d \in A_{\infty} \cap B' \) be a positive contraction.

Then, there are pairwise orthogonal positive contractions

\[ d^{(0)}, \ldots, d^{(k)} \in A_{\infty} \cap B' \cap \{d\}' \]

satisfying

\[ \tau(d^{(i)} f(d)b) \geq \gamma_{m,\bar{m},k} \cdot \tau(f(d)b) \]

for all \( b \in B_+ \), \( f \in C_0((0, 1])_+ \), \( \tau \in T_{\infty}(A) \) and \( i = 0, \ldots, k \).

**Proof**  Let us first assume that \( k = 2^l - 1 \) for some \( l \in \mathbb{N} \). For \( k \) of this specific form, we verify the statement by induction over \( l \).
For $l = 0$, the statement is trivial with $\gamma_{m,\tilde{m},0} = 1$ and $d^{(0)} = 1_{A_\infty}$. Suppose now the statement has been verified for $k = 2^l - 1$ for some $l \in \mathbb{N}$, with elements

$$\tilde{d}^{(0)}, \ldots, \tilde{d}^{(2^l - 1)} \in A_\infty \cap B' \cap \{d\}' .$$

Set

$$\tilde{B}^{(i)} := \mathbb{C}^*(B, \tilde{d}^{(i)}) \subset A_\infty$$

and note that for each $i \in \{0, \ldots, 2^l - 1\}$, $\tilde{B}^{(i)}$ is a quotient of $\mathcal{C}_0([0, 1]) \otimes \tilde{B}$ by 2.1(i), whence

$$\dim_{\text{nuc}} \tilde{B}^{(i)} \leq 2m + 1$$

by [43]. Set

$$\gamma_{m,\tilde{m},2^l+1-1} := \beta_{2m+1,\tilde{m}} \cdot \gamma_{m,\tilde{m},2^l-1},$$

where $\beta_{2m+1,\tilde{m}}$ comes from 5.7.

For each $i \in \{0, \ldots, 2^l - 1\}$, apply Proposition 5.7 (with $2m + 1$ in place of $m$ and $\tilde{B}^{(i)}$ in place of $B$) to obtain orthogonal positive contractions

$$d^{(i,j)} \in A_\infty \cap B' \cap \{d\}' \cap \{\tilde{d}^{(i)}\}' , \quad j = 0, 1,$$

satisfying

$$\tau(d^{(i,j)} \tilde{d}^{(i)} db) \geq \beta_{2m+1,\tilde{m}} \cdot \tau(\tilde{d}^{(i)} f(d)b)$$

$$\geq \beta_{2m+1,\tilde{m}} \cdot \gamma_{m,\tilde{m},2^l-1} \cdot \tau(f(d)b)$$

$$= \gamma_{m,\tilde{m},2^l+1-1} \cdot \tau(f(d)b)$$

for all $b \in B_+$, $f \in \mathcal{C}_0((0, 1])_+$ and $\tau \in T_\infty(A)$. This yields the assertion for $k = 2^l+1 - 1$, where the $2^{l+1}$ pairwise orthogonal positive contractions are given by

$$d^{(i,j)} \tilde{d}^{(i)} \in A_\infty \cap B' \cap \{d\}'$$

for $i = 0, \ldots, 2^l - 1$ and $j = 0, 1$.

Induction yields the statement of the proposition for any $k$ of the form $2^l - 1$, $l \in \mathbb{N}$. This clearly also implies the case of arbitrary $k \in \mathbb{N}$.

**Proposition 5.9** For any $m, \tilde{m}, k \in \mathbb{N}$, there is $\gamma_{m,\tilde{m},k} > 0$ such that the following holds:

Let $A$ be a separable, simple, unital $\mathbb{C}^*$-algebra with tracial $\tilde{m}$-almost divisibility, and let $B \subset A_\infty$ be a separable, unital $\mathbb{C}^*$-subalgebra with $\dim_{\text{nuc}} B \leq m$. Let $l \in \mathbb{N}$ and let

$$\Phi : M_l \to A_\infty \cap B'$$
be a c.p.c. order zero map.

Then, there are pairwise orthogonal positive contractions

\[ d^{(0)}, \ldots, d^{(k)} \in A_\infty \cap B' \cap \Phi(M_l)' \]

satisfying

\[ \tau(d^{(i)} f(\Phi)(x)b) \geq \gamma_{m,\tilde{m},k} \cdot \tau(f(\Phi)(x)b) \]

for all \( x \in (M_l)_+, f \in C_0((0,1])_+, b \in B_+, \tau \in T_\infty(A) \) and \( i = 0, \ldots, k \).

**Proof** Apply Proposition 5.8 (with \( \Phi(e_{11}) \)) in place of \( d \) to obtain pairwise orthogonal positive contractions

\[ \tilde{d}^{(0)}, \ldots, \tilde{d}^{(k)} \in A_\infty \cap B' \cap \{\Phi(e_{11})\}' \]

such that

\[ \tau(\tilde{d}^{(i)} f(\Phi(e_{11}))b) \geq \gamma_{m,\tilde{m},k} \cdot \tau(f(\Phi(e_{11}))b) \quad (135) \]

for \( i = 0, \ldots, k, f \in C_0((0,1])_+, b \in B_+ \) and \( \tau \in T_\infty(A) \).

For \( q \in \mathbb{N} \), define pairwise orthogonal positive contractions

\[ \tilde{d}_q^{(i)} := \tilde{d}^{(i)} g_{0,\frac{1}{q+1}}(\Phi(e_{11})) \in A_\infty \cap B' \cap \{\Phi(e_{11})\}' \quad (136) \]

then

\[ \tau(\tilde{d}_q^{(i)} f(\Phi)(e_{11})b) \geq \gamma_{m,\tilde{m},k} \cdot \tau(f(\Phi)(e_{11})b) \quad (136),(135) \]

for \( i = 0, \ldots, k, \) for all positive contractions \( b \in B \), for all \( f \in C_0((\frac{1}{q+1},1])_+ \) and for all \( \tau \in T_\infty(A) \).

Set

\[ d_q^{(i)} := \sum_{j=1}^{l} \pi_{\Phi}(e_{j1}) \tilde{d}_q^{(i)} \pi_{\Phi}(e_{1j}), \quad i = 0, \ldots, k, \]

where \( \pi_{\Phi} \) denotes the supporting \(*\)-homomorphism of \( \Phi \); one checks that these are pairwise orthogonal positive contractions, that

\[ d_q^{(i)} \in A_\infty \cap B' \cap \Phi(M_l)' \]

and that

\[ \tau(d_q^{(i)} f(\Phi)(x)b) \geq \gamma_{m,\tilde{m},k} \cdot \tau(f(\Phi)(x)b) \]

for all positive contractions \( x \in M_l \) and \( b \in B \), for all \( f \in C_0((\frac{1}{q+1},1])_+ \) and for all \( \tau \in T_\infty(A) \).
After taking positive contractions

\[ \tilde{d}_q^{(i)} \in \prod_{\mathbb{N}} A \]

and a c.p.c. order zero map

\[ \Phi : M_l \to \prod_{\mathbb{N}} A \]

lifting the \( d_q^{(i)} \) and \( \Phi \), respectively, a standard diagonal sequence argument yields orthogonal positive contractions

\[ d^{(i)} \in A_\infty \cap B' \cap \Phi(M_l)', \quad i = 0, \ldots, k, \]

such that the assertion of the Proposition holds. \( \square \)

**Proposition 5.10** For any \( m, \bar{m} \in \mathbb{N} \), there is \( \alpha_{m, \bar{m}} > 0 \) such that the following holds:

Let \( A \) be a separable, simple, unital C*-algebra with tracial \( \bar{m} \)-almost divisibility, and let \( B \subset A \) be a unital C*-subalgebra with \( \dim_{\text{nuc}} B \leq m \). Let \( k, l \in \mathbb{N} \) and let

\[ \Phi : M_l \to A_\infty \cap B' \]

be a c.p.c. order zero map.

Then, there is a c.p.c. order zero map

\[ \Psi : M_k \to A_\infty \cap B' \cap \Phi(M_l)' \]

satisfying

\[ \tau(\Psi(1_k)\Phi(1_l)b) \geq \alpha_{m, \bar{m}} \cdot \tau(\Phi(1_l)b) \]

for all \( b \in B_+ \) and \( \tau \in T_\infty(A) \).

**Proof** Set

\[ \alpha_{m, \bar{m}} := \frac{\gamma_{m, \bar{m}, m} \cdot \gamma_{m, \bar{m}, 1}}{4(\bar{m} + 1)^2}, \]

where \( \gamma_{m, \bar{m}, l} \) comes from Proposition 5.9.

Let \( A, B, k, l, \Phi \) be as in the proposition. Let \( (G_p)_{p \in \mathbb{N}} \) be a nested sequence of finite subsets (each containing \( 1_B \)) of the unit ball of \( B_+ \) with dense union.

By Proposition 4.2, there is a system of \( m \)-decomposable c.p. approximations for \( B \)

\[ (F_p = F_p^{(0)} \oplus \cdots \oplus F_p^{(m)}, \psi_p, \varphi_p)_{p \in \mathbb{N}} \]
such that the
\[ \psi_p : B \to F_p \]
are c.p.c. maps, the
\[ \varphi_p^{(i)} : F_p^{(i)} \to B \] (139)
are c.p.c. order zero maps, and such that
\[ \| \varphi_p \psi_p (b) - b \| < \frac{1}{p+1}, \] (140)
\[ \| [\varphi_p \psi_p (b), \varphi_p^{(i)} (\psi_p^{(i)} (1_B))] \| < \frac{1}{p+1} \]
and
\[ \| b \varphi_p^{(i)} (\psi_p^{(i)} (1_B)) - \varphi_p^{(i)} \psi_p^{(i)} (b) \| < \frac{1}{p+1} \] (141)
for all \( b \in G_p, i \in \{0, \ldots, m\} \) and \( p \in \mathbb{N} \).

Let
\[ h_{p,1}^{(j)}, \ldots, h_{p,p+1}^{(j)} \in C_0((0,1]), \quad j = 0, 1, \]
be a standard partition of unity for \((0,1] \) as in 2.5, and note that
\[ \left\| \sum_{j=0,1; r=1,\ldots,p+1} \frac{r}{p+1} \cdot h_{p,r}^{(j)} - \text{id}_{(0,1]} \right\| \leq \frac{1}{p+1} \] (142)
(where \( \text{id}_{(0,1]} \) denotes the identity function on \((0,1]) \) and
\[ \left\| \bigoplus_{r=1}^{p+1} h_{p,r}^{(j)} (\Phi)(1) \Phi(z) - \bigoplus_{r=1}^{p+1} \frac{r}{p+1} \cdot h_{p,r}^{(j)} (\Phi)(z) \right\| \leq \frac{1}{p+1} \cdot \|z\| \] (143)
for \( j = 0, 1 \) and \( z \in (M_1)_+ \).

Use Proposition 5.9 to find orthogonal positive contractions
\[ \tilde{d}_p^{(0)}, \tilde{d}_p^{(1)} \in A_\infty \cap B' \cap \Phi(M_1)' \] (144)
such that
\[ \tau \left( \tilde{d}_p^{(j)} \left( \bigoplus_{r=1}^{p+1} \frac{r}{p+1} \cdot h_{p,r}^{(j)} (\Phi)(z) b \right) \right) \]
\[ \geq \gamma_{m,\tilde{m},1} \cdot \tau \left( \bigoplus_{r=1}^{p+1} \frac{r}{p+1} \cdot h_{p,r}^{(j)} (\Phi)(z) b \right) \] (145)
for all \( p \in \mathbb{N}, \ z \in (M_i)_+, \ b \in B_+, \ \tau \in T_\infty(A) \) and \( j = 0, 1 \).

For each \( p \in \mathbb{N}, \ j \in \{0, 1\} \) and \( r \in \{1, \ldots, p + 1\} \) note that

\[
\tilde{d}_p^{(j)} h_p^{(j)}(\Phi)(\cdot)
\]

is a c.p.c. order zero map, so that we may apply Proposition 5.9 once more to find pairwise orthogonal positive contractions

\[
d_{p,r}^{(0,j)}, \ldots, d_{p,r}^{(m,j)} \in A_\infty \cap B' \cap (\tilde{d}_p^{(j)} h_p^{(j)}(\Phi)(M_i))'
\]

satisfying

\[
\tau(d_{p,r}^{(i,j)} \tilde{d}_p^{(j)} h_p^{(j)}(\Phi)(z)b) \geq \gamma_{m,\tilde{m},m} \cdot \tau(\tilde{d}_p^{(j)} h_p^{(j)}(\Phi)(z)b)
\]

for all \( i = 0, \ldots, m, \ z \in (M_i)_+, \ b \in B_+ \) and \( \tau \in T_\infty(A) \).

Use (137), (139), (144) and (146) to check that

\[
x \otimes z \mapsto \psi_p^{(i)}(x) d_{p,r}^{(i,j)} \tilde{d}_p^{(j)} h_p^{(j)}(\Phi)(z)
\]

defines a c.p.c. order zero map

\[
F_p^{(i)} \otimes M_l \rightarrow A_\infty
\]

and apply Proposition 5.5 to obtain c.p.c. order zero maps

\[
\Psi_{p,r}^{(i,j)} : M_k \rightarrow A_\infty \cap (\psi_p^{(i)}(F_p^{(j)}) d_{p,r}^{(i,j)} \tilde{d}_p^{(j)} h_p^{(j)}(\Phi)(M_l))'
\]

satisfying

\[
\tau(\psi_p^{(i)}(x) d_{p,r}^{(i,j)} \tilde{d}_p^{(j)} h_p^{(j)}(\Phi)(z) \Psi_{p,r}^{(i,j)}(1_k)) \geq \frac{1}{4(\tilde{m} + 1)^2} \cdot \tau(\psi_p^{(i)}(x) d_{p,r}^{(i,j)} \tilde{d}_p^{(j)} h_p^{(j)}(\Phi)(z))
\]

for all \( x \in (F_p^{(i)})_+, \ z \in (M_l)_+, \ i \in \{0, \ldots, m\}, \ j \in \{0, 1\}, \ r \in \{1, \ldots, p + 1\} \) and \( \tau \in T_\infty(A) \).

Note that

\[
\psi_p^{(i)}(\psi_p^{(i)}(b)) d_{p,r}^{(i,j)} \tilde{d}_p^{(j)} h_p^{(j)}(\Phi)(z) \Psi_{p,r}^{(i,j)}(y) = \Psi_{p,r}^{(i,j)}(y) h_p^{(j)}(\Phi)(z) \bar{d}_p^{(j)} d_{p,r}^{(i,j)} \varphi_p^{(i)}(\psi_p^{(i)}(b))
\]

for all \( i \in \{0, \ldots, m\}, \ j \in \{0, 1\}, \ r \in \{1, \ldots, p + 1\}, \ b \in B, \ z \in M_l \) and \( y \in M_k \) (using (137), (139), (144), (146) and (148)).
Define c.p. maps

\[ \Psi_p : M_k \to A_\infty \]

by

\[ \Psi_p(\cdot) := \sum_{i,j,r} \varphi_p^{(i)}(\psi_p^{(i)}(1_B))d_{p,r}^{(i,j)} \tilde{d}_p^{(j)} h_{p,r}(\Phi)(1_l)\psi_p(i,j) \cdot \cdot . \quad (150) \]

We may write \( \Psi_p \) as a direct sum

\[ \Psi_p(\cdot) = \bigoplus_{j=0,1} \left( \bigoplus_{i=0}^{m} \sum_{i,j,r} \varphi_p^{(i)}(\psi_p^{(i)}(1_B))d_{p,r}^{(i,j)} \tilde{d}_p^{(j)} h_{p,r}(\Phi)(1_l)\psi_p(i,j) \cdot \cdot . \right) \quad (151) \]

with c.p.c. order zero summands, which entails that \( \Psi_p \) itself is a c.p.c. order zero map (for (151) we have used (144), (146) and the facts that

\[ \tilde{d}_p^{(0)} \perp \tilde{d}_p^{(1)}, \]

that for fixed \( p, j \) and \( r \), the \( d_{p,r}^{(i,j)} \), \( i = 0, \ldots, m \), are pairwise orthogonal, and that for fixed \( p \) and \( j \) the \( h_{p,r}^{(j)} \) are pairwise orthogonal).

We estimate

\[ \tau(\Phi(1_l)b\Psi_p(1_k)) \]

\[ \geq \tau(\Phi(1_l) \sum_{i,j,r} \varphi_p^{(i)}(\psi_p^{(i)}(b)) \cdot d_{p,r}^{(i,j)} \tilde{d}_p^{(j)} h_{p,r}(\Phi)(1_l)\psi_p(i,j) (1_k) - \frac{1}{p+1} \]

\[ \geq \tau(\sum_{i,j,r} \frac{r}{p+1} \cdot \varphi_p^{(i)}(\psi_p^{(i)}(b)) \cdot d_{p,r}^{(i,j)} \tilde{d}_p^{(j)} h_{p,r}(\Phi)(1_l)\psi_p(i,j) (1_k) - \frac{2}{p+1} \]

\[ \geq \frac{1}{4(\bar{m}+1)^2} \cdot \tau(\sum_{i,j,r} \frac{r}{p+1} \cdot \varphi_p^{(i)}(\psi_p^{(i)}(b)) \cdot d_{p,r}^{(i,j)} \tilde{d}_p^{(j)} h_{p,r}(\Phi)(1_l) - \frac{2}{p+1} \]

\[ \geq \frac{\gamma m, \bar{m}, m}{4(\bar{m}+1)^2} \cdot \tau(\sum_{i,j,r} \frac{r}{p+1} \cdot \varphi_p^{(i)}(\psi_p^{(i)}(b)) \]
for all \( b \in G_p \) and \( \tau \in T_\infty(A) \).

Next we compute

\[
\| \Phi(z)b\Psi_p(y) - \sum_{i,j,r} \frac{r}{p+1} \cdot \varphi_p^{(i)}(\psi_p^{(i)}(b))d_p^{(i,j)}h_p^{(j)}(\Phi)(z)\Psi_p^{(i,j)}(y) \|
\]

\[
\leq \left\| \Phi(z) \sum_{i,j,r} \frac{r}{p+1} \cdot \varphi_p^{(i)}(\psi_p^{(i)}(b))d_p^{(i,j)}h_p^{(j)}(\Phi)(z)\Psi_p^{(i,j)}(y) \right\| - \frac{1}{p+1}
\]

\[
\leq \frac{2}{p+1}
\]

for \( b \in G_p, y \in (M_k)_+ \) and \( z \in (M_l)_+ \) of norm at most one. Taking adjoints, we see from (148), (146), (144) and (137) that

\[
\|[\Phi(z)b, \Psi_p(y)]\| \leq \frac{4}{p+1}
\]

for \( b \in G_p, y \in (M_k)_+ \) and \( z \in (M_l)_+ \) of norm at most one.
The usual diagonal sequence argument now yields a c.p.c. order zero map
\[ \tilde{\Psi} : M_k \to A_\infty \cap (B \Phi(M_l))' \]
such that
\[ \tau(\tilde{\Psi}(1_k) \Phi(1_l)b) \geq \alpha_{m, \tilde{m}} \cdot \tau(\Phi(1_l)b) \]
for all \( b \in B_+ \) and \( \tau \in T_\infty(A) \); it is not hard to check that in fact we have
\[ \tilde{\Psi}(M_k) \subset (B f(\Phi(M_l)))' \]
for any \( f \in C_0((0, 1])_+ \).

For \( p \in \mathbb{N} \), we may now define c.p.c. order zero maps
\[ \tilde{\Psi}_p : M_k \to A_\infty \cap (B \Phi(M_l))' \]
by
\[ \tilde{\Psi}_p(\cdot) := \tilde{\Psi}(\cdot)g_{0, \frac{1}{p+1}}(\Phi(1_l)); \]
one checks that
\[ \tilde{\Psi}_p(M_k) \subset A_\infty \cap B' \cap \Phi(M_l)' \]
and that
\[ \tau(\tilde{\Psi}_p(1_k) \Phi(1_l)b) \geq \alpha_{m, \tilde{m}} \cdot \tau(\Phi(1_l)b) - \frac{1}{p+1} \]
for all positive contractions \( b \in B_+ \) and \( \tau \in T_\infty(A) \).

Another straightforward application of the diagonal sequence argument now yields a c.p.c. order zero map
\[ \Psi : M_k \to A_\infty \cap B' \cap \Phi(M_l)' \]
with the desired properties. \( \square \)

5.11

We are finally prepared to prove the main technical result of this section, which will provide large almost central order zero maps into simple, unital C*-algebras with locally finite nuclear dimension and tracial \( \tilde{m} \)-almost divisibility. In spirit this is very similar to [40, Lemma 3.10]; the main difference is that there almost divisibility was ensured by finite decomposition rank, whereas here we make it a separate hypothesis.
Lemma Let $A$ be a separable, simple, unital $C^*$-algebra with tracial $\bar{m}$-almost divisibility, and let $B \subset A$ be a unital $C^*$-subalgebra with $\dim_{\text{nuc}} B \leq m$ for some $\bar{m}, m \in \mathbb{N}$. Let $E \subset B$ be a compact subset of elements of norm at most one and let $k \in \mathbb{N}$ and $\gamma > 0$ be given.

Then, there is a c.p.c. order zero map

$$\Phi : M_k \to A$$

such that

$$\|\left[\Phi(x), b\right]\| \leq \gamma \cdot \|x\|$$

and

$$\tau(\Phi(1_k)) \geq 1 - \gamma$$

for all $x \in M_k$, $b \in E$ and $\tau \in T(A)$.

Proof Obtain

$$0 < \gamma_{m,\bar{m},1} < 1$$

from Proposition 5.9 and

$$0 < \alpha_{m,\bar{m}} < 1$$

from Proposition 5.10 and choose $L \in \mathbb{N}$ such that

$$\alpha_{m,\bar{m}} \cdot \gamma_{m,\bar{m},1} \cdot \sum_{j=0}^{L} (1 - \alpha_{m,\bar{m}} \cdot \gamma_{m,\bar{m},1})^j > \left(1 - \frac{\gamma}{3}\right).$$

Set

$$\eta := \frac{\gamma}{6L}.$$ (153)

Apply Proposition 5.10 (with $l = 1$ and $\Phi : \mathbb{C} \to A_\infty \cap B'$ given by $\Phi(1) := 1_{A_\infty}$) to obtain a c.p.c. order zero map

$$\Phi_0 : M_k \to A_\infty \cap B'$$

such that

$$\tau(\Phi_0(1_k)b) \geq \alpha_{m,\bar{m}} \cdot \tau(b) \geq \alpha_{m,\bar{m}} \cdot \gamma_{m,\bar{m},1} \cdot \tau(b)$$

for all $b \in B_+$ and $\tau \in T_\infty(A)$.

Now suppose that for some $i \in \{0, \ldots, L - 1\}$ we have constructed a c.p.c. order zero map

$$\Phi_i : M_k \to A_\infty \cap B'$$
such that
\[
\tau(\Phi_i(1_k)b) \geq \alpha_{m,\tilde{m}} \cdot \gamma_{m,\tilde{m},1} \cdot \sum_{j=0}^{i} (1 - \alpha_{m,\tilde{m}} \cdot \gamma_{m,\tilde{m},1})^j \cdot \tau(b) - 2i \eta \tag{154}
\]
for all positive contractions \(b \in B\) and for all \(\tau \in T_\infty(A)\).

Define c.p.c. order zero maps
\[
\hat{\Phi}_i := (g_{0,\eta} - g_{2\eta})(\Phi_i), \quad \tilde{\Phi}_i := (g_{\eta,2\eta} - g_{2\eta,3\eta})(\Phi_i) : M_k \to A_\infty \cap B'. \tag{155}
\]
Apply Proposition 5.9 (with \(k\) in place of \(l\), 1 in place of \(k\) and \(\Phi_i\) in place of \(\Phi\)) to obtain positive contractions
\[
d^{(0)} \perp d^{(1)} \in A_\infty \cap B' \cap \Phi_i(M_k)' \tag{156}
\]
such that
\[
\tau(d^{(j)} f(\Phi_i)(1_k)b) \geq \gamma_{m,\tilde{m},1} \cdot \tau(f(\Phi_i)(1_k)b) \tag{157}
\]
for all \(j \in \{0, 1\}, b \in B_+, f \in C_0((0, 1])_+\) and \(\tau \in T_\infty(A)\).

Define a c.p.c. order zero map
\[
\check{\Phi}_i : M_k \to A_\infty \cap B'
\]
by
\[
\check{\Phi}_i := g_{2\eta,3\eta}(\Phi_i) + d^{(0)} \tilde{\Phi}_i. \tag{158}
\]
This is clearly c.p.c.; \(\check{\Phi}_i\) has order zero since
\[
\check{\Phi}_i(x) \leq \pi_{\Phi_i}(x) \quad \text{for } x \in (M_k)_+,
\]
where \(\pi_{\Phi_i}\) is the supporting \(*\)-homomorphism of \(\Phi_i\).

Define a positive contraction
\[
h \in A_\infty \cap B'
\]
by
\[
h := d^{(1)} \hat{\Phi}_i(1_k) + (1_{A_\infty} - g_{0,\eta}(\Phi_i(1_k))). \tag{159}
\]
Observe that
\[
h \perp \check{\Phi}_i(1_k) \tag{160}
\]
by (155), (156), (158) and (159).
Apply Proposition 5.10 (with \( l = 1 \) and \( \Phi : M_1 \to A_\infty \cap B' \) given by \( l \mapsto h \)) to obtain a c.p.c. order zero map

\[
\hat{\Phi}_i : M_k \to A_\infty \cap B' \cap \{ h \}'
\]
such that

\[
\tau(\hat{\Phi}_i(1_k)hb) \geq \alpha_{m,\bar{m}} \cdot \tau(hb)
\]
for all \( b \in B_+ \) and \( \tau \in T_\infty(A) \).

Define a c.p.c. order zero map

\[
\tilde{\Phi}_i : M_k \to A_\infty \cap B'
\]
by

\[
\tilde{\Phi}_i(\cdot) := h \hat{\Phi}_i(\cdot).
\]

Next, define a c.p. map

\[
\Phi_{i+1} : M_k \to A_\infty \cap B'
\]
by

\[
\Phi_{i+1} := \tilde{\Phi}_i + \hat{\Phi}_i
\]
and observe that \( \Phi_{i+1} \) is c.p.c. and has order zero since \( \tilde{\Phi}_i \) and \( \hat{\Phi}_i \) are c.p.c. order zero maps with orthogonal images (using (162) and (160)).

For a positive contraction \( b \in B_+ \) and \( \tau \in T_\infty(A) \) we now estimate

\[
\tau(\Phi_{i+1}(1_k)b)
\]

\[
\overset{(163)}{=} \tau(\hat{\Phi}_i(1_k)b) + \tau(\tilde{\Phi}_i(1_k)b)
\]

\[
\overset{(162),(161),(158),(157)}{\geq} \alpha_{m,\bar{m}} \cdot \tau(hb) + \tau(g_{2\eta,3\eta}(\Phi_i(1_k))b) + \gamma_{m,\bar{m},1} \cdot \tau(\hat{\Phi}_i(1_k)b)
\]

\[
\overset{(159),(157)}{\geq} \alpha_{m,\bar{m}} \cdot \gamma_{m,\bar{m},1} \cdot \tau(\hat{\Phi}_i(1_k)b) + \alpha_{m,\bar{m}} \cdot \tau((1_A \cap g_{0,\eta}(\Phi_i(1_k)))b) + \tau(g_{2\eta,3\eta}(\Phi_i(1_k))b) + \gamma_{m,\bar{m},1} \cdot \tau(\hat{\Phi}_i(1_k)b)
\]

\[
\overset{(155)}{=} \tau(g_{\eta,2\eta}(\Phi_i(1_k))b) + \alpha_{m,\bar{m}} \cdot \gamma_{m,\bar{m},1} \cdot \tau((1_A \cap g_{0,2\eta}(\Phi_i(1_k)))b) = \alpha_{m,\bar{m}} \cdot \gamma_{m,\bar{m},1} \cdot \tau(b) + (1 - \alpha_{m,\bar{m}} \cdot \gamma_{m,\bar{m},1}) \cdot \tau(g_{2\eta,3\eta}(\Phi_i(1_k))b)
\]
\[ \geq \alpha_m,\bar{m} \cdot \gamma_m,\bar{m},1 \cdot \tau(b) + (1 - \alpha_m,\bar{m} \cdot \gamma_m,\bar{m},1) \cdot \tau(\Phi_i(1_k)b) - 2\eta \]

\[ (154) \geq (1 - \alpha_m,\bar{m} \cdot \gamma_m,\bar{m},1) \cdot \alpha_m,\bar{m} \cdot \gamma_m,\bar{m},1 \]

\[ \cdot \sum_{j=0}^i (1 - \alpha_m,\bar{m} \cdot \gamma_m,\bar{m},1)^j \cdot \tau(b) \]

\[ + \alpha_m,\bar{m} \cdot \gamma_m,\bar{m},1 \cdot \tau(b) - 2i \eta - 2\eta \]

\[ = \alpha_m,\bar{m} \cdot \gamma_m,\bar{m},1 \cdot \sum_{j=0}^{i+1} (1 - \alpha_m,\bar{m} \cdot \gamma_m,\bar{m},1)^j \cdot \tau(b) \]

\[ - 2(i + 1) \eta. \]

Induction yields c.p.c. order zero maps
\[ \Phi_0, \ldots, \Phi_L : M_k \to A_\infty \cap B' \]

such that
\[ \Phi_L : M_k \to A_\infty \cap B' \]

satisfies
\[ \tau(\Phi_L(1_k)b) \geq \alpha_m,\bar{m} \cdot \gamma_m,\bar{m},1 \cdot \sum_{j=0}^L (1 - \alpha_m,\bar{m} \cdot \gamma_m,\bar{m},1)^j \cdot \tau(b) - 2L\eta \]

\[ (152),(153) \geq \left( 1 - \frac{\gamma}{3} \right) \cdot \tau(b) - \frac{\gamma}{3} \geq \tau(b) - \frac{2}{3}\gamma \]

for all positive contractions \( b \in B_+ \) and all \( \tau \in T_\infty(A) \).

Lift \( \Phi_L \) to a c.p.c. order zero map
\[ \tilde{\Phi} : M_k \to \prod_{N} A \]

(with c.p.c. order zero components \( \tilde{\Phi}_\nu, \nu \in \mathbb{N} \)), then it is straightforward to show that there is \( N \in \mathbb{N} \) such that
\[ \Phi := \tilde{\Phi}_N \]

has the desired properties. \( \square \)
Remark 5.12 If, in the preceding result, the elements of $\mathcal{E}$ are assumed to be positive, then the proof shows that we even obtain
\[
\tau(\Phi(1_k)b) \geq \tau(b) - \gamma
\]
for all $b \in \mathcal{E}$ and $\tau \in T(A)$.

6 Almost central dimension drop embeddings

The main result of this section is Proposition 6.5 (which in turn refines Proposition 6.3); this will provide the final missing ingredient to prove $\mathcal{Z}$-stability in very much the same way as Proposition 4.5 did in [40]. In fact, Proposition 6.5 is only a small modification of [40, 4.5], and it is proven in almost the same manner. The essential difference is that in [40] strong tracial $m$-comparison was provided by [40, Corollary 3.12], whereas in the present setup it is built into the hypotheses.

In the subsequent Sect. 7 we will then assemble our results to construct $\phi'$ and $v'$ as in Proposition 2.14, which in turn gives $\mathcal{Z}$-stability. While existence of the map $\phi'$ essentially follows from Lemma 5.11, the element $v'$ will be provided by Proposition 6.5.

6.1

Let us recall Proposition 4.1 from [40].

Proposition Let $A$ be a unital C*-algebra, and let $a \in A_+$ be a positive element of norm at most one satisfying $\tau(a) > 0$ for any $\tau \in T(A)$.

Then, for any $k \in \mathbb{N}$ and $0 < \beta < 1$ there is $\gamma > 0$ such that the following holds: If $h \in A_+$ is another positive element of norm at most one satisfying
\[
\tau(h) > 1 - \gamma
\]
for any $\tau \in T(A)$, then
\[
d_\tau((a^{\frac{1}{2}}(1_A - h)a^{\frac{1}{2}}) - \beta)_+ < \frac{1}{k} \cdot \tau(a)
\]
for any $\tau \in T(A)$.

6.2

The next result is essentially [40, Proposition 4.2]. There, the assumption of decomposition rank at most $m$ was only used to have [40, Corollary 3.12]
available. But that Corollary just says that decomposition rank at most \( m \) implies strong tracial \( m \)-comparison, which we now make a hypothesis; the conclusions are essentially the same. In the present setup we use two dimension constants, \( m \) and \( \bar{m} \), to emphasize the different roles played by dimension—these will eventually enter in terms of comparison and in terms of divisibility.

**Proposition** Let \( A \) be a separable, simple, unital \( C^* \)-algebra for which every quasitrace is a trace and which has strong tracial \( \bar{m} \)-comparison for some \( \bar{m} \in \mathbb{N} \). Let \( m, n \in \mathbb{N} \), let

\[
F = M_{r_1} \oplus \cdots \oplus M_{r_s}
\]

be a finite-dimensional \( C^* \)-algebra, and let

\[
\psi : F \to A
\]

be a c.p.c. order zero map. For any \( 0 < \bar{\gamma}_0 \) and \( 0 < \zeta < 1 \) there are \( 0 < \bar{\gamma}_1 \) and \( 0 < \bar{\beta} \) such that the following holds:

If \( i_0 \in \{0, \ldots, m\} \) and

\[
\Phi : M_{m+1} \otimes M_n \otimes M_2 \to A
\]

is a c.p.c. order zero map satisfying

\[
\|[\psi(x), \Phi(y)]\| \leq \bar{\beta} \|x\| \|y\|
\]

for any \( x \in F \) and \( y \in M_{m+1} \otimes M_n \otimes M_2 \) and

\[
\tau(\Phi(1_{M_{m+1} \otimes M_n \otimes M_2})) > 1 - \bar{\beta},
\]

(164)

then there are \( \tilde{v}(j) \in A, \ j = 1, \ldots, s \), of norm at most one with the following properties:

(i) \( \| (\tilde{v}(j)^* - \psi(j)(f_{11}^{(j)})^{1/2}(1 - \Phi(1_{M_{m+1} \otimes M_n \otimes M_2}))\psi(j)(f_{11}^{(j)})^{1/2}) \| < \bar{\gamma}_0 \)

(ii) \( \| [\tilde{v}(j), \psi(j)(f_{11}^{(j)})] \| < \bar{\gamma}_0 \)

(iii) \( \| \tilde{v}(j) - \tilde{g}_0(\psi(j)(f_{11}^{(j)}))\tilde{v}(j) \| < \bar{\gamma}_0 \)

(iv) \( \tilde{v}(j) = g_{\zeta, \xi + \bar{\gamma}_1}(\Phi(d_{i_0i_0} \otimes e_{11} \otimes 1_{M_2}))\tilde{v}(j) \).

Here, \( \{f_{ij}^{(j)} \mid j = 1, \ldots, s; i, i' = 1, \ldots, r_j\} \) and \( \{d_{ii'} \mid i, i' = 1, \ldots, m + 1\} \) denote sets of matrix units for \( F \) and \( M_{m+1} \), respectively; the canonical matrix units for \( M_n \) and \( M_2 \) are both denoted by \( \{e_{ii'}\} \).

**Proof** The proof is the same as that of [40, 4.2] almost verbatim, with only three changes:
In (200) of [40] we take \((m + 1)(\tilde{m} + 1)\) in place of \((m + 1)^2\), in (212) of [40] we take \(\frac{1}{m+1}\) in place of \(\frac{1}{m+1}\), and instead of using [40, Corollary 3.12] on page 285 in [40], we use strong tracial \(\tilde{m}\)-comparison of \(A\). □

6.3

The next result is a slight variation of [40, Proposition 4.3], in pretty much the same manner as Proposition 6.2 varies [40, Proposition 4.2].

**Proposition** Let \(A\) be a separable, simple, unital \(\mathrm{C}^\ast\)-algebra for which every quasitrace is a trace and which has strong tracial \(\tilde{m}\)-comparison for some \(\tilde{m} \in \mathbb{N}\). Let \(m \in \mathbb{N}\), let \(F\) be a finite-dimensional \(\mathrm{C}^\ast\)-algebra, \(n \in \mathbb{N}\) and \(\psi : F \to A\) a c.p.c. order zero map. For any \(0 < \theta\) and \(0 < \zeta < 1\) there is \(\beta > 0\) such that the following holds:

If \(i_0 \in \{0, \ldots, m\}\) and \(\Phi : M_{m+1} \otimes M_n \otimes M_2 \to A\) is a c.p.c. order zero map satisfying

\[
\|[\psi(x), \Phi(y)]\| \leq \beta \|x\| \|y\| \quad (165)
\]

for any \(x \in F\) and \(y \in M_{m+1} \otimes M_n \otimes M_2\) and

\[
\tau(\Phi(1_{M_{m+1} \otimes M_n \otimes M_2})) > 1 - \beta \forall \tau \in T(A),
\]

then there is \(v \in A\) of norm at most one such that

\[
\|v^*v - (1_A - \Phi(1_{M_{m+1} \otimes M_n \otimes M_2}))\frac{1}{2}\psi(1_F)(1_A - \Phi(1_{M_{m+1} \otimes M_n \otimes M_2}))\| < \theta,
\]

\[
v^*v \in \left(\Phi(d_{i_0i_0} \otimes e_{11} \otimes 1_{M_2}) - \zeta\right)_+ A \left(\Phi(d_{i_0i_0} \otimes e_{11} \otimes 1_{M_2}) - \zeta\right)_+
\]

(where \(\{d_{kl} \mid k, l = 0, \ldots, m\}\) and \(\{e_{kl} \mid k, l = 1, \ldots, n\}\) denote sets of matrix units for \(M_{m+1}\) and \(M_n\), respectively) and

\[
\|[\psi(x), v]\| \leq \theta \|x\|
\]

for all \(x \in F\).

**Proof** The proof is essentially that of [40, Proposition 4.3]. The main difference is that Proposition 6.2 is used instead of [40, Proposition 4.2], but I mention another necessary adjustment, pointed out to me by D. Archey:
The elements $v^{(j)}$ and $v$ defined in [40, equations (241) and (242)] are not necessarily in $A$; they should be replaced by
\[
\tilde{v}^{(j)} := \sum_{k=1}^{r^{(j)}} \pi^{(j)}(f_{k1}^{(j)})g_0, \tilde{\gamma}_1(\psi^{(j)}(f_{11}^{(j)}))\tilde{v}^{(j)}g_0, \tilde{\gamma}_1(\psi^{(j)}(f_{11}^{(j)}))\pi^{(j)}(f_{1k}^{(j)})
\]
and
\[
\tilde{v} := g_{\xi,\xi} + \tilde{\gamma}_1(\Phi(d_{ij}|_0 \otimes e_{11} \otimes 1_{M_2})) \sum_{j=1}^{s} \tilde{v}^{(j)}.
\]
Then, the $\tilde{v}^{(j)}$ and $\tilde{v}$ clearly are in $A$, and one checks that $\|v^{(j)} - \tilde{v}^{(j)}\| < 2\tilde{\gamma}_0^{1/2}$ and $\|v - \tilde{v}\| < 2s\tilde{\gamma}_0^{1/2}$ (using [40, 4.2(i) and 4.2(iii)]). The rest of the proof works in the same manner, provided one chooses $0 < \tilde{\gamma}_0 < 1$ such that $16s \max\{r^{(j)}\}^{1/2} < \theta$ (cf. [40, inequality (236)]).

\[\square\]

6.4

Below we essentially repeat [40, Proposition 4.4], with the additional observation that there it would not have been quite necessary to assume all of $A$ (as opposed to a C∗-subalgebra $B$) to have finite decomposition rank, and that one can replace decomposition rank by nuclear dimension with only a little bit of extra effort.

**Proposition** Let $A$ be a simple, separable, unital C∗-algebra, and let $B \subset A$ be a unital C∗-subalgebra with $\dim_{nu} B \leq m < \infty$. Given a finite subset $F \subset B$, a positive normalized function $\tilde{h} \in C_0((0,1])$ and $\delta > 0$, there are a finite subset $G \subset B$ and $\alpha > 0$ such that the following holds:

Suppose
\[
(F = F^{(0)} \oplus \cdots \oplus F^{(m)}, \sigma, \varrho)
\]
is an $m$-decomposable c.p. approximation (for $B$) of $G$ to within $\alpha$, with c.p.c. maps $\sigma : B \rightarrow F$ and $\varrho^{(i)} : F^{(i)} \rightarrow B$, and such that the composition $\varrho \sigma$ is contractive. For $i = 0, \ldots, m$, let $v_i \in A$ be normalized elements satisfying
\[
\|[\varrho^{(i)}(x), v_i]\| \leq \alpha \|x\| \quad (166)
\]
for all $x \in F^{(i)}$.

Then,
\[
v := \sum_{i=0}^{m} v_i \tilde{h}(\varrho^{(i)}(\sigma^{(i)}(1_A))) \quad (167)
\]

\[\square\]
satisfies
\[ \| [v, a] \| < \delta \]
for all \( a \in \mathcal{F} \).

Proof The proof is essentially the same as that of [40, Proposition 4.4], but since there are several modifications, we revisit the full proof in detail.

We may assume the elements of \( \mathcal{F} \) to be positive and normalized. For convenience, we set
\[ \tilde{\delta} := \frac{\delta}{9(m + 1)}. \] (168)
Take \( \tilde{h} \in C([0, 1]) \) such that
\[ \| \text{id}_{[0, 1]} \cdot \tilde{h} - \bar{h} \| < \tilde{\delta}. \] (169)
By using [40, Proposition 1.8], there is
\[ 0 < \beta < \frac{\tilde{\delta}}{\| \tilde{h} \|} \] (170)
such that, whenever \( b, c \in A \) are elements of norm at most one with \( c \) positive, and satisfying
\[ \| [b, c] \| \leq \beta, \] (171)
we have
\[ \| [b, \tilde{h}(c)] \| \leq \bar{\delta}. \] (172)
Using [40, Lemma 1.10], we find \( \alpha > 0 \) and a finite subset \( \mathcal{G} \subset B \) such that, whenever \((F, \hat{\sigma}, \hat{\varrho})\) is a c.p.c. approximation (for \( B \)) of \( \mathcal{G} \) to within \( \alpha \), we have
\[ \| \hat{\varrho}(x)\hat{\sigma}(a) - \hat{\varrho}(x\hat{\sigma}(a)) \| \leq \frac{\beta}{2} \| x \| \] (173)
for all \( x \in F \) and \( a \in \mathcal{F} \). We may assume that
\[ \alpha < \beta, \tilde{\delta}, \frac{\tilde{\delta}}{\| \tilde{h} \|}. \] (174)
Now let a c.p. approximation \((F, \sigma, \varrho)\) (for \( B \)) and \( v_i \in A \) as in the proposition be given. Define maps
\[ \hat{\sigma}(\cdot) := \sigma(1_A)^{-1/2} \sigma(\cdot) \sigma(1_A)^{-1/2} \]
and
\[
\hat{\varrho}(\cdot) := \varrho(\sigma(1_A)^{1/2} \cdot \sigma(1_A)^{1/2})
\]
(with inverses taken in the respective hereditary subalgebra), then
\[
\hat{\varrho}^{(i)} \hat{\sigma}^{(i)} = \varrho^{(i)} \sigma^{(i)}
\]
for all \(i\) and \((F, \hat{\sigma}, \hat{\varrho})\) is a c.p.c. approximation of \(G\) to within \(\alpha\).

Note that we have
\[
\| \tilde{h}(\varrho^{(i)}(\sigma^{(i)}(1_A))) - \varrho^{(i)}\sigma^{(i)}(1_A) \tilde{h}(\varrho^{(i)}(1_F^{(i)})) \| \\
\leq \| \pi_{\varrho^{(i)}}(\sigma^{(i)}(1_A)) \tilde{h}(\varrho^{(i)}(1_F^{(i)})) - \varrho^{(i)}\sigma^{(i)}(1_A) \tilde{h}(\varrho^{(i)}(1_F^{(i)})) \| \\
+ \| \varrho^{(i)}\sigma^{(i)}(1_A) \tilde{h}(\varrho^{(i)}(1_F^{(i)})) \| + \tilde{\delta}
\]
and
\[
\| \varrho^{(i)}\sigma^{(i)}(1_A) \varrho\sigma(a) - \varrho^{(i)}\sigma^{(i)}(a) \| \\
\leq \frac{\bar{\delta}}{\| \tilde{h} \|}
\]
for \(i = 0, \ldots, m\) and \(a \in F\) (the elements of \(F\) are normalized). We obtain
\[
\|[v_i \tilde{h}(\varrho^{(i)})(\sigma^{(i)}(1_A))), a]\| \\
\leq \| [v_i \varrho^{(i)}\sigma^{(i)}(1_A) \tilde{h}(\varrho^{(i)}(1_F^{(i)}))), \varrho\sigma(a)]\| + 2\alpha + \bar{\delta}
\]
\[
\leq \| [v_i \varrho^{(i)}\sigma^{(i)}(1_A) \tilde{h}(\varrho^{(i)}(1_F^{(i)}))), \varrho\sigma(a)]\| + 2\alpha + \bar{\delta}
\]
\[
\leq \| [v_i \varrho^{(i)}\sigma^{(i)}(1_A) \tilde{h}(\varrho^{(i)}(1_F^{(i)}))), \varrho\sigma(a)]\| + 2\alpha + 3\bar{\delta}
\]
\[
\leq \| [v_i \tilde{h}(\varrho^{(i)}(1_F^{(i)}))), \varrho^{(i)}\sigma^{(i)}(a)]\| + 2\alpha + 5\bar{\delta}
\]
\[
\leq \frac{\bar{\delta}}{m + 1}
\]
for $i = 0, \ldots, m$ and $a \in \mathcal{F}$. It follows that

$$
\|[v, a]\| \overset{(167)}{=} \left\| \sum_{i=0}^{m} v_i \tilde{h}(\varrho^{(i)}(\sigma^{(i)}(1_A)), a) \right\| < (m + 1) \frac{\delta}{m + 1} = \delta
$$

for $a \in \mathcal{F}$.

6.5

We are now prepared to state the main technical result of this section, which again is a modification of a result from [40], Proposition 4.5.

**Proposition** Let $A$ be a separable, simple, unital $C^*$-algebra with strong tracial $\bar{m}$-comparison for some $\bar{m} \in \mathbb{N}$ and such that every quasi-trace on $A$ is a trace. Let $B \subset A$ be a unital $C^*$-subalgebra with $\dim_{\text{nuc}} B \leq m < \infty$. Given a finite subset $\mathcal{F} \subset B$, $0 < \delta$, $0 < \zeta < 1$ and $n \in \mathbb{N}$, there are $\mathcal{G} \subset B$ finite and $\alpha > 0$ such that the following holds:

Whenever $(F, \sigma, \varrho)$ is an $m$-decomposable c.p.c. approximation (for $B$) of $\mathcal{G}$ to within $\alpha$, with c.p.c. maps $\sigma : B \to F$ and $\varrho^{(i)} : F^{(i)} \to B$, and such that the composition $\varrho \sigma$ is contractive, there is $\gamma > 0$ satisfying the following: If

$$
\Phi : M_{m+1} \otimes M_n \otimes M_2 \to A
$$

is a c.p.c. order zero map satisfying

$$
\|[\varrho(x), \Phi(y)]\| \leq \gamma \|x\| \|y\| \quad (178)
$$

for all $x \in F$, $y \in M_{m+1} \otimes M_n \otimes M_2$ and

$$
\tau(\Phi(1_{M_{m+1} \otimes M_n \otimes M_2})) \geq 1 - \gamma \quad (179)
$$

for all $\tau \in T(A)$, then

$$
\|[\Phi(y), a]\| \leq \delta \|y\|
$$

for all $y \in M_{m+1} \otimes M_n \otimes M_2$, $a \in \mathcal{F}$, and there is $v \in A$ such that

$$
\|v^* v - (1_A - \Phi(1_{M_{m+1} \otimes M_n \otimes M_2}))\| < \delta,
$$

$$
v v^* \in (\Phi(1_{M_{m+1} \otimes e_{11} \otimes M_2}) - \zeta) + A(\Phi(1_{M_{m+1} \otimes e_{11} \otimes M_2}) - \zeta)_+
$$

and

$$
\|[a, v]\| < \delta
$$

for all $a \in \mathcal{F}$.
Proof Again, the proof is essentially the same as that of [40, Proposition 4.5], but since there are some modifications and to avoid confusion, we state the full proof explicitly.

We may clearly assume that the elements of $\mathcal{F}$ are normalized. Set

$$\tilde{h} := g_{0,\delta/(6(m+1))}$$  \hspace{1cm} (180)

and take $\tilde{h} \in C([0, 1])$ such that

$$\text{id}_{[0, 1]} \cdot \tilde{h} = \bar{h};$$  \hspace{1cm} (181)

note that

$$\|\tilde{h}\| = \frac{6(m+1)}{\delta}.$$  

From Proposition 6.4, obtain a finite subset $\mathcal{G} \subset B$ and $\alpha > 0$ such that the assertion of 6.4 holds. We may assume that

$$\alpha < \frac{\delta}{6}$$  \hspace{1cm} (182)

and that

$$1_A \in \mathcal{G}.$$  \hspace{1cm} (183)

Now suppose

$$(F, \sigma, \varrho)$$  \hspace{1cm} (184)

is a c.p. approximation (for $B$) of $\mathcal{G}$ to within $\alpha$ as in the proposition, such that $\varrho$ is $m$-decomposable with respect to $F = F^{(0)} \oplus \cdots \oplus F^{(m)}$.

Similarly as in (176), observe that

$$\tilde{h}(\varrho^{(i)})(\sigma^{(i)}(1_A)) = \varrho^{(i)}(\sigma^{(i)}(1_A))\tilde{h}(\varrho^{(i)}(1_{F^{(i)}})).$$  \hspace{1cm} (185)

Choose some

$$0 < \theta < \frac{\alpha}{3(m+1)}.$$  \hspace{1cm} (186)

Obtain $\beta > 0$ from Proposition 6.3 so that the assertion of 6.3 holds for each

$$\varrho^{(i)} := \varrho|_{F^{(i)}},$$  

$i = 0, \ldots, m$, in place of $\psi$. Using [40, Proposition 1.8], we may choose some

$$0 < \gamma < \min\{\theta, \beta\}$$  \hspace{1cm} (187)
such that, if \( b \) and \( c \) are elements of norm at most one in some \( \mathbb{C}^* \)-algebra, with \( c \geq 0 \) and

\[
\|[b, c]\| < \gamma,
\]

then

\[
\|[b^\frac{1}{2}, c]\|, \|[b^\frac{1}{2}, \tilde{h}(c)]\|, \|[b^\frac{1}{2}, c^\frac{1}{2}]\| < \theta \cdot \frac{\delta}{6(m + 1)}.
\] (188)

Now if \( \Phi \) is as in the assertion of Proposition 6.5, then (note (187)) Proposition 6.3 yields elements \( v_i \in A \) of norm at most one, \( i = 0, \ldots, m \), such that

\[
\|v_i^* v_i - (1_A - \Phi(1_{M_{m+1} \otimes M_n \otimes M_2})^\frac{1}{2} Q^{(i)}(1_F^{(i)})(1_A - \Phi(1_{M_{m+1} \otimes M_n \otimes M_2})^\frac{1}{2})\| < \theta,
\] (189)

\[
v_i v_i^* \in (\Phi(d_{ii} \otimes e_{11} \otimes 1_{M_2}) - \zeta)_+ A (\Phi(d_{ii} \otimes e_{11} \otimes 1_{M_2}) - \zeta)_+
\] (190)

and

\[
\|\tilde{h}(Q^{(i)}(x), v_i)\| \leq \theta \|x\| \quad \text{for all } x \in F^{(i)}
\] (191)

for each \( i \). Note that

\[
v_i v_i^* \perp v_{i'} v_{i'}^*
\] (192)

by (190) if \( i \neq i' \), since \( \Phi \) has order zero. Set

\[
v := \sum_{i=0}^{m} v_i \tilde{h}(Q^{(i)})(\sigma^{(i)}(1_A)),
\] (193)

then

\[
\|v^* v - (1_A - \Phi(1_{M_{m+1} \otimes M_n \otimes M_2}))\|
\] (193),(192)

\[\leq \sum_{i=0}^{m} \tilde{h}(Q^{(i)})(\sigma^{(i)}(1_A)) v_i^* v_i \tilde{h}(Q^{(i)})(\sigma^{(i)}(1_A))
\]

\[- (1_A - \Phi(1_{M_{m+1} \otimes M_n \otimes M_2}))\|
\] (189),(183)

\[\leq \sum_{i=0}^{m} \tilde{h}(Q^{(i)})(\sigma^{(i)}(1_A))(1_A - \Phi(1_{M_{m+1} \otimes M_n \otimes M_2})^\frac{1}{2} Q^{(i)}(1_F^{(i)}))
\]

\[- (1_A - \Phi(1_{M_{m+1} \otimes M_n \otimes M_2})^\frac{1}{2} \tilde{h}(Q^{(i)})(\sigma^{(i)}(1_A))
\]

\[- (1_A - \Phi(1_{M_{m+1} \otimes M_n \otimes M_2})^\frac{1}{2} Q^{(i)}(1_A))
\]
\[
\cdot (1_A - \Phi(1_{M_{m+1} \otimes M_n \otimes M_2}))^{\frac{1}{2}}
+ (m + 1)\theta + \alpha
\]

\[(178),(188),(185)\leq \frac{m}{\sum_{i=0} m (1_A - \Phi(1_{M_{m+1} \otimes M_n \otimes M_2}))^{\frac{1}{2}}}
\cdot (\tilde{h}(Q^{(i)}(\sigma^{(i)}(1_A))Q^{(i)}(1_{F^{(i)}}))\tilde{h}(Q^{(i)}(\sigma^{(i)}(1_A)))
- Q^{(i)}\sigma^{(i)}(1_A))
\cdot (1_A - \Phi(1_{M_{m+1} \otimes M_n \otimes M_2}))^{\frac{1}{2}}
+ (m + 1)\theta + \alpha + 4(m + 1)\theta
\]

\[(185),(181),(180)\leq (m + 1)\theta + \alpha + 4(m + 1)\theta + 2(m + 1) \frac{\delta}{6(m + 1)}
\]

\[(186),(182)\leq \delta.
\]

Moreover,

\[
u v^* (193),(190)\in \operatorname{her} \left( \sum_{i=0}^m (\Phi(d_{ii} \otimes e_{11} \otimes 1_{M_2}) - \zeta)_+ \right)
\]

\[2.1 \equiv \operatorname{her}((\Phi(1_{M_{m+1} \otimes e_{11} \otimes 1_{M_2}} - \zeta)_+).
\]

We have

\[\|[v, a]\| < \delta \text{ for all } a \in \mathcal{F}\]

by Proposition 6.4 and our choice of \(\alpha\), using (191) and (186). Finally, for \(y \in M_{m+1} \otimes M_n \otimes M_2\) and \(a \in \mathcal{F}\) we have

\[\|[\Phi(y), a]\| \leq \|[\Phi(y), \varphi \sigma(a)]\| + 2\alpha\|y\|
\]

\[\leq \gamma\|y\| + 2\alpha\|y\|
\]

\[\leq \delta\|y\|.
\]

\[\square\]
7 The main result and its consequences

We are now in a position to prove the main result and its corollaries. As was already mentioned, the structure of the proof of Theorem 7.1 is similar to that of [40, Theorem 5.1].

**Theorem 7.1** Let $A$ be a separable, simple, unital $C^*$-algebra with locally finite nuclear dimension; suppose $A$ is tracially $(\bar{m}, \tilde{m})$-pure for some $\bar{m}, \tilde{m} \in \mathbb{N}$.

Then, $A$ is $\mathcal{Z}$-stable.

**Proof** We check that $A$ satisfies the hypotheses of Proposition 2.14. This follows very much the same pattern as the proof of [40, Theorem 5.1].

So, let $n \in \mathbb{N}$, $\mathcal{F} \subset A$ finite and $0 < \delta < 1$ and $0 < \zeta < 1$ be given. We have to find $\psi' : M_n \to A$ and $v' \in A$ as in 2.14.

Since $A$ has locally finite nuclear dimension, we may assume that $\mathcal{F} \subset B$, where $B \subset A$ is a unital $C^*$-subalgebra with $\dim_{\text{nuc}} B = m < \infty$. Moreover, by Remark 4.3 every quasitrace on $A$ is a trace.

By Proposition 6.5, there are a finite subset $1_A \in \mathcal{G} \subset B$ and $\alpha > 0$, such that, if $(F, \sigma, \varrho)$ is an $m$-decomposable c.p. approximation (for $B$) of $\mathcal{G}$ to within $\alpha$ as in 6.5, there is $\gamma > 0$ such that the following holds: If

$$\Phi : M_{m+1} \otimes M_n \otimes M_2 \to A$$

is a c.p.c. order zero map satisfying (178) and (179) of 6.5, then there is $v' \in A$ which, together with

$$\psi' := \Phi|_{1_{M_{m+1}} \otimes M_n \otimes 1_{M_2}},$$

satisfies the hypotheses of 2.14.

So, choose an $m$-decomposable c.p. approximation $(F, \sigma, \varrho)$ (for $B$) of $\mathcal{G}$ to within $\alpha$ as in Proposition 6.5. The existence of $\Phi$ now follows from Lemma 5.11 with $(m + 1) \cdot n \cdot 2$ in place of $k$ and $\mathcal{E} := \varrho(B_1(F))$ (where $B_1(F)$ denotes the unit ball of $F$). \(\square\)

7.2

Let us note the following combination of Corollary 3.10 and Theorem 7.1 explicitly.

**Corollary** Let $A$ be a separable, simple, unital $C^*$-algebra with locally finite nuclear dimension. Suppose $A$ is $(m, \tilde{m})$-pure for some $m, \tilde{m} \in \mathbb{N}$. (For example, $A$ could be pure.)

Then, $A$ is $\mathcal{Z}$-stable.
Corollary 7.3 Let $A$ be a separable, simple, nonelementary, unital $C^*$-algebra with finite nuclear dimension.

Then, $A$ is $\mathcal{Z}$-stable.

Proof This combines Corollary 4.9 and Theorem 7.1.

Corollary 7.4 Let $A$ be a separable, simple, nonelementary, unital $C^*$-algebra with locally finite nuclear dimension.

Then, $A \cong A \otimes \mathcal{Z}$ if and only if $W(A) \cong W(A \otimes \mathcal{Z})$.

Proof If $W(A) \cong W(A \otimes \mathcal{Z})$, then $A$ is pure by Proposition 3.7, so $A$ is $\mathcal{Z}$-stable by Corollary 7.2. The forward implication is trivial.

Corollary 7.5 Let $A$ be a separable, simple, nonelementary, unital, approximately subhomogeneous $C^*$-algebra.

Then, $A$ is $\mathcal{Z}$-stable if and only if it has slow dimension growth (in the sense of [30]).

Proof If $A$ has slow dimension growth, we have $W(A) \cong W(A \otimes \mathcal{Z})$ by [30, Theorem 1.2], so $A$ is $\mathcal{Z}$-stable by the preceding Corollary. ($A$ has locally finite decomposition rank, hence locally finite nuclear dimension, see [22].) Conversely, if $A$ is $\mathcal{Z}$-stable, it has slow dimension growth by [30].

Corollary 7.6 Let $\mathcal{A}$ denote the class of separable, simple, nonelementary, unital, approximately subhomogeneous $C^*$-algebras with slow dimension growth and for which projections separate traces.

Then, $\mathcal{A}$ is classified by ordered $K$-theory.

Proof The members of $\mathcal{A}$ are $\mathcal{Z}$-stable by Corollary 7.5, so they are classified by [21, Corollary 5.5] (cf. also [38, Corollary 8.3]).

7.7

Let us note the following corollary explicitly. This is mostly repeating known results; the only new implication is (iii)$\Rightarrow$(iv), which follows from Corollary 7.5 and settles an important technical question left open in [11].

Corollary Let $A$ be a separable, simple, nonelementary, unital, approximately homogeneous $C^*$-algebra.

Then, the following are equivalent:

(i) $A$ has no dimension growth (as an AH algebra)
(ii) $A$ has very slow dimension growth (as an AH algebra)
(iii) $A$ has slow dimension growth (as an AH algebra)

(iv) $A$ has slow dimension growth (as an AH algebra)
(iv) $A$ is $\mathcal{Z}$-stable.

(v) $A$ has finite decomposition rank.

The class of such $C^*$-algebras is classified by the Elliott invariant.

Proof Implications (i)$\Rightarrow$(ii)$\Rightarrow$(iii) and (i)$\Rightarrow$(v) are trivial; (iii)$\Rightarrow$(iv) follows from Corollary 7.5, and (v)$\Rightarrow$(iv) is (a special case of) [40, Theorem 5.1]. The classification result for $\mathcal{Z}$-stable AH algebras was obtained in [20]; this also yields (iv)$\Rightarrow$(i), cf. [20, Corollary 11.13].

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