Schwinger–Dyson equations and line integrals

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Abstract
The complex Langevin (CL) method sometimes shows convergence to the wrong limit, even though the Schwinger–Dyson equations (SDE) are fulfilled. We analyze this problem in a more general context for the case of one complex variable. We prove a theorem that shows that under rather general conditions not only the equilibrium measure of CL but any linear functional satisfying the SDE on a space of test functions is given by a linear combination of integrals along paths connecting the zeroes of the underlying measure and noncontractible closed paths. This proves rigorously a conjecture stated long ago by one of us (L. L. S.) and explains a fact observed in nonergodic cases of CL.

Keywords: sign problem, complex Langevin, Schwinger–Dyson equations

1. Introduction

The notorious sign problem arises in many contexts, for instance in the functional integral of real time quantum mechanics or in imaginary time quantum field theory at nonzero chemical potential. The problem comes from the fact that even in finite dimensional approximations we are dealing with a complex measure whose sign, or rather phase is fluctuating wildly.

A popular method to attack this problem is the complex Langevin method [1, 2]. It is well known that the method, even when it converges, i.e. defines a stationary probability measure on the (complexified) configuration space, may not reproduce the desired integrals based on a complex measure.

In this paper we try to give an answer to the question: what does such an ‘incorrect’ stationary measure represent then? It is known that the expectation values under stationary measures obtained from a Complex Langevin process obey certain equations closely related to (but a little weaker than) the so-called Schwinger–Dyson equations characterizing the original complex measure.
In order to find an answer we look at a slightly more general problem: how can we characterize linear functionals on the space of ‘observables’ (functions on configuration space) that satisfy the Schwinger–Dyson equations? It has been conjectured long ago by one of us [3] that those functionals are always linear combinations of line integrals, where the lines are connecting zeroes of the original complex density or are noncontractible closed loops.

In this paper we rigorously prove this conjecture for the one-dimensional case under some rather general conditions.

2. Motivation

The purpose of this section is to motivate the theorem of section 3, so it is of descriptive character. Precise definitions are postponed to sections 3 and 4. A one-dimensional setting is assumed throughout. For definiteness we discuss the nonperiodic setting although everything can be repeated for the periodic case.

Let $\rho(x)$ be a complex distribution (to be referred to as density) defined on the real line such that $\int \rho(x) \, dx = 1$. Given a complex test function $f(x)$, to extract ‘expectation values’ of the type

$$\langle f \rangle = \int_{\mathbb{R}} \rho(x) f(x) \, dx$$

by means of a Monte Carlo approach constitutes the well-known sign (or in the present case phase) problem, since plain importance sampling does not apply to a complex weight $\rho$.

The Schwinger–Dyson equations (SDE) characterize the density $\rho$ via the simple identity

$$\langle fv \rangle + \langle f' \rangle = 0,$$

where

$$v(x) = \frac{\rho'(x)}{\rho(x)}.$$  \hspace{1cm} (3)

(2) follows from integration by parts under some weak decay assumptions.

One way to attack the sign problem when $\rho$ and $f$ admit a holomorphic extension to $\mathbb{C}$ (possibly with isolated singularities or branching points for $\rho$) is the complex Langevin (CL) approach [1, 2]. In its simplest version an ensemble of walkers move on $\mathbb{C}$ with $v(x + iy)$ as drift and with real Gaussian noise normalized such that the probability density of walkers, $P(x, y; t)$, follows a Fokker–Planck equation

$$\frac{\partial P(x, y; t)}{\partial t} = \frac{\partial^2 P}{\partial x^2} - \frac{\partial v_x P}{\partial x} - \frac{\partial v_y P}{\partial y},$$

where $v = v_x + iv_y$, and $t$ is the evolution time parameter.

When the stochastic process reaches a stationary state, $P(x, y) \equiv P(x, y; \infty)$ defines a linear form $T_P$ on a domain of sufficiently well-behaved holomorphic test functions as

$$\langle T_P f \rangle = \int_{\mathbb{C}} dz \, P(x, y) f(z).$$

The equations refer to $\langle \cdot \rangle$, not $f$. The SDE is the statement that the linear form defined through $f \mapsto \langle (\partial_t + v)f \rangle$ vanishes identically for some suitably chosen domain of test functions.
Multiplying the stationary Fokker–Planck equation by \( f(z) \) and using the Cauchy–Riemann equations for \( f \), it is straightforward to establish the relation
\[
0 = P(\partial_z + v)\partial_z f + \partial_x A_x + \partial_y A_y.
\]
(6)
The explicit form of \( A_x \) and \( A_y \) is also easily obtained but not needed here. Therefore, when these boundary terms do not contribute upon integration on \( \mathbb{C} \), the linear form \( T_P \) fulfills the equations
\[
0 = (T_P, (\partial_z + v)\partial_z f).
\]
(7)
(7) also follows from using Ito calculus on \( f(z(t)) \), averaging over the noise and sending \( t \to \infty \) (see for instance [4]). They characterize the equilibrium behavior, i.e. that expectation values do not change under the CL process, so we call them ‘convergence conditions’ (CC) (see [5]).

The boundary terms may refer to \( \infty \) as an isolated point on the extended complex plane and also to isolated singularities on \( \mathbb{C} \). In applications the boundary terms are not guaranteed to drop [5, 6], but they do in some cases such as \( \rho(z) \) of the type Gaussian times polynomial, and others [7, 8].

Clearly the linear form \( T_P \) defined by
\[
(T_P, f) = \langle f \rangle
\]
not only complies with the the SDE, but the CC as well, for sufficiently convergent \( \rho \) and \( f \):
\[
\langle (\partial_z + v)\partial_z f \rangle = \int_R \rho(f'' + \frac{\rho'}{\rho} f') = \int_R (\rho f')' = \rho f' \bigg|_{-\infty}^{+\infty}.
\]
(9)
Obviously the SDE imply the CC, provided the test function space is closed under taking derivatives. The converse is true if the test function space contains with each function also its primitive, for instance if it consists of the polynomials; an informal argument for this was given in [5]. For the periodic case this argument does not hold, since the constant function is not the derivative of a periodic function, so the space of linear functionals satisfying the CC has one dimension more than the one satisfying the SDE. In this paper we will, however, always assume that the SDE hold.

The idea of the CL approach is that, provided the boundary terms can be dropped and the solution of the CC is unique, \( T_P \) must coincide with \( T_P \) (for sufficient conditions guaranteeing this see [9]). Hence, the expectation values \( \langle f \rangle \) can be computed as averages of \( f(z) \) weighted with \( P(x, y) \), thereby solving the sign problem. For a more detailed justification of CL, involving finite evolution times and possible boundary terms arising there, see [5].

As follows from (1), the form \( T_P \) relies upon integration along the real axis of \( \rho f \). There are well-known instances in CL in which the integral does not use the whole real range. For instance
\[
\rho(x) \propto (x - a)e^{-x^2/2}
\]
(10)
for real \( a \). If the stochastic process starts at \( x_0 \in \mathbb{R} \) it remains real and the segregation theorem applies [10], one stationary solution \( P_+(z) \propto \theta(x - a)\rho(x)\delta(y) \) is obtained if \( x_0 > a \) while another \( P_-(z) \propto \theta(a - x)\rho(x)\delta(y) \) obtains if \( x_0 < a \). Therefore the stochastic process is not ergodic [8, 11]. The two corresponding linear forms \( T_{P \pm} \) can be expressed as
\[
(T_{P \pm}, f) = \mathcal{N}_\pm \int_{\gamma_{\pm}} \rho(z)f(z)
\]
(11)
where $\gamma_{\pm}$ are the paths from $a$ to $\pm\infty$ along the real axis, and $N_{\pm}$ takes care of the normalization. Both linear forms are solutions of the SDE as is readily verified and $T_{\rho}$ can be recovered as a linear combination of them.

Numerical experiments presented in [3] for this $\rho$ indicate that the linear form $T_{\rho}$ obtained from CL is a linear combination of $T_{\rho_{\pm}}$ when $a$ lies outside the real axis, even though the stochastic process is ergodic in this case.

Coming back to a generic $\rho$, one can generalize the integral over $\mathbb{R}$ present in $\langle f \rangle$ to other continuous paths $\gamma$ on $\mathbb{C}$ and define the corresponding linear form as

$$ (T_{\gamma}, f) = \int_{\gamma} d\gamma \rho(z) f(z). $$

Due to Cauchy’s theorem, $T_{\gamma}$ does not depend on deformations of the path, provided no singularities are crossed.

Hence, there are two cases in which the rhs vanishes and $T_{\gamma}$ fulfills the SDE. One is when $\gamma$ connects two zeroes of $\rho$, and such zeroes are not overruled by the factor $f$. This is always true for finite zeroes and entire holomorphic $f$. For a zero at $\infty$, this condition puts restrictions on the space of test functions. The other case is when $\gamma$ is a closed path; the form $T_{\gamma}$ is non null if (non removable) singularities of $\rho$ are enclosed by the path. Denoting $\Gamma$ the set of paths of the two types just noted, it is clear that (finite) linear combinations

$$ T = \sum_{\gamma \in \Gamma} a_\gamma T_{\gamma} $$

also satisfy the SDE.

Based partially on numerical experiments with CL simulations and partially on the naturalness of the linear forms $T_{\gamma}$ in the present context of holomorphic weights and test functions, it was conjectured in [3] that the space $T$ of linear forms defined in (14) actually saturates the set of solutions of the SDE for a given complex density $\rho$.

To illustrate the idea we can use a density analyzed in [8] in the light of CL, namely,

$$ \rho(z) = N(z-i)^2 e^{-\beta z^2}, \quad \beta = 1.6, \quad N = -0.9634. $$

Let $\gamma_{\pm}$ be paths starting at $z = i$ and ending at $z = \pm\infty$. These generate the set $\Gamma$ of paths. One can define the linear forms

$$ (T_{\pm}, f) = \int_{\gamma_{\pm}} d\gamma (z-i)^2 e^{-\beta z^2} f(z) = \int_{0}^{\pm\infty} dt t^2 e^{-\beta (t+i)^2} f(t+i), $$

with normalizations $(T_{\pm}, 1) = \mp 0.4817 - i 0.2228$. These form a basis of the space $T$ of linear forms. Let $\hat{T}_{\pm}$ denote the corresponding normalized versions.

In table 1 ‘expectation values’ of the type $(T, f)$ are displayed for several choices of $T$ and $f$. The column labeled CL corresponds to $T_{\rho}$ of the CL numerical stochastic process. The labels of the other columns are self-explanatory. Several ‘observables’ $f$ are computed, namely, $x^m$ for $m = 1, 2, 3, 4$, and $e^{ikx}$ for $k = \pm 1, \pm 2$. The CL expectation values are reproduced with

4 In this case the paths $\gamma_{\pm}$ go from $a$ to $\pm\infty$ parallel to the real axis.

5 Here we are assuming that when $\gamma$ is an open path with finite endpoints, these are not affected by the deformation. For endpoints at infinity a sufficient condition is to restrict the deformation to a bounded region of $\mathbb{C}$. In many cases this is unnecessarily restrictive.
Table 1. For \( \rho(x) = N(x - i)^2 e^{-\beta x^2} \) with \( \beta = 1.6 \), and several test functions \( f \). The columns ‘CL’, ‘\( \tilde{T}_\pm \)’, ‘\( T_\rho \)’ correspond to CL numerical results, \((\tilde{T}_\pm, f)\), and \((f)\), respectively. \( a_+ T_+ + a_- T_- \) gives the linear combination of \( \tilde{T}_\pm \) with coefficients \( a_\pm = 0.5 \mp i 0.0243(8) \) obtained from a best fit to ‘CL’.

| \( f(x) \) | CL | \( a_+ \tilde{T}_+ + a_- \tilde{T}_- \) | \( \tilde{T}_\pm \) | \( T_\rho \) |
|-----------------|-----|----------------------------------|-----------------|----------|
| \( x \)         | \( i 0.5244(2) \) | \( i 0.5247 \) | \( \pm 0.7521 \mp i 0.5613 \) | \( i 0.9091 \) |
| \( x^2 \)       | \( 0.4129(9) \)  | \( 0.4122 \)  | \( 0.3763 \pm i 0.7521 \) | \( 0.0284 \) |
| \( x^3 \)       | \( i 0.7562(9) \) | \( i 0.7563 \) | \( \pm 0.1880 \mp i 0.7653 \) | \( i 0.8523 \) |
| \( x^4 \)       | \( 0.2147(20) \)  | \( 0.2161 \)  | \( 0.1733 \pm i 0.8931 \) | \( -0.2397 \) |
| \( e^{-ix} \)   | \( 1.2100(6) \)  | \( 1.2100 \)  | \( 1.2626 \mp i 1.0634 \) | \( 1.7544 \) |
| \( e^{ix} \)    | \( 0.3940(2) \)  | \( 0.3942 \)  | \( 0.3754 \pm i 0.3808 \) | \( 0.1993 \) |
| \( e^{-2ix} \)  | \( 0.6109(21) \) | \( 0.6110 \)  | \( 0.7272 \mp i 1.23470 \) | \( 1.8126 \) |
| \( e^{2ix} \)   | \( -0.0064(3) \) | \( -0.0063 \) | \( -0.0186 \pm i 0.2491 \) | \( -0.1338 \) |

\[
T_\rho = a_+ \tilde{T}_+ + a_- \tilde{T}_-, \quad a_\pm = 0.5 \mp i 0.0243(8).
\] (17)

The relation \( a_+ + a_- = 1 \) is required by normalization. A single independent parameter allows to reproduce the various observables obtained from CL. This is in agreement with the conjecture, and incidentally it checks that the CL solution does fulfill the SDE for this density. Also noteworthy is the equivalent form

\[
T_\rho = \frac{b}{2} (\tilde{T}_+ + \tilde{T}_- ) + (1 - b) T_\rho, \quad b = 1.105(3).
\] (18)

Using an overcomplete set of paths, namely, \( \gamma_\pm \) and \( \mathbb{R} \), and their normalized linear forms, it has been possible to choose the weights real (but not nonnegative) for this \( \rho \). This fact goes beyond the conjecture.

Many more similar numerical experiments have been done with other \( \rho(z) \). In all cases the forms \( T_\gamma \) reproduce the CL numerical calculation.

The \( \rho \) considered include

\[
\rho(z) = (z - z_p)^n e^{-\beta z^2} \quad \text{Re}(\beta) > 0
\] (19)

for various \( z_p \) and \( \beta \) (including \( \beta = 1 + i \), but mostly \( \beta > 0 \), and \( n = 1, 2 \), but also \( n = -1, -2 \) and \( n = 1/2 \)).

For integer positive \( n \) the case \( \rho(x) = (x - i)^2 \exp(-x^2/2) \) is particularly interesting, since the normalization of \( \rho \) vanishes in this case, \( (T_\rho, 1) = 0 \), yet \( T_\rho \) (from CL, which is always normalized) is still a combination of the two \( T_{\gamma_\pm} \).

The negative \( n \) introduce poles and in this case the set \( \Gamma \) contains noncontractible closed paths. On the other hand, for noninteger rational \( n \) the complex drift \( v(z) \) is still univalent so the CL stochastic process takes place on \( \mathbb{C} \). \( \rho \) itself is not univalent and the paths \( \gamma \) should be considered to wander on the Riemann surface. Nevertheless, for a \( \rho(z) \) of the type \( (z - z_p)^n \sigma(z) \) with \( \sigma(z) \) a univalent function, paths on different branches but with the same projection on \( \mathbb{C} \) give the same integrals up to a phase, which can be absorbed in the coefficients \( a_\gamma \). Therefore the number of independent paths is still finite, namely, \( 2 \) for \( \rho(z) \) in (19) with \( n \) any positive rational number.
Actually, there is no evidence against the conjecture even for more general Riemann surfaces, however when $\Gamma$ is not finitely generated ($\rho$ having an infinite number of zeroes or singularities, or the Riemann surface having an infinite number of sheets) the conjecture is rendered non predictive, as there would be an infinite number of coefficients to be adjusted to reproduce any given linear form $T$ fulfilling the SDE. Also in the case of not finitely generated $\Gamma$, one would have to enter into functional analysis (topological vector spaces), something we want to avoid here.

Another case analyzed is
\[
\rho(z) = (z - z_1)(z - z_2)e^{-\beta z^2}, \quad \beta > 0, \quad z_1 \neq z_2.
\]
with two finite zeroes, besides the zeroes at $z = \pm\infty$. Therefore $\Gamma$ is generated by three independent paths and $\dim T = 3$. Another example is
\[
\rho(z) = \exp\left(-\frac{z^2}{2} - \frac{n}{z - z_s}\right),
\]
with various $z_s$ and $n = 1, 2$. This $\rho$ has a finite zero at $z_s$ which is an essential singularity. Hence there are two independent paths connecting zeroes, plus a closed path around the essential singularity; again $\dim T = 3$.

An example of a periodic $\rho(x)$ is provided by the density analyzed in [8],
\[
\rho(z) = (1 + \kappa \cos(z - i\mu))^{n_p} \exp(\beta \cos(z)), \quad \kappa = 2, \quad \beta = 0.3, \quad \mu = 1, \quad n_p = 2.
\]
Due to the periodicity of density and test functions, the manifold is effectively a cylinder, $[0, 2\pi] \times \mathbb{R}$. The CL process has two ergodic components for this $\rho$. There are no (finite) isolated singularities (hence the only closed paths in $\Gamma$ are the ones winding around the cylinder) and two finite zeroes at $z = \pm 2\pi/3 + i\mu$, which on the cylinder can be connected in two inequivalent ways (hence $\dim T = 2$), and can be combined to form a winding path. These correspond to the two ergodic components in the present case.

Note however, that in general $\dim T > 1$ may coexist with an ergodic CL process as the example (15) shows; another example is $\rho = \exp(ix^2)$ studied by Guralnik and Pehlevan [12].

3. A general theorem

3.1. Nonperiodic case

We first need a number of definitions.

Let $G_\epsilon \subset \mathbb{C}$ be a domain, i.e. a nonempty open connected subset of $\mathbb{C}$. We require that the fundamental group $\Pi_1(G_\epsilon)$ of $G_\epsilon$ is finitely generated. This implies that $G_\epsilon = \mathbb{C} \setminus \cup_{i=1}^{n_r} R_i$ ($n_r < \infty$), where the $R_i$ are connected and simply connected closed disjoint subsets of $\mathbb{C}$ and $\mathbb{C} = \mathbb{C} \cup \{\infty\}$ is the Riemann sphere.

**Remark 1.** $G_\epsilon \subset \mathbb{C}$ implies that $\infty$ is always an element of one of the $R_i$. The $R_i$ can be single points.

**Definition 1 (Set of densities).** $\mathcal{R}(G_\epsilon)$ is the set of functions (‘densities’) $\rho$ which are meromorphic on $G_\epsilon(\rho)$, with a set $P(\rho)$ of poles. We denote by $G(\rho) = G_\epsilon(\rho) \setminus P(\rho)$ the domain of holomorphy of $\rho$. 
\( \rho \) will in general have zeroes in \( G(\rho) \); we denote the set of zeroes by \( N_0(\rho) \); The sets \( P(\rho) \) and \( N_0(\rho) \) are required to be finite.

**Remark 2.** \( P \) being finite, the fundamental group \( \Pi_1(G) \) of \( G \) is also finitely generated.

**Definition 2 (Boundary zeroes).** Let \( \partial G \) be the boundary of \( G \) in \( \mathcal{C} \). We say that \( \rho \) has a boundary zero at \( b \in \partial G \) if there is a smooth path \( \gamma \) in \( G \) such that
\[
\lim_{z \to b \in \gamma} \rho(z) = 0.
\]
(23)

We denote the set of boundary zeroes by \( N_e(\rho) \) and the total set of generalized zeroes by \( N(\rho) \equiv N_0(\rho) \cup N_e(\rho) \).

**Remark 3.** \( \infty \) can be at the boundary of \( G_e \), hence it can be an element of \( N_e \). Essential singularities of \( \rho \) are instances of boundary zeroes, \( R_i \) being a single point in those cases.

Next we define a set of paths as follows:

**Definition 3 (Maximal set of paths).** Given \( \rho \in \mathcal{R}(G) \), the set \( \Gamma_m(\rho) \) consists of all oriented open and closed noncontractible paths \( \gamma \) in \( G \) such that

- the closed noncontractible paths are oriented rectifiable curves, differentiably parameterized by arc length: \([s_-, s_+] \ni s \mapsto z_\gamma(s) \) with \( z_\gamma(s_-) = z_\gamma(s_+) \).
- the arcs \( \gamma \) are oriented curves, differentiably parameterized by arc length:
  \( (s_-, s_+) \ni s \mapsto z_\gamma(s) \) where now \( s_\pm \) may be \( \pm \infty \) and the endpoints are elements of \( N(\rho) \), that is, the limits \( z_\pm(\gamma) \equiv \lim_{s \to s_\pm} z_\gamma(s) \) exist in \( \mathcal{C} \) and \( z_\pm(\gamma) \in N(\rho) \).

**Definition 4 (Weak and strong decay).** Let \( f \) be holomorphic in \( G_e \) and \( z_0 \in N(\rho) \) be a generalized zero of \( \rho \in \mathcal{R}(G_e) \); let \( \gamma \in \Gamma_m(\rho) \) be such that \( \lim_{s \to s_{\gamma\pm}} z_\gamma(s) = z_0 \). \( f \) decays weakly at \( z_0 \) iff
\[
\lim_{s \to s_{\gamma\pm}} |s| \alpha f(z_\gamma(s)) \rho(z_\gamma(s)) = 0
\]
and it decays strongly iff
\[
\lim_{s \to s_{\gamma\pm}} |s| \alpha f(z_\gamma(s)) \rho(z_\gamma(s)) = 0 \quad \forall \alpha \in \mathbb{R}.
\]
(25)

Analogous definitions apply when \( z_0 = \lim_{s \to s_{\gamma\pm}} z_\gamma(s) \).

**Definition 5 (Space of test functions).** Let \( \Gamma' \) be a subset of \( \Gamma_m(\rho) \). Then we define \( \mathcal{D}(\Gamma') \) as the set of functions holomorphic in \( G_e \) which are of weak decay at the finite endpoints of \( \gamma \in \Gamma' \) and of strong decay at the infinite generalized zeroes of \( \rho \) which are endpoints of a \( \gamma \in \Gamma' \).

**Definition 6 (Schwinger–Dyson operator).** We define the SD operator \( A \) on \( f \in \mathcal{D}(\Gamma') \) by
\[
(Af)(z) \equiv \frac{\rho'(z)}{\rho(z)} \partial_z f(z) = \frac{1}{\rho} \partial_z (\rho(z) f(z))
\]
(26)

Note that \( Af \) is in general no longer holomorphic, but meromorphic in \( G(\rho) \), so as codomain of \( A \) we take the space of meromorphic functions on \( G_e \).
Definition 7 (SD space). We define the ‘Schwinger–Dyson space’ by
\[ H \equiv D + AD. \] (27)

Definition 8 (SDE). We say that a linear functional \( T \) on \( H \) satisfies the Schwinger–Dyson equations iff
\[ (T, Af) = 0 \quad \forall f \in D. \] (28)

Definition 9 (Linear functionals given by paths). For any path \( \gamma \in \Gamma \) and \( f \in H(\Gamma) \) we define
\[ (T_\gamma, f) \equiv \int_\gamma d\rho(z) f(z); \] (29)
We define \( \mathcal{T}(\Gamma) \) to be the linear span of the \( T_\gamma, \gamma \in \Gamma \).

Remark 4. The linear forms in \( \mathcal{T}(\Gamma) \) fulfill the SDE, that is, they annihilate \( AD(\Gamma) \).

Now we can formulate

Theorem 1. Let \( \Gamma = \{\gamma_1, \ldots \gamma_n\} \) be a finite subset of \( \Gamma_m(\rho) \), such that
(i) the set of open paths in \( \Gamma \) forms a connected network,
(ii) each zero in \( N_0(\rho) \) is the endpoint of an open path in \( \Gamma \), and
(iii) the set of closed paths in \( \Gamma \) generates the fundamental group \( \Pi_1(G) \) of \( G \).

Then any linear functional \( T \) on \( H \) satisfying the SDE (28) is given by a linear combination of integrals of \( f \rho \) over paths in \( \Gamma \), i.e.
\[ (T, f) = \sum_{i=1}^n a_i (T_{\gamma_i}, f) \] (30)
with some coefficients \( a_i \in \mathbb{C} \).

To prove the theorem we need the following

Lemma 1. Let \( \Gamma \) be as above and \( f \in H \) such that
\[ (T_\gamma, f) = 0 \quad \forall \gamma \in \Gamma, \] (31)
then \( f \in AD \).

Proof. Because \( H = D + AD \) and \( T_\gamma \) annihilates \( AD \), it is sufficient to consider the case \( f \in D \). It has to be shown that \( f \rho = (h \rho)' \) for some \( h \in D \). Let \( N' \) be the set of generalized zeroes connected by some path in \( \Gamma \). By assumption \( N_0 \subset N' \subset N \).

Let us assume first that the set \( N' \) is empty. Let \( z_0 \in G \) and
\[ F(z) = \int_{z_0}^z \frac{dz'}{f(z')} \rho(z'), \] (32)
so that \( h = F/\rho \) is a solution of the differential equation \( f \rho = (h \rho)' \). The condition that the closed paths in \( \Gamma \) generate \( \Pi_1(G) \) implies that \( F \), and hence \( h \), is univalent because
\[ \oint f(z) \rho(z) \, dz = 0 \]  \hspace{1cm} (33)

for closed paths \( \gamma \in \Gamma' \) around each \( R_i \) and each pole of \( \rho \). In addition, where \( \rho \) has a pole of order \( m \), \( F \) has at most a pole of order \( m - 1 \), hence \( h \) is holomorphic in \( G_e \).

When \( N' \) is not empty we define \( F \) as above with \( z_0 \in N' \). \( F \) vanishes at \( z_0 \) and also at all other generalized zeroes in \( N' \) since they are connected directly or indirectly to \( z_0 \) by open paths \( \gamma \in \Gamma \) fulfilling

\[ \oint f(z) \rho(z) \, dz = 0. \]  \hspace{1cm} (34)

It follows that \( h = F/\rho \) has weak decay at the finite zeroes of \( N' \). \( h \) also has strong decay at the infinite (generalized) zeroes of \( N' \), because the integral of a strongly decaying function approaches its limit faster than any power. Finally, \( h \) is holomorphic at \( N_0(\rho) \) because where \( \rho \) has a zero of order \( m \), \( F \) has a zero of order at least \( m + 1 \). Therefore \( h \in D \). The solution \( h \) is unique iff \( N' \) is nonempty. \( \square \)

**Proof of theorem 1.** We need some preparation:

Let \( \mathcal{K} \) be the subspace of \( \mathcal{H} \) consisting of the \( f \in \mathcal{K} \) satisfying

\[ (T_\gamma, f) = 0 \quad \forall \gamma \in \Gamma. \]  \hspace{1cm} (35)

\( \mathcal{K} \) is determined by \( n \) not necessarily linearly independent linear conditions. We pick a subset \( \Gamma_0 \) of \( \Gamma \) such that the set \( \{ T_\gamma | \gamma \in \Gamma_0 \} \) is a basis of \( T(\Gamma) \). \( |\Gamma_0| = n_0 \leq n \) is then the dimension of the quotient space \( \mathcal{H}/\mathcal{K} \). It follows that \( \mathcal{H} \) can be written as a direct sum

\[ \mathcal{H} = \mathcal{K} \oplus \mathcal{G} \]  \hspace{1cm} (36)

with \( \dim \mathcal{G} = \dim \mathcal{H}/\mathcal{K} = n_0 \). \( \mathcal{G} \) is of course not unique. So any \( f \in \mathcal{H} \) can written as

\[ f = f_\mathcal{K} + f_\mathcal{G} \]  \hspace{1cm} (37)

with \( f_\mathcal{K} \in \mathcal{K} \) and \( f_\mathcal{G} \in \mathcal{G} \). Once \( \mathcal{G} \) has been chosen, the dual space \( \mathcal{H}^* \) of \( \mathcal{H} \) can be split accordingly:

\[ \mathcal{H}^* = \mathcal{K}^* \oplus \mathcal{G}^* \]  \hspace{1cm} (38)

and any \( T \in \mathcal{H}^* \) can be written as

\[ T = T_\mathcal{K} + T_\mathcal{G} \]  \hspace{1cm} (39)

with \( T_\mathcal{K} \in \mathcal{K}^* \) and \( T_\mathcal{G} \in \mathcal{G}^* \), such that

\[ (T_\mathcal{G}, f_\mathcal{K}) = 0 = (T_\mathcal{K}, f_\mathcal{G}). \]  \hspace{1cm} (40)

Note that we understand here \( \mathcal{K}^* \) and \( \mathcal{G}^* \) as spaces of linear functionals acting on \( \mathcal{H} \). \( \mathcal{G}^* \) is then the subspace of \( \mathcal{H}^* \) annihilating \( \mathcal{K} \). We now state

**Lemma 2.** \( \mathcal{G}^* \) is spanned by the set \( \{ T_\gamma | \gamma \in \Gamma_0 \} \), so \( \mathcal{G}^* = T(\Gamma) \).

**Proof.** The set \( \{ T_\gamma | \gamma \in \Gamma_0 \} \) annihilates \( \mathcal{K} \) by definition, so it is a subset of \( \mathcal{G}^* \); since it consists of \( n_0 \) linearly independent elements it spans all of \( \mathcal{G}^* \). \( \square \)

To complete the proof of the theorem we note that, by lemma 1, \( \mathcal{K} \subset AD \). Obviously also \( AD \subset \mathcal{K} \), so we have
\[ \mathcal{K} = AD. \tag{41} \]

Now let \( T \) obey the SDE, i.e. equation (28) holds. But then also

\[ \langle T, f \rangle = 0 \quad \forall f \in \mathcal{K}. \tag{42} \]

i.e. \( T \in \mathcal{G}^* \). So theorem 1 follows by using lemma 2.

**Remark 5.** We can enlarge \( \Gamma \) considerably by allowing homotopic deformations in \( G(\rho) \) of the paths \( \gamma \), provided they leave \( \mathcal{D} \) and \( T_\gamma \) invariant. This is guaranteed if the deformation takes place outside an open neighborhood (in \( \hat{\mathbb{C}} \)) of the endpoints when they are boundary zeroes of \( \rho \) (see remark 8 below).

**Remark 6.** We may also enlarge \( D(\rho) \) by completing this space with respect to some (semi) norms. For instance we can introduce an \( L^1 \) norm on \( D(\rho) \) by

\[ ||f||_1 = \sum_{\gamma \in \Gamma_0} \int_{\gamma} |f(z)\rho(z)| dz; \tag{43} \]

which is finite for \( f \in \mathcal{D} \), and define the completion \( \hat{\mathcal{D}} \) with respect to \( ||.||_1 \). Clearly \( T(\Gamma) \) will be continuous in that norm. Theorem 1 remains true if we replace \( H^* \) (the algebraic dual of \( H \)) by \( H' \) (the topological dual of \( H \)).

**Remark 7.** Lemma 1 does not require \( \Gamma \) to be finite. Hence theorem 1 would still hold for infinite sets \( \Gamma \), provided \( T(\rho) \) is finite-dimensional (required for lemma 2).

**Remark 8.** The paths in \( \Gamma \), a subset of \( \Gamma_m(\rho) \), can be bundled into equivalence classes through the relation \( \gamma_1 \equiv \gamma_2 \) iff \( T_{\gamma_1} = T_{\gamma_2} \). However, that an open path is contractible\(^6\) does not imply it is null when the endpoint is a boundary zero of \( \rho \). Likewise, two homotopic open paths need not be equivalent if any of the endpoints is a boundary zero. The two paths will be equivalent if they coincide in an open neighborhood (with respect to \( \hat{\mathbb{C}} \)) of their endpoints.

### 3.2. Periodic case

Let \( \mathbb{T} \) be the cylinder \([0, 2\pi] \times \mathbb{R}\) with 0 and \(2\pi\) identified. We can compactify \( \mathbb{T} \) to \( \hat{\mathbb{T}} \) by adding two points \(+i\infty\) and \(-i\infty\) to obtain a topological sphere. Let \( G_\epsilon \) be a nonempty open connected subset of \( \mathbb{T} \) such that \( \Pi_1(G_\epsilon) \) is finitely generated.

**Definition 1′ (Set of periodic densities).** \( \mathcal{R}_p(G_\epsilon) \) is the set of functions (‘densities’) which are meromorphic on \( G_\epsilon(\rho) \), with a finite set \( P(\rho) \) of poles and a finite set \( N_0(\rho) \) of zeroes. We denote by \( G(\rho) = G_\epsilon(\rho) \setminus P(\rho) \) the domain of holomorphy of \( \rho \).

**Remark 9.** By definition the densities \( \rho \in \mathcal{R}_p(G_\epsilon) \) can be viewed as functions on (a subset of) \( \hat{\mathbb{C}} \) satisfying

\[ \rho(z + 2\pi) = \rho(z). \tag{44} \]

\(^6\)Contractible here means that the two endpoints coincide and the closed path in \( \hat{\mathbb{C}} \) obtained by adding the endpoint, \( z_0 \), is contractible within \( G \cup \{z_0\} \).
We have to consider the periodic case separately because of our finiteness assumptions made for the nonperiodic case. They would be violated if we considered the periodic case simply as a special case of the nonperiodic one.

**Definition 2′ (Boundary zeroes).** Let \( \partial G_e \) be the boundary of \( G_e \) in \( \overline{T} \). We say that \( \rho \) has a boundary zero at \( b \in \partial G_e \) iff there is a smooth path \( \gamma \) in \( G \) such that

\[
\lim_{z \to b} \rho(z) = 0.
\]

We denote the set of boundary zeroes by \( N_e(\rho) \) and the total set of generalized zeroes by \( N(\rho) \equiv N_0(\rho) \cup N_e(\rho) \).

We then have the periodic analogue of theorem 1:

**Theorem 1′.** Let \( \Gamma = \{\gamma_1, \ldots, \gamma_n\} \) be a finite subset of \( \Gamma_m(\rho) \), such that

(i) the set of open paths in \( \Gamma \) forms a connected network,
(ii) each zero in \( N_0(\rho) \) is the endpoint of an open path in \( \Gamma \), and
(iii) the set of closed paths in \( \Gamma \) generates the fundamental group of \( G \).

Then any linear functional \( T \) on \( H \) satisfying the SDE (28) is given by a linear combination of integrals of \( f \rho \) over paths in \( \Gamma \),

\[
(T, f) = \sum_{i=1}^{n} a_i (T_{\gamma_i}, f)
\]

with some coefficients \( a_i \in \mathbb{C} \).

**Proof.** The proof of theorem 1 applies with some obvious adjustments. \( \square \)

### 4. Special instances of the theorem

#### 4.1. Preliminaries

In the previous section we have constructed a large class of triples \( (\rho, \Gamma, D) \) for which the fulfillment of the SDE (28) implies that the functional is a combination of pathwise functionals: if \( T \in \mathcal{H}^* \) such that \((T, AD) = 0\) then \( T \in \mathcal{T}(\Gamma) \). Equivalently

\[
\text{Ann}(AD) \subset \mathcal{T},
\]

where \( \text{Ann} \) denotes the annihilator space of a set \( S \subset \mathcal{H} \), i.e.

\[
\text{Ann}(S) \equiv \{T \in \mathcal{H}^* \mid (T, f) = 0 \forall f \in S \}.
\]

The method to prove this started by showing that \( \mathcal{K} = AD \) (lemma 1) where \( \mathcal{K} \equiv \cap_{\gamma \in \Gamma} \ker T_\gamma \), and then showing that \( \text{Ann}(\mathcal{K}) = \mathcal{T} \) when \( \dim \mathcal{T} < \infty \). In turn, lemma 1 is based on \( D \cap \mathcal{K} \subset AD \), or in words, if \( f \in D \) fulfills the constraints \((T_\gamma, f) = 0 \forall \gamma \in \Gamma \), the equation \( f = Ah \) has a solution \( h \in D \).

It is natural to seek other triples of density, set of paths and domain of \( A \) fulfilling the relation (47) which can be regarded as a corollary of the theorem. A way to do this is as follows. Given \( \rho \in \mathcal{R} \) and \( \Gamma \) a finite subset of \( \Gamma_m(\rho) \), one can consider triples \( (\rho, \Gamma, D_1) \), where now a subspace \( D_1 \subset D \) is used as domain of \( A \). (This introduces the corresponding \( \mathcal{H}_1 = D_1 + AD_1 \),
The triple will continue to be consistent in the sense that $T_1 \subset \text{Ann} (AD_1)$ but in general $\text{Ann} (AD_1) \subset T_1$ will not hold for such smaller domain. The impediment comes from lemma 1: in general, $\text{Ann} (AD_1)$ will not hold for such smaller domain. The impediment comes from lemma 1: in general, $f$ of $D_1$ fulfilling the constraints $(T, f) = 0 \forall \gamma \in \Gamma$ implies that $f = Ah$ for some $h \in D$ but not necessarily $h \in D_1$. In some sense $D_1$ is not complete. One can remedy this by enlarging the domain to $D_2$ by simply adding all the new $h$ so obtained. The new $D_2$ still will be not complete in general, but the process of enlarging the domain can be iterated until reaching a fixed point $D_\infty$. This is the smallest domain with $D_1 \subset D_\infty \subset D$ such that the triple $(\rho, \Gamma, D_\infty)$ is complete, i.e. $\text{Ann} (AD_\infty) \subset T_\infty$ holds.

The domain $D_\infty$ can be $D$ itself, but not necessarily. In this section we will show that for large classes of $\rho$ simple domains, like polynomials in the noncompact case or Fourier modes in the periodic case, yield triples which are complete.

4.2. Densities of rational type

We make the following definitions:

Definition 10 (Nonperiodic densities of rational type). A density $\rho \in \mathcal{R}$ is called of rational type iff $\rho'(z)/\rho(z)$ is a rational function.

Definition 10’ (Periodic densities of rational type). A density $\rho \in \mathcal{R}_p$ is called of rational type iff $\rho'(z)/\rho(z)$ is a rational function of $\omega = \exp(iz)$.

For these classes of densities, with rather natural choices of $D$ and $\Gamma$, it is possible to determine explicitly the numbers $N_{\text{SDE}}$ and $N_\Gamma$ and show that they are equal. We provide the general, rather elaborate discussion in appendix A and limit ourselves here to three examples illustrating how the new strategy (of proof) works and also to point out the difference with the previous strategy.

Example 1. This is an extremely simplified case, in the periodic setting: the triple $(\rho, \Gamma, D)$ is composed of $\rho(z) = e^{-i z}$, $\Gamma$ contains just the path $\gamma$ winding once around the cylinder $[0, 2\pi] \times \mathbb{R}$ (in the positive real direction) and $D$, the domain of $A$, is the space spanned by the Fourier modes:

$$D = \text{span} \{ e^{ik}, k \in \mathbb{Z} \}. \tag{51}$$

Applying the SD operator $A$ on a basis of $D$,

\[ T_1 \subset H^*_1, \text{ etc.} \]
\[ Ae^{ikz} = i(k - 1)e^{ikz} \quad k \in \mathbb{Z}, \]  

(52)

yields the image of \( A \)

\[ AD = \text{span}\{e^{ikz}, k \in \mathbb{Z} \setminus \{1\}\}. \]  

(53)

In this example \( \mathcal{H} = \mathcal{D} + AD \) is just \( \mathcal{D} \), and the dual space \( \mathcal{H}^* \) consists of all sequences of complex numbers. As it is readily verified \((T_\gamma, f) = 0\) for all \( f \in AD \), hence \( T \subset \text{Ann}(AD) \).

Also the space \( \mathcal{K} = \ker T_\gamma = \{ f \in \mathcal{H} | \oint_{\gamma} df = 0 \} \)  

(54)

coincides with \( AD \) (so lemma 1 is satisfied). To check that \( \text{Ann}(\mathcal{K}) = T \) (\( T \) being finite-dimensional) it is sufficient to verify that their dimensions coincide, i.e.

\[ \text{dim} \text{Ann}(\mathcal{K}) = 1 \]  

in this case. This is equivalent to

\[ \text{dim} \mathcal{H}/\mathcal{K} = 1. \]  

And indeed, one can write the decomposition

\[ \forall k \in \mathbb{Z} \quad e^{ikz} = (1 - \delta_{k,1})e^{ikz} + \delta_{k,1}e^z \]  

(55)

where the first function in the r.h.s. is in \( \mathcal{K} \) and the second one is in a one-dimensional complementary space of \( \mathcal{K} \) (denoted \( \mathcal{G} \) in the proof of theorem 1). From here the theorem \( T \in \text{Ann}(AD) \Rightarrow T = \lambda T_\gamma \) follows:

\[ (T, e^{ikz}) = (T, e^z) \delta_{k,1} \quad (T_\gamma, e^{ikz}) = 2\pi \delta_{k,1} \]  

(56)

hence \((T, f) = \lambda(T_\gamma, f)\) with \( \lambda = (T, e^z)/(2\pi) \).

In the alternative proof \( \mathcal{K} \) is not explicitly used; \( N_T = 1 \) as there is just one path, and \( N_{\text{SDE}} \) is directly computed by looking for the most general \( T \in \mathcal{H}^* \) fulfilling the SDE:

\[ 0 = (T, Ae^{ikz}) = i(k - 1)(T, e^{ikz}). \]  

(57)

Clearly, all the components \((T, e^{ikz})\) of \( T \) except \( k = 1 \) are determined by the SDE (namely, they vanish in this case). Therefore \( N_{\text{SDE}} = 1 \), which coincides with \( N_T \), implying \( \text{Ann}(AD) = T \) (using \( T \subset \text{Ann}(AD) \)).

\[ \square \]

Example 2. This is a slightly more complicated case in the the noncompact setting. Let

\[ \rho(z) = \frac{(z - a)^3}{(z - b)^2} e^{-z} e^{-1/z^2} \quad (a \neq b) \]  

(58)

with \( \mathcal{D} \) the space of polynomials,

\[ \mathcal{D} = \text{span}\{z^k, k \in \mathbb{N}_0\}. \]  

(59)

The image \( AD \), as well as \( \mathcal{H} = \mathcal{D} + AD \) is living in the space of rational functions. There is an essential singularity at \( z = 0 \) and a pole at \( z = b \) unless \( b = 0 \). So the number of closed paths is \( N_c = 1 \) if \( b = 0 \) and \( N_c = 2 \) otherwise. (It is important to note that singularities at infinity do not add any closed paths.) There are zeroes at \( z = 0, a, \infty \). But the zero at infinity is reached in four different (path-in equivalent) ways giving rise to an effective number of four zeroes. Similarly, the zero at the essential singularity at \( z = 0 \) can be reached in two
inequivalent ways, namely, \( z \to 0^+ \) or \( z \to 0^- \), so from the point of view of \( N_T \) it counts as two zeroes\(^8\). Effectively the number of zeroes is \( N_z = 7 \), unless \( a = 0 \) in which case \( N_z = 6 \). This produces \( N_z - 1 \) linearly independent paths connecting them. Hence, \( N_T = N_z + N_e - 1 \). One last observation is that \( N_z - 1 \) is the number of open paths because \( N_z > 0 \). When there are no zeroes the number of open paths connecting them is itself zero. As for the evaluation of \( \mathcal{N}_{\text{SDE}} \) for the density in (58), it is not simpler than for the general case. The computation in appendix A.1 shows that \( \mathcal{N}_{\text{SDE}} \) indeed coincides with \( N_T \).

\[ \square. \]

**Example 3.** We consider a less trivial periodic example, with the same domain as in (51):

\[
\rho(z) = e^{\sigma i z} \exp(e^{2i z})
\]

with \( \sigma = \pm 1 \). There are two zeroes at \( z = -i \infty \) from \( e^{2i z} \), hence \( N_T = 2 \).

It can be noted that when \( \sigma = 1 \) \( \rho \) has an additional zero at \( z = +i \infty \) which however does not contribute to \( N_T \) because it fails to be a zero common to all functions in \( \rho D \). \( N_T \) counts the number of independent paths \( \gamma \) for which \( T_\gamma \) fulfills the SDE as linear forms on a given domain \( D \) of test functions. For paths connecting zeroes, SDE requires \( \rho f \) to vanish at the zeroes for all \( f \in D \) (the vanishing of \( \rho \) is not sufficient). Clearly a factor \( f = e^{ikz} \) with \( k \leq -1 \) cancels the zero of \( \rho \) at \( z = +i \infty \) for \( \sigma = 1 \) in the product \( \rho f \).

Let \( \omega \equiv e^{i z} \). Application of the SD operator on a generic element of the basis produces

\[-i A \omega^\rho = (n + \sigma) \omega^n + 2 \omega^{n+2} \]

If \( T \) complies with the SDE, \( 0 = (T, A \omega^\rho) \), this gives

\[0 = (n + \sigma) E_n + 2 E_{n+2} \quad \forall n \in \mathbb{Z},\]

where \( E_n \equiv (T, \omega^n) \). Clearly, all \( E_n \) with even \( n \) are determined by \( E_0 \). On the other hand for \( \sigma = 1 \) \( E_n = 0 \) for all positive odd \( n \) and all \( E_n \) with odd \( n < -1 \) are determined by \( E_{-1} \), hence there are two arbitrary coefficients and \( \mathcal{N}_{\text{SDE}} = 2 \), matching \( N_T \). For \( \sigma = -1 \) the situation is similar.

These examples, while being nongeneric, still serve as the paradigms for the treatment of the generic cases of rational type, described in appendix A.

**5. Conclusions and outlook**

Motivated by observations in the study of complex Langevin simulations we proved under rather general conditions that for one-dimensional systems with a complex density \( \rho \) any linear functional on the ‘observables’ satisfying the SDE is a linear combination of line integrals, where the lines either connect zeroes of \( \rho \) or are noncontractible closed loops.

For the nonperiodic case, this implies that the stationary expectation values obtained by the complex Langevin process, which satisfy superficially weaker conditions (the ‘convergence conditions’ CC) are also given by such linear combinations.

For the periodic case we can only prove a weaker statement, because in this case the CC do not imply the SDE. The weaker statement is that those expectation values are linear combinations of the line integrals as above plus one extra linear functional which is not of that form.

\[8\text{For instance, the path } z(t) = i + e^{it}, -\pi/2 < t < 3\pi/2, \text{ defines a linear form } T_\gamma \not\equiv 0 \text{ on the space of polynomial test functions, regardless of the values of } a \text{ and } b.\]
The single mode that in the periodic case is not a derivative is the constant mode \( f = 1 \), hence the SD condition that is not a consequence of CC in the steady CL solution is \( 0 = (T_P,A) = (T_P,v) \). As shown in appendix B under reasonable conditions flux conservation guarantees the property \( 0 = \text{Im}(T_P,v) \), which is close to what would be needed. The correct relation \( (T_P,v) = 0 \) is checked for the density \( \rho(x) = \exp(imx + \beta \cos(x)) \) \((m \in \mathbb{Z})\) analyzed in [6] in two cases, \( \beta \in \mathbb{C} \) with \( m = 0 \), and \( \beta \in \mathbb{R} \) with \( m \neq 0 \). While in the former case \( (T_P,v) \) vanishes due to parity, the vanishing in the second case is not trivial\(^9\).

It should be remarked that in the study of complex Langevin we do not know of a single case in which such an extra linear functional actually appears. A possible reason is the fact observed in [5] that a linear functional satisfying the CC together with some bounds in terms of a sup norm of the observables on a path imply (via the Riesz–Markov theorem) that the functional is indeed given by the corresponding line integral. But the issue is far from being settled and needs further study.

An important open question concerns the generalization of our results to higher dimensions, in particular group manifolds, which form the configuration space of lattice gauge theories. We think that this is probably possible, but not quite trivial since it will involve the theory of analytic functions of several complex variables.

It can be noted that the SDE operator \( A = \partial_z + v(z) \) has the form of a (gauge) covariant derivative. In fact a kind of gauge symmetry is present in the problem analyzed here; letting \( \omega(z) \) be a holomorphic function without zeroes (hence \( \omega^{-1} \) being also holomorphic), the space of test functions is unchanged under the transformation \( f \mapsto \omega f \). In this case, the identity \( \int f \rho \, dx = \int (f \omega)(\omega^{-1}\rho) \, dx \), indicates that \( \tilde{\rho} = \omega^{-1}\rho \) is gauge equivalent to \( \rho \) and this property is preserved by the SDE: \( 0 = \int dx \partial f(\rho f) = \int dx \partial \tilde{\rho} \tilde{f} \). The upshot is that if the statement in equation (14) must hold for a given \( \rho \), it must hold too for any other complex density \( \tilde{\rho} \) in the same gauge orbit. Such gauge copies share in common the position of zeroes and singularities. Going a bit further one can consider the transformation \( f \rightarrow \rho f \), so that \( \tilde{\rho} = 1 \) and the SDE operator becomes simply \( \tilde{A} = \partial_z \). The domain of test functions becomes \( \tilde{\mathcal{D}} = \rho \mathcal{D} \) (\( \mathcal{D} \) being the original domain) and in this setting all the information about the gauge orbit is contained in the domain \( \tilde{\mathcal{D}} \), which is composed of analytic functions with some prescribed zeroes and singularities. That structure, being common to all functions, can be regarded as a property of the manifold itself, and the property \( \tilde{A} = \partial_z \) suggests the speculation that the problem analyzed in this work could be reformulated as one of determining the cohomological properties of that manifold.

This idea is reinforced (through the presence of the exterior derivative) when another symmetry, general covariance, is considered. The basic structure \( \langle f \rangle = \int f \rho \, d^nx \) (in a general \( n \)-dimensional case) has a geometric, i.e. coordinate independent, meaning where \( f \) is a 0-form and \( \tilde{\rho} \equiv \rho \, d^n x \) is an \( n \)-form. However the SDE are not automatically general covariant: for each coordinate \( x^i \), \( i = 1, \ldots, n \), one can consider an SD operator \( A_i \) defined by \( A_i f = \rho^{-1} \partial_i (\rho f) \), and indeed \( \langle A_i f \rangle = \int d^nx \partial_i (\rho f) \) must vanish, assuming sufficient convergence at the boundary, but the \( A_i \) depend on the coordinate system. A covariant operator can be written as

\[
A_i f = \rho^{-1} \partial_i (\xi^i f),
\]

\(^9\)For these densities \( v(z) = im - \beta \sin(z) \) and the integral \( \int d^2z P(z) v(z) \) is absolutely convergent, as shown in [6]. This is no longer guaranteed for higher Fourier modes, \( e^{ikx} \) with \( |k| > 1 \).
where $\xi^i(x)$ is a vector field, leading to $\langle A_\xi f \rangle = 0$. The SDE express the invariance of $\int f \rho \, d^n x$ under rearrangements of the integrand and the field $\xi^i$ represents one such deformation. The connection with the exterior derivative comes through Stokes’ theorem

$$
\int_M d(\xi \cdot \hat{\rho} f) = \int_{\partial M} \xi \cdot \hat{\rho} f = 0,
$$

where the last equality assumes sufficient convergence at the boundary or an empty boundary, and $\xi \cdot \hat{\rho}$ denotes the $(n-1)$-form obtained from the interior product of the vector field $\xi$ with the $n$-form $\hat{\rho}$. This is related to the operator $A_\xi$ through $d(\xi \cdot \hat{\rho} f) = \frac{1}{\rho} A_\xi f$.

A vector field $\xi$ was not present in the original problem of evaluating $\langle f \rangle$ but its introduction is needed for a geometric formulation of the SDE. The same phenomenon occurs in other situations, such as the problem of finding the extrema of a function $f$ by the steepest descent. Assuming that $f$ is a scalar, the equation $\dot{x}^i = -\partial_i f$ leads to different trajectories when applied in different coordinates systems. A coordinate-independent formulation requires instead $\dot{x}^i = -g^{ij} \partial_j f$, where $g^{ij}(x)$ transforms as the contravariant components of a metric field. The steepest descent method is closely related to CL, so the introduction of a metric is also needed in a geometric formulation of that stochastic process [3]. While the previous geometric considerations have played no role in the one-dimensional case, they are likely to be more relevant for higher dimensions.

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### Appendix A. Explicit counting of dimensions

#### A.1. Nonperiodic densities of rational type

In this appendix we will study densities $\rho$ of rational type. Since $\rho'/\rho$ is rational, $\rho$ is holomorphic on $\mathbb{C}$ up to isolated singularities. $\rho$ can be written as

$$
\rho(z) = P(z) \exp(R(z))
$$

with $P$ and $Q$ rational functions. As domain we take the space of polynomials

$$
\mathcal{D} = \text{span} \{z^n, n \in \mathbb{N}_0\}.
$$

For this domain, the paths connecting generalized zeroes can be naturally grouped into equivalence classes and $\Gamma$ will include one path for each class, plus one path encircling each isolated singularity. Certainly the weak and strong decay conditions in definition 4 are fulfilled by polynomials, at finite zeroes and at infinity when the latter is an essential singularity of $\rho$.

In (A.1) $P(z)$ is a general rational function (general up to a global factor) which we write as

$$
P(z) = \prod_{\ell=1}^{N_\rho} p_\ell(z)^{\alpha_\ell}, \quad p_\ell(z) = z - a_\ell, \quad \alpha_\ell \in \mathbb{Z}\{0\}, \quad N_\rho \geq 0
$$

and the $a_\ell$ are pairwise different. On the other hand, $R(z)$ is also a general rational function, but expressed differently, for convenience:

$$
R(z) = Q(z) + R_s(z),
$$

where $Q(z)$ and $R_s(z)$ are...
where \( Q(z) \) is a polynomial of degree \( N_q \)
\[
Q(z) = \sum_{k=0}^{N_q} c_k z^k, \quad N_q \geq 0, \quad c_{N_q} \neq 0,
\]
(A.5)
and \( R_s(z) \) contains the principal parts:
\[
R_s(z) = \sum_{m=1}^{N_s} \sum_{r=1}^{\beta_m} \frac{d_{m,r}}{q_m(z)^r}, \quad N_s \geq 0,
\]
with
\[
q_m(z) = z - b_m, \quad \beta_m > 0, \quad d_{m,\beta_m} \neq 0,
\]
(A.6)
and the \( b_m \) are pairwise different.

Due to invariance under translations, we can assume without loss of generality that the \( a_\ell \) and the \( b_m \) are all different from zero.

In order to prove the equality
\[
N_{\Gamma} = N_{\text{SDE}},
\]
the first task is to compute the number \( N_{\Gamma} \) of independent paths in \( \Gamma \). Independence means here that the corresponding linear functionals \( T_\gamma \) form a maximal linear independent set. Some of the issues involved were illustrated with in example 2 in section 4.

Considering now the general case (A.1), let \( N'_p \) be the number of \( a_\ell \) which are different from the \( b_m \), \( N'_p \leq N_p \). The positive \( \alpha_\ell \) correspond to zeroes and the negative ones to poles. Further, there are \( \beta_m \) zeroes from each \( b_m \) in \( R_s(z) \) and \( N_q \) zeroes at infinity from \( Q(z) \), plus \( N_s \) essential singularities from \( R_s(z) \). So the total number of zeroes plus finite singularities is
\[
N_g = N'_p + N_q + \sum_{m=1}^{N_s} (\beta_m + 1).
\]
(A.7)
This gives
\[
N_{\Gamma} = \begin{cases} 
N_g - 1 & \rho \text{ has zeroes} \\
N_g & \rho \text{ has no zeroes}
\end{cases}
\]
(A.8)
\( \rho \) has no zeroes iff \( N_q = N_s = 0 \) and all \( \alpha_\ell < 0 \). One final remark is that in the special case of \( N_s = 0, P(z) \), and hence \( \rho(z) \), vanishes at infinity if \( \sum_\ell \alpha_\ell < 0 \). Such a zero is ineffective in the domain of polynomials because \( \lim_{z \to \infty} z^n \rho \) would not be zero for all \( n \). Equivalently, only zeroes common to all functions in \( \rho D \) are effective. Thus such would-be zero does not contribute to \( N_{\Gamma} \).

To compute \( N_{\text{SDE}} \) we consider a \( T \in \mathcal{H}^* \) and impose the SDE
\[
0 = \langle A z^n \rangle = \langle n z^{n-1} + v(z) z^n \rangle, \quad n \geq 0
\]
(A.9)
where we have used the notations
\[
\langle f \rangle \equiv (T, f)
\]
(A.10)
and
\[
v(z) \equiv \frac{\rho'}{\rho} = \sum_{\ell=1}^{N_p} \frac{\alpha_\ell}{P_\ell} + \sum_{k=1}^{N_q} k c_k z^{k-1} - \sum_{m=1}^{N_s} \sum_{r=1}^{\beta_m} \frac{d_{m,r}}{q_m(z)^r},
\]
(A.11)
One can see that the coefficients of \( T \) involved in the SDE (A.11) are of the type

\[
E_n \equiv \langle z^n \rangle, \quad F_n \equiv \langle 1/p_\ell(z) \rangle, \quad G_{m,r} \equiv \langle 1/q_m(z) \rangle.
\]  
(A.14)

The coefficients \( E_n \) are sufficient for \( T \) acting on \( D \) but \( F_n \) and \( G_{m,r} \) are required on \( \mathcal{H} = D + AD \).

The expressions involved can be analyzed making use of the expansion

\[
\frac{z^n}{q(z)} = \sum_{j=0}^{\infty} \binom{n}{j} q^{-r-j} b^{n-j}, \quad q(z) \equiv z - b, \quad r \geq 1, \quad n \geq 0, \quad b \neq 0.
\]  
(A.15)

or, schematically, simplifying the notation by not writing the coefficients of the powers of \( z \) and \( q \):

\[
\frac{z^n}{q(z)} \sim \left\{ \frac{1}{q} + \cdots + \frac{1}{q^r}, \quad \frac{1}{q} + \cdots + \frac{1}{q^r} + 1 + z + \cdots + z^{n-r} \right\} \quad \text{for} \quad r \geq 1, \quad n \geq 0.
\]  
(A.16)

Hence, schematically, the various terms in \( \langle Az^n \rangle \) produce,

\[
\langle n z^{n-1} \rangle = n E_{n-1},
\]

\[
\langle \alpha_t z^n/p_t \rangle \sim F_{t} + E_0 + \cdots + E_{n-1},
\]

\[
\langle k c_z z^{n+k-1} \rangle = k c E_{n+k-1},
\]

\[
\langle r d_{m,r} z^n/q_{m}^{n+1} \rangle \sim \begin{cases} G_{m,r+1} + \cdots + G_{m,r-n+1} & \text{for} \quad n \leq r, \\ G_{m,r+1} + \cdots + G_{m,1} + E_0 + \cdots + E_{r-1} & \text{for} \quad n > r. \end{cases}
\]  
(A.17)

Clearly, for large enough \( n \) just one new term appears when \( n \) increases by one unit, thus we definitely obtain a proper recursion from the SDE, i.e. a finite-dimensional \( \text{AnnAD} \).

Momentarily we assume generic values for the parameters in \( \rho \). In particular, we assume that no \( a_t \) coincides with any \( b_m \), so \( N_p^\rho = N_p \) and all the \( F_t \) are different from the \( G_{m,1} \).

Expanding \( \langle Az^n \rangle \) with (A.15), one can see from (A.17) that for \( n = 0 \), the following expectation values are involved in the recursion

\[
\langle n = 0 \rangle = \{ F_0 \}, \quad \{ E_0, \ldots, E_{N_q-1} \}, \quad \{ G_{m,2}, \ldots, G_{m,\beta_{m,n+1}} \}. \]  
(A.18)

The number of coefficients is \( N_p^\rho + N_q + \sum_m \beta_m \), with one constraint among them, hence \( N_p^\rho + N_q + \sum_m \beta_m - 1 \) free parameters. For \( n = 1 \), the coefficients related by the recursion are

\[
\langle n = 1 \rangle = \{ F_0 \}, \quad \{ E_0, \ldots, E_{N_q} \}, \quad \{ G_{m,1}, \ldots, G_{m,\beta_{m,n+1}} \}. \]  
(A.19)

This introduces anew \( E_{N_q} \) and all the \( G_{m,1} \) \( (N_q \) values), and one constraint, hence \( 1 + N_q - 1 \) new free parameters. In general for \( n \geq 1 \), the formula involves

\[
\langle n \geq 1 \rangle = \{ F_0 \}, \quad \{ E_0, \ldots, E_{N_q+n-1} \}, \quad \{ G_{m,1}, \ldots, G_{m,\beta_{m,n+1}} \}. \]  
(A.20)

For \( n \geq 2 \), each time a new coefficient is introduced (namely \( E_{N_q+n-1} \)) which is fixed by the new constraint. The number of degrees of freedom is thus (from \( n = 0 \) + \( n = 1 \) + \( n = 2 \) + \( n = 3 \) + \( n = 4 \) + \( n \geq 1 \)...

\[
N_p + N_q + \sum_m (\beta_m - 1) + [1 + \beta_1] + [1 + \beta_2] + [1 + \beta_3] + \cdots + [1 + \beta_{N_q}] + \cdots
\]

\[
= N_p + N_q + \sum_{m=1}^{N_q} (\beta_m + 1) - 1 = N_q - 1.
\]  
(A.21)
i.e.
\[ N_{\text{SDE}} = N_{\rho} - 1 \]  \hspace{1cm} (A.22)
which coincides with the counting of \( N_T \) independent paths. This analysis covers the case of
generic \( \rho \) of the class rational function times exponential of rational function.

For the non generic cases, when some of the \( a_\ell \) coincide with some of the \( b_m \), hence
\( F_\ell = G_{m,1} \) for some pairs \((\ell, m)\), those \( G_{m,1} \) appear already at \( n = 0 \) so they do not count
as new coefficients in \( n = 1 \). This reduces \( N_\rho \) to \( N_\rho' \) in the counting, which again implies
\( N_{\text{SDE}} = N_T \).

We do not make an exhaustive analysis of all particular cases but will analyze in more detail
the cases of the type \( \rho(z) = P(z) \), i.e. a rational function. For these densities \( N_\rho = N_\rho' = 0 \) and
\( N_{\rho} = N_{\rho}' = N_\rho \). The SDE give
\[
\begin{align*}
(n = 0) & \quad 0 = \sum_\ell a_\ell F_\ell, \\
(n = 1) & \quad 0 = (1 + \sum_\ell a_\ell) E_0 + \sum_\ell a_\ell a_\ell F_\ell, \\
(n = 2) & \quad 0 = (2 + \sum_\ell a_\ell) E_1 + \sum_\ell a_\ell a_\ell E_0 + \sum_\ell a_\ell a_\ell F_\ell, \quad (A.23) \\
& \quad \ldots .
\end{align*}
\]
The equation for \( n = 0 \) gives \( N_\rho - 1 \) free parameters in the recursion and in general the
remaining equations do not change this. There are two cases:

1. First \( \sum_\ell a_\ell \geq 0 \) (hence some \( a_\ell \) must correspond to zeroes instead of poles). The coeffi-
cient in front of \( E_{n-1} \) in the \( n \)th equation, namely, \( n + \sum_\ell a_\ell \), is not vanishing and the value of \( E_{n-1} \) is determined from the recursion, for all \( n \). Thus no new free parameters are
introduced and \( N_{\text{SDE}} = N_\rho - 1 \), which coincides with \( N_T \) in this case.

2. The other possibility is \( \sum_\ell a_\ell < 0 \). In this case the coefficient of \( E_{n-1} \) vanishes for the
equation with \( n = - \sum_\ell a_\ell \). Generically that equation eliminates (fixes) one free param-
eter (among the \( F_\ell \) and \( E_k \), \( 0 \leq k \leq n - 2 \)) but \( E_{n-1} \) is a new free parameter, hence still
\( N_{\text{SDE}} = N_\rho - 1 \). This is correct when not all \( a_\ell \) are negative, therefore \( \rho \) has zeroes and
\( N_T = N_\rho - 1 \). However, when all \( a_\ell \) are negative the equation for \( n = - \sum_\ell a_\ell \) turns
out to be redundant (a consequence of the previous ones)\(^{10} \) and no free parameter is
eliminated, hence \( N_{\text{SDE}} = N_\rho \), which again matches \( N_T \), since \( \rho \) has only poles and no
zeroes.

### A.2. Periodic densities of rational type

Here we consider periodic densities of rational type. The separate treatment is needed as some
differences appear in the present case.

The class of densities considered is as follows
\[
\rho(z) = \omega^\gamma P(\omega) \exp(R(\omega)), \quad \omega \equiv e^{i\gamma}. \quad (A.24)
\]
Here \( \gamma \in \mathbb{Z} \), and \( P(\omega) \) is the same function (with respect to \( \omega \)) as in (A.3) with the added condition that the \( a_\ell \) cannot be zero (the factor \( \omega^\gamma \) takes care of that). Further, \( R(\omega) \) obeys (A.4)

\(^{10}\) We do not have a closed proof of this. It is trivial when \( N_\rho = 1 \), and it is also easily proven when all \( a_\ell = -1 \) (i.e. simple poles only) but it holds in all cases analyzed by us.
where $R_i(\omega)$ there is as in (A.6), with the new condition that all the $b_m \neq 0$. On the other hand, $Q(\omega)$ is slightly more general:

$$Q(\omega) = \sum_{k=N_0}^{N_+} c_k \omega^k, \quad N_+ \in \mathbb{Z}, \quad N_0 \leq N_+ \quad c_{N_0} \neq 0.$$  \hspace{1cm} (A.25)

The domain will be the space spanned by the Fourier modes in $[0, 2\pi]$

$$D = \text{span} \{ e^{i\omega n} | n \in \mathbb{Z} \} = \text{span} \{ \omega^n | n \in \mathbb{Z} \},$$  \hspace{1cm} (A.26)

and $\Gamma$ contains one path of each equivalence class.

$N\Gamma$ is the total number of closed paths plus the number of zeroes minus one, unless there are no zeroes. This number can be computed as follows. The (finite) singularities come from $R_i(\omega)$ and $P(\omega)$. This gives a number of closed paths

$$N\Gamma = N_p + \sum_{\ell=1}^{N_p} \Theta(\alpha_\ell < 0) + 1, \hspace{1cm} (A.27)$$

where the function $\Theta(x)$ takes the value 1 (resp. 0) when the proposition $x$ is true (resp. false) and the prime indicates to exclude the term if $\alpha_\ell$ equals some $b_m$ in $R_i(\omega)$. The plus one counts the closed path encircling once the cylinder.

The number of finite zeroes is

$$N_{\text{finite}} = \sum_{m=1}^{N_z} \beta_m + \sum_{\ell=1}^{N_p} \Theta(\alpha_\ell > 0). \hspace{1cm} (A.28)$$

Let us count the number of zeroes of $\rho$ at $z = -i\infty$ (i.e. $\omega = \infty$). The factor $Q(\omega)$ gives $(N_+^+)_+$ zeroes, where $(\chi)_+ \equiv \max(x, 0)$. The factor $\omega^\gamma P(\omega)$ would give one zero when $\gamma + \sum \alpha_\ell < 0$ which could be effective only if $(N_+^+)_+ = 0$ (otherwise the essential singularity dominates). However, such zero from $\omega^\gamma P(\omega)$ does not contribute even when $(N_+^+)_+ = 0$ because $\omega^n \rho$ would not go to zero for sufficiently large $n$. Thus

$$N_{\text{c, }-i\infty} = (N_+^+)_+. \hspace{1cm} (A.29)$$

Similarly for $z = +i\infty$ ($\omega = 0$)

$$N_{\text{c, }+i\infty} = (-N_-^+)_+. \hspace{1cm} (A.30)$$

Collecting the various contributions

$$N\Gamma = (N_+^+)_+ + (-N_-^+)_+ + N_p' \sum_{m=1}^{N_z} (\beta_m + 1) + 1 - \Theta(N\Gamma > 0), \hspace{1cm} (A.31)$$

where $N\Gamma = N_{\text{finite}} + N_{\text{c, }-i\infty} + N_{\text{c, }+i\infty}$ is the total number of zeroes. As before in the non-compact setting, $N_0'$ denotes the number of $\alpha_\ell$ in $P(\omega)$ different from any $b_m$ in $R_i(\omega)$. It is noteworthy that our treatment is asymmetric, after choosing to express $\rho(z)$ in terms of $\omega = e^{i\xi}$ instead of $e^{-i\xi}$, yet $N\Gamma$ is unchanged under $z \rightarrow -z$, as it should.

In order to compute $N_{\text{SDE}}$, let us apply $A$ to a generic element of the basis,

$$-iA\omega^n = (n + \gamma)\omega^n + \sum_{\ell=1}^{N_\gamma} \alpha_\ell \frac{\omega^{n+\ell}}{p_\ell} + \sum_{k=N^-_0}^{N_+} kc_k \omega^{n+k} = \sum_{m=1}^{N_z} \beta_m \sum_{r=1}^{q_{m+1}} \omega^{n+1} q_{m+1}. \hspace{1cm} (A.32)$$
For $T \in \text{Ann } AD$, the SDE imply $\forall n \in \mathbb{Z} \ (-i A \omega^n) = 0$. As in the noncompact case, the action of $T$ is fully determined in terms of the basic coefficients

$$E_n \equiv \langle \omega^n \rangle, \quad F_\ell \equiv \langle 1/p_\ell(\omega) \rangle, \quad G_{m,r} \equiv \langle 1/q_m(\omega)^r \rangle. \quad \text{(A.33)}$$

To do the reduction, for positive $n$ the expansion in (A.15) directly applies (with $\omega$ instead of $z$) while for negative $n$ the following expansion applies

$$q(\omega) = \sum_{j=0}^{n_r} (-b)^{n-j} \omega^j + \sum_{j=1}^{r} \left(\frac{n}{r-j}\right) b^{n-j} \omega^j.$$

Using these expansions, from inspection of (A.32) one obtains the following general structure

$$\langle \omega^n \rangle, \langle E_n \rangle, \ldots, E_{n+\mathcal{N}^+} \rangle, \quad \text{(A.35)}$$

from the first and third terms in the r.h.s. of (A.32). The second term gives

$$n \geq 0 \quad \{F_\ell, E_0, \ldots, E_n\},$$

$$n = -1 \quad \{F_\ell\},$$

$$n \leq -2 \quad \{F_\ell, E_{n+1}, \ldots, E_{-1}\},$$

and the fourth term gives (1 \leq r \leq \beta_m)

$$n \geq r \quad \{G_{m,r+1}, \ldots, G_{m,1}, E_0, \ldots, E_{n-r}\},$$

$$-1 \leq n < r \quad \{G_{m,r+1}, \ldots, G_{m,n}\},$$

$$n \leq -2 \quad \{G_{m,r+1}, \ldots, G_{m,1}, E_{n+1}, \ldots, E_{-1}\}. \quad \text{(A.37)}$$

Again, as in the noncompact case, for large $|n|$ the number of new terms increases at the same rate as $|n|$ hence this is a proper recursion, with $N_{\text{SDE}} < \infty$.

Rather than analyzing all cases, let us consider a density $\rho$ with parameters taking generic values, plus the explicit simplifying assumption $N_q^- < 0 < N_q^+$. For $n = 0$ the coefficients involved are

$$(n = 0) \quad \{F_\ell, E_{n^-}, \ldots, E_{n^+}, G_{m,1}, \ldots, G_{m,\beta_m+1}\}, \quad \text{(A.38)}$$

therefore the number of free parameters is

$$N_p + (N_q^+ - N_q^- + 1) + \sum_{m=1}^{N_r} (\beta_m + 1) - 1. \quad \text{(A.39)}$$

For $n = 1, 2, \ldots$ or $n = -1, -2, \ldots$ one finds that each time just one new coefficient is introduced, which gets determined by the equation. Thus no new parameters are generated and

$$N_{\text{SDE}} = N_p + N_q^+ - N_q^- + N_k + \sum_{m=1}^{N_r} \beta_m. \quad \text{(A.40)}$$

As expected, this number matches $N_{\Gamma}$ in (A.31).

The examples 1 and 3 analyzed in section 3 illustrate non generic cases not covered by this discussion.
Appendix B. Vanishing of the net flux through the boundary of a region in a steady state

Let \( P(z) \) be the stationary state of the CL process for a density \( \rho(z) \). \( P \) is normalized and we assume that \( P \) obeys the Fokker–Planck equation (4); if \( v \) has singularities, (4) has to be understood as a weak solution (see below). (4) is just the continuity equation of the CL process:

\[
0 = \partial_t P(z; t) + \partial_x j_x + \partial_y j_y
\]

with density current

\[
j_x = v_x P - \partial_x P, \quad j_y = v_y P.
\]

By the divergence theorem for any closed curve \( \gamma \) that is the boundary of a region \( G \) free of singularities of \( v \), \( \gamma = \partial G \) and avoids any singularities of \( v \), the net flux of \( j \) through \( \gamma \) vanishes.

The same holds in the case of a periodic \( \rho \) for any closed loop \( \gamma \) homotopic to \( z(t) = t + iy_0 \) (\( y_0 \) a constant value and \( t \in [0, 2\pi] \)) and avoiding any singularities of \( v \). In this case the manifold is the cylinder \([0, 2\pi] \times \mathbb{R} \) and \( \gamma \) divides the cylinder in two disconnected regions \( C_+ = \{(x,y), y > y_0\} \) and \( C_- = \{(x,y), y < y_0\} \) with \( \gamma \) as common boundary. Each region carries a fraction of the normalization of \( P \), denoted \( \alpha_\pm \), respectively, with \( 0 \leq \alpha_\pm \leq 1 \) and \( \alpha_+ + \alpha_- = 1 \). The CL process is a driven Brownian process, hence its sample paths can be chosen to be almost everywhere continuous. Therefore the CL walkers can only enter or leave the regions \( C_\pm \) by crossing the boundary \( \gamma \). If the net flux through \( \gamma \) were not zero, the fractions \( \alpha_\pm \) would change under time evolution, which is not possible for a stationary state. As a consequence, \( \int_0^{2\pi} dx j_x(x,y_0) = 0 \) and likewise for any path homotopic to the one written.

If the path \( \gamma \) encloses a singularity of \( v \), we need the assumption that \( P \) is a weak solution of the Fokker–Planck equation, which makes the vanishing of the net flux through \( \gamma \) obvious.

To justify the assumption, consider a small closed path enclosing a singularity; based on experience from CL simulations (see for instance [8]), we expect \( P \) to vanish at least linearly at the singularity; this implies that the current \( j \) remains bounded there and \( P \) is indeed a weak solution of (4). Letting the enclosing path shrink to zero then implies that there is no net flux coming in or out of singularities.

The argument can be extended to conservative processes in any number of dimensions to state that the flux through a closed hypersurface (codimension 1) must vanish in a stationary state. The conditions to obtain such a statement are (i) the stochastic process must have almost everywhere continuous sample paths, (ii) the stationary density must be normalizable, and (iii) the hypersurface must divide the manifold in two regions.

Obviously continuity is needed, even to have a well-defined density current. The normalizability condition is also relevant. Consider for instance the cylinder of the periodic case as the manifold and a constant density there with a constant density current in the \( y \) direction. The flux through the line \( y = 0 \) is not zero even if the system is conservative. The statement does not apply because the fractions \( \alpha_\pm \) are undefined when \( P \) is not normalizable. Finally, the condition that the hypersurface divides the manifold is also relevant. Consider for instance a two-dimensional torus as manifold, from compactification of \([0, 2\pi] \times [-Y, Y] \). Again one can have a constant density and a constant flux in the \( y \) direction without violating flux conservation, yet the flux through the line \([0, 2\pi] \) does not vanish. In this case the statement does not apply because the line \([0, 2\pi] \) does not separate the torus into two regions and the quantities \( \alpha_\pm \) are meaningless.
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