Preparation of edge states by shaking boundaries

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Preparing topological states of quantum matter, such as edge states, is one of the most important direction in condensed matter physics. In this work, we present a proposal to prepare edge states in Aubry-André-Harper (AAH) model with open boundaries. The proposal applies Lyapunov control to design operations (or shaking scheme) exerted on the boundaries. We show that the edge states can be obtained with almost arbitrary initial states. The dependence of the performance on the system size is examined and discussed. An numerical optimization for the control is performed. The merit of this proposal is that the shaking exerts only on the boundaries of the model. As a by-product, an topological entangled state is achieved by elaborately designing the shaking scheme.

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I. INTRODUCTION

Topological insulators (TIs) [1, 2] are new states of quantum matter with profound physical features that have their origin in topology. They have a bulk insulating gap but are distinct from ordinary insulators at stable gapless edge states. The quantum Hall system [3], which is an insulator without any kind of spontaneous symmetry breaking, is the first example of topological insulators.

It is believed that there only exists topological trivial phase in 1D systems due to the lack of symmetries. Recently, it has been shown that the 1D quasiperiodic system [4] shares the same topological non-trivial phase emerged in the 2D integer quantum Hall effect [3]. Indeed, the edge state that characterizes topological property in this system has been observed [4]. And most recently, topologically protected edge state has been demonstrated in the 1D commensurate off-diagonal Aubry-André-Harper model [5], interpreted by the Z2 topological index of the Kitaev model [7]. It is worth noticing that these models can be realized in optical lattices [8–25] and photonic system [26, 27].

State steering is an important task in quantum control. Many protocols [28] have been proposed for this purpose, including the optimal control technique, adiabatic control, the technique of stimulated Raman scattering involving adiabatic passage (STIRAP), and open loop controls based on Lyapunov functions. Among them, the Lyapunov based control has its own advantage due to the effectiveness of the control fields designing. To apply the Lyapunov control, a function called Lyapunov function has to be specified. By the use of this function, control fields can be designed to guarantee the decrease of Lyapunov function. As a consequence, the system would evolve to the target state asymptotically. This control scheme has been studied by several researchers [29–36] and has been applied to diverse fields in physics.

In this paper, by shaking the boundaries, we steer an initial state into the edge states in an one-dimensional optical lattice described by AAH model. The shaking scheme can be established by the Lyapunov control. By choosing different Lyapunov functions, we show distinct convergence behaviors for the system. We find that the edge state can be obtained with arbitrary given initial states by only shaking the on-site energy of the boundaries. In addition, we also explore a feasible way to realize the boundary-boundary steady state entanglement [37–50].

The paper is organized as follows. In Sec. II, we present a general formalism for the Lyapunov control. In Sec. III, we first apply the general method to steer fermions in an optical lattice by shaking only the energy of boundary sites, and study the dynamical behaviors of control system with distinct Lyapunov functions. Then we show how to optimize the Lyapunov function in terms of fidelity and control time. An exploration on the effect of errors on the fidelity of final state is given in Sec. IV. By the use of the elaborately designing control field, we study the behavior of boundary-boundary steady state entanglement in Sec. V. Finally, we conclude in Sec. VI.

II. LYAPUNOV CONTROL

Although the problem of quantum control might be formulated in different ways, the final purpose is mostly to steer a quantum system from an (arbitrary) initial state to a target state by control fields. We start with the dynamics of a closed quantum system under control, governed by the Schrödinger equation ($\hbar = 1$)

\[ |\psi(t)\rangle = -i[H_0 + \sum_k f_k(t)H_k]|\psi\rangle, \]

(1)
where $H_0$ and $H_k$ denote the free and control Hamiltonians of the system and $f_k(t)$ are the control fields. As mentioned in the introduction, the design of the control fields are different from proposal to proposal. Here, we will use the Lyapunov control technique to design the control fields. The essence of the Lyapunov control is to choose a Lyapunov function, which is required to be positive and reaches its minimum (maximum) when the system arrive at the target state. Obviously, the following form of Lyapunov function
\[
V_1 = 1 - |\langle \psi_T | \psi \rangle|^2,
\]
meets the requirement. Here $|\psi_T\rangle$ denotes the target state which is conventionally an eigenstate of the free Hamiltonian. The time derivative of $V_1$ yields
\[
\dot{V}_1 = -2 \sum_k f_k(t) \text{Im}[\langle \psi | \psi_T \rangle \langle \psi_T | H_k | \psi \rangle] = -2 \sum_k |\langle \psi | \psi_T \rangle|^2 f_k(t) \text{Im}[e^{i \text{arg}\langle \psi | \psi_T \rangle} \langle \psi_T | H_k | \psi \rangle],
\]
where $\text{Im}[\cdot]$ stands for the imaginary part of $[\cdot]$ and $\text{arg}\langle \psi | \psi_T \rangle$ is the angle between states $|\psi\rangle$ and $|\psi_T\rangle$. Thus, the condition $\dot{V}_1 \leq 0$ can be satisfied naturally if we choose the control fields $f_k(t) = -A_{1k} \text{Im}[e^{i \text{arg}\langle \psi | \psi_T \rangle} \langle \psi_T | H_k | \psi \rangle]$ with $A_{1k} > 0$. $A_{1k}$ can be used to adjust the amplitude of control field and controls the convergence time. As the Lyapunov function is not unique, it can be constructed distinctly even though the control problem is same.

For example, we may define another Lyapunov function by
\[
V_2 = \langle \psi | P | \psi \rangle,
\]
where the operator $P$ is hermitian and time independent. Additionally, $P$ is also assumed to be positive semidefinite operator acting on the Hilbert space spanned by the eigenvectors of free Hamiltonian $H_0$. With this definition, the time derivative of $V_2$ becomes
\[
\dot{V}_2 = \langle \psi | [H_0, P] | \psi \rangle + \sum_k f_k(t) \langle \psi | [H_k, P] | \psi \rangle.
\]
The Lyapunov control requires that the operator $P$ should be constructed properly to guarantee \[31\],
\[
[H_0, P] = 0.
\]
A straightforward way to construct the operator $P$ is
\[
P = p_f | \lambda_f \rangle \langle \lambda_f | + \sum_{i \neq f} p_i | \lambda_i \rangle \langle \lambda_i |,
\]
where $| \lambda_i \rangle$ ($i = 1, ..., N$) are the eigenstates of free Hamiltonian $H_0$ with corresponding eigenvalues $\lambda_i$, and $| \lambda_f \rangle$ is the target state. Of course, the values of $p_f$ and $p_i$ ($i = 1, ..., N, i \neq f$) can be chosen arbitrarily while the only necessary condition is $p_f < p_i$ \[51\]. Clearly, if we select the control fields $f_k(t) = -A_{2k} \langle \psi | i[H_k, P] | \psi \rangle$ with $A_{2k} > 0$, the condition $\dot{V}_2 \leq 0$ is satisfied naturally. Discussions on the Lyapunov function are in order. The Lyapunov function given in equation (1) with equation (3) is quite general for unitary systems. In fact, the Lyapunov function $V_2$ covers the Lyapunov function $V_1$ \[40\]. This suggests that the optimization over the Lyapunov function reduces to searching a set of $\{p_j\}$ that maximizes the fidelity of final state or minimizes the control time, etc., depending on the optimal function. We will carry out this optimization latter.

According to the Lyapunov control theory, the Lyapunov function $V_k$ will converge to its extremum while the state of system converges to a LaSalle’s invariant set given by $E = \{V_k = 0\}$. When the dimension of LaSalle’s invariant space is more than one, it becomes complicate to control a quantum system from an arbitrary initial state to a given target state. Nonetheless, by elaborately designing the Lyapunov function, we can still steer a quantum system to evolve into a desired state. We will illustrate this point in the next section.

### III. Preparation of Edge State in AAH Model

Consider an one-dimensional Fermi gas loaded in an optical lattice described by,
\[
H_0 = -t \sum_{i=1}^{N-1} (c_i^\dagger c_{i+1} + h.c.) + V \sum_{i=1}^{N} \cos(2\pi\alpha i + \delta) c_i^\dagger c_i,
\]
where $N$ denotes the total number of lattice sites and $t$ is the hopping amplitude and has been set to be 1 throughout this paper. $c_i$ ($c_i^\dagger$) is the fermionic annihilation (creation) operator at lattice site $i$. $V$ denotes the strength of commensurate potential with an rational number $\alpha$. When $\alpha$ is an irrational number, $V$ represents the strength of incommensurate potential which is the well-known Aubry-André model \[8\], namely, all eigenstates are extended (localized) for $V < 2$ ($V > 2$) with a single excitation. Physically, we note that it is impossible to distinguish between an irrational and an rational approximation. Thus we just take $\alpha = 1/3$ for numerical calculation in the following sections. It is also suitable in the quasiperiodic system when $\alpha$ takes irrational number. Other parameters are chosen as $V = 1.5, \delta = 2\pi/3$.

For our system with the open boundary condition, the origin of edge states \{|$\text{Edge}_i$\rangle\} stem from the non-trivial properties of the system, which can be envisioned by mapping to the 2D quantum Hall effects in the Hofstadter problem \[4\]. Especially, eigenstates localized at one of the boundaries can be found by resolving the eigenvalue equation,
\[
H_0 |\lambda_i\rangle = \lambda_i |\lambda_i\rangle, i = 1, ..., N.
\]
The distribution of the two edge states are shown in figure \[4\]. With this knowledge and according to the spirit of
Lyapunov control, it is possible to steer the system from an arbitrary given initial state to an edge state. One of the key points in the control is how to chose the control Hamiltonian. A control easy to implement in experiment is to manipulate the on-site energy of the boundary sites. Hence we restrict the control only on the boundaries of optical lattice here. The control Hamiltonian then can be written as

\[ H_k = c_k^* c_k, \quad k = 1, N, \]

(10)

and the Lyapunov function can be chosen as

\[ V_1 = 1 - |\langle \psi_T | \psi \rangle|^2, \]

(11)

which results in the control fields \( f_k(t) = -A_1 k \text{Im}[e^{i \arg \langle \psi_T | \psi \rangle} \langle \psi_T | H_k | \psi \rangle] \). To be specific, we will consider the optical lattice with \( N = 29 \) sites and \( A_1 k = 1 \) as an example. Our goal is to steer an arbitrary given initial state to one of the edge states, for instance, \( |\psi_T\rangle = |\text{Edge}_N\rangle \).

From the expressions in equation (3) or (5), the Lyapunov functions are time dependent via \( |\psi(t)\rangle \), which implies that the control fields would depend on the initial state at the beginning. So, to design the control fields, we have to know the initial state \( |\psi_0\rangle \), though it might be arbitrary. As the single particle states (i.e., the single excitation occupies only a site \( n \), denoted by \( |n\rangle (n = 1, ..., N) \)) are more easily to prepare, we will consider these states as initial states in the following. Of course, this method is also applicable for an arbitrary superposition of the single particle states. Figure 2 shows the time evolution of the fidelity defined by \( F_{\psi(t)\text{Edge}_N} = |\langle \psi(t) | \text{Edge}_N \rangle| \) and the amplitude of control fields \( f_k(t) \) while the initial state of system is chosen as \( |\psi_0\rangle = |3\rangle \). We find that the control fields \( f_k(t) \) steer the initial state to the edge state eventually. In addition to this, it can also be observed that the control field \( f_2(t) \) plays an important role in the evolution, since the control field \( f_1(t) \) is comparatively small. The physics behinds this result can be understood as follow. The excitation mainly occupies the site 29 for the target state \( |\text{Edge}_N\rangle \), thus the control Hamiltonian \( H_2 \) dominates during the time evolution, as a result, the corresponding control field plays an important role.

Next, we investigate the fidelity \( F_{\psi, \text{Edge}_N} \) (denotes the fidelity between the final state \( |\psi_\lambda\rangle \) and the target state \( |\text{Edge}_N\rangle \)) at the different initial states \( |\psi_0\rangle = |n\rangle (n = 1, ..., N) \), which are illustrated in figure 3. Observing figure 3, it is an intuitive hypothesis that the fidelity \( F_{\psi, \text{Edge}_N} \) might be related to the fidelity \( F_{\psi_0, \text{Edge}_N} \) (denotes the fidelity between the initial state \( |\psi_0\rangle \) and the edge state \( |\text{Edge}_1\rangle \)). In order to confirm it, figure 4(a) demonstrates the relation between the fidelity \( F_{\psi_0, \text{Edge}_N} \) and the fidelity \( F_{\psi, \text{Edge}_N} \). We can see in figure 4(a) that steering the system to the target state becomes more difficult with the increasing of \( F_{\psi_0, \text{Edge}_N} \) because the edge state \( |\text{Edge}_1\rangle \) is also an element of the LaSalle’s invariant set (we will demonstrate it in the following). Therefore, small value of \( F_{\psi_0, \text{Edge}_1} \) benefits the control process when
we set the edge state $|\text{Edge}_N\rangle$ as the target state. This observation also holds true when the initial state is an arbitrary superposition of the other single particle states, which is not shown in figure 4(a). Those results can be explained as follow. As the initial state can be rewritten as a superposition of the eigenstates of free Hamiltonian $H_0$, i.e.,

$$|\psi_0\rangle = a(0)|\text{Edge}_1\rangle + \sum_{i=1}^{29} a_i(0)|\lambda_i\rangle, \quad i \neq 10. \quad (12)$$

Since the edge state $|\text{Edge}_1\rangle$ is an element of the largest invariant set, the amplitude of the edge state $|\text{Edge}_1\rangle$ remains almost unchanged during the control process and the other eigenstates would be steered into the target state. Consequently, it can not be steered into the target state perfectly if the initial state contains the component of the edge state $|\text{Edge}_1\rangle$. The details of proof can be found in the appendix.

As the choice of the Lyapunov function is not unique, we can choose another Lyapunov function to design the control fields for the system, which we have elucidated in Sec. II. By elaborately designing the value of $p_i$ ($i = 1, ..., N$) in equation (7), we can obtain a high fidelity with an arbitrary given initial state. Based on several trials, we choose the operator $P$ as

$$P_1 = p_f|\text{Edge}_N\rangle\langle\text{Edge}_N| + \sum_{i \neq f} p_i|\lambda_i\rangle\langle\lambda_i|, \quad (13)$$

where we have set $p_i = \lambda_i$ and $p_f = -3$ to guarantee $p_f < p_i$. The corresponding control fields are specified to

$$f_k(t) = -A_{2k}^1\langle\psi|H_k, P_1|\psi\rangle$$

with $A_{2k}^1 = 5$. From figure 4(b), we observe that a high steady fidelity of final state can be obtained with an (almost) arbitrarily given initial state. More generally, figure 5 shows a collection of fidelity with an arbitrary given initial state in the form $|\psi_0\rangle = \cos \theta|n\rangle + \sin \theta|m\rangle$, where $n$ and $m$ are stochastic integer created between 1 and 29, while $\theta$ is randomly created in $[0, 2\pi]$. The results are a collection of 500 random initial states.

The above numerical calculations imply that different Lyapunov functions lead to different fidelity and behaviors of convergence due to the distinct largest invariant set. To be more specific, the largest invariant set with Lyapunov function $V_1$ and operator $P_1$. The initial state is $|\psi_0\rangle = \cos \theta|n\rangle + \sin \theta|m\rangle$, where $n$ and $m$ are stochastic integer created between 1 and 29, while $\theta$ is randomly created in $[0, 2\pi]$. We find that it can approach 98.74% on average. Here $n$ and $m$ are integers stochastically created between 1 and 29. This observation argues that the value of $F_{\psi, \text{Edge}_N}$ has slightly effects on the fidelity of final state when the operator $P$ is elaborately constructed.

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The fidelity of final state to the edge state $|\text{Edge}_N\rangle$ as a function of coefficients $p$ and $p_f$. We have chosen the initial state $|\psi_0\rangle = |3\rangle$, the control time $t = 1000$, and the Lyapunov function $V_2$ with the operator $P_1$.

FIG. 6: The fidelity of final state to the edge state $|\text{Edge}_N\rangle$ as a function of coefficients $p$ and $p_f$. We have chosen the initial state $|\psi_0\rangle = |3\rangle$, the control time $t = 1000$, and the Lyapunov function $V_2$ with the operator $P_1$.

FIG. 7: The control time is a function of $p_f$ (pink line with stars) while the fidelity reaches 0.97 and we have fixed $p = 5$. The other parameters are the same as in figure 6.

IV. ROBUSTNESS

In practice, errors in the implementation of the control fields are unavoidable. For example, the Lyapunov control requires to know the initial state, but the information of the initial state acquired may be different from the exact one. The errors may also occur in the application of control fields. In the following, we will explore the effect of these errors on the fidelity of final state. We characterize the errors by $\delta$. Namely, the theoretical exact control fields are $f_k(t)$, while the actual control fields applied in the control are $f'_k(t) = (1 + \delta)f_k(t)$. As to the errors in the initial state, we randomly create a state, and mix it with the initial state (say $|3\rangle$) such that $\delta = 1 - \langle \psi_0 | 3 \rangle$, where $|\psi_0\rangle$ is the ideal initial state and $|3\rangle$ is the ideal initial state. With these considerations, we simulate the fidelity of final state and show the results in figure 8. We find that the unperfect preparation of initial state $|3\rangle$ does not have a serious effect on the fidelity of final state. The control field $f_1(t)$ has a slight influence on the fidelity of final state, but the control field $f_2(t)$ does affect the fidelity of final state. The reason is that the edge state $|\text{Edge}_N\rangle$ is mainly localized on the site $N$ in the optical lattice, then the control field $f_2(t)$ dominates in the control process.

It can be found in figure 9(a) that different initial states manifest distinct dynamics behaviors and convergence time in the control process since the Lyapunov functions rely on the initial state $|\psi(0)\rangle$ through $|\psi(t)\rangle$. In addition, the length of optical lattice can also affect the control time and the time for edge state preparation is approximate linearly proportional to the length of the optical lattice, as shown in figure 9(b).
where $C_n(t)$ represents the probability amplitude of a single particle located at the $n$-th site of optical lattice. In order to get a high degree of boundary-boundary steady entangled states, we deform the Lyapunov function as

$$V_3 = 1 - |(\text{Edge}_1 | \psi \rangle|^2 - |(\text{Edge}_N | \psi \rangle|^2,$$  \hspace{1cm} (17)

and the time derivative of $V_3$ yields

$$\dot{V}_3 = -2 \sum_{k=1,2} f_k(t) \sum_{m=1,N} |(\langle \psi | \text{Edge}_m) \times \text{Im}[e^{i\arg \langle \psi | \text{Edge}_m) \langle \text{Edge}_m | H_k | \psi \rangle].$$  \hspace{1cm} (18)

Hence, the control field can be chosen as $f_k(t) = -A_{3k} \sum_{m=1,N} \text{Im}[e^{i\arg \langle \psi | \text{Edge}_m) \langle \text{Edge}_m | H_k | \psi \rangle]$ with $A_{3k} = 1$.

When investigating the entangled states between sites 1 and sites $N$ in the optical lattice, the remaining sites should be traced out, namely $\rho_{1N} = \text{Tr}_{2...N-1}(\langle \psi(t) | \psi(t) \rangle)$. In the Hilbert space spanned by $\{|00\rangle, |10\rangle, |01\rangle, |11\rangle\}$, we arrive at the following expression for reduced density matrix $\rho_{1N}(t)$

$$\rho_{1N}(t) = \begin{pmatrix}
  a & 0 & 0 & 0 \\
  0 & b & d & 0 \\
  0 & d^* & c & 0 \\
  0 & 0 & 0 & 0
\end{pmatrix}$$  \hspace{1cm} (19)

with $a = \sum_{n=2}^{N-1} |C_n|^2$, $b = |C_1|^2$, $c = |C_N|^2$, $d = C_1 C_N^*$. Then we use the concurrence $\text{C}^*$ to measure the degree of entanglement,

$$\text{C}(\rho) \equiv \max \{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\},$$  \hspace{1cm} (20)

where $\lambda_i$ ($i = 1, 2, 3, 4$) are the square roots of eigenvalues of the matrix $\rho(\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)$ in decreasing order, and $\rho^*$ is the complex conjugate of $\rho$. According to the expression of $\rho_{1N}$, it is straightforward to find that

$$\text{C}_{1N} = \max \{0, 2\sqrt{bc}, 2|d|\}.$$  \hspace{1cm} (21)

Figure 10 illustrates the dynamics behavior of concurrence with initial state $|\psi_0\rangle = |3\rangle$, which shows that the concurrence $\text{C}_{1N}$ approaches to a fixed value 0.74.

FIG. 8: We have assumed that the perfect initial state is $|3\rangle$, where the errors caused by unperfect preparation is defined by $\delta = 1 - |\langle \psi_0 | 3 \rangle|$. The red solid line and blue dash line shows the fidelity of final state versus the errors caused by mismatching control fields $f_1(t)$ and $f_2(t)$ respectively. The other parameters are the same as in figure $[3]$.

FIG. 9: (a) The dynamics evolution of fidelity with different initial states $|\psi_0\rangle = |n\rangle$, $n = 1, ..., 29$. The Lyapunov function is $V_2$ with the operator $P_1$. (b) The control time versus different length $N$ of optical lattice when the fidelity is 0.99. The initial state is $|3\rangle$.

V. BOUNDARY ENTANGLEMENT IN STEADY STATE

For the system under the Lyapunov control, we have shown that the LaSalle’s invariant set is spanned by $\{|\text{Edge}_1\rangle, |\text{Edge}_N\rangle\}$ with Lyapunov function $V_1$, as a result the state of the system would asymptotically converge to

$$|\psi(t)\rangle = \alpha |\text{Edge}_1\rangle + \beta |\text{Edge}_N\rangle.$$  \hspace{1cm} (15)

Equivalently, it can be rewritten in the following form

$$|\psi(t)\rangle = \sum_{n=1}^{N} C_n(t) |n\rangle,$$  \hspace{1cm} (16)
This is shown in figure 11 where the initial state is approximate a superposition of the two states. Although the boundary-boundary entanglement value of concurrence with respect to the boundary-boundary steady entangled state is small, it is symmetry protected and therefore maybe useful in quantum information processing.

VI. DISCUSSIONS AND CONCLUSION

Adiabatic evolution can also be used to realize state preparation \[28\]. We now compare the adiabatic state preparation with the present one. The idea of adiabatic state preparation is as follows: design a Hamiltonian \( H'_1 \) whose ground state is the target state \( |\Psi_T\rangle \) and the ground state \( |\Psi_0\rangle \) of Hamiltonian \( H'_0 \) is easily to prepared. Consider a quantum system governed by the following Hamiltonian,

\[
H' = [1 - \varepsilon(t/T)]H'_0 + \varepsilon(t/T)H'_1,
\]

where \( \varepsilon(t) \) is a slowly varying function of evolution time \( t \) with \( \varepsilon(0) = 0 \) and \( \varepsilon(1) = 1 \). \( T \) is the time needed to finish the preparation. According to the adiabatic theorem, the quantum system of Hamiltonian \( H' \) evolves adiabatically from the initial state (ground state) \( |\Psi_0\rangle \) to the target state (ground state) \( |\Psi_T\rangle \) at time \( t = T \).

Although the target state is a stationary state in both the adiabatic state preparation and the Lyapunov control, the initial states are quite different. It needs a specific initial state (typically the ground state) in the adiabatic control, whereas an arbitrary initial state (except the state in the invariant set) can be steered into the target state in Lyapunov control. In particular, the adiabatic state preparation requires very slow time-evolution. If this is not satisfied, non-adiabaticity would induce a population transfer between the eigenstates, leading to the failure of state preparation. On contrast, it is not required to meet the adiabatic condition in the Lyapunov control.

In conclusion, we have proposed a scheme to prepare edge states in the optical lattice by shaking the boundaries. The work is motivated by the fact that the edge states exhibit interesting physical properties. The shaking scheme is designed by Lyapunov control. The Lyapunov function lies at the heart of the control design. By choosing different Lyapunov function, we show that different control would lead to different fidelity and convergence feature. The control Hamiltonian we use is restricted to the boundaries of the optical lattice. We find that the edge state we obtain depends on the initial states with Lyapunov function \( V_1 \) while it can be obtained for almost arbitrary initial states when using Lyapunov function \( V_2 \) with operator \( P_1 \). In addition, we have discussed the influence of the errors on the fidelity of edge state and analyze the difference between Lyapunov control and adiabatic control. By this proposal, we can also prepare boundary-boundary steady entangled state which is actually the superposition of the two edge states.

APPENDIX

In this appendix, we show that the amplitude of state \( |\text{Edge}_1\rangle \) remains approximately unchanged under the control with Lyapunov function \( V_1 \). In following, we will use the same notions as those in the main text.

Suppose the initial state of this system is

\[
|\psi_0\rangle = a(0)|\lambda_{10}\rangle + \sum_{i=1}^{29} a_i(0)|\lambda_i\rangle,
\]

\[
|\lambda_{10}\rangle = |\text{Edge}_1\rangle, i \neq 10.
\]
The state at time $t$ becomes
\[
|\psi(t)\rangle = a(t)|\lambda_1\rangle + \sum_{i=1}^{29} a_i(t)|\lambda_i\rangle, \quad i \neq 10. \tag{24}
\]
By Schrödinger equation, we can deduce the differential equation for the probability amplitude of the edge state $|\text{Egde}_1\rangle$ ($\hbar = 1$),
\[
 i \cdot \dot{a}(t) = \langle \lambda_{10} | [H_0 + \sum_{k=1}^{2} f_k(t)H_k] |\psi(t)\rangle
\]
\[
= \lambda_{10} \cdot a(t) + \gamma_{10} \cdot a(t) + \sum_{j=1}^{29} \gamma_j \cdot a_j(t), \tag{25}
\]
where $\gamma_j = \langle \lambda_{10} | [f_1(t)H_1 + f_2(t)H_2] |\lambda_j\rangle$, $j = 1, \ldots, 29$. Subsequently, one can estimate the order of magnitude in $H_1|\lambda_{20}\rangle$ and $H_2|\lambda_{10}\rangle$, i.e.,
\[
H_1|\lambda_{20}\rangle = H_1|\psi_T\rangle \sim 10^{-5} \cdot (1, 0, \ldots, 0)^T, \tag{26}
\]
\[
H_2|\lambda_{10}\rangle \sim 10^{-5} \cdot (0, \ldots, 0, 1)^T,
\]
where the superscript $T$ represents transposition and the order of magnitude in the function $f_1(t)$ can be estimated as well:
\[
f_1(t) = \Im \left[ e^{\arg(\langle \psi_T | H_1 |\psi\rangle)} \langle \psi_T | H_1 |\psi\rangle \right] \sim 10^{-5}. \tag{27}
\]
Those estimations give the order of magnitude in $\gamma_j$:
\[
|\gamma_j| = |f_1(t)(\lambda_{10}|H_1|\lambda_j) + f_2(t)(\lambda_{10}|H_2|\lambda_j)| \
\sim 10^{-5} \ll |\lambda_{10}| = 1.13. \tag{28}
\]
Therefore the differential equation of $a(t)$ can be calculated approximately
\[
i \cdot \dot{a}(t) \simeq \lambda_{10} \cdot a(t), \tag{29}
\]
leading to
\[
|a(t)|^2 \simeq \text{constant} = |a(0)|^2. \tag{30}
\]
Hence, the amplitude of state $|\text{Egde}_1\rangle$ remains approximately unchanged during the control process with Lyapunov function $V_1$.

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