ON THE BARTNIK MASS OF NON-NEGATIVELY CURVED CMC SPHERES

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ABSTRACT. Let \( g \) be a smooth Riemannian metric on \( S^2 \) and \( H > 0 \) a constant. We establish an upper bound for the corresponding Bartnik mass \( m_B(S^2, g, H) \) assuming that the Gauss curvature \( K_g \) is non-negative. Our upper bound approaches the Hawking mass \( m_H(S^2, g, H) \) when either \( g \) becomes round or else \( H \to 0^+ \), the bound is zero for \( H \) sufficiently large, and in any case the bound is not more than \( r/2 = m_H(S^2, g, 0) \). We obtain upper bounds on \( m_B(S^2, g, H) \) as well in the case when \( g \) is arbitrary and \( H \) is sufficiently large depending on \( g \).

1. Introduction

Let \( g \) be a smooth metric and \( H \) a smooth function on \( S^2 \). The Bartnik mass \([2]\) of the triple \((S^2, g, H)\) is defined as

\[
m_B(S^2, g, H) := \inf_{(M, \gamma)} \{m_{ADM}(M, \gamma) : (M, \gamma) \text{ is an admissible extension of } (S^2, g, H) \}
\]

where \( m_{ADM}(M, \gamma) \) is the ADM mass \([1]\) and admissibility means that \( M = [0, \infty) \times S^2 \) and \( \gamma \) is a smooth asymptotically flat Riemannian metric having non-negative scalar curvature such that \( \gamma_{\partial M} = g \), and \( \partial M \) has mean curvature \( H \) and is outer minimizing in \( M \). The Bartnik mass associates a notion of quasi-local mass to a closed compact 3 dimensional region \( \Omega \) of an asymptotically flat space-like hypersurface in a Lorentzian 4-manifold, where \( \partial \Omega = S^2 \) and \( g \) and \( H \) are respectively the metric and mean curvature induced from the associated Riemannian metric \( g_\Omega \) on \( \Omega \). Assuming the dominant energy condition and time symmetric setting, \( g_\Omega \) will have non-negative scalar curvature, and if \((M, \gamma)\) is further required to smoothly extend \((\Omega, g_\Omega)\) in \([1]\) as in the original formulation in \([2]\), then \( m_B(S^2, g, H) \) will be non-negative by the positive mass theorem \([13]\).

Another such notion of quasi-local mass associated to the triple \((S^2, g, H)\) is the Hawking mass:

\[
m_H(S^2, g, H) := \sqrt{\frac{\text{Area}(S^2, g)}{16\pi}} \left( 1 - \int_{S^2} H^2 \right).
\]

The proof of the Riemann Penrose inequality in \([7]\) implies that the Bartnik mass is bounded below by the Hawking mass thus providing a positive lower bound for the Bartnik mass when \( m_B(S^2, g, H) > 0 \).

In this article we will focus on the problem of bounding the Bartnik mass from above. In \([9]\) Mantoulidis and Schoen showed that when the operator \(-\Delta_g + K_g\) has positive first eigenvalue and \( H = 0 \), then the Bartnik mass actually attains the above lower bound as given by the Riemann Penrose inequality so that one has

\[
m_B(S^2, g, 0) = m_H(S^2, g, 0) = \sqrt{\text{Area}(S^2)/16\pi}.
\]

In \([5]\) the authors of this paper extended the above result to the degenerate case when the operator \(-\Delta_g + K_g\) has zero first eigenvalue. In \([4, 11, 8]\) various upper bounds for Bartnik mass were obtained under the assumption that the Gauss curvature \( K_g \) is strictly positive and that \( H \) is a positive constant. In \([12]\) assuming the Gauss curvature \( K_g \) is strictly positive and that \( H \) is a positive function, Miao-Xie adapted the method in \([14]\) and its variation in \([10]\) to construct admissible extensions with zero scalar curvature and ADM mass arbitrarily close to \( \sqrt{\text{Area}(S^2)/16\pi} \) from above, thus establishing the bound

\[
m_B(S^2, g, H) \leq \sqrt{\text{Area}(S^2)/16\pi}.
\]

In Theorem \([11]\) below, assuming the Gauss curvature \( K_g \) is non-negative and that \( H \) is a positive constant we obtain an upper bound for the Bartnik mass where the bound is not more than \( \sqrt{\text{Area}(S^2)/16\pi} \) and is in fact equal to zero for all \( H \) sufficiently large, while the bound approaches the Hawking mass as \( g \) converges smoothly to a round metric and the Hawking mass is non-negative. Theorem \([11]\) follows in part

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from Theorem 1.2 which also implies an upper bound for the Bartnik mass for an arbitrary metric \( g \) provided \( H \) is sufficiently large depending on \( g \), where again the upper bound equals zero for \( H \) sufficiently large.

**Theorem 1.1.** Let \( g \) a smooth Riemannian metric on \( S^2 \) with \( K_g \geq 0 \) and \( H > 0 \) a constant. Then if \( D = D(g, H) \) is the constant in Definition 2 we have

\[
 m_B(S^2, g, H) \leq \max \left( \min \left( \frac{r\sqrt{1+D}}{2} \left[ 1 - \frac{r^2H^2}{4(1+D)} \right], \frac{r}{2} \right), 0 \right)
\]

where \( r = \sqrt{\text{Area}(S^2, g)}/4\pi \). Moreover, \( D(g, H) \) satisfies \( D \to 0 \) if either \( H \to 0^+ \) or \( g \) converges smoothly to a round metric.

**Theorem 1.2.** Let \( g \) a smooth Riemannian metric on \( S^2 \) and \( H > 0 \) a constant. Suppose there exists a \((g, H)\)-admissible path \( g(t) \) as in Definition 2. Then if \( D = D(g, H) \) is the constant in Definition 2 we have

\[
 m_B(S^2, g, H) \leq \max \left( \frac{r\sqrt{1+D}}{2} \left[ 1 - \frac{r^2H^2}{4(1+D)} \right], 0 \right)
\]

**Remark 1.1.** The expression \( \frac{r\sqrt{1+D}}{2} \left[ 1 - \frac{r^2H^2}{4(1+D)} \right] \) approaches the Hawking mass \( m_H(S^2, g, H) \) as \( D \to 0 \).

**Remark 1.2.** Given any smooth metric \( g \), it is proven in Proposition 2.1 that a \((g, h)\)-admissible path exists when either \( K_g \geq 0 \) or else when \( H \) is sufficiently large depending on \( g \).

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## 2. Admissible paths and the definition of \( D(g, H) \)

Given a smooth metric \( g \) on \( S^2 \) we define

\[
 r_g := \sqrt{\text{Area}(S^2, g)}/4\pi \\
 K_g := \text{Gauss curvature of } g
\]

Now we define the following notions of admissible paths of metrics starting from \( g \), and the corresponding constant \( D(g, H) \) in Theorem 1.1.

**Definition 1.** Given a smooth metric \( g \) on \( S^2 \), a smooth path of metrics \( \zeta = g(t) \) for \( t \in [0, 1] \) is called \( g \)-admissible if

- a) \( g(0) = g \) and \( g(1) \) is round (has constant curvature),
- b) \( \text{tr}_g g' \equiv 0 \) for all \( t \).

**Definition 2.** Given a smooth metric \( g \) on \( S^2 \) and a constant \( H > 0 \). A \( g \)-admissible path \( \zeta = g(t) \) is called \((g, H)\)-admissible if there exists a positive constant \( C > 0 \) for which

\[
 \frac{4C^2}{H^2} K_{g(t)}(1 + C\sqrt{t}) - 2t|g'|^2(1 + C\sqrt{t})^2 + C^2 > 0 \quad \text{on } S^2 \times [0, 1].
\]

Defining \( C_\zeta \) to be the infimum of all such constants \( C \), we define \( D(g, H) \) as

\[
 D(g, H) := \inf_{(g, H)\text{-admissible paths } \zeta} C_\zeta.
\]

The following proposition confirms the existence of admissible paths when \( K_g \geq 0 \) or else \( H \) is sufficiently large depending on \( g \), in which cases lower bounds for \( D(g, H) \) are also provided.

**Proposition 2.1.** Let \( g \) be a smooth metric and \( H > 0 \).
1. If $K_g \geq 0$ then $(g, H)$-admissible paths $\zeta = g(t)$ exist, one can be chosen to satisfy
\[
\frac{4K_g(t) + H^2}{H^2} - \sqrt{2t^2|g'(t)|^2} > 0 \quad \text{on } S^2 \times [0, 1],
\]
and for any such path we have the estimate:
\[
D(g, H) \leq \max_{S^2 \times [0, 1]} \sqrt{\frac{2t^2|g'(t)|^2}{4K_g(t) + H^2}} - \sqrt{\frac{2t^2|g'(t)|^2}{H^2}}.
\]
2. If $H$ is sufficiently large depending on $g$ then $(g, H)$-admissible paths exist. In particular, a $g$-
admissible path $\zeta = g(t)$ (independent of $H$) can always be chosen to satisfy
\[
1 - 2t^2|g(t)|^2 > 0 \quad \text{on } S^2 \times [0, 1],
\]
and if we then choose any constant $C$ satisfying
\[
C > \max_{S^2 \times [0, 1]} \frac{\sqrt{2t^2|g'(t)|^2}}{1 - \sqrt{2t^2|g'(t)|^2}},
\]
then $\zeta$ will be $(g, H)$-admissible provided $H$ satisfies
\[
H \geq \max_{S^2 \times [0, 1]} \sqrt{\frac{4K_g(t)(1 + C\sqrt{t})C^2}{2t|g'(t)|^2(1 + C\sqrt{t})^2 - C^2}}
\]
(where the definition of $C$ guarantees the positivity of the denominator) and in this case we have the
estimate $D(g, H) \leq C$.

Proof. Given any smooth metric $g$, we may write $g = e^{u(x)}g_*$ for a round metric $g_*$ with area $4\pi$. The
construction in [9] (Proposition 1.1 and Lemma 1.2) shows that given any smooth non-increasing function
$\alpha(t) : [0, 1] \to [0, 1]$ with $\alpha(0) = 1$ and $\alpha(1) = 0$, a function $a(t)$ and a family of diffeomorphisms $\xi_t : S^2 \to S^2$
can be chosen so that $\zeta = h(t) := \xi_t^*(e^{2\alpha(t)u(x)} + a(t))g_*$ is a $g$-admissible path. In particular, choosing $\alpha(t) = 1$
in a neighborhood of $t = 0$ ensures that $a(t)$ must be constant, that $\xi_t$ remains constant by its construction,
and thus $h$ remains constant for all $t > 0$ sufficiently small. Fix such a path and consider the modified path
of metrics $h_c(t)$ for $t \in [0, 1]$, defined as
\[
h_c(t) = \begin{cases} 
  g & 0 < t \leq e^{-\frac{c}{2}} \\
  h(c \log t + 1) & e^{-\frac{c}{2}} \leq t \leq 1
\end{cases}
\]
First, we note that since $h(t)$ is assumed to be constant in a neighborhood of $t = 0$ it follows that $h_c(t)$
is smooth on $S^2 \times [0, 1]$ for all $c > 0$. By the facts that $h(t)$ itself is admissible, and that $h_c(t)$ is just a
re-parametrization of $h(t)$ we see that $h_c(t) = g$ admissible as in Definition [1]. We now proceed to complete
the proof of each part of the Proposition separately.

Part 1. In this case we shrink $c$ if necessary so that
\[
0 < c < \sqrt{\min_{S^2 \times [0, 1]} \frac{4K_{h_c(t)} + H^2}{2H^2 \max_{S^2 \times [0, 1]} |h'(s)|^2}}
\]
which gives
\[
\max_{S^2 \times [0, 1]} 2t^2|h_c'(t)|^2 = \max_{S^2 \times [e^{-\frac{c}{2}}, 1]} 2t^2|h_c'(t)|^2 = \max_{S^2 \times [0, 1]} c^2 2|h'(s)|^2 < \min_{S^2 \times [0, 1]} \frac{4K_{h_c(t)} + H^2}{H^2}
\]
and thus allows us to define
\[
C := \max_{S^2 \times [0, 1]} \sqrt{\frac{4K_{h_c(t)} + H^2}{H^2} - \sqrt{\frac{2t^2|h'(t)|^2}}}
\]
which in turn implies

\[ 0 \leq K_{h_c(t)} \frac{4C^2}{H^2} - 2t|h_c'(h_c)\|^2(1 + C\sqrt{t})^2 + C^2 \quad \text{on } S^2 \times [0, 1] \]

which is in fact a stronger inequality than (5) thus making \( h_c(t) \) a \((g, H)\) admissible path and also establishing the estimate for \( D(g, H) \) in (7).

**Part 2.** In this case, by the calculation in (11) we see that we may shrink \( c \) if necessary so that the \( g\)-admissible path \( h_c(t) \) also satisfies (3). If we take \( C \) as in (9), then

\[ -2t|h_c'(h_c)|^2(1 + C\sqrt{t})^{-2}C^2 > 0 \quad \text{on } S^2 \times [0, 1] \]

and it follows that when \( H \) satisfies (10) we have

\[ K_{h_c(t)}(1 + C\sqrt{t})^{-1} \frac{4C^2}{H^2} - 2t|h_c'(h_c)|^2(1 + C\sqrt{t})^{-2}C^2 > 0 \quad \text{on } S^2 \times [0, 1] \]

making \( h_c(t) \) a \((g, H)\)-admissible path and also establishing the stated estimate for \( D(g, H) \).

\[ \square \]

**Remark 2.1.** The construction of the \((g, H)\)-admissible path \( \zeta = h(t) \) from [9] (Proposition 1.1 and Lemma 1.2) referred to in the proof of Proposition 2.1 implies that

1. if \( g \) is arbitrarily close to a fixed round metric on \( S^2 \) (in each \( C^k \) norm relative to a fixed metric), then \( |g(t)|^2 \) will be arbitrarily close to 0 on \( S^2 \times [0, 1] \).
2. if \( K_g \geq 0 \), then \( K_{h(t)} > 0 \) for all \( t > 0 \).

It follows that if either \( g \) is arbitrarily close to a fixed round metric on \( S^2 \), or \( H \) is arbitrarily large, the inequality

\[ \frac{4C^2}{H^2} K_{g(t)}(1 + C\sqrt{t}) - 2t|g'|^2(1 + C\sqrt{t})^2 + C^2 > 0 \quad \text{on } S^2 \times [0, 1] \]

will be satisfied for \( C > 0 \) arbitrarily small, and thus \( D(g, H) \) from definition 2 will be arbitrarily close to 0.

**Proposition 2.2.** Let \( g \) be a smooth metric and \( H > 0 \) a constant such that there exists a \((g, H)\)-admissible path. Then given any \( \epsilon > 0 \), a \((g, H)\)-admissible path \( \zeta = g(t) \) can be chosen so that

\[ C_\zeta \leq D(g, H) + \epsilon \]

and also \( g(t) = g(1) \) for all \( t \in [1 - \theta, 1] \) for some \( \theta > 0 \). Moreover,

**Proof.** Let \( \epsilon > 0 \) be given and consider a \((g, H)\)-admissible path \( \zeta = g(t) \) satisfying (5) for some \( C \leq D(g, H) + \epsilon \). Namely,

\[ (13) \quad \frac{4C^2}{H^2} K_{g(t)}(1 + C\sqrt{t}) - 2t|g'|^2(1 + C\sqrt{t})^2 + C^2 > 0 \quad \text{on } S^2 \times [0, 1] \]

We will construct a family of \((g, H)\)-admissible paths \( g_\theta(t) \) satisfying \( g_\theta(t) = g_\theta(1) \) for \( t \in [1 - \theta, 1] \) so that for all \( \theta > 0 \) sufficiently small, (13) is satisfied with the same constant \( C \) but with \( g(t) \) replaced with \( g_\theta(t) \). For each \( \theta \in (0, 1/3) \), we consider a smooth auxiliary function \( \sigma_\theta : [0, 1] \to [0, 1] \) satisfying

\[
\begin{align*}
\sigma_\theta(t) &= \frac{1}{1 - 2\theta}, \quad \forall t \in [0, 1 - 3\theta] \\
\sigma_\theta(t) &= 1, \quad \forall t \in [1 - \theta, 1] \\
0 &\leq \sigma_\theta'(t) \leq \frac{1}{1 - 2\theta}, \quad \forall t \in [0, 1].
\end{align*}
\]

Such a function can be constructed by mollification as discussed in [4]. Then the path \( \zeta_\theta = g_\theta(t) := g(\sigma_\theta(t)) \) satisfies \( g_\theta(t) = g(1) \) for all \( t \in [1 - \theta, 1] \). Moreover, we have

\[ 2t|g_\theta'(t)|^2 \frac{g'(g_\theta(t))}{g'(\sigma_\theta(t))^2} \frac{(2\sigma_\theta(t))^2}{\sigma_\theta(t)} \leq \frac{t}{\sigma_\theta(t)(1 - 2\theta)^2} (2\sigma_\theta(t)|g'(\sigma_\theta(t))^2|g'(g_\theta(t))^2) \]

\[ (14) \]
and thus letting \( s = \sigma_\theta(t) \), we have
\[
\frac{4C^2}{H^2} K_{g_\theta(t)}(1 + C \sqrt{t}) - 2|g_\theta'(t)|^2_{g(\sigma_\theta(t))}(1 + C \sqrt{t})^2 + C^2 \geq 0.
\]
(15)

On the other hand, as \( t/\sigma_\theta(t) \to 1 \) uniformly for \( t \in [0, 1] \) as \( \theta \to 0 \) it follows from (13) that for \( \theta > 0 \) sufficiently small, the expression on the LHS of (15) is positive for \( t \in [0, 1] \), thus making \( g_\theta(t) \) a \((g, H)\)-admissible path.

\[\square\]

3. Collar Metrics on \((S^2 \times [0, 1], \gamma)\) and an Extension Result

The approach used in \cite{9} to constructing admissible extensions of a given metric \( g \) on \( S^2 \) is to first construct a suitable collar metric \( \gamma \) on \( S^2 \times [0, 1] \) which extends \( g \), then glue this isometrically to a Riemannian manifold/extension \((S^2 \times (0, \infty), \bar{g})\) for \( 0 < b < 1 \) so that the ADM mass of the extension is suitably controlled. We will follow this approach here.

We first collect some basic formulas we will use for our collar construction. Let \( t \mapsto g(t) \) be a \( g \)-admissible path of metrics. Let \( M = (S^2 \times [0, 1], \gamma) \) where
\[ \gamma = E(t)g(t) + \Phi(t)^2 dt^2. \]

Write \( h(t) = E(t)g(t) \) to simplify our notation to \( \gamma = h(t) + \Phi(t)^2 dt^2. \)

**Mean Curvature:** Fix any \( t \in [0, 1] \). The mean curvature of the sphere \( S^2 \times \{t\} \) as a submanifold of \((\Sigma, \gamma)\) is \( H_t = \text{tr}_{h(t)}(\rho) \) where \( \rho = (N, II) \gamma \). Here \( N = -\frac{\partial}{\partial \gamma(t)} \overline{\gamma} \) is the unit normal and \( II \) is the second fundamental form. To calculate this, let \( E_1, E_2 \) be a local coordinate frame on \( S^2 \times \{t\} \). Then
\[ \rho(t)_{ij} = (N, II(E_i, E_j)) \gamma = \gamma^{ab} N^a ((\nabla E_E, E_j) \gamma)^b = \gamma^{ab} N^a ((\nabla^E, E_j)^a)^b \]
\[ = \gamma^{\alpha\mu} \frac{1}{\Phi(t)} \Gamma^{t}_{ij} = \gamma^{\alpha\mu} \frac{1}{\Phi(t)} \left( \frac{1}{2} \gamma^{tt} \gamma_{ij} \right) = \frac{1}{2\Phi(t)} h_{ij}. \]

Since \( h(t) = E(t)g(t) \), we have
\[ \dot{h} = E'(t)g(t) + E(t)\dot{g}(t) \implies \text{tr}_\gamma h = E^{-1}\text{tr}_g \dot{h} = 2E'(t)E^{-1}(t). \]

Note that we used \( \text{tr}_g \gamma = 0 \) in the above calculation. Therefore on \( \Sigma_t := S^2 \times \{t\} \), the foliating sphere at time \( t \) we have
\[ H_t = \text{tr}_{h(t)}(\rho) = \frac{1}{\Phi(t)} \text{tr}_\gamma(h) = \frac{1}{\Phi(t)} E'(t)E^{-1}(t). \]

**Scalar curvature:** The scalar curvature of \( \gamma \) is given by the formula \cite{12}:
\[ R_\gamma = 2K_{h(t)} + \Phi^{-2} \left[ -\text{tr}_h h'' - \frac{1}{4}(\text{tr}_h h')^2 + \frac{3}{4}|h'|^2_h + \frac{\partial_t \Phi}{\Phi} \text{tr}_h h' \right]. \]

By some basic calculations, we have
\[ \text{tr}_h h' = 2E^{-1}E', \quad \text{and} \quad |h'|^2_h = E^{-2}(E')^2 + E^2|g'|_g^2. \]

Using this, we have
\[ \frac{1}{4}(\text{tr}_h h')^2 + \frac{3}{4}|h'|^2_h = -E^{-2}(E')^2 + \frac{3}{2}E^{-2}(E')^2 + \frac{3}{4}|g'|_g^2 = \frac{1}{2}E^{-2}(E')^2 + \frac{3}{4}|g'|_g^2. \]

We know that \( K_{h(t)} = E(t)^{-1}K_g(t) \), and we also have
\[ \text{tr}_h h'' = 2E^{-1}E'' + \text{tr}_g g'' \quad \text{and} \quad 0 = [(\text{tr}_g g')'] = \text{tr}_g g'' - |g'|_g^2 \implies \text{tr}_g g'' = |g'|_g^2. \]
This gives
\[
R_{\gamma} = 2K_{h(t)} + \Phi^{-2} \left[ -\text{tr}_h h'' - \frac{1}{4} (\text{tr}_h h')^2 + \frac{3}{4} |h'|_g^2 + \frac{\partial_t \Phi}{\Phi} \text{tr}_h h' \right] \\
= 2E^{-1}K_{g(t)} + \Phi^{-2} \left[ -2E^{-1}E'' - \text{tr}_g g'' + \frac{1}{2} E^{-2}(E')^2 + \frac{3}{4} |g'|_g^2 + 2E^{-1}E' \Phi \frac{\partial_t \Phi}{\Phi} \right] \\
= \Phi^{-2} \left[ 2E^{-1}K_{g(t)}(t)^2 - 2E^{-1}E'' - \frac{1}{4} |g'|_g^2 + \frac{1}{2} E^{-2}(E')^2 + 2E^{-1}E' \Phi \frac{\partial_t \Phi}{\Phi} \right] \\
= E^{-1}\Phi^{-2} \left[ 2K_{g(t)}(t)^2 - 2E'' - \frac{1}{4} E|g'|_g^2 + \frac{1}{2} E^{-1}(E')^2 + 2E' \Phi \frac{\partial_t \Phi}{\Phi} \right].
\]

(Bartnik Mass): Recall that the Hawking mass of a foliating sphere \( \Sigma_t \) is given by
\[
m_H(\Sigma_t, h(t), H_t) = \frac{\sqrt{|\Sigma_t|}}{16\pi} \left( 1 - \frac{1}{16\pi} \int_{\Sigma_t} H_t^2 \right).
\]

Now using the facts that (1) the path \( g(t) \) has constant pointwise area form, and (2) scaling a metric by a constant factor scales the area by the same constant factor, we have \(|\Sigma_t| = 4\pi r_g^2 E(t)\) for each \( t \in [0,1] \). From our mean curvature calculations above, we have
\[
H_t = \frac{E'(t)E(t)^{-1}}{\Phi(t)}.
\]

Then
\[
m_H(\Sigma_0) = \frac{\sqrt{|\Sigma_0|}}{16\pi} \left( 1 - \frac{1}{16\pi} \int_{\Sigma_0} H_0^2 \right) = \frac{r_g}{2} \left( 1 - \frac{r_g^2 H_0^2}{4} \right).
\]

Similarly,
\[
m_H(\Sigma_1) = \frac{\sqrt{|\Sigma_1|}}{16\pi} \left( 1 - \frac{1}{16\pi} \int_{\Sigma_1} H_1^2 \right) = \frac{r_g}{2} \sqrt{E(1)} \left( 1 - \frac{r_g^2 E'(1)^2 E(1)^{-1}}{4\Phi(1)^2} \right).
\]

After constructing suitable collars, we will use the following gluing/extension result from [4] to produce admissible extensions with control on their ADM mass.

**Proposition 3.1.** (Proposition 2.1 in [4]) Consider a smooth metric \( \gamma = f(t)g(t) + dt^2 \) on a cylinder \( (\mathbb{S}^2 \times [0,1], \gamma) \). Suppose

1. \( \gamma \) has positive scalar curvature
2. \( g(t) = g_s \) (the standard round metric) and \( f'(t) > 0 \) for all \( s \in [a,1] \) for some \( 0 < a < 1 \)
3. \( \mathbb{S}^2 \times \{1\} \) has positive mean curvature \( H_1 \)
4. \( m_H(\mathbb{S}^2 \times \{1\}, f(1)g(1), H_1) \geq 0 \).

Then given any \( \epsilon > 0 \) there exists a smooth rotationally symmetric metric \( \tilde{\gamma} \) on the manifold with boundary \( \mathbb{S}^2 \times [a,\infty) \) such that

1. for sufficiently large \( c > a \), \( (\mathbb{S}^2 \times (c,\infty), \tilde{\gamma}) \) is isometric to an exterior region of the Schwarzschild manifold of mass \( m := m_H(\mathbb{S}^2 \times \{1\}, f(1)g(1), H_1) + \epsilon \):
   \[
   (\mathbb{S}^2 \times (2m,\infty), r^2 g_s + \frac{1}{1 - \frac{2m}{r}} dr^2)
   \]
2. \( \tilde{\gamma} = \gamma \) on \( \mathbb{S}^2 \times [a,(a+1)/2] \)

and thus in particular we have
\[
m_B(\mathbb{S}^2 \times \{0\}, f(0)g(0), H_0) \leq m_H(\mathbb{S}^2 \times \{1\}, f(1)g(1), H_1)
\]
4. Proof of Theorem 1.2

Theorem 1.2 will follow immediately from the following Theorem, the definition of $D(g,H)$ in Definition 2 and Proposition 2.2.

**Theorem 4.1.** Let $g$ be a Riemannian metric on $S^2$ and $H > 0$ a constant and let $\zeta = g(t)$ be an admissible path, so that in particular,

$$
\frac{4C^2}{H^2} K_{g(t)}(1 + C \sqrt{t}) - 2t|g'|^2(1 + C \sqrt{t})^2 + C^2 > 0 \quad \text{on } S^2 \times [0,1]
$$

for some $C > 0$. Suppose further that $g(t) = g(1)$ for $t < 1$ sufficiently close to 1.

Then

$$
m_B(S^2, g, H) \leq \max \left( \frac{r_g \sqrt{1 + C}}{2}, 0 \right).
$$

**Proof.** Let $g, H$ and $g(t)$ be as in the Theorem. Consider the smooth Riemannian 3 manifold with boundary $\tilde{M} = (S^2 \times [0,1], \gamma)$ where

$$
\gamma = (1 + 2Hs)g \left( \frac{s^2}{4A^2} \right) + ds^2
$$

where $C$ is defined in the Theorem and $A = \frac{C}{2\sqrt{t}}$. We will show that $R_\gamma \geq 0$ on $\tilde{M}$ and that the mean curvature of $S^2 \times \{t\}$ in $\tilde{M}$ is positive and approaches $H$ as $t \to 0$. We will do this after changing variables to $t = s^2/(4A^2)$ on $M = (S^2 \times (0,1])$ in which case $\gamma$ takes the form

$$
\gamma = E(t)g(t) + \Phi(t)^2 dt^2
$$

where $E(t) = 1 + C \sqrt{t}$ and $\Phi(t) = A/\sqrt{t}$. In these coordinates, we will show that $R_\gamma \geq 0$ on $M$ while the mean curvature $H_t$ of the foliating spheres in $M$ are positive and approach $H$ as $t \to 0$.

**Claim 1** (mean curvature): $H_t > 0$ is positive and approaches $H$ as $t \to 0$:

By (16) and our definitions of $E$ and $\Phi$ we have

$$
H_t = \frac{E'(t)}{\Phi(t) E(t)} = \frac{C}{2\sqrt{t}} \frac{E_{\pm}^2}{(1 + C \sqrt{t})} = \frac{H}{1 + C \sqrt{t}}
$$

Clearly $H_t > 0$ for all $t \in (0,1]$ and $H_t \to H$ uniformly as $t \searrow 0$.

**Claim 2** (scalar curvature): $R_\gamma \geq 0$ on $M$.

From our definition of $E$ and $\Phi$ we have

$$
-2E^{-1}E'' + 2E^{-1}E' \frac{\partial \Phi}{\Phi} = E^{-1} \left[ -2 \left( \frac{C}{4} t^{-3/2} \right) + 2 \left( \frac{1}{2} C t^{-1/2} \right) \left( \frac{-At^{-3/2}}{2At^{-1/2}} \right) \right] \equiv 0.
$$

and it follows from (18) that

$$
R_\gamma \geq 2E^{-1}K_{g(t)} + \Phi^{-2} \left[ -\frac{1}{4} |g'|^2 + \frac{E^{-2}(E')^2}{2} \right]
$$

$$
= 2(1 + C \sqrt{t})^{-1} K_{g(t)} + \frac{4H^2t}{C^2} \left[ -\frac{1}{4} |g'|^2 + \frac{1}{2} \left( 1 + C \sqrt{t} \right)^{-2} C^2 \right]
$$

$$
= (1 + C \sqrt{t})^{-2} \frac{H^2}{2C} \left( \frac{4C^2}{H^2} K_{g(t)}(1 + C \sqrt{t}) - 2t|g'|^2(1 + C \sqrt{t})^2 + C^2 \right).
$$

$$
\geq 0
$$

on $S^2 \times [0,1]$ by (20).

**Hawking mass of** $(\Sigma_1 := S^2 \times \{1\})$: From (19) and our definition of $E$ and $\Phi$ we have

$$
m_H(\Sigma_1) = \sqrt{\frac{|\Sigma_1|}{16\pi}} \left( 1 - \frac{1}{16\pi} \int_{\Sigma_1} H_1^2 \right) = \frac{r_g \sqrt{1+C}}{2} \left( 1 - \frac{r_g^2 E'(1)^2 E(1)^{-1} - 1}{4\Phi(1)^2} \right) = \frac{r_g \sqrt{1+C}}{2} \left( 1 - \frac{r_g^2 H^2}{4(1+C)} \right)
$$
In conclusion, we have established that the metric
\[ \gamma = (1 + \frac{C}{2A}s)g \left( \frac{s^2}{4A^2} \right) + ds^2 \]
on \((S^2 \times [0, 1], \gamma)\), introduced at the beginning of the proof, satisfies the hypothesis of Proposition 3.1 provided \( H < \frac{2\sqrt{1+C}}{r_g} \), in which case we conclude
\[ m_B(S^2, g, H) \leq m_H(\Sigma_1) = \frac{r_g \sqrt{1+C}}{2} \left( 1 - \frac{r_g^2 H^2}{4(1+C)} \right). \]
Now suppose \( H \geq \frac{2\sqrt{1+C}}{r_g} \) and thus
\[ c := \frac{H^2 r_g^2}{4} - 1 \geq C > 0 \]
Then if we consider the smooth Riemannian 3 manifold with boundary \( \widetilde{M} = (S^2 \times [0, 1], \gamma) \) where
\[ \gamma = (1 + 2Hs)g \left( \frac{s^2}{4A^2} \right) + ds^2 \]
and \( A = \frac{C}{2H} \), the exact same proof as above show that \( \gamma \) satisfies Claims 1 & 2 while our definition of \( c \) gives
\[ m_H(\Sigma_1) = \frac{r_g \sqrt{1+c}}{2} \left( 1 - \frac{r_g^2 H^2}{4(1+c)} \right) = 0, \]
and again using Proposition 3.1 we conclude \( m_B(S^2, g, H) \leq 0 \).

This concludes the proof of the Theorem.

\[ \square \]

5. PROOF OF THEOREM 1.1

Proof of Theorem 1.1. If \( K_g \geq 0 \) and \( H > 0 \) is arbitrary, then the pair \( (g, H) \) will satisfy the hypothesis of Theorem 1.2 by part 1 of Proposition 2.2. Moreover, in this case the constant \( D(g, H) \to 0 \) as either \( H \to 0^+ \) or else \( g \) converges smoothly to a round metric as pointed out in remark 2.1. Theorem 1.1 then follows from Theorem 5.1 below.

\[ \square \]

Theorem 5.1. Let \( g \) be a Riemannian metric on \( S^2 \) with \( K_g \geq 0 \) and let \( H > 0 \) be a constant. Then
\[ m_B(S^2, g, H) \leq r/2 \]
Proof. Fix any \( \epsilon \leq 1 \) and consider the smooth Riemannian 3 manifold with boundary \( M = (S^2 \times [0, 1], \gamma) \) where
\[ \gamma = E(t)g(t) + \Phi(t)dt^2 \]
where \( g(t) \) is an \( g \)-admissible path, \( E(t) = (1 + \epsilon t) \) and
\[ \Phi(t) = \begin{cases} \frac{At}{\theta} & t \leq \frac{1}{2} \\ \frac{A}{\theta} + \frac{t}{\theta} & \frac{1}{2} < t \leq \frac{3}{2} \\ \frac{A}{\theta} + \frac{3}{\theta} - 1 & \frac{3}{2} \leq t \end{cases} \]
for a constant \( A \) to be chosen sufficiently large below. As described in the proof of Proposition 2.1, the \( g \)-admissible path can be taken to have the form \( \xi_t(\epsilon^{2a(t)w(x)} + \alpha(t)g_s) \) where \( \alpha(t) : [0, 1] \to [0, 1] \) is smooth and non-increasing, \( g = w(x)g_s \) for a round metric \( g_s \) with area \( 4\pi \), and \( \xi_t \) is a family of diffeomorphisms so that in particular, choosing \( \alpha(t) \) constant for \( t \) sufficiently close to \( 1 \) implies that \( g(t) \) is also constant for \( t > 0 \) sufficiently close to \( t = 1 \). Moreover, we can also bound \( K_{g(t)} \) from below linearly in \( t \) as \( K_{g(t)} \geq ct \) on \( S^2 \times [0, 1] \) for some \( c > 0 \). To see why this is true, recall the relationship of Gauss curvature of conformal metrics:
\[ K_{g(t)} \circ \xi_t = K_{e^{2a(t)w(x)} - 2a(t)g_s} = e^{2a(t)}K_{e^{2a(t)w(x)} + \alpha(t)g_s} = e^{2a(t)}K_{e^{2a(t)w(x)}(1 - \alpha(t)\Delta_x w)}. \]
We know that \( \Delta_x w \leq 1 \) on \( S \) as \( K_{g(0)} \geq 0 \). Letting \( B = \inf_{t,x} e^{2a(t)-2a(t)w(x)} \) gives
\[ K_{g(t)} \circ \xi_t \geq B(1 - \alpha(t)) = B(1 - \alpha'(0)t + O(t^2)) \geq B\alpha'(0)t. \]
Now (18) implies that in order to ensure $R_γ \geq 0$ it suffices to show

$$ct \Phi(t)^2 - 2C + \frac{1}{4}E^{-1}(E')^2 + 4E \frac{\partial_t \Phi}{\Phi} \geq ct \Phi(t)^2 - C + \frac{\partial_t \Phi}{\Phi} \geq 0$$

where $C := \max_{S^2 \times [0,1]} \frac{1}{4} |g'|^2$. To this end, let $\delta = \min \left\{ \frac{1}{2}, \frac{C}{2H} \right\}$. Then since $\delta \leq 1/4$ we have $\Phi(t) \leq \Phi(\delta)$ for all $t \in [0, \delta]$. If we choose $A \geq 2C/H$ we may then estimate for all $t \in [0, \delta]$ as

$$\frac{\partial_t \Phi(t)}{\Phi(t)} \geq A = \frac{A}{A+\frac{C}{H}} \geq \frac{A}{2C + \frac{C}{H}} = \frac{C}{\epsilon}$$

thus implying $R_γ \geq 0$ for $t \in [0, \delta]$. Now if we further choose $A \geq \sqrt{\frac{C}{\epsilon}}$, then for $t \in [\delta, 1]$,

$$ct \Phi(t)^2 = ct \left( A\delta + \frac{\epsilon}{H} \right)^2 \geq c\delta^3 A^2 \geq c\delta^3 \sqrt{\frac{C}{\epsilon}} \sqrt{\frac{C}{\epsilon}} = C$$

thus implying $R_γ \geq 0$ for $t \in [\delta, 1]$. So whenever $A$ is sufficiently large (as described), we have $R_γ \geq 0$ on $[0, 1]$.

By (19) we have that the mean curvature $H_t$ of the sphere $S^2 \times t$ in $M$ is given by

$$H_t = \frac{E'(t)E(t)^{-1}}{\Phi(t)}$$

which is clearly positive by our choice of $E(t)$ and $\Phi(t)$, and is equal to $\frac{s(1+t)^{-1}}{A+\frac{C}{H}}$ for $t \leq 1/4$ and thus approaches $H$ as $t \to 0$.

Moreover, by (19) and our choice of $E(t)$ and $\Phi(t)$ the Hawking mass of $Σ_1 = S^2 \times 1$ is given by

$$m_H(Σ_1) = \frac{r_g \sqrt{E(1)}}{2} \left( 1 - \frac{r_g^2 E'(1)^2 E(1)^{-1}}{4E(1)^2} \right)$$

$$= \frac{r_g \sqrt{(1+\epsilon)}}{2} \left( 1 - \frac{r_g^2 E'(1)^2 E(1)^{-1}}{4(1+\epsilon)(A/4 + \epsilon/H + 1)^2} \right)$$

(22)

Noting that in the above construction, $\epsilon$ could have been chosen arbitrarily small while $A$ could have been chosen arbitrarily large, it follows from (22) and Proposition 3.1 that $m_B(S^2, g, H) \leq r_g/2$. 

□

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