κ-Deformed quantum and classical mechanics for a system with position-dependent effective mass

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We present the quantum and classical mechanics formalisms for a particle with position-dependent mass in the context of a deformed algebraic structure (named κ-algebra), motivated by the Kappa-statistics. From this structure we obtain deformed versions of the position and momentum operators, which allow to define a point canonical transformation that maps a particle with constant mass in a deformed space into a particle with position-dependent mass in the standard space. We illustrate the formalism with a particle confined in an infinite potential well and the Mathews-Lakshmanan oscillator, exhibiting uncertainty relations depending on the deformation.

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Minimum length scales are of crucial importance in several areas of physics like quantum gravity, string theory, relativity, fundamentally due to the techniques developed for removing divergences in field theories maintaining the parameters lengths as universal constants of the theory in question (for a review see for instance Ref. 1). In this sense, the seek for these minimum lengths in quantum mechanics has been translated into generalizations of the standard commutation relationship between position and momentum.\textsuperscript{2} Further studies in noncommuting quantum spaces led to a Schrödinger equation with a position-dependent effective mass (PDM).\textsuperscript{3} Along the last decades the PDM systems have attracted attention because of their wide range of applicability in semiconductor theory,\textsuperscript{4–7} nonlinear optics,\textsuperscript{8} quantum liquids,\textsuperscript{9,10} inversion potential for NH\textsubscript{3} in density functional theory,\textsuperscript{11} particle physics,\textsuperscript{12} many body theory,\textsuperscript{13} molecular physics,\textsuperscript{14} Wigner functions,\textsuperscript{15} relativistic quantum mechanics,\textsuperscript{16} superintegrable systems,\textsuperscript{17} nuclear physics,\textsuperscript{18} magnetic monopoles,\textsuperscript{19,20} astrophysics,\textsuperscript{21} nonlinear oscillations,\textsuperscript{22–31} factorization methods and supersymmetry,\textsuperscript{32–36} coherent states,\textsuperscript{37–39} etc.

Complementarily, it has been found that the mathematical foundations of the PDM systems rely on the assumption of the noncommutativity between the mass operator \( m(\hat{x}) \) and the linear momentum operator \( \hat{p} \), thus giving place to the ordering problem for the kinetic energy operator,\textsuperscript{4,40–47} In addition, the development of generalized translation operators motivated the introduction of a position-dependent linear momentum for characterizing a particle with a PDM\textsuperscript{7,48–56} that can be related to a generalized algebraic structure (called \( q \)-algebra\textsuperscript{57}) inherited from the mathematical background of nonextensive statistics.\textsuperscript{58} Concerning these formal structures, the \( \kappa \)-deformed statistics, originated from the \( \kappa \)-exponential and \( \kappa \)-logarithm functions, allows to develop an algebraic structure, called \( \kappa \)-algebra,\textsuperscript{59–74} with similar properties to the those of the \( q \)-algebra. In particular, the \( \kappa \)-statistics has been employed in plasma physics,\textsuperscript{75} astrophysics,\textsuperscript{76} paramagnetic systems,\textsuperscript{77} nonlinear diffusion,\textsuperscript{78} social systems,\textsuperscript{79} complex networks,\textsuperscript{80} analysis of human DNA,\textsuperscript{81} blackbody radiation,\textsuperscript{82} quantum entanglement,\textsuperscript{83} etc.

In this work we employ the \( \kappa \)-algebra for generalizing classical and quantum mechanics with the aim of studying the properties of the resulting noncommuting space originated by the deformation. Between these properties we found that the \( \kappa \)-deformed space, classical and
quantum, allows to characterize a PDM system with the mass being univocally determined by the \( \kappa \)-algebra. The work is organized as follows. In Section II, we review the properties of the \( \kappa \)-algebra that are used in the forthcoming sections. Next, we present in Section III the dynamics resulting from a generic PDM and then we specialize with the mass function \( m(x) \) associated to the \( \kappa \)-algebra. Here we obtain the Schrödinger equation associated to the \( \kappa \)-derivative and we show that all the standard properties remain to be valid in the deformed structure such as the continuity equation, the wave-function normalization, the classical limit, etc. In Section IV we illustrate our proposal with a particle in an infinite potential well. In Section V, we use the \( \kappa \)-deformed formalism to revisit the problem of the Mathews-Lakshmanan oscillator.\(^{22-31}\) Finally, in Section VI we draw some conclusions and outline future perspectives.

II. REVIEW OF THE \( \kappa \)-ALGEBRA

The \( \kappa \)-statistics emerges from a generalization of the Boltzmann-Gibbs entropy derived by means of a kinetic interaction principle, that allows to characterize nonlinear kinetics in particle systems (see, for instance, Ref. 59 for more details). In the last two decades several theoretical developments have shown that the \( \kappa \)-formalism preserve features as Legendre transform in thermodynamics,\(^{62}\) H-theorem,\(^{63}\) Lesche stability,\(^{64}\) composition law of the \( \kappa \)-entropy,\(^{65}\) among others. The mathematical background of the \( \kappa \)-deformed formalism is based on generalizations of the standard exponential and logarithm functions, from which it is possible to introduce deformed versions of algebraic operators and calculus,\(^{59-61}\) trigonometric and hyperbolic functions,\(^{66,67}\) Fourier transform,\(^{68}\) Gaussian law of error,\(^{69}\) Stirling approximation and Gamma function,\(^{70}\) Cantor set,\(^{71}\) Lambert \( W \) function,\(^{72}\) information geometry,\(^{73}\) and other possible exponential and logarithm functions,\(^{74}\) etc.

More specifically, the so-called \( \kappa \)-exponential is a deformation of the ordinary exponential function, defined by\(^{59-61}\)

\[
\exp_\kappa u \equiv \left( \kappa u + \sqrt{1 + \kappa^2 u^2} \right)^{1/\kappa} = \exp \left( \frac{1}{\kappa} \text{arcsinh}(\kappa u) \right), \quad (\kappa \in \mathbb{R}).
\]  

The inverse function of the \( \kappa \)-exponential is the \( \kappa \)-logarithm, given by

\[
\ln_\kappa u \equiv \frac{u^\kappa - u^{-\kappa}}{2\kappa} = \frac{1}{\kappa} \sinh(\kappa \ln u), \quad (u > 0).
\]
In the limit $\kappa \to 0$, the ordinary exponential and logarithmic functions are recovered, i.e. $\exp_0 x = \exp x$ and $\ln_0 x = \ln x$. These functions satisfy the properties $\exp_\kappa(a) \exp_\kappa(b) = \exp_\kappa(a \oplus b)$, $\exp_\kappa(a) / \exp_\kappa(b) = \exp_\kappa(a \ominus b)$, $\ln_\kappa(ab) = \ln_\kappa(a) \oplus \ln_\kappa(b)$ and $\ln_\kappa(ab) = \ln_\kappa(a) \ominus \ln_\kappa(b)$, where the symbol $\oplus$ represents the $\kappa$-addition operator defined by $a \oplus b \equiv a\sqrt{1 + \kappa^2b^2} + b\sqrt{1 + \kappa^2a^2}$, and $\ominus$ represents the $\kappa$-subtraction, $a \ominus b \equiv a\sqrt{1 + \kappa^2b^2} - b\sqrt{1 + \kappa^2a^2}$.\textsuperscript{59,66}

A $\kappa$-deformed calculus has been introduced in Ref. 59 from the deformed differential

$$d_\kappa u \equiv \lim_{u' \to u} u' \overset{\kappa}{\ominus} u = \frac{du}{\sqrt{1 + \kappa^2u^2}} + O((du)^2).$$  \hfill (3)

The definition of a deformed variable $u_\kappa$ (also named deformed $\kappa$-number) is

$$u_\kappa \equiv \frac{1}{\kappa} \text{arcsinh}(\kappa u) = \ln[\exp_\kappa(u)],$$  \hfill (4)

implies $d_\kappa u = du_\kappa$, i.e., the deformed differential of an ordinary variable $u$ can be rewritten as with the ordinary differential of a deformed variable $u_\kappa$. In this way, one defines the $\kappa$-derivative operator by

$$D_\kappa f(u) \equiv \lim_{u' \to u} \frac{f(u') - f(u)}{u' \overset{\kappa}{\ominus} u} = \sqrt{1 + \kappa^2u^2} \frac{df(u)}{du},$$  \hfill (5)

with the $\kappa$-exponential an eigenfunction of $D_\kappa$, $D_\kappa \exp_\kappa u = \exp_\kappa u$. Similarly, the dual $\kappa$-derivative is defined by

$$\widetilde{D}_\kappa f(u) \equiv \lim_{u' \to u} \frac{f(u') \overset{\kappa}{\ominus} f(u)}{u' - u} = \frac{1}{\sqrt{1 + \kappa^2[f(u)]^2}} \frac{df(u)}{du},$$  \hfill (6)

which satisfies $\widetilde{D}_\kappa \ln_\kappa u = 1/u$. These operators obey $\widetilde{D}_\kappa x(y) = [D_\kappa y(x)]^{-1}$. In particular, we have $D_\kappa u = (\widetilde{D}_\kappa u)^{-1} = \sqrt{1 + \kappa^2u^2}$. From Eqs. (5) and (6) we see that the standard derivative is recovered as $\kappa \to 0$. The deformed derivative operator (5) can be seen as the variation of the function $f(u)$ with respect to a nonlinear variation of the independent variable $u$, i.e., $D_\kappa f(u) = df(u)/du_\kappa$. On the other hand, the dual deformed derivative operator (6) is the rate of change of a nonlinear variation of the function $f(u)$ with respect to the standard variation of the independent variable $u$, $\widetilde{D}_\kappa f(u) = d_\kappa f(u)/du$. The deformed second derivatives satisfy

$$D^2_\kappa f(u) = \sqrt{1 + \kappa^2u^2} \frac{d}{du} \left[ \sqrt{1 + \kappa^2u^2} \frac{df}{du} \right],$$  \hfill (7)
and
\[
\tilde{D}^2_{\kappa} f(u) = \frac{1}{\sqrt{1 + \kappa^2 f(u)^2}} \frac{d}{du} \left\{ \frac{1}{\sqrt{1 + \kappa^2 f(u)^2}} \frac{df}{du} \right\}.
\] (8)

These rules can be extended to deformed derivatives of higher order.

III. $\kappa$-DEFORMED DYNAMICS OF A SYSTEM WITH POSITION-DEPENDENT MASS

A. $\kappa$-Deformed classical formalism

Let us first consider the problem of a particle with a position-dependent mass (PDM) $m(x)$ in 1D for the classical formalism. The Hamiltonian of the system is
\[
\mathcal{H}(x, p) = \frac{p^2}{2m(x)} + V(x),
\] (9)
whose the linear momentum is $p = m(x)\dot{x}$, leads to the equation of motion
\[
m(x)\ddot{x} + \frac{1}{2}m'(x)\dot{x}^2 = F(x)
\] (10)
with $F(x) = -dV/dx$ the force acting on the particle, where $\dot{x} = dx/dt$, $\ddot{x} = d^2x/dt^2$ and $m'(x) = dm/dx$ give velocity, acceleration and mass gradient, respectively. The point canonical transformation (PCT)
\[
\eta = \int_{x}^{x} \sqrt{\frac{m(y)}{m_0}} dy \quad \text{and} \quad \Pi = \sqrt{\frac{m_0}{m(x)}} p,
\] (11)
maps the Hamiltonian (9) of a particle with PDM $m(x)$ in the usual phase space $(x, p)$ into another Hamiltonian of a particle with a constant mass $m_0$ represented in the deformed phase space $(\eta, \Pi)$,
\[
\mathcal{K}(\eta, \Pi) = \frac{1}{2m_0} \Pi^2 + U(\eta),
\] (12)
with $U(\eta) = V(x(\eta))$ the potential expressed in the deformed space-coordinate $\eta$. When $m(x) = m_0$, both representations coincide.

Let us consider in particular the mass function
\[
m(x) = \frac{m_0}{1 + \kappa^2 x^2},
\] (13)
where the parameter $\kappa$ has units of inverse length and controls the dependence of the mass with position, where $\kappa = 0$ corresponds to the standard case. Thus equation of motion (10) becomes

$$m_0 \left[ \frac{\ddot{x}}{1 + \kappa^2 x^2} - \frac{\kappa^2 \dot{x}^2}{(1 + \kappa^2 x^2)^2} \right] = F(x).$$

This equation can be compactly rewritten in the form of a deformed Newton’s second law

$$m_0 \tilde{D}_\kappa^2 x(t) = F(x).$$

Moreover, for the mass function (13) the $\kappa$-deformed spatial coordinate and its conjugated linear momentum are

$$\eta = \frac{1}{\kappa} \arcsinh(\kappa x) \equiv x_\kappa,$$

and

$$\Pi = \sqrt{1 + \kappa^2 x^2} p \equiv \Pi_\kappa,$$

with Poisson brackets $\{x_\kappa, \Pi_\kappa\}_{x,p} = 1$. The deformed displacement $d_\kappa x$ of a particle with the non-constant mass $m(x)$, given in Eq. (13), is mapped into the usual displacement $dx_\kappa$ in a deformed space $x_\kappa$ provided with a constant mass $m_0$: $d_\kappa x \equiv (x + dx) \tilde{\kappa} x = dx/\sqrt{1 + \kappa^2 x^2}$, up to first order. The time evolution of the system is governed by the dual derivative, i.e.

$$\tilde{D}_\kappa x(t) = \dot{x}/\sqrt{1 + \kappa^2 x^2}.$$

B. $\kappa$-Deformed quantum formalism

In the quantization of a PDM system an ordering ambiguity arises for defining the kinetic energy operator in terms of the mass operator $m(\hat{x})$ and the linear momentum $\hat{p}$. There are several ways to define a Hermitian kinetic energy operator, and a general two-parameter form is given by

$$\hat{T} = \frac{1}{4} \left\{ [m(\hat{x})]^{-\alpha} \hat{p}[m(\hat{x})]^{-1+\alpha+\beta} \hat{p}[m(\hat{x})]^{-\beta} + [m(\hat{x})]^{-\beta} \hat{p}[m(\hat{x})]^{-1+\alpha+\beta} \hat{p}[m(\hat{x})]^{-\alpha} \right\}. \quad (17)$$

For more details see the discussions, for instance, of von Roos, Lévy-Leblond, and others. Among many particular cases in the literature, we point out the proposals by Ben Daniel and Duke ($\alpha = \beta = 0$), Gora and Williams ($\alpha = 1, \beta = 0$), Zhu and Kroemer ($\alpha = \beta = \frac{1}{2}$), Li and Kuhn ($\alpha = \frac{1}{2}, \beta = 0$). Morrow and Brownstein have shown that only the case $\alpha = \beta$ satisfies the conditions of continuity of the wave-function at the boundaries of a
heterojunction in crystals. In particular, Mustafa and Mazharimousavi\textsuperscript{46} have shown that the case $\alpha = \beta = \frac{1}{4}$ allows the mapping of a quantum Hamiltonian with PDM into a Hamiltonian with constant mass by means a PCT. More precisely, considering the quantum Hamiltonian

$$\hat{H}(\hat{x}, \hat{p}) = \frac{1}{2} [m(\hat{x})]^{-\frac{1}{4}} \hat{p}[m(\hat{x})]^{-\frac{1}{4}} \hat{p}[m(\hat{x})]^{-\frac{1}{4}} + V(\hat{x}), \quad (18)$$

the Schrödinger equation $i\hbar \frac{\partial}{\partial t} |\Psi \rangle = \hat{H} |\Psi \rangle$ in the position representation $\{|\hat{x}\rangle\}$ reads

$$i\hbar \frac{\partial \Psi(x,t)}{\partial t} = \left( -\frac{\hbar^2}{2m_0} \sqrt{\frac{m_0}{m(x)}} \frac{\partial}{\partial x} \sqrt{\frac{m_0}{m(x)}} \frac{\partial}{\partial x} \sqrt{\frac{m_0}{m(x)}} + V(x) \right) \Psi(x,t), \quad (19)$$

with $\Psi(x,t) = \psi(x)e^{-iEt/\hbar}$ and $E$ the eigenvalue corresponding to the eigenfunction $\psi(x)$ of $\hat{H}$. It is straightforwardly verified that the probability density $\rho(x,t) \equiv |\Psi(x,t)|^2$ satisfies the continuity equation

$$\frac{\partial \rho(x,t)}{\partial t} = -\frac{\partial J(x,t)}{\partial x}, \quad (20)$$

where the probability current is

$$J(x,t) \equiv \text{Re} \left\{ \Psi^*(x,t) \left( \frac{\hbar}{i} \frac{\partial}{\partial x} \right) \left[ \frac{1}{m(x)} \Psi(x,t) \right] \right\}. \quad (21)$$

Equation (19) can be conveniently rewritten by means of the transformation $\Psi(x,t) = \sqrt[4]{m(x)/m_0} \Phi(x,t)$, as

$$i\hbar \frac{\partial \Phi(x,t)}{\partial t} = \left[ -\frac{\hbar^2}{2m_0} \left( \frac{\partial}{\partial x} \sqrt{\frac{m_0}{m(x)}} \frac{\partial}{\partial x} \sqrt{\frac{m_0}{m(x)}} \right)^2 + V(x) \right] \Phi(x,t). \quad (22)$$

Let us consider in particular the mass function (13). The modified wave-function $\Phi(x,t) = \sqrt[4]{1 + \kappa^2 x^2} \Psi(x,t)$ obeys a $\kappa$-deformed Schrödinger wave-equation

$$i\hbar \frac{\partial \Phi(x,t)}{\partial t} = -\frac{\hbar^2}{2m_0} D_\kappa^2 \Phi(x,t) + V(x) \Phi(x,t) \quad (23)$$

with $D_\kappa = \sqrt[4]{1 + \kappa^2 x^2} \partial_x$, which is the analog of the $\kappa$-derivative operator (5). Using Eq. (8), we obtain

$$i\hbar \frac{\partial \Phi(x,t)}{\partial t} = -\frac{\hbar^2(1 + \kappa^2 x^2)}{2m_0} \partial_x^2 \Phi(x,t) - \frac{\hbar^2 \kappa^2 x}{2m_0} \partial_x \Phi(x,t) + V(x) \Phi(x,t). \quad (24)$$

Equation (23) is indeed equivalent to a Schrödinger-like equation for $\Phi(x,t)$ with the non-Hermitian Hamiltonian operator

$$\hat{H}_\kappa = \frac{1}{2m_0} \partial_x^2 + V(\hat{x}), \quad (25)$$
where \( \hat{p}_\kappa \equiv -i\hbar D_\kappa = \sqrt{1 + \kappa^2 \hat{x}^2} \) stands for a \( \kappa \)-deformed non-Hermitian momentum operator, and obeys the commutation relation

\[
[\hat{x}, \hat{p}_\kappa] = i\hbar \sqrt{1 + \kappa^2 \hat{x}^2}.
\]

This leads to generalized uncertainty principle \( \Delta x \Delta p_\kappa \geq \frac{\hbar}{2} \langle \sqrt{1 + \kappa^2 \hat{x}^2} \rangle \). We notice that if the standard wave-function \( \Psi(x,t) \) is normalized, then \( \Phi(x,t) \) is normalized under a \( \kappa \)-deformed integral. Indeed, we have

\[
\int_{x_i}^{x_f} \Psi^*(x,t) \Psi(x,t) \, dx = \int_{x_i}^{x_f} \Phi^*(x,t) \Phi(x,t) \, d_\kappa x = 1.
\]

Besides, we obtain the \( \kappa \)-deformed continuity equation

\[
\frac{\partial \rho(x,t)}{\partial t} + D_\kappa J(x,t) = 0,
\]

with \( \rho(x,t) = |\Phi(x,t)|^2 \) and

\[
J(x,t) \equiv \text{Re} \left\{ \Phi^*(x,t) \left( \frac{\hbar}{i} D_\kappa \right) \left[ \frac{\Phi(x,t)}{m_0} \right] \right\}.
\]

It is worth noting that there is an equivalence between the Schrödinger equation for the Hermitian system (18) with the mass function \( m(x) \) given by (13) and the non-Hermitian one (25) expressed in terms of a \( \kappa \)-deformed momentum operator, where \( \Psi(x,t) \) must be replaced by \( \Phi(x,t) = \sqrt{1 + \kappa^2 x^2} \Psi(x,t) \). Moreover, we see that in the description of quantum systems with the mass function (13) in terms of the modified wave-function \( \Phi(x,t) \), the usual derivative and integral with respect to the variable \( x \) are replaced by their corresponding \( \kappa \)-deformed versions. Analogous features apply in the classical formalism, with the motion equation expressed in terms of the dual \( \kappa \)-derivative (see Eq. (15)).

Using the change of variable \( x \rightarrow x_\kappa = \ln[\exp_\kappa(x)] \) (see Eq. (4)), then Eq. (23) can be rewritten in the \( \kappa \)-deformed space as

\[
i\hbar \frac{\partial \Lambda(x_\kappa,t)}{\partial t} = -\frac{\hbar^2}{2m_0} \frac{\partial^2 \Lambda(x_\kappa,t)}{\partial x^2_\kappa} + U(x_\kappa) \Lambda(x_\kappa,t),
\]

with \( \Lambda(x_\kappa,t) = \Phi(x(x_\kappa),t) \) and \( U(x_\kappa) = V(x(x_\kappa)) \) a modified potential in terms of the original one \( V \) and the inverse transformation \( x = x(x_\kappa) \). Therefore, the wave-equation for \( \Psi(x,t) \) of a system with PDM (13) with the potential \( V(x) \) in the standard space \( \{|x\} \) is mapped into an equation for \( \Lambda(x_\kappa,t) \) with the potential \( U(x_\kappa) = V(x(x_\kappa)) \) in the deformed space \( \{ |\hat{x}_\kappa\rangle \} \). The quantum Hamiltonian associated with the Schrödinger wave-equation
(30) is \( \hat{K}(\hat{x}_\kappa, \hat{\Pi}_\kappa) = \frac{1}{2m_0} \hat{\Pi}_\kappa^2 + U(\hat{x}_\kappa) \), that can be obtained by applying the point canonical transformation \((\hat{x}, \hat{p}) \rightarrow (\hat{x}_\kappa, \hat{\Pi}_\kappa)\) on the quantum Hamiltonian (18) where

\[
\hat{x}_\kappa = \frac{1}{\kappa} \arcsinh(\kappa \hat{x}), \quad (31a)
\]

\[
\hat{\Pi}_\kappa = \sqrt{1 + \kappa^2 \hat{x}^2} \hat{p} \sqrt{1 + \kappa^2 \hat{x}^2} = \frac{1}{2}(\hat{p}_\kappa^\dagger + \hat{p}_\kappa), \quad (31b)
\]

with \([\hat{x}_\kappa, \hat{\Pi}_\kappa] = i\hbar \hat{1}\). Also, we have that \( \hat{\Pi}_\kappa \) is in accordance with the definition of a PDM pseudo-momentum operator introduced in Ref. 46. Thus, the dynamical variables (11) are the classical counterparts of the Hermitian operators (31).

From the eigenvalue equation \( \hat{\Pi}_\kappa |k\rangle = \hbar k |k\rangle \), the eigenfunctions in the representation \( \{|\hat{x}\rangle\} \) result

\[
\psi_k(x) = \frac{C}{\sqrt{1 + \kappa^2 x^2}} \exp \left[ \frac{ik}{\kappa} \arcsinh(\kappa x) \right], \quad (32)
\]

where \( C \) is a constant. As in the non deformed case \((\kappa = 0)\), the function \( \psi_k(x) \) is not normalizable. Even though, a deformed wave-packet can be defined from the \( \kappa \)-deformed Fourier transform

\[
\psi(x) = \frac{1}{\sqrt{1 + \kappa^2 x^2}} \int_{-\infty}^{+\infty} g(k) e^{\frac{ik}{\kappa} \arcsinh(\kappa x)} dk, \quad (33)
\]

where \( g(k) \) is the distribution function of the wave-vectors \( k \). It is verified straightforwardly that the corresponding wave-packet of the operator \( \hat{p}_\kappa \) is \( \varphi(x) = \int_{-\infty}^{+\infty} g(k) \exp(\kappa x) e^{ikx} dk \).

The wave-packet in the representation of the deformed space is \( \phi(x_\kappa) = \varphi(x(x_\kappa)) = \sqrt{1 + \kappa^2 x^2} \psi(x(x_\kappa)) = \int_{-\infty}^{+\infty} g(k) e^{ikx_\kappa} dk \). From the Plancherel theorem, we have

\[
g(k) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{\phi(x_\kappa) e^{-ikx_\kappa} dx_\kappa}{\sqrt{1 + \kappa^2 x^2}}
= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \varphi(x) e^{-ikx} dx
= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{\psi(x) e^{-ikx}}{\sqrt{1 + \kappa^2 x^2}} dx. \quad (34)
\]

IV. PARTICLE IN AN INFINITE POTENTIAL WELL

In Secs. IV and V we illustrate the quantum and classical \( \kappa \)-deformed formalism with two paradigmatic examples.
A. Classical case

First we consider the problem of a particle confined in an infinite square potential well between \( x = 0 \) and \( x = L \). If \( \mathcal{H}(x, p) = E \) is the energy of the classical particle, then the linear momentum is \( p(x) = \pm \sqrt{2m_0E/(1 + \kappa^2x^2)} \) and the velocity is \( v(x) = \pm v_0\sqrt{1 + \kappa^2x^2} \) with \( v_0 = \sqrt{2E/m_0} \). For \( v(0) = v_0 \) and \( 0 < x < L \), the position as a function of time is \( x(t) = \ln\kappa[\exp(v_0t)] \). Hence, the classical probability density \( \rho_{\text{classic}}(x)dx \propto dx/v_0 \) to find the particle within the interval \( [x, x+dx] \) is

\[
\rho_{\text{classic}}(x)dx = \frac{\kappa}{\ln(\kappa L + \sqrt{1 + \kappa^2L^2})} \frac{dx}{\sqrt{1 + \kappa^2x^2}},
\]

from which the uniform distribution \( \rho_{\text{classic}}(x) = 1/L \) is recovered when \( \kappa \to 0 \). The first and the second moments of the position and the linear momentum for the classical distribution (35) are

\[
\begin{align*}
\frac{x}{L} &= \frac{\sqrt{1 + \kappa^2L^2} - 1}{\kappa L \ln(\kappa L + \sqrt{1 + \kappa^2L^2})}, \\
\frac{x^2}{L^2} &= \frac{1}{2\kappa^2L^2} \left[ \frac{\kappa L \sqrt{1 + \kappa^2L^2}}{\ln(\kappa L + \sqrt{1 + \kappa^2L^2})} - 1 \right], \\
\bar{p} &= 0, \\
\bar{p}^2 &= 2m_0E \left[ \frac{\kappa L}{\sqrt{1 + \kappa^2L^2} \ln(\kappa L + \sqrt{1 + \kappa^2L^2})} \right].
\end{align*}
\]

We can verify that \( \lim_{\kappa \to 0} \frac{x}{L} = L/2 \), \( \lim_{\kappa \to 0} \frac{x^2}{L^2} = L^2/3 \) and \( \lim_{\kappa \to 0} \bar{p}^2 = 2m_0E \). From the change of variable \( x \to x_\kappa \) the PDM particle confined in an interval \([0, L]\) is mapped into a particle with constant mass in \([0, L_\kappa]\), where \( L_\kappa = \text{arcsinh}(\kappa L)/\kappa \) corresponds to the length of the box in the deformed space.

B. Quantum case

Let us now analyze the problem in the \( \kappa \)-deformed quantum formalism. Considering \( \Phi(x, t) = \varphi(x)e^{-iEt/\hbar} \), this leads to the time-independent Schrödinger-like equation

\[-\frac{\hbar^2}{2m_0}D^2_\kappa \varphi(x) = E \varphi(x), \]

whose eigenfunctions are given by

\[
\varphi_n(x) = \sqrt{1 + \kappa^2x^2} \psi_n(x) = C_\kappa \sin \left[ \frac{k_{\kappa,n}}{\kappa} \text{arcsinh}(\kappa x) \right]
\]
for $0 \leq x \leq L$, and $\varphi_n(x) = 0$ elsewhere, with $C_\kappa^2 = 2/L_\kappa$ and $k_{\kappa,n} = n\pi/L_\kappa$, where $n$ is an integer number and $L_\kappa = \kappa^{-1}\operatorname{arcsinh}(\kappa L)$. The energy levels corresponding to these eigenfunctions are

$$E_n = \frac{\hbar^2 \pi^2 n^2 \kappa^2}{2m_0 \operatorname{arcsinh}^2(\kappa L)} = \varepsilon_0 \left[ \frac{\kappa L}{\operatorname{arcsinh}(\kappa L)} \right]^2 n^2$$

(38)

with $\varepsilon_0 = \hbar^2 \pi^2 / (2m_0 L^2)$. The effect of the deformation parameter $\kappa$ corresponds to a contraction of the space ($L_\kappa < L$ for $\kappa \neq 0$), and consequently this leads to an increase of the energy levels of the particle. In Fig. 1 we illustrate the energy levels of the particle as a function of the quantum number for different values of $\kappa$.

![Energy Levels of a Particle](image)

**FIG. 1.** (Color online) Energy levels of a particle with PDM $m(x) = m_0/(1 + \kappa^2 x^2)$ in an infinite square well of size $L$, for different quantum numbers $n$ and values of $\kappa$, given in terms of the nondeformed fundamental energy $\varepsilon_0 = \frac{k^2 \pi^2}{2m_0 \kappa^2}$. The values of the energies are discrete, and the solid lines help for guiding the eyes.

The probability densities of the stationary states in position space are

$$\rho_n(x) = |\psi_n(x)|^2 = \frac{2\kappa}{\operatorname{arcsinh}(\kappa L)} \frac{1}{\sqrt{1 + \kappa^2 x^2}} \sin^2 \left[ \frac{k_{\kappa,n}}{\kappa} \operatorname{arcsinh}(\kappa x) \right].$$

(39)

Substituting Eq. (37) into the inverse Fourier transform (34), we obtain the eigenfunctions for the particle confined in a box in momentum space $k$

$$g_n(k) = n \sqrt{\frac{L_\kappa}{2}} \left[ 1 + (-1)^{n+1} e^{-i k L_\kappa} \right].$$

(40)

Consequently, its associated probability density results

$$\gamma_n(k) = |g_n(k)|^2 = n^2 L_\kappa \frac{1 - \cos(n\pi) \cos(k L_\kappa)}{[(k L_\kappa)^2 - (n\pi)^2]^2}.$$
Interestingly, the eigenfunctions \((40)\) and the probability densities \((41)\) have the same form as in the case of a particle with constant mass, but with \(L_\kappa\) instead of \(L\). In Fig. 2 we plot the eigenfunctions \(\psi_n(x)\) and their probability densities in the coordinate and momentum spaces, \(\rho_n(x)\) and \(\gamma_n(k)\), for the three states of lower energy and for some values of the deformation parameter \(\kappa\). We can see that as \(\kappa\) increases, \(\rho_n(x)\) becomes more asymmetric and \(\gamma_n(k)\) more spread along its domain. In Fig. 3 we show that the average value of the quantum probability density \(\rho_n(x)\) approaches to the classical probability density \(\rho_{\text{class}}(x)\) (illustrated here for \(n = 20\)) in accordance with the correspondence principle. The distribution \(\gamma_n(k)\) is also shown for the same state \(n = 20\).

FIG. 2. (Color online) Eigenfunctions \(\psi_n(x)\) ((a)-(c)), probability densities \(\rho_n(x) = |\psi_n(x)|^2\) ((d)-(f)) and \(\gamma_n(k) = |g_n(k)|^2\) ((g)-(i)) for a particle with PDM \(m(x) = m_0/(1+\kappa^2 x^2)\) and confined in an infinite square well for different parameters \(\kappa L\) (the usual case, \(\kappa L = 0\), is shown for comparison). [(a), (d) and (g)] \(n = 1\) (ground state), [(b), (e) and (h)] \(n = 2\) (first excited state), [(c), (f) and (i)] \(n = 3\) (second excited state).

The eigenfunctions \((37)\) constitute an orthonormal set of functions that obey the inner
FIG. 3. (Color online) Probability densities (a) $\rho_n(x) = |\psi_n(x)|^2$ and (b) $\gamma_n(k) = |g_n(k)|^2$ of a particle with PDM confined in an infinite square well for $\kappa L = 3.0$ and for the eigenstate $n = 20$. In the panel (a), the classical distribution [Eq. (35)] is shown for comparison, and the dotted upper line is $2\kappa L/[\text{arcsinh}(\kappa x)\sqrt{1 + \kappa^2 x^2}]$.

The product $\int_0^L \varphi_n(x)\varphi_{n'}(x) d_\kappa x = \delta_{n,n'}$, so that any continuous function in the interval $[0, L]$ can be written as a linear combination

$$f(x) = \sum_{n=1}^{\infty} c_n \sin \left[ \frac{n\pi \arcsinh(\kappa x)}{\arcsinh(\kappa L)} \right], \quad (42)$$

with the coefficients $c_n$ of the series given by

$$c_n = \frac{2\kappa}{\arcsinh(\kappa L)} \int_0^L f(x) \sin \left[ \frac{n\pi \arcsinh(\kappa x)}{\arcsinh(\kappa L)} \right] d_\kappa x. \quad (43)$$

Concerning the Sturm-Liouville problem, Braga et al.\textsuperscript{54} have introduced a Fourier series in terms of deformed trigonometric functions that emerge from the formalism studied in Ref. 48. Likewise, we have that the $\kappa$-deformed Fourier series (42) has the same structure like the proposed by Scarfone in Ref. 66, considering the $\kappa$-deformed mathematics. For the particular case $f(x) = 1$, we have $f(x) = \lim_{N \to \infty} f_N(x)$ with

$$f_N(x) = \frac{4}{\pi} \sum_{l=0}^{N} \frac{1}{2l+1} \sin \left[ (2l+1)\pi \frac{\arcsinh(\kappa x)}{\arcsinh(\kappa L)} \right]. \quad (44)$$

Similarly as was done in Ref. 54, we consider as a quantitative measure of the error the function defined by $R(N) = \int_0^L [f(x) - f_N(x)]^2 d_\kappa x$. In Fig. 4 we show that when $N$ becomes large, the partial sum $f_N(x)$ converges to $f(x) = 1$, as well as $R(N)$ goes to zero.

Expected values of $\hat{x}$ and $\hat{\Pi}_x$ for stationary states can be obtained from usual internal products of the eigenfunctions $\psi_n(x)$ or, equivalently, from the deformed internal products of the modified eigenfunctions $\varphi(x)$, i.e.,

$$\langle \hat{x} \rangle = \int \psi_n^*(x)\hat{x}\psi_n(x) dx = \int \varphi_n^*(x)\hat{x}\varphi_n(x) d_\kappa x$$

and
\[
\langle \hat{\Pi} \rangle = \int \psi_n^*(x)\hat{\Pi}\psi_n(x)dx = \int \varphi_n^*(x)p_\lambda \varphi_n(x)d\kappa x, \text{ which is a positive integer. The expectation values } \langle \hat{x} \rangle, \langle \hat{x}^2 \rangle, \langle \hat{p} \rangle, \text{ and } \langle \hat{p}^2 \rangle \text{ for the eigenstates of the particle in a one dimensional infinite potential well are respectively }
\]

\[
\frac{\langle \hat{x} \rangle}{L} = \frac{(\sqrt{1 + \kappa^2L^2} - 1)(2\pi n)^2}{\kappa L \ln(\kappa L + \sqrt{1 + \kappa^2L^2}) \left[ \ln^2(\kappa L + \sqrt{1 + \kappa^2L^2}) + (2\pi n)^2 \right]} \tag{45a}
\]

\[
\frac{\langle \hat{x}^2 \rangle}{L^2} = \frac{1}{2\kappa^2L^2} \left\{ \frac{\kappa L \sqrt{1 + \kappa^2L^2}(n\pi)^2}{\ln(\kappa L + \sqrt{1 + \kappa^2L^2}) \left[ \ln^2(\kappa L + \sqrt{1 + \kappa^2L^2}) + (n\pi)^2 \right]} - 1 \right\} \tag{45b}
\]

\[
\langle \hat{p} \rangle = 0, \tag{45c}
\]

\[
\langle \hat{p}^2 \rangle = \hbar^2 \left[ k_{n,n}^2 I_{1,0}(1) + \kappa^2 \left( \frac{1}{2} I_{1,0}(1) - \frac{5}{4} I_{1,1}(1) - I_{3,0}(1) + 5I_{3,1}(1) \right) \right] \tag{45d}
\]

with \( I_{j,l}(z) = 2\int_0^z \text{sech}^2(\lambda_\kappa u)\tanh^2(\lambda_\kappa u) \sin^2(n\pi u)du \) and \( \lambda_\kappa = \kappa L_\kappa \). The analytical form of the functions \( I_{j,l}(z) \) is expressed by means of the Appell hypergeometric function of two variables (http://functions.wolfram.com/ElementaryFunctions/Sech/21/01/14/01/10/01/0001/) and due to its complicated expression, it becomes convenient to write the expectation value (45d) in terms of \( I_{j,l}(z) \).

We can see that in the limit \( n \to \infty \), the Eqs. (45) coincide with the Eqs. (36), which expresses the consistency of the classical limit. We can also verify that in the limit \( \kappa \to 0 \) we recover the usual results \( \langle \hat{x} \rangle \to \frac{L}{2} , \langle \hat{x}^2 \rangle \to \frac{L^2}{3} - \frac{L^2 \kappa^2}{2n^2\pi^2} \) and \( \langle \hat{p}^2 \rangle \to \hbar^2 k_n^2 \) with \( E_n = \hbar^2 k_n^2 / 2m_0 \) \( (k_n \equiv k_{0,n} = n\pi / L) \). It is straightforwardly to verify that the expectation values of the
pseudo-momentum satisfy
\[\langle \hat{\Pi}_\kappa \rangle = \hbar \langle k \rangle = 0, \tag{46a}\]
\[\langle \hat{\Pi}_\kappa^2 \rangle = \hbar \langle k^2 \rangle = \left(\frac{n\pi\hbar}{L_\kappa}\right)^2 \tag{46b}\]

with \(\langle \hat{\Pi}_\kappa^2 \rangle\) and \(\langle \hat{p}^2 \rangle\) different for \(\kappa \neq 0\). In Fig. 5 we plot the uncertainty relation for different values of \(\kappa\). Once the operators \(\hat{x}\) and \(\hat{p}\) are Hermitian and canonically conjugated, the uncertainty relation is satisfied for different values of \(\kappa\), i.e., \(\Delta x \Delta p \geq \frac{\hbar}{2}\). We can also see that position and wave-vector satisfy the uncertainty relation \(\Delta x \Delta k \geq \frac{1}{2}\). In both curves (c) and (d), the minimum of the uncertainty relation is attained for \(\kappa = 0\). Similar features have been observed in other system provided with PDM. In Ref. 56 the Cramér-Rao, Fisher-Shannon and LópezRuiz-Mancini-Calbet (LMC) complexities have been investigated for the problem of a particle with a PDM and confined in an infinite potential well within the framework of the \(q\)-algebra. In the context of these complexities, the conjugated variables exhibit a behavior similar to the standard Heisenberg uncertainty principle. For different states, the uncertainty relation associated to the Cramér-Rao, Fisher-Shannon and LMC complexities exhibits a minimum lower bound when the mass of the particle is constant (i.e., with a null space deformation). This result is expectedly reasonable since the \(q\)-exponential\(^{58}\) and the \(\kappa\)-exponential functions present a similar behavior when their deformation parameters recover the standard exponential \((q \to 1\) and \(\kappa \to 0\)).

V. \(\kappa\)-DEFORMED OSCILLATOR WITH POSITION-DEPENDENT MASS

A. \(\kappa\)-Deformed classical oscillator

Now we consider a particle with the position-dependent mass (13) subjected to the potential \(V(x) = \frac{1}{2}m(x)\omega_0^2x^2\). This problem is known as the Mathews-Lakshmanan oscillator,\(^{22}\) where the classical Hamiltonian is given by
\[\mathcal{H}(x, p) = \frac{1 + \kappa^2 x^2}{2m_0} \frac{p^2}{2} + \frac{m_0\omega_0^2 x^2}{2(1 + \kappa^2 x^2)} \tag{47}\]
The deformed second Newton’s law (15) for this oscillator becomes
\[\tilde{D}_{\kappa^2 x}(t) = -\frac{\omega_0^2 x}{(1 + \kappa^2 x^2)^2}, \tag{48}\]
FIG. 5. (Color online) Uncertainty in function of $\kappa L$ of (a) the position $\Delta x$, (b) the momentum $\Delta p$ along with the uncertainty relations ((c) and (d)) $\Delta x \Delta p$ and $\Delta x \Delta k$ for a particle with a PDM confined in a box for the ground state and the first two excited ones.

or more explicitly,

$$ (1 + \kappa^2 x^2) \ddot{x} + \omega_0^2 x - \kappa^2 \dot{x}^2 x = 0. $$

(49)

The solution of Eq. (48) (or equivalently (49)) is

$$ x(t) = A_{\kappa} \cos(\Omega_{\kappa} t + \delta_0), $$

(50)

with $A_{\kappa} = A_0 / \sqrt{1 - \kappa^2 A_0^2}$ the amplitude of the oscillation, $\Omega_{\kappa} = \omega_0 \sqrt{1 - \kappa^2 A_0^2}$ the angular frequency and $A_0^2 = 2E/m_0\omega_0^2$. The potential of this oscillator has a finite well depth $W_{\kappa} = m_0\omega_0^2/2\kappa^2$. Since $E/W_{\kappa} = \kappa^2 A_0^2$, the oscillator has a closed (open) path in the phase space for $0 < \kappa^2 A_0^2 < 1$ ($\kappa^2 A_0^2 > 1$), according to Ref. 22. The PCT (16) maps the Hamiltonian (47) into the corresponding to the anharmonic oscillator, i.e.

$$ \mathcal{K}(x_{\kappa}, \Pi_{\kappa}) = \frac{1}{2m_0} \Pi_{\kappa}^2 + W_{\kappa} \tanh^2(\kappa x_{\kappa}), $$

(51)

with $\kappa$ a continuous parameter that controls the anharmonicity of the potential. In Fig. 6 we plot the phase spaces $(x, p)$ and $(x_{\kappa}, \Pi_{\kappa})$ for different values of $\kappa A_0$. The bounded motion in the interval $-A_{\kappa} < x < A_{\kappa}$ of the standard space turns out into the interval $-x_{\kappa,\text{max}} < x_{\kappa} < x_{\kappa,\text{max}} = \kappa^{-1}\text{atanh}(\kappa A_0)$ in the deformed space. Besides, the unbounded motion has
the interval of the linear momentum \(0 < |p| < m_0\omega_0 A_0\) turned into \(m_0\omega_0 A_0\sqrt{1 - \frac{1}{\kappa^2 A_0^2}} < |\Pi_\kappa| < m_0\omega_0 A_0\). As the dimensionless parameter \(\kappa A_0\) increases from 0 to 1.1 within the interval \([0.9, 1.1]\) it is observed that the horizontal axe of the ellipses become infinite, thus giving place to an unbounded motion.

![FIG. 6. (Color online) Phase spaces of the \(\kappa\)-deformed oscillator in the (a) usual canonical coordinates \((x, p)\) and the (b) deformed canonical ones \((x_\kappa, \Pi_\kappa)\) for \(\kappa A_0 = 0, 0.5, 0.9\) and 1.1.](image)

By means of the WKB approximation we can obtain the energy levels of the corresponding quantum system. Using this method, we have

\[
\left( n + \frac{1}{2} \right) \frac{\hbar}{2} = \frac{1}{2\pi} \int_{-A_\kappa}^{A_\kappa} p(x) dx = \frac{m_0\omega_\kappa}{2\pi} \int_{-A_\kappa}^{A_\kappa} \frac{\sqrt{A_\kappa^2 - x^2}}{1 + \kappa^2 x^2} dx
\]

\[
= \frac{m_0\omega_\kappa A_\kappa^2}{4\pi} \int_{0}^{2\pi} \frac{\sin^2 \theta_\kappa}{1 + \kappa^2 A_\kappa^2 \cos^2 \theta_\kappa} d\theta_\kappa
\]

\[
= \frac{m_0\omega_\kappa}{2\kappa^2} \left( \sqrt{1 + \kappa^2 A_\kappa^2} - 1 \right)
\]

with \(n = 0, 1, 2, \ldots\). Since \(E = \frac{1}{2} m_0\Omega_\kappa^2 A_\kappa^2\) we obtain

\[
E_n = \hbar \omega_0 \left( n + \frac{1}{2} \right) - \frac{\hbar^2 \kappa^2}{2 m_0} \left( n + \frac{1}{2} \right)^2,
\]

which corresponds to the energy levels of an anharmonic oscillator.

From Eq. (50), the classical density probability of finding the particle between \(x\) and \(x + dx\) results \(\rho_{\text{classic}}(x) = \frac{1}{\pi \sqrt{A_\kappa^2 - x^2}}\). The first and second moments of the position and the
linear momentum in terms of the amplitude or the energy for the deformed oscillator are

\[
\begin{align*}
\overline{x} &= 0, \quad (54a) \\
\overline{x^2} &= \frac{A_x^2}{2} = \frac{E}{m_0\omega_0^2 \left(1 - \frac{2E\kappa^2}{m_0\omega_0^2}\right)}, \quad (54b) \\
\overline{p} &= 0, \quad (54c) \\
\overline{p^2} &= \frac{m_0\omega_0^2A_x^2}{2(1 + \kappa^2 A_x^2)^{3/2}} = m_0E \sqrt{1 - \frac{2E\kappa^2}{m_0\omega_0^2}}. \quad (54d)
\end{align*}
\]

The mean values of the kinetic and potential energies satisfy the relationship

\[
\overline{T} = E - \overline{V} = \frac{m_0\omega_0^2}{2\kappa^2} \frac{1}{\sqrt{1 + \kappa^2 A_x^2}} \left(1 - \frac{1}{\sqrt{1 + \kappa^2 A_x^2}}\right),
\]

with \(\overline{V} = \int \rho_{\text{classic}}(x) V(x) dx\). Since \(\overline{V} = \overline{T}/\sqrt{1 - \kappa^2 A_0^2}\), we have that the virial theorem \((\overline{V} = \overline{T})\) is satisfied only for \(\kappa A_0 = 0\), which implies \(\kappa = 0\).

**B. \(\kappa\)-Deformed quantum oscillator**

The corresponding \(\kappa\)-deformed time-independent Schrödinger equation for the PDM oscillator is

\[
-\frac{\hbar^2}{2m_0} D^2_\kappa \phi(x) + \frac{1}{2} \frac{m_0\omega_0^2x^2}{(1 + \kappa^2x^2)^2} \phi(x) = E \phi(x).
\]

Making the change of variable \(x \to x_\kappa = \kappa^{-1} \text{arcsinh}(\kappa x)\) (see Eq. (4)) we obtain a particle with constant mass \(m_0\) subjected to the Pöschl-Teller potential

\[
-\frac{\hbar^2}{2m_0} \frac{d^2\phi(x_\kappa)}{dx^2_\kappa} - \frac{\hbar^2\kappa^2}{m_0} \frac{\nu(\nu + 1)}{2} \text{sech}^2(\kappa x_\kappa) \phi(x_\kappa) = \epsilon \phi(x_\kappa),
\]

with \(\epsilon = E - \hbar\omega_0/2\kappa^2 a_0^2\), \(\nu(\nu + 1) = 1/\kappa^2 a_0^4\) and \(a_0^2 = \hbar/m_0\omega_0\). The solutions of the Eq. (57) are

\[
\phi(x_\kappa) = \sqrt{\frac{\kappa \mu(\nu - \mu)!}{(\nu + \mu)!}} P^\mu_\nu(\tanh(\kappa x_\kappa)),
\]

where \(\mu = \nu - n\), \(n\) is an integer and \(P^\mu_\nu\) are the associated Legendre polynomials. Then, the eigenfunctions for the \(\kappa\)-deformed oscillator in the space representation \(x\) are

\[
\psi_n(x) = \sqrt{\frac{\kappa(\nu - n)n!}{(2\nu - n)!}} \frac{1}{\sqrt{1 + \kappa^2 x^2}} P^{\nu-n}_{\nu-n} \left(\frac{\kappa x}{\sqrt{1 + \kappa^2 x^2}}\right).
\]

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The energy levels are given by
\[ E_n = \hbar \omega_\kappa \left( n + \frac{1}{2} \right) - \frac{\hbar^2 \kappa^2}{2m_0} \left( n + \frac{1}{2} \right)^2 - \frac{\hbar^2 \kappa^2}{8m_0} \]
with \( \omega_\kappa = \omega_0 \sqrt{1 + \frac{\hbar^2 \kappa^2}{4m_0^2 \omega_0^2}} \). It should be noted that the quantum energy levels differ from those obtained using the WKB approximation (Eq. (53)) by the constant term \( -\frac{\hbar^2 \kappa^2}{8m_0} \) and the frequency of small oscillations \( \omega_0 \) replaced by \( \omega_\kappa \). This modification in the frequency is associated with the symmetrization problem of the classical Hamiltonian in order to construct its corresponding Hamiltonian operator in the quantum formalism (see Ref. 23 for more details). However, in the limit \( \hbar \to 0 \) with \( n \gg 1 \), the Eq. (60) recovers the semi-classical approximation, Eq. (53). In Fig. 7 an illustration of the potential \( V(x) = \frac{m_0 \omega_0^2 x^2}{2(1+\kappa^2 x^2)} \) along with the energy levels for some values of \( \kappa A_0 \), is shown. In Fig. 8 we show the wavefunctions and the probability densities for the four lower energy states and for some values of \( \kappa a_0 \). The values of \( \kappa a_0 \) chosen are such that \( \nu(\nu + 1) = 1/\kappa^4 a_0^4 \) is satisfied with \( \nu \) integer. We consider \( \nu = 4, 5, 10 \) and \( \infty \) in such a way that the corresponding values of \( \kappa a_0 \) are \( 20^{-1/4}, 30^{-1/4}, 110^{-1/4} \) and 0.

From the Legendre differential equation
\[ (1-u^2) \frac{d^2 P_\nu(u)}{du^2} - 2u \frac{dP_\nu(u)}{du} + \left[ \nu(\nu + 1) - \frac{\mu^2}{1-u^2} \right] P_\nu(u) = 0, \]
the identities (see Eqs. (2) and (3) in page 965 of the Ref. 84), we obtain the expectation
FIG. 8. (Color online) Eigenfunctions $\psi_n(x)$ (upper line) and probability densities $\rho_n(x) = |\psi_n(x)|^2$ (bottom line) for a $\kappa$-deformed oscillator particle for the values of $\kappa a_0$ such that $\nu(\nu + 1) = 1/(\kappa a_0)^4$ with $\nu = 4, 5, 10$ and $\infty$ in such a way that the corresponding values of $\kappa a_0$ are $20^{-1/4}, 30^{-1/4}, 110^{-1/4}$ and 0. (a) and (b): $n = 0$ (ground state), (c) and (d): $n = 1$ (first excited state), (e) and (f): $n = 2$ (second excited state), (g) and (h): $n = 3$ (third excited state).
values of $\langle \hat{x} \rangle$, $\langle \hat{x}^2 \rangle$, $\langle \hat{p} \rangle$, and $\langle \hat{p}^2 \rangle$ are

\[
\langle \hat{x} \rangle = 0, \quad \langle \hat{x}^2 \rangle = \frac{E_n + \frac{\hbar^2 \kappa^2}{2m_0}}{m_0\omega_0^2 \left(1 - \frac{2E_n\kappa^2}{m_0\omega_0^2} - \frac{\hbar^2 \kappa^4}{m_0^2\omega_0^2}\right)} \equiv \frac{\hbar}{m_0\omega_0} \left\{ \frac{\omega_{\kappa}}{\omega_0} \left(n + \frac{1}{2}\right) - \frac{\kappa^2 a_0^2}{2} (n^2 + n - \frac{1}{2}) \right\}, \quad \text{(62b)}
\]

\[
\langle \hat{p} \rangle = 0, \quad \langle \hat{p}^2 \rangle = m_0 \left( E_n - \frac{\hbar^2 \kappa^2}{4m_0} \right) \frac{z^2 - (2n + 1)z}{z^2 - 4} \equiv m_0\hbar\omega_0 \left[ \frac{\omega_\kappa}{\omega_0} \left(n + \frac{1}{2}\right) - \frac{\kappa^2 a_0^2}{2} (n^2 + n + 1) \right] \frac{z^2 - (2n + 1)z}{z^2 - 4}, \quad \text{(62d)}
\]

with $z \equiv 2\nu + 1 = \sqrt{1 + \frac{4m_0^2\omega_0^2}{\hbar^2 \kappa^2}}$.

In the limit $\kappa \to 0$, i.e. $z \to \infty$, the usual cases $\langle \hat{x}^2 \rangle = \frac{\hbar}{m_0\omega_0} (n + \frac{1}{2})$ and $\langle \hat{p}^2 \rangle = m_0\hbar\omega_0 \left(n + \frac{1}{2}\right)$ are recovered. According to the principle of correspondence, in the limit of large quantum numbers (or equivalently $\hbar \to 0$), we have $E_n \to E$ and $\omega_\kappa \to \omega_0$, and it is immediately verified that Eqs. (62b) and (62d) coincide with Eqs. (54b) and (54d), respectively. Indeed, when $\hbar \to 0$ one obtains that $z \approx 2m_0\omega_0/\hbar \kappa^2 \gg 1$, and we have

\[
\lim_{\hbar \to 0} \langle \hat{p}^2 \rangle = \lim_{\hbar \to 0} m_0E_n \left(1 - \frac{2n + 1}{z}\right) = \lim_{\hbar \to 0} m_0E_n \sqrt{1 - \frac{2\kappa^2 E_n}{m_0\omega_0^2 \kappa^4} - \frac{\hbar^2 \kappa^4}{4m_0\omega_0^2}} = m_0E_n \sqrt{1 - \frac{2\kappa^2 E_n}{m_0\omega_0^2 \kappa^4} + \mathcal{O}(\hbar^2)}. \quad \text{(63)}
\]

The expectation values of the kinetic and potential energies satisfy

\[
\langle \hat{T} \rangle = E_n - \langle \hat{V} \rangle = \frac{m_0\omega_0^2}{2\kappa^2} \frac{1}{\sqrt{1 + \kappa^2 a_{n,\kappa}^2}} \left(\frac{\omega_0}{\omega_\kappa} - \frac{1}{\sqrt{1 + \kappa^2 a_{n,\kappa}^2}}\right), \quad \text{(64)}
\]

with $E_n = \frac{m_0\omega_0^2 a_{n,\kappa}^2}{2(1 + \kappa^2 a_{n,\kappa}^2)}$ and the quantum amplitude

\[
a_{n,\kappa} = a_0 \left\{ \frac{\omega_{\kappa}}{\omega_0} (2n + 1) - \kappa^2 a_0^2 (n^2 + n + \frac{1}{2}) \right\}^{-1/2} \left[ \frac{\omega_{\kappa}}{\omega_0} (2n + 1) - \kappa^2 a_0^2 (n^2 + n + \frac{1}{2}) \right]^{1/2}. \quad \text{(65)}
\]

In the classical limit, one has that $a_{n,\kappa} \to A_{\kappa}$ once $E_n \to E$, so the expectation value (64) recovers its classical average value (55). In Fig. 9 we show the uncertainty relation of the
κ-deformed oscillator, along with the uncertainties \( \Delta x \) and \( \Delta p \) of \( x \) and \( p \), for the ground state and the first three excited ones. As expected, while \( \Delta x \) increases as the dimensionless deformation parameter \( \kappa a_0 \) varies within the interval \([-1, 1]\), \( \Delta p \) decreases and viceversa. In turn, this implies a generalized \( \kappa \)-uncertainty inequality (Fig. 9 (c)) which is an increasing function of the quantum number \( n \) and it also grows fast as \( \kappa a_0 \) varies. The symmetry exhibited around the axis \( \kappa a_0 = 0 \) in the curves of the Fig. 9 are a consequence of the invariance of the mass function (and then of the Hamiltonian too) given by Eq. (13) against the transformation \( \kappa \to -\kappa \).

![FIG. 9. (Color online) Uncertainty of (a) the position \( \Delta x \), (b) the momentum \( \Delta p \), and of the product (c) \( \Delta x \Delta p \), as function of \( \kappa a_0 \), for the quantum states with \( n = 0, 1, 2 \) and 3. The standard uncertainty relation \( \Delta x \Delta p = (n + \frac{1}{2})\hbar \) is recovered for \( \kappa a_0 \to 0 \).]

VI. CONCLUSIONS

We have presented the quantum and the classical mechanics that results from assuming a position-dependent mass related to the \( \kappa \)-algebra, which is the mathematical background underlying \( \kappa \)-statistics. Indeed, we have characterized both the quantum and classical formalism of a particle with a PDM determined univocally by the \( \kappa \)-algebra. The consistency of the \( \kappa \)-deformed formalism is manifested in the following arguments.

The \( \kappa \)-deformed Schrödinger equation turns out to be equivalent to a Schrödinger-like equation for a deformed wave-function provided with a \( \kappa \)-deformed non-Hermitian momentum operator. Within the \( \kappa \)-formalism one can define deformed versions of the continuity equation, the Fourier transform, etc. In particular, a deformed Newton’s second law in terms of the deformed dual \( \kappa \)-derivative (Eq. (15)) follows in the classical limit.

We have illustrated the approach with the problems of a particle confined in an infinite
potential well and a \( \kappa \)-deformed oscillator which is equivalent to the Mathews-Lakshmanan oscillator (in the standard space) or to the Pösch-Teller potential problem (in the \( \kappa \)-deformed space), provided with the change of variable \( x \rightarrow x_\kappa \). We have obtained the distributions for the classical case as well as the eigenfunctions and eigenenergies for the quantum case. Although we have applied the mapping approach to a \( \kappa \)-deformed space in order to study the quantum Mathews-Lakshmanan oscillator, it is important to mention that other equivalent approaches can be found in the literature. For instance, factorization methods, supersymmetry and coherent states have also been investigated for this nonlinear oscillator (see 34–38 and references therein).

Analogously to the quantum oscillator and the Hermite polynomials, the eigenvalues equation for the \( \kappa \)-deformed quantum oscillator is expressed in terms of the Legendre polynomials. Expectedly, in both examples we have reported the localization and delocalization of the probability density functions corresponding to the conjugated variables \( x \) and \( p \), from which the uncertainty relation follows (Figs. 5 and 9), with the particularity that the lower bound is an increasing function of the deformation parameter \( \kappa \), satisfied by the ground state and the first three excited ones. This could be physically interpreted as if the quantum role of the deformation (or equivalently, of the non-constant mass) is to increase the intrinsic correlation between the conjugated operators \( \hat{x} \) and \( \hat{p} \). Also, for the case of the \( \kappa \)-deformed oscillator we have studied the effect of the deformation parameter \( \kappa \) on the phase space in the usual coordinates \( (x, p) \) and in the deformed ones \( (x_\kappa, \Pi_\kappa) \). It is verified that for a certain range of values of \( \kappa \) the motion is unbounded (Fig. 6).

We consider that the techniques employed in this work could stimulate the seek of other generalizations of classical and quantum mechanical aspects, as has been reported in recent researches by means of the \( q \)-algebra.\(^{7,48–56}\)

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DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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