CONSTRUCTION OF CUSP FORMS USING RANKIN-COHEN
BRACKETS

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Abstract. For a fix modular form $g$ and a non negative ineteger $\nu$, by using Rankin-Cohen bracket we first define a linear map $T_{g,\nu}$ on the space of modular forms. We explicitly compute the adjoint of this map and show that the $n$-th Fourier coefficients of the image of the cusp form $f$ under this map is, upto a constant a special value of Rankin-Selberg convolution of $f$ and $g$. This is a generalization of the work due to W. Kohnen (Math. Z., 207, (1991), 657-660) and S. D. Herrero (Ramanujan J., 36(2014), no.3, 529-536) in the case of integral weight modular forms to half integral weight modular forms. As a consequence we get non-vanishing of special value of certain Rankin- Selberg convolution of modular forms.

1. Introduction

W. Kohnen [7] constructed cusp forms whose Fourier coefficients are given by special values of certain Dirichlet serise by computing the adjoint of the product map by a fixed cusp form with respect to the usual Petersson scalar product. This result has been generalized by several authors to other automorphic forms (see the list [1, 8, 9, 10, 12]). The work of Kohnen has been generalized by S. D. Herrero [3], where the author constructed the cusp forms by computing the adjoint of the map constructed using the Rankin-Cohen brackets by a fixed cusp form instead of the product map. Recently, the work of S. D. Herrero has been generalised by first author and B. Sahu to the case of Jacobi forms [5] which also generalises the result of H. Sakata [10]. In this article we extend the work of S. D. Herrero to the case of half integral weight modular forms. We apply this result to get non-vanishing of special value of certain Rankin- Selberg convolution of modular forms.

2. Preliminaries

2.1. Elliptic Modular Forms. Let $\mathcal{H}$ be the complex upper half-plane and $\Gamma$ be a congruence subgroup of the full modular group $SL_2(\mathbb{Z})$. For $k \in \mathbb{Z}$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, define the slash operator as follows;

$$f \mid_k \gamma(z) := (cz + d)^{-k} f(\gamma z), \text{ where } \gamma z = \frac{az + b}{cz + d}.$$

Let $M_k(\Gamma, \chi)$ (respectively $S_k(\Gamma, \chi)$) denote the space of modular forms (resp. cusp forms) of integral weight $k$ and character $\chi$ for $\Gamma$, i.e., for every $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, $f \mid_k \gamma(z) = \ldots$
\( \chi(d)f(z) \), and holomorphic at cusps of \( \Gamma \).

We define the Petersson scalar product on \( S_k(\Gamma, \chi) \) as follows:

\[
\langle f, g \rangle = \int_{\mathcal{H}} f(z) \overline{g(z)} (Im(z))^k d^*z,
\]

where \( z = x + iy \) and \( d^*z = \frac{dx dy}{y^2} \) is an invariant measure under the action on \( \Gamma \) on \( \mathcal{H} \). For more details on the theory of modular forms, we refer to [6].

2.2. Poincaré series.

**Definition 2.1.** Let \( n \) be a positive integer. The \( n \)-th Poincaré series of integer weight \( k \) is defined by

\[ P_{k,n}(z) := \sum_{\gamma \in \Gamma_\infty \setminus \Gamma} e^{2\pi inz|_{k\gamma}} \]

where \( \Gamma_\infty := \left\{ \pm \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \mid t \in \mathbb{Z} \right\} \). It is well known that \( P_{k,n} \in S_k(\Gamma) \) for \( k > 2 \).

This series has the following property.

**Lemma 2.2.** Let \( f \in S_k(\Gamma) \) with Fourier expansion \( f(z) = \sum_{m=1}^{\infty} a(m)q^m \). Then

\[ \langle f, P_{k,n} \rangle = \alpha_{k,n} a(n), \quad \text{where} \quad \alpha_{k,n} = \frac{\Gamma(k-1)}{(4\pi n)^{k-1}} \]

2.3. Modular Forms of Half Integral Weight. Let \( \Gamma = \Gamma_0(4) \). For \( k \in \mathbb{Z} \) and \( \gamma = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \), define the slash operator as follows;

\[ f|_{k+\frac{1}{2}} \gamma(z) := \left( \frac{c}{d} \right)^{-\frac{k}{2}} (cz + d)^{-k-\frac{1}{2}} f(\gamma z), \]

where \( \left( \frac{c}{d} \right) \) is the Kronecker symbol.

Let \( M_{k+\frac{1}{2}}(\Gamma, \chi) \) (resp. \( S_{k+\frac{1}{2}}(\Gamma, \chi) \)) denote the space of modular forms (resp. cusp forms) of weight \( k + \frac{1}{2} \) and character \( \chi \) for \( \Gamma \), i.e., for every \( \gamma = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \in \Gamma \), \( f|_{k+\frac{1}{2}} \gamma(z) = \chi(d) f(z) \), and holomorphic (resp. vanish) at cusps of \( \Gamma \).

We define the Petersson scalar product on \( S_{k+\frac{1}{2}}(\Gamma, \chi) \) as follows;

\[ \langle f, g \rangle = \int_{\mathcal{H}} f(z) \overline{g(z)} (Im(z))^{k+\frac{1}{2}} d^*z, \]

where \( z = x + iy \). The spaces \( S_k(\Gamma, \chi) \) and \( S_{k+\frac{1}{2}}(\Gamma, \chi) \) are finite dimensional Hilbert spaces. For more details on the theory of modular forms of half integral weight, we refer to [6] and [11].
2.4. Poincaré series of half integral weight.

**Definition 2.3.** Let \( n \) be a positive integer. The \( n \)-th Poincaré series of weight \( k + \frac{1}{2} \), where \( k \in \mathbb{Z} \) is defined by

\[
P_{k + \frac{1}{2}, n}(z) := \sum_{\gamma \in \Gamma \setminus \mathbb{H}} e^{2\pi i n z} |k + \frac{1}{2}|. \tag{3}
\]

It is well known that \( P_{k + \frac{1}{2}, n} \in S_{k + \frac{1}{2}}(\Gamma) \) for \( k > 2 \).

This series has the following property.

**Lemma 2.4.** Let \( f \in S_{k + \frac{1}{2}}(\Gamma) \) with Fourier expansion \( f(z) = \sum_{m=1}^{\infty} a(m)q^m \). Then

\[
\langle f, P_{k + \frac{1}{2}, n} \rangle = \tilde{\alpha}_{k,n} a(n), \quad \text{where} \quad \tilde{\alpha}_{k,n} = \frac{\Gamma(k - \frac{1}{2})}{(4\pi n)^{k - \frac{1}{2}}}. \tag{4}
\]

The following lemmas tell about the growth of the Fourier coefficients of a modular form.

**Lemma 2.5.** [4] If \( f \in M_k(\Gamma, \chi) \) with Fourier coefficients \( a(n) \), then

\[
a(n) \ll |n|^{k-1+\epsilon},
\]

and moreover, if \( f \) is a cusp form, then

\[
a(n) \ll |n|^{\frac{k}{2}+\epsilon}.
\]

**Lemma 2.6.** If \( f \in M_{k + \frac{1}{2}}(\Gamma, \chi) \) with Fourier coefficients \( a(n) \), then

\[
a(n) \ll |n|^{k-\frac{1}{2}+\epsilon},
\]

and moreover, if \( f \in S_{k + \frac{1}{2}}(\Gamma, \chi) \) is a cusp form, then

\[
a(n) \ll |n|^{\frac{k}{2}+\epsilon}.
\]

2.5. Rankin-Cohen Brackets. Let \( k \) and \( l \) be real numbers and \( \nu \geq 0 \) be an integer. Let \( f \) and \( g \) be two complex valued holomorphic functions on \( \mathbb{H} \). Define the \( \nu \)-th Rankin-Cohen bracket of \( f \) and \( g \) by

\[
[f, g]_\nu := \sum_{r=0}^{\nu} C_r(k, l; \nu) D^r f D^{\nu-r} g,
\]

where \( D^r f = \frac{1}{(2\pi i)^r} \frac{d^r f}{dz^r} \) and \( C_r(k, l; \nu) = (-1)^{\nu-r} \binom{\nu}{r} \frac{\Gamma(k+\nu)\Gamma(l+\nu)}{\Gamma(k+r)\Gamma(l+\nu-r)} \) and \( \Gamma(x) \) is the usual Gamma function.

**Remark 2.1.** It is easy to verify that

\[
[f|k\gamma, g|l\gamma]_\nu = [f, g]_{k+l+2\nu}\gamma, \quad \forall \gamma \in \Gamma.
\]

**Remark 2.2.** We note that the 0-th Rankin-Cohen bracket is the usual product of modular forms i.e., \([f, g]_0 = fg\).
Theorem 2.7. [2] Let \( \nu \geq 0 \) be an integer and \( f \in M_k(\Gamma, \chi_1) \) and \( g \in M_l(\Gamma, \chi_2) \). Then
\[
[f, g]_\nu \in M_{k+l+2\nu}(\Gamma, \chi_1\chi_2\chi),
\]
where \( \chi = \begin{cases} 
1, & \text{if both } k, l \in \mathbb{Z}, \\
\chi_4^k, & \text{if } k \in \mathbb{Z} \text{ and } l \in \mathbb{Z} + \frac{1}{2}, \\
\chi_4^l, & \text{if } k \in \mathbb{Z} + \frac{1}{2} \text{ and } l \in \mathbb{Z}, \\
\chi = \chi_{-4}^{k+l}, & \text{if both } k, l \in \mathbb{Z} + \frac{1}{2},
\end{cases}
\]
Moreover if \( \nu > 0 \), then \([f, g]_\nu \in S_{k+l+2\nu}(\Gamma, \chi_1\chi_2\chi)\). In fact, \([ \cdot, \cdot ]_\nu\) is a bilinear map from \( M_k(\Gamma, \chi_1) \times M_l(\Gamma, \chi_2) \) to \( M_{k+l+2\nu}(\Gamma, \chi_1\chi_2\chi)\). Here \( \chi_{-4} \) is the character defined by \( \chi_{-4}(x) = \left( \frac{-4}{x} \right) \).

Let \( k, l \in \frac{7}{2} \) and \( \nu \geq 0 \) be integers and \( \Gamma \) be a congruence subgroup of the full modular group \( SL_2(\mathbb{Z}) \). Also assume that that \( \Gamma \subseteq \Gamma_0(4) \) if either of \( k \) or \( l \) is non integer. For a fixed \( g \in M_l(\Gamma, \chi_2) \), we define the map
\[
T_{g, \nu} : S_k(\Gamma) \to S_{k+l+2\nu}(\Gamma, \chi_2)
\]
defined by \( T_{g, \nu}(f) = [f, g]_\nu \). \( T_{g, \nu} \) is a \( \mathbb{C} \)-linear map of finite dimensional Hilbert spaces and therefore has an adjoint map \( T_{g, \nu}^* : S_{k+l+2\nu}(\Gamma, \chi_2) \to S_k(\Gamma) \) such that
\[
\langle f, T_{g, \nu}(h) \rangle = \langle T_{g, \nu}^*(f), h \rangle, \quad \forall f \in S_{k+l+2\nu}(\Gamma, \chi_2) \text{ and } h \in S_k(\Gamma).
\]
In [3] S.D. Herrero computed the adjoint map for the case when \( k, l \in \mathbb{Z}, \Gamma = SL_2(\mathbb{Z}) \) and \( \chi_2 \) is the trivial character.

Theorem 2.8. [3] Let \( k \geq 6 \) and \( l \) be natural numbers and \( \nu \geq 0 \). Let \( g \in M_l(SL_2(\mathbb{Z})) \) with Fourier expansion
\[
g(z) = \sum_{m=0}^{\infty} b(m) q^m.
\]
Suppose that either (a) \( g \) is a cusp form or (b) \( g \) is not cusp form and \( l < k - 3 \). Then the image of any cusp form \( f \in S_{k+l+2\nu}(SL_2(\mathbb{Z})) \) with Fourier expansion
\[
f(z) = \sum_{m=1}^{\infty} a(m) q^m
\]
under \( T_{g, \nu}^* \) is given by
\[
T_{g, \nu}^*(f)(z) = \sum_{n=1}^{\infty} c(n) q^n,
\]
where
\[
c(n) = \beta(k, l, \nu; n) L_{f, g, \nu, n}(\gamma),
\]
where \( L_{f, g, \nu, n} \) is the \( L \)- function associated with \( f \) and \( g \), defined by for \( s \in \mathbb{C} \),
\[
L_{f, g, \nu, n}(s) = \sum_{m=1}^{\infty} \frac{a(n + m)b(m)}{(n + m)^s} \alpha(k, l, \nu, n, m)
\]
with
\[ \alpha(k, l, \nu, n, m) = \sum_{r=0}^{\nu} (-1)^{\nu-r} \Gamma(k + \nu \Gamma(l + \nu) \Gamma(k + r) \Gamma(l + \nu - r) n^r m^{\nu-r} \]

and
\[ \gamma = k + l + 2\nu - 1, \ \beta(k, l, \nu; n) = \frac{\Gamma(k + l + 2\nu - 1)}{\Gamma(k - 1)(4\pi)^{l+2\nu}}. \]

Remark 2.3. One can prove the similar result for the case when \( \Gamma \) is a congruence subgroup of level \( N \) and \( \chi_2 \) is any character mod \( N \) using the technique used in proof of Theorem 3.1.

3. Statement of the Theorem

Consider the following maps:

1. \( T_{g,\nu} : S_{k+\frac{1}{2}}(\Gamma) \to S_{k+l+2\nu+1}(\Gamma, \chi_2\chi), \) with \( g \in M_{l+\frac{1}{2}}(\Gamma, \chi_2) \),
2. \( T_{g,\nu} : S_k(\Gamma) \to S_{k+l+2\nu+1}(\Gamma, \chi_2\chi), \) with \( g \in M_{l+\frac{1}{2}}(\Gamma, \chi_2) \),
3. \( T_{g,\nu} : S_{k+\frac{1}{2}}(\Gamma) \to S_{k+l+2\nu+\frac{1}{2}}(\Gamma, \chi_2\chi), \) with \( g \in M_l(\Gamma, \chi_2) \)

We exhibit explicitly the Fourier coefficients of \( T_{g,\nu}^*(f) \) for \( f \in S_{k+l+2\nu+1}(\Gamma, \chi_2\chi) \) in (1) and by using the same method, we can find the analogous maps in (2) and (3) (see the remark 3.1). These involve special values of certain Dirichlet series of Rankin-Selberg type associated to \( f \) and \( g \). We now state the main theorem.

Theorem 3.1. Let \( k \) and \( l \) be natural numbers and \( \nu \geq 0 \). Let \( g \in M_{l+\frac{1}{2}}(\Gamma, \chi_2) \) with Fourier expansion
\[ g(z) = \sum_{m=0}^{\infty} b(m)q^m. \]

Suppose that either (a) \( g \) is a cusp form and \( k > 2 \) or (b) \( g \) is not cusp form and \( l < k - \frac{3}{2} \). Then the image of any cusp form \( f \in S_{k+l+2\nu+1}(\Gamma, \chi_2\chi) \) with Fourier expansion
\[ f(z) = \sum_{m=1}^{\infty} a(m)q^m \]

under \( T_{g,\nu}^* \) is given by
\[ T_{g,\nu}^*(f)(z) = \sum_{n=1}^{\infty} c(n)q^n, \]

where
\[ c(n) = \beta(k, l, \nu; n)L_{f,g,\nu,n}(\gamma), \]

where
\[ \gamma = k + l + 2\nu, \ \beta(k, l, \nu; n) = \frac{\Gamma(k + l + 2\nu)}{\Gamma(k - \frac{1}{2})(4\pi)^{l+2\nu+\frac{1}{2}}} \]

and \( L_{f,g,\nu,n}(\gamma) \) is defined in Theorem 2.8.
Remark 3.1. We have the similar results for the map in (2) with
\[ \gamma = k + l + 2\nu - \frac{1}{2}, \quad \text{and} \quad \beta(k, l, \nu; n) = \frac{\Gamma(k + l + 2\nu - \frac{1}{2}) n^{k-1}}{\Gamma(k - 1) (4\pi)^{l+2\nu+\frac{1}{2}}}, \]
and for the map in (3) with
\[ \gamma = k + l + 2\nu - \frac{1}{2}, \quad \text{and} \quad \beta(k, l, \nu; n) = \frac{\Gamma(k + l + 2\nu - \frac{1}{2}) n^{k-\frac{1}{2}}}{\Gamma(k - \frac{1}{2}) (4\pi)^{l+2\nu}}. \]
with the assumption that either (a) \( g \) is a cusp form and \( k > 3 \) or (b) \( g \) is not cusp form and \( l < k - 2 \).

Remark 3.2. Using Lemma 2.5 and Lemma 2.6 one can show that the series appearing in (8) converges.

4. Proof of Theorem 3.1

We need the following lemma to proof the main theorem.

Lemma 4.1. Using the same notation in Theorem 3.1, we have
\[ \sum_{\gamma \in \Gamma_{\infty} \setminus \Gamma \setminus \mathcal{H}} \int_{\Gamma \setminus \mathcal{H}} | f(z) \left[ e^{2\pi i nz} | k \gamma, g \right]_{\nu} (Im(z))^{k+l+2\nu+1} | \ d^{*}z \]
converges.

Proof. The proof is similar to Lemma 1 in [3]. \( \square \)

Now we give a proof of Theorem 3.1. Put
\[ T_{g,\nu}^{*}(f)(z) = \sum_{n=1}^{\infty} c(n)q^{n}. \]
Now, we consider the \( n \)-th Poincaré series of weight \( k + \frac{1}{2} \) as given in (3). Then using the Lemma 2.4, we have
\[ \langle T_{g,\nu}^{*}f, P_{k+\frac{1}{2}, n} \rangle = \tilde{\alpha}_{k,n} c(n), \]
where
\[ \tilde{\alpha}_{k,n} = \frac{\Gamma(k - \frac{1}{2})}{(4\pi n)^{k-\frac{1}{2}}}. \]
On the other hand, by definition of the adjoint map we have
\[ \langle T_{g,\nu}^{*}f, P_{k+\frac{1}{2}, n} \rangle = \langle f, T_{g,\nu}(P_{k+\frac{1}{2}, n}) \rangle = \langle f, [P_{k+\frac{1}{2}, n}, g]_{\nu} \rangle. \]
Hence we get
\[ c(n) = \frac{(4\pi n)^{k-\frac{1}{2}}}{\Gamma(k - \frac{1}{2})} \langle f, [P_{k+\frac{1}{2}, n}, g]_{\nu} \rangle. \] (9)
By definition,

\[
\langle f, [P_{k+\frac{1}{2}, n}, g]_\nu \rangle = \int_{\Gamma \backslash \mathcal{H}} f(z) \overline{\left[ P_{k+\frac{1}{2}, n}, g \right]_\nu (z)} (\text{Im}(z))^{k+l+2\nu+1} d^* z
\]

\[
= \int_{\Gamma \backslash \mathcal{H}} f(z) \left[ \sum_{\gamma \in \Gamma \backslash \mathcal{H}} e^{2\pi i n z} \chi_{\nu}(z) (\text{Im}(z))^{k+l+2\nu+1} d^* z \right]
\]

\[
= \int_{\Gamma \backslash \mathcal{H}} \sum_{\gamma \in \Gamma \backslash \mathcal{H}} f(z) \left[ e^{2\pi i n z} \chi_{\nu}(z) (\text{Im}(z))^{k+l+2\nu+1} d^* z \right].
\]

By Lemma 4.1, we can interchange the sum and integration in \( \langle f, [P_{k,n}, g]_\nu \rangle \). Hence we get,

\[
\langle f, [P_{k+\frac{1}{2}, n}, g]_\nu \rangle = \sum_{\gamma \in \Gamma \backslash \mathcal{H}} \int_{\Gamma \backslash \mathcal{H}} f(z) \overline{\left[ e^{2\pi i n z} \chi_{\nu}(z) (\text{Im}(z))^{k+l+2\nu+1} d^* z \right]}.
\]

Since \( g \in M_{l+1}(\Gamma, \chi_2), \) \( g_{l+\frac{1}{2}} = \chi_2(d)g(z) \), for every \( \gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma \). Therefore

\[
\langle f, [P_{k+\frac{1}{2}, n}, g]_\nu \rangle = \sum_{\gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma \backslash \mathcal{H}} \int_{\Gamma \backslash \mathcal{H}} f(z) \overline{\left[ e^{2\pi i n z} \chi_{\nu}(z) (\text{Im}(z))^{k+l+2\nu+1} d^* z \right]}.
\]

Using the change of variable \( z \) to \( \gamma^{-1}z \) in each integral, \( \langle f, [P_{k+\frac{1}{2}, n}, g]_\nu \rangle \) equals

\[
\sum_{\gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma \backslash \mathcal{H}} \left( \frac{(d^{-1})^{k+l+1}}{\chi_2(d)} \right) \int_{\Gamma \backslash \mathcal{H}} f(\gamma^{-1}z) \overline{\left[ e^{2\pi i n z} \chi_{\nu}(z) (\text{Im}(\gamma^{-1}z))^{k+l+2\nu+1} d^* (\gamma^{-1}z) \right]}.
\]

Since \( f \in S_{k+l+2\nu+1}(\Gamma, \chi_2), \) \( f(\gamma^{-1}z) = \chi_2(d)\chi(d)(cz+d)^{k+l+2\nu+1}f(z) \), for every \( \gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma \), and hence

\[
\langle f, [P_{k+\frac{1}{2}, n}, g]_\nu \rangle = \sum_{\gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma \backslash \mathcal{H}} \left( \frac{(d^{-1})^{k+l+1}}{\chi_2(d)} \right) \int_{\Gamma \backslash \mathcal{H}} \chi_2(a)\chi(a)(-cz+a)^{k+l+2\nu+1}f(z)
\]

\[
= \left( \frac{(d^{-1})^{k+l+1}}{\chi_2(d)} \right) \sum_{\gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma \backslash \mathcal{H}} \chi_2(a)\chi(a)(-cz+a)^{k+l+2\nu+1}f(z)
\]

\[
\times \left( \frac{(d^{-1})^{k+l+2\nu+1}}{|cz+a|^2} \right) \int_{\Gamma \backslash \mathcal{H}} (\text{Im}(z))^{k+l+2\nu+1} d^* z.
\]
Now using (6), we get

$$\langle f, [P_{k+\frac{1}{2}, n}, g]_\nu \rangle = \sum_{\gamma = (a \ b \ c \ d) \in \Gamma_\infty \setminus \Gamma} \frac{(a \ b)}{\chi_2(d)} \chi_2(a) \chi(a) \int_{\gamma \ H} f(z) \left[ e^{2\pi i m z}, g \right]_\nu (Im(z))^{k+\nu+1} d^*_z.$$ 

The quantity appearing before integral is equals to 1, for all \((a \ b) \in \Gamma_\infty \setminus \Gamma\), hence we get

$$\langle f, [P_{k+\frac{1}{2}, n}, g]_\nu \rangle = \sum_{\gamma \in \Gamma_\infty \setminus \Gamma} \int_{\gamma \ H} f(z) \left[ e^{2\pi i m z}, g \right]_\nu (Im(z))^{k+\nu+1} d^*_z.$$ 

Now using Rankin unfolding argument, we have

$$\langle f, [P_{k+\frac{1}{2}, n}, g]_\nu \rangle = \int_{\Gamma_\infty \setminus \Gamma} f(z) \left[ e^{2\pi i m z}, g \right]_\nu (Im(z))^{k+\nu+1} d^*_z$$

$$= \int_{\Gamma_\infty \setminus \Gamma} f(z) \sum_{r=0}^{\nu} C_r(k, l; \nu) \overline{D^r(e^{2\pi i m z})} \overline{D^{\nu-r}(g)} (Im(z))^{k+\nu+1} d^*_z \tag{10}$$

Now replacing \(f\) and \(g\) by their Fourier series in (10), \(\langle f, [P_{k+\frac{1}{2}, n}, g]_\nu \rangle\) equals

$$\sum_{r=0}^{\nu} C_r(k, l; \nu) \int_{\Gamma_\infty \setminus \Gamma} \left( \sum_s a(s) e^{2\pi i z s} \right) n^r e^{2\pi i m z} m^{\nu-r} \overline{b(m)} e^{2\pi i m z} (Im(z))^{k+\nu+1} d^*_z$$

$$= \int_{\Gamma_\infty \setminus \Gamma} \sum_s \sum_m a(k, l, \nu, n, m) a(s) \overline{b(m)} e^{2\pi i z s} e^{2\pi i m z} e^{2\pi i m z} (Im(z))^{k+\nu+1} d^*_z$$

$$= \sum_s \sum_m a(k, l, \nu, n, m) a(s) \overline{b(m)} \int_{\Gamma_\infty \setminus \Gamma} e^{2\pi i z s} e^{2\pi i m z} e^{2\pi i m z} (Im(z))^{k+\nu+1} d^*_z.$$

A fundamental domain for the action of \(\Gamma_\infty\) on \(\mathbb{H}\) is given by \([0, 1] \times [0, \infty)\). Integrating on this region after substituting \(z = x + iy\),

$$\langle f, [P_{k, n}, g]_\nu \rangle = \sum_s \sum_m a(k, l, \nu, n, m) a(s) \overline{b(m)} \int_0^1 \int_0^\infty e^{2\pi i (s-n-m)x} e^{-2\pi (a+n+m)y} y^{k+\nu+1} dxdy$$

$$= \sum_m a(k, l, \nu, n, m) \overline{b(m)} \int_0^\infty e^{-4\pi (a+n+m)y} y^{k+\nu+1} dy$$

$$= \frac{\Gamma(k + l + 2\nu)}{(4\pi)^{k+l+2\nu}} \sum_m a(n + m) \overline{b(m)} \alpha(k, l, \nu, n, m).$$
Now substituting the above value of $\langle f, [P_{k+\frac{1}{2}, n}, g]_\nu \rangle$ in (9), we get the required expression for $c(n)$ given in Theorem 3.1.

5. Applications

Consider the linear map $T_{g,\nu}^* \circ T_{g,\nu}$ on $S_k(\Gamma)$ with $g(z) \in M_l(\Gamma, \chi_2)$. If $\lambda$ is a eigenvalue of $T_{g,\nu}^* \circ T_{g,\nu}$, then $\lambda \geq 0$. Suppose that $S_k(\Gamma)$ is one dimensional space generated by $f(z) = \sum_{m} a(m)q^m$. Then $T_{g,\nu}^* \circ T_{g,\nu}(h) = \lambda f$, $\forall h \in S_k(\Gamma)$. In particular, $T_{g,\nu}^* \circ T_{g,\nu}(f) = \lambda f$ with $\lambda \geq 0$ and if we write $T_{g,\nu}^* \circ T_{g,\nu}(f) = \sum_n c(n)q^n$ then

$$c(n) = \frac{\Gamma(k + l + 2\nu - 1)}{\Gamma(k - 1)} \frac{n^{k-\frac{1}{2}}}{(4\pi)^{l+2\nu}} \sum_{m=1}^{\infty} a_{T_{g,\nu}(f)}(m + n) \overline{b(m)} \frac{\alpha(k, l, \nu, n, m)}{(n + m)^{k + l + 2\nu - 1}}.$$ 

where $a_{T_{g,\nu}(f)}(n)$ is the $n$-th Fourier coefficient of $T_{g,\nu}(f) = [f, g]_\nu$. If $a(m_0)$ is the first non-zero Fourier coefficient of $f$ then by comparing the Fourier coefficients in $T_{g,\nu}^* \circ T_{g,\nu}(f) = \lambda f$, we have

$$\lambda = \frac{\Gamma(k + l + 2\nu - 1)}{a(m_0)\Gamma(k - 1)} \frac{m_0^{k-\frac{1}{2}}}{(4\pi)^{l+2\nu}} \sum_{m=1}^{\infty} a_{T_{g,\nu}(f)}(m_0 + m) \overline{b(m)} \frac{\alpha(k, l, \nu, m_0, m)}{(m_0 + m)^{k + l + 2\nu - 1}} \geq 0.$$ 

In particular, if we take $l = 0$, $k = 6$ and $\nu = 0$ with $g(z) = \theta(z) = \sum_{n} q^{n^2}$ and the unique newform $\Delta_{4,6}(z) = \sum_n \tau_{4,6}(n)q^n \in S_0(\Gamma_0(4))$, in case (2) then $m_0 = 1$, $\alpha(k, l, \nu, m_0, m) = 1$, and

$$\lambda = \frac{\Gamma(\frac{11}{2})}{\Gamma(5)2\sqrt{\pi}} \sum_{m=1}^{\infty} a_{T_{\theta,0}(\Delta_{4,6})}(m + 1) \frac{\overline{b(m)}}{(m + 1)^{\frac{11}{2}}} > 0,$$

or

$$\sum_{m=1}^{\infty} a_{T_{\theta,0}(\Delta_{4,6})}(m + 1) \frac{\overline{b(m)}}{(m + 1)^{\frac{11}{2}}} > 0. \tag{11}$$

Now $a_{T_{\theta,0}(\Delta_{4,6})}(m + 1)$ is the $(m + 1)$-th Fourier coefficient of $\theta(z)\Delta_{4,6}(z)$ and equals to $\sum_{r=1}^{m+1} b(r)\tau_{4,6}(m + 1 - r)$. Putting the value of $a_{T_{\theta,0}(\Delta_{4,6})}(m + 1)$ in (11), we have

$$\sum_{m=1}^{\infty} \frac{\left(\sum_{r=1}^{m+1} b(r)\tau_{4,6}(m + 1 - r)\right) \overline{b(m)}}{(m + 1)^{\frac{11}{2}}} > 0,$$

or

$$\sum_{m=1}^{\infty} \frac{\left(\sum_{r=1}^{m+1} \tau_{4,6}(m^2 + 1 - r^2)\right)}{(m^2 + 1)^{\frac{11}{2}}} > 0.$$


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