PS-Hollow Representations of Modules over Commutative Rings *

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Abstract
Let $R$ be a commutative ring and $M$ a non-zero $R$-module. We introduce the class of pseudo strongly hollow submodules (PS-hollow submodules, for short) of $M$. Inspired by the theory of modules with secondary representations, we investigate modules which can be written as finite sums of PS-hollow submodules. In particular, we provide existence and uniqueness theorems for the existence of minimal PS-hollow strongly representations of modules over Artinian rings.

Introduction
This paper is part of our continuing project of investigating the different notions of primeness and coprimeness for (sub)modules of a given a non-zero module $M$ over a (commutative) ring $R$ in their natural context as prime (coprime) elements in the lattice $\text{Sub}_R(M)$ of $R$-submodules with the canonical action of the poset $\text{Ideal}(R)$ of ideals of $R$. This approach proved to be very appropriate and enabled use to prove several results in this general setting and to provide more elegant and shorter proofs of our results. Moreover, it enabled us to generalize several notions and dualize them in a more systematic and elegant way.

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Generalizing the notion of a strongly hollow element in a lattice, we introduce for a lattice with an action of a poset the notion of a pseudo strongly hollow element. The two notions are equivalent in case the lattice is multiplication. Considering the lattice $\text{Sub}_R(M)$ of a non-zero module $M$ over a commutative ring $R$, we obtain new class of modules, which we call pseudo strongly hollow modules. We study this class of $R$-modules, as well as modules which can be written as finite sums of their pseudo strongly hollow submodules. In particular, we provide existence and uniqueness theorems of such representation over Artinian rings.

This paper consists of two sections. In Section 1, we define, for a bounded lattice $\mathcal{L} = (L, \wedge, \vee, 0, 1)$, several notions of primeness for elements in $L \setminus \{1\}$ as well as several coprimeness notions for elements in $L \setminus \{0\}$. In Theorem 1.13, we prove that the spectrum $\text{Spec}^c(\mathcal{L})$ of coprime elements in $L$ is nothing but the spectrum $\text{Spec}^c(\mathcal{L}^0)$ of second elements in the dual bounded lattice $\mathcal{L}^0 := (L, \vee, \wedge, 1, 0)$.

In Section 2, we apply the results of Section 1 to the lattice $\mathcal{L} := \text{Sub}_R(M)$ of submodules of a non-zero module $M$ over a commutative ring $R$. We present the notion of a pseudo strongly hollow submodule (PS-hollow submodule for short) $N \leq M$ as dual to the pseudo strongly irreducible submodules. Modules which are finite sums of PS-hollow submodules are said to be PS-hollow representable. Proposition 2.10 asserts the existence of minimal PS-hollow representations for PS-hollow representable modules over Artinian rings. The First and the Second Uniqueness Theorems of minimal pseudo strongly hollow representations are provided in Theorems 2.15 and 2.16, respectively. Sufficient conditions for $rM$ to have a PS-hollow representation are given in Proposition 2.22. Finally, Theorem 2.27 investigates semisimple modules each PS-hollow submodules of which is simple.

# 1 Primeness and Coprimeness Conditions for Lattices

In this section, we provide some preliminaries and study several notions of primeness and coprimeness for elements in a complete lattice $\mathcal{L} := (L, \wedge, \vee, 0, 1)$ attaining an action of a poset $(S, \leq)$.

Throughout, $S = (S, \leq)$ is a non-empty poset and $S^0 = (S, \geq)$ is the dual poset.

1.1. ([12]) A lattice $\mathcal{L}$ is a poset $(L, \leq)$ closed under two binary commutative, associative and idempotent operations: $\wedge$ (meet) and $\vee$ (join), and we write $\mathcal{L} = (L, \wedge, \vee)$; we say that $\mathcal{L}$ is a bounded lattice iff there exist $0, 1 \in L$ such that $0 \leq x \leq 1$ for all $x \in L$. We say that a lattice $(L, \wedge, \vee)$ is a complete lattice iff $\bigwedge_{x \in H} x$ and $\bigvee_{x \in H} x$ exist in $L$ for any $H \subseteq L$. Every complete lattice is bounded with $0 = \bigwedge_{x \in L} x$ and $1 = \bigvee_{x \in L} x$.

For two (complete) lattices $\mathcal{L} = (L, \wedge, \vee)$ and $\mathcal{L}' = (L', \wedge', \vee')$, a homomorphism of (complete) lattices from $\mathcal{L}$ to $\mathcal{L}'$ is a map $\varphi : L \rightarrow L'$ that preserves finite (arbitrary) meets and finite (arbitrary) joins.

The notion of a strongly hollow submodule was introduced by Abuhlail in [6], as dual to that of strongly irreducible submodules. The notion was generalized to general lattices and investigated by Abuhlail and Lomp in [3].

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1.2. Let $\mathcal{L} = (L, \wedge, \vee, 0, 1)$ be a bounded lattice.

(1) An element $x \in L \setminus \{1\}$ is said to be:

- irreducible (or uniform) iff for any $a, b \in L$ with $a \wedge b = x$, we have $a = x$ or $b = x$;
- strongly irreducible iff for any $a, b \in L$ with $a \wedge b \leq x$, we have $a \leq x$ or $b \leq x$.

(2) An element $x \in L \setminus \{0\}$ is said to be:

- hollow iff whenever for any $a, b \in L$ with $x = a \vee b$, we have $x = a$ or $x = b$;
- strongly hollow iff for any $a, b \in L$ with $x \leq a \vee b$, we have $x \leq a$ or $x \leq b$.

We denote the set of irreducible (resp. strongly irreducible, hollow, strongly hollow) elements in $L$ by $I(\mathcal{L})$ (resp. $SI(\mathcal{L}), H(\mathcal{L}), SH(\mathcal{L})$).

We say that $\mathcal{L}$ is a hollow lattice (resp. uniform lattice) iff $1$ is hollow (0 is uniform).

1.3. Let $\mathcal{L} = (L, \wedge, \vee)$ be a lattice. An $S$-action on $\mathcal{L}$ is a map $\mapsto : S \times L \rightarrow L$ satisfying the following conditions for all $s, s_1, s_2 \in S$ and $x, y \in L$:

(1) $s_1 \leq_S s_2 \Rightarrow s_1 \mapsto x \leq s_2 \mapsto x$.

(2) $x \leq y \Rightarrow s \mapsto x \leq s \mapsto y$.

(3) $s \mapsto x \leq x$.

A bounded lattice $\mathcal{L} = (L, \wedge, \vee, 0, 1)$ with an $S$-action is multiplication iff for every element $x \in L$, there is some $s \in S$ such that $x = s \mapsto 1$.

**Example 1.4.** Let $M$ be an $R$-module. The complete lattice $\text{LAT}(rM)$ of $R$-submodule has an $\text{Ideal}(R)$-action defined by the canonical product $\text{IN}$ of an ideal $I \leq R$ and a submodule $N \leq M$.

**Remark 1.5.** Let $\mathcal{L} = (L, \wedge, \vee, 0, 1)$ a bounded lattice with an $S$-action $\mapsto : S \times L \rightarrow L$. The dual lattice $\mathcal{L}^0$ has an $S^0$-action given by

$$s \mapsto^0 x = (s \mapsto 1) \lor x, \text{ for all } s \in S \text{ and } x \in L.$$  

We generalized the notion of a strongly hollow element of a lattice investigated by Abuhlail and Lomp in [3] to a strongly hollow element of a lattice with an action from a poset. Moreover, we introduced its dual notion of a pseudo strongly irreducible element which is a generalization of the notion of a strongly irreducible element.

**Definitions 1.6.** Let $(\mathcal{L}, \mapsto)$ a bounded lattice with an $S$-action. We say that:

(1) $x \in L \setminus \{1\}$ is

- pseudo strongly irreducible iff for all $y \in L$ and $s \in S$:

$$s \mapsto 1 \wedge y \leq x \Rightarrow s \mapsto 1 \leq x \text{ or } y \leq x; \quad (1)$$

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prime iff for all \( y \in L \) and \( s \in S \) with

\[
s \to y \leq x \implies s \to 1 \leq x \quad \text{or} \quad y \leq x.
\]

(2)

\[s \to 1 \leq x \quad \text{or} \quad (s \to 1) \lor x = 1\]  

(3)

\[
(2) \quad x \in L \setminus \{0\} \text{ is pseudo strongly hollow (or PS-hollow for short) iff for all } s \in S :
\]

\[
z \leq s \to x + y \implies z \leq s \to 1 \text{ or } z \leq y.
\]

(4)

second iff for all \( s \in S \):

\[
s \to x = x \quad \text{or} \quad s \to x = 0
\]

(5)

first iff for all \( y \in L \) and \( s \in S \) with

\[
s \to y = 0 \quad \text{and} \quad y \leq x \implies s \to x = 0 \quad \text{or} \quad y = 0.
\]

(6)

The spectrum of pseudo strongly irreducible (resp. prime, coprime, pseudo strongly hollow, second, first) elements of \( L \) is denoted by \( \text{Spec}^{\text{psh}}(L) \) (resp. \( \text{Spec}^p(L), \text{Spec}^c(L), \text{Spec}^s(L), \text{Spec}^f(L) \)).

**Lemma 1.7.** Let \( L = (L, \wedge, \lor, 0, 1) \) be a bounded lattice with an \( S \)-action and define

\[
s \to^* x = (s \to 1) \wedge x
\]

for all \( s \in S \) and \( x \in L \). Then \( ((L, \to^0)0 = (L, \to^*) \).  

**Proof.** It is clear that \( \to^* \) is an \( S \)-action on \( L \). For all \( s \in S \) and all \( x \in L \) we have

\[
s(\to^0)x = (s \to^0 1^0) \lor^0 x = ((s \to 1) \lor^0 0) \land x = (s \to 1) \land x = s \to^* x.
\]

\[\Box\]

**Remarks 1.8.** Let \( (L, \to) = (L, \wedge, \lor, 0, 1) \) a bounded lattice with an \( S \)-action.

(1) 0 is prime if and only if 1 is first.

(2) \( \text{SH}(L) \subseteq \text{Spec}^p(L^0). \)
(3) If \((\mathcal{L}, \rightarrow)\) is multiplication, then

\[
\text{Spec}^\text{op}(\mathcal{L}) = SH(\mathcal{L}) = \text{Spec}^p(\mathcal{L}^0)
\]

Assume that \((\mathcal{L}, \rightarrow)\) is multiplication. The first equality follow from the definitions.

Let \(x \in \text{Spec}^p(\mathcal{L}^0)\). Suppose that \(x \leq y \lor z\) for some \(y, z \in L\). Since \((\mathcal{L}, \rightarrow)\) is multiplication, \(y = s \rightarrow 1\) for some \(s \in S\), and so \(x \leq (s \rightarrow 1) \lor z\), i.e. \(s \rightarrow 0 z \leq 0 x\). Since \(x \in \text{Spec}^p(\mathcal{L}^0)\), we have \(s \rightarrow 0 1^0 \leq 0 x\) or \(z \leq 0 x\) and so \(x \leq s \rightarrow 1 = y\) or \(x \leq z\). So, \(\text{Spec}^p(\mathcal{L}^0) \subseteq SH(\mathcal{L})\). The inverse inclusion follows by (2).

(4) \(x \in L \setminus \{1\}\) is prime in \((\mathcal{L}, \rightarrow^*)\) if and only if \(x\) is pseudo strongly irreducible in \((\mathcal{L}, \rightarrow)\).

(5) \(x \in L \setminus \{1\}\) is coprime in \((\mathcal{L}, \rightarrow)\) if and only if \(x\) is coprime in \((\mathcal{L}, \rightarrow^*)\).

(6) \(x \in L \setminus \{0\}\) is first if and only if 0 is prime in \([0,x]\).

\((\Rightarrow)\) Let \(x \in L \setminus \{0\}\) be first. Observe that the maximum element in the sublattice \([0,x]\) is \(x\).

Suppose that \(s \rightarrow y = 0\) for some \(y \leq x\). Since \(x\) is first, \(y = 0\) or \(s \rightarrow x = 0\). So 0 is prime in \([0,x]\).

\((\Leftarrow)\) Let 0 be prime in \([0,x]\). Suppose that \(s \rightarrow y = 0\) for some \(y \leq x\). Since \(y \in [0,x]\), we have \(y = 0\) or \(s \rightarrow x = 0\) as \(x\) is the maximum element of \([0,x]\).

(7) \(x \in L \setminus \{0\}\) is second if and only if 0 is coprime in the interval \([0,x]\).

The notion of top-lattices was introduced by Abuhlail and Lomp [2]:

1.9. Let \((\mathcal{L}, \rightarrow) = (L, \land, \lor, 0, 1)\) a complete lattice and \(X \subseteq L \setminus \{1\}\). For \(a \in L\), we define the variety of \(a\) as \(V(a) := \{p \in X \mid a \leq p\}\) and set \(V(\mathcal{L}) := \{V(a) \mid a \in L\}\). Indeed, \(V(\mathcal{L})\) is closed under arbitrary intersections (in fact, \(\bigcap_{a \in A} V(a) = V(\bigvee_{a \in A}(a))\) for any \(A \subseteq L\)). The lattice \(\mathcal{L}\) is called \(X\)-top (or a topological lattice iff \(V(\mathcal{L})\) is closed under finite unions.

Many results in the literature for prime, coprime, second, first, and other types of spectra of submodules of a module can be generalized to a top-lattices with actions from posets. For example, we have the following generalization of [17, Theorem 3.5].

Lemma 1.10. Let \((\mathcal{L}, \rightarrow)\) be a complete lattice with an action from a poset \(S\). If \(\mathcal{L}\) is multiplication, then \(\mathcal{L}\) is \(\text{Spec}^p(\mathcal{L})\)-top.

Proof. This follows from the fact that we have \(V(s \rightarrow 1) \cup V(y) = V((s \rightarrow 1) \land y)\) for all \(s \in S\) and \(y \in L\). Indeed, by definition of prime elements and the axioms of the \(S\)-action, and noting that \(V(-)\) is an order reversing map, we have:

\[
V(s \rightarrow y) \subseteq V(s \rightarrow 1) \cup V(y) \subseteq V((s \rightarrow 1) \land y) \subseteq V(s \rightarrow y)
\]
Definition 1.11. Let \( \mathcal{L} = (L, \wedge, \vee) \) be a lattice. Let \( x, y, z \in L \), with \( x \leq y \) and \( x \leq z \). We define \( y \sim z \) iff for all \( y' \leq y \), there exists \( z' \leq z \) such that \( y' \vee x = z' \vee x \), and for all \( z' \leq z \), there exists \( y' \leq y \) such that \( y' \vee x = z' \vee x \). It is clear that \( \sim \) is an equivalence relation. Denote the equivalence class of \( y \geq x \) by \( y/x \), and define

\[
L/x := \{ y/x \mid y \in L \text{ and } x \leq y \}.
\]

Define \( y/x \leq_L z/x \) iff for all \( y' \leq y \), there exists \( z' \leq z \) such that \( y' \vee x = z' \vee x \). Then \( \mathcal{L}/x = (L/x, \wedge^q, \vee^q) \) is a lattice, called the quotient lattice, where the meet \( \wedge^q \) and the join \( \vee^q \) on \( L/x \) are defined by:

\[
y/x \wedge^q z/x := (y \wedge z)/x \text{ and } y/x \vee^q z/x := (y \vee z)/x.
\]

If \( \mathcal{L} = (L, \wedge, \vee, 0, 1) \) is a complete lattice, then \( \mathcal{L}/x = (L/x, \wedge^q, \vee^q) \) is a complete lattice, where

\[
\bigwedge_{i \in A}^q (x_i/x) = (\bigwedge_{i \in A} x_i)/x \text{ and } \bigvee_{i \in A}^q (x_i/x) = (\bigvee_{i \in A} x_i)/x.
\]

Remark 1.12. Let \( \mathcal{L}/x \) a lattice with an \( S \)-action. Define for all \( s \in S \) and \( y/x \in \mathcal{L}/x \):

\[
s \rightarrow^q y/x = (s \rightarrow y) \vee x
\]

Then \( \mathcal{L}/x \) is a lattice with an \( S \)-action.

Theorem 1.13. Let \( \mathcal{L}/x = (L, \wedge, \vee, 0, 1) \) a complete lattice with an \( S \)-action.

1. \( \text{Spec}^c(\mathcal{L}) = \text{Spec}^c(\mathcal{L}^0) \).
2. \( \text{Spec}^c(\mathcal{L}^0) = \text{Spec}^c(\mathcal{L}^+) \).
3. If \( x \in L \setminus \{1\} \) is prime, then

\[
\text{Spec}^f(\mathcal{L}/x) = (\mathcal{L}/x) \setminus \{x/x\}.
\]

4. Assume that the following additional condition is satisfied for our action:

\[
s \rightarrow (y \vee z) = s \rightarrow y \vee s \rightarrow z \text{ for all } s \in S \text{ and } y, z \in L
\]

Then \( x \in L \setminus \{1\} \) is prime \( \iff \text{Spec}^f(\mathcal{L}/x) = (\mathcal{L}/x) \setminus \{x/x\} \).

Proof. (1) \( p \in \text{Spec}^c(\mathcal{L}) \iff s \rightarrow 1 \leq p \text{ or } (s \rightarrow 1) \vee p = 1 \text{ for all } s \in S \)

\[
\iff s \rightarrow 1 \vee p = p \text{ or } s \rightarrow^0 p = 0^0 \text{ for all } s \in S
\]

\[
\iff p \in \text{Spec}^c(\mathcal{L}^0).
\]

(2) \( p \in \text{Spec}^c(\mathcal{L}^0) \iff s \rightarrow^0 1^0 \leq p \text{ or } (s \rightarrow^0 1^0) \vee^0 p = 1^0 \)

\[
\iff (s \rightarrow 1) \vee 0 \geq p \text{ or } ((s \rightarrow 1) \vee 0) \wedge p = 0 \text{ for all } s \in S
\]

\[
\iff (s \rightarrow 1) \wedge p = p \text{ or } (s \rightarrow 1) \wedge p = 0 \text{ for all } s \in S
\]

\[
\iff s \rightarrow^*_p = p \text{ or } s \rightarrow^*_p = 0 \text{ for all } s \in S
\]

\[
\iff p \in \text{Spec}^c(\mathcal{L}^+).
\]
(3) Let \( x \in L \setminus \{1\} \) be prime. **Claim:** \( y/x \in \mathcal{L}/x \) is first.

Let \( s \rightarrow^q z/x = x/x \) and \( z/x \leq^q y/x \) and suppose that \( z/x \nleq x/x \). Then \( ((s \rightarrow z) \vee x)/x = x/x \). It follows that \((s \rightarrow z) \vee x = x\), and hence \((s \rightarrow z) \leq x\). Since \( x \) is prime, \((s \rightarrow 1) \leq x\) or \( z \leq x\). But \( z \leq x \) implies that \( z = x \), and so \( z/x \leq x/x \). Therefore, \((s \rightarrow 1) \leq x\), and so \((s \rightarrow 1) \vee x = x\). Hence \( s \rightarrow^q 1/x = x/x \). Therefore, \( s \rightarrow^q y/x = x/x \).

(4) Assume that the additional condition (11) is satisfied and that \( \text{Spec}^\ell (\mathcal{L}/x) = (\mathcal{L}/x) \setminus \{x/x\} \).

**Claim:** \( x \) is prime in \( \mathcal{L} \).

Suppose that \( s \dashv y \leq x \) and \( y \nleq x \). Since \( s \dashv y \leq x \), we have \((s \dashv y) \vee x = x\). It follows by (11) that \( s \dashv (y \vee x) = s \dashv y \vee s \rightarrow x\). Since \( s \rightarrow x \leq x \), we have

\[
s \rightarrow (y \vee x) = s \rightarrow y \vee s \rightarrow x \leq (s \rightarrow y) \vee x = x.
\]

Therefore \( (s \rightarrow (y \vee x)) \vee x = x/x \), whence \( s \rightarrow^q (y \vee x)/x = x/x \). But \( 1/x \) is first in \( \mathcal{L}/x \), whence \( (y \vee x)/x = x/x \) or \( s \rightarrow^q 1/x = x/x \). Notice that \( (y \vee x)/x = x/x \) cannot happen as \( y \nleq x \). Thus \( s \rightarrow^q 1/x = x/x \). Whence \( s \rightarrow 1 \vee x = x \), i.e. \( s \rightarrow 1 \leq x \).

**Remark 1.14.** Let \((\mathcal{L}, \rightarrow) = (L, \wedge, \vee, 0, 1)\) a complete lattice with an \( S \)-action. Since \( \text{Spec}^\varepsilon (\mathcal{L}) = \text{Spec}^\varepsilon (\mathcal{L}^0) \) by 1.13 (2), the result on the second spectrum can be dualized to the coprime spectrum.

### 2 PS-Hollow Representation

Throughout this Section, \( R \) is a commutative ring with unity and \( M \) is a non-zero \( R \)-module.

We consider the poset \( \mathcal{I} = (\text{Ideal}(R), \subseteq) \) of ideals of \( R \) acting on the lattice \( \mathcal{L} = \text{Sub}_R(M) \) of \( R \)-submodules of \( M \) in the canonical way. We say that a proper \( R \)-module of \( M \) is irreducible (resp. strongly irreducible, pseudo strongly irreducible, prime, coprime) if it is so as an element of \( \text{Sub}_R(M) \). On the other hand, we say that a non-zero \( R \)-submodule of \( M \) is hollow (resp. strongly hollow, pseudo strongly hollow, second, first) iff it is so as an element of \( \text{Sub}_R(M) \). For such notions for modules one might consult [4], [5], [6], [19], [18],[20]).

In [1], we introduced and investigated modules attaining second representations, i.e. modules which are finite sums of second submodules (see [8], [11]). Since every second submodule is secondary, modules with secondary representations can be considered as generalizations of such modules. Secondary modules can be considered, in some sense, as dual to those of primary submodules.

In this section, we consider modules with pseudo strongly hollow representations, i.e. which are finite sums of pseudo strongly hollow submodules. Assuming suitable conditions, we prove existence and uniqueness theorems for modules with such representations (called PS-hollow representable modules). This work is inspired by the theory of primary and secondary decompositions of modules over commutative rings (e.g. Ann2002).

2.1. A proper \( R \)-submodule \( N \leq M \) is called primary [7] iff whenever \( rx \in N \) for some \( r \in R \) and \( x \in M \), either \( x \in N \) or \( r^nM \subseteq N \) for some \( n \in \mathbb{N} \). We say that \( M_R \) has a primary decomposition [7] iff there are primary submodules \( N_1, \cdots, N_n \) of \( M \) with \( M = \bigcap_{i=1}^n N_i \).
Dually, an $R$-submodule $N \leq M$ is said to be a secondary submodule ([14], [16]) iff for any $r \in R$ we have $rN = N$ or $r^2N = 0$ for some $n \in \mathbb{N}$. An $R$-module $M$ has a secondary representation iff $M = \sum_{i=1}^{n} N_i$, where $N_1, \ldots, N_n$ are secondary $R$-submodules of $M$.

The notion of a primary submodule can be dualized in different ways. Instead of considering such notions, we consider the exact dual of a pseudo strongly irreducible submodule (defined in 1). Recall that, the pseudo strongly irreducible elements in $(\text{Sub}_R(M), \rightarrow)$ are exactly the prime elements in $(\text{Sub}_R(M), \rightarrow^*)$ (defined in 4).

Strongly irreducible submodules (ideal) have been considered by several authors (e.g. [15], [9], [10]). The dual notion of a strongly hollow submodule was investigated by Abuhlail and Lomp in [3]. In this section we consider the more general notion of a pseudo strongly hollow submodule. For the convenience of the reader, we restate the definition in the special case of the lattice $\text{Sub}_R(M)$.

**Definition 2.2.** We say that an $R$-submodule $N \leq M$ is pseudo strongly hollow submodule (or PS-hollow for short) iff for any ideal $I \leq R$ and any $R$-submodule $L \leq M$, we have

$$N \subseteq IM + L \Rightarrow N \subseteq IM \text{ or } N \subseteq L.$$  

(12)

We say that $_RM$ is a pseudo strongly hollow module (or PS-hollow for short) iff $M$ is a PS-hollow submodule of itself, that is, $M$ is PS-hollow iff for any ideal $I \leq R$ and any $R$-submodule $L \leq M$, we have

$$M = IM + L \Rightarrow M = IM \text{ or } M = L.$$  

(13)

**Example 2.3.** Let $_RM$ be second. Every $R$-submodule $N \leq M$ is a PS-hollow submodule of $M$. Indeed, suppose that $N \subseteq IM + L$ for some $L \leq M$ and $I \leq R$. Since $_RM$ is second, either $IM = 0$ whence $N \subseteq L$, or $IM = M$ whence $N \subseteq IM$. In particular, every second module is a PS-hollow module.

**Remark 2.4.** It is clear that any strongly hollow submodule of $M$ is PS-hollow; the converse holds in case $_RM$ is multiplication.

**Example 2.5.**

(1) There exists an $R$-module $M$ which is not multiplication but all of its PS-hollow submodules are strongly hollow. Consider the Prüfer group $M = \mathbb{Z}(p^\infty)$ as a $\mathbb{Z}$-module. Notice that $\mathbb{Z}M$ is not a multiplication module, however every $\mathbb{Z}$-submodule of $M$ is strongly hollow).

(2) A PS-hollow submodule $N \leq M$ need not be hollow. Consider $M = \mathbb{Z}[x]$ as a $\mathbb{Z}$-module. Set $N := x\mathbb{Z}[x]$, and $L := (x+1)\mathbb{Z}[x]$. Then $N, L \leq M$ and $M = L + N$ is PS-hollow which is not hollow. Indeed, $x^i = x^{i-1}(x+1) - x^{i-2}(x)$ for all $i \geq 2$ and $1 = (x+1) - x$. On the other hand, $IM = M$ or $IM = 0$ for every $I \leq \mathbb{Z}$.

**Lemma 2.6.** Let $N \leq M$ be a PS-hollow submodule. If $I$ is minimal in $A := \{I \leq R \mid N \subseteq IM\}$, then $I$ is a hollow ideal of $R$.  

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Proof. Let \( I = J + K \) for some ideals \( J, K \leq R \). Notice that \( N \subseteq IM = (J + K)M = JM + KM \), whence \( N \subseteq JM \) or \( N \subseteq KM \), i.e. \( J \in A \) or \( K \in A \). By the minimality of \( I \), it follows that \( J = I \) or \( K = I \). Therefore, \( I \) is hollow. \( \blacksquare \)

2.7. Let \( N \leq M \) be a PS-hollow submodule and set

\[
A_N := \{ I \leq R \mid N \subseteq IM \}, \quad H_N := \text{Min}(A) \quad \text{and} \quad \text{In}(N) := \bigcap_{I \in H_N} IM.
\]

Notice that \( A_N \) is non-empty as \( R \in A \), while \( H_N \) might be empty and in this case \( \text{In}(N) := M \) (however \( H_N \neq \emptyset \) if \( R \) is Artinian). When \( N \) is clear from the context, we drop it from the index of the above notations. We say that \( N \) is an \( H \)-\( \text{PS-hollow submodule} \) of \( M \). Every element in \( H \) is called an \( \text{associated hollow ideal of } M \). We write \( \text{Ass}^h(M) \) to denote the set of all associated hollow ideals of \( M \).

**Proposition 2.8.** Let \( R \) be an Artinian ring, \( N \) and \( L \) be incomparable PS-hollow submodules of \( M \) and \( H \subseteq \text{Ass}^h(M) \). Then \( N + L \) is \( H \)-PS-hollow if and only if \( N \) and \( L \) are \( H \)-PS-hollow.

**Proof.** \( (\Leftarrow) \) Let \( N \leq M \) and \( L \leq M \) be \( H \)-PS-hollow submodules.

**Claim 1:** \( H_{N+L} = H_{N} = H \).

Consider \( I \in H_N = H_L \). Clearly, \( I \in A_{N+L} \). If \( I \notin H_{N+L} := \text{Min}(A_{N+L}) \), then there is \( I' \subset I \) such that \( N \subseteq N + L \subseteq I'M \) which contradicts the minimality of \( I \) in \( A_N \).

Conversely, let \( I \in H_{N+L} \). Clearly, \( I \in A_N \cap A_L \). If \( I \notin H_N \), then there is \( I' \in H_N = H_L \) with \( I' \subset I \) and therefore \( N + L \subseteq I'M \), whence \( I = I' \) since \( I' \in A_{N+L} \). Therefore, \( H_{N+L} = H_N = H \).

**Claim 2:** \( N + L \) is \( H \)-PS-hollow.

Suppose that \( N + L \subseteq JM + K \) for some ideal \( J \leq R \) and some submodule \( K \leq M \). Then \( N \subseteq N + L \subseteq JM + K \) and so \( N \subseteq JM \) or \( N \subseteq K \). Similarly \( L \subseteq N + L \subseteq JM + K \) and so \( L \subseteq JM \) or \( L \subseteq K \). Suppose that \( N \subseteq JM \), whence there is \( I \in H \) such that \( N \subseteq IM \) and \( I \subseteq J \) (as \( R \) is Artinian) and so \( L \subseteq IM \subseteq JM \). Therefore, either \( N + L \subseteq JM \) or \( N + L \subseteq K \). Hence \( N + L \) is PS-hollow.

\( (\Rightarrow) \) Assume that \( N + L \) is \( H \)-PS-hollow. It is clear that \( H_{N+L} \subseteq H_L \). Assume that \( L \subseteq IM \). Then \( N + L \subseteq IM + L \) and \( N + L \nsubseteq L \) as \( N \) and \( L \) are incomparable, whence \( N + L \subseteq IM \) and so \( H_L \subseteq H_{N+L} \). Therefore, \( H_L = H_{N+L} \). Similarly, \( H_N = H_{N+L} \). \( \blacksquare \)

2.9. We say that a module \( M \) is **PS-hollow representable** iff \( M \) can be written as a *finite* sum of PS-hollow submodules. A module \( M \) is called **directly PS-hollow representable** (or DPS-hollow representable, for short) iff \( M \) is a *finite direct* sum of PS-hollow submodules. A module \( M \) is called **semi-pseudo strongly hollow representable** (or **SPS-hollow representable**, for short) iff \( M \) is a sum of PS-hollow submodules. We call \( M = \sum_{i=1}^{n} N_i \), where each \( N_i \) is \( H_i \)-PS-hollow, a **minimal PS-hollow representation** for \( M \) iff the following conditions are satisfied:

(1) \( H_1, H_2, \ldots, H_n \) are distinct.

(2) \( N_j \nsubseteq \sum_{i=1, i \neq j}^{n} N_i \) for all \( j \in \{1, \ldots, n\} \).
If such a minimal PS-hollow representation for \( M \) exists, then we call each \( N_i \) a main PS-hollow submodule of \( M \) and the elements of \( H_1, H_2, \ldots, H_n \) are called main associated hollow ideals of \( M \); the set of the main associated hollow ideals of \( M \) is denoted by \( \text{ass}^H(M) \).

**Proposition 2.10.** (Existence of minimal PS-hollow representation) If \( R \) is an Artinian ring, then every PS-hollow representable \( R \)-module has a minimal PS-hollow representation.

**Proof.** Let \( M = \sum_{i \in A} K_i \), where \( A \) is finite and \( K_i \) is an \( H_i \)-PS-hollow submodule \( \forall i \in A \).

**Step 1:** Remove the redundant submodules \( K_j \subseteq \sum_{i \neq j} K_i \). This is possible by the finiteness of \( A \).

**Step 2:** Gather all submodules \( K_m \) that share the same \( H \) to construct an \( H \)-PS-hollow \( N \leq M \) as a sum of such \( H \)-PS-hollow submodules (this is possible by Proposition 2.8).

**Remark 2.11.** Let \( R \) be Artinian and \( N \leq M \) be an \( H \)-PS-hollow submodule. If \( \text{In}(N) \) is PS-hollow, then \( \text{In}(N) \) is \( H \)-PS-hollow. To show this, observe that for any ideal \( I \leq R \), we have \( N \subseteq IM \) if and only if there exists \( I' \in H \) such that \( N \subseteq IM \) with \( I' \subseteq I \) (as \( R \) is Artinian), whence \( \text{In}(N) \subseteq IM \) if and only if \( N \subseteq IM \).

**Lemma 2.12.** Let \( R \) be Artinian, \( N \leq M \) be an \( H \)-PS-hollow submodule and \( \text{In}(N) \leq L \) whenever \( N \leq L \leq M \). Then \( \text{In}(N) \) is \( H \)-PS-hollow.

**Proof.** Let \( K = \text{In}(N) := \bigcap_{I \in H} IM \). Suppose that \( K \subseteq JM + L \) for some \( J \leq R \) and \( L \leq M \). If \( K \not\subseteq JM \), then \( N \not\subseteq JM \) and so \( N \subseteq L \), whence \( K \subseteq L \). Therefore \( K \) is PS-hollow. Thus, by the Remark 2.11, \( \text{In}(N) \) is \( H \)-PS-hollow.

**Example 2.13.** If \( R \) is Artinian, then every multiplication \( R \)-module \( M \) satisfies the conditions of Lemma 2.12 and so \( \text{In}(N) \) is \( H \)-PS-hollow for every \( H \)-PS-hollow \( N \leq M \) (in fact, \( \text{In}(N) = N \) in this case).

**Remark 2.14.** Let \( R \) be Artinian and \( M \) a multiplication \( R \)-module. It is easy to see that there is a unique minimal PS-hollow representation of \( M \) up to the order, i.e. if \( \sum_{i=1}^{n} N_i = M = \sum_{j=1}^{m} K_j \) are two minimal PS-representations such that each \( N_i \) is \( H_i \)-PS-hollow and each \( K_j \) is \( H'_j \)-PS-hollow, then \( n = m \) and \( \{N_1, \ldots, N_n\} = \{K_1, \ldots, K_n\} \).

**Theorem 2.15.** (First uniqueness theorem of PS-hollow representation) Let \( R \) be Artinian and \( \sum_{i=1}^{n} N_i = M = \sum_{j=1}^{m} K_j \) be two minimal PS-representations for \( R \)-module \( M \) such that \( N_i \) is \( H_i \)-PS-hollow for each \( i \in \{1, \ldots, n\} \) and \( K_j \) is \( H'_j \)-PS-hollow for each \( j \in \{1, \ldots, m\} \). Then \( n = m \), \( \{H_1, \ldots, H_n\} = \{H'_1, \ldots, H'_m\} \) and \( \text{In}(N_i) = \text{In}(K_j) \) whenever \( H_i = H'_j \).

**Proof.** Set \( N'_i = \text{In}(N_i) \) and \( K'_j = \text{In}(K_j) \) for \( i \in \{1, \ldots, n\} \) and \( j \in \{1, \ldots, m\} \).

**Claim:** For any \( i \in \{1, \ldots, n\} \), there is \( j \in \{1, \ldots, m\} \) such that \( N'_i = K'_j \).
Step 1: Suppose that there exists some \( i \in \{1, \ldots, n\} \) for which \( N_i \nsubseteq K'_j \) for all \( j \in \{1, \ldots, m\} \).

Then for any \( j \in \{1, \ldots, m\} \), there is \( J'_j \in H'_j \) such that \( N_i \nsubseteq J'_j M \). But \( N_i \subseteq M = \sum_{j=1}^{n} K_j \subseteq \sum_{j=1}^{m} J'_j M \), whence \( N_i \subseteq J'_j M \) for some \( j \) (a contradiction). So, \( N_i \subseteq K'_j \) for some \( j \in \{1, \ldots, m\} \).

Step 2: We show that \( N'_i \subseteq K'_j \).

Since \( N_i \subseteq K'_j \), we have \( N_i \subseteq IM \) for all \( I \in H'_j \). Since \( R \) is Artinian, there is a minimal ideal \( J_I \subseteq I \) such that \( N_i \subseteq J_I M \) and so

\[
N'_i = In(N_i) = \bigcap_{I \in H_i} IM \subseteq \bigcap_{I \in H'_j} J_I M \subseteq K'_j.
\]

Similarly, for any \( j \in \{1, \ldots, m\} \), there is some \( i \in \{1, \ldots, n\} \) such that \( K'_j \subseteq N'_i \). Therefore, for any \( i \in \{1, \ldots, n\} \), there is some \( j \in \{1, \ldots, m\} \) such that \( N'_i = K'_j \) as \( N'_1, N'_2, \ldots, N'_n \) are incomparable.

Claim: \( H_i = H'_j \) whenever \( N'_i = K'_j \).

Let \( N'_i = K'_j \). Pick any \( I \in H_i \). Then \( N_i \subseteq IM \), whence \( K'_j = N'_i \subseteq IM \). Since \( R \) is Artinian, there is a minimal ideal \( I' \in H'_j \) such that \( I' \subseteq I \), and therefore \( I' = I \) as \( I \) is minimal with respect to \( N_i \subseteq IM \). Hence \( H_i \subseteq H'_j \). One can prove similarly that \( H'_j \subseteq H_i \). So, \( H_i = H'_j \).

Theorem 2.16. (Second uniqueness theorem of PS-hollow representation) Let \( R \) be Artinian, \( M \) be an \( R \)-module with two minimal \( PS \)-hollow representations \( \sum_{i=1}^{n} N_i = M = \sum_{j=1}^{m} K_j \) with \( N_i \) is \( H_i \)-PS-hollow for each \( i \in \{1, \ldots, n\} \) and \( K_j \) is \( H_j \)-PS-hollow for each \( j \in \{1, \ldots, n\} \). If \( H_m \) is minimal in \( \{H_1, H_2, \ldots, H_n\} \), then either \( N_m = K_m \) or \( In(N_m) \) is not \( PS \)-hollow.

Proof. Let \( H_m \) be minimal in \( \{H_1, H_2, \ldots, H_n\} \) such that \( In(N_m) \) is \( PS \)-hollow. For any \( j \neq m \), there is \( I_j \in H_j \setminus H_m \). But \( \sum_{j \neq m} I_j M + N_m = M \) and so \( In(N_m) \subseteq \sum_{j \neq m} I_j M + N_m \). Since \( I_j \in H_j \setminus H_m \), it follows that \( In(N_m) \nsubseteq I_j M \) for all \( j \in \{1, \ldots, n\} \setminus \{m\} \) and so \( In(N_m) \subseteq N_m \), whence \( In(N_m) = N_m \). One can prove similarly that \( In(K_m) = K_m \). It follows that

\[
N_m = In(N_m) \xRightarrow{Theorem 2.15} In(K_m) = K_m.
\]

Corollary 2.17. Let \( R \) be Artinian and \( \sum_{i=1}^{n} N_i = M = \sum_{i=1}^{n} K_i \) be two minimal \( PS \)-hollow representations of \( \mathcal{R} M \) such that \( N_i \) is \( H_i \)-PS-hollow for \( i \in \{1, \ldots, n\} \) and \( K_i \) is \( H_i \)-PS-hollow for \( i \in \{1, \ldots, n\} \). If \( In(N) \) is \( PS \)-hollow whenever \( N \) is a main \( PS \)-hollow submodule of \( M \), then \( N_i = K_i \) for all \( i \in \{1, \ldots, n\} \).

Proof. Apply Theorem 2.16 and observe that \( H_i \) is minimal in \( \{H_1, H_2, \ldots, H_n\} \) for each \( i \in \{1, \ldots, n\} \) as \( In(N_i) \) is \( PS \)-hollow: otherwise, \( H_j \subsetneq H_i \) for some \( i \neq j \) and \( In(N_j) \) can replace \( N_i + N_j \) whence \( \sum_{i=1}^{n} N_i \) is not minimal (a contradiction).
2.18. We say that an $R$-module $M$ is pseudo distributive iff for all $L, N \leq M$ and every $I \leq R$ we have
\[
L \cap (IM + N) = (L \cap IM) + (L \cap N).
\]
(14)
Every distributive $R$-module is indeed pseudo distributive. The two notions coincide for multiplication modules.

Example 2.19. A pseudo distributive module need not be distributive. Consider $M := \mathbb{Z}_2[x]$ as a $\mathbb{Z}$-module. Let $N := xM, L := (x+1)M$ and $K = \mathbb{Z}_2$. Then $N, L, K \leq M$ are $R$-submodules and
\[
(K \cap L) + (K \cap N) = 0 \neq K = K \cap (L + N).
\]
Notice that $M$ is pseudo distributive as $IM = 0$ or $IM = M$ for every $I \leq R$.

Remark 2.20. Assume that $M$ is a (directly) hollow representable module for which every maximal hollow is PS-hollow. Then $M$ is (directly) PS-hollow representable.

In [1], we introduced the notion of $s$-lifting modules:

2.21. Recall that an $M$ is a lifting $R$-module iff any $R$-submodule $N \leq M$ contains a direct summand $X \leq M$ such that $N/X$ is small in $M/X$ (e.g. [13, 22.2]). We call $RM$ s-lifting iff $RM$ is lifting and every maximal hollow submodule of $M$ is second.

Proposition 2.22. (1) If $RM$ is pseudo distributive, then every hollow submodule of $M$ is PS-hollow.

(2) If $RM$ is s-lifting, then every maximal hollow submodule of $M$ is PS-hollow.

Proof. (1) Let $M$ is pseudo distributive. Let $N \leq M$ be hollow. Suppose that $N \subseteq IM + L$, whence $N = (IM + L) \cap N = (IM \cap N) + (L \cap N)$ as $M$ is pseudo distributive. Since $N$ is hollow, $N = IM \cap N$ or $N = L \cap N$, therefore $N \subseteq IM$ or $N \subseteq L$. So, $N$ is PS-hollow.

(2) Let $RM$ be s-lifting. Suppose that $K \leq M$ is a maximal hollow submodule of $M$ and that $K \leq IM + L$. Since $M$ is s-lifting, there exists $K' \subseteq K$ and $N \leq M$ such that $K' \oplus N = M$ and $K/K'$ is small in $M/K'$.

Case 1: $K' = 0$: i.e. $M = N$. Since $K$ is second, we have $K = IK \subseteq IN = IM$.

Case 2: $K' \neq 0$: We claim that $K = K'$. To prove this, let $x \in K$. Then there are $y \in K'$ and $z \in N$ such that $x = y + z$. But $y \in K$, whence $z \in K$. Therefore, $K \subseteq K' \oplus (K \cap N)$, but $K$ hollow implies that $K = K'$ or $K = K \cap N$. But $K' \neq 0$, whence $K = K'$; otherwise, $K' \cap N \neq 0$. Therefore, $M = K \oplus N$. Now, it is easy to show that
\[
IM + L \leq (IM \cap K + L \cap K) \oplus (IM \cap N + L \cap N),
\]
and so
\[
K \leq (IM \cap K + L \cap K) \oplus (IM \cap N + L \cap N),
\]
whence $K \leq IM \cap K + L \cap K$. Since $IM \cap K + L \cap K \leq K$, it follows that $K = IM \cap K + L \cap K$ and so $K = IM \cap K$ or $K = L \cap K$ which implies that $K \leq IM$ or $K \leq L$.\[\]
Examples 2.23. (1) Every (directly) hollow representable pseudo distributive module is (directly) PS-hollow representable.

(2) Every s-lifting module with finite hollow dimension is directly PS-hollow representable.

(3) The $\mathbb{Z}$-module $M = \mathbb{Z}_n$ is PS-hollow representable. To see this, consider the prime factorization $n = p_1^{m_1} \cdots p_k^{m_k}$, and set $n_i = \frac{n}{p_i^{m_i}}$ for $i \in \{1, \cdots, k\}$. Then $M = \sum_{i=1}^{k} (n_i)$ is a minimal PS-hollow representation for $M$, and $(n_i)$ is $H_i$-PS-hollow where $H_i = \{(n_i)\}$ for $i \in \{1, \cdots, k\}$.

(4) The $\mathbb{Z}$-module $M = \mathbb{Z}_{12}$ is PS-hollow representable ($M = 4\mathbb{Z}_{12} + 3\mathbb{Z}_{12}$), but $M$ is not second representable. Observe that $M$ is not semisimple and is even not s-lifting as $3\mathbb{Z}_{12} \leq \mathbb{Z}_{12}$ is a maximal hollow $\mathbb{Z}$-subsemimodule but not second.

(5) Any Noetherian semisimple $R$-module is directly PS-hollow representable.

(6) Any Artinian semisimple $R$-module is directly PS-hollow representable.

Lemma 2.24. Let $N \leq M$ be an $H$-PS-hollow submodule such that every non-small submodule $K$ of $M$ is of the form $JM$ for some ideal $J \leq R$. Every non-small submodule $K \leq N$ is $H$-PS-hollow submodule; Moreover, for any ideal $I \leq R$, we have: $K \subseteq IM$ if and only if $N \subseteq IM$.

Proof. Let $N \leq M$ be an $H$-PS-hollow submodule and $K \leq N$ be a non-small submodule. Suppose that $K \subseteq IM + L$ and $K \not\subseteq L$. Notice that $N \not\subseteq L$. Since $K$ is not small in $N$, there is a proper submodule $K'$ of $N$ such that $N = K + K' \subseteq IM + L + K'$. If $N \subseteq L + K'$, then $K' = JM$ for some $J \leq R$ (notice that $K'$ not small in $N$) and therefore $N \subseteq K'$ (a contradiction). Hence, $N \subseteq IM$ and so $K \subseteq IM$, whence $K$ is PS-hollow.

Claim: $A_H = A_K$.

Assume that $K \subseteq IM$ for some $I \leq R$. Then $N = K + K' \subseteq IM + K'$. Since $N$ is PS-hollow and $K' \neq N$, we have $N \subseteq IM$.

Example 2.25. Consider $M = \mathbb{Z}_{12}$ as a $\mathbb{Z}$-module. Then $K_1 = 3\mathbb{Z}_{12}$ and $K_2 = 4\mathbb{Z}_{12}$ satisfy the assumptions of Lemma 2.24. Notice that $\mathbb{Z}M$ is not semisimple.

2.26. A module $_RM$ is called comultiplication [5] iff for every submodule $K \leq M$, we have $K = (0 :_M (0 :_R K))$.

Theorem 2.27. Let $_RM$ be semisimple, $B$ the set of all maximal second submodules of $M$, and assume that $\text{Ann}(M) \neq \bigcap_{K \in B \setminus \{N\}} \text{Ann}(K)$ for any $N \in B$. The following conditions are equivalent:

(1) $_RM$ is multiplication.

(2) Every PS-hollow submodule of $M$ is simple.

(3) Every second submodule of $M$ is simple.
(4) $RM$ is comultiplication.

**Proof.** Let $M = \bigoplus_{S \in A} S$, where $S$ is a simple submodule of $M$ for all $S \in A$.

(1) $\Rightarrow$ (2): Assume that $RM$ is multiplication. Suppose that there is an $H$-PS-hollow submodule $N \leq M$, which is not simple. Then $N$ contains properly a simple submodule $S' \in A$. Since $S'$ is not small in $N$, Lemma 2.24 implies that $S'$ is $H$-PS-hollow. But there is another simple submodule $S''$ of $N$ (as $N$ is not simple). Let $I := Ann(S'')$. It follows that $S' \subseteq IM$ while $N \nsubseteq JM$ (which contradicts Lemma 2.24).

(2) $\Rightarrow$ (3): Assume that every PS-hollow submodule of $M$ is simple.

**Claim:** Every second submodule of $M$ is PS-hollow, whence simple.

Let $N = \bigoplus_{i \in A} S_i$ be a second submodule of $M$ and suppose that $N \subseteq IM + L$ for some ideal $I$ of $R$ and some $R$-submodule $N$ of $M$.

**Case 1:** $I \subseteq Ann(N)$. In this case, $N \cap IM = 0$, and it follows that $N \subseteq L$.

**Case 2:** $I \nsubseteq Ann(N)$. Since $N$ is second, $N = IN \subseteq IM$.

(3) $\Rightarrow$ (1): Assume that every second submodule of $M$ is simple. Consider a submodule $K = \bigoplus_{S \in C \subseteq A} S$ of $M$ and set $I := \bigcap_{S \in A \setminus C} Ann(S)$. Notice that $K = IM$, otherwise, $I \subseteq Ann(S)$ for some $S \in C$ whence $Ann(M) = \bigcap_{S \in A \setminus C} Ann(S)$ (a contradiction). Since $K$ is an arbitrary submodule of $M$, we conclude that $MR$ is multiplication.

(4) $\Rightarrow$ (3): Let $RM$ be comultiplication. Let $K \leq M$ be second. For any simple $S \leq K$ we have $K = (0 :_M (0 :_R K)) = (0 :_M (0 :_R S)) = S$, (15)
i.e. $RK$ is simple. ■

**Example 2.28.** Consider the $\mathbb{Z}$-module $M = \prod_{i=1}^{\infty} \mathbb{Z}_{p_ip'_i}$, where $p_i$ and $p'_i$ are primes and $p_i \neq p_j$, $p'_i \neq p'_j$ for all $i \neq j \in \mathbb{N}$ and $p'_i \neq p_j$ for any $i$ and $j$. Let the simple $\mathbb{Z}$-modules $K_{p_i}$ and $K_{p'_i}$ be such that $(0 : K_{p_i}) = (p_i)$ and $(0 : K_{p'_i}) = (p'_i)$, so

$$M = \bigoplus_{i=1}^{\infty} K_{p_i} \oplus \bigoplus_{i=1}^{\infty} K_{p'_i}.$$ 

Every second $\mathbb{Z}$-submodule of $M$ is simple, while $\mathbb{Z}M$ is not multiplication. Notice that the assumption on $Ann(M)$ in Theorem 2.27 is not satisfied for this $\mathbb{Z}$-module, which shows that this condition cannot be dropped.
Recall from [1] that an $R$-module $M$ is second representable iff $M = \bigoplus_{i=1}^{n} K_i$, where $K_i$ is a second $R$-submodule of $M$ for all $i = 1, \cdots, n$. If this second representation is minimal, the set of main second attached primes of $M$ is given by $\text{att}^s(M) = \{\text{Ann}(K_i) \mid i = 1, \cdots, n\}$.

**Corollary 2.29.** If $R M$ is semisimple second representable with $\text{att}^s(M) = \text{Min}(\text{att}^s(M))$. The following are equivalent:

1. $M$ is multiplication.
2. Every PS-hollow submodule of $M$ is simple.
3. Every second submodule of $M$ is simple.
4. $M$ is comultiplication.

**Proof.** Since $M$ is second representable, the set $B$ defined in Theorem 2.27 is finite. Since $\text{Ann}(S_i)$ is prime for every $i \in A$ and $\text{att}^s(M) = \text{Min}(\text{att}^s(M))$ (i.e. different annihilators of simple submodules of $M$ are incomparable), we have $\text{Ann}(M) \neq \bigcap_{K \in B \setminus \{N\}} \text{Ann}(K)$ for every $N \in B$. The result follows now from Theorem 2.27.

**Example 2.30.** Consider $M = \mathbb{Z}_{30}[x]$ as a $\mathbb{Z}$-module. Let $K_i = (10x^i)$, $N_i = (15x^i)$ and $L_i = (6x^i)$. Set $K := \bigoplus_{i=1}^{\infty} K_i$, $N := \bigoplus_{i=1}^{\infty} N_i$ and $L := \bigoplus_{i=1}^{\infty} L_i$. Notice that

$$M = K + N + L.$$ 

It is clear that $M$ is second representable semisimple with infinite length, and

$$\text{att}^s(M) = \text{Min}(\text{att}^s(M)) = \{(2), (3), (5)\}.$$ 

Since $K$ is second but not simple, $\mathbb{Z} M$ is not comultiplication by Theorem 2.27 (notice also that $\mathbb{Z} M$ is not multiplication).

**Example 2.31.** Consider $M = \mathbb{Z}_{30} = (10) + (6) + (15)$. It is clear that $M$ is a second representable, multiplication, comultiplication and semisimple $\mathbb{Z}$-module in which $\text{att}^s(M) = \text{Min}(\text{att}^s(M))$ and every second submodule of $M$ is simple. By Corollary 2.29, every PS-hollow submodule of $M$ is simple, and so $(10), (6)$ and $(15)$ are the only PS-hollow submodules of $M$.

**Theorem 2.32.**

1. If $M = \bigoplus_{i=1}^{n} K_i$ is a minimal second representation of $M$ with $\text{att}^s(M) = \text{Min}(\text{att}^s(M))$ and $K_i \cap \bigoplus_{j \neq i} K_j$ is PS-hollow in $M$ for all $i \in \{1, \cdots, n\}$, then $M = \bigoplus_{i=1}^{n} K_i$ if and only if $K_i \cap K_j = 0$ for all $i \neq j$.

2. Let $R M$ be distributive and $M = \bigoplus_{i=1}^{n} K_i$ be a minimal PS-hollow representation such that every submodule of $K_i$ is zero or strongly irreducible or $H_i$-PS-hollow. Then $M = \bigoplus_{i=1}^{n} K_i$. 

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Proof. (1) Assume that $K_i \cap K_j = 0$ for all $i \neq j$ in $\{1, \cdots, n\}$. Set $I_i = \bigcap_{j \neq i} \text{Ann}(K_i)$. Since $\text{att}^i(M) = \text{Min} (\text{att}^i(M))$, we have $I_i M = K_i$. Also, $K_i \cap \sum_{j \neq i} K_j \subseteq K_i$. Since $K_i \cap \sum_{j \neq i} K_j$ is PS-hollow and each $K_j = I_j M$ for all $j \neq i$, we have $K_i \cap \sum_{j \neq i} K_j \subseteq \sum_{j \neq i} K_j$ implies that $K_i \cap \sum_{j \neq i} K_j \subseteq K_i$ for some $l \neq i$, whence $K_i \cap \sum_{j \neq i} K_j \subseteq K_i \cap K_i = 0$.

(2) Since $R\!M$ is distributive, it is enough to prove that $K_i \cap K_j = 0$ for all $i \neq j$ in $\{1, \cdots, n\}$. Suppose that $K_i \cap K_j \neq 0$ for some $i \neq j$. But $0 \neq K_i \cap K_j \subseteq K_i$, whence $K_i \cap K_j$ is strongly irreducible or $H_j$-PS-hollow. Suppose that $K_i \cap K_j$ is strongly irreducible. Since $K_i \cap K_j \subseteq K_i \cap K_j$, it follows that $K_i \subseteq K_i \cap K_j$ or $K_j \subseteq K_i \cap K_j$ and so $K_i \subseteq K_j$ or $K_j \subseteq K_i$ which contradicts the minimality of $\sum_{i=1}^n K_i$. So, $K_i \cap K_j$ is $H_1$-PS-hollow and at the same time $H_j$-PS-hollow, which contradicts the minimality of $\sum_{i=1}^n K_i$. Therefore $K_i \cap K_j = 0$ for all $i \neq j$ in $\{1, \cdots, n\}$.■

Examples 2.33. (1) Every second representable semisimple module satisfies the assumptions of Theorem 2.32 (2).

(2) $M = \mathbb{Z}_n$, considered as a $\mathbb{Z}$-module, $M$ satisfies all assumptions of Theorem 2.32 ((1) and (2)).

Theorem 2.34. Let $R$ be Artinian and $M = \sum_{i=1}^n K_i$ be a minimal PS-hollow representation of $R\!M$. Suppose that the submodules of $K_i$ are PS-hollow $\forall i \in \{1, \cdots, n\}$. If $\text{In}(K_i) \cap \text{In}(K_j) = 0$ for all $i \neq j$ in $\{1, \cdots, n\}$, then $M = \bigoplus_{i=1}^n K_i$.

Proof. Assume that $\text{In}(K_i) \cap \text{In}(K_j) = 0$ for all $i \neq j$ in $\{1, \cdots, n\}$. For each $j \in \{1, \cdots, n\}$, set $N_j := K_j \cap \sum_{i \neq j} K_i$. Then $N_j \subseteq \text{In}(K_i)$ for some $i \neq j$. Otherwise, $N_j \not\subseteq \text{In}(K_i)$ for all $i \neq j$, and so for all $i \neq j$ there is $I_i \in H_i$ such that $N_j \not\subseteq I_i M$. But $N_j \subseteq \sum_{i \neq j} K_i \subseteq I_i M$ and $N_j$ is a PS-hollow submodule by assumption, whence $N_j \subseteq I_i M$ for some $i \neq j$ in $\{1, \cdots, n\}$ (a contradiction).

Observe that $N_j \subseteq K_j \subseteq \text{In}(K_j)$ and so $N_j \subseteq \text{In}(K_i) \cap \text{In}(K_j)$ for some $i \neq j$ in $\{1, \cdots, n\}$. It follows that $N_j = 0$ for all $j \in \{1, \cdots, n\}$ and therefore $M = \bigoplus_{i=1}^n K_i$.■

Corollary 2.35. Let $R$ be Artinian and $M = \sum_{i=1}^n K_i$ a minimal PS-hollow representation of $R\!M$. Suppose that the nonzero submodules of $\text{In}(K_i)$ are $H_i$-PS-hollow for all $i \in \{1, \cdots, n\}$, where $K_i$ is $H_i$-PS-hollow for each $i \in \{1, \cdots, n\}$. Then $M = \bigoplus_{i=1}^n K_i$.

Proof. Suppose that $\text{In}(K_i) \cap \text{In}(K_j) \neq 0$ for some $i \neq j$ in $\{1, \cdots, n\}$. Then $\text{In}(K_i) \cap \text{In}(K_j)$ is $H_i$-PS-hollow, and at the same time $\text{In}(K_i) \cap \text{In}(K_j)$ is $H_j$-PS-hollow, which is a contradiction since $H_i \neq H_j$ as $M = \sum_{i=1}^n K_i$ is a minimal PS-hollow representation. Therefore $\text{In}(K_i) \cap \text{In}(K_j) = 0$. The result is obtained by Theorem 2.34.■
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