Abstract

We compute the nuclear force in a holographic model of QCD on the basis of a D4-D8 brane configuration in type IIA string theory. The repulsive core of nucleons is important in nuclear physics, but its origin has not been well understood in strongly coupled QCD. We find that the string theory via gauge/string duality deduces this repulsive core at a short distance between nucleons. Since baryons in the model are realized as solitons given by Yang-Mills instanton configuration on flavor D8-branes, ADHM construction of two instantons probes well the nucleon interaction at short scale, which provides the nuclear force quantitatively. We obtain a central force, as well as a tensor force, which is strongly repulsive as suggested in experiments and lattice results. In particular, the nucleon-nucleon potential $V(r)$ (as a function of the distance) scales as $r^{-2}$, which is peculiar to the holographic model. We compare our results with the one-boson exchange model using the nucleon-nucleon-meson coupling obtained in our previous paper.\[^{19}\]
§1. Introduction

Nuclear force, the force between nucleons, exhibits a repulsive core of nucleons at short distances. This repulsive core is quite important for large varieties of physics of nuclei and nuclear matter. For example, the well-known presence of nuclear saturation density is essentially due to this repulsive core. However, from the viewpoint of strongly coupled QCD, the physical origin of this repulsive core has not been well understood. Despite the long history of the problem, it was rather recent\textsuperscript{1)\textdagger} that lattice QCD could reach the problem,\textsuperscript{\ast} and of course, any understanding of it based on analytic computations is quite helpful for revealing the basic nature of nuclear and hadron physics.\textsuperscript{\ast\ast}

The recent rapid progress in applying gauge/string duality\textsuperscript{4)–7)} to QCD, holographic QCD, has been surprising. Now, it has been made possible to compute various observables in hadron physics such as spectra of mesons/baryons/glueballs and the interactions among them. Although most of the works rely on the supergravity approximation that works for large $N_c$ and large ’t Hooft coupling $\lambda$, it turned out that the holographic QCD reproduces quite well the properties of hadrons not only qualitatively but also quantitatively.

We apply this gauge/string duality to the problem of nuclear force. In our previous paper,\textsuperscript{8)} we computed nucleon-nucleon-meson couplings, using the holographic QCD on the basis of a D4-D8 brane configuration in type IIA string theory,\textsuperscript{9), 10)} which incorporates chiral quark dynamics. This amounts in principle to computing the large distance behavior of nuclear force, given that the potential between two nucleons can be understood as an exchange of mesons between them. In this paper, we take one step further: by directly solving the two-nucleon system in the D4-D8 model of the holographic QCD, we find a short distance scale of the nuclear force. In fact, we find the repulsive core of nucleons.

First, let us briefly summarize what has been computed for baryons in the D4-D8 model of the holographic QCD. The D4-D8 model\textsuperscript{9), 10)} of the holographic QCD describes a strong coupling regime of massless QCD at low energy, in the large $N_c$ limit with large ’t Hooft coupling $\lambda$, for a fixed number $N_f$ of flavors.\textsuperscript{\dag} The low-energy degrees of freedom on the flavor D8-branes in the holographic geometry of Ref.\textsuperscript{12)}, which are basically the Yang-Mills (YM) fields in five-dimensional curved space-time, give Kaluza-Klein towers of mesons, while instantons in the YM theory correspond to baryons in low-energy QCD\textsuperscript{9)} (this is based on the baryon vertices in gauge/string duality\textsuperscript{13), 14)} and the fact that branes inside branes are

\textsuperscript{\ast)} See also Ref.\textsuperscript{2)} for a study of the interactions between nucleons and hyperons using lattice QCD.

\textsuperscript{\ast\ast)} For a review on the theoretical aspects of nuclear force including that of short distance, see for example Ref.\textsuperscript{3)}.

\textsuperscript{\ast\ast\ast)} See also Refs.\textsuperscript{19), 21–24) for closely related works.

\textsuperscript{\dag)} Introducing massive quarks in the model has been discussed in Ref.\textsuperscript{11)}.
represented by solitonic instantons. Here, in our terminology, the instanton is a gauge configuration that is localized in spatial four dimensions in the five-dimensional space-time. The baryon number is identified as the instanton number in four-dimensional space. Since it is conserved in the time direction and localized in the spatial directions, it behaves as point particles that are interpreted as baryons. Quantization of a single instanton a la moduli space approximation gives rise to a spectrum of baryons including nucleons. In our previous paper, we computed the static properties of the baryons by evaluating the chiral currents in the presence of the instanton: charge radius, magnetic moments, form factors etc. were computed, in addition to the nucleon-nucleon-meson couplings. This analysis is reminiscent of that by Adkins et. al. for Skyrmions. In fact, the holographic description mimics the relation between the Skyrmion and instantons found by Atiyah and Manton. The physics of finite baryon density and nuclear matter has been explored in many papers recently, and we do not describe them in detail here.

Next, we briefly outline our method. The one-instanton analysis given in Refs. revealed that the desired configuration with instanton number 1 can be obtained simply by considering corrections to the BPST instanton in four-dimensional flat space. The corrections are due to (i) overall $U(1)$ part of the YM gauge fields coupled to the instanton density, and (ii) curved space-time along the extra dimension $x^4$ in the five-dimensional space-time. These corrections induce a small potential in the instanton moduli space, fix the size of the instanton to be of order $1/(\sqrt{\lambda}M_{KK})$ (where $M_{KK}$ is the only parameter with mass dimension and gives the meson mass scale), and give the quantization of the instanton in the moduli space approximation. This type of analysis can be extended to our case of two baryons. If the two baryons sit close to each other so that the distance $r$ satisfies $r < O(1/M_{KK})$, we can use two-instanton configuration in the flat space as a starting point, since the effects of the curved space are small. The properties of the two-instanton moduli space are known, concerning not only its construction via renowned ADHM (Atiyah-Drinfeld-Hitchin-Manin) method, but also the metric in its moduli space (see Refs. for some of the papers relevant to our computations). We use them explicitly as a basis in a manner similar to the one-instanton case, to explore the physics of the nuclear force, i.e., the interaction between two baryons sitting close to each other.

We compute the additional potential induced in the moduli space, due to the presence

***) To describe the nuclear force, a two-instanton configuration was used for this Atiyah-Manton ansatz for Skyrmions (see for example Ref. ).

---

1) By using five-dimensional spinor fields introduced as nucleon fields on the D8-brane, in Refs., the static quantities of baryons were computed. See also Refs.

2) The analysis of the Skyrmions based on the four-dimensional meson effective action derived from the D4-D8 model is given in Ref.

3) To describe the nuclear force, a two-instanton configuration was used for this Atiyah-Manton ansatz for Skyrmions (see for example Ref. ).
of the two instantons. The analytic form of the moduli Lagrangian can be obtained in the asymptotic expansion of $r$. This includes corrections to the kinetic term, coming from the metric of two-instanton moduli space. Specifying the two-nucleon states by tensor product of single-baryon states obtained in Ref. [18], we can evaluate the nuclear force for given nucleon states.

Note that this “asymptotics” means a large distance in the region $r < \mathcal{O}(1/M_{\text{KK}})$. Therefore, in the standard terminology for the nuclear force, our result is for short distances. In addition, we use the asymptotic expansion in $r$, so our analytic formula of the nucleon-nucleon potential is not for nucleons on top of each other. However, this is sufficient for seeing the repulsive core of the nucleons.

We find that our final expression for the nucleon-nucleon potential, (4.45), is repulsive, and has $1/r^2$ dependence. This $r$-dependence is peculiar to the four-dimensional space, not the three-dimensional harmonic potential. The appearance of the $1/r^2$ potential is due to the extra holographic dimension, thus typical in holographic description. Physically speaking, the Kaluza-Klein summation of all the meson states in the tower produces this new behavior.

The main reason why the force is repulsive is that the instantons carry electric charge of the overall $U(1)$ part of the YM fields on the D8-branes. This electric charge is supplied by a Chern-Simons (CS) coupling on the $N_f$ D8-branes, and is nothing but the baryon number. The $U(1)$ force is repulsive since the instantons have the $U(1)$ charge of the same sign. There are some contributions from the $SU(2)$ gauge field that give attractive potential, but this $SU(2)$ force turns out not to be strong enough to cancel the $U(1)$ repulsive force. The Kaluza-Klein decomposition of the $U(1)$ part of the gauge fields provides a mass tower starting with $\omega$ meson as the lightest vector meson [9], and so, our computation shows that the repulsive force is partly due to the $\omega$ meson exchange. Not only the $\omega$ meson but also the whole massive mesons participate in the nuclear force, and as a result, the nucleon-nucleon potential becomes $1/r^2$.

One might wonder whether it is reasonable to sum up the contributions from all the massive mesons, since the model deviates from QCD at the energy scale higher than $M_{\text{KK}}$. However, there are some lines of evidence suggesting that the results obtained by summing up the infinite tower of massive mesons are better than those obtained by only taking into account the first few modes. For example, in our previous paper [5] we showed that the electromagnetic form factors for the nucleon are very close to the dipole profile observed in the experiment. This result is obtained by summing up the contributions from all the massive vector mesons. If we only take into account the rho meson, the form factors can never be close to the dipole profile.

The organization of this paper is as follows. First, in §2, we describe our strategy,
together with a brief review of the instantons in the D4-D8 model. In §3, we obtain an effective Hamiltonian for moduli parameters of the two instantons in the model. In §4, using the wave functions for nucleon states, we evaluate the nucleon nucleon interaction potential. We decompose it to a central force and a tensor force. In §5 we compare our results with one-boson-exchange potential evaluated using the nucleon-nucleon-meson coupling obtained in our previous paper. Section 6 is for a brief summary. In the appendices, we review the ADHM construction of two instantons and summarize the necessary formulas used in this paper.

§2. Nuclear force in holographic QCD

In this section, we briefly summarize the treatment of the single baryon in the D4-D8 model of the holographic QCD following Ref. [18] and describe our strategy for obtaining the nuclear force. Our first goal is to obtain a quantum mechanics Hamiltonian for a two-nucleon system. The total Hamiltonian consists of one-body canonical kinetic terms for each nucleon, potential terms for each nucleon, plus interactions. One generically has an interaction potential as well as a correction to the kinetic term. Then, secondly, we evaluate the inter-baryon energy using the Hamiltonian. This provides an explicit nuclear force that is dependent on nucleon states labeled by spin and isospin.

The concrete calculations of the Hamiltonian will be given in §3 and its evaluation with explicit nucleon states will be presented in detail in §4.

2.1. Review: single baryon in the model

First, we review briefly the single baryon case in the holographic QCD proposed in Refs. [9], [10]. The notation of our paper follows that of Ref. [18].

Our starting point is the meson effective action derived in Refs. [9], [10], which is given by the following five-dimensional $U(N_f)$ Yang-Mills-Chern-Simons (YMCS) theory in a curved background:

$$S = S_{YM} + S_{CS} \, ,$$

$$S_{YM} = -\kappa \int dt dz \, \text{tr} \left[ \frac{1}{2} h(z) F_{\mu\nu}^2 + k(z) F_{\mu z}^2 \right] \, , \quad S_{CS} = \frac{N_c}{24\pi^2} \int_{M^4 \times \mathbb{R}} \omega_5(A) \, . \quad (2.1)$$

Here $\mu, \nu = 0, 1, 2, 3$ are four-dimensional Lorentz indices, and $z$ is the coordinate of the fifth dimension. The field strength is defined as $F = \frac{1}{2} F_{\alpha\beta} dx^\alpha \wedge dx^\beta = dA + iA \wedge A$ with the $U(N_f)$ gauge field $A = A_\alpha dx^\alpha = A_\mu dx^\mu + A_z dz \ (\alpha = 0, 1, 2, 3, z)$, and the front factor $\kappa$ is related to the ’t Hooft coupling $\lambda$ and the number of colors $N_c$ as

$$\kappa = \frac{\lambda N_c}{216 \pi^3} \equiv a \lambda N_c \, . \quad (2.2)$$
The action (2.1) is written in the unit $M_{KK} = 1$, where $M_{KK}$ is the only dimensionful parameter in the model. The functions $h(z)$ and $k(z)$ appearing as the “metric” in the action (2.1) are given by $h(z) = (1 + z^2)^{-1/3}$ and $k(z) = 1 + z^2$, while, in the second term, $\omega_5(\mathcal{A})$ is the CS 5-form (here, we omit the $\wedge$ product, e.g., $\mathcal{A}F^2 = \mathcal{A} \wedge F \wedge F$)

$$\omega_5(\mathcal{A}) = \text{tr} \left( \mathcal{A}F^2 - \frac{i}{2} \mathcal{A}^3F - \frac{1}{10} \mathcal{A}^5 \right). \quad (2.3)$$

In the two-flavor case ($N_f = 2$) that we focus on in this paper, the $U(2)$ gauge fields $\mathcal{A}$ are decomposed as

$$\mathcal{A} = A + \hat{A} \frac{1_2}{2} = A^\alpha \tau^a \frac{1}{2} + \hat{A} \frac{1_2}{2} = \sum_{C=0}^{3} \mathcal{A}^C \tau^C \frac{1}{2}, \quad (2.4)$$

where $\tau^a (a = 1, 2, 3)$ are Pauli matrices and $\tau^0 = 1_2$ is a unit matrix of size 2.

This action is obtained from the low-energy effective action on $N_f$ D8-branes in the curved ten-dimensional geometry corresponding to $N_c$ D4-branes wrapped on a circle with an antiperiodic boundary condition for fermions. At low energy, this D-brane configuration provides $U(N_c)$ QCD with $N_f$ massless quarks and the action (2.1) describes the dynamics of mesons and baryons. The action (2.1) is written in $(1+4)$ dimensions, and the space along the extra dimension $x^4 (\equiv z)$ is curved. Once the gauge fields are decomposed into their Kaluza-Klein states concerning the $z$ direction, each mass eigenstate corresponds to a meson, and the action (2.1) describes the whole spectra/interactions of the mesons. By contrast, baryons are solitons with nonzero instanton number in the four-dimensional space parameterized by $x^M = (\vec{x}, z) \; (M = 1, 2, 3, z)$. As they are localized in the four-dimensional space in the five-dimensional space-time, they behave as pointlike particles. The instanton number is identified with the baryon number and these particles are interpreted as baryons. We will see more of the details below.

The single-baryon solution was found to have the size of order $\lambda^{-1/2}$.[13, 14] It is helpful to rescale the coordinates as

$$\tilde{x}^M = \lambda^{1/2} x^M, \; \tilde{x}^0 = x^0,$$

$$\tilde{A}_0(t, \tilde{x}) = A_0(t, \tilde{x}), \; \tilde{A}_M(t, \tilde{x}) = \lambda^{-1/2} A_M(t, \tilde{x}), \quad (2.5)$$

to see the consistent $1/\lambda$ expansion of the equations of motion and the total energy of the single baryon. Hereafter, we omit the tilde for simplicity. Then, for large $\lambda$, the YM part of

---

* In Refs. [9] and [10], these two parameters are chosen as $M_{KK} = 949$ MeV, $\kappa = 0.00745$ to fit the experimental values of the $\rho$ meson mass $m_\rho \simeq 776$ MeV and the pion decay constant $f_\pi \simeq 92.4$ MeV.
the action is
\[ S_{\text{YM}} = -aN_c \int d^4xz \, \text{tr} \left[ \frac{\lambda}{2} F_{MN}^2 + \left( -\frac{z^2}{6} F_{ij}^2 + z^2 F_{iz}^2 - F_{0M}^2 \right) + \mathcal{O}(\lambda^{-1}) \right] \]
\[ - \frac{aN_c}{2} \int d^4xz \left[ \frac{\lambda}{2} \hat{F}_{MN}^2 + \left( -\frac{z^2}{6} \hat{F}_{ij}^2 + z^2 \hat{F}_{iz}^2 - \hat{F}_{0M}^2 \right) + \mathcal{O}(\lambda^{-1}) \right], \quad (2.6) \]

while the total equations of motion are
\[ D_M F_{0M} + \frac{1}{64\pi^2a^2} \epsilon_{MNPQ} \hat{F}_{MN} F_{PQ} + \mathcal{O}(\lambda^{-1}) = 0, \quad (2.7) \]
\[ D_N F_{MN} + \mathcal{O}(\lambda^{-1}) = 0. \quad (2.8) \]
\[ \partial_M \hat{F}_{0M} + \frac{1}{64\pi^2a^2} \epsilon_{MNPQ} \left\{ \text{tr}(F_{MN} F_{PQ}) + \frac{1}{2} \hat{F}_{MN} \hat{F}_{PQ} \right\} + \mathcal{O}(\lambda^{-1}) = 0, \quad (2.9) \]
\[ \partial_N \hat{F}_{MN} + \mathcal{O}(\lambda^{-1}) = 0. \quad (2.10) \]

Therefore, at the leading order, the warp factors \( h(z) \) and \( k(z) \) are approximated by 1, so the \( SU(2) \) part of the equations is nothing but the standard YM equation in flat space. It is solved by a BPST instanton located around \( z \sim 0 \). The electric \( U(1) \) part is sourced by the instanton density, as seen in (2.9). The explicit solution is
\[ A_M^i = -if(\xi)g \partial_M g^{-1}, \quad \hat{A}_0^i = \frac{1}{8\pi^2a} \frac{1}{\xi^2} \left[ 1 - \frac{\rho^4}{(\rho^2 + \xi^2)^2} \right], \quad A_0 = \hat{A}_M = 0, \quad (2.11) \]
with the BPST instanton profile
\[ f(\xi) = \frac{\xi^2}{\xi^2 + \rho^2}, \quad g(x) = \frac{(z - Z) + i(\vec{x} - \vec{X}) \cdot \vec{r}}{\xi}, \quad \xi = \sqrt{(z - Z)^2 + |\vec{x} - \vec{X}|^2}. \quad (2.12) \]

\( \rho \) is the size of the instanton, while \( X^M = (X^1, X^2, X^3, Z) = (\vec{X}, Z) \) is the position of the soliton in the four-dimensional space.

Quantization of this soliton has been carried out in Ref. [18]. It is basically the same as the quantization of a YM instanton in the moduli space approximation [19, 20] except for the additional potential in the moduli space induced by the presence of the subleading terms in the action (2.6). The moduli space for a single YM instanton is \( \mathcal{M}_1 \simeq \mathbb{R}^4 \times \mathbb{R}^4/\mathbb{Z}_2 \) parameterized by \( (\vec{X}, Z) \) and \( y^I \ (I = 1, 2, 3, 4) \) with the \( \mathbb{Z}_2 \) action \( y^I \rightarrow -y^I \). The radial component \( \rho \equiv \sqrt{(y^I)^2} \) of \( y^I \) gives the instanton size and the angular components \( a^I = y^I / \rho \) parameterize the \( SU(2) \) orientation of the instanton. The quantization of the soliton is described by quantum mechanics on this moduli space, with the Lagrangian
\[ L = \frac{m_X}{2} \dot{X}^2 + \frac{m_Z}{2} \dot{Z}^2 + \frac{m_y}{2} (y^I)^2 - U(\rho, Z), \quad (2.13) \]
where the “mass” for each moduli is given by
\[ m_X = m_Z = \frac{m_y}{2} = 8\pi^2aN_c , \] (2.14)
and the potential
\[ U(\rho, Z) = M_0 + 8\pi^2aN_c \left( \frac{\rho^2}{6} + \frac{1}{5(8\pi^2a)^2} \frac{1}{\rho^2} + \frac{Z^2}{3} \right) \] (2.15)
is obtained by substituting the solution (2.11) to the action (2.6). \( M_0 \equiv 8\pi^2\kappa \) is the classical mass at the leading order in the \( 1/\lambda \) expansion. The potential is classically minimized at
\[ \rho_{\text{cl}}^2 = \frac{1}{8\pi^2a} \sqrt{\frac{6}{5}} , \quad Z_{\text{cl}} = 0 , \] (2.16)
which shows that in fact the soliton has the size of order \( \lambda^{-1/2} \) when it is rescaled back to the original coordinates by (2.5). The Hamiltonian is given by
\[ H = -\frac{1}{2m_X} \left( \frac{\partial}{\partial X^2} \right)^2 + -\frac{1}{2m_Z} \left( \frac{\partial}{\partial Z} \right)^2 + -\frac{1}{2m_y} \left( \frac{\partial}{\partial y^I} \right)^2 + U(\rho, Z) . \] (2.17)
This system has an \( SO(4) \cong (SU(2)_I \times SU(2)_J)/\mathbb{Z}_2 \) rotational symmetry acting on \( y^I \). Here \( SU(2)_I \) and \( SU(2)_J \) are interpreted as the isospin and spin rotations, respectively, and they act on \( y \equiv y^I + iy^a\tau^a \) as
\[ y \rightarrow g_I y g_J \] (2.18)
with \( (g_I, g_J) \in SU(2)_I \times SU(2)_J \). The isospin and spin operators are given by
\[ I^a = \frac{i}{2} \left( y^4 \frac{\partial}{\partial y^a} - y^a \frac{\partial}{\partial y^4} - \epsilon^{abc} y^b \frac{\partial}{\partial y^c} \right) , \]
\[ J^a = \frac{i}{2} \left( -y^4 \frac{\partial}{\partial y^a} + y^a \frac{\partial}{\partial y^4} - \epsilon^{abc} y^b \frac{\partial}{\partial y^c} \right) , \] (2.19)
respectively. From this, we have \( \vec{I}^2 = \vec{J}^2 \) and, hence, only baryons with \( I = J \) appear in this approach.

Quantum states of the baryon can be labeled using quantum numbers of isospin/spin \( I = J \equiv l/2 \), \( (l = 1, 3, 5, \cdots) \), the eigenvalues of the third components of isospin and spin operators \( I^3 \) and \( J^3 \), and the quantum numbers \( n_\rho = 0, 1, 2, \cdots \) and \( n_\rho = 0, 1, 2, \cdots \), which label the excitation numbers of (almost) harmonic oscillators in \( \rho \) and \( Z \), respectively. For example, the proton and neutron have quantum numbers \( (l, I_3, n_\rho, n_\rho) = (1, 1/2, 0, 0) \) and
\((l, I_3, n_\rho, n_z) = (1, -1/2, 0, 0)\), respectively. The corresponding wavefunctions are normalized spin/isospin states\(^\text{\textsuperscript{25}}\)

\[ |p \uparrow\rangle = \frac{1}{\pi} (y^1 + iy^2)/\rho , \quad |p \downarrow\rangle = -\frac{i}{\pi} (y^1 - iy^3)/\rho , \]
\[ |n \uparrow\rangle = \frac{i}{\pi} (y^4 + iy^3)/\rho , \quad |n \downarrow\rangle = -\frac{1}{\pi} (y^1 - iy^2)/\rho , \]

multiplied by the following \(\rho\) and \(Z\) wavefunctions,

\[ R(\rho) = \rho \tilde{l} e^{-\frac{m_\rho^2}{2} \rho^2} , \quad \psi_Z(Z) = e^{-\frac{m_Z^2}{2} Z^2} , \]

with \(\tilde{l} = -1 + 2\sqrt{1 + N_\sigma^2/5}, \quad \omega_\rho = 1/\sqrt{6}, \quad \text{and} \quad \omega_Z = \sqrt{2/3}. \) The functions \(R(\rho)\) and \(\psi_Z(Z)\) should be multiplied by normalization factors.

2.2. Our strategy

Our strategy for the calculation of the nuclear force consists of three steps:

1) Construction of generic two-baryon solution of the YMCS theory \(^{\text{\textsuperscript{21}}}\),
2) Computation of the quantum-mechanical Hamiltonian for the moduli parameters, and
3) Evaluation of the Hamiltonian with specified nucleon states.

This is a direct generalization of the single-baryon case to the two-baryon case. In the following, we describe each step in more detail.

2.2.1. Construction of two-baryon solution

The case of two baryons, which is our concern, can be considered by solving the equations of motion of the original action \(^{\text{\textsuperscript{21}}}\) with the constraint that the instanton number is 2. As we have seen, the rescaled variables are useful for seeing the properties of the single baryon. There, one can start with a BPST instanton solution in flat space, since the size of the instanton is smaller than the scale of the curved background geometry. When we have two baryons, the situation is different. If the distance between the two is larger than \(O(1/M_{KK}^\ast)\) \(^{\text{\textsuperscript{4}}\text{) or} O(\sqrt{\lambda}/M_{KK}) \text{in the rescaled coordinate), the effect of the curved space-time comes into play, thus a similar analysis cannot be performed. In this paper, we concentrate on the case where the two baryons are close to each other, i.e., the distance is smaller than \(O(1/M_{KK}).\)

It is well-known that one can explicitly construct generic two-instanton solutions of Euclidean four-dimensional YM theory in flat space. We use ADHM construction of the instantons for our purpose. The construction is reviewed in Appendix \(^{\text{\textsuperscript{B}}}\).

The two-instanton moduli space is parameterized by four quaternionic parameters \(X_1, X_2, y_1, y_2\). We summarize our notation for the quaternion in Appendix \(^{\text{\textsuperscript{A}}}\). The quaternion has

\(^{\text{\textsuperscript{4}}}\) As a reference, if we use the value of \(M_{KK}\) that is fixed by the rho meson mass, we have \(1/M_{KK} \simeq 0.208\) fm.
a representation by $2 \times 2$ complex matrices as in \((A.3)\). In this notation, these moduli parameters can be written as

$$X_i = Z_i + i \vec{X}_i \cdot \vec{\tau}, \quad y_i = y^1_i + i \vec{y}_i \cdot \vec{\tau}, \quad (i = 1, 2) \quad (2.22)$$

with $\vec{X}_i = (X^1_i, X^2_i, X^3_i)$ and $\vec{y}_i = (y^1_i, y^2_i, y^3_i)$. When the separation between the two instantons is large, the two-instanton solution can be approximated with a superposition of two one-instanton configurations with moduli parameters $(X_i, y_i)$ $(i = 1, 2)$. Here, $X^M_i = (\vec{X}_i, Z_i)$ corresponds to the position of the instanton in the four-dimensional space, $\rho_i \equiv \sqrt{y^I_i y^I_i}$ is the size, and $a_i \equiv y_i / \rho_i$ is the $SU(2)$ orientation of the instanton.

Defining $r^M \equiv X^M_1 - X^M_2$ $(M = 1, 2, 3, z)$ as the relative position of the two instantons and $|r| = \sqrt{r^M r^M}$ as the distance between them in the four-dimensional space, the requirement for the flat space approximation to be valid amounts to

$$|r| < O\left(\sqrt{\lambda}/M_{KK}\right). \quad (2.23)$$

Note that this is written in the rescaled coordinates (2.22).

As seen from the structure of the equations of motion in the $1/\lambda$ expansion, the only nonzero quantities at leading order are $A_M$ and $\hat{A}_0$, as in the case of the single baryon. The equation of motion (2.19) shows that the $U(1)$ part of the gauge field is again sourced by the instanton density, now with two maxima at the location of the separated baryons. The explicit solution of the $SU(2)$ two-instanton solution and the $U(1)$ part will be presented in \[3\] with the help of the ADHM construction reviewed in Appendix B.

2.2.2. Calculation of quantum-mechanical Hamiltonian for two-baryon moduli

The next task is to obtain the classical potential $U(y^1_I, y^2_I, \vec{X}_1 - \vec{X}_2, Z_1, Z_2)$. We substitute the two-instanton configuration into the action (2.6). As we mentioned, the nonzero fields at the leading order are only $A_M(x)$ (the spatial components of the $SU(2)$) and $\hat{A}_0(x)$ (the temporal component of the $U(1)$). Therefore, in the rescaled action (2.6), nonzero contributions are

$$U = 2M_0 + H^{SU(2)}_{\text{pot}} + H^{U(1)}_{\text{pot}} + O\left(\lambda^{-1}\right), \quad (2.24)$$

$$H^{SU(2)}_{\text{pot}} \equiv a N_c \int d^3 x d z \, \text{tr} \left[ -\frac{z^2}{6} F^2_{ij} + z^2 F^2_{iz} \right] = a N_c \int d^3 x d z \, \text{tr} \left[ z^2 F^2_{MN} \right], \quad (2.25)$$

$$H^{U(1)}_{\text{pot}} \equiv \frac{a N_c}{2} \int d^3 x d z \left[ \hat{F}^2_{0M} \right]. \quad (2.26)$$

The first term of the total potential (2.24) is the energy contribution from the first term in the action (2.6), which gives the leading order term in the $1/\lambda$ expansion. The important part is the subleading order potential, $H^{SU(2)}_{\text{pot}}$ and $H^{U(1)}_{\text{pot}}$, which, in the case of the single
baryon, was used to determine the size of the instanton classically and was responsible for the quantum effect. We compute these for the cases with two baryons, with explicit dependence on the moduli parameters. In the last equation of (2.25), we have used the self-dual equation for the instanton.

The computation of $H_{\text{pot}}^{(SU(2))}$ is straightforward although it is technically involved, which will be presented in §3.1. On the other hand, it turns out that $H_{\text{pot}}^{(U(1))}$ is not easy to compute. One can evaluate it numerically, but numerical results are not useful for our purpose, as we will later need to make a moduli integration of it with the baryon wavefunction. Thus, we need the explicit analytic expression for the moduli dependence of the potential. For this purpose, we concentrate on the case with a large inter-baryon distance,

$$O(1/M_{\text{KK}}) < |r| \ ,$$

where the left-hand side is the size of the single instanton, (2.16), in the rescaled coordinate. In this region, since there is only a slight overlap of the instantons, one can obtain an analytic expression for $H_{\text{pot}}^{(U(1))}$. The evaluation will be presented in §3.2.

Together with the constraint (2.23), in this paper, we consider the separation of the baryon satisfying

$$O(1/M_{\text{KK}}) < |r| < O(\sqrt{\lambda}/M_{\text{KK}}) \quad (2.28)$$

in the rescaled coordinates (2.5).

The quantum mechanics of the moduli parameters consists of the potential term $U$ and the kinetic term. The kinetic term of the quantum mechanics of the moduli is given by the moduli space metric. As opposed to the single-instanton case, the moduli space metric of the two-instanton configuration is complicated. It is found that the asymptotic form in the case of a large separation takes the form

$$ds^2 = ds_0^2 + ds_1^2 + O(|r|^{-3}) \ ,$$

where $ds_0^2 = 2(dy_I^1)^2 + (dX^1_M)^2 + 2(dy_I^2)^2 + (dX^2_M)^2$ is just two copies of the metric for the single instanton, and $ds_1^2$ is $O(|r|^{-2})$, which is our concern. This contributes to the quantum mechanics as a $O(|r|^{-2})$ correction to the kinetic term of the moduli dynamics. We explicitly compute this correction in §3.3.

All together, the analytic expressions of the $O(|r|^{-2})$ terms of the quantum mechanics are computed, and this is the Hamiltonian for the two-baryon interaction. The total expression is summarized in §3.4.
2.2.3. Evaluation of nucleon-nucleon potential

The final step is to evaluate the energy with the Hamiltonian of the quantum mechanics. The interaction Hamiltonian is given in $1/|\vec{r}|$ expansion, and we treat this as a perturbation. The state of our baryons is specified at infinite separation, and we use simply the tensor product of two copies of a single-baryon wavefunction. This can be justified at leading order in the perturbation of quantum mechanics.\[1\]

Since we have an analytic expression for the Hamiltonian, the integration of the moduli parameter with the given wavefunctions is straightforward. The result is the nucleon-nucleon potential, in particular if we choose the baryon wavefunctions to be that of a nucleon. This integration will be presented in §4. The distance between the baryons in the three-dimensional space, $|\vec{r}| = \sqrt{(X_1^1 - X_2^1)^2 + (X_1^2 - X_2^2)^2 + (X_1^3 - X_2^3)^2}$, is related to the four-dimensional distance $|\vec{r}|$ as $|\vec{r}| = \sqrt{|\vec{r}|^2 + (Z_1 - Z_2)^2}$. Note that we fix one of the moduli $|\vec{r}|$ and perform the integration of the other moduli, $Z_1, Z_2, y_1, y_2$. This is because we are interested in the potential in the scattering problem, rather than the computation of the bound state energy.

Our final result for the nucleon-nucleon potential is given in (4.45) and (4.46). The central force (4.45) shows that the nucleons have a repulsive core. We also obtain the tensor force (4.46). All the potentials have the form $|\vec{r}|^{-2}$, which is peculiar to four-dimensional space, as described in the introduction.

To illustrate the properties of our nuclear force (4.45) and (4.46), we next compute the one-boson-exchange potential among nucleons. In our previous paper, we derived the nucleon-nucleon-meson coupling in the D4-D8 model of the holographic QCD. By using this coupling, the summing up of all types of mesons propagating among the nucleons should provide a certain aspect of the nuclear force. This computation can be carried out for arbitrary distances between the nucleons, as long as the nucleon radii do not overlap each other. The resultant nuclear force, at larger distances, exhibits the standard properties of the nuclear force, such as scalar/tensor forces due to pion/$\rho$-meson/other-meson exchanges. The computation will be presented in §5.

We will find there that, in the region (2.28), this one-boson-exchange potential does not coincide with our nuclear force (4.45) and (4.46) derived using the ADHM construction of two instantons. The reason is that when nucleons are close to each other the nucleon itself is deformed by the effect of the other nucleon. In deriving the one-meson-exchange potential, this effect cannot be taken into account. Thus, naive computation based on the one-boson exchange is not sufficient to capture the complete picture of the nuclear force at

\[\text{For describing the deuteron system, this perturbation is not the way to proceed. One needs a minimum of the whole potential of moduli including } \vec{r}, \text{ to obtain quantized energy of a bound state of two baryons. Our interest in this paper is the potential force appearing in the scattering process of the two baryons.}\]
3. Effective Hamiltonian for two baryons

In this section, we calculate the effective Hamiltonian for the quantum mechanics of the two-baryon state following the strategy described in the previous section. The system is described as a quantum mechanics of a particle living in the two-instanton moduli space.

3.1. Potential from SU(2) part

Let us first evaluate the contribution of the SU(2) part of the gauge field to the potential. As explained in §2, the leading term in the $1/\lambda$ expansion can be obtained by substituting the two-instanton solution in the flat space-time into the action (2.6). The two-instanton solution can be obtained using the ADHM construction. See Appendices A and B for our notation and a brief review of the ADHM construction.

The leading term in the SU(2) part is obtained by evaluating (2.25). This integral can be calculated using the formula (C.7) obtained in Appendix C. Substituting (B.14) into (C.7), we obtain

$$\int d^3xdz z^2 \text{tr} F_{MN}^2 = 8\pi^2 (\rho_1^2 + \rho_2^2 + 2(Z_1^2 + Z_2^2) + 4(w^4)^2),$$

where $w^4$ is the real part of (B.17), which is given by

$$w^4 = -\frac{r^a}{|r|^2} \cdot (\vec{y}_1 y_2^4 - \vec{y}_2 y_1^4 + \vec{y}_2 \times \vec{y}_1) = \frac{\rho_1 \rho_2}{2} \frac{r^a}{|r|^2} \text{tr} (i\tau^a a_2^{-1} a_1) .$$

Therefore the potential from the SU(2) part is given by

$$H_{pot}^{(SU(2))} = \frac{4\pi^2 a N_c}{3} \left( \rho_1^2 + \rho_2^2 + 2(Z_1^2 + Z_2^2) + 4(w^4)^2 \right)$$

$$= \frac{4\pi^2 a N_c}{3} \left( \rho_1^2 + \rho_2^2 + 2(Z_1^2 + Z_2^2) + \rho_1^2 \rho_2^2 |r|^{-4} \text{tr} (i\tau^a a_2^{-1} a_1) \text{tr} (i\tau^b a_2^{-1} a_1) \right).$$

The leading $r$-independent terms reproduce the contribution of SU(2) part in the one-instanton potential (2.15) for the two instantons. The next-to-leading term gives the interaction between the two baryons. It is of order $1/|r|^2$ as expected in a five-dimensional gauge theory.

3.2. Potential from U(1) part

The field strength of the SU(2) part of the gauge field satisfies (C.3) and then it is easy to see that

$$\hat{A}_0 = \frac{1}{32\pi^2a} \Box \log \det L$$

(3.4)
is the regular solution of the equation of motion (2.9), which vanishes at infinity. Here, $L$ is
given by (B.2) and (B.14). By using the constraint (B.16), it can be written as

$$L(x) = \begin{pmatrix} f_1(x) & e(x) \\ e(x) & f_2(x) \end{pmatrix},$$

(3.5)

where

$$f_i(x) = \rho_i^2 + |x - X_i|^2 + |w|^2, \quad (i = 1, 2)$$

(3.6)

and

$$e(x) = (y_1 \cdot y_2) + (w \cdot (X_1 + X_2 - 2x)).$$

(3.7)

We evaluate the energy contribution of this $U(1)$ gauge field when the separation between
the two instantons is large. Then, it can be confirmed that the energy density is mainly
concentrated around $x \sim X_i$ ($i = 1, 2$), where the two instantons are located. When $x$ is
close to $X_1$, we can make the expansion

$$x \sim X_1, \quad |x - X_1| \ll |x - X_2|.$$  

(3.8)

Without losing generality, we can choose $X_1 = 0$ and $X_2$ to be very far away from the origin,
and so we choose $x$ around the origin. The order estimate is

$$|X_2| \sim |x - X_2| \sim |r| \equiv |X_2 - X_1| \gg |x - X_1| \sim |x|.$$  

(3.9)

We can evaluate the matrix $L$ in this expansion, and obtain the following expression:

$$\Box \log \det L = \frac{4}{|x|^2} \left( 1 - \frac{\rho_1^4}{(|x|^2 + \rho_1^2)^2} \right)
+ \frac{4}{|r|^2} \left( 1 + \frac{2(y_1 \cdot y_2)^2 \rho_1^2}{(|x|^2 + \rho_1^2)^3} \right) - \frac{8\rho_1^2}{(|x|^2 + \rho_1^2)^3} |w|^2 + O(|r|^{-3}).$$  

(3.10)

In this expression, note that $w = O(|r|^{-1})$. More explicitly, from (B.17), we have

$$|w|^2 = \frac{1}{|r|^2} |y_2 \times y_1|^2.$$  

(3.11)

Then, the gauge field around the position of one of the two instantons $x \sim X_1$ is expanded as

$$\tilde{A}_0 = \frac{1}{8\pi^2a} \left[ \frac{1}{|x|^2} \left( 1 - \frac{\rho_2^4}{(|x|^2 + \rho_1^2)^2} \right) + \frac{1}{|r|^2} \left( 1 + \frac{2Y \rho_1^2}{(|x|^2 + \rho_1^2)^3} \right) + O(|r|^{-3}) \right],$$

(3.12)

where

$$Y \equiv (y_1 \cdot y_2)^2 - |y_2 \times y_1|^2 = 2(y_1 \cdot y_2)^2 - |y_1|^2 |y_2|^2 = \rho_1^2 \rho_2^2 (2(a_1 \cdot a_2)^2 - 1).$$

(3.13)
The leading term in the $1/|r|$ expansion reproduces the result (2.11) for the single instanton. We are interested in the next-to-leading term of order $|r|^{-2}$. Note that the first term in the next-to-leading terms,

$$
\frac{1}{|r|^2} \cdot 1,
$$

is precisely the leading contribution of the five-dimensional Coulomb interaction with the second instanton located at $x \sim X_2$.

Let us compute the potential energy, with this expression for the gauge field. The $U(1)$ part of the energy (2.26) can be written as

$$
H^{(U(1))}_{\text{pot}} = -\frac{a N_c}{2} \int d^3x dz \hat{A}_0 \square \hat{A}_0.
$$

Note that to obtain this expression we performed a partial integration, it is harmless at this stage because the gauge field $\hat{A}_0$ decays fast asymptotically. Then, we substitute the expanded expression (3.12) to this energy formula. This procedure is slightly ambiguous, since the integration does not commute with the $1/|r|$ expansion in (3.12). We give a more systematic way of evaluating the integral in Appendix E. Here, we present an easy way to obtain the correct answer. Substituting the expansion (3.12) into (3.15), we obtain

$$
\frac{N_c}{40 \pi^2 a} \left( \frac{1}{\rho_1^2} + \frac{1}{\rho_2^2} \right)
$$

as the leading term. This is just the sum of the energy contribution of the single instanton in (2.15). The next-to-leading order that we are interested in is the cross terms,

$$
-\frac{a N_c}{2} \int d^3x dz \frac{1}{(8 \pi^2 a)^2} \left[ \left( \frac{1}{|x|^2} \left( 1 - \frac{\rho_1^4}{(|x|^2 + \rho_1^2)^2} \right) \right) \square \left( \frac{1}{|r|^2} \left( 1 + \frac{2 Y \rho_1^2}{(|x|^2 + \rho_1^2)^3} \right) \right) \right]
$$

$$
-\frac{a N_c}{2} \int d^3x dz \frac{1}{(8 \pi^2 a)^2} \left[ \left( \frac{1}{|r|^2} \left( 1 + \frac{2 Y \rho_1^2}{(|x|^2 + \rho_1^2)^3} \right) \right) \square \left( \frac{1}{|x|^2} \left( 1 - \frac{\rho_1^4}{(|x|^2 + \rho_1^2)^2} \right) \right) \right]
$$

$$
+(y_1 \leftrightarrow y_2).
$$

Here, $(y_1 \leftrightarrow y_2)$ denotes the contribution from the integration around $x \sim X_2$, which is obtained by exchanging $y_1$ and $y_2$ in the first and second terms of (3.17). Note that the first term in (3.17) is different from the second term. This is because, at this stage, the partial integration suffers from a surface term, owing to the constant $1/|r|^2$. Anyway, we can perform the integration analytically, and the result is

$$
H^{(U(1))}_{\text{pot}} \simeq \frac{N_c}{40 \pi^2 a} \left( \frac{1}{\rho_1^2} + \frac{1}{\rho_2^2} \right) + \frac{N_c}{8 \pi^2 a |r|^2} \left[ \frac{1}{2} + \frac{2(a_1 \cdot a_2)^2 - 1}{5} \left( \frac{\rho_2^2}{\rho_1^2} + \frac{\rho_1^2}{\rho_2^2} \right) \right] + O(|r|^{-3}).
$$

(3.18)
3.3. **Kinetic term**

The kinetic term of the quantum mechanics of the two-baryon states is given by

\[
\frac{m}{2} g_{\alpha\beta} \dot{X}^\alpha \dot{X}^\beta, \tag{3.19}
\]

where \( X^\alpha = (X^M_i, y^I_i) \) \((\alpha = 1, 2, \ldots, 16)\) are the coordinates of the two-instanton moduli space that are promoted to time-dependent variables and \( \dot{X}^\alpha = \frac{d}{dt} X^\alpha \). \( g_{\alpha\beta} \) is the metric of the two-instanton moduli space.

The line element of the two-instanton moduli space for large separation is obtained in Ref. [36] as

\[
ds^2 = ds_0^2 + ds_1^2 + \mathcal{O}(|r|^{-3}) \tag{3.20}
\]

where

\[
ds_0^2 = (dX_1 \cdot dX_1) + (dX_2 \cdot dX_2) + 2(dy_1 \cdot dy_1) + 2(dy_2 \cdot dy_2),
\]

\[
ds_1^2 = \frac{2}{|r|^2} \left[ \rho_1^2(dy_1 \cdot dy_1) + \rho_2^2(dy_2 \cdot dy_2) + 2(y_1 \cdot dy_1)(y_2 \cdot dy_2) - (y_2 \cdot dy_1)^2 - (y_1 \cdot dy_2)^2 - 2(y_1 \cdot y_2)(dy_1 \cdot dy_2) \right. \\
\left. + 2 \epsilon_{IJKL} y^I_1 y^J_2 dy^K_1 dy^L_2 - ((y_2 \cdot dy_1) - (y_1 \cdot dy_2))^2 \right]. \tag{3.21}
\]

The leading terms \( ds_0^2 \) in (3-21) is just a sum of the metric for each instanton, which gives the canonical kinetic term (2.13). The next-to-leading terms \( ds_1^2 \) contribute to the potential of order \( 1/|r|^2 \).

The kinetic term of the Hamiltonian is given by the Laplacian \( \nabla^2 \) of the two-instanton moduli space as

\[
H_{\text{kin}} = -\frac{1}{2m_X} \nabla^2. \tag{3.22}
\]

The outline of the calculation is summarized in Appendix D. The result is

\[
\nabla^2 = \nabla_0^2 + \nabla_1^2 + \mathcal{O}(|r|^{-3}), \tag{3.23}
\]

where

\[
\nabla_0^2 = \left( \frac{\partial}{\partial X^M_i} \right)^2 + \left( \frac{\partial}{\partial X^M_j} \right)^2 + \frac{1}{2} \left( \frac{\partial}{\partial y^I_1} \right)^2 + \frac{1}{2} \left( \frac{\partial}{\partial y^I_2} \right)^2, \tag{3.24}
\]

\[
\nabla_1^2 = -\frac{1}{|r|^2} \left[ \frac{\rho_1^2}{2} \left( \frac{\partial}{\partial y^I_1} \right)^2 + \frac{\rho_2^2}{2} \left( \frac{\partial}{\partial y^I_2} \right)^2 - (y^I_1 \frac{\partial}{\partial y^I_1})^2 - (y^I_2 \frac{\partial}{\partial y^I_2})^2 + y^I_1 \frac{\partial}{\partial y^I_1} + y^I_2 \frac{\partial}{\partial y^I_2} \\
+ \epsilon_{IJKL} y^I_1 y^J_2 \frac{\partial}{\partial y^K_1} \frac{\partial}{\partial y^L_2} + (y^I_1 y^J_2 - (y_1 \cdot y_2) \delta^{IJ}) \frac{\partial}{\partial y^I_1} \frac{\partial}{\partial y^J_2} \right], \tag{3.25}
\]

16
Using the spin/isospin operators (2.19), it can also be written as
\[ \nabla^2_0 = \left( \frac{\partial}{\partial X_1} \right)^2 + \left( \frac{\partial}{\partial X_2} \right)^2 + \frac{1}{2} \sum_{i=1,2} \left[ \frac{1}{\rho_i^3} \frac{\partial}{\partial \rho_i} \left( \rho_i^3 \frac{\partial}{\partial \rho_i} \right) - \frac{4}{\rho_i^2} \rho_i^2 I_i^a I_i^a \right], \]  
(3.26)

\[ \nabla^2_1 = -\frac{1}{|r|^2} \left[ \frac{\rho_1^2}{\rho_1^2 - \rho_2^2} \left( \frac{1}{\rho_1^2} \frac{\partial}{\partial \rho_1} \left( \rho_1^3 \frac{\partial}{\partial \rho_1} \right) - \frac{4}{\rho_1^2} \rho_1^2 I_1^a I_1^a \right) \right. 
+ \left. \frac{\rho_2^2}{\rho_1^2 - \rho_2^2} \left( \frac{1}{\rho_2^2} \frac{\partial}{\partial \rho_2} \left( \rho_2^3 \frac{\partial}{\partial \rho_2} \right) - \frac{4}{\rho_2^2} \rho_2^2 I_2^a I_2^a \right) \right. 
+ \left. 4I_1^a I_2^a + \rho_1 \rho_2 \frac{\partial}{\partial \rho_1} \frac{\partial}{\partial \rho_2} + \rho_1 \frac{\partial}{\partial \rho_1} + \rho_2 \frac{\partial}{\partial \rho_2} - \left( \frac{y_1^a}{\rho_1^2} \frac{\partial}{\partial \rho_1} \right)^2 - \left( \frac{y_2^a}{\rho_2^2} \frac{\partial}{\partial \rho_2} \right)^2 \right]. \]  
(3.27)

Here, \( a, b, c = 1, 2, 3 \) are the indices for the \( SU(2) \) adjoint representation and the subscripts \( i = 1, 2 \) label the two instantons.

3.4. Summary

By collecting (3.3), (3.18), and (3.22) with (3.25) or (3.27), the total Hamiltonian is obtained as
\[ H = H_0 + H_1. \]  
(3.28)

The leading order Hamiltonian \( H_0 \) is just two copies of the Hamiltonian for one baryon (2.17) obtained in Ref. 18
\[ H_0 = \sum_{i=1,2} \left[ -\frac{1}{2m_X} \left( \frac{\partial}{\partial X_i} \right)^2 + \frac{1}{2m_y} \left( \frac{\partial}{\partial y_i^a} \right)^2 + \frac{1}{2m_Z} \left( \frac{\partial}{\partial Z_i} \right)^2 + U(\rho_i, Z_i) \right], \]  
(3.29)

where \( U(\rho_i, Z_i) \) is the potential given in (2.15).

The final term \( H_1 \) gives the \( \mathcal{O}(|r|^{-2}) \) interaction between the two baryons,
\[ H_1 = H^{(U(1))}_1 + H^{(SU(2))}_1 - \frac{1}{2m_X} \nabla^2_1, \]  
(3.30)

where \( m_X = 8\pi^2 a N_c \) and
\[ H^{(U(1))}_1 = \frac{N_c}{8\pi^2 a} \frac{1}{|r|^2} \left[ \frac{1}{2} + \frac{2(a_1 \cdot a_2)^2 - 1}{5} \left( \frac{\rho_1^2}{\rho_2^2} + \frac{\rho_2^2}{\rho_1^2} \right) \right], \]  
(3.31)

\[ H^{(SU(2))}_1 = \frac{4\pi^2 a N_c}{3} \rho_1^2 \rho_2^2 \frac{r_1 r_2}{|r|^4} \text{tr}(i\tau^a a_2^{-1} a_1) \text{tr}(i\tau^b a_2^{-1} a_1), \]  
(3.32)

and \( \nabla^2_1 \) is defined in (3.27).

§4. Nucleon-nucleon interaction

We are ready for the evaluation of the nucleon-nucleon interaction potential at short distances, using the quantum mechanical Hamiltonian (3.28) obtained in the previous section. This section is devoted to demonstrating these manipulations in detail.
We evaluate the interaction Hamiltonian $H_1$ with the nucleon wave function obtained in Ref. 18. The wavefunctions for consistent quantum states of two asymptotic nucleons are given in §4.1 and with the wavefunctions, the computation of the expectation value of $H_1$ follows in §4.2. Finally, the decomposition of $\langle H_1 \rangle$ into a central force and a tensor force is given in §4.3. Our final result for the nucleon-nucleon potential at short distances is (4.45) and (4.46).

4.1. Wavefunctions of two-nucleon states

We have two baryons, and the wavefunction for them is given by a tensor product of the wavefunctions (2.20) and (2.21). We can arrange (2.20) in a simple form,

$$\frac{1}{\pi} (\tau^2 a)_{IJ} = \begin{pmatrix} \left| p \uparrow \rightangle \left| p \downarrow \rightangle \\ \left| n \uparrow \rightangle \left| n \downarrow \rightangle \end{pmatrix}_{IJ},$$

(4.1)

where $\tau^2$ is the Pauli matrix $\tau^a$ with $a = 2$. Here, we specify the matrix element using the indices $I, J$ that take values in $\{\pm 1/2\}$. From (4.1), we see that $(I, J)$ component of the matrix $\tau^2 a$ is directly related to the wavefunction of the spin/isospin state with $(I^3, J^3) = (I, J)$. For the two nucleons specified by $(I^3_1, J^3_1, I^3_2, J^3_2)$, our wavefunction is

$$\frac{1}{\pi^2} (\tau^2 a_1)_{I_1J_1} (\tau^2 a_2)_{I_2J_2},$$

(4.2)

where we have omitted the upper index 3 in $I^3_i$ and $J^3_i$ for simplicity.

The wavefunction for the instanton size variable $\rho$ is given by a tensor product of (2.21), which is $R(\rho_1)R(\rho_2)$. To reduce the computational effort for the evaluation of the $\rho$ integral, we use the following simplified wavefunction instead of (2.21),

$$R(\rho_1)R(\rho_2), \quad \text{where} \quad R(\rho) \equiv \rho^I \exp \left[ -\rho^2 \right].$$

(4.3)

Hereafter, we mean $R(\rho)$ using this expression. This is obtained by a rescaling $(1/2)m_y \omega \rho \rho^2 \rightarrow \rho^2$. Note that, among the terms in $H_1$ (3.30), $H_1^{(U(1))}$ and $\nabla_1^2$ are invariant under the rescaling, so we can just replace the wavefunction by this simplified one. However, $H_1^{(SU(2))}$ has a scaling dimension that needs to be taken into account. The overall normalization of (4.3) will be taken care of later.

The wavefunction for the $Z$ modulus, for the nucleon states of $n_Z = 0$, is again a tensor product of (2.21) and given by

$$\psi_Z(Z_1)\psi_Z(Z_2).$$

(4.4)

This is, again, up to a normalization constant.
4.2. Spin/isospin matrix elements of interaction Hamiltonian

Let us evaluate the expectation value of the interaction Hamiltonian \( H_1 \), for given wavefunctions for bra and ket, \((4.2), (4.3), \) and \((4.4)\). Since the moduli \( Z_1 \) and \( Z_2 \) are decoupled from the rest, we treat them later independently. Here, first, we integrate the moduli \( y_i^I \) and \( y_i^J \). The integration measure is \( d^4y_1^I d^4y_2^J = \rho_1^3 \rho_2^3 d\rho_1 d\rho_2 d\Omega_1 d\Omega_2 \), where \( d\Omega_i \) is the integration over the angular coordinates \( a_i^I \) in \( SU(2) \sim S^3 \). The normalization is given by \( \int d\Omega_i = 2\pi^2 \), which is the volume unit \( S^3 \). For the integral \( d\Omega_i \), the following formulas for the integration over \( g \in SU(2) \) are useful:

\[
\int d\Omega \; \frac{1}{g_{mm} g_{pq}^{-1}} = \frac{\pi^2}{3} \left[ 2 \left( \delta_{il} \delta_{jq} \delta_{np} + \delta_{iq} \delta_{ml} \delta_{nk} - \delta_{jk} \delta_{np} \delta_{il} \right) \right], \\
\int d\Omega \; g_{ilk}^{-1} = \pi^2 \delta_{il} \delta_{jk}.
\]

\[\text{(4.5)}\]

Evaluation of \( H_1^{(U(1))} \)

Let us evaluate the expectation value of \( H_1^{(U(1))} \) \((3.31)\). In \( H_1^{(U(1))} \), the spatial coordinates \( X_i^M \) do not couple to \( y_i^I \), so it provides only a central force, and does not yield a tensor force.

As for the integration over the \( \rho \) variables, only the following three integrals appear:

\[
\int_0^\infty d\rho \; \rho^3 R(\rho)^2, \quad \int_0^\infty d\rho \; \rho^5 R(\rho)^2, \quad \int_0^\infty d\rho \; \rho^7 R(\rho)^2.
\]

\[\text{(4.6)}\]

On the other hand, to compute the expectation values, we need to include the normalization of the wavefunctions. Considering both, we only need the following ratios for our computation:

\[
\left( \int_0^\infty d\rho \; \rho^5 R(\rho)^2 \right) / \left( \int_0^\infty d\rho \; \rho^3 R(\rho)^2 \right) = 1 + \frac{l}{2}, \\
\left( \int_0^\infty d\rho \; \rho R(\rho)^2 \right) / \left( \int_0^\infty d\rho \; \rho^3 R(\rho)^2 \right) = \frac{2}{1 + l}.
\]

\[\text{(4.7)}\]

These can be derived by partial integrations. Using these, we obtain

\[
\left( \frac{\rho_2^2}{\rho_1^2} + \frac{\rho_1^2}{\rho_2^2} \right) = \frac{\int d\rho_1 d\rho_2 \; \rho_1^3 \rho_2^3 \left( \frac{\rho_1^2}{\rho_1^2} + \frac{\rho_2^2}{\rho_2^2} \right) R(\rho_1)^2 R(\rho_2)^2}{\int d\rho_1 d\rho_2 \; \rho_1^3 \rho_2^3 R(\rho_1)^2 R(\rho_2)^2} = \frac{2}{l + 1}.
\]

\[\text{(4.8)}\]

Note that we could use \((4.3)\) because \( \frac{\rho_2^2}{\rho_1^2} + \frac{\rho_1^2}{\rho_2^2} \) is scale-invariant.

Next, let us perform the \( a \) integration. In \( H_1^{(U(1))} \), the nontrivial term is only \( (a_1 \cdot a_2)^2 \), so we compute the matrix element of this. We first note

\[
(a_1 \cdot a_2)^2 = \frac{1}{4} \text{tr} \left[ a_1^\dagger a_2 \right] \text{tr} \left[ a_1 a_2^\dagger \right] = \frac{1}{4} (a_1^\dagger)_{KL} (a_1)_{MN} (a_2)_{LK} (a_2^\dagger)_{NM}.
\]

\[\text{(4.9)}\]
Thus, with the wavefunction \( |\psi\rangle \), the matrix element is
\[
\langle (a_1 \cdot a_2)^2 \rangle_{(l_1', l_1', l_2', l_2')} = \frac{1}{4\pi^3} \int d\Omega_1 \int d\Omega_2 \left( a_1^\dagger a_2^\dagger \tau a_2 a_1 \right)_{(l_1', l_1', l_2', l_2')} .
\]

Here, we have used \( (\tau^a a)^* = (a^\dagger \tau^a) a \). For the integral \( (4.10) \), we can use the formula \( (4.5) \) to obtain
\[
\langle (a_1 \cdot a_2)^2 \rangle_{(l_1', l_1', l_2', l_2')} = \frac{1}{4\pi^3} \left[ 5\delta_{l_1' l_1} \delta_{l_2' l_2} - \delta_{l_1' l_2} \delta_{l_1' l_1} \delta_{l_2' l_2} - \delta_{l_1' l_1} \delta_{l_2' l_2} \delta_{l_1' l_2} + 2\delta_{l_1' l_2} \delta_{l_1' l_1} \delta_{l_2' l_2} \delta_{l_2' l_2} \right] = \frac{1}{4} \frac{\tilde{r}_{l_1' l_1} \cdot \tilde{r}_{l_2' l_2}}{\delta_{l_1' l_2}} \delta_{l_2' l_2} .
\]

To obtain the last expression of \( (4.11) \), we have used the identity
\[
2\delta_{l_1' l_2} \delta_{l_1' l_1} = \delta_{l_1' l_2} + \tilde{r}_{l_1' l_1} \cdot \tilde{r}_{l_2' l_2} .
\]

Combining \( (4.11) \) with \( (4.8) \), we obtain the matrix element for \( H_1^{(U(1))} \) as
\[
\langle H_1^{(U(1))} \rangle_{(l_1', l_1', l_2', l_2')} = \frac{N_c}{8\pi^2a} \left[ \frac{3\tilde{r}}{10} \delta_{l_1' l_1} \delta_{l_2' l_2} + \frac{\tilde{r}_{l_1' l_1} \cdot \tilde{r}_{l_2' l_2}}{45} \delta_{l_1' l_2} \right] .
\]

The \( Z_1, Z_2 \) dependence in \(|r|^2\) will be integrated later, together with the other terms in \( H_1 \).

**Evaluation of \( H_1^{(SU(2))} \)**

Next, we consider \( (3.32) \). First we make the integration over \( \rho_1 \) and \( \rho_2 \). In the present case, the integral is not invariant under the scaling of \( \rho \) as opposed to the previous examples. This time, after the scaling \( (1/2)m_\gamma \Omega^2 \to \rho^2 \), we obtain the multiplicative new factor \( 4/(m_\gamma \Omega^2)^2 \). Therefore, using \( (4.7) \), we obtain
\[
\langle \rho_1^2 \rho_2^2 \rangle = \frac{4}{(m_\gamma \Omega^2)^2} \int d\rho_1 \rho_1^2 R(\rho_1)^2 \int d\rho_2 \rho_2^2 R(\rho_2)^2 = \frac{4}{(m_\gamma \Omega^2)^2} \left( 1 + \frac{\tilde{r}}{2} \right)^2 .
\]

Next, we consider the integration over \( a_1 \) and \( a_2 \). It is obvious that it proceeds in the same manner as \( H_1^{(U(1))} \). To use the formulas \( (4.5) \), we bring the relevant part of \( H_1^{(SU(2))} \) to the form
\[
\text{tr}(\tau^a a_2^{-1} a_1) \text{tr}(\tau^b a_2^{-1} a_1) = \text{tr}(\tau^a a_2^{-1} a_1) \text{tr}(\tau^b a_2^{-1} a_1) = \langle a_1 \rangle_{ij} \langle a_2 \rangle_{pq} \langle \tau^a \rangle_{1i} \langle a_1 \rangle_{j} \langle \tau^b \rangle_{p} \langle \tau^b \rangle_{q}. \]

\[20\]
Then, using \((4.15)\), we obtain

\[
\langle \text{tr}(i\sigma^a a_2^{-1} a_1) \text{ tr}(i\sigma^b a_2^{-1} a_1) \rangle_{(J_1', J_2', J_3')(I_1, J_1, J_2, J_3)} = \frac{1}{9} \left[ 2\delta_{ab}(4\delta_{I_1 I_1'}\delta_{I_2 I_2'}\delta_{J_1 J_1'}\delta_{J_2 J_2'} + \delta_{I_1 I_2}\delta_{I_2 I_1'}\delta_{J_1 J_1'}\delta_{J_2 J_2'}) + (\tau^a\tau^b)_{J_1 J_1'}(\delta_{I_1 I_1'}\delta_{I_2 I_2'}\delta_{J_1 J_1'} - 2\delta_{I_1 I_2}\delta_{I_2 I_1'}\delta_{J_1 J_1'}) + \frac{1}{2}(\tau^a_{J_1 J_1'}(\delta_{I_1 I_1'}\delta_{I_2 I_2'}\delta_{J_1 J_1'} - 2\delta_{I_1 I_2}\delta_{I_2 I_1'}\delta_{J_1 J_1'})) \right].
\]

To simplify this expression, we use \((4.12)\) and the Fierz transformation

\[
(\tau^a)_{J_1 J_1'}(\delta_{I_1 I_1'}\delta_{I_2 I_2'}\delta_{J_1 J_1'} - 2\delta_{I_1 I_2}\delta_{I_2 I_1'}\delta_{J_1 J_1'}) = \frac{1}{9}(\bar{\tau}_{I_1 I_1'} \cdot \bar{\tau}_{I_2 I_2'})(2(\bar{\tau} \cdot \bar{\tau})_{J_1 J_1'} - \delta_{ab}(\bar{\tau}_{J_1 J_1'} \cdot \bar{\tau}_{J_1 J_1'})).
\]

Then, \((4.16)\) becomes

\[
\langle \text{tr}(i\sigma^a a_2^{-1} a_1) \text{ tr}(i\sigma^b a_2^{-1} a_1) \rangle_{(J_1', J_2', J_3')(I_1, J_1, J_2, J_3)} = \delta_{ab}\delta_{I_1 I_1'}\delta_{I_2 I_2'}\delta_{J_1 J_1'}\delta_{J_2 J_2'} + \frac{1}{9}(\bar{\tau}_{I_1 I_1'} \cdot \bar{\tau}_{I_2 I_2'})(2(\bar{\tau} \cdot \bar{\tau})_{J_1 J_1'} - \delta_{ab}(\bar{\tau}_{J_1 J_1'} \cdot \bar{\tau}_{J_1 J_1'})).
\]

Combining \((4.14)\) and \((4.18)\), we arrive at

\[
\langle H_1^{(SU(2))} \rangle_{(I_1', J_1', J_2', J_3'),(I_1, J_1, J_2, J_3)} = \frac{(\hat{I} + 2)^2}{32\pi^2 a N_c} \left[ \delta_{I_1 I_1'}\delta_{I_2 I_2'}\delta_{J_1 J_1'}\delta_{J_2 J_2'} + \frac{1}{9}(\bar{\tau}_{I_1 I_1'} \cdot \bar{\tau}_{I_2 I_2'})(2(\bar{\tau} \cdot \bar{\tau})_{J_1 J_1'} - \delta_{ab}(\bar{\tau}_{J_1 J_1'} \cdot \bar{\tau}_{J_1 J_1'})) \right],
\]

where \(\hat{\tau} = \frac{\bar{\tau}}{|\bar{\tau}|}\).

**Evaluation of \(\nabla_1^2\)**

Finally, we evaluate \(\nabla_1^2\) in \(H_1\). In \((5.47)\), only the last two terms \((y J_1 \frac{\partial}{\partial y J_1})^2 + (y J_2 \frac{\partial}{\partial y J_2})^2\) have the angular dependence. Thus, as for the terms other than these, we only have to perform the \(\rho\) integral. For example, using

\[
\frac{1}{\rho^2} \frac{\partial}{\partial \rho} \left( \rho^2 \frac{\partial}{\partial \rho} R(\rho) \right) = \left( 4\rho^2 + (-8 - 4\hat{I}) + (\hat{I}^2 + 2\hat{I}) \frac{1}{\rho^2} \right) R(\rho),
\]

21
the ratio formulas (4.7) can be applied, and this results in

$$
\langle \nabla^2_{l_1}(l_1', l_1, l_2, j_2), (l_1, j_1, l_2, j_2) \rangle
= \frac{1}{|r|^2} \left( \frac{(\tilde{l} + 2) (\tilde{l} + 5)}{\tilde{l} + 1} \delta_{l_1, l_1'} \delta_{J_2, J_2} - \tilde{\tau}_{l_1 J_1} \cdot \tilde{\tau}_{l_2 J_2} \right) \delta_{l_1, l_1'} \delta_{J_2, J_2}
+ \frac{1}{|r|^2} \left[ \left( y_1' \frac{\partial}{\partial y_2} \right)^2 + \left( y_2' \frac{\partial}{\partial y_1} \right)^2 \right] \right)_{(l_1, j_1, l_1', J_1), (l_1, j_1, l_2, J_2)} .
$$

(4.21)

We have used \(4J_1^a J_2^a = \tilde{\tau}_{l_1 J_1} \cdot \tilde{\tau}_{l_2 J_2} \).

To perform the integral in the last term in (4.21), we first perform a partial integration

$$
\int d^4 y_1 \int d^4 y_2 \left( \psi' \right)^*_{(l_1', l_1, l_2, j_2)} \left( y_1' \frac{\partial}{\partial y_2} \right)^2 \psi_{(l_1, l_1, l_2, j_2)}
= - \int d^4 y_1 \int d^4 y_2 \left( y_1' \frac{\partial}{\partial y_2} \psi'_{(l_1, l_1, l_2, j_2)} \right)^* y_1' \frac{\partial}{\partial y_2} \psi_{(l_1, l_1, l_2, j_2)} .
$$

(4.22)

Using the wavefunctions (4.2) and (4.3), we find

$$
y_1^a \frac{\partial}{\partial y_1} \psi_{(l_1, l_1, l_2, j_2)} = \frac{1}{\pi^2} \left[ \left( \tilde{a}_1^2 \tilde{a}_2 \right)_{l_1, l_1} \left( \tilde{a}_1^2 \tilde{a}_2 \right)_{l_2, l_2} \frac{\partial}{\partial \rho_1} R(\rho_1) R(\rho_2)
+ \left( \tilde{a}_1^2 \tilde{a}_2 \right)_{l_1, l_1} \left( \tilde{a}_1^2 \tilde{a}_2 \right)_{l_2, l_2} \rho_1 \rho_2 \frac{\partial}{\partial \rho_1} \left( \frac{R(\rho_1)}{\rho_1} \right) R(\rho_2) \right] .
$$

(4.23)

Thus, in (4.22), the following integrals for \(\rho\) appear:

$$
\frac{\int \rho_1^3 d\rho_1 \int \rho_2^3 d\rho_2 \frac{\partial}{\partial \rho_1} R(\rho_1)^2 R(\rho_2)^2}{\int \rho_1^3 d\rho_1 \int \rho_2^3 d\rho_2 R(\rho_1)^2 R(\rho_2)^2} = \frac{\tilde{l} + 2}{\tilde{l} + 1} ,
$$

(4.24)

and

$$
\frac{\int \rho_1^3 d\rho_1 \int \rho_2^3 d\rho_2 \frac{\partial}{\partial \rho_1} R(\rho_1)^2 R(\rho_2)^2}{\int \rho_1^3 d\rho_1 \int \rho_2^3 d\rho_2 R(\rho_1)^2 R(\rho_2)^2} = \frac{\tilde{l} + 2}{\tilde{l} + 1} .
$$

(4.25)

Here, again, we have used (4.7). Then, straightforward calculations with the formulas (4.5) and (4.11) show that

$$
\left\langle \left[ \left( y_1' \frac{\partial}{\partial y_2} \right)^2 + \left( y_2' \frac{\partial}{\partial y_1} \right)^2 \right] \right\rangle_{(l_1, j_1, l_1', J_1), (l_1, j_1, l_2, J_2)}
= - \left( \frac{\tilde{l} + 2}{\tilde{l} + 5} \right) \left\{ \frac{1}{2} \delta_{l_1, l_1'} \delta_{J_2, J_2} \delta_{l_2, l_2} \delta_{J_2, J_2} + \frac{1}{18} (\tilde{\tau}_{l_1 J_1} \cdot \tilde{\tau}_{l_2 J_2})(\tilde{\tau}_{l_1 J_1} \cdot \tilde{\tau}_{l_2 J_2}) \right\} .
$$

(4.27)
As a result, the matrix elements of the third term in (3.30) are obtained from (4.21) and (4.22) as

\[
\langle H_3 \rangle_{(I_1', J_1', I_2', J_2'), (I_1, J_1, I_2, J_2)} = \frac{1}{2m_X} \langle \nabla^2 \rangle_{(I_1', J_1', I_2', J_2'), (I_1, J_1, I_2, J_2)} = \frac{1}{32\pi^2 a N_c |\vec{r}|^2} \left[ 2\langle \vec{r}_{I_1'} \cdot \vec{r}_{I_2'} \rangle \delta_{I_1' J_1} \delta_{I_2' J_2} - \frac{(\vec{r}_{I_1'} \cdot \vec{r}_{I_2'})}{\vec{r}_1' \cdot \vec{r}_2'} \langle \vec{r}_{I_1 J_1} \cdot \vec{r}_{I_2 J_2} \rangle \right].
\]

Since \( \tilde{l} \sim \mathcal{O}(N_c) \), this term is \( \mathcal{O}(1) \) in the \( 1/N_c \) expansion, while the other terms (4.13) and (4.19) are \( \mathcal{O}(N_c) \). Therefore, this term is subleading, compared with the contributions from \( H_1^{(U(1))} \) and \( H_1^{(SU(2))} \).

Summary of the result

Summing up all the results (4.13), (4.19), and (4.28), we have the spin/isospin matrix elements of the interaction Hamiltonian

\[
\langle H_1 \rangle_{(I_1', J_1', I_2', J_2'), (I_1, J_1, I_2, J_2)} = c_1 \frac{1}{|\vec{r}|^2} + c_2 \frac{\vec{r}_1'^2}{|\vec{r}|^4},
\]

where the coefficients are given by

\[
c_1 = \frac{N_c}{8\pi^2 a} \left[ \frac{3\tilde{l} + 1}{10} \frac{1}{l + 1} \delta_{I_1' I_1} \delta_{I_2' I_2} \delta_{J_1' J_1} \delta_{J_2' J_2} + \frac{\tilde{l} + 2}{45 l + 1} (\vec{r}_{I_1'} \cdot \vec{r}_{I_2'}) \langle \vec{r}_{I_1 J_1} \cdot \vec{r}_{I_2 J_2} \rangle \right] + \frac{1}{32\pi^2 a N_c} \left[ 2\langle \vec{r}_{I_1'} \cdot \vec{r}_{I_2'} \rangle \delta_{I_1' J_1} \delta_{I_2' J_2} - \frac{(\vec{r}_{I_1'} \cdot \vec{r}_{I_2'})}{\vec{r}_1' \cdot \vec{r}_2'} \langle \vec{r}_{I_1 J_1} \cdot \vec{r}_{I_2 J_2} \rangle \right],
\]

\[
c_2 = \frac{(\tilde{l} + 2)^2}{32\pi^2 a N_c} \left[ \delta_{I_1' I_1} \delta_{I_2' I_2} \delta_{J_1' J_1} \delta_{J_2' J_2} + \frac{1}{9} (\vec{r}_{I_1'} \cdot \vec{r}_{I_2'}) \left( 2(\vec{r} \cdot \vec{r})_{I_1' J_1} (\vec{r} \cdot \vec{r})_{I_2' J_2} - \vec{r}_{I_1' J_1} \cdot \vec{r}_{I_2' J_2} \right) \right].
\]

4.3. Final result for nuclear force at short distance

We finally evaluate this Hamiltonian (4.29) with the wavefunction for \( Z_i, \) (4.3), and take the leading term in the large \( N_c \) expansion.

In our result (4.29), the \( Z \) dependence is included in the four-dimensional distance, \(|\vec{r}| = \sqrt{\vec{r}_1'^2 + (Z_1 - Z_2)^2}\). We fix the baryon distance \(|\vec{r}|\) in the real space, and perform the
integration over $Z_1$ and $Z_2$. To perform the integration, we rewrite the wavefunction (4.31) as
\[
\exp \left[ -\frac{1}{2} m_Z \omega_Z \((Z_1)^2 + (Z_2)^2) \right] = \exp \left[ -m_Z \omega_Z \((Z_1 - Z_2)/2)^2 + ((Z_1 + Z_2)/2)^2) \right].
\]
(4.32)

Since (4.29) consists of two terms, $1/|r|^2$ and $1/|r|^4$, we use the following formula for integration,
\[
\int_{-\infty}^{\infty} dx \, e^{-\beta x^2 + \alpha^2} = \frac{\pi e^{\alpha^2 \beta^2 (1 - \text{Err}[\alpha])}}{\alpha},
\]
(4.33)
\[
\int_{-\infty}^{\infty} dx \, e^{-\beta x^2 + \alpha^2} = \frac{2\sqrt{\pi} \alpha \beta + \pi (1 - 2\alpha^2 \beta^2) e^{\alpha^2 \beta^2 (1 - \text{Err}[\alpha])}}{2\alpha^3}.
\]
(4.34)

Here Err is the error function,
\[
\text{Err}[x] \equiv \frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{-t^2} \, dt.
\]
(4.35)

Then we obtain
\[
\left\langle \frac{1}{|r|^2} \right\rangle_Z = \sqrt{\pi} \beta e^{2|\vec{r}|^2/4} (1 - \text{Err}[\beta|\vec{r}|/2]) / 2|\vec{r}|,
\]
(4.36)
\[
\left\langle \frac{1}{|r|^4} \right\rangle_Z = \beta |\vec{r}| + \sqrt{\pi} (1 - \beta^2 |\vec{r}|^2/2) e^{2|\vec{r}|^2/4} (1 - \text{Err}[\beta|\vec{r}|/2]) / 4|\vec{r}|^3,
\]
(4.37)

where $\beta = \sqrt{2m_Z \omega_Z} = (16(2/3)^{1/2} \pi^2 a_{N_c})^{1/2}$.

For large $N_c$, the argument of the error function, $\beta|\vec{r}|$ is very large for nonzero $|\vec{r}|$. We can use the following asymptotic formula for the error function,
\[
1 - \text{Err}(x) = \frac{1}{\sqrt{\pi}} e^{-x^2} \left[ \frac{1}{x} - \frac{1}{2x^3} + O(x^{-4}) \right].
\]
(4.38)

By using this expression, the $Z$-integral results are markedly simplified,
\[
\left\langle \frac{1}{|r|^2} \right\rangle_Z = \frac{1}{|\vec{r}|^2} + O(1/N_c), \quad \left\langle \frac{1}{|r|^4} \right\rangle_Z = \frac{1}{|\vec{r}|^4} + O(1/N_c).
\]
(4.39)

This expression can be easily guessed since this is just a substitution of $Z_1 = Z_2 = 0$. The value $Z_1 = Z_2 = 0$ is the classical value of the location of the instanton in the $z$ space, and the large $N_c$ limit should reproduce the classical result.

On the other hand, the coefficients $c_1$ and $c_2$ in (4.29) are evaluated in the large $N_c$ expansion as
\[
c_1 = \pi N_c \left( \frac{81}{10} \delta_{\vec{I}_1 \vec{I}_2} \delta_{\vec{J}_1 \vec{J}_2} \delta_{\vec{I}_1 \vec{J}_2} + \frac{3}{5} (\vec{r}_{\vec{I}_1 \vec{J}_1} \cdot \vec{r}_{\vec{I}_1 \vec{J}_2}) (\vec{r}_{\vec{J}_1 \vec{J}_1} \cdot \vec{r}_{\vec{J}_1 \vec{J}_2}) \right) + O(1),
\]
(4.40)
Here, for deriving these, we used \( \tilde{r}^2 = (4/5)N_c^2 + \mathcal{O}(1) \). It is interesting that in fact all the contributions from \( \nabla^2 \tilde{r} \) drop off, since they are subleading compared with the terms from \( H_1^{(U(1))} \) and \( H_1^{SU(2)} \) in this large \( N_c \) expansion.

Therefore, the leading term of the interaction Hamiltonian is

\[
\langle H_1 \rangle_{(J_1',J_2',J_3'),(J_1,J_2,J_3)} = \frac{\pi N_c}{\lambda |\tilde{r}|^2} \left( \frac{27}{2} \delta_{J_1'J_1} \delta_{J_2'J_2} \delta_{J_3'J_3} + \frac{6}{5} (\tilde{r}_{12} \cdot \tilde{r}_{12}) (\tilde{r}_{12} \cdot \tilde{r}_{12}) (\tilde{r}_{12} \cdot \tilde{r}_{12}) \right) \, \mathcal{O}(1). \tag{4.41}
\]

We rescaled \( \tilde{r} \) back to the original coordinate (remember the rescaling (2.5)). This is the nucleon-nucleon potential at a short distance in the large \( N_c \) limit.

Let us decompose this force into a central force \( V_C(|\tilde{r}|) \) and a tensor force \( V_T(|\tilde{r}|) \),

\[
V = V_C(|\tilde{r}|) + S_{12} V_T(|\tilde{r}|) \, . \tag{4.42}
\]

Here, the tensor operator \( S_{12} \) is defined by

\[
S_{12} \equiv 3(\tilde{\sigma}_1 \cdot \tilde{r})(\tilde{\sigma}_2 \cdot \tilde{r}) - \tilde{\sigma}_1 \cdot \tilde{\sigma}_2 = 12(\tilde{J}_1 \cdot \tilde{r})(\tilde{J}_2 \cdot \tilde{r}) - 4 \tilde{J}_1 \cdot \tilde{J}_2 \, , \tag{4.43}
\]

where \( \tilde{\sigma}_i = (\sigma^1_i, \sigma^2_i, \sigma^3_i) = 2\hat{J}_i \) are the Pauli-spin operators (it is just twice the spin operator).

Applying the decomposition (4.43) to our result (4.42), we obtain

\[
V_C(|\tilde{r}|) = \pi \left( \frac{27}{2} + \frac{32}{5} (I_1^a I_2^a)(J_1^b J_2^b) \right) \frac{N_c}{\lambda |\tilde{r}|^2} \, , \tag{4.44}
\]

\[
V_T(|\tilde{r}|) = \frac{8\pi}{5} I_1^a I_2^a N_c \frac{1}{\lambda |\tilde{r}|^2} \, . \tag{4.45}
\]

We have derived the nuclear force at a short distance. As already mentioned, the expression is valid in the region (2.28). We have found that there is a strong repulsive core in the central force (4.43). Our finding is quite important, as an analytic computation of the repulsive core of the nuclear force, based on strongly coupled QCD.

The tensor force (4.46) is found to be negative for \( I = 0 \) (since \( I_1^a I_2^a = -3/4 \)), which is consistent with the experimental observation and also with lattice QCD calculations. It is intriguing that the nuclear force has \( |\tilde{r}|^{-2} \) dependence. This \( |\tilde{r}|^{-2} \) is peculiar to the physics with one extra spatial dimension, and thus it is a direct consequence of the holographic approach to QCD. It would be interesting to fit the lattice and experimental data of the nuclear force with our result.
§5. Comparison with one-boson-exchange potential

In this section, for a comparison, we compute the nucleon-nucleon potential using the one-boson-exchange picture. This can be done by integrating the meson propagator with the nucleon-nucleon-meson coupling obtained in our previous paper\textsuperscript{8} in the D4-D8 model in the holographic QCD.\textsuperscript{3} We find some discrepancy between the two pictures. The reason for this discrepancy is basically the fact that in the one-boson-exchange picture the deformation of the nucleon by the other nucleon has not been taken into account, while our potential obtained in §4 includes this effect via the ADHM construction in the previous section. In sum, we find that the one-boson-exchange potential captures merely a part of the nucleon-nucleon potential.

5.1. Interaction potential

In Ref.\textsuperscript{8}, the nucleon-nucleon-meson couplings were computed, by extracting the asymptotic behavior of the one-baryon solution.\textsuperscript{∗∗} The one-boson-exchange potential is obtained by just evaluating the energy of a superposition of two asymptotic baryons, which are regarded as point particles sourcing the meson fields propagating between them. The distance $r$ should be larger than the instanton size $\rho$, of course, because we use the asymptotic form of the solution (which was given in §2.3 in Ref.\textsuperscript{8}) as a baryon solution.

Before getting into the details of the evaluation of the potential energy of our system, it is instructive to consider a simple example of a two-electron system in classical electrodynamics with the action

$$S = \int d^4 x \left( -\frac{1}{4} F_{\mu\nu}^2 - j^\mu A_\mu \right), \quad (5.1)$$

where $j_\mu$ is a current of an external source. Here, we work in the Lorenz gauge $\partial_\mu A^\mu = 0$ and consider a static configuration. Then, the equation of motion $\Delta A^\mu = j^\mu$ is solved by

$$A^\mu(\vec{x}) = (\Delta^{-1} j^\mu)(\vec{x}) \equiv \int d^3 y \Delta^{-1}(\vec{x},\vec{y}) j^\mu(\vec{y}), \quad (5.2)$$

where $\Delta^{-1}(\vec{x},\vec{y}) \equiv \frac{1}{4\pi |\vec{x}-\vec{y}|}$ is the Green’s function for the Laplacian $\Delta$. The on-shell action is evaluated as

$$S = -\frac{1}{2} \int d^4 x A^\mu A_\mu = -\frac{1}{2} \int d^4 x j^\mu A_\mu = -\frac{1}{2} \int d^4 x j^\mu \Delta^{-1} j_\mu. \quad (5.3)$$

\textsuperscript{*)} In §5 only, we use the upper (or lower) index “(1)” and “(2)” to label the gauge fields and the currents for the two instantons, and we do not use the rescaled coordinates (2,5). This is for making use of the notation of Ref.\textsuperscript{8}.

\textsuperscript{**) In Refs.\textsuperscript{[19, 20]}, the couplings were computed by a different approach, using baryon spinor fields in the bulk curved space-time.
Let us consider the current associated with two electrons placed at $\vec{X}_i$ ($i = 1, 2$):

$$j_\mu = j^{(1)}_\mu + j^{(2)}_\mu ,$$

$$j^{(i)}_\mu (\vec{x}) = e \delta^3 (\vec{x} - \vec{X}_i) , \quad j^{(i)}_1 = j^{(i)}_2 = j^{(i)}_3 = 0 . \quad (i = 1, 2) \quad (5.4)$$

Then, the solution of the equation of motion is given by

$$A_0(x) = A^{(1)}_0(x) + A^{(2)}_0(x) , \quad A_1 = A_2 = A_3 = 0 ,$$

$$A^{(i)}_\mu = e \Delta^{-1} (\vec{x}, \vec{X}_i) . \quad (i = 1, 2) \quad (5.5)$$

Substituting this in the on-shell action (5.3) and picking up the cross term, we obtain the potential due to the interaction of the two sources as

$$V = \int d^3 x A^{(1)}_\mu \Delta A^{(2)\mu} = \int d^3 x A^{(1)}_\mu j^{(2)\mu} = - e^2 \Delta^{-1} (\vec{X}_2, \vec{X}_1) .$$

(5.6)

Let us follow this line of argument for our action (2.1) in the curved space-time. It was shown in Ref. [8] that nonlinear terms in the equations of motion can be neglected in the asymptotic region. Therefore, we are allowed to consider the linearized equations of motion in the curved space-time:

$$h(z) \partial^2_{\mu} A_\mu + \partial_z (k(z) \partial_z A_\mu) = 0 , \quad \partial^2_{\mu} A_\mu + \partial_z (h(z)^{-1} \partial_z (k(z) A_\mu)) = 0 ,$$

with the gauge condition

$$h(z) \partial^\mu A_\mu + \partial_z (k(z) A_\mu) = 0 .$$

(5.7)

(5.8)

The potential energy analogous to (5.6) is

$$V = 2 \kappa \int d^3 x dz \text{tr} \left[ - A^{(1)}_0 j^{(2)}_0 + A^{(1)}_i j^{(2)}_i + A^{(1)}_z j^{(2)}_z \right] ,$$

(5.9)

where the “current” $j^{(2)}_\alpha$ is defined as

$$j^{(2)}_0 = (h(z) \partial_\mu \partial_\mu + \partial_z k(z) \partial_z) A^{(2)}_0 ,$$

$$j^{(2)}_i = (h(z) \partial_\mu \partial_\mu + \partial_z k(z) \partial_z) A^{(2)}_i ,$$

$$j^{(2)}_z = k(z) \left( \partial_\mu \partial_\mu + \partial_z h(z)^{-1} \partial_z k(z) \right) A^{(2)}_z .$$

(5.10)

Here, $A^{(i)}_\mu$ ($i = 1, 2$) are the asymptotic solutions for a single baryon located at $x^M = X^M_i$ that satisfy the linearized equations of motion (5.7) and the gauge condition (5.8). They are explicitly obtained in Ref. [8]. The “current” (5.10) behaves as the pointlike source placed at $X^M_2$, which corresponds to the delta function source $j^{(2)}_\mu$ in (5.4) in the previous example.
Of course, the asymptotic solutions can only be trusted in the asymptotic region, since the nonlinear terms in the equations of motion will become important near the position of the instanton $X_i^M$. However, here, we simply assume that the baryons can be treated as point particles to obtain the one-boson-exchange potential.

For simplicity, we consider the static configuration. By noting that our focus is on the leading order behavior of the large $\lambda$ and large $N_c$ limit, the leading contribution to the currents turns out to come only from

$$\hat{A}_0, A_i, A_z \ , \ (5.11)$$

which are of order $O(\lambda^{-1})$ while the next-to-leading order is by the component $A_0$, which is of order $O(\lambda^{-1}N_c^{-1})$. All the other components are of order $O(\lambda^{-2}N_c^{-1})$. Thus, let us consider only the leading order components.

The solutions and the “current” are

$$\hat{A}_0^{(1)} = -\frac{1}{2\alpha\lambda} G(\bar{x}, z; \bar{X}_1, Z_1) \ ,$$

$$\hat{J}_0^{(2)} = -\frac{1}{2\alpha\lambda} \delta^3(\bar{x} - \bar{X}_2)\delta(z - Z_2) \ , \ (5.12)$$

$$A_i^{(1)b} = -2\pi^2(\rho_1)^2 \text{tr} \left( \tau^b a_1 \tau^a(a_1)^{-1} \right) \left( \epsilon_{ij} \frac{\partial}{\partial X_1^i} - \delta_{ia} \frac{\partial}{\partial Z_1} \right) G(\bar{x}, z; \bar{X}_1, Z_1) \ ,$$

$$j_i^{(2)b} = -2\pi^2(\rho_2)^2 \text{tr} \left( \tau^b a_2 \tau^a(a_2)^{-1} \right) \left( \epsilon_{ij} \frac{\partial}{\partial X_2^i} - \delta_{ia} \frac{\partial}{\partial Z_2} \right) \delta^3(\bar{x} - \bar{X}_2)\delta(z - Z_2) \ , \ (5.13)$$

$$A_z^{(1)b} = -2\pi^2(\rho_1)^2 \text{tr} \left( \tau^b a_1 \tau^a(a_1)^{-1} \right) \frac{\partial}{\partial X_1^a} H(\bar{x}, z; \bar{X}_1, Z_1) \ ,$$

$$j_z^{(2)b} = -2\pi^2(\rho_2)^2 \text{tr} \left( \tau^b a_2 \tau^a(a_2)^{-1} \right) \frac{\partial}{\partial X_2^a} \delta^3(\bar{x} - \bar{X}_2)\delta(z - Z_2) \ . \ (5.14)$$

Note that as in Ref. [8], these expressions are written in terms of the original variables without the rescaling [25], keeping them in the order of $O(1)$. Here, $G$ and $H$ are the Green’s functions associated with the linearized equations of motion [5,7]. They are obtained in Ref. [8] as

$$G = \kappa \sum_{n=1}^{\infty} \psi_n(Z_2)\psi_n(Z_1)Y_n(|\vec{r}|) \ , \ H = \kappa \sum_{n=0}^{\infty} \phi_n(Z_2)\phi_n(Z_1)Y_n(|\vec{r}|) \ , \ (5.15)$$

where $\{\psi_n\}_{n=1,2,\ldots}$ is a complete set of the eigenfunctions satisfying the eigenequation

$$-h(z)^{-1}\partial_z(k(z)\partial_z\psi_n) = \lambda_n\psi_n \ , \ (5.16)$$

* This amounts to throwing away momentum-dependent nuclear force such as $L \cdot S$ force.
with eigenvalue $\lambda_n$ and the normalization condition
\[ \kappa \int dz \, h(z) \psi_n \psi_m = \delta_{nm} , \] (5.17)
and \( \{ \phi_n \}_{n=0,1,\ldots} \) is defined as
\[ \phi_n(z) = \frac{1}{\sqrt{\lambda_n}} \partial_z \psi_n(z) , \quad \phi_0(z) = \frac{1}{\sqrt{\kappa \pi}} k(z) . \] (5.18)
The eigenfunction $\psi_n(z)$ is an even (odd) function for odd (even) $n$. When the five-dimensional gauge field is expanded by \( \{ \psi_n(z) \} \), the coefficient fields are interpreted as the vector (for odd $n$) and axial-vector (for even $n$) meson fields, and the eigenvalues $\lambda_n$ are proportional to the mass squared of the corresponding mesons. The function $Y_n(|\vec{r}|)$ is the Yukawa potential associated with the vector/axial-vector meson of mass $\sqrt{\lambda_n}$:
\[ Y_n(|\vec{r}|) = -\frac{1}{4\pi} e^{-\sqrt{\lambda_n}|\vec{r}|} . \] (5.19)

We substitute these expressions to the interaction potential (5.9) to obtain
\[ V = \kappa \left[ -\frac{1}{4a^2 \lambda^2} G(\vec{X}_2, Z_2; \vec{X}_1, Z_1) 
+ 4\pi^4 (\rho_1)^2 (\rho_2)^2 \text{tr} (\tau^b a_1 \tau^a (a_1)^{-1}) \text{tr} (\tau^b a_2 \tau^c (a_2)^{-1}) \times \left( \epsilon^{iaj} \frac{\partial}{\partial X^j_1} - \delta^{ia} \frac{\partial}{\partial Z_1} \right) \right] . \] (5.20)

This is the inter-baryon potential energy that we want to evaluate as the nuclear force, in the one-boson-exchange approximation.

### 5.2. Short distance behavior

When the distance between the solitons is smaller than $1/M_{KK}$, the Green’s functions $G$ and $H$ can be approximated by their flat-space analogue as explained in our previous paper:
\[ G = H = -\frac{1}{4\pi^2} \frac{1}{|\vec{X}_1 - \vec{X}_2|^2 + (Z_1 - Z_2)^2} . \] (5.21)
Substituting this expression to the inter-instanton potential energy (5.20), we can easily see that only the first term in (5.20) remains, while the other terms cancel each other and vanish,
\[ V = \kappa \frac{1}{16\pi^2 a^2 \lambda^2} \frac{1}{|\vec{X}_1 - \vec{X}_2|^2 + (Z_1 - Z_2)^2} . \] (5.22)
This is a harmonic potential in four-dimensional space. Once the \( z \)-part of the wavefunction for the nucleon, \( \langle 4.4 \rangle \), is taken into account, the expectation value is given just by substituting the classical value \( Z_1 = Z_2 = 0 \) to the above potential (5.22),

\[
V = \frac{27\pi N_c}{2} \frac{1}{\lambda |\vec{r}|^2}.
\]  

(5.23)

Here, \( |\vec{r}| \) is the distance between the nucleons. We find that only the central force appears.

This expression (5.23) is apparently different from our previous results (4.45) and (4.46). In fact, in (5.23), there is no isospin dependence, and no tensor force. The reason is that the one-boson-exchange description is not sufficient to capture the entire nuclear force. In fact, the two-instanton solution in ADHM construction is not just a superposition of two BPST instantons. In particular, there is a deformation of the instanton that causes additional contribution. The one-boson-exchange potential (5.23) only captures a part of the complete results (4.45) and (4.46) obtained in the previous subsection.

In Appendix F, we try to evaluate the potential height for nucleons overlapping each other in real space, in the one-boson-exchange picture.

5.3. Large distance behavior

Let us look at the large distance behavior of the nucleon-nucleon potential obtained from (5.20). The large distance means \( |X_1 - X_2| \gg 1 \) in the unit \( M_{KK} = 1 \). In this limit, essentially only the pion dominates, since only the pion is the zero mode while others have Yukawa potential that decays exponentially fast.

The contribution of the pion comes from \( n = 0 \) component of the Green’s function \( H \) defined in (5.15). Therefore, in the potential (5.20), we are interested in the last term, and the function \( H \) is now approximated by a massless Green’s function in three dimensions:

\[
V \simeq \kappa \cdot 4\pi^4 \langle \rho_1^2 \rangle \langle \rho_2^2 \rangle \text{tr} \left( \tau^b a_1 \tau^a (a_1)^{-1} \right) \text{tr} \left( \tau^b a_2 \tau^c (a_2)^{-1} \right) \\
\left. \times \frac{\partial}{\partial X^a_1} \frac{\partial}{\partial X^b_2} \kappa \phi_0(Z_2) \phi_0(Z_1) \right|_{-1} \frac{1}{4\pi |X_1 - X_2|}. 
\]  

(5.24)

For spin 1/2 baryons, the trace part can be easily evaluated as

\[
\text{tr} \left( \tau^b a_1 \tau^a (a_1)^{-1} \right) \text{tr} \left( \tau^b a_2 \tau^c (a_2)^{-1} \right) = \frac{64}{9} (I_1^b I_2^b) J_1^a J_2^c, 
\]  

(5.25)

where \( J_i \) and \( I_i \) are spin and isospin operators, respectively. The potential energy is then

\[
V \simeq -\kappa \pi^2 \left( \frac{\langle \rho_1^2 \rangle}{k(Z_1)} \right) \left( \frac{\langle \rho_2^2 \rangle}{k(Z_2)} \right) \frac{64}{9} (I_1^b I_2^b) J_1^a J_2^c \frac{\partial}{\partial X^a_1} \frac{\partial}{\partial X^b_2} \frac{1}{|X_1 - X_2|} \\
= \frac{16\kappa \pi^2}{9} \left( \frac{\langle \rho_1^2 \rangle}{k(Z_1)} \right) \left( \frac{\langle \rho_2^2 \rangle}{k(Z_2)} \right) (I_1^b I_2^b) S_{12} \left. \frac{1}{|\vec{r}|^3} \right|, 
\]  

(5.26)
where $S_{12}$ is defined in (4.44). Here, we have used the relation
\[
\frac{\partial}{\partial X_1^a} \frac{\partial}{\partial X_2^c} \frac{1}{|X_1 - X_2|} = \left( \frac{\delta^{ac}}{3} - \frac{\bar{r}^a \bar{r}^c}{|\bar{r}|^3} \right) \frac{3}{|\bar{r}|^3}
\]  
(5.27)
with $\bar{r} \equiv X_1 - X_2$ and $\hat{\bar{r}} \equiv \bar{r}/|\bar{r}|$.

This expression can be compared with the well-known one-pion-exchange potential
\[
V^{(\pi)} = \frac{1}{\pi} \left( \frac{g_A}{f_{\pi}} \right)^2 \left( I_1^b I_2^b \right) J_1^a J_2^c \frac{\partial}{\partial r^a} \frac{\partial}{\partial r^c} e^{-m_\pi|\bar{r}|} \\
= \frac{1}{\pi} \left( \frac{g_A}{f_{\pi}} \right)^2 \left( I_1^b I_2^b \right) \left( \frac{m_\pi^2}{3} (J_1^a J_2^a) + \frac{S_{12}}{4} \left( \frac{m_\pi^2}{|\bar{r}|^2} + \frac{1}{|\bar{r}|^2} \right) \right) e^{-m_\pi|\bar{r}|}.  
\]  
(5.28)

In the chiral limit $m_\pi \to 0$, only the tensor force remains, and it agrees with (5.26) when
\[
\frac{g_A}{f_{\pi}} = \frac{8\pi \sqrt{\kappa \rho}}{3} \left\langle \frac{\rho^2}{3} \frac{1}{k(Z)} \right\rangle,
\]  
(5.29)
which is exactly the relation found in Ref. 8).

If we use the classical values for the above expectation values,
\[
\left\langle \frac{(\rho_1)^2}{k(Z_1)} \right\rangle_{(1)} = \left\langle \frac{(\rho_2)^2}{k(Z_2)} \right\rangle_{(2)} \approx \rho_{cl}^2 = \frac{1}{8\pi^2 a \lambda} \sqrt{\frac{6}{5}},
\]  
(5.30)
then the potential (5.26) becomes
\[
V \approx -\frac{2N_c}{15\pi^2 a \lambda} \left( I_1^b I_2^b \right) J_1^a J_2^c \frac{\partial}{\partial X_1^a} \frac{\partial}{\partial X_2^c} \frac{1}{|X_1 - X_2|} = \frac{N_c}{30\pi^2 a \lambda} \left( I_1^b I_2^b \right) S_{12} \frac{1}{|\bar{r}|^3}.  
\]  
(5.31)
In a quantum evaluation for the expectation values, we will have roughly $\times (1.05)^2$ times the classical value above, after substituting the numerical values.

5.4. Intermediate distance behavior

As the baryons approach each other from asymptotics, there appear effects of the massive meson exchange. At this intermediate distance, the potential (5.20) becomes
\[
V \approx \kappa^2 \left[ \frac{1}{4a^2 \lambda^2} \sum_{n=1, \text{odd}}^{\infty} \psi_n(Z_2) \psi_n(Z_1) Y_n(|\bar{r}|) \right.
+ \frac{256\pi^4}{9} (\rho_1)^2 (\rho_2)^2 \left( I_1^b I_2^b \right) J_1^a J_2^c
\times \left( \epsilon^{ijk} \epsilon^{ikl} \frac{\partial}{\partial X_1^i} \frac{\partial}{\partial X_2^l} + \delta^{ca} \frac{\partial}{\partial Z_1^a} \frac{\partial}{\partial Z_2^c} \right) \sum_{n=1}^{\infty} \psi_n(Z_2) \psi_n(Z_1) Y_n(|\bar{r}|) \\
+ \frac{256\pi^4}{9} (\rho_1)^2 (\rho_2)^2 \left( I_1^b I_2^b \right) J_1^a J_2^c \frac{\partial}{\partial X_1^a} \frac{\partial}{\partial X_2^c} \sum_{n=0, \text{even}}^{\infty} \phi_n(Z_2) \phi_n(Z_1) Y_n(|\bar{r}|) \right],
\]  
(5.32)
31
where we have used (5.15) and also dropped the terms including $(\partial/\partial X)(\partial/\partial Z)$, $\psi_{2k}(Z)$, and $\phi_{2k-1}(Z)$ with $k = 1, 2, \cdots$, because they vanish when they are evaluated with baryon wavefunction, owing to the parity property of the $\psi_n(Z)$ functions. The potential can be summarized to the following form:

$$V \simeq \kappa^2 \left[ \left( -\frac{1}{4a^2\lambda^2} + \frac{256\pi^4}{9}(\rho_1)^2(\rho_2)^2(1_{I_1}^b I_2^b)J_1^a J_2^c \varepsilon^{aie} \varepsilon^{jck} \frac{\partial}{\partial X_1^i} \frac{\partial}{\partial X_2^j} \right) \sum_{n=1, odd}^{\infty} \psi_n(Z_2)\psi_n(Z_1)Y_n(|\vec{r}|) \right. \right.
$$

$$\left. \left. + \frac{256\pi^4}{9}(\rho_1)^2(\rho_2)^2(1_{I_1}^b I_2^b)J_1^a J_2^c \sum_{n=0, even}^{\infty} \phi_n(Z_2)\phi_n(Z_1)Y_n(|\vec{r}|) + \sum_{n=2, even}^{\infty} \lambda_n \phi_n(Z_2)\phi_n(Z_1)Y_n(|\vec{r}|) \right]\right]. \quad (5.33)$$

The first term in the first line, $1/(4a^2\lambda^2)$, corresponds to the contribution of the vector mesons that appear from the $U(1)$ part of the gauge field. Among them, the $\omega$ meson exchange is at the lowest order ($n = 1$). The second term in the first line is the contribution of the vector meson in the $SU(2)$ part of the gauge field, whose lowest order ($n = 1$) corresponds to the $\rho$ meson. The third line gives the contribution of the pion ($n = 0$) and axial-vector mesons ($n \geq 1$), whose lowest order is the $a_1$ meson, in the $SU(2)$ part of the gauge field. Note that the contributions from the $U(1)$ part of the axial-vector and pseudo-scalar mesons are subleading in the $1/N_c$ expansion.

To divide this expression into the central force and tensor force, we use the following formula for the Yukawa potential $Y_n(|\vec{r}|)$,

$$\frac{\partial}{\partial X_1^i} \frac{\partial}{\partial X_2^j} Y_n(|\vec{r}|) = \left( \delta^{ac} - \hat{r}^a \hat{r}^c \right) \left( \frac{3}{|\vec{r}|^2} + \frac{3\sqrt{\lambda_n}}{|\vec{r}|} + \lambda_n \right) Y_n(|\vec{r}|) - \frac{\delta^{ab}}{3} \lambda_n Y_n(|\vec{r}|). \quad (5.34)$$

Then, we obtain the potential energy due to the central force and tensor force,

$$V = V_C + S_{12}V_T, \quad (5.35)$$

$$V_C = \kappa^2 \left[ -\frac{1}{4a^2\lambda^2} \sum_{n=1, odd}^{\infty} \psi_n(Z_2)\psi_n(Z_1)Y_n(|\vec{r}|) \right. \right.
$$

$$\left. \left. + \frac{256\pi^4}{9}(\rho_1)^2(\rho_2)^2(1_{I_1}^b I_2^b)J_1^a J_2^c \sum_{n=0, even}^{\infty} \phi_n(Z_2)\phi_n(Z_1)Y_n(|\vec{r}|) \right]\right], \quad (5.36)$$

$$V_T = \kappa^2 \frac{64\pi^4}{27}(\rho_1)^2(\rho_2)^2(1_{I_1}^b I_2^b) \left( \sum_{n=1, odd}^{\infty} \psi_n(Z_2)\psi_n(Z_1) \left( \frac{3}{|\vec{r}|^2} + \frac{3\sqrt{\lambda_n}}{|\vec{r}|} + \lambda_n \right) Y_n(|\vec{r}|) \right. \right.
$$

$$\left. \left. - \sum_{n=0, even}^{\infty} \phi_n(Z_2)\phi_n(Z_1) \left( \frac{3}{|\vec{r}|^2} + \frac{3\sqrt{\lambda_n}}{|\vec{r}|} + \lambda_n \right) Y_n(|\vec{r}|) \right]\right]. \quad (5.37)$$
We note that these expressions can be reproduced from the leading order of the one-boson-exchange potential computed from tree-level Feynman diagram with the nucleon-nucleon-meson couplings obtained in Ref. 8). See Appendix G for detail.

Let us look at the contribution from light mesons. The ω meson gives only the central force, the first line in $V_C$ with $n = 1$. It is

$$V_C^{(ω)} = -\kappa^2 \frac{1}{4a^2 \lambda^2} \langle ψ_1(Z_1) \rangle_{(1)} \langle ψ_1(Z_2) \rangle_{(2)} Y_1(|\vec{r}|) .$$  \hspace{1cm} (5.38)

For the potential between the same types of baryons, this expression of course reduces to

$$V_C^{(ω)} = -\kappa^2 \frac{1}{4a^2 \lambda^2} \langle ψ_1(Z) \rangle^2 Y_1(|\vec{r}|) .$$  \hspace{1cm} (5.39)

Since the Yukawa potential $Y_n$ defined in \((5.19)\) is negative, the ω meson exchange gives a strong repulsion force in the central force. The ω meson is the lightest vector meson that comes from the $U(1)$ part of the five-dimensional gauge field. The instanton is electrically charged under this $U(1)$, so the baryons should have this universal repulsive force. The ω meson exchange manifests its lowest term in the KK decomposition of the higher-dimensional “electric” repulsion.

The ρ meson exchange can be seen in $SU(2)$ components of $n = 1$. The central force is

$$V_C^{(ρ)} = \kappa^2 \frac{512\pi^4}{27} (ρ_1)^2 (ρ_2)^2 (I^b_1 I^b_2) (J^b_1 J^b_2) \psi_1(Z_2) \psi_1(Z_1) (-\lambda_1) Y_n(|\vec{r}|) ,$$  \hspace{1cm} (5.40)

while the tensor force is

$$V_T^{(ρ)} = \kappa^2 \frac{64\pi^4}{27} (ρ_1)^2 (ρ_2)^2 (I^b_1 I^b_2) \psi_1(Z_2) \psi_1(Z_1) \left( \frac{3}{|\vec{r}|^2} + \frac{3\sqrt{λ_1}}{|\vec{r}|} + λ_1 \right) Y_1(|\vec{r}|) .$$

When two baryons are of the same type, these reduce to

$$V_C^{(ρ)} = \kappa^2 \frac{512\pi^4}{27} (ρ^2)^2 (I^b_1 I^b_2) (J^b_1 J^b_2) \langle ψ_1(Z) \rangle^2 (-\lambda_1) Y_1(|\vec{r}|) ,$$

$$V_T^{(ρ)} = \kappa^2 \frac{64\pi^4}{27} (ρ^2)^2 (I^b_1 I^b_2) \langle ψ_1(Z) \rangle^2 \left( \frac{3}{|\vec{r}|^2} + \frac{3\sqrt{λ_1}}{|\vec{r}|} + λ_1 \right) Y_1(|\vec{r}|) .$$  \hspace{1cm} (5.41)

The strength of this tensor force can be compared with the strength of the pion tensor force \((5.26)\). The ratio of the front coefficients is given by $-\langle ψ_1(Z) \rangle^2 / ⟨φ_0(Z)⟩^2$. The classical evaluation of this gives

$$\frac{-⟨ψ_1(Z)⟩^2}{⟨φ_0(Z)⟩^2} = -\kappa \pi ⟨ψ_1(Z)⟩^2 ≃ -π (0.597)^2 \sim -1 .$$  \hspace{1cm} (5.42)

Let us see how this ρ meson exchange may change the result of the π exchange. At the length scale $r \sim 1$ fm $\sim 200$ MeV$^{-1}$, the pion behaves as a massless particle while the ρ meson is
massive. The above ratio denotes that the coefficient is of the same order, while the sign is opposite to each other. The normalization of the Yukawa potential gives the following rough ratio

$$\frac{e^{-m_{\rho}|\vec{r}|}}{e^{-m_{\pi}|\vec{r}|}} \sim O(0.1). \quad (5.43)$$

This means that the $\rho$ meson exchange does not contribute much to the tensor force at the distance scale $\sim 1$ fm.

§6. Summary

In this paper, we have deduced the nuclear force at short distance in large $N_c$ strongly coupled QCD, by applying gauge/string duality.

In the D4-D8 model\textsuperscript{[9,10]} of holographic QCD, baryons are instantons in (1+4)-dimensional YMCS theory. We have explicitly constructed a two-instanton solution in the theory by employing ADHM construction of instantons. The analytic expression for the potential energy plus kinetic terms of the baryon, \textit{i.e.}, the effective Hamiltonian of quantum mechanics for two-baryon system, has been derived, for the distance $O(1/\sqrt{\lambda M_{KK}}) < r < O(1/M_{KK})$.

The evaluation of this interaction Hamiltonian for specific two-nucleon states, labeled by spin $J_i^3$ and isospin $I_i^3$ with $i = 1, 2$, provides the nuclear force at the short distance scale. We have obtained a central force \textsuperscript{(4.45)} as well as a tensor force \textsuperscript{(4.46)}. The central force exhibits a strong repulsive core of a nucleon. As the repulsive core has been mysterious from the viewpoint of strongly coupled QCD, our result is of importance as a derivation of the repulsive core from the “first principle” of QCD, in the large $N_c$ expansion and also with the gauge/string duality.

The obtained nucleon-nucleon potential at short distances has $r^{-2}$ behavior. Technically speaking, this comes from the harmonic potential in four spatial dimensions including the holographic extra dimension. It would be interesting to fit this peculiar behavior with the experimental observation or the recent lattice result\textsuperscript{[1]}

As our result is for the short distance, it is important to generalize our analysis to a larger distance scale. For $r > O(1/M_{KK})$, the effect of the curvature along $z$ becomes indispensable, thus, one needs to construct a two-instanton solution in curved space. This may lead to an analysis of a deuteron system in holographic QCD. For this, the inclusion of quark mass to the model\textsuperscript{[11]} that should make the pion massive may be important. On the other hand, the height of the nucleon-nucleon potential at $r = 0$ is of interest, but its derivation has turned out to be difficult, as we explained in \textsuperscript{[2,2]} Further effort along these directions may reveal some more interesting physics in QCD, via the holography.
Acknowledgements

We would like to thank H. Fujii, T. Hatsuda, K. Itakura, T. Matsui, T. Nakatsukasa, M. Nitta, S. Sasaki, and Y. Yamaguchi for valuable discussions. The work of K.H. and T.S. is partially supported by a Grant-in-Aid for Young Scientists (B), MEXT, Japan. The work of S.S. is supported in part by JSPS Grant-in-Aid for Creative Scientific Research No. 19GS0219 and also by World Premier International Research Center Initiative (WPI Initiative), MEXT, Japan. We would like to thank the Yukawa Institute for Theoretical Physics at Kyoto University, where we discussed this topic during the workshop YITP-W-08-04 on “Development of Quantum Field Theory and String Theory”.

Appendix A

--- Notation for quaternion ---

A quaternion \( q \in \mathbb{H} \) is given as a linear combination of the form

\[
q = q^4 - q^1 I - q^2 J - q^3 K , \quad (q^m \in \mathbb{R}, \ m = 1, 2, 3, 4)
\]  

(A-1)

where \( I, J, \) and \( K \) satisfy

\[
I^2 = J^2 = K^2 = -1 , \quad IJ = -JI = K , \quad JK = -KJ = I , \quad KI = -IK = J .
\]  

(A-2)

Since \((-i\tau^1, -i\tau^2, -i\tau^3)\) satisfy the same algebra as \((I, J, K)\), a quaternion can be represented as a \(2 \times 2\) complex matrix of the form

\[
q = q^4 + iq^a \tau^a .
\]  

(A-3)

The conjugate and norm of a quaternion \( q \) are defined as \( q^\dagger = q^4 + q^1 I + q^2 J + q^3 K \) and \(|q| \equiv \sqrt{q^\dagger q} = \sqrt{qq^\dagger} = \sqrt{q^m q^m} \), respectively.

The product of two quaternions \( q, w \in \mathbb{H} \) follows from the relation (A-2) and it can be written in terms of the \(2 \times 2\) complex matrix representation (A-3)

\[
qw = q^4 w^4 - \vec{q} \cdot \vec{w} + i(q^4 \vec{w} + w^4 \vec{q} - \vec{q} \times \vec{w}) \cdot \vec{\tau} ,
\]  

(A-4)

where \( \vec{q} = (q^1, q^2, q^3) \), etc. We also use the following notation:

\[
(q \cdot w) \equiv \frac{1}{2}(q^\dagger w + w^\dagger q) = q^m w^m ,
\]  

(A-5)

\[
(q \times w) \equiv \frac{1}{2}(q^\dagger w - w^\dagger q) = i(q^4 \vec{w} - w^4 \vec{q} + \vec{q} \times \vec{w}) \cdot \vec{\tau} .
\]  

(A-6)
An element of $Sp(n)$ is defined as a $n \times n$ quaternionic matrix $Q$ satisfying $Q^\dagger Q = 1_n$.\footnote{Sp(n) in this paper is the unitary symplectic group, which is also written as USp(2n).} In particular, an element of $Sp(1)$ is a quaternion satisfying
\[
q^\dagger q = q^m q^m = 1 , \tag{A.7}
\]
which is equivalent to the condition for an element of $SU(2)$ in the $2 \times 2$ complex matrix representation \([A.3]\).

**Appendix B**

---

**ADHM construction**

### B.1. ADHM construction for $Sp(n)$ instantons

Here, we briefly review the ADHM construction.\cite{31} (See for example Ref.\cite{32} for a review.) Since $SU(2) = Sp(1)$, the ADHM construction for the $Sp(n)$ instantons is useful for our purpose.

We define an $(n+k) \times k$ quaternionic matrix of the form
\[
\Delta(x) = a + b x , \tag{B.1}
\]
where $a$ and $b$ are $(n+k) \times k$ quaternionic matrices and $x = x^4 - x^1 I - x^2 J - x^3 K \in \mathbb{H}$ is a quaternion composed of the coordinate of the four-dimensional space $(x^1, x^2, x^3, x^4) = (\vec{x}, z)$. The matrices $a$ and $b$ are required to satisfy that $a^\dagger a$ and $b^\dagger b$ are $k \times k$ real symmetric matrices, and $a^\dagger b$ is a $k \times k$ symmetric quaternionic matrix. These conditions for the matrices $a$ and $b$ are equivalent to the constraint that the matrix $\Delta$ satisfies
\[
\Delta^\dagger \Delta = L(x) \tag{B.2}
\]
with a $k \times k$ real symmetric matrix $L(x)$. We also implicitly assume that the matrices $a$ and $b$ are generic and they are matrices of rank $k$.

The ADHM gauge field for the $k$-instanton configuration is given by
\[
A_m(x) = -i U(x)^\dagger \partial_m U(x) , \tag{B.3}
\]
where $U(x)$ is $(n+k) \times n$ quaternionic matrix satisfying
\[
\Delta^\dagger U = 0 , \quad U^\dagger U = 1_n . \tag{B.4}
\]
Note that the matrix $U(x)$ is defined up to a transformation
\[
U(x) \rightarrow U(x) g(x) , \quad (g(x) \in Sp(n)) , \tag{B.5}
\]
which acts as a gauge transformation for the gauge field $\mathbf{(B.3)}$.

The gauge field $\mathbf{(B.3)}$ as well as the constraint $\mathbf{(B.2)}$ is invariant under

$$
\Delta(x) \to Q \Delta(x) R, \quad L(x) \to R^T L(x) R, \quad (B.6)
$$

where $Q \in Sp(n + k)$ and $R \in GL(k, \mathbb{R})$. By using this transformation, the matrix $b$ can be fixed as

$$
b = \begin{pmatrix}
0 \\
-1_k
\end{pmatrix}, \quad (B.7)
$$

and then $\Delta$ is of the canonical form

$$
\Delta(x) = \begin{pmatrix}
Y \\
X - x 1_k
\end{pmatrix}, \quad (B.8)
$$

where $X$ is a $k \times k$ symmetric quaternionic matrix and $Y$ is an $n \times k$ quaternionic matrix such that $Y^\dagger Y + X^\dagger X$ is a $k \times k$ symmetric real matrix. Note that we have not completely used the transformation $\mathbf{(B.6)}$. In fact, a transformation $\mathbf{(B.6)}$ with $R \in O(k)$ and

$$
Q = \begin{pmatrix}
q \\
R^T
\end{pmatrix}, \quad (q \in Sp(n)) \quad (B.9)
$$

leaves $\mathbf{(B.7)}$ invariant.

**B.2. $Sp(1) = SU(2)$ one-instanton**

As an exercise, let us consider the $n = k = 1$ case. Using the canonical form $\mathbf{(B.8)}$, we have

$$
\Delta(x) = \begin{pmatrix}
y \\
X - x
\end{pmatrix}, \quad (B.10)
$$

with $y, x, X \in \mathbb{H}$. In this case, $\mathbf{(B.2)}$ is satisfied without imposing further constraints.

The condition $\mathbf{(B.4)}$ is solved by

$$
U^\dagger = \frac{1}{\sqrt{\xi^2 + \rho^2}} (y(x - X)y^{-1}, y), \quad (B.11)
$$

where $\xi \equiv \sqrt{|x - X|^2}$ and $\rho \equiv \sqrt{|y|^2}$.

Then the ADHM gauge field $\mathbf{(B.3)}$ is

$$
A_m = -i \alpha(f(\xi) g \partial_m g^{-1}) \alpha^{-1}, \quad (B.12)
$$
where
\[ f(\xi) = \frac{\xi^2}{\xi^2 + \rho^2}, \quad g = \frac{x - X}{\xi}, \] (B.13)
and \( a \equiv y/\rho \) is an element of \( Sp(1) = SU(2) \).

This is the BPST instanton solution. The moduli parameters \( X, \rho, \) and \( a \) correspond to the position, size, and gauge orientation of the instanton. The \( a \)-dependence of the gauge field can be eliminated by a global gauge transformation. However, it is known that this degree of freedom is also physically relevant when we quantize the system via moduli space approximation method.

**B.3. \( Sp(1) = SU(2) \) two-instanton**

For \( n = 1 \) and \( k = 2 \), the ansatz (B.8) can be written as
\[ Y = (y_1, y_2), \quad X = \begin{pmatrix} X_1 & w \\ w & X_2 \end{pmatrix}, \quad \Delta(x) = \begin{pmatrix} y_1 & y_2 \\ X_1 - x & w \\ w & X_2 - x \end{pmatrix}. \] (B.14)

The constraint (B.2) requires
\[ Y^\dagger Y + X^\dagger X = \begin{pmatrix} |y_1|^2 + |X_1|^2 + |w|^2 & y_1^\dagger y_2 + X_1^\dagger w + w^\dagger X_2 \\ y_2^\dagger y_1 + X_2^\dagger w + w^\dagger X_1 & |y_2|^2 + |X_2|^2 + |w|^2 \end{pmatrix} \] (B.15)
to be a real symmetric matrix and hence
\[ y_1^\dagger y_2 - y_2^\dagger y_1 + r^\dagger w - w^\dagger r = 0, \] (B.16)
where we have defined \( r = X_1 - X_2 \). This equation is solved when
\[ w = \frac{r}{|r|^2}(y_2 \times y_1) + \alpha r, \] (B.17)
with \( \alpha \in \mathbb{R} \). This parameter \( \alpha \) can be eliminated by the residual \( O(2) \) symmetry in (B.9) with \( q = 1 \) and \( R \in O(2) \). We will set \( \alpha = 0 \) in this paper.

After all, we have 4 quaternionic parameters \( y_1, y_2, X_1, \) and \( X_2 \) to parameterize the two instanton moduli space. It can be shown that when the separation of the two instantons is sufficiently large, the two-instanton configuration can be approximated by the superposition of two 1-instanton configurations. Then, \( X_i (i = 1, 2) \) represents the positions of the two instantons, \( \rho_i \equiv \sqrt{|y_i|^2} \) corresponds to their size, and \( a_i \equiv y_i / \rho_i \) is their \( SU(2) \) orientation. This fact can be explicitly seen in the effective Hamiltonian (3.28).
In this section, we compute
\[ \int d^4x \, z^2 \, \text{tr} \, F_{mn}^2 , \] (C.1)
where \( z = x^4 \). We will follow the strategy of Ref. [34], in which
\[ \int d^4x \, |x|^2 \, \text{tr} \, F_{mn}^2 , \] (C.2)
is calculated.

To evaluate \( \text{tr} \, F_{mn}^2 \), we use the useful formula [33]
\[ \text{tr} \, F_{mn}^2 = \frac{1}{2} \epsilon_{mnpq} \, \text{tr} \, F_{mn} \, F_{pq} = -\Box \log \det L , \] (C.3)
where \( \Box \equiv \partial_m \partial_m \) and \( L(x) \) is defined in (B.2).

In general, \( L(x) \) can be written as
\[ L(x) = A|x|^2 - 2\gamma_m x^m + A , \] (C.4)
where \( A = b^\dagger b \) and \( A = a^\dagger a \) are positive definite \( k \times k \) real symmetric matrices, and \( \gamma_m \)
\( (m = 1, 2, 3, 4) \) are \( k \times k \) real symmetric matrices. For the canonical form (B.8), they are
given by
\[ A = 1 , \quad \gamma_m = X_m , \quad A = Y^\dagger Y + X^\dagger X , \] (C.5)
where \( X_m \) is the \( k \times k \) real symmetric matrix satisfying \( X = X_4 - X_1 I - X_2 J - X_3 K \).

The result we are going to prove is
\[ \int d^4x \, z^2 \, \text{tr} \, F_{mn}^2 = 8\pi^2 \, \text{tr} \left[ (\gamma_4 A^{-1})^2 - (\gamma_1 A^{-1})^2 - (\gamma_2 A^{-1})^2 - (\gamma_3 A^{-1})^2 + AA^{-1} \right] . \] (C.6)
For the canonical form with (C.5), we obtain
\[ \int d^4x \, z^2 \, \text{tr} \, F_{mn}^2 = 8\pi^2 \, \text{tr} \left( 2(X_4)^2 + Y^\dagger Y \right) . \] (C.7)

Note that this formula implies
\[ \int d^4x \, |x|^2 \, \text{tr} \, F_{mn}^2 = 16\pi^2 \, \text{tr} \left( 2Y^\dagger Y + X^\dagger X \right) , \] (C.8)
which reproduces the result in Ref. [34].
To show (C.6), we can set $A = 1$ without loss of generality. The $A$ dependence can easily be recovered by the transformation (B.6). Integrating by parts, we obtain

$$
\int d^4 x \ z^2 \ tr F_{mn}^2 = - \lim_{R \to \infty} \int_{S^3} d^3 \Omega \ R^2 x^m \left[ z^2 \partial_m \Box \log \det L - 2z \partial_m \Box \log \det L + 2 \partial_m \log \det L \right],
$$

(C.9)

where $R^2 = |x|^2$ and $d^3 \Omega$ is the volume element of unit $S^3$. Here, we have used Gauss’s law

$$
\int d^4 x \partial_m W_m = \lim_{R \to \infty} \int_{S^3} d^3 \Omega \ R^3 n_m W_m,
$$

(C.10)

where $n_m = x^m / R$ is a unit normal vector of the $S^3$.

We can easily check the following formulas

$$
\partial_m \log \det L = tr(V_m),
$$

(C.11)

$$
\Box \log \det L = 8 \ tr(L^{-1}) - tr(V_m^2),
$$

(C.12)

$$
\partial_m \Box \log \det L = -12 \ tr(L^{-1} V_m) + 2 \ tr(V_m V_n^2),
$$

(C.13)

where $V_m = L^{-1} \partial_m L$. Using these, (C.9) becomes

$$
\int d^4 x \ z^2 \ tr F_{mn}^2 = - \lim_{R \to \infty} \int_{S^3} d^3 \Omega \ R^2 tr \left[ -12 z^2 L^{-1} x^m V_m + 2 z^2 x^m V_m V_n^2 
\right.

\left. - 16 z^2 L^{-1} + 2 z^2 V_n^2 + 2 x^m V_m \right].
$$

(C.14)

Inserting the relation $V_m = L^{-1} \partial_m L = 2L^{-1}(x^m - \gamma_m)$, we obtain

$$
\int d^4 x \ z^2 \ tr F_{mn}^2 = - \lim_{R \to \infty} \int_{S^3} d^3 \Omega \ R^2 tr \left[ 4(R^2 - 4 z^2) L^{-1} - 4 x^m \gamma_m L^{-1} - 24 z^2 (R^2 - x^m \gamma_m) L^{-2}
\right.

\left. + 8 z^2 \left( L + 2(R^2 - x^m \gamma_m) \right) \left( R^2 L^{-2} + L^{-1} \gamma_m L^{-1} \gamma_m - 2L^{-2} x^m \gamma_m \right) \right].
$$

(C.15)

We are only interested in the $O(R^0)$ terms in the integrand of the right-hand side of this equation. Then, recalling $L = O(R^2)$, we obtain

$$
\int d^4 x \ z^2 \ tr F_{mn}^2 = - \lim_{R \to \infty} \int_{S^3} d^3 \Omega \ tr \left[ P_1 + P_2 + P_3 + P_4 + P_5 \right],
$$

(C.16)

where

$$
P_1 = R^2 \cdot 4L^{-3}(R^2 - 4 z^2), \quad P_2 = R^4 \cdot 4 x^m \gamma_m L^{-3}(-5R^2 + 14z^2),
$$

$$
P_3 = R^{-4} \cdot 16(x^m \gamma_m)^2 (2R^2 - 3z^2), \quad P_4 = R^{-2} \cdot 8A(R^2 - 6z^2),
$$

$$
P_5 = R^{-2} \cdot 24 \gamma_m^2 z^2.
$$

(C.17)
Useful formulas to evaluate the integral are

\[ \int_{S^3} d^3 \Omega = 2 \pi^2 , \]
\[ \int_{S^3} d^3 \Omega \, x^m x^n = \frac{\pi^2}{2} R^2 \delta_{mn} , \]
\[ \int_{S^3} d^3 \Omega \, (x^m)^2 (x^n)^2 = \frac{\pi^2}{12} R^4 (1 + 2 \delta_{mn}) . \]
(no sum for \( m, n \)) \hspace{1cm} (C.18)

To evaluate the integral of \( P_1 \) in (C.17), we expand \( L^{-3} \)

\[ L^{-3} = \frac{1}{R^6} \left( 1 + 6 \frac{x^m \gamma_m}{R^2} - 3 \frac{A}{R^2} + 24 \frac{(x^m \gamma_m)^2}{R^4} + \mathcal{O}(R^{-3}) \right) \hspace{1cm} (C.19) \]

and

\[- \lim_{R \to \infty} \int_{S^3} d^3 \Omega \, \text{tr} \, P_1 = - \lim_{R \to \infty} \int_{S^3} d^3 \Omega \, \text{tr} \left[ R^6 4L^{-3} (R^2 - 4z^2) \right] \]
\[ = - \lim_{R \to \infty} \int_{S^3} d^3 \Omega \, \text{tr} \left[ 4(R^2 - 4z^2) + 4(R^2 - 4z^2) \cdot \frac{6x^m \gamma_m}{R^2} \right. \]
\[ + 4(R^2 - 4z^2) \left( -\frac{3}{R^2} A + \frac{24}{R^4} (x^m \gamma_m)^2 \right) \right] . \] \hspace{1cm} (C.20)

Using the formulas (C.18), we see that the first term vanishes. The second term also vanishes since it is odd under \( x \to -x \). From the third term, we obtain

\[- \lim_{R \to \infty} \int_{S^3} d^3 \Omega \, \text{tr} \, P_1 = -16 \pi^2 \text{tr}(\gamma_1^2 + \gamma_2^2 + \gamma_3^2 - 3 \gamma_4^2) . \] \hspace{1cm} (C.21)

The integrals for the other \( P_2, \cdots, P_5 \) are calculated in a similar manner. The results are

\[- \lim_{R \to \infty} \int_{S^3} d^3 \Omega \, \text{tr} \, P_2 = 8 \pi^2 \text{tr} \left( 4(\gamma_1^2 + \gamma_2^2 + \gamma_3^2) - 3 \gamma_4^2 \right) , \] \hspace{1cm} (C.22)

\[- \lim_{R \to \infty} \int_{S^3} d^3 \Omega \, \text{tr} \, P_3 = -4 \pi^2 \text{tr} \left( 3(\gamma_1^2 + \gamma_2^2 + \gamma_3^2) + \gamma_4^2 \right) , \] \hspace{1cm} (C.23)

\[- \lim_{R \to \infty} \int_{S^3} d^3 \Omega \, \text{tr} \, P_4 = \pi^2 \text{tr}(A) , \] \hspace{1cm} (C.24)

\[- \lim_{R \to \infty} \int_{S^3} d^3 \Omega \, \text{tr} \, P_5 = -12 \pi^2 \text{tr}(\gamma_m^2) . \] \hspace{1cm} (C.25)

Summing up all these, we finally obtain

\[ \int d^4 x \, z^2 \, \text{tr} \, P_{mn}^2 = 8 \pi^2 \text{tr} \left( -\left( \gamma_1^2 + \gamma_2^2 + \gamma_3^2 \right) + \gamma_4^2 + A \right) . \] \hspace{1cm} (C.26)
Appendix D

Laplacian of the two-instanton moduli space

Here, we outline the derivation of the expression (3.25) and (3.27).

Consider a metric given as

$$ g_{\alpha\beta} = \overline{g}_{\alpha\beta} + h_{\alpha\beta} , \quad (D.1) $$

where $\overline{g}_{\alpha\beta}$ is a constant metric and $h_{\alpha\beta} \ll 1$ is a small perturbation. Then, omitting the $O(h^2)$ terms, we have

$$ g_{\alpha\beta} = \overline{g}_{\alpha\beta} - h_{\alpha\beta} , \quad \sqrt{g} = \sqrt{\overline{g}} \left( 1 + \frac{1}{2} h^{\alpha}_{\alpha} \right) , \quad (D.2) $$

$$ \nabla^2 = \frac{1}{\sqrt{g}} \partial_{\alpha} \sqrt{g} g^{\alpha\beta} \partial_{\beta} = \nabla^2_0 - h_{\alpha\beta} \partial_{\alpha} \partial_{\beta} + \frac{1}{2} (\partial_{\beta} h^{\alpha}_{\alpha}) \partial_{\beta} - (\partial_{\alpha} h^{\alpha}_{\beta}) \partial_{\beta} , \quad (D.3) $$

where $(\overline{g}^{\alpha\beta})$ is the inverse matrix of $\overline{g}_{\alpha\beta}$, $\nabla^2_0 = \overline{g}_{\alpha\beta} \partial_{\alpha} \partial_{\beta}$, $g = \det(\overline{g}_{\alpha\beta})$, $\overline{g} = \det(\overline{g}_{\alpha\beta})$. Here, raising and lowering the indices is done with $\overline{g}_{\alpha\beta}$ and $\overline{g}^{\alpha\beta}$. We apply these formulas to the metric (3.21), in which $ds^2_0$ and $ds_1$ correspond to $\overline{g}_{\alpha\beta}$ and $h_{\alpha\beta}$, respectively.

A little algebra shows

$$ h^{\alpha}_{\alpha} = \frac{2}{|r|^2} (|y_1|^2 + |y_2|^2) , \quad (D.4) $$

$$ (\partial_{\beta} h^{\alpha}_{\alpha}) \partial^{\beta} = \frac{2}{|r|^2} \left[ \left( y_1 \cdot \frac{\partial}{\partial y_1} \right) + \left( y_2 \cdot \frac{\partial}{\partial y_2} \right) \right] + O(|r|^{-3}) , \quad (D.5) $$

$$ (\partial_{\alpha} h^{\alpha\beta}) \partial_{\beta} = \frac{2}{|r|^2} \left[ \left( y_1 \cdot \frac{\partial}{\partial y_1} \right) + \left( y_2 \cdot \frac{\partial}{\partial y_2} \right) \right] + O(|r|^{-3}) , \quad (D.6) $$

$$ h^{\alpha\beta} \partial_{\alpha} \partial_{\beta} = \frac{1}{|r|^2} \left[ \rho_1^2 \left( \frac{\partial}{\partial y_1} \cdot \frac{\partial}{\partial y_1} \right) + \rho_2^2 \left( \frac{\partial}{\partial y_2} \cdot \frac{\partial}{\partial y_2} \right) + \left( y_1 \cdot \frac{\partial}{\partial y_1} \right) \left( y_2 \cdot \frac{\partial}{\partial y_2} \right) - \left( y_2 \cdot \frac{\partial}{\partial y_1} \right)^2 - \left( y_1 \cdot \frac{\partial}{\partial y_2} \right)^2 - (y_1 \cdot y_2) \left( \frac{\partial}{\partial y_1} \cdot \frac{\partial}{\partial y_2} \right) + \epsilon_{IJKL} y_I^I y_J^J \frac{\partial}{\partial y_I^K} \frac{\partial}{\partial y_J^L} + y_I^I y_J^J \frac{\partial}{\partial y_I^K} \frac{\partial}{\partial y_J^L} \right] . \quad (D.7) $$

From these, we can easily obtain (3.25).

The following formulas are useful to obtain the expression in (3.27):

$$ \left( y_i \cdot \frac{\partial}{\partial y_i} \right) = \rho_i \frac{\partial}{\partial \rho_i} , \quad (D.8) $$

42
\[ (\frac{\partial}{\partial y_i} \cdot \frac{\partial}{\partial y_j}) = \frac{\partial^2}{\partial y_i^2} + \frac{3}{\rho_i} \frac{\partial}{\partial \rho_i} - \frac{4}{\rho_i^2} I_i^i = \frac{\partial^2}{\partial y_i^2} + \frac{3}{\rho_i} \frac{\partial}{\partial \rho_i} - \frac{4}{\rho_i^2} J_i^a J_i^a, \quad (D.9) \]

\[ I_1^a I_2^a = -\frac{1}{4} \left[ (y_1 \cdot y_2) \left( \frac{\partial}{\partial y_1} \frac{\partial}{\partial y_2} \right) - y_1^i y_2^j \frac{\partial}{\partial y_1^i} \frac{\partial}{\partial y_2^j} - \epsilon_{ijk} y_1^i y_2^j \frac{\partial}{\partial y_1^k} \frac{\partial}{\partial y_2^l} \right], \quad (D.10) \]

\[ J_1^a J_2^a = -\frac{1}{4} \left[ (y_1 \cdot y_2) \left( \frac{\partial}{\partial y_1} \frac{\partial}{\partial y_2} \right) - y_1^i y_2^j \frac{\partial}{\partial y_1^i} \frac{\partial}{\partial y_2^j} + \epsilon_{ijk} y_1^i y_2^j \frac{\partial}{\partial y_1^k} \frac{\partial}{\partial y_2^l} \right]. \quad (D.11) \]

**Appendix E**

**Evaluation of \( H_{\text{pot}}^{(U(1))} \)**

Here, we rederive the result (3.18) in a more systematic way. From the expression in (3.5), we obtain

\[ \log \det L = \log f_1 + \log f_2 + \log f_3, \quad (E.1) \]

where

\[ f_i(x) = \rho_i^2 + |x - X_i|^2 + |w|^2, \quad (i = 1, 2) \quad (E.2) \]

\[ f_3(x) = 1 - \frac{e(x)^2}{f_1(x) f_2(x)} \quad (E.3) \]

with

\[ e(x) = (y_1 \cdot y_2) + (w \cdot (X_1 + X_2 - 2x)) \quad (E.4) \]

Substituting (3.4) into (2.26), we obtain

\[ H_{\text{pot}}^{(U(1))} = \frac{a N_c}{2} \frac{1}{(32 \pi^2 a)^2} \int d^3 x d z \left( (\partial_M \Box) (\log f_1 + \log f_2 + \log f_3) \right)^2 \]

\[ = \frac{a N_c}{2} \frac{1}{(32 \pi^2 a)^2} \int d^3 x d z \left[ \sum_{i=1,2} (\partial_M \Box \log f_i)^2 + (\partial_M \Box \log f_3)^2 + \right. \]

\[ \left. + 2(\partial_M \Box \log f_1)(\partial_M \Box \log f_2) + 2 \sum_{i=1,2} (\partial_M \Box \log f_i)(\partial_M \Box \log f_3) \right]. \quad (E.5) \]

The following formulas are useful for evaluating this integral:

\[ \Box \log f_i = \frac{4(|x - X_i|^2 + \rho_i^2 + |w|^2))}{(|x - X_i|^2 + \rho_i^2 + |w|^2)^2}, \quad (E.6) \]

\[ \Box \Box \log f_i = -\frac{96(\rho_i^2 + |w|^2)^2}{(|x - X_i|^2 + \rho_i^2 + |w|^2)^4}, \quad (E.7) \]

\[ \Box \Box \Box \log f_i = -\frac{1536(\rho_i^2 + |w|^2)^2(3|x - X_i|^2 - 2(\rho_i^2 + |w|^2)))}{(|x - X_i|^2 + \rho_i^2 + |w|^2)^6}, \quad (E.8) \]
for \( i = 1, 2 \).

Then, the first term in (E.5) is evaluated as

\[
\int d^3x d^3z (\partial_M \Box \log f_1)^2 = - \int d^3x d^3z (\Box \log f_1)(\Box \log f_1)
\]

\[
= \frac{1}{\rho_i^2 + |w|^2} \int_0^\infty du \ 2\pi^2 u^3 \frac{4(u^2 + 2)}{(u^2 + 1)^2 (u^2 + 1)^4} \frac{96}{256\pi^2} \frac{1}{5} \rho_i^2 + \frac{|w|^2}{5}
\]

\[
\simeq \frac{256\pi^2}{5} \left( \frac{1}{\rho_i^2} - \frac{|w|^2}{\rho_i^4} + \mathcal{O}(|r|^{-4}) \right), \quad (E.9)
\]

where we have used

\[
u \equiv \frac{x - X_1}{\sqrt{\rho_i^2 + |w|^2}}, \quad (E.10)
\]

and \( u \equiv |u| \). A similar formula for \( f_2 \) is obtained by replacing \( \rho_1 \) with \( \rho_2 \) in (E.9).

The third term in (E.5) is evaluated as

\[
\int d^3x d^3z (\partial_M \Box \log f_1)(\partial_M \Box \log f_2) = - \int d^3x d^3z (\Box \log f_2)(\Box \log f_1)
\]

\[
= \frac{1}{|r|^2} \int d^4u \frac{4(|V_i u + \hat{r}|^2 + 2V_i^2)}{(|V_i u + \hat{r}|^2 + V_i^2)^2} \frac{96}{(u^2 + 1)^4}. \quad (E.11)
\]

where \( r = X_1 - X_2 \) and we have defined

\[
\hat{r} = \frac{r}{|r|}, \quad V_i = \frac{\sqrt{\rho_i^2 + |w|^2}}{|r|}. \quad (i = 1, 2)
\]

To evaluate the leading term in the \( 1/|r| \) expansion, we consider the limit \( V_i \to 0 \). Although the integrand of (E.11) is divergent at \( u = -\hat{r}/V_i \) when \( V_2 = 0 \), the integral around \( u = -\hat{r}/V_i \) is convergent. Besides, there is a suppression factor \( 1/(u^2 + 1)^4 \) that makes the contribution around \( u = -\hat{r}/V_i \) in the integral vanish in the \( V_i \to 0 \) limit. Therefore, we can safely take the \( V_i \to 0 \) limit and using

\[
\int d^4u \frac{4 \cdot 96}{(u^2 + 1)^4} = 64\pi^2, \quad (E.13)
\]

we obtain

\[
\int d^3x d^3z (\partial_M \Box \log f_1)(\partial_M \Box \log f_2) = \frac{64\pi^2}{|r|^2} + \mathcal{O}(|r|^{-4}) \quad . \quad (E.14)
\]
The last term in (E.5) is given by

\[\int d^3 x dz \left( \partial_M \Box \log f_1 \right) \left( \partial_M \Box \log f_3 \right) = -\int d^3 x dz \left( \Box \Box \log f_1 \right) \log f_3 = \frac{1}{\rho_1^2 + |\mathbf{w}|^2} \int d^4 u \frac{1536(3u^2 - 2)}{(u^2 + 1)^6} \log f_3, \tag{E.15}\]

with

\[f_3 = 1 - \frac{((\mathbf{y}_1 \cdot \mathbf{y}_2) - 2V_1 |\mathbf{r}|(\mathbf{w} \cdot \mathbf{u}))^2}{|\mathbf{r}|^4V_1^2(u^2 + 1)(|\mathbf{r} + V_1\mathbf{u}|^2 + V_2^2)}. \tag{E.16}\]

Note that we have used the relation \(\mathbf{w} \cdot \mathbf{r} = 0\), which follows from the definition (B.17) with \(\alpha = 0\). Again, to obtain the leading order terms in the \(O(|\mathbf{r}|^{-1})\) expansion, it is allowed to pick up the leading term in the integrand as

\[\log f_3 \simeq -\frac{(\mathbf{y}_1 \cdot \mathbf{y}_2)^2}{|\mathbf{r}|^4V_1^2(u^2 + 1)} + O(|\mathbf{r}|^{-3}). \tag{E.17}\]

Using the formula

\[\int d^4 u \frac{1536(3u^2 - 2)}{(u^2 + 1)^6} \frac{1}{u^2 + 1} = -\frac{128\pi^2}{5}, \tag{E.18}\]

we obtain

\[\int d^3 x dz \left( \partial_M \Box \log f_1 \right) \left( \partial_M \Box \log f_3 \right) \simeq \frac{128\pi^2}{5} \frac{1}{|\mathbf{r}|^2 \rho_1^2} (\mathbf{a}_1 \cdot \mathbf{a}_2)^2 + O(|\mathbf{r}|^{-3}), \tag{E.19}\]

and similarly

\[\int d^3 x dz \left( \partial_M \Box \log f_2 \right) \left( \partial_M \Box \log f_3 \right) \simeq \frac{128\pi^2}{5} \frac{1}{|\mathbf{r}|^2 \rho_2^2} (\mathbf{a}_1 \cdot \mathbf{a}_2)^2 + O(|\mathbf{r}|^{-3}). \tag{E.20}\]

As one can see in (E.17), \(\log f_3\) is \(O(|\mathbf{r}|^{-2})\) and hence the second term in (E.5) does not contribute to the leading \(O(|\mathbf{r}|^{-2})\) terms in the potential. Collecting (E.9), (E.14), (E.19), and (E.20), we reproduce the potential (3.18).

\textbf{Appendix F}

---

\textbf{Height of one-boson-exchange potential}

---

In this appendix, we try to evaluate the height of the nucleon-nucleon potential, in the one-boson-exchange approximation. Note that as shown in §5.3 the one-boson-exchange model does not describe correctly the short distance behavior. Thus, this appendix is only
for an illustration of what will happen in general when two baryons are on top of each other in real space.

If the instantons are located within the distance of \( O(1/M_{\text{KK}}) \), the one-boson-exchange potential is \( (5.22) \). When the instantons are located at \( Z_i \neq 0 \), there is an additional classical potential coming from the self-energy part \( (2.15) \), so in total, the inter-instanton potential energy is

\[
V = \frac{N_c}{16\pi^2a\lambda} \frac{1}{|\vec{X}_1 - \vec{X}_2|^2 + (Z_1 - Z_2)^2} + 8\pi^2a\lambda N_c \left[ \frac{(Z_1)^2}{3} + \frac{(Z_2)^2}{3} \right]. \tag{F.1}
\]

This classical potential exhibits an interesting structure. Let us find a minimum of this potential for fixed inter-baryon distance in real space, \(|\vec{X}_1 - \vec{X}_2| = |\vec{r}|\). We employ a classical approximation for \( Z_i \), by taking the large \( N_c \) limit, for simplicity. Owing to the exchange symmetry \( Z_1 \leftrightarrow Z_2 \), the potential is minimized at \( Z_1 = -Z_2 = r_4/2 \). Thus, the minimization problem is for the potential

\[
V = \frac{N_c}{16\pi^2a\lambda} \frac{1}{|\vec{r}|^2 + r_4^2} + 4\pi^2a\lambda N_c \frac{r_4^2}{3}. \tag{F.2}
\]

The minimization condition is

\[
\frac{\partial V}{\partial r_4} = -\frac{N_c}{8\pi^2a\lambda (|\vec{r}|^2 + r_4^2)^2} + \frac{8\pi^2a\lambda N_c}{3} r_4 = 0. \tag{F.3}
\]

This is solved with

\[
|\vec{r}|^2 + r_4^2 = \frac{\sqrt{3}}{8\pi^2a\lambda}. \tag{F.4}
\]

This forms a three-dimensional sphere around the origin in the four-dimensional space. For a fixed \(|\vec{r}|\), we obtain nonzero \( r_4 \) to minimize the classical potential. The instantons go away from \( Z = 0 \) axis, to minimize the potential energy. This can be understood as follows. The instantons have the overall \( U(1) \) electric charge, so they try to be away from each other. But at the same time there is an effect of the curved space-time, which tries to bring the instanton toward the \( Z = 0 \) axis. The balance of these two effects results in the minimization at \( (F.4) \). The minimum energy for a fixed inter-baryon distance is

\[
V = \frac{N_c}{\sqrt{3}} - \frac{4\pi^2a\lambda N_c}{3} |\vec{r}|^2. \tag{F.5}
\]

Thus, the inter-baryon potential height is maximized at \(|\vec{r}| = 0\), with the height value

\[
V_{\text{max}} = \frac{N_c}{\sqrt{3}}. \tag{F.6}
\]
The sphere \(|\mathbf{r}|^2 > \frac{\sqrt{2}}{8\pi a \lambda}\) does not reach the region \(|\mathbf{r}|^2 > \frac{\sqrt{2}}{8\pi a \lambda}\). In fact, in this region, the potential energy is minimized by \(r_4 = 0\) in (F.22), which results in the previous result (5.23). It is smoothly connected with (F.4) at \(|\mathbf{r}|^2 = \frac{\sqrt{2}}{8\pi a \lambda}\).

For the sphere (F.4) to make sense, its radius should be larger than the classical radius of the instanton, \(|\mathbf{r}|^2 = \frac{\sqrt{2}}{8\pi a \lambda}\). Unfortunately, both are of the same order, \(\sim O(1/\sqrt{8\pi a \lambda})\), so one cannot trust this sphere radius. However, we find an interesting picture of the instantons, where the generic feature of the potential structure suggests that instantons do not overlap in the spatial four dimensions although they look overlapped in the spatial three dimensions.

Appendix G

One-boson-exchange potential revisited

In this appendix, we rederive the one-boson-exchange potential of §5 by summing up an infinite number of one-boson-exchange diagrams explicitly, in the standard field-theoretical computation. A key ingredient for this computation is the nucleon-nucleon-meson cubic couplings obtained in Ref. 8). It is found that the results are in total agreement with those derived in §5. See §4.3 of Ref. 8) for the definition of the couplings and the effective Lagrangian that we use to obtain the Feynman rule.

G.1. Pion exchange

The Feynman rule for the Yukawa coupling among a pion, nucleon \(N\) and nucleon \(N\) reads

\[
ig_{\pi NN} \gamma_5 \tau^a \quad (\text{isotriplet sector}) , \quad i\hat{g}_{\pi NN} \gamma_5 \tau^0 \quad (\text{isosingle sector}) . \tag{G.1}
\]

Consider the scattering process where two initial nucleons with \((p_1, s_1, I_1)\) and \((p_2, s_2, I_2)\) scatter to the final state composed of the two nucleons labeled as \((p'_1, s'_1, I'_1)\) and \((p'_2, s'_2, I'_2)\) by exchanging a single pion. Here, \(p_1, p_2, p'_1,\) and \(p'_2\) are the on-shell momenta with the nucleon mass given by \(m_B\). \(s_1, s_2, s'_1,\) and \(s'_2\) specify the third components of the spin of the nucleons, and \(I_1, I_2, I'_1,\) and \(I'_2\) stand for the third components of the isospin. It turns out that the scattering amplitudes due to the isotriplet and isosinglet pseudo-scalar meson exchange are given by

\[
\mathcal{M}^{SU(2)}_\pi = (ig_{\pi NN})^2 \sqrt{2E_1} \sqrt{2E_2} \sqrt{2E'_1} \sqrt{2E'_2} \tau^a_{I_1 I'_1} \tau^a_{I_2 I'_2} \times \overline{u}(p'_1, s'_1) \gamma_5 u(p_1, s_1) \frac{1}{k^2 + m^2_{\pi}} \overline{u}(p'_2, s'_2) \gamma_5 u(p_2, s_2) ,
\]

\(^*)\) In Ref. 8), the cubic couplings involving excited baryons with \(I = J = 1/2\) are also calculated. The extension of the computation of the nucleon-nucleon potential in this appendix to such excited states is straightforward.
\[ M_{\pi}^{U(1)} = M_\pi^{SU(2)} \bigg|_{g_{\pi NN} \rightarrow \tilde{g}_{\pi NN}, \tau \rightarrow \tau^0}, \tag{G.2} \]
respectively. Here, \( k = p_1 - p'_1 = p'_2 - p_2 \) and
\[ \tau^a_{\pi I} = \chi^{(I')}^\dagger \tau^a \chi^{(I)}, \tag{G.3} \]
with \( \chi^{(I=1/2)} = (1, 0)^T \) and \( \chi^{(I=-1/2)} = (0, 1)^T \) being the isospin wavefunctions. The same notation will be used for the spin matrices \( \sigma^a_{s's} \). For the definition of the Dirac spinors, see Appendix B.2 in Ref. [3]. Note also that we regard the pion as being massive with \( m_\pi \neq 0 \) for the moment although the pion is massless in our model.

In the large \( N_c \) and large \( \lambda \) limit, the nucleon mass \( m_B \) scales as \( \mathcal{O}(\lambda N_c) \) so that the nonrelativistic approximation is valid by considering the momenta to be of order one. Then
\[ E_1 = E'_1 = E_2 = E'_2 \simeq m_B, \]
\[ k^0 \simeq \frac{1}{2m_B}(\vec{p}_1^2 - \vec{p}_1'^2) = \mathcal{O}(m_B^{-1}), \quad k^2 \simeq \vec{k}^2. \tag{G.4} \]
Furthermore, it can be shown that
\[ \bar{\pi}(p'_1, s'_1)\gamma_5 u(p_1, s_1) \simeq \frac{1}{2m_B} (p_1 - p'_1)_a \sigma^a_{s's_1}. \tag{G.5} \]
Hence,
\[ M_\pi^{SU(2)} \simeq + (2m_B)^2 \frac{g_{\pi NN}^2}{(2m_B)^2} \left( \tilde{\tau}_1 \cdot \tilde{\tau}_2 \right) \left( \vec{k} \cdot \vec{\sigma}_1 \right) \left( \vec{k} \cdot \vec{\sigma}_2 \right) \frac{1}{k^2 + m_\pi^2}, \tag{G.6} \]
and a similar expression holds for the \( U(1) \) part. Here, in abbreviation,
\[ \tilde{\tau}_i = \tilde{\tau}^{I_i I_i}, \quad \vec{\sigma}_i = \vec{\sigma}_{s'_i s_i}, \quad (i = 1, 2) \tag{G.7} \]
To estimate the order of the amplitudes in \( \lambda \) and \( N_c \), recall that the Yukawa couplings are given in Ref. [3] as
\[ \tilde{g}_{\pi NN} = \frac{m_B}{f_\pi} N_c \left\langle \frac{1}{k(Z)} \right\rangle, \quad g_{\pi NN} = \frac{m_B}{f_\pi} \frac{16\pi \kappa}{3} \left\langle \frac{\rho^2}{k(Z)} \right\rangle, \tag{G.8} \]
with \( f_\pi^2 = (4/\pi)\kappa \). Here, \( \left\langle \right\rangle \) denotes the expectation value with respect to the nucleon wavefunction given in [2]. This implies that
\[ g_{\pi NN} = \mathcal{O}(\lambda^{1/2} N_c^{3/2}), \quad \tilde{g}_{\pi NN} = \mathcal{O}(\lambda^{-1/2} N_c^{1/2}), \tag{G.9} \]
showing that the \( SU(2) \) part dominates the \( U(1) \) part.
The effective action of the nucleons is defined to reproduce the amplitude computed above. In particular, the effective potential should be equated with

\[ -\tilde{V}_\pi = \frac{g^2_{\pi NN}}{(2m_B)^2} \left( \vec{\tau}_1 \cdot \vec{\tau}_2 \right) \left( \vec{k} \cdot \vec{\sigma}_1 \right) \left( \vec{k} \cdot \vec{\sigma}_2 \right) \frac{1}{\vec{k}^2 + m^2_\pi}. \]  

(G-10)

Note that the overall factor \((2m_B)^2\) is removed from (G.6) because this comes from the wavefunctions assigned to the four external lines. By Fourier-transforming this, we obtain

\[ V_\pi(\vec{x}) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k} \cdot \vec{x}} \tilde{V}_\pi. \]  

(G-11)

Using the formula

\[ \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k} \cdot \vec{x}} \frac{1}{\vec{k}^2 + m^2} = \frac{1}{4\pi} \frac{e^{-mr}}{r}, \]  

(G-12)

with \(r = |\vec{x}|\), and also for any function \(g(r)\),

\[ \left( \vec{\sigma}_1 \cdot \vec{\nabla} \right) \left( \vec{\sigma}_2 \cdot \vec{\nabla} \right) g(r) = \frac{1}{3} \left( \vec{\sigma}_1 \cdot \vec{\sigma}_2 \right) \vec{\nabla}^2 g(r) + \frac{1}{3} S_{12} \left( \partial^2 g - \frac{1}{r} \partial_r g \right), \]  

(G-13)

with

\[ S_{12} = 3 \left( \vec{\sigma}_1 \cdot \vec{r} \right) \left( \vec{\sigma}_2 \cdot \vec{r} \right) r^2 - \left( \vec{\sigma}_1 \cdot \vec{\sigma}_2 \right) \]  

(G-14)

being the tensor operator, we find

\[ V_\pi(\vec{x}) = \frac{g^2_{\pi NN}}{4\pi} \frac{1}{(2m_B)^2} \left( \vec{\tau}_1 \cdot \vec{\tau}_2 \right) \left[ S_{12} \frac{e^{-m_\pi r}}{r} \left( \frac{m^2_\pi}{3} + \frac{m_\pi}{r} + \frac{1}{r^2} \right) + \frac{m^2_\pi}{3} \left( \vec{\sigma}_1 \cdot \vec{\sigma}_2 \right) \frac{e^{-m_\pi r}}{r} \right]. \]  

(G-15)

Using the Goldberger-Treiman relation

\[ g_A = \frac{f_\pi g_{\pi NN}}{m_B}, \]  

(G-16)

we find that the one-pion-exchange potential (G.15) agrees with the expression (5.28) used in §5. As discussed in §5, the central force vanishes when \(m_\pi = 0\) and the potential (G.15) reproduces (5.26), and the \(n = 0\) component of the tensor force (5.37). For \(m_\pi \neq 0\), it is standard in the literature to define the coupling

\[ f^2 = \frac{g^2_{\pi NN}}{4\pi} \left( \frac{m_\pi}{2m_B} \right)^2, \]  

(G-17)

with which

\[ V_\pi(\vec{x}) = m_\pi \frac{f^2}{3} \left( \vec{\tau}_1 \cdot \vec{\tau}_2 \right) \left[ S_{12} \left( 1 + \frac{3}{m_\pi r} + \frac{3}{m^2_\pi r^2} \right) + \left( \vec{\sigma}_1 \cdot \vec{\sigma}_2 \right) \right] \frac{e^{-m_\pi r}}{m_\pi r}. \]  

(G-18)
G.2. **Axial-vector meson exchange**

The Feynman rule for the nucleon-nucleon-axial-vector-meson couplings is

\( g_{a^a NN} i\gamma_5\gamma^\mu \frac{r^a}{2} \) (isotriplet sector), \( \widehat{g}_{a^a NN} i\gamma_5\gamma^\mu \frac{1}{2} \) (isosinglet sector). \( \text{(G.19)} \)

Here, \( a^a (n = 1, 2, \cdots) \) is the axial-vector meson associated with the wavefunction \( \psi_{2n} \) with the mass squared given by \( \lambda_{2n} \). The propagator for a massive (axial-)vector boson of mass \( m \) is given by

\[ \frac{1}{k^2 + m^2} \left( \eta_{\mu\nu} + \frac{k_\mu k_\nu}{m^2} \right). \] \( \text{(G.20)} \)

From these, the amplitude of the two nucleons exchanging an isotriplet axial-vector meson, summed over the species of the exchanged mesons, becomes

\[ M_{SU}^{(2)}(a) \approx \sqrt{2} E_1 \sqrt{2} E_2 \sqrt{2} E'_1 \sqrt{2} E'_2 \frac{1}{4} (\vec{r}_1 \cdot \vec{r}_2) \]

\[ \times \sum_{n \geq 1} -\frac{g_{a^a NN}^2}{k^2 + \lambda_{2n}} \left( \eta_{\mu\nu} + \frac{k_\mu k_\nu}{\lambda_{2n}} \right) (\bar{u}(p'_1, s'_1) \gamma_5 \gamma^\mu u(p_1, s_1)) (\bar{u}(p'_2, s'_2) \gamma_5 \gamma^\nu u(p_2, s_2)) , \] \( \text{(G.21)} \)

and we obtain a similar expression for the isosinglet case. Note that the cubic couplings are computed in Ref. [8] as

\[ \widehat{g}_{a^a NN} = \frac{N_c}{32\pi^2 \kappa} \langle \partial_Z \psi_{2n}(Z) \rangle, \quad g_{a^a NN} = \frac{8\pi^2 \kappa}{3} \langle \rho^2 \rangle \langle \partial_Z \psi_{2n}(Z) \rangle. \] \( \text{(G.22)} \)

This shows that

\[ g_{a^a NN} = O(\lambda^{-1/2} N_c^{-1/2}), \quad \widehat{g}_{a^a NN} = O(\lambda^{-1/2} N_c^{-1/2}) , \] \( \text{(G.23)} \)

and therefore the isosinglet sector is negligible compared with the isotriplet sector, as in the pion exchange case.

In the nonrelativistic limit, where

\[ \bar{u}(p', s') \gamma_5 \gamma^0 u(p, s) = O(m_B^{-1}), \quad \bar{u}(p', s') \gamma_5 \gamma^0 u(p, s) = i\sigma_{s's}^j + O(m_B^{-2}) , \] \( \text{(G.24)} \)

the scattering amplitude is dominated by the spatial component of the axial-vector fields so that

\[ M_{SU}^{(2)}(a) \approx (2m_B)^2 \frac{1}{4} (\vec{r}_1 \cdot \vec{r}_2) \sum_{n \geq 1} +\frac{g_{a^a NN}^2}{k^2 + \lambda_{2n}} \left( \vec{\sigma}_1 \cdot \vec{\sigma}_2 \right) + \frac{1}{\lambda_{2n}} \left( \vec{k} \cdot \vec{\sigma}_1 \right) \left( \vec{k} \cdot \vec{\sigma}_2 \right) \]

\[ = -(2m_B)^2 \bar{V}_{a}^{SU(2)}. \] \( \text{(G.25)} \)
This yields the effective potential due to the axial-vector-meson exchange:

\[ V_{a}^{SU(2)}(\vec{x}) = -\frac{1}{4} (\vec{r}_1 \cdot \vec{r}_2) \sum_{n \geq 1} g_{a}^{2} \int \frac{d^{3}k}{(2\pi)^{3}} e^{i\vec{k}\cdot\vec{x}} \left[ \frac{\langle \vec{\sigma}_1 \cdot \vec{\sigma}_2 \rangle + \frac{1}{\lambda_{2n}} \frac{\vec{k} \cdot \vec{\sigma}_1}{\vec{k}^2 + \lambda_{2n}} \left( \vec{k} \cdot \vec{\sigma}_2 \right) + \frac{2}{3} (\vec{\sigma}_1 \cdot \vec{\sigma}_2) \right] Y_{2n}(r) \]

= \frac{1}{4} (\vec{r}_1 \cdot \vec{r}_2) \sum_{n \geq 1} g_{a}^{2} \int \frac{d^{3}k}{(2\pi)^{3}} e^{i\vec{k}\cdot\vec{x}} \left[ \frac{1}{\lambda_{2n}} \frac{1}{r^2} \left( 1 + \sqrt{\lambda_{2n}} r + \frac{3}{\lambda_{2n}} r^2 \right) \right] Y_{2n}(r) \]

= V_{C}^{(a)} + S_{12}V_{T}^{(a)}, \quad (G.26)

where

\[ V_{C}^{(a)} = \frac{1}{6} (\vec{r}_1 \cdot \vec{r}_2) (\vec{\sigma}_1 \cdot \vec{\sigma}_2) \sum_{n \geq 1} g_{a}^{2} Y_{2n}(r), \quad (G.27) \]

\[ V_{T}^{(a)} = -\frac{1}{12} (\vec{r}_1 \cdot \vec{r}_2) \sum_{n \geq 1} g_{a}^{2} \int \frac{d^{3}k}{(2\pi)^{3}} e^{i\vec{k}\cdot\vec{x}} \left[ \frac{1}{\lambda_{2n}} \left( \frac{3}{r^2} + \frac{3\sqrt{\lambda_{2n}}}{r} + \lambda_{2n} \right) \right] Y_{2n}(r). \quad (G.28) \]

Using (G.22), it is easy to show that this agrees with the \( n = 2, 4, 6, \cdots \) components of (5.36) and (5.37).

G.3. Vector meson exchange

The Feynman rule states that for the nucleon-nucleon-\( \nu^n \) cubic couplings, we assign

\[ i \frac{\pi}{2} \left( g_{\nu NN} \gamma^\mu - \frac{\hbar_{\nu NN}}{2m_B} \sigma^{\mu\nu} k_\nu \right), \quad i \frac{1}{2} \left( \tilde{g}_{\nu NN} \gamma^\mu - \frac{\hbar_{\nu NN}}{2m_B} \sigma^{\mu\nu} k_\nu \right), \quad (G.29) \]

for the isotriplet and isosinglet cases, respectively. Here, \( \nu^n \) (\( n = 1, 2, \cdots \)) is the vector meson associated with the wavefunction \( \psi_{2n-1} \), whose mass squared is equal to \( \lambda_{2n-1} \), and \( k \) is the momentum of the vector meson flowing outwards from the vertex. It follows that the two-nucleon scattering amplitude due to the exchange of an isotriplet vector meson, summed over the infinite tower of the vector meson species, is given by

\[ M^{SU(2)}_{\nu} = \sqrt{2E_1} \sqrt{2E_2} \sqrt{2E_1'} \sqrt{2E_2'} \frac{1}{4} (\vec{r}_1 \cdot \vec{r}_2) \]

\[ \times \sum_{n \geq 1} \left[ \pi(p_1', s_1') \left( g_{\nu NN} \gamma^\mu - \frac{\hbar_{\nu NN}}{2m_B} \sigma^{\mu\nu} k_\nu \right) u(p_1, s_1) \frac{1}{k^2 + \lambda_{2n-1}} (\eta_{\mu\nu} + \frac{\kappa_{\mu\nu} k_\nu}{\lambda_{2n-1}}) \right] \]

\[ \times \pi(p_2', s_2') \left( g_{\nu NN} \gamma^\nu + \frac{\hbar_{\nu NN}}{2m_B} \sigma^{\nu\sigma} k_\sigma \right) u(p_2, s_2) \]. \quad (G.30) \]

As before, a similar expression follows for the \( U(1) \) part.

In the nonrelativistic limit, we have

\[ \pi(p', s') \gamma^0 u(p, s) = -i\delta_{s's} + \mathcal{O}(m_B^{-2}) \]

51
Furthermore, we note that the cubic coupling constants obtained in Ref. [8] are given by

\begin{equation}
\bar{u}(p', s') \gamma^j u(p, s) = -\frac{i}{2m_B} \left[ (p + p')_j \delta_{ss'} + i \epsilon_{jla}(p - p')_l \sigma^n_{ss'} \right] + \mathcal{O}(m_B^2),
\end{equation}

\begin{equation}
\bar{u}(p', s') \sigma^{0j} u(p, s) = -\frac{i}{2m_B} \left[ (p - p')_j \delta_{ss'} + i \epsilon_{jla}(p + p')_l \sigma^n_{ss'} \right] + \mathcal{O}(m_B^2),
\end{equation}

\begin{equation}
\bar{u}(p', s') \sigma^{jk} u(p, s) = -\epsilon^{jka} \sigma^n_{ss'} + \mathcal{O}(m_B^2).
\end{equation}

Furthermore, we note that the cubic coupling constants obtained in Ref. [8] are given by

\begin{equation}
g_{v,NN} = \langle \psi_{2n-1}(Z) \rangle, \quad h_{v,NN} = \frac{16 \pi^2 \kappa m_B}{3} \langle \rho^2 \rangle \langle \psi_{2n-1}(Z) \rangle,
\end{equation}

\begin{equation}
\tilde{g}_{v,NN} = N_c \langle \psi_{2n-1}(Z) \rangle, \quad \tilde{h}_{v,NN} = N_c \left( \frac{m_B}{16 \pi^2 \kappa} - 1 \right) \langle \psi_{2n-1}(Z) \rangle,
\end{equation}

which imply

\begin{equation}
g_{v,NN} = \mathcal{O}(\lambda^{-1/2} N_c^{-1/2}), \quad h_{v,NN} = \mathcal{O}(\lambda^{1/2} N_c^{3/2}),
\end{equation}

\begin{equation}
\tilde{g}_{v,NN} = \mathcal{O}(\lambda^{-1/2} N_c^{1/2}), \quad \tilde{h}_{v,NN} = \mathcal{O}(\lambda^{-1/2} N_c^{1/2}).
\end{equation}

Then, among the spinor bilinear forms appearing in the amplitudes, the leading ones for large \( \lambda \) and large \( N_c \) are

\begin{equation}
\bar{u}(p', s') \left( g_{v,NN} \gamma^j - \frac{h_{v,NN}}{2m_B} \sigma^{jp}(p - p')_\rho \right) u(p, s) \simeq \frac{h_{v,NN}}{2m_B} \epsilon_{jla}(p - p')_l \sigma^n_{ss'} + \cdots,
\end{equation}

\begin{equation}
\bar{u}(p', s') \left( \tilde{g}_{v,NN} \gamma^0 - \frac{\tilde{h}_{v,NN}}{2m_B} \sigma^{0p}(p - p')_\rho \right) u(p, s) \simeq -i \tilde{g}_{v,NN} \delta_{ss'} + \cdots.
\end{equation}

This shows that for the isotriplet vector mesons, the spatial components dominate the amplitude, while for the isosinglet vector mesons, the time component does. Consequently,

\begin{equation}
\mathcal{M}_{v}^{SU(2)} \sim (2m_B)^2 \frac{1}{4} \left( \bar{\tau}_1 \cdot \bar{\tau}_2 \right) \epsilon_{jla} k_l \sigma^n_{s's1} \epsilon_{jmb} k_m \sigma^n_{s's2} \sum_{n \geq 1} \frac{1}{k_2^2 + \lambda_{2n-1}} \left( \frac{h_{v,NN}}{2m_B} \right)^2,
\end{equation}

\begin{equation}
\mathcal{M}_{v}^{U(1)} \sim -(2m_B)^2 \frac{1}{4} \sum_{n \geq 1} \frac{\tilde{g}_{v,NN}^2}{k_2^2 + \lambda_{2n-1}} = -(2m_B)^2 \tilde{V}_{v}^{U(1)}.
\end{equation}

Fourier-transforming the effective potentials gives

\begin{equation}
V_v(\bar{x}) = \int \frac{d^3 k}{(2\pi)^3} e^{i \bar{x} \cdot \bar{k}} \left( \tilde{V}_{v}^{SU(2)} + \tilde{V}_{v}^{U(1)} \right) = V_C^{(v)}(\bar{x}) + S_1 V_T^{(v)}(\bar{x}),
\end{equation}

with

\begin{equation}
V_C^{(v)}(\bar{x}) = -\frac{1}{6} \left( \bar{\tau}_1 \cdot \bar{\tau}_2 \right) \left( \bar{\sigma}_1 \cdot \bar{\sigma}_2 \right) \sum_{n \geq 1} \lambda_{2n-1} \left( \frac{h_{v,NN}}{2m_B} \right)^2 Y_{2n-1}(r) - \frac{1}{4} \sum_{n \geq 1} \tilde{g}_{v,NN}^2 Y_{2n-1}(r),
\end{equation}

\begin{equation}
V_T^{(v)}(\bar{x}) = \frac{1}{12} \left( \bar{\tau}_1 \cdot \bar{\tau}_2 \right) \sum_{n \geq 1} \left( \frac{h_{v,NN}}{2m_B} \right)^2 \left( \lambda_{2n-1} + \frac{3 \sqrt{\lambda_{2n-1}}}{r} + \frac{3}{r^2} \right) Y_{2n-1}(r).
\end{equation}
Again, this is in agreement with the $n = 1, 3, 5, \cdots$ components of (5.36) and (5.37).

References

1) N. Ishii, S. Aoki and T. Hatsuda, “The nuclear force from lattice QCD,” Phys. Rev. Lett. 99 (2007), 022001, [nucl-th/0611096]. “Nuclear Force from Monte Carlo Simulations of Lattice Quantum Chromodynamics,” [arXiv:0805.2462].

H. Nemura, N. Ishii, S. Aoki and T. Hatsuda, “Hyperon-nucleon force from lattice QCD,” [arXiv:0806.1094].

S. Aoki, J. Balog, T. Hatsuda, N. Ishii, K. Murano, H. Nemura and P. Weisz, “Energy dependence of nucleon-nucleon potentials,” [arXiv:0812.0673].

2) S. R. Beane, P. F. Bedaque, K. Orginos and M. J. Savage, “Nucleon nucleon scattering from fully-dynamical lattice QCD,” Phys. Rev. Lett. 97 (2006), 012001, [hep-lat/0602010].

S. R. Beane, P. F. Bedaque, T. C. Luu, K. Orginos, E. Pallante, A. Parreno and M. J. Savage (NPLQCD Collaboration), “Hyperon nucleon scattering from fully-dynamical lattice QCD,” Nucl. Phys. A 794 (2007), 62, [hep-lat/0612026].

S. R. Beane, K. Orginos and M. J. Savage, “Hadronic Interactions from Lattice QCD,” Int. J. Mod. Phys. E 17 (2008), 1157, [arXiv:0805.4629].

3) F. Myhrer and J. Wroldsen, “THE NUCLEON-NUCLEON FORCE AND THE QUARK DEGREES OF FREEDOM,” Rev. Mod. Phys. 60 (1988), 629.

4) J. M. Maldacena, “The large N limit of superconformal field theories and supergravity,” Adv. Theor. Math. Phys. 2 (1998), 231 [Int. J. Theor. Phys. 38 (1999), 1113], [hep-th/9711200].

5) S. S. Gubser, I. R. Klebanov and A. M. Polyakov, “Gauge theory correlators from non-critical string theory,” Phys. Lett. B 428 (1998), 105, [hep-th/9802109].

6) E. Witten, “Anti-de Sitter space and holography,” Adv. Theor. Math. Phys. 2 (1998), 253, [hep-th/9802150].

7) O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, “Large N field theories, string theory and gravity,” Phys. Rept. 323 (2000), 183, [hep-th/9905111].

8) K. Hashimoto, T. Sakai and S. Sugimoto, “Holographic Baryons: Static Properties and Form Factors from Gauge/String Duality,” Prog. Theor. Phys. 120 (2008), 1093, [arXiv:0806.3122].

9) T. Sakai and S. Sugimoto, “Low energy hadron physics in holographic QCD,” Prog. Theor. Phys. 113 (2005), 843, [hep-th/0412141].

10) T. Sakai and S. Sugimoto, “More on a holographic dual of QCD,” Prog. Theor. Phys.
114 (2005), 1083, [hep-th/0507073]

11) O. Aharony and D. Kutasov, “Holographic Duals of Long Open Strings,” Phys. Rev. D 78 (2008), 026005, [arXiv:0803.3547]

K. Hashimoto, T. Hirayama, F. L. Lin and H. U. Yee, “Quark Mass Deformation of Holographic Massless QCD,” J. High Energy Phys. 07 (2008), 089, [arXiv:0803.4192]

R. McNees, R. C. Myers and A. Sinha, “On quark masses in holographic QCD,” J. High Energy Phys. 11 (2008), 056, [arXiv:0807.5127]

P. C. Argyres, M. Edalati, R. G. Leigh and J. F. Vazquez-Poritz, “Open Wilson Lines and Chiral Condensates in Thermal Holographic QCD,” [arXiv:0811.4617]

See also R. Casero, E. Kiritsis and A. Paredes, “Chiral symmetry breaking as open string tachyon condensation,” Nucl. Phys. B 787 (2007), 98, [hep-th/0702155]

K. Hashimoto, T. Hirayama and A. Miwa, “Holographic QCD and pion mass,” J. High Energy Phys. 06 (2007), 020, [hep-th/0703024]

N. Evans and E. Threlfall, “Quark Mass in the Sakai-Sugimoto Model of Chiral Symmetry Breaking,” [arXiv:0706.3285]

O. Bergman, S. Seki and J. Sonnenschein, “Quark mass and condensate in HQCD,” J. High Energy Phys. 12 (2007), 037, [arXiv:0708.2839]

A. Dhar and P. Nag, “Sakai-Sugimoto model, Tachyon Condensation and Chiral symmetry Breaking,” J. High Energy Phys. 01 (2008), 055, [arXiv:0708.3233]

A. Dhar and P. Nag, “Tachyon condensation and quark mass in modified Sakai-Sugimoto model,” Phys. Rev. D 78 (2008), 066021, [arXiv:0804.4807]

12) E. Witten, “Anti-de Sitter space, thermal phase transition, and confinement in gauge theories,” Adv. Theor. Math. Phys. 2 (1998), 505, [hep-th/9803131]

13) D. J. Gross and H. Ooguri, “Aspects of large N gauge theory dynamics as seen by string theory,” Phys. Rev. D 58 (1998), 106002, [hep-th/9805129]

14) E. Witten, “Baryons and branes in anti de Sitter space,” JHEP 9807 (1998), 006, J. High Energy Phys. 07 (1998), 006, [hep-th/9805112]

15) M. R. Douglas, “Branes within branes,” [hep-th/9512077]

16) J. L. Gervais and B. Sakita, “Extended particles in quantum field theories,” Phys. Rev. D 11 (1975), 2943.

17) N. S. Manton, “A Remark On The Scattering Of Bps Monopoles,” Phys. Lett. B 110 (1982), 54.

18) H. Hata, T. Sakai, S. Sugimoto and S. Yamato, “Baryons from instantons in holographic QCD,” Prog. Theor. Phys. 117 (2007), 1157, [hep-th/0701280]

19) D. K. Hong, M. Rho, H. U. Yee and P. Yi, “Chiral dynamics of baryons from string theory,” Phys. Rev. D 76 (2007), 061901, [hep-th/0701276]. “Dynamics of Baryons
from String Theory and Vector Dominance,” J. High Energy Phys. 09 (2007), 063, arXiv:0705.2632; “Nucleon Form Factors and Hidden Symmetry in Holographic QCD,” Phys. Rev. D 77 (2008), 014030, arXiv:0710.4615.

20) J. Park and P. Yi, “A Holographic QCD and Excited Baryons from String Theory,” J. High Energy Phys. 06 (2008), 011, arXiv:0804.2926.

21) H. Hata, M. Murata and S. Yamato, “Chiral currents and static properties of nucleons in holographic QCD,” Phys. Rev. D 78 (2008), 086006, arXiv:0803.0180.

22) K. Y. Kim and I. Zahed, “Electromagnetic Baryon Form Factors from Holographic QCD,” J. High Energy Phys. 09 (2008), 007, arXiv:0807.0033.

23) G. Panico and A. Wulzer, “Nucleon Form Factors from 5D Skyrmions,” arXiv:0811.2211.

24) S. Seki and J. Sonnenschein, arXiv:0810.1633.

25) G. S. Adkins, C. R. Nappi and E. Witten, “Static Properties Of Nucleons In The Skyrme Model,” Nucl. Phys. B 228 (1983), 552.

26) T. H. R. Skyrme, “A Nonlinear field theory,” Proc. Roy. Soc. Lond. A 260 (1961), 127; “Particle states of a quantized meson field,” Proc. Roy. Soc. Lond. A 262 (1961), 237; “A Unified Field Theory Of Mesons And Baryons,” Nucl. Phys. 31 (1962), 556.

27) K. Nawa, H. Suganuma and T. Kojo, “Baryons in Holographic QCD,” Phys. Rev. D 75 (2007), 086003, [hep-th/0612187]; “Brane-induced Skyrmion on $S^3$: baryonic matter in holographic QCD,” Phys. Rev. D 79 (2009), 026005, [arXiv:0810.1005 [hep-th]].

28) M. F. Atiyah and N. S. Manton, “Skyrmions from instantons,” Phys. Lett. B 222 (1989), 438.

29) A. Hosaka, S. M. Griffies, M. Oka and R. D. Amado, “Two skyrmion interaction for the Atiyah-Manton ansatz,” Phys. Lett. B 251 (1990), 1.

A. Hosaka, M. Oka and R. D. Amado, “Skyrmions and their interactions using the Atiyah-Manton construction,” Nucl. Phys. A 530 (1991), 507.

N. R. Walet, R. D. Amado and A. Hosaka, “Skyrmions and the nuclear force,” Phys. Rev. Lett. 68 (1992), 3849.

N. R. Walet, “The Kinetic Energy And The Geometric Structure In The B = 2 Sector Of The Skyrme Model: A Study Using The Atiyah-Manton Ansatz,” Nucl. Phys. A 586 (1995), 649, [hep-ph/9410254].

R. A. Leese, N. S. Manton and B. J. Schroers, “Attractive Channel Skyrmions And The Deuteron,” Nucl. Phys. B 442 (1995), 228, [hep-ph/9502405].

30) A. A. Belavin, A. M. Polyakov, A. S. Shvarts and Yu. S. Tyupkin, “Pseudoparticle solutions of the Yang-Mills equations,” Phys. Lett. B 59 (1975), 85.
31) M. F. Atiyah, N. J. Hitchin, V. G. Drinfeld and Y. I. Manin, “Construction of instantons,” Phys. Lett. A 65 (1978), 185.
32) E. Corrigan and P. Goddard, “Construction Of Instanton And Monopole Solutions And Reciprocity,” Annals Phys. 154 (1984), 253.
33) H. Osborn, “Calculation Of Multi - Instanton Determinants,” Nucl. Phys. B 159 (1979), 497.
34) A. Maciocia, “Metrics on the moduli spaces of instantons over Euclidean four space,” Commun. Math. Phys. 135 (1991), 467.
35) N. Dorey, V. V. Khoze and M. P. Mattis, “Multi-Instanton Calculus in N=2 Supersymmetric Gauge Theory,” Phys. Rev. D 54 (1996), 2921, hep-th/9603136.
36) K. Peeters and M. Zamaklar, “Motion on moduli spaces with potentials,” J. High Energy Phys. 12 (2001), 032, hep-th/0107164.
37) K. Y. Kim and I. Zahed, “Nucleon-Nucleon Potential from Holography,” arXiv:0901.0012.
38) Y. Kim, S. Lee and P. Yi, “Holographic Deuteron and Nucleon-Nucleon Potential,” arXiv:0902.4048.

Note added: When preparing this paper, we became aware of Ref. 37), which overlaps partly with our strategy.

Note added in the second version: The added appendix G has some overlaps with Ref. 38), which appeared when we were preparing the revised version.