Analytic Solutions of the Ultra-relativistic Thomas-Fermi Equation

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It is well known that the ultra-relativistic Thomas-Fermi equation, amply adopted in the study of heavy nuclei, admits an exact solution for a constant proton distribution within a spherical core of radius \( R_c \). Here exact solutions of a generalized ultra-relativistic Thomas-Fermi equation are presented, assuming a Wood-Saxon-like proton distribution and its further generalizations. These solutions present an overcritical electric field close to their surface. The variation of the electric fields as a function of the generalized Wood-Saxon parameters are studied.

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INTRODUCTION

To study the electrodynamic properties of the bulk matter at nuclear densities a step proton distribution has been chosen \[1, 2\]. Using the Migdal et. al. approximation, \[3\], the ultra-relativistic Thomas-Fermi model, which governs this problem, reads

\[
\frac{d^2 \phi(x)}{dx^2} = \phi(x)^3 - \theta(-x),
\]

where the proton density, \( n_p \), and the Coulomb potential at the center, \( V(0) \), are given by

\[
x = k(r-R_c),
\]

\[
k = 2\sqrt{\alpha}(\pi/6)^{1/6}n_p^{1/3},
\]

\[
eV(0) = (3\pi^2n_p)^{1/3}.
\]

The equation (1) admits the exact solution

\[
\phi(x) = \begin{cases} 
1 - 3 & \left[1 + 2^{-1/2}\sinh(a - \sqrt{3}x)\right]^{-1}, \\
\frac{\sqrt{\pi}e}{(x+b)}, & x > 0,
\end{cases}
\]

where integration constants \( a \) and \( b \) are: \( \sinh a = 11\sqrt{2} \), \( a = 3.439; b = (4/3)\sqrt{2}. \[3\].

GENERALIZED ULTRA-RELATIVISTIC THOMAS-FERMI EQUATION

In this section we want to look for exact solutions to a generalized ultra-relativistic Thomas-Fermi equation

\[
\frac{d^2 \phi(x)}{dx^2} = \phi(x)^3 - f_p \theta(-x),
\]

where

\[
\begin{cases} 
f_p(x_b) \to 0, & 0 \leq x_b \leq \infty \\
f_p(-\infty) \to 1, \\
f'_p(x) \leq 0, & \text{for all } x
\end{cases}
\]

It is possible to write several distinct infinite \( b \)-dependent sets of analytic solutions to the Thomas-Fermi Eq. (6).
\[-\text{Set 1}\]

\[\phi(x; b) = \begin{cases} \frac{1}{2} - \frac{1}{\pi} \arctan(bx), & \text{for } x < x_b, \\ \frac{1}{b + x}, & \text{for } x > x_b, \end{cases}\]  

(8)

and leads to the following set of proton profiles

\[f_p(x; b) = \begin{cases} \frac{1}{\pi}(\left(\frac{1}{2} - \arctan(bx)\right) - \frac{2b^2x}{\pi(1 + bx^2)}), & \text{for } x < x_b, \\ 0, & \text{for } x > x_b, \end{cases}\]  

(9)

where \(\alpha, \beta\) are real constants given by

\[\begin{align*}
\alpha &= \frac{\pi}{b}(\phi(x_b; b))^2, \\
\beta &= \frac{\pi}{b}\phi(x_b; b) - x_b,
\end{align*}\]  

(10)

because of the continuity of \(\phi(x; b), \phi'(x; b)\) in \(x_b\).

The electric field for \(x < x_b\), is given by

\[E(x; b) = \frac{2}{(3\pi)^{1/2}}e^2V(0)^2\frac{1}{\pi} \frac{b}{1 + (bx)^2}.\]  

(11)

The parameter \(b\) describes the width \(2\delta\) (in cm) of the transition layer near the edge of the core

\[b = \frac{1}{k\delta}.\]  

(12)

Precisely \(2\delta\) is the width of the transition layer of the core in which the electric field goes from its maximum to the half of its maximum. Now, let \(b_c\) be the value of \(b\) such that the electric field \(E(x = 0; b_c)\) is equal to the critical field \(E_c\). Then

\[b_c;\text{Set 1} \approx \frac{1}{0.8} \frac{E_c}{E_{\text{max}}},\]  

(13)

and

\[\delta_c;\text{Set 1} = \left[\frac{1}{229/6} \frac{27}{5} \frac{h}{mc}\right] a_0 n_p^{1/3}(cm).\]  

(14)

where \(E_{\text{max}}\) is the electric field at \(x = 0\) to the step-proton distribution. We see that \(\delta_c\) can be of the order of the Bohr radius \(a_0\) i.e. of order of \(10^3\) electron Compton length.

\[-\text{Set 2}\]

\[\phi(x; b) = \begin{cases} \frac{1}{2} - \frac{1}{\alpha}\tanh(bx), & \text{for } x < x_b, \\ \frac{1}{b + x}, & \text{for } x > x_b, \end{cases}\]  

(15)

and leads to the following set of proton profiles

\[f_p(x; b) = \begin{cases} \left(\frac{1}{8}(1 - \tanh(bx))^3 - b^2\tanh(bx)(1 - \tanh(bx)^2)\right), & \text{for } x < x_b, \\ 0, & \text{for } x > x_b, \end{cases}\]  

(16)

where \(\alpha, \beta\) are real constants given by

\[\begin{align*}
\alpha &= \frac{1}{\pi}\left(\frac{1}{2} - \frac{\tanh(bx)}{1 + \tanh(bx)^2}\right), \\
\beta &= \left(\frac{b + 1}{b - 1 - \tanh(bx)^2}\right) - x_b,
\end{align*}\]  

(17)

because of the continuity of \(\phi(x; b), \phi'(x; b)\) in \(x_b\).

The electric field for \(x < x_b\), is given by

\[E(x; b) = \frac{2}{(3\pi)^{1/2}}e^2V(0)^2\frac{1}{2\cosh^2(bx)}.\]  

(18)
The parameter $b$ as above, describes the width $2\delta$ (in cm) of the transition layer near the edge of the core. Precisely

\[ b_{SET2} = b_{SET1} \frac{\ln(2\sqrt{2} + 3)}{2}. \]  

(19)

Now, let $b_c$ be the value of $b$ such that the electric field $E(x = 0; b_c)$ is equal to the critical field $E_c$. Then

\[ b_{c; SET2} \approx \frac{1}{1.3} \frac{E_c}{E_{max}}, \]  

(20)

and

\[ \delta_{c; SET2} = \frac{\ln(2\sqrt{2} + 3)}{2} \delta_{c; SET1}(cm). \]  

(21)

We note that

\[ \frac{1}{2} - \frac{1}{2} \tanh(bx) = \frac{1}{1 + e^{2bx}} \]  

(22)

which is well known in nuclear physics as Wood-Saxon profile.

The Wood-Saxon profile can be generalized by

\[ \phi(x; a, b) = \begin{cases} 
  1 - \frac{1}{1 + ae^{-bx}} & \text{for } x < x_b, \\
  \frac{a}{\alpha + x} & \text{for } x > x_b,
\end{cases} \]  

(23)
with the following set of proton profiles

\[
    f_p(x; a, b) = \begin{cases} \left(1 - \frac{1}{1 + ae^{-bx}}\right)^3 + \frac{b^2 e^{-bx} (e^{-bx} - 1)}{(1 + ae^{-bx})^2 (1 + ae^{-bx} - 1)} & \text{for } x > x_b, \\
    0, & \text{for } x > x_b,
\end{cases}
\]

(24)

where \( \alpha, \beta \) are real constants given by

\[
    \left\{ \begin{array}{l}
    \alpha = \left[ 1 - \frac{1}{(1 + ae^{-bx})^{1/\alpha}} \right] \frac{[(1 + ae^{-bx})^{1/\alpha} - 1]}{[(1 + ae^{-bx})^{1/\alpha} - 1] + be^{-bx}}, \\
    \beta = -x_b + \frac{be^{-bx}}{[(1 + ae^{-bx})^{1/\alpha} - 1] + be^{-bx}}.
\end{array} \right.
\]

(25)

because of the continuity of \( \phi(x; a, b), \phi'(x; a, b) \) in \( x_b \).

We have

\[
\phi'(x; a, b) = -\frac{b e^{-bx}}{(1 + ae^{-bx})^{1/\alpha}(1 + ae^{-bx})},
\]

(26)

hence the maximum of \( E(x; a, b) \) is

\[
E(x = 0; a, b) = \frac{b}{(1 + a)^{1/\alpha}(1 + a)} E_{\text{max}},
\]

(27)

- Set 3

\[
\phi(x; b) = \begin{cases} \frac{1}{2} - \frac{1}{2} \sinh^{-1}(bx), & \text{for } x < x_b, \\
\frac{1}{2} + \frac{1}{2} \sinh^{-1}(bx), & \text{for } x > x_b,
\end{cases}
\]

(28)

and leads to the following set of proton profiles

\[
f_p(x; b) = \begin{cases} \left(\frac{1}{2} - \frac{1}{2} \sinh^{-1}(bx)\right)^3 - \frac{b^2 x}{12(1 + b^2 x^2)^{1/2}}, & \text{for } x < x_b, \\
0, & \text{for } x > x_b,
\end{cases}
\]

(29)

where \( \alpha, \beta \) are real constants.

The Set 1, Set 3 of analytic solutions to the Thomas-Fermi equation (II) belong to the more general following set

\[
\phi(x; a, b) = \begin{cases} c_1 \Phi(x; a, b) + c_2, & \text{for } x < x_b, \\
\frac{1}{2} - \frac{1}{2} \sinh^{-1}(bx), & \text{for } x > x_b,
\end{cases}
\]

(30)

where \( \Phi(x, a, b), c_1, c_2 \) are given by

\[
\left\{ \begin{array}{l}
\Phi(x; a, b) = -\frac{x}{2x(a - 1)} F_{1/2}(1/2, a; 3/2, -b^2 x^2), \\
c_1 = (\lim_{x \to -k R_e} \Phi(x, a, b) + \lim_{x \to -\infty} \Phi(x, a, b))^{-1}, \quad c_2 = \lim_{x \to -\infty} \Phi(x, a, b).
\end{array} \right.
\]

(31)

and \( F_{1/2} \) is the Gauss hyper-geometric function. For positive integer \(( \geq 2) \) or positive half-integer \(( \geq 3/2) \) values of \( a, F_{1/2} \) can be written in terms of elementary functions (Table II).

Also the Set 2 belongs to the more general set given by

\[
\phi(x; a, b) = \begin{cases} c_1 \Phi(x; a, b) + c_2, & \text{for } x < x_b, \\
\frac{1}{2} + \frac{1}{2} \sinh^{-1}(bx), & \text{for } x > x_b,
\end{cases}
\]

(32)

where

\[
\left\{ \begin{array}{l}
\Phi(x; a, b) = \int_{-k R_e}^{x} \frac{1}{2x} (1 - \tanh(by))^2 dy, \\
c_1 = (\lim_{x \to -k R_e} \Phi(x, a, b) + \lim_{x \to -\infty} \Phi(x, a, b))^{-1}, \quad c_2 = \lim_{x \to -\infty} \Phi(x, a, b).
\end{array} \right.
\]

(33)

For positive integer \(( \geq 1) \) or positive half-integer \(( \geq 1/2) \) values of \( a, F_{1/2} \) can be written in terms of elementary functions (Table II).

These results, obtained by explicit analytic formulae, complement the numerical results presented in [4].

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TABLE I: Generalized exact solutions to Set 1 and Set 3

| a   | Φ(x; a, b)                                                                 |
|-----|------------------------------------------------------------------------------|
| 3/2 | - sinh^{-1}(bx)                                                              |
| 2   | - \frac{1}{2b} \text{arctan}(bx)                                             |
| 5/2 | - \frac{x}{\sqrt{3(1+b^2x^2)}}                                              |
| 3   | - \frac{x}{(4+2b^2x^2)^{3/2}} - \frac{1}{16b} \text{arctan}(bx)             |
| 7/2 | - \frac{x}{(15+4b^2x^2)^{3/2}} - \frac{1}{16b} \text{arctan}(bx)             |
| 4   | \frac{x}{(24(1+b^2x^2)^2)^{3/2}} - \frac{1}{16b} \text{arctan}(bx)          |
| 9/2 | \frac{x}{(15+6b^2x^2)^{3/2}} - \frac{1}{16b} \text{arctan}(bx)              |

TABLE II: Generalized exact solutions to Set 2

| a   | Φ(x; a, b)                                                                 |
|-----|------------------------------------------------------------------------------|
| 1/2 | - sinh^{-1}(\text{tanh}(bx))                                               |
| 1   | - \frac{1}{2} \text{tanh}(bx) - \frac{1}{4} \ln(\text{tanh}(bx) - 1) + \frac{1}{4} \ln(\text{tanh}(bx) + 1) |
| 3/2 | - \frac{1}{4} \text{tanh}(bx) \sqrt{1 - \text{tanh}(bx)^2} - \frac{1}{4} \sinh^{-1}(\text{tanh}(bx)) |
| 2   | \frac{1}{12} \text{tanh}(bx)^3 - \frac{1}{2} \text{tanh}(bx) - \frac{1}{4} \ln(\text{tanh}(bx) - 1) + \frac{1}{4} \ln(\text{tanh}(bx) + 1) |
| 5/2 | - \frac{1}{48} \text{tanh}(bx) \sqrt{1 - \text{tanh}(bx)^2} - \frac{1}{48} \text{tanh}(bx)(1 - \text{tanh}(bx)^2)^{3/2} - \frac{1}{48} \sinh^{-1}(\text{tanh}(bx)) |

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