REGULARITY FOR SOLUTIONS OF NON LOCAL, NON SYMMETRIC EQUATIONS

HÉCTOR CHANG LARA AND GONZALO DÁVILA

ABSTRACT. We study the regularity for solutions of fully nonlinear integro-differential equations with respect to nonsymmetric kernels. More precisely, we assume that our operator is elliptic with respect to a family of integro differential linear operators where the symmetric part of the kernels have a fixed homogeneity $\sigma$ and the skew symmetric part have strictly smaller homogeneity $\tau$. We prove a weak ABP estimate and $C^{1,\alpha}$ regularity. Our estimates remain uniform as we take $\sigma \to 2$ and $\tau \to 1$ so that this extends the regularity theory for elliptic differential equations with dependence on the gradient.

1. Introduction

We are interested in studying integro differential equations that arise when studying discontinuous stochastic processes. By the Lévy-Khintchine formula, the generator of an $n$-dimensional Lévy process is given by

$$Lu(x) = \sum_{ij} a_{ij} \delta_{ij} u + \sum_i b_i \delta_i u$$

$$+ \int_{\mathbb{R}^n} (u(x+y) - u(x) - \nabla u(x) \cdot y \chi_{B_1}(y)) d\mu(y),$$

where $\mu$ is a positive measure such that $\int |y|^2/(|y|^2 + 1) d\mu(y) < \infty$. The first and second term corresponds to the diffusion and drift part, and the third one correspond to the jump. The effect of first term is already well understood as it regularizes the solution. The type of equations that we will study come from processes with only the jump part,

$$Lu(x) = \int_{\mathbb{R}^n} (u(x+y) - u(x) - \nabla u(x) \cdot y \chi_{B_1}(y)) d\mu(y).$$

More general than the linear operator are the fully non linear ones also important in stochastic control as seen in [13]. For example, a convex type of equation takes the form,

$$Iu(x) = \sup_{\alpha} L_\alpha u(x).$$

Equation (1.2) can be seen as a one player game, for which he can choose different strategies at each step to maximize the expected value of some
function at the first exit point of the domain. A natural extension for (1.2), when there are two players competing is

$$Iu(x) = \inf_{\beta} \sup_{\alpha} L_{\alpha \beta} u(x).$$

We are mainly interested in studying interior regularity for solutions of

$$(1.3) \quad Iu(x) = f(x), \quad \text{in } \Omega,$$

for $f$ continuous, $\Omega$ a given domain and $I$ a fully non linear operator of fractional order to be defined in the next section. In [11] the regularity for this type of problem was already established by using analytic techniques. However those estimates blow up as the order of the equation goes to the classical one, so it was expected that better estimates could be possible. Those results are more elaborated and presented in [3], [4] and [5] in the case that the kernels are symmetric. We remove this symmetry hypothesis of the kernel and are able to obtain $C^\alpha$ regularity and $C^{1,\alpha}$ regularity if $f$ is constant.

The paper is divided as follows. In Section 2 we will give a precise definition of the equation and kernels we are dealing with. We will introduce the notion of viscosity solution and ellipticity for nonlocal integro differential operators by defining some suitable maximal operators. In Section 3 we state the main results of this work, which is $C^\alpha$ and $C^{1,\alpha}$ regularity for solutions of equations of the form (1.3) under different hypothesis on the kernels. In section 4 we study the basic stability properties of the elliptic integro differential operators, state the standard comparison principle and prove existence of the solution of the Dirichlet problem by using Perron’s method. Sections 5 and 6 are the core of this paper. In section 5 we prove a very weak ABP estimate which combined with a rescale argument will allow us to prove in Section 6 an $L^\varepsilon$ lemma. Section 7 and 8 deal with $C^\alpha$ and $C^{1,\alpha}$ regularity by applying the previous point estimates. Finally in Section 9 we make some comments on what type of results are recovered in the limits when $\sigma$ or $\tau$ go to two and one. In Section 10 we discuss future research and some open problems.

2. Viscosity Solutions and Preliminaries

In this work we restrict ourselves to measures $d\mu = K(y)dy$. From equation (1.1) we formally can write

$$Lu(x) = \int_{\mathbb{R}^n} \delta_e(u, x, y)K_e(y)dy + \int_{\mathbb{R}^n} \delta_o(u, x, y)K_o(y)dy + \nabla u(x) \cdot b,$$

where

$$\delta_e(u, x, y) = u(x + y) + u(x - y) - 2u(x),$$

$$\delta_o(u, x, y) = u(x + y) - u(x - y).$$
$K_{e,o}$ are the even and odd part of $K$ and $b$ is a constant vector given by

$$b = \int_{B_1} K_o(y)ydy.$$  

Notice that if the total kernel $K$ is even the last two terms in (2.4) disappear. This was convenient in [3] as these bring additional difficulties with the scaling as can be noticed in [9].

The second term can be considered as a drift term, in the sense that has a "direction", by $K_o$ being odd. If the singularity of $K_o$ at the origin is of order $n + \tau$, with $\tau \to 1^-$, then this integral becomes a gradient term. For this reason, one can consider studying the regularizing effect of the first two terms. Then, the linear operators we are interested take the form

$$Lu(x) = PV \int_{\mathbb{R}^n} (u(x + y) - u(x))K(y)dy. \tag{2.5}$$

The kernel, when decomposed in its symmetric and skew symmetric parts, $K = K_e + K_o$ respectively, must satisfy a standard integrability assumption,

$$\int_{\mathbb{R}^n} \frac{|y|^2}{|y|^2 + 1}|K_{e,o}(y)|dy < \infty. \tag{2.6}$$

Now we want to define what is a fully non linear non local operator. Keep in mind examples as the linear ones in (2.5) or also the following ones,

$$\begin{align*}
(I\text{-sup type}) & \quad Iu(x) = \inf_{\alpha} \sup_{\beta} L_{\alpha,\beta} u(x), \\
(\text{Maximal}) & \quad M^+_{\mathcal{L}} u(x) = \sup_{L \in \mathcal{L}} Lu(x), \\
(\text{Minimal}) & \quad M^-_{\mathcal{L}} u(x) = \inf_{L \in \mathcal{L}} Lu(x). \tag{2.7-2.9}
\end{align*}$$

**Definition 2.1.** We say that $I$ is a non local fully non linear operator if it satisfies the following:

(i) If $u$ is any bounded $C^{1,1}(x)$ function then $Iu(x)$ is well defined.

(ii) If $u \in C^2(\Omega)$ for some open set $\Omega \subseteq \mathbb{R}^n$, then $Iu(x)$ is a continuous function in $\Omega$.

Recall that $u$ is $C^{1,1}(x)$ if and only if $|\delta_{e}(u,x,y)| = O(|y|^2)$, as $y$ goes to zero.

An important family, that will be used for the study of regularity, is given by $\mathcal{L}_0$ with operators $L$, where the kernels $K_{e,o}$ are comparable to those the $\sigma$ fractional Laplacian and some derivation of order $\tau$, with $0 < \sigma < 2$ and $0 < \tau < \min(1, \sigma)$.

$$\begin{align*}
(2 - \sigma)\frac{\lambda}{|y|^{n+\sigma}} \leq K_e \leq (2 - \sigma)\frac{\lambda}{|y|^{n+\sigma}}, \tag{2.10}

|K_o| \leq (1 - \tau)\frac{b}{|y|^{n+\tau}}. \tag{2.11}
\end{align*}$$
In this family the operators (2.8), (2.9) take the explicit form
\begin{equation}
M^+_L v(x) = M^+_\sigma v(x) + b(1 - \tau) \int_{\mathbb{R}^n} \frac{\delta_o(v, x, y)}{|y|^{n+\sigma}} \, dy,
\end{equation}
\begin{equation}
M^-_L v(x) = M^-_\sigma v(x) - b(1 - \tau) \int_{\mathbb{R}^n} \frac{\delta_o(v, x, y)}{|y|^{n+\sigma}} \, dy,
\end{equation}
where $M^+_\sigma$ and $M^-_\sigma$ are the extremal operators found in [3], i.e,
\begin{align*}
M^+_\sigma v(x) &= (2 - \sigma) \int_{\mathbb{R}^n} \frac{\Lambda \delta_e(v, x, y)^+ - \lambda \delta_e(v, x, y)^-}{|y|^{n+\sigma}} \, dy, \\
M^-_\sigma v(x) &= (2 - \sigma) \int_{\mathbb{R}^n} \frac{\lambda \delta_e(v, x, y)^+ - \Lambda \delta_e(v, x, y)^-}{|y|^{n+\sigma}} \, dy.
\end{align*}
For ease of notation we introduce what we call the maximal $\tau$ derivative $|D^\tau v(x)|$ given by
\begin{equation}
|D^\tau v(x)| = (1 - \tau) \int_{\mathbb{R}^n} \frac{\delta_o(v, x, y)}{|y|^{n+\tau}} \, dy,
\end{equation}
so that we can rewrite the operators as
\begin{align*}
M^+_L v(x) &= M^+_\sigma v(x) \pm b|D^\tau v(x)|. \\
The factors $(2 - \sigma)$ and $(1 - \tau)$ become important as $\sigma \to 2$, and $\tau \to 1$, as they will allows us to recover second order differential equations with gradient terms as limits of integro differential equations. We will elaborate on this on Section 9 with more detail.

We have that for any subfamily $\mathcal{L} \subseteq \mathcal{L}_0$
\[ M^-_{\mathcal{L}_0} u \leq M^-_L u \leq M^+_L u \leq M^+_{\mathcal{L}_0} u. \]
This will be an important tool as we have already shown explicit expressions for $M^+_L$.

We introduce now the concept of ellipticity for a general family $\mathcal{L}$ of linear operators.

**Definition 2.2.** Let $\mathcal{L}$ be a class of linear integro differential operators satisfying (2.6). We say that a fully non linear operator $I$ is elliptic with respect to the class $\mathcal{L}$ if
\begin{equation}
M^-_L (u - v)(x) \leq Iu(x) - Iv(x) \leq M^+_L (u - v)(x).
\end{equation}
Viscosity solutions provide the right framework to study fully non linear equations, as seen in the local case in [2], and also in the non local case, see for example [1].

**Definition 2.3.** A bounded function $u : \mathbb{R}^n \to \mathbb{R}$, upper (lower) semicontinuous in $\Omega$, is said to be a sub solution (super solution) to $Iu = f$, and we write $Iu \geq f$ ($Iu \leq f$), if every time $\varphi$ is a function $C^2$ in a neighborhood $N$ of $x \in \Omega$ such that
(i) \( \varphi(x) = u(x) \)  
(ii) \( \varphi(y) > u(y) \) (\( \varphi(y) < u(y) \)) for every \( x \in N \setminus \{x\} \),  
then \( Iv(x) \geq f(x) \) (\( Iv(x) \leq f(x) \)), for \( v \) defined as  
\[
v = \begin{cases} 
\varphi & \text{in } N, \\
u & \text{in } \mathbb{R}^n \setminus N.
\end{cases}
\]

In the definition we could have used test functions that are only \( C^{1,1} \). This provides a weaker definition of viscosity solution a priori, by enlarging the set of test functions. As in [3] and [9] these definitions are equivalent for operators of the inf-sup (or sup-inf) type, if the operators are elliptic with respect to a class with non negative kernels.

Here are some properties that follow from the definition.

**Lemma 2.1.** Let \( I \) be an elliptic operator with respect to a class \( \mathcal{L} \) of non negative kernels. Let \( u, v \) be smooth functions such that \( u \geq v \) in a neighborhood of \( x \) and \( u(x) = v(x) \). Then \( Iu(x) \geq Iv(x) \)

**Lemma 2.2.** Let \( I \) be an elliptic operator with respect to a class \( \mathcal{L} \) of non negative kernels. Let \( u, v \) be viscosity solutions of \( Iu \leq f \), then \( w = \min(u, v) \) is also a super solution.

### 3. Statement of Results

In this section we state the main results obtained in this paper. An important tool used to prove the following theorems is a point estimate, also known as \( L^\varepsilon \) Lemma. This comes from a partial ABP inequality similar to the one in [3] and a scaling argument which decreases the effect of the lower order term \( \tau \).

In order to prove our regularity results we will need to impose some assumptions on \( \sigma \) and \( \tau \). Given \( \sigma_0, \tau_0, m, A_0 > 0 \) we will assume that the following holds.

(H1) \( 2 \sigma \geq \sigma_0 > 0, 1 \tau \geq \tau_0 > 0 \),  
(H2) \( \sigma - \tau \geq m > 0 \),  
(H3) \( \lambda A_0 (2 - \sigma) \geq b(1 - \tau) \).

**Theorem 3.1.** Let \( \sigma_0, \tau_0, m, A_0 > 0 \) and assume that H1, H2 and H3 holds. Let \( u \) be a bounded function in \( \mathbb{R}^n \) such that in \( B_1 \),  
\[
\mathcal{M}_{\mathcal{L}_0}^{+} u \geq -C_0 \quad \text{and} \quad \mathcal{M}_{\mathcal{L}_0}^{-} u \leq C_0,
\]
in the viscosity sense. Then there exists a universal exponent \( \alpha > 0 \) such that \( u \in C^\alpha(B_{1/2}) \) and  
\[
\|u\|_{C^\alpha(B_{1/2})} \leq C(\|u\|_\infty + 1)^2(\|u\|_\infty + C_0)
\]
for some universal constant \( C > 0 \).

An immediate corollary is the following.
Corollary 3.2. Let $\sigma_0, \tau_0, m, A_0 > 0$ as before and let $u$ be a bounded function in $\mathbb{R}^n$ such that in $B_1$, $Iu = f$ in the viscosity sense, with $I$ elliptic respect to $L_0$. Assume that $H1$, $H2$ and $H3$ holds, then there exists a universal exponent $\alpha > 0$ such that $u \in C^{\alpha}(B_{1/2})$ and
\[ \|u\|_{C^{\alpha}(B_{1/2})} \leq C(\|u\|_{\infty} + 1)^2(\|u\|_{\infty} + \|f\|_{\infty}) \]
for some universal constant $C > 0$.

Coming back to Theorem 3.1, even though we don’t recover the full standard estimate
\[ [u]_{C^{\alpha}_{B_{1/2}}} \leq C(\|u\|_{\infty} + C_0), \]
our bounds still remain uniform as $\sigma \to 2$ and $\tau \to 1$, which allows us to recover Hölder regularity for equations with bounded measurable coefficients including gradient terms. We also note that $\alpha$ deteriorates as $\tau \to \sigma$, which is known as the critical case. Note that in this critical case $\sigma = \tau$, both terms in the equation are of the same order and rescaling the equation doesn’t have any effect on the $\tau$ derivative, hence our argument doesn’t work. The case $\sigma = 1$ and $\tau = 1$ has been recently studied by L. Silvestre in the parabolic setting in [12].

To get higher regularity we need to add two extra assumptions, which are a modulus of continuity of $K_e$ and $K_o$ in measure and positivity of the total kernel. More precisely, given $\rho_0$ we define the class $L_1$ by the operators $L$ with kernels $K = K_e + K_o \geq 0$ such that $K_e$ and $K_o$ satisfy (2.10) and (2.11) respectively and
\[ \int_{\mathbb{R}^n \setminus B_{\rho_0}} \frac{|K(y) - K(y - h)|}{|h|} dy \leq C \]
for every $|h| \leq \rho_0/2$. We note that a sufficient condition for (3.15) is that $|\nabla K(y)| \leq \Lambda/|y|^{n+1+\sigma}$. In this smaller class we are able to get $C^{1,\alpha}$ by studying the incremental quotients of solutions and using the a priori $C^{\alpha}$ estimates given by Theorem 3.1. The proof follows the ideas of [2] and [3].

Theorem 3.3. Let $\sigma_0, \tau_0, m, A_0 > 0$ and assume that $H1$, $H2$ and $H3$ holds. There is $\rho_0 > 0$ small enough so that if $I$ is a nonlocal elliptic with respect to $L_1$ and $u$ is a bounded viscosity solution of $Iu = 0$ in $B_1$, then there is a universal $\alpha > 0$ such that $u \in C^{1,\alpha}(B_{1/2})$ and
\[ \|u\|_{C^{1,\alpha}(B_{1/2})} \leq C(\|u\|_{\infty} + 1)^2(\|u\|_{\infty} + I0) \]
for some universal $C > 0$ and where $I0$ means the value of the operator $I$ applied to the constant function zero.

In the proofs of our regularity results the odd part doesn’t have to be of a fixed order. We could ask for example
\[ |K_o| \leq b \max \left( \frac{1 - \tau_1}{|y|^{n+\tau_1}}, \frac{1 - \tau_2}{|y|^{n+\tau_2}} \right) \]
with $0 < \tau_1 \leq \tau_2 < \min(1, \sigma)$. The reason is that the proofs will treat the lower order term as a perturbation term that can be made small enough after a dilation large enough. For the sake of keeping the exposition simpler we decided to restrict to the case of $\tau_1 = \tau_2 = \tau$.

4. Stability Properties, Comparison Principle and Existence

This section is devoted to prove basic results that concern the definition of viscosity solution for our class of integro differential equations. Most of the results are the same as the ones in [3] but we will state them here for sake of completion. We also note that most of the proofs are minor modifications of the original ones in [3] and we will just point out the differences in these cases.

The notion of viscosity solution given in (2.3) is strong enough that will allow us to compute $Iu$ every time we have a paraboloid touching from above or below. More precisely, we will prove the following lemma.

**Lemma 4.1.** Let $I$ be an elliptic operator, with respect the class $L_0$, of the inf-sup (or sup-inf) type (in particular the maximal operators of any subfamily of $L_0$). Given a viscosity solution of $Iu \leq f$ in $\Omega$ and $\phi$ is a $C^2$ function that touches $u$ from below at a point $x \in \Omega$, then $Iu(x)$ is defined in the classical sense and we have $Iu(x) \leq f(x)$.

To prove Lemma 4.1 we need an interpolation result that will allow us to replace the $\tau$ derivative by the $\sigma$ derivative and a residue term evaluated at a given function $\phi$ touching $u$ by below. This result is also useful when the function touching by below is the convex envelope as $\delta_e(\phi) = 0$.

**Lemma 4.2.** Let $x \in B_1$ and $u$ defined in $B_2$ such that the following are finite,

$$
\int_{B_2} \frac{\delta^+(u, x; y)}{|y|^{n+\sigma}} dy \quad \text{and} \quad \int_{B_2} \frac{\delta_0(u, x; y)}{|y|^{n+\tau}} dy.
$$

Let $r \leq 1$ and $\varphi$ a function defined in $B_r(x)$ and touching $u$ by below at $x$. Then

$$
\int_{B_r} \lambda(2 - \sigma) \frac{\delta^+(u, x; y)}{|y|^{n+\sigma}} - b(1 - \tau) \frac{\delta_0(u, x; y)}{|y|^{n+\tau}} dy \geq \int_{B_r} \alpha \lambda(2 - \sigma) \frac{\delta^-_e(\varphi, x; y)}{|y|^{n+\sigma}} + b(1 - \tau) \frac{\delta^-_e(\varphi, x; y) - |\delta_0(\varphi, x; y)|}{|y|^{n+\tau}} dy,
$$

for $\alpha \in (0, 1)$ given that

$$
r \leq \left( \frac{(1 - \alpha)\lambda(2 - \sigma)}{b(1 - \tau)} \right)^{1/(\sigma-\tau)}.
$$

**Proof.** Since $\varphi$ touches $u$ by below, we have that for every $y \in B_r$,

$$
\delta^+_e(u - \varphi) \geq |\delta_0(u - \varphi)|,
$$

where
and also,
\[ \delta^+(u) \geq \delta^+(u - \varphi) + \delta^-(\varphi), \]
\[ |\delta_0(u - \varphi)| \geq |\delta_0(u)| - |\delta_0(\varphi)|, \]
so that
\[ \delta^+(u) - |\delta_0(u)| \geq \delta^-(\varphi) - |\delta_0(\varphi)|. \]

Now we can replace \(|\delta_0|\) by \(\delta^+ \) in the integral,
\[ \int_{B_r} \lambda(2 - \sigma) \frac{\delta^+(u)}{|y|^{n+\sigma}} - b(1 - \tau) \frac{|\delta_0(u)|}{|y|^{n+\tau}} dy \geq \]
\[ \int_{B_r} \delta^+(u) \left\{ \frac{\lambda(2 - \sigma)}{|y|^{n+\sigma}} - \frac{b(1 - \tau)}{|y|^{n+\tau}} \right\} + b(1 - \tau) \frac{\delta^-(\varphi) - |\delta_0(\varphi)|}{|y|^{n+\tau}} dy. \]

By using that \( r \leq \left( \frac{(1 - \alpha)\lambda(2 - \sigma)}{b(1 - \tau)} \right)^{1/(\sigma - \tau)} \),
and that \( \sigma > \tau \) we can substitute the difference of the fractions by \( \alpha \) times \(|y|^{-n+\sigma}\),
\[ \int_{B_r} \delta^+(u) \left\{ \frac{\lambda(2 - \sigma)}{|y|^{n+\sigma}} - \frac{b(1 - \tau)}{|y|^{n+\tau}} \right\} dy \geq \alpha \lambda(2 - \sigma) \int_{B_r} \frac{\delta^+(u)}{|y|^{n+\sigma}}. \]

**Proof of Lemma 4.1** Let, for \( r \leq 1 \),
\[ v_r(y) = \begin{cases} u(y) & \text{for } |y| > r, \\ \varphi(y) & \text{for } |y| \leq r. \end{cases} \]
Then \( \delta^-(v_r, x, y) \leq \delta^-(\varphi, x, y) \) and the functions \( \delta^-(v_r, x, y) \) decrease to \( \delta^-(u, x, y) \) as \( r \) goes to zero. So that,
\[ \int_{\mathbb{R}^n} \frac{\delta^-(u, x, y)}{|y|^{n+\sigma}} dy, \]
is well defined and equal to the limit, as \( r \) goes to zero, of
\[ \int_{\mathbb{R}^n} \frac{\delta^-(v_r, x, y)}{|y|^{n+\sigma}} dy. \]
By using that \( u \) is bounded we can find a finite constant \( M \), independent of \( r \), such that
\[ M \geq \int_{B_1} \lambda(2 - \sigma) \frac{\delta^+(v_r, x, y)}{|y|^{n+\sigma}} - b(1 - \tau) \frac{|\delta_0(v_r, x, y)|}{|y|^{n+\tau}} dy. \]

We see that we can replace the factor \(|\delta_0(v_r)|\) by \( \delta^+(v_r) \) by using Lemma 4.2 with \( \alpha = 1/2 \) if \( r \) is sufficiently small. We have that for a different constant \( M \) still independent of \( r \)
\[ M \geq \int_{B_1} \frac{\delta^+(v_r, x, y)}{|y|^{n+\sigma}} dy. \]
Then by taking the limit as $r \to 0$ we get that $\delta_e(u, x, y)/|y|^{n+\sigma}$ is also integrable around the origin. Going back to the $\tau$ derivative we can deduce now that,

$$M \geq \int_{B_1} \frac{|\delta_o(v_r, x, y)|}{|y|^{n+\sigma}} dy,$$

for some $M$ independent of $\tau$. By Fatou we get that $|\delta_o(u, x, y)/|y|^{n+\sigma}$ is integrable around the origin too. Therefore we have shown that for every pair of indexes $\alpha, \beta$, $L_{\alpha, \beta} u$ is well defined and $Iu(x)$ is computable by being an inf-sup combination of $L_{\alpha, \beta}$.

Now to see that $Iu(x) \leq f(x)$ we use the ellipticity,

$$Iu(x) \leq Iv_r(x) + M^{+}_I(u - v_r)$$

$$\leq f(x) + M^{+}_\sigma(u - v_r)(x) + b|D\tau(u - v_r)|(x).$$

Both integrals go to zero because the integrands are positive and decreasing to zero. □

We are interested now in studying limit of sub or super solutions. To state the result we need first to recall the definition of $\Gamma$ convergence.

**Definition 4.1.** We say that a sequence of lower-semicontinuous functions $u_k$ $\Gamma$-converge to $u$ in a set $\Omega$ if the two following conditions hold

(i) For every sequence $x_k \to x$ in $\Omega$, $\liminf_{k \to \infty} u_k(x_k) \geq u(x)$.

(ii) For every $x \in \Omega$, there is a sequence $x_k \to x$ in $\Omega$ such that

$$\limsup_{k \to \infty} u_k(x_k) = u(x).$$

The next result is the stability of the solutions of fully non linear equations. It is used in [3] to prove the comparison principle. We will omit the proof since it’s just a mild modification of the one found in [3].

**Lemma 4.3.** Let $I$ be an elliptic operator in the sense of Definition 2.2 with respect to $\mathcal{L}_0$ and $u_k$ be a sequence of functions that are uniformly bounded in $\mathbb{R}^n$ and lower-semicontinuous in $\Omega \subseteq \mathbb{R}^n$ such that

(i) $Iu_k \leq f_k$ in $\Omega$

(ii) $u_k \to u$ in the $\Gamma$ sense in $\Omega$,

(iii) $u_k \to u$ a.e. in $\mathbb{R}^n$ and

(iv) $f_k \to f$ locally uniformly un $\Omega$ for some continuous function $f$.

Then $Iu \leq f$ in $\Omega$.

We can also deduce the same result for sub solution instead of super solution.

Now we will prove that the barrier constructed in [4] is still a super solution in our case. To do so we will need to assume that the total kernel is positive.
Lemma 4.4. Let \( \varphi(x) = \min(1, C(|x| - 1)^{\varepsilon}_+), \) where \( C \) and \( \varepsilon \) has been chosen as in [4]. Then for any pair \( \sigma, \tau \) satisfying \( H1 \) we have
\[
\mathcal{M}_L^+ \varphi(x) \leq 0, \quad x \in \mathbb{R}^n \setminus B_1,
\]
where \( \mathcal{L} \) consists of the class of linear operators in \( \mathcal{L}_0 \) with non negative kernels \( K = K_e + K_o \geq 0 \). Moreover,
\[
\mathcal{M}_L^+ \varphi(x) \leq -\delta < 0, \quad x \in B_2 \setminus B_1.
\]

Proof. Thanks to Lemma 2.2 we only need to prove that there exist \( r \) such that
\[
\mathcal{M}_L^+ v(x) \leq 0, \quad x \in B_1 + r \setminus B_1,
\]
where \( v(x) = (|x| - 1)^{\varepsilon}_+ \).

Let \( r_0 \) be the radius from Lemma 3.1 in [4]. We know that \( \mathcal{M}_L^+ v(x_0) = -1 \) for \( |x_0| - 1 = r_0 \) and also \( |D\tau v(x_0)| = d < \infty \). Let \( s \in (0, 1) \) and \( |x| - 1 = sr_0 \), we have for every \( L \in \mathcal{L} \)
\[
L v(x) = s^{\varepsilon} L_s v_s(x_0) \leq s^{\varepsilon} L_s v(x_0),
\]
where \( v_s \) is a translation of \( s^{-\varepsilon} v(sx) \) such that \( v_s \leq v \) and \( v_s(x_0) = v(x_0) \). Note that \( L_s \) belongs to the class \( \mathcal{L}_0^0 = \mathcal{L}_0(s^{-\sigma}\lambda, s^{-\sigma}\Lambda, s^{-\tau}b) \), so we have then that for each \( L \in \mathcal{L} \)
\[
L v(x_0) \leq s^{\varepsilon - \sigma} \left\{ \mathcal{M}_L^+ v(x_0) + b^{\sigma - \tau} |D\tau v(x_0)| \right\}.
\]
We conclude the desired result by taking \( s \) small enough and supremum over \( L \in \mathcal{L} \). \( \square \)

We state now the comparison principle theorem.

Theorem 4.5. Let \( I \) be an elliptic operator with respect to a class \( \mathcal{L} \subseteq \mathcal{L}_0 \) of non negative kernels. Let \( \Omega \) be a bounded open set and \( u, v \) two functions such that
(i) \( u, v \) are bounded in \( \mathbb{R}^n \),
(ii) \( u \) is upper-semicontinuous and \( v \) is lower-semicontinuous at every point in \( \Omega \),
(iii) \( Iu \geq f \) and \( Iv \leq f \) in \( \Omega \) for some \( f \in C(\Omega) \),
(iv) \( u \leq v \) in \( \mathbb{R}^n \setminus \Omega \).

Then \( u \leq v \) in \( \Omega \).

Remark 4.6. Note first that we need a global control outside the domain, due to the nonlocal behavior of the equation. Also note that the proof of this result it’s a direct consequence of the definitions when one of the functions is \( C^{1,1} \). Finally we would like to point out that the difficulty in proving this theorem comes from the definition of viscosity solution and that we could have discontinuities outside \( \Omega \).
The proof of Theorem 4.5 is done in several steps as the proof of [3]. In our case, we use the barrier from Lemma 4.4 instead of the function from Assumption 5.1. We will omit the proof of this result and refer to [3] for the reader interested in the details.

Finally we also have available an existence result for the Dirichlet problem.

**Theorem 4.7.** Let $\Omega \subset \mathbb{R}^n$ be an open bounded set satisfying the exterior ball condition. Let $g : \mathbb{R}^n \setminus \Omega \to \mathbb{R}$ be a function which is globally bounded and continuous on $\partial \Omega$. Let $I$ be an elliptic operator with respect the class $L \subseteq L^0$ of non negative kernels and assume $\sigma$ and $\tau$ satisfy $H1$ and $I$ is zero if evaluated at constants functions, then there exist a viscosity solution $u$ of

$$Iu(x) = 0, \text{ in } \Omega,$$

$$u = g, \text{ in } \mathbb{R}^n \setminus \Omega.$$

**Proof.** By using stability and the comparison principle we can apply the Perron method to prove that,

$$u = \sup \{ v \in USC(\Omega) : -\|g\|_{\infty} \leq v \leq \|g\|_{\infty} \text{ in } \mathbb{R}^n \text{ s.t. } Iv \leq 0 \text{ in } \Omega \text{ and } v \geq g \text{ in } \mathbb{R}^n \setminus \Omega\},$$

solves $Iu = 0$. The problem is to see that we actually attain the boundary values in a continuous way. The argument only brings the additional feature that we also have to take care of the values in the interior of $\mathbb{R}^n \setminus \Omega$. In any case we have to show that for any $x \in \mathbb{R}^n \setminus \Omega$ and any $\varepsilon > 0$ we can find a sub a super solutions, $v, w$ respectively, such that,

$$w \geq g \text{ in } \mathbb{R}^n \setminus \Omega,$$

$$w(x) \leq g(x) + \varepsilon,$$

and similarly for $v$. Let’s just prove it for $w$.

If $x$ belongs to the interior of $\mathbb{R}^n \setminus \partial \Omega$ then a function $w$ which is equal to $\|g\|_{\infty}$ for every $y \neq x$ and equal to $g(x)$ for $y = x$ is in $USC(\Omega)$ and is a super solution, i.e. $Iw \leq \mathcal{M}^+w = 0$. On the other side, if $x \in \partial \Omega$ then there is a ball $B_r(x + r_0\eta)$ such that $B_r(x + r_0\eta) \cap \partial \Omega = \{x\}$ and where $\eta$ is a unitary vector and $r_0$ less than one. Let

$$w(y) = \|g\|_{\infty} \varphi((y - (x + r\eta))/r) + g(x) + \varepsilon$$

with $\varphi$ from Lemma 4.4 and some $r < r_0$. There is some $\delta$ such that $|g(y) - g(x)| \leq \varepsilon$ whenever $|x - y| \leq \delta$. Then we need to take $r$ such that $B_r(x + r\eta) \subseteq B_\delta(x)$ to see that $w$ works as an upper barrier. □

5. Partial ABP Estimates

The classical ABP theorem states that for a super solution, positive in $\partial B_3$, the supremum of $u^-$ is controlled by the $L^n$ norm of the right hand side, integrated only over the contact set for the convex envelope. These
estimates are useful to get lower bounds in the measure of the contact set which are then needed to get an $L^\infty$ Lemma.

We denote by $\Gamma$ the convex envelope supported in $B_3$. In the next lemma we see that we can almost put a paraboloid above $\Gamma$, with the opening controlled by $f(x)$, the supremum of $u$ outside $B_1$ and the $\tau$ derivative of $\Gamma$ at $x$.

**Lemma 5.1.** Let $u \geq 0$ in $\mathbb{R}^n \setminus B_1$ a globally bounded viscosity solution of,

\[ \mathcal{M}^-_0 u(x) \leq f \text{ in } B_1, \]

and $x \in \{u = \Gamma\}$. Assume

\[ \frac{\beta}{1 - 2^{-2(2-\sigma)}} = \frac{(2 - \sigma)\lambda - 2\sigma - (1 - \tau)b}{1 - 2^{-2(2-\sigma)}} \geq \delta > 0, \]

and let $\rho_0 = 1/(8\sqrt{n})$, $r_k = \rho_0 2^{-1/(2-\sigma)} \cdot k$ and $R_k = B_{r_k}(x) \setminus B_{r_{k+1}}(x)$. Then there is a constant $C_0$ such that for any $M > 0$ there is a $k$ such that

\[ |R_k \cap \{u(y) > u(x) + (y - x) \cdot \nabla \Gamma(x) + Mr_k^2\}| \leq \frac{C_0}{\delta} \frac{\star}{M} |R_k(x)|, \]

where

\[ \star = f(x) + (1 - \tau)b \int_{B_2} \frac{\delta_0(\Gamma)}{|y|^{n+\tau}} dy + \frac{1 - \tau}{\tau} b \|u^+\|_{L^\infty(\mathbb{R}^n \setminus B_1)}. \]

**Remark 5.2.** So far, we can not say that $\delta$ is independent of $\sigma$ and $\tau$. However, when $\tau = 1$ then $\delta > 0$ exists and it is universal because the quotient $(2 - \sigma)/(1 - 2^{-2(2-\sigma)})$ is bounded away from zero uniformly.

In general, in order to have $\delta$ a universal constant, we could ask the additional hypothesis that $b(1 - \tau) \ll \lambda(2 - \sigma)$ uniformly. In fact, by the following bounds

\[ 3/8(\sigma - 2) \leq 1 - 2^{-2(2-\sigma)} \leq \ln 2(2 - \sigma), \]

the restriction on $\beta$ is satisfied if

\[ \frac{\lambda}{\ln 2} - \frac{32}{3} \frac{1 - \tau}{2 - \sigma} b \geq \delta > 0. \]

In the following sections we will be rescaling solutions, such that they satisfy an equation with ellipticity coefficient $\tilde{b}$ sufficiently small to satisfy the previous condition.

**Proof.** Because we have a convex test function touching by below at $x \in B_1$ we have that $\delta^-_e(u, x, \cdot) \equiv 0$. By Lemma 4.1 we can evaluate $\mathcal{M}^- u$ at the
point \( x \), hence
\[
\int_{B_2} (2 - \sigma) \frac{\delta^+_{\Gamma}(u, x, y)}{|y|^{n+\sigma}} - (1 - \tau) b \frac{|\delta_\sigma(u, x, y)|}{|y|^{n+\tau}} dy \leq f(x) + (1 - \tau) b \int_{\mathbb{R}^n \setminus B_2} \frac{|\delta_\sigma(u, x, y)|}{|y|^{n+\tau}} dy \leq f(x) + C(n) b \frac{1 - \tau}{\tau} \|u^+\|_{L^\infty(\mathbb{R}^n \setminus B_1)}.
\]
By Lemma 4.2 using \( \Gamma \) as the test function,
\[
\int_{B_2} (2 - \sigma) \frac{\delta^+_{\Gamma}(u, x, y)}{|y|^{n+\sigma}} - (1 - \tau) b \frac{|\delta_\sigma(u, x, y)|}{|y|^{n+\tau}} dy \geq \beta \int_{B_{p_0}} \frac{\delta^+_{\Gamma}(u, x, y)}{|y|^{n+\sigma}} dy - (1 - \tau) b \int_{B_2} \frac{|\delta_\sigma(\Gamma, x, y)|}{|y|^{n+\tau}} dy.
\]
Adding what we have so far
\[
\beta \int_{B_{p_0}} \frac{\delta^+_{\Gamma}(u, x, y)}{|y|^{n+\sigma}} dy \leq C*
\]
The rest of the proof goes as in [3] by splitting the ball \( B_{p_0} \) into dyadic rings.

The following is just a modification of the previous lemma. The aim is to replace the second term in * by \( \|u\|_{\infty} \).

**Remark 5.3.** By the intermediate value theorem, for each \( x \in B_1 \) and \( y \in B_2 \), \( |\delta_\sigma(\Gamma, x, y)| \) is equal to \( 2|\nabla \Gamma(x')|\|y| \) for \( x' \) an intermediate point in the segment between \( x + y \) and \( x - y \). So that
\[
\int_{B_2(x)} \frac{|\delta_\sigma(\Gamma, x, y)|}{|y|^{n+\tau}} dy \leq C(n) \frac{1}{1 - \tau} \|\nabla \Gamma\|_{\infty}.
\]
By the geometry of the convex envelope \( \|\nabla \Gamma\|_{\infty} \leq 1/2 \|u^-\|_{L^\infty(B_1)} \). Finally could have also considered that \( \tau > \tau_0 > 0 \) for a universal constant \( \tau_0 \), so that * can be simplified to
\[
* = f(x) + b \|u\|_{\infty}.
\]
Notice that we haven’t absorb the constant \( b \) into the universal constants of the estimate. The importance of this will be seen in the results of the next sections.

Here is the weak ABP type estimate we are able to obtain.

**Theorem 5.4.** Let \( u \) and \( \Gamma \) as in Lemma 5.1. There is a disjoint family of cubes \( Q_j \) with diameters \( d_j \leq p_0 2^{-1/(2-\sigma)} \) \((p_0 = 1/(8\sqrt{n}))\) which covers the contact set \( \{\Gamma = u\} \) such that the following holds
(i) \( \{u = \Gamma\} \cap Q_j \neq \emptyset \) for any \( Q_j \).
\[|\nabla \Gamma(\bar{Q}_j)| \leq \frac{C}{\delta^n} \left( \max_{Q_j \cap \{\Gamma = u\}} * \right)^n |Q_j|.\]

(iii)

\[\left\{ y \in 8\sqrt{n}Q_j : u(y) < \Gamma(y) + \frac{C}{\delta} \left( \max_{Q_j \cap \{\Gamma = u\}} * \right) d_j^2 \right\} \geq \mu |Q_j|,\]

where

\[* = f(x) + (1 - \tau)b \int_{B_2} \frac{\delta_0(\Gamma)}{|y|^{n+\sigma}} dy + \frac{1 - \tau}{\tau} b \|u^+\|_{L^\infty(\mathbb{R}^n \setminus B_1)}.\]

The proof is the same as in [3] by using Lemma 5.1 instead.

As \(\tau\) and \(\sigma\) go to one and two respectively in a controlled way (see Remark 5.2) this theorem recovers a sufficient step to complete the proof of the classical ABP estimate. However, to prove regularity for \(u\) it will be sufficient to use a weaker version where \(* = f(x) + \|u\|_{L^\infty} (\text{see Remark 5.3}).\)

6. Point Estimates

In this section we will prove a partial \(L^\epsilon\) lemma. We will use the scaling property of the operator extensively.

**Remark 6.1.** Define, for \(x \in Q_1\), \(\tilde{u}(x) = u(\kappa x)\). Where \(u\) is a super solution of \(M^-u \leq f\) in \(Q_{\kappa}\). Then we have that, in \(Q_1\), \(\tilde{u}\) satisfies

\[M^-\tilde{u} - \kappa^{\sigma-\tau} b |D_{\tau}\tilde{u}| \leq \kappa^\sigma f(\kappa x)\]

When \(0 < \kappa \ll 1\), and if \(\sigma - \tau > m > 0\), then the equation behaves more and more like a pure \(\sigma\) order one. The importance of taking \(m > 0\) independent of \(\sigma\) and \(\tau\) is to preserve the results if one takes limits in \(\sigma\) and \(\tau\).

We will also need to consider the special function of Section 9 in [3]. This implies an additional set of hypothesis. From this point on we will always assume that, for \(\sigma_0, \tau_0, m, A_0 > 0\) given, the set of hypothesis H1, H2 and H3 hold.

We recall the special function constructed in [3].

**Lemma 6.2.** Let \(2 > \sigma_0 > 0\), there is a function \(\Phi\) such that,

(i) \(\Phi\) is continuous in \(\mathbb{R}^n\),

(ii) \(\Phi(x) = 0\) for \(x\) outside \(B_{2\sqrt{\pi}}\),

(iii) \(\Phi(x) < -2\) for \(x\) in \(Q_3\), and

(iv) \(M^+\Phi \leq \psi(x)\) in \(\mathbb{R}^n\) for some non negative function \(\psi(x)\) supported in \(\bar{B}_{1/4}\)

for every \(\sigma > \sigma_0\).

Now we are able to prove the following lemma.
**Lemma 6.3.** Let $\sigma_0, \tau_0, m, A_0 > 0$, there exist constants $1 > \mu$, $\varepsilon_0 > 0$ and $M > 1$, such that if

(i) $u \geq 0$ in $\mathbb{R}^n$,
(ii) $\inf_{Q_{3\kappa}} u \leq 1$,
(iii) $\mathcal{M}_{L_0}^- u \leq 1$ in $Q_{4\sqrt{\tau}\kappa}$,

then

$$|\{u \leq M\} \cap Q_\kappa| > \mu|Q_\kappa|,$$

for

$$\kappa = \frac{\varepsilon_0}{(1 + \|u\|_\infty)^{1/(\sigma-\tau)}}.$$

**Proof.** Consider $\tilde{u}(x) = u(\kappa x)$ and note that by Remark 6.1 $\tilde{u}$ satisfies (i), (ii) in the cube of side 3 and

$$\mathcal{M}_{\sigma}^- \tilde{u} - \tilde{b}|D_\tau \tilde{u}| \leq \kappa^\sigma \text{ in } Q_{4\sqrt{\tau} \kappa}.$$

for $\tilde{b} = \kappa^{\sigma-\tau} b$. We will prove that the Lemma holds for $\tilde{u}$ in $Q_1$, which implies the desired result. The proof follows as in [3] but we need to be careful because the ABP type results that we have are different.

First thing we require from $\varepsilon_0^m$ is to be small enough, with respect to $A_0$, such that the condition in the Remark 5.2 holds and then be able to use the ABP estimate, now with $\delta$ universal.

Consider $v = \tilde{u} + \Phi$, where $\Phi$ is the special function given in [3]. We have that $v$ satisfies in $Q_{4\sqrt{\tau}}$

$$\mathcal{M}_{\sigma}^- v - \tilde{b}|D_\tau v| \leq \kappa^\sigma + \mathcal{M}_{\sigma}^+ \Phi + \kappa^{\sigma-\tau} b|D_\tau \Phi|,$$

$$\leq \kappa^\sigma + \psi + \kappa^{\sigma-\tau} bC,$$

for a universal constant $C$.

Let $\Gamma$ be the concave envelope of $v$ supported in the ball $B_{2\sqrt{\tau}}$. Let $Q_j$ be the cubes from a rescaled version of our partial ABP estimate. Notice that by changing the constant $b$ by $\tilde{b}$, a smaller one, we still satisfy the restriction [5.16] and the final result is true with the same universal constants. Then we have

$$\max_{B_{2\sqrt{\tau}}} v^- \leq |\nabla \Gamma(B_{2\sqrt{\tau}})|^{1/n} \leq C \left( \sum_j |\nabla \Gamma(Q_j)| \right)^{1/n},$$

$$\leq C \left( \sum_j \left( \max_{Q_j} \psi + \kappa^\sigma + \kappa^{\sigma-\tau} b (1 + \|v\|_\infty) \right)^n \right)|Q_j|^{1/n}.$$

We can make the terms $\kappa^\sigma + \kappa^{\sigma-\tau} b (1 + \|v\|_\infty)$ small enough by choosing $\varepsilon_0$ small enough,

$$\kappa^\sigma + \kappa^{\sigma-\tau} b (1 + \|v\|_\infty) \leq C(\varepsilon_0^\sigma + \varepsilon_0^{\sigma-\tau}) \leq C\varepsilon_0^m.$$
Using that $\Phi \leq -2$ in $Q_3$

$$1 \leq C \varepsilon_0^m + C \left( \sum_j \left( \max_{Q_j} \psi^+ \right)^n |Q_j| \right)^{1/n},$$

which implies then, for $\varepsilon_0$ small enough, the following inequality,

$$\frac{1}{2} \leq C \left( \sum_j \left( \max_{Q_j} \psi^+ \right)^n |Q_j| \right)^{1/n}.$$

Since $\psi$ is supported in $\bar{B}_{1/4}$ and is bounded, we get

$$C \left( \sum_{Q_j \cap \bar{B}_{1/4} \neq \emptyset} |Q_j| \right)^{1/n} \geq c,$$

which provides a bound for the sum of the volumes of the cubes inside $B_{1/4}$, where $c$ is a universal constant. Now, the diameters of all cubes $Q_j$ are bounded by $\rho_0 2^{-1/(2-\sigma)}$, which is smaller than $\rho_0 = 1/(8\sqrt{n})$. So, every time we have that $Q_j$ intersects $B_{1/4}$ the cube $4\sqrt{n}Q_j$ will be contained in $B_{1/2}$.

Note, since $\varepsilon_0$ is universal, the partial ABP estimates translate into

$$\frac{1}{n} | \{ x \in 4\sqrt{n}Q_j : v(x) \leq \Gamma(x) + C d_j^2 \} | \geq c |Q_j|,$$

for $C$ universal and $Cd_j^2 < C \rho_0^2$. Let us consider now the cubes $4\sqrt{n}Q_j$ for every $Q_j$ that intersects $B_{1/4}$. This provides an open cover of the union of the corresponding cubes $Q_j$ and it is contained in $B_{1/2}$. Taking a subcover with finite overlapping and using (6.17) and (6.18) we get

$$| \{ x \in B_{1/2} : v(x) \leq \Gamma(x) + C \rho_0^2 \} | \geq c.$$

Hence, if we let $-M_0 = \min_{B_{1/2}} \Phi$ we get

$$| \{ x \in B_{1/2} : \tilde{u}(x) \leq M_0 + C \rho_0^2 \} | \geq c.$$

Finally let $M = M_0 + C \rho_0^2$ and note that since $B_{1/2} \subset Q_1$ we get

$$| \{ x \in Q_1 : \tilde{u}(x) \leq M \} | \geq c,$$

which concludes the result for $\tilde{u}$. \hfill \Box

**Remark 6.4.** Note that in the previous proof the scaling is necessary to have, for some universal constant $C$:

(i) $\tilde{b}(1 + \|u\|_\infty) \leq \varepsilon_0^m$,

(ii) Right hand side smaller than $\varepsilon_0^m$.

In future references we will use that if these identities hold, then the conclusion also holds without any further scaling.

This result it is not scaling invariant because $\kappa$ depends on $\|u\|_\infty$. However it can be still applied after any dilation.
Remark 6.5. Consider for $u$ and $\tilde{u}$ as before and $0 < r \leq 1$, $v(x) = \tilde{u}(rx)$. Then $v$ satisfies,

$$\mathcal{M}_\sigma v(x) - b(r\kappa)^{\sigma - \tau} |D_\tau v| \leq (r\kappa)^\sigma.$$  

Then we recall the previous remark and check that $b(r\kappa)^{\sigma - \tau}(1 + \|v\|_\infty) \leq \varepsilon_0^m$ and that the right hand side is also smaller or equal than $\varepsilon_0^m$ if $r \leq 1$.

In particular, the transformations required to prove the full $L^\varepsilon$ lemma are of the form

$$v(y) = \frac{u(x_0 + 2^{-i}y)}{M^k}.$$  

The previous lemma can still be applied and we can iterate by means of a Calderón Zygmund decomposition as in [2].

Lemma 6.6. Let $\sigma_0, \tau_0, m, A_0 > 0$ and $u \geq 0$, $\tilde{u}$ as in Lemma 6.3. Then we have

$$|\{\tilde{u} > M^k\} \cap Q_1| \leq (1 - \mu)^k,$$

for $k = 1, 2, \ldots$ where $M$ and $\mu$ are as in Lemma 6.3. As a consequence we have the following inequality,

$$|\{\tilde{u} \geq t\} \cap Q_1| \leq dt^{-\varepsilon}, \quad \forall t > 0,$$

where $d$ and $\varepsilon$ are positive universal constants.

By standard covering arguments one can pass from cubes to balls.

Corollary 6.7. Let $\sigma_0, \tau_0, m, A_0 > 0$ and $u \geq 0$, $\tilde{u}$ as in Lemma 6.3 with $u$ super solution of $\mathcal{M}_{L_0}^- u \leq 1$ in $B_{2\kappa}$ and $u(0) \leq 1$. Then we have

$$|\{\tilde{u} \geq t\} \cap B_1| \leq ct^{-\varepsilon}, \quad \forall t > 0,$$

where $c$ and $\varepsilon$ are positive universal constants.

By using Remark 6.5 one more time we can prove a rescaled version of the Corollary 6.7.

Corollary 6.8. Let $\sigma_0, \tau_0, m, A_0 > 0$, $u \geq 0$, $\tilde{u}$ as in Lemma 6.3 with $u$ super solution of $\mathcal{M}_{L_0}^- u \leq C_0$ in $B_{2\kappa}$, for $r \leq 1$. Then we have

$$|\{\tilde{u} \geq t\} \cap B_r| \leq C r^n(u(0) + C_0 r^m)^{\varepsilon} t^{-\varepsilon}, \quad \forall t > 0,$$

where $C$ and $\varepsilon$ are positive universal constants.

Proof. Consider

$$v(x) = \frac{\tilde{u}(rx)}{u(0) + C_0 r^m},$$

and note that if $r \leq 1$ the conditions in Remark 6.5 are satisfied. Apply Corollary 6.7 and rescale back to conclude. $\square$
7. Hölder Regularity

Our aim in this section is to prove Hölder regularity for solutions of our operator. We assume still the general hypothesis of the previous section.

Lemma 7.1. Let $\sigma_0, \tau_0, m, A_0 > 0$ and $u$ be a function such that:

(i) $-\frac{1}{2} \leq u \leq \frac{1}{2}$ in $\mathbb{R}^n$;

(ii) $\mathcal{M}^+_\kappa_0 u \geq -1$ and $\mathcal{M}^-_\kappa_0 u \leq 1$ in $B_\kappa$.

Then there are universals $\alpha, C > 0$ such that

$$|\tilde{u}(x) - u(0)| \leq C|x|^{\alpha}$$

Again our proof relies in noticing that a dilation powerful enough puts us in the same situation as in the proof of [3]. There is still a difficulty which is that the rescaling considered in such proof consists of a dilation of the domain, which as we already saw are good for our situation, times some constants that grow geometrically. We want to see that by making $\alpha$ small enough we can make the effect of this second multiplication controlled.

Proof. Recall that $\tilde{u}$ is defined in terms of the dilation factor $\kappa$ which depends on a parameter $\varepsilon_0$. We will let $\varepsilon_0$ to be a free parameter small enough, to begin with, such that the previous estimates are valid and in the course of the proof we will see that $\varepsilon_0$ may have to be chosen even smaller than in the previous results.

We will show that there exists sequences $m_k$ and $M_k$ such that $m_k \leq \tilde{u} \leq M_k$ in $B_1$ and

$$M_k - m_k = 4^{-\alpha k}$$

so that result holds for $C = 4^\alpha$.

For $k = 0$ we choose $m_0 = -1/2$ and $M_0 = 1/2$ and by (i) we have $m_0 \leq \tilde{u} \leq M_0$ in $\mathbb{R}^n$. We will construct the sequence by induction. Assume then that we have the sequences up to $k$, we want to find $m_{k+1}$ and $M_{k+1}$. In the ball $B_{4^{-(k+1)}}$, either $\tilde{u} \geq (M_k + m_k)/2$ in at least half the points (in measure) or we have the other inequality. Let’s assume that

$$\left|\left\{\tilde{u} \geq \frac{M_k + m_k}{2}\right\} \cap B_{4^{-(k+1)}}\right| \geq \frac{|B_{4^{-(k+1)}}|}{2}.$$ 

Consider now

$$v(x) = \frac{\tilde{u}(4^{-k}x) - m_k}{(M_k - m_k)/2},$$

so that $v(x) \geq 0$ in $B_1$ and $|\{v \geq 1\} \cap B_{1/4}| \geq |B_{1/4}|/2$. 


From the inductive hypothesis, we have that for any \( j \geq 1 \)
\[
v \geq \frac{m_{k-j} - m_k}{(M_k - m_k)/2} \geq \frac{m_{k-j} - M_{k-j} + M_k - m_k}{(M_k - m_k)/2}
\geq -2 \cdot 4^{\alpha j} + 2
\geq 2(1 - 4^{\alpha j}), \quad \text{in } B_{2j}.
\]
Therefore \( v(x) \geq -2(|4x|^\alpha - 1) \) outside \( B_1 \). Let \( w(x) = v^+(x) = \max(v, 0) \).

\( w \) satisfies
\[
M_\sigma^- w - 4^{-k(\sigma-\tau)}\kappa^{\sigma-\tau} b|D_\tau w| \leq 4^{-k\alpha} \kappa^\sigma + M_\sigma^+ v^- + 4^{-k(\sigma-\tau)}\kappa^{\sigma-\tau} b|D_\tau v^-|.
\]
We still have \( |\{ w \geq 1 \} \cap B_{1/4} | \geq |B_{1/4}|/2. \) Use the other bound \( v^- \leq 2(|4x|^\alpha - 1) \) outside \( B_1 \), also proved by induction, and \( v^- = 0 \) in \( B_1 \) to get that the right hand side can be made smaller than \( \varepsilon_0^m \) by choosing a small exponent \( \alpha \). We recall the conditions in the Remark 6.4. So far we have shown the second one which is satisfied with a right hand side \( 2\varepsilon_0^m \). For the first condition note that
\[
4^{-k(\sigma-\tau)}\kappa^{\sigma-\tau} b(1 + \|v\|_{\infty}) \leq 4^{-k((\sigma-\tau)-\alpha)}C\varepsilon_0^m.
\]
so we just have to choose \( \alpha \leq m \).

Now, given any \( x \in B_{1/4} \) we can apply Corollary 6.8 in \( B_1(x) \) to get
\[
C(w(x) + \varepsilon_0)^\tilde{\tau} \geq |\{ w > 1 \} \cap B_{1/2}(x) | \geq \frac{1}{2} |B_{1/4}|,
\]
hence, since \( \varepsilon_0 \) is small, we conclude \( w \geq \theta > 0 \) in \( B_{1/4} \) for some \( \theta > 0 \). If we let \( M_{k+1} = M_k \) and \( m_{k+1} = m_k + \theta(M_k - m_k)/2 \) for the inductive step
\[
m_{k+1} \leq \tilde{u} \leq M_{k+1}, \quad \text{in } B_{4^{-k-1}}.
\]
Moreover \( M_{k+1} - m_{k+1} = (1 - \theta/2)4^{-ak}, \) so choosing \( \alpha \) and \( \theta \) such that
\[
1 - \theta/2 = 4^{-\alpha}
\]
we conclude \( M_{k+1} - m_{k+1} = 4^{-\alpha(k+1)}. \)

As a result we get that \( \tilde{u} \) is \( C^\alpha \). If we want to go back to \( u \) the constant degenerates by the factor \( (\|u\|_{\infty} + 1)^{\sigma-\tau} \leq (\|u\|_{\infty} + 1)^2. \) This proves the Theorem 6.1 and the Corollary 6.2.

8. \( C^{1,\alpha} \) Regularity

For translation invariant equations, \( C^{1,\alpha} \) regularity comes by proving \( C^\alpha \) regularity for the incremental quotients of a given solution. This procedure allows to improve the regularity from \( C^\alpha \) to \( C^{2\alpha} \) and so forth all the way up to \( C^{n,1} \) and then to \( C^{1,\alpha} \), see [2]. We need to use the comparison principle to see that these incremental quotients satisfy a uniformly elliptic equation with bounded measurable coefficients and zero right hand side, for which we already have \( C^\alpha \) estimates. The difficulty in this case is that we need, in each step, these incremental quotients to be uniformly bounded in \( \mathbb{R}^n \). The previous regularity only guarantees this on \( B_{r-\delta} \), given that the equation is satisfied in \( B_r \).
Recall the class $\mathcal{L}_1$ of kernels $K = K_\epsilon + K_\sigma \geq 0$ such that,

$$\int_{\mathbb{R}^n \setminus B_{\rho_0}} \frac{|K(y) - K(y - h)|}{|h|} dy \leq C \text{ every time } |h| < \frac{\rho_0}{2},$$

with the same ellipticity constants $\lambda, \Lambda$ and $b$ as usual.

**Theorem 8.1.** Let $\sigma_0, \tau_0, m, A_0 > 0$ and assume that $H1$, $H2$ and $H3$ holds. There is $\rho_0 > 0$ small enough so that if $I$ is a nonlocal elliptic with respect to $\mathcal{L}_1$ and $u$ a bounded viscosity solution of $Ju = 0$ in $B_1$, then there is a universal $\alpha > 0$ such that $u \in C^{1,\alpha}(B_{1/2})$ and

$$\|u\|_{C^{1,\alpha}(B_{1/2})} \leq C(\|u\|_\infty + 1)^2(\|u\|_\infty + I0)$$

for some universal $C > 0$ and where $I0$ means the value of the operator $I$ applied to the constant function zero.

**Proof.** Let say that we want to go from $C^\alpha(B_{3/4})$ to $C^{2\alpha}(B_{3/4-\delta})$ where $\delta$ is universal given by $\delta[1/\alpha] = 1/4$. The idea in $[3]$ is to localize $u$ by multiplying it by a smooth cutoff function $\eta$ supported in $B_{3/4-\delta/4}$ and equal to one in $B_{3/4-\delta/2}$. We want to see if the $L^\infty$ norm of the incremental quotients,

$$w^h_1(x) = \frac{\eta u(x + h) - \eta u(x)}{|h|^{\alpha}},$$

are uniformly bounded, for $|h| < \delta/8$. If $x \in B_{3/4-\delta/8}$, $|w^h_1(x)|$ is bounded above by $C(\|u\|_\infty + 1)^2\|u\|_\infty$ with $C$ depending also on $\eta$, which is universal. If $x \in \mathbb{R}^n \setminus B_{3/4-\delta/8}$ then $w^h_1(x)$ just cancels. $w^h_1(x)$ satisfy a different equation,

$$(8.20) \quad M^+_{\mathcal{L}_1} w^h_1 \geq I0 - M^-_{\mathcal{L}_1} w^h_2 \quad \text{ and } \quad M^-_{\mathcal{L}_1} w^h_1 \leq I0 - M^+_{\mathcal{L}_1} w^h_2,$$

for

$$w^h_2 = \frac{(1 - \eta)u(x + h) - (1 - \eta)u(x)}{|h|^{\alpha}}.$$

and $I0$ is the value of the operator $I$ applied to the constant function zero.

For $x \in B_{3/4-3\delta/4}$ and $|h| \leq \delta/8$ the terms $|M^\pm_{\mathcal{L}_1} w^h_2|$ are controlled by $\|u\|_\infty$ by using that

$$\int_{\mathbb{R}^n \setminus B_{\rho_0}} \frac{|K(y) - K(y - h)|}{|h|} dy \leq C \text{ every time } |h| < \frac{\rho_0}{2}$$

with $\rho_0 = \delta/8$.

Finally, when iterating this argument and applying Lemma 5.6 of $[2]$ we have to see that the incremental quotients are not bounded by $(\|u\|_\infty + I0)$ as in $[3]$ but by $(\|u\|_\infty + 1)^2(\|u\|_\infty + I0)$. This is what brings the different right hand side in the estimate. $\square$
9. Limit Case as $\sigma \to 2$ and $\tau \to 1$

The aim of this section is to recover second order equations with gradient terms as $\sigma \to 2$ and $\tau \to 1$. This also can be seen as an extension of Section 6 of [3].

As in [3] it is well known that we can recover any linear second order elliptic equation as a limit of integro differential equations when we let $\sigma \to 2$. This could take the form

$$\sum a_{ij}u_{ij}(x) = \lim_{\sigma \to 2} \int \frac{c_n(2 - \sigma)}{\det A\det A^{-1}y^{n+\sigma}} \delta_\varepsilon(u, x, y)dy,$$

where $a_{ij} = (AA^T)_{ij}$. A similar thing happens with the odd part of the kernel when $\tau$ goes to one. More precisely, given any vector $v \in \mathbb{R}^n$ we have the following identity

$$Du(x) \cdot v = \lim_{\tau \to 1} c_n|v|(1 - \tau) \int \frac{\delta_0(u, x, y)}{|y|^{n+\tau}} \cos(\theta)dy,$$

where $\theta$ is the angle between $v$ and $y$.

In a similar way we can also reconstruct any fully non linear operator $F(M, p)$, Lipschitz in $M$ and $p$ and monotone in the space of symmetric positive definite matrices. Here

$$F(M, p) = \lim_{\sigma \to 2, \tau \to 1} \inf_{\alpha} \sup_{\beta} (\cdot),$$

where

$$\cdot = \int \left( \frac{c_n(2 - \sigma)}{\det A\det A^{-1}y^{n+\sigma}} + \frac{c_n|v_{\alpha\beta}|(1 - \tau)\delta_0}{|y|^{n+\tau}} \cos(\theta_{\alpha\beta}) \right) dy + c_{\alpha\beta}.$$

The results of this paper are still valid in the limits as soon as the restriction in Remark 5.2 holds and $\sigma$ and $\tau$ are uniformly separated. This says that, given $b > 0$, $\sigma$ can’t go to two if $\tau$ doesn’t go to one too and at a comparable rate, the restriction of being uniformly separated is clearly eventually satisfied. Another limits which are also interesting are when $\tau$ goes to one but $\sigma$ stays less than two and bigger than one, these are studied in [6] and [8]. In [12] we see that the condition of uniform separation between $\sigma$ and $\tau$ can be removed.

10. Open Problems and Future Research

In this section we discuss briefly open problems and future research related to this topic.

**Full ABP estimate.**

One clear open question left behind is if we can improve Theorem 5.3 in the sense of [3]. This would allow for an easier proof of the regularity results and also would allow us to recover the standard $L^\infty$ dependence on $u$. Moreover, the authors are convinced that with a better ABP estimate one can also prove a Harnack inequality.
Recently in [7], N. Guillen and R. Schwab were able to prove a very general ABP type estimates for integro differential equations with purely symmetric kernels that have quadratic structure.

\[ K = K_o = (2 - \sigma) \frac{y^T A y}{|y|^{n+2+\sigma}}, \]

for symmetric matrices \( A \), such that \( \lambda \leq A \leq \Lambda \). We note that this class is smaller than the class of symmetric operators considered in [3] but large enough to recover the Pucci extremal operators as \( \sigma \to 2 \), see for example [2].

Another interesting question is if the perturbation techniques used here would also work to prove similar results as in [7]. The difficulty here would be to study the effect of the Riesz transform in the \( \tau \) derivative and the interaction with the fractional envelope.

**Pertubative Methods for non translation invariant equations.**

As in [4] one could study fully nonlinear integro differential equations with kernels \( K_e \) and \( K_o \) that are not necessarily translation invariant. More precisely, we are interested in proving regularity for solutions of equations elliptic with respect to the class of linear operators \( L_{\alpha\beta} \) is of the form

\[ L_{\alpha\beta} u(x) = \int_{\mathbb{R}^n} \delta_o(u, x, y) K_e^{\alpha\beta}(x, y) dy + \int_{\mathbb{R}^n} \delta_o(u, x, y) K_o^{\alpha\beta}(x, y) dy. \]

with non constant right hand side. The authors believe that an adaptation of the perturbative methods found in [4] combined with our results could be apply in order to obtain \( C^{1,\alpha} \) regularity under some extra assumptions. This will also allow to use the already known \( C^{\sigma,\alpha} \) regularity in the case of the symmetric kernels (see [5]) to prove the same regularity for non necessarily symmetric kernel by treating the lower order term as a perturbation.

**Regularity for Fully Nonlinear Integro Differential Parabolic Equations.**

The authors are currently studying equations of the form

\[ u_t = I u, \]

where \( I \) is an elliptic operator with respect some class \( \mathcal{L} \). We are interested in recovering a parabolic ABP estimate like the one found in [14], or [15] and proving regularity by the means of a parabolic Harnack inequality.

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University of Texas at Austin, Department of Mathematics, 1 University Station C1200, Austin, TX 78712
E-mail address: hchang@math.utexas.edu

University of Texas at Austin, Department of Mathematics, 1 University Station C1200, Austin, TX 78712
E-mail address: gdavila@math.utexas.edu