THE DISTRIBUTION OF $H_8$-EXTENSIONS OF QUADRATIC FIELDS

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ABSTRACT. We compute all the moments of a normalization of the function which counts unramified $H_8$-extensions of quadratic fields, where $H_8$ is the quaternion group of order 8, and show that the values of this function determine a point mass distribution. Furthermore we propose a similar modification to the non-abelian Cohen-Lenstra heuristics for unramified $G$-extensions of quadratic fields for $G$ in a large class of 2-groups, which we conjecture will give finite moments which determine a distribution. Our method additionally can be used to determine the asymptotics of the unnormalized counting function, which we also do for unramified $H_8$-extensions.

1. Introduction

The Cohen-Lenstra conjectures describe the distributions of $p$-class groups of number fields in certain families [4]. By class field theory the class group is the Galois group of the maximal abelian unramified extension. Thus there is a natural non-abelian generalization of this question to describe the distribution of the Galois groups of maximal unramified extensions.

Let $D_X^\pm$ be the set of fundamental discriminants $0 < \pm d < X$. For any function $f$ on quadratic number fields define

$$E^\pm(f) = \lim_{X \to \infty} \frac{\sum_{K, \text{disc} K \in D_X^\pm} f(K)}{\sum_{K, \text{disc} K \in D_X^\pm} 1}.$$

The Cohen-Lenstra conjecture for quadratic fields is equivalent to: for all finite abelian $p$-groups $A$

$$E^\pm(|\text{Sur} (Cl_{K,p}, A)|) = \frac{1}{|A|^u}$$

where $u = 0, 1$ in imaginary and real cases respectively and $Cl_{K,p}$ is the $p$ part of the class group over $K$. Thus in the non-abelian case it is natural to study, for any non-abelian group $G$,

$$E^\pm(|\text{Sur} (\text{Gal}(K^{un}/K), G)|).$$

Note that this is equivalent to determining the number of unramified extensions $L/K$ with $\text{Gal}(L/K) = G$.

We will refine this question further. Fix a finite group $G'$ and a subgroup $G$. Let $f(K)$ be the number of unramified extensions $L/K$ with $\text{Gal}(L/K) = G$ and $\text{Gal}(L/Q) = G'$. For now we will consider extensions unramified only at the finite primes. We will call such $L/K$ an unramified $(G', G)$-extension. Define $E^\pm(G', G) = E^\pm(f(K))$. In [2] Bhargava asked about the value of $E^\pm(G', G)$ and proved several cases.
**Theorem 1** (Bhargava). For $n = 3, 4, 5$
\[ E^\pm (S_n \times C_2, S_n) = \infty \]
\[ E^+ (S_n, A_n) = \frac{1}{n!} \]
\[ E^- (S_n, A_n) = \frac{1}{2(n-2)!}. \]

Wood conjectured an answer to this question and proved some results in function fields which support it [12] (we refer there for the precise definitions).

**Conjecture 2** (Wood). Suppose there is a unique conjugacy class $c$ of order 2 elements of $G'$ which are not contained in $G$. Then
\[ E^\pm (G', G) = \frac{|H_2 (G', c)[2]|}{|c|^u \text{Aut}_{G'} (G)} \]
where $u = 0, 1$ in the imaginary and real cases respectively. Otherwise $E^\pm (G', G) = \infty$.

Alberts verified [1] the case of this conjecture $E^\pm (H_8 \rtimes C_2, H_8)$ where $H_8 \rtimes C_2$ is isomorphic to the quotient of $D_4 \oplus C_4$ obtained by identifying their Frattini subgroups and is the unique group that can occur as Gal ($L/Q$). Throughout the rest of the paper we will let $H_8 \rtimes C_2$ denote this group.

In this paper we consider $E^\pm (H_8 \rtimes C_2, H_8)$ with the important modification that $f(K)$ is appropriately normalized. We show all of the $k$th moments of this new function are finite, and in fact determine the point mass distribution.

We will now change the definition of $f(K)$ to count extensions unramified everywhere, including at the infinite prime, and call the corresponding extensions unramified everywhere $(G', G)$-extensions. We prove our theorems in this case- the case of extensions unramified at only finite primes is simpler and follows easily from our work (see Proposition [10]).

Our main result is

**Theorem 3.** Let $(G', G) = (H_8 \rtimes C_2, H_8)$. For a quadratic field $K$ with odd discriminant $d$ let $f(d)$ be the number of unramified everywhere $(G', G)$-extensions of $K$ and let $g(d) = 3^{\omega(d)}$. Then for all $k \in \mathbb{Z}_{\geq 1}$
\[ E^- ((f/g)^k) = \left( \frac{1}{32} \right)^k \]
and
\[ E^+ ((f/g)^k) = \left( \frac{1}{192} \right)^k \]
Thus the function $f(d)/g(d)$ determines the point mass distribution on $\mathbb{R}$.

We mean by this last statement that the sequence of measures
\[ \mu_n (U) = \frac{1}{|D_n^\pm|} |\{ f(d)/g(d) \in U \mid d < n \}| \]
on \( \mathbb{R} \) converges to the point mass \( \mu_c \) in distribution, where \( c = 1/32 \) and \( 1/192 \) in the complex and real case respectively.

As a corollary we have the following result which ties back to the question of the distribution of the Galois group of the maximal unramified extension.

**Corollary 4.** The density of quadratic fields \( K \) with \( \text{Gal}(K^{un}/K) = H_8^m \) is equal to 0 for any positive \( m \in \mathbb{Z} \).

Additionally, we prove an analogous unnormalized result which directly generalizes the aforementioned results due to Alberts [1]. Let \( H_8^k \rtimes_\sigma C_2 \) denote the group where the action of \( \sigma \) on each coordinate gives \( H_8 \rtimes C_2 \) according to our definition.

**Theorem 5.** Let \( k \in \mathbb{Z}_{\geq 1}, G = H_8^k \), and \( G' = H_8^k \rtimes C_2 \). Define \( \text{Sur}_\sigma(\text{Gal}(K^{un}/K),G) \) be the set of surjections which lift to a surjection \( \text{Gal}(K^{un}/\mathbb{Q}) \to G' \). Then

\[
\sum_{d \in D_\chi < X} |\text{Sur}_\sigma(\text{Gal}(K^{un}/K),G)| = \left( \frac{1}{4} \right)^k \left( \sum_{d \in D_\chi} 3^{k\omega(d)} \right) + O \left( X (\log X)^{3k-2+\epsilon} \right)
\]

and

\[
\sum_{d \in D_+ \chi} |\text{Sur}_\sigma(\text{Gal}(K^{un}/K),G)| = \left( \frac{1}{24} \right)^k \left( \sum_{d \in D_+ \chi} 3^{k\omega(d)} \right) + O \left( X (\log X)^{3k-2+\epsilon} \right).
\]

The proof of these results relies on a condition for the existence of \( H_8 \)-extensions and explicit construction thereof, due to Lemmermeyer [10] (see Proposition 10 below). We use this obtain a formula for the number of such extensions of any quadratic field. Then we build on the methods of Fouvry and Klüners from [5] to study the asymptotic growth of this expression. The constant of the main term in this expression is then obtained using combinatorial arguments with vector spaces over \( \mathbb{F}_2 \). The proof is separated into cases depending on the sign and congruence class of \( d \) modulo 8. The cases are all qualatatively similar, so we only present the case \( d < 0 \) and \( d \equiv 1 \mod 4 \) in the main body of the paper. We include the computations necessary for the remaining cases in an appendix for the sake of completeness.

For \( 1/3 \leq a \) let \( D_\chi^X,n,m \) be the set of discriminants \( d \) such that \( \pm d > 0 \) and \( d \equiv n \mod m \) and

\[
S_k^\pm(X,n,m) = \sum_{d \in D_\chi^X,n,m} \left( a^{\omega(d)} f(d) \right)^k.
\]

Both of the above theorems follow from the next result, the proof of which constitutes the bulk of this paper.

**Theorem 6.** Let \( (G',G) = (H_8 \rtimes C_2,H_8) \). For a quadratic field \( K \) with discriminant \( d \) let \( f(d) \) be the number of unramified everywhere \( (G',G) \)-extensions of \( K \). Then for
all $k \in \mathbb{Z}_{\geq 1}$, $a \geq 1/3$, and $(n, m) \in \{(4, 8), (0, 8)\}$

$$S_k^-(X, 1, 4) = \frac{1}{2^{5k}} \left( \sum_{d \in \mathcal{D}_{-X, 1, 4}} (3a)^{k\omega(d)} \right) + O \left( X (\log X)^{(3a)^k-a^k-1+\epsilon} \right)$$

$$S_k^-(X, n, m) = \frac{1}{a^k 2^{5k}} \left( \sum_{d \in \mathcal{D}_{-X,n,m}} (3a)^{k\omega(d)} \right) + O \left( X (\log X)^{(3a)^k-a^k-1+\epsilon} \right)$$

And for all $(n, m) \in \{(4, 8), (0, 8)\}$

$$S_k^+(X, 1, 4) = \frac{1}{3^{2k}2^{5k}} \left( \sum_{d \in \mathcal{D}_{+X, 1, 4}} (3a)^{k\omega(d)} \right) + O \left( X (\log X)^{(3a)^k-a^k-1+\epsilon} \right)$$

$$S_k^+(X, n, m) = \frac{1}{(3a)^k 2^{5k}} \left( \sum_{d \in \mathcal{D}_{+X,n,m}} (3a)^{k\omega(d)} \right) + O \left( X (\log X)^{(3a)^k-a^k-1+\epsilon} \right)$$

In fact Lemmermeyer exhibits conditions for the existence of a multitude of unramified $(G', G)$-extensions [11], all of which are 2-groups with $|G' : G| = 2$. However it should be noted that from that list only $H_8$ and $D_4$ can occur as the Galois group of an unramified extension of quadratic field which is Galois over $\mathbb{Q}$.

However it seems plausible that there are other examples of the form $(G', G)$ for which the corresponding extensions can be counted using similar expression built out of Legendre symbols. In such cases we would expect similar methods to work in computing finite extensions with the appropriate normalizations, though it is not clear what distributions to expect. We make the following conjecture for all unramified $(G', G)$-extensions, and which we make more specific in the case of $(D_4 \times C_2, D_4)$ using the above condition of Lemmermeyer combined with a heuristic which we detail in Section 7.

**Conjecture 7.** For a quadratic field $K$ with discriminant $d$, and an admissible pair of 2-groups $(G', G)$ as defined in [12], let $f(d)$ be the number of $(G', G)$-extensions of $K$. Then there exists a number $0 < a < 1$ such that

$$\mathbf{E} \left( (a^{\omega(d)} f(d))^k \right) < \infty$$

and these moments determine a distribution.

In the case of $(G', G) = (D_4 \times C_2, D_4)$ we have $a = 1/4$.

What the value of $a$ and the form of the distribution should be is unclear at this time. The result of Theorem [3] does not seem to fit with the finite case of Conjecture 2.

In each case $a$ seems to be at least partially related to the genus field $K^{gen}$ of $K$. For example in Theorem [3] where $a = 1/3$ it can be seen that $g(d) = 3^{\omega(d)}/6$ is the number of subfields $K^{gen}$ of the form $K(\sqrt{d_1}, \sqrt{d_2}, \sqrt{d_3})$ with $d = d_1d_2d_3$. All unramified
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$H_8$-extensions of $K$ intersect $K^{gen}$ in one of these, though not all factorizations can occur.

We remark that our method of normalizing by $g$ is reminiscent of Gerth’s idea \cite{6,7} of replacing $Cl_K[p]$ by $Cl_K[p]/Cl_K[p]^G$ to obtain finite moments in the abelian setting in the case when $p$ divides the degree of $K$. There $Cl_K[p]^G$ corresponds to the whole genus field. Gerth’s conjecture was proven by Fouvry and Klüners for $p = 2$ \cite{5} and for $p = 3$ by the second author also utilizing similar methods \cite{9}. Thus the case studied in this paper should be considered the non-abelian analog of that situation.

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2. Counting $H_8$ extensions

First we restate the results of Lemmermeyer from \cite{10} which are key to our expression for the number of unramified everywhere $(H_8 \rtimes C_2, H_8)$ extensions of a quadratic field $K$.

**Theorem 8** (Lemmermeyer). Let $K$ be a quadratic field with discriminant $d$. There exists an unramified $(H_8 \rtimes C_2, H_8)$-extension of $K$ if and only if there exists a nontrivial factorization $d = d_1d_2d_3$ into relatively prime discriminants such that

$$
\left(\frac{d_1d_2}{p_3}\right) = \left(\frac{d_2d_3}{p_1}\right) = \left(\frac{d_3d_1}{p_2}\right) = 1
$$

for all $p_i | d_i$.

Let $K^{gen} = K\left(\sqrt{p_1}, \ldots, \sqrt{p_r}\right)$ where $p_i$ are the prime fundamental discriminants dividing $d$. Any unramified $H_8$-extension $L$ of $K$ will satisfy $L \cap K^{gen} = K\left(\sqrt{d_1}, \sqrt{d_2}, \sqrt{d_3}\right)$ where the $d_i$ are coprime discriminants and $d = d_1d_2d_3$. Let $L^g = L \cap K^{gen}$. Then $L = L^g\left(\sqrt{\mu}\right)$ for some $\mu \in L^g$.

**Proposition 9** (Lemmermeyer). Suppose $L$ is an unramified $(H_8 \rtimes C_2, H_8)$-extension of $K$ such that $L = L^g\left(\sqrt{\mu}\right)$. Then all the other such extensions $M$ of $K$ with $M \cap K^{gen} = L^g$ are exactly given by $L^g\left(\sqrt{\delta\mu}\right)$ for each $\delta \in \mathbb{Z}$ a discriminant such that $\delta \mid d$.

We use these results to prove the next proposition.

**Proposition 10.** The number of unramified everywhere $(H_8 \rtimes C_2, H_8)$-extensions of $K$ is

$$
f(d) = \beta(d)2^{-3-\alpha(d)} \sum_{d = d_1d_2d_3 \prod p \mid d_3} \prod_{p \mid d_2} \left(1 + \left(\frac{d_1d_2}{p}\right)\right) \prod_{p \mid d_1} \left(1 + \left(\frac{d_2d_3}{p}\right)\right)\prod_{p \mid d_1} \left(1 + \left(\frac{d_3d_1}{p}\right)\right)
$$
Where the sum is over nontrivial factorizations into coprime fundamental discriminants \( d = d_1d_2d_3 \) up to permutation, and

\[
\alpha(d) = \begin{cases} 
1 & d > 0 \text{ and } \exists p \mid d, p \equiv 3 \mod 4 \\
0 & \text{else} 
\end{cases}
\]

\[
\beta(d) = \begin{cases} 
1 & d < 0 \text{ or } \exists p \mid d, p \equiv 3 \mod 4 \\
between 0 \text{ and } 1 & \text{else} 
\end{cases}
\]

Proof. We have that

\[
2^{-\omega(d)} \prod_{p\mid d_3} \left(1 + \left(\frac{d_1d_2}{p}\right)\right) \prod_{p\mid d_2} \left(1 + \left(\frac{d_1d_3}{p}\right)\right) \prod_{p\mid d_1} \left(1 + \left(\frac{d_2d_3}{p}\right)\right)
\]

is 1 if \( d = d_1d_2d_3 \) is an admissible factorization and corresponds to an extension \( L/K \) unramified at all finite places, and 0 otherwise. We have for each factorization \( 2^{\omega(d)-3} \) such extensions unramified at all finite places. For \( d < 0 \), this is the same as unramified everywhere and the result follows immediately.

Suppose \( d > 0 \). Then we have \( d_1, d_2, d_3 > 0 \), and a map

\[ \varphi : Cl(K)[2]/\langle d_1, d_2, d_3 \rangle \to \{+1, -1\} \]

sending \( d' \mapsto d'/|d'| \). If there exists a prime \( p \mid d \) with \( p \equiv 3 \mod 4 \), then \(-p \mid d\) is a discriminant, making \( \varphi \) surjective. If \( L^g(\sqrt{p})/K \) is an extension unramified at all finite places, but ramified at the infinite place then \( L^g(\sqrt{-p})/K \) is unramified everywhere. We then have that the number of extensions unramified everywhere is the size of the kernel, \( 2^{\omega(d)-4} \). If there are no primes dividing \( d \) congruent to 3 \mod 4 \) then \( \varphi \) is the trivial map. It follows similarly to the last case that either every extension \( L/K \) corresponding to the factorization \( d = d_1d_2d_3 \) is unramified at infinity or none of them are, so there are \( \beta(d)2^{\omega(d)-3} \) such extensions for some \( 0 \leq \beta(d) \leq 1 \) denoting the fraction of such factorizations. \( \square \)

We can replace the conditions of Proposition 10 by \( \beta(d) = 1 \) for all \( d \) and \( \alpha(d) = 1 \) if \( d > 0 \) and \( 0 \) if \( d < 0 \). This is because discriminants not divisible by some \( p \equiv 3 \mod 4 \) do not contribute to the main term, which can be seen by an argument similar to Corollary 1 from [5] which we sketch here.

Lemma 11. Let

\[
\tilde{f}(d) = 2^{3-\tilde{\alpha}(d)} \sum_{d = d_1d_2d_3} \prod_{p \mid d_1} \left(1 + \left(\frac{d_1d_2}{p}\right)\right) \prod_{p \mid d_2} \left(1 + \left(\frac{d_1d_3}{p}\right)\right) \prod_{p \mid d_1} \left(1 + \left(\frac{d_2d_3}{p}\right)\right),
\]

\[
\tilde{\alpha}(d) = \begin{cases} 
1 & d > 0 \\
0 & d < 0.
\end{cases}
\]

If Theorem \( \Box \) holds for \( \tilde{f}(d) \), then it also holds for \( f(d) \).
Proof. We have by Landau’s Theorem and by Hölder’s inequality with $1/b + 1/c = 1$ that

$$
\sum_{d \in D^\pm_{\pm 1, 1, 4, \text{p} | d \Rightarrow p \equiv 1 \mod 4}} (a^{\omega(d)} f(d))^k = \left( \sum_{d \in D^\pm_{\pm 1, 1, 4, \text{p} | d \Rightarrow p \equiv 1 \mod 4}} 1 \right)^{1/b} \left( \sum_{d \in D^\pm_{\pm 1, 1, 4}} (a^{\omega(d)} f(d))^k \right)^{1/c}
$$

$$
\ll \left( \frac{X}{\sqrt{\log X}} \right)^{1/b} \left( X (\log X)^{(3a)^k - 1} \right)^{1/c}.
$$

The result follows by noting that we can find values $b, c > 1$ such that $-\frac{1}{2b} + \frac{(3a)^k - 1}{c} \leq (3a)^k - a^k - 1 + \epsilon$, forcing the only terms where the two sums differ into the error term (for example, taking $b = a^{-k((3a)^k - 1/2)} > 1$).

Thus in the remainder of the paper we will let write $f(d)$ to denote $\tilde{f}(d)$. We will proceed with the proof of Theorem 6. Our first step will be to express

$$
S_k^\pm = \sum_{d \in D^\pm_X} (a^{\omega(d)} f(d))^k
$$

as a character sum. We note that below the factorizations will not be assumed to be nontrivial and the sums will be over all permutations of the $d_i$.

As previously mentioned we will first prove the results in the case of negative discriminants congruent to $1 \mod 4$. The computations necessary to treat the remaining cases are handled in the appendix.

For a quadratic field $K$ with fundamental discriminant $d$ define $f(d)$ to be the number of unramified $H_8$-extensions of $K$ which are Galois over $\mathbb{Q}$. Define functions

$$
f_1(d) = 2^{-\frac{1}{2}} \sum_{d = -d_1 d_2 d_3 \text{p} | d_1} \prod_{p | d_1} \left( 1 + \left( \frac{-d_1 d_2}{p} \right) \right) \prod_{p | d_2} \left( 1 + \left( \frac{d_2 d_3}{p} \right) \right) \prod_{p | d_3} \left( 1 + \left( \frac{-d_3 d_1}{p} \right) \right)
$$

and

$$
f_2(d) = 2^{-3} \sum_{d = -d_1 d_2 \text{p} | d_1} \prod_{p | d_1} \left( 1 + \left( \frac{d_2}{p} \right) \right) \prod_{p | d_2} \left( 1 + \left( \frac{-d_1}{p} \right) \right)
$$

where the factorization $d = -d_1 d_2 d_3$ is into integers such that $d_1 \equiv -1 \mod 4$ and $d_2, d_3 \equiv 1 \mod 4$ and similarly for $d = d_1 d_2$ is into integers such that $d_1 \equiv -1 \mod 4$ and $d_2 \equiv 1 \mod 4$.

Then we have

$$
f(d) = f_1(d) - f_2(d) + 2^{\omega(d) - 3}.
$$

The congruence conditions above are imposed since the condition from [10] requires that the factorization be into fundamental discriminants, but factorizations where more than one term is negative don’t contribute to the count (that is the expression in [10] evaluates to 0). So we can assume $-d_1$ is the negative factor in the factorization of $d$. 
The expression contains $f_2$ and $2^{\omega(d)-1}$ to account for trivial factorizations (where at least one of the $d_i$ equals 1). Note this implies $f(d) = 0$ when $\omega(d) \leq 2$.

Expanding these expressions gives

\[
\begin{align*}
f_1(d) &= \frac{1}{2^4} \sum_{d = -d_1d_2d_3} \left( \sum_{a|d_3} \left( \frac{-d_1d_2}{a} \right) \right) \left( \sum_{b|d_1} \left( \frac{-d_2d_3}{b} \right) \right) \left( \sum_{c|d_2} \left( \frac{-d_3d_1}{c} \right) \right) \\
&= \frac{1}{2^4} \sum_{d = -d_1d_2d_3} \left( \sum_{D_4D_1 = d_3} \left( \frac{-D_0D_3D_2D_5}{D_4} \right) \right) \left( \sum_{D_0D_3 = d_1} \left( \frac{D_2D_5D_4D_1}{D_0} \right) \right) \\
&\times \left( \sum_{D_2D_5 = D_2} \left( \frac{-D_4D_1D_0D_3}{D_2} \right) \right) \\
&= \frac{1}{2^4} \sum_{d = -D_0D_1D_2D_3D_4D_5} \left( \frac{-1}{D_4} \right) \left( \frac{-1}{D_2} \right) \\
&\times \left( \frac{D_0D_3D_2D_5}{D_4} \right) \left( \frac{D_2D_5D_4D_1}{D_0} \right) \left( \frac{D_4D_1D_0D_3}{D_2} \right) \\
\end{align*}
\]

(2.1)

and

\[
\begin{align*}
f_2(d) &= \frac{1}{2^3} \sum_{d = -d_1d_2} \left( \sum_{b|d_1} \left( \frac{d_2}{b} \right) \right) \left( \sum_{c|d_2} \left( \frac{-d_1}{c} \right) \right) \\
&= \frac{1}{2^3} \sum_{d = -d_1d_2} \left( \sum_{E_0E_1 = d_1} \left( \frac{E_2E_4}{E_0} \right) \right) \left( \sum_{E_2E_3 = d_2} \left( \frac{-E_0E_1}{E_2} \right) \right) \\
&= \frac{1}{2^3} \sum_{d = -E_0E_1E_2E_3} \left( \frac{-1}{E_2} \right) \left( \frac{E_2E_3}{E_0} \right) \left( \frac{E_0E_1}{E_2} \right). \\
\end{align*}
\]

(2.2)

where the sums are over factorizations which satisfy the congruences

\[
\begin{align*}
D_0D_3 &\equiv -1 \mod 4 \\
D_2D_5 &\equiv 1 \mod 4 \\
D_4D_1 &\equiv 1 \mod 4
\end{align*}
\]

and

\[
\begin{align*}
E_0E_1 &\equiv -1 \mod 4 \\
E_2E_3 &\equiv 1 \mod 4.
\end{align*}
\]

For two indices $u, v$ define the function $\Phi(u, v) \in \mathbb{F}_2$ to be 1 if and only if the symbol $\left( \frac{D_u}{D_v} \right)$ appears in (2.1) and similarly define $\lambda(u) \in \mathbb{F}_2$ to be 1 if and only if $\left( \frac{-1}{D_u} \right)$ appears. Also define the function $\Psi(u, v) \in \mathbb{F}_2$ to be 1 if and only if the symbol $\left( \frac{E_u}{E_v} \right)$ appears in (2.2) and similarly define $\gamma(u) \in \mathbb{F}_2$ to be 1 if and only if $\left( \frac{-1}{E_u} \right)$ appears.
Taking the \( k \)th power of \( f(d) \) gives
\[
(f(d))^k = \sum_{j_1+j_2+j_3=k} \binom{k}{j_1, j_2, j_3} f_1^{j_1}(d) (-1)^{j_2} f_2^{j_2}(d) 2^{j_3(\omega(d)-4)}
\]

Thus with the above notations we can write
\[
\prod_{u} D_u \prod_{v} D_v \equiv \begin{cases} 
-1 & \text{mod 4} \, \text{if} \, (u_i, v_i) = (0, 3) \\
1 & \text{mod 4} \, \text{if} \, (u_i, v_i) = (2, 5), (4, 1) 
\end{cases}
\]

We will define some notation to rewrite \( f_{j_1,j_2}(d) \) in a form suitable to application of analytic techniques. In the expression for \( f_{j_1,j_2}(d) \) we have \( j_1 \) different factorizations of \( d \) into \( 6 \) variables and \( j_2 \) factorizations of \( d \) into \( 4 \) variables, with \( j_1 + j_2 = l \). Write these factorizations of \( d \) as
\[
d = \prod_{u_1} D^{(1)}_{u_1} \cdots \prod_{u_{j_1}} D^{(j_1)}_{u_{j_1}} = \prod_{u_{j_1+1}} E^{(j_1+1)}_{u_{j_1+1}} \cdots \prod_{u_l} E^{(l)}_{u_l}
\]
where each index \( u_i \) runs from 0 to \( 5 \) for \( 1 \leq i \leq j_1 \) and from 0 to \( 3 \) for \( j_1 + 1 \leq i \leq l \). From this we obtain a further factorization of each \( D^{(h)}_{u_h} \) by
\[
D^{(h)}_{u_h} = \prod_{1 \leq i \leq l, i \neq h} \text{gcd} \left( D^{(1)}_{u_1}, \ldots, D^{(j_1)}_{u_{j_1}}, E^{(j_1+1)}_{u_{j_1+1}}, \ldots, E^{(l)}_{u_l} \right).
\]

Define
\[
D_{u_1,\ldots,u_{j_2}} = \text{gcd} \left( D^{(1)}_{u_1}, \ldots, D^{(j_1)}_{u_{j_1}}, E^{(j_1+1)}_{u_{j_1+1}}, \ldots, E^{(l)}_{u_l} \right).
\]

To simplify notation we let \( u = (u_1, \ldots, u_l), v = (v_1, \ldots, v_l) \), and define the functions
\[
\Phi_{j_1}(u, v) = \sum_{i=1}^{j_1} \Phi(u_i, v_i)
\]
and
\[
\Psi_{j_2}(u, v) = \sum_{i=j_1}^{l} \Psi(u_i, v_i).
\]

Thus with the above notations we can write
\[
f_{j_1,j_2}(d) = \frac{1}{2^{j_1+3j_2}} \sum_{c(D_u)} \prod_{u} \left( -\frac{1}{D_u} \right)^{\lambda_{j_1}(u) + \gamma_{j_2}(u)} \prod_{u,v} \left( \frac{D_u}{D_v} \right)^{\Phi_{j_1}(u,v) + \Psi_{j_2}(u,v)}
\]
where now the sum is over \( 6^{j_1}4^{j_2} \) tuples of integers \((D_u)\) which satisfy \( \prod_u D_u = d \) and the following congruence conditions:

for all \( 1 \leq i \leq j_1 \) and \((u_i, v_i) \in \{(0, 3), (2, 5), (4, 1)\}\)

\[
(2.3) \quad \prod_u D_u \prod_v D_v \equiv \begin{cases} 
-1 & \text{mod 4} \, \text{if} \, (u_i, v_i) = (0, 3) \\
1 & \text{mod 4} \, \text{if} \, (u_i, v_i) = (2, 5), (4, 1) 
\end{cases}
\]
and for all \( j_1 + 1 \leq i \leq l \) and \( (u_i, v_i) \in \{(0, 1), (2, 3)\} \)

(2.4) \[ \prod_u D_u \prod_v D_v \equiv \begin{cases} -1 \text{ mod } 4 & \text{if } (u_i, v_i) = (0, 1) \\ 1 \text{ mod } 4 & \text{if } (u_i, v_i) = (2, 3) \end{cases} \]

where the above products are over all \( u \) with \( u_i \) in the \( i \)th position and all \( v \) with \( v_i \) in the \( i \)th position.

Thus multiplying by \( a^{\kappa(d)} \) and summing over discriminants \( d < 0 \) with \( d \equiv 1 \mod 4 \) we get

(2.5) \[ \sum_{d < X} 2^{j_1 \omega(d)} a^{\kappa(d)} f_{j_1,j_2} (d) = \frac{1}{2^{4j_1+3j_2}} \sum_{(D_u)} \mu^2 \left( \prod_u D_u \right) 2^{j_3 \omega(d)} a^{\kappa(d)} \]

\[ \times \prod_u \left( \frac{-1}{D_u} \right) \lambda_{j_1} (u) + \gamma_{j_2} (u) \prod_{u,v} \left( \frac{D_u}{D_v} \right) \phi_{j_1} (u,v) + \psi_{j_2} (u,v) \]

where the sum is over \( 6^{j_1} 4^{j_2} \) tuples of integers \((D_u)\) which satisfy \( \prod_u D_u < X \) and the conditions \((2.3)\) and \((2.4)\).

3. Bounding the Error Term

So far we have written

\[ S_k (X) = \sum_{j_1+j_2+j_3=k} \binom{k}{j_1,j_2,j_3} (-1)^{j_2} a^{\kappa(d)} 2^{j_3 \omega(d)-3-\alpha(d)} f_{j_1,j_2} (d). \]

Where \( \alpha(d) = 1 \) if \( d < 0 \) or there exists a prime \( p \equiv 3 \mod 4 \) with \( p \mid d \), and \( \alpha(d) = 0 \) otherwise. We will analyse each term separately. Let

\[ S_{j_1,j_2,j_3} (X) = \sum_{d < X} a^{\kappa(d)} 2^{j_3 \omega(d)-3-\alpha(d)} f_{j_1,j_2} (d). \]

The right hand side was computed in the previous section and is given by \( (2.5) \).

We want to separate \((2.5)\) into a main term and error term. The methods of Fouvry-Klüners from [5] apply with minor modifications to accomplish this. We state the key points of their argument and refer to [5] for the proofs.

Fix \( k \in \mathbb{Z}_{\geq 1} \) and let \( \Delta = 1 + \log^{-\theta(a)} X \). Define \( \mathbf{A} \) to be a tuple \((A_i)_{i=0}^{6^{j_1} 4^{j_2}}\) of variables with each \( A_i \) corresponding to \( D_i \), and each \( A_i \) is \( \Delta^j \) for some \( j \geq 0 \). We can partition \( S_{j_1,j_2,j_3} (X) \) according to the various \( \mathbf{A} \), by letting \( S_{j_1,j_2,j_3} (X, \mathbf{A}) \) be the sum \((2.5)\) but now restricted to tuples \((D_i)\) for which \( A_i \leq D_i \leq \Delta A_i \) and \( \prod_i D_i < X \). Hence

\[ S_{j_1,j_2,j_3} (X) = \sum_{\mathbf{A}} S_{j_1,j_2,j_3} (X, \mathbf{A}). \]

Note that if \( \Delta = 1 + \log^{-\theta(a)} X \) then there are \( O \left( (\log X)^{6^{j_1} 4^{j_2} \left(1+(\alpha))^k\right)} \right) \) possible \( \mathbf{A} \) with \( S_{j_1,j_2} (X, \mathbf{A}) \) not empty. This is since there are \( O \left( (\log X)^{\left(1+(\alpha)^k\right)} \right) \) choices for each \( 1 < A_i \leq X \).

Let \( \Omega = e(a)^k (\log \log X + B_0) \). Noting \( a > 1/6 \), we have
Lemma 12 (Fouvry-Klüners [5]). Let $S$ be the sum of the terms in (2.5) which satisfy: at least one $d_i$ has $\omega(d_i) > \Omega$. Then

$$S = O \left( X (\log X)^{-1} \right).$$

Lemma 13 (Fouvry-Klüners [5]). Let $\mathcal{F}_1$ be the set of $A$ which satisfies $\prod_i \Delta A_i > X$. Then

$$\sum_{A \in \mathcal{F}_1} S_{j_1,j_2,j_3} (X, A) = O \left( X (\log X)^{-1} \right).$$

For the next two lemmas we will need to define

$$X^\dagger = \log^{3(1+6^{j_1}4^{j_2}(1+(a6)^k))} X$$

$$X^\ddagger = \exp \left( \log^{\eta(k)} X \right)$$

for some small $\eta(k)$ depending only on $k$.

Now let $Y_1 = \{0, 1, 2, 3, 4, 5\}$ and let $Y_2 = \{0, 1, 2, 3\}$. The indices of the variables lie in $Y = Y_1^{j_1} \times Y_2^{j_2}$. We define two indices $u, v \in Y$ to be linked if $\Phi(u, v) + \Psi(u, v) + \Phi(v, u) + \Psi(v, u) = 1$. This means that exactly one of the symbols $\left( \frac{D_k}{D_c} \right)$ and $\left( \frac{D_k}{D_a} \right)$ appears in (2.5).

Define

$$U(j_3) = \begin{cases} 3^{k-j_3} + 1 & \text{if } j_3 > 0 \\ 3^k & \text{if } j_3 = 0 \end{cases}$$

Lemma 14 (Fouvry-Klüners [5]). Let $\mathcal{F}_2$ be the set of $A$ satisfying: at most $U(j_3) - 1$ variables are $A_i > X^\dagger$. Then

$$\sum_{A \in \mathcal{F}_2} S_{j_1,j_2,j_3} (X, A) \ll X (\log X)^{2^{j_1}a^k(3^{k-j_3} - 1 + \eta(k)6^{j_1}4^{j_2}) - 1}$$

Lemma 15 (Fouvry-Klüners [5]). Let $\mathcal{F}_3$ be the set of $A$ satisfying: there exist two linked indices $i$ and $j$ with $A_i \geq X^\ddagger$ and $A_j \geq 2$. Then

$$\sum_{A \in \mathcal{F}_3} S_{j_1,j_2,j_3} (X, A) = O \left( X (\log X)^{-1} \right).$$

Consider the set of $A$ satisfying

(3.1) $A$ is not in $\mathcal{F}_1$ for any $i = 0, 1, 2, 3$.

Combining the above lemmas we reduce our expression to

$$S_{j_1,j_2,j_3} (X) = \sum_A' S_{j_1,j_2,j_3} (X, A) + O \left( X (\log X)^{2^{j_1}a^k3^{k-j_3} - 2^{j_3}a^k - 1 + \epsilon} \right)$$

where the sum is over $A$ satisfying (3.1), after taking $\eta(k) = \epsilon 6^{-j_1}4^{-j_2}2^{-j_3}a^{-k}$.

Note this condition implies that there are at least $U(j_3)$ variables $A_i > X^\ddagger$ and they are all unlinked. In the next section we will show that a maximal unlinked set in $Y$ is exactly of size $3^{j_1}2^{j_2}$ and this is strictly less than $U(j_3)$ unless $j_1 = k$. 

4. Maximal unlinked sets

Consider the set of indices \( Y = Y_1^{j_1} \times Y_2^{j_2} \). As in the previous section, we call two indices \( u, v \) linked if \( \Phi_{j_1}(u, v) + \Psi_{j_2}(u, v) + \Phi_{j_1}(v, u) + \Psi_{j_2}(v, u) = 1 \) and unlinked otherwise. When \( j_1 = 0 \) the maximal unlinked subsets of \( Y \) are determined [5] and are of size \( 2^{j_2} \). We will now determine the largest maximal unlinked sets when \( j_2 = 0 \).

**Proposition 16.** Let \( A = \{1, 3, 5\} \) and \( B = \{0, 2, 4\} \). Let \( S = \{A, B\}^k \). The largest maximal unlinked sets are all of size \( 3^k \) and correspond bijectively to elements of \( S \).

The set corresponding to \( s \in S \) is

\[
U_s = \{ u \in Y^k \mid |u| = s \}.
\]

**Proof.** Define a graph \( G_k \) with vertices \( \{0, 1, 2, 3, 4, 5\}^k \) and adjacency matrix given by \([G_k] \equiv [B_k(u, v)] \mod 2\), where we define \( B_k(u, v) = \Phi_k(u, v) + \Phi_k(v, u) \). Unlinked sets are exactly the independent sets of \( G_k \). Notice for \( k = 1 \) that \( G_1 \) is a cyclic graph with 6 vertices, and has largest maximal independent sets given by \( A \) and \( B \). We use this as a base case for induction.

Suppose the theorem holds true for \( k - 1 \), and let \( U \subseteq G_k \) be independent, and partition it into \( U = \bigcup_{i=0}^{5} C_i \) where \( C_i = \{(u, i) \in U : u \in G_{k-1}\} \). Call \( c_i = |C_i| \), so that we have \(|U| = \sum c_i \). We know that

\[
[G_k] \equiv [B_k(u, v)] = [B_{k-1}((u_2, ..., u_k), (v_2, ..., v_k))] + [B_1(u_1, v_1)]
\]

The subgraph induced by \( U \) inside of \( G_k \) corresponds to a submatrix \([U] = 0\) in \([G_k]\) along the vertices of \( U \). In particular,

\[
[B_{k-1}((u_2, ..., u_k), (v_2, ..., v_k))] = [B_1(i, j)]
\]

for all \( u \in C_i, v \in C_j \). Ordering indices lexicographically, with the \( k^{th} \) entire weighted highest, we then have

\[
[B_{k-1}|U] = \\
\begin{pmatrix}
0_{c_0 \times c_0} & 1_{c_0 \times c_1} & 0_{c_0 \times c_2} & 0_{c_0 \times c_3} & 0_{c_0 \times c_4} & 1_{c_0 \times c_5} \\
1_{c_1 \times c_0} & 0_{c_1 \times c_1} & 1_{c_1 \times c_2} & 0_{c_1 \times c_3} & 0_{c_1 \times c_4} & 0_{c_1 \times c_5} \\
0_{c_2 \times c_0} & 1_{c_2 \times c_1} & 0_{c_2 \times c_2} & 1_{c_2 \times c_3} & 0_{c_2 \times c_4} & 0_{c_2 \times c_5} \\
0_{c_3 \times c_0} & 0_{c_3 \times c_1} & 1_{c_3 \times c_2} & 0_{c_3 \times c_3} & 1_{c_3 \times c_4} & 0_{c_3 \times c_5} \\
0_{c_4 \times c_0} & 1_{c_4 \times c_1} & 0_{c_4 \times c_2} & 1_{c_4 \times c_3} & 0_{c_4 \times c_4} & 1_{c_4 \times c_5} \\
1_{c_5 \times c_0} & 0_{c_5 \times c_1} & 0_{c_5 \times c_2} & 0_{c_5 \times c_3} & 1_{c_5 \times c_4} & 0_{c_5 \times c_5}
\end{pmatrix}
\]

Where \( 0_{n \times m} \) and \( 1_{n \times m} \) are block matrices of dimension \( n \times m \) with all 0 and 1 entries respectively.

**Lemma 17.**

\[
c_{i-1} + c_{i+1} \leq \begin{cases} 
2 \cdot 3^{k-1} & c_i = 0, c_{i+2} = c_{i+4} = 0 \\
3^{k-1} & c_i = 0; \text{else}
\end{cases}
\]

\[
2(3^{k-1} - 1) & c_i \neq 0, c_{i+2} = c_{i+4} = 0 \\
3^{k-1} - 1 & c_i \neq 0; \text{else}
\]

**Proof.** Consider a complete bipartite induces subgraph \( K_{V,W} \subset G_{k-1} \). By the inductive hypothesis \(|V| \leq 3^{k-1}\) with equality if and only if \( V = U_t \) for some \( t \in S \), and similarly
for $W$. Suppose $V \neq \emptyset$, then for any $u \in V$ of $t \in S$ we have that there exists a $v \in U_s$ for any $s \in S$ defined

$$u_j = \begin{cases} 
  v_{j+2m} & t_j = s_j \\
  v_{j+3} & t_j \neq s_j
\end{cases}$$

Then $B_{k-1}(u, v) = 0$, so $v \notin W$. Thus $W \neq U_s$ and so $|W| \leq 3^{k-1} - 1$. By symmetry the same is true for $V$ if $W \neq \emptyset$.

Define $p : G_k \to G_{k-1}$ to be the projection forgetting the $k^{th}$ coordinate. Notice that $p|_{C_i}$ is injective for all $i$ values.

Suppose $c_i = 0, c_{i+2} = c_{i+4} = 0$. Then we use the trivial bound: the submatrix on vertices in $C_j$ is a block zero matrix, so that $p(C_j)$ is an independent set of $G_{k-1}$. Thus, $c_j \leq 3^{k-1}$, so $c_{i-1} + c_{i+1} \leq 2 \cdot 3^{k-1}$ by the inductive hypothesis.

Suppose $c_i = 0$ and without loss of generality $c_{i+2} \neq 0$. Then for $(u, i-1) \in C_{i-1}$ and $(v, i+1) \in C_{i+1}$, choose some $(w, i+2) \in C_{i+2}$. We have $B_{k-1}(u, w) = B(i-1, i+2) = 0$ and $B_{k-1}(v, w) = B(i+1, i+2) = 1$, implying $u \neq v$. Then we have $p(C_{i-1} \cap p(C_{i+1}) = \emptyset$ and $p(C_{i-1}) \cup p(C_{i+1})$ is an independent set of $G_{k-1}$, by $B_{k-1}(u, v) = B(i-1, i+1) = 0$.

Thus $c_{i-1} + c_{i+1} \leq 3^{k-1}$ by the inductive hypothesis.

Suppose $c_i \neq 0, c_{i+2} = c_{i+4} = 0$. Then $p(C_{i+1} \cup p(C_i)$ is a complete bipartite induced subgraph of $G_{k-1}, K_{V,W}$ for $|V| = c_{i+1}$ and $|W| = c_i$. Similarly for $c_{i-1}, c_i$. Then $c_{i-1} + c_{i+1} \leq 3^{k-1} - 1$ and the result follows.

Suppose $c_i \neq 0$ and without loss of generality $c_{i+2} \neq 0$. We can similarly prove $p(C_{i-1}) \cap p(C_{i+1}) = \emptyset$ and $p(C_{i-1}) \cup p(C_{i+1}) \cup p(C_i)$ is an induce bipartite subgraph of $G_{k-1}, K_{V,W}$ with $V = p(C_{i-1}) \cup p(C_{i+1})$ and $W = p(C_i)$. So a combination of the previous two results gives us this case. 

\[ \square \]

Let $I = \{ i : c_i = 0 \}$ we will separate cases based on the size of $I$.

If $|I| \geq 4$ then we have

\[
2|U| = \sum_i c_{i-1} + c_{i+1} = \sum_{i \notin I} 2c_i \\
\leq 2(6 - |I|)3^{k-1} \\
\leq 4 \cdot 3^{k-1} \\
< 2 \cdot 3^k
\]

And is not maximum.

If $|I| = 3$ we must separate into two cases. If $I = \{0, 2, 4\}$ or $\{1, 3, 5\}$ then $U \leq 3^k$ with equality iff $p(C_j)$ is of maximum size for an indeendent set in $G_{k-1}$ for all $j \notin I$. But maximum implies it equals some $U_s$, and $I = \{0, 2, 4\}$ or $\{1, 3, 5\}$ implies we can extend the type for $k-1$ to a type for $k$ with $U = U_s$.

Otherwise, by symmetry we can assume $I \cap \{0, 2, 4\} = \{0, 2\}$. Let $j$ be the third element of $I$. Then we have by the above lemma $c_1 + c_3, c_1 + c_5 \leq 3^{k-1}$. We also have $c_0 + c_2 = 0$ and $c_0 + c_4 = c_2 + c_4 = c_4 \leq 3^{k-1} - 1$ by at least one of $c_5, c_3$ nonzero.
Lastly \( c_3 + c_5 \leq 2 \cdot 3^{k-1} \). Thus we have

\[
2|U| = \sum_i \left( c_{i-1} + c_{i+1} \right) \\
\leq 4 \cdot 3^{k-1} + 2(3^{k-1} - 1) \\
\leq 2 \cdot 3^k - 2 \\
< 2 \cdot 3^k
\]

And is then not maximum.

If \(|I| = 2\) we need two cases. First, if \( I \subset \{0,2,4\} \) or \( \{1,3,5\} \). Then there exists a \( j \) such that \( I = \{j-1,j+1\} \). Without loss of generality suppose \( j = 1 \). Then we have \( c_0 + c_2 = 0 \) and \( c_2 + c_4 = c_0 + c_4 \leq 3^{k-1} - 1 \). We also have \( c_1 + c_3, c_1 + c_5 \leq 3^{k-1} \) and \( c_3 + c_5 \leq 2 \cdot (3^{k-1} - 1) \). Thus we have

\[
2|U| = \sum_i \left( c_{i-1} + c_{i+1} \right) \\
\leq 2 \cdot (3^{k-1} - 1) + 2 \cdot 3^{k-1} + 2(3^{k-1} - 1) \\
\leq 2 \cdot 3^k - 4 \\
< 2 \cdot 3^k
\]

And is not maximum.

Otherwise, \( I \neq \{j-1,j+1\} \). Then we have \( c_{j-1} + c_{j+1} < 3^{k-1} \) if \( j \in I \) and \( 3^{k-1} - 1 \) is \( j \notin I \). Thus we have

\[
2|U| = \sum_i \left( c_{i-1} + c_{i+1} \right) \\
\leq 2 \cdot 3^{k-1} + 4(3^{k-1} - 1) \\
\leq 2 \cdot 3^k - 4 \\
< 2 \cdot 3^k
\]

And is not maximum.

If \(|I| = 1\) suppose without loss of generality that \( 0 \in I \). Then \( c_0 + c_2, c_0 + c_4, c_1 + c_5 \leq 3^{k-1} \) and \( c_1 + c_3, c_2 + c_4, c_3 + c_5 \leq 3^{k-1} - 1 \). Thus we have

\[
2|U| = \sum_i \left( c_{i-1} + c_{i+1} \right) \\
\leq 3^k + 3(3^{k-1} - 1) \\
\leq 2 \cdot 3^k - 3 \\
< 2 \cdot 3^k
\]

And is not maximum.
If \(|I| = 0\) then \(c_i - 1 + c_{i+1} \leq 3^{k-1} - 1\). Thus we have

\[
2|U| = \sum_i c_i - 1 + c_{i+1} \\
\leq 6(3^{k-1} - 1) \\
\leq 2 \cdot 3^k - 6 \\
< 2 \cdot 3^k
\]

And is not maximum. \(\square\)

To simplify notation we will refer to the largest maximal unlinked set corresponding to \(s \in S\) as being of type \(s\) or simply as a type.

We now combine these results to determine the largest maximal unlinked sets for all \(j_1, j_2 > 0\).

**Proposition 18.** The largest maximal unlinked sets in \(Y\) are of the form \(V \times W\) where \(V\) is a type in \(Y_1^{j_1}\) and \(W\) is a maximal unlinked set in \(Y_2^{j_2}\). Thus the largest maximal unlinked sets of \(Y\) are of size \(3^{j_1}2^{j_2}\).

**Proof.** We fix \(j_1 > 0\) and prove this by induction on \(j_2\). Let \(G_{j_1,j_2} = Y_1^{j_1} \times Y_2^{j_2}\). The base case \(j_2 = 0\) is Theorem 16.

Let \(U \subset G_{j_1,j_2}\) be a largest maximal unlinked set. Now let \(C_i = \{(u, i) \in U \mid u \in G_{j_2-1}\}\) and \(c_i = |C_i|\) for \(i \in Y_2\), as above. Let \(p : G_{j_1,j_2} \rightarrow G_{j_1,j_2-1}\) be the projection dropping the last coordinate. Suppose \(i\) and \(j\) are unlinked, and \(c_i, c_j, c_k \neq 0\) for \(i, j, k\) all distinct. Let \((u, i) \in C_i, (v, j) \in C_j, (w, k) \in C_k\). \(k\) is linked to exactly one of \(i, j\) by the pigeonhole principle, as \(i, j, k \in \{0, 1, 2, 3\}\) and the linked pairs are \(\{0, 1\}\) and \(\{2, 3\}\), so without loss of generality say \(k\) and \(i\) are linked. Then it follows that \(B_{j_1,j_2-1}(u, w) = B_{0,1}(i, k) = 1\) and \(B_{j_1,j_2-1}(v, w) = B_{0,1}(j, k) = 0\). So we must have \(u \neq v\), which implies \(p\) is injective on \(C_i \cup C_j\).

We consider several cases:

**Case 1:** Suppose only one \(c_i\) is nonzero. Note \(p(C_i)\) is unlinked for any \(i\). Hence \(c_i \leq 3^{j_2}2^{j_2-1} < 3^{j_2}2^{j_2}\) by the inductive hypothesis.

**Case 2:** Suppose exactly two \(c_i\) and \(c_j\) are non-zero. Suppose \(i\) and \(j\) are linked. If \(c_i = 3^{j_1}2^{j_2-1}\) then by the induction hypothesis \(p(C_i)\) is a maximal unlinked set in \(G_{j_1,j_2-1}\) of the form \(V \times W\) with \(V\) a type. This implies that for every \(u \in G_{j_1,j_2-1}-p(C_i)\) the set \(p(C_i)\) contains both an element which is linked with \(u\) and an element which is unlinked with \(u\) (see the construction in the previous proof). But every element in \(p(C_j)\) is linked with every element in \(p(C_i)\) by \(B_{j_1,j_2-1}(p(u), p(v)) = B_{0,1}(i, j) = 1\), which is a contradiction. Hence \(c_i < 3^{j_1}2^{j_2-1}\). Thus \(|U| < 3^{j_1}2^{j_2}\).

Now suppose \(i\) and \(j\) are unlinked. Then \(|U| = 3^{j_1}2^{j_2}\) if and only if \(p(C_i)\) and \(p(C_j)\) are maximal unlinked sets. But these must also be unlinked with each other and hence \(p(C_i) = p(C_j)\) by maximality. We have then that \(p(U) = p(C_i) = V \times W\) for \(V\) a type and \(W\) a maximal unlinked set. Thus \(U = V \times (W \times \{i, j\})\).

**Case 3:** Suppose at least three of the \(c_i\) are non-zero. Then for any unlinked pair of indices \(\{i, j\}\), there exists \(k\) such that \(c_k \neq 0\) and at least one of \(\{i, k\}, \{j, k\}\) are unlinked. So it follows that \(p\) is injective on \(C_i \cup C_j\) so that \(c_i + c_j \leq 3^{j_1}2^{j_2-1}\) with
equality if and only if $p(C_i \cup C_j) = V \times W$ is an unlinked set of maximal size in $G_{j_1,j_2-1}$ by the inductive hypothesis. Then $|U| = (c_0 + c_2) + (c_1 + c_3) \leq 3^{j_2}2^{j_2}$ with equality if and only if $p(C_0 \cup C_2)$ and $p(C_1 \cup C_3)$ are unlinked of maximal size in $G_{j_1,j_2-1}$. But we also have that $|U| = (c_0 + c_3) + (c_1 + c_2) \leq 3^{j_2}2^{j_2}$ with equality if and only if $p(C_0 \cup C_3)$ and $p(C_1 \cup C_2)$ are unlinked of maximal size in $G_{j_1,j_2-1}$. Suppose we have equality, and by the inductive hypothesis suppose $p(C_0 \cup C_3) = V_{0,3} \times W_{0,3}$ and $p(C_0 \cup C_2) = V_{0,2} \cup W_{0,2}$. Then $p(C_0) \subset p(C_0 \cup C_1) \cap p(C_0 \cup C_2)$, so it follows that $V_{0,3}$ and $W_{0,2}$ must have the same type, as their intersection is non trivial and the types are mutually disjoint. In particular $V_{0,3} = V_{0,2}$. By symmetry, we find that $V_{0,3} = V_{1,3} = V_{1,2} = V_{0,2}$. In particular, since $p(U) = V_{0,3} \times W_{0,3} \cup V_{1,2} \times W_{1,2}$ we find that $p(U) = V \times W$ for $V = V_{0,3}$ a type and $W$ some unlinked set. But $|W| = 2^{j_2}$, and so it is a maximal unlinked set in $Y_{j_2}^2$.

Combining this with the results of Section 3 we have shown:

Proposition 19. For $j_1 < k$ we have

$$S_{j_1,j_2,j_3}(X) = O(X(\log X)^{2j_3a k^{3j_3-j_3-2j_3a k-1+\epsilon}}).$$

5. Computing the Moments

It now remains to consider the contribution to $S_k(X)$ from $\sum_{A} S_{k,0,0}(X,A)$. Fix $A$ as above. Then by (5.1) there will be exactly $3^k$ unlinked variables $A_u$ greater than $X^4$ and all the remaining ones will satisfy $A_u = 1$. This combined with quadratic reciprocity reduces $S_{k,0,0}(X,A)$ to the following expression, which we are further partitioning by the congruence classes of each $D_u$.

$$(5.1) \quad S_{k,0,0}(X,A) = \sum_{(h_u) \in \frac{1}{24k} \sum_{(D_u)} \mu^2(d) a^{k \omega(d)} \left[ \prod_{u} (-1)^{\lambda_k(u)(u) \frac{h_u-1}{2}} \right] \times \left[ \prod_{u,v} (-1)^{\varphi_k(u,v) \frac{h_u-1}{2}} \right] + O(X(\log X)^{(3a)k-a k-1+\epsilon})$$

where $h_u$ denotes the congruence class of $D_u \mod 4$. Next we would like to remove the congruence condition on the inner sum over the $(D_u)$.

We will use the following result from [5], again referring there for the proof.

Lemma 20 (Fouvry-Kl"uners [5]). For any fixed tuple $(h_u)$ with $h_u \equiv \pm 1 \mod 4$ we have

$$\sum_{(D_u \equiv h_u \mod 4)} \mu^2(d) a^{k \omega(d)} = \frac{1}{24k} \sum_{D_u} \mu^2(2d) a^{k \omega(d)} + O(X(\log X)^{-16k(1+2^k)}).$$

Fix a maximal unlinked set of indices $\mathcal{U}$. We will call any $A$ satisfying (3.1) admissible for $\mathcal{U}$ if $A_u > X^4$ exactly when $u \in \mathcal{U}$. Applying Lemma 20 to (5.1) and rearranging
summations and also summing over all $A$ admissible for $U$ we get

$$\sum_{A \text{ admissible for } U} S_{k,0,0} (X, A) = \frac{1}{2^{4k+3^k}} \left( \sum_{(D_u)} \mu^2 (2d) a^{k\omega(d)} \right) \left( \sum_{(h_u)} \prod_{u} (-1)^{\lambda_k(u) \frac{h_u-1}{2}} \right) \times \left( \prod_{u,v} (-1)^{\Phi_k(u,v) \frac{h_u-1}{2} \frac{h_u-1}{2}} \right) + O \left( X \left( \log X \right)^{(3a)^k-a^k-1+\epsilon} \right).$$

Note the summation is over $(D_u)$ such that there is some $A$ admissible for $U$ with $A_u \leq D_u \leq \Delta A_u$. However we can include the missing terms to extend the range to $1 \leq D_u < X$ at the cost an error of $O \left( X \left( \log X \right)^{(3a)^k-a^k-1+\epsilon} \right)$ by Lemma 14.

Summing over all maximal unlinked sets $U$ we get

$$S_{k,0,0} (X) = \frac{1}{2^{4k+3^k}} \left( \sum_{U} \gamma (U) \right) \left( \sum_{n<X} \mu^2 (2n) (3a)^{k\omega(n)} \right) + O \left( X \left( \log X \right)^{(3a)^k-a^k-1+\epsilon} \right)$$

where we define

$$\gamma (U) = \sum_{(h_u)} \left[ \prod_{u} (-1)^{\lambda_k(u) \frac{h_u-1}{2}} \right] \left[ \prod_{u,v} (-1)^{\Phi_k(u,v) \frac{h_u-1}{2} \frac{h_u-1}{2}} \right].$$

Recall (see (2.3) and (2.4)) that we are allowing all possible congruence classes $(h_u)$ satisfying the conditions: for all $1 \leq i \leq k$ and $(u_i, v_i) \in \{(0,3), (2,5), (4,1)\}$

$$(5.2) \prod_{u} h_u \prod_{v} h_v \equiv \begin{cases} -1 \mod 4 & \text{if } (u_i, v_i) = (0,3) \\ 1 \mod 4 & \text{if } (u_i, v_i) = (2,5), (4,1) \end{cases}$$

where the above products are over all $u$ with $u_i$ in the $i$th position and all $v$ with $v_i$ in the $i$th position.

So far we have shown

$$S_{k,0,0} (X) = \frac{1}{2^{4k+3^k}} \left( \sum_{U} \gamma (U) \right) \left( \sum_{n<X} \mu^2 (2n) (3a)^{k\omega(n)} \right) + O \left( X \left( \log X \right)^{(3a)^k-a^k-1+\epsilon} \right)$$

We will now prove

**Proposition 21.**

$$\sum_{U} \gamma (U) = 2^{3^k-k-1}.$$
$M_k = \begin{pmatrix}
\vec{1} & \vec{0} & \vec{0} \\
\vec{0} & \vec{1} & \vec{0} \\
\vec{0} & \vec{0} & \vec{1} \\
M_{k-1} & M_{k-1} & M_{k-1}
\end{pmatrix}$

where $M_1 = I_3$ is the identity matrix and $\vec{0}, \vec{1}$ are row vectors of 0s and 1s respectively.

For example for $k = 2$ we get

$$M_2 = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1
\end{pmatrix}.$$ 

Then the $y \in \mathbb{F}_2^k$ satisfying condition (5.2) are solutions of $M_k y = w$ for an appropriate $w \in \mathbb{F}_2^k$. This set of solutions is the coset $y + \ker M_k$.

Now we will prove (21) by combining the following two lemmas.

**Lemma 22.** For all $k \geq 1$

$$\sum_{U} \gamma(U) = 2^{k + \dim \ker M_k}.$$ 

**Proof.** Recall

$$\gamma(U) = \sum_{(h_u)} \left[ \prod_u (-1)^{\lambda_k(u) \frac{h_u-1}{2}} \right] \left[ \prod_{\{u,v\}} (-1)^{\Phi_k(u,v) \frac{h_u-1}{2}} \right]$$

By the discussion above, for any $(h_u)$ we let $x \in \mathbb{F}_2^k$ be the vector corresponding to it by (5.3). Then $x$ belongs to a coset of $\ker M_k$, call it $y + \ker M_k$. In particular, rephrasing the congruence conditions (5.2) shows us that $\sum_{u, u_i = a} x_u = 1$ if and only if $a \in \{0, 3\}$.

Now consider that $\Phi(u, v) = 0$ if $v \in A = \{1, 3, 5\}$. $U$ is a largest maximal unlinked set, and so has a type $s \in S$, so it follows that

$$\Phi_k(u, v) = \sum_i \Phi(u_i, v_i) = \sum_{i: s_i = B} \Phi(u_i, v_i)$$

And similarly, $\lambda|_A = 0$ so that

$$\lambda(u) = \sum_i \lambda(u_i) = \sum_{i: s_i = B} \lambda(u_i)$$
This way we show that
\[
\sum_{\mathcal{U}} \gamma(\mathcal{U}) = \sum_{s \in S} \sum_{x \in \ker M_k} \left[ \prod_u (-1)^{\sum_{i:s_i = B} \lambda(u_i)x_u} \right] \left[ \prod_{\{u,v\}} (-1)^{\sum_{i:s_i = B} \lambda(u_i)x_u + \sum_{\{u,v\}} \sum_{i:s_i = B} \Phi(u_i,v_i)x_u x_v} \right]
\]
\[
= \sum_{s \in S} \sum_{x \in \ker M_k} (-1)^{\sum_{i:s_i = B} \lambda(u_i)x_u + \sum_{\{u,v\}} \sum_{i:s_i = B} \Phi(u_i,v_i)x_u x_v}.
\]

Call $\mathcal{U}_B = B^k$. Interchanging summations we can apply the binomial theorem to the sum over types $s \in S$ to show
\[
\sum_{\mathcal{U}} \gamma(\mathcal{U}) = \sum_{x \in \ker M_k} \prod_{j=1}^k \left( 1 - (-1)^{\sum_{u \in \mathcal{U}_B} \lambda(u)x_u + \sum_{\{u,v\} \subset \mathcal{U}_B} \Phi(u,v)x_u x_v} \right).
\]
Notice that for all $j = 1, ..., k$ we have
\[
\sum_{\{u,v\} \subset \mathcal{U}_B} \Phi(u_j, v_j)x_u x_v = \sum_{u_j = 0, v_j = 2} x_u x_v + \sum_{u_j = 0, v_j = 4} x_u x_v + \sum_{u_j = 2, v_j = 4} x_u x_v
\]
\[
= \sum_{u_j = 0} \sum_{v_j = 2} x_u x_v + \sum_{u_j = 0} \sum_{v_j = 4} x_u x_v + \sum_{u_j = 2} \sum_{v_j = 4} x_u x_v
\]
and
\[
\sum_{u \in \mathcal{U}_B} \lambda(u_j)x_u = \sum_{u_j = 2} \sum_{u = 1} x_u + \sum_{u_j = 4} \sum_{u = 1} x_u
\]
and it follows that these are 0 from the conditions (5.2). Thus
\[
\sum_{\mathcal{U}} \gamma(\mathcal{U}) = \sum_{x \in \ker M_k} \prod_{j=1}^k 2 = 2^{k + \dim \ker M_k}.
\]

**Lemma 23.** $\dim \ker M_k = 3^k - 2k - 1$.

**Proof.** Without loss of generality suppose $U$ has type $s$ with $s_1 = \{0, 2, 4\}$. For $x \in \mathbb{F}_2^{3^k}$ and $j \in s_1$ let $p_j(x)$ be the projection onto $\mathbb{F}_2^{3^k-1}$ of the coordinates $x_u$ of $x$ where $u_1 = j$.

Now $x \in \ker M_k$ if and only if $\sum_{j \in s_1} p_j(x) \in \ker M_{k-1}$ and $\alpha_{k-1}(p(x_j)) = 0$ for all $j \in s_1$, with $\alpha_{k-1} : \mathbb{F}_2^{3^k-1} \rightarrow \mathbb{F}_2$ the augmentation map defined by $v \mapsto 1 \cdot v$.

It is clear that $\ker M_{k-1} \subset \ker \alpha_{k-1}$. So we have that
\[
\ker M_k = \left\{ x \mid \sum_{j \in s_1} p_j(x) \in \ker M_{k-1}, p(x_j) \in \ker \alpha_{k-1} \right\}.
\]

There are $|\ker \alpha_{3^k-1}|$ choices for $p_0(x)$ and $p_2(x)$. Then we have
\[
p_4(x) \in (p_0(x) + p_2(x) + \ker M_{k-1}) \cap \ker \alpha_{k-1}.
\]
That is $p_4(x)$ belongs to a coset of $\ker M_{k-1} \subset \ker \alpha_{k-1}$, so there are $|\ker M_{k-1}|$ choices. So we have

$$|\ker M_k| = |\ker \alpha_{k-1}|^2 |\ker M_{k-1}| = 2^{2(3^{k-1} - 1)} |\ker M_{k-1}|.$$ 

Clearly $M_1$ is the identity map, and so $|\ker M_k| = 0$. Then a simple induction shows that $|\ker M_k| = 2^{3^k - 2^{k-1}}$ which completes the proof. □

Combining Lemma 22 and Lemma 23 immediately proves Proposition 21. 

In summary we have shown that

$$S_k(X) = \frac{1}{2^{5k+1}} \left( \sum_{n < X} \mu^2(2n)(3a)^{k\omega(n)} \right) + O \left( X (\log X)^{3a^k - a^{k-1} + \epsilon} \right)$$

and the first case of Theorem 6 follows by noting that

$$\sum_{n < X} \mu^2(2n)(3a)^{k\omega(n)} = 2 \sum_{d \in \mathcal{D}_{X,1,4}} (3a)^{k\omega(d)} + o(X).$$

6. Main results and implications

We now have the tools to prove Theorems 24 and 25 as corollaries to Theorem 6. We restate the theorems here followed by their proofs:

**Theorem 24.** Let $(G', G) = (H_8 \rtimes C_2, H_8)$. For a quadratic field $K$ with discriminant $d$ let $f(d)$ be the number of unramified everywhere $(G', G)$-extensions of $K$. Let $g(d) = 3^{\omega(d)}$. Then for all $k \in \mathbb{Z}_{\geq 1}$,

$$E^-( (f(d)/g(d))^k ) = \left( \frac{1}{32} \right)^k$$

and

$$E^+ ( (f(d)/g(d))^k ) = \left( \frac{1}{192} \right)^k$$

Thus the function $f(d)/g(d)$ determines the point mass distribution at $1/32$ (resp. $1/192$) on $\mathbb{R}$.

**Proof.** This follows immediately from setting $a = 1/3$ in Theorem 6 (and multiplying by $3^{-k}$ for the even cases). □

Using this theorem we can show the sequence $f(d)/g(d)$ determines a distribution in the following sense. Clearly $\mu_n(U) = \frac{1}{|\mathcal{D}|} \{ f(d)/g(d) \in U \mid d < n \}$ is a probability measure on $\mathbb{R}$ for all $n$. Let $c = 1/32$. By Theorem 24 $\lim_{n \to \infty} \mathbb{E}_{\mu_n} (x^k) = c^k$ for all $k$. Let $\mu_c$ be the point-mass at $c$. The measure $\mu_c$ has moments $c^k$ and is determined by its moments. Then it is a standard fact (see for instance [3]) that the $\mu_n$ converge to $\mu_c$ in distribution, as $n \to \infty$.

Recall that $H_k \rtimes \sigma C_2$ denotes the group where the action of $\sigma$ on each coordinate gives $H_8 \rtimes C_2$ according to our definition.
Theorem 25. Let \( k \in \mathbb{Z}_{\geq 1} \), \( G = H^k_8 \), and \( G' = H^k_8 \times C_2 \). Define \( \text{Sur}_\sigma (\text{Gal} (K^{un}/K), G) \) be the set of surjections which lift to a surjection \( \text{Gal} (K^{un}/\mathbb{Q}) \to G' \). Then

\[
\sum_{d \in \mathcal{D}_X} |\text{Sur}_\sigma (\text{Gal} (K^{un}/K), G)| = \left( \frac{1}{4} \right)^k \left( \sum_{d \in \mathcal{D}_X^+} 3^{k\omega(d)} \right) + O \left( X (\log X)^{3k-2+\varepsilon} \right)
\]

and

\[
\sum_{d \in \mathcal{D}_X^+} |\text{Sur}_\sigma (\text{Gal} (K^{un}/K), G)| = \left( \frac{1}{24} \right)^k \left( \sum_{d \in \mathcal{D}_X^+} 3^{k\omega(d)} \right) + O \left( X (\log X)^{3k-2+\varepsilon} \right)
\]

Proof. As a notational convenience, let \( G^{un}_K = \text{Gal} (K^{un}/K) \) through the proof. Clearly taking \( a = 1 \) and noting that \( \#\text{Sur}_\sigma (G^{ur}_d, H_8) = 8f(d) \) we see that

\[
\sum_{|d| < X} |\text{Sur}_\sigma (G^{ur}_d, H_8)^k| = \left\{ \begin{array}{ll}
\left( \frac{1}{4} \right)^k \left( \sum_{\text{odd} \ |d| < X} 3^{k\omega(d)} \right) + O \left( X (\log X)^{3k-2+\varepsilon} \right) & \text{imaginary} \\
\left( \frac{1}{24} \right)^k \left( \sum_{\text{odd} \ |d| < X} 3^{k\omega(d)} \right) + O \left( X (\log X)^{3k-2+\varepsilon} \right) & \text{real}
\end{array} \right.
\]

Consider that we have

\[
|\text{Sur}_\sigma (G^{ur}_d, H_8)| = \sum_{A \leq H^k_8} \mu_{H^k_8} (A) |\text{Hom}_\sigma (G^{ur}_d, A)|
\]

as in \( \mathbb{S} \). A simple exercise in group theory shows that all subgroups of \( H^k_8 \) are isomorphic to \( H^j_8 \times C_4^{j_2} \times C_2^{j_3} \) for some \( j_1 + j_2 + j_3 \leq k \). So we have that

\[
|\text{Sur}_\sigma (G^{ur}_d, H_8)^k| = \sum_{j_1+j_2+j_3 \leq k} \mu_{H^k_8} (H^j_8 \times C_4^{j_2} \times C_2^{j_3}) \\
\quad \times |\text{Hom}_\sigma (G^{ur}_d, H_8)^{j_1} | \text{Hom}_\sigma (G^{ur}_d, C_4)^{j_2} | \text{Hom}_\sigma (G^{ur}_d, C_4)^{j_3} .
\]

A similar argument shows that the \( j_1 = k \) term is the main term, and then

\[
\sum_{|d| < X} |\text{Sur}_\sigma (G^{ur}_d, H_8)^k| = \sum_{|d| < X} |\text{Hom}_\sigma (G^{ur}_d, H_8)|^k + O \left( X (\log X)^{3k-2+\varepsilon} \right)
\]

\[
= \sum_{|d| < X} (|\text{Sur}_\sigma (G^{ur}_d, H_8)| - 3 |\text{Sur}_\sigma (G^{ur}_d, C_4)| + 2 |\text{Sur}_\sigma (G^{ur}_d, C_2)|)^k \\
\quad + O \left( X (\log X)^{3k-2+\varepsilon} \right)
\]

\[
= \sum_{|d| < X} \sum_{i_1+i_2+i_3} \binom{k}{i_1, i_2, i_3} |\text{Sur}_\sigma (G^{ur}_d, H_8)|^{i_1} \\
\quad \times 3^{i_2} |\text{Sur}_\sigma (G^{ur}_d, C_4)|^{i_2} 2^{i_3} |\text{Sur}_\sigma (G^{ur}_d, C_2)|^{i_3} \\
\quad + O \left( X (\log X)^{3k-2+\varepsilon} \right)
\]

Where again a similar argument shows the \( i_1 = k \) term is the main term, so that

\[
\sum_{|d| < X} |\text{Sur}_\sigma (G^{ur}_d, H_8)| = \sum_{|d| < X} |\text{Sur}_\sigma (G^{ur}_d, H_8)|^k + O \left( X (\log X)^{3k-2+\varepsilon} \right).
\]
This additionally answers a conjecture of Wood [12] for nonabelian Cohen-Lenstra heuristics in the case of \( G = H_k^8 \) and \([G' : G] = 2\), which says we expect the sum in this corollary to be \( \gg X \).

**Corollary 26.** The density of quadratic fields \( K \) with \( \text{Gal}(K^{un}/K) = H^m_8 \) is equal to 0 for any positive \( m \in \mathbb{Z} \).

**Proof.** Repeating the proof of Theorem 25 with \( a = 1/3 \) gives

\[
\sum_{|d| < X} \frac{|\text{Sur}_\sigma(G^ur_d, H^m_8)|}{g(d)^m} = \sum_{|d| < X} \left( \frac{|\text{Sur}_\sigma(G^ur_d, H^m_8)|}{g(d)} \right)^m + o(X),
\]

for any \( m \), from which it is clear that the \( k \)th moments of the function \( |\text{Sur}_\sigma(G^ur_d, H^m_8)| / g(d)^m \) will be \( m \)th powers of the \( k \)th moments of \( |\text{Sur}_\sigma(G^ur_d, H^m_8)| / g(d) \), which are given by Theorem 24. Thus the distribution of the values of \( |\text{Sur}_\sigma(G^ur_d, H^m_8)| / g(d)^m \) will again by a point mass supported at some positive real number \( c \). By definition this means that for any fixed \( m \in \mathbb{Z} \) and \( \epsilon > 0 \), one hundred percent of quadratic fields satisfy \( |\text{Sur}_\sigma(G^ur_d, H^m_8)| / g(d)^m - c \ < \epsilon \). In particular \( |\text{Sur}_\sigma(G^ur_d, H^m_8)| \) is non-zero one hundred percent of the time, which means there is at least one \( H^m_8 \) extension. But this holds for any \( m \). The corollary follows. \( \square \)

7. **Future work and conjectures**

As mentioned in the introduction [11] contains conditions for the existence of unramified \((G', G)\)-extensions in several other cases. In the case of \((G', G) = (D_4 \times C_2, D_4)\) a proof similar to Proposition 10 gives the formula

\[
f_{(D_4 \times C_2, D_4)}(d) = \frac{1}{2} \sum_{d = d_1d_2d_3} \prod_{p|d_1} 2^{\omega(d_1)-1} \prod_{p|d_2} 2^{\omega(d_2)} \prod_{p|d_3} \left( 1 + \left( \frac{d_2}{p} \right) \right) \prod_{p|d_1} \left( 1 + \left( \frac{d_1}{p} \right) \right).
\]

Our choice of normalizing constant in Conjecture [7] is based on the following simple heuristic. Consider the case \((D_4 \times C_2, D_4)\). If we assume each residue symbol takes values \( \pm 1 \) with probability \( 1/2 \) then the expected value of the product is

\[
\left( \frac{1}{2} \right)^{\omega(d_1d_2)} \cdot 2^{\omega(d_1) - 1} \cdot 1 = 1.
\]

Hence on average one expects \( f \) to be on the order of

\[
\sum_{d = d_1d_2d_3} 2^{\omega(d_1)} = \sum_{i=0}^{\omega(d)} 2^{\omega(i)} 2^{\omega(d) - i} \binom{\omega(d)}{i} = 4^{\omega(d)}.
\]

The heuristic applies similarly to all such expressions.
8. Appendix

We maintain all the notation from the previous sections, notably the matrix \( M_k \).

8.1. The case \( d < 0 \) and \( d \equiv 4 \mod 8 \). Next we consider fundamental discriminants \( d < 0 \) and \( d \equiv 4 \mod 8 \). The number of \( H_8 \) extensions of a quadratic field \( k \) Galois over \( \mathbb{Q} \) with such a discriminant is

\[
f (d) = f_1 (d) - f_2 (d) + 2^{\omega (d) - 4}
\]

where we define

\[
f_1 (d) = \frac{1}{2} \sum_{d = d_1 d_2 d_3} \left( 1 + \left( \frac{d_2 d_3}{2} \right) \right) \prod_{p \mid d_1} 2^{\omega (d_1) - 1} \prod_{p \mid d_2} \left( 1 + \left( \frac{-d_1 d_2}{p} \right) \right)
\]

\[
\times \prod_{p \mid d_1} \left( 1 + \left( \frac{d_2 d_3}{p} \right) \right) \prod_{p \mid d_2} \left( 1 + \left( \frac{-d_3 d_1}{p} \right) \right)
\]

\[
+ \sum_{d = d_1 d_2 d_3} \left( 1 + \left( \frac{-d_2 d_3}{2} \right) \right) \prod_{p \mid d_1} 2^{\omega (d_1) - 1} \prod_{p \mid d_2} \left( 1 + \left( \frac{-d_1 d_2}{p} \right) \right) \prod_{p \mid d_3} \left( 1 + \left( \frac{-d_2 d_3}{p} \right) \right)
\]

\[
\times \prod_{p \mid d_1} \left( 1 + \left( \frac{-d_2 d_3}{p} \right) \right) \prod_{p \mid d_2} \left( 1 + \left( \frac{d_3 d_1}{p} \right) \right)
\]

and

\[
f_2 (d) = \sum_{d = d_1 d_2} \prod_{i} 2^{\omega (d_i) - 1} \left( 1 + \left( \frac{d_i}{2} \right) \right) \prod_{p \mid d_1} \left( 1 + \left( \frac{d_2}{p} \right) \right) \prod_{p \mid d_2} \left( 1 + \left( \frac{-d_1}{p} \right) \right)
\]

\[
+ \sum_{d = d_1 d_2} \prod_{i} 2^{\omega (d_i) - 1} \left( 1 + \left( \frac{-d_2}{2} \right) \right) \prod_{p \mid d_1} \left( 1 + \left( \frac{-d_2}{p} \right) \right) \prod_{p \mid d_2} \left( 1 + \left( \frac{-d_1}{p} \right) \right)
\]

where the factorization \( d = d_1 d_2 d_3 \) is into integers such that \( d_1 \equiv -1 \mod 4 \) and \( d_i \equiv 1 \mod 4 \), and each sum corresponds to \( d_1 < 0 \) and \( d_i < 0 \) for \( i \neq 1 \). Otherwise this follows from the same reasoning as in the previous subsections. As before this gives

\[
f_1 (d) = \frac{1}{2^3} \sum_{m = 0}^{1} \sum_{d = -4}^{d} \left( 1 + \left( \frac{D_1 D_2 D_3 D_4 D_5}{2} \right) \right) \left( \frac{1}{2} \left( \frac{-1}{D_2 D_4} \right) m + \left( \frac{-1}{D_0 D_4} \right) (1 - m) \right)
\]

\[
\times \left( \frac{D_0 D_3 D_2 D_5}{D_1} \right) \left( \frac{D_2 D_5 D_3 D_1}{D_0} \right) \left( \frac{D_4 D_1 D_0 D_3}{D_2} \right)
\]

and

\[
f_2 (d) = \frac{1}{2^3} \sum_{m = 0}^{1} \sum_{d = -4}^{d} \left( 1 + \left( \frac{E_2 E_3}{2} \right) \right) \left( \frac{-1}{E_2} \right) m + \left( \frac{-1}{E_0} \right) (1 - m) \left( \frac{E_2 E_3}{E_0} \right) \left( \frac{E_0 E_1}{E_2} \right)
\].
where the sum is over factorizations which satisfy the congruences

\[
\begin{align*}
D_0 D_3 & \equiv (-1)^{m+1} \mod 4 \\
D_2 D_5 & \equiv (-1)^{m+1} \mod 4 \\
D_4 D_1 & \equiv 1 \mod 4 \\
E_0 E_1 & \equiv (-1)^{m+1} \mod 4 \\
E_2 E_3 & \equiv (-1)^{m+1} \mod 4
\end{align*}
\]

As before we compute

\[
f_{j_1, j_2}(d) = \frac{1}{2^{3j_1+3j_2}} \sum_{(D_u) \subset \{1, \ldots, j_1+j_2\}} \prod_{u} \left( \frac{D_u}{2} \right)^{\sum_{i \in C} Q(u_i)} \prod_{u,v} \frac{1}{D_u} \\
\times \left( -1 \right)^{j_1+j_2} \lambda^1(u_i) v \chi_j (i) + \lambda^2(u_i)(1-v \chi_j (u_i)) \gamma^1(u_i) v \chi_j (i) + \gamma^2(u_i)(1-v \chi_j (i)) \\
\times \prod_{u,v} \left( \frac{D_u}{D_v} \right) \Phi_{j_1}(u,v) + \Psi_{j_2}(u,v)
\]

For \(Q(u) = 0\) if \(u \in \{0, 3\}\) and 1 otherwise, \(\lambda^1(u) = 1\) iff \(u = 2, 4\), \(\lambda^2(u) = 1\) iff \(u = 0, 4\), \(\gamma^1(u) = 1\) iff \(u = 2\), and \(\gamma^2(u) = 1\) iff \(u = 0\). The sum is over \(6^{j_1} 4^{j_2}\) tuples of integers \((D_u)\) which satisfy \(\prod_{u} D_u = d\) and the congruence conditions: for all \(1 \leq i \leq j_1\) and \((u_i, v_i) \in \{(0, 3), (2, 5), (4, 1)\}\) and all \(j_1 + 1 \leq i \leq j_1 + j_2\) and \((u_i, v_i) \in \{(0, 1), (2, 3)\}\)

\[
(8.1) \prod_{u} D_u \prod_{v} D_v \equiv \begin{cases} 
(-1)^{|j_1|+1} \mod 4 & (u_i, v_i) = (0, 3), (2, 5), i \leq j_1 \\
(-1)^{|j_2|+1} \mod 4 & i \geq j_2 \\
1 \mod 4 & \text{else}
\end{cases}
\]

where the above products are over all \(u\) with \(u_i\) in the \(i\)th position and all \(v\) with \(v_i\) in the \(i\)th position.

Thus multiplying by \(2^{k_1 \omega(d)} d^{k_2 \omega(d)}\) and summing over discriminants \(d < X\) with \(d \equiv 1 \mod 4\) we get

\[
\sum_{d < X} 2^{k_1 \omega(d)} d^{k_2 \omega(d)} f_{j_1, j_2}(d) = \frac{1}{3^{j_1} 2^{3j_1+3j_2}} \sum_{(D_u) \subset \{1, \ldots, j_1+j_2\}} \mu^2 \left( \prod_{u} D_u \right) d^{k_2 \omega(d)} 2^{k_1 \omega(d)} \\
\times \sum_{C \subset \{1, \ldots, j_1+j_2\}} \sum_{J_1 \subset \{1, \ldots, j_1\}} \sum_{J_2 \subset \{j_1+1, \ldots, j_1+j_2\}} \prod_{u} \left( \frac{D_u}{2} \right)^{\sum_{i \in C} Q(u_i)} \\
\times \frac{1}{2^{|j_2|}} \left( -1 \right)^{\sum_{i=1}^{|j_1+j_2|} \lambda^1(u_i) v \chi_j (i) + \lambda^2(u_i)(1-v \chi_j (u_i)) \gamma^1(u_i) v \chi_j (i) + \gamma^2(u_i)(1-v \chi_j (i))} \\
\times \prod_{u,v} \left( \frac{D_u}{D_v} \right) \Phi_{j_1}(u,v) + \Psi_{j_2}(u,v)
\]
where the sum is over $6^{j_1} 4^{j_2}$ tuples of integers $(D_u)$ which satisfy $\prod_u D_u < X$ and the conditions (??).

In this case, we have

$$
\sum_{\mathbf{A} \text{ admissible for } \mathcal{U}} S_{k,0,0}(X, \mathbf{A}) = \frac{1}{2^{3k}} \left( \sum_{(D_u)} \mu^2(d/2) a^{k_0(d/4)} \right) 
\times \sum_{(h_u)} \left[ \prod_{C \subset \{1, \ldots, k\}} (-1) \sum_i Q(u_i) \chi_C(i) \frac{J_{u_i}^2 - 1}{8} \right] 
\times \sum_{J \subset \{1, \ldots, k\}} \frac{1}{2^{2j+1}} \prod_u (-1)^{\frac{h_u-1}{2}} \sum_{i=1}^k \lambda_1(u_i) \chi_J(i) + \lambda_2(u_i)(1-\chi_J(u_i)) 
\times \prod_{u,v} (-1)^{\Phi(u,v)\frac{h_u-1}{2}} \prod_{u} (-1)^{\Phi(u,v)\frac{h_u-1}{2}} + O \left( X (\log X)^{(3a)^k - a^k - 1 + \epsilon} \right).
$$

For $h_u$ the congruence class of $D_u$ mod 8 and $d = 4 \prod D_u$ and we grouped the 4 factor with the first discriminant in the factorization (i.e. for $k = 1$ the factorization is $d = (4D_0D_3)(D_1D_4)(D_2D_3)$).

Then, noting that $h_u$ is one out of four choices for odd numbers mod 8, we get

$$
S_{k,0,0}(X) = \frac{1}{2^{3k+2} 3^k} \left( \sum_{\mathcal{U}} \gamma(\mathcal{U}) \right) \left( \sum_{4n<X} \mu^2(2n)(3a)^{k_0(n)} \right) + O \left( X (\log X)^{(3a)^k - a^k - 1 + \epsilon} \right)
$$

where we define

$$
\gamma(\mathcal{U}) = \sum_{(h_u)} \left[ \prod_{C \subset \{1, \ldots, k\}} (-1) \sum_i Q(u_i) \chi_C(i) \frac{J_{u_i}^2 - 1}{8} \right] \left[ \prod_{u,v} (-1)^{\Phi(u,v)\frac{h_u-1}{2}} \right] 
\times \sum_{J \subset \{1, \ldots, k\}} \frac{1}{2^{2j+1}} \prod_u (-1)^{\frac{h_u-1}{2}} \sum_{i=1}^k \lambda_1(u_i) \chi_J(i) + \lambda_2(u_i)(1-\chi_J(u_i))
$$

allowing odd congruence classes $h_u$ mod 8 satisfying the following conditions: for all $1 \leq i \leq k$

$$
(8.2) \quad \prod_u h_u \prod_v h_v \equiv \begin{cases} 1 & \text{mod 4} \\ (-1)^{|j|+1} & \text{mod 4} \end{cases} \quad (u_i, v_i) \in \{(1, 4)\} \quad (u_i, v_i) \in \{(0, 3), (2, 5)\}
$$

where the above products are taken over all $u$ with $u_i$ in the $i$th position and $v$ with $v_i$ in the $i$th position. (Recalling that the odd parts of even discriminants are $-1$ mod 4 if $8 \nmid d$). If we call $x \equiv \frac{h_u-1}{2}$ mod 2 an element of $F_2^{k_0}$, then $x \equiv y + \ker M_k$ a coset of $\ker M_k$.

Now consider that $\Phi(u, v) = 0$ if $u, v \in \{1, 3, 5\}$. $\mathcal{U}$ is a largest maximal unlinked set, and so has a type $s \in S$, so it follows that

$$
\Phi_k(u, v) = \sum_i \Phi(u_i, v_i) = \sum_{i: s_i = B} \Phi(u_i, v_i)
$$
This way we show that

\[ \sum_{U} \gamma(U) = \sum_{s \in S} \sum_{x \in y + \ker M_k} \left[ \left( \sum_{\substack{u \mod 2 C \subset \{1, \ldots, k\}}} \prod_{\{u, v\}} (-1)^{\sum_{i \in C} Q(u)} \left( \frac{h_u^{2-1}}{8} \right) \right) \times \left( \prod_{\{u, v\}} -\sum_{\{u, v\}} \frac{1}{2^{2^{|J|}}} \prod_{\{u, v\}} (-1)^{\sum_{i \in C} \lambda^2(u)} \right) \right] \]

Notice that for each \( x_u \), there are two choices of \( h_u \mod 8 \) such that \( h_u^{2-1} \equiv x_u \mod 2 \), because \( h_u \) and \( 5h_u \) give the same image. If we fix an \( (h(x)_u) \) satisfying this property without loss of generality also satisfying \( h_u^{2-1} \equiv 0 \mod 2 \), then we have

\[ \sum_{(h_u): h_u^{2-1} \equiv x_u \mod 2 C \subset \{1, \ldots, k\}} \prod_{\{u, v\}} (-1)^{\sum_{i \in C} Q(u)} \left( \frac{h_u^{2-1}}{8} \right) = \sum_{T \subset U \ C \subset \{1, \ldots, k\}} \prod_{\{u, v\}} (-1)^{\sum_{i \in C} Q(u)} \left( \frac{h_{tu}^{2-1}}{8} \right) \]

Now, \( 5^2-1 \equiv 1 \mod 2 \). So it follows that \( \frac{5^2 h_{tu}^{2-1}}{8} \equiv (\chi_T(u)) \mod 2 \) so we have

\[ \sum_{(h_u): h_u^{2-1} \equiv x_u \mod 2 C \subset \{1, \ldots, k\}} \prod_{\{u, v\}} (-1)^{\sum_{i \in C} Q(u)} \left( \frac{h_u^{2-1}}{8} \right) = \sum_{T \subset U \ C \subset \{1, \ldots, k\}} \prod_{\{u, v\}} (-1)^{\sum_{i \in C} Q(u)} \chi_T(u) \]

By using the binomial theorem in two different ways. Now, for \( s_i \) being \( A \) or \( B \), we can find a \( u \in U \) such that \( \sum_{i \in C} Q(u_i) \equiv 1 \mod 2 \) for any \( C \neq \emptyset \), by making \( u \in \{0, 3\} \) for all except one value of \( i \in C \), and \( u \in \{1, 2, 4, 5\} \) for exactly one such value. So we have the only nonzero term in this sum is for \( C = \emptyset \), which must be equal to \( 2^{h_{TB}} \) independent of the type.
So then we have

$$
\sum \gamma(U) = 2^{3k} \sum_{s \in S} \sum_{x \in y + \ker M_k} \left[ \prod_{(u,v)} (-1)^{\sum_{i=B} \Phi(u_i, v_i) x_u x_v} \right] \\
\times \left[ \sum_{J \subset \{1, \ldots, k\}} \frac{1}{2^{|J|}} \prod_u (-1)^{x_u \sum_{i=1}^k \lambda^1(u_i) \chi_J(i) + \lambda^2(u_i)(1-\chi_J(u_i))} \right] \\
= 2^{3k} \sum_{s \in S} \sum_{x \in y + \ker M_k} (-1)^{\sum_{i=B} \Phi(u_i, v_i) x_u x_v} \\
\times \left[ \sum_{J \subset \{1, \ldots, k\}} \frac{1}{2^{|J|}} (-1)^{x_u \sum_{i=1}^k \lambda^1(u_i) \chi_J(i) + \lambda^2(u_i)(1-\chi_J(u_i))} \right]
$$

Call $U_B = B^k$. Then we can write every $U = \{(s, u) : u \in U_B\}$ where clearly $\Phi((s, u), (s, v)) = \sum_{i:j_i=B} \Phi(u_i, v_i)$. The same holds for $\lambda^1, \lambda^2$ similarly. We can apply the binomial theorem to the sum over types $s \in S$ to show

$$
\sum \gamma(U) = 2^{3k} \sum_{x \in y + \ker M_k} \sum_{J \subset \{1, \ldots, k\}} \frac{1}{2^{|J|}} \\
\times k \left( 1 + (-1)^{\sum_{i,B} \Phi(u_i, v_i) x_u x_v + \sum_{u \in U_B} x_u \lambda^1(u_i) \chi_J(i) + x_u \lambda^2(u_i)(1-\chi_J(u_i))} \right)
$$

Notice we have

$$
\sum_{\{u,v\} \subset U_B} \Phi(u_1, v_1) x_u x_v = \sum_{u_1=0, v_1=2} x_u x_v + \sum_{u_1=0, v_1=1} x_u x_v + \sum_{u_1=2, v_1=0} x_u x_v + \sum_{u_1=2, v_1=1} x_u x_v \\
= \sum_{u_1=0} x_u \sum_{v_1=2} x_v + \sum_{u_1=0} x_u \sum_{v_1=1} x_v + \sum_{u_1=2} x_u \sum_{v_1=0} x_v + \sum_{u_1=2} x_u \sum_{v_1=1} x_v \\
= \sum_{u_1=0} x_u \lambda^1(u_1) \chi_J(1) + \sum_{u_1=0} x_u \lambda^2(u_1)(1-\chi_J(1)) = \begin{cases} \\
\sum_{u_1=2} x_u + \sum_{u_1=4} x_u & 1 \in J \\
\sum_{u_1=0} x_u + \sum_{u_1=4} x_u & 1 \notin J 
\end{cases}
$$

Notice that we have $\sum_{u_1=m} x_u \equiv |J| + 1 \mod 2$ if $m = 0, 2$ and $\equiv 0 \mod 2$ otherwise. It then follows that

$$
\sum_{\{u,v\} \subset U_B} \Phi(u_1, v_1) x_u x_v = (|J| + 1)^2 \equiv |J| + 1 \mod 2 \\
\sum_{u \in U_B} x_u \lambda^1(u_1) \chi_J(1) + x_u \lambda^2(u_1)(1-\chi_J(1)) = |J| + 1
$$
Noting that squaring integers mod 2 does nothing. By symmetry the same is true for all \( j = 1, \ldots, k \), so we now have

\[
\sum_{\mathcal{U}} \gamma(\mathcal{U}) = 2^{3^k} \sum_{x \in y + \ker M_k} \sum_{J \subset \{1, \ldots, k\}} \frac{1}{2^{|J|}} \prod_{j=1}^k (1 + (-1)^0)
\]

\[
= 2^{3^k} \sum_{x \in y + \ker M_k} 2^k \sum_{J \subset \{1, \ldots, k\}} \frac{1}{2^{|J|}}
\]

\[
= 2^{\dim \ker M_k + 3^k + k} \left( \frac{3}{2} \right)^k
\]

Recall that \( \dim M_k = 3^k - 2k - 1 \). So we have

\[
\sum_{\mathcal{U}} \gamma(\mathcal{U}) = 2^{3^k - 2k - 1} 3^k
\]

Thus implying

\[
S_{k,0,0}(X) = \frac{3^k}{2^{5k+1}} \left( \sum_{4n<X} \mu^2(2n)(3a)^{k\omega(n)} \right) + O \left( X(\log X)^{(3a)^k - a^k - 1 + \epsilon} \right)
\]

We then have that

\[
\sum_{4n<X} \mu^2(2n)(3a)^{k\omega(n)} = \frac{2}{(3a)^k} \sum_{d \in \mathcal{D}_{4,8,X}} (3a)^{k\omega(d)}
\]

implies the result.

8.2. The case \( d < 0 \) and \( d \equiv 0 \mod 8 \). Next we consider fundamental discriminants \( d < 0 \) and \( d \equiv 4 \mod 8 \). The number of \( H_8 \) extensions of a quadratic field \( k \) Galois over \( \mathbb{Q} \) with such a discriminant is

\[
f(d) = f_1(d) - f_2(d) + 2^{\omega(d) - 4}
\]

where we define

\[
f_1(d) = \frac{1}{2} \sum_{d = d_1d_2d_3} \left( 1 + \left( \frac{d_2d_3}{2} \right) \right) \prod_{i=1}^3 \frac{2^{\omega(d_i) - 1}}{2^{\omega(d)}} \prod_{p|d_i} \left( 1 + \left( \frac{-2d_1d_2}{p} \right) \right)
\]

\[
\times \prod_{p|d_1} \left( 1 + \left( \frac{d_2d_3}{p} \right) \right) \prod_{p|d_2} \left( 1 + \left( \frac{-2d_3d_1}{p} \right) \right)
\]

\[
+ \sum_{d = d_1d_2d_3} \left( 1 + \left( \frac{-d_2d_3}{2} \right) \right) \prod_{i=1}^3 \frac{2^{\omega(d_i) - 1}}{2^{\omega(d)}} \prod_{p|d_i} \left( 1 + \left( \frac{-2d_1d_2}{p} \right) \right)
\]

\[
\times \prod_{p|d_1} \left( 1 + \left( \frac{-d_2d_3}{p} \right) \right) \prod_{p|d_2} \left( 1 + \left( \frac{2d_3d_1}{p} \right) \right)
\]
and
\[
 f_2(d) = \sum_{d = d_1d_2} \prod_i \frac{2^{\omega(d_i) - 1}}{2^\omega(d)} \left( 1 + \frac{d_i}{2} \right) \prod_{p|d_i} \left( 1 + \frac{d_i}{p} \right) \prod_{p|d_2} \left( 1 + \frac{-2d_1}{p} \right) 
 + \sum_{d = d_1d_2} \prod_i \frac{2^{\omega(d_i) - 1}}{2^\omega(d)} \left( 1 + \frac{-d_2}{2} \right) \prod_{p|d_i} \left( 1 + \frac{-d_2}{p} \right) \prod_{p|d_2} \left( 1 + \frac{2d_1}{p} \right)
\]
where the factorization \( d = d_1d_2d_3 \) is into integers such that \( d_1 \equiv -1 \mod 4 \) and \( d_i \equiv 1 \mod 4 \), and each sum corresponds to \( d_1 < 0 \) and \( d_i < 0 \) for \( i \neq 1 \). Otherwise this follows from the same reasoning as in the previous subsections. As before this gives
\[
f_1(d) = \frac{1}{2^d} \sum_{m=0}^1 \sum_{d=-4E_0E_1E_2E_3} \left( 1 + \frac{E_2E_3}{2} \right) \left( 2 \right) 
\times \left( \frac{-1}{E_2} \right) m + \left( \frac{-1}{E_0} \right) (1 - m) \left( \frac{E_2E_3}{E_0} \right) \left( \frac{E_0E_1}{E_2} \right)
\]
where the sum is over factorizations which satisfy the congruences
\[
D_2D_5 \equiv (-1)^{m+1} \mod 4 \\
D_4D_1 \equiv 1 \mod 4
\]
and
\[
E_2E_3 \equiv (-1)^{m+1} \mod 4
\]
As before we compute
\[
f_{j_1,j_2}(d) = \frac{1}{2^{j_1+j_2}} \sum_{(D_u) \subset \{1, \ldots, j_1+j_2\}} \prod_u \left( \frac{D_u}{2} \right) \sum_{i \in C} Q(u_i) \sum_{J_1 \subset \{1, \ldots, j_1\}} \sum_{J_2 \subset \{j_1+1, \ldots, j_1+j_2\}} \frac{1}{2^{j_1}} 
\times \left( \frac{-1}{D_u} \right)^{\sum_{i=1}^{j_1} \lambda_i(u_i) \chi_j(i) + \lambda^2(u_i)(1-\chi_j(u)) + \gamma^1(u_i) \chi_j(i)^2 + \gamma^2(u_i)(1-\chi_j(i))} 
\times \left( \frac{2}{D_u} \right)^{\sum_{i=1}^{j_1} \lambda_i(u_i) + \gamma^1(u_i)} \prod_{u,v} \left( \frac{D_u}{D_v} \right)^{\Phi_{j_1}(u,v)+\Phi_{j_2}(u,v)}
\]
For \( Q(u) = 0 \) if \( u \in \{0, 3\} \) and 1 otherwise, \( \lambda^1(u) = 1 \) iff \( u = 2, 4 \), \( \lambda^2(u) = 1 \) iff \( u = 0, 4 \), \( \gamma^1(u) = 1 \) iff \( u = 2 \), and \( \gamma^2(u) = 1 \) iff \( u = 0 \). The sum is over \( 6^{j_1}4^{j_2} \) tuples of integers.
\((D_u)\) which satisfy \(\prod_u D_u = d\) and the congruence conditions: for all \(1 \leq i \leq j_1\) and \((u_i, v_i) \in \{(0, 3), (2, 5), (4, 1)\}\) and all \(j_1 + 1 \leq i \leq j_1 + j_2\) and \((u_i, v_i) \in \{(0, 1), (2, 3)\}\)

\((8.3)\) \[ \prod_u D_u \prod_v D_v \equiv \begin{cases} (-1)^{|J_1|+1} \mod 4 & (u_i, v_i) = (0, 3), (2, 5), i \leq j_1 \\ (-1)^{|J_2|+1} \mod 4 & (u_i, v_i) = (0, 1), i \geq j_2 \\ 1 \mod 4 & \text{else} \end{cases} \]

where the above products are over all \(u\) with \(u_i\) in the \(i\)th position and all \(v\) with \(v_i\) in the \(i\)th position.

Thus multiplying by \(2^{k_1\omega(d)}d^{k_2\omega(d)}\) and summing over discriminants \(d < 0\) with \(d \equiv 1 \mod 4\) we get

\[
\sum_{d < X} 2^{j_1\omega(d)}d^{\omega(d)} f_{j_1,j_2}(d) = \frac{1}{3^{j_1}2^{j_1+3j_2}} \sum_{(D_u)} \mu^2 \left( \prod_u D_u \right) d^{k_1\omega(d)}2^{k_2j_2\omega(d)} \times \sum \sum \prod \left( \frac{D_u}{2} \right)^{\sum_{i \in C} Q(u_i)} \times \left( \frac{2}{D_u} \right)^{\sum_{i=1}^{j_1+j_2} \lambda^3(u_i)+\gamma(u_i)} \prod_{u,v} \left( \frac{D_u}{D_v} \right)^{\Phi_{j_1}(u,v)+\Psi_{j_2}(u,v)}
\]

where the sum is over \(6^{j_1}4^{j_2}\) tuples of integers \((D_u)\) which satisfy \(\prod_u D_u < X\) and the conditions \((??)\).

In this case, we have

\[
\sum_{\mathbf{A} \text{ admissible for } \mathbf{U}} S_{k,0,0}(X, \mathbf{A}) = \frac{1}{2^{3k}} \left( \sum_{(D_u)} \mu^2 \left( \frac{d}{2} \right) d^{k_1\omega(d/4)} \right) \times \sum_{(h_u)} \left[ \sum_{C \subseteq \{1,\ldots,k\}} \prod_{u} (-1)^{\sum_i Q(u_i) \chi_C(i) \frac{h_u^3-1}{8}} \right] \times \sum_{J \subseteq \{1,\ldots,k\}} \frac{1}{2^{j_1}} \prod_{u} (-1)^{\frac{h_u^2-1}{2}} \sum_{i=1}^{h_u} \lambda^3(u_i) \chi_J(i) + \lambda^2(u_i) (1 - \chi_J(u_i)) \times \prod_{u} (-1)^{\sum_i \lambda^3(u_i) \frac{h_u^2-1}{8}} \times \left[ \prod_{u,v} (-1)^{\Phi_k(u,v) \frac{h_u^2-1}{2}} \right] + O \left( X (\log X)^{3k^1-a^k-1+\epsilon} \right). \]
For $h_u$ the congruence class of $D_u \mod 8$ and $d = 4 \prod D_u$ and we grouped the 4 factor with the first discriminant in the factorization (i.e. for $k = 1$ the factorization is $d = (4D_0 D_3)(D_1 D_4)(D_2 D_5)$).

Then, noting that $h_u$ is one out of four choices for odd numbers mod 8, we get

$$S_{k,0,0}(X) = \frac{1}{2^{4k}2 \cdot 2 \cdot 3^k} \left( \sum_{U} \gamma(U) \right) \left( \sum_{4 \nu < X} \mu^2(2n)(3a)^{k \omega(n)} \right) + O \left( X(\log X)^{(3a)^k - a^k - 1 + \epsilon} \right)$$

where we define

$$\gamma(U) = \sum_{(h_u)} \sum_{C \subseteq \{1, \ldots, k\}} \prod_{u} (-1)^{\sum_i Q(u_i) \chi_C(i) + \lambda^1(u_i) \frac{h_u^{2k} - 1}{8}} \prod_{u,v} (-1)^{\Phi_k(u,v) h_u v^{h_u - 1}}$$

allowing odd congruence classes $h_u \mod 8$ satisfying the following conditions: for all $1 \leq i \leq k$

$$(8.4) \prod_{u} h_u \prod_{v} h_v \equiv \begin{cases} 1 \mod 4 & (u_i, v_i) = (1, 4) \\ (-1)^{[J] + 1} \mod 4 & (u_i, v_i) = (2, 5) \\ (-1)^{[J] + 1} \prod_{u} h_u \prod_{v} h_v & (u_i, v_i) = (u_j', v_j') = (0, 3) & \text{for some } j \neq i \end{cases}$$

where the above products are taken over all $u$ with $u_i$ in the $i$th position and $v$ with $v_i$ in the $i$th position. (Recalling that the odd parts of even discriminants are $-1 \mod 4$ if $8 \nmid d$). If we call $x \equiv \left( \frac{h_u - 1}{2} \right) \mod 2$ an element of $\mathbb{F}_2$, then $x \in y + \ker M_k$ some coset of $\ker M_k$. There are two such cosets depending on the congruence class of $\prod_{u} h_u \prod_{v} h_v$ for $(u_i, v_i) = (0, 3)$.

Now consider that $\Phi(u, v) = 0$ if $u, v \in A = \{1, 3, 5\}$. $U$ is a largest maximal unlinked set, and so has a type $s \in S$, so it follows that

$$\Phi_k(u, v) = \sum_{i} \Phi(u_i, v_i) = \sum_{i : s_i = B} \Phi(u_i, v_i)$$

This way we show that

$$\sum_{U} \gamma(U) = \sum_{s \in S} \sum_{y \in y + \ker M_k} \sum_{(h_u, h_v) \equiv \left( \frac{h_u - 1}{2} \right) \mod 2} \sum_{C \subseteq \{1, \ldots, k\}} \prod_{u} (-1)^{\sum_{\nu \in C} Q(u_i) \left( \frac{h_u^{2k} - 1}{8} \right)}$$

$$\times \prod_{u} (-1)^{\sum_{u} \lambda^1(u_i) \frac{h_u^{2k} - 1}{8}} \prod_{(u,v)} (-1)^{\sum_{\nu \in B} \Phi(u_i, v_i) x_u x_v}$$

$$\times \sum_{J \subseteq \{1, \ldots, k\}} \frac{1}{2^{[J]}} \prod_{u} (-1)^{x_u \sum_{\nu \in J} \lambda^1(u_i) \chi_J(i) + \lambda^2(u_i) (1 - \chi_J(u_i))}$$
Notice that for each \( x_u \), there are two choices of \( h_u \mod 8 \) such that \( \frac{h_u - 1}{2} \equiv x_u \mod 2 \), because \( h_u \) and \( 5h_u \) give the same image. If we fix an \( (h(x)u) \) satisfying this property without loss of generality also satisfying \( \frac{h_u^2 - 1}{8} \equiv 0 \mod 2 \), then we have

\[
\sum_{(h_u)\frac{h_u - 1}{2} \equiv x_u \mod 2} \sum_{C \subseteq \{1,...,k\}} \prod_u (-1)^i (Q(u_i)C(i) + \lambda^1(u_i)) \left( \frac{h_u^2 - 1}{8} \right) = \sum_{T \subseteq U} \sum_{C \subseteq \{1,...,k\}} \prod_u (-1)^i (Q(u_i)C(i) + \lambda^1(u_i)) \left( \frac{h_u^2 - 1}{8} \right)
\]

\[
= \sum_{T \subseteq U} \sum_{C \subseteq \{1,...,k\}} \prod_u (-1)^i (Q(u_i)C(i) + \lambda^1(u_i)) \chi_T(u) = \sum_{T \subseteq U} \sum_{C \subseteq \{1,...,k\}} (-1)^i (Q(u_i)C(i) + \lambda^1(u_i)) \chi_T(u)
\]

\[
= \sum_{T \subseteq U} \prod_u \left( 1 + (-1)^i \sum_{u_i} Q(u_i)C(i) + \lambda^1(u_i) \right)
\]

By using the binomial theorem in two different ways.

We then have

\[
Q(u_i)C(i) + \lambda(u_i) = \begin{cases} 
0 & u_i \in \{0, 3, 2, 4\}, i \in C \\
1 & u_i \in \{1, 5\}, i \in C \\
0 & u_i \in \{0, 3, 1, 5\}, i \notin C \\
1 & u_i \in \{2, 4\}, i \notin C
\end{cases}
\]

Suppose that \( C \not\subset \{i : s_i = B\} \), then fix some \( j \in C \) such that \( s_j = A \). Choose \( u \in U \) such that

\[
u_i \in \begin{cases} 
\{1\} & i = j \\
\{0, 3\} & i \neq j
\end{cases}
\]

Then we get the summands are zero corresponding to these \( C \). Suppose next that \( \{i : s_i = B\} \not\subset C \), then fix \( j \notin C \) such that \( s_j = B \) and choose \( u \in U \) such that

\[
u_i \in \begin{cases} 
\{2\} & i = j \\
\{0, 3\} & i \neq j
\end{cases}
\]
Then these summands give us zero as well. The only summand remaining is \( C = \{ i : s_i = B \} \), which clearly gives \( 2^k \).

So then we have

\[
\sum_{\mathcal{U}} \gamma(\mathcal{U}) = 2^k \sum_{s \in S} \sum_{y} \sum_{x \in y + \ker M_k} \left[ \prod_{\{u,v\}} (-1)^{\sum_{i,j=0} s_i = B} \phi(u_i, v_i) x_u x_v \right]
\]

\[
\times \left[ \sum_{J \subset \{1, \ldots, k\}} \frac{1}{2^{\mid J \mid}} \prod_{u} (-1)^{x_u \sum_{i=1}^{k} \lambda^1(u_i) \chi_J(i) + \lambda^2(u_i)(1 - \chi_J(u_i))} \right]
\]

\[
= 2^k \sum_{s \in S} \sum_{y} \sum_{x \in y + \ker M_k} (-1)^{\sum_{i,j=0} s_i = B} \phi(u_i, v_i) x_u x_v
\]

\[
\times \left[ \sum_{J \subset \{1, \ldots, k\}} \frac{1}{2^{\mid J \mid}} (-1)^{\sum_{i=0}^k \sum_{i,j=0} x_u \sum_{u \in \mathcal{U}_B} \lambda^1(u_i) \chi_J(i) + \lambda^2(u_i)(1 - \chi_J(u_i))} \right]
\]

Call \( \mathcal{U}_B = B^k \). Then we can write every \( \mathcal{U} = \{(s, u) : u \in \mathcal{U}_B\} \) where clearly \( \phi((s, u), (s, v)) = \sum_{i,j=0}^k \phi(u_i, v_i) \). The same holds for \( \lambda^1, \lambda^2 \) similarly. We can apply the binomial theorem to the sum over types \( s \in S \) to show

\[
\sum_{\mathcal{U}} \gamma(\mathcal{U}) = 2^k \sum_{s \in S} \sum_{y} \sum_{x \in y + \ker M_k} \frac{1}{2^{\mid J \mid}} \sum_{j=1}^k \left( 1 - (-1)^{\sum_{i,j=0} x_u \sum_{i,j=0} x_u \phi(u_i, v_j) x_u x_v} \sum_{u \in \mathcal{U}_B} x_u \lambda^1(u_j) \chi_J(j) + x_u \lambda^2(u_j)(1 - \chi_J(j)) \right)
\]

Notice we have

\[
\sum_{\{u,v\} \subset \mathcal{U}_B} \phi(u_1, v_1) x_u x_v = \sum_{u_1=0, v_1=2} x_u x_v + \sum_{u_1=0, v_1=4} x_u x_v + \sum_{u_1=2, v_1=4} x_u x_v
\]

\[
= \sum_{u_1=0} x_u \sum_{v_1=2} x_v + \sum_{u_1=0} x_u \sum_{v_1=4} x_v + \sum_{u_1=2} x_u \sum_{v_1=4} x_v
\]

\[
\sum_{u \in \mathcal{U}_B} x_u \lambda^1(u_1) \chi_J(1) + x_u \lambda^2(u_1)(1 - \chi_J(1)) = \begin{cases} 
\sum_{u_1=2} x_u + \sum_{u_1=4} x_u & 1 \in J \\
\sum_{u_1=0} x_u + \sum_{u_1=4} x_u & 1 \notin J 
\end{cases}
\]

Notice that we have \( \sum_{u_1=m} x_u \equiv |J| + 1 \mod 2 \) if \( m = 0, 2 \) and \( \equiv 0 \mod 2 \) otherwise. It then follows that

\[
\sum_{\{u,v\} \subset \mathcal{U}_B} \phi(u_1, v_1) x_u x_v = (|J| + 1)^2 \equiv |J| + 1 \mod 2
\]

\[
\sum_{u \in \mathcal{U}_B} x_u \lambda^1(u_1) \chi_J(1) + x_u \lambda^2(u_1)(1 - \chi_J(1)) = |J| + 1
\]
Noting that squaring integers mod 2 does nothing. By symmetry the same is true for all \( j = 1, \ldots, k \), so we now have

\[
\sum_{\mathcal{U}} \gamma(\mathcal{U}) = 2^{3k} \sum_{y} \sum_{x \in y + \ker M_k} \sum_{J \subseteq \{1, \ldots, k\}} \frac{1}{2 |J|} \prod_{j=1}^{k} (1 + (-1)^0)
\]

\[
= 2^{3k} \sum_{y} \sum_{x \in y + \ker M_k} \sum_{J \subseteq \{1, \ldots, k\}} \frac{1}{2 |J|}
\]

\[
= 2^{\dim \ker M_k + 3^k + k + 1} \left( \frac{3}{2} \right)^k
\]

Noting that there are two choices for \( y \). Recall that \( \dim M_k = 3^k - 2k - 1 \). So we have

\[
\sum_{\mathcal{U}} \gamma(\mathcal{U}) = 2^{2 \cdot 3^k - 2k + k}
\]

Thus implying

\[
S_{k,0,0}(X) = \frac{3^k}{2^{5k}} \left( \sum_{8n < X} \mu^2(2n)(3a)^{\Omega(n)} \right) + O \left( X(\log X)^{(3a)^k - a^k - 1 + \epsilon} \right)
\]

We then have that

\[
\sum_{8n < X} \mu^2(2n)(3a)^{\Omega(n)} = \frac{1}{(3a)^k} \sum_{d \in D_{0,8,X}} (3a)^{\omega(d)}
\]

implies the result.

### 8.3. The case \( d > 0 \) and \( d \equiv 1 \mod 4 \)

Next we consider fundamental discriminants \( d > 0 \) and \( d \equiv 1 \mod 4 \). The number of \( H_8 \) extensions of a quadratic field \( k \) Galois over \( \mathbb{Q} \) with such a discriminant is

\[
f(d) = f_1(d) - f_2(d) + 2^{\omega(d) - 4}
\]

where we define

\[
f_1(d) = \frac{1}{6} \cdot 2^{-4} \sum_{d \equiv d_1 d_2 d_3 \mod 4} \prod_{p | d_1} \left( 1 + \left( \frac{d_1 d_2}{p} \right) \right) \prod_{p | d_2} \left( 1 + \left( \frac{d_2 d_3}{p} \right) \right) \prod_{p | d_3} \left( 1 + \left( \frac{d_3 d_1}{p} \right) \right)
\]

and

\[
f_2(d) = \frac{1}{2} \cdot 2^{-4} \sum_{d \equiv d_1 d_2 d_3 \mod 4} \prod_{p | d_1} \left( 1 + \left( \frac{d_2}{p} \right) \right) \prod_{p | d_2} \left( 1 + \left( \frac{d_1}{p} \right) \right)
\]

where the factorization \( d = d_1 d_2 d_3 \) is into integers such that each \( d_i \equiv 1 \mod 4 \) by the same reasoning as in the previous subsection, and \( \alpha(d) = 0 \) if all primes \( p | d \) satisfy \( p \equiv 1 \mod 4 \) and \( \alpha(d) = 1 \) otherwise. As before this gives

\[
f_1(d) = \frac{1}{3 \cdot 2^5} \sum_{d \equiv D_0 D_1 D_2 D_3 D_4 D_5} \left( \frac{D_0 D_3 D_2 D_5}{D_4} \right) \left( \frac{D_2 D_5 D_4 D_1}{D_0} \right) \left( \frac{D_4 D_1 D_0 D_3}{D_2} \right)
\]
and

\[ f_2(d) = \frac{1}{2^4} \sum_{d = E_0E_1E_2E_3} \left( \frac{E_2E_3}{E_0} \right) \left( \frac{E_0E_1}{E_2} \right). \]

where the sum is over factorizations which satisfy the congruences

\[ D_0D_3, D_2D_5, D_1D_1 \equiv 1 \pmod{4} \]

\[ E_0E_1, E_2E_3 \equiv 1 \pmod{4}. \]

As before we compute

\[ f_{j_1,j_2}(d) = \frac{1}{3^j 2^{j_1+4j_2}} \sum_{(D_u)} \prod_{u,v} \left( \frac{D_u}{D_v} \right)^{\Phi_{j_1}(u,v) + \Psi_{j_2}(u,v)} \]

where now the sum is over 6\(^{j_1}4^{j_2}\) tuples of integers \((D_u)\) which satisfy \(\prod_u D_u = d\) and the congruence conditions: for all \(1 \leq i \leq j_1\) and \((u_i, v_i) \in \{(0, 3), (2, 5), (4, 1)\}\) and all \(j_1 + 1 \leq i \leq j_1 + j_2\) and \((u_i, v_i) \in \{(0, 1), (2, 3)\}\)

\[ (8.5) \quad \prod_u D_u \prod_v D_v \equiv 1 \pmod{4} \]

where the above products are over all \(u\) with \(u_i\) in the \(i\)th position and all \(v\) with \(v_i\) in the \(i\)th position.

Thus multiplying by \(2^{j_3}4^{j_2}\alpha^k\omega(d)\) and summing over discriminants \(d < 0\) with \(d \equiv 1 \pmod{4}\) we get

\[ (8.6) \quad \sum_{d < X} 2^{j_3}4^{j_2}\alpha^k\omega(d) f_{j_1,j_2}(d) = \]

\[ \frac{1}{3^j 2^{j_1+3j_2+k}} \sum_{(D_u)} \mu^2 \left( \prod_u D_u \right) \alpha^k\omega(d) 2^{j_3}4^{j_2} \sum_{u,v} \left( \frac{D_u}{D_v} \right)^{\Phi_{j_1}(u,v) + \Psi_{j_2}(u,v)} \]

where the sum is over \(6^{j_1}4^{j_2}\) tuples of integers \((D_u)\) which satisfy \(\prod_u D_u < X\) and the conditions \((8.5)\).

Here \(S_{k,0,0}(X, A)\) reduces to

\[ S_{k,0,0}(X, A) = \sum_{(h_u)} \frac{1}{3^k 2^{2k}} \sum_{(D_u)} 2^{-k} \mu^2(d) \alpha^k\omega(d) \left[ \prod_{u,v} (-1)^{\Phi_{k}(u,v) h_u h_v - 1} \right] + O \left( X \log X \right) \]

and by the same procedure as in Section ?? we obtain

\[ S_{k,0,0}(X) = \frac{1}{3^k 2^{2k+3k}} \left( \sum_{U} \gamma(U) \right) \left( \sum_{n < X} \mu^2(2n)(3a)^\omega(n) \right) + O \left( X \log X \right) \]
where now define
\[ \gamma(U) = \sum_{(h_u)} \left[ \prod_{u,v} (-1)^{\Phi_k(u,v)\frac{h_u - 1}{2} \frac{h_v - 1}{2}} \right]. \]

Recall (see \(8.3\)) that we are allowing all possible congruence classes \((h_u)\) satisfying the conditions: for all \(1 \leq i \leq k\) and \((u_i, v_i) \in \{(0, 3), (2, 5), (4, 1)\}\)

(8.7) \[ \prod_u h_u \prod_v h_v \equiv 1 \mod 4 \]
where the above products are over all \(u\) with \(u_i\) in the \(i\)th position and all \(v\) with \(v_i\) in the \(i\)th position.

As in Section ?? we identify tuples \((h_u)\) with elements \(x \in F_{3k}^2\). In this case the \(x\) satisfying condition (8.7) are exactly the kernel of the matrix \(M_k\) defined in the previous section. The following proposition is proved almost identically to Proposition 21.

**Proposition 27.** For all \(k \geq 1\)
\[ \sum_U \gamma(U) = 2^{3^k - k - 1}. \]

Thus we have proven:

(8.8) \[ S_{k,0,0}(X) = \frac{1}{3^k 2^{6k+1}} \left( \sum_{n<X} 2^2 \sum_{d|n} \mu^2(2n)(3a)^{k\omega(n)} \right) + O \left( X (\log X)^{(3a)^k - a^k - 1 + \epsilon} \right). \]

from which the corresponding case of Theorem 6 follows by noting that
\[ \sum_{n<X} \mu^2(2n)(3a)^{k\omega(n)} = 2 \sum_{d \in D_{X,1,4}^i} (3a)^{k\omega(d)} + o(X) \]

8.4. **The case \(d > 0\) and \(d \equiv 4 \mod 8\).** Next we consider fundamental discriminants \(d > 0\) and \(d \equiv 4 \mod 8\). The number of \(H_8\) extensions of a quadratic field \(k\) Galois over \(\mathbb{Q}\) with such a discriminant is
\[ f(d) = f_1(d) - f_2(d) + 2\omega(d)^{-4} \]
where we define

\[
\begin{align*}
f_1(d) &= \frac{1}{2} 2^{-4} \sum_{d=4d_1d_2d_3} \left( \prod_{p|d_3} \left( 1 + \left( \frac{d_2d_3}{2} \right) \right) \prod_{p|d_2} \left( 1 + \left( \frac{d_3d_2}{p} \right) \right) \prod_{p|d_1} \left( 1 + \left( \frac{d_2d_1}{p} \right) \right) \right) \\
& \quad \times \prod_{p|d_1} \left( 1 + \left( \frac{d_2d_3}{p} \right) \right) \prod_{p|d_2} \left( 1 + \left( \frac{d_3d_1}{p} \right) \right)
\end{align*}
\]

and
\[
\begin{align*}
f_2(d) &= \frac{1}{2} 2^{-4} \sum_{d=4d_1d_2} \left( \prod_{p|d_3} \left( 1 + \left( \frac{d_2}{2} \right) \right) \prod_{p|d_2} \left( 1 + \left( \frac{d_3}{p} \right) \right) \prod_{p|d_1} \left( 1 + \left( \frac{d_1}{p} \right) \right) \right)
\end{align*}
\]
where the factorization $d = d_1d_2d_3$ is into integers such that $d_1 \equiv -1 \mod 4$ and $d_i \equiv 1 \mod 4$ otherwise by the same reasoning as in the previous subsections, noting that $\alpha(d) = 1$. As before this gives

$$f_1(d) = \frac{1}{2^5} \sum_{d = D_0D_1D_2D_3D_4D_5} \left(1 + \left(\frac{D_1D_2D_4D_5}{2}\right)\right) \left(\frac{D_0D_3D_2D_5}{D_4}\right) \left(\frac{D_2D_5D_4D_1}{D_0}\right) \left(\frac{D_4D_1D_0D_3}{D_2}\right)$$

and

$$f_2(d) = \frac{1}{2^4} \sum_{d = E_0E_1E_2E_3} \left(1 + \left(\frac{E_2E_3}{2}\right)\right) \left(\frac{E_2E_3}{E_0}\right) \left(\frac{E_0E_1}{E_2}\right).$$

where the sum is over factorizations which satisfy the congruences

$$D_0D_3 \equiv -1 \mod 4$$
$$D_2D_5, D_1D_4 \equiv 1 \mod 4$$
$$E_0E_1 \equiv -1 \mod 4$$
$$E_2E_3 \equiv 1 \mod 4.$$

Now define where

$$Q(u) = \begin{cases} 1 & u \in \{1, 2, 4, 5\} \\ 0 & u \in \{0, 3\} \end{cases}$$

and for any $B \subset \{1, \ldots, j_1\}$ let $Q_B(u) = \sum_{i \in B} Q(u_i)$. There is a similar definition for a function $S_C(u)$ for any $C \subset \{j_1 + 1, \ldots, j_2\}$ which appears in the following expression but the exact form is not important as we will eventually only be concerned with the case $j_2 = 0$. As before we compute

$$f_{j_1,j_2}(d) = \frac{1}{2^{5j_1+4j_2}} \sum_{(D_u) \subset \{1, \ldots, j_1\}} \sum_{(C_v) \subset \{j_1+1, \ldots, j_2\}} \prod_u \left(\frac{D_u}{2}\right)^{Q_B(u)+S_C(u)}$$
\[ \times \prod_{u,v} \left(\frac{D_u}{D_v}\right)^{\Phi_{j_1}(u,v)+\Psi_{j_2}(u,v)} \]

where the sum is over $6^{j_1}4^{j_2}$ tuples of integers $(D_u)$ which satisfy $\prod_u D_u = d$ and the congruence conditions: for all $1 \leq i \leq j_1$ and $(u_i, v_i) \in \{(0, 3), (2, 5), (4, 1)\}$$

(8.9) \quad \prod_u D_u \prod_v D_v \equiv \begin{cases} -1 \mod 4 & \text{if } (u_i, v_i) = (0, 3) \\ 1 \mod 4 & \text{if } (u_i, v_i) = (2, 5), (4, 1) \end{cases}

and all $j_1 + 1 \leq i \leq j_1 + j_2$ and $(u_i, v_i) \in \{(0, 1), (2, 3)\}$

(8.10) \quad \prod_u D_u \prod_v D_v \equiv \begin{cases} -1 \mod 4 & \text{if } (u_i, v_i) = (0, 1) \\ 1 \mod 4 & \text{if } (u_i, v_i) = (2, 3) \end{cases}

and the above products are over all $u$ with $u_i$ in the $i$th position and all $v$ with $v_i$ in the $i$th position.
Thus summing over discriminants \( d < 0 \) with \( d \equiv 1 \mod 4 \) we get
\[
\sum_{d < X} 2^{j_1 \omega(d)} a^{\omega(d)} f_{j_1,j_2}(d) = \frac{1}{3^{j_1} 2^{j_2 + 1 + j_2}} \sum_{(D_u)} \mu^2 \left( \prod_u D_u \right) a^{k \omega(d)} 2^{j_1 \omega(d)}
\]
\[
\times \sum_{B \subset \{1, \ldots, j_1\}} \sum_{C \subset \{j_1 + 1, \ldots, j_2\}} \prod_u \left( \frac{D_u}{2} \right)^{Q_B(u) + S_C(u)}
\]
\[
\times \prod_{u,v} \left( \frac{D_u}{D_v} \right)^{\Phi_{j_1}(u,v) + \Phi_{j_2}(u,v)}
\]
where the sum is over \( 6^{j_1} 4^{j_2} \) tuples of integers \((D_u)\) which satisfy \( \prod_u D_u < X \) and the conditions \((8.9)\).

Here \( S_{k,0,0}(X, A) \) reduces to
\[
\sum_{\mathbf{A} \text{ admissible for } \mathcal{U}} S_{k,0,0}(X, A) = \frac{1}{25k} \left( \sum_{(D_u)} \mu^2 (d/2) a^{k \omega(d/4)} \right)
\]
\[
\times \sum_{(h_u)} \left[ \sum_{C \subset \{1, \ldots, k\}} \prod_{u} (-1)^{\sum_i Q(u_i) \chi_C(i) \frac{k^2 - 1}{8}} \right]
\]
\[
\times \left[ \prod_{u,v} (-1)^{\Phi_{k}(u,v) \frac{h_u - 1}{2} \frac{h_v - 1}{2}} \right] + O \left( X (\log X)^{(3a)^k - a^k - 1 + \epsilon} \right).
\]

For \( h_u \) the congruence class of \( D_u \mod 8 \) and \( d = 4 \prod D_u \) and we grouped the 4 factor with the first discriminant in the factorization (i.e. for \( k = 1 \) the factorization is \( d = (4D_0D_3)(D_1D_4)(D_2D_5) \)).

Then, noting that \( h_u \) is one out of four choices for odd numbers \( \mod 8 \), we get
\[
S_{k,0,0}(X) = \frac{1}{25k} \left( \sum_{\mathcal{U}} \gamma(\mathcal{U}) \right) \left( \sum_{4n<X} \mu^2(2n)(3a)^{k \omega(n)} \right) + O \left( X (\log X)^{(3a)^k - a^k - 1 + \epsilon} \right)
\]
where we define
\[
\gamma(\mathcal{U}) = \sum_{(h_u)} \left[ \sum_{C \subset \{1, \ldots, k\}} \prod_{u} (-1)^{\sum_i Q(u_i) \chi_C(i) \frac{k^2 - 1}{8}} \right] \left[ \prod_{u,v} (-1)^{\Phi_{k}(u,v) \frac{h_u - 1}{2} \frac{h_v - 1}{2}} \right]
\]
allowing odd congruence classes \( h_u \mod 8 \) satisfying the following conditions: for all \( 1 \leq i \leq k \)
\[
(8.11) \quad \prod_u h_u \prod_v h_v \equiv \begin{cases} 1 \mod 4 & (u_i, v_i) \in \{(1, 4), (2, 5)\} \\ -1 \mod 4 & (u_i, v_i) \in \{(0, 3)\} \end{cases}
\]
where the above products are taken over all \( u \) with \( u_i \) in the \( i \)th position and \( v \) with \( v_i \) in the \( i \)th position. (Recalling that the odd parts of even discriminants are \(-1 \mod 4\) if \( 8 \nmid d \)). If we call \( x \equiv \left( \frac{h_u - 1}{2} \right) \mod 2 \) an element of \( \mathbb{F}_2^k \), then \( x \in y + \ker M_k \) a coset of \( \ker M_k \).
Now consider that $\Phi(u, v) = 0$ if $u, v \in A = \{1, 3, 5\}$. $\mathcal{U}$ is a largest maximal unlinked set, and so has a type $s \in S$, so it follows that

$$\Phi_k(u, v) = \sum_i \Phi(u_i, v_i) = \sum_{i, s_i = B} \Phi(u_i, v_i)$$

This way we show that

$$\sum_{\mathcal{U}} \gamma(\mathcal{U}) = \sum_{s \in S} \sum_{x \in y + \ker \mathcal{M}_k} \left[ \sum_{(h_u): \frac{h_u - 1}{2} \equiv x_u \mod 2} \sum_{C \subseteq \{1, \ldots, k\}} \prod_u (-1)^{\sum_{i \in C} Q(u_i) \left(\frac{h_u^2 - 1}{8}\right)} \right] 
\times \left[ \prod_{\{u, v\}} (-1)^{\sum_{i \in C} \Phi(u_i, v_i) x_u x_v} \right]$$

Notice that for each $x_u$, there are two choices of $h_u \mod 8$ such that $\frac{h_u - 1}{2} \equiv x_u \mod 2$, because $h_u$ and $5h_u$ give the same image. If we fix an $(h(x)_u)$ satisfying this property without loss of generality also satisfying $\frac{h_u^2 - 1}{8} \equiv 0 \mod 2$, then we have

$$\sum_{(h_u): \frac{h_u - 1}{2} \equiv x_u \mod 2} \sum_{C \subseteq \{1, \ldots, k\}} \prod_u (-1)^{\sum_{i \in C} Q(u_i) \left(\frac{h_u^2 - 1}{8}\right)} = \sum_{T \subseteq \mathcal{U}} \sum_{C \subseteq \{1, \ldots, k\}} \prod_u (-1)^{\sum_{i \in C} Q(u_i) \left(\frac{5^2 \chi_T(u) - 1}{8}\right)}$$

Now, $\frac{5^2 - 1}{8} \equiv 1 \mod 2$. So it follows that $\left(\frac{5^2 \chi_T(u) - 1}{8}\right) \equiv \left(\chi_T(u)\right) \mod 2$ so we have

$$\sum_{(h_u): \frac{h_u - 1}{2} \equiv x_u \mod 2} \sum_{C \subseteq \{1, \ldots, k\}} \prod_u (-1)^{\sum_{i \in C} Q(u_i) \left(\frac{h_u^2 - 1}{8}\right)} = \sum_{T \subseteq \mathcal{U}} \sum_{C \subseteq \{1, \ldots, k\}} \prod_u (-1)^{\sum_{i \in C} Q(u_i) \chi_T(u)}$$

By using the binomial theorem in two different ways. Now, for $s_i$ being $A$ or $B$, we can find a $u \in \mathcal{U}$ such that $\sum_{i \in C} Q(u_i) \equiv 1 \mod 2$ for any $C \neq \emptyset$, by making $u \in \{0, 3\}$ for all except one value of $i \in C$, and $u \in \{1, 2, 4, 5\}$ for exactly one such value. So we have the only nonzero term in this sum is for $C = \emptyset$, which must be equal to $2^k$ independent of the type.
So then we have

$$\sum_{\mathcal{U}} \gamma(\mathcal{U}) = 2^{3k} \sum_{s \in S} \sum_{x \in y + \ker M_k} \left[ \prod_{\{u, v\}} (-1)^{\sum_{i \in B} \Phi(u_i, v_i)x_u x_v} \right]$$

$$= 2^{3k} \sum_{s \in S} \sum_{x \in y + \ker M_k} (-1)^{\sum_{i \in B} \sum_{x_i = B} \Phi(u_i, v_i)x_u x_v}$$

Call $\mathcal{U}_B = B^k$. Then we can write every $\mathcal{U} = \{(s, u) : u \in \mathcal{U}_B\}$ where clearly $\Phi((s, u), (s, v)) = \sum_{i : s_i = B} \Phi(u_i, v_i)$. We can apply the binomial theorem to the sum over types $s \in S$ to show

$$\sum_{\mathcal{U}} \gamma(\mathcal{U}) = 2^{3k} \sum_{x \in y + \ker M_k} \prod_{j = 1}^k \left( 1 + (-1)^{\sum_{\{u, v\} \subset \mathcal{U}_B} \Phi(u_j, v_j)x_u x_v} \right)$$

Notice we have

$$\sum_{\{u, v\} \subset \mathcal{U}_B} \Phi(u_1, v_1)x_u x_v = \sum_{u_1 = 0, v_1 = 2} x_u x_v + \sum_{u_1 = 0, v_1 = 4} x_u x_v + \sum_{u_1 = 2, v_1 = 4} x_u x_v$$

$$= \sum_{u_1 = 0} x_u \sum_{v_1 = 2} x_v + \sum_{u_1 = 0} x_u \sum_{v_1 = 4} x_v + \sum_{u_1 = 2} x_u \sum_{v_1 = 4} x_v$$

Which are both 0 for any $x \in W + \ker M_k$, as the only possible nonzero terms $\sum_{u_1 = 0} x_u$ are multiplied by another term going to 0. By symmetry the same is true for all $j = 1, ..., k$, so we now have

$$\sum_{\mathcal{U}} \gamma(\mathcal{U}) = 2^{3k} \sum_{x \in y + \ker M_k} 2^k$$

$$= 2^{\dim M_k + 3^k + k}$$

Recall that $\dim M_k = 3^k - 2k - 1$. Also it follows that $W$ has a basis of vectors $w$ such that $\sum_{u : u_i = j} w_u = 1$ for exactly one pair of $1 \leq i \leq k$ and $j \in \{0, 3\}$ (corresponding to the type). In other words, $\dim W = k$. So we have

$$\sum_{\mathcal{U}} \gamma(\mathcal{U}) = 2^{3^k - k - 1}$$

Thus implying

$$S_{k, 0, 0}(X) = \frac{1}{2^{6k+1}} \left( \sum_{4n < X} \mu^2(2n)(3a)^{k_\omega(n)} \right) + O \left( X (\log X)^{(3a)^k - k - 1 + \epsilon} \right)$$

from which the corresponding case of follows by noting

$$\sum_{4n < X} \mu^2(2n)(3a)^{k_\omega(n)} = \frac{2}{(3a)^k} \sum_{d \in \mathcal{D}_{X, 4, 8}} (3a)^{k_\omega(d)}$$
8.5. **The case $d > 0$ and $d \equiv 0 \mod 8$.** Next we consider fundamental discriminants $d > 0$ and $d \equiv 0 \mod 8$. The number of $H_8$ extensions of a quadratic field $k$ Galois over $\mathbb{Q}$ with such a discriminant is

$$f(d) = f_1(d) - f_2(d) + 2^{\omega(d)-4}$$

where we define

$$f_1(d) = \frac{1}{2} \cdot 2^{-4} \sum_{d=8d_1d_2d_3} \left(1 + \left(\frac{d_2d_3}{2}\right)\right) \prod_{p|d_1} \left(1 + \left(\frac{d_1d_2}{p}\right)\right) \prod_{p|d_2} \left(1 + \left(\frac{2d_3}{p}\right)\right)$$

and

$$f_2(d) = 2^{-4} \sum_{d=8d_1d_2} \left(1 + \left(\frac{d_2}{2}\right)\right) \prod_{p|d_1} \left(1 + \left(\frac{d_2}{p}\right)\right) \prod_{p|d_2} \left(1 + \left(\frac{2d_1}{p}\right)\right)$$

where the factorization $d = d_1d_2d_3$ is into integers such that $d_1 \equiv -1 \mod 4$ and $d_i \equiv 1 \mod 4$ otherwise by the same reasoning as in the previous subsections, where $\alpha(d) = 0$ is all odd primes dividing $d$ satisfy $p \equiv 1 \mod 4$ and $\alpha(d) = 1$ otherwise. As before this gives

$$f_1(d) = \frac{1}{2^5} \sum_{d=D_0D_1D_2D_3D_4D_5} \left(1 + \left(\frac{D_1D_2D_3D_4D_5}{2}\right)\right) \left(\frac{2}{D_2D_4}\right) \left(\frac{D_0D_3D_2D_5}{D_4}\right) \left(\frac{D_2D_5D_4D_1}{D_0}\right) \left(\frac{D_4D_1D_0D_3}{D_2}\right)$$

and

$$f_2(d) = \frac{1}{2^4} \sum_{d=E_0E_1E_2E_3} \left(1 + \left(\frac{E_2E_3}{2}\right)\right) \left(\frac{2}{E_2}\right) \left(\frac{E_2E_3}{E_0}\right) \left(\frac{E_0E_1}{E_2}\right)$$

where the sum is over factorizations which satisfy the congruences

$$D_2D_5, D_1D_4 \equiv 1 \mod 4$$

$$E_2E_3 \equiv 1 \mod 4.$$
The sum is over $6^{j_1}4^{j_2}$ tuples of integers $(D_u)$ which satisfy $\prod_u D_u = d$ and the congruence conditions: for all $1 \leq i \leq j_1$ and $(u_i, v_i) \in \{(2, 5), (4, 1)\}$ and all $j_1 + 1 \leq i \leq j_1 + j_2$ and $(u_i, v_i) \in \{(2, 3)\}$

(8.12) \[
\prod_u D_u \prod_v D_v \equiv 1 \mod 4
\]

where the above products are over all $u$ with $u_i$ in the $i$th position and all $v$ with $v_i$ in the $i$th position.

Thus summing over discriminants $d < 0$ with $d \equiv 1 \mod 4$ we get

\[
\sum_{d<X} 2^{j_3 \omega(d) - j_3} a^{\omega(d)} f_{j_1, j_2}(d) = \frac{1}{3j_1 2^{j_1 + 4j_2 + k}} \sum_{(D_u)} \mu^2 \left( \prod_u D_u \right) a^{k \omega(d)} 2^{j_3 \omega(d)} \prod_{B \subseteq \{1, \ldots, j_1\}} \prod_{C \subseteq \{j_1 + 1, \ldots, j_2\}} \prod_u \left( \frac{D_u}{2} \right)^{Q_B(u) + S_C(u)} \prod_{u, v} \left( \frac{D_u}{D_v} \right)^{\Phi_{j_1}(u,v) + \Psi_{j_2}(u,v)}
\]

where the sum is over $6^{j_1}4^{j_2}$ tuples of integers $(D_u)$ which satisfy $\prod_u D_u < X$ and the conditions \[8.12\].

In this case, as in the $d > 0$ and $d \equiv 1 \mod 4$ case, we have

\[
\sum_{\mathbf{A} \ \text{admissible for } \mathcal{U}} S_{k,0,0}(X, \mathbf{A}) = \frac{1}{2^{5k}} \left( \sum_{(D_u)} 2^{-k} \mu^2 \left( \frac{d}{2} \right) a^{k \omega(d/4)} \right)
\]

\[
\times \sum_{(h_u)} \left[ \sum_{C \subseteq \{1, \ldots, k\}} \prod_u (-1)^{\sum_i Q(u_i) \chi_C(i)} \lambda(u) \frac{h_u - 1}{2} \right]
\]

\[
\times \left[ \prod_{u,v} (-1)^{\Phi(u,v) \frac{h_u - 1}{2}} \right] + O \left( X(\log X)^{(3a)^k - a^k - 1 + \epsilon} \right).
\]

For $h_u$ the congruence class of $D_u$ mod 8 and $d = 8 \prod D_u$ and we grouped the 8 factor with the first discriminant in the factorization (i.e. for $k = 1$ the factorization is $d = (8D_0D_3)(D_1D_4)(D_2D_5)$).

Noting that $h_u$ is one out of four choices for odd numbers mod 8, we get

\[
S_{k,0,0}(X) = \frac{1}{2^{6k} 2^{2-3^k}} \left( \sum_{\mathcal{U}} \gamma(\mathcal{U}) \right) \left( \sum_{8n<X} \mu^2(2n)(3a)^{k \omega(n)} \right) + O \left( X(\log X)^{(3a)^k - a^k - 1 + \epsilon} \right)
\]

where we define

\[
\gamma(\mathcal{U}) = \sum_{(h_u)} \left[ \sum_{C \subseteq \{1, \ldots, k\}} \prod_u (-1)^{\sum_i Q(u_i) \chi_C(i)} \frac{\lambda(u) \frac{h_u - 1}{2}}{8} \right]
\]

\[
\times \left[ \prod_{u,v} (-1)^{\Phi(u,v) \frac{h_u - 1}{2}} \right]
\]

\[
\times \left[ \prod_{u,v} (-1)^{\Phi(u,v) \frac{h_u - 1}{2}} \right]
\]

\[
\times \left[ \prod_{u,v} (-1)^{\Phi(u,v) \frac{h_u - 1}{2}} \right]
\]
allowing odd congruence classes \( h_u \mod 8 \) satisfying the following conditions: for all \( 1 \leq i \leq k \):

\[
\prod_u h_u \prod_v h_v \equiv \begin{cases} 
1 \mod 4 & (u_i, v_i) \in \{(1,4), (2,5)\} \\
\prod_{u'} h_{u'} \prod_{v'} h_{v'} & (u_i, v_i) = (u'_i, v'_i) = (0,3) 
\end{cases}
\]

where the above products are taken over all \( u \) with \( u_i \) in the \( i \)th position and \( v \) with \( v_i \) in the \( i \)th position.

If we call \( x \equiv \left( \frac{h_u-1}{2} \right) \mod 2 \) an element of \( \mathbb{F}_2^k \), then \( x \in y + \ker M_k \) one of two cosets of \( \ker M_k \) depending on the congruence class of \( \prod_u h_u \prod_v h_v \) for \( (u_i, v_i) = (0,3) \).

Now consider that \( \Phi(u, v) = 0 \) if \( u, v \in A = \{1,3,5\} \). \( U \) is a largest maximal unlinked set, and so has a type \( s \in S \), so it follows that

\[
\Phi_k(u, v) = \sum_i \Phi(u_i, v_i) = \sum_{i,s_i=B} \Phi(u_i, v_i)
\]

This way we show that

\[
\sum_U \gamma(U) = \sum_S \sum_y \sum_{x \in y + \ker M_k} \left[ \sum_{(h_u): \frac{h_u-1}{2} \equiv x \mod 2} \sum_{C \subseteq \{1,\ldots,k\}} \prod_{u \in \text{C}} (-1)^{\sum_{i \in \text{C}} Q(u_i)} \left( \frac{h_u^2-1}{8} \right) \right] \\
\times \prod_{u \in \text{C}} (-1)^{\sum_{s_i=B} \lambda(u_i) \frac{h_u^2-1}{8}} \\
\times \prod_{\{u,v\}} (-1)^{\sum_{s_i=B} \Phi(u_i,v_i) x_u x_v}
\]

Where the sum of \( y \) is over \( 2^k \) cosets of \( \ker M_k \) to account for the missing condition on \( u_i \in \{0,3\} \). Notice that for each \( x_u \), there are two choices of \( h_u \mod 8 \) such that \( \frac{h_u-1}{2} \equiv x_u \mod 2 \), because \( h_u \) and \( 5h_u \) give the same image. If we fix an \( (h(x) u) \) satisfying this property without loss of generality also satisfying \( \frac{h_u^2-1}{8} \equiv 0 \mod 2 \), then we have

\[
\sum_{(h_u): \frac{h_u-1}{2} \equiv x \mod 2} \sum_{C \subseteq \{1,\ldots,k\}} \prod_{u \in \text{C}} (-1)^{\sum_{i \in \text{C}} Q(u_i)+\lambda_k(u)} \left( \frac{h_u^2-1}{8} \right)
\]

\[
= \sum_{T \subseteq U} \sum_{C \subseteq \{1,\ldots,k\}} \prod_{u \in \text{C}} (-1)^{\sum_{i \in \text{C}} Q(u_i)+\lambda_k(u)} \left( \frac{(h_u T(u))^{2}-1}{8} \right)
\]

\[
= \sum_{T \subseteq U} \sum_{C \subseteq \{1,\ldots,k\}} \prod_{u \in \text{C}} (-1)^{\sum_{i \in \text{C}} Q(u_i)+\lambda_k(u)} \left( \frac{2^{\chi T(u)-1}}{8} \right)
\]
Now, $\frac{5^2 - 1}{8} \equiv 1 \mod 2$. So it follows that $\left(\frac{5^{2xT(u)} - 1}{8}\right) \equiv (\chi_T(u)) \mod 2$ so we have

\[
\sum_{(h_u): \frac{h_u - 1}{2} \equiv x_u \mod 2} \sum_{C \subset \{1, \ldots, k\}} \prod_{u} (-1)^{\left(\sum_{i \in C} Q(u_i) + \lambda_k(u)\right)} \left(\frac{5^2 - 1}{8}\right)
\]

\[
= \sum_{T \subset U} \sum_{C \subset \{1, \ldots, k\}} \prod_{u} (-1)^{\left(\sum_{i \in C} Q(u_i) + \lambda_k(u)\right)\chi_T(u)}
\]

\[
= \sum_{T \subset U} \sum_{C \subset \{1, \ldots, k\}} (-1)^{\sum_{u, i} Q(u_i)\chi_C(i)\chi_T(u) + \lambda_k(u)\chi_T(u)}
\]

\[
= \sum_{C \subset \{1, \ldots, k\}} \prod_{u} \left(1 + (-1)^{\sum_{i} Q(u_i)\chi_C(i) + \lambda(u_i)}\right)
\]

We then have

\[
Q(u_i)\chi_C(i) + \lambda(u_i) = \begin{cases} 
0 & u_i \in \{0, 3, 2, 4\}, i \in C \\
1 & u_i \in \{1, 5\}, i \in C \\
0 & u_i \in \{0, 3, 1, 5\}, i \notin C \\
1 & u_i \in \{2, 4\}, i \notin C
\end{cases}
\]

Suppose that $C \notin \{i : s_i = B\}$, then fix some $j \in C$ such that $s_j = A$. Choose $u \in U$ such that

\[
u_i \in \begin{cases} 
\{1\} & i = j \\
\{0, 3\} & i \neq j
\end{cases}
\]

Then we get the summands are zero corresponding to these $C$. Suppose next that $\{i : s_i = B\} \notin C$, then fix $j \notin C$ such that $s_j = B$ and choose $u \in U$ such that

\[
u_i \in \begin{cases} 
\{2\} & i = j \\
\{0, 3\} & i \neq j
\end{cases}
\]

Then these summands give us zero as well. The only summand remaining is $C = \{i : s_i = B\}$, which clearly gives $2^3k$.

So then we have

\[
\sum_{\mathcal{U}} \gamma(\mathcal{U}) = 2^{3k} \sum_{s \in S} \sum_{x \in y + \ker M_k} \left[ \prod_{\{u, v\}} (-1)^{\sum_{i = B} \Phi(u_i, v_i) x_u x_v} \right]
\]

\[
= 2^{3k} \sum_{y \in S} \sum_{x \in y + \ker M_k} (-1)^{\sum_{\{u, v\}} \sum_{i = B} \Phi(u_i, v_i) x_u x_v}
\]

Noting that there are 2 choices for $y$, the same proof for the case $d > 0, d \equiv 4 \mod 8$ implies

\[
S_{k,0,0}(X) = \frac{1}{2^{6k}} \left( \sum_{8n < X} \mu^2(2n)(3\omega(n)) \right) + O \left( X (\log X)(3a)^k - a^{k-1} + \epsilon \right)
\]
Then the corresponding case of $\mathbb{Z}$ follows from
\[
\sum_{8n<X} \mu^2(2n)(3a)^k\omega(n) = \frac{1}{(3a)^k} \sum_{d \in \mathcal{D}_{X,6}} (3a)^k\omega(d) + o(X).
\]

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