COORDINATE RINGS OF SOME SL₂-CHARACTER VARIETIES

VICENTE MUÑOZ AND JESÚS MARTÍN OVEJERO

Abstract. We determine generators of the coordinate ring of SL₂-character varieties. In the case of the free group \(\mathcal{F}_3\) we obtain an explicit equation of the SL₂-character variety. For free groups \(\mathcal{F}_k\) we find transcendental generators. Finally, for the case of the 2-torus, we get an explicit equation of the SL₂-character variety and use the description to compute their \(E\)-polynomials.

1. Introduction

Let \(\Gamma\) be a finitely generated group and let \(G\) be an algebraic group over an algebraically closed field \(k\). The character variety \(\mathcal{X}(\Gamma, G)\) parametrizes isomorphism classes of representations \(\rho : \Gamma \to G\). Character varieties are rich objects that contain geometric information linking distant areas in mathematics. An important instance is when we take \(\Gamma_g = \pi_1(\Sigma_g)\) to be the fundamental group of the compact orientable surface of genus \(g \geq 1\). In this case, these character varieties are one of the three incarnations of the moduli space of Higgs bundles, as stated by the celebrated non-abelian Hodge correspondence [2, 8, 20]. For this reason, character varieties of surface groups have been widely studied, particularly regarding the computation of some algebraic invariants like their \(E\)-polynomial.

Character varieties also play a prominent role in the topology of 3-manifolds, starting with the foundational work of Culler and Shalen [3], where the authors used algebro-geometric properties of SL₂(\(\mathbb{C}\))-character varieties to provide new proofs of remarkable results, as Thurston theorem that says that the space of hyperbolic structures on an acylindrical 3-manifold is compact, or the Smith conjecture. Character varieties of 3-manifolds have been also used to study knots \(K \subset S^3\), by analyzing the character variety associated to the fundamental group of their complement, \(\Gamma_K = \pi_1(S^3 - K)\). For instance, the geometry of these knot character varieties has been studied in [6, 11] for trivial links (i.e. when \(\Gamma\) is a free group), and in [9, 15] for the torus knot, among others.

Fix \(G = \text{SL}_r\), the group of matrices of size \(r\) with trivial determinant. If we have a presentation \(\Gamma = \langle x_1, \ldots, x_k | r_1, \ldots, r_s \rangle\), then \(\mathcal{X}(\Gamma, G)\) parametrizes \(k\)-tuples \((A_1, \ldots, A_k)\) of matrices in \(G\) subject to the relations \(r_j(A_1, \ldots, A_k) = I\). In the same vein as the isomorphism class of a matrix \(A\) is determined by the traces of its powers, \(\text{tr}(A^i), 1 \leq i \leq r - 1\), the \(k\)-tuples \((A_1, \ldots, A_k)\) are determined by the traces of suitable products of the matrices. In this paper, we focus on the group \(\text{SL}_2\) and work out many identities with traces. This serves to find coordinates for the character variety \(\mathcal{X}(\Gamma, \text{SL}_2)\). Note that there is a natural embedding \(\mathcal{X}(\Gamma, \text{SL}_2) \subset \mathcal{X}(\mathcal{F}_k, \text{SL}_2)\), where \(\mathcal{F}_k\) is the free group of \(k\) elements, and \(k\) is the number of generators of \(\Gamma\). Therefore it is natural to look initially to the case of the free group.

The structure of SL₂(\(\mathbb{C}\))-character varieties of free groups has been well-understood for some time. There is a modern treatment in [7], where historical references can be found. We

2020 Mathematics Subject Classification. Primary: 14M35. Secondary: 20G05, 14D20.
Key words and phrases. Characters, representations, trace of matrix.
thank Sean Lawton for pointing this out to us. More generally, there is an effective algorithm to compute the coordinate ring of any \( SL_2(\mathbb{C}) \)-character variety for any finitely presentable group in \[1\]. Here we obtain in an alternative way some of these coordinate rings, and connect with results of \[13\] about their \( E \)-polynomials. We start with the following:

**Theorem 1.1.** Let \( A_1, \ldots, A_k \in SL_2 \), then the character variety \( \mathcal{X}_k = \mathcal{X}(F_k, SL_2) \) is parametrized by \( T_{i_1 \ldots i_p} := t_{A_{i_1}} \cdots t_{A_{i_p}} \), \( i_1 < \ldots < i_p \) with \( p = 1, 2, 3 \).

For the situation of \( k = 3 \), we can obtain an explicit equation. By Theorem \[1.1\] the coordinates of \( \mathcal{X}_3 = \mathcal{X}(F_3, SL_2) \) are \((x, y, z, u, v, w, P) = (t_A, t_B, t_C, t_{BC}, t_{AC}, t_{AB}, t_{ABC}) \), where \((A, B, C) \in \mathcal{X}_3 \). We have the following:

**Theorem 1.2.** The character variety \( \mathcal{X}_3 \subset \mathbb{k}^7 \) is a hypersurface defined by the equation

\[
P^2 = (wz + vy + ux - xyz)P - x^2 - y^2 - z^2 + uyz + vxz + wxy - uvw - w^2 - v^2 - w^2 + 4.
\]

Next, we look at the case of the character varieties of a compact orientable surface \( \Sigma_g \) of genus \( g \geq 1 \). Its fundamental group is

\[
\pi_1(\Sigma_g) = \langle a_1, b_1, \ldots, a_g, b_g \mid \prod_{i=1}^g [a_i, b_i] = 1 \rangle.
\]

Take a conjugacy class \([\xi] \) determined by an element \( \xi \in SL_2 \), then we define as in \[13\]

\[
\mathcal{M}_\xi = \{(A_1, B_1, \ldots, A_g, B_g) \in (SL_2)^{2g} \mid \prod_{i=1}^g [A_i, B_i] = \xi \} \mod \text{Stab}(\xi)
\]

(1)

There are five different types of conjugacy classes, namely \([I], [-I], [J_+], [J_-] \) and \([\xi] \), where \( J_\pm = \begin{pmatrix} \pm 1 & 0 \\ 0 & \mp 1 \end{pmatrix} \) are the Jordan types, and \( \xi_t = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \), \( t = \lambda + \lambda^{-1} \), \( \lambda \in \mathbb{C} - \{0, \pm 1\} \), are the diagonal types. For \( k = \mathbb{C} \), these varieties have been studied in \[13\] and the \( E \)-polynomials are computed in a series of papers \[13, 16, 17\]. For \( g = 1 \), we have the following result from \[13, \text{Theorem 1.1}\].

**Theorem 1.3.** For the 2-torus \( \Sigma_1 = T^2 \), the \( E \)-polynomials of \( \mathcal{M}_\xi \) are as follows:

\[
e(\mathcal{M}_I) = q^2 + 1,
\]
\[
e(\mathcal{M}_{-I}) = 1,
\]
\[
e(\mathcal{M}_{J_+}) = q^2 - 2q - 3,
\]
\[
e(\mathcal{M}_{J_-}) = q^2 + 3q,
\]
\[
e(\mathcal{M}_{\xi_t}) = q^2 + 4q + 1,
\]

where \( q = uv \), \( e(\mathcal{M}_\xi) \in \mathbb{Z}[u, v] \).

We look at the character varieties \( \mathcal{M}_\xi \) more closely by working out trace identities of commutators of two matrices. First, for matrices \((A, B)\), we determine the equation for \((x, y, z) = (t_A, t_B, t_{AB})\),

\[
F(x, y, z) = t_{[A, B]} = x^2 + y^2 + z^2 - xyz - 2.
\]
which produces the character varieties:
\[ \mathcal{X}_t = F^{-1}(t) = \{(A, B) \in (\text{SL}_2)^2 \mid \text{tr}(A, B) = t\} \text{ // SL}_2, \]
for \( t \in \mathbb{C} \). Then

- \( \mathcal{X}_t = \mathcal{M}_{\xi_t} \) for \( t \neq \pm 2 \),
- \( \mathcal{X}_2 = \mathcal{M}_I \cup \mathcal{M}_{J_1} \),
- \( \mathcal{X}_{-2} = \mathcal{M}_{I} \cup \mathcal{M}_{J_1} \).

We study the geometry of the character varieties \( \mathcal{X}_t \), and recover the results of Theorem 1.3. More specifically,

**Theorem 1.4.** Let \( t \in \mathbb{C} \). We have the following:

- For \( t \neq \pm 2 \), the character variety \( \mathcal{X}_t \subset \mathbb{C}^3 \) is a smooth surface, and \( e(\mathcal{X}_t) = q^2 + 4q + 1 \).
- For \( t = 2 \), the character variety \( \mathcal{X}_2 \subset \mathbb{C}^3 \) has 4 ordinary double points. We have \( \mathcal{M}_{J_1} \subset \mathcal{M}_I \), \( \mathcal{X}_2 = \mathcal{M}_I \), and \( e(\mathcal{X}_2) = q^2 + 1 \).
- For \( t = -2 \), the character variety \( \mathcal{X}_{-2} \subset \mathbb{C}^3 \) has only one singular point which is an ordinary double point. We have \( \mathcal{X}_{-2} = \mathcal{M}_{I} \cup \mathcal{M}_{J_1} \), and \( e(\mathcal{X}_{-2}) = q^2 + 3q + 1 \).

**Acknowledgements.** We thank Sean Lawton for useful comments and references. The first author is partially supported by Project MCI (Spain) PID2020-118452GB-I00. The second author is supported by the Spanish Government research project PGC2018-099599-B-I00. The second author would like to express his gratitude to the University of Málaga, for giving him the opportunity to carry out a research stay with the first author.

2. Moduli of representations and character varieties

Let \( \Gamma \) be a finitely presented group, and let \( G < \text{GL}_r \) be an algebraic group over an algebraic closed field \( k \). A representation of \( \Gamma \) in \( G \) is a homomorphism \( \rho : \Gamma \rightarrow G \). Consider a presentation \( \Gamma = \langle x_1, \ldots, x_k | r_1, \ldots, r_s \rangle \). Then \( \rho \) is determined by the \( k \)-tuple \( (A_1, \ldots, A_k) = (\rho(x_1), \ldots, \rho(x_k)) \) subject to the relations \( r_j(A_1, \ldots, A_k) = I \), \( 1 \leq j \leq s \). The space of representations is

\[ R(\Gamma, G) = \text{Hom}(\Gamma, G) = \{(A_1, \ldots, A_k) \in G^k \mid r_j(A_1, \ldots, A_k) = I, 1 \leq j \leq s\}. \tag{2} \]

Therefore \( R(\Gamma, G) \) is an affine algebraic set.

We say that two representations \( \rho \) and \( \rho' \) are equivalent if there exists \( P \in G \) such that \( \rho'(g) = P^{-1}\rho(g)P \), for every \( g \in G \). This corresponds to a change of basis in \( k^r \). Note that the action of \( G \) descends to an action of the projective group \( \text{PGL}_r \) on \( R(\Gamma, G) \). The moduli space of representations is the GIT quotient

\[ \mathcal{M}(\Gamma, G) = R(\Gamma, G) \text{ // } G. \]

Recall that by definition of GIT quotient for an affine variety, if we write \( R(\Gamma, G) = \text{Spec} \mathcal{O} \), then \( \mathcal{M}(\Gamma, G) = \text{Spec} \mathcal{O}^G \).

Suppose from now on that \( G = \text{SL}_r \). A representation \( \rho \) is reducible if there exists some proper subspace \( V \subset k^r \) such that for all \( g \in G \) we have \( \rho(g)V \subset V \); otherwise \( \rho \) is irreducible. Note that if \( \rho \) is reducible, then let \( V \subset k^r \) an invariant subspace, and consider a complement \( k^r = V \oplus W \). Let \( \rho_1 = \rho|_V \) and \( \rho_2 \) the induced representation on the quotient space \( W = k^r/V \). Then we can write \( \rho = \begin{pmatrix} \rho_1 & 0 \\ f & \rho_2 \end{pmatrix} \), where \( f : \Gamma \rightarrow \text{Hom}(W, V) \). Take \( P_t = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \).
which is independent of the chosen collection. The natural algebraic map $H^*$ and $\mathfrak{M}(\Gamma, G)$.

2.1. The ring of invariant polynomials is generated by characters. We introduce the basic definitions. Here the ground field is $\mathbb{C}$, which is an invariant of a complex algebraic variety constructed as an Euler characteristic $\chi$.

There is a character map $\tilde{\rho} : R(\Gamma, G) \to k^{\Gamma}$, $\rho \mapsto \chi_{\rho}$, whose image

$$\mathfrak{X}(\Gamma, G) = \chi(R(\Gamma, G))$$

is called the character variety of $\Gamma$. Let us give $\mathfrak{X}(\Gamma, G)$ the structure of an algebraic variety. The traces $\chi_{\rho}$ span a subring $B \subset A$. Clearly $B \subset A^G$. As $A$ is noetherian, we have that $B$ is a finitely generated $k$-algebra. Hence there exists a collection $g_1, \ldots, g_a$ of elements of $G$ such that $\chi_{\rho}$ is determined by $\chi_{\rho}(g_1), \ldots, \chi_{\rho}(g_a)$, for any $\rho$. Such collection gives a map

$$\tilde{\chi} : R(\Gamma, G) \to k^a, \quad \tilde{\chi}(\rho) = (\chi_{\rho}(g_1), \ldots, \chi_{\rho}(g_a)),$$

and $\mathfrak{X}(\Gamma, G) \cong \tilde{\chi}(R(\Gamma, G))$. This endows $\mathfrak{X}(\Gamma, G)$ with the structure of an algebraic variety, which is independent of the chosen collection. The natural algebraic map

$$\mathfrak{M}(\Gamma, G) \to \mathfrak{X}(\Gamma, G)$$

is an isomorphism (see Chapter 1 in [1]). This is the same as to say that $B = A^G$, that is, the ring of invariant polynomials is generated by characters.

2.1. Hodge structures and $E$-polynomials. Later we will need the notion of $E$-polynomial, which is an invariant of a complex algebraic variety constructed as an Euler characteristic of its Hodge numbers. We introduce the basic definitions. Here the ground field is $k = \mathbb{C}$. A pure Hodge structure of weight $k$ consists of a finite dimensional complex vector space $H$ with a real structure, and a decomposition $H = \bigoplus_{k-p+q} H^{p,q}$ such that $H^{p,q} = \overline{H^{q,p}}$, the bar meaning complex conjugation on $H$. A Hodge structure of weight $k$ gives rise to the so-called Hodge filtration, which is a descending filtration $F^p = \bigoplus_{s \geq p} H^{s,k-s}$. We define $Gr^p_F(H) := F^p / F^{p+1} = H^{p,k-p}$.

A mixed Hodge structure consists of a finite dimensional complex vector space $H$ with a real structure, an ascending (weight) filtration $\cdots \subset W_{k-1} \subset W_k \subset \cdots \subset H$ (defined over $\mathbb{R}$) and a descending (Hodge) filtration $F$ such that $F$ induces a pure Hodge structure of weight $k$ on each $Gr^W_k(H) = W_k / W_{k-1}$. We define $H^{p,q} := Gr^p_F Gr^{W}_{p+q}(H)$ and write $h^{p,q}$ for the Hodge number $h^{p,q} := \dim H^{p,q}$.

Let $Z$ be any quasi-projective algebraic variety (possibly non-smooth or non-compact). The cohomology groups $H^k(Z)$ and the cohomology groups with compact support $H^k_c(Z)$ are endowed with mixed Hodge structures [3]. We define the Hodge numbers of $Z$ by $h^{p,q}_c(Z) = h^{p,q}(H^k_c(Z)) = \dim Gr^p_F Gr^{W}_{p+q} H^k_c(Z)$. The $E$-polynomial is defined as

$$e(Z) := \sum_{p,q,k} (-1)^k h^{p,q}_c(Z) u^p v^q.$$

The key property of Hodge-Deligne polynomials that permits their calculation is that they are additive for stratifications of $Z$. If $Z$ is a complex algebraic variety and $Z = \bigsqcup_{i=1}^n Z_i$, where all $Z_i$ are locally closed in $Z$, then $e(Z) = \sum_{i=1}^n e(Z_i)$. Also $e(X \times Y) = e(X)e(Y)$. 

\[
\begin{pmatrix}
t I_V & 0 \\
0 & I_W
\end{pmatrix}, \text{ where } k = \dim V. \text{ Then } P_t^{-1} \rho P_t = \begin{pmatrix}
\rho_1 & 0 \\
t P_2 & \rho_2
\end{pmatrix} \to \rho' = \begin{pmatrix}
\rho_1 & 0 \\
0 & \rho_2
\end{pmatrix}, \text{ when } t \to 0. 
\]
3. The character variety for free groups

Now we focus on the case of a free group. Let \( \Gamma = F_k := \langle x_1, x_2, \ldots, x_k \rangle \) be the free group generated by \( k \) elements. Then, the space of representations of \( F_k \) in \( \text{SL}_r \) is just

\[
\text{Hom}(F_k, \text{SL}_r) = (\text{SL}_r)^k = \{(A_1, A_2, \ldots, A_k) | A_i \in \text{SL}_r\},
\]

the space of \( k \)-tuples of matrices in \( \text{SL}_r \). The moduli space of \( k \)-tuples of matrices up to conjugation is

\[
\mathfrak{M}(F_k, \text{SL}_r) = (\text{SL}_r)^k \sslash \text{SL}_r.
\]

As we said in Section 2, this is isomorphic to the character variety \( \mathcal{X}(F_k, \text{SL}_r) \). This implies that there are finitely many \( g_1, \ldots, g_a \in F_k \) such that a character \( \chi_\rho \in \mathcal{X}(F_k, \text{SL}_r) \) is determined by \( \chi_\rho(g_1), \ldots, \chi_\rho(g_a) \). Denote \( \rho(x_i) = A_i \in \text{SL}_r \), and also

\[
\text{tr}(A) = t_A
\]

for the trace of a matrix. For an element \( g_j = x_{i_{j_1}} \cdots x_{i_{j_l}} \in F_k \), we have

\[
\chi_\rho(g) = \text{tr}(\rho(g)) = \text{tr}(\rho(x_{i_{j_1}}) \cdots \rho(x_{i_{j_l}})) = \text{tr}(A_{i_{j_1}} \cdots A_{i_{j_l}}) = t_{A_{i_{j_1}} \cdots A_{i_{j_l}}}.
\]

This implies that \( \mathcal{X}(F_k, \text{SL}_r) \) is parametrized by the above traces for \( j = 1, \ldots, a \), that is

\[
\begin{align*}
\mathfrak{M}(F_k, \text{SL}_r) &\rightarrow \mathcal{X}(F_k, \text{SL}_r) \subset \mathbb{k}^a \\
(A_1, \ldots, A_k) &\mapsto (t_{A_{i_{11}} \cdots A_{i_{1l_1}}}, \ldots, t_{A_{i_{a1}} \cdots A_{i_{al_a}}})
\end{align*}
\]

(4)

is a parametrization of the character variety.

**Proposition 3.1.** If \( k \geq 2 \), the dimension of the character variety \( \mathcal{X}(F_k, \text{SL}_r) \) is

\[
\dim \mathcal{X}(F_k, \text{SL}_r) = (r^2 - 1)(k - 1).
\]

If \( k = 1 \) then

\[
\dim \mathcal{X}(F_1, \text{SL}_r) = (r - 1).
\]

**Proof.** Let us assume that \( k \geq 2 \). The action of \( \text{SL}_r \) on irreducible representations has finite stabilizer, so the action has generic orbits of dimension \( \dim \text{SL}_r \). This means that

\[
\dim \mathcal{X}(F_k, \text{SL}_r) = \dim (\text{SL}_r)^k - \dim \text{SL}_r = (k - 1) \dim \text{SL}_r = (r^2 - 1)(k - 1).
\]

On the other hand, if \( k = 1 \), the character variety \( \mathcal{X}(F_1, \text{SL}_r) = \text{SL}_r \sslash \text{SL}_r \) is canonically isomorphic to \( \mathbb{k}^{r-1} \), as it is proved in [10], so it has dimension \( r - 1 \).

From now on we focus on the case of rank 2, that is, the group \( \text{SL}_2 \). We want to determine how many traces are needed in (4). First, we demonstrate some useful matrix identities.

**Lemma 3.2.** Let \( P, Q \in \text{SL}_2 \). Then the following holds

\[
QP = (t_{PQ} - t_{P}t_{Q})I + t_{P}Q + t_{Q}P - PQ.
\]

(5)
Proof. First of all, let us recall that $t_{A^{-1}} = t_A$ for every $A \in \text{SL}_2$. On the other hand, the relation $t_{AB} = t_{BA}$ holds for every pair of square matrices $A, B$ of the same size. Given $A \in \text{SL}_2$, the following relation is given by the characteristic polynomial of $A$,

$$A^2 = t_A A - I$$

and therefore, the following holds

$$A^{-1} = t_A I - A$$

By (7), we can write $(PQ)^{-1}$ as

$$Q^{-1} P^{-1} = (PQ)^{-1} = t_{PQ} I - PQ.$$ 

Applying (7) to $Q^{-1}$ and $P^{-1}$ on the above equation we obtain the following

$$(t_Q I - Q)(t_P I - P) = t_{PQ} I - PQ,$$

therefore $QP = (t_{PQ} - t_{PQ} t_Q I + t_P Q + t_Q P - PQ$, as required. \hfill \Box

**Proposition 3.3.** Let $A, B, C, P, Q \in \text{SL}_2$, then the following holds.

(i) $t_I = 2$

(ii) $t_{AB} = t_{BA}$

(iii) $t_{A^2} = t_A^2 - 2$

(iv) $t_{ABAB} = t_{AB}^2 - 2$

(v) $t_{PBAQ} = t_{PQ} t_{AB} - t_{PQ} t_A t_B + t_A t_{PBQ} + t_B t_{PAQ} - t_{PABQ}$

(vi) $t_{PA^2Q} = t_A t_{PAQ} - t_{PQ}$

(vii) $t_{PA^{-1}Q} = t_A t_{PQ} - t_{PAQ}$

(viii) $t_{ABC} = t_A t_{BC} + t_{BT} t_{AC} + t_C t_{AB} - t_A t_B t_C - t_{ABC}$

Proof. (i) and (ii) Immediate.

(iii) Since $A \in \text{SL}_2$, the result follows from Equation (6) taking traces.

(iv) It follows from (iii) by just observing that $ABAB = (AB)^2$.

(v) Use the formula of Lemma 3.2 multiplying in the left by $P$ and in the right by $Q$, to get

$$PBAQ = P((t_{AB} - t_A t_B) I + t_A B + t_B A - AB)Q$$

and take traces to obtain the sought formula.

(vi) Start with Equation (6), and multiply in the left by $P$ and in the right by $Q$, to get

$$PA^2 Q = t_{PAQ} Q - PQ.$$

Finally take traces we get the required formula.

(vii) From Equation (7), we get

$$PA^{-1} Q = t_{PAQ} - PQ,$$

and take traces.

(viii) In (v), take $P = I, Q = C$ to get $t_{BAC} = t_C t_{AB} - t_C t_A t_B + t_A t_{BC} + t_B t_{AC} - t_{ABC}$, which rewrites as the above. \hfill \Box

**Theorem 3.4.** Let $A_1, \ldots, A_k \in \text{SL}_2$, take a monomial $x = A_{i_1}^{r_1} \cdots A_{i_m}^{r_m}$, with $r_j \in \mathbb{Z}$ and $1 \leq i_1, \ldots, i_m \leq k$. Then $t_x$ has a (polynomial) expression in terms of

$$T_{i_1 \ldots i_p} := t_{A_{i_1} \cdots A_{i_p}}, \quad 1 \leq i_1 < \ldots < i_p \leq k.$$
Therefore the ring of functions of $\mathcal{X}_k = \mathfrak{X}(F_k, SL_2) = \text{Spec} \mathcal{O}_{\mathcal{X}_k}$ is given as

$$\mathcal{O}_{\mathcal{X}_k} = \mathbb{k}[\{T_{i_1, \ldots, i_p}\}_{1 \leq i_1 < \ldots < i_p \leq k}] / \mathcal{I},$$

for some ideal $\mathcal{I}$ of relations.

**Proof.** If $r_j < 0$, we use Proposition 3.3(vii) to write $t_x$ in terms of traces of monomials in which $r_j \geq 0$. Now, if $r_j \geq 2$, we use Proposition 3.3(vi) to write $t_x$ in terms of traces of monomials in which $r_j$ is smaller. Repeating we can reach an expression with $r_j = 0, 1$. Doing this for all indices, we finally get a polynomial expression in terms of $A_{i_1, \ldots, i_p}, 1 \leq i_1, \ldots, i_p \leq k$, where $i_j \neq i_{j+1}$. That is, in the monomial two consecutive matrices are distinct.

Now suppose that $i_j > i_{j+1}$. Then we use Proposition 3.3(v) to get an expression in which the traces appearing have either less number of matrices, or $A_{i_j}, A_{i_{j+1}}$ are swapped. In the first case, we can work by induction on the number of matrices involved to get to the result (note that the other operations do not increase the number of matrices in a given monomial). In the second case, now we get an expression $PA_{i_{j+1}}A_{i_j}Q$ with $i_{j+1} < i_j$. If now there are two consecutive matrices repeated (that is, a square), we use Proposition 3.3(vi) again. Otherwise, we have managed to reorder two matrices. We can permute the matrices with this process until $i_1$ is the lowest index, so that $i_1 < i_2, \ldots, i_p$. We continue in this fashion until $i_1 < i_2 < \ldots < i_p$. □

The number of monomials of the form $A_{i_1, \ldots, i_p}$ with $1 \leq i_1 < \ldots < i_p \leq k$ described in Theorem 3.4 is

$$\sum_{p=1}^{k} \binom{k}{p} = 2^k - 1.$$  

**Corollary 3.5.** For $k = 2$, the character variety $\mathcal{X}_2 = \mathfrak{X}(F_2, SL_2)$ is isomorphic to $\mathbb{k}^3$, and it is parametrized by $(t_A, t_B, t_{AB})$, for $(A, B) \in \mathcal{X}_2$.

**Proof.** We have by Proposition 3.1 that $\dim \mathfrak{X}(F_2, SL_2) = 3$. By Theorem 3.4, the traces $t_A, t_B, t_{AB}$ parametrize. Therefore the result. □

### 4. Equation of the character variety $\mathfrak{X}(F_3, SL_2)$

Since a general algorithm to compute the ideal $\mathcal{I}$ described in Theorem 3.4 is unknown, let us start by looking at the free group generated by three elements $F_3$. The aim of this section is to study the character variety

$$\mathcal{X}_3 = \mathfrak{X}(F_3, SL_2) = \{(A, B, C) | A, B, C \in SL_2 \} // SL_2.$$  

By Proposition 3.1 we have that $\dim \mathfrak{X}(F_3, SL_2) = 6$. By Theorem 3.4, the traces $t_A, t_B, t_{C}, t_{AB}, t_{AC}, t_{BC}, t_{ABC}$ generate the ring of functions of $\mathfrak{X}(F_3, SL_2)$. These are 7 variables, hence there is an embedding

$$\mathfrak{X}(F_3, SL_2) \subset \mathbb{k}^7,$$
and the character variety is a hypersurface defined by a single equation. To find such equation, we work as follows. For the sake of clarity, let us set the following variables

\[
x = t_A, \quad y = t_B, \quad z = t_C \\
u = t_{BC}, \quad v = t_{AC}, \quad w = t_{AB} \\
P = t_{ABC}
\]

Now we complete Theorem 1.2

**Theorem 4.1.** The character variety \(X_3 \subset \mathbb{k}^7\) is a hypersurface defined by the equation

\[
P^2 = (wz + vy + ux - xyz)P - x^2 - y^2 - z^2 + uyz + vzx + wxy - uwv - u^2 - v^2 - w^2 + 4.
\]

**Proof.** Since \(A, B, C \in \text{SL}_2\), by Proposition 3.3(iv), the following holds

\[
t_{ABCABC} = t_{ABC}^2 - 2
\]

Then

\[
t_{ABCABC} = t_{ABBC}(t_{AC} - t_{AtC}) + t_{AtBC} + t_{ctABABC} - t_{ABACBC} \quad \text{(Prop. 3.3(v))}
\]

\[
\begin{aligned}
&= t_{ABBC}(t_{AC} - t_{AtC}) + t_A(t_{ABBC} - t_Bt_{ABC}) + t_Bt_{ABC} + t_{ctABABC} - t_{ABBCC} \\
&\quad + t_C(t_{AB} - t_{AtB}) + t_{AtABC} + t_{ABBC} - t_{AABBCC} \\
&\quad - (t_{ABBC}(t_{AB} - t_{AtB}) + t_Bt_{ABC} + t_{ABBC} - t_{AABBCC}) \quad \text{(Prop. 3.3(v))}
\end{aligned}
\]

\[
= -z^2 - x^2 + vzx - v^2 + (wz + vy + ux - xyz)P \\
- (t_{ACBC}t_{AB} - t_Bt_{BC} + t_{BCBC}) \\
= -z^2 - x^2 + vzx - v^2 + (wz + vy + ux - xyz)P \\
- \left((t_{AC}(t_{BC} - t_Bt_C) + t_{ctABC} + t_Bt_{ACC} - t_{ABC})t_{AB} \\
\quad - t_B(t_c(t_{BC} - t_Bt_C) + t_{BC} + t_{ctBC} - t_{BC}) \quad \text{(Prop. 3.3(v))}
\end{aligned}
\]

\[
+ t_{BC}^2 - 2 \right)
\]

\[
= -z^2 - x^2 + vzx - v^2 + (wz + vy + ux - xyz)P \\
- \left((t_{AC}(t_{BC} - t_Bt_C) + t_{ctABC} + t_B(t_{ACtC} - t_A) - t_{ABctC} + t_{AB})t_{AB} \\
\quad - t_B(t_c(t_{BC} - t_Bt_C) + t_{BC}^2 - 2 + t_{ctBC} - t_{BCtC} + t_B) \quad \text{(Prop. 3.3(vi))}
\end{aligned}
\]

\[
+ t_{BC}^2 - 2 \right)
\]

\[
= -x^2 - y^2 - z^2 + uyz + vzx + wxy - uwv \\
- u^2 - v^2 - w^2 + 2 + (wz + vy + ux - xyz)P.
\]
Theorem 4.1 can be rewritten as the following equality with traces for triple of matrices $A, B, C \in \text{SL}_2$,

$$
t_{ABC}^2 = (t_At_BC + t_Bt_AC + t_Ct_AB - t_At_Bt_C)t_{ABC} - t_A^2 - t_B^2 - t_C^2 + t_{ABC}t_{ABC} - t_{ABC}^2 - t_{ABC}^2 - t_{ABC}^2 - t_{ABC}t_{ABC} + 4.
$$

(10)

Corollary 4.2. The variety $X_3 = X(F_3, \text{SL}_2)$ is a ramified double cover of the plane $k^6$. The variables $t_A, t_B, t_C, t_{AB}, t_{AC}, t_{BC}$ are transcendental generators and the ring of functions $\mathcal{O}_{X_3}$ is a degree 2 extension of $k[t_A, t_B, t_C, t_{AB}, t_{AC}, t_{BC}]$.

By Theorem 4.1, $X_3$ is a double cover over $k^6$ ramified over $V(\Delta)$, where $\Delta = Y^2 - 4X$ is the discriminant, with $X, Y$ defined as in (12). This is a sextic in $k^6$. The singularities of $X_3 \subset k^7$ are at the points $(x, y, z, u, v, w) \in V(\Delta)$, $P = \frac{1}{2}X$, which are singular points of $V(\Delta)$. The singular locus of $X_3$ is determined in [5], and it is equal to the reducible locus $X_3^{\text{red}}$, which consists of representations $(A, B, C)$ which can be put, in a suitable basis as

$$
A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \quad B = \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix}, \quad C = \begin{pmatrix} \nu & 0 \\ 0 & \nu^{-1} \end{pmatrix}.
$$

This is a 3-dimensional subspace. Equivalently $A, B, C$ are pairwise commuting, which by Equation (15), it amounts to the equations

$$
x^2 + y^2 + w^2 = xyw - 2, \quad x^2 + z^2 + v^2 = xzv - 2, \quad y^2 + z^2 + u^2 = yzu - 2, \quad P = \frac{1}{2}X.
$$

5. Generators of the ring of $X(F_k, \text{SL}_2)$ for $k \geq 4$

Now we give an expression for the trace of matrices that extends Theorem 3.1 to products of more than 4 matrices.

Theorem 5.1. Let $A, B, C, D \in \text{SL}_2$. The trace of $ABCD$ can be expressed as a polynomial expression in terms of $t_A, t_B, t_C, t_{AB}, t_{AC}, t_{AD}, t_{BD}, t_{CD}, t_{ABC}, t_{ABD}, t_{ACD}$ and $t_{BCD}$. More specifically,

$$
t_{ABCD} = \frac{1}{2} \left( t_A t_{BCD} + t_B t_{ACD} + t_C t_{ABD} + t_D t_{ABC} + t_{AD} t_{BCD} - t_{ACT} t_{BD} + t_{ABC} t_{CD} - t_{ABCD} t_{AT} + t_{AB} t_{BCD} \right).
$$

(11)

Proof. By Proposition 3.3(v), we have that

$$
t_{ABCD} = t_{AD} t_{BC} - t_{Bt_C} + t_{Bt_ACD} + t_{Ct_ABD} - t_{ACBD}
$$

$$
t_{ACBD} = t_{CBDA} = t_{CA} (t_{BD} - t_{Bt_D}) + t_{BT} CDA + t_{D} t_{CBA} - t_{CDBA}
$$

$$
t_{CDBA} = t_{DBCA} = t_{DC} (t_{BA} - t_{Bt_A}) + t_{BT} DBC + t_{A} t_{DBC} - t_{DABC}
$$

Substituting the expression of each equation into the previous one, and using the cyclicity of the traces, namely $t_{DABC} = t_{ABCD}$, we obtain:

$$
t_{ABCD} = t_{AD} (t_{BC} - t_{Bt_C}) + t_{Bt_ACD} + t_{Ct_ABD} - (t_{CA} (t_{BD} - t_{Bt_D})
$$

$$
+ t_{BT} CDA + t_{D} t_{CBA}) + t_{DC} (t_{BA} - t_{Bt_A}) + t_{BT} DBC + t_{A} t_{DBC} - t_{ABCD},
$$
and hence
\[ t_{ABCD} = \frac{1}{2} \left( t_{AD}(t_{BC} - t_{BT}) + t_{BT}t_{AD} + t_{CT}t_{ABD} - (t_{AC}(t_{BD} - t_{BT}) + t_{BT}t_{ACD} \right. \\
+ t_{CD}(t_{AB} - t_{ATB}) + t_{BT}t_{ACD} + t_{ATBCD}) \right) \\
= \frac{1}{2} \left( t_{AD}(t_{BC} - t_{BT}) + t_{BT}t_{AD} + t_{CT}t_{ABD} - t_{AC}(t_{BD} - t_{BT}) - t_{BT}t_{ACD} \right.

\left. - t_D(t_{AT} + t_{BT} + t_{CT} - t_{ATBC} - t_{ABCD}) + t_{CD}(t_{AB} - t_{ATB}) \right. \\
\left. + t_{BT}t_{ACD} + t_{ATBCD}) \right) \quad \text{(Prop. 3.3 (viii))} \]
\[ = \frac{1}{2} \left( t_{AT}t_{BCD} + t_{BT}t_{AD} + t_{CD}t_{ABD} + t_{DT}t_{ABC} - t_{AC}t_{BD} + t_{ABCD} \right.

\left. - t_{AT}t_{BC} - t_{BT}t_{CD} - t_{AT}t_{CD} - t_{CD}t_{AT} + t_{AT}t_{CD}D \right). \quad \text{(Simplifying)} \]

As a consequence, the ring of functions of \( \mathfrak{X}_k = \mathfrak{X}(F_k, \text{SL}_2) \) is generated by traces of the product of at most three matrices. This completes the proof of Theorem 1.1.

**Corollary 5.2.** Let \( A_1, \ldots, A_k \in \text{SL}_2 \), take a monomial \( x = A_{i_1}^{r_1} \cdots A_{i_m}^{r_m} \), with \( 1 \leq i_1, \ldots, i_m \leq k \) and \( r_j \in \mathbb{Z} \). Then \( t_x \) has a (polynomial) expression in terms of
\[ T_{i_1 \cdots i_p} := t_{A_{i_1} \cdots A_{i_p}}, 1 \leq i_1 < \cdots < i_p \leq k, \]
with \( p \leq 3 \). Therefore the ring
\[ \mathcal{O}_x = k \left[ \{ T_{i_1 \cdots i_p} \}_{1 \leq i_1 < \cdots < i_p \leq k, 1 \leq p \leq 3} \right] / \mathcal{I}, \]
for some ideal \( \mathcal{I} \) of relations.

**Proof.** By Theorem 3.4 \( T_{i_1 \cdots i_p} \) with \( 1 \leq p \leq k \), gives generators of the ring of all \( t_x \). Now by Theorem 5.1 \( t_{A_{i_1} \cdots A_{i_4}} \) is expressible in terms of all traces of one, two and three matrices among \( A_{i_1}, \ldots, A_{i_4} \). In general, for \( p \geq 4 \),
\[ t_{A_{i_1} \cdots A_{i_p}} = t_{A_{i_1}A_{i_2}A_{i_3}(A_{i_4} \cdots A_{i_p})} \]
is expressible in terms of the traces of products of one, two and three matrices among \( A_{i_1}, A_{i_2}, A_{i_3} \), and \( Q := A_{i_4} \cdots A_{i_p} \). These are traces of products of less than \( p - 1 \) matrices. By induction, we get the result. \( \square \)

In virtue of Corollary 5.2 the number of generators of the ring \( \mathcal{O}_{\mathfrak{X}_k} \) is
\[ k + \binom{k}{2} + \binom{k}{3}. \]

If we look at the case \( k = 4 \), to parametrize
\[ \mathfrak{X}_4 = \mathfrak{X}(F_4, \text{SL}_2) = \{(A, B, C, D) | A, B, C, D \in \text{SL}_2\} \parallel \text{SL}_2, \]
Corollary 5.2 says that we need the traces
\[ t_A, t_B, t_C, t_D, t_{AB}, t_{AC}, t_{AD}, t_{BC}, t_{BD}, t_{CD}, t_{ABC}, t_{ABD}, t_{ACD}, t_{BCD} \]
giving an embedding \( \mathcal{X}(F_1, \text{SL}_2) \subset k^{14} \). By Proposition 3.3, \( \dim \mathcal{X}(F_1, \text{SL}_2) = 9 \), so 5 of the above traces are algebraically dependent of the others. Letting

\[
X(x, y, z, u, v, w) := wz + vy + ux - xyz,
Y(x, y, z, u, v, w) := -x^2 - y^2 - z^2 + uy + vx + wxy - uuv - u^2 - v^2 - w^2 + 4,
\]

we have an equation

\[
t_{ABCD}^2 = X(t_A, t_B, t_C, t_{BC}, t_{AC}, t_{AB}) \cdot t_{ABC} + Y(t_A, t_B, t_C, t_{BC}, t_{AC}, t_{AB}),
\]

and similarly for the others \( t_{ABD}, t_{ACD}, t_{BCD} \). This gives four algebraic dependent variables.

**Proposition 5.3.** The trace \( t_{CD} \) is algebraically dependent with \( t_A, t_B, t_C, t_D, t_{AB}, t_{AC}, t_{BC}, t_{AD} \) and \( t_{BD} \). Therefore \( t_A, t_B, t_C, t_D, t_{AB}, t_{AC}, t_{BC}, t_{AD}, t_{BD} \) are transcendental generators of \( \mathcal{O}_{\mathcal{X}_1} \).

**Proof.** Clearly there is an algebraic dependence relation between all these variables, as the transcendental degree of the field that they generate is 9. The variables \( t_A, t_B, t_C, t_D \) are clearly algebraically independent. Therefore, there is one of the other variables that depends algebraically on the rest. Permuting the order of the matrices, we can assume that it is \( t_{CD} \).

\[ \square \]

It is not easy to find out an explicit algebraic equation satisfied by \( t_{CD} \) in Proposition 5.3. This can be done as follows. Consider the element \( t_{ABCD} = t_{(AB)CD} \) and apply Equation (10) to get

\[
t_{ABCD}^2 = X(t_A, t_B, t_{CD}, t_{ABD}, t_{ACD}, t_{ABC})t_{ABCD} + Y(t_A, t_B, t_C, t_{BCD}, t_{ABD}, t_{ABC}),
\]

with the expressions \( X, Y \) appearing in (12). Now use Equation (11) to substitute \( t_{ABCD} \) in the above. This gives an equation involving \( t_A, \ldots, t_{CD}, t_{ABD}, t_{ACD}, t_{(AB)CD} \). Using Theorem 4.1, we have algebraic equations for \( t_{ABCD}, \ldots, t_{BCD} \) in terms of the traces \( t_A, t_B, t_C, t_D, t_{AB}, t_{AC}, t_{AD}, t_{BC}, t_{BD}, t_{CD} \). This will yield an equation involving all required traces. Note that we can also work out an equation like (14) for \( t_{ABCD} = t_{(AB)CD} \) or \( t_{ABCD} = t_{(AB)CD} \). This can serve to eliminate \( t_{ABCD} \).

**Corollary 5.4.** In \( \mathcal{X}_k = \mathcal{X}(F_k, \text{SL}_2) \), we have parameters for \( (A_1, \ldots, A_k) \) given by \( t_{A_i A_j}, t_{A_i A_j A_k}, i < j < k \). The parameters

\[
t_{A_1 A_2}, t_{A_1 A_2}, t_{A_1 A_2}, \ldots, t_{A_j A_1 A_j}, t_{A_1 A_2}, \ldots, j \geq 3,
\]

are transcendental generators of \( \mathcal{O}_{\mathcal{X}_k} \).

**Proof.** First, by Proposition 3.1 the dimension of \( \mathcal{X}_k \) is \( 3(k - 1) \). Now \( t_{A_1}, t_{A_2}, t_{A_1 A_2} \) generate \( \mathcal{O}_{\mathcal{X}_2} \) by Corollary 3.5. For \( k = 3 \), Corollary 4.2 says that \( t_{A_1}, t_{A_2}, t_{A_3}, t_{A_1 A_2}, t_{A_1 A_3}, t_{A_2 A_3} \) are transcendental generators of \( \mathcal{O}_{\mathcal{X}_3} \). For \( k \geq 4 \), we use Proposition 5.3 applied to \( (A_1, A_2, A_i, A_j) \) to get an algebraic equation of \( t_{A_i A_j} \) in terms of \( t_{A_1}, t_{A_2}, t_{A_1 A_j}, t_{A_1 A_2}, t_{A_1 A_3}, t_{A_2 A_3}, t_{A_1 A_4}, t_{A_3 A_4}, t_{A_2 A_4} \). Therefore the given set of traces are transcendental generators. There cannot be less than they are because \( \dim \mathcal{X}_k = 3k - 3 \), which is the number of parameters in the list.

\[ \square \]
6. Character variety of the 2-torus

Now we are going to focus on the 2-torus $T^2$ and the space of representations of its fundamental group $\Gamma = \pi_1(T^2) = \langle x, y \mid [x, y] = 1 \rangle$ in $\text{SL}_2$. By the general description in [2], we have that the character variety of a finitely generated group embeds as a subvariety of the character variety of the free group $F_k$, where $k$ is the number of generators of the group. In this situation,

$$\mathcal{X}_{T^2} = \mathcal{X}(T^2, \text{SL}_2) \subset \mathcal{X}(F_2, \text{SL}_2) = \mathbb{k}^3,$$

the last equality by Corollary [3.5] Then $\mathcal{X}_{T^2}$ is parametrized by $(t_A, t_B, t_{AB})$, and there will be an equation describing this variety. To find it, we work out a relation for the trace of a commutator.

**Lemma 6.1.** For matrices $A, B \in \text{SL}_2$, we have

$$t_{[A,B]} = t_A^2 + t_B^2 + t_{AB}^2 - t_A t_B t_{AB} - 2. \quad (15)$$

**Proof.** We compute

$$A A^{-1} B^{-1} = A (t_A I - A) B^{-1} \quad \text{(by Eqn. (7))}$$

$$= t_A A - A B A B^{-1} \quad \text{(by Eqn. (8))}$$

$$= t_A A - (t_{AB} - t_A t_B) I + t_A B + t_B A - A B B^{-1} + A^2 \quad \text{(Eqn. (9))}$$

$$= - (t_{AB} - t_A t_B) A (t_B I - B) - t_B (t_A A - I) (t_B I - B) + t_A A - I$$

$$= - t_{AB} t_B A + t_A t_B B + t_A A - I$$

Taking traces,

$$t_{[A,B]} = - t_{AB} t_B A + t_A A - I - t_B B + t_A A - I.$$

producing the result. □

From now on, we fix the ground field $k = \mathbb{C}$. Take a conjugacy class $[\xi] = \text{SL}_2 \cdot \xi$ determined by an element $\xi \in \text{SL}_2$ (the action of $\text{SL}_2$ by conjugation). We have the *twisted* moduli space of representations as defined in [1],

$$\mathcal{M}_\xi = \{ (A, B) \in (\text{SL}_2)^2 \mid [A, B] \in [\xi] \} \setminus \text{SL}_2.$$

There are five different types of conjugacy classes. We have $[I], [-I], [J_+], [J_-]$ and $[\xi]$, where $J_\pm = \left( \begin{array}{cc} \pm 1 & 0 \\ 1 & \pm 1 \end{array} \right)$ are the Jordan types, and $\xi_t = \left( \begin{array}{cc} \lambda & 0 \\ 0 & \lambda^{-1} \end{array} \right)$, $t = \lambda + \lambda^{-1}$, $\lambda \in \mathbb{C} - \{0, \pm 1\}$, are the diagonal types. Consider the trace map

$$\text{tr} : \text{SL}_2 \to \mathbb{C}$$

and note that

$$\text{tr}^{-1}(t) = [\xi_t], \quad t \in \mathbb{C} - \{\pm 2\},$$

$$\text{tr}^{-1}(2) = [I] \sqcup [J_+],$$

$$\text{tr}^{-1}(-2) = [-I] \sqcup [J_-].$$
Using the variables \( x = t_A, y = t_B, z = t_{AB} \), Lemma 6.1 gives the function

\[
F(x, y, z) = t_{[A,B]} = x^2 + y^2 + z^2 - xyz - 2.
\]

Then we have the following twisted character varieties:

\[
\mathfrak{X}_t = F^{-1}(t) = \{(A, B) \in \text{SL}_2 \mid \text{tr}([A, B]) = t\} / \text{SL}_2,
\]

for \( t \in \mathbb{C} \). Then

- \( \mathfrak{X}_t = \mathcal{M}_{\xi_t} \), for \( t \neq \pm 2 \),
- \( \mathfrak{X}_2 = \mathcal{M}_I \cup \mathcal{M}_{J_1} \),
- \( \mathfrak{X}_{-2} = \mathcal{M}_{-I} \cup \mathcal{M}_{J_-} \).

**Remark 6.2.** Note the symmetry of equation (16). This is given by the change of generators \( (A, B) \mapsto (AB, B^{-1}) \), that changes \( (t_A, t_B, t_{AB}) \mapsto (t_{AB}, t_B, t_A) \).

We study the geometry of the character varieties \( \mathfrak{X}_t \), and recover results of Theorem 1.3. Now we prove Theorem 1.3.

**Theorem 6.3.** Let \( t \in \mathbb{C} \). We have the following:

- For \( t \neq \pm 2 \), the character variety \( \mathfrak{X}_t \subset \mathbb{C}^3 \) is a smooth surface. The E-polynomial of \( \mathfrak{X}_t \) is \( e(\mathfrak{X}_t) = q^2 + 4q + 1 \).
- For \( t = 2 \), the character variety \( \mathfrak{X}_2 \subset \mathbb{C}^3 \) has 4 ordinary double points. \( \mathcal{M}_{J_1} \subset \mathcal{M}_I \), \( \mathfrak{X}_2 = \mathcal{M}_I \), and \( e(\mathfrak{X}_2) = q^2 + 1 \).
- For \( t = -2 \), the character variety \( \mathfrak{X}_{-2} \subset \mathbb{C}^3 \) has only one singular point which is an ordinary double point. Now \( \mathfrak{X}_{-2} = \mathcal{M}_{-I} \cup \mathcal{M}_{J_-} \), and \( e(\mathfrak{X}_{-2}) = q^2 + 3q + 1 \).

**Proof.** We start analysing the singular points of \( \{F(x, y, z) = t\} \). We compute the derivatives of \( F \),

\[
\left( \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right) = (2x - yz, 2y - xz, 2z - xy).
\]

For a singular point \( 2x = yz, 2y = xz, 2z = xy \). From this, we get \( x^2 = y^2 = z^2 = \frac{1}{2}xyz \) and hence \( F = \frac{1}{2}xyz - 2 \).

- For \( t = -2 \), we have \( x^2 = y^2 = z^2 = \frac{1}{2}xyz = 0 \), so there is a singular point \( (x, y, z) = (0, 0, 0) \). The leading term of \( F \) is \( x^2 + y^2 + z^2 \), hence the point is an ordinary double point.
- For \( t = 2 \), we have \( x^2 = y^2 = z^2 = \frac{1}{2}xyz = 4 \). Therefore, the singular points are \( (2, 2, 2), (2, -2, -2), (-2, 2, -2) \) and \( (-2, -2, 2) \). Let us focus on one of them, say \( (2, 2, 2) \), then the Hessian of \( F \) is

\[
H_F(2, 2, 2) = \begin{pmatrix}
2 & -z & -y \\
-z & 2 & -x \\
-y & -x & 2
\end{pmatrix}_{(2,2,2)} = \begin{pmatrix}
2 & -2 & -2 \\
-2 & 2 & -2 \\
-2 & -2 & 2
\end{pmatrix},
\]

which is non-degenerate, hence it is an ordinary double point. The other singular points are similar.
- For \( t \neq \pm 2 \), we have \( x^2 = y^2 = z^2 = \frac{1}{2}xyz = 2 + t \neq 0 \). Then \( x = \pm y, x = \pm z \), hence \( 2x = yz = \pm x^2 \), and \( x = \pm 2 \). This implies that \( x^2 = 4 = 2 + t \), hence \( t = 2 \). So for \( t \neq \pm 2 \), the surface \( \mathfrak{X}_t \) is smooth.
To proceed, consider the completion of $V(F - t) \subset \mathbb{P}^3$. This is given by the homogeneous polynomial

$$\tilde{F}_t = x^2u + y^2u + z^2u - xyz - (2 + t)u^2,$$

for projective coordinates $[x, y, z, u]$. We compute the derivatives

$$\left( \frac{\partial \tilde{F}_t}{\partial x}, \frac{\partial \tilde{F}_t}{\partial y}, \frac{\partial \tilde{F}_t}{\partial z}, \frac{\partial \tilde{F}_t}{\partial u} \right) = (2xu - yz, 2yu - xz, 2zu - xy, x^2 + y^2 + z^2 - 3(2 + t)u^2),$$

and look at a point at infinity, that is, $u = 0$. Then the above derivatives reduce to $(-yz, -xz, -xy, x^2 + y^2 + z^2)$. This cannot vanish, because for this to be zero, two coordinates should vanish, and hence $x^2 + y^2 + z^2 \neq 0$. This means that $V(\tilde{F}_t)$ is smooth at the points at infinity.

Now take $t \neq \pm 2$. Then $V = V(\tilde{F}_t) \subset \mathbb{P}^3$ is a smooth surface of degree 3. By [12, Example 9.11], we have the Hodge numbers of $V$ to be $h^{1,0} = h^{0,1} = 0$, $h^{2,0} = h^{0,2} = 0$ and $h^{1,1} = 7$. Hence, the $E$-polynomial is $e(V) = q^2 + 7q + 1$, where $q = uv$. Now, the intersection $V_\infty := V \cap \{ u = 0 \} = \ell_1 \cup \ell_2 \cup \ell_3$, consists of 3 lines, which has $E$-polynomial $e(V_\infty) = \sum e(\ell_i) - \sum e(\ell_i \cap \ell_j) = 3(q + 1) - 3 = 3q$. Therefore

$$e(X_t) = e(V) - e(V_\infty) = q^2 + 4q + 1.$$

Denote $f : SL_2^2 \to SL_2, f(A, B) = [A, B]$. As in [13, Section 4] we denote $X_0 = f^{-1}(I)$, $X_1 = f^{-1}(-I)$, $X_2 = f^{-1}(\{ J_+ \})$, $X_3 = f^{-1}(\{ J_- \})$, so that $f^{-1}(\text{tr}^{-1}(2)) = X_0 \cup X_2$ and $f^{-1}(\text{tr}^{-1}(-2)) = X_1 \cup X_3$. By [13, Section 4.3], the representations of

$$\tilde{X}_2 = \left\{ (A, B) \left| (A, B) = J_+ = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right. \right\}$$

are of the form $A = \begin{pmatrix} a & 0 \\ b & a^{-1} \end{pmatrix}$ and $B = \begin{pmatrix} x & 0 \\ y & x^{-1} \end{pmatrix}$. Conjugating by $\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}$, we get the matrices

$$A_t = \begin{pmatrix} a & 0 \\ tb & a^{-1} \end{pmatrix}, B_t = \begin{pmatrix} x & 0 \\ ty & x^{-1} \end{pmatrix}, [A_t, B_t] = J_{t,+} = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}.$$ 

Taking $t \to 0$, we get in the limit representations in $X_0$. Therefore $X_2$ is contained in the closure of $X_0$. This implies that $\mathcal{M}_{J_+} \subset \mathcal{M}_I$. Hence $\tilde{X}_2 = \mathcal{M}_I$.

To compute the $E$-polynomial note that $X = V(\tilde{F}_2) \subset \mathbb{P}^3$ appears as a degeneration of $V(\tilde{F}_t)$ when $t \to 2$. Such degeneration produces 4 singularities which are ordinary double points. Each of them reduces the Betti number $b_2$ by one, hence $b_2(X) = 3$. Therefore the $E$-polynomial of $X$ is $e(X) = q^2 + 3q + 1$. Now

$$e(X_2) = e(X) - e(V_\infty) = q^2 + 1.$$

Finally, we look at the case $t = -2$. By [13, Section 4.2], $\mathcal{M}_{-I}$ consists of one point, which has representative $A = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, with $(t_A, t_B, t_{AB}) = (0, 0, 0)$, the singular point of $X_{-2}$. Regarding $\mathcal{M}_{J_-}$, according to [13, Section 4.4] the matrices of

$$\tilde{X}_3 = \left\{ (A, B) \left| (A, B) = J_- = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix} \right. \right\}$$
are of the form $A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ and $B = \begin{pmatrix} x & z \\ y & w \end{pmatrix}$ with $z = 2(x + w)$, $c = -2(a + d)$, $cy + 2dw + bz = 0$. This implies that it cannot be $(t_A, t_B) = (z, c) = (0, 0)$. So $M_{f_\cdot} = \mathcal{X}_{-2} - M_{f_\cdot} I$, and hence $\mathcal{X}_{-2} = M_{f_\cdot} \cup M_{f_\cdot} I$.

To compute the $E$-polynomial note that $Y = V(\hat{F}_{-2}) \subset \mathbb{P}^3$ appears as a degeneration of $V(\hat{F}_1)$ when $t \to -2$. Such degeneration produces one ordinary double point, hence $b_2(Y) = 6$ and $e(Y) = q^2 + 6q + 1$. Thus

$$e(\mathcal{X}_{-2}) = e(Y) - e(V_{\infty}) = q^2 + 3q + 1 = e(M_{f_\cdot}) + e(M_{f_\cdot} I).$$

\[\square\]

Note that the results of the $E$-polynomials of Theorem 6.3 agree with those of Theorem 15.3

7. Character variety of the genus 2 surface

We look at the character variety

$$\mathcal{M}_2 = \mathcal{X}(\pi_1(\Sigma_2), SL_2) = \{(A, B, C, D) \in (SL_2)^4 \mid [A, B] [C, D] = I \} \parallel SL_2 .$$

The dimension of $\mathcal{M}_g = \mathcal{X}(\pi_1(\Sigma_g), SL_2)$, for the orientable compact surface $\Sigma_g$ of genus $g$, is $\dim \mathcal{M}_g = 6g - 6$. Therefore $\dim \mathcal{M}_2 = 6$.

**Proposition 7.1.** The ring $\mathcal{O}_{\mathcal{M}_2}$ has transcendental generators $t_A, t_B, t_C, t_D, t_{AB}, t_{AC}$.

**Proof.** We have that $\mathcal{M}_2 \subset \mathcal{X}_4 = \mathcal{X}(F_4, SL_2)$ and $\dim \mathcal{X}_4 = 9$, generated by $t_A, t_B, t_C, t_D, t_{AB}, t_{AC}, t_{BC}, t_{AD}, t_{BD}$, where $t_{CD}$ is algebraically dependent on the previous ones, by Corollary 5.4. Using $[A, B] = [C, D]^{-1}$, we have $t_{[A, B]} = t_{[C, D]}$, and using Equation (15),

$$t_A^2 + t_B^2 + t_{AB}^2 - t_A t_B t_{AB} = t_C^2 + t_D^2 + t_{CD}^2 - t_C t_D t_{CD},$$

thereby $t_{CD}$ is algebraically dependent on $t_A, t_B, t_C, t_D, t_{AB}$.

Using Equation (16), we get

$$t_{[A, B]} = -t_A t_B t_{AC} + t_{AB} t_{ABC} + t_B^2 t_C - t_B t_{BC} + t_A t_{AC} - t_C$$

From $[A, B] [C, D] = I$, we rewrite $[A, B] C = D C D^{-1}$ which implies that

$$-t_A t_B t_{AC} + t_{AB} t_{ABC} + t_B^2 t_C - t_B t_{BC} + t_A t_{AC} - t_C = t_C$$

Using (13), which is an algebraic dependence of $t_{ABC}$ on $t_A, t_B, t_C, t_{AB}, t_{AC}, t_{BC}$, we get unravelling the above an algebraic equation, and then we isolate $t_{BC}$ as algebraic dependent of $t_A, t_B, t_C, t_{AB}, t_{AC}$.

Now we use the equation $[A, B] [C, D] = I$, to get $[C, D] A = B A B^{-1}$, and working as before we get an algebraic dependence of $t_{AD}$ on $t_A, t_C, t_D, t_{CD}, t_{AC}$. But using the dependence of $t_{CD}$ on $t_{AB}$, we get that $t_{AD}$ is algebraically dependent of $t_A, t_B, t_C, t_D, t_{AB}, t_{AC}$.

Finally take $[A, B] [C, D] = I$, to rewrite $D^{-1}[A, B] = CD^{-1} C^{-1}$, and work analogously to get an algebraic dependence of $t_{BD}$ in terms of $t_A, t_B, t_D, t_{AB}, t_{AD}$. Using the previous paragraph, we get that $t_{BD}$ is algebraic dependent on $t_A, t_B, t_C, t_D, t_{AB}, t_{AC}$. \[\square\]
References

[1] C. Ashley, J-P. Burelle and S. Lawton, Rank 1 character varieties of finitely presented groups, Geom. Dedicata 192 (2017) 1–19.
[2] K. Corlette, Flat G-bundles with canonical metrics, J. Diff. Geom. 28 (1988) 361–382.
[3] M. Culler and P. Shalen, Varieties of group representations and splitting of 3-manifolds, Annals Math. (2) 117 (1983) 109–146.
[4] P. Deligne, Théorie de Hodge II, Publ. Math. I.H.E.S. 40 (1971) 5–57.
[5] C. Florentino and S. Lawton, Singularities of free group character varieties, Pac. J. Math. (1) 260 (2012) 149–179.
[6] C. Florentino, A. Nozad and A. Zamora, Serre polynomials of $SL_n$- and $PGL_n$-character varieties of free groups, J. Geom. Physics 161 (2020) 104008.
[7] W. Goldman, Trace Coordinates on Fricke spaces of some simple hyperbolic surfaces, Handbook of Teichmüller theory. Vol. II, 611–684, IRMA Lect. Math. Theor. Phys., 13, Eur. Math. Soc., Zürich, 2009.
[8] N. Hitchin, The self-duality equations on a Riemann surface, Proc. London Math. Soc. (3) 55 (1987) 59–126.
[9] T. Kitano and T. Morifuji, Twisted Alexander polynomials for irreducible $SL(2,\mathbb{C})$-representations of torus knots, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 11 (2012) 395–406.
[10] S. Lawton, Minimal affine coordinates for $SL(3,\mathbb{C})$-character varieties of free groups, J. Algebra 320 (2008) 3773–3810.
[11] S. Lawton and V. Muñoz, $E$-polynomial of the $SL(3,\mathbb{C})$-character variety of free groups, Pac. J. Math. 282 (2016) 173–202.
[12] J. Lewis, A Survey of the Hodge Conjecture, CRM Monograph Series, Vol. 10, AMS, 1999, 2nd Edition.
[13] M. Logares, V. Muñoz and P.E. Newstead, Hodge polynomials of $SL(2,\mathbb{C})$-character varieties for curves of small genus, Rev. Mat. Complut. 26 (2013) 635–703.
[14] A. Lubotzky and A. Magid, Varieties of representations of finitely generated groups, Mem. Amer. Math. Soc. 58 (1985).
[15] J. Martín-Morales and A-M. Oller-Marcén, On the varieties of representations and characters of a family of one-relator subgroups, Topol. Appl. 156 (2009) 2376–2389.
[16] J. Martínez and V. Muñoz, $E$-polynomial of the $SL(2,\mathbb{C})$-character variety of a complex curve of genus 3, Osaka J. Math. 53 (2016) 645–681.
[17] J. Martínez and V. Muñoz, $E$-polynomials of the $SL(2,\mathbb{C})$-character varieties of surface groups, Internat. Math. Research Notices 2016 (2016) 926–961.
[18] V. Muñoz, The $SL(2,\mathbb{C})$-character varieties of torus knots, Rev. Mat. Complut. 22 (2009) 489–497.
[19] V. Muñoz and J. Porti, Geometry of the $SL(3,\mathbb{C})$-character variety of torus knots, Algebraic Geometric Topology 16 (2016) 397–426.
[20] C. Simpson, Moduli of representations of the fundamental group of a smooth projective variety. I, Inst. Hautes Études Sci. Publ. Math. 79 (1994) 47–129.

Departamento de Álgebra, Geometría y Topología, Facultad de Ciencias, Universidad de Málaga, Campus de Teatinos s/n, 29071 Málaga, Spain

Email address: vicente.munoz@uma.es

Departamento de Matemáticas, Facultad de Ciencias, Universidad de Salamanca, Plaza de la Merced 1, 37008 Salamanca, Spain

Email address: lemurx@usal.es