ANCIENT SOLUTIONS TO THE RICCI FLOW WITH PINCHED CURVATURE

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1. Introduction

In this note, we study ancient solutions to the Ricci flow on compact manifolds. Recall that a one-parameter family of metrics \( g(t) \) on a compact manifold \( M \) evolves by the Ricci flow if

\[
\frac{\partial}{\partial t} g = -2 \text{Ric}_g(t).
\]

A solution to the Ricci flow is called ancient if it is defined on a time interval \(( -\infty, T )\). Ancient solutions typically arise in the study of singularities to the Ricci flow (see e.g. [15], [16], [19], [20]).

P. Daskalopoulos, R. Hamilton, and N. Šešum [11] have recently obtained a complete classification of all ancient solutions to the Ricci flow in dimension 2. (See also [10], where the analogous question for the curve shortening flow is studied.) V. Fateev [12] has constructed an interesting example of an ancient solution in dimension 3. L. Ni [18] showed that any ancient solution to the Ricci flow which is of Type I, \( \kappa \)-noncollapsed, and has positive curvature operator has constant sectional curvature.

In this note, we show that any ancient solution to the Ricci flow in dimension \( n \geq 3 \) which satisfies a suitable curvature pinching condition must have constant sectional curvature. In dimension 3, we require a uniform lower bound for the Ricci tensor:

**Theorem 1.** Let \( M \) be a compact three-manifold, and let \( g(t), t \in ( -\infty, 0 ) \), be an ancient solution to the Ricci flow on \( M \). Moreover, suppose that there exists a uniform constant \( \rho > 0 \) such that

\[
\text{Ric}_{g(t)} \geq \rho \text{scal}_{g(t)} g(t) \geq 0
\]

for all \( t \in ( -\infty, 0 ) \). Then the manifold \( ( M, g(t) ) \) has constant sectional curvature for each \( t \in ( -\infty, 0 ) \).

Fateev’s example shows that the pinching condition for the Ricci tensor cannot be removed. The proof of Theorem I relies on a new interior estimate for the Ricci flow in dimension 3 (cf. Proposition 3 below). The proof of this estimate relies on the maximum principle, and will be presented in Section 2.

In dimension \( n \geq 4 \), we prove the following result:
Theorem 2. Let $M$ be a compact manifold of dimension $n \geq 4$, and let $g(t), t \in (-\infty, 0)$, be an ancient solution to the Ricci flow on $M$. Moreover, suppose that there exists a uniform constant $\rho > 0$ with the following property: for each $t \in (-\infty, 0)$, the curvature tensor of $(M, g(t))$ satisfies
\begin{align*}
R_{g(t)}(e_1, e_3, e_1, e_3) + \lambda^2 R_{g(t)}(e_1, e_4, e_1, e_4) \\
+ R_{g(t)}(e_2, e_3, e_2, e_3) + \lambda^2 R_{g(t)}(e_2, e_4, e_2, e_4) \\
- 2\lambda R_{g(t)}(e_1, e_2, e_3, e_4) \geq \rho \text{scal}_{g(t)} \geq 0
\end{align*}
for all orthonormal four-frames $\{e_1, e_2, e_3, e_4\}$ and all $\lambda \in [0, 1]$. Then the manifold $(M, g(t))$ has constant sectional curvature for each $t \in (-\infty, 0)$.

Theorem 2 again follows from pointwise curvature estimates which are established using the maximum principle (see Corollary 7 below). In dimension $n \geq 4$, the evolution equation for the curvature tensor is much more complicated, and our estimates are not as explicit as in the three-dimensional case. In order to handle the higher dimensional case, we use the invariant curvature conditions introduced in [3] and [6]. These ideas also play a key role in the proof of the Differentiable Sphere Theorem (cf. [6], [8]).

2. Proof of Theorem

Proposition 3. Let $M$ be a compact three-manifold, and let $g(t), t \in [0, T)$, be a solution to the Ricci flow on $M$. Moreover, suppose that there exists a uniform constant $\rho \in (0, 1)$ such that
\[ \text{Ric}_{g(t)} \geq \rho \text{scal}_{g(t)} g(t) \geq 0 \]
for each $t \in [0, T)$. Then, for each $t \in (0, T)$, the curvature tensor of $(M, g(t))$ satisfies the pointwise estimate
\[ |\text{Ric}_{g(t)}| \leq \left( \frac{3}{2t} \right)^{\sigma} \text{scal}^{2-\sigma}_{g(t)}, \]
where $\sigma = \rho^2$.

Proof. The assertion is trivial if $(M, g(0))$ is Ricci flat. Hence, it suffices to consider the case that $(M, g(0))$ is not Ricci flat. By the maximum principle, the manifold $(M, g(t))$ has strictly positive scalar curvature for all $t \in (0, T)$.

We next define a function $f : M \times (0, T) \to \mathbb{R}$ by
\[ f = \text{scal}^{\sigma-2} |\text{Ric}|^2, \]
where $\sigma = \rho^2$. It is easy to see that $f \leq \text{scal}^{\sigma}$. Moreover, it follows from Lemma 10.5 in [13] that
\[ \frac{\partial}{\partial t} f \leq \Delta f + \frac{2(1-\sigma)}{\text{scal}} \partial_k \text{scal} \partial^k f + 2 \text{scal}^{\sigma-3} \left[ \sigma |\text{Ric}|^2 |\text{Ric}|^2 - 2P \right], \]
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where $P$ is a polynomial expression in the eigenvalues of the Ricci tensor. By assumption, we have $\text{Ric} \geq \rho \text{scal} \ g$. Hence, it follows from Lemma 10.7 in [13] that

$$P \geq \sigma |\text{Ric}|^2 |\text{Ric}|^2.$$

This implies

$$2P - \sigma |\text{Ric}|^2 |\text{Ric}|^2 \geq \sigma |\text{Ric}|^2 |\text{Ric}|^2$$

$$\geq \frac{1}{3} \sigma \text{scal}^2 |\text{Ric}|^2$$

$$= \frac{1}{3} \sigma \text{scal}^4 - \sigma f$$

$$\geq \frac{1}{3} \sigma \text{scal}^4 - \sigma f^{1 + \frac{1}{2}}.$$

Putting these facts together, we conclude that

$$\frac{\partial}{\partial t} f \leq \Delta f + \frac{2(1 - \sigma)}{\text{scal}} \partial_k \text{scal} \partial^k f - \frac{2}{3} \frac{\partial}{\partial t} f^{1 + \frac{1}{2}}.$$

Using the maximum principle, we obtain

$$f \leq \left(\frac{3}{2t}\right)^{\sigma}.$$

This completes the proof.

**Corollary 4.** Let $M$ be a compact three-manifold, and let $g(t)$, $t \in (-\infty, 0)$, be an ancient solution to the Ricci flow on $M$. Moreover, suppose that there exists a uniform constant $\rho \in (0, 1)$ such that

$$\text{Ric}_{g(t)} \geq \rho \text{scal}_{g(t)} g(t) \geq 0$$

for each $t \in (-\infty, 0)$. Then the manifold $(M, g(t))$ has constant sectional curvature for each $t \in (-\infty, 0)$.

**Proof.** It follows from Proposition 3 that $|\text{Ric}_{g(t)}|^2 = 0$ for each $t \in (-\infty, 0)$. Therefore, the manifold $(M, g(t))$ has constant sectional curvature for each $t \in (-\infty, 0)$.

3. The higher dimensional case

In this section, we develop some general tools that will be used in proof of Theorem 2. To that end, we fix an integer $n \geq 4$. Moreover, we denote by $\mathcal{C}_B(\mathbb{R}^n)$ the space of algebraic curvature tensors on $\mathbb{R}^n$. Given any algebraic curvature tensor $R \in \mathcal{C}_B(\mathbb{R}^n)$, we define an algebraic curvature tensor $Q(R) \in \mathcal{C}_B(\mathbb{R}^n)$ by

$$Q(R)_{ijkl} = \sum_{p,q=1}^{n} R_{ijpq} R_{klpq} + 2 \sum_{p,q=1}^{n} (R_{ipkq} R_{jplq} - R_{iplq} R_{jpkq}).$$
The expression $Q(R)$ arises naturally in the evolution equation for the curvature tensor under Ricci flow (cf. \cite{14}; see also \cite{5}, Section 2.3). The ordinary differential equation $\frac{d}{dt} R = Q(R)$ on the space $\mathcal{C}_B(\mathbb{R}^n)$ will be referred to as the Hamilton ODE.

We next consider a cone $C \subset \mathcal{C}_B(\mathbb{R}^n)$. We say that the cone $C$ has property ($\ast$) if the following conditions are met:

(i) $C$ is closed, convex, and $O(n)$-invariant.

(ii) $C$ is transversally invariant under the Hamilton ODE $\frac{d}{dt} R = Q(R)$.

(iii) Every algebraic curvature tensor $R \in C \setminus \{0\}$ has positive scalar curvature.

(iv) The curvature tensor $I_{ijkl} = \delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}$ lies in the interior of $C$.

In the remainder of this section, we assume that $C \subset \mathcal{C}_B(\mathbb{R}^n)$ is a cone satisfying ($\ast$). Then $Q(R)$ lies in the interior of the tangent cone $T_RC$ for all $R \in C \setminus \{0\}$.

For each $t \in [0, \delta]$, we define a subset $F(t) \subset \mathcal{C}_B(\mathbb{R}^n)$ by

$$F(t) = \{R \in C : R + (1 - t \text{scal}(R)) I \in C\}.$$ 

Clearly, $F(t)$ is closed, convex, and $O(n)$-invariant. Moreover, $F(0) = C$.

**Lemma 5.** Suppose that $R$ is an algebraic curvature tensor on $\mathbb{R}^n$ such that $R \in C$ and $R + (1 - t \text{scal}(R)) I \in C$ for some $t \in [0, \delta]$. Then

$$Q(R) - \text{scal}(R) I - 2t |\text{Ric}(R)|^2 I$$

lies in the interior of the tangent cone to $C$ at the point $R + (1 - t \text{scal}(R)) I$.

**Proof.** If $t \text{scal}(R) < 1$, then the sum $R + (1 - t \text{scal}(R)) I$ lies in the interior of $C$. In this case, the assertion is trivial.

Hence, it suffices to consider the case $t \text{scal}(R) \geq 1$. For abbreviation, let $S = R + (1 - t \text{scal}(R)) I \in C$.

Since $t \in [0, \delta]$, we have

$$\text{scal}(S) > (1 - n(n - 1)t) \text{scal}(R) \geq \frac{1}{2} \text{scal}(R).$$

Hence, if we put

$$\alpha = \frac{t \text{scal}(R) - 1}{\text{scal}(S)},$$

then we have $0 \leq \alpha < 2t \leq \alpha_0$. Since $S \in C \setminus \{0\}$, it follows that

$$Q(S + \alpha \text{scal}(S) I - \alpha_0^2 \text{scal}(S) I \in T_SC.$$
by definition of $\alpha_0$. We next observe that
\[ S + \alpha \text{scal}(S) I = R \]
and
\[ \alpha_0^2 \text{scal}(S)^2 > \frac{\alpha_0^2}{4} \text{scal}(R)^2 \geq (1 + 2\Lambda^2) t \text{scal}(R)^2 \geq \text{scal}(R) + 2t |\text{Ric}(R)|^2. \]
Putting these facts together, we conclude that
\[ Q(R) - \text{scal}(R) I - 2t |\text{Ric}(R)|^2 I \]
lies in the interior of the tangent cone $T_S C$. This completes the proof.

**Proposition 6.** Suppose that $R(t)$ is a solution of the Hamilton ODE
\[ \frac{d}{dt} R(t) = Q(R(t)) \]
which is defined on some time interval $[t_0, t_1] \subset [0, \delta]$. If $R(t_0) \in F(t_0)$, then $R(t) \in F(t)$ for all $t \in [t_0, t_1]$.

**Proof.** By assumption, we have $R(t_0) \in C$. Since $C$ is invariant under the Hamilton ODE, we conclude that $R(t) \in C$ for all $t \in [t_0, t_1]$. Hence, it suffices to show that $R(t) + (1 - t \text{scal}(R(t))) I \in C$ for all $t \in [t_0, t_1]$. For abbreviation, let
\[ S(t) = R(t) + (1 - t \text{scal}(R(t))) I \]
for all $t \in [t_0, t_1]$. Since $R(t)$ is a solution of the Hamilton ODE, we have
\[ \frac{d}{dt} S(t) = Q(R(t)) - \text{scal}(R(t)) I - 2t |\text{Ric}(R(t))|^2 I \]
for all $t \in [t_0, t_1]$. We claim that $S(t) \in C$ for all $t \in [t_0, t_1]$. Suppose this false. We define a real number $\tau$ by
\[ \tau = \inf \{ t \in [t_0, t_1] : S(t) \notin C \}. \]
By definition of $\tau$, we have $\tau \in [0, \delta]$ and $S(\tau) \in C$. Furthermore, we have $R(\tau) \in C$. Hence, Lemma 5 implies that the derivative $\frac{d}{dt} S(t)|_{t=\tau}$ lies in the interior of the tangent cone $T_{S(\tau)} C$. By Proposition 5.4 in [5], there exists a real number $\varepsilon > 0$ such that $S(t) \in C$ for all $t \in [\tau, \tau + \varepsilon)$. This contradicts the definition of $\tau$.

**Corollary 7.** Let $\delta$ be defined as above. Moreover, let $g(t)$, $t \in [0, \delta]$, be a solution to the Ricci flow on a compact $n$-dimensional manifold $M$. Finally, we assume that the curvature tensor of $(M, g(0))$ lies in the cone $C$ for all points $p \in M$. Then
\[ R_{g(t)} + (1 - t \text{scal}_{g(t)}) I \in C \]
for all points $(p, t) \in M \times [0, \delta]$. 

Proof. By assumption, the curvature tensor of \((M, g(0))\) lies in the set \(F(0)\) for all points \(p \in M\). Using Proposition \(\boxplus\) and the maximum principle (cf. \([9]\), Theorem 3), we conclude that the curvature tensor of \((M, g(t))\) lies in the set \(F(t)\) for all points \((p, t) \in M \times [0, \delta]\). This proves the assertion.

Corollary 8. Let \(g(t), t \in (-\infty, 0)\), be an ancient solution to the Ricci flow on a compact \(n\)-dimensional manifold \(M\). Moreover, suppose that the curvature tensor of \((M, g(t))\) lies in the cone \(C\) for all \(t \in (-\infty, 0)\). Then

\[ R_{g(t)} - \delta \, \text{scal}_{g(t)} \, I \in C \]

for all points \((p, t) \in M \times (-\infty, 0)\).

Proof. Fix a time \(\tau \in (-\infty, 0)\) and a real number \(\sigma > 0\). We define a one-parameter family of metrics \(\tilde{g}(t), t \in [0, \delta]\), by

\[ \tilde{g}(t) = \sigma \, g \left( \frac{t - \delta}{\sigma} + \tau \right). \]

Clearly, the metrics \(\tilde{g}(t), t \in [0, \delta]\), form a solution to the Ricci flow. By assumption, the curvature tensor of \((M, \tilde{g}(0))\) lies in the cone \(C\) for all points \(p \in M\). Hence, it follows from Corollary 7 that

\[ R_{\tilde{g}(\delta)} + (1 - \delta \, \text{scal}_{\tilde{g}(\delta)}) \, I \in C \]

for all points \(p \in M\). This implies

\[ R_{g(\tau)} + (\sigma - \delta \, \text{scal}_{g(\tau)}) \, I \in C \]

for all points \(p \in M\). Taking the limit as \(\sigma \to 0\), we conclude that

\[ R_{g(\tau)} - \delta \, \text{scal}_{g(\tau)} \, I \in C \]

for all points \(p \in M\). Since \(\tau \in (-\infty, 0)\) is arbitrary, the assertion follows.

Theorem 9. Let \(C(s), s \in [0, 1]\), be a family of cones in \(\mathcal{C}_B(\mathbb{R}^n)\) satisfying property \((\ast)\). Moreover, suppose that the cones \(C(s)\) vary continuously in \(s\). Finally, let \(g(t), t \in (-\infty, 0)\), be an ancient solution to the Ricci flow on a compact \(n\)-dimensional manifold \(M\) such that \(R_{g(t)} \in C(0)\) for all points \((p, t) \in M \times (-\infty, 0)\). Then \(R_{g(t)} \in C(1)\) for all \((p, t) \in M \times (-\infty, 0)\).

Proof. Let \(\mathcal{S}\) denote the set of all real numbers \(s \in [0, 1]\) with the property that \(R_{g(t)} \in C(s)\) for all points \((p, t) \in M \times (-\infty, 0)\). We claim that \(\mathcal{S} = [0, 1]\).

Clearly, \(\mathcal{S}\) is closed and non-empty. We next show that \(\mathcal{S}\) is an open subset of \([0, 1]\). To that end, we fix a real number \(s_0 \in \mathcal{S}\). Then \(R_{g(t)} \in C(s_0)\) for all points \((p, t) \in M \times (-\infty, 0)\). By Corollary \(\boxplus\) there exists a real number \(\delta > 0\) such that

\[ R_{g(t)} - \delta \, \text{scal}_{g(t)} \, I \in C(s_0) \]

for all points \((p, t) \in M \times (-\infty, 0)\). Since the cones \(C(s)\) vary continuously in \(s\), there exists a real number \(\varepsilon > 0\) such that \(R_{g(t)} \in C(s)\) for all points
$$(p, t) \in M \times (-\infty, 0)$$ and all $s \in [s_0 - \varepsilon, s_0 + \varepsilon] \cap [0, 1]$. Consequently, we have $[s_0 - \varepsilon, s_0 + \varepsilon] \cap [0, 1] \subset \mathcal{J}$. This shows that $\mathcal{J}$ is an open subset of $[0, 1]$. Thus, we conclude that $\mathcal{J} = [0, 1]$, as claimed.

4. Proof of Theorem 2

We now describe the proof of Theorem 2. As in the previous section, we fix an integer $n \geq 4$. We denote by $\hat{C}$ and $\check{C}$ the cones introduced in [3] and [6]. The cone $\hat{C}$ consists of all algebraic curvature tensors $R \in \mathcal{C}_B(\mathbb{R}^n)$ satisfying

$$R(e_1, e_3, e_1, e_3) + \lambda^2 R(e_1, e_4, e_1, e_4) + R(e_2, e_3, e_2, e_3) + \lambda^2 R(e_2, e_4, e_2, e_4) - 2\lambda R(e_1, e_2, e_3, e_4) \geq 0$$

for all orthonormal four-frames $\{e_1, e_2, e_3, e_4\} \subset \mathbb{R}^n$ and all $\lambda \in [0, 1]$. Similarly, the cone $\check{C}$ consists of all algebraic curvature tensors $R \in \mathcal{C}_B(\mathbb{R}^n)$ satisfying

$$R(e_1, e_3, e_1, e_3) + \lambda^2 R(e_1, e_4, e_1, e_4) + \mu^2 R(e_2, e_3, e_2, e_3) + \lambda^2 \mu^2 R(e_2, e_4, e_2, e_4) - 2\lambda \mu R(e_1, e_2, e_3, e_4) \geq 0$$

for all orthonormal four-frames $\{e_1, e_2, e_3, e_4\} \subset \mathbb{R}^n$ and all $\lambda, \mu \in [0, 1]$. The cones $\hat{C}$ and $\check{C}$ are both invariant under the Hamilton ODE $\frac{d}{dt}R = Q(R)$. A detailed discussion of these cones can be found in [5], Chapter 7.

We next describe a family of invariant curvature cones interpolating between the cone $\hat{C}$ and the cone $\check{C}$. For each $s \in (0, \infty)$, we denote by $\check{C}(s)$ the set of all algebraic curvature tensors $R \in \mathcal{C}_B(\mathbb{R}^n)$ such that

$$R(e_1, e_3, e_1, e_3) + \lambda^2 R(e_1, e_4, e_1, e_4) + \mu^2 R(e_2, e_3, e_2, e_3) + \lambda^2 \mu^2 R(e_2, e_4, e_2, e_4) - 2\lambda \mu R(e_1, e_2, e_3, e_4) + \frac{1}{s} (1 - \lambda^2) (1 - \mu^2) \text{scal}(R) \geq 0$$

for all orthonormal four-frames $\{e_1, e_2, e_3, e_4\} \subset \mathbb{R}^n$ and all $\lambda, \mu \in [0, 1]$. Clearly, $\check{C}(s)$ is a closed, convex cone, which is invariant under the natural action of $O(n)$. Moreover, we have $\hat{C} \subset \check{C}(s) \subset \check{C}$ for each $s \in (0, \infty)$. The following result is an immediate consequence of Proposition 10 in [3]:

**Proposition 10.** For each $s \in (0, \infty)$, the cone $\check{C}(s)$ is invariant under the Hamilton ODE $\frac{d}{dt}R = Q(R)$.

**Proof.** Let us fix a real number $s \in (0, \infty)$. Moreover, let $R(t)$, $t \in [0, T)$, be a solution of the Hamilton ODE such that $R(0) \in \check{C}(s)$. We claim that $R(t) \in \check{C}(s)$ for all $t \in [0, T)$. Without loss of generality, we may assume
that \( \text{scal}(R(0)) = s \). This implies

\[
R(0)(e_1, e_3, e_1, e_3) + \lambda^2 R(0)(e_1, e_1, e_4)
+ \mu^2 R(0)(e_2, e_3, e_2, e_3) + \lambda^2 \mu^2 R(0)(e_2, e_4, e_2, e_4)
- 2\lambda \mu R(0)(e_1, e_2, e_3, e_4) + (1 - \lambda^2) (1 - \mu^2) \geq 0
\]

for all orthonormal four-frames \( \{ e_1, e_2, e_3, e_4 \} \subset \mathbb{R}^n \) and all \( \lambda, \mu \in [0, 1] \).

Hence, Proposition 10 in [3] implies that

\[
R(t)(e_1, e_3, e_1, e_3) + \lambda^2 R(t)(e_1, e_1, e_4)
+ \mu^2 R(t)(e_2, e_3, e_2, e_3) + \lambda^2 \mu^2 R(t)(e_2, e_4, e_2, e_4)
- 2\lambda \mu R(t)(e_1, e_2, e_3, e_4) + (1 - \lambda^2) (1 - \mu^2) \geq 0
\]

for all orthonormal four-frames \( \{ e_1, e_2, e_3, e_4 \} \subset \mathbb{R}^n \), all \( \lambda, \mu \in [0, 1] \), and all \( t \in [0, T) \). Since \( \text{scal}(R(t)) \geq \text{scal}(R(0)) = s \), we conclude that \( R(t) \in \hat{C}(s) \) for all \( t \in [0, T) \).

After these preparations, we now present the proof of Theorem 2.

**Theorem 11.** Assume that \( g(t), t \in (-\infty, 0) \), is an ancient solution to the Ricci flow on a compact \( n \)-dimensional manifold \( M \). Moreover, we assume there exists a uniform constant \( \rho > 0 \) such that

\[
R_g(t) - \rho \text{scal}_g(t) I \in \hat{C}
\]

for all points \((p, t) \in M \times (-\infty, 0)\). Then the manifold \((M, g(t))\) has constant sectional curvature for each \( t \in (-\infty, 0) \).

**Proof.** Consider the one-parameter family of cones \( \hat{C}(s) \), \( s \in (0, \infty) \), defined in [3]. It is shown in [3] that the cone \( \hat{C}(s) \) has property (s) for each \( s \in (0, \infty) \). Furthermore, the cones \( \hat{C}(s) \) vary continuously in \( s \).

By assumption, there exists a real number \( s_0 \in (0, \infty) \) such that \( R_g(t) \in \hat{C}(s_0) \) for all points \((p, t) \in M \times (-\infty, 0)\). Using Theorem 11 we conclude that \( R_g(t) \in \hat{C}(s) \) for all points \((p, t) \in M \times (-\infty, 0)\) and all \( s \in (0, \infty) \). Consequently, the manifold \((M, g(t))\) has constant sectional curvature for each \( t \in (-\infty, 0) \). This completes the proof of Theorem 11.

**Theorem 12.** Assume that \( g(t), t \in (-\infty, 0) \), is an ancient solution to the Ricci flow on a compact \( n \)-dimensional manifold \( M \). Moreover, we assume there exists a uniform constant \( \rho > 0 \) such that

\[
R_g(t) - \rho \text{scal}_g(t) I \in \hat{C}
\]

for all points \((p, t) \in M \times (-\infty, 0)\). Then the manifold \((M, g(t))\) has constant sectional curvature for each \( t \in (-\infty, 0) \).

**Proof.** By assumption, we have

\[
R_g(t) - \rho \text{scal}_g(t) I \in \hat{C}
\]
for all points \((p, t) \in M \times (-\infty, 0)\). Hence, we can find a real number \(s_0 \in (0, \infty)\) such that
\[
R_{g(t)} - \frac{1}{2} \rho_{g(t)} I \in \tilde{C}(s_0)
\]
for all points \((p, t) \in M \times (-\infty, 0)\).

We next consider a pair of real numbers \(a, b\) such that \(2a = 2b + (n - 2)b^2\) and \(b \in \left(0, \sqrt{\frac{2n(n-2)+4-2}{n(n-2)}}\right]\). Following [2], we define a linear transformation \(\ell_{a,b} : \mathcal{C}_B(\mathbb{R}^n) \to \mathcal{C}_B(\mathbb{R}^n)\) by
\[
\ell_{a,b}(R) = R + b \text{Ric}(R) \otimes \text{id} + \frac{1}{n} (a - b) \text{scal}(R) \text{id} \otimes \text{id},
\]
where \(\otimes\) denotes the Kulkarni-Nomizu product; see e.g. [1], Definition 1.110.

If we choose \(b \in \left(0, \sqrt{\frac{2n(n-2)+4-2}{n(n-2)}}\right]\) sufficiently small, then
\[
R_{g(t)} \in \ell_{a,b}(\tilde{C}(s_0))
\]
for all points \((p, t) \in M \times (-\infty, 0)\).

By Proposition [11] the cone \(\tilde{C}(s)\) is invariant under the Hamilton ODE for each \(s \in (0, \infty)\). Consequently, the cone \(\ell_{a,b}(\tilde{C}(s))\) is transversally invariant under the Hamilton ODE for each \(s \in (0, \infty)\) (cf. [2], Proposition 3.2). Therefore, the cone \(\ell_{a,b}(\tilde{C}(s))\) has property (*) for each \(s \in (0, \infty)\). Using Theorem [11] we conclude that \(R_{g(t)} \in \ell_{a,b}(\tilde{C}(s))\) for all points \((p, t) \in M \times (-\infty, 0)\) and all \(s \in (0, \infty)\). Taking the limit as \(s \to \infty\), we obtain \(R_{g(t)} \in \ell_{a,b}(\tilde{C})\) for all points \((p, t) \in M \times (-\infty, 0)\). Hence, it follows from Theorem [11] that \((M, g(t))\) has constant sectional curvature for each \(t \in (-\infty, 0)\).

5. Ancient solutions satisfying a diameter bound

In this final section, we study ancient solutions to the Ricci flow satisfying a suitable diameter bound. Throughout this section, we assume that \(M\) is a compact manifold of dimension \(n\), and \(g(t), \ t \in (-\infty, 0)\), is a solution to the Ricci flow on \(M\). The following proposition is a consequence of the differential Harnack inequality established in [4].

Lemma 13. Suppose that the curvature tensor of \((M, g(t))\) lies in the cone \(\tilde{C}\) for each \(t \in (-\infty, 0)\). Then
\[
\inf_M \text{scal}_{g(\tau/2)} \geq \exp \left( -\frac{\text{diam}(M, g(\tau))^2}{|\tau|} \right) \sup_M \text{scal}_{g(\tau)}.
\]
for all \(\tau \in (-\infty, 0)\).

Proof. Fix an arbitrary pair of points \(p, q \in M\). We can find a smooth path \(\gamma : [\tau, \tau/2] \to M\) such that \(\gamma(\tau) = p, \gamma(\tau/2) = q\), and
\[
|\gamma'(t)|_{g(\tau)} = \frac{2 d_{g(\tau)}(p, q)}{|\tau|}.
\]
This implies
\[ |\gamma'(t)|_{g(t)} \leq \frac{2d_{g(\tau)}(p,q)}{|\tau|} \]
for all \( t \in [\tau, \tau/2] \). Using the trace Harnack inequality (cf. [4], Proposition 13), we obtain
\[ \partial \frac{\partial}{\partial t} \text{scal} + 2 \partial_i \text{scal} v^i \geq -2 \text{Ric}(v,v) \]
for every tangent vector \( v \). Putting \( v = \frac{1}{2} \gamma'(t) \) gives
\[ \frac{d}{dt} \text{scal}_{g(t)}(\gamma(t)) \geq -\frac{1}{2} \text{Ric}_{g(t)}(\gamma'(t), \gamma'(t)) \]
\[ \geq -\frac{1}{2} \text{scal}_{g(t)}(\gamma(t)) |\gamma'(t)|_{g(t)}^2 \]
\[ \geq -\frac{2d_{g(\tau)}(p,q)^2}{|\tau|^2} \text{scal}_{g(t)}(\gamma(t)) \]
for all \( t \in [\tau, \tau/2] \). Thus, we conclude that
\[ \text{scal}_{g(\tau/2)}(q) \geq \exp \left( -\frac{d_{g(\tau)}(p,q)^2}{|\tau|} \right) \text{scal}_{g(\tau)}(p). \]
Since \( p, q \in M \) are arbitrary, the assertion follows.

**Proposition 14.** Suppose that the curvature tensor of \((M, g(t))\) lies in the cone \( \hat{C} \) for each \( t \in (-\infty, 0) \). Moreover, suppose that
\[ \limsup_{\tau \to -\infty} \frac{1}{\sqrt{|\tau|}} \text{diam}(M, g(\tau)) < \infty. \]
Then
\[ \limsup_{\tau \to -\infty} \left[ |\tau| \sup_M \text{scal}_{g(\tau)} \right] < \infty. \]

**Proof.** Since the solution \( g(t) \) is defined until time 0, we have
\[ \inf_M \text{scal}_{g(\tau/2)} \leq \frac{n}{|\tau|} \]
for each \( \tau \in (-\infty, 0) \) (see e.g. [5], Proposition 2.19). Using Lemma 13, we deduce that
\[ \sup_M \text{scal}_{g(\tau)} \leq \frac{n}{|\tau|} \exp \left( \frac{\text{diam}(M, g(\tau))^2}{|\tau|} \right). \]
From this, the assertion follows.

Finally, we recall the following result due to B. Kostant (see [17], Corollary 2.2):

**Proposition 15.** Let \((N, h)\) be a compact, simply connected Riemannian manifold of dimension \( n \neq 5 \) which is, topologically, a rational homology sphere. Then the holonomy representation of \((N, h)\) is complete; that is, \((N, h)\) has holonomy group \( \text{SO}(n) \).
Theorem 16. Let $g(t), t \in (-\infty, 0)$, be an ancient solution to the Ricci flow on a compact, even-dimensional manifold $M$. Suppose that the curvature tensor of $(M, g(t))$ lies in the interior of the cone $\hat{C}$ for each $t \in (-\infty, 0)$. Moreover, suppose that

$$\limsup_{\tau \to -\infty} \frac{1}{\sqrt{|\tau|}} \text{diam}(M, g(\tau)) < \infty.$$ 

Then $(M, g(t))$ has constant sectional curvature for each $t \in (-\infty, 0)$.

Proof. Suppose the assertion is false. By Theorem 11, we can find a sequence of points $(p_k, \tau_k) \in M \times (-\infty, 0)$ such that $\lim_{k \to \infty} \tau_k = -\infty$ and

$$R_{g(\tau_k)} - \frac{1}{k} \text{scal}_{g(\tau_k)} I \notin \hat{C}$$

at $p_k$. For each $k$, we consider the rescaled metrics

$$\tilde{g}_k(t) = \frac{1}{|\tau_k|} g(|\tau_k| t), \quad t \in (-2, -\frac{1}{2}).$$

For each $k$, the metrics $\tilde{g}_k(t), t \in (-2, -\frac{1}{2})$, form a solution to the Ricci flow on $M$. By assumption, the diameter of $(M, \tilde{g}_k(t))$ has uniformly bounded diameter; moreover, it has uniformly bounded curvature by Proposition 14. Since $M$ is even-dimensional, we conclude that the injectivity radius of $(M, \tilde{g}_k(t))$ is uniformly bounded from below.

Hence, after passing to a subsequence if necessary, the sequence $(M, \tilde{g}_k(t))$ converges in the Cheeger-Gromov sense to some limiting solution $(M, \bar{g}(t))$ to the Ricci flow. This limiting solution is defined for all $t \in (-2, -\frac{1}{2})$. Clearly, the curvature tensor of $(M, \bar{g}(t))$ lies in the cone $\hat{C}$ for each $t \in (-2, -\frac{1}{2})$. Moreover, it follows from (2) that the curvature tensor of $(M, \bar{g}(-1))$ lies on the boundary of the cone $\hat{C}$ for some point $q \in M$. By Proposition 9 in [7], the manifold $(M, \bar{g}(-1))$ has non-generic holonomy group, i.e. $\text{Hol}^0(M, \bar{g}(-1)) \neq \text{SO}(n)$. On the other hand, it follows from the Differentiable Sphere Theorem that the universal cover of $(M, \bar{g}(-1))$ is diffeomorphic to $S^n$ (cf. [6], Theorem 3). By Proposition 15 the universal cover of $(M, \bar{g}(-1))$ has holonomy group $\text{SO}(n)$. This is a contradiction.

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