Perennials and the Group-Theoretical Quantization of a Parametrized Scalar Field on a Curved Background

P. Hájíček

Institute for Theoretical Physics
University of Bern
Sidlerstrasse 5, CH-3012 Bern, Switzerland

C.J. Isham

Theoretical Physics Group, Blackett Laboratory
Imperial College of Science, Technology & Medicine
South Kensington, London SW7 2BZ, U.K.

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Abstract

The perennial formalism is applied to the real, massive Klein-Gordon field on a globally-hyperbolic background space-time with compact Cauchy hypersurfaces. The parametrized form of this system is taken over from the accompanying paper. Two different algebras $S_{\text{can}}$ and $S_{\text{loc}}$ of elementary perennials are constructed. The elements of $S_{\text{can}}$ correspond to the usual creation and annihilation operators for particle modes of the quantum field theory, whereas those of $S_{\text{loc}}$ are the smeared fields. Both are shown to have the structure of a

*email: hajicek@butp.unibe.ch

†email: c.isham@ic.ac.uk
Heisenberg algebra, and the corresponding Heisenberg groups are described. Time evolution is constructed using transversal surfaces and time shifts in the phase space. Important roles are played by the transversal surfaces associated with embeddings of the Cauchy hypersurface in the space-time, and by the time shifts that are generated by space-time isometries. The automorphisms of the algebras generated by this particular type of time shift are calculated explicitly.

The construction of the quantum theory using the perennial formalism is shown to be equivalent to the Segal quantization of a Weyl system if the time shift automorphisms of the algebra $S_{\text{can}}$ are used. In this way, the absence of any timelike Killing vector field in the background space-time leads naturally to the ‘problem of time’ for quantum field theory on a background space-time. Within the perennial formalism, this problem is formally identical to the problem of time for any parametrized system, including general relativity itself. Two existing strategies—the ‘scattering’ approach, and the ‘algebraic’ approach—for dealing with this problem in quantum field theory on a background space-time are translated into the language of the perennial formalism in the hope that this may give some insight into how the general problem can be solved. The non-unitary time evolution typical of the Hawking effect is shown to be due to global properties of the corresponding phase space: specifically, the time shifts map a global transversal surface to a non-global one. Thus, the existence of this effect is closely related to the global time problem.
I. INTRODUCTION

In the canonical approach to quantum gravity, much emphasis is placed on three particular issues: the conceptual problems that arise in the interpretation of the theory, especially the problem of time; the role of the space-time diffeomorphism group; and the construction of non-perturbative quantization methods (for reviews, see [1–3]). The Dirac method of imposing operator constraints on the allowed state vectors (which, in the case of gravity, leads to the Wheeler-DeWitt equation) has become particularly popular because of its manifest relation to the idea of invariance under the action of space-time diffeomorphisms.

However, diffeomorphism invariance can also be secured by adopting a method in which only ‘gauge-invariant’ objects are quantized; Dirac himself laid the foundations [1] for a powerful approach of this sort. A generalization of the Dirac method to include all finite-dimensional, first-class parametrized systems was presented by Hájíček [5]. The resulting theory, which combines the group-theoretic [6] and algebraic [7] methods of quantization with that of Dirac, is called the ‘perennial formalism’, following the notation introduced by Kuchař in his analysis [3] of the problem of time in canonical quantum gravity. The key idea of this method is to find an algebra of phase-space functions whose Poisson brackets with all the first-class constraints vanish; such functions are therefore constant on the phase-space orbits of the (function) group generated by the constraints. Furthermore, this algebra is required to be large enough to generate all gauge-invariant functions in an appropriate sense.

Quantization of the system then consists in finding irreducible self-adjoint representations of this algebra of ‘physical observables’ or, essentially equivalently, finding irreducible, unitary representations of the associated ‘canonical’ group.

In the present paper we develop the perennial formalism in the context of a field system, namely a linear, massive scalar field propagating on a fixed, globally-hyperbolic space-time with compact Cauchy hypersurfaces.

Various motivations lie behind such a study. To begin with, infinite-dimensional systems are qualitatively different from finite-dimensional ones, and it is an important—and
mathematically non-trivial—challenge to see how the perennial formalism can be extended to this case. We shall show how the scalar field theory can be rewritten in such a way as to become a simple example of a system with perennials. However, quantum field theory in a fixed background has been much studied in the past using standard methods, and hence it provides a useful model for exploring the perennial formalism for an infinite-dimensional system. As we shall see, new problems do appear, the most important of which concerns the choice of the operator representation for the group/algebra generated by the perennials.

In the standard approach to quantum field theory on a background space-time, the normal way of addressing the problem of operator representation makes extensive use of the classical time evolution of the system. Thus the time evolution and associated Hamiltonian are considerably more important for infinite-dimensional systems than they are in the finite-dimensional case. Specifically, if there is no timelike isometry group then the physical representation for the quantum system is not determined and—at the same time—the time evolution and Hamiltonian of the classical system are not well-defined (by the perennial formalism). Such a lack of a Hamiltonian was identified in [5] as the general form taken by the problem of time within the perennial formalism. However, studies of quantum field theory on a background space-time have produced several possible strategies for dealing with this problem, and one can hope that some of these ideas may be applicable to other situations, especially if a common language—in our case, that of the perennial formalism—has been developed.

The perennial formalism enables us to reformulate the dynamics of a scalar field in a background space-time in terms of properties of the system’s phase space. In this reformulation, an important role is played by the idea of a transversal surface, defined in general for a system with a gauge group as a submanifold in the phase space that cuts orbits of the gauge group transversally; the domain of a transversal surface is the set of points in the constraint submanifold that can be joined to the transversal surface by the orbits of the constraints, see [4]. Global problems may arise: for example, there may not exist any global transversal surface (i.e., one whose domain is the whole constraint surface); or there might be a sym-
metry transformation that maps a global transversal surface onto a transversal surface that is not global. Indeed, we shall show that the latter situation arises in the particular case of a space-time with a black hole. This leads to an information loss in which the quantum evolution associated with the symmetry sends pure states to mixed states (the Hawking effect). A situation with somewhat similar features can occur if a system has no global transversal surfaces (the so-called ‘global’ time problem); in particular, an analogous non-unitarity is exhibited by the quantum time evolution. Toy models exhibiting such behaviour were studied in [13,14]. Thus, the ideas developed in the present paper may be of use in finding a physical interpretation in general situations in which the global time problem arises.

Another motivation for our present work is that our understanding of quantum field theory on a fixed background might itself profit from the use of the language of perennials. For example, the symmetries and transversal surfaces that have been employed in the past in studies of such quantum field theories are of a very special nature: namely, transformations of phase space that are generated in a particular way by space-time transformations, and surfaces associated with space-time hypersurfaces. The perennial formalism permits more general types of symmetry and transversal surface, and suggests how these can be found. The question then is if these symmetries can be utilized in the quantum theory, and—if not—why not.

The plan of the paper is as follows. In section II, we summarize the results of Ref. [15] where the dynamics and the symplectic geometry of the parametrized scalar field was cast into the standard form of a first-class parametrized system so that the perennial formalism can be applied. In section III, two different kinds of perennials are constructed and each of them is shown to form an algebra of elementary perennials, both of which are versions of the infinite Heisenberg algebra. We show that each isometry of the background space-time defines a map of the phase space that is a symmetry. We calculate the action of these symmetries on the elementary perennials and find that they define automorphisms of the algebras. Then we show how the theory of time evolution as described in [3] can be applied to the present case.
In the final section we show how the group and algebraic quantization that is performed as the next step reduces to the familiar problem of finding a physically appropriate representation of the Weyl group. We briefly summarise the relevant results of the theory of Weyl systems in connection with quantum field theory on a background space-time, and we show that our quantization method leads to a quantum theory that is equivalent to the usual one. Finally, we discuss the problem of time in a quantum field theory in a fixed background. We study a ‘scattering approach’ to quantization that makes use of an isolated symmetry that is defined on only a small subset of the phase space. In the context of the perennial formalism, such a generalized symmetry is sufficient to define a time evolution. We show that the resulting quantum evolution is non-unitary if the symmetry does not preserve the domains of the transversal surfaces involved in the construction, and we apply the results to the Hawking effect. We also briefly describe the ‘algebraic approach’ to quantization in which the states are defined as functionals on the algebras of peripherals.

II. STRUCTURE OF THE EXTENDED PHASE SPACE

In this section, we shall summarize the results of the companion work \[15\] so that the present paper becomes self-contained. For more details, one should consult \[15\].

We work with a curved background space-time \((\mathcal{M}, g)\) and assume that it is \(C^\infty\) and globally hyperbolic; the Cauchy surface \(\Sigma\) is assumed to be compact. The real scalar field \(\phi\) satisfies the Klein-Gordon equation

\[
|\det g|^{-1/2}\partial_\mu(|\det g|^{1/2}g^{\mu\nu}\partial_\nu\phi) + m^2\phi = 0. \tag{1}
\]

Consider a \(C^\infty\) embedding \(X: \Sigma \to \mathcal{M}\) that is spacelike with respect to the metric \(g\). Let \(\mathcal{E}\) denote the space of all such embeddings. Each embedding \(X\) determines a positive-definite metric \(\gamma_X\) on \(\Sigma\) and a unit normal vector field \(n_X\) to \(X(\Sigma)\) in \(\mathcal{M}\). The embedding \(X\) also defines a Cauchy datum for the field \(\phi\) along the hypersurface \(X(\Sigma)\). This is a pair \((\varphi, \pi)\) of fields on \(\Sigma\), where the scalar \(\varphi\) and the density (of weight \(w = 1\)) \(\pi\) are defined by
\[ \varphi(x) := \phi(X(x)), \quad (2) \]
\[ \pi(x) := (\det \gamma)^{1/2}(X(x)) n^\mu(X(x)) \partial_\mu \phi(X(x)). \quad (3) \]

The space of all \( C^\infty \) Cauchy data will be denoted by \( \Gamma_{\varphi} \)—a linear space that can be equipped with a Sobolev structure (for details, see [15]). The dynamical equation (1) defines a mapping between Cauchy data corresponding to different embeddings. Specifically, let \( X \) and \( X' \) be two arbitrary spacelike embeddings and let \( (\varphi, \pi) \in \Gamma_{\varphi} \). Then there is a unique solution \( \phi \) of equation (1) whose Cauchy datum at \( X(\Sigma) \) is \( (\varphi, \pi) \), and this induces a well-defined Cauchy datum \( (\varphi', \pi') \) on \( X'(\Sigma) \). Thus we get a map \( (\varphi, \pi) \rightarrow (\varphi', \pi') \), which we denote by \( \rho_{XX'} \).

One can show that \( \rho_{XX'} \) is an automorphism of the Sobolev space \( \Gamma_{\varphi} \).

The space \( \Gamma_{\varphi} \) is the phase space of the (non-constrained) scalar field \( \phi \) on the curved background \((M, g)\). If we extend this space by adding all spacelike embeddings \( X \) and their conjugate momenta \( P \), and if we impose suitable constraints, we obtain a constrained system that is dynamically equivalent to the original one. The points of the resulting extended phase space \( \Gamma \) are collections of fields, \( x \mapsto (\varphi(x), \pi(x), X(x), P(x)) \) on \( \Sigma \). These fields can be characterised by their transformation properties with respect to a pair of local charts

\[ (U, h) \text{ of } \Sigma \text{ and } (V, \bar{h}) \text{ of } M \quad (4) \]

(where \( X(U) \cap \bar{V} \neq \emptyset \)). In particular, \( P_\mu(x) \) is a covector with respect to the transformation of \( (V, \bar{h}) \) on \( M \) and a quadruple of scalar densities with respect to the transformation of \( (U, h) \) on \( \Sigma \). Quantities of this type were called ‘\( e \)-tensor densities’ by Kuchař [9].

The phase space \( \Gamma \) can be given a structure of an infinite-dimensional differentiable manifold with tangent and cotangent vectors described as follows. Consider a curve \( \lambda \rightarrow (\varphi_\lambda, \pi_\lambda, X_\lambda, P_\lambda) \) whose tangent vector components \( (\Phi, \Pi, V, W) \equiv (\dot{\varphi}_\lambda, \dot{\pi}_\lambda, \dot{X}_\lambda, \dot{P}_\lambda) \) can be calculated by differentiating with respect to \( \lambda \) the coordinate representatives associated with the pair of charts [4]. The fields \( \Phi(x), \Pi(x), V(x) \) and \( W(x) \) are again characterized by their transformation properties: the first three are \( e \)-tensors, but the fourth transforms in a more complicated way (see [15]). The space \( T_{(\varphi, \pi, X, P)} \Gamma \) of all such vectors can be given
the $C^\infty$ structure of a Fréchet space. A cotangent vector at a point $(\varphi, \pi, X, P)$ of $\Gamma$ will be a quadruple $(A_\varphi, A_\pi, A, B)$ of fields on $\Sigma$ such that the pairing

$$\langle (A_\varphi, A_\pi, A, B), (\Phi, \Pi, V, W) \rangle := \int_\Sigma d^3x (A_\varphi \Phi + A_\pi \Pi + A_\mu V^\mu + B_\mu W^\mu).$$  \hspace{1cm} (5)$$

with vectors from $T^*_{(\varphi, \pi, X, P)}\Gamma$ gives a coordinate independent number. This requirement determines the transformation properties of the fields $(A_\varphi, A_\pi, A, B)$.

One can show that $(A_\varphi, -A_\pi, B, -A_X)$ transforms as a tangent vector. Thus, there is a map $J : T^*_{(\varphi, \pi, X, P)}\Gamma \to T^{*}_{(\varphi, \pi, X, P)}\Gamma$ given by $J(A_\varphi, A_\pi, A_X, A_P) := (A_\pi, -A_\varphi, A_P, -A_X)$. The $C^\infty$ structure of the space $T^*_{(\varphi, \pi, X, P)}\Gamma$ can be chosen in such a way that $J$ is an isomorphism. Using this isomorphism, a symplectic structure $\Omega$ on $\Gamma$ can be defined as follows. If $v_1$ and $v_2$ are two vectors in $T_{(\varphi, \pi, X, P)}\Gamma$ then

$$\Omega(v_1, v_2) := -\langle J^{-1}v_1, v_2 \rangle.$$  \hspace{1cm} (6)$$

It follows at once that (i) $\Omega(v_1, v_2) = -\Omega(v_2, v_1)$; (ii) $\Omega$ is weakly non-degenerate (see [16]); and (iii) $\Omega$ is not only closed but also exact.

As $\Omega$ is only a weak symplectic form, not every differentiable function on $\Gamma$ will have an associated Hamiltonian vector field. The class of functions that do can be characterized as follows. If $F : \Gamma \to \mathbb{R}$, we say that $F$ has a gradient if the following two conditions are satisfied:

1. the Fréchet derivative, $DF|_{(\varphi, \pi, X, P)} : T_{(\varphi, \pi, X, P)}\Gamma \to \mathbb{R}$ is a bounded linear map;

2. there exists $\text{grad} F \in T^*_{(\varphi, \pi, X, P)}\Gamma$ such that $\langle \text{grad} F, v \rangle = DF|_{(\varphi, \pi, X, P)}(v)$ for all $v \in T_{(\varphi, \pi, X, P)}\Gamma$.

The quantity $\text{grad} F$ is calculated from $DF$ as usual by integration by parts (if $F$ contains derivatives). The ‘components’ of this gradient will be denoted by the collection of functions $(\text{grad}_\varphi F, \text{grad}_\pi F, (\text{grad}_X F)_\mu, (\text{grad}_P F)^\nu)$.

For a differentiable function with a gradient, we can define the associated ‘Hamiltonian vector field’. Specifically, if $F$ is such a function, then $\xi_F \in T_{(\varphi, \pi, X, P)}\Gamma$ is defined by the
relation \( \langle \text{grad } F, v \rangle = \Omega(v, \xi_F) \), for all \( v \in T_{(\varphi, \pi, X, P)}\Gamma \). Hence, because of Eq. (3), we see that \( \langle \text{grad } F, v \rangle = \langle J^{-1}\xi_F, v \rangle \) for all \( v \), and so \( \xi_F = J(\text{grad } F) \). Finally, the Poisson bracket of a pair of differentiable functions \( F \) and \( G \) is defined as \( \{ F, G \} := -\Omega(\xi_F, \xi_G) \), and we see immediately that

\[
\{ F, G \} = \langle \text{grad } F, \xi_G \rangle. \tag{7}
\]

This Poisson bracket is antisymmetric and, since \( \Omega \) is closed, it satisfies the Jacobi identity.

The constraints that must be imposed on the extended phase space \( \Gamma \) are

\[
\mathcal{H}_\mu = P_\mu + \mathcal{H}_\mu^\phi, \tag{8}
\]

\[
\mathcal{H}_\mu^\phi = -\mathcal{H}_\mu^\phi n_\mu + \mathcal{H}_k^\phi X_k^\phi, \tag{9}
\]

where \( \mathcal{H}_\mu^\phi = \frac{1}{2}(\det \gamma)^{1/2}\left( \frac{x^2}{\det \gamma} + \gamma^{kl} \varphi_{,k} \varphi_{,l} + m^2 \varphi^2 \right) \), and \( \mathcal{H}_k^\phi = \pi \varphi_{,k} \). One can show that the constraint set \( \tilde{\Gamma} \) defined by \( \mathcal{H}_\mu = 0 \) is a smooth submanifold in a neighbourhood of any of its points. Moreover, \( \tilde{\Gamma} = \tilde{C}(\Gamma \phi \times \mathcal{E}) \), where \( \tilde{C} : \Gamma \phi \times \mathcal{E} \rightarrow \Gamma \) is defined by \( \tilde{C}(\varphi, \pi, X) := (\varphi, \pi, X, -\mathcal{H}^\phi(\varphi, \pi, X)) \). We shall often refer to \( \tilde{\Gamma} \) as \( \Gamma \phi \times \mathcal{E} \).

If we smear \( \mathcal{H}_\mu \) to get \( \mathcal{H}_N := \int_\Sigma d^3x \, N^\mu(x)\mathcal{H}_\mu(x) \), where \( N \in TX \mathcal{E} \) (i.e., \( N(x) \in TX(x)\mathcal{M} \)), we obtain a differentiable function with a gradient. Such a function defines a Hamiltonian vector field, and one can proceed in complete analogy with the theory of finite-dimensional systems. For example, let \( \phi \) satisfy the Klein-Gordon equation (1) on \( \mathcal{M} \). Then for each curve \( \lambda \mapsto X_\lambda \) with the tangent vector field \( N \) on \( \mathcal{E} \), the initial data \( (\varphi_\lambda, \pi_\lambda) \) for \( \phi \) on \( X_\lambda(\Sigma) \) satisfy the evolution equation

\[
(\dot{\varphi}, \dot{\pi}, \dot{X}, \dot{P}) = J(\text{grad } \mathcal{H}_N). \tag{10}
\]

Thus the Hamiltonian vector fields \( J(\text{grad } \mathcal{H}_N) \) of the functions \( \mathcal{H}_N \) are tangential to \( \tilde{\Gamma} \), and the system is first class according to the definition given in [5]. The pull-back of the vector field (10) to \( \Gamma \phi \times \mathcal{E} \) is given by

\[
\dot{\varphi} = \text{grad}_\pi \mathcal{H}_N, \tag{11}
\]

\[
\dot{\pi} = -\text{grad}_\varphi \mathcal{H}_N, \tag{12}
\]

\[
\dot{X} = N. \tag{13}
\]
Let us denote the space of longitudinal vectors at \((\varphi, \pi, X) \in \Gamma_\varphi \times \mathcal{E}\) by \(\Xi_{(\varphi, \pi, X)}\), i.e.,

\[
\Xi_{(\varphi, \pi, X)} := \{(\Phi, \Pi, V) \in T_{(\varphi, \pi, X)} \tilde{\Gamma} \mid \Phi = \text{grad}_\pi \mathcal{H}_N, \Pi = -\text{grad}_\varphi \mathcal{H}_N, V = N \in T_X \mathcal{E}\}. \tag{14}
\]

The space \(\Xi_{(\varphi, \pi, X)}\) is a closed subspace of \(T_{(\varphi, \pi, X)} \tilde{\Gamma}\). Moreover, there is a submanifold of \(\tilde{\Gamma}\) whose tangent space coincides with \(\Xi_{(\varphi, \pi, X)}\) at each point of the submanifold. Let us denote the maximal submanifold of this kind passing through a point \((\varphi, \pi, X) \in \tilde{\Gamma}\) by \(\gamma_{(\varphi, \pi, X)}\). This subset \(\gamma_{(\varphi, \pi, X)}\) is called a ‘\(c\)-orbit through \((\varphi, \pi, X)\)’, in complete analogy with the situation for a finite-dimensional system.

Finally, the pull-back \(\tilde{\Omega}\) of the form \(\Omega\) to \(\tilde{\Gamma}\) is given by the formula

\[
\tilde{\Omega}((\Phi_1, \Pi_1, V_1), (\Phi_2, \Pi_2, V_2)) = \int_\Sigma d^3x \left[ (\Pi_1 + \text{grad}_\varphi \mathcal{H}_{V_1})(\Phi_2 - \text{grad}_\pi \mathcal{H}_{V_2}) - (\Pi_2 + \text{grad}_\varphi \mathcal{H}_{V_2})(\Phi_1 - \text{grad}_\pi \mathcal{H}_{V_1}) \right]. \tag{15}
\]

Clearly, \(\tilde{\Omega}\) is a presymplectic form and \(\Xi_{(\varphi, \pi, X)}\) is its singular subspace.

### III. PERENNIALS, SYMMETRIES AND TIME EVOLUTION

In section II we showed that the geometrical structure of the phase space, constraint submanifold and the \(c\)-orbits of our infinite-dimensional system are all analogous to those of the corresponding objects in a finite-dimensional system as studied, for example, in [5]. The application of the perennial formalism is now straightforward.

A crucial role in the perennial formalism is played by quantities that are reparametrization and gauge invariant. In particular, a perennial is defined as a function \(o : \Gamma \to \mathbb{R}\) that is constant along the \(c\)-orbits; or, equivalently,

\[
\{o, \mathcal{H}_N\}_{\tilde{\Gamma}} = 0, \text{ for all } N. \tag{16}
\]

In most physical applications of a field theory, one deals with a restricted class of functions on the phase space—the so-called ‘local functionals’. Each local functional has the form \(\int_\Sigma d^3x F(x)\), where the value of \(F(x)\) at \(x \in \Sigma\) is a polynomial function of values of the
fields and their $x$-derivatives taken at the same point. It is easy to show that (i) all local functionals possess gradients; (ii) the Poisson bracket of two local functionals is again a local functional, so that multiple Poisson brackets are well-defined; and (iii) they satisfy the Jacobi identity. Since the smeared constraint $H_N$ is itself a local functional it follows that the set of all local functionals that are perennials forms a Poisson algebra $P_{lf}$.

Let us now construct a particular class of local functional perennials for the case of scalar field theory on a background space-time. The idea is to associate a perennial $o_\phi$ with each maximal solution $\phi$ of the classical field equations. Specifically, let $(\xi, \eta, X, P)$ be an arbitrary point of $\Gamma$ and let $(\varphi, \pi)$ be the Cauchy datum of $\phi$ at $X$. Then $o_\phi$ is defined by

$$
o_\phi(\xi, \eta, X, P) := \int_\Sigma d^3x (\varphi \eta - \xi \pi). \tag{17}$$

The main task is to show that $o_\phi$ is constant along $c$-orbits.

Let $\gamma(\xi, \eta, X)$ be a $c$-orbit and let $\psi$ be the associated maximal classical solution, i.e., $(\xi, \eta)$ is the Cauchy datum of $\psi$ on $X(\Sigma)$. Let $(\xi', \eta', X', P') \in \gamma(\xi, \eta, X)$ be an arbitrary point on the $c$-orbit. Then

$$
o_\phi(\xi', \eta', X', P') = \int_\Sigma d^3x (\varphi \eta' - \xi' \pi'), \tag{18}$$

where $(\varphi, \pi')$ and $(\xi', \eta')$ are the Cauchy data of $\phi$ and $\psi$ respectively at $X'$. Using Eqs. (2) and (3), we can rewrite this as

$$
o_\phi(\xi', \eta', X', P') = \int_\Sigma d^3x (\det \gamma')^{1/2} n'\mu (\phi \psi, \mu - \psi \phi, \mu)_{X'(\Sigma)} \tag{19}$$

where $\gamma'$ is the induced metric on $X'(\Sigma)$, and $n'\mu$ is the unit normal vector to $X'(\Sigma)$. However, the integral on the right hand side is just the familiar ‘Klein-Gordon inner product’ $(\phi, \psi)_{KG}$ of the two solutions $\phi$ and $\psi$, and this is well-known to be independent of the Cauchy surface $X'(\Sigma)$. Hence $o_\phi$ is constant, as claimed.

The perennial $o_\phi$ has the following properties.

1. Suppose that $o_\phi$ is constant along the whole of the constraint set $\tilde{\Gamma}$. Then $\phi$ must have the same Klein-Gordon product with any other solution, which is only possible if $\phi = 0$. 

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2. Let $\phi$ and $\psi$ be two maximal solutions with corresponding perennials $o_\phi$ and $o_\psi$. Then the definition (17) implies immediately that

\[ o_\phi + o_\psi = o_{\phi+\psi}, \quad (20) \]

\[ r o_\phi = o_{r\phi} \text{ for all } r \in \mathbb{R}, \quad (21) \]

where the linearity of the field equation (1) guarantees that the solutions $\phi + \psi$ and $r\phi$ are again maximal.

3. The Poisson bracket of $o_\phi$ and $o_\psi$ can be obtained from Eq. (7). To calculate it we need the gradients, and using Eqs. (11) and (12) we obtain

\[
\langle \text{grad} o_\phi |_{(\xi,\eta,X,P)} , (\Phi,\Pi,V,W) \rangle = \\
\int_{\Sigma} d^3x \left[ \varphi \Pi - \pi \Phi + \xi \text{grad}_\varphi H_V |_{(\varphi,\pi,X,P)} + \eta \text{grad}_\pi H_V |_{(\varphi,\pi,X,P)} \right].
\]

(22)

Thus,

\[
\text{grad}_\varphi o_\phi |_{(\xi,\eta,X,P)} = -\pi,
\]

\[
\text{grad}_\pi o_\phi |_{(\xi,\eta,X,P)} = \varphi,
\]

\[
\text{grad}_P o_\phi |_{(\xi,\eta,X,P)} = 0
\]

where $(\varphi,\pi)$ is the Cauchy datum of $\phi$ at $X$. It follows that

\[
J(\text{grad} o_\phi) = (\varphi,\pi,0,A),
\]

(23)

where $A_\mu$ are functions of $\xi, \eta, X, P$ and $\phi$. Then

\[
\{o_\phi, o_\psi\} |_{(\xi,\eta,X,P)} = \int_{\Sigma} d^3x (\det \gamma)^{1/2} n^\mu (\phi \psi_{,\mu} - \psi \phi_{,\mu}) |_{X(\Sigma)} = (\phi, \psi)_{KG},
\]

(24)

where $\gamma$ and $n^\mu$ are the induced metric and unit normal vector at $X(\Sigma)$. Thus the Poisson bracket is independent of $(\xi,\eta,X,P)$ and is hence a constant real function on the phase space $\Gamma$. 

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4. Let $\phi$ and $\psi$ be two different maximal solutions representing two different orbits $\gamma_\phi$ and $\gamma_\psi$. Then there is a third solution, $\chi$, such that $(\chi, \phi - \psi)_{KG} \neq 0$, and hence $o_\chi$ has different values at $\gamma_\phi$ and $\gamma_\psi$.

Let $S_\phi$ denote the set of perennials of the form $o_\phi$ where $\phi$ runs over the set of all $C^\infty$ solutions to the field equations. Then, because of the second property above, $S_\phi$ is a linear space. Let $\mathcal{R} \cong \mathbb{R}$ denote the set of the constant real functions on $\Gamma$, and consider the linear space $S_{\text{can}} := S_\phi \oplus \mathcal{R}$. This space $S_{\text{can}}$ is closed with respect to Poisson bracket operations because of the third property above. Moreover, $S_{\text{can}} \subset \mathcal{P}_{\text{fr}}$. Thus $S_{\text{can}}$ is a Lie subalgebra of $\mathcal{P}_{\text{fr}}$. According to the fourth property above, it separates the $c$-orbits in $\tilde{\Gamma}$. It follows that $S_{\text{can}}$ can play the role of an ‘algebra of elementary perennials’ for our system—the basic ingredient in the ‘algebraic method of quantization’ in which the quantum theory is associated with a self-adjoint representation of $S_{\text{can}}$ on a Hilbert space (see [3]). The relations (21), (22) and (24) imply that $S_{\text{can}}$ is an infinite-dimensional Heisenberg algebra. Specifically, as a linear space it is a direct sum of $\mathcal{R}$ and the space $S_\phi$ that is equipped with the (weakly) non-degenerate skew-symmetric form $(\cdot, \cdot)_{KG}$; the Lie bracket is then defined by

$$[(\phi_1, r_1), (\phi_2, r_2)] := (0, (\phi_1, \phi_2)_{KG}).$$

(25)

The corresponding Lie group is the so-called ‘Heisenberg group’ $G_{\text{can}}$ defined on $S_\phi \times \mathbb{R}$ by the group law

$$(\phi_1, r_1) \cdot (\phi_2, r_2) := (\phi_1 + \phi_2, r_1 + r_2 + \frac{1}{2}(\phi_1, \phi_2)_{KG}).$$

(26)

The action of $G_{\text{can}}$ on $\Gamma$ can be deduced from the action of its generators $o_\phi$ given by equation (17): namely, the point $(\xi, \eta, X, P)$ maps to $(\xi + \varphi, \eta + \pi, X, P')$, where $(\varphi, \pi)$ is the initial datum of $\phi$ at $X(\Sigma)$ and $P'$ is a function of $\xi, \eta, X, P$, and $\phi$ such that $(\xi + \varphi, \eta + \pi, X, P') \in \tilde{\Gamma}$ if $(\xi, \eta, X, P) \in \tilde{\Gamma}$. The action is not faithful since the subgroup $(0, \mathbb{R})$ acts trivially, and hence the group $G_{\text{can}}$ is a central extension of a group of symmetries on $\Gamma$ (see
Thus $\mathcal{G}$ satisfies all the conditions for a so-called ‘first-class canonical group’ whose irreducible, unitary representations can be associated with a quantization of the system [3].

There is an alternative choice for the algebra of elementary perennials in which the perennials are associated with ‘smeared fields’; as such, they form the basis for a different (but ultimately equivalent) quantization of the scalar field. The construction goes as follows. Let $D(\mathcal{M})$ be the space of $C^\infty$ test functions with compact support on the space-time $\mathcal{M}$, let $f \in D(\mathcal{M})$ and $(\varphi, \pi, X, P) \in \Gamma$. Then there is a unique maximal classical solution $\phi$ with the Cauchy datum $(\varphi, \pi)$ at $X(\Sigma)$, and we define the perennial $\kappa_f : \Gamma \to \mathbb{R}$ by the equation

$$\kappa_f(\varphi, \pi, X, P) := \int_{\mathcal{M}} d^4y \sqrt{|\det g|} \phi f. \quad (27)$$

Note that $\kappa_f$ does not depend on $P$, and it is a perennial because the same classical solution leads to the same value of $\kappa_f$. Let us list some important properties of this type of perennial.

1. Clearly, $\kappa_f$ can be constant along $\bar{\Gamma}$ only if $f = 0$, and then $\kappa_f = 0$.

2. Let $f$ and $f'$ be two elements of $D(\mathcal{M})$ with corresponding perennials $\kappa_f$ and $\kappa_{f'}$. Then the definition (27) implies immediately that

$$\kappa_f + \kappa_{f'} = \kappa_{f+f'},$$

$$r\kappa_f = \kappa_{rf} \text{ for all } r \in \mathbb{R}.$$

3. We can find an explicit expression for $\kappa_f$ if we use the Cauchy propagator $G(x, y)$ for the equation (1) ($G(x, y)$ is sometimes known as the ‘Pauli-Jordan function’). The existence and uniqueness of such a propagator for space-times of the type with which we are dealing was shown by Choquet-Bruhat [17]. The basic properties of the Cauchy propagator (for example, see [18]) are (i) $G(x, y) = -G^r(x, y) + G^a(x, y)$, where $G^r$ and $G^a$ are respectively the retarded and advanced propagators; (ii) $G(x, y)$ is real and skew-symmetric in $x$ and $y$; and (iii) $G(x, y)$ satisfies the identity

$$\phi(x) = (G(x, \cdot), \phi(\cdot))_{KG}, \quad (28)$$
where $\phi(x)$ is any $C^\infty$ solution to equation (1). From Eq. (28), it follows immediately that

$$(G(x, \cdot), G(\cdot, y))_{\text{KG}} = G(x, y).$$

(29)

If the Klein-Gordon product on the right hand side of Eq. (28) is written out along $\Sigma$, we obtain an expression for the solution $\phi$ at any point $y \in \mathcal{M}$ in terms of its Cauchy data at $X(\Sigma)$. Substituting for $\phi(y)$ in Eq. (27) from Eq. (28) then gives the desired formula:

$$\kappa_f = -\int_\Sigma d^3x (\det \gamma)^{1/2} n^\mu \left. \frac{\partial G(f,y)}{\partial y^\mu} \right|_{X(x)} \varphi(x) + \int_\Sigma d^3x G(f, X(x)) \pi(x),$$

(30)

where

$$G(f, x) := \int_\mathcal{M} d^4y \left| \det g \right|^{1/2} f(y) G(y, x).$$

(31)

Thus, $\kappa_f$ belongs to the class of local functionals.

4. We have the relation $\{\kappa_f, \kappa'_f\} = -G(f, f')$ whose derivation is simple: read off the gradient of $\kappa_f$ from the formula (30), insert it in the equation (7), and use the identity (29).

The smeared perennials generate a Lie algebra, which we denote by $\mathcal{S}_{\text{loc}}$. Properties 2 and 4 above imply that $\mathcal{S}_{\text{loc}}$ is a Heisenberg algebra on $D(\mathcal{M}) \times \mathbb{R}$ with the skew-symmetric form $-G(f, f')$. The corresponding Heisenberg group $\mathcal{G}_{\text{loc}}$ can be used as a first-class canonical group for the system.

The next important step is to consider the role played by symmetries, where—in complete analogy with the finite-dimensional case (see [5])—a symmetry is defined as a symplectic diffeomorphism of $\Gamma$ that preserves the constraint surface $\tilde{\Gamma}$. In particular, it can be shown that each symmetry maps $c$-orbits onto $c$-orbits.

We shall describe a particular class of symmetries that play an important role in the study of quantum field theory on a curved space-time. Any isometry $\vartheta: \mathcal{M} \to \mathcal{M}$ defines
a map \( \theta : \Gamma \rightarrow \Gamma \) as follows. Let \((\varphi, \pi, X, P) \in \Gamma \) be arbitrary and set \( X' := \vartheta \circ X \). Since \( \vartheta \) is an isometry, the embedding \( X' \) is spacelike, and hence a Cauchy surface for \( \mathcal{M} \). The fields \( \varphi, \pi \) and \( P \) are \( e \)-tensor densities at \( X \in \mathcal{E} \), and so can be considered as \( \mathcal{M} \)-tensors at points in \( X(\Sigma) \) (note that \( \varphi(x) \) and \( \pi(x) \) are scalars, and \( P(x) \) is a covector). Set \((\varphi', \pi', P') := (\vartheta^* \varphi, \vartheta^* \pi, \vartheta^* P)\), where \( \vartheta^* \) is the usual pull-back of differential forms on \( \mathcal{M} \). Finally, define \( \theta(\varphi, \pi, X, P) := (\varphi', \pi', X', P') \).

A simple way of showing that \( \theta \) is a symmetry is to use the \( \vartheta \)-shifted chart. Each pair of local charts of the type \((U, h), (\bar{V}, \bar{h})\) in Eq. (4) can be ‘shifted’ by \( \vartheta \) to become the chart \((U, h)\) on \( \Sigma \) and the chart \((\vartheta(V), \vartheta(h) \circ \vartheta^{-1})\) on \( \mathcal{M} \). The functions that represent \((\varphi', \pi', X', P')\) in the shifted charts coincide numerically with those that represent \((\varphi, \pi, X, P)\) in the original charts. Moreover, the metric \( g_{\mu\nu}' \) in \( \vartheta(V) \) coincides with \( g_{\mu\nu} \) in \( \vartheta(V) \). Thus, if \((\varphi, \pi, X, P)\) satisfies the constraints, then \((\varphi', \pi', X', P')\) will also do so. Furthermore, any curve \( \lambda \mapsto (\varphi_\lambda, \pi_\lambda, X_\lambda, P_\lambda) \) on \( \Gamma \) defines a curve \( \lambda \mapsto \theta(\varphi_\lambda, \pi_\lambda, X_\lambda, P_\lambda) \) that has the same form in the respective coordinate systems. Thus the tangent vectors of these two curves must have the same components. It follows that \( \theta \) is differentiable and—moreover—symplectic since the values of the symplectic form at \((\varphi, \pi, X, P)\) and at \((\varphi', \pi', X', P')\) must coincide numerically in the respective coordinate systems. Hence \( \theta \) is a symmetry.

Let us list some of the important properties of \( \theta \).

1. The restriction \( \bar{\theta} : \Gamma_\phi \times \mathcal{E} \rightarrow \Gamma_\phi \times \mathcal{E} \) of \( \theta \) to \( \bar{\Gamma} \), is given by

\[
\bar{\theta}(\varphi, \pi, X) = (\varphi, \pi, \vartheta \circ X).
\]

This follows immediately from the idea of a shifted chart and the fact that \( \varphi \) and \( \pi \) are scalar fields. Thus we obtain the same Cauchy datum at the shifted Cauchy surface.

2. Let \( \phi \) be a global solution of the field equation, with a Cauchy datum in the Sobolev space \( \Gamma_\phi \). Then \( \phi \circ \vartheta^{-1} \) is again such a solution since \( \vartheta \) is an isometry. If \( \phi \) has the Cauchy datum \((\varphi, \pi)\) at \( X \), then \( \phi \circ \vartheta^{-1} \) has the datum \((\varphi, \pi)\) at \( \vartheta \circ X \). It follows that the map \( \rho_{XX'} \) defined in theorem 1 satisfies \( \rho_{\theta X \theta X'}(\varphi, \pi) = \rho_{XX'}(\varphi, \pi) \).
3. The definition of $\theta$ implies immediately that it maps $c$-orbits onto $c$-orbits. In particular, if $\gamma_\phi$ is an orbit corresponding to a maximal solution $\phi$, then $\theta(\gamma_\phi) = \gamma_{\phi \circ \theta^{-1}}$.

4. Let $o_\phi$ be a perennial in $\mathcal{S}_{\text{can}}$. The $\theta$-shifted perennial $s_\theta(o_\phi)$ was defined in [5] as $s_\theta(o_\phi) := o_\phi \circ \theta^{-1}$. Then we have the relation

$$s_\theta(o_\phi) = o_{\phi \circ \theta^{-1}}. \quad (33)$$

Indeed, $o_\phi(\theta^{-1}(\xi, \eta, X, P)) = o_\phi(\xi, \eta, X', P') = \int_\Sigma d^3x (\varphi' \eta - \pi' \xi)$ where $(\varphi', \pi')$ is the Cauchy datum of $\phi$ at $X' = \vartheta^{-1} \circ X$. However, $(\varphi', \pi')$ is also the Cauchy datum of $\phi \circ \vartheta^{-1}$ at $X$. Thus, $o_\phi(\theta^{-1}(\xi, \eta, X, P)) = o_{\phi \circ \theta^{-1}}(\xi, \eta, X, P)$, and this is equivalent to Eq. (33). A straightforward calculation gives the $\theta$-shift for $\mathcal{S}_{\text{loc}}$ as $s_\theta(\kappa f) = \kappa_{f \circ \theta}$.

The relation in Eq. (33) implies that $s_\theta$ is an automorphism of the algebra $\mathcal{S}_{\text{can}}$ since the map $\phi \mapsto \phi \circ \vartheta^{-1}$ is a linear transformation of solutions that preserves the Klein-Gordon product, and the constant functions on $\Gamma$ are left invariant by $s_\theta$. Similarly, $s_\theta$ is an automorphism of the algebra $\mathcal{S}_{\text{loc}}$ because the map $f \mapsto f \circ \vartheta$ is linear and preserves the quadratic form $G(\cdot, \cdot)$. In fact, $s_\theta$ induces a transformation of perennials from $\mathcal{S}_{\text{can}}$ that is directly related to the Bogoliubov transformations that arise in the study of quantum field theory on a curved background. Indeed, if we choose a complex orthonormal basis $\{\phi_m, \phi_m^*\}$ for the space of solutions (for example, see [18]), then the coefficients $\{a_m, a_m^*\}$ of the expansion of any solution $\phi$ in terms of this basis have the form of our perennials: namely $a_m = (\phi^*_m, \phi)_{\text{KG}}$. The transformation $s_\theta$ of perennials thus defines new coefficients $a_m$, and the expansion of these in terms of the old ones is what is normally called a ‘Bogoliubov transformation’.

Let us observe that the perennial formalism allows a more general type of symmetry that is not necessarily associated with transformations of space-time. For example, the group $\mathcal{G}_{\text{can}}$ is a group of such symmetries. This raises the interesting question of whether other symmetries that are not associated with space-time transformations can be found, and—if so—if they can be helpful in the study of quantum field theory on a curved background.
Finally, let us construct the time evolution of the system. In the finite-dimensional case, such a construction is based on a transversal surface $\Gamma_0$ and a one-dimensional symmetry group \{h(t)\} that moves $\Gamma_0$ (see [3]). A transversal surface $\Gamma_0$ is defined to be a smooth submanifold of $\tilde{\Gamma}$ that (i) intersects each c-orbit $\gamma$ in at most one point $p = \Gamma_0 \cap \gamma$; and (ii) has the property that each such intersection is transversal, i.e., $T_p\Gamma_0 \cap T_p\gamma = \{0\}$, where 0 is the zero vector. A transversal surface is said to be ‘global’ if it intersects each c-orbit. All these definitions can be extended without change to the infinite-dimensional case.

Similarly, the projections of perennials and of symmetries can be defined as for finite-dimensional systems. Thus, let $i_0 : \Gamma_0 \to \Gamma$ be the submanifold injection, and let $\pi_0 : \tilde{\Gamma} \to \Gamma_0$ be the projection that is defined by $\pi_0(p) := \gamma_p \cap \Gamma_0$, where $\gamma_p$ is the c-orbit through the point $p \in \tilde{\Gamma}$. If $o$ is a perennial, then its projection $a_0 : \Gamma_0 \to R$ is defined by

$$a_0 := i_0^*o = o \circ i_0 = o|_{\Gamma_0}. \quad (34)$$

If $\psi : \Gamma \to \Gamma$ is a symmetry, then $a_0(\psi) : \Gamma_0 \to \Gamma_0$ is defined by

$$a_0(\psi) := \pi_0 \circ \psi|_{\Gamma_0}. \quad (35)$$

One can easily show that $i_0^*$ is a Poisson algebra isomorphism, and that $a_0(\psi)$ is a symmetry of $\Gamma_0$.

We shall use a special type of transversal surface that is associated with embeddings as follows. For any spacelike embedding $X : \Sigma \to M$, define the subset $\Gamma_X \subset \tilde{\Gamma}$ by

$$\Gamma_X := \{ (\varphi, \pi, X) \in \tilde{\Gamma} \mid (\varphi, \pi) \in \Gamma_\phi \}. \quad (36)$$

In the following steps we shall show that $\Gamma_X$ is a transversal surface.

1. The subset $\Gamma_X$ can be considered as the image of the map $i_X : \Gamma_\phi \to \Gamma$ defined by $i_X(\varphi, \pi) := (\varphi, \pi, X, P)$ where $P$ is given by Eq. (9). Note that $Di_X|_{(\varphi, \pi)}$ is given by

$$Di_X|_{(\varphi, \pi)}(\Phi, \Pi, V, W) = (\Phi, \Pi, 0, W) \in T_{\tilde{C}(\varphi, \pi, X)}\tilde{\Gamma}, \quad (37)$$

and hence the linear map $Di_X|_{(\varphi, \pi)}$ is injective and splits as
T_{\tilde{\Gamma}}^{(\varphi,\pi,X)} = \{(\Phi, \Pi, 0, W) \mid (\Phi, \Pi) \in T_{(\varphi,\pi)}\Gamma_{\phi}, \ W = -DH^{\phi}(\Phi, \Pi, 0)\}
\times \{(0, 0, V, W) \mid V \in H^s, \ W = -DH^{\phi}(0, 0, V)\}.

(38)

Hence, $\Gamma_X$ is a smooth submanifold of $\Gamma$.

2. Any tangent vector to $\Gamma_X$ at $(\varphi, \pi, X) \in \Gamma_{\phi} \times E$ has the form $(\Phi, \Pi, 0)$, where $(\Phi, \Pi) \in T_{(\varphi,\pi)}\Gamma_{\phi} \simeq \Gamma_{\phi}$. The tangent space to $\gamma_{(\varphi,\pi, X)}$ at $(\varphi, \pi, X)$ is the space $\Xi_{(\varphi,\pi, X)}$ given by Eq. (14), and the only vector $(\Phi, \Pi, V)$ in $\Xi_{(\varphi,\pi, X)}$ with $V = 0$ is the zero vector. Thus the condition for transversality is satisfied.

3. Any c-orbit $\gamma_{\phi}$ intersects $\Gamma_X$, and the point of intersection is $(\varphi, \pi, X)$ where $(\varphi, \pi)$ is the (unique) Cauchy datum of $\phi$ at $X$. Thus $\Gamma_X$ is a global transversal surface.

Note that the injection $i_X$ gives $\Gamma_X$ the structure of a linear (Fréchet) space. Hence we can identify $T_{(\varphi,\pi)}\Gamma_X$ with $\Gamma_X$ itself.

The pull-back $\Omega_X$ of $\tilde{\Omega}$ by $i_X|_{\tilde{\Gamma}}$ can easily be calculated from Eq. (15) as

$$\Omega_X((\Phi_1, \Pi_1), (\Phi_2, \Pi_2)) = \int_{\Sigma} d^3x (\Phi_2 \Pi_1 - \Phi_1 \Pi_2).$$

(39)

This is a constant, weakly non-degenerate form on $T_{(\varphi,\pi)}\Gamma_{\phi} \times T_{(\varphi,\pi)}\Gamma_{\phi}$ that can be identified with the following one on $\Gamma_{\phi} \times \Gamma_{\phi}$:

$$\Omega_X((\varphi_1, \pi_1), (\varphi_2, \pi_2)) = \int_{\Sigma} d^3x (\pi_1 \varphi_2 - \varphi_1 \pi_2).$$

(40)

This form can be used to equip $\Gamma_{\phi}$ with the structure of a linear, weak-symplectic space.

Note that the perennial formalism allows for more general transversal surfaces that are not necessarily associated with surfaces in space-time. An intriguing—and open—question is if an explicit example of such a surface can be found and, if so, if it can be used to construct a quantum field theory on a generic space-time with no timelike Killing vectors (see later).

Let us suppose next that there is a one-dimensional group of isometries $\vartheta(t)$ in the space-time $\mathcal{M}$ such that, for all $t$, $\vartheta(t)(X(\Sigma)) \neq X(\Sigma)$ and $\vartheta(t)$ is generated by an everywhere
timelike Killing vector in $M$. The corresponding one-dimensional group $\{\theta(t)\}$ of symmetries of $\Gamma$, together with the transversal surface $\Gamma_X$, form a basis for the construction of an ‘auxiliary rest frame’ with ‘time levels’ given by $\Gamma_t := \theta(t)\Gamma_X$ and ‘rest trajectories’ given by $\theta$-orbits $\{\theta(t)p\}$, $p \in \Gamma_0$. Any $c$-orbit $\gamma$ defines a curve, $t \mapsto \eta_\gamma(t) := \Gamma_t \cap \gamma$, and the motion with respect to the auxiliary rest frame can be defined in a complete analogy to the finite-dimensional case by comparing the curve $\eta_\gamma(t)$ with the rest trajectories $\theta(t)p$.

‘The same measurement at different times’ can again be defined as the set of time shifted perennials $o \rightarrow o_t := s_{\theta(t)}o$, and the construction of the classical Schroedinger or Heisenberg pictures by means of the projection to $\Gamma_0$ is straightforward (for details see [5]).

However, note that the procedure described here differs in one respect from that described in [5]. Namely, the symmetry group $\{\theta(t)\}$ we have chosen to generate the time evolution is not a subgroup of the first-class canonical group $G_{\text{can}}$ or $G_{\text{loc}}$. Thus, the construction of the quantum mechanical time evolution as given in [5] has to be generalized. This will be done in the next section.

**IV. QUANTUM THEORY**

In this section, the construction of the quantum theory described in [5] for finite-dimensional systems will be extended to the scalar field on a fixed background. The construction uses a representation of the first-class canonical group $\mathcal{G}$ by unitary operators $R(g)$, $g \in \mathcal{G}$, on a Hilbert space $\mathcal{K}$. The generators of $\mathcal{G}$—the elements of the Lie algebra $\mathcal{S}$—are represented by self-adjoint operators on $\mathcal{K}$. Then the automorphism $s_{\theta(t)}$ of the algebra $\mathcal{S}$ defines an automorphism $\hat{s}_{\theta(t)}$ of the corresponding operator algebra by the commutative diagram:

$$
\begin{array}{ccc}
\mathcal{S} & \xrightarrow{s_{\theta(t)}} & \mathcal{S} \\
\downarrow R & & \downarrow R \\
R(\mathcal{S}) & \xrightarrow{\hat{s}_{\theta(t)}} & R(\mathcal{S})
\end{array}
$$ (41)
We arrive at a unitary evolution if we can implement the automorphism \( \hat{s}_{\vartheta(t)} \) by a unitary map \( U(t) : \mathcal{K} \to \mathcal{K} \); that is, \( \hat{s}_{\vartheta(t)}(\hat{O}) = U^{-1}(t) \hat{O} U(t) \).

The classical constructions in the previous sections—in particular, the choice of the algebras of elementary perennials—were performed in such a way that the rules of the algebraic or group-theoretical approaches to quantization as described above lead directly to well-known approaches to the quantization of a scalar field on a fixed space-time background. In particular, \( \mathcal{S}_{\text{can}} \) leads to the Segal theory (for example, see [19] and [20]).

Let us concentrate on \( \mathcal{S}_{\text{can}} \). As was explained in the previous section, \( \mathcal{S}_{\text{can}} \) is an infinite-dimensional version of the Heisenberg algebra, and it determines an abstract infinite-dimensional Heisenberg group \( \mathcal{G}_{\text{can}} \) (in fact, a ‘nuclear group’, see [21]) that is a central extension of the corresponding symmetry group of the phase space \( \Gamma \) and which acts transitively on the \( c \)-orbits. The group \( \mathcal{G}_{\text{can}} \), together with a unitary representation (which must satisfy certain additional conditions in order to guarantee the existence of ‘quantum observables’, see [19]) is called a ‘Weyl system’ in the literature. If we apply the theory of Weyl systems to the present case, we can draw the following conclusions:

1. The group-theoretical approach to the quantization of an infinite-dimensional system differs significantly from the finite-dimensional case in the following respects. As a rule, a finite-dimensional Lie group has only relatively few representations—indeed, the Heisenberg group of \( n \) dimensions has just one (up to unitary equivalence) for any \( n \). Many finite-dimensional canonical groups arise naturally as semi-direct products in which the ‘non-abelian’ factor is sufficiently large that there are only a few inequivalent orbits in the dual of the abelian factor (see [6]). However, an infinite-dimensional Heisenberg group has a huge number of non-equivalent representations. Many of these have no obvious physical application, while others have a meaning in relation to external parameters. For example, in quantum field theory at a finite temperature each value of the temperature is associated with a particular representation (and non-zero temperature representations are not even irreducible).
2. The most difficult part of the construction of a linear quantum field theory is therefore the choice of a ‘physical’ representation. In the Segal theory, the key object on which such a choice is based is the time evolution automorphism \( s_{\vartheta(t)} \) of the Heisenberg algebra \( S_{\text{can}} \). For example, a cyclic state (generating the representation by a Gel’fand-Neumark-Segal construction) might be selected using the Kubo-Martin-Schwinger condition with \( \hat{s}_{\vartheta(t)} \). Another approach based on \( s_{\vartheta(t)} \) is described in [20].

Thus a new problem arises here, analogous perhaps to the ‘Hilbert space problem’ of Kuchar’s classification [3] of different aspects of the problem of time in canonical quantum gravity. In fact, quantum field theory in a curved background has a time problem of its own: most interesting background space-times do not possess a one-dimensional group of timelike isometries \( \vartheta(t) \) (i.e., there is no time-like Killing vector), so that \( s_{\vartheta(t)} \) is not available. However, several methods have been developed for (at least, partly) bypassing this problem and thereby enabling a number of interesting questions to be addressed. These methods are not as mathematically rigorous as those based on a timelike Killing vector but, nevertheless, they may give some hints about the problem of time in the full theory of canonical quantum gravity. We shall consider two different strategies that we shall call the ‘scattering approach’ and the ‘algebraic approach’ to quantization. Let us describe how they can be applied in the context of the perennial formalism.

A. Scattering approach

The scattering approach to quantization is based on an isolated symmetry whose domain is a small subset of the phase space \( \Gamma \). Let us consider first an (idealized) example in which \((\mathcal{M}, g)\) is a space-time that satisfies the conditions of section [1] and which contains open subsets \( U' \) and \( U'' \) with the following properties:

1. Both \( U' \) and \( U'' \) are locally stationary: i.e., there are local flows \( \vartheta'(t, X) \) and \( \vartheta''(t, X) \) generated by timelike Killing generators that are defined everywhere on \( U' \) and \( U'' \) respectively.
2. Both $U'$ and $U''$ contain Cauchy hypersurfaces: i.e., there are spacelike embeddings $X'$ and $X''$ such that $X'(\Sigma) \subset U'$ and $X''(\Sigma) \subset U''$.

3. There is an isometry $\vartheta : U' \to U''$ such that $X'' = \vartheta \circ X'$.

Finally, let $\mathcal{S}_{\text{can}}$ denote the algebra of elementary perennials as discussed in section [11].

The local flows $\vartheta'$ and $\vartheta''$ may not induce global symmetries of $\Gamma$, but they will define perennials $h'$ and $h''$ in some neighbourhood of $\Gamma_{X'}$ and $\Gamma_{X''}$ that correspond to the generators of $\vartheta'$ and $\vartheta''$ respectively. These perennials can be used to construct representations $(R', \mathcal{K}')$ and $(R'', \mathcal{K}'')$ of $\mathcal{S}_{\text{can}}$ such that $-h'$ and $-h''$ are represented by positive, self-adjoint operators $\hat{H}'$ and $\hat{H}''$ (see, e.g. [22]). Following the procedure described in [13], one could now try to implement the map $\rho_{X',X''} : \Gamma_{X'} \to \Gamma_{X''}$, which is a symplectic diffeomorphism (see section [11]), by a unitary map $U(\rho) : \mathcal{K}' \to \mathcal{K}''$. This would leave us with only one Hilbert space (a 'pasting' of $\mathcal{K}'$ and $\mathcal{K}''$).

However, the literature on the quantum theory of a scalar field has proceeded in a different direction that can be related to the Heisenberg picture in the perennial formalism, as described in [3]. The first observation is that the discrete isometry $\vartheta$ induces a symmetry $\theta$ that is defined in some neighbourhood of $\Gamma_{X'}$ in $\Gamma$; in turn, $\theta$ determines a well-defined automorphism $s_{\theta} : \mathcal{S}_{\text{can}} \to \mathcal{S}_{\text{can}}$ of the space of perennials. Indeed, for this it is sufficient that $\theta$ maps a globally transversal surface $\Gamma_{X'}$ onto another such $\Gamma_{X''}$. The $\theta$-shifted perennials are then completely determined by their values on $\Gamma_{X''}$, and these are given by the $\theta$-maps of the restrictions of the original perennials to $\Gamma_{X'}$. Note that that $\theta$ is not a symmetry in the sense of Ref. [3] (it is not globally defined); we shall refer to such a map as a 'time shift'.

The next step is to define the map $\hat{s}_\theta : R'(\mathcal{S}_{\text{can}}) \to R'(\mathcal{S}_{\text{can}})$ by the obvious analogue of the commutative diagram [11], and then to see whether or not it can be implemented by a unitary map $U(\vartheta) : \mathcal{K}' \to \mathcal{K}'$. If the Cauchy hypersurface is compact, this is always possible [23]. Thus one can again work with just a single Hilbert space. The interpretation of the various mathematical objects is then that $R'(\mathcal{S}_{\text{can}})$ contains the Heisenberg observables at the 'time' $\Gamma_{X'}$; $R'(s_\theta(\mathcal{S}_{\text{can}}))$ contains those at the 'time' $\Gamma_{X''}$; the elements of $\mathcal{K}'$ are the
Heisenberg states; and $U(\vartheta)$ is the unitary scattering matrix.

If $U(\vartheta)$ does not exist, a Heisenberg-picture dynamics can still be used to calculate the expectation values of time-shifted operators that are well-defined in certain states. For example, in this way one can calculate the number of particles within a given finite energy range and a finite volume that are created from the vacuum of $\mathcal{K}'$ in the region between $X'(\Sigma)$ and $X''(\Sigma)$, even though the total number of created particles diverges.

Note that the scattering approach will work even if there is only the ‘rudiments’ of a symmetry, but it will give only information on what comes ‘out’ if we let something go ‘in’; what happens ‘inside’ remains quite undertermined.

**B. The Hawking effect**

An example of the scattering approach is the calculation of the Hawking effect \[22\]. In this section, we shall reformulate this calculation in terms of the perennial formalism. Our motivation is not to present a new and conceptually better derivation of the effect but rather to use this model of the scalar field on a black-hole background to suggest a possible meaning of a time shift that does not preserve the domains of transversal surfaces.

In a general system, a transversal surface $\Gamma_X$ will not be global (\emph{i.e.}, it will not cut all the $\mathcal{c}$-orbits transversally), and the time shift that is available will not preserve the domains of these surfaces (the domain $\mathcal{D}(\Gamma_1)$ of a transversal surface $\Gamma_1$ is the subset of $\tilde{\Gamma}$ such that the $\mathcal{c}$-orbit through any point of $\mathcal{D}(\Gamma_1)$ intersects $\Gamma_1$, see \[4\]).

For example, the toy models studied in \[13\] and \[14\] do not possess global transversal surfaces, but there are some that are almost global in the sense that the closure of the domain contains the whole constraint surface $\tilde{\Gamma}$. The results of \[13\] suggest that there is no difference between global and almost global surfaces as far as the quantum theory is considered. In \[14\], two almost global surfaces $\Gamma_1$ and $\Gamma_2$ were chosen, each with two components, $\Gamma_1^\pm$ and $\Gamma_2^\pm$ respectively, giving a total of four, closed transversal surfaces. There is a discrete symmetry $\vartheta$ that maps $\Gamma_1^+$ onto $\Gamma_2^+$, but $\mathcal{D}(\Gamma_1^+) \neq \mathcal{D}(\Gamma_2^+)$. The Hilbert spaces $\mathcal{K}_1$ and $\mathcal{K}_2$ corresponding to
the transversal surfaces $\Gamma_1^+$ and $\Gamma_2^+$ were constructed, and a unitary map $U(\vartheta) : K_1 \to K_2$ of these Hilbert spaces was found that corresponds to the classical map $\theta$. However, the pasting map $\rho : \Gamma_1^+ \to \Gamma_2^+$ is defined (by the $c$-orbits, see \cite{13}) only between some proper subsets of $\Gamma_1^+$ and $\Gamma_2^+$; the corresponding map $U(\rho)$ is defined only on a proper subspace $K_{12} \subset K_1$, and $K_{21} = U(\rho)K_{12}$ is a proper subset of $K_2$. Although the map $U(\rho) : K_{12} \to K_{21}$ itself is unitary, the corresponding time evolution operator $U(\rho) \circ U^{-1}(\vartheta) : K_1 \to K_1$ is defined only on the subspace $K_{12}$ and so it is not a unitary operator on $K_1$. This leads to a time evolution that can change the norms of states, the only physical interpretation of which is that the system can be ‘lost’ or ‘found’ during the time evolution—this can in fact happen already in the classical theory of this (bizarre) system.

A similar situation can arise in the context of quantum field theory on a curved background. Suppose first that there is a Cauchy surface $\Sigma$ that consists of two components $\Sigma_1$ and $\Sigma_2$, so that $\Sigma = \Sigma_1 \cup \Sigma_2$. Then both $\Sigma_1$ and $\Sigma_2$ are closed surfaces in $\mathcal{M}$, and the space $\Gamma_\phi$ of Cauchy data on $\Sigma$ splits into the direct sum of $\Gamma_{\phi_1}$ and $\Gamma_{\phi_2}$, where

$$\Gamma_{\phi_i} = \{ (\varphi, \pi) \in \Gamma_\phi \mid \text{supp}(\varphi, \pi) \subset \Sigma_i \}, \quad (42)$$

$i = 1, 2$. Both $\Gamma_{\phi_1}$ and $\Gamma_{\phi_2}$ are Fréchet spaces, and $\Gamma_\phi = \Gamma_{\phi_1} \otimes \Gamma_{\phi_2}$. For a given spacelike embedding $X : \Sigma_1 \cup \Sigma_2 \to \mathcal{M}$, a pair of transversal surfaces $\Gamma_{X_i}, i = 1, 2$, can be defined by

$$\Gamma_{X_i} := \{ (\varphi, \pi, X) \in \bar{\Gamma} \mid \text{supp}(\varphi, \pi) \subset \Sigma_i \}. \quad (43)$$

The proof that $\Gamma_{X_i}$ is transversal is analogous to that for $\Gamma_X$ in section \textsection I\textsection with the relation Eq. (38) being replaced by

$$T_{\mathcal{C}(\varphi, \pi, X)} \bar{\Gamma} = \{ (\Phi, \Pi, 0, W) \mid \Phi, \Pi \in \Gamma_{\phi_1}, W = -DH^\phi(\Phi, \Pi, 0) \} \times \{ (\Phi, \Pi, 0, W) \mid \Phi, \Pi \in \Gamma_{\phi_2}, W = -DH^\phi(\Phi, \Pi, 0) \} \times \{ (0, 0, V, W) \mid V \in T_X \mathcal{E}, W = -DH^\phi(0, 0, V) \}. \quad (44)$$

Of course, the surface $\Gamma_{X_1}$ is not even almost globally-transversal: we have the relation $\Gamma_X = \Gamma_{X_1} \times \Gamma_{X_2}$, and only $\Gamma_X$ is a global transversal surface.
The Poisson algebra $\mathcal{P}$ of perennials contains ideals $\mathcal{P}_1$ and $\mathcal{P}_2$ of perennials associated with the transversal surfaces $\Gamma_{X_1}$ and $\Gamma_{X_2}$, where

$$\mathcal{P}_i := \{ o \in \mathcal{P} | \text{supp } o \subset D(\Gamma_{X_i}) \}. \quad (45)$$

These ideals $\mathcal{P}_1$ and $\mathcal{P}_2$ generate $\mathcal{P}$. Similarly, the Lie algebra of elementary perennials $\mathcal{S}$ contains two ideals $\mathcal{S}_1$ and $\mathcal{S}_2$ defined by analogous equations, and $\mathcal{S} = \mathcal{S}_1 \oplus \mathcal{S}_2$ is a Lie algebra decomposition. Indeed, each function $o \in \mathcal{S}$ has a restriction $o_{X_1}$ to $\Gamma_{X_1}$, and there are unique $o_{X_1} \in \mathcal{S}_1$ and $o_{X_2} \in \mathcal{S}_2$ such that $o_X = o_{X_1} + o_{X_2}$. Observe that $o_{X_1}$ vanishes at $\Gamma_{X_1}$, so that we have $\{o_{X_1}, o_{X_2}\} = 0$ as desired.

Suppose that the physical representations of the algebras $\mathcal{S}$, $\mathcal{S}_1$ and $\mathcal{S}_2$ on Hilbert spaces $\mathcal{K}$, $\mathcal{K}_1$ and $\mathcal{K}_2$ respectively have the property $\mathcal{K} = \mathcal{K}_1 \otimes \mathcal{K}_2$, where `\otimes` denotes the symmetrized tensor product. Let $|a\rangle$ be an arbitrary element of $\mathcal{K}$. Then it is well-known that there is a density operator $\hat{a}_1$ in $\mathcal{K}_1$ such that

$$\langle a|\hat{a}_1|a\rangle = \text{tr}(\hat{a}_1\hat{a}_1), \quad \text{for all } \hat{a}_1 \in L(\mathcal{K}_1) \quad (46)$$

(for example, see [22]). Suppose finally that there is an isometry $\vartheta : X'(\Sigma) \to X(\Sigma_1)$, where $X'(\Sigma)$ is a Cauchy hypersurface and $X(\Sigma_1)$ is defined as above. The corresponding time shift $\theta : \Gamma_{X'} \to \Gamma_{X_1}$ maps a global transversal surface onto a non-global one. Let us define $s_{\vartheta}$ by

$$s_{\vartheta}o|_{\Gamma_{X_1}} = o \circ \theta^{-1},$$
$$s_{\vartheta}o|_{\Gamma_{X_2}} = 0 \quad \text{for all } o \in \mathcal{S}.$$\nonumber$$

We see immediately that $s_{\vartheta}(\mathcal{S}) = \mathcal{S}_1$.

Now we can apply the Heisenberg picture method as described in subsection [V.A]. The elements of $\mathcal{K}$ are considered as Heisenberg states, and the algebra $R(\mathcal{S})$ contains the Heisenberg observables at the ‘time’ $\Gamma_{X'}$ and $R(\mathcal{S}_1)$ contains those at the ‘time’ $\Gamma_{X_1}$. The result is that the time evolution operator $\hat{s}_{\vartheta}$ maps the algebra $R(\mathcal{S})$ onto its own proper subalgebra $R(\mathcal{S}_1)$ so that the representation $R$ of $\mathcal{S}_1$ is not irreducible, and a Heisenberg state $|a\rangle$ that
is pure with respect to the algebra $R(S)$ is a mixed state $\hat{a}_1$ with respect to the time shifted algebra $R(S_1)$.

An example in which $\vartheta$ maps a global transversal surface onto a non-global one is the Hawking radiation produced by the spherically-symmetric, asymptotically flat space-time associated with a collapsing star (see [22]) (actually, it is a limiting case of the procedure above; moreover, the assumption must be made that the theory in [15] can be generalized to an asymptotically-flat spacetime). In this example, the scalar field is chosen to have a vanishing mass-parameter $m$, and hence the dynamics is determined completely by the conformal structure of the space-time $(M, g)$. The past and future null infinities $I^+$ and $I^-$ are null hypersurfaces in the conformal completion, $\bar{M}$, of $M$. The hypersurfaces $I^-$ and $I^+ \cup H$ are Cauchy hypersurfaces for a zero rest mass field, where $H$ is the event horizon in $M$ (see [24]). They can be considered as limits of time-like Cauchy hypersurfaces. Both $H$ and $I^+$ are closed hypersurfaces in $\bar{M}$. Note that $\Sigma$ and $X_\pm$ can be chosen such that $X_-(\Sigma) = I^-$ and $X_+(\Sigma) = I^+$. The maps $\vartheta$, $\vartheta_+$ and $\vartheta_-$ which were (effectively) used in [22] can be described by means of the Eddington-Finkelstein coordinates $(u, r, \alpha, \beta)$ and $(v, r, \alpha, \beta)$ in some neighbourhoods of $I^-$ and $I^+$ as follows: $\vartheta_-$ is defined by $(v, \infty, \alpha, \beta) \rightarrow (v + t, \infty, \alpha, \beta)$, $\vartheta_+$ by $(u, \infty, \alpha, \beta) \rightarrow (u + t, \infty, \alpha, \beta)$, and $\vartheta$ by $u(v, \alpha, \beta) = v$, $r = \infty$, $dr \rightarrow -dr$, $\alpha(v, \alpha, \beta) = \alpha$, and $\beta(v, \alpha, \beta) = \beta$ (in-coming modes are mapped into out-going ones). This time shift $\vartheta$ is not uniquely determined because $u$ and $v$ are defined up to an additive constant, but most of the physically interesting results do not depend on the choice made. In this situation, the considerations above are applicable, and the result is again a non-unitary evolution that sends pure states into mixed states. This time, the normalization of states is preserved: some information is lost, but the system itself is not.

**C. Algebraic approach**

Hilbert spaces play a less direct role in this approach in which the basic objects are elements of some algebra of local observables on which states are defined as linear functionals.
One can reformulate the algebraic approach in terms of the perennial formalism using the algebra of the smeared fields, $S_{\text{loc}}$. We shall not go into detail here, but just sketch the main ideas.

The local observables are polynomials in the smeared field operators

$$\hat{\kappa}_f, \quad \hat{\kappa}_f \hat{\kappa}_h, \ldots,$$

as well as the regularized stress-energy tensor components $\hat{T}^{\mu\nu}(p)$ at arbitrary points $p$ of the space-time $M$. The stress-energy tensor has an immediate physical interpretation whereas the smeared field operators play only an auxiliary role.

The states are defined as certain linear functionals on the above algebra, with the value of such a state $\sigma$ on an operator $\hat{o}$ having the physical meaning of the expected value. Attention is restricted to so-called ‘quasifree Hadamard states’, whose value on any polynomial of the smeared fields is determined by its value on the following second-order polynomial

$$G_\sigma(f, h) = \sigma(\hat{\kappa}_f \hat{\kappa}_h + \hat{\kappa}_h \hat{\kappa}_f).$$

The bilinear form $G_\sigma(f, h)$ has a kernel $G_\sigma(y_1, y_2)$, so that

$$G_\sigma(f, h) = \int_M \text{d}^4 y_1 \text{d}^4 y_2 G_\sigma(y_1, y_2) f(y_1) h(y_2),$$

which satisfies the field equation in each argument $y_1$ and $y_2$. This leads to the crucial observation that the state can be ‘calculated’ by solving the wave equation. For a Hadamard state, the short-distance behaviour of $G_\sigma(y_1, y_2)$ as $y_1 \to y_2$ is such that the expected value in the state $\sigma$ of the stress-energy tensor is well-defined and can be calculated from $G_\sigma(y_1, y_2)$.

The quasifree Hadamard states do not form a Hilbert space: neither a scalar product—nor a linear combination—of a pair of them is well-defined. However, to obtain a physical interpretation of such a state it is only necessary to calculate the expected value of the stress-energy tensor, and this is feasible. For more details see [25].
V. CONCLUSIONS

We have found an intriguing result: canonical quantization of a system can lead to a non-unitary time evolution. The result has been derived by a careful analysis of global properties of the physical phase space: an aspect that has been rather neglected heretofore. However, more work is necessary to understand the relations between the global properties and the time evolution in some generality. Also, the physical interpretation of the time evolution in parametrized systems needs to be developed further.

In the course of our calculations we have seen that the central commandment of the perennial formalism—to work only with perennials—is in reality not too restrictive since almost everything can be viewed as a perennial. In particular, the usual particle variables of the quantum field theory—for example, creation and annihilation operators—can be considered as perennials ($\mathcal{S}_{\text{can}}$), as can the more local, smeared fields in $\mathcal{S}_{\text{loc}}$.

However, these and other insights gained in our paper have only a relative value in so far as their derivation exploited two special features of our model—the linearity of the field equations, and the existence of a background space-time. A question left for future research is if these structure can be replaced with something that will work in more complicated cases.

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