Pseudospectral method with linear damping effect and de-aliasing technique in solving nonlinear PDEs

Azwani Alias, Nik Nur Amiza Nik Ismail, and Fatimah Noor Harun
School of Informatics and Applied Mathematics, Universiti Malaysia Terengganu, 21030 Kuala Nerus, Terengganu, Malaysia.
E-mail: azwani.alias@umt.edu.my

Abstract. Pseudospectral method is an alternative of finite differences and finite elements method to solve nonlinear partial differential equations (PDEs), especially in nonlinear waves. The Pseudospectral method is very efficient because it use the fast fourier transform to calculate discrete Fourier transform in the algorithm. In this paper, the Pseudospectral scheme is modified by adding the linear damping effect and de-aliasing technique, and has been tested in Ostrovsky equation, where Ostrovsky equation is a modified of Korteweg-de Vries equation with an addition of background Earth’s rotation. The addition of the linear damping is to prevent the possibility of radiated waves re-entering from the boundaries and disturbing the main wave structure. Most of the numerical simulations occur with the aliasing errors due to pollution of numerically calculated Fourier transform by higher frequencies component because of the truncation of the series. To prevent this, the de-aliasing technique is implemented on the nonlinear term and linear damping region by setting of the amplitudes to be zero at the end of both boundaries. Therefore, the simulation results of Pseudospectral method will be smooth without any high frequency errors even for the high amplitude of the waves from initial condition.

1. Introduction
A well known equation that arises in the study of nonlinear waves is the Korteweg-de Vries (KdV) equation. From previous studies, KdV-type equation such as forced KdV, variable KdV and perturbed KdV equation was numerically successfull by pseudospectral method independently without any addition of linear damping effect and de-aliasing technique [1, 2, 3, 4]. However, when an extra term is added to the original KdV equation, particularly a background rotation term, KdV equation becomes modified KdV equation, which also known as Ostrovsky equation in (1),

\[(u_t + \alpha uu_x + \lambda u_{xxx})_x = \gamma u,\]  

with some constant \(\alpha, \lambda\) and \(\gamma\) [5, 6, 7]. Here, \(u(x, t)\) is denote as the amplitude of wave, while \(x\) an \(t\) are space and time variables, respectively. A constant \(\gamma\) is the effect of the rotation while \(\alpha\) and \(\lambda\) are the coefficients of nonlinear and dispersive coefficients, respectively. Equation (1) will become KdV equation when the value of \(\gamma\) is zero.

It is known that the Ostrovsky equation does not support steady solitary wave solutions. The Earth’s rotation effect give a significant result for wave evolution where the long time effect of rotation will destroy the initial internal solitary wave and eventually the nonlinear wave packet...
emerges [8, 9, 10]. By concerning the value \( \lambda \gamma > 0 \) in equation (1), an unsteady wave packet emerges, whereas when \( \lambda \gamma < 0 \), a family of steady wave packet exists [11, 12].

In this paper, an unsteady wave packet is generate from numerical simulations initiated with the KdV solitary wave between amplitude 2 and 32. However, pseudospectral method is no longer suitable for high values of amplitude, particularly for \( a = 8,16 \) and 32. Therefore, in the numerical simulation, we need to increase the number of discretization, \( N \) and decrease the time discretization, \( \Delta t \) to make sure that the numerical simulation will successfully generate smooth results. However, this will lead to more computational time to obtain the results. Whereas, with the low values of \( N \) and high value of amplitude, the numerical simulation will produce an error which is due to the pollution of the numerically calculated Fourier transform by higher frequencies. Therefore, to reduce the computational simulation time and remove the aliasing error of high frequency and noise of simulation, Ostrovsky equation has been solve using pseudospectral method with de-aliasing technique. The addition of linear damping effect at each end of the domain is to prevent the possibility of radiated waves re-entering the region of interest and interfering with the main wave structure.

2. Pseudospectral Method

In this section, we directly focused on the Pseudospectral method in solving Ostrovsky equation by considering the de-aliasing technique and the linear damping effect. Pseudospectral method is an alternative to the finite difference or finite element methods to solve nonlinear PDEs. Generally, in pseudospectral method, the used of Fast Fourier Transform and Discrete Fourier Transform (DFT) are very efficient for the simulations. Basically, DFT convert the real space (time-domain discrete sequence) into Fourier space (frequency-domain discrete spectrum) and substitute the temporal derivative by finite difference approximation.

By assuming periodic boundary conditions for \( u(x,t) \) at \( x = -L \) to \( L \), equation (1) is transformed from \( u(x,t) \) to \( 2\pi \) periodicity dependent variables \( v(\xi,t) \) by introducing \( \xi = sx + \pi \), where \( s = \frac{2\pi}{L} \) as

\[
v_t + \alpha sv_x + \lambda s^3 v_{\xi\xi\xi} = \frac{\gamma}{s} v. \quad (2)
\]

Before we proceed, the nonlinear term should be firstly evaluated by letting \( w(\xi,t) = \frac{1}{2} \alpha s v^2 \). The equation (2) is now be transformed to

\[
v_t + w_{\xi} + \lambda s^3 v_{\xi\xi\xi} = \frac{\gamma}{s} v. \quad (3)
\]

Then, the interval \([0, 2\pi]\) is discretised by \( N + 1 \) equidistant points as in Figure 1. We let \( \xi_0 = 0, \xi_1, \xi_2, \ldots, \xi_N = 2\pi \), so that \( \Delta \xi = \frac{2\pi}{N} \). In this case, \( N \) is always even and is to be a power of 2. So we let \( m = \frac{N}{2} \).

The Discrete Fourier Transform (DFT) of \( v(\xi_j,t) \) for \( j = 0,1,2,\ldots,N-1 \) is denoted by \( \hat{v}(p,t) \) and is given by

\[
\hat{v}(p,t) = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} v(\xi_j,t)e^{-\frac{2\pi ipj}{N}}, \quad (4)
\]

where \( p = -m, -m + 1, -m + 2, \ldots, m - 1 \).
The inverse Fourier Transform of $\hat{v}(p,t)$ for $p = -m, -m + 1, -m + 2, \ldots, m - 1$ is denoted by $v(\xi_j, t)$ and written as

$$v(\xi_j, t) = \frac{1}{\sqrt{N}} \sum_{p=-m}^{m-1} \hat{v}(p,t)e^{i(2\pi p \xi_j / N)},$$

(5)

where $j = 0, 1, 2, \ldots, N - 1$ and $i = \sqrt{-1}$ is the imaginary number.

Therefore the DFT of equation (3) with respect to $\xi$ gives

$$\hat{v}_t(p,t) + ip\hat{w}(p,t) - i\lambda p^3 s^3 \hat{v}(p,t) = -\frac{i\gamma}{ps} \hat{v}(p,t).$$

(6)

At each time step, $\hat{w}(p,t)$ will be known when $\hat{v}(p,t)$ is known. For instance, by using the following finite difference approximations,

$$\hat{v}_t(p,t) \approx \frac{\hat{v}(p,t + \Delta t) - \hat{v}(p,t - \Delta t)}{2 \Delta t},$$

$$\hat{v}(p,t) \approx \frac{\hat{v}(p,t + \Delta t) + \hat{v}(p,t - \Delta t)}{2},$$

and by denoting $\hat{v}(p,t + \Delta t)$ by $\hat{v}_{mt}$, $\hat{v}(p,t - \Delta t)$ by $\hat{v}_{mt}$, our forward scheme for Ostrovsky equation is

$$\hat{v}_{pt} = \frac{1}{1 - is^3 p^3 \Delta t + \frac{i\gamma \Delta t}{ps}}[(1 + is^3 p^3 \Delta t - \frac{i\gamma \Delta t}{ps})\hat{v}_{mt} - 2ip \Delta t \hat{w}(p,t)].$$

(7)

This is a two-step scheme, and in particular to get $\hat{w}(p,t)$, the value of $\hat{v}(p,t)$ is needed as initial condition. The values of $N$ and $\Delta t$ were obtained through numerical experimentation. The solution in physical space is obtained by the inverse discrete Fourier transform from equation (5).

2.1. Linear damping

Now, the Ostrovsky equation (1) is implemented with a linear damping term, $r(x)$ as follows,

$$(u_t + \mu uu_x + \lambda u_{xxx} + r(x) u)_x = \gamma u,$$

where

$$r(x) = \nu \frac{\kappa}{2} \left( (1 + \tanh\kappa(x - 7L/8)) + (1 - \tanh\kappa(x + 7L/8)) \right).$$

for some constants $\nu, \kappa$. Choose $\kappa$ so that $r(x) \approx 0$ for $-3L/4 < x < 3L/4$, that is $\tanh\kappa L/8 \approx 1$, or $\kappa L = 24$. 

\[ \]
We choose the value of $\nu$ so that the damping occurs quickly, for instance the solution in the sponge layer is reduced by 1% in a short time interval $T$, so that $\nu T \approx 0.01$. Thus, $T = 1$ gives $\nu = 0.01$, or $T = 10, \nu = 0.001$. This can be tested to find the best value for the simulations. This additional linear damping also implemented in the coupled nonlinear wave equations [14, 15].

2.2. De-aliasing technique

In order to generate the smooth numerical simulations and removing the aliasing error, the forward scheme is then implemented with de-aliasing using the truncation rule [13]. This error is due to the pollution of the numerically calculated Fourier transform by higher frequencies. The basic idea of de-aliasing is by setting the amplitudes to be zero at one-third part of interval at the end of each boundaries. This setting should be done priorly before solving nonlinear multiplication term in physical space. Therefore, it becomes important in dealing with the nonlinear $uu_x$ and the sponge layer $r(x)u$ terms in the equations.

After implemented with both linear damping term and de-aliasing technique, the forward equation (7) becomes,

$$\hat{v}_{pt} = \frac{1}{1 - is^3p^3 \Delta t + \frac{i\gamma \Delta t}{ps}}[(1 + is^3p^3 \Delta t - \frac{i\gamma \Delta t}{ps})\hat{v}_{mt} - 2ip \Delta t\hat{w}(p,t) - 2R \Delta t].$$

(8)

where $R = r(\xi)\hat{v}$ and $r(\xi)$ is a DFT of linear damping term.

3. Results and Discussions

All the numerical simulations of Ostrovsky equation starts with initial condition as a solitary wave solution from the KdV equation when $\gamma = 0$ such as,

$$u(x,0) = a \text{sech}^2(x/D), \quad \nu a D^2 = 12\lambda,$$

(9)

with the different amplitudes, $a = 2, 4, 8, 16, 32$, and $\nu, \lambda = 1$. The comparison of numerical results between pseudospectral method with and without linear damping term and de-aliasing technique will be considered. Table 1 shows the simulation data contains the value of amplitude in (9), discretization of interval $N$, the boundary $L$, and time discretazation $\Delta t$. Notice that all the constant in equation (1) are unity.

| $a$ | $N$  | $L$  | $\Delta t$ |
|-----|------|------|------------|
| 2   | 1024 | 800  | 0.0001     |
| 4   | 2048 | 800  | 0.0001     |
| 8   | 4096 | 800  | 0.0001     |
| 16  | 4096 | 500  | 0.0001     |
| 32  | 8192 | 600  | 0.0001     |

Numerical results of Ostrovsky equation will be shown by comparing them without an additional de-aliasing technique and a linear damping and vice versa. Figure 2 and Figure 3 refer to the simulation when $a = 2, 4$, respectively when $\alpha = \lambda = \gamma = 1$ using initial condition (9). As we can see, the difference between two plots, where the right plot in Figure 2 and Figure 3
show non-smooth simulation results due to the high frequencies in the Fourier scheme. However, when the scheme is added with both techniques using the same data in Table 1, we obtained a clearer graph and we can observed the unstable wave packet, however this unstable wave packet does not completely separate from the trailing radiation at the end of the simulation.

Figure 4, 6 and 8 refer to amplitudes, $a = 8$, 16 and 32, respectively. Here, the separation of the unsteady wave packet is becoming more clearer than the previous graphs. The right plot in Figure 5 shows the simulation when the scheme is not de-aliased and linear damping is not added for the initial amplitude when $a = 8$ at $t = 40s$. We clearly observed that the results are completely interrupted with the radiated waves and high frequencies of Fourier mode in pseudospectral method. However, the left plot in Figure 5 clearly showed the unsteady wave packet eventually exists from the initial solitary waves after long simulation. The results showed the same behaviour as when $a = 16$ and 32 in Figure 7 and 9, respectively.
Figure 4. Numerical simulation when $a = 8$ with de-aliasing and linear damping term in numerical scheme.

Figure 5. Numerical simulation of $a = 8$ at $t = 40s$. Left: with the additional linear damping de-aliasing technique. Right: without the additional linear damping de-aliasing technique.

Figure 6. Numerical simulation when $a = 16$ with de-aliasing and linear damping term in numerical scheme.
Figure 7. Numerical simulation of $a = 16$ at $t = 5s$. Left: with the additional linear damping de-aliasing technique. Right: without the additional linear damping de-aliasing technique.

Figure 8. Numerical simulation when $a = 32$ with de-aliasing and linear damping term in numerical scheme.

Figure 9. Numerical simulation of $a = 32$ at $t = 2.5s$. Left: with the additional linear damping de-aliasing technique. Right: without the additional linear damping de-aliasing technique.
4. Conclusion
In this paper, Pseudospectral method with the implementation of linear damping effect and de-aliasing technique was used for solving the Ostrovsky equation (1) with solitary wave solution of KdV equation as an initial condition. By adding both, the simulation time can be reduced by choosing the value of \( N \) only up to 8192, compared with the simulation experiments without addition of linear damping and de-aliasing technique, where the value of \( N \) is increased to 16384 to make the numerical runs smoothly. However, \( N = 16384 \) was only valid when the amplitude, \( a = 2, 4, 8 \) and not for \( a = 16, 32 \). For the higher amplitude, particularly for \( a = 16, 32 \), the value of \( N \) should be increased and the value of \( \Delta t \) decreased, but the numerical simulation will took a longer time to generate results. Therefore, to reduce the computational time, firstly, a linear damping effect was added to prevent the possibility of radiated waves from re-entering the domain of interval and interrupt the dominant wave packet. Secondly, de-aliasing technique was introduced by letting some part of the amplitudes to be zero. Significantly, this technique helped to avoid high frequencies from the aliasing error in numerical simulations and the noise in propagation. These techniques could help the researcher to obtain good numerical results with less computational time in solving nonlinear PDEs.

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