ANTIPRISMLESS, 
OR: REDUCING COMBINATORIAL EQUIVALENCE TO PROJECTIVE EQUIVALENCE IN REALIZABILITY PROBLEMS FOR POLYTOPES

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Abstract. This article exhibits a polytope that has no antiprism, answering a question posed by Bernt Lindström. This uses a result by Below that says, given any polytope with vertices having algebraic coordinates, there is a combinatorial “stamp” polytope with a specified face that is projectively equivalent to the given polytope in all realizations. This article provides a different proof of this result. The stamp polytope provides a general method for solving certain types of realizability problems, and this method is first presented for polytopes with algebraic coordinates and then with real coordinates.

1. Introduction

The combinatorial type of a polytope is defined by the partial ordering of its face lattice. We generally “see” a partial ordering by drawing its Hasse diagram (see Figure 1). If we draw the Hasse diagram of the face lattice of a polytope, we may observe that it resembles the 1-skeleton of a larger polytope. For example, the Hasse diagram of a simplex’s face lattice is the 1-skeleton of a hypercube balancing on a vertex. The 1-skeleton alone does not uniquely determine the combinatorial type of a polytope, but there is a natural extension of the Hesse diagram that does, the intervals of a poset ordered by inclusion. When the original poset is a combinatorial polytope, the resulting poset of intervals shares some basic properties with combinatorial polytopes, such as being a lattice and satisfying Euler’s formula [12].

In 1971 Lindström asked whether the intervals of a polytope’s face lattice is again a combinatorial polytope [13]. This article answers, no. Moreover, it will construct a 4-polytope such that the poset of intervals of its face lattice is not the combinatorial type of

Figure 1. A triangle and a Hasse diagram of its face lattice.
any polytope. An equivalent question was considered by Broadie in [6] to better understand linear optimization algorithms, and appears in Grünbaum’s textbook [10]. The question is “Does every polytope have an antiprism?” An antiprism is the combinatorial dual of the interval polytope (see Figure 2). Anders Björner later announced that the answer is yes in 3 dimensions; every 3-polytope does have an antiprism [5].

![Image of antiprism of a cube](image)

**Figure 2.** The antiprism of a cube.

Broadie gave sufficient conditions for a polytope to have an antiprism [6]. These conditions ask for a perfectly centered realization of the original polytope, a term coined later in [8]. Perfectly centered has a nice physical interpretation; it says a polytope (with a point in its interior specified as the center of mass) can rest on any face without falling over. Hence every 3-polytope has a realization such that it can rest on any face, but this fails for every realization of the combinatorial 4-polytope constructed here. This article will give necessary and sufficient conditions for the realizability of a polytope’s antiprism. Then to find a 4-polytope without an antiprism, we will use the stamp of a polytope.

In $\mathbb{R}^3$ it is known that faces of polyhedra are prescribable [2]. That is, given a realization of a face of a combinatorial 3-polytope, it is always possible to extend this to a realization of the entire polytope. This does not hold in higher dimension [11][18], and a stamp gives the strongest possible violation of this for polytopes in general. A stamp is a combinatorial polytope that forces a specified face to have a fixed projective type in all realizations. In his unpublished thesis, Below constructed a stamp for any projective type of polytope having vertices with coordinates in the real algebraic completion of the rationals, $\mathbb{R}_{alg}$ [4]. This article will give a different stamp construction.

![Images of three combinatorially equivalent polytopes](image)

**Figure 3.** Three combinatorially equivalent polytopes. Only the left two are projectively equivalent.

Stamps can be used to answer the following type of questions. Does a certain geometric property hold for some realization of every combinatorial type of polytope? Such questions
are made difficult by the universality theorem for polytopes \[14\]. The universality theorem states that for any primary basic semialgebraic set \( S \), there exists a combinatorial 4-polytope with realization space (up to isometry) that is homotopy equivalent to \( S \). The proof of this universality theorem shows us how to encode a system of polynomial equations and inequalities into a polytope in such a way that a realization of the polytope corresponds to a solution of the given system. The stamp construction here uses similar techniques to force the coordinates of certain vertices of a polytope to satisfy a system of polynomial equations and inequalities.

Unlike 2 and 3 dimensions where combinatorial polytopes have a nice characterization, as a consequence of universality for 4-polytopes, there is no such nice characterization in dimensions 4 and higher. Furthermore, searching for a realization of a certain combinatorial polytope that satisfies a certain geometric property, can be as hard as searching for a point in a semialgebraic set. This may be difficult, since a semialgebraic set may be disconnected or have holes or other unwanted features for a search space.

For a geometric property that is general enough to be relevant in any dimension, if the property holds for a polytope, then in many cases, it holds for the polytope’s faces as well. When this happens we say it is a property that faces inherit. An example of such a property is, “The polytope’s vertices have rational coordinates”. Trivially, if this is true of a polytope then it is also true of its faces. This article will show that the problem of determining whether a property (of polytopes in \( \mathbb{R}_{\text{alg}}^d \) that faces inherit) holds for some realization of every combinatorial type can be reduced to determining whether the property holds for some realization of every projective type. This is a considerable improvement since, in contrast to the realization spaces of polytopes with fixed combinatorial type, the space of polytopes (up to isometry) with fixed projective type is convex. We will also see that, when such a realization does not always exist, there is a gap of at most 2 between the highest dimension where this fails for combinatorial types and where it fails for projective types. Returning to the example property of having rational coordinates, this reduces the question “Can every combinatorial 4-polytope be realized with rational coordinates?” to the question “Does every polygon (in \( \mathbb{R}_{\text{alg}}^2 \)) have a projective transformation making its coordinates rational?”. Answering the first question was a considerable hurdle that paved the way for Gale duality and the universality theorem \[19\]. The second question, however, is easily answered, no. Just consider the regular pentagon.

![Pentagon](image)

**Figure 4.** For any projective copy of a regular pentagon, the cross-ratio \((a, b/c, d) = \frac{1 + \sqrt{5}}{2}\) is irrational, so the vertices cannot have rational coordinates.

Usually we are interested in polytopes in \( \mathbb{R}^d \) rather than \( \mathbb{R}_{\text{alg}}^d \), and this article will also prove an analogous reduction theorem in \( \mathbb{R}^d \). If we were to naively replace \( \mathbb{R}_{\text{alg}}^d \) in the algebraic version of the reduction theorem with \( \mathbb{R}^d \), then the resulting statement would be false. To see why, consider the property “Every vertex coordinate is algebraic”. This is a property that faces inherit, and every combinatorial type of polytope has a realization where this property
holds, but there are polytopes such that no projective copy has algebraic coordinates. This is not just an arbitrary example where the $\mathbb{R}^d_{algebraic}$ version of the reduction fails in $\mathbb{R}^d$; it is the essential example. The $\mathbb{R}^d$ version of the reduction theorem replaces the restriction that the space be algebraic to a restriction that the property be algebraic. More precisely, the property must be formulated for each combinatorial type of polytope in the language of real closed fields. This follows easily from basic model theory; see [2, Section IV.23].

Section 2 deals with the reduction from combinatorial equivalence to projective equivalence in $\mathbb{R}^d_{algebraic}$. The analogous reduction in $\mathbb{R}^d$ is left to Section 3. Section 3 proves some basic results about prismoids, then shows that the existence of a balanced pair is equivalent to the existence of an antiprism, and finally presents a polytope without an antiprism. Section 4 constructs the stamp of a polytope. Both Section 3 and Section 5 depend on Section 2, and Section 2 depends on Section 4, but Section 4 is placed later in the text to spare the reader the extensive details of the stamp construction until motivated by its use. Finally, Section 6 leaves the reader with some remaining questions.

In this article, we assume that a polytope’s faces are indexed by a poset according to the polytope’s combinatorial type. We will frequently deal with several polytopes indexed by the same poset and the implied correspondences between their faces. We should keep in mind that some polytopes have non-trivial symmetry, and some of the properties that we will consider depend on the indexing of the faces in a way that may not be preserved by these symmetries. A brief glossary of notation follows. More on notation and a list of basic propositions are given in Appendix A.

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Later we will see how to construct the stamp of an algebraic polytope, but in this section we consider the consequences of its existence. For now, a stamp is the pair $(S_P, f_P)$ implied by the following theorem.
Theorem 1. Given an algebraic $d$-polytope $P$, there exists a combinatorial $(d+2)$-polytope $S_P$ with a specified face $f_P \in S_P$ such that for any realization $S$ of $S_P$, the specified face is projectively equivalence to the given polytope, face$(S, f_P) \sim_P P$.

Such a polytope was constructed in [4, p. 134] and later in [7]. The stamp helps us answer questions about properties that faces inherit, or more generally the following class of predicates. Let $\psi$ be a predicate of several algebraic polytopes of the same combinatorial type. We say the face $f \in \mathcal{P}$ projectively inherits $\psi$ when, if $P_1, \ldots, P_n$ are realizations of $\mathcal{P}$ such that $\psi(P_1, \ldots, P_n)$ holds, then there are projectivities $\pi_1, \ldots, \pi_n$ such that $\psi(\pi_1(\text{face}(P_1, f)), \ldots, \pi_n(\text{face}(P_n, f)))$ holds. Recall that a ridge of a polytope is a face of co-dimension 2.

Corollary 2. Let $\psi$ be a predicate of several algebraic polytopes of the same combinatorial type that ridges projectively inherit. Then, $\psi$ holds for some realizations of every combinatorial type of polytope if and only if it holds for some realizations of every algebraic projective type. Moreover, there can be a gap of at most 2 dimensions,

$$
\forall P \in \mathbb{R}_\text{alg}^{d+2} \exists P_1, \ldots, P_n \in [P]_{\text{comb.}} \psi(P_1, \ldots, P_n)
= \forall P \in \mathbb{R}_\text{alg}^d \exists P_1, \ldots, P_n \in [P]_{\text{proj.}} \psi(P_1, \ldots, P_n)
= \forall P \in \mathbb{R}_\text{alg}^d \exists P_1, \ldots, P_n \in [P]_{\text{comb.}} \psi(P_1, \ldots, P_n).
$$

Proof. Since projective equivalence is finer than combinatorial equivalence, we have the ‘if’ direction trivially. For the other direction, let $\psi$ be a predicate that ridges projectively inherit and suppose $\psi$ holds for some realizations of every combinatorial polytope. Consider an algebraic polytope $P$. Since the stamp $S_P$ has realizations where $\psi$ holds, it must also hold for some projective copies of the face $f_P$ of each of these realizations, and these faces are all projectively equivalent to $P$ by Theorem 1. Thus, $\psi$ holds for some projective copies of $P$. Also, since $S_P$ is 2 dimensions higher than $P$, if $\psi$ holds for some realizations of every combinatorial polytope up to dimension $d + 2$, then it holds for some projective copies of every algebraic polytope up to dimension $d$.

3. Antiprisms

A polytope is a prismoid when every vertex of the polytope is in one of two nonintersecting faces, which we call the bases of the prismoid. That is, every prismoid $P$ is of the form

$$
P = B_0 \cup B_1 = \{t_0B_0 + t_1B_1 : t_i \geq 0, t_0 + t_1 = 1\}
$$

where $B_0, B_1$ are disjoint faces. The sides of the prismoid are faces that are not contained in either base, along with the trivial side $\perp$. When a combinatorial polytope is a prismoid, we call it a combinatorial prismoid. Some examples of prismoids are pyramids, tents, prisms, and antiprisms.

We define a purely combinatorial construction, called an abstract prismoid. The definition is motivated by the fact that a face of a prismoid is determined by its intersection with each of the bases. For bounded posets $B_0, B_1$, an abstract prismoid $\mathcal{P}$ with these bases is a bounded subposet of the categorical product $B_0^{\text{cat}} \times B_1$ such that the bases themselves are included as $(f_0, \perp), (\perp, f_1) \in \mathcal{P}$ for all $f_i \in B_i$. All other faces and $\perp = (\perp, \perp)$ are sides, denoted side($\mathcal{P}$).
Lemma 3. Every combinatorial prismoid is isomorphic to an abstract prismoid. And, an abstract prismoid $P \subset B_0 \times B_1$ can be realized if and only if there are realizations $B_i$ of $B_i$ such that the dual of the common refinement of their normal fans is indexed by the sides $\text{indx}(\text{nfan}(B_0) \cup \text{nfan}(B_1))^* = \text{side}(P)$, in which case $[B_0] \cup [B_1]$ realizes $P$.

![Diagram of a prismoid with triangular and square base](image)

**Figure 5.** Top: A prismoid with a triangular and square base. Bottom: The prismoid bases and a horizontal slice with the common refinement of the normal fans of the bases.

**Proof.** For the first part, Every combinatorial prismoid is a bounded poset, and every face can be uniquely identified by its intersection with the bases, and the bases are faces, so every combinatorial prismoid is isomorphic to an abstract prismoid. The second part follows from the fact that we can project a prismoid so that its bases are in parallel hyperplanes, in which case a horizontal slice between these hyperplanes is a weighted Minkowski sum of the bases, and the normal fan of the Minkowski sum of a pair of polytopes is the common refinement of their normal fans [13, Proposition 7.12].

Specifically, for the ‘if’ direction of the second part, suppose we have such realizations $B_0$, $B_1$. Recall that every face of a polytope (in particular $P = [B_0] \cup [B_1] \subset \mathbb{R}^d$) is the solution set of some linear optimization problem. Specifically, for a face $f$ these are the optimization problems with linear objective function in the relative interior of the normal cone of $f$. Notice that the optimal solutions in $[B_i]$ depend only on the restriction of the linear objective function $v^* = [w^* c]$ to the first $d - 1$ coordinates. For $q_i \in [B_i]$, we have $v^*q_0 = v^*q_0$ and $v^*q_1 = v^*q_1 + c$. For optimal solutions $q_i \in B_i$ to objective $w^*$, setting $c = w^*(q_0 - q_1)$ we get $v^*q_0 = v^*q_1$. Hence, a non-trivial pair of faces $(f_0, f_1)$ defines a face of $P$ if and only if there is a vector in the relative interior of the normal cone of the faces $f_0$ of $B_0$ and $f_1$ of $B_1$. Therefore, $\text{side}(\text{indx}(P))$ is the common refinement of the normal fans of $B_0$ and $B_1$. 
For the ‘only if’ direction suppose we have a realization $P$ of some abstract prismoid $P \subset B_0 \times B_1$. Let $B_0 = \text{face}(P, (\top, \bot))$ and $B_1 = \text{face}(P, (\bot, \top))$. Then there is some projective transformation $\pi$ sending $B_i$ into the hyperplane $\{x : x_d = i\}$. Hence $\pi(P) = [B_0'] \cup [B_1']$ for some realizations $B_i'$ of $B_i$, and again side($P$) is the common refinement of the normal fans of $B_0'$ and $B_1'$.

In the introduction we defined an (antiprism) interval polytope of $P$ by giving its combinatorial type as the intervals of $\text{indx}(P)$ ordered by (reverse) inclusion. We now define the antiprism as the order isomorphic abstract prismoid. The abstract antiprism of a bounded poset $P$ is the poset $\{(g, f^*) \in P^{\text{cat}} \times P^* : g \leq f\}$. When an abstract antiprism can be realized we call it a combinatorial antiprism.

We call a pair of polytopes $P_1, P_2$ balanced, when they are centered, have the same combinatorial type, and the relative open normal cone of a face $g$ of $P_1$ intersects the relative open face $f$ of $P_2$ if and only if $f$ is greater than $g$.

\[
\text{balance}(P_1, P_2) := \forall g, f. \ (\exists x \in \text{ncone}(P_1, g)^o \cap \text{face}(P_2, f)^o \iff g \leq f).
\]

Observe that balance($P_1, P_2$) implies balance($P_2^*, P_1^*$), but that balance is not a symmetric relation, see Figure 6.

**Figure 6.** Left: A pair of 4-gons that are balanced.
Right: The same pair of 4-gons in reverse order is not balanced.

**Theorem 4.** A combinatorial polytope has an antiprism if and only if it has a balanced pair.

**Proof.** By Lemma 3, $P$ has an antiprism if and only if there is are realizations of $P$ of $P$ and $Q$ of $P^*$ such that the common refinement of their normal fans are indexed by minimal pairs $(g, f^*) \in P^{\text{cat}} \times P^*$ such that $g \leq f$. The common refinement of the normal fans consists of minimal non-empty intersections, so this is equivalent to the statement, “$\text{ncone}(P, g)^o$ and $\text{ncone}(Q, f^*)^o$ intersect if and only if $g \leq f$”. Since $\text{ncone}(Q, f^*)^o = \text{cone}(Q^*, f)^o = \mathbb{R}_{>0} \text{face}(Q^*, f)^o$ and $Q^*$ realizes $P$, this is exactly the conditions for the pair to be balanced balance($P, Q^*$).

We say a polytope is **perfectly centered** when the orthogonal projection of the origin into the affine closure of each face is in the relative interior of that face. We now see that this is equivalent to the polytope being balanced with itself.

**Lemma 5.** A polytope is perfectly centered if and only if it is balanced with itself.
Proof. If balance($P, P$), then the orthogonal projection of the origin into the affine span of face($P, f$) is exactly ncone($P, f^\circ \cap face(P, f^\circ)$, so $P$ is perfectly centered. Hence the ‘if’ direction of the lemma holds. It remains to show the ‘only if’ direction.

Consider a perfectly centered polytope $P$. With this, span(face($P^*, g^*$)) intersects face($P, f^\circ$) if $g \leq f$, and for $g = f$ this intersection is a single point $x_g$, which implies span(face($P^*, g^*$)) is disjoint from face($P, g^*^\circ$) for $g' < g$.

Consider a facet $g$. Now span(face($P^*, g^*$)) is a line, and $x_g \neq 0$ is on this line. If $x_g \notin$ ncone($P, g$), then we would have $-x_g \in$ ncone($P, g$), but from the definition of normal cone this would give $(-x_g, x_g) \geq 0$, which is impossible. Thus, ncone($P, g)^\circ$ intersects face($P, g)^\circ$.

Consider a proper face $g$, other than a facet. If ncone($P, g)^\circ$ were disjoint from face($P, f')^\circ$ for some $f' \geq g$, then we could extend span(face($P, f'))$ to a homogeneous hyperplane $h$ such that face($P^*, g^*)^\circ$ is on one side. If all facets $f \geq g$ were on one side of $h$, then $h$ would be a supporting hyperplane, but a centered polytope cannot have a supporting homogeneous hyperplane. Hence, there is some facet $f \geq g$ such that $h$ separates face($P, f)^\circ$ from face($P^*, g^*^\circ$). But, face($P^*, g^*)^\circ$ contains face($P, f)^\circ$, so this would make face($P, f)^\circ$ disjoint from ncone($P, f)^\circ$, which we have just seen to be impossible. Thus, the perfectly centered condition implies the ‘if’ direction of the balanced condition.

Now, the boundary of the normal cone $\partial$ ncone($P, g$) intersects the boundary of the slice $\partial P \cap$ span(face($P^*, g^*$))) in a topological sphere that separates ncone($P, g)^\circ$ from all other faces intersecting span(face($P^*, g^*$)). Thus ncone($P, g)^\circ$ does not intersect face($P, g')$ for $g' \notin [g, 1]$, which means the perfectly centered condition implies the ‘only if’ direction of the balanced condition is aswell.

Unlike 4 dimensions, where the problem of realizing polytopes has no easy solution, the situation is much simpler in 3 dimensions [14]. Every 3-polytope has a particularly nice realization called a midscribed polytope [17]. Recall that a 3-polytope is midscribed when every edge is tangent to the unit sphere.

**Theorem 6.** Midscribed polytopes are perfectly centered. Hence, every combinatorial 3-polytope has an antiprism.

Proof. Each edge of a midscribed polytope is tangent to the unit sphere, so the orthogonal projection of the origin into the line spanning that edge is exactly the point of tangency. Hence, the perfectly condition holds for edges. For each facet of a midscribed polytope, the plane spanning that facet intersects the unit ball in a disk, and each edge of the facet is tangent to the disk. In other words, each facet circumscribes the disk where it intersect the unit ball. The orthogonal projection of the origin into the plane spanning the facet is the center of the disk. Hence, the condition holds for facets. For other faces the condition is trivial. Thus midscribed polytopes are perfectly centered, and since every combinatorial 3-polytope has such a realization, every combinatorial 3-polytope has an antiprism.

We now set about constructing a polytope without an antiprism. We only consider dimension 4, since this immediately implies the result for higher dimensions. By Lemma 3 this is equivalent to finding a combinatorial 4-polytope without a balanced pair. We use Corollary 2 to reduce this problem to finding an algebraic polygon that cannot be balanced by projective transformations. To use Corollary 2, however, we need faces to projectively inherit balance. To get some intuition why they do, notice that the interval polytope of $P$ has among its faces all of the interval polytopes of the faces of $P$. By Lemma 3 if a combinatorial polytope $P$ has a balanced pair, then it has an interval polytope, which means
Lemma 7. Faces projectively inherit the predicate balance.

Proof. Let \( P_1, P_2 \) be a pair of centered realizations of \( \mathcal{P} \) such that balance\((P_1, P_2)\). By Lemma 3 these are bases of an antiprism \( A = [\begin{array}{c} P^*_1 \\ -1 \end{array}] \cup [\begin{array}{c} P^*_2 \\ -1 \end{array}] \). Note that the faces of \( A \) are

\[
\text{face}(A, (g, f^*)) = \text{face}\left([\begin{array}{c} P^*_1 \\ 1 \end{array}], g\right) \cup \text{face}\left([\begin{array}{c} P^*_2 \\ -1 \end{array}], f^*\right)
\]

for \( g \leq f \), and the cones over \( \text{face}(P^*_1, f) \) and \( \text{face}(P^*_2, f^*) \) are in complementary linear subspaces.

Let \( A_f := \text{orgn}(p, \text{face}(A^*, (1^*, f)))^\circ \) where \( p \) is some point in the relative interior of \( \text{face}(A^*, (1^*, f)) \), and index the faces of \( A_f \) by \( (h, g^*) \in [1, f] \) \( \text{cat} \ [f^*, 1^*] \in \mathcal{P} \) such that \( h \leq g \). We claim that for any \( f \in \mathcal{P} \), \( A_f \) is an antiprism of the combinatorial polytope \([1, f]\), and its bases \( F_1 = \text{face}(A_f, (f, f^*)) \) and \( F_2^* = \text{face}(A_f, (1, 1^*)) \) are projective copies of \( \text{face}(P_1, f) \) and \( \text{face}(P_2^*, f^*) \) respectively. For the base \( F_1 \), the claim holds by the following calculation, using the basic propositions in Appendix A.

\[
\text{cone}(F_1) = \text{cone}(A_f, (f, f^*)) = \text{cone}(\text{orgn}(p, \text{face}(A^*, (1^*, f)))^\circ, (f, f^*)) = \text{rncone}(\text{orgn}(p, \text{face}(A^*, (1^*, f))), (f^*, f)) = \text{orgn}(p, p + \text{rncone}(\text{face}(A^*, (1^*, f)), (f^*, f))^\circ) = \text{orgn}(p, p + \text{face}(\text{cone}(A, (f, f^*))^*, (1^*, f)))^\circ = \text{orgn}(p, p + \text{face}((\text{cone}\left([\begin{array}{c} P^*_1 \\ 1 \end{array}], f) \cup \text{cone}\left([\begin{array}{c} P^*_2 \\ -1 \end{array}], f^*)^\circ, (1^*, f)))^\circ = \text{orgn}(p, p + \text{face}(\text{cone}\left([\begin{array}{c} P^*_1 \\ 1 \end{array}], f^* \oplus \tilde{0})^\circ, (1^*, f)))^\circ = \text{orgn}(p, p + \text{cone}(\left([\begin{array}{c} P^*_1 \\ 1 \end{array}], f\right)^\circ, (1^*, f)))^\circ = p + \text{cone}(\left([\begin{array}{c} P^*_1 \\ 1 \end{array}], f\right)^\circ, (1^*, f)))^\circ
\]

Therefore, \( F_1 \) is projectively equivalent to \( \text{face}(P_1, f) \). The same calculation applied to \( \text{cone}(F_2^*) \) gives that \( F_2^* \) is projectively equivalent to \( \text{face}(P_2^*, f^*) \). By Lemma 3 again, we have balance\((F_1, F_2)\). Thus, faces projectively inherit balance. \( \square \)

Now consider the problem of balancing two projective copies of a polygon. Since applying a projective transformation to one and the polar transformation to the other preserves balance (see Proposition A.3), we can reduce the problem to applying a projective transformation to only one polygon and keeping the other polygon fixed. Furthermore, a projective transformation consists of an affine part and a perspectivity, but a perspectivity applied to a single vector only scales that vector. To see this, compare the matrix representation of a projective transformation to that of the affine part of the same transformation.

\[
\begin{bmatrix}
A & b \\
c^* & 1
\end{bmatrix} (x) = A x + b \\
c^* x + 1
\]

\[
\begin{bmatrix}
A & b \\
0 & 1
\end{bmatrix} (x) = A x + b
\]

all of its faces also have interval polytopes. Using Lemma 3 in the other direction, for any face of \( \mathcal{P} \) we get a balanced pair. But to show projective inheritance, we need such a pair to be projective copies of the original polytopes' faces.
Since scaling vectors by positive values does not change the cone of positive linear combinations of those vectors, and balance depends on the intersection of cones, we can reduce the problem further to balancing a polygon by applying an affine transformation to one copy.

We will now see informally why an affine transformation cannot always balance polygons. Consider the direction vectors of the vertices of a polygon. Two polygons are balanced if and only if the vectors alternate around the unit circle between belonging to one polygon and the polar of the other. For an affine transformation to balance one polygon with another, it may have to change some of these direction vectors to make them alternate. If we think of moving a transformation continuously from the identity to one that balances the pair, along the way the vectors will turn clockwise or counter-clockwise on the circle. The obstruction we will see requires an affine transformation to turn too many direction vectors alternately clockwise and counter-clockwise. To get an idea of how many direction vectors is too many we decompose an affine transformation into parts and see how many vectors each part can handle. Consider the orthogonal linear part (rotating), symmetric positive definite linear part (stretching), and translational part of an orientation preserving affine transformation. The orthogonal part turns all vectors in the same way. The spd part divides the circle into 4 quadrants where direction vectors turn alternately clockwise and counter-clockwise. And, the translational part divides the circle into 2 halves where direction vectors turn the opposite way. Naively adding this up we get 7 regions. The polygon we construct will require an affine transformation to turn 8 vectors alternately to balance a polygon with a copy of the original.

![Figure 7](image)

**Figure 7.** From the left, the way the directions of vectors turn by rotating, stretching, translating.

With these limitations in mind, let $G$ be the polygon with vertices $(8, 5), (7, 7)$ and all permutations and changes of sign: $(8, 5), (7, 7), (5, 8), (-5, 8), (-7, 7), (-8, 5), (-8, -5), (-7, -7), (-5, -8), (5, -8), (7, -7), (8, -5)$.

**Lemma 8.** No two projective copies of $G$ are balanced.

**Proof.** The polar $G^*$ has vertices $(\frac{1}{5}, 0), (\frac{3}{27}, \frac{1}{27})$ and all permutations and changes of sign. Balance is invariant under positive scaling, so we scale $G^*$ by 21 for convenience, giving vertices $(\frac{21}{5}, 0), (2, 1)$ instead. We see that $G$ is not perfectly centered since that would require the slope $m$ of the outward normal vector of the edge between $(8, 5)$ and $(7, 7)$ to be between the slopes of the vertices, $\frac{5}{8} < m < 1$, but the slope is $m = \frac{1}{7} < \frac{5}{8}$ as seen from the vertex $(2, 1)$ of $21G^*$. By construction, the reflection group of $G$ and $G^*$ is the same as that
Figure 8. A polygon that cannot be balanced by projective transformations.

of the unit square, and is given by the matrices
\[
\begin{bmatrix}
1 & 0 \\
0 & 1 \\
\end{bmatrix}, \quad
\begin{bmatrix}
0 & 1 \\
1 & 0 \\
\end{bmatrix}, \quad
\begin{bmatrix}
0 & -1 \\
1 & 0 \\
\end{bmatrix}, \quad
\begin{bmatrix}
-1 & 0 \\
0 & 1 \\
\end{bmatrix}, \quad
\begin{bmatrix}
-1 & 0 \\
0 & -1 \\
\end{bmatrix}, \quad
\begin{bmatrix}
0 & -1 \\
-1 & 0 \\
\end{bmatrix}, \quad
\begin{bmatrix}
1 & 0 \\
0 & -1 \\
\end{bmatrix}, \quad
\begin{bmatrix}
1 & 0 \\
0 & -1 \\
\end{bmatrix}.
\]

These transformations give us a total of 8 places where the perfectly centered condition is violated. Consider an affine transformation \( T \) acting on \( G^* \) by
\[
T \begin{bmatrix} x \\ y \end{bmatrix} := \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} s \\ t \end{bmatrix}.
\]

For balance \((G, T^{-1}G) = \text{balance}(T(21G^*)^*, G)\) to hold, we must have \(a, d > 0\). For \(a > 0\), this is because \(T \left( \frac{21}{8}, 0 \right)\) must point to the right and \(T \left( -\frac{21}{8}, 0 \right)\) must point to the left,
\[
T \left( \frac{21}{8}, 0 \right) = \frac{21}{8}a + s > 0 \quad T \left( -\frac{21}{8}, 0 \right) = -\frac{21}{8}a + s < 0.
\]

If \(s \leq 0\) then the first inequality implies \(a > 0\), and if \(s \geq 0\) then the second inequality implies \(a > 0\). The same holds for \(d\) because of the corresponding inequalities in the 2\(^{nd}\) coordinate.

Additionally, the image of \((2,1)\) must have slope greater than \(\frac{5}{8}\), and this must also be the case for \(T\) conjugated by all elements of the reflection group. The image of \((2,1)\) by all conjugates is
\[
\begin{bmatrix}
a2 + b1 + s \\
c2 + d1 + t
\end{bmatrix}, \quad
\begin{bmatrix}
d2 + c1 + t \\
b2 + a1 + s
\end{bmatrix}, \quad
\begin{bmatrix}
d2 - c1 - t \\
-b2 + a1 + s
\end{bmatrix}, \quad
\begin{bmatrix}
a2 - b1 - s \\
-c2 + d1 + t
\end{bmatrix}, \quad
\begin{bmatrix}
a2 + b1 - s \\
c2 + d1 - t
\end{bmatrix}, \quad
\begin{bmatrix}
d2 + c1 - t \\
b2 + a1 - s
\end{bmatrix}, \quad
\begin{bmatrix}
d2 - c1 + t \\
-b2 + a1 - s
\end{bmatrix}, \quad
\begin{bmatrix}
a2 - b1 + s \\
-c2 + d1 - t
\end{bmatrix}.
\]
For the first of these vectors the slope requirement is given by the following inequality.

$$\frac{T(2,1)_2}{T(2,1)_1} = \frac{c2 + d1 + t}{a2 + b1 + s} > \frac{5}{8}$$

Equivalently, $-10a - 5b + 16c + 8d - 5s + 8t > 0$. Putting the inequalities we get from all these slope requirements with the sign requirements of $a,d$ together we get the matrix inequality

$$\begin{bmatrix}
-10 & -5 & 16 & 8 & -5 & 8 \\
8 & 16 & -5 & -10 & 8 & -5 \\
8 & -16 & 5 & -10 & 8 & 5 \\
-10 & 5 & -16 & 8 & 5 & 8 \\
-10 & -5 & 16 & 8 & 5 & -8 \\
8 & 16 & -5 & -10 & -8 & 5 \\
8 & -16 & 5 & -10 & -8 & -5 \\
-10 & 5 & -16 & 8 & -5 & -8 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0
\end{bmatrix}\begin{bmatrix}
a \\
b \\
c \\
d \\
s \\
t
\end{bmatrix} > 0.$$
4.1. Gluing and Whittling.

Throughout this section we will build up large combinatorial polytopes by combining smaller combinatorial polytopes. The way we combine these is by gluing, and to a lesser extent whittling. Given a pair of combinatorial polytopes (or more generally bounded posets) $P_0, P_1$, a facet (coatom) $f_i \in P_i$ of each, and an isomorphism $\varphi : [1, f_0] \to [1, f_1]$ between the respective face lattices of these facets, we glue $P_0$ and $P_1$ along these facets by removing the facets from each and identifying corresponding faces by the given isomorphism,

$$P_0\#_\varphi P_1 := (P_0 \setminus f_0) \cup (P_1 \setminus f_1)/\sim_\varphi$$

where $g_0 \sim_\varphi g_1$ when $\varphi(g_0) = g_1$ and $\top \sim_\varphi \top$, and the partial order on these equivalence classes is given by $G \leq F$ when $\exists g \in G, \exists f \in F. g \leq f$. We indicate this by a gluing diagram, which consists of a box for each poset and an edge between posets that are glued together,

$$\begin{array}{c}
\boxtimes P_0 \quad \boxtimes P_1
\end{array} := P_0\#_\varphi P_1.$$

We see examples of polytopes glued together in 3 dimensions in Figure 13. We call the dual operation **whittling**, denoted $P_0\#^*_\varphi P_1 := (P_0^*\#_\varphi P_1^*)^*$, and we say the vertex $v$ of $P_0$ was **whittled** when $[v, \top]$ is the domain of $\varphi$. We will generally glue combinatorial polytopes along facets that are necessarily flat. A $d$-polytope is **necessarily flat** when any realization of its $(d-1)$-skeleton in a space of arbitrarily high dimension will be contained in a subspace of dimension $d$. The following results are shown in [14, Section 3.2].

![Figure 9. Top: A pair of polytopes and the result of gluing the pair together Bottom: A Schlegel diagram of each polytope.](image)

**Lemma 10.** Any pair of polytopes with projectively equivalent facets, can be glued along those facets.

Formally, for any facets $F_i$ of $P_i$ such that $F_1 \lproj F_2$, there exists a projectivity $\pi$ such that

$$\indx(\pi P_1 \cup P_2) = \indx(P_1)\#_\varphi \indx(P_1)$$

where $\varphi : \indx(F_1) \to \indx(F_2)$ is the isomorphism induced by $\pi$. 

Lemma 11. Any realization of a pair of combinatorial polytopes glued along a necessarily flat facet of each can be decomposed into the union of realizations of the given pair that intersect along the given facets.

Formally, for any facets $f_i$ of $\mathcal{P}_i$ and isomorphism $\varphi : [\bot, f_1] \to [\bot, f_2]$, if $[\bot, f_1]$ is necessarily flat then for all realizations $P$ of $\mathcal{P}_0 \# \mathcal{P}_1$, $P = P_1 \cup P_2$ where $P_1$ realizes $\mathcal{P}_i$ and $P_1 \cap P_2 = \text{face}(P_1, f_1) = \text{face}(P_2, f_2)$.

Lemma 12. Pyramids and Prisms of dimension at least 3 are necessarily flat.

4.2. Completion Conditions.

The defining property of a stamp polytope is that a certain face is rigid up to projective transformations. We will construct stamps from other polytopes with faces that are less constrained by gluing these polytopes together in ways that combine these constraints. We call such constraints, completion conditions. Informally, the completion condition from a face to a polytope is the condition a realization of the face must satisfy to be completed to a realization of the entire polytope. Formally, for a bounded poset $\mathcal{P}$ and $f \in \mathcal{P}$, we say a realization $F$ of $[\bot, f]$ can be completed from $f$ to $\mathcal{P}$ when there exists a realization $P$ of $\mathcal{P}$ such that $F = \text{face}(P, f)$, and we call any sequence of statements $\Gamma$ about a polytope $F$ the completion condition for $F$ from $f$ to $\mathcal{P}$ when $\Gamma$ holds if and only if $F$ can be completed from $f$ to $\mathcal{P}$. For example, the completion conditions for $F$ from $f_P$ to the stamp $S_P$ is $F \text{proj}^P \mathcal{P}$. For another example, if $\mathcal{P}$ is a bounded poset and $[\bot, f]$ is a combinatorial polytope, but $\mathcal{P}$ is not a combinatorial polytope, then the completion condition from $f$ to $\mathcal{P}$ is the logical sentence, ‘False’.

Similarly we define completion conditions for a collection of faces. Since we will only be concerned with the geometric properties of the faces, and not how they are positioned relative to eachother, and since projective equivalence is finer than combinatorial equivalence, we only consider each face up to its projective type. For $f_1, \ldots, f_n \in \mathcal{P}$, we say respective realizations $F_1, \ldots, F_n$ of $[\bot, f_1], \ldots, [\bot, f_n]$ can be completed to $\mathcal{P}$ when there exists a realization $P$ of $\mathcal{P}$ and projectivities $\pi_i$ such that $\pi_i(F_i) = \text{face}(P, f_i)$, and we call any sequence of statements $\Gamma$ about a collection of polytopes $F_1, \ldots, F_n$ the completion condition from $f_1, \ldots, f_n$ to $\mathcal{P}$ when $\Gamma$ holds if and only if these polytopes can be completed to $\mathcal{P}$.

4.3. Visibility.

The completion conditions that we will use can be understood geometrically as saying that certain faces of a polytope are visible from certain points in space around that polytope. We define the visibility of a face from a point in terms of the convex join. For polytopes in affine space, we say a face $f$ of $P$ is visible from a point $p \notin P$ when the relative interior of the convex join of the face with the point is disjoint from the polytope,

$$\text{vis}(p, P, f) := \emptyset x \in (p \cup \text{face}(P, f))^\circ \cap P.$$ 

When a face is not visible we say it is obscured. Note that a face is always obscured from a point in the affine span of the face.

Generally, polytopes are assumed to be in a vector space or affine space, but for us it will be more convenient to consider polytopes in projective space. Recall that the complement of a hyperplane (called the horizon) in projective space (together with projective transformations preserving the horizon) is an affine space. We consider a subset of real projective space to be convex, when it is a convex subset of such an affine space. Unlike affine space, where convex
sets ordered by inclusion form a complete lattice, in projective space a pair of convex sets may be contained in two distinct minimal convex sets. For further details see [15].

Since a polytope in projective space and a point outside the polytope have a pair of convex joins, whether a face is visible from the point depends on which convex join we use in the definition of visibility. Let $p^+$ denote a point $p \notin P$ together with a fixed choice of convex join denoted $p^+ \cup P$. We say $p^+$ is oriented with respect to $P$, and we denote by $p^-$ the opposite choice. This distinguishes two definitions of visibility in projective space that are analogous to visibility in affine space; $f$ is **front visible** from $p^+$ when $\nexists x \in (p^+ \cup \text{face}(P, f))^c \cap P$, and **back visible** from $p^+$ when it is front visible from $p^-$. All together, this gives four possible answers to whether a face is visible from a point: a face may be doubly visible $\text{vis}(p^+, P, f) = \ast$, front only visible $\text{vis}(p^+, P, f) = +$, back only visible $\text{vis}(p^+, P, f) = -$, or doubly obscured $\text{vis}(p^+, P, f) = 0$.

We will use visibility from oriented points on a hyperplane as a projective analog of a polytope’s normal vectors. For a polytope $P$, hyperplane $h$ disjoint from $P$, and a co-dimension 2 flat $g \subset h$, a **sky** $s$ over $P$ is the set $h \setminus g$ together with a choice of convex join with $P$. Observe that this choice defines an orientation for points in $s$. We say a sky $s$ is visible from a face of $P$ when the face is front visible from every oriented point in $s$. We denote the set of skies in $h$ that are visible from a face $f$ by $\text{sky}(h, P, f)$. The **sky complex** $\text{sky}(h, P) = \{\text{sky}(h, P, f) : f \in \text{indx}(P)\}$ is the collection of sets of skies in $h$ visible from each face of $P$. We see from the following lemma that the sky complex corresponds to the normal fan of $P$.

**Lemma 13.** For polytopes $P$ in an affine space given by choice of horizon in projective space, there is a bijection between the unit normal vectors and skies on the horizon such that a sky is visible from a face if and only if it is the image of an outward normal of that face.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{normal_vector_example.png}
\caption{Normal vector of an edge (Left) and of a vertex (Right) with their corresponding skies. Orientations in the sky are indicated by shading.}
\end{figure}

**Proof.** Briefly, skies are given in homogeneous coordinates by the open halfspaces of a hyperplane, and for a vector to be normal to a face is equivalent to the vector being the inward normal of an open halfspace given by a sky visible from that face.

To make this explicit, choose coordinates in projective space so $P \subset \mathbb{R}^d \subset \mathbb{RP}^d$. In homogeneous coordinates, the polytope is given by a cone $C = \mathbb{R}_{\geq 0}[P] \subset \mathbb{R}^{d+1}$, and the horizon is
given by $h = \{ x : x_{d+1} = 0 \}$. For $v \in \mathbb{R}^d$ let $\sigma(v) = \{ x \in h : \langle v, x \rangle > 0 \}/\mathbb{R}_{\geq 0}$ be a sky over $P$ in homogeneous coordinates. That is, for an oriented point $x^+ \in \sigma(v)$ given by a ray $r$ in homogeneous coordinates, the positive convex join $x^+ \cup P$ in projective space, is given by the corresponding convex join $r \cup C$ in homogeneous coordinates. We have immediately that $\psi$ is a bijection from unit vectors in $\mathbb{R}^d$ to skies on the horizon of $\mathbb{R}P^d$. It remains to show that $v$ is a normal vector of a face $f$ if and only if $\psi(v)$ is visible from $f$.

Suppose $v$ is a normal vector of $f$ and consider a point $x^+ \in \psi(v)$ given by $\mathbb{R}_{\geq 0}[v_0]$ in homogeneous coordinates. Then, $\forall p \in P, q \in \text{face}(P, f)$ we have $b = \langle v, q \rangle \geq \langle v, p \rangle$. Any point $q' \in x^+ \cup \text{face}(P, f)$ is of the form $q' = q + ty$ for $t \geq 0$. This gives $b \geq \langle v, q' \rangle$ with equality if and only if $t = 0$, in which case $q'$ is on the boundary of $x^+ \cup \text{face}(P, f)$. Otherwise, if $t > 0$ then $b > \langle v, q' \rangle$ and $q' \notin P$. Thus, $(x^+ \cup \text{face}(P, f))^\circ$ and $P$ are disjoint, so $f$ is front visible from $x^+$.

Now suppose $f$ is front visible from every $x^+ \in \psi(v)$. Assume further that there is a pair of points $p \in P, q \in \text{face}(P, f)$ such that $\langle v, q \rangle < \langle v, p \rangle$. Then $\mathbb{R}_{\geq 0}[v_q]$ would give an oriented point $x^+$ in $\psi(v)$, but $f$ is front visible from $x^+$, so $q + (p - q) = p \notin P$, which is a contradiction. Therefore, no such pair of points $p, q$ exist, which implies $v$ is a normal vector of $f$. $\blacksquare$

### 4.4. Prismoids.

Lemma $3$ of the previous section gave a condition for a pair of combinatorial polytopes to be the bases of a prismoid. We now make this into a completion condition from the bases to the prismoid and we state this condition in terms of visibility. Let $\text{flat}(P)$ denote the smallest flat containing $P$.

**Lemma 14.** The completion condition for polytopes $B_0, B_1 \subset \mathbb{R}P^d$ from the respective bases $(\tau, 1), (1, \tau)$ to an abstract prismoid $\mathcal{P} \subset B_0 \times B_1$ is that there be hyperplanes $h_i$ disjoint from $B_i$ and an oriented projectivity $\phi : \mathbb{R}P^d \to \mathbb{R}P^d$ such that $\phi(h_1) = h_0$ and the dual of the common refinement of the sky complex of $B_0$ in $h_0$ and the image of the sky complex of $B_1$ in $h_1$ by $\phi$ is indexed by the sides of $\mathcal{P}$

$$\text{indx}(\text{sky}(h_0, B_0) \land \phi(\text{sky}(h_1, B_1)))^\circ = \text{side}(\mathcal{P}).$$

**Proof.** Briefly, Lemma $3$ says the common refinement of the normal fans of a pair of polytopes must be ordered by the sides of an abstract prismoid for them to be the bases, and Lemma $13$ says this is equivalent to the common refinement of their sky complexes.

To see the condition is sufficient, suppose for polytopes $B_0, B_1 \subset \mathbb{R}P^d$ there is such an oriented projectivity $\phi$. By Lemma $13$ the respective normal fans of $B_0$ and $\phi B_1$ (in the affine space with $h_0$ as the horizon) are in bijection with the sky complexes on $h_0$, so the common refinement of the normal fans has the same indexing as the common refinement of the sky complexes, which is dual to the sides of $\mathcal{P}$. Let $\pi_0$ be some projectivity sending $h_0$ to the horizon of $\mathbb{R}^d \subset \mathbb{R}P^d$ and $\pi_1 = \pi_0 \phi$. By Lemma $3$ $[\pi_0 B_0] \cup [\pi_1 B_1]$ realizes $\mathcal{P}$.

To see the condition is necessary, suppose for polytopes $B_0$ and $B_1$ there is a realization $P$ of $\mathcal{P}$ and projective transformations $\pi$, such that $\pi_0 B_0 = \text{face}(P, (\tau, 1))$ and $\pi_1 B_1 = \text{face}(P, (1, \tau))$. From the definition of a prismoid, for the appropriate choice of $\psi$, this means $\pi_0 B_0 \cup \pi_1 B_1 = P$. Let $\pi$ be a projectivity sending $h = \text{flat}(\pi_0 B_0) \land \text{flat}(\pi_1 B_1)$ (but not $P$) to the horizon of the affine space $\mathbb{R}^{d+1} \subset \mathbb{R}P^{d+1}$. By Lemma $3$ the common refinement of the normal fans of $\pi_0 B_0$ and $\pi_1 B_1$ is dual to the sides of $\text{indx}(\pi_0 B_0 \cup \pi_1 B_1)$ which are also the sides of $\mathcal{P}$. Therefore, by Lemma $13$ the common refinement of the sky
complexes of $B_0$ and $B_1$ in $h$ is dual to the sides of $\mathcal{P}$ as well. This makes $\phi = \pi_0^{-1}\pi_1$ the desired projectivity.

4.5. Tents.

When one base of an prismoid is an edge, we call the prismoid a tent. In this case we call the edge the apex of the tent, and we refer to the other base face exclusively as the base of the tent. Likewise, an abstract prismoid $\mathcal{T}$ is an abstract tent when it has an edge as an apex $A$ and a face of the base forms a side with the apex if and only if it forms a side with either both apex vertices or neither apex vertex.

$\mathcal{T} \subset \mathcal{B} \times \mathcal{A}$

$A := \{+, -, \top, \bot\}, \bot \leq \pm \leq \top$

$(f, \tau) \in \mathcal{T} \iff ((f, +) \in \mathcal{T} \iff (f, -) \in \mathcal{T})$

An abstract tent is determined by the vertices of the apex that each base face forms a side with. As such we define an abstract tent from a function $\chi$ that takes a base face and returns a value indicating which apex vertices form a side with that base. For this, let $\chi : \mathcal{B} \to \{+, -, 0, *\}$ and

$\text{tent}(\chi) := \{(f, a) : f \in \mathcal{B}, a \in A_f\}$ where

$A_f = \begin{cases} 
\{+, \bot\} & \chi(f) = + \\
\{-, \bot\} & \chi(f) = - \\
\{\top, \bot\} & \chi(f) = 0 \\
\{+, -, \top, \bot\} & \chi(f) = * 
\end{cases}$

An equivalent definition of abstract tent is the range of tent over functions $\chi$ with $\chi(\bot) = *$.

We will use the combinatorics of a tent to impose completion conditions on the realizations of its base. Specifically the combinatorics of the tent determines the visibility of the faces of the base $B$ from a point $p \notin B$. In this context, a tent is sometimes called a Lawrence extension [3]. The following result appears in [14, Section 3.3] in slightly different language with further details of its history and use.

Lemma 15. The completion condition for $B$ from the base to an abstract tent $\text{tent}(\chi)$ is that there be an oriented point $p^*$ such that $\text{vis}(p^*, B, f) = \chi(f)$ for each face $f$ of $B$.

Proof. In a halfspace $h$ disjoint from a segment $A$, there is a unique point $p \in h$ such that from an orientation $p^*$ only a single vertex of $A$ is front visible and only the other vertex is back visible, and from any other point of $h$ all faces are doubly visible. As such, the sky complex of $A$ in $h$ consists of the set of skies that include $p^*$, the set of skies that include $p^-$, and between these the set of skies that do not include $p$ with either orientation. Therefore, the common refinement of the sky complex of $B$ that of a segment depends only on the visibility of each face from some point $p$. The result then follows from Lemma [14] \[\square\]

4.6. Transmitters.

The purpose of a transmitter polytope is to impose a relationship between two of its faces through the completion conditions from those faces. The most restrictive example of this is the full transmitter, which forces two faces to be projectively equivalent to eachother. A slightly more general transmitter that will also be widely used is the forgetful transmitter $\mathcal{T}_{B_0, B_1}$. The forgetful transmitter is similar to the full transmitter except one base $B_0$ is a copy of the other base $B_1$ with some simple (degree $d$) vertices whittled. We can think
of whittling a vertex as introducing linear constraints to a polytope that truncate that vertex. The forgetful transmitter would force one base to be a projective copy of the other, but it “forgets” these extra linear constraints. The constructions of the full and forgetful transmitters given in this article are modified versions of those constructed in the proof of [14, Theorem 5.1.1 and Theorem 5.3.1], which are very close to the following lemma. Note that ‘full transmitters’ were simply called ‘transmitters’ in [14], but here ‘transmitter’ will refer to something more general.

**Lemma 16.** For respective realizations $B_0, B_1$ of $\mathcal{B}_0, \mathcal{B}_1$ where $\mathcal{B}_0$ is a copy of $\mathcal{B}_1$ whittled at vertices $\mathcal{V}$, the completion condition from the bases to a forgetful transmitter $\mathcal{T}_{\mathcal{B}_0, \mathcal{B}_1}$ is that there be a projectivity $\phi$ such that $\text{face}(B_0, f) \subset \text{face}(\phi B_1, f)$ for $f \in \mathcal{B}_1 \setminus \mathcal{V}$.

In particular, for full transmitters the completion condition is that the bases be projectively equivalent, $\mathcal{B}_0 \overset{\text{proj}}{\sim} \mathcal{B}_1$.

Before dealing with full and forgetful transmitters, we define a general transmitter. For polytopes $B_0, B_1 \in \mathbb{R}^d$, let

$$\text{trans}(B_0, B_1) := \begin{bmatrix} P \\ 0 \end{bmatrix} \cup \begin{bmatrix} A \\ 1 \end{bmatrix}$$

where $P = \begin{bmatrix} B_0 \\ 0 \end{bmatrix} \cup \begin{bmatrix} B_1 \\ 1 \end{bmatrix}$ and $A = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \cup \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

A transmitter is any polytope in the range of ‘trans’ over pairs of polytopes, or any polytope of that combinatorial type. For an abstract prismoid $\mathcal{P}$ with bases $\mathcal{B}_0$ and $\mathcal{B}_1$, and a subset
of the faces of each base $\mathcal{V} = (\mathcal{V}_0, \mathcal{V}_1)$, where $\mathcal{V}_i \in \mathcal{B}_i \setminus \top$ with $\bot \in \mathcal{V}_i$, we overload the definition of ‘trans’ by letting

$$\text{trans}(\mathcal{P}, \mathcal{V}) := \text{tent}(\chi) \quad \text{where}$$

$$\chi(f, g) = \begin{cases} 
- f \in \mathcal{V}_0, \ g \notin \mathcal{V}_1 \\
+ f \notin \mathcal{V}_0, \ g \in \mathcal{V}_1 \\
0 \ f \notin \mathcal{V}_0, \ g \notin \mathcal{V}_1 \\
* \ f \in \mathcal{V}_0, \ g \in \mathcal{V}_1
\end{cases}.$$ 

An abstract transmitter is any poset in the range of ‘trans’ over abstract prismoids and subsets of their base faces. Observe that $\mathcal{T} = \text{trans}(\mathcal{P}, \mathcal{V})$ is an abstract tent with base $\mathcal{P}$. We call $\mathcal{P}$ the prismoid of the transmitter, and we call the bases of this prismoid, $\mathcal{B}_0$ and $\mathcal{B}_1$, the bases of the transmitter. We refer to sides of $\mathcal{T}$ as transmitter sides and the sides of $\mathcal{P}$ as prismoid sides. Note that any transmitter realizes an abstract transmitter.

As a tent, any realization of a transmitter $\mathcal{T}$ defines an oriented point $p^+$ such that one base $B_0$ is front only visible from $p^+$ and the other base $B_1$ is back only visible from $p^+$. Consequently, all faces of $B_0$ and $B_1$ are respectively front and back visible from $p^+$. Whether a face of $B_0$ is front visible or a face $B_1$ is front visible from $p^+$ is determined by the set $\mathcal{V}$, which tells us what faces of $B_0$ and $B_1$ are doubly visible from $p^+$. Note that the visibility of a prismoid side is determined by the visibility of its intersection with each base, and $\chi$ is defined so a side is front visible when its intersection with $B_1$ is in $\mathcal{V}$, and similarly back visible when its intersection with $B_0$ is in $\mathcal{V}$.

A transmitter is full when it has the same combinatorial type as $\text{trans}(B, B)$ for any polytope $B$, and an abstract full transmitter with base $\mathcal{B}$ is

$$\mathcal{I}_B := \text{trans}(\text{pris}(\mathcal{B}), \emptyset) \quad \text{where}$$

$$\text{pris}(\mathcal{B}) := (\mathcal{B} \times \bot) \cup (\bot \times \mathcal{B}) \cup \{(f, f) : f \in \mathcal{B}\} \rightarrow \mathcal{B} \setminus \top \mathcal{B}.$$ 

Equivalently, we can define an abstract full transmitter of a poset $\mathcal{B}$ as a prism over a pyramid $\mathcal{T}_B := \text{pris} \circ \text{pyr}(\mathcal{B})$ where $\text{pyr}(\mathcal{B}) := \{\bot, \top\} \setminus \top \mathcal{B}$. Intuitively, we can think of a realization $T$ of $\mathcal{T}_B$ as determining an oriented point $p^+$ where a light source cast on one base $B_0$ would form the other base $B_1$ as a shadow on some hyperplane.

A transmitter is forgetful when it has the same combinatorial type as $\text{trans}(B_0, B_1)$ and $B_0$ is the intersection of $B_1$ and some collection of halfspaces that each truncate a single simple vertex of $B_1$. For a combinatorial polytope $\mathcal{B}_1$ (or bounded poset of height $d + 2$) and simple vertices $\{v_1, \ldots, v_n\} = \mathcal{V} \subset \mathcal{B}_1$ (or atoms covered by $d$ elements each), and $\mathcal{B}_0 = \mathcal{B}_1 \setminus \varphi_1 \mathcal{W}_1 \setminus \cdots \setminus \varphi_n \mathcal{W}_n$ where $\varphi_i : [v_i, \top] \rightarrow \mathcal{W}$ is an order embedding, the abstract forgetful transmitter with bases $\mathcal{B}_0$, $\mathcal{B}_1$ is

$$\mathcal{T}_{\mathcal{B}_0, \mathcal{B}_1} := \text{trans}(\mathcal{P}, \mathcal{V}) \quad \text{where}$$

$$\mathcal{P} = (\mathcal{B}_0 \times \bot) \cup (\bot \times \mathcal{B}_1) \cup \{(f, f) : f \in \mathcal{B}_1 \setminus \mathcal{V}\} \cup \bigcup_{i=1}^n (\mathcal{W}_i \setminus \varphi_i([v_i, \top]) \times v_i).$$
Lemma 17. The completion condition for $B_0, B_1 \subset \mathbb{RP}^d$ from the bases to an abstract transmitter $\text{trans}(\mathcal{P}, \mathcal{V})$ is that there be a projectivity $\phi : \mathbb{RP}^d \to \mathbb{RP}^d$ and hyperplanes $h_0 = \phi h_1$ that satisfy the completion condition from bases to the prismoid $\mathcal{P}$, and

for each face $f$ of $B_0$, there is a strictly supporting hyperplane $h_f$, $B_0 \cap h_f = \text{face}(B_0, f)$, such that $\phi(B_1)$ is in the same component of $\mathbb{RP}^d \setminus (h_0 \cup h_f)$ as $B_0 \setminus h_f$ if and only if $f \in \mathcal{V}$,

for each face $f$ of $B_1$, there is a strictly supporting hyperplane $h_f$, $B_1 \cap h_f = \text{face}(B_1, f)$, such that $B_0$ is in the same component of $\mathbb{RP}^d \setminus (h_0 \cup \phi h_f)$ as $\phi(B_1 \setminus h_f)$ if and only if $f \notin \mathcal{V}$.

Proof. The completion condition of the bases of a transmitter must include that of its prismoid, since the prismoid is realized as a face of the transmitter. For the rest of the proof we only consider the rest of the condition.

To see that the condition is necessary, suppose we have a completion for $B_0$ and $B_1$ from the bases to a realization $T$ of trans$(\mathcal{P}, \mathcal{V})$, with apex $A$, prismoid $P$, and $p = \text{flat}(P) \wedge \text{flat}(A)$. By Lemma 15 there is an orientation of $p$ such that $\text{vis}(p^+, P, (f, g)) = \chi(f, g)$ where $\chi$ is defined from $\mathcal{V}$ as in the definition of trans. Let $\pi_i$ be the projectivity sending $B_i$ to a base of $T$, and $U_i = \pi_i(\mathbb{RP}^d) = \text{flat}(\pi_i B_i)$. From the proof of Lemma 14 we can let $h = \pi_0(h_0) = \pi_1(h_1)$. If $p \in U_0$ then $\pi_0 B_0$ would be doubly obscured, but $\pi_0 B_0$ is front visible, so $p \notin U_0$ and similarly $p \notin U_1$. Let $\pi(x) = (x \lor p) \wedge U_0$. We claim $\phi := \pi_0^{-1} \pi_1$ is the desired projectivity.

Consider $f \in \mathcal{V}_1$ and $F = \text{face}(\pi_1 B_1, f)$. Since $F$ is doubly visible from $p^+$, both sets in $F \cup p$ are disjoint from $\pi_0 B_0$. Hence, $(F \lor p) \cap P = F$, so $F \lor p$ can be extended to a hyperplane $h' \ni p$ such that $h' \cap P = F$. Therefore, $h_f = \pi_1^{-1}(h' \cap U_1)$ is a hyperplane that strictly supports the face $f$ of $B_1$. Also, $\pi_0 B_0$ is in the same component of $U_0 \setminus (h \lor h')$ as $\pi_1 B_1$. Thus, $h_f$ and $\phi$ satisfy the condition of the lemma, $B_0$ is in the same component of $\mathbb{RP}^d \setminus (h_0 \cup \phi h_f)$ as $\phi(B_1 \setminus h_f)$.

Now consider $f \notin \mathcal{V}_1$. With this, $F \lor p^+$ intersects $P$ at some point $q \notin \pi_1 B_1$. Since $\pi_i B_i$ are the bases of $P$, $q = tb_i + (1-t)b_0$ for some $b_i \in \pi_i B_i$ and $t \in (0, 1]$. Let $h_f$ be a hyperplane strictly supporting the face $f$ of $B_1$. With this, $h' = h_f \lor p \supset F \lor p^+ \ni q$, so either $b_0$ and $b_1$ are both in $h'$ or they are strictly separated by $h'$. In the first case $B_0$ intersects $\phi h_f$, and
in the second case $B_0$ and $\phi B_1$ have a point in each of the components of $\mathbb{R}P^d \setminus (h_0 \cup \phi h_f)$. Either way, this satisfies the condition of the lemma. We have only considered faces of $\mathcal{B}_1$, but by symmetry, the cases for faces of $\mathcal{B}_0$ are the same.

To see that the conditions are sufficient, suppose we have such a projectivity $\phi$. Let $T = \text{trans}(B_0, \phi B_1)$ with $P$ and $A$ as in the definition of ‘trans’ above, where convex join is defined with respect to $\text{flat}(h_0 \times \mathbb{R}^2)$ as the horizon.

For a face $f \in \mathcal{B}_1$, the existence of a strictly supporting hyperplane $h_f$ such that $B_0$ is in the same component of $\mathbb{R}P^d \setminus (h_0 \cup \phi h_f)$ as $\phi(B_1 \setminus h_f)$ is equivalent to $F \times \mathbb{R} \cap P = [F]_1$ where $F = \text{face}(\phi B_1, f)$, which is equivalent to the face $f$ of $[\phi B_1]_0$ being doubly visible from $p = \text{flat}(P) \cap \text{flat}(A)$. Likewise for a face $f \in \mathcal{B}_0$, the existence of a strictly supporting hyperplane $h_f$ such that $\phi B_1$ is in the same component of $\mathbb{R}P^d \setminus (h_0 \cup h_f)$ as $B_0 \setminus h_f$ is equivalent to $f$ of $[B_0]_1$ being doubly visible from $p$. Hence, by Lemma 15, $T$ has combinatorial type $\text{trans}(\mathcal{P}, \mathcal{V})$.

We now apply this to full and forgetful transmitters.

**Proof of Lemma 16.** The condition is necessary by Lemma 17 and the fact that the faces $(f, f)$ for $f \in \mathcal{B}_1 \setminus \mathcal{V}$ are side of the prismoid. This implies $\text{flat}(B_0, f) = \text{flat}(\phi B_1, f)$, and since this includes all facets of $B_1$, $\text{face}(B_0, f) \subseteq \text{face}(\phi B_1, f)$.

To see that the condition is sufficient, observe that the common refinement of the sky complexes of a $B_0$ and $\phi B_1$ on the horizon of $\mathbb{R}^d$ is indexed by the sides of the prismoid. As such, the completion condition of the prismoid is satisfied, and we may choose $h_0$ to be the horizon of $\mathbb{R}^d$. We have already that the condition of this lemma implies the condition of Lemma 17 for faces $f \in \mathcal{B}_1 \setminus \mathcal{V}$ of either base. The condition of this lemma further implies that for any face $f \in \mathcal{B}_0 \setminus \mathcal{B}_1$ and any strictly supporting hyperplane $h_f$ of $f$, $\phi(B_1)$ intersects $h_f$. For $v \in \mathcal{V}$ however, the vertex $\text{face}(\phi B_1, v)$ is bounded away from $B_0$, so there is a strictly supporting hyperplane $h_v$ such that $B_0$ is in the same component of $\mathbb{R}^d \setminus h_v$ as $\phi(B_1 \setminus h_v)$. Since the set of whittled vertices is also the set of doubly visible faces $\mathcal{V}$ in the definition of a transmitter, the condition of the lemma implies the condition of Lemma 17.

4.7. **Combining Completion Conditions.**

Recall that a full transmitter has two pyramid facets that are projectively equivalent. Suppose we are given a polytope and a certain completion condition for a pyramid facet of that polytope. Then, if we glue a transmitter along this pyramid facet, we get a new polytope with the same completion condition from the transmitter’s other pyramid facet. So far this does not give us anything new, but if we use connectors instead then we get more. Connectors serve the same purpose as full transmitters, but may have more than just 2 pyramid facets that are projectively equivalent. Suppose now we are given several polytopes and completion conditions from a pyramid facet to each of these polytopes. If we glue the pyramid facets of these polytopes along the pyramids of a connector then the completion condition for any the connector’s remaining pyramids will be the conjunction of the given completion conditions.

We now define the **connector** $\text{conn}(n, \mathcal{B})$ with $n$ facets of type $\text{pyr}(\mathcal{B})$. For $n = 2$, this is just the full transmitter, and for $n = 4$, this consists of two copies of the full transmitter glued together along the prism of each

$$\text{conn}(2, \mathcal{B}) := \text{trans}(\text{pris}(\mathcal{B}), \emptyset)$$

$$\text{conn}(4, \mathcal{B}) := \text{conn}(2, \mathcal{B}) \#_{\id_{\text{pris}(\mathcal{B})} \times 1} \text{conn}(2, \mathcal{B}),$$
see Figure 13. For $n > 4$ even, this consists of $\frac{n}{2} - 1$ copies of $\text{conn}(4, \mathcal{B})$ glued together along their pyramids, and for $n$ odd we just disregard a pyramid of $\text{conn}(n+1, \mathcal{B})$. The choices of pyramids to glue along or disregard is irrelevant. Generally, the value $n$ will be the number of other polytopes that are glued to a connector, so we denote $\text{conn}(n, \mathcal{B})$ simply by $\text{conn}(\mathcal{B})$ in this case. This is a variant of the connector in [14, Section 5.2].

**Figure 13.** Left: Two Schlegel diagrams of full a transmitter. Right: A Schlegel diagram of the connector with 4 pyramids that results from gluing these transmitters. (Compare with Figure 9 and Figure 12 Left)

**Lemma 18.** The completion condition from $n$ pyramids to $\text{conn}(n, \mathcal{B})$ are that all the pyramids be projectively equivalent.

*Proof.* First recall that two pyramids are projectively equivalent if and only if their bases are projectively equivalent. For $n = 2$, the condition is the same as Lemma 16 for full transmitters applied to the bases of the pyramids. By Lemma 10, identical copies of a realization of $\text{conn}(2, \mathcal{B})$ can always be glued together as in the definition of $\text{conn}(n, \mathcal{B})$, so the condition is sufficient. By Lemma 11, any realization of $\text{conn}(4, \mathcal{B})$ is the union of two realizations of $\text{conn}(2, \mathcal{B})$ intersecting along the prism of each, since prisms are necessarily flat, so the condition is necessary for $n = 4$, or 3. Likewise for $n > 4$, any realization of $\text{conn}(n, \mathcal{B})$ is the union of a realization of $\text{conn}(4, \mathcal{B})$ and of $\text{conn}(n-2, \mathcal{B})$ intersecting along a pyramid of each, and all pyramids of both pieces are projectively equivalent to the pyramid where they intersect, so the condition is necessary in this case. □

While gluing connectors and transmitters together, we may in some cases want to glue along a lower dimensional face instead of a facet. For this we can repeatedly stellate a facet containing this face until we have a facet that is an iterated pyramid over this face. We can then glue along this facet instead. These repeated stellations are equivalent to gluing a single polytope to the facet, which we call an *adapter* [14, p. 149]. We only consider gluing along some facets of type $\text{pyr}(\mathcal{B})$. For a facet $g \in \mathcal{B}$, the adapter between $\mathcal{B}$ and $g$ is a pyramid...
over the polytope $\text{adapt}(\mathcal{B}, g) := \text{pyr}^2(\mathcal{B})$. We only include $g$ to indicate what face we intend to glue along. When $g$ is a lower dimensional face we define the adapter inductively. For this we choose some facet $f \in \mathcal{B}$ such that $g < f$, and let

$$\text{adapt}(\mathcal{T}, g) := \text{adapt}(\mathcal{B}, f) \# \text{id}_{\text{pyr}(\mathcal{T}, f)} \times \text{pyr}(\text{adapt}(\mathcal{T}, f), g).$$

Note that this depends on choosing a chain of faces $g \prec \cdots \prec f$, but this choice of is irrelevant, so we leave it implicit.

**Lemma 19.** The completion condition for $\mathcal{B}, F$ from $\tau_\mathcal{B}, f$ to $\text{adapt}(\mathcal{B}, f)$ is that $F$ be projectively equivalent to face$(\mathcal{B}, f)$.

**Proof.** This is immediate from construction. \qed

4.8. Stamp of the Cube.

We now have a way to combine the completion conditions of several polytopes’ faces in a single polytope. As a simple example, we construct our first stamp, the stamp of the unit $d$-cube $\mathcal{G}^d$. In the general stamp construction, we will use this cube as a scaffolding to which we fix points, and to give us a coordinate system in projective space.

The **unit cube stamp** $S_{\mathcal{G}^d}$ consists of a connector $\mathcal{C}_{\mathcal{G}^d} = \text{conn}(d+1, \mathcal{G}^d)$ with one of its pyramids unglued. For each opposite pair of facets $f_{i,0}$ and $f_{i,1}$ of $\mathcal{G}^d$, we glue a pyramid over the full transmitter $\mathcal{T}_i = \text{trans}(\mathcal{G}^{d-1}, \mathcal{G}^{d-1})$ with bases $f_{i,0}$ and $f_{i,1}$ to the connector. Specifically, one of the connector’s pyramids is glued to $\text{pyr}(\mathcal{T}_i)$ along the pyramid over the prism of $\mathcal{T}_i$. These pieces are glued together as illustrate in Figure 14. The completion condition of transmitter $\mathcal{T}_i$ forces the lines spanning a set of parallel edges of the unit cube to meet at a point in projective space. Let $f_{\mathcal{G}}$ be the base of the unglued pyramid of $\mathcal{C}_{\mathcal{G}^d}$ in $S_{\mathcal{G}^d}$. We may omit $d$ for brevity.

![Figure 14. Gluing diagram for the stamp the unit cube.](image)

**Lemma 20.** The completion condition for $F$ from $f_{\mathcal{G}}$ to $S_{\mathcal{G}^d}$ is that $F$ be projectively equivalent to the unit $d$-cube.

For the proof we will make use of the fact that the unit cube determines a coordinate system for an affine part of real projective space. Recall that a choice of origin $\mathbf{0}$, linear basis vectors $e_i$, and horizon $h_\infty$ in a projective space $X$ are a basis. This extends to a unique projective map from $X \setminus h_\infty$ to $\mathbb{R}^d$, which determines a coordinate system.

**Proof of Lemma 20.** In the construction of $S_{\mathcal{G}}$, we only glue along pyramids, which are necessarily flat, so by Lemmas 10 and 11 $F$ can be completed from $f_{\mathcal{G}}$ to $S_{\mathcal{G}}$ if and only if we can realize each of the facets that we glue along such that they satisfy the completion condition of each of the pieces and $F$ is the realization of the specified face $f_{\mathcal{G}}$. By Lemma 18, this is equivalent to $F$ satisfying the completion condition to each transmitter $\mathcal{T}_i$ from its prismoid.

We now see that the combined completion conditions for $F \circ \mathcal{G}^b_{\mathcal{G}}$ from the prismoid to $\mathcal{T}_i$ for $i = 1, \ldots, d$ is equivalent to $F \circ \mathcal{G}^b_{\mathcal{G}^d}$. These conditions are that the lines through the edges
between any opposite pair of facets of the cube all meet at a common point. The condition is sufficient, since the unit cube satisfies all of these theses conditions. Given a realization $S$ of $S^d$ with $F = \text{face}(S, f_i)$, we use $F$ to define a coordinate system on its projective closure. For this, choose the vertex in the faces $f_{1,0}, \ldots, f_{d,0}$ to be the origin $0$ and each of its neighboring vertices to be the standard basis vectors $e_1, \ldots, e_d$. Each of the transmitters $\mathcal{T}_i$ is a Lawrence extension at the point $\infty_i$ implied by Lemma 15. To complete the projective basis we have $h_\infty = \infty_1 \vee \cdots \vee \infty_d$ as the horizon. This gives $F_{i,0} = \text{face}(S, f_{i,0}) \in \{p : (p)_i = 0\}$, since $F_{i,0}$ contains $0$ and $e_j$ for $j \neq i$, and gives $F_{i,1} = \text{face}(S, f_{i,1}) \in \{p : (p)_i = 1\}$, since $F_{i,1}$ contains $e_i$ and $\infty_j$ for $j \neq i$. Hence $F$ the unit cube in this coordinate system, which means the condition is necessary.

Note that we can use the cube to express the coordinates of a point in terms of the operations meet, join, and cross ratio. Recall that for points $p$, $p_0$, $p_1$, $p_\infty$ on a line in real projective space, the cross ratio $(p, p_1 | p_0, p_\infty)$ is the value of $\phi p$ in $\mathbb{RP}^1$ given by the unique projectivity sending $\phi p_0 = 0$, $\phi p_1 = 1$, $\phi p_\infty = \infty$. Let, $h_i,x = \text{flat}(\text{face}(\emptyset, f_{i,x}))$ be the facet supporting hyperplanes of the unit cube, and $h_\infty = (h_{i,0} \land h_{i,1}) \lor (h_{j,0} \land h_{j,1})$ for any $i \neq j$ be the horizon, and $\infty_i = (0 \lor e_i) \land \infty$ be the point on the horizon in the direction of $e_i$. We project a point $p$ into the facet supporting hyperplanes by $\pi_{i,x}(p) := (p \lor \infty_i) \land h_{i,x}$. For $p \notin h_\infty$ the $i$th coordinate is $(p)_i := (p, \pi_{i,1}(p) | \pi_{i,0}(p), \infty_i)$.

4.9. Lamppost Polytopes.

The purpose of a lamppost polytope is to impose certain incidences between lines and points and faces of a polytope. The lamppost polytope is defined by a pair of faces of the base and a visibility function on the base. We can think of this visibility function as requiring a light-source to be on a line through the specified pair of faces, as in Figure 15. For a poset $\mathcal{P}$ with a pair of incomparable faces $f_0$, $f_1$ and $\chi : \mathcal{P} \to \{+, -, 0, \ast\}$, let

$$\text{lamp}(\chi, f_0, f_1) := \text{trans}(\text{tent}(\chi), \mathcal{P} \setminus ([f_0, \top] \cup [f_1, \top])).$$

An abstract lamppost polytope with base $\mathcal{P}$ is any poset of the form $\text{lamp}(\chi, f_0, f_1)$, and a realization is called a lamppost polytope. This is a generalization of “polytope $X$” from [14, Section 5.4].

**Lemma 21.** The completion condition for $P$ from the base $(\top_\mathcal{P}, \bot)$ to lamp$(\chi, f_0, f_1)$ is there be an oriented point $p^+$ such that $\text{vis}(p^+, P, f) = \chi(f)$ for all $f \in \mathcal{P}$ and there be a line $l$ through face $(P, f_0)^\circ$, face $(P, f_1)^\circ$, and $p$.

**Proof.** By Lemma 15, $P$ can be completed to a realization of $\text{tent}(\chi)$ if and only if the first part of the condition holds, $\text{vis}(p^+, P, f) = \chi(f)$. Let $A$ be the apex of this tent. By Lemma 17, the tent can be completed to lamp$(\chi, f_0, f_1)$ if and only if there is a projective transformation $\phi$ sending flat$(A)$ to flat$(P)$ such that, (i) $\phi$ preserves some $(d-1)$-flat $\triangledown p = \text{flat}(A) \land \text{flat}(P)$, (ii) no supporting hyperplane of a vertex $v$ of $\phi A$ together with $h$ separates $\phi(A \setminus v)$ from $P$, (iii) the faces $f$ of $P$ with a strictly supporting hyperplane $h_f$ that together with $h$ separates $P \setminus h_f$ from $\phi A$ are precisely the faces that do not contain $f_0$ or $f_1$. Condition (ii) is equivalent to $\phi A \subset P$. Condition (iii) implies that no supporting hyperplane of $f_0$ is disjoint from $\phi A$, since $\phi A \subset P$, so $\phi A$ must intersect face $(P, f_0)$. If $\phi A$ intersected face $(P, f_0)$ on its boundary in a face $g < f_0$, then any supporting hyperplane of $g$ would also intersect $\phi A$, violating condition (iii), so $\phi A$ must intersect face $(P, f_0)^\circ$ and
likewise intersect $\text{face}(P, f_1)^\circ$. Since these faces are incomparable, this is the case if and only if $\phi A$ has a vertex in $\text{face}(P, f_0)^\circ$ and $\text{face}(P, f_0)^\circ$ each, in which case condition (iii) is satisfied for all other faces of $P$. Finally, condition (i) says that $p$ is on the line $l$ containing $\phi A$.

**4.10. Anchor Polytopes.**

The purpose of the anchor polytope $\text{anch}(\alpha)$ is to fix a point on an edge of a square through the completion condition from a pentagonal face. We choose coordinates so this pentagon results from truncating a vertex of the unit square. Truncating a vertex of the unit square introduces two new vertices, and the anchor polytope fixes the coordinates of one of these vertices. Later the anchor polytope will be used to “pin” or “anchor” a hyperplane containing this vertex.

**Lemma 22.** For any algebraic number $0 < \alpha < 1$, the completion condition for a pentagon $F$ from $f_5$ to the anchor polytope $\text{anch}(\alpha)$ is that $F$ be a projective copy of the unit square with $(1, 1)$ truncated that has a vertex at $p = (1, \alpha)$.

The construction of the anchor polytope will make use of another polytope $\mathcal{R}_\alpha$ to encode the value $\alpha$ in a computational frame. A computational frame is a $2k$-gon $G$ that satisfies the following. For each opposite pair of edges $e_i, e_{i'}$ of $G$ let $p_i = \text{flat}(e_i) \wedge \text{flat}(e_{i'})$. A $2k$-gon $G$ is a **computational frame** when the points $p_1, \ldots, p_k$ are colinear, see Figure 16 Right. We say that a computational frame represents the values $\alpha_1, \ldots, \alpha_k$ when there is a projectivity $\pi$ sending $\pi p_i = \alpha_i$ on $\mathbb{R}P^1$. Note that, as along as the images of three points are fixed, this determines the other values represented by the computational frame. We will often choose three such points to represent the values $\pi p_0 = 0, \pi p_1 = 1,$ and $\pi p_\infty = \infty$. In this case any other point $p$, determined by a pair of edges, represents the value $\pi p = (p, p_1 | p_0, p_\infty)$. The following lemma is close to being a special case of [14, Theorem 8.1.1], but some modifications are needed.

**Lemma 23.** For any positive algebraic number $\alpha$, the completion condition for $G$ from $g_\alpha$ to $\mathcal{R}_\alpha$ is that $G$ be a computational frame representing $0, \alpha, 1, \infty$.

We will now construct the anchor polytope $\text{anch}(\alpha)$ using $\mathcal{R}_\alpha$ and prove Lemma 22. Later we will construct $\mathcal{R}_\alpha$ and prove Lemma 23. The **combinatorial anchor polytope** $\text{anch}(\alpha)$
Figure 16. Left: The truncated unit square $F$ with vertex at $p = (1, \alpha)$.
Right: A computational frame representing the values $0, \alpha, 1, \infty$.

consists of combinatorial polytopes glued together as in Figure 17. Let $\mathcal{E}$, $\mathcal{F}$, and $\emptyset$ be a combinatorial 9-gon, 5-gon, and 8-gon, and label their edges consecutively as follows

| $\mathcal{E}$ | $\mathcal{F}$ | $\emptyset$ |
| --- | --- | --- |
| $\infty$, $0$, $\alpha$, $1$, $\infty'$, $h$, $0'$, $\alpha'$, $1'$ | $\infty$, $0$, $\infty'$, $h$, $0'$, $\alpha'$, $1'$ | $\infty$, $0$, $\alpha$, $1$, $\infty'$, $0'$, $\alpha'$, $1'$ |

We may consider $\mathcal{E}$ as a copy of $\mathcal{F}$ with the vertices $0 \land \infty'$ and $0' \land \infty$ whittled by the edges $\alpha$, $1$ and $\alpha'$, $1'$ respectively. This defines a forgetful transmitter $T_{\mathcal{F}, \mathcal{E}}$ between $\mathcal{E}$ and $\mathcal{F}$ that “forgets” these extra edges. Similarly, let $T_{\mathcal{E}, \emptyset}$ be the forgetful transmitter that “forgets” the edge $h$. Let $\mathcal{C}_\mathcal{E} = \text{conn}(4, \mathcal{E})$. One pyramid of $\mathcal{C}_\mathcal{E}$ is glued to $T_{\mathcal{F}, \mathcal{E}}$ and two other pyramids are each glued to a copy of $T_{\mathcal{E}, \emptyset}$. One copy of $T_{\mathcal{E}, \emptyset}$ is glued to $R_\alpha$ such that pairs of edges $\{0, 0'\}$, $\{\alpha, \alpha'\}$, $\{1, 1'\}$, and $\{\infty, \infty'\}$ of $\emptyset$ are identified with the pairs of $R_\alpha$ representing the values $0$, $\alpha$, $1$, and $\infty$ respectively. Note that how the edges in each pair are identified is irrelevant, as long as the edges are in the same consecutive order. The other copy of $T_{\mathcal{E}, \emptyset}$ is glued to a lamppost polytope $X_1 = \text{lamp}(\chi_1, v_0, v_1)$ where $v_0 = \infty \land 0$ and $v_1 = \infty' \land 0'$ and $\chi_1(f_1) = \chi_1(f_1') = 0$ and for all other faces $f$, $\chi_1(f) = +$ for $f$ after $1$ and $\chi_1(f) = -$ for $f$ after $1'$. The remaining 9-gonal pyramid of $\mathcal{C}_\mathcal{E}$ is glued to another lamppost polytope $X_\alpha = \text{lamp}(\chi_\alpha, v_0, v_\alpha)$ where $v_0 = \infty' \land l_h$ and $\chi_\alpha(f_\alpha) = \chi_\alpha(f_\alpha') = 0$ and for all other faces $f$, $\chi_\alpha(f) = +$ after $\alpha$ and $\chi_\alpha(f) = -$ after $\alpha'$. The specified face $f_\alpha$ is the 5-gonal base $\mathcal{F}$ of $T_{\mathcal{F}, \mathcal{E}}$. Observe that $X_1$ and $X_\alpha$ impose the colinearities seen in Figure 16 Left.

Figure 17. Gluing diagram for an anchor polytope of $\alpha$. 

\[
\text{anch}(\alpha) :=
\]

\[
\begin{array}{c}
\text{anch}(\alpha) \\
\text{ann}(\alpha)
\end{array}
\]

\[
\begin{array}{c}
T_{\mathcal{F}, \mathcal{E}} \\
\mathcal{E}
\end{array}
\]

\[
\begin{array}{c}
\mathcal{F} \\
\mathcal{E}
\end{array}
\]

\[
\begin{array}{c}
\emptyset \\
\mathcal{E}
\end{array}
\]

\[
\begin{array}{c}
\mathcal{F} \\
\mathcal{E}
\end{array}
\]

\[
\begin{array}{c}
R_\alpha \\
\mathcal{E}
\end{array}
\]

\[
\begin{array}{c}
X_1 \\
\mathcal{E}
\end{array}
\]

\[
\begin{array}{c}
X_\alpha \\
\mathcal{E}
\end{array}
\]
Proof of Lemma 22. In the construction of anch(α) we always glue along pyramids so by Lemmas 10 and 11 a pentagon $F$ can be completed from $f_0$ if and only if we can find respective realizations $E$ and $O$ of $E$ and $O$ such that these polygons, together with $F$ realizing $\mathcal{F}$, satisfy the completion conditions of the pieces. Let $l_i$ be the line supporting the edge labeled $i$, and let $p_l = l_i \wedge l_r$.

To see the given condition is sufficient let $F$ be the unit square with $(1,1)$ truncated by a line $l_h$ passing though $(1,\alpha)$. Let $E$ be a 9-gon that results from $F$ when the vertices $(1,0)$ and $(0,1)$ are each truncated by a pair of lines with slope 1 and slope $\alpha$. Let $O$ be the 8-gon that results when the linear constraint imposed on $E$ by the supporting line $l_h$ is dropped. With this $p_0$ is the point that meets all horizontal lines, $p_\alpha$ meets all lines with slope $\alpha$, $p_1$ meets all lines with slope 1, and $p_\infty$ meets all vertical lines. As such $O$ is a computational frame representing $0,\alpha,1,\infty$, the points $(0,0),(1,1)$, and $p_1$ are collinear, and the points $(0,0),(1,\alpha)$, and $p_\alpha$ are colinear. Thus, all the completion conditions of the pieces are satisfied, which means the given condition is sufficient.

To see that the given condition is necessary, suppose we have a realization of anch($\alpha$). We start by choosing a projective coordinate system so $F$ is a truncated unit square with $(p)_1 = 1$. For this we let the region bounded by $l_x$, $l_y$, $l'_x$, $l'_y$ be the unit square with $l_x = \{(x,y) : x = 0\}$, $l_y = \{(x,y) : y = 0\}$, $l'_x = \{(x,y) : x = 1\}$, $l'_y = \{(x,y) : y = 1\}$.

The polytope $R_\alpha$ forces the points $p_0$, $p_\alpha$, $p_1$, and $p_\infty$ to be on a line $\Delta$, which is the horizon in this coordinate system, and $R_\alpha$ forces $(p_\alpha,p_1|p_0,p_\infty) = \alpha$. The lamppost polytope $X_1$ forces $w_0 = (0,0)$, $w_1 = (1,1)$, $p_1$ to be collinear, and $X_\alpha$ forces $w_0 = (0,0)$, $p$, $p_\alpha$ to be collinear. The effect of these colinearities is that the projection from $l_x'$ to $\Delta$ through $w_0$ sends $(1,0)$, $p$, $(1,1)$, $p_\infty$ to $p_0$, $p_\alpha$, $p_1$, $p_\infty$ respectively, which implies $(p)_2 = (p,(1,1)|(1,0),p_\infty) = (p_\alpha,p_1|p_0,p_\infty) = \alpha$.

Thus, the given condition is necessary.

To construct $R_\alpha$ we use certain combinatorial 4-polytopes that encode the basic arithmetic relations (addition and multiplication) in a computational frame through its completion condition. The construction of $R_\alpha$ will be similar to that of the polytope in [14] Theorem 8.1.1, where these arithmetic polytopes were introduced and combined to encode a system of polynomial relations in a computational frame through its completion condition. Care was needed there to avoid imposing any additional relations on the values represented, but here we can be more direct, since we are only representing a single fixed value $\alpha$.

Lemma 24. There exist combinatorial 4-polytopes $R_{2x}$, $R_{x+y}$, $R_{x^2}$, $R_{xy}$ that each have a pyramid facet with base $g$ such that the completion condition of $G$ from $g$ to each of these is that there exist $x,y,z \in \mathbb{R}$ such that $G$ is a computational frame representing the sequence values,

\[
0 < x < 2x < \infty \text{ for } R_{2x} \\
0 < x < y < x+y < \infty \text{ for } R_{x+y} \\
0 < 1 < x < x^2 < \infty \text{ for } R_{x^2} \\
0 < 1 < x < y < xy < \infty \text{ for } R_{xy}
\]

The construction and proof of Lemma 24 is given by [14] Lemmas 6.2.1, 7.1.1, 7.2.1, and 7.2.2]. Note that, since values are represented in a computational frame by edges that appear in a specified order, this imposes a total ordering on these values. We can, however, apply certain affine transformations to $\mathbb{RP}^1$ and make substitutions, so that the values represented
satisfy a different arithmetic relation or a different total ordering. For example, if we apply 
$$\phi(t) = t - x$$ to the values represented by the computational frame in \(R_{2x}\), and make the 
substitution \(x' = \phi(2x)\), we get a computational frame representing 
\(-x' < 0 < x' < \infty\). In this 
way, we can use these 4 basic arithmetic polytopes to represent the respective arithmetic 
relations on values ordered as in the following table, together with the value \(\infty\).

| \(R_{2x}\) | \(R_{x+y}\) | \(R_{x^2}\) | \(R_{xy}\) |
|---|---|---|---|
| 0 \(xy\) 2x | 0 \(xy\) \(x+y\) | 0 1 \(xy\) \(x^2\) | 0 1 \(xy\) \(x\) \(y\) |
| \(-x\) 0 \(x\) | \(x\) 0 \(x+y\) \(y\) | 0 \(x^{-1}\) 1 \(x\) | 0 \(x\) 1 \(xy\) \(y\) |
| 2x \(x\) 0 | \(x\) \(x+y\) 0 \(y\) | 0 \(x^2\) 1 \(x\) | 0 \(xy\) 1 \(y\) |
| \(x+y\) \(x\) \(y\) 0 | 0 \(xy\) \(x\) \(y\) 1 |  |  |

The values that a computational frame represents correspond to pairs of edges, and an im-
portant feature of those in the table above, is that the particular pair of edges corresponding 
to \(\infty\) never changes. For each of the combinatorial polytopes \(R_{2x}, R_{x+y}, R_{x^2}, \) and \(R_{xy}\), we fix 
one and for all the pairs of edges of the specified polygon \(g\) that represent \(\infty\), but leave the 
values represented by the other edges to depend on context. Later we will glue basic arith-
metic polytopes along computational frames when we construct \(R_\alpha\), and to do so, we will 
need to specify an isomorphism between the faces of its specified polygon \(g\) and the polygon 
that we glue it to. Both of these polygons will represent certain values, as computational 
frames. These values will always include \(\infty\), and the pairs of edges corresponding to \(\infty\) will 
always be identified by the gluing. This, with the total ordering of edges, defines the gluing. 
Moreover in the case of \(R_{2x}\) and \(R_{x+y}\), the values 0 and \(\infty\) will always be represented, and 
in the case of \(R_{x^2}\) and \(R_{xy}\), the values 0, 1, and \(\infty\) will be. It is also important to observe 
that the order of these values among the others unambiguously determines which arithmetic 
relation each polytope imposes. For example, if \(R_{x+y}\) is glued along a computational frame 
representing \(s < t < 0 < u < \infty\), with the pairs of edges representing \(\infty\) identified, then the 
arithmetic relation imposed on these variables is \(t = s + u\), since 0 is the third value.

We will combine the completion conditions of basic arithmetic polytopes to force a com-
putational frame to represent \(0, \alpha, 1, \infty\) along with several other values. Afterwards, we will 
use a forgetful transmitter to “forget” these other values. Let \(p(x) = \Sigma_{i=0}^{n} c_i x^i\), be the min-
imal polynomial of \(\alpha\), and let \(b_1, b_2 \in \mathbb{N}\) such that \(\alpha\) is the only real root of \(p\) bounded by 
\(b_1 - 1 < b_2 \alpha < b_1\). Let \(x_1 < \cdots < x_m\) be the sequence of values consisting of: the integers from 
0 to the largest among \(b_i\) or \(|c_i|\), and the powers of \(\alpha\) up to \(\alpha^n\), and the absolute values of 
all monomials \(|c_i\alpha^i|\), and the values of all partial sums \(\Sigma_{i=1}^{k} c_i \alpha^i\), and \(b_2 \alpha\). We now define 
collections of sets of indices \(R_{2x}, R_{x+y}, R_{x^2}, R_{xy}\) corresponding to the arithmetic relations 
that determine the values \(x_i\). Let these collections contain precisely the following sets of 
indices.

\[
\begin{align*}
\{i, j\} & \in R_{2x} \quad \text{for} \quad (x_i, x_j) = (1, 2) \\
\{i, j, k\} & \in R_{x+y} \quad \text{for} \quad (x_i, x_j, x_k) = (1, 1, t + 1) \\
\{i, j, k\} & \in R_{xy} \quad \text{for} \quad (x_i, x_j, x_k) = (b_2, \alpha, b_2 \alpha) \\
\{i, j\} & \in R_{x^2} \quad \text{for} \quad (x_i, x_j) = (\alpha, \alpha^2) \\
\{i, j, k\} & \in R_{xy} \quad \text{for} \quad (x_i, x_j, x_k) = (\alpha, \alpha^t, \alpha^{t+1}) \\
\{i, j, k\} & \in R_{xy} \quad \text{for} \quad (x_i, x_j, x_k) = (|c_t|, \alpha^t, |c_t\alpha^t|) \\
\{i, j, k\} & \in R_{x+y} \quad \text{for} \quad (x_i, x_j, x_k) = (|c_t\alpha^t|, \Sigma_{s=1}^{t-1} c_s \alpha^s, \Sigma_{s=1}^{t} c_s \alpha^s) \\
\{i, j\} & \in R_{2x} \quad \text{for} \quad (x_i, x_j) = (|c_n \alpha^n|, \Sigma_{s=1}^{n-1} c_s \alpha^s) 
\end{align*}
\]
Let $x_o = 0$, $x_a = \alpha$, and $x_u = 1$. Let $\mathcal{G}$ be a combinatorial polygon with $2m + 2$ edges indexed consecutively $1, \ldots, m, \infty, 1', \ldots, m', \infty'$, and $\mathcal{C}_G = \text{conn}(\mathcal{G})$. Let $\mathcal{T}_{i,j}$ be a forgetful transmitter between $\mathcal{G}$ and an 8-gon with edges corresponding to $\{o, o', \infty, \infty', i, i', j, j'\}$, and let $\mathcal{T}_{i,j,k}$ and $\mathcal{T}_{i,j,k,l}$ be similarly defined forgetful transmitters. Finally, let $\mathcal{R}_\alpha$ be given by the gluing diagram in Figure 18.

![Figure 18. Gluing diagram for $\mathcal{R}_\alpha$](image)

**Proof of Lemma 23.** In the construction of $\mathcal{R}_\alpha$ we always glue along pyramids so by Lemmas 10 and 11 a polygon $G_\alpha$ can be completed from $g_\alpha$ to $\mathcal{R}_\alpha$ if and only if each of the pieces glued together in the construction can be realized such that each pair of facets glued along is a projectively equivalent pair and the polygonal face $g_\alpha$ is $G_\alpha$. By Lemmas 16 and 18 and by Lemma 21 this is equivalent to the existence of a polygon $G$ with edge supporting lines $l_1, \ldots, l_m, l_\infty, l_1', \ldots, l_m', l_\infty'$ such that the polygon bounded by $l_o, l_a, l_u, l_\infty, l_o', l_a', l_u', l_\infty'$ is $G_\alpha$, and for each set of indices in $\mathcal{R}_{2x}, \mathcal{R}_{x+y}, \mathcal{R}_{x+2}$, and $\mathcal{R}_{xy}$ the polygon bounded by the corresponding lines is a computational frame representing values that satisfy the corresponding arithmetic relation. Let $p_i = l_i \land l_i'$ and $\Delta = p_o \lor p_\infty$. Every index $i \neq o, \infty$ appears in at least one of the given arithmetic relations, and since the corresponding polygon is a computational frame with $o, o', \infty, \infty'$ among its edges, this puts $p_i \in \Delta$. Therefore, this is equivalent to $G$ being a computational frame representing values $y_1, \ldots, y_m, y_\infty$ where $y_o = 0$, $y_a = 1$, $y_\infty = \infty$ and all the given arithmetic relations are satisfied.

In any realization of $\mathcal{R}_\alpha$, the value $y_\alpha$ in particular must satisfy $p(y_\alpha) = 0$ and $b_1 - 1 < b_2 y_\alpha < b_1$. These were chosen to leave only one possibility $y_\alpha = \alpha$. Hence, $G_\alpha$ must be a computational frame representing $0, \alpha, 1, \infty$, and the given condition is necessary.

Suppose $G_\alpha$ is a computational frame representing $0, \alpha, 1, \infty$. Then $G_\alpha$ can be truncated to produce a computational frame representing $x_1, \ldots, x_m, x_\infty$. These values satisfy all of the given arithmetic relations, so the given condition is sufficient.

Thus, $G_\alpha$ can be completed from $g_\alpha$ to $\mathcal{R}_\alpha$ if and only if it satisfies the given condition. □

### 4.11. Stamps.

We now construct the stamp $S_P$ of a polytope $P \subset \mathbb{R}^d_{\text{alg}}$ from Theorem 11. Our goal is to fix a face $F_P = \text{face}(S, f_P)$ up to projectivity for all realizations $S \in [S_P]_{\text{real}}$ such that $F_P \cong P$. We do so by combining a sequence of completion conditions such that the conjunction of
all these conditions is equivalent to the condition $F_P \sim P$, conditions that determine $P$ up to projectivity. For every facet $f$ of $P$, we give a sequence of conditions that determine the supporting hyperplane $h_f$ of the facet $F = \text{face}(P, f)$. For this, we give the coordinates of a set of $d$ points in $h_f$ in general position. We will assume $P$ is positioned “nicely” in the sense that each of these points $p$ is the intersection of $h_f$ with an edge $e$ of the unit cube. The edge $e$ determines all coordinates of $p$ except one, and to fix this last coordinate as part of the completion conditions of $F_P$ we use, an anchor polytope. An anchor polytope fixes a coordinate of a vertex of a 5-gon. The following lemmas ensures we can always find an appropriate 5-gon.

**Lemma 25.** If $H$ is a closed halfspace such that the following holds

- $H$ does not contain the unit $d$-cube $\mathcal{Q}$,
- $H$ intersects every facet of $\mathcal{Q}$,
- no vertices of $\mathcal{Q}$ are on the boundary $h = \partial H$,

then $Q_H := \mathcal{Q} \cap H$ is a polytope with at least $d$ distinct vertices on the hyperplane $h$ that are each a vertex of some 5-gonal face of $Q_H$.

Let $\mathbf{F}$ be the facets of $P$, and for each $f \in \mathbf{F}$, and let $H_f$ be the halfspace that contains $P$ and strictly supports $F = \text{face}(P, f)$.

**Lemma 26.** There is a rational affine transformation such that for all facets $f \in \mathbf{F}$, $H_f$ satisfies the hypothesis of Lemma 25.

Let $Q = \mathcal{Q} \cap P$ and $\mathcal{C}_Q = \text{conn} (\text{indx}(Q))$ and $\mathcal{T}_{P,Q}$, $\mathcal{T}_{Q,\mathcal{Q}}$ be the combinatorial types of trans$(P, Q)$, trans$(Q, \mathcal{Q})$ respectively. Let $\mathcal{Q}_f$ be the triples $(\alpha, p, \circ)$ where $(p, \circ)$ are the (vertex, 5-gonal face) pairs of $Q_f = \mathcal{Q} \cap H_f$ implied by Lemma 25, and $0 < \alpha < 1$ is the only coordinate of $p$ that is not 0 or 1. Let $\mathcal{T}_{Q,Q_f}$ be the combinatorial type of trans$(Q, Q_f)$, and $\mathcal{A}_{Q_f,(\alpha, p, \circ)}$ be pyr$^{d-2}$ anch$(\alpha)$ glued by adapt$(\text{indx}(Q_f), \circ)$ to $\circ$, so that the unit square defining coordinates for the anchor polytope is identified with a face of $\mathcal{Q}$, and $p$ is the vertex with coordinates fixed by the anchor polytope. Finally let $\mathcal{S}_P$ be the combinatorial polytope we get by gluing these pieces together according to Figure 19.

\[
\begin{align*}
\mathcal{S}_P & := \mathcal{T}_{P,Q} \\
& \quad \mathcal{C}_Q \\
& \quad \mathcal{T}_{Q,\mathcal{Q}} \\
& \quad \mathcal{T}_{Q,Q_f} \\
& \quad \mathcal{A}_{Q_f,(\alpha, p, \circ)} \\
\end{align*}
\]

**Figure 19.** Gluing diagram for a stamp of $P$.

**Proof of Lemma 26.** First choose a coordinate system that is generic with respect to $P$, and let $v_i$ and $w_i$ be the vertices of $P$ with the respectively least and greatest $i$th coordinate. Note that some vertices may be repeated. For each such vertex $u \in \{v_1, \ldots, w_d\}$, let $\bar{u} \in \text{tcone}(P, u)^\circ$
such that \( \bar{u} \) is rational, \( \bar{u} \) is sufficiently close to \( u \) that the \( i^{th} \) coordinate all vertices of \( P \) except \( v_i \) and \( w_i \) are in the interval \( I_i = \left[ (\bar{v}_i), (\bar{w}_i) \right] \), and no vertex of \( C = \prod_{i=1}^{d} I_i \) is in the supporting hyperplane of any facet of \( P \). We now get the desired affine transformation defined by sending \( C \) to \( \mathbb{R} \).

\[ \text{□} \]

**Proof of Lemma 25.** We index the nonempty faces of \( \mathbb{R}^d \) by \( \{0, 1, *\}^d \) ordered by the \( d \)-fold product of 0 ≤ * and 1 ≤ * with 0,1 incomparable. For this, 0 and 1 indicate coordinates that are constant on a face and * indicates the coordinates of a face that can vary. Let \( C_H \) be the set of maximal faces of \( \mathbb{R}^d \) that do not intersect \( H \). We can assume without loss of generality that \( C_H \subset \{1,*\}^d \). If this is not the case, we can relabel vertices of the cube so this assumption holds.

Each such (vertex, 5-gonal face) pair \((p, \varnothing)\) of \( Q_H \) can be identified by the (edge, square) pair \((e, s)\) of \( \mathbb{R}^d \) such that \( e \) contains \( p \) and \( s \) contains \( \varnothing \). We label each such pair \((p, \varnothing)\) by the map \( f = (x, y) \mapsto \{0, 1, x, y\}^d \) where \( f(1,*) = e \) and \( f(*,*) = s \).

Now let \( I := \bigcap_{c \in C_H} \{ i : c_i = * \} \) be the set of coordinate where all faces of \( C_H \) have the value *. We claim that for each \( c \in C_H \) and \( J \subset I \) and \( i_x, i_y \in \{ i : c_i = 1 \} \) with \( i_x \neq i_y \) there is a pair \((p, \varnothing)\) labeled by \( f = f_{c,J,i_x,i_y} \) where

\[
\begin{align*}
\text{f}(x, y)_i = \begin{cases} 
  x & i = i_x \\
  y & i = i_y \\
  0 & c_i = *, \ i \notin J \\
  1 & \text{else}
\end{cases}
\end{align*}
\]

For this we just have to see that exactly one of the vertices of the corresponding square \( f(*,*) \) of \( \mathbb{R}^d \) is not in \( H \). Specifically, face(\( \mathbb{R}^d \), \( f(1,1) \)) \( \notin H \) and face(\( \mathbb{R}^d \), \( v \)) \( \notin H \) for \( v = f(0,1), f(1,0), f(0,0) \). For every coordinate \( 1 \leq i \leq d \), if \( c_i = 1 \) then \( f(1,1)_i = 1 \), so \( f(1,1) \) is a vertex of \( c \in C_H \subset \{1,*\}^d \), which implies face(\( \mathbb{R}^d \), \( f(1,1) \)) \( \notin H \). For any other vertex \( v \) of \( f(*,*) \), either \( v_{i_x} = 0 \) or \( v_{i_y} = 0 \), but \( c_{i_x} = c_{i_y} = 1 \), so \( v \) is not a vertex of \( c \). For any other \( c' \in C_H \), \( c' \neq c \), there is some coordinate \( 1 \leq i \leq d \) such that \( c'_i = 1 \) but \( c_i = * \), since the faces in \( C_H \) are incomparable. Since \( c'_i \neq * \) We have \( i \notin I \), so \( i \notin J \), which implies \( v_i = 0 \), so \( v \) not a vertex of \( c' \). We have now that \( v \) is not a vertex of any face in \( C \), so face(\( \mathbb{R}^d \), \( v \)) \( \in H \).

We now find \( d \) distinct points where \( h \) intersects a distinct edge of \( \mathbb{R}^d \) of the form \( f(1,*) \). Observe that for any \( c \in C_H \), there is always at least two coordinates \( i_x, i_y \) such that \( c_{i_x} = c_{i_y} = 1 \), since \( c \) would otherwise be a facet of \( \mathbb{R}^d \) disjoint from \( H \), or be \((*,\ldots,*)\) \( \in \tau \). For \( 1 \leq k \leq d \), select \( f^k = f(c^k, J^k, i^k_x, i^k_y) \) in the following way. For \( k \in I \) let \( c^k \) be any element of \( C_H \) and \( i^k_x, i^k_y \) be any appropriate coordinate and \( J^k = \{k\} \). For \( k \notin I \) let \( c^k \) be such that \( c^k_k = 1 \) and \( i^k_y = k \) and \( i^k_x \) be any appropriate index and \( J^k = \emptyset \).

For \( k \in I \) and any \( j \neq k \), we have \( f^k(1,*)_k = 1 \) but \( f^j(1,*)_k = 0 \), so \( f^k(1,*) \) and \( f^j(1,*) \) are distinct edges. Alternatively for \( k, j \notin I \) with \( j \neq k \), we have \( f^k(1,*)_k = * \) but \( f^j(1,*)_k \neq * \) since \( f^j(1,*)_j = * \), so again \( f^k(1,*) \) and \( f^j(1,*) \) are distinct edges.

\[ \text{□} \]

**Proof of Theorem 17.** First we see that \( S_P \) is always realizable. As long as we can realize each piece such that every pair of facets we glue along is projectively equivalent, by Lemma 10 we can actually glue them together to get a realization of \( S_P \). All of these facets are pyramids, so a pair is projectively equivalent if and only if a their respective bases, which are ridges of \( S_P \), are projectively equivalent. In defining \( S_P \) we start with polytopes \( P \) and \( \mathbb{R} \), and define the combinatorics of the various projective transmitters to be realizable with projective copies of these, so the projective transmitters \( T_{P,Q}, T_{Q,\mathbb{R}}, T_{\mathbb{R},Q} \) can be realized with the specified
bases projectively equivalent to $P, Q, \mathcal{F}, Q_f$. The stamp of the cube can be realized with the specified ridge projectively equivalent to $\mathcal{F}$, and for each pentagon in $\mathcal{O}\tau$, the anchor polytope anch($\alpha$) can be realized so its specified ridge is projectively equivalent to the corresponding 5-gonal face of $Q_f$.

For the other direction we see that in every realization $S$ the specified facet $F_h = \text{face}(S, f_S)$ is projectively equivalent to $P$. Note that, since each facet we glue along to construct $S_h$ is a pyramid, these facets are necessarily flat, which by Lemma 11 implies $S$ can be decomposed into the union of realizations of these pieces such that each adjacent pair of pieces intersects in the facet of each where they are glued together. Moreover, the projective transformations implied by the completion conditions of the connectors in Lemma 18 and transmitters in Lemma 17 send a projective basis among the faces of one ridge to that of another ridge, so there is a unique projective transformation between these pairs ridges. Among these, the ridge $F_\mathcal{F} = \text{face}(S_\mathcal{F}, f_\mathcal{F})$ of the piece realizing $S_\mathcal{F}$ must be projectively equivalent to $\mathcal{F}$. This determines a unique projective transformation $\phi_\mathcal{F} : \text{flat}(F_\mathcal{F}) \rightarrow \mathbb{R}_{\text{alg}}^d$ such that $\phi_\mathcal{F}(F_\mathcal{F}) = \mathcal{F}$, and $\phi_\mathcal{F}$ in turn determines a unique projective transformation $\phi : \text{flat}(F_P) \rightarrow \mathbb{R}_{\text{alg}}^d$ by composition with the projective transformations implied by Lemmas 13 and 17 for the intermediate pieces $\mathcal{T}_{P,Q}, \mathcal{C}_Q, \mathcal{T}_{Q, F_\mathcal{F}}$. For each facet supporting hyperplane $h_f, f \in \mathcal{F}$ of $F_P$, each of the specified points in $\mathcal{O}_h$ must have the same coordinates as the corresponding point in the corresponding facet supporting hyperplane of $P$, since it is on an edge of $\mathcal{F}$, which determines $d-1$ coordinates, and appears as the specified point of an anchor polytope anch($\alpha$), which by Lemma 22 sets the remaining coordinate to $\alpha$. Since this determines $d$ points of $h_f$ that are not contained in any affine space of lower dimension, $\phi(h_f)$ must be a supporting hyperplane of face($P, f$). Hence $F_P \equiv_{\text{proj}}^\mathcal{F} P$. □

5. From Algebraic to Real Polytopes

Generally it is common to consider polytopes in $\mathbb{R}^d$ rather than $\mathbb{R}_{\text{alg}}^d$, so it would be nicer if Corollary 2 were not restricted to polytopes in $\mathbb{R}_{\text{alg}}^d$. If we simply remove this condition, the resulting claim would be false. Instead, we can replace this restriction on the space of polytopes to further restrictions on the kind of properties considered. Specifically, we require the predicate $\psi$ to be in the language of real closed fields. This is the first order theory with vocabulary $(+, \cdot, 0, 1, \leq)$ and axioms:

- the axioms of fields
- the axioms of total orderings
- $\forall x. (0 \leq x \Leftrightarrow \exists y. (x = y^2))$
- for $n$ odd, $\forall a_0, \ldots, a_n, \exists x. (a_0 + a_1 x + \ldots + a_n x^n = 0)$.

Recall that a model of a theory is a set together with functions and relations that satisfy the axioms of the theory, and in particular $\mathbb{R}$ and $\mathbb{R}_{\text{alg}}$ are models of the theory real closed field. For a model $\mathcal{M}$ and formula $\phi$ with free variables $x_1, \ldots, x_n$ and constants $c_1, \ldots, c_n \in \mathcal{M}$, $\mathcal{M} \models \phi(c_1, \ldots, c_n)$ denotes $\phi$ is true in $\mathcal{M}$ with $c_i$ substituted for $x_i$. We say $\psi$ is a predicate on $K$ polytopes of the same combinatorial type in the language of real closed fields when, for each combinatorial type of polytope $\mathcal{P}$, the restriction of $\psi$ to $\mathcal{P}$ is definable as a predicate on the vertex coordinates. That is, for $\mathcal{P}$ with $m$ vertices, there is some formula $\psi_\mathcal{P}$ in the language of real closed fields with free variables $v_{1,1}, \ldots, v_{d,m,K}$ such that for any model $\mathcal{M}$ and polytopes $P_1, \ldots, P_K$ of type $\mathcal{P}$ with vertices $\bar{c}_{1,1}, \ldots, \bar{c}_{m,K} \in \mathcal{M}^d$, $\psi(P_1, \ldots, P_K) \Leftrightarrow (\mathcal{M} \models \psi_\mathcal{P}(\bar{c}_{1,1}, \ldots, \bar{c}_{m,K}))$. 

Theorem 27. If \( \psi \) is a predicate in the language of real closed fields of several polytopes of the same combinatorial type such that ridges always projectively inherit \( \psi \), then \( \psi \) holds for some realizations of every combinatorial type of polytope if and only if it holds for some realizations of every projective type. Moreover, there can be a gap of at most 2 dimensions.

Theorem 27 is easily seen to hold for any real closed field, not just \( \mathbb{R} \), but as polytopes are conventionally assumed to be in \( \mathbb{R}^d \), other models may currently be of secondary interest.

Proof. Briefly, Theorem 27 follows from the fact that \( \mathbb{R} \) and \( \mathbb{R}_{alg} \) are elementarily equivalent \( (\mathbb{R} \models \phi \iff \mathbb{R}_{alg} \models \phi) \text{[16]} \), and both realizability and projective equivalence are definable in the language of real closed fields.

Note that ‘and’ will be denoted \( \land \). While this does overlap with the use of \( \land \) as the meet of elements in a lattice, which logical conjunction is, we will use \( \land \) exclusively for logical conjunction here. For a formula \( \phi \) with free variable \( x \), and a formula \( \theta, \phi[\theta/x] \) denotes the formula where \( x \) is replaced by \( \theta \). We will also use conventional notation in \( \mathbb{R}^d \) as abbreviations for the formulas that can easily be written in the language of real closed fields. For example, \( x - y = z \) should be understood as \( x = z + y \).

First we write a formula \( \psi'_{\mathcal{P}} \) that says there are realizations of \( \mathcal{P} \) where \( \psi_{\mathcal{P}} \) holds. For this, write a formula \( \rho_{\mathcal{P}} \) with \( dmK \) free variables \( v_{i,j,k} \) for each coordinate of each vertex \( \vec{v}_{j,k} \) of each polytope \( P_k \) that says these are indeed vertex coordinates of a polytope of type \( \mathcal{P} \). Let \( \mathcal{F}_1, \ldots, \mathcal{F}_n \) be the set vertices in each facet of \( \mathcal{P} \).

\[
\rho_{\mathcal{P}} := \exists \vec{a}_{1,1} \ldots \exists \vec{a}_{n,K} \cdot \nu_{\mathcal{P}}
\]

\[
\nu_{\mathcal{P}} := \bigwedge_{k=1}^K \left( \bigwedge_{v_j \in \mathcal{F}_i} \langle \vec{a}_{i,k}, \vec{v}_{j,k} - \vec{r}_k \rangle = 1 \right) \land \left( \bigwedge_{v_j \notin \mathcal{F}_i} \langle \vec{a}_{i,k}, \vec{v}_{j,k} - \vec{r}_k \rangle < 1 \right)
\]

\[\vec{r}_k := \frac{1}{m} \sum_{j=1}^m \vec{v}_{j,k}\]

Note that \( \vec{r}_k \) is a formula for a translation vector that centers \( P_k \). The formula \( \nu_{\mathcal{P}} \) includes free variables for the halfspaces supporting each facet of a centered translation of the polytope, and says that the vertices of a facet are on the boundary of its supporting halfspace and the rest of the vertices are in the interior of this halfspace.

To write a formula \( \psi'_{\mathcal{P}} \) asserting the existence of a realization where \( \psi_{\mathcal{P}} \) holds, we existentially quantify the free variables in \( \rho_{\mathcal{P}} \land \psi_{\mathcal{P}} \).

\[\psi'_{\mathcal{P}} := \exists \vec{v}_{1,1} \ldots \exists \vec{v}_{m,K} \cdot \rho_{\mathcal{P}} \land \psi_{\mathcal{P}}\]

Next, write a formula \( \chi_{\mathcal{P}} \) saying for \( K \) polytopes with combinatorial type \( \mathcal{P} \) that there is a projective copy where predicate \( \psi_{\mathcal{P}} \) holds. For this we represent projectivities on \( \mathbb{R}^d \) by a \((d+1) \times (d+1)\) matrix \( M := [A_{ij}] \) acting on homogeneous coordinates, and quantify over projectivities by quantifying over the entries of this matrix, except for \((M)_{d+1,d+1} = 1 \) which is fixed.

\[\chi_{\mathcal{P}} := \exists M_1 \ldots \exists M_K \exists x_{1,1} \ldots \exists x_{m,K} \cdot \mu_{\mathcal{P}}\]

\[\mu_{\mathcal{P}} := \bigwedge_{j,k=1}^{m_{\mathcal{P}},K} (x_{j,k} \cdot ((\vec{c}_{k}, \vec{w}_{j,k}) + 1) = 1) \land (\rho_{\mathcal{P}} \land \psi_{\mathcal{P}})[x_{j,k} \cdot (A_k \vec{w}_{j,k} + \vec{b}_k)/\vec{v}_{j,k}]
\]

We get the formula \( \mu_{\mathcal{P}} \) by replacing each free variable of \( \rho_{\mathcal{P}} \land \psi_{\mathcal{P}} \) that is a coordinate of \( \vec{v}_{j,k} \) with a formula for that coordinate of \( \pi_k(\vec{v}_{j,k}) := u_{d+1}^{-1} \vec{u}_{1:d} \) where \( \vec{u} = M_k [\vec{w}_{j,k}] \), and asserting the existence of a multiplicative inverse \( x \) for \( u_{d+1} \) to appear in this formula.
To write a formula $\chi'_P$ asserting that every realization has a projective copy where $\psi_P$ holds, we universally quantify the free variables in $(\rho_P \Rightarrow \chi_P)$

$$\chi'_P := \forall \bar{v}_{11} \ldots \forall \bar{v}_{m_K, K}, \rho_P \Rightarrow \chi_P.$$  

In both models it immediately holds that the existence of a realization of every projective type where the predicate holds implies the existence of that of every combinatorial type. For the other direction, suppose there is some combinatorial $d$-polytope $P$ with realizations in $\mathbb{R}^d$ such that $\psi$ does not hold for any projective copies in $\mathbb{R}^d$. That is, $\mathbb{R} \models \neg \chi'_P$. Then, $\mathbb{R}_{\text{alg}} \models \neg \chi'_P$, which asserts the existence of an algebraic polytope $P$ where $\psi$ does not hold for any algebraic projective copies $\pi_1(P), \ldots, \pi_K(P)$. Let $S_P$ be the combinatorial $(d+2)$-polytope that is the stamp of $P$. Then, $\mathbb{R}_{\text{alg}} \models \neg \psi'_S$, and therefor $\mathbb{R} \models \neg \psi'_S$. Thus, we have found a combinatorial $(d+2)$-polytope such that $\psi$ does not hold for any realization in $\mathbb{R}^d$.  

\[6. \text{ Questions} \]

This article was initially motivated by the question, "Is the Hasse diagram of the face lattice of any polytope the 1-skeleton of some other polytope?" but we have not actually answered that question. Theorem 9 says that if such a polytope exists, it is not the natural candidate, the interval polytope.

A sufficient condition was already known for a polytope to have an antiprism, that some realization is perfectly centered. Here we saw a condition that is necessary and sufficient, that some pair of realizations is balanced, but it is not immediately clear that this new condition is actually weaker. Does there exist a combinatorial polytope that has a pair of balanced realizations, but does not have a perfectly centered realization?

Perhaps the most obvious question is, what further applications does the stamp have? Among geometric properties of polytopes that are being studied, which of these do faces inherit, and can Corollary 2 or Theorem 27 be applied? Such methods have recently been employed in [1].

The stamp fixes a face of co-dimension 2 up to projectivity, but what about co-dimension 1. We saw in the introduction a 3-polytope does not impose any completion condition on its facets, but such a stamp may exist in higher dimensions. Is there a $d_0$ such that for any polytope $P$ of dimension $d \geq d_0$ there is a combinatorial polytope of dimension $d + 1$ such that in all realizations a specified facet is projectively equivalent to $P$? Or, are there other properties $P$ could to satisfy to guarantee that such a combinatorial $(d+1)$-polytope exists?

We have seen a variety of polytopes with various completion conditions. What sort of condition can be the completion condition of a face of a polytope? Ideally, this question would be answered by giving a formal language, consisting of ground types, a vocabulary of symbols, and a type signature for each symbol, together with a semantic interpretation that includes polytopes among the types, and this language would satisfy the following. Given a set of realizations $R$ of a combinatorial polytope $\mathcal{P}$, there is another combinatorial polytope $\mathcal{Q}$ such that $R$ is the restriction of realizations of $\mathcal{Q}$ to a certain face, $R = \{\text{face}(Q, f) : \text{index}(Q) = \mathcal{Q}\}$ if and only if there exists a predicate $\psi$ in this language such that $R$ is the set where the predicate holds $R = \{P : \psi(P) = \text{True}\}$. Of course, it would have to be possible to formulate the completion conditions already given as a predicate in such a language. In particular it must be possible to say that a polytope is fixed up to projectivity. In the other direction, any
predicate of this language would have to be projectively invariant. A starting point for ridges would be to consider the vocabulary: vis, face, flat, ⊥, ∧, ∨, cross ratio, (+), (·), 1, 0, (≤).

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APPENDIX A. Notation

We generally use capital script letters to denote a poset $\mathcal{P}$, and lowercase letters to denote an element of a poset $f \in \mathcal{P}$. A poset is bounded when it has a greatest element $\top$ and a least element $\bot$. A poset is a lattice when every pair of elements $f, g \in \mathcal{P}$ has a unique greatest common lower bound $f \wedge g$ and a unique least common upper bound $f \vee g$. A lattice is complete when any subset $\mathcal{X}$ of elements has a unique greatest common lower bound $\bigwedge \mathcal{X}$ and least common upper bound $\bigvee \mathcal{X}$. Examples of complete lattices we will encounter here are, the faces of a polytope, the cones of a vector space, the convex subsets of an affine space, and the flats of a projective space.

The intervals of a poset are denoted $[h, f] = \{g : h \leq g \leq f\}$. The dual poset $\mathcal{P}^\ast$ is the set $\{f^\ast : f \in \mathcal{P}\}$ ordered by $g^\ast \leq f^\ast$ when $g \geq f$. For posets $\mathcal{P}_1, \mathcal{P}_2$, the categorical product $\mathcal{P}_1 \otimes \mathcal{P}_2$ is the set $\mathcal{P}_1 \times \mathcal{P}_2$ ordered by $(g_1, g_2) \leq (f_1, f_2)$ when $g_1 \leq f_1$ and $g_2 \leq f_2$. The common refinement of two sublattices $\mathcal{P}_1, \mathcal{P}_2 \subset \mathcal{P}$ is the pair-wise meet of their members

$$\mathcal{P}_1 \wedge \mathcal{P}_2 = \{f_1 \wedge f_2 : f_i \in \mathcal{P}_i\}.$$ 

In general an element of the common refinement can be expressed as the meet of more than one possible pair of elements from each sublattice. For complete sublattices, we associate each element $g \in \mathcal{P}_1 \wedge \mathcal{P}_2$ of the common refinement to the minimal such pair

$$g \equiv (\bigwedge \{f \in \mathcal{P}_1 : g \leq f\}, \bigwedge \{f \in \mathcal{P}_2 : g \leq f\}).$$

This puts $\mathcal{P}_1 \wedge \mathcal{P}_2 \rightarrow \mathcal{P}_1 \otimes \mathcal{P}_2$.

We generally use capital letters to denote a polytope $P$ or cone $C$. Here a polytope is the convex hull of finitely many points in an affine space, and a cone is the positive hull of finitely many vectors in a vector space. When the vector (or affine) space is $\mathbb{R}^d_{\text{alg}}$, we call a polytope or cone algebraic. We denote the convex join (convex hull of the union of sets) by $X_1 \cup X_2 := \text{conv}(X_1 \cup X_2)$.

For a polytope $P$ with indexed faces, let $F = \text{face}(P, f)$ denote the face of $P$ with index $f \in \mathcal{P}$. We say the combinatorial type of a polytope $P$ is the poset $\mathcal{P}$ when the faces of $P$ are indexed by $\mathcal{P}$ and $\text{face}(P, f) \subset \text{face}(P, g)$ if and only if $f \leq g$. In this case we say that $\mathcal{P}$ is a combinatorial polytope and $P$ is a realization of $\mathcal{P}$. We denote this poset by $\text{indx}(P) = \mathcal{P}$. This defines an equivalence relation $(Q \text{ comb } P) := (\text{indx}(Q) = \text{indx}(P))$, and we denote the equivalence class by $[P]_{\text{comb}} := \{Q : Q \text{ comb } P\}$. We index the faces of a polytope’s face by $\text{indx}(\text{face}(P, f)) = [1, f]$ where $\text{face}(\text{face}(P, f), g) = \text{face}(P, g)$ for $g \leq f$. The dimension of a face $f$ is $\text{rank}(f) - 1$.

We say a polytope is centered when it contains the origin $\vec{0} \in P$. For a vector space $\mathbb{V}$, let $\mathbb{V}^\ast$ denote the space of linear functionals on $\mathbb{V}$. We denote the evaluation of $x \in \mathbb{V}^\ast$ at $v \in \mathbb{V}$ by $\langle x, v \rangle$ and identify $\mathbb{V}^{\ast \ast} = \mathbb{V}$ by $v \equiv (x \mapsto \langle x, v \rangle)$. For a centered polytope $P$ or a cone $C$ we denote corresponding polar objects by

$$P^\ast := \{x \in \mathbb{V}^\ast : \forall p \in P. \langle x, p \rangle \leq 1\}$$

$$\text{face}(P^\ast, f^\ast) := \{x \in P^\ast : \forall q \in \text{face}(P, f). \langle x, q \rangle = 1\}$$

$$C^\ast := \{x \in \mathbb{V}^\ast : \forall p \in C. \langle x, p \rangle \leq 0\}$$

$$\text{face}(C^\ast, f^\ast) := \{x \in C^\ast : \forall q \in \text{face}(C, f). \langle x, q \rangle = 0\}$$

For a subspace $V \subset \mathbb{V}$, let $V^\circ$ be the restriction of functions in $\mathbb{V}^\ast$ to only act on $V$. For complimentary subspaces $V_1, V_2$ with $v_i \in V_i$ and $x_i \in V_i^\circ$, let $\langle x_1 \oplus x_2, v_1 + v_2 \rangle = \langle x_1, v_1 \rangle + \langle x_2, v_2 \rangle$. 


Note that $V^\circ$ is not a subset of $V^*$, and the operation $\oplus$ depends on the spaces $V_i$. We denote the cone over a face and the normal cone of a face of a polytope respectively by

$$\text{cone}(P, f) := \{ sq : q \in \text{face}(P, f), s \geq 0 \}$$
$$\text{ncone}(P, f) := \{ x \in \text{span}(P)^\circ : \forall q \in \text{face}(P, f), \forall p \in P, \langle x, q \rangle \geq \langle x, p \rangle \}$$

We can think of the normal cone of a face as the linear objective functions that are optimized over the polytope at that face. We similarly denote the face fan and normal fan respectively by

$$\text{fan}(P) := \{ \text{cone}(P, f) : f \in \text{indx}(P) \}$$
$$\text{nfan}(P) := \{ \text{ncone}(P, f) : f \in \text{indx}(P) \}$$

And we index these fans by $\text{indx}(P)$ and $\text{indx}(P)^*$ respectively.

**Proposition A.1.** For cones $C_1, \ldots, C_n$ in complementary subspaces,

$$(C_1 + \cdots + C_n)^* = C_1^\circ \oplus \cdots \oplus C_n^\circ$$

$$\text{indx}(C_1 + \cdots + C_n) = \text{indx}(C_1)^\text{cat} \times \cdots \times \text{cat} \text{indx}(C_1)$$

$$\text{face}(C_1 + \cdots + C_n, (f_1, \ldots, f_n)) = \text{face}(C_1, f_1) + \cdots + \text{face}(C_n, f_n).$$

**Proposition A.2.** For a centered polytope $P$ with face indices $g \leq f$,

$$\text{ncone}(P, f) = \text{cone}(P^*, f^*)$$
$$\text{rncone}(P, f) = \text{cone}(P^*, f^*)$$
$$\text{indx}(\text{cone}(P, f)) = [1, f]$$
$$\text{face}(\text{cone}(P, f), g) = \text{cone}(\text{face}(P, g))$$
$$\text{rncone}(\text{face}(P, f), g) = \text{face}(\text{ncone}(P, g)^*, f)^\circ.$$ 

We represent a projective transformation $\pi$ acting on a vector $x \in \mathbb{R}^d$ by a matrix $M = [\begin{array}{cc} A & b \\ c^* & 1 \end{array}]$ using homogenous coordinates as

$$\pi(x) = \begin{bmatrix} A & b \\ c^* & 1 \end{bmatrix}(x) = \frac{Ax + b}{c^*x + 1}$$

For $\pi$ as above, let

$$\pi^* = \begin{bmatrix} -I & 0 \\ 0 & 1 \end{bmatrix}M^* \begin{bmatrix} -I & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} A^* - c \\ -b^* & 1 \end{bmatrix}.$$ 

We call $\pi^{-*} := (\pi^*)^{-1} = (\pi^{-1})^*$ the polar transformation.

**Proposition A.3.** For a centered polytope $P$ and a projectivity $\pi$ such that $\pi(P)$ is centered, $\pi(P)^* = \pi^{-*}(P^*)$.

For an affine space $\mathbb{A}$, let $\text{trans}(\mathbb{A})$ denote the vector space of translations in $\mathbb{A}$. Recall that an affine space is equipped with operations

$$+: \mathbb{A} \times \text{trans}(\mathbb{A}) \to \mathbb{A}, \quad -: \mathbb{A} \times \mathbb{A} \to \text{trans}(\mathbb{A}),$$

where $(\cdot)$ applies a translation to a point and $(-)$ finds the translation sending one point to another. For $p \in \mathbb{A}$, let $\text{orgn}(p)$ denote the vector space where $p$ is fixed as the origin, and for $P \subset \mathbb{A}$, let $\text{orgn}(p, P)$ denote $P$ embedded in the vector space $\text{orgn}(p)$ by the identity. A cone in an affine space is any set $C$ such that $\text{orgn}(p, C)$ is a cone for some $p$. We define the tangent cone and the normal cone of a face $f$ of a polytope $P \subset \mathbb{A}$ respectively by

$$\text{tcone}(P, f) := \{ g + sv : q \in \text{face}(P, f), q + v \in P, s \geq 0 \}.$$ 

$$\text{ncone}(P, f) := \{ x \in \text{trans}(\text{flat}(P))^* : \forall p \in P, \forall q \in \text{face}(P, f), \langle x, q - p \rangle \geq 0 \}.$$
The tangent cone of a face is the set of points that can be reach from the face by a ray passing through the polytope. If we think of the normal cone in a vector space as a set of optimization problems, then the normal cone in affine space consist of functions comparing inputs to these optimization problems.

**Proposition A.4.** For a polytope $P$ in affine space with face index $f$ and point $p$,
\[
\text{ncone}(\text{orgn}(p, P), f) = \text{orgn}(p, p + \text{ncone}(P, f)^*)^*
\]
\[
\text{rncone}(\text{orgn}(p, P), f) = \text{orgn}(p, p + \text{rncone}(P, f)^*)^\circ.
\]

**Proposition A.5.** For a polytope $P$ in affine space with face indices $g \leq f$
\[
\text{tcone}(P, f) = \text{flat}(\text{face}(P, f)) + \text{rncone}(P, f)^\circ
\]
\[
\text{indx(tcone}(P, g)) = [g, \top]
\]
\[
\text{face(tcone}(P, g), f) = \text{tcone(\text{face}(P, f), g)}
\]