FILTERED HIRSCH ALGEBRAS

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Abstract. Motivated by the cohomology theory of loop spaces we consider a concept of a special class of higher order homotopy commutative differential graded algebras. For such an algebra, \( A \), the so-called filtered Hirsch model is constructed. For \( A \) over the integers, we show that for any \( a \in H(A) \) with \( a^2 = 0 \) the symmetric Massey products \( \langle a \rangle^n, n \geq 3 \) (whenever defined) have a finite order that immediately implies their annihilation for a field of characteristic zero. For an odd dimensional \( a \) and \( p \) an odd prime, the Kraines formula \( \langle a \rangle^p = -\beta P_1(a) \) is lifted to \( H^\ast(A \otimes \mathbb{Z}_p) \).

1. Introduction

We investigate a special class of homotopy commutative algebras, the so-called Hirsch algebras (\cite{15}). In the case a Hirsch algebra componentwise agrees with a homotopy G-algebra (hga) it may differ from it by the pre-Jacobi axiom \([6, 7, 14, 26]\); in other words, we do not require the induced product on the bar construction to be associative necessarily. The loop space cohomology theory suggests such a choice: In general, given a space, it is impossible to construct a small model for the loop space in the category of hga’s. The investigation mentioned above relies on the perturbation theory that extends such a theory already well developed for differential graded modules, differential graded algebras (dga’s), etc. \([2, 8, 12, 10, 21]\). A difficulty for developing a theory of homological algebra for Hirsch algebras is that the Steenrod cochain product \( a \cdot b \) does not lift automatically on the cohomology level, since it is not a cocycle even for cocycles \( a \) and \( b \). We introduce here filtered Hirsch algebras for controlling the above difficulties. Such objects can be thought of as a specialization of distinguished resolutions in the sense of \([9]\)(see also \([13]\)). On the other hand, the filtered Hirsch model \((RH, d + h)\) of a Hirsch algebra \( A \) is itself a Hirsch algebra with structural operations \( E_{p,q} : RH^{\otimes p} \otimes RH^{\otimes q} \longrightarrow RH \) being formally determined only by the commutative graded algebra (cga) structure of \( H = H(A, d_A) \); furthermore, the perturbation \( h : RH \rightarrow RH \) of the resolution differential \( d \) appears by means of the Hirsch algebra structure on \( A \) (Theorem\([1]\)). So that, ignoring the operations \( E_{p,q} \), we simply get that \( (RH, d) \rightarrow (H, 0) \) is a multiplicative resolution of the cga \( H \) (thought of as a non-commutative version of the Tate-Jozefiak resolution of a cga) and \((RH, d + h) \rightarrow (A, d_A)\) is the filtered model of the dga \( A \) \([21]\); in the category of cdga’s over a field of characteristic zero such a filtered model was constructed in \([10]\).
We also introduce the notion of a *quasi-homotopy commutative* Hirsch algebra (qha). A reason is that a Hirsch resolution automatically admits a binary operation \( \cup_2 \) like Steenrod’s \(-2\)-cochain product up to action of the differential on a diagonal pair of variables. The notion of a qha just involves this difference to describe canonically certain syzygies in a Hirsch resolution in particular. On the other hand, in general to construct a qha model for a space is obstructive, since a non-free action of the Steenrod cohomology operation \( Sq_1 \) on the cohomology in question.

Obviously, a cdga \( H \) can be trivially considered as a Hirsch algebra, i.e., when all operations \( E_{p,q}, p,q \geq 1, \) on \( H \) are assumed to be identically zero; however, we give examples of commutative algebras thought of as the cohomology ones on which there exists a non-trivial Hirsch algebra structure purely determined by \( Sq_1 \).

For a Hirsch algebra \( A \) over the integers, we establish certain formulas relating the structural operations \( E_{p,q} \) with some syzygies in \((RH,d)\) arising from a single element \( a \in H(A) \) with \( a^2 = 0 \). Whence the \( n \)-fold symmetric Massey product \( \langle a \rangle^n, n \geq 3 \), is defined \([18],[17]\); as a single class of \( H(A) \) in our case), these formulas in particular imply that it has a finite order; consequently, \( \langle a \rangle^n, n \geq 3 \), vanish for \( A \) over a field of characteristic zero. Furthermore, for a qha \( A \) over the integers we obtain a formula of the same form as in \([18]\):

\[
-\beta P_1(a) = \langle a \rangle^p, \quad a \in H^{2m+1}(A \otimes \mathbb{Z}_p),
\]

this time \( P_1 : H^{2m+1}(A \otimes \mathbb{Z}_p) \to H^{2mp+1}(A \otimes \mathbb{Z}_p) \) is a cohomology operation canonically determined by the iteration of the \(-1\)-product on \( A \otimes \mathbb{Z}_p \) and \( \beta \) is the Bockstein operator (dually, see \([17]\) in which case for \( A \) to be the singular chains of a third loop space, \( P_1 \) can be identified with the Dyer-Lashof operation).

Finally, for a space \( X = BF_4 \), the classifying space of the exceptional group \( F_4 \), some perturbations in the filtered model of \( X \) are explicitly pointed out enough to recover the above equality on \( H^*(X; \mathbb{Z}_3) \).

Some applications of the filtered Hirsch algebras an earlier version of the paper was dealing with now are considered in \([23],[24]\) (see also \([22]\)).

2. The category of Hirsch algebras

We denote by \( k \) a commutative ring with 1; the special cases we essentially apply are the integers \( \mathbb{Z} \) and the finite fields \( \mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z} \) for \( p \) prime. Graded modules \( A^* \) over \( k \) (non-negatively or non-positively) are assumed to be connected, \( A^0 = k \). A non-negatively graded, connected module \( A \) is 1-reduced if \( A^1 = 0 \).

For a module \( A \), let \( T(A) \) be the tensor module of \( A \), i.e., \( T(A) = \bigoplus_{i=0}^{\infty} A^{\otimes i} \). An element \( a_1 \otimes \ldots \otimes a_n \in A^{\otimes n} \) is denoted by \( [a_1 \cdots a_n] \) or by \( a_1 \cdots a_n \) when \( T(A) \) is viewed as the tensor coalgebra or the tensor algebra respectively. We denote by \( s^{-1}A \) the desuspension of \( A \), i.e., \( (s^{-1}A)^i = A^{i+1} \).

Let \((A,d_A,\mu)\) be a 1-reduced differential graded algebra (dga). The (reduced) bar construction \( BA \) on \( A \) is the tensor coalgebra \( T(A) \), \( \tilde{A} = s^{-1}(A^{>0}) \), with differential \( d = d_1 + d_2 \) given for \( [\bar{a}_1 \cdots \bar{a}_n] \in T^n(\tilde{A}) \) by

\[
d_1[\bar{a}_1 \cdots \bar{a}_n] = - \sum_{i=1}^n (-1)^{i-1} [\bar{a}_1 \cdots \bar{d}_A(a_i) \cdots \bar{a}_n],
\]

and

\[
d_2[\bar{a}_1 \cdots \bar{a}_n] = - \sum_{i=1}^{n-1} (-1)^i [\bar{a}_1 \cdots \bar{a}_i \bar{a}_{i+1} \cdots \bar{a}_n],
\]
where $\epsilon_i = |a_1| + \cdots + |a_i| + i$.

Recall the definition of a Hirsch algebra [15] (compare [26], [14]). Let $A$ be a dga and consider the dg module $(\text{Hom}(BA \otimes BA, A), \nabla)$ with differential $\nabla$. The $\cdot$-product induces a dga structure on it (the tensor product $BA \otimes BA$ is a dgc with the standard coalgebra structure).

**Definition 1.** A Hirsch algebra is a 1-reduced associative dga $A$ equipped with multilinear maps

$$E_{p,q} : A^{\otimes p} \otimes A^{\otimes q} \to A,$$

satisfying the following conditions:

(i) $E_{p,q}$ is of degree $1 - p - q$;

(ii) $E_{1,0} = \text{id} = E_{0,1}$ and $E_{p>1,0} = 0 = E_{0,q>1}$;

(iii) The homomorphism $E : BA \otimes BA \to A$ defined by

$$E([a_1| \cdots |a_p] \otimes [b_1| \cdots |b_q]) = E_{p,q}(a_1, \ldots, a_p; b_1, \ldots, b_q)$$

is a twisting cochain in the dga $(\text{Hom}(BA \otimes BA, A), \nabla, \cdot)$, i.e., it satisfies

the equality $\nabla E = -E \cdot E$.

A morphism $f : A \to B$ between two Hirsch algebras is a dga map $f$ commuting with all $E_{p,q}$.

Condition (iii) implies that $\mu_E : BA \otimes BA \to BA$ is a chain map; thus $BA$ becomes a dg Hopf algebra with not necessarily associative multiplication $\mu_E$ (compare [7], [26], [4], [16], [20]); by this, $\mu_{E_{p+q,E_{p,q}}}$ coincides with the shuffle product on $BA$. It is useful to write the above equality for $E$ componentwise:

(2.1) \[ dE_{p,q}(a_1, \ldots, a_p; b_1, \ldots, b_q) = \sum_{1 \leq i \leq p} (-1)^{\epsilon_{i-1}} E_{p,q}(a_1, \ldots, da_i, \ldots, a_p; b_1, \ldots, b_q) \]

\[ + \sum_{1 \leq i \leq p} (-1)^{\epsilon_{i} + \epsilon_{i-1}} E_{p,q}(a_1, \ldots, a_p; b_1, \ldots, db_i, \ldots, b_q) \]

\[ + \sum_{1 \leq i \leq p} (-1)^{\epsilon_{i}} E_{p-1,q}(a_1, \ldots, a_i a_{i+1}, \ldots, a_p; b_1, \ldots, b_q) \]

\[ + \sum_{1 \leq i \leq p} (-1)^{\epsilon_{i} + \epsilon_{j+1}} E_{p-1,q}(a_1, \ldots, a_p; b_1, \ldots, b_j b_{j+1}, \ldots, b_q) \]

\[ - \sum_{0 \leq i \leq p} \sum_{0 \leq j \leq q} (-1)^{\epsilon_{i} + \epsilon_{j}} E_{i,j}(a_1, \ldots, a_i; b_1, \ldots, b_j) \cdot E_{p-i,q-j}(a_1, \ldots, a_p; b_{j+1}, \ldots, b_q), \]

\[ \epsilon_{x_{i}} = |x_{1}| + \cdots + |x_{i}| + i, \]

\[ \epsilon_{a_{ij}} = \epsilon_{i} + \epsilon_{j} + \epsilon_{p} + \epsilon_{q}. \]

In particular, the operation $E_{1,1}$ satisfies conditions similar to Steenrod’s cochain $\sim_{1}$-product:

$$dE_{1,1}(a;b) - E_{1,1}(da;b) + (-1)^{\epsilon_{i}} E_{1,1}(a;db) = (-1)^{\epsilon_{i}} ab - (-1)^{\epsilon_{i}} (ba),$$

so it measures the non-commutativity of the $\cdot$ product on $A$. Thus, a Hirsch algebra with $E_{p,q} = 0$ for $p, q \geq 1$ is just a commutative differential graded algebra (cdga). The following two special cases are also important for us, so that we write them explicitly. Since the above formula we use below the notation $a \sim_{1} b = E_{1,1}(a;b)$ interchangeably.
The Hirsch formulas up to homotopy:

\[ dE_{2,1}(a; b; c) \equiv E_{2,1}(da; b; c) - (-1)^{\lfloor a \rfloor}E_{1,2}(a; db; c) + (-1)^{\lfloor a \rfloor + \lfloor b \rfloor}E_{2,1}(a; b; dc) - (-1)^{\lfloor a \rfloor}(ab) \sim_1 c + (-1)^{\lfloor a \rfloor + \lfloor b \rfloor + \lfloor c \rfloor}(a \sim_1 c)b + (-1)^{\lfloor a \rfloor}a(b \sim_1 c) \]

and

\[ dE_{1,2}(a; b; c) \equiv E_{1,2}(da; b; c) - (-1)^{\lfloor a \rfloor}E_{1,2}(a; db; c) - (-1)^{\lfloor a \rfloor + \lfloor b \rfloor}E_{1,2}(a; b; dc) + (-1)^{\lfloor a \rfloor + \lfloor b \rfloor}a \sim_1 (bc) - (-1)^{\lfloor a \rfloor + \lfloor b \rfloor}(a \sim_1 b)c - (-1)^{\lfloor b \rfloor - 1}b(a \sim_1 c) \]

that means that the deviations of the binary operation \( \sim_1 \) from the right and left derivation with respect to the \( \cdot \) product are measured by the operations of three variables \( E_{2,1} \) and \( E_{1,2} \) respectively.

**Definition 2.** A quasi-homotopy commutative Hirsch algebra \((qha)\) is a Hirsch algebra \( A \) equipped with a binary product \( \cup_2 : A \otimes A \to A \) such that

\[ (2.2) \quad d(a \cup_2 b) = da \cup_2 b + (-1)^{\lfloor a \rfloor}a \cup_2 db + (-1)^{\lfloor a \rfloor}a \sim_1 b + (-1)^{\lfloor a \rfloor + 1}b \sim_1 a - q(a; b), \]

where \( q(a; b) \) satisfies:

\[ (2.2)_1 \text{ Leibnitz rule. } dq(a; b) = q(da; b) + (-1)^{\lfloor a \rfloor}q(a; db) \]

\[ (2.2)_2 \text{ Acyclicity. } \quad [q(a, b)] = 0 \in H(A, d) \text{ for } da = db = 0. \]

Note that discarding the parameter \( q(a; b) \) the above formula just becomes the Steenrod formula for the \( \sim_2 \)-cochain product:

\[ (2.3) \quad d(a \sim_2 b) = da \sim_2 b + (-1)^{\lfloor a \rfloor}a \sim_2 db + (-1)^{\lfloor a \rfloor}a \sim_1 b + (-1)^{\lfloor a \rfloor + 1}b \sim_1 a. \]

However, \( q(-; -) \) may be non-zero when passing to models constructed by means of the cohomology below. First consider the following

**Example 1.** For topological spaces \( X \), main examples of Hirsch algebras provide the cubical or simplicial cochain complexes \([13], [14], [16]\). Note that in the last case one can choose \( E_{p,q} = 0 \) for \( q \geq 2 \), to obtain a homotopy G-algebra (hga) structure on the simplicial chains \( C^*(X) \) (see also [13]). In particular, the product \( \mu_E \) on \( BC^*(X) \) is geometric, i.e., gives the multiplicative structure of the loop space cohomology \( H^*(\Omega X) \); here the cochain complex \( C^*(X) \) of a space \( X \) is 1-reduced, since by definition \( C^*(X) = C^*(Sing^1 X)/C^{>0}(Sing x) \), in which \( Sing^1 X \subset Sing X \) is the Eilenberg 1-subcomplex generated by the singular simplices that send the 1-skeleton of the standard \( n \)-simplex \( \Delta^n \) to the base point \( x \) of \( X \). These Hirsch algebras are also \( qha \)'s by setting \( \cup_2 = \sim_2 \) and \( q(-; -) = 0 \).

**Example 2.** Let \( (H, d = 0) \) be a free cga generated by a graded set \( V^* \). Then any map of sets \( \tilde{E}_{p,q} : V^p \times V^q \to H \) of degree \( 1 - p - q \) extends to a Hirsch algebra structure \( \tilde{E}_{p,q} : H^p \otimes H^q \to H \) on \( H^* \). Indeed, using formula \( (2.1) \) the construction goes by induction on the sum \( p + q \). In particular, if only \( \tilde{E}_{1,1} \) is non-zero then the image of \( E_{p,q} \) for \( p, q \geq 1 \) is into the submodule of \( H \) spanned on the monomials of the form \( \tilde{E}_{1,1}(a_1; b_1) \cdots \tilde{E}_{1,1}(a_k; b_k) \cdot x \) for \( a_i, b_i \in V, x \in H, k \geq 1 \).

**Example 3.** The argument of the previous example suggests how to lift a Hirsch \( \mathbb{Z}_2 \)-algebra structure from the cochain level to the cohomology one. Given a Hirsch algebra \( A \) and a cocycle \( a \in A \), one has \( d_A E_{1,1}(a, a) = 0 \) and denote \( Sq_a = [E_{1,1}(a, a)] \in H = H^*(A) \). A trick here is to convert the Hirsch formulas up to
homotopy on $A$ to the Cartan formula $Sq_1(ab) = Sq_1a \cdot Sq_0b + Sq_0a \cdot Sq_1b$ on $H$. Namely, given a set of multiplicative generators $U \subset H$, define first the map $\tilde{Sq}_{1,1} : U \times U \rightarrow H$ by

$$\tilde{Sq}_{1,1}(a;b) = \begin{cases} Sq_1a, & \text{if } a = b, \ a \in U, \\ 0, & \text{otherwise} \end{cases}$$

and extend it to the operation $Sq_{1,1} : H \otimes H \rightarrow H$ as a (both side) derivation with respect to the $\cdot$ product, while to $Sq_{p,q} : H^{\otimes p} \otimes H^{\otimes q} \rightarrow H$ for $p, q > 1$ by means of (2.1) (this time $E_{p,q}$ is denoted by $\overline{Sq}_{p,q}$); in particular, $Sq_{1,1}(u;u) = Sq_1u$ for all $u \in H$.

However, such an extension maybe incorrect in general, since the non-freeness of the multiplicative structure on $H$.

Note that, for the following cases the above procedure actually gives the Hirsch algebra structure $\overline{Sq}_{p,q}$ on the cohomology algebra $H^\ast$:

(i) The algebra $H$ has trivial multiplication (e.g. the cohomology of a suspension).

(ii) The algebra $H$ is polynomial.

(iii) The algebra $H$ has the property that for each relation of the form $a \cdot b = 0$, one has $Sq_1a \cdot b = 0 = Sq_1a \cdot Sq_1b$ for $a, b \in H$.

Obviously we have the following proposition:

**Proposition 1.** A morphism $f : A \rightarrow A'$ of Hirsch algebras induces a Hopf dga map of the bar constructions

$$Bf : BA \rightarrow BA'$$

being a homology isomorphism, if $f$ does and the modules $A, A'$ are $k$-free.

In particular, this proposition is helpful to apply special models for a Hirsch algebra $A$ for calculating the cohomology algebra $H^\ast(BA)(= \text{Tor}^A(k,k))$, and, consequently, the loop space cohomology $H^\ast(\Omega X;k)$ when $A = C^\ast(X;k)$, the cochain complex of a topological space $X$ with coefficients in $k$ (see, for example, [22]).

Given a Hirsch algebra $A$, here we construct its model starting with the cohomology $H = H(A)$. By this note that we assume no Hirsch algebra higher order operations $E_{p,q}$ on $H$, unless its commutativity. Nevertheless one can build a special multiplicative resolution $(RH, d)$ of $H$ endowed with Hirsch algebra operations $E_{p,q}$ and then at the cost of the perturbed resolution differential $d_h$ to obtain a Hirsch algebra model $(RH, d_h)$ of $A$ as desired.

2.1. **Hirsch resolution.** First recall that given a graded algebra $H^\ast$, its multiplicative resolution $(R^\ast H^\ast, d)$ is a bigraded tensor algebra $T(V)$ generated by a bigraded free $k$-module

$$V = \bigoplus_{j, m} V^{-j,m}, \ V^{-j,m} \subset R^{-j}H^m, \ j, m \geq 0,$$

where $d$ is of bidegree $(1,0)$, together with a map of bigraded algebras $\rho : (RH, d) \rightarrow H$ inducing an isomorphism $\rho^* : H^\ast(RH, d) \xrightarrow{\simeq} H^\ast$ ($H^\ast$ is bigraded via $H^{0,*} = H^\ast, H^{<0,*} = 0$ and the total degree of $R^{-j}H^m$ is the sum $-j + m$) [21] (compare [10], [12]).

Next we give the following
Definition 3. Given a cga $H^*$, its Hirsch resolution is a multiplicative resolution

$$\rho : R^* H^* \to H^*, \quad RH = T(V), \quad V = \langle \mathcal{V} \rangle,$$

endowed with the Hirsch algebra structural operations

$$E_{p,q} : RH^{\otimes p} \otimes RH^{\otimes q} \to V \subset RH$$

such that $V$ is decomposed as $V^{*,*} = \mathcal{E}^{*,*} \oplus U^{*,*}$ in which $\mathcal{E}^{*,*} = \{ E_{p,q} \}$ is distinguished by an isomorphism of modules

$$E_{p,q} : \otimes_{r=1}^{p-n} H^{k_r} \otimes \otimes_{n=1}^{q} R^{j_n} H^{t_n} \xrightarrow{\cong} \mathcal{E}_{p,q}^{s-p+1, t} \subset V^{*,*}$$

with $(s,t) = (\sum_{r=1}^{p} i_r + \sum_{n=1}^{q} j_n, \sum_{r=1}^{p} k_r + \sum_{n=1}^{q} \ell_n)$ for $p, q \geq 1$.

Proposition 2. Any cga $H^*$ has a Hirsch resolution $\rho : R^* H^* \to H^*$.

Proof. We build a Hirsch resolution of $H^*$ by induction on the resolution degree. Choose a set of multiplicative generators $V^{0,0}$ of $H^*$, span on it the free $k$-module $V^{0,0} = \langle \gamma \rangle$ and take the free (tensor) graded algebra $R^0 H^* = T(V^{0,0})$. Obviously, we get an epimorphism $\rho^0 : R^0 H^* \to H^*$. Assume by induction that we have constructed $R^n H^* = T(V^{n,0} \oplus \cdots \oplus V^{0,n})$, $n \geq 0$, with a dga map $\rho(n) : R^{(n)} H^* \to H^*$ such that $V^{n,*} = \mathcal{E}^{n,*} \oplus U^{n,*}$, $\mathcal{E}^{n,*} = \{ E_{p,q} \}_{n>0}$ with $\mathcal{E}^{n,*}$ spanned by the set of (formal) expressions $E_{p,q}(a_1, \ldots, a_p; b_1, \ldots, b_q)$, $a_r \in R^{-i_r} H^*$, $b_m \in R^{-j_m} H^*$, $n = \sum_{r=1}^{p} i_r + \sum_{m=1}^{q} j_m$, $p, q \geq 1$, and $d : R^{-i} H^* \to R^{-i+1} H^*$ is a differential satisfying (2.1) on $\mathcal{E}^{(n-1),*}$ and acyclic in resolution degrees $-i$, $0 < i < n$. Then set $V^{n-1,*} = \mathcal{E}^{n-1,*} \oplus U^{n-1,*}$ where $\mathcal{E}^{n-1,*} = \{ E_{p,q} \}$ with $\mathcal{E}^{n-1,*}$ spanned on the set of expressions $E_{p,q}(a_1, \ldots, a_p; b_1, \ldots, b_q)$, $a_r \in R^{-i_r} H^*$, $b_m \in R^{-j_m} H^*$, $n + 1 = \sum_{r=1}^{p} i_r + \sum_{m=1}^{q} j_m$, $p, q \geq 1$; Define differential $d$ on $\mathcal{E}^{n-1,*}$ by formula (2.1), while, as usual, $U^{n-1,*}$ appears as the domain of $d : U^{n-1,*} \to R^{-i} H^*$ to achieve the acyclicity in the resolution degree $-i$. Then put $\rho^{n+1}(R^{-n} H^*) = 0$ and $\rho(n+1)|_{R^{-n} H^*} = \rho(n)$ to complete the inductive step.

Set $R^* H^* = \oplus_n R^{-n} H^*$ with $V^{*,*} = \langle \mathcal{V}^{*,*} \rangle$, $\mathcal{E}^{*,*} = \oplus_n \mathcal{E}^{-n,*}$, $U^{*,*} = \oplus_n U^{-n,*}$, $\rho|_{R^* H^*} = \rho^0$ and $\rho|_{R^{-n} H^*} = 0$ for $n > 0$ to obtain a resolution map $\rho : RH \to H$ as desired. \hfill \Box

Given an arbitrary triple $(a; b; c) = (a_1, \ldots, a_k; b_1, \ldots, b_t; c_1, \ldots, c_r)$, denote

$$R_{k,t,r}(a; b; c) = \sum_{k_1 + \cdots + k_p = k} \sum_{l_1 + \cdots + l_q = l} (-1)^{p+r} E_{p,r}(a_{k_1}, \ldots, a_{k_p}; b_{l_1}, \ldots, b_{l_r}),$$

$$\ldots, E_{k_r}(a_{k_{r-p+1}}, \ldots, a_k; b_{l_{r-p+1}}, \ldots, b_{l_t}); c_1, \ldots, c_r$$

and

$$R_{k,t,r}(a; b; c) = \sum_{k_1 + \cdots + k_p = k} \sum_{l_1 + \cdots + l_q = l} (-1)^{p+r} E_{k,q}(a_{k_1}, \ldots, a_k; b_{l_1}, \ldots, b_{l_r}; c_1, \ldots, c_r),$$

$$\ldots, E_{q,r}(b_{l_{r-q+1}}, \ldots, b_{l_t}; c_{r-q+1}, \ldots, c_r),$$

where the signs $\varepsilon$ and $\delta$ are caused by permutations of symbols $a_i, b_j, c_k$ as in [20]. For example, the expression $R_{k,t,r}(a; b; c) - R_{k,t,r}(a; b; c)$ is a cocycle in $U^{1-k-\ell-r,*}$ for the variables from $R^0 H$, and then it should be killed by some
where

\[ (2.9) \quad a \ni V \quad \ni V \quad a, b \]

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\[ (2.5) \quad d(a \cup_2 b) = \begin{cases} (-1)^{|a|} a \ni_1 b + (-1)^{|a|} b \ni_1 a, & \text{if } a \neq b \\ a \ni_1 a, & \text{if } a = b, |a| \text{ is even} \\ 0, & \text{otherwise.} \end{cases} \]

Thus, the deviation of the above equality from (2.2) is measured by the parameter

\[ q(a; a) = a \ni_1 a \text{ for } |a| \text{ even.} \]

In order to define the operation \( \cup_2 \) on the whole \( RH \), it is convenient to do this by means of the operation \( \ni_2 \) on \( RH \) satisfying (2.3). Namely, these operations are in fact introduced simultaneously by induction on the resolution degree satisfying the following formulas:

\[ a \ni_2 b = \begin{cases} 2a \cup_2 a, & \text{if } a = b, |a| \text{ is even, } a \in V^{0,*}, \\ 0, & \text{if } a = b, |a| \text{ is odd, } a \in V, \\ a \cup_2 b, & \text{if } a \neq b, \quad a, b \in V^{0,*}, \end{cases} \]

\[ a \ni_2 b \cdot c = (a \ni_2 b) \cdot c + (-1)^{|a||b|} b (a \ni_2 c) + (-1)^{|b|} E_{1,2}(a; b, c) - (-1)^{|a|(|b|+|c|)+|c|} E_{2,1}(b, c; a), \quad b \in V, a, c \in RH, \]

\[ a \ni_2 E_{p,q}(b_1, \ldots, b_p; b_{p+1}, \ldots, b_{p+q}) = \sum_{i=1}^{p+q} (-1)^{|a|} E_{p,q}(b_1, \ldots, b_{i-1}, a \ni_2 b_i, b_{i+1}, \ldots, b_p; b_{p+1}, \ldots, b_{p+q}), \quad a, b \in RH, \]

\[ b \ni_2 c = (-1)^{|a||b|} a \ni_2 b \ni_2 c + b \cup_3 d a, \quad b \in V, a, c \in RH, \]

where

\[ b \cup_3 z = \begin{cases} (-1)^{|b|+|u|}(E_{2,2}(u, v; b, c) - E_{2,2}(b, c; u, v)) + (-1)^{|u|(|b|+|c|)}(|u|+1) \\ (1 - (1)|b| \ni_2 u)(v \ni_1 c) + (-1)^{|c|+1} b \ni_1 u)(c \ni_2 v), \quad z = uv, \\ 0, \quad z \in V \text{ or } z = b \cdot c, |z| \text{ is even,} \end{cases} \]
where the summation is over unshuffles $(i)$

\[
    z_1 \cup_4 z_2 = \begin{cases} 
    \begin{aligned}
    ( -1 )^{( l| | + |y| ) ( |y| + 1 ) + ( |y| + 1 ) |c|} ( x \prec_2 u ) ( y \prec_2 v ), & z_1 = xy, z_2 = uv, \\
    0, & z_i \in V, \text{ some } i, \text{ or } z_1 = z_2, |z_1| \text{ is odd},
    \end{aligned}
    \end{cases}
\]

(2.11) \hspace{1cm} a \prec_2 ( b \prec_2 c ) = ( a \prec_2 b ) \prec_2 c , \hspace{0.5cm} a, b, c \in V \setminus E ,

(2.12) \hspace{1cm} a_1 \prec_2 \cdots \prec_2 a_n = n_1! \cdots n_k! ( a_1 \cup_2 \cdots \cup_2 a_n ) , \hspace{0.5cm} a_i \in V \setminus E , \hspace{0.5cm} n \geq 2 ,

\hspace{1cm} a_1 \cup_2 \cdots \cup_2 a_n , a_1 \prec_2 \cdots \prec_2 a_n \in U ,

where each $n_i$ is the maximal number of the same generators of even degree among $a_i \in V , 1 \leq i \leq n$ , so that the expression $d ( a_1 \prec_2 \cdots \prec_2 a_n )$ is formally divisible by $n_1! \cdots n_k!$. In particular, when all $a_i$ are different to each other one gets the equality $a_1 \prec_2 \cdots \prec_2 a_n = a_1 \cup_2 \cdots \cup_2 a_n$.

Finally, for any $a, b \in RH$, define $a_1 \cup_2 b \in RH$ by

\[
    d ( a_1 \cup_2 b ) = \frac{1}{n} d ( a \prec_2 b )
\]

where $n \geq 1$ is the maximal integer that divides $d ( a \prec_2 b )$; for $n = 1$, let $a_1 \cup_2 b = a \prec_2 b$.

For example, given $a, b \in RH$, $d a = d b = 0$, with $|a| , |b|$ even, apply (2.7)–(2.11) together with (2.11) to check that the expression $a b \prec_2 a b - d E_{2 , 2} ( a ; b ; a , b ) + ( - 1 )^{|a||b|} ( a \prec_1 a ) ( b \prec_1 b )$ is formally divisible by 2 and then one can set

\[
    a b \cup_2 a b = \frac{1}{2} \left[ a b \prec_2 a b - d E_{2 , 2} ( a ; b ; a , b ) + ( - 1 )^{|a||b|} ( a \prec_1 a ) ( b \prec_1 b ) \right].
\]

Also $a b \cup_2 a b$ can be chosen to be zero for $|a| , |b|$ odd and $d a = d b = 0$, though $a b \prec_2 a b \neq 0$.

Furthermore, for $a_i \in RH$ with $d a_i = 0$ we have

\[
    d ( a_1 \cup_2 \cdots \cup_2 a_n ) = \sum_{(i , j)} ( -1 )^{ |a_i| + \cdots + |a_k| } ( a_{i_1} \cup_2 \cdots \cup_2 a_{i_k} ) \prec_1 ( a_{j_1} \cup_2 \cdots \cup_2 a_{j_l} ) ,
\]

where the summation is over unshuffles $(i , j)$ and

(2.13) \hspace{1cm} d ( a \cup_2 b ) = \frac{1}{n} d ( a \prec_2 b )

where $n \geq 1$ is the maximal integer that divides $d ( a \prec_2 b )$; for $n = 1$, let $a_1 \cup_2 b = a \prec_2 b$.

Now let $n \geq 0$ for some $n \geq 2$. Let $n > 2$. Then $\cup_2$ is introduced on $RH$ again by means of formulas (2.6)–(2.13) only with a remark that (2.12) is valid when the left hand side is non-zero, while the right hand side is always non-trivially defined. When $n = 2$, $\cup_2$ is defined on $V$ by

\[
    d ( a \cup_2 b ) = \begin{cases} 
    \begin{aligned}
    d a \cup_2 b + a \cup_2 d b + a \prec_1 b + b \prec_1 a, & a \neq b, \hspace{0.5cm} a , b \in V , \\
    a \cup_2 d a + a \prec_1 a & a = b, \hspace{0.5cm} a \in V \setminus E ,
    \end{aligned}
    \end{cases}
\]

and extended on $RH$ by replacing $\prec_2$ by $\cup_2$ in (2.8)–(2.11), while (2.7) is replaced by

\[
    a \cup_2 b c = ( a \prec_2 b ) c + b ( a \prec_2 c ) + E_{1 , 2} ( a ; b , c ) + E_{2 , 1} ( b , c ; a ) , \hspace{0.5cm} b \in V , a , c \in RH ,\]

where $n \geq 1$ is the maximal integer that divides $d ( a \prec_2 b )$; for $n = 1$, let $a_1 \cup_2 b = a \prec_2 b$.
with a convention
\[ u \sim_2 v = \begin{cases} u \cup_2 v, & u \neq v, \quad u, v \in V, \\ 0, & u = v, \quad u \in V. \end{cases} \]

For example, we obtain for \( a \in V^{0,*} \) the following mod 2 equalities: \( a^2 \cup_2 a = a \cup_2 a^2 = E_{12}(a; a, a) + E_{21}(a, a; a) \) and \((a \sim_1 a) \cup_2 a = a \cup_2 (a \sim_1 a) = (a \cup_2 a) \sim_1 a + a \sim_1 (a \cup_2 a).\)

Define a submodule \( T \subset U \) as
\[ T = \langle a_1 \cup_2 a_2 \cup_2 \cdots \cup_2 a_n | a_i \in V \setminus E, \ n \geq 2 \rangle. \]

Then fix a decomposition
\[ V = E \oplus U = E \oplus T \oplus \mathcal{N} \]
in which, whence \( \mathcal{N} \) is chosen, the summand \( E \oplus T \) is canonically determined.

A Hirsch resolution \((RH, d)\) is minimal if
\[ d(U) \subset E + D + \mu \cdot V, \]
where \( D^{*,*} \subset R^H \) denotes the submodule of decomposables \( RH^+ \). \( RH^+ \) and \( \mu \in \mathbb{k} \) is non-invertible; in particular, \( \mu \neq \pm 1 \) for \( \mathbb{k} = \mathbb{Z} \) and \( \mu = 0 \) for \( \mathbb{k} \) a field.

For example, the minimal Hirsch resolution for a polynomial algebra is completely determined by the \( \sim_1 \) and \( \cup_2 \)-operations [23].

Note that a minimal Hirsch resolution is not minimal in the category of dga’s, i.e., the resolution differential to be sending the multiplicative generators into \( D \), even for \( \mathbb{k} \) a field.

![Geometrical interpretation of some canonical syzygies in the Hirsch resolution RH.](image-url)
Note that in the above figure the symbol ‘=’ assumes equality \(2.3\); the picture for \(a \cup b \cup c\) is in fact 4-dimensional and must be understood as follows: Whence \(a \cup b\) corresponds to the 2-ball, the boundary of \(a \cup b \cup c\) consists of six 3-balls each of them is subdivided into four cells by fixing two equations (these cells just correspond to the four summand components of the differential evaluated on the compositions of the \([-1\)- and \(2\)-products]), and then, given a 3-ball, two cells from these four cells is glued to the ones of the boundary of the (diagonally) opposite 3-ball, while the other cells to the ones of the boundaries of the neighboring 3-balls according to the relation

\[
x \sim_1 (y \sim_1 z) + (x \sim_1 y) \sim_1 z = y \sim_1 (x \sim_1 z) + (y \sim_1 x) \sim_1 z.
\]

2.2. Filtered Hirsch model. Recall that a dga \((A^*, d)\) is multialgebra if it is bigraded \(A^n = \oplus_{i+j=n} A^{i,j}, i \leq 0, j \geq 0, \) and \(d = d^0 + d^1 + \cdots + d^n + \cdots\) with \(d^n : A^{p,q} \to A^{p+n,q-n+1}\). Given a dga \(A\), we consider it as bigraded via \(A^{0,*} = A^*\) and \(A^{i,*} = 0\) for \(i \neq 0\); consequently, it becomes a multialgebra.

A multialgebra \(A\) is called homological if \(d^0 = 0\) and \(H^i(A^{1,*}, d^1) = 0, i < 0\). For a homological multialgebra the sum \(d^2 + d^3 + \cdots + d^n + \cdots\) is called a perturbation of \(d^1\). In the sequel we always consider homological multialgebras in which case denote \(d^1, d^2, \ldots\) by \(h\) and the sum \(h^2 + h^3 + \cdots + h^n + \cdots\) by \(h\). A multialgebra is free if it is the tensor algebra over a bigraded \(k\)-module. Given \(r \geq 2\), the map \(h^r|_{A^{-r,*}} : A^{-r,*} \to A^{0,*}\) is referred to as the transgressive component of \(h\) and is denoted by \(h^{tr}\). A multialgebra \(A\) with a Hirsch algebra structure

\[
E_{p,q} : \bigotimes_{r=1}^p A^i_r \otimes A^j_q \to A^{r-p-q+1,p}\]

with \((s, t) = (\sum_{r=1}^p i_r + \sum_{n=1}^q j_n, \sum_{r=1}^p k_r + \sum_{n=1}^q \ell_n)\), \(p, q \geq 1\), is called Hirsch multialgebra.

**Definition 4.** A free Hirsch homological multialgebra \((A, E_{p,q}, d + h)\) is filtered Hirsch algebra if it has the following additional properties:

(i) In \(A = T(V)\) a decomposition

\[
V^{*,*} = E^{*,*} \oplus U^{*,*}
\]

is fixed where \(E^{*,*} = \{E_{p,q}^{<0,*}\}_{p,q \geq 1}\) is distinguished by an isomorphism of modules

\[
E_{p,q} : \bigotimes_{r=1}^p A^0_r \otimes A^{q} \cong E_{p,q} \subset V, p, q \geq 1;
\]

(ii) The restriction of the perturbation \(h\) to \(E\) has no transgressive components \(h^{tr}\), i.e., \(h^{tr}|_E = 0\).

Given a Hirsch algebra \(B\), its filtered Hirsch model is a filtered Hirsch algebra \(A\) together with a Hirsch algebra map \(A \to B\) that induces an isomorphism on cohomology.

A multialgebra morphism \(\zeta : A \to B\) between two multialgebras \(A\) and \(B\) is a dga map of total degree zero that preserves the column (resolution) filtration, so that \(\zeta\) has the components \(\zeta = \zeta^0 + \cdots + \zeta^t + \cdots\), \(\zeta^t : A^{s,t} \to B^{s+t,t-1}\).

A homotopy between two morphisms \(f, g : A \to B\) of multialgebras is an \((f, g)\)-derivation homotopy \(s : A \to B\) of total degree \(-1\) that lowers the column filtration by \(1\).

A homotopy between two morphisms \(f, g : A \to A'\) of Hirsch (multi)algebras is a homotopy \(s : A \to A'\) of underlying (multi)algebras and
Proposition 3. Adams-Hilton type statement. Theorem 1. Let $\xi : \mathbb{C} \to C$ be a map of (filtered) Hirsch algebras that induces an isomorphism on cohomology. Then for a filtered Hirsch algebra $A$, there is a bijection on the sets of homotopy classes of (filtered) Hirsch algebra maps $A \to C$. Namely, for each inductive step of constructing of a chain homotopy $s_1, \ldots, s_q \to \mathbb{C}$, define a perturbation $s : \mathbb{C} \to C$ between two multiplicative maps $f, g : A \to C$ in question we can choose $s$ on $\mathbb{C}^{i,j}$ satisfying formula (2.14). Then there is an isomorphism $\phi : \mathbb{C} \to C$.

Proof. Discarding the Hirsch algebra structures the proof goes by induction on the column grading and is similar to that of Theorem 2.5 in [11]. The Hirsch algebra structure only specifies a choice of a homotopy $s$ on the multiplicative generators $E \subset V$. Namely, for each inductive step of constructing of a chain homotopy $s_1, \ldots, s_q \to \mathbb{C}$, we can choose $s$ on $\mathbb{C}^{i,j}$ satisfying formula (2.14).

The basic examples of a filtered Hirsch algebra are provided by the following theorem - the main result of the paper about Hirsch algebras.

Theorem 1. Let $H$ be a cga and $\rho : (RH,d) \to H$ its Hirsch resolution. Suppose that for a Hirsch algebra $A$ there is an isomorphism $i_A : H \approx H(A,d)$. Then there is a pair $(h,f)$ where $h : RH \to RH$ is a perturbation of the resolution differential $d$ on $RH$ and

$$f : (RH,d+h) \to A$$

is a filtered Hirsch model of $A$ such that $(f|_H^H)^* = i_A\rho|_H^H : R^0H \to H(A)$.

Uniqueness. If there is another pair $(h,f)$, $f : (RH,d+h) \to A$, with the above properties, then there is an isomorphism of filtered Hirsch models $\zeta : (RH,d+h) \approx (RH,d+h)$ such that $\zeta$ has the form $\zeta = Id + \zeta^1 + \cdots + \zeta^r + \cdots$, $\zeta^r : R^{-s}H^r \to R^{-s+r}H^{r-r}$, and $f$ is homotopic to $f \circ \zeta$.

Proof. Existence. Let $RH = T(V)$ with $V = E \oplus U$. We define a perturbation $h$ and a Hirsch algebra map $f : (RH,d+h) \to (A,d)$ by induction on resolution (column) grading (compare [10], [21]). Since $R^0H$ is a free algebra, define a dga
map \( f^0 : R^0H \to (A, d) \) with \((f^0)^* = i_{A\partial}|_{p=0} : R^0H \to H(A) \). Then there is 
\( f^1 : V^{-1,*} \to A^{*-1} \) with \( f^0d = df^1 \); in particular, choose \( f^1 \) on \( \mathcal{E}^{-1,*} (= \mathcal{E}^{1,-*}) \) to be defined by the formula \( f^1(a \sim_1 b) = f^0a \sim_1 f^0b \), \( a, b \in V^{0,*} \). Then extend the map \( f^0 + f^1 \) from \( V^{(-1),*} = \mathcal{E}^{1,-*} \oplus U^{-1,*} \oplus U^{0,*} \) to obtain a dga map \( f^{(1)} : (R^{(-1)}H, d) \to (A, d) \).

Assume by induction that we have constructed a pair \((h^{(n)}, f^{(n)})\) satisfying the following conditions:

1. \( h^{(n)} = h^2 + \cdots + h^n \) is a derivation on \( RH \),
2. Equality (2.1) holds on \( RH \) for \( d + h^{(n)} \) in which

\[
\begin{align*}
(h^n)^* & = h^2 + \cdots + h^n, \\
f^{(n)} & = (f^0 + \cdots + f^n)
\end{align*}
\]

(3) \( df^n + h^n d + \sum_{i+j=n+1} h^i h^j = 0, \)
(4) \( f^{(n)}|_{V^{(-n),*}} = (f^0 + \cdots + f^n)|_{V^{(-n),*}} \),
(5) \( f^{(n)} : R^{(-n)}H \to A \) is a dga map,
(6) \( f^{(n)}(d + h^{(n)}) = df^{(n)} \) on \( R^{(-n)}H \),
(7) \( f^{(n)} \) is compatible with the maps \( E_{p,q} \) on \( \mathcal{E}^{(-n),*} \).

Next consider

\[
f^{(n)}(d + h^{(n)})|_{V^{(-n-1),*}} : V^{(-n-1),*} \to A^{*-n-1}.
\]

Clearly \( df^{(n)}(d + h^{(n)}) = 0 \). Define a linear map \( h^{n+1} : U^{(-n-1),*} \to R^0H^{*-n} \) with \( \rho h^{n+1} = i_A \left[ f^{(n)}(d + h^{(n)}) \right] \). Then extend \( h^{n+1} \) on \( RH \) as a derivation (denoting by the same symbol) with

\[
dh^{n+1} + h^{n+1} d + \sum_{i+j=n+2} h^i h^j = 0
\]

and

\[
h^{n+1} E_{p,q}(a_1, \ldots, a_p; b_1, \ldots, b_q) = \sum_{i=1}^p (-1)^{\epsilon_i} E_{p,q}(a_1, \ldots, h^{n+1} a_i, \ldots, a_q; b_1, \ldots, b_q)
\]

\[
+ \sum_{j=1}^q (-1)^{\epsilon_j} E_{p,q}(a_1, \ldots, a_p; h^{n+1} b_j, \ldots, b_q).
\]

Hence, there is \( f^{n+1} : V^{(-n-1),*} \to A^{*-n-1} \) with

\[
f^{(n)}(d + h^{(n+1)}) = df^{n+1}.
\]

Extend the restriction of \( f^{(n)} + f^{n+1} \) to \( V^{(-n-1),*} \) to obtain a multiplicative map

\[
f^{(n+1)} : R^{(-n-1)}H \to A
\]

being compatible with \( E_{p,q} \) on \( \mathcal{E}^{(-n-1),*} \). Thus, we get the pair \((h^{(n+1)}, f^{(n+1)})\) that finishes the inductive step. Finally, a perturbation \( h \) and a Hirsch algebra map \( f \) defined by \( h = h^2 + \cdots + h^n + \cdots \) and \( f|_V = (f^0 + \cdots + f^n + \cdots)|_V \) respectively are as desired.

**Uniqueness.** Using Proposition 3 we find a multialgebra morphism

\[
\zeta : (RH, d + h) \to (RH, d + \tilde{h}),
\]
ζ = ζ^0 + ζ^1 + · · · , with ̃f ◦ ζ ≈ f. It is easy, in addition, to choose ζ with ζ^0 = Id. □

Given a qha A (e.g. A = C^*(X; k)) in the above theorem, we can refine the perturbation h as follows. Consider a decomposition of the submodule T ⊂ U as

\[ T = T_2 \oplus T', \]

where T_2 consists of the iteration of the \( \cup_2 \)-product evaluated on different variables.

**Proposition 4.** Suppose that A is a qha in the hypotheses of Theorem 1. Then in the filtered Hirsch model f : (RH, d + h) → A:

(i) The perturbation h can be chosen with \( h(a \cup_2 b) = ha \cup_2 b + (-1)^{|a|} a \cup_2 hb \) and \( h^{tr}|_{T_2} = 0 \) for \( a, b \in V \setminus E, a \neq b \).

(ii) If, in addition, H is evenly graded, the perturbation h can be chosen with \( h^{tr}|_{T} = 0 \).

**Proof.** (i) By the inductive construction of a pair \((h, f)\) in the proof of Theorem 1 we can define f on \( T_2 \) by the formula \( f(a \cup_2 b) = fa \cup_2 fb \) which guarantees a choice of the perturbation h as required.

(ii) Follows from the obvious fact that obstructions to annihilation of \( h^{tr} \) on T lay in the odd degrees of \( H^* \). □

Note that in the proof of Theorem 1 the perturbation h can be also chosen with

\[ hq(a; b) = q(ha; b) + (-1)^{|a|} q(a; hb), \]

where \( q(a; b) \) is the parameter for \( a \cup_2 b \) in (2.2) with respect to the resolution differential d, then regarding \( d_h = d + h \), the product \( a \cup_2 b \) again satisfies (2.2) and (2.2)_1, but this time involving the parameter \( q_h(a; b) = q(a; b) + h^{tr}(a \cup_2 b) \). In the case of a qha A item (2.2)_2 is also satisfied in the filtered model of A, so that \( (RH, d_h) \) admits a qha structure, too (compare Example 4 below).

Remark also that we can not achieve f to be a qha map in Proposition 4 in general. The matter is that there appears an obstruction to compatibility of parameters \( q(-; -) \) under f, since a non-free action of the operation \( Sq_1 \) on H for \( k = \mathbb{Z}_2 \).

**Example 4.** It is known that the Hochschild cochain complex \( C^*(B; B) \) of an associative algebra B has an hga structure (see [26] for references), a particular case of Hirsch algebra. View the Hochschild cohomology \( H = HC^*(B; B) \) as a cga and apply Theorem 1 for A = \( C^*(B; B) \) to obtain the filtered Hirsch model f : (RH, d + h) → \( C^*(B; B) \). Given \( a, b \in V^{0,*} \), we have that \( h^2(a \cup_2 b) = \rho a + \rho b \), the product by the G-algebra multiplication on the Hochschild cohomology H. In other words, the non-triviality of the G-algebra structure on H implies the non-triviality of perturbation \( h^2 \) restricted to the submodule T ⊂ U. Consequently, the operation \( a \cup_2 b \) with \( q(a, b) \) satisfying item (2.2)_2 does not exist on a filtered Hirsch model of \( C^*(B; B) \) in general.
3. SOME EXAMPLES AND APPLICATIONS

As it is already mentioned in the introduction certain applications of the above material are given in [23, 24]. Here we consider the following.

3.1. Symmetric Massey products. Given a sequence of relations in (minimal) Hirsch resolution \((RH, d)\) of the following form

\[
du_i = a_i a_{i+1} + \lambda v_i, \quad a_i, u_i, v_i \in V, \quad \lambda \in k, \quad 1 \leq i < n,
\]

there are corresponding generators \(u_{a_1, \ldots, a_k} \in V, 3 \leq k \leq n\), appearing by those syzygies that mimic the definition of \(k\)-fold Massey products arising from \(k\)-tuples \((a_1, \ldots, a_k)\) [13]. In particular, for \(a_1 \in V^{0,*} \cup V^{-1,*}\), we have that \(u_{a_1} \in V\) is defined by

\[
du_{a_1, a_2, \ldots, a_n} = \sum_{0 \leq i < n} (-1)^{\varepsilon_i} u_{a_1, a_i} u_{a_{i+1}, \ldots, a_n} + \lambda v_{a_1, \ldots, a_n}, \quad \varepsilon_i = |a_1| + \cdots + |a_i| + i,
\]

\[
dv_{a_1, a_2, \ldots, a_n} = \sum_{0 \leq i < n} (-1)^{\varepsilon_i} u_{a_1, a_{i+1}, \ldots, a_n},
\]

where \(u_{a_i}\) and \(u_{a_i, a_{i+1}}\) denote \(a_i\) and \(-(-1)^{|a_i}|u_i\) respectively.

We are interested in a special case of (3.1), namely, when \(a_1 = \cdots = a_n\). More precisely, let \(A\) be a Hirsch algebra over \(Z\). Let \(a \in H^l(A), l \geq 2\) with \(a^2 = 0\). Then we have the corresponding relation

\[
dx_1 = -(-1)^{|x_0}|x_0^2
\]

in \((RH, d)\) where \(x_0 \in V^{0,l}, \rho x_0 = a\) and \(x_1 \in V^{-1,2l}\).
The above relation generates the following sequence of relations in \((RH, d)\) with \(x_n \in V^{−n,l(n+1)}\):

\[
(3.2) \quad dx_n = \sum_{i+j=n-1} (-1)^{|x_i|x_j}, \quad n \geq 1.
\]

Furthermore, the generators \(x_n\) take part in the following family of relations involving elements \(b_{k,\ell} \in V^{−(k+\ell),r(k+\ell)}\) with \(k, \ell \geq 1\). Denote \(i_{(n)} = i_1 + \cdots + i_n + n\). Then for \(|x_0|\) odd,

\[
(3.3) \quad db_{k,\ell} = (-1)^{k+\ell} \binom{k+\ell}{\ell} x_{k+\ell-1}
\]

\[
- \sum_{i_{(p)=k, j_{(q)}=\ell}} (-1)^{k+\ell+p+q} E_{p,q}(x_{i_1}, \ldots, x_{i_p}; x_{j_1}, \ldots, x_{j_q})
\]

\[
- \sum_{0 \leq \ell < k, 0 \leq m < \ell, \ i_{(s)}=r, j_{(t)}=m} (-1)^{r+m} \binom{r+m}{m} b_{k-r,\ell-m}
\]

in which the first equalities are:

\[
db_{1,1} = 2x_1 + x_0 \sim x_0,
\]

\[
 db_{2,1} = -3x_2 + E_{2,1}(x_0, x_0; x_0) - x_1 \sim x_0 - x_0 b_{1,1} + b_{1,1} x_0,
\]

\[
 db_{1,2} = -3x_2 + E_{1,2}(x_0; x_0, x_0) - x_0 \sim x_1 - x_0 b_{1,1} + b_{1,1} x_0.
\]

And for \(|x_0|\) even,

\[
(3.4) \quad db_{k,\ell} = (-1)^{k+\ell} \alpha_{k,\ell} x_{k+\ell-1}
\]

\[
- \sum_{i_{(p)=k, j_{(q)}=\ell}} (-1)^{k+\ell+p+q} E_{p,q}(x_{i_1}, \ldots, x_{i_p}; x_{j_1}, \ldots, x_{j_q})
\]

\[
- \sum_{0 \leq \ell < k, 0 \leq m < \ell, \ i_{(s)}=r, j_{(t)}=m} \left( (-1)^{k+\ell+r+1} E_{s,t}(x_{i_1}, \ldots, x_{i_p}; x_{j_1}, \ldots, x_{j_q}) b_{k-r,\ell-m}ight)
\]

\[
+ (-1)^{k+\ell+r(\ell+m)} \alpha_{r,m} b_{k-r,\ell-m} x_{r+m-1}
\]

\[
\alpha_{i,j} = \begin{cases} \binom{(i+j)/2}{j/2}, & i, j \text{ are even}, \\ \binom{(i+j-1)/2}{j/2}, & i \text{ is odd, } j \text{ is even}, \\ 0, & i, j \text{ are odd}, \end{cases}
\]

in which the first equalities are:

\[
 db_{1,1} = x_0 \sim x_0 \quad \text{(i.e., } b_{1,1} = x_0 \cup_2 x_0),
\]

\[
 db_{2,1} = -x_2 + E_{2,1}(x_0, x_0; x_0) - x_1 \sim x_0 - x_0 b_{1,1} - b_{1,1} x_0,
\]

\[
 db_{1,2} = -x_2 + E_{1,2}(x_0; x_0, x_0) - x_0 \sim x_1 + x_0 b_{1,1} + b_{1,1} x_0.
\]

Of course, for the sake of minimality one can choose only some \(b_{k,\ell}\) above. For example, for \(|x_0|\) even, choose \(x_2 = E_{1,2}(x_0; x_0, x_0) - x_1 \sim x_0 + x_0(x_0 \cup_2 x_0) + (x_0 \cup_2 x_0) x_0\) in (3.2) and then put \(b_{1,2} = 0\) to eliminate the last relation. Furthermore,
given \( n \geq 1 \), denote \( b_n = b_{1,n} \) and set \((k, \ell) = (1, n)\) in (3.3) to obtain
\[
(3.5) \quad db_n = -(-1)^n ((n + 1)x_n - \sum_{\substack{j(n) = n \\ 1 \leq q \leq n}} (-1)^q E_{1,q}(x_0; x_j, \ldots, x_{j_q})) + \sum_{i+j=n-1} (-1)^j (b_{j,x_i} - x_i b_j).
\]
Note that for an hga \( A \) (e.g. \( A = C^*(X; \mathbb{Z}) \)) the above formula simplifies by removing the operations \( E \) since the second Hirsch formula up to homotopy from Section 2 is the strict one.

Next recall [18], [17] the definition of the \( n \)-fold symmetric Massey product \( \langle a \rangle^n \). Note that as in Lemma 9 [17] \( \langle a \rangle^n \) contains only a single class of \( H(A) \) for a Hirsch algebra \( A \), too.

Now if \( n \geq 2 \) is the first integer such that \( h^{tr}x_n \neq 0 \), then from (3.2) and \( d^2_n = 0 \) we deduce that \( h \) can be chosen in the filtered model \( (RH, d_h) \) so that \( hx_i = 0 \) for \( 0 \leq i < n \) and \( hx_n = h^{tr}x_n \) (see also (3.9) below). Consequently, we immediately get
\[
(3.6) \quad [h^{tr}x_n] = \langle a \rangle^{n+1} \in H^*(A).
\]
We have the following

\textbf{Theorem 2.} Let \( A \) be a Hirsch algebra over \( k = \mathbb{Z} \) and \( a \in H(A) \) an element with \( a^2 = 0 \). If for \( n \geq 3 \) the symmetric Massey product \( \langle a \rangle^n \in H(A) \) is defined, then it has a finite order; consequently, for \( k \) a field of characteristic zero, \( \langle a \rangle^n \) is defined and equals to zero for all \( n \).

\textbf{Proof.} Given \( n \geq 3 \), let \( \langle a \rangle^n \) be defined. From formulas (3.3)–(3.4) and (3.6) immediately follows that \( \langle a \rangle^{n+1} \) has a finite order: namely, if \( |x_0| \) odd, take \((k, \ell) = (1, n)\) to obtain \((n + 1)|h^{tr}x_n| = 0 \) (cf. (3.5)), while for \( |x_0| \) even, take \((k, \ell) = (2, n-1)\) to obtain \( r|h^{tr}x_n| = 0 \) or \((r+1)|h^{tr}x_n| = 0 \) for \( n = 2r \) or \( n = 2r+1, r \geq 1 \), respectively. \( \square \)

\subsection{3.2. The Kraines formula.} Given an odd dimensional element \( a \in H^{2m+1}(X; \mathbb{Z}_p) \), \( p \) is an odd prime, the following formula is established in [18] (for the dual case see [17]):

\[
(3.7) \quad \langle a \rangle^p = -\beta P_1(a)
\]
in which \( P_1 : H^{2m+1}(X; \mathbb{Z}_p) \to H^{2m+1}(X; \mathbb{Z}_p) \) is Steenrod’s cohomology operation and \( \beta \) is the Bockstein cohomology homomorphism. We generalize the above equality for a qha of the form \( A \otimes \mathbb{Z}_p \), where \( A \) is a \( \mathbb{Z} \)-free qha, as follows. Let \( a \in H^{2m+1}(A) \) and \( x \in A^{2m+1} \) its representative cocycle. Given \( n \geq 2 \), take (the right most) \( n^{th} \)-power of \( x \in A \) under the \( \mu_E \) product on \( BA \) and then consider its component, \( s^{-1}(x^{\mu_n}) \in \bar{A} \), in which \( x^{\mu_n} \) has the form
\[
x^{\mu_n} = x^{-1}n + Q_n \in A^{2mn+1},
\]
where \( Q_2 = 0 \) and \( Q_n \) is expressed in terms of \( E_{1,k} \) for \( 1 \leq k \leq n - 2 \). For example, \( Q_3 = 2E_{1,2}(x;x,x) \). We have that \( dx^{\mu_n} \) is divided by an integer \( p \geq 2 \) if and only if \( p \) is a prime and \( n = p^i \), some \( i \geq 1 \). Consequently, the homomorphism
\[
P_1 : H^{2m+1}(A \otimes \mathbb{Z}_p) \to H^{2mp+1}(A \otimes \mathbb{Z}_p), \quad a \to [x^{\mu_p}],
\]
is well defined. Now let \( x_0 \in V^{0,2m+1} \) be as in subsection 3.2. Assume that \( p \) is an odd prime and denote \( c_2 = b_1, c_k = (k - 1)!b_{k-1} + x_0 \sim_1 c_{k-1}, k > 1 \), to obtain for \( k = p \) from (3.5) a relation of the following form

\[
d c_p = -p!x_p-1 + x_0^{w_0} + p u_p, \quad u_p \in D.
\]

For example,

\[
d c_3 = d(2b_2 + x_0 \sim b_1) = -6x_2 + x_0^{\sim 1.3} + 2E_{1,2}(x_0; x_0) - 3(x_0b_1 - b_1x_0).
\]

Consider \((RH, d_h)\) and the action of the perturbation \( h \) on (3.8) (see Fig. 2). For this it suffices to consider such an action on (3.2) and (3.5). Given \( n, r \geq 2 \) and \( 1 \leq i_1 \leq \ldots \leq i_r < n \), denote

\[
w_{i_1, \ldots, i_r} = (-1)^r h^{tr} x_{i_1} \cup_2 \ldots \cup_2 h^{tr} x_{i_r}, \quad \text{and}
\]

\[
w_{i_1, \ldots, i_r} = (-1)^r \zeta x_{i_1} \cup_2 h^{tr} x_{i_2} \cup_2 \ldots \cup_2 h^{tr} x_{i_r},
\]

where

\[
\zeta x_{i_1} = (-1)^{n+1} \frac{n+1}{i_1+1} x_{i_1} + h^{tr} b_{i_1}, \quad \text{with} \quad \xi x_{i_k} \in V^{-1,*}, \quad d \xi x_{i_k} = (-1)^k (k + 1) h^{tr} x_k;
\]

thus, \( w_{i_1, \ldots, i_r} = 0 \) if some \( i_j = 1 \). Note also that just the qha structure on \( A \) guarantees a choice of \( h \) on \( RH \) so that \( h^{tr} w_{i_1, \ldots, i_r} = 0 = h^{tr} w_{i_1, \ldots, i_r} \) (compare Proposition 4) for example, for \( A \) to be the singular chains of a second loop space the elements \( h^{tr} w_{i_1, \ldots, i_r} \) may be non-trivial and are related with the Browder operation (compare [74]); details are left to the interested reader. Then we can choose \( h \) to be defined for \( x_n \) and \( b_n \) by

\[
h x_n = h^{tr} x_n + \sum_{i+q \equiv n \atop 1 \leq q \leq n} (-1)^q E_{1,q} (h^{tr} x_i; x_{j_1}, \ldots, x_{j_q})
\]

\[
\quad + \sum_{i(r) + q(r) \equiv n+1 \atop n \geq 1} (-1)^q E_{1,q}(w_{i_1, \ldots, i_r}; x_{j_1}, \ldots, x_{j_q}) - \sum_{i(r) \equiv n+1} w_{i_1, \ldots, i_r}
\]

and

\[
h b_n = \zeta x_n + \sum_{i+q \equiv n \atop 1 \leq q \leq n} (-1)^q E_{1,q}(\zeta x_{i_1}; x_{j_1}, \ldots, x_{j_q}) - (-1)^n x_0 \cup_2 h x_{n-1}
\]

\[
\quad + \sum_{i(r) + q(r) \equiv n+1 \atop n \geq 1} (-1)^q E_{1,q} (w_{i_1, \ldots, i_r}; x_{j_1}, \ldots, x_{j_q}) - \sum_{i(r) \equiv n+1} w_{i_1, \ldots, i_r}
\]

\[
\quad + \sum_{i+j = n-1} (-1)^i (h b_j \sim_1 x_i - b_j \sim_1 h x_i).
\]

In particular, \( h b_2 = h^2 b_2 + h^3 b_2 = \xi x_2 + h^3 b_2 + h^2 b_1 \sim_1 x_0 \). Furthermore, consider

\[
h^p c_p = (p-2)! \left((p-1)(\xi x_p-1 + h^{p-1} b_{p-2} \sim_1 x_0) + h^{p-1} b_{p-2} \sim_1 x_0, x_0\right).
\]

Then \((h h)(b_n) |_{\mathfrak{gl}^{tr}} = 0 \) and \( d_h^2 = 0 \) imply the following equality in \( R^0 H^* \):

\[
d h^p c_p = p h^{tr} x_{p-1} - p(p-2)(x_0 h^{tr} b_{p-2} + (h^{tr} b_{p-2})x_0).
\]

From (3.10) follows that \((k+1)[h^{tr} x_k] = 0 \mod p \) for \( 2 \leq k \leq p-2 \). Consequently, \([h^{tr} x_{p-1}] = (a)^p \mod p \). On the other hand, from (3.8) and (3.10) we get \([\xi x_{p-1}] = \)
\( P_1(a) \mod p. \) Finally, since \((p-1)! = -1 \mod p\) and \([x_0(h^{tr}b_{p-2}) + (h^{tr}b_{p-2})x_0] = 0 \in H(A)\), we obtain from \((3.11)\) formula \((3.17)\) this time in \(H(A \otimes \mathbb{Z}_p)\).

Note that the case of \(a \in H(A \otimes \mathbb{Z}_p)\) in \((3.17)\) corresponding to the "Kraines formula" \(a^2 = (a)^2 = \beta S_{q_1}(a)\) is analogous (see also the following example).

When \(p = 2\) the relation \(d(x_0 \sim_1 x_0) = -2x_0^2\) implies the Adem relation \(S_q(0)(a) = Sq^1Sq_1(a)\) in \(H(A \otimes \mathbb{Z}_2)\) thought of as the "Kraines formula" \(a^2 = (a)^2 = \beta S_{q_1}(a)\).

Denote \( \tilde{V}_k = s^{-1}(V_k^{-1})) \oplus k \) and define the differential \(d_{\tilde{h}}\) on \( \tilde{V}_k \) by the exception of \(d + h\) on \(V_k\) to obtain the cochain complex \( (\tilde{V}_k, d_{\tilde{h}}) \). There are isomorphisms \([5]\) (recall also Example \([4]\) )

\[ H^*(\tilde{V}_k, d_{\tilde{h}}) \approx H^*(BC^*(X; k), d_{BC}) \approx H^*(\Omega X; k). \]

**Example 5.** Let we are given a single relation

\[ (3.12) \]

\[ da = \lambda b, \quad a \in V^{-1,2k+1}, b \in V^{0,2k+1}, \lambda \geq 2, k \geq 1. \]

Then we deduce from it the following relations in \((RH, d)\). First,

\[ (3.13) \]

\[ dc = \begin{cases} 
ab + \frac{\lambda}{2}b \sim_1 b, & \lambda \text{ is even} \\
2ab + \lambda b \sim_1 b, & \lambda \text{ is odd}
\end{cases} \]

Let \( \lambda \) be odd, and denote \( u_{2b,b} = 2ab + (\lambda - 1)b \sim_1 b \) \((= 2b_1; cf. \((3.2)\))\), \( u_{2a,b} = -c, \) \( u_{2a,2a} = c - 2a \sim_1 b \) to obtain

\[ (3.14) \]

\[ du_{a,a} = -a^2 + \lambda (c - a \sim_1 b), \]
\[ du_{a,2b,b} = -au_{2b,b} - u_{2a,2b} + \lambda v_{a,2b,b} \]
\[ = -2a^2b - (\lambda - 1)a(b \sim_1 b) + cb + \lambda u_{b,2b,b}, \]
\[ du_{b,2a,b} = bu_{2a,b} - u_{b,2a}b + \lambda v_{b,2a,b} = bc - (c - 2a \sim_1 b)b + \lambda u_{b,2b,b}, \]
\[ du_{a,2a,b} = -au_{2a,b} + u_{2a,2b} + \lambda (u_{2a,b} - u_{a,2b,h}), \]

where \(-v_{b,2a,b} = v_{a,2b,h} = u_{b,2b,b} = 2u_{b,b,b} = 2b_2\) (recall that \(u_{b,b,b} = b_2\)). Then in the filtered model \((RH, d_{\tilde{h}})\) we establish the following action of the perturbation \(h\) on the above relations:

\[ dh^2u_{a,a} = \lambda h^2c, \]
\[ dh^2u_{a,2b,b} = h^2c \cdot b + \lambda h^2u_{b,2b,b}, \]
\[ dh^2u_{b,2a,b} = -b \cdot h^2c - h^2c \cdot b + \lambda h^2u_{b,2b,b}, \]
\[ dh^2u_{a,2b,b} = a \cdot h^2c + 2h^2u_{a,a} \cdot b + \lambda h^2(ub_{2a,b} - u_{a,2b,b}), \]
\[ dh^3u_{a,2a,b} = h^3u_{a,a} \cdot b + \lambda h^3(u_{2a,b} - u_{a,2b,h}) + h^2h^2u_{a,2a,b}, \]

and, in particular,

\[ d \left( h^2u_{2a,2a,b} - b \sim_1 h^2c \right) = -\lambda h^2u_{b,2b,b} = -2\lambda h^2b_2. \]

Now let \( \lambda = 3 \), and compare the last equality with \((3.11)\) for \(p = 3\) to deduce that

\[ (3.15) \]

\[ h^2u_{2a,2a,b} - b \sim_1 h^2c = 2\xi_{b_2}. \]

Furthermore, from \((3.15)\) follows that \([h^2c] = 2[a][b]\), and if one has in \(H^*(A \otimes \mathbb{Z}_3)\)

\[ (3.16) \]

\[ [a][b] = 0 \quad \text{and} \quad [h^3u_{b,2a,b}] = 0, \]

then from \((3.15)\) together with the third equality of \((3.14)\) immediately follows that \([\xi_{b_2}] = 0 \in H(V \otimes \mathbb{Z}_3, d_{\tilde{h}})\), i.e., \( P_1(b) = 0 \in H(V \otimes \mathbb{Z}_3, d_{\tilde{h}})\).
For example, for $A = C^*(BF_4; \mathbb{Z}_3)$, the cochain complex of the classifying space $BF_4$ of the exceptional group $F_4$, equalities (3.16) hold in $H(BF_4; \mathbb{Z}_3)$. More precisely, in the notations of [25] some multiplicative generators $x_i$ of $H^i(BF_4; \mathbb{Z}_3)$ and some relations among them are: $x_4, x_9, x_20, x_{21}, x_{25}, x_{26}$ with $x_8x_9 = 0 = x_4x_{21}, \delta x_8 = x_9, \delta x_{25} = x_{26}$. The knowledge both of $H(BF_4; \mathbb{Z}_3)$ and $H(F_4; \mathbb{Z}_3)$ together with a simple dimension argument enables us to deduce in the filtered Hirsch model of $BF_4$ the following: In our notations $[a] = x_8, [b] = x_9, P_1[b] = [\xi_2] = x_{25}$ and $(b)^3 = [h^2b_2] = x_{26}$. And the relations $P^3(x_9) = x_{21}$ and $P^3(x_{21}) = x_{25}$ established in [25] too, together with $\delta x_{25} = x_{26}$ can be combined into the Kraines formula:

$$x_{26} = -\beta P_1(x_9).$$

Finally remark that this formula becomes to be trivial under the loop suspension map $\sigma : H^*(X; \mathbb{Z}_3) \to H^{*-1}(\Omega X; \mathbb{Z}_3)$, since a general fact about Massey products [18], [19] (compare $P_1(i_3)$ for $i_3 \in H^3(K(\mathbb{Z}_3; 3); \mathbb{Z}_3)$.

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