Leveraging concepts from state machine refinement proofs, we use prophecy variables, which predict information about the future program execution, to enable forward reasoning for backward dataflow analyses. Drawing prophecy and history variables (concepts from the dynamic execution of the program) from the same lattice as the static program analysis results, we require the analysis results to satisfy both the dataflow equations and the transition relations in the operational semantics of underlying programming language. This approach eliminates explicit abstraction and concretization functions and promotes a more direct connection between the analysis and program executions, with the connection taking the form of a bisimulation relation between concrete executions and an augmented operational semantics over the analysis results. We present several classical dataflow analyses with this approach (live variables, very busy expressions, defined variables, and reaching definitions) along with proofs that highlight how this approach can enable more streamlined reasoning. To the best of our knowledge, we are the first to use prophecy variables for dataflow analysis.

1 INTRODUCTION

Dataflow analysis is a classic field. Originally developed to enable compiler optimizations [3, 6, 11, 27–29, 37], over the last several decades it has evolved to solve problems in a wide range of fields including, for example, program verification [8, 15, 21, 22, 36, 38], program understanding [10, 19, 20, 30, 41], and computer security [2, 17, 18, 40, 42].

Early in the history of the field the question of the relationship between the analysis results and program executions arose. One answer to this question developed as follows [3, 5, 12, 27, 29, 32]. First, define an operational semantics in which the program executes commands $c$ that read and write program states $\sigma$ to produce a sequence of states $\sigma_0, \ldots, \sigma_i, \ldots$, with each state storing the values of variables at a corresponding program point $l_i$ (i.e., the program location $l_i$ before the command $c_i$ executes). Second, define a lattice $\langle S, \leq \rangle$ of dataflow facts $s \in S$ along with an abstraction function $\alpha (s = \alpha (\sigma))$ that maps each program state $\sigma$ to a corresponding dataflow fact $s$ and a concretization function $\gamma$ (where $\sigma \in \gamma (s)$) that maps each dataflow fact $s$ to the set of program states $\sigma$ that it abstracts. Together $\alpha$ and $\gamma$ form a Galois connection [12].

A sound dataflow analysis guarantees the property that for all program states $\sigma$, $\alpha (\sigma) \leq s$, where $s$ is the result that the analysis produces at the corresponding program point for $\sigma$ (this property essentially requires the analysis to take all execution paths into account). A natural way to prove an analysis sound is by forward reasoning, operating by induction on the length of the program execution, with the induction step proved via a case analysis on the last command to execute [32].

There are several anomalies with this approach. First, many classic program analyses (for example, live variables and very busy expressions [3]) are backward analyses that maintain information not about the past execution but about the future execution. Forward reasoning is often ineffective for reasoning about these analyses or proving their soundness. Standard presentations of dataflow analysis therefore typically focus on forward analyses, with backward analyses introduced later as a kind of dual of forward analyses [3, 4].

Second, many classic dataflow analyses (such as, for example, reaching definitions or available expressions [3]) maintain information about the past execution of the program that is not present in the program states $\sigma$ that the standard operational semantics maintains. For example, the standard operational semantics for simple imperative languages maintains only the current values of
variables. These semantics leave no record of which command produced the current value. Reaching definitions extracts information about which commands produce values read by subsequent variable uses. This fact makes it impossible to construct an abstraction function $\alpha(\sigma)$ that operates on the standard program state $\sigma$, which records only variable values — the reaching definition information is not available in $\sigma$. A solution to this problem is to develop an instrumented semantics that maintains this past information explicitly in $\sigma$ to enable the construction of the abstraction function [13, 32].

1.1 History and Prophecy Variables

The relationship between a concrete and abstract perspective on a computation has also been explored in the context of using refinement mappings to prove forward simulation relations for verifying the correctness of a (concrete) implementation with respect to an (abstract) specification [1]. In this context the specification and implementation are both modeled as state machines, with simulation proofs (proving that each implementation action correctly simulates some specification action) verifying that the implementation correctly implements the specification.

Stating the appropriate correctness conditions that the specification and implementation must preserve often involves reasoning about the past execution of the specification and/or implementation. To enable this reasoning, the formal framework uses history variables, i.e., additional state components that do not affect the externally visible behavior of the state machine but rather simply record information about the past execution that can then be used to state and prove relevant correctness conditions. History variables were initially developed in the context of program verification [39] and have since been widely used under a variety of names (e.g., auxiliary variables, ghost variables) in a range of communities including the programming languages and program verification communities [16, 24, 39, 49].

In some situations, the specification and implementation make (typically nondeterministic) choices at different points in their execution, with, for example, a natural specification making the choice before the implementation. In these situations it is often not possible to prove that the implementation correctly implements the specification using the standard history variable and forward simulation proof mechanisms [1]. One solution to this problem is to introduce prophecy variables, which make predictions about the future executions of state machines (typically the specification) to enable the correctness properties to be stated and proven [1, 34].

1.2 History and Prophecy Variables for Dataflow Analysis

Inspired by the use of prophecy and history variables for proving simulation relations as well as the unsatisfying treatment of backward and forward analyses in the standard dataflow analysis framework, we use prophecy and history variables to formalize a new treatment of both backward and forward dataflow analyses.

Backward analyses augment the standard operational semantics of the underlying programming language with prophecy variables that (typically nondeterministically) predict any information about the future execution of the program required to establish the correspondence between the analysis and the execution. Because some of these predictions may be incorrect, the analysis also augments the semantics with prophecy variable preconditions that check prediction correctness to filter out any executions with incorrect predictions. Forward analyses augment the standard operational semantics of the underlying programming language with history variables that record any information about the past execution required to establish the correspondence between the analysis and the execution.
With this formulation, the standard semantics operates on states \( \langle l, \sigma \rangle \) and the augmented semantics operates on augmented states \( \langle l, \sigma, \pi \rangle \), where \( l \) is the label of the next command to execute. \( \sigma \) records the standard state of the program (for example, the values of the variables that the program manipulates), and \( \pi \) is the prophecy or history variable from the analysis. The standard operational semantics involves a transition relation \( \langle l, \sigma \rangle \rightarrow \langle l', \sigma' \rangle \); the augmented operational semantics involves a transition relation \( \langle l, \sigma, \pi \rangle \Rightarrow \langle l', \sigma', \pi' \rangle \). The introduction of the prophecy or history variable \( \pi \) produces, in effect, two executions of the program that run together in lock-step — the standard execution that operates on \( \sigma \) and another execution that runs on top of the standard execution, may read both \( \sigma \) and \( \pi \), but only updates \( \pi \).

The dataflow analysis produces, for every program point \( \bullet l \) (the program point before the command at label \( l \) executes) and \( l \bullet \) (the program point after the command at label \( l \) executes), analysis results \( \beta_\bullet l \) and \( \beta_l \). These analysis results are drawn from the same lattice as the prophecy or history variables \( \pi \), making it possible to substitute the analysis results directly into the augmented operational semantics to obtain transitions \( \langle l, \sigma, \beta_\bullet l \rangle \Rightarrow \langle l', \sigma', \beta_\bullet l' \rangle \), where \( l' \) is the label of the command that executes next after the command at \( l \).

This setup enables us to formulate the soundness criteria that the dataflow analysis must preserve via two properties that establish the correspondence between the dataflow analysis and the program execution:

- **Preservation**: \( \langle l, \sigma, \pi \rangle \Rightarrow \langle l', \sigma', \pi' \rangle \) implies \( \langle l, \sigma \rangle \rightarrow \langle l', \sigma' \rangle \). Preservation ensures that the augmented semantics does not produce any new executions.
- **Progress**: \( \langle l, \sigma \rangle \rightarrow \langle l', \sigma' \rangle \) implies \( \langle l, \sigma, \beta_\bullet l \rangle \Rightarrow \langle l', \sigma', \beta_\bullet l' \rangle \). Progress requires prophecy variables to correctly predict all possible future executions. In particular, proving \( \langle l, \sigma, \beta_\bullet l \rangle \Rightarrow \langle l', \sigma', \beta_\bullet l' \rangle \) requires the analysis to produce analysis results \( \beta_\bullet l \) that satisfy the prophecy variable preconditions that filter out incorrect prophecy variable predictions. For analyses that use history variables, Progress requires the history variables to correctly summarize all past executions.

To satisfy these properties, the analysis must produce analysis results \( \beta \) that satisfy both the dataflow equations and the transition relations in the augmented operational semantics. The analysis results \( \beta \) therefore tie the analysis and concrete executions together via the prophecy and history variables, with prophecy and history variable properties formalizing a direct connection between the analysis, the augmented semantics, and the standard semantics. This connection is reflected in the fact that, together, Preservation and Progress induce a bisimulation relation \([35, 47]\) between the standard semantics and the augmented semantics running on the analysis results \( \beta_\bullet l \). This bisimulation relation formalizes the guaranteed correspondence between the analysis and the standard execution of the program.

This connection can then be used to prove, via forward reasoning for both backward and forward analyses, additional analysis properties. These analysis properties may, for example, enable semantics-preserving program transformations such as dead variable elimination (Section 4.1), code hoisting (Section 4.2), or constant propagation (Section 5.2), or to check for program correctness properties such as the absence or presence of undefined variables (Section 5.1).

### 1.3 Contributions

This paper makes the following contributions:

- **Prophecy Variables for Backward Dataflow Analysis**: Prophecy variables were originally developed to prove forward simulation relations between state machines that take corresponding actions at different times. Leveraging the ability of prophecy variables to
predict information about the future execution, we use prophecy variables to develop a unified treatment of backward and forward dataflow analyses. In this treatment, backward analyses deliver accurate information about the future execution, with prophecy variables enabling the statement and proof of precise conditions that the analysis must satisfy to 1) accurately predict information about the future (as checked by the prophecy variable preconditions) while 2) remaining consistent with the prophecy variable predictions. To the best of our knowledge we are the first to use prophecy variables for dataflow analysis.

**Mechanisms:** Drawing the prophecy and history variables from the same lattice as the analysis results eliminates explicit abstraction and concretization functions from the treatment, including the elimination of abstraction and concretization functions from any proofs involving the analysis or the analysis results. The proofs instead work with analysis-specific properties over the prophecy and history variables as induced by the prophecy variable preconditions, prophecy variable predictions, and history variable updates. These properties directly relate concrete executions and the analysis via a bisimulation induced by the analysis results (Theorems 4.5, 4.9, 4.13, 4.17, 5.2, 5.3, 5.7, and 5.9).

Replacing traditional collecting and/or instrumented semantics with explicit prophecy or history variables leaves the standard operational semantics intact, separated from the prophecy and history variables in the augmented operational semantics. The result is a more direct connection between the analysis and the concrete execution and the elimination of the need to work through the instrumented and/or collecting semantics to state and prove properties of standard program executions.

**Dataflow Analyses and Proofs:** We present several classical dataflow analyses (live variables, Section 4.1; very busy expressions, Section 4.2; defined variables, Section 5.1; and reaching definitions, Section 5.2) with prophecy variables and history variables along with proofs that establish the relevant bisimulations and proofs of analysis correctness properties for semantics-preserving program transformations. These proofs highlight the features of our treatment, including the ability of prophecy variables to deliver a more unified treatment of backward and forward dataflow analyses to enable forward reasoning for both backward and forward analyses. They also highlight how the use of the same lattice for the prophecy variables, history variables, and analysis results enables more streamlined reasoning.

The remainder of the paper is structured as follows. Section 2 presents an overview of the basic concepts in our treatment, including the Preservation and Progress properties that together establish the bisimulation. Section 3 presents the core imperative language that we use to present the dataflow analyses. Section 4 presents two backward analyses, live variables and very busy expressions, including prophecy variable preconditions, prophecy variable predictions, proofs that establish the bisimulation between the analysis results and program executions, and proofs that establish relevant analysis correctness properties. Section 5 similarly presents two forward dataflow analyses, defined variables and reaching definitions. We discuss related work in Section 6 and conclude in Section 7.

## 2 OVERVIEW

We work with programs $P$ that contain labeled commands of the form $l : c \in P$, where $l, g \in L$, $c \in C$. \texttt{labels}(P) = \{l.l : c \in P\} is the set of labels in $P$. Labels are unique — no two labeled commands in $P$ have the same label $l$. An executing program operates on states $\sigma \in \Sigma$. The standard operational semantics is modeled by a program execution transition relation $\langle l, \sigma \rangle \rightarrow \langle l', \sigma' \rangle$. 
Execution starts at \( \langle l_0, \sigma_0 \rangle \), where \( l_0 = \text{first}(P) \) is the label of the first command to execute and \( \sigma_0 \) is the initial state. Execution terminates if it encounters an \( l : \text{halt} \) command.

The standard operational semantics is typically defined by a set of program execution rules. Each rule starts with a standard program configuration \( \langle l, \sigma \rangle \) to produce a next configuration \( \langle l', \sigma' \rangle \). Each rule has a set of preconditions that must be satisfied for the rule to execute. If the execution encounters an error, at least one of the relevant preconditions will not be satisfied and the execution will become stuck in a configuration \( \langle l, \sigma \rangle \) such that \( \langle l, \sigma \rangle \not\rightarrow \).

Each program analysis augments the state with a prophecy or history variable \( \pi \in \Pi \), where \( \langle \Pi, \leq \rangle \) is a lattice ordered by \( \leq \) with least upper bound \( \lor \) and greatest lower bound \( \land \). The analysis also defines an augmented operational semantics by updating the program execution rules to define an augmented program execution relation \( \langle l, \sigma, \pi \rangle \rightarrow \langle l', \sigma', \pi' \rangle \).

If \( \pi \) is a prophecy variable, the updated program execution rules use \( \pi \) to predict information about the future program execution, with incorrect predictions filtered out by new prophecy variable preconditions that check that the prediction was correct (with the execution becoming stuck if the prediction was not correct). If \( \pi \) is a history variable, the updated rules use \( \pi \) to record information about the past program execution.

We require the augmented program execution relation \( \langle l, \sigma, \pi \rangle \rightarrow \langle l', \sigma', \pi' \rangle \) not to introduce new executions. More precisely, we require the augmented program execution relation to satisfy the following preservation property:

**Definition 2.1.** (Preservation): If \( \langle l, \sigma, \pi \rangle \rightarrow \langle l', \sigma', \pi' \rangle \), then \( \langle l, \sigma \rangle \rightarrow \langle l', \sigma' \rangle \).

Verifying the preservation property is typically straightforward as the updated program execution rules in the augmented operational semantics typically have the same preconditions over \( l \) and \( \sigma \) and generate the same \( l' \) and \( \sigma' \) as the corresponding program execution rules in the standard operational semantics.

For each labeled command \( l : c \in P \), there are two program points: \( \bullet l \) (the program point before \( l : c \in P \) executes) and \( l \bullet \) (the program point after \( l : c \in P \) executes). The program analysis produces an analysis result \( \beta \in \Pi \) (drawn from same lattice \( \langle \Pi, \leq \rangle \) as the prophecy and history variables \( \pi \)) at each program point. Given a labeled command \( l : c \in P \), \( \beta \bullet l \) is the program analysis result at the program point before \( l : c \in P \) executes; \( \beta l \bullet \) is the program analysis result at the program point after \( l : c \in P \) executes.

Conceptually, the analysis is consistent with the augmented operational semantics if it produces an analysis result that enables a corresponding transition in the augmented operational semantics for each transition in the standard operational semantics. We formalize this requirement with the following Progress property:

**Definition 2.2.** (Progress): If \( \langle l, \sigma \rangle \rightarrow \langle l', \sigma' \rangle \), then \( \langle l, \sigma, \beta \bullet l \rangle \rightarrow \langle l', \sigma', \beta l \bullet \rangle \).

For analyses with prophecy variables, the progress property requires the analysis to produce correct predictions about all possible future executions (in the sense that the analysis results satisfy the prophecy variable preconditions that check for incorrect predictions). For analyses with history variables, the progress property requires the analysis to produce results that take all possible past executions into account. It is the responsibility of the analysis developer to ensure that the preservation and progress properties hold, typically by proving corresponding preservation and progress theorems for the analysis.

The Progress property is typically verified by local reasoning, usually by a case analysis on the command that generated the \( \langle l, \sigma \rangle \rightarrow \langle l', \sigma' \rangle \) transition. For backward program analyses the Progress property can flip the direction of causality to enable forward reasoning — reasoning from a chosen point in the computation forward along the potential program execution paths to verify
a relationship between the analysis and the execution of the program. Examples of this forward reasoning for backwards analyses that use prophecy variables include Theorems 4.10, 4.18, 4.19. Because forward reasoning is typically straightforward for forward analyses, the Progress property can effectively unify reasoning approaches for forward and backward analyses.

We note that if Preservation (Definition 2.1) and Progress (Definition 2.2) both hold, then the relation \( \sim \) defined by \( \langle l, \sigma \rangle \sim \langle l', \sigma', \beta \rangle \) is a bisimulation relation \([35, 47]\) between standard and augmented program configurations.

For analyses with prophecy variables \( \pi \), the following downward closure metarule is often helpful in ensuring the progress property holds:

\[
\frac{\langle l, \sigma, \pi \rangle \Rightarrow \langle l', \sigma', \pi' \rangle \quad \pi'' \leq \pi'}{\langle l, \sigma, \pi \rangle \Rightarrow \langle l', \sigma', \pi'' \rangle}
\]

Conceptually, moving down the lattice from \( \pi' \) to \( \pi'' \) takes fewer program executions into account, which happens 1) when the prophecy variable makes a prediction about a future execution and 2) at program control flow split points for backward program analyses (which typically use prophecy variables).

For analyses with history variables, the following upward closure metarule is often helpful in ensuring the progress property holds:

\[
\frac{\langle l, \sigma, \pi \rangle \Rightarrow \langle l', \sigma', \pi' \rangle \quad \pi' \leq \pi''}{\langle l, \sigma, \pi \rangle \Rightarrow \langle l', \sigma', \pi'' \rangle}
\]

Conceptually, moving up the lattice from \( \pi' \) to \( \pi'' \) takes more program executions into account, which happens at program control flow join points for forward program analyses (which typically use history variables).

3 CORE PROGRAMMING LANGUAGE

We present program analyses for a core programming language inspired by Glynn Winskell’s imperative language IMP \([48]\). Notable differences include the introduction of labels for all commands and the use of variables \( V \) instead of locations \((L)\).

\( n, m \in \mathbb{N} \) is the set of integers. \( t \in T = \{\text{true}, \text{false}\} \) is the set of truth values. Programs work with variables \( v, w \in V \), arithmetic expressions \( e \in E \), and boolean expressions \( b \in B \) defined as follows:

\[
\begin{align*}
E & ::= n|v|E_0 + E_1|E_0 - E_1|E_0 \times E_1 \\
B & ::= \text{true}|\text{false}|E_0 = E_1|E_0 \leq E_1|\text{not } B|B_0 \text{ and } B_1|B_0 \text{ or } B_1 \\
C & ::= \text{skip}|v := E|\text{if } B \text{ then } g|g \text{ goto } g|\text{halt}|\text{done}
\end{align*}
\]

\( \text{vars}(e) \) is the set of variables \( v \) that \( e \) reads, \( \text{vars}(b) \) is the set of variables \( v \) that \( b \) reads, and \( \text{vars}(c) \) is the set of variables that \( c \) reads.

Each program \( P \) is a sequence of labeled commands of the form \( l : c \), where \( l, g \in L, c \in C \). Given a program \( P \) and label \( l \in \text{labels}(P) \), \( l' = \text{next}(l) \) is the label \( l' \) of the next command (in the sequential execution order) in \( P \) after \( l : c \in P \). Conceptually, when the program executes a \( l : \text{halt} \in P \) command, the program stops executing in the \( \text{done} \) state. We therefore require if \( l : \text{halt} \in P \) and \( l' = \text{next}(l) \), then \( l' : \text{done} \in P \). We also require \( \text{next}(l) = g \) if \( l : c = \text{if } l : \text{goto } g \in P \) (but typically reference the branch target \( g \) explicitly instead of \( \text{next}(l) \)). For each labeled command \( l : c \in P \), there are two program points: \( \bullet l \) (the program point before \( l : c \in P \)
A program in execution maintains a state \( \sigma \) (under subset inclusion) over \( B \). State \( \sigma \) satisfies the arithmetic expression evaluation rules in Figure 1. Given a boolean expression \( e \), we define the boolean expression evaluation relation \( \langle e, \sigma \rangle \rightarrow t \) as the smallest relation (under subset inclusion) over \( B \times \Sigma \times T \) that satisfies the boolean expression evaluation rules in Figure 2.

Given a program \( P \), the standard operational semantics works with configurations of the form \( \langle l, \sigma \rangle \), where \( l \) is the label of a labeled command \( l : c \in P \) and \( \sigma : V \rightarrow N \) is an environment that maps variables \( v \in V \) to values \( n \in N \). We define the program execution relation \( \langle l, \sigma \rangle \rightarrow \langle l', \sigma' \rangle \) as the smallest relation (under subset inclusion) over \( L \times \Sigma \times L \times \Sigma \) that satisfies the program execution rules in Figure 3.
program. In this case augmented executions of \( P \) semantics to appropriately update and/or check the prophecy or history variable \( \pi \). Each analysis then updates one or more of the rules from the baseline augmented operational semantics to simply thread \( \pi \). Each program analysis typically updates only a few program execution rules, with the remaining rules never becoming stuck because of the augmentation.

3.2 Baseline Augmented Operational Semantics

Each program analysis typically updates only a few program execution rules, with the remaining rules simply threading the prophecy or history variable \( \pi \) through the execution unchanged. We therefore define a baseline augmented operational semantics by updating all of the rules from the standard operational semantics (Figures 1, 2, and 3) to simply thread \( \pi \) through the execution unchanged (by changing \( \sigma \) to \( \sigma, \pi \) in each rule) (only for completeness, presented in Figures 4, 5, and 6). Each analysis then updates one or more of the rules from the baseline augmented operational semantics to appropriately update and/or check the prophecy or history variable \( \pi \) as appropriate for that analysis.

Analyses that use history variables \( \pi \) typically record actions taken during the execution of the program. In this case augmented executions of \( P \) never become stuck because of the augmentation.
Analyses that use prophecy variables \( \pi \), on the other hand, typically make nondeterministic predictions that are validated later in the execution. Executions involving invalid predictions become stuck at the preconditions that validate the predictions.

4 BACKWARD ANALYSES AND PROPHECY VARIABLES

We next present several backward analyses, including the use of prophecy variables to formulate and prove properties characterizing the relationship between the analysis and program execution.

4.1 Live Variables

Live variables analysis (conservatively) determines, for each program point, the variables that are live at that program point, i.e., variables that may be read before they are written in the future program execution. The analysis uses a backward dataflow analysis to reason about the future execution of the program. It therefore augments the standard operational semantics with a prophecy variable \( \pi \subseteq V \). \( \pi \) predicts which variables are live; i.e., which variables will be read by some future command before the variable is reassigned. The prophecy variable \( \pi \) is drawn from the live variables program analysis lattice \( \langle \Pi, \subseteq \rangle \), where \( \Pi = \mathcal{P}(E) \) is ordered under subset inclusion (\( \subseteq \)), with least upper bound \( \cup \) and greatest lower bound \( \cap \).

4.1.1 Augmented Operational Semantics. Starting with the baseline augmented operational semantics (Section 3.2), which passes the prophecy variable \( \pi \) unchanged through all commands, the analysis updates the program execution rules that read variables (i.e., the rules for commands \( l : v := e \in P \) and \( l : \text{if } b \text{ then } g \in P \)) to include new prophecy variable predictions that predict which variables will be live after the variable reads. Conceptually, the rule for \( l : v := e \in P \) adds \( v \) to the set of predicted live variables, then predicts that some subset of \( \pi \cup \{v\} \) will no longer be live after \( l : v := e \in P \). Some variables may become dead either because \( e \) contained the last read to a variable before the variable is reassigned or because \( v \) itself is not read before it is reassigned. Similarly, the rule for \( l : \text{if } b \text{ then } g \in P \) predicts that some subset of the predicted live variables \( \pi \) before \( l : \text{if } b \text{ then } g \in P \) will no longer be live after \( l \), for example because \( b \) contained the last access to a variable before the variable is reassigned. We do not apply the downward closure metarule (Definition 2.3).

\[
\begin{align*}
\text{if } b \text{ then } g \in P & \quad \langle a, \sigma, \pi \rangle \Rightarrow n & \quad \pi' \subseteq \pi \cup \{v\} \\
\langle l, \sigma, \pi \rangle & \Rightarrow \langle \text{next}(l), \sigma[v \mapsto n], \pi' \rangle \\
\text{if } b \text{ then } g \in P & \quad \langle b, \sigma, \pi \rangle \Rightarrow \text{false} & \quad \pi' \subseteq \pi \\
\langle l, \sigma, \pi \rangle & \Rightarrow \langle \text{next}(l), \sigma, \pi' \rangle \\
\text{if } b \text{ then } g \in P & \quad \langle b, \sigma, \pi \rangle \Rightarrow \text{true} & \quad \pi' \subseteq \pi \\
\langle l, \sigma, \pi \rangle & \Rightarrow \langle g, \sigma, \pi' \rangle
\end{align*}
\]

Of course, it is possible for the program execution rules to mispredict which variables will be dead after executing \( l : v := e \in P \) or \( l : \text{if } b \text{ then } g \in P \). The augmented operational semantics therefore updates the variable read rule with the prophecy variable precondition \( v \in \pi \), which requires that every variable \( v \) read during expression evaluation must be predicted live. With this precondition, all executions that mispredict a live variable become stuck at the command that attempts to read the mispredicted variable. Here \( \text{dom} \ \sigma \) is the domain of \( \sigma \) viewed as a function — the set of variables \( v \) for which \( \sigma(v) \) is defined.
\[ v \in \text{dom} \sigma \quad \sigma \in \pi \quad \langle v, \sigma, \pi \rangle \Rightarrow \sigma(v) \]

We next state some lemmas and the Preservation theorem. These lemmas and theorems essentially leverage/formalize the fact that the introduction of the prophecy variable \( \pi \) into the augmented operational semantics (conceptually) preserves the standard operational semantics as long as the prophecy variable preconditions \( v \in \pi \) encountered during the evaluation of an expression are satisfied:

**Lemma 4.1.** If \( \langle e, \sigma, \pi \rangle \Rightarrow n \), then \( \langle e, \sigma \rangle \Rightarrow n \). If \( \langle b, \sigma, \pi \rangle \Rightarrow t \), then \( \langle b, \sigma \rangle \Rightarrow t \).

**Proof:** The updated expression evaluation rules in the augmented operational semantics have the same preconditions (with the exception of the prophecy variable preconditions \( v \in \pi \), which are not present in the standard operational semantics) and produce the same expression values as the corresponding rules from the standard operational semantics.

**Lemma 4.2.** If \( \langle e, \sigma \rangle \Rightarrow n \) and \( \text{vars}(e) \subseteq \pi \), then \( \langle e, \sigma, \pi \rangle \Rightarrow n \). If \( \langle b, \sigma \rangle \Rightarrow t \) and \( \text{vars}(b) \subseteq \pi \), then \( \langle b, \sigma, \pi \rangle \Rightarrow t \).

**Proof:** \( \text{vars}(e) \subseteq \pi \) and \( \text{vars}(b) \subseteq \pi \) ensure that any prophecy variable precondition \( v \in \pi \) from the local variable read rule is satisfied during the evaluation of \( e \) or \( b \), so \( \langle e, \sigma, \pi \rangle \Rightarrow n' \) for some \( n' \) and \( \langle b, \sigma, \pi \rangle \Rightarrow t' \) for some \( t' \). By Lemma 4.1, \( n' = n \) and \( t' = t \).

**Lemma 4.3.** If \( \langle l, \sigma, \pi \rangle \Rightarrow \langle l', \sigma', \pi' \rangle \) and \( l : c \in P \), then \( \text{vars}(c) \subseteq \pi \).

**Proof:** Consider any variable \( v \in \text{vars}(c) \). If \( v \) is not in the set of predicted live variables \( \pi \), the prophecy variable precondition \( v \in \pi \) of the local variable read rule will not be satisfied during the evaluation of an expression \( e \) or \( b \) in \( c \), the evaluation of \( e \) or \( b \) will become stuck, the execution of \( l : c \in P \) will become stuck, and \( \langle l, \sigma, \pi \rangle \not\Rightarrow \).

**Lemma 4.4.** If \( \langle l, \sigma, \pi \rangle \Rightarrow \langle l', \sigma', \pi' \rangle \), \( v \not\in \pi \), and \( v \in \pi' \), then \( l : v := e \in P \):

**Proof:** Case analysis on \( l : c \in P \):

- \( l : c = l : \text{if} \ b \text{ then} \ g \in P \): In this case \( \pi' \subseteq \pi \), so there is no \( v \) such that \( v \not\in \pi \) and \( v \in \pi' \).
- \( l : c = l : \text{goto} \ g \in P, l : c = l : \text{skip} \in P \) or \( l : c = l : \text{halt} \in P \): In this case \( \pi' = \pi \), so there is no \( v \) such that \( v \not\in \pi \) and \( v \in \pi' \).
- \( l : c = l : v := e \in P \): This is the only remaining case. In this case \( \pi' \subseteq \pi \cup \{v\} \) and it is possible for \( v \not\in \pi \) and \( v \in \pi' \).

**Theorem 4.5.** (Preservation) If \( \langle l, \sigma, \pi \rangle \Rightarrow \langle l', \sigma', \pi' \rangle \) then \( \langle l, \sigma \rangle \Rightarrow \langle l', \sigma' \rangle \).

**Proof:** All rules that generate a transition \( \langle l, \sigma, \pi \rangle \Rightarrow \langle l', \sigma', \pi' \rangle \) in the augmented operational semantics have the same preconditions over \( l : c \in P \) and \( \sigma \) and produce the same values for \( l' \) and \( \sigma' \) as the corresponding rules from the standard operational semantics.

4.1.2 Live Variables Analysis. The live variables analysis is a backward dataflow analysis that propagates variable liveness information backward against the flow of control. For each command \( l : c \in P \), the analysis produces \( \beta_{c} \subseteq V \) (the set of variables live at the program point before \( l : c \in P \)) and \( \beta_{c} \subseteq V \) (the set of variables live after \( l : c \in P \)). The analysis obtains the \( \beta_{l} \) and \( \beta_{c} \) by formulating and solving, using standard least fixed-point techniques, the following set of
backward dataflow equations:

\[
\begin{align*}
\beta_{l \cdot} &= \emptyset \text{ if } l : \text{halt} \in P \\
\beta_{l \cdot} &= \cup \beta_{g}, \text{ where } l : c \in P \text{ and } g \in \text{succ}(l) \\
\beta_{s l} &= f(l, \beta_{l \cdot})
\end{align*}
\]

where \( f \) is the transfer function for the analysis defined as follows:

\[
f(l, \beta) = \begin{cases} 
(\beta - \{v\}) \cup \text{vars}(e) & \text{if } l : v := e \in P \\
\beta \cup \text{vars}(b) & \text{if } l : b \text{ then } l' \in P \\
\beta & \text{otherwise}
\end{cases}
\]

Note that these dataflow equations ensure that every variable \( v \) read by a command \( c \) is live when the command executes:

**Lemma 4.6.** vars(\( c \)) \( \subseteq \beta_{s l} \) where \( l : c \in P \).

**Proof:** Case analysis on \( l : c \in P \):

- \( l : c = l : v := e \in P \):
  
  Then \( \beta_{s l} = (\beta_{l \cdot} - \{v\}) \cup \text{vars}(e) \) and vars(\( c \)) = vars(\( e \)) \( \subseteq \beta_{s l} \).

- \( l : c = l : \text{if } b \text{ then } g \in P \):
  
  Then \( \beta_{s l} = (\beta_{l \cdot} - \{v\}) \cup \text{vars}(b) \) and vars(\( c \)) = vars(\( b \)) \( \subseteq \beta_{s l} \).

- \( l : c = l : \text{goto } g \in P, l : c = l : \text{skip} \in P, \) or \( l : c = l : \text{halt} \in P \):
  
  Then vars(\( c \)) = \emptyset \( \subseteq \beta_{s l} \).

### 4.1.3 Prophecy Variable Predictions and Dataflow Analyses with Sets of Elements and Subset Inclusion

Live variables is an instance of a more general class of backward dataflow analyses in which the dataflow facts \( \beta \in \Pi \) are sets of elements with the dataflow lattice (\( \Pi, \subseteq \)) ordered by subset inclusion (\( \subseteq \)), with least upper bound \( \cup \), greatest lower bound \( \cap \), and dataflow equations of the following form:

\[
\begin{align*}
\beta_{l \cdot} &= \cup \beta_{g}, \text{ where } l : c \in P \text{ and } g \in \text{succ}(l) \\
\beta_{s l} &= (\beta_{l \cdot} - D_l) \cup U_l
\end{align*}
\]

where \( D_l \in \Pi \) is the definition set for \( l : c \in P \) and \( U_l \in \Pi \) is the use set for \( l : c \in P \). Note that the transfer function for \( l : c \in P \) is \( f(l, \beta) = (\beta \text{ or } D_l) \cup U_l \) and the equations ensure \( \beta_{s l'} \subseteq \beta_{l \cdot} \) for \( l' \in \text{succ}(l) \). For live variables \( D_l = \{v\} \) and \( U_l = \text{vars}(e) \) when \( l : v := e \in P; D_l = \emptyset \) and \( U_l = \text{vars}(b) \) when \( l : b \text{ then } l' \in P \). For \( l : c = l : \text{goto } g \in P, l : c = l : \text{skip} \in P, \) and \( l : c = l : \text{halt} \in P, D_l = \emptyset \) and \( U_l = \emptyset \).

One of the proof obligations required to show \( \langle l, \sigma, \beta_{s l} \rangle \Rightarrow \langle l', \sigma', \beta_{s l'} \rangle \) is establishing that the analysis results \( \beta_{s l} \) and \( \beta_{s l'} \) are consistent with the prophecy variable predictions. We next show that prophecy variable predictions \( \pi' \subseteq \pi \cup D_l \) are consistent with these analyses:

**Lemma 4.7.** If \( \beta_{s l} = (\beta_{l \cdot} - D_l) \cup U_l \) and \( \beta_{s l'} \subseteq \beta_{l \cdot} \), then \( \beta_{s l'} \subseteq \beta_{s l} \cup D_l \).

**Proof:**

- Known facts from dataflow analysis: \( \beta_{s l} = (\beta_{l \cdot} - D_l) \cup U_l \) and \( \beta_{s l'} \subseteq \beta_{l \cdot} \).

- Then \( \beta_{s l} \supseteq (\beta_{s l'} - D_l) \cup U_l, \beta_{s l} \supseteq (\beta_{s l'} - D_l), \beta_{s l} \cup D_l \supseteq (\beta_{s l'} - D_l) \cup D_l, \beta_{s l} \cup D_l \supseteq \beta_{s l'} \cup D_l, \) and \( \beta_{s l} \cup D_l \supseteq \beta_{s l'} \).
If $D_l = U_l = \emptyset$ so that $f(l, \beta) = \beta$ and $\{l'\} = \text{succ}(l)$, then $\beta_{l*} = \beta_{l*} = \beta_{l*'}$. For live variables this is the case for $l : c = l : \text{goto} \; g \in P$, $l : c = l : \text{skip} \in P$, and $l : c = l : \text{halt} \in P$. In this case the analysis results are consistent with prophecy variable predictions $\pi' = \pi$ and $\pi' \subseteq \pi$ (as in, for example, analyses that use the downward closure metarule):

**Lemma 4.8.** If $\beta_{l*} = \beta_{l*}$ and $\beta_{l*} = \beta_{l*'}$, then $\beta_{l*} = \beta_{l*'}$ and $\beta_{l*'} \subseteq \beta_{l*}$.

### 4.1.4 Live Variables Progress Theorem

We next state and prove the Progress theorem for the live variables analysis. First, the proof must ensure that the analysis results $\beta_{l*}$ satisfy the prophecy variable precondition $v \in \pi$ in the augmented expression evaluation rules for all variables $v$ in evaluated expressions $e$ and $b$. This property shows up as proof obligations of the form $\text{vars}(e) \subseteq \beta_{l*}$ and $\text{vars}(b) \subseteq \beta_{l*}$ for expressions $e$ and $b$ that appear in commands $l : c \in P$. These proof obligations are immediately discharged because the transfer function $f$ explicitly places $\text{vars}(e)$ and $\text{vars}(b)$ in $\beta_{l*}$ for commands $l : c \in P$ that contain $e$ or $b$ — in other words, the prophecy variable precondition proof obligations are immediately discharged regardless of the values of related program analysis results $\beta_{l*}$ and $\beta_{l*'}$ where $l' \in \text{succ}(l)$.

The proof must also ensure that the analysis results are consistent with the prophecy variable predictions in the augmented operational semantics. For commands $l : c = l : v := e \in P$ this property shows up as proof obligations $\beta_{l*'} \subseteq \beta_{l*} \cup \{v\}$. For $l : c = l : \text{if} \; b \; \text{then} \; g \in P$ this property shows up as proof obligations $\beta_{l*'} \subseteq \beta_{l*}$. For $l : c = l : \text{goto} \; g \in P$, $l : c = l : \text{skip} \in P$, and $l : c = l : \text{halt} \in P$, this property shows up as proof obligations $\beta_{l*'} = \beta_{l*}$. Unlike the prophecy variable preconditional proof obligations, these prophecy variable prediction proof obligations do depend on the relationship between $\beta_{l*}$, $\beta_{l*}$, and $\beta_{l*'}$ where $l' \in \text{succ}(l)$. They can therefore be discharged by pushing the analysis result $\beta_{l*'}$ through the transfer function $f$ for $l : c \in P$ to check that the analysis related analysis results $\beta_{l*}$, $\beta_{l*}$, and $\beta_{l*'}$ are consistent with the prophecy variable predictions.

For live variables, the analysis and prophecy variable prediction conform to the requirements of Lemmas 4.7 and 4.8. So the prophecy variable prediction proof obligations for these commands are immediately discharged by applying these lemmas.

At a higher level, these properties ensure that the analysis never spontaneously takes a variable that is not live and makes it live. The augmented operational semantics uses the prophecy variable $\pi$ to enforce this property, which must be preserved by the static analysis for the analysis to produce an analysis result consistent with the prophecy variable predictions and in which the prophecy variable preconditions hold.

**Theorem 4.9.** (Progress) If $\langle l, \sigma \rangle \rightarrow \langle l', \sigma' \rangle$ then $\langle l, \sigma, \beta_{l*} \rangle \Rightarrow \langle l', \sigma', \beta_{l*'} \rangle$.

**Proof:** If all of the prophecy variable preconditions are satisfied, the standard and augmented program execution rules for $l : c \in P$ define the same values for $l'$, $\sigma'$, and all evaluated expressions $e$ or $b$. The following case analysis on $l : c \in P$ shows that the prophecy variable preconditions (which require all variables $v$ read in evaluated expressions $e$ and $b$ to be predicted live) are satisfied and that $\beta_{l*}$ and $\beta_{l*'}$ satisfy $\langle l, \sigma, \beta_{l*} \rangle \Rightarrow \langle l', \sigma', \beta_{l*'} \rangle$:

- $l : c = l : v := e \in P$:
  - Facts from dataflow equations:
    - $\beta_{l*} = (\beta_{l*} \setminus \{v\}) \cup \text{vars}(e)$ (from transfer function $f$ for $l : v := e \in P$) and
    - $\beta_{l*} = \beta_{l*'}$ (because $\{l'\} = \text{succ}(l)$).
By Lemma 4.2, \( (l, \sigma) \rightarrow (l', \sigma') \) and \( \text{vars}(e) \subseteq \beta_{s|} \) imply \( \langle e, \sigma, \beta_{s|} \rangle \Rightarrow n \), where \( \langle e, \sigma \rangle \rightarrow n \).

- Prove prophecy variable precondition \( \forall v \in \text{vars}(e).v \in \beta_{s|}, \) i.e., prove \( \text{vars}(e) \subseteq \beta_{s|}: \text{vars}(e) \subseteq (\beta_{s|} - \{v\}) \cup \text{vars}(e) = \beta_{s|} \).

- Prove consistent with prophecy variable prediction \( \beta_{s|'} \subseteq \beta_{s|} \cup \{v\} \): Lemma 4.7.

By program execution rule for \( l : v := e \in P \), with \( l' = \text{next}(l) \), \( \sigma' = \sigma[v \leftrightarrow n] \), \( \pi = \beta_{s|} \), and \( \pi' = \beta_{s|'} \), \( \langle l, \sigma, \beta_{s|} \rangle \Rightarrow \langle l', \sigma', \beta_{s|'} \rangle \).

\[ l : c = l : \text{if } b \text{ then } g \in P : \]
- Facts from dataflow equations:

\[ \beta_{s|} = \beta_{i\*} \cup \text{vars}(b) \text{ (from transfer function } f \text{ for } l : \text{if } b \text{ then } g \in P \) and \( \beta_{s|'} \subseteq \beta_{s|} \text{ (because } l' \in \text{succ}(l)) \).

By Lemma 4.2, \( (l, \sigma) \rightarrow (l', \sigma') \) and \( \text{vars}(e) \subseteq \beta_{s|} \) imply \( \langle b, \sigma, \beta_{s|} \rangle \Rightarrow t \), where \( \langle b, \sigma \rangle \rightarrow t \).

- Prove prophecy variable precondition \( \forall v \in \text{vars}(b).v \in \beta_{s|}, \) i.e., prove \( \text{vars}(b) \subseteq \beta_{s|}: \text{vars}(b) \subseteq \beta_{i\*} \cup \text{vars}(b) = \beta_{s|} \).

- Prove consistent with prophecy variable prediction \( \beta_{s|'} \subseteq \beta_{s|} \): Lemma 4.7.

- By program execution rule for \( l : \text{if } b \text{ then } g \in P \) with \( l' = g \) if \( \sigma(b) = \text{true} \) or \( l' = \text{next}(l) \) if \( \sigma(b) = \text{false} \), \( \sigma' = \sigma, \pi = \beta_{s|} \supseteq \beta_{s|'} \), and \( \langle l, \sigma, \beta_{s|} \rangle \Rightarrow \langle l', \sigma', \beta_{s|'} \rangle \).

\[ l : c = l : \text{goto } g \in P, l : c = l : \text{skip } \in P, \text{ or } l : c = l : \text{halt } \in P : \]
- Facts from dataflow equations:

\[ \beta_{s|} = \beta_{i\*} \text{ (from transfer function } f \text{ for } l : \text{goto } g \in P, l : \text{skip } \in P, \text{ or } l : \text{halt } \in P \) and \( \beta_{s|'} \subseteq \beta_{i\*} \text{ (because } l' \in \text{succ}(l)) \).

- Prove prophecy variable precondition:

There is no prophecy variable precondition for \( l : \text{goto } g \in P, l : \text{skip } \in P, \text{ or } l : \text{halt } \in P \).

- Prove consistent with prophecy variable prediction \( \beta_{s|'} = \beta_{s|} \): Lemma 4.8.

- By the program execution rule for:

\[ l : \text{goto } g \in P \text{ with } l' = g, \sigma' = \sigma, \text{ and } \pi' = \beta_{s|'} = \beta_{s|} = \pi, \langle l, \sigma, \beta_{s|} \rangle \Rightarrow \langle l', \sigma', \beta_{s|'} \rangle. \]

\[ l : \text{skip } \in P \text{ with } l' = \text{next}(l), \sigma' = \sigma, \text{ and } \pi' = \beta_{s|'} = \beta_{s|} = \pi, \langle l, \sigma, \beta_{s|} \rangle \Rightarrow \langle l', \sigma', \beta_{s|'} \rangle. \]

\[ l : \text{halt } \in P \text{ with } l' = \text{next}(l), \sigma' = \sigma, \text{ and } \pi' = \beta_{s|'} = \beta_{s|} = \pi, \langle l, \sigma, \beta_{s|} \rangle \Rightarrow \langle l', \sigma', \beta_{s|'} \rangle. \]

Note that \( \langle l, \sigma \rangle \rightarrow \langle l', \sigma' \rangle \) implies \( \text{next}(l) : \text{done } \in P \).

### 4.1.5 Live Variables Correctness and Optimization Theorems

We next state and prove Theorem 4.10, which characterizes a relationship between the analysis and the program execution. Specifically, the theorem states that if a variable \( v \) is not live at some point in the execution and it is read in some future point in the execution, then there is an intervening write to \( v \) before it is read. Theorem 4.10 ensures, for example, that if \( v \) is not live immediately after an assignment (i.e., \( l : v := e \in P \) and \( v \notin \beta_{l|} \), it is possible to remove \( l : v := e \in P \) without changing the result that the computation produces. Note that Theorem 4.10 leverages the Progress theorem (Theorem 4.9) to use forward reasoning even though the key live variable properties deal with information about future program executions and the analysis itself is a backward analysis.

**Theorem 4.10.** If \( \langle l_1, \sigma_1 \rangle \rightarrow \cdots \rightarrow \langle l_j, \sigma_j \rangle, v \notin \beta_{l_j|}, v \in \text{vars}(c) \) where \( l_j : c \in P \), then \( \exists i \leq k < j. l_k : v := e' \in P \):
Proof: Find a $k$ that satisfies the theorem.

By Lemma 4.6, $\text{vars}(c) \subseteq \beta_s|_l$. Then $v \in \text{vars}(c)$ implies $v \in \beta_s|_l$.

$v \notin \beta_s|_l$ and $v \in \beta_s|_l$ imply $\exists i \leq k < j, v \notin \beta_i|_l$ and $v \in \beta_i|_{l+1}$.

By Progress (Theorem 4.17), $\langle l_k, \sigma_k, \beta_s|_{l_k} \rangle \Rightarrow \langle l_{k+1}, \sigma_{k+1}, \beta_s|_{l_{k+1}} \rangle$.

By Lemma 4.4 $l_k : v := e' \in P$. ■

4.1.6 Live Variables with Downward Closure Metarule. The augmented operational semantics in Section 4.1.1 does not use the downward closure metarule (Definition 2.3). It is possible to formulate the analysis using this rule. Starting with the baseline augmented operational semantics from Section 3.2, update the rule for $l : v := e \in P$ so that the prophecy variable $\pi$ predicts $v$ to be live after the assignment. The downward inference metarule then implements any predictions that a variable $v$ becomes not live (specifically by removing $v$ from $\pi$ after the execution of a command).

$$l : v := e \in P \quad \langle a, \sigma, \pi \rangle \Rightarrow n$$

$$\langle l, \sigma, \pi \rangle \Rightarrow \langle \text{next}(l), \sigma[v \mapsto n], \pi \cup \{v\} \rangle$$

The only other update is to update the variable reference rule to include the prophecy variable precondition:

$$v \in \text{dom} \sigma \quad v \in \pi$$

$$\langle v, \sigma, \pi \rangle \Rightarrow \sigma(v)$$

With this change the only difference between the augmented semantics with and without the downward closure metarule is that, with the downward closure metarule, the prophecy variable $\pi$ can predict that any command, including an $l : \text{goto} \ g \in P, l : \text{skip} \in P$ or $l : \text{halt} \in P$ command, may transition a variable $v$ from live ($v \in \pi$) to not live ($v \notin \pi'$). Without the downward closure metarule only an $l : v := e$ or $l : \text{if} \ b \text{ then} \ g \text{ command can transition a variable } v \text{ from live (} v \in \pi \text{) to not live (} v \notin \pi' \text{).}$ With the downward inference metarule, the proofs in Sections 4.1.1 through 4.1.5 go through unchanged — the only difference is that, in the Progress proof (Theorem 4.9), the dataflow facts for commands $l : v := e \in P, l : \text{goto} \ g \in P, l : \text{skip} \in P$ or $l : \text{halt} \in P$ include $\beta_{1n} \supseteq \beta_{1n'}$, instead of $\beta_{1n} = \beta_{1n'}$, so the proofs for these commands leverage a different case of Lemma 4.8.

4.2 Very Busy Expressions

The very busy expressions analysis (conservatively) determines, for each program point, the expressions that are very busy at that program point, i.e., expressions $e$ that, in every terminating execution, must be evaluated before some $v \in \text{vars}(e)$ is written. The analysis augments the standard operational semantics with a prophecy variable $\pi \subseteq E$ that predicts which expressions are very busy. The prophecy variable $\pi$ is drawn from the very busy expressions program analysis lattice $\langle \Pi, \sqsupseteq \rangle$, where $\Pi = \mathcal{P}(E)$, ordered under reverse subset inclusion ($\sqsupseteq$), with least upper bound $\cap$ and greatest lower bound $\cup$. We use the notation $\text{subs}(e)$ is the set of subexpressions in $e$, $\text{subs}(b)$ is the set of subexpressions in $b$, and and $\text{subs}(e)$ is the set of subexpressions in $c$, all defined recursively over the structure of $e$, $b$, or $c$.

4.2.1 Augmented Operational Semantics. Starting with the baseline augmented operational semantics (Section 3.2), the analysis updates the program execution rule for $l : v := e \in P$ to include a prophecy variable precondition to check for incorrect very busy expression predictions. Specifically, the check requires that the predicted very busy expressions $\pi$ can contain an expression $e'$ only if 1) $e'$ is evaluated during the evaluation of $e$, i.e., $e' \in \text{subs}(e)$ or 2) $e'$ does not read $v$, i.e., $v \notin \text{subs}(e')$. The prophecy variable then predicts that some new set $\pi'$ of expressions will be very
busy after the execution of \( l : v := e \in P \), with the constraint that \( \pi' \) must include all previously predicted very busy expressions not evaluated during the evaluation of \( e \), i.e., \( \pi' \supseteq \pi - \text{subs}(e) \). This last condition reflects the fact that any expression that is very busy before \( l : v := e \in P \) and not evaluated by \( e \) during the execution of \( l : v := e \in P \) must also be very busy after \( l : v := e \in P \).

\[
\begin{align*}
\text{if } b \text{ then } g \in P \quad &\Rightarrow \quad \pi' \supseteq \pi - \text{subs}(b) \\
\end{align*}
\]

The analysis similarly updates the rules for \( l : \text{if } b \text{ then } g \in P \) to predict new very busy expressions \( \pi' \supseteq \pi - \text{subs}(b) \). Note that the predicted very busy expressions may increase after the execution of \( l : \text{if } b \text{ then } g \in P \) because of the control flow split — there may be more very busy expressions after the split than before (because there are fewer paths after the split than before) and different very busy expressions along the different control flow paths.

\[
\begin{align*}
\text{if } b \text{ then } g \in P \quad &\Rightarrow \quad \pi' \supseteq \pi - \text{subs}(b) \\
\end{align*}
\]

Finally the analysis also introduces a prophecy variable precondition for \( l : \text{halt} \) that requires that \( \pi = \emptyset \), i.e., that there are no predicted very busy expressions \( \pi \) when the program halts:

\[
\begin{align*}
l : \text{halt} \in P \quad &\Rightarrow \quad \text{next}(l) : \text{done} \in P \\
&\Rightarrow \quad \pi = \emptyset \\
\end{align*}
\]

Unlike the live variables analysis, the very busy expressions analysis introduces no prophecy variable preconditions into the expression evaluation rules, so expression evaluation is identical in the standard and augmented semantics.

**Lemma 4.11.** If \( \langle e, \sigma, \pi \rangle \Rightarrow n \), then \( \langle e, \sigma \rangle \Rightarrow n \). If \( \langle b, \sigma, \pi \rangle \Rightarrow t \), then \( \langle b, \sigma \rangle \Rightarrow t \). If \( \langle e, \sigma \rangle \Rightarrow n \), then \( \langle e, \sigma, \pi \rangle \Rightarrow n \). If \( \langle b, \sigma \rangle \Rightarrow t \), then \( \langle b, \sigma, \pi \rangle \Rightarrow t \).

**Proof:** The updated expression evaluation rules in the augmented operational semantics have the same preconditions and produce the same expression values as the corresponding rules from the standard operational semantics. ■

The following lemma characterizes the conditions under which an expression \( e' \) may leave the set of predicted very busy expressions \( \pi \), specifically when the expression \( e' \) is evaluated during the execution of a command \( l : c \in P \). The only commands \( l : c \in P \) that remove expressions \( e' \) from the predicted very busy expressions \( \pi \) (i.e., commands for which \( \pi \subseteq \pi \)) are \( l : v := e \in P \) and \( l : \text{if } b \text{ then } g \in P \).

**Lemma 4.12.** If \( \langle l, \sigma, \pi \rangle \Rightarrow \langle l', \sigma', \pi' \rangle \), \( e' \in \pi \), and \( e' \notin \pi' \), then \( e' \in \text{subs}(c) \) where \( l : c \in P \).

**Proof:** Case analysis on \( l : c \in P \):

- \( l : c = l : v := e \in P \): In this case \( \pi' \supseteq \pi - \text{subs}(e) \), so \( e' \in \pi \) and \( e' \notin \pi' \) implies \( e' \in \text{subs}(e) \).
- \( l : c = l : \text{if } b \text{ then } g \in P \): In this case \( \pi' \supseteq \pi - \text{subs}(b) \), so \( e' \in \pi \) and \( e' \notin \pi' \) implies \( e' \in \text{subs}(b) \).
- \( l : c = l : \text{skip} \in P \) or \( l : c = l : \text{goto} \ g \in P \): In this case \( \pi = \pi' \), so there is no \( e' \) such that \( e' \in \pi \) and \( e' \notin \pi' \).
Theorem 4.13. (Preservation) If \( \langle l, \sigma, \pi \rangle \Rightarrow \langle l', \sigma', \pi' \rangle \) then \( \langle l, \sigma \rangle \Rightarrow \langle l', \sigma' \rangle \).

Proof: All rules that generate a transition \( \langle l, \sigma, \pi \rangle \Rightarrow \langle l', \sigma', \pi' \rangle \) in the augmented operational semantics have the same preconditions over \( l : c \in P \) and \( \sigma \) and produce the same values for \( l' \) and \( \sigma' \) as the corresponding rules from the standard operational semantics. □

4.2.2 Very Busy Expressions Analysis. For each command \( l : c \in P \), the analysis produces \( \beta_{\star l} \subseteq E \) (the set of very busy expressions at the program point before \( l : c \in P \)) and \( \beta_{\star l} \subseteq E \) (the set of very busy expressions after \( l : c \in P \)). The analysis obtains the \( \beta_{\star l} \) and \( \beta_{\star l} \) by formulating and solving, using standard least fixed-point techniques, the following set of backward dataflow equations:

\[
\begin{align*}
\beta_{\star l} &= \emptyset \text{ if } l : \text{done} \in P \\
\beta_{\star l} &= \cap \beta_{\star g}, \text{ where } l : c \in P \text{ and } g \in \text{succ}(l) \\
\beta_{\star l} &= f(l, \beta_{\star l})
\end{align*}
\]

where \( f \) is the transfer function for the analysis defined as follows:

\[
f(l, \beta) = \begin{cases} 
(\beta - \{e' \in \beta.\upsilon \in \upsilon(e')\}) \cup \text{subs}(e) & \text{if } l : \upsilon := e \in P \\
\beta \cup \text{subs}(b) & \text{if } l : \text{if } b \text{ then } l' \\
\emptyset & \text{otherwise}
\end{cases}
\]

The next lemma states that if a variable \( \upsilon \) is one of the variables referenced in an expression \( e' \), then the analysis determines that \( e' \) is not very busy before an assignment \( l : \upsilon := e \) unless \( e' \) is evaluated as part of the evaluation of \( e \):

Lemma 4.14. If \( l : \upsilon := e \in P, \upsilon \in \upsilon(e'), \text{ and } e' \notin \text{subs}(e), \text{ then } e' \notin \beta_{\star l}. \)

Proof: By definition of transfer function \( f \) for \( l : \upsilon := e \in P, \beta_{\star l} = (\beta_{\star l} - \{e' \in \beta_{\star l}.\upsilon \in \upsilon(e')\}) \cup \text{subs}(e) \). Then if \( e \in \upsilon(e') \) and \( e' \notin \text{subs}(e) \), then \( e' \notin \beta_{\star l}. \)

4.2.3 Prophecy Variable Predictions and Dataflow Analyses with Sets of Elements and Reverse Subset Inclusion. Very busy expressions is an instance of a more general class of backward dataflow analyses in which the dataflow facts \( \beta \in \Pi \) are sets of elements with the dataflow lattice \( \langle \Pi, \leq \rangle \) ordered by reverse subset inclusion (\( \subseteq \)), with least upper bound \( \cap \), greatest lower bound \( \cup \), and dataflow equations of the following form:

\[
\begin{align*}
\beta_{\star l} &= \cap \beta_{\star g}, \text{ where } l : c \in P \text{ and } g \in \text{succ}(l) \\
\beta_{\star l} &= (\beta_{\star l} - D_l) \cup U_l
\end{align*}
\]

Note that these equations ensure \( \beta_{\star l'} \supseteq \beta_{\star l} \) for \( l' \in \text{succ}(l) \). For very busy expressions \( D_l = \{e' \in \beta.\upsilon \in \upsilon(e')\} \) and \( U_l = \text{subs}(e) \) when \( l : \upsilon := e \in P; D_l = \emptyset \) and \( U_l = \text{subs}(b) \) when \( l : \text{if } b \text{ then } l' \in P \). For \( l : c = l : \text{goto } g \in P, l : c = l : \text{skip } e \in P \), or \( l : c = l : \text{halt } e \in P, D_l = U_l = \emptyset \).

We next show that prophecy variable predictions \( \pi ' \supseteq \pi - U_l \) are consistent with the results that these analyses produce:

Lemma 4.15. If \( \beta_{\star l} = (\beta_{\star l} - D_l) \cup U_l \) and \( \beta_{\star l'} \supseteq \beta_{\star l} \), then \( \beta_{\star l'} \supseteq \beta_{\star l} - U_l. \)

Proof:

- Known facts from dataflow analysis: \( \beta_{\star l} = (\beta_{\star l} - D_l) \cup U_l \) and \( \beta_{\star l'} \supseteq \beta_{\star l}. \)
- Then \( \beta_{\star l} \subseteq (\beta_{\star l'} - D_l) \cup U_l, \beta_{\star l} \subseteq \beta_{\star l'} \cup U_l, \beta_{\star l} - U_l \subseteq (\beta_{\star l'} \cup U_l) - U_l, \beta_{\star l} - U_l \subseteq \beta_{\star l'} - U_l, \), and \( \beta_{\star l} - U_l \subseteq \beta_{\star l'} - U_l. \)
If $D_l = U_l = \emptyset$ so that $f(l, \beta) = \beta$ and $\{l'\} = \text{succ}(l)$, then $\beta_{s l} = \beta_{t s} = \beta_{s l'}$. For live variables this is the case for $l : c = l : \text{goto } g \in P, l : c = l : \text{skip} \in P$, and $l : c = l : \text{halt} \in P$. In this case the analysis results are consistent with prophecy variable predictions $\pi' = \pi$ and $\pi' \supseteq \pi$ (as in, for example, analyses that use the downward closure metarule):

**Lemma 4.16.** If $\beta_{s l} = \beta_{t s}$ and $\beta_{t s} = \beta_{s l'}$, then $\beta_{s l} = \beta_{s l'}$ and $\beta_{s l'} \supseteq \beta_{s l}$.

### 4.2.4 Very Busy Expressions Progress Theorem

We next state and prove the Progress theorem for the very busy expressions analysis. As in the Live Variables Progress theorem (Theorem 4.9), the prophecy variable preconditions apply to $\beta_{s l}$ and are immediately satisfied by the transfer function $f$ regardless of the values of $\beta_{t s}$ and $\beta_{s l'}$. As in the Live Variables Progress theorem (Theorem 4.9), the proof discharges the proof obligations required to show that the analysis results $\beta_{s l}$, $\beta_{t s}$, and $\beta_{s l'}$ are consistent with the prophecy variable predictions by pushing the analysis result $\beta_{t s}$ through the transfer function for $l : c e \in P$.

Because the analysis and prophecy variable predictions conform to the requirements of Lemmas 4.15 and 4.16, the prophecy variable prediction proof obligations for these commands are immediately discharged by applying these lemmas.

**Theorem 4.17.** (Progress) If $\langle l, \sigma \rangle \rightarrow \langle l', \sigma' \rangle$ then $\langle l, \sigma, \beta_{s l} \rangle \Rightarrow \langle l', \sigma', \beta_{s l'} \rangle$.

**Proof:** If all of the prophecy variable preconditions are satisfied, the standard and augmented program execution rules for $l : c e \in P$ define the same values for $l', \sigma'$, and all evaluated expressions $e$ or $b$. The following case analysis on $l : c e \in P$ shows that the prophecy variable preconditions are satisfied and that $\beta_{s l}$ and $\beta_{s l'}$ satisfy $(l, \sigma, \beta_{s l}) \Rightarrow (l', \sigma', \beta_{s l'})$:

- **Case 1: $l : c = l : v := e P$**
  - Facts from dataflow equations:
    $\beta_{s l} = (\beta_{t s} - \{e' e \in \beta_{t s} \text{ and } e \in \text{vars}(e')\}) \cup \text{subs}(e)$ (from transfer function $f$ for $l : v := e \in P$ and $\beta_{t s} = \beta_{s l'}$ (because $\{l' = \text{succ}(l)\}$).
  - By Lemma 4.11, $(l, \sigma) \rightarrow \langle l', \sigma' \rangle$ implies $(e, \sigma, \beta_{s l}) \Rightarrow n$, where $(e, \sigma) \rightarrow n$.
  - Prove prophecy variable precondition $e' e \in \beta_{s l}$ implies $e' e \in \text{subs}(e)$ or $v \notin \text{subs}(e')$:
    - Consider any $e' e \in \beta_{s l}$. Because $\beta_{s l} = (\beta_{t s} - \{e' e \in \beta_{t s} \text{ and } e \in \text{vars}(e')\}) \cup \text{subs}(e)$, either $e' e \in \text{subs}(e)$ or $v \notin \text{subs}(e')$.
    - Prove consistent with prophecy variable prediction $\beta_{s l'} \supseteq \beta_{s l} - \text{subs}(e)$: Lemma 4.15.
    - By program execution rule for $l : v := e \in P$, with $l' = \text{next}(l), \sigma' = \sigma[v \mapsto n], \pi = \beta_{s l}$, and $\pi' = \beta_{s l'}$, $(l, \sigma, \beta_{s l}) \Rightarrow (l', \sigma', \beta_{s l'})$.

- **Case 2: $l : c = l : b \text{ then } g P$**
  - Facts from dataflow equations:
    $\beta_{s l} = \beta_{t s} \cup \text{subs}(b)$ (from transfer function $f$ for $l : b \text{ then } g \in P$ and $\beta_{t s} \subseteq \beta_{s l'}$ (because $l' \in \text{succ}(l)$).
  - By Lemma 4.11, $(l, \sigma) \rightarrow \langle l', \sigma' \rangle$ implies $(b, \sigma, \beta_{s l}) \Rightarrow t$, where $(e, \sigma) \rightarrow t$.
  - Prove prophecy variable precondition:
    - There is no prophecy variable precondition for $l : b \text{ then } g P$.
    - Prove consistent with prophecy variable prediction $\beta_{s l'} \supseteq \beta_{s l} - \text{subs}(b)$: Lemma 4.15.
    - By program execution rule for $l : b \text{ then } g \in P$ with $l' = g$ if $\sigma(b) = \text{true}$ or $l' = \text{next}(l)$ if $\sigma(b) = \text{false}$, $\sigma' = \sigma, \pi = \beta_{s l} \supseteq \beta_{s l'} \cup \text{subs}(b)$, and $\pi' = \beta_{s l'}$, $(l, \sigma, \beta_{s l}) \Rightarrow (l', \sigma', \beta_{s l'})$.

- **Case 3: $l : c = l : \text{goto } g P$ or $l : c = l : \text{skip} P$**.
- Facts from dataflow equations:
  $\beta_{\mathbf{e}} = \beta_{\mathbf{e}}(\mathbf{f})$ (from transfer function $\mathbf{f}$ for $l : \text{goto } g \in P$, $l : \text{skip } \in P$, or $l : \text{halt } \in P$) and $\beta_{\mathbf{e}} = \beta_{\mathbf{e}}'$ (because $\{l'\} = \text{succ}(l)$).
- Prove prophecy variable precondition:
  There is no prophecy variable precondition for $l : \text{goto } g \in P$, $l : \text{skip } \in P$, or $l : \text{halt } \in P$.
- Prove consistent with prophecy variable prediction $\beta_{\mathbf{e}} = \beta_{\mathbf{e}}$: Lemma 4.16.
- By the program execution rule for:
  \[
  \begin{align*}
  \ast l : \text{goto } g \in P &\text{ with } l' = g, \sigma' = \sigma, \text{ and } \pi' = \beta_{\mathbf{e}} = \beta_{\mathbf{e}} = \pi, \langle l, \sigma, \beta_{\mathbf{e}} \rangle \Rightarrow \langle l', \sigma', \beta_{\mathbf{e}} \rangle. \\
  l : \text{skip } \in P &\text{ with } l' = \text{next}(l), \sigma' = \sigma, \text{ and } \pi' = \beta_{\mathbf{e}} = \beta_{\mathbf{e}} = \pi, \langle l, \sigma, \beta_{\mathbf{e}} \rangle \Rightarrow \langle l', \sigma', \beta_{\mathbf{e}} \rangle. \\
  l : \text{halt } \in P &\text{ with } l' = \text{next}(l), \sigma' = \sigma, \text{ and } \pi' = \beta_{\mathbf{e}} = \beta_{\mathbf{e}} = \pi, \langle l, \sigma, \beta_{\mathbf{e}} \rangle \Rightarrow \langle l', \sigma', \beta_{\mathbf{e}} \rangle. 
  \end{align*}
\]
  Note that $\langle l, \sigma \rangle \Rightarrow \langle l', \sigma' \rangle$ implies $\text{next}(l) : \text{done } \in P$.

- $l : c = l : \text{halt } \in P$:
  - Facts from dataflow equations:
    $\beta_{\mathbf{e}} = \beta_{\mathbf{e}} = \emptyset$ (from transfer function $\mathbf{f}$ for $l : \text{halt } \in P$) and $\beta_{\mathbf{e}} = \beta_{\mathbf{e}}'$ (because $\{l'\} = \text{succ}(l)$).
  - Prove prophecy variable precondition $\beta_{\mathbf{e}} = \emptyset$:
    Because $\beta_{\mathbf{e}} = \beta_{\mathbf{e}} = \emptyset$, $\beta_{\mathbf{e}} = \emptyset$.
  - Prove consistent with prophecy variable prediction $\beta_{\mathbf{e}} = \beta_{\mathbf{e}}$:
    $\beta_{\mathbf{e}} = \beta_{\mathbf{e}}$ and $\beta_{\mathbf{e}} = \beta_{\mathbf{e}}$ imply $\beta_{\mathbf{e}} = \beta_{\mathbf{e}}$.
  - By the program execution rule for $l : \text{halt } \in P$ with $l' = \text{next}(l), \sigma' = \sigma, \pi' = \beta_{\mathbf{e}} = \beta_{\mathbf{e}} = \pi, \langle l, \sigma, \beta_{\mathbf{e}} \rangle \Rightarrow \langle l', \sigma', \beta_{\mathbf{e}} \rangle$. Note that $\langle l, \sigma \rangle \Rightarrow \langle l', \sigma' \rangle$ implies $\text{next}(l) : \text{done } \in P$.

\[\blacksquare\]

4.2.5 Very Busy Expressions Correctness Theorems. The following two theorems establish correctness properties of the very busy expressions analysis. The first states that, in all executions, all very busy expressions $e'$ are evaluated before any of the variables $v \in \text{vars}(e')$ is reassigned. The second states that no execution halts before evaluating all very busy expressions $e'$. These are the correctness properties required to establish the soundness of, for example, standard code hoisting optimizations that use very busy expressions [11].

**Theorem 4.18.** If $\langle l_i, \sigma_i \rangle \rightarrow \cdots \rightarrow \langle l_j, \sigma_j \rangle$, $l_j : v := e \in P$, $e' \in \beta_{\mathbf{e}}$, $v \in \text{vars}(e')$, then $\exists i \leq k \leq j. e' \in \text{subs}(c)$ where $l_k : c \in P$.

**Proof:** Find a $k$ that satisfies the theorem.
If $e' \in \text{subs}(e)$, then $k = j$.
If $e' \notin \text{subs}(e)$, then by Lemma 4.14, $e' \notin \beta_{\mathbf{e}}$.
$e' \in \beta_{\mathbf{e}}$ implies $\exists i \leq k < j. e' \in \beta_{\mathbf{e}}$ and $e' \notin \beta_{\mathbf{e}}$.
By Progress (Theorem 4.17), $\langle l_k, \sigma_k, \beta_{\mathbf{e}} \rangle \Rightarrow \langle l_{k+1}, \sigma_{k+1}, \beta_{\mathbf{e}} \rangle$.
By Lemma 4.12 $e' \in \text{subs}(c)$ where $l_k : c \in P$. \[\blacksquare\]

**Theorem 4.19.** If $\langle l_i, \sigma_i \rangle \rightarrow \cdots \rightarrow \langle l_j, \sigma_j \rangle$, $e' \in \beta_{\mathbf{e}}$, and $l_j : \text{done } \in P$, then $\exists i \leq k < j. e' \in \text{subs}(c)$ where $l : c \in P$.

**Proof:** Find a $k$ that satisfies the theorem.
$l_j : \text{done } \in P$ implies $\beta_{\mathbf{e}} = \beta_{\mathbf{e}} = \emptyset$. Then $e' \notin \beta_{\mathbf{e}}$.
$e' \in \beta_{\mathbf{e}}$ implies $\exists i \leq k < j. e' \in \beta_{\mathbf{e}}$ and $e' \notin \beta_{\mathbf{e}}$.
By Progress (Theorem 4.17), \( \langle l_k, \sigma_k, \beta_{\bullet l_k} \rangle \Rightarrow \langle l_{k+1}, \sigma_{k+1}, \beta_{\bullet l_{k+1}} \rangle \).
By Lemma 4.12 \( e' \in \text{subs}(c) \) where \( l_k : c \in P \).

### 4.3 Prophecy Variables, Dataflow Analyses, and Complemented, Distributive Lattices

We next generalize Lemmas 4.7 and 4.15 to arbitrary complemented, distributive lattices \( \langle \Pi, \leq \rangle \) ordered by \( \leq \), with least upper bound \( \lor \), greatest lower bound \( \land \), greatest element \( \top \), and least element \( \bot \). For each \( \beta \in \Pi \), \( \overline{\beta} \) is the complement of \( \beta \), i.e., the unique lattice element \( \overline{\beta} \in \Pi \) such that \( \beta \lor \overline{\beta} = \top \) and \( \beta \land \overline{\beta} = \bot \). Note that if the \( \beta \in \Pi \) are sets of elements from some underlying set \( S \) with \( \Pi = \mathcal{P}(S) \), then the lattice \( \langle \Pi, \subseteq \rangle \) and the lattice \( \langle \Pi, \supseteq \rangle \) are both complemented, distributive lattices where \( \overline{\beta} \) is set complement; i.e. \( \overline{\beta} = S - \beta \). For \( \langle \Pi, \subseteq \rangle \), \( \beta - D = \beta \land \overline{D} \). For \( \langle \Pi, \supseteq \rangle \), \( \beta - D = \beta \lor \overline{D} \).

We consider two cases for arbitrary complemented, distributive lattices:

\[
\begin{align*}
\beta_{l\bullet} &= \lor \beta_{g\bullet}, \text{ where } l : c \in P \text{ and } g \in \text{succ}(l) \\
\beta_{a\bullet} &= (\beta_{l\bullet} \land D_l) \lor U_l
\end{align*}
\]

where the analysis results are consistent with the prophecy variable prediction \( \pi' \leq \pi \lor D_l \) and

\[
\begin{align*}
\beta_{l\bullet} &= \lor \beta_{g\bullet}, \text{ where } l : c \in P \text{ and } g \in \text{succ}(l) \\
\beta_{a\bullet} &= (\beta_{l\bullet} \lor D_l) \land U_l
\end{align*}
\]

where the analysis results are consistent with the prophecy variable prediction \( \pi' \leq \pi \lor U_l \).

**Lemma 4.20.** If \( \beta_{a\bullet} = (\beta_{l\bullet} \land D_l) \lor U_l \) and \( \beta_{a\bullet}' \leq \beta_{l\bullet} \), then \( \beta_{a\bullet}' \leq \beta_{a\bullet} \lor D_l \).

**Proof:**
- Known facts from dataflow analysis: \( \beta_{a\bullet} = (\beta_{l\bullet} \land D_l) \lor U_l \) and \( \beta_{a\bullet}' \leq \beta_{l\bullet} \).
- Then \( \beta_{a\bullet} \geq (\beta_{a\bullet}' \land D_l) \lor U_l, \beta_{a\bullet} \geq (\beta_{a\bullet}' \land D_l), \beta_{a\bullet} \lor D_l \geq (\beta_{a\bullet}' \land D_l) \lor D_l, \beta_{a\bullet} \lor D_l \geq (\beta_{a\bullet}' \lor D_l), \beta_{a\bullet} \lor D_l \geq (\beta_{a\bullet}' \lor D_l) \land \top, \beta_{a\bullet} \lor D_l \geq (\beta_{a\bullet}' \lor D_l), \) and \( \beta_{a\bullet} \lor D_l \geq \beta_{a\bullet}' \).

**Lemma 4.21.** If \( \beta_{a\bullet} = (\beta_{l\bullet} \lor D_l) \land U_l \) and \( \beta_{a\bullet}' \leq \beta_{l\bullet} \), then \( \beta_{a\bullet}' \leq \beta_{a\bullet} \lor U_l \).

**Proof:**
- Known facts from dataflow analysis: \( \beta_{a\bullet} = (\beta_{l\bullet} \lor D_l) \land U_l \) and \( \beta_{a\bullet}' \leq \beta_{l\bullet} \).
- Then \( \beta_{a\bullet} \geq (\beta_{a\bullet}' \lor D_l) \land U_l, \beta_{a\bullet} \geq (\beta_{a\bullet}' \lor D_l), \beta_{a\bullet} \lor U_l \geq (\beta_{a\bullet}' \lor U_l) \lor U_l, \beta_{a\bullet} \lor U_l \geq (\beta_{a\bullet}' \lor U_l) \land \top, \beta_{a\bullet} \lor U_l \geq (\beta_{a\bullet}' \lor U_l), \) and \( \beta_{a\bullet} \lor U_l \geq \beta_{a\bullet}' \).

### 5 FORWARDED ANALYSES AND HISTORY VARIABLES

We next present several forward analyses that use history variables to record information about the past execution of the program.

#### 5.1 Defined Variables Analysis

The standard operational semantics in Figures 1 – 3 will become stuck if an expression reads the value of an undefined variable. We next present an analysis that computes, for each program point, the variables that are defined on all program execution paths to that program point. This analysis can be used to (conservatively) check if any program execution can become stuck because it attempts to access an undefined variable.
5.1.1 Defined Variables Augmented Operational Semantics. Starting with the baseline augmented operational semantics (Section 3.2), the analysis augments the baseline semantics with a history variable \( \pi \subseteq V \). \( \pi \) records (a subset of) the variables \( v \in V \) that are defined in each configuration \( \langle l, \sigma, \pi \rangle \). The program analysis lattice \( \langle \Pi, \supseteq \rangle \) is ordered under reverse subset inclusion \( (\supseteq) \) with least upper bound \( \cap \) and greatest lower bound \( \cup \). The augmented operational semantics for this analysis updates the program execution rule for commands \( l : v := e \in P \) to update the history variable \( \pi \) to record \( v \) as one of the defined variables. All other rules remain unchanged and the analysis applies the upward closure metarule (Definition 2.4). \( \pi_0 = \emptyset \) is the initial value of the history variable \( \pi \).

\[
\begin{align*}
l : v := e \in P \quad & \quad \langle e, \sigma, \pi \rangle \Rightarrow n \\
\langle l, \sigma, \pi \rangle \Rightarrow \langle \text{next}(l), \sigma[v \mapsto n], \pi \cup \{v\} \rangle
\end{align*}
\]

We next state and prove a lemma that the history variable \( \pi \) records a subset of the defined variables at every point in the execution. Without the upward closure metarule \( \pi = \text{dom} \sigma \). The upward closure metarule enables the augmented operational semantics to drop defined variables \( v \) from \( \pi \) so that \( \pi \subseteq \text{dom} \sigma \).

**Lemma 5.1.** \( \langle l_0, \sigma_0, \pi_0 \rangle \Rightarrow \cdots \Rightarrow \langle l_i, \sigma_i, \pi_i \rangle \) implies \( \pi_i \subseteq \text{dom} \sigma_i \).

**Proof** (induction on \( i \)):

- Base Case: \( (i = 0) : \pi_0 = \emptyset \not\subseteq \text{dom} \sigma_0 \).
- Induction Step: (assume for \( i \), prove for \( i + 1 \)):
  \( \langle l_i, \sigma_i, \pi_i \rangle \Rightarrow \langle l_{i+1}, \sigma_{i+1}, \pi_{i+1} \rangle \)

  - Case analysis on \( l_i : c \in P \):
    - Known facts:
      * \( \pi_{i+1} = \pi_i[c \mapsto n] \) where \( \langle e, \sigma_i, \pi_i \rangle \Rightarrow n \).
      * \( \pi_{i+1} \subseteq \pi_i \cup \{v\} \).
      * \( \pi_i \subseteq \text{dom} \sigma_i \).
    - Then \( \text{dom} \sigma_{i+1} = \text{dom} \sigma_i \cup \{v\} \) and \( \pi_{i} \cup \{v\} \subseteq \text{dom} \sigma_i \cup \{v\} \). So \( \pi_{i} \cup \{v\} \subseteq \text{dom} \sigma_{i+1} \) and \( \pi_{i+1} \subseteq \text{dom} \sigma_{i+1} \).
    - \( l_i : c = l_i \) if \( b \) then \( g \), \( l_i : c = l_i \) : \text{goto} \( g \), \( l_i : c = l_i \) : \text{skip}, or \( l_i : c = l_i \) : \text{halt}:
      - Then \( \pi_{i+1} = \pi_i \) and \( \sigma_{i+1} = \sigma_i \). By induction hypothesis \( \pi_i \subseteq \text{dom} \sigma_i \), so \( \pi_{i+1} \subseteq \text{dom} \sigma_{i+1} \).

The Preservation theorem is straightforward as the defined variables analysis introduces no rule preconditions or changes to \( l \) or \( \sigma \).

**Theorem 5.2.** (Preservation): If \( \langle l, \sigma, \pi \rangle \Rightarrow \langle l', \sigma', \pi' \rangle \), then \( \langle l, \sigma \rangle \Rightarrow \langle l', \sigma' \rangle \).

**Proof:** All rules that generate a transition \( \langle l, \sigma, \pi \rangle \Rightarrow \langle l', \sigma', \pi' \rangle \) in the augmented operational semantics have the same preconditions over \( l : c \in P \) and \( \sigma \) and produce the same values for \( l' \) and \( \sigma' \) as the corresponding rules from the standard operational semantics.

5.1.2 Defined Variables Analysis. The defined variables analysis obtains the analysis results \( \beta_{\ast l} \) and \( \beta_{l \bullet} \) by formulating and solving, using standard least fixed-point techniques, the following set of forward dataflow equations:

\[
\begin{align*}
\beta_{\text{first}(l)} &= \emptyset \\
\beta_{\ast l} &= \bigcap \beta_{g \bullet} \quad \text{where} \ l : c \in P \text{ and } g \in \text{pred}(l) \\
\beta_{l \bullet} &= f(l, \beta_{\ast l})
\end{align*}
\]
where $f$ is the transfer function for the analysis defined as follows:

$$f(l, \beta) = \begin{cases} 
\beta \cup \{v\} & \text{if } l : v := e \in P \\
\beta & \text{otherwise}
\end{cases}$$

5.1.3 Defined Variables Progress Theorem. We next state and prove the Progress theorem for the defined variables analysis.

**Theorem 5.3. (Progress):** If $\langle l, \sigma \rangle \Rightarrow \langle l', \sigma' \rangle$, then $\langle l, \sigma, \beta_{si} \rangle \Rightarrow \langle l', \sigma', \beta_{si'} \rangle$.

**Proof:** The following case analysis on $l : c \in P$ shows that the analysis results are consistent with the history variable updates. so that $\beta_{si}$ and $\beta_{si'}$ satisfy $\langle l, \sigma, \beta_{si} \rangle \Rightarrow \langle l', \sigma', \beta_{si'} \rangle$:

- **$l : c = l : v := e \in P$:**
  - Facts from dataflow equations:
    $\beta_{si} \cup \{v\} = \beta_{si}$ (from transfer function $f$ for $l : v := e \in P$)
    $\beta_{si} \supseteq \beta_{si'}$ (because $l \in \text{pred}(l')$)
  - Prove history variable consistency (prove $\beta_{si} \cup \{v\} \supseteq \beta_{si'}$):
    $\beta_{si} \cup \{v\} = \beta_{si}$ and $\beta_{i*} \supseteq \beta_{si}$ imply $\beta_{si} \cup \{v\} \supseteq \beta_{si'}$.
  - By the program execution rule for $l : v := e \in P$, with $l' = \text{next}(l)$, $\sigma' = \sigma[v \mapsto n]$, $\pi = \beta_{si}$, and $\pi' = \beta_{si'}, \langle l, \sigma, \beta_{si} \rangle \Rightarrow \langle l', \sigma', \beta_{si'} \rangle$.

- **$l : c = l : b$ then $g \in P, l : \text{skip} \in P, l : \text{goto} g \in P$:**
  - Facts from dataflow equations:
    $\beta_{si} = \beta_{ib}$ (from transfer function $f$ for $l : c \in P$)
    $\beta_{ib} \supseteq \beta_{ib'}$ (because $l \in \text{pred}(l')$)
  - Prove history variable consistency (prove $\beta_{ib} \supseteq \beta_{ib'}$):
    $\beta_{ib} = \beta_{ib}$ and $\beta_{i*} \supseteq \beta_{ib'}$ imply $\beta_{si} \supseteq \beta_{si'}$.
  - By program execution rule for $l : b$ then $g \in P$:
    * if $\langle l, \sigma, b \rangle \Rightarrow \text{true}, l' = g, \sigma' = \sigma, \pi = \beta_{ib}$ and $\pi' = \beta_{ib'}, \langle l, \sigma, \beta_{si} \rangle \Rightarrow \langle l', \sigma', \beta_{si'} \rangle$.
    * if $\langle l, \sigma, b \rangle \Rightarrow \text{false}, l' = \text{next}(l), \sigma' = \sigma, \pi = \beta_{ib}$ and $\pi' = \beta_{ib'}, \langle l, \sigma, \beta_{si} \rangle \Rightarrow \langle l', \sigma', \beta_{si'} \rangle$.
  - By program execution rule for $l : \text{skip} \in P$, with $l' = \text{next}(l)$, $\sigma' = \sigma, \pi = \beta_{ib}$, and $\pi' = \beta_{ib'}, \langle l, \sigma, \beta_{si} \rangle \Rightarrow \langle l', \sigma', \beta_{si'} \rangle$.
  - By program execution rule for $l : \text{goto} g \in P$, with $l' = g, \sigma' = \sigma, \pi = \beta_{ib}$, and $\pi' = \beta_{ib'}, \langle l, \sigma, \beta_{si} \rangle \Rightarrow \langle l', \sigma', \beta_{si'} \rangle$.

5.1.4 Defined Variables Correctness Theorem. We now state and prove a correctness theorem for the defined variable analysis. At a high level, this theorem states that the analysis (conservatively) computes an under approximation of the variables that are defined at any point in the execution of the program $P$ — if the analysis says that a variable is defined, then it is defined in all executions.

**Theorem 5.4.** $\langle l_0, \sigma_0 \rangle \rightarrow \cdots \rightarrow \langle l_i, \sigma_i \rangle$ implies $\beta_{si_i} \subseteq \text{dom } \sigma_i$.

**Proof:** By Progress (Theorem 5.3), $\langle l_0, \sigma_0, \beta_{si_0} \rangle \Rightarrow \cdots \Rightarrow \langle l_i, \sigma_i, \beta_{si_i} \rangle$, where $\beta_{si_0} = \pi_0 = \emptyset$. By Lemma 5.1, $\beta_{si_i} \subseteq \text{dom } \sigma_i$. 

In the standard program execution semantics (Figures 1-3), a program execution becomes stuck at $l$ if the evaluation of an expression $e$ in $l : v := e \in P$ or $b$ in $l : \text{if } b \text{ goto } g \in P$ attempts to read an undefined variable $v$ (i.e., a variable $v \notin \text{dom } \sigma$). But if the analysis determines that all
variables in \( e \) or \( b \) are defined, then the execution will not become stuck at the evaluation of \( e \) or \( b \) because of an attempt to read an undefined variable. We formalize this reasoning as follows:

**Theorem 5.5.** If \( \langle l_0, \sigma_0 \rangle \rightarrow \cdots \rightarrow \langle l_i, \sigma_i \rangle \) and \( \text{vars}(e) \subseteq \beta_{\pi_l} \), then \( \langle e, \sigma_i \rangle \rightarrow n \).

**Theorem 5.6.** If \( \langle l_0, \sigma_0 \rangle \rightarrow \cdots \rightarrow \langle l_i, \sigma_i \rangle \) and \( \text{vars}(b) \subseteq \beta_{\pi_l} \), then \( \langle b, \sigma_i \rangle \rightarrow t \).

**Proof:** By Theorem 5.4, \( \beta_{\pi_l} \subseteq \text{dom} \ \sigma_i \), so \( \text{vars}(e) \subseteq \beta_{\pi_l} \subseteq \text{dom} \ \sigma_i \), which ensures that all variables are defined during the evaluation of \( e \). Similarly \( \text{vars}(b) \subseteq \beta_{\pi_l} \subseteq \text{dom} \ \sigma_i \) ensures that all variables are defined during the evaluation of \( b \).

### 5.2 Reaching Definitions

Reaching definitions is a classic program analysis used, for example, in constant propagation and other compiler optimizations [11]. The analysis augments the standard operational semantics with a history variable \( \pi : V \rightarrow \mathcal{P}(L) \). \( \pi \in \Pi \) records the most recent definition of a given variable \( v \in V \) by recording, for each variable \( v \), the label \( l \) of the most recent assignment to \( v \). The program analysis lattice \( (\Pi, \leq) \) is ordered under element-wise subset inclusion (i.e., \( \pi_1 \leq \pi_2 \) if \( \forall v \in V. \pi_1(v) \subseteq \pi_2(v) \)) with least upper bound \( \vee \) (i.e., \( \pi_1 \vee \pi_2 = \lambda v \in V. \pi_1(v) \cup \pi_2(v) \)) and greatest lower bound \( \wedge \) (i.e., \( \pi_1 \wedge \pi_2 = \lambda v \in V. \pi_1(v) \cap \pi_2(v) \)). The augmented operational semantics updates the program execution rule for commands \( l : v := e \in P \) to record the fact that \( l \) is the current definition of \( v \) (i.e., \( \pi(v) = \{l\} \)). All other rules remain unchanged and we apply the upward closure metarule. \( \pi_0 = \lambda v. \emptyset \) is the initial value for \( \pi \).

\[
\langle l, \sigma, \pi \rangle \Rightarrow \langle \text{next}(l), \sigma[v \mapsto n], \pi[v \mapsto \{l\}] \rangle
\]

**Theorem 5.7.** (Preservation): If \( \langle l, \sigma, \pi \rangle \Rightarrow \langle l', \sigma', \pi' \rangle \), then \( \langle l, \sigma \rangle \Rightarrow \langle l', \sigma' \rangle \).

**Proof:** All rules that generate a transition \( \langle l, \sigma, \pi \rangle \Rightarrow \langle l', \sigma', \pi' \rangle \) in the augmented operational semantics have the same preconditions over \( l : c \in P \) and \( \sigma \) and produce the same values for \( l' \) and \( \sigma' \) as the corresponding rules from the standard operational semantics.

We next prove a lemma related to the relationship between reaching definitions and the values recorded in the program execution states \( \sigma \), specifically that if a state \( \sigma \) records \( n \) as the value of \( v \), then one of the labels recorded in the corresponding history variable \( \pi(v) \) is the label of an executed assignment statement that assigned the value \( n \) to \( v \). Note that (in the absence of any defined variable information as could be computed, for example, by the defined variables analysis from Section 5.1) there is no guarantee that any executed assignment command \( l : v := e \in P \) assigned a value to \( v \) and no guarantee that \( v \) is defined.

**Lemma 5.8.** If \( \langle l_0, \sigma_0, \pi_0 \rangle \Rightarrow \cdots \Rightarrow \langle l_i, \sigma_i, \pi_i \rangle \), then \( v \in \text{dom} \ \sigma_i \) implies \( \exists 0 \leq k < i. l_k : v := e \in P, \langle e, \sigma_k, \pi_k \rangle \Rightarrow \sigma_l(v), \text{and } l_k \in \pi_i \).

**Proof:** (induction on \( i \))

Base case \( (i = 0) \): If \( i = 0 \), \( \text{dom} \ \sigma_0 = \emptyset \) so \( v \notin \text{dom} \ \sigma_0 \).

Induction step (assume for \( i \), prove for \( i + 1 \)): \( \langle l_i, \sigma_i, \pi_i \rangle \Rightarrow \langle l_{i+1}, \sigma_{i+1}, \pi_{i+1} \rangle \).

Case analysis on \( l_i : c \in P \):
- \( l_i : c = l_i : w := e \in P \):
  - Known facts:
    * \( \sigma_i[w \mapsto n] = \sigma_{i+1} \) where \( \langle e, \sigma_i, \pi_i \rangle \Rightarrow n \).
    * \( \pi_i[w \mapsto \{l_i\}] \leq \pi_{i+1} \) by the augmented operational semantics.
- Must show: \( \forall v \in \text{dom } \sigma_{i+1}, \exists 0 < k < i + 1. l_k : v := e \in P, \langle e, \sigma_k, \pi_k \rangle \Rightarrow \sigma_{i+1}(v) \), and \( l_k \in \pi_{i+1} \).
  - For \( v = w, k = i : v \in \text{dom } \sigma_{i+1}, l_i : v := e \in P, \sigma_{i+1}(v) = n \) where \( \langle e, \sigma_i, \pi_i \rangle \Rightarrow n \), and \( l_i \in \pi_{i+1} \).
  - For \( v \neq w \), if \( v \in \text{dom } \sigma_{i+1} \), then \( v \in \text{dom } \sigma_i, \sigma_{i+1}(v) = \sigma_i(v) \), \( \pi_{i+1} = \pi_i(v) \) and the theorem holds by the induction hypothesis.

- \( l_i : c = l_i : \text{if } b \text{ then } g \in P, \ l_i : c = l_i : \text{goto } g \in P, \ l_i : c = l_i : \text{skip} \in P \), or \( l_i : c = l_i : \text{halt} \in P \): Then \( \pi_{i+1} = \pi_i \) and \( \sigma_{i+1} = \sigma_i \) and the theorem holds by the induction hypothesis.

5.2.1 Reaching Definitions Analysis. The program analysis obtains the analysis results \( \beta_{\star l} \) and \( \beta_l \) by formulating and solving, using standard least fixed-point techniques, the following set of forward dataflow equations:

\[
\begin{align*}
\beta_{\text{first}(P)} &= \lambda v. \emptyset \\
\beta_{\star l} &= \bigvee_{l \in \text{pred}(l)} \beta_g, \text{ where } l : c \in P \text{ and } g \in \text{pred}(l) \\
\beta_l &= f(l, \beta_{\star l})
\end{align*}
\]

where \( f \) is the transfer function for the analysis defined as follows:

\[
f(l, \beta) = \begin{cases} 
\beta[v \mapsto \{l\}] & \text{if } l : v := e \in P \\
\beta & \text{otherwise}
\end{cases}
\]

Note that because the analysis is a may analysis (it only computes definitions that may reach program points), it does not attempt to determine if that any definition will reach any specific program point — it only verifies that if a definition does reach a program point, it will be one of the definitions recorded in the corresponding analysis result at that program point. It is, of course, possible to combine the reaching definitions analysis with the defined variables analysis (Section 5.1) to obtain a guarantee that 1) a variable \( v \) is always defined at a program point and 2) therefore in all executions one of the recorded definitions reaches that program point.

Theorem 5.9. (Progress): If \( \langle l, \sigma \rangle \rightarrow \langle l', \sigma' \rangle \), then \( \langle l, \sigma, \beta_{\star l} \rangle \Rightarrow \langle l', \sigma', \beta_{\star l'} \rangle \).

Proof: The standard and augmented program execution rules for \( l : c \in P \) have the same preconditions over and define the same values for \( l' \) and \( \sigma' \). The following case analysis on \( l : c \in P \) shows that \( \beta_{\star l} \) and \( \beta_{\star l'} \) satisfy the history variable conditions in the augmented operational semantics so that \( \langle l, \sigma, \beta_{\star l} \rangle \Rightarrow \langle l', \sigma', \beta_{\star l'} \rangle \):

- \( l : c = l : v := e \in P \):
  - Facts from dataflow equations:
    \( \beta_{\star l}[v \mapsto \{l\}] = \beta_{\star l} \) (from transfer function \( f \) for \( l : v := e \in P \))
    \( \beta_{\star l} \leq \beta_{\star l'} \) (because \( l \in \text{pred}(l') \))
  - Prove history variable consistency (prove \( \beta_{\star l}[v \mapsto \{l\}] \leq \beta_{\star l'} \)):
    \( \beta_{\star l}[v \mapsto \{l\}] = \beta_{\star l} \) and \( \beta_{\star l} \leq \beta_{\star l'} \) imply \( \beta_{\star l}[v \mapsto \{l\}] \leq \beta_{\star l'} \).
  - By the program execution rule for \( l : v := e \in P \), with \( l' = \text{next}(l) \), \( \sigma' = \sigma[v \mapsto n] \), \( \pi = \beta_{\star l} \), and \( \pi' = \beta_{\star l'} \): \( \langle l, \sigma, \beta_{\star l} \rangle \Rightarrow \langle l', \sigma', \beta_{\star l'} \rangle \).
- \( l : c = l : \text{if } b \text{ then } g \in P, l : \text{goto } g \in P, l : \text{skip} \in P, l : \text{halt} \in P \):
  - Facts from dataflow equations:
    \( \beta_{\star l} = \beta_{l} \) (from transfer function \( f \) for \( l : c \in P \))
    \( \beta_{l} \leq \beta_{l'} \) (because \( l \in \text{pred}(l') \))
− Prove history variable consistency (prove \( \beta_\ast \leq \beta_{\ast l} \)):
  \( \beta_\ast = \beta_{\ast l} \) and \( \beta_{\ast l} \leq \beta_{\ast l'} \) imply \( \beta_\ast \leq \beta_{\ast l'} \)

− By program execution rule for \( l : \text{if } b \text{ then } g \in P \):
  * if \( \langle l, \sigma, b \rangle \Rightarrow \text{true}, l' = g, \sigma' = \sigma, \pi = \beta_{\ast l} \) and \( \pi' = \beta_{\ast l'}, \langle l, \sigma, \beta_{\ast l} \rangle \Rightarrow \langle l', \sigma', \beta_{\ast l'} \rangle \)
  * if \( \langle l, \sigma, b \rangle \Rightarrow \text{false}, l' = \text{next}(l), \sigma' = \sigma, \pi = \beta_{\ast l} \) and \( \pi' = \beta_{\ast l'}, \langle l, \sigma, \beta_{\ast l} \rangle \Rightarrow \langle l', \sigma', \beta_{\ast l'} \rangle \)

By program execution rule for \( l : \text{goto } g \in P \), with \( l' = g, \sigma' = \sigma, \pi = \beta_{\ast l} \), and \( \pi' = \beta_{\ast l'}, \langle l, \sigma, \beta_{\ast l} \rangle \Rightarrow \langle l', \sigma', \beta_{\ast l'} \rangle \)

− By program execution rule for \( l : \text{skip} \in P \) or \( l : \text{halt} \in P \) with \( l' = \text{next}(l), \sigma' = \sigma, \pi = \beta_{\ast l} \), and \( \pi' = \beta_{\ast l'}, \langle l, \sigma, \beta_{\ast l} \rangle \Rightarrow \langle l', \sigma', \beta_{\ast l'} \rangle\)

\[\blacksquare\]

### 5.2.2 Reaching Definitions Correctness Theorem

We next use the Progress theorem (Theorem 5.9) to illustrate the application of reaching definitions to constant propagation. The theorem states that if all of the definitions of a variable \( v \) that reach a given program point are from assignments of \( v \) to the same constant \( n \), then in any execution of the program at that point, if the value of \( v \) is defined, then the value of \( v \) is \( n \):

**Theorem 5.10.** If \( \langle l_0, \sigma_0 \rangle \rightarrow \ldots \rightarrow \langle l_i, \sigma_i \rangle, v \in \text{dom } \sigma_i \), and \( \forall g \in \beta_{\ast l_i}(v).g : v := n \in P \), then \( \sigma_i(v) = n \).

**Proof:** If \( \langle l_0, \sigma_0 \rangle \rightarrow \ldots \rightarrow \langle l_i, \sigma_i \rangle \), then by Progress (Theorem 5.9), \( \langle l_0, \sigma_0, \beta_{\ast l_0} \rangle \Rightarrow \ldots \Rightarrow \langle l_i, \sigma_i, \beta_{\ast l_i} \rangle \) where \( \beta_{\ast l_0} = \pi_0 \). By Lemma 5.8, \( v \in \text{dom } \sigma_i \) implies \( \exists 0 \leq k < i, l_k : v := e \in P \), \( l_k \in \beta_{\ast l_i} \), and \( (e, \sigma_k) \rightarrow \sigma_i(v) \). Consider \( l_k \). Because \( l_k \in \beta_{\ast l_i}, l_k : v := n \in P \), \( (e, \sigma_k) \rightarrow n \), and \( \sigma_i(v) = n \). \[\blacksquare\]

The defined variables analysis (Section 5.1) is designed to determine if a variable \( v \) is always defined at a given program point. If so, the value of \( v \) at that point is always given by one of the definitions identified by the reaching definitions analysis.

### 6 RELATED WORK

Simulation relations, and techniques for proving that simulation relations exist, have been extensively explored in the context of establishing simulation relations between state machines [33, 34]. The developed theory includes a range of proof techniques and mechanisms, including forward and backward proof techniques with refinement mappings, abstraction functions, and abstraction relations. Prophecy variables were initially developed for the purpose of proving that implementations satisfy specifications via refinement mappings with forward simulations, specifically in the case when the specification makes a choice before the implementation [1]. The addition of prophecy variables to the framework of refinement mappings with history variables and forward simulation proofs enabled a completeness result for the ability to prove trace inclusions of implementations within specifications [1]. It is, of course, known that backward simulation is an alternative to forward simulation with prophecy variables [34]. In general, there are a number of alternatives when choosing a formal framework for proving simulation properties, with the appropriate framework depending on pragmatic issues such as the convenience and conceptual difficulty of working with the concepts in the framework. In general, approaches that reason forward in time seem to be more attractive and intuitive than approaches that reason backward against time, as can be seen, for example, in pedagogical presentations of dataflow analyses, which invariably present forward analyses first, then backward analyses second as a kind of dual of forward analyses [3–6, 11, 28, 37].
Many of the concepts that appear in simulation relation proofs for state machines also appear in the program verification, dataflow analysis, and abstract interpretation literature. For example, history variables were first introduced in the program verification literature [39], abstraction functions, originally introduced in the program verification literature [23], can be seen as a form of refinement mappings, and program analyses can be seen as establishing a simulation relation between an abstract interpretation of the program (which plays the role of the specification) and concrete executions of the program (which play the role of the implementation) [12, 13]. It is also known that, in this context, backward or reverse simulation relations can be used to establish the correspondence between backward analyses (which extract information about the future execution) and program executions [13, 44].

In this paper we introduce prophecy variables to enable forward reasoning about program analysis properties that involve the future execution of the program. To the best of our knowledge, we are the first to introduce prophecy variables for this purpose (here we contrast with the recent use of prophecy variables for program verification [25, 46, 50] as well as the traditional use of prophecy variables for proving forward simulation relations between state machines [1]). In this context prophecy variables enable a unified treatment of forward and backward dataflow analyses and support forward reasoning to establish correctness properties that involve backward analysis results (Theorems 4.10, 4.18, 4.19).

We also exploit aspects of the program analysis context to specialize the more general state machine simulation relation framework to the program analysis context. The result is a simpler and more tractable framework as appropriate in this context:

- Drawing the prophecy and history variables \( \pi \) and the analysis results \( \beta \) from the same lattice eliminates the need to work with an explicit abstraction function or refinement mapping \( \alpha \) to establish a connection between the analysis and program executions. The resulting direct connection between the analysis and the execution eliminates the abstraction function/refinement mapping from proofs that connect the analysis with the execution and from any subsequent correctness proofs involving the analysis results.
- Instead of using a refinement mapping or abstraction function to establish a one-way simulation relation between a specification and an implementation or between concrete and abstract executions of the program, in our approach correct analysis results establish a two-way bisimulation between the standard semantics and the augmented semantics over the analysis results \( \beta \).
- Augmenting the standard operational semantics with prophecy or history variables \( \pi \) eliminates the need to work with traditional instrumented or collecting semantics — the prophecy or history variable updates (which typically parallel the updates to the standard program state \( \sigma \)) directly extract this information as the program executes. It is possible to see the prophecy and history variable mechanism in this context as replacing the combination of a traditional instrumented/collecting semantics plus an abstraction function with a single unified mechanism.

Prophecy variables have recently been applied for program verification in a Hoare program logic based on separation logic [25]. Our purpose is different, specifically to use prophecy variables to enable forward reasoning in the context of backward dataflow analysis algorithms. Instead of using prophecy variables to enable forward reasoning about properties of complex parallel algorithms and data structures, we use prophecy variables to prove properties of algorithms that analyze sequential programs (as well as properties of the analysis results that they produce).

Cobalt enables compiler developers to specify a range of dataflow optimizations (such as constant propagation and partial dead assignment elimination) [31]. Each optimization is specified by
a transformation pattern whose guard specifies a condition over sequences of actions in paths in
the program representation that must hold for the transformation to be legal. Cobalt has separate
constructs for specifying forward and backward optimizations — forward guards reason about
forward properties, backward guards reason about backward properties.
Rhodium implements soundness proofs for dataflow analyses [32]. It largely automates a standard
dataflow analysis setup, with de facto abstraction functions (expressed as predicates over concrete
program states) establishing the connection between concrete program states and dataflow facts
and state extensions (a form of instrumented semantics) to support analyses that extract informa-
tion about the past execution of the program not present in standard concrete program states. Like
Cobalt, Rhodium has separate support for forward and backward analyses; subsequent work on
automatically inferring correct propagation rules supports only forward rules [43].

We note that dataflow analysis and abstract interpretation are large fields with a long history
of technical development. In this work we aspire only to rework the treatment of some of the
basic concepts in the field. We note that integrating backward and forward information via alter-
ating backward and forward analyses is a known technique [14], including transformations of
analyzed systems of Horn clauses to effectively convert combined backward and forward Horn
clause analysis problems into forward analysis problems [7, 26]. It remains to be seen what, if
any, role prophecy variables may usefully play in combining these kinds of backward and forward
analysis problems.

Researchers have also formulated dataflow correctness properties via temporal logic [44, 45],
which can be seen as specifying properties about paths that connect relevant program actions,
such as writing or reading a variable, in the representation of the program. The approach can
therefore eliminate the need for an instrumented operational semantics that explicitly carries in-
formation about the past execution through the program representation. In our approach this kind
of information (when required) is stored explicitly in prophecy and history variables, propagated
locally, and updated by the augmented operational semantics.

The CompCert verified compiler contains an implementation of a generally standard dataflow
analysis framework for supporting traditional compiler optimizations such as constant propaga-
tion and common subexpression elimination [9]. The formulation includes lattices of dataflow
facts, abstraction functions for mapping register values to lattice values, and a forward and back-
ward implementation of Kildall’s fixed point algorithm for solving dataflow equations. Example
dataflow domains record when registers contain constant values (for constant propagation) or the
expressions for register values (for common subexpression elimination).

7 CONCLUSION

Dataflow analysis has been the focus of intensive research for decades. Despite this focus, and
despite conceptual similarities between many problems that arise in program analysis and state
machine refinement proofs, prophecy variables (originally developed to support forward state ma-
chine simulation relation proofs) have seen little to no application to program analysis problems.
By showing how to use prophecy variables to enable forward reasoning for backward dataflow
analyses, as well as developing a streamlined treatment of both backward and forward dataflow
analyses based on prophecy and history variables, we hope to promote the use of these mecha-
nisms as appropriate to productively revisit basic concepts in the field and obtain a more unified
and effective approach to a range of program analysis problems.

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