Abstract The notion of \textit{p-adic multiresolution analysis} (MRA) is introduced. We discuss a “natural” refinement equation whose solution (a refinable function) is the characteristic function of the unit disc. This equation reflects the fact that the characteristic function of the unit disc is a sum of \textit{p} characteristic functions of mutually disjoint discs of radius \textit{p}^{-1}. This refinement equation generates a MRA. The case \textit{p} = 2 is studied in detail. Our MRA is a 2-adic analog of the real Haar MRA. But in contrast to the real setting, the refinable function generating our Haar MRA is 1-periodic, which never holds for real refinable functions. This fact implies that there exist \textit{infinity many different} 2-adic orthonormal wavelet bases in \(L^2(\mathbb{Q}_2)\) generated by the same Haar MRA. All of these new bases are described. We also constructed \textit{infinity many different} multidimensional 2-adic Haar orthonormal wavelet bases for \(L^2(\mathbb{Q}_2^n)\) by means of the tensor product of one-dimensional MRAs. We also study connections between wavelet analysis and spectral analysis of pseudo-differential operators. A criterion for multidimensional \textit{p}-adic wavelets to be eigenfunctions for a pseudo-differential operator (in the Lizorkin space) is derived. We proved also that these wavelets are eigenfunctions of the Taibleson multidimensional fractional operator. These facts create the necessary prerequisites for intensive using our wavelet bases in applications. Our results related to the pseudo-differential operators develop the investigations started in Albeverio et al. (J. Fourier Anal. Appl. 12(4):393–425, 2006).
1 Introduction

According to the well-known Ostrovsky theorem, any nontrivial valuation on the field $\mathbb{Q}$ is equivalent either to the real valuation $|\cdot|$ or to one of the $p$-adic valuations $|\cdot|_p$. This $p$-adic norm $|\cdot|_p$ is defined as follows: if an arbitrary rational number $x \neq 0$ is represented as $x = p^\gamma \frac{m}{n}$, where $\gamma = \gamma(x) \in \mathbb{Z}$ and the integers $m, n$ are not divisible by $p$, then

$$|x|_p = p^{-\gamma}, \quad x \neq 0, \quad |0|_p = 0. \quad (1.1)$$

The norm $|\cdot|_p$ satisfies the strong triangle inequality $|x + y|_p \leq \max(|x|_p, |y|_p)$ and is non-Archimedean. The field $\mathbb{Q}_p$ of $p$-adic numbers is defined as the completion of the field of rational numbers $\mathbb{Q}$ with respect to the norm $|\cdot|_p$.

Thus there are two equal in rights universes: the real universe and the $p$-adic one. The latter has a specific and unusual properties. Nevertheless, there are a lot of papers where different applications of $p$-adic analysis to physical problems, stochastics, cognitive sciences and psychology are studied [4, 5, 8, 12–14, 19, 23, 37–39] (see also the references therein). In view of the Ostrovsky theorem, such investigations are not only of great interest in itself, but lead to applications and better understanding of similar problems in usual mathematical physics.

Since there exists a $p$-adic analysis connected with the mapping $\mathbb{Q}_p$ into $\mathbb{Q}_p$ and an analysis connected with the mapping $\mathbb{Q}_p$ into the field of complex numbers $\mathbb{C}$, one considers two corresponding types of $p$-adic physical models. For the $p$-adic analysis related to the mapping $\mathbb{Q}_p \rightarrow \mathbb{C}$, the operation of differentiation is not defined, and as a result, large number of models connected with $p$-adic differential equations use pseudo-differential operators (see the above-mentioned papers and books). In particular, fractional operator $D^\alpha$ are extensively used in applications. A very important fact that the eigenfunctions of a one-dimensional fractional operator $D^\alpha$ form an orthonormal basis for $L^2(\mathbb{Q}_p)$ was observed by Vladimirov et al. [38]. Later Kozyrev [20] found an orthonormal compactly supported $p$-adic wavelet basis in $L^2(\mathbb{Q}_p)$

$$\theta_{k,j;a}(x) = p^{-j/2} \chi_p \left( p^{-1} k(p^j x - a) \right) \Omega \left( |p^j x - a|_p \right), \quad x \in \mathbb{Q}_p, \quad (1.2)$$

$k = 1, 2, \ldots, p - 1, \quad j \in \mathbb{Z}, \quad a \in I_p = \mathbb{Q}_p / \mathbb{Z}_p$, which is an analog of the real Haar basis. Wavelets (1.2) are also eigenfunctions of the one-dimensional fractional operator $D^\alpha$ (see (6.5)). Multidimensional $p$-adic bases obtained by direct multiplying out the wavelets (1.2) were considered in [1], but, generally speaking, this is not a wavelet basis. The authors of [17, 18], found the following new type of $p$-adic wavelet basis:

$$\theta_{s;j;a}^{(m)}(x) = p^{-j/2} \chi_p \left( s(p^j x - a) \right) \Omega \left( |p^j x - a|_p \right), \quad x \in \mathbb{Q}_p, \quad (1.3)$$
where \( m \geq 1 \) is a fixed positive integer; \( s = p^{-m}(s_0 + s_1 p + \cdots + s_{m-1} p^{m-1}) \), \( s_r = 0, 1, \ldots, p - 1 \), \( r = 0, 1, \ldots, m - 1 \), \( s_0 \neq 0 \); \( j \in \mathbb{Z}, a \in I_p \). In contrast to (1.2), the number of generating wavelet functions is not minimal, for example, if \( p = 2 \), then we have \( 2^{m-1} \) wavelet functions (instead of one wavelet function as it is for (1.2) and for real wavelet bases obtained by the MRA scheme). The Haar wavelet basis (1.2) was extended to the ultrametric spaces by Kozyrev [21, 22], Khrennikov and Kozyrev [15, 16]. It turned out that the bases mentioned above are eigenfunctions of pseudo-differential operators [1–3, 15–18, 20–22]. So, wavelet bases is expected to be useful for application and give a new powerful technique for solving \( p \)-adic problems. Nevertheless, in the cited papers, there was no attempt to create a theory describing common properties of \( p \)-adic wavelet bases and giving methods for their finding.

Benedetto and Benedetto [6], Benedetto [7] suggest a method for finding wavelet bases on the locally compact abelian groups with compact open subgroups, which includes the \( p \)-adic setting. They did not develop the MRA approach, their method is based on the theory of wavelet sets. Only the functions whose Fourier transforms are the characteristic functions of some sets may be wavelet functions obtained by this method [6, Proposition 5.1.].

The main goal of this paper is to start the development the MRA theory in \( L^2(\mathbb{Q}_p) \) and to find new \( p \)-adic wavelet bases. Another goal of this paper is to study connections between wavelet analysis and spectral analysis of pseudo-differential operators.

It’s interesting to compare appearing first wavelets in \( p \)-adic analysis with the history of the wavelet theory in real analysis. In 1910 Haar [11] constructed an orthogonal basis for \( L^2(\mathbb{R}) \) consisting of the dyadic shifts and scales of one piecewise constant function:

\[
\psi_{jn}(t) = 2^{-j/2} \psi(2^{-j} x - n), \quad t \in \mathbb{R}, \ j \in \mathbb{Z}, \ n \in \mathbb{Z},
\]

where

\[
\psi(t) = \begin{cases} 
1, & 0 \leq t < \frac{1}{2}, \\
-1, & \frac{1}{2} \leq t < 1, \\
0, & t \notin [0, 1). 
\end{cases}
\]

A lot of mathematicians actively studied Haar basis (1.4), different kinds of generalizations were introduced, but during almost the whole century nobody could find another wavelet function (a function whose shifts and scales form an orthogonal basis). Only in early nineties a general scheme for construction of wavelet functions was developed. This scheme is based on the notion of multiresolution analysis (MRA in the sequel) introduced by Meyer and Mallat [28, 29, 31]. Smooth compactly supported wavelet functions were found in this way, which has been very important for various engineering applications. In the present paper we introduce MRA in \( L^2(\mathbb{Q}_p) \) and study a concrete MRA for \( p = 2 \) being an analog of Haar MRA in \( L^2(\mathbb{R}) \). The same scheme as in the real setting leads to a Haar basis. It turned out that this Haar basis coincides with Kozyrev’s wavelet system (1.2). However, 2-adic Haar MRA is not an identical copy of its real analog. We proved that, in contrast to Haar MRA in \( L^2(\mathbb{R}) \), there exist infinity many different Haar orthogonal bases for \( L^2(\mathbb{Q}_2) \) generated by the same MRA. It remains to point out that Kozyrev’s wavelet basis (3.14)
(which is one of our bases) can be constructed in the framework of the approach [6, Proposition 5.1.] developed by Benedetto and Benedetto. Other our bases given by (4.4), (4.5) can not be constructed by their method. For example, it is easy to see that the Fourier transform of our generating wavelet-function \( \psi^{(1)} \) defined by (3.17) and its all shifts are not characteristic functions (see Remark 3).

Note that a MRA theory was developed for the Cantor dyadic group [24, 25] and for the half-line with the dyadic addition [9]. These setting may seem very similarly to the \( p \)-adics at first sight, but the existence of infinity many different orthogonal wavelet bases generated by the same MRA does not hold there. This effect is provided by the non-Archimedean metric in \( \mathbb{Q}_2 \).

The paper is organized as follows.

In Sect. 2, we recall some facts from the \( p \)-adics. We shall systematically use the notations and the results from [38] and [10, Chap. II]. Some results from [1] is also intensively used in our paper.

In Sect. 3, a notion of \( p \)-adic MRA is introduced (Definition 1). In Sect. 3.2, we discuss refinement equation (3.7):

\[
\phi(x) = \sum_{r=0}^{p-1} \phi\left(\frac{1}{p} x - \frac{r}{p}\right), \quad x \in \mathbb{Q}_p,
\]

whose solution (a refinable function) \( \phi \) is the characteristic function \( \Omega(|x|_p) \) of the unit disc. The conjecture to use the above equation as a refinement equation was proposed in [17]. The above refinement equation is natural and reflects the fact (see (2.6)) that the characteristic function of the unit disc \( B_0 = \{ x : |x|_p \leq 1 \} \) is represented as a sum of \( p \) characteristic functions of the mutually disjoint discs

\[
B_{-1}(r) = \left\{ x : \left| \frac{x}{p} - \frac{r}{p} \right|_p \leq 1 \right\}
\]

of radius \( p^{-1} \) with the centers at the points \( r, \ r = 0, 1, \ldots, p - 1 \). This geometric fact is the result of the ultrametric structure of the \( p \)-adic field \( \mathbb{Q}_p \). In Sect. 3.3, the 2-adic Haar MRA is constructed. Namely, we proved that the refinable function \( \phi(x) = \Omega(|x|_2) \) generates a MRA, which is an analog of the classical Haar MRA. It is shown that a 2-adic analog of the real wavelet function generated by Haar MRA generates an orthonormal basis (3.14) for \( L^2(\mathbb{Q}_2) \). This basis coincides with Kozyrev’s one (1.2) for \( p = 2 \). We proved that Kozyrev’s basis is not a unique orthonormal wavelet basis generated by 2-adic Haar MRA.

In Sect. 4, infinity many new 2-adic Haar wavelet functions (4.4) generating different orthonormal bases for \( L^2(\mathbb{Q}_2) \) are found. Explicit formulas for Haar wavelet functions \( \psi^{(s)} \), \( s \in \mathbb{N} \), (Theorem 1) are given in Sect. 4.1. All real wavelet functions are described for \( s = 1, 2 \) in Sect. 4.2.

In Sect. 5, we study multivariate Haar bases. A general scheme for construction of 2-adic multidimensional separable MRA is described in Sect. 5.1. According to this scheme, infinity many new separable 2-adic Haar wavelets (5.7) in \( L^2(\mathbb{Q}_2^n) \) are constructed in Sect. 5.2. Our wavelets belong to the Lizorkin space of test functions \( \Phi(\mathbb{Q}_2^n) \).
Section 6 is devoted to the spectral theory of pseudo-differential operators. In Sects. 6.1, 6.2 we recall some facts [1] related to the multidimensional pseudo-differential operators (6.1) and the Taibleson fractional operator (6.4) in the $p$-adic Lizorkin spaces of test functions $\Phi(Q^n_p)$ and distributions $\Phi'(Q^n_p)$. The Lizorkin space $\Phi'(Q^n_p)$ is a “natural” domain for pseudo-differential operators (6.1), since $\Phi'(Q^n_p)$ is invariant under them. It is appropriate to mention here that the class of our operators (6.1) includes the pseudo-differential operators studied in [19, 40, 41]. In Sect. 6.3, we show that spectral analysis of $p$-adic pseudo-differential operators is a wavelets analysis. Namely, we prove Theorems 5, 6 which present a criterion for pseudo-differential operators (6.1) to have multidimensional wavelets (5.5) or (5.7) as eigenfunctions. In particular, according to Corollaries 1, 2, the multidimensional wavelets (5.5) and (5.7) are eigenfunctions of the multidimensional Taibleson fractional operator (6.4).

2 Preliminary Results in $p$-Adics

2.1 $p$-Adics Numbers

Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{C}$ be the sets of positive integers, integers, complex numbers, respectively, $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$. The field of $p$-adic numbers is denoted by $\mathbb{Q}_p$. The canonical form of any $p$-adic number $x \neq 0$ is

$$x = p^\gamma (x_0 + x_1 p + x_2 p^2 + \cdots),$$  \hfill (2.1)

where $\gamma = \gamma(x) \in \mathbb{Z}, x_j = 0, 1, \ldots, p - 1, x_0 \neq 0, j = 0, 1, \ldots$. The series is convergent in the $p$-adic norm (1.1), and one has $|x|_p = p^{-\gamma}$. By means of representation (2.1), the fractional part $\{x\}_p$ of a number $x \in \mathbb{Q}_p$ is defined as follows

$$\{x\}_p = \begin{cases} 0, & \text{if } \gamma(x) \geq 0 \text{ or } x = 0, \\ p^\gamma (x_0 + x_1 p + x_2 p^2 + \cdots + x_{|\gamma|-1} p^{|\gamma|-1}), & \text{if } \gamma(x) < 0. \end{cases}$$ \hfill (2.2)

The space $\mathbb{Q}^n_p := \mathbb{Q}_p \times \cdots \times \mathbb{Q}_p$ consists of points $x = (x_1, \ldots, x_n)$, where $x_j \in \mathbb{Q}_p, j = 1, 2, \ldots, n, n \geq 2$. The $p$-adic norm on $\mathbb{Q}^n_p$ is

$$|x|_p = \max_{1 \leq j \leq n} |x_j|_p, \quad x \in \mathbb{Q}^n_p,$$ \hfill (2.3)

where $|x_j|_p$ is defined by (1.1). Denote by $B^a_n(a) = \{x \in \mathbb{Q}^n_p : |x - a|_p \leq p^{\gamma}\}$ the ball of radius $p^{\gamma}$ with the center at a point $a = (a_1, \ldots, a_n) \in \mathbb{Q}^n_p$ and by $S^a_n(a) = \{x \in \mathbb{Q}^n_p : |x - a|_p = p^{\gamma}\} = B^a_n(a) \setminus B^a_{n-1}(a)$ its boundary (sphere), $\gamma \in \mathbb{Z}$. For $a = 0$, we set $B^0_n(0) = B^0_n$ and $S^0_n(0) = S^0_n$. For the case $n = 1$, we will omit the upper index $n$. It is clear that

$$B^a_n(a) = B^a_1(1) \times \cdots \times B^a_n(a_n),$$ \hfill (2.4)

where $B^a_1(a_j) = \{x_j : |x_j - a_j|_p \leq p^{\gamma}\} \subset \mathbb{Q}_p$ is the disc (one-dimensional ball) of radius $p^{\gamma}$ with the center at a point $a_j \in \mathbb{Q}_p, j = 1, 2, \ldots, n$. Any two balls in $\mathbb{Q}^n_p$ either are disjoint or one contains the other. Every point of a ball is its center.
According to [38, I.3., Examples 1, 2.], a one-dimensional disc $B_{\gamma}$ is represented as a sum of $p^{r-\gamma'}$ mutually disjoint discs $B_{\gamma'}(a)$, $\gamma' < \gamma$:

$$B_{\gamma} = B_{\gamma} \cup \left( \bigcup_{a} B_{\gamma'}(a) \right),$$

(2.5)

where $a = 0$ and $a = a_{-r} p^{-r} + a_{-r+1} p^{-r+1} + \cdots + a_{-\gamma'} p^{-\gamma'-1}$ are the centers of the discs $B_{\gamma'}(a)$, $r = \gamma, \gamma - 1, \gamma - 2, \ldots, \gamma' + 1$, $0 \leq a_{j} \leq p - 1$, $a_{-r} \neq 0$. In particular, the disc $B_{0}$ is represented as a sum of $p$ mutually disjoint discs

$$B_{0} = B_{-1} \cup \left( \bigcup_{r=1}^{p-1} B_{-1}(r) \right),$$

(2.6)

where $B_{-1}(r) = \{ x \in S_{0} : x_{0} = r \} = r + p\mathbb{Z}_{p}$, $r = 1, \ldots, p - 1$; $S_{0} = \{ |x|_{p} = 1 \} = \bigcup_{r=1}^{p-1} B_{-1}(r)$. Coverings (2.5) and (2.6) are called the canonical covering of the discs $B_{0}$ and $B_{\gamma}$, respectively.

2.2 $p$-Adic Functions and Distributions

There exists the Haar measure $dx$ on $\mathbb{Q}_{p}$. This measure is positive, invariant under the shifts, i.e., $d(x + a) = dx$, and normalized by $\int_{|x|_{p} \leq 1} dx = 1$. The invariant measure $dx$ on the field $\mathbb{Q}_{p}$ is extended to an invariant measure $d^{n}x = dx_{1} \cdots dx_{n}$ on $\mathbb{Q}_{p}^{n}$ in the standard way.

The function

$$\chi_{p}(\xi x) = e^{2\pi i \{ \xi x \}_{p}}$$

(2.7)

for every fixed $\xi \in \mathbb{Q}_{p}$ is an additive character of the field $\mathbb{Q}_{p}$, where $\{ \cdot \}_{p}$ is a fractional part (2.2).

A complex-valued function $f$ defined on $\mathbb{Q}_{p}^{n}$ is called locally-constant if for any $x \in \mathbb{Q}_{p}^{n}$ there exists an integer $l(x) \in \mathbb{Z}$ such that

$$f(x + y) = f(x), \quad y \in B_{l(x)}. $$

Let $\mathcal{E}(\mathbb{Q}_{p}^{n})$ and $\mathcal{D}(\mathbb{Q}_{p}^{n})$ be the linear spaces of locally-constant $\mathbb{C}$-valued functions on $\mathbb{Q}_{p}^{n}$ and locally-constant $\mathbb{C}$-valued functions with compact supports (so-called test functions), respectively [38, VI.1.2.]. If $\varphi \in \mathcal{D}(\mathbb{Q}_{p}^{n})$, according to Lemma 1 from [38, VI.1.1.], there exists $l \in \mathbb{Z}$, such that

$$\varphi(x + y) = \varphi(x), \quad y \in B_{l}, \quad x \in \mathbb{Q}_{p}^{n}. $$

The largest of such numbers $l = l(\varphi)$ is called the parameter of constancy of the function $\varphi$. Let us denote by $\mathcal{D}_{N}^{l}(\mathbb{Q}_{p}^{n})$ the finite-dimensional space of test functions from $\mathcal{D}(\mathbb{Q}_{p}^{n})$ having supports in the ball $B_{N}^{n}$ and with parameters of constancy $\geq l$ [38, VI.2.]. The following embedding holds: $\mathcal{D}_{N}^{l}(\mathbb{Q}_{p}^{n}) \subset \mathcal{D}_{N'}^{l'}(\mathbb{Q}_{p}^{n})$, $N \leq N'$, $l \geq l'$. Thus $\mathcal{D}(\mathbb{Q}_{p}^{n}) = \lim_{N \to \infty} \lim_{l \to -\infty} \mathcal{D}_{N}^{l}(\mathbb{Q}_{p}^{n})$. The space $\mathcal{D}(\mathbb{Q}_{p}^{n})$ is a complete
locally convex vector space. According to [38, VI, (5.2')], any function \( \varphi \in \mathcal{D}_N^1(\mathbb{Q}_p^n) \) is represented in the following form

\[
\varphi(x) = \sum_{v=1}^{p^n(N-1)} \varphi(c^v) \Omega(p^{-l}|x - c^v|_p), \quad x \in \mathbb{Q}_p^n, \tag{2.8}
\]

where \( \Omega(p^{-l}|x - c^v|_p) \) are the characteristic functions of the mutually disjoint balls \( B_l(c^v) \), and the points \( c^v = (c_1^v, \ldots, c_n^v) \in B_N^n \) do not depend on \( \varphi \).

Denote by \( \mathcal{D}'(\mathbb{Q}_p^n) \) the set of all linear functionals \( (p\text{-adic distributions}) \) on \( \mathcal{D}(\mathbb{Q}_p^n) \) [38, VI.3.].

Let us introduce in \( \mathcal{D}(\mathbb{Q}_p^n) \) a canonical \( \delta \)-sequence \( \delta_k(x) = p^{nk} \Omega(p^k|x|_p) \), and a canonical \( 1 \)-sequence \( \Omega(p^{-k}|x|_p) \) (the characteristic function of the ball \( B_k^n \), \( k \in \mathbb{Z}, \ x \in \mathbb{Q}_p^n \), where

\[
\Omega(t) = \begin{cases} 
1, & 0 \leq t \leq 1, \\
0, & t > 1, \ t \in \mathbb{R}.
\end{cases}
\tag{2.9}
\]

It is clear that \( \delta_k \to \delta, \ k \to \infty \) in \( \mathcal{D}'(\mathbb{Q}_p^n) \), where \( \delta \) is the Dirac delta function, and \( \Omega(p^{-k}| \cdot |_p) \to 1, \ k \to \infty \) in \( \mathcal{E}(\mathbb{Q}_p^n) \) (see [38, VI.3., VII.1.]).

The Fourier transform of \( \varphi \in \mathcal{D}(\mathbb{Q}_p^n) \) is defined by the formula

\[
F[\varphi](\xi) = \int_{\mathbb{Q}_p^n} \chi_p(\xi \cdot x) \varphi(x) \, d^n x, \quad \xi \in \mathbb{Q}_p^n,
\]

where \( \chi_p(\xi \cdot x) = \chi_p(\xi_1 x_1) \cdots \chi_p(\xi_n x_n) = e^{2\pi i \sum_{j=1}^n (\xi_j x_j)} \); \( \xi \cdot x \) is the scalar product of vectors and \( \chi_p(\xi_j x_j) \) are additive characters (2.7). The Fourier transform is a linear isomorphism \( \mathcal{D}(\mathbb{Q}_p^n) \) into \( \mathcal{D}(\mathbb{Q}_p^n) \). The Fourier transform a distribution \( f \in \mathcal{D}'(\mathbb{Q}_p^n) \) is defined by the relation \( \langle F[f], \varphi \rangle = \langle f, F[\varphi] \rangle \), for all \( \varphi \in \mathcal{D}(\mathbb{Q}_p^n) \).

Let \( A \) be a matrix and \( b \in \mathbb{Q}_p^n \). Then for a distribution \( f \in \mathcal{D}'(\mathbb{Q}_p^n) \) the following relation holds [38, VII, (3.3)]:

\[
F[f(Ax + b)](\xi) = |\det A|_p^{-1} c_A \chi_p\left(-A^{-1}b \cdot \xi\right) F[f(x)](\xi), \tag{2.10}
\]

where \( \det A \neq 0 \). According to [38, IV, (3.1)],

\[
F[\Omega(p^{-k}| \cdot |_p)](x) = \delta_k(x), \quad k \in \mathbb{Z}, \ x \in \mathbb{Q}_p^n. \tag{2.11}
\]

In particular, \( F[\Omega(| \cdot |_p)](x) = \Omega(|x|_p) \).

A space of the \( p \)-adic Lizorkin test functions was introduced in [1, 2] by the formula

\[
\Phi(\mathbb{Q}_p^n) = \{ \varphi : \varphi = F[\psi], \ \psi \in \Psi(\mathbb{Q}_p^n) \},
\]

where \( \Psi(\mathbb{Q}_p^n) = \{ \psi(\xi) \in \mathcal{D}(\mathbb{Q}_p^n) : \psi(0) = 0 \} \). The space \( \Phi(\mathbb{Q}_p^n) \) can be equipped with the topology of the space \( \mathcal{D}(\mathbb{Q}_p^n) \) which makes it a complete space. The Lizorkin space \( \Phi(\mathbb{Q}_p^n) \) admits the following characterization:

\[
\phi \in \Phi(\mathbb{Q}_p^n) \iff \int_{\mathbb{Q}_p^n} \phi(x) \, d^n x = 0, \quad \phi \in \mathcal{D}(\mathbb{Q}_p^n). \tag{2.12}
\]
Let $\Phi'(\mathbb{Q}_p^n)$ and $\Psi'(\mathbb{Q}_p^n)$ denote the topological dual of the spaces $\Phi(\mathbb{Q}_p^n)$ and $\Psi(\mathbb{Q}_p^n)$, respectively. We call $\Phi'(\mathbb{Q}_p^n)$ the space of $p$-adic Lizorkin distributions. We define the Fourier transform of $f \in \Phi'(\mathbb{Q}_p^n)$ and $g \in \Psi'(\mathbb{Q}_p^n)$ respectively by formulas

$$\langle F[f], \psi \rangle = \langle f, F[\psi] \rangle,$$

for all $\psi \in \Psi(\mathbb{Q}_p^n)$, and $\langle F[g], \phi \rangle = \langle g, F[\phi] \rangle$, for all $\phi \in \Phi(\mathbb{Q}_p^n)$. It is clear that $F[\Phi'(\mathbb{Q}_p^n)] = \Psi'(\mathbb{Q}_p^n)$ and $F[\Psi'(\mathbb{Q}_p^n)] = \Phi'(\mathbb{Q}_p^n)$ [1].

Recall that the usual Lizorkin spaces were studied in the excellent papers of Lizorkin [26, 27] (see also [33, 34]).

3 Multiresolution Analysis (One-Dimensional Case)

3.1 $p$-Adic Multiresolution Analysis

Consider the set

$$I_p = \left\{ a = p^{-\gamma}(a_0 + a_1 p + \cdots + a_{\gamma-1} p^{\gamma-1}) : \gamma \in \mathbb{N}; a_j = 0, 1, \ldots, p-1; j = 0, 1, \ldots, \gamma - 1 \right\}. \quad (3.1)$$

This set can be identified with the factor group $\mathbb{Q}_p / \mathbb{Z}_p$.

It is well known that $\mathbb{Q}_p = B_0 \cup \bigcup_{\gamma=1}^{\infty} S_\gamma$, where $S_\gamma = \{ x \in \mathbb{Q}_p : |x|_p = p^\gamma \}$. Due to (2.1), $x \in S_\gamma, \gamma \geq 1$, if and only if $x = x_{-\gamma} p^{-\gamma} + x_{-\gamma+1} p^{-\gamma+1} + \cdots + x_{-1} p^{-1} + \xi$, where $x_{-\gamma} \neq 0, \xi \in B_0$. Since $x_{-\gamma} p^{-\gamma} + x_{-\gamma+1} p^{-\gamma+1} + \cdots + x_{-1} p^{-1} \in I_p$, we have a “natural” decomposition of $\mathbb{Q}_p$ to a union of mutually disjoint discs:

$$\mathbb{Q}_p = \bigcup_{a \in I_p} B_0(a).$$

So, $I_p$ is a “natural” set of shifts for $\mathbb{Q}_p$, which will be used in the sequel.

**Definition 1** A collection of closed spaces $V_j \subset \mathcal{L}^2(\mathbb{Q}_p), j \in \mathbb{Z}$, is called a multiresolution analysis (MRA) in $\mathcal{L}^2(\mathbb{Q}_p)$ if the following axioms hold

(a) $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$;
(b) $\bigcup_{j \in \mathbb{Z}} V_j$ is dense in $\mathcal{L}^2(\mathbb{Q}_p)$;
(c) $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$;
(d) $f(\cdot) \in V_j \iff f(p^{-1}\cdot) \in V_{j+1}$ for all $j \in \mathbb{Z}$;
(e) There exists a function $\phi \in V_0$ such that the system $\{\phi(\cdot - a), a \in I_p\}$ is an orthonormal basis for $V_0$.

The function $\phi$ from axiom (e) is called refinable or scaling. It follows immediately from axioms (d) and (e) that the functions $p^{j/2}\phi(p^{-j}\cdot-a), a \in I_p$, form an orthonormal basis for $V_j, j \in \mathbb{Z}$.

According to the standard scheme (see, e.g., [32, Sect. 1.3]) for construction of MRA-based wavelets, for each $j$, we define a space $W_j$ (wavelet space) as the orthogonal complement of $V_j$ in $V_{j+1}$, i.e.,

$$V_{j+1} = V_j \oplus W_j, \quad j \in \mathbb{Z}, \quad (3.2)$$
where $W_j \perp V_j$, $j \in \mathbb{Z}$. It is not difficult to see that

$$f \in W_j \iff f(p^{-1} \cdot) \in W_{j+1}, \quad \text{for all } j \in \mathbb{Z}$$

(3.3)

and $W_j \perp W_k$, $j \neq k$. Taking into account axioms (b) and (c), we obtain

$$\bigoplus_{j \in \mathbb{Z}} W_j = L^2(\mathbb{Q}_p) \quad (\text{orthogonal direct sum}).$$

(3.4)

If now we find a finite number of functions $\psi(k) \in W_0$, $k = 1, \ldots, K$, such that the system $\{\psi(k)(x - a), a \in I_p, k = 1, \ldots, K\}$ is an orthonormal basis for $W_0$, then, due to (3.3) and (3.4), the system $\{p^{j/2}\psi(k)(p^{-j} \cdot -a), a \in I_p, j \in \mathbb{Z}, k = 1, \ldots, K\}$, is an orthonormal basis for $L^2(\mathbb{Q}_p)$. Such a functions $\psi(k)$, $k = 1, \ldots, K$, are called a wavelet functions and the basis is a wavelet basis.

### 3.2 $p$-Adic Refinement Equation

Let $\phi$ be a refinable function for a MRA. As was mentioned above, the system $\{p^{1/2}\phi(p^{-1} \cdot -a), a \in I_p\}$, is a basis for $V_1$. It follows from axiom (a) that

$$\phi = \sum_{a \in I_p} \alpha_a \phi(p^{-1} \cdot -a), \quad \alpha_a \in \mathbb{C}.$$  

(3.5)

We see that the function $\phi$ is a solution of a special kind of functional equation. Such equations are called refinement equations. Investigation of refinement equations and their solutions is the most difficult part of the wavelet theory in real analysis.

A natural way for construction of a MRA (see, e.g., [32, Sect. 1.2]) is the following. We start with an appropriate function $\phi$ whose integer shifts form an orthonormal system and set $V_j = \text{span}\{\phi(p^{-j} \cdot -a) : a \in I_p\}$, $j \in \mathbb{Z}$. It is clear that axioms (d) and (e) of Definition 1 are fulfilled. Of course, not any such a function $\phi$ provides axiom (a). In the real setting, the relation $V_0 \subset V_1$ holds if and only if the refinable function satisfies a refinement equation. Situation is different in $p$-adics. Generally speaking, a refinement equation (3.5) does not imply the including property $V_0 \subset V_1$. Indeed, we need all the functions $\phi(\cdot - b)$, $b \in I_p$, to belong to the space $V_1$, i.e., the identities $\phi(x - b) = \sum_{a \in I_p} \alpha_{a,b} \phi(p^{-1}x - a)$ should be fulfilled for all $b \in I_p$. Since $p^{-1}b + a$ is not in $I_p$ in general, we can not state that

$$\phi(x - b) = \sum_{a \in I_p} \alpha_{a,b} \phi(p^{-1}x - p^{-1}b - a) \in V_1$$

for all $b \in I_p$. Nevertheless, some refinable equations imply including property, which may happen because of different causes.

The refinement equation reflects some “self-similarity”. The structure of the space $\mathbb{Q}_p$ has a natural “self-similarity” property which is given by formulas (2.5), (2.6). By (2.6), the characteristic function $\Omega(|x|_p)$ of the unit disc $B_0$ is represented as a sum of
\( p \) characteristic functions of the mutually disjoint discs \( B_{-1}(r), r = 0, 1, \ldots, p - 1 \), i.e.,

\[
\Omega(|x|_p) = \sum_{r=0}^{p-1} \Omega(p|x - r|_p) = \sum_{r=0}^{p-1} \left( \frac{|x - r|}{p|_p} \right), \quad x \in \mathbb{Q}_p. \tag{3.6}
\]

Thus, in \( p \)-adics, we have a natural refinement equation (3.5):

\[
\phi(x) = \sum_{r=0}^{p-1} \phi \left( \frac{1}{p} x - \frac{r}{p} \right), \quad x \in \mathbb{Q}_p, \tag{3.7}
\]

whose solution is \( \phi(x) = \Omega(|x|_p) \). This equation is an analog of the refinement equation generating the Haar MRA in real analysis.

3.3 Construction of 2-Adic Haar Multiresolution Analysis

Using the refinement equation (3.7) for \( p = 2 \)

\[
\phi(x) = \phi \left( \frac{1}{2} x \right) + \phi \left( \frac{1}{2} x - \frac{1}{2} \right), \quad x \in \mathbb{Q}_2, \tag{3.8}
\]

and its solution, the refinable function \( \phi(x) = \Omega(|x|_2) \), we construct a 2-adic multiresolution analysis.

Set

\[
V_j = \text{span}\{ \phi(2^{-j} x - a) : a \in I_2 \}, \quad j \in \mathbb{Z}. \tag{3.9}
\]

It is clear that axioms (d) and (e) of Definition 1 are fulfilled and the system \( \{2^{l/2}\phi(2^{-j} \cdot -a), a \in I_p\} \) is an orthonormal basis for \( V_j, j \in \mathbb{Z} \). Since the numbers \( 2^{-1}b, 2^{-1}b + 2^{-1} \) are in \( I_2 \) for all \( b \in I_2 \), it follows from the refinement equation (3.8) that \( V_0 \subset V_1 \). By the definition (3.9) of the spaces \( V_j \), this yields axiom (a). Due to the refinement equation (3.8), we obtain that \( V_j \subset V_{j+1} \), i.e., the axiom (a) from Definition 1 holds.

Note that the characteristic function of the unit disc \( \Omega(|x|_2) \) has a wonderful feature: \( \Omega(|\cdot + \xi|_2) = \Omega(|\cdot|_2) \), for all \( \xi \in \mathbb{Z}_2 \) because the \( p \)-adic norm is non-Archimedean. In particular, \( \Omega(|\cdot|_2) = \Omega(|\cdot|), \) i.e.,

\[
\phi(x \pm 1) = \phi(x), \quad \forall x \in \mathbb{Q}_2. \tag{3.10}
\]

Thus \( \phi \) is a 1-periodic function.

**Proposition 1** The axiom (b) of Definition 1 holds, i.e., \( \bigcup_{j \in \mathbb{Z}} V_j = L^2(\mathbb{Q}_2) \).

**Proof** According to (2.8), any function \( \varphi \in \mathcal{D}(\mathbb{Q}_2) \) belongs to one of the spaces \( \mathcal{D}_N(\mathbb{Q}_2) \), and consequently, is represented in the form

\[
\varphi(x) = \sum_{v=1}^{2^{N-l}} \varphi(c^v)\Omega(2^{-l}|x - c^v|_2), \quad x \in \mathbb{Q}_2, \tag{3.11}
\]
where $c^v \in B_N$, $v = 1, 2, \ldots, 2^{N-I}$; $l = l(\varphi)$, $N = N(\varphi)$; $l \in \mathbb{Z}$. Taking into account that $\Omega(2^{-l}|x-c^v|_2) = \Omega(2^l|x-2^lc^v|_2) = \phi(2^l|x-2^lc^v|)$, we can rewrite (3.11) as

$$
\varphi(x) = \sum_{v=1}^{2^{N-I}} \alpha_v \phi(2^l x - 2^l c^v), \quad x \in \mathbb{Q}_2, \ c^v \in B_N, \ \alpha_v \in \mathbb{C}.
$$

Since any number $2^l c^v$ can be represented in the form $2^l c^v = a^v + b^v$, $a^v \in I_2$, $b^v \in \mathbb{Z}_2$, using (3.10), we have

$$
\varphi(x) = \sum_{v=1}^{2^{N-I}} \alpha_v \phi(2^l x - a^v), \quad x \in \mathbb{Q}_2, \ a^v \in I_2, \ \alpha_v \in \mathbb{C},
$$
i.e., $\varphi(x) \in V_{-l}$. Thus any test function $\varphi$ belongs to one of the space $V_j$, where $j = j(\varphi)$, $j \in \mathbb{Z}$.

Since the space $\mathcal{D}(\mathbb{Q}_2)$ is dense in $L^2(\mathbb{Q}_2)$ [38, VI.2], approximating any function from $L^2(\mathbb{Q}_2)$ by test functions $\varphi \in \mathcal{D}(\mathbb{Q}_2)$, we prove our assertion. \hfill \Box

**Proposition 2** The axiom (c) of Definition 1 holds, i.e., $\cap_{j \in \mathbb{Z}} V_j = \{0\}$.

**Proof** Assume that $\cap_{j \in \mathbb{Z}} V_j \neq \{0\}$. Then there exists a function $f \in \mathcal{D}(\mathbb{Q}_2)$ such that $\|f\| \neq 0$ and $f \in V_j$ for all $j \in \mathbb{Z}$. Hence, due to (3.9), we have $f(x) = \sum_{a \in I_2} c_{ja} \phi(2^{-j} x - a)$ for all $j \in \mathbb{Z}$.

Let $x = 2^{-N}(x_0 + x_1 2 + x_2 2^2 + \cdots)$. If $j \leq -N$, then $2^{-j}x \in \mathbb{Z}_2$, which implies that $|2^{-j}x - a|_2 > 1$ for all $a \in I_2, a \neq 0$. Thus, $\phi(2^{-j} x - a) = 0$ for all $a \in I_2, a \neq 0$, and $\phi(2^{-j} x) = 1$, whenever $j \leq -N$. Since, we have $f(x) = c_{j0}$ for all $j \leq -N$. Similarly, for another $x' = 2^{-N'}(x_0' + x_1' 2 + x_2' 2^2 + \cdots)$, we have $f(x') = c_{j0}$ for all $j \leq -N'$. This yields that $f(x) = f(x')$. Consequently, $f(x) \equiv C$, where $C$ is a constant. However, if $C \neq 0$, then $f \not\in L^2(\mathbb{Q}_2)$. Thus, $C = 0$ what was to be proved. \hfill \Box

According to the above scheme, we introduce the space $W_0$ as the orthogonal complement of $V_0$ in $V_1$.

Set

$$
\psi^{(0)}(x) = \phi\left(\frac{x}{2}\right) - \phi\left(\frac{x - 1}{2}\right).
$$

(3.12)

**Proposition 3** The shift system $\{\psi^{(0)}(\cdot - a), a \in I_2\}$, is an orthonormal basis of the space $W_0$.

**Proof** Let us prove that $W_0 \perp V_0$. It follows from (3.8), (3.12) that

$$
(\psi^{(0)}(\cdot - a), \phi(\cdot - b)) = \int_{\mathbb{Q}_2} \psi^{(0)}(x-a) \phi(x-b) \, dx
$$

$$
= \int_{\mathbb{Q}_2} \left(\phi\left(\frac{x-a}{2}\right) - \phi\left(\frac{x-a - 1}{2}\right)\right) \phi\left(\frac{x-b}{2}\right) \, dx
$$
φ((x^2 − 1/2 − b^2)) dx

for all \( a, b \in I_2 \). Let \( a \neq b \). Since it is impossible \( a = b + 1 \) or \( b = a + 1 \), taking into account that the functions \( 2^{1/2} \phi(2^{-1} \cdot c) \), \( c \in I_2 \) are mutually orthogonal, we conclude that \( (\psi^{(0)}(x - a), \phi(x - b)) = 0 \). If \( a = b \), again due to the orthonormality of the system \( \{2^{1/2} \phi(2^{-1} \cdot c) \}, c \in I_2 \) \( a \notin I_2 \), we have

\[
(\psi^{(0)}(\cdot - a), \phi(\cdot - a)) = \int_{Q^2} \phi^2(\cdot - a) \, dx - \int_{Q^2} \phi^2(\cdot - 1/2 - a/2) \, dx = 0.
\]

Thus, \( \psi^{(0)}(\cdot + a) \perp \phi(\cdot + b) \) for all \( a, b \in I_2 \).

Similarly, computing the integrals

\[
(\psi^{(0)}(\cdot - a), \psi^{(0)}(\cdot - b)) = \int_{Q^2} \psi^{(0)}(x - a) \psi^{(0)}(x - b) \, dx
\]

we establish that the system \( \{\psi^{(0)}(\cdot - a) \}, a \in I_2 \) is orthonormal.

It follows from (3.8) and (3.12) that

\[
\phi\left(\frac{x}{2}\right) = \frac{1}{2}\left(\phi(x) + \psi^{(0)}(x)\right), \quad \phi\left(\frac{x}{2} - 1/2\right) = \frac{1}{2}\left(\phi(x) - \psi^{(0)}(x)\right).
\]

If \( a \in I_2 \), then either \( a = \frac{1}{2} b \), \( b \in I_2 \), or \( a = \frac{1}{2} + \frac{1}{2} b \), \( b \in I_2 \). Hence,

\[
\phi\left(\frac{x}{2} - a\right) = \frac{1}{2}\left(\phi(x - b) + \psi^{(0)}(x - b)\right), \quad b \in I_2,
\]

whenever \( a = \frac{1}{2} b \), and

\[
\phi\left(\frac{x}{2} - a\right) = \frac{1}{2}\left(\phi(x - b) - \psi^{(0)}(x - b)\right), \quad b \in I_2,
\]

whenever \( a = \frac{1}{2} + \frac{1}{2} b \). Since \( \{2^{1/2} \phi(2^{-1} \cdot -a) : a \in I_2\} \) is a basis for \( V_1 \), we obtain that the system \( \{\phi(\cdot - b), \psi^{(0)}(\cdot - b) \}, b \in I_2 \) is also a basis for \( V_1 \), i.e., the functions \( \psi^{(0)}(\cdot - b), b \in I_2 \), form a basis for the space \( W_0 = V_1 \oplus V_0 \).

Thus according to Propositions 1, 2, 3, the collection \( \{V_j : j \in \mathbb{Z}\} \) is a MRA in \( L^2(Q_2) \) and the function \( \psi^{(0)} \) defined by (3.12) is a wavelet function. This MRA
is a 2-adic analog of the real Haar MRA and the wavelet basis generated by $\psi(0)$ is an analog of the real Haar basis. But in contrast to the real setting, the refinable function $\phi$ generating our Haar MRA is periodic with the period 1 (see (3.10)), which never holds for real refinable functions. It will be shown below that due this specific property of $\phi$, there exist infinity many different orthonormal wavelet bases in the same Haar MRA (see Sect. 4).

Due to (2.7), (2.6), the function $\psi(0)$ can be rewritten in the form

$$\psi(0)(x) = \chi_2(2^{-1}x)\Omega(|x|_2).$$

(3.13)

Thus the Haar wavelet basis is

$$\psi_{ja}(x) = 2^{-j/2}\psi(0)(2^j x - a)$$

$$= 2^{-j/2}\chi_2(2^{-1}(2^j x - a))\Omega(|2^j x - a|_2), \quad x \in \mathbb{Q}_2, \ j \in \mathbb{Z}, \ a \in I_2.$$ (3.14)

It is clear that a locally-constant function $\psi_{ja}(x)$ satisfies the relation

$$\int_{\mathbb{Q}_2} \psi_{ja}(x) dx = 0,$$

i.e., according to (2.12), $\psi_{ja}(x)$ belongs to the Lizorkin space $\Phi_1(\mathbb{Q}_2)$, $j \in \mathbb{Z}$, $a \in I_2$.

Remark 1 The Haar wavelet basis (3.14) coincides with the Kozyrev wavelet basis (1.2) for the case $p = 2$. In present paper we restrict ourself by constructing the Haar wavelets only for $p = 2$. Since Haar refinement equation (3.7) was presented for all $p$, a similar construction may be easily realized in the general case. Moreover, it is not difficult to see that Kozytev’s wavelet function $\theta_j(x)$ from (1.2) can be expressed in terms of the refinable function $\phi(x)$ as

$$\theta_k(x) = \chi_p(p^{-1}kx)\Omega(|x|_p) = \sum_{r=0}^{p-1} h_r \phi\left(1/p x - r/p\right), \quad x \in \mathbb{Q}_p,$$

(3.15)

where $h_r = e^{2\pi i (kr/p)}$, $r = 0, 1, \ldots, p - 1$, $k = 1, 2, \ldots, p - 1$.

Remark 2 Because of periodicity (3.10) of the refinable function $\phi$, one can use the shifts $\psi(0)(\cdot + a)$, $a \in I_2$, instead of $\psi(0)(\cdot - a)$, $a \in I_2$.

Now we show that there is another function $\psi^{(1)}$ whose shifts form an orthonormal basis for $W_0$ (different from the basis generated by $\psi(0)$). Set

$$\psi^{(1)}(x) = \frac{1}{\sqrt{2}} \left( \phi\left(\frac{x}{2}\right) + \phi\left(\frac{x}{2} - \frac{1}{2}\right) - \phi\left(\frac{x}{2} - \frac{1}{2}\right) - \phi\left(\frac{x}{2} - \frac{1}{2}\right) \right)$$

(3.16)

and prove that the functions $\psi^{(1)}(\cdot + a)$, $a \in I_2$, are mutually orthonormal. Evidently, it suffices to check that $\psi^{(1)} \perp \psi^{(1)}(\cdot - a)$ for any $a \in I_2 \setminus \{0\}$ (because if $a, b \in I_2$, then either $b - a \in I_2$ or $a - b \in I_2$). If $a \in I_2$, $a \neq 0$, $\frac{1}{2}$, then each of the numbers 0, $\frac{1}{2}, \frac{1}{2}$, $\frac{1}{2}$ differs modulo 1 from each of the numbers $\frac{a}{2}, \frac{a}{2}, \frac{a}{2}, \frac{a}{2}$.
Due to orthonormality of the system \( \{2^{1/2}\phi(2^{-1}x-a), a \in I_2\} \) and (3.10), it follows that \( \psi^{(1)}(\cdot - a) \) is orthogonal to \( \psi^{(1)}(\cdot - a) \) whenever \( a \in I_2, a \neq 0, \frac{1}{2} \). Again due to orthonormality of the system \( \{2^{1/2}\phi(2^{-1}x-a), a \in I_2\} \) and (3.10), we have

\[
(\psi^{(1)}, \psi^{(1)}(\cdot - 2^{-1})) = \int_{Q_2} \psi^{(1)}(x) \psi^{(1)}(x - 2^{-1}) \, dx
\]

\[
= \frac{1}{2} \int_{Q_2} \left( -\phi^2 \left(\frac{x}{2}\right) + \phi^2 \left(\frac{x - 1}{2^2}\right) - \phi^2 \left(\frac{x - 1}{2}\right) \right) \, dx = 0,
\]

\[
(\psi^{(1)}, \psi^{(1)}) = \int_{Q_2} \psi^{(1)}(x) \psi^{(1)}(x) \, dx
\]

\[
= \frac{1}{2} \int_{Q_2} \left( \phi^2 \left(\frac{x}{2}\right) + \phi^2 \left(\frac{x - 1}{2}\right) + \phi^2 \left(\frac{x - 1}{2^2} - \frac{1}{2}\right) \right) \, dx = 1.
\]

Thus we proved that the system \( \{\psi^{(1)}(\cdot + a), a \in I_2\} \) is orthonormal. It is not difficult to see that

\[
\psi^{(1)}(x) = \frac{1}{\sqrt{2}} \left( \psi^{(0)}(x) + \psi^{(0)} \left( x - \frac{1}{2} \right) \right),
\]

\[
\psi^{(1)} \left( x - \frac{1}{2} \right) = \frac{1}{\sqrt{2}} \left( -\psi^{(0)}(x) + \psi^{(0)} \left( x - \frac{1}{2} \right) \right).
\]

This yields that

\[
\psi^{(0)}(x) = \frac{1}{\sqrt{2}} \left( \psi^{(1)}(x) - \psi^{(1)} \left( x - \frac{1}{2} \right) \right).
\]

Since the system \( \{\psi^{(0)}(\cdot - a), a \in I_2\} \) is a basis for \( W_0 \), it follows that the system \( \{\psi^{(1)}(\cdot - a), a \in I_2\} \), is another orthonormal basis for \( W_0 \).

So, we showed that a wavelet basis generated by the Haar MRA is not unique.

**Remark 3** Note that, according to (3.17), (3.13), and (2.10), (2.11), we have

\[
F[\psi^{(1)}](\xi) = \frac{1}{\sqrt{2}} \Omega(|\xi + 1/2|_2)(1 + \chi_2(2^{-1}\xi))
\]

\[
= \frac{1}{\sqrt{2}} \begin{cases} 
1 - i, & \xi \in B_{-1}(3/2), \\
1 + i, & \xi \in B_{-1}(1/2), \\
0, & \text{otherwise}.
\end{cases}
\]

Thus, this function is not a characteristic function. It is clear that the Fourier transform of any \( I_2 \)-shift of \( \psi^{(1)} \) is also not a characteristic function because \( F[\psi^{(1)}(\cdot -
a) \[(\xi) = \chi(a\xi)F[\psi^{(1)}](\xi),\] the function \(\chi(a\xi)\) is not constant on \(B_{-1}(1/2)\) whenever \(|a|_2 > 2\) and
\[
\chi\left(\frac{1}{2}\xi\right) = \begin{cases} 
-i, & \xi \in B_{-1}(3/2), \\
i, & \xi \in B_{-1}(1/2).
\end{cases}
\]
Consequently, the wavelet basis corresponding to the generating wavelet function \(\psi^{(1)}\) can not be constructed by the method of [6].

4 Description of One-Dimensional 2-Adic Haar Bases

4.1 Wavelet Functions

Now we are going to show that there exist infinitely many different wavelet functions \(\psi^{(s)}, s \in \mathbb{N}, \) in \(W_0\) generating different bases for \(L^2(\mathbb{Q}_2)\).

In what follows, we shall write the 2-adic number \(a = 2^{-s}(a_0 + a_1 2 + \cdots + a_{s-1} 2^{s-1}) \in I_2, a_j = 0, 1, \) \(j = 0, 1, \ldots, s - 1,\) in the form \(a = \frac{m}{2^r},\) where \(m = a_0 + a_1 2 + \cdots + a_{s-1} 2^{s-1} .\)

Since the refinable function \(\phi\) of the Haar MRA is 1-periodic (see (3.10)), evidently, the wavelet function \(\psi^0\) has the following property:
\[
\psi^0(x \pm 1) = -\psi^0(x).
\]

Before we prove a general result, let us consider a simple special case. Set
\[
\psi^{(1)}(x) = \alpha_0 \psi^0(x) + \alpha_1 \psi^0\left(x - \frac{1}{2}\right), \quad \alpha_0, \alpha_1 \in \mathbb{C},
\]
and find all the complex numbers \(\alpha_0, \alpha_1\) for which \(\{\psi^{(1)}(x - a), a \in I_2\}\) is an orthonormal basis for \(W_0\).

Taking into account orthonormality of the system \(\{\psi^0(\cdot - a), a \in I_2\}\) and (4.1), we can easily see that \(\psi^{(1)}\) is orthogonal to \(\psi^{(1)}(\cdot - a)\) whenever \(a \in I_2, a \neq 0, 1/2\).

Thus the system \(\{\psi^{(1)}(x - a), a \in I_2\}\) is orthonormal if and only if the system consisting of the functions (4.2) and
\[
\psi^{(1)}\left(x - \frac{1}{2}\right) = -\alpha_1 \psi^0(x) + \alpha_0 \psi^0\left(x - \frac{1}{2}\right)
\]
is orthonormal, which is equivalent to the unitary property of the matrix
\[
D = \begin{pmatrix} \alpha_0 & \alpha_1 \\ -\alpha_1 & \alpha_0 \end{pmatrix}.
\]

It is clear that \(D\) is a unitary matrix whenever \(|\alpha_0|^2 + |\alpha_1|^2 = 1, \alpha_0 \bar{\alpha}_1 = \alpha_1 \bar{\alpha}_0 .\) In this case the system \(\{\psi^{(1)}(\cdot - a), a \in I_2\}\) is a basis for \(W_0\) because \(\{\psi^0(\cdot - a), a \in I_2\}\) is a basis for \(W_0\) and we have
\[
\psi^0(x) = \bar{\alpha}_0 \psi^{(1)}(x) - \bar{\alpha}_1 \psi^{(1)}\left(x - \frac{1}{2}\right).
\]
So, $\psi^{(1)}$ is a Haar wavelet function if and only if $|\alpha_0|^2 + |\alpha_1|^2 = 1$, $\alpha_0\overline{\alpha_1} = \alpha_1\overline{\alpha_0}$. In particular, we obtain (3.17) for $\alpha_0 = \alpha_1 = \frac{1}{\sqrt{2}}$.

**Theorem 1** Let $s = 1, 2, \ldots$. The function

$$
\psi^{(s)}(x) = \sum_{k=0}^{2^s-1} \alpha_k \psi^{(0)}(x - \frac{k}{2^s}), \quad (4.4)
$$

is a wavelet function for the Haar MRA if and only if

$$
\alpha_k = 2^{-s}(-1)^k \sum_{r=0}^{2^s-1} \gamma_r e^{-i\pi \frac{2r+1}{2^s} k}, \quad k = 0, \ldots, 2^s - 1, \quad (4.5)
$$

where $\gamma_r \in \mathbb{C}$, $|\gamma_r| = 1$.

**Proof** Let $\psi^{(s)}$ be defined by (4.4). Since $\{\psi^{(0)}(\cdot - a), a \in I_2\}$ is an orthonormal system (see Sect. 3.3), taking into account (4.1), we see that $\psi^{(s)}$ is orthogonal to $\psi^{(s)}(\cdot - a)$ whenever $a \in I_2$, $a \neq \frac{k}{2^s}, k = 0, 1, \ldots, 2^s - 1$. Thus the system $\{\psi^{(s)}(x - a), a \in I_2\}$ is orthonormal if and only if the system consisting of the functions

$$
\psi^{(s)}\left(x - \frac{r}{2^s}\right)
$$

is orthonormal. Set

$$
\Xi^{(0)} = \left(\psi^{(0)}, \psi^{(0)}\left(\cdot - \frac{1}{2^s}\right), \ldots, \psi^{(0)}\left(\cdot - \frac{2^s - 1}{2^s}\right)\right)^T,
$$

$$
\Xi^{(s)} = \left(\psi^{(s)}, \psi^{(s)}\left(\cdot - \frac{1}{2^s}\right), \ldots, \psi^{(s)}\left(\cdot - \frac{2^s - 1}{2^s}\right)\right)^T.
$$

By (4.6), we have $\Xi^{(s)} = D \Xi^{(0)}$, where

$$
D = \begin{pmatrix}
\alpha_0 & \alpha_1 & \alpha_2 & \ldots & \alpha_{2^s-2} & \alpha_{2^s-1} \\
-\alpha_{2^s-1} & \alpha_0 & \alpha_1 & \ldots & \alpha_{2^s-3} & \alpha_{2^s-2} \\
-\alpha_{2^s-2} & -\alpha_{2^s-1} & \alpha_0 & \ldots & \alpha_{2^s-4} & \alpha_{2^s-3} \\
& & & & & \\
-\alpha_2 & -\alpha_3 & -\alpha_4 & \ldots & \alpha_0 & \alpha_1 \\
-\alpha_1 & -\alpha_2 & -\alpha_3 & \ldots & -\alpha_{2^s-1} & \alpha_0
\end{pmatrix}, \quad (4.7)
$$
Due to orthonormality of \( \{ \psi^{(a)}(\cdot - a), a \in I_2 \} \), the coordinates of \( \Xi^{(s)} \) form an orthonormal system if and only if the matrix \( D \) is unitary.

Let \( u = (\alpha_0, \alpha_1, \ldots, \alpha_{2^s-1})^T \) be a vector and

\[ A = \begin{pmatrix}
0 & 0 & \ldots & 0 & 0 & -1 \\
1 & 0 & \ldots & 0 & 0 & 0 \\
0 & 1 & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \ldots & 1 & 0 & 0 \\
0 & 0 & \ldots & 0 & 1 & 0 \\
\end{pmatrix} \]

be a \( 2^s \times 2^s \) matrix. It is not difficult to see that

\[ A^r u = (-\alpha_{2^s-r}, -\alpha_{2^s-r+1}, \ldots, -\alpha_{2^s-1}, \alpha_0, \alpha_1, \ldots, \alpha_{2^s-r-1})^T, \]

where \( r = 1, 2, \ldots, 2^s - 1 \). Thus \( D = (u, Au, \ldots, A^{2^s-1}u)^T \). Hence, to describe all unitary matrices \( D \), we should find all vectors

\[ u = (\alpha_0, \alpha_1, \ldots, \alpha_{2^s-1})^T \]

such that the system of vectors \( \{ A^r u, r = 0, \ldots, 2^s - 1 \} \) is orthonormal. We have already one such a vector \( u_0 = (1, 0, \ldots, 0, 0)^T \) because the matrix \( D_0 = (u_0, Au_0, \ldots, A^{2^s-1}u_0)^T \) is the identity matrix. Let us prove that the system \( \{ A^r u_0, r = 0, 1, \ldots, 2^s - 1 \} \) is orthonormal if and only if \( u = Bu_0 \), where \( B \) is a unitary matrix such that \( AB = BA \). Indeed, let \( u = Bu_0 \), \( B \) is a unitary matrix, \( AB = BA \).

Then \( A^r u = BA^r u_0, r = 0, 1, \ldots, 2^s - 1 \). Since the system \( \{ A^r u_0, r = 0, 1, \ldots, 2^s - 1 \} \) is orthonormal and the matrix \( B \) is unitary, the vectors \( A^r u, r = 0, 1, \ldots, 2^s - 1 \) are also orthonormal. Conversely, if the system \( A^r u, r = 0, 1, \ldots, 2^s - 1 \) is orthonormal, taking into account that \( \{ A^r u_0, r = 0, 1, \ldots, 2^s - 1 \} \) is also an orthonormal system, we conclude that there exists a unitary matrix \( B \) such that \( A^r u = B(A^r u_0), r = 0, 1, \ldots, 2^s - 1 \). Since \( A^2 u = -u \), \( A^{2^s} u_0 = -u_0 \), we obtain additionally \( A^{2^s} u = BA^{2^s} u_0 \). It follows from the above relations that \( (AB - BA)(A^r u_0) = 0, r = 0, 1, \ldots, 2^s - 1 \). Since the vectors \( A^r u_0, r = 0, 1, \ldots, 2^s - 1 \) form a basis in the \( 2^s \)-dimensional space, we conclude that \( AB = BA \).

Thus all unitary matrices \( D \) are given by

\[ D = (Bu_0, BAu_0, \ldots, BA^{2^s-1}u_0)^T, \]

where \( B \) is a unitary matrix such that \( AB = BA \). It remains to describe all such matrices \( B \). It is not difficult to see that the eigenvalues of \( A \) and the corresponding normalized eigenvectors are respectively

\[ \lambda_r = -e^{i \pi \frac{2^s+1}{2^s}}, \quad (4.8) \]

and \( v_r = ((v_r)_0, \ldots, (v_r)_{2^s-1})^T \), where

\[ (v_r)_l = 2^{-s/2}(-1)^l e^{-i \pi \frac{2^s+1}{2^s}}, \quad l = 0, 1, 2, \ldots, 2^s - 1, \quad (4.9) \]
$r = 0, 1, \ldots, 2^s - 1$. Hence the matrix $A$ can be represented as $A = C\tilde{A}C^{-1}$, where

$$
\tilde{A} = \begin{pmatrix}
\lambda_0 & 0 & \cdots & 0 \\
0 & \lambda_1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \lambda_{2^s-1}
\end{pmatrix}
$$

is a diagonal matrix, $C = (v_0, \ldots, v_{2^s-1})$ is a unitary matrix. It follows that the matrix $B = C\tilde{B}C^{-1}$ is unitary if and only if $\tilde{B}$ is unitary. On the other hand, $AB = BA$ if and only if $\tilde{A}\tilde{B} = \tilde{B}\tilde{A}$. Moreover, since according to (4.8), $\lambda_k \neq \lambda_l$ whenever $k \neq l$, all unitary matrices $\tilde{B}$ such that $\tilde{A}\tilde{B} = \tilde{B}\tilde{A}$, are given by

$$
\tilde{B} = \begin{pmatrix}
\gamma_0 & 0 & \cdots & 0 \\
0 & \gamma_1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \gamma_{2^s-1}
\end{pmatrix},
$$

where $\gamma_k \in \mathbb{C}$, $|\gamma_k| = 1$. Hence all unitary matrices $B$ such that $AB = BA$, are given by $B = CBC^{-1}$. Using (4.9), one can calculate

$$
\alpha_k = (Bu_0)_{k} = (C\tilde{B}C^{-1}u_0)_{k} = \sum_{r=0}^{2^s-1} \gamma_r (v_r)_k (\overline{v}_r)_0
$$

$$
= 2^{-s}(-1)^k \sum_{r=0}^{2^s-1} \gamma_r e^{-i\pi \frac{2s+1}{2^s} r}, \quad k = 0, 1, \ldots, 2^s - 1,
$$

where $\gamma_k \in \mathbb{C}$, $|\gamma_k| = 1$.

It remains to prove that $\{\psi^{(s)}(\cdot - a), a \in I_2\}$ is a basis for $W_0$ whenever $\psi^{(s)}$ is defined by (4.4), (4.5). Since $\{\psi^{(0)}(\cdot - a), a \in I_2\}$ is a basis for $W_0$, it suffices to check that any function $\psi^{(0)}(\cdot - c), c \in I_2$, can be decomposed with respect to the functions $\psi^{(s)}(\cdot - a), a \in I_2$. Any $c \in I_2$, $c \neq 0$, can be represented in the form $c = \frac{r}{2^s} + b$, where $r = 0, 1, \ldots, 2^s - 1$, $|b|_2 \geq 2^{s+1}$. Taking into account that $\Xi^{(0)} = D^{-1}\Xi^{(s)}$, i.e.,

$$
\psi^{(0)}(x - \frac{r}{2^s}) = \sum_{k=0}^{2^s-1} \beta^{(r)}_k \psi^{(s)}(x - \frac{k}{2^s}), \quad r = 0, 1, \ldots, 2^s - 1,
$$

we have

$$
\psi^{(0)}(x - c) = \psi^{(0)}(x - \frac{r}{2^s} - b) = \sum_{k=0}^{2^s-1} \beta^{(r)}_k \psi^{(s)}(x - \frac{k}{2^s} - b),
$$

and $\frac{k}{2^s} + b \in I_2$, $k = 0, 1, \ldots, 2^s - 1$. \hfill \square

Since we have $\int_{\mathbb{Q}_2} \psi^{(0)}(2^j x - a) \, dx = 0$, $j \in \mathbb{Z}$, $a \in I_2$, due to (2.12), any function $\psi^{(s)}(2^j \cdot - a)$ belongs to the Lizorkin space $\Phi(\mathbb{Q}_2)$. 
4.2 Real Wavelet Functions

Using formulas (4.5), one can extract all real wavelet functions (4.4).

Let $s = 1$. According to (4.2), (4.3),

$$\psi^{(1)}(x) = \cos \theta \psi^{(0)}(x) + \sin \theta \psi^{(0)} \left(x - \frac{1}{2}\right)$$

is the real wavelet function.

Let $s = 2$. Set $\gamma_r = e^{i\theta_r}, r = 0, 1, \ldots, 2^s - 1$. It follows from (4.5) that the wavelet function $\psi^{(1)}$ is real if and only if

$$\sin \theta_1 + \sin \theta_2 + \sin \theta_3 + \sin \theta_4 = 0,$$
$$\cos \theta_1 - \cos \theta_2 + \cos \theta_3 - \cos \theta_4 = 0,$$
$$\sin \theta_1 - \sin \theta_2 - \sin \theta_3 + \sin \theta_4 = \cos \theta_1 + \cos \theta_2 - \cos \theta_3 - \cos \theta_4,$$
$$\sin \theta_1 - \sin \theta_2 - \sin \theta_3 + \sin \theta_4 = - (\cos \theta_1 + \cos \theta_2 - \cos \theta_3 - \cos \theta_4).$$

These relations are equivalent to the system

$$\sin \theta_1 = - \sin \theta_4, \quad \cos \theta_1 = \cos \theta_4,$$
$$\sin \theta_2 = - \sin \theta_3, \quad \cos \theta_2 = \cos \theta_3.$$

Thus the real wavelet functions (4.4) with $s = 2$ are given by

$$\psi^{(1)}(x) = \frac{1}{2} (\cos \theta_1 + \cos \theta_2) \psi^{(0)}(x)$$
$$+ \frac{1}{2 \sqrt{2}} (\cos \theta_1 - \cos \theta_2 + \sin \theta_1 + \sin \theta_2) \psi^{(0)} \left(x - \frac{1}{2^2}\right)$$
$$+ \frac{1}{2} (\sin \theta_1 - \sin \theta_2) \psi^{(0)} \left(x - \frac{1}{2}\right)$$
$$+ \frac{1}{2 \sqrt{2}} (\cos \theta_1 - \cos \theta_2 - \sin \theta_1 - \sin \theta_2) \psi^{(0)} \left(x - \frac{1}{2^2} - \frac{1}{2}\right).$$

In particular, for the special cases $\theta_1 = \theta_2 = \theta, \theta_1 = - \theta_2 = \theta, \theta_1 = \theta_2 + \frac{\pi}{2} = \theta$, we obtain respectively the following one-parameter families of real wavelet functions

$$\psi^{(1)}(x) = \cos \theta \psi^{(0)}(x) + \sin \theta \psi^{(0)} \left(x - \frac{1}{2}\right),$$
$$\psi^{(1)}(x) = \cos \theta \psi^{(0)}(x) + \frac{1}{\sqrt{2}} \sin \theta \psi^{(0)} \left(x - \frac{1}{2^2}\right)$$
$$- \frac{1}{\sqrt{2}} \sin \theta \psi^{(0)} \left(x - \frac{1}{2^2} - \frac{1}{2}\right).$$
\[ \psi^{(1)}(x) = \frac{1}{2} (\cos \theta - \sin \theta) \psi^{(0)}(x) + \frac{1}{2\sqrt{2}} (\cos \theta + \sin \theta) \psi^{(0)} \left( x - \frac{1}{2} \right) \]

\[ - \frac{1}{2} (\cos \theta - \sin \theta) \psi^{(0)} \left( x - \frac{1}{2} \right). \]

(4.12)

5 Description of Multidimensional 2-Adic Haar Bases

5.1 \( p \)-Adic Separable Multidimensional MRA

Here we describe multidimensional wavelet bases constructed by means of a tensor product of one-dimensional MRAs. This standard approach for construction of multivariate wavelets was suggested by Meyer [30] (see, e.g., [32, Sect. 2.1]).

Let \( \{V_j^{(v)}\}_{j \in \mathbb{Z}} \), \( v = 1, \ldots, n \), be one-dimensional MRAs (see Sect. 3.1). We introduce subspaces \( V_j \), \( j \in \mathbb{Z} \), of \( L^2(\mathbb{Q}_p^n) \) by

\[ V_j = \bigotimes_{v=1}^n V_j^{(v)} = \operatorname{span}\{ F = f_1 \otimes \ldots \otimes f_n, \ f_v \in V_j^{(v)} \}. \]

(5.1)

Let \( \phi^{(v)} \) be a refinable function of \( v \)-th MRA \( \{V_j^{(v)}\}_{j} \). Set

\[ \Phi = \phi^{(1)} \otimes \ldots \otimes \phi^{(n)}. \]

(5.2)

Since the system \( \{\phi^{(v)}(\cdot - a)\}_{a_v \in I_p} \) is an orthonormal basis for \( V_0^{(v)} \) (axiom (e) of Definition 1) for any \( v = 1, \ldots, n \), it is clear that

\[ V_0 = \operatorname{span}\{ \Phi(\cdot - a) : a = (a_1, \ldots, a_n) \in I_p^n \}, \]

where \( I_p^n = I_p \times \ldots \times I_p \) is the direct product of \( n \) sets \( I_p \), and the system \( \Phi(\cdot - a) \), \( a \in I_p^n \), is an orthonormal basis for \( V_0 \). It follows from Definition (5.1) and axiom (d) of Definition 1 that \( f \in V_0 \) if and only if \( f(2^{-j} \cdot) \in V_j \) for all \( j \in \mathbb{Z} \). Since axiom (a) from Definition 1 holds for any one-dimensional MRA \( \{V_j^{(v)}\}_{j} \), it is easy to see that \( \Phi(p^{-j} \cdot - a) \in V_{j+1} \) for any \( a \in I_p^n \). Thus, \( V_j \subset V_{j+1} \). It is not difficult to check that the axioms of completeness and separability for the spaces \( V_j \) hold. Thus we have the following statement.

**Theorem 2** Let \( \{V_j^{(v)}\}_{j \in \mathbb{Z}}, v = 1, \ldots, n \), be KMAs in \( L^2(\mathbb{Q}_p^n) \). Then the subspaces \( V_j \) of \( L^2(\mathbb{Q}_p^n) \) defined by (5.1) satisfy the following properties:

(a) \( V_j \subset V_{j+1} \) for all \( j \in \mathbb{Z} \);
(b) \( \bigcup_{j \in \mathbb{Z}} V_j \) is dense in \( L^2(\mathbb{Q}_p^n) \);
(c) \( \bigcap_{j \in \mathbb{Z}} V_j = \{0\} \);
(d) \( f(\cdot) \in V_j \iff f(p^{-j} \cdot) \in V_{j+1} \) for all \( j \in \mathbb{Z} \);
(e) The system \( \{\Phi(x - a), a \in I_p^n\} \), is an orthonormal basis for \( V_0 \), where \( \Phi \in V_0 \) is defined by (5.2).
Similarly to Definition 1, the collection of spaces $V_j$, $j \in \mathbb{Z}$, which satisfies conditions (a)–(e) of Theorem 2 is called a multiresolution analysis in $L^2(\mathbb{Q}_p^n)$, the function $\Phi$ from axiom (e) is called refinable.

Next, following to the standard scheme (see, for example, [32, Sect. 2.1]), we define the wavelet spaces $W_j$ as the orthogonal complement of $V_j$ in $V_{j+1}$, i.e.,

$$W_j = V_{j+1} \ominus V_j, \quad j \in \mathbb{Z}.$$

Since

$$V_{j+1} = \bigotimes_{v=1}^{n} V_{j+1}^{(v)} = \bigotimes_{v=1}^{n} (V_j^{(v)} \oplus W_j^{(v)}) = V_j \oplus \bigoplus_{e \subset \{1, \ldots, n\}, e \not= \emptyset} \left( \bigotimes_{\nu \in e} W_j^{(v)} \right) \left( \bigotimes_{\mu \not\in e} V_j^{(\mu)} \right).$$

So, the space $W_j$ is a direct sum of $2^n - 1$ subspaces $W_{j,e}$, $e \subset \{1, \ldots, n\}$, $e \not= \emptyset$. Let $\psi^{(v)}$ be a wavelet function, i.e. a function whose shifts (with respect to $a \in I_p$) form an orthonormal basis for $W_0^{(v)}$. It is clear that the shifts (with respect to $a \in I_p^n$) of the function

$$\Psi_e = \left( \bigotimes_{\nu \in e} \psi^{(v)} \right) \left( \bigotimes_{\mu \not\in e} \phi^{(\mu)} \right), \quad e \subset \{1, \ldots, n\}, \ e \not= \emptyset,$$

form an orthonormal basis for $W_{0,e}$. So, we have

$$L^2(\mathbb{Q}_p^n) = \bigoplus_{j \in \mathbb{Z}} W_j = \bigoplus_{j \in \mathbb{Z}} \left( \bigoplus_{e \subset \{1, \ldots, n\}, e \not= \emptyset} W_{j,e} \right),$$

and the functions $p^{-nj/2} \Psi_e(p^j \cdot + a)$, $e \subset \{1, \ldots, n\}$, $e \not= \emptyset$, $j \in \mathbb{Z}$, $a \in I_p^n$, form an orthonormal basis for $L^2(\mathbb{Q}_p^n)$.

5.2 Construction of Multidimensional 2-Adic Haar MRA

Let us apply the above construction taking the 2-adic Haar MRA as $v$-th one-dimensional multiresolution analysis $(V_j^{(v)})_{j \in \mathbb{Z}}$, $v = 1, \ldots, n$.

To construct multivariate wavelet functions (5.3), we choose $\psi^{(0)}$ as a wavelet function for each one-dimensional MRA. Thus we have the following $2^n - 1$ multi-
dimensional wavelet functions

\[
\Psi_{\{1,\ldots,n\}}^{(0)} = \psi^{(0)}(x_1) \psi^{(0)}(x_2) \cdots \psi^{(0)}(x_{n-1}) \psi^{(0)}(x_n),
\]
\[
\Psi_{\{1,\ldots,n-1\}}^{(0)} = \psi^{(0)}(x_1) \psi^{(0)}(x_2) \cdots \psi^{(0)}(x_{n-1}) \phi(x_n),
\]
\[
\Psi_{\{2,\ldots,n\}}^{(0)} = \phi(x_1) \psi^{(0)}(x_2) \cdots \psi^{(0)}(x_{n-1}) \psi^{(0)}(x_n),
\]
\[
\Psi_1^{(0)} = \psi^{(0)}(x_1) \phi(x_2) \cdots \phi(x_{n-1}) \psi^{(0)}(x_n),
\]
\[
\Psi_n^{(0)} = \phi(x_1) \phi(x_2) \cdots \phi(x_{n-1}) \psi^{(0)}(x_n).
\]

Let \( e \subset \{1, \ldots, n\}, \ e \neq \emptyset \). Denote by \( k_e = ((k_e)^1, \ldots, (k_e)^n) \) the vector whose coordinates are given by

\[
(k_e)_v = \begin{cases} 
1, & v \in e, \\
0, & v \notin e, 
\end{cases} \quad v = 1, \ldots, n.
\]

Since \( \phi(x_v) = \Omega(|x_v|) \) and \( \psi^{(0)}(x_v) = \chi_2(2^{-1}x_v) \Omega(|x_v|), x_v \in \mathbb{Q}_p, v = 1, 2, \ldots, n, \) (see (3.8) and (3.13)), it follows from (2.4), (3.8), that the wavelet function \( \Psi_e^{(0)} \) can be rewritten as

\[
\Psi_e^{(0)}(x) = \chi_2(2^{-1}k_e \cdot x) \Omega(|x|), \quad x = (x_1, \ldots, x_n) \in \mathbb{Q}_2^n.
\]

According to the above consideration, we have we the following statement

**Theorem 3** The system of functions

\[
\Psi_{e;ja}^{(0)}(x) = 2^{-nj/2} \psi^{(0)}_e(2^j(x - a))
\]
\[
= 2^{-nj/2} \chi_2(2^{-1}k_e \cdot (2^j x - a)) \Omega(|2^j x - a|), \quad x \in \mathbb{Q}_2^n.
\]

\( e \subset \{1, \ldots, n\}, \ e \neq \emptyset, \ j \in \mathbb{Z}, \ a \in \mathbb{I}_2^n, \) is an orthonormal basis for \( L^2(\mathbb{Q}_2^n) \).

Now we construct multidimensional wavelet bases using different one-dimensional Haar wavelet bases (see Sect. 4.1). Namely, we apply the construction of Sect. 5.1 taking again the Haar MRA as \( \nu \)-th one-dimensional multiresolution analysis \( \{V_j^{(\nu)}\}_{j \in \mathbb{Z}}, \ \nu = 1, \ldots, n, \) and choosing wavelet functions \( \psi^{(\nu)} \) for construction of multivariate wavelet functions (5.3).
Let \( s = (s_1, \ldots, s_n) \), where \( s_\nu \in \mathbb{N}_0, \nu = 1, 2, \ldots, n \). We have the following \( 2^n - 1 \) wavelet functions

\[
\psi_{[1,\ldots,n]}(x) = \psi(s_1)(x_1)\psi(s_2)(x_2)\cdots\psi(s_{n-1})(x_{n-1})\psi(s_n)(x_n),
\]

\[
\psi_{[1,\ldots,n-1]}(x) = \psi(s_1)(x_1)\psi(s_2)(x_2)\cdots\psi(s_{n-1})(x_{n-1})\phi(x_n),
\]

\[
\psi_{[2,\ldots,n]}(x) = \phi(x_1)\psi(s_2)(x_2)\cdots\psi(s_{n-1})(x_{n-1})\psi(s_n)(x_n),
\]

\[
\psi_{[1]}(x) = \psi(s_1)(x_1)\phi(x_2)\cdots\phi(x_{n-1})\psi(s_n)(x_n),
\]

\[
\psi_{[n]}(x) = \phi(x_1)\phi(x_2)\cdots\phi(x_{n-1})\psi(s_n)(x_n).
\]

Set \( \alpha_1 = \alpha_r \), where \( \alpha_r \) is given by (4.5), \( r = 0, 1, \ldots, 2^s - 1 \), and \( \alpha_0 = 1, \alpha_1^0 = \cdots \alpha_{2^s-1}^0 = 0 \). Since \( \phi(x_\nu) = \Omega(|x_\nu|_2), x_\nu \in \mathbb{Q}_2 \), and \( \psi(s) \) is given by (4.4), (4.5), \( \nu = 1, 2, \ldots, n \), due to (2.4), the wavelet functions \( \psi_e(s) \) can be rewritten as

\[
\psi_e(s)(x) = \sum_{r_1=0}^{2^{s_1}-1} \cdots \sum_{r_n=0}^{2^{s_n}-1} \alpha_{r_1}^{(k_e)} \cdots \alpha_{r_n}^{(k_e)} \psi_e^{(0)}(x - \left( \frac{r_1}{2^{s_1}}(k_e)_1, \ldots, \frac{r_n}{2^{s_n}}(k_e)_n \right)).
\]

where \( \psi_e^{(0)} \) is defined by (5.4), \( e \subset \{1, \ldots, n\}, e \neq \emptyset, x \in \mathbb{Q}_2^n \).

According to the above consideration, we have the following statement.

**Theorem 4** The system of functions

\[
\psi_{e:j,a}(x) = \sum_{r_1=0}^{2^{s_1}-1} \cdots \sum_{r_n=0}^{2^{s_n}-1} \alpha_{r_1}^{(k_e)} \cdots \alpha_{r_n}^{(k_e)} \psi_e^{(0)}(2^j x - a - \left( \frac{r_1}{2^{s_1}}(k_e)_1, \ldots, \frac{r_n}{2^{s_n}}(k_e)_n \right)),
\]

\( x \in \mathbb{Q}_2^n, e \subset \{1, \ldots, n\}, e \neq \emptyset, j \in \mathbb{Z}, a \in I^n_2 \) forms an orthonormal basis for \( L^2(\mathbb{Q}_2^n) \).

It is easy to see that a wavelet function \( \psi_{e:j,a}^{(s)}(x) \) satisfies the relation

\[
\int_{\mathbb{Q}_2^n} \psi_{e:j,a}^{(s)}(x) d^n x = 0, \text{ i.e., in view of (2.12), } \psi_{e:j,a}^{(s)} \in \Phi(\mathbb{Q}_2^n).
\]
6 $p$-Adic Wavelets as Eigenfunctions of Pseudo-Differential Operators

6.1 Pseudo-Differential Operators in the Lizorkin Space

Let us consider a class of pseudo-differential operators introduced and studied in [1].

We introduce a pseudo-differential operator $A$ with a symbol $A \in \mathcal{E}(\mathbb{Q}_p^n \setminus \{0\})$ in the Lizorkin space of the test functions $\Phi(\mathbb{Q}_p^n)$ by the following relation:

$$(A\phi)(x) = F^{-1}[A F[\phi]](x) = \int_{\mathbb{Q}_p^n} \int_{\mathbb{Q}_p^n} \chi_p((y - x) \cdot \xi) A(\xi) \phi(y) d^n \xi d^n y, \quad \phi \in \Phi(\mathbb{Q}_p^n). \quad (6.1)$$

For the Lizorkin distribution $f \in \Phi'(\mathbb{Q}_p^n)$ this operator is defined as

$$Af = F^{-1}[A F[f]] \in \Phi'(\mathbb{Q}_p^n), \quad (6.2)$$

**Lemma 1** [1] *The Lizorkin spaces $\Phi(\mathbb{Q}_p^n)$, $\Phi'(\mathbb{Q}_p^n)$ are invariant under the pseudo-differential operator (6.1).*

6.2 The Taibleson Fractional Operator

In a symbol of operator (6.1) is given by a formula $A(\xi) = |\xi|^\alpha_p$, $\alpha \in \mathbb{C}$, we have a fractional operator $D^\alpha$. For $\alpha \neq -n$ this fractional operator was introduced by Taibleson [35, Sect. 2], [36, III.4.] as operator in the space of distributions $D'(\mathbb{Q}_p^n)$. In [1], the Taibleson fractional operator was defined in the Lizorkin space by definitions (6.1), (6.2):

$$(D^\alpha f)(x) = F^{-1}[| \cdot |^\alpha_p F[f](\cdot)](x), \quad f \in \Phi'(\mathbb{Q}_p^n), \quad \alpha \in \mathbb{C}. \quad (6.3)$$

According to [1], in the Lizorkin space of test functions $\phi \in \Phi(\mathbb{Q}_p^n)$ the relation (6.3) can be rewritten in the following form

$$(D^\alpha \phi)(x) \overset{\text{def}}{=} \kappa_{-\alpha}(x) \ast \phi(x) = \langle \kappa_{-\alpha}(\xi), \phi(\cdot - \xi) \rangle, \quad x \in \mathbb{Q}_p^n, \quad (6.4)$$

where the multidimensional *Riesz kernel*

$$\kappa_\alpha(x) = \begin{cases} |x|^{\alpha-n}_{p}, & \alpha \neq 0, n, \\ \frac{\Gamma^{(n)}_p(\alpha)}{\Gamma^{(\alpha)}_p(\alpha)}, & \alpha = 0, \\ \lim_{\alpha \to n} \kappa_\alpha(x) = \delta(x), & x \in \mathbb{Q}_p^n, \\ \lim_{\alpha \to -n} \kappa_\alpha(x) = -\frac{1}{\log p} \log |x|_p, & \alpha = n, \end{cases}$$

$|x|_p$ is defined by (2.3), $\Gamma^{(n)}_p(\alpha)$ is the $n$-dimensional $\Gamma$-function (for details, see [35], [36, III], [38, VIII], [1]).

According to Lemma 1, the Lizorkin spaces $\Phi(\mathbb{Q}_p^n)$ and $\Phi'(\mathbb{Q}_p^n)$ are invariant under the Taibleson fractional operator $D^\alpha$. Moreover, we have $D^\alpha(\Phi'(\mathbb{Q}_p^n)) = \Phi'(\mathbb{Q}_p^n)$ (see [1]).
6.3 Spectral Analysis of $p$-Adic Pseudo-Differential Operators as Wavelets

Analysis

It is typical that elements of wavelet bases are eigenfunctions of $p$-adic pseudo-differential operators. Kozyrev [20] proved that one-dimensional $p$-adic wavelets (1.2) are eigenfunctions of the Vladimirov fractional operator $D^\alpha$, $\alpha > 0$:

$$D^\alpha \theta_{k; ja}(x) = p^{\alpha(1-j)} \theta_{k; ja}(x), \quad x \in \mathbb{Q}_p.$$  \hspace{1cm} (6.5)

Since wavelets (1.2) satisfy (2.12) they belong to the Lizorkin space $\Phi(\mathbb{Q}_p^n)$, due to Lemma 1, in fact, this statement holds for all $\alpha \in \mathbb{C}$ (see [1]). A criterion for pseudo-differential operators (6.1) to have wavelets (1.2) and (1.3) as eigenfunctions was found in [1, 17]. Now we consider a similar problem for 2-adic wavelets (5.5) and (5.7). The below results generalize the technique developed in our previous paper [1].

**Theorem 5** Let $A$ be a pseudo-differential operator (6.1) with a symbol $A \in \mathcal{E}(\mathbb{Q}_2^n \setminus \{0\})$, $e \subset \{1, \ldots, n\}$, $e \neq \emptyset$, $j \in \mathbb{Z}$, $a \in I_2^n$. Then the function

$$\Psi_{e; ja}(x) = 2^{-nj/2} \chi_2(2^{-1}k_e \cdot (2^j x - a))\Omega(|2^j x - a|_2), \quad x \in \mathbb{Q}_2^n,$$

is an eigenfunction of $A$ if and only if

$$A(2^j(-2^{-1}k_e + \eta)) = A(-2^{-1}k_e), \quad \forall \eta \in \mathbb{Z}_2^n.$$  \hspace{1cm} (6.6)

The corresponding eigenvalue is $\lambda = A(-2^{-1}k_e)$, i.e.,

$$A\Psi_{e; ja}^{(0)} = A(-2^{-1}k_e)\Psi_{e; ja}^{(0)}.$$  \hspace{1cm} (6.7)

**Proof** Combining (2.10), (2.11), (2.4) with (5.5), we obtain

$$F[\Psi_{e; ja}^{(0)}](\xi) = 2^{-nj/2} F[\Psi_{e}^{(0)}(2^j x - a)](\xi) = 2^{nj/2} \chi_2(2^{-j}a \cdot \xi)F[\Psi_{e}^{(0)}(x)](2^{-j}\xi)$$

$$= 2^{nj/2} \chi_2(2^{-j}a \cdot \xi) F\left[ \prod_{v=1}^n \chi_2(2^{-1}(k_e)_v x_v)\Omega(|x_v|_2) \right] (2^{-j}\xi)$$

$$= 2^{nj/2} \chi_2(2^{-j}a \cdot \xi) \prod_{v=1}^n F[\Omega(|x_v|_2)](2^{-1}(k_e)_v + 2^{-j}x_v)$$

$$= 2^{nj/2} \chi_2(2^{-j}a \cdot \xi) \Omega(|2^{-1}k_e + 2^{-j}x|_2).$$  \hspace{1cm} (6.7)

It is clear that $\Omega(|2^{-1}(k_e)_v + \xi|_2) \neq 0$ only if $\xi_k = -2^{-1}(k_e)_v + \eta_v$, where $\eta_v \in \mathbb{Z}_2$, $v = 1, 2, \ldots, n$. This yields $\xi = -2^{-1}k_e + \eta$, $\eta \in \mathbb{Z}_2^n$.

If condition (6.6) is satisfied, then, using (6.1), (6.7), we have

$$A \Psi_{e; ja}^{(0)}(x) = F^{-1}[A(\xi)F[\Psi_{e; ja}^{(0)}](\xi)](x)$$

$$= 2^{nj/2} F^{-1}[A(\xi)\chi_2(2^{-j}a \cdot \xi)\Omega(|2^{-1}k_e + 2^{-j}x|_2)](x).$$  \hspace{1cm} (6.8)
Making the change of variable $\xi \to 2^j (-2^{-1}k_e + \eta)$ and using (2.11), we obtain

$$A\Psi_e^{(0)}_{e; ja}(x) = 2^{-nj/2} \int_{\mathbb{Q}_2^n} \chi_2((-2^j x - a) \cdot (-2^{-1}k_e + \eta))$$

$$\times \Delta(2^j (-2^{-1}k_e + \eta)) \Omega(|\eta|_2) d^n \eta$$

$$= 2^{-nj/2} A(-2j^{-1} j) \chi_2(-2^j x - a) \int_{B_0^n} \chi_2((-2^j x - a) \cdot \eta) d^n \eta$$

$$= A(-2j^{-1} j) \Psi_e^{(0)}_{e; ja}(x).$$

Consequently, $A\Psi_e^{(0)}_{e; ja}(x) = \lambda \Psi_e^{(0)}_{e; ja}(x)$, where $\lambda = A(-2j^{-1} k_e)$.

Conversely, if $A\Psi_e^{(0)}_{e; ja} = \lambda \Psi_e^{(0)}_{e; ja}$, $\lambda \in \mathbb{C}$, taking the Fourier transform from both the left and the right hand sides of this identity and using (6.7), (6.8), we have

$$(A(\xi) - \lambda) \chi_2((-2^{-1} a \cdot \xi)) \Omega(|\xi|_2) = 0, \quad \xi \in \mathbb{Q}_2^n. \quad (6.9)$$

If now $\eta \in \mathbb{Z}_2^n$, $\xi = 2^j (-2^{-1}k_e + \eta)$, then $2^{-1}k_e + 2^{-j} \xi = \eta \in \mathbb{Z}_2^n$. Since $\Omega((2^{-1}k_e + 2^{-j} \xi)_2) \neq 0$ and $\chi_2(2^{-j} a \cdot \xi) \neq 0$, it follows from (6.9) that $\lambda = A(\xi)$. Thus $\lambda = A(2^j (-2^{-1}k_e + \eta))$ for any $\eta \in \mathbb{Z}_2^n$. In particular, $\lambda = A(-2j^{-1} j)$, and, consequently, (6.6) holds.

In particular, we have the following statement.

**Corollary 1** Let $e \subset \{1, \ldots, n\}$, $e \neq \emptyset$, $j \in \mathbb{Z}$, $a \in I^n_2$. Then the function $\Psi_e^{(0)}_{e; ja}$ is an eigenfunction of the fractional operator (6.4). The corresponding eigenvalue is $\lambda = 2^{\alpha(1-j)}$, i.e.,

$$D^\alpha \Psi_e^{(0)}_{e; ja} = 2^{\alpha(1-j)} \Psi_e^{(0)}_{e; ja}, \quad \alpha \in \mathbb{C}.$$

**Proof** The symbol $A(\xi) = |\xi|_2^\alpha$ of the fractional operator $D^\alpha$ satisfies condition (6.6):

$$A(2^j (-2^{-1}k_e + \eta)) = |2^j (-2^{-1}k_e + \eta)|_2^\alpha = 2^{-ja} \left( \max_{1 \leq v \leq n} |2^j (-2^{-1}k_e)_v + \eta|_2 \right)^\alpha$$

$$= 2^{\alpha(1-j)} \left( \max_{1 \leq v \leq n} |2^j (-2^{-1}k_e)_v|_2 \right)^\alpha = A(-2j^{-1} k_e)$$

for all $\eta \in \mathbb{Z}_2^n$, $e \subset \{1, \ldots, n\}$, $e \neq \emptyset$. Here we take into account that $(k_e)_v = 0, 1; v = 1, 2, \ldots, n; (k_e)_1 + \cdots + (k_e)_n \neq 0$. Thus, by Theorem 5, $\Psi_e^{(0)}_{e; ja}$ is an eigenfunction and the corresponding eigenvalue is $\lambda = 2^{\alpha(1-j)}$.

Similarly to Theorem 5 and Corollary 1, using representation (5.7), it is not difficult to prove the following statements.

**Theorem 6** Let $A$ be a pseudo-differential operator (6.1) with a symbol $A(\xi) \in \mathcal{E}(\mathbb{Q}_p^n \setminus \{0\})$, $s = (s_1, \ldots, s_n)$, where $s_v \in \mathbb{N}_0$, $e \subset \{1, \ldots, n\}$, $e \neq \emptyset$, $j \in \mathbb{Z}$, $a \in I^n_2$. 


Then the function $\Psi^{(s)}_{e, ja}$ is an eigenfunction of $A$ if and only if (6.6) holds. The corresponding eigenvalue is $\lambda = A(-2^{j-1} k_e)$, i.e.,

$$A \Psi^{(s)}_{e, ja} = A(-2^{j-1} k_e) \Psi^{(s)}_{e, ja}.$$

Corollary 2 Let $s = (s_1, \ldots, s_n)$, where $s_v \in \mathbb{N}_0$, $e \subset \{1, \ldots, n\}$, $e \neq \emptyset$, $j \in \mathbb{Z}$, $a \in I^n_2$. Then the function $\Psi^{(s)}_{e, ja}$ is an eigenfunction of the fractional operator (6.4). The corresponding eigenvalue is $\lambda = 2^{\alpha(1-j)}$, i.e.,

$$D^\alpha \Psi^{(s)}_{e, ja}(x) = 2^{\alpha(1-j)} \Psi^{(s)}_{e, ja}(x), \quad \alpha \in \mathbb{C}.$$

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