Membrane Topology

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Abstract

We construct membrane homology groups $\mathcal{H}(M)$ associated with each compact connected oriented smooth manifold, and show that $\mathcal{H}(M)$ is matrix graded algebra.

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1 Introduction

In this paper we continue our research of homological quantum field theories HLQFT initiated in [2] focusing our attention in the two dimensional situation. Our definition of HLQFT is based on several sources. The leading actor is the category $\text{Cob}_d$ of $d$ dimensional cobordisms. Objects in $\text{Cob}_d$ are boundaryless compact oriented smooth manifolds. Morphisms in $\text{Cob}_d$, called cobordisms, from $P$ to $Q$ are diffeomorphisms classes of compact oriented $d-1$ dimensional manifolds with boundary $M$ together with a diffeomorphism from $(P^- \sqcup Q) \times [0,1)$ onto an open neighborhood of $\partial M$. Composition is given by gluing of cobordisms along their boundaries.

Gradually it has become clear that the geometric background for quantum fields are monoidal representations of the category of cobordisms, i.e., monoidal functors $F: \text{Cob}_d \to \text{vect}$. Different types of field theories correspond with different data on objects and morphisms in the cobordisms category. For example in full quantum field theories objects and cobordisms are given Riemmanian or Lorentzian metrics. Similarly, in conformal field theory [13] cobordisms are endowed with Riemmanian metrics defined up to conformal equivalence.

Topological quantum field theories, an important tool in modern algebraic topology, are objects in the category of monoidal functors $F: \text{Cob}_d \to \text{vect}$ introduced by Atiyah [1]. Turaev in [15] and [16] introduced the notion of homotopical quantum field theory. Fix a compact connected smooth manifold $M$. Objects in the category $\text{HCob}_d^M$ of homotopically extended cobordisms in $M$ are $d-1$ dimensional smooth compact manifolds $N$ together with a homotopy class of maps $f: N \to M$. A morphism in $\text{HCob}_d^M$ from $N_0$ to $N_1$ is a cobordism $P$ connecting $N_0$ and $N_1$ together with a homotopy class of maps $g: P \to M$ such that its restriction to the boundary gives the homotopy classes associated with the boundary maps. A

1There are additional constrains for a realistic quantum field theory other than those imposed by the fact that they yield monoidal representations of $\text{Cob}_d$.
homotopical quantum field theory is a monoidal functor $\text{HCob}_d^M \to \text{vect}$.

In this note we work within the context of homological quantum field theory HLQFT, i.e., monoidal representations of $\text{Cob}_d^M$ the category of homological extended cobordisms in $M$. Objects in $\text{Cob}_d^M$ are $d-1$ dimensional manifolds $N$ together with a map sending each boundary component of $N$ into an oriented embedded submanifold of $M$. Morphisms are cobordisms together with an homology class of maps\(^2\) from the cobordism into $M$. Composition of morphisms in $\text{Cob}_d^M$ combines the usual composition of cobordisms with a new sort of techniques introduced in the context of string topology.

Chas and Sullivan in their seminal paper \cite{ChasSullivan1994} showed that the homology of the space of free loops on compact connected oriented manifolds, after a degree shift, carries the structure of a Batalin-Vilkovisky algebra. In order to define the loop product, the $\Delta$ operator, and the bracket a sort of intersection product on the homology of infinite dimensional manifolds is required. Cohen and Jones, see \cite{CohenJones1997}, \cite{CohenJones1999}, and \cite{CohenJones2001}, showed that the construction of the intersection product for certain infinite dimensional manifolds reduces to the construction of the umkehr map $F_i : H(N) \to H(M)$, for $F$ a smooth map between infinite dimensional manifolds $N$ and $M$. They proved that the umkehr map exists if $F$ is a regular embedding of finite codimension, which occurs if $F$ fits into a pullback square diagram

$$
\begin{array}{c}
N \\
p \downarrow \\
N
\end{array}
\quad \xrightarrow{f} \quad
\begin{array}{c}
M \\
q \downarrow \\
M
\end{array}
\quad \xrightarrow{\quad q \quad}
\begin{array}{c}
F \\
f \downarrow \\
M
\end{array}
\quad \xrightarrow{\quad F \quad}
\begin{array}{c}
M
\end{array}
$$

where $N$ and $M$ are finite dimensional manifolds, $f$ is an embedding and $q$ is a fiber bundle.

In this paper we use this fundamental fact repeatedly, and refer to it as the Cohen and Jones technique.

Our goal in this paper is to study homological quantum field theories, i.e., monoidal functors $\text{Cob}_d^M \to \text{vect}$ using the Cohen and Jones technique. In Section 2 we formally defined and give an example of HLQFT. In Section 3 we describe one dimensional HLQFT using Cohen and Jones technique and show that one can construct examples from connections on fiber bundles. In Section 4 we associate a two dimensional HLQFT with each $B$-field defined on a connected oriented smooth manifold. In Section 5 we construct membrane homology groups $\mathcal{H}(M)$ associated with each compact connected oriented smooth manifold and show that it is matrix graded algebra equipped with a natural representation.

### 2 Homological quantum field theory

Homology groups of space $M$ are denoted by $H(M)$. Let $H(M) = H(M)[d]$ be the homology of $M$ with degrees shifted by $d \in \mathbb{N}$, i.e., $H_i(M) = H_{i+d}(M)$. Let $D(M)$ be the set of connected

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\(^2\)Each map should constant on a neighborhood of each boundary component, and mapping each boundary component into its associated embedded submanifold of $M$. 

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oriented embedded submanifolds of $M$. The empty set is assumed to be a $d$-dimensional manifold for $d \in \mathbb{N}$.

Objects in the category $\text{Cob}_d^M$ of homologically extended cobordisms are triples $(N, f, <)$ such that $N$ is a compact oriented manifold of dimension $d - 1$, $f : \pi_0(N) \to D(M)$ is a map, and $<$ is a linear ordering on $\pi_0(N)$. We use the notation $\mathcal{T} = \prod_{c \in \pi_0(N)} f(c)$.

For objects $(N_0, f_0, <)$ and $(N_1, f_1, <)$ in $\text{Cob}_d^M$ we set

$$\text{Cob}_d^M((N_0, f_0, <), (N_1, f_1, <)) = \text{Cob}_d^M((N_0, f_0, <), (N_1, f_1, <)) / \sim,$$

where $\text{Cob}_d^M((N_0, f_0, <), (N_1, f_1, <))$ is the set of triples $(P, \alpha, c)$ such that

- $P$ is a compact oriented $d$-manifold with boundary.
- $\alpha : N_0 \bigcup N_1 \times [0, 1) \to \text{im}(\alpha) \subseteq P$ is a diffeomorphism such that $\alpha|_{N_0}$ reverses orientation and $\alpha|_{N_1}$ preserves orientation.
- $c \in H(M_{f_0, f_1}^P) = H(M_{f_0, f_1}^P)[\dim(\mathcal{T}_1)]$, where $M_{f_0, f_1}^P$ denotes the space of smooth maps $g : P \to M$ such that $g$ is constant on a neighborhood of each connected component of $\partial P$.

Triples $(P, \alpha, \xi)$ and $(P', \alpha', \xi')$ in $\text{Cob}_d^M((N_0, f_0, <), (N_1, f_1, <))$ are $\sim$ equivalent if there exists an orientation preserving diffeomorphism $\varphi : P_1 \to P_2$ such that $\varphi \circ \alpha = \alpha'$ and $\varphi_* (\xi) = \xi'$.

**Theorem 1.** ($\text{Cob}_n^M$, $\sqcup$, $\emptyset$) is a monoidal category with product $\sqcup$ and unit $\emptyset$.

**Proof.** Assume we are given morphisms $(P, \alpha, c) \in \text{Cob}_d^M((N_0, f_0, <), (N_1, f_1, <))$ and $(Q, \beta, d) \in \text{Cob}_d^M((N_1, f_1, <), (N_2, f_2, <))$. The composition morphism

$$(P, \alpha, c) \circ (Q, \beta, d) \in \text{Cob}_d^M((N_0, f_0, <), (N_2, f_2, <))$$

is the triple $(P \circ Q, \alpha \circ \beta, c \circ d)$ where $P \circ Q = P \bigcup_{N_1} Q$, $\alpha \circ \beta = \alpha |_{N_2} \sqcup \beta |_{N_0}$, and $c \circ d$ is constructed from the pull back diagram

$$
\begin{array}{ccc}
M_{f_0, f_1}^P \times M_{f_1, f_2}^Q & \xrightarrow{d} & M_{f_0, f_1}^P \times M_{f_1, f_2}^Q \\
\downarrow{\epsilon} & & \downarrow{\epsilon \times \epsilon} \\
M_{f_0, f_1}^{P} & \xrightarrow{i} & M_{f_0, f_1}^{P} \times M_{f_1, f_2}^{Q} \\
\end{array}
$$

as the composition of maps

$$H(M_{f_0, f_1}^P) \otimes H(M_{f_1, f_2}^Q) \xrightarrow{d} H(M_{f_0, f_1}^P \times M_{f_1, f_2}^Q) \xrightarrow{i} H(M_{f_0, f_2}^{P \sqcup N_1 Q}).$$

where $i : M_{f_0, f_1}^P \times M_{f_1, f_2}^Q \to M_{f_0, f_2}^{P \sqcup N_1 Q}$ sends a pair $(x, y)$ into the map $i(x, y) : P \sqcup N_1 Q \to M$ whose restriction to $P$ is $x$ and whose restriction to $Q$ is $y$. Associativity is proved as in the case of string topology [3]. The identity morphism is $(N \times [0, 1], \alpha, 1_N) \in \text{Cob}_d^M((N, f, <), (N, f, <))$, where $1_N$ is defined as follows: consider the map $c : N \to M_{f, f}^{N \times [0, 1]}$ sending $n \in N$ to the map constantly equal to $n$, then $1_N = c_*([N])$. □
Definition 2. \( (\text{Cob}_n^M, \sqcup) \) is the full monoidal subcategory of \( \text{Cob}_n^M \) without unit.

Given monoidal categories \( \mathcal{C} \) and \( \mathcal{D} \) we let \( \text{MFuc}(\mathcal{C}, \mathcal{D}) \) be the category of monoidal functors from \( \mathcal{C} \) to \( \mathcal{D} \).

Definition 3. The category of \( d \) dimensional homological quantum field theories is given by \( \text{HLQFT}_{d}(\mathcal{M}) = \text{MFuc}(\text{Cob}_{n}^{M}, \text{vect}). \) The category of the \( d \) dimensional restricted homological quantum field theories is \( \text{HLQFT}_{d,r}(\mathcal{M}) = \text{MFuc}(\text{Cob}_{n}^{M}, \text{vect}). \)

Let us construct an example of restricted homological quantum field theory.

Theorem 4. The map \( H: \text{Cob}^M_{d,r} \rightarrow \text{vect} \) given on objects by \( H(N, f, <) = H(\mathcal{F}) \) defines a restricted homological quantum field theory.

Proof. Fix \( P \) a cobordism between \( N_0 \) and \( N_1 \). We need a map \( H(M^P_{f_0,f_1}) \rightarrow \text{Hom}(H(\mathcal{F}_0), H(\mathcal{F}_1)) \), or equivalently, an adjoint map \( H(\mathcal{F}_0) \otimes H(M^P_{f_0,f_1}) \rightarrow H(f_1) \). The pullback diagram

\[
\begin{array}{ccc}
\mathcal{F}_0 \times_{\mathcal{F}_0} M^P_{f_0,f_1} & \xrightarrow{d} & \mathcal{F}_0 \times M^P_{f_0,f_1} \\
e & & \downarrow \Delta \\
\mathcal{F}_0 & \xrightarrow{\Delta} & \mathcal{F}_0 \times \mathcal{F}_0 
\end{array}
\]

induces the desired map through the compositions

\[
H(\mathcal{F}_0) \otimes H(M^P_{f_0,f_1}) \xrightarrow{\eta} H(\mathcal{F}_0 \times_{\mathcal{F}_0} M^P_{f_0,f_1}) \xrightarrow{t} H(\mathcal{F}_1).
\]

Units and associativity are constructed as in the previous theorem.

\[\square\]

3 One dimensional homological quantum field theory

In this section we study HLQFT in dimension one using the Cohen and Jones technique. For a manifold \( N \) we let \( \pi_0(N) \) be the set of connected components of \( N \), and we set \( \overline{N} = \prod_{c \in \pi_0(N)} c \). Objects in open string category \([14]\) are embedded submanifolds of \( M \). For \( N_0 \) and \( N_1 \) embedded submanifolds of \( M \) the space of morphisms is \( H(M^I_{N_0,N_1}) \), where \( M^I_{N_0,N_1} \) is the space of smooth maps \( x : I \rightarrow M \) constant on neighborhoods of 0 and 1, respectively. Let \( H(M^I_{N_0,N_1}) = H(M^I_{N_0,N_1})[\text{dim}(N_1)]. \) Composition of morphisms is defined as follows. We have a pullback diagram

\[
\begin{array}{ccc}
M^I_{N_0,N_1} \times N_1 & M^I_{N_1,N_2} & M^I_{N_0,N_1} \times M^I_{N_1,N_2} \\
e & & \downarrow \Delta \\
N_1 & \xrightarrow{\Delta} & N_1 \times N_1 
\end{array}
\]

and a map \( i : M^I_{N_0,N_1} \times N_1 M^I_{N_1,N_2} \rightarrow M^I_{N_0,N_2} \) sending a pair \((x, y)\) to the path that runs trough \( x \) in half the time and then trough \( y \) in the other half. Consider the following map

\( \bullet : H(M^I_{N_0,N_1}) \otimes H(M^I_{N_1,N_2}) \rightarrow H(M^I_{N_0,N_2}) \) given through compositions

\[
H(M^I_{N_0,N_1}) \otimes H(M^I_{N_1,N_2}) \xrightarrow{\eta} H(M^I_{N_0,N_1} \times_{N_1} M^I_{N_1,N_2}) \xrightarrow{i_*} H(M^I_{N_0,N_2}).
\]
Let us now consider HLQFT in dimension one. An object $f$ in $\text{Cob}^M_{1,r}$ is a map $f: [n] \to D(M)$ where For $n \in \mathbb{N}^+$ we set $[n] = \{1, \ldots, n\}$. We use the notation $\mathcal{T} = \prod_{i \in [n]} f(i)$. The space of morphisms in $\text{Cob}^M_{1,r}$ from $f$ to $g$ is

$$\text{Cob}^M_{1,r}(f, g) = \bigoplus_{\sigma \in S_n} \bigotimes_{i=1}^n H(P(f(i), g(\sigma(i)))).$$ 

Composition of morphisms in $\text{Cob}^M_{1,r}$ is given by

$$\begin{align*}
\text{Cob}^M_{1,r}(f, g) \otimes \text{Cob}^M_{1,r}(g, h) &= \bigoplus_{\sigma, \tau \in S_n} \bigotimes_{i=1}^n H(P(f(i), g(\sigma(i)))) \otimes \bigotimes_{j=1}^n H(P(g(j), h(\tau(j)))) \\
&\quad \downarrow^s \\
&\bigoplus_{\sigma, \tau \in S_n} \bigotimes_{i=1}^n H(P(f(i), g(\sigma(i)))) \otimes H(P(g(\sigma(i)), h(\tau(\sigma(i))))) \\
&\quad \downarrow^\bullet \\
&\bigoplus_{\rho \in S_n} \bigotimes_{i=1}^n H(P(f(i), h(\rho(i)))) \\
&\quad \downarrow \\
&\text{Cob}^M_{1,r}(f, h)
\end{align*}$$

the map $s$ permutes order in the tensor products.

Let $G$ a compact Lie group and $\pi: P \to M$ a principal $G$ bundle over $M$. Let $\mathcal{A}_P$ denote the space of connections on $P$ and $\Lambda \in \mathcal{A}_P$. If $\gamma: I \to M$ is a smooth curve on $M$ and $x \in P$ is such that $\pi(x) = \gamma(0)$, then we let $P_{\Lambda}(\gamma, x)$ be $\tilde{\gamma}(1)$ where $\tilde{\gamma}$ is the horizontal lift of $\gamma$ with respect to $\Lambda$ such that $\tilde{\gamma}(0) = x$. Our next goal is to prove the following result.

**Proposition 5.** There is a natural map $H: \mathcal{A}_P \to \text{HLQFT}_{1,r}(M)$.

For each connection $\Lambda \in \mathcal{A}_P$ we construct a functor $H_\Lambda: \text{Cob}^M_{1,r} \to \text{vect}$ given on an object $f$ by

$$H_\Lambda(f) = H(P |_f) = H(P |_f)[\dim(\mathcal{T})],$$

where $P |_{f(i)}$ denotes the restriction of $P$ to $f(i) \subseteq M$ and $P |_f = \prod_{i \in [n]} P |_{f(i)}$. Proposition 5 follows from the next result.

**Theorem 6.** The map $H_\Lambda: \text{Cob}^M_{1,r} \to \text{vect}$ sending $f$ into $H_\Lambda(f)$ defines an one dimensional restricted homological quantum field theory.

**Proof.** In order to define $H_\Lambda: \text{Cob}^M_{1,r}(f, g) \to \text{Hom}(H(P |_f), H(P |_g))$ we construct the adjoint map $H(P |_f) \otimes \text{Cob}^M_{1,r}(f, g) \to H(P |_g)$. The pullback diagram

$$\begin{array}{ccc}
\bigcup_{\sigma \in S_n} \prod_{i=1}^n P |_{f(i)} \times_{f(i)} P(f(i), g(\sigma(i))) & \xrightarrow{d} & \bigcup_{\sigma \in S_n} \prod_{i=1}^n P |_{f(i)} \times P(f(i), g(\sigma(i))) \\
\downarrow^\delta & & \downarrow^\delta \\
\mathcal{T} \times \mathcal{T} & \xrightarrow{\Delta} & \mathcal{T} \times \mathcal{T}
\end{array}$$


together with the map
\[
i: \bigcup_{\sigma \in S_n} \prod_{i=1}^n P \mid_{f(i) \times f(i)} P(f(i), g(\sigma(i))) \xrightarrow{P_s} \bigcup_{\sigma \in S_n} \prod_{i=1}^n P \mid_{g(\sigma(i))}^{s} \rightarrow \prod_{i=1}^n P \mid_{g(i)}
\]
allow us to define the desired map through the compositions

\[
\begin{align*}
H(P \mid f) \otimes \text{Cob}^M_{1,r}(f, g) \downarrow^k \\
\bigoplus_{\sigma \in S_n} \bigotimes_{i=1}^n H(P \mid f_i \times f(i) P(f(i), g(\sigma(i)))) \downarrow^{d!} \\
\bigoplus_{\sigma \in S_n} \bigotimes_{i=1}^n H(P \mid f_i \times P(f(i), g(\sigma(i)))) \downarrow^{i*} \\
H(P \mid g)
\end{align*}
\]

There is a remarkable analogy between HLQFT in dimension one and the algebra of matrices. In [3] this analogy is studied, and several well-known constructions for matrices are generalized to the homological context, among them the notion of Schur algebras. Representations of homological Schur algebras are deeply related with one dimensional quantum field theories. Further examples of HLQFT in dimension one are considered in [4].

4 Two dimensional homological quantum field theory

Let $M$ be a compact oriented smooth manifold. According to Segal [12] a $B$ field, also known as a gerbe with connection, is a complex Hermitian line bundle $L$ on $M^{S^1}$, loops in $M$, equipped with a string connection. A string connection is a rule that assigns to each surface with boundaries $\Sigma$ and each map $y: \Sigma \rightarrow M$ a transport operator $B_y: L_{\partial(\Sigma)} \rightarrow L_{\partial(\Sigma)}$, where the extensions of $L$ to $(M^{S^1})^n$ is defined by the rule $L_{(x_1, \ldots, x_n)} = L_{x_1} \otimes \ldots \otimes L_{x_n}$. The assignment $y \rightarrow B_y$ is assumed to have the following properties

- It is a continuous map taking values in unitary operators. Therefore we have induced maps $B_y: L^1_{\partial(\Sigma)} \rightarrow L^1_{\partial(\Sigma)}$ between the corresponding circle bundles.
- It is transitive with respect to the gluing of surfaces.
- It is parametrization invariant.

Our next goal is to prove the following

**Proposition 7.** There is a natural map from $B$ fields on $M$ to HLQFT$_{2,r}(M)$.

Thus for each $B$ field we need a functor $H_B: \text{Cob}^M_{2,r} \rightarrow \text{vect}$. An object in $\text{Cob}^M_{2,r}$ is an disjoint union of $n$ ordered circles together with a map $f: [n] \rightarrow D(M)$. The functor $H_B$ is defined by the rule $H_B(f) = H(L^1_f) = H(L^1_f)[\dim(f)]$, where $H(L^1_f) = H(L^1_{f(1) \times \ldots \times f(n)})$. The notation $L^1_{f(1) \times \ldots \times f(n)}$ makes sense since $f(1) \times \ldots \times f(n) \subseteq M \times \ldots \times M \subseteq M^{S^1} \times \ldots \times M^{S^1}$.
Theorem 8. The map $H_B: \text{Cob}^M_{2,r} \to \text{vect}$ sending $f$ into $H(L_f^1)$ defines a two dimensional restricted homological quantum field theory.

Proof. Consider the pullback diagram

$$
\begin{array}{ccc}
L_f^1 \times \mathcal{M}^\Sigma_{f,g} & \xrightarrow{d} & L_f^1 \times \mathcal{M}^\Sigma_{f,g} \\
\downarrow e & & \downarrow \pi \times e \\
\triangledown & \xrightarrow{\mathcal{F}} & \mathcal{F} \times \mathcal{F}
\end{array}
$$

For $f$ and $g$ objects in $\text{Cob}^M_{2,r}$, the $B$ field induces a map $L_{f_0}^1 \times \mathcal{M}^\Sigma_{f,g} \xrightarrow{e_B} L_g^1$. We need a map $H(\mathcal{M}^\Sigma_{f,g}) \to \text{Hom}(H(L_f^1), H(L_g^1))$, its adjoint map is given by the compositions

$$H(L_f^1) \otimes H(\mathcal{M}^\Sigma_{f,g}) \xrightarrow{d} H(L_f^1 \times \mathcal{M}^\Sigma_{f,g}) \xrightarrow{e_B^*} H(L_g^1).$$

5 Membrane topology

Let us take a closer look at objects in the category $\text{Cob}^M_{2,r}$. We focus our attention on objects $f: [n] \to D(M)$ such that $f$ is constantly equal to $M$, and so objects are just integers. A morphism from $n$ to $m$ is a homology class of $M\Sigma$ where $\Sigma$ is a compact oriented surface with $n$ incoming boundary components and $m$ outgoing boundary components. We further restrict our attention to connected surfaces $\Sigma$.

Definition 9. For integers $n, m \geq 1$, let $\Sigma^{m}_{n,g}$ be a Riemann surface of genus $g$ with $n$ incoming numbered marked points and $m$ outgoing numbered marked points.

Let $M^{\Sigma^{m}_{n,g}}$ be the space of smooth maps $x: \Sigma^{m}_{n,g} \to M$ constant in a neighborhood of each marked point. If $\Sigma$ is a genus $g$ surface with $n$ incoming boundaries and $m$ outgoing boundaries, then the spaces $M^{\Sigma}$ and $M^{\Sigma^{m}_{n,g}}$ are homotopically equivalent, see Figure 1 and therefore $H(M^{\Sigma}) = H(M^{\Sigma^{m}_{n,g}})$.

![Figure 1: An element in $M^{\Sigma}$ and the corresponding element in $M^{\Sigma^{m}_{n,g}}$.](image)

Let us introduce an algebraic notion.
**Definition 10.** Algebra \((A, m)\) is a matrix graded if \(A = \bigoplus_{n,m=1}^\infty A^n_m\), \(m : A^n_m \otimes A^k_m \to A^k_n\), and \(m |_{A^n_m \otimes A^k_h} = 0\) if \(p \neq m\).

We are ready to define membrane homology groups.

**Definition 11.** Membrane homology of a compact oriented manifold \(M\) is given by \(\mathcal{H}(M) = \bigoplus_{n,m=1}^\infty \mathcal{H}^m_n(M)\), where \(\mathcal{H}^m_n(M) = \bigoplus_{g=0}^\infty \mathcal{H}^m_{n,g}(M)\) and \(\mathcal{H}^m_{n,g}(M) = H(M^{\Sigma_{n,g}})[m \dim M]\).

**Theorem 12.** \(\mathcal{H}(M)\) is a matricially graded algebra.

**Proof.** There is a pullback square diagram

\[
\begin{array}{ccc}
M^{\Sigma_{n,g}} \times M^m & \xrightarrow{d} & M^{\Sigma_{n,g}} \times M^{\Sigma_{m,h}} \\
\downarrow e & & \downarrow e_1 \times e_2 \\
M^m & \xrightarrow{\Delta} & M^m \times M^m
\end{array}
\]

and a natural map \(i : M^{\Sigma_{n,g}} \times M^m M^{\Sigma_{m,h}} \to M^{\Sigma_{n,g+h,m-1}}\) which is better explained by Figures 2, 3 and 4 below, where a pair \((x, y)\) in \(M^{\Sigma_{3,2}} \times M^m M^{\Sigma_{2,1}}\) is shown as well as the induced element \(i(x, y) \in M^{\Sigma_{3,4}}\).

![Figure 2: Element in \(M^{\Sigma_{3,2}}\).](image)

![Figure 3: Element in \(M^{\Sigma_{2,1}}\).](image)

![Figure 4: Element \(i(x, y) \in M^{\Sigma_{3,4}}\).](image)
From the pullback diagram and the map $i$ above, we define a product on $\mathcal{H}(M)$ via the composition of maps
\[
\mathcal{H}(M_{\Sigma_{n,g}}) \otimes \mathcal{H}(M_{\Sigma_{m,h}}) \xrightarrow{d^i} \mathcal{H}(M_{\Sigma_{n,g}} \times M_{\Sigma_{m,h}}) \xrightarrow{i^*} \mathcal{H}(M_{\Sigma_{n+1}g+m+1})
\]

Next we show that membrane homology comes equipped with a natural representation. For a vector space $V$ we let $T_+(V) = \bigoplus_{n=1}^{\infty} V^\otimes n$.

**Theorem 13.** $T_+(\mathcal{H}(M))$ is a representation of $\mathcal{H}(M)$.

**Proof.** The pullback diagram
\[
\begin{array}{ccc}
M^n \times M^n & \xrightarrow{d} & M^n \times M^n \\
\downarrow e & & \downarrow (\times e) \\
M^n & \xrightarrow{\triangle} & M^n \times M^n
\end{array}
\]

and the map $i : M^n \times M^n \to M^m$, induce an action of $\mathcal{H}(M)$ on $T_+(\mathcal{H}(M))$ via the composition of maps
\[
\mathcal{H}(M)^\otimes n \otimes \mathcal{H}(M_{\Sigma_{n,g}}) \xrightarrow{d^i} \mathcal{H}(M^m \times M^n \times M_{\Sigma_{n,g}}) \xrightarrow{i^*} \mathcal{H}(M)^\otimes m
\]

As we have seen membrane topology is an interesting algebraic structure associated with any oriented manifold. It would be interesting to compute it explicitly for familiar spaces, and also to study its relations with other types of two dimensional field theories, such as topological conformal field theories in the sense of [11] and [10].

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