Quantum Computing Approaches for Mission Covering Optimization

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We study quantum computing algorithms for solving certain constrained resource allocation problems we coin as Mission Covering Optimization (MCO). We compare formulations of constrained optimization problems using Quantum Annealing techniques and the Quantum Alternating Operator Ansatz\textsuperscript{[1]} (a generalized algorithm of the Quantum Approximate Optimization Algorithm\textsuperscript{[2]}) on D-Wave and IBM machines respectively using the following metrics: cost, timing, constraints held, and qubits used. We provide results from two different MCO scenarios and analyze results.

I. INTRODUCTION

Quantum computer hardware is quickly developing\textsuperscript{[3]} together with the theory of quantum computing algorithms\textsuperscript{[4]}. Problems historically solved classically have been translated into quantum computing formulations\textsuperscript{[5–8]}. Problems formulated for a particular quantum computing hardware have a completely different formulation for different hardware. In this work we attempt to understand how well constraint optimization can be implemented between two well known quantum computing approaches, (i) quantum annealing, and (ii) the gate-based model, that reflect on two problem formulations: Quadratic Unconstrained Binary Optimization (QUBO) and Quantum Approximate Optimization Algorithm (QAOA). The goal of this paper is to assess the performance of the algorithm when solving a particular problem related to Mission Covering Optimization (MCO).

MCO is a variant of the Generalized Assignment Problem\textsuperscript{[9]}. The goal is to perform a set of missions given constraints on the available resources. Missions are organized efforts to achieve an objective. Examples of mission types include industrial, logistic, and militaristic, where the problem can be anything from getting packages to destinations, to efficiently producing and maintaining equipment devices\textsuperscript{[10, 11]}. To assure that a mission is successful, a set of resources performing different tasks are used. The global solutions of such problem is computationally challenging. It is a complex optimization problem\textsuperscript{[12]}, often non-convex. Heuristic approaches are used to find feasible solutions, but they depend on domain knowledge and problem decomposition, leading to inaccurate solutions. So this category of problems is a good candidate for quantum acceleration. In this effort, we will address an abstraction of a real MCO problem.

We formulate the problem for a quantum annealer, specifically the D-Wave quantum computer, as well as investigate formulation for a gate-based model approach, specifically the IBM quantum computer. Quantum annealer is a type of quantum discrete optimization algorithm that exploits the power of tunneling to reach a global minimum\textsuperscript{[13]}. Gate-based quantum computer solve problems applying a sequence of unitary operators to qubits in superposition. From using results from the QPU and through simulations, we present strengths and weaknesses found with both models, and outline fundamentals of this new type of optimization problem.

This paper is divided into 4 main sections. Section II describes MCO in greater formal detail. Section III outlines two different scenarios of MCO that are tested in this study. Section IV details how each scenario is implemented using three different algorithmic techniques. Section V describes the results of each scenario and compares the different algorithmic techniques by cost, timing, and constraint-holding metrics. In Section VI we summarize and conclude our findings of this work. Lastly, Section VII discusses research directions for future work.

II. FORMALISM FOR MISSION COVERING OPTIMIZATION

The objective of an MCO problem is to allocate a set of resources to missions such that each resource is assigned to at most one mission, and each mission’s requirements are satisfied.

A MCO problem is described by a 7-tuple:

\[(M, R, Q, C, REQ_M, REQ_R, f_{obj})\]

1. \(M\) is the set of missions
2. \(R\) is the set of resources
3. \(Q\) is the set of qualifications
4. \(C\) is the capability function
5. \(REQ_M\) is the mission’s requirements function
6. \(REQ_R\) is the resource’s requirements function
7. \(f_{obj}\) is the objective function that scores problem solutions

In the rest of this section these terms will be explained in detail.
A. Missions

Missions are operations to achieve specific results. They require resources for specific tasks. For example, a mission could be transporting children to school. The resources are the buses and the bus drivers. In manufacturing, a mission could represent the process of building an asset from the design to delivery. It includes design, engineering, sourcing, suppliers, production, control, and packaging. \( N_M \) denotes the number of missions. In an MCO, missions need to be completed at the same time; we denote the mission set as \( M = \{m_1, m_2, \ldots, m_{N_M}\} \).

B. Resources

\( R \) represents the set of all resources. Resources are all the assets needed for mission completion. Resources can be objects, machines or people. For example, a given MCO may define its resources as four planes, four pilots and two engineers. \( R = \{r_1, r_2, \ldots, r_{N_R}\} \) is the set of \( N_R \) resources within the MCO.

C. Qualifications

\( Q \) represents the set of all qualifications. Resources are specialized in the sense that they have qualifications and they can be assigned to the mission that requests a resource with such qualification. For example ‘Can transport’, ‘Can pilot’, and ‘Can troubleshoot’ are qualifications a mission may require. \( Q = \{q_1, \ldots, q_{N_Q}\} \) is the number of \( N_Q \) qualifications in the MCO problem.

D. Capabilities

\( C \) represents the capability function. Each resource is scored for its qualifications. Capabilities are integer number that represent how well a resource can perform a certain qualification. The higher the number, the more qualified the resource is. For example, consider the resource set composed by a senior engineer, intern engineer, and an HR Manager and with capabilities of 2, 1, 0 respectively to a certain qualification titled ‘Troubleshoot carburetor’. This reflects the fact that the senior engineer has a higher capability than the intern engineer to troubleshoot carburetor because he/she is more experienced in the field. Conversely, the HR manager has no qualification capability to ‘Troubleshoot carburetor’, therefore has a score of zero. Capability is the function

\[
C: R \times Q \rightarrow \mathbb{Z}_{\geq 0},
\]

which returns a zero if and only if the resource is not qualified for the specified qualification. When this happens, we say that this resource has no capability for that qualification.

Missions require resources with specific qualifications. For example, mission \( m_1 \) requires \( n \) resources with qualification \( q_1 \) where \( m_1 \in M, n \in \mathbb{N} \) and \( q_1 \in Q \), which represents ‘mission 1 requires 10 resources that can fly a plane’. We assume that resources with higher capability for certain qualifications are chosen.

E. Mission’s Requirements

The mission’s requirement function is described as

\[
\text{REQ}_M : M \times Q \rightarrow \mathbb{N},
\]

and represents the qualification requirement for each mission. This function returns the number of required resources that satisfy the qualification needed for that mission.

F. Resource’s Requirements

The resource’s requirement function is described as

\[
\text{REQ}_R : R \times Q \rightarrow \mathbb{N},
\]

and represents the qualification requirement for each mission. This function returns the number of required resources that satisfy the qualification needed for that resource. This will be used on scenario 2.

G. Solutions and Solution Score

A solution of the MCO is defined as a function that associates resources to missions

\[
f_s : R \rightarrow M,
\]

If \( N_F \) is the number of functions that are the solutions of the MCO, then \( F = \{f_1, f_2, \ldots, f_{N_F}\} \) is the set of all possible solutions. The goal is to find the best solution in which all missions meet requirements as well as possible.

The objective function maps solutions to real numbers, which measures how far off each solution is from meeting requirements. The objective function represents a cost function:

\[
f_{\text{obj}} : f_s \rightarrow \mathbb{R}
\]
s.t.

\[
F_s = \{\{(m, r) \in M \times R \mid m = f_s(r)\} \mid f_s : R \rightarrow M\}.
\]

Therefore, the best solution to an MCO is achieved by minimizing the objective function and retrieving the minimizer.
III. MCO SCENARIOS

The MCO optimization problem involves the parameters (missions, resources, mission require, resource require). From this, there are different metrics of cost that can be used to represent the objective function. Two specific MCO scenarios are formalized and implemented on different quantum computing hardware machines. These scenarios are not computationally difficult to compute classically. However, these scenario’s design is intentional for comparing results in this study, as brute force methods for finding the absolute best cost-minimizers can be used to without being throttled by the exponential time complexity since the scenario’s are designed to permutationally symmetric. The primary focus of each of these readily solvable MCO scenarios is to identify how well each algorithm implementation performs when constraints are introduced. Results are compared in terms of cost, timing, and constraints met. In the general MCO formulation, resources can be assigned to at most one mission, so that resources could be unused. In this paper, in both scenarios, it is assumed that each resource is assigned to one and only one mission. This is done by introducing an additional mission that includes the unused resources. The following subsections describe each scenario.

A. Scenario 1: Primary and Secondary Resources

The first scenario involves two categories of resources: primary and secondary. Primary resources are ready to be used for mission completion. Secondary resources are allocated when primary resources are not able to perform their duty. Primary resources that cannot be allocated to missions are removed from the set of primary resources. For example, the mission covering optimization solution shown in figure Fig. 1 is composed by three missions requiring three, two and two pilots respectively for mission completion. Suppose in the planning phase that seven pilots were allocated to cover the three missions; they are on-duty (in the primary resource set), and three other pilots are on-call (in the secondary resource set). Suppose at a certain point in time, two of the pilots got sick and they were removed from the resource group. This is an emergency, unforeseen situation that requires to re-run an optimization algorithm to cover the missions including secondary resources. The pilots are pulled from the set of five pilots on-duty, ready to cover the missions and two pilots from the secondary resource set should be also included. The challenge is to find the best allocation of pilots to missions using all the pilots on-duty and two pilots on-call. The goal of our effort is to use quantum computing to solve this problem and analyse results from different quantum computing hardware implementations.

The following outlines all the rules in this specific scenario:

1. The set of missions in the problem is \( M = \{m_1, \ldots, m_{N_m} - 1, U\} \).
2. The set of resources is \( R = \{r_1, \ldots, r_{N_r}\} \).
3. There only exists one qualification, which is the ability to fly a plane, \( Q = \{q_1\} \).
4. Resources can have a capability of 1 or 2. So the capability function is then \( C: R \times Q \rightarrow \{1, 2\} \). This is a way to represent how ready the resource is to be allocated to a mission. It can be though of as an ordering for the allocation of resources: resources with capability 2 should be allocated before ones with 1. Resources that have capability 2 are referred to as primary resources, which are assigned to missions first. Resources that have capability 1 are referred to as secondary resources, which are assigned to missions only if primary resources become unavailable.
5. \( REQ_M \) is the mission’s requirement. For example:
   - \( REQ_M(m_1, q_1) = 3 \).
   - \( REQ_M(m_2, q_1) = 2 \).
   - \( REQ_M(m_3, q_1) = 2 \).
6. There are no resource requirements. Therefore, the resource requirement function is \( REQ_R: R \times Q \rightarrow \{0\} \).
7. \( f_{obj} \) is the function used to score the different problem solutions in terms of cost

The mission cost includes two parts:

- The mission cost represents how many of the mission’s requirements are not met. It is a penalty introduced every time a mission is not able to accomplish its goal due to a lack of resources.
- The precedence cost measures how well each solution allocates resources with higher capabilities before others to missions. In the specific example, it means that it is desirable that primary pilots are allocated to missions before secondary pilots.

As discussed previously, MCO covers a broad spectrum of problems distinguishable only by it’s objective function. In the scenario described here, the objective function reflects the cost which is the sum of the mission cost and precedence cost. All solutions are measured in terms of the cost in order to find the optimal solution. This scenario can be described by using a matrix arrangement of Boolean variables, shown in Fig. 1. Each row of the table in the figure represents a mission, and each column represents a resource.

The final row of the table represents an artificial mission created for the purpose of having all the resources utilized. When resources are not allocated to any of the
other missions, it has to be allocated this special mission. It will be referred to as the unallocated mission (U mission for short) and has no mission cost associated.

A separate column specifies the required amount of qualifying resources for each mission. Since there is only one qualification in the scenario, this number just represents how many resources should be allocated to the mission. The example in Fig. 1 shows a solution where each mission satisfies its requirements, but only after using all primary resources before it uses any of the secondary ones. Since each resource can only be assigned to one mission, no more than one Boolean variable may be true in a single column of the matrix representation of a MCO. In general, the column constraint assures that a resource can be assigned at most to one mission. This is a hard constraint.

B. Scenario 2: The Buddy System

This scenario was designed to see how different algorithms deal with additional constraints. There are two groups of resources of different types. If a resource from one group is assigned to a mission, it is required that a resource from the other group joins the same mission. Consider the previous example where a set of planes and pilots must be allocated to a set of aerial missions. A pilot and plane are required to complete a mission. And every time a plane is chosen, a pilot needs to be chosen. From an application standpoint, this scenario highlights the modeling of resource dependencies (planes depend on pilots for allocation). This scenario contains an additional row constraint. The list of constraints is as follows:

- Column constraint: a resource can be assigned at most to one mission. This is the hard constraint of the problem.
- Row constraint: if a type of resource is chosen then a resource of the second type must also be chosen. We call it the buddy constraint. This is the additional constraint added in this scenario.

The entire set of rules for this scenario is as follows:

1. The set of missions in the problem is $M = \{m_1, \ldots, m_{N_m-1}, U\}$.
2. The set of resources is $R = \{r_1, \ldots, r_{N_r}\}$.
3. There are two qualifications $Q = \{q_1, q_2\}$, which means that resources are divided in two groups: $R_1$ and $R_2$.
4. Resource’s capability is $C: R \times Q \rightarrow \{0, 1\}$, which means that all resources that are qualified have the same capability. Resources have only one qualification:
   - The set of all resources that have capability 1 for qualification $q_1$ is notated as $R_1$.
   - The set of all resources that have capability 1 for qualification $q_2$ is notated as $R_2$.
   - The sets $R_1$ and $R_2$ do not contain the same resource: $R_1 \cap R_2 = \emptyset$.
   - The sets $R_1$ and $R_2$ contain all resources: $R_1 \cup R_2 = R$.
   - The number of resources in $R_1$ and $R_2$ are the same: $|R_1| = |R_2|$.
5. Missions require resources with qualification $q_1$ and not $q_2$. Therefore, the mission require function can be formally described as: $REQ_M: M \times \{q_1\} \rightarrow \mathbb{Z}_{\geq 0}$.
6. The resource’s requirement is called the buddy requirement. Every resource when allocated requires another resource with the opposite qualification.
   - $REQ_R(r, 1) = 0 \land REQ_R(r, 2) = 1 \ \forall r \in R_1$
   - $REQ_R(r, 1) = 1 \land REQ_R(r, 2) = 0 \ \forall r \in R_2$
7. As in the previous scenario, the objective function measures the total mission cost.

The mission cost formulation is exactly the same as the secondary resources scenario. However, the way we view the problem in its matrix representation is slightly different, as shown in Fig. 2.

The primary difference between this solution and the prior scenario’s solution is that now there are buddy constraints present along the rows of the matrix (for this reason it’s also called the row constraint). For example, in Fig. 2, the first row related to the first mission has two resources allocated from $R_1$. In order for the buddy constraint to not be violated, two resources from

\[\begin{array}{cccccc}
\text{Missions} & 1 & 2 & 3 & 4 & 5 \\
\text{Primary} & & & & & \\
\text{Secondary} & & & & & \\
1 & 3 & \circ & \circ & \circ & \circ \\
2 & 2 & \circ & \circ & \circ & \circ \\
3 & 2 & \circ & \circ & \circ & \circ \\
U & - & \circ & \circ & \circ & \circ \\
\end{array}\]

FIG. 1. Matrix view for Solution: This represents a table representing the solution of a problem with three missions, five primary resources and three secondary resources. A purple circle symbolize a Boolean variable $x_{m,r} \in \{0, 1\}$ where $m$ represents the mission (row) while $r$ represents the resource (column). A circle filled in with orange represents $x_{m,r} = 1$, and the fact that resource $r$ is allocated to mission $m$.\[\text{Resources}\]
All three approaches use similar means for encoding the objective function but differ in the way they implement their constraints and what machines support them.

1. Quadratic Unconstrained Binary Optimization

Quantum Annealing is an quantum optimization methodology performed on a specific implementation of quantum computers such as D-Wave quantum annealing machines. Quantum Annealing is an adiabatic quantum computing technique that capitalizes on a unique feature of quantum mechanics, quantum tunneling, which is the capability to surf and dive though a energy landscape until it hits a minimum energy level[13]. This property of quantum mechanics is engineered to solve Quadratic Unconstrained Binary Optimization problems[15]. QUBO, is a type of optimization where the solution is a binary vector that minimizes an objective function described with terms up to a quadratic degree. These problems are unconstrained, but there are methods to incorporate the behavior of constraints[16] into the cost function. To encode an MCO onto a quantum annealing machine, we must translate the objective function into a QUBO.

2. Quantum Alternating Operator Ansatz

Quantum Alternating Operator Ansatz QAOA is method for solving combinatorial optimization problems on NISQ devices[2]. The algorithm is supported by gate-based quantum computers such as the IBM, Rigetti, IonQ or Xanadu machines. QAOA In this effort we focus on implementing the problem on an IBM quantum computer[17]. At the current time, IBM’s Qiskit implementation of QAOA uses Variational Quantum Eigensolver (VQE) to find the expectance of a parameterized ansatz eigenstate. This quantity is then used to calculate the minimum of the cost function which is embedded in a cost Hamiltonian $H_C$. Like QA, it is limited to solving binary quadratic problems[18].

3. Quantum Alternating Operator Ansatz with Constraints (QAOAH)

In QAOA, the starting state is in an equal superposition of all computational basis states, and the mixing Hamiltonian is a sum of Pauli-X operators acting on each qubit. In this configuration, every possible classical-state solution can be traversed to making it ideal for unconstrained problems where there are no states to be filtered out. However, it is possible to incorporate constraints by altering the default mixing Hamiltonian $H_M$ and the initial quantum state $|\psi\rangle$. Since the mixing operator (the exponentiated mixing Hamiltonian) specifies how to explore the solution space, it is possible to specify a mixing
operator that determines how to move to another solution within the constraints of the problem. Consider an initial quantum state $|\psi\rangle$ which represents a solution that does not violate the constraints of the optimization problem, but is not necessarily the minimizer. Consider the mixing operator $H_M$ that is constructed such that after an application on a state within the constrained space, the output state is also guaranteed to be within the constrained space. The idea is to construct $H_M$ and $|\psi\rangle$ so that when stopping QAOA at any iteration, the final encoded solution is always within the constrained solution space. If this mixing operator can mix such that it allows the algorithm to reach every possible classical-state in the constrained solution space, it is then possible to obtain the minimizer in that same space \[1]\.

Since QA and QAOA are both derived from the same formulation, the implementation details are described together. In QAOAH, constraints are embedded into the Mixing Hamiltonian which brings to a different formulation. The following sub-sections outline the implementation for both scenarios using the three approaches described above.

A. Scenario 1 (One Constraint)

1. QA and QAOA

The goal of a MCO problem is to minimize the total cost. Various ways are used to define mission cost. The following formulation takes into account limitations of QUBO. In the rest of the paper QA denotes QUBO problems performed on quantum annealing machines while QAQA and QAOAH denote QUBO problems executed on gate-based architectures.

The optimal mission cost occurs when all the resources required are allocated. Or, formally, consider mission $m$; the optimal mission cost occurs when:

$$
\sum_{r \in \mathbf{R}} x_{m,r} = \text{REQ}_M(m, 1). \tag{1}
$$

The $x$ values represent Booleans (an alternative representation of the solution $f_x$), such that when indexed by $m \in \mathbf{M}$ and $r \in \mathbf{R}$, e.g. $x_{m,r}$, it represents whether or not the solution mapped resource $r$ to mission $m$.

As an example, suppose there are three resources, $\mathbf{R} = \{1, 2, 3\}$ and the first mission requires two of them, then the optimal mission cost for the mission occurs when:

$$
x_{1,1} + x_{1,2} + x_{1,3} = 2. \tag{2}
$$

Eq. (2) is true when 2 out of the 3 variables are true. The mission cost represents the penalty added any time a mission lacks of one or more needed resources. The penalty is higher when more resources are missing. A squared error is used to represent the mission cost:

$$
\text{MC}(x, m) = \left( \sum_{r \in \mathbf{R}} x_{m,r} - \text{REQ}_M(m, 1) \right)^2, \tag{3}
$$

where $\text{MC}(x, m)$ is the cost associated to mission $m$. It can be seen from Eq. (3) that the minimum mission cost, when $\text{MC}(x, m) = 0$, yields the optimal case as described in Eq. (1). The mission cost is quadratic. The example that yielded the optimal case in Eq. (2) in terms of the mission cost function is:

$$
\text{MC}(x, 1) = (x_{1,1} + x_{1,2} + x_{1,3} - 2)^2. \tag{4}
$$

For this optimization problem, secondary resources should be allocated only after primary resources are allocated first. Therefore, we introduce precedence cost for resources. Eq. (5) shows the ideal condition regarding the allocation of secondary resources.

$$
\sum_{m \in \mathbf{M}, m \neq U} x_{m,r} = \text{C}(r, 1) - 1 \quad \forall r \in \mathbf{R} \tag{5}
$$

The precedence cost may or may not be met when optimizing, so just like mission cost, precedence cost is the squared error of this equality:

$$
\text{PC}(x, r) = \left( \sum_{m \in \mathbf{M}, m \neq U} x_{m,r} - \text{C}(r, 1) + 1 \right)^2. \tag{6}
$$

Finally, the objective function is then the total mission cost and the total precedence cost in the MCO:

$$
f_{\text{obj}}(x) = \sum_{m \in \mathbf{M}} \text{MC}(x, m) + \frac{1}{|\mathbf{R}|} \sum_{r \in \mathbf{R}} \text{PC}(x, r). \tag{7}
$$

We weight the precedence cost by $\frac{1}{|\mathbf{R}|}$ to ensure that the mission cost is minimized before precedence cost. The total cost is the sum of the mission cost and the precedence cost and it is reflected in the objective function.

The constraint that a resource must be paired to exactly one mission is formulated:

$$
\sum_{m \in \mathbf{M}} x_{m,r} = 1 \quad \forall r \in \mathbf{R}. \tag{8}
$$

The constraint function used for this scenario is defined as

$$
\text{CONSTR}(x, r) = \left( \sum_{m \in \mathbf{M}} x_{m,r} - 1 \right)^2. \tag{9}
$$

Since the QUBO is used to solve problems without constraints, we must add it to the objective function so that when it minimizes, the constraints will be met. The method of Lagrange multipliers is a strategy for finding the local maxima and minima of functions subject to equality constraints.
If $f_{\text{obj}}(x)$ is the objective function to be minimized, the lagrangian function is

$$L(x, \lambda) = f_{\text{obj}}(x) + \lambda \cdot \text{CONSTR}(x).$$

(10)

And the solution to the original constrained problem is always a saddle point of this function. Setting a large value for $\lambda$, the term related to the constraint, will have the greatest impact on the optimization problem. And the solution will minimize the constraint first and then the cost.

The new objective function that includes the constraints is:

$$f_{\text{obj}}(x) = L(x, \lambda) = \sum_{m \in M} MC(x, m) + \frac{1}{|R|} \sum_{r \in R} PC(x, r) + \lambda \cdot \sum_{r \in R} \text{CONSTR}(x, r).$$

(11)

QAOA for this method uses an equal superposition for the starting state $|\Psi\rangle$ over $N$ states:

$$|\Psi\rangle = \frac{1}{\sqrt{N}} \sum_{i=0}^{N-1} |i\rangle.$$  

(12)

The value $N$ is equal to $2^n$ where $n$ is the number of qubits used. For this problem, the number of qubits used is $N_M \cdot M_R$, the number of missions times the number of resources.

The mixing operator is constructed using a Hamiltonian, which is the sum of Pauli-X as follows:

$$X_i = I \otimes \cdots \otimes X \otimes \cdots \otimes I,$$

(13)

$$H_M = \sum_{i=1}^{n} X_i.$$  

(14)

2. QAOAH

In the last section, Lagrange multipliers are used to encode constraints into the QUBO problem. Alternatively, by choosing the appropriate mixing Hamiltonian and initial state, we can constrain the solution space outside of the QUBO formulation in QAOA.[1]

The initial state must be within the constrained solution space. A trivial starting configuration that is known not to violate the constraint is when all resources are set to the unallocated mission as shown in Fig. 3.

Thus, a single resource mixing $\text{MIX}(r)$ is:

$$\text{MIX}(r) = \sum_{m \in M \neq U} \text{SWAP}_{(U, r), (m, r)}.$$  

(16)

where $(m, r)$ encodes the index of the qubit representing a mapping of resource $r$ and mission $m$. The $U$ in
\((U, r)\) represents the unallocated mission. The total mixing operator \(H_m\) is the sum of \(\text{MIX}(r)\) on each resource with the identity operation:

\[
H_m = I^\otimes n + \sum_{r \in R} \text{MIX}(r). \tag{17}
\]

In order to embed the mixing operation as a Hamiltonian to run on the IBM machines, it must be described as a composition of tensored Pauli gates. Each \(\text{SWAP}\) gate can be decomposed in terms of Pauli gates:

\[
\text{SWAP} = \frac{1}{2} (I \otimes I + X \otimes X + Y \otimes Y + Z \otimes Z). \tag{18}
\]

When \(\text{SWAP}\) is indexed by \(i\) and \(j\), the Pauli gates fall on the \(i^{th}\) and \(j^{th}\) qubits respectively:

\[
\text{SWAP}_{i,j} = \frac{1}{2} (I^\otimes n + X_{i,j} + Y_{i,j} + Z_{i,j}), \tag{19}
\]

\[
X_{i,j} = I \otimes \cdots \otimes X \otimes \cdots \otimes X \otimes \cdots \otimes I. \tag{20}
\]

Each Pauli gate used in Eq. (19) \((X_{i,j}, Y_{i,j}, Z_{i,j})\) is defined similarly to what is defined in Eq. (20). For clarity, Eq. (20) places the corresponding Pauli gate only at the specified indices \(i\) and \(j\) in a tensor product of identities \(I\).

### B. Scenario 2 (Two Constraints)

#### 1. QA and QAOA

In this scenario, the objective function measures just mission cost, as opposed to the previous scenario that also measures precedence cost. Therefore, the objective function is the sum of mission cost and the constraint function:

\[
\text{f}_{\text{obj}}(x) = \sum_{m \in M} \text{MC}(x, m) + \lambda \cdot \text{CONST}(x). \tag{21}
\]

Two different constraints are embedded into the objective function using Lagrange multipliers. The first one was discussed in the previous scenario assures that resources are allocated to no more than one mission. It is formulated in a similar way as before:

\[
\text{CONST}_{1}(x, r) = \left(\sum_{m \in M} x_{m,r} - 1\right)^2. \tag{22}
\]

The additional constraint (buddy constraint) requires that the amount of resources allocated from set \(R_1\) must be the same as the number of resources allocated from set \(R_2\). This hard equality is formulated as:

\[
\text{CONST}_{2}(x, m) = \left(\sum_{r \in R_1} x_{m,r} - \sum_{r \in R_2} x_{m,r}\right)^2. \tag{23}
\]

The total constraint function expressed in Eq. (21) can be expressed as the sum of both of these constraints:

\[
\text{CONST}(x) = \sum_{r \in R} \text{CONST}_{1}(x, r) + \sum_{m \in M} \text{CONST}_{2}(x, m). \tag{24}
\]

For QAOA, the starting state and the Mixing Hamiltonian are the same as defined in Eq. (12) and Eq. (14).

#### 2. QAOAH

For this scenario, it is more challenging to construct the mixing Hamiltonian to describe how to move in the constrained space in QAOA. Unlike the Lagrange multiplier’s case, it is not easy to linearly combine two constraint encodings to get the final constraint Hamiltonian. In other words, one cannot add \(H_{m1} + H_{m2}\) to get \(H_{m1+m2}\), where \(H_{m1}\) and \(H_{m2}\) are mixing Hamiltonian’s representing 1st, 2nd and 1st & 2nd constraints respectively. This scenario presents two challenges:

1. If a mixing operator allowed a resource in \(R_1\) to move from an unallocated state to an allocated state by pairing it with a mission, then it must also move a resource from \(R_2\) to the same mission

2. the mixing operator must operate such that it is possible after multiple applications to visit every classical state from the starting quantum state. These two obstacles require a slightly different mixing operator and more qubits.

The strategy for creating a two-constraint mixing operator is to reallocate resources in pairs - one from \(R_1\) and one from \(R_2\). These pairs will always move together. However, the problem is that not every possible classical state can be produced from the mixing. For example, let’s say that there are resources \(R_1 = \{1, 2, 3, 4\}\) and \(R_2 = \{5, 6, 7, 8\}\). If resource 1 and resource 5 move together, then it’s never possible to see resource 2 just be paired with resource 5 without also being paired with 1. A way to resolve this problem is to introduce an additional mixing operator to swap entire columns within just \(R_1\) or \(R_2\) resources. However, every time columns are swapped, the classical state must remember what mappings are paired with which others so that if another reallocation is done, the buddy constraint won’t be violated.
For this reason new qubits are introduced to each column. These qubits represent the pair ID that is present in the columns for resources in \( R_1 \) and \( R_2 \). Resources with the same ID are reallocated together. When columns are swapped, the pair IDs of the columns are also swapped.

Consider a MCO with 3 missions (plus the unallocated mission) and 8 resources evenly split between \( R_1 \) and \( R_2 \). Our initialized state is shown in Fig. 5a. Each column has 2 extra ID qubits (lowest two rows) with a unique bit-encoding that matches another column from the opposite qualification type. This means that these columns are paired together.

For example, consider the case where the mixer operation reallocates resource 2 from the unallocated mission to mission 2. Since this column has an ID of ‘10b’ (top ID qubit true, bottom one false) it is paired with resource 6 because it has the same ID. In order to respect this pairing, both resource 2 and 6 are swapped together using a dual swap gate \( \text{C-DSWAP} \) as shown in Fig. 5b.

Consider the situation when the mixing operator performs a column swap between columns 2 and 4 (see Fig. 5c). Note that the IDs of these columns are also swapped. If the mixer chooses to move resource 6 back to the unallocated state, it also would move resource 4 into the unallocated state since they have the same ID. This can be ensured if the mixer uses the dual swap operation once again (see Fig. 5d).

The added qubits to represent the IDs of each of the columns are notated as \(|\text{ID}\rangle\) and defined as:

\[
|\text{ID}\rangle = \bigotimes_{j_1=0}^{\text{ID}_{\text{max}}} |j_1\rangle \bigotimes_{j_2=0}^{\text{ID}_{\text{max}}} |j_2\rangle.
\]

(25)

The starting state \(|\psi\rangle\) is:

\[
|\psi\rangle = |\text{ID}\rangle \bigotimes_{(m,r) \in M \times R} \begin{cases} 
|1\rangle & m = U \\
|0\rangle & m \neq U
\end{cases}
\]

(26)

and the mixing Hamiltonian \( H_m \) becomes

\[
H_m = \sum_{p \in R_1 \times R_2 \times M} \sum_{j=0}^{\text{ID}_{\text{max}}} \text{C-DSWAP}(p,j)
\]

\[
+ \frac{1}{2} \sum_{r_1 \in R_1} \sum_{r_1' \in R_1} \sum_{r_1 \neq r_1'} \text{COL-SWAP}(r_1,r_1')
\]

\[
+ \frac{1}{2} \sum_{r_2 \in R_2} \sum_{r_2' \in R_2} \sum_{r_2 \neq r_2'} \text{COL-SWAP}(r_2,r_2').
\]

(27)

The constant \( \text{ID}_{\text{max}} \) represents the maximum required binary states to represent all columns. This is the number of resources in either \( R_1 \) or \( R_2 \) and it is represented by \( \text{ID}_{\text{max}} \).

\[
|\text{ID}_{\text{max}}\rangle = |R_1| = |R_2|.
\]

(28)

The column-swap gate, notated as \( \text{COL-SWAP}(r,r') \), swaps the columns represented by resources \( r \) and \( r' \). Its decomposition is trivial as it employs many swaps tensored together.

The control dual-swap, notated as \( \text{C-DSWAP}(p,j) \), has parameters \( p \) and \( j \). The parameter \( p = (r_1,r_2,m) \) is a tuple composing of a resource from \( R_1 \), a resource from \( R_2 \), and a mission \( m \) in \( M \). This gate applies a control swap gate to resource \( r_1 \) and \( r_2 \) between the mission \( m \) and the unallocated mission \( U \). The parameter \( j \) is an ID which represents how to control the swap gate. For example, \( 10b \) is applying a control-true, control-false gate to both IDs in the columns represented by \( r_1 \) and \( r_2 \).

To implement the algorithm on the IBM machine, we must decompose \( H_m \) such that it is a sum of tensored Pauli gates. The column-swap gates \( \text{COL-SWAP}(r,r') \) can be decomposed, knowing that they are made up of swap gates, as from Eq. (18).

Decomposing a generalized version of the control dual-swap gate is tedious, so we provided an example decomposition for \( \text{C-DSWAP}(*,10b) \), which is the gate used in Fig. 5a and Fig. 5d. First, we present the Pauli-decomposition of the control-true and control-false unitary operations shown in Eq. (29) and Eq. (30), respectively. Unitary \( A \) is arbitrarily acting on \( m \) qubits.

\[
\text{C-T}(A) = \frac{1}{2} \left( (I + Z) \otimes I^{\otimes m} + (I - Z) \otimes A \right).
\]

(29)

\[
\text{C-F}(A) = \frac{1}{2} \left( (I - Z) \otimes I^{\otimes m} + (I + Z) \otimes A \right).
\]

(30)

Now, \( \text{C-DSWAP}(*,10b) \) can be represented in terms of control-true and control-false unitaries and the dual-swap gate \( \text{DSWAP} \):

\[
\text{C-DSWAP}(*,10b) = \text{C-T} \left( \text{C-F} \left( \text{C-T} \left( \text{C-F} \left( \text{DSWAP} \right) \right) \right) \right).
\]

(31)

Following this, the dual-swap gate \( \text{DSWAP} \) is two swap gates tensored together:

\[
\text{DSWAP} = \text{SWAP} \otimes \text{SWAP}.
\]

(32)

The \( \text{DSWAP} \) can further be decomposed using Eq. (18). As mentioned in the previous section, our two-constraint MCO problem has permutation symmetry between the resources in the same set/group. So the column swap terms can be effectively removed, and \( H_m \) becomes

\[
H_m = \sum_{p \in R_1 \times R_2 \times M} \sum_{j=0}^{\text{ID}_{\text{max}}} \text{C-DSWAP}(p,j).
\]

(33)
### B. Scenario 2 (Two Constraints)

FIG. 5. (a) Initial State Layout. The initial state is similar to the QAOA case except with additional ID qubits to be used to support the constraint mixer. Each column of ID qubits form a unique binary number for that resource within its resource set. (b) First Control Dual-Swap Operation. Through mixing, resource 2 is allocated to mission 2. In order to not violate the row constraint, resource 6 from $R_2$ is allocated using a Dual-Swap gate with controls on the ID qubits. (c) Column Swap. The mixing operator can never produce the valid solution where just resource 4 and 6 are paired to mission 2 using just Controlled Dual-Swap Operations. Therefore, column-swap operations are permitted, which swap any two columns in the table within its resource set. Here, resource 2 and 4 in resource set $R_1$ have their entire columns swapped. (d) Second Control Dual-Swap Operation. This operation is exactly the same as (b), but using the Control Dual-Swap operator to unallocate resource 4 from mission 2 to mission $U$. Notice that the Control Dual-Swap operator has its control configuration identical on both resource 4 and 6 ID-qubit columns; this is to ensure that the row constraint cannot be violated. Violation occurs when leaving resource 6 with no pair on mission 2.

This effectively shrinks the search space from the total constraint space. However, because of the symmetry, it is known that the optimal solution still lies inside the smaller subspace. When different capabilities are introduced to each resource, this optimization technique cannot be done, since the permutation symmetry is not guaranteed.

### V. ANALYSES OF RESULTS

In this section we compare the results of the different MCO implementations. Employing the implementation methods discussed above, the MCO problem was run on the D-Wave and on IBM machines, capturing several key metrics:

- Number of qubits
- Quantum processor time
- Cost

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- Number of constraints violated

Table I (top) indicates the execution status (quantum hardware or simulation) of the MCO algorithms, while Table I (bottom) shows the average number of constraints violated per implementation.

For both scenarios, 100 random MCO configurations were generated using up to 27-qubits (200 different MCO configurations in total). Quantum Annealing, QAOA, QAOAH, and Brute Force (BF) methods were run for each generated configuration. For Quantum Annealing, the DW_2000Q machine was used, while ibmq_toronto, ibm_hanoi, ibm_cairo, ibmq_mumbai, and ibmq_montreal machines were used for running QAOA and QAOAH. The Lagrange multipliers were set to 5 for both scenarios, and a $p$-value of 2 is used for QAOA (this parameter is discussed in the original paper[2]). The Quantum annealing runs each sampled the anneal 50 times, while each IBM job sampled the state-vector 1000 times. These parameters were chosen based off of a good balance of timing, cost, and constraints satisfied found by preliminary results not discussed in this paper.

### A. Scenario 1

Fig. 6 shows the timing averages for QA, QAOA, and QAOAH, respectively, versus the problem qubit size. Because the timing differed by different levels of magnitude, three $y$-axes with different scales are displayed. QA run time outperforms the other methods by having an overall constant run-time of 0.012 seconds, while QAOA can use the QPU for 5 hours across all jobs in certain worse-case instances (this excludes queue-time and creation time on QPU for IBMQ devices). For all methods, timing was calculated based on qpu time (not wall-clock time). For the QA method, ‘qpu_sampling_time’ was used to calculate the total qpu time, while ‘running time’ is used for QAOA and QAOAH.

Fig. 7 (left) shows the average costs for all MCO configurations versus the number of qubits it took to encode each scenario 1 run. These costs do not include the cost accumulated from the embedded constraint functions Eq.(9) and Eq.(24) as these are used to make the constraints hold. Also, all the costs plotted are actually the cost for that particular run minus the best possible cost it can receive. This best possible cost is computed by using brute force search methods in simulation. This difference is referred to as the relative cost. Therefore, the best possible relative cost a run could have is zero. For most runs, it can be seen that QA and QAOA runs have a relative cost around zero, but as the number of qubits increase, QAOA becomes less optimal compared to QA. The QAOAH approach, when using large amount of qubits, actually had a negative relative cost indicating that it must have violated some constraints in order to achieve a cost below the solution found with brute-force.

In Fig. 7b, the average number of constraints violated is plotted against qubit size. The number of constraints is calculated by counting the amount of resources that were assigned to more than one mission. For example, if resource 1 was assigned to two extra missions, and resource 2 was assigned to three extra missions, then the number of constraints violated is 5. QA mostly had no constraints violated at any sized qubit problem, while QAOA and QAOAH suffered constraint violations when the problem size increased.

### B. Scenario 2

For scenario 2, the QAOAH method was calculated via IBM’s state-vector numerical simulation tool due to the long running-time of runs in scenario 1. Because of this shift from running on actual hardware (scenario 1), to
Scenario 2

FIG. 7. QPU Results (a) Average Relative Cost versus the number of qubits used. The relative cost is the mission plus the precedence cost obtained minus the lowest possible mission plus precedence cost that can be achieved (while keeping constraints) for the problem; The lowest possible cost is found by brute force: Relative Cost = Algorithm’s Solution Cost - Brute Force Solution’s Cost. Negative relative costs imply that the solution violated constraints. (b) Average number of constraints violated versus the number of qubits used. The number of constraints counts up how many extras qubit are on within a column in the matrix view representation. This count includes the amount of columns that do not have any qubits on at all.

FIG. 8. QPU & Simulated Results QA & QAOA QPU Times and Simulated QAOAH Wall-Clock times for Scenario 2. QA timing includes the anneal time of each 50 samples, each 20 microseconds per sample. QAOA ran with 1000 shots. QAOAH ran in undermined amount of jobs using statevector-simulator to get results. Unlike Scenario 1, the times recorded are total wall-clock times for QAOAH.

In scenario 2, QA times are quite faster than QAOA methods. It can be seen that the QAOAH method has far less flexibility in terms of qubit-range. This is because in our MCO algorithm implementation, extra qubits are required to represent each resource’s ID. To keep consistency with the QA and QAOA methods, the x-axis in each plot in this section represents the number of qubits used minus the amount used to represent the IDs.

As in scenario 1, the relative costs for each method is plotted in Fig. 9a. Both QAOA and QA methods have zero relative costs. However, QAOA by itself did not exhibit a positive relative cost. For both cases in this subplot, the data is insufficient to deduce whether or not these violated constraints.

In Fig. 9b, the average number of constraints violated is plotted against the number of qubits the problem encodes. The number of constraints violated is calculated similarly to scenario 1, but now including all violations of the second constraint. For example, if four resources of type-1 and 2 of type-2 were allocated to a mission, then the number of constraints violated is $4 - 2 = 2$. Fig. 9b shows that QA and QAOAH did not violate constraints at any qubit size, while QAOA on average did. QAOA violated constraints mostly likely because it found a solution where the mission cost exceeded the cost incurred by the constraint function. QAOAH in simulation, however, did not violate constraints because the mixing operator that was used transforms solutions without leaving the constrained space. These results contrast with the QPU runs for QAOAH in Scenario 1 where it did violate constraints. Since the simulation ran without any noise profiles, it is expected that QAOAH shouldn’t violate constraints in theory, but this is not the case when running on the actual quantum machine. The source of noise on actual hardware is most likely due to the gate noise and noise from measurement.
VI. SUMMARY AND CONCLUSION

In this paper, we introduced Mission Covering Optimization (MCO), implemented three different constraint optimization techniques (QA, QAOA, and QAOAH) to find solutions for two scenarios of MCO, and discussed results after running implementations on the IBM machine, D-Wave Machine, and on a state vector simulator. Results were compared based on timing, relative cost, qubits used, and constraints violated. From the 200 tests performed on each scenario, QA achieved the quickest results while using the least number of qubits and violating the least number of constraints. We conjecture that QAOA and QAOAH approaches may have taken longer for the gradient descent algorithm to be convinced that an optimal solution was found because of the abundance of noise in current hardware. It was found that it is non-trivial to engineer in multiple constraints by embedding into the Mixing-Hamiltonian, especially when compared to the ease of using Lagrange multipliers, in simulation, where adding constraints entails simply adding terms together. The study conducted here suggests that the additional complexity in the QAOAH approach poses potential scalability challenges (due to the additional qubits required to ID the constraints) for problems similar to MCO with multiple types of unique constraints.

VII. FUTURE WORK

Adding further capabilities to different resources in Scenario 2 would make for a more interesting/realistic optimization problems for quantum computers to solve. This is just one of the many alterations that can be done to MCO to increase the complexity of the optimization problem.

In this paper, resources dependencies are modeled within MCO, however there may be missions with different priorities along with mission dependencies. Also, in this work resources were only shown to possess only one type of qualification. At the heart of MCO, it is an optimization problem concerning allocation of resources invariant to time. An interesting direction for future study is how well this type of optimization problem can be ported to an extension of a Job shop problem.

Although error mitigation is not a focus of this study, research for mitigating error for QAOA are being studied by others[19]. Employing error mitigation techniques for MCO is another interesting direction of study.

Lastly, another interesting solution method might entail the use of a “bang-bang” strategy for multiple constraints. Here, for each constraint $C_i$ associated with a constraint Hamiltonian $H_i$, one might randomly cycle through applications of individual $H_i$ for each time step, as opposed to the application of the joint Hamiltonian $H = \sum H_i$. While each $H_i$ only preserves constraints $C_i$, the supposition is that the application of $H_i$ might only partially violated constraints $C_{j \neq i}$, if applied for a short time, and randomly. This solution approach to MCO-like problems will be investigated in future work.

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