Higher algebraic $K$-groups and $\mathcal{D}$-split sequences

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Abstract

In this paper, we use $\mathcal{D}$-split sequences and derived equivalences to provide formulas for calculation of higher algebraic $K$-groups (or mod-$p$ $K$-groups) of certain matrix subrings which cover tiled orders, rings related to chains of Glaz-Vasconcelos ideals, and some other classes of rings. In our results, we do not assume any homological requirements on rings and ideals under investigation, and therefore extend sharply many existing results of this type in the algebraic $K$-theory literature to a more general context.

1 Introduction

One of the fundamental questions in the algebraic $K$-theory of rings is to understand and calculate higher algebraic $K$-groups $K_n$ of rings, which were deeply developed in a very general context by Quillen in [19] for exact categories and by Waldhausen in [24] for Waldhausen categories. On the one hand, the usual methods for computing $K_n$ may be the fundamental theorem, splitting morphisms, or certain long exact sequences of $K_n$-groups, namely, Mayer-Vietoris sequences, localization sequences or excision. In this direction there is a lot of literature (for example, see [8, 15, 18, 25, 26, 27], and others). On the other hand, we know that derived-equivalent rings share many common homological and numerical features, in particular, they have the isomorphic higher algebraic $K_n$-groups for all $n \geq 0$ (see [6]). This means that, in order to understand the higher $K$-groups $K_n$ of a ring, one might refer to another ring which is derived-equivalent to the given one, and which may hopefully have a simple form so that its $K_n$-groups can be determined easily. This idea, however, seems not much to be benefited in the study of higher algebraic $K$-theory of rings, especially in dealing with calculation of $K_n$-groups.

In the present note, we shall use ring extensions and derived equivalences as reduction techniques to investigate the higher algebraic $K_n$-groups of certain matrix subrings which include many maximal orders, hereditary orders, tiled orders, endomorphism rings of chains of Glaz-Vasconcelos ideals, and other classes of rings. To produce such derived equivalences, we shall employ $\mathcal{D}$-split sequences defined in [10]. In this way, we reduce our calculation inductively to that of certain triangular matrix rings. The advantage of our method is: We not only drop all homological conditions on rings and ideals under investigation, but also extend many existing results (see [1, 8, 13]) of this type in the literature to a more general context. Our main results in this note can be stated as follows.

Theorem 1.1. Suppose that $p \geq 2$, $m$ and $s$ are positive integers such that $s$ is divisible by $p$. Let $R$ be a $\mathbb{Z}/p^m\mathbb{Z}$-algebra with identity, and let $I, I_i$ and $I_{ij}$ be (not necessarily projective) ideals of $R$. We denote by $K_*(R)$ the $*$-th algebraic $K$-group of $R$ with $* \in \mathbb{N}$.

1. If $I_{ij} \subseteq I$ for all $i, j$, $I_{kj} \subseteq I_{ij}$ for $k \leq i$, $I_{ki} \subseteq I_{kj}$ for $j \leq i$ and $I_{ki}I_{kj} \subseteq I_{ij}$ for $i < k < j$, then

\[
S := \begin{pmatrix}
R & I_{12} & I_{13} & \cdots & I_{1n} \\
I & R & I_{23} & \cdots & I_{2n} \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
I & \cdots & I & R & I_{n-1n} \\
I & \cdots & I & I & R
\end{pmatrix}
\]

is a ring, and

\[
K_*(S) \otimes_{\mathbb{Z}} \mathbb{Z}_{\frac{1}{s}} \cong K_*(R) \otimes_{\mathbb{Z}} \mathbb{Z}_{\frac{1}{s}} \oplus \bigoplus_{j=2}^{n} K_*(R/I_{j-1j}) \otimes_{\mathbb{Z}} \mathbb{Z}_{\frac{1}{s}}.
\]

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(2) For $2 \leq i \leq n$, suppose that $R_i$ is a subring of $R$ with the same identity. If $I_{i+1} \subseteq I_i \subseteq R_i$ for all $i$, $I_j \subseteq I_{ij} \subseteq I$ for all $i, j$, and $I_k I_{kj} \subseteq I_{ij}$ for $j < k < i$, then

$$T := \begin{pmatrix} R & I_2 & I_3 & \cdots & I_{n} \\ I & R_2 & I_3 & \cdots & I_{n} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ I & I_{n2} & \cdots & I_{n-1} & I_n \end{pmatrix}$$

is a ring, and

$$K_s(T) \otimes \mathbb{Z}[\frac{1}{s}] \simeq K_s(R) \otimes \mathbb{Z}[\frac{1}{s}] \oplus \bigoplus_{j=2}^{n} K_s(R_j/I_j) \otimes \mathbb{Z}[\frac{1}{s}].$$

As pointed out in Section 6 below, Theorem 1.1 holds true for the mod-$p$ $K$-groups $K_s(-, \mathbb{Z}/p\mathbb{Z})$ if we assume in Theorem 1.1 that $R$ is a $\mathbb{Z}[\frac{1}{p}]$-algebra and $p \neq 2 \pmod{4}$, that is, under these two assumptions, one can replace $K_s(-, \mathbb{Z}/p\mathbb{Z})$ by $K_s(-, \mathbb{Z}/p\mathbb{Z})$ in Theorem 1.1.

The proof of the above result is based on the following observation. Note that the assumptions in Theorem 1.2 below is weaker than the ones in Theorem 1.1 above.

**Theorem 1.2.** Let $R$ be a ring with identity, and let $I_i$ be (not necessarily projective) ideals of $R$. We denote by $K_s(R)$ the $s$-th algebraic $K$-group of $R$ with $s \in \mathbb{N}$.

(1) If $I_{ij} \subseteq I_j$ for $k \leq i$, $I_i \subseteq I_{ij}$ for $j \leq i$ and $I_k I_{kj} \subseteq I_{ij}$ for $i < k < j$, then

$$S := \begin{pmatrix} R & I_2 & I_3 & \cdots & I_{n} \\ R & R & I_3 & \cdots & I_{n} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ R & R & R & R & I_{n-1} \\ R & R & R & R & R \end{pmatrix}$$

is a ring, and

$$K_s(S) \simeq K_s(R) \oplus \bigoplus_{j=2}^{n} K_s(R/I_{j-1} I_j).$$

(2) For $2 \leq i \leq n$, suppose that $R_i$ is a subring of $R$ with the same identity, that $I_j \subseteq R_i$ is a right ideal of $R_i$, and that $I_i$ is a left ideal of $R$. If $I_{i+1} \subseteq I_i$ for all $i$, $I_j \subseteq I_{ij}$ for all $i, j$, and $I_k I_{kj} \subseteq I_{ij}$ for $j < k < i$, then

$$T := \begin{pmatrix} R & I_2 & I_3 & \cdots & I_{n} \\ R & R_2 & I_3 & \cdots & I_{n} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ R & I_{n2} & \cdots & I_{n-1} & I_n \end{pmatrix}$$

is a ring, and

$$K_s(T) \simeq K_s(R) \oplus \bigoplus_{j=2}^{n} K_s(R_j/I_j).$$

The strategy of our proofs of the theorems is first to use ring extensions, which are motivated from [23], and then to combine $K$-groups in Mayer-Vietoris sequences with $K$-groups of rings which are linked by derived equivalences produced from certain $\varnothing$-split sequences.

This note is organized as follows. In Section 2 we recall some definitions and elementary facts on derived equivalences needed in the later proofs. In Section 3 we construct $\varnothing$-split sequences by ring extensions and calculate the endomorphism rings of tilting modules related to these sequences. In Section 4 we prove the main results.
and state some of its consequences. Our proofs of the above results also give an explanation of the multiplicity factor $n - 1$ in the isomorphisms of $K_n$-groups of the rings in \cite{13} and \cite{11}. In Section 5 we calculate $K_0$ and $K_1$ for some matrix subrings which are not covered by the main results. In Section 6 we show that the main result Theorem 1.1 holds for mod-$p$ $K$-theory by outlining the key ingredients of its proof. In Section 7 we give some examples to show how our method can work, here GV-ideals in commutative rings enter into our play. These examples demonstrate also that the matrix rings studied in Section 3 really occur, as the endomorphism rings of chains of GV-ideals, in the field of commutative algebra.

2 Preliminaries

Let $A$ be a ring with identity. By an $A$-module we mean a left $A$-module. Let $A$-Mod (respectively, $A$-mod) denote the category of all (respectively, finitely generated) left $A$-modules. Similarly, by $A$-Proj (respectively, $A$-proj) we denote the full subcategory of all (finitely generated) projective $A$-modules in $R$-Mod. For an $A$-module $M$, we denote by $\text{proj.dim}(M)$ the projective dimension of $M$. Let $\mathcal{X}^b(A$-proj) be the homotopy category of the additive category $A$-proj. The unbounded derived category of $A$-Mod is denoted by $\mathcal{D}(A)$, whereas the bounded derived category of $A$-Mod is denoted by $\mathcal{D}^b(A)$. We say that two rings $A$ and $B$ are derived-equivalent if $\mathcal{D}(A)$ and $\mathcal{D}(B)$ are derived-equivalent as triangulated categories. It is well-known that if $\mathcal{D}^b(A)$ and $\mathcal{D}^b(B)$ are derived-equivalent as triangulated categories then $\mathcal{D}(A)$ and $\mathcal{D}(B)$ are derived-equivalent as triangulated categories.

Given an additive category $C$ and an object $X$ in $C$, we denote by $\text{add}(X)$ the full subcategory of $C$ consisting of all objects which are direct summands of direct sums of finitely many copies of $X$.

For derived equivalences, Rickard’s Morita theory \cite{22} is very useful.

Theorem 2.1. \cite{22} For two rings $A$ and $B$ with identity, the following are equivalent:

(a) $\mathcal{D}^b(A)$ and $\mathcal{D}^b(B)$ are equivalent as triangulated categories.
(b) $\mathcal{X}^b(A$-proj) and $\mathcal{X}^b(B$-proj) are equivalent as triangulated categories.
(c) $B \simeq \text{End}_{\mathcal{X}^b(A$-proj)}(T^\bullet)$, where $T^\bullet$ is a complex in $\mathcal{X}^b(A$-proj) satisfying

1. $\text{Hom}(T^\bullet, T^\bullet[i]) = 0$ for $i \neq 0$, and
2. $\text{add}(T^\bullet)$ generates $\mathcal{X}^b(A$-proj) as a triangulated category.

For derived equivalences, it is shown in \cite{6} that the algebraic $K$-theory is an invariant. Recall that, for a ring $A$ with identity, $K_n(A)$ denotes the $n$-th homotopy group of a certain space $K(A)$ produced by one’s favorite $K$-theory defined for each $n \in \mathbb{N}$ (see \cite{19, 24, 27}).

Theorem 2.2. \cite{6} If two rings $A$ and $B$ with identity are derived-equivalent, then their algebraic $K$-groups are isomorphic: $K_n(A) \simeq K_n(B)$ for all $n \in \mathbb{N}$.

As is known, Morita equivalences are derived equivalences. Thus, if $A$ and $B$ are Morita equivalent, then their algebraic $K$-groups are isomorphic.

Another special class of derived equivalences can be constructed by tilting modules initialled from the representation theory of finite-dimensional algebras (for example, see \cite{2}). Recall that a module $T$ over a ring $A$ is called a tilting module if the following three conditions are satisfied:

1. $T$ has a finite projective resolution $0 \to P_n \to \cdots \to P_0 \to T \to 0$, where each $P_i$ is a finitely generated projective $A$-module;
2. $\text{Ext}^i_A(T, T) = 0$ for all $i > 0$, and
3. there is an exact sequence $0 \to A \to T^0 \to \cdots \to T^m \to 0$ of $A$-modules with each $T^i$ in $\text{add}(T)$ which is a full subcategory of $A$-mod.

Note that, for a tilting module $T$, the projective resolution $P^\bullet$ of $T$ satisfies (1) and (2) of Theorem 2.1(c). Thus, if $A$ is a tilting $A$-module then $A$ and $\text{End}_A(T)$ are derived-equivalent. To produce tilting modules, one may use the notion of $\mathcal{D}$-split sequences. Now let us recall the definition of $\mathcal{D}$-split sequences from \cite{10}.

Let $\mathcal{C}$ be an additive category and $\mathcal{D}$ a full subcategory of $\mathcal{C}$. A sequence

$$X \xrightarrow{f} M \xrightarrow{g} Y$$

of morphisms between objects in $\mathcal{C}$ is called a $\mathcal{D}$-split sequence if

1. $M \in \mathcal{D}$,
(2) $f$ is a left $\mathcal{D}$-approximation of $X$, that is, $\text{Hom}_C(f, D') : \text{Hom}_C(M, D') \to \text{Hom}_C(X, D')$ is surjective for all $D' \in \mathcal{D}$, and $g$ is a right $\mathcal{D}$-approximation of $Y$, that is, $\text{Hom}_C(D', g) : \text{Hom}_C(D', M) \to \text{Hom}_C(D', Y)$ is surjective for all $D' \in \mathcal{D}$, and

(3) $f$ is a kernel of $g$, and $g$ is a cokernel of $f$.

Examples of $\mathcal{D}$-split sequences include Auslander-Reiten sequences and short exact sequences of the form $0 \to X \to P \to Y \to 0$ in $A$-$\text{Mod}$ with $P$ projective-injective. A non-example is the sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ of abelian groups, that is, this sequence is not an $\text{add}(\mathbb{Z}/\mathbb{Q})$-split sequence. For more examples, one may find in [10], and also in the next section as well as in the last section of the present paper.

Given a $\mathcal{D}$-split sequence $X \to M' \to Y$, with $\mathcal{D} = \text{add}(M)$ for $M$ an object in $C$, it is shown in [10] that there is a tilting module $T$ over $\text{End}_C(X \oplus M)$ of projective dimension at most 1 such that $\text{End}(T)$ is isomorphic to $\text{End}_C(M \oplus Y)$. Thus $\text{End}_C(X \oplus M)$ and $\text{End}_C(M \oplus Y)$ are derived-equivalent, and have the isomorphic algebraic $K$-theory by Theorem 2.2.

## 3 Ring extensions and derived equivalences

Ring extensions were used in [28] to study the finitistic dimensions of algebras. In this section, we shall use ring extensions to construct $\mathcal{D}$-split sequences which will be applied to calculation of the algebraic $K$-groups of rings in the next section.

We first establish the following general fact.

**Lemma 3.1.** Let $B \subseteq A$ be an extension of rings with the same identity.

1. If $\text{Ext}_B^1(BA, BA) = 0$, then the sequence

$$0 \to \text{Hom}_B(BA, B) \to \text{Hom}_B(BA, A) \to \text{Hom}_B(A, A/B) \to \text{Ext}_B^1(A, BA)$$

is an $\text{add}(BA)$-split sequence in $B$-$\text{Mod}$. Thus $\text{End}_B(BA \oplus BA)$ and $\text{End}_B(BA \oplus A/B)$ are derived-equivalent.

2. If $BA$ is projective, then the above sequence is an $\text{add}(BA)$-split sequence.

3. Suppose that $\text{Ext}_B^1(BA, BA) = 0$. If $BA$ is finitely presented with $\text{proj.dim}(BA) \leq 1$ (for instance, $BA$ is projective and finitely generated), then $A \oplus A/B$ is a tilting $B$-module of projective dimension at most 1. In particular, $\text{End}_B(A \oplus A/B)$ is derived-equivalent to $B$.

*Proof.* (1) We have the following exact sequence

$$0 \to \text{Hom}_B(BA, B) \to \text{Hom}_B(BA, A) \to \text{Hom}_B(A, A/B) \to \text{Ext}_B^1(A, BA).$$

The condition $\text{Ext}_B^1(A, BA) = 0$ implies that the canonical surjection $A \to A/B$ is a right $\text{add}(BA)$-approximation of $A/B$. To see that the inclusion $B \to A$ is a left $\text{add}(BA)$-approximation of $B$, we note that each homomorphism from $BA$ to $BA$ is given by an element $a$ in $A$. Thus it can be extended to a homomorphism from $BA$ to $BA$ by the right multiplication of $a$. Clearly, one can check that this is also true for any homomorphism from $BA$ to a direct summand of $BA$. Thus we see that the inclusion map from $B$ to $A$ is a left $\text{add}(BA)$-approximation of $B$. Thus (* ) is an $\text{add}(BA)$-split sequence in $B$-$\text{Mod}$, and therefore $\text{End}_B(BA \oplus BA)$ and $\text{End}_B(BA \oplus A/B)$ are derived-equivalent by [10] Theorem 1.1. This finishes the proof of Lemma 3.1.

(2) is a special case of (1).

(3) Let $T := BA \oplus A/B$. Since $BA$ is finitely presented of projective dimension at most one, there is an exact sequence $0 \to P_1 \to P_0 \to BA \to 0$ such that $P_i$ are finitely generated projective $B$-modules and the following diagram is commutative:

$$
\begin{array}{cccccc}
0 & 0 & \\
\downarrow & & \\
P_1 & P_0 & \\
\downarrow & & \\
0 & \to & P & \to & P_0 & \to & A/B & \to & 0 \\
\downarrow & \downarrow & & \| & \\
0 & \to & B & \to & A & \to & A/B & \to & 0.
\end{array}
$$
From this diagram we see that $T$ is finitely presented of projective dimension at most one. Thus the conditions (1) and (3) in the definition of tilting modules are satisfied. It remains to show $\text{Ext}_B^1(A \oplus A/B, A \oplus A/B) = 0$. This is equivalent to that $\text{Ext}_B^1(A, A/B) = 0$, $\text{Ext}_B^1(A/B, A/B) = 0$ and $\text{Ext}_B^2(A/B, A) = 0$ since $\text{Ext}_B^1(A, A) = 0$ by assumption.

Indeed, we have seen that the inclusion map $A$ from $B$ into $A$ is always a left add($\mu A$)-approximation of $\mu B$. Thus the induced map $\lambda^* := \text{Hom}_B(\lambda, A)$ is surjective. Hence, by applying $\text{Hom}_B(\lambda, -)$ to the canonical exact sequence ($\ast$), we get an exact sequence

$$0 \rightarrow \text{Hom}_B(A/B, A) \rightarrow \text{Hom}_B(B, A) \rightarrow \text{Ext}_B^1(A/B, A) \rightarrow \text{Ext}_B^2(A, A),$$

which shows $\text{Ext}_B^1(A/B, A) = 0$. If we apply $\text{Hom}_B(A/B, -)$ to the canonical exact sequence, then we get an exact sequence:

$$\text{Ext}_B^1(A/B, B) \rightarrow \text{Ext}_B^1(A/B, A) \rightarrow \text{Ext}_B^1(A/B, A/B) \rightarrow 0$$

since the projective dimension of $A/B$ is at most 1. This implies $\text{Ext}_B^2(A/B, A/B) = 0$. Similarly, applying $\text{Hom}_B(A, -)$ to the canonical exact sequence ($\ast$), we can deduce $\text{Ext}_B^1(A, A/B) = 0$. Thus we complete the proof of (3). □

**Remark.** Sometimes the following observation is useful for getting $D$-split sequences: Suppose that $e$ and $f$ are idempotent elements in a ring $R$ and $a \in eRf$. Then the right multiplication map $Re \rightarrow Rf$, defined by $x \mapsto xa$ for $x \in Re$, is a left add($Rf$)-approximation of $Re$ if and only if $eRf = afRf$. Thus, if the right multiplication map is injective, then $0 \rightarrow Re \rightarrow Rf \rightarrow Rf/Rea \rightarrow 0$ is an add($Rf$)-split sequence if and only if $eRf = afRf$. For instance, the sequence $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow 0$ is not an add($\mathbb{Z}$)-split sequence.

Let us mention an example of ring extensions which satisfy the conditions in Lemma 3.1. Recall that an extension $B \subseteq A$ of rings is called a quasi-Frobenius extension if $\mu A$ is finitely generated and projective, and the bimodule $\mu AB$ is isomorphic to a direct summand of the direct sum of finitely many copies of $\mu \text{Hom}_B(\mu A, B)$. Thus each quasi-Frobenius extension $B \subseteq A$ provides an add($\mu A$)-split sequence

$$0 \rightarrow B \rightarrow A \rightarrow A/B \rightarrow 0,$$

and a tilting $B$-module $A \oplus A/B$ by Lemma 3.1.

Now we consider some consequences of Lemma 3.1 which are needed in the next section.

Let $R$ be a ring with identity and $I_{ij}$ ideals in $R$ with $1 \leq i < j \leq n$, such that

1. $I_{kj} \subseteq I_{ij}$ for $k \leq i$,
2. $I_{ki} \subseteq I_{kj}$ for $j \leq i$, and
3. $I_{ik} I_{kj} \subseteq I_{ij}$ for $i < k < j$.

Then

$$B := \begin{pmatrix}
R & I_{12} & I_{13} & \cdots & I_{1n} \\
R & R & I_{23} & & I_{2n} \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
R & R & & \cdots & I_{n-1,n} \\
R & R & & & R
\end{pmatrix}$$

is a ring. The rings of this form include tiled triangular orders and maximal orders [21]. They occur also as the endomorphism rings of chains of Glaz-Vasconcelos ideals of rings, see Section 7.

The following lemma shows that we may use derived equivalences to simplify the ring $B$.

**Lemma 3.2.** Let $B$ be the ring defined above. Then $B$ is derived-equivalent to

$$C := \begin{pmatrix}
R & I_{12} & I_{13} & \cdots & I_{1n-1} & I_{1n-1}/I_{1n} \\
R & R & I_{23} & \cdots & I_{2n-2} & \cdots & I_{2n-1}/I_{2n} \\
\vdots & \vdots & \ddots & \ddots & \vdots & \ddots & \vdots \\
R & R & R & \cdots & R & R/I_{n-1,n} \\
0 & 0 & \cdots & \cdots & 0 & R/I_{n-1,n}
\end{pmatrix}.$$
Proof. We make the following conventions on notations. Let $S = M_n(R)$, the $n \times n$ matrix ring over $R$. Let $e_i$ be the $n \times n$ matrix with $1_{R}$ in $(i,i)$-entry and zero in other entries. For convenience, we denote by $e_{i,j}(x)$ the matrix with $x$ in $(i,j)$-position, and zero in other positions, and by $B_{ij}$ the $(i,j)$-component of the matrix subring $B$ of $S$, that is, the set of $(i,j)$-entries of all matrices in $B$. We define

$$A := \begin{pmatrix} R & I_{12} & \cdots & I_{1,n-1} & I_{1,n-1} \\ R & R & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & I_{n-2,n-1} & I_{n-2,n-1} \\ R & R & \cdots & R & R \\ R & R & \cdots & R & R \end{pmatrix}.$$ 

Note that the only difference between $A$ and $B$ is the last column. We can verify that $A$ is a ring containing $B$ as a subring.

Clearly, as a left $B$-module, $BA \simeq Be_1 \oplus \cdots \oplus Be_{n-1} \oplus Be_n$. Thus $BA$ is finitely generated and projective. Furthermore, it follows that $B$ is Morita equivalent to $\operatorname{End}_B(B \oplus BA)$ and that the latter is derived-equivalent to $\operatorname{End}_B(BA \oplus A/B)$ by Lemma 3.1. Thus $B$ is derived-equivalent to $\operatorname{End}_B(Be_1 \oplus \cdots \oplus Be_{n-1} \oplus Ae_n/Be_n)$. For simplicity, we denote by $Q$ the $B$-module $Ae_n/Be_n$. Note that $Ae_n \simeq Be_{n-1}$ as $B$-modules, and that we have a canonical exact sequence:

$$0 \longrightarrow Be_n \overset{\lambda}{\longrightarrow} Be_{n-1} \overset{\pi}{\longrightarrow} Q \longrightarrow 0,$$

where $\lambda$ is the composition of the inclusion of $Be_n$ into $Ae_n$ with the right multiplication $\cdot e_n$, and $\pi$ is the composition of the right multiplication of $e_{n-1} \cdot e_n$ with the canonical surjective map $Ae_n \to Ae_n/Be_n$.

In the following, we shall prove that $\operatorname{End}_B(Be_1 \oplus \cdots \oplus Be_{n-1} \oplus Q)$ is isomorphic to $C$.

First, we define a map $\varphi : R \longrightarrow \operatorname{Hom}_B(Q,Q)$ as follows: For $b \in R$, let $e_nbe_n$ be the right multiplication map from $Be_n$ to $Be_n$. This is well-defined by our assumptions. Also, $e_{n-1}be_n$ be the right multiplication map from $Be_{n-1}$ to itself. Then we see that $\lambda(e_{n-1}be_n) = (e_nbe_n)\lambda$. So, there is a unique $\alpha \in \operatorname{Hom}_B(Q,Q)$ making the following diagram commutative:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & Be_n & \overset{\lambda}{\longrightarrow} & Be_{n-1} & \overset{\pi}{\longrightarrow} & Q & \longrightarrow & 0 \\
& & \downarrow{e_nbe_n} & & \downarrow{e_{n-1}be_n} & & \downarrow{\alpha} & \\
0 & \longrightarrow & Be_n & \overset{\lambda}{\longrightarrow} & Be_{n-1} & \overset{\pi}{\longrightarrow} & Q & \longrightarrow & 0.
\end{array}
$$

Hence, we can define the image of $b$ under $\varphi$ is $\alpha$. Clearly, if $b, b' \in R$, then $(b + b')\varphi = (b)\varphi + (b')\varphi$. Since $e_n(b'b')e_n = e_ne_nb'be_n$, we also have $(b'b')\varphi = (b\varphi)(b'\varphi)$. Thus $\varphi$ is a homomorphism of rings.

Now, we calculate the kernel of $\varphi$. Suppose $b \in R$ such that $\alpha = b\varphi = 0$. Then the map $\cdot e_{n-1}be_n$ factorizes through $\lambda$. This means that there is an element $r \in B_{n-1,n}$ such that $e_{n-1}be_n = (e_{n-1}re_n)\lambda$ and $e_nbe_n = \lambda(e_{n-1}re_n)$. Hence $b = r \in B_{n-1,n}$. Thus $\ker(\varphi) \subseteq B_{n-1,n}$. Since any map $e_nbe_n$ from $Be_n$ to $Be_n$ with $b \in B_{n-1,n}$ factorizes through $\lambda$, the corresponding $\alpha$ is zero. Hence $\ker(\varphi)$ is $B_{n-1,n}$.

Given an element $\alpha \in \operatorname{Hom}_B(Q,Q)$, we may form the following commutative diagram in $B$-Mod:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & Be_n & \overset{\lambda}{\longrightarrow} & Be_{n-1} & \overset{\pi}{\longrightarrow} & Q & \longrightarrow & 0 \\
& & \downarrow{b} & & \downarrow{a} & & \downarrow{\alpha} & \\
0 & \longrightarrow & Be_n & \overset{\lambda}{\longrightarrow} & Be_{n-1} & \overset{\pi}{\longrightarrow} & Q & \longrightarrow & 0.
\end{array}
$$

Note that the homomorphism $a$ exists and makes the right square of the above diagram commutative. Thus we have a homomorphism $b$ making the left square commutative. We may identify $a$ with an element in $B_{n-1,n-1}$, say $a = e_{n-1,n-1}(r)$ with $r \in B_{n-1,n-1}$, and identify $b$ with an element in $B_{mn}$, say $b = e_{n,m}(s)$ with $s \in B_{mn}$. The commutativity of the left square means that $r = s \in B_{mn}$. This means that $\varphi$ is surjective. Thus $\operatorname{End}_B(Q) \simeq R/I_{n-1,n}$.

If we apply $\operatorname{Hom}_B(-, Be_j)$ to $(*)$ for $1 \leq j \leq n-1$ and use Lemma 3.1.3, we have the following exact commutative diagram with $e_{n-1,n-1}$: an isomorphism:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{Hom}_B(Q,Be_j) & \overset{(\pi)j}{\longrightarrow} & \operatorname{Hom}_B(Be_{n-1},Be_j) & \overset{(\lambda)j}{\longrightarrow} & \operatorname{Hom}_B(Be_n,Be_j) & \longrightarrow & 0 \\
& & \downarrow{\varepsilon_{n-1}Be_j} & & \downarrow{e_{n-1}Be_j} & & \downarrow{e_nBe_j} & & \downarrow{\varepsilon_{n-1}Be_j} & & \downarrow{e_nBe_j}.
\end{array}
$$
Thus \( \text{Hom}_B(Q,Be_j) = 0 \) for all \( 1 \leq j \leq n - 1 \).

If we apply \( \text{Hom}_B(Be_j,-) \) to the exact sequence \((\ast)\) for \( 1 \leq j \leq n - 1 \), we get an exact sequence

\[
0 \to \text{Hom}_B(Be_j,Be_n) \to \text{Hom}_B(Be_j,B_{n-1}) \to \text{Hom}_B(Be_j,Q) \to 0,
\]

which shows that \( \text{Hom}_B(Be_j,Q) \approx B_{jn-1}/B_{jn} = B_{jn-1}/I_{jn-1} \).

Now we identify \( \text{Hom}_B(Be_j,Be_i) \) with \( e_i Be_i \) for all \( 1 \leq i, j \leq n - 1 \), and \( \text{Hom}_B(Be_j,Q) \) with \( B_{jn-1}/B_{jn} = B_{jn-1}/I_{jn-1} \). Then we can see that \( \text{End}_B(Be_1 \oplus \cdots \oplus Be_{n-1} \oplus Q) \) is isomorphic to \( C \). This finishes the proof of Lemma 3.2.

A special case of Lemma 3.2 is the ring considered in [8] under certain homological assumptions and finiteness condition. Here we start with a more general setting and drop all homological conditions on ideals as well as finiteness condition on quotients.

Let \( R \) be a ring with identity, and \( I \) an arbitrary ideal in \( R \). We consider the ring of the following form

\[
B := \begin{pmatrix}
R & P_{12} & \cdots & P_{1n} \\
R & R & \cdots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
R & R & \cdots & R
\end{pmatrix},
\]

where \( t_{ij} \) are positive integers. Note that the conditions for \( B \) to be a ring are

1. \( t_{ij} \leq t_{i+1,j}, t_{i,j+1} \leq t_{ij} \) for \( i < j \), and
2. \( t_{ij} \leq t_k + t_{kj} \) for \( i < k < j \).

The next result follows immediately from Lemma 3.2.

**Lemma 3.3.** Assume that the above defined \( B \) is a ring. Then \( B \) is derived-equivalent to

\[
C := \begin{pmatrix}
R & P_{12} & \cdots & P_{1n-1} & P_{1n-1}/P_{1n} \\
R & R & \cdots & \vdots & \vdots \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
R & R & \cdots & R & R/P_{1n-1} \\
0 & 0 & 0 & 0 & R/P_{1n-1}
\end{pmatrix}.
\]

Next, we consider a variation of the ring \( B \) in Lemma 3.2, which was considered in [14, 5] and cover some tiled orders in [21], and many other cases, for example, rings in [14], and some Auslander-regular, Cohen-Macaulay rings (not necessarily maximal orders, see [23]).

Let \( R \) be a ring with identity. Suppose that \( R_i \) is a subring of \( R \) with the same identity for \( 2 \leq i \leq n \), that \( I_i \) is a left ideal of \( R \) for all \( 2 \leq i \leq n \), and that \( I_i \) is ideal of \( R \), with \( 2 \leq j < i \leq n \), which satisfies the following conditions:

1. \( I_i \subseteq R_i \) is a right ideal of \( R_i \) for all \( i 
2. \( I_n \subseteq I_{n-1} \subseteq \cdots \subseteq I_2 
3. \( I_j \subseteq I_i \) for all \( i, j \), and
4. \( I_k I_j \subseteq I_j \) for \( j < k < i \).

Here we do not assume that \( I_i \) is projective as a left \( R \)-module, nor that \( I_i \) is an ideal of \( R \). Nevertheless one can check that

\[
B := \begin{pmatrix}
R & I_2 & I_3 & \cdots & I_{n-1} & I_n \\
R & R_2 & I_3 & \cdots & I_{n-1} & I_n \\
R & I_{32} & R_3 & \cdots & I_{n-1} & I_n \\
R & I_{32} & I_{33} & \cdots & I_{n-1} & I_n \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
R & I_{n2} & I_{n3} & \cdots & I_{n-1} & R_n
\end{pmatrix}, 
A := \begin{pmatrix}
R & I_3 & \cdots & I_{n-1} & I_n \\
R & R & I_3 & \cdots & I_{n-1} & I_n \\
R & R & R_3 & \cdots & I_{n-1} & I_n \\
R & R & I_{n3} & \cdots & I_{n-1} & I_n \\
R & R & I_{n-1} & \cdots & I_{n-1} & I_n \\
R & R & I_{n-1} & \cdots & I_{n-1} & I_n
\end{pmatrix}.
\]
Lemma 3.4. The above defined rings $B$ and $C$ are derived-equivalent.

An alternative proof of Lemma 3.4 can be found in [5, Theorem 5.2], where $A$ is replaced by the $n \times n$ matrix ring over $R$.

4 Higher algebraic $K$-theory of matrix subrings

In the algebraic $K$-theory of rings, the calculation of higher algebraic $K$-groups $K_n$ seems to be one of the interesting and hard problems. In this section, we shall provide formulas for computation of the $K_n$-groups of certain rings by applying the results in the previous section. Our computation is based on the philosophy that derived equivalences of rings preserve the $K$-theory and $G$-theory (see [9]), thus one can transfer the calculation of $K_n$ of a ring to that of another ring which is derived-equivalent to and may be much more simpler than the original one. In the literature, there are many papers dealing with $K_n$-groups by exploiting excision, Mayer-Vietoris exact sequences or other related sequences (for example, see [8], [15], [18], [26], [27]). However, it seems that there are few papers using derived equivalences to calculate the higher algebraic $K$-groups. In the present section we shall show that sometimes our philosophy works powerfully though it may be difficult to find derived equivalences in general. For some new advances in constructing derived equivalences, we refer the reader to recent papers [9][11].

Let $R$ be a ring with identity. We denote by $K_n(R)$ the series of algebraic $K$-groups of $R$ with $* \in \{0, 1, 2, \cdots, \}$. The algebraic $K$-theory of matrix-like rings has been of interest since a long time. In [1], Barrick and Keating showed the following result.

Lemma 4.1. If $R_i$ is a ring with identity for $i = 1, 2$, and if $M$ is an $R_1$-$R_2$-bimodule, then, for the triangular matrix ring

$$S = \begin{pmatrix} R_1 & M \\ 0 & R_2 \end{pmatrix},$$

there is an isomorphism of $K$-groups: $K_n(S) \simeq K_n(R_1) \oplus K_n(R_2)$ for all integers $n \in \mathbb{Z}$.

Moreover, this isomorphism is induced from the canonical inclusion of $R_1 \oplus R_2$ into $S$.

For $n = 0$, this is classical. For $n = 1, 2$, this was already shown by Dennis and Geller in 1976. We remark that Lemma 4.1 can be used to calculate the higher algebraic $K$-groups of algebras associated to finite $EI$-categories, or more generally, of “triangular” Artin algebras. Recall that an Artin algebra $A$ over a commutative Artin ring is said to be triangular if the set of non-isomorphic indecomposable projective $A$-modules can be ordered as $P_1, P_2, \cdots, P_n$ such that $\text{Hom}_A(P_j, P_i) = 0$ for all $j > i$. In this case, we have $K_n(A) \simeq \bigoplus_{j+i}^n K_n(\text{End}_A(P_j))$ by Lemma 4.1.

For a matrix ring of the form

$$C := \begin{pmatrix} R_2/I_2 & 0 & 0 & \cdots & 0 & 0 \\ R/I_2 & R & I_3 & I_4 & \cdots & I_n \\ R/I_{32} & R & I_3 & I_4 & \cdots & I_n \\ R/I_{42} & R & R_3 & I_4 & \cdots & I_n \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ R/I_{n-1,2} & R & I_{n-1,3} & I_4 & \cdots & I_n \\ R/I_{n2} & R & I_{n3} & I_4 & \cdots & I_{n-1} & I_n \end{pmatrix}$$

with the usual matrix addition and multiplication form three rings with identity. Note that only the second column of $A$ is different from the one of $B$.

We define a $B$-module $Q$ as follows:

$$0 \to Be_2 \xrightarrow{\lambda} Be_1 \xrightarrow{\pi} Q \to 0,$$

where $\lambda$ is a composition of the inclusion $Be_2 \to Ae_2$ with the isomorphism $Ae_2 \simeq Be_1$ as $B$-modules and where $\pi$ is the cokernel of $\lambda$.

Now, we consider the endomorphism ring $\text{End}_B(Q \oplus Be_1 \oplus Be_3 \oplus \cdots \oplus Be_n)$. By a proof similar to that of Lemma 5.2, one can show that the following lemma is true. We leave the details of its proof to the reader.

Lemma 3.4. The above defined rings $B$ and $C$ are derived-equivalent.
where $R$ is a ring and $I$ is an ideal in $R$ such that the $R$-modules $RI$ and $IR$ are projective, it was shown by Keating in [13] that there is an isomorphism of $K$-theory:

$$K_*(T) \simeq K_*(R) \oplus (n-1)K_*(R/I).$$

In [13], the author also considered the so-called trivial extension of a ring by a bimodule. It was shown that if $T$ is the trivial extension of a ring $R$ by an $R$-bimodule $M$, then $K_*(T) \simeq K_*(R)$ provided that $M$ has finite projective dimension as a left $T$-module. Here the condition on $M$ in this statement is necessary. See the counterexample $T := k[x]/(x^2)$ which is the trivial extension of $k$ by $k$, where $k$ is any field.

Recently, as a kind of generalization of the above result of Keating, the authors of [8] consider the following matrix ring: Let $B$ be the matrix ring $\begin{pmatrix} R & I & \cdots & I \\ R & R & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ R & R & \cdots & R \end{pmatrix}$. Let $B$ be the ring and $S$ be any abelian group.

We shall use our results in the previous section to extend all results on matrix rings mentioned above without any homological conditions on rings and ideals under investigation. Our proofs also explain the reason why the multiplicity $n-1$ appears in the above mentioned isomorphisms of the higher algebraic $K$-theory.

**Lemma 4.2.** Let $B$ be the matrix ring

$$B := \begin{pmatrix} R & I_{12} & I_{13} & \cdots & I_{1n} \\ R & R & I_{23} & \cdots & I_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ R & R & \cdots & R & I_{n-1n} \\ R & R & \cdots & R \end{pmatrix}$$

defined in Lemma 3.2. Then

$$K_*(B) \simeq K_*(R) \oplus \bigoplus_{j=1}^{n-1} K_*(R/I_{j-1j+1}).$$

**Proof.** We show this lemma by induction on $n$. By Theorem 2.2 (see [9]), the algebraic $K$-theory and $G$-theory are invariant under derived equivalences. So, by Lemma 3.2, we have $K_*(B) \simeq K_*(C)$ (for notation see Section 3). Now it follows from Lemma 4.1 that $K_*(C) \simeq K_*(B/I_{n-1n}) \oplus K_*(B_{n-1})$, where $B_{n-1}$ is the $(n-1) \times (n-1)$ left upper corner matrix subring of $B$. By induction, we have $K_*(B_{n-1}) \simeq K_*(R) \oplus K_*(R/I_{12}) \oplus \cdots \oplus K_*(R/I_{n-2n-1})$. Hence

$$K_*(B) \simeq K_*(R) \oplus K_*(R/I_{12}) \oplus \cdots \oplus K_*(R/I_{n-2n-1}) \oplus K_*(R/I_{n-1n}).$$

This finishes the proof of Lemma 4.2. □

In particular, as a consequence of Lemma 4.2, we can strengthen the result in [8] as the following corollary, here we drop all assumptions on rings and ideals.
Corollary 4.3. Let $R$ be an arbitrary ring with identity and $I$ an arbitrary ideal in $R$. Then, for a ring of the following form
\[
S = \begin{pmatrix}
R & P_{12} & \cdots & P_{1n} \\
R & R & \ddots & \vdots \\
\vdots & \vdots & \ddots & P_{n-1,n} \\
R & R & \cdots & R
\end{pmatrix},
\]
where $t_{ij}$ are positive integers, we have
\[
K_*(S) \simeq K_*(R) \oplus \bigoplus_{j=2}^{n} K_*(R/I^{j-1,j}).
\]

As a special case of Corollary 4.3, we get the following result of [13] without the assumption that $R/I$ and $I_R$ are projective.

Corollary 4.4. Let $R$ be a ring with identity and $I$ an ideal in $R$. Suppose that $t_j$ is a positive integers with $t_j \leq t_{j+1}$ for $j = 2, \ldots, n - 1$. Let
\[
T = \begin{pmatrix}
R & P_2 & \cdots & P_n \\
R & R & \ddots & \vdots \\
\vdots & \vdots & \ddots & P_n \\
R & R & \cdots & R
\end{pmatrix}.
\]
Then $T$ is a ring and
\[
K_*(T) \simeq K_*(R) \oplus \bigoplus_{i=2}^{n} K_*(R/I^i).
\]

Let us remark that if $I$ is a nilpotent ideal in $R$ with identity then $K_0(R) \simeq K_0(R/I)$. In general, this is not true for higher $K_*$-groups with $n \geq 1$. Thus, for $K_0$, we may replace the direct summands $K_0(R/I^i)$ by $K_0(R/I)$ in Corollary 4.4 and get $K_0(T) \simeq K_0(R) \oplus (n - 1) K_0(R/I)$.

Similarly, we have the following result on the groups $K_n$ of the ring defined in Lemma 3.4.

Lemma 4.5. Let $B$ be the ring
\[
B := \begin{pmatrix}
R & I_2 & I_3 & \cdots & I_{n-1} & I_n \\
R & R_2 & I_3 & \cdots & I_{n-1} & I_n \\
R & I_{32} & R_3 & \cdots & I_{n-1} & I_n \\
R & I_{42} & I_{43} & \cdots & I_{n-1} & I_n \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
R & I_{n-2} & I_{n-3} & \cdots & I_{nn-1} & R_n
\end{pmatrix}
\]
defined in Lemma 3.4. Then
\[
K_*(B) \simeq K_*(R) \oplus \bigoplus_{j=2}^{n} K_*(R/I_j).
\]

This result shows that the abelian group $K_n(B)$ of the ring $B$ is independent of the choice of the ideals $I_j$ in $R$ for all $n \geq 0$.

As a direct consequence of Lemma 4.5, we have the following corollary.

Corollary 4.6. Let $R$ be a ring with identity, and let $I_j$ be an ideal of $R$ with $2 \leq j \leq n$ such that $I_j \subseteq I_{j-1}$ for all $j$. Then, for the rings
Here is the text in a more readable format:

Let us remark that we can also use our method in this section to calculate some corner rings $eB_e$, though, in general, we cannot get an add$(eB_eAe)$-split sequence

$$0 \longrightarrow eB_e \longrightarrow eAe \longrightarrow eAe/eB_e \longrightarrow 0,$$

with $e$ an idempotent in $B$, from a given add$(B)$-split sequence

$$0 \longrightarrow B \longrightarrow A \longrightarrow A/B \longrightarrow 0.$$  

For example, suppose that $B$ is the ring defined in Lemma 4.2. If $e$ is an idempotent in $R$, then, for the corner ring

$$B_1 := \begin{pmatrix}
  eRe & eI_{12}e & eI_{13}e & \cdots & eI_{1n}e \\
  eRe & eRe & eI_{23}e & \cdots & eI_{2n}e \\
  \vdots & \vdots & \ddots & \cdots & \vdots \\
  eRe & eRe & eRe & \cdots & eI_{n-1n}e \\
  eRe & eRe & eRe & \cdots & eRe \\
\end{pmatrix}$$

of $B$, we have

$$K_\ast(B_1) \simeq K_\ast(eRe) \oplus \bigoplus_{j=1}^{n-1} K_\ast(eRe/eI_{j+1}e).$$

Also, we remark that, for any ring $R$, the $R$-duality Hom$_R(\cdot, R)$ is an equivalence between the category $R$-proj and the category $R^{op}$-proj, where $R^{op}$ is the opposite ring of $R$. Thus, for each $n \geq 0$, we have $K_n(R) \simeq K_n(R^{op})$. From this fact, or from Lemma 3.1(3) for right modules, we can see that if $S'$ is a ring of the form

$$S' := \begin{pmatrix}
  R & I_1 & I_1 & \cdots & I_1 \\
  I_2 & R & I_2 & \cdots & I_2 \\
  \vdots & \vdots & \ddots & \cdots & \vdots \\
  I_{n-1} & \cdots & I_{n-1} & R & I_{n-1} \\
  R & \cdots & R & R & R \\
\end{pmatrix},$$

where $R$ is a ring with identity and $I_j$ is an ideal of $R$ for each $1 \leq j < n$, then

$$K_\ast(S') \simeq K_\ast(R) \oplus \bigoplus_{j=1}^{n-1} K_\ast(R/I_j).$$

Note that $S'$ is closely related to the ring $S$ in Corollary 4.6(1).

Now, recall that a pullback diagram of rings:

$$\begin{array}{ccc}
R & \xrightarrow{f_1} & R_1 \\
\downarrow{h_2} & & \downarrow{h_1} \\
R_2 & \xrightarrow{f_2} & R_0
\end{array}$$

we have

$$K_\ast(S) \simeq K_\ast(R) \oplus \bigoplus_{j=2}^{n} K_\ast(R/I_j) \simeq K_\ast(T).$$
is called a Milnor square if one of \( f_2 \) and \( h_1 \) is surjective.

An example of Milnor squares is the following case: Let \( R \subseteq S \) be an extension of rings with the same identity. If there is an ideal \( J \) of \( S \) such that \( J \subseteq R \), then there is a canonical Milnor square

\[
\begin{array}{ccc}
R & \longrightarrow & S \\
\downarrow & & \downarrow \\
R/J & \longrightarrow & S/J.
\end{array}
\]

Let \( R \) be the product \( R_1 \times \cdots \times R_n \) of finitely many rings \( R_i \) with \( 1 \leq i \leq n \). A subdirect product of ring \( R \) is a subring \( S \subseteq R \) for which each projection \( S \rightarrow R_i \) carries \( S \) onto \( R_i \) for each \( i \). In this case we say that the inclusion \( S \subseteq R \) is an inclusion of a subdirect product.

The following lemma is useful and well-known for calculation of higher \( K \)-groups of rings.

**Lemma 4.7.** For a given Milnor square \((\ast)\), the following are true:

1. (See [17, Theorem 3.3]) There is a Mayer-Vietoris exact sequence:

\[
K_i(R) \left(\frac{\left(f_1, (h_2)\right)}{\left(f_2, (h_1)\right)}\right) K_i(R_1) \oplus K_i(R_2) \rightarrow K_i(R_0) \rightarrow K_0(R) \rightarrow K_0(R_1) \oplus K_0(R_2) \rightarrow K_0(R_0),
\]

where \( f_\ast \) denotes the homomorphism induced by \( f \).

2. (See [4, 25, Theorem 5.5]) Suppose that \((\ast)\) is a Milnor square of \( \mathbb{Z}/p^m\mathbb{Z} \)-algebras, where \( p \geq 2 \) and \( m \) are fixed positive integers. Let \( s \) be a non-zero integer such that \( p \) divides \( s \). Then there is an exact sequence of \( K \)-groups, that is, the Mayer-Vietoris sequence:

\[
\cdots \rightarrow K_{i+1}(R_1) \otimes \mathbb{Z}[\frac{1}{p}] \oplus K_{i+1}(R_2) \otimes \mathbb{Z}[\frac{1}{p}] \rightarrow K_{i+1}(R_0) \otimes \mathbb{Z}[\frac{1}{p}] \rightarrow K_i(R)
\]

\[
\left(\frac{\left(f_1, (h_2)\right)}{\left(f_2, (h_1)\right)}\right) K_i(R_1) \otimes \mathbb{Z}[\frac{1}{p}] \oplus K_i(R_2) \otimes \mathbb{Z}[\frac{1}{p}] \rightarrow K_i(R_0) \otimes \mathbb{Z}[\frac{1}{p}] \rightarrow \cdots.
\]

3. Suppose that \((\ast)\) is a Milnor square of \( \mathbb{Z}/p^m\mathbb{Z} \)-algebras, where \( p \geq 2 \) and \( m \) are fixed positive integers. Let \( s \) be a non-zero integer such that \( p \) divides \( s \). If the induced homomorphism \( f_1 \) in (2) is a split epimorphism for all \( \ast \in \mathbb{N} \), then there is an exact sequence for all \( \ast \in \mathbb{N} \):

\[
0 \rightarrow K_*(R) \otimes \mathbb{Z}[\frac{1}{p}] \rightarrow K_*(R_1) \otimes \mathbb{Z}[\frac{1}{p}] \oplus K_*(R_2) \otimes \mathbb{Z}[\frac{1}{p}] \rightarrow K_*(R_0) \otimes \mathbb{Z}[\frac{1}{p}] \rightarrow 0.
\]

In particular, if the induced homomorphism \( f_1 \) in (2) is an isomorphism for all \( \ast \in \mathbb{N} \), then so is the induced homomorphism \( f_1 \).

4. (16, Theorem 13.33, 17) If both \( h_1 \) and \( f_1 \) are surjective, or if \( h_1 \) is surjective and \( f_1 \) is the inclusion of a subdirect product, then there is an exact sequence

\[
K_2(R) \left(\frac{\left(f_1, (h_2)\right)}{\left(h_2, (f_1)\right)}\right) K_2(R_1) \oplus K_2(R_2) \rightarrow K_2(R_0) \rightarrow K_1(R) \rightarrow K_1(R_1) \oplus K_1(R_2) \rightarrow K_1(R_0).
\]

Remark that there is a dual statement of (3) for \( f_1 \) to be a split monomorphism for each \( \ast \in \mathbb{N} \).

Now we turn to proof of Theorem 11.1. Observe that the argument in our proof below is actually a combination of the previous results with Mayer-Vietoris exact sequences, and works also for many other cases. Here we prove only Theorem 11.1(1), and leave the details of the proof of Theorem 11.1(2) to the reader.
**Proof of Theorem 1.1 (1):** Let
\[ J := \begin{pmatrix} I & I_{12} & \cdots & I_{1n} \\ I & I & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ I & \cdots & \cdots & I_{n-1} \\ I & \cdots & \cdots & I \\ I & I & \cdots & I \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ I & \cdots & \cdots & I_{n-1} \\ I & \cdots & \cdots & I \\ I & I & \cdots & I \end{pmatrix}, \quad B := \begin{pmatrix} R & I_{12} & \cdots & I_{1n} \\ R & I & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ R & \cdots & \cdots & I_{n-1} \\ R & \cdots & \cdots & I \\ R & I & \cdots & I \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ R & \cdots & \cdots & I_{n-1} \\ R & \cdots & \cdots & I \\ R & I & \cdots & I \end{pmatrix}, \quad A := \begin{pmatrix} R & I_{12} & \cdots & I_{1n} \\ R & R & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ R & \cdots & \cdots & I_{n-1} \\ R & \cdots & \cdots & I \\ R & I & \cdots & I \end{pmatrix}. \]

By the assumptions in Theorem 1.1 (1), we can verify that \( A \) and \( B \) are rings and that \( J \) is an ideal of \( A \). Thus \( J \) is also an ideal of \( B \). Note that \( B \) is a subalgebra of \( A \). Let \( f \) be the inclusion of \( B \) into \( A \). If we define
\[ B' := B/J = \begin{pmatrix} R/I & 0 & \cdots & 0 \\ 0 & R/I & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & R/I \end{pmatrix}, \quad A' := A/J = \begin{pmatrix} R/I & 0 & \cdots & 0 \\ R/I & R/I & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ R/I & \cdots & R/I & R/I \end{pmatrix}, \]
then we have a Milnor square
\[ B \xrightarrow{f} A \downarrow \quad \quad B' \xrightarrow{f'} A', \]
where \( g' \) and \( g' \) are the canonical surjective maps, and where \( f' \) is the injective map induced from \( f \). Since the map \( f_1^*: K_*(B') \to K_*(A') \) is an isomorphism for \( * = 0, 1, 2, \cdots \), it follows that \( f_1^* \otimes \mathbb{Z}[\frac{1}{n}] : K_*(B') \otimes \mathbb{Z}[\frac{1}{n}] \to K_*(A') \otimes \mathbb{Z}[\frac{1}{n}] \) is an isomorphism. Thus we see from Lemma 4.5 (3) that \( f_1 \otimes \mathbb{Z}[\frac{1}{n}] \) is also an isomorphism. It then follows from Corollary 4.5 (2) that
\[ K_*(B) \otimes \mathbb{Z}[\frac{1}{n}] \cong K_*(A) \otimes \mathbb{Z}[\frac{1}{n}] \cong K_*(R) \otimes \mathbb{Z}[\frac{1}{n}] \otimes \bigoplus_{j=1}^{n-1} K_*(R/I_{j+1}) \otimes \mathbb{Z}[\frac{1}{n}]. \]

This finishes the proof of Theorem 1.1 (1).

If we define
\[ J := \begin{pmatrix} I & I_2 & I_3 & \cdots & I_n \\ I & I_2 & I_3 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ I & I_2 & I_3 & \cdots & I_n \\ I & I_2 & I_3 & \cdots & I_n \end{pmatrix}, \]
then the proof of Theorem 1.1 (2) can be carried out similarly since we have Lemma 4.5. \( \square \)

Now we mention the following corollary of the proof of Theorem 1.1

**Corollary 4.8.** Suppose that \( p \geq 2 \) and \( m \) are positive integers. Let \( R \) be a \( \mathbb{Z}/p^m \mathbb{Z} \)-algebra with identity, and let \( I \) and \( J \) be two arbitrary ideals of \( R \). Define
\[ S := \begin{pmatrix} R & I & \cdots & I \\ J & R & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ J & \cdots & J & R \end{pmatrix}_{n \times n}. \]

Then \( S \) is a ring, and if, in addition, \( I^2 \subseteq J \) (for example, \( I^2 = 0, \) or \( I \subseteq J \)), we have
\[ K_*(S) \otimes \mathbb{Z}[\frac{1}{n}] \cong K_*(R) \otimes \mathbb{Z}[\frac{1}{n}] \oplus (n-1)K_*(R/I) \otimes \mathbb{Z}[\frac{1}{n}] \]
for all non-zero integer \( s \) such that \( s \) is divisible by \( p \).
Proof. We define
\[
A := \begin{pmatrix} R & I & \cdots & I \\
I + J & R & \ddots & \vdots \\
\vdots & \ddots & \ddots & I \\
I + J & \cdots & I + J & R \end{pmatrix}_{n \times n}, \quad J' := \begin{pmatrix} I & I & \cdots & I \\
J & I & \ddots & \vdots \\
\vdots & \ddots & \ddots & I \\
J & \cdots & J & 1 \end{pmatrix}_{n \times n}.
\]
Then one can verify that \( A \) is a ring and \( J' \subseteq S \) is an ideal in \( A \). Note that \( S \) is a subring of \( A \). Now, let \( B := S, \ B' := B/J' \) and \( A' := A/J' \). Then we may use the same argument as in the proof of Theorem 1.1 to show \( K_*(B) \otimes \mathbb{Z}[\frac{1}{p}] \simeq K_*(A) \otimes \mathbb{Z}[\frac{1}{p}] \). But for the latter, we have \( K_*(A) \otimes \mathbb{Z}[\frac{1}{p}] \simeq K_*(R) \otimes \mathbb{Z}[\frac{1}{p}] \oplus (n-1)K_*(R/I) \otimes \mathbb{Z}[\frac{1}{p}] \) by Theorem 1.1. Thus Corollary 4.8 follows. \( \square \)

Let us illustrate how the argument in the above proof of Theorem 1.1 can be applied to other cases.

Again, suppose that \( p \geq 2 \) and \( m \) are positive integers. Let \( R \) be a \( \mathbb{Z}/p^m \mathbb{Z} \)-algebra with identity and \( I \) an arbitrary ideal of \( R \). For each finite partially ordered set \( P \), we associate a ring \( B := B(R, I, P) \) which is a subring of the matrix ring over \( R \) with indexing set \( P \), it is defined as follows: Let \( B = (B_{ij})_{i,j \in P} \) with \( B_{ij} = R \) if \( i \geq j \), and \( B_{ij} = I \) otherwise. We may assume that \( P = \{a_1, \ldots, a_n\} \) such that \( a_i \leq a_j \) implies \( i \leq j \). Under this assumption we see that \( J' := M_s(I) \) is an ideal of \( B \), which is also an ideal of

\[
A := \begin{pmatrix} R & I & \cdots & I \\
R & R & \ddots & \vdots \\
\vdots & \ddots & \ddots & I \\
R & \cdots & R & R \end{pmatrix}_{n \times n}.
\]

Note that \( B \) is a subring of \( A \). Let \( B' := B/J' \) and \( A' := A/J' \). We define \( C \) to be the diagonal matrix ring with the principal diagonal entries \( R/I \). Then \( C \) is a subring of both \( B' \) and \( A' \). Using this ring \( C \), we can see that the inclusion \( f' \) of \( B' \) into \( A' \) induces an isomorphism \( f'_s \) for all \( s \in \mathbb{N} \). Then we may use the same argument as the above to show that, for any \( s \) divisible by \( p \),

\[
K_*(B) \otimes \mathbb{Z}[\frac{1}{p}] \simeq K_*(A) \otimes \mathbb{Z}[\frac{1}{p}] \simeq K_*(R) \otimes \mathbb{Z}[\frac{1}{p}] \oplus (n-1)K_*(R/I) \otimes \mathbb{Z}[\frac{1}{p}].
\]

We end this section by a couple of remarks concerning Theorem 1.1

1. In Theorem 1.1, if \( R \) is a \( \mathbb{Z}/p \mathbb{Z} \)-algebra instead of a \( \mathbb{Z}/p^m \mathbb{Z} \)-algebra, and if \( R/I, R/I_i \) and \( R/I_{ij} \) are finite rings for all \( i, j \), then Theorem 1.1 still holds true. Indeed, in this case we can use Charney’s excision at the end of the paper [4] since \( I \otimes \mathbb{Z}[\frac{1}{p}] \) is a unit. This is due to \( \text{Tor}_2^R(-, \mathbb{Z}[\frac{1}{p}]) = 0 \) and to the fact that the quotient rings \( R/I, R/I_i \) and \( R/I_{ij} \) are \( \mathbb{Z}/p \mathbb{Z} \)-algebras for some \( m > 0 \). Indeed, we have an exact sequence

\[
\text{Tor}_2^R(R/I, \mathbb{Z}[\frac{1}{s}]) \longrightarrow I \otimes \mathbb{Z}[\frac{1}{s}] \longrightarrow R \otimes \mathbb{Z}[\frac{1}{s}] \longrightarrow (R/I) \otimes \mathbb{Z}[\frac{1}{s}].
\]

Clearly, the first and last terms vanish, this implies \( I \otimes \mathbb{Z}[\frac{1}{s}] \simeq R \otimes \mathbb{Z}[\frac{1}{s}] \). So, the condition of Charney’s result in [4] is satisfied. I thank X. J. Guo for explaining this fact.

2. A crucial fact of our proofs of the main results is: Given an extension \( B \subseteq A \) of rings with the same identity such that \( pA \) is finitely generated and projective, we have \( K_*(B) \simeq K_*(\text{End}_B(A \oplus A/B)) \) for all \( s \in \mathbb{N} \). Moreover, we may also compare the algebraic K-theory of \( B \) with that of \( A \). For this purpose, we define \( \Omega \) to be the kernel of the multiplication map \( A \otimes_B A \to A \). It follows from the Additivity Theorem (see [19] Corollary 1, Section 3) that the exact sequence of the exact functors

\[
0 \longrightarrow \Omega \otimes_A \longrightarrow A \otimes_B \longrightarrow id \longrightarrow 0
\]
on the category of finitely generated projective \( A \)-modules gives rise to three homomorphisms of abelian groups:

\[
r_* : K_*(A) \to K_*(B), \quad t_* : K_*(B) \to K_*(A) \quad \text{and} \quad \omega_* : K_*(A) \to K_*(A) \quad \text{such that} \quad r_* t_* = 1_{K_*(A)} + \omega_*. \]

If, in addition, the \( n \)-fold tensor product of \( \Omega \) over \( A \) vanishes for some natural number \( n \), that is, \( \Omega^\otimes n = 0 \) (for example, \( \Omega = 0 \) in case the inclusion \( B \subseteq A \) is an injective ring epimorphism), then the map \( t_* \) is split surjective, and \( K_*(A) \) is a direct summand of \( K_*(B) \). In general, neither \( t_* \) nor \( r_* \) is an isomorphism.
5 Lower $K$-theory for matrix subrings

In this section we consider the algebraic $K$-groups $K_0$ and $K_1$ for matrix subrings. Our results in this section are not covered by the main results in the previous sections.

We first consider the group $K_0$. In this case, we have the following result in which we do not assume that the rings considered are $\mathbb{Z}/p^n\mathbb{Z}$-algebras or $\mathbb{Z}[\frac{1}{p}]$-algebras.

**Proposition 5.1.** Let $R$ be an arbitrary ring with identity, and let $I, J$ and $I_j$ be ideals in $R$.

1. For the rings $S$ and $T$ defined in Theorem 1.1, we have

$$K_0(S) \simeq K_0(R) \oplus \bigoplus_{j=2}^{n} K_0(R/I_{j-1}), \quad K_0(T) \simeq K_0(R) \oplus \bigoplus_{j=2}^{n} K_0(R/I_j).$$

2. For the ring $S$ defined in Corollary 4.4 with $I^2 \subseteq J$, we have

$$K_0(S) \simeq K_0(R) \oplus (n - 1)K_0(R/I).$$

The proof of this proposition is actually a combination of Corollary 4.5 and Lemma 4.7(1) and (3), and we leave the details of the proof to the interested reader.

Here arises an open question: We do not know, at moment, if Proposition 5.1 is true for higher algebraic $K$-groups $K_n$ with $n \geq 1$. But, for $K_1$, we do have some partial answers. Before stating our result, we first prove the following lemma.

**Lemma 5.2.** Let $B \subseteq A$ be an extension of rings with the same identity. Suppose that $I$ is an idempotent ideal of $A$ contained in $B$. If the inclusion $B \subseteq A$ induces an isomorphism $\gamma_i : K_i(B/I) \to K_i(A/I)$ for $i = 1, 2$, then $K_1(B) \simeq K_1(A)$.

**Proof.** Let $K_i(B,I)$ denote the $i$-th relative $K$-group of the canonical surjective map $B \to B/I$. Then we may form the following commutative diagram of abelian groups with exact rows:

$$
\begin{array}{cccccccc}
K_2(B/I) & \longrightarrow & K_1(B,I) & \longrightarrow & K_1(B) & \longrightarrow & K_1(B/I) & \longrightarrow & K_0(B,I) \\
\gamma & \downarrow & \gamma & \downarrow & \gamma & \downarrow & n & \simeq \\
K_2(A/I) & \longrightarrow & K_1(A,I) & \longrightarrow & K_1(A) & \longrightarrow & K_1(A/I) & \longrightarrow & K_0(A,I).
\end{array}
$$

Here we use the fact that $K_0(B,I)$ is always independent of $B$. Now, by the Five Lemma in homological algebra, we know that the map $\beta$ is isomorphic if $\gamma$ is isomorphic. However, this follows from a result of Vaserstein (see [27], Chapter III, Section 2, Remark 2.2.1), which states that if $J$ is an ideal in a ring $R$ with identity, then $K_1(R/J)$ is independent of $R$ if and only if $J^2 = J$. Thus $\gamma$ is an isomorphism. $\square$

We should notice that, in general, $K_n(R,I)$ depends on $R$ for $n \geq 1$. This is why the conclusions in Theorem 1.1 are localized.

So, with Lemma 5.2 in mind, we can prove the following proposition for $K_1$.

**Proposition 5.3.** Let $R$ be a ring with identity, and let $I \subseteq J$ be ideals in $R$. If $I$ is idempotent, then, for the ring

$$B := \begin{pmatrix} R & I & \cdots & I \\ J & R & \ddots & ; \\ \vdots & \ddots & \ddots & I \\ J & \cdots & J & R \end{pmatrix}_{n \times n},$$

we have

$$K_1(B) \simeq K_1(R) \oplus (n - 1)K_1(R/I).$$
In this section, we shall point out that our main result, Theorem 1.1, holds true for the mod-
under the assumption that algebras considered are
6 Higher mod-
calculation of lower
By Lemma 4.7(4), we can show that
Thus one has a Milnor diagram
Lemma 6.2. [26, Corollary 1.3]
and
Finally, we mention another type of matrix rings: Let R and S are rings with identity and \(rM_S\) and \(sN_R\) are bimodules. We define a ring
\[
A := \begin{pmatrix} R & M \\ N & S \end{pmatrix}, \quad \begin{pmatrix} r & m \\ n & s \end{pmatrix}' \begin{pmatrix} r' & m' \\ n' & s' \end{pmatrix} = \begin{pmatrix} rr' & rm' + ms' \\ nr' + sn' & ss' \end{pmatrix}
\]
for \(r, r' \in R, s, s' \in S, m, m' \in M\) and \(n, n' \in N\). Note that \(M' := \begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix}\) and \(N' := \begin{pmatrix} 0 & 0 \\ N & 0 \end{pmatrix}\) are two ideals in A.
Thus one has a Milnor diagram
\[
\begin{array}{ccc}
A & \longrightarrow & A/M' \\
\downarrow & & \downarrow \\
A/N' & \longrightarrow & A/(M' + N').
\end{array}
\]
By Lemma 4.7(4), we can show that \(K_i(A) \simeq K_i(R) \oplus K_i(S)\) for \(i = 0, 1\). This result can be used to reduce the calculation of lower \(K\)-groups of finite-dimensional algebras with radical-square-zero to local algebras.

6 Higher mod-\(p\) \(K\)-theory

In this section, we shall point out that our main result, Theorem 1.1 holds true for the mod-\(p\) \(K\)-theory \(K_*(\ldots, \mathbb{Z}/p\mathbb{Z})\) under the assumption that algebras considered are \(\mathbb{Z}_{1/p}\)-algebras and \(p \not\equiv 2\) (mod 4), where \(p \geq 2\) is any positive integer.

Let \(R\) be a ring with identity. In [3], Browder developed \(K\)-theory with coefficients \(\mathbb{Z}/p\mathbb{Z}\). This is the so-called mod-\(p\) \(K\)-theory \(K_*(R, \mathbb{Z}/p\mathbb{Z})\) for \(* \in \mathbb{Z}\). Note that \(K_0(R, \mathbb{Z}/p\mathbb{Z}) = K_0(R) \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z}\) and \(K_i(R, \mathbb{Z}/p\mathbb{Z}) = 0\) if \(i < 0\) (see [3] p. 45). Later, Weibel observed in [26] that excision holds and that Mayer-Vietoris sequences exist if the rings involved are \(\mathbb{Z}_{1/p}\)-algebras. The mod-\(p\) \(K\)-theory is closely related to the usual \(K\)-theory in the following manner.

Lemma 6.1. Universal Coefficient Theorem (see [3] and [26]):
Let \(R\) be a \(\mathbb{Z}_{1/p}\)-algebra with identity. For all \(* \in \mathbb{N}\), there is a short exact sequence of abelian groups
\[
0 \longrightarrow K_*(R) \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z} \longrightarrow K_*(R, \mathbb{Z}/p\mathbb{Z}) \longrightarrow \text{Tor}^\mathbb{Z}_1(K_{*-1}(R), \mathbb{Z}/p\mathbb{Z}) \longrightarrow 0.
\]
If \(p \not\equiv 2\) (mod 4), then this sequence splits (not naturally), so that \(K_*(R, \mathbb{Z}/p\mathbb{Z})\) is a \(\mathbb{Z}/p\mathbb{Z}\)-module. If \(p \equiv 2\) (mod 4), then \(K_*(R, \mathbb{Z}/p\mathbb{Z})\) is a \(\mathbb{Z}/2p\mathbb{Z}\)-module.

Thus, if \(p \not\equiv 2\) (mod 4), we see that \(K_*(R, \mathbb{Z}/p\mathbb{Z})\) is completely determined by the usual \(K\)-groups \(K_*(R)\).

Another result which we need is a Mayer-Vietoris sequence for mod-\(p\) \(K\)-groups.

Lemma 6.2. [26, Corollary 1.3] For a Milnor square (\(*\)) of \(\mathbb{Z}_{1/p}\)-algebras, there is a long exact sequence of abelian groups for all integers \(*\):
\[
\cdots \longrightarrow K_{*+1}(R_1, \mathbb{Z}/p\mathbb{Z}) \oplus K_{*+1}(R_2, \mathbb{Z}/p\mathbb{Z}) \longrightarrow K_{*+1}(R_0, \mathbb{Z}/p\mathbb{Z}) \longrightarrow \]
\[
K_*(R, \mathbb{Z}/p\mathbb{Z}) \longrightarrow K_*(R_1, \mathbb{Z}/p\mathbb{Z}) \oplus K_*(R_2, \mathbb{Z}/p\mathbb{Z}) \longrightarrow K_*(R_0, \mathbb{Z}/p\mathbb{Z}) \longrightarrow \cdots.
\]
Now it follows from the above two lemmas and Theorem [1.2] that Theorem [1.1] holds true for the mod-$p$ $K$-groups $K_*(R,\mathbb{Z}/p\mathbb{Z})$ if $R$ is a $\mathbb{Z}^{1/p}$-algebra and if $p \neq 2$ (mod 4), since the argument there in the proof of Theorem [1.1] works in our new situation.

If $p \equiv 2$ (mod 4), we do not know whether $K_*(R,\mathbb{Z}/p\mathbb{Z})$ can be fully controlled by the first and last terms in Lemma [6.1]. In general, extensions of fixed abelian groups may not be isomorphic, for instance, the cyclic group of order 4 and the Klein group $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$ both are extensions of the cyclic group of order 2 by itself, but they are not isomorphic.

7 Examples: GV-ideals

In this section we shall give some examples related to our results. The first one is constructed from a $\mathcal{D}$-split sequence which is induced by a surjective ring homomorphism.

Let $B$ be a ring with identity and $J$ an ideal of $B$. We define $A = B/J$. Then we have an exact sequence in $B$-Mod:

$$0 \rightarrow J \rightarrow B \xrightarrow{\pi} A \rightarrow 0,$$

where $\pi$ is the canonical surjection.

For this sequence to be an add$(B/B)$-split sequence, we have to assume $\text{Ext}^1_B(A,B) = 0$. This happens often in commutative algebra. For example, if $B$ is a commutative noetherian ring, and $J$ is an ideal of $B$ such that $J$ contains a regular sequence on $B$ of length 2, then $\text{Ext}^1_B(A,B) = 0$ for $i = 0, 1$ (see [12 p. 101]). Another example is the so-called GV-ideals in integral domains. Here we will state the following general definition of GV-ideals.

Let $R$ be an arbitrary ring with identity. Recall that an ideal $I$ of $R$ is called a GV-ideal (after the names Glaz and Vasconcelos, see [29, 71]) if the induced map $\mu_I : R \rightarrow \text{Hom}_R(I,R)$, given by $r \mapsto (x \mapsto xr)$ for $x \in I$, is an isomorphism of $R$-bimodule. This is equivalent to $\text{Ext}^i_R(R/I,R) = 0$ for $i = 0, 1$. Thus $R$ is a GV-ideal of $R$. Note that $p\mathbb{Z}$ is not a GV-ideal of $\mathbb{Z}$ for any $p \in \mathbb{Z}$ with $|p| \neq 1$, even though we have $\mathbb{Z} \simeq \text{Hom}_\mathbb{Z}(p\mathbb{Z},\mathbb{Z})$. We remark that the above definition of GV-ideals is more general than that in commutative rings where it is required that $R/I$ is finitely generated (see [29]).

Let $GV(R)$ be the set of all GV-ideals of $R$. For ideals $I$ and $J$ of $R$, we denote by $(I : J) := \{ x \in R \mid Ix \subseteq J \}$. (This notation is different from what was usually used in ring theory, but soon we will see its convenience when elements compose). Clearly, $(I : R) = R$, $(R : I) = I$, and $(I : J)$ is an ideal of $R$.

The following lemma shows some properties of GV-ideals, which are of interest for our proofs.

Lemma 7.1. Let $B$ be a ring with identity, and let $J$ be a GV-ideal in $B$. Then

1. the sequence $0 \rightarrow J \rightarrow B \xrightarrow{\pi} A \rightarrow 0$ is an add$(B/B)$-split sequence in $B$-Mod. Thus $\text{End}_B(B \oplus J)$ is derived-equivalent to $\left( \begin{array}{c} B \\ B/J \\ 0 \end{array} \right)$.

2. $\text{End}_B(J) \simeq B$ (as rings and as $B$-bimodules).

3. If $I$ is an ideal in $B$, then $\text{Hom}_B(J,I) \simeq (J : I)$ as $B$-bimodules. In particular, if $J \subseteq I$, then $\text{Hom}_B(J,I) \simeq B$.

4. If $x \in B$ such that $Jx = 0$, then $x = 0$.

5. If $I$ is an ideal in $B$ with $J \subseteq I$, then $I \subseteq GV(J)$.

6. If $I,J \in GV(B)$, then $IJ \subseteq GV(B)$.

Proof. (1) is clear by [10, Theorem 1.1] and the definition of GV-ideals.

(2) By the definition of GV-ideals, the induced map $\mu_J : B \rightarrow \text{Hom}_B(gJ,B)$ is an isomorphism, this means that every homomorphism $f$ from $gJ$ to $gB$ is given by the right multiplication of an element in $B$. Since $J$ is an ideal in $R$, $f$ is in fact an endomorphism of the module $gJ$. Conversely, if $f \in \text{End}_B(J)$, then $f$ is a restriction of a right multiplication of an element of $B$. Hence $\text{End}_B(J) \simeq B$.

(3) We define a map $\varphi : \text{Hom}_B(J,I) \rightarrow (J : I)$ as follows: For $f \in \text{Hom}_B(J,I)$, there is a unique element $b \in B$ such that the composition of $f$ with the inclusion $\lambda : I \rightarrow B$ is the right multiplication map $\cdot b$ since $J \subseteq GV(B)$. This means that $f\lambda = b$ and $b \in (J : I)$. So, we define $f \mapsto b$. As $(J : I)$ is an ideal of $B$, it has a commutative bimodule structure. Now one can check that $\varphi$ is an isomorphism of $B$-bimodules.

(4) This is a trivial consequence of the induced isomorphism $\mu_J : B \simeq \text{Hom}_B(J,B)$.

(5)-(6) These statements were already proved in detail in [29] for commutative rings, the ideas of their proofs are as follows: It follows from (4) that $\text{Hom}_B(J/I,B) = 0$. Further, by the isomorphism $\mu_J$ and the fact $\mu_J = \mu_Ji_e$ where $i_e : \text{Hom}_B(I,B) \rightarrow \text{Hom}_B(J,B)$ is induced from the inclusion $i : J \rightarrow I$, one can check that $\mu_J$ is an isomorphism of $B$-bimodules. This proves (5).
Let $I, J \in GV(B)$. It follows from (4) that $\mu_{IJ} : B \to \text{Hom}_B(II, B)$ is injective. We show that it is also surjective. In fact, since the composition of the maps $B \to \text{Hom}_B(J, B) \to \text{Hom}_B(J, \text{Hom}_B(I, B)) \to \text{Hom}_B(I \otimes_B J, R)$ is an isomorphism of $B$-bimodules, which is the composition of $\mu_{IJ}$ with the injective map $m_* : \text{Hom}_B(II, B) \to \text{Hom}_B(I \otimes_B J, B)$ induced from the surjective multiplication map $I \otimes_B J \to IJ$, we see that $m_*$ is surjective, thus it is an isomorphism of $B$-bimodules. This implies that $\mu_{IJ}$ is surjective, and therefore (6) holds. □

From Lemma [7.1] we have the following

**Proposition 7.2.** Let $B$ be a ring with identity. Suppose that $I_n \subseteq I_{n-1} \subseteq \cdots \subseteq I_2 \subseteq I_1$ is a chain of ideals in $B$. If $I_n$ is a GV-ideal in $B$, then

1. $\text{End}_B(I_1 \oplus \cdots \oplus I_n)$ is isomorphic to

$$C := \begin{pmatrix} B & (I_1 : I_2) & (I_1 : I_3) & \cdots & (I_1 : I_n) \\ B & B & (I_2 : I_3) & \cdots & (I_2 : I_n) \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ B & B & \cdots & B & (I_{n-1} : I_n) \\ B & B & \cdots & B & B \end{pmatrix}.$$  

2. $K_*(\text{End}_B(\bigoplus_{j=1}^n I_j)) \simeq K_*(B) \oplus \bigoplus_{j=1}^{n-1} K_*(B/(I_j : I_{j+1}))$ for all $* \in \mathbb{N}$.

**Proof.** (1) Note that $\text{End}_B(I_1 \oplus I_2 \oplus \cdots \oplus I_n)$ is the matrix ring with the entries $\text{Hom}_B(I_i, I_j)$ for $1 \leq i, j \leq n$. Since $I_n$ is a GV-ideal in $B$, every ideal $I_j$ in the chain is a GV-ideal of $B$ by Lemma [7.1]. Now (1) follows from Lemma [7.1] immediately.

(2) This is a direct consequence of (1) and Lemma [4.12]. □

As a consequence of Proposition [7.2] and Lemma [7.1], we have the following corollary.

**Corollary 7.3.** If $I$ is a GV-ideal in a ring $B$ with identity, then, for any positive integer $n$,

$$K_*(\text{End}_B(\bigoplus_{j=1}^n I_j)) \simeq K_*(B) \oplus \bigoplus_{j=1}^{n-1} K_*(B/(I_j : I_{j+1})).$$

If we take $I_1 = B$, we have the following corollary.

**Corollary 7.4.** Let $B$ be a ring with identity. Suppose that $I_n \subseteq I_{n-1} \subseteq \cdots \subseteq I_2 \subseteq I_1 = B$ is a chain of GV-ideals in $B$. Then

$$K_*(\text{End}_B(B \oplus \bigoplus_{j=2}^n B/I_j)) \simeq K_*(B) \oplus K_*(B/I_2) \oplus \bigoplus_{j=2}^{n-1} K_*(B/(I_j : I_{j+1})).$$

**Proof.** For each $j$, we have an add($B$)-split sequence by Lemma [7.11]:

$$0 \to I_j \to B \to B/I_j \to 0.$$

This yields another add($B$)-split sequence

$$0 \to \bigoplus_{j=1}^n I_j \to \bigoplus_{j=1}^n B \to \bigoplus_{j=1}^n B/I_j \to 0.$$

Hence $\text{End}_B(B \oplus \bigoplus_{j=2}^n I_j)$ and $\text{End}_B(B \oplus \bigoplus_{j=2}^n B/I_j)$ are derived-equivalent by [10, Theorem 1.1], and have the isomorphic algebraic $K$-groups $K_*$. By Lemma [7.2] we see that $K_*(\text{End}_B(B \oplus \bigoplus_{j=2}^n B/I_j)) \simeq K_*(B) \oplus \bigoplus_{j=2}^{n-1} K_*(B/(I_j : I_{j+1}))$ for all $* \in \mathbb{N}$. □

As a concrete example, we consider the polynomial ring $B := \mathbb{Z}[x]$ over $\mathbb{Z}$ in one variable $x$ and its ideal $J := (p, x)$ with $p$ a prime number in $\mathbb{Z}$. It is known that $J$ is a GV-ideal in $B$. Thus, for the ring $R := \text{End}_{\mathbb{Z}[x]}(\mathbb{Z}[x] \oplus J)$, by Proposition [7.2] we have

$$K_*(R) \simeq K_*(\mathbb{Z}[x]) \oplus K_*(\mathbb{Z}/p\mathbb{Z}).$$

Since $\mathbb{Z}$ is a left noetherian ring of global dimension one, the Fundamental Theorem in algebraic $K$-theory says that the above isomorphism can be rewritten as

$$K_*(R) \simeq K_*(\mathbb{Z}) \oplus K_*(\mathbb{Z}/p\mathbb{Z}).$$
By [20], we get

\[ K_0(R) \simeq \mathbb{Z} \oplus \mathbb{Z}, \quad K_1(R) \simeq \mathbb{Z}/2\mathbb{Z} \oplus (\mathbb{Z}/p\mathbb{Z})^\times, \]
\[ K_{2m}(R) = K_{2m}(\mathbb{Z}) \quad \text{for} \quad m \geq 1, \quad K_{2m-1}(R) \simeq K_{2m-1}(\mathbb{Z}) \oplus \mathbb{Z}/(p^m - 1)\mathbb{Z} \quad \text{for} \quad m \geq 2, \]

where \((\mathbb{Z}/p\mathbb{Z})^\times\) denotes the set of all non-zero elements of \(\mathbb{Z}/p\mathbb{Z}\). Note that \(J\) is not a projective \(\mathbb{Z}[x]\)-module. In fact, we have a non-split exact sequence

\[ 0 \longrightarrow \mathbb{Z}[x] \xrightarrow{\lambda} \mathbb{Z}[x] \oplus \mathbb{Z}[x] \xrightarrow{\pi} J \longrightarrow 0, \]

where \(\lambda\) sends \(f(x)\) to \((xf(x), -pf(x))\), and \(\pi\) sends \((f(x), g(x))\) to \(pf(x) + xg(x)\) for all \(f(x), g(x) \in \mathbb{Z}[x]\). So, the result in [13] cannot be applied to \(R\). However, the one in this note is applicable.

Finally, we mention the radical-full extensions in [23]. Recall that an extension \(B \subseteq A\) of rings with the same identity is said to be left radical-full if \(\operatorname{rad}(B)\) is a left ideal of \(A\) and \(\operatorname{rad}(A) = \operatorname{rad}(B)A\), where \(\operatorname{rad}(A)\) stands for the Jacobson radical of \(A\). So, given a left radical-full extension \(B \subseteq A\) of rings, we may form the ring \(C := \begin{pmatrix} A & \operatorname{rad}(B) \\ A & B \end{pmatrix}\). It follows from our results in this note that \(K_n(C) \simeq K_n(A) \oplus K_n(B/\operatorname{rad}(B))\) for all \(n \geq 0\) since for any ring extension \(S \subseteq R\) and any ideal \(I\) in \(S\), if \(I\) is a left ideal in \(R\) then the rings \(\left( \begin{array}{cc} R & I \\ R & S \end{array} \right)\) and \(\left( \begin{array}{cc} R/I & 0 \\ R/I & R \end{array} \right)\) are derived-equivalent by Lemma [3,4].

Related to the last example, we have the following open question:

**Question:** Suppose that \(I\) and \(J\) are two arbitrary ideals in a ring \(R\) with identity. For the ring \(S := \left( \begin{array}{cc} R & I \\ J & R \end{array} \right)\), can one give a formula for \(K_n(S)\) similar to the one in Theorem [1,2] for \(n \in \mathbb{N}\)? (See also the question mentioned in Section 5.)

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