Calculation of the Casimir energy at zero and finite temperature: some recent results

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Abstract

This survey summarizes briefly results obtained recently in the Casimir energy studies devoted to the following subjects: i) account of the material characteristics of the media in calculations of the vacuum energy (for example, Casimir energy of a dilute dielectric ball); ii) application of the spectral geometry methods for investigating the vacuum energy of quantized fields with the goal to gain some insight, specifically, in the geometrical origin of the divergences that enter the vacuum energy and to develop the relevant renormalization procedure; iii) a universal method for calculating the high temperature dependence of the Casimir energy in terms of heat kernel coefficients. A special attention is payed to the mathematical tools applied in this field, namely, to the spectral zeta function method and heat kernel technique.

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I. INTRODUCTION

In 1948 Casimir [1] proceeding from the general principles of quantum theory of electromagnetic field has shown that two uncharged perfectly conducting plates, placed in vacuum, should attract each other with the force per unit area

\[ F = -\frac{\pi^2 c \hbar}{240 a^4}, \]  

where \( a \) is the distance between the plates. Later this prediction was verified experimentally [2]. For dielectric bodies the Casimir force was also measured with reasonable agreement to the theory [3].

The Casimir force is very weak, however it increases rapidly as the separation \( a \) decreases and it becomes measurable when \( a \sim 1 \mu m \) or less. For plates 1 cm in area with \( a = 0.5 \mu m \), the Casimir force is about 0.2 dyn. Firstly the Casimir force has been measured, due to technical reasons, between a conducting plane and a sphere with progressively higher precision using torsion balances [4], atomic force microscopes [5], and capacitance bridges [6]. Only recently this force was measured between parallel metallic surfaces in the range 0.5 – 3.0 \( \mu m \) at the 15% precision level using cantilevers [7]. The Casimir force may be relevant in nanotechnology [6,8,9]. At a separation of 10 nm, this force is \( \sim 1 \) atm. Now the temperature dependence of the Casimir forces becomes a feasible task for experiment [10,11]. The juxtaposition of the experimental results and theoretical calculations with detailed analysis of errors and precision achieved in this field can be found in ref. [11]. An interesting proposal was made recently to measure variations of the Casimir energy by making use a superconducting cavity [12].

The Casimir effect is, in fact, the first macroscopic effect predicted in the framework of the relativistic quantum field theory, namely, in quantum electrodynamics. The macroscopic feature of the Casimir effect implies that it is observed in the experiments with macroscopic bodies, instead of the systems of atomic size. In this respect, it is analogous to superconductivity and superfluidity.

This effect was always an interesting subject of both theoretical and experimental studies. Now the Casimir effect is treated in a more general way, namely, as the influence on the physical characteristics of a quantum field system of the external conditions (boundary conditions, background fields and so on) that lead to the restriction of the configuration space of the system under study.

Calculations of the Casimir forces for configurations more complicated than two parallel plates proved to be difficult, and the progress in this field is rather slow. Till now one has no intuition even as to whether the force should be attractive or repulsive for any given boundary geometry, not to mention the estimation of its value.

In the case of perfectly conducting surfaces placed in vacuum the Casimir energy is calculated exactly, safe for parallel plates [1], for spheres [13–18] and for circular infinite cylinders [19–21].

Dielectric properties of the media separated by plane boundaries did not add new mathematical difficulties [22–24] as compared with the Casimir pioneer paper [1]. However the first result on the calculation of the Casimir energy for the non-flat boundaries was obtained only in 1968. By computer calculations, lasted 3 years, Boyer found the Casimir energy of
a perfectly conducting spherical shell [13]. Account of dielectric and magnetic properties of
the media in calculations of the vacuum energy for nonflat interface leads to new difficulties
or, more precisely, to a new structure of divergencies.

The calculation of the Casimir energy in a special case, when both the material media
have the same velocity of light, turns out to be, from the mathematical standpoint, exactly
the same as for perfectly conducting shells placed in vacuum and having the shape of the
interface between these media [19,25–36].

The first attempt to calculate the Casimir energy of a dielectric compact ball has been
undertaken by Milton in 1980 [37]. And only just recently the final result was obtained for
a \textit{dilute} dielectric ball at zero [38–45] and finite [46,47] temperature.

Situation with a dielectric compact cylinder proves to be much more complicated. The
main reason of this is that here there is no decoupling between transverse-electric and
transverse-magnetic modes [19]. It was shown that the summation of van der Waals in-
teractions inside a purely dielectric cylinder in the dilute-dielectric approximation lead to a
surprising null result [19]. First it was derived independently by Romeo and Milonni (un-
published calculations) and by Barton in ref. [48]. Bordag and Pirozhenko [49] have found
the first few heat kernel coefficients for this configuration. Their results imply, in particular,
that zeta function technique should supply a finite value for vacuum energy of a dielectric
cylinder in the \textit{dilute} approximation. And only very recently Cavero-Pelaez and Milton [50]
have shown rigorously (in quantum field theory approach) that the Casimir energy of an
infinite circular dielectric cylinder vanishes through the second order in the deviation of
the permittivity from its vacuum value. This result is of a five-year-long effort! The same
vanishing value for this energy was derived anew in paper [51] by making use of the mode
summation method.

For the force between a plane and a sphere the exact result is not known, and only an
estimation valid when both are close enough is available. This estimation is based on the
so called proximity theorem [52,53]. Similar estimations have been recently obtained for
the force between two spheres [54]. The problems of two concentric spheres [45,55] and two
concentric cylinders [56] have also been considered recently. For both cases expressions for
the Casimir energy have been obtained using the Abel-Plana formalism. However a detailed
numerical calculation and analysis of the results are still missing. The authors of ref. [57]
studied the Casimir interactions, due to the massless scalar field fluctuations, of two surfaces
which are close to each other. Two close co-axial cylinders, co-centric spheres co-axial cones
and two tori were considered. The first correction to the parallel plates result was calculated.

For dielectric-diamagnetic media with a continuous velocity of light at the interface the
exact results are known for a sphere [25,27,29–36] and cylinder [19,26,35].

For arbitrary dielectric media the exact results are obtained for parallel plates (Lifshitz
formula [22–24]), for a dilute dielectric ball [38–43,46] and for a dilute dielectric cylin-
der [50,51]. A single surface immersed in an inhomogeneous medium also experiences the
Casimir force [58].

Recently there appear comprehensive reviews [59,60] and excellent book [61] which are
devoted to the Casimir effect. Previous survey publications in this field [62–67] remain
valuable. Many references concerning vacuum energy can be found in [68]. Of special
interest are the Proceedings of regularly organized, every 3 years since 1989, Workshops on
Quantum Field Theory Under the Influence of External Conditions [69–72].
In this situation a new survey in this field should be naturally concerned with particular problems to investigation of which a certain contribution has been done by present authors. First of all, we are going to give a lucid and at the same time comprehensive introduction into the mathematical methods that are used in current Casimir calculations. We hope that the review will provide a sufficient material in order to apply these methods to coinciding problems in diverse branches of theoretical physics.

The layout of the review is as follows. In sect. II we consider the basic terms used in the Casimir studies, specifically, the notion of zero point energy. To our mind employment of this term is not only convenient but also physically justified in this field. In sect. III the mathematical methods applied in Casimir calculations are considered, namely, the spectral zeta functions and heat kernel technique. In sect. IV the spectral zeta functions for simplest boundary configurations are calculated and the relevant Casimir energies are found. Section V is devoted to calculation of the vacuum energy of electromagnetic field connected with a dielectric ball. This problem proved to be nontrivial one. We are trying to reveal the principal drawbacks encountered here and summarize the experience obtained in these calculations. The physical origin of the divergencies in vacuum energy of quantized field and their relation to the boundary properties in considered in sect. VI. In sect. VII a universal method of constructing the high temperature asymptotics in the Casimir calculations is discussed. It is substantially based on the heat kernel technique, more precisely, it uses the coefficients of the asymptotic expansion of the heat kernel. In sect. VIII an attempt is undertaken to summarize the lessons gained in the course of quiet long studies in this field and to formulate the present status of our physical understanding of the Casimir effect. In the Appendices A, B and C the mathematical details of the calculations involved are placed.

II. ZERO POINT ENERGY IN QUANTUM MECHANICS AND IN QUANTUM FIELD THEORY

The quantum theory forbids the particle to possess simultaneously a definite value of its coordinate and momentum. This results, in particular, in the following. The particle placed in a smooth potential with a local minimum cannot acquire the value of energy which corresponds to the potential minimum. In fact, smooth potential can be approximated around its minimum by the potential of the harmonic oscillator with the known spectrum

$$E_n = \hbar \omega \left( n + \frac{1}{2} \right), \quad n = 0, 1, 2, \ldots$$

Thus the particle cannot possess the energy less than \(E_0 = \hbar \omega/2\), which has obtained the name of zero point energy. It is interesting that the notion of the zero point energy has first appeared before the development of quantum mechanics, in Planck paper of 1912, where he has suggested the “second theory” of the black body spectrum [73]. By making use of zero point energy, Einstein and Stern derived the Planck distribution following practically the classical consideration [74].

The prediction of zero point energy, obtained in the framework of the quantum mechanics dealing with the systems of a finite number of degrees of freedom, is directly verified in experiment. For example, by making use of the Roentgen rays or neutrons scattering by a
crystal lattice or by liquid helium one can convince that, when temperature tends to zero, the atoms in these systems indeed occupy the state with the energy $E_0$ and they cannot possess lower energy. In actual fact these experiments provide the Debye-Waller factor [75], calculation of which in the framework of the quantum mechanics takes into account the zero point oscillations of the atoms.

A direct justification of the reality of zero point oscillations (or zero point energy), in the case of a quantum mechanical systems, is afforded by the observation of vibration spectra of diatomic molecules. It is interesting that it has been also done prior to the development of quantum mechanics [76].

The zero point energy is also manifested in the temperature dependence of the specific heat of rigid bodies. For example, in the model suggested by Einstein [77] the crystal lattice is considered as the system of $3N$ independent oscillators, $N$ being the total number of lattice sites. All the oscillators have the same frequency (a single phenomenological parameter of the model). In this model the area between the curve $c(T)$ describing the temperature dependence of the specific heat and the straight line $y = c(\infty)$ is proportional to the zero point energy of all the oscillators. In a more realistic model of specific heat of crystal bodies proposed by Debye [78] the zero point energy is also taken into account without problem.

Drastically different situation concerning the zero point energy arises when one considers the system of infinite number of oscillators. It is this case that is encountered in quantum field theory. The Hamiltonian of the free (noninteracting) quantum field is an infinite sum of the Hamiltonians of the harmonic oscillators. When the field is considered in unbounded space this sum is substituted by the integral

$$H = \frac{1}{2} \int d^3k \omega(k) [a^\dagger(k)a(k) + a(k)a^\dagger(k)],$$

(2.2)

where the frequency $\omega(k)$ is equal to the energy of the particle with momentum $k$. For simplicity we are considering uncharged spinless particles (bosons)

$$[a(k), a^\dagger(k')] = \delta^{(3)}(k - k').$$

(2.3)

The equidistant energy spectrum of the harmonic oscillator (2.1) enables one to interpret the quantum field dynamics in terms of particles, namely, the transition of one of the oscillators in the Hamiltonian (2.2) to the next higher (lower) level is treated as creation (annihilation) of the particle with the energy $\omega(k)$. In the framework of this interpretation, the zero point energy of quantum field $E_0$ is treated as the vacuum energy, because in this case there are no excited oscillators (all them occupy the energy level $E_0$). Hence there are no real particles. For any physically acceptable dispersion law $\omega(k)$ this energy is, obviously, unbounded

$$E_0 = \frac{1}{2} \int d^3k \omega(k) = \infty.$$  

(2.4)

Naturally, question arises here how to treat this energy.

Divergences are typical for any realistic quantum field theory because the Hamiltonian of quantum field (see, for example, the free field Hamiltonian (2.2)) involves the oscillators with arbitrary large frequencies $\omega(k)$ when $k \to \infty$. Physically it is clear that such quanta cannot be relevant in any concrete calculation. However, quantum field theory does not
provide us with the mechanism that could suppress the contribution of such quanta, and the
problem of divergencies in relativistic quantum field theory requires suitable renormalization
procedures.

If the energy in the system under consideration is conserved (for example, the scattering
of elementary particles possessing the same total energy in the initial and final states is
considered) then the infinite vacuum energy $E_0$ can be ignored by shifting the origin of the
energy scale by $E_0$. Formally this subtraction is accomplished by transition to the normal
ordering of the operators in the field Hamiltonian (2.2)

\[ H = \frac{1}{2} \int d^3k \omega(k) \{ a^\dagger(k)a(k) + a(k)a^\dagger(k) \} = \int d^3k \omega(k) a^\dagger(k)a(k). \] (2.5)

However when the energy of the quantum field is not conserved then such a procedure
of removing $E_0$ cannot be justified. This issue is of a special importance in the gravitation
theory because the energy of matter fields is the source of the gravitation field.

In quantum field theory there is only one receipt of a consistent, from the mathematical
standpoint, treatment of the divergencies. It is the renormalization procedure, i.e., transi-
tion from the initial parameters of the theory, which turn out to be, as a rule, infinite and
therefore unobservable, to finite physical parameters [79,80]. On principle, the renormaliza-
tion procedure in quantum field theory should be done always independently of the presence
of divergencies. If the theory does have the infinities then they are removed in the course of
renormalization “by the way”. Certainly, it is true only for renormalizable field theories.

A consistent renormalization procedure is developed only for the scattering processes in
unbounded Minkowski space (for example, the Bogoliubov $R$ operation). If the field occupies
a bounded region in space, the relevant renormalization scheme is absent. For example, the
renormalizability of quantum electrodynamics considered in a compact space region, by
imposing the corresponding boundary conditions, is not proved explicitly. However there
is no reason to suspect this renormalizability. Indeed, the boundary conditions, confining
the fields to a compact region, arise due to transition from a microscopic description of the
problem at hand (in terms of charges and electromagnetic field by making use of standard
renormalizable quantum electrodynamics in unbounded space) to a macroscopic description
by imposing boundary conditions on electromagnetic field and introducing phenomenological
characteristics (permittivity and permeability) specifying the materials the boundary made
up.

With lacking general renormalization algorithm for nontrivial boundary conditions, one
has to develop this procedure for each concrete configuration a new. In fact, it is the main
drawback in the Casimir calculations for boundaries of diverse geometries. Undoubtedly,
comparison of the theoretical predictions with the experimental observations should play
here the crucial part. By now only the simplest configuration, plane plates, is feasible for
experimental study (see Introduction). In the next sect. we consider different mathematical
methods for removing the divergencies in the Casimir calculations.

It is interesting to note that in the case of fermions the vacuum energy has opposite sign
in comparison with bosons. Really, the Hamiltonian function for noninteracting fermionic
oscillators is given by [81]

\[ H_F = \frac{1}{2} \int d^3k \omega(k) [b^\dagger(k)b(k) - b(k)b^\dagger(k)]. \] (2.6)
In quantum theory the amplitudes \( b(k) \) and \( b^\dagger(k) \) obey the anticommutation relation
\[
[b(k), b^\dagger(k)]_+ = \delta^{(3)}(k - k'),
\] (2.7)
which enables one to cast eq. (2.6) to the form
\[
H_F = \int d^3k \omega(k) b^\dagger(k)b(k) + E_{0F},
\] (2.8)
where \( E_{0F} \) is the fermionic zero point energy
\[
E_{0F} = -\frac{1}{2} \int d^3k \omega(k).
\] (2.9)

As known, the opposite signs in eqs. (2.4) and (2.9) result in substantial conciliation of divergencies in field theories symmetric with respect to permutation of bosons and fermions, i.e., in supersymmetric theories.

The Casimir effect can be considered without introducing the notion of the vacuum fields and its zero-point energy, for example, in the framework of the Schwinger’s source theory [82–84]. In this connection one can meet the opinion [65], according to which there is no sense to insist that “... observable phenomena like the Casimir effect strongly suggest that the vacuum electromagnetic field and its zero-point energy are real physical entities and not mere artifices of the quantum formalism”. And it is largely “matter of taste” to use for the interpretation of this effect vacuum or source fields.

A similar assertion has been done in a recent paper [85] by R. L. Jaffe: “... no known phenomenon, including the Casimir effect, demonstrates that zero point energies are “real”. The author stressed once more the well known fact that the Casimir force originates in the forces between charged particles in the metal plates or, more precisely, between fluctuating dipoles inside the different plates [22,61,86]. In view of this, it is obvious that in the microscopic theory, describing the atomic structure of the plates, the Casimir force vanishes as \( \alpha \), the fine structure constant, goes to zero. The independent of \( \alpha \) result (1.1) obtained in macroscopic approach corresponds to the limit \( \alpha \to \infty \) in the microscopic consideration. It is clear that in microscopic description the notion of zero point energy does not appear.

However we think that denying the reality of the zero point energy of physical fields mentioned above is not sufficiently justified. Here it is worthy to recall that the usual classical mechanics may be reformulated without using the notion of the force, as has been proposed by Hertz [87] (see the evaluation of this approach due to Sommerfeld [88]). However, now nobody suspects the reality of forces, as the physical reasons that cause the alteration of the velocity of a test particle. Therefore we believe that it is well-grounded to treat the experimentally observed Casimir forces as the direct manifestation of the zero point energy of the relevant vacuum fields [61] because the use of this term enables one to accomplish the theoretical analysis of the Casimir effect in the most simple and clear way.

Closing this section it is interesting to note that the notion of the zero point energy was also considered and used in the framework of the classical theory of electromagnetic field (see the papers [89–91] and references therein).
III. RENORMALIZATION OF THE VACUUM ENERGY

A. Physical background for removing the divergencies

In the general case one cannot assign the physical meaning to the vacuum energy of quantum field (2.4) unlike to the zero point energy of quantum mechanical systems with finite degrees of freedom. However if the configuration space $\mathcal{M}$ is bounded (the Casimir effect), then one can get a finite value for the vacuum energy subtracting appropriate counter terms, i.e., by making the renormalization.

Certainly, in the experiment one can observe only the Casimir forces that are derived from the vacuum energy by differentiation with respect to relevant distance. We shall always bare in mind this when considering the vacuum energy problem.

When the field occupies a bounded region $\mathcal{M}$, its eigenfrequencies are discrete and their values are determined by the geometry of the boundary $\partial \mathcal{M}$. As a result, the vacuum energy also depends on the form of the boundary

$$E_0(\partial \mathcal{M}) = \frac{1}{2} \sum_n \omega_n.$$ (3.1)

Following the basic line of the quantum field theory formalism, one can expect that an observable value of the vacuum energy is obtained by subtracting from eq. (3.1) the same expression calculated for the boundary of a special form. In the simplest case, when electromagnetic field is confined between two parallel perfectly conducting plates, it is sufficient to subtract from eq. (3.1) the contribution of unbounded Minkowski space $E_0(0)$

$$E_0 = E_0(\partial \mathcal{M}) - E_0(0).$$ (3.2)

Preliminary both expressions for $E_0$ should be regularized, for example, by introducing a smooth function, that suppresses the contributions due to large momenta. In this way the Casimir energy for parallel plates is calculated, for example, in the text book on quantum field theory [92].

The prescription (3.2) for obtaining the physical value of the vacuum energy may be interpreted in the following way. In the case of infinite number of degrees of freedom (quantum field theory) the observable quantity is not the zero point energy itself, but only its excess, caused by boundaries or inhomogeneities of the space, compared with the zero point energy of quantum field occupying unbounded homogeneous space. It is the main point that differs treating the zero point energy in quantum mechanics and in quantum field theory.

However, the subtraction of the Minkowski space contribution turns out to be insufficient in order to obtain finite vacuum energy in the case of more complicated boundaries, for example, for perfectly conducting sphere. Unfortunately, there are no general rules that could enable one to construct the subtraction procedure for the boundary of arbitrary form, i.e., , to find the corresponding counter terms. In view of this, in calculation of vacuum energy the analytical regularization, or zeta regularization, proved to be very useful because it simultaneously accomplishes the renormalization without introducing the relevant counter terms. Here we are dealing with renormalization of the field energy. This problem is central one in quantum field theory on nonflat background [93].
There is a few surveys and books concerned with the spectral zeta function technique and its application to the Casimir calculations [60,94–102]. Therefore we present here only the basic formulae and stress the points that are not sufficiently elucidated in the mathematical literature.

B. Spectral zeta functions

Let the field $\varphi(t, x)$ is governed by the following equation

$$
\left( L + \frac{\partial^2}{c^2 \partial t^2} \right) \varphi(t, x) = 0,
$$

(3.3)

where $L$ is an elliptic differential operator of the second order acting only on the spatial variables $x$. For example, in the case of a free scalar massless field $L = -\Delta$, where $\Delta$ is the Laplace operator.

Upon separating the time dependence $\varphi(t, x) = e^{\pm i\omega t} \varphi_n(x)$ the field equation (3.3) and the relevant boundary conditions with respect the variable $x$ generate the spectral problem

$$
L \varphi_n(x) = \lambda_n \varphi_n(x) \quad \text{or} \quad L |n> = \lambda_n |n>, \quad \lambda_n = \omega_n^2/c^2.
$$

(3.4)

It is convenient to use here the Dirac bracket notation. We assume that the spectral problem is well posed, i.e., all the equations written below have sense.

The completeness relation of the vector set $\{|n>\}$ can be represented (formally) in the following way

$$
I = \sum_n |n><n|,
$$

(3.5)

where $I$ is a unity operator acting in the linear space of the vectors $|n>$. In view of this we have for the inverse operator $L^{-1}$

$$
L^{-1} = \sum_n \frac{|n><n|}{\lambda_n}.
$$

(3.6)

It can be easily checked with allowance for eq. (3.4). In the same way we have for the $s$th power of the inverse operator $L^{-s}$

$$
L^{-s} = \sum_n \frac{|n><n|}{\lambda_n^s}.
$$

(3.7)

The local spectral zeta function $\zeta_L(s; x)$ of the operator $L$ is a diagonal element of the operator $L^{-s}$

$$
\zeta_L(s; x) = \sum_n \frac{|n><n|}{\lambda_n^s} = \sum_n \lambda_n^{-s} \varphi^*_n(x) \varphi_n(x).
$$

(3.8)

The global spectral zeta function [103,104] is defined by
\[ \zeta_L(s) = \text{Tr} L^{-s} = \sum_n \lambda_n^{-s}. \] (3.9)

It is obtained by integration of \( \zeta_L(s; x) \) over the whole space
\[ \zeta_L(s) = \int \zeta_L(s; x) \, dx. \] (3.10)

The definition of the spectral zeta function (3.9) is a direct extension of the Riemann zeta function \[ \zeta_R(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \text{Re} \, s > 1 \] (3.11)
to the spectrum of the operator \( L \).

In the case of the \( d \)-dimensional space the index \( n \) in eq. (3.9) stands for the set of \( d \) indices \( n = \{n_1, n_2, \ldots, n_d\} \) and summation should be carried out with respect to each of them in respective ranges. In the majority of the problems, having physical application,
\[ \lambda_n \sim n_1^2 + n_2^2 + \ldots + n_d^2, \quad n_i \to \infty, \quad 1 \leq i \leq d. \] (3.12)
In this case the series (3.9), defining the global zeta function, converges in the domain \( \text{Re} \, s > d/2 \), where it represents an analytical function \( \zeta_L(s) \) of a complex variable \( s \). This function can be analytically continued into the left half-plane \( \text{Re} \, s < d/2 \) except for simple separate poles, for example, at the point \( s = d/2 \). This fact is a mathematical basis for employment of the spectral zeta functions with the aim of obtaining finite expressions when treating the divergencies.

In the framework of the zeta function technique the vacuum energy \( E_0 \) of the quantum field \( \varphi(t, x) \) is defined by [99]
\[ E_0 = \frac{1}{2} \zeta_L \left( s = -\frac{1}{2} \right). \] (3.13)

In view of eqs. (3.10) and (3.13), the quantity \((1/2)\zeta_L(-1/2; x)\) can be interpreted as the vacuum energy density. Besides this, it is easy to show that the vacuum expectation value of the canonical energy-momentum tensor \( T_{00}(x) \) for the field \( \varphi(t, x) \) can be represented in the form
\[ \langle 0| T_{00}(x) |0 \rangle = \frac{1}{2} \sum_n \lambda_n^{1/2} \varphi_n^\ast(x) \varphi_n(x) = \frac{1}{2} \zeta_L \left( -\frac{1}{2}; x \right). \] (3.14)

In order to construct the global zeta function, the spectrum of the problem under consideration should be known. Certainly, it is feasible only for boundaries with high symmetry, for example, with spherical or cylindrical symmetry. In the Casimir calculations with \( d = 3 \) and \( n = \{n_1, n_2, n_3\} \), the configurations are usually considered when, for fixed values of \( n_1 \) and \( n_2 \), the eigenfrequencies \( \omega_{n_3} \) are the roots of the equation
\[ f_{n_1n_2}(\omega) = 0 \] (3.15)
with a known function $f_{n_1n_2}(z)$. The sum of these roots $\omega_{n_3}$ can be found by making use of a contour integration in complex frequency plane [105]. It leads to the following compact representation for the spectral zeta function

$$\zeta(s) = \frac{1}{2\pi i} \sum_{n_1, n_2} \oint_C z^{-s} \frac{d}{dz} f_{n_1n_2}(z) \, dz, \quad (3.16)$$

where the contour $C$ encloses, counterclockwise, all the positive roots of eq. (3.15). This representation proves to be convenient for analytical continuation of the right hand side of eq. (3.16) to the left half plane, specifically, at the point $s = -1/2$. However, it may happens that the zeta function $\zeta(s)$ has a pole at this point. It will imply that the zeta regularization does not afford a finite value for the vacuum in the problem at hand. There is a rigorous mathematical criterion that enables one to predict when this happens. In order to formulate it we need another spectral function, heat kernel.

Before to go over to consideration of this function it is worthy to make a general comment concerning the spectral functions. In the theory of differential operators these functions play in fact the same part as characteristic polynomials do for matrix operators of a finite dimension, namely, they store all the information about the spectrum of a given operator in an “encoded” form. When the spectrum consists of a finite number of eigenvalues $\lambda_n$, $n = 1, 2, \ldots, N$, it is natural to construct on this basis the polynomial of $N$th degree the roots of which give this spectrum: $P_N(\lambda) = \prod_{n=1}^{N}(\lambda - \lambda_n), \quad \lambda \in \mathbb{C}$. The matrix equation $P_N(L_N) = 0$ is used, as known, for constructing the functions of the matrix $L_N$. In the case of an infinite spectrum a concrete form of the relevant spectral function is not so definite. Ultimately, all is determined by the problem under study. For example, when dealing with the spectral sums, it is convenient to use the zeta function (3.9).

The procedure of constructing the local spectral zeta function (3.8) proves to be much more complicated as compared with that for the global zeta function. Besides the spectrum of the operator $L$ we have to use the respective natural modes. As far as we know the local zeta function has been derived only for the scalar Laplace operator defined on a cone or in wedge. This function was used in studies of black hole physics [106,107], cosmic strings [108] and in calculation of the Casimir effect for wedge [109].

The relation of the zeta function technique to more transparent regularizations is discussed, for example, in refs. [110,111].

### C. Heat kernel technique

In Casimir calculations also useful is another spectral function of the operator $L$ introduced in eq. (3.3), namely, the heat kernel of this operator

$$K(\tau) = \text{Tr} \left( e^{-\tau L} \right) = \sum_n e^{-\lambda_n \tau}, \quad (3.17)$$

where $\tau$ is an auxiliary variable ranging from 0 to $+\infty$. Such a name of this function is due to the following. By making use of the unity operator (3.5) one can write
\[ e^{-\tau L} = \sum_n e^{-\tau \lambda_n} |n\rangle < n| \quad (3.18) \]

The matrix element of this operator
\[
K(x, y; \tau) \equiv <x| e^{-\tau L} |y> = \sum_n e^{-\tau \lambda_n} <x| |n\rangle <n|y> = \sum_n e^{-\tau \lambda_n} \varphi_n^*(x) \varphi_n(y) \quad (3.19)
\]
is the Green function of the heat conduction equation with the operator \( L = L_x + \frac{\partial}{\partial \tau} \)
\[
\left( L_x + \frac{\partial}{\partial \tau} \right) K(x, y; \tau) = 0, \quad (3.20)
\]
\[
K(x, y; \tau) = \delta(x, y), \quad \tau \to +0. \quad (3.21)
\]

For the functions \( K(\tau) \) and \( K(x, y; \tau) \) the relation analogous to (3.10) holds
\[
K(\tau) = \int dx K(x, x; \tau). \quad (3.22)
\]

The integrated heat kernel (3.17) and the spectral zeta function (3.9) are connected by the Mellin transform. Indeed, from the definition of the gamma function [112] it follows that
\[
\frac{1}{\lambda_n^s} = \frac{1}{\Gamma(s)} \int_0^{\infty} d\tau \tau^{s-1} e^{-\lambda_n \tau}, \quad \text{Re} \ s > 0. \quad (3.23)
\]

Upon summation in both sides of this equation we obtain
\[
\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} d\tau \tau^{s-1} K(\tau), \quad \text{Re} \ s > \frac{d}{2}. \quad (3.24)
\]

To the left half-plane \( \text{Re} \ s < d/2 \) eq. (3.24) should be continued analytically.

Thus, from the mathematical point of view, both the spectral functions \( \zeta(s) \) and \( K(\tau) \) are equivalent. However in practical use it is difficult to succeed in finding the spectral zeta function or the heat kernel in an exact form, and one has to deal with the approximate expressions. Therefore different initial formulae either (3.9) or (3.17) and (3.20) – (3.22) prove to be useful.

In physical applications are important the coefficients in the asymptotic expansion of the heat kernel, when \( \tau \to +0 \)
\[
K(\tau) = \sum_n e^{-\lambda_n \tau} = (4\pi \tau)^{-d/2} \sum_{n=0,1,2,...}^{\infty} \tau^{n/2} B_{n/2} + \text{ES}. \quad (3.25)
\]

In this expansion \( d \) is the dimension of the configuration space manifold \( \mathcal{M} \) in the problem at hand, \( \text{ES} \) stands for the exponentially small corrections as \( \tau \to +0 \). The first two coefficients in expansion (3.25) are determined by the volume \( V \) of the manifold \( \mathcal{M} \) and by the area of
the boundary $\partial \mathcal{M}$. For example, if $L = -\Delta$, where $\Delta$ is the Laplacian, acting on a massless scalar field, then

$$B_0 = V, \quad B_{1/2} = \mp \frac{\sqrt{\pi}}{2} S.$$  \hspace{1cm} (3.26)

The upper sign in this equation is for the Dirichlet boundary conditions and the lower sign is for the Neumann conditions.

For flat manifolds all the coefficients $B_{n/2}$, $n \geq 1$ are due to the boundary contributions, namely, they are defined by the integrals over the boundary with the integrands expressed in terms of geometrical invariants of the boundary [60,94,95,97]. The boundary is expected to be sufficiently smooth.

We give here the properties of the heat kernel coefficients which are especially important when calculating the vacuum energy. The proof of these statements can be found in the literature (see, for example [60,94,95]). The first few coefficients $B_{n/2}$ yield the ultraviolet divergencies of the vacuum energy (3.13). For $d = 3$ it is the coefficients up to $B_2$. If the coefficients $B_2$ is equal to zero, then the zeta regularization gives a finite value for the vacuum energy according to the formula (3.13). For $d = 2$ the role of such an indicator plays the coefficient $B_{3/2}$ (see below). The heat kernel coefficients also specify conformal anomalies taking place in a concrete field theory model, the high temperature behavior of the thermodynamic functions (see sect. VII) and so on [94].

The heat kernel coefficients are determined by the residua of the product $\Gamma(s)\zeta(s)$ at corresponding points. Indeed, substituting the expansion (3.25) into eq. (3.24) and integrating over $\tau$ in the vicinity of the origin we obtain

$$\frac{B_{n/2}}{(4\pi)^{d/2}} = \lim_{s \to \frac{d-n}{2}} (s + \frac{n-d}{2}) \Gamma(s)\zeta(s), \quad n = 0, 1, 2, \ldots.$$  \hspace{1cm} (3.27)

In order to use this formula the spectral zeta function $\zeta(s)$ should be known in the vicinity of the following points

$$s = \frac{d}{2}, \frac{d-1}{2}, \ldots.$$  \hspace{1cm} (3.28)

For this one has to do analytical continuation of the initial formula (3.9) to the left half-plane $\text{Re } s < d/2$.

If $d = 3$ and $B_2 \neq 0$ or for a given $d$ the coefficient $B_{(d+1)/2}$ does not vanish then according to eq. (3.27) the zeta function has a pole at the point $s = -1/2$. As a result in these cases the renormalization by zeta function does not provide a finite value for the vacuum energy $E_0$ (see the definition (3.13)).

There is another technique for calculating the heat kernel coefficients that enables one to avoid the procedure of analytical continuation [113–116]. It is close to the method presented above but sometimes calculations become simpler.

Let us consider the spectral zeta function depending on a parameter $x^2$

$$\zeta(s, x^2) = \sum_n (\lambda_n + x^2)^{-s}.$$  \hspace{1cm} (3.29)
It may be regarded as an extension of the Epstein-Hurwitz zeta function
\[ \zeta_{EH}(s, a^2) = \sum_{n=1}^{\infty} (n^2 + a^2)^{-s} \]
to the general spectral problem (3.4).

It turns out that the heat kernel coefficients \( B_{n/2} \) can be found from the expansion of the function \( \zeta(s, x^2) \) in terms of inverse powers of \( x \) developed for a certain value of \( s \). It is convenient to choose this value to be equal to \( 1 + d/2 \). In fact, from the definition of the gamma function (3.23) it follows that
\[ \Gamma(s) (\lambda + x^2)^{-s} = \int_0^\infty dt \, t^{s-1} e^{-(\lambda + x^2)t}, \quad \text{Re } s > 0. \] (3.30)

For \( s = 1 + d/2 \) eq. (3.30) gives
\[ \Gamma \left(1 + \frac{d}{2}\right) \sum_n (\lambda_n + x^2)^{-1-d/2} = \int_0^\infty dt \, t^{d/2} e^{-x^2t} \sum_n e^{-\lambda_n t} \]
\[ = \int_0^\infty dt \, t^{d/2} e^{-x^2t} K(t). \] (3.31)

On substituting the asymptotic expansion (3.25) in eq. (3.31) we obtain
\[ \Gamma \left(1 + \frac{d}{2}\right) \zeta \left(1 + \frac{d}{2}, x^2\right) \simeq \sum_{n=0}^\infty \frac{B_{n/2}}{(4\pi)^{d/2}} \Gamma \left(1 + \frac{n}{2}\right) x^{-n-2} \]
\[ = \frac{1}{(4\pi)^{d/2}} \left[ \frac{B_0}{x^2} + \frac{B_{1/2}\Gamma(3/2)}{x^3} + \frac{B_1\Gamma(2)}{x^4} + \frac{B_{3/2}\Gamma(5/2)}{x^5} \right. \]
\[ + \frac{B_2\Gamma(3)}{x^6} + \frac{B_{5/2}\Gamma(7/2)}{x^7} + \frac{B_3\Gamma(4)}{x^8} \left. + \mathcal{O}(x^{-9}) \right]. \] (3.32)

Let \( F(z) = 0 \) be the frequency equation which determines the spectrum \( \lambda_n \) in the problem under consideration. We also suppose that the function \( F(z) \) allows one to rewrite this equation in the form
\[ \prod_n (\lambda_n - z^2) = 0. \] (3.33)

Taking into account the equality
\[ \frac{1}{(\lambda_n + x^2)^m} = \frac{(-1)^m}{\Gamma(m)} \left( \frac{d}{2x \, dx} \right)^m \ln(\lambda_n + x^2), \quad z = ix, \] (3.34)
we recast the left-hand side of eq. (3.32) to the form
\[ \Gamma \left(1 + \frac{d}{2}\right) \zeta \left(1 + \frac{d}{2}, x^2\right) = - \left( \frac{1}{2x \, dx} \right)^{1+d/2} \ln F(ix). \] (3.35)
Obviously formula (3.35) is applicable only to the manifolds of even dimension. Now we have to substitute the left-hand side of eq. (3.32) by the expansion of eq. (3.35) for large $x^2$.

After that the heat kernel coefficients $B_{n/2}$ are obtained by comparing the coefficients of $x^2$ in both sides of eq. (3.32).

This technique can be extended to the odd values of $d$ by calculating the zeta function $\zeta(s, x^2)$ at the point $s = (1 + d)/2$. Instead of eqs. (3.32) and (3.35) we have now

$$
\Gamma\left(\frac{d+1}{2}\right) \zeta\left(\frac{d+1}{2}, x^2\right) = -\left(-\frac{1}{2x} \frac{d}{dx}\right)^{(d+1)/2} \ln F(ix) \\
= \sum_{n=0}^{\infty} \frac{B_{n/2}}{(4\pi)^{d/2}} \Gamma\left(\frac{n+1}{2}\right) x^{-n-1}.
$$

(3.36)

Again comparing the coefficients of $x$ in both sides of eq. (3.36) we obtain the heat kernel coefficients for odd $d$.

**IV. SPECTRAL ZETA FUNCTIONS AND VACUUM ENERGY FOR SIMPLEST BOUNDARIES**

Practically every problem on calculation of the Casimir energy (or force) has been considered often with employment of more and more effective and elaborated mathematical methods. For example, the first calculation of the Casimir energy of a perfectly conducting spherical shell carried out by T.H. Boyer in 1968 [13] has required computer calculations during 3 years [117]. Later this problem was considered in many papers [14–16]. By making use of the modern methods [118] it can be solved almost without numerical calculations (with a precision of a few percent). It requires only the application of the uniform asymptotic expansion for the Bessel functions. The zeta function regularization enables one to calculate the vacuum energy in a consistent way without dealing with the manifestly infinite expressions.

**A. Parallel perfectly conducting plates**

As known, for example, from the theory of waveguides and resonators [119] the vectors of electric and magnetic fields in the problem at hand are expressed in terms of the electric ($\Pi'$) and magnetic ($\Pi''$) Hertz vectors, each having only one nonzero component $\Pi'_z$ and $\Pi''_z$ satisfying, respectively, Dirichlet and Neumann conditions on the internal surface of the plates. The functions $\Pi'_z$ and $\Pi''_z$ obey the equations

$$
\left(\frac{\partial^2}{\partial z^2} + \nabla^2\right) \Pi'_z = \frac{\omega^2}{c^2} \Pi'_z, \quad \left(\frac{\partial^2}{\partial z^2} + \nabla^2\right) \Pi''_z = \frac{\omega^2}{c^2} \Pi''_z,
$$

(4.1)

where $\omega$ is the frequency of electromagnetic oscillations, $\nabla^2$ stands for the two-dimensional Laplace operator for the variables $(x, y) = x$. The separation of variables results in the following solution.
\[ \Pi'_z(x, z) = \exp(i k x) \sin \left( \frac{n \pi z}{a} \right), \quad n = 1, 2, \ldots, \]
\[ \Pi''_z(x, z) = \exp(i k x) \cos \left( \frac{n \pi z}{a} \right), \quad n = 0, 1, 2, \ldots, \]
\[ \omega_n^2(k) = c^2 \left[ k^2 + \left( \frac{n \pi}{a} \right)^2 \right], \quad (4.2) \]

where \( a \) is the distance between the plates. Hence, the states of electromagnetic field with the energy \( \hbar \omega_n \), \( n \geq 1 \), are doubly degenerate, while the state with the energy \( \hbar \omega_0 = \hbar c k \) is nondegenerate.

With allowance for this the zeta function in the problem under consideration is given by
\[ \zeta(s) = \frac{L_x L_y}{c^{2s}} \int \frac{d^2 k}{(2\pi)^2} \left\{ 2 \sum_{n=1}^{\infty} \left[ k^2 + \left( \frac{n \pi}{a} \right)^2 \right]^{-s} + (k^2 + \mu^2)^{-s} \right\}, \quad (4.3) \]

where \( L_x \) and \( L_y \) are the dimensions of the plates.

For a correct definition of the integral in this formula in the small \( k \) region the photon mass \( \mu \) is introduced (infrared regularization). At the final step of calculations one should put \( \mu = 0 \). On integrating in eq. (4.3) and substituting the sum over \( n \) by the Riemann zeta function one arrives at the result
\[ \zeta(s) = \frac{L_x L_y}{2\pi c^{2s}} \left[ \left( \frac{\pi}{a} \right)^{2-2s} \zeta_R(2s-2) + \frac{1}{2} s^{-1} \right]. \quad (4.4) \]

The zeta function (4.4) leads to the well-known value for the Casimir energy
\[ E_0 = \frac{\hbar}{2} \zeta \left( -\frac{1}{2} \right) = -c \hbar \frac{\pi^2}{120} \frac{L_x L_y}{a^3} \quad (4.5) \]

or for its density
\[ \frac{E_0}{V} = -\frac{c \hbar \pi^2}{720 a^4}, \quad \text{where} \quad V = a L_x L_y. \quad (4.6) \]

Differentiation of the vacuum energy (4.5) with respect to the distance \( a \) gives the Casimir force (1.1).

**B. Sphere**

Let us consider a solid ball of radius \( a \), consisting of a material which is characterized by permittivity \( \varepsilon_1 \) and permeability \( \mu_1 \). The ball is assumed to be placed in an infinite medium with permittivity \( \varepsilon_2 \) and permeability \( \mu_2 \). In the case of spherical symmetry the solutions to Maxwell equations are expressed in terms of two scalar Debye potentials \( \psi \) (see, for example, [120])
\[ \mathbf{E}^{TM}_{lm} = \nabla \times \nabla \times (r \psi^{TM}_{lm}), \quad \mathbf{H}^{TM}_{lm} = -i \omega \nabla \times (r \psi^{TM}_{lm}) \quad (E\text{-modes}), \]
\[ \mathbf{E}^{TE}_{lm} = i \omega \nabla \times (r \psi^{TE}_{lm}), \quad \mathbf{H}^{TE}_{lm} = \nabla \times \nabla \times (r \psi^{TE}_{lm}) \quad (H\text{-modes}). \quad (4.7) \]
These potentials obey the Helmholtz equation, have the indicated angular dependence, and are regular at the origin

$$(\nabla^2 + k_i^2)\psi_{lm} = 0, \quad k_i^2 = \varepsilon_i\mu_i\frac{\omega_i^2}{c^2}, \quad i = 1, 2; \quad (r \neq a); \quad \psi_{lm}(r) = \phi(r)Y_{lm}(\Omega). \quad (4.8)$$

Here we have to make a note concerning the formulation of the spectral problem for differential elliptic operators defined on unbounded manifolds. First of all, the question arises what conditions should be imposed on the eigenfunctions at the spatial infinity. In order for the uniqueness theorem to hold for the solutions looked for, the radiation conditions at infinity should be introduced. It is a well known fact in classical mathematical physics (see, for example, [121]). However, in this case there are no solutions with real frequencies and one has to consider the complex frequencies and respective natural modes, i.e., quasi-normal modes. Certainly, it is absolutely unclear how to use these modes when quantizing the fields. Here one can proceed in two different ways. The quantum field under consideration can be placed inside a sphere of a large radius $R$ imposing on field functions reasonable conditions on the internal surface of the sphere [18]. All outside the sphere is ignored. The spectrum of oscillations in this case is discrete, infinite and real because we are dealing simply with standing waves. On this basis the spectral zeta function can be constructed in a standard way and ultimately the radius of an auxiliary sphere $R$ should be allowed to go to infinity.

More appealing, from the physical point of view, is an approach that uses at infinity the conditions on field functions taken from the scattering problem instead of the radiation conditions. The frequency spectrum in this case is real, positive but continuous. The drawbacks due to unnormalizable natural modes can be overcome by introducing into consideration the relevant spectral density. This density is expressed in terms of the phase shifts, or more precisely, through the Jost function of the corresponding scattering problem.

Both these approaches lead to the same final results. Furthermore, it turns out, that one can also deal with the radiation conditions at infinity and consequently with complex frequencies, ignoring the problem of treating the corresponding quasi-normal modes. Point is that the equation for real frequencies in the case with additional large sphere reduces, when $R \to \infty$, to the frequency equation in the approach which uses the radiation condition at infinity. And further, the Jost function mentioned above is determined by the left-hand side of the equation for complex frequencies.

In view of all this, we shall impose at infinite the radiation conditions on the function $\phi(r)$ in eq. (4.8) and deal with the frequency equation with complex roots keeping in mind that it is in fact the relevant Jost function.

At the ball surface the tangential components of electric and magnetic fields are continuous. As a result, the eigenfrequencies of electromagnetic field for this configuration are determined by the frequency equation for the TE–modes

$$\Delta_i^{TE}(a\omega) \equiv \sqrt{\varepsilon_i\mu_2} \bar{s}_i(k_1a) \hat{e}_i(k_2a) - \sqrt{\varepsilon_2\mu_1} \bar{s}_i(k_1a) \hat{e'}_i(k_2a) = 0, \quad (4.9)$$

and the analogous equation for the TM–modes

$$\Delta_i^{TM}(a\omega) \equiv \sqrt{\varepsilon_2\mu_1} \bar{s}_i(k_1a) \hat{e}_i(k_2a) - \sqrt{\varepsilon_1\mu_2} \bar{s}_i(k_1a) \hat{e'}_i(k_2a) = 0, \quad (4.10)$$

where $k_i = \sqrt{\varepsilon_i\mu_i}\omega, \quad i = 1, 2$ are the wave numbers inside and outside the ball, respectively [122]. Here $\bar{s}_i(x)$ and $\hat{e}_i(x)$ are the Riccati–Bessel functions.
\[ \tilde{s}_l(x) = \sqrt{\frac{\pi}{2x}} J_{l+1/2}(x), \quad \tilde{e}_l(x) = \sqrt{\frac{\pi}{2x}} H^{(1)}_{l+1/2}(x), \] (4.11)

and prime stands for the differentiation with respect to their arguments, \( k_1a \) or \( k_2a \). The orbital momentum \( l \) in eqs. (4.9) and (4.10) assumes the values \( 1, 2, \ldots \). The roots of the frequency equations (4.9) and (4.10), including the complex ones, have been investigated by Debye in his PhD thesis concerned with the light pressure on a material ball [123].

The material presented below is based in main part on the paper [124]. When considering the boundaries with spherical and cylindrical symmetries, it is convenient to subtract in the definition of the spectral zeta function (3.9) the contribution of the same configuration under the condition \( a \to \infty \) with frequencies \( \bar{\omega}_p \).

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The material presented below is based in main part on the paper [124]. When considering the boundaries with spherical and cylindrical symmetries, it is convenient to subtract in the definition of the spectral zeta function (3.9) the contribution of the same configuration under the condition \( a \to \infty \) with frequencies \( \bar{\omega}_p \).

\[ \zeta(s) = \sum_p (\omega_p^{-s} - \bar{\omega}_p^{-s}). \] (4.12)

As usual when one is dealing with an analytic continuation, it is useful to represent the sum (4.12) in terms of the contour integral by making use of eq. (3.16)

\[ \zeta_{\text{ball}}(s) = \sum_{l=1}^{\infty} \frac{2l + 1}{2\pi i} \lim_{\mu \to 0} \int_C \frac{dz}{2(z^2 + \mu^2)^s} \frac{d}{dz} \ln \frac{\Delta^\text{TE}_l(az) \Delta^\text{TM}_l(az)}{\Delta^\text{TE}_l(\infty) \Delta^\text{TM}_l(\infty)}, \] (4.13)

where the contour \( C \) surrounds, counterclockwise, the roots of the frequency equations (4.9) and (4.10) in the right half-plane. For brevity we write in (4.13) simply \( \Delta_l(\infty) \) instead of \( \lim_{a \to \infty} \Delta_l(az) \). Transition to the complex frequencies \( z \) in eq. (4.13) is accomplished by introducing the unphysical photon mass \( \mu \)

\[ \omega \to (z^2 + \mu^2)^{1/2}|_{\mu \to 0}. \] (4.14)

Extension to the complex \( z \)-plane of the frequency equations \( \Delta^\text{TE}_l(az) \) and \( \Delta^\text{TM}_l(az) \) should be done in the following way. In the upper (lower) half-plane the Hankel functions of the first (second) kind \( H^{(2)}_\nu(az) \) \( H^{(1)}_\nu(az) \) must be used. If we are considering only outgoing waves it is natural to use the solutions of the Maxwell equations which are finite (or at least do not grow infinitely) in the future. Such solutions should be proportional to \( \exp(-i\omega t)H^{(1)}_\nu(\omega r) \) when \( \omega \) lies in the lower half-plane. For the upper half-plane \( \omega \), the solutions describing outgoing waves and finite in the future should contain the factor \( \exp(i\omega t)H^{(2)}_\nu(\omega r) \).

Location of the roots of eqs. (4.9) and (4.10) enables one to deform the contour \( C \) into a segment of the imaginary axis \((-i\Lambda, i\Lambda)\) and a semicircle of radius \( \Lambda \) in the right half-plane. When \( \Lambda \) tends to infinity the contribution along the semicircle into \( \zeta_{\text{ball}}(s) \) vanishes because the argument of the logarithmic function in the integrand tends in this case to 1. As a result we obtain

\[ \zeta_{\text{ball}}(s) = -a^{2s} \sum_{l=1}^{\infty} \frac{2l + 1}{2\pi i} \lim_{\mu \to 0} \int_{-i\infty}^{+i\infty} \frac{dz}{(z^2 + \mu^2)^s} \frac{d}{dz} \ln \frac{\Delta^\text{TE}_l(z) \Delta^\text{TM}_l(z)}{\Delta^\text{TE}_l(i\infty) \Delta^\text{TM}_l(i\infty)}. \] (4.15)

Now we impose the condition that the velocity of light inside and outside the ball is the same.
\[ \varepsilon_1 \mu_1 = \varepsilon_2 \mu_2 = c^{-2}. \] (4.16)

The physical motivation for this is the following. The constancy condition for the velocity of gluonic field when crossing the interface between two media is used, for example, in a dielectric vacuum model (DVM) of quark confinement [67,125,126]. This model has many elements in common with the bag models [127], but among the other differences, in DVM there is no explicit condition of the field vanishing outside the bag. It proves to be important for calculation of the Casimir energy contribution to the hadronic mass in DVM. The point is that in the case of boundaries with nonvanishing curvature there happens a considerable (not full, however) mutual cancellation of the divergences from the contributions of internal and external (with respect to the boundary) regions. If only the field confined inside the cavity is considered, as in the bag models [128–131], then there is no such a cancellation, and one has to remove some divergences by means of renormalization of the phenomenological parameter in the model defining the QCD vacuum energy density.

From a physical point of view the vanishing of the field or its normal derivative precisely on the boundary is an unsatisfactory condition, because, due to quantum fluctuations, it is impossible to measure the field as accurately as desired at a certain point of the space [132].

Under condition (4.16) the argument of the logarithm in (4.15) can be simplified considerably [25] with the result

\[ \zeta_{\text{ball}}(s) = \left( \frac{a}{c} \right)^{2s} \frac{\sin(\pi s)}{\pi} \sum_{l=1}^{\infty} (2l + 1) \int_{0}^{\infty} \frac{dy}{y^{2s}} \frac{d}{dy} \ln[1 - \xi^2 \sigma_l^2(y)], \] (4.17)

where

\[ \xi^2 = \left( \frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + \varepsilon_2} \right)^2 = \left( \frac{\mu_1 - \mu_2}{\mu_1 + \mu_2} \right)^2, \quad \sigma_l(y) = \frac{d}{dy} [s_l(y) e_l(y)]. \] (4.18)

Here \( s_l(y) \) and \( e_l(y) \) are the modified Riccati–Bessel functions

\[ s_l(x) = \sqrt{\frac{\pi x}{2}} I_{\nu}(x), \quad e_l(x) = \sqrt{\frac{2x}{\pi}} K_{\nu}(x), \quad \nu = l + 1/2. \] (4.19)

More details concerning the contour integral representation of the spectral \( \zeta \) functions can be found in [130,133–135].

Further the analytic continuation of eq. (4.17) is accomplished by expressing the sum over \( l \) in terms of the Riemann zeta function. This cannot be done in a closed form. Making use of the uniform asymptotic expansion (UAE) for the Bessel functions\(^1\) in increasing powers

\(^1\)In physical literature this expansion is usually referred to as the Debye expansion. However we did not find these formulae in Debye papers. Furthermore in mathematical handbook [136] the Debye type expansions for the Bessel functions are defined as those for large argument \( x \) and large order \( \nu \) both real and positive such that \( \nu/x \) is fixed and simultaneously \( |x - \nu| = \mathcal{O}(x^{1/3}) \). The uniform asymptotic expansions were derived in ref. [137]
of $1/\nu$ enables one to construct the analytic continuation looked for in the form of the series, the terms of which are expressed through the Riemann zeta function. The problem of the convergence of this series does not arise because its terms go down very fast, at least when calculating the vacuum energy.

We demonstrate this keeping only two terms in UAE for the product of the modified Bessel functions $I_{\nu}(\nu z)K_{\nu}(\nu z)$ [138]

$$I_{\nu}(\nu z)K_{\nu}(\nu z) \simeq \frac{t}{2\nu} \left[ 1 + \frac{t^2(1 - 6t^2 + t^4)}{8\nu^2} + \ldots \right], \quad t = \frac{1}{\sqrt{1 + z^2}}. \quad (4.20)$$

After changing the integration variable $y = \nu z$ in eq. (4.17) we substitute (4.20) into this formula and expand the logarithm function up to the order $\nu^{-3}$ keeping at the same time only the terms linear in $\xi^2$. The last assumption is not crucial. It is introduced for simplicity and in order to have possibility of a direct comparison with the results of ref. [25]. Thus we have

$$\frac{d}{dz} \ln \left\{ 1 - \xi^2 \left[ \frac{d}{dz}(zI_{\nu}(\nu z)K_{\nu}(\nu z)) \right]^2 \right\} = \frac{3}{2\nu^2} \frac{\xi^2}{z^8} \ln \left( 1 + \frac{\xi^2}{16\nu^4} z^8(-12 + 216t^2 - 600t^4 + 420t^6) + O(\nu^{-6}) \right).$$

Integration over $z$ can be done by making use of the formula [112]

$$\int_0^\infty z^{-\alpha - 1}e^{\beta z}dz = \frac{1}{2} \frac{\Gamma\left(\frac{\alpha + \beta}{2}\right)\Gamma\left(-\frac{\alpha}{2}\right)}{\Gamma\left(\frac{\beta}{2}\right)}, \quad \operatorname{Re} \alpha < 0, \quad \operatorname{Re} (\alpha + \beta) > 0. \quad (4.22)$$

Also the properties of the $\Gamma$ function

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}, \quad \Gamma(1+z) = z\Gamma(z) \quad (4.23)$$

prove to be useful. After simple calculations we arrive at the result

$$\zeta_{\text{ball}}(s) \simeq \frac{\xi^2}{4} \left( \frac{a}{c} \right)^{2s} s(1 + s)(2 + s) \left( \sum_{l=1}^\infty \nu^{-1-2s} + p(s) \sum_{l=1}^\infty \nu^{-3-2s} + \ldots \right), \quad (4.24)$$

where

$$p(s) = -\frac{1}{2} \left[ 1 - \frac{9}{2}(3 + s) + \frac{5}{2}(3 + s)(4 + s) - \frac{7}{24}(3 + s)(4 + s)(5 + s) \right]. \quad (4.25)$$

The first term in eq. (4.24) is defined in the region $0 < \operatorname{Re} s < 1/2$ and the second one exists, when $-1 < \operatorname{Re} s < 1/2$, the left boundaries being given by the sums over the values of the angular momentum $l$ and the right limits are due to the integral formula (4.22). Thus, this expression can be used for its analytical continuation outside these regions. Obviously, it
can be done by accepting the integration formula (4.22) outside the domain indicated there and by expressing the sums over angular momentum $l$ through the Riemann zeta function according to the formula [112]

$$\sum_{l=1}^{\infty} \nu^{-s} = (2^s - 1)\zeta_R(s) - 2^s, \quad \nu = l + 1/2. \quad (4.26)$$

As a result one gets

$$\zeta_{\text{ball}}(s) \simeq \frac{\xi^2}{4} \left( \frac{a}{c} \right)^{2s} s(1 + s)(2 + s)\{(2^{1+2s} - 1)\zeta_R(1 + 2s) - 2^{1+2s} + p(s)[(2^{3+2s} - 1)\zeta_R(1 + 2s) - 2^{3+2s}] + \ldots \}. \quad (4.27)$$

The singularities in eq. (4.24) are transformed in (4.27) into the poles of the Riemann zeta functions at the points $s = -k, \quad k = 0, 1, 2, \ldots$

$$\zeta_R(1 + 2s) \simeq \frac{1}{2s} + \gamma + \ldots, \quad s \to 0,$$

$$\zeta_R(3 + 2s) \simeq \frac{1}{2s + 2} + \gamma + \ldots, \quad s \to -2,$$

$$\zeta_R(5 + 2s) \simeq \frac{1}{2s + 4} + \gamma + \ldots, \quad s \to -4,$$

$$\ldots$$

where $\gamma$ is the Euler constant. The first three poles are annihilated by the multipliers in front of the curly brackets in eq. (4.27). The nearest surviving singularity (simple pole) appears only at the point $s = -3$. Thus formula (4.27) affords the required analytic continuation of the function $\zeta_{\text{ball}}(s)$ into the region $\text{Re} \ s < 0$. In view of eq. (3.13) we are interested in the point $s = -1/2$ where $\zeta_{\text{ball}}(s)$ given by (4.27) is regular

$$E_{\text{ball}} = \frac{1}{2} \zeta_{\text{ball}} \left( -\frac{1}{2} \right) = \frac{3\xi^2 c}{64a} \left[ 1 + \frac{9}{128} \left( \frac{\pi^2}{2} - 4 \right) + \ldots \right]. \quad (4.29)$$

It is exactly the first two terms in eq. (3.10) of ref. [25]. The procedure of analytic continuation presented above can be extended in a straightforward way to the arbitrary order of the uniform asymptotic expansion (4.20). Certainly in this case analytical calculations should be done by making use of Mathematica or Maple.

In sect. V the exact in the $\xi^2$-approximation expression for the Casimir energy (4.29) will be obtained by making use of another approach (see eq. (5.27)).

The problem under consideration with $\xi = 1$ is of a special interest because in this case it gives the Casimir energy of a perfectly conducting spherical shell. As it was noted above, this configuration has been considered by many authors. We present here the basic formulae which afford the analytical continuation of the corresponding spectral zeta function. We again content ourselves with two terms in the UAE (4.20). It is impossible to put simply $\xi = 1$ in the next formula (4.21). One has to do the expansion here anew keeping all the terms $\sim 1/\nu^4$. This gives
\[
\frac{d}{dz} \ln \left\{ 1 - \left[ \frac{d}{dz} (zI_\nu(\nu z)K_\nu(\nu z)) \right]^2 \right\} = \\
= \left[ \frac{3}{2\nu^2} z^{t^8} + \frac{3}{4\nu^4} z^{t^8} (-1 + 18t^2 - 50t^4 + 35t^6) + O(\nu^{-6}) \right]. \tag{4.30}
\]

After integration and elementary simplifications we arrive at the following result for the spectral function in hand

\[
\zeta_{\text{shell}}(s) \simeq \frac{a^{2s}}{4} s(1 + s)(2 + s) \left\{ \sum_{l=1}^\infty \nu^{-1-2s} + q(s) \sum_{l=1}^\infty \nu^{-3-2s} + \ldots \right\}, \tag{4.31}
\]

where

\[
q(s) = \frac{1}{480} (60 + 217s + 252s^2 + 71s^3). \tag{4.32}
\]

The analytic continuation of eq. (4.31) is accomplished in the same way as it has been done with eq. (4.24), i.e., , by making use of eqs. (4.22) and (4.26)

\[
\zeta_{\text{shell}}(s) \simeq \frac{a^{2s}}{4} s(1 + s)(2 + s) \left\{ (2^{1+2s} - 1)\zeta_R(1 + 2s) - 2^{1+2s} + \\
+q(s)\left[(2^{3+2s} - 1)\zeta_R(3 + 2s) - 2^{3+2s} \right] + \ldots \right\}. \tag{4.33}
\]

The nearest singularity in this formula is simple pole at \( s = -3 \). As above it is originated in the term \( \sim 1/\nu^7 \) in the UAE (4.20). At the point \( s = -1/2 \) the spectral zeta function \( \zeta_{\text{shell}}(s) \) is regular and gives the following value for the Casimir energy of a perfectly conducting spherical shell

\[
E_{\text{shell}} = \frac{1}{2} \zeta_{\text{shell}} \left( -\frac{1}{2} \right) = \frac{3}{64a} \left[ 1 - \frac{3}{256} \left( \frac{\pi^2}{2} - 4 \right) + \ldots \right] = \frac{1}{a} 0.046361 \ldots \tag{4.34}
\]

Without considering the analytic continuation and do not carrying out the analysis of the singularities in the complex \( s \) plane this result has been obtained in [118]. For nanometer size, that is for \( a = 10^{-7} \text{cm} \), the energy (4.34) is (in \( \hbar = c = 1 \) unit, 1eV \( \simeq 0.5 \cdot 10^5 \text{cm}^{-1} \))

\[
E_{\text{shell}} \simeq 10 \text{eV} \text{ which is of a considerable magnitude.}
\]

Unlike the Casimir energy for parallel conducting plates (4.5) both the Casimir energies for dielectric-diamagnetic ball (4.29) and for perfectly conducting spherical shell (4.34) prove to be positive. It implies that the Casimir forces in these cases are repulsive. In view of this the Casimir idea [139] to construct an extended model for electron failed because the vacuum forces cannot stabilize it.

In principle, it is possible to get a negative Casimir energy for a sphere, however for this purpose we have to consider the scalar massless field obeying on inner and outer surfaces of the sphere the Neumann boundary conditions [118]

\[
E_N = -\frac{1}{a} 0.223777 \ldots. \tag{4.35}
\]
The same field obeying the Dirichlet conditions on a sphere has positive vacuum energy \[118\] 

\[ E^D = \frac{1}{a} 0.002819 \ldots \]  

(4.36)

When obtaining the Casimir energies (4.35) and (4.36) the summation over the values of the angular momentum \( l \) starts from \( l = 0 \), unlike the electromagnetic field, where \( l \) assumes the values 1, 2, \ldots.

Undoubtedly, the calculation of the Casimir energy for a material ball with \( \varepsilon_1 \mu_1 \neq \varepsilon_2 \mu_2 \) by a rigorous zeta function method is also of a special interest. However, in this case the very definition of the spectral zeta function should probably be changed in order to incorporate the contact terms which seem to be essential in this problem [38,40,41,140] (see sect. V).

C. Cylinder

Calculation of the Casimir energy of an circular infinite cylinder [20,19] proves to be a more involved problem in comparison with that for sphere. Here the spectral zeta function \( \zeta_{cyl}(s) \) for this configuration will be constructed, its analytical continuation into the left half-plane of the complex variable \( s \) will be carried out, and relevant singularities will be analyzed.

Thus we are considering an infinite cylinder of radius \( a \) which is placed in an uniform unbounded medium. The permittivity and the permeability of the material making up the cylinder are \( \varepsilon_1 \) and \( \mu_1 \), respectively, and those for surrounding medium are \( \varepsilon_2 \) and \( \mu_2 \). We assume again that the condition (4.16) is fulfilled. In this case the electromagnetic oscillations can be divided into the transverse-electric (TE) modes and transverse-magnetic (TM) modes. In terms of the cylindrical coordinates \((r, \theta, z)\) the eigenfunctions of the given boundary value problem contain the multiplier 

\[ \exp(\pm i\omega t + ik_z z + in\theta) \]  

(4.37)

and their dependence on \( r \) is described by the Bessel functions \( J_n \) for \( r < a \) and by the Hankel functions \( H_n^{(1)} \) or \( H_n^{(2)} \) for \( r > a \). The frequencies of TE- and TM-modes are determined, respectively, by the equations [122]

\[ \Delta^\text{TE}_n(\lambda, a) \equiv \lambda a [\mu_1 J'_n(\lambda a) H_n(\lambda a) - \mu_1 J_n(\lambda a) H'_n(\lambda a)] = 0, \]  

(4.38)

\[ \Delta^\text{TM}_n(\lambda, a) \equiv \lambda a [\varepsilon_1 J'_n(\lambda a) H_n(\lambda a) - \varepsilon_1 J_n(\lambda a) H'_n(\lambda a)] = 0, \]  

(4.39)

where \( n = 0, \pm 1, \pm 2, \ldots \). Here \( \lambda^2 \) is the eigenvalue of the corresponding transverse (membrane-like) boundary value problem

\[ \lambda^2 = \frac{\omega^2}{c^2} - k_z^2. \]  

(4.40)

Decoupling of TE- and TM-modes in this problem is due to the condition (4.16).

In a complete analogy with the ball the Casimir energy per unit length of the cylinder is defined by the relevant spectral zeta function according to eq. (3.13). Let \( \lambda_{nr} \) be the roots
of the frequency equations (4.38) and (4.39), then the function \( \zeta_{\text{cyl}}(s) \) is introduced in the following way

\[
\zeta_{\text{cyl}}(s) = c^{-2s} \int_{-\infty}^{+\infty} \frac{dk_z}{2\pi} \sum_{n,r} \left[ (\lambda^2_{nr}(a) + k_z^2)^{-s} - (\lambda^2_{nr}(\infty) + k_z^2)^{-s} \right].
\]

(4.41)

In terms of the contour integral (see eq. (3.16)) it can be represented in the form

\[
\zeta_{\text{cyl}}(s) = c^{-2s} \int_{-\infty}^{+\infty} \frac{dk_z}{2\pi} \sum_{n=-\infty}^{+\infty} \oint_C (\lambda^2 + k_z^2)^{-s} d\lambda \ln \frac{\Delta_{n}^{\text{TE}}(\lambda a) \Delta_{n}^{\text{TM}}(\lambda a)}{\Delta_{n}^{\text{TE}}(\lambda a) \Delta_{n}^{\text{TM}}(\lambda a)}.
\]

(4.42)

Again we can take the contour \( C \) to consist of the imaginary axis \((+i\infty,-i\infty)\) closed by a semicircle of an infinitely large radius in the right half–plane. Continuation of the expressions \( \Delta_{n}^{\text{TE}}(\lambda a) \) and \( \Delta_{n}^{\text{TM}}(\lambda a) \) into the complex plane \( \lambda \) should be done in the same way as in the preceding section, i.e., by using \( H_{n}^{(1)}(\lambda) \) for \( \text{Im} \ \lambda < 0 \) and \( H_{n}^{(2)}(\lambda) \) for \( \text{Im} \ \lambda > 0 \). On the semicircle the argument of the logarithm in eq. (4.42) tends to 1. As a result this part of the contour \( C \) does not give any contribution to the zeta function \( \zeta_{\text{cyl}}(s) \). When integrating along the imaginary axis we choose the branch line of the function \( f(\lambda) = (\lambda^2 + k_z^2)^{-s} \) to run between \(-ik_z\) and \(+ik_z\), where \( k_z = +\sqrt{k_z^2} > 0 \) and use further that branch of this function which assumes real values when \(|y| < k_z\), where \( y = \text{Im} \ \lambda \). More precisely we have

\[
f(iy) = \begin{cases} 
\ e^{-i\pi s}(y^2 - k_z^2)^{-s}, & y > k_z, \\
\ (k_z^2 - y^2)^{-s}, & |y| < k_z, \\
\ e^{i\pi s}(y^2 - k_z^2)^{-s}, & y < -k_z.
\end{cases}
\]

(4.43)

Employment of the Hankel functions \( H_{n}^{(1)}(\lambda) \) and \( H_{n}^{(2)}(\lambda) \) by extending the expressions \( \Delta_{n}^{\text{TE}}(\lambda) \) and \( \Delta_{n}^{\text{TM}}(\lambda) \) into the complex plane \( \lambda \), as it was noted above, gives rise to the argument of the logarithm function depending only on \( y^2 \) on the imaginary axis. It means that the derivative of the logarithm function is odd function of \( y \). As a result the segment of the imaginary axis \((-ik_z, +ik_z)\) gives zero, and eq. (4.42) acquires the form

\[
\zeta_{\text{cyl}}(s) = \frac{\sin \pi s}{c^{2s} \pi^2} \sum_{n=-\infty}^{+\infty} \int_{0}^{\infty} dk_z \int_{k_z}^{\infty} (y^2 - k_z^2)^{-s} dy \ln \frac{\Delta_{n}^{\text{TE}}(ay) \Delta_{n}^{\text{TM}}(ay)}{\Delta_{n}^{\text{TE}}(i\infty) \Delta_{n}^{\text{TM}}(i\infty)}.
\]

(4.44)

Changing the order of integration of \( k_z \) and \( y \) and taking into account the value of the integral \([112]\)

\[
\int_{0}^{y} dk_z (y^2 - k_z^2)^{-s} = \frac{\sqrt{\pi}}{2} y^{1-2s} \frac{\Gamma(1-s)}{\Gamma\left(\frac{3}{2} - s\right)}, \quad \text{Re} \ s < 1 ,
\]

(4.45)

we obtain after the substitution \( ay \to y \)

\[
\zeta_{\text{cyl}}(s) = \frac{1}{2\sqrt{\pi} a \Gamma(s) \Gamma\left(\frac{3}{2} - s\right)} \left(\frac{a}{c}\right)^{2s} \sum_{n=-\infty}^{+\infty} \int_{0}^{\infty} dy \ y^{1-2s} \frac{d}{dy} \ln[1 - \xi^2 \mu_n^2(y)],
\]

(4.46)
where
\[ \mu_n(y) = y[I_n(y)K_n(y)]', \quad (4.47) \]
and the parameter \( \xi^2 \) was defined in eq. (4.18). We shall again content ourselves with the first two terms in the uniform asymptotic expansion (4.20) and take into account only the terms linear in \( \xi^2 \). In this approximation, upon changing the integration variable \( y = nz, n = \pm 1, \pm 2, \ldots \), we have
\[
\ln \left\{ 1 - \xi^2 \left[ z \frac{d}{dz} (I_n(nz)K_n(nz)) \right]^2 \right\} = \quad (4.48)
\]
\[
= -\xi^2 z^4 t^6 \left[ 1 + t^2 \left( \frac{3}{4n^2} \right) + \mathcal{O}(n^{-4}) \right].
\]
Now we substitute (4.48) into all the terms in (4.46) with \( n \neq 0 \). The term with \( n = 0 \) in this sum will be treated by subtracting and adding to the logarithmic function the quantity
\[ -\xi^2 \frac{y^4}{4 (1 + y^2)^3}. \quad (4.49) \]
As a result the zeta function \( \zeta_{cyl}(s) \) can be presented now as the sum of three terms
\[ \zeta_{cyl}(s) = Z_1(s) + Z_2(s) + Z_3(s), \quad (4.50) \]
where
\[
Z_1(s) = \frac{\left( \frac{a}{c} \right)^{2s}}{2\sqrt{\pi a} \Gamma(s) \Gamma \left( \frac{3}{2} - s \right)} \int_0^\infty dy y^{1-s} \frac{d}{dy} \left\{ \ln[1 - \xi^2 \mu_0^2(y)] + \frac{\xi^2}{4} y^4 t^6 \right\}, \quad (4.51)
\]
\[
Z_2(s) = -\xi^2 \left( \frac{a}{c} \right)^{2s} \sum_{n=1}^{\infty} n^{-2s-1} \frac{2}{8\sqrt{\pi a} \Gamma(s) \Gamma \left( \frac{3}{2} - s \right)} \int_0^\infty dz z^{1-s} \frac{d}{dz} (z^4 t^6), \quad (4.52)
\]
\[
Z_3(s) = -\xi^2 \frac{\left( \frac{a}{c} \right)^{2s} \sum_{n=1}^{\infty} n^{-3-2s}}{32\sqrt{\pi a} \Gamma(s) \Gamma \left( \frac{3}{2} - s \right)} \int_0^\infty dz z^{1-2s} \frac{d}{dz} [z^4 t^6 (3 - 30t^2 + 35t^4)]. \quad (4.53)
\]
In these equations \( Z_1(s) \) has accumulated the term with \( n = 0 \) from eq. (4.46) subtracted by (4.49); \( Z_2(s) \) involves the contribution of the term of order \( 1/n^2 \) in expansion (4.48) and the added expression (4.49); \( Z_3(s) \) is generated by the terms of order \( 1/n^4 \) in the expansion (4.48).

Taking into account that
\[ \mu_0^2(y)|_{y \to 0} \to 1 \quad \text{and} \quad \mu_0^2(y)|_{y \to \infty} \to \frac{1}{4y^2} + \frac{3}{16y^4}, \quad (4.54) \]

the integration by parts in eq. (4.51) can be done for \(-3/2 < \Re s < 1/2\) with the result
\[ Z_1(s) = \frac{(2s - 1)\alpha^{2s-1}}{2\sqrt{\pi} \epsilon^2 \Gamma(s) \Gamma \left( \frac{3}{2} - s \right)} \int_0^\infty \frac{dy}{y^{2s}} \left\{ \ln[1 - \xi^2 \mu_0^2(y)] + \frac{\xi^2}{4} y^4 \ell^6(y) \right\}. \quad (4.55) \]

With allowance for (4.54) one infers easily that the function \(Z_1(s)\) is an analytic function of the complex variable \(s\) in the region \(-3/2 < \Re s < 1/2\). In the linear order of \(\xi^2\) it reduces to
\[ Z_{1,\text{lin}}(s) = \xi^2 \frac{2s - 1}{2\sqrt{\pi} a \Gamma(s) \Gamma \left( \frac{3}{2} - s \right)} \left( \frac{\alpha}{c} \right)^{2s} \int_0^\infty \frac{dy}{y^{2s}} \left[ \frac{y^2}{4(1 + y^2)^3} - \mu_0^2(y) \right]. \quad (4.56) \]

This function is also analytic in the region \(-3/2 < \Re s < 1/2\). Integration in eq. (4.52) can be accomplished exactly by making use of the formula [112]
\[ \int_0^\infty dz z^{1-2s} \frac{dz}{dz}(z^4 \ell^6) = \frac{2s - 1}{4} \Gamma \left( \frac{1}{2} + s \right) \Gamma \left( \frac{5}{2} - s \right), \quad -\frac{1}{2} < \Re s < \frac{5}{2}. \quad (4.57) \]

This gives for \(Z_2(s)\) in (4.52)
\[ Z_2(s) = \xi^2 \left( \frac{\alpha}{c} \right)^{2s} \frac{(1 - 2s)(3 - 2s)}{64\sqrt{\pi} a} \left( 2 \sum_{n=1}^\infty n^{-2s-1} + 1 \right) \frac{\Gamma \left( \frac{1}{2} + s \right)}{\Gamma(s)}. \quad (4.58) \]

In view of the sum over \(n\) in (4.58) the function \(Z_2(s)\) is defined only for \(s > 0\).

For simplicity we apply in eq. (4.53) the integration by parts which is correct for \(-3/2 < \Re s < 1\) and leads to the result
\[ Z_3(s) = \xi^2 \left( \frac{\alpha}{c} \right)^{2s} \frac{(1 - 2s)(3 - 2s)(28s^2 - 8s - 27)}{6144\sqrt{\pi} a} \left( 2 \sum_{n=1}^\infty n^{-2s-3} + 1 \right) \frac{\Gamma \left( \frac{3}{2} + s \right)}{\Gamma(s)} \sum_{n=1}^\infty n^{-1}. \quad (4.59) \]

Again the sum over \(n\) in (4.59) gives the restriction \(\Re s > -1\) for the definition of the function \(Z_3(s)\).

Thus the spectral zeta function \(\zeta_{\text{cyl}}(s)\) in the linear approximation with respect to \(\xi^2\) and with allowance for the first two terms in the UAE (4.48) is given by
\[ \zeta_{\text{cyl}}(s) = Z_{1,\text{lin}}(s) + Z_2(s) + Z_3(s), \quad (4.60) \]

where the \(Z\)'s are presented in eqs. (4.56), (4.58) and (4.59), respectively. Summing up all the restrictions on the complex variable \(s\) which have been imposed when deriving these equations we infer that \(\zeta_{\text{cyl}}(s)\) is defined in the strip \(0 < \Re s < 1/2\). In order to continue this function into the surroundings of the point \(s = -1/2\), it is sufficient to express the
sum in eq. (4.59) in terms of the Riemann zeta function and treat the right-hand side of eq. (4.57) as an analytic continuation of its left-hand side over all the complex plane $s$. This gives

$$Z_2(s) = \xi^2 \left(\frac{a}{c}\right)^{2s} \frac{(1 - 2s)(3 - 2s)}{64\sqrt{\pi a}} [2\zeta(2s + 1) + 1] \frac{\Gamma\left(\frac{1}{2} + s\right)}{\Gamma(s)}. \quad (4.61)$$

It is left now to take the limit $s \to -1/2$ in eqs. (4.56), (4.58) and (4.61). A special care should be paid when calculating this limit in eq. (4.61) in view of the poles of the function $\Gamma(1/2 + s)$ at this point. Using the formulae

$$\zeta_R(0) = -\frac{1}{2}, \quad \zeta'_R(0) = -\frac{1}{2} \ln 2\pi, \quad \Gamma(x) = \frac{1}{x} - \gamma + O(x) \quad x \to 0, \quad (4.62)$$

one derives

$$\lim_{s \to -1/2} [2\zeta(1 + 2s) + 1] \frac{\Gamma\left(\frac{1 + 2s}{2}\right)}{\Gamma(1 + 2s)} = \lim_{s \to -1/2} [2\zeta(0) + 2\zeta'_R(0)(1 + 2s) + O((1 + 2s)^2) + 1] \frac{2}{1 + 2s} + \gamma + O(1 + 2s) = -2 \ln(2\pi). \quad (4.63)$$

Making allowance for this, we obtain from (4.61)

$$Z_2(-1) = \frac{\ln(2\pi)}{8\pi} \frac{cc^2}{a^2}. \quad (4.64)$$

The appearance of the finite term proportional to $\ln(2\pi)$ is remarkable for the problem under consideration. It is derived here in a consistent way by making use of an analytic continuation of the relevant spectral zeta function. In ref. [19] it was calculated in a more transparent though not rigorous way.

Gathering together eqs. (4.56), (4.59) with $s = -1/2$ and eq. (4.64) we obtain

$$\zeta_{cyl}(-1/2) = \frac{cc^2}{2\pi a^2} \left\{ \int_0^\infty y dy \left[ \frac{y^4}{4(1 + y^2)^3} - \mu_0^2(y) \right] + \frac{1}{48} \sum_{n=1}^{+\infty} \frac{1}{n^2} + \frac{1}{4} \ln(2\pi) \right\} = (-0.490878 + 0.034269 + 0.459469) \frac{cc^2}{2\pi a^2} = 0.002860 \frac{cc^2}{2\pi a^2}. \quad (4.65)$$

This result is not the final answer in the problem in hand. Point is that in view of severe cancellations in eq. (4.65) the contribution of the next term in the UAE (4.48) proves to be essential. Its account gives [19]

$$\zeta_{cyl}(-1/2) = 0. \quad (4.66)$$

Thus the Casimir energy of a compact cylinder possessing the same speed of light inside and outside proves to be zero in the $c^2$-approximation. The consideration presented in this section can be extended to the next term of order $\sim 1/n^6$ in the UAE (4.48) in a straightforward way. Therefore we shall not present here these rather cumbersome expressions. More precisely, in
ref. [19] not only the next term in UAE (4.48) has been taken into account but also the first 5 terms in the sum in (4.53) were taken exactly instead of using their asymptotics $n^{-3-2s}$. This turns out to be essential for reaching the required precision. However this point is not critical for our consideration because $Z_3(s)$ does not need analytic continuation. The exact zero result for this energy was obtained in paper [35] by making use of the addition formulae for the Bessel functions instead of the UAE.

However the Casimir energy of a compact cylinder possessing the same speed of light inside and outside does not vanish in higher approximations with respect to the parameter $\xi^2$ [26,35]. For example, up to the $\xi^4$ term it is defined by the formula

$$E(\xi^2) = -0.0955275 \frac{c \xi^4}{4\pi a^2} = -0.007602 \frac{c \xi^4}{a^2}. \quad (4.67)$$

In contrast to the Casimir energy of a compact ball with the same properties (see eq.(4.29))

$$E_{\text{ball}} \simeq \frac{3}{64a}c \xi^2 = 0.046875 \frac{c \xi^2}{a} \quad (4.68)$$

the Casimir energy of a cylinder under consideration turned out to be negative. Consequently, the Casimir forces strive to contract the cylinder. The numerical coefficient in eq. (4.67) proved really to be small, for example, in comparison with the analogous coefficient in eq. (4.68). Probably it is a manifestation of the vanishing of the Casimir energy of a pure dielectric cylinder noted in the Introduction. Numerically the Casimir energy $E(\xi^2)$ was calculated for arbitrary values of the parameter $\xi^2$ in paper [26].

Now we address to the consideration of a special case when $\xi = 1$. It corresponds to a perfectly conducting cylindrical shell [19]. Instead of the expansion (4.48) we have

$$\ln \left\{ 1 - \left[ z \frac{d}{dz} (I_n(nz)K_n(nz)) \right]^2 \right\} = -\frac{z^4 t_6}{4n^2} \left[ 1 + \frac{t^2}{4n^2} \left( 3 - 30t^2 + 35t^4 + \frac{1}{2} z^4 t^4 \right) + \mathcal{O}(n^{-4}) \right]. \quad (4.69)$$

Proceeding in the same way as above we obtain for the spectral zeta function concerned

$$\zeta_{\text{cyl}}^\text{shell}(s) = Z_1(s) + Z_2(s) + Z_3(s), \quad (4.70)$$

where $Z_1(s)$ is given by eq. (4.55) with $\xi = 1$, $Z_2(s)$ is the same as in eq. (4.61), and $Z_3(s)$ now is

$$Z_3(s) = \frac{(1-2s)(3-2s)(284s^2-104s-235)}{61440 \sqrt{\pi a^{1-2s}}} \frac{\Gamma \left( \frac{3}{2} + s \right)}{\Gamma(s)} \sum_{n=1}^{+\infty} n^{-3-2s}. \quad (4.71)$$

At the point $s = -1/2$ it acquires the value

$$\zeta_{\text{cyl}}^\text{shell} \left( -\frac{1}{2} \right) = -0.6517 - \frac{1}{2\pi a^2} + \frac{1}{2\pi a^2} \frac{7}{480} \sum_{n=1}^{+\infty} \frac{1}{n^2} + \frac{1}{8\pi a^2} \ln(2\pi)$$

$$= (-0.6517 + 0.0240 + 0.4595) \frac{1}{2\pi a^2} = -0.0268 \frac{1}{a^2}. \quad (4.72)$$
This exactly reproduce the contribution of the first two terms in calculations of the Casimir energy for perfectly conducting circular cylindrical shell in ref. [19]. With higher accuracy this energy is given by [19, 20]

\[ E_{\text{shell}} = -0.01356 \frac{1}{a^2}. \]  

(4.73)

The Casimir energy of a massless field subjected to Dirichlet and Neumann conditions on internal and external surfaces of an infinite circular cylindrical shell are also finite but of opposite signs [141]

\[ E_{\text{cyl}}^D = 0.000606 \frac{1}{a^2}, \]  

(4.74)

\[ E_{\text{cyl}}^N = -0.01417 \frac{1}{a^2}. \]  

(4.75)

The sum of these two energies gives the vacuum energy of electromagnetic field for this configuration (4.73).

However the zeta regularization does not give a finite value for the Casimir energy in the case of the two-dimensional version of the configuration under consideration, i.e., for a circle of radius \( a \) placed on a plane. In paper [141] these energies were calculated by making use of the relation between the zeta functions for a cylinder and a circle of the same radius

\[ \zeta_{\text{cir}}(s) = 2\sqrt{\pi} \frac{\Gamma(s + 1/2)}{\Gamma(s)} \zeta_{\text{cyl}}(s + 1/2). \]  

(4.76)

Having calculated the zeta function for an infinite cylinder in the region \(-1/2 \leq \Re s \leq 0\) one obtaines immedeatly the Casimir energy for a cylinder via \( \zeta_{\text{cyl}}(-1/2) \) and the Casimir energy for a circle in terms of \( \zeta_{\text{cyl}}(0) \)

\[ E_{\text{cir}}^D = \frac{1}{a} \left( 0.0023595 - \frac{1}{256} s \bigg|_{s \to 0} \right), \]  

(4.77)

\[ E_{\text{cir}}^N = -\frac{1}{a} \left( 0.3131 + \frac{5}{256} s \bigg|_{s \to 0} \right). \]  

(4.78)

The Casimir energy for the sum of these two fields is also infinite

\[ E_{\text{cir}}^{D+N} = -\frac{1}{a} \left( 0.2895 + \frac{3}{128} s \bigg|_{s \to 0} \right). \]  

(4.79)

On a plane electromagnetic field is reduced to a scalar massless field obeying the Neumann boundary conditions [142]. It was shown, that the zeta function regularization does not lead to a finite answer for the vacuum energy for spheres in spaces of arbitrary even dimensions [142–145]. As usual the coefficients in front of the pole-like contributions calculated by different methods coincide, but the finite parts of the answers differ [142, 143]. Thus for obtaining here physical results additional renormalization is needed.

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In ref. [21] the vacuum energy of a perfectly conducting cylindrical surface has been calculated to much higher accuracy by making use of another version of the zeta function technique. By integrating over $dk_2$ directly in eq. (4.41) the authors reduced this problem to investigation of the zeta function for circle, which has been considered earlier by introducing the partial wave zeta functions for interior and exterior region separately. In this respect our approach dealing only with one spectral zeta function for given boundary conditions proves to be more simple and straightforward.

Many results on calculation of the Casimir energies for spherically symmetric cavities by making use of the zeta function technique can be found in ref. [145].

V. CASIMIR ENERGY OF A DIELECTRIC BALL

Calculation of the vacuum energy of electromagnetic field connected with a dielectric ball proves to be a quite involved problem. The first attempt to calculate this energy has been undertaken by K. A. Milton in 1980 [37]. And only just recently the vacuum electromagnetic energy of a dilute dielectric ball was found at zero [38–45] and finite [46,47] temperature. It is worth noting that the first rough and not rigorous estimation of the Casimir energy of a dilute dielectric ball has been done in ref. [25]. The difficulties encountered here are due to the following mathematical peculiarities of the problem in question.

As it was noted in subsect. IVB the solutions to the Maxwell equations with spherical symmetry are expressed in terms of two scalar Debye potentials obeying Helmholtz equations (4.8). In the Casimir calculations these equations should be treated as one eigenvalue problem, hence we have to rewrite them in the form

$$-c_i^2 \Delta \psi_{lm}(r) = \omega^2 \psi_{lm}(r), \quad c_i^2 = \frac{c^2}{\varepsilon_i\mu_i}, \quad i = 1 \text{ for } r < a, \quad i = 2 \text{ for } r > a. \quad (5.1)$$

Here $c_1$ and $c_2$ are the velocities of light respectively inside and outside the ball. Thus the coefficient in front of the differential operator $-\Delta$ is discontinuous, it has a jump at the surface of the ball. It turns out that the heat kernel coefficient $B_2$ for the spectral problem (5.1) does not vanish but is a function of $c_1 - c_2$ [146]. If this difference is small (a dilute dielectric ball) it was shown that

$$B_2 \sim (c_1 - c_2)^3. \quad (5.2)$$

Thus the coefficient $B_2$ vanishes up to terms $(c_1 - c_2)^2$. It is this approximation that have been used in all the calculations of the Casimir energy of a dielectric ball.

In sect. III we have tried to show that the zeta function technique is the most consistent, from the mathematical standpoint, method for calculation of vacuum energy. However one did not manage to use this technique in a direct way for finding the Casimir energy of a dielectric ball. Here we describe briefly the methods practically used in these calculations.

In ref. [38] the Casimir energy of a dilute dielectric ball was derived by summing up the van den Wails forces between the individual molecules inside the compact ball. Barton [39] found this energy in the framework of a special perturbation theory, in which the dielectric ball was treated as a perturbation when treating the electromagnetic field in unbounded
empty space. Milton, Brevik and Marachevsky [40,41,43] started from the Green’s function of the quantized Maxwell field with an explicit account of the so called contact terms, on the stage of the numerical calculations the uniform asymptotic expansion for the Bessel functions and the zeta regularization technique being applied. We have considered this problem [42] by direct mode summation method with use of the addition theorem for the Bessel functions. In calculations without employment of the uniform asymptotic expansions for the Bessel functions [38,39,42] the exact (in the \(\Delta n^2\) approximation) result for the Casimir energy at hand was obtained. Brevik et al [44,45] considered this problem in a statistical approach.

First we present the calculation of this Casimir energy at zero temperature and further we extended this approach to a finite temperature.

A. Zero temperature

As in subsect. IV B, a solid ball of a radius \(a\) placed in an unbounded uniform medium is considered but the condition (4.16) is not imposed now. Unfortunately, the zeta regularization cannot be used here in a direct way. The reason for this will be seen below.

We shall proceed from the standard definition of the vacuum energy as the sum over the eigenfrequencies of electromagnetic oscillations

\[
E = \frac{1}{2} \sum_s (\omega_s - \bar{\omega}_s).
\] (5.3)

Here \(\omega_s\) are the frequencies defined by eqs. (4.9) and (4.10), and the frequencies \(\bar{\omega}_s\) correspond to a certain limiting boundary conditions that will be specified below.

In the case of the plane geometry of boundaries or when considering the Casimir effect for distinct bodies it is sufficient to subtract in eq. (5.3) the contribution of the Minkowski space. In the problem at hand it implies taking the limit \(a \to \infty\), i.e., that the medium 1 tends to fill the entire space. But it turns out that this subtraction is not sufficient because the linear in \(\varepsilon_1 - \varepsilon_2\) contribution into the vacuum energy retains. Further, we assume that the difference \(\varepsilon_1 - \varepsilon_2\) is small and content ourselves only with the \((\varepsilon_1 - \varepsilon_2)^2\)-terms.

The necessity to subtract the contributions to the vacuum energy linear in \(\varepsilon_1 - \varepsilon_2\) is justified by the following consideration. The Casimir energy of a dilute dielectric ball can be thought of as the net result of the van der Waals interactions between the molecules making up the ball [38]. These interactions are proportional to the dipole momenta of the molecules, i.e., to the quantity \((\varepsilon_1 - 1)^2\). Thus, when a dilute dielectric ball is placed in the vacuum, then its Casimir energy should be proportional to \((\varepsilon_1 - 1)^2\). It is natural to assume that when such a dielectric ball is surrounded by an infinite dielectric medium with permittivity \(\varepsilon_2\), then its Casimir energy should be proportional to \((\varepsilon_1 - \varepsilon_2)^2\). The physical content of the contribution into the vacuum energy linear in \(\varepsilon_1 - \varepsilon_2\) has been investigated in the framework of the microscopic model of the dielectric media (see paper [147] and references therein). It has been shown that this term represents the self-energy of the electromagnetic field attached to the polarizable particles or, in more detail, it is just the sum of the individual atomic Lamb shifts. Certainly this term in the vacuum energy should be disregarded when calculating the Casimir energy which is originated in the electromagnetic interaction between different polarizable particles or atoms [39,40,44,45,148].
Further, we put for sake of symmetry
\[ \sqrt{\varepsilon_1} = n_1 = 1 + \frac{\Delta n}{2}, \quad \sqrt{\varepsilon_2} = n_2 = 1 - \frac{\Delta n}{2}. \] (5.4)

Here \( n_1 \) and \( n_2 \) are the refractive indices of the ball and of its surroundings, respectively, and it is assumed that \( \Delta n \ll 1 \). From here it follows, in particular, that
\[ \varepsilon_1 - \varepsilon_2 = (n_1 + n_2)(n_1 - n_2) = 2\Delta n. \] (5.5)

Thus, using the definition (5.3) we shall keep in mind that really two subtractions should be done: first the contribution, obtained in the limit \( a \to \infty \), has to be subtracted and then all the terms linear in \( \Delta n \) should also be removed.

We present the vacuum energy defined by eq. (5.3) in terms of the contour integral in the complex frequency plane. Upon the contour deformation one gets
\[
E = -\frac{1}{2\pi} \sum_{l=1}^{\infty} (2l + 1) \int_{0}^{\infty} dy \frac{d}{dy} \ln \left( \frac{\Delta_{l}^{\text{TE}}(iay)\Delta_{l}^{\text{TM}}(iay)}{\Delta_{l}^{\text{TE}}(i\infty)\Delta_{l}^{\text{TM}}(i\infty)} \right),
\] (5.6)

where \( \Delta_{l}^{\text{TE}}(iay) \) and \( \Delta_{l}^{\text{TM}}(iay) \) are defined in eqs. (4.9) and (4.10).

In eq. (5.6) we have introduced cutoff \( K \) in integration over the frequencies. This regularization is natural in the Casimir problem because physically it is clear that the photons of a very short wave length do not contribute to the vacuum energy since they do not “feel” the boundary between the media with different permittivities \( \varepsilon_1 \) and \( \varepsilon_2 \). In the final expression the regularization parameter \( K \) should be put to tend to infinity, the divergencies, that may appear here, being cancelled by appropriate counter terms.

Taking into account the asymptotics of the Riccati-Bessel functions we obtain
\[
\Delta_{l}^{\text{TE}}(i\infty)\Delta_{l}^{\text{TM}}(i\infty) = -\frac{(n_1 + n_2)^2}{4} e^{2(n_1-n_2)y}. \quad (5.7)
\]

Upon substituting eqs. (4.9), (4.10) and (5.7) into eq. (5.6) and changing the integration variable \( ay \to y \), we cast eq. (5.6) into the form (see ref. [25])
\[
E = -\frac{1}{2\pi a} \sum_{l=1}^{y_0} (2l + 1) \int_{0}^{y_0} dy \frac{d}{dy} \ln \left( \frac{4e^{-2(n_1-n_2)y}}{(n_1 + n_2)^2} \right) \left[ n_1n_2(s'_l e_l)^2 + (s_l e'_l)^2 - (n_1^2 + n_2^2)s_l s'_l e_l e'_l \right], \quad (5.8)
\]

where \( s_l \equiv s_l(n_1y), \quad e_l \equiv e_l(n_2y), \quad y_0 = aK \).

It should be noted here that in eq. (5.8) only the first subtraction is accomplished, which removes the contribution to the vacuum energy obtained when \( a \to \infty \). As noted above, for obtaining the final result all the terms linear in \( \Delta n \) should be discarded also.

Further it is convenient to rewrite eq. (5.8) in the form
\[
E = E_1 + E_2 \quad (5.9)
\]

with
\[
E_1 = \frac{\Delta n}{2\pi a} \sum_{l=1}^{\infty} (2l + 1) \int_0^{y_0} y \, dy,
\]
(5.10)

\[
E_2 = -\frac{1}{2\pi a} \sum_{l=1}^{\infty} (2l + 1) \int_0^{y_0} dy \, y \, d \frac{d}{dy} \ln \left[ \frac{W_l^2(n_1 y, n_2 y) - \frac{\Delta n^2}{4} P_l^2(n_1 y, n_2 y)}{4} \right],
\]
(5.11)

where

\[
W_l(n_1 y, n_2 y) = s_l(n_1 y)e_l(n_2 y) - s_l'(n_1 y)e_l'(n_2 y),
\]
(5.12)

\[
P_l(n_1 y, n_2 y) = s_l(n_1 y)e_l'(n_2 y) + s_l'(n_1 y)e_l(n_2 y).
\]
(5.13)

The term \(E_1\) accounts for only the expression \(\exp(-2\Delta n y)\) in the argument of the logarithm function in eq. (5.8) and it appears as a result of subtracting the Minkowski space contribution to the Casimir energy (the sum with \(\bar{\omega}_s\) in eq. (5.3) and the denominator in eq. (5.6)).

It is worth noting that the term \(E_1\) is exactly the Casimir energy considered by Schwinger in his attempt to explain the sonoluminescence [149]. Really, introducing the cutoff \(K\) for frequency integration and the cutoff \(y = \omega/a\) for the angular momentum summation we arrive at the result

\[
E_1 = \frac{\Delta n}{\pi a} \int_0^{aK} y \, dy \sum_{l=1}^{\infty} \left( l + \frac{1}{2} \right) \sim \frac{\Delta n}{2\pi a} \int_0^{aK} y^3 \, dy = \Delta n \frac{K^4 a^3}{8\pi}.
\]
(5.14)

We have substituted here the summation over \(l\) by integration. Up to the multiplier \((-2/3)\) it is exactly the Schwinger value for the Casimir energy of a ball \((\varepsilon_1 = 1)\) in water \((\sqrt{\varepsilon_2} \simeq 4/3)\) [140]. The term linear in \(\Delta n\) and of the same structure was also derived in papers [39,44,148]. As it was explained above the energy \(E_1\) should be discarded.

In our calculation, we content ourselves with the \(\Delta n^2\)-approximation. Hence, in eq. (5.11) one can put \(P_l^2(n_1 y, n_2 y) \sim P_l^2(y, y)\) and keep in expansion of the logarithm function only the terms proportional to \(\Delta n^2\). In this approximation, the contributions of \(W_l^2\) and \(P_l^2\) into the vacuum energy are additive

\[
E^\text{ren} = E_W^\text{ren} + E_P^\text{ren}.
\]
(5.15)

In the Appendix A it is shown that for obtaining the \(\Delta n^2\)-contribution into the Casimir energy of the function \(W_l^2\) in the argument of the logarithm in eq. (5.11), it is sufficient to calculate the \(\Delta n^2\)-contribution of the function \(W_l^2\) alone but changing the sign of this contribution to the opposite one (see eq. (A20)). Hence,

\[
E_W = \frac{1}{2\pi a} \sum_{l=1}^{\infty} (2l + 1) \int_0^{y_0} dy \, y \, d \frac{d}{dy} W_l^2(n_1 y, n_2 y),
\]
(5.16)

and only the \(\Delta n^2\)-term being preserved in this expression.

For \(E_P\) we have

\[
E_P = \frac{\Delta n^2}{8\pi a} \sum_{l=1}^{\infty} (2l + 1) \int_0^{y_0} dy \, y \, d \frac{d}{dy} P_l^2(n_1 y, n_2 y).
\]
(5.17)
Usually, when calculating the vacuum energy in the problem with spherical symmetry, the uniform asymptotic expansion of the Bessel functions is used (see sect. IV). As a result, an approximate value of the Casimir energy can be derived, the accuracy of which depends on the number of terms preserved in the asymptotic expansion.

We shall persist in another way employing the technique of the paper \[33\]. By making use of the addition theorem for the Bessel functions \[138\], we first do the summation over the angular momentum \(l\) in eq. (5.11) and only after that we will integrate over the imaginary frequency \(y\). As a result, we obtain an exact (in the \(\Delta n^2\)-approximation) value of the Casimir energy in the problem involved.

The addition theorem for the Bessel functions is given by the formula \[138\]

\[
\sum_{l=0}^{\infty} (2l + 1) s_l(\lambda r) e_l(\lambda \rho) P_l(\cos \theta) = \frac{\lambda r \rho}{R} e^{-\lambda R} \equiv D, \tag{5.18}
\]

where

\[
R = \sqrt{r^2 + \rho^2 - 2r \rho \cos \theta}. \tag{5.19}
\]

Differentiating the both sides of eq. (5.18) with respect to \(\lambda r\) and squaring the result we deduce

\[
\sum_{l=0}^{\infty} (2l + 1) [s'_l(\lambda r)e_l(\lambda \rho)]^2 = \frac{1}{2r \rho} \int_{r-\rho}^{r+\rho} \left(\frac{1}{\lambda} \frac{\partial D}{\partial r}\right)^2 R \, dR. \tag{5.20}
\]

Here the orthogonality relation for the Legendre polynomials

\[
\int_{-1}^{+1} P_l(x) P_m(x) \, dx = \frac{2\delta_{lm}}{2l + 1}
\]

has been taken into account. Now we put

\[
\lambda = y, \quad r = n_1 = 1 + \frac{\Delta n}{2}, \quad \rho = n_2 = 1 - \frac{\Delta n}{2}. \tag{5.21}
\]

Applying eq. (5.20) and analogous ones, we derive

\[
\sum_{l=1}^{\infty} (2l + 1) W_l^2(n_1 y, n_2 y) = \frac{1}{2r \rho \lambda^2} \int_{r-\rho}^{r+\rho} R \, dR \left(\mathcal{D}_r - \mathcal{D}_\rho\right)^2 - e^{2\Delta n y}
\]

\[
= \frac{\Delta n^2}{8} \int_{\Delta n}^{2} \frac{e^{-2yR}}{R^5} \left(4 + R^2 + 4y R - y R^3\right)^2 dR - e^{2\Delta n y}, \tag{5.22}
\]

\[
\sum_{l=1}^{\infty} (2l + 1) P_l^2(y, y) = \frac{1}{2} \int_{0}^{2} \left[\frac{\partial}{\partial y} \left(\frac{y}{R} e^{-y R}\right)\right]^2 R \, dR - e^{-4y}. \tag{5.23}
\]

Here \(\mathcal{D}_r\) and \(\mathcal{D}_\rho\) stand for the results of the partial differentiation of the function \(D\) in eq. (5.18) with respect to the corresponding variables and with the subsequent substitution of (5.21). The last terms in eqs. (5.22) and (5.23) are \(W_0^2(n_1 y, n_2 y)\) and \(P_0^2(y, y)\), respectively.
As it was stipulated before, in eq. (5.22) we have to keep only the terms proportional to $\Delta n^2$ and in eq. (5.23) we have put $\Delta n = 0$.

The calculation of the contribution $E_P$ to the Casimir energy is straightforward. Upon differentiation of the right-hand side of eq. (5.23) with respect to $y$, the integration over $dR$ can be done here. Substitution of this result into eq. (5.17) gives

$$E_P = -\frac{\Delta n^2}{2\pi a} \left( -\frac{1}{4} \right) \int_0^{y_0} dy \left[ e^{-4y} \left( 2y^2 + 2y + \frac{1}{2} \right) - \frac{1}{2} \right]. \quad (5.24)$$

The term $(-1/2)$ in the square brackets in eq. (5.24) gives rise to the divergence when $y_0 \to \infty$

$$E_{P,\text{div}} = -\frac{\Delta n^2}{16\pi a} y_0. \quad (5.25)$$

Therefore we have to subtract it with the result

$$E_{P,\text{ren}} = E_P - E_{P,\text{div}} = \frac{5}{128} \frac{\Delta n^2}{\pi a}. \quad (5.26)$$

It is worth noting here that eq. (5.26) after substitution $\Delta n^2/4 = \xi^2$ gives the exact, in the $\xi^2$-approximation, value for the Casimir energy of a compact ball with the same velocity of light inside and outside the ball [33,42,46,15]

$$E_{\text{ball}} = \frac{5}{32} \frac{\xi^2 c}{\pi a} \quad (5.27)$$

(compare this formula with eq. (4.29)). In subsect. IV B this energy has been calculated by making use of the zeta function technique (see eq. (4.29)). As this zeta function is not known exactly, we have derived there only an approximation for the exact result (5.51).

As far as the expression (5.22), it is convenient to substitute it into eq. (5.16), to do the integration over $y$ and only after that to address the integration over $dR$

$$\frac{\Delta n^2}{8} \int_{\Delta n}^{2} dR \int_0^\infty dy y \frac{dy}{R^5} \left[ e^{-2yR} \left( 4 + R^2 + 4yR - yR^3 \right)^2 \right] =$$

$$= -\frac{\Delta n^2}{4} \int_{\Delta n}^{2} \left( \frac{10}{R^6} + \frac{1}{R^4} + \frac{1}{8R^2} \right) dR$$

$$= \frac{1}{8} \left( \frac{\Delta n^2}{3} - \frac{4}{\Delta n^3} - \frac{2}{3\Delta n} - \frac{\Delta n}{4} \right). \quad (5.28)$$

We have put here $y_0 = \infty$ without getting the divergencies. As it is explained in the Appendix A, in eq. (5.28) we have to pick up only the term proportional to $\Delta n^2$. Remarkably

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2This divergence has the same origin as those arising in summation over $l$ when the uniform asymptotic expansions of the Bessel functions are used [40,41]. The technique employed here is close to the multiple scattering expansion [15], where these divergencies are also subtracted.
this term is finite. It is an essential advantage of our approach. The rest of the terms in this equation are irrelevant to our consideration. Thus the counter term for $E_W$ vanishes due to the regularizations employed (see the Appendix A). In view of this we have

$$E_W^\text{ren} = E_W = \frac{1}{2\pi a} \frac{1}{8} \frac{\Delta n^2}{3} = \frac{1}{48} \frac{\Delta n^2}{\pi a}.$$  \hfill (5.29)

Finally we arrive at the following result for the Casimir energy of a dilute dielectric ball

$$E^\text{ren} = E_W^\text{ren} + E_P^\text{ren} = \frac{\Delta n^2}{\pi a} \left( \frac{1}{48} + \frac{5}{128} \right) = \frac{23}{384} \frac{\Delta n^2}{\pi a}.$$  \hfill (5.30)

Taking into account the relation (5.5) between $\varepsilon_i$ and $n_i, i = 1, 2$, we can write

$$E^\text{ren} = \frac{23}{1536} \frac{(\varepsilon_1 - \varepsilon_2)^2}{\pi a}.$$  \hfill (5.31)

At the first time, this value for the Casimir energy of a dilute dielectric ball has been derived in ref. [38] by summing up the van der Waals interactions between individual molecules making up the ball ($\varepsilon_2 = 1$). The result (5.31) was obtained also by treating a dilute dielectric ball as a perturbation in the complete Hamiltonian of the electromagnetic field for relevant configuration [39]. In papers [40,41], the value close to the exact one, eq. (5.31), has been obtained by employing the uniform asymptotic expansion of the Bessel functions.

In ref. [25] the estimation of the Casimir energy of a dilute dielectric ball has been done taking into account, as it is clear now, only the second term in eq. (5.30). And nevertheless it was not so bad having the accuracy about 35%.

**B. Finite temperature**

An essential advantage of the calculation of the Casimir energy of a dilute dielectric ball, carried out above by the mode summation method, is the possibility for its straightforward generalization to the finite temperature. The employment of the addition theorem for the Bessel functions again enables one to carry out the summation over the angular momentum in a closed form. As a result, the exact (in the $\Delta n^2$ approximation) value for internal and free energies, as well as for entropy of a dilute dielectric ball are derived for finite temperature also. The divergencies, inevitable in such studies, are removed by making use of the renormalization procedure developed for calculation of the relevant Casimir energy at zero temperature (see subsect. VA). The thermodynamic characteristics are presented as the sum of the respective quantity for a compact ball with uniform velocity of light and an additional term which is specific only for a pure dielectric ball. The behavior of the thermodynamic characteristics in the low and high temperature limits is investigated.

Practically the extension to finite temperature $T$ is accomplished by substituting the $y$-integration in eqs. (5.11) by summation over the Matsubara frequencies $\omega_n = 2\pi n T$. Doing in this way we obtain the internal energy of a dielectric ball
\[ U(T) = -T \sum_{l=1}^{\infty} (2l + 1) \sum_{n=0}^{\infty} w_n \frac{d}{dw_n} \ln \left[ W_l^2(n_1w_n, n_2w_n) \right. \]
\[ \left. - \frac{\Delta n^2}{4} P_l^2(n_1w_n, n_2w_n) \right], \tag{5.32} \]

where
\[ W_l(n_1w_n, n_2w_n) = s_l(n_1w_n)e_l(n_2w_n) - s_l'(n_1w_n)e_l(n_2w_n), \tag{5.33} \]
\[ P_l(n_1w_n, n_2w_n) = s_l(n_1w_n)e_l(n_2w_n) + s_l'(n_1w_n)e_l(n_2w_n), \tag{5.34} \]
and we have introduced the dimensionless Matsubara frequencies
\[ w_n = a\omega_n = 2\pi n a T, \quad n = 0, 1, 2, \ldots. \tag{5.35} \]

The prime on the summation sign in eq. (5.32) means that the \( n = 0 \) term is counted with half weight. In what follows it turns out to be important, for example, when removing the divergencies, that we proceed from the expression for the internal energy instead of from the analogous formula for free energy.

When considering the low temperature behavior of the thermodynamic functions of a dielectric ball the term proportional to \( T^3 \) in our paper [46] was lost. It was due to the following. We have introduced the summation over the Matsubara frequencies in eq. (3.20) under the sign of the \( R \)-integral. Here we show how to do this summation in a correct way.

In the \( \Delta^2 \)-approximation the last term in eq. (3.20) from the article [46]
\[ \mathcal{U}_W(T) = 2T \Delta n^2 \sum_{n=0}^{\infty} w_n^2 \int_{\Delta n}^{2} \frac{e^{-2w_nR}}{R} dR, \quad w_n = 2\pi n a T \tag{5.36} \]
can be represented in the following form
\[ \mathcal{U}_W(T) = -2T \Delta n^2 \sum_{n=0}^{\infty} w_n^2 E_1(4w_n), \tag{5.37} \]
where \( E_1(x) \) is the exponential-integral function [138]. Now we accomplish the summation over the Matsubara frequencies by making use of the Abel-Plana formula
\[ \sum_{n=0}^{\infty} f(n) = \int_{0}^{\infty} f(x) dx + i \int_{0}^{\infty} \frac{f(ix) - f(-ix)}{e^{2\pi x} - 1} dx. \tag{5.38} \]
The first term in the right-hand side of this equation gives the contribution independent of the temperature, and the net temperature dependence is produced by the second term in this formula. Being interested in the low temperature behavior of the internal energy we substitute into the second term in eq. (5.38) the following expansion of the function \( E_1(z) \)
\[ E_1(z) = -\gamma - \ln z - \sum_{k=1}^{\infty} \frac{(-1)^k z^k}{k \cdot k!}, \quad |\arg z| < \pi, \tag{5.39} \]
where $\gamma$ is the Euler constant [138]. The contribution proportional to $T^3$ is produced by the logarithmic term in the expansion (5.39). The higher powers of $T$ are generated by the respective terms in the sum over $k$ in this formula ($t = 2\pi aT$)

$$U_W(T) = \frac{\Delta n^2}{\pi a} \left( -\frac{1}{96} + \frac{\zeta_R(3)}{4\pi^2} t^3 - \frac{1}{30} t^4 + \frac{8}{567} t^6 - \frac{8}{1125} t^8 + \mathcal{O}(t^{10}) \right). \quad (5.40)$$

All these terms, safe for $2\zeta(3)\Delta n^2 a^2 T^3$, are also reproduced by the last term in eq. (3.31) in our paper [46] (unfortunately additional factor 4 was missed there)

$$\Delta n^2 \cdot 8 T \cdot 4 t^2 \int_{\Delta n} \frac{dR \coth(t R)}{R \sinh^2(t R)}.$$

Taking all this into account we arrive at the following low temperature behavior of the internal Casimir energy of a dilute dielectric ball

$$U(T) = \frac{\Delta n^2}{\pi a} \left( \frac{23}{384} + \frac{\zeta_R(3)}{4\pi^2} t^3 - \frac{7}{360} t^4 + \frac{22}{2835} t^6 - \frac{46}{7875} t^8 + \mathcal{O}(t^{10}) \right). \quad (5.41)$$

The relevant thermodynamic relations give the following low temperature expansions for free energy

$$F(T) = \frac{\Delta n^2}{\pi a} \left( \frac{23}{384} - \frac{\zeta_R(3)}{8\pi^2} t^3 - \frac{7}{1080} t^4 + \frac{22}{14175} t^6 + \frac{46}{55125} t^8 + \mathcal{O}(t^{10}) \right) \quad (5.42)$$

and for entropy

$$S(T) = -\frac{\partial F}{\partial T} = \Delta n^2 \left( \frac{3\zeta_R(3)}{4\pi^2} t^2 - \frac{7}{135} t^3 + \frac{88}{4725} t^5 - \frac{736}{55125} t^7 + \mathcal{O}(t^9) \right). \quad (5.43)$$

The $T^3$ term in the free energy (5.42) does not give contribution to the Casimir force exerted on the surface of a dielectric ball, however, it proves to be important for insuring the positive entropy (5.43) at low temperatures.

The range of applicability of the expansions (5.41), (5.42), and (5.43) can be roughly estimated in the following way. The curve $S(T)$ defined by eq. (5.43) monotonically goes up when the dimensionless temperature $t = 2\pi aT$ changes from 0 to $t \sim 1.0$. After that this curve sharply goes down to the negative values of $S$. It implies that eqs. (5.41) – (5.43) can be used in the region $0 \leq t < 1.0$. The $T^3$-term in eqs. (5.41) and (5.42) proves to be principal because it gives the first positive term in the low temperature expansion for the entropy (5.43). It is worth noting, that the exactly the same $T^3$-term, but with opposite sign, arises in the high temperature asymptotics of free energy in the problem at hand (see eq. (4.30) in ref. [150]).

For large temperature $T$ we found [46]

$$U(T) \simeq \frac{\Delta n^2}{8} T, \quad F(T) \simeq -\frac{\Delta n^2}{8} T [\ln(aT) - c], \quad S(T) \simeq \frac{\Delta n^2}{8} [\ln(aT) + c + 1], \quad (5.44)$$

where $c$ is a constant [25,39] $c = \ln 4 + \gamma - 7/8$. Analysis of eqs. (3.20) and (3.31) from the paper [46] shows that there are only exponentially suppressed corrections to the leading terms (5.44).
In the course of calculation of the thermodynamical functions of a dilute dielectric ball these functions were first obtained for a compact ball with the same velocity of light inside and outside [46]. There we have derived the following exact, in the $\xi^2$-approximation, expression for the internal energy

$$U(T) = \frac{\xi^2}{2} T \left[ t^2 \frac{\coth(2t)}{\sinh^2(2t)} + \frac{t}{\sinh^2(2t)} + \frac{1}{2} \coth(2t) \right]. \quad (5.45)$$

In the small $T$ region eq. (5.45) gives

$$U(T) = \frac{\xi^2}{\pi a} \left( \frac{5}{32} + \frac{1}{90} t^4 + \frac{8}{945} t^6 - \frac{8}{325} t^8 + O(t^{10}) \right). \quad (5.46)$$

Integration of the thermodynamic relation

$$U(T) = \frac{\partial}{\partial \beta} [\beta F(T)], \quad \beta = T^{-1} \quad (5.47)$$

enables one to get the free energy

$$F(T) = -T \int \frac{U(T)}{T^2} dT + C T, \quad (5.48)$$

where $C$ is a constant. Substituting the asymptotics (5.46) into eq. (5.48) we obtain the respective free energy in the low temperature region

$$F(T) = \frac{\xi^2}{\pi a} \left( \frac{5}{32} - \frac{1}{135} t^4 - \frac{8}{4725} t^6 + \frac{512}{3675} t^8 + O(t^{10}) \right). \quad (5.49)$$

Here the linear in $T$ term $C T$ has been dropped in view of the requirement that the entropy $S(T)$ should vanish at $T = 0$ gives [36]. Indeed, the thermodynamic relation (7.14) gives in this limit

$$S(0) = \lim_{T \to 0} T^{-1} (U(T) - F(T)) = C = 0. \quad (5.50)$$

Hence, at low temperature the expansions both for the internal energy (5.46) and for the free energy (5.49) involve only even powers of the temperature beginning from $T^4$. At zero temperature we have

$$U(0) = F(0) = E_{\text{ball}} = \frac{5\xi^2}{32\pi a}, \quad (5.51)$$

where $E_{\text{ball}}$ is the exact, in the $\xi^2$-approximation, Casimir energy of a compact ball with the same velocity of light inside and outside the ball (see eq. (5.27)).

The entropy in the problem at hand is obtained by differentiation of the free energy (5.49) (see eq. (7.14))

$$S(T) = \pi a \left( \frac{4}{135} t^3 + \frac{96}{4725} t^5 - \frac{128}{3675} t^7 + O(t^9) \right). \quad (5.52)$$
The exact formula (5.45) leads to the following high temperature asymptotics of the internal energy

\[ U(T) \simeq \frac{\xi^2}{4} T, \quad T \to \infty. \]  

Substituting this asymptotics into eq. (5.48) we arrive at the high temperature limit for the free energy \( F(T) \)

\[ F(T) \simeq -\frac{\xi^2}{4} T \left[ \ln(aT) + \alpha \right], \quad T \to \infty. \]  

The constant \( \alpha \) will be calculated in subsect. VII C 2 (see eq. (7.42))

\[ \alpha = \gamma + \ln 4 - \frac{5}{4}. \]

This constant has been found in ref. [36] too. The asymptotics (5.44) (5.53) and (5.54) contain terms the Planck constant, thus it is pure classical contributions (see also sect. VII).

Summarizing we conclude that now there is a complete agreement between the results of calculation of the Casimir thermodynamic functions for a dilute dielectric ball carried out in the framework of two different approaches: by the mode summation method [42,46] and by perturbation theory for quantized electromagnetic field, when dielectric ball is considered as a perturbation in unbounded continuous surroundings [39].

VI. NON-SMOOTHNESS OF THE BOUNDARY AND DIVERGENCIES OF VACUUM ENERGY

A. Semi-circular cylinder

The spectrum of electromagnetic oscillations, as well as oscillations of other fields, is determined by the boundary form and by the conditions imposed on the field functions on the boundary. In view of this, one could anticipate that the divergencies in vacuum energy are also connected with the boundary geometry. In general it is true but this relation turns out to be very complicated and it is still far from being clear. We are going to show this by calculating the Casimir energy for a simple configuration, namely, for a semi-circular cylindrical shall by making use of the zeta function technique. This shall is obtained by crossing an infinite circular cylindrical shell by a plane passing through the symmetry axes of the cylinder. All the surfaces, including the infinite cutting plane, are assumed to be perfectly conducting. Obviously it is sufficient to consider only a half of this configuration (left or right) which we shall refer to as a semi-circular cylindrical shall or, for sake of shortening, as a semi-circular cylinder (see fig. 1). The internal boundary value problem for this configuration is nothing else as a semi-cylindrical waveguide. In the theory of waveguides [119] it is well known that a semi-circular waveguide has the same eigenfrequencies as the cylindrical one but without degeneracy (without doubling) and safe for one frequency series (see below). Notwithstanding the very close spectra, the zeta function technique does not give a finite result for a semi-circular cylinder unlike for a circular one (see subsect. IV C).
We start with considering the natural modes of electromagnetic field for circular and semi-circular cylinders. The construction of the solutions to the Maxwell equations with boundary conditions given on closed surfaces proves to be nontrivial problem. Mainly it is due to the vector character of the electromagnetic field [119,122,151]. In the case of cylindrical symmetry the electric \( E \) and magnetic \( H \) fields are expressed in terms of the electric (\( \Pi' \)) and magnetic (\( \Pi'' \)) Hertz vectors having only one non-zero component

\[
\Pi' = e_z \Phi(r, \varphi) e^{\pm ik' z}, \\
\Pi'' = e_z \Psi(r, \varphi) e^{\pm ik'' z}.
\]

Here the cylindrical coordinate system \( r, \varphi, z \) is used with \( z \) axes directed along the cylinder axes. The common time-dependent factor \( e^{i \omega t} \) is dropped. The scalar functions \( \Phi(r, \varphi) \) and \( \Psi(r, \varphi) \) are the eigenfunctions of the two-dimensional transverse Laplace operator and meet, respectively, the Dirichlet and Neumann conditions on the boundary \( \partial \Gamma \)

\[
(\nabla_{\perp}^2 + \gamma'^2) \Phi(r, \varphi) = 0, \quad \Phi(r, \varphi)|_{\partial \Gamma} = 0,
\]

\[
(\nabla_{\perp}^2 + \gamma''^2) \Psi(r, \varphi) = 0, \quad \frac{\partial \Psi(r, \varphi)}{\partial n}|_{\partial \Gamma} = 0,
\]

where \( \nabla_{\perp}^2 \) is the transverse part of the Laplace operator

\[
\nabla_{\perp}^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2}
\]

and

\[
\gamma'^2 = \omega^2 - k'^2, \quad \gamma''^2 = \omega^2 - k''^2.
\]

First we consider a cylindrical shell. In this case the functions \( \Phi(r, \varphi) \) and \( \Psi(r, \varphi) \) should be \( 2\pi \)-periodic in angular variable \( \varphi \). As a result the Dirichlet boundary value problem (6.3) has the following unnormalized eigenfunctions (\( E \)-modes)

\[
\Phi_{nm}(r, \varphi) = \sin n \varphi \begin{cases} J_n(\gamma'_{nm} r), & r < a, \\
H_n^{(1)}(\gamma'_{nm} r), & r > a,
\end{cases}
\]

where \( a \) is the cylinder radius, \( J_n(x) \) are the Bessel functions, \( H_n^{(1)}(x) \) are the Hankel functions of the first kind, and \( \gamma'_{nm}, \tilde{\gamma}'_{nm} \) stand for the roots of the frequency equations

\[
J_n(\gamma'_{nm} a) = 0, \quad H_n^{(1)}(\gamma'_{nm} a) = 0,
\]

\[
n = 0, 1, 2, \ldots, \quad m = 1, 2, \ldots.
\]

For the Neumann boundary value problem (6.4) we have the \( H \)-modes

\[
\Psi_{nm}(r, \varphi) = \sin n \varphi \begin{cases} J_n(\gamma''_{nm} r), & r < a, \\
H_n^{(1)}(\gamma''_{nm} r), & r > a,
\end{cases}
\]
where \( \gamma_{nm}'' \) and \( \tau_{nm}'' \) are the roots of the equations

\[
\frac{d}{dr} J_n(\gamma_{nm}'' r) \bigg|_{r=a} = 0, \quad \frac{d}{dr} H^{(1)}_n(\tau_{nm}'' r) \bigg|_{r=a} = 0,
\]

\( n = 0, 1, 2, \ldots, \quad m = 1, 2, \ldots. \) \hfill (6.10)

As usual, it is assumed that for \( r > a \) the eigenfunctions should satisfy the radiation condition.

It is important to note that each root

\[
\gamma_{nm}', \quad \tau_{nm}', \quad \gamma_{nm}'', \quad \tau_{nm}''', \quad n \geq 1, \quad m \geq 1
\]

is doubly degenerate since, according to eqs. (6.7), (6.9), because there are two eigenfunctions which are proportional to either \( \sin(n\varphi) \) or \( \cos(n\varphi) \). The frequencies with \( n = 0 \)

\[
\gamma_{0m}', \quad \tau_{0m}', \quad \gamma_{0m}'', \quad \tau_{0m}''', \quad m = 1, 2, \ldots
\]

(6.11)

are independent on \( \varphi \), and the degeneracy disappears.

For given Hertz vectors \( \Pi' \) and \( \Pi'' \) the electric and magnetic fields are constructed by the formulas

\[
\begin{align*}
E &= \nabla \times \nabla \times \Pi', \quad H = -i\omega \nabla \times \Pi' \quad (E\text{-modes}), \\
E &= i\omega \nabla \times \Pi'', \quad H = \nabla \times \nabla \times \Pi'' \quad (H\text{-modes}).
\end{align*}
\]

(6.13)

It has been proved [152] that the superposition of these modes gives the general solution to the Maxwell equations in the problem under consideration. An essential merit of using the Hertz polarization vectors is that in this approach the necessity to satisfy the gauge conditions does not arise.

Now we turn to a waveguide which is obtained by cutting the infinite cylindrical shell by a plane passing through the symmetry axes of the cylinder (see fig. 1). All the surfaces are assumed to be perfectly conducting. In this case the boundary value problems (6.3) and (6.4) for the Hertz electric (\( \Pi' \)) and magnetic (\( \Pi'' \)) vectors have the following eigenfunctions

\[
\begin{align*}
\Phi_{nm}(r, \varphi) &= \sin(n\varphi) \begin{cases} J_n(\gamma_{nm}' r), & r < a, \\ H^{(1)}_n(\tau_{nm}' r), & r > a, \end{cases} \quad n = 1, 2, \ldots, \quad m = 1, 2, \ldots, \\
\Psi_{nm}(r, \varphi) &= \cos(n\varphi) \begin{cases} J_n(\gamma_{nm}'' r), & r < a, \\ H^{(1)}_n(\tau_{nm}'' r), & r > a, \end{cases} \quad n = 0, 1, 2, \ldots, \quad m = 1, 2, \ldots.
\end{align*}
\]

(6.14)

and

\[
\begin{align*}
\Phi_{nm}(r, \varphi) &= \sin(n\varphi) \begin{cases} J_n(\gamma_{nm}' r), & r < a, \\ H^{(1)}_n(\tau_{nm}' r), & r > a, \end{cases} \quad n = 1, 2, \ldots, \quad m = 1, 2, \ldots, \\
\Psi_{nm}(r, \varphi) &= \cos(n\varphi) \begin{cases} J_n(\gamma_{nm}'' r), & r < a, \\ H^{(1)}_n(\tau_{nm}'' r), & r > a, \end{cases} \quad n = 0, 1, 2, \ldots, \quad m = 1, 2, \ldots.
\end{align*}
\]

(6.15)

The frequencies \( \gamma_{nm}', \tau_{nm}', \gamma_{nm}'', \) and \( \tau_{nm}'' \) are determined by the same equations (6.8) and (6.10). However the new spectral problem has two essential distinctions: i) the frequencies (6.11) are now nondegenerate, and ii) two series of eigenfrequencies

\[
\gamma_{0m}', \quad \tau_{0m}', \quad m = 1, 2, \ldots
\]

(6.16)
FIG. 1. The cross section of an infinite semi-circular cylindrical shell of radius \( a \). All the surfaces (bold-faced lines) are assumed to be perfectly conducting. At the same time this picture presents the two-dimensional (plane) version of the problem under consideration, i.e., the semi-circular boundaries for massless fields defined on the plane.

are absent. At first sight one could expect that such a change of the spectrum cannot influence drastically on the ultraviolet behavior of the relevant spectral density. However, as it will be shown below, the zeta function for a semi-circular cylinder, unlike for a circular one, does not provide a finite answer for the Casimir energy in the problem in question.

In view of all above-mentioned the zeta function for electromagnetic field obeying the boundary conditions on the surface of the semi-circular cylinder is the sum of two zeta functions for scalar massless fields satisfying the Dirichlet and Neumann conditions on the lateral of this cylinder.

By making use of the contour integration and analytical continuation discussed in subsect. IV C one can construct the spectral zeta functions for a semi-circular cylinder also. In ref. [153] it has been done first for scalar massless fields obeying, on internal and external surfaces of a semi-circle infinite cylinder, the Dirichlet and Neumann conditions. As a result the following expressions were obtained for relevant vacuum energies

\[
E^{\text{D}}_{s\text{-cyl}} = \frac{1}{2} \zeta^{\text{D}}_{s\text{-cyl}} \left( -\frac{1}{2} \right) = \frac{1}{2a^2} \left( 0.000523 - 0.004974 \frac{1}{s + 1/2} \Bigg|_{s \to -1/2} \right), \tag{6.17}
\]

\[
E^{\text{N}}_{s\text{-cyl}} = \frac{1}{2} \zeta^{\text{N}}_{s\text{-cyl}} \left( -\frac{1}{2} \right) = \frac{1}{2a^2} \left( -0.04345 - 0.0149 \frac{1}{s + 1/2} \Bigg|_{s \to -1/2} \right). \tag{6.18}
\]

The sum of these formulae gives the Casimir energy of electromagnetic field for semi-circular cylinder

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\[ E_{s-cyl}^{EM} = \frac{1}{2} \zeta_{s-cyl}^{D+N} \left( -\frac{1}{2} \right) = \frac{1}{2a^2} \left( -0.0439 - 0.0199 \frac{1}{s+1/2} \right) \bigg|_{s \to -1/2}. \] 

(6.19)

Pole singularities in eqs. (6.17), (6.18) and (6.19) imply that the zeta function technique does not provide a finite value of vacuum energy for this configuration and further renormalization is needed.

The same situation takes place also in the two-dimensional version of the problem under study, i.e., when a semi-circle is considered on a plane. In fact, the relevant zeta functions can be obtained in a straightforward way by making use the relation (4.76). It gives [153]

\[ E_{s-cir}^{D} = \frac{1}{2} \zeta_{s-cir}^{D} \left( -\frac{1}{2} \right) = \frac{1}{2a^2} \left( 0.038127 - \frac{1}{256} \frac{1}{s} \right) \bigg|_{s \to 0}. \] 

(6.20)

\[ E_{s-cir}^{N} = \frac{1}{2} \zeta_{s-cir}^{N} \left( -\frac{1}{2} \right) = \frac{1}{2a^2} \left( -0.237103 - 0.0062 \frac{1}{s} \right) \bigg|_{s \to 0}. \] 

(6.21)

The sum of these energies is also infinite.

Keeping in mind that the zeta function regularization does nor supply a finite value for the Casimir energy also in the case of spheres in even dimensional spaces (see subsect. IV C) we have thus a quite long list of boundary conditions for which the vacuum energy cannot be calculated by making use of the known mathematical methods. It is worth elucidating the geometrical peculiarities of the boundaries that are responsible for this failure. In order to do this we have to address the corresponding heat kernel coefficients.

B. Mutual cancellation of divergencies in vacuum energy

The relevant analysis of the heat kernel coefficients (up to \( B_{5/2} \)) has been accomplished in ref. [116]. We are interested in the coefficients \( B_2 \) and \( B_{3/2} \). As it has been noted in subsect. III C, the nonzero value of \( B_2 \) \((B_{3/2})\) implies that it is impossible to derive a finite value for the vacuum energy in the problem at hand, when the dimension of the configuration space \( d = 3 \) \((d = 2)\). From the asymptotic expansion (3.25) it follows that the heat kernel coefficients introduced in this way are the same for the three-dimensional cylindrical-like boundaries and for the corresponding plane problem obtained by crossing the former boundary by a transverse plane. Certainly, it is correct only for the Laplace operator \(-\Delta\). In fact, the respective eigenvalues in these two problems are related by

\[ \lambda_n(d = 3) = k^2 + \lambda_n(d = 2), \quad 0 \leq k < \infty. \] 

(6.22)

Hence

\[ K_{d=3}(t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{-k^2 t} K_{d=2}(t) = \frac{1}{\sqrt{4\pi t}} K_{d=2}(t). \] 

(6.23)

Taking into account the definition (3.25) one can easily deduce form (6.23) that the heat kernel coefficients \( B_{n/2} \) in the two eigenvalue boundary value problems mentioned above, are equal.
### Table I. The contribution of different parts of boundary to heat kernel coefficients; D and N stands for the Dirichlet and Neumann boundary conditions; the upper (lower) sign refers to internal (external) region.

|         | Curvature | Corners |
|---------|-----------|---------|
| $B_{3/2}$ | D         | $\pm \sqrt{\frac{\pi}{64R^4}}$ |
|         | N         | $\pm \frac{5\sqrt{\pi}}{64R}$ |
| $B_2$   | D         | $\pm \frac{4\pi}{315R^2}$     |
|         | N         | $\pm \frac{4\pi}{45R^2}$       |
| $B_{5/2}$ | D         | $\pm \frac{37\sqrt{\pi}}{8192R^3}$ |
|         | N         | $\pm \frac{269\sqrt{\pi}}{8192R^3}$ |

The heat kernel coefficients we are interested in are contributed by the following peculiarities of the boundary (see fig. 1): the curvature of semi-circle 1–2 and four right-angled corners at points 1 and 2. It is worthy to remind that we are considering both internal ($D_{in}$) and external ($D_{ex}$) regions. All these contributions are presented in table I. The contributions of the curvature of the semi-circle 1–2 to the coefficients $B_2$ for internal and external regions are of the same absolute value but they have opposite signs. As a result they are mutually cancelled in the net $B_2$ coefficient. Unlike this the contributions to $B_2$ due to four right-angled corners at the points 1 and 2 are added

$$B_D^2 = 2\frac{\pi}{8R^2} = \frac{\pi}{4R^2}, \quad B_N^2 = 2\frac{3\pi}{8R^2} = \frac{3\pi}{4R^2}, \quad B_{EM}^2 = B_D^2 + B_N^2 = \frac{\pi}{R^2}. \quad (6.24)$$

This fact has been noted at first time in ref. [154]. In the case of the coefficient $B_{3/2}$ the situation is opposite, i.e., the contributions of the corners from internal and external regions are mutually cancelled, while the contributions of the curvature of the arc 1–2 from internal and external regions are added

$$B_D^{3/2} = 2\frac{\pi}{64R} = \frac{\pi}{32R}, \quad B_N^{3/2} = 2\frac{5\pi}{64R} = \frac{5\pi}{32R}, \quad B_{D+N}^{3/2} = \frac{3\pi}{16R}. \quad (6.25)$$

Different geometrical origins of the zeta function failure to provide a finite value of the vacuum energy in the two- and three-dimensional versions of the boundary value problem in question probably imply the impossibility of obtaining a finite and unique value of this quantity by taking advantage of the atomic structure of the boundary [155] or its quantum fluctuations [156]. It is clear because any physical reason of the Casimir energy divergences should be valid simultaneously in the two- and three-dimensional versions of the boundary configuration under consideration.
The alternating sign changes of contributions due to curvature and corners to heat kernel coefficients are presumably true for all the coefficients $B_n$ with $n = 1, 3/2, 2, \ldots$, at least it is the case for $B_1$, $B_{3/2}$, $B_2$ and $B_{5/2}$ (see table I and also ref. [116]). The analogous situation takes place for spheres in spaces of even and odd dimensions, namely, the curvature contributions to respective heat kernel coefficients are mutually cancelled for odd dimension of ambient space and they are added in spaces of even dimension (see subsect. IV C). In the former case the zeta regularization gives a finite value for vacuum energy, in the latter situation it is not the case, and further renormalization of vacuum energy is needed in the problem at hand.

The corners at points 1 and 2 of the boundary considered here (see fig. 1) are obtained by intersection of a straight line and an arc of a circle possessing nonzero curvature $(1/a)$. They contribute to all the heat kernel coefficients starting with $n = 1$. These corners should be distinguished from those formed by crossing two straight lines, for example, from the corners of a rectangle. Such corners contribute only to the coefficient $B_1$, the contribution being the same for Dirichlet and Neumann boundary conditions

$$c(\alpha) = \frac{\pi^2 - \alpha^2}{6\alpha},$$  \hspace{1cm} (6.26)

where $\alpha$ is the angle of the corner. Such corner singularity of the boundary proves to be very close to the conical singularity, which is important in many areas of mathematical physics [157,158]. Lately it has been investigated in connection with studies of quantum fields on the background of black holes [159] and cosmic strings [160,161].

This general assertion concerning the corner contribution to the heat kernel coefficients can be illustrated by a known heat kernel expansion for a rectangle with sides $a$ and $b$

$$K(t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \exp \left[ - \left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \right) \pi^2 t \right] = \left( \frac{a}{\sqrt{4\pi t}} - \frac{1}{2} \right) \left( \frac{b}{\sqrt{4\pi t}} - \frac{1}{2} \right) + ES$$

$$= \frac{ab}{4\pi t} - \frac{a + b}{4\sqrt{\pi t}} + \frac{1}{4} + ES,$$  \hspace{1cm} (6.27)

where ES denotes the exponentially small corrections as $t \to +0$ (see, for example, [162]). Here the scalar operator $-\Delta$ with Dirichlet boundary conditions is considered. For a rectangle the coefficient $B_1$ is obviously equal to the contributions of four right-angled corners. Indeed, the third term in the expansion (6.27) can be represented in the form

$$\frac{1}{4} = \frac{1}{4\pi} B_1 = \frac{1}{4\pi} 4c(\alpha = \pi/2),$$

where $c(\alpha)$ is given in eq. (6.26). Besides these corners the boundary of a rectangle has no other singularities, therefore the heat kernel coefficients $B_n$ with $n = 3/2, 2, \ldots$ vanish in this problem.

These rules for obtaining the heat kernel coefficients are directly generalized to an arbitrary polygon with the angles $\alpha_i$. The first two coefficients $B_0$ and $B_{1/2}$ are defined by eq. (3.26), where $V$ is the area of the polygon and $S$ is its perimeter. The third coefficient $B_1$ is equal to the sum of the contributions due to the angles $\alpha_i$. 

\hspace{1cm} 46
\( B_1 = \sum_i c(\alpha_i) \).

The rest of the coefficients \( B_n, n \geq 1 \) vanish both for internal and external regions. In particular, it implies that the zeta function technique should provide a finite value of the Casimir energy for a polygon on a plane \((B_{3/2} = 0)\) and for a cylindrical generalization of the polygon spectral problem \((B_2 = 0)\). These subjects have been discussed earlier in papers [163]. The calculations of vacuum energy have been accomplished only for parallelepiped geometries [164–173] and with allowance for only internal region. For such configurations the spectrum of the Laplace operator is known exactly and the zeta regularization works well. For example, the Casimir energy for the cube with sides \( a \) is negative

\[
E_{\text{cube}} = \frac{\pi}{2a} \sum_{n,m,k=1}^{\infty} \sqrt{n^2 + m^2 + k^2} \approx -\frac{0.015}{a}. \tag{6.29}
\]

However Milton [59] comments these results in the following way: here “divergences occur which cannot be legitimately removed, which nevertheless are artificially removed by zeta function methods. It is the view of the author that such finite results are without meaning”. With regard to external region the wave equation is not separable outside a cube or a rectangular solid.

Recently the image method has been extended to a set of geometries with planar boundaries and without rectangular angles [174]. In this approach the Casimir energy of a massless scalar field was calculated inside a triangle with angles proportional to \( \pi/N, N \geq 3 \). However the final expression for the vacuum energy is not ready for numerical estimation. Further this technique was applied for calculating the Casimir energy of a conical wedge and a conical cavity [175].

**VII. HIGH TEMPERATURE BEHAVIOR OF THE VACUUM ENERGY**

The influence of temperature on the Casimir effect was an important topic since its first experimental demonstration [2] which had been done at room temperature. It was first shown in ref. [176] that the temperature influence was just below what had been measured (see, also, paper [177]). Now it is expected that the temperature contributions will be seen in the upcoming series of experiments [10]. The temperature dependence of the vacuum energy proves to be important practically in all problems where the Casimir effect is taken into account, for example, in hadron physics (quark deconfinement in the framework of bag models [178]) and in cosmology [179].

The Casimir calculations at finite temperature is a nontrivial problem specifically for boundary conditions with nonzero curvature. Investigation of the high temperature limit, i.e., the classical limit, in this problem is of independent interest [180]. For this goal a powerful method of the zeta function technique and the heat kernel expansion can be used. It is important that for obtaining the high temperature asymptotics of the thermodynamic characteristics it is sufficient to know the heat kernel coefficients and the determinant for the spatial part of the operator governing the field dynamics. This is an essential merit of this approach.
A. Heat kernel coefficients and high temperature expansions

In quantum field theory, finite temperature effects can be described at equilibrium in the Matsubara formalism by imposing periodic (resp. antiperiodic for fermions) boundary conditions in the imaginary time coordinate \[181\]. Let the dynamics of quantum field \( \varphi(t, x) \) be defined, as before, by eq. (3.3). The Helmholtz free energy \( F \) of this field is determined by the functional integral \[181,182\]

\[
e^{-\beta F} = \int D\varphi \exp \left( - \int dt \ dx \varphi^*(t, x)L_T\varphi(t, x) \right).
\]

Here \( \beta \) is inverse temperature \( \beta = T^{-1} \). For simplicity the Boltzmann constant \( k_B \) is assumed to be equal to 1. Therefore the temperature \( T \) is measured in energy units. For the Euclidean time and for the Euclidean version of the field \( \varphi(t, x) \) we use the former notations. The operator \( L \) in eq. (7.1) is the Euclidean version of the full differential operator in the field equation (3.3)

\[
L_T = L - \frac{\partial^2}{\partial t^2}
\]

with the eigenfunctions

\[
\phi_{mn}(t, x) = e^{i\Omega_m t} \varphi_n(x)
\]

and the eigenvalues

\[
L_T \phi_k(t, x) = \lambda_k^T \phi_k(t, x), \quad k = \{m, n\}, \quad \lambda_k^T = \Omega_m^2 + \omega_n^2,
\]

where \( \Omega_m = \frac{2\pi m T}{\hbar}, \quad m = 0, \pm 1, \pm 2, \ldots \) are the Matsubara frequencies, and \( \omega_n \) are defined in eq. (3.4).

Functional integration in eq. (7.1) yields

\[
e^{-\beta F} = (\det L_T)^{-1/2}.
\]

The determinant of the operator \( L_T \) is expressed in terms of its spectral zeta function

\[
\zeta_T(s) = \sum_k (\lambda_k^T)^{-s} = \sum_{m=-\infty}^{\infty} \sum_n (\Omega_m^2 + \omega_n^2)^{-s}.
\]

Indeed

\[
\det L_T = \prod_k \lambda_k^T, \quad \ln(\det L_T) = \sum_k \ln \lambda_k^T.
\]

Differentiation of the zeta function \( \zeta_T(s) \) at the point \( s = 0 \) gives

\[
\zeta_T'(0) = -\sum_k \ln \lambda_k^T.
\]
From eqs. (7.6), (7.7) and (7.8) we deduce finally

\[ F = -\frac{T}{2} \zeta_T'(0). \]  

(7.9)

The characteristics of the quantum field (3.3) at zero temperature are determined by the zeta function \( \zeta(s) \) associated with the operator \( L \) (see eq. (3.9)). From the mathematical point of view the zeta function \( \zeta(s) \) corresponding to the space part of the full wave equation (3.3) is, undoubtedly, a simpler object than the complete zeta function \( \zeta_T(s) \) because the definition (7.6) involves an additional sum over the Matsubara frequencies. Here a natural question arises whether one can gain knowledge of the quantum field at nonzero temperature possessing only the zeta function \( \zeta(s) \). In ref. [183] it was shown that proceeding from the zeta function \( \zeta(s) \) one can deduce the high temperature asymptotics of the thermodynamic functions such as Helmholtz free energy, internal energy, and entropy. Let us remind briefly the derivation of these asymptotics. By making use of the formula (3.24) the zeta function (3.23) can be represented in the form

\[
\zeta_T(s) = \frac{1}{\Gamma(s)} \int_0^\infty dt \, t^{s-1} \sum_{m=-\infty}^{\infty} e^{-\Omega_m^2 t} \sum_k e^{-\omega_k^2 t}.
\]  

(7.10)

The term with \( m = 0 \) in this formula gives the zeta function (3.9). In the remaining terms we substitute the heat kernel \( K(t) \) of the operator \( L \) by its asymptotic expansion (3.25). As a result we arrive at the following asymptotic representation for the complete zeta function \( \zeta_T(s) \)

\[ \zeta_T(s) \simeq \zeta(s) \]

\[ + \frac{2}{(2\pi)^{3/2}} \sum_{n=0,1/2,...} B_n \left( \frac{\hbar}{2\pi T} \right)^{2s-3+2n} \Gamma(s-3/2+n) \Gamma(s-3/2+n) \zeta_R(2s+2n-3), \]  

(7.11)

where \( \zeta_R(s) \) is the Riemann zeta function (3.11). Taking the derivative of the right hand side of eq. (7.11) at the point \( s = 0 \) and substituting the result into eq. (7.9) one obtains the high temperature expansion for the free energy

\[
F(T) \simeq -\frac{T}{2} \zeta'(0) + \frac{T^4}{90} \pi^2 - B_{1/2} \frac{T^3}{4\pi^3/2\hbar^2} \zeta_R(3) - \frac{B_1}{24} \frac{T^2}{\hbar} + \frac{B_{3/2}}{(4\pi)^{3/2}} T \ln \frac{\hbar}{T} + \frac{B_2}{16\pi^2} \hbar \left[ \ln \left( \frac{\hbar}{4\pi T} \right) + \gamma \right] - \frac{B_{5/2}}{(4\pi)^{3/2}} \frac{\hbar^2}{24T} - T \sum_{n\geq 3} \frac{B_n}{(4\pi)^{3/2}} \left( \frac{\hbar}{2\pi T} \right)^{2n-3} \Gamma(n-3/2) \zeta_R(2n-3), \quad T \to \infty.
\]  

(7.12)

Here \( \gamma \) is the Euler constant. The argument of the logarithms in expansion (7.12) are dimensional, but upon collecting similar terms with account for the logarithmic ones in \( \zeta'(0) \) it is easy to see that finally the logarithm function has a dimensionless argument, at least for \( B_2 = 0 \). It is worth noting that the auxiliary evolution variable \( \tau \) in eq. (3.25) has the dimension [length]².
The asymptotic expansions for the internal energy $U(T)$ and the entropy $S(T)$ are deduced from eq. (7.12) employing the thermodynamic relations

$$U(T) = -T^2 \frac{\partial}{\partial T} (T^{-1} F(T)), \quad (7.13)$$

$$S(T) = T^{-1} (U(T) - F(T)) = -\frac{\partial F}{\partial T}. \quad (7.14)$$

Substituting the expansion (7.12) into eqs. (7.13) and (7.14) one arrives at the asymptotics

$$U(T) \simeq B_0 \frac{T^4 \pi^2}{h^3 30} + B_{1/2} \frac{T^3}{h^2} \frac{\zeta_R(3)}{2 \pi^{3/2}} + B_1 \frac{T^2}{24 h} + \frac{B_{3/2}}{(4\pi)^{3/2}} T$$

$$- B_2 \frac{h}{16\pi^2} \left[ \ln \left( \frac{h}{4\pi T} \right) + \gamma + 1 \right] - \frac{B_{5/2}}{(4\pi)^{3/2}} \frac{h^2}{12 T}$$

$$- \frac{T}{4 \pi^{3/2}} \sum_{n \geq 3} B_n \left( \frac{h}{2\pi T} \right)^{2n-3} \Gamma(n-1/2) \zeta_R(2n-3), \quad (7.15)$$

$$S(T) \simeq \frac{1}{2} \zeta'(0) + B_0 \frac{T^3}{h^3} \frac{2\pi^2}{45} + B_{1/2} \frac{T^2}{h^2} \frac{3 \zeta_R(3)}{4 \pi^{3/2}} + B_1 \frac{T}{12 h}$$

$$+ \frac{B_{3/2}}{(4\pi)^{3/2}} \left( 1 - \ln \frac{h}{T} \right) - B_2 \frac{h}{16\pi^2 T} - \frac{B_{5/2}}{(4\pi)^{3/2}} \frac{h^2}{24 T^2}$$

$$- \frac{1}{4 \pi^{3/2}} \sum_{n \geq 3} B_n \left( \frac{h}{2\pi T} \right)^{2n-3} (n-2) \Gamma(n-3/2) \zeta_R(2n-3). \quad (7.16)$$

In eq. (7.15) the term proportional to $B_2$ contains the logarithm of dimensional quantity: $[h/T] = [\text{time}]^{-1}$. This is the result of the arbitrariness arising in from the ultraviolet divergences in the case of $B_2 \neq 0$ (see ref. [146] for a more detailed discussion). Unlike this situation, collecting the logarithm functions in the $B_{3/2}$-term and in $\zeta'(0)$ in eq. (7.16) leads to a dimensionless argument of the logarithm in the final expression.

It is worth noting that the zeta determinant of the operator $L$, i.e., $\zeta'(0)$, does not enter the asymptotic expansion for the internal energy (7.15). Therefore this high temperature expansion is completely defined only by the heat kernel coefficients. In view of this, the first term in the asymptotics of the free energy in eq. (7.12) is referred to as a pure entropic contribution. Its physical origin is till now not elucidated.

**B. Perfectly conducting parallel plates in vacuum**

First we demonstrate the application of the high temperature expansions (7.12), (7.15) and (7.16) to a simple problem of electromagnetic field confined between two perfectly conducting parallel plates in vacuum.

Substituting eq. (4.4) into eq. (3.27) we obtain for perfectly conducting parallel plates only one nonzero coefficient $B_0$

$$B_0 = \frac{2 V}{c^2}, \quad (7.17)$$
where \( V = L_x L_y a \) is the volume of the space bounded by the plates. In subsect. IV A it was noted that for parallel conducting plates electromagnetic field is reduced to two massless scalar fields obeying Dirichlet and Neumann conditions on internal surface of the plates. As a result we have the multiplier 2 in eq. (7.17) and vanishing coefficient \( B_{1/2} \) (see eq. (3.26)). It should be noted here that we are considering only electromagnetic field confined between the plates and do not take into account that field outside the plates.

From eqs. (7.15) and (7.17) it follows that the density of internal energy has the following high temperature asymptotics

\[
\frac{U(T)}{V} \simeq 4 \frac{\sigma}{c} T^4, \quad T \to \infty,
\]

where \( \sigma \) is the Stefan-Boltzmann constant

\[
\sigma = \frac{\pi^2 k_B^4}{60 c^2 \hbar^3}.
\]

Recall that in our formulae we put \( k_B = 1 \), that is, the temperature is measured in energy units. The transition to degrees is performed by the substitution \( T \to k_B T \).

When calculating the high temperature asymptotics of the free energy (7.12) and the entropy (7.16) one needs to derive \( \zeta'(0) \) for the zeta function (4.4). Keeping in mind that \( \zeta_R(-2) = 0 \) it is convenient to use here the Riemann reflection formula [112]

\[
2^{1-s} \Gamma(s) \zeta_R(s) \cos(\pi s/2) = \pi^2 \zeta_R(1 - s)
\]

which yields

\[
\zeta_R(2s - 2) \simeq -s \frac{\zeta_R(3)}{2\pi^2} + O(s^2).
\]

From here we deduce

\[
\zeta'(0) = \frac{L_x L_y}{4 \pi a^2} \zeta_R(3) = \frac{V}{4 \pi a^3} \zeta_R(3).
\]

Insertion of eqs. (7.17) and (7.22) into eq. (7.12) gives the following high temperature behavior for the density of free energy

\[
\frac{F}{V} \simeq -\frac{T}{8 \pi a^3} \zeta_R(3) - \frac{T^4}{c^3 \hbar^3} \frac{\pi^2}{90}.
\]

As was noted above, we are considering only electromagnetic field between the plates. Therefore when calculating the Casimir forces one should drop the last term in eq. (7.23) since its contribution is cancelled by the pressure of the black body radiation on the outward surfaces.

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\(^3\)For obtaining the vanishing \( B_{1/2} \) coefficient it is important to take into account the second term in eq. (4.4) which depends on the photon mass \( \mu \).
of the plates. As a result the high temperature asymptotics of the Casimir force, per unit surface area, attracting two perfectly conducting plates in vacuum is

\[ \mathcal{F} \simeq -\frac{T}{4\pi a^3} \zeta_R(3). \]  

(7.24)

Usually in the Casimir calculations the contribution of the free black body radiation is subtracted from the very beginning [67].

It is interesting to note that the Casimir force (7.24) and the first term on the right hand side of eq. (7.23) are pure classical quantities because they do not involve the Planck constant \( \hbar \). These classical asymptotics seem to be derivable without appealing to the notion of quantized electromagnetic field. The classical limit of the theory of the Casimir effect is discussed in a recent paper [180].

Employing eqs. (7.14) and (7.23) one arrives at the high temperature behavior of the entropy density

\[ \frac{S(T)}{V} \simeq \frac{\zeta_R(3)}{8\pi a^3} + \frac{2T^3 \pi^2}{45c^3 \hbar^3}. \]  

(7.25)

Corrections to eqs. (7.18), (7.23) and (7.25) are exponentially small.

C. Sphere

We consider electromagnetic field subjected to three types of boundary conditions on the surface of a sphere: i) an infinitely thin and perfectly conducting spherical shell; ii) the surface of a sphere delimits two material media with the same velocity of light; iii) a dielectric ball placed in unbounded dielectric medium. In order to obtain the heat kernel coefficients determining the high temperature asymptotics (7.12), (7.15) and (7.16) it is convenient to use the explicit representation of the relevant spectral zeta functions in terms of the Riemann zeta function. These formulae were derived in subsec. IV B by taking into account the first two terms of the uniform asymptotic expansion for the product of the modified Bessel functions \( I_\nu(\nu z) K_\nu(\nu z) \).

1. Perfectly conducting spherical shell

The corresponding spectral zeta function is given in eq. (4.33). The terms omitted in this equation are of the form

\[ q_k(s) \left[ (2^{2(k+s)+1} - 1) \zeta_R(2k + 2s + 1) - 2^{2(k+s)+1} \right], \quad k = 2, 3, 4, \ldots , \]  

(7.26)

where \( q_k(s) \) stand for some polynomials in \( s \).

Analysis of eqs. (4.33) and (4.32) shows that the zeta function for a perfectly conducting spherical shell enables one to find the exact values of the first six heat kernel coefficients, namely:

\[ B_0 = 0, \quad B_{1/2} = 0, \quad B_1 = 0, \quad B_{3/2} = 2\pi^{3/2}, \quad B_2 = 0, \quad B_{5/2} = \frac{\pi^{3/2}}{20} \frac{c^2}{a^2}. \]  

(7.27)
Taking into account the structure of the omitted terms (7.26) it is easy to see that
\[ B_j = 0, \quad j = 3, 4, 5, \ldots \]  
(7.28)

Having obtained the heat kernel coefficients (7.27) and (7.28) we are in position to construct the high temperature asymptotics of the internal energy of electromagnetic field by making use of eq. (7.15)
\[ U(T) \approx \frac{T}{4} - \left( \frac{c \hbar}{R} \right)^2 \frac{1}{1920T} + \mathcal{O}(T^{-3}). \]  
(7.29)

Applying the technique developed in ref. [184] more terms to this expansion can be easily added.

In order to write the asymptotic expansions (7.12) and (7.16) the derivative of the zeta function at the point \( s = 0 \) should be calculated. Equation (4.33) gives an approximate value for \( \zeta'(0) \)
\[ \zeta'_{\text{shell}}(0) = \frac{\gamma}{2} + \ln 2 + \frac{7}{16} \zeta_R(3) - \frac{9}{8} + \frac{1}{2} \ln \frac{a}{c} = 0.38265 + \frac{1}{2} \ln \frac{a}{c}. \]  
(7.30)

The terms omitted in (4.33) will render precise only the first term in the final form of this expression, while the second term \((1/2) \ln(a/c)\) will not change. The exact value of \( \zeta'_{\text{shell}}(0) \) is calculated in Appendix B
\[ \zeta'_{\text{shell}}(0) = \frac{1}{2} - \frac{\gamma}{2} + \frac{7}{6} \ln 2 + 6 \zeta_R'(1) + \left( -\frac{5}{8} + \frac{1}{2} \ln \frac{a}{c} + \ln 2 + \frac{\gamma}{2} \right) \]
\[ = 0.38429 + \frac{1}{2} \ln \frac{a}{c}. \]  
(7.31)

It is worth noting that the expression in the round parentheses, being multiplied by \( \xi^2 \), is exactly the value of \( \zeta'_{\text{ball}}(0) \) for a compact ball with continuous velocity of light on its surface (see eq. (7.41) in the next subsect.). As a result we have the following high temperature asymptotics of the free energy and the entropy in the problem in question
\[ F(T) \approx -\frac{T}{4} \left( \ln \frac{aT}{\hbar c} + 0.76858 \right) - \left( \frac{\hbar c}{a} \right)^2 \frac{1}{3840T} + \mathcal{O}(T^{-3}), \]  
(7.32)
\[ S(T) \approx 0.44215 + \frac{1}{4} \ln \frac{aT}{\hbar c} - \frac{1}{3840} \left( \frac{\hbar c}{aT} \right)^2 + \mathcal{O}(T^{-4}). \]  
(7.33)

The expression (7.32) exactly reproduces the asymptotics obtained in ref. [15] (1978) by making use of the multiple scattering technique (see eq. (8.39) in that paper). We have not calculated the coefficient \( B_{7/2} \), therefore we do not know the sign of the \( T^{-3} \)-correction in (7.32). In ref. [15] it is noted that this term is negative.

In eqs. (7.29), (7.32) and (7.33) the large expansion parameter is actually a dimensionless ‘temperature’ \( \tau = aT/(\hbar c) \). Therefore the same formulae describe the behavior of the thermodynamic functions when \( a \to \infty \) and temperature \( T \) is fixed.

The high temperature asymptotics of the thermodynamic functions derived by making use of the general expansions (7.12), (7.15) and (7.16) contain the terms independent of the
Planck constant $\hbar$ or, in other words, classical contributions (see eqs. (7.29), (7.32) and (7.33)). This is also true for the high temperature limit of the Casimir force calculated per unit area of a sphere

$$\mathcal{F}(T) \simeq -\frac{1}{4\pi R^2} \frac{\partial F(T)}{\partial a} = \frac{T}{16\pi a^3} \left( \frac{\hbar c}{a} \right)^2 \frac{1}{4\pi a^3} \frac{1}{1920 T} + \mathcal{O}(T^{-3}).$$

(7.34)

The leading classical term in the asymptotics (7.34) describes the Casimir force that seeks to expand the sphere. The quantum correction in this formula stands for the Casimir pressure exerted on the sphere surface.

In eqs. (7.29), (7.32) and (7.33) the Stefan-Boltzmann terms proportional to $T^4$ are absent because the contribution of the Minkowski space was subtracted from the very beginning in our calculations (see eq. (4.12)). As a result we obtain the vanishing heat kernel coefficient $B_0$ which, in general case, is determined by the volume of the system under study (see eq. (7.17)). Therefore our results describe only the deviation from the Stefan-Boltzmann law caused by the perfectly conducting sphere.

The vanishing of the coefficients $B_{1/2}$ and $B_1$ in the problem at hand can be explained by taking into account the general properties of the heat kernel coefficients [60] and by making use of the results obtained in ref. [184]. As known [122] the solutions to the Maxwell equations with allowance for a perfectly conducting sphere are expressed in terms of the two scalar functions that satisfy the Laplace equation with the Dirichlet and Robin boundary conditions on internal and external surfaces of the sphere. In view of this one can write

$$B_n = B^D_{n+} + B^D_{n-} + B^R_{n+} + B^R_{n-}, \quad n = 1/2, 1, \ldots ,$$

(7.35)

where the subscript plus (minus) corresponds to internal (external) region and the rest notations are obvious. In ref. [184] it was found that

$$B^D_{1/2+} = -2\pi^{3/2}a^2 = B^D_{1/2-}, \quad B^R_{1/2+} = 2\pi^{3/2}a^2 = B^R_{1/2-},$$

$$B^D_{1\pm} = \pm \frac{8\pi}{3} a, \quad B^R_{1\pm} = \mp \frac{16\pi}{3} a.$$

(7.36)

As a result we have for electromagnetic field inside and outside a perfectly conducting sphere

$$B_{1/2} = B_1 = 0.$$

(7.37)

Having calculated the corrections to the Stefan-Boltzmann law one should naturally discuss the possibility of their detection. The ratio of the leading term in eq. (7.29) to the internal energy of black body radiation in unbounded space given by the Stefan-Boltzmann law (7.18) is proportional to $\tau^{-3}$. Already for $\tau \sim 10$ the corrections prove to be of order $10^{-3}$. The same value of $\tau$ can be reached by varying the scale of length $a$ in the problem under consideration or by respective choice of the temperature $T$. Keeping in mind the value of the conversion coefficient $c\hbar = 197.326\text{MeV} \cdot \text{fm} = 0.229\text{K} \cdot \text{cm}$ [185] we obtain the following estimations. For $a \sim 10^{-13}\text{cm}$ (a typical hadron size) the temperature $T$ should satisfy the inequality $T \gg 200\text{MeV}$ in order to apply the asymptotics found. For $a \sim 1\text{cm}$ we have $T \gg 0.229\text{K}$ and for $a \sim 7 \cdot 10^{10}\text{cm}$ (radius of the Sun) the range of applicability of the asymptotics at hand extends practically to any temperature value $T \gg 10^{-10}\text{K}$. Here we shall not go into the details of a concrete experimental equipment that enables one to observe the calculated corrections to the Stefan-Boltzmann law confining ourselves to the estimations presented above.
2. Compact ball with equal velocities of light inside and outside

The spectral zeta function for this configuration is given in eqs. (4.27) and (4.25). It affords the exact heat kernel coefficients up to $B_{5/2}$

$$B_0 = 0, \quad B_{1/2} = 0, \quad B_1 = 0, \quad B_{3/2} = 2 \pi^{3/2} \xi^2, \quad B_2 = 0,$$

$$B_{5/2} = \xi^2 \frac{\zeta^2}{a^2} \pi \sqrt{\pi} p(-1) = 0. \quad (7.38)$$

With allowance of the structure of the omitted terms in eq. (4.27) we can again deduce that

$$B_j = 0, \quad j = 3, 4, 5, \ldots. \quad (7.39)$$

Substitution of these coefficients into eq. (7.15) gives the following high temperature behavior of the internal energy in the problem under consideration

$$U(T) \simeq \xi^2 \frac{T}{4} + O(T^{-3}). \quad (7.40)$$

The value of $\zeta'_{\text{ball}}(0)$ is calculated in the Appendix B2

$$\zeta'(0) = \xi^2 \left(-\frac{5}{8} + \frac{1}{2} \ln \frac{a}{c} + \ln 2 + \frac{\gamma}{2}\right) = \xi^2 \left(0.35676 + \frac{1}{2} \ln \frac{a}{c}\right). \quad (7.41)$$

It is this value that is supplied by eq. (4.24) with allowance for that $p(-1) = 0$.

By making use of eqs. (7.12), (7.38) and (7.41) we deduce the high temperature asymptotics for free energy

$$F(T) = -\xi^2 \frac{T}{4} \left(\gamma + \ln 4 - \frac{5}{4}\right) + \frac{\xi^2}{4} T \ln \frac{hc}{aT} + O(T^{-3}),$$

$$= -\xi^2 \frac{T}{4} 0.71352 + \frac{\xi^2}{4} T \ln \frac{hc}{aT} + O(T^{-3}). \quad (7.42)$$

The entropy in the present case has the following high temperature behavior

$$S(T) = \frac{\xi^2}{4} \left(1 + \gamma + \ln 4 - \frac{5}{4} - \ln \frac{hc}{aT}\right) + O(T^{-4}),$$

$$= \frac{\xi^2}{4} \left(1.71352 - \ln \frac{hc}{aT}\right) + O(T^{-4}). \quad (7.43)$$

The asymptotics (7.40) and (7.42) completely coincide with the analogous formulae obtained in subsect. V B by the mode summation method combined with the addition theorem for the Bessel functions (see also ref. [36]).

The exact expression for the internal energy in the problem at hand (see eq. (5.45)) gives only exponentially suppressed corrections to the leading term (7.40)

$$U(T) \simeq \xi^2 \frac{T}{4} \left[1 + 2(4t^2 + 4t + 1)e^{-4t}\right]. \quad (7.44)$$
The asymptotics (7.44) implies in particular that in reality in eq. (7.40) there are no corrections proportional to the inverse powers of the temperature \( T \). From here it follows immediately that all the heat kernel coefficients with integer and half integer numbers equal or greater than 3 should vanish

\[ B_j = 0, \quad j = 3, \frac{7}{2}, 4, \frac{5}{2}, 5, \ldots \]  

(compare with eq. (7.39)). In view of this the sign denoting the omitted terms in eqs. (7.40), (7.42) and (7.43) should be substituted by \( \mathcal{O}(e^{-8\pi a T}) \).

3. Dielectric ball in unbounded dielectric medium

The zeta function for electromagnetic field in the background of a pure dielectric ball \( (\mu_1 = \mu_2 = 1, \varepsilon_1 \neq \varepsilon_2) \) has not been obtained in an explicit form. In ref. [146] the heat kernel coefficients up to \( B_2 \) in this problem were found. Here we use the results of this paper confining ourselves to the \( \Delta n^2 \)-approximation, where \( \Delta n = n_1 - n_2 = n_1 n_2 (c_2 - c_1)/c \simeq (c_2 - c_1)/c \), \( n_i \) and \( c_i \) are the refractive index and the velocity of light inside \( (i = 1) \) and outside \( (i = 2) \) the ball, and \( c \) is the velocity of light in the vacuum: \( n_i = \sqrt{\varepsilon_i}, \quad c_i = c/n_i, \quad i = 1, 2 \).

It is assumed that \( c_1 \) and \( c_2 \) differ from \( c \) slightly, therefore \( c_2 - c_1 \) and \( \Delta n \) are small quantities. In view of this we have

\[ B_0 = \frac{8}{3} \pi a^3 \frac{c_2^3 - c_1^3}{c_1^3 c_2^3} \simeq 8\pi a^3 \left( \frac{\Delta n + 2 \Delta n^2}{c^3} \right), \]

\[ B_{1/2} = -2\pi^{3/2} a^2 \frac{(c_2^2 - c_1^2)^2}{c_1^2 c_2 (c_1 + c_2)^2} \simeq -4\pi^{3/2} \frac{a^2}{c^2} \Delta n^2, \]

\[ B_1 \simeq 0, \quad B_{3/2} = \pi^{3/2} \frac{(c_2^2 - c_1^2)^2}{(c_1^2 + c_2^2)} \simeq \pi^{3/2} \Delta n^2, \quad B_2 \simeq 0. \]  

(7.46)

The coefficients \( B_1 \) and \( B_2 \) equal zero only in the \( \Delta n^2 \)-approximation considered here. In the general case they contain terms proportional to \( \Delta n^k \), where \( k \geq 3 \).

Allowance for one more term in the uniform asymptotic expansion of the modified Bessel functions, as compared with the calculations in ref. [146], gives the next heat kernel coefficient

\[ \frac{B_{3/2}}{(4\pi)^{3/2}} = \frac{25}{2688} \frac{c^4}{a^2} \Delta n^4. \]  

(7.47)

Correcting the mistake made in [186] we state that this coefficient has no contributions proportional to \( \Delta n^2 \), and in the \( \Delta n^2 \)-approximation one has to put

\[ B_{5/2} \simeq 0. \]  

(7.48)

Making use of the technique developed in ref. [187,188] one obtains the following expression for the derivative of the zeta function for a pure dielectric ball at the point \( s = 0 \) (see ref. [150])

\[ \zeta'(0) = \frac{\Delta n^2}{4} \left( -\frac{7}{8} + \ln \frac{a}{c} + \ln 4 + \gamma \right). \]

(7.49)
Before turning to the construction of the high temperature asymptotics in the problem at hand by making use of the general formulæ (7.12), (7.15) and (7.16) a physical remark should be done. When considering the electromagnetic field in the background of a dielectric body in the formalism of quantum electrodynamics of continuous media, as a matter of fact one is dealing with a system consisting of two objects: electromagnetic field plus a continuous dielectric body. It is important that this body is described (phenomenologically) only by respective permittivity without introducing into the Hamiltonian special additional dynamical variables. As a result the zeta function and the relevant heat kernel coefficients calculated in this formalism also describe both electromagnetic field and dielectric body. When we are interested in the Casimir thermodynamic functions in such problems we have obviously to separate in the general expressions the contributions due to the dielectric body itself [189].

Let us turn to such separation procedure in the high temperature asymptotics for a dielectric ball. Following the reasoning of refs. [47,148] we divide the Helmholtz free energy of a material body with volume $V$ and the surface area $S$ into the parts

$$F = Vf + S\sigma + F_{\text{Cas}},$$

where $f$ is the free energy of a unit volume of a ball, $\sigma$ denotes the surface tension, and $F_{\text{Cas}}$ is referred to as the Casimir free energy of electromagnetic field connected with this body and having the temperature $T$. In this way we obtain the following high temperature behavior of the free energy $F(T)$ in the problem at hand

$$F(T) \simeq B_0 \frac{T^4}{\hbar^3} \frac{\pi^2}{90} - B_{1/2} \frac{T^3}{4\pi^{3/2}\hbar^2} \zeta_R(3) + F_{\text{Cas}}(T),$$

(7.51)

where $B_0$ and $B_{1/2}$ are defined in eq. (7.46) and

$$F_{\text{Cas}}(T) \simeq -\frac{\Delta n^2}{8} T \left( \ln \frac{4T a}{\hbar c} + \gamma - \frac{7}{8} \right) + O(T^{-2}).$$

(7.52)

The high temperature asymptotics for the Casimir internal energy and for the Casimir entropy can be derived by making use of the respective thermodynamical relations (7.15), (7.16)

$$U_{\text{Cas}}(T) \simeq \frac{\Delta n^2}{8} T + O(T^{-2}),$$

(7.53)

$$S_{\text{Cas}}(T) \simeq \frac{\Delta n^2}{8} \left( \frac{1}{8} + \gamma + \ln \frac{4aT}{\hbar c} \right) + O(T^{-3}).$$

(7.54)

It is worth comparing these results with analogous asymptotics obtained by different methods. In our paper [46] at the beginning of calculations the first term of expansion of internal energy (7.53) was derived. The subsequent integration of the thermodynamic relation (7.13) gave the correct coefficient of the logarithmic term in the asymptotics of free energy (7.52) (see eq. (5.44)). In paper [47] Barton managed to deduce the asymptotics (7.52) – (7.54). One should keep in mind that our parameter $\Delta n$ corresponds to $2\pi\alpha n$ in the notations of ref. [39].
The asymptotics (7.52)–(7.54) contain the $a$-independent terms. As far as we know the physical meaning of such terms remains unclear.

Preliminary analysis of a complete expression for the internal energy of a dielectric ball (see eqs. (3.20) and (3.31) in ref. [46]) shows that probably there are only exponentially suppressed corrections to the leading term (7.53). In that case in addition to eq. (7.48) all the heat kernel coefficients with number greater than 3 should vanish in the $\Delta n^2$-approximation.

D. Cylinder

The calculation of the vacuum energy of electromagnetic field with boundary conditions defined on a cylinder, to say nothing of the temperature corrections, turned out to be technically a more involved problem than the analogous one for a sphere. Therefore the Casimir problem for a cylinder has been considered only in a few papers [15,19–21,26,49,124]. We again examine three cases: i) perfectly conducting cylindrical shell; ii) solid cylinder with $c_1 = c_2$; iii) dielectric cylinder when $c_1 \neq c_2$. Here we shall use the results of our previous papers [49,124].

1. Perfectly conducting cylindrical shell

The zeta function for this problem is given by eqs. (4.70), (4.55), (4.61) and (4.71).

The function $Z_1(s)$ is defined in the strip $-3/2 < \text{Re} \, s < 1/2$, while the functions $Z_2(s)$ and $Z_3(s)$ are analytic functions in the whole complex plane $s$ except for the points, where $\Gamma(s)$ and $\zeta_{\text{R}}(s)$ have simple poles. In order to find the heat kernel coefficients $B_0$, $B_{1/2}$, and $B_1$ thorough the relation (3.27) one needs the zeta function defined in the region $1/2 + \varepsilon \leq \text{Re} \, s \leq 3/2 + \varepsilon$ with $\varepsilon$ being a positive infinitesimal. However in this region eq. (4.55) is not applicable directly due to the bad behavior of the integral at the upper limit. In the most simple way we can overcome this difficulty as in the case of perfectly conducting plates by introducing the photon mass $\mu$ at the very beginning of the calculation and making then the analytic continuation of the zeta function to the points $s = 1/2, 1, 3/2$. Upon taking the residua at these points one should put $\mu = 0$.

With regard to all this and using the relation (3.27) we find the heat kernel coefficients

$$B_0 = 0, \quad B_{1/2} = 0, \quad B_1 = 0, \quad B_2 = 0.$$  \hspace{1cm} (7.55)

The vanishing heat kernel coefficient $B_2$ implies that the zeta regularization gives a finite value for the vacuum energy in the problem at hand (see eq. (4.72) and ref. [19]). The coefficient $B_{3/2}$ is determined by the function $Z_2(s)$ only (see eq. (4.61))

$$\frac{B_{3/2}}{(4\pi)^{3/2}} = \frac{3}{64 \, a}. \hspace{1cm} (7.56)$$

The coefficient $B_{5/2}$ is defined by the function $Z_3(s)$ given in eq. (4.71)

$$\frac{B_{5/2}}{(4\pi)^{3/2}} = \frac{153 \, c^2}{8192 \, a^3}. \hspace{1cm} (7.57)$$
The calculation of the next heat kernel coefficients $B_3$, $B_{7/2}, \ldots$ would demand a knowledge of the additional terms in the expansion of the spectral zeta function in the problem under consideration in terms of the Riemann zeta function. These terms are proportional to $\zeta_R(2k + 2s + 1)$ with $k = 2, 3, \ldots$, and may be obtained employing the technique presented in subsect. IV C. Analyzing the position of poles for these Riemann zeta functions it is easy to show that, as well as in the spherical case, we have

$$B_j = 0, \quad j = 3, 4, 5 \ldots.$$  

The zeta determinant entering the high temperature asymptotics of Hlemholtz free energy (7.12) and entropy (7.16) is calculated in the Appendix C 1

$$\zeta'(0) = \frac{0.45847}{a} + \frac{3}{32a} \ln \frac{a}{2c}. \quad (7.58)$$

Now we are able to construct the high temperature expansions of the thermodynamic functions in the problem under consideration. For the free energy we have

$$F(T) \simeq -0.22924 \frac{T}{a} - 0.3 \frac{T}{64a} \ln \frac{aT}{2hc} - \frac{51}{65536} \frac{\hbar^2 c^2}{a^3 T} + \mathcal{O}(T^{-3}). \quad (7.59)$$

When comparing eq. (7.59) with results of other authors one should remember that all the thermodynamic quantities that we obtained in this section are related to a cylinder of unit length. The high temperature asymptotics of the electromagnetic free energy in presence of perfectly conducting cylindrical shell was investigated in ref. [15]. To make the comparison handy we rewrite their result as follows

$$F(T) \simeq -0.10362 \frac{T}{a} - 0.3 \frac{T}{64a} \ln \frac{aT}{2hc}. \quad (7.60)$$

The discrepancy between the terms linear in $T$ in eqs. (7.59) and (7.60) is due to the double scattering approximation used in ref. [15] (see also the next subsection). Our approach provides an opportunity to calculate the exact value of this term (see eq. (7.59)).

And finally, making use of the general formulae (7.15) and (7.16) we derive

$$U(T) \simeq \frac{3T}{64a} - \frac{153}{98304} \frac{c^2 \hbar^2}{a^3 T} + \mathcal{O}(T^{-3}), \quad (7.61)$$

$$S(T) \simeq \frac{0.27612}{a} + \frac{3}{64a} \ln \frac{RT}{2hc} - \frac{153}{196608} \frac{c^2 \hbar^2}{a^3 T^2} + \mathcal{O}(T^{-4}). \quad (7.62)$$

2. Compact cylinder with $c_1 = c_2$ and with $c_1 \neq c_2$

Here we consider the boundary conditions for electromagnetic field of two types: i) a compact infinite cylinder with uniform velocity of light on its lateral surface, ii) a pure dielectric cylinder with $c_1 \neq c_2$.

The zeta function for the former configuration is given by eqs. (4.60), (4.56), (4.58) and (4.59). It enables one to find the following heat kernel coefficients
As before we are considering the $\xi^2$-approximation. The heat kernel coefficients (7.63) lead to the following high temperature behavior of the internal energy in the problem at hand

$$U(T) = \frac{3\xi^2 T}{64 R} \left(1 - \frac{5}{512} \frac{c^2 h^2}{R^2 T^2}\right) + \mathcal{O}(T^{-3}).$$

(7.64)

The corresponding zeta determinant is calculated in Appendix C2

$$\zeta'(0) = \frac{\xi^2}{a} \left(0.20699 + \frac{3}{32} \ln \frac{a}{2c}\right).$$

(7.65)

Now we can write the high temperature asymptotics for free energy

$$F(T) = -\frac{\xi^2 T}{a} \left[0.10350 + \frac{3}{64} \ln \frac{T a}{2 hc} + \frac{15}{65536} \frac{c^2 h^2}{a^2 T^2}\right] + \mathcal{O}(T^{-3}).$$

(7.66)

and for entropy

$$S(T) = \frac{\xi^2}{a} \left[0.10350 + \frac{3}{64} \left(1 + \ln \frac{a T}{2 hc}\right) - \frac{15}{65536} \frac{c^2 h^2}{T^2 a^2}\right] + \mathcal{O}(T^{-4}).$$

(7.67)

Putting in these equations $\xi^2 = 1$ we arrive at the double scattering approximation for a perfectly conducting cylindrical shell (see eq. (7.60)). A slight distinction between the linear in $T$ terms in eq. (7.60) and eq. (7.66) is due to a finite error inherent in the numerical methods employed in both the approaches.

In the case of a pure dielectric cylinder ($\mu_1 = \mu_2 = 1$, $\varepsilon_1 \neq \varepsilon_2$) the first four heat kernel coefficients are different from zero even in the dilute approximation [49] (small difference between the velocities of light inside and outside the cylinder)

$$B_0 = 0, \quad B_{1/2} = 0, \quad B_1 = 0, \quad B_{3/2} = \frac{3 \pi \sqrt{\pi} \xi^2}{8 a}, \quad B_2 = 0,$$

$$B_{5/2} = \frac{45 \pi \sqrt{\pi}}{1024}, \quad B_j = 0, \quad j = 3, 4, 5, \ldots.$$

(7.68)

It should be noted that the coefficient $B_2$ vanishes only in the $\xi^2$-approximation. As a matter of fact $B_2$ contains nonvanishing $(c_1 - c_2)^2$-terms and those of higher order [49]. Therefore the zeta regularization provides a finite answer for the vacuum energy of a pure dielectric cylinder only in the $(c_1 - c_2)^2$-approximation even at zero temperature.

The contribution to the asymptotic expansions of the first three heat kernel coefficients should be involved into the relevant phenomenological parameters in the general expression of the classical energy of a dielectric cylinder (in the same way as it has been done for a pure
dielectric ball). By making use of the coefficients $B_{3/2}$ and $B_{5/2}$ we get the high temperature asymptotics of the internal energy in the problem at hand

$$U(T) = \Delta n^2 \frac{3}{128} T \left( 1 - \frac{857 c^2 \hbar^2}{17280 T^2 R^2} \right) + \mathcal{O}(T^{-2}).$$

(7.69)

where $\Delta n = n_1 - n_2 \simeq (c_2 - c_1)/c$.

In view of considerable technical difficulties we shall not calculate the zeta function determinant for a pure dielectric cylinder. We recover the respective asymptotics of free energy by integrating the thermodynamic relation (7.13) and of entropy by using the relation (7.14). Pursuing this way we introduce a new constant of integration $\alpha$ that remains undetermined in our consideration

$$F(T) = -\Delta n^2 \frac{3}{128} a \left( \alpha + \ln \frac{a T}{\hbar c} + \frac{857 c^2 \hbar^2}{34560 T^2 a^2} \right) + \mathcal{O}(T^{-2}),$$

(7.70)

$$S(T) = \Delta n^2 \frac{3}{128} \left( 1 + \alpha + \ln \frac{a T}{\hbar c} - \frac{857 c^2 \hbar^2}{34560 T^2 a^2} \right) + \mathcal{O}(T^{-3}).$$

(7.71)

E. Summary of sect. VII

We have demonstrated efficiency and universality of the high temperature expansions in terms of the heat kernel coefficients for the Casimir problems with spherical and cylindrical symmetries. All the known results in this field are reproduced in a uniform approach and in addition a few new asymptotics are derived (for a compact ball with $c_1 = c_2$ and for a pure dielectric infinite cylinder).

As the next step in the development of this approach one can try to retain the terms exponentially decreasing when $T \to \infty$. These corrections are well known, for example, for thermodynamic functions of electromagnetic field in the presence of perfectly conducting parallel plates [67] (see also eq. (7.44)). In order to reveal such terms, first of all the exponentially decreasing corrections should be retained in the asymptotic expansion (3.25) for the heat kernel.

It is worth noting that in the framework of the method employed the high temperature asymptotics can also be constructed in the problems when the zeta regularization does not provide a finite value of the vacuum energy at zero temperature, i.e., when the heat kernel coefficient $B_2$ does not vanish. Such problems were considered, for example, in ref. [190] by making use of the asymptotic energy densities (see also paper [191]).

In ref. [180] it was argued that in the high temperature limit the behavior of the Casimir thermodynamic quantities should be the following. In the case of disjoint boundary pieces the free energy tends to minus infinity, the entropy approaches a constant, and the internal energy vanishes. Contributions to the Casimir thermodynamic quantities due to each individual connected component of the boundary exhibits logarithmic deviations in temperature from the behavior just described. In our consideration we were obviously dealing with an
individual connected component of the boundary (a sphere or cylinder). Our results corroborate the relevant conclusions of ref. [180] concerning the free energy and entropy. However the internal energy in our calculations tends to infinity like \( T \) instead to vanish, this increase being caused by the respective logarithmic terms in the high temperature asymptotics of free energy.

It is worth developing the methods that enable one to reveal the \textit{low temperature asymptotics} of Casimir energy by making use of the spectral zeta functions and heat kernel technique.

**VIII. CONCLUSIONS**

The reality of Casimir forces and consequently the reality of relevant vacuum energy of quantized fields is now well founded experimentally. Therefore the task of the theory in this area is to develop mathematically consistent methods for calculating these quantities. Unfortunately we have to assert that till now one didn’t succeed in casting the definition of the Casimir energy (3.2) on a rigorous mathematical footing for boundaries of an arbitrary geometry. It is this reason that causes the controversies persisting here for a long time (see, for example, papers [61,192–197] and references therein).

In accordance with the general concept of the quantum field theory, the subtraction procedure (3.2) could be rigorously specified in the framework of proving the renormalizability of the quantum field model under study. However it is very unlikely to implement this by a straightforward generalization of the renormalizability of quantum electrodynamics treated in unbounded Minkowski space. Point is that the boundaries or nonhomogeneous characteristics of the configuration space drastically complicate the propagators [198] which are the kernels of the Schwinger-Dayson integral equations for complete Green’s functions. The proof of the renormalizability is substantially based on the analysis of these equations [92,199].

The following consideration is also important here. The proof of renormalization theorem is carried out, as a matter of fact, in the framework of perturbation theory because the complete kernels of the Schwinger-Dayson equations are known only in the form of perturbation series. In order to anticipate the convergence of resulting series one has to assume the interaction in the theory under consideration to be small. The boundaries can also be treated as an interaction with appropriate classical fields. However the latter cannot be considered as a weak one [85]. In this situation application to the calculation of the vacuum energy of the renormalization group technique is interesting [200,201].

To our opinion, the present status of the Casimir calculations can be appropriately described by the following words due to Milton [193]: ”Obviously we are still at the early stages of understanding quantum field theory. The nature of divergences in vacuum energy calculations is still not understood. However, there are a few established peaks that rise above the murky clouds of ignorance, and we should not abandon them lightly because the rest is obscure.”

In this situation the spectral zeta function method and heat kernel technique are distinguished here because they relay on sound mathematics, reproduce the results obtained in other approaches and are applicable, at least in principle, to a broad range of physical
problems.

An urgent task in the Casimir studies is the development of methods that enable one to do calculations for boundaries without high symmetry. It is worth noting the proposal to employ in this field the world line formalism in QFT [202] and optical approach to the Casimir effect [203]. Certainly, one can anticipate here only approximate solutions. In order to preserve the rigorous treatment of the divergencies we suppose that it is worthy to use here the chain of relations between the Casimir energy, the spectral zeta function (3.13) and the relevant heat kernel (3.24). For the latter one can construct the integral equations [204] which provide us at least with perturbation series. The procedure of analytic continuation should be done here for each term of these series separately.

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APPENDIX A: ANALYSIS OF THE DIVERGENCIES GENERATED BY $W^2_l$

Here we reveal an important relation between linear and quadratic in $\Delta n$ terms in $W^2_l$ defined in eq. (5.12).

Let us put

$$x_1 = y \left(1 + \frac{\Delta n}{2}\right), \quad x_2 = y \left(1 - \frac{\Delta n}{2}\right), \quad \Delta x = \Delta n y. \quad (A1)$$

The Taylor expansion yields

$$W_l(x_1, x_2) = s_l(x_1)e'_l(x_2) - s'_l(x_1)e_l(x_2)$$

$$= -1 + (2s'_le'_l - s'_le''_l - s''_le_l) \frac{\Delta x}{2}$$

$$+ \left[ \frac{1}{2} (s'e''_l - s''_le_l) + \frac{3}{2} (s''_le'_l - s''_le''_l) \right] \frac{\Delta x^2}{4} + O(\Delta x^3). \quad (A2)$$

For brevity we have dropped the argument $y$ of the function $s_l$ and $e_l$, and have used the value of the Wronskian

$$W\{s_l(y), e_l(y)\} = s_l'e_l - s_l' e_l = -1. \quad (A3)$$

By making use of the equation for the Riccati–Bessel functions
\[ w_i''(y) - L(l, y) w_i(y) = 0, \quad L(l, y) \equiv 1 + \frac{l(l + 1)}{y^2}, \quad (A4) \]

we obtain
\[ s''_l e_l - s_l e'''_l = L(l, y), \]
\[ s''_l e'_l - s'_l e''_l = -L(l, y). \quad (A5) \]

Substitution of (A5) into (A2) gives
\[ W_l(x_1, x_2) = -1 + [s'_l e'_l - L(l, y)s_l e_l] \Delta x - \frac{1}{2} L(l, y) \Delta x^2 + O(\Delta x^3). \quad (A6) \]

Squaring eq. (A6) one gets
\[ W_l^2(x_1, x_2) = 1 + A_l \Delta n + B_l \Delta n^2 + O(\Delta n^3), \quad (A7) \]

where
\[ A_l = y(s''_l e_l + s_l e''_l - 2s'_l e'_l) = 2y \left[ 2L(l, y)s_l e_l - \frac{1}{2} (s_l e_l)'' \right], \quad (A8) \]
\[ B_l = y^2 L(l, y) + \frac{1}{4} A_l^2. \quad (A9) \]

In terms of these notations we can write
\[ \ln \left( \frac{W_l^2 - \Delta n^2}{4} P_l^2 \right) = A_l \Delta n + \left( B_l - \frac{A_l^2}{2} \right) \Delta n^2 - \frac{\Delta n^2}{4} P_l^2 + O(\Delta n^3). \quad (A10) \]

The terms quadratic in \( \Delta n \) in eq. (A10) exactly reproduce the function \( F_l(y) \) in eq. (9) of the paper [40]. It is this function that affords the whole finite value of the Casimir energy in the problem under consideration. Unlike the papers [40,41,43] we didn’t introduce the contact terms in the definition of the Casimir energy and nevertheless we have reproduced the key function \( F_l(y) \). It implies that the contact terms do not give a contribution into the finite part of the Casimir energy in this problem. They merely cancel the terms \( A_l \Delta n \) in eq. (A10).

Now we show, without invoking the contact terms, that the \( A_l \) terms in eq. (A10) do not contribute into the vacuum energy.

Using eq. (5.18) with \( \theta = 0 \) we introduce the notation
\[ \sum_{l=1}^{\infty} (2l + 1)s_l(yr)e_l(y \rho) + 1 = \frac{yr \rho}{|r - \rho|} e^{-y|r - \rho|} \equiv \mathcal{D}(r, \rho, y). \quad (A11) \]

Taking into account the explicit form of the coefficients \( A_l \) defined in eq. (A8) one can write
\[ \Delta n \sum_{l=1}^{\infty} (2l + 1)A_l = y \Delta n \left( \frac{\partial^2}{\partial r^2} - 2\frac{\partial}{\partial r} + \frac{\partial^2}{\partial \rho^2} \right) \mathcal{D}(r, \rho, y) \bigg|_{r=\rho=1} + 1. \quad (A12) \]
When \( r = \rho = 1 \) the derivatives of the function \( \mathcal{D} \) in eq. (A12) tend to infinity. Therefore a preliminary regularization should be introduced here in order to put our consideration on a rigorous mathematical footing. To this end we define the right-hand side of eq. (A12) in the following way

\[
\Delta n \sum_{l=1}^{\infty} (2l + 1) A_l = \Delta n \lim_{\varepsilon \to 0} \left( \mathcal{D}_{rr} - 2\mathcal{D}_{rp} + \mathcal{D}_{\rho\rho} \right) \bigg|_{r=1+\varepsilon/2, \rho=1-\varepsilon/2} + 1, \tag{A13}
\]

where the positive constant \( \varepsilon \) is a regularization parameter. From the explicit form of the function \( \mathcal{D}(r, \rho, y) \) (see eq. (A11)) it follows immediately

\[
\lim_{\varepsilon \to 0} \left. \left( \mathcal{D}_{rr} - \mathcal{D}_{\rho\rho} \right) \right|_{r=1+\varepsilon/2, \rho=1-\varepsilon/2} = 0. \tag{A14}
\]

The analogous limit for the differences

\[
\mathcal{D}_{rr} - \mathcal{D}_{rp} \quad \text{and} \quad \mathcal{D}_{\rho\rho} - \mathcal{D}_{rp} \tag{A15}
\]

also vanishes. Hence in the regularization introduced above the sum under consideration has the following value

\[
\Delta n \sum_{l=1}^{\infty} (2l + 1) A_l = 1. \tag{A16}
\]

It implies immediately that the term linear in \( \Delta n \), which encounters eq. (A10) does not contribute to the vacuum energy \( E_2 \) defined in eq. (5.11).

Now we show that the contributions to the Casimir energy given by \( \sum_l B_l \) and by \((1/4) \sum_l A_l^2 \) are the same. In other words, \( y^2 L(l, y) \) in eq. (A9) does not give any finite contribution to the vacuum energy. In order to prove this, we consider the expression

\[
I = \sum_{l=1}^{\infty} \nu \int_0^{\infty} y^2 \, dy, \quad \nu = l + \frac{1}{2}. \tag{A17}
\]

Instead of the cutoff regularization we shall use here the analytical regularization presenting (A17) in the following form

\[
I = \lim_{s \to 0} \sum_{l=1}^{\infty} \nu \int_0^{\infty} y^{2-s} \, dy = \lim_{s \to 0} \sum_{l=1}^{\infty} \nu^{4-s} \int_0^{\infty} z^{2-s} \, dz
\]

\[
= \lim_{s \to 0} \lim_{\mu^2 \to 0} \sum_{l=1}^{\infty} \nu^{4-s} \int_0^{\infty} (z^2 + \mu^2)^{1-s/2} \, dz. \tag{A18}
\]

Here the change of integration variable \( y = \nu z \) is done and the photon mass \( \mu \) is introduced. Further we have
\[ I = \lim_{s \to 0} \lim_{\mu^2 \to 0} \left[ (2^{-4+s} - 1)\zeta(s - 4) - 2^{-4+s} \right] \frac{\mu^{3-s}}{2} \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{s}{2} - 1\right)} \]
\[
= -\frac{\pi}{24} \lim_{s \to 0} \lim_{\mu^2 \to 0} \frac{\mu^2}{\Gamma\left(\frac{s}{2} - 1\right)} \to 0. \quad \text{(A19)}
\]

In view of all this, we are left with the following scheme for calculating the Casimir energy of a dielectric ball in the \( \Delta n^2 \)-approximation. First, the \( \Delta n^2 \)-contribution should be find, which is given by the sum \( \sum_i W_i^2 \). Upon changing its sign to the opposite one, we obtain the contribution generated by \( W_i^2 \), when this function is in the argument of the logarithm. Obviously, this result would be deduced directly if one could find in a closed form the sum \( \sum_i W_i^2 W_i^2 \) \[33\]. This assertion can be represented by a symbolic formula
\[
\ln \left( W_i^2 - \frac{\Delta n^2}{4} P_i^2 \right) \sim -\Delta n^2 B_i - \frac{\Delta n^2}{4} P_i^2 + O(\Delta n^3). \quad \text{(A20)}
\]

The sign \( \sim \) means here the equality subject to the regularizations described above are employed.

**APPENDIX B: ZETA FUNCTION DETERMINANTS FOR SPHERICALLY SYMMETRIC BOUNDARIES**

1. Perfectly conducting sphere

We shall use here the integral representation for the zeta function given in eq. (4.17) with \( \xi^2 = 1 \). The analytic continuation of this expression to the region \( \Re s < 0 \) is performed by adding and subtracting from the integrand its uniform asymptotics at large \( \nu \)
\[
\sigma_i^2(\nu z) \simeq \frac{t^6(z)}{4\nu^2}, \quad t(z) = \frac{1}{\sqrt{1 + z^2}}. \quad \text{(B1)}
\]

As a result we obtain
\[
\zeta(s) = Z(s) + \zeta_{as}(s), \quad \text{(B2)}
\]

where
\[
Z(s) = \left(\frac{a}{c}\right)^{2s} \frac{\sin(\pi s)}{2\pi} \sum_{l=1}^{\infty} \nu^{1-2s} \int_0^\infty \frac{dz}{z^{2s}} d\nu \left\{ \ln[1 - \sigma_i^2(\nu z)] + \frac{1}{4\nu^2 (1 + z^2)^2} \right\}, \quad \text{(B3)}
\]
\[
\zeta_{as}(s) = \left(\frac{a}{c}\right)^{2s} \frac{3 \sin(\pi s)}{4\pi} \sum_{l=1}^{\infty} \nu^{-1-2s} \int_0^\infty dz z^{1-2s} t^8(z)
\]
\[
= \frac{1}{4} \left(\frac{a}{c}\right)^{2s} s(1 + s)(2 + s) \left[ (2^{1+2s} - 1)\zeta_R(1 + 2s) - 2^{1+2s} \right]. \quad \text{(B4)}
\]
When calculating $\zeta'(0)$ one can put in eq. (B3) $s = 0$ everywhere except for $\sin(\pi s)$, the latter function being substituted simply by $\pi s$. In view of this the integral in eq. (B3) is evaluated easy if one takes into account the limits
\[
\lim_{z \to 0} \sigma^2(z) = \left[ \frac{1}{2\Gamma(\nu)} \right]^2 = \frac{1}{4\nu^2}, \quad \lim_{z \to 0} t^6(z) = \frac{1}{4\nu^2}
\] (B5)
and the asymptotics (B1) at large $z$. As a result we obtain
\[
Z'(0) = -2\sum_{l=1}^{\infty} \nu \left[ \ln \left( 1 - \frac{1}{4\nu^2} \right) + \frac{1}{4\nu^2} \right].
\] (B6)
Differentiation of eq. (B4) with respect to $s$ at the point $s = 0$ gives
\[
\zeta'(0) = -\frac{5}{8} + \frac{1}{2} \ln a + \ln 2 + \frac{\gamma}{2}.
\] (B7)
In order to calculate the sum over $l$ in eq. (B6) we consider an auxiliary sum
\[
S(q) = -\sum_{l=1}^{\infty} 2\nu \left[ \ln \left( 1 - \frac{q^2}{4\nu^2} \right) + \frac{q^2}{4\nu^2} \right], \quad S(0) = 0, \quad S(1) = Z'(0),
\] (B8)
where $q$ is a parameter. Derivative of this sum with respect to $q$ can be rewritten in the form
\[
S'(q) = -\frac{q}{2} \sum_{l=1}^{\infty} \left[ \frac{1}{l} - \frac{1}{l+1/2} - \frac{1}{l+(1+q)/2} + \frac{1}{l+1/2} - \frac{1}{l+(1-q)/2} \right].
\] (B9)
The summation in eq. (B9) is accomplished by making use of the following relations [112]
\[
\sum_{k=1}^{\infty} \left( \frac{1}{y+k} - \frac{1}{x+k} \right) = \frac{1}{x} - \frac{1}{y} + \psi(x) - \psi(y),
\]
\[
\psi(x+1) = \psi(x) + \frac{1}{x}, \quad \psi \left( \frac{1}{2} \right) = -\gamma - 2\ln 2,
\] (B10)
where $\psi(x)$ is the digamma function (the Euler $\psi$ function): $\psi(x) = (d/dx)\ln \Gamma(x)$. This gives
\[
S'(q) = q \left( 2 - \gamma - 2\ln 2 \right) - \frac{q}{2} \left[ \psi \left( \frac{3}{2} + \frac{q}{2} \right) + \psi \left( \frac{3}{2} - \frac{q}{2} \right) \right].
\] (B11)
Now we integrate the both sides of eq. (B11) over $a$ from 0 to 1 by making use of ‘Maple’
\[
S(1) = Z'(0) = \frac{1}{2} - \frac{\gamma}{2} + \frac{7}{6} \ln 2 + 6\zeta'_R(-1).
\] (B12)
From eqs. (B2), (B7) and (B12) it follows that
\[
\zeta'(0) = \frac{1}{2} - \frac{\gamma}{2} + \frac{7}{6} \ln 2 + 6\zeta'_R(-1) + \left( -\frac{5}{8} + \frac{1}{2} \ln \frac{a}{c} + \ln 2 + \frac{\gamma}{2} \right)
\]
\[
= \frac{1}{8} + \frac{13}{6} \ln 2 + 6\zeta'_R(-1) + \frac{1}{2} \ln \frac{a}{c} = 0.38429 + \frac{1}{2} \ln \frac{a}{c}.
\] (B13)
2. Material ball with \( c_1 = c_2 \)

The complete zeta function in this problem is given in eq. (4.17) (without expanding in powers of the parameter \( \xi^2 \)). Adding and subtracting under the integral sign in this equation the uniform asymptotics of the integrand at large \( \nu \) we get

\[
\zeta(s) = \left( \frac{a}{c} \right)^{2s} \frac{\sin(\pi s)}{2\pi} \sum_{l=1}^{\infty} \nu^{1-2s} \int_0^\infty \frac{dz}{z^{2s}} \frac{d}{dz} \left\{ \ln[1 - \xi^2 \sigma_l^2(\nu z)] + \frac{\xi^2}{4\nu^2} \left( 1 + z^2 \right)^{-3} \right\} + \xi^2 \zeta_{as}(s),
\]

where the function \( \zeta_{as}(s) \) was introduced in eq. (B4). Proceeding in the same way as in the previous subsection we obtain for the derivative of the function \( \zeta(s) \) at the point \( s = 0 \)

\[
\zeta'(0) = S(\xi^2) + \xi^2 \zeta'_{as}(0),
\]

where the function \( S(\xi^2) \) is defined in eq. (B8). For small values of the argument \( \xi^2 \) we deduce from eq. (B8)

\[
S(\xi^2) = \frac{\xi^4}{16} \sum_{l=1}^{\infty} \frac{1}{\nu^3} = \frac{\xi^4}{16} \left[ 7 \zeta_R(3) - 8 \right] + O(\xi^6).
\]

Therefore restricting ourselves to the first order of \( \xi^2 \) we arrive at the final result

\[
\zeta'(0) = \xi^2 \zeta'_{as}(0) = \xi^2 \left( -\frac{5}{8} + \frac{1}{2} \ln \frac{a}{c} + \ln 2 + \frac{\gamma}{2} \right).
\]

APPENDIX C: ZETA FUNCTION DETERMINANTS FOR CYLINDRICALLY SYMMETRIC BOUNDARIES

1. Perfectly conducting cylindrical shell

The spectral zeta function for this configuration has been considered in subsect. IV C. This function was used there for obtaining a finite value for the relevant Casimir energy. For this the value of the zeta function at the point \( s = -1/2 \) was calculated. Now we are interested in the value of \( \zeta'(0) \). Therefore it is convenient to represent the initial formula (4.46) with \( \xi^2 = 1 \) in a slightly different way as compared with the subsect. IV C, namely

\[
\zeta(s) = \frac{a^{2s-1}}{2\sqrt{\pi c^{2s}}} \Gamma(s) \frac{1}{\Gamma(3/2 - s)} \int_0^\infty dy y^{1-2s} \frac{d}{dy} \ln[1 - \mu_0^2(y)]
\]

\[
+ \frac{a^{2s-1}}{\sqrt{\pi c^{2s}}} \frac{1}{\Gamma(3/2 - s)} \sum_{n=1}^{\infty} n^{1-2s} \int_0^\infty dy y^{1-2s} \frac{d}{dy} \ln[1 - \mu_n^2(ny)],
\]

where \( \mu_n(y) \) was defined in eq. (4.47). The first term on the right hand side of eq. (C1) is an analytic function of the complex variable \( s \) in the strip \(-1/2 < \text{Re} \, s < 1/2\). Therefore
there is no need in analytic continuation of this expression when calculating \( \zeta'(0) \). As regard to the second term in eq. (C1) its analytic continuation to the region \( \text{Re} \, s < 0 \) can be accomplished in a standard way. We add and subtract here the uniform asymptotics of the integrand when \( n \) tends to infinity

\[
\ln[1 - \mu_n^2(ny)] \simeq -\frac{y^4 t^6(y)}{4 n^2} + \mathcal{O}(n^{-4}), \quad t(y) = \frac{1}{\sqrt{1 + y^2}}. \tag{C2}
\]

As a result we obtain

\[
\zeta(s) = \frac{a^{2s - 1}}{2\sqrt{\pi} c^{2s} \Gamma(s)} \int_0^\infty \frac{dy}{y^{2s-1}} \frac{d}{dy} \ln[1 - \mu_0^2(y)] \\
+ \frac{a^{2s - 1}}{\sqrt{\pi} c^{2s} \Gamma(s)} \sum_{n=1}^\infty n^{1-2s} \int_0^\infty \frac{dy}{y^{2s-1}} \left\{ \ln[1 - \mu_n^2(ny)] + \frac{y^4 t^6}{4 n^2} \right\} \\
- \frac{a^{2s - 1}}{32 \sqrt{\pi} c^{2s}} (1 - 2s)(3 - 2s) \zeta_R(2s + 1) \frac{\Gamma(1/2 + s)}{\Gamma(s)}. \tag{C3}
\]

Keeping in mind the behavior of the gamma function at the origin \( \Gamma(s) \simeq s^{-1} \) one can easily find the derivative of \( \zeta(s) \) at the point \( s = 0 \)

\[
\zeta'(0) = \frac{1}{\pi a} \int_0^\infty dy y \frac{d}{dy} \ln[1 - \mu_0^2(y)] \\
+ \frac{2}{\pi a} \sum_{n=1}^\infty n \int_0^\infty dy y \left\{ \ln[1 - \mu_n^2(ny)] + \frac{y^4 t^6}{4 n^2} \right\} + \frac{1}{32 a} \left( 3\gamma - 4 - 3 \ln \frac{2c}{a} \right). \tag{C4}
\]

Unlike the spherically symmetric boundaries, the integration is not removed in the formula obtained for \( \zeta'(0) \). Therefore the first two terms in eq. (C4) can be calculated only numerically

\[
-\frac{1}{\pi a} \int_0^\infty dy \ln[1 - \mu_0^2(y)] = \frac{0.53490}{a}. \tag{C5}
\]

Applying the FORTRAN subroutine that approximates the Bessel functions by Chebyshev’s polynomials we evaluate the first 30 terms in the sum in eq. (C4)

\[
-\frac{2}{\pi a} \sum_{n=1}^\infty n \int_0^\infty dy \left\{ \ln[1 - \mu_n^2(ny)] + \frac{y^4 t^6}{4 n^2} \right\} = -\frac{0.00554}{a}. \tag{C6}
\]

Finally gathering together all these results we have

\[
\zeta'(0) = \frac{0.45847}{a} + \frac{3}{32 a} \ln \frac{a}{2c}. \tag{C7}
\]
2. Compact infinite cylinder with \( c_1 = c_2 \)

Now we turn to a compact cylinder placed into unbounded medium such that the velocity of light is uniform on the lateral surface of the cylinder. Proceeding as in the case of a cylindrical shell we rewrite the initial equation (4.46) in the linear approximation with respect to \( \xi^2 \) in the form

\[
\zeta(s) = - \frac{a^{2s-1} \xi^2}{2 \sqrt{\pi c^2 \Gamma(s) \Gamma(3/2 - s)}} \int_0^\infty dy y^{1-2s} \frac{d}{dy} \mu_0^2(y) \\
- \frac{a^{2s-1} \xi^2}{\sqrt{\pi c^2 s \Gamma(s) \Gamma(3/2 - s)}} \sum_{n=1}^\infty \int_0^\infty dy y^{1-2s} \frac{d}{dy} \mu_n^2(y), \tag{C8}
\]

where \( \mu_n(y) \) is defined in eq. (4.47). The analytic continuation to the region Re \( s < 0 \) is needed only for the second term in eq. (C8). Adding and subtracting here the uniform asymptotics of the integrand for large \( n \)

\[
-\mu_n^2(ny) \simeq -\frac{y^4 t^6(ny)}{4n^2} + O(n^{-4}), \tag{C9}
\]

we obtain

\[
\zeta'(0) = -\frac{\xi^2}{\pi a} \int_0^\infty dy y \frac{d}{dy} \mu_0^2(y) + \frac{2\xi^2}{\pi a} \sum_{n=1}^\infty \int_0^\infty dy y \frac{d}{dy} \left[ -\mu_n^2(ny) + \frac{y^4 t^6}{4n^2} \right] \\
+ \frac{\xi^2}{32a} \left( 3\gamma - 4 - 3 \ln \frac{2c}{a} \right). \tag{C10}
\]

The first two terms in eq. (C10) can again be calculated only numerically

\[
\frac{\xi^2}{\pi a} \int_0^\infty dy \mu_0^2(y) = \frac{\xi^2}{a} 0.28428, \tag{C11}
\]

\[
-\frac{2\xi^2}{\pi a} \sum_{n=1}^\infty \int_0^\infty dy \left[ -\mu_n^2(ny) + \frac{y^4 t^6}{4n^2} \right] = -\frac{\xi^2}{a} 0.00640. \tag{C12}
\]

The final result reads

\[
\zeta'(0) = \frac{\xi^2}{a} \left[ 0.28428 - 0.00640 + \frac{1}{32} \left( 3\gamma - 4 - 3 \ln \frac{2c}{a} \right) \right] \\
= \frac{\xi^2}{a} \left( 0.20699 + \frac{3}{32} \ln \frac{a}{2c} \right). \tag{C13}
\]
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