Antiferromagnetic systems with spin gap: exact results

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Abstract

We describe some antiferromagnetic systems which exhibit spin gaps. We also discuss the effect of doping one such system, namely, the spin-ladders, with holes. Some model antiferromagnetic systems with spin gap are reviewed for which exact results are available. Exact results for a doped spin-ladder model are also mentioned.

I. Introduction

Recently, several antiferromagnetic (AFM) systems with spin gap have been discovered. These systems exhibit a variety of phenomena a full understanding of which is as yet lacking. The effect of doping on the spin gap, with either hole or magnetic and non-magnetic impurities, has also been studied. The AFM Hamiltonian \( H \) is generally of the Heisenberg-type with nearest-neighbour (n.n.) as well as frustrating further-neighbour interactions included.

\[
H = \sum_{<ij>} J_{ij} \vec{S}_i \cdot \vec{S}_j \tag{1}
\]

\( \vec{S}_i \) is the spin operator at lattice site \( i \) and \( J_{ij} \) is the exchange integral for interaction between spins located at sites \( i \) and \( j \).

Some theorems and rigorous results are available for the Hamiltonian (1). The Lieb-Mattis theorem \([1]\) has proved for general spin and for all dimensions that for a bipartite lattice, the entire eigenvalue spectrum satisfies the inequality

\[
E_0(S) \leq E_0(S + 1) \tag{2}
\]

where \( E_0(S) \) is the minimum energy state corresponding to total spin \( S \). The weak inequality becomes a strict inequality for a ferromagnetic (FM)
exchange coupling between spins of the same sublattice. The theorem is
valid for any range of exchange coupling and the proof does not require the
existence of periodic boundary conditions.
Marshall’s sign rule \[2\] specifies the structure of the ground state of the
spin-$1/2$ AFM on a bipartite lattice with n.n. interaction. The proof can be
extended to spin $S$, next-nearest-neighbour (n.n.n.) FM interaction but not
to n.n.n. AFM interaction. The ground state wave function $\Psi$ is given by

$$\Psi = \sum_\mu C_\mu | \mu \rangle$$

where $| \mu \rangle$ is one of the $2^N$ basis states. According to Marshall’s sign rule,
the coefficient $C_\mu$ is of the form

$$C_\mu = (-1)^{p_\mu} a_\mu$$

where $a_\mu$ is real and $\geq 0$ and $p_\mu$ is the number of up spins on the A sublattice
(the bipartite lattice consists of two sublattices A and B).

The Lieb-Mattis (LSM) theorem\[3\] says that the excitation spectrum of
the $S = 1/2$ Heisenberg AFM linear chain with n.n. interaction is gapless. The
proof can be extended to any half-odd integer spin \[4\] but not to integer
spins giving rise to the famous ‘Haldane’s conjecture’ which we will discuss
shortly.

The Mermin Wagner theorem \[5\] proves that there cannot be any AFM
long range order (LRO) at finite temperature $T$ in dimensions $d = 1$ and 2.
Various proofs \[6, 7\] exist to show that LRO exists in the ground state ($T = 0$) for $S \geq 1$ in $d = 2$ and for $S \geq 1/2$ in $d = 3$. The proof for the case $S = 1/2$, $d = 2$ is not, however, known but approximate calculations have shown the
existence of LRO in the ground state.

The Hamiltonian which describes AFMs doped with holes is the t-J
model:

$$H_{t-J} = -t \sum_{i,j,\sigma} (1 - n_{i-\sigma}) C_{i,\sigma}^\dagger C_{j,\sigma} (1 - n_{j-\sigma}) + J \sum_{<ij>} \vec{S}_i \cdot \vec{S}_j$$

$C^\dagger$, $C$ are the electron creation and annihilation operators, $i$, $j$ denote the
lattice sites, $\sigma$ is the spin index and $n_{i,\sigma}$ is the occupation number of the
ith site with spin $\sigma$. The first term in the Hamiltonian is the kinetic energy
term and operates only in the subspace of states in which there is no double
occupancy. The last term is the well-known AFM Heisenberg Hamiltonian. The t-J model is derived from the well-known Hubbard model in the limit of strong correlation [8]. It is this strong correlation which prohibits the double occupancy of a site. In the half-filled limit, when there is one electron at each site, the first term of H is ineffective and the Hamiltonian reduces to the AFM Heisenberg Hamiltonian. The t-J model has been extensively used to study the properties of doped high-$T_c$ cuprate systems [9]. In this review, we describe some AFM systems with spin gap and also a doped AFM system, namely, the spin-ladder. In Section II, a general description of AFM systems with spin gap is given. Some examples of doped AFMs are also given. In Sections III and IV, some AFM models are described for which exact results are available. Section IV contains exact results on the hole dynamics in a spin-ladder model. The review of exact results is by no means exhaustive and describes mainly the Author’s own work.

II. Antiferromagnetic systems with spin gap

In this Section, we describe real AFM systems with spin gap. The LSM theorem, as we have already mentioned, is applicable only to half-odd-integer spins. Haldane in 1983 [10] made the conjecture that integer spin chains have a gap in the excitation spectrum. Experimental realizations of the Haldane-gap systems include the compounds $Ni(C_2H_8N_2)_2NO_2ClO_4$ (NENP), $Ni(C_3H_{10}N_2)_2NO_2ClO_4$ (NINO) and more recently $Y_2BaNiO_5$ [11]. In 1993, Hase et al [12] showed that the $S=1/2$ AFM Heisenberg spin chain compound $CuGeO_3$ undergoes a spin-Peierls (SP) transition at a temperature $T_{SP} \approx 14$ K. In the SP phase, the ground state becomes dimerized in which successive pairs of sites are brought close together. The spins in a pair mainly interact with each other forming singlets ($S=0$). There is thus an alternation of exchange interaction strengths $J_1$ and $J_2$. The ground state is non-magnetic and a finite energy gap exists in the S=1 spin excitation spectrum. $CuGeO_3$ is the first example of an inorganic system exhibiting the SP transition.

Next, we turn to the discussion of spin-ladders [13]. The spins have magnitude 1/2. The simplest spin-ladder consists of two chains coupled by rungs and interpolates between 1d and 2d AFMs. The Hamiltonian is given
by
\[ H_L = J_\parallel \sum_{\text{chains}} \vec{S}_i \cdot \vec{S}_j + J_\perp \sum_{\text{rungs}} \vec{S}_i \cdot \vec{S}_j \]  
(6)
where \( J_\parallel \) and \( J_\perp \) are the exchange interactions along the chains and between them. The ladder has a gap in the excitation spectrum even in the isotropic coupling limit \( J_\parallel = J_\perp \). The ground state consists of singlets along the rungs. An excitation is created by replacing one of the singlets by a triplet and then letting it propagate. The triplet excitation spectrum exhibits a gap.

Recent inelastic neutron scattering experiments [14] have verified that the S=1/2 AFM vanadyl pyrophosphate \((VO)_2P_2O_7\) is an accurate realization of the spin-ladder system.

A general spin ladder consists of \( n \) chains. One example of such a system is \( Sr_{n-1}Cu_{n+1}O_{2n} \ (n = 3, 5, 7, \ldots) \) [14] which consists of ladders of \((n+1)/2\) chains with frustrated "trellis" coupling between the ladders. A ladder with an odd number of chains has properties similar to that of a single chain, namely, gapless excitation spectrum and a power-law decay of the spin-spin correlation function. A ladder with an even number of chains has a spin gap and an exponential decay of the spin-spin correlation function. The significant difference between the properties of odd and even-chain ladders has been verified in a number of experiments [13]. The system \( La_{4+4n}Cu_{8+2n}O_{14+8n} \) has also a ladder-like structure. Another compound of interest is \( LaCuO_{2.5} \).

Initial susceptibility experiments were interpreted as showing a gap in the excitation spectrum but subsequent \( \mu s r \) and NMR experiments indicate an AFM transition below \( T_N \sim 110K \) [16]. Hiroi and Takano [17] synthesized the first doped ladder system by replacing some of the La ions of \( LaCuO_{2.5} \) by Sr. They observed an insulator to metal transition on doping but no superconductivity was found down to 5 K. The theoretical studies on ladders, on the other hand, predict strong superconducting pairing correlations [7].

The compound \( Cu_2(C_5H_{12}N_2)_2Cl_4 \) is another example of a two-chain ladder compound [18]. Magnetic susceptibility results indicate the presence of weak FM diagonal interactions in the ladder. The compound \( Sr_{14}Cu_{24}O_{41} \) is composed of layers containing two-chain ladders alternating with layers of \( CuO_2 \) chains. Experiments have been carried out both on this system as well as on the system doped with Ca ions: \( (Sr_{0.8}Ca_{0.2})_{14}Cu_{24}O_{41} \) [14, 21]. Spin gaps have been seen in the excitation spectra of both the chains and the ladders. A recent exciting development is the observation of superconductivity...
in the ladder compound $\text{Sr}_{0.4}\text{Ca}_{13.6}\text{Cu}_{24}\text{O}_{41.84}$ under a pressure of 3 to 4.5 GPa [21]. The superconducting transition temperature is 12 and 9 K at 3 and 4.5 GPa, respectively. The discovery of superconductivity is in conformity with theoretical predictions.

The compound $\text{CaCuGe}_2\text{O}_6$ can be described in terms of isolated dimers (singlets) [22]. There is a finite energy gap separating the singlet $S = 0$ ground state from the excited $S = 1$ triplet. The compound $\text{BaCuSi}_2\text{O}_6$ is a quasi-2d AFM with a bilayer structure [23]. Experiments show the existence of a spin gap. Dimers predominantly form between the layers and are weakly-interacting. A recent addition to the list of AFM systems exhibiting spin gap is the compound $\text{CaV}_4\text{O}_9$ [24]. The lattice structure of this compound corresponds to the 1/5-depleted square lattice. In this lattice, 1/5 of the original lattice sites of the square lattice are missing (Fig. 1).

Lastly, mention should be made of the fact that some of the high-$T_c$ cuprate systems also exhibit spin gap. A good example is the bilayer yttrium-barium-copper-oxide compound [25]. Detailed discussion of the nature of the spin gap in the cuprate systems is, however, beyond the scope of this review.

III. Exactly-solvable models with spin gap

The first example that comes to mind is the celebrated Majumdar-Ghosh (MG) chain [26]. The $S=1/2$ AFM Hamiltonian includes both n.n. and n.n.n. interactions of strength $J$ and $J/2$ respectively.

$$H_{\text{MG}} = J \sum_i \vec{S}_i \cdot \vec{S}_{i+1} + \frac{J}{2} \sum_i \vec{S}_i \cdot \vec{S}_{i+2}$$

(7)

The first term in the Hamiltonian is the usual Heisenberg Hamiltonian for which the ground state energy can be calculated exactly using the Bethe Ansatz [27]. The ground state is non-degenerate and disordered. The structure of the ground state is not known explicitly enough so that calculation of correlation functions is not possible. For the MG Hamiltonian, the ground state, with periodic boundary conditions, is doubly degenerate and has a simple structure

$$\phi_1 = [12][34]...[N-1N]$$

$$\phi_2 = [23][45]...[N1]$$

(8)
where \([\text{lm}]\) is the singlet \((S=0)\) with spin configuration \(\frac{1}{\sqrt{2}} (\alpha(l) \beta(m) - \beta(l) \alpha(m))\), \(\alpha, \beta\) being the spin-up and down spins and \(l, m\) are the lattice sites. The ground state energy \(E_g = -\frac{3NJ}{8}\) where \(N\) is the number of spins in the chain. The ground state has total spin \(S = 0\). Translational symmetry is broken in the ground state and the two-spin correlation function has an exponential decay, i.e., there is no conventional LRO.

The MG chain with simple ground states has motivated a large number of studies of AFM models with similar ground states \([28, 29, 30]\). In all these models the proof of exact ground state is obtained using the method of ‘divide and conquer’ \([31]\). Suppose one is able to construct an exact eigenstate of a spin Hamiltonian with energy \(E_1\). Let \(E_g\) and \(\Psi_g\) be the exact ground state energy and wave function. Then

\[
E_g \leq E_1
\]  

(9)

The Hamiltonian \(H\) is divided into sub-Hamiltonians \(H_i\)’s for which the ground state energy \(E_{ig}\) can be determined exactly. Then from the variational theorem

\[
E_g = \langle \Psi_g | H | \Psi_g \rangle = \sum_i \langle \Psi_g | H_i | \Psi_g \rangle \geq \sum_i E_{ig}
\]  

(10)

Thus from (9) and (10) one obtains the relation

\[
\sum_i E_{ig} \leq E_g \leq E_1
\]  

(11)

If one can show that \(E_1 = \sum_i E_{ig}\), then the exact eigenstate is also the exact ground state. For the MG chain, the states \(\phi_1\) and \(\phi_2\) can be shown to be the exact ground states by the use of the spin identity

\[
S_n \cdot (S_1 + S_m)[lm] \equiv 0
\]  

(12)

The energy of the exact eigenstate is \(E_1 = -\frac{3JN}{8}\). The Hamiltonian \(H_{MG}\) is divided into cluster sub-Hamiltonians \(H_i\)’s, each \(H_i\) describing a triplet of successive spins \((123, 234, 345, \ldots\) etc.). Each spin in the three-spin cluster (a triangle) interacts with the other two spins with the strength \(J/2\). The ground state energy \(E_{ig}\) is \(-\frac{3J}{8}\) corresponding to a singlet along one of the bonds in the triangle. Since, there are \(N\) sub-Hamiltonians, \(\sum_i E_{ig} = -\frac{3JN}{8}\).
which is equal to the energy $E_1$ of the exact eigenstate. Thus this state is also
the exact ground state. In adding up the sub-Hamiltonians, the n.n. bonds
are counted twice and so have the strength $J$ in the total Hamiltonian. The
n.n.n. bond which is counted once has strength $\frac{J}{2}$.

Recently, there has been a large number of studies on frustrated quantum
AFMs in 2d which includes both n.n. as well as further-neighbour interac-
tions [32]. The frustrated models are described as $J_1 - J_2$ and $J_1 - J_2 - J_3$
models where $J_1$, $J_2$ and $J_3$ are the strengths of the n.n., diagonal (d) and
n.n.n. exchange interactions. The ground states of these models in certain
parameter regimes are expected to be spin-disordered. The ground states,
though lacking in conventional LRO, can be characterised by novel order
parameters. Four candidate ground states that have been proposed are [33]
chiral, twisted, strip or collinear and columnar dimer (CD) states. In the
fourth type of state, the ground state consists of dimers (singlets) arranged
in columns. For the square lattice, four such states are possible. Bose and
Mitra [34] have constructed the $S = 1/2$ AFM $J_1 - J_2 - J_3 - J_4 - J_5$ model
on the square lattice for which the CD states are the exact eigenstates when

$$J_1 : J_2 : J_3 : J_4 : J_5 = 1 : 1 : \frac{1}{2} : \frac{1}{2} : \frac{1}{4}$$

(13)

The proof of exact eigenstate can be obtained using the spin-identity (12).
The CD states are presumably also the exact ground states. The ‘divide and
conquer’ proof, based on three-spin sub-Hamiltonians, works only for the
case when $J_1 = 7J$ and all the other interaction strengths are in the ratio
given in (13). Also, only one of the CD states for which the dimer bonds
have the strength $7J$ is the exact ground state. Mean-field theory based on
the bond-operator formalism shows [35] that the CD states are the ground
states when the ratio of interaction strengths satisfies (13).

Bose and Gayen [36] have constructed a spin-ladder model (Fig.2) which
includes diagonal interactions besides n.n. intra-chain and rung exchange
interactions. The rung exchange interaction has strength $J'$ and the other
exchange interactions are of equal strength $J$. For $J' \geq 2J$, the exact ground
state consists of singlets along the rungs. This is in conformity with the
approximate ground state of the usual spin-ladder which does not have diag-
onal exchange interactions. Inclusion of these interactions makes it possible
to determine the exact ground state. An excitation is created in the model
ladder system by replacing one of the singlets by a triplet. The triplet excita-
tion is localized and separated by an energy gap from the ground state. The localized triplet excitation is an artefact of the special interaction strengths assumed in the spin-ladder model. Ghosh and Bose [37] have generalized the two-chain ladder system of Bose and Gayen to a n-chain ladder system. In this system, alternate two-chain ladders have diagonal interactions. The chain of rungs in the vertical direction has both n.n. as well as n.n.n. interactions. The n.n.n. interactions have finite and zero strengths in an alternate manner. The exact ground state can be determined for n both odd and even. It can also be rigorously shown that for n odd (even) the excitation spectrum is gapless (with gap). This is in conformity with the experimental results on the ladder system $Sr_{n-1}Cu_{n+1}O_{2n}$ mentioned in Section II.

Recently, Bose and Ghosh [38] have considered the $S = 1/2$ AFM Heisenberg Hamiltonian on the 1/5-depleted lattice which describes the compound $CaV_4O_9$ mentioned in Section II. They have constructed a model Hamiltonian which includes both n.n. as well as further-neighbour interactions. For a particular ratio of interaction strengths, the plaquette resonating-valence-bond (PRVB) state has been shown to be the exact ground state. Various studies [39, 40, 41] have indicated that the PRVB state might be the ground state of $CaV_4O_9$. The experimentally-observed spin gap in $CaV_4O_9$ can be naturally linked to the PRVB state. The model proposed by Bose and Ghosh shows that the PRVB state is an exact ground state. The ground state is unique with total spin $S = 0$ and does not break any lattice symmetry. The PRVB state consists of a RVB-type spin configuration in each plaquette (marked ‘A’ in Fig.1) of the 1/5-depleted lattice. The RVB state is a linear superposition of two dimer states. In one dimer state, the spin singlets (dimers or valence bonds) form along the horizontal bonds. In the other dimer state, the singlets are along the vertical bonds.

We have so far been discussing spin models with spin $S = 1/2$. For a spin-1 chain, Affleck et al [42] proposed a model, known as the AKLT model, for which the ground state can be determined exactly. It can further be rigorously shown that the excitation spectrum has a gap, thus verifying Haldane’s conjecture. The ground state is unique with no broken translational symmetry in contrast to the MG chain. To construct the ground state, regard each spin 1 as a pair of spin-$1/2$’s. Couple all the n.n. spin-$1/2$’s into singlets. This state does not have $S = 1$ at each site but this can be remedied by symmetrizing the wave function at each site. In the final state, each n.n. bond has no longer spin $S = 0$ but it does have the property that there is no $S =$
2 component. The Hamiltonian $H_{AKLT}$ is written as a sum over projections onto spin 2 ($P_2$) of successive pairs of spins, i.e.,

\[ H_{AKLT} = \sum_i P_2 (\vec{S}_i + \vec{S}_{i+1}) \] (14)

\[ = \sum_i \left( \frac{1}{2} \vec{S}_i \cdot \vec{S}_{i+1} + \frac{1}{6} (\vec{S}_i \cdot \vec{S}_{i+1})^2 + \frac{1}{3} \right) \] (15)

$H_{AKLT}$ acting on the state described above gives the value zero as there is no $S = 2$ component for each n.n. pair of spins. Since the projection operator has zero or positive eigenvalue, the state considered is the exact ground state with eigenvalue zero. Extension of the S=1 AKLT model to general spin $S$ and to higher dimensional lattices is possible [43].

IV. Exact results for doped spin ladders

The study of doped AFMs has acquired added significance in connection with high-$T_c$ cuprate systems. The cuprates in their undoped state are AFMs and Mott insulators. On doping with a few percent of holes the AFM LRO is rapidly destroyed giving rise to a spin-disordered state. The study of frustrated AFMs with spin-disordered state as ground states is of relevance in this context. The cuprates have a rich phase diagram as the temperature and dopant concentrations are varied. The holes predominantly move in a background of antiferromagnetically interacting spins residing in the copper-oxide planes, a common structural ingredient of all the cuprate systems. The phase diagram exhibits, besides the insulating phase, metallic as well as superconducting (SC) phases. In the SC phase, charge transport occurs through the motion of bound pairs of holes. A large number of studies exists to understand the dynamics of holes in an antiferromagnetically interacting spin background. As mentioned in Section III, Bose and Gayen [44] have constructed a spin-ladder model (Fig.2) for which the exact ground state consists of singlets along the rungs. The dynamics of a single hole and a pair of holes have been studied in this model [36, 44, 45] and several exact results have been obtained. In the following, we give a brief description of these results. For a general discussion of hole dynamics in the usual spin-ladder model (ladder model not containing diagonal interactions), one should refer to [13, 46].
The doped spin ladder model is described by the t-J Hamiltonian given by

\[
H_{t-t'-J} = - \sum_{i,j,\sigma} t_{ij} C_i^\dagger C_j + h.c. + \sum_{<ij>} J_{ij} \vec{S}_i \cdot \vec{S}_j
\]

\[= H_t + H_{t'} + H_J \] (16)

The constraint that no site can be doubly occupied is implied in the model. The hopping integral \(t_{ij}\) has value \(t\) for n.n. hopping within a chain and also for diagonal transfer between chains (solid lines in Fig.2). The corresponding spin-spin interactions, \(J_{ij}\), are of strength \(J\). The spins have magnitude \(\frac{1}{2}\). The hopping integral across vertical links (dotted lines) connecting two chains has strength \(t'\). The corresponding spin-spin interaction strength \(J_{ij}\) is assumed to be \(2J\) though the exact results derived below hold true also for other interaction strengths. In the following, we assume \(t\) and \(t'\) to be positive. In the half-filled limit, i.e., in the absence of holes, the \(t - t' - J\) Hamiltonian in (16) reduces to \(H_J\). The exact ground state \(\Psi_g\) of \(H_J\) consists of singlets along the vertical bonds with energy \(E_g = -\left(\frac{3J^2}{2}\right)N\), where \(2N\) is the number of sites in the system. For \(J' < 2J\), the exact ground state is still the same; however, for \(J' > 2J\), the state, though an exact eigenstate, may not be the ground state. We now introduce a single hole into the system. Let \(\Psi(m)\) denote a spin configuration when the single hole is located in the \(m\)th column (rung) of the ladder model.

\[
\Psi(m) = \frac{1}{\sqrt{2}} \left( \Psi_m(p) + \Psi_m(q) \right)
\] (17)

In \(\Psi_m(p)\) and \(\Psi_m(q)\), the hole is located in the top and bottom rows, respectively, on the \(m\)th column. The other site in the \(m\)th column is occupied by an up spin. The spin configurations on all the other vertical links are the same as in \(\Psi_g\), namely, singlets. The wave function

\[
\Psi = \frac{1}{\sqrt{N}} \sum_{m=1}^{N} e^{ikm} \Psi(m)
\] (18)

is an exact eigenfunction of the total t-J Hamiltonian \(H\) with eigenvalue

\[
E = 2t \cos(k) - t' - \frac{3J}{2} (N - 1)
\] (19)
Eqn.(17) describes the bonding combination of hole states in a rung (momentum wave vector $k_y = 0$). The anti-bonding hole state ($k_y = \pi$) can be constructed as

$$
\Psi'(m) = \frac{1}{\sqrt{2}} (\Psi_m(q) - \Psi_m(p))
$$

When $H_{t-t'-J}$ in (16) operates on $\Psi'(m)$, the hole accompanied by a free spin-$\frac{1}{2}$ moves one lattice constant leaving behind a triplet excitation in column $m$. On further operating with the $t - t' - J$ Hamiltonian, a closed subspace of states is generated in each of which the triplet excitation is localized in the $m$th column and the hole quasi-particle (hole + up-spin) moves one lattice constant to the left or right. One can write down a set of exact eigenvalue equations for the propagating hole.

For $J = 0$, the eigenvalue equations are similar to those for a single hopping electron in a 1d chain of atoms with the atom number ‘zero’ being an impurity atom. The other atoms are located at positions 1, 2, 3,...and -1, -2, -3,... The electron can hop from one atom to its nearest neighbours with amplitude $t$. The site energy of the impurity atom is different from that of the other atoms. The problem has been extensively discussed in the Feynman Lectures, vol. III [47] and provides physical insight for our eigenvalue problem. In the case of the ladder model, the localized triplet excitation is the ‘impurity’ atom, the hole accompanied by a free spin-$\frac{1}{2}$ constitutes the propagating object and the singlets along the vertical links are the ‘other atoms’ of the lattice.

The eigenvalue equations can also be solved exactly for the case $J \neq 0$. The hole quasi-particle (QP) can be in a scattering state or form bound and anti-bound (localized but with energy greater than that of the scattering state) states with the triplet excitation. For $0 \leq J/t < 0.05$, the bound state energy is less than the lowest energy corresponding to the bonding band (Eq.(19)). For $J/t \geq 0.05$, the bonding band has a lower energy. Thus there is a localization-to-delocalization transition. The propagating hole has a QP character with charge + e and spin $S = \frac{1}{2}$. There is thus no spin-charge separation, a hallmark of the interacting electron systems in 1d, the so-called Luttinger liquids (LLs). Strongly correlated systems in dimension $d \approx 1$ have been conjectured to be LLs. The doped spin ladder, though strongly correlated, is not a LL but has properties similar to those of the Luther-Emery model with gapless charge excitations and a spin gap [46].

We next turn to the case of the spin-ladder doped with a pair of holes.
A brief description of the exact results obtained in [40] is given (for details refer to the original paper). The holes are introduced in the ground state of the model in two different vertical links so that the dimers along the links are broken. The two free spins from the broken dimers combine to make the total spin of the system either $S = 1$, i.e., a triplet, or $S = 0$, i.e., a singlet. Consider first the case $S = 1$ with $S^z = +1$. The states $S = 1, S^z = 0,-1$ are degenerate with the $S^z = +1$ state. Let the holes be located in the columns denoted by $m_1$ and $m_2$ respectively, where $m_1 < m_2$. The eigenfunction $\psi$ of the $t-t'-J$ Hamiltonian is given by

$$\psi = \sum_{m_1 < m_2} a(m_1, m_2) \psi(m_1, m_2)$$

(21)

where the basis function $\psi(m_1, m_2)$ is given by

$$\psi(m_1, m_2) = \frac{1}{2} [\cdots |_{(m_1-1)} \left( O + \uparrow \right)_{m_1} |_{(m_1+1)} \cdots | \left( O + \uparrow \right)_{m_2} | \cdots].$$

(22)

The solid vertical lines represent singlets, the arrows stand for up-spins and the open circles denote holes. We have to solve the eigenvalue problem $H_t-t'-J \psi = E \psi$. The state $\psi(m_1, m_2)$ belongs to a closed subspace of states within which $H_t-t'-J$ operates. The subspace does not contain the state in which the two holes are located in the same column. This fact reduces the ladder problem basically to a 1d one so that the exact Bethe Ansatz [27] technique can be applied. For the general situation of $r$ holes, the eigenfunction is a linear combination of the $N \cdot C_r$ functions $\psi(m_1, \cdots, m_r)$:

$$\psi = \sum_{m} a(m_1, m_2, \cdots, m_r) \psi(m_1, m_2, \cdots, m_r)$$

(23)

Each of the numbers $m_1, \ldots, m_r$ runs over the possible values 1 to $N$ subject to the condition $m_1 < m_2 < \ldots < m_r$. The general BA for the $r$-hole state can be written as

$$a(m_1, m_2, \cdots, m_r) = \sum_P \exp \left[ i \left( \sum_{l=1}^{r} k_{P_l} m_l + 1/2 \sum_{l<n}^{1,r} \phi_{P_l, P_n} \right) \right]$$

(24)

$P$ is any permutation of $r$ numbers 1,2,\ldots,$r$. $P_l$ is the number obtained by operating $P$ on $l$. The wave vectors $k_l$’s (r in number) are determined by applying the periodic boundary condition which leads to r equations.
The phase shifts are determined by demanding that the same BA (Eq. (24)) satisfies the amplitude equations when holes occupy n.n. links. This leads to $\frac{r(r-1)}{2}$ equations for the same number of phase shifts $\phi'$ s ($\phi_{ij} = -\phi_{ji}$). Thus there are $\frac{r(r+1)}{2}$ equations in as many unknowns which can be solved in an appropriate manner. The eigenvalues finally obtained correspond to scattering states as well as anti-bound states of holes.

We next consider the situation when the free spins from the broken dimers (after the introduction of two holes in the ground state) form a singlet with total spin $S = 0$. In this case, the reduction of the ladder problem to a 1d situation is not possible as the subspace of states now includes the state in which two holes occupy the same vertical link, leading to the possibility of exchange of holes. The BA technique, unique to 1d systems, can now no longer be applied. The subspace of states includes states of the type

\[
\phi(m_1, m_2) = \frac{1}{2\sqrt{2}} \| \cdots | \left( \begin{array}{c} \uparrow \\downarrow \end{array} \right)^{m_1} | \cdots | \left( \begin{array}{c} \uparrow \\downarrow \end{array} \right)^{m_2} | \cdots | \left( \begin{array}{c} \uparrow \\downarrow \end{array} \right)^{m_1} | \cdots | \left( \begin{array}{c} \uparrow \\downarrow \end{array} \right)^{m_2} | \cdots \|
\]

and

\[
\phi(m_1, m_2) = \| \cdots \begin{array}{c} \uparrow \\downarrow \end{array}^{m_1} | \cdots \begin{array}{c} \uparrow \\downarrow \end{array}^{m_2} | \cdots \|
\]

Exact eigenvalue equations can be written down in this case and solved exactly and analytically. The solutions include scattering states of two holes, bound as well as anti-bound states. The method of solution cannot be extended to the case of more than two holes. The rigorous demonstration of the binding of two holes is of significance in the context of superconductivity both in the ladder as well as high-$T_c$ cuprate systems. The exact results obtained also show that the spin gap of magnitude $\frac{3J}{2}$ is reduced on doping with holes (see Eq. (19)). Exact results are also available [18] for the case when the spin-ladder model considered here is doped with a single magnetic or non-magnetic impurity. The spin-ladder model under discussion yields a set of exact results for the dynamics of holes, which are in conformity with the results for conventional spin ladders, based on approximate calculations or exact diagonalization of small systems [13, 46]. In the calculations described, quantum fluctuations and the constraint of no-double-occupancy have been explicitly and exactly taken into account. Thus the results obtained are characteristic of both strong correlation and quantum effects. The possibility of
superconductivity in a ladder system with spin gap has been experimentally realized as mentioned in Section II. The crucial question which remains to be settled is whether the spin gap disappears in the SC phase or survives even for temperature $T < T_c$. Experimentally, a reduction of the spin gap with increasing dopant concentration has been observed.

In summary, we have described in this review varied AFM systems which exhibit spin gaps. Some exactly-solvable models with spin gaps have been determined. Exact results for the hole dynamics in a ladder model with spin gap have been discussed. The review is by no means complete and should be supplemented by the references mentioned. In the last few years, new AFM systems with spin gaps have been discovered. These systems exhibit a variety of phenomena like the spin-Peierls transition, metal-insulator transition and superconductivity, a full understanding of which is as yet lacking. We may expect that in the coming years new AFM systems with spin gap will be discovered and a lot of insight gained about such systems.
Figure Captions

**Fig. 1** The 1/5-depleted lattice of $CaV_4O_9$. A and B represent four-spin plaquettes and dimer bonds connecting plaquettes, respectively.

**Fig. 2** The spin ladder model described by the $t-t'-J$ Hamiltonian (Eqn. 16).
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Fig. 1
Fig. 2