Homogenization and influence of fragmentation in a biological invasion model

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Dedicated to Professor Masayasu Mimura for his 65th birthday

Abstract

In this paper, some properties of the minimal speeds of pulsating Fisher-KPP fronts in periodic environments are established. The limit of the speeds at the homogenization limit is proved rigorously. Near this limit, generically, the fronts move faster when the spatial period is enlarged, but the speeds vary only at the second order. The dependence of the speeds on habitat fragmentation is also analyzed in the case of the patch model.

1 Introduction and main hypotheses

In homogeneous environments, the probably most used population dynamics reaction-diffusion model is the Fisher-KPP model \cite{13, 23}. In a one-dimensional space, it corresponds to the following equation

\begin{equation}
\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2} + u \left( \mu - \nu u \right), \quad t > 0, \quad x \in \mathbb{R}.
\end{equation}

The unknown $u = u(t, x)$ is the population density at time $t$ and position $x$, and the positive constant coefficients $D$, $\mu$ and $\nu$ respectively correspond to the diffusivity (mobility of the individuals), the intrinsic growth rate and the susceptibility to crowding effects.

A natural extension of this model to heterogeneous environments is the Shigesada-Kawasaki-Teramoto model \cite{32},

\begin{equation}
\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left( a_L(x) \frac{\partial u}{\partial x} \right) + u \left( \mu_L(x) - \nu_L(x) u \right), \quad t > 0, \quad x \in \mathbb{R},
\end{equation}

where the coefficients depend on the space variable $x$ in a $L$-periodic fashion:
Definition 1.1 (L-periodicity) Let $L$ be a positive real number. We say that a function $h : \mathbb{R} \to \mathbb{R}$ is $L$-periodic if
\[ \forall x \in \mathbb{R}, \; h(x + L) = h(x). \]

In this paper, we are concerned with the general equation:
\[ \frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left( a_L(x) \frac{\partial u}{\partial x} \right) + f_L(x, u), \quad t \in \mathbb{R}, \; x \in \mathbb{R}. \tag{1.3} \]
The diffusion term $a_L$ satisfies
\[ a_L(x) = a(x/L), \]
where $a$ is a $C^{2,\delta}(\mathbb{R})$ (with $\delta > 0$) $1$-periodic function that satisfies
\[ \exists \; 0 < \alpha_1 < \alpha_2, \; \forall \; x \in \mathbb{R}, \; \alpha_1 \leq a(x) \leq \alpha_2. \tag{1.4} \]
On the other hand, the reaction term satisfies $f_L(x, \cdot) = f(x/L, \cdot)$, where $f := f(x, s) : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ is $1$-periodic in $x$, of class $C^{1,\delta}$ in $(x, s)$ and $C^2$ in $s$. In this setting, both $a_L$ and $f_L$ are $L$-periodic in the variable $x$. Furthermore, we assume that:
\[ \begin{align*}
&\forall \; x \in \mathbb{R}, \; f(x, 0) = 0, \\
&\exists \; M \geq 0, \; \forall \; s \geq M, \; \forall \; x \in \mathbb{R}, \; f(x, s) \leq 0, \\
&\forall \; x \in \mathbb{R}, \; s \mapsto f(x, s)/s \text{ is decreasing in } s > 0. \tag{1.5}
\end{align*} \]
Moreover, we set
\[ \mu(x) := \lim_{s \to 0^+} f(x, s)/s, \]
and
\[ \mu_L(x) := \lim_{s \to 0^+} f_L(x, s)/s = \mu \left( \frac{x}{L} \right). \]
The growth rate $\mu$ may be positive in some regions (favorable regions) or negative in others (unfavorable regions).

The stationary states $p(x)$ of (1.3) satisfy the equation
\[ \frac{\partial}{\partial x} \left( a_L(x) \frac{\partial p}{\partial x} \right) + f_L(x, p) = 0, \; x \in \mathbb{R}. \tag{1.6} \]
Under general hypotheses including those of this paper, and in any space dimension, it was proved in [4] that a necessary and sufficient condition for the existence of a positive and bounded solution $p$ of (1.6) was the negativity of the principal eigenvalue $\rho_{1,L}$ of the linear operator
\[ \mathcal{L}_0 : \Phi \mapsto -(a_L(x)\Phi')' - \mu_L(x)\Phi, \tag{1.7} \]
with periodicity conditions. In this case, the solution $p$ was also proved to be unique, and therefore $L$-periodic. Actually, it is easy to see that the map $L \mapsto \rho_{1,L}$ is nonincreasing in $L > 0$, and even decreasing as soon as $a$ is not constant (see the proof of Lemma 3.1).
Furthermore, $\rho_{1,L} \to -\int_0^1 \mu(x)dx$ as $L \to 0^+$. In this paper, in addition to the above-mentioned hypotheses, we make the assumption that

$$\int_0^1 \mu(x)dx > 0.$$  \hspace{1cm} (1.8)

This assumption then guarantees that

$$\forall L > 0, \quad \rho_{1,L} < 0,$$

whence, for all $L > 0$, there exists a unique positive periodic and bounded solution $p_L$ of (1.6). Notice that assumption (1.8) is immediately fulfilled if $\mu(x)$ is positive everywhere.

In this work, we are concerned with the propagation of pulsating traveling fronts which are particular solutions of the reaction-diffusion equation (1.3). Before going further on, we recall the definition of such solutions:

**Definition 1.2 (Pulsating traveling fronts)** A function $u = u(t, x)$ is called a pulsating traveling front propagating from right to left with an effective speed $c \neq 0$, if $u$ is a classical solution of:

$$\begin{cases}
\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left( a_L(x) \frac{\partial u}{\partial x} \right) + f_L(x, u), & t \in \mathbb{R}, \ x \in \mathbb{R}, \\
\forall k \in \mathbb{Z}, \ \forall (t, x) \in \mathbb{R} \times \mathbb{R}, \quad u(t, x + kL) = u(t, x + kL), \\
0 \leq u(t, x) \leq p_L(x), \\
\lim_{x \to -\infty} u(t, x) = 0 \text{ and } \lim_{x \to +\infty} u(t, x) - p_L(x) = 0,
\end{cases}$$

where the above limits hold locally in $t$.

This definition has been introduced in [31, 32]. It has also been extended in higher dimensions with $p_L \equiv 1$ in [1] and [35], and with $p_L \neq 1$ in [5].

Under the above assumptions, it follows from [5] that there exists $c_L^* > 0$ such that pulsating traveling fronts satisfying (1.9) with a speed of propagation $c$ exist if and only if $c \geq c_L^*$. Moreover, the pulsating fronts (with speeds $c \geq c_L^*$) are increasing in time $t$. Further uniqueness and qualitative properties are proved in [14, 15]. The value $c_L^*$ is called the minimal speed of propagation. We refer to [2, 3, 11, 18, 25, 27, 28, 34] for further existence results and properties of the minimal speeds of KPP pulsating fronts. For existence, uniqueness, stability and further qualitative results for combustion or bistable nonlinearities in the periodic framework, we refer to [6, 7, 12, 16, 17, 19, 24, 26, 35, 36, 37, 38].

In the particular case of the Shigesada et al model (1.2), when $a(x) \equiv 1$, the effects of the spatial distribution of the function $\mu_L$ on the existence and global stability of a positive stationary state $p_L$ of equation (1.2) have been investigated both numerically [30, 31] and theoretically [4, 8, 29]. In particular, as already noticed, enlarging the scale of fragmentation, i.e. increasing $L$, was proved to decrease the value of $\rho_{1,L}$. Biologically, this result means
that larger scales have a positive effect on species persistence, for species whose dynamics is modelled by the Shigesada et al model.

The effects of the spatial distribution of the functions $a_L$ and $\mu_L$ on the minimal speed of propagation $c_L^*$ have not yet been investigated rigorously. This is a difficult problem, since the known variational formula for $c_L^*$ bears on non-self-adjoint operators, and therefore, the methods used to analyze the dependence of $\rho_{1,L}$ on fragmentation cannot be used in this situation. However, in the case of model (1.2), when $a_L \equiv 1$, $\nu_L \equiv 1$ and $\mu_L(x) = \mu(x/L)$, for a 1-periodic function $\mu$ taking only two values, Kinezaki et al [22] numerically observed that $c_L^*$ was an increasing function of the parameter $L$. For sinusoidally varying coefficients, the relationships between $c_L^*$ and $L$ have also been investigated formally by Kinezaki, Kawasaki, Shigesada [21]. The case of a rapidly oscillating coefficient $a_L(x)$, corresponding to small $L$ values, and the homogenization limit $L \to 0$, have been discussed in [19] and [38] for combustion and bistable nonlinearities $f(u)$.

The first aim of our work is to analyze rigorously the dependence of the speed of propagation $c_L^*$ with respect to $L$, under the general setting of equation (1.3), for small $L$ values. We determine the limit of the minimal speeds $c_L^*$ as $L \to 0^+$ (the homogenization limit), and we also prove that near the homogenization limit, the species tends to propagate faster when the spatial period of the environment is enlarged. Next, in the case of an environment composed of patches of “habitat” and “non-habitat”, we consider the dependence of the minimal speed with respect to habitat fragmentation. We prove that fragmentation decreases the minimal speed.

2 Main results

In this section, we describe the main results of this paper. Unless otherwise mentioned, we make the assumptions of Section 1. The first theorem gives the limit of $c_L^*$ as $L$ goes to 0.

**Theorem 2.1** Let $c_L^*$ be the minimal speed of propagation of pulsating traveling fronts solving (1.9). Then,

$$\lim_{L \to 0^+} c_L^* = 2\sqrt{<a>_H <\mu>_A},$$

(2.1)

where

$$<\mu>_A = \int_0^1 \mu(x)dx \text{ and } <a>_H = \left(\int_0^1 (a(x))^{-1}dx\right)^{-1} = <a^{-1}>^{-1}_A$$

denote the arithmetic mean of $\mu$ and the harmonic mean of $a$ over the interval $[0, 1]$.

Formula (2.1) was derived formally in [33] for sinusoidally varying coefficients. Theorem 2.1 then provides a generalization of the formula in [33] and a rigorous analysis of the homogenization limit for general diffusion and growth rate profiles.

**Remark 2.2** The previous theorem gives the limit of $c_L^*$ as $L \to 0$ when the space dimension is 1. Theorem 3.3 of El Smaily [11] answered this issue in any dimensions $N$, but under an additional assumption of free divergence of the diffusion field (in the one-dimensional case considered here, this assumption reduces to $da/dx = 0$ in $\mathbb{R}$). Lastly, we refer to [6, 7, 16] for other homogenization limits with combustion-type nonlinearities.
Our second result describes the behavior of the function $L \mapsto c^*_L$, for small $L$ values.

**Theorem 2.3** Let $c^*_L$ be the minimal speed of propagation of pulsating traveling fronts solving (1.9). Then, the map $L \mapsto c^*_L$ is of class $C^\infty$ in an interval $(0, L_0)$ for some $L_0 > 0$. Furthermore,

$$\lim_{L \to 0^+} \frac{dc^*_L}{dL} = 0$$

(2.2)

and

$$\lim_{L \to 0^+} \frac{d^2c^*_L}{dL^2} = \gamma \geq 0.$$

(2.3)

Lastly, $\gamma > 0$ if and only if the function

$$\frac{\mu}{<\mu>_A} + \frac{<a>_H}{a}$$

is not identically equal to 2.

**Corollary 2.4** Under the notations of Theorem 2.3, it follows that if $a$ is constant and $\mu$ is not constant, or if $\mu$ is constant and $a$ is not constant, then $\gamma > 0$ and the speeds $c^*_L$ are increasing with respect to $L$ when $L$ is close to 0.

**Remark 2.5** The question of the monotonicity of the map $L \mapsto c^*_L$ had also been studied under different assumptions in [11] (see Theorem 5.3). The author answered this question for a reaction-advection-diffusion equation over a periodic domain $\Omega \subseteq \mathbb{R}^N$, under an additional assumption on the diffusion coefficient (like in Remark 2.2, this assumption would mean again in our present setting that the diffusion coefficient $a(x)$ is constant over $\mathbb{R}$). Our result gives the behavior of the minimal speeds of propagation near the homogenization limit for general diffusion and growth rate coefficients. The condition $\gamma > 0$ is generically fulfilled, which means that, roughly speaking, the more oscillating the medium is, the slower the species moves. But the speeds vary only at the second order with respect to the period $L$. Based on numerical observations which have been carried out in [21] for special types of diffusion and growth rate coefficients, we conjecture that the monotonicity of $c^*_L$ holds for all $L > 0$.

Lastly, we give a first theoretical evidence that habitat fragmentation, without changing the scale $L$, can decrease the minimal speed $c^*$. We here fix a period $L_0 > 0$.

We assume that $a \equiv 1$, and that $\mu_{L_0} := \mu_z$ takes only the two values 0 and $m > 0$, and depends on a parameter $z$. More precisely:

$$\mu_z \equiv m \text{ on } [0, l/2) \cup [l/2 + z, l + z),$$

$$\mu_z \equiv 0 \text{ on } [l/2, l/2 + z) \cup [l + z, L_0].$$

(2.4)

With this setting, the region where $\mu_z$ is positive, which can be interpreted as “habitat” in the Shigesada et al model, is of Lebesgue measure $l$ in each period cell $[0, L_0]$. For $z = 0$, this region is simply an interval. However, whenever $z$ is positive, this region is fragmented into two parts of same length $l/2$ (see Figure 1). Our next result means that this fragmentation into two parts reduces the speed $c^*$. 5
Theorem 2.6 Let $c_z^*$ be the minimal speed of propagation of pulsating traveling fronts solving (1.9), with $a_{L_0} \equiv 1$ and $\mu_{L_0} = \mu_z$ defined by (2.4). Assume that $l \in (3L_0/4, L_0)$. Then $z \mapsto c_z^*$ is decreasing in $[0, (L_0 - l)/2]$, and increasing in $[(L_0 - l)/2, L_0 - l]$.

Remark 2.7 Note that, whenever $z > (L_0 - l)/2$, the two habitat components in the period cell $[l/2 + z, L_0 + l/2 + z]$ are at a distance smaller than $(L_0 - l)/2$ from each other. In fact, Theorem 2.6 proves that, when $z$ varies in $(0, L_0 - l)$, $c_z^*$ is all the larger as the minimal distance separating two habitat components is small, that is as the maximal distance between two consecutive habitat components is large.

Remark 2.8 Here, the function $\mu_z$ does not satisfy the general regularity assumptions of Section 1. However, $c_z^*$ can still be interpreted as the minimal speed of propagation of weak solutions of (1.9), whose existence can be obtained by approaching $\mu_z$ with regular functions.

The main tool of this paper is a variational formulation for $c_L^*$ involving elliptic eigenvalue problems which depend strongly on the coefficients $a$ and $f$. Such a formulation was given in any space dimension in [3] in the case where the bounded stationary state $p$ of the equation (1.3) is constant, and in [5] in the case of a general nonconstant bounded stationary state $p(x)$.

3 The homogenization limit: proof of Theorem 2.1

This proof is divided into three main steps.

Step 1: a rough upper bound for $c_L^*$. For each $L > 0$, the minimal speed $c_L^*$ is positive and, from [5] (see also [3] in the case when $p \equiv 1$), it is given by the variational formula

$$c_L^* = \min_{\lambda > 0} \frac{k(\lambda, L)}{\lambda} = \frac{k(\lambda_L^*, L)}{\lambda_L^*},$$

(3.1)

where $\lambda_L^* > 0$ and, for each $\lambda \in \mathbb{R}$ and $L > 0$, $k(\lambda, L)$ denotes the principal eigenvalue of the problem

$$(a_L \psi_{\lambda,L})' + 2\lambda a_L \psi_{\lambda,L} + \lambda a_L' \psi_{\lambda,L} + \lambda^2 a_L \psi_{\lambda,L} + \mu_L \psi_{\lambda,L} = k(\lambda, L) \psi_{\lambda,L} \text{ in } \mathbb{R},$$

(3.2)

with $L$-periodicity conditions. In (3.2), $\psi_{\lambda,L}$ denotes a principal eigenfunction, which is of class $C^{2,\delta}(\mathbb{R})$, positive, unique up to multiplication by a positive constant, and $L$-periodic.
Furthermore, it follows from Section 3 of [5] that the map $\lambda \mapsto k(\lambda, L)$ is convex and that $\frac{\partial k}{\partial \lambda}(0, L) = 0$ for each $L > 0$. Therefore, for each $L > 0$, the map $\lambda \mapsto k(\lambda, L)$ is nondecreasing in $\mathbb{R}_+$ and

$$\forall \lambda \geq 0, \forall L > 0, \quad k(\lambda, L) \geq k(0, L) = -\rho_{1,L} > 0$$

(3.3)

under the notations of Section 1.

Multiplying (3.2) by $\psi_{\lambda,L}$ and integrating by parts over $[0, L]$, we get, due to the $L$-periodicity of $a_L$ and $\psi_{\lambda,L}$:

$$k(\lambda, L) \int_0^L \psi_{\lambda,L}^2 = -\int_0^L a_L (\psi_{\lambda,L}')^2 + \lambda^2 \int_0^L a_L \psi_{\lambda,L}^2 + \int_0^L \mu_L \psi_{\lambda,L}^2,$$

for all $\lambda > 0$ and for all $L > 0$. Consequently,

$$\forall \lambda > 0, \forall L > 0, \quad k(\lambda, L) \leq \lambda^2 a_M + \mu_M,$$

(3.4)

where

$$a_M = \max_{x \in \mathbb{R}} a(x) > 0 \text{ and } \mu_M = \max_{x \in \mathbb{R}} \mu(x) > 0.$$

Using (3.1), we get that

$$\forall L > 0, \quad 0 < c_*^{*L} \leq 2 \sqrt{a_M \mu_M}. \quad (3.5)$$

Step 2: the sharp upper bound for $c_*^{*L}$. For any $\lambda > 0$ and $L > 0$, consider the functions

$$\varphi_{\lambda,L}(x) := e^{\lambda x} \psi_{\lambda,L}(x), \quad x \in \mathbb{R}.$$ 

Since $\psi_{\lambda,L}$ is unique up to multiplication, we will assume in this step 2 that

$$\int_0^2 \varphi_{\lambda,L}^2(x) dx = 1. \quad (3.6)$$

The above choice ensures that

$$\int_0^2 \psi_{\lambda,L}^2(x) dx \leq \int_0^2 e^{2\lambda x} \psi_{\lambda,L}^2(x) dx = \int_0^2 \varphi_{\lambda,L}^2(x) dx = 1. \quad (3.7)$$

We are now going to prove that the families $(\psi_{\lambda,L})_{\lambda,L}$ and $(\varphi_{\lambda,L})_{\lambda,L}$ remain bounded in $H^1(0, 1)$ for $L$ small enough and as soon as $\lambda$ stays bounded. For each $L > 0$, we call

$$M_L = \lfloor 1/L \rfloor + 1 \in \mathbb{N},$$

where $\lfloor 1/L \rfloor$ stands for the integer part of $1/L$. Multiplying (3.2) by $\psi_{\lambda,L}$ and integrating by parts over $[0, M_L L ]$, we get that

$$- \int_0^{M_L L} a_L \psi_{\lambda,L}'^2 + \int_0^{M_L L} \lambda^2 a_L \psi_{\lambda,L}^2 + \int_0^{M_L L} \mu_L \psi_{\lambda,L}^2 = k(\lambda, L) \int_0^{M_L L} \psi_{\lambda,L}^2.$$
Using (1.4), (3.3) and (3.4), it follows that
\[
0 \leq \int_0^{M_L L} \psi_{\lambda,L}'^2 \leq \frac{1}{\alpha_1} \times (\lambda^2 a_M + \mu_M) \times \int_0^{M_L L} \psi_{\lambda,L}^2.
\]
Since \(1 < M_L L \leq 1 + L\) for all \(L > 0\), we have that \(1 < M_L L \leq 2\) for all \(L \leq 1\). Thus, for all \(0 < L \leq 1\),
\[
\int_0^1 \psi_{\lambda,L}'^2 \leq \int_0^{M_L L} \psi_{\lambda,L}'^2 \text{ and } \int_0^{M_L L} \psi_{\lambda,L}^2 \leq \int_0^2 \psi_{\lambda,L}^2 \leq 1
\]
from (3.7). It follows now that
\[
\forall \lambda > 0, \forall 0 < L \leq 1, \quad \int_0^1 \psi_{\lambda,L}'^2 \leq \frac{\lambda^2 a_M + \mu_M}{\alpha_1}. \tag{3.8}
\]
From (3.7) and (3.8), we conclude that, for any given \(\Lambda > 0\), the family \((\psi_{\lambda,L})_{0 < \lambda \leq \Lambda, 0 < L \leq 1}\) is bounded in \(H^1(0, 1)\). On the other hand,
\[
\varphi'_{\lambda,L}(x) = \lambda \varphi_{\lambda,L}(x) + e^{\lambda x} \psi_{\lambda,L}'(x).
\]
Owing to (3.6) and (3.8), we get:
\[
\forall \lambda > 0, \forall L \leq 1, \quad ||\varphi'_{\lambda,L}||_{L^2(0,1)} \leq \lambda ||\varphi_{\lambda,L}||_{L^2(0,1)} + e^{\lambda} ||\psi_{\lambda,L}'||_{L^2(0,1)} \leq \lambda + e^{\lambda} \times \sqrt{\frac{\lambda^2 a_M + \mu_M}{\alpha_1}}. \tag{3.9}
\]
From (3.6) and (3.9), we obtain that, for any given \(\Lambda > 0\), the family \((\varphi_{\lambda,L})_{0 < \lambda \leq \Lambda, 0 < L \leq 1}\) is bounded in \(H^1(0, 1)\) and that the family \((a_L \varphi'_{\lambda,L})_{0 < \lambda \leq \Lambda, 0 < L \leq 1}\) is bounded in \(L^2(0, 1)\) (due to (1.4)). Moreover,
\[
(a_L \varphi'_{\lambda,L})' = \lambda^2 a_L e^{\lambda x} \varphi_{\lambda,L} + 2a_L \lambda e^{\lambda x} \psi_{\lambda,L} + \lambda a_L e^{\lambda x} \varphi_{\lambda,L} + e^{\lambda x} a_L \varphi'_{\lambda,L} + e^{\lambda x} a_L \psi_{\lambda,L}.
\]
Multiplying (3.2) by \(e^{\lambda x}\), we then get
\[
(a_L \varphi'_{\lambda,L})' + \mu_L \varphi_{\lambda,L} = k(\lambda, L) \varphi_{\lambda,L} \text{ in } \mathbb{R}. \tag{3.10}
\]
Let
\[
v_{\lambda,L}(x) = a_L(x) \varphi'_{\lambda,L}(x)
\]
for all \(\lambda > 0, \ L > 0\) and \(x \in \mathbb{R}\). Pick any \(\Lambda > 0\). One already knows that the family \((v_{\lambda,L})_{0 < \lambda \leq \Lambda, 0 < L \leq 1}\) is bounded in \(L^2(0, 1)\). Furthermore,
\[
v_{\lambda,L}' + \mu_L v_{\lambda,L} = k(\lambda, L) \varphi_{\lambda,L} \text{ in } \mathbb{R}. \tag{3.11}
\]
Notice that the family \((k(\lambda, L))_{0 < \lambda \leq \Lambda, 0 < L \leq 1}\) is bounded from (3.3) and (3.4). From (3.6) and (3.11), it follows that the family \((v_{\lambda,L}')_{0 < \lambda \leq \Lambda, 0 < L \leq 1}\) is bounded in \(L^2(0, 1)\). Eventually, \((v_{\lambda,L})_{0 < \lambda \leq \Lambda, 0 < L \leq 1}\) is bounded in \(H^1(0, 1)\).
Pick now any sequence \((L_n)_{n \in \mathbb{N}}\) such that \(0 < L_n \leq 1\) for all \(n \in \mathbb{N}\), and \(L_n \to 0^+\) as \(n \to +\infty\). Choose any \(\lambda > 0\) and any sequence \((\lambda_n)_{n \in \mathbb{N}}\) of positive numbers such that \(\lambda_n \to \lambda\) as \(n \to +\infty\). We claim that

\[
k(\lambda_n, L_n) \to \lambda^2 <a>_H + <\mu>_A \text{ as } n \to +\infty,
\]

where \(<a>_H = \left( \int_0^1 (a(x))^{-1} dx \right)\) and \(<\mu>_A = \int_0^1 \mu(x) dx\). To do so, call

\[
\psi_n = \psi_{\lambda_n, L_n}, \quad \varphi_n = \varphi_{\lambda_n, L_n} \text{ and } v_n = v_{\lambda_n, L_n}.
\]

It follows from the above computations that the sequences \((\psi_n)\) and \((v_n)\) are bounded in \(H^1(0, 1)\). Hence, up to extraction of a subsequence,

\[
\psi_n \to \overline{\psi} \text{ and } v_n \to w \text{ as } n \to +\infty,
\]

strongly in \(L^2(0, 1)\) and weakly in \(H^1(0, 1)\). By Sobolev injections, the sequence \((\psi_n)\) is bounded in \(C^{0,1/2}([0, 1])\). But since each function \(\psi_n\) is \(L_n\)-periodic (with \(L_n \to 0^+\)), it follows from Arzela-Ascoli theorem that \(\overline{\psi}\) has to be constant over \([0, 1]\). Moreover, the boundedness of the sequence \((k(\lambda_n, L_n))_{n \in \mathbb{N}}\) implies that, up to extraction of another subsequence,

\[
k(\lambda_n, L_n) \to \overline{k}(\lambda) \in \mathbb{R} \text{ as } n \to +\infty.
\]

We denote this limit by \(\overline{k}(\lambda)\), we will see later that indeed it depends only on \(\lambda\). It follows now, from (3.11) after replacing \((\lambda, L)\) by \((\lambda_n, L_n)\) and passing to the limit as \(n \to +\infty\), that

\[
w' + <\mu>_A e^{\lambda x} \overline{\psi} = \overline{k}(\lambda) \overline{\psi} e^{\lambda x} \text{ a.e. in } (0, 1).
\]

Notice indeed that \(\mu_L \to <\mu>_A\) as \(L \to 0^+\) in \(L^2(0, 1)\) weakly. Meanwhile,

\[
\varphi'_n = \lambda_n e^{\lambda x} \psi'_n + e^{\lambda x} \lambda_n \psi' = \frac{v_n}{a_{L_n}} \to <a^{-1}>_A \text{ in } (0, 1),
\]

where \(<a^{-1}>_A = \int_0^1 (a(x))^{-1} dx\). Thus, we obtain

\[
w = <a^{-1}>_A \lambda e^{\lambda x} \overline{\psi} = <a>_H \lambda e^{\lambda x} \overline{\psi}.
\]

Consequently,

\[
\lambda^2 <a>_H \overline{\psi} + <\mu>_A \overline{\psi} = \overline{k}(\lambda) \overline{\psi}.
\]

Actually, since the functions \(\psi_n\) are \(L_n\)-periodic (with \(L_n \to 0^+\)) and converge to the constant \(\overline{\psi}\) strongly in \(L^2(0, 1)\), they converge to \(\overline{\psi}\) in \(L^2_{loc}(\mathbb{R})\). But

\[
1 = \int_0^2 \varphi_n^2 \leq e^{4\lambda_n} \int_0^2 \psi_n^2 \leq e^{4M} \int_0^2 \psi_n^2,
\]

where \(M = \sup_{n \in \mathbb{N}} \lambda_n\). Hence, \(\overline{\psi} \neq 0\) and

\[
\lambda^2 <a>_H + <\mu>_A = \overline{k}(\lambda),
\]

(3.13)
By uniqueness of the limit, one deduces that the whole sequence \((k(\lambda_n, L_n))_{n \in \mathbb{N}}\) converges to this quantity \(\overline{k}(\lambda)\) as \(n \to +\infty\), which proves the claim (3.12).

Now, take any sequence \(L_n \to 0^+\) such that \(c^*_L \to \limsup_{L \to 0^+} c^*_L\) as \(n \to +\infty\). For each \(\lambda > 0\) and for each \(n \in \mathbb{N}\), one has

\[
c^*_L \leq \frac{k(\lambda, L_n)}{\lambda}
\]

from (3.1), whence

\[
\limsup_{L \to 0^+} c^*_L = \lim_{n \to +\infty} c^*_L \leq \frac{\overline{k}(\lambda)}{\lambda} = \lambda < a_H + \frac{\langle \mu \rangle_A}{\lambda}.
\]

Since this holds for all \(\lambda > 0\), one concludes that

\[
\limsup_{L \to 0^+} c^*_L \leq 2\sqrt{\langle a_H \rangle A H < \mu > A}.
\] (3.14)

Step 3: the sharp lower bound for \(c^*_L\). The aim of this step is to prove that

\[
\liminf_{L \to 0^+} c^*_L \geq 2\sqrt{\langle a_H \rangle A H < \mu > A}
\]

which would complete the proof of Theorem 2.1.

For each \(L > 0\), the minimal speed \(c^*_L\) is given by (3.1) and the map \((0, +\infty) \ni \lambda \mapsto k(\lambda, L)/\lambda\) attains its minimum at \(\lambda^*_L > 0\). We will prove that, for \(L\) small enough, the family \((\lambda^*_L)\) is bounded from above and from below by \(\overline{\lambda} > 0\) and \(\underline{\lambda} > 0\) respectively. Namely, one has

**Lemma 3.1** There exist \(L_0\) and \(0 < \underline{\lambda} \leq \lambda^*_L \leq \overline{\lambda} < +\infty\) such that

\[
\underline{\lambda} \leq \lambda^*_L \leq \overline{\lambda} \quad \text{for all} \quad 0 < L \leq L_0.
\]

The proof is postponed at the end of this section. Take now any sequence \((L_n)_n\) such that \(0 < L_n \leq L_0\) for all \(n\), and \(L_n \to 0^+\) as \(n \to +\infty\). From Lemma 3.1, there exists \(\lambda^* > 0\) such that, up to extraction of a subsequence, \(\lambda^*_L \to \lambda^*\) as \(n \to +\infty\). One also has

\[
c^*_L = \frac{k(\lambda^*_L, L_n)}{\lambda^*_L} \to \frac{\overline{k}(\lambda^*)}{\lambda^*} = \lambda^* < a_H + \frac{\langle \mu \rangle_A}{\lambda^*} \geq 2\sqrt{\langle a_H \rangle A H < \mu > A}
\]

from (3.12) and (3.13). Therefore, \(\liminf_{L \to 0^+} c^*_L \geq 2\sqrt{\langle a_H \rangle A H < \mu > A}\). Eventually,

\[
\lim_{L \to 0^+} c^*_L = 2\sqrt{\langle a_H \rangle A H < \mu > A}
\]

and the proof of Theorem 2.1 is complete. \(\square\)

**Proof of Lemma 3.1.** Observe first that, for \(\lambda = 0\) and for any \(L > 0\), \(k(0, L)\) is the principal eigenvalue of the problem

\[
(a_L \phi'_L) + \mu_L \phi_L = k(0, L) \phi_L \quad \text{in} \quad \mathbb{R},
\]
and we denote \( \phi_L = \psi_{0,L} \) a principal eigenfunction, which is \( L \)-periodic, positive and unique up to multiplication. In other words, \( k(0, L) = -\rho_{1,L} \) under the notations of Section 1. Dividing the above elliptic equation by \( \phi_L \) and integrating by parts over \([0, L]\), one gets
\[
k(0, L) = \frac{1}{L} \int_0^L \frac{a_L \phi'_L}{\phi_L^2} dx + \int_0^1 \mu(x) dx \geq <\mu>_A > 0.
\]
On the other hand, as already recalled, \( \frac{\partial k}{\partial \lambda}(0, L) = 0 \) and the map \( \lambda \mapsto k(\lambda, L) \) is convex for all \( L > 0 \). Therefore,
\[
\forall \lambda > 0, \forall L > 0, \ k(\lambda, L) \geq k(0, L) \geq <\mu>_A > 0.
\]
Assume here that there exists a sequence \((L_n)_{n \in \mathbb{N}}\) of positive numbers such that \( L_n \to 0^+ \) and \( \lambda_{L_n}^* \to 0^+ \) as \( n \to +\infty \). One then gets
\[
\epsilon_{L_n}^* = \frac{k(\lambda_{L_n}^*, L_n)}{\lambda_{L_n}^*} \geq <\mu>_A \to +\infty \quad \text{as} \quad n \to +\infty.
\]
This is contradiction with (3.14). Thus, for \( L > 0 \) small enough, the family \((\lambda_L^*)_L\) is bounded from below by a positive constant \( \Lambda > 0 \) (actually, these arguments show that the whole family \((\lambda_L^*)_L>0\) is bounded from below by a positive constant).

It remains now to prove that \((\lambda_L^*)_L\) is bounded from above when \( L \) is small enough. We assume, to the contrary, that there exists a sequence \( L_n \to 0^+ \) as \( n \to +\infty \) such that \( \lambda_{L_n}^* \to +\infty \) as \( n \to +\infty \). Call
\[
k_n = k(\lambda_{L_n}^*, L_n), \quad \psi_n(x) = \psi_{\lambda_{L_n}^*, L_n}(x) \quad \text{and} \quad \varphi_n(x) = \varphi_{\lambda_{L_n}^*, L_n}(x) = e^{\lambda_{L_n}^* x} \psi_n(x)
\]
for all \( n \in \mathbb{N} \) and \( x \in \mathbb{R} \). Rewriting (3.10) for \( \lambda = \lambda_{L_n}^* \) and for \( L = L_n \), one consequently gets
\[
\forall n \in \mathbb{N}, \quad (a_L \varphi_n')' + \mu_L \varphi_n = k_n \varphi_n \quad \text{in} \quad \mathbb{R}.
\]
Owing to the positivity and the \( L_n \)-periodicity of the \( C^2(\mathbb{R}) \) eigenfunction \( \psi_n \), it follows that
\[
\forall n \in \mathbb{N}, \exists \theta_n \in [0, L_n], \quad \psi_n(\theta_n) = \max_{x \in \mathbb{R}} \psi_n(x) = \max_{x \in [0,L_n]} \psi_n(x),
\]
whence
\[
\forall n \in \mathbb{N}, \quad \psi_n'(\theta_n) = 0.
\]
For each \( n \in \mathbb{N} \), let \( M_{L_n} = [1/L_n] + 1 \in \mathbb{N} \). Thus,
\[
\forall n \in \mathbb{N}, \quad \varphi_n'(\theta_n + M_{L_n} L_n) = \lambda_{L_n}^* e^{\lambda_{L_n}^* (\theta_n + M_{L_n} L_n)} \psi_n(\theta_n).
\]
Multiplying (3.15) by \( \varphi_n \) and integrating by parts over the interval \([\theta_n, \theta_n + M_{L_n} L_n]\), one then obtains
\[
-\int_{\theta_n}^{\theta_n+M_{L_n} L_n} a_L \varphi_n^2 + \int_{\theta_n}^{\theta_n+M_{L_n} L_n} \mu_L \varphi_n^2 = k_n \int_{\theta_n}^{\theta_n+M_{L_n} L_n} \varphi_n^2.
\]
But, for each \( n \in \mathbb{N} \), \( M_{L_n} \in \mathbb{N} \) while \( a_{L_n} \) and \( \psi_n \) are \( L_n \)-periodic. Hence, \( a_{L_n}(\theta_n + M_{L_n} L_n) = a_{L_n}(\theta_n) \), \( \psi_n(\theta_n + M_{L_n} L_n) = \psi_n(\theta_n) \), and \( \psi_n'(\theta_n + M_{L_n} L_n) = \psi_n'(\theta_n) = 0 \). Then,

\[
A(n) = a_{L_n}(\theta_n) \lambda_{L_n}^* \psi_n^2(\theta_n) \left( e^{2 \lambda_{L_n}^* (\theta_n + M_{L_n} L_n)} - e^{2 \lambda_{L_n}^* \theta_n} \right) \\
\geq \frac{\alpha_1}{2} \lambda_{L_n}^* \psi_n^2(\theta_n) e^{2 \lambda_{L_n}^* (\theta_n + M_{L_n} L_n)} \quad (\alpha_1 > 0 \text{ is given by } (1.4))
\]

whenever \( n \) is large enough so that \( 2 \leq e^{2 \lambda_{L_n}^* M_{L_n} L_n} \) (remember that \( \lambda_{L_n}^* \to +\infty \) as \( n \to +\infty \), by assumption). Meanwhile, for all \( n \in \mathbb{N} \),

\[
|C(n)| \leq \int_{\theta_n}^{\theta_n + M_{L_n} L_n} \mu(x) \left| e^{2 \lambda_{L_n}^* \psi_n^2(x)} \right| dx \leq \mu_\infty \times \frac{\psi_n^2(\theta_n)}{2 \lambda_{L_n}^*} \times e^{2 \lambda_{L_n}^* (\theta_n + M_{L_n} L_n)},
\]

(3.18)

where \( \mu_\infty = \max_{x \in \mathbb{R}} |\mu(x)| \). On the other hand, (3.1) and (3.5) yield

\[
k_n \leq 2 \sqrt{a_M \mu_M} \times \lambda_{L_n}^*
\]

for all \( n \in \mathbb{N} \), whence

\[
k_n \int_{\theta_n}^{\theta_n + M_{L_n} L_n} \psi_n^2(x) = k_n \int_{\theta_n}^{\theta_n + M_{L_n} L_n} e^{2 \lambda_{L_n}^* \psi_n^2(x)} dx \leq \sqrt{a_M \mu_M} \times \psi_n^2(\theta_n) \times e^{2 \lambda_{L_n}^* (\theta_n + M_{L_n} L_n)}.
\]

(3.19)

Now, the term \( B(n) \) can be estimated as follows

\[
B(n) = \sum_{j=0}^{M_{L_n}-1} \int_{\theta_n + jL_n}^{\theta_n + (j+1)L_n} a_{L_n} e^{2 \lambda_{L_n}^* \psi_n^2(x)} \left( \psi_n'(x) + \lambda_{L_n}^* \psi_n(x) \right)^2 dx
\]

\[
\leq \sum_{j=0}^{M_{L_n}-1} \alpha_2 \lambda_{L_n}^* \psi_n^2(\theta_n) \int_{\theta_n + jL_n}^{\theta_n + (j+1)L_n} \left( \psi_n'(x) + \lambda_{L_n}^* \psi_n(x) \right)^2 dx \quad \text{[from (1.4)]}
\]

\[
= \sum_{j=0}^{M_{L_n}-1} \alpha_2 \lambda_{L_n}^* \psi_n^2(\theta_n) \int_{0}^{L_n} \left( \psi_n'(x) + \lambda_{L_n}^* \psi_n(x) \right)^2 dx
\]

since \( \psi_n \) is \( L_n \)-periodic. One has

\[
\int_{0}^{L_n} \left( \psi_n'(x) + \lambda_{L_n}^* \psi_n(x) \right)^2 dx \leq \psi_n^2(\theta_n) \int_{0}^{L_n} \left( \frac{\psi_n'(x)}{\psi_n(x)} + \lambda_{L_n}^* \right)^2 dx.
\]

We refer now to equation (3.2). Taking \( \lambda = \lambda_{L_n}^* \), dividing this equation (3.2) by the \( L_n \)-periodic function \( \psi_n \) and then integrating by parts over the interval \([0, L_n]\), we get

\[
\int_{0}^{L_n} a_{L_n} \left( \frac{\psi_n'}{\psi_n} \right)^2 + 2 \lambda_{L_n}^* \int_{0}^{L_n} a_{L_n} \psi_n' \psi_n + \lambda_{L_n}^* \int_{0}^{L_n} a_{L_n} \psi_n + \int_{0}^{L_n} \mu_{L_n} = k_n L_n
\]

for all \( n \in \mathbb{N} \). Thus,

\[
\int_{0}^{L_n} a_{L_n} \left( \frac{\psi_n^2}{\psi_n} + \lambda_{L_n}^* \right)^2 + \int_{0}^{L_n} \mu_{L_n} = k_n L_n \leq 2 \sqrt{a_M \mu_M} \times \lambda_{L_n}^* L_n.
\]
Owing to (1.4), it follows that
\[ \forall \ n \in \mathbb{N}, \ \int_0^{L_n} \left( \frac{\psi_n'}{\psi_n} + \lambda_n^* \right)^2 \leq \frac{2 \sqrt{a M \mu M}}{\alpha_1} \times \lambda_n^* \times L_n. \]

Putting the above result into \( B(n) \), we obtain, for all \( n \in \mathbb{N} \),
\[
B(n) \leq \frac{2 \alpha_2 \sqrt{a M \mu M}}{\alpha_1} \times \lambda_n^* \times L_n \psi_n^2(\theta_n) \sum_{j=0}^{Mn-1} e^{2\lambda_n^*(\theta_n + jL_n)} \\
= \frac{2 \alpha_2 \sqrt{a M \mu M}}{\alpha_1} \times \lambda_n^* \times L_n \psi_n^2(\theta_n) e^{2\lambda_n^*(\theta_n + L_n)} \times \frac{e^{2\lambda_n^*(\theta_n + M_n L_n)} - 1}{e^{2\lambda_n^*(\theta_n + M_n L_n)} - 1} \tag{3.20}
\]
\[
= \frac{2 \alpha_2 \sqrt{a M \mu M}}{\alpha_1} \times \psi_n^2(\theta_n) \times \frac{\lambda_n^* L_n e^{2\lambda_n^*(\theta_n + M_n L_n)}}{e^{2\lambda_n^*(\theta_n + M_n L_n)} - 1} \times e^{2\lambda_n^*(\theta_n + M_n L_n)} \times (\lambda_n^* L_n + 1),
\]
where \( \beta = \left( \frac{2 \alpha_2 \sqrt{a M \mu M}}{\alpha_1} \right) \times C \) and \( C \) is a positive constant such that
\[ \forall x \geq 0, \ \frac{xe^{2x}}{e^{2x} - 1} \leq C \times (x + 1). \]

Lastly, let us rewrite equation (3.16) as
\[ \forall \ n \in \mathbb{N}, \ A(n) + C(n) - k_n \int_{\theta_n}^{\theta_n + M_n L_n} \varphi_n^2 = B(n). \]

Together with (3.17), (3.18), (3.19) and (3.20), one concludes that there exists \( n_0 \in \mathbb{N} \) such that for \( n \geq n_0 \),
\[
\frac{\alpha_1}{2} \times \lambda_n^* \psi_n^2(\theta_n) e^{2\lambda_n^*(\theta_n + M_n L_n)} - \mu_\infty \times \frac{\psi_n^2(\theta_n)}{2\lambda_n^*} e^{2\lambda_n^*(\theta_n + M_n L_n)} \\
= \sqrt{a M \mu M} \times \psi_n^2(\theta_n) e^{2\lambda_n^*(\theta_n + M_n L_n)} \leq \beta \times \psi_n^2(\theta_n) e^{2\lambda_n^*(\theta_n + M_n L_n)} \times (\lambda_n^* L_n + 1). \tag{3.21}
\]

Divide (3.21) by \( \lambda_n^* \psi_n^2(\theta_n) e^{2\lambda_n^*(\theta_n + M_n L_n)} \). Then
\[ \forall \ n \geq n_0, \ \frac{\alpha_1}{2} - \frac{\mu_\infty}{2(\lambda_n^*)^2} - \frac{\sqrt{a M \mu M}}{\lambda_n^*} \leq \beta \times \left( L_n + \frac{1}{\lambda_n^*} \right). \]

Passing to the limit as \( n \to +\infty \), one has \( L_n \to 0^+ \) and \( \lambda_n^* \to +\infty \), whence \( \alpha_1 \leq 0 \), which is impossible.

Therefore the assumption that \( \lambda_n^* \to +\infty \) as \( L_n \to 0^+ \) is false and consequently the family \( (\lambda_n^*)_L \) is bounded from above by some positive \( \lambda > 0 \) whenever \( L \) is small (i.e. \( 0 < L \leq L_0 \)). This completes the proof of Lemma 3.1. \( \Box \)
Remark 3.2 From Theorem 2.1, one concludes that the map \((0, +\infty) \ni L \mapsto c_L^*\) can be extended by continuity to the right at \(L = 0^+\). Furthermore, for any sequence \((L_n)_n\) of positive numbers such that \(L_n \to 0^+\) as \(n \to +\infty\), one claims that the positive numbers \(\lambda_{L_n}^*\) given in (3.1) converge to \(\sqrt{<a>^{-1}_H<\mu>_A} = \sqrt{<a^{-1}>_A<\mu>_A}\) as \(n \to +\infty\). Indeed

\[
\forall n \in \mathbb{N}, \quad c_{L_n}^* = \frac{k(\lambda_{L_n}^*, L_n)}{\lambda_{L_n}^*}
\]

and Lemma 3.1 implies that, up to extraction of a subsequence, \(\lambda_{L_n}^* \to \lambda^* > 0\). Passing to the limit as \(n \to +\infty\) in the above equation and due (3.13) together with Step 2 of the proof of Theorem 2.1, one gets

\[
2\sqrt{<a>_H<\mu>_A} = \frac{E(\lambda^*)}{\lambda^*} = \lambda^* <a>_H + <\mu>_A, \lambda^* \quad \text{and} \quad \lambda^* = \sqrt{<a>-1_H<\mu>_A}. \quad \text{Since the limit does not depend on any subsequence, one concludes that the limit of} \lambda^*_L, \text{as} \, L \to 0^+, \text{exits and}
\]

\[
\lim_{L \to 0^+} \lambda_L^* = \sqrt{<a>-1_H<\mu>_A} = \sqrt{<a^{-1}>_A<\mu>_A}.
\]

The sharp lower bound of \(\liminf_{L \to 0^+} c_L^*\) from the homogenized equation. In the following, we are going to derive the homogenized equation of (1.3), which will lead to the sharp lower bound of \(\liminf_{L \to 0^+} c_L^*\). However, to furnish this goal we will only consider for the sake of simplicity a particular type of nonlinearities among those satisfying (1.5). In fact, the following ideas can be generalized to a wider family of nonlinearities which satisfy (1.5), but the proof requires technical extra-arguments which will be the purpose of a forthcoming paper.

For each \(L > 0\), let \(u_L\) be a pulsating travelling front with minimal speed \(c_L^*\) for the reaction-diffusion equation

\[
\begin{align*}
\frac{\partial u_L}{\partial t} &= \frac{\partial}{\partial x} \left( a_L(x) \frac{\partial u_L}{\partial x} \right) + \mu \frac{x}{L} g(u_L), \quad t \in \mathbb{R}, \, x \in \mathbb{R}, \\
\forall (t, x) &\in \mathbb{R} \times \mathbb{R}, \quad 0 < u_L(t + \frac{L}{c_L}, x) = u_L(t, x + L) < 1, \\
\lim_{x \to -\infty} u_L(t, x) &= 0 \quad \text{and} \quad \lim_{x \to +\infty} u_L(t, x) = 1,
\end{align*}
\]

(3.22)

where \(a_L(x) = a(x/L)\), \(a\) is a \(C^{2,\delta}(\mathbb{R})\) 1-periodic function satisfying (1.4), \(\mu\) is a \(C^{1,\delta}(\mathbb{R})\) positive 1-periodic function and \(g\) is a \(C^2(\mathbb{R}_+)\) function such that \(g(0) = g(1) = 0\) and \(u \mapsto g(u)/u\) is decreasing in \((0, +\infty)\). Up to a shift in time, one can assume that

\[
\forall L > 0, \quad \int_{(0,1) \times (0,1)} u_L(t, x) \, dt \, dx = \frac{1}{2}.
\]

(3.23)
For each $L > 0$, set $f_L(x, u) := f(x/L, u) = \mu(x/L)g(u)$. In this setting, there holds $p_L \equiv 1$. From standard parabolic estimates, each function $u_L$ is (at least) of class $C^2(\mathbb{R} \times \mathbb{R})$. Denote

$$v_L(t, x) = a_L(x) \frac{\partial u_L}{\partial t}(t, x) \quad \text{and} \quad w_L(t, x) = \frac{\partial u_L}{\partial t}(t, x) \quad \text{in} \quad \mathbb{R} \times \mathbb{R}.$$ 

As already underlined, it follows from [1] that $w_L = \frac{\partial u_L}{\partial x} > 0$ in $\mathbb{R} \times \mathbb{R}$ for each $L > 0$. Under the notations of the beginning of this section, it follows from (1.4) and (3.2) that $k(\lambda, L) \geq \lambda^2 \alpha_1 + \mu_m$ for all $L > 0$ and $\lambda \in \mathbb{R}$, where $\mu_m = \min_{\mathbb{R}} \mu > 0$. Hence, $c'_L \geq 2\sqrt{\alpha_1 \mu_m}$ for each $L > 0$ and $\liminf_{L \to 0^+} c'_L \geq 2\sqrt{\alpha_1 \mu_m} > 0$.

We shall now establish some estimates for the functions $u_L$, $v_L$ and $w_L$ which are independent of $L$, in order to pass to the limit as $L \to 0^+$. Notice first that standard parabolic estimates and the $(t, x)$-periodicity satisfied by the functions $u_L$ imply that, for each $L > 0$, $u_L(-\infty, x) = 0$ and $u_L(+\infty, x) = 1$ in $C^1_{\text{loc}}(\mathbb{R})$, and $w_L(\pm \infty, x) = 0$ in $C^1_{\text{loc}}(\mathbb{R})$.

Let $k \in \mathbb{N}\setminus\{0\}$ be given. Integrating the first equation of (3.22) by parts over $\mathbb{R} \times (-kL, kL)$, one obtains

$$\forall L > 0, \quad \int_{\mathbb{R} \times (-kL,kL)} f(\frac{x}{L}, u_L) \, dt \, dx = 2kL. \quad (3.24)$$

Multiplying the first equation of (3.22) by $u_L$ and integrating by parts over $\mathbb{R} \times (-kL, kL)$, one then gets

$$\forall L > 0, \quad kL = -\int_{\mathbb{R} \times (-kL,kL)} a_L(x) \left( \frac{\partial u_L}{\partial x} \right)^2 \, dx + \int_{\mathbb{R} \times (-kL,kL)} f(\frac{x}{L}, u_L) u_L \, dt \, dx. \quad (3.25)$$

Notice that the last integral in (3.25) converges because of (3.24) and $0 \leq f(x/L, u_L)u_L \leq f(x/L, u_L)$. Together with (1.4), one concludes that for each $L > 0$, the first integral in (3.25) converges and

$$\forall L > 0, \quad \int_{\mathbb{R} \times (-kL,kL)} \left( \frac{\partial u_L}{\partial x} \right)^2 \, dx \leq \frac{kL}{\alpha_1}. \quad (3.26)$$

Multiply the first equation of (3.22) by $\frac{\partial u_L}{\partial t}$ and integrate by parts over $\mathbb{R} \times (-kL, kL)$. Since

$$\int_{\mathbb{R} \times (-kL,kL)} \frac{\partial}{\partial t} \left( a_L(x) \frac{\partial u_L}{\partial x} \right) \frac{\partial u_L}{\partial t} = -\frac{1}{2} \int_{\mathbb{R} \times (-kL,kL)} \frac{\partial}{\partial t} \left( a_L(x) \left( \frac{\partial u_L}{\partial x} \right)^2 \right) = 0,$$

one obtains that

$$\forall L > 0, \quad \int_{\mathbb{R} \times (-kL,kL)} \left( \frac{\partial u_L}{\partial t} \right)^2 \, dx = \int_{(-kL,kL)} F(\frac{x}{L}, 1) \, dx = 2kL \times \int_0^1 \mu \times \int_0^1 g, \quad (3.26)$$

where $F(y, s) = \int_0^s f(y, \tau) \, d\tau$. It follows from the above estimates that for each compact subset $K$ of $\mathbb{R}$,

$$\forall 0 < L < 1, \quad \int_{\mathbb{R} \times K} \left[ \left( \frac{\partial u_L}{\partial t} \right)^2 + \left( \frac{\partial u_L}{\partial x} \right)^2 \right] \, dt \, dx \leq C(K), \quad (3.27)$$
where $C(K)$ is a positive constant depending only on $K$.

In particular, for each compact $K$ of $\mathbb{R}$ and for each $L > 0$, $\|w_L\|_{L^2(\mathbb{R} \times K)} \leq \sqrt{C(K)}$.

Now, differentiate the first equation of (3.22) with respect to $t$ (actually, from the regularity of $f$, the function $w_L$ is of class $C^2$ with respect to $x$). There holds

$$\frac{\partial w_L}{\partial t} = \frac{\partial}{\partial x} \left( a_L(x) \frac{\partial w_L}{\partial x} \right) + \mu\left( \frac{x}{L} \right) g'(u_L)w_L \quad \text{in } \mathbb{R} \times \mathbb{R}.$$ 

Multiply the above equation by $w_L$ and integrate by parts over $\mathbb{R} \times (-kL, kL)$. From (1.4) and (3.26), it follows that

$$\int \int_{\mathbb{R} \times (-kL, kL)} \left( \frac{\partial w_L}{\partial x} \right)^2 \, dt \, dx \leq \frac{2kL\eta}{\alpha_1},$$

where $\eta$ is the positive constant defined by

$$\eta = \max_{x \in \mathbb{R}} \mu(x) \max_{u \in [0,1]} |g'(u)| |F(x, 1)| \geq \frac{1}{2kL} \int \int_{\mathbb{R} \times (-kL, kL)} \mu\left( \frac{x}{L} \right) g'(u_L)w_L^2 \, dt \, dx > 0.$$ 

Then, for each compact $K \subset \mathbb{R}$, there exists a constant $C'(K) > 0$ depending only on $K$ such that

$$\forall 0 < L \leq 1, \int \int_{\mathbb{R} \times K} \left( \frac{\partial w_L}{\partial x} \right)^2 \, dt \, dx \leq C'(K). \quad (3.28)$$

Let $(L_n)_{n \in \mathbb{N}}$ be a sequence of real numbers in $(0, 1)$ such that $L_n \to 0$ and $c^*_L \to \liminf_{L \to 0^+} c^*_L > 0$ as $n \to +\infty$. It follows from (3.27) and the bounds $0 < u_{L_n} < 1$ that there exists $u_0$ in $H^1_{loc}(\mathbb{R} \times \mathbb{R})$ such that, up to extraction of a subsequence, $u_{L_n} \to u_0$ strongly in $L^2_{loc}(\mathbb{R} \times \mathbb{R})$ and almost everywhere in $\mathbb{R} \times \mathbb{R}$, and

$$\left( \frac{\partial u_{L_n}}{\partial t}, \frac{\partial u_{L_n}}{\partial x} \right) \rightharpoonup \left( \frac{\partial u_0}{\partial t}, \frac{\partial u_0}{\partial x} \right) \quad \text{weakly in } L^2_{loc}(\mathbb{R} \times \mathbb{R}) \text{ as } n \to +\infty.$$ 

Remember that $v_{L_n} = a_{L_n} \frac{\partial u_{L_n}}{\partial x}$ and $0 < \alpha_1 \leq a_{L_n} \leq \alpha_2$ for each $n \in \mathbb{N}$. Thus, (3.27) yields that for each compact $K$ of $\mathbb{R}$ and for each $n \in \mathbb{N}$, $\|v_{L_n}\|_{L^2(\mathbb{R} \times K)} \leq \alpha_2 C(K)$. Furthermore, (3.22) implies that

$$\forall n \in \mathbb{N}, \frac{\partial v_{L_n}}{\partial x} = \frac{\partial u_{L_n}}{\partial t} - f\left( \frac{x}{L_n}, u_{L_n} \right) \quad \text{in } \mathbb{R} \times \mathbb{R},$$

while $0 \leq f(x/L_n, u_{L_n}(t, x)) \leq \kappa$ in $\mathbb{R} \times \mathbb{R}$ where $\kappa = \max_{\mathbb{R}} \mu \times \max_{[0,1]} g > 0$ is independent of $n$. Together with (3.27), one concludes that the sequence $(\frac{\partial u_{L_n}}{\partial x})_{n \in \mathbb{N}}$ is bounded in $L^2_{loc}(\mathbb{R} \times \mathbb{R})$. On the other hand, $\frac{\partial v_{L_n}}{\partial x} = a_{L_n} \frac{\partial u_{L_n}}{\partial x} = a_{L_n} \frac{\partial u_{L_n}}{\partial x}$. Owing to (1.4) and (3.28), the sequence $(\frac{\partial v_{L_n}}{\partial x})_{n \in \mathbb{N}}$ is bounded in $L^2_{loc}(\mathbb{R} \times \mathbb{R})$. Consequently, up to extraction of another subsequence, there exists $v_0 \in H^1_{loc}(\mathbb{R} \times \mathbb{R})$ such that $v_{L_n} \to v_0$ strongly in $L^2_{loc}(\mathbb{R} \times \mathbb{R})$ and

$$\left( \frac{\partial v_{L_n}}{\partial t}, \frac{\partial v_{L_n}}{\partial x} \right) \rightharpoonup \left( \frac{\partial v_0}{\partial t}, \frac{\partial v_0}{\partial x} \right) \quad \text{weakly in } L^2_{loc}(\mathbb{R} \times \mathbb{R}) \text{ as } n \to +\infty.
However, \( a_{L_n}^{-1} \to < a^{-1} >_A = < a >_H^{-1} \) in \( L^\infty(\mathbb{R}) \) weak-* as \( n \to +\infty \). Thus,

\[
\frac{\partial u_{L_n}}{\partial x} = \frac{v_{L_n}}{a_{L_n}} \to \frac{v_0}{< a >_H} \text{ weakly in } L^2_{loc}(\mathbb{R} \times \mathbb{R}) \text{ as } n \to +\infty.
\]

By uniqueness of the limit, one gets \( v_0 = < a >_H \frac{\partial u_0}{\partial x} \). Passing to the limit as \( n \to +\infty \) in the first equation of (3.22) with \( L = L_n \) implies that \( u_0 \) is a weak solution of the equation

\[
\frac{\partial u_0}{\partial t} = < a >_H \frac{\partial^2 u_0}{\partial x^2} + < \mu >_A g(u_0) \text{ in } \mathcal{D}'(\mathbb{R} \times \mathbb{R}).
\]

From parabolic regularity, the function \( u_0 \) is then a classical solution of the homogenous equation

\[
\frac{\partial u_0}{\partial t} = < a >_H \frac{\partial^2 u_0}{\partial x^2} + < \mu >_A g(u_0) \text{ in } \mathbb{R} \times \mathbb{R},
\]

such that \( 0 \leq u_0 \leq 1 \) and \( \frac{\partial u_0}{\partial x} \geq 0 \) in \( \mathbb{R} \times \mathbb{R} \). Lastly, \( \int_{(0,1)^2} u_0(t, x) \, dt \, dx = \frac{1}{2} \) from (3.23). On the other hand, it follows from the second equation of (3.22) and (3.27) that

\[
\forall \gamma \in \mathbb{R}, \quad u_0(t + \frac{\gamma}{c}, x) = u_0(t, x + \gamma) \text{ in } \mathbb{R} \times \mathbb{R},
\]

where \( c = \lim \inf_{L \to 0^+} c^*_L = \lim_{n \to +\infty} c^*_{L_n} > 0 \). In other words, \( u_0(t, x) = U_0(x + ct) \), where \( U_0 \) is a classical solution of the equation

\[
cU_0' = < a >_H U_0'' + < \mu >_A g(U_0), \quad 0 \leq U_0 \leq 1 \text{ in } \mathbb{R}
\]

that satisfies \( U_0' \geq 0 \) in \( \mathbb{R} \) and

\[
\int_0^1 \left( \int_{cs}^{cs+1} U_0 \right) \, ds = \frac{1}{2}.
\]

Standard elliptic estimates imply that \( U_0 \) converges as \( s \to \pm \infty \) in \( C^2_{loc}(\mathbb{R}) \) to two constants \( U_0^\pm \in [0,1] \) such that \( < \mu >_A g(U_0^\pm) = 0 \), that is \( g(U_0^\pm) = 0 \). The monotonicity of \( U_0 \) and the assumption on \( g \) imply that \( U_0^- = 0 \) and \( U_0^+ = 1 \). In other words, \( U_0 \) is a usual travelling front for the homogenized equation (3.29) with speed \( c \) and limiting conditions 0 and 1 at infinity. Since the minimal speed for this problem is equal to \( 2\sqrt{< a >_H < \mu >_A} \), one concludes that

\[
\lim \inf_{L \to 0^+} c^*_L = c \geq 2\sqrt{< a >_H < \mu >_A}.
\]

### 4 Monotonicity of the minimal speeds \( c^*_L \) near the homogenization limit

This section is devoted to the proof of Theorem 2.3. Before going further in the proof, we recall that for each \( L > 0 \), the minimal speed \( c^*_L \) is given by the variational formula

\[
c^*_L = \min_{\lambda > 0} \frac{k(\lambda, L)}{\lambda} = \frac{k(\lambda^*_L, L)}{\lambda^*_L},
\]

where
where $\lambda^*_L > 0$ and $k(\lambda, L)$ is the principal eigenvalue of the elliptic equation (3.2). Notice that $k(\lambda, L)$ can be defined for all $\lambda \in \mathbb{R}$ and $L > 0$.

Step 1: properties of $k(\lambda, L)$ and definition of $\tilde{k}(\lambda, L)$. The principal eigenfunction $\psi_{\lambda,L}$ of (3.2) is $L$-periodic, positive and unique up to multiplication. Denote

$$
\phi_{\lambda,L}(x) = \psi_{\lambda,L}(Lx)
$$

for all $L > 0$, $\lambda \in \mathbb{R}$ and $x \in \mathbb{R}$. Each function $\phi_{\lambda,L}$ is 1-periodic, positive and it is the principal eigenfunction of

$$(a\phi'_{\lambda,L})'' + 2L\lambda a\phi'_{\lambda,L} + L\lambda a\phi_{\lambda,L} + L^2\lambda^2 a\phi_{\lambda,L} + L^2\mu \phi_{\lambda,L} = L^2k(\lambda, L)\phi_{\lambda,L},$$

associated to the principal eigenvalue $L^2k(\lambda, L)$. But the above problem can be defined for all $\lambda \in \mathbb{R}$ and $L \in \mathbb{R}$. That is, for each $(\lambda, L) \in \mathbb{R}^2$, there exists a unique principal eigenvalue $\tilde{k}(\lambda, L)$ and a unique (up to multiplication) principal eigenfunction $\tilde{\phi}(\lambda, L)$ of

$$(a\tilde{\phi}'_{\lambda,L})'' + 2L\lambda a\tilde{\phi}'_{\lambda,L} + L\lambda a\tilde{\phi}_{\lambda,L} + L^2\lambda^2 a\tilde{\phi}_{\lambda,L} + L^2\mu \tilde{\phi}_{\lambda,L} = \tilde{k}(\lambda, L)\tilde{\phi}_{\lambda,L}.$$  \quad(4.1)

Furthermore, $\tilde{\phi}_{\lambda,L}$ is 1-periodic, positive and it can be normalized so that

$$
\int_0^1 \tilde{\phi}_{\lambda,L}^2(x)dx = 1
$$  \quad(4.2)

for all $(\lambda, L) \in \mathbb{R}^2$. By uniqueness of the principal eigenelements, it follows that

$$
\forall \ L > 0, \ \forall \ \lambda \in \mathbb{R}, \ \tilde{k}(\lambda, L) = L^2k(\lambda, L)
$$

and $\tilde{\phi}_{\lambda,L}$ and $\phi_{\lambda,L}$ are equal up to multiplication by positive constants for each $L > 0$ and $\lambda \in \mathbb{R}$.

Some useful properties of $k(\lambda, L)$ as $L \to 0^+$ shall now be derived from the study the function $\tilde{k}$. Notice first that, since the coefficients of the left-hand side of (4.1) are analytic in $(\lambda, L)$, the function $\tilde{k}$ is analytic, and from the normalization (4.2), the functions $\tilde{\phi}_{\lambda,L}$ also depend analytically in $H^2_{loc}(\mathbb{R})$ on the parameters $\lambda$ and $L$ (see [10, 20]). In particular, the function $k$ is analytic in $\mathbb{R} \times (0, +\infty)$. Observe also that

$$
\tilde{k}(\lambda, 0) = 0 \text{ and } \tilde{\phi}_{\lambda,0} = 1 \text{ for all } \lambda \in \mathbb{R}.
$$

Lastly, when $\lambda$ is changed into $-\lambda$ or when $L$ is changed into $-L$, then the operator in (4.1) is changed into its adjoint. But since the principal eigenvalues of the operator and its adjoint are identical, it follows that

$$
\forall \ (\lambda, L) \in \mathbb{R}^2, \ \tilde{k}(\lambda, L) = \tilde{k}(\lambda, -L) = \tilde{k}(-\lambda, L).
$$

In particular, it follows that

$$
\forall \ (i, j) \in \mathbb{N}^2, \ \frac{\partial^i \tilde{k}}{\partial \lambda^i} (\lambda, 0) = \frac{\partial^i \tilde{k}^{2j+1}}{\partial L^{2j+1}} (\lambda, 0) = 0.
$$  \quad(4.3)
Therefore, for all $\bar{\lambda} \in \mathbb{R}$,

$$k(\lambda, L) = \frac{\tilde{k}(\lambda, L)}{L^2} \to \frac{1}{2} \times \frac{\partial^2 \tilde{k}}{\partial L^2}(\bar{\lambda}, 0) \text{ as } (\lambda, L) \to (\bar{\lambda}, 0^+) .$$

But since this limit is equal to $\bar{k}(\bar{\lambda}) = \bar{\lambda}^2 < a >_H + < \mu >_A$ from Step 2 of the proof of Theorem 2.1, one then gets that

$$\frac{1}{2} \times \frac{\partial^2 \tilde{k}}{\partial L^2}(\bar{\lambda}, 0) = \bar{\lambda}^2 < a >_H + < \mu >_A \text{ for all } \bar{\lambda} \in \mathbb{R} . \tag{4.4}$$

It also follows from (4.3) that

$$\frac{\partial^2 k}{\partial \lambda^2}(\lambda, L) = \frac{1}{L^2} \times \frac{\partial^2 \tilde{k}}{\partial \lambda^2}(\lambda, L) \to \frac{1}{2} \times \frac{\partial^4 \tilde{k}}{\partial \lambda^2 \partial L^2}(\bar{\lambda}, 0) \text{ as } (\lambda, L) \to (\bar{\lambda}, 0^+) . \tag{4.5}$$

From (4.4) and (4.5), one deduces that

$$\frac{\partial^2 k}{\partial \lambda^2}(\lambda, L) \to 2 < a >_H = 0 \text{ as } (\lambda, L) \to (\bar{\lambda}, 0^+) . \tag{4.6}$$

Similarly, as $(\lambda, L) \to (\bar{\lambda}, 0^+)$,

$$\begin{align*}
\frac{\partial k}{\partial L}(\lambda, L) &= \frac{\partial}{\partial L} \left( \frac{\tilde{k}(\lambda, L)}{L^2} \right) \to \frac{1}{6} \times \frac{\partial^3 \tilde{k}}{\partial L^3}(\bar{\lambda}, 0) = 0 \\
\frac{\partial^2 k}{\partial \lambda \partial L}(\lambda, L) &= \frac{\partial}{\partial L} \left( \frac{1}{L^2} \times \frac{\partial \tilde{k}}{\partial \lambda}(\lambda, L) \right) \to \frac{1}{6} \times \frac{\partial^4 \tilde{k}}{\partial \lambda \partial L^3}(\bar{\lambda}, 0) = 0 \tag{4.7} \\
\frac{\partial^2 k}{\partial L^2}(\lambda, L) &= \frac{\partial^2}{\partial L^2} \left( \frac{\tilde{k}(\lambda, L)}{L^2} \right) \to \frac{1}{12} \times \frac{\partial^4 \tilde{k}}{\partial L^4}(\bar{\lambda}, 0)
\end{align*}$$

**Remark 4.1** As a byproduct of the fact that $\tilde{k}$ and $k$ are even in $\lambda$, it follows that the minimal speed of pulsating fronts propagating from right to left (as in Definition 1.2) is the same as that of fronts propagating from left to right.

**Step 2:** properties of $c^*_L$ and $\lambda^*_L$ in the neighbourhood of $L = 0^+$. Let us first prove that, for each fixed $L > 0$, the positive real number $\lambda^*_L > 0$ given in (3.1) is unique. Indeed, if there are $0 < \lambda_1 < \lambda_2$ such that

$$c^*_L = \frac{k(\lambda_1, L)}{\lambda_1} = \frac{k(\lambda_2, L)}{\lambda_2} = \min_{\lambda > 0} \frac{k(\lambda, L)}{\lambda} ,$$

then $k(\lambda, L) = c^*_L \lambda$ for all $\lambda \in [\lambda_1, \lambda_2]$ since $k$ is convex with respect to $\lambda$. Then $k(\lambda, L) = c^*_L \lambda$ for all $\lambda \in \mathbb{R}$ by analyticity of the map $\mathbb{R} \ni \lambda \mapsto k(\lambda, L)$. But $k(0, L) = -\rho_{1,L} > 0$, which
In this step, we fix $L > 0$, $\lambda_L^*$ is the unique minimum of the map $(0, +\infty) \ni \lambda \mapsto k(\lambda, L)/\lambda$.

Furthermore, we claim that $L \mapsto \lambda_L^*$ and $L \mapsto c_L^*$ are of class $C^\infty$ in a right neighbourhood of $L = 0$. Indeed, by definition, $\lambda_L^*$ satisfies

$$F(\lambda_L^*, L) := \frac{\partial k}{\partial \lambda}(\lambda_L^*, L) \times \lambda_L^* - k(\lambda_L^*, L) = 0.$$  \hfill (4.8)

The function $(\lambda, L) \mapsto F(\lambda, L)$ is of class $C^\infty$ on $\mathbb{R} \times (0, +\infty)$ and $\frac{\partial F}{\partial \lambda}(\lambda, L) = \frac{\partial^2 k}{\partial \lambda^2}(\lambda, L) \times \lambda$. But

$$\lambda_L^* \to \lambda^* = \sqrt{<a>_H^{-1}<\mu>_A} > 0 \text{ as } L \to 0^+$$

from Remark 3.2, and

$$\frac{\partial^2 k}{\partial \lambda^2}(\lambda_L^*, L) \to 2 <a>_H > 0 \text{ as } L \to 0^+$$

from (4.6). Therefore, from the implicit function theorem, the map $L \mapsto \lambda_L^*$ is of class $C^\infty$ in an interval $(0, L_0)$ for some $L_0 > 0$. As a consequence of formula (3.1), the map $L \mapsto c_L^*$ is also of class $C^\infty$ on $(0, L_0)$.

For each $L \in (0, L_0)$, one has

$$\frac{dc_L^*}{dL} = \left( \frac{1}{\lambda_L^*} \times \frac{\partial k}{\partial \lambda}(\lambda_L^*, L) - \frac{k(\lambda_L^*, L)}{(\lambda_L^*)^2} \right) \times \frac{d\lambda_L^*}{dL} + \frac{1}{\lambda_L^*} \times \frac{\partial k}{\partial L}(\lambda_L^*, L) = \frac{1}{\lambda_L^*} \times \frac{\partial k}{\partial L}(\lambda_L^*, L)$$

by definition of $\lambda_L^*$ and formula (3.1). But $\lambda_L^* \to \lambda^* > 0$ and $\frac{\partial k}{\partial L}(\lambda_L^*, L) \to 0$ as $L \to 0^+$ from (4.7). Thus,

$$\frac{dc_L^*}{dL} \to 0 \text{ as } L \to 0^+.$$

On the other hand, it follows from (4.6), (4.7) and (4.8) that

$$\frac{d\lambda_L^*}{dL} = \frac{1}{\lambda_L^*} \times \frac{\partial^2 k}{\partial \lambda^2}(\lambda_L^*, L) \times \left( \frac{\partial k}{\partial L}(\lambda_L^*, L) \right) \to 0 \text{ as } L \to 0^+.$$

Therefore,

$$\frac{d^2 c_L^*}{dL^2} = \frac{d\lambda_L^*}{dL} \times \left( -\frac{1}{(\lambda_L^*)^2} \times \frac{\partial k}{\partial L}(\lambda_L^*, L) + \frac{1}{\lambda_L^*} \times \frac{\partial^2 k}{\partial \lambda \partial L}(\lambda_L^*, L) \right) + \frac{1}{\lambda_L^*} \times \frac{\partial^2 k}{\partial L^2}(\lambda_L^*, L)$$

$$\to -\frac{1}{12\lambda^*} \times \frac{\partial^4 k}{\partial L^4}(\lambda^*, 0) \text{ as } L \to 0^+,$$  \hfill (4.9)

from (4.7).

Step 3: calculation of $\frac{\partial^4 k}{\partial L^4}(\lambda^*, 0)$. In this step, we fix $\lambda^* = \sqrt{<a>_H^{-1}<\mu>_A}$. Since the functions $\tilde{\phi}_{\lambda^*, L}$ depend analytically on $L \in \mathbb{R}$ in $H^2_{loc}(\mathbb{R})$, the expansion

$$\tilde{\phi}_{\lambda^*, L} = 1 + L\phi_1 + L^2\phi_2 + L^3\phi_3 + L^4\phi_4 + \ldots$$
is valid in $H^2_{loc}(\mathbb{R})$ in a neighbourhood of $L = 0$, where $1 = \tilde{\phi}_{\lambda^*,0}$ and

$$\phi_i = \frac{1}{i!} \times \frac{\partial^i \tilde{\phi}_{\lambda^*,L}}{\partial L^i} \bigg|_{L=0}$$

for each $i \geq 1$. We now put this expansion into

$$(a\tilde{\phi}_{\lambda^*,L})' + 2L\lambda^* a\tilde{\phi}_{\lambda^*,L} + L\lambda^* a' \tilde{\phi}_{\lambda^*,L} + L^2(\lambda^*)^2 a\tilde{\phi}_{\lambda^*,L} + L^2 \mu \tilde{\phi}_{\lambda^*,L} = \tilde{k}(\lambda^*, L)\tilde{\phi}_{\lambda^*,L}$$

and remember that

$$\tilde{k}(\lambda^*, 0) = \frac{\partial \tilde{k}}{\partial L}(\lambda^*, 0) = \frac{\partial^3 \tilde{k}}{\partial L^3}(\lambda^*, 0) = 0$$

and

$$\frac{\partial^2 \tilde{k}}{\partial L^2}(\lambda^*, 0) = 2 \times [(\lambda^*)^2 <a>_H + <\mu>_A] = 4 <\mu>_A$$

from (4.3) and (4.4). Since both $\tilde{\phi}_{\lambda^*,L}$ and $\tilde{k}(\lambda^*, L)$ depend analytically on $L$, it follows in particular that

$$\begin{cases}
(a\phi_1')' + \lambda^* a' = 0, \\
(a\phi_2')' + 2\lambda^* a\phi_1' + \lambda^* a' \phi_1 + (\lambda^*)^2 a + \mu = 2 <\mu>_A, \\
(a\phi_3')' + 2\lambda^* a\phi_2' + \lambda^* a' \phi_2 + (\lambda^*)^2 a \phi_1 + \mu \phi_1 = 2 <\mu>_A \phi_1, \\
(a\phi_4')' + 2\lambda^* a\phi_3' + \lambda^* a' \phi_3 + (\lambda^*)^2 a \phi_2 + \mu \phi_2 = 2 <\mu>_A \phi_2 + \frac{1}{24} \times \frac{\partial^4 \tilde{k}}{\partial L^4}(\lambda^*, 0)
\end{cases}$$

in $\mathbb{R}$. Furthermore, each function $\phi_i$ is 1-periodic and, by differentiating the normalization condition $\|\tilde{\phi}_{\lambda^*,L}\|_{L^2(0,1)}^2 = 1$ with respect to $L$ at $L = 0$, it follows especially that

$$\int_0^1 \phi_1 = 0 \text{ and } \int_0^1 \phi_2 = -\frac{1}{2} \int_0^1 \phi_1^2.$$

It is then found that, for all $x \in \mathbb{R}$,

$$\phi_1(x) = \lambda^* \times \left(-x + <a>_H \int_0^x \frac{1}{a(y)} dy - \frac{1}{2} - <a>_H \int_0^1 \frac{y}{a(y)} dy \right)$$

and

$$\phi_2(x) = <\mu>_A \times \left[\frac{x}{a(x)} - \int_0^1 \frac{y}{a(y)} dy - \int_0^x \frac{1}{a(y)} dy - \frac{<a>_H}{a(x)} \int_0^1 \frac{y}{a(y)} dy \right]$$

$$+ \frac{1}{a(x)} \times \left[<a>_H \int_0^1 \left( \frac{1}{a(y)} \int_0^y \mu(z) dz \right) dy - \int_0^x \mu(y) dy \right] + (\lambda^*)^2 \left(x + \frac{1}{2}\right).$$

Moreover, it follows from the third equation of (4.10) that, for all $x \in \mathbb{R}$,

$$a(x)\phi_3(x) = -2\lambda^* \int_0^x a(y)\phi_2(y) dy - \lambda^* \int_0^x a'(y)\phi_2(y) dy$$

$$- (\lambda^*)^2 \int_0^x a(y)\phi_1(y) dy - \int_0^x \mu(y)\phi_1(y) dy + 2 <\mu>_A \int_0^x \phi_1(y) dy + <a>_H c,$$
where
\[
c = \int_0^1 \left[ \frac{1}{a(y)} \times \left( 2\lambda^* \int_0^y a(z) \phi'_2(z)dz + \lambda^* \int_0^y a'(z) \phi_2(z)dz + (\lambda^*)^2 \int_0^y a(z) \phi_1(z)dz \right) + \int_0^y \mu(z) \phi_1(z)dz - 2<\mu>_A \int_0^y \phi_1(z)dz \right] dy
\]

On the other hand, by integrating the fourth equation of (4.10) over the interval \([0, 1]\), one gets that
\[
\frac{1}{24} \times \frac{\partial^4 \tilde{k}}{\partial L^4}(\lambda^*, 0) = -2<\mu>_A \int_0^1 \phi_2 + \lambda^* \int_0^1 a \phi'_3 + (\lambda^*)^2 \int_0^1 a \phi_2 + \int_0^1 \mu \phi_2.
\] (4.11)

Now, put all the previous calculations into (4.11). After a lengthy sequence of integrations by parts, it is finally found that
\[
\frac{1}{24} \times \frac{\partial^4 \tilde{k}}{\partial L^4}(\lambda^*, 0) = \int_0^1 A(x) a(x) dx - <a>_H \left( \int_0^1 \frac{A(x)}{a(x)} dx \right)^2,
\]
where
\[
A(x) = \int_0^x \mu(y) dy + <\mu>_A <a>_H \int_0^x \frac{1}{a(y)} dy - 2<\mu>_A x.
\]

From (4.9), it follows that
\[
d^2 c^*_L dL^2 \rightarrow \gamma := 2\sqrt{<a>_H <\mu>_A^{-1}} \times \left[ \int_0^1 \frac{A(x)^2}{a(x)} dx - <a>_H \left( \int_0^1 \frac{A(x)}{a(x)} dx \right)^2 \right] \text{ as } L \rightarrow 0^+.
\]

Cauchy-Schwarz inequality yields \(\gamma \geq 0\). Furthermore, \(\gamma = 0\) if and only if \(A\) is constant. But since \(A(0) = 0\), the condition \(\gamma = 0\) is equivalent to \(A'(x) = 0\) for all \(x\), which means that
\[
\frac{\mu(x)}{<\mu>_A} + <a>_H \frac{a(x)}{A(x)} = 2 \text{ for all } x \in \mathbb{R}.
\]

In particular, if \(\mu\) is constant and \(a\) is not constant (resp. if \(a\) is constant and \(\mu\) is not constant), then this condition is not satisfied, whence \(\lim_{L \rightarrow 0^+} \frac{d^2 c^*_L}{dL^2} > 0\) in this case. That completes the proofs of Theorem 2.3 and Corollary 2.4. \(\square\)

**Remark 4.2** In the case when \(<\mu>_A = 0\) and \(\mu \neq 0\), then \(\rho_{1,L} < 0\) for each \(L > 0\), and the minimal speed \(c^*_L\) of pulsating traveling fronts is well-defined and it is still positive. From the arguments developed in this section and in the previous one, one can check that, in this case,
\[
c^*_L \rightarrow 0^+, \quad \lambda^*_L \rightarrow 0^+, \quad \frac{d\lambda^*_L}{dL} \rightarrow \sqrt{\frac{\beta}{<a>_H}} > 0 \quad \text{and} \quad \frac{dc^*_L}{dL} \rightarrow 2\sqrt{\beta <a>_H} > 0 \text{ as } L \rightarrow 0^+,
\]
where
\[
\beta = \int_0^1 \frac{A(x)^2}{a(x)} dx - <a>_H \left( \int_0^1 \frac{A(x)}{a(x)} dx \right)^2 > 0
\]
and $A(x) = \int_0^x \mu(y)dy$. Therefore, the speeds $c^*_L$ are increasing in a right neighbourhood of $L = 0$ but, in this case, the variation is of the first order. Notice that the formula $\lim_{L \to 0^+} \frac{dc^*_L}{dL} = 2\sqrt{\beta <a>_H}$ is coherent with the numerical calculations done by Kinezaki, Kawasaki and Shigesada in [21] (see Figure 3b with $<\mu>_A = 0$, that is $A = 0$ under the notations of [21]).

5 Proof of Theorem 2.6

As in the proofs of the previous theorems, we use the following formula for the minimal speed:

$$c^*_z = \min_{\lambda > 0} \frac{k_z(\lambda)}{\lambda} = \frac{k_z(\lambda^*_z)}{\lambda^*_z},$$

(5.1)

where $k_z(\lambda)$ is defined as the unique real number such that there exists a positive $L_0$-periodic function $\psi$ satisfying:

$$\psi'' + 2\lambda \psi' + \lambda^2 \psi + \mu_z(x)\psi = k_z(\lambda)\psi \text{ in } (0, L_0).$$

(5.2)

Setting $\varphi(x) = e^{\lambda x}\psi(x)$, the above equation and periodicity conditions become equivalent to:

$$\begin{cases}
\varphi'' + \mu_z(x)\varphi = k_z(\lambda)\varphi \text{ in } (0, L_0), \\
\varphi(L_0) = e^{\lambda L_0}\varphi(0), \\
\varphi'(L_0) = e^{\lambda L_0}\varphi'(0),
\end{cases}$$

(5.3)

which therefore admits, for every positive $\lambda$, a unique solution $(\varphi, k_z(\lambda))$ with $\varphi > 0$ satisfying the normalisation condition $\varphi(0) = 1$.

Let $\lambda > 0$ be fixed. System (5.3), together with the normalization condition $\varphi(0) = 1$, is equivalent to:

$$\begin{cases}
\varphi'' = (k_z(\lambda) - m)\varphi \text{ on } [0, l/2), \\
\varphi'' = k_z(\lambda)\varphi \text{ on } [l/2, l/2 + z), \\
\varphi'' = (k_z(\lambda) - m)\varphi \text{ on } [l/2 + z, l + z), \\
\varphi'' = k_z(\lambda)\varphi \text{ on } [l + z, L_0), \\
\varphi(0) = 1, \varphi(L_0) = e^{\lambda L_0}\varphi(0), \varphi'(L_0) = e^{\lambda L_0}\varphi'(0).
\end{cases}$$

(5.4)

For each $z \in [0, L_0 - l]$, let $\lambda^*_z$ be defined by the formula (5.1). We have the following lemma:

**Lemma 5.1** Assume that $l > 3L_0/4$. Then, for all $z \in [0, L_0 - l]$, we have $k_z(\lambda^*_z) > m$.

**Proof of Lemma 5.1.** Let us divide equation (5.2) by $\psi$ and integrate by parts over $[0, L_0]$. Using the $L_0$-periodicity of $\psi$, we obtain:

$$\int_0^{L_0} \frac{|\psi'|^2}{\psi^2} dx + L_0\lambda^2 + \int_0^{L_0} \mu_z(x)dx = L_0k_z(\lambda).$$

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Thus,
\[ k_z(\lambda) \geq \lambda^2 + \frac{1}{L_0} \int_0^{L_0} \mu_z(x) dx = \lambda^2 + m \frac{l}{L_0}. \]  
(5.5)

From (5.1) and (5.5) we get:

\[ (\lambda^*_z)^2 + m \frac{l}{L_0} \leq k_z(\lambda^*_z) \leq 2 \lambda^*_z \sqrt{m}. \]

Thus, \((\lambda^*_z)^2 - 2 \lambda^*_z \sqrt{m} + ml/L_0 \leq 0\), which implies that

\[ \lambda^*_z \geq \sqrt{m} - \sqrt{m - ml/L_0}. \]

Using (5.5), we finally get

\[ k_z(\lambda^*_z) \geq 2m(1 - \sqrt{1 - l/L_0}) > m, \]
as soon as \(l > 3L_0/4\). \(\square\)

We now turn to the proof of Theorem 2.6 and we assume that \(l \in (3L_0/4, L_0)\). Using the fact that \(\varphi \in C^1(\mathbb{R})\), a straightforward but lengthy computation shows that, whenever \(k_z(\lambda) > m\), system (5.4) is equivalent to

\[ F(z, \lambda, k_z(\lambda)) = 0, \]

where \(F\) and \(G\) are two functions, defined respectively in \([0, L_0 - l] \times (0, +\infty) \times [m, +\infty)\) and \([0, L_0 - l] \times [m, +\infty)\) by:

\[
F(z, \lambda, s) = 4(2s - m) \sqrt{s} \sqrt{s - m} \sinh(l \sqrt{s - m}) \sinh(\alpha \sqrt{s}) \\
+ m^2 \cosh(\beta \sqrt{s})(1 - \cosh(l \sqrt{s - m})) \\
+ 8(s^2 - ms)[\cosh(l \sqrt{s - m}) \cosh(\alpha \sqrt{s}) - \cosh(\lambda L_0)] \\
+ m^2 \cosh(\alpha \sqrt{s}) \cosh(\lambda L_0) \right) + m \sqrt{s - m} \sinh(l \sqrt{s - m}) \cosh(\alpha \sqrt{s}) \right] \frac{4s}{m} - 1 + \frac{\cosh(\beta \sqrt{s})}{\cosh(\alpha \sqrt{s})}, \]

(5.6)

and

\[
G(z, s) = m \sqrt{s} \cosh(l \sqrt{s - m}) [4 \sinh(\alpha \sqrt{s})(s/m - 1) \\
+ (\sinh(\alpha \sqrt{s}) - \sinh(\beta \sqrt{s})) \left(1 - 1/ \cosh(l \sqrt{s - m}) \right) \\
+ m \sqrt{s - m} \sinh(l \sqrt{s - m}) \cosh(\alpha \sqrt{s}) \left(4s/m - 1 + \frac{\cosh(\beta \sqrt{s})}{\cosh(\alpha \sqrt{s})} \right), \]

(5.7)

with \(\alpha := L_0 - l\) and \(\beta := L_0 - l - 2z\).

Each factor in the expression (5.7) is positive, as soon as \(s > m\), for \(z \in [0, L_0 - l]\). Thus, whenever \(k_z(\lambda) > m\), system (5.4) is equivalent to the simpler equation

\[ F(z, \lambda, k_z(\lambda)) = 0. \]

(5.8)

Furthermore, from Krein-Rutman theory, since the eigenfunction \(\psi\) in (5.2) is positive, \(k_z(\lambda)\) is the largest real eigenvalue of the operator \(\psi \mapsto \psi'' + 2\lambda \psi' + \lambda^2 \psi + \mu_z(x) \psi\). This result, implies that, for each \(z \in [0, L_0 - l]\), and each \(\lambda > 0\), \(k_z(\lambda)\) is the largest real root of equation (5.8), as soon as \(k_z(\lambda) > m\).
From equation (5.6), we easily see that
\[
\lim_{s \to +\infty} F(z, \lambda, s) = +\infty, \tag{5.9}
\]
for all \(z \in [0, L_0 - l]\) and \(\lambda > 0\). Moreover, differentiating (5.6) with respect to \(z\), we obtain
\[
\frac{\partial F}{\partial z}(z, \lambda, s) = 2m^2 \sqrt{s} \sinh(\sqrt{s}(L_0 - l - 2z)) \left[ \cosh(l\sqrt{s - m}) - 1 \right].
\]
Thus, for all \(s > m\), and \(\lambda > 0\),
\[
\frac{\partial F}{\partial z}(z, \lambda, s) > 0 \text{ for } z \in [0, (L_0 - l)/2], \tag{5.10}
\]
and
\[
\frac{\partial F}{\partial z}(z, \lambda, s) < 0 \text{ for } z \in ((L_0 - l)/2, L_0 - l].
\]

Now, take \(z_1 < z_2\) in \([0, (L_0 - l)/2]\), and assume that \(c^*_z \leq c^*_z\). It follows from formula (5.1) that \(k_{z_2}(\lambda) \geq c^*_z \lambda\) for all \(\lambda > 0\). In particular,
\[
k_{z_2}(\lambda^*_z) \geq c^*_z \lambda^*_z \geq c^*_z \lambda^*_z = k_{z_1}(\lambda^*_z). \tag{5.11}
\]
From Lemma 5.1, we know that \(k_{z_1}(\lambda^*_z) > m\). Thus, (5.11) implies \(k_{z_2}(\lambda^*_z) > m\).

From the above discussion, \(k_{z_2}(\lambda^*_z)\) is therefore the largest real root of the equation \(F(z_2, \lambda^*_z, k_{z_2}(\lambda^*_z)) = 0\), and, similarly, \(k_{z_1}(\lambda^*_z)\) is the largest real root of \(F(z_1, \lambda^*_z, k_{z_1}(\lambda^*_z)) = 0\). Using (5.9) and (5.10), and since \(0 \leq z_1 < z_2 \leq (L_0 - l)/2\), we obtain \(k_{z_2}(\lambda^*_z) < k_{z_1}(\lambda^*_z)\), which contradicts (5.11). Therefore, \(c^*_z\) is a decreasing function of \(z\) in \([0, (L_0 - l)/2]\). Similar arguments imply that \(c^*_z\) is an increasing function of \(z\) in \(((L_0 - l)/2, L_0 - l]\). This concludes the proof of Theorem 2.6.

\[\square\]

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