Rotation Curves of Spiral Galaxies and Large Scale Structure of Universe under Generalized Einstein Action

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ABSTRACT

We consider an addition of the term which is a square of the scalar curvature to the Einstein-Hilbert action. Under this generalized action, we attempt to explain i) the flat rotation curves observed in spiral galaxies, which is usually attributed to the existence of dark matter, and ii) the contradicting observations of uniform cosmic microwave background and non-uniform galaxy distributions against redshift. For the former, we attain the flatness of velocities, although the magnitudes remain about half of the observations. For the latter, we obtain a solution with oscillating Hubble parameter under uniform mass distributions. This solution leads to several peaks of galaxy number counts as a function of redshift with the first peak corresponding to the Great Wall.

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1. Introduction

One of the most straightforward generalizations of the Einstein-Hilbert action is the addition of a square terms of the scalar curvature and Ricci tensor. This action was introduced as counter terms to regulate ultraviolet divergences of the Einstein theory.\(^1\) It was also used to obtain a bounce universe, which avoids the initial singularity of the big bang cosmology.\(^2\)

In the present work we consider this action in order to overcome the difficulties encountered by the standard Einstein theory in explaining certain astrophysical and cosmological observations. The famous astrophysical observations that appear to be in conflict with our expectations from the Newton’s theory of gravity is the flat rotation curves of spiral galaxies.\(^3\) The rotation velocities are constant as a function of the distance from the center of galaxy, while a naive application of the \(1/r^2\) force law implies a decline in the velocity function. This observation is usually accounted for by the incorporation of dark matter.\(^4\) In the present work we attempt to obtain flat rotation curves from the generalized Einstein action without relying upon dark matter. As for the cosmological observations, the recent technological developments in observations have yielded contradictory results: highly uniform cosmic microwave background\(^5\) and non-uniform galaxy distributions in the scales of hundreds of Mpc\(^6,7\). In order to resolve this difficulty, Morikawa introduced a non-conformal scalar field in the action and obtained a theory with oscillating Hubble constant.\(^8\) According to this theory, the mass distribution of the universe is uniform at any moment (hence, uniform cosmic microwave background), but the galaxy distributions as a function of redshift become periodic (hence, non-uniform large scale structure of the universe) due to the oscillation in the expansion rate of
the universe. In the present work, we follow the same idea under the generalized Einstein action without introducing the scalar matter.

The paper is organized as follows. In sect. 2 we discuss rotation curves of spiral galaxies under the generalized Einstein action. In sect. 3 we consider galaxy number counts under the generalized Einstein action. A concluding remark is given in sect. 4.

2. Rotation Curves of Spiral Galaxies under the Generalized Einstein Action

In this section, we consider a modification of Newtonian theory of gravitation under the generalized Einstein action and apply this force law to the rotation curves of spiral galaxies. The generalized Einstein action is given by

\[ I_g = -\frac{1}{16\pi G} \int d^4x \sqrt{g}(R + 2\Lambda + c_1 R^2 + c_2 R_{\mu\nu} R^{\mu\nu}) , \]  

(2.1)

where \( G \) is the gravitational constant, \( R \) is the scalar curvature, \( R_{\mu\nu} \) is the Ricci tensor, \( \Lambda \) is the cosmological constants, and \( g \) is the negative determinant of the metric tensor. Throughout this paper we follow the conventions of Ref. 9 and, in particular, the speed of light is set equal to unity. The Einstein equation in this theory is then written as,\[^2\]

\[ R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - \Lambda g_{\mu\nu} + c_1 J_{\mu\nu} + c_2 K_{\mu\nu} = -8\pi G T_{\mu\nu} , \]  

(2.2)

where \( J_{\mu\nu} \) and \( K_{\mu\nu} \) are defined by

\[ J_{\mu\nu} = 2R(R_{\mu\nu} - \frac{1}{4} R g_{\mu\nu}) + 2(R_{;\mu\nu} - g_{\mu\nu} \Box R) , \]  

(2.3)
\( K_{\mu\nu} = R_{;\mu\nu} - \square R_{\mu\nu} - \frac{1}{2}(\square R + R_{\alpha\beta} R^{\alpha\beta}) g_{\mu\nu} + 2 R^{\alpha\beta} R_{\mu\alpha\nu\beta} \), \hspace{1cm} (2.4)

with

\[
R_{;\mu\nu} \equiv \nabla_\mu \nabla_\nu R, \quad \square R \equiv g^{\alpha\beta} R_{\alpha\beta} .
\] \hspace{1cm} (2.5)

In order to obtain the modification to the Newton’s \(1/r^2\) law under this theory, we consider the following static, weak field limit:

\[
g_{00} \cong -(1 + 2\phi) ,
\] \hspace{1cm} (2.6)

\[
g_{ij} \cong \delta_{ij} (1 + 2\psi) ,
\] \hspace{1cm} (2.7)

where \(\phi\) and \(\psi\) are functions of spatial coordinates only. Note that \(\phi\) corresponds to the gravitational potential field. The Ricci tensor and scalar curvature are then given by

\[
R_{00} \cong -\Delta \phi ,
\] \hspace{1cm} (2.8)

\[
R_{ij} \cong (\partial_i^2 \phi + \partial_j^2 \psi + \Delta \psi) \delta_{ij} ,
\] \hspace{1cm} (2.9)

\[
R \cong 2 \Delta \phi + 4 \Delta \psi .
\] \hspace{1cm} (2.10)

Substituting these into the time-time component of Eq. (2.2), we obtain the equation for \(\phi\) and \(\psi\):

\[
\Delta[\phi + 2c_1 \Delta \phi + (4c_1 + 2c_2) \Delta \psi] = 4\pi G \rho ,
\] \hspace{1cm} (2.11)

where \(\rho\) is the mass density. The trace of Eq. (2.2), on the other hand, gives
another equation for $\phi$ and $\psi$:

$$[1 + (6c_1 + 2c_2)\Delta] \Delta (\phi + 2\psi) = -4\pi G\rho . \tag{2.12}$$

We have to solve (2.11) and (2.12) simultaneously. For the point source with

$$\rho = M\delta^{(3)}(\vec{r}) , \tag{2.13}$$

we obtain the following solution: \cite{10,11}

$$\phi = GM \left[ -\frac{1}{r} - \frac{1}{3} \frac{e^{-m_1 r}}{r} + \frac{4}{3} \frac{e^{-m_2 r}}{r} \right] , \tag{2.14}$$

$$\psi = GM \left[ \frac{1}{r} - \frac{1}{3} \frac{e^{-m_1 r}}{r} - \frac{2}{3} \frac{e^{-m_2 r}}{r} \right] , \tag{2.15}$$

where

$$m_1^2 = -\frac{1}{6c_1 + 2c_2} , \quad m_2^2 = \frac{1}{c_2} . \tag{2.16}$$

Note that the first term in (2.14) is the usual Newton potential and that the second and third terms in (2.14) correspond to its corrections. The third term in (2.14) is, however, undesirable, since its existence does not yield the usual attractive Newton’s force law in the limit $r \to 0$. We thus set

$$c_2 = 0 . \tag{2.17}$$

As for the constant $c_1$, a positive $c_1$ gives

$$\phi = -GM \left[ \frac{1}{r} + \frac{1}{3} \left( \cos \mu r \frac{\cos \mu r}{r} + y \sin \mu r \right) \right] , \tag{2.18}$$
where \( y \) is an arbitrary constant and

\[
\mu = \frac{1}{\sqrt{6c_1}}, \quad (2.19)
\]

while a negative \( c_1 \) gives

\[
\phi = -GM \left[ \frac{1}{r} + \frac{1}{3} e^{-\frac{m_1}{3}r} \right]. \quad (2.20)
\]

Both cases give the correct \( r \to 0 \) limit of the observed gravitational constant \( G_0 \) with

\[
G_0 = \frac{4G}{3}. \quad (2.21)
\]

Eq. (2.20), however, implies more decline of force than the \( 1/r^2 \) law as a function of \( r \), and this is opposite to what is expected from the rotation curves of galaxies. We thus adopt the solution of Eq. (2.18) and calculate rotational velocities of typical spiral galaxies. Taking into account the exponential mass distribution implied by the surface photometry data, \(^{[12]}\) we approximate a spiral galaxy by a truncated disk with mass density

\[
\rho(\vec{r}) = \begin{cases} 
\sigma_0 e^{-\frac{z}{a}} \delta(z), & (r \leq r_g) \\
0, & (r > r_g)
\end{cases} \quad (2.22)
\]

where we take

\[
r_g = 4.3a \text{ [kpc]} , \quad a = 2, 3, 4, 5, 6 , \quad (2.23)
\]

for the truncation radius. \(^{[13]}\) The value of \( \sigma_0 \) was also taken from Ref.11. Namely, we impose the condition that the total mass is \( 2.2 \times 10^{10} M_\odot \) for \( a = 2 \), where \( M_\odot \)
is the solar mass. The free parameter $\mu$ and $y$ in Eq. (2.18) were chosen so that the velocity curve becomes flat. Our choice is

$$\mu = \frac{1}{50} \left[ \frac{1}{\text{kpc}} \right], \quad (2.24)$$

$$y = 3. \quad (2.25)$$

We thus obtain the square of the rotation velocity as a function of the distance $r$ from the center of galaxy by

$$v^2(r) = \frac{\sigma_0}{4\pi} r \int_0^{r_g} dr' \int_0^{2\pi} d\theta' r'e^{-r'/a} \frac{\partial}{\partial r} \phi_0(|\vec{r} - \vec{r}'|), \quad (2.26)$$

where $\phi_0$ is the Green’s function of Eq. (2.18) with $M = 1$, and we only consider $\vec{r}$ on the plane of the galaxy disk.

In Fig. 1, we show the results of numerical integration of Eq. (2.26) for each value of $a$ with $r$ in the range of typical galaxy radius, i.e., 10—100 kpc. For a comparison, the result for $a = 2$ under usual Newtonian gravity is also shown by the broken curve in the figure. As is clear from the figure, the rotation velocities become flat for $r \gtrsim 50$ kpc, while that of Newtonian gravity steadily decreases as $r$ increases. However, the magnitudes of velocities are about half of what are observed for each value of $a$ in Eq. (2.23) for $r \gtrsim 50$ kpc.
In this section we study the cosmology under the generalized Einstein action. The action we consider is given by Eq. (2.1) with \( c_2 = 0 \). Assuming the homogeneous and isotropic universe, we employ the Robertson-Walker metric:

\[
ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2d\theta^2 + r^2\sin^2\theta d\phi^2\right), \tag{3.1}
\]

and the energy-momentum tensor \( T^{\mu\nu} \) of the form

\[
T^{\mu\nu} = pg^{\mu\nu} + (p + \rho)U^\mu U^\nu. \tag{3.2}
\]

The time-time component and space-space component of the Einstein equation, Eq. (2.2), are then respectively written as

\[
H^2 + K - \frac{\Lambda}{3} - 6c_1(-\ddot{H} + K^2 + 6\dot{H}H^2 - 2KH^2 + 2H\ddot{H}) = \frac{8\pi G}{3}\rho, \tag{3.3}
\]

and

\[
-2\dot{H} - 3H^2 - K + \Lambda + 6c_1(2\partial^3H + 12H\dddot{H} + 9\dot{H}^2 + 18H\dot{H}^2 - 4K\ddot{H} - 2KH^2 - K^2) = 8\pi Gp, \tag{3.4}
\]

where

\[
H = \frac{\dot{a}}{a}, \tag{3.5}
\]

and the dots stand for time derivatives. We now introduce a few dimensionless
parameters as follows;

\[ \tau \equiv H_0 t , \quad (3.6) \]

\[ \Omega_0 \equiv \frac{\rho_0}{\rho_c} , \quad (3.7) \]

\[ k_0 \equiv \frac{k}{H_0^2} , \quad (3.8) \]

\[ \lambda_0 \equiv \frac{\Lambda}{3H_0^2} , \quad (3.9) \]

where \( H_0 \) is the Hubble parameter at the present time, and the critical density \( \rho_c \) is defined by

\[ \rho_c \equiv \frac{3}{8\pi G} H_0^2 . \quad (3.10) \]

The density parameter \( \Omega_0 \) have two components:

\[ \Omega_0 \equiv \Omega_{N_0} + \Omega_{R_0} , \quad (3.11) \]

where \( \Omega_{N_0} \) and \( \Omega_{R_0} \) are respectively non-relativistic and relativistic components.

We can then write

\[ \frac{8\pi G}{3H_0^2} \rho = \frac{\Omega_0}{\rho_0} \rho = \frac{\Omega_{N_0}}{a^3} + \frac{\Omega_{R_0}}{a^4} , \quad (3.12) \]

where we have set the present value of the scale factor \( a_0 \) to unity:

\[ a_0 \equiv a(0) = 1 . \quad (3.13) \]

By (3.6) \( \sim (3.9) \), (3.12) and the energy-momentum conservation

\[ \dot{\rho} = -3 \left( \frac{\dot{a}}{a} \right) (\rho + p) , \quad (3.14) \]
we finally reduce (3.3) and (3.4) to

$$
\frac{a'''}{a} = - \frac{1}{12c_0} \left( \frac{a}{a'} \right) \left[ \frac{\Omega_{N_0}}{a^3} + \frac{\Omega_{R_0}}{a^4} - \left( \frac{a'}{a} \right)^2 - \frac{k_0}{a^2} + \lambda_0 \right] \\
- \frac{1}{2} \left( \frac{a}{a'} \right) \left[ 2 \frac{a'' a'}{a^3} - \left( \frac{a''}{a} \right) - 3 \left( \frac{a'}{a} \right)^4 \right] \\
+ \frac{k_0}{a^2} \left( \frac{k_0}{a^2} - 2 \left( \frac{a'}{a} \right)^2 \right),
$$

and

$$
\frac{a''''}{a} = - \frac{1}{12c_0} \left[ \frac{\Omega_{R_0}}{a^4} + 2 \frac{a''}{a} + \left( \frac{a'}{a} \right)^2 + \frac{k_0}{a^2} - 3\lambda_0 \right] \\
- \frac{1}{2} \left[ 4 \frac{a' a''}{a^2} - 12 \frac{a^2 a''}{a^3} + 3 \left( \frac{a''}{a} \right)^2 + 3 \left( \frac{a'}{a} \right)^4 \right] \\
- \frac{k_0}{a^2} \left( \frac{k_0}{a^2} + 4 \frac{a''}{a} - 2 \left( \frac{a'}{a} \right)^2 \right),
$$

respectively, where

$$
c_0 \equiv c_1 H_0^2,
$$

and the prime stands for derivative with respect to $\tau$. These are third-order and fourth-order differential equations for $a(\tau)$. In particular, we consider solving Eq. (3.16) numerically. For this we need initial condition of $a, a', a''$, and $a'''$. We have chosen $a_0 = 1$ (Eq. (3.13)). It then follows that

$$
a'_0 = 1,
$$

$$
a''_0 = -q_0,
$$
where $q_0$ is the deacceleration parameter

$$ q_0 \equiv -\frac{a_0''a_0}{a_0'^2}. \quad (3.20) $$

By choosing values of $\Omega_0, k_0, \lambda_0,$ and $q_0$ which are observationally acceptable, we can determine $a_0'''$ for each value of $c_0$ from Eq. (3.15) evaluated at the present time:

$$ \Omega_0 - k_0 + \lambda_0 + 6c_0 \left[ 2a_0'' - (a_0'')^2 + 2a_0'' - 3 + k_0^2 - 2k_0 \right] = 1. \quad (3.21) $$

In the present work, we set

$$ k_0 = 0, \quad (3.22) $$

$$ \lambda_0 = 0, \quad (3.23) $$

$$ \Omega_{R_0} = 10^{-5}, \quad (3.24) $$

for simplicity and use only positive $q_0$. (Non-zero values of $k_0$ and $\lambda_0$ or negative $q_0$ gave similar results unless their values are taken to be unnaturally large.) This leaves $\Omega_{N_0}, q_0(= -a_0'''),$ and $c_0$ as free parameters. In Fig. 2 we show the results of numerical integration of Eq. (3.16) for several sets of values of these parameters. For $c_0 > 0$, we have a de Sitter-like expansion (Fig. 2(a)), a bounce (Curve $f$ in Fig. 2(b)) which is essentially the bounce solution of Ref. 2, an expansion followed by an eventual collapse (Fig. 2(c)), or an expansion, shrink, and eventual de Sitter-like expansion (Curve $c$ in Fig. 2(d)). For $c_0 < 0$, on the other hand, we generally have oscillation: oscillating expansion or oscillating bounce (Fig. 2(e) and (f)).
This last solution, Curve $e$ in Fig. 2(e) in particular, is of interest, since it might explain the observed large scale structure of galaxy number counts as a function of red shift.\cite{6,7} The number of galaxies $dN$ which are located between the comoving distance $r$ and $r + dr$ at a fixed solid angle $d\Omega$ with absolute luminosity between $L$ and $L + dL$ is given by

$$dN = n(L,t)a^3 r^2 dr d\Omega dL,$$  

(3.25)

where $n(L,t)$ is the number density at time $t$ with absolute luminosity $L$, satisfying

$$n(L,t) = \left(\frac{a_0}{a}\right)^3 n(L,0).$$  

(3.26)

Since the redshift $z$ and scale factor $a$ are related by

$$z = \frac{a_0}{a} - 1,$$  

(3.27)

we have

$$r = \int_{t}^{0} \frac{dt}{a} = \int_{t}^{0} \frac{z(t') + 1}{a_0} dt'.$$  

(3.28)

By substituting (3.26) and (3.28) into (3.25), we have\cite{8}

$$\frac{dN}{dz d\Omega dL} \propto \left[\int_{t}^{0} (z(t') + 1) dt'\right]^2 \frac{1}{H(t)}.$$  

(3.29)

Note that Eq. (3.27) gives a differential equation for $z(t)$:

$$\frac{dz}{dt} = -(1 + z)H(t).$$  

(3.30)

Hence, by solving Eqs.(3.16) and (3.30) simultaneously, we obtain the number
count in Eq. (3.29). The results for

\[ \Omega_{N_0} = 0.1 \], \quad (3.31) \\
\[ a''_0 = -1.0 \], \quad (3.32) \\
\[ c_0 = -0.0002 \], \quad (3.33) \\

are shown in Fig. 3 together with the scale factor \( a(\tau) \). Note that there exist four peaks in the galaxy count for \( z < 0.5 \) with the first one corresponding to the location of Great Wall\(^6\). Even though the peaks are not separated with equal interval, overall features are remarkably similar to observations\(^7\).

4. Concluding Remark

In this paper we discussed a possible generalization of the Einstein theory by adding a square term, \( c_1 R^2 \), of scalar curvature to the action. Choosing appropriate values of the coefficients, we found successful explanation of the flat rotation curves of spiral galaxies \( (c_1 H_0^2 \approx 5 \times 10^{-11}) \) and the large scale structure of universe \( (c_1 H_0^2 \approx -2 \times 10^{-4} ) \) at least at a qualitative level. Our theory, however, is not all faultless. It turned out that the coefficient \( \gamma \) of the Robertson expansion is \( 1/2 \), and this disagrees strongly with, for instance, the radar echo experiments from Mercury, which implies that \( \gamma \) is very close to 1 in accord with the Einstein theory\(^{14}\). Originally \( R^2 \) term was introduced as a result of quantum corrections\(^1\) so that the coefficient \( c_1 \) will have the logarithmic \( r \) dependence as \( c_1 = (\text{const.}) \times \log(r/r_0) \). This interpretation may lead to a consistent explanation of both the rotation curves.
of spiral galaxies and the large scale structure of the universe from the single effective Lagrangian with the running coefficient $c_1$ that depends on the distance scale of interest. The difficulty of $\gamma = 1/2$ may also be avoided if $r_0$ takes a value of the order of solar radius.

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**FIGURE CAPTIONS**

1) Rotation curves of spiral galaxies under the generalized Einstein action. The numbers on the curves indicate the values of $a$ in Eq. (2.23). The prediction of Newtonian theory for $a = 2$ is also given by a broken curve for comparison.

2) Scale factor of universe as a function of $\tau$ for various values of parameters $c_0$, $\Omega_{N_0}$, and $a_0''$. The parameter $c_0$ is positive for (a) $\sim$ (d) and negative for (e) and (f). The values of $|c_0|$ are indicated in the figure by the letters $a$, $b$, $c$, $d$, $e$, $f$ for $10^0$, $10^{-1}$, $10^{-2}$, $10^{-3}$, $10^{-4}$, $10^{-5}$, respectively.

3) (a) Scale factor of universe as a function of $\tau$ for the set of parameters in (3.31) $\sim$ (3.33) and (b) number count of galaxies as a function of $z$ for the same set of parameters. The scale of the ordinate for (b) is arbitrary.