Generalized Lipkin–Meshkov–Glick models of Haldane–Shastry type

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Received 27 June 2017
Accepted for publication 24 August 2017
Published 16 October 2017

Abstract. We introduce a class of generalized Lipkin–Meshkov–Glick (gLMG) models with $\text{su}(m)$ interactions of Haldane–Shastry type. We compute the partition function of these models in closed form by exactly evaluating the partition function of the restriction of a spin chain Hamiltonian of Haldane–Shastry type to subspaces with well-defined magnon numbers. As a byproduct of our analysis, we obtain strong numerical evidence of the Gaussian character of the level density of the latter restricted Hamiltonians, and study the distribution of the spacings of consecutive unfolded levels. We also discuss the thermodynamic behavior of a large family of $\text{su}(2)$ and $\text{su}(3)$ gLMG models, showing that it is qualitatively similar to that of a two-level system.

Keywords: integrable spin chains and vertex models, solvable lattice models
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1. Introduction

One of the first, and still one of the few, quantum mechanical many-body models that has been solved in the literature is the Lipkin–Meshkov–Glick (LMG) model [1–3], which describes a system of \( N \) fermions with two \( N \)-fold degenerate one-particle levels. The original motivation for introducing this model was testing the validity of different approximation schemes from solid state physics or field theory in the context of nuclear physics. Over the years, the LMG model has appeared in connection with a wide range of problems of physical interest, including shape transitions in nuclei [4], trapped ion and optical cavity experiments [5, 6], two-modes Bose–Einstein condensates [7–9], and quantum information theory [10–14]. In particular, it has been shown that the von Neumann entanglement entropy of its ground state grows logarithmically with the size of the subsystem, as is the case for one-dimensional critical systems [15–18] (although this model is actually not critical [19]).

As already noted in the original papers, the key to the solvability of the LMG model is the fact that it can be mapped to a system of \( N \) spin-1/2 particles with constant long-range interactions of XY type in an external transverse magnetic field. In the isotropic (XX) case the Hamiltonian of this effective model is a polynomial in \( \mathbf{J}^2 \) and \( J_z \), where \( \mathbf{J} \) is the the total spin operator, and can thus be exactly solved for arbitrary \( N \). The general (non-isotropic) LMG model can be solved in principle via the Bethe ansatz [20, 21], though in practice this is less efficient than brute-force numerical diagonalization.

https://doi.org/10.1088/1742-5468/aa8c14
In the thermodynamic limit, however, the density of states of the latter model in the highest spin sector \( J = N/2 \) has been derived by means of a spin-coherent-state formalism \([22, 23]\).

A wide family of models with long-range interactions of \( \text{su}(m) \) type generalizing the isotropic LMG model was recently introduced in \([19]\). In analogy with the latter model, the non-degenerate ground state of these novel models is given by a Dicke state whose reduced density matrix for a subsystem of \( L < N \) spins can be computed in closed form, which in turn yields the entanglement entropy in the thermodynamic limit \( N \to \infty \) with \( L/N = \alpha \) finite. Although both the von Neumann and the Rényi entanglement entropies grow logarithmically with the size \( L \) of the subsystem, the corresponding prefactor is independent of the Rényi parameter, which implies that none of these models can be critical. Interestingly, for \( m > 3 \) there is at least one quantum phase whose Tsallis entanglement entropy \([24, 25]\) becomes extensive for a suitable value of the Tsallis parameter. However, the full spectrum of these models in general cannot be evaluated in closed form.

In this paper we introduce a family of generalized Lipkin–Meshkov–Glick (gLMG) models, with interactions governed by an \( \text{su}(m) \) integrable spin chain of Haldane–Shastry type. The latter chains are the celebrated Haldane–Shastry (HS) \( \text{su}(m) \) spin chain \([26–28]\), which describes a circular array of equispaced spins with two-body long-range interactions inversely proportional to the square of the (chord) distance, and its rational \([29, 30]\) and hyperbolic \([31]\) analogues. Although the HS chain was originally introduced as a model whose exact ground state coincides with Gutzwiller’s variational wave function for the Hubbard model in the limit of large on-site interaction \([32, 33]\), it soon proved of interest per se in condensed matter and theoretical physics. Indeed, as pointed out by Haldane \([34]\), the spinon excitations of this chain provide one of the simplest examples of a quantum system featuring fractional statistics (see also \([28, 35, 36]\)). The HS chain is closely connected to important conformal field theories like the \( k = 1 \) Wess–Zumino–Novikov–Witten model \([34, 37]\), and has recently been related to infinite matrix product states \([38]\). Integrable extensions of the Haldane–Shastry chain with long-range interactions involving more than two spins also play a key role for describing non-perturbatively the spectrum of planar \( N = 4 \) gauge theory in the context of the AdS-CFT correspondence \([39, 40]\). The interest in spin chains of HS type has been further reinforced by recent developments in quantum simulation, as witnessed by the proposal of an experimental realization of the HS chain using two internal atomic states of atoms trapped in a photonic crystal waveguide \([41]\).

One of the key features of spin chains of Haldane–Shastry type is the fact that their partition functions can be exactly computed for any number of spins \([42–44]\) by exploiting their connection with a corresponding spin dynamical model of Calogero–Sutherland type \([45–48]\) through a mechanism known as the Polychronakos ‘freezing trick’ \([42]\). This has made it possible to check the validity of several fundamental conjectures on the characterization of quantum chaos versus integrability \([49, 50]\). In particular, it has been shown that spin chains of HS type do not behave as expected for a ‘generic’ integrable system, in the sense that the distribution of the spacings between consecutive levels is not Poissonian \([43, 44, 51]\).

The gLMG models that we introduce in this paper can also be regarded as a deformation of the \( \text{su}(m) \) spin chains of HS type. More precisely, we add to the HS-type
Hamiltonian a term depending on the generators of the standard $\text{su}(m)$ Cartan subalgebra, which commutes with the former Hamiltonian. In particular, when this extra term is linear in the Cartan generators it can be interpreted as an $\text{su}(m)$ external magnetic field, and the corresponding models are the ones studied in reference [52]. Likewise, when the extra term is a suitable quadratic combination of the Cartan generators we recover the models introduced in reference [19], which include the isotropic LMG model. We shall see that the Hilbert space of a general gLMG model decomposes as a direct sum of subspaces with fixed magnon numbers, which are separately invariant under the action of both the original HS-type Hamiltonian and the new term. By suitably adapting the freezing trick, we shall be able to compute the partition function of the restriction of the Hamiltonians of the three spin chains of HS type to the latter invariant subspaces. This in turn yields the partition function of the full gLMG Hamiltonian, since the Cartan generators are proportional to the identity on these subspaces. The knowledge of the partition function of the gLMG models of HS type, as well as the restricted partition functions of the corresponding spin chains, enables one to study several statistical properties of the spectrum of the latter models. In particular, we have obtained strong numerical evidence that the level density of the restriction of the HS-type chain Hamiltonians to subspaces with fixed magnon numbers follows a Gaussian distribution in the large $N$ limit, as is known to be the case for the full spectrum of these models [53, 54]. We have also studied the distribution of the spacings between consecutive levels of the restrictions of these models to the invariant subspaces, showing that it follows the characteristic law for an approximately equispaced spectrum with normally distributed energy levels [44, 51]. Finally, we have numerically computed the thermodynamic functions of gLMG models of HS type whose extra term is quadratic in the Cartan generators, comparing them with the exact results for the original (HS-type) chains in the thermodynamic limit derived in reference [52].

2. The models

The models we shall study in this paper are deformations of $\text{su}(m)$ spin chains with Hamiltonians of the form

$$H_0 = \sum_{1 \leq i < j \leq N} h_{ij} (1 - \epsilon S_{ij}), \quad \epsilon = +, -,$$

with $h_{ij} \in \mathbb{R}$. In the latter equation $S_{ij}$ is the operator permuting the $\text{su}(m)$ spins of the $i$th and $j$th particles, whose action on the canonical $\text{su}(m)$ spin basis

$$S = \{ |s_1 \rangle \otimes \cdots \otimes |s_N \rangle \equiv |s_1, \ldots, s_N \rangle | s_i = 1, \ldots, m, \; 1 \leq i \leq N \},$$

is given by

$$S_{ij} |s_1, \ldots, s_i, \ldots, s_j, \ldots, s_N \rangle = |s_1, \ldots, s_j, \ldots, s_i, \ldots, s_N \rangle.$$

These operators can be expressed in terms of the local (Hermitian) generators $t^a_k$ ($a = 1, \ldots, m^2 - 1$) of the fundamental representation of the $\text{su}(m)$ algebra acting on the $k$th site (with the normalization $\text{tr}(t^a_k t^b_k) = \frac{1}{2} \delta_{ab}$) as

$$S_{ij} = \sum_{k=1}^N t^a_k (1 - \epsilon \omega_k),$$

where $\omega_k$ is the Pauli matrix in the $k$th site.
\[ S_{ij} = \frac{1}{m} + 2 \sum_{a=1}^{m^2-1} t_i^a t_j^a \equiv \frac{1}{m} + 2 \mathbf{t}_i \cdot \mathbf{t}_j. \]  

(2.3)

We can thus write\(^1\)

\[ H_0 = -\epsilon \sum_{i \neq j} h_{ij} \mathbf{t}_i \cdot \mathbf{t}_j + E_0, \]

with \( E_0 = (1 - \frac{\epsilon}{m^2}) \sum_{i<j} h_{ij}. \) In particular, for \( m = 2 \) we have \( \mathbf{t}_k = \frac{1}{2} \sigma_k \), where \( \sigma_k = (\sigma^1_k, \sigma^2_k, \sigma^3_k) \) are the three Pauli matrices acting on the \( k \)th site.

Let \( N_a \) denote the \( a \)th magnon number operator defined by

\[ N_a |s_1, \ldots, s_N\rangle = N_a |s_1, \ldots, s_N\rangle, \quad 1 \leq a \leq m, \]

(2.4)

where\(^2\)

\[ N_a = |\{ k = 1, \ldots, N \mid s_k = a \}|. \]

(2.5)

The latter operators are related to the Hermitian generators of the standard Cartan subalgebra of the Lie algebra \( su(m) \), as we shall now explain. Indeed, let \( J^a_k \) denote the operator whose action on the Hilbert space of the \( k \)th particle is given by

\[ J^a_k |s_k\rangle = (\delta_{a,s_k} - \delta_{m,s_k}) |s_k\rangle, \quad 1 \leq a \leq m - 1. \]

(2.6)

The \( m - 1 \) commuting operators \( iJ^a_k \) generate the standard Cartan subalgebra\(^3\) of \( su(m) \) at each site \( k \). We then define the global (Hermitian) Cartan generators

\[ J^a \equiv \sum_{k=1}^{N} J^a_k, \quad 1 \leq a \leq m - 1. \]

From equation (2.6) it then follows that

\[ J^a = N_a - N_m, \quad 1 \leq a \leq m - 1. \]

Summing over \( a \) and taking into account that \( \sum_{a=1}^{m} N_a = N \) we obtain

\[ \sum_{a=1}^{m-1} J^a = N - m N_m, \quad 1 \leq a \leq m - 1. \]

Using the last two equations we can express the magnon number operators in terms of the Cartan subalgebra generators as

\[ N_a = J^a(1 - \delta_{am}) - \frac{N}{m}, \quad 1 \leq a \leq m, \]

(2.7)

where

\(^1\) Here and throughout the paper, all sums and products run from 1 to \( N \) unless otherwise specified.

\(^2\) We shall denote in what follows by \(|A|\) the cardinal of the set \( A \).

\(^3\) This choice of the generators of the standard Cartan subalgebra of \( su(m) \) is simply a matter of convenience. Note, however, that these generators are not orthogonal with respect to the usual Killing–Cartan scalar product, i.e. \( \text{tr}(J^a f^b) \neq 0 \) for \( a \neq b \).
We shall consider in what follows deformations $H = H_0 + H_1$ of (2.1) in which

$$H_1 = h(N_1, \ldots, N_m)$$

(2.8)
is an analytic function of the magnon number operators $N_a$. Note, first of all, that the previous expression for $H_1$ is not ambiguous, since $[N_a, N_b] = 0$ for $1 \leq a, b \leq m$. It is also clear that $iH_1$ lies in the enveloping algebra of the $su(m)$ Cartan subalgebra on account of equation (2.7). For this reason, we shall say that

$$H = H_0 + H_1 = \sum_{i<j} h_{ij}(1 - \epsilon S_{ij}) + h(N_1, \ldots, N_m)$$

(2.9)
is an $su(m)$ generalized Lipkin–Meshkov–Glick (gLMG) model. In particular, when $h_{ij} > 0$ for all $i < j$, $\epsilon = +$ and $h$ is the quadratic polynomial

$$h(x_1, \ldots, x_m) = \sum_{a=1}^{m-1} c_a(x_a - x_m - Nh_a)^2, \quad \text{with } h_a \in \mathbb{R}, \ c_a > 0,$$

we obtain the models whose ground state entanglement entropy was computed in closed form in reference [19]. The latter models include the original (su(2), isotropic) LMG model when $h_{ij} = 2/N$ for all $i < j$ and $c_1 = 1/(2N)$, up to a constant energy.

One of the fundamental properties of the Hamiltonian (2.9) is that it preserves the subspaces of the Hilbert space $\mathcal{H} \equiv (\mathbb{C}^m)^{\otimes N}$ with a fixed magnon configuration. Indeed, let us denote by $\mathcal{H}(\mathbf{N})$, where $\mathbf{N} = (N_1, \ldots, N_m)$ and $|\mathbf{N}| = N_1 + \cdots + N_m = N$, the subspace of $\mathcal{H}$ whose elements are linear combinations of basis states $|s_1, \ldots, s_N\rangle \equiv |\mathbf{s}\rangle$ with magnon numbers $N_a$ (see equation (2.5)). Clearly $H_0$ leaves $\mathcal{H}(\mathbf{N})$ invariant, since each permutation operator $S_{ij}$ does. On the other hand, $N_a|\mathbf{s}\rangle = N_a|\mathbf{s}\rangle$ on $\mathcal{H}(\mathbf{N})$ by construction, and therefore

$$H_1 = h(\mathbf{N}) \quad \text{on } \mathcal{H}(\mathbf{N}).$$

Thus $H = H_0 + H_1$ preserves $\mathcal{H}(\mathbf{N})$, as stated. It is also clear from the above discussion that $[H_0, H_1] = 0$, and that the eigenvalues of $H^\mathbf{N} \equiv H|\mathcal{H}(\mathbf{N})\rangle$ can be expressed as

$$E_i^\mathbf{N}(\mathbf{N}) + h(\mathbf{N}), \quad 1 \leq i \leq \text{dim } \mathcal{H}(\mathbf{N}),$$

where $\{E_i^\mathbf{N}(\mathbf{N})\}_{1 \leq i \leq \text{dim } \mathcal{H}(\mathbf{N})}$ is the spectrum of $H_0^\mathbf{N} \equiv H_0|\mathcal{H}(\mathbf{N})\rangle$. Hence the partition function $Z^\mathbf{N}(T)$ of $H^\mathbf{N}$ is given by

$$Z^\mathbf{N}(T) = q^{h(\mathbf{N})} \sum_{i=1}^\text{dim } \mathcal{H}(\mathbf{N}) q^{E_i^\mathbf{N}(\mathbf{N})} \equiv q^{h(\mathbf{N})} Z_0^\mathbf{N}(T), \quad q \equiv e^{-1/k_B T},$$

where $Z_0^\mathbf{N}(T)$ is the partition function of $H_0^\mathbf{N}$. Since

$$\mathcal{H} = \bigoplus_{|\mathbf{N}|=N} \mathcal{H}(\mathbf{N}),$$

https://doi.org/10.1088/1742-5468/aa8c14
the partition function of \( H \) is given by
\[
Z(T) = \sum_{\{N\}=N} Z^N(T) = \sum_{\{N\}=N} q^{h(N)} Z_0^N(T).
\] (2.10)

Thus the partition function of the model (2.9) is completely determined by the partition functions \( Z_0^N(T) \) of the restrictions of the spin chain Hamiltonian \( H_0 \) to each of the subspaces \( \mathcal{H}(N) \). We shall see in the following sections that the latter partition functions can be computed in closed form when \( H_0 \) is the Hamiltonian of one of the three spin chains of HS type, namely the Haldane–Shastry [26, 27], Polychronakos–Frahm (PF) [29, 30] and Frahm–Inozemtsev (FI) [31] chains. The chain sites of these integrable spin chains can be expressed as
\[
z_k = \begin{cases} 
  k\pi/N, & \text{for the HS chain} \\
  \zeta_k, & \text{for the PF chain} \\
  e^{2\xi_k}, & \text{for the FI chain}
\end{cases}
\] (2.11)

where \( \zeta_k \) and \( \xi_k \) respectively denote the \( k \)th zero of the Hermite polynomial of degree \( N \) and the generalized Laguerre polynomial \( L_{\beta-2N+1}^{2N+1} \) with \( \beta > 2(N-1) \). In all three cases, the interaction strength is a function \( h_{ij} = h(z_i - z_j) \) of the difference \( z_i - z_j \), namely
\[
h(x) = \begin{cases} 
  \frac{1}{2} \sin^{-2} x, & \text{for the HS chain} \\
  x^{-2}, & \text{for the PF chain} \\
  \frac{1}{2} \sinh^{-2} x, & \text{for the FI chain}
\end{cases}
\] (2.12)

Remarkably, the (total) partition function \( Z_0(T) = \sum_{\{N\}=N} Z_0^N(T) \) of all of these models can be computed in closed form by exploiting their close connection with their associated spin Calogero–Sutherland models (see, e.g. [42–44]). In the following sections we shall adapt this technique, known in the literature as Polychronakos’s freezing trick [42], to evaluate the restricted partition functions \( Z_0^N(T) \).

3. The freezing trick

In this section we shall outline the computation of the restricted partition function \( Z_0^N \) for the Haldane–Shastry spin chain, which is the best known of these models and presents certain technical subtleties stemming from its translation invariance. To this end, we first recall that in this case \( H_0 \) is related to the strong interaction limit of the spin Sutherland model
\[
H_{sp} = -\Delta + a \sum_{i \neq j} \sin^{-2}(x_i - x_j)(a - \epsilon S_{ij}), \quad a > 0,
\]
where \( \Delta \equiv \sum_i \partial^2_{x_i} \). Indeed, we can write
\[
H_{sp} = H_{sc} + 4a \hat{H}_0(x),
\]
where \( x = (x_1, \ldots, x_N) \).
\[ H_{sc} = -\Delta + a(a - 1) \sum_{i \neq j} \sin^{-2}(x_i - x_j) \]

is the scalar Sutherland model and
\[ \hat{H}_0(\mathbf{x}) = \frac{1}{2} \sum_{i < j} \sin^{-2}(x_i - x_j)(1 - \epsilon S_{ij}) \]

is obtained from \( H_0 \) replacing the chain sites \( z_i \) by the dynamical variables \( x_i \). Since \( H_{sp} \) and \( H_{sc} \) are translation invariant, the total momentum is conserved and can be set to zero by working in the center of mass frame. In the strong interaction limit \( a \to \infty \) the eigenfunctions of \( H_{sp} \) become sharply peaked at the coordinates of the minimum of the scalar potential
\[ U(\mathbf{x}) = \sum_{i \neq j} \sin^{-2}(x_i - x_j) \]

in the configuration space (\( A_{N-1} \) Weyl chamber)
\[ A = \{ \mathbf{x} \in \mathbb{R}^N \mid x_1 < \cdots < x_N \}, \]

which (up to an overall translation) coincide with the chain sites \( z_k = k\pi/N \). Thus, when \( a \gg 1 \) the eigenvalues of \( H_{sp} \) are approximately given by
\[ E_{ij} \simeq E_{ij}^{sc} + 4aE_0^j, \quad a \gg 1, \]

where \( E_{ij}^{sc} \) and \( E_0^j \) respectively denote two arbitrary eigenvalues of \( H_{sc} \) and \( H_0 \). From the latter equation it immediately follows that the partition function \( Z_0(T) \) of the Haldane–Shastry chain is given by the freezing trick formula
\[ Z_0(T) = \lim_{a \to \infty} \frac{Z_{sp}(4aT)}{Z_{sc}(4aT)}. \tag{3.1} \]

This is the basis for the computation of \( Z_0(T) \) in reference [43]. We shall now show that the same procedure can be carried out to compute the restricted partition functions \( Z_0^N(T) \). Essentially, this is due to the fact that the spin Hamiltonian \( H_{sp} \) preserves the subspaces \( L^2(A) \otimes \mathcal{H}(N) \) of its Hilbert space \( L^2(A) \otimes \mathcal{H} \). Thus, \( Z_0^N \) can be obtained from the analogue of equation (3.1), namely
\[ Z_0^N(T) = \lim_{a \to \infty} \frac{Z_{sp}^N(4aT)}{Z_{sc}(4aT)}. \tag{3.2} \]

where \( Z_{sp}^N \) is the partition function of \( H_{sp}^N = H_{sp}|_{L^2(A) \otimes \mathcal{H}(N)} \).

To begin with, note that the Hamiltonian \( H_{sp} \) is equivalent to its symmetric/antisymmetric extension to the Hilbert space \( \Lambda_{\pm}(L^2(\mathbb{R}^N) \otimes \mathcal{H}) \), where \( \Lambda_+ \) (resp. \( \Lambda_- \)) is the symmetrizer (resp. antisymmetrizer) with respect to permutations of the particles’ coordinates and \( su(m) \) spin variables. This is basically due to the fact that any point \( \mathbf{x} \in \mathbb{R}^N \) not lying on the singular hyperplanes \( x_i - x_j = 0 \) can be mapped in a unique way to a point in \( A \) by a suitable permutation. As we shall see below, it shall be convenient for what follows to identify \( H_{sp} \) with its symmetric (resp. antisymmetric) extension when \( \epsilon = 1 \) (resp. \( \epsilon = -1 \)). With this identification, it can be shown [43] that
$H_{sp}$ is represented by an upper triangular matrix in the appropriately ordered (non-orthonormal) basis with elements

$$|p, s\rangle = \Lambda_i \left( e^{2ip \times \prod_{i<j} \sin(x_i - x_j)^a} |s\rangle \right),$$

(3.3)

where $|s\rangle \in S$ and $p \equiv (p_1, \ldots, p_N) \in \mathbb{R}^N$ satisfy the following conditions:

(i) The differences $n_i \equiv p_i - p_{i+1}$ ($1 \leq i \leq N - 1$) are nonnegative integers.

(ii) If $p_i = p_{i+1}$ then $s_i < s_{i+1}$.

(iii) The total momentum of the state $|p, s\rangle$ vanishes, i.e. $\sum_i p_i = 0$.

In the second condition, the notation $s_i < s_j$ stands for $s_i < s_j$ when $\epsilon = -1$ and $s_i \leq s_j$ when $\epsilon = 1$. The first condition is justified in [43], the second one can be arranged due to the symmetric/antisymmetric nature of the states (3.3), while the last one simply reflects that we are working in the center of mass frame. As shown in the latter reference, the states $|p, s\rangle$ should be ordered in such a way that $|p, s\rangle$ precedes $|p', s'\rangle$ whenever $p < p'$, where the last notation means that $p$ precedes $p'$ in the lexicographic order. With this partial order, the action of $H_{sp}$ on the basis (3.3) is upper triangular. More precisely [43],

$$H_{sp}|p, s\rangle = E(p)|p, s\rangle + \sum_{p' < p; s'} c(p', s') |p', s'\rangle,$$

(3.4)

with $c(p', s') \in \mathbb{C}$ and

$$E(p) = \sum_i \left[ 2p_i + a(N + 1 - 2i) \right]^2.$$  

(3.5)

Since $H_{sp}$ preserves $\mathcal{H}(N)$, if the vector $s$ in equation (3.4) is such that $|s\rangle \in \mathcal{H}(N)$ then $|s'\rangle \in \mathcal{H}(N)$ for all vectors $s'$ appearing in the RHS of the latter equation. In other words, $H_{sp}^N$ is also upper triangular with respect to the basis (3.3), where $|s\rangle \in S \cap \mathcal{H}(N)$ and the quantum numbers $(p, s)$ satisfy conditions (i)–(iii) above, ordered as previously explained. Moreover, by equation (3.4) the eigenvalues of $H_{sp}^N$ are given by equation (3.5). Expanding the latter equation in powers of $a$ we obtain

$$E(p) = E_{GS} + 4a \sum_i p_i (N + 1 - 2i) + O(1),$$

where

$$E_{GS} = \frac{a^2}{3} N(N^2 - 1)$$

is the ground state energy of the ferromagnetic model ($\epsilon = 1$). Thus in the limit $a \to \infty$ we have

$$\lim_{a \to \infty} q^{-E_{GS}/4a} Z_{sp}^N(4aT) = \sum_{p, s} q^{\sum_i p_i (N + 1 - 2i)},$$

https://doi.org/10.1088/1742-5468/aa8c14
where the sum is extended to all \((p, s)\) satisfying conditions (i)–(iii) above with \(|s| \in S \cap H(N)\). Since the exponent is independent of the spin variables \(s\), the sum over \(s\) can be immediately carried out, namely

\[
\lim_{a \to \infty} q^{-E_{GS}/4a} Z_{sp}^N(4aT) = \sum_{p} d(p, N, \epsilon) q^{\sum \rho_i(N + 1 - 2i)},
\]

(3.6)

where the spin degeneracy factor \(d(p, N, \epsilon)\) is the number of multiindices \(s\) satisfying condition (ii) above for a given \(p\) such that \(|s| \in S \cap H(N)\). In other words,

\[
d(p, N, \epsilon) = |S(p, \epsilon) \cap H(N)|
\]

(3.7)

where

\[
S(p, \epsilon) \equiv \{|s| \in S | p_i = p_{i+1} \Rightarrow s_i < s_{i+1}, 1 \leq i \leq N - 1\}.
\]

(3.8)

In order to evaluate the sum in equation (3.6), we note that by conditions (i) and (iii) above we can write the multiindex \(p\) as

\[
p = (p_1, \ldots, p_1, \ldots, p_r, \ldots, p_r),
\]

(3.9)

with

\[
k_1 + \cdots + k_r = N, \quad k_1 \rho_1 + \cdots + k_r \rho_r = 0, \quad \rho_i > \rho_{i+1}, \quad \rho_i - \rho_{i+1} \in \mathbb{N}.
\]

(3.10)

Thus the multiindex \(p\) consists of \(r\) blocks of lengths \(k_1, \ldots, k_r\). Calling

\[
K_i = \sum_{j=1}^{i} k_j, \quad 0 \leq i \leq r,
\]

(3.11)

we have

\[
\sum_{i} p_i(N + 1 - 2i) = \sum_{i=1}^{r} \rho_i \sum_{j=K_{i-1}+1}^{K_i} (N + 1 - 2j) = \sum_{i=1}^{r} \rho_i k_i(N - 2K_i + k_i).
\]

Since \(d(p, N, \epsilon)\) obviously depends on \(p\) only through \(k \equiv (k_1, \ldots, k_r)\), we can rewrite equation (3.6) as

\[
\lim_{a \to \infty} q^{-E_{GS}/4a} Z_{sp}^N(4aT) = \sum_{p} \sum_{k \in \mathcal{P}_N} d(p, N, \epsilon) \sum_{\rho_1 > \cdots > \rho_r, \rho_1 - \rho_{1+1} \in \mathbb{N}} q^{\sum_{i=1}^{r} \rho_i k_i(N - 2K_i + k_i)},
\]

(3.12)

where \(\mathcal{P}_N\) denotes the set of all partitions of the integer \(N\) in \(r\) parts with order taken into account. The inner sum in equation (3.12) was evaluated in [43], with the result

\[
\sum_{\rho_1 > \cdots > \rho_r, \rho_1 - \rho_{1+1} \in \mathbb{N}} q^{\sum_{i=1}^{r} \rho_i k_i(N - 2K_i + k_i)} = \prod_{i=1}^{r-1} \frac{q^{E(K_i)}}{1 - q^{E(K_i)}},
\]

(3.13)

where

\[
E(k) = k(N - k).
\]

(3.14)
Substituting equations (3.13) into (3.12) we obtain
\[
\lim_{a \to \infty} q^{-E_{\text{GS}}/4a} Z_{\text{sp}}(4aT) = \sum_{r=1}^{N} \sum_{k \in P_N^r} d(p, N, \epsilon) \prod_{i=1}^{r-1} \frac{q^{E(K_i)}}{1 - q^{E(K_i)}},
\]
(3.15)
where \( p \) is any multiindex of the form (3.9). The partition function for the scalar Hamiltonian was also evaluated in [43] in the large \( a \) limit, namely
\[
\lim_{a \to \infty} q^{-E_{\text{GS}}/4a} Z_{\text{sp}}(4aT) = \prod_{i=1}^{N} (1 - q^{E(i)})^{-1}.
\]
(3.16)
Combining equations (3.15) and (3.16) with (3.2) we finally obtain the following explicit formula for the restricted partition function \( Z_0^N(T) \):
\[
Z_0^N(T) = \sum_{r=1}^{N} \sum_{k \in P_N^r} d(p, N, \epsilon) \prod_{i=1}^{r-1} q^{E(K_i)} \cdot \prod_{j=1}^{N-r} (1 - q^{E(K'_j)}),
\]
(3.17)
where
\[
\{K'_1, \ldots, K'_{N-r}\} = \{1, \ldots, N - 1\} - \{K_1, \ldots, K_{r-1}\}
\]
and \( p \) is determined by \( k \) through equation (3.9). Following a similar procedure for the PF and FI chains we again obtain equation (3.17), but with \( E(k) \) in equation (3.14) respectively given by \( k \) and \( k(\beta - 2N + k + 1) \) (see [44, 51] for more details). In summary, the restricted partition function \( Z_0^N(T) \) for the three chains of HS type is given by equation (3.17), with dispersion relation
\[
E(k) = \begin{cases} 
  k(N - k), & \text{for the HS chain} \\
  k, & \text{for the PF chain} \\
  k(\beta - 2N + k + 1), & \text{for the FI chain} 
\end{cases}
\]
(3.18)
Equations (2.10)–(3.17) yield an explicit formula for the partition function of the \( \text{su}(m) \) gLMG model (2.9) with interactions \( h_{ij} = h(z_i - z_j) \) given by equations (2.11) and (2.12), once the degeneracy factor \( d(p, N, \epsilon) \) is known.

4. Degeneracy factor

As we have seen in the previous section, in order to evaluate the partition function \( Z(T) \) of an \( \text{su}(m) \) gLMG model of HS type through equations (2.10)–(3.17), we only need to determine the degeneracy factor \( d(p, N, \epsilon) \) defined in equation (3.7). To this end, let us fix \( p \) in equation (3.9) (with \( k \in P_N^r \)) and take \( N = (N_1, \ldots, N_m) \) such that \( N_a \in [0, \infty)) \equiv N \cup \{0\} \) and \( |N| = N \). The degeneracy factor \( d(p, N, \epsilon) \) is obviously much easier to compute in the antiferromagnetic case \( (\epsilon = -1) \), since by Pauli’s principle the \( \text{su}(m) \) spins in each block of length \( k_1, \ldots, k_r \) in which the components of \( p \) are equal must all be different (in fact, arranged in a strictly increasing sequence according to condition (ii) in the previous section).
4.1. Anti-ferromagnetic case

Let us define the vector \( \mathbf{r} = (r_1, \ldots, r_m) \) by

\[
r_i \equiv \left| \{ j = 1, \ldots, r \mid k_j = i \} \right| \in \mathbb{N}_0, \quad 1 \leq i \leq m,
\]

so that

\[
r_1 + \cdots + r_m = r, \quad r_1 + 2r_2 + \cdots + mr_m = N. \quad (4.1)
\]

In other words, \( r_i \) is the number of blocks of length \( i \) in the expression (3.9) for \( \mathbf{p} \). Obviously \( d(\mathbf{p}, \mathbf{N}, -) \equiv D_m(\mathbf{r}, \mathbf{N}) \), where \( D_m(\mathbf{r}, \mathbf{N}) \) denotes the number of ways one can distribute \( N_i \) spins \( |1\rangle \), \( N_2 \) spins \( |2\rangle \), \( N_m \) spins \( |m\rangle \) in \( r_1 \) blocks of one site, \( r_2 \) blocks of two sites, etc., \( r_m \) blocks of \( m \) sites, with all spins different in each block.

For \( 1 \leq i, j \leq m \), let us denote by \( N_{ij} \in \mathbb{N}_0 \) the number of spins \( |i\rangle \) in the \( r_j \) blocks of \( j \) sites, and define \( \mathbf{N}_i = (N_{i,1}, \ldots, N_{i,m}) \) such that \( \left| \mathbf{N}_i \right| = N_i \) for \( 1 \leq i \leq m \). We can find an expression for the degeneracy factor by counting the number of ways one can fill the pattern of blocks so that all the spins in each block are different. To this end, we start with an empty pattern and fill it as follows:

(i) **Fill all the** \( r_m \) **blocks of** \( m \) **sites.**

In the \( r_m \) blocks of \( m \) sites there must be \( r_m \) spins of each type. We are left with \( N_1 - r_m \) spins \( |1\rangle \), \( N_2 - r_m \) spins \( |2\rangle \), \( N_m - r_m \) spins \( |m\rangle \) and a pattern of \( r_1 \) blocks of one site, \( r_2 \) blocks of two sites, etc., \( r_m - 1 \) blocks of \( m - 1 \) sites.

(ii) **Distribute the remaining** \( N_m - r_m \) **spins of type** \( |m\rangle \) **in the** \( r - r_m \) **empty blocks left.**

As in the previous step, we next fix a vector

\[
\mathbf{x} = (x_1, \ldots, x_{m-1})
\]

with \( x_i \equiv N_{m,i} \in \mathbb{N}_0 \) and \( |\mathbf{x}| = N_m - r_m \). Clearly, the number of ways of distributing the \( N_m - r_m \) spins \( |m\rangle \) in the available \( r - r_m \) blocks is given by the product of binomial coefficients \( \prod_{i=1}^{m-1} \binom{r_i}{x_i} \).

(iii) **For each** \( \mathbf{x} \) **in step (ii), we are left with a new pattern** \( \mathbf{\hat{r}} \in \mathbb{N}_0^{m-1} \) **and new spins of types** \( 1, \ldots, m - 1 \) **with magnon numbers** \( (\hat{N}_1, \ldots, \hat{N}_{m-1}) \equiv \mathbf{\hat{N}} \).

Remarkably, the new pattern \( \mathbf{\hat{r}} \) has no blocks of \( m \) sites and the new vector \( \mathbf{\hat{N}} \) has no spins \( |m\rangle \). More precisely, for \( i = 1, \ldots, m - 1 \) there are now \( \hat{r}_i = r_i - x_i + x_{i+1} \) blocks of \( i \) sites, i.e. the previous \( r_i \) minus the occupied blocks of \( i \) sites plus the occupied blocks of \( i + 1 \) sites (note that we must take \( x_m = 0 \), since all the blocks of \( m \) sites were filled up in the first step). Thus, the new pattern \( \mathbf{\hat{r}} \equiv \mathbf{\hat{r}}(\mathbf{x}) \) and magnon vector \( \mathbf{\hat{N}} \) are given by

\[
\hat{r}_i = r_i - x_i + x_{i+1}, \quad 1 \leq i \leq m - 2; \quad \hat{r}_{m-1} = r_{m-1} - x_{m-1}, \quad (4.2)
\]

\[
\hat{N}_i = N_i - r_m, \quad 1 \leq i \leq m - 1, \quad (4.3)
\]
and therefore
\[
D_m(r, N) = \sum_{|x|=N_m-r_m} \prod_{i=1}^{m-1} \left( \frac{r_i}{x_i} \right) \cdot D_{m-1}(\hat{r}(x), \hat{N}).
\] (4.4)

Note that the new vectors \( \hat{N} \) and \( \hat{r} \) satisfy a relation analogous to the last equation (4.1), namely (by equations (4.2) and (4.3))
\[
|\hat{N}| = N - N_m - (m-1)r_m = \hat{r}_1 + 2\hat{r}_2 + \cdots + (m-1)\hat{r}_{m-1}.
\] (4.5)

(iv) Iterate the process described above.

By equation (4.4), we can express the degeneracy factor
\[
D_m(r, N) \equiv D_m(r^{(m)}, N^{(m)})
\]
as a linear combination of degeneracy factors
\[
D_{m-1}(\hat{r}, \hat{N}) \equiv D_{m-1}(r^{(m-1)}, N^{(m-1)}).
\]

This process can be iterated, by expressing each term \( D_{m-1} (r^{(m-1)}, N^{(m-1)}) \) in equation (4.4) in terms of degeneracy factors
\[
D_{m-2} (r^{(m-1)}, N^{(m-1)}) \equiv D_{m-2} (r^{(m-2)}, N^{(m-2)}),
\]
and so on. We thus obtain the recursion relation
\[
D_k(r^{(k)}, N^{(k)}) = \sum_{|x|=N^{(k)}_k-r^{(k)}_k} \prod_{i=1}^{k-1} \left( \frac{r^{(k)}_i}{x_i} \right) \cdot D_{k-1}(r^{(k-1)}(x), N^{(k-1)}),
\] (4.6)

where
\[
r^{(k)}_i, N^{(k)}_i \in \mathbb{N}_0^k, \quad x, r^{(k-1)}(x), N^{(k-1)} \in \mathbb{N}_0^{k-1},
\]
with
\[
r^{(k-1)}_i(x) = r^{(k)}_i - x_i + x_{i+1} \quad (x_k \equiv 0), \quad N^{(k-1)}_i = N^{(k)}_i - r^{(k)}_i
\]
and \( r^{(m)} \equiv r, N^{(m)} \equiv N \). The above recursion relation, together with the obvious initial condition \( D_1 = 1 \), fully determines \( D_m(r, N) \).

In section 5 we shall illustrate the above procedure for computing the degeneracy factor \( d(p, N, -) \equiv D_m(r, N) \) with several examples. Once \( D_m \) is determined, the restricted partition \( Z_0^{N,(-)}(T) \) in the antiferromagnetic case is obtained from equation (3.17), namely
Generalized Lipkin–Meshkov–Glick models of Haldane–Shastry type

\[ Z_0^{N,(T)}(T) = \sum_{r=0}^{N} \sum_{k \in \mathbb{P}^r_N} D_m(r, N) \prod_{i=1}^{r-1} q^{E(K_i)} \cdot \prod_{j=1}^{N-r} (1 - q^{E(K'_j)}), \]  
(4.7)

where the range of the last sum comes from condition (ii) above, since in the antiferromagnetic case the lengths \( k_i \) of the blocks in equation (3.9) are all at most equal to \( m \). The partition function of the corresponding gLMG model of HS type (2.9) can then be computed from equation (2.10), with the result

\[ Z^{(-)}(T) = \sum_{|N|=N} q^{h(N)} \sum_{r=N/m}^{N} \sum_{k \in \mathbb{P}^r_N} D_m(r, N) \prod_{i=1}^{r-1} q^{E(K_i)} \cdot \prod_{j=1}^{N-r} (1 - q^{E(K'_j)}). \]  
(4.8)

### 4.2. Ferromagnetic case

A similar procedure could be followed in principle to compute the degeneracy factor \( d(p, N, +) \) in the ferromagnetic case \( \epsilon = 1 \). The main difference is that now each value of the \( \text{su}(m) \) spin can be used more than once to fill the blocks of length \( k_1, \ldots, k_r \) determined by the multiindex \( p \) in equation (3.9), which considerably complicates matters.

In practice, it is much easier to derive the ferromagnetic partition function \( Z^{(+)} \) from the antiferromagnetic one \( Z^{(-)} \) computed in the previous subsection by means of the identity

\[ H_0^{(+)} + H_0^{(-)} = \sum_{i \neq j} h_{ij} \equiv E_{\text{max}}(N), \]
(4.9)

where \( H_0^{(\pm)} \) denotes the Hamiltonian (2.1) with \( \epsilon = \pm \). The constant \( E_{\text{max}}(N) \), which is the maximum energy of \( H_0^{(-)} \), can be easily computed in closed form for each of the interactions (2.12) and (2.11) taking into account the identity [55]

\[ E_{\text{max}}(N) = \sum_{i=1}^{N-1} E(i), \]
(4.10)

namely

\[
E_{\text{max}} = \begin{cases} 
\frac{N}{6} (N^2 - 1), & \text{for the HS chain} \\ 
\frac{N}{2} (N - 1), & \text{for the PF chain} \\ 
\frac{N}{6} (N - 1)(3\beta - 4N + 2), & \text{for the FI chain} 
\end{cases}
\]

From equation (4.9) it immediately follows that the restricted partition functions \( Z_0^{N,(\pm)} \) of \( H_0^{(\pm)} \) are related by

\[ Z_0^{N,(+)}(q) = q^{E_{\text{max}}(N)} Z_0^{N,(-)}(q^{-1}). \]

Using equations (3.17) and (4.10) we easily obtain

https://doi.org/10.1088/1742-5468/aa8c14
\[ Z_0^{N,(+)}(T) = q_1^{E_{\text{max}}(N)} \sum_{r=[N/m]}^N \sum_{k \in \mathcal{P}_N} D_m(r, N) \prod_{i=1}^{r-1} q^{i\mathcal{E}(K_i)} \prod_{j=1}^{N-r} (1 - q^{\mathcal{E}(K'_j)}) \]

\[ = \sum_{r=[N/m]}^N (-1)^{N-r} \sum_{k \in \mathcal{P}_N} D_m(r, N) \prod_{i=1}^{N-r} (1 - q^{\mathcal{E}(K'_i)}), \quad (4.11) \]

where \( D(r, N) \) is the antiferromagnetic degeneracy factor computed in the previous subsection. By equation (2.10), the partition function of the ferromagnetic Hamiltonian \( H^{(+)} \equiv H_{0}^{(+)} + H_1 \) is given by

\[ Z^{(+)}(T) = \sum_{|N|=N} q^{h(N)} \sum_{r=[N/m]}^N (-1)^{N-r} \sum_{k \in \mathcal{P}_N} D_m(r, N) \prod_{i=1}^{N-r} (1 - q^{\mathcal{E}(K'_i)}). \quad (4.12) \]

5. Examples

5.1. \( su(2) \)

In this case \( r = (r_1, r_2), \ N = (N_1, N_2), \) and the recursion relation (4.6) with \( D_1 = 1 \) immediately yields

\[ D_2(r, N) = \begin{pmatrix} r_1 \\ N_2 - r_2 \end{pmatrix}. \]

Expressing \( r_1, r_2 \) in terms of \( r \) and \( N \) by means of the relations \( r = r_1 + r_2, \ N = N_1 + N_2 = r_1 + 2 r_2 \) we finally obtain

\[ D_2(r, N) = \begin{pmatrix} 2r - N \\ N - r_1 \end{pmatrix} = \begin{pmatrix} 2r - N \\ r - N_2 \end{pmatrix}. \]

Thus the restricted partition function of the \( su(2) \) chains (2.1) of HS type is given by

\[ Z_0^N(T) = \sum_{r_1=1}^{N} (-1)^{N-r_1} \sum_{k \in \mathcal{P}_N} \left( \begin{array}{c} 2r - N \\ r - N_1 \end{array} \right) q^{\frac{1}{2} \sum_{i=1}^{r_1} \mathcal{E}(K_i)} \prod_{i=1}^{N-r} (1 - q^{\mathcal{E}(K'_i)}). \]

By equation (2.10), the partition function of the corresponding \( su(2) \) gLMG model reads

\[ Z(T) = \sum_{N_1=0}^{N} q^{h(N_1, N-N_1)} \sum_{r_1=1}^{N} (-1)^{N-r_1} \sum_{k \in \mathcal{P}_N} \left( \begin{array}{c} 2r - N \\ r - N_1 \end{array} \right) q^{\frac{1}{2} \sum_{i=1}^{r_1} \mathcal{E}(K_i)} \prod_{i=1}^{N-r} (1 - q^{\mathcal{E}(K'_i)}). \]

5.2. \( su(3) \)

Let \( r = (r_1, r_2, r_3) \) and \( N = (N_1, N_2, N_3) \) such that \( r = r_1 + r_2 + r_3, \ N = N_1 + N_2 + N_3 \) and \( r_1 + 2 r_2 + 3 r_3 = N. \) We then have

https://doi.org/10.1088/1742-5468/aa8c14
where we have used the identities \( \hat{r} = r - r_3 - x_1 \), \( \hat{N} = N - N_3 - 2 r_3 \) and \( \hat{N}_2 = N_2 - r_3 \).

### 5.3. su(4)

Let \( r = (r_1, r_2, r_3, r_4) \) and \( N = (N_1, N_2, N_3, N_4) \) such that \( r = r_1 + r_2 + r_3 + r_4 \), \( N = N_1 + N_2 + N_3 + N_4 = r_1 + 2 r_2 + 3 r_3 + 4 r_4 \). Using equation (5.1) with \( (\hat{r}, \hat{N}) \equiv (r^{(3)}, N^{(3)}) \) in place of \( (r, N) \) we easily obtain

\[
D_4(r, N) = \sum_{x_1 + x_2 + x_3 = N_1 - r_4} \binom{r_1}{x_1} \binom{r_2}{x_2} \binom{r_3}{x_3} D_3(\hat{r}, \hat{N})
\]

\[
= \sum_{x_1 + x_2 + x_3 = N_1 - r_4} \binom{r_1}{x_1} \binom{r_2}{x_2} \binom{r_3}{x_3} \sum_{y_1 + y_2 = N_3 - r_3 - x_3} \binom{r_1 - x_1 + x_2}{y_1} \binom{r_2 - x_2 + x_3}{y_2} \binom{2 r_2 - 2 x_1 - 2 y_1 - N_1 - N_2}{r_1 - x_1 - N_2}.
\]

### 6. The LMG-PF model

When \( H_0 \) is the Hamiltonian of the PF chain the restricted partition function \( Z_0^N(T) \), and hence the partition function \( Z(T) \) of the corresponding LMG-PF model (2.9), can be considerably simplified. Indeed, in this case

\[
H_{\text{sp}} = -\Delta + ar^2 + \sum_{i \neq j} a(a - \epsilon S_{ij}) \frac{1}{(x_i - x_j)^2} = H_{\text{sc}} + 2aH_0(x),
\]

where \( r^2 \equiv \sum_{i \neq j} x_i^2 \),

\[
H_{\text{sc}} = -\Delta + ar^2 + \sum_{i \neq j} a(a - 1) \frac{1}{(x_i - x_j)^2}
\]

is the scalar Calogero model and

\[
H_0(x) = \sum_{i < j} \frac{1 - \epsilon S_{ij}}{(x_i - x_j)^2}
\]

is obtained from \( H_0 \) by the formal substitution \( z_i \mapsto x_i \). Proceeding as in section 3 we obtain the analogue of equation (3.2), namely

\[
Z_0^N(T) = \lim_{a \to \infty} \frac{Z_{\text{sp}}^N(2aT)}{Z_{\text{sc}}(2aT)},
\]

where the partition function \( Z_{\text{sc}}(2aT) \) of the scalar Calogero model is given by [51]

\[
Z_{\text{sc}}(2aT) = q^{E_{\text{GS}}/2a} \prod_i (1 - q^i)^{-1}, \quad E_{\text{GS}} \equiv a^2 N (N - 1) + a N.
\]
In order to compute $Z_{sp}^N(2aT)$, we note [51] that the Hamiltonian (6.1) of the spin Calogero model is upper triangular in the basis with elements

$$\vert p, s \rangle = e^{-ar^2/2} \prod_{i<j} \vert x_i - x_j \rangle^{a} \Lambda_e \left( \prod_i x_i^{p_i} \vert s \rangle \right)$$

(6.4)

partially ordered by the total degree $|p|$, with corresponding eigenvalues

$$E(p) = 2a(p_1 + \cdots + p_N) + E_{GS}. \quad \text{(6.5)}$$

Of course, we must choose the quantum numbers $(p, s)$ in such a way that the states (6.4) are actually a basis. The main difference with the HS and FI models is that only in this case $E(p)$ and the admissible partial order of the basis states (6.4) do not depend on the ordering of the components of $p$ [43, 44, 51]. As a consequence, we can choose the quantum numbers $(p, s)$ in each subspace $\Lambda_e [L^2(\mathbb{R}^N) \otimes \mathcal{H}(N)]$ as follows:

(i) We first order the components of the spin quantum number $s$ increasingly, so that

$$s = \left( \overbrace{1, \ldots, 1}^{N_1}, \ldots, \overbrace{m, \ldots, m}^{N_m} \right)$$

is now fixed.

(ii) In each block of $s$ with fixed magnon number $|a\rangle$ we order the corresponding components of the vector $p$ also increasingly, so that $p = (\rho^1, \ldots, \rho^m)$ with

$$\rho^j \equiv (\rho^1_j, \ldots, \rho^N_j)$$

and $\rho^i_j < \rho^i_{j+1}$.

We thus have

$$E(p) = E_{GS} + 2a \sum_{i=1}^N p_i = E_{GS} + 2a \sum_{j=1}^m \sum_{i=1}^{N_j} \rho^i_j,$$

and therefore

$$q^{-E_{GS}/2a} Z_{sp}^N(2aT) = \sum_{\rho^i_1 < \cdots < \rho^i_{K_j}} \prod_{j=1}^m q^{\rho^i_1 + \cdots + \rho^i_{K_j}} = \prod_{j=1}^m \sum_{0 < \rho^i_1 < \cdots < \rho^i_{K_j}} q^{\rho^i_1 + \cdots + \rho^i_{K_j}}.$$

The inner sum in the latter formula can be computed in closed form, with the result

$$\sum_{0 < \rho^i_1 < \cdots < \rho^i_{K_j}} q^{\rho^i_1 + \cdots + \rho^i_{K_j}} = q^{\frac{1}{2}N_j(N_j-1)} \prod_{i=1}^{N_j} (1 - q^i)^{-1} = q^{\frac{1}{2}N_j(N_j-1)} (q^{-1})^{N_j}.$$
and thus

$$Z_\text{sp}^N(2\alpha T) = \prod_{j=1}^m q^{1/2} \sum_{N_j}^N (N_j - 1)(q)_{N_j}^{-1} = q^{1/2} \prod_{j=1}^m \sum_{N_j}^N (N_j - 1)(q)_{N_j}^{-1}. $$

From this equation and equations (6.2) and (6.3) we obtain the following closed-form expression for the restricted partition function of the PF chain:

$$Z_0^N(T) = q^{1/2} \sum_{N_j}^N (q)_N \prod_{N_j=1}^m (q)_{N_j}. $$

Finally, by equation (2.10) the partition function of the LMG-PF model is given by

$$Z(T) = \sum_{N_1 + \ldots + N_m = N} q^{h(N)+ \frac{1}{2} \sum_{j=1}^m N_j(N_j - 1)} \prod_{j=1}^m (q)_{N_j} \equiv \sum_{N_1 + \ldots + N_m = N} q^{h(N)+ \frac{1}{2} \sum_{j=1}^m N_j(N_j - 1)} \left[ \begin{array}{c} N \\ N_1, \ldots, N_m \end{array} \right] q. $$

(6.6)

In particular, for \( h = 0 \) we recover the well-known formula for the partition function of the PF chain in reference [42].

7. Analysis of the spectrum and thermodynamics

In this section we shall take advantage of the knowledge of the restricted partition function of the gLMG models (2.9) to study several statistical properties of their spectrum and analyze the behavior of their thermodynamic functions for large \( N \). To begin with, we have examined the level density of the restriction of the Hamiltonian to subspaces with a fixed magnon content. Since \( H_1 \) is constant on these subspaces, this is of course equivalent to studying the level density of the corresponding spin chains of HS type. It is well-known in this respect [53, 54] that the level density of the complete spectrum of the latter models becomes normally distributed in the \( N \to \infty \) limit, essentially due to the existence of a description of the spectrum in terms of Haldane’s motifs [28, 55]. We have computed the spectrum of the HS chain for up to \( N = 26 \) for \( su(2) \) and \( N = 24 \) for \( su(3) \) in the largest subspace \( \mathcal{H}(N) \) (with \( N_i = N/m \) for all \( i \)). Our results clearly indicate that the spectrum of the restriction of \( H_0 \) to this subspace is also normally distributed (see figure 1, left), with parameters \( \mu \) and \( \sigma \) given by the mean and standard deviation of the restricted spectrum. For the FI and PF chains we have obtained similar results. This fact suggests [54] that in all three cases there might be a formula for the energies in each sector of the spectrum with fixed magnon numbers in terms of motifs.

Since the continuous part of the cumulative level density in each sector can be well approximated by a Gaussian distribution, the energies of the ‘unfolded’ spectrum [56] can be taken as

$$\eta_i = \int_{-\infty}^{E_i} g(t)dt, \quad g(E) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(E-\mu)^2}{2\sigma^2}}, \quad i = 1, \ldots, n.$$
According to a long-standing conjecture due to Berry and Tabor [49], the distribution of the (normalized) spacings between consecutive levels of the unfolded spectrum, defined as

\[ s_i = (n - 1) \frac{\eta_{i+1} - \eta_i}{\eta_n - \eta_1}, \quad i = 1, \ldots, n - 1, \]

is expected to be Poissonian for ‘generic’ integrable systems. On the other hand, for a chaotic system the well-known Bohigas–Giannoni–Schmit conjecture posits that this distribution should be given by the Wigner distribution corresponding to the appropriate ensemble of random matrices [50]. In references [43, 44, 51] it was observed that the distribution \( p(s) \) of the spacings between consecutive levels of the whole spectrum of all three chains of HS type follows none of the above distributions, but is typically given by the ‘square root of a logarithm’ law

\[ P(s) = 1 - \frac{2}{\sqrt{\pi \, s_{\text{max}}}} \sqrt{\log \left( \frac{s_{\text{max}}}{s} \right)}, \quad (7.1) \]

where \( P(s) = \int_0^s p(s')ds' \) is the cumulative distribution and \( s_{\text{max}} \) is the maximum spacing. As shown in [57, 58], this is due to the fact that the raw spectrum of the latter chains is approximately equispaced and normally distributed. We have computed the distribution of consecutive (normalized) spacings in the subspaces mentioned above for the HS, PF and FI chains for \( m = 2 \) and \( 3 \). In all cases, the cumulative spacings distribution \( P(s) \) fits equation (7.1) with remarkable accuracy (see the insets figure 1, right, for the HS chain). This clearly suggests that the (raw) spectrum of the restriction of the three HS-type chains to subspaces with fixed magnon content is also approximately equispaced. We have also verified that this conclusion is indeed correct for all three chains of HS type. For instance, for the \( \text{su}(2) \) HS chain with \( N_1 = N_2 = 13 \) (see figure 1, top right) 93.8\% of the spacings between consecutive levels of the raw spectrum are equal to 1, while for the \( \text{su}(3) \) HS chain with \( N_1 = N_2 = N_3 = 8 \) (see figure 1, bottom right) the predominant spacing is again 1 and occurs 95.7\% of the times.

We shall next analyze the thermodynamics of a class of LMG models of HS type whose deformation Hamiltonian (2.8) is given by

\[ h(x_1, \ldots, x_m) = \frac{1}{N} \sum_{a=1}^{m} (x_a - n_a N)^2, \quad (7.2) \]

where the parameters \( n_a (1 \leq a \leq m) \) are assumed to lie in the interval \((0,1)\) and \( n_1 + \cdots + n_m = 1 \). These parameters thus represent the magnon densities of the ground state in the ferromagnetic case (\( \epsilon = 1 \)). The motivation for considering a quadratic deformation Hamiltonian is, first of all, that in the original, isotropic LMG model the external term \( H_1 \) is precisely of this form. More recently, generalized LMG models with a quadratic external term have proved of interest in the context of quantum information theory, since they are some of the few systems for which the bipartite entanglement entropy of the ground state can be computed in closed form [11, 13, 19]. Using the exact formulas (4.8)–(4.12) and (6.6), we have evaluated the partition function of this class of models for a relatively large number of spins, of the order of 100 (resp. 50) for the \( \text{su}(2) \) (resp. \( \text{su}(3) \)) ferromagnetic LMG-PF models. From the resulting
expression, we have computed the free energy $f$, the internal energy $u$, the entropy $s$ and the specific heat $c$ (per spin, in all cases) via the formulas

$$ f(T) = -\frac{T}{N} \log Z(T), \quad u(T) = \frac{T^2}{N} \frac{\partial \log Z(T)}{\partial T}, $$

$$ s(T) = \frac{\partial}{\partial T} \left( \frac{T}{N} \log Z(T) \right), \quad c(T) = \frac{2T}{N} \frac{\partial \log Z(T)}{\partial T} + \frac{T^2}{N} \frac{\partial^2 \log Z(T)}{\partial T^2}, $$

(7.3, 7.4)

where we have taken Boltzmann’s constant $k_B = 1$. We have first verified that the thermodynamic functions are practically independent of $N$ for $N \lesssim 100$ (in the su(2) case) and $N \lesssim 50$ (in the su(3) case). Thus the thermodynamic functions for $N = 100$ (in the su(2) case) and $N = 50$ (in the su(3) case) can be regarded as a reasonable approximation of their $N \to \infty$ counterparts. As an additional check, we have compared the results for the su(2) PF chain with no deformation Hamiltonian and $N = 100$ spins with the exact $N \to \infty$ formulas derived in [52], finding them in excellent agreement (see figure 2). In particular, the extensive behavior of the thermodynamic entropy contrasts with the logarithmic growth of the ground-state entanglement entropy of the ferromagnetic ‘quadratic’ gLMG models studied in reference [19].

In figures 2 and 3 we present the plots of the free and internal energies, the entropy and the specific heat (per spin) respectively of the su(2) and su(3) models (2.9)–(7.2).
Figure 2. Thermodynamic functions for the $\text{su}(2)$ ferromagnetic LMG-PF model with $h(N_1, N_2) = [(N_1 - n_1 N)^2 + (N_2 - n_2 N)^2]/N$ for $N = 100$ spins. The red, blue and green lines correspond respectively to the magnon densities $(n_1, n_2) = (1/8, 7/8)$, $(1/4, 3/4)$ and $(1/2, 1/2)$, while the continuous gray line represents the $h = 0$, $N = \infty$ exact result. (In all cases, we have used natural units $\hbar = 2M = k_B = 1$.)

Figure 3. Thermodynamic functions for the $\text{su}(3)$ ferromagnetic LMG-PF model with $h(N_1, N_2, N_3) = \sum_{i=1}^3 (N_i - n_i N)^2/N$ for $N = 50$ spins. The red, blue and green lines correspond respectively to the magnon densities $(n_1, n_2, n_3) = (1/8, 1/4, 5/8)$, $(1/4, 1/4, 1/2)$ and $(1/3, 1/3, 1/3)$. (In all cases, we have used natural units $\hbar = 2M = k_B = 1$.)
in the PF case. It is apparent from these figures that both the \(\text{su}(2)\) and the \(\text{su}(3)\) thermodynamic functions qualitatively behave like those of a two-level system, as for instance the one-dimensional Ising model at zero magnetic field or a paramagnetic spin 1/2 ion [59]. In particular, from figures 2 and 3 we see that the specific heat exhibits the Schottky peak characteristic of the latter systems. Finally, it may seem surprising that the entropy per spin does not appear to vanish at \(T = 0\) in some cases, especially when \(h = 0\) (see, e.g. figure 3). Of course, the explanation for this behavior is that the number of spins \(N\) is finite (though large), so that \(s(0) = (\log d(m, N))/N\), where \(d(m, N)\) is the ground state degeneracy. In the ferromagnetic case under consideration, it follows from equation (2.9) that when \(h = 0\) the ground states are the symmetric states, so that

\[
d(m, N) = \binom{N + m - 1}{m - 1} \simeq \frac{N^{m-1}}{(m-1)!},
\]

and thus \(s(0) \simeq (m - 1)(\log N)/N\) is small but nonzero. On the other hand, when \(h\) does not vanish identically the \(H_1\) term in equation (2.9) breaks the ground state degeneracy almost completely (the more so in the less symmetric cases, in which the densities \(n_a\) are all different), so that \(s(0)\) is significantly smaller than its \(h = 0\) counterpart.

8. Conclusions

We shall finish this paper with a brief summary of its main results. We have introduced a family of generalized \(\text{su}(m)\) Lipkin–Meshkov–Glick models whose interacting term is a spin chain of Haldane–Shastry type, which can be equivalently regarded as the deformation of a spin chain of HS type \(H_0\) by the addition of a term \(H_1\) in the enveloping algebra of the Cartan subalgebra of \(\text{su}(m)\). The Hilbert space of the system is a direct sum of subspaces \(H(N)\) with fixed magnon numbers, in which the action of the deformation term is diagonal, so that the model’s partition function decomposes as in equation (2.10). By a suitable adaptation of Polychronakos’ freezing trick, we have been able to compute in a closed form the partition functions of the restrictions of the spin chain Hamiltonian \(H_0\) to the subspaces \(H(N)\). In view of the previous remarks, this immediately yields the partition function of the associated gLMG model. In particular, when \(H_0\) is the Hamiltonian of the Polychronakos–Frahm spin chain we have obtained an alternative, simpler expression for the partition function akin to Polychronakos’s formula [42] for the case \(H_1 = 0\). This closed-form expression for the partition function of the restriction of \(H_0\) to the subspaces \(H(N)\) has been used in numerical calculations to provide strong evidence that the level density of the latter restriction is Gaussian when the number of spins tends to infinity. In view of the results of [54], this suggests that there exists a description of the spectrum of \(H_0\) restricted to \(H(N)\) in terms of motifs, a fact that deserves further investigation. We have also numerically studied the distribution of the spacings of consecutive unfolded levels of \(H_0\) restricted to \(H(N)\), showing that it follows the same characteristic law previously found for the complete spectrum. As a final application, we have computed the free and internal energies, the entropy and the specific heat per spin of a class of \(\text{su}(2)\) and \(\text{su}(3)\) gLMG models with quadratic \(H_1\). We have checked
that these functions are virtually independent of the number of spins $N$ when this number is sufficiently large, which indicates that they yield reasonable approximations to their respective thermodynamic limits. Our analysis shows that the thermodynamic functions of these models are qualitatively similar to those of a two-level system, as already observed in [52] for the $\text{su}(2)$ chains of HS type. In the latter chains, this similarity is ultimately due to the existence of a description of the spectrum in terms of motifs, which leads to simple closed formulas for the thermodynamic functions in terms of the dispersion relation. This again suggests that such a description should also exist for the more general models studied in this paper.

Acknowledgments

This work was partially supported by Spain’s MINECO under research grant no. FIS2015-63966-P. JAC would also like to acknowledge the financial support of the Universidad Complutense de Madrid through a 2015 predoctoral scholarship.

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