Properties of Schrödinger Black Holes from AdS Space

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Abstract

Properties of Schrödinger black holes are derived from those of AdS black holes expressed in light-cone coordinates with a particular normalization. Unlike the usual construction from an AdS black hole using a null Melvin twist, an AdS black hole in light-cone is simple and has a well-defined Brown-York procedure with the standard counterterms. Our procedure is demonstrated by the computation of the DC conductivity and the derivation of the R-charged black hole thermodynamic properties.

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1 Introduction

The exploration of non-relativistic generalizations of the AdS/CFT correspondence [3, 4, 5], initiated by Son [1] and by Balasubramanian and McGreevy [2], has become an active research area. The original works focused on a geometric realization of the non-relativistic conformal symmetry, also known as Schrödinger symmetry, and the proposed geometry (“Schrödinger space”) successfully and naturally fitted into the usual scheme of the AdS/CFT correspondence. Soon after, the system was generalized to non-zero temperature by incorporating black holes in the geometry [6, 7, 8]. In addition, it was realized that the five dimensional Schrödinger space can be obtained from the AdS$_5 \times S^5$ solution of type IIB supergravity by applying a series of transformations known as a null Melvin twist [9, 10]. Subsequently, the analysis of Schrödinger space, especially through the null Melvin twist, has become mainstream as a geometric realization of Schrödinger symmetry.

However, before the finite temperature generalization was considered, it was found by Goldberger [11] and by Barbon and Fuertes [12] that the Schrödinger symmetry can be geometrically realized in the pure AdS space, without a deformation. Their procedure is very similar to the light-front quantization of relativistic field theories, which can be reduced to Galilean invariant non-relativistic analogues [13, 14, 15]. The key difference, however, is that the authors of References [11, 12] project the theory onto a fixed momentum in one of the light-cone directions and identify the other light-cone coordinate as the time of the resulting non-relativistic theory. These two key procedures differentiate the system from those of the infinite momentum frame, in which the momentum projection were not done. Moreover, light-cone directions do not have the interpretation as time and space, even though one of them is sometimes called “light-cone time.” It is just a convenient frame to work for some cases, and in fact, the formalism is Lorentz invariant.

In addition to these two important operations, we use the light-cone coordinates with a particular normalization [7]

$$x^+ = b(t + x), \quad x^- = \frac{1}{2b}(t - x),$$

even for the zero temperature case. This transformation can be thought as a two-step procedure: a boost in the $x$-direction with rapidity $\log b$ followed by going to light-cone coordinates. To ensure a definite dynamical exponent $z=2$, we assign $[b]$ (a scaling dimension in the unit of mass) as $-1$, and thus $[x^+] = -2$ and $[x^-] = 0$. $x^-$ is invariant under the scaling transformation, which is crucial for the system to have special conformal invariance in its symmetry group. The parameter $b$ exactly matches the extra parameter generated in the null Melvin twist. Incidentally, the light-cone coordinates [11] have been adopted in the previous works with the null Melvin twist because it is the unique normalization that makes the metric of Schrödinger space independent of the parameter $b$ in its asymptotic boundary. But it has not been emphasized that this is originated from a boost parameter. Thus the resulting physical quantities, described by the light-cone coordinates with modified scaling dimensions $[x^+] = -2$ and $[x^-] = 0$, are different from those of the original AdS space.

Therefore, assigning different scaling dimensions to the light-cone coordinates, the momentum projection and the re-interpretation of time completely separate the system from the infinite momentum frame. The purpose of the infinite momentum frame is to investigate the question: “Can one write Schrödinger (Galilean-invariant) theories at infinite momentum that completely describe interacting relativistic system?” In particular, a theory (at infinite momentum) demands Poincaré invariance, unitarity, and positivity of mass spectrum [14].

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the relativistically causal theory. One should not carry over relativistic reasoning, such as causal structure from the original AdS spacetime. This implies that pure AdS space in light-cone coordinates with the three procedures can rightly be a conceivable dual of a non-relativistic field theory. The technique adopted in References [11, 12, 7], in a sense, can be regarded as the AdS/CFT gravitational counterpart to the field theory in the light-front, but the system is truly non-relativistic, rather than a non-relativistic analogue, as a result of the three key procedures mentioned above.

Despite interesting aspects and its sheer simplicity, the approach of Goldberger, Barbon and Fuertes has become less popular and non-zero temperature generalizations along this line have not been pursued explicitly. One exception is Reference [7] where it is briefly mentioned that the thermodynamic quantities of a planar Schrödinger black hole can be obtained also from a planar AdS black hole expressed in the light-cone coordinates with a particular normalization given in equation (1).

In this paper, we pursue the direction of References [11, 12, 7]. We explicitly demonstrate that the introduction of the light-cone coordinates (1) to a relativistic AdS system of interest, with the three key procedures, is enough to reproduce the thermodynamic and transport properties of the Schrödinger counterpart. The advantages of this approach are the following. Most importantly, the well-defined nature of the boundary permits the computation of thermodynamic quantities through the Brown-York procedure [17] with the standard counterterms of Balasubramanian and Kraus [18]. This is carefully laid out in Section 2 by taking the planar AdS$_5$ black hole as a prototype example. The Brown-York procedure completely fails when applied to the Schrödinger black holes — due to the unusual boundary structure — and the derivations of the thermodynamic quantities have been very awkward and ad hoc. In particular, all the derivations effectively assume the first law of thermodynamics and the independent check of the first law for the derived quantities has not been possible. (See, however, the work of Ross and Saremi [20] in which the thermodynamic quantities are successfully derived by adopting modified definition of the stress tensor.)

It is worthwhile to mention here that there are two technical modifications in computing conserved quantities along the two light-cone coordinates. This is required due to the cross term $dx^+ dx^-$ in AdS in light-cone compared to its original AdS space. We systematically explain them in equations (14), (15) and (16) of Section 2 following carefully [17]. Identifying this modifications is possible because of our clear procedure with a well-defined boundary structure.

Another important advantage of our construction is its simplicity. Generating an asymptotically Schrödinger spacetime through the null Melvin twist is a fairly complicated procedure. This is especially so when the original metric has non-vanishing off-diagonal components in time and space directions, and the resulting RR-potentials becomes even more complicated. One such intricate example is an R-charged black hole. However, the procedure that we adopt is no more complicated than usual relativistic calculations, and in Section 3, thermodynamic quantities of the Schrödinger R-charged black hole are derived for the planar, spherical and hyperbolic cases including the finite Liu-Sabra counter terms [21] for the first law of thermodynamics to be satisfied. In that section, we also discuss

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2 This observation was adopted and the result was used in the discussion of the hydrodynamics from the Schrödinger black holes in Reference [16].

3 See [19] for the earlier attempt to resolve the puzzles of the holographic renormalization for the spacetime with unusual boundary structure including the Schrödinger and Lifshitz spacetime using anisotropic scaling.
phase diagrams of the black holes in the planar and spherical cases.

In Section 4, we demonstrate that our method is not only effective in deriving thermodynamic quantities but also Minkowski two-point correlators and conductivities. In particular, we derive the non-relativistic counterpart of the DC conductivity computed by Karch and O’Bannon [22]. This result is compared to the conductivity calculated directly from the Schrödinger space by Ammon et al. [23].

In Appendix A, we explore the well-known connection between the light-front formalism and the system in the infinitely boosted frame. Susskind in Reference [13] considered the system of free bosons in the infinitely boosted frame and found that it is possible to consistently extract finite quantities which found in non-relativistic systems. In particular, the resulting Hamiltonian generates motion in one of the light-like directions, i.e., one of the light-cone coordinates is the analog of the non-relativistic time. Subsequently, Bardakci and Halpern [14] and Chang and Ma [15] argued that the infinite-momentum limit is equivalent to adopting light-cone coordinates. Then we expect to extract the properties of Schrödinger black holes from the counterparts of AdS black hole in the infinitely boosted frame. This is demonstrated in the appendix for the case of the planar black hole. By doing so we rederive the results of Section 2.

We exclusively work on the five dimensional AdS spacetime for concreteness. However, the method adopted here to generate the quantities of Schrödinger space can be trivially generalized to other dimensions.

2 The Brown-York Procedure

In this section, we carefully work out the Brown-York procedure [17] for the planar AdS5 black hole in the special light-cone coordinates mentioned in the introduction. This simple system serves as a prototype for more complex black holes.

The action of the gravitational system in concern is

\[
I = \frac{1}{16\pi G_5} \int d^5x \sqrt{-g} \left( R + \frac{12}{R^2} \right) - \frac{1}{8\pi G_5} \int d^4x \sqrt{-\gamma} \left( K + \frac{3}{R^2} + \frac{R}{4} R_4 \right),
\]

where the symbols \( g, R \) and \( R \) are the determinant of the metric, the scalar curvature and the length scale of the theory that is related to the cosmological constant, respectively. We have included the boundary terms and \( \gamma \) denotes the determinant of the boundary metric. The first boundary term is the Gibbons-Hawking term [24] with the trace of the boundary second-fundamental form \( K \), and the second and third terms are the standard five-dimensional counterterms of Balasubramanian and Kraus [18] with the (intrinsic) scalar curvature of the boundary \( R_4 \).

The planar black hole solution to the equations of motion following from the action can be written as

\[
ds^2 = \left( \frac{r}{R} \right)^2 (-h dt^2 + dx^2 + dy^2 + dz^2) + \left( \frac{R}{r} \right)^2 h^{-1} dr^2 \quad \text{with} \quad h := 1 - \frac{r_H^4}{r^4},
\]

where \( r = r_H \) is the location of the horizon and we set the boundary at some large (with respect to \( r_H \)) fixed value of \( r \). Now, we introduce the light-cone coordinates mentioned in the introduction,

\[
x^+ = b(t + x), \quad x^- = \frac{1}{2b}(t - x),
\]
where we modified the scaling dimension of $b$ as $[b] = -1$ in the mass unit, and thus $[x^+] = -2$ and $[x^-] = 0$ to have manifest dynamical exponent $z=2$. The metric takes the form

$$ds^2 = \left( \frac{r}{R} \right)^2 \left\{ \frac{1 - h}{4b^2} dx^2 + (1 + h) dx^+ dx^- + (1 - h) b^2 dx^2 + dy^2 + dz^2 \right\} + \left( \frac{R}{r} \right)^2 h^{-1} dr^2 .$$

One can check that this is nothing but the boosted form of the metric (3) transformed into usual light-cone coordinates with modified scaling dimensions of light-cone coordinates. Thus, the parameter $b$ signifies the boosted system of the Lorentz non-invariant solution, and in the end the metric (5) describes non-relativistic setup with dynamical exponent $z=2$.

In the spirit of the infinite-boost limit of Reference [13] mentioned in the introduction, the non-relativistic “limit” corresponds to re-interpreting one of the light-cone coordinates, say $x^+$, as the time. Then the ADM form of the metric is

$$ds^2 = \left( \frac{r}{R} \right)^2 \left\{ - \frac{h}{1 - h} b^{-2} dx^2 + (1 - h) b^2 \left( dx^- - \frac{1}{2b^2} \frac{1 + h}{1 - h} dx^+ \right)^2 + dy^2 + dz^2 \right\} + \left( \frac{R}{r} \right)^2 h^{-1} dr^2 .$$

From the ADM form, we can read off the lapse function $N$, the shift function $V^i$ and the horizon coordinate velocity in the $x^-$ direction $\Omega_H$, which can be interpreted as chemical potential associated with the conserved quantities along the $x^-$ direction, as

$$N = \left( \frac{r}{R} \right) \sqrt{\frac{h}{1 - h} b^{-1}} , \quad V^- = - \frac{1}{2b^2} \frac{1 + h}{1 - h} \quad \text{and} \quad \Omega_H = \frac{1}{2b^2} .$$

Notice that we have alluded two kinds of hypersurfaces; the timelike boundary surface at a large fixed $r$ and the spacelike surface at a fixed time $x^+$ whose time development is described by the ADM form. Moreover, the spacelike surface at the intersection of those two hypersurfaces plays a crucial role for the definition of the Brown-York conserved quantities. Since we have the non-trivial shift function, we must pay careful attention to the projection tensors onto those surfaces. The definition of the projections, of course, requires the normals to the surfaces, and we take the unit normal of the timelike boundary surface as

$$n_\mu := \left( \frac{R}{r} \right) h^{-1/2} (0, 0, 0, 0, 1) ,$$

where the components are ordered according to $(+, -, y, z, r)$, and following Brown and York [17], we define the normal of the spacelike surface by

$$u_\mu := -N (1, 0, 0, 0, 0) ,$$

\footnote{In Appendix A we show how to derive the results from the infinite-boost limit, rather than adopting light-cone coordinates.}
where $N$ is the lapse function given in Equation (7). The projections onto the four-dimensional timelike boundary hyperspace and the three-dimensional spacelike intersection surface are given respectively as

$$
\gamma_{\mu\nu} := g_{\mu\nu} - n_\mu n_\nu \quad \text{and} \quad \sigma_{\mu\nu} := g_{\mu\nu} - n_\mu n_\nu + u_\mu u_\nu .
$$

(10)

Since they are projection operators, they do not have inverses and the five-dimensional indices are raised and lowered by the metric $g_{\mu\nu}$. However, if we restrict them to the appropriate components, namely, $\gamma_{ij}$ with $i, j \in \{+, -, y, z\}$ and $\sigma_{ab}$ with $a, b \in \{-, y, z\}$, then they have well-defined inverses and can be defined as the metric on the respective surfaces.

The key object in deriving the Brown-York conserved quantities is the stress-energy-momentum tensor. It is defined as the on-shell value of the variation

$$
T_{ij} := -\frac{2}{\sqrt{-\gamma}} \frac{\delta I_{bd}}{\delta \gamma_{ij}} ,
$$

(11)

where $I_{bd}$ is the boundary terms in the action (2). Explicitly, we have

$$
T_{ij} = \frac{1}{8\pi G_5} \left( K_{ij} - \gamma_{ij} K - \frac{3}{R} \gamma_{ij} + \frac{R}{2} G_{4ij} \right),
$$

(12)

where $G_{4ij}$ is the Einstein tensor with respect to the metric $\gamma_{ij}$. This $T_{ij}$ is evaluated at the boundary of the solution (5), and then the boundary is removed to infinity. We remark that the projected second fundamental form $K_{ij}$ is defined as

$$
K_{ij} := \gamma_{\mu}^i \gamma_{\nu}^j K_{\mu\nu} \quad \text{with} \quad K_{\mu\nu} := -n_{\nu\mu}.
$$

(13)

Given the stress-energy-momentum tensor, we can proceed to compute the mass of the black hole as instructed by Brown and York. We suggest that the appropriate mass for our system is not the quasilocal energy, but the Hamiltonian that generates the unit time translation

$$
M := \int d^3x \sqrt{\sigma} (N \epsilon - V^a j_a) ,
$$

(14)

where as defined in Reference [17], we have

$$
\epsilon := u_i u_j T^{ij} \quad \text{and} \quad j_a := -\sigma_{ai} u_j T^{ij}.
$$

(15)

The reader should pay careful attention to the indices where proper projections must be done. For example, $V^a$ is the projection of $V^i$ onto the intersection surface, the indices of $T_{ij}$ are raised by the metric $\gamma_{ij}$ and also we have $u_i := \gamma_i^\mu u_\mu$. The momentum in the $x^-$ direction can be also computed following Brown and York. Taking the Killing vector field $\phi^a = (1, 0, 0)$, they tell us to compute

$$
J = \int d^3x \sqrt{\sigma} j_a \phi^a ,
$$

(16)

---

5 The four-dimensional Einstein tensor obviously vanishes for our boundary of the planar black hole in discussion.
which can be interpreted as the total particle number. We define the entropy, \( S \), of the system as a quarter of the volume given at the fixed time \( (x^+) \) and at the horizon \( (r = r_H) \). The temperature can be obtained by requiring the smoothness of the Euclidean geometry where the time \( x^+ \) is analytically continued to \( ix^+ \). In doing so, the ADM form must be employed, as in the case of the Kerr black hole. Alternatively, the same result can be derived through the computation of the surface gravity.

The thermodynamic quantities computed as described are identical to the ones derived from the planar Schrödinger black hole [6, 7, 25], namely,

\[
\begin{align*}
\Omega_H &= \frac{1}{2b^2}, \quad M = \frac{r_H^4 V_3}{16\pi G_5 R_5}, \quad J = -\frac{r_H^4 V_3}{4\pi G_5 R_5} b^2, \quad S = \frac{r_H^3 V_3}{4G_5 R_5} b, \quad \beta = \frac{\pi R^2}{r_H} b, \\
\end{align*}
\]

where we have defined \( V_3 := \int dx^- dydz \). However, we would like to emphasize that we have computed the quantities through the usual well-defined procedure, as the IAS group has done so already in Reference [7]. On the other hand, the other derivations involve ambiguous counterterms and necessity to assume the validity of the first law, rather than independently checking it. Also notice the simplicity of the method, as compared to the ones that are employed for the Schrödinger black holes. Taking advantage of the simplicity, we are going to apply this method to the complicated systems of R-charged black holes in the next section.

### 3 Thermodynamics of R-Charged Black Holes

This section can be safely skipped for the reader who is more interested in the computation of the conductivity in Section 4.

The R-charged black holes have three independent charges in general. The Schrödinger version with the three equal charges has been obtained in References [26, 27] through the complicated null Melvin twist. The equal-charge configuration is a very special case because the scalars decouple from the theory and the effective five dimensional action becomes simple Einstein-Maxwell with the negative cosmological constant [28]. Applying the null Melvin twist to the general-charge configuration is a very cumbersome task. Here, we see that the thermodynamic properties can be obtained easily for the general-charge configuration by adopting the light-cone coordinates in the AdS R-charged black hole systems. There are planar, spherical and hyperbolic black hole solutions in the AdS space and we are going to introduce them simultaneously. Then we discuss the phase diagrams of the planar and spherical black holes.

One way to describe the five-dimensional R-charged black holes is to regard them as the solutions of the five dimensional \( N = 2 \) gauged \( U(1)^3 \) supergravity [29]. The bosonic part of the action is

\[
I = \frac{1}{16\pi G_5} \int d^5 x \sqrt{-g} \left[ R - \frac{1}{2} \sum_{i=1}^3 (\partial_\mu \ln X_i)(\partial^\mu \ln X_i) - \frac{1}{4} \sum_{i=1}^3 X_i^{-2} F_{i\mu\nu} F_i^{\mu\nu} + \frac{V}{R^2} \right] \\
+ \frac{1}{16\pi G_5} \int F_1 \wedge F_2 \wedge A_3 - \frac{1}{8\pi G_5} \int d^4 x \sqrt{-\gamma} \left( K + \frac{3}{R} + \frac{R}{4} R_4 + \frac{1}{2R} \partial^2 \right). 
\]

The fields \( X_i \) with \( i = 1, 2, 3 \) are the scalars and they are subject to the constraint

\[
X_1 X_2 X_3 \equiv 1 ,
\]
so there are actually two independent scalar fields. The potential $V$ is defined as

$$V := 4 \sum_{i=1}^{3} X_i^{-1}. \quad (20)$$

The one-form fields $A_i$ correspond to the $U(1)^3$ gauged symmetry and $F_i := dA_i$. As in Section 2, we have included the boundary terms but with extra $\bar{\phi}^2$, which is the Liu-Sabra finite counterterm [21] and this will be defined shortly.

The black hole solutions to the equations of motion that follow from the action can be written as

$$ds^2 = \left(\frac{r}{R}\right)^2 H(r)^{1/3} \left\{ -h(r)dt^2 + \eta_0^2 + dX_0^2 \right\} \left(\frac{R}{r}\right)^2 H(r)^{-2/3} h(r)^{-1} dr^2,$$

$$X_i(r) = H(r)^{1/3} H_i(r) \quad \text{and} \quad A_i(r) = \left(\frac{g_i}{r_H^2 + q_i} - \frac{g_i}{r^2 + q_i}\right) dt. \quad (21)$$

The function $H$ is

$$H = H_1 H_2 H_3 \quad \text{with} \quad H_i := 1 + \frac{q_i}{r^2}, \quad (22)$$

where $q_i$ are the parameters related to the $U(1)^3$ charges $g_i$ via

$$g_i = \sqrt{q_i(r_0^2 + kq_i)}. \quad (23)$$

The parameter $r_0$ is the non-extremality parameter and the integer $k$ will be explained momentarily. The blackening factor $h(r)$ is given by

$$h := 1 + \left(\frac{R}{r}\right)^2 \left\{ k - \left(\frac{r_0}{r}\right)^2 \right\} H^{-1} \quad \text{and} \quad r_0 = r_H \sqrt{k + \left(\frac{r_H}{R}\right)^2 H_H}, \quad (24)$$

where the parameter $r_H$ is the location of the horizon in the $r$ coordinate and is the largest root of $h = 0$. We have expressed the non-extremality parameter in terms of $r_H$ and the short hand notation $H_H := H(r_H)$ is introduced. The parameter $k$ represents three possible values, $k = 0, +1$ and $-1$, and they correspond to the planar, spherical and hyperbolic black holes, respectively. According to the value of the parameter $k$, we have

$$\eta_0 := dx \quad \text{and} \quad dX_0^2 := dy^2 + dz^2,$$

$$\eta_{+1} := \left(\frac{R}{2}\right) (d\psi + \cos \theta d\phi) \quad \text{and} \quad dX_{+1}^2 := \left(\frac{R}{2}\right)^2 (d\theta^2 + \sin^2 \theta d\phi^2),$$

$$\eta_{-1} := \left(\frac{R}{2}\right) (d\psi + \cosh \chi d\phi) \quad \text{and} \quad dX_{-1}^2 := \left(\frac{R}{2}\right)^2 (d\theta^2 - \sinh^2 \chi d\phi^2). \quad (25)$$

The one-forms $\eta_{\pm 1}$ are introduced in Reference [25] for the Schrödinger black holes.

Following that reference, we introduce the coframe

$$\omega^+ := b(dt + \eta_k), \quad \omega^- := \frac{1}{2b}(dt - \eta_k), \quad \bar{\omega}_k \quad \text{and} \quad \omega^r = dr, \quad (26)$$
where

\[
\vec{\omega}_0 := (dy,dz), \quad \vec{\omega}_{+1} := \left( \frac{R}{2} d\theta, \frac{R}{2} \sin \theta d\phi \right) \quad \text{and} \quad \vec{\omega}_{-1} := \left( \frac{R}{2} d\chi, \frac{R}{2} \sinh \chi d\phi \right).
\] (27)

For the cases \( k = \pm 1 \), they are non-coordinate bases and their dual frames, \( \{ e_\mu \} \), and Lie brackets are given in Appendix B. Also recall that the normalization of \( \omega^\pm \) with the parameter \( b \) is the boost for the planar case. This physical interpretation clearly fails when \( k = \pm 1 \), because \( \eta^\pm_1 \) are not translationally invariant directions. We comment them further when we discuss their thermodynamics separately.

In these bases, the metric takes the form

\[
ds^2 = \left( \frac{r}{R} \right)^2 H^{1/3} \left\{ \frac{1 - h}{4b^2} \omega^+ \omega^+ - (1 + h) \omega^+ \omega^- + (1 - h)b^2 \omega^- \omega^- + \vec{\omega}_k^2 \right\} \\
+ \left( \frac{R}{r} \right)^2 H^{-2/3} h^{-1} \omega^2 ,
\] (28)

where \( \vec{\omega}_k^2 \) are defined as

\[
\vec{\omega}_0^2 := \omega^y^2 + \omega^z^2, \quad \vec{\omega}_{+1}^2 := \omega^\theta^2 + \omega^\phi^2 \quad \text{and} \quad \vec{\omega}_{-1}^2 := \omega^\chi^2 - \omega^\phi^2 .
\] (29)

One notices that the coframes are chosen so that the metric appears similar to the one in Equation (5). As in Section 2 and in Reference [25], we propose that the non-relativistic counterpart is obtained by re-interpreting \( \omega^+ \) as the non-relativistic time direction. Then the ADM form is

\[
ds^2 = \left( \frac{r}{R} \right)^2 H^{1/3} \left\{ \frac{1}{1 - h} b^{-2} \omega^2 + (1 - h)b^2 \left( \omega^- - \frac{1 + h}{2b^2} \frac{1}{1 - h} \omega^+ \right)^2 + \vec{\omega}_k^2 \right\} \\
+ \left( \frac{R}{r} \right)^2 H^{-2/3} h^{-1} \omega^2 .
\] (30)

From this, we obtain the lapse function, the shift function and the horizon coordinate velocity in the \( \omega^- \) direction

\[
N = \left( \frac{r}{R} \right) H^{1/6} \sqrt{\frac{1}{1 - h} b^{-1}}, \quad V^- = - \frac{1}{2b^2} \frac{1 + h}{1 - h} \quad \text{and} \quad \Omega_H = \frac{1}{2b^2} .
\] (31)

We are almost ready to compute the thermodynamic quantities, except that the Liu-Sabra finite counterterm in the action (18) needs to be defined. This is the term introduced in Reference [21], and unlike other counterterms, it is not necessary to cancel the divergences of the on-shell action. Being plainly square of the two (independent) scalar fields \( \vec{\phi} = (\phi_1, \phi_2) \), this is the simplest finite non-vanishing matter field term. It can be explicitly written in terms of the functions \( H_i \) as

\[
\vec{\phi}^2 = \frac{1}{6} \left\{ 3 \left( \ln \frac{H_1}{H_2} \right)^2 + \left( \ln \frac{H_1 H_2}{H_3} \right)^2 \right\} .
\] (32)

It is commonly considered that the addition of finite counterterms corresponds to changes in the renormalization scheme in the dual field theory. However, our main motivation to include this term comes from the fact that only with the finite counterterm, does the Brown-York procedure yield thermodynamic quantities that satisfy the first law.
3.1 Thermodynamic Quantities

The thermodynamic quantities of the R-charged black holes are obtained in a completely similar manner as in Section 2. The masses and momenta are calculated by the Brown-York procedure with the normal vectors of the hypersurfaces and the projection tensors defined similarly as before. For the cases \( k = \pm 1 \), the stress-energy-momentum tensors should be computed with the geometric quantities in the non-coordinate bases summarized in Appendix B. The entropy and temperature are computed from the metric expressed in the special coframes above.

In addition, we have the \( U(1)^3 \) R-charges and the conjugate chemical potentials. The charges \( N_i \) are computed through Gauss’ law,

\[
N_i = \lim_{r \to \infty} \frac{1}{16\pi G_5} \int \omega^3 \sqrt{\sigma} n_\mu u_\nu F^\mu_\nu, \tag{33}
\]

where \( n_\mu \) and \( u_\nu \) are the normals of the timelike and spacelike hypersurfaces introduced in Section 2 and \( \omega^3 := \omega^- \wedge \omega^0 \wedge \omega^2 \) for the \( k = 0 \) case and similarly defined for the \( k = \pm 1 \) cases. Recall that Gauss’ law follows from the action by taking the variation of the Lagrange multiplier, i.e., the time component of the gauge fields, \( A_i \). This immediately implies that the conjugate chemical potentials, \( \mu_i \), are exactly the multipliers. However, there are two issues to which we must pay attention. First, the chemical potentials must be the difference of the time component \( A_i \) between the boundary and the horizon. Since we have set \( A_i(r_H) = 0 \) as required by the regularity of the vector field, the chemical potential should be the value at \( r = \infty \). The second issue is that our system has the horizon velocity \( \Omega_H = 1/(2b^2) \), so it is more adequate to write the gauge fields as

\[
A_i(r = \infty) = \frac{g_i}{r_H^2 + q_i} \left\{ b^{-1} \omega^+ + b(\omega^- - \frac{1}{2b^2} \omega^+) \right\}, \tag{34}
\]

and take the first term as the chemical potentials. This definition of the chemical potentials is equivalent to the (difference) value of the gauge fields in the co-moving frame with respect to the horizon. Changing to the co-moving frame is necessary to avoid the ill behavior of the timelike killing vector field and the associated time component of the gauge fields inside the ergo-region, whose existence can be clearly seen in the ADM form \([30]\).

We collect the thermodynamic quantities described above;

\[
\begin{align*}
\Omega_H &= \frac{1}{2b^2}, \quad M = \frac{V_3}{16\pi G_5 R^3} \left( r_0^2 + \frac{2}{3} k \sum_i q_i + |k| \frac{R^2}{4} \right) \\
J &= -\frac{V_3}{4\pi G_5 R^3} \left( r_0^2 + \frac{2}{3} k \sum_i q_i + |k| \frac{R^2}{4} \right) b^2, \quad S = \frac{r_H^3 H_{1/2} V_3 b}{4G_5 R^3} \\
\beta &= \frac{H_{1/2}^{1/2} \pi R^2}{Q_k r_H}, \quad N_i = \frac{g_i V_3}{8\pi G_5 R^3 b}, \quad \mu_i = \frac{g_i}{r_H^2 + q_i} b^{-1}, \tag{35}
\end{align*}
\]

where we have defined \( V_3 := \int \omega^3 \) and

\[
Q_k := 1 + \frac{k R^2 + q_1 + q_2 + q_3}{2r_H^2} - \frac{q_1 q_2 q_3}{2r_H^6}. \tag{36}
\]

More discussion on this point can be found in Reference [30]. As will be mentioned shortly, this issue is overlooked in Reference [26] and consequently, there is an extra factor of 2 in their chemical potential. In Reference [27], this chemical potential is not discussed.
One can check that the quantity

\[
\beta \left( M - \beta^{-1}S - \Omega_H J - \sum_i \mu_i N_i \right) = \frac{(\beta V_3) r_H^2}{16\pi G_5 R^3} \left\{ k + \frac{3}{4} |k| \left( \frac{R}{r_H} \right)^2 - \left( \frac{r_H}{R} \right)^2 H_H \right\},
\]

precisely is the on-shell value of the action with the boundary removed to infinity, and this serves as a non-trivial check on our procedure.

As noted in the foregoing sections, our derivation of the thermodynamic quantities do not assume the first law, while the previous derivations do depend on it and the independent check of the law has not been possible. Therefore it is a worthwhile digression here to discuss our results in comparison with the known ones.

The planar case without the charges was discussed already in Section 2 and our results are identical to the previously derived quantities. Imeroni and Sinha in Reference [26] discuss a special charged case with \((q_1, q_2, q_3) = (q, q, q)\). Their derivation involves type IIB supergravity action with unconventional counterterms. The counterterms are not unique and the coefficients are determined through a number of (reasonable) assumptions. The thermodynamic quantities are obtained from the regulated action, by effectively assuming the first law. However, they identify the entropy as a quarter of the horizon area with the original relativistic coordinates \((t\) and \(x\)), and this is not an appropriate identification for the non-relativistic system with the time direction \(\omega^+\). Moreover, they define the chemical potential without taking into account of the horizon velocity and the existence of the ergo-region, which is not adequate for the system. These lead to the mismatch with our results, but their results agree with ours after correcting the entropy and the chemical potential.

The spherical and hyperbolic cases without the charges were discussed in Reference [25]. The derivation of the on-shell action was based on the background subtraction method with an unusual boundary matching, which is highly ad hoc. The thermodynamic quantities were derived, again, by effectively assuming the first law. However, in our thermodynamic quantities with \(k = \pm 1\), the first law is not satisfied with respect to the variable \(b\). From the viewpoint of our procedure, this is reasonable, because the interpretation of the variable \(b\) as a boost parameter is only possible for the \(k = 0\) case. Hence the parameter \(b\) (for \(k = \pm 1\)) is not the variable that determines the thermodynamic equilibrium. In other words, we cannot take the variation with respect to \(b\) to extremize the action. This fact leads to the discrepancy between our results and those of Reference [25], because the latter assumes that \(b\) is a variable for the extremization. We are unable to determine which is the correct identification of the thermodynamic quantities, for the methods are completely different. However, we will see that the emerging phase diagrams turn out to be identical (for the uncharged setting).

### 3.2 Planar Black Hole Phase Diagram

Let us specialize to the planar black hole (with \(k = 0\)) and discuss its phase structure. As usual in the system of AdS black holes, the first interest is the phase transition between the black hole and the AdS space without the black hole, i.e., the generalization of the

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7 To compare their results with ours, one must bring their charge \(Q\) to our \(g = g_i\) (for all \(i\)) and their \(r\) to our \(\sqrt{r^2 + q}\) (consequently their \(r_0\) to \(\sqrt{r_H^2 + q}\)).

8 They identify the temperature, the horizon velocity and the chemical potential using the coordinate system \(\omega^\pm\), so it is inconsistent.
Hawking-Page phase transition \cite{Hawking:1982dh} to the finite charges and chemical potentials. The phases are determined by comparing the values of the action evaluated on the respective solutions. The on-shell value of the action for the black hole is given in Equation (37). The other solution of interest can be obtained by setting \( r_0 = 0 = q_i \) in the black hole solution, and the value of the action \cite{Gubser:2008px} with respect to this solution turns out to be zero. Since the black hole solution always gives negative on-shell action, the black hole is always preferred and there is no Hawking-Page phase transition for this planar case.

Another interesting structure in the phase diagram is the thermodynamic stability threshold. We determine the threshold line by following Reference \cite{Hartnoll:2008kx}. We calculate the Hessian matrix of the left-hand side in Equation (37) with respect to the variables \( r_H \), \( q_i \) and \( b \), but with fixed \( \beta \), \( \Omega_H \) and \( \mu_i \). Then the determinant of the matrix is evaluated with the on-shell values of the fixed quantities, and the zero of this quantity determines the threshold. We have carried out this analysis for the charge configurations \((q, 0, 0)\), \((q, q, 0)\) and \((q, q, q)\). The analysis reveals that the stability thresholds in the \( T - \mu \) phase diagram are given as straight lines, as shown in Figure 1. The phase diagram is identical to the relativistic case, as worked out in Reference \cite{Hartnoll:2008kx}. It turns out that the critical lines are independent of the parameter \( b \), and as long as \( \mu \) and \( T \) are in the stable region, any value of \( b \) is allowed. The similarity of the Schrödinger black hole properties to the relativistic counterpart has been pointed out since References \cite{Gubser:2008px, Gubser:2008wv}, and the situation does not change with the inclusion of the R charges.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{phase_diagram.png}
\caption{The stability threshold lines in \( T - \mu \) phase diagram. The solid, dashed and dotted lines are for the configurations \((q, 0, 0)\), \((q, q, 0)\) and \((q, q, q)\), respectively. The slopes are \( \pi/\sqrt{2} \), \( \pi \) and \( 2\pi \). We have set \( R = 1 \).}
\end{figure}

\footnote{For the case \((q, q, 0)\), it is important to note that the Hessian must be computed with respect to the two charge parameters \( q_1 \) and \( q_2 \), and then compute the determinant of it with the constraint \( q := q_1 = q_2 \). Similarly for the configuration \((q, q, q)\).}
3.3 Spherical Black Hole Phase Diagram

Let us now focus on the spherical black hole with $k = +1$. We can determine the Hawking-Page phase transition as described in the previous subsection. We have the difference action

$$\Delta I = \frac{(\beta V_0) r_H^2}{16\pi G_5 R^5} \left( 1 - \frac{r_H^2}{R^2} H_H \right).$$

(38)

When $\Delta I$ is negative, the black hole is preferred to the AdS space without the black hole, and when it is positive, the preference is the other way around. The thermodynamic stability can also be analyzed as before using the Hessian matrix with respect to the variables $r_H$ and $q_i$.

In Figure 2, the $T-\mu$ phase diagram for the charge configuration $(q, 0, 0)$ is shown. This is identical to the relativistic counterpart as worked out in Reference [32]. (See also [30].)

![Figure 2: The $k = +1$ phase diagram in $T-\mu$ parameter space with the charge configuration $(q, 0, 0)$. The temperature and the chemical potential are scaled with the parameter $b$. The solid lines are the stability thresholds and the dashed curve is the Hawking-Page phase transition line. The curves merge and terminate at $(bT, b\mu) = (1/\pi, 1)$. We have set $R = 1$.](image)

The curves in the diagram merge and terminate at $(bT, b\mu) = (1/\pi, 1)$ in the units of $R$. From the discussion so far, the region of the phase diagram $bT < 1/\pi$ is completely unclear because the black hole saddle point in the action does not exist. However, it is shown in Reference [30] that there is a metastability line running at $b\mu = 1$ for all values of $T$, and it is argued that the metastability line below $bT < 1/\pi$ is the natural continuation of the stability threshold, but representing the stability of the AdS space without the black hole. The physical cause of this instability is hard to see in the five dimensional action (18). However, when describing the action as the $S^5$ compactification of type IIB supergravity with rotations in the five sphere, the critical line corresponds to the point where the speed of the rotation exceeds the speed of light [28 32]. Thus it appears that the thermodynamics of the five dimensional theory reflects the misbehavior in the higher dimensional theory. The instability line at $b\mu = 1$ can also be demonstrated from the dual field theory side. The field theory dual of the critical line at $b\mu = 1$ is where the chemical potentials become greater than the mass of the scalar fields, induced by the conformal coupling of the fields to
the curvature of the space, $S^3$. This phenomenon is shown in Reference [33] for the weak coupling case and the authors argue that the critical line persists to the strong coupling regime due to the supersymmetry. Therefore, in what follows, we assume the critical line at $b\mu = 1$ for all values of temperature.

The $T-\mu$ phase diagram of the charge configuration $(q, q, 0)$ is similar to the one-charge configuration just discussed, except that the curves of the phase diagram merge and terminate at $(bT, b\mu) = (1/2\pi, 1)$. The configuration $(q, q, q)$ also have a similar phase diagram but as is well-known, the black hole with this charge configuration can become zero temperature with finite entropy, so the curves merge at $(bT, b\mu) = (0, 1)$.

Let us now consider the $T-\Omega_H$ phase diagram with fixed $\mu$. In Figure 3 the phase diagrams for the uncharged case and the charged case with $(q_1, q_2, q_3) = (q, 0, 0)$ are shown. For

![Figure 3](image-url)

**Figure 3:** The $k = +1$ phase diagram in $T-\Omega_H$ parameter space. The left diagram is the uncharged case and the right is for the charge configuration $(q, 0, 0)$ with $\mu = 1$, where we have set $R = 1$. The solid lines are the stability thresholds and the dashed curve is the Hawking-Page phase transition line. The curves merge at $(T, \Omega_H) = (0, 0)$ for the left diagram and at $(1/\pi, 0.5)$ for the right.

the uncharged black hole phase diagram, the Hawking-Page phase transition and the stability threshold are given by the curves $\Omega_H = (2\pi^2/9)T^2$ and $\Omega_H = (\pi^2/4)T^2$, respectively. (We are adopting the units of $R$.) As mentioned before, this phase diagram is identical to the one in Reference [25], despite the differences in each thermodynamic quantities.

For the charged case, recall that we have the threshold at $bT = 1/\pi$ and $b\mu = 1$, that is, at $T = \sqrt{2\Omega_H/\pi}$ and $\mu = \sqrt{2\Omega_H}$. In the right diagram of Figure 3, we have chosen $\mu = 1.0$ as an example, then the critical value of $\Omega_H$ is 0.5 and the lowest possible temperature is $1/\pi$, as one sees in the diagram. Similar phase diagrams are observed for the other charge configurations $(q, q, 0)$ and $(q, q, q)$.

We now consider the $\mu-\Omega_H$ phase diagram with fixed $T$. Figure 4 shows the phase diagram for the charge configuration $(q, 0, 0)$. For this plot, we have chosen $T = 0.5$. Then, translating the critical point $(bT, b\mu) = (1/\pi, 1)$, we deduce that the curves merge at $(\mu, \Omega_H) = (\pi/2, \pi^2/8)$, as observed in the diagram. We have also plotted the critical line $b\mu = 1$, i.e., $\Omega_H = \mu^2/2$ after the other curves merge and terminate. The case $(q, q, 0)$
is similar to Figure 4 and for \((q, q, q)\), the curves merge at infinity along the critical curve \(\Omega_H = \mu^2/2\).

4 DC Conductivity

In this section, we demonstrate that the introduction of the light-cone coordinates (4) is enough to compute non-relativistic quantities other than the thermodynamic ones presented in the previous sections. In particular, we compute a two point correlator and the DC conductivity.

Since the planar black hole horizon has translationally invariant directions, it is natural to consider two-point correlators and shear viscosity in terms of their Fourier modes, following the prescription of Son and Starinets [34]. The computation is similar to theirs and sketched in Appendix C with the same results as the ones previously derived from the Schrödinger black hole [6, 7, 8]. In particular, the viscosity-entropy ratio is identical to the relativistic counterpart.

As another demonstration of the utility we consider the DC conductivity, which is an interesting physical observables in condensed matter system such as high \(T_c\) superconductor. At optimal doping, resistivity reveals an intriguing universal behavior as \(\rho \sim T\). In holographic approach, there exist a few systems which show this interesting property. These include the AdS [37], Lifshitz [38] and charged dilatonic black holes [39].

In this section we work out the DC conductivity, which is the non-relativistic counterpart of the computation by Karch and O’Bannon [22]. The non-relativistic DC conductivity from the Schrödinger black hole spacetime has been obtained in Reference [23] and we shall compare the results. For the sake of the comparison, the discussion here closely follows Reference [23].
We start with the full (asymptotically) AdS\(_5 \times S^5\) spacetime expressed in the light-cone coordinates
\[
ds^2 = g_{++} dx^+ dx^+ + 2 g_{+-} dx^+ dx^- + g_{--} dx^- dx^- + g_{yy} dy^2 + g_{zz} dz^2 + g_{rr} dr^2 + (d\chi + A)^2 + ds^2_{\mathbb{CP}^2},
\]
where the AdS part of the metric is given in (5) and the \(S^5\) metric is expressed as a Hopf fibration over \(\mathbb{CP}^2\), with \(\chi\) the Hopf fiber direction. The one-form \(A\) gives the Kähler form \(J\) of \(\mathbb{CP}^2\) via \(dA = 2J\). To write the metric of \(\mathbb{CP}^2\) and \(A\) explicitly, we introduce \(\mathbb{CP}^2\) coordinates \(\alpha_1, \alpha_2, \alpha_3, \) and \(\theta\) and define the \(SU(2)\) left-invariant forms
\[
\sigma_1 := \frac{1}{2} (\cos \alpha_2 \, d\alpha_1 + \sin \alpha_1 \sin \alpha_2 \, d\alpha_3),
\sigma_2 := \frac{1}{2} (\sin \alpha_2 \, d\alpha_1 - \sin \alpha_1 \cos \alpha_2 \, d\alpha_3),
\sigma_3 := \frac{1}{2} (d\alpha_2 + \cos \alpha_1 \, d\alpha_3),
\]
so that the coordinate expression of \(\mathbb{CP}^2\) and \(A\) are
\[
ds^2_{\mathbb{CP}^2} = d\theta^2 + \cos^2 \theta \left( \sigma_1^2 + \sigma_2^2 + \sin^2 \theta \, \sigma_3^2 \right), \quad A = \cos^2 \theta \, \sigma_3. \tag{41}
\]

As in References \[22, 23\], we introduce \(N_f\) probe D7-branes — but in the AdS\(_5 \times S^5\) expressed in light-cone coordinates. We are working with the probe limit, \(N_f \ll N_c\), which suppresses the back-reaction of the D7-branes on the gravity background.

The D7-branes are extended to the AdS\(_5\) and the angular directions \(\alpha_{1,2,3}\) of \(S^5\) at some values of the coordinates \(\theta\) and \(\chi\). The embedding implies that there are two world volume scalars \(\theta\) and \(\chi\). We choose the scalar \(\chi\) to be trivial so that the D7-branes sit at a fixed value of \(\chi\), but we take the scalar \(\theta\), which is dual to the mass operator of the \(\mathcal{N} = 2\) theory, as a function of \(r\). We consider the diagonal \(U(1)\) worldvolume gauge fields \(A_\mu\), which are dual to \(U(1)\) current \(J_\mu\) of the dual field theory. Recall that in the relativistic setup of Reference \[22\], the constant background electric field, \(F_{by} = -E\), was introduced. To obtain the counterpart, we express this in the light-cone coordinates \[41\] and get the corresponding gauge fields
\[
A_+ = E_b y + h_+(r), \quad A_- = 2 b^2 E_b y + h_-(r), \quad A_y = A_y(r), \tag{42}
\]
where we have included the fluctuation fields that depend only on \(r\) and redefined the electric field \(E_b := E/(2b)\), following the convention of Reference \[23\]. Note that this gauge-field configuration is identical to the one adopted in Reference \[23\].

Since we do not have NSNS B-field nor the coupling to RR-potentials, the DBI action of the probes takes the form
\[
S_{D7} = -N_f T_{D7} \int d^8\xi e^{-\Phi} \sqrt{-\det(g_{D7} + (2\pi\alpha') F)}, \tag{43}
\]
where \(T_{D7}, \xi, F\) and \(\Phi\) are the D-brane tension, worldvolume coordinates, the \(U(1)\) field strength and the dilaton, respectively. The metric \(g_{D7}\) is the pullback of the spacetime

\[10\] The sign difference in \(A_-\) stems from the difference in the definition of light-cone coordinates.
metric with respect to the aforementioned embedding map. Explicitly,
\[
    ds^2_{DT} = g_{++} dx^+ dx^+ + 2g_{+-} dx^+ dx^- + g_{--} dx^- dx^- + g_{yy} dy^2 + g_{zz} dz^2 \\
    + g^{DT}_{rr} dr^2 + \cos^2 \theta (\sigma_1^2 + \sigma_2^2 + \sigma_3^2),
\]
where \( g^{DT}_{rr} := g_{rr} + \theta'(r)^2 \) and the prime denotes the derivative with respect to \( r \). With our gauge fields, the action has the form
\[
    S_{DT} = -N \int dr \sqrt{-\det M},
\]
where both sides are divided by the infinite volume of the field theory directions and we are working with the action density. We have defined \( N := 2\pi^2 N_f T_{DT} \) with \( 2\pi^2 \) from the trivial integration of \( S^3 \) and\(^{11} \)
\[
    \det M = G_{\alpha_2 \alpha_3} g_{\alpha_1 \alpha_1}(r) g_{yy}(r) \left[ g^{DT}_{rr}(r) \left( \tilde{E}_b^2 G_3 + G_{+-} g_{yy}(r) \right) + G_{+-} \tilde{A}'_y(r) \right] \\
    + g_{--}(r) g_{yy}(r) \left( \tilde{h}'_+(r) - \tilde{h}'_-(r) \right)^2 + \tilde{E}_b^2 \left( 2\tilde{h}'_+(r) - \tilde{h}'_-(r) \right)^2 \right].
\]
Here the tildes indicate that the quantities are scaled with the factor of \( 2\pi \alpha' \), such as \( \tilde{F} = (2\pi \alpha') F \). The sub-matrix determinants we need are
\[
    G_{+-} = g_{++}(r) g_{--}(r) - g_{+-}(r)^2, \quad G_3 = 4b^4 g_{++}(r) - 4b^2 g_{+-}(r) + g_{--}(r), \\
    G_{\alpha_2 \alpha_3} = \frac{g_{\alpha_2 \alpha_2}(r) g_{\alpha_3 \alpha_3}(r) - g_{\alpha_2 \alpha_3}(r)^2}{\sin^2 \alpha_1}. \quad (47)
\]
The equations of motion of the gauge fields are constants of motion because the action consists of their derivatives only with respect to \( r \). Thus there are three constants of motion
\[
    \langle J^+ \rangle := \frac{\delta \mathcal{L}}{\delta \tilde{h}'_+} = H \left( [4\tilde{E}_b^2 b^4 + g_{--}(r) g_{yy}(r)]\tilde{h}'_+(r) - [2\tilde{E}_b^2 b^2 + g_{+-}(r) g_{yy}(r)]\tilde{h}'_-(r) \right) \\
    \langle J^- \rangle := \frac{\delta \mathcal{L}}{\delta \tilde{h}'_-} = -H \left( [2\tilde{E}_b^2 b^2 + g_{--}(r) g_{yy}(r)]\tilde{h}'_+(r) - [\tilde{E}_b^2 + g_{+-}(r) g_{yy}(r)]\tilde{h}'_-(r) \right) \\
    \langle J^y \rangle := \frac{\delta \mathcal{L}}{\delta \tilde{A}'_y} = H G_{+-} \tilde{A}'_y(r) \quad (48)
\]
with \( H := N \frac{G_{\alpha_2 \alpha_3} g_{\alpha_1 \alpha_1}(r) g_{yy}(r)}{\sqrt{-\det M}} \), where \( \mathcal{L} \) is the Lagrangian of the DBI action. The quantities have their physical meanings as light-cone charge density \( \langle J^+ \rangle \), light-cone current along \( x^- \) direction \( \langle J^- \rangle \), which is not independent but directly connected to \( \langle J^+ \rangle \), and current along \( y \) direction \( \langle J^y \rangle \).

After solving the equations and plugging those solutions back into the action, we have the on-shell action
\[
    S_{DT} = -2\pi \alpha' N^2 \int dr \ G_{\alpha_2 \alpha_3} g_{\alpha_1 \alpha_1}(r) \sqrt{g^{DT}_{rr}(r) g_{yy}(r)} \sqrt{\frac{\tilde{E}_b^2 G_3 + G_{+-} g_{yy}(r)}{U(r) - V(r)}},
\]
\(^{11}\) The contribution from the coordinates \( \alpha_{2,3} \) in \( G_{\alpha_2 \alpha_3} \) decouples from the others and becomes a multiplicative factor of the DBI action, while the factor \( G_{\alpha_2 \alpha_3} \) in the Schrödinger case \(^{23} \) couples with the \( B_{\mu\nu} \) fields and contributes to the conductivity calculation in a nontrivial way.
where

\[ U(r) = \frac{(J_y)^2}{G_{++}} - \hat{N}^2 G_{a_2a_3} g_{a_1a_1}(r) g_{yy}(r), \]  

\[ V(r) = \frac{\hat{E}_b^2 \left((J^+) + 2(J^-) b^2\right)^2 + \left((J^+)^2 g_{++}(r) + (J^-)(2(J^+) g_{+-}(r) + (J^-) g_{--}(r))\right) g_{yy}(r)}{g_{yy}(r) \left(\hat{E}_b^2 G_3 + G_{+-} g_{yy}(r)\right)}. \]  

We concentrate on the last square root factor of the action \(^{(19)}\) and demand this to be real all the way from the horizon to the boundary \(^{(22)}\). In the square root, the numerator changes sign somewhere between the horizon and the boundary. This can be easily seen by the explicit expression of the numerator with the metric \(^{(39)}\),

\[ \hat{E}_b^2 G_3 + G_{+-} g_{yy}(r) = \frac{r^2 \left(-r^4 + r_H^4 + 4\hat{E}_b^2 R^4 b^2\right)}{R^6}. \]  

At horizon \( r = r_H \), this quantity is positive and it changes sign as \( r \) is increased. We assign the value of \( r \) where this numerator changes sign as \( r_* \) and we have

\[ \left[\hat{E}_b^2 G_3 + G_{+-} g_{yy}(r)\right]_{r=r_*} = 0. \]  

For the on-shell action \(^{(19)}\) to be real, the denominator should also vanish at \( r = r_* \), so we demand \( U(r_*) - V(r_*) = 0 \). For this to happen, the numerator of the function \( V \) should vanish at \( r = r_* \) at least as fast as \(^{(52)}\). Setting the numerator of \( V \) to be zero at \( r = r_* \), we get

\[ \langle J^- \rangle = -\frac{2\hat{E}_b^2 b^2 + g_{+-}(r) g_{yy}(r)}{4\hat{E}_b^2 b^4 + g_{--}(r) g_{yy}(r)} \bigg|_{r=r_*} \langle J^+ \rangle, \]  

which reduced to \( \langle J^- \rangle = \langle J^+ \rangle/(2\beta^2) \) in the absence of the electric field.

By plugging this condition in the equation \( V(r_*) = U(r_*) \), we obtain the expression of the current along \( y \)-direction as

\[ \langle J_y \rangle^2 = \frac{\hat{E}_b^2 G_3}{g_{yy}(r)} \left[\hat{N}^2 G_{a_2a_3} g_{a_1a_1}(r) g_{yy}(r) + \frac{\langle J^+ \rangle^2}{4\hat{E}_b^2 b^4 + g_{--}(r) g_{yy}(r)}\right]\bigg|_{r=r_*}. \]  

Using Ohm’s law, we get

\[ \sigma = 2\pi a' \sqrt{\frac{G_3}{g_{yy}(r_*)}} \left[\hat{N}^2 G_{a_2a_3} g_{a_1a_1}(r_*) g_{yy}(r_*) + \frac{\langle J^+ \rangle^2}{4\hat{E}_b^2 b^4 + g_{--}(r_*) g_{yy}(r_*)}\right]^{\frac{1}{2}} \]  

\[ = 2\pi a' \sqrt{\frac{\hat{N}^2 b^2 \cos^6 \theta(r_*)}{16}} \sqrt{4\hat{E}_b^2 b^2 + R^4 \pi^4 T^4 b^4} + \frac{4\langle J^+ \rangle^2}{4\hat{E}_b^2 b^2 + R^4 \pi^4 T^4 b^4}, \]  

where we used the relation \( r_H = R^2 \pi T b \). This is the final expression for conductivity and is the analogue of the equation (3.27) of Reference \(^{(23)}\). In the reference, the origin of the first term was identified as the Schwinger pair production. Also notice that this is a dimensionless quantity, which is appropriate for the conductivity in \( 2+1 \) dimensional field theory.
Our result and that of Reference [23] are different. However, this is not inconsistent, because while we have derived the conductivity which is exactly the counterpart of Karch-O’Bannon [22], the setup of Reference [23] may not be so. In fact, we have checked that the null Melvin twist on the system of Karch and O’Bannon (with the constant background electric field) yields very different supergravity solution from the setup of Reference [23]. Therefore, it appears that despite the same gauge field configuration in Equation (42), the DC conductivities belong to different non-relativistic systems. We, however, demonstrate in the rest of this section that they become identical in a few important limiting cases.

Let us take the limit \( E_b \ll b(RT)^2 \), for weak electric fields compared to temperature. The conductivity becomes

\[
\sigma \approx 2\pi\alpha' \sqrt{\frac{4\langle J^+ \rangle^2}{\pi^4 b^4(RT)^4} + \frac{N^2 \cos^6 \theta(r_*)}{16} \pi^2 b^4(RT)^2}. 
\] (55)

Note that there is a difference in the \( \cos \theta \) factor compared to Reference [23]. For \( \cos \theta(r_*) \approx 0 \) or at low temperature, the second term of (55) is suppressed and the conductivity is

\[
\sigma \approx 2\pi\alpha' \frac{2\langle J^+ \rangle}{\pi^2 b^2(RT)^2}, 
\] (56)

and this is identical to the result of Reference [23].

For the opposite limit \( E_b \gg b(RT)^2 \), we have

\[
\sigma \approx 2\pi\alpha' \sqrt{\frac{\langle J^+ \rangle^2}{b^2 E_b^2} + \frac{N^2 \cos^6 \theta(r_*)}{8} b^3 E_b}. 
\] (57)

In this limit, the conductivity is identical to the (3.30) in [23] without further assumptions to the field \( \theta \). If we further take a limit with a small density \( \langle J^+ \rangle \approx 0 \) and small \( \theta(r_*) \), the conductivity, \( \sigma \approx (2\pi\alpha') \sqrt{\frac{N^2}{8} E_b b^3} \), is mainly from the Schwinger pair production. On the other hand, in the opposite limit with a large density \( \langle J^+ \rangle \approx 0 \) and \( \theta(r_*) \approx \pi/2 \) in the equation (57), we get \( \sigma \approx (2\pi\alpha') \frac{\langle J^+ \rangle}{E_b b} \), which is identical to the previous result in the same limit [23].

Thus we see that our result and that of Reference [23] are identical in some limiting cases, especially when the limits bring the expressions independent of the scalar profile \( \theta(r) \). This is remarkable considering the drastic differences in the backgrounds, where one of them even involves non-vanishing NSNS B-field.

5 Concluding Remarks

In this paper, we have taken less discussed approach to the non-relativistic generalization of the AdS/CFT, essentially by following References [11, 12, 7]. The procedure is to simply adopt special light-cone coordinates and re-interpret one of them as the non-relativistic time. We have shown that various black hole properties can be obtained in the same way as the relativistic counterparts, and our results are consistent with the ones directly derived from the Schrödinger black holes. We hope that we have demonstrated and conveyed the simplicity and well-defined nature of the procedure. In particular, we have emphasized that
the standard boundary of the geometry allowed us to utilize the usual Brown-York procedure to obtain the thermodynamic quantities of non-relativistic systems. We also derived the scalar two-point correlator and the associated shear viscosity, as well as the DC conductivity which is the non-relativistic counterpart of Reference [22]. All those computations are no more complicated than the relativistic counterparts.

While we think that our procedure is very powerful in dealing with the geometry with Schrödinger symmetry, it is unclear how to apply or generalize the procedure to different non-relativistic systems, such as Lifshitz spacetime. Schrödinger and Lifshitz systems are special cases of scale invariant non-relativistic systems, and they are characterized by the relative scaling factor of time and space, known as the dynamical exponent. One way to incorporate the dynamical exponent for the Schrödinger space is demonstrated explicitly using the parameter $b$ [40].

The calculation of DC conductivity is carried out with the gauge fields that is corresponding to the electric field, $F_{ty}$, in the relativistic AdS space. This is not really natural from the point of view of the non-relativistic theory, because as we have seen, it gives rise to nonzero $F_{-y}$ as well as $F_{+y}$. The reason we take this particular form of gauge fields is to compare directly to the known results. It is more appropriate to have background field only along the $F_{+y}$ to calculate the DC conductivity, which seems to give us more interesting DC conductivity results, while the same gauge field does not give any instability to the Schrödinger space. This is discussed recently in [40].

It is of a great interest to figure out the similarities and differences between the two different geometric realizations of the Schrödinger geometry, Schrödinger background and AdS in light-cone, in detail.

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**A Infinite Boost**

In this appendix, we demonstrate that the quantities of Schrödinger black holes can be extracted from the AdS black hole counterparts in infinite momentum frame. We reproduce the results of Section 2 as a prototype example of this technique. The motivation comes from Susskind’s work in 1967 [13], where he shows that a theory (a system of free scalar bosons) on a infinitely boosted frame resembles a non-relativistic system. We have a gravitational system but the procedure is somewhat similar to Susskind’s.
We start from the metric (3) and apply the Lorentz transformation
\[
\begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} \cosh \zeta & \sinh \zeta \\ \sinh \zeta & \cosh \zeta \end{pmatrix} \begin{pmatrix} t' \\ x' \end{pmatrix}.
\] (58)

Then the metric becomes
\[
ds_5^2 = \left( \frac{r}{R} \right)^2 \left\{ (\sinh^2 \zeta - h \cosh^2 \zeta) dt^2 + 2(1 - h) \sinh \zeta \cosh \zeta dtdx \\
+ (\cosh^2 \zeta - h \sinh^2 \zeta) dx^2 + dy^2 + dz^2 \right\} + \left( \frac{R}{r} \right)^2 h^{-1} dr^2,
\]
where we have omitted the primes on the coordinates \( t \) and \( x \). The ADM form of the metric is
\[
ds_5^2 = \left( \frac{r}{R} \right)^2 \left[ -\frac{h dt^2}{\cosh^2 \zeta - h \sinh^2 \zeta} + (\cosh^2 \zeta - h \sinh^2 \zeta) \left\{ dx + \frac{(1 - h) \sinh \zeta \cosh \zeta dt}{\cosh^2 \zeta - h \sinh^2 \zeta} \right\}^2 \\
+ dy^2 + dz^2 \right] + \left( \frac{R}{r} \right)^2 h^{-1} dr^2.
\] (59)

From this, one obtains the entropy, horizon velocity and temperature in terms of the rapidity \( \zeta \). Moreover, the mass and the momentum in \( x \)-direction can be computed through the Brown-York procedure as detailed in Section 2. The result is
\[
S = \frac{V_3}{4G_5} \left( \frac{r_H}{R} \right)^3 \cosh \zeta = \frac{V_3}{8G_5} \left( \frac{r_H}{R} \right)^3 \left( \frac{1}{\epsilon} + \epsilon \right)
\]
\[
\Omega_H = -\tanh \zeta = -1 + 2\epsilon^2 - 2\epsilon^4 + \mathcal{O}(\epsilon^6)
\]
\[
\beta = \frac{\pi R^2}{r_H} \cosh \zeta = \frac{\pi R^2}{2r_H} \left( \frac{1}{\epsilon} + \epsilon \right)
\]
\[
M = \frac{3r_H^4 V_3}{16\pi G_5 R^5} \left[ 4 \cosh^2 \zeta - 1 \right] = \frac{r_H^4 V_3}{16\pi G_5 R^5} \left( \frac{1}{\epsilon^2} + 1 + \epsilon^2 \right)
\]
\[
J_x = -\frac{r_H^4 V_3}{4\pi G_5 R^5} \frac{\sinh \zeta \cosh \zeta}{3} = -\frac{r_H^4 V_3}{16\pi G_5 R^5} \left( \frac{1}{\epsilon^2} - \epsilon^2 \right),
\]
where we have defined \( \epsilon := e^{-\zeta} \) and we are interested in the limit \( \epsilon \to 0 \).

As a non-relativistic system, we pick the most singular terms in \( S \), \( \beta \) and \( J_x \). This procedure is similar to the scaling done by Susskind [13]. For the horizon velocity \( \Omega_H \), we drop the speed of light \((-1)\) and take the next term in the \( \epsilon \) expansion. This is because there is an infinite contribution to the mass from the momentum in \( x \)-direction (the most singular term in \( \Omega_H J_x \)) and following Reference [13], we chose to drop this infinite contribution. Consequently, we must choose the finite term in the quantity \( M \). After the replacement \( \epsilon \to 1/2b \), we find that the extracted non-relativistic quantities are identical to Equation (17).

### B Geometric Quantities in Non-Coordinate Frames

We summarize the frames adopted for the spherical \((k = +1)\) and hyperbolic \((k = -1)\) black holes and remind the reader of the geometric quantities in a non-coordinate basis.
B.1 Frame for $k = +1$

In Section 3, following coframe is introduced;

$$\omega^+ = b(dt + \eta_{+1}), \quad \omega^- = \frac{1}{2b}(dt - \eta_{+1}),$$

$$\omega^\theta = \frac{R}{2}d\theta, \quad \omega^\phi = \frac{R}{2}\sin \theta d\phi, \quad \omega^r = dr,$$

where $\eta_{+1} := (R/2)(d\psi + \cos \theta d\phi)$. The dual frame $\{e_\mu\}$ that satisfy $\omega^\mu e_\nu = \delta^\mu_\nu$ is

$$e_+ = \frac{1}{2b}(\partial_t + \frac{2}{R}\partial_\psi), \quad e_- = b(\partial_t - \frac{2}{R}\partial_\psi),$$

$$e_\theta = \frac{2}{R}\partial_\theta, \quad e_\phi = \frac{2}{R}(-\cot \theta \partial_\psi + \frac{1}{\sin \theta} \partial_\phi), \quad e_r = \partial_r.$$  \hspace{1cm} (63)

This is a non-coordinate basis and we have the non-vanishing Lie bracket

$$[e_\theta, e_\phi] = \frac{2b}{R}e_+ - \frac{1}{bR}e_- - \frac{2}{R}\cot \theta e_\phi.$$ \hspace{1cm} (64)

B.2 Frame for $k = -1$

For this case, the coframe introduced is

$$\omega^+ = b(dt + \eta_{-1}), \quad \omega^- = \frac{1}{2b}(dt - \eta_{-1}),$$

$$\omega^\chi = \frac{R}{2}d\chi, \quad \omega^\phi = \frac{R}{2}\sinh \chi d\phi, \quad \omega^r = dr,$$

where $\eta_{-1} := (R/2)(d\psi + \cosh \chi d\phi)$. The dual frame is

$$e_+ = \frac{1}{2b}(\partial_t + \frac{2}{R}\partial_\psi), \quad e_- = b(\partial_t - \frac{2}{R}\partial_\psi),$$

$$e_\chi = \frac{2}{R}\partial_\chi, \quad e_\phi = \frac{2}{R}(-\coth \chi \partial_\psi + \frac{1}{\sinh \chi} \partial_\phi), \quad e_r = \partial_r.$$ \hspace{1cm} (66)

The non-vanishing Lie bracket is

$$[e_\chi, e_\phi] = -\frac{2b}{R}e_+ + \frac{1}{bR}e_- - \frac{2}{R}\coth \chi e_\phi.$$ \hspace{1cm} (67)

B.3 Geometric Quantities

The geometric quantities are defined without referring to a frame, especially in mathematical literature. Coordinate expressions are popular among physicists but the usual formulas must be modified when a non-coordinate basis is adopted. The formulas below are taken from MTW \[35\].

Given a frame whose Lie algebra product is

$$[e_\alpha, e_\beta] = c^{\gamma}_{\alpha \beta} e_\gamma,$$ \hspace{1cm} (68)

the connection coefficients are

$$\Gamma^{\gamma}_{\alpha \beta} = \frac{1}{2}(g_{\alpha \beta, \gamma} + g_{\alpha, \beta \gamma} - g_{\beta, \gamma \alpha} + c_{\alpha \beta \gamma} + c_{\alpha \gamma \beta} - c_{\beta \gamma \alpha}),$$ \hspace{1cm} (69)
where the comma \(\frac{\partial}{\partial \mu}\) implies the derivation with respect to \(e_\mu\). Notice that the usual symmetry exchanging the indices \(\beta\) and \(\gamma\) is not necessarily true. The covariant derivative is given with respect to this connection. So for instance,

\[ F_{\mu\nu} := A_{\nu,\mu} - A_{\mu,\nu}, \]

and since the exchanging symmetry is lost, the semicolons here cannot be replaced by just colons.

Also with the covariant derivative, the definition of the second fundamental form \([13]\) is not modified.

The Riemann tensor is

\[ R^\alpha_{\beta\gamma\delta} = \Gamma^\alpha_{\beta\delta,\gamma} - \Gamma^\alpha_{\beta\gamma,\delta} + \Gamma^\alpha_{\mu\gamma} \Gamma^\mu_{\beta\delta} - \Gamma^\alpha_{\mu\delta} \Gamma^\mu_{\beta\gamma} - \Gamma^\alpha_{\beta\mu} e^\gamma_{\delta}. \] (70)

The Ricci tensor and the scalar curvature are given from this Riemann tensor.

### C Scalar Two-Point Correlator and Viscosity-Entropy Ratio

Using the metric \([5]\) with the time coordinate \(x^+\), we can exactly follow the procedure of Son and Starinets \([34]\) to compute two-point correlators. In particular, we need the correlator of the stress tensor in \(yz\) components to obtain the viscosity. However, as shown in Reference \([36]\), this is equivalent to the correlator of a minimally coupled scalar in the background. Therefore, we sketch the calculation of the scalar two point correlator here.

First, we introduce a new coordinate in the metric \([5]\), according to

\[ u := \frac{r^2}{r_H^2}. \] (71)

Then the action for the scalar field becomes

\[ K \int d^4x \int_0^1 du \sqrt{-g} \left[ g^{uu}(\partial_u \phi)^2 + g^{\mu\nu}(\partial_\mu \phi)(\partial_\nu \phi) + m^2 \phi^2 \right] \]

\[ = (KR^3) \left( \frac{r_H^2}{2r_H^2} \right) \int d^4x \int_0^1 du u^{-2} \left[ \frac{4r_H^2}{R^4} hu(\partial_u \phi)^2 + (\partial_\mu \phi)^2 + b^2(1 - h^{-1})(\partial_\phi)^2 \right. \]

\[ - (1 + h^{-1})(\partial_\phi)^2 + \frac{1}{4b^2}(1 - h^{-1})(\partial_\phi)^2 + m^2 \left( \frac{r_H}{R} \right)^2 u^{-1} \phi^2 \bigg], \] (72)

where we have \(h = 1 - u^2\) and through the AdS/CFT dictionary, \(KR^3 = -N_c^2/(16\pi^2)\).

After Fourier decomposing as

\[ \phi(u, x^+, x^-, \vec{y}) = \int \frac{d^4k}{(2\pi)^4} e^{-i\omega x^+ + ik_- x^- + ik \cdot \vec{y}} f_k(u) \phi_0(k), \] (73)

the equations of motion read

\[ f_k'' + (\ln[u^{-1}h])' f_k' \]

\[ - \frac{1}{u h} \left( q_i^2 + (1 - h^{-1})w^2 + 2(1 + h^{-1})w q_- + (1 - h^{-1)}q_-^2 \right) + \left( \frac{mR}{2} \right)^2 u^{-1} \bigg] f_k = 0, \] (74)

where the primes denote the derivatives with respect to the variable \(u\) and we have defined

\[ w := \frac{b R^2}{2r_H^2} \omega, \quad q_- := \frac{1}{2b} \frac{R^2}{2r_H^2} k_- \quad \text{and} \quad q_i := \frac{R^2}{2r_H^2} k_i. \] (75)
The differential equation is singular at the horizon \( u = 1 \) and the idea is to extract the regular part of the function and solve for it. For this purpose, we set

\[
f_k(u) = (1 - u)^\nu F_k(u) ,
\]

and plug this into the differential equation. For the massless case \( m = 0 \), the regularity for the function \( F_k(u) \) near \( u = 1 \) determines that

\[
\nu = \pm \frac{i}{2} (w - q_-) .
\]

Since we are interested in the limit \( q_- \to 0 \), we pick the negative sign for the incoming wave solution. With this value of \( \nu \) (and with \( m = 0 \)), the differential equation for the regular part \( F_k(u) \) is

\[
F_k'' + \left\{ - \frac{1 + u^2}{u(1 - u^2)} + \frac{i(w - q_-)}{1 - u} \right\} F_k' - \left\{ \frac{(w - q_-)^2}{4(1 + u)^2} + \frac{2q_-^2 + 8q_-w + i(w - q_-)}{2u(1 - u^2)} \right\} F_k = 0 .
\]

By assuming \( w, q \ll 1 \), we can solve this differential equation order by order in those parameters, and the solution is

\[
F_k(u) = 1 - \left\{ \frac{i}{2} (w - q_-) + q_i^2 \right\} \ln \frac{1 + u}{2} + \text{higher orders} ,
\]

where the regularity of the solution at \( u = 1 \) and the boundary condition \( F_k(1) = 1 \) were imposed, by following Reference [34].

To obtain the viscosity through Kubo’s formula, we set \( q_- = 0 = q_i \) and write the solution as

\[
f_k(u) = (1 - u^2)^{-iw/2} + \text{higher orders} ,
\]

where the factor \( 2^{iw/2} \) was extracted from \( \phi_0(k) \) in Equation (73), in order to satisfy the required boundary condition \( f_k(0) = 1 \). Then the Son-Starinets prescription yields the retarded Green’s function

\[
G^R = \frac{ibN^2_c r_H^3}{8\pi^2 R^6} \omega .
\]

This is the same result as Reference [34], except for the parameter \( b \). However, since the same parameter also appears in the entropy (17), the viscosity-entropy ratio is identical to the relativistic case.

\[\text{12} \] We could have avoided this awkward extraction of the factor \( 2^{iw/2} \) by changing the boundary condition \( F_k(1) = 1 \), but we chose to have the congruence with Son and Starinets [34].
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