Information theoretic limits of learning a sparse rule

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Abstract

We consider generalized linear models in regimes where the number of nonzero components of the signal and accessible data points are sublinear with respect to the size of the signal. We prove a variational formula for the asymptotic mutual information per sample when the system size grows to infinity. This result allows us to heuristically derive an expression for the minimum mean-square error (MMSE) of the Bayesian estimator. We then find that, for discrete signals and suitable vanishing scalings of the sparsity and sampling rate, the MMSE displays an all-or-nothing phenomenon, namely, the MMSE sharply jumps from its maximum value to zero at a critical sampling rate. The all-or-nothing phenomenon has recently been proved to occur in high-dimensional linear regression. Our analysis goes beyond the linear case and applies to learning the weights of a perceptron with general activation function in a teacher-student scenario. In particular we discuss an all-or-nothing phenomenon for the generalization error with a sublinear set of training examples.

1 Introduction

Modern tasks in statistical analysis, signal processing and learning require solving high-dimensional inference problems with a very large number of parameters. This arises in areas as diverse as learning with neural networks [1], high-dimensional regression [2] or compressed sensing [3, 4]. In many situations, there appear barriers to what is possible to estimate or learn when the data becomes too scarce or too noisy. Such barriers can be of algorithmic nature, but they can also be intrinsic to the very nature of the problem. A celebrated example is the impossibility of reconstructing a noisy signal when the noise is beyond the so-called Shannon capacity of the communication channel [5]. A large amount of interdisciplinary work has shown that these intrinsic barriers can be understood as static phase transitions (in the sense of physics) when the system size tends to infinity (see [6, 7, 8]).

When the problem can be formulated as an (optimal) Bayesian inference problem the mathematically rigorous theory of these phase transitions is now quite well developed. Progress initially came from applications of the Guerra-Toninelli interpolation method developed for the Sherrington-Kirkpatrick spin-glass model [9], to coding and communication theory [10] [11] [12] [13] [14] [15], and more recently to low-rank matrix and tensor estimation [16] [17] [18] [19] [20] [21] [22] [23] [24], compressive sensing and high-dimensional regression [25] [26] [27] [28], and generalized linear models [29] [30]. In particular, for all these problems it has been possible to reduce the asymptotic mutual information to a low-dimensional variational expression, and deduce from its solution relevant error measures (e.g., minimum mean-square and generalization errors). All these works consider the traditional regime of

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However, there exist other interesting regimes for which many of the above mentioned problems also display fundamental intrinsic limits akin to phase transitions. Consider for example the problem of compressive sensing. An interesting regime is one where both the number of nonzero components and of samples scale in a sublinear manner as the system size tends to infinity. In this case we would like to identify the phase transition, if there is any, and its nature. This question has first been addressed recently in the framework of compressed sensing for binary Bernoulli signals by [31, 32, 33]. An all-or-nothing phenomenon is identified, that is, in an appropriate sparse regime, the minimum mean-square error (MMSE) sharply drops from its maximum possible value (no reconstruction) for “too small” sampling rates to zero (perfect reconstruction) for “large enough” sampling rates. The interest of such regime is not limited to estimation problems. It is also relevant from a learning point of view, e.g., it corresponds to learning scenarios where we have access to a high number of features but only a sublinear number of them – unknown to us – are relevant for the learning task at hand.

Examples abound where the “bet on sparsity principle” [34, 35] is of utmost importance for the interpretability of a high-dimensional model. Let us mention the MNIST handwritten digit database, where each digit can be presented as a 784-dimensional binary vector representing the pixels whereas, in effect, the digits live in a space of the order of 10’s of dimensions [36, 37]. Another example of effective sparsity comes from natural images which are often sparse in a wavelet basis [38].

The model can be interpreted as either an estimation problem or a learning problem: when is it possible to achieve a low estimation or generalization error of the model? Then, a fundamental question is “when is it possible to achieve a low estimation or generalization error with a sublinear amount of samples (sublinear with respect to the total number of features)?”.

In this contribution we address this question for a mathematically simple, but precise and tractable, setting. We consider generalized linear models in the regime of vanishing sparsity and sample rate, or equivalently, of sublinear number of data samples and nonzero signal components. As explained below these models can be used for estimation as well as learning, and we uncover in the sublinear regime intrinsic statistical barriers to these tasks in the form of sharp phase transitions. These statistical barriers are computed exactly and thus provide precise benchmarks to which algorithmic performance can be compared.

Let us outline the mathematical setting (see Section 2 for more precisions). In a probabilistic setting the unknown signal vector $X^* \in \mathbb{R}^n$ has entries drawn independently at random from a distribution $P_{0,n} := \rho_n P_0 \| \mathbf{0} \|$ with $P_0$ a fixed distribution. The parameter $\rho_n$ controls the sparsity of the signal so that $X^*$ has $k_n := n \rho_n$ nonzero components on average. We observe the data $Y = \varphi(\Phi X^* / \sqrt{\rho_n}) \in \mathbb{R}^m$ obtained by first multiplying the signal with a known $m_n \times n$ random matrix $\Phi$ whose entries are independent standard Gaussian random variables, and then applying $\varphi$ component-wise. The number of data points is controlled by the sampling rate $\alpha_n$, i.e., $m_n := \alpha_n n$. We consider the regime $(\rho_n, \alpha_n) \to (0, 0)$ and, especially, the case $\alpha_n = \gamma \rho_n \ln \rho_n$ for which the sharp all-or-nothing phase transition appears as $n$ goes to infinity. Note that both $m_n$ and $k_n$ scale sublinearly as $n \to +\infty$.

The model can be interpreted as either an estimation problem or a learning problem:

- **In the estimation interpretation**, we assume a purely Bayesian (or optimal) setting. We know the model, the activation function $\varphi$, the prior $P_{0,n}$ as well as the measurement matrix $\Phi$. Our goal is then to determine what is the lowest reconstruction error that we can achieve, i.e., what is the average minimum mean-square error $k_n^{-1} \mathbb{E} \| X^* - \mathbb{E} [X^* | Y, \Phi] \|^2$ when $n$ gets large.

- **In the learning interpretation**, we consider a teacher-student scenario in which a teacher hands out training samples $\{ (Y_{\mu}, (\Phi_{\mu i})_{i=1}^{m_n}) \}_{\mu=1}^{n_{new}}$ to a student. The teacher produces the output label $Y_{\mu}$ by feeding the input $(\Phi_{\mu i})_{i=1}^{m_n}$ to its own one-layer neural network with activation function $\varphi$ and weights $X^* = (X^* i)_{i=1}^{m_n}$. The student – who is given the model and the prior – has to learn the weights $X^*$ of the teacher one-layer neural network by minimizing the empirical training error of the $m_n$ training samples. For example, the binary perceptron corresponds to $\varphi = \text{sign}$ and $Y_{\mu} \in \{ \pm 1 \}$. Of particular interest is the generalization error. Given a new – previously unseen – random pattern $\Phi_{\text{new}} = (\Phi_{\text{new},i})_{i=1}^{m_n}$ whose true label is $Y_{\text{new}}$ (generated by the teacher’s neural network), the optimal generalization error is the error made when estimating $Y_{\text{new}}$ in a purely Bayesian way: $\mathbb{E} [ (Y_{\text{new}} - \mathbb{E} [\varphi(k_n^{-1/2} \Phi_{\text{new}} \cdot X^*) | Y, \Phi, \Phi_{\text{new}} ] ]^2$. 

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Let us summarize informally our results in the case of Bernoulli signals which form our main interest here, i.e., \( P_{0,n} := (1 - \rho_n)\delta_0 + \rho_n\delta_1 \). We set \( \alpha_n = \gamma \rho_n |\ln \rho_n| \) where \( \gamma \) stays fixed as \( n \to +\infty \). We first rigorously determine the mutual information \( m_n^{-1}I(\mathbf{X}^*; \mathbf{Y}|\Phi) \) in terms of a low-dimensional variational problem, see Theorem 1 which also provides a precise control of the finite size fluctuations. This theorem has interesting consequences which are not made fully rigorous here. Using I-MMSE type formulas\(^\text{[39]}\), we deduce the MMSE and generalization errors from the solution to this variational problem. Our analysis then shows that both errors display an all-or-nothing behaviour as \( n \to +\infty \), with a sharp transition at a threshold \( \gamma = \gamma_c \) explicitly computed. These findings are illustrated and their significance discussed in Section 3.

In our work the generalized linear model is treated by entirely different methods than the linear model in\(^\text{[31][32]}\). Importantly, the sparsity regime treated by our method requires the sparsity \( \rho_n \) to go to zero slower than \( n^{-1/2} \), while it has to go to zero faster than \( n^{-1/2} \) in the results of\(^\text{[32]}\) for the linear case. From this angle, both results complement each other. Our proof technique for Theorem 1 exploits the adaptive interpolation method (see\(^\text{[40][41]}\)) that is a powerful improvement over the Guerra-Toninelli interpolation and allows to prove replica symmetric formulas for Bayesian inference problems. We adapt the analysis of\(^\text{[30]}\) in a non-trivial way in order to consider the new scaling regime of our problem where \( \alpha_n = \gamma \rho_n |\ln \rho_n| \) and \( \rho_n \to 0 \) as \( n \) gets large, instead of fixed. We show that the adaptive interpolation can still be carried through, which requires a more refined control of the error terms compared to\(^\text{[30]}\). It is interesting, and not a priori obvious, that this can be done since this is not the usual statistical mechanics extensive regime. For example, the mutual information has to be normalized by the subextensive quantity \( m_n = \mathcal{O}(n) \). Quite remarkably, with this suitable normalization, the asymptotic mutual information, MMSE and generalization error have a similar form to those famously found in ordinary thermodynamic regimes in physics\(^\text{[42][43][44][45]}\).

In Section 2 we present the setting and state our main rigorous result (Theorem 1) on the mutual information in the sublinear regime. In Section 3 we use the latter result to uncover the all-or-nothing phase transitions for general activation functions. In Section 4 we give an overview of the adaptive interpolation method used to prove Theorem 1. The full proof is given in the supplementary material.

## 2 Problem setting and results

### 2.1 Generalized linear estimation of low sparsity signals at low sampling rates

Let \( n \in \mathbb{N}^* \) and \( m_n := \alpha_n n \) with \( \alpha_n \in \mathbb{N}^* \), a decreasing sequence of positive sampling rates. Let \( P_0 \) be a probability distribution with finite second moment \( \mathbb{E}_{X \sim P_0} [X^2] \). Let \( (X_i^*)_i \overset{iid}{\sim} P_{0,n} \) be the components of a signal vector \( \mathbf{X}^* \) (this is also denoted \( \mathbf{X}^* \overset{iid}{\sim} P_{0,n} \)), where

\[
P_{0,n} := \rho_n P_0 + (1 - \rho_n)\delta_0.
\]

The parameter \( \rho_n \in (0, 1) \) controls the sparsity of the signal; the latter being made of \( k_n := \rho_n n \) nonzero components in expectation. We will be interested in low sparsity regimes where \( k_n = \mathcal{O}(n) \).

Let \( k_A \in \mathbb{N} \). We consider a measurable function \( \varphi : \mathbb{R} \times \mathbb{R}^{k_A} \to \mathbb{R} \) and a probability distribution \( P_A \) over \( \mathbb{R}^{k_A} \). The \( m_n \) data points \( \mathbf{Y} := (Y_{\mu})_{\mu=1}^{m_n} \) are generated as

\[
Y_{\mu} := \varphi\left( \frac{1}{\sqrt{k_n}} (\Phi \mathbf{X}^*)_\mu, A_\mu \right) + \sqrt{\Delta} Z_\mu, \quad 1 \leq \mu \leq m_n,
\]

where \( (\Phi \mathbf{X}^*)_\mu \overset{iid}{\sim} P_A, (Z_\mu)_{\mu=1}^{m_n} \overset{iid}{\sim} \mathcal{N}(0, 1) \) is an additive white Gaussian noise (AWGN), \( \Delta > 0 \) is the noise variance, and \( \Phi \) is a \( m_n \times n \) measurement (or data) matrix with independent entries having zero mean and unit variance. Note that the noise \( (Z_\mu)_{\mu=1}^{m_n} \) can be considered as part of the model, or as a “regularising noise” needed for the analysis but that can be set arbitrarily small. Typically, and as \( n \) gets large, \( (\Phi \mathbf{X}^*)_\mu/\sqrt{k_n} = \Theta(1) \).

The estimation problem is to recover \( \mathbf{X}^* \) from the knowledge of \( \mathbf{Y} \), \( \Phi \), \( Z \), \( \varphi \), \( P_{0,n} \) and \( P_A \) (the realization of the random stream \( A \) itself, if present in the model, is unknown). It will be helpful to think of the measurements as the outputs of a channel:

\[
Y_{\mu} \sim P_{\text{out}} \left( \cdot \bigg| \frac{1}{\sqrt{k_n}} (\Phi \mathbf{X}^*)_\mu \right), \quad 1 \leq \mu \leq m_n.
\]

The transition kernel \( P_{\text{out}} \) admits a transition density with respect to Lebesgue’s measure given by:

\[
P_{\text{out}}(y|x) = \frac{1}{2\pi \Delta} \int dP_A(a) e^{-\frac{1}{2\Delta}(y - \varphi(x,a))^2}.
\]
If $A$ is absent of (2) ($k_A = 0$) then the associated integral $\int dP_A(a)$ in (1) disappears. The random variable $A$ represents any source of randomness in the model. For example, the logistic regression $P(Y_\mu = 1) = f(\Phi^* X_\mu / \sqrt{\pi n})$, with $f(x) = (1 + e^{-\lambda x})^{-1}$, is modeled by considering a teacher that generates i.i.d. uniform numbers $A_\mu \sim \mathcal{U}[0, 1]$ and then obtains the labels through $Y_\mu = 1(A_\mu \leq f(\Phi^* X_\mu / \sqrt{\pi n})) - 1(A_\mu > f(\Phi^* X_\mu / \sqrt{\pi n}))$ with $1(\cdot)$ the indicator function. Our numerical experiments in Section 3 are for deterministic activations but our main theoretical result Theorem 1 includes this more generic setting.

We have presented the problem from an estimation point of view. In this case, the important quantity whose weights are important for the proof of Theorem 1 (stated below). Both $\rho > 0$ and $q \in [0, \rho]$.

### 2.2 Main result

The mutual information $I(X^*; Y|\Phi)$ between the signal $X^*$ and the data $Y$ given the matrix $\Phi$ is the main quantity of interest in our work. Before stating Theorem 1 on the value of this mutual information, we first introduce two scalar denoising models that play a key role.

The first model is an additive Gaussian channel. Let $X^* \sim P_{0,n}$ be a scalar random variable. We observe $Y^{(r)} := \sqrt{r} X^* + Z$ where $r \geq 0$ plays the role of a signal-to-noise ratio (SNR) and the noise $Z \sim \mathcal{N}(0, 1)$ is independent of $X^*$. The mutual information $I_{P_{0,n}}(r) := I(X^*; Y^{(r)})$ between the signal of interest $X^*$ and $Y^{(r)}$ depends on $\rho_n$ through the prior $P_{0,n}$, and it reads:

$$I_{P_{0,n}}(r) = \frac{r \rho_n E_{X^*} X^2}{2} - \mathbb{E} \ln \int dP_{0,n}(x) e^{rX^* + \sqrt{r} x}/2 .$$

The second scalar channel is linked to the transition kernel $P_{\text{out}}$ defined by (4). Let $V, W^*$ be two independent standard Gaussian random variables. In this scalar estimation problem we want to infer $W^*$ from the knowledge of $V$ and the observation $Y^{(q,\rho)} \sim P_{\text{out}}(\cdot | \sqrt{q} V + \sqrt{\rho - q} W^*)$ where $\rho > 0$ and $q \in [0, \rho]$. The conditional mutual information $I_{P_{\text{out}}}(q, \rho) := I(W^*; Y^{(q,\rho)} | V)$ is:

$$I_{P_{\text{out}}}(q, \rho) = \mathbb{E} \ln P_{\text{out}}(Y^{(q,\rho)} | \sqrt{\rho} V) - \mathbb{E} \ln \int dw e^{-w^2 / 2\pi} P_{\text{out}}(Y^{(q,\rho)} | \sqrt{q} V + \sqrt{\rho - q} w) .$$

Both $I_{P_{0,n}}$ and $I_{P_{\text{out}}}$ have nice monotonicity, Lipschitzianity and concavity properties that are important for the proof of Theorem 1 (stated below).

We use the mutual informations (5) and (6) to define the (replica-symmetric) potential:

$$i_{\text{RS}}(q, r; \alpha_n, \rho_n) := \frac{1}{\alpha_n} I_{P_{0,n}}(\alpha_n r) + I_{P_{\text{out}}}(q, E_{P_0}[X^2]) - \frac{r [E_{P_0}[X^2] - q]}{2} .$$

Our main result links the extrema of this potential to the mutual information of our original problem.

**Theorem 1** (Mutual information of the GLM at sublinear sparsity and sampling rate). Suppose that $\Delta > 0$ and that the following hypotheses hold:

(H1) There exists $S > 0$ such that the support of $P_{0}$ is included in $[-S, S]$.

(H2) $\varphi$ is bounded, and its first and second partial derivatives with respect to its first argument, are bounded and continuous. They are denoted $\partial_x \varphi, \partial_{xx} \varphi$.

(H3) $(\Phi_\mu) \overset{\text{d}}{\sim} \mathcal{N}(0, 1)$.

Let $\rho_n = \Theta(n^{-\lambda})$ with $\lambda \in [0, 1/9)$ and $\alpha_n = \gamma \rho_n |\ln \rho_n|$ with $\gamma > 0$. Then for all $n \in \mathbb{N}^*$:

$$\left| \frac{I(X^*; Y|\Phi)}{m_n} - \inf_{q \in [0, E_{P_0}[X^2]]} \sup_{r \geq 0} i_{\text{RS}}(q, r; \alpha_n, \rho_n) \right| \leq \frac{\sqrt{C} |\ln n|^{1/6}}{n^{1/2} - \frac{\Delta}{4}} ,$$

where $C$ is a polynomial in $(S; \| \varphi \|_{\infty}, \| \partial_x \varphi \|_{\infty}, \| \partial_{xx} \varphi \|_{\infty}, \lambda, \gamma)$ with positive coefficients.
We place ourselves in a regime where signals whose entries are either Bernoulli random variables, i.e., we extend their results in two ways: i) for the estimation error of a generalized linear model, and ii) for the generalization error of a perceptron neural network with general activation function \( \varphi \). Our analysis is based on non-rigorous but analytical arguments which follow from Theorem 1.

| Activation \( \varphi(x) \) | \( \Delta \) | \( \gamma_c \) |
|--------------------------|--|---|
| \( x \) | \( > 0 \) | \( 2/\ln(1 + \Delta^{-1}) \) |
| \( = 0 \) | \( 0 \) |
| \( \text{sign}(x) \) | \( > 0 \) | \( \left( \ln 2 - \mathbb{E} \ln(1 + e^{-2/\Delta + 2/\sqrt{\Delta}}) \right)^{-1} \) with \( Z \sim N(0, 1) \) |
| \( = 0 \) | \( \frac{1}{\ln 2} \) |
| \( \max(0, x) \) | \( > 0 \) | \( \left( \frac{1}{4\Delta} - \mathbb{E}[h_{\Delta}(Z) \ln h_{\Delta}(Z)] \right)^{-1} \) with \( Z \sim N(0, 1) \), \( h_{\Delta}(Z) = \frac{1}{2} + \sqrt{\frac{\Delta}{1+\Delta}} e^{\frac{\Delta}{2(1+\Delta)}} \int_{-\infty}^{\frac{\Delta}{2}} dt e^{-t^2/2} \) |
| \( = 0 \) | \( 0 \) |

Table 1: Closed-formed formulas of \( \gamma_c \) for different activation functions.

Hence the asymptotic mutual information is given to leading order by a variational problem. We remark that, because \( C \) depends only polynomially on its arguments, it is possible to weaken the assumptions of the theorem with more technical work. For example, one could extend it to distributions \( P_0 \) having infinite support but with finite (first few) moments, to unbounded activation functions that do not grow too fast or to some nondifferentiable activation functions. This includes the linear activation \( \varphi(x) = x \), the usual perceptron \( \varphi(x) = \text{sign}(x) \) and the ReLU \( \varphi(x) = \max(0, x) \).

### 3 The all-or-nothing phenomenon

We now highlight interesting consequences of Theorem 1 for the MMSE of the estimation problem, as well as the generalization error of the learning problem in the teacher-student scenario. Reeves et al. [32] have proved the existence of an all-or-nothing phenomenon for the linear model and here we extend their results in two ways: i) for the estimation error of a generalized linear model, and ii) for the generalization error of a perceptron neural network with general activation function \( \varphi \). Our analysis is based on non-rigorous but analytical arguments which follow from Theorem 1.

We consider signals whose entries are either Bernoulli random variables, i.e., \( P_{0,n} := (1 - \rho_n)\delta_0 + \rho_n P_0 \) with \( P_0 = \delta_1 \), or Bernoulli-Rademacher random variables, i.e., \( P_{0,n} := (1 - \rho_n)\delta_0 + \rho_n P_0 \) with \( P_0 = (\delta_1 + \delta_{-1})/2 \). In both cases \( \mathbb{E}_{P_0}[X^2] = 1 \) (we can always assume the latter by rescaling the noise).

We place ourselves in a regime where \( \alpha_n = \gamma \rho_n \ln \rho_n \) for some fixed \( \gamma > 0 \) and \( \rho_n \to 0 \) in the high-dimensional limit \( n \to +\infty \). Finally, we denote \( q_n^*(\gamma) := \arg \min_{q \in [0,1]} \mathbb{E}_{P_0}[h_{\Delta}(Z) \ln h_{\Delta}(Z)] \) the minimizer of the potential of Theorem 1. In this regime and for such signals we will observe an all-or-nothing phenomenon: there exists \( \gamma_c > 0 \) such that, in the high-dimensional limit, perfect reconstruction is possible \( (q_n^*(\gamma) \to 0) \) for \( \gamma > \gamma_c \), while it is impossible to do better than a random guess \( (q_n^*(\gamma) \to 0) \) for \( \gamma < \gamma_c \). To locate this all-or-nothing phase transition we use the analytical method described in the supplementary material which predicts (\( H_b : x \mapsto x \ln \left|x + (1-x)\right| \ln(1-x) \)) is the binary entropy, \( H(P_0) \) is the entropy of \( P_0 \):

\[
\gamma_c = \frac{1}{I_{P_0}(0, 1)}, \quad \alpha_c(\rho_n) = \gamma_c H_b(\rho_n) + \rho_n H(P_0) \sim \gamma_c \rho_n \ln |\rho_n|.
\]

(9)

Remember that \( I_{P_0}(0, 1) := I(W^*; \varphi(W^\ast, A) + \sqrt{\Delta} Z) \) where \( W^*, Z \sim N(0, 1) \perp A \sim P_A \) is simply computed. The factor \( \gamma_c \) is thus fully determined by the activation function and the amount of noise. In Figure 1 we draw \( \gamma_c \) for \( \varphi(x) = x \), \( \varphi(x) = \text{sign}(x) \), \( \varphi(x) = \max(0, x) \) and noise variance \( \Delta \in [0, 0.5] \). The corresponding formulas for \( \gamma_c \) are given in Table 1. We see that for \( \Delta \) small enough the ReLU activation requires less training samples to learn the sparse rule than the linear one; it is the opposite once \( \Delta \) becomes large enough. When \( \Delta \) diverges both the linear and sign activations have the asymptote \( \gamma_c \sim 2\Delta \) while the ReLU activation has another steeper asymptote \( \gamma_c \sim \alpha \Delta, \alpha \approx 5.87 \). Figure 1 also gives the potential (once it has been maximized over \( r \)) as a function of \( q \) for two different configurations and several sample rates \( \alpha_n = (\gamma/\gamma_c)\alpha_c(\rho_n) \). We see that the potential is minimized for \( q \) close to 0 when \( \gamma \) is less than \( \gamma_c \) and close to 1 when \( \gamma \) is greater than \( \gamma_c \). Exactly at \( \gamma = \gamma_c \), the potential reaches similar level at \( q = 0 \) and \( q = 1 \). This numerically confirms the heuristic condition used in the supplementary material in order to locate the phase transition.
We thus find that the generalization error also displays an all-or-nothing phenomenon. More precisely, as a function of \( \alpha \)

\[
\text{MMSE}(X^*, Y, \Phi) := \frac{1}{\sigma_n} \mathbb{E} \|X^* - \mathbb{E}[X^*|Y, \Phi]\|^2 = 1 - q_n^*(\gamma) + o_n(1). \tag{10}
\]

We note that the link \( \text{(10)} \) between the MMSE and the extremizer of the variational formula for the mutual information in the high-dimensional limit has been proved in a number of cases, e.g., [46, 20].

**Generalization error** When learning in a (matched) teacher-student scenario, the components of \( X^* \) correspond to the unknown weights of the teacher’s one-layer neural network. The student is given the model and training samples \( \{Y_{\text{tr}} \sim \mathcal{P}_{\Phi_{\mu,i}}\}_{i=1}^{m_{\text{tr}}} \). Then, the optimal generalization error is the MMSE for predicting the output \( Y_{\text{tr}} \sim \mathcal{P}_{\Phi_{\mu,i}} \). We have (in what follows \( V, W \sim \mathcal{N}(0, 1) \)

\[
\text{MMSE}(Y_{\text{tr}}, Y, \Phi_{\text{new}, i}) := \mathbb{E}[(Y_{\text{tr}} - \mathbb{E}[Y_{\text{tr}}|Y, \Phi_{\text{new}, i}])^2] = \Delta + \mathbb{E}[(\varphi(V, A) - \mathbb{E}[\varphi(\sqrt{q_n^*(\gamma)} W, A)|Y])^2] + o_n(1). \tag{11}
\]

We thus find that the generalization error also displays an all-or-nothing phenomenon. More precisely, as a function of \( \alpha \), the generalization error equals \( \Delta + \mathbb{E}[(\varphi(V, A) - \mathbb{E}[\varphi(\sqrt{q_n^*(\gamma)} W, A)|Y])^2] \).

**Illustration of the all-or-nothing phenomenon** In Figure 2, we draw the MMSE according to \( \text{(10)} \) for both priors Bernoulli and Bernoulli-Rademacher and the activation functions \( \varphi(x) = x, \varphi(x) = \text{sign}(x) \). The curves are obtained by optimizing the potential \( \text{I}_{\text{R}}(q, r; \alpha_n, \rho_n) \) for different values of \( \alpha_n = (\gamma/\gamma_n) \alpha_n(\rho_n) \) with \( \gamma \) lying in a region surrounding \( \gamma_c \).

![Figure 1: Top: factor \( \gamma_c \) of the all-or-nothing phase transition for different activation functions. Bottom: potential \( \sup_{\rho_n \geq 0} \text{I}_{\text{R}}(q, r; \alpha_n, \rho_n) \) as a function of \( q \) for \( \gamma \) below/above the transition and \( \rho_n := \frac{1 - \rho_n \delta_0 + \rho_n \delta_1}{} \). Bottom left: \( \varphi(x) = x, \rho_n = 10^{-6}, \Delta = 0.1 \). Bottom right: \( \varphi(x) = \text{sign}(x), \rho_n = 10^{-8}, \Delta = 0 \).](image)
which such techniques have been developed. They have been used in a regime where the number of
measurements and sparsity are linear in \( n \).

An important aspect of our results is to provide a definitive statistical benchmark allowing to measure
the quality of algorithms with respect to the minimal amount of sparse data needed to estimate or
learn. This benchmark is provided by non-trivial formulas \( \gamma_c \) for the threshold \( \gamma_c \) given for several
examples in Table 1. We note that such precise benchmarks are quite rarely obtained in traditional
machine learning approaches.

**Further remarks**  Algorithmic aspects are beyond the scope of this paper. However we make a few
remarks about generalized approximate message passing algorithms (GAMP) whose state evolution
equations, that precisely track their asymptotic performance, are linked to the fixed point equation
\( \partial I_{\alpha_n} / \partial q \approx 0 \) – to obtain both the mean-square and generalization errors of
GAMP algorithms. These errors are represented with dotted lines in Figures 2 and 3. We observe an
algorithmic-to-statistical gap, that is, the dotted lines corresponding to the algorithmic performance
do not drop to zero at \( \gamma_c \) but at a higher *algorithmic threshold*. We conjecture that the all-or-nothing
behaviour is also generally true at an algorithmic level for GAMP algorithms. This has been shown
rigorously for a linear activation \( \phi \) but it would be highly desirable to obtain a proof for other
activations and derive the corresponding thresholds.

4  Overview of the proof of Theorem 1

The interested reader will find the proof of Theorem 1 in the supplementary material. In this section
we give an outline of the proof and its main ideas. The proof is based on the adaptive interpolation
method \( \theta \) whose main difference with the canonical interpolation method \( \theta \) is the increased flexibility
given to the path followed by the interpolation between its two extremes. The method has been developed separately for symmetric rank-one tensor problems where the spike has
i.i.d. components \( \xi \), and for one-layer GLMs whose input signal has again i.i.d. components
\( \eta \). The sparse regime of the problem studied in this contribution differs of the usual scaling for
which such techniques have been developed. They have been used in a regime where the number of
measurements and sparsity are linear in \( n \) as in \( \theta \). Working in the sparse regime requires writing
more refined concentration bounds and proving that the key steps of the adaptive interpolation can
still be carried through.

1. Interpolating estimation problem  To simplify the presentation we assume that \( \Delta = 1 \) and
\( \mathbb{E}_X \text{Var}(X) = 1 \). The proof starts by introducing an interpolating inference problem that depends
on a parameter \( t \in [0,1] \) and two continuous interpolation functions \( R_1, R_2 : [0,1] \to \mathbb{R} \), with

\[
\frac{\partial I_{\alpha_n} / \partial q}{\alpha_n} \approx 0, \quad q = -\frac{2}{\rho_n} I_{\alpha_n} \left( \frac{\alpha_n r}{\rho_n} \right),
\]

Finally, the fixed point \( q^* \) yielding the lowest potential \( I_{\alpha_n} \) is used to determine
the MMSE thanks to \( \theta \). In all configurations the MMSE indeed jumps from a value close to 1
to \( \approx 0 \) as \( \gamma \) increases past \( \gamma_c \). As \( \rho_n \) gets closer to 0, this jump becomes sharper with the MMSE
approaching 0 or 1 depending on which side of \( \gamma_c \) we are. Note that for the random linear model
\( \varphi(x) = x \), the threshold \( \alpha_c(\rho_n) = 2\rho_n / \ln(1 + 1 / \gamma) \) is in agreement with the sample rate \( n^* \) for
which \( \gamma_c \) prove that weak recovery is impossible below it while strong recovery is possible above.

In Figure 3 we plot the optimal generalization error for the Bernoulli prior and the same activation
functions. The curves are obtained similarly to the ones in Figure 2 but using \( \theta \) instead of \( \theta \). In
all configurations the generalization error jumps from a value close to \( \Delta + \text{Var}(\varphi(V)) \) to \( \approx \Delta \) as
\( \gamma \) increases past \( \gamma_c \) (note that the activations are deterministic so there is no \( A \) contribution in the
error). The value \( \Delta \) is as good as the optimal generalization error can get, i.e., it is equal to the noise
variance which is the squared error we would get if we were given the true weights \( X^* \). Again, the
all-or-nothing phase transition gets sharper as \( \rho_n \) approaches 0.

The all-or-nothing behaviour of the MMSE and generalization error is quite striking. Indeed, in the
limit of vanishing sparsity and sampling rate either estimation or learning is as good as it can get or
as bad as a random guess. This purely dichotomic behaviour only occurs in the truly sparse limit, and
is shown here to be pretty general in the sense that it occurs for a wide variety of activation functions.
An important aspect of our results is to provide a definitive statistical benchmark allowing to measure
the quality of algorithms with respect to the minimal amount of sparse data needed to estimate or
learn. This benchmark is provided by non-trivial formulas \( \gamma_c \) for the threshold \( \gamma_c \) given for several
examples in Table 1. We note that such precise benchmarks are quite rarely obtained in traditional
machine learning approaches.

\[
\frac{\partial I_{\alpha_n} / \partial q}{\alpha_n} \approx 0, \quad q = -\frac{2}{\rho_n} I_{\alpha_n} \left( \frac{\alpha_n r}{\rho_n} \right),
\]

Finally, the fixed point \( q^* \) yielding the lowest potential \( I_{\alpha_n} \) is used to determine
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learn. This benchmark is provided by non-trivial formulas \( \gamma_c \) for the threshold \( \gamma_c \) given for several
examples in Table 1. We note that such precise benchmarks are quite rarely obtained in traditional
machine learning approaches.
Figure 2: MMSE for $\alpha_n/\alpha_c(\rho_n)$ around the all-or-nothing phase transition. Dotted lines are for algorithmic performance (iterating (12) from $q = 10^{-10}$). Left panels: Bernoulli prior. Right panels: Bernoulli-Rademacher prior. From top to bottom: $\varphi(x) = x$, $\Delta = 0$ $\varphi(x) = \text{sign}(x)$, $\Delta = 0$ $\varphi(x) = \max(0,x)$, $\Delta = 0.5$.

Figure 3: Generalization error for $\alpha_n/\alpha_c(\rho_n)$ around the all-or-nothing phase transition. Dotted lines are for algorithmic performance (iterating (12) from $q = 10^{-10}$). Top left: random linear model $\varphi(x) = x$, $\Delta = 0$. Top right: perceptron $\varphi(x) = \text{sign}(x)$, $\Delta = 0$. Bottom: ReLU $\varphi(x) = \max(0,x)$, $\Delta = 0.5$.

$R_1(0) = R_2(0) = 0$. Let $X^* \overset{iid}{\sim} P_{0,n}$, $\Phi := (\Phi_{\mu}) \overset{iid}{\sim} \mathcal{N}(0,1)$, $V := (V_{\mu})_{\mu=1}^{m} \overset{iid}{\sim} \mathcal{N}(0,1)$ and $W^* := (W^*_{\mu})_{\mu=1}^{m} \overset{iid}{\sim} \mathcal{N}(0,1)$. We define for all $t \in [0,1]$ an “interpolating pre-activation”:

$$S_{\mu}^{(t)} := \sqrt{(1-t)/k_n} (\Phi X^*)_\mu + \sqrt{R_2(t)} V_{\mu} + \sqrt{t - R_2(t)} W^*_\mu.$$  

The inference problem at a fixed $t$ is to recover both unknowns $X^*, W^*$ from the knowledge of $V$, $\Phi$ and the data

$$\begin{aligned}
Y^{(t)}_{\mu} &\sim P_{\text{out}}(\cdot | S^{(t)}_{\mu}) , \quad 1 \leq \mu \leq m_n; \\
\tilde{Y}^{(t)}_i &\sim \sqrt{R_1(t)} X^*_i + \tilde{Z}_i , \quad 1 \leq i \leq n;
\end{aligned}$$

where $Z_\mu, \tilde{Z}_i \overset{iid}{\sim} \mathcal{N}(0,1)$. The corresponding interpolating mutual information is:

$$i_n(t) := m_n^{-1} I((X^*, W^*); (Y^{(0)}, \tilde{Y}^{(t)})(\Phi, V)) .$$

2. Fundamental sum-rule Note that at $t = 0$ we recover the original problem of interest and $i_n(0) = I(X^*; Y|\Phi)_{m_n}$. At the other extreme $t = 1$, the mutual information can be written in terms of the simple mutual informations $I_{P_0,n}$ and $I_{P_{\text{out}}}$ that is, $i_n(1) = I_{P_0,n}(R_1(1))/\alpha_n + I_{P_{\text{out}}}(R_2(1), 1)$. We link the mutual information at both extremes by computing the derivative $i'_n(\cdot)$ of $i_n(\cdot)$ and then
using the fundamental identity \(i_n(0) = i_n(1) - \int_0^1 i'_n(t) dt\). It yields the sum-rule:

\[
\frac{I(X^*; Y|\Phi)}{m_n} = \frac{1}{\alpha_n} I_{P_{in}}(R_1(1)) + I_{P_{out}}(R_2(1), 1) - \frac{\rho_n}{2\alpha_n} \int_0^1 R'_1(t)(1 - R'_2(t)) dt + R_n.
\]

The last term \(R_n\) is a remainder whose absolute value we want to control in order to get Theorem 1.

3. Controlling the remainder This is done by plugging two different choices of interpolation functions \((R_1, R_2)\) in the sum-rule. One choice yields an upper bound on the difference in the left-hand side of (8), while another yields a lower bound. Each choice of interpolation functions \((R_1, R_2)\) is defined implicitly as the solution to a second order ordinary differential equation. Remarkably, under these two choices, the remainder \(R_n\) can be controlled using precise concentration results.

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A Proof of Theorem 1 with the adaptive interpolation method

Note that it is the same to observe (3) or their rescaled versions \( \frac{1}{\sqrt{\Delta}} q \) or \( \frac{1}{\sqrt{\Delta}} \) positive real number. For all \( \epsilon = (\epsilon_1, \epsilon_2) \in \mathcal{B}_s \), we define the interpolation functions

\[
R_1(\cdot, \epsilon) : t \in [0, 1] \mapsto \epsilon_1 + \int_0^t r_1(x)dv \quad \text{and} \quad R_2(\cdot, \epsilon) : t \in [0, 1] \mapsto \epsilon_2 + \int_0^t q_2(x)dv ,
\]

where \( q_1 : [0, 1] \to [0, 1] \text{ and } r_1 : [0, 1] \to [0, \frac{\alpha}{\rho_0} \rho_{\max}] \) are two continuous functions. We say that the families of functions \( (q_i)_{i \in \mathcal{B}_n} \) and \( (r_i)_{i \in \mathcal{B}_n} \) are regular if \( \forall t \in [0, 1] : \epsilon \mapsto \left( R_1(t, \epsilon), R_2(t, \epsilon) \right) \) is a \( C^1 \) diffeomorphism from \( \mathcal{B}_s \) onto its image whose Jacobian determinant is greater than, or equal, to one. This property will reveal important later in our proof. Let \( X^* \overset{\text{iid}}{\sim} P_{0,n}, \Phi := (\Phi_{\mu}) \overset{\text{iid}}{\sim} \mathcal{N}(0, 1), V := (V_{\mu})_{\mu=1}^{m_n} \overset{\text{iid}}{\sim} \mathcal{N}(0, 1) \text{ and } W^* := (W_{\mu})_{\mu=1}^{m_n} \overset{\text{iid}}{\sim} \mathcal{N}(0, 1) \). We define:

\[
S^{(t, \epsilon)}_{\mu} = S^{(t, \epsilon)}_{\mu}(X^*, W^*) := \sqrt{1 - \frac{t}{k_n}} (\Phi X^*)_\mu + \sqrt{1 + 2s_n - R_2(t, \epsilon)} W^* .
\]

Consider the following observations coming from two types of channels:

\[
\begin{align*}
Y^{(t, \epsilon)}_{\mu} & \sim S^{(t, \epsilon)}_{\mu}, \quad 1 \leq \mu \leq m_n ; \\
\hat{Y}^{(t, \epsilon)}_i & = \sqrt{R_1(t, \epsilon)} X^*_i + \hat{Z}_i, \quad 1 \leq i \leq n ;
\end{align*}
\]

where \( (\hat{Z}_i)_{i=1}^{m_n} \overset{\text{iid}}{\sim} \mathcal{N}(0, 1) \). The inference problem (at time \( t \)) is to recover both unknowns \( X^*, W^* \) from the knowledge of \( V, \Phi \) and the observations \( Y^{(t, \epsilon)} := (Y^{(t, \epsilon)}_{\mu})_{\mu=1}^{m_n}, \hat{Y}^{(t, \epsilon)} := (\hat{Y}^{(t, \epsilon)}_i)_{i=1}^{n} \). The joint posterior
density of \((X^*, W^*)\) given \((Y^{(t, s)}, \tilde{Y}^{(t, s)}, \Phi, V)\) reads:

\[
\begin{align*}
    dP(x, w|Y^{(t, s)}, \tilde{Y}^{(t, s)}, \Phi, V) := & \frac{1}{2\pi} \prod_{i=1}^{n} dP_{0,n}(x_i) e^{-\frac{1}{4} \left( \sqrt{M_1(x_i)} {z}_i - \tilde{y}^{(t, s)}_i \right)^2} \prod_{\mu=1}^{m_0} \frac{dw_\mu}{\sqrt{2\pi}} e^{-\frac{w_\mu^2}{2} P_{\text{out}}(y_\mu^{(t, s)}) |s_\mu^{(t, s)}},
\end{align*}
\]

where \(z_i^{(t, s)} := S_0^{(t, s)}(x_i, w_\mu)\) and \(Z_{t,s} \equiv Z_{t,s}(Y^{(t, s)}, \tilde{Y}^{(t, s)}, \Phi, V)\) is the normalization. The interpolating mutual information is:

\[
i_{n,e}(t) := \frac{1}{m_n} I((X^*, W^*); (Y^{(t, s)}, \tilde{Y}^{(t, s)}), \Phi, V).
\]

The perturbation \(\epsilon\) only induces a small change in mutual information. In particular, at \(t = 0:\)

**Lemma 1.** Suppose that (H1)(H2)(H3) hold, that \(\Delta = \mathbb{E}_{Y} |X^2| = 1\) and that there exist real positive numbers \(M_n, M_{\rho/\alpha}\) such that \(\forall n \in \mathbb{N} :\) \(\alpha_n \leq M_n\) and \(\rho_n/\alpha_n \leq M_{\rho/\alpha}\). For all \(\epsilon \in \mathbb{B}_n:\)

\[
\left| i_{n,e}(0) - \frac{I((X^*, W^*); (Y^{(t, s)}, \tilde{Y}^{(t, s)})|\Phi, V)}{m_n} \right| \leq \sqrt{C} \frac{\delta_n}{\sqrt{\rho_n}},
\]

where \(C\) is a polynomial in \((S, \|\varphi\|_\infty, \|\partial_x \varphi\|_\infty, M_n, M_{\rho/\alpha})\) with positive coefficients.

We prove Lemma 1 in Appendix D.2. By the chain rule for mutual information and the Lipschitzianity of \(I_{P_{0,n}}, I_{P_{\text{out}}}\) (see Lemma 3 and 8 in Appendix C), at \(t = 1\) we have for all \(\epsilon \in \mathbb{B}_n:\)

\[
i_{n,e}(1) = \frac{I((X^*, \tilde{Y}^{(t,s)}; \Phi) + I(W^*; Y^{(t,s)}, \tilde{Y}^{(t,s)})|\Phi, V)}{m_n} = I_{P_{0,n}}(R_1(1, \epsilon)) + I_{P_{\text{out}}}(R_2(1, \epsilon), 1 + 2s_n)
\]

\[
= \frac{1}{\alpha_n} I_{P_{0,n}} \left( \int_0^1 r_x(t)dt \right) + I_{P_{\text{out}}} \left( \int_0^1 q_x(t)dt, 1 \right) + O(s_n),
\]

assuming there exists \(M_{\rho/\alpha} > 0\) such that \(\forall n \in \mathbb{N} :\) \(\rho_n/\alpha_n \leq M_{\rho/\alpha}\), \(O(s_n)\) is a quantity whose absolute value is bounded by \(CS_n\) where \(C\) is a polynomial in \((S, \|\varphi\|_\infty, \|\partial_x \varphi\|_\infty, \|\partial_x \varphi \|_\infty, M_n, M_{\rho/\alpha})\) with positive coefficients.

### A.2 Fundamental sum rule

We want to compare the original model of interest (model at \(t = 0\)) to the purely scalar one (\(t = 1\)). To do so, we use \(i_{n,e}(0) = i_{n,e}(1) - \int_0^1 i'_{n,e}(t)dt\) where \(i'_{n,e}(\cdot)\) is the derivative of \(i_{n,e}(\cdot)\). Once combined with Lemma 1 and 17, it yields (note that \(O(s_n) = O(s_n/\sqrt{\rho_n})\) since \(0 < \rho_n < 1\):

\[
\frac{I((X^*, W^*); Y^{(t,s)}|\Phi, V)}{m_n} = O \left( \frac{s_n}{\sqrt{\rho_n}} \right) + \frac{1}{\alpha_n} I_{P_{0,n}} \left( \int_0^1 r_x(t)dt \right) + \int_0^1 q_x(t)dt, 1 \right) - \int_0^1 i'_{n,e}(t)dt.
\]

From now on let \((x, w) \in \mathbb{R}^n \times \mathbb{R}^{m_n}\) be a pair of random vectors sampled from the joint posterior distribution \(\mathcal{P}_{X \times \mathcal{W}}\). The angular brackets \((\cdot)_{n,t,s}\) denote an expectations w.r.t. the distribution \(\mathcal{P}_{X \times \mathcal{W}}\), i.e., \((g(x, w))_{n,t,s} := \int g(x, w) dP(x, w|X^{(t,s)}, \tilde{Y}^{(t,s)}, \Phi, V)\) for every integrable function \(g\). We define the scalar overlap \(Q := \frac{1}{m_n} \sum_{n=1} X_n x_n\). The computation of \(i'_{n,e}(\cdot)\) is found in Appendix D.1.

**Proposition 1.** Suppose that (H1)(H2)(H3) hold and that \(\Delta = \mathbb{E}_{X \sim \mathbb{P}_1} |X^2| = 1\). Further assume that there exist real positive numbers \(M_n, M_{\rho/\alpha}\) such that \(\forall n \in \mathbb{N} :\) \(\alpha_n \leq M_n\) and \(\rho_n/\alpha_n \leq M_{\rho/\alpha}\). Define \(u_x(x) := \ln P_{\text{out}}(y|x)\) and \(u'_{x}(\cdot)\) its derivative w.r.t. \(x\). For all \((t, \epsilon) \in [0, 1] \times \mathbb{B}_n:\)

\[
i'_{n,e}(t) = O \left( \frac{1}{\rho_n \alpha_n} \right) + \frac{\rho_n}{2\alpha_n} r_x(t)(1 - q_x(t))
\]

\[
+ \frac{1}{2} \mathbb{E} \left( \left(Q - q_x(t) \right) \left( \sum_{\mu=1}^{m_0} u'_{\mu}(t) \left( S_\mu^{(t)}(x) \right) u'_{\mu}(t) \left( S_\mu^{(t)}(x) \right) - \frac{\rho_n}{\alpha_n} r_x(t) \right) \right)_{n,t,s},
\]

where \(\mathbb{O}(\frac{1}{\rho_n \alpha_n})\leq \sqrt{\frac{C}{\rho_n \alpha_n}}, \text{ with } C \ \text{a polynomial in } (S, \|\varphi\|_{\infty}, \|\partial_x \varphi\|_{\infty}, \|\partial_x \varphi \|_{\infty}, M_n, M_{\rho/\alpha})\) with positive coefficients, uniformly in \((t, \epsilon)\).

The next key result states that the overlap concentrates on its expectation. This behavior is called replica symmetric in statistical physics. Similar results have been obtained in the spin glass literature [80][51]. In this work we use a formulation tailored to Bayesian inference problems as developed in the context of LDPC codes, random linear estimation [85] and Nishimori symmetric spin glasses [11][14][16].
Proposition 2 (Overlap concentration). Suppose that $\{H1\}$, $\{H2\}$, $\{H3\}$ hold, that $\Delta = E_{P_{\mu}}[X^2] = 1$ and that the family of functions $(r_\mu)_\mu \in \mathbb{B}_n$ is regular. Further assume that there exist real positive numbers $M_\mu$, $M_{\mu/\alpha}$, $m_\mu$ such that for all $\mu \in \mathbb{N}^*$: $\alpha_\mu \leq M_\mu$ and $\frac{m_\mu}{\alpha_\mu} < \frac{\rho_\mu}{\alpha_\mu} \leq M_{\mu/\alpha}$. Let $M_\mu := \left(\frac{2_n \rho_\mu (\alpha_\mu m_\mu)^{-1/2}}{\alpha_\mu m_\mu} \right)^{1/4} - \left(\frac{2_n \rho_\mu}{\alpha_\mu} \right)^{-1} > 0$. We have for all $t \in [0, 1]$:
\[
\int_{\mathbb{B}_n} \frac{d\mathbb{E}}{d\mathbb{P}} \int_0^1 dt \mathbb{E} \left( (Q - E(Q)_{n,t})^2 \right)_{n,t,\epsilon} \leq CM_n \tag{20}
\]
where $C$ is a polynomial in $(S, ||\varphi||, ||\varphi||, ||\varphi||, ||\varphi||, ||\varphi||, M_\mu, M_{\mu/\alpha}, m_\mu)$ with positive coefficients.

We prove Proposition 2 in Appendix C. We can now prove the fundamental sum rule.

**Proposition 3 (Fundamental sum rule).** Suppose that $\forall (t, \epsilon) \in [0, 1] \times \mathbb{B}_n : q_\epsilon(t) = E(Q)_{n,t,\epsilon}$. Under the assumptions of Proposition 2 we have:
\[
I(X^*, Y(\Phi))_{m_\mu} = O(\sqrt{M_n}) + O\left(\frac{s_n}{\sqrt{n}}\right) + \int_{\mathbb{B}_n} \frac{d\mathbb{E}}{d\mathbb{P}} \int_0^1 dt \mathbb{E} \left( (Q - q_\epsilon(t)) \left( \frac{1}{m_n} \sum_{\mu=1}^{m_n} u_{\mu}^{(t,\epsilon)}(s_{\mu}^{(t,\epsilon)}(r_{\mu})(t)) - \frac{\rho_\mu}{\alpha_\mu} r_{\mu}(t) \right) \right)_{n,t,\epsilon} \tag{21}
\]
\[
\leq \int_{\mathbb{B}_n} \frac{d\mathbb{E}}{d\mathbb{P}} \int_0^1 dt \mathbb{E} \left( \left( \frac{1}{m_n} \sum_{\mu=1}^{m_n} u_{\mu}^{(t,\epsilon)}(s_{\mu}^{(t,\epsilon)}(r_{\mu})(t)) - \frac{\rho_\mu}{\alpha_\mu} r_{\mu}(t) \right)^2 \right)_{n,t,\epsilon} + \int_{\mathbb{B}_n} \frac{d\mathbb{E}}{d\mathbb{P}} \int_0^1 dt \mathbb{E} \left( (Q - q_\epsilon(t))^2 \right)_{n,t,\epsilon}.
\]

The first factor on the right-hand side of this inequality is bounded by a constant that depends polynomially on $||\varphi||$, $||\varphi||$. Since $\forall (t, \epsilon) \in [0, 1] \times \mathbb{B}_n : q_\epsilon(t) = E(Q)_{n,t,\epsilon}$, the second term is in $O(M_n)$ (see Proposition 2). Therefore, by Proposition 2
\[
E_n,t_i n_i(t) = O(\sqrt{M_n}) + O\left(\frac{1}{\rho_n \sqrt{n}}\right) + E_{n,n,t} \frac{\rho_n}{2\epsilon_n} r_n(t) (1 - q_\epsilon(t)).
\]

Note that $1/\rho_n \sqrt{n} = O(\sqrt{M_n})$. Integrating (13) over $\epsilon \in \mathbb{B}_n$ and making use of (21) give the result. □

### A.3 Matching bounds

To prove Theorem 1, we will lower and upper bound $I(X^*, Y(\Phi))_{m_n}$ by the same quantity, up to a small error. To do so we will plug two different choices of interpolation functions $R_1(\cdot, \epsilon), R_2(\cdot, \epsilon)$ in the sum-rule of Proposition 3. In both cases, the interpolation functions will be the solutions of a second-order ordinary differential equation (ODE). We now describe these ODEs.

Fix $t \in [0, 1]$ and $R = (R_1, R_2) \in [0, +\infty) \times [0, t + 2s_n]$. Consider the observations:
\[
\begin{align*}
Y_{\mu}^{(t,R_2)} & \sim P_{out}(\cdot | S_{\mu}^{(t,R_2)}) , \ 1 \leq \mu \leq m_n; \\
\tilde{Y}_{\mu}^{(t,R_1)} & \sim \sqrt{R_1} X_{\mu} + \tilde{Z}_{\mu} , \ 1 \leq \mu \leq n;
\end{align*}
\]
where $S_{\mu}^{(t,R_2)} = S_{\mu}^{(t,R_2)}(X^*, W_{\mu}^*) := \sqrt{(t - \epsilon)} (\Phi X^*)_\mu + \sqrt{R_2} V_{\mu} + \sqrt{t + 2s_n - R_2} W_{\mu}^*$. The joint posterior density of $(X^*, W^*)$ given $(Y_{\mu}^{(t,R_2)}, \tilde{Y}_{\mu}^{(t,R_1)}, \Phi, V)$ is:
\[
dP(x, w | Y_{\mu}^{(t,R_2)}, \tilde{Y}_{\mu}^{(t,R_1)}, \Phi, V)
\]
\[
\propto \prod_{i=1}^{n} dP_0,\nu(x_i) e^{-\frac{1}{2} \left( \sqrt{\epsilon_n z_i} - \tilde{Y}_{\mu}^{(t,R_1)}(x, w) \right)^2} \prod_{\mu=1}^{m_n} d\nu_{\mu} e^{-\frac{\nu_{\mu}^2}{2\pi} P_{out}(Y_{\mu}^{(t,R_2)} | S_{\mu}^{(t,R_2)}(x, w_{\mu})).
\]

2 Remember that $r_n$ takes its values in $[0, \frac{s_n}{\sqrt{n}} \rho_{\max}]$. Besides, under $\{H2\} u_{\mu}^{(t,\epsilon)}(s_{\mu}^{(t,\epsilon)}(r_{\mu})(t))$ is upper bounded by $(|Y_{\mu}^{(t,\epsilon)}| + ||\varphi||_\infty) \Delta^{-1} ||\varphi||_\infty = (\sqrt{\Delta} |Z_{\mu}| + 2 ||\varphi||_\infty) \Delta^{-1} ||\varphi||_\infty$ (see the inequality 13 in Appendix C). The noise $Z_{\mu}$ is averaged over thanks to the expectation.
The angular brackets \( \langle \cdot \rangle_{n,t,R} \) denote the expectation w.r.t. this posterior. Let \( r \in [0, r_{\text{max}}] \), \( F_2^{(n)}(t, R) := \frac{-2\alpha_n}{\rho_n} \frac{\partial h(q)}{\partial q} \big|_{q=E(Q)_{n,t,R}, \rho=1} \). We will consider the two following second-order ODEs with initial value \( \epsilon \in [s_n, 2s_n]^2 \):

\[
y'(t) = \left( \frac{\alpha_n}{\rho_n} r, F_2^{(n)}(t, y(t)) \right), \; y(0) = \epsilon;
\]

\[
y'(t) = \left( F_1^{(n)}(t, y(t)), F_2^{(n)}(t, y(t)) \right), \; y(0) = \epsilon.
\]

The next proposition sums up useful properties on the solutions of these two ODEs, i.e., our two kinds of interpolation functions. The proof is given in Appendix [A].

**Proposition 4.** Suppose that \([H1],[H2],[H3]\) hold and that \( \Delta = E_X \rho_0 [X^2] = 1 \). For all \( \epsilon \in B_{\alpha} \), there exists a unique global solution \( R(\cdot, \epsilon) : [0, 1] \rightarrow [0, +\infty)^2 \) to (24). This solution is continuously differentiable and its derivative \( R'(\cdot, \epsilon) \) satisfies \( R'(0,\epsilon) \subseteq [0, \alpha_{n,max}/\rho_n] \times [0, 1] \). Besides, for all \( t \in [0, 1] \), \( R(t, \cdot) \) is a \( C^1 \)-diffeomorphism from \( B_{\alpha} \) onto its image whose Jacobian determinant is greater than, or equal to, one. Finally, the same statement holds if we consider (25) instead.

**Proposition 5 (Upper bound).** Suppose that \([H1],[H2],[H3]\) hold, that \( \Delta = E_X \rho_0 [X^2] = 1 \) and that \( \forall n \in \mathbb{N}^* : \alpha_n \leq M_{\alpha}, \frac{m_{\alpha/n}}{\alpha_n} \leq M_{\rho/\alpha}, \frac{m_{\alpha/n}}{\rho_n} \leq M_{\rho/\alpha} \). Then:

\[
\forall n \in \mathbb{N}^* : \frac{I(X^*; Y|\Phi)}{m_n} \leq \inf_{\epsilon \in [0,r_{\text{max}}]} \sup_{q \in [0,1]} i_{\text{RS}}(q, r; \alpha_n, \rho_n) + O\left(\sqrt{M_n}\right) + O\left(\frac{s_n}{\sqrt{\rho_n}}\right). \tag{25}
\]

**Proof.** Fix \( r \in [r_{\text{max}}] \). For all \( \epsilon \in B_{\alpha} \), \( (R_1(\cdot, \epsilon), R_2(\cdot, \epsilon)) \) is the unique solution to the ODE (23) (see Proposition 4). Let \( q_\epsilon(t) := R_2(t, \epsilon) = E(Q)_{n,t,\epsilon} \), \( r_\epsilon(t) := R_1(t, \epsilon) = \frac{-2\alpha_n}{\rho_n} \frac{\partial h(q)}{\partial q} \big|_{q=E(Q)_{n,t,\epsilon}, \rho=1} \). By Proposition 4, the families of functions \( (q_\epsilon)_{\epsilon \in B_{\alpha}}, (r_\epsilon)_{\epsilon \in B_{\alpha}} \) are regular. We can now apply Proposition 1 to get:

\[
\frac{I(X^*; Y|\Phi)}{m_n} = \int_{B_{\alpha}} \frac{d\epsilon}{s_n} \sup_{q \in [0,1]} i_{\text{RS}}(q, r; \alpha_n, \rho_n) + O\left(\sqrt{M_n}\right) + O\left(\frac{s_n}{\sqrt{\rho_n}}\right).
\]

The inequality (25) holds for all \( r \in [0, r_{\text{max}}] \) and the constant factors in the quantities \( O\left(\sqrt{M_n}\right), O\left(\frac{s_n}{\sqrt{\rho_n}}\right) \) are uniform in \( r \). Hence the inequality (25) with the infimum over \( r \). \( \square \)

**Proposition 6 (Lower bound).** Under the same hypotheses than Proposition 5, we have:

\[
\forall n \in \mathbb{N}^* : \frac{I(X^*; Y|\Phi)}{m_n} \leq \inf_{\epsilon \in [0,r_{\text{max}}]} \sup_{q \in [0,1]} i_{\text{RS}}(q, r; \alpha_n, \rho_n) + O\left(\sqrt{M_n}\right) + O\left(\frac{s_n}{\sqrt{\rho_n}}\right). \tag{27}
\]

**Proof.** By Proposition 4, the families of functions \( (q_\epsilon)_{\epsilon \in B_{\alpha}}, (r_\epsilon)_{\epsilon \in B_{\alpha}} \) are regular. Let \( \forall \epsilon \in B_{\alpha} \):

\[
\frac{1}{\alpha_n} I_{P_{0,n}} \left( \int_0^1 r_\epsilon(t) dt \right) + I_{P_{out}} \left( \int_0^1 q_\epsilon(t) dt, 1 \right) - \frac{\rho_n}{2\alpha_n} \int_0^1 r_\epsilon(t) (1 - q_\epsilon(t)) dt
\]

\[
\geq \int_0^1 \left\{ \frac{1}{\alpha_n} I_{P_{0,n}} (r_\epsilon(t)) + I_{P_{out}} (q_\epsilon(t), 1) - \frac{\rho_n}{2\alpha_n} r_\epsilon(t) (1 - q_\epsilon(t)) \right\} dt
\]

\[
= \int_0^1 \left\{ \sup_{q \in [0,1]} \frac{1}{\alpha_n} I_{P_{0,n}} (r_\epsilon(t)) + I_{P_{out}} (q, 1) - \frac{\rho_n}{2\alpha_n} r_\epsilon(t) (1 - q) \right\} dt
\]

\[
= \int_0^1 \sup_{q \in [0,1]} i_{\text{RS}} \left( q, \frac{\rho_n}{\alpha_n} r_\epsilon(t); \alpha_n, \rho_n \right) dt \tag{28}
\]

\[
\geq \inf_{r \in [0,r_{\text{max}}]} \sup_{q \in [0,1]} i_{\text{RS}} (q, r; \alpha_n, \rho_n). \tag{29}
\]

The first inequality is an application of Jensen’s inequality to the concave functions \( I_{P_{0,n}}, I_{P_{out}} (\cdot, 1) \) (see Lemmas 3 and 4). The subsequent equality is because the global maximum of the concave function \( h : q \in [0, 1] \rightarrow I_{P_{out}} (q, 1) - \frac{\rho_n}{2\alpha_n} r_\epsilon(t) (1 - q) \) is reached at \( q_\epsilon(t) \) since \( h'(q_\epsilon(t)) = 0 \). The equality (28) follows from the definition (7) of \( i_{\text{RS}} \). Finally, the inequality (29) is because \( r_\epsilon(t) \in [0, \frac{\alpha_n}{\rho_n} r_{\text{max}}] \) and we simply lowerbound the integrand in (28) by a quantity independent of \( t \in [0, 1] \). We now apply Proposition 5 and make use of (29) to obtain the inequality (27). \( \square \)
To prove Theorem 1, it remains to combine Propositions 5 and 6 with the identity
\[ \inf_{q \in [0,r_{\max}]} \sup_{q \in [0,1]} \text{irs}(q,r) = \inf_{q \in [0,1]} \sup_{r \geq 0} \text{irs}(q,r), \] (30)
and the choice \( \rho_0 = \Theta(n^{-\lambda}), \alpha_n = \gamma \rho_0 n, \) and \( s_n = \Theta(n^{-\beta}) \) with \( \lambda \in [0,1/6), \gamma > 0 \) and \( \beta \in (1/2,1/6 - \lambda). \) Optimizing over \( \beta \) to maximize the convergence rate of
\[ O(\sqrt{n} \ln n) + O\left( \frac{s_n}{\sqrt{\rho_0}} \right) = O\left( \max \left\{ \frac{1}{n^{\beta/2} \rho_0}, \frac{\ln n}{n^{1/6 - \lambda - \beta}} \right\} \right) \]
yields Theorem 1. The identity (30) has been proved in [45] Proposition 7 and Corollary 7 in SI.

B Heuristic derivation of the phase transition

In this section, we analyze the potential function \( \text{irs} \) defined in (7) in order to heuristically locate the information theoretic transition in the special case of a prior \( P_{0,n} := (1 - \rho_0)\delta_0 + \rho_0 P_0 \) where \( P_0 \) is a distribution over a discrete set. If we are to witness an all-or-nothing phenomenon then \( \arg \min_{q \in [0,\mathbb{E}_{X \sim P_0}[X^2]]} \text{irs}(q,r) \) should drop abruptly from \( q \) close to \( \mathbb{E}_{X \sim P_0}[X^2] \) to \( q \) close to 0. Therefore, we first evaluate \( \sup_{r \geq 0} \text{irs}(q,r) \) for \( q \in \{0,\mathbb{E}_{X \sim P_0}[X^2]\} \).

**Potential at** \( q = \mathbb{E}_{X \sim P_0}[X^2] \): We have for all \( r \in [0,\infty) \):
\[
\text{irs}(\mathbb{E}_{X \sim P_0}[X^2],r) := \frac{1}{\alpha_n} I_{P_{0,n}} \left( \frac{\alpha_n}{\rho_n} r \right) + I_{P_{\text{out}}} \left( \mathbb{E}_{X \sim P_0}[X^2], \mathbb{E}_{X \sim P_0}[X^2] \right)
\]
where we used that \( I_{P_{\text{out}}}(\mathbb{E}_{X \sim P_0}[X^2],\mathbb{E}_{X \sim P_0}[X^2]) = 0 \). By Lemma 5 \( I_{P_{0,n}} \) is nonincreasing so \( \sup_{r \geq 0} \text{irs}(\mathbb{E}_{X \sim P_0}[X^2],r) \) is attained for \( r \to +\infty \).

Let \( X^* \sim P_{0,n}, Z \sim \mathcal{N}(0,1) \) be two independent random variables. When \( r \) goes to \( +\infty \), the mutual information \( I_{P_{0,n}}(r) := I(X^*; \sqrt{r} X^* + Z) \) converges to the entropy \( H(X^*) \) since the noise \( Z \) disappears. Hence:
\[
\sup_{r \geq 0} \text{irs}(\mathbb{E}_{X \sim P_0}[X^2],r) = \lim_{r \to +\infty} \frac{1}{\alpha_n} I_{P_{0,n}} \left( \frac{\alpha_n}{\rho_n} r \right) = \frac{H(X^*)}{\alpha_n} = \frac{H_b(\rho_n) + \rho_n H(P_0)}{\alpha_n}, \] (31)
where \( H_b : x \to -x \ln(x) - (1-x) \ln(1-x) \) denotes the binary entropy function and \( H(P_0) \) is the entropy of the probability distribution \( P_0 \).

**Potential at** \( q = 0 \): Let \( X^* \sim P_{0,n}, Z \sim \mathcal{N}(0,1) \) be two independent random variables. Note that:
\[
I_{P_{0,n}}(r) := I(X^*; \sqrt{r} X^* + Z) = \frac{r \rho_n \mathbb{E}_{X \sim P_0}[X^2]}{2} - \psi_{P_{0,n}}(r)
\]
where \( \psi_{P_{0,n}}(r) := \mathbb{E} \ln \int dP_{0,n}(x) e^{r x X^* + \sqrt{r} x Z - \frac{x^2}{2}} \). Therefore, for all \( r \in [0,\infty) \):
\[
\text{irs}(0,r) := \frac{1}{\alpha_n} I_{P_{0,n}} \left( \frac{\alpha_n}{\rho_n} r \right) + I_{P_{\text{out}}} (0, \mathbb{E}_{X \sim P_0}[X^2]) = \frac{r \mathbb{E}_{X \sim P_0}[X^2]}{2}
\]
\[
= I_{P_{\text{out}}} (0, \mathbb{E}_{X \sim P_0}[X^2]) - \frac{1}{\alpha_n} \psi_{P_{0,n}} \left( \frac{\alpha_n}{\rho_n} r \right).
\]
The function \( \psi_{P_{0,n}} : r \to [0,\infty) \) being nondecreasing (see [30] Appendix B.1, Proposition 17)), it directly comes:
\[
\sup_{r \geq 0} \text{irs}(0,r) = \frac{1}{\alpha_n} I_{P_{\text{out}}} (0, \mathbb{E}_{X \sim P_0}[X^2]) - \frac{1}{\alpha_n} \psi_{P_{0,n}} (0) = I_{P_{\text{out}}} (0, \mathbb{E}_{X \sim P_0}[X^2]). \] (32)

Using (32) and (31) we see that in the regime \( \alpha_n = \gamma (H_b(\rho_n) + \rho_n H(P_0)) \), the threshold \( \gamma_c := 1/I_{P_{\text{out}}} (0, \mathbb{E}_{X \sim P_0}[X^2]) \) is such that:

- for \( \gamma < \gamma_c \), \( \sup_{r \geq 0} \text{irs}(0,r) \) is smaller than \( \sup_{r \geq 0} \text{irs}(\mathbb{E}_{X \sim P_0}[X^2],r) \);

- for \( \gamma > \gamma_c \), \( \sup_{r \geq 0} \text{irs}(\mathbb{E}_{X \sim P_0}[X^2],r) \) is smaller than \( \sup_{r \geq 0} \text{irs}(0,r) \).

Of course, in this regime, \( q_0(\gamma) := \arg \min_{q \in [0,\mathbb{E}_{X \sim P_0}[X^2]]} \sup_{r \geq 0} \text{irs}(q,r) \) is not necessarily (close to) 0 or \( \mathbb{E}_{X \sim P_0}[X^2] \), but if it is then \( \gamma_c \) is the threshold at which we will observe the all-or-nothing phase transition. In practice, for the examples studied in Section 3 of the main paper, for \( n \) large we indeed observe that \( q_0(\gamma) \to \mathbb{E}_{X \sim P_0}[X^2] \) for \( \gamma > \gamma_c \) and \( q_0(\gamma) \to 0 \) for \( \gamma < \gamma_c \) when \( \rho_n \) vanishes. Thus we observe an all-or-nothing phase transition that becomes sharper and sharper as \( \rho_n \) becomes negligible.
C Properties of the mutual informations of the scalar channels

This appendix gives important properties on the mutual informations of the scalar channels defined in Section 2.

Lemma 2 (Nishimori identity). Let \( (X, Y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \) be a pair of jointly distributed random vectors. Let \( k \geq 1 \). Let \( X^{(1)}, \ldots, X^{(k)} \) be \( k \) independent samples drawn from the conditional distribution \( P(X = \cdot | Y) \), independently of every other random variables. The angular brackets \( \langle \cdot \rangle \) denote the expectation operator with respect to \( P(X = \cdot | Y) \), while \( \mathbb{E} \) denotes the expectation with respect to \( (X, Y) \). Then, for every integrable function \( g \) the two following quantities are equal:

\[
\mathbb{E} \langle g(Y, X^{(1)}, \ldots, X^{(k-1)}, X^{(k)}) \rangle := \mathbb{E} \int g(Y, x^{(1)}, \ldots, x^{(k-1)}, x^{(k)}) \prod_{i=1}^{k} dP(x^{(i)}|Y) ;
\]

\[
\mathbb{E} \langle g(Y, X^{(1)}, \ldots, X^{(k-1)}, X) \rangle := \mathbb{E} \int g(Y, x^{(1)}, \ldots, x^{(k-1)}, x) \prod_{i=1}^{k-1} dP(x^{(i)}|Y) .
\]

Proof. This is a simple consequence of Bayes’ formula. It is equivalent to sample the pair \((X, Y)\) according to its joint distribution, or to first sample \( Y \) according to its marginal distribution and to then sample \( X \) conditionally to \( Y \) from its conditional distribution \( P(X = \cdot | Y) \). Hence the \((k + 1)\)-tuple \((Y, X^{(1)}, \ldots, X^{(k)})\) is equal in law to \((Y, X^{(1)}, \ldots, X^{(k-1)}, X)\).

Lemma 3. Let \( X \sim P_X \) be a real random variable with finite second moment. Let \( Z \sim \mathcal{N}(0,1) \) be independent of \( X \). Define \( I_X(Z) := I(X; Y^{(r)}) \) the mutual information between \( X \) and \( Y^{(r)} := \sqrt{r}X + Z \). Then \( I_X \) is continuously differentiable, concave, nondecreasing and Lipschitz – with Lipschitz constant \( 2 \mathbb{E}[X^2]/2 - \mathbb{E}[X^4] \) on \([0, +\infty)\).

Proof. Note that \( I_{P_X}(r) = r \mathbb{E}[X^2]/2 - \psi_{P_X}(r) \) where \( \psi_{P_X}(r) := \mathbb{E} \ln \int dP_X(x) e^{r\sqrt{2}Y} - \frac{r^2}{2} \). The lemma then directly follows from \([30]\) Appendix B.1, Proposition 17] which lists properties of the function \( \psi_{P_X} \). \( \square \)

Lemma 4. Let \( \Delta \in (0, +\infty) \). Let \( \varphi : \mathbb{R} \times \mathbb{R}^{k_a} \rightarrow \mathbb{R} \) be a bounded measurable function. Further assume that the first and second partial derivatives of \( \varphi \) with respect to its first argument, denoted \( \partial_{x} \varphi \) and \( \partial_{xx} \varphi \), exist and are bounded.

Let \( W^*, V, Z \sim \mathcal{N}(0,1) \) and \( A = P_A - P_A \) is a probability distribution over \( \mathbb{R}^{k_A} \) – be independent random variables. Define \( I_{P_{out}}(q, \rho) := I(W^*; \tilde{Y}^{(q, \rho)}|V) \) the conditional mutual information between \( W^* \) and \( \tilde{Y}^{(q, \rho)} \) conditional to \( V \). Then:

- For all \( q \in (0, +\infty) \) the function \( q \mapsto I_{P_{out}}(q, \rho) \) is continuously twice differentiable, concave and nonincreasing on \([0, \rho]\);

- For all \( q \in (0, +\infty) \), the function \( q \mapsto I_{P_{out}}(q, \rho) \) is Lipschitz on \([0, \rho]\) with Lipschitz constant \( C_1(\|Q_{m}\|_{\infty}, \|\partial_{x}Q_{m}\|_{\infty}, \|\partial_{xx}Q_{m}\|_{\infty}) \) where:

\[
C_1(a, b) := (4a^2 + 1)b^2 .
\]

- For all \( q \in [0, +\infty) \), the function \( \rho \mapsto I_{P_{out}}(q, \rho) \) is Lipschitz on \([q, +\infty)\) with Lipschitz constant \( C_2(\|Q_{m}\|_{\infty}, \|\partial_{x}Q_{m}\|_{\infty}, \|\partial_{xx}Q_{m}\|_{\infty}) \) where:

\[
C_2(a, b, c) := b^2 (12a^2 + 27) + c (16a^4 + 4\sqrt{2}) .
\]

Proof. Let \( P_{out}(y|x) = \int \frac{dP_{X}(a)}{\sqrt{2}\pi} \exp(-\frac{1}{2}(y-\varphi(x,a))^2) \). The posterior density of \( W^* \) given \((V, \tilde{Y}^{(q, \rho)})\) is

\[
dP(w|V, \tilde{Y}^{(q, \rho)} := \frac{1}{Z_{q, \rho}(V, \tilde{Y}^{(q, \rho)})} \frac{dw}{2\pi} P_{out}(\tilde{Y}^{(q, \rho)}|\sqrt{\rho - q}w + \sqrt{q}V) \),
\]

where \( Z(q, \rho) := \int \frac{dw}{2\pi} e^{-\frac{w^2}{2}} P_{out}(\tilde{Y}^{(q, \rho)}|\sqrt{\rho - q}w + \sqrt{q}V) \). The normalization factor is

\[
I_{P_{out}}(q, \rho) = \mathbb{E} \left[ \ln P_{out}(Y^{(q, \rho)}|\sqrt{\rho - q}W^* + \sqrt{q}V) \right] - \mathbb{E} \ln Z(q, \rho)
\]

\[
= \mathbb{E} \ln Z(q, \rho) - \mathbb{E} \ln Z(q, \rho) .
\]

It is shown in \([30]\) Appendix B.2, Proposition 18] that, for all \( q \in (0, +\infty) \), \( q \mapsto I_{P_{out}}(q, \rho) \) is continuously twice differentiable, concave and nondecreasing on \([0, \rho]\), i.e., \( q \mapsto I_{P_{out}}(q, \rho) \) is continuously twice differentiable, concave and nonincreasing on \([0, \rho]\).
We prove the second point of the lemma by upper bounding the partial derivative of $I_{P_{out}}$ with respect to $q$. The Lipschitzianity will then follow directly from the mean-value theorem. We denote an expectation with respect to the posterior distribution (33) using the angular brackets $\langle \cdot \rangle_{q, \rho}$, i.e., $\langle g(w) \rangle_{q, \rho} := \int g(w) dP(w|V, Y^q, \rho)$. Let $u_y(x) := \ln P_{out}(y|x)$. We know from (30) Appendix B.2, Proposition 18 that $\forall q \in (0, +\infty), \forall q \in [0, \rho)$:

$$
\frac{\partial I_{P_{out}}}{\partial q} \bigg|_{q, \rho} = - \frac{\partial}{\partial q} \ln Z \bigg|_{q, \rho} = - \frac{1}{2} \mathbb{E} \left[ (u'_{Y(q, \rho)}(\sqrt{\rho - q} W + \sqrt{q} V))^2 \right].
$$

(35)

By Jensen’s inequality and Nishimori identity, it directly follows from (35):

$$
\left| \frac{\partial I_{P_{out}}}{\partial q} \bigg|_{q, \rho} \right| \leq \frac{1}{2} \mathbb{E} \left[ (u'_{Y(q, \rho)}(\sqrt{\rho - q} W + \sqrt{q} V))^2 \right] = \frac{1}{2} \mathbb{E} \left[ u'^2_{Y(q, \rho)} \left( \sqrt{\rho - q} W + \sqrt{q} V \right)^2 \right].
$$

(36)

Remember that $\partial_x \varphi, \partial_x \varphi$ denote the first and second partial derivatives of $\varphi$ with respect to its first coordinate. The infinity norms $\|\varphi\|_\infty$ and $\|\partial_x \varphi\|_\infty$ are finite by assumptions. Note that $\forall x \in \mathbb{R}$:

$$
|u'_{\rho}(x)| \leq \frac{|y| + \|\varphi\|_\infty}{\Delta} \|\partial_x \varphi\|_\infty
$$

(37)

Then $|u'_{\rho}(x)| \leq \frac{\|\varphi\|_\infty + \Delta}{\Delta^2}$, this upper bound combined with (36) yields:

$$
\left| \frac{\partial I_{P_{out}}}{\partial q} \bigg|_{q, \rho} \right| \leq \frac{4\|\varphi\|_\infty^2 + \Delta}{\Delta^2 \Delta^4} \|\partial_x \varphi\|_\infty^2,
$$

(39)

which implies the second point of the lemma thanks to the mean-value theorem.

To prove the third, and last, point of the lemma we will now upper bound the partial derivative of $I_{P_{out}}$ with respect to $\rho$. Note that

$$
\mathbb{E} \ln Z(q, \rho) = \mathbb{E} \left[ \int dy e^{u_y(\sqrt{\rho - q} W + \sqrt{q} V)} \ln \int \frac{dw}{\sqrt{2\pi}} e^{u_y(\sqrt{\rho - q} W + \sqrt{q} V) - \frac{w^2}{2}} \right].
$$

Therefore:

$$
\frac{\partial \mathbb{E} \ln Z}{\partial \rho} \bigg|_{q, \rho} = \mathbb{E} \left[ \frac{W^*}{2\sqrt{\rho - q}} \int dy (u_y(x)e^{u_y(x)}) \bigg|_{x=\sqrt{\rho - q} W + \sqrt{q} V} \ln \int \frac{dw}{\sqrt{2\pi}} e^{u_y(\sqrt{\rho - q} W + \sqrt{q} V) - \frac{w^2}{2}} \right]
$$

$$
+ \mathbb{E} \left[ \frac{W^*}{2\sqrt{\rho - q}} u'_{Y(q, \rho)}(\sqrt{\rho - q} W + \sqrt{q} V) \right]
$$

$$
= \mathbb{E} \left[ \frac{W^*}{2\sqrt{\rho - q}} \int dy (u_y(x)e^{u_y(x)}) \bigg|_{x=\sqrt{\rho - q} W + \sqrt{q} V} \ln \int \frac{dw}{\sqrt{2\pi}} e^{u_y(\sqrt{\rho - q} W + \sqrt{q} V) - \frac{w^2}{2}} \right]
$$

$$
+ \mathbb{E} \left[ \frac{W^*}{2\sqrt{\rho - q}} u'_{Y(q, \rho)}(\sqrt{\rho - q} W + \sqrt{q} V) \right]
$$

$$
= \frac{1}{2} \mathbb{E} \left[ \left( u'_{Y(q, \rho)}(x) + u''_{Y(q, \rho)}(x) \right)^2 \bigg|_{x=\sqrt{\rho - q} W + \sqrt{q} V} \ln Z(q, \rho) \right]
$$

$$
+ \frac{1}{2} \mathbb{E} \left[ u''_{Y(q, \rho)}(\sqrt{\rho - q} W + \sqrt{q} V) \right]
$$

$$
= \frac{1}{2} \mathbb{E} \left[ \left( u'_{Y(q, \rho)}(x) + u''_{Y(q, \rho)}(x) \right)^2 \bigg|_{x=\sqrt{\rho - q} W + \sqrt{q} V} (\ln Z(q, \rho) + 1) \right]
$$

(40)

The second equality follows from Nishimori identity and the third one from integrating by parts with respect to $W^*$. We now define $\forall \rho \in [0, +\infty)$ : $h(\rho) := \mathbb{E} \ln Z(\rho, \rho) = \mathbb{E} \left[ \int dy e^{u_y(\sqrt{\rho} V)} u_y(\sqrt{\rho} V) \right]$. We have:

$$
h'(\rho) = \mathbb{E} \left[ \frac{V}{\sqrt{2\rho}} \int dy e^{u_y(\sqrt{\rho} V)} u_y(\sqrt{\rho} V) + 1) \right] u'_y(\sqrt{\rho} V)
$$

$$
= \frac{1}{2} \mathbb{E} \left[ \int dy e^{u_y(\sqrt{\rho} V)} (u'_y(\sqrt{\rho} V) + u'_y(\sqrt{\rho} V)^2) u_y(\sqrt{\rho} V) + 1 \right]
$$

$$
+ \frac{1}{2} \mathbb{E} \left[ \int dy e^{u_y(\sqrt{\rho} V)} u'_y(\sqrt{\rho} V) \right]
$$

(41)
Combining (34), (40) and (41) yields
\[
\frac{\partial I_{\text{out}}}{\partial \rho} \big|_{\rho, \varphi} = \frac{1}{2} \mathbb{E} \left[ (u_{\rho}''(\rho, \varphi)(x) + u_{\varphi}''(\rho, \varphi)(x))^2 \right]_{x = \sqrt{\frac{1}{2\pi}}} (\ln Z(\rho, \varphi) + 1)
\]
\[
- \frac{1}{2} \mathbb{E}\left[ (u_{\rho}''(\rho, \varphi)(x) + u_{\varphi}''(\rho, \varphi)(x))^2 \right]_{x = \sqrt{\frac{1}{2\pi}}} (\ln Z(q, \varphi) + 1)
\]
\[
+ \frac{1}{2} \mathbb{E} \left[ u_{\rho}''(\rho, \varphi)(\sqrt{\rho} V)^2 \right] + \frac{1}{2} \mathbb{E} \left[ u_{\varphi}''(\rho, \varphi)(\sqrt{\varphi} W)^2 \right].
\]
(42)
The last two summands on the right-hand side of (42) are upper bounded by \(\frac{4\|x\|_{\infty}^2 + \Delta}{\Delta^2} \|\partial_x \varphi\|_{\infty}^2\) (see the proof of the second point of the lemma). The first two summands on the right-hand side of (42) involve the function \((x, y) \mapsto u''_{\rho}(x) + u''_{\varphi}(x)^2\). We have:
\[
\int_{-\infty}^{+\infty} (u''_{\rho}(x) + u''_{\varphi}(x)^2) e^{u_{\rho}(x)} dy
\]
\[
= \int dP_{A} \int_{-\infty}^{+\infty} (y - \varphi(x, a)) \partial_x \varphi(x, a) + \Delta \partial_x \varphi(x, a)(y - \varphi(x, a)) e^{-\frac{(y - \varphi(x, a))^2}{2\pi}} dy
\]
\[
= \int dP_{A} \int_{-\infty}^{+\infty} (y^2 - 1) \partial_x \varphi(x, a) + \Delta \partial_x \varphi(x, a) e^{-\frac{y^2}{2\pi}} dy
\]
\[
= 0.
\]
(44)
Therefore:
\[
\mathbb{E}\left[ (u''_{\rho}(\rho, \varphi)(x) + u''_{\varphi}(\rho, \varphi)(x))^2 \right]_{x = \sqrt{\frac{1}{2\pi}}} (\ln Z(\rho, \varphi) + 1)
\]
\[
= \mathbb{E}\left[ \int_{-\infty}^{+\infty} (u''_{\rho}(x) + u''_{\varphi}(x)^2) e^{u_{\rho}(x)} dy \right]_{x = \sqrt{\frac{1}{2\pi}}} (\ln Z(\rho, \varphi) + 1)
\]
\[
= 0.
\]
This directly implies:
\[
\mathbb{E}\left[ (u''_{\rho}(\rho, \varphi)(x) + u''_{\varphi}(\rho, \varphi)(x))^2 \right]_{x = \sqrt{\frac{1}{2\pi}}} (\ln Z(q, \varphi) + 1)
\]
\[
= \mathbb{E}\left[ (u''_{\rho}(\rho, \varphi)(x) + u''_{\varphi}(\rho, \varphi)(x))^2 \right]_{x = \sqrt{\frac{1}{2\pi}}} (\ln Z(q, \varphi) + 1) + \frac{\ln(2\pi\Delta)}{2}
\]
\[
= 0.
\]
(45)
We use the formula (43) for \(u''_{\rho}(x) + u''_{\varphi}(x)^2\) to get the upper bound:
\[
|u''_{\rho}(\rho, \varphi)(x) + u''_{\varphi}(\rho, \varphi)(x)|^2 \leq \left( \frac{4\|\varphi\|_{\infty} + \sqrt{\Delta}|Z|}{\Delta^2} \right)^2
\]
\[
\left( \frac{4\|\varphi\|_{\infty} + \sqrt{\Delta}|Z|}{\Delta^2} \right)^2
\]
\[
\frac{\ln(2\pi\Delta)}{2}
\]
(46)
Trivially, \(P_{\text{out}}(\varphi|x) \leq 1/\sqrt{2\pi\Delta}\). This implies
\[
\ln Z(q, \rho) = \ln \int \frac{d\varphi}{\sqrt{2\pi}} e^{-\frac{\varphi^2}{2}} P_{\text{out}}(\tilde{\varphi}(q, \rho)|\sqrt{\rho} w + \sqrt{\varphi} V) \leq \frac{-\ln(2\pi\Delta)}{2},
\]
while, by Jensen's inequality, we have
\[
\ln Z(q, \rho) = \ln \int \frac{d\varphi}{\sqrt{2\pi}} e^{-\frac{\varphi^2}{2}} dP_{A}(a) \frac{1}{\sqrt{2\pi\Delta}} e^{-\frac{1}{2\pi} (\tilde{\varphi}(q, \rho) - \varphi(x, a))^2}
\]
\[
\geq \int \frac{d\varphi}{\sqrt{2\pi}} e^{-\frac{\varphi^2}{2}} dP_{A}(a) \left( -\frac{\ln(2\pi\Delta)}{2} - \frac{(\tilde{\varphi}(q, \rho) - \varphi(x, a))^2}{2\Delta} \right)
\]
\[
\geq -\frac{\ln(2\pi\Delta)}{2} \leq \frac{2\|\varphi\|_{\infty} + \sqrt{\Delta}|Z|}{2\Delta}.
\]
Hence
\[
\ln Z(q, \rho) + \frac{\ln(2\pi\Delta)}{2} \leq \frac{(2\|\varphi\|_{\infty} + \sqrt{\Delta}|Z|)^2}{2\Delta}.
\]
(47)
Combining (45), (46), (47) yields the following upper bound of the second term on the right-hand side of (42):

\[
\left| \frac{1}{2} E \left[ \left( u_{\lambda}(y, \rho) (x) + u_{\lambda}''(y, \rho) (x)^2 \right) \right]_{x = \sqrt{\frac{\varphi}{\Delta}} W^{*} + \sqrt{\varphi} \nu (\ln Z(\eta, \rho) + 1)} \right| \leq C \left( \left\| \frac{\varphi}{\Delta} \right\|_\infty, \left\| \frac{\partial_x \varphi}{\Delta} \right\|_\infty, \left\| \frac{\partial_{xx} \varphi}{\Delta} \right\|_\infty \right),
\]

(48)

where \( C(a, b, c) := b^2 (6a^4 + 6a^2 + 13.5) + c (8a^3 + 2 \sqrt{\frac{\pi}{2}}) \). This upper bound holds for all \( q \in [0, \rho] \). In particular, it holds for the first term on the right-hand side of (42) where \( q = \rho \). We now have an upper bound for each summand on the right-hand side of (42) and we can combine them to get:

\[
\left. \frac{\partial I_{\text{Pout}}}{\partial \mu} \right|_{\eta, \rho} \leq 2C \left( \left\| \frac{\varphi}{\Delta} \right\|_\infty, \left\| \frac{\partial_x \varphi}{\Delta} \right\|_\infty, \left\| \frac{\partial_{xx} \varphi}{\Delta} \right\|_\infty \right) + 2 \left( \frac{\left\| \frac{\varphi}{\Delta} \right\|_\infty^2 + 1}{\left\| \frac{\varphi}{\Delta} \right\|_\infty} \right). \]

We can conclude the proof of the third point of the lemma using this last upper bound and the mean-value theorem.

**D  Properties of the interpolating mutual information**

We recall that \( u_{\eta}(x) := \ln P_{\text{out}}(y|x) \), and that \( u_{\eta}'(\cdot) \) and \( u_{\eta}''(\cdot) \) are the first and second derivatives of \( u_{\eta}(\cdot) \). We denote \( P_{\text{out}}^\mu(y|x) \) and \( P_{\text{out}}^{\mu'}(y|x) \) the first and second derivatives of \( x \mapsto P_{\text{out}}(y|x) \). Finally, the scalar overlap is \( Q := \frac{1}{n} \sum_{t=1}^{n} X_i^t x_i \).

**D.1 Derivative of the interpolating mutual information**

**Proposition (extended).** Suppose that \( \Delta > 0 \) and that all of \([H1], [H2]\) and \([H3]\) hold. Further assume that \( \mathbb{E}_{\Psi \sim p_{\Psi}} X^2 \equiv 1 \). The derivative of the interpolating mutual information \( I_{\mu} \) with respect to \( t \) satisfies for all \( (t, \epsilon) \in [0, 1] \times \mathcal{B}_n \):

\[
i_{\eta, \epsilon}'(t) = \mathcal{O} \left( \left\| \frac{1}{\sqrt{n} \rho_n} \right\| \right) + \mathcal{O} \left( \left\| \frac{\alpha_n \sqrt{\varphi}}{\rho_n} \ln \frac{\varphi_{\delta, \epsilon}}{m_n} \right\| + \frac{\rho_n}{2 \alpha_n} r_t(t)(1 - q_t(t)) \right) + \frac{1}{2} \mathbb{E} \left\{ (Q - q_t(t)) \left( \frac{1}{m_n} \sum_{t=1}^{m_n} u_{\lambda}''(\epsilon) \left( S_{\mu}^{(\epsilon)} \right) u_{\lambda}'(\epsilon) \left( S_{\mu}^{(\epsilon)} \right) - \frac{\rho_n}{2 \alpha_n} r_t(t) \right) \right\}_{n, t, \epsilon},
\]

(49)

where

\[
\mathcal{O} \left( \frac{1}{\sqrt{n} \rho_n} \right) \leq \frac{S^2 C}{\sqrt{n} \rho_n} \quad \text{and} \quad \mathcal{O} \left( \left\| \frac{\alpha_n \sqrt{\varphi}}{\rho_n} \ln \frac{\varphi_{\delta, \epsilon}}{m_n} \right\| \right) \leq \frac{S^2}{D} \frac{\alpha_n \sqrt{\varphi}}{\rho_n} \ln \frac{\varphi_{\delta, \epsilon}}{m_n};
\]

with \( \partial_x \varphi \) and \( \partial_{xx} \varphi \) denote the first and second partial derivatives of \( \varphi \) with respect to its first argument:

\[
C := \left\| \frac{\partial_x \varphi}{\Delta} \right\|_\infty^2 \left( 64 \left\| \frac{\varphi}{\Delta} \right\|_\infty^4 + 2 \left\| \frac{\varphi}{\Delta} \right\|_\infty^2 + 12.5 \right) + \left\| \frac{\partial_{xx} \varphi}{\Delta} \right\|_\infty \left( 8 \left\| \frac{\varphi}{\Delta} \right\|_\infty^2 + 2 \sqrt{\frac{\pi}{2}} \right);
\]

\[
D := \left\| \frac{\partial_x \varphi}{\Delta} \right\|_\infty^4 + \frac{4}{2} \left\| \frac{\partial_{xx} \varphi}{\Delta} \right\|_\infty^2.
\]

In addition, if both sequences \( \left( \alpha_n \right)_{n} \) and \( \left( \rho_n / \alpha_n \right)_{n} \) are bounded, i.e., if there exist real positive numbers \( M_{\alpha}, M_{\rho/\alpha} \) such that \( \forall n \in \mathbb{N}^* : \alpha_n \leq M_{\alpha}, \rho_n / \alpha_n \leq M_{\rho/\alpha} \) then for all \( (t, \epsilon) \in [0, 1] \times \mathcal{B}_n \):

\[
i_{\eta, \epsilon}'(t) = \mathcal{O} \left( \left\| \frac{1}{\sqrt{n} \rho_n} \right\| \right) + \frac{\rho_n}{2 \alpha_n} r_t(t)(1 - q_t(t)) + \frac{1}{2} \mathbb{E} \left\{ (Q - q_t(t)) \left( \frac{1}{m_n} \sum_{t=1}^{m_n} u_{\lambda}''(\epsilon) \left( S_{\mu}^{(\epsilon)} \right) u_{\lambda}'(\epsilon) \left( S_{\mu}^{(\epsilon)} \right) - \frac{\rho_n}{2 \alpha_n} r_t(t) \right) \right\}_{n, t, \epsilon},
\]

(50)

where

\[
\mathcal{O} \left( \left\| \frac{1}{\sqrt{n} \rho_n} \right\| \right) \leq \frac{S^2 C + S^2 \sqrt{D} (\tilde{C}_1 + M_{\rho/\alpha} \tilde{C}_2 + M_{\alpha} \tilde{C}_3)}{\sqrt{n} \rho_n}.
\]

Here \( \tilde{C}_1, \tilde{C}_2, \tilde{C}_3 \) are the polynomials in \( \mathcal{S} \) defined in Proposition 7.
Proof. We recall that $Z_{t,\epsilon}$ is the normalization to the joint posterior density of $(X^*, W^*)$ given $(Y^{(t,\epsilon)}, \Phi, V)$. We define the average interpolating free entropy $f_{n,\epsilon}(t) := E \ln Z_{t,\epsilon}/m_n$. Note that $i_{n,\epsilon}(t) := \langle (X^*, W^*), (Y^{(t,\epsilon)}, \Phi, V) \rangle |Z_{t,\epsilon}/m_n$ satisfies:

$$i_{n,\epsilon}(t) = -\frac{1}{m_n} E \ln Z_{t,\epsilon} + \frac{1}{m_n} E \left[ \ln \left( e^{-\frac{\|X^*\|^2}{k_n} + t + 2s_n} \right) \right] = -f_{n,\epsilon}(t) - \frac{1}{2\alpha_n} + E \left[ \ln P_{out}(Y^{(t,\epsilon)}|s^{(t,\epsilon)}) \right]$$

Given $X^*, S_1^{(t,\epsilon)} \sim N(0, V(t))$ where $\rho(t) := \frac{1}{k_n} ||X^*||^2 + t + 2s_n$. Then:

$$E \ln P_{out}(Y^{(t,\epsilon)}|s^{(t,\epsilon)}) = E E[\ln P_{out}(Y^{(t,\epsilon)}|s^{(t,\epsilon)})] = \mathbb{E}[\mathbb{E}[h(\rho(t))]],$$

where $h : \rho \in [0, +\infty) \mapsto \mathbb{E}_{V \sim N(0,1)} \int u_y(\sqrt{\rho} V e^{\epsilon x} dV) dy$. All in all, we have:

$$i_{n,\epsilon}(t) = E[h(\rho(t))] - f_{n,\epsilon}(t) - \frac{1}{2\alpha_n}. \quad (51)$$

We directly obtain for the derivative of $i_{n,\epsilon}(\cdot)$:

$$i_{n,\epsilon}'(t) = -E \left[ \frac{\|X^*\|^2}{k_n} - 1 \right] - f_{n,\epsilon}'(t), \quad (52)$$

where $h', f_{n,\epsilon}'$ are the derivatives of $h, f_{n,\epsilon}$. In Lemma 3 of Appendix C we compute $h'$ and show:

$$\forall \rho \in [0, +\infty) : |h'(\rho)| \leq C := C \left( \alpha \sqrt{\beta} \right) \left( \varphi_{\epsilon} \sqrt{\alpha}, \varphi_{\epsilon} \sqrt{\alpha}, \varphi_{\epsilon} \sqrt{\alpha} \right)$$

with $C(a, b, c) := b^2(64a^4 + 2a^2 + 12.5) + c(8a^3 + 2\sqrt{\frac{a}{2}})$. The first term on the right-hand side of (52) thus satisfies:

$$\left| E \left[ h'(\rho(t)) \right] \right| \leq C \sqrt{n \max \left( \frac{\|X^*\|^2}{k_n} \right)} = \frac{C}{k_n} \sqrt{n \max \left( \frac{\|X^*\|^2}{k_n} \right)} = C S^2 \sqrt{n \max \left( \frac{\|X^*\|^2}{k_n} \right)} . \quad (53)$$

We now turn to the computation of $f_{n,\epsilon}'$.

**Derivative of the average interpolating free entropy**

Note that

$$f_{n,\epsilon}(t) = \frac{1}{m_n} E \left[ \int \frac{dy}{\sqrt{2\pi}} e^{-H_{t,\epsilon}(X^*, W^*, y, \Phi, V) \ln \int dP_{b,\epsilon}(x) dW e^{-H_{t,\epsilon}(x, y, \Phi, V)}} \right] \quad (54)$$

where the expectation is over $X^*, \Phi, V, W^*$, $\mathcal{D}w := \frac{d\mathcal{D}w}{\|w\|^2}$ and the Hamiltonian $H_{t,\epsilon}$ is:

$$H_{t,\epsilon}(x, y, \Phi, V) := -\sum_{\mu=1}^{m_n} \ln P_{out}(y, \sigma^{(t,\epsilon)}), + \frac{1}{2} \sum_{i=1}^{n} \left( \bar{y}_i - \sqrt{R_i(t, \epsilon)} x_i \right)^2 . \quad (55)$$

We will need its derivative $H_{t,\epsilon}'$ with respect to $t$:

$$H_{t,\epsilon}'(x, y, \Phi, V) := -\sum_{\mu=1}^{m_n} \frac{\partial \sigma^{(t,\epsilon)}}{\partial t} u'_{y_\mu}(\sigma^{(t,\epsilon)}) - \frac{r_\epsilon(t)}{2 \sqrt{R_\epsilon(t, \epsilon)}} \sum_{i=1}^{n} x_i (\bar{y}_i - \sqrt{R_i(t, \epsilon)} x_i) . \quad (56)$$

The derivative of $f_{n,\epsilon}$ can be obtained by differentiating (54) under the expectation:

$$f_{n,\epsilon}'(t) = -\frac{1}{m_n} E \left[ H_{t,\epsilon}'(X^*, W^*, Y^{(t,\epsilon)}, \Phi, V) \right] \ln Z_{t,\epsilon}$$

$$= -\frac{1}{m_n} E \left[ H_{t,\epsilon}'(x, y; Y^{(t,\epsilon)}, \Phi, V) \right]_{n, t, \epsilon}$$

$$= -\frac{1}{m_n} E \left[ H_{t,\epsilon}'(x, W^*, Y^{(t,\epsilon)}, \Phi, V) \right] \ln Z_{t,\epsilon}$$

$$= -\frac{1}{m_n} E \left[ H_{t,\epsilon}'(X^*, W^*, Y^{(t,\epsilon)}, \Phi, V) \right] . \quad (57)$$

The last equality follows from the Nishimori identity

$$E \langle H_{t,\epsilon}'(x, y; Y^{(t,\epsilon)}, \Phi, V) \rangle_{n, t, \epsilon} = E \left[ H_{t,\epsilon}'(X^*, W^*, Y^{(t,\epsilon)}, \Phi, V) \right] .$$
Evaluating \( f(6) \) at \( (x, w; y, \tilde{y}, \Phi, V) = (X^*, W^*, Y^{(t, e)}, \tilde{Y}^{(t, e)}, \Phi, V) \) yields:

\[
\mathcal{H}_t^{(t, e)}(X^*, W^*, Y^{(t, e)}, \tilde{Y}^{(t, e)}, \Phi, V) = - \sum_{\mu=1}^{m_n} \frac{\partial S^{(t, e)}_{\mu}}{\partial t} u_{Y^{(t, e)}_{\mu}}(S^{(t, e)}_{\mu}) - \frac{r_\mu(t)}{2 \sqrt{R_{3}(t, e)}} \sum_{i=1}^{n} X^*_i \tilde{Z}_i. \tag{58}
\]

The expectation of (55) is zero:

\[
E \mathcal{H}_t^{(t, e)}(X^*, W^*, Y^{(t, e)}, \tilde{Y}^{(t, e)}, \Phi, V) = - \sum_{\mu=1}^{m_n} \mathbb{E} \left[ \frac{\partial S^{(t, e)}_{\mu}}{\partial t} u_{Y^{(t, e)}_{\mu}}(S^{(t, e)}_{\mu}) \right] = 0.
\]

The last equality is because for all \( x \):

\[
\int P_{\text{out}}(y | x) dy = \int dP_{x}(a) \partial_x \varphi(x, a) \frac{e^{-(y - \varphi(x, a))^2}}{\sqrt{2\pi}} dy = 0.
\]

The expectation of (55) being zero, the identity (57) reads:

\[
f_{n,e}(t) = \frac{1}{m_n} \sum_{\mu=1}^{m_n} \mathbb{E} \left[ \frac{\partial S^{(t, e)}_{\mu}}{\partial t} u_{Y^{(t, e)}_{\mu}}(S^{(t, e)}_{\mu}) \ln Z_{t,e} \right] + \frac{1}{m_n} \frac{r_\mu(t)}{2 \sqrt{R_{3}(t, e)}} \sum_{i=1}^{n} \mathbb{E} \left[ X^*_i \tilde{Z}_i \ln Z_{t,e} \right]. \tag{59}
\]

First, we compute the first kind of expectation on the right-hand side of (59), \( \forall \mu \in \{1, \ldots, m_n\} \):

\[
\mathbb{E} \left[ \frac{\partial S^{(t, e)}_{\mu}}{\partial t} u_{Y^{(t, e)}_{\mu}}(S^{(t, e)}_{\mu}) \ln Z_{t,e} \right] = \frac{1}{2} \mathbb{E} \left[ \left( -\frac{\Phi X^*_{\mu}}{\sqrt{k_n(1-t)}} + \frac{q(t) V_{n}}{\sqrt{R_{2}(t, e)}} + \frac{(1-q(t)) W_{n}}{\sqrt{T+2k_n-R_{2}(t, e)}} \right) u_{Y^{(t, e)}_{\mu}}(S^{(t, e)}_{\mu}) \ln Z_{t,e} \right]. \tag{60}
\]

An integration by parts w.r.t. the independent standard Gaussians \( (\Phi_{t, e})_{t=1}^{n} \) yields:

\[
\mathbb{E} \left[ \frac{(\Phi X^*_{\mu})_{t}}{\sqrt{k_n(1-t)}} u_{Y^{(t, e)}_{\mu}}(S^{(t, e)}_{\mu}) \ln Z_{t,e} \right] = \sum_{i=1}^{n} \mathbb{E} \left[ \frac{X^*_{i}}{k_n} \left( u_{Y^{(t, e)}_{\mu}}(S^{(t, e)}_{\mu}) + u_{Y^{(t, e)}_{\mu}}(S^{(t, e)}_{\mu}) \right) \ln Z_{t,e} + \frac{X^*_i u_{Y^{(t, e)}_{\mu}}(S^{(t, e)}_{\mu})}{k_n} \right], \tag{61}
\]

where, in the last equality, we used the identity \( u_{Y^{(t, e)}_{\mu}}(x) + u_{Y^{(t, e)}_{\mu}}(x)^2 = \frac{\rho_{\text{out}}(y | x)}{\rho_{\text{out}}(y | y)} \). Another Gaussian integration by parts, this time with respect to \( V_{n} \sim \mathcal{N}(0,1) \), gives:

\[
\mathbb{E} \left[ \frac{q(t) V_{n}}{\sqrt{R_{2}(t, e)}} u_{Y^{(t, e)}_{\mu}}(S^{(t, e)}_{\mu}) \ln Z_{t,e} \right] = \mathbb{E} \left[ \frac{q(t) V_{n}}{\sqrt{R_{2}(t, e)}} \int dy dy' u_{Y^{(t, e)}_{\mu}}(S^{(t, e)}_{\mu}) \ln Z_{t,e} + \mathbb{E} \left[ Q u_{Y^{(t, e)}_{\mu}}(S^{(t, e)}_{\mu}) u_{Y^{(t, e)}_{\mu}}(S^{(t, e)}_{\mu}) \right] \right], \tag{62}
\]
Finally, a Gaussian integration by part w.r.t. $W_\mu^* \sim \mathcal{N}(0, 1)$ gives:

$$
\mathbb{E} \left[ \frac{1 - q_\epsilon(t)}{\sqrt{t + 2s_n - R_\epsilon(t)}} \int u_{Y_\mu(t, \epsilon)}^{(t, \epsilon)}(S_{\mu}^{(t, \epsilon)}) \ln Z_{t, \epsilon} \right] = \mathbb{E} \left[ \frac{1 - q_\epsilon(t)}{P_{\text{out}}(Y_\mu^{(t, \epsilon)} | S_{\mu}^{(t, \epsilon))})} \ln Z_{t, \epsilon} \right] .
$$

(63)

Plugging (61), (62) and (63) back in (60), we obtain:

$$
\mathbb{E} \left[ \frac{\partial S_{\mu}^{(t, \epsilon)}}{\partial t} u_{Y_\mu(t, \epsilon)}^{(t, \epsilon)}(S_{\mu}^{(t, \epsilon)}) \ln Z_{t, \epsilon} \right] = -\frac{1}{2} \mathbb{E} \left[ \frac{P_{\text{out}}(Y_\mu^{(t, \epsilon)} | S_{\mu}^{(t, \epsilon)})}{P_{\text{out}}(Y_\mu^{(t, \epsilon)} | S_{\mu}^{(t, \epsilon)})} \left( \frac{\|X^*\|^2}{k_n} - 1 \right) \ln Z_{t, \epsilon} \right] 
$$

$$
- \frac{1}{2} \mathbb{E} \left( (Q - q_\epsilon(t)) u_{\mu(t, \epsilon)}^{(t, \epsilon)}(S_{\mu}^{(t, \epsilon)}) u_{\mu(t, \epsilon)}^{(t, \epsilon)}(S_{\mu}^{(t, \epsilon)}) \right)_{n, t, \epsilon} .
$$

(64)

It remains to compute the first kind of expectation on the right-hand side of (59), i.e.,

$$
\mathbb{E} \left[ X_\epsilon^n \ln Z_{t, \epsilon} \right] = \mathbb{E} \left[ X_\epsilon^n \ln \int dP_{\text{out}}(x) P_{\text{out}}(Y_\mu^{(t, \epsilon)} | S_{\mu}^{(t, \epsilon)}) e^{-\sum_{k=1}^{\infty} \sqrt{R_\epsilon(t) (X_\epsilon^* - x_k)^2 + Z_\epsilon^2}} \right] 
$$

$$
= -\mathbb{E} \left[ X_\epsilon^n \langle \ln \sqrt{R_\epsilon(t) (X_\epsilon^* - x_k) + Z_\epsilon^2} \rangle \right]_{n, t, \epsilon} 
$$

$$
= -\mathbb{E} \left[ X_\epsilon^n \langle \ln (\sqrt{R_\epsilon(t)} (X_\epsilon^* - x_k) + Z_\epsilon) \rangle \right]_{n, t, \epsilon} .
$$

(65)

The second equality follows from a Gaussian integration by parts w.r.t. $\tilde{Z}_\epsilon \sim \mathcal{N}(0, 1)$. Plugging the two simplified expectations (64) and (65) back in (59) yields:

$$
f_{n, \epsilon}(t) = -\frac{\rho_\epsilon}{2\alpha_n} r_\epsilon(t) (1 - q_\epsilon(t)) - \frac{1}{2} \mathbb{E} \left[ \sum_{k=1}^{m_n} \frac{P_{\text{out}}(Y_\mu^{(t, \epsilon)} | S_{\mu}^{(t, \epsilon)})}{P_{\text{out}}(Y_\mu^{(t, \epsilon)} | S_{\mu}^{(t, \epsilon)})} \left( \frac{\|X^*\|^2}{k_n} - 1 \right) \ln Z_{t, \epsilon} \right] 
$$

$$
- \frac{1}{2} \mathbb{E} \left( (Q - q_\epsilon(t)) \left( \frac{1}{m_n} \sum_{k=1}^{m_n} u_{\mu(t, \epsilon)}^{(t, \epsilon)}(S_{\mu}^{(t, \epsilon)}) u_{\mu(t, \epsilon)}^{(t, \epsilon)}(S_{\mu}^{(t, \epsilon)}) - \frac{\rho_\epsilon}{\alpha_n} r_\epsilon(t) \right) \right)_{n, t, \epsilon} .
$$

(66)

The last step to end the proof of the proposition is to upper bound

$$
A_{n, \epsilon}^{(t, \epsilon)} := \mathbb{E} \left[ \sum_{k=1}^{m_n} \frac{P_{\text{out}}(Y_\mu^{(t, \epsilon)} | S_{\mu}^{(t, \epsilon)})}{P_{\text{out}}(Y_\mu^{(t, \epsilon)} | S_{\mu}^{(t, \epsilon)})} \left( \frac{\|X^*\|^2}{k_n} - 1 \right) \ln Z_{t, \epsilon} \right] 
$$

which appears on the right-hand side of (66).

**Upper bounding the quantity (67)** Remember that $u_\mu^p(x) + u_\mu^q(x)^2 = \frac{P_{\text{out}}(y \mid x)}{P_{\text{out}}(y \mid x)}$ and $P_{\text{out}}(y \mid x) = e^u_\mu(x)$. Therefore, $\forall x$:

$$
\int_{-\infty}^{+\infty} P_{\text{out}}(y \mid x) \ dy = \int_{-\infty}^{+\infty} (u_\mu^p(x) + u_\mu^q(x)^2) e^{u_\mu(x)} \ dy = 0 ,
$$

where the second equality follows from the direct computation (64) in Lemma G of Appendix C. Consequently, using the tower property of the conditional expectation, for all $\mu \in \{1, \ldots, m\}$:

$$
\mathbb{E} \left[ \sum_{k=1}^{m_n} \frac{P_{\text{out}}(Y_\mu^{(t, \epsilon)} | S_{\mu}^{(t, \epsilon)})}{P_{\text{out}}(Y_\mu^{(t, \epsilon)} | S_{\mu}^{(t, \epsilon)})} \left( \frac{\|X^*\|^2}{k_n} - 1 \right) \ln Z_{t, \epsilon} \right] = \mathbb{E} \left[ \left( \frac{\|X^*\|^2}{k_n} - 1 \right) \sum_{k=1}^{m_n} \left( \frac{P_{\text{out}}(Y_\mu^{(t, \epsilon)} | S_{\mu}^{(t, \epsilon)})}{P_{\text{out}}(Y_\mu^{(t, \epsilon)} | S_{\mu}^{(t, \epsilon)})} \right) X^*, S^{(t, \epsilon)} \right] 
$$

$$
= \mathbb{E} \left[ \left( \frac{\|X^*\|^2}{k_n} - 1 \right) \sum_{k=1}^{m_n} \int_{-\infty}^{+\infty} P_{\text{out}}(y \mid S_{\mu}^{(t, \epsilon)}) \ dy \right] = 0 .
$$

(68)

Making use of (68) and Cauchy-Schwarz inequality, we have:

$$
|A_{n, \epsilon}^{(t, \epsilon)}| = \left| \mathbb{E} \left[ \sum_{k=1}^{m_n} \frac{P_{\text{out}}(Y_\mu^{(t, \epsilon)} | S_{\mu}^{(t, \epsilon)})}{P_{\text{out}}(Y_\mu^{(t, \epsilon)} | S_{\mu}^{(t, \epsilon)})} \left( \frac{\|X^*\|^2}{k_n} - 1 \right) \ln Z_{t, \epsilon} - f_{n, \epsilon}(t) \right] \right| 
$$

$$
\leq \mathbb{E} \left[ \left( \sum_{k=1}^{m_n} \frac{P_{\text{out}}(Y_\mu^{(t, \epsilon)} | S_{\mu}^{(t, \epsilon)})}{P_{\text{out}}(Y_\mu^{(t, \epsilon)} | S_{\mu}^{(t, \epsilon)})} \left( \frac{\|X^*\|^2}{k_n} - 1 \right)^2 \right)^{\frac{1}{2}} \sqrt{\text{Var} \ln Z_{t, \epsilon} / m_n} \right] .
$$

(69)

Using again the tower property of the conditional expectation gives:

$$
\mathbb{E} \left[ \sum_{k=1}^{m_n} P_{\text{out}}(Y_\mu^{(t, \epsilon)} | S_{\mu}^{(t, \epsilon)}) \left( \frac{\|X^*\|^2}{k_n} - 1 \right)^2 \right] 
$$

$$
= \mathbb{E} \left[ \left( \frac{\|X^*\|^2}{k_n} - 1 \right)^2 \mathbb{E} \left[ \sum_{k=1}^{m_n} P_{\text{out}}(Y_\mu^{(t, \epsilon)} | S_{\mu}^{(t, \epsilon)}) \left( \frac{\|X^*\|^2}{k_n} - 1 \right) \right]^2 \right] .
$$

(70)
We now use the formula (43) for Appendix E) gives:

\[ \text{Combining the identity (74) with the upper bounds (53) and (73) yields (49).} \]

Putting everything together: proofs of (49) and (50) Combining (52) and (66) yields the following formula for the derivative of \( i_{n,e} \) (remember the definition (67) of \( A_{n_e}^{(t,e)} \)):

\[ i'_{n,e}(t) = \frac{A_{n,e}^{(t,e)}}{2} - E \left[ h'(\rho^{(t)}) \left( \frac{\|X^e\|^2}{k_n} - 1 \right) \right] + \frac{\rho_n}{2\alpha_n r_n(t)} (1-q_e(t)) \]

Combining the identity (74) with the upper bounds (53) and (73) yields (59).

It remains to prove the identity (50) that holds under the additional assumption that \( \forall n : \alpha_n \leq M_s, \rho_n/\alpha_n \leq M_{\rho/s} \). Combining (73) with the upper bound (59) on the variance of \( \text{Var}(\ln Z_{t,e}/m_n) \) (see Proposition \( E \) of Appendix C) gives:

\[ \left| A_{n,e}^{(t,e)} \right| \leq S^2 \sqrt{D(\tilde{C}_1 + M_{\rho/s} \tilde{C}_2 + M_{\rho/s} \tilde{C}_3)} / \sqrt{n \rho_n} \]

The constants \( \tilde{C}_1, \tilde{C}_2, \tilde{C}_3 \) are defined in Proposition \( E \) while \( D \) has been defined earlier in the proof. Besides, as \( \rho_n \leq 1 \), we have \( \frac{1}{\sqrt{m_n}} \leq \frac{1}{\sqrt{m_{\text{m}}}} \) and we can loosen the upper bound (53): \( E \left[ h'(\rho^{(t)}) \left( \frac{\|X^e\|^2}{k_n} - 1 \right) \right] \leq \frac{C_s^2}{\sqrt{m_{\text{m}}}} \).

Then, the term \( A_{n,e}^{(t,e)} / 2 = E \left[ h'(\rho^{(t)}) \left( \frac{\|X^e\|^2}{k_n} - 1 \right) \right] \) on the right-hand side of (74) is in \( O(1/\sqrt{n \rho_n}) \) and this proves the identity (50).
D.2 Proof of Lemma 1

Proof. At \( t = 0 \) the functions \( r_i \) and \( q_i \) do not play any role in the observations \([14] \) since \( R_1(t, \epsilon) = r_1 \) and \( R_2(t, \epsilon) = q_2 \). While in the main text we restricted \( \epsilon \) to be in \( B_n := [s_n, 2s_n]^2 \), we can define observations \((Y^{(0, \epsilon)}, \widetilde{Y}^{(0, \epsilon)})\) using \((14)\) for \( t = 0 \) and \( \epsilon \in [0, 2s_n]^2 \). We then extend the interpolating mutual information at \( t = 0 \) to all \( \epsilon \in [0, 2s_n]^2 \):

\[
in_{n, \epsilon}(0) := \frac{1}{m_n} I((X^*, W^*); (Y^{(0, \epsilon)}, \widetilde{Y}^{(0, \epsilon)}); \Phi, V) .
\]

Note that the variation we want to control in this lemma satisfies:

\[
\left| i_{n, \epsilon}(0) - \frac{I(X^*; Y_{\tilde{\Phi}})}{m_n} \right| \leq \left| i_{n, \epsilon}(0) - \frac{I(X^*; Y_{\Phi})}{m_n} \right| + \left| \frac{I(X^*; Y_{\Phi})}{m_n} - \frac{I(X^*; \Phi)}{m_n} \right|. \tag{75}
\]

We will upper bound the two terms on the right-hand side of \((75)\) separately.

1. By the I-MMSE relation (see \([39]\)), we have for all \( \epsilon \in [0, 2s_n]^2 \):

\[
\left| \frac{\partial i_{n, \epsilon}(0)}{\partial \epsilon} \right| = \frac{1}{2\alpha_n} E \left[ \left( I(X^*_n, \epsilon) - \langle x_1 \rangle_{n, 0, s} \right)^2 \right] \leq \frac{E[\|Z_1\|^2]}{2\alpha_n} = \frac{\rho_n}{2\alpha_n} . \tag{76}
\]

To upper bound the absolute value of the partial derivative with respect to \( \epsilon_2 \), we use that \( \epsilon \in [0, 2s_n]^2 \):

\[
\left| \frac{\partial i_{n, \epsilon}(0)}{\partial \epsilon_2} \right| \leq \frac{1}{2} E \left[ (2\|\varphi\| + |Z_1|)^2 \right] \leq (4\|\varphi\|^2 + 1)\|\varphi\|^2 . \tag{77}
\]

By the mean value theorem, and the upper bounds \((76)\) and \((77)\), we have:

\[
\left| i_{n, \epsilon}(0) - i_{n, \epsilon(0, 0)}(0) \right| \leq \frac{\rho_n}{2\alpha_n} |\epsilon_1| + (4\|\varphi\|^2 + 1)\|\varphi\|^2 |\epsilon_2| \leq \left( \frac{\rho_n}{2\alpha_n} + (4\|\varphi\|^2 + 1)\|\varphi\|^2 \right) 2s_n \leq (M_{\rho/\alpha} + 2(4\|\varphi\|^2 + 1)\|\varphi\|^2)2s_n . \tag{78}
\]

2. It remains to upper bound the second term on the right-hand side of \((75)\). Define the following observations where \( X^* \overset{iid}{\sim} P_{0, n}, \Phi := (\phi_\mu) \overset{iid}{\sim} N(0, 1), W^* := (W^*_\mu)_{\mu=1}^{m_n} \overset{iid}{\sim} N(0, 1) \) and \( \eta \in [0, +\infty) \):

\[
Y^{(n)} \sim P_{n, 0}((\Phi X^*)_{\mu} \sqrt{k_n} + \sqrt{\eta} W^*_\mu) + Z_\mu , 1 \leq \mu \leq m_n . \tag{79}
\]

The joint posterior density of \((X^*, W^*)\) given \((Y^{(n)}, \Phi)\) reads:

\[
dP(x, w|Y^{(n)}, \Phi) := \frac{1}{Z_n} dP_{0, n}(x) \prod_{\mu=1}^{m_n} \frac{dw_\mu}{2\pi} e^{-\frac{w_\mu^2}{2} P_{\text{out}}(Y^{(n)}|\frac{(\Phi X^*)_{\mu}}{\sqrt{k_n} + \sqrt{\eta} w_\mu})} , \tag{80}
\]

where \( Z_n \) is the normalization factor. Define the average free entropy \( f_n(\eta) := \frac{1}{m_n} I((X^*, W^*); Y^{(n)}; \Phi) \) satisfies:

\[
i_n(\rho) = \mathbb{E} [h(\frac{\|X^*\|^2}{k_n} + \eta)] - f_n(\rho) = \frac{1}{2\alpha_n} ,
\]

where \( h : \rho \in [0, +\infty) \mapsto \mathbb{E}_{v \sim N(0, 1)} \int w v e^{w v + (\sqrt{\rho} v)^2} dv \). The identity \((81)\) can be obtained exactly as the identity \((51)\) in Appendix \([14]\). Under the assumptions of the lemma, all the hypotheses of domination are reunited to make sure that \( \eta \mapsto i_n(\eta) \) is continuous on \([0, 2s_n]\) and differentiable on \((0, 2s_n)\). Therefore, by the mean-value theorem, there exists \( \eta^* \in (0, 2s_n) \) such that:

\[
\left| i_{n, \epsilon(0, 0)}(0) - \frac{I(X^*; Y_{\Phi})}{m_n} \right| = \left| i_n(2s_n) - i_n(0) \right| = \left| i_n(\eta^*) \right| 2s_n . \tag{82}
\]
Again, in a similar fashion to the computation of the derivative of $i_{n,s}(\cdot)$ in Appendix D.1 we can show that
\[ i'_{n}(\rho) = E \left[ h' \left( \frac{\|X^\star\|^2}{k_n} + \eta \right) \right] - f'_{n}(\rho) ; \quad (83) \]
\[ f'_{n}(\rho) = \frac{1}{2} E \left[ \sum_{\mu=1}^{m_n} \mu_{\omega_{t}}(Y_{\mu}) \left[ \frac{\partial \nu^*}{\partial \xi} \right]_{\mu} + \sqrt{\eta} W_{\mu}^* \right] \ln Z_{\rho} \right] . \quad (84) \]

In Lemma 4 of Appendix C we compute $h'$ and show:
\[ \forall \rho \in [0, +\infty) : \left| h'(\rho) \right| \leq C := C \left( \left( \frac{\varphi}{\sqrt{\Delta}} \right)^{\infty}, \left\| \frac{\partial \psi}{\sqrt{\Delta}} \right\|_{\infty}, \left\| \frac{\partial_{x} \varphi}{\sqrt{\Delta}} \right\|_{\infty} \right) \]
\[ \text{with } C(a, b, c) := b^2(64a^4 + 2a^2 + 12.5) + c(8a^3 + 2\sqrt{\frac{2}{\pi}}) . \]
\[ \text{The first term on the right-hand side of (83) thus satisfies:} \]
\[ \left| E \left[ h' \left( \frac{\|X^\star\|^2}{k_n} + \eta \right) \right] \right| \leq C . \quad (85) \]
\[ \text{The second term, i.e., } f'_{n}(\rho) \text{ is similar to the quantity } A_{n,s}^{(t)} \text{ defined in (67). We upper bound } A_{n,s}^{(t)} \text{ in the last part of the proof in Appendix D.1 We can follow the same steps as for upper bounding } A_{n,s}^{(t)} \text{ and obtain:} \]
\[ \left| f'_{n}(\eta) \right| \leq \frac{\langle Dm_n \rangle \text{Var} \ln Z_{\rho}}{m_n} . \quad (86) \]
\[ \text{Note that } Z_{n} = Z_{n} \cdot Z_{n} = Z_{n} \cdot Z_{n} . \]
\[ \text{We have } \text{Var} \left[ \ln Z_{n} \right] \leq \frac{\tilde{C}}{m_n \rho_n} \text{ where } \tilde{C} \text{ is a polynomial in } S, \left\| \varphi \right\|_{\infty}, \left\| \partial_{x} \varphi \right\|_{\infty}, \left\| \partial_{x} \varphi \right\|_{\infty}, M_n, \rho_n \text{ with positive coefficients. In fact, this upper bound holds for all } \eta \in [0, 2s_n], \text{i.e.,} \]
\[ \forall \eta \in [0, 2s_n] : \text{Var} \left[ \ln Z_{n} \right] \leq \frac{\tilde{C}}{m_n \rho_n} . \]
\[ \text{The proof of this uniform bound on } \text{Var} \left[ \ln Z_{n} \right] \text{ is the same as the one of Proposition 7 only that it is simpler because there is no second channel similar to } Y^{(t)} , \text{ We now combine (82), (83), (85), (86) to finally obtain:} \]
\[ \left| i_{n,s}(0) - I(X^\star; Y | \Phi) \right| \leq \left( C + \sqrt{\frac{D \tilde{C}}{\rho_n}} \right) 2s_n . \quad (87) \]

3. We now plug (87) back in (75) and use that $\rho_n \in (0, 1]$ to end the proof of the lemma:
\[ \left| i_{n,s}(0) - I(X^\star; Y | \Phi) \right| \leq \left( M_{\rho_n} + 2(4 \left\| \varphi \right\|_{\infty}^2 + 1) \left\| \partial_{x} \varphi \right\|_{\infty}^2 + 2C + \sqrt{D \tilde{C}} \right) \frac{2s_n}{\sqrt{\rho_n}} . \]

\[ \square \]

### E Concentration of the free entropy

In this appendix we show that the log-partition function per data point, or free entropy, of the interpolating model studied in Section A.1 concentrates around its expectation.

**Proposition 7** (Free entropy concentration). Suppose that $\Delta > 0$ and that all of (H1), (H2) and (H3) hold. Further assume that $E_{X \sim \rho_n} [X^2] = 1$. We have for all $(t, \epsilon) \in [0, 1] \times B_n$:
\[ \text{Var} \left[ \ln Z_{n} \right] \leq \frac{1}{m_n \rho_n} \left( \tilde{C}_1 + \frac{\tilde{C}_2}{\alpha_n} + \alpha_n \tilde{C}_3 \right) , \quad (88) \]

where $(\partial_{x} \varphi$ and $\partial_{x} \varphi$ denote the first and second partial derivatives of $\varphi$ with respect to its first argument):
\[ \tilde{C}_1 := 1.5 + 4 \left( \frac{\varphi}{\sqrt{\Delta}} \right)^{\infty} + 8S^2 \left( 4 \left( \frac{\varphi}{\sqrt{\Delta}} \right)^{\infty} + 1 \right) \left\| \frac{\partial_{x} \varphi}{\sqrt{\Delta}} \right\|_{\infty}^2 \]
\[ + \left( 2 \frac{\varphi}{\sqrt{\Delta}} \right)^{\infty} + \sqrt{\frac{2}{\pi}} \left( \left( \frac{\varphi}{\sqrt{\Delta}} \right)^{\infty} + 16 + 4S^2 \right) \left\| \frac{\partial_{x} \varphi}{\sqrt{\Delta}} \right\|_{\infty}^2 ; \]
\[ \tilde{C}_2 := 1.5 + 12S^2 ; \]
\[ \tilde{C}_3 := 8S^2 \left( 3 \left( \frac{\partial_{x} \varphi}{\sqrt{\Delta}} \right)^{\infty} + \frac{\varphi}{\sqrt{\Delta}} \right) \left\| \frac{\partial_{x} \varphi}{\sqrt{\Delta}} \right\|_{\infty} + 12 \left\| \frac{\partial_{x} \varphi}{\sqrt{\Delta}} \right\|_{\infty}^2 + 2 \sqrt{\frac{2}{\pi}} \frac{\varphi}{\sqrt{\Delta}} \left\| \frac{\partial_{x} \varphi}{\sqrt{\Delta}} \right\|_{\infty}^2 \right)^2 . \]
In addition, if both sequences \((\alpha_n)\) and \((v_n/\alpha_n)\) are bounded, i.e., if there exist real positive numbers \(M, \rho, \alpha\) such that \(\forall n \in \mathbb{N}^*: \alpha_n \leq M, v_n/\alpha_n \leq M/\rho\), then for all \((t, \epsilon) \in [0, 1] \times \mathcal{B}_n\):

\[
\var\left(\frac{\ln Z_{t, \epsilon}}{m_n}\right) \leq \frac{C}{m_n \rho} \tag{89}
\]

where \(C := \tilde{C}_1 + M/\rho \tilde{C}_2 + \alpha \tilde{C}_3\).

To lighten notations, we define \(k_1 := \sqrt{2R_2(t, \epsilon)}, k_2 := \sqrt{1 + 2s/2 - R_2(t, \epsilon)}\). Let \(X^* \sim P_{0,n}, \Phi := (\Phi_i) \sim \mathcal{N}(0, 1), V := (V_i)_{i=1}^m \sim \mathcal{N}(0, 1)\) and \(W^* := (W_i)_{i=1}^m \sim \mathcal{N}(0, 1)\). Remember that

\[
S_{\mu}^{(t, \epsilon)} := \sqrt{1 - \frac{\epsilon}{k_0}} (\Phi X^*)_\mu + k_1 V_\mu + k_2 W_\mu, \tag{90}
\]

and that, in the interpolation problem, we observe:

\[
\begin{align*}
\mathbb{E}_t \left( S_{\mu}^{(t, \epsilon)} \right) &\sim \varphi(S_{\mu}^{(t, \epsilon)}, \mathcal{A}_\mu) + \sqrt{\Delta} Z_\mu, \quad 1 \leq \mu \leq m_n; \\
\mathbb{E}_t \left( \hat{S}_{\mu}^{(t, \epsilon)} \right) &\sim \sqrt{\Delta} X^*_\mu + \tilde{Z}_\mu, \quad 1 \leq i \leq n,
\end{align*}
\]

where \((Z_\mu)_{\mu=1}^m, (\tilde{Z}_i)_{i=1}^n \sim \mathcal{N}(0, 1)\) and \((\mathcal{A}_\mu)_{\mu=1}^m \sim \mathcal{P}_A\). \(Z_{t, \epsilon}\) is the normalization to the joint posterior density of \((X^*, \mathcal{W}^*)\) given \((\mathcal{Y}^{(t, \epsilon)}, \mathcal{Y}^{(t, \epsilon)}), \Phi, V\), i.e.,

\[
Z_{t, \epsilon} := \int \mathcal{D}P_{0,n}(x) e^{-\frac{\sqrt{\Delta} z}{2} - \frac{\Delta}{2} (\mathcal{Y}^{(t, \epsilon)} - \Phi z)^2} P_{out}(\mathcal{Y}_{\mathcal{A}}^{(t, \epsilon)} | \mathcal{Y}^{(t, \epsilon)}) ,
\]

where \(\mathcal{D}w := \frac{dw}{\sqrt{2\pi} e^{-n/2}}\) and \(s_{\mu}^{(t, \epsilon)} := \sqrt{1 - \frac{\epsilon}{k_0}} (\Phi x)_\mu + k_1 V_\mu + k_2 w_\mu\). We define:

\[
\Gamma_{\mu}^{(t, \epsilon)} := \varphi(S_{\mu}^{(t, \epsilon)}, \mathcal{A}_\mu) - \varphi(s_{\mu}^{(t, \epsilon)}, \mathcal{A}_\mu) \frac{\Delta}{\epsilon} + \frac{\varphi}{\epsilon} (\epsilon^{(t, \epsilon)} + Z_\mu)^2. \tag{91}
\]

By definition, \(P_{out}(\mathcal{Y}_{\mathcal{A}}^{(t, \epsilon)} | \mathcal{Y}^{(t, \epsilon)}) = \int \mathcal{D}P_A(a) e^{-\frac{1}{\sqrt{2\pi} \Delta} (\mathcal{Y}^{(t, \epsilon)} - \mathcal{A}^{(t, \epsilon)})^2} (\mathcal{Y}^{(t, \epsilon)} - \mathcal{A}^{(t, \epsilon)}) = \frac{1}{\sqrt{2\pi} \Delta} e^{-\frac{1}{2} (\mathcal{Y}^{(t, \epsilon)} + Z_\mu)^2}. \)

Therefore, the interpolating free entropy satisfies:

\[
\frac{\ln Z_{t, \epsilon}}{m_n} = \frac{1}{2} \ln(2\pi \Delta) - \frac{1}{2} \frac{m_n}{m_n} \sum_{\mu=1}^{m_n} Z_\mu^2 - \frac{1}{2m_n} \sum_{i=1}^{m_n} Z_i^2 + \ln \tilde{Z}_{t, \epsilon} \tag{92}
\]

where

\[
\tilde{Z}_{t, \epsilon} := \int \mathcal{D}P_{0,n}(x) \mathcal{D}w P_A(a) e^{-\mathcal{H}_{t, \epsilon}(x, w, a)} ;
\]

\[
\mathcal{H}_{t, \epsilon}(x, w, a) := \frac{1}{2} \sum_{\mu=1}^{m_n} (\Gamma_{\mu}^{(t, \epsilon)})^2 + 2Z_\mu \Gamma_{\mu}^{(t, \epsilon)} + \frac{1}{2} \sum_{i=1}^{m_n} R_1(t, \epsilon)(X^*_i - x_i)^2 + 2Z_\mu' \sqrt{R_1(t, \epsilon)'}(X^*_i - x_i) . \tag{93}
\]

From (92), it follows directly that:

\[
\begin{align*}
\var\left(\frac{\ln Z_{t, \epsilon}}{m_n}\right) &\leq 3 \var\left(\frac{1}{2m_n} \sum_{\mu=1}^{m_n} Z_\mu^2\right) + 3 \var\left(\frac{1}{2m_n} \sum_{i=1}^{m_n} Z_i^2\right) + 3 \var\left(\frac{\ln \tilde{Z}_{t, \epsilon}}{m_n}\right) \\
&= \frac{3}{2\alpha_n n} + \frac{3}{2\alpha_n n} + 3 \var\left(\frac{\ln \tilde{Z}_{t, \epsilon}}{m_n}\right) \tag{95}
\end{align*}
\]

In order to prove Proposition 8, it remains to show that \(\ln \tilde{Z}_{t, \epsilon}/m_n\) concentrates. We recall here the classical variance bounds that we will use. We refer to [3] Chapter 3 for detailed proofs of these statements.

**Proposition 8 (Gaussian Poincaré inequality).** Let \(U = (U_1, \ldots, U_N)\) be a vector of \(N\) independent standard normal random variables. Let \(g : \mathbb{R}^N \to \mathbb{R}\) be a \(C^1\) function. Then

\[
\var(g(U)) \leq \mathbb{E}[\|\nabla g(U)\|^2] . \tag{96}
\]

**Proposition 9 (Bounded difference).** Let \(U \subset \mathbb{R}\). Let \(g : \mathbb{U}^N \to \mathbb{R}\) a function that satisfies the bounded difference property, i.e., there exists some constants \(c_1, \ldots, c_N \geq 0\) such that

\[
\sup_{u_1, \ldots, u_N, v_1, \ldots, u_N} |g(u_1, \ldots, u_i, \ldots, u_N) - g(u_1, \ldots, u_i, \ldots, u_N)| \leq c_i \quad \text{for all} \quad 1 \leq i \leq N.
\]

Let \(U = (U_1, \ldots, U_N)\) be a vector of \(N\) independent random variables that take values in \(U\). Then

\[
\var(g(U)) \leq \frac{1}{2} \sum_{i=1}^{N} c_i^2 . \tag{97}
\]
We first show the concentration w.r.t. all Gaussian variables $\rho \in g$.

**Proposition 10** (Efron-Stein inequality). Let $U \subset \mathbb{R}$, and a function $g : U^N \to \mathbb{R}$. Let $u = (U_1, \ldots, U_N)$ be a vector of $N$ independent random variables with law $P_U$ that take values in $U$. Let $U^{(i)}$ a vector which differs from $U$ only by its $i$-th component, which is replaced by $U'_i$ drawn from $P_U$ independently of $U$. Then

$$\operatorname{Var}(g(U)) \leq \frac{1}{2} \sum_{i=1}^{N} \mathbb{E}_U[\mathbb{E}_{U'_i}[(g(U) - g(U^{(i)}))^2]] .$$

(98)

We finally the one w.r.t. $A$ and finally the one w.r.t. $X^*$. The order in which we prove the concentration does matter.

We will denote $\partial_{x_i}\varphi$ and $\partial_{xx_i}\varphi$ the first and second partial derivatives of $\varphi$ with respect to its first argument. Note that $|R_1| \leq 2s_n + \frac{n}{\rho_n} n_{\max}$ and, by the inequality (39) in Lemma 7 of Appendix C, $r_{\max} := 2\left|\frac{\partial I_{\rho n}}{\partial y}\right|_{1,1} \leq 2C_1(\|\varphi\|_\infty, \|\frac{\partial^2 \varphi}{\partial x^2}\|_\infty)$ with $C_1(a, b) := (4a^2 + 1)b^2$. Then, the quantity

$$K_n := 2\left(s_n + \frac{n}{\rho_n} C_1(\|\varphi\|_\infty, \|\frac{\partial^2 \varphi}{\partial x^2}\|_\infty)\right)$$

upper bounds $|R_1|$. Besides, $|R_2|$ is upper bounded by 2.

**Concentration with respect to the Gaussian random variables**

**Lemma 5.** Let $E_{Z, \tilde{Z}}$ be the expectation w.r.t. $(Z, \tilde{Z})$ only. Under the assumptions of Theorem 1, we have for all $(t, \epsilon) \in [0,1] \times B_n$:

$$E\left[\left(\frac{\ln \tilde{Z}_{t,\epsilon}}{m_n} - \frac{1}{m_n} E_{Z, \tilde{Z}} \ln \tilde{Z}_{t,\epsilon}\right)^2\right] \leq \frac{C_2}{n\alpha_n \rho_n} + \frac{C_3}{n\alpha_n^2},$$

(99)

where $C_2 := 4\|\varphi\|_\infty^2 + 8S^2 C_1(\|\varphi\|_\infty, \|\frac{\partial^2 \varphi}{\partial x^2}\|_\infty)$ and $C_3 = 4S^2$.

**Proof.** In this proof we see $g := \ln \tilde{Z}_{t,\epsilon}/m_n$ as a function of $Z$ and $\tilde{Z}$, and we work conditionally on all other random variables. We have $\|\nabla g\|^2 = \|\nabla Z g\|^2 + \|\nabla \tilde{Z} g\|^2$. Each partial derivative has the form

$$\partial_{\mu i} g = m_n^{-1} (\partial_{\mu i} \tilde{Z}_{t,\epsilon}, t, \epsilon).$$

We find:

$$\|\nabla Z g\|^2 = n^{-2} \sum_{\mu = 1}^{m_n} (\Phi(t, \epsilon))_{\mu}^2 \leq 4m_n^{-1} \|\varphi\|_\infty^2,$$

$$\|\nabla \tilde{Z} g\|^2 = n^{-2} R_1(t, \epsilon) \sum_{i=1}^{n} (X_i^* - X_i)_{t, \epsilon}^2 \leq 4K_n S^2 m_n^{-4} n.$$

So $\|\nabla g\|^2 \leq 4m_n^{-1} (\|\varphi\|_\infty^2 + \frac{K_n S^2}{\alpha_n})$. Applying Proposition 9 yields:

$$E_{Z, \tilde{Z}}\left[\left(\frac{\ln \tilde{Z}_{t,\epsilon}}{m_n} - \frac{1}{m_n} E_{Z, \tilde{Z}} \ln \tilde{Z}_{t,\epsilon}\right)^2\right] \leq \frac{4}{n\alpha_n} \left(\|\varphi\|_\infty^2 + \frac{K_n S^2}{\alpha_n}\right)$$

$$= \frac{4}{n\alpha_n} \left(\|\varphi\|_\infty^2 + \frac{2S^2 s_n}{\alpha_n} + \frac{2S^2}{\rho_n} C_1(\|\varphi\|_\infty, \|\frac{\partial^2 \varphi}{\partial x^2}\|_\infty)\right)$$

$$\leq \frac{4}{n\alpha_n \rho_n} \left(\|\varphi\|_\infty^2 + 2S^2 C_1(\|\varphi\|_\infty, \|\frac{\partial^2 \varphi}{\partial x^2}\|_\infty)\right) + \frac{4S^2}{n\alpha_n^2}.$$

The last inequality follows from $\rho_n \leq 1$ and $2s_n \leq 1$. Taking the expectation on both sides of this last inequality gives the lemma.

**Lemma 6.** Let $E_{\Phi}$ denotes the expectation w.r.t. $(Z, \tilde{Z}, V, W^*, \Phi)$ only. Under the assumptions of Theorem 1, we have for all $(t, \epsilon) \in [0,1] \times B_n$:

$$E\left[\left(\frac{\ln \tilde{Z}_{t,\epsilon}}{m_n} - \frac{1}{m_n} E_{Z, \tilde{Z}} \ln \tilde{Z}_{t,\epsilon}\right)^2\right] \leq \frac{C_4}{n\alpha_n \rho_n} .$$

(100)

where $C_4 := (4\|\varphi\|_\infty^2 + 2\sqrt{2\pi})^2 (4 + S^2)\|\varphi\|_\infty^2$.

**Proof.** In this proof we see $g = E_{Z, \tilde{Z}} \ln \tilde{Z}_{t,\epsilon}/m_n$ as a function of $V, W^*, \Phi$ and we work conditionally on $A, X^*$. Once again each partial derivative has the form $\partial_{\mu i} g = m_n^{-1} (\partial_{\mu i} \tilde{Z}_{t,\epsilon}, t, \epsilon)$. We first compute the partial
derivatives of \( g \) w.r.t. \( \{V_\mu\}_{\mu=1}^{m_n} \):

\[
\left| \frac{\partial g}{\partial V_\mu} \right| = m_n^{-1} E_{\tilde{Z}, \tilde{Z}} \left( (\Gamma^{(t,\varepsilon)}_\mu + Z_\mu) \frac{\partial \Gamma^{(t,\varepsilon)}_\mu}{\partial V_\mu} \right) \leq m_n^{-1} E_{\tilde{Z}, \tilde{Z}} \left[ \left( 2 \left\| \frac{\varphi}{\sqrt{\Delta}} \right\|_\infty + |Z_\mu| \right) 2 \sqrt{2} \left\| \partial_V \varphi \right\|_\infty \right] = m_n^{-1} \left( 4 \left\| \frac{\varphi}{\sqrt{\Delta}} \right\|_\infty + 2 \frac{2}{\pi} \sqrt{2} \left\| \partial_V \varphi \right\|_\infty \right).
\]

The same inequality holds for \( |\frac{\partial g}{\partial \Phi_{\mu,i}}| \). To compute the derivative w.r.t. \( \Phi_{\mu,i} \), we first remark that:

\[
\frac{\partial \Gamma^{(t,\varepsilon)}_\mu}{\partial \Phi_{\mu,i}} = \sqrt{\frac{1 - t}{\Delta_n}} \left\{ X_i^t \partial_x \varphi \left( \sqrt{\frac{1 - t}{k_n}} (\Phi X^t)_\mu + k_1 V_\mu + k_2 W_\mu, A_\mu \right) \right. - x_i \partial_x \varphi \left( \sqrt{\frac{1 - t}{k_n}} (\Phi x)_\mu + k_1 V_\mu + k_2 w_\mu, a_\mu \right) \bigg\}.
\]

Therefore:

\[
\left| \frac{\partial g}{\partial \Phi_{\mu,i}} \right| = m_n^{-1} E_{\tilde{Z}, \tilde{Z}} \left( (\Gamma^{(t,\varepsilon)}_\mu + Z_\mu) \frac{\partial \Gamma^{(t,\varepsilon)}_\mu}{\partial \Phi_{\mu,i}} \right) \leq \frac{1}{m_n \mu \rho_n} E_{\tilde{Z}, \tilde{Z}} \left[ \left( 2 \left\| \frac{\varphi}{\sqrt{\Delta}} \right\|_\infty + |Z_\mu| \right) 2 S \left\| \partial_V \varphi \right\|_\infty \right] \leq \frac{1}{m_n \mu \rho_n} \left( 4 \left\| \frac{\varphi}{\sqrt{\Delta}} \right\|_\infty + 2 \frac{2}{\pi} \sqrt{2} \left\| \partial_V \varphi \right\|_\infty \right) \left( 4 + S^2 \right) \left\| \partial_V \varphi \right\|_\infty^2.
\]

Putting together these inequalities on the partial derivatives of \( g \), we find:

\[
\|\nabla g\|^2 = \sum_{\mu=1}^{m_n} \left| \frac{\partial g}{\partial V_\mu} \right|^2 + \sum_{\mu=1}^{m_n} \sum_{i=1}^{n} \left| \frac{\partial g}{\partial \Phi_{\mu,i}} \right|^2 \leq \frac{4}{m_n} \left( 4 \left\| \frac{\varphi}{\sqrt{\Delta}} \right\|_\infty^2 + 2 \frac{2}{\pi} \sqrt{2} \left\| \partial_V \varphi \right\|_\infty^2 \right) + \frac{1}{m_n \mu \rho_n} \left( 4 \left\| \frac{\varphi}{\sqrt{\Delta}} \right\|_\infty^2 + 2 \frac{2}{\pi} \sqrt{2} \left\| \partial_V \varphi \right\|_\infty^2 \right) \left( 4 + S^2 \right) \left\| \partial_V \varphi \right\|_\infty^2.
\]

In the last inequality we used that \( \rho_n \leq 1 \). To end the proof of the lemma it remains to apply Proposition \( \text{[8]} \) as we did in Lemma \( \text{[5]} \).

**Concentration with respect to the random stream** We now apply the variance bound of Proposition \( \text{[9]} \) to show that \( E_{\tilde{Z}_{t,\varepsilon}} \ln \tilde{Z}_{t,\varepsilon}/m_n \) concentrates w.r.t. \( A \).

**Lemma 7.** Let \( E_A \) denotes the expectation w.r.t. \( A \) only. Under the assumptions of Theorem \( \text{[7]} \) we have for all \( (t, \varepsilon) \in [0, 1] \times B_n \):

\[
E \left[ \left( \frac{E_{\tilde{Z}_{t,\varepsilon}} \ln \tilde{Z}_{t,\varepsilon}}{m_n} - \frac{E_{\tilde{Z}_{t,\varepsilon}} \ln \tilde{Z}_{t,\varepsilon}}{m_n} \right) \right] \leq C_5 \frac{m_n}{m_n}.
\]

where \( C_5 := \left( 2 \left\| \frac{\varphi}{\sqrt{\Delta}} \right\|_\infty + \sqrt{2} \right) \left\| \frac{\varphi}{\sqrt{\Delta}} \right\|_\infty^2 \).

**Proof.** We see \( g = E_{\tilde{Z}_{t,\varepsilon}} \ln \tilde{Z}_{t,\varepsilon}/m_n \) as a function of \( A \) only. Let \( \nu \in \{1, \ldots, m_n\} \). We want to estimate the difference \( g(A) - g(A^{(\nu)}) \) corresponding to two configurations \( A \) and \( A^{(\nu)} \) such that \( A^{(\nu)}_\mu = A_\mu \) for \( \mu \neq \nu \) and \( A^{(\nu)} \sim P_A \) independently of everything else. We will denote \( \tilde{H}^{(\nu)}_{t,\varepsilon} \) and \( \Gamma^{(t,\varepsilon)(\nu)}_{\mu} \) the quantities \( \tilde{H}_{t,\varepsilon} \) and \( \Gamma^{(t,\varepsilon)}_{\mu} \) when \( A \) is replaced by \( A^{(\nu)} \). By Jensen’s inequality, we have:

\[
\frac{1}{m_n} E_{\tilde{Z}_{t,\varepsilon}} (\tilde{H}^{(\nu)}_{t,\varepsilon} - \tilde{H}_{t,\varepsilon}) \leq g(A) - g(A^{(\nu)}) \leq \frac{1}{m_n} E_{\tilde{Z}_{t,\varepsilon}} (\tilde{H}^{(\nu)}_{t,\varepsilon} - \tilde{H}_{t,\varepsilon}) \quad \text{(102)}
\]

where the angular brackets \( \langle - \rangle_{t,\varepsilon} \) and \( \langle - \rangle_{t,\varepsilon}^{(\nu)} \) denote expectation with respect to the distributions \( \propto dP_{0,n}(x) D\nu dP_A(a_\mu) e^{-\tilde{H}_{t,\varepsilon}(x,w,a)} \) and \( \propto dP_{0,n}(x) D\nu dP_A(a_\mu) e^{-\tilde{H}_{t,\varepsilon}^{(\nu)}(x,w,a)} \), respectively. From the definition \( \text{[52]} \) of \( \tilde{H}_{t,\varepsilon} \),

\[
\tilde{H}^{(\nu)}_{t,\varepsilon} - \tilde{H}_{t,\varepsilon} = \frac{1}{2} \left( (\Gamma^{(t,\varepsilon)(\nu)}_{\mu} - \Gamma^{(t,\varepsilon)}_{\mu})^2 + 2 Z_{\nu} (\Gamma^{(t,\varepsilon)(\nu)}_{\mu} - \Gamma^{(t,\varepsilon)}_{\mu}) \right).
\]

Note that:

\[
\left( \left( \Gamma^{(t,\varepsilon)(\nu)}_{\mu} - \Gamma^{(t,\varepsilon)}_{\mu} \right)^2 + 2 Z_{\nu} (\Gamma^{(t,\varepsilon)(\nu)}_{\mu} - \Gamma^{(t,\varepsilon)}_{\mu}) \right) \leq 8 \left\| \frac{\varphi}{\sqrt{\Delta}} \right\|_\infty^2 + 4 |Z_{\nu}| \left\| \frac{\varphi}{\sqrt{\Delta}} \right\|_\infty.
\]
We thus conclude that \( g \) satisfies the bounded difference property:

\[
\forall \nu \in \{1, \ldots, m_n\} : |g(A) - g(A^{(\nu)})| \leq \frac{2}{m_n} \left( \frac{\varphi}{\sqrt{\Delta}} \right)_\infty + \sqrt{\frac{2}{\pi}} \left( \frac{\varphi}{\sqrt{\Delta}} \right)_\infty .
\]  

(103)

To end the proof of Lemma 7, we just need to apply Proposition 9. \( \square \)

**Concentration with respect to the signal** Let \( \mathbb{E}_{\sim X^*} \equiv \mathbb{E}_{A,G} \) denote the expectation w.r.t. all quenched variables except \( X^* \). It remains to bound the variance of \( \mathbb{E}_{\sim X^*} \ln \tilde{Z}_{t,\epsilon}/m_n \) (which only depends on \( X^* \)).

**Lemma 8.** Under the assumptions of Theorem 7, we have for all \((t, \epsilon) \in [0, 1] \times \mathcal{B}_n: \)

\[
\mathbb{E} \left( \left( \frac{\mathbb{E} \ln \tilde{Z}_{t,\epsilon}|X^*}_{m_n} - \frac{\mathbb{E} \ln \tilde{Z}_{t,\epsilon}}{m_n} \right)^2 \right) \leq \frac{C_6}{np_n} + \frac{C_7 p_n}{n \alpha_n^2}
\]

where \( C_7 := 8S^2 \)

\[
C_6 := 8S^2 \left( 3 \left\| \frac{\partial \varphi}{\sqrt{\Delta}} \right\|_2^2 + \left\| \frac{\varphi}{\sqrt{\Delta}} \right\|_\infty \left\| \frac{\partial_x \varphi}{\sqrt{\Delta}} \right\|_\infty + 12 \left\| \frac{\partial^{2} \varphi}{\sqrt{\Delta}} \right\|_\infty \left\| \frac{\varphi}{\sqrt{\Delta}} \right\|_\infty + 2 \sqrt{\frac{2}{\pi}} \left\| \frac{\varphi}{\sqrt{\Delta}} \right\|_\infty \left\| \frac{\partial_x \varphi}{\sqrt{\Delta}} \right\|_\infty \right)^2
\]

**Proof.** \( g = \mathbb{E} \ln \tilde{Z}_{t,\epsilon}|X^*/m_n \) is a function of \( X^* \). For \( j \in \{1, \ldots, n\} \), we have:

\[
\frac{\partial g}{\partial X_j} = -\frac{1}{m_n} \mathbb{E} \left[ \frac{\partial \hat{H}_{t,\epsilon}}{\partial X_j} \right]_{n,t,\epsilon} \mid X^*
\]

\[
= -\frac{1}{m_n} \sqrt{\frac{1 - t}{\Delta k_n}} \sum_{\mu = 1}^{m_n} \mathbb{E} \left[ \Phi_{\mu,j} \partial_x \varphi(S_{\mu}^{(t,\epsilon)}, A_{\mu}) (\Gamma_{\mu}^{(t,\epsilon)})_{n,t,\epsilon} + Z_\mu \right] \mid X^*
\]

\[
+ \frac{1}{m_n} \mathbb{E} \left[ (R_1(t, \epsilon)X_j^* - x_j) + \sqrt{R_1(t, \epsilon)}Z_j \right]_{n,t,\epsilon} \mid X^*
\]

\[
= -\frac{1}{m_n} \sqrt{\frac{1 - t}{\Delta k_n}} \sum_{\mu = 1}^{m_n} \mathbb{E} \left[ \Phi_{\mu,j} \partial_x \varphi(S_{\mu}^{(t,\epsilon)}, A_{\mu}) (\Gamma_{\mu}^{(t,\epsilon)})_{n,t,\epsilon} + Z_\mu \right] \mid X^*
\]

\[
+ \frac{R_1(t, \epsilon)}{m_n} \mathbb{E} \left[ (X_j^* - \langle x_j \rangle_{n,t,\epsilon}) \right] \mid X^* \]  

(104)

To get the last equality we use \( \mathbb{E}[\Phi_{\mu,j} \partial_x \varphi(S_{\mu}^{(t,\epsilon)}, A_{\mu}) Z_\mu | X^*] = \mathbb{E}[\Phi_{\mu,j} \partial_x \varphi(S_{\mu}^{(t,\epsilon)}, A_{\mu}) | X^*] \mathbb{E}[Z_\mu] = 0 \) and \( \mathbb{E}\sqrt{R_1(t, \epsilon)Z_j | X^*} = 0 \). An integration by parts with respect to \( \Phi_{\mu,j} \) yields:

\[
\mathbb{E} \left[ \Phi_{\mu,j} \partial_x \varphi(S_{\mu}^{(t,\epsilon)}, A_{\mu}) (\Gamma_{\mu}^{(t,\epsilon)})_{n,t,\epsilon} \mid X^* \right]
\]

\[
= \sqrt{\frac{1 - t}{\Delta k_n}} \mathbb{E} \left[ \left( X_j^* \partial_x \varphi^2 + \varphi \partial_x \varphi \right)(S_{\mu}^{(t,\epsilon)}, A_{\mu}) \mid X^* \right]
\]

\[
- \sqrt{\frac{1 - t}{\Delta k_n}} \mathbb{E} \left[ \left( \partial_x \varphi \frac{\partial_x \varphi^2}{\partial x} \right)(S_{\mu}^{(t,\epsilon)}, A_{\mu}) \mid X^* \right]
\]

\[
- \sqrt{\frac{1 - t}{\Delta k_n}} \mathbb{E} \left[ \partial_x \varphi (S_{\mu}^{(t,\epsilon)}, A_{\mu}) (x_j \partial_x \varphi(S_{\mu}^{(t,\epsilon)}, A_{\mu}))_{n,t,\epsilon} \mid X^* \right]
\]

\[
+ \sqrt{\frac{1 - t}{\Delta k_n}} \mathbb{E} \left[ \partial_x \varphi (S_{\mu}^{(t,\epsilon)}, A_{\mu}) (\varphi(S_{\mu}^{(t,\epsilon)}, A_{\mu}))_{n,t,\epsilon} \right]
\]

\[
(X_j^* \partial_x \varphi(S_{\mu}^{(t,\epsilon)}, A_{\mu}) - x_j \partial_x \varphi(S_{\mu}^{(t,\epsilon)}, A_{\mu})) (\Gamma_{\mu}^{(t,\epsilon)} + Z_\mu)_{n,t,\epsilon} \mid X^* \]

\[
- \sqrt{\frac{1 - t}{\Delta k_n}} \mathbb{E} \left[ \partial_x \varphi (S_{\mu}^{(t,\epsilon)}, A_{\mu}) (\varphi(S_{\mu}^{(t,\epsilon)}, A_{\mu}))_{n,t,\epsilon} \right]
\]

\[
(\langle X_j^* \partial_x \varphi(S_{\mu}^{(t,\epsilon)}, A_{\mu}) - x_j \partial_x \varphi(S_{\mu}^{(t,\epsilon)}, A_{\mu}) \rangle_{\Gamma_{\mu}^{(t,\epsilon)} + Z_\mu})_{n,t,\epsilon} \mid X^* \]

It directly follows that:

\[
\mathbb{E} \left[ \Phi_{\mu,j} \partial_x \varphi(S_{\mu}^{(t,\epsilon)}, A_{\mu}) (\Gamma_{\mu}^{(t,\epsilon)})_{n,t,\epsilon} \mid X^* \right] \leq \sqrt{\frac{2}{\Delta k_n}} C_6 \]

\[
\tilde{C}_6 := 2S \left( \left\| \frac{\partial \varphi}{\sqrt{\Delta}} \right\|_\infty \left\| \frac{\varphi}{\sqrt{\Delta}} \right\|_\infty \left\| \frac{\partial_x \varphi}{\sqrt{\Delta}} \right\|_\infty + 4 \left\| \frac{\partial \varphi}{\sqrt{\Delta}} \right\|_\infty \left\| \frac{\varphi}{\sqrt{\Delta}} \right\|_\infty + 2 \sqrt{\frac{2}{\pi}} \left\| \frac{\varphi}{\sqrt{\Delta}} \right\|_\infty \left\| \frac{\partial_x \varphi}{\sqrt{\Delta}} \right\|_\infty \right)^2 .
\]

30
We used \( E \) (see Lemma 2). Proposition 2 is then a direct consequence of the following:

\[
\frac{\partial g}{\partial X_j} \leq \frac{C_0}{\kappa_n} + \frac{2SK_n}{m_n} = \frac{C_0}{\kappa_n} + \frac{2S}{m_n} \left( 2s_n + \frac{2\sigma_n}{\rho_n} \right) C_1 \left( \left\| \varphi \right\|_{\infty}, \left\| \frac{\partial \varphi}{\Delta} \right\|_{\infty} \right) + \frac{2S}{n\alpha_n}.
\]

(105)

For a fixed \( j \in \{1, \ldots, n\} \), let \( X^{(j)} \) be a vector such that \( X^{(j)} \sim X^* \) for \( i \neq j \) and \( X^{(j)} \sim P_{0,n} \) independently of everything else. By the mean-value theorem and thanks to (105), we have:

\[
\mathbb{E}_X \cdot \mathbb{E}_{X^{(j)}} \left[ (g(X^*) - g(X^{(j)}))^2 \right]
\]

\[
\leq \left( \frac{1}{n\rho_n} \right) \left( C_0 + 4SC_1 \left( \left\| \varphi \right\|_{\infty}, \left\| \frac{\partial \varphi}{\Delta} \right\|_{\infty} \right) \right) + \frac{2S}{n\alpha_n} \mathbb{E}[ (X^*_j - X^{(j)}_j)^2 ]
\]

(106)

\[
\leq \frac{4}{n^2\rho_n} \left( C_0 + 4SC_1 \left( \left\| \varphi \right\|_{\infty}, \left\| \frac{\partial \varphi}{\Delta} \right\|_{\infty} \right) \right) + \frac{16S^2\rho_n}{n^2\alpha_n}.
\]

We used \( \mathbb{E}[ (X^*_j - X^{(j)}_j)^2 ] = 2\rho_n \mathbb{E}_{X \sim P_0}[ X^2 ] - 2\rho_n^2 \mathbb{E}_{X \sim P_0}[ X ]^2 \leq 2\rho_n \mathbb{E}_{X \sim P_0}[ X^2 ] = 2\rho_n \) and Jensen’s inequality \((a+b)^2 \leq 2a^2 + 2b^2\) to get the last inequality. To end the proof it now suffices to apply Proposition [10].

\(\square\)

**Proof of Proposition** Combining Lemmas 5, 6, 7, and 8 yields:

\[
\text{var} \left( \frac{\ln Z_{t,e}}{m_n} \right) \leq \frac{C_2 + C_4}{n\alpha_n\rho_n} + \frac{C_3 + C_7\rho_n}{n\alpha_n^2} + \frac{C_5}{n\alpha_n} + \frac{C_6}{n\rho_n}.
\]

(106)

Plugging (106) back in (105) gives:

\[
\text{var} \left( \frac{\ln Z_{t,e}}{m_n} \right) \leq \frac{C_2 + C_4}{n\alpha_n\rho_n} + \frac{C_3 + C_7\rho_n + 1.5}{n\alpha_n^2} + \frac{C_5 + 1.5}{n\alpha_n} + \frac{C_6}{n\rho_n}
\]

\[
\leq \frac{C_2 + C_4 + C_5 + 1.5}{n\alpha_n\rho_n} + \frac{C_3 + C_7 + 1.5}{n\alpha_n^2} + \frac{C_6}{n\rho_n}
\]

(107)

\[
= \frac{1}{n\alpha_n\rho_n} \left( C_2 + C_4 + C_5 + 1.5 + \frac{\rho_n}{\alpha_n} (C_3 + C_7 + 1.5) + \alpha_n C_6 \right).
\]

The second inequality follows from \( \rho_n \leq 1 \). It ends the proof of Proposition 7.

**F Concentration of the overlap**

In this appendix we prove Proposition 3. Define the average free entropy \( f_{n,t}(t) := \frac{1}{m_n} \ln Z_{t,e} \). In this section we think of it as a function of \( R_1 = R_1(t,\epsilon) \) and \( R_2 = R_2(t,\epsilon) \), i.e., \((R_1, R_2) \mapsto f_{n,t}(t)\). Similarly, we also view the free entropy for a realization of the quenched variables as a function

\((R_1, R_2) \mapsto F_{n,t}(t) \equiv \frac{1}{m_n} \ln Z_{t,e}(Y_t, Y'_t, \Phi, V)\).

In this appendix, to lighten the notations, we drop the indices of the angular brackets \((-)_{n,t,e}\) and simply write (\(-\)). We denote with \( \cdot \) the scalar product between two vectors. We define:

\[
\mathcal{L} := \frac{1}{\kappa_n} \left( \frac{\| x \|^2}{2} - x \cdot X^* - \frac{x \cdot \tilde{Z}}{2\sqrt{R_1}} \right).
\]

The fluctuations of the overlap \( Q := \frac{1}{\kappa_n} X^* \cdot x \) and those of \( \mathcal{L} \) are related through the inequality:

\[
\frac{1}{4} \mathbb{E}[ (Q - \mathbb{E}(Q))^2 ] \leq \mathbb{E}[ (\mathcal{L} - \mathbb{E}(\mathcal{L}))^2 ].
\]

(108)

The proof of (108) is based on integrations by parts with respect to \( \tilde{Z} \) and a repeated use of the Nishimori identity (see Lemma 2). Proposition 2 is then a direct consequence of the following:

**Proposition 11** (Concentration of \( \mathcal{L} \) on \( \mathbb{E}(\mathcal{L}) \)). Suppose that \( \Delta > 0 \), that all of \([H1],[H2],[H3]\) hold, that \( \mathbb{E}_{X \sim P_0}[X]^2 = 1 \) and that the family of functions \((r_{t,e})_{t \in B_n}, (q_{t,e})_{t \in B_n}\) are regular. Further assume that there exist real positive numbers \( M_n, M_{\rho/\alpha}, m_{\rho/\alpha} \) such that \( \forall n \in \mathbb{N}^+ \):

\[
\alpha_n \leq M_n \quad \text{and} \quad \frac{m_{\rho/\alpha}}{n} < \frac{\rho_n}{\alpha_n} \leq M_{\rho/\alpha}.
\]
Let \((s_n)_{n \in \mathbb{N}}\) be a sequence of real numbers in \((0, 1/2]\). Define \(B_n := [s_n, 2s_n]^2\). We have \(\forall t \in [0, 1]\):
\[
\int_{B_n} \frac{dE}{\rho_n} \left( \frac{\|L - \mathbb{E}(L)\|_2}{\rho_n} \right)^\frac{4}{3} C
\]
(109)
where \(C\) is a polynomial in \((S, \|\mathcal{F}\|_{\infty}, \|\mathcal{F}\|_{\infty}, \|\mathcal{F}\|_{\infty}, M_\alpha, M_{\rho, \alpha}, m_{\rho, \alpha})\) with positive coefficients.

Because \(E(\|L - \mathbb{E}(L)\|^2) = E(\langle L - \mathbb{E}(L) \rangle^2) + E(\langle (L - \mathbb{E}(L))^2 \rangle\), Proposition 2 follows directly from the next two lemmas.

**Lemma 9** (Concentration of \(L\) on \(\mathbb{E}(L)\)). Under the assumptions of Proposition 11 \(\forall t \in [0, 1]\):
\[
\int_{B_n} \frac{dE}{\rho_n} \left( \frac{\|L - \mathbb{E}(L)\|_2}{\rho_n} \right)^\frac{4}{3} \leq \frac{1}{n \rho_n}.
\]

The second lemma states that \(L\) concentrates w.r.t. the realizations of quenched disorder variables. It is a consequence of the concentration of the free entropy (see Proposition 7 in Appendix E).

**Lemma 10** (Concentration of \(\langle L \rangle\) on \(\mathbb{E}(\langle L \rangle)\)). Under the assumptions of Proposition 2 \(\forall t \in [0, 1]\):
\[
\int_{B_n} \frac{dE}{\rho_n} \left( \frac{\|L - \mathbb{E}(L)\|_2}{\rho_n} \right)^\frac{4}{3} \leq \frac{1}{n \rho_n}.
\]

We now turn to the proof of Lemmas 9 and 10. The main ingredient will be a set of formulas for the first two partial derivatives of the free entropy w.r.t. \(L_1 = R_1(t, \epsilon)\).

From (114) we have:
\[
dF_{n, \epsilon}(t) = \frac{\rho_n}{\alpha_n} \left( \frac{\|X\|^2}{\|X\|^2} + \frac{X^* \cdot Z}{\|X\|^2} \right),
\]
\[
\frac{1}{m_n} \frac{d^2F_{n, \epsilon}(t)}{dt^2} = \left( \frac{\rho_n}{\alpha_n} \right)^2 \left( \langle L^2 \rangle - \langle L \rangle^2 \right) - \frac{1}{4m_n^2 R_1^2} \langle Z \cdot (X^* - \langle x \rangle) \rangle.
\]

Averaging (111) yields:
\[
df_{n, \epsilon}(t) = \frac{\rho_n}{\alpha_n} \left( \mathbb{E}(L) + \frac{1}{2} \right) = \frac{\rho_n}{2\alpha_n} \left( \mathbb{E}(\|x\|^2) \right) - 1.
\]

To obtain the second equality we simplified \(\mathbb{E}(\langle L \rangle)\) by using an integration by parts w.r.t. the standard Gaussian vector \(Z\) and \(\mathbb{E}(x \cdot X^*) = \mathbb{E}(\|x\|^2)\) by Nishimori identity, see Lemma 2. Averaging (112) and integrating by parts w.r.t. the standard Gaussian random vector \(Z\) gives:
\[
1 \frac{d^2f_{n, \epsilon}(t)}{dt^2} = \left( \frac{\rho_n}{\alpha_n} \right)^2 \mathbb{E}(\|L^2 \rangle - \langle L \rangle^2) \left( \frac{1}{4m_n^2 R_1^2} \mathbb{E}(\|x\|^2) - \mathbb{E}(\|x\|^2) \right).
\]

**Proof of Lemma 9** From (114) we have:
\[
\mathbb{E}(\langle L - \langle L \rangle \rangle^2) = \left( \frac{\alpha_n}{\rho_n} \right)^2 \frac{1}{m_n} \frac{d^2f_{n, \epsilon}(t)}{dt^2} + \left( \frac{\alpha_n}{\rho_n} \right)^2 \frac{1}{4m_n^2 R_1^2} \mathbb{E}(\|x\|^2) - \mathbb{E}(\|x\|^2)
\]
\[
\leq \frac{\alpha_n}{\rho_n} \frac{d^2f_{n, \epsilon}(t)}{dt^2} \left( \frac{1}{4m_n^2 R_1^2} + \frac{4m_n^2 R_1^2}{4m_n^2 R_1^2} \right) = \frac{\alpha_n}{\rho_n},
\]
(115)
where we used \(\mathbb{E}(\|x\|^2) = \mathbb{E}(\|X\|^2) = \frac{1}{n \rho_n}\) by the Nishimori identity and \(R_1 \geq 1\). Recall \(B_n := [s_n, 2s_n]^2\).

By assumption the families of functions \((q_n)_{n \in \mathbb{N}}\) and \((r_n)_{n \in \mathbb{N}}\) are regular. Therefore, \(R^t : (\epsilon_1, \epsilon_2) \mapsto (R_1(t, 1), R_2(t, \epsilon))\) is a \(C^1\)-diffeomorphism whose Jacobian determinant \(|J_{R^t}|\) satisfies \(\forall \epsilon \in B_n : |J_{R^t}(\epsilon)| \geq 1\).

Integrating (115) over \(\epsilon \in B_n\) yields:
\[
\int_{B_n} dE \mathbb{E}(\langle L - \langle L \rangle \rangle^2) \leq \frac{\alpha_n}{\rho_n} \frac{1}{m_n} \int_{B_n} dE \left[ \frac{1}{R_{\epsilon_1} R_{\epsilon_2}} \mathbb{E}(\|x\|^2) - \mathbb{E}(\|x\|^2) \right] \frac{dR_{\epsilon_1}}{dt^2} \left( \frac{1}{4m_n^2 R_1^2} + \frac{4m_n^2 R_1^2}{4m_n^2 R_1^2} \right) + \frac{1}{4m_n^2 R_1^2} \ln 2.
\]
(116)

Note that \(R^t(B_n) \subset [s_n, 2s_n + \frac{m}{m_n}] \times [s_n, 2s_n + 1]\) by definition of the interpolation functions. Thus:
\[
\int_{B_n} dE \mathbb{E}(\langle L - \langle L \rangle \rangle^2) \leq \frac{\alpha_n}{\rho_n} \frac{1}{m_n} \int_{s_n}^{2s_n + 1} dR_{\epsilon_1} \left[ \frac{dE}{\rho_n} \right] R_{\epsilon_1 = s_n} \left( \frac{1}{4m_n^2 R_1^2} + \frac{4m_n^2 R_1^2}{4m_n^2 R_1^2} \right) + \frac{1}{4m_n^2 R_1^2} \ln 2 \leq \frac{1}{n \rho_n}.
\]
(117)
Also, the last term on the right hand side of (123) satisfies:
\[
\begin{align*}
\var\left( \frac{\|X^*\|^2}{k_n} + \frac{X^* \cdot \tilde{Z}}{k_n \sqrt{R_1}} \right) &= \var\left( \frac{\|X^*\|^2}{k_n} \right) + \var\left( \frac{X^* \cdot \tilde{Z}}{k_n \sqrt{R_1}} \right) \\
&= \frac{n}{k_n^2} \var\left( X^* \right)^2 + \frac{n}{k_n^2 R_1} \var\left( X^* \cdot \tilde{Z} \right) \\
&\leq \frac{n}{k_n^2} \var\left( X^* \right)^2 + \frac{1}{n \rho_n R_1} .
\end{align*}
\]
Therefore:

\[ \mathbb{E}[(\mathcal{L} - \mathbb{E}(\mathcal{L}))^2] \leq \frac{18C_{\alpha n}}{n \rho n_\delta^2} + \frac{S^4}{4n \rho n_\delta^2} + \left( \frac{\alpha_n}{\rho n_\delta} \right)^2 C_\delta(R_1)^2 + \frac{S^2}{n \rho n_\delta} \left( \frac{12}{\delta} + \frac{36R_1}{\delta^2} \right) + \frac{S^2 + 0.25}{n \rho n_\delta R_1} \cdot \tag{127} \]

The next step is to integrate both sides of (127) over \( B_n := [s_n, 2s_n]^2 \). By assumption the families of functions \( (g_\varepsilon), \varepsilon \in B_n \) and \( (r_\varepsilon), \varepsilon \in B_n \) are regular. Therefore, \( R' : (\varepsilon_1, \varepsilon_2) \mapsto (R_1(\varepsilon_1, \varepsilon_2), R_2(\varepsilon_1, \varepsilon_2)) \) is a \( C^1 \)-diffeomorphism whose Jacobian determinant \( |J_{R'}| \) satisfies \( \forall \varepsilon \in B_n : |J_{R'}(\varepsilon)| \geq 1 \). Besides, \( R'(B_n) \subseteq [s_n, \mathcal{K}_n] \times [s_n, 2s_n + 1] \) where \( \mathcal{K}_n := 2s_n + \frac{\alpha_n}{\rho n_{\max}} \). Therefore:

\[
\int_{B_n} d\varepsilon \frac{S^2}{n \rho n_\delta} \left( \frac{12}{\delta} + \frac{36R_1(\varepsilon_1, \varepsilon_2)}{\delta^2} \right) \leq \frac{12S^2}{n \rho n_\delta} \int_{B_n} d\varepsilon \left( \frac{1}{\delta} + \frac{3K_n}{\delta^2} \right)
= \frac{12S^2}{n \rho n_\delta} s_n \left( \frac{1}{\delta} + \frac{3K_n}{\delta^2} \right) \leq 12S^2 \left( 3.5M_{\rho/\alpha} + 3r_{\max} \right) \frac{\alpha_n s_n^2}{n \rho n_{\delta}^2}. \tag{128} \]

To get the last equality we used that \( \delta + 3K_n = (\delta + 6s_n) \frac{\rho n}{\alpha n} + 3r_{\max} \frac{\rho n}{\alpha n} \leq (3.5M_{\rho/\alpha} + 3r_{\max}) \frac{\rho n}{\alpha n} \) because \( \delta < s_n \leq \frac{1}{2} \) and \( \frac{\rho n}{\alpha n} \leq M_{\rho/\alpha} \). By the change of variables \( \varepsilon \to (R_1, R_2) = R'(\varepsilon) \), we get:

\[
\int_{B_n} d\varepsilon \frac{S^2 + 0.25}{n \rho n_\delta R_1(\varepsilon_1, \varepsilon_2)} = \frac{S^2 + 0.25}{n \rho n_\delta} \int_{R'(B_n)} \frac{dR_1dR_2}{|J_{R'}((R')^{-1}(R_1, R_2))|} \frac{1}{R_1} \leq \frac{(S^2 + 0.25)(1 + s_n)}{n \rho n_\delta} \int_{s_n}^{2s_n + 1} dR_2 \int_{s_n}^{2s_n + \frac{\alpha_n}{\rho n_{\max}}} dR_1 \frac{1}{R_1} = \frac{(S^2 + 0.25)(1 + s_n)}{n \rho n_\delta} \ln(K_n) \leq \frac{1.5(S^2 + 0.25)r_{\max}\alpha_n}{n \rho n_\delta^2}. \tag{129} \]

The last inequality follows from \( \ln(K_n) \leq \ln(1 + r_{\max}\alpha_n/\rho_n) \leq r_{\max}\alpha_n/\rho_n \). It remains to upper bound the integral of \( C_\delta(R_1)^2 \). We recall that \( (C_\delta(R_1)) = C_\delta(R_1) = \tilde{f}(R_1 + \delta) - \tilde{f}(R_1 - \delta) \). We have:

\[
|\tilde{f}(R_1)| \leq \frac{\rho_n}{2\alpha_n} + \frac{S}{\alpha n \sqrt{R_1}} \mathbb{E}[\tilde{Z}_1] \leq \frac{\rho_n}{2\alpha_n} + \frac{S}{\alpha n \sqrt{s_n}}. \tag{130} \]

The first inequality uses the definition (119) and the upper bound (118). The second inequality uses the definition (118) and the upper bound (118). We have (uniformly in \( R_2 \)):

\[
\int_{B_n} d\varepsilon C_\delta(R_1(\varepsilon_1, \varepsilon_2))^2 \leq \frac{1}{\alpha n} \left( \rho_n + \frac{2S}{\sqrt{s_n - \delta}} \right) \int_{B_n} d\varepsilon C_\delta(R_1(\varepsilon_1, \varepsilon_2)) \leq \frac{1}{\alpha n} \left( \rho_n + \frac{2S}{\sqrt{s_n - \delta}} \right) \int_{R'(B_n)} \frac{dR_1dR_2}{|J_{R'}((R')^{-1}(R_1, R_2))|} C_\delta(R_1) \leq \frac{1}{\alpha n} \left( \rho_n + \frac{2S}{\sqrt{s_n - \delta}} \right) \int_{s_n}^{2s_n + 1} dR_2 \int_{s_n}^{2s_n + \frac{\alpha_n}{\rho n_{\max}}} dR_1 C_\delta(R_1) \leq \frac{1}{\alpha n} \left( \rho_n + \frac{2S}{\sqrt{s_n - \delta}} \right) \int_{s_n}^{2s_n + 1} dR_2 \left( \tilde{f}(K_n + \delta) - \tilde{f}(K_n - \delta) + \tilde{f}(s_n - \delta) - \tilde{f}(s_n + \delta) \right). \]

By the mean value theorem and the upper bound (118), we have:

\[
|\tilde{f}(R_1 - \delta) - \tilde{f}(R_1 + \delta)| \leq \frac{2\delta}{\alpha n} \left( \rho_n + \frac{2S}{\sqrt{s_n - \delta}} \right). \]

Therefore:

\[
\int_{B_n} d\varepsilon \left( \frac{\alpha_n}{\rho n_\delta} \right)^2 C_\delta(R_1(\varepsilon_1, \varepsilon_2))^2 \leq \frac{4(1 + s_n)\delta}{\alpha n} \left( \rho_n + \frac{2S}{\sqrt{s_n - \delta}} \right)^2 \leq \frac{4(1 + s_n)\delta}{\alpha n} \left( \frac{1 + 2S}{\sqrt{s_n - \delta}} \right)^2 \leq 6 \left( 1 + 2S \right)^2 \frac{\delta}{\alpha n^2 (s_n - \delta)}. \tag{131} \]

Integrating (127) over \( \varepsilon \in B_n \) and making use of (128), (129), (131) yields (using \( \int_{B_n} d\varepsilon = s_n^2 \)):

\[
\int_{B_n} d\varepsilon \mathbb{E}[(\mathcal{L} - \mathbb{E}(\mathcal{L}))^2] \leq \frac{\alpha_n s_n^2}{n \rho n_\delta^2} \left( 18C + 12S^2 \left( 3.5M_{\rho/\alpha} + 3r_{\max} \right) + \frac{S^4}{4} \frac{\alpha_n^2}{\rho n_\delta^2} + 1.5(S^2 + 0.25)r_{\max} \frac{\rho n_\delta^2}{s_n^2} + \frac{6(1 + 2S)^2}{\rho n_\delta^2 (s_n - 1)} \right). \]
Note that $\delta^2 \rho_n^2 / \alpha_n \leq M_{\rho/\alpha}$ (because $\rho_n/\alpha_n \leq M_{\rho/\alpha}$, $\alpha_n \leq 1$ and $\delta \leq 1$) and $\rho_n \delta^2 / \alpha_n^2 \leq 1$ (because $\rho_n \leq 1$ and $\delta / \alpha_n \leq 1$). Hence, the last upper bound implies:

$$\int_{B_n} d\mathcal{E} \left[ (\langle L \rangle - E(L))^2 \right] \leq C_1 \frac{\alpha_n \delta^2}{\rho_n^2 \delta^2} + C_2 \frac{1}{\rho_n^2 \left( \frac{\delta}{\rho_n} - 1 \right)},$$

where $C_1 := 18C + 12S^2 \left( 3.5M_{\rho/\alpha} + 3r_{\text{max}} \right) + \frac{\alpha_n}{\rho_n} M_{\rho/\alpha} + \frac{1}{\rho_n^2} M_{\rho/\alpha} + 1.5(S^2 + 0.25)r_{\text{max}}$ and $C_2 := 6(1 + 2S)^2$. If $\delta / \alpha_n$ vanishes when $n$ goes to infinity (which is required if we want the second term on the right-hand side of (132) to vanish) then $\frac{1}{\rho_n^2 \left( \frac{\delta}{\rho_n} - 1 \right)} = \Theta \left( \frac{\delta}{\rho_n} \right)$. Further choosing $\delta \propto \left( \frac{\alpha_n}{\rho_n} \right) ^{1/2} s_n$ yields $\delta / \rho_n = \Theta \left( \frac{\alpha_n \delta^2}{\rho_n^2 \delta^2} \right)$, i.e., both terms on the right-hand side of (132) are equivalent. Note that we can choose $\delta \propto \left( \frac{\alpha_n}{\rho_n} \right) ^{1/2} s_n$ and make sure that $\forall n \in \mathbb{N}^* : \delta \in (0, s_n)$ because there exists $m_{\rho/\alpha}$ such that $\forall n \in \mathbb{N}^* : \rho_n / \alpha_n > m_{\rho/\alpha} / n$. Plugging the choice $\delta = \left( \frac{m_{\rho/\alpha}}{\rho_n} \right) ^{1/2} s_n$ back in (132) ends the proof of the lemma:

$$\int_{B_n} d\mathcal{E} \left[ (\langle L \rangle - E(L))^2 \right] \leq C_1 \frac{1}{\rho_n^2 \left( \frac{\alpha_n}{\rho_n} \right) ^{1/2} + C_2 \frac{1}{\rho_n^2 \left( \frac{\alpha_n}{\rho_n} \right) ^{1/2} - \rho_n^2} \leq C_1 \frac{1}{\rho_n^2 \left( \frac{\alpha_n}{\rho_n} \right) ^{1/2} + C_2 \frac{1}{\rho_n^2 \left( \frac{\alpha_n}{\rho_n} \right) ^{1/2} - \rho_n^2}. $$

Proof of Proposition 4

Before proving the proposition, we recall a few definitions for reader’s convenience. We suppose that (H1), (H2), (H3) hold and that $\Delta = \mathbb{E}_{X \sim P_0} [X^2] = 1$. For all $n \in \mathbb{N}^*$, we define the interval $B_n := [s_n, 2s_n]$ where $(s_n)_{n \in \mathbb{N}^*}$ is a sequence that takes its values in $(0, 1/2]$. Let $r_{\text{max}} := -2 \partial^2 \mathcal{L}_{\text{out}} / \partial \theta^2 |_{q=1, \rho=1}$ a nonnegative real number. We have $X_i \sim P_0, A_i \sim P_A$, and $\Phi_{\mu}, \Phi_{\mu}^*, W_{\mu}, W_{\mu}^*, Z_i \sim \mathcal{N}(0, 1)$ for $i = 1 \ldots n$ and $\mu = 1 \ldots m_n$. For fixed $t \in [0, 1]$ and $R = (R_1, R_2) \in [0, +\infty) \times [0, t + 2s_n]$, consider the observations:

$$\begin{align*}
Y^{(t, R_2)}_\mu &= \varphi \left( S^{(t, R_2)}_\mu, A_\mu \right) + Z_\mu, \quad 1 \leq \mu \leq m_n \\
\bar{Y}^{(t, R_1)}_i &= \sqrt{r_{\text{R}_1}} X_i + \bar{Z}_i, \quad 1 \leq i \leq n
\end{align*}$$

where $S^{(t, R_2)}_\mu = S^{(t, R_2)}_\mu (X^*, W^*_\mu) := \sqrt{r_{\text{R}_2}} (\Phi X^*_\mu) + \sqrt{r_{\text{R}_2}} W_{\mu} + \sqrt{t + 2s_n - R_2} W^*_\mu$. The joint posterior density of $(X^*, W^*)$ given $(Y^{(t, R_2)}, \bar{Y}^{(t, R_1)}, \Phi, V)$ is:

$$dP(x, w | Y^{(t, R_2)}, \bar{Y}^{(t, R_1)}, \Phi, V) = \frac{1}{Z_{t,R}} \prod_{i=1}^n dP_{\text{out}}(x_i) e^{-\frac{1}{2} \left( \sqrt{r_{\text{R}_1}} X_i - \bar{Y}^{(t, R_1)}_i \right)^2} \prod_{\mu=1}^{m_n} dP_{\text{out}}(Y^{(t, R_2)}_\mu | S^{(t, R_2)}_\mu (x, w_\mu)),$$

where $Z_{t,R}$ is the normalization. The angular brackets $\langle \cdot \rangle_{n,t,R}$ denotes the expectation w.r.t. this posterior. The scalar overlap is the quantity $Q := \frac{1}{n} \sum_{i=1}^n X^*_i x_i$. We define:

$$F^{(n)}_2(t, R) := \mathbb{E}_{Q_{n,t,R}} \quad \text{and} \quad F^{(n)}_1(t, R) := -2 \frac{\partial^2 \mathcal{L}_{\text{out}}}{\partial \theta^2} \bigg|_{q=1, \rho=1}.$$

We now repeat and prove Proposition 4

Proposition 4 (extended). Suppose that (H1), (H2), (H3) hold and that $\Delta = \mathbb{E}_{X \sim P_0} [X^2] = 1$. For all $\epsilon \in B_n$, there exists a unique global solution $R(\cdot, \epsilon) : [0, 1] \to [0, +\infty)$ to the second-order ODE:

$$y'(t) = \left( F^{(n)}_1(t, y(t)), F^{(n)}_2(t, y(t)) \right), \quad y(0) = \epsilon.$$

This solution is continuously differentiable and its derivative $R'(\cdot, \epsilon)$ satisfies:

$$R'(0, \epsilon) \leq \left[ 0, \frac{\alpha_n}{\rho_n} r_{\text{max}} \right] \times [0, 1].$$

Besides, for all $t \in [0, 1]$, $R(t, \cdot)$ is a $C^1$-diffeomorphism from $B_n$ onto its image whose Jacobian determinant is greater than, or equal to, one:

$$\forall \epsilon \in B_n : \det J_{R(t, \cdot)}(\epsilon) \geq 1.$$
where \( J_{F(t, \cdot)} \) denotes the Jacobian matrix of \( R(t, \cdot) \).

Finally, the same statement holds if, for a fixed \( r \in [0, r_{\text{max}}] \), we instead consider the second-order ODE:

\[
y'(t) = \left( \frac{\alpha n}{\rho n} r, F_2^{(n)}(t, y(t)) \right), \quad y(0) = \epsilon.
\]

**Proof.** We only give the proof for the ODE \( y' = (F_1^{(n)}(t, y), F_2^{(n)}(t, y)) \) since the one for the ODE \( y' = \left( \frac{\alpha n}{\rho n} r, F_2^{(n)}(t, y) \right) \) is simpler and follows the same arguments.

By Jensen's inequality and Nishimori identity (see Lemma 2):

\[
\mathbb{E}(Q)_{t, R} = \frac{\mathbb{E}(\|\mathbf{x}\|^2)_{t, R}}{k_n} \leq \frac{\mathbb{E}(\|\mathbf{x}\|^2)}{k_n} = 1,
\]

i.e., \( \mathbb{E}(Q)_{t, R} \in [0, 1] \). By Lemma 1, the function \( q \mapsto I_{P_{\text{out}}}(q, 1) \) is continuously twice differentiable, concave and nonincreasing on \([0, 1]\). Therefore, \( q \mapsto -2 \partial I_{P_{\text{out}}}/\partial q \) is nonnegative and nondecreasing on \([0, 1]\), which implies \( -2 \partial I_{P_{\text{out}}}/\partial q \) is in \([0, r_{\text{max}}]\). We have thus shown that the function \( F : (t, R) \mapsto (F_1^{(n)}(t, R), F_2^{(n)}(t, R)) \) is defined on all

\[
D_n := \left\{ (t, R_1, R_2) \in [0, 1] \times [0, +\infty)^2 : R_2 \leq t + 2s_n \right\},
\]

and takes its values in \([0, \alpha n r_{\text{max}}/\rho n] \times [0, 1] \).

To invoke Cauchy-Lipschitz theorem, we have to check that \( F \) is continuous in \( t \) and uniformly Lipschitz continuous in \( R \) (meaning the Lipschitz constant is independent of \( t \)). We can show that \( F \) is continuous on \( D_n \) and that, for all \( t \in [0, 1] \), \( F(t, \cdot) \) is differentiable on \([0, +\infty) \times (0, t + 2s_n) \) thanks to the standard theorems of continuity and differentiation under the integral sign. The domination hypotheses are indeed verified because we assume that \( \text{[H1][H2]} \) hold. To check the uniform Lipschitzianity, we show that the Jacobian matrix \( J_{F(t, \cdot)}(R) \) of \( F(t, \cdot) \) is uniformly bounded in \( (t, R) \). For all \( (R_1, R_2) \in [0, +\infty) \times (0, t + 2s_n) \), we have:

\[
J_{F(t, \cdot)}(R) = \begin{bmatrix} c(t, R) & 1 \\ 1 & c(t, R) \end{bmatrix},
\]

and

\[
\left. \frac{\partial F_2^{(n)}}{\partial R_1} \right|_{t, R} = \frac{1}{k_n} \sum_{i,j=1}^n \mathbb{E}[\langle (x_i x_j)_{n, t, R} - \langle x_i \rangle_{n, t, R} \langle x_j \rangle_{n, t, R} \rangle^2];
\]

\[
\left. \frac{\partial F_2^{(n)}}{\partial R_2} \right|_{t, R} = \frac{1}{k_n} \sum_{i=1}^n \mathbb{E}[\| \langle u_{i, \mu}(t, R) \rangle_{n, t, R} - \langle u_{i, \mu}(t) \rangle \langle \rho(T_i) \rangle \|_{n, t, R}^2].
\]

The function \( u_{i, \mu}(\cdot) \) is the derivative of \( u_{i, \mu} : x \mapsto \ln P_{\text{out}}(y|x) \). Both \( \partial F_2^{(n)}/\partial R_1 \) and \( \partial F_2^{(n)}/\partial R_2 \) are clearly nonnegative. Using the assumption \( \text{[H1]} \) we easily obtain from (134) that

\[
0 \leq \left. \frac{\partial F_2^{(n)}}{\partial R_1} \right|_{t, R} \leq 4S^4 n.
\]

In the proof of Lemma 4 under the hypothesis \( \text{[H2]} \) we obtain the upper bound (138) on \( |u_{i, \mu}(x)| \). It yields \( \| \cdot \|_{\infty} + |Z_{\mu}| \| \partial \varphi \|_{\infty} \). Then, we easily see from (134) that

\[
0 \leq \left. \frac{\partial F_2^{(n)}}{\partial R_2} \right|_{t, R} \leq 8S^2(4\|\varphi\|_{\infty}^2 + 1)\|\partial \varphi\|_{\infty}^2 \frac{\alpha n}{\rho n}.
\]

Finally, by Lemma 4, \( q \mapsto \frac{\partial I_{P_{\text{out}}}}{\partial q} \) is nonnegative continuous on the interval \([0, 1]\), so it is bounded by a constant \( C \) and \( c(t, R) \in [0, 2C\alpha n/\rho n] \). Combining the later with (133), (136) and (137) shows that \( J_{F(t, \cdot)}(R) \) is uniformly bounded in \((t, R) \in \left\{ (t, R_1, R_2) \in [0, 1] \times (0, +\infty)^2 : R_2 < t + 2s_n \right\} \). By the mean-value theorem, this implies that \( F \) is uniformly Lipschitzcontinuous in \( R \).

By the Cauchy-Lipschitz theorem, for all \( \epsilon \in \mathbb{B}_n \) there exists a unique solution to the initial value problem \( y' = F(t, y), y(0) = \epsilon \) that we denote \( R(\cdot, \cdot) : [0, \delta] \rightarrow [0, +\infty)^2 \). Here \( \delta \in [0, 1] \) is such that \([0, \delta] \) is the maximal interval of existence of the solution. Because \( F \) has its image in \([0, \alpha n r_{\text{max}}/\rho n] \times [0, 1] \), we have that \( \forall t \in [0, \delta] : R(t, \epsilon) \in [s_n, 2s_n + \alpha n r_{\text{max}}/\rho n] \times [s_n, 2s_n + 1] \), which means that \( \delta = 1 \) (the solution never leaves the domain of definition of \( F \)).
Each initial condition $\epsilon \in B_n$ is tied to a unique solution $R(\cdot, \epsilon)$. This implies that the function $\epsilon \mapsto R(t, \epsilon)$ is injective. Its Jacobian determinant is given by Liouville’s formula [53, Chapter V, Corollary 3.1]:

$$
\det J_{R(t, \cdot)}(\epsilon) = \exp \int_0^t ds \left( \left. \frac{\partial F_1^{(n)}}{\partial R_1} + \frac{\partial F_2^{(n)}}{\partial R_2} \right|_{s, R(s, \epsilon)} \right) = \exp \int_0^t ds \left( c(s, R(s, \epsilon)) \frac{\partial F_2^{(n)}}{\partial R_1} \bigg|_{s, R(s, \epsilon)} + \frac{\partial F_2^{(n)}}{\partial R_2} \bigg|_{s, R(s, \epsilon)} \right).
$$

This Jacobian determinant is greater than, or equal to, one since we saw that all of $c(t, R)$, $\partial F_1^{(n)}/\partial R_1$ and $\partial F_2^{(n)}/\partial R_2$ are nonnegative. The fact that the Jacobian determinant is bounded away from 0 uniformly in $\epsilon$ implies by the inverse function theorem that the injective function $\epsilon \mapsto R(t, \epsilon)$ is a $C^1$-diffeomorphism from $B_n$ onto its image. \qed