Serre-Swan theorem for non-commutative C*-algebras.  
Revised edition\(^1\)

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**Abstract**

We generalize the Serre-Swan theorem to non-commutative C*-algebras. For a Hilbert C*-module \(X\) over a C*-algebra \(A\), we introduce a hermitian vector bundle \(E_X\) associated to \(X\). We show that there is a linear subspace \(\Gamma_X\) of the space of all holomorphic sections of \(E_X\) and a flat connection \(D\) on \(E_X\) with the following properties: (i) \(\Gamma_X\) is a Hilbert \(A\)-module with the action of \(A\) defined by \(D\), (ii) the C*-inner product of \(\Gamma_X\) is induced by the hermitian metric of \(E_X\), (iii) \(E_X\) is isomorphic to an associated bundle of an infinite dimensional Hopf bundle, (iv) \(\Gamma_X\) is isomorphic to \(X\).

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1 Introduction

The Serre-Swan theorem [9, 15, 16] is described as follows:

**Theorem 1.1** Let \(\Omega\) be a connected compact Hausdorff space and let \(C(\Omega)\) be the algebra of all complex-valued continuous functions on \(\Omega\). Assume that \(X\) is a module over \(C(\Omega)\). Then \(X\) is finitely generated projective iff there is a complex vector bundle \(E\) over \(\Omega\) such that \(X\) is isomorphic onto the module of all continuous sections of \(E\).

By Theorem [11] finitely generated projective modules over the commutative C*-algebra \(C(\Omega)\) and complex vector bundles over \(\Omega\) are in one-to-one correspondence up to isomorphism. In non-commutative geometry [6, 17], a certain module over a non-commutative C*-algebra \(A\) is treated as a non-commutative vector bundle over the non-commutative space \(A\), generalizing Theorem [11] in a sense of point-less geometry. Therefore both a non-commutative space and a non-commutative vector bundle are invisible even if one desires to look hard.

\(^1\)Original paper [11]. The essential mathematical statement is same as before.

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On the other hand, for a unital generally non-commutative C*-algebra $A$, the functional representation on a certain geometrical space is studied by [4]. We review it as follows.

**Definition 1.2** A triplet $(P,p,B)$ is the uniform Kähler bundle associated with $A$ if $P (=\text{Pure}A)$ is the set of all pure states of $A$, endowed with the $w^*$-uniformity, i.e. the uniformity which induces the $w^*$-topology, $B (=\text{Spec}A)$ is the spectrum of $A$, the set of all equivalence classes of irreducible representations of $A$, and $p$ is the natural projection from $P$ onto $B$ by the GNS representation.

For each $b \in B$, the fiber $P_b \equiv p^{-1}(b)$ is a Kähler manifold (Appendix D in [4]). Especially, if $A$ is commutative, then $P \cong B$ and it is a compact Hausdorff space. In this case, each fiber of $(P,p,B)$ is a 0-dimensional Kähler manifold. Define $C^\infty(P)$ the set of all fiberwise-smooth complex-valued functions on $P$. The product $\ast$ on $C^\infty(P)$ is defined by

$$l \ast m \equiv l \cdot m + \sqrt{-1}X_ml \quad (l,m \in C^\infty(P)) \quad (1.1)$$

where $X_l$ is the holomorphic part of the complex Hamiltonian vector field of $l$ with respect to the Kähler form on $P$. Then $C^\infty(P)$ is a * algebra with the unit 1 and the involution * by complex conjugation, which is not associative in general. Define the subset $C^\infty_u(P)$ of $C^\infty(P)$ consisting of uniformly continuous functions on $P$.

**Theorem 1.3** For a unital non-commutative C*-algebra $A$, the Gel’fand representation

$$f_A(\rho) \equiv \rho(A) \quad (A \in A, \rho \in P), \quad (1.2)$$

gives an injective * homomorphism $f$ from $A$ into $C^\infty(P)$ where $C^\infty(P)$ is endowed with the *-product in [4][4]. The norm $\|\cdot\|$ on $f(A)$ defined by

$$\|l\| \equiv \sup_{\rho \in P} |(\bar{l} \ast l)(\rho)|^{\frac{1}{2}} \quad (l \in f(A)), \quad (1.3)$$

is a C*-norm on the associative * subalgebra $f(A)$.

Furthermore $f(A)$ is precisely the subset $K_u(P)$ of $C^\infty_u(P)$ defined by

$$K_u(P) \equiv \{l \in C^\infty_u(P) : \bar{l} \ast l, l \ast \bar{l} \in C^\infty_u(P), D^2l = \bar{D}^2l = 0\} \quad (1.4)$$

where $D, \bar{D}$ are the holomorphic and anti-holomorphic part, respectively, of covariant derivative of Kähler metric defined on each fiber of $P$. In consequence, the following equivalence of C*-algebras holds:

$$A \cong K_u(P).$$
Proof. See Proposition 3.2 in [4].

By Theorem 1.3 it seems that there exists a geometry consisting of points associated with not only a commutative C*-algebra but also a non-commutative one. According to Theorem 1.3, we introduce a representation of a Hilbert C*-module as the sections of a vector bundle over \( \mathcal{P} \).

A vector space \( X \) is a Hilbert C*-module over a C*-algebra \( A \) if \( X \) is a right \( A \)-module with an \( A \)-valued inner product \( \langle \cdot | \cdot \rangle \) which satisfies

\[
\langle \eta | [\xi a] \rangle = \langle \eta | \xi \rangle a \quad \text{for each } \eta, \xi \in X \text{ and } a \in A,
\]

and \( X \) is complete with respect to the norm \( \| \cdot \| \) defined by

\[
\| \xi \| = \| \langle \xi | \xi \rangle \|^{1/2}
\]

for \( \xi \in X \).

**Definition 1.4** The triplet \((E_X, \Pi_X, \mathcal{P})\) is the atomic bundle associated with a Hilbert C*-module \( X \) over a unital C*-algebra \( A \) if it is the fiber bundle with the base space \( \mathcal{P} \) and the total space \( E_X \):

\[
E_X \equiv \bigcup_{\rho \in \mathcal{P}} E_{X,\rho}
\]

where \( \Pi_X \) is the natural projection from \( E_X \) onto \( \mathcal{P} \), and the fiber \( E_{X,\rho} \) for \( \rho \in \mathcal{P} \) is the Hilbert space defined as follows: Define the quotient vector space \( E_{X,\rho}^0 \equiv X/N_\rho \) where \( N_\rho \) is the closed subspace of \( X \) defined by \( N_\rho \equiv \{ \xi \in X : \rho(\| \xi \|^2) = 0 \} \). Define the inner product \( \langle \cdot | \cdot \rangle_\rho \) on \( E_{X,\rho}^0 \) by

\[
\langle [\xi]_\rho | [\eta]_\rho \rangle_\rho \equiv \rho(\langle \xi | \eta \rangle) \quad ([\xi]_\rho, [\eta]_\rho \in E_{X,\rho}^0)
\]

where \([\xi]_\rho \equiv \xi + N_\rho \in E_{X,\rho}^0\) for \( \xi \in X \). Let \( E_{X,\rho} \) denote the completion of \( E_{X,\rho}^0 \) by the norm \( \| \cdot \|_\rho \) associated with \( \langle \cdot | \cdot \rangle_\rho \).

We show the property of \( E_X \). Let \( \mathcal{H} \) denote a complex Hilbert space with \( 1 \leq \dim \mathcal{H} \leq \infty \). A triplet \((S(\mathcal{H}), \mu, \mathcal{P}(\mathcal{H}))\) is the Hopf (fiber) bundle over \( \mathcal{H} \) if the projective Hilbert space \( \mathcal{P}(\mathcal{H}) \) and the Hilbert sphere \( S(\mathcal{H}) \) are defined by

\[
\mathcal{P}(\mathcal{H}) \equiv (\mathcal{H} \setminus \{0\})/\mathbb{C}^\times, \quad S(\mathcal{H}) \equiv \{ z \in \mathcal{H} : \| z \| = 1 \}
\]

and the projection \( \mu \) from \( S(\mathcal{H}) \) onto \( \mathcal{P}(\mathcal{H}) \) is defined by \( \mu(z) \equiv [z] \) for \( z \in S(\mathcal{H}) \).

**Theorem 1.5** For \( b \in B \) (= Spec\( A \)), let \( \mathcal{H}_b \) be a representative of \( b \), \( E_X^b \equiv \Pi_X^{-1}(\mathcal{P}_b) \) and \( \Pi_X^b \equiv \Pi_X|_{E_X^b} \). Then \((E_X^b, \Pi_X^b, \mathcal{P}_b)\) is a locally trivial vector bundle which is isomorphic to the associated bundle of \((S(\mathcal{H}_b), \mu, \mathcal{P}(\mathcal{H}_b))\) by a certain Hilbert space \( F_X^b \).
One of our aims is a geometric realization of a Hilbert $C^\ast$-module. We illustrate the two-step fibration structure of the atomic bundle as follows:

Next, we reconstruct $X$ from $E_X$. Define the space of bounded sections

$$
\Gamma(E_X) \equiv \{ s: \mathcal{P} \to E_X \mid \Pi_X \circ s = id, \|s\| < \infty \}
$$

where the norm $\| \cdot \|$ is defined by

$$
\|s\| \equiv \sup_{\rho \in \mathcal{P}} \|s(\rho)\|_{\rho}.
$$

By these preparations, we state the following theorem which is a version of the Serre-Swan theorem generalized to non-commutative $C^\ast$-algebras.

**Theorem 1.6** Let $\mathcal{A}$ be a unital $C^\ast$-algebra with $(\mathcal{P}, p, B)$ in Definition 1.2, $f$ in (1.2) and $\mathcal{K}_u(\mathcal{P})$ in (1.4). Let $X$ be a Hilbert $\mathcal{A}$-module with $(E_X, \Pi_X, \mathcal{P})$ in Definition 1.4 and $H$ in (1.8). Then the following holds:
Let $X \times \mathcal{P}$ be the trivial bundle over $\mathcal{P}$ and define the linear map $(P_X)_* \colon \Gamma(X \times \mathcal{P}) \to \Gamma(\mathcal{E}_X)$ by \((P_X)_*(s))(\rho) \equiv [s(\rho)]_\rho\) for $s \in \Gamma(X \times \mathcal{P})$, $\rho \in \mathcal{P}$. Define the subspace $\Gamma_X$ of $\Gamma(\mathcal{E}_X)$ by
\[
\Gamma_X \equiv (P_X)_*(\Gamma_{\text{const}}(X \times \mathcal{P}))
\]
where $\Gamma_{\text{const}}(X \times \mathcal{P})$ is the set of all constant sections of $X \times \mathcal{P}$. Then any element in $\Gamma_X$ is holomorphic.

(ii) There is a flat connection $D$ on $\mathcal{E}_X$ such that $\Gamma_X$ is a Hilbert $\mathcal{K}_a(\mathcal{P})$-module with respect to the following right $*$-action
\[
s \cdot l \equiv s \cdot l + \sqrt{-1}D_X s \quad (s, l) \in \Gamma_X \times \mathcal{K}_a(\mathcal{P}) \tag{1.9}
\]
and the $C^*$-inner product $H|_{\Gamma_X \times \Gamma_X}$.

(iii) Under the identification $\mathcal{K}_a(\mathcal{P})$ with $\mathcal{A}$ by $f$, the Hilbert $\mathcal{A}$-module $\Gamma_X$ is isomorphic to $X$.

Here we summarize correspondences between geometry and algebra.

| Gel’fand representation | Serre-Swan theorem |
|-------------------------|--------------------|
| space                  | algebra            |
| $C(\Omega)$            | $\mathcal{E}_X \to \mathcal{P}$ |
| $\mathcal{K}_a(\mathcal{P})$ | $\Gamma_X$ |

where we call respectively, CG = commutative geometry as a geometry associated with commutative $C^*$-algebras, and NCG = non-commutative geometry as a geometry associated with non-commutative $C^*$-algebras according to [5]. In this way, NCG’s are realized as visible geometries with points.

In §2 we review the Hopf bundle and the uniform Kähler bundle. In §2.3 we review [4] more closely. In §3 we show Theorem 1.5. In §4 we prove Theorem 1.6.

2 Hopf bundle and uniform Kähler bundle

2.1 The Hopf bundle and its associated bundle

We review the Hopf bundle and its associated bundle. Let $\mathcal{S} \equiv (S(\mathcal{H}), \mu, \mathcal{P}(\mathcal{H}))$ be the Hopf (fiber) bundle over a Hilbert space $\mathcal{H}$ in [4]. The space $S(\mathcal{H})$ is
a real submanifold of \( \mathcal{H} \) in the relative topology. We give \( \mathcal{P}(\mathcal{H}) \) the quotient topology from \( \mathcal{H} \setminus \{0\} \subset \mathcal{H} \) by the natural projection. Then \( \mu \) is continuous and open.

We define local trivial neighborhoods of the Hopf bundle according to Appendix C in [4]. For \( h \in S(\mathcal{H}) \), define

\[
\begin{align*}
  \mathcal{V}_h &\equiv \{ [z] \in \mathcal{P}(\mathcal{H}) : \langle h | z \rangle \neq 0 \}, \\
  \mathcal{H}_h &\equiv \{ z \in \mathcal{H} : \langle h | z \rangle = 0 \}, \\
  \beta_h : \mathcal{V}_h &\rightarrow \mathcal{H}_h ; \quad \beta_h([z]) \equiv \langle h | z \rangle^{-1} \cdot z - h \quad ([z] \in \mathcal{V}_h).
\end{align*}
\]

On the holomorphic tangent space \( T_h\mathcal{P}(\mathcal{H}) \) at the local coordinate \((\mathcal{V}_h, \beta_h, \mathcal{H}_h)\) and \( \beta_h(\rho) = z \), we define the Kähler metric \( g \) and the Kähler form \( \omega \) on \( \mathcal{P}(\mathcal{H}) \) by

\[
\begin{align*}
  g^h_{\bar{v}}(\bar{v}, u) &\equiv w_z \cdot \langle v | u \rangle - w^2_z \cdot \langle v | \bar{z} \rangle \langle \bar{z} | u \rangle, \\
  \omega^h_{\bar{v}}(\bar{v}, u) &\equiv \sqrt{-1} \{ -w_z \cdot \langle v | u \rangle + w^2_z \cdot \langle v | \bar{z} \rangle \langle \bar{z} | u \rangle \}, \\
  \omega^h_{\bar{v}}(\bar{v}, u) &\equiv -\omega^h_{\bar{v}}(\bar{v}, u)
\end{align*}
\]

for \( v, u \in \mathcal{H}_h \) where \( w_z \equiv 1/(1 + ||z||^2) \) and \( \bar{x} \in H^*_h \) means the dual vector of \( x \in \mathcal{H}_h \). Then \( \mathcal{P}(\mathcal{H}) \) is a Kähler manifold with the holomorphic atlas \( \{(\mathcal{V}_h, \beta_h, \mathcal{H}_h)\}_{h \in S(\mathcal{H})} \). For \( l \in C^\infty(\mathcal{P}(\mathcal{H})) \), define the holomorphic Hamiltonian vector field \( X_l \) of \( l \) by the equation

\[
\omega^\rho((X_l)_\rho, Y_\rho) = \partial_{\bar{z}}(\overline{Y_\rho}) \quad (Y_\rho \in T_\rho \mathcal{P}(\mathcal{H}), \rho \in \mathcal{P}(\mathcal{H}))
\]

where \( \partial_{\bar{z}} \) is the anti-holomorphic differential operator on \( C^\infty(\mathcal{P}(\mathcal{H})) \) and \( T_\rho \mathcal{P}(\mathcal{H}) \) denotes the anti-holomorphic tangent space of \( \mathcal{P}(\mathcal{H}) \) at \( \rho \in \mathcal{P}(\mathcal{H}) \).

The family \( \{\mathcal{V}_h\}_{h \in S(\mathcal{H})} \) is a system of local trivial neighborhoods for \( S \) by the family \( \{\psi_h\}_{h \in S(\mathcal{H})} \) of maps defined by \( \psi_h : \mu^{-1}(\mathcal{V}_h) \rightarrow \mathcal{V}_h \times U(1) ; \)

\[
\psi_h(z) \equiv ([z], \phi_h(z)), \quad \phi_h(z) \equiv \langle z | h \rangle \cdot |\langle h | z \rangle|^{-1}.
\]

Furthermore we can verify that \( S \) is a principal \( U(1) \)-bundle.

Assume that \( F \) is a complex vector space. The fibration \( F \equiv (S(\mathcal{H}) \times U(1), F, \pi_F, \mathcal{P}(\mathcal{H})) \) is called the associated bundle of \( S \) by \( F \) if \( S(\mathcal{H}) \times U(1) \) is the set of all \( U(1) \)-orbits in the product space \( S(\mathcal{H}) \times F \) where the \( U(1) \)-action is defined by

\[
(z, f) \cdot c \equiv (cz, cf) \quad (c \in U(1), (z, f) \in S(\mathcal{H}) \times F),
\]

and the projection \( \pi_F \) from \( S(\mathcal{H}) \times U(1) \) onto \( \mathcal{P}(\mathcal{H}) \) is defined by \( \pi_F([[(x, f)]] \equiv \mu(x) \) where we denote \([[(x, f)]] \) the element in \( S(\mathcal{H}) \times U(1) \) containing \((x, f) \). The topology of \( S(\mathcal{H}) \times U(1) \) \( F \) is induced from \( S(\mathcal{H}) \times F \) by the natural projection.
For $h \in S(\mathcal{H})$, the local trivialization $\psi_{F,h}$ of $F$ at $V_h$ is defined as the map $\psi_{F,h}$ from $\pi_F^{-1}(V_h)$ to $V_h \times F$ by

$$\psi_{F,h}([z,f]) \equiv (\mu(z), \phi_{F,h}([z,f]))$$

$$\phi_{F,h}([z,f]) \equiv \phi_h(z)f.$$  

The definition of $\psi_{F,h}$ is independent of the choice of $(z,f)$.

2.2 Connection

Let $F = (S(\mathcal{H}) \times_U (1), \pi_F, \mathcal{P}(\mathcal{H}))$ be the associated bundle of the Hopf bundle $S$ by $F$ in §2.1. Let $\Gamma_{\infty}(F)$ be the linear space of all smooth sections of $F$. A connection on $F$ is a $C$-bilinear map $D$ from $\mathfrak{X}(\mathcal{P}(\mathcal{H})) \times \Gamma_{\infty}(F)$ to $\Gamma_{\infty}(F)$ which is $C^\infty(\mathcal{P}(\mathcal{H}))$-linear with respect to $\mathfrak{X}(\mathcal{P}(\mathcal{H}))$ and satisfies the Leibniz law with respect to $\Gamma_{\infty}(F)$:

$$D_Y(s \cdot l) = \partial_Y l \cdot s + l \cdot D_Y s \quad (Y \in \mathfrak{X}(\mathcal{P}(\mathcal{H})), s \in \Gamma_{\infty}(F), l \in C^\infty(\mathcal{P}(\mathcal{H}))).$$

For $Y \in \mathfrak{X}(\mathcal{P}(\mathcal{H}))$, $h \in S(\mathcal{H})$ and $\rho \in V_h$, we denote $Y^h_\rho$ the corresponding tangent vector at $\rho$ in a local chart. Assume that a connection $D$ on $F$ is written as

$$D = \partial + A.$$ 

According to the notation at the local chart, we obtain families $\{A^h_{Y,\rho} : Y \in \mathfrak{X}(\mathcal{P}(\mathcal{H})), h \in S(\mathcal{H}), \rho \in V_h\}$ of linear maps on $F$ such that $\partial_Y^{h,\rho} + A^h_{Y,\rho} = (\partial_Y + A_Y)^h_{\rho} = (\partial + A)^h_{Y,\rho}$. Then we can verify that $D$ is a connection on $F$ if and only if the following holds for each $h, h' \in S(\mathcal{H})$ with $<h|h'> \neq 0$:

$$A^h_{Y,\rho} = -\frac{\langle h|Y \rangle}{2 \langle h|z + h' \rangle} + A^h_{Y,\rho'} \quad (\rho \in V_{h',h} \cap V_h) \quad (2.5)$$

where $Y$ is a holomorphic tangent vector of $\mathcal{P}(\mathcal{H})$ at $\rho$ which is realized on $\mathcal{H}_{h'}$ and $z = \beta_{h'}(\rho)$.

A connection $D$ on $F$ is flat if the curvature $R$ of $F$ with respect to $D$ defined by $R_{Y,Z} \equiv [D_Y, D_Z] - D_{[Y,Z]}$, $(Y, Z \in \mathfrak{X}(\mathcal{P}(\mathcal{H})))$, vanishes.

**Proposition 2.1.** For $h \in S(\mathcal{H})$ and the chart $(V_h, \beta_h, \mathcal{H}_h)$ at $\rho \in \mathcal{P}(\mathcal{H})$ in §2.1, we consider the trivializing neighborhood $V_h$ for the Hopf bundle. For $Y \in \mathfrak{X}(\mathcal{P}(\mathcal{H}))$, define the operator $D_Y$ on $\Gamma_{\infty}(F)$ by

$$(D_Y s)(\rho) \equiv (\partial_Y s)(\rho) + (A_{Y,\rho} s)(\rho) \quad (\rho \in \mathcal{P}(\mathcal{H}))$$

where $A_{Y,\rho}$ is defined as the family $\{A^h_{Y,\rho} : h \in S(\mathcal{H}), \rho \in V_h\}$ of linear operators on $F$ at $(V_h, \beta_h, \mathcal{H}_h)$, by

$$A^h_{Y,\rho} v \equiv -\frac{\langle \beta_h(\rho)|Y^h \rangle}{2 + \|\beta_h(\rho)\|^2} \cdot v \quad (v \in F).$$
Then this defines a flat connection $D$ on $\mathbf{F}$.

Proof. We can verify (2.5) for \( \{ A^b_{Y, \rho} \} \). Hence $D$ is a connection. Furthermore it is straightforward to show that the curvature of $D$ vanishes.

2.3 Uniform Kähler bundle

We show a geometric characterization of the set of all pure states and the spectrum of a C*-algebra according to [4].

Definition 2.2 A triplet \((E, \mu, M)\) is called a uniform Kähler bundle if $E$ and $M$ are topological spaces and $\mu$ is an open, continuous surjection from $E$ to $M$ such that (i) the topology of $E$ is induced by a given uniformity, (ii) each fiber $E_m \equiv \mu^{-1}(m)$ is a Kähler manifold.

The local triviality of uniform Kähler bundle is not assumed. In general, the topological space $M$ is neither compact nor Hausdorff.

For uniform spaces, see Chapter 2 in [2]. Two uniform Kähler bundles \((E, \mu, M)\) and \((E', \mu', M')\) are isomorphic if there is a pair \((\beta, \phi)\) of a uniform homeomorphism $\beta$ from $E$ to $E'$ and a homeomorphism $\phi$ from $M$ to $M'$, such that $\mu' \circ \beta = \phi \circ \mu$ and any restriction $\beta|_{\mu^{-1}(m)} : \mu^{-1}(m) \to (\mu')^{-1}(\phi(m))$ is a holomorphic Kähler isometry for any $m \in M$. We call \((\beta, \phi)\) a uniform Kähler isomorphism from \((E, \mu, M)\) to \((E', \mu', M')\).

Let \((\mathcal{H}_b, \pi_b)\) be an irreducible representation of $\mathcal{A}$ belonging to $b \in B$. Then $\rho \in \mathcal{P}_b$ corresponds $[x_\rho] \in \mathcal{P}(\mathcal{H}_b)$ where $\rho = \langle x_\rho | \pi_b(\cdot) x_\rho \rangle$. Define the bijection $\tau^b$ from $\mathcal{P}_b$ onto $\mathcal{P}(\mathcal{H}_b)$ by

\[
\tau^b(\rho) \equiv [x_\rho] \quad (\rho \in \mathcal{P}_b). \quad (2.6)
\]

Then $\mathcal{P}_b$ has a Kähler manifold structure induced by $\tau^b$. Furthermore the following holds.

Theorem 2.3 (i) For a unital $C^*$-algebra $\mathcal{A}$, let \((\mathcal{P}, p, B)\) be as in Definition 1.2 and assume that $B$ is endowed with the Jacobson topology [13]. Then \((\mathcal{P}, p, B)\) is a uniform Kähler bundle.

(ii) Let $\mathcal{A}_i$ be a $C^*$-algebra with the associated uniform Kähler bundle \((\mathcal{P}_i, p_i, B_i)\) for $i = 1, 2$. Then $\mathcal{A}_1$ and $\mathcal{A}_2$ are * isomorphic if and only if \((\mathcal{P}_1, p_1, B_1)\) and \((\mathcal{P}_2, p_2, B_2)\) are isomorphic as uniform Kähler bundle.
Proof. (i) See [1, 4]. (ii) See Corollary 3.3 in [4].

By Theorem 2.3 (ii), the uniform Kähler bundle \((\mathcal{P}, p, B)\) associated with \(\mathcal{A}\) is uniquely determined up to uniform Kähler isomorphism.

By the above results, we obtain a fundamental correspondence between algebra and geometry as follows:

\[
\text{unital commutative } \mathcal{C}^*\text{-algebra} \leftrightarrow \text{compact Hausdorff space}
\]

\[
\cap
\]

\[
\text{unital generally non-commutative } \mathcal{C}^*\text{-algebra} \leftrightarrow \text{uniform Kähler bundle associated with a } \mathcal{C}^*\text{-algebra}
\]

The upper correspondence above is just the Gel’fand representation of unital commutative \(\mathcal{C}^*\)-algebras. By these correspondences, we show the infinitesimal version of the Takesaki duality of Hamiltonian vector fields on a symplectic manifold [10].

3 Proof of Theorem [1.5]

In this section, we construct the typical fiber \(F^b_X\) of \(\mathcal{E}_X\) in Theorem [1.5] and show the isomorphism among vector bundles.

In order to construct the typical fiber \(F^b_X\) of \(\mathcal{E}_X\), we define the action \(T = (t, \chi)\) of the group \(G \equiv \mathcal{U}(\mathcal{A})\) of all unitaries in \(\mathcal{A}\) on \((\mathcal{E}_X, \Pi_X, \mathcal{P})\) as follows: The action \(\chi\) of \(G\) on the base space \(P\) is defined by

\[
\chi_u(\rho) \equiv \rho \circ \text{Ad}u^* \quad (u \in G, \rho \in \mathcal{P}).
\]

The action \(t\) of \(G\) on the total space \(\mathcal{E}_X\) is defined by

\[
t_u([\xi]_\rho) \equiv [\xi u^*]_{\chi_u(\rho)} \quad (u \in G, [\xi]_\rho \in \mathcal{E}_X^\rho).
\]

It is well-defined on the whole \(\mathcal{E}_X\). We see that \(T = (t, \chi)\) is an action of \(G\) on \((\mathcal{E}_X, \Pi_X, \mathcal{P})\) by bundle automorphism. This action also preserves B-fibers \((\mathcal{E}_X^b, \Pi_X^b, \mathcal{P}_b)\) for each \(b \in B\).

For \(b \in B\), let \((\mathcal{H}, \pi)\) be a representative of \(b\). We identify \(\mathcal{P}_b\) with \(\mathcal{P}(\mathcal{H})\) by \(\tau_b\) in [2.6]. Furthermore we identify \(\pi(u)\) with \(u\) for each \(u \in G\). For the atomic bundle \((\mathcal{E}_X^b, \Pi_X^b, \mathcal{P}_b)\) and the Hopf bundle \((S(\mathcal{H}), \mu_b, \mathcal{P}_b)\) in [1.6], define their fiber product \(\mathcal{E}_X^b \times_{\mathcal{P}_b} S(\mathcal{H})\) by

\[
\mathcal{E}_X^b \times_{\mathcal{P}_b} S(\mathcal{H}) = \{(x, h) \in \mathcal{E}_X^b \times S(\mathcal{H}) : \Pi_X^b(x) = \mu_b(h)\}.
\]
Thus the action $\sigma^b$ of $G$ on $\mathcal{E}_X^b \times \mathcal{P}_b S(\mathcal{H})$ is defined by

$$
\sigma^b_u(x, h) \equiv (t_u(x), \pi_b(u)h) \quad ((x, h) \in \mathcal{E}_X^b \times \mathcal{P}_b S(\mathcal{H}), u \in G).
$$

Define

$$
F_X^b \quad \text{the set of all orbits of } G \text{ in } \mathcal{E}_X^b \times \mathcal{P}_b S(\mathcal{H})
$$

and let $O(x, h) \in F_X^b$ be the orbit of $G$ containing $(x, h) \in \mathcal{E}_X^b \times \mathcal{P}_b S(\mathcal{H})$. We see that $O(0, h) = \{(0, h') : h' \in S(\mathcal{H})\}$. We introduce the Hilbert space structure on $F_X^b$ as follows: For $h \in S(\mathcal{H})$, define the sum and the scalar product on $F_X^b$ by

$$
aO(x, h) + bO(y, h) \equiv O(ax + by, h) \quad (a, b \in \mathbb{C}, x, y \in \mathcal{E}_X^b).
$$

Then this operation is independent in the choice of $x, y$ and $h$. For $h \in S(\mathcal{H})$, define the inner product $\langle \cdot | \cdot \rangle$ on the vector space $F_X^b$ by

$$
\langle O(x, h) | O(y, h) \rangle \equiv \langle x | y \rangle_\rho \quad (x, y \in \mathcal{E}_X^b)
$$

where $\rho = \mu_b(h)$. Then $\langle O(x, h) | O(y, h) \rangle$ is independent in the choice of $x, y, \rho$ and $h$. For $h_0 \in S(\mathcal{H})$ with $\mu_b(h_0) = \rho$, define the map $R_\rho$ from $\mathcal{E}_{X, \rho}$ to $F_X^b$ by $R_\rho(x) \equiv O(x, h_0)$ for $x \in \mathcal{E}_{X, \rho}$. Then $R_\rho$ is a unitary from $\mathcal{E}_{X, \rho}$ to $F_X^b$ for each $\rho \in \mathcal{P}_b$. In this way, $F_X^b$ is a Hilbert space.

We introduce the Hilbert bundle isomorphism in Theorem 1.5. Let $F_X^b \equiv (S(\mathcal{H}) \times U(1) F_X^b, \pi_{F_X^b}, \mathcal{P}(\mathcal{H}))$ be the associated bundle of $(S(\mathcal{H}), \mu_b, \mathcal{P}(\mathcal{H}))$ by $F_X^b$.

**Lemma 3.1** Any element of $S(\mathcal{H}) \times U(1) F_X^b$ can be written as $[(h, O(x, h))]$ where $O(x, h) \in F_X^b$.

**Proof.** By definition of the associated bundle in §2.1 an element of $S(\mathcal{H}) \times U(1) F_X^b$ is the $U(1)$-orbit $[(h, O(y, k))]$. Because $(\mathcal{H}, \pi)$ is an irreducible representation of $\mathcal{A}$, the action of $G$ on $S(\mathcal{H})$ is transitive. By this and definition of $O(y, k)$, there is $u \in G$ such that $h = uk$ and $(t_u(y), h) \in O(y, k)$. Denote $x \equiv t_u(y)$. Then $O(x, h) = O(y, k)$. Hence $[(h, O(y, k))] = [(h, O(x, h))]$. $\blacksquare$

**Proof of Theorem 1.5** By Lemma 3.1 we shall denote

$$
[h, x] \equiv [(h, O(x, h))] \in S(\mathcal{H}) \times U(1) F_X^b \quad (h \in S(\mathcal{H}), x \in \mathcal{E}_X^b).
$$

Define the map $\Phi^b$ from $\mathcal{E}_X^b$ to $S(\mathcal{H}) \times U(1) F_X^b$ by

$$
\Phi^b(x) \equiv [h_x, x] \quad (x \in \mathcal{E}_X^b)
$$
where \( h_x \in \mu_b^{-1}(\Pi^b_X(x)) \). By definition of \( F^b_X \), the map \( \Phi^b \) is bijective. We obtain a set-theoretical isomorphism \((\Phi^b, \tau^b)\) of fibrations between \((E^b_X, \Pi^b_X, P_b)\) and \( F^b_X \) such that any restriction \( \Phi^b|_{E_X, \rho} \) of \( \Phi^b \) at a fiber \( E_{X, \rho} \) is a unitary from \( E_{X, \rho} \) to \( \pi^{-1}_X(\rho) \) for \( \rho \in P_b \). This unitary induces the Hilbert bundle isomorphism from \((E^b_X, \Pi^b_X, P_b)\) to \( F^b_X \).

4 Proof of Theorem 1.6

Let us summarize our notations. Let \( A \) be a unital \( C^* \)-algebra with the uniform Kähler bundle \((P, p, B)\) and let \( X \) be a Hilbert \( C^* \)-module over \( A \) with the atomic bundle \( E_X = (E_X, \Pi_X, P) \).

Fix \( b \in B \) and assume that \((\mathcal{H}, \pi)\) is a representative of \( b \). For the Hilbert space \( \mathcal{H} \), let \( \{V_h, \beta_h, \mathcal{H}_h\}_{h \in S(\mathcal{H})} \) be as in (2.1). For \( \rho \in \mathcal{V}_h \), define the vector \( \Omega^h_\rho \) in \( \mathcal{H} \) by

\[
\Omega^h_\rho \equiv \{1 + \|\beta_h(\rho)\|^2\}^{-1/2} \cdot \{\beta_h(\rho) + h\}.
\]

Then \( \rho = \langle \Omega^h_\rho | \pi(\cdot) \Omega^h_\rho \rangle \) and \( \langle h | \Omega^h_\rho \rangle > 0 \). We prepare two lemmata to prove Theorem 1.6.

**Lemma 4.1** For \( s \in \Gamma(\mathcal{E}_X) \), assume that there is a family \( \{\xi_\rho \in X : \rho \in P\} \) such that \( s(\rho) = [\xi_\rho]_\rho \in \mathcal{E}_X, \rho \) for each \( \rho \in P \) and we identify \( 
\mathcal{E}^b_X \) with \( S(\mathcal{H}) \times_{U(1)} F^b_X \) by Theorem 1.5. Let \( z = \beta_h(\rho) \) for \( h \in S(\mathcal{H}) \) such that \( \rho \in \mathcal{V}_h \). Define \( w_z \equiv 1/(1 + \|z\|^2) \) and let \( \phi_{F,h} \) be as in (2.4) for \( F = F^b_X \). Then the following equations hold:

\[
\langle e | \phi_{F,h}(s(\rho)) \rangle = \sqrt{w_z} \cdot \langle \Omega^h_\rho | \pi([\xi^\prime_\rho]_\rho) (z + h) \rangle,
\]

(4.1)

\[
\partial_Y \phi_{F,h}(s(\rho)) = \mathcal{O}(\partial_Y \xi^\prime_\rho + \xi_\rho (K^h_{Y,\rho} - 2^{-1} w_z (z^Y))|_\rho, h)
\]

(4.2)

for \( e = \mathcal{O}([\xi^\prime_\rho]_\rho, h) \in F^b_X \) where \( K^h_{Y,\rho} \in A \) is defined by

\[
\pi(K^h_{Y,\rho})(h + z) = Y
\]

(4.3)

and \( [\partial_Y \xi^\prime_\rho]_\rho \in \mathcal{E}_{X,\rho} \) is defined by \( \langle \eta|_\rho | [\partial_Y \xi^\prime_\rho]_\rho \rangle \equiv \rho(\partial_Y \eta|_{\xi^\prime_\rho}) \) for \( \eta|_\rho \in \mathcal{E}_{X,\rho} \).

**Proof.** By definition, we have that \( \phi_{F,h}(s(\rho)) = c_{z,h} \cdot \mathcal{O}([\xi^\prime_\rho]_\rho, z) \) where \( c_{z,h} \equiv \langle z|h\rangle \cdot |\langle h|z\rangle|^{-1} \). We have

\[
\langle e | \phi_{F,h}(s(\rho)) \rangle = c_{z,h} \langle \mathcal{O}([\xi^\prime_\rho]_\rho, h) | \mathcal{O}([\xi_\rho]_\rho, z) \rangle.
\]
Let \( u \in G \) such that \( \pi(u^*)z = h = \Omega^h_{\rho'} \). Then \( \mathcal{O}([\xi_\rho], z) = \mathcal{O}([\xi_\rho, u^*], \pi(u^*)z) \).
By this,
\[
\langle \mathcal{O}([\xi_\rho'], h) | \mathcal{O}([\xi_\rho], z_\rho) \rangle = \langle \Omega^h_{\rho'} | \pi_h(\langle \xi_\rho' | \xi_\rho \rangle) \pi_h(u) \Omega^h_{\rho'} \rangle = \langle \Omega^h_{\rho'} | \pi_h(\langle \xi_\rho' | \xi_\rho \rangle) z_\rho \rangle.
\]
Because \( z_\rho = c_{h,z} \Omega^h_{\rho}, \) (4.1) is verified.
By (4.1), we get
\[
\langle e | \partial_Y \phi_{F,h}(s(\rho)) \rangle = \sqrt{w_z} \cdot \left[ \langle \Omega^h_{\rho'} | \pi(\partial_Y(\langle \xi_\rho | \xi_\rho \rangle)(z + h)) \rangle + \langle \Omega^h_{\rho'} | \pi((\langle \xi_\rho | \xi_\rho \rangle)Y) \rangle \right] - 2^{-1} w_z^{-3/2} \cdot \langle \Omega^h_{\rho'} | \pi((\langle \xi_\rho | \xi_\rho \rangle)(z + h)) \rangle \langle z | Y \rangle.
\]
Hence we obtain (4.2).

For \( \xi \in X \), define the section \( s_\xi \) of \( \mathcal{E}_X \) by \( s_\xi(\rho) \equiv [\xi]_\rho \) for \( \rho \in \mathcal{P} \). Then \( \|s_\xi\| = \|\xi\| \) for every \( \xi \in X \). Define the linear isometry \( \Psi \) from \( X \) into \( \Gamma(\mathcal{E}_X) \) by
\[
\Psi(\xi) \equiv s_\xi \quad (\xi \in X).
\]

**Lemma 4.2**

(i) For each \( \xi \in X \), \( \Psi(\xi) \) belongs to \( \Gamma_\infty(\mathcal{E}_X) \) and is holomorphic.

(ii) According to Theorem 1.3, define the connection \( D \) on \( \mathcal{E}_X \) by the one in Proposition 2.1 at each fiber. Let * be as in (1.9) with respect to \( D \). Then \( \Psi(\xi) * f_A = \Psi(\xi \cdot A) \) for \( \xi \in X \) and \( A \in \mathcal{A} \).

*Proof.* Let \( \rho \in \mathcal{P}_b \) for \( b \in B \). Choose as a representative for \( b \) an irreducible representation \( (\mathcal{H}, \pi) \). Fix \( h \in S(\mathcal{H}) \) and, using the notations in (2.1), take the local trivialization \( \psi_{F,h} \) of the Hopf bundle at \( (V_h, \beta_h, \mathcal{H}_h) \) with \( \rho \in V_h \).

Let \( z \equiv \beta_h(\rho) \in \mathcal{H}_h \) and \( w_z \equiv 1/(1 + \|z\|^2) \).

(i) Applying (1.2) for \( s = s_\xi \), we obtain
\[
\partial_Y \phi_{F,h}(s_\xi(\rho)) = \mathcal{O}(\partial_Y \xi + \xi(K_{Y,\rho}^h - 2^{-1} w_z \cdot \langle z | Y \rangle) \rangle)_{\rho, h}.
\]
Owing to (1.3), the right-hand side of (4.4) is smooth with respect to \( z \). Hence \( s_\xi \) is smooth at \( \mathcal{P}_b \) for each \( b \in B \). For \( \rho_0 \in \mathcal{P}_b \), we can choose \( h_0 \in S(\mathcal{H}) \) such that \( \rho_0 = \langle h_0 | \pi(\cdot) \rangle h_0 \). Then \( \beta_{h_0}(\rho_0) = 0 \). According to the proof of Lemma 1.1, we have
\[
\langle e | \phi_{F,h_0}(s_\xi(\rho)) \rangle = \sqrt{w_z} \langle \Omega^h_{\rho'} | \pi((\langle \xi_\rho | \xi_\rho \rangle)(z + h_0)) \rangle
\]
for $z = \beta_{h_0}(\rho)$, $\rho \in \mathcal{V}_{h_0}$. For an anti-holomorphic tangent vector $\bar{Y}$ of $\mathcal{P}_b$, we have
\[ \bar{\partial}_Y \phi_{F,h}(s_\xi(\rho)) = \mathcal{O}([-2^{-1}w_2(Y) \cdot \xi_\rho, h) \]
from which follows $\bar{\partial}_Y \phi_{F,h}(s_\xi(\rho))(s_\xi(\rho))|_{z=0} = 0$. We see that the anti-holomorphic derivative of $s_\xi$ vanishes at each point in $\mathcal{P}_b$. Hence $s_\xi$ is holomorphic.
(ii) For $z \in \mathcal{H}_h$, we have
\[ \{f_\Lambda \circ \beta^{-1}_h(z) = w_z \cdot \langle(z + h)|\pi(A)(z + h). \]
Then the representation $X^h_{f_\Lambda}$ of the Hamiltonian vector field $X_{f_\Lambda}$ of $f_\Lambda$ at $(\mathcal{V}_h, \beta, \mathcal{H}_h)$ is
\[ (X^h_{f_\Lambda})_z = -\sqrt{-1}\{\pi(A)(z + h) - \langle h|\pi(A)(z + h)(z + h)\} \quad (z \in \mathcal{H}_h). \]
If we take $h$ such that $\beta_h(\rho_0) = 0$, then it holds that
\[ (X^h_{f_\Lambda})_0 = -\sqrt{-1}\{\pi(A)h - \langle h|\pi(A)h)h}. \]
The connection $D$ satisfies $\langle v|(D_{X_{f_\Lambda}}s)(\rho_0)\rangle_{\rho_0} = \partial_{\rho_0}(\langle v|s(\cdot)\rangle_{\rho_0})(X_{f_\Lambda})$ for $v \in \mathcal{E}_{X,\rho_0}$ and $s \in \Gamma_\infty(\mathcal{E}_X)$. Hence we have $(D_{X_{f_\Lambda}}s_\xi)(\rho_0) = [\xi a_{X_{f_\Lambda},0}]_{\rho_0}$ where $a_{X_{f_\Lambda},0} \in \mathcal{A}$ satisfies that
\[ \pi(a_{X_{f_\Lambda},0})h = X_{f_\Lambda} = -\sqrt{-1}(\pi(A) - \langle h|\pi(A)h)h. \]
Therefore we have $\sqrt{-1}(D_{X_{f_\Lambda}}s_\xi)(\rho_0) = s_{\xi A}(\rho_0) - s_\xi(\rho_0)f_\Lambda(\rho_0)$ from which follows
\[ (s_\xi \ast f_\Lambda)(\rho_0) = s_{\xi A}(\rho_0)f_\Lambda(\rho_0) + \sqrt{-1}(D_{X_{f_\Lambda}}s_\xi)(\rho_0) = s_{\xi A}(\rho_0). \]
Therefore we obtain the statement.

Finally, we come to prove Theorem 1.6.

Proof of Theorem 1.6
(i) By definition, we see that $\Gamma_X = \Psi(X)$. Therefore the statement follows from Lemma 4.2 (i).
(ii) Because $\Gamma_X = \Psi(X)$, $\mathcal{K}_u(\mathcal{P}) = f(\Lambda)$ and Lemma 4.2 (ii) for $D$, the linear space $\Gamma_X$ is a right $\mathcal{K}_u(\mathcal{P})$-module.

Because $\rho(\langle\xi|\xi'\rangle) = f_{\langle\xi|\xi'\rangle}(\rho)$, we see that $H(\Psi(\xi), \Psi(\xi')) = f_{\langle\xi|\xi'\rangle} \in \mathcal{K}_u(\mathcal{P})$. Hence $H(s, s') \in \mathcal{K}_u(\mathcal{P})$ for each $s, s' \in \Gamma_X$. For $\xi, \eta \in X$ and $A \in \mathcal{A}$, we can verify that $H(\rho \circ \eta, \xi \ast f_\Lambda) = \{H(\rho \circ \eta, \xi) \ast f_\Lambda\}(\rho)$ where we use $H(\rho(\Psi(\xi), \Psi(\eta)) = \rho(\langle\xi|\eta\rangle)$ for $\xi, \eta \in X$ and $\rho \in \mathcal{P}$. Hence $H(s, s' \ast l) =}$
$H(s, s') \ast l$ for each $s, s' \in \Gamma_X$ and $l \in K_u(P)$. From the property of the $\mathcal{A}$-valued inner product of $X$ and by the proof of Lemma 4.2 (i), we obtain $\|H(s, s)\|^{1/2} = \|s\|$ for each $s \in \Gamma_X$ where the norm of $H(s, s)$ is the one defined in (1.3). Hence the statement holds.

(iii) Because $H(\Psi(\xi), \Psi(\xi')) = f(\xi|\xi')$, the map $\Psi$ is an isometry from $X$ onto $\Gamma_X$. Rewrite module actions $\phi$ and $\psi$ on $X$ and $\Gamma_X$, respectively, by

$$\phi(\xi, A) \equiv \xi A, \quad \psi(s, l) \equiv s \ast l \quad (\xi \in X, A \in \mathcal{A}, s \in \Gamma_X, l \in K_u(P)).$$

Then we obtain that $\psi \circ (\Psi \times f) = \Psi \circ \phi$ by Lemma 4.2 (ii). Hence the statement holds.

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Appendix

A Example of uniform Kähler bundle

Example A.1 Assume that $\mathcal{H}$ is a separable infinite dimensional Hilbert space.

(i) Let $\mathcal{A} \equiv \mathcal{L}(\mathcal{H})$ be the $C^*$-algebra of all bounded linear operators on $\mathcal{H}$. The uniform Kähler bundle of $\mathcal{A}$ is $(\mathcal{P}(\mathcal{H}) \cup \mathcal{P}_-, p, 2^{[0,1]} \cup \{b_0\})$ where $\mathcal{P}(\mathcal{H})$ is the projective Hilbert space of $\mathcal{H}$, $\mathcal{P}_-$ is the union of a family of projective Hilbert spaces indexed by the power set of the closed interval $[0, 1]$ and $\{b_0\}$ is the one-point set corresponding to the equivalence class of identity representation $(\mathcal{H}, id_{\mathcal{L}(\mathcal{H})})$ of $\mathcal{L}(\mathcal{H})$ on $\mathcal{H}$. Since the primitive spectrum of $\mathcal{L}(\mathcal{H})$ is a two-point set, the topology of $2^{[0,1]} \cup \{b_0\}$ is equal to $\{\emptyset, 2^{[0,1]}, \{b_0\}, 2^{[0,1]} \cup \{b_0\}\}$ $\mathbb{S}$. In this way, the base space of the uniform Kähler bundle is not always a singleton when the $C^*$-algebra is type I.

(ii) For the $C^*$-algebra $\mathcal{A}$ generated by the Weyl form of the 1-dimensional canonical commutation relation $U(s)V(t) = e^{\sqrt{-ts}}V(t)U(s)$ for $s, t \in \mathbb{R}$, its uniform Kähler bundle is $(\mathcal{P}(\mathcal{H}), p, \{1pt\})$. The spectrum is a one-point set $\{1pt\}$ from von Neumann uniqueness theorem $\mathbb{M}$.

(iii) The CAR algebra $\mathcal{A}$ is a UHF algebra with the nest $\{M_{2^n}(\mathbb{C})\}_{n \in \mathbb{N}}$. The uniform Kähler bundle has the base space $2^\mathbb{N}$ and each fiber on
$2^\mathbb{N}$ is a separable infinite dimensional projective Hilbert space where $2^\mathbb{N}$ is the power set of the set $\mathbb{N}$ of all natural numbers with trivial topology, that is, the topology of $2^\mathbb{N}$ is just $\{\emptyset, 2^\mathbb{N}\}$. In general, the Jacobson topology of the spectrum of a simple $\mathrm{C}^*$-algebra is trivial.

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