FRACTAL GROWTH WITH QUENCHED DISORDER

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Abstract. In this lecture we present an overview of the physics of irreversible fractal growth process, with particular emphasis on a class of models characterized by quenched disorder. These models exhibit self-organization, with critical properties developing spontaneously, without the fine tuning of external parameters. This situation is different from the usual critical phenomena, and requires the introduction of new theoretical methods. Our approach to these problems is based on two concepts, the Fixed Scale Transformation, and the quenched-stochastic transformation, or Run Time Statistics (RTS), which maps a dynamics with quenched disorder into a stochastic process. These methods, combined together, allow us to understand the self-organized nature of models with quenched disorder and to compute analytically their critical exponents. In addition, it is also possible characterize mathematically the origin of the dynamics by avalanches and compare it with the continuous growth of other fractal models. A specific application to Invasion Percolation will be discussed. Some possible relations to glasses will also be mentioned.

1 INTRODUCTION

The introduction of the fractal geometry (Mandelbrot (1982)) has changed the way physicists look at a vast class of natural phenomena which produce irregular structures. Many models have been introduced since the early eighties trying to relate these structures to well defined physical phenomena. These are the Diffusion Limited Aggregation (DLA) (Witten and Sander (1981)), the Dielectric Breakdown Model (DBM) (Niemeyer and Pietronero (1984)), the Invasion Percolation (IP) (Wilkinson and Willemsen (1983)), the Sandpile (Bak, Tang and Wiesenfeld (1987)), the Bak and Sneppen model (BS) (Bak and Sneppen (1993)), just to give some examples. All these models lead spontaneously (for a broad range of parameters) to the development of critical properties and fractal structures.

In the last years these has been a great interest on fractal models characterized by quenched disorder. These models are generally characterized by an intermittent dynamics, with bursts of activity of any size concentrated in a region of the system (avalanches), and by memory effects induced by the presence of quenched disorder and by the dynamical rules (Paczuski, Bak and Maslov...
These memory effects in fractal growth processes with quenched disorder, may resemble an element characteristic of the spin glasses and glass dynamics. In fact, also in spin glasses, memory effects (aging) are due to the presence of quenched disorder, and are relied to the typical Kohrausch stretched exponential relaxation dynamics. However, at the moment it is hard to develop such an analogy in a more concrete way.

The study of physical phenomena leading to fractal structures can be classified by three different levels:

– **Mathematical Level: Fractal Geometry.** This is a descriptive level, at which one simply recognizes the fractal nature of the phenomena and estimates the fractal dimension $D$.
– **Physical Models.** One develops a model of fractal growth based on the physical process. This level is the analogue of the Ising model in equilibrium statistical mechanics.
– **Physical Theories.** This level corresponds to a fully understanding of the origin of fractals in nature, their self-organization etc. The corresponding level for phase transitions is the Renormalization Group.

The analogy we make with phase transitions is quite natural, because, like ordinary critical phenomena, fractal growth models are scale invariant. However, some profound differences, like irreversible, non-equilibrium dynamics and SOC, make unavoidable the development of new theoretical concepts.

Fractal physical models can be classified into two main groups:

1. Irreversible stochastic models
2. Irreversible quenched models

In the next section we discuss these models in relation with standard critical phenomena.

## 2 PHYSICAL MODELS FOR SELF-SIMILAR GROWTH

We briefly mention below some examples of these two classes of models, with a particular emphasis on models with quenched disorder.

1) Irreversible stochastic models

– **Diffusion Limited Aggregation (DLA)** (Witten and Sander (1981)). This is the first physical model of fractal growth. Particles performing a Brownian motion aggregate and form complex fractal structures.
– **Dielectric Breakdown Model (DBM)** (Niemeyer and Pietronero (1984)). Is a generalization of DLA via the relation between potential theory and random walk.
– *Sandpile Models* (Bak, Tang and Wiesenfeld (1987)). These models are inspired by the marginal stability of sandpiles. The random addition of sand grains drives the system into a stationary state with a scale invariant distribution of avalanches.

2) Irreversible Quenched Models

– *Invasion Percolation (IP)* (Wilkinson and Willemsen (1983)). This model was developed to simulate the capillary displacement of a fluid in a porous medium. The porous medium is represented by a lattice where to each bond $i$ is assigned a quenched value $x_i$ of its conductance. At each time step the dynamics of the fluid evolves by occupying the bond with the smallest conductance between all its perimeter bonds. We call this kind of dynamics *extremal dynamics*. IP is known to reproduce asymptotically the Percolation cluster of standard critical Percolation.

The main characteristics of this model are:

1. *Deterministic dynamics*. Once a realization of the quenched disorder is chosen, the dynamical rule selects in a deterministic way the bond to be invaded.

2. *Self-organization*. The process spontaneously develops scale-invariant structures and critical properties. In the limit $t \to \infty$ both long range space and time correlations appear.

3. *Avalanches*. The asymptotic dynamical evolution consists of local, scale-invariant macro-events, composed by elementary growth steps spatially and causally connected, called *avalanches*. When an avalanche stops, the activity is transferred to another region of the perimeter.

– *Bak and Sneppen Model (BS)* (Bak and Sneppen (1993)). This model is similar to IP and has the same properties exposed in points 1-3. In fact, points 1-3 are characteristic of the whole class of SOC models with quenched disorder and extremal dynamics (for a review see Paczuski, Bak and Maslov (1996)). The BS model has been introduced to describe scale free events in biological evolution.

– *Quenched models with an external modulating field*. This is a particular class of models, the prototype of which is the *Quenched Dielectric Breakdown Model (QDBM)* (Family and Zhang (1986); De Arcangelis, Hansen, Herrmann and Roux (1989)). This model is a sort of combination between IP (quenched) and DBM (stochastic). To each bond of a lattice is assigned a quenched random number $x_i$, representing the local resistivity. In addition, an external electric field $E$ is introduced. The ratio $y_i = x_i/E_i$ between quenched disorder and local electric field is the inverse of the current flowing in the bonds. At each time step the dynamics breaks the bond with the smallest $y_i$, that is to say with the biggest current. The QDBM modelizes the dielectric breakdown of a disordered solid. A similar model can be formulated to modelize the propagation of fractures in a inhomogeneous solid (Hansen, Hinrichsen, Roux, Herrmann and De Arcangelis (1990)).
All the models in these classes share some important properties, that differentiates them from ordinary critical phenomena. First of all they are characterized by an \textit{irreversible dynamics}, so that they cannot be described in hamiltonian terms, and the statistical weight of a configuration depends on the complete growth history (see. fig. 1). Quenched models have an additional problem, in that they have a deterministic dynamics, and the stochasticity enters only in the choice of the realization of the disorder. So, it is difficult to define transition probabilities for the dynamics.

Another fundamental difference, is that these models are self- organized. Their dynamics evolves spontaneously in the phase space towards an \textit{attractive} fixed point. No fine tuning of any parameter is needed. In addition, quenched models with extremal dynamics have an \textit{avalanche dynamics}, that is to say the system tends to concentrate its activity in a well localized region of the perimeter, during an avalanche. When an avalanche stops, the activity transfers to another region of the perimeter and a new avalanche starts. On the contrary, in stochastic models, like DLA, there is a continuous growth process, that is to say for large systems the probability to have two nearby subsequent growth events tends to zero. The dynamical activity is diffused at each time on the whole growth interface.

In table 1 we propose a scheme of comparison between the properties of ordinary critical phenomena and the most popular stochastic and quenched fractal growth models.
3 NEW THEORETICAL CONCEPTS

The application of the standard theoretical methods of statistical physics (field theory and renormalization group) is, in general, not possible for the main fractal growth problems, like DLA, DBM, Invasion Percolation, the sandpile model, which are characterized by an intrinsically irreversible dynamics.

Here we discuss two theoretical methods we have developed in the past few years, as a step towards the construction of a physical theory for self-organized fractal growth processes. The exposition will be colloquial, and mainly devoted to the theoretical analysis of quenched models. For more details readers should refer to the bibliography.

3.1 FIXED SCALE TRANSFORMATION (FST)

This approach combines a technique of lattice path integral, to take into account the irreversible dynamics, with the study of the scale invariant dynamics inspired by the RG theory. It permits a description of the scale invariant properties of fractal growth models.

The method focuses on the dynamics at a given scale and analyzes the nearest neighbours correlations at this scale using lattice path integral approach by which one can calculate the elements of a probability matrix, the FST matrix. The fixed point of the FST matrix gives the nearest neighbour correlations, at that scale. If one uses the scale invariant dynamics of the system, that can be obtained by Real Space Renormalization Group approaches (Cafiero, Vespignani and Pietronero (1993)), one can generalize these correlations to all scales and compute the fractal dimension (Erzan, Pietronero and Vespignani (1995)).

The basic point of the FST is the separation of the long time limit \( t \to \infty \) for the dynamical process at a given scale, from the large scale limit \( r \to \infty \), that defines the scale invariant dynamics. The interesting feature of FST is that it works at a fixed scale, so it is possible to include the fluctuation of the boundary conditions, that in systems with long range interactions, like Diffusion Limited Aggregation (DLA), have a great influence on the fractal properties. For this reason, the FST approach allows to reach a remarkable level of accuracy in the calculation of the fractal dimension.

At the moment, the FST framework, eventually combined with the RTS method that we discuss below, seems to be the only general approach to understand the self-organized critical nature of a broad class of models going from DLA, to Percolation, to sandpile, to Invasion Percolation (Erzan, Pietronero and Vespignani (1995); Cafiero, Gabrielli, Marsili, and Pietronero (1996)). This situation therefore supports the idea that these models pose new questions for which one would like to develop a common theoretical scheme.

3.2 QUENCHED-STOCHASTIC TRANSFORMATION (RTS)

As we mentioned above, quenched models with extremal dynamics, like Invasion Percolation (IP), have a deterministic dynamics. This makes it impossible to ad-
Table 1. Comparison between the properties of ordinary critical phenomena represented by the Ising model and the most popular stochastic and quenched models generating fractal or scale invariant structures in a self-organized way.

| SELF-SIMILARITY: PHYSICAL MODELS | Ising-Type (70’s) | DLA-DBM (81) | Invasion Percolation (82-83) |
|----------------------------------|------------------|--------------|-----------------------------|
| Equilibrium                      |                  |              | NON LINEAR, IRREVERSIBLE DYNAMICAL EVOLUTION. |
| Statistical Mechanics            |                  |              | Statistical weight of configurations depend on the whole growth history. In extremal models they are difficult to compute because of deterministic dynamics |
| Ergodicity                       |                  |              | CRITICAL BEHAVIOUR IS SELF-ORGANIZED |
| Boltzmann Weight                 |                  |              | ATTRACTIVE FIXED POINT |
| Standard Critical behaviour      |                  |              | Asymptotically frozen fractal structure |
| Fine Tuning: T=T_c               |                  |              | Fractal structure |
| Repulsive Fixed Point            | (T-T_c)^ν        | Long range interactions (Laplacian) | Avalanche dynamics driven by extremal statistics, with avalanches of all sizes |
| Approach to the critical point   |                  |              | Cognitive time memory |
| ξ = (T-T_c)^ν                    |                  |              | Problems: origin of SOC and avalanche dynamics. |
| Anomalous dimension exactly at T=T_c | Problem: understand the origin of fractal properties and computation of the fractal dimension D | Computation of D |
| Theory: Renormalization Group    |                  |              | distribution of avalanche sizes P(s) = s^{-τ} |
|                                  |                  |              | THEORY: NEW CONCEPTS ARE NEEDED, LIKE FST AND RTS |

dress directly these class of model with the FST method or any other microscopic theory. Here we describe a general theoretical approach which addresses the basic problems of extremal models: (i) the understanding of the scale-invariance and self-organization; (ii) the origin of the avalanche dynamics and (iii) the computation of the independent critical exponents. We will discuss in particular a specific application to Invasion Percolation (IP) (Caferio, Gabrielli, Marsili, and Pietronero (1996)), but these ideas can be easily extended to other models of this type like the Bak and Sneppen model (Marsili (1994b)).

In order to overcome the problem represented by quenched disorder, we introduced a mapping of the quenched extremal dynamics into a stochastic one.
with cognitive memory, the *quenched-stochastic transformation*, also called Run Time Statistics (RTS) (Marsili (1994a)). This approach was improved in various steps (Pietronero, Schneider and Stella (1990); Pietronero and Schneider (1990); Marsili (1994a); Marsili (1994b)) and now we can develop it into a general theoretical scheme, that we call RTS-FST method (Cafero, Gabrielli, Marsili, and Pietronero (1996)). Its essential points are:
- Quenched-stochastic transformation.
- Identification of the microscopic fixed point dynamics. This point clarifies the *SOC nature* of the problem.
- Identification of the scale invariant dynamics for block variables. This elucidates the origin of fractal structures.
- Definition of local growth rules for the extremal model. This clarifies the origin of avalanche dynamics.
- Use of the above elements in a real space scheme, like the FST, to compute analytically the relevant exponents of the model.

A general stochastic process is based on the following elements: a) a set of time dependent dynamical variables \{η_i,t\}; b) a Growth Probability Distribution (GPD) for the single growth step \{µ_i,t\}, obtained from the \{η_i,t\}; c) a rule for the evolution of the dynamical variables \(η_i,t \rightarrow η_i,t+1\).

Therefore, in order to map IP onto a stochastic process we have to: a) find the correct dynamical variables (the \{η_i,t\}'s); b) determine the GPD \{µ_i,t\} in terms of these variables; c) find the evolution rule of the \{η_i,t\}.

A simple example can be useful to get an insight into the essence of the problem. Consider two independent random variables \(X_1, X_2\), with uniform distribution \(p_0(x) = 1\) in \([0,1]\) and let us eliminate the smallest, for example \(X_2\). Clearly the probability that \(X_2 < X_1\) is 1/2. At the second “time step”, we compare the surviving variable \(X_1\) with a third, uniform, random variable \(X_3\) just added to the game and, again, we eliminate the smallest one. At first sight one might think that, since both variables are independent, the probability that \(X_1\) survives again is 1/2, but this is actually incorrect. In this case we indeed need to calculate the probability \(µ_3\) that \(X_3 < X_1\) given that \(X_2 < X_1\). This, using the rules of conditional probability, reads:

\[
µ_3 = \hat{P}(X_3 < X_1) = P(X_3 < X_1 | X_2 < X_1) = \frac{P(X_3 < X_1 \cap X_2 < X_1)}{P(X_2 < X_1)} = \frac{2}{3},
\]

where \(P(A|B)\) is the probability of the event \(A\), given that \(B\) occurred, and \(P(A \cap B)\) is the probability of occurrence of both \(A\) and \(B\). The point is that the distribution of the variable \(X_1\) is no longer uniform when it is compared with \(X_3\), even though they are independent. The information that \(X_2 < X_1\) changes in a conditional way the effective probability density \(p_1(x)\) of \(X_1\). Indeed the probability that \(x < X_1 < x + dx\) must now account for the fact that \(X_2 < x\). By imposing this condition, we get: \(p_1(x) = 2x\). An analogous calculation for the distribution of \(X_2\) gives \(p_2(x) = 2(1-x)\). Qualitatively, the event \(X_2 <
$X_1$ decreases the probability that $X_1$ has small values. On the contrary, the probability that $X_2$ is small is enhanced.

The above example contains the essential idea of the *quenched-stochastic transformation*. Extremal dynamics establishes, at each time step $t$, an order relation between quenched variables ($X_2 < X_1$ in the example). This information on the statistical properties of the variables involved in the process (active variables) can be conditionally stored in the form of their effective densities. Variables which have experienced the same dynamical history, will have the same effective density, irrespectively of their spatial position. This memory is represented by the *age* $k = t - t_0$, where $t$ is the actual time and $t_0$ is the time at which the variable became active. The effective densities $p_{k,t}(x)$ ($p_{0,t}(x) = p_{0,0}(x) = 1$) of variables of age $k$ at time $t$ are the dynamical variables of the stochastic process we are looking for.

A generalization of our simple example (Eq. 1) leads to the following equation for the growth probability $\mu_{k,t}$ of a variable of age $k$ at time $t$ (GPD):

$$
\mu_{k,t} = \int_0^1 dx \ p_{k,t}(x) \prod_{\theta} (1 - P_{\theta,t}(x))^{n_{\theta,t} - \delta_{\theta,k}},
$$

where $P_{\theta,t}(x) = \int_0^x dy p_{\theta,t}(y)$, the product is intended over all the ages of the active variables and $n_{\theta,t}$ is the number of active variables of age $\theta$ at time $t$. The meaning of this expression is that the product inside the integral takes into account of the competition of the selected variable with each of the other active variables. The density $m_{k,t}(x)$ of this (smallest) variable after its growth is conditioned by the information that it has grown, and can be computed from Eq. 2. The temporal evolution of the densities of the still active variables is then given by:

$$
p_{\theta+1,t+1}(x) = p_{\theta,t}(x) \int_0^x \frac{m_{k,t}(y)}{1 - P_{\theta,t}(y)} dy.
$$

Equations 2, 3 accomplish our goal to describe a quenched extremal process as a stochastic process with time memory. The presence of memory is enlightened by the dependence of the GPD on the parameter $k$. A mean field like expansion of Eq.(2) in the limit $t \to \infty$ gives (Marsili (1994a)): $\mu_{k,\infty} \sim \frac{1}{(k+1)^\alpha}$. This result is also confirmed by simulations (Marsili, Caldarelli and Vendruscolo (1996)). Memory is at the origin of screening effects in the GPD $\{\mu_{k,t}\}$. The power law behaviour of $\mu_{k,t}$ guarantees that screening is preserved at all scales, which is the condition to generate holes of all sizes in a growing pattern, leading to fractal structures (Cafiero, Vespignani and Pietronero (1993)).

This mapping, applied to models like IP, allows us to characterize mathematically the self-organization. In fact, the following histogram equation can be derived by the RTS equations 2, 3:

$$
\partial_t \Phi_t(x) = \beta \Omega_t \Phi_t^2(x) \left[ 1 - \frac{\omega_t}{\omega_t + 1} \Phi_t(x) \right]
$$

(4)
where \( \omega_t = \langle N_{t+1} - N_t \rangle \), \( \Omega_t = \langle N_t \rangle \). \( N_t \) is the number of interface variables at time \( t \), and \( \beta \) is the solution of: \( \beta = 1 - e^{-\beta \Omega_t} \). This equation describes the time evolution of the distribution \( \Phi_t(x) \) of quenched disorder on the growth interface. The solution of eq. 4 becomes asymptotically (fig.2) (Marsili (1994a)):

\[
\lim_{t \to \infty} \Phi_t(x) = \frac{1}{1 - p_c} \theta(x - p_c)
\]

where \( p_c \) is a critical threshold of the original extremal dynamics \( (p_c = 1/2 \text{ for } 2 \text{d bond IP}) \), in agreement with numerical simulations (Wilkinson and Willemensen (1983)). Note that, in order to obtain the asymptotic behaviour 5, no fine tuning of any parameter is needed. This clarifies the SOC nature of the problem.

The scale invariant dynamics can be shown to coincide with the microscopic one (Cafiero, Gabrielli, Marsili, and Pietronero (1996)). The RTS approach permits also to characterize the origin of avalanche dynamics and to write down a set of equations describing the evolution of a single avalanche, by a straightforward modification of equations 2, 3 (Cafiero, Gabrielli, Marsili, and Pietronero (1996)). From simulations (Maslov (1995)), and from the histogram equation, one deduces that, asymptotically, each avalanche starts with a variable (initiator) equal to the threshold \( p_c \) \( (p_c = 1/2 \text{ for } 2 \text{d bond IP}) \). All other variables in the avalanche have values smaller than \( p_c \). In view of the above arguments, the RTS equations for the local avalanche dynamics are obtained from Eqs. (2, 3) by
taking into account only the variables which become active after the initiator’s growth and by integrating in Eq.(2) only in \([0, p_c]\).

By using the equations for the local avalanche dynamics together with the FST method we have been able to compute with a very good accuracy (tipically 1−2\%, depending on approximations), the relevant critical exponents of Invasion Percolation, that is to say the fractal dimension and the avalanche exponent (Cafiero, Gabrielli, Marsili, and Pietronero (1996)). The method can be applied successfully also to the Bak and Sneppen model (Marsili (1994b)). In table 2 we show the theoretical values of the exponents of IP, that we have computed with the RTS-FST method, compared with numerical simulations.

| Table 2. Theoretical values of the fractal dimension of IP, with \((D_f^{trap})\) and without trapping \((D_f)\) and of directed IP \((D_f^{DIP})\), and of the avalanche exponent \(\tau\). These values are compared with numerical simulations. |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| \(D_f\)       | \(D_f^{trap}\)  | \(D_f^{DIP}\)  | \(\tau^{IP}\)  | \(\tau^{trap}\) |
| **RTS−FST**  | 1.8879          | 1.8544          | 1.7444          | 1.5832          | 1.5463          |
| **simul.**    | \(\sim 1.89\)  | \(\sim 1.86\)  | \(\sim 1.75\)  | \(\sim 1.60\)  | \(\sim 1.53\)  |

Recently, we have extended the RTS-FST scheme to QDBM, and we have obtained interesting, although preliminar, results, which allow us to elucidate some important characteristics of the class of models to which QDBM belongs (Cafiero, Gabrielli, Marsili, Torosantucci and Pietronero (1996)).

4 FURTHER DEVELOPMENTS

In this lecture we have discussed two recently introduced theoretical methods, the FST and the RTS. These approaches have been applied sucessfully to many models for fractal growth, and allow to make a significative step towards the formulation of a common theoretical scheme for the physics of self-organized fractal growth.

At the moment, we are studying the application of the RTS mapping to interface dynamics in quenched disorder (Sneppen (1992)), and to glassy type dynamics. An interesting work, for what concerns the last point, is and RTS-type analysis of the statistical and dynamical properties of the random walk in quenched disorder (RRW) (Vendruscolo and Marsili (1996)), which has been studied by many authors as a toy model for localization (Tosatti, Zannetti and Pietronero (1988)), depinning transitions (Bouchaud, Comtet, Georges and Le Doussal (1990)), and aging effects (Marinari and Parisi (1993)). In this work, the authors map, by using the RTS method, the RRW dynamics into a stochastic dynamics with cognitive memory and recover all the characteristics of the original model. This suggests a link between stochastic dynamics with memory and the realizations of a dynamics with quenched disorder.
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