Fast Pricing of Energy Derivatives with Mean-reverting Jump Processes

Nicola Cufaro Petroni†
Dipartimento di Matematica and TIRES, Università di Bari
INFN Sezione di Bari
via E. Orabona 4, 70125 Bari, Italy

Piergiacomo Sabino‡
Quantitative Modelling
E.ON SE
Brüsseler Platz 1, 45131 Essen, Germany

Abstract
The law of a mean-reverting (Ornstein-Uhlenbeck) process driven by a compound Poisson with exponential jumps is investigated in the context of the energy derivatives pricing. The said distribution turns out to be related to the self-decomposable gamma laws, and its density and characteristic function are here given in closed-form. Algorithms for the exact simulation of such a process are accordingly derived with the advantage of being significantly faster (at least 30 times) than those available in the literature. They are also extended to more general cases (bilateral exponential jumps, and time-dependent intensity of the Poisson process). These results are finally applied to the pricing of gas storages and swings under jump-diffusion market models, and the apparent computational advantages of the proposed procedures are emphasized.

1 Introduction and Motivation

The mathematical modeling of the day-ahead price in commodity and energy markets is supposed to capture some peculiarities like mean-reversion, seasonality and

†The views, opinions, positions or strategies expressed in this article are those of the authors and do not necessarily represent the views, opinions, positions or strategies of, and should not be attributed to E.ON SE.
‡cufaro@ba.infn.it
‡piergiacomo.sabino@eon.com
jumps. A typical approach consists in resorting to price processes driven either by a generalized Ornstein-Uhlenbeck (OU) process, or by a regime switching process. The present literature is very rich of model suggestions: Lucia and Schwartz [24], for instance, propose a one-factor Gaussian-OU with application to the Nordic Power Exchange, while a two factor version can be found in Schwartz and Smith [30] with an additional Brownian Motion (BM). Models that go beyond the Gaussian world can be found among others, in Benth et al. [1], Meyer-Brandis and Tankov [25] and Cartea and Figueroa [7]. The first two papers investigate the use of generalized OU processes, while the last one studies the modeling with a jump-diffusion OU process.

In the present paper we first analyze the properties of a mean-reverting OU process driven by a compound Poisson process with exponential jumps superposed to a standard Gaussian OU process. This combination has been investigated also by other authors: for instance Deng [13], Kluge [20] and Kjaer [19], or even Benth and Pircalabu [3] in the context of modelling wind power futures.

Our contribution consists then in the derivation of the closed-form for both the density and the cumulative distribution of such a process. In its turn this main result enables us to obtain fast algorithms for their exact simulation, along with an unbiased transition density that can be used for parameter estimation, at variance with previous biased discretization schemes. To this end, following Barndorff-Nielsen and Shephard [2], we consider a Lévy process $Z(t)$ and the generalized OU process defined by the SDE

$$dX(t) = -kX(t)dt + dZ(t) \quad X(0) = X_0 \quad P\text{-a.s.} \quad k > 0 \quad (1)$$

Here $Z(t)$ is called the Backward Driving Lévy Process (BDLP), and we will adopt the following notation: if $\mathcal{D}$ is the stationary law of $X(t)$ we will say that $X(t)$ is a $\mathcal{D}$-OU process; if on the other hand $Z(1)$ (namely the BDLP at time $t = 1$) is distributed according the id (infinitely divisible) law $\widetilde{\mathcal{D}}$, then we will say that $X(t)$ is an $\mathcal{D}$-\widetilde{\mathcal{D}} process. Now a well known result (see for instance Cont and Tankov [8], Sato [29]) is that, a given one-dimensional distribution $\mathcal{D}$ always is the stationary law of a suitable $\mathcal{D}$-\widetilde{\mathcal{D}} process if and only if $\mathcal{D}$ is self-decomposable.

We recall that a law with probability density (pdf) $f(x)$ and characteristic function (chf) $\varphi(u)$ is said to be self-decomposable (sd) (see Sato [29] or Cufaro Petroni [9]) when for every $0 < a < 1$ we can find another law with pdf $g_a(x)$ and chf $\chi_a(u)$ such that

$$\varphi(u) = \varphi(au)\chi_a(u) \quad (2)$$

We will accordingly say that a random variable (rv) $X$ with pdf $f(x)$ and chf $\varphi(u)$ is sd when its law is sd: looking at the definition this means that for every $0 < a < 1$ we can always find two independent rv’s, a $Y$ (with the same law of $X$) and a $Z_a$ (here called $a$-remainder, with pdf $g_a(x)$ and chf $\chi_a(u)$) such that

$$X \overset{d}{=} aY + Z_a \quad (3)$$
We will show in the following that the law of an $X(t)$ solution of (1) when $Z(t)$ is a compound Poisson with exponential jumps (hereafter denoted $OU-CPE$) coincides with the law of the $a$-remainder $Z_a$ of a $sd$ gamma distribution. Although a similar result has yet to be proved for other generalized $OU$ processes, in our particular case it allows to find the pdf and the chf of $X(t)$ in closed-form because the $a$-remainder of a gamma distribution turns out to be a manageable mixture of other elementary distributions. As a consequence we can write down efficient and fast algorithms to exactly simulate $OU-CPE$ processes, outperforming in so doing every other existing alternative. Moreover, with little additional effort we can extend these results both to the case of positive and negative jumps, by using either a bilateral exponential or a Laplace law, and to that of a time-dependent Poisson intensity.

We will finally illustrate the applications of these new algorithms to the pricing of gas storages and swings under a price dynamics driven by a standard $OU$ plus an $OU-CPE$ with positive and negative jumps and time-dependent intensity reflecting the concentration of spikes in certain period of the year. We show in particular that our approach can have a remarkable impact and can reduce the path simulation time of a factor larger than 30. The paper is structured as follows: the Section 2 introduces the concept of generalized $OU$ process and the standard algorithms available for the exact simulation of an $OU-CPE$ process. In the Section 3, relying on the properties of the $sd$ gamma and Erlang laws, we prove the results that turn out to be instrumental in producing the simulation algorithms detailed in the subsequent Section 4. The Section 5 consists then of their financial applications to the pricing of gas storages and swings: here we also extend our results to the case of bilateral exponential, Laplace distributed jumps with time-dependent Poisson intensity. The Section 6 finally concludes the paper with an overview of future inquiries and possible further applications.

2 $OU$ process with Compound Poisson noise

Consider a Lévy process $Z(t)$, with $Z(1)$ distributed as $\tilde{D}$, and acting as the BDLP for the generalized $OU-\tilde{D}$ process $X(t)$ solution of the SDE (1), namely

$$X(t) = x_0 e^{-kt} + \int_0^t e^{-k(t-s)} dZ(s). \quad (4)$$

According to an already recalled result, given a distribution $\mathcal{D}$ we can find an id $\tilde{\mathcal{D}}$ such that the $OU-\tilde{\mathcal{D}}$ process $X(t)$ is also $\mathcal{D}$-$OU$(i.e. admits $\mathcal{D}$ as stationary distribution), if and only if $\mathcal{D}$ is $sd$

Consider now as the BDLP of (1) a compound Poisson process $Z(t)$ with intensity $\lambda_P$ of the number process $N(t)$, and identically distributed exponential jumps $J_n \sim \mathcal{E}_1(\lambda_J)$

$$Z(t) = \sum_{n=0}^{N(t)} J_n \quad J_0 = 0 \quad P-a.s.$$
a process that we will synthetically dub $OU-CPE(k, \lambda_P, \lambda_J)$ to recall its parameters. It turns out that $Z(t)$ is a subordinator, and that the solution (4) reads now as

$$X(t) = x_0 e^{-kt} + \sum_{n=0}^{N(t)} e^{-k(t-\tau_n)} J_n$$

(5)

where $\tau_n$ ($\tau_0 = 0, P$-a.s.) are the jump times of the Poisson process $N(t)$. Following Cont and Tankov [8] and Kluge [20] it results that the chf of $X(t)$, with $X(0) = 0$ $P$-a.s., is

$$\varphi(u, t) = \left(\frac{\lambda_J - iue^{-kt}}{\lambda_J - iu}\right)^{\lambda_P}$$

(6)

As will be discussed in the next section, this coincides with the chf of the $e^{-kt}$-remainder of the gamma law $\mathcal{G}(\lambda_P/k, \lambda_J)$ which is famously sd, while its stationary distribution

$$\varphi_s(u) = \left(\frac{\lambda_J}{\lambda_J - iu}\right)^{\lambda_P}$$

is instead recovered for $t \to +\infty$ and it coincides with the chf of the previous gamma law (see also Barndorff-Nielsen and Shephard [2], Grigelionis [17]).

Now, at variance with a simple Poisson process, the simulation of an $OU$-Compound Poisson over a time grid $t_0, t_1, \ldots, t_M$ ($\Delta t_m = t_m - t_{m-1}, m = 1, \ldots, M$) requires the explicit simulation of the jump times. The exact sequential simulation of a $OU-CPE$ process can be achieved modifying Algorithm 6.2 pag. 174 in Cont and Tankov [8] as follows

**Algorithm 1.** For $m = 1, \ldots, M$:

- Generate $N(\Delta t_m)$ with intensity $\lambda_P$.
- Given $N(\Delta t_m) = n$, generate $n$ uniforms $u = (u_1, \ldots, u_n) \sim \mathcal{U}([0, 1]^n)$.
- Sort $u$, $(u[1] < \cdots < u[n])$, and set $\tau_i = \Delta t_m u[i], i = 1, \ldots, n$.
- Generate $n$ independent $J_n \sim \mathcal{E}_1(\lambda_J)$.
- Set $X(t_m) = X(t_{m-1}) e^{-k\Delta t_m} + \sum_{i=1}^{n} e^{-k(\Delta t_m - \tau_i)} J_i$.

This algorithm does not rely directly on the statistical properties described by the chf (6), but it is rather based on the process definition (5): the aim of the present paper is instead to show that it is possible to explicitly find the law of such a process and therefore generate rv’s with the same distribution of the $OU-CPE$ with apparent computational advantages.

Considering indeed for simplicity an equally-spaced time grid, one might be tempted (as often done in the framework of the previous algorithm) to adopt the
following Euler discretization scheme of (1) with the assumption that only one jump can occur within each time step with probability $\lambda P \Delta t$:

$$X_{t_m} = X_{t_{m-1}} (1 - k \Delta t) + B_m (1) Y_m, \quad m = 1, \ldots M,$$

(7)

where $B_m (1) \sim \mathcal{B}(1, \lambda P \Delta t)$ are $m$ independent Bernoulli rv’s. Taking then for simplicity $b = 1 - \lambda P \Delta t$, the chf of $B_m (1) Y_m$ is

$$\varphi_m (u, t) = b + \lambda \frac{1 - b}{\lambda - iu} = \frac{\lambda - i(1 - \lambda P \Delta t)u}{\lambda - iu}.$$

This chf however could be considered as a first order approximation of (6) only if $k = \lambda P$. On the other hand a reduction of the time step would by no means provide an improvement, and hence any calibration, or pricing of derivatives relying on the simulation of an OU-CPE with the assumption that only one jump can occur per time step would lead to wrong and biased results.

Finally, it is also worthwhile noticing that in the literature several simulation algorithms based on the knowledge of the chf are available (see for instance Devroye [15] pag 695, Devroye [14] and Barabesi and Pratelli [1]). Unfortunately however all these algorithms require some regularity conditions on the chf (absolutely integrability, absolutely continuity and absolutely integrability of first two derivatives), that are not fulfilled by the chf (6).

### 3 Self-Decomposable Laws

#### 3.1 Definitions

We have already mentioned that a law with pdf $f(x)$ and chf $\varphi(u)$ is said to be sd when for every $0 < a < 1$ we can find another law with pdf $g_a (x)$ and chf $\chi_a (u)$ such that (2) is verified. Remark that this is not at all a trivial requirement because, albeit for every $\varphi(u)$ and $0 < a < 1$ it would be always possible to take

$$\chi_a (u) = \frac{\varphi(u)}{\varphi(au)}$$

(8)

it is by no means apparent that the right-hand side would always be a good chf: this happens only under special circumstances which give meaning to the definition of self-decomposability and select an important family of laws with many relevant properties. Remark however that, while the chf $\chi_a (u)$ can be explicitly expressed in terms of $\varphi(u)$, the corresponding pdf $g_a (x)$ can not be given in a general, elementary form from $f(x)$.

We also agreed to call sd a rv $X$ when its law is sd. This also entails that for every $0 < a < 1$ we can find two independent rv’s, a $Y$ (with the same law of $X$) and a $Z_a$ (its $a$-remainder), such that (3) holds in distribution. We can look at this,
however, also from a reverse point of view: if \( Y \) is sd, and to the extent that, for \( 0 < a < 1 \), a suitable independent \( a \)-remainder \( Z_a \) is known, we can define the \( rv \) \( X \)

\[
X = aY + Z_a \quad P\text{-a.s.}
\]

being sure that it will have the same sd law of \( Y \).

### 3.2 Gamma Laws \( \mathfrak{G}(\alpha, \lambda) \)

The laws of the two gamma family \( \mathfrak{G}(\alpha, \lambda) \) (\( \alpha > 0, \lambda > 0 \)) have the following pdf and chf

\[
f_{\alpha,\lambda}(x) = \frac{\lambda}{\Gamma(\alpha)} (\lambda x)^{\alpha-1} e^{-\lambda x} \quad x > 0 \tag{9}
\]

\[
\varphi_{\alpha,\lambda}(u) = \left( \frac{\lambda}{\lambda - iu} \right)^\alpha \tag{10}
\]

In particular \( \mathfrak{G}(k, \lambda) \), with \( \alpha = k = 1, 2, \ldots \) a natural number, are the Erlang laws \( \mathfrak{E}_k(\lambda) \), and \( \mathfrak{G}(1, \lambda) \) is the usual exponential law \( \mathfrak{E}_1(\lambda) \). The \( \mathfrak{G}(\alpha, \lambda) \) laws are sd (see Grigelionis [17]), so that from (8) the law of their \( a \)-remainder \( Z_a \) has the chf

\[
\chi_a(u; \alpha, \lambda) = \frac{\varphi_{\alpha,\lambda}(u)}{\varphi_{\alpha,\lambda}(au)} \tag{11}
\]

It is apparent now from (11) that the chf (6) of an \( OU\text{-}CPE(k, \lambda_P, \lambda_J) \) is that of the \( a \)-remainder of a \( \mathfrak{G}(\alpha, \lambda) \) when we take \( a = e^{-kt} \), \( \lambda = \lambda_J \) and \( \alpha = \lambda_P/k \). Before going ahead we also recall here that the (symmetric) bilateral gamma laws \( b\mathfrak{G}(\alpha, \lambda) \) of the difference \( X - Y \) with iid \( X, Y \sim \mathfrak{G}(\alpha, \lambda) \), are sd too and have the following pdf and chf

\[
f_{\alpha,\lambda}(x) = \sqrt{\frac{2}{\pi}} \frac{\lambda}{2^\alpha \Gamma(\alpha)} (\lambda|x|)^{\alpha-1/2} K_{\alpha-\frac{1}{2}}(\lambda|x|) \tag{12}
\]

\[
\varphi_{\alpha,\lambda}(u) = \left( \frac{\lambda^2}{\lambda^2 + u^2} \right)^\alpha \tag{13}
\]

where \( K_{\alpha}(z) \) denotes the modified Bessel function of the third kind, while the law of their \( a \)-remainder \( Z_a \) has the chf

\[
\chi_a(u; \alpha, \lambda) = \left( \frac{\lambda^2 + a^2 u^2}{\lambda^2 + u^2} \right)^\alpha \tag{14}
\]

### 3.3 Polya mixtures

Consider now a \( rv \) \( S \) distributed according to a negative binomial, or Polya distribution, denoted hereafter \( \mathfrak{B}(\alpha, p) \), \( \alpha > 0, 0 < p < 1 \), namely such that

\[
P\{S = k\} = \binom{\alpha + k - 1}{k} (1-p)^{\alpha} p^k \quad k = 0, 1, \ldots
\]
Remark that, when $\alpha = n = 1, 2, \ldots$ is a natural number, the Polya distribution $\mathfrak{B}(n, p)$ coincides with the so called Pascal distribution, and in particular $\mathfrak{B}(1, p)$ is nothing else than the usual geometric distribution $(1 - p)p^k$. From the generalized binomial formula it is possible to see now that its chf is

$$\varphi_S(u) = \sum_{k=0}^{\infty} \binom{\alpha + k - 1}{k} (1 - p)^{\alpha} p^k e^{iuk} = \left( \frac{1 - p}{1 - pe^{iu}} \right)^\alpha.$$  

where the series – that certainly converges because $|p e^{iu}| = p < 1$ – has the form of an infinite Polya $\mathfrak{B}(\alpha, p)$-weighted mixture of degenerate laws.

As observed for instance in Panjer and Wilmott [26], this result can also be extended by taking the rv’s

$$Z = \sum_{j=0}^{S} X_j$$  

sums of a random number $S \sim \mathfrak{B}(\alpha, p)$ of iid rv’s $X_j$ with the common chf $\varphi_X(u)$, and $X_0 = 0$, $P$-a.s.: in this case we have indeed

$$\varphi_Z(u) = E[e^{iuZ}] = E \left[ E[e^{iuZ} | S] \right]$$  

$$= \sum_{k=0}^{\infty} \binom{\alpha + k - 1}{k} (1 - p)^{\alpha} p^k E \left[ e^{iu \sum_{j=0}^{k} X_j} \right]$$  

$$= (1 - p)^{\alpha} \sum_{k=0}^{\infty} \binom{\alpha + k - 1}{k} p^k \varphi_X(u)^k = \left( \frac{1 - p}{1 - p \varphi_X(u)} \right)^\alpha$$  

where again the series converges because $|p \varphi_X(u)| \leq p < 1$. This shows that the law of $Z$ is again an infinite Polya $\mathfrak{B}(\alpha, p)$-weighted mixture of laws $\varphi_X(u)^k$: if these laws also have a known pdf, then the law of $Z$ too has an explicit representation as a mixture of pdf’s.

By taking now $p = 1 - a$ and $X \sim \mathcal{E}_1(\lambda/a)$ an exponential with chf

$$\varphi_X(u) = \frac{\lambda}{\lambda - iau}$$  

it is easy to see from (11) and (15) that

$$\left( \frac{\lambda - iau}{\lambda - iu} \right)^\alpha = \left( \frac{a(\lambda - iau)}{(\lambda - iau) - (1 - a)\lambda} \right)^\alpha = \left( \frac{a}{1 - (1 - a)\lambda} \right)^\alpha$$  

$$= \sum_{k=0}^{\infty} \binom{\alpha + k - 1}{k} a^\alpha (1 - a)^k \left( \frac{\lambda}{\lambda - iau} \right)^k$$  

namely that the law of a gamma $a$-remainder $Z_a$ is an infinite Polya $\mathfrak{B}(\alpha, 1 - a)$-weighted mixture of Erlang laws $\mathcal{E}_k(\lambda/a)$. This distribution can also be considered
either as an Erlang law $\mathcal{E}_S(\lambda/a)$ with a Polya $\mathcal{B}(\alpha, 1-a)$-distributed random index $S$, or even as that of a sum of a Polya random number of $iid$ exponentials

$$\sum_{j=0}^{S} X_j \sim \mathcal{B}(\alpha, 1-a) \quad X_j \sim \mathcal{E}_1(\lambda/a) \quad X_0 = 0, \ P\text{-a.s.}$$

Since on the other hand from (9) the pdf’s of the Erlang laws $\mathcal{E}_k(\lambda/a)$ are known, also the pdf of the $a$-remainder $Z_a$ of a gamma law $\mathcal{G}(\alpha, \lambda)$ is the following explicit mixture plus a degenerate in $x = 0$

$$g_a(x) = a^\alpha \delta(x) + \sum_{k=1}^{\infty} \binom{\alpha + k - 1}{k} a^\alpha (1-a)^k f_{k, \lambda/a}(x) \quad x > 0 \quad (16)$$

Of course, in the practical applications, some series truncation rule must be adopted: it can however be easily fine tuned, and will above all not produce the unwanted biases linked for instance with the Euler discretization procedure recalled in the Section 2.

### 3.4 Binomial mixtures

It follows in particular from the previous subsection that for $\alpha = n = 1, 2, \ldots$ the $a$-remainder of the Erlang laws $\mathcal{G}(n, \lambda) = \mathcal{E}_n(\lambda)$ is an infinite mixture of Erlang $\mathcal{E}_k(\lambda/n)$ with Pascal weights $\mathcal{B}(n, 1-a)$, while for $n = 1$ the $a$-remainder of the exponential law $\mathcal{G}(1, \lambda) = \mathcal{E}_1(\lambda)$ is an infinite mixture of Erlang $\mathcal{E}_k(\lambda)$ with geometric weights $\mathcal{B}(1, 1-a)$. In these two cases, however, it is easy to see that there is an alternative decomposition of the $a$-remainder law into a finite, binomial mixture of Erlang laws: for $\alpha = n$ we have indeed from (11)

$$\left(\frac{\lambda - iau}{\lambda - iu}\right)^n = \left(a + (1-a)\frac{\lambda}{\lambda - iu}\right)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} (1-a)^k \left(\frac{\lambda}{\lambda - iu}\right)^k \quad (17)$$

namely a finite mixture of Erlang $\mathcal{E}_k(\lambda)$ with binomial weights $\mathcal{B}(n, 1-a)$, or in other words an Erlang law $\mathcal{E}_S(\lambda)$ with a binomial $\mathcal{B}(n, 1-a)$-distributed random index $S$, that is a sum

$$Z_a \overset{d}{=} \sum_{j=0}^{S} X_j \quad S \sim \mathcal{B}(n, 1-a)$$

of $S$ $iid$ exponentials $X_j \sim \mathcal{E}_1(\lambda)$ with $X_0 = 0, \ P\text{-a.s.}$ This ambiguity in the mixture representation of a law is apparently allowed because in general a mixture decomposition is not unique.

The said binomial decomposition, however, while legitimate for $\alpha = n$, cannot be extended to the general case of $\alpha > 0$. While indeed – always from the generalized
binomial formula – the following infinite decomposition of $\chi_a(u, \alpha, \lambda)$ in (11)

$$
\left( \frac{\lambda - iau}{\lambda - iu} \right)^\alpha = a^\alpha \left( 1 + \frac{1 - a}{a} \frac{\lambda}{\lambda - iu} \right)^\alpha
= a^\alpha \sum_{k=0}^{\infty} \frac{(\alpha)}{k} \left( \frac{1 - a}{a} \frac{\lambda}{\lambda - iu} \right)^k
= \sum_{k=0}^{n} \omega_k(a, \alpha) \left( \frac{\lambda}{\lambda - iu} \right)^k
$$

(18)

$$
\omega_k(a, \alpha) = \left( \frac{\alpha}{k} \right) a^{\alpha-k} (1-a)^k
$$

looks again as another infinite mixture of Erlang laws $\mathcal{E}_k(\lambda)$, we must remark that first this expansion definitely converges exclusively when it is

$$
\left| \frac{1 - a}{a} \frac{\lambda}{\lambda - iu} \right| \leq \frac{1 - a}{a} < 1
$$

which, for $0 < a < 1$, only happens if $\frac{1}{2} \leq a < 1$; and second, and mainly, that although the infinite sequence of the $\omega_k(a, \alpha)$ sums up to one, the generalized binomial coefficients

$$
\left( \frac{\alpha}{k} \right) = \frac{\alpha(\alpha-1) \ldots (\alpha-k+1)}{k!} \quad \left( \frac{\alpha}{0} \right) = 1
$$

take also negative values for $k > \alpha + 1$, and hence the $\omega_k(a, \alpha)$ not always constitute a legitimate probability distribution. As a consequence the decomposition (18) is not in general a true mixture, even if it holds mathematically whenever it converges.

In other words (as an alternative to (16)) the pdf of the $a$-remainder $Z_a$ can always be represented also as the following combination – let us call it a pseudo-mixture – of Erlang pdf’s

$$
g_a(x) = a^n \delta(x) + \sum_{k=1}^{\infty} \omega_k(a, \alpha) f_{k, \lambda}(x), \quad \frac{1}{2} \leq a < 1
$$

(19)

that can be interpreted as a true mixture only when $\alpha$ is an integer and the sum is cut down to a finite number of terms.

4 Simulation Algorithms

The results of the previous sections show that the chf of an $OU-CPE(k, \lambda_P, \lambda_J)$ coincides with that of the $a$-remainder $Z_a$ of a gamma law $\mathcal{G}(\alpha, \lambda)$ by simply taking $a = e^{-k\Delta t}$, $\lambda = \lambda_J$ and $\alpha = \lambda_P/k$. As a consequence, in alternative to the Algorithm 1 in Section 2 our procedure to generate an $OU-CPE(k, \lambda_P, \lambda_J)$ is

Algorithm 2. For $m = 1, \ldots, M$

- generate $Z_a^{(m)}$: with $a = e^{-k\Delta t_m}$, $\alpha = \lambda_P/k$, $\lambda = \lambda_J$ and $0 \leq a < 1$
– take a Polya number \( B \sim \mathbb{B}(\alpha, 1 - a); \)
– conditional on \( B \), generate \( M \) Erlang \( Z_a^{(m)} \sim \mathcal{E}_B(\lambda/a); \)

• set \( X(t_m) = X_{t_{m-1}} e^{-k\Delta t_m} + Z_a^{(m)}. \)

The simulation of \( Z_a \) is very simple and it is applicable with no parameter constraints. When in particular \( \lambda P/k = \alpha \) is an integer \( n \), the sub-procedure is either

– take a binomial number \( B \sim \mathbb{B}(n, 1 - a); \)
– conditional on \( B \), generate \( M \) Erlang \( Z_a^{(m)} \sim \mathcal{E}_B(\lambda); \)

or else

– take a negative binomial number \( B \sim \mathbb{B}(n, 1 - a); \)
– conditional on \( B \), generate \( M \) Erlang \( Z_a^{(m)} \sim \mathcal{E}_B(\lambda/a); \)

Of course the assumption \( \lambda P/k = n \) becomes acceptable for a fairly large \( n \), namely for an OU-CPE with either a low mean-reversion rate or a high number of jumps. In other words this approximation could be used if \( \lambda P \gg k \), or better when the integer part \( \lfloor \lambda P/k \rfloor \) is much larger than its remainder. On the other hand such a conjecture is justified by the fact that in practice every estimation procedure presents estimation errors.

The simulation of the \( Z_a^{(m)} \), moreover, could also be implemented starting from the representation (19) of their density. Over the usual time grid \( t_0, t_1, \ldots, t_M \) \( (\Delta t_m = t_m - t_{m-1} m = 1, \ldots, M) \) the constraint \( \frac{1}{2} \leq a < 1 \) implies that \( k < \log 2/\Delta t_m \). In energy markets and financial applications it is common to assume \( \Delta t_m < 1/365 \) or \( \Delta t_m < 1/252 \) that correspond to \( k < 253 \) or \( k < 175 \) respectively, values that virtually cover all the realistic market conditions.

Under this parameter constraint we can conceive an acceptance-rejection procedure based on the method of Bignami and de Matteis \[5\] for pseudo-mixtures with non positive terms (see also Devroye \[15\] pag. 74). Denoting indeed \( \omega_k(a, \alpha)^+ = \max\{\omega_k(a, \alpha), 0\} \) and \( \omega_k(a, \alpha)^- = \min\{\omega_k(a, \alpha), 0\} \), so that \( \omega_k(a, \alpha) = \omega_k(a, \alpha)^+ + \omega_k(a, \alpha)^- \), the approach of Bignami and de Matteis relies on the remark that from (19) we have

\[
g_a(x) \leq \sum_{k \geq 0} \omega_k(a, \alpha)^+ f_{k, \lambda}(x) = \mathcal{g}(x) = cg(x)
\]

where

\[
1 < c = \sum_{k \geq 0} \omega_k(a, \alpha)^+ < \infty \quad p_k = \frac{\omega_k(a, \alpha)^+}{c} \quad g(x) = \sum_{k \geq 0} p_k f_{k, \lambda}(x)
\]

so that \( g(x) \) turns out to be a true mixture of Erlang laws, namely the pdf of

\[
V = \sum_{i=0}^{S} X_i \sim \mathcal{E}_S(\lambda) \quad X_i \sim \mathcal{E}_1(\lambda) \quad P\{S = k\} = p_k
\]
and this leads finally to the following procedure

**Algorithm 3.** Taking for simplicity $\lambda = 1$, and only the first $N$ terms of $g_a(x)$, the generation of $Z_a$ can be achieved by the following acceptance-rejection algorithm:

- generate $S$ with $P\{S = k\} = p_k$, $k = 0, \ldots, N$
- repeat the following sub-procedure until $U \leq \frac{g_a(Z)}{\overline{g}(Z)}$
  - generate a uniform $U \sim U[0, 1]$
  - generate an Erlang $Z \sim E(1)$
- return $Z$ as the accepted value for $Z_a$

The computational performance of this algorithm can be assessed by observing that for relatively small values of $\alpha$ the probability $P\{S\} = 0$ is high, hence $V$ and $Z_a$ turn out to be degenerate, so that $Z_a$ can be set to 0 as well because the acceptance condition is always satisfied. Since on the other hand the efficiency of the acceptance-rejection algorithm depends of the constant $c$ in (21), and $1/c$ roughly represents the probability of accepting $E(1)$, it is also preferable to have $c$ as close to 1 as possible.

Remark that for $0 < \alpha \leq 1$ and $\frac{1}{2} \leq a < 1$ we always have $\omega_0(a, \alpha) = a^\alpha \geq 0.5$ with the minimum value 0.5 attained for $a = 0.5$, $\alpha = 1$, which coincides with the simulation of $Z \sim E_{B}(1, 1-a)$ (see Cufaro Petroni and Sabino [10]). This means that the concentration of the weights $\omega_k(a, \alpha)$ is mainly around $\omega_0(a, \alpha)$ (which is a positive number) because in the said range of $a, \alpha$ the negative coefficients $\omega_k(a, \alpha)$ are rather negligible; for instance, setting $N = 40$, we find

| $\alpha$ | $c$          | $1/c$          | $\omega$          |
|---------|--------------|----------------|-------------------|
| 0.1     | $c \simeq 1.1311$ | $1/c \simeq 0.8841$ | when $a = 0.5$   |
|         | $c \simeq 1.0006$ | $1/c \simeq 0.9995$ | when $a = 0.9$   |
| 0.9     | $c \simeq 1.0348$ | $1/c \simeq 0.9663$ | when $a = 0.5$   |
|         | $c \simeq 1.0005$ | $1/c \simeq 0.9994$ | when $a = 0.9$   |

It is apparent then that for $0 < \alpha < 1$ the acceptance-rejection method is very efficient because the law of $Z_a$ is similar to that of $V$. If on the other hand $\alpha > 1$, taking $n = \lceil \alpha \rceil$ and $\beta$ its remainder, $Z_a$ can be also seen (and generated) as the sum of $Z_1 + Z_2$ with $Z_1 \sim E_{B}(n, 1-a)$ and $Z_2$ with chf in equation (11). In any case our numerical experiments will show that $c$ is very close to 1 also for $\alpha > 1$.

In the Table 2 we compare to their exact values the first 5 moments obtained with the Algorithms 1, 2, and 3 when the parameters are chosen as in the Table 1; the Figures 1a and 1b show instead their respective computational times. Throughout the present paper the calculations are performed using *MATLAB* on a 64-bit Intel Core i5-6300U CPU, 8GB. As a further validation the comparisons of the simulation computational times have also been performed with *R* and *Python* to the same avail.
$\lambda_p \quad \lambda_f \quad k \quad \alpha \quad \Delta t_m \quad a \quad N$

| Order | Exact | Algorithm 1 | Algorithm 2 | Algorithm 3 |
|-------|-------|-------------|-------------|-------------|
| 1     | 0.0261| 0.0260      | 0.0262      | 0.0261      |
| 2     | 0.0504| 0.0501      | 0.0503      | 0.0569      |
| 3     | 0.1462| 0.1437      | 0.1428      | 0.1434      |
| 4     | 0.5658| 0.5433      | 0.5719      | 0.5351      |
| 5     | 2.7389| 2.5223      | 2.8930      | 2.4177      |

Table 1: Parameters used in the MC simulation in Table 2 with $10^6$ scenarios

Table 2: Comparison of the moments calculated with Algorithms 1, 2 and 3

All these algorithms seem to be equally unbiased, although it is quite common that the MC estimations deteriorate for higher moments. The Algorithm 2 looks by far as the best and provides a remarkable improvement to the time required to simulate an $OU$-$CPE$, its computational cost being about 50 times smaller than that of the other two alternatives. Although based on a rejection method, even the Algorithm 3 is faster than the Algorithm 1 because – as explained above – the likelihood of accepting the draw of a rv with dominating density is very high. Of course, the superior performances of the Algorithm 2 w.r.t. the alternatives become even more remarkable when simulating an entire trajectory with, let say, 365 points. We also repeated these comparisons with several parameter combinations always to the same avail; the parameter choice of the Table 1 was indeed motivated by the fact that these are realistic values used for the pricing of energy facilities: similar values will also be adopted in the examples of the Sections 5.3 and 5.4.

5 Financial Applications

We apply the results of this study to the pricing of energy contracts namely swings and storages. The pricing of such contracts is very often tackled with Monte Carlo (MC) methods whose implementation needs to be unbiased and with a low computational burden, especially if the solution is meant for real-time pricing.

From the financial modelling perspective, it is well-known that day-ahead prices exhibit seasonality, mean reversion and jumps that have to be captured in the mathematical modelling. The literature is indeed very rich of alternatives: Lucia and Schwartz [24] for instance, propose a one-factor Gaussian-$OU$ with application to the Nordic Power Exchange while a two factor version can be found in Schwartz and Smith [30] where an additional Brownian Motion ($BM$) is introduced. Models that go beyond the Gaussian world can be found among the others, in Benth et al. [4] and Meyer-Brandis and Tankov [25], Cartea and Figueroa [7]. The first two
papers investigate the use of generalized OU processes while the last one studies the modelling with a jump-diffusion OU process. In particular, the assumption that the discontinuous component is modelled by a OU-CPE (with positive and negative jumps) is detailed in Deng [13], Kluge [20] and Kjaer [19] or in Benth and Pircalabu [3] in the context of modelling wind power futures.

By summarizing, we assume that the spot price can be modeled by the following dynamics with independent factors

$$S(t) = F(0, t) \exp \left\{ h(t) + \sum_{d=1}^{D} X_d(t) + \sum_{j=1}^{J} Y_j(t) \right\} = F(0, t) e^{h(t)+H(t)}$$

$$= S_{\text{season}}(t) \cdot S_{\text{diffusion}}(t) \cdot S_{\text{jumps}}(t)$$

(22)

where

$$\varphi_H(u, t) = \prod_{d=1}^{D} \varphi_{X_d}(u, t) \prod_{j=1}^{J} \varphi_{Y_j}(u, t) = \varphi_{\text{diffusion}}(u, t) \cdot \varphi_{\text{jumps}}(u, t)$$

(23)

Using the risk-neutral arguments of the Lemma 3.1 in Hambly et al. [18] we get the deterministic function $h(t)$ consistent with forward curve

$$h(t) = - \log \varphi_H(-i, t).$$

(24)

In our numerical experiments we will adopt the following two settings for the spot price.
prices: first

\[ G(t) = F(0, t) \exp \left\{ h(t) + \sigma_D \int_0^t e^{-k_D(t-s)} dW(s) + \sum_{n=1}^{N(t)} e^{-k_N(t-\tau_n)} U_n - \sum_{m=1}^{M(t)} e^{-k_M(t-\tau_m)} D_m \right\} \]  

(25)

where \( N(t) \) and \( M(t) \) are two independent Poisson processes and \( D_n \) and \( U_n \) are independent rv’s with laws \( \mathcal{E}_1(\lambda^-) \) and \( \mathcal{E}_1(\lambda^+) \) respectively; and second

\[ G(t) = F(0, t) \exp \left\{ h(t) + \sigma_D \int_0^t e^{-k_D(t-s)} dW(s) + \sum_{n=1}^{N(t)} e^{-k_N(t-\tau_n)} J_n \right\} \]  

(26)

where now \( J_n \) are centered Laplace rv’s with parameter \( \lambda_J \). We decided to assign different mean-reversion rates to the jump and to the diffusive components not only to better capture the spikes, but also to avoid a discretization bias as explained in the Section 2. With respect to the parameter settings used in Deng [13] and Kjaer [19] the mean-reversion rates of our jump components will be larger than that of their diffusion counterparts. The parameter combination in Kjaer [19] assumes indeed that the process \( G(t) \) has just one – and small – mean-reversion rate with a high \( \lambda_P \), so that \( \lambda_P/k \simeq 7 \) and one could implement the simplified version of the Algorithm 2 with \( \alpha \) an integer number.

5.1 Bilateral exponential jumps

Normally financial markets display positive and negative jumps whose size is often modeled with a compound Poisson with bilateral exponential rv’s and asymmetric parameters. As also observed in Cummins et al [12] and Kjaer [19], this feature can also be modeled splitting the compound process into the difference of two (independent) compound Poisson with exponential jumps: the chf’s of the jumps laws can indeed be written respectively as

\[ \varphi_J(u) = \left( \frac{\lambda^+}{\lambda^- - iu} \right) \left( \frac{\lambda^-}{\lambda^+ + iu} \right) \text{ asymmetric} \]  

(27)

\[ \varphi_J(u) = \frac{\lambda^2}{\lambda^2 + u^2} \text{ symmetric} \]  

(28)

Of course if the law is symmetric the bilateral exponential turns into a Laplace law (centered in zero) with the chf (28). On the other hand, because of Lemma 3.4.2 in Kluge [20], the logarithmic characteristic of the OU-compound Poisson is

\[ \log \varphi(u, t) = \lambda_P \int_0^t [\varphi_J(ue^{-ks}) - 1] \, ds \]  

(29)
and when $\varphi_J(u)$ is the chf of the symmetric Laplace law we get

$$\varphi(u, t) = \left(\frac{\lambda^2_j + u^2 e^{-2kt}}{\lambda^2_j + u^2}\right)^{\frac{\lambda_P}{\lambda}}. \quad (30)$$

In the context of financial mathematics Küchler and Tappe [21] have also studied the use of bilateral gamma distributions where the laws described by the equations (27) and (28) are the special case with shape parameter equal to 1. We consider hereafter the case where the bilateral gamma is a gamma difference, and the scale and shape parameters of the positive and negative distributions coincide.

Retracing the path of the Section 3.2, we find that the chf of the $a$-remainder of a symmetric bilateral gamma is

$$\chi_a(u) = \left(\frac{\lambda^2 + a^2 u^2}{\lambda^2 + u^2}\right)^\alpha = \left(\frac{a^2}{1 - a^2}\right)^\alpha \left(\frac{\lambda^2}{\lambda^2 + a^2 u^2}\right)^\alpha,$$

$$= \sum_{k=0}^{\infty} \binom{\alpha + k - 1}{k} a^{2\alpha} (1 - a^2)^k \left(\frac{\lambda^2}{\lambda^2 + a^2 u^2}\right)^k,$$

once more an infinite Polya $\mathcal{B}(\alpha, 1 - a^2)$-weighted mixture of bilateral Erlang laws with parameter $\lambda/a$. As a consequence, taking $a = e^{-kt}$ and $\alpha = \frac{\lambda_P}{2k}$, the chf of a OU-compound Poisson with (centered) Laplace jumps again coincides with the chf of the $a$-remainder law of a gamma difference.

It is then straightforward to adapt all the results presented in the Section 3.2 and the simulation algorithms of the Section 4. The procedure consists in simply replacing the chf’s of gamma and Erlang laws and their mixtures with those of their bilateral counterparts, complying moreover with the constraint $k \leq \log 2/\Delta t$. Accordingly, the functions $f_{n,\lambda}$ representing the gamma densities in equations (16), (19) and (20) have to be replaced by the densities of the bilateral gamma in equation (12). Remark though that the functions $f_{n,\lambda}$ are relative to the Erlang difference therefore the Bessel function $K_{n+\frac{1}{2}}(z)$ can be simplified in terms of elementary functions (see 8.468 in Gradshteyn and Ryzhik [16])

$$K_{n+\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}} e^{-z} \sum_{k=0}^{n} \frac{(n + k)!}{k!(n - k)!(2z)^k}.$$

### 5.2 Time-dependent Poisson Intensity

Jumps are often concentrated in clusters, for instance energy markets are very seasonal and jumps occur more often either during a period of high demand, or through a cold spell. A more realistic approach could then be to consider a non-homogeneous Poisson process with time-dependent intensity $\lambda_P(t)$ and $\Lambda_P(t) = \int_0^t \lambda_P(s)ds$. In this case the new Poisson process and its relative compound version have independent,
but non-stationary increments. In addition, the modelling becomes more challenging for a OU-compound Poisson because the Lemma 3.4.2 in Kluge [20] is not applicable and the chf in (6) somehow depends on the specific intensity function. In any case one could consider a time grid \( t_0, t_1, \ldots, t_M \) (\( \Delta t_m = t_m - t_{m-1}, m = 1, \ldots, M \)) such that the non-homogeneous Poisson process has a step-wise intensity, \( \lambda_P(s) = \lambda_m1_{s \in \Delta t_m} \). Because the non-homogeneous Poisson has independent increments, it behaves at time \( t \) as the sum of different independent Poisson processes each with a constant intensity. The main consequence of this simple assumption is that at each step \( m \) in Algorithm 3 the generation of \( Z_m^{(m)} \) requires a different parameter \( \alpha_m = \lambda_m/k, m = 1, \ldots M \).

5.3 Numerical Experiments: Gas Storages

Denote by \( C(t) \) the volume of a (virtual) gas storage at time \( t \) with \( C_{\min} \leq C(t) \leq C_{\max} \). The holder of such an energy asset is faced with a timing problem that consists in deciding when to inject, to withdraw or to do-nothing. One standard approach to the pricing of gas storages is a modified version of the Least-Squares Monte Carlo (LSMC), introduced in Longstaff-Schwartz [23], detailed in Boogert and C. de Jong [6].

Denoting \( J(t, x, c) \) the value of a gas storage at time \( t \) given \( S(t) = x, C(t) = c \), one can write:

\[
J(t, x, c) = \sup_{u \in \mathcal{U}} \mathbb{E} \left[ \int_t^T \phi_u(S(s)) \, ds + q(S(T), C(T)) \bigg| S(t) = x, C(t) = c \right],
\]

where \( \mathcal{U} \) denotes the set of the admissible strategies, \( u(t) \in \{-1, 0, 1\} \) is the regime at time \( t \) such that

\[
\begin{align*}
\phi_{-1}(S(t)) &= -S(t) - K_{in}a_{in}, \quad \text{injection} \\
\phi_0(S(t)) &= -K_N, \quad \text{do nothing} \\
\phi_{-1}(S(t)) &= S(t) - K_{out}a_w, \quad \text{withdrawal}
\end{align*}
\]

where \( a_{in} \) and \( a_w \) are the injection and withdrawal rates, \( K_{in}, K_{out} \) and \( K_N \) respectively represent the costs of injection, do-nothing and withdrawal, and \( q \) takes into account the possibility of final penalties.

We then perform a few numerical experiments selecting a three-factors spot model as in (25) with one OU diffusion \( (D = 1) \) plus the difference of two OU-CPE in order to capture asymmetric jumps (we set \( H(0) = 0 \)): in this case, because of (24), for \( \lambda^+J > 1 \) it results

\[
h(t) = \frac{\sigma^2_D}{4k_D} \left( 1 - e^{-2k_D t} \right) - \ln \left( \frac{\lambda^+_J - e^{-k_N t}}{\lambda^+_J - 1} \right) \frac{\lambda^+_N}{\lambda^+_J} - \ln \left( \frac{\lambda^-_J - e^{-k_M t}}{\lambda^-_J - 1} \right) \frac{\lambda^-_M}{\lambda^-_J}.
\]

This model can also be extended to cover correlated Poisson processes. For instance, in Cufaro Petroni and Sabino [10] and [11] we used once more the concept
Table 3: Parameters for Spot (day-ahead) dynamics

| \( S_0 \) | \( k_D \) | \( \sigma_D \) | \( k_N \) | \( k_M \) | \( \lambda_N \) | \( \lambda_M \) | \( \lambda_j^+ \) | \( \lambda_j^- \) |
|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| 22     | 67     | 0.25   | 50     | 40     | 20     | 20     | 10     | 20     |

Table 4: Fast Storage Specification.

of \( sd \) to produce correlated Poisson processes with a time-delay mechanism among jumps, and we discussed an application to the pricing of spread options. In the present paper, however, we cannot squarely rely on the Algorithm 2 because we should first design sequences of correlated \( Z_a \)'s.

Going back to the initial problem, we assume that the units of \( C(0), C(T) \) and \( C_{max} \) are MWh, those of the injection and withdrawal rates are MWh/day, while \( S_0 \) can be taken in \( \text{€}/\text{MWh} \); in addition we suppose a flat forward curve. The remaining model parameters are shown in Table 3 and can be considered realistic. Although the calibration is not the focus of this paper, we remark that the explicit knowledge of the transition density of the \( \text{OU-CPE} \) gives the possibility (at least in terms of convolution) to write down the overall transition density, and hence the likelihood function. Therefore the eventual estimated parameters would not be affected by the approximations implicit in any discretization scheme (besides truncating the infinite series). Of course this statement holds when the jumps follow a Laplace distribution with the relative parameter constraints.

We consider finally a one-year fast-churn storage with the parameters shown in Table 4 such that 20 days are required to fill or empty the storage as shown in Figure 2a. As displayed in the Figure 3a, the three types of implementation apparently return comparable gas storage values. Besides this, the numerical results in Figure 3b show that the Algorithm 2 is by far the most performing solution, and that even the Algorithm 3 is slightly faster than the Algorithm 1 though the overall computational burden is mainly driven by the stochastic optimization. The overall computational cost is in fact the sum of two terms: that required to simulate the price trajectories, and the cost of the stochastic optimization. The latter component, however, is not influenced by the particular simulation algorithm, and in our experiments it amounts to roughly 70% of the overall computational time using the Algorithm 1. Looking then at the right axis of Figure 3b we see that the computational time of the price simulation is drastically reduced by the Algorithm 2 that w.r.t. the Algorithm 1 cuts down the overall time to barely that required for the stochastic optimization.

Remark finally that we relied on the sequential simulation of the price trajectory forward in time. In combination with LSMC methods, this is not the optimal approach because the entire set of trajectories and simulations are stored in memory.
Figure 2: Gas assets.

(a) Feasible Volumes of the fast churn storage
(b) Sample Trajectory

Figure 3: Gas Storage Results.

(a) Values Gas Storage.
(b) Ratio Computational Times.
with a risk of memory allocation issues. For instance, in Pellegrino and Sabino [27] and Sabino [28] we have shown that the backward simulation is preferable with LSMC. Unfortunately, however, while we know the law of the standard Gaussian diffusion $OU$ bridge, we do not know the law of the $OU$-$CPE$ bridge which will be one of the topics of our future studies.

### 5.4 Numerical Experiments: Swings

A swing option is a type of contract used by investors in energy markets that lets the option holder buy a predetermined quantity of energy at a predetermined price (strike) while retaining a certain degree of flexibility in both the amount purchased and the price paid. Such a contract can also be seen as a simplified gas storage where $a_{in} = 0$, $K_N = 0$ and $K_w$ is the strike of the contract. We consider a 120-120 swing with the specifications of Table 5 and Figure 5a: it can be seen as plugging $C(0) = 120$, $C(T) = 0$, $a_{in} = 0$, $a_w = 1$, $C_{max} = 120$ into (31) with an injection cost equal to the strike.

At variance with the example of the previous subsection, we now choose a two-factors spot model in (26) with an $OU$ diffusion ($D = 1$) and an $OU$-compound Poisson with Laplace jumps (once more we set $H(0) = 0$). We also consider the following step-wise daily approximation of the time-dependent intensity

$$
\lambda_P(t) = \frac{2\theta}{1 + |sin(\pi\omega(t-\tau))|}
$$

so that for $m = 1, \ldots, M$ and $\lambda_J > 1$ we have

$$
h(t_m) = \frac{\sigma_X^2}{4k_X} (1 - e^{-2k_D t_m}) - \ln \left( \frac{\lambda_J^2 - e^{-2k_J t_m}}{\lambda_J^2 - 1} \right) \frac{\Delta t}{X}
$$

with the parameters of Table 5 again with a flat forward curve. The value of $\theta$ is such that the average number of jumps per year is about 40 as in the storage example.

While, as mentioned in the previous section, the parameter calibration is not the focus of this paper, the transition density function, under the usual parameter constraints, could be calculated in closed-form and be used to explicitly write the likelihood function. The discretization schemes often assume that the Poisson process can have only one jump in one unit of time: this approximation could however be unreliable when the jumps occurs in clusters, as captured by a time-dependent
\[
\begin{array}{cccccccc}
S_0 & k_D & \sigma_D & k_N & \theta & \omega & \tau & \lambda_J \\
22 & 67 & 0.25 & 50 & 32 & 2 & 0.25 & 20 \\
\end{array}
\]

Table 6: Parameters for Spot (day-ahead) dynamics

Figure 4: Market Model.  
(a) Time-dependent Intensity  
(b) Sample Trajectory.

Figure 5: Market Model.  
(a) Feasible Volumes of a 120-120 Swing.  
(b) Swing Values and Ratio of Computational times.
intensity, while instead our approach allows to relax this assumption. In contrast to the gas storage case we did not implement the Algorithm 3 and we focused instead on the comparison of the other two alternatives. The conclusion from these numerical experiments are very much in line with what observed in the case of gas storages.

As expected the estimated values of the swing obtained with the two types of implementation are similar and equal to 120.68 € for $10^6$ simulations. As shown in Figure 5b, however, once more the computational times of the Algorithm 2 are far lower resulting in a competitive advantage of about 40% on the overall computational cost. This factor becomes even higher if one focuses on the time required to simulate the price paths: the contribution of the stochastic optimization step to the overall computational burden is again of about 75% using the Algorithm 1 while instead with the Algorithm 2 the path generation step becomes almost negligible compared to the total cost.

6 Conclusions and future inquiries

In this paper we have studied the law of the mean-reverting compound Poisson process with exponential, bilateral exponential and Laplace distributed jumps. Based on the results of Barndorff-Nielsen and Shephard [2], we know that the stationary law of a generalized Ornstein-Uhlenbeck process has to be self-decomposable if the background driving process is Lévy. For the cases under study, we proved that in the transient regime the law of such a process is exactly the law of the $a$-remainder $rv$ of the gamma law which is the stationary law of our compound Poisson with exponential jumps (or their bilateral counterpart).

With the explicit knowledge of the law in closed-form one can avoid the usual assumption of having at most one jump per unit of time: we have shown indeed that this last hypothesis leads to biased results, unless the mean-reversion rate and the intensity of the Poisson process coincide. We are able instead to design algorithms for the exact simulation of each point of the trajectory of the mean-reverting compound Poisson, requiring in so doing a significantly lower computational effort compared to the cost of those available in the literature. Moreover – even if it is not the focus of our numerical examples – the knowledge of the transition density in closed-form can give the possibility to write the correct likelihood function for the parameter estimation, at least for a pure jump process.

We illustrated the applications of our findings in the context of pricing gas storages and swings adopting the Least-Squares Monte Carlo method introduced in Boogert and de Jong [6] in conjunction with jump-diffusion price models. The overall computational burden depends on the cost of simulating the price trajectories and the stochastic optimization (this last step is not influenced by the particular simulation algorithm). Our numerical experiments have shown that our strategy has a remarkable computational advantage and cuts the simulation time down by a
factor larger than 30. The role of the mathematical notion of self-decomposability, on the other hand, is of fundamental importance in understanding the law of the mean-reverting compound Poisson with exponential jumps. From the mathematical point of view, it would be interesting to study if – and under which conditions – these results hold for other generalized Ornstein-Uhlenbeck processes used in financial applications and in energy markets (see for instance Cummins et al. [12]).

In a primarily economic and financial perspective, the future studies could cover the extension to a multidimensional setting with correlated Poisson processes as those introduced for instance in Lindskog and McNeil [22] or in Cufaro Petroni and Sabino [10]. A last topic deserving further investigation is a possible enhancement of the computational speed relying on backward simulations generalizing the results of Pellegrino and Sabino [27] and Sabino [28] to the case of the mean reverting compound Poisson processes.

References

[1] L. Barabesi and L. Pratelli. A note on a Universal Random Variate Generator for Integer-valued Random Variables. Statistics and Computing, 24(4):589–596, 2014.

[2] O.E. Barndorff-Nielsen and N. Shephard. Non-Gaussian Ornstein-Uhlenbeck-based models and some of their uses in financial economics. Journal of the Royal Statistical Society: Series B, 63(2):167–241, 2001.

[3] F.E. Benth and A. Piricalabu. A non-gaussian ornstein-uhlenbeck model for pricing wind power futures. Applied Mathematical Finance, 25(1), 2018.

[4] F.E. Benth and J. Kallsen T. Meyer-Brandis. A non-gaussian ornstein-uhlenbeck process for electricity spot price modeling and derivatives pricing. Applied Mathematical Finance, 14(2):153–169, 2007.

[5] A. Bignami and A. de Matteis. A Note on Sampling from Combination of Distribution. Journal of the Institute of Mathematics and its Applications, 8:80–81, 1971.

[6] A. Boogert and C. de Jong. Gas Storage Valuation Using a Monte Carlo Method. Journal of Derivatives, 15:81–91, 2008.

[7] A. Cartea and M. Figueroa. Pricing in Electricity Markets: a Mean Reverting Jump Diffusion Model with Seasonality. Applied Mathematical Finance, No. 4, December 2005, 12(4):313–335, 2005.

[8] R. Cont and P. Tankov. Financial Modelling with Jump Processes. Chapman and Hall, 2004.
[9] N. Cufaro-Petroni. Self-decomposability and Self-similarity: a Concise Primer. *Physica A, Statistical Mechanics and its Applications*, 387(7-9):1875–1894, 2008.

[10] N. Cufaro-Petroni and P. Sabino. Coupling Poisson Processes by Self-decomposability. *Mediterranean Journal of Mathematics*, 14(2):69, 2017.

[11] N. Cufaro-Petroni and P. Sabino. Pricing exchange options with correlated jump diffusion processes. *Quantitative Finance*, pages 1–13, 2018.

[12] M. Cummins, G. Kiely, and B. Murphy. Gas Storage Valuation under Lévy Processes using Fast Fourier Transform. *Journal of Energy Markets*, 4:43–86, 2017.

[13] S. Deng. Stochastic Models of Energy Commodity Prices and Their Applications: Mean-reversion with Jumps and Spikes. Citeseer, 2000.

[14] L. Devroye. On the Computer Generation of Random Variables with a given Characteristic Function. *Computers & Mathematics with Applications*, 7(6):547–552, 1981.

[15] L. Devroye. *Non-Uniform Random Variate Generation*. Springer-Verlag, 1986.

[16] I. S. Gradshteyn and I. M. Ryzhik. *Table of integrals, series, and products*. Elsevier/Academic Press, Amsterdam, seventh edition, 2007.

[17] B. Grigelionis. On the Self-Decomposability of Euler’s Gamma Function. *Lithuanian Mathematical Journal*, 43(3):295–305, 2003.

[18] B. Hambly, S. Howison, and T. Kluge. Information-Based Models for Finance and Insurance. *Quantitative Finance*, 9(8):937–949, 2009.

[19] M. Kjaer. Pricing of Swing Options in a Mean Reverting Model with Jumps. *Applied Mathematical Finance*, 15(5-6):479–502, 2008.

[20] T. Kluge. Pricing Swing Options and other Electricity Derivatives. Technical report, 2006. PhD Thesis, Available at http://perso-math.univ-mlv.fr/users/bally.vlad/publications.html.

[21] U. Küchler and S. Tappe. Bilateral Gamma Distributions and Processes in Financial Mathematics. *Stochastic Processes and their Applications*, 118(2):261–283, 2008.

[22] F. Lindskog and J. McNeil. Common poisson shock models: applications to insurance and credit risk modelling. *ASTIN Bulletin*, 33(2):209–238, 2003.
[23] F. A. Longstaff and E.S. Schwartz. Valuing American Options by Simulation: a Simple Least-Squares Approach. *Review of Financial Studies*, 14(1):113–147, 2001.

[24] J.J. Lucia and E.S. Schwartz. Electricity prices and power derivatives: Evidence from the nordic power exchange. *Review of Derivatives Research*, 5(1):5–50, Jan 2002.

[25] T. Meyer-Brandis and P. Tankov. Multi-factor Jump-diffusion Models of Electricity Prices. *International Journal of Theoretical and Applied Finance*, 11(5):503–528, 2008.

[26] H.H. Panjer and G.E. Willmot. Finite sum evaluation of the negative binomial-exponential model. *ASTIN Bulletin*, 12(2):133137, 1981.

[27] T. Pellegrino and P. Sabino. Enhancing least squares monte carlo with diffusion bridges: an application to energy facilities. *Quantitative Finance*, 15(5):761–772, 2015.

[28] P. Sabino. Forward or Backward Simulations? A Comparative Study. Technical report, 2019. Available at www.ssrn.com.

[29] K. Sato. *Lévy Processes and Infinitely Divisible Distributions*. Cambridge U.P., Cambridge, 1999.

[30] P. Schwartz and J.E. Smith. Short-term Variations and Long-term Dynamics in Commodity Prices. *Management Science*, 46(7):893–911, 2000.