Rate of convergence of a two-scale expansion for some “weakly” stochastic homogenization problems

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Abstract

We establish a rate of convergence of the two scale expansion (in the sense of homogenization theory) of the solution to a highly oscillatory elliptic partial differential equation with random coefficients that are a perturbation of periodic coefficients.

1 Introduction and presentation of the main result

This article focuses on establishing a rate of convergence of the two scale expansion (in the sense of homogenization theory) of the solution to a highly oscillatory partial differential equation with random coefficients. We begin our exposition by briefly discussing the same question in a deterministic setting, before turning to the stochastic setting.

Consider the highly oscillatory problem

\[
\begin{align*}
-\text{div} \left[ A_{\text{per}} \left( \frac{x}{\varepsilon} \right) \nabla u_0^\varepsilon \right] &= f & \text{in } \mathcal{D}, \\
u_0^\varepsilon &= 0 & \text{on } \partial \mathcal{D},
\end{align*}
\]

(1)

where \( \mathcal{D} \) is a regular bounded domain of \( \mathbb{R}^d \), \( f \in L^2(\mathcal{D}) \), and \( A_{\text{per}} \) is a \( Q \)-periodic elliptic bounded matrix, with \( Q = (-1/2, 1/2)^d \). For simplicity, we manipulate henceforth symmetric matrices, but the arguments carry over to non-symmetric matrices up to slight modifications. It is well known (see e.g. the classical textbooks [7, 11, 17], and also [14] for a general, numerically oriented presentation) that \( u_0^\varepsilon \) converges, weakly in \( H^1(\mathcal{D}) \) and strongly in \( L^2(\mathcal{D}) \), to the solution \( u_0^\star \) to

\[
\begin{align*}
-\text{div} \left[ A_{\text{per}}^\star \nabla u_0^\star \right] &= f & \text{in } \mathcal{D}, \\
u_0^\star &= 0 & \text{on } \partial \mathcal{D},
\end{align*}
\]

(2)

where the homogenized matrix is given by

\[
(A_{\text{per}}^\star)_{ij} = \int_Q (e_i + \nabla w_{e_j}^0(y))^T A_{\text{per}}(y)(e_j + \nabla w_{e_j}^0(y)) \, dy,
\]

(3)

where, for any \( p \in \mathbb{R}^d \), \( w_p^0 \) is the unique (up to the addition of a constant) solution to the corrector problem associated to the periodic matrix \( A_{\text{per}}^p \):

\[
\begin{align*}
-\text{div} \left[ A_{\text{per}}^p \nabla w_p^0 \right] &= 0, \\
w_p^0 \text{ is } Q\text{-periodic.}
\end{align*}
\]

(4)
The corrector function allows to compute the homogenized matrix, and it also allows to obtain a convergence result in the $H^1$ strong norm. Indeed, in dimension $d > 1$, under some regularity assumptions recalled below, we have

$$\left\| u_0^\varepsilon - \left[ u_0^{\varepsilon_0} + \varepsilon \sum_{i=1}^{d} w_{ei}^0 \left( \cdot \frac{\varepsilon}{\varepsilon} \right) \frac{\partial u_0^\varepsilon}{\partial x_i} \right] \right\|_{H^1(D)} \leq C \sqrt{\varepsilon}$$

for a constant $C$ independent of $\varepsilon$ (in dimension $d = 1$, the difference is of order $\varepsilon$ rather than $\sqrt{\varepsilon}$).

Note that $u_0^\varepsilon = u_0^\varepsilon + \varepsilon \sum_{i=1}^{d} w_{ei}^0 \left( \cdot \frac{\varepsilon}{\varepsilon} \right) \frac{\partial u_0^\varepsilon}{\partial x_i}$ is a function of order 1 in the $H^1$ norm. At first sight, one could thus expect that the difference between $u_0^\varepsilon$ and $v_0^\varepsilon$ is of order $\varepsilon$, rather than $\sqrt{\varepsilon}$. This lower order (in dimension $d > 1$) is due to an inconsistency of the boundary conditions. Note indeed that, by definition, $u_0^\varepsilon = 0$ on $\partial D$, which is not the case of $v_0^\varepsilon$. Note also that the (lower than expected) rate in (5) is not specific to the choice of homogeneous Dirichlet boundary conditions in (1), and also holds for Neumann boundary conditions, as stated in [17, p. 29] (see also [22]).

The order of approximation improves if we ignore the difference between $u_0^\varepsilon$ and $v_0^\varepsilon$ at the boundary of the domain (see [1, Theorem 2.3]). Alternatively, one can build functions, the so-called boundary layers, that correct $v_0^\varepsilon$ in the neighborhood of $\partial D$, to eventually improve the accuracy of the approximation of $u_0^\varepsilon$ so obtained, in the complete domain $D$. We refer to [1, 21] and to [13, Appendix B] (see also [2, Chap. 5] for the study of the same question in a time-dependent, parabolic setting). On another note, we refer to [26] for studies on the rate of convergence of $u_0^\varepsilon$ to $u_0^\varepsilon$ in the $L^\infty(D)$ norm (see also [14] and references therein, and [10] for extensions to some nonlinear cases), and to [21, 22] for similar studies on the lowest eigenvalue $\lambda_0^\varepsilon$ of the operator $L^\varepsilon = -\text{div} \left[ A_{\text{per}} \left( \frac{\varepsilon}{\varepsilon} \right) \nabla \right]$.

The result (5) is interesting from the theoretical viewpoint. It is also helpful for proving numerical analysis results. In particular, this result is a key ingredient to prove error bounds for the Multiscale Finite Element Method (MsFEM). This numerical approach aims at approximating the solution $u_0^\varepsilon$ to the highly oscillatory problem (1) (for a small, but non vanishing small scale $\varepsilon$), and does so by performing a variational approximation of (1) using pre-computed basis functions that are adapted to the problem. Consequently, the MsFEM approach yields an accurate approximation of $u_0^\varepsilon$ using only a limited number of degrees of freedom, in contrast to a standard Finite Element Method approach. In addition, the MsFEM approach is applicable in general situations, and is not limited to the case when the highly oscillatory coefficient of the equation reads $A^\varepsilon(x) \equiv A_{\text{per}} \left( \frac{\varepsilon}{\varepsilon} \right)$ for a fixed periodic matrix $A_{\text{per}}$. See [13] and references therein. As described below, our motivation for this work stems from our work [19], where we suggest a possible extension of the MsFEM approach to weakly stochastic settings. Again, a key ingredient for proving error bounds on the approach we propose there is to have a rate of convergence of the type (5).

Let us now turn to the stochastic case. As will be seen below, less precise results are known than in the deterministic, periodic case. The highly oscillatory problem reads

$$\begin{cases}
-\text{div} \left[ A_\eta \left( \frac{\cdot}{\varepsilon}, \omega \right) \nabla u_\eta^\varepsilon(\cdot, \omega) \right] = f & \text{in } D, \\
u_\eta^\varepsilon(\cdot, \omega) = 0 & \text{on } \partial D,
\end{cases}$$

where the matrix $A_\eta$ is now a stationary symmetric matrix, uniformly elliptic and bounded (see (8) below for a precise definition of stationarity, which is the common assumption in stochastic homogenization). The role of the parameter $\eta$ will be made precise in (9) below. It can momentarily be ignored. Again, as in the periodic case, it is well known (see for instance [17]) that $u_\eta^\varepsilon$ converges, almost surely, weakly in $H^1(D)$ and strongly in
$L^2(D)$, to $u^*_\eta$, solution to the homogenized equation

$$
\begin{cases}
-\text{div} \left[ A^*_\eta \nabla u^*_\eta \right] = f & \text{in } D, \\
u^*_\eta = 0 & \text{on } \partial D,
\end{cases}
$$

where the homogenized matrix is given by

$$(A^*_\eta)_{ij} = E \left( \int_Q (e_i + \nabla w^p_{\varepsilon_i}(y,\cdot))^T A_\eta(y,\cdot)(e_j + \nabla w^p_{\varepsilon_j}(y,\cdot)) \, dy \right),$$

where, for any $p \in \mathbb{R}^d$, $w^p_\eta$ is the unique (up to the addition of a random constant) solution to the stochastic corrector problem

$$
\begin{align*}
-\text{div} \left[ A_\eta(\cdot,\omega) \left( p + \nabla w^p_\eta(\cdot,\omega) \right) \right] &= 0 \text{ in } \mathbb{R}^d, \\
\nabla w^p_\eta \text{ is stationary in the sense of (8) below,} \\
E \left( \int_Q \nabla w^p_\eta(y,\cdot) \, dy \right) &= 0.
\end{align*}
$$

As in the periodic case, the corrector function $w^p_\eta$ allows to obtain a convergence result in the $H^1$ norm (see [24, Theorem 3]):

$$
E \left[ \left\| u^*_\eta(\cdot,\omega) - \left[ u^*_\eta + \varepsilon \sum_{i=1}^d w^p_{\varepsilon_i}(\cdot,\omega) \frac{\partial u^*_\eta}{\partial x_i} \right] \right\|_{H^1(D)}^2 \right] \text{ converges to } 0 \text{ as } \varepsilon \to 0. \quad (7)
$$

However, in contrast to the periodic case, the rate of convergence is generally not known, in dimensions higher than one. In the one-dimensional case, this question has been addressed in [9, 6]. It is shown there that the rate can be arbitrary small, depending on the rate with which the correlations of the random coefficient in (5) vanish. The only assumptions of stationarity and ergodicity do not allow for a precise rate. See also [20] for the study of a similar question for a variant of stochastic homogenization, again in the one-dimensional case, and [5] for results in the multi-dimensional case, for a different equation.

The aim of this article is to show that, in a weakly stochastic case (the precise sense of which is given below), a convergence rate for (7) can be obtained (in the same spirit as [3]). As in the deterministic case, this result is interesting from the theoretical viewpoint, and somewhat complements the one-dimensional results of [9, 6, 20]. It is also useful from a numerical analysis viewpoint. In [19], we propose an extension of the MsFEM approach to weakly stochastic settings, and we use there the homogenization result that we prove in this work (see Theorem 2 below) to obtain error bounds (see [19, Theorem 10]).

Before presenting our result, let us briefly recall the basic setting of stochastic homogenization. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. For a random variable $X \in L^1(\Omega, d\mathbb{P})$, we denote by $E(X) = \int_\Omega X(\omega) \, d\mathbb{P}(\omega)$ its expectation value. We assume that the group $(\mathbb{Z}^d, +)$ acts on $\Omega$. We denote by $(\tau_k)_{k \in \mathbb{Z}^d}$ this action, and assume that it preserves the measure $\mathbb{P}$, i.e.

$$
\forall k \in \mathbb{Z}^d, \quad \forall A \in \mathcal{F}, \quad \mathbb{P}(\tau_k A) = \mathbb{P}(A).
$$

We assume that $\tau$ is ergodic, that is,

$$
\forall A \in \mathcal{F}, \quad \left( \forall k \in \mathbb{Z}^d, \quad \tau_k A = A \right) \Rightarrow (\mathbb{P}(A) = 0 \text{ or } 1).
$$

We define the following notion of stationarity: any $F \in L^1_{\text{loc}}(\mathbb{R}^d, L^1(\Omega))$ is said to be stationary if

$$
\forall k \in \mathbb{Z}^d, \quad F(x + k, \omega) = F(x, \tau_k \omega) \text{ almost everywhere, almost surely.} \quad (8)
$$
Note that we have chosen to present the theory in a discrete stationary setting, which is more appropriate for our specific purpose, which is to consider a setting close to periodic homogenization. Random homogenization is more often presented in the continuous stationary setting. This is only a matter of small modifications. We refer to the bibliography for the latter.

We now precisely describe the weakly stochastic setting we consider. We assume that the matrix $A_\eta$ in (6) reads

$$A_\eta(x, \omega) = A_{\text{per}}(x) + \eta A_1(x, \omega),$$

where $\eta \in \mathbb{R}$ is small deterministic parameter, $A_{\text{per}}$ is a symmetric uniformly elliptic bounded $Q$-periodic matrix, and $A_1$ is a symmetric matrix, stationary in the sense of (8), and bounded: $|A_1(x, \omega)| \leq C$ almost everywhere in $\mathbb{R}^d$, almost surely. We also assume that $A_\eta$ is uniformly elliptic and bounded, in the sense that, for all $\eta \in \mathbb{R}$,

$$A_\eta(\cdot, \omega) \in (L^\infty(\mathbb{R}^d))^{d \times d} \text{ a.s.}$$

and there exists $c > 0$ such that

$$\forall \xi \in \mathbb{R}^d, \quad \xi^T A_\eta(x, \omega) \xi \geq c \xi^T \xi \quad \text{a.s., a.e. on } \mathbb{R}^d.$$  

We furthermore assume that $A_1$ is of the form

$$A_1(x, \omega) = \sum_{k \in \mathbb{Z}^d} 1_{Q+k}(x) X_k(\omega) B_{\text{per}}(x),$$

where $(X_k(\omega))_{k \in \mathbb{Z}^d}$ is a sequence of i.i.d. scalar random variables such that

$$\exists C, \forall k \in \mathbb{Z}^d, \quad |X_k(\omega)| \leq C \quad \text{almost surely},$$

and $B_{\text{per}} \in (L^\infty(\mathbb{R}^d))^{d \times d}$ is a $Q$-periodic matrix. Finally, we assume that

$$A_{\text{per}} \text{ is Hölder continuous,}$$

$$B_{\text{per}} \text{ is Hölder continuous.}$$

As pointed out above, the symmetry assumption is not essential, and our arguments below carry over to non-symmetric matrices up to slight modifications. Likewise, the assumption (10) can be relaxed. What is important in (10) is that $A_1$ is a sum of direct products of a function depending on $x$ with a random variable, depending only on $\omega$.

In contrast, it is difficult to weaken assumptions (11) and (12), which are used to obtain some regularity on the correctors $u_p^0$ and $\psi_p$, solutions to (4) and (16) below, respectively. We indeed recall that, under assumption (11), we have $u_p^0 \in W^{1,\infty}(\mathbb{R}^d)$ for any $p \in \mathbb{R}^d$ (see e.g. [15] Theorem 8.22 and Corollary 8.36), and similarly for $\psi_p$, under assumption (12). In the sequel, we will use the fact that $u_p^0$ and $\psi_p$ belong to $W^{1,\infty}(\mathbb{R}^d)$, which is a standard assumption when proving convergence rates of two-scale expansions (see e.g. [17] p. 28).

We also note that, following [3], the assumption (11) is useful to characterize the asymptotic behavior of the Green function associated to the operator $L = -\text{div}[A_{\text{per}} \nabla \cdot]$ on the domain $D/\varepsilon$ (with homogeneous Dirichlet boundary conditions). This Green function will be used in the sequel.

**Remark 1.** There are several ways to formalize a notion of "weakly" stochastic setting, and (9) is only one of them. We refer to [18, 4] for other examples.

Our main result is the following.
Theorem 2. Assume that the dimension $d$ is strictly higher than 1. Let $u^\varepsilon_\eta$ be the solution to (1), and assume that $A_{\eta}$ satisfies (9)-(10)-(11)-(12). Let $A^*_{\text{per}}$, $w^0_p$ and $u^0_\eta$ be defined by (3), (4) and (5). Let $B \in \mathbb{R}^{d \times d}$ and $\mathbf{\nu}_1 \in H^1_0(\mathcal{D})$ be defined by

$$\forall 1 \leq i, j \leq d, \quad \mathbf{B}_{ij} = \int_Q (e_i + \nabla w^0_{e_j})^T B_{\text{per}}(e_j + \nabla w^0_{e_j})$$

and

$$\begin{cases}
- \text{div} [A^*_{\text{per}} \nabla \mathbf{\nu}_1] = \text{div} [B \nabla u^\varepsilon_\eta] & \text{in } \mathcal{D}, \\
\mathbf{\nu}_1 = 0 & \text{on } \partial \mathcal{D}.
\end{cases}$$

Introduce $v^\varepsilon_\eta$ defined by

$$v^\varepsilon_\eta(\cdot, \omega) = u^0_\eta + \eta \mathbb{E}(X_0) \mathbf{\nu}_1^\varepsilon + \varepsilon \sum_{p=1}^{d} \left[ w^0_p \left( \frac{\cdot}{\varepsilon} \right) \left( \partial_p u^*_\eta + \eta \mathbb{E}(X_0) \partial_p \mathbf{\nu}_1 \right) \right]$$

$$+ \eta \mathbb{E}(X_0) \psi_p \left( \frac{\cdot}{\varepsilon} \right) \partial_p u^*_\eta + \eta \sum_{k \in I_\varepsilon} (X_k(\omega) - \mathbb{E}(X_0)) \chi_p \left( \frac{\cdot}{\varepsilon} - k \right) \partial_p u^*_\eta,$$

where $\partial_p u^*_\eta$ denotes the partial derivative $\frac{\partial u^*_\eta}{\partial x_p}$,

$$I_\varepsilon = \left\{ k \in \mathbb{Z}^d \text{ such that } \varepsilon(Q + k) \cap \mathcal{D} \neq \emptyset \right\},$$

and where, for any $p \in \mathbb{R}^d$, $\psi_p$ is the solution (unique up to the addition of a constant) to

$$\begin{cases}
- \text{div} [A^*_{\text{per}} \nabla \psi_p] = \text{div} \left[ B_{\text{per}} \left( p + \nabla w^0_p \right) \right], \\
\psi_p \text{ is } Q\text{-periodic},
\end{cases}$$

and $\chi_p$ is the unique solution to

$$\begin{cases}
- \text{div} [A^*_{\text{per}} \nabla \chi_p] = \text{div} [1_Q B_{\text{per}} \left( p + \nabla w^0_p \right)] & \text{in } \mathbb{R}^d, \\
\chi_p \in L^2_{\text{loc}}(\mathbb{R}^d), \\
\nabla \chi_p \in L^2(\mathbb{R}^d)^d, \\
\lim_{|x| \to \infty} \chi_p(x) = 0.
\end{cases}$$

We assume that $u^*_\eta \in W^{2,\infty}(\mathcal{D})$ and $\mathbf{\nu}_1 \in W^{2,\infty}(\mathcal{D})$. Then

$$\sqrt{\mathbb{E}} \left[ \|u^\varepsilon_\eta - v^\varepsilon_\eta\|_{H^1(\mathcal{D})}^2 \right] \leq C \left( \sqrt{\varepsilon} + \eta \sqrt{\varepsilon \ln(1/\varepsilon)} + \eta^2 \right),$$

where $C$ is a constant independent of $\varepsilon$ and $\eta$.

We wish to point out that the assumption $u^*_\eta \in W^{2,\infty}(\mathcal{D})$ (and subsequently $\mathbf{\nu}_1 \in W^{2,\infty}(\mathcal{D})$) is a standard assumption when proving convergence rates of two-scale expansions (see e.g. [11, Theorem 2.1] and [17, p. 28]). Note that, in view of (2), this assumption implies that the right hand side $f$ in (6) belongs to $L^\infty(\mathcal{D})$. We also note that $v^\varepsilon_\eta$ is not uniquely defined, since $w^0_p$ and $\psi_p$ are only defined up to an additive constant. However, adding a constant to any of these functions does not change the order of convergence in (18) with respect to $\varepsilon$ and $\eta$, but only the constant $C$. Choosing the best constants in $w^0_p$ and $\psi_p$ is hence irrelevant here, although it is an important matter from the practical viewpoint. Lastly, the existence and uniqueness of a function $\chi_p$ satisfying (17) is shown in Lemma [10] below, in dimension $d > 1$. In dimension $d = 1$, the boundary conditions of (17) need to be modified for this problem to have a solution. The one-dimensional version of Theorem 2 is as follows:
Theorem 3. Assume that the dimension $d$ is equal to one. Let $u^\varepsilon_\eta$ be the solution to \((\ref{2})\) in the domain $\mathcal{D}$ with $f \in L^2(\mathcal{D})$, and assume that $A_\eta$ satisfies \((\ref{11})-\(\ref{13}\)). Let $v^\varepsilon_\eta$ be defined by \((\ref{15})\), where the definition \((\ref{17})\) of the function $\chi$ is replaced by

$$
\begin{cases}
- [A_{per}\chi]' = [1_{(0,1)}B_{per}(1+(w^0)')]' & \text{in } \mathbb{R}, \\
\chi \in L^2_{loc}(\mathbb{R}), & \chi' \in L^2(\mathbb{R}), 
\end{cases}
$$

where $w^0$ solves \((\ref{3})\). Then

$$
\begin{align*}
\sqrt{\mathbb{E} \left[ \|u^\varepsilon_\eta - v^\varepsilon_\eta\|_{H^1(\mathcal{D})}^2 \right]} & \leq C (\varepsilon + \eta \sqrt{\varepsilon} + \eta^2), \\
\sqrt{\mathbb{E} \left[ \|u^\varepsilon_\eta - v^\varepsilon_\eta\|_{L^\infty(\mathcal{D})}^2 \right]} & \leq C (\varepsilon + \eta \sqrt{\varepsilon} + \eta^2),
\end{align*}
$$

where $C$ is a constant independent of $\varepsilon$ and $\eta$.

Note that, in dimension $d = 1$, we do not need to assume \((\ref{11})\) and \((\ref{12})\). In dimensions $d > 1$, as pointed out above, these assumptions are used to have that the correctors $w^0_p$ and $\psi_p$, solutions to \((\ref{4})\) and \((\ref{16})\) respectively, both belong to $W^{1,\infty}(\mathbb{R}^d)$. In dimension $d = 1$, the coercivity assumption on $A_{per}$ and the boundedness assumption on $B_{per}$ are enough to show that $w^0$ and $\psi$ both belong to $W^{1,\infty}(\mathbb{R})$. Likewise, when $d > 1$, we assumed that $u^0_{\eta} \in W^{2,\infty}(\mathcal{D})$ and $\pi^1_{\eta} \in W^{2,\infty}(\mathcal{D})$ (which implies that $f \in L^\infty(\mathcal{D})$). When $d = 1$, the assumption $f \in L^2(\mathcal{D})$ is enough.

On another note, we notice that $\chi$ is now only defined up to an additive constant. Again, changing $\chi$ by a constant does not change the order of convergence in \((\ref{13})-\(\ref{14}\)) with respect to $\varepsilon$ and $\eta$, but only changes the constant $C$.

In addition to its theoretical interest, Theorem 2 has also interesting numerical counterparts. Indeed, to compute $v^\varepsilon_\eta$, one needs to solve problems set on a bounded domain (either with Dirichlet or periodic boundary conditions), and to solve for $\chi_p$, solution to the problem \((\ref{17})\), set on the entire space. However, the right hand side in \((\ref{17})\) is the divergence of a compactly supported function, and we will see that $\chi_p(x)$ quickly vanishes when $x$ is sufficiently large (see Lemma \((\ref{10})\) below). Hence, in practice, it is possible to approximate \((\ref{17})\) by using Dirichlet boundary conditions on a domain of limited a size.

The proof of Theorem 2 consists of two steps. The first one is to expand $u^\varepsilon_\eta$ with respect to $\eta$. This is performed in Section 2 below (see Lemma \((\ref{4})\)). Each term of the expansion of $u^\varepsilon_\eta$ is found to be the unique solution of a partial differential equation with a deterministic, highly oscillating coefficient, to which is associated a homogenized equation. The second step of the proof consists in successively estimating, for each of the terms of the expansion in $\eta$, the rate of convergence of their two scale expansion in $\varepsilon$. Corresponding results are stated in Section 3 (and proved in Section 4). Collecting these results, we are then in position to prove our main result, Theorem 2 (see Section 4), where we also prove Theorem 3.

2 Expansion in powers of $\eta$

In this section, we expand the solution $u^\varepsilon_\eta$ to \((\ref{3})\) with respect to $\eta$.

Lemma 4. Let $u^\varepsilon_\eta$ be the solution to \((\ref{4})\). Under the assumption \((\ref{14})\), it can be expanded in powers of $\eta$ as follows:

$$
u^\varepsilon_\eta = u^\varepsilon_0 + \eta u^\varepsilon_1 + \eta^2 v^\varepsilon_\eta,$$

where $u^\varepsilon_0$ is solution to the deterministic problem \((\ref{11})\), $u^\varepsilon_1$ is solution to

$$
\begin{align*}
-\text{div} \left[ A_{per} \left( \frac{\cdot}{\varepsilon} \right) \nabla u^\varepsilon_1(\cdot, \omega) \right] &= \text{div} \left[ A_1 \left( \frac{\cdot}{\varepsilon}, \omega \right) \nabla u^\varepsilon_0 \right] & \text{in } \mathcal{D}, \\
u^\varepsilon_1(\cdot, \omega) &= 0 & \text{on } \partial\mathcal{D},
\end{align*}
$$

(21)
and $r_\eta^\varepsilon$ is solution to

\[
\begin{aligned}
-\text{div} \left[ A_\eta \left( \frac{\cdot}{\varepsilon}, \omega \right) \nabla r_\eta^\varepsilon(\cdot, \omega) \right] &= \text{div} \left[ A_1 \left( \frac{\cdot}{\varepsilon}, \omega \right) \nabla u_1^\varepsilon(\cdot, \omega) \right] \quad \text{in } \mathcal{D}, \\
{r_\eta^\varepsilon}(\cdot, \omega) &= 0 \quad \text{on } \partial \mathcal{D}.
\end{aligned}
\]

In addition, we have, almost surely,

\[
\|u_0^\varepsilon\|_{H^1(\mathcal{D})} \leq C, \quad \|u_1^\varepsilon(\cdot, \omega)\|_{H^1(\mathcal{D})} \leq C, \quad \|r_\eta^\varepsilon(\cdot, \omega)\|_{H^1(\mathcal{D})} \leq C,
\]

where $C$ is a deterministic constant independent of $\varepsilon$ and $\eta$.

**Proof.** The relation (21) is a simple consequence of the linearity of the considered equation. The bounds (23) follow from the uniform ellipticity of the matrices $A_\eta$ and $A_{\text{per}}$, and the boundedness of $A_1$.

For the sequel, it is useful to further decompose $u_1^\varepsilon$ in a deterministic part and a stochastic part of vanishing expectation.

**Lemma 5.** Under assumptions (11) - (13), the solution $u_1^\varepsilon$ to (22) writes

\[
u_1^\varepsilon = \mathbb{E}(X_0)\pi_1^\varepsilon + \sum_{k \in \mathbb{Z}^d} (X_k(\omega) - \mathbb{E}(X_0))\phi_k^\varepsilon, \tag{24}\]

where $\pi_1^\varepsilon$ is the unique solution to

\[
\begin{aligned}
-\text{div} \left[ A_{\text{per}} \left( \frac{\cdot}{\varepsilon} \right) \nabla \pi_1^\varepsilon \right] &= \text{div} \left[ B_{\text{per}} \left( \frac{\cdot}{\varepsilon} \right) \nabla u_0^\varepsilon \right] \quad \text{in } \mathcal{D}, \\
\pi_1^\varepsilon &= 0 \quad \text{on } \partial \mathcal{D},
\end{aligned} \tag{25}\]

and $\phi_k^\varepsilon$ is the unique solution to

\[
\begin{aligned}
-\text{div} \left[ A_{\text{per}} \left( \frac{\cdot}{\varepsilon} \right) \nabla \phi_k^\varepsilon \right] &= \text{div} \left[ B_{\text{per}} \left( \frac{\cdot}{\varepsilon} \right) \nabla u_0^\varepsilon \right] \quad \text{in } \mathcal{D}, \\
\phi_k^\varepsilon &= 0 \quad \text{on } \partial \mathcal{D}. \tag{26}\end{aligned}
\]

In addition, there exists $C$, independent of $\varepsilon$, such that

\[
\mathbb{E} \left[ \left\| \sum_{k \in \mathbb{Z}^d} |X_k - \mathbb{E}(X_0)|\phi_k^\varepsilon \right\|_{H^1(\mathcal{D})}^2 \right] \leq C. \tag{27}\]

Assume furthermore that (11) holds, and that

\[
f \in L^q(\mathcal{D}) \quad \text{for some } \quad q > d. \tag{28}\]

Then there exists $C$, independent of $k$ and $\varepsilon$, such that

\[
\|\phi_k^\varepsilon\|_{L^\infty(\mathcal{D})} \leq C\varepsilon. \tag{29}\]

**Proof.** We note that, if $k \in \mathbb{Z}^d$ is such that $\varepsilon(Q + k) \cap \mathcal{D} = \emptyset$, then (26) writes

\[
\begin{aligned}
-\text{div} \left[ A_{\text{per}} \left( \frac{\cdot}{\varepsilon} \right) \nabla \phi_k^\varepsilon \right] &= 0 \quad \text{in } \mathcal{D}, \\
\phi_k^\varepsilon &= 0 \quad \text{on } \partial \mathcal{D},
\end{aligned} \tag{28}\]

and $r_\eta^\varepsilon$ is solution to

\[
\begin{aligned}
-\text{div} \left[ A_\eta \left( \frac{\cdot}{\varepsilon}, \omega \right) \nabla r_\eta^\varepsilon(\cdot, \omega) \right] &= \text{div} \left[ A_1 \left( \frac{\cdot}{\varepsilon}, \omega \right) \nabla u_1^\varepsilon(\cdot, \omega) \right] \quad \text{in } \mathcal{D}, \\
{r_\eta^\varepsilon}(\cdot, \omega) &= 0 \quad \text{on } \partial \mathcal{D}.
\end{aligned} \tag{29}\]
the solution of which is obviously $\phi_k^\varepsilon \equiv 0$. The sum in (24) hence only contains a finite number of terms, and the proof of the decomposition (24) goes by linearity of the equation (22). Note however that the number of terms in (24) depends on $\varepsilon$, and diverges when $\varepsilon \to 0$.

We now prove the bound (27). Using that $A_{per}$ is coercive, we infer from (26) that

$$
\alpha \|\phi_k^\varepsilon\|^2_{H^1(\mathcal{D})} \leq \int_{\mathcal{D}} (\nabla \phi_k^\varepsilon)^T A_{per} \left(\frac{\cdot}{\varepsilon}\right) \nabla \phi_k^\varepsilon \leq \int_{\mathcal{D}} (\nabla \phi_k^\varepsilon)^T B_{per} \left(\frac{\cdot}{\varepsilon}\right) \nabla u_0^\varepsilon \\
\leq \|B_{per}\|_{L^\infty} \|\phi_k^\varepsilon\|_{H^1(\mathcal{D})} \|u_0^\varepsilon\|_{H^1(\varepsilon(Q+k))},
$$

where $\alpha > 0$ is some constant that only depends on the coercivity constant of $A_{per}$ and the Poincaré constant of the domain $\mathcal{D}$. Thus

$$
\|\phi_k^\varepsilon\|^2_{H^1(\mathcal{D})} \leq \alpha^{-2} \|B_{per}\|_{L^\infty} \|u_0^\varepsilon\|^2_{H^1(\varepsilon(Q+k))}.
$$

Using that $\phi_k^\varepsilon \equiv 0$ as soon as $\varepsilon(Q + k) \cap \mathcal{D} = \emptyset$, we obtain

$$
\sum_{k \in \mathbb{Z}^d} \|\phi_k^\varepsilon\|^2_{H^1(\mathcal{D})} \leq \alpha^{-2} \|B_{per}\|_{L^\infty} \|u_0^\varepsilon\|^2_{H^1(\mathcal{D})}.
$$

We deduce from that bound and the assumption that the random variables $X_k$ are i.i.d. that

$$
\mathbb{E} \left[ \left\| \sum_{k \in \mathbb{Z}^d} (X_k - \mathbb{E}(X_0)) \phi_k^\varepsilon \right\|^2_{H^1(\mathcal{D})} \right] = \text{Var}(X_0) \sum_{k \in \mathbb{Z}^d} \|\phi_k^\varepsilon\|^2_{H^1(\mathcal{D})} \leq \text{Var}(X_0) \alpha^{-2} \|B_{per}\|_{L^\infty} \|u_0^\varepsilon\|^2_{H^1(\mathcal{D})} \leq C,
$$

where $C$ is independent of $\varepsilon$ (we have used (23) to bound $u_0^\varepsilon$). We thus have shown (27).

We finally turn to the proof of (29). Let us define $\overline{\phi}_k(x) = \phi_k^\varepsilon(x)$ on $\mathcal{D}/\varepsilon$. In view of (26), we see that $\overline{\phi}_k$ solves

$$
\begin{align*}
-\text{div} \left[ A_{per} \nabla \overline{\phi}_k \right] &= \varepsilon \text{div} \left[ 1_{Q+k} B_{per} \nabla u_0^\varepsilon \right] \quad \text{in } \mathcal{D}/\varepsilon, \\
\overline{\phi}_k &= 0 \quad \text{on } \partial(\mathcal{D}/\varepsilon).
\end{align*}
$$

Introduce now the Green function $\Gamma_\varepsilon(x, y)$ associated to the operator $L = -\text{div} \left[ A_{per} \nabla \cdot \right]$ on the domain $\mathcal{D}/\varepsilon$, with homogeneous Dirichlet boundary conditions. We recall that $\Gamma_T \varepsilon(x, y, z) := \phi_\varepsilon(y, x)$ is the Green function associated to the adjoint operator $L_T = -\text{div} \left[ A_{per} \nabla \cdot \right]$ on the domain $\mathcal{D}/\varepsilon$, with homogeneous Dirichlet boundary conditions (a proof of this fact is given in [10] Theorem 1.3 and [12] Theorem 1) in the case $d \geq 3$, and this proof carries over to the case $d = 2$). Consequently, we have $\Gamma_\varepsilon(x, y, z) = 0$ as soon as $x$ or $y$ belongs to the boundary $\partial(\mathcal{D}/\varepsilon)$. We can thus write

$$
\overline{\phi}_k(x) = \varepsilon \int_{\mathcal{D}/\varepsilon} \Gamma_\varepsilon(x, y) \text{div}_y \left[ 1_{Q+k} B_{per}(y) \nabla u_0^\varepsilon(y) \right] dy
$$

$$
= -\varepsilon \int_{Q+k} \nabla_y \Gamma_\varepsilon(x, y) B_{per}(y) \nabla u_0^\varepsilon(y) dy.
$$

Hence, for any $x \in \mathcal{D}$, we have

$$
\phi_k^\varepsilon(x) = -\varepsilon^{1-d} \int_{\varepsilon(Q+k)} \nabla_y \Gamma_\varepsilon \left( \frac{x - y}{\varepsilon} \right) B_{per} \left( \frac{y}{\varepsilon} \right) \nabla u_0^\varepsilon(y) dy.
$$
Using the fact (see [3 Proposition 8]) that, under assumption (11), the Green function \( \Gamma_\varepsilon \) on the domain \( D/\varepsilon \) satisfies
\[
\forall x \in D/\varepsilon, \quad \forall y \in D/\varepsilon, \quad |\nabla_x \Gamma_\varepsilon(x, y)| + |\nabla_y \Gamma_\varepsilon(x, y)| \leq \frac{C}{|x - y|^{d-1}}
\] (30)
for a constant \( C \) independent of \( \varepsilon \), we have
\[
|\phi_\varepsilon^\varepsilon(x)| \leq C \|B_\text{per}\|_{L^\infty} \|\nabla u_0^\varepsilon\|_{L^\infty(D)} \int_{\varepsilon(Q+k)} \frac{1}{|x - y|^{d-1}} dy.
\] (31)
We will show in the sequel that (28) implies that there exists \( C \) such that, for all \( \varepsilon \),
\[
\|\nabla u_0^\varepsilon\|_{L^\infty(D)} \leq C.
\] (32)
We are thus left with bounding the integral in (31). To this aim, we distinguish two cases. If \( |x - \varepsilon k| \geq \varepsilon \), then any \( x \in \varepsilon(Q + k) \) satisfies \( 7d \leq |x - y| \) for a constant \( 7d \) that only depends on \( d \) (in dimension \( d = 2 \), \( 7d = 1 + \sqrt{2}/2 \)). For those \( x \), we have
\[
\int_{\varepsilon(Q+k)} \frac{1}{|x - y|^{d-1}} dy \leq \int_{B(x, 7d \varepsilon)} \frac{1}{|x - y|^{d-1}} dy \leq C \varepsilon, \quad C \text{ independent of } \varepsilon.
\] (33)
Otherwise, if \( |x - \varepsilon k| \geq \varepsilon \), then any \( y \in \varepsilon(Q + k) \) satisfies \( 7d \leq |x - y| \) for a constant \( 7d \) that only depends on \( d \) (in dimension \( d = 2 \), \( 7d = 1 - \sqrt{2}/2 \)). For those \( y \), we have
\[
\int_{\varepsilon(Q+k)} \frac{1}{|x - y|^{d-1}} dy \leq \int_{\varepsilon(Q+k)} \frac{1}{(|7d| \varepsilon)^{d-1}} \int_{\varepsilon(Q+k)} dy \leq C \varepsilon, \quad C \text{ independent of } \varepsilon.
\] (34)
Thus, collecting (31), (32), (33) and (34), we obtain that
\[
\|\phi_\varepsilon^\varepsilon\|_{L^\infty(D)} \leq C \varepsilon, \quad C \text{ independent of } \varepsilon.
\]
Proving (29) therefore amounts to now proving (32). Again using the Green function \( \Gamma_\varepsilon(x, y) \) associated to the operator \( L = -\text{div} (A_\text{per} \nabla \cdot) \) on the domain \( D/\varepsilon \), with homogeneous Dirichlet boundary conditions, we write
\[
\nabla u_0^\varepsilon(x) = \varepsilon^{1-d} \int_D \nabla_x \Gamma_\varepsilon \left( \frac{x}{\varepsilon}, \frac{y}{\varepsilon} \right) f(y) dy.
\]
Using the bound (31), we deduce that there exists \( C \) independent of \( \varepsilon \) such that
\[
\forall x \in D, \quad |\nabla u_0^\varepsilon(x)| \leq C \int_D \frac{|f(y)|}{|x - y|^{d-1}} dy.
\]
In view of assumption (28), we have \( f \in L^p(D) \) for some \( q > d \). Using Hölder inequality, we write
\[
\forall x \in D, \quad |\nabla u_0^\varepsilon(x)| \leq C \|f\|_{L^p(D)} \left\| \frac{1}{|x - |x|^d} \right\|_{L^q(\varepsilon)} = C \frac{1}{q} + \frac{1}{q^*} = 1.
\]
The function \( y \mapsto |x - y|^{1-d} \) belongs to \( L^p(D) \) for any \( p < d/(d - 1) \). Since \( q > d \), we have \( q^* < d/(d - 1) \), and the norm in \( L^q^* \) of \( y \mapsto |x - y|^{1-d} \) is independent of \( x \). The above estimate thus yields (32). This concludes the proof of Lemma (5).
3 Two scale expansions in powers of $\varepsilon$

Collecting (21) and (24), we have obtained that

$$u_\eta^\varepsilon(x, \omega) = u_0^\varepsilon(x) + \eta \left[ \mathbb{E}(X_0)\tau_1^\varepsilon(x) + \sum_{k \in \mathbb{Z}^d} (X_k(\omega) - \mathbb{E}(X_0))\phi_k^\varepsilon(x) \right] + \eta^2 r_\eta^\varepsilon(x, \omega), \quad (35)$$

where $r_\eta^\varepsilon$ is bounded in $H^1(D)$ uniformly in $\varepsilon$, $\eta$ and $\omega$ (see (23)).

We now consider successively each term of the above series and show a rate of convergence on the difference between $u_\eta^\varepsilon$, $\tau_1^\varepsilon$ and $\phi_k^\varepsilon$ and their respective two-scale expansions. For clarity, the proofs of our results are postponed until Section 5.

We start by $u_0^\varepsilon$ solution to (1). Note that this problem is a classical periodic homogenization problem, the limit of which, when $\varepsilon \to 0$, is well-known. The following result, giving a rate of convergence of $u_0^\varepsilon$ to its homogenized limit, is also classical (see e.g. [17, p. 28]).

**Proposition 6.** Let $u_\eta^\varepsilon$ and $u_0^\varepsilon$ be the solution to (1) and (2), respectively. For any $p \in \mathbb{R}^d$, we assume that the solution $w_0^p$ to (1) satisfies $w_0^p \in W^{1,\infty}(\mathbb{R}^d)$. We also assume that $u_0^\varepsilon \in W^{2,\infty}(D)$. We then have

$$u_0^\varepsilon = u_0^\varepsilon + \varepsilon \sum_{i=1}^d w_{0,i}^p \left( \frac{x}{\varepsilon} \right) \partial_i u_0^\varepsilon + \varepsilon \theta_0^\varepsilon, \quad (36)$$

where $\theta_0^\varepsilon$ satisfies

$$\|\varepsilon \theta_0^\varepsilon\|_{H^1(D)} \leq C\sqrt{\varepsilon} \quad (37)$$

for a constant $C$ independent of $\varepsilon$.

We recall that, under assumption (11), we indeed have that $w_0^p \in W^{1,\infty}(\mathbb{R}^d)$ for any $p \in \mathbb{R}^d$ (see e.g. [15, Theorem 8.22 and Corollary 8.36]).

We now consider successively each term of the above series and show a rate of convergence on the difference between $u_\eta^\varepsilon$, $\tau_1^\varepsilon$ and $\phi_k^\varepsilon$ and their respective two-scale expansions. For clarity, the proofs of our results are postponed until Section 5.

We now turn to $\tau_1^\varepsilon$ solution to (25). This problem is not a classical homogenization problem, since its right-hand side also varies at the scale $\varepsilon$, and only weakly converges in $H^{-1}(D)$ when $\varepsilon \to 0$. We first proceed formally, using the two-scale ansatz approach, to identify the homogenized equation. We next state a precise homogenization result, and finally evaluate the rate of convergence of the two scale expansion.

To derive formally the homogenized equation associated to (25), we make the classical two-scale ansatz

$$\tau_1^\varepsilon(x) = \tau_1 \left( x, \frac{x}{\varepsilon} \right) + \varepsilon \tau^0_1 \left( x, \frac{x}{\varepsilon} \right) + \varepsilon^2 \tau^0_1 \left( x, \frac{x}{\varepsilon} \right) + \cdots,$$

where each term of the above expansion is assumed to be periodic with respect to the second variable. Inserting this ansatz in (25) and using the two scale expansion (36) of $u_0^\varepsilon$ (where we neglect the remainder $\varepsilon \theta_0^\varepsilon$), we can easily derive a hierarchy of equations. We deduce from the equation of order $\varepsilon^{-2}$ that $\tau_1^\varepsilon$ is independent of its second variable: $\tau_1^\varepsilon(x,y) \equiv \tau_1^\varepsilon(x)$. The equation of order $\varepsilon^{-1}$ reads

$$-\text{div} \left[ A_{\text{per}}(y) \left( \nabla_x \tau_1^\varepsilon(x) + \nabla_y \tau_1^\varepsilon(x, y) \right) \right] = \sum_{i=1}^d \partial_i u_0^\varepsilon(x) \text{div} \left[ B_{\text{per}}(y) \left( \epsilon_i + \nabla_y w_0^\varepsilon(\epsilon_i) \right) \right].$$

Using the functions $w_0^p$ and $\psi_p$ defined by (11) and (16), we thus see that

$$\tau_1^\varepsilon(x,y) = \tau(x) + \sum_{i=1}^d \partial_i u_0^\varepsilon(x) \psi_{\epsilon_i}(y) + \partial_i \tau_1^\varepsilon(x) w_0^\varepsilon(\epsilon_i), \quad (38)$$

where each term of the above series and show a rate of convergence on the difference between $u_\eta^\varepsilon$, $\tau_1^\varepsilon$ and $\phi_k^\varepsilon$ and their respective two-scale expansions. For clarity, the proofs of our results are postponed until Section 5.
where $\tau$ is an undetermined function that only depends on $x$. We are now in position to use the equation of order $\varepsilon^0$, which reads (recall we have neglected the remainder $\varepsilon\theta_0$ in (36))

\[- \text{div}_x \left[ A_{\text{per}}(y) \left( \nabla_x \bar{u}_1^\varepsilon(x) + \nabla_y \bar{u}_1^\varepsilon(x,y) \right) \right] - \text{div}_y \left[ A_{\text{per}}(y) \left( \nabla_x \bar{u}_1^\varepsilon(x,y) + \nabla_y \bar{u}_1^\varepsilon(x,y) \right) \right] = \sum_{i=1}^d \text{div}_x \left[ B_{\text{per}}(y) \left( e_i + \nabla_y w_{e_i}^0(y) \right) \partial_i u_0^\varepsilon \right] + \text{div}_y \left[ \sum_{i=1}^d w_{e_i}^0(y) B_{\text{per}}(y) \nabla_x \partial_i u_0^\varepsilon(x) \right].\]

We close the hierarchy by integrating the above equation over the variable $y \in Q$, using that $y \mapsto \bar{u}_1^\varepsilon(x,y)$ is $Q$-periodic. Using (38) and the expression (3), we then obtain that $\bar{u}_1^\varepsilon$ satisfies

\[
\begin{cases}
- \text{div} \left[ A_{\text{per}}^* \nabla \bar{u}_1^\varepsilon \right] = \text{div} \left[ \tilde{B} \nabla u_0^\varepsilon \right] & \text{in } \mathcal{D}, \\
\bar{u}_1^\varepsilon = 0 & \text{on } \partial \mathcal{D},
\end{cases}
\]

with

\[
\forall 1 \leq i, j \leq d, \quad \tilde{B}_{ij} = \int_Q e_i^T A_{\text{per}} \nabla \psi_{e_j} + \int_Q e_j^T B_{\text{per}}(e_j + \nabla w_{e_j}^0).
\]

Mutiplying (10) (for $p = e_j$) by $w_{e_i}^0$ and integrating over $Q$, we find that

\[
\forall 1 \leq i, j \leq d, \quad \int_Q (\nabla w_{e_i}^0)^T A_{\text{per}} \nabla \psi_{e_j} = - \int_Q (\nabla w_{e_i}^0)^T B_{\text{per}}(e_j + \nabla w_{e_j}^0).
\]

Inserting this relation in (10), we deduce that the matrix $\tilde{B}$ is equal to the matrix $\overline{B}$ defined by (13). We hence deduce from (39) that $\bar{u}_1^\varepsilon$ indeed satisfies (14).

These formal computations are formalized in a rigorous way in the following Propositions:

**Proposition 7.** Assume that, for any $p \in \mathbb{R}^d$, the corrector $w_p^0$ solution to (1) satisfies $w_p^0 \in W^{1,\infty}(\mathbb{R}^d)$, and that the solution $u_0^\varepsilon$ to (2) satisfies $u_0^\varepsilon \in W^{2,\infty}(\mathcal{D})$. Then the function $\bar{u}_1^\varepsilon$ solution to (25) converges, weakly in $H^1(\mathcal{D})$ and strongly in $L^2(\mathcal{D})$, to the unique solution $\bar{u}_1^\varepsilon$ to (14).

The regularity assumptions on $w_p^0$ and $u_0^\varepsilon$ ensure that $\nabla u_0^\varepsilon$ in the right-hand side of (25) can be controlled in the appropriate norm.

**Proposition 8.** Let $\bar{u}_1^\varepsilon$ be the solution to (25), $\bar{u}_1^\varepsilon$ be the solution to (14) and $u_0^\varepsilon$ be the solution to (2). For any $p \in \mathbb{R}^d$, let $w_p^0$ be the solution to (1) and $\psi_p$ be the solution to (10).

Introduce $\bar{u}_1^\varepsilon$ defined by

\[
\bar{u}_1^\varepsilon = \bar{u}_1^\varepsilon + \varepsilon \sum_{i=1}^d \left( w_{e_i}^0 \left( \frac{\cdot}{\varepsilon} \right) \partial_i \bar{u}_1^\varepsilon + \psi_{e_i} \left( \frac{\cdot}{\varepsilon} \right) \partial_i u_0^\varepsilon \right),
\]

and assume that $u_0^\varepsilon \in W^{2,\infty}(\mathcal{D})$, $\bar{u}_1^\varepsilon \in W^{2,\infty}(\mathcal{D})$, and that, for any $p \in \mathbb{R}^d$, we have $w_p^0 \in W^{1,\infty}(\mathbb{R}^d)$ and $\psi_p \in W^{1,\infty}(\mathbb{R}^d)$. We then have

\[
\| \bar{u}_1^\varepsilon - \bar{u}_1 \|_{H^1(\mathcal{D})} \leq C \varepsilon
\]

for a constant $C$ independent of $\varepsilon$.

Again, under assumptions (11) and (12), we have $w_p^0 \in W^{1,\infty}(\mathbb{R}^d)$ and $\psi_p \in W^{1,\infty}(\mathbb{R}^d)$ for any $p \in \mathbb{R}^d$ (see e.g. [15] Theorem 8.22 and Corollary 8.36).

We finally turn to $\phi_k^\varepsilon$ solution to (26), namely

\[
\begin{cases}
- \text{div} \left[ A_{\text{per}} \left( \frac{\cdot}{\varepsilon} \right) \nabla \phi_k^\varepsilon \right] = \text{div} [c_k^\varepsilon] & \text{in } \mathcal{D}, \\
\phi_k^\varepsilon = 0 & \text{on } \partial \mathcal{D},
\end{cases}
\]
with
\[ c_k^\varepsilon(x) = 1_{Q+k}(\varepsilon) B_{\text{per}}(\varepsilon) \nabla u_0^\varepsilon. \]

Assume momentarily that the sequence \( \nabla u_0^\varepsilon \) is bounded in \( L^\infty(D) \) (we have proved such a bound above, see (42)), under the strong assumptions (28) and (11)). Then, for any \( k \in \mathbb{Z}^d \), \( c_k^\varepsilon \) converges to 0 in \( L^2(D) \). Using the coercivity of \( A_{\text{per}} \), this implies that \( \phi_k^\varepsilon \) converges to 0 in \( H^1(D) \). We thus have the following result, which will be rigourously proved in Section 5 below:

**Proposition 9.** Let \( \phi_k^\varepsilon \) be the solution to (26), and let \( u_0^\varepsilon \) and \( w_0^\varepsilon \) be the solutions to (2) and (4). Assume that \( u_0^\varepsilon \in W^{2,\infty}(D) \) and that, for any \( p \in \mathbb{R}^d \), we have \( w_0^\varepsilon \in W^{1,\infty}(\varepsilon^2) \). Then, \( \phi_k^\varepsilon \) converges to 0 in \( H^1(D) \).

To describe more precisely the behavior of \( \phi_k^\varepsilon \), we need to introduce the auxiliary function \( \chi_p \) defined by (41) below. Recall first that \( Q = (-1/2, 1/2)^d \). Following the same arguments as in [9, Lemma 4], we have the following result, which will be useful in the sequel.

**Lemma 10.** For any \( p \in \mathbb{R}^d \), the problem
\[
\begin{cases}
-\text{div}[A_{\text{per}} \nabla \chi_p] = \text{div}[1_{Q} B_{\text{per}}(p + \nabla w_0^\varepsilon)] & \text{in } \mathbb{R}^d, \\
\chi_p \in L^2_{\text{loc}}(\mathbb{R}^d), \quad \nabla \chi_p \in (L^2(\mathbb{R}^d))^d,
\end{cases}
\]
has a solution which is unique up to the addition of a constant. In addition, under assumption (11), there exists a solution of (41) and a constant \( C > 0 \) such that
\[
\forall x \in \mathbb{R}^d \text{ with } |x| \geq 1, \quad |\nabla \chi_p| \leq \frac{C}{|x|^d},
\]
\[
\forall x \in \mathbb{R}^d, \quad |\chi_p| \leq \frac{C}{1 + |x|^{d-1}}.
\]
In the sequel, we will always refer to that particular solution of (41).

We are now in position to make precise the behavior of \( \phi_k^\varepsilon \) in the \( H^1 \) norm. Let us first argue formally. Introduce the matrix \( E_k = 1_{Q+k} B_{\text{per}} \). Using the periodicity of \( A_{\text{per}} \), \( B_{\text{per}} \) and \( w_0^\varepsilon \), and after changing variables, we recast (41) as
\[-\text{div} \left[ A_{\text{per}} \left( \frac{x}{\varepsilon} \right) \nabla \chi_p \left( \frac{x}{\varepsilon} - k \right) \right] = \text{div} \left[ E_k \left( \frac{x}{\varepsilon} \right) \left( p + \nabla w_0^\varepsilon \left( \frac{x}{\varepsilon} \right) \right) \right].
\]
In turn, the problem (26) reads
\[-\text{div} \left[ A_{\text{per}} \left( \frac{x}{\varepsilon} \right) \nabla \phi_k^\varepsilon \right] = \text{div} \left[ E_k \left( \frac{x}{\varepsilon} \right) \nabla u_0^\varepsilon \right] \approx \sum_{i=1}^d \text{div} \left[ E_k \left( \frac{x}{\varepsilon} \right) \partial_i u_0^\varepsilon \left( \frac{x}{\varepsilon} - k \right) \right],
\]
where we have used the expansion (30) of \( u_0^\varepsilon \) (in which we have only kept the highest order terms). Assuming that, in the above equation, \( x \) and \( x/\varepsilon \) are independent variables, we thus see that \( \nabla \phi_k^\varepsilon(x) \approx \sum_{i=1}^d \partial_i u_0^\varepsilon(x) \nabla \chi_{\varepsilon_i} \left( \frac{x}{\varepsilon} - k \right) \), and thus \( \phi_k^\varepsilon(x) \approx \sum_{i=1}^d \partial_i u_0^\varepsilon(x) \chi_{\varepsilon_i} \left( \frac{x}{\varepsilon} - k \right) \). These formal manipulations motivate the following result, the rigorous proof of which is postponed until Section 6.

**Proposition 11.** Let \( \phi_k^\varepsilon \) be the solution to (26) and \( \chi_{\varepsilon_i} \) be the solution to (41), for \( 1 \leq i \leq d \). Introduce
\[
I_\varepsilon = \left\{ k \in \mathbb{Z}^d \text{ such that } \varepsilon(Q + k) \cap D \neq \emptyset \right\}, \quad \text{Card}(I_\varepsilon) \sim \varepsilon^{-d},
\]
(44)
and
\[ \Pi_k = \varepsilon \sum_{i=1}^{d} \chi_{e_i} \left( \frac{\cdot}{\varepsilon} - k \right) \partial_i u^*_0, \]
where \( u^*_0 \) is solution to (2). Assume that \( u^*_0 \in W^{2,\infty}(\mathcal{D}) \), and that (11) holds. We then have
\[ \sum_{k \in I_{\varepsilon}} \| \phi^*_k - \Pi_k \|_{H^1(\mathcal{D})}^2 \leq C\varepsilon \ln(1/\varepsilon), \]
where \( C \) is a constant independent of \( \varepsilon \).

4 Proofs of Theorems 2 and 3

Proof of Theorem 2. We have shown above (see (35)) that
\[ u^*_\varphi(x, \omega) = u^*_0(x) + \eta \mathbb{E}(X_0)\Pi_1(x) + \eta \sum_{k \in I_{\varepsilon}} (X_k(\omega) - \mathbb{E}(X_0))\phi^*_k(x) + \eta^2 r^*_\varphi(x, \omega), \]
where the set \( I_{\varepsilon} \) is defined by (14) (recall that \( \phi^*_k \equiv 0 \) whenever \( k \in \mathbb{Z}^d \) is such that \( k \not\in I_{\varepsilon} \)). Using the fact that \( X_k \) are i.i.d. scalar random variables, we have
\[ \mathbb{E} \left[ \| u^*_\varphi - u^*_\eta \|_{H^1(\mathcal{D})}^2 \right] \leq C \left[ D_0^2 + D_1^2 + D_2^2 + D_3^2 \right], \]
where
\[
D_0 = \left\| u^*_0 - u^*_\varphi - \varepsilon \sum_{p=1}^{d} w^0_p \left( \frac{\cdot}{\varepsilon} \right) \partial_p u^*_0 \right\|_{H^1(\mathcal{D})},
\]
\[
D_1 = \eta \mathbb{E}(X_0) \left\| \Pi_1 - u^*_\varphi - \varepsilon \sum_{p=1}^{d} \left( w^0_\varphi \left( \frac{\cdot}{\varepsilon} \right) \partial_p \Pi_1 + \psi_\varphi \left( \frac{\cdot}{\varepsilon} \right) \partial_p u^*_0 \right) \right\|_{H^1(\mathcal{D})},
\]
\[
D_2 = \eta \sqrt{\text{Var}(X_0)} \sqrt{\sum_{k \in I_{\varepsilon}} \left\| \phi^*_k - \varepsilon \sum_{p=1}^{d} X_p \left( \frac{\cdot}{\varepsilon} - k \right) \partial_p u^*_0 \right\|_{H^1(\mathcal{D})}^2},
\]
\[
D_3 = \eta^2 \sqrt{\mathbb{E} \left[ \| r^*_\varphi \|_{H^1(\mathcal{D})}^2 \right]}.
\]
We have shown in Propositions 6, 8 and 11 that \( D_0 \leq C\sqrt{\varepsilon} \), \( D_1 \leq C\eta\sqrt{\varepsilon} \) and \( D_2 \leq C\eta\sqrt{\varepsilon \ln(1/\varepsilon)} \) respectively, for a constant \( C \) independent of \( \varepsilon \) and \( \eta \) (note that all assumptions of these propositions are satisfied since, in view of (11) and (12), we have \( u^*_0 \in W^{1,\infty}(\mathbb{R}^d) \) and \( \psi_\varphi \in W^{1,\infty}(\mathbb{R}^d) \) for any \( p \in \mathbb{R}^d \)). Next, using Lemma 3, we see that \( D_3 \leq C\eta^2 \) for a constant \( C \) independent of \( \varepsilon \) and \( \eta \). This concludes the proof of (18).

Proof of Theorem 3. To fix the idea, we choose \( \mathcal{D} = (0,1) \). We again argue on the basis of (35). Tedious but straightforward computations show that, in dimension one, the estimates of Propositions 6, 8 and 11 read
\[
\left\| \frac{d\theta^0_\varphi}{dx} \right\|_{L^2(0,1)} \leq C\varepsilon, \quad \left\| \frac{d\Pi_1}{dx} - \frac{d\Pi_1}{dx} \right\|_{L^2(0,1)} \leq C\varepsilon, \quad \sum_{k \in I_{\varepsilon}} \left\| \frac{d\phi^*_k}{dx} - \frac{d\Pi_k}{dx} \right\|_{L^2(0,1)}^2 \leq C\varepsilon.
\]
We thus obtain
\[ \sqrt{E \left[ \left\| \frac{du_\eta}{dx} - \frac{dv_\eta}{dx} \right\|^2_{L^2(0,1)} \right]} \leq C \left( \varepsilon + \eta \sqrt{\varepsilon + \eta^2} \right). \] (45)

We next write that, almost surely,
\[ \left\| u_\eta^\varepsilon(\cdot, \omega) - v_\eta^\varepsilon(\cdot, \omega) \right\|_{L^\infty(0,1)} \leq \left\| \frac{du_\eta}{dx}(\cdot, \omega) - \frac{dv_\eta}{dx}(\cdot, \omega) \right\|_{L^2(0,1)} + \left| u_\eta^\varepsilon(0, \omega) - v_\eta^\varepsilon(0, \omega) \right| . \] (46)

Using that \( u_\eta^\varepsilon(0, \omega) = u_0^\varepsilon(0) = \psi^0(0) = 0 \) and that \( w^0 \) and \( \psi \) belong to \( L^\infty(\mathbb{R}) \), we obtain that
\[ \left| u_\eta^\varepsilon(0, \omega) - v_\eta^\varepsilon(0, \omega) \right| \leq C \varepsilon + C \eta \varepsilon \left( u_0^\varepsilon(\cdot, \omega) \right) \sum_{k \in I_\varepsilon} \left( X_k(\omega) - \mathbb{E}(X_0) \right) \chi(-k), \]

hence, using that \( \chi \in L^\infty(\mathbb{R}) \), we have
\[ E \left[ \left| u_\eta^\varepsilon(0, \omega) - v_\eta^\varepsilon(0, \omega) \right|^2 \right] \leq C \varepsilon^2 + C \text{Var}(X_0) \varepsilon^2 \eta^2 \sum_{k \in I_\varepsilon} \chi^2(-k) \leq C \varepsilon^2 + C \eta^2 \varepsilon. \]

Collecting this result with (45) and (46) yields the bound (20). Likewise, collecting (20) and (45), we obtain the bound (19). This concludes the proof of Theorem 3. \( \square \)

5 Proofs of the two scale expansions

We collect in this section the proofs of the results stated in Section 3. The following technical result, already present in [17, p. 27], and that we recall here for the sake of completeness, will be useful.

**Lemma 12.** Let \( D \) be a bounded open set of \( \mathbb{R}^d \). Consider \( Z \in (L^2_{\text{loc}}(\mathbb{R}^d))^d \) a \( Q \)-periodic vector field such that

\[ \text{div} \ (Z) = 0 \quad \text{and} \quad \int_Q Z = 0. \]

Then, for any \( v \in W^{1,\infty}(D) \), we have
\[ \left\| \text{div} \left[ Z \left( \frac{\cdot}{\varepsilon} \right) v \right] \right\|_{H^{-1}(D)} \leq C \varepsilon \left\| \nabla v \right\|_{L^\infty(D)}, \]

where \( C \) is a constant independent of \( \varepsilon \) and \( v \).

Note that, as \( Z \) is divergence free, we have
\[ \text{div} \left[ Z \left( \frac{\cdot}{\varepsilon} \right) v \right] = Z \left( \frac{\cdot}{\varepsilon} \right) \cdot \nabla v. \]

Since \( Z \) is \( Q \)-periodic, this quantity converges weakly in \( L^2(D) \) to \( \langle Z \rangle \cdot \nabla v = 0 \), as the average of \( Z \) vanishes. The above result hence shows that, in the \( H^{-1}(D) \) norm, the above quantity vanishes at the rate \( \varepsilon \).

**Proof.** In view of the assumptions of \( Z \), there exists (see [17, p. 6]) a skew symmetric matrix \( J \) such that,
\[ \forall 1 \leq j \leq d, \quad Z_j = \sum_{i=1}^d \frac{\partial J_{ij}}{\partial x_i} \]

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and
\[ \forall 1 \leq i, j \leq d, \quad J_{ij} \in H^1_{\text{loc}}(\mathbb{R}^d), \quad J_{ij} \text{ is } Q\text{-periodic, } \int_Q J_{ij} = 0. \]
The \( j \)-th coordinate of the vector \( Z\left(\frac{x}{\varepsilon}\right) v \) reads
\[
\left[ Z\left(\frac{x}{\varepsilon}\right) v(x) \right]_j = \sum_{i=1}^d \frac{\partial J_{ij}}{\partial x_i} \left(\frac{x}{\varepsilon}\right) v(x) \\
= \varepsilon \sum_{i=1}^d \frac{\partial}{\partial x_i} \left( J_{ij} \left(\frac{x}{\varepsilon}\right) v(x) \right) - \varepsilon \sum_{i=1}^d J_{ij} \left(\frac{x}{\varepsilon}\right) \frac{\partial v}{\partial x_i}(x) \\
= \varepsilon \tilde{B}_j(x) - \varepsilon B_j(x),
\]
where
\[
B_j(x) = \sum_{i=1}^d J_{ij} \left(\frac{x}{\varepsilon}\right) \frac{\partial v}{\partial x_i}(x) \quad \text{and} \quad \tilde{B}_j(x) = \sum_{i=1}^d \frac{\partial}{\partial x_i} \left( J_{ij} \left(\frac{x}{\varepsilon}\right) v(x) \right).
\]
The vector \( \tilde{B}(x) \) is divergence free as \( J \) is skew symmetric. For any \( \phi \in H^1_{\text{loc}}(\mathcal{D}) \), we thus have
\[
\left\langle \text{div} \left[ Z\left(\frac{x}{\varepsilon}\right) v \right], \phi \right\rangle = -\varepsilon \left\langle \text{div} [B], \phi \right\rangle \\
= \varepsilon \int_{\mathcal{D}} B \cdot \nabla \phi \\
= \varepsilon \sum_{i,j=1}^d \int_{\mathcal{D}} \partial_j \phi J_{ij} \left(\frac{x}{\varepsilon}\right) \partial_i v,
\]
hence
\[
\left| \left\langle \text{div} \left[ Z\left(\frac{x}{\varepsilon}\right) v \right], \phi \right\rangle \right| \leq \varepsilon \| \nabla v \|_{L^\infty(\mathcal{D})} \| \phi \|_{H^1(\mathcal{D})} \sum_{i,j=1}^d \left\| J_{ij} \left(\frac{x}{\varepsilon}\right) \right\|_{L^2(\mathcal{D})} \\
\leq \varepsilon \| \nabla v \|_{L^\infty(\mathcal{D})} \| \phi \|_{H^1(\mathcal{D})} \sum_{i,j=1}^d \| J_{ij} \|_{L^2(Q)}.
\]
As the above bound holds for any \( \phi \in H^1_{\text{loc}}(\mathcal{D}) \), we deduce that there exists \( C \) such that, for any \( v \in W^{1,\infty}(\mathcal{D}) \) and any \( \varepsilon \), we have
\[
\left\| \text{div} \left[ Z\left(\frac{x}{\varepsilon}\right) v \right] \right\|_{H^{-1}(\mathcal{D})} \leq C\varepsilon \| \nabla v \|_{L^\infty(\mathcal{D})}.
\]
This concludes the proof.

5.1 Two scale expansion of \( \pi_1 \)

In this section, we prove Propositions 7 and 8.

Proof of Proposition 7. This homogenization result is proved using the method of oscillating test functions [23, 25]. The variational formulation of (46) reads
\[
\forall v \in H^1(\mathcal{D}), \quad A_\varepsilon(\pi_1, v) = -L_\varepsilon(v),
\]

(47)
where, for any $u$ and $v$ in $H^1_0(D)$,

$$A_\varepsilon(u, v) = \int_D (\nabla v)^T A_{per} \left( \frac{\cdot}{\varepsilon} \right) \nabla u \quad \text{and} \quad L_\varepsilon(v) = \int_D (\nabla v)^T B_{per} \left( \frac{\cdot}{\varepsilon} \right) \nabla u_0^\varepsilon.$$  

Using the coercivity of $A_{per}$, the boundedness of $B_{per}$ and (23), and taking $v = \mathbf{\Pi}_1$ as a function test in (47), we obtain that $\mathbf{\Pi}_1$ is bounded in $H^1_0(D)$. Thus, using the Rellich Theorem, we deduce that there exists $\mathbf{\Pi}_1 \in H^1_0(D)$ such that, up to the extraction of a subsequence,

$$\mathbf{\Pi}_1 \text{ converges to } \mathbf{\Pi}_1, \text{ weakly in } H^1_0(D) \text{ and strongly in } L^2(D).$$

For any function $\varphi \in C_0^\infty(D)$, define the test function

$$v^\varepsilon = \varphi + \varepsilon \sum_{i=1}^d w_0^{i} \left( \frac{\cdot}{\varepsilon} \right) \partial_i \varphi,$$

which obviously belongs to $H^1_0(D)$. In view of (47), we have

$$A_\varepsilon(\mathbf{\Pi}_1, v^\varepsilon) = -L_\varepsilon(v^\varepsilon). \quad (48)$$

We now expand both sides of (48) in powers of $\varepsilon$:

$$A_\varepsilon(\mathbf{\Pi}_1, v^\varepsilon) = A^0_\varepsilon(\mathbf{\Pi}_1, \varphi) + \varepsilon A^1_\varepsilon(\mathbf{\Pi}_1, \varphi), \quad L_\varepsilon(v^\varepsilon) = L^0_\varepsilon(\varphi) + \varepsilon L^1_\varepsilon(\varphi), \quad (49) \quad (50)$$

where

$$A_\varepsilon^0(\mathbf{\Pi}_1, \varphi) = \int_D \left( \nabla \varphi + \sum_{i=1}^d \nabla w_0^i \left( \frac{\cdot}{\varepsilon} \right) \partial_i \varphi \right)^T A_{per} \left( \frac{\cdot}{\varepsilon} \right) \nabla \mathbf{\Pi}_1,$$

$$A_\varepsilon^1(\mathbf{\Pi}_1, \varphi) = \int_D \sum_{i=1}^d w_0^i \left( \frac{\cdot}{\varepsilon} \right) (\nabla \partial_i \varphi)^T A_{per} \left( \frac{\cdot}{\varepsilon} \right) \nabla \mathbf{\Pi}_1,$$

$$L_\varepsilon^0(\varphi) = \int_D \left( \nabla \varphi + \sum_{i=1}^d \nabla w_0^i \left( \frac{\cdot}{\varepsilon} \right) \partial_i \varphi \right)^T B_{per} \left( \frac{\cdot}{\varepsilon} \right) \nabla u_0^\varepsilon,$$

$$L_\varepsilon^1(\varphi) = \int_D \sum_{i=1}^d w_0^i \left( \frac{\cdot}{\varepsilon} \right) (\nabla \partial_i \varphi)^T B_{per} \left( \frac{\cdot}{\varepsilon} \right) \nabla u_0^\varepsilon.$$

We now successively study the limit of these four quantities as $\varepsilon \to 0$. Using (23), the fact that $w_0^i \in W^{1,\infty}(\mathbb{R}^d)$, that $\mathbf{\Pi}_1$ is bounded in $H^1(D)$ and the boundedness of $A_{per}$ and $B_{per}$, we obtain

$$|A_\varepsilon^1(\mathbf{\Pi}_1, \varphi)| \leq C \quad \text{and} \quad |L_\varepsilon^1(\varphi)| \leq C, \quad C \text{ independent of } \varepsilon. \quad (51)$$

We now turn to $L_\varepsilon^0$. Using the two scale expansion (36) of $u_0^\varepsilon$, we see that

$$L_\varepsilon^0(\varphi) = L_\varepsilon^{00}(\varphi) + L_\varepsilon^{01}(\varphi) + L_\varepsilon^{02}(\varphi), \quad (52)$$
where
\[ L_ε^{00}(φ) = \sum_{i,j=1}^{d} \int_{D} (e_i + \nabla w_{ε_i}^0 \left( \frac{\cdot}{ε} \right))^T B_{per} \left( \frac{\cdot}{ε} \right) (e_j + \nabla w_{ε_j}^0 \left( \frac{\cdot}{ε} \right)) \partial_i φ \partial_j u_0^*, \]
\[ L_ε^{01}(φ) = ε \sum_{i,j=1}^{d} \int_{D} (e_i + \nabla w_{ε_i}^0 \left( \frac{\cdot}{ε} \right))^T B_{per} \left( \frac{\cdot}{ε} \right) w_{ε_j}^0 \left( \frac{\cdot}{ε} \right) (\nabla \partial_j u_0^*) \partial_i φ, \]
\[ L_ε^{02}(φ) = \sum_{i=1}^{d} \int_{D} (e_i + \nabla w_{ε_i}^0 \left( \frac{\cdot}{ε} \right))^T B_{per} \left( \frac{\cdot}{ε} \right) ε\nabla θ_0^* \partial_i φ. \]

Using (57), \( w_{ε_i}^0 \in W^{1,∞}(\mathbb{R}^d) \) and \( u_0^* \in W^{2,∞}(D) \), we obtain that
\[ |L_ε^{02}(φ)| ≤ C\|θ_0^*\|_{H^1(D)} ≤ C\sqrt{ε} \quad \text{and} \quad |L_ε^{01}(φ)| ≤ Cε, \tag{53} \]
where \( C \) is a constant independent of \( ε \). Turning to \( L_ε^{00} \), we see, using that \( B_{per} \) and \( w_{ε_i}^0 \) are \( Q \)-periodic, that
\[ \left( e_i + \nabla w_{ε_i}^0 \left( \frac{\cdot}{ε} \right) \right)^T B_{per} \left( \frac{\cdot}{ε} \right) \left( e_j + \nabla w_{ε_j}^0 \left( \frac{\cdot}{ε} \right) \right) → B_{ij} \quad \text{weakly-* in } L^∞, \]
where \( B \) is defined by (13). Thus
\[ L_ε^{00}(φ) → \int_{D} (∇φ)^T B∇ u_0^* \quad \text{as } ε \to 0. \tag{54} \]

Collecting (52), (53) and (54), we obtain that
\[ L_ε^0(φ) → \int_{D} (∇φ)^T B∇ u_0^* \quad \text{as } ε \to 0. \tag{55} \]

We next turn to \( A_ε^0 \). Using that \( \text{div} \left[ A_{per} \left( \frac{\cdot}{ε} \right) \left( e_i + \nabla w_{ε_i}^0 \left( \frac{\cdot}{ε} \right) \right) \right] = 0 \) and that \( A_{per} \) is symmetric, we obtain that
\[ A_ε^0(\overline{u}_1, φ) = -\sum_{i=1}^{d} \int_{D} \overline{u}_1^i (∇θ_i φ)^T A_{per} \left( \frac{\cdot}{ε} \right) \left( e_i + \nabla w_{ε_i}^0 \left( \frac{\cdot}{ε} \right) \right). \tag{56} \]

Recall now that \( \overline{u}_1 \to \overline{u}_1^\ast \) strongly in \( L^2(D) \) and that, as \( A_{per} \) and \( w_{ε_i}^0 \) are \( Q \)-periodic, we have
\[ A_{per} \left( \frac{\cdot}{ε} \right) \left( e_i + \nabla w_{ε_i}^0 \left( \frac{\cdot}{ε} \right) \right) → ∫_Q A_{per} \left( e_i + \nabla w_{ε_i}^0 \left( \frac{\cdot}{ε} \right) \right) = A_{per}^* e_i \quad \text{weakly-* in } L^∞, \]
where \( A_{per}^* \) is defined by (14). We thus deduce from (56) that
\[ A_ε^0(\overline{u}_1, φ) → -\sum_{i=1}^{d} \int_{D} \overline{u}_1^i (∇∂_i φ)^T A_{per}^* e_i \quad \text{as } ε \to 0. \]

Collecting (48), (49), (50), (51), the above limit and (55), we obtain that \( \overline{u}_1 \) satisfies
\[ -\sum_{i=1}^{d} \int_{D} \overline{u}_1^i (∇∂_i φ)^T A_{per}^* e_i = -\int_{D} (∇φ)^T B∇ u_0^* \]
for any \( φ \in C_0^∞(D) \). This shows that \( \overline{u}_1 \) solves (14) (which has a unique solution) and thus concludes the proof of Proposition 7. \( \square \)
Proof of Proposition 8. The proof mostly goes by using the coercivity of $A_{\text{per}}$ and showing that, in some appropriate norm, $-\text{div}[A_{\text{per}}(\nabla u_{\varepsilon} - \nabla v_{\varepsilon})]$ is small. However, a technical difficulty comes from the fact that $\nabla u_{\varepsilon} \notin H^1_0(D)$, as it does not vanish on $\partial D$. A preliminary step (Step 1 below) thus consists in approximating $\nabla u_{\varepsilon}$ by a function (namely $g_{\varepsilon}^1$ defined by (57) below) that is equal to $\nabla v_{\varepsilon}$ away from the boundary $\partial D$, but vanishes on the boundary. Step 2 consists in estimating the difference $\nabla u_{\varepsilon} - g_{\varepsilon}^1$.

Step 1: Truncation of $\nabla v_{\varepsilon}$
Let us define $\tau_{\varepsilon} \in C^\infty_0(D)$ such that $0 \leq \tau_{\varepsilon}(x) \leq 1$ for all $x \in D$, $\tau_{\varepsilon}(x) = 1$ when $\text{dist}(\partial D, x) \geq \varepsilon$ and $\varepsilon\|\nabla \tau_{\varepsilon}\|_{L^\infty(D)} \leq C$, where $C$ is a constant independent of $\varepsilon$. We denote by $D_\varepsilon \subset D$ the set of $\mathbb{R}^d$ defined by

$$D_\varepsilon := \{ x \in D \text{ such that } \text{dist}(\partial D, x) \geq \varepsilon \}$$

and we note that

$$|D \setminus D_\varepsilon| \leq C\varepsilon.$$  

Introduce now $g_{\varepsilon}^1 \in H^1_0(D)$ defined by

$$g_{\varepsilon}^1 = \nabla v_{\varepsilon} + \varepsilon \tau_{\varepsilon} \sum_{i=1}^d \left( w_{e_i}^0 \left( \frac{x}{\varepsilon} \right) \partial_i \nabla v_{\varepsilon} + \psi_{e_i} \left( \frac{x}{\varepsilon} \right) \partial_i \psi_{e_i}^0 \right),$$

(57)

where $\nabla v_{\varepsilon}$ is the solution to (14), $u_{e_i}^0$ is the solution to (2), and $w_{e_i}^0$ and $\psi_{e_i}$ are solutions (with $p = e_i$) to (4) and (10), respectively. Note that $g_{\varepsilon}^1 = \nabla v_{\varepsilon}$ except in a neighborhood of $\partial D$. In the sequel, we estimate $\nabla u_{\varepsilon} - g_{\varepsilon}^1$.

By definition,

$$\nabla \nabla \nabla v_{\varepsilon} - \nabla g_{\varepsilon}^1 = e_{\varepsilon}^\varepsilon - e_0^\varepsilon + \varepsilon e_{\varepsilon}^\varepsilon,$$

(58)

where

$$e_0^\varepsilon = (1 - \tau_{\varepsilon}) \sum_{i=1}^d \left( \nabla w_{e_i}^0 \left( \frac{x}{\varepsilon} \right) \partial_i \nabla v_{\varepsilon} + \nabla \psi_{e_i} \left( \frac{x}{\varepsilon} \right) \partial_i \psi_{e_i}^0 \right),$$

$$e_1^\varepsilon = \varepsilon \nabla \tau_{\varepsilon} \sum_{i=1}^d \left( w_{e_i}^0 \left( \frac{x}{\varepsilon} \right) \partial_i \nabla v_{\varepsilon} + \psi_{e_i} \left( \frac{x}{\varepsilon} \right) \partial_i \psi_{e_i}^0 \right),$$

$$e_2^\varepsilon = (1 - \tau_{\varepsilon}) \sum_{i=1}^d \left( w_{e_i}^0 \left( \frac{x}{\varepsilon} \right) \nabla (\partial_i \nabla v_{\varepsilon}) + \psi_{e_i} \left( \frac{x}{\varepsilon} \right) \nabla (\partial_i \psi_{e_i}^0) \right).$$

We now bound from above successively the $L^2$ norm of $e_{\varepsilon}^\varepsilon$, $e_1^\varepsilon$ and $e_2^\varepsilon$. First, as $u_{e_i}^0 \in W^{2,\infty}(D)$, $\nabla v_{\varepsilon} \in W^{2,\infty}(D)$, $\psi_{e_i} \in W^{1,\infty}(\mathbb{R}^d)$, $w_{e_i}^0 \in W^{1,\infty}(\mathbb{R}^d)$ and $0 \leq \tau_{\varepsilon} \leq 1$, we have

$$\|e_{\varepsilon}^\varepsilon\|_{L^2(D)} \leq C, \quad C \text{ independent of } \varepsilon.$$

(59)

The same arguments lead to

$$\|e_1^\varepsilon\|_{L^2(D)}^2 = \int_D \left[ \sum_{i=1}^d \left( w_{e_i}^0 \left( \frac{x}{\varepsilon} \right) \partial_i \nabla v_{\varepsilon} + \psi_{e_i} \left( \frac{x}{\varepsilon} \right) \partial_i \psi_{e_i}^0 \right) \right]^2 |\varepsilon \nabla \tau_{\varepsilon}|^2$$

$$\leq C \|D \setminus D_\varepsilon\|$$

$$\leq C\varepsilon,$$

(60)
for a constant $C$ independent of $\varepsilon$. We next write

$$\|\varepsilon_0\|^2_{L^2(D)} \leq |D \setminus D_\varepsilon| \sum_{i=1}^d \left( \nabla u_{e_i}^0 \left( \frac{\cdot}{\varepsilon} \right) \partial_i \bar{w}_1^0 + \nabla \psi_{e_i} \left( \frac{\cdot}{\varepsilon} \right) \partial_i \bar{u}_0 \right) \leq C \varepsilon. \quad (61)$$

Collecting (58), (59), (60) and (61), we have

$$\|\nabla \bar{w}_1^0 - \nabla g_1^0\|^2_{L^2(D)} \leq C \varepsilon, \quad C \text{ independent of } \varepsilon.$$

Observing that

$$\|\bar{w}_1^0 - g_1^0\|^2_{L^2(D)} \leq 2d \varepsilon^2 \sum_{i=1}^d \left( \|w_{e_i}^0\|^2_{L^\infty} \|\bar{w}_1^0\|^2_{H^1(D)} + \|\psi_{e_i}\|^2_{L^\infty} \|u_0^\ast\|^2_{H^1(D)} \right) \leq C \varepsilon^2,$$

we obtain that

$$\|\bar{w}_1^0 - g_1^0\|^2_{H^1(D)} \leq C \sqrt{\varepsilon}, \quad C \text{ independent of } \varepsilon. \quad (62)$$

**Step 2:** We next turn to estimating $\bar{w}_1^0 - g_1^0$. Using that $A_{per}$ is coercive and the fact that $\bar{w}_1^0 - g_1^0 \in H_0^1(D)$, we have

$$\alpha \|\bar{w}_1^0 - g_1^0\|^2_{H^1(D)} \leq \int_D (\nabla \bar{w}_1^0 - \nabla g_1^0) \cdot A_{per} \left( \frac{\cdot}{\varepsilon} \right) (\nabla \bar{w}_1^0 - \nabla g_1^0)
\leq \int_D (\nabla \bar{w}_1^0 - \nabla g_1^0) \cdot A_{per} \left( \frac{\cdot}{\varepsilon} \right) (\nabla \bar{w}_1^0 - \nabla g_1^0) + \int_D (\nabla \bar{w}_1^0 - \nabla g_1^0) \cdot A_{per} \left( \frac{\cdot}{\varepsilon} \right) (\nabla \bar{w}_1^0 - \nabla g_1^0)
\leq \|\bar{w}_1^0 - g_1^0\|_{H^1(D)} \left( \|\nabla \bar{w}_1^0 - \nabla \bar{v}_1^0\|_{H^{-1}(D)} + \|A_{per}\|_{L^\infty} \|\bar{w}_1^0 - g_1^0\|_{H^1(D)} \right). \quad (63)$$

where the constant $\alpha > 0$ only depends on the coercivity constant of $A_{per}$ and the Poincaré constant of the domain $D$. In the sequel, we bound from above $\|\nabla \bar{w}_1^0 - \nabla \bar{v}_1^0\|_{H^{-1}(D)}$.

By definition of $\bar{w}_1^0$, we have

$$\bar{w}_1^0 = \bar{v}_1^0 + \bar{v}_1^0,$$

with

$$\bar{v}_1^0 = \bar{w}_1^0 + \varepsilon \sum_{i=1}^d w_{e_i}^0 \left( \frac{\cdot}{\varepsilon} \right) \partial_i \bar{w}_1^0 \quad \text{and} \quad \bar{v}_1^0 = \varepsilon \sum_{i=1}^d \psi_{e_i} \left( \frac{\cdot}{\varepsilon} \right) \partial_i \bar{u}_0.$$

Using the equation (25) on $\bar{w}_1^0$ and the relation (14) between $\bar{w}_1^0$ and $u_0^\ast$, we compute

$$\nabla A_{per} \left( \frac{\cdot}{\varepsilon} \right) (\nabla \bar{w}_1^0 - \nabla \bar{v}_1^0) = \nabla A_{per} \left( \frac{\cdot}{\varepsilon} \right) \nabla \bar{v}_1^0 + \nabla A_{per} \left( \frac{\cdot}{\varepsilon} \right) \nabla \bar{v}_1^0 - \nabla A_{per} \left( \frac{\cdot}{\varepsilon} \right) \nabla \bar{v}_1^0 + \nabla A_{per} \left( \frac{\cdot}{\varepsilon} \right) \nabla \bar{v}_1^0 + \nabla A_{per} \left( \frac{\cdot}{\varepsilon} \right) \nabla \bar{v}_1^0 - \nabla A_{per} \left( \frac{\cdot}{\varepsilon} \right) \nabla \bar{v}_1^0 + \nabla A_{per} \left( \frac{\cdot}{\varepsilon} \right) \nabla \bar{v}_1^0 + \nabla A_{per} \left( \frac{\cdot}{\varepsilon} \right) \nabla \bar{v}_1^0 - \nabla A_{per} \left( \frac{\cdot}{\varepsilon} \right) \nabla \bar{v}_1^0,$$

where

$$D_0 = \sum_{i=1}^d \left( \nabla A_{per} \left( \frac{\cdot}{\varepsilon} \right) (e_i + \nabla w_{e_i}^0 \left( \frac{\cdot}{\varepsilon} \right) - A_{per}^\ast e_i) \partial_i \bar{w}_1^0 \right),$$

$$D_1 = \sum_{i=1}^d \left( \nabla A_{per} \left( \frac{\cdot}{\varepsilon} \right) \nabla \psi_{e_i} \left( \frac{\cdot}{\varepsilon} \right) + B_{per} \left( \frac{\cdot}{\varepsilon} \right) (e_i + \nabla w_{e_i}^0 \left( \frac{\cdot}{\varepsilon} \right) - \bar{B} e_i) \partial_i \bar{u}_0 \right),$$

$$D_2 = \nabla B_{per} \left( \frac{\cdot}{\varepsilon} \right) \nabla \theta_0^\ast + \sum_{i=1}^d \left( A_{per} \left( \frac{\cdot}{\varepsilon} \right) (u_{e_i}^0 \left( \frac{\cdot}{\varepsilon} \right) \nabla \partial_i \bar{w}_1 + \psi_{e_i} \left( \frac{\cdot}{\varepsilon} \right) \nabla \partial_i \bar{u}_0 + B_{per} \left( \frac{\cdot}{\varepsilon} \right) w_{e_i}^0 \left( \frac{\cdot}{\varepsilon} \right) \nabla \partial_i \bar{u}_0 \right).$$
We now bound from above these three quantities. As $A_{\text{per}}$ and $B_{\text{per}}$ are bounded, we see that

$$
\|D_2\|_{H^{-1}(D)} \leq C\|\theta_0^1\|_{H^1(D)} + C\sum_{i=1}^{d} \left[ \|w^0_{e_i}\|_{L^\infty(\Omega)} \|\nabla \psi_i\|_{H^2(D)} + \|\psi_i\|_{L^\infty(\Omega)} \|u^0_0\|_{H^2(D)} + \|u^0_0\|_{L^\infty(\Omega)} \|u^0_0\|_{H^2(D)} \right],
$$

from which we infer, in view of (37), that

$$
\|D_2\|_{H^{-1}(D)} \leq C\sqrt{\epsilon}, \quad C \text{ independent of } \epsilon. \tag{65}
$$

Let us now turn to $D_0$. Consider, for any $1 \leq i \leq d$, the vector-valued function

$$
Z(y) = A_{\text{per}}(y) (e_i + \nabla w^0_{e_i}(y)) - A^*_{\text{per}} e_i.
$$

We observe that $Z \in (L^2_{\text{loc}}(\mathbb{R}^d))^d$ is divergence free, $Q$-periodic and of vanishing mean. Since $\partial_i \psi_1^1 \in W^{1,\infty}(D)$, we can use Lemma 12, and we obtain

$$
\|D_0\|_{H^{-1}(D)} \leq C_{\epsilon}, \quad C \text{ independent of } \epsilon. \tag{66}
$$

Turning now to $D_1$, we likewise consider, for any $1 \leq i \leq d$, the vector-valued function

$$
\overline{Z}(y) = A_{\text{per}}(y) \nabla \psi_i, (y) + B_{\text{per}}(y) (e_i + \nabla w^0_{e_i}(y)) - B e_i.
$$

By construction, $\overline{Z} \in (L^2_{\text{loc}}(\mathbb{R}^d))^d$ is $Q$-periodic and divergence free, in view of the definition (16) of $\psi_i$. In addition, the mean of $\overline{Z}$ vanishes. Indeed, for any $1 \leq j \leq d$, using (13), (16), the symmetry of $A_{\text{per}}$ and (1), we have

$$
\int_Q \overline{Z} \cdot e_j = \int_Q e_j^T A_{\text{per}} \nabla \psi_i + \int_Q e_j^T B_{\text{per}} (e_i + \nabla w^0_{e_i}) - \int_Q (e_j + \nabla w^0_{e_i})^T B_{\text{per}} (e_i + \nabla w^0_{e_i})
$$

$$
= \int_Q e_j^T A_{\text{per}} \nabla \psi_i - \int_Q (\nabla w^0_{e_i})^T B_{\text{per}} (e_i + \nabla w^0_{e_i})
$$

$$
= \int_Q e_j^T A_{\text{per}} \nabla \psi_i + \int_Q (\nabla w^0_{e_i})^T A_{\text{per}} \nabla \psi_i
$$

$$
= \int_Q (\nabla \psi_i)^T A_{\text{per}} (e_j + \nabla w^0_{e_i})
$$

$$
= 0.
$$

Since $\partial_i u^0_0 \in W^{1,\infty}(D)$, we have that $\overline{Z}$ and $\partial_i u^0_0$ satisfy the assumptions of Lemma 12 hence

$$
\|D_1\|_{H^{-1}(D)} \leq C \epsilon, \quad C \text{ independent of } \epsilon. \tag{67}
$$

Collecting (63), (65), (66) and (67), we have

$$
\left\| \text{div} \left[ A_{\text{per}} \left( \frac{\psi_1}{\epsilon} \right) (\nabla \psi_1^1 - \nabla \psi_1^1) \right] \right\|_{H^{-1}(D)} \leq C\sqrt{\epsilon}, \tag{68}
$$

where $C$ is a constant independent of $\epsilon$. We now infer from (62), (63) and (68) that

$$
\alpha \|\psi_1^1 - g_1^1\|_{H^1(D)}^2 \leq C \|\psi_1^1 - g_1^1\|_{H^1(D)} \sqrt{\epsilon},
$$

hence

$$
\|\psi_1^1 - g_1^1\|_{H^1(D)} \leq C \sqrt{\epsilon}, \quad C \text{ independent of } \epsilon.
$$

**Step 3: Conclusion**

Collecting the above bound with (62), we deduce that

$$
\|\psi_1^1 - \psi_1^1\|_{H^1(D)} \leq C \sqrt{\epsilon}, \quad C \text{ independent of } \epsilon.
$$

We thus have proved the claimed bound, and this concludes the proof of Proposition 8.
5.2 Two-scale expansion of $\phi^\varepsilon_k$

In this section, we prove Propositions 9 and 11.

**Proof of Proposition 9.** Introducing

$$c^\varepsilon_k(x) = 1_{Q+k} \left( \frac{x}{\varepsilon} \right) B_{\text{per}} \left( \frac{x}{\varepsilon} \right) \nabla u^\varepsilon_0(x),$$

the problem \ref{eq:26} writes

$$\begin{cases}
- \text{div} \left[ A_{\text{per}} \left( \frac{\cdot}{\varepsilon} \right) \nabla \phi^\varepsilon_k \right] = \text{div} \left[ c^\varepsilon_k \right] \quad \text{in } D, \\
\phi^\varepsilon_k = 0 \quad \text{on } \partial D.
\end{cases}$$

Multiplying this equation by $\phi^\varepsilon_k$, integrating over $D$, and using the coercivity of $A_{\text{per}}$, we obtain that there exists $C$ independent of $k$ and $\varepsilon$ such that

$$\| \phi^\varepsilon_k \|_{H^1(D)} \leq C \| c^\varepsilon_k \|_{L^2(D)}.$$  \tag{69}$$

Let us now show that $c^\varepsilon_k$ converges to 0 in $L^2(D)$. Using the expansion \ref{eq:36}, we write

$$\nabla u^\varepsilon_0 = T^\varepsilon + \varepsilon \nabla \theta^\varepsilon_0,$$

with

$$T^\varepsilon = \sum_{i=1}^d \partial_i u^\varepsilon_0 \left( e_i + \nabla w^\varepsilon_0 \left( \frac{\cdot}{\varepsilon} \right) \right) + \varepsilon \sum_{i=1}^d \nabla (\partial_i u^\varepsilon_0) w^\varepsilon_0 \left( \frac{\cdot}{\varepsilon} \right).$$

Using the fact that $w_0^\varepsilon \in W^{1,\infty}(\mathbb{R}^d)$ and $u^\varepsilon_0 \in W^{2,\infty}(D)$, we see that $T^\varepsilon$ is bounded in $L^\infty(D)$. We next write

$$\| c^\varepsilon_k \|_{L^2(D)}^2 \leq \| B_{\text{per}} \|_{L^\infty(\mathbb{R}^d)}^2 \int_{\varepsilon(Q+k)} |\nabla u^\varepsilon_0|^2 \leq C \varepsilon^d + C \int_{\varepsilon(Q+k)} |\varepsilon \nabla \theta^\varepsilon_0|^2 \leq C \varepsilon^d + C \| \varepsilon \theta^\varepsilon_0 \|_{H^1(D)}^2.$$ \tag{69}$$

Using the bound \ref{eq:37}, we deduce that $c^\varepsilon_k$ converges to 0 in $L^2(D)$. In view of \ref{eq:69}, this implies that $\phi^\varepsilon_k$ converges to 0 in $H^1_0(D)$. This concludes the proof.

**Proof of Proposition 11.** As in the proof of Proposition 9, the proof falls in two steps. We first truncate $\pi_k^\varepsilon$ in a function $\tilde{\pi}^\varepsilon_k$ (defined by \ref{eq:70} below) that vanishes on $\partial D$. We next estimate the difference between $\tilde{\pi}^\varepsilon_k$ and $\phi^\varepsilon_k$.

**Step 1: Truncation of $\pi_k^\varepsilon$.**

Let us define $\tau_\varepsilon \in C_0^\infty(D)$ such that $0 \leq \tau_\varepsilon(x) \leq 1$ for all $x \in D$, $\tau_\varepsilon(x) = 1$ when $\text{dist}(\partial D, x) \geq \varepsilon$ and $\varepsilon \| \nabla \tau_\varepsilon \|_{L^\infty(D)} \leq C$, where $C$ is a constant independent of $\varepsilon$. We introduce

$$D_\varepsilon := \{ x \in D \text{ such that } \text{dist}(\partial D, x) \geq \varepsilon \}, \quad |D \setminus D_\varepsilon| \sim \varepsilon,$$

$$J_\varepsilon := \left\{ k \in \mathbb{Z}^d \text{ such that } \varepsilon(Q+k) \cap D \setminus D_\varepsilon \neq \emptyset \right\}, \quad \text{Card}(J_\varepsilon) \sim \varepsilon^{1-d},$$

and the function $\tilde{\pi}^\varepsilon_k \in H^1_0(D)$ defined by

$$\tilde{\pi}^\varepsilon_k = \varepsilon \tau_\varepsilon \sum_{i=1}^d \chi_{e_i} \left( \frac{\cdot}{\varepsilon} - k \right) \partial_i u^\varepsilon_0,$$ \tag{70}$$

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where \( u_0^\ast \) is solution to (2) and \( \chi_{e_i} \) is solution to (41). Note that \( \tilde{v}_{\epsilon}^e = v_{\epsilon}^e \) except in the neighborhood of the boundary of \( D \). In the sequel, we estimate \( \sum_{k \in I_\epsilon} \| \phi_k^\epsilon - \tilde{v}_k^\epsilon \|_{H^1(D)}^2 \), and, in the next Step, we estimate \( \sum_{k \in I_\epsilon} \| \phi_k^\epsilon - \tilde{v}_k^\epsilon \|_{H^1(D)} \), where, we recall (see (44)),

\[
I_\epsilon = \left\{ k \in \mathbb{Z}^d \text{ such that } \varepsilon(Q + k) \cap D \neq \emptyset \right\}, \quad \text{Card}(I_\epsilon) \sim \varepsilon^{-d}.
\]

Recall also that, whenever \( k \notin I_\epsilon \), we have \( \phi_k^\epsilon \equiv 0 \).

By definition,

\[
\nabla \tilde{v}_k^\epsilon - \nabla \tilde{v}_k^\epsilon = e_{0,k}^{\epsilon} - e_{1,k}^{\epsilon} + e_{2,k}^{\epsilon},
\]

where

\[
e_{0,k}^{\epsilon} = (1 - \tau_\epsilon) \sum_{i=1}^d \nabla \chi_{e_i} \left( \frac{\cdot}{\varepsilon} - k \right) \partial_i u_0^\ast,
\]

\[
e_{1,k}^{\epsilon} = \varepsilon \tau_\epsilon \sum_{i=1}^d \chi_{e_i} \left( \frac{\cdot}{\varepsilon} - k \right) \partial_i u_0^\ast,
\]

\[
e_{2,k}^{\epsilon} = \varepsilon (1 - \tau_\epsilon) \sum_{i=1}^d \chi_{e_i} \left( \frac{\cdot}{\varepsilon} - k \right) \nabla \partial_i u_0^\ast.
\]

We now bound from above successively the \( L^2 \) norm of \( e_2,k^{\epsilon} \), \( e_1,k^{\epsilon} \) and \( e_0,k^{\epsilon} \). To this aim, the following computation will be useful: for any \( 1 \leq i \leq d \), we have

\[
\sum_{k \in I_\epsilon} \int_{D \setminus D_\epsilon} \chi_{e_i}^2 \left( \frac{\cdot}{\varepsilon} - k \right) \leq \sum_{k \in I_\epsilon} \int_{Q+j} \chi_{e_i}^2 \left( \frac{\cdot}{\varepsilon} - k \right) \leq \sum_{j \in I_\epsilon} \int_{Q+j} \chi_{e_i}^2 \left( \frac{\cdot}{\varepsilon} - k \right).
\]

There exists \( \rho \) such that

\[
\forall \varepsilon, \forall j \in J_\epsilon, \forall k \in I_\epsilon, \quad Q + j - k \subset B(0, \rho/\varepsilon).
\]

We thus obtain that

\[
\sum_{k \in I_\epsilon} \int_{D \setminus D_\epsilon} \chi_{e_i}^2 \left( \frac{\cdot}{\varepsilon} - k \right) \leq \sum_{j \in J_\epsilon} \varepsilon^d \int_{B(0,\rho/\varepsilon)} \chi_{e_i}^2 \leq \varepsilon \int_{B(0,\rho/\varepsilon)} \chi_{e_i}^2.
\]

We next infer from (43) that

\[
\int_{B(0,\rho/\varepsilon)} \chi_{e_i}^2 \leq \int_{B(0,\rho/\varepsilon)} \frac{C}{\left(1 + |y|^{d-1}\right)^2} dy \leq C + C \int_1^{\rho/\varepsilon} \frac{1}{r^{d-1}} dr \leq CR_{d,\epsilon},
\]

where \( C \) is a constant independent of \( \varepsilon \) and

\[
R_{d,\epsilon} := \left\{ \begin{array}{ll} 1 + \ln(1/\varepsilon) & \text{if } d = 2, \\ 1 & \text{if } d > 2. \end{array} \right.
\]

Collecting (72) and (73), we deduce that

\[
\sum_{k \in I_\epsilon} \int_{D \setminus D_\epsilon} \chi_{e_i}^2 \left( \frac{\cdot}{\varepsilon} - k \right) \leq C \varepsilon R_{d,\epsilon}.
\]
We now bound $e_2^{k,\varepsilon}$. As $u_0^* \in W^{2,\infty}(D)$, and using (75), we have

$$
\sum_{k \in I_\varepsilon} \|e_2^{k,\varepsilon}\|_{L^2(D)}^2 = \sum_{k \in I_\varepsilon} \varepsilon^2 \int_D \left(1 - \tau_\varepsilon\right) \sum_{i=1}^d \chi_{\varepsilon_i} \left(\frac{\cdot}{\varepsilon} - k\right) \nabla (\partial_i u_0^*)^2 \\
\leq C \varepsilon^2 \|\nabla^2 u_0^*\|_{L^\infty}^2 \sum_{k \in I_\varepsilon} \sum_{i=1}^d \int_{D\setminus D_\varepsilon} \chi_{\varepsilon_i}^2 \left(\frac{\cdot}{\varepsilon} - k\right) \\
\leq C \varepsilon^3 R_{d,\varepsilon},
$$

(76)

We next turn to $e_1^{k,\varepsilon}$. The same arguments and the fact that $\varepsilon \|\nabla \tau_\varepsilon\|_{L^\infty} \leq C$ lead to

$$
\sum_{k \in I_\varepsilon} \|e_1^{k,\varepsilon}\|_{L^2(D)}^2 \leq \varepsilon \|\nabla \tau_\varepsilon\|_{L^\infty}^2 \sum_{k \in I_\varepsilon} \int_{D\setminus D_\varepsilon} \left[ \sum_{i=1}^d \chi_{\varepsilon_i} \left(\frac{\cdot}{\varepsilon} - k\right) \partial_i u_0^* \right]^2 \\
\leq C \sum_{i=1}^d \sum_{k \in I_\varepsilon} \int_{D\setminus D_\varepsilon} \chi_{\varepsilon_i}^2 \left(\frac{\cdot}{\varepsilon} - k\right) \\
\leq C \varepsilon R_{d,\varepsilon},
$$

(77)

where we have again used (75). Turning to $e_0^{k,\varepsilon}$, we have, using $\nabla \chi_{\varepsilon_i} \in (L^2(\mathbb{R}^d))^d$,

$$
\sum_{k \in I_\varepsilon} \|e_0^{k,\varepsilon}\|_{L^2(D)}^2 \leq C \|\nabla u_0^*\|_{L^\infty}^2 \sum_{i=1}^d \sum_{k \in I_\varepsilon} \int_{D\setminus D_\varepsilon} |\nabla \chi_{\varepsilon_i} (\frac{\cdot}{\varepsilon} - k)|^2 \\
\leq C \sum_{i=1}^d \sum_{j \in J_\varepsilon} \varepsilon^d \sum_{k \in I_\varepsilon} \int_{Q+j-k} |\nabla \chi_{\varepsilon_i}|^2 \\
\leq C \sum_{i=1}^d \sum_{j \in J_\varepsilon} \varepsilon^d \|\nabla \chi_{\varepsilon_i}\|_{L^2(\mathbb{R}^d)}^2 \\
\leq C \varepsilon.
$$

(78)

Collecting (77), (76), (77) and (78), we deduce that

$$
\sum_{k \in I_\varepsilon} \|\nabla \bar{v}_k - \nabla \tilde{v}_k\|_{L^2(D)}^2 \leq C \left(\varepsilon + \varepsilon R_{d,\varepsilon} + \varepsilon^3 R_{d,\varepsilon}\right),
$$

where $C$ is a constant independent of $\varepsilon$. Observing that

$$
\sum_{k \in I_\varepsilon} \|\bar{v}_k - \tilde{v}_k\|_{L^2(D)}^2 \leq C \varepsilon^2 \|\nabla u_0^*\|_{L^\infty}^2 \sum_{k \in I_\varepsilon} \sum_{i=1}^d \int_{D\setminus D_\varepsilon} \chi_{\varepsilon_i}^2 \left(\frac{\cdot}{\varepsilon} - k\right) \leq C \varepsilon^3 R_{d,\varepsilon},
$$

we obtain that

$$
\sum_{k \in I_\varepsilon} \|\bar{v}_k - \tilde{v}_k\|_{H^1(D)}^2 \leq C \left(\varepsilon + \varepsilon R_{d,\varepsilon} + \varepsilon^3 R_{d,\varepsilon}\right) \leq \left\{ \begin{array}{ll} C \varepsilon \left[1 + \ln(1/\varepsilon)\right] & \text{if } d = 2, \\
C \varepsilon & \text{if } d > 2, \end{array} \right.
$$

(79)

where $C$ is a constant independent of $\varepsilon$. 

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**Step 2:** We next turn to estimating \( \sum_{k \in L_\varepsilon} \| \phi_k^\varepsilon - \tilde{\nu}_k^\varepsilon \|^2_{H^1(D)} \). Using that \( A_{\text{per}} \) is coercive and the fact that \( \phi_k^\varepsilon - \tilde{\nu}_k^\varepsilon \in H^1_0(D) \), we have

\[
\alpha \| \phi_k^\varepsilon - \tilde{\nu}_k^\varepsilon \|^2_{H^1(D)} \leq \int_D (\nabla \phi_k^\varepsilon - \nabla \tilde{\nu}_k^\varepsilon)^T A_{\text{per}} \left( \frac{.}{\varepsilon} \right) (\nabla \phi_k^\varepsilon - \nabla \tilde{\nu}_k^\varepsilon) = D_{00}^{k,\varepsilon} + D_{01}^{k,\varepsilon},
\]

where the constant \( \alpha > 0 \) only depends on the coercivity constant of \( A_{\text{per}} \) and the Poincaré constant of the domain \( D \), and where

\[
D_{00}^{k,\varepsilon} = \int_D (\nabla \phi_k^\varepsilon - \nabla \tilde{\nu}_k^\varepsilon)^T A_{\text{per}} \left( \frac{.}{\varepsilon} \right) (\nabla \phi_k^\varepsilon - \nabla \tilde{\nu}_k^\varepsilon),
\]

\[
D_{01}^{k,\varepsilon} = \int_D (\nabla \phi_k^\varepsilon - \nabla \tilde{\nu}_k^\varepsilon)^T A_{\text{per}} \left( \frac{.}{\varepsilon} \right) (\nabla \tilde{\nu}_k^\varepsilon).
\]

We successively bound \( D_{00}^{k,\varepsilon} \) and \( D_{01}^{k,\varepsilon} \) from above. We begin with \( D_{00}^{k,\varepsilon} \). Observe that, in view of [26],

\[
-\text{div} \left[ A_{\text{per}} \left( \frac{.}{\varepsilon} \right) (\nabla \phi_k^\varepsilon - \nabla \tilde{\nu}_k^\varepsilon) \right] = \text{div} \left[ 1_{Q+k} \left( \frac{.}{\varepsilon} \right) B_{\text{per}} \left( \frac{.}{\varepsilon} \right) \left( \nabla u_0^\varepsilon - \sum_{p=1}^d \left( e_p + \nabla w_{\varepsilon p}^0 \left( \frac{.}{\varepsilon} \right) \right) \partial_p u_0^\varepsilon \right) \right] + \sum_{p=1}^d \text{div} \left[ Z_k \left( \frac{.}{\varepsilon} \right) \partial_p u_0^\varepsilon \right] + \varepsilon \text{div} \left[ A_{\text{per}} \left( \frac{.}{\varepsilon} \right) \nabla (\partial_p u_0^\varepsilon) \chi_p \left( \frac{.}{\varepsilon} - k \right) \right],
\]

where the vector-valued function \( Z_k \) is defined by

\[
Z_k(y) = 1_{Q+k}(y) B_{\text{per}}(y) \left( e_p + \nabla w_{\varepsilon p}^0(y) \right) + A_{\text{per}}(y) \nabla \chi_p(y - k).
\]

Note that, in view of [13], \( Z_k \) is a divergence free vector, hence \( \text{div} \left[ Z_k \left( \frac{.}{\varepsilon} \right) \partial_p u_0^\varepsilon \right] = Z_k \left( \frac{.}{\varepsilon} \right) \cdot \nabla \partial_p u_0^\varepsilon \). We can thus rewrite \( D_{00}^{k,\varepsilon} \) as

\[
D_{00}^{k,\varepsilon} = D_{001}^{k,\varepsilon} + D_{01}^{k,\varepsilon} + D_{02}^{k,\varepsilon},
\]

where

\[
D_{001}^{k,\varepsilon} = -\int_D (\nabla \phi_k^\varepsilon - \nabla \tilde{\nu}_k^\varepsilon)^T 1_{Q+k} \left( \frac{.}{\varepsilon} \right) B_{\text{per}} \left( \frac{.}{\varepsilon} \right) \left( \nabla u_0^\varepsilon - \sum_{p=1}^d \left( e_p + \nabla w_{\varepsilon p}^0 \left( \frac{.}{\varepsilon} \right) \right) \partial_p u_0^\varepsilon \right),
\]

\[
D_{01}^{k,\varepsilon} = \sum_{p=1}^d \int_D (\phi_k^\varepsilon - \tilde{\nu}_k^\varepsilon) \cdot Z_k \left( \frac{.}{\varepsilon} \right) \cdot \nabla (\partial_p u_0^\varepsilon),
\]

\[
D_{02}^{k,\varepsilon} = -\varepsilon \sum_{p=1}^d \int_D (\nabla \phi_k^\varepsilon - \nabla \tilde{\nu}_k^\varepsilon)^T A_{\text{per}} \left( \frac{.}{\varepsilon} \right) \nabla (\partial_p u_0^\varepsilon) \chi_p \left( \frac{.}{\varepsilon} - k \right).
\]

We successively bound these three quantities. Since \( \chi_p \in L^\infty(\mathbb{R}^d) \) (see [13]) and \( u_0^\varepsilon \in W^{1,\infty}(D) \), we have \( \| \tilde{\nu}_k^\varepsilon \|_{L^\infty(D)} \leq C \varepsilon \). We also have that \( \| \phi_k^\varepsilon \|_{L^\infty(D)} \leq C \varepsilon \), in view of [23] (recall indeed that \( u_0^\varepsilon \in W^{2,\infty}(D) \) implies that \( f \in L^\infty(D) \), in view of [2]; assumptions of Lemma [3] are thus satisfied). Using that \( u_0^\varepsilon \in W^{2,\infty}(D) \), we
now bound from above $D_{01}^{k,\varepsilon}$:

$$\|D_{01}^{k,\varepsilon}\| \leq \sum_{p=1}^{d} \|\phi_k^e - \tilde{v}_{k}^e\|_{L^\infty(D)} \|\nabla^2 u_0^e\|_{L^\infty(D)} \|Z_k(\bar{z})\|_{L^1(D)}$$

$$\leq C\varepsilon \sum_{p=1}^{d} \left[ \|B_{per}\|_{L^\infty} \int_D 1_{Q+k}(\frac{\bar{z}}{\varepsilon}) |e_p + \nabla w^0_{ep} \left(\frac{\bar{z}}{\varepsilon}\right)| + \|A_{per}\|_{L^\infty} \int_D |\nabla\chi_{ep} \left(\frac{\bar{z}}{\varepsilon} - k\right)| \right]$$

$$\leq C\varepsilon \sum_{p=1}^{d} \left[ \varepsilon^d |e_p + \nabla w^0_{ep}|_{L^2(Q)} + \varepsilon^d \int_{D/\varepsilon-k} |\nabla\chi_{ep}| \right]$$

$$\leq C\varepsilon^{d+1} \sum_{p=1}^{d} \left[ 1 + \left( \int_{B(0,1)} |\nabla\chi_{ep}| + \int_{B(0,\rho/\varepsilon)\setminus B(0,1)} |\nabla\chi_{ep}| \right) \right].$$

Using that $\nabla\chi_{ep} \in (L^2(\mathbb{R}^d))^d$ (see Lemma 10) and the bound (42), we deduce that

$$|D_{01}^{k,\varepsilon}| \leq C\varepsilon \sum_{p=1}^{d} \left[ 1 + \left( \int_{B(0,1)} |\nabla\chi_{ep}| + C \int_{1}^{1/\varepsilon} \frac{1}{r^d} \right) \right]$$

$$\leq C\varepsilon^{d+1} [1 + \ln(1/\varepsilon)],$$

where $C$ is a constant independent of $\varepsilon$. We thus get

$$\sum_{k \in I_\varepsilon} \|D_{01}^{k,\varepsilon}\| \leq C\varepsilon \left[ 1 + \ln(1/\varepsilon) \right]. \quad (82)$$

We now turn to $D_{02}^{k,\varepsilon}$. Using (73), we observe that, for any $k \in I_\varepsilon$,

$$\left\|\chi_{ep} \left(\frac{\bar{z}}{\varepsilon} - k\right)\right\|_{L^2(D)}^2 = \left\|\int_D \chi_{ep} \left(\frac{\bar{z}}{\varepsilon} - k\right) \right\|^2 = \varepsilon^d \int_{D/\varepsilon-k} |\chi_{ep}|^2 \leq \varepsilon^d \int_{B(0,\rho/\varepsilon)} |\chi_{ep}|^2 \leq C\varepsilon^d R_{d,\varepsilon}.$$}

We thus can bound from above $D_{02}^{k,\varepsilon}$,

$$\sum_{k \in I_\varepsilon} \|D_{02}^{k,\varepsilon}\| \leq \varepsilon \|\nabla^2 u_0^e\|_{L^\infty(D)} \|A_{per}\|_{L^\infty} \sum_{p=1}^{d} \sum_{k \in I_\varepsilon} \|\nabla\phi_k^e - \nabla v_k^e\|_{L^2(D)} \left\|\chi_{ep} \left(\frac{\bar{z}}{\varepsilon} - k\right)\right\|_{L^2(D)}$$

$$\leq C\varepsilon \sqrt{\sum_{k \in I_\varepsilon} \|\nabla\phi_k^e - \nabla v_k^e\|_{L^2(D)}^2} \sqrt{\sum_{p=1}^{d} \sum_{k \in I_\varepsilon} \left\|\chi_{ep} \left(\frac{\bar{z}}{\varepsilon} - k\right)\right\|_{L^2(D)}^2}$$

$$\leq C\varepsilon \sqrt{\sum_{k \in I_\varepsilon} \|\nabla\phi_k^e - \nabla v_k^e\|_{L^2(D)}^2} \sqrt{R_{d,\varepsilon}} \quad (83)$$

where $C$ is a constant independent of $\varepsilon$. We next turn to $D_{00}^{k,\varepsilon}$. Using the bound (37) on the two scale expansion
of \( u_0^\varepsilon \), we have

\[
\sum_{k \in I_e} |D_{00}^{k,\varepsilon}| \leq \|B_{\text{per}}\|_{L^\infty} \sum_{k \in I_e} \|\nabla \phi_k^\varepsilon - \nabla \tilde{v}_k^\varepsilon\|_{L^2(D)} \left\| \nabla u_0^\varepsilon - \sum_{p=1}^d \left( e_p + \nabla w_{\varepsilon}^0 \left( \frac{\cdot}{\varepsilon} \right) \right) \partial_p u_0^\varepsilon \right\|_{L^2(\varepsilon(Q + k))},
\]

\[
\leq C \sqrt{\sum_{k \in I_e} \|\nabla \phi_k^\varepsilon - \nabla \tilde{v}_k^\varepsilon\|_{L^2(D)}^2} \left\| \nabla u_0^\varepsilon - \sum_{p=1}^d \left( e_p + \nabla w_{\varepsilon}^0 \left( \frac{\cdot}{\varepsilon} \right) \right) \partial_p u_0^\varepsilon \right\|_{L^2(\varepsilon(Q + k))}.
\]

Collecting (81), (82), (83) and (84), we obtain that

\[
\sum_{k \in I_e} |D_{00}^{k,\varepsilon}| \leq C \left( \sqrt{\varepsilon} + \varepsilon \sqrt{R_{d,\varepsilon}} \right) \sqrt{\sum_{k \in I_e} \|\nabla \phi_k^\varepsilon - \nabla \tilde{v}_k^\varepsilon\|_{L^2(D)}^2} + \varepsilon \ln(1/\varepsilon),
\]

(85)

We now turn to \( D_1^{k,\varepsilon} \). Using (79), we have

\[
\sum_{k \in I_e} |D_1^{k,\varepsilon}| \leq C \sqrt{\sum_{k \in I_e} \|\nabla \phi_k^\varepsilon - \nabla \tilde{v}_k^\varepsilon\|_{L^2(D)}^2} \sqrt{\sum_{k \in I_e} \|\nabla \tilde{u}_k^\varepsilon - \nabla \tilde{v}_k^\varepsilon\|_{L^2(D)}^2} \sqrt{\varepsilon R_{d,\varepsilon}},
\]

(86)

Collecting (80), (85) and (86), we obtain

\[
\alpha \sum_{k \in I_e} \|\phi_k^\varepsilon - \tilde{v}_k^\varepsilon\|_{H^1(D)}^2 \leq C \left( \varepsilon \ln(1/\varepsilon) + \left( \sqrt{\varepsilon R_{d,\varepsilon}} + \varepsilon \sqrt{R_{d,\varepsilon}} \right) \sqrt{\sum_{k \in I_e} \|\nabla \phi_k^\varepsilon - \nabla \tilde{v}_k^\varepsilon\|_{L^2(D)}^2} \right)
\]

with, in view of (74), \( R_{d,\varepsilon} = 1 + \ln(1/\varepsilon) \) if \( d = 2 \), and \( R_{d,\varepsilon} = 1 \) if \( d > 2 \). This implies that

\[
\sum_{k \in I_e} \|\phi_k^\varepsilon - \tilde{v}_k^\varepsilon\|_{H^1(D)}^2 \leq C \varepsilon \ln(1/\varepsilon), \quad C \text{ independent of } \varepsilon.
\]

Collecting this bound with (79), we obtain the claimed bound. This concludes the proof of Proposition 11.

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