Some Harmonic Number Identities involving certain Reciprocals

M.J. Kronenburg

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Abstract

Some finite series of harmonic numbers involving certain reciprocals are evaluated. Products of such reciprocals are expanded in a sum of the individual reciprocals, leading to a computer program. A list of examples is provided.

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1 Definitions and Basic Identities

The generalized harmonic numbers used in this paper are:

\[ H_{n}^{(m)} = \sum_{k=1}^{n} \frac{1}{k^m} \]  

from which follows that \( H_{0}^{(m)} = 0 \). The traditional harmonic numbers are:

\[ H_n = H_{n}^{(1)} \]

A well known identity is \(2, 3, 7, 8\):

\[ \sum_{k=1}^{n} \frac{1}{k} H_k = \frac{1}{2} (H_{n}^{2} + H_{n}^{(2)}) \]  

and \(3, 4\):

\[ \sum_{k=0}^{n} \frac{1}{k+1} H_k = \frac{1}{2} (H_{n+1}^{2} - H_{n+1}^{(2)}) \]

and \(5\):

\[ \sum_{k=1}^{n} \frac{1}{k} H_{n-k} = H_{n}^{2} - H_{n}^{(2)} \]

\[ \sum_{k=0}^{n} \frac{1}{k+1} H_{n-k} = H_{n+1}^{2} - H_{n+1}^{(2)} \]
When \( a \leq b \) are two integers and \( \{x_k\} \) and \( \{y_k\} \) are two sequences of complex numbers, and \( \{s_k\} \) the sequence of complex numbers defined by:

\[
s_k = \sum_{i=a}^{k} x_i
\]

then there is the following summation by parts formula [4]:

\[
\sum_{k=a}^{b-1} x_k y_k = s_{b-1} y_b - \sum_{k=a}^{b-1} s_k (y_{k+1} - y_k)
\]

### 2 Harmonic Number Identities with a Reciprocal

**Theorem 2.1.** For nonnegative integer \( n \) and integer \( p > 0 \):

\[
\sum_{k=0}^{n} \frac{1}{k + p} H_k = H_{n+p}(H_{n+1} + H_{p-1}) - \frac{1}{2}[(H_{n+1} + H_{p-1})^2 + H_{n+1}^{(2)} + H_{p-1}^{(2)}] - \sum_{k=0}^{p-2} \frac{1}{n + k + 2} H_k
\]

**Proof.** Summation by parts (1.8) with \( x_k = 1/(k + p) \) and \( y_k = H_k \) yields:

\[
\sum_{k=1}^{n} \frac{1}{k + p} H_k = (H_{n+p} - H_p)H_{n+1} - \sum_{k=1}^{n} \frac{1}{k + 1} (H_{k+p} - H_p)
\]

Using:

\[
H_{k+p} = H_k + \sum_{s=1}^{p} \frac{1}{k + s}
\]

and for \( s > 1 \):

\[
\frac{1}{(k + s)(k + 1)} = \frac{1}{s - 1} \left( \frac{1}{k + 1} - \frac{1}{k + s} \right)
\]

yields:

\[
\frac{1}{k + 1} H_{k+p} = \frac{1}{k + 1} H_k + \frac{1}{(k + 1)^2} + \sum_{s=2}^{p} \frac{1}{s - 1} \left( \frac{1}{k + 1} - \frac{1}{k + s} \right)
\]

Performing the summation over \( n \) and using (1.4) yields:

\[
\sum_{k=1}^{n} \frac{1}{k + p} H_k = H_{n+p} H_{n+1} - \frac{1}{2} (H_{n+1}^2 + H_{n+1}^{(2)}) - H_p + 1 + \sum_{s=1}^{p-1} \frac{1}{s} (H_{n+s+1} - H_{n+1} - H_{s+1} + 1)
\]
Using
\[ H_{n+s+1} - H_{n+1} = \sum_{k=1}^{s} \frac{1}{n+k+1} \] (2.7)
and changing the order of summation over \( s \) and \( k \):
\[
\sum_{s=1}^{p-1} \frac{1}{s} \sum_{k=1}^{s} \frac{1}{n+k+1} = \sum_{k=1}^{p-1} \frac{1}{n+k+1} \sum_{s=k}^{p-1} \frac{1}{s} \\
= \sum_{k=1}^{p-1} \frac{1}{n+k+1} (H_{p-1} - H_{k-1}) \\
= H_{p-1}(H_{n+p} - H_{n+1}) - \sum_{k=0}^{p-2} \frac{1}{n+k+2} H_k
\] (2.8)
and using \( H_{s+1} = H_s + 1/(s+1) \) and (1.3) and \( 1/(s+1) = 1/s - 1/(s+1) \) yields the theorem.

**Theorem 2.2.** For nonnegative integer \( n \) and integer \( p > 0 \):
\[
\sum_{k=0}^{n} \frac{1}{k+p} H_{n-k} = H_{n+1}(H_{n+p} - H_{p-1}) - \frac{1}{2} [H_{n+1}^2 - H_{n+p}^2 + H_{n+1}^{(2)} + H_{n+p}^{(2)}] - \sum_{k=0}^{p-2} \frac{1}{n+k+2} H_k
\] (2.9)

**Proof.** Summation by parts (1.8) with \( x_k = 1/(n+p-k) \) and \( y_k = H_k \) yields:
\[
\sum_{k=0}^{n} \frac{1}{k+p} H_{n-k} = \sum_{k=1}^{n} \frac{1}{n+p-k} H_k \\
= (H_{n+p-1} - H_{p-1})H_{n+1} - \sum_{k=1}^{n} \frac{1}{k+1} (H_{n+p-1} - H_{n+p-k-1}) \\
\] (2.10)
Using:
\[
H_{n+p-k-1} = H_{n-k} + \sum_{s=1}^{p-1} \frac{1}{n + s - k}
\] (2.11)
and for \( s > 0 \):
\[
\frac{1}{(n + s - k)(k + 1)} = \frac{1}{n + s + 1} \left( \frac{1}{k + 1} + \frac{1}{n + s - k} \right)
\] (2.12)
yields:
\[
\frac{1}{k+1} H_{n+p-k-1} = \frac{1}{k+1} H_{n-k} + \sum_{s=1}^{p-1} \frac{1}{n + s + 1} \left( \frac{1}{k + 1} + \frac{1}{n + s - k} \right)
\] (2.13)
Performing the summation over $n$ and using (1.6) yields:

$$\sum_{k=1}^{n} \frac{1}{k + p} H_{n-k} = H_{n+p-1} - H_{p-1} H_{n+1} + H_{n+1}^2 - H_{n+1}^{(2)} - H_n$$

$$+ \sum_{s=1}^{p-1} \frac{1}{n + s + 1} (H_{n+s-1} + H_{n+1} - H_{s-1} - 1) \quad (2.14)$$

Using

$$\sum_{s=1}^{p-1} \frac{1}{n + s + 1} (H_{n+1} - 1) = (H_{n+1} - 1)(H_{n+p} - H_{n+1}) \quad (2.15)$$

and $H_{n+s-1} = H_{n+s} - 1/(n+s)$ and $1/((n+s)(n+s+1)) = 1/(n+s) - 1/(n+s+1)$ and with (1.6):

$$\sum_{s=0}^{p-2} \frac{1}{n + s + 2} H_{n+s+1} = \sum_{s=0}^{p-1} \frac{1}{k + 1} H_k - \sum_{s=0}^{n} \frac{1}{k + 1} H_k$$

$$= \frac{1}{2}(H_{n+p}^2 - H_{n+p}^{(2)} - H_{n+1}^2 + H_{n+1}^{(2)}) \quad (2.16)$$

yields the theorem. \qed

**Theorem 2.3.** For nonnegative integer $n$ and integer $0 \leq p \leq n$:

$$\sum_{k=p+1}^{n} \frac{1}{k - p} H_k = \frac{1}{2}[(H_{n-p+1} + H_p)^2 + H_{n-p+1}^{(2)} + H_p^{(2)}] - H_{n+1}(H_p + \frac{1}{n - p + 1})$$

$$+ \sum_{k=0}^{p-1} \frac{1}{n - p + k + 2} H_k \quad (2.17)$$

Proof. Summation by parts (1.8) with $x_k = 1/(k - p)$ and $y_k = H_k$ yields:

$$\sum_{k=p+1}^{n} \frac{1}{k - p} H_k = H_{n+1} H_{n-p} - \sum_{k=p+1}^{n} \frac{1}{k + 1} H_{k-p}$$

$$= H_{n+1} H_{n-p} - \sum_{k=1}^{n-p} \frac{1}{k + p + 1} H_k \quad (2.18)$$

The last sum is (2.1) with $p$ replaced by $p + 1$ and $n$ by $n - p$, which yields the theorem. \qed

**Theorem 2.4.** For nonnegative integer $n$ and integer $0 \leq p \leq n$:

$$\sum_{k=p+1}^{n} \frac{1}{k - p} H_{n-k} = H_{n-p}^2 - H_{n-p}^{(2)} \quad (2.19)$$
Proof.

\[
\sum_{k=p+1}^{n} \frac{1}{k-p} H_{n-k} = \sum_{k=0}^{n-p-1} \frac{1}{k+1} H_{n-p-k-1}
\] (2.20)

The last sum is \(\text{(1.6)}\) with \(n\) replaced by \(n-p-1\), which yields the theorem. □

### 3 Products of Reciprocals

A finite product of these reciprocals with different \(p\)'s can be written as a sum of the individual reciprocals. The formula for two reciprocals is, where \(p_1 \neq p_2\):

\[
\frac{1}{(k + p_1)(k + p_2)} = \frac{1}{p_1 - p_2} \left( \frac{1}{k + p_1} - \frac{1}{k + p_2} \right)
\] (3.1)

The formula for three reciprocals is, where \(p_1 \neq p_2 \neq p_3\):

\[
\frac{1}{(k + p_1)(k + p_2)(k + p_3)} = \frac{1}{p_1 - p_2} \left( \frac{1}{p_2 - p_3} - \frac{1}{p_1 - p_3} \right) \frac{1}{k + p_3}
- \frac{1}{p_2 - p_3} \frac{1}{k + p_2} + \frac{1}{p_1 - p_3} \frac{1}{k + p_1}
\] (3.2)

The recursion formula for \(m\) reciprocals in terms of the formula for \(m-1\) reciprocals is:

\[
\prod_{i=1}^{m-1} \frac{1}{k+p_i} = \sum_{i=1}^{m-1} \alpha_i \frac{1}{k+p_i}
\] (3.3)

\[
\prod_{i=1}^{m} \frac{1}{k+p_i} = \sum_{i=1}^{m-1} \alpha_i \frac{1}{k+p_m} \frac{1}{k+p_i}
- \sum_{i=1}^{m-1} \alpha_i \frac{1}{p_i - p_m} \frac{1}{k+p_i}
+ \frac{1}{k+p_m} \sum_{i=1}^{m-1} \alpha_i \frac{1}{p_i - p_m}
\] (3.4)

This recursion formula means that starting with \(m = 1\) and \(\alpha_1 = 1\), in each pass for certain \(m > 1\) the \(\alpha_i\) for \(i = 1 \ldots m-1\) are divided by \(p_m - p_i\), after which \(\alpha_m\) is minus the sum of the new \(\alpha_i\) for \(i = 1 \ldots m-1\). This way the recursion formula reduces to a double iteration, and it is also clear from this that for \(m > 1\):

\[
\sum_{i=1}^{m} \alpha_i = 0
\] (3.5)
When the $\alpha_i$ have been computed, each individual reciprocal can be summed using the appropriate formula in the previous section, where the following substitutions are made:

$$H_{n+p}^{(m)} = H_{n+1}^{(m)} + \sum_{k=2}^{p} \frac{1}{(n+k)^m}$$  \hspace{1cm} (3.6)$$

$$H_{n-p+1}^{(m)} = H_{n+1}^{(m)} - \sum_{k=0}^{p-1} \frac{1}{(n-k+1)^m}$$  \hspace{1cm} (3.7)$$

After these substitutions the coefficient of $H_{n+1}^2$ in the formula for each individual reciprocal is identical, and therefore, by equation (3.5), the coefficient of $H_{n+1}^2$ in the resulting formula for $m > 1$ is zero, which means that for these products of these reciprocals only terms linear in harmonic numbers remain.

### 4 Examples

$$\sum_{k=0}^{n} \frac{1}{k+1} H_k = \frac{1}{2} (H_{n+1}^2 - H_{n+1}^{(2)})$$  \hspace{1cm} (4.1)$$

$$\sum_{k=0}^{n} \frac{1}{k+2} H_k = \frac{1}{2} (H_{n+1}^2 - H_{n+1}^{(2)}) + \frac{1}{n+2} H_{n+1} - \frac{n+1}{n+2}$$  \hspace{1cm} (4.2)$$

$$\sum_{k=0}^{n} \frac{1}{k+3} H_k = \frac{1}{2} (H_{n+1}^2 - H_{n+1}^{(2)}) + \frac{2n+5}{(n+2)(n+3)} H_{n+1} - \frac{(n+1)(7n+20)}{4(n+2)(n+3)}$$  \hspace{1cm} (4.3)$$

$$\sum_{k=1}^{n} \frac{1}{k} H_k = \frac{1}{2} (H_n^2 + H_n^{(2)})$$  \hspace{1cm} (4.4)$$

$$\sum_{k=2}^{n} \frac{1}{k-1} H_k = \frac{1}{2} (H_{n+1}^2 + H_{n+1}^{(2)}) - \frac{2n+1}{n(n+1)} H_{n+1} + \frac{n}{n+1}$$  \hspace{1cm} (4.5)$$

$$\sum_{k=3}^{n} \frac{1}{k-2} H_k = \frac{1}{2} (H_{n+1}^2 + H_{n+1}^{(2)}) - \frac{3n^2-1}{(n-1)n(n+1)} H_{n+1} + \frac{7n^2-n-2}{4n(n+1)}$$  \hspace{1cm} (4.6)$$

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} H_k = H_{n+1}^{(2)} - \frac{1}{n+1} H_{n+1}$$  \hspace{1cm} (4.7)$$

$$\sum_{k=0}^{n} \frac{1}{(k+1)(k+2)} H_k = \frac{n+1}{n+2} - \frac{1}{n+2} H_{n+1}$$  \hspace{1cm} (4.8)$$

$$\sum_{k=2}^{n} \frac{1}{k(k-1)} H_k = \frac{2n+1}{n+1} - \frac{1}{n} H_{n+1}$$  \hspace{1cm} (4.9)$$

$$\sum_{k=3}^{n} \frac{1}{(k-1)(k-2)} H_k = \frac{9n^2+5n-2}{4n(n+1)} - \frac{1}{n-1} H_{n+1}$$  \hspace{1cm} (4.10)$$
\[
\sum_{k=1}^{n} \frac{1}{k(k+1)(k+2)} H_k = \frac{1}{2} H_{n+1}^{(2)} - \frac{1}{2(n+1)(n+2)} H_{n+1} - \frac{n+1}{2(n+2)} \tag{4.11}
\]
\[
\sum_{k=2}^{n} \frac{1}{(k+1)k(k-1)} H_k = \frac{5n+3}{4(n+1)} - \frac{1}{2(n+1)} H_{n+1} - \frac{1}{2} H_{n+1}^{(2)} \tag{4.12}
\]
\[
\sum_{k=3}^{n} \frac{1}{k(k-1)(k-2)} H_k = \frac{2n^2 + 2n - 1}{4n(n+1)} - \frac{1}{2n-1} H_{n+1} \tag{4.13}
\]
\[
\sum_{k=2}^{n} \frac{1}{(k+2)(k+1)(k-1)} H_k = \frac{23n^2 + 57n + 28}{36(n+1)(n+2)} - \frac{1}{3n(n+1)(n+2)} H_{n+1} - \frac{1}{3} H_{n+1}^{(2)} \tag{4.14}
\]
\[
\sum_{k=3}^{n} \frac{1}{(k+1)(k-1)(k-2)} H_k = \frac{1}{6} H_{n+1}^{(2)} - \frac{1}{3(n-1)n(n+1)} H_{n+1} - \frac{2n^2 + 1}{12n(n+1)} \tag{4.15}
\]
\[
\sum_{k=0}^{n} \frac{1}{k+1} H_{n-k} = H_{n+1}^2 - H_{n+1}^{(2)} \tag{4.16}
\]
\[
\sum_{k=0}^{n} \frac{1}{k+2} H_{n-k} = H_{n+1}^2 - H_{n+1}^{(2)} - \frac{n}{n+2} H_{n+1} \tag{4.17}
\]
\[
\sum_{k=0}^{n} \frac{1}{k+3} H_{n-k} = H_{n+1}^2 - H_{n+1}^{(2)} - \frac{3n^2 + 7n - 2}{2(n+2)(n+3)} H_{n+1} - \frac{n+1}{(n+2)(n+3)} \tag{4.18}
\]
\[
\sum_{k=1}^{n} \frac{1}{k} H_{n-k} = H_{n}^2 - H_{n}^{(2)} \tag{4.19}
\]
\[
\sum_{k=2}^{n} \frac{1}{k-1} H_{n-k} = H_{n+1}^2 - H_{n+1}^{(2)} - \frac{2(2n+1)}{n(n+1)} H_{n+1} + \frac{2(3n^2 + 3n + 1)}{n^2(n+1)^2} \tag{4.20}
\]
\[
\sum_{k=3}^{n} \frac{1}{k-2} H_{n-k} = H_{n+1}^2 - H_{n+1}^{(2)} - \frac{2(3n^2 - 1)}{(n-1)n(n+1)} H_{n+1} + \frac{2(6n^4 - 3n^2 + 1)}{(n-1)^2n^2(n+1)^2} \tag{4.21}
\]
\[
\sum_{k=1}^{n} \frac{1}{k(k+1)} H_{n-k} = \frac{n-1}{n+1} H_{n+1} - \frac{n-1}{(n+1)^2} \tag{4.22}
\]
\[
\sum_{k=0}^{n} \frac{1}{(k+1)(k+2)} H_{n-k} = \frac{n}{n+2} H_{n+1} \tag{4.23}
\]
\[
\sum_{k=2}^{n} \frac{1}{k(k-1)} H_{n-k} = \frac{n-2}{n} H_{n+1} - \frac{(n-2)(2n+1)}{n^2(n+1)} \tag{4.24}
\]

\[
\sum_{k=3}^{n} \frac{1}{(k-1)(k-2)} H_{n-k} = \frac{n-3}{n-1} H_{n+1} - \frac{(n-3)(3n^2-1)}{(n-1)^2 n(n+1)} \tag{4.25}
\]

\[
\sum_{k=1}^{n} \frac{1}{k(k+1)(k+2)} H_{n-k} = \frac{n^2+3n-2}{4(n+1)(n+2)} H_{n+1} - \frac{3n-1}{4(n+1)^2} \tag{4.26}
\]

\[
\sum_{k=2}^{n} \frac{1}{(k+1)(k+2)} \frac{1}{k(k-1)} H_{n-k} = \frac{n^2-n-4}{4n(n-1)} H_{n+1} - \frac{(n^2-2n-1)(5n^2+3n-4)}{4(n-1)^2 n^2(n+1)} \tag{4.27}
\]

\[
\sum_{k=2}^{n} \frac{1}{(k+2)(k+1)k(k-1)} H_{n-k} = \frac{n^3+3n^2+2n-12}{18n(n+1)(n+2)} H_{n+1} - \frac{7n^3+18n^2-25n-12}{36n^2(n+1)^2} \tag{4.28}
\]

\[
\sum_{k=3}^{n} \frac{1}{(k+1)k(k-1)(k-2)} H_{n-k} = \frac{n^3-n-12}{18(n-1)n(n+1)} H_{n+1} - \frac{9n^5+9n^4-41n^3-81n^2+32n+24}{36(n-1)^2 n^2(n+1)^2} \tag{4.29}
\]

\[
\sum_{k=3}^{n} \frac{1}{(k+1)(k-1)(k-2)} H_{n-k} = \frac{n^3-n-12}{18(n-1)n(n+1)} H_{n+1} - \frac{9n^5+9n^4-41n^3-81n^2+32n+24}{36(n-1)^2 n^2(n+1)^2} \tag{4.30}
\]

## 5 Computer Program

The Mathematica\textsuperscript{®} [9] program used to compute the expressions is given below.

\[
\text{HarmNumPlus[p_,m_] := HarmonicNumber[n+1,m] + Sum[1/(n+k)^m, \{k,2,p\}]}
\]

\[
\text{HarmNumMinus[p_,m_] := HarmonicNumber[n+1,m] - Sum[1/(n-k+1)^m, \{k,0,p-1\}]}
\]

\[
\text{HarmSumPPos[p_,d_] := Simplify[HarmNumPlus[p,1] (HarmonicNumber[n+1] + HarmonicNumber[p-1]) - \frac{1}{2} ((HarmonicNumber[n+1] + HarmonicNumber[p])^2 + HarmonicNumber[n+1,2] + HarmonicNumber[p-1,2]) - Sum[1/(n+k+2)HarmonicNumber[k], \{k,0,p-1\}] - Sum[1/(k+p)HarmonicNumber[k], \{k,0,d-1\}] ]}
\]

\[
\text{HarmSumPNeg[p_,d_] := Simplify[1/2 ((HarmonicNumber[n+1] + HarmonicNumber[p])^2 - HarmonicNumber[n+1,2] - HarmonicNumber[p-1,2]) - Sum[1/(n+k+2)HarmonicNumber[k], \{k,0,p-2\}] - Sum[1/(n+p+k+2)HarmonicNumber[k], \{k,0,p-1\}] - Sum[1/(k+p)HarmonicNumber[k], \{k,0,d-1\}] ]}
\]

\[
\text{HarmSumP[p_,d_] := If[p \leq 0, HarmSumPNeg[-p,d], HarmSumPPos[p,d]]}
\]
HarmSumQPos[p_,d_] := Simplify[
  HarmonicNumber[n+1] (HarmNumPlus[p,1] - HarmonicNumber[p-1])
  -1/2 (HarmonicNumber[n+1]^2 - HarmNumPlus[p,1]^2
  + HarmonicNumber[n+1,2] + HarmNumPlus[p,2])
  - Sum[1/(n+k+2) HarmonicNumber[k], {k,0,p-2}]
  - Sum[1/(k+p) HarmNumMinus[k+1,1], {k,0,d-1}]
]

HarmSumQNeg[p_,d_] := Simplify[
  HarmNumMinus[p+1,1] - HarmNumMinus[p+1,2]
  - Sum[1/(k-p) HarmNumMinus[k+1,1], {k,p+1,d-1}]
]

HarmTable[m_] := Table[HarmonicNumber[n+1,i], {i,m}]

HarmSumPQ[s_Integer, f_] := Module[
  {d, u, t = HarmTable[2]},
  d = If[s <= 0, -s + 1, 0];
  u = Factor[CoefficientArrays[f[s,d], t]];
  u[[1]] + Dot[u[[2]], t] + Dot[Dot[u[[3]], t], t]
]

HarmSumPQ[s_, f_] := Module[
  {facs, d, u, funs = 0, l = Length[s], t = HarmTable[2]},
  facs = Table[0, {l}]; facs[[1]] = 1;
  Do[Do[facs[[i]] = (s[[i]] - s[[j]])/facs[[j]], {j,1,i-1}], {i,2,l}];
  d = Min[s];
  u = Factor[CoefficientArrays[funs, t]];
  u[[1]] + Dot[u[[2]], t]
]

HarmonicSumP[s_] := If[Length[s] == 1, HarmSumPQ[s[[1]], HarmSumP], HarmSumPQ[s, HarmSumP]]

HarmonicSumQ[s_] := If[Length[s] == 1, HarmSumPQ[s[[1]], HarmSumQ], HarmSumPQ[s, HarmSumQ]]

(* Compute some examples *)
HarmonicSumP[3] // TraditionalForm
HarmonicSumP[{2, 1, 0, -1}] // TraditionalForm
HarmonicSumQ[{-2}] // TraditionalForm
HarmonicSumQ[{0, -1, -2}] // TraditionalForm

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