Noncritical Dimensions for Critical String Theory: Life beyond the Calabi–Yau Frontier

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ABSTRACT

A recently introduced framework for the compactification of supersymmetric string theory involving noncritical manifolds of complex dimension $2k + D_{\text{crit}}, k \geq 1$, is reviewed. These higher dimensional manifolds are spaces with quantized positive Ricci curvature and therefore do not, a priori, describe consistent string vacua. It is nevertheless possible to derive from these manifolds the massless spectra of critical string groundstates. For a subclass of these noncritical theories it is also possible to explicitly construct Calabi–Yau manifolds from the higher dimensional spaces. Thus the new class of theories makes contact with the standard framework of string compactification. This class of manifolds is more general than that of Calabi–Yau manifolds because it contains spaces which correspond to critical string vacua with no Kähler deformations, i.e. no antigenerations, hence providing mirrors of rigid Calabi–Yau manifolds. The constructions reviewed here lead to new insight into the relation between exactly solvable models and their mean field theories on the one hand and Calabi–Yau manifolds on the other, leading, for instance, to a modification of Gepner’s conjecture. They also raise fundamental questions about the Kaluza–Klein concept of string compactification, in particular regarding the rôle played by the dimension of the internal theories.

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1. Introduction

String theory remains the only viable candidate for a unified theory of quantum gravity. One of the attractions of this theory is the fact that it describes a rather tight framework. A consequence is that there are severe restrictions on the internal part of the theory which to a large extent determines the observable low energy physics in four dimensions. Based on the conventional framework formulated in [1] it is believed that in left–right symmetric compactifications without torsion the internal space of the heterotic string is described by a space which has to be a compact manifold which is

- complex,
- Kähler, and admits a
- covariantly constant spinor,

i.e. has vanishing first Chern class, so–called Calabi–Yau manifolds.

Such manifolds are particularly simple, a fact that is encoded concisely in the spectrum of the theory described, in part, by the cohomology of the space. Because the space is complex the real cohomology can be decomposed, via the Hodge decomposition, into complex cohomology groups. Thus the Betti numbers \( b_i = \dim_{\mathbb{R}} H^i(M, \mathbb{R}) \) can be expressed in terms of the Hodge numbers \( h^{p,q} = \dim_{\mathbb{C}} H^{p,q}(M, \mathbb{C}) \):

\[
b_i = \sum_{p+q=i} h^{p,q}. \tag{1}
\]

Because the manifold is Kähler the Hodge numbers are symmetric, \( h^{p,q} = h^{q,p} \), and because the first Chern class vanishes it follows that \( h^{1,0} = 0 = h^{0,1} \) for \( p = 1, 2 \) and \( h^{3,0} = 1 = h^{0,3} \). Hence the cohomology of the internal space, summarized in the Hodge diamond

\[
\begin{array}{ccccccc}
& & & 1 & & & \\
& & 0 & & 0 & & \\
1 & & h^{(1,1)} & & h^{(2,1)} & & 1 \\
0 & & h^{(2,1)} & & h^{(2,1)} & & 0 \\
& & & 0 & & & \\
& & 1 & & 0 & & 0 \\
& & & 0 & & & \\
& & & 0 & & & \\
1 & & & & & & \\
\end{array}
\]

contains only two independent elements \( h^{(1,1)} = h^{(2,2)} \) and \( h^{(2,1)} = h^{(1,2)} \) which parametrize the number of antigenerations and generations, respectively, that are observed in low energy physics.
It is also believed that this class of string vacua features an unexpected symmetry, mirror symmetry, which has been discovered in the context of Landau–Ginzburg vacua in [2]. Independent evidence for this symmetry has been found in the context of orbifolds of exactly solvable tensor models by Greene and Plesser [3]. The effect of this symmetry is that for each string vacuum with some number $h^{(1,1)}$ of antigenerations and some number $h^{(2,1)}$ of generations there exists a mirror vacuum for which these number are exchanged: the spectrum of the mirror vacuum consists of $h^{(2,1)}$ antigenerations and $h^{(1,1)}$ generations. Mirror symmetry thus flips the Hodge diamond along the offdiagonal.

Mirror symmetry is by now well established: beyond the class of exactly solvable models discussed in [3], in which mirror symmetry is understood best, lies the much larger class of Landau–Ginzburg theories constructed explicitly in [2] which clearly indicates that mirror symmetry is a property of string theory. That this symmetry is not accidental in this wider context has been proven in [4] where it was shown that by a combination of orbifolding and fractional transformations a mirror construction can be established between a priori independent pairs of Landau–Ginzburg theories with opposite spectrum. Mirror symmetry is at present being used as a hypothesis to obtain results in algebraic geometry and has been shown to be correct in all computations that have been performed sofar [10, 11, 12].

Mirror symmetry creates a puzzle. There exist well-known Calabi–Yau vacua which are rigid, i.e. they do not have string modes corresponding to complex deformations of the manifold. Since mirror symmetry exchanges complex deformations and Kähler deformations of a manifold it would seem that the mirror of a rigid Calabi–Yau manifold cannot be Kähler and hence does not describe a consistent string vacuum. In fact, it appears, using Zumino’s result [13] that $N = 2$ supersymmetry of a $\sigma$–model requires that the target manifold is Kähler, that the mirror vacuum cannot even be $N = 1$ spacetime supersymmetric. It follows that the class of Calabi–Yau manifolds is not the appropriate setting by a long shot in which to discuss mirror symmetry and the question arises what the proper framework might be.

In this review I discuss recent work [14] which shows the existence of a new class of manifolds which generalizes the class of Calabi–Yau spaces of complex dimension $D_{\text{crit}}$ in a natural way. The manifolds involved are of complex dimension $(2k + D_{\text{crit}})$ and have a positive first Chern class which is quantized in multiples of the degree of the manifold. Thus they do not describe, a priori, consistent string groundstates. Surprisingly however, it is possible to derive from these higher dimensional manifolds the spectrum of critical string vacua. This can be done not only for the generations but also for the antigenerations. For particular types of these

\footnote{The construction of all quasihomogeneous $N = 2$ Landau–Ginzburg with an isolated singularity was recently completed in [3, 8].}
new manifolds it is also possible to construct the corresponding $D_{\text{crit}}$–dimensional Calabi–Yau manifold directly from the $(2k + D_{\text{crit}})$–dimensional space.

This new class of manifolds is, however, not in one to one correspondence with the class of Calabi–Yau manifolds as it also contains manifolds which describe string vacua that do not contain massless modes corresponding to antigenerations. It is precisely this new type of manifold that is needed in order to construct mirrors of rigid Calabi–Yau manifolds without generations.

2. Higher Dimensional Manifolds with Quantized Positive First Chern Class

Consider the class of manifolds of complex dimension $N$ embedded in a weighted projective space $\mathbb{P}_{(k_1,\ldots,k_{N+2})}$ as hypersurfaces

$$M_{N,d} \equiv \{p(z_1,\ldots,z_{N+2}) = 0\} \cap \mathbb{P}_{(k_1,\ldots,k_{N+2})}$$

defined as the zero set of some transverse polynomial $p$ of degree $d$. Here the $k_i$ are the weights of the ambient weighted projective space. The set of hypersurfaces determined by such polynomials will be denoted by

$$\mathbb{P}_{(k_1,k_2,\ldots,k_{N+2})}[d]$$

and called a configuration. Assume that for the hypersurfaces (3) the weights $k_i$ and the degree $d$ are related via the constraint

$$\sum_{i=1}^{N+2} k_i = Qd,$$

where $Q$ is a positive integer. Relation (4) is the defining property of the class of spaces to be considered below. It is a rather restrictive condition in that it excludes many types of varieties which are transverse and even smooth but are not of physical relevance.

A simple example is the Fermat hypersurface

$$\mathbb{P}_{(420,280,210,168,140,120,105)}[840] \ni \{p = \sum_{i=1}^{7} z_{i+1}^{i+1} = 0\}$$

which is a rather nice, transverse, i.e. quasismooth manifold. It is also interesting from a different point of view. A curious aspect of Calabi–Yau hypersurfaces is that they are automatically what is called well formed, i.e. they do not contain orbifold singularities that are surfaces (in

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\[2\]It will become clear below that this definition is rather natural in the context of the theory of Landau–Ginzburg string vacua with an arbitrary number of scaling fields. A particular simple manifold in this class, the cubic sevenfold $\mathbb{P}_8[3]$, has been the subject of recent investigations. 

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the case of threefolds). More generally this fact translates into the statement that the only resolutions that have to be performed are so-called small resolutions, i.e. the singular set are of codimension larger than one. The same is true for the higher dimensional manifolds defined above whereas the manifold (3) contains the singular 4–fold \( S = \mathbb{P}_{(210,140,105,84,70,60)} \).

Alternatively, manifolds of the type above may be characterized via a curvature constraint. Because of (3) the first Chern class is given by

\[
c_1(M_{N,d}) = (Q - 1) \ c_1(\mathcal{N})
\]

(6)

where \( c_1(\mathcal{N}) = dh \) is the first Chern class of the normal bundle \( \mathcal{N} \) of the hypersurface \( M_{N,d} \) and \( h \) is the pullback of the Kähler form \( H \in H^{(1,1)}(\mathbb{P}_{(k_1,\ldots,k_{N+2})}) \) of the ambient space. Hence the first Chern class is quantized in multiples of the degree of the hypersurface \( M_{N,d} \). For \( Q = 1 \) the first Chern class vanishes and the manifolds for which (3) holds are Calabi–Yau manifolds, defining consistent groundstates of the supersymmetric closed string. For \( Q > 1 \) the first Chern class is nonvanishing and therefore these manifolds cannot possibly describe vacua of the critical string, or so it seems.

It turns out that these spaces are closely related to string vacua of complex critical dimension

\[
D_{crit} = N - 2(Q - 1)
\]

(7)
i.e. the critical dimension is offset by twice the coefficient of the first Chern class of the normal bundle of the hypersurface. The evidence for this is twofold. First it is possible to derive from these higher dimensional manifolds the massless spectrum of critical vacua. Furthermore it is shown that for certain subclasses of hypersurfaces of type (3) it is possible to construct Calabi–Yau manifolds \( M_{CY} \) of dimension \( D_{crit} \) and complex codimension

\[
codim_{\mathbb{C}}(M_{CY}) = Q
\]

(8)
directly from these manifolds. In terms of the critical dimension and the codimension the class of manifolds to be investigated below can be described as the projective configurations

\[
\mathbb{P}_{(k_1,\ldots,k(D_{crit}+2Q))}\left[\frac{1}{Q} \sum_{i=1}^{D_{crit}+2Q} k_i\right].
\]

(9)

As mentioned already in the introduction the class of spaces defined by (3) contains manifolds with no antigenerations and hence it is necessary to have some way other than Calabi–Yau manifolds to represent string groundstates if one wants to compare them with the higher dimensional manifolds. One possible way to do this is to relate them to Landau–Ginzburg theories:
any manifold of type (3) can be viewed as a projectivization via a weighted equivalence defined on an affine noncompact hypersurface defined by the same polynomial

\[ \mathcal{C}_{(k_1, \ldots, k_{N+2})}[d] \ni \{p(z_1, \ldots, z_{N+2}) = 0\}. \tag{10} \]

Because the polynomial \( p \) is assumed to be transverse in the projective ambient space the affine variety has a very mild singularity: it has an isolated singularity at the origin defining what is called a catastrophe in the mathematics literature.

The complex variables \( z_i \) parametrizing the ambient space are to be viewed as the field theoretic limit \( \varphi_i(z, \bar{z}) = z_i \) of the lowest components of the order parameters \( \Phi_i(z_i, \theta_i^+, \bar{\theta}_i^+) \), described by chiral \( N = 2 \) superfields of a 2–dimensional Landau–Ginzburg theory defined by the action

\[ \int d^2z d^2\theta d^2\bar{\theta} \ K(\Phi_i, \bar{\Phi}_i) + \int d^2z d^2\theta \ W(\Phi_i) + c.c. \tag{11} \]

where \( K \) is the Kähler potential and \( W \) is the superpotential. It was the important insight of Martinec [15] and Vafa and Warner [16] that such Landau–Ginzburg theories are useful for the understanding of string vacua and also that much information about such groundstates is already encoded in the associated catastrophe (10). A crucial piece of information about a vacuum, e.g., is its central charge. Using a result from singularity theory, it is easy to derive that the central charge of the conformal fixed point of the LG theory is

\[ c = 3 \sum_{i=1}^{N+2} \left( 1 - 2q_i \right), \tag{12} \]

where \( q_i = k_i/d \) are the U(1) charges of the superfields. It is furthermore possible to derive the massless spectrum of the GSO projected fixed of the LG theory, defining the string vacuum, directly from the catastrophe (10) via a procedure described by Vafa [17].

The manifolds (9) therefore correspond to LG theories of central charge

\[ c = 3(N - 2(Q - 1)) = 3D_{\text{crit}} \tag{13} \]

where the relation (7) has been used.

In certain benign situations the subring of monomials of charge 1 in the chiral ring describes the generations of the vacuum [15]. For this to hold at all it is important that the GSO projection is the canonical one with respect to the cyclic group \( \mathbb{Z}_d \), the order of which is the degree \( d \) of the superpotential [9]. Thus the generations are easily derived for this subclass of theories in

\footnote{It does not hold for projections that involve orbifolds with respect to different groups such as those discussed in [19]. This is to be expected as these modified projections can be understood as orbifolds of canonically constructed vacua. The additional moddings generate singularities the resolution of which introduces, in general, additional modes in both sectors, generations and antigenerations.}
because the polynomial ring is identical to the chiral ring of the corresponding Landau–Ginzburg theory. In general a more sophisticated analysis, involving the resolution of higher dimensional singularities, will have to be done [20].

It remains to extract the second cohomology. In a Calabi–Yau manifold there are no holomorphic 2–forms and hence all of the second cohomology is in $H^{(1,1)}$. Because of Kodaira’s vanishing theorem the same is true for manifolds with positive first Chern class and therefore for the manifolds under discussion. At first sight it might appear hopeless to find a construction corresponding to the analysis of (2,1)–forms because of the following example which involves the orbifold of a 3–torus.

Consider the orbifold $T^3_1/\mathbb{Z}_3$ where the two actions are defined as $(z_1, z_4) \rightarrow (\alpha z_1, \alpha^2 z_4)$, all other coordinates invariant and $(z_1, z_7) \rightarrow (\alpha z_1, \alpha^2 z_7)$, all other invariant. Here $\alpha$ is the third root of unity. The resolution of the singular orbifold leads to a Calabi–Yau manifold with 84 antigenerations and no generations [21]. This is precisely the mirror flipped spectrum of the exactly solvable tensor model $1^9$ of 9 copies of $N=2$ superconformal minimal models at level $k=1$ [22] which can be described in terms of the Landau–Ginzburg potential $W = \sum z_i^3$ which belongs to the configuration $\mathbf{C}^{(1,1,1,1,1,1)}[3]$. After imposing the GSO projection by modding out a $\mathbb{Z}_3$ symmetry this Landau–Ginzburg theory leads to the same spectrum as the $1^9$ theory.

This Landau–Ginzburg theory clearly is a mirror candidate for the resolved torus orbifold just mentioned [7][8][9] and the question arises whether a manifold corresponding to this LG potential can be found. Since the theory does not contain modes corresponding to (1,1)–forms it appears that the manifold cannot be Kähler and hence not projective. Thus it appears that the 7–dimensional manifold $\mathbb{I} \mathbb{P}_8[3]$ whose polynomial ring is identical to the chiral ring of the LG theory is merely useful as an auxiliary device in order to describe one sector of the critical LG string vacuum. Even though there exists a precise identity between the Hodge numbers in the middle cohomology group of the higher dimensional manifold and the middle dimension of the cohomology of the Calabi–Yau manifold this is not the case for the second cohomology group.

3. Noncritical Manifolds and Critical Vacua

It turns out however, that by looking at the manifolds [9] in a slightly different way it is nevertheless possible to extract the second cohomology in a canonical manner (even if there is none). The way this works is as follows: the manifolds of type [9] will, in general, not be described by smooth spaces but will have singularities which arise from the projective identification. The basic idea now is to associate the existence of antigenerations in a critical string vacuum with
the existence of singularities in these higher dimensional \textit{noncritical} spaces.

Because the structure of these geometrical singularities depends on the precise form of the polynomial constraint it is difficult to prove the correctness of this idea in full generality. vacua Instead I will, in the following, make this idea more precise and illustrate how it works with a few particularly simple classes of theories, leaving a more detailed investigation of other types of manifolds for a more extensive discussion \cite{20}. As an unexpected bonus this derivation will provide new insight into the Landau–Ginzburg/Calabi–Yau connection.

Consider again the simple example related to the tensor model $^9$. Its LG theory is described by $C^*_{\bar{9}}[3]$ the naive compactification of which leads to

$$\mathbb{P}_8[3] \ni \{ \sum_{i=1}^{9} z_i^3 = 0 \}. \quad (14)$$

Counting monomials leads to the spectrum of 84 generations found previously for the corresponding string vacuum and because this manifold is smooth \textit{no} antigenerations are expected in this model! Hence there does not exist a Calabi–Yau manifold that describes the groundstate $^4$. A second theory in the space of all LG vacua with no antigenerations is

$$(2^6)^{(0,90)} \equiv C^*_{(1,1,1,1,1,1,2)}[4] \ni \{ \sum_{i=1}^{6} z_i^4 + z_7^2 = 0 \} \quad (15)$$

with an obviously smooth manifold $\mathbb{P}_{(1,1,1,1,1,1,2)}[4]$.

Vacua without antigenerations are rather exceptional however; the generic groundstate will have both sectors, generations and antigenerations. The idea described above to derive the antigenerations works for higher dimensional manifolds corresponding to different types of critical vacua but in the following we will illustrate it with two types of such manifolds. A more detailed analysis can be found in \cite{20}.

To be concrete consider the exactly solvable tensor theory $(1 \cdot 16^3)_{A_{2} \otimes E_{7}^{1}}$ with 35 generations and 8 antigenerations which corresponds to a Landau–Ginzburg theory belonging to the configuration

$$C^*_{(2,3,2,3,2,3,3)}[9]^{(8,35)} \quad (16)$$

and which induces, via projectivization, a 5–dimensional weighted hypersurface

$$\mathbb{P}_{(2,2,2,3,3,3,3)}[9] \ni \{ p = \sum_{i=1}^{3} (y_i^3 x_i + x_i^3) + x_4^3 = 0 \}. \quad (17)$$

\footnote{It would seem that a generalization of this 7–dimensional smooth manifold is the infinite class of models $C_{(1,1,1,1,1,1,1,1+3q)}[3+q]$, but since the manifolds \cite{4} are required to be transverse the only possibility is $q = 0$.}
with orbifold singularities

\[
\mathbb{Z}_3 : \mathbb{P}_3[3] \ni \{ p_1 = \sum_{i=1}^{4} x_i^3 = 0 \}
\]
\[
\mathbb{Z}_2 : \mathbb{P}_2.
\] (18)

The \( \mathbb{Z}_3 \)-singular set is a smooth cubic surface which supports seven \((1,1)\)-form as can be easily shown. The \( \mathbb{Z}_2 \) singular set is just the projective plane and therefore adds one further \((1,1)\)-form. Hence the singularities induced on the hypersurface by the singularities of the ambient weighted projective space give rise to a total of eight \((1,1)\)-forms. A simple count leads to the result that the subring of monomials of charge 1 is of dimension 35. Thus we have derived the spectrum of the critical theory from the noncritical manifold (17).

It is presumably possible to derive this result via a surgery process on the singular space (17) but more important is, at this point, that the idea introduced above of relating the spectrum of the string vacuum to the singularity structure of the noncritical manifold also makes it possible to derive from these higher dimensional manifolds the Calabi–Yau manifold of critical dimension! Thus a canonical prescription is obtained which also allows to pass from the Landau–Ginzburg theory to its geometrical counterpart.

This works as follows: Recall that the structure of the singularities of the weighted hypersurface just involved part of the superpotential, namely the cubic polynomial \( p_1 \) which determined the \( \mathbb{Z}_3 \) singular set described by a surface. The superpotential thus splits naturally into the two parts

\[
p = p_1 + p_2
\] (19)

where \( p_2 \) is the remaining part of the polynomial. The idea now is to consider the product \( \mathbb{P}_3[3] \times \mathbb{P}_2 \) where the factors are determined by the singular sets of the higher dimensional space and to impose on this 4–dimensional space a constraint described by the remaining part of the polynomial which did not take part in constraining the singularities of this ambient space. In the case at hand this leaves a polynomial of bidegree \((3,1)\) and hence we are lead to a manifold embedded in

\[
\begin{array}{c}
\mathbb{P}_2 \begin{bmatrix} 3 & 0 \\ 1 & 3 \end{bmatrix} \\
\mathbb{P}_3
\end{array}
\] (20)

defined by polynomials

\[
p_1 = y_1^3 x_1 + y_2^3 x_2 + y_3^3 x_3
\]
\[
p_2 = \sum_{i=1}^{4} x_i^3
\] (21)
which is precisely the manifold constructed in [23], the exactly solvable model of which was later found in [24]. Thus we have found how to construct from the noncritical manifold (17) the critical Calabi–Yau manifold.

A subclass of manifolds of a different type which can be discussed in this framework rather naturally is defined by

\[ I \mathbb{P}^{1}(2k, K - k, 2k, K - k, 2k, 2k, 2k) [2K] \]

where \( K = k + k_3 + k_4 + k_5 \) and it is assumed, for simplicity, that \( K/k \) and \( K/k_i \) are integers. The potentials are

\[ W = \sum_{i=1}^{2} (x_i^{K/k} + x_i y_i^{2}) + x_3^{K/k_3} + x_4^{K/k_4} + x_5^{K/k_5}. \]

The singularities in these manifolds are of two types,

\[ \mathbb{Z}_2 : \mathbb{P}^{1}(k, k, k_3, k_4, k_5) [K] \]
\[ \mathbb{Z}_{K-k} : \mathbb{P}_1. \]

where the constraint of degree \( K \) is given by

\[ p_1 = \sum_{i=1}^{5} x_i^{K/k_i}. \]

The \( \mathbb{Z}_2 \)-singular set is 3-fold with positive first Chern class embedded in weighted \( \mathbb{P}_4 \) whereas the \( \mathbb{Z}_{K-k} \) singular set is just the sphere \( S^2 \sim \mathbb{P}_1 \).

To construct the corresponding critical manifolds note that the structure of the singularities of the weighted hypersurface just involved part of the superpotential, namely the quartic polynomial \( p_1 \) which determined the \( \mathbb{Z}_2 \) singularity set described by a 3-fold. The superpotential thus splits naturally into the two parts \( p = p_1 + p_2 \) where \( p_2 \) is the remaining part of the polynomial. The idea now is to consider again the product \( \mathbb{P}^{1}(k, k, k_3, k_4, k_5) [K] \times \mathbb{P}_1 \) of singular sets of the higher dimensional space and to impose on this 4-dimensional space a constraint described, as before, by the remaining part \( p_2 \) of the polynomial which did not take part in constraining the singularities of this ambient space. In the case at hand this leaves a polynomial of bidegree \( (k, 2) \) and hence we are lead to a manifold embedded in

\[ \mathbb{P}_1 \]
\[ \mathbb{P}^{1}(k, k, k_3, k_4, k_5) \]

defined by polynomials

\[ p_1 = y_1^2 x_1 + y_2^2 x_2 \]
\[ p_2 = x_1^{K/k} + x_2^{K/k} + x_3^{K/k_3} + x_4^{K/k_4} + x_5^{K/k_5}. \]
That this correspondence is in fact correct can be inferred from the work of \[25\] where it was shown that these codimension–2 weighted CICYs correspond to \( N = 2 \) minimal exactly solvable tensor models of the type
\[
\left[ 2 \left( \frac{K}{k} - 1 \right) \right]^2_D \cdot \prod_{i=3}^5 \left( \frac{K}{k_i} - 2 \right)_A .
\] (28)
where the subscripts indicate the affine invariants chosen for the individual levels.

The general picture that emerges from these constructions then is the following: embedded in the higher dimensional manifold is a submanifold which is fibered where the base and the fibres are determined by the singular sets of the ambient manifold. The Calabi–Yau manifold itself is a hypersurface embedded in this fibered submanifold. A heuristic sketch of the geometry is shown in the Figure 1.

The examples above illustrate the simplest situation that can appear. In more complicated manifolds the singularity structure will consist of hypersurfaces whose fibers and/or base themselves are fibered, leading to an iterative procedure. The submanifold to be considered will, in those cases, be of codimension larger than one and the Calabi–Yau manifold will be described by a submanifold with codimension larger than one as well. To illustrate this point consider the 7-fold
\[
\mathbb{P}_{(1,1,6,6,2,2,2,2,2)}[8] \ni \left\{ \sum_{i=1}^2 (x_i^2 y_i + y_i z_i + z_i^4) + z_3^4 + z_4^4 + z_5^4 = 0 \right\}
\] (29)
which leads to the \( \mathbb{Z}_2 \) fibering \( \mathbb{P}_1 \times \mathbb{P}_{(3,3,1,1,1,1)}[4] \) which in turn leads to the \( \mathbb{Z}_3 \) fibering \( \mathbb{P}_1 \times \mathbb{P}_1 \times \mathbb{P}_4[4] \). Following the splits of the potential thus leads to the Calabi–Yau configuration
\[
\begin{bmatrix}
2 & 0 & 0 \\
1 & 1 & 0 \\
0 & 1 & 4
\end{bmatrix}
\ni \begin{cases}
p_1 = \sum_{i=1}^2 x_i^2 y_i = 0 \\
p_2 = \sum_{i=1}^2 y_i z_i = 0 \\
p_3 = \sum_{j=1}^5 z_i^4 = 0
\end{cases}
\] (30)
which is of codimension 3. This example also shows that there are nontrivial relations between these higher dimensional manifolds. The way to see this is via the process of splitting and contraction of Calabi–Yau manifolds introduced in ref. [26]. It can be shown in fact that the Calabi–Yau manifold (30) is an ineffective split of a Calabi–Yau manifold in the class (22). Thus there also exists a corresponding relation between the higher dimensional manifolds.

4. Generalization to Arbitrary Critical Dimensions

Even though the examples discussed in the previous section were all concerned with 6–dimensional Calabi–Yau manifolds and the way they are embedded in the new class of spaces, it should be clear that the ideas presented are not specific to this dimension. Instead of considering compactifications down to the physical dimension, 4, one might contemplate compactifying down
to 2, 6 or 8 dimensions, or else, discuss the class of manifolds above not in the context of compactification at all.

To illustrate this point consider the infinite class of manifolds

$$\mathbb{P}_{(2,n-1,2,n-1,2,...,2)\{2n\}}$$

of complex dimension $n + 1$, defined by polynomials

$$p = \sum_{i=1}^{2} (x_i^n + x_i y_i^2) + x_3^n + \cdots + x_{n+1}^n. \quad (32)$$

According to the considerations above these spaces are related to Calabi–Yau manifolds embedded in

$$\mathbb{P}_1 \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \mathbb{P}_n \begin{bmatrix} 1 \\ n \end{bmatrix} \quad (33)$$

via the equations

$$p_1 = y_1^2 x_1 + y_2^2 x_2$$
$$p_2 = \sum_{i=1}^{n+1} x_i^n. \quad (34)$$

The simplest example is, of course, the case $n = 2$ where the higher dimensional manifold is a 3–fold described by

$$\mathbb{P}_{(2,1,2,1,2)\{4\}} \ni \{ \sum_{i=1}^{2} (z_i^2 + z_i y_i^2) + z_3^2 = 0 \} \quad (35)$$

with a $\mathbb{Z}_2$ singular set isomorphic to the sphere $\mathbb{P}_2[2] \sim \mathbb{P}_1$ which contributes one (1,1)–form, the remaining one being provided by the $\mathbb{P}_1$ defined by the remaining coordinates. The singularity structure of the 3–fold then relates this space to the complex torus described by the algebraic curve

$$\mathbb{P}_1 \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \mathbb{P}_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}. \quad (36)$$

It should be remarked that the Landau–Ginzburg theory corresponding to this theory derives from an exactly solvable tensor model $(2^2)_{D2}$ described by two $N = 2$ superconformal minimal theories at level $k = 2$ equipped with the affine D–invariant.

It is of interest to consider the cohomology groups of the 3–fold itself. With the third Chern class $c_3 = 2h^3$ the Euler number of the singular space is

$$\chi_s = \int c_3 = 1 \quad (37)$$
and hence the Euler number of the resolved manifold is
\[ \tilde{\chi} = 1 - \left(\frac{2}{2}\right) + 2 \cdot 2 = 4. \]  \tag{38}

Since the singular set is a sphere its resolution contributes just one \((1,1)\)-form and hence the second Betti number becomes \(b_2 = 2\). With \(\tilde{\chi} = 2(1 + h^{(1,1)}) - 2h^{(2,1)}\) it follows that
\[ h^{(2,1)} = 1. \]  \tag{39}

The case \(n = 3\) is particularly illuminating because it involves a higher dimensional manifold that is smooth
\[ \mathbb{P}^5[3] \]  \tag{40}
and hence it is easy to determine the cohomology groups of this space and to compare it with the spectrum of \(K3\)
\[ K3 = \begin{bmatrix} \mathbb{P}_1 & 2 \\ \mathbb{P}_3 & 0 \end{bmatrix}, \]  \tag{41}
which consists of \(h^{(0,0)} = h^{(2,2)} = h^{(2,0)} = h^{(0,2)} = 1\) and \(h^{(1,1)} = 20\), all other Hodge numbers are zero. Hence the Euler number becomes \(\chi(K3) = 24\).

The Euler number for the noncritical manifold is easily computed to be \(\chi = 27\). Since the manifold is Kähler \(h^{(p,q)} = h^{(q,p)}\) and because of Poincaré duality \(b_p = b_{8-p}\). Because the manifold has positive first Chern class it follows from Kodaira’s vanishing theorem that \(h^{(p,0)} = 0\) for \(p \neq 0\) and via Lefshetz’ hyperplane theorem it is known that below the middle dimension all the cohomology is inherited from the ambient space and therefore the only nonvanishing cohomology groups lead to \(h^{(0,0)} = h^{(1,1)} = 1\). It can be shown that \(h^{(3,1)} = h^{(1,3)} = 1\) \footnote{I’m grateful to P.Candelas and T.Hübsch for explanations regarding this computation.} and therefore the only remaining cohomology is in \(H^{(2,2)}\). Since
\[ \chi = 2(b_0 + b_2) + b_4 = 6 + h^{(2,2)} \]  \tag{42}
it follows that \(h^{(2,2)} = 21 = 20 + 1\). Thus we have obtained the spectrum of \(K3\) plus one additional mode which always appears in this construction.

This example is also useful because it indicates a generalization of the considerations of the previous section. The surprising new feature of this manifold is that even though the higher dimensional manifold did not have any orbifold singularities it was nevertheless possible to split it in such a way as to construct a Calabi–Yau manifold from it. This was possible because the defining equation was not of Fermat type but involved couplings between the fields. Because of this the manifold featured a new \(\mathbb{Z}_2\) symmetry not present in the Fermat hypersurface and it is
this new symmetry that dictated how to perform the split. This indicates that even for smooth
higher dimensional manifolds it is possible to relate them to Calabi–Yau manifolds once one
moves away from the symmetric point. Table 1 contains a few other examples of how to relate
different singular 4–folds to K3 representations.

| ECFT       | Projective Manifold | Singularities | CY              |
|------------|---------------------|---------------|-----------------|
| $(1^6)_{D^2}@A^4$ | $\mathbb{P}_5[3]$   | $\mathbb{Z}_2 : \mathbb{P}_1$ | $\mathbb{P}_3 \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$ |
| $(6^2 \cdot 2)_{D^2}@A$ | $\mathbb{P}_1(2,3,2,3,2,4)[8]$ | $\mathbb{Z}_2 : \mathbb{P}_1(1,1,1,2)[4]$ | $\mathbb{P}_1(1,1,1,2) \begin{bmatrix} 2 & 0 \\ 1 & 4 \end{bmatrix}$ |
| $(10^2 \cdot 1)_{D^2}@A$ | $\mathbb{P}_1(2,5,2,5,4,6)[12]$ | $\mathbb{Z}_2 : \mathbb{P}_1(1,1,2,3)[6]$ | $\mathbb{P}_1(1,1,2,3) \begin{bmatrix} 2 & 0 \\ 1 & 6 \end{bmatrix}$ |

Table 1: Examples of dimension $D^C = 2$

An example involving a 4–dimensional critical manifold of a different type is defined by the
polynomial

$$p = \sum_{i=1}^{3} (x_i^3 + x_i y_i^3) + \sum_{j=4}^{5} x_i^6$$

which corresponds to the tensor model $(16^3 \cdot 4)_{E^3@D^2}$ with central charge $c = 12$ and belongs
to the configuration

$$\mathbb{P}_{(6,4,6,4,6,4,3,3)}[18].$$

(44)

The critical manifold derived from this 6–fold belongs to the configuration class

$$\mathbb{P}_2 \begin{bmatrix} 3 & 0 \\ 2 & 6 \end{bmatrix}$$

(45)

which is indeed a Calabi–Yau deformation class.

Again it should be emphasized that the construction is not restricted to the infinite series
defined in (E2) as the final example illustrates. A five–dimensional critical vacuum is obtained
by considering the Landau–Ginzburg potential

$$W = \sum_{j=1}^{2} (u_i^3 + u_i v_i^2) + \sum_{i=3}^{5} (u_i^3 + u_i w_i^3)$$

(46)

which corresponds to the exactly solvable model $(16^3 \cdot 4^2)_{E^3@D^2}$. The nine–dimensional non-
critical manifold

$$\mathbb{P}_{(3,2,3,2,3,2,3,3,3,3)}[9]$$

(47)
leads, via its singularity structure, to the five–dimensional critical manifold

\[
\begin{bmatrix}
2 & 0 & 0 \\
0 & 3 & 0 \\
1 & 1 & 3
\end{bmatrix}
\]

It is again crucial that a non–Fermat polynomial was chosen for the last four coordinates in the noncritical manifold.

5. Conclusion

Mirror symmetry cannot be understood in the framework of Calabi–Yau manifolds. Thus, beyond the class of such spaces, there must exist a space of a new type of noncritical manifolds which contain information about critical vacua, such as the mirrors of rigid Calabi–Yau manifolds. Mirrors of spaces with both sectors, antigeneration and generations, are again of Calabi–Yau type and hence the noncritical manifolds which correspond to such groundstates should make contact with Calabi–Yau manifolds in some manner.

What has been shown in [14] is that the class (9) of higher dimensional Kähler manifolds with positive first Chern class, quantized in a particular way, generalizes the framework of Calabi–Yau vacua in the desired way: For particular types of such noncritical manifolds Calabi–Yau manifolds of critical dimension are embedded algebraically in a fibered submanifold. For string vacua which cannot be described by Kähler manifolds and which are mirror candidates of rigid Calabi–Yau manifolds the higher dimensional manifolds still lead to the spectrum of the critical vacuum and a rationale emerges that explains why a Calabi–Yau representation is not possible in such theories. Thus these manifolds of dimension \( c/3 + 2k \) define an appropriate framework in which to discuss mirror symmetry.

There are a number of important consequences that follow from the results of the previous sections. First it should be realized that the relevance of noncritical manifolds suggests the generalization of a conjecture regarding the relation between (2,2) superconformal field theories of central charge \( c = 3D \), \( D \in \mathbb{N} \), with N=1 spacetime supersymmetry on the one hand and Kähler manifolds of complex dimension \( D \) with vanishing first Chern class on the other. It was suggested by Gepner that this relation is 1–1. It follows from the results above that instead superconformal theories of the above type are in correspondence with Kähler manifolds of dimension \( c/3 + 2k \) with a first Chern class quantized in multiples of the degree.

A second consequence is that the ideas of section 3 lead, for a large class of Landau–Ginzburg theories, to a new canonical prescription for the construction of the critical manifold, if it exists, directly from the 2D field theory.
Recently Batyrev [27] introduced a new construction of mirrors of Calabi–Yau manifolds based on dual polyhedra. His method appears to apply only to manifolds defined by one polynomial in a weighted projective space or products thereof. The method of toric geometry that is used in [27] is however not restricted to Calabi–Yau manifolds [28] and therefore the constructions described in sections 3 and 4 lead to the exciting possibility of extending Batyrev’s results to Calabi–Yau manifolds of codimension larger than one by proceeding via noncritical manifolds.

A final remark is that in this framework the role played by the dimension of the manifolds becomes of secondary importance. This is as it should be, at least for an effective theory, which tests only matter content and couplings. It is, however, somewhat mysterious that via ineffective splittings manifolds of different dimension describe one and the same critical vacuum.

It is clear that the emergence in string theory of manifolds with quantized first Chern class should be understood better. The results described here are a first step in this direction. They indicate that these manifolds are not just auxiliary devices but may be as physical as Calabi–Yau manifolds of critical dimension. In order to probe the structure of these models in more depth it is important to get further insight into the complete spectrum of these theories and to compute the Yukawa couplings of the fields. It is clear from the results presented here that the spectra of the higher dimensional manifolds contain additional modes beyond those that are related to the generations and antigenerations of the critical vacuum and the question arises what physical interpretation these fields have.

A better grasp on the complete spectrum of these spaces should also give insight into a different, if not completely independent, approach toward a deeper understanding of these higher dimensional manifold, which is to attempt the construction of consistent $\sigma$–models defined via these spaces. Control of the complete spectrum will shed light on the precise relation between the $\sigma$–models associated to the noncritical manifolds and critical $\sigma$–models.

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