Variational principle for subadditive sequence of potentials in bundle RDS

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Abstract
The topological pressure is defined for subadditive sequence of potentials in bundle random dynamical systems. A variational principle for the topological pressure is set up in a very weak condition. The result may have some applications in the study of multifractal analysis for random version of nonconformal dynamical systems.

Key words: Random dynamical system, subadditive sequence of potentials, variational principle

\textit{2000 MSC:} 37D35, 37A35, 37H99

1. Introduction
The topological pressure for single potential was first presented by Ruelle \cite{24} for expansive maps. Walters \cite{26} generalized it to general continuous maps. The theory about the topological pressure, variational principle and equilibrium states plays a fundamental role in statistical mechanics, ergodic theory and dynamical systems (\cite{11, 9, 25, 8, 23, 12}). Falconer \cite{15} introduced the topological pressure for subadditive sequence of potentials on mixing repellers. Cao \cite{10} extended this notion to general compact dynamical systems. The topological pressure for nonadditive sequence of potentials

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Preprint submitted to Elsevier September 14, 2009
has proved valuable tool in the study of multifractal formalism of dimension theory, especially for nonconformal dynamical systems [15, 3, 2].

In random dynamical systems (RDS), the topological pressure is also important in the study of chaotic properties of random transformations [13, 16, 6, 18, 20]. The earlier work on the topological pressure for single potential was due to Ledrappier [21] and Bogenschütz [4]. Bogenschütz [7] also established the random version of the Bowen-Ruelle formula for expanding almost conformal bundle RDS. Later, Kifer [19] generalized this notion to general bundle RDS and set up the corresponding variational principle. Thus it is a natural question if there exists a random version of thermodynamic formalism for subadditive sequence of potentials, which probably have some potential applications in the study of multifractal formalism of nonconformal RDS.

In this paper, we give the definition of topological pressure for subadditive sequence of potentials and derive a variational principle for the topological pressure. In fact, we formulate a variational principle between the topological pressure, measure-theoretic entropies of RDS and some functions about the invariant measure. Our conditions for this principle are very weak. We only assume that the topological pressure is not $-\infty$. As to the case of $-\infty$, the condition $\Phi^*(\mu) = -\infty$ for all invariant measure $\mu$ is equivalent to that the topological pressure is $-\infty$. The result generalizes both Kifer’s additive variational principle to subadditive case and Cao’s result in deterministic dynamical systems to bundle RDS. The method we used is still in the framework of Kifer’s approach [19], which is the generalization of Misiurewicz’s elegant proof of the nonadditive variational principle [22]. However, since the technique for tackling the subadditive sequence is different from the additive case, we make some changes.

This paper is organized as follows. In section 2 we give a short description of the definitions of bundle RDS, the measure-theoretic entropies of RDS and the topological pressure for the subadditive sequence of potentials together with a corollary. In section 3, we state the variational principle for the subadditive sequence of potentials and give the proof. A required lemma is also given there.

2. Preliminary

Let $(\Omega, \mathcal{F}, P)$ be a probability space together with an invertible $P$-preserving transformation $\vartheta$, where $\mathcal{F}$ is assumed to be complete, countably generated
and to separate points. Let \((X,d)\) be a compact metric space together with the Borel \(\sigma\)-algebra \(\mathcal{B}\). A set \(\mathcal{E} \subset \Omega \times X\) is measurable with respect to the product \(\sigma\)-algebra \(\mathcal{F} \times \mathcal{B}\) and such that the fibers \(\mathcal{E}_\omega = \{x \in X : (\omega,x) \in \mathcal{E}\}\), \(\omega \in \Omega\), are compact. A continuous bundle random dynamical system (RDS) over \((\Omega,\mathcal{F},P,\vartheta)\) is generated by map \(T_\omega : \mathcal{E}_\omega \to \mathcal{E}_{\vartheta \omega}\) with iterates \(T^n_\omega = T_{\vartheta^{n-1} \omega} \cdots T_{\vartheta \omega} T_\omega\), \(n \geq 1\), so that the map \((\omega,x) \to T_\omega x\) is measurable and the map \(x \to T_\omega x\) is continuous for \(P\)-almost all \((a.a)\) \(\omega\). The map \(\Theta : \mathcal{E} \to \mathcal{E}\) defined by \(\Theta(\omega,x) = (\vartheta \omega,T_\omega x)\) is called the skew product transformation.

Let \(\mathcal{P}_{\mathcal{P}}(\mathcal{E}) = \{\mu \in \mathcal{P}_{\mathcal{P}}(\Omega \times X) : \mu(\mathcal{E}) = 1\}\), where \(\mathcal{P}_{\mathcal{P}}(\Omega \times X)\) is the space of probability measures on \(\Omega \times X\) having the marginal \(\mathcal{P}\) on \(\Omega\). Any \(\mu \in \mathcal{P}_{\mathcal{P}}(\mathcal{E})\) on \(\mathcal{E}\) can be disintegrated as \(d\mu(\omega,x) = d\mu_\omega(x) d\mathcal{P}(\omega)\) (See [14]), where \(\mu_\omega\) are regular conditional probabilities with respect to the \(\sigma\)-algebra \(\mathcal{F}_\mathcal{E}\) formed by all sets \((A \times X) \cap \mathcal{E}\) with \(A \in \mathcal{F}\). Let \(\mathcal{M}_{\mathcal{P}}(\mathcal{E},T)\) be the set of \(\Theta\)-invariant measures \(\mu \in \mathcal{P}_{\mathcal{P}}(\mathcal{E})\). \(\mu\) is \(\Theta\)-invariant if and only if the disintegrations \(\mu_\omega\) of \(\mu\) satisfy \(T_\omega \mu_\omega = \mu_{\vartheta \omega}\) \(P\)-a.s. [1]. Let \(\mathcal{Q} = \{\mathcal{Q}_i\}\) be a finite measurable partition of \(\mathcal{E}\), and \(\mathcal{Q}(\omega) = \{\mathcal{Q}_i(\omega)\}\), where \(\{\mathcal{Q}_i(\omega)\} = \{x \in \mathcal{E}_\omega : (\omega,x) \in \mathcal{Q}_i\}\), is a partition of \(\mathcal{E}_\omega\). The conditional entropy of \(\mathcal{Q}\) given the \(\sigma\)-algebra \(\mathcal{F}_\mathcal{E}\) is defined by

\[
H_\mu(\mathcal{Q} | \mathcal{F}_\mathcal{E}) = -\int \sum_i \mu(Q_i) \log \mu(Q_i | \mathcal{F}_\mathcal{E}) d\mathcal{P} = \int H_{\mu_\omega}(\mathcal{Q}(\omega)) d\mathcal{P}(\omega),
\]

where \(H_{\mu_\omega}(A)\) denotes the usual entropy of a partition \(A\). The entropy \(h^{(r)}_\mu (T)\) of the RDS \(T\) with respect to \(\mu\) is defined by the formula

\[
h^{(r)}_\mu (T) = \sup_{\mathcal{Q}} h^{(r)}_\mu (T, \mathcal{Q}), \quad \text{where} \quad h^{(r)}_\mu (T, \mathcal{Q}) = \lim_{n \to \infty} \frac{1}{n} H_{\mu} \left( \bigvee_{i=0}^{n-1} (\Theta^i)^{-1} \mathcal{Q} | \mathcal{F}_\mathcal{E} \right),
\]

the supremum is taken over all finite measurable partitions \(\mathcal{Q} = \{\mathcal{Q}_i\}\) of \(\mathcal{E}\) with finite conditional entropy \(H_{\mu}(\mathcal{Q} | \mathcal{F}_\mathcal{E}) < \infty\). It should be noted that the supremum can be taken only over partitions \(\mathcal{Q}\) of \(\mathcal{E}\) into sets \(Q_i\) of the form \(Q_i = (\Omega \times P_i) \cap \mathcal{E}\), where \(\mathcal{P} = \{P_i\}\) is a partition of \(X\) into measurable sets, so that \(Q_i(\omega) = P_i \cap \mathcal{E}_\omega\) (See [4, 17, 5]). By (1), the limit can be also expressed as

\[
h^{(r)}_\mu (T, \mathcal{Q}) = \lim_{n \to \infty} \frac{1}{n} \int H_{\mu_\omega} \left( \bigvee_{i=0}^{n-1} (T^i_\omega)^{-1} \mathcal{Q}(\vartheta^i \omega) \right) d\mathcal{P}(\omega).
\]
For each measurable in \((\omega, x)\) and continuous in \(x \in \mathcal{E}_\omega\) function \(f\) on \(\mathcal{E}\), let
\[
\|f\| = \int \|f(\omega)\|_{\infty} d\mathbb{P}, \quad \text{where} \quad \|f(\omega)\|_{\infty} = \sup_{x \in \mathcal{E}_\omega} |f(\omega, x)|,
\]
and \(L^1_\omega(\Omega, \mathcal{C}(X))\) be the space of such functions \(f\) with \(\|f\| < \infty\) and identify \(f\) and \(g\) provided \(\|f - g\| = 0\), then \(L^1_\omega(\Omega, \mathcal{C}(X))\) is a Banach space with the norm \(\| \cdot \|\).

Let \(\Phi = \{f_n\}_{n=1}^\infty\) be a sequence functions on \(\mathcal{E}\) such that each \(f_n\) is measurable in \(\omega\) and continuous in \(x\) on \(\mathcal{E}\). These functions are measurable in \((\omega, x)\) in view of Lemma III.14 from [17]. \(\Phi\) is called subadditive if for any \((\omega, x) \in \mathcal{E}\) and \(m, n \in \mathbb{N}\),
\[
f_{n+m}(\omega, x) \leq f_n(\omega, x) + f_m(\Theta^n(\omega, x)).
\]
If \(f_1 \in L^1_\omega(\Omega, \mathcal{C}(X))\) and the above inequality is satisfied, then a simple calculation indicates that each \(f_n \in L^1_\omega(\Omega, \mathcal{C}(X))\). In the sequel we always assume \(\Phi\) satisfies these conditions.

For any \(\Theta\)-invariant measure \(\mu\), denote
\[
\Phi^*(\mu) = \lim_{n \to \infty} \frac{1}{n} \int f_n \, d\mu.
\]
Existence of the limit follows from the well-known subadditive argument. If we denote \(\Phi^k = \{f_{kn}\}_{n=1}^\infty\) for any \(k \in \mathbb{N}\), then \((\Phi^k)^*(\mu) = k \Phi^*(\mu)\).

For each \(n \in \mathbb{N}\) and \(\epsilon > 0\), a family of metrics \(d^\epsilon_n\) on \(\mathcal{E}_\omega\) is defined as
\[
d^\epsilon_n(x, y) = \max_{0 \leq k < n} (d(T^k_\omega x, T^k_\omega y)), \quad x, y \in \mathcal{E}_\omega,
\]
where \(T^0_\omega\) is the identity map. For each \(n \in \mathbb{N}\) and \(\epsilon > 0\), a set \(F \subset \mathcal{E}_\omega\) is said to be \((\omega, \epsilon, n)\)-separated if \(x, y \in F\), \(x \neq y\) implies \(d^\epsilon_n(x, y) > \epsilon\).

For \(\Phi = \{f_n\}_{n=1}^\infty\), \(\epsilon > 0\) and an \((\omega, \epsilon, n)\)-separated set \(F \subset \mathcal{E}_\omega\), denote
\[
\pi_T(\Phi)(\omega, \epsilon, n) = \sup \{ \sum_{x \in F} \exp(f_n(\omega, x)) : F \text{ is an}(\omega, \epsilon, n)\)-separated subset of \(\mathcal{E}_\omega\}\).

Obviously, the supremum can be taken only over all maximal \((\omega, \epsilon, n)\)-separated subsets. By replacing the function \(S_n f\) in Lemma 1.2 of [19] with \(f_n\), a completely similar argument can give the following result, which provides basic measurable properties we needed. In fact, for any measurable function \(g\) on the \(\mathcal{E}_\omega\), the result is also correct.
Lemma 1. For any \( n \in \mathbb{N} \) and \( \epsilon > 0 \) the function \( \pi_T(\Phi)(\omega, \epsilon, n) \) is measurable in \( \omega \), and for each \( \delta > 0 \) there exists a family of maximal \((\omega, \epsilon, n)\)-separated set \( G_\omega \subset E_\omega \) satisfying

\[
\sum_{x \in G_\omega} \exp(f_n(\omega, x)) \geq (1 - \delta)\pi_T(\Phi)(\omega, \epsilon, n)
\]  

and depending measurably on \( \omega \) in the sense that \( G = \{(\omega, x) : x \in G_\omega\} \in \mathcal{F} \times \mathcal{B} \). In particular, the supremum in the definition of \( \pi_T(\Phi)(\omega, \epsilon, n) \) can be taken only measurable in \( \omega \) families of \((\omega, \epsilon, n)\)-separated sets.

In view of this lemma, for \( \Phi = \{f_n\}_{n=1}^{\infty} \) as above and \( \epsilon > 0 \), we can denote

\[
\pi_T(\Phi)(\epsilon) = \limsup_{n \to \infty} \frac{1}{n} \int \log \pi_T(\Phi)(\omega, \epsilon, n) \, dP(\omega).
\]

(5)

The topological pressure of \( \Phi \) is defined as

\[
\pi_T(\Phi) = \lim_{\epsilon \to 0} \pi_T(\Phi)(\epsilon),
\]

since \( \pi_T(\Phi)(\epsilon) \) is a monotone decreasing function in \( \omega \), the limit exists and the limit in fact equals to \( \sup_{\epsilon > 0} \pi_T(\Phi)(\epsilon) \).

For a given \( n \in \mathbb{N}_+ \), through replacing \( \vartheta \) by \( \vartheta^n \) we can consider the bundle RDS \( T^k \) defined by \((T^k)^\omega = T^k_{\vartheta(n-1)k^\omega} \cdots T^k_{\vartheta^k \omega} T^k_\omega \).

Corollary 2. If \( \Phi = \{f_n\}_{n=1}^{\infty} \) is a subadditive sequence of functions, each \( f_n \) is measurable in \( \omega \) and continuous in \( x \) on \( E \) and \( f_1 \in L^1(\Omega, C(X)) \), then for any \( n \in \mathbb{N}_+ \), \( \pi_{T^k}(\Phi^k) = k\pi_T(\Phi) \).

Proof. Since each \((\omega, \epsilon, n)\)-separated set for \( T^k \) is also a \((\omega, \epsilon, kn)\) for \( T \), then \( \pi_T(\Phi)(\omega, \epsilon, kn) \geq \pi_{T^k}(\Phi^k)(\omega, \epsilon, n) \) and \( \pi_{T^k}(\Phi^k) \leq k\pi_T(\Phi) \) follows. For any \( \epsilon > 0 \), by the continuity of \( T_\omega \), there exists some small enough \( \delta > 0 \) such that if \( d(x, y) \leq \delta \), \( x, y \in E_\omega \) then \( d^k(x, y) < \epsilon \). For any positive integer \( m \), there exists some integer \( n \) such that \( kn \leq m < k(n + 1) \). It is easy to see that any \((\omega, \epsilon, m)\)-separated set of \( T \) is also an \((\omega, \delta, n)\)-separated set of \( T^k \). In view of \( f_m(\omega, x) \leq f_{kn}(\omega, x) + f_{m-kn}(T^{kn}(\omega, x)) \) and \( f_{m-kn}(T^{kn}(\omega, x)) \leq \sum_{i=k}^{m-1} f_1(T^i(\omega, x)) \), we have

\[
\pi_T(\Phi)(\omega, \epsilon, m) = \sup \{\sum_{x \in F} \exp f_m(\omega, x) : F \text{ is an } (\omega, \epsilon, m)\text{-separated set of } T\}
\]

5
\[
\leq \sup\left\{ \sum_{x \in F} \exp(f_{kn}(\omega, x)) + \sum_{i=k}^{m-1} f_i(T^i(\omega, x)) \right\} : \\
F \text{ is an } (\omega, \epsilon, m)\text{-separated set of } T
\]

\[
\leq \sum_{i=k}^{m-1} \|f_1(\vartheta^i\omega)\|\sup_{x \in F} \sum_{x \in F} \exp(f_{kn}(\omega, x)) : \\
F \text{ is an } (\omega, \delta, n)\text{-separated set of } T^k
\]

Since \( f_1 \in L^1(\Omega, \mathcal{C}(X)) \), so \( \int \sum_{i=k}^{m-1} \|f_1(\vartheta^i\omega)\|\sup_{x \in F} \sum_{x \in F} \exp(f_{kn}(\omega, x)) \, d\mathbf{P}(\omega) < \infty \), then by (5), \( k\pi_T(\Phi)(\epsilon) \leq \pi_{T^k}(\Phi^k)(\delta) \). If \( \epsilon \to 0 \), then \( \delta \to 0 \), so the inequality opposite follows from the definition of the topological pressure.

In the argument of the variational principle, only the first part of the corollary is used. However, for integrability, we give the other part, which shows the similarity with the usual additive situation, i.e., \( f_n = \sum_{i=0}^{n-1} f_1 T^i(\omega, x) \).

### 3. The variational principle for subadditive sequence of potentials

First we give the following Lemma which we need in the proof of our main theorem.

**Lemma 3.** For a sequence probability measures \( \{\mu_n\}_{n=1}^{\infty} \) in \( \mathcal{P}(\mathcal{E}) \), where \( \mu_n = \frac{1}{n} \sum_{i=0}^{n-1} \Theta^i \nu_n \) and \( \{\nu_n\}_{n=1}^{\infty} \subset \mathcal{P}(\mathcal{E}) \), if \( \{n_i\} \) is some subsequence of natural numbers \( \mathbb{N} \) such that \( \mu_{n_i} \to \mu \in \mathcal{M}(\mathcal{E}, T) \), then for any \( k \in \mathbb{N} \),

\[
\limsup_{i \to \infty} \frac{1}{n_i} \int f_{n_i}(\omega, x) \, d\nu_{n_i} \leq \frac{1}{k} \int f_k \, d\mu. \tag{6}
\]

In particular, the left part is no more than \( \Phi^*(\mu) \).

**Proof.** For \( 0 \leq j < k \) and \( n \geq 2k \), by the subadditivity of \( \Phi \),

\[
f_n \leq f_{n-j} \Theta^j + f_j \leq (f_k + f_k \Theta^k + \cdots + f_k \Theta^{[\frac{n-j-1}{k}]} + f_{n-j-[\frac{n-j-1}{k}]} \Theta^{[\frac{n-j-1}{k}]}) \Theta^j + f_j \]

\[
= \sum_{l=0}^{[\frac{n-j-1}{k}]} f_k \Theta^{lk+j} + (f_{n-j-[\frac{n-j-1}{k}]} \Theta^{[\frac{n-j-1}{k}]k+j} + f_j)
\]
Summing over $j$ and dividing by $k$, we have

$$f_n \leq \frac{1}{k} \sum_{j=0}^{k-1} \frac{[n-j-1]}{k} f_{k^{j+1}} + \frac{1}{k} \sum_{j=0}^{k-1} (f_{n-j-[n-k]} k^{j+1} + f_j)$$

$$= \frac{1}{k} \sum_{s=0}^{n-k} f_k^{s} + \frac{1}{k} \sum_{j=0}^{k-1} (f_{n-j-[n-k]} k^{j+1} + f_j).$$

Since $n - j - \lceil n/k \rceil \leq k$ and $f_r \leq \sum_{t=0}^{r-1} f_1 \Theta^t$, where $1 \leq r \leq k$, integrating this inequality, then by $\int f_{k} d\mu'_{n} = \int f_{k} d\mu$, we get

$$\int f_n d\nu_n \leq \frac{1}{k} \int f_k d\mu'_{n} + 2k \int \|f(\omega)\| dP(\omega)$$

$$= n - k + 1 \int f_k d\mu_n + 2k \int \|f(\omega)\| dP(\omega), \quad (7)$$

where $\mu'_{n} = \frac{1}{n-k+1} \sum_{s=0}^{n-k} \Theta^s \nu_n$. Since for any $f \in L^k_1(\Omega, \mathcal{C}(X))$,

$$n \int f d\mu_n - (n - k + 1) \int f d\mu_n'$$

$$= \sum_{i=n-k+1}^{n-1} \int f \Theta^i d\nu_n \leq k \int \|f(\omega)\| dP(\omega).$$

Dividing by $n$ and letting $n \to \infty$, we get

$$\lim_{n \to \infty} \int f d\mu_n = \lim_{n \to \infty} \int f d\mu_n'.$$

Observing that $\lim_{i \to \infty} \mu'_{n_i} = \mu$, which follows from $\{\mu_n\} \to \mu$, we have

$$\lim_{i \to \infty} \int f_k d\mu'_{n_i} = \int f_k d\mu. \quad (8)$$

Replacing $n$ by $n_i$ in (7), dividing by $n_i$ and passing to $\limsup_{i \to \infty}$, (6) follows by (8). Letting $k \to \infty$, the result holds.
Theorem 4. Let $T$ be a continuous bundle RDS on $\mathcal{E}$, $\Phi = \{f_n\}_{n=1}^{\infty}$ is subadditive, $f_1 \in L^1(\Omega, \mathcal{C}(X))$, and each $f_n$ be measurable in $\omega$ and continuous in $x$. If $\pi_T(\Phi) > -\infty$, then

$$\pi_T(\Phi) = \sup \{h^{(r)}_{\mu}(T) + \Phi^*(\mu) : \mu \in \mathcal{M}^1_p(\mathcal{E}, T) \text{ and } \Phi^*(\mu) > -\infty\}. $$

Proof. Let $\mu \in \mathcal{M}^1_p(\mathcal{E}, T)$, $\Phi^*(\mu) > -\infty$, $\mathcal{P} = \{P_1, \cdots, P_k\}$, be a finite measurable partition of $X$, and $\epsilon$ be a positive number with $\epsilon k \log k < 1$. Denote by $\mathcal{P}(\omega) = \{P_1(\omega), \cdots, P_k(\omega)\}$, $P_i(\omega) = P_i \cap \mathcal{E}_\omega, i = 1, \cdots, k$, the corresponding partition of $\mathcal{E}_\omega$. By the regularity of $\mu$, we can find compact sets $Q_i \subset P_i, 1 \leq i \leq k$, such that

$$\mu(P_i \setminus Q_i) = \int \mu_\omega(P_i(\omega) \setminus Q_i(\omega)) d\mathcal{P}(\omega) < \epsilon,$$

where $Q_i(\omega) = Q_i \cap \mathcal{E}_\omega$. Let $\mathcal{Q}(\omega) = \{Q_0(\omega), \cdots, Q_k(\omega)\}$ be the partition of $\mathcal{E}_\omega$, where $Q_0(\omega) = \mathcal{E}_\omega \setminus \bigcup_{i=1}^{k} Q_i(\omega)$. Then by the results of Kifer [17] and Bogenschütz [4], the following inequality holds (See [19] for details),

$$h^{(r)}_{\mu}(T, \Omega \times \mathcal{P}) \leq h^{(r)}_{\mu}(T, \Omega \times \mathcal{Q}) + 1, \quad (9)$$

where $\Omega \times \mathcal{P}$ (respectively by $\Omega \times \mathcal{Q}$) denotes the partition of $\Omega \times X$ into sets $\Omega \times P_i$ (respectively by $\Omega \times Q_i$). Set

$$\mathcal{Q}_n(\omega) = \bigvee_{i=0}^{n-1} (T^i_\omega)^{-1} \mathcal{Q}((\partial^i \omega), \quad f^*_n(\omega, C) := \sup \{f_n(\omega, x) : x \in C\}

for $C \in \mathcal{Q}_n(\omega)$. Then by the well-known inequality (See [27]) $\sum_{1 \leq i \leq m} p_i (a_i - \log p_i) \leq \sum_{1 \leq i \leq m} \exp a_i$, where each $a_i$ is a real number, $p_i \geq 0$ and $\sum_{i=1}^{k} p_i = 1$, we have

$$H_{\mu_\omega}(\mathcal{Q}_n(\omega)) + \int_{\mathcal{E}_\omega} f_n(\omega) d\mu_\omega \leq \log \sum_{C \in \mathcal{Q}_n(\omega)} \exp f^*_n(\omega, C). \quad (10)$$

Let $\mathcal{R} = \{Q_0 \cup Q_1, \cdots, Q_0 \cup Q_k\}$ be the open cover set of $X$, and $\delta$ be the Lebesgue number for $\mathcal{R}$. Then for every $\omega$, $\mathcal{R}_\omega = \{Q_0(\omega) \cup Q_1(\omega), \cdots, Q_0(\omega) \cup Q_k(\omega)\}$ is the open cover of $\mathcal{E}_\omega$ and $\delta$ is also a Lebesgue number. Let $x(C)$ be the point in $\overline{C}$ such that $f^*_n(\omega, x(C)) = f^*_n(\omega, C)$. If $d(x(C), x(D)) < \delta$, then $x(C)$ and $x(D)$ are in the same element of $\mathcal{R}_\omega$, say $Q_0(\omega) \cup Q_j(\omega), 0 \leq j \leq k$. By the compactness of $\overline{C}$, we have

$$d(x(C), x(D)) < \delta \quad \text{for all } C, D \in \mathcal{R}_\omega.$$
$j < k + 1$. Hence for each $C$, there are at most $2^n$ elements $D$ of $Q_n(\omega)$ such that
\[
d_n^\omega(x(C), x(D)) = \max \{d(T_j^\omega (x(C)), T_j^\omega (x(D)) \} < \delta.
\]
Now an $(\omega, \delta, n)$-separated set $E$ can be constructed such that
\[
\sum_{C \in Q_n(\omega)} f_n^\ast(\omega, C) \leq 2^n \sum_{y \in E} f_n(\omega, y) \tag{11}
\]
We first select the point $x(C_1)$ such that $f_n^\ast(\omega, C_1) = \max_{C \in Q_n(\omega)} f_n^\ast(\omega, C)$, then select the second point $x(C_2)$ such that $f_n^\ast(\omega, C_2) = \max_{C' \in Q_n(\omega)} f_n^\ast(\omega, C')$,
\[
d_n^\omega(x(C_1), x(C')) \geq \delta
\]
the third point $x(C_3)$ such that
\[
f_n^\ast(\omega, C_3) = \max_{C'' \in Q_n(\omega)} f_n^\ast(\omega, C''),
\]
continue this process, a finite step $m$ can complete this selection since $Q_n(\omega)$ is finite. Let $E = \{x(C_1), \ldots, x(C_m)\}$. Obviously $E$ is an $\omega, \delta, n$-separated set. By the above analysis, for each step, we delete at most $2^n$ elements of $Q_n(\omega)$, so the inequality (11) holds.

From (10) and (11), we get
\[
H_{\mu_\omega}(Q_n(\omega)) + \int_{E_\omega} f_n(\omega) d\mu_\omega \leq n \log 2 + \log \pi_T(\Phi)(\omega, \delta, n).
\]
Integrating this inequality through $\mathcal{P}$, dividing by $n$ and letting $n \to \infty$, from (5), (9) and $\Phi^\ast(\mu) > -\infty$, we have
\[
H^{(r)}_\mu(T, \Omega \times \mathcal{P}) + \Phi^\ast(\mu) \\
\leq 1 + H^{(r)}_\mu(T, \Omega \times Q)\Phi^\ast(\mu) \leq 1 + \log 2 + \pi_T(\Phi)(\delta).
\]
By the arbitrariness of $\mathcal{P}$ and $\delta$,
\[
h^{(r)}_\mu(T) + \Phi^\ast(\mu) \leq 1 + \log 2 + \pi_T(\Phi).
\]
Replacing \( T \) and \( \Phi \) by \( T^n \) and \( \Phi^n \), respectively, then by the equality \( h^{(n)}(T^n) = nh^{(n)}(T) \) (See [4, 17]) and \( (\Phi^n)^*(\mu) = n\Phi^*(\mu) \), we obtain

\[
n(h^{(n)}(T) + \Phi^*(\mu)) \leq 1 + \log 2 + \pi T^n(\Phi^n).
\]

Using lemma 2, dividing by \( n \) and letting \( n \to \infty \), we get the first part

\[
h^{(n)}(T) + \Phi^*(\mu) \leq \pi T(\Phi).
\]

In the opposite direction, choose some small \( \epsilon > 0 \) with \( \pi T(\Phi, \epsilon) > -\infty \), and a family of measurable in \( \omega \) maximal \( (\omega, \epsilon, n) \)-separated sets \( G(\omega, \epsilon, n) \subset \mathcal{E}_\omega \) by Lemma 1 such that

\[
\sum_{x \in G(\omega, \epsilon, n)} \exp f_n(\omega, x) \geq \frac{1}{e} \pi T(\Phi)(\omega, \epsilon, n). \tag{12}
\]

Let \( \{\nu^{(n)}_\omega\} \) be a family of atomic measures on \( \mathcal{E}_\omega \) such that they are measurable disintegrations of some probability measure \( \nu^{(n)} \), i.e., \( d\nu^{(n)}(\omega, x) = d\nu^{(n)}_\omega(x) dP(\omega) \), where

\[
\nu^{(n)}_\omega = \frac{\sum_{x \in G(\omega, \epsilon, n)} \exp f_n(\omega, x) \delta_x}{\sum_{y \in G(\omega, \epsilon, n)} \exp f_n(\omega, y)}.
\]

Denote

\[
\mu^{(n)} = \frac{1}{n} \sum_{i=0}^{n-1} \Theta^i \nu^{(n)}.
\]

By (5) and Lemma 2.1 (i)-(ii) in [17], choose a subsequence \( \{n_j\} \) satisfying the following two limits simultaneously,

\[
\lim_{j \to \infty} \pi T(\Phi)(\epsilon) = \frac{1}{n_j} \int \log \pi T(\Phi)(\omega, \epsilon, n_j) dP(\omega),
\]

\[
\lim_{j \to \infty} \mu^{(n_j)} = \mu \quad \text{for some} \quad \mu \in \mathcal{M}_1^1(\mathcal{E}, T). \tag{13}
\]

Choose a partition \( \mathcal{P} = \{P_1, \cdots, P_k\} \) of \( X \) with \( \text{diam} \mathcal{P} \leq \epsilon \) and \( \int \mu_\omega(\partial P_i) dP(\omega) = 0 \) for all \( 1 \leq i \leq k \), where \( \partial \) denotes the boundary. Set \( \mathcal{P}(\omega) = \{P_1(\omega), \cdots, P_k(\omega)\} \), \( P_i(\omega) = P_i \cap \mathcal{E}_\omega \). Since each element of \( \bigvee_{i=0}^{n-1} (T^i_\omega)^{-1} \mathcal{P}(\vartheta^i \omega) \) contains at most
one element of $G(\omega, \epsilon, n)$, by (12), we have
\[
H_{\nu_{\omega}}(n) \left( \bigvee_{i=0}^{n-1} \mathcal{P}(\vartheta^i\omega) \right) + \int f_n(\omega) \, d\nu_{\omega}(n) = \log \left( \sum_{x \in G(\omega, \epsilon, n)} \exp f_n(\omega, x) \right) \geq \log \pi T(\Phi)(\omega, \epsilon, n) - 1.
\]

Let $\mathcal{Q} = \{Q_1, \cdots, Q_k\}$, $Q_i = (\Omega \times P_i) \cap \mathcal{E}$; then $\mathcal{Q}$ is a partition of $\mathcal{E}$ and $Q_i(\omega) = \{x \in \mathcal{E}_\omega : (\omega, x) \in Q_i\} = P_i(\omega)$. Integrating the inequality against $P$, by the definition of the conditional entropy, we get
\[
H_{\mu(\omega)} \left( \bigvee_{i=0}^{n-1} \mathcal{Q} \mid \mathcal{F}_\omega \right) + \int f_n(\omega, x) \, d\nu_{\omega}(n) \geq \int \log \pi T(\Phi)(\omega, \epsilon, n) \, dP(\omega) - 1.
\]

Setting $q, n \in \mathbb{N}$, $1 < q < n$, using the usual method as in [27] and using the subadditivity of conditional entropy [17, 5] and Lemma 3.2 in [21], we have the following inequality (See [19] for the detail)
\[
q H_{\mu(\omega)} \left( \bigvee_{m=0}^{n-1} \mathcal{Q} \mid \mathcal{F}_\omega \right) \leq n H_{\mu(\omega)} \left( \bigvee_{i=0}^{q-1} \mathcal{Q} \mid \mathcal{F}_\omega \right) + 2q^2 \log k.
\]

Then by the above two inequalities,
\[
q \int \log \pi T(\Phi)(\omega, \epsilon, n) \, dP(\omega) - q \leq n H_{\mu(\omega)} \left( \bigvee_{i=0}^{q-1} \mathcal{Q} \mid \mathcal{F}_\omega \right) + 2q^2 \log k + \int f_n(\omega, x) \, d\nu_{\omega}(n).
\]

Since $\mu \in \mathcal{M}_P^1(\omega, T)$ and $\bigvee_{i=0}^{q-1} (T_\omega^i)^{-1}\mathcal{P}(\vartheta^i\omega) \subseteq \bigcup_{i=0}^{q-1} (T_\omega^i)^{-1}\mathcal{P}(\vartheta^i\omega)$, it is easy to see that $\mu_\omega(\partial \bigvee_{i=0}^{q-1} (T_\omega^i)^{-1}\mathcal{P}(\vartheta^i\omega)) = 0P$-a.s. Dividing by $n$, passing to the limit along a subsequence $n_j \to \infty$ satisfying (13) and taking into account Lemma 3 and Lemma 2.1 (iii) in [17], it follows in view of the choice of the partition $P$ that
\[
q \pi T(\Phi)(\epsilon) \leq H_{\mu} \left( \bigvee_{i=0}^{q-1} \mathcal{Q} \mid \mathcal{F}_\omega \right) + q \Phi^*(\mu),
\]
then $\Phi^*(\mu) > -\infty$ since $\pi_T(\Phi) > -\infty$. Dividing this inequality by $q$ and letting $q \to \infty$, so $\pi_T(\Phi)(\epsilon) \leq h_{\mu}^{(r)}(T, \mathcal{Q}) + \Phi^*(\mu)$. Hence $\pi_T(\Phi)(\epsilon) \leq h_{\mu}^{(r)}(T) + \Phi^*(\mu)$ and letting $\epsilon \to 0$, the required inequality follows, then the results holds.

**Remark 5.** If $\pi_T(\Phi) \geq \sup\{h_{\mu}^{(r)}(T, \mathcal{Q}) + \Phi^*(\mu)\}$, then obviously $\pi_T(\Phi) > -\infty$ by $\Phi^*(\mu) > -\infty$. So the condition we give is only used in the opposite direction " $\leq "$ . In fact, by the above argument, it is not hard to see that $\pi_T(\Phi) = -\infty$ is equivalent to $\Phi^*(\mu) = -\infty$ for all invariant measure $\mu$.

**Acknowledgements**

The first author is supported by a grant from Postdoctoral Science Research Program of Jiangsu Province (0701049C). The second author is partially supported by the National Natural Science Foundation of China (10571086) and National Basic Research Program of China (973 Program) (2007CB814800).

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