Interaction of gated and ungated plasmons in two-dimensional electron systems

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Unique properties of plasmons in two-dimensional electron systems (2DESs) have been studied for many years. There are two basic types of 2D plasmons: gated and ungated ones. Many efforts were spent on investigations of a more realistic case, partly gated 2DES. We present an analytic theory of interaction of gated and ungated plasmons in such a system. It is considered that the gate is formed by a metallic stripe, which is parallel to a plane hosting an infinite homogeneous 2DES. Our theory demonstrates the appearance of a series of 1D gapped modes with a linear dispersion, confined under the gate and running along it. Surprisingly, the fundamental plasmon mode is not all like other modes. In fact, it is a hybrid of gated and ungated plasmons, having gapless square root dispersion. This ”gated-ungated” plasmon is localized near the gate and was not considered early.

INTRODUCTION

Plasma oscillations (plasmons) in 2DESs strongly differ from 3D plasma oscillations. If 2DES is embedded in dielectric medium with permittivity $\varepsilon$, then 2D plasmons have square root dispersion law $[1]$ as follows:

$$\omega(q) = \sqrt{\frac{2\pi ne^2 q}{\varepsilon m}}, \quad q = \sqrt{q_x^2 + q_y^2},$$

(1)

where $n$ is the 2D electron concentration, $m$ is the electron effective mass, and $q$ is the 2D wave vector of the plasmon.

Metallic gate, which is situated near 2DES and parallel to it, screens electron-electron ($e-e$) interaction in 2DES and softens the frequency of ungated plasmon $[1]$ by the factor $\sqrt{2d}$, where $d$ is the distance between the gate and 2DES. As a result, the gated plasmons at $qd \ll 1$ have linear dispersion law as follows $[2]$:

$$\omega_g(q) = qV_p, \quad V_p = \sqrt{\frac{4\pi ne^2 d}{m\varepsilon}},$$

(2)

$V_p$ is the velocity of gated plasmons.

For the first time 2D plasmons were observed in 2D systems of electrons on the liquid helium surface $[3]$ and in silicon inversion layers $[4, 5]$. By now, plasmons were investigated in different 2D electron systems, including semiconductor heterojunctions and quantum wells $[6-14]$, graphene $[15-18]$, topological isolators $[19-21]$, transition metal dichalcogenide monolayers $[22]$, etc.

Dependence of the plasmon frequency on the 2D electron concentration $n$ provides the opportunity to vary easily the frequency of 2D plasmons in a wide range by changing the gate voltage. That is why 2D gated structures are promising not only for fundamental studies of physics of collective excitations, but also as detectors and emitters of electromagnetic radiation in terahertz range $[23-30]$.

If restricted 2DES (with or without gate) is placed in the perpendicular magnetic field then two types of plasmons appear – gapped bulk 2D magnetoplasmons and gapless one-way edge magnetoplasmons (EMP$s$). If 2DES has the semi-plane form, then the exact solutions $[31, 32]$ and a model $[33]$ solutions show that EMP$s$ exist in any magnetic field with or without metallic gate. In classically strong magnetic field, EMP$s$ damp weakly even in low-mobility samples. That is why, EMP$s$ attract researches so far $[34-39]$. Recently it was shown $[40]$ that the gated magnetoplasmons may belong to one of 2D classes in frames of topological classification of bosons $[41]$, and EMP$s$ are topologically protected. Topological arguments also generate much interest to 2D plasmon physics.

In the case of the confined 2DES, for example, in form of semi-plane, stripe or disk, the connection between charge and induced electric field is nonlocal. To calculate the spectrum of confined plasmons we have to solve a complicated integral equation. This equation is solved analytically only in several simplest cases $[42-49]$.

In this Letter we consider analytically the influence of the finite gate on the 2D plasmon spectrum in infinite 2DES. We assume that metallic gate has stripe shape, see Fig. 1. Such a geometry is interesting in itself, besides, it is the elementary part of a metallic grating, which is applied to excite 2D plasmons, beginning with pioneering paper $[4]$. Finally, similar designs are often used to detect electromagnetic waves when developing devices. As a rule, numerical methods are used for this aims $[50-55]$. Generally, the absorption of the electromagnetic wave, whose electric field is directed across the...
long metallic stripe, is calculated. In this case, however, plasmon modes with gapless dispersion law, if they exist, do not manifest. We prove that it is the fundamental mode that has this feature.

In the system under consideration it is convenient to assume that $e^{-e}$ interaction of gate electrons is screened by 2D plasmons, in contrast to the case of finite 2DES with infinite gate, where $e^{-e}$ interaction of electrons in 2DES was rather screened by the gate electrons. It is also convenient to solve the integral equation for the charge electron density in the gate. We find analytical solution of the problem using two realistic assumptions: 1) the sought-for frequency of new plasmon mode is small compared to the frequency of the ungated plasmon, 2) the distance between the gate and 2DES $d$ should be small compared to the gate size $L_x$ and the plasmon wavelength ($d \ll L_x, q^{-1}$). Under these approximations (and assuming that gate conductivity is infinite), the exact integral equation for the plasmon charge density can be reduced to the differential equation with boundary conditions at the edges of the gate. The solutions describe plasmons, which are confined under and near the gate and run along it. Their spectrum consists of a series of 1D subbands $\omega_N(q_y)$, see Fig. 2 where $N$ has a sense of the number of half-waves across the stripe, $N = 0, 1, 2, ...$

The fundamental mode ($N = 0$) at $|q_y| L_x \ll 1$ has spectrum as follows:

$$\omega_0(q_y) = \sqrt{\frac{8\pi e^2 n d |q_y|}{m \omega}} \frac{1}{L_x}; \quad (3)$$

note, that $\omega_0(0, q_y) \ll \omega_0(q_y) \ll \omega_0(0, q_y)$. The dispersion of the fundamental mode is unusual: it combines the behavior of ungated [1] and gated [2] plasmons. We shall call the mode [3] as hybrid "gated-ungated" plasmon. Eq. [3] is one of the key results of this Letter.

**PLASMONS IN 2DES WITH GATE IN FORM OF A STRIPE**

Consider infinite 2DES that is situated in plane $z = 0$. At distance $d$ above the 2DES is placed metallic gate, which is infinite along $y$-axis, and occupies interval $[-L_x/2, L_x/2]$ along $x$-axis, see Fig. 1. Gate has the conductivity $\sigma_g$, to find spectra of plasmons we consider the limit $\sigma_g \to \infty$.

We look for solutions in the form of wave running along the gate exp($i q_y y - i \omega t$). We consider the spectra in the long-wavelength limit $|q_y| \ll k_F$, where $h k_F$ is the Fermi momentum, so we use classical approach (Ohm’s law with the collisionless Drude model for conductivity of 2DES) to describe the electron dynamics. We also neglect electromagnetic retardation effects.

Poisson equation for plasmon potential $\varphi(x, z)$ is as follows:

$$(\partial_x^2 + \partial_z^2 - q_y^2)\varphi(x, z) = -\frac{4\pi}{\omega} [\rho(x)\delta(z) + \rho_g(x)\delta(z - d)], \quad (4)$$

where we assume that 2DES and metallic gate are $\delta$-thin, $\rho(x)$ and $\rho_g(x)$ are plasmon charge density in 2DES and in the gate correspondingly; $\rho_g(x)$ equals zero outside the stripe $[-L_x/2, L_x/2]$.

Using Green’s function approach one can write down Eq. [1] in the following form:

$$\varphi(x, z) = \int_{-\infty}^{\infty} G(x - x', 0) \rho(x') dx' +$$

$$\int_{-\infty}^{\infty} G(x - x', d) \rho_g(x') dx', \quad (5)$$

where $G(x, z) = 2 K_0(|q_y| \sqrt{x^2 + z^2})/x$, $K_0(x)$ is the modified Bessel function of the second kind and zero order.

Let us define the plasmon potential in 2DES $\varphi(x) = \varphi(x, 0)$ and in the gate $\varphi_g(x) = \varphi(x, d)$. Using Fourier transform one can find from Eq. [5]:

$$\varphi(q_x) = \frac{2\pi}{\sqrt{q_x^2 + q_y^2}} \left( \rho(q_x) + \rho_g(q_x) e^{-d \sqrt{q_x^2 + q_y^2}} \right), \quad (6)$$

$$\varphi_g(q_x) = \frac{2\pi}{\sqrt{q_x^2 + q_y^2}} \left( \rho(q_x) e^{-d \sqrt{q_x^2 + q_y^2}} + \rho_g(q_x) \right).$$

Now we use Ohm’s law and the continuity equation to find the relation between $\varphi(q_x)$ and $\rho(q_x)$:

$$i \omega \rho(q_x) = \sigma(q_x^2 + q_y^2) \varphi(q_x), \quad (7)$$

where $\sigma = \sigma(\omega)$ is the dynamical conductivity of 2DES.

Excluding $\varphi(q_x)$ and $\rho(q_x)$ from Eqs. [6] and [7], after inverse Fourier transform one can obtain:

$$\varphi_g(x) = \frac{\alpha}{\omega} \int_{-\infty}^{\infty} e^{i q_x x - 2d \sqrt{q_x^2 + q_y^2}} \rho_g(q_x) dq_x +$$

$$\frac{2}{\omega} \int_{-L_x/2}^{L_x/2} K_0(|q_y||x - x'|) \rho_g(x') dx', \quad (8)$$

where we use the collisionless Drude model for conductivity $\sigma = e^2 n/(-i \omega m)$, and introduce the notation $\alpha = 2\pi e^2 n/(\pi m \omega^2)$.

It should be mentioned, that we look for plasmons coupled with the gate; spectra of the plasmons lie “outside”
the spectra of bulk plasmons existing far from the gate. More precisely, this means that frequency of the plasmons under consideration \( \omega \) is less than frequency of the bulk plasmon \( \omega_p(q_y) \) at the same wave vector \( q_y \). This condition can be rewritten as \( \alpha|q_y| \gg 1 \). Thus the denominator in the first integral in the right-hand side of Eq. (5) does not become zero.

At \( \alpha|q_y| \gg 1 \) we introduce the following expansion:

\[
\frac{\alpha}{1 - \alpha \sqrt{q_y^2 + q_x^2}} = \sum_{M=0}^{\infty} \frac{-1}{\alpha^M(q_y^2 + q_x^2)^{(M+1)/2}}. \tag{9}
\]

In the series we can leave only the two largest terms \( M = 0, 1 \). After this approximation, Eq. (5) takes form as follows:

\[
\varphi_g(x) + \frac{1}{\pi \alpha} \int_{-\infty}^{\infty} e^{iq_y x} \rho_g(q_x) \frac{e^{-2d\sqrt{q_y^2 + q_x^2}}}{\alpha(q_y^2 + q_x^2)} dq_x = \frac{2}{\pi} \int_{-L_x/2}^{L_x/2} \Delta K(x - x') \rho_g(x') dx', \tag{10}
\]

where

\[
\Delta K(x) = K_0(|q_y||x|) - K_0(|q_y|\sqrt{x^2 + 4d^2}). \tag{11}
\]

Now we make the second approximation: \( d \ll L_x, |q_y|d \ll 1 \). Firstly, in this limit \( \Delta K(x) \) becomes \( \delta \)-function: \( C\delta(x) \) [2], where the coefficient \( C \) is defined by the integrated area of \( \Delta K(x) \). One can find that \( C = 2\pi d \) (it is so-called local capacity approximation). Secondly, we neglect the factor \( \exp(-2dq_x) \approx 1 \) in the left-hand side of Eq. (10).

After these assumptions, Eq. (10) takes form as follows:

\[
\varphi_g(x) + \frac{1}{\pi \alpha} \int_{-\infty}^{\infty} e^{iq_y x} \frac{\rho_g(q_x)}{q_y^2 + q_x^2} dq_x = \frac{4\pi d}{\alpha} \rho_g(x), \tag{12}
\]

where \(-L_x/2 \leq x \leq L_x/2\).

Finally, expressing \( \rho_g(q_x) \) via \( \rho_g(x) \), Eq. (12) can be rewritten in the convenient form:

\[
\frac{4\pi d}{\alpha} \rho_g(x) = \frac{\pi}{|q_y|} \int_{-L_x/2}^{L_x/2} e^{-|q_y||x-x'|} \rho_g(x') dx' + \varphi_g(x). \tag{13}
\]

Now we assume that gate conductivity is infinite. This means that the amplitude of potential \( \varphi_g(x) \) equals zero inside the gate \(-L_x/2 \leq x \leq L_x/2\). Therefore, Eq. (13) becomes the integral equation for charge density inside the gate \( \rho_g(x) \).

Obtained integral equation can be solved exactly as it reduces to the following differential equation [54]:

\[
\left( \partial_x^2 - q_y^2 \frac{\omega_p^2}{V_p^2} \right) \rho_g(x) = 0, \tag{14}
\]

with boundary conditions

\[
\left. \partial_x \rho_g(x) - |q_y| \rho_g(x) \right|_{x=-L_x/2} = 0, \quad \left. \partial_x \rho_g(x) + |q_y| \rho_g(x) \right|_{x=L_x/2} = 0. \tag{15}
\]

Instead of the non-conserved transverse wave vector \( q_x \) we introduce the effective transverse wave number \( k \):

\[
k^2 = \omega_p^2 \frac{\omega_p^2}{V_p^2} - q_y^2, \tag{16}
\]

which, as shown below, takes discrete values for given \( q_y \).

The solutions of Eqs. (14) and (15) have a certain parity. Even and odd solutions have form \( \cos kx \) and \( \sin kx \) correspondingly, where \( k \) is determined by the dispersion equation as follows:

\[
k \left( \tan \frac{kL_x}{2} \right)^{\pm 1} = \pm |q_y|, \tag{17}
\]

where upper signs \( '+' \) correspond to the even modes, lower sings \( '-' \) correspond to the odd modes.

Note that the dispersion equation for odd modes has formal solution \( k = 0 \), but it corresponds to zero charge density \( \rho_g(x) = 0 \), so we do not take into consideration \( k = 0 \).

From Eqs. (17) we obtain the discrete series of plasmon modes with frequencies \( \omega_N(q_y) \) and transverse wave numbers \( k_N(q_y) \), where \( N = 0, 1, 2, ... \) is the number of a mode, see Figs. 2 and 3. At \( |q_y|L_x \ll 1 \) the fundamental \( N = 0 \) mode has unusual for gated 2DES square root dependence of frequency as a function of wave vector \( q_y \) [3], in contrast to the linear dependence in 2DES with infinite gate [2]. Note that this mode have frequency lower than \( \omega_p(0, q_y) \), therefore condition \( \alpha|q_y| \gg 1 \) is satisfied.
Modes with $N > 0$ at $q_y = 0$ have nonzero frequencies, thus they lie inside the bulk spectrum [1] and strongly interact with the continuum of ungated plasmons. Analogous, for instance, to the case of inter-edge magneto-plasmons [57], such an interaction results in the decay of the gated modes and the appearance of the small imaginary correction to the plasmon frequency, corresponding to the finite lifetime of exited modes. 

If one neglects this correction then modes with $N > 0$ at $|q_y|L_x \ll 1$ have asymptotics as follows: $\omega_N^2/V_p^2 = \pi^2 N^2/L_x^2 + 4|q_y/L_x|$, i.e. at $q_y = 0$ the frequency $\omega_N$ equals the frequency of gated plasmon [2] with usual “quantization rule” $q_x \rightarrow \pi N/L_x$. 

At $|q_y|L_x \gg 1$ modes have asymptotics as follows: $\omega_N^2/V_p^2 = \pi^2(N+1)^2/L_x^2 + q_y^2$, i.e. the frequency of $N$th mode tends to the frequency of gated plasmon [2] with quantization rule in $x$-direction: $q_x \rightarrow \pi(N+1)/L_x$. 

Predicted spectrum $\omega_N(q_y)$ can be understood in terms of relation [2], in which $q_x$ should be replaced by effective transverse wave number $k_N$, defined by Eqs. [17]. Dependence of $k_N$ on $q_y$ is shown in Fig. 3. In dimensionless variables $(kL_x, q_yL_x)$ the curves in Fig. 3 do not depend on parameters of the system.

Plasmon charge density in 2DES and in the gate and electric field $E_x$ in 2DES for modes $N = 0, 1$ are shown in Fig. 3. Actually, the entire plasmon charge density in 2DES is localized under the gate. At $|x| \gg L_x$ and $|q_y|L_x \ll 1$ the field decreases with the characteristic length of the order of $|q_y|^{-1}$. The spatial distribution of the electromagnetic energy density strongly depends on the mode number. It can be shown that the main part of energy of excited mode $N = 1$ is localized under the gate. However, the energy of fundamental mode at $|q_y|L_x \ll 1$ is situated mainly out of the gate. This gives one more reason to consider the mode $N = 0$ as hybrid gated-ungated plasmon.

**CONCLUSION**

We have considered so far the system without external magnetic field. If the system is placed in the perpendicular constant magnetic field $B$ (but gate conductivity $\sigma_g$ does not depend on magnetic field), then usual frequency gap appears at zero wave vector, and the spectrum becomes as follows: $\omega_N(B) = \sqrt{\omega_N^2 + \omega_c^2}$, where $\omega_c$ is the electron cyclotron frequency $\omega_c = |eB|/mc$.

Thus, we show that metallic gate itself confines plasmons, even without any changes of electron density in 2DES, i.e. when 2DES is homogeneous. This conclusion is in qualitative agreement with results of Ref. [50] obtained numerically for several excited modes in graphene with metallic grating.

In conclusion, we have analytically considered plasmon modes in the infinite homogeneous 2DES with metallic gate in the form of the stripe situated in the vicinity of 2DES. Plasmon spectrum is characterized by the mode number $N = 0, 1, 2, \ldots$ and the wave vector along the gate $q_y$. The fundamental mode (gated-ungated plasmon) has peculiar spectrum [3]. In fact, it is the hybrid of gated ($\omega_0 \propto \sqrt{d}$) and ungated ($\omega_0 \propto \sqrt{|q_y|}$) plasmons. The obtained spectra can be understood in terms of quantized wave number $k_N$, see Fig. 3. $k_N$ lies between $\pi N/L_x$ and $\pi(N+1)/L_x$. Our findings are promising for...
possible application in integral THz-optics and nanoplasmonics.

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