NON-NEGATIVE SPECTRAL MEASURES AND INTEGRAL REPRESENTATIONS OF UNBOUNDED ∗-REPRESENTATIONS

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Abstract. Regular normalized $B(W_1, W_2)$-valued non-negative spectral measures introduced in [8] are in one-to-one correspondence with unital algebra homomorphisms $\rho : C(X, \mathbb{C}) \otimes W_1 \to W_2$ such that $\rho(F^*) = \rho(F^*)^*$ for every $F \in C(X, \mathbb{C}) \otimes W_1$, where $X$ stands for a compact Hausdorff space and $W_1, W_2$ stand for von Neumann algebras. In this paper we show that for every realcompact and $\sigma$-compact space $X$, and a special algebra homomorphism $\rho : C(X, \mathbb{C}) \otimes A_1 \to L(K)$ there exists a unique representing $B(A_1, B(K))$-valued non-negative spectral measure, where $A_1$ stands for a $C^*$-algebra and $K$ stands for a Hilbert space.

1. Introduction

Our main result is the following theorem on ∗-representations of the form $\rho : C(X, \mathbb{C}) \otimes A_1 \to L(K)$, where $X$ is a realcompact and $\sigma$-compact space, $A_1 \subseteq B(H)$ a $C^*$-algebra, $B(H)$ the Banach space of bounded linear operators on a Hilbert space $H$ and $L(K)$ is a vector space of all linear operators on a Hilbert space $K$ (not necessarily everywhere defined). For a dense subspace $D_0$ in $K$ we denote by $L(D_0, K)$ a vector space of all linear operators mapping $D_0$ into $K$.

Theorem 1.1. Let $X$, $A_1$, $L(K)$, $D_0$ and $L(D_0, K)$ be as above, $\text{Bor}(X)$ be a Borel $\sigma$-algebra on $X$ and

\[ \rho : C(X, \mathbb{C}) \otimes A_1 \to L(K) \]

a ∗-representation on a subspace $D_0$ of a Hilbert space $K$, such that for every function $f \in C(X, \mathbb{C})$ the map

\[ \rho_f : A_1 \to L(D_0, K), \quad \rho_f(A) = \rho(f \otimes A) \]

is continuous relative to the operator topology on $A_1$ and the strong operator topology on $L(D_0, K)$. Then there exists a unique regular normalized non-negative spectral measure

\[ M : \text{Bor}(X) \to B(A_1, B(K)) \]

such that

\[ \rho(F)x = \left( \int_X F \, dM \right)x \]

holds for every $x \in D_0$ and every $F \in C(X, \mathbb{C}) \otimes A_1$.

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Remark 1.2. The strong operator topology on $L(D_0, K)$, where $D_0$ is a dense subspace in a Hilbert space $K$, is a topology determined by the family of seminorms

$$L(D_0, K) \to [0, \infty), \quad T \mapsto \|Tx\|$$

for every $x \in D_0$.

Let $D(T)$ denote the domain of an operator $T \in L(K)$. A function $\alpha : C(X, \mathbb{C}) \to [0, \infty)$ is an absolute value if $\alpha$ is symmetric (i.e., $\alpha(\overline{f}) = \alpha(f)$), $\alpha(1) = 1$ and $\alpha(fg) \leq \alpha(f)\alpha(g)$ for all $f, g \in C(X, \mathbb{C})$. The family of all absolute values is denoted by $A(C(X, \mathbb{C}))$.

We call a map $\rho : C(X, \mathbb{C}) \otimes A_1 \to L(K)$ a *-representation on a subspace $D_0$ of a Hilbert space $K$, if for every $\alpha, \beta \in \mathbb{C}$, every $F, G \in C(X, \mathbb{C}) \otimes A_1$, every continuous functions $f, g \in C(X, \mathbb{C})$ and every hermitian projection $P \in A_1$, we have:

(i) $D_0$ is contained in the domain $D(\rho(F))$ and is a core for $\rho(F)$,
(ii) $\rho(1 \otimes \text{Id}_K) = \text{Id}_K$,
(iii) $\rho(F^*) \subseteq \rho(F)^*$,
(iv) $\rho(\alpha F + \beta G) \subseteq \alpha \rho(F) + \beta \rho(G)$,
(v) $\rho(FG) \subseteq \rho(F)\rho(G)$,
(vi) $\rho(f \otimes P)^* = \rho(\overline{f} \otimes P)$,
(vii) $\rho(f \otimes P)\rho(g \otimes P) \subseteq \rho(fg \otimes P)$ and
$$D(\rho(f \otimes P)\rho(g \otimes P)) = D(\rho(fg \otimes P)) \cap D(\rho(f \otimes P)),$$
(viii) $\rho(f \otimes P)\rho(g \otimes P) = \rho(fg \otimes P)$,
(ix) The subspace

$$D\rho := \bigcup_{\alpha \in A(C(X, \mathbb{C}))} \left\{ x \in \bigcap_{f \in C(X, \mathbb{C})} D(\rho(f \otimes \text{Id}_K)) : \|\rho(f \otimes \text{Id}_K)x\| \leq \alpha(f)\|x\| \text{ for all } f \in C(X, \mathbb{C}) \right\}.$$

is dense in $K$.

The definition of a *-representation and Theorem 1.1 are the non-commutative generalizations of the special case of [3] Definition 1.1 and [6] Theorem 1.2, i.e., instead of a commutative, unital semigroup $S = C(X, \mathbb{C})$ we take the non-commutative, unital algebra $S = C(X, \mathbb{C}) \otimes A_1$. Theorem 1.2 is also an extension of [3] Theorem 9.1. from the case of bounded *-representations $\rho : C(X, \mathbb{C}) \otimes W_1 \to W_2$, where $X$ is a compact space and $W_1, W_2$ are von Neumann algebras, to the case of unbounded *-representations $\rho : C(X, \mathbb{C}) \otimes A_1 \to L(K)$, where $X$ is a realcompact and $\sigma$-compact space and $A_1$ a $C^*$-algebra.

Non-negative spectral measure $M : \text{Bor}(X) \to B(A_1, B(K))$ is a set function, if for every hermitian projection $P \in A_1$ the set functions $M_P$ are spectral measures and the equality

$$M_P(\Delta_1)M_Q(\Delta_2) = M_{PQ}(\Delta_1 \cap \Delta_2)$$

holds for all hermitian projections $P, Q \in A_1$ and all sets $\Delta_1, \Delta_2 \in \text{Bor}(X)$.

The paper is structured in the following way. In Subsection 2.1 we present the non-negative spectral measures, which we introduced in [8] to prove a theorem on the integral representation of the *-representation of a $C^*$-algebra (see Theorem 2.3). In Subsection 2.2 we present a spectral theory of unbounded functions on
a Hilbert space. In Subsection 2.3 we define a \( * \)-representation of a commutative semigroup with an involution and state a theorem on the integral representation of the \( * \)-representation of the commutative semigroup with an involution (see Theorem 2.8). In Section 3 we introduce an integral of an unbounded measurable function with respect to a non-negative spectral measure. In Section 4 we firstly define a \( * \)-representation of the algebra \( C(X, \mathbb{C}) \otimes \mathcal{A}_1 \) on a dense subspace in a Hilbert space, secondly prove that the non-negative spectral integral is a \( * \)-representation (see Theorem 4.2) and finally prove Theorem 1.1 (see Theorem 4.3).

2. Preliminaries

2.1. Non-negative Measures and \( * \)-Representations of \( C^* \)-algebras. Let \( (X, \text{Bor}(X), \mathcal{A}_1, \mathcal{W}_2) \) be a measure space, i.e., \( X \) is a topological space, \( \text{Bor}(X) \) a \( \sigma \)-algebra on \( X \), \( \mathcal{A}_1 \subseteq \mathcal{B}(\mathcal{H}) \) a \( C^* \)-algebra and \( \mathcal{W}_2 \subseteq \mathcal{B}(\mathcal{K}) \) a von Neumann algebra, where \( \mathcal{B}(\mathcal{H}), \mathcal{B}(\mathcal{K}) \) denote the bounded linear operators on Hilbert spaces \( \mathcal{H}, \mathcal{K} \). We denote by \( \mathcal{A}_p, \mathcal{A}_+ \) the subsets of all hermitian projections and all positive operators of a \( C^* \)-algebra \( \mathcal{A} \). By a hermitian projection we mean an operator \( P \), which satisfies \( P = P^* = P^2 \) and by a positive operator we mean a hermitian operator \( A \), such that \( \langle Ah, h \rangle \geq 0 \) for every \( h \in \mathcal{H} \), where \( \mathcal{H} \) is a Hilbert space with an inner product \( \langle \cdot, \cdot \rangle \), such that \( A \subseteq \mathcal{B}(\mathcal{H}) \). Non-negative spectral measure \( M : \text{Bor}(X) \to \mathcal{B}(\mathcal{A}_1, \mathcal{W}_2) \) is a set function, if for every hermitian projection \( P \in (\mathcal{A}_1)_p \) the set functions \( M_P \) are spectral measures and the equality

\[
M_P(\Delta_1)M_Q(\Delta_2) = M_{PQ}(\Delta_1 \cap \Delta_2)
\]

holds for all hermitian projections \( P, Q \in (\mathcal{A}_1)_p \) and all sets \( \Delta_1, \Delta_2 \in \text{Bor}(X) \).

Theorem 8.1 is a characterization of non-negative spectral measures:

**Theorem 2.1.** Let \( (X, \text{Bor}(X), \mathcal{A}_1, \mathcal{W}_2) \) be a measure space, \( \{F_P\}_{P \in (\mathcal{A}_1)_p} \) a family of spectral measures \( F_P : \text{Bor}(X) \to \mathcal{W}_2 \). There is a unique non-negative spectral measure \( M \) such that

\[
M_P = F_P
\]

for all hermitian projections \( P \in (\mathcal{A}_1)_p \), iff the following conditions hold.

(1) \[
\sum_{i=1}^n \lambda_i F_{P_i}(\Delta) = \sum_{j=1}^m \mu_j F_{Q_j}(\Delta)
\]

for all hermitian projections \( P_i, Q_j \in (\mathcal{A}_1)_p \), all real numbers \( \lambda_i, \mu_j \in \mathbb{R} \), and all sets \( \Delta \in \text{Bor}(X) \) such that \( \sum_{i=1}^n \lambda_i P_i = \sum_{j=1}^m \mu_j Q_j \), for each set \( \Delta \in \text{Bor}(X) \) there exists a constant \( k_\Delta \in \mathbb{R}^{>0} \) such that

(2) \[
\|F_P(\Delta)\| \leq k_\Delta
\]

for all hermitian projections \( P \in (\mathcal{A}_1)_p \), and for all hermitian projections \( P, Q \in (\mathcal{A}_1)_p \) and all sets \( \Delta_1, \Delta_2 \in \text{Bor}(X) \)

(3) \[
F_P(\Delta_1)F_Q(\Delta_2) = \lim_{\ell \to \infty} \sum_{k=1}^{n_\ell} \lambda_{k,\ell} R_{k,\ell}(\Delta_1 \cap \Delta_2)
\]

holds for every sequence \( T_\ell := \sum_{k=1}^{n_\ell} \lambda_{k,\ell} R_{k,\ell} \) such that \( \lim_{\ell \to \infty} \|PQ - T_\ell\| = 0 \), where \( \lambda_{k,\ell} \in \mathbb{R} \) are real numbers and \( R_{k,\ell} \in (\mathcal{A}_1)_p \) hermitian projections.
REMARK 2.2. [8] Theorem 8.1 deals with a measure space of the form \((X, \text{Bor}(X), \mathcal{W}_1, \mathcal{W}_2)\), where \(\mathcal{W}_1, \mathcal{W}_2\) are von Neumann algebras. Without any changes in the proof \(\mathcal{W}_1\) can be replaced by a \(C^\ast\)-algebra \(A_1\).

Let \(X\) be a topological space and \(\text{Bor}(X)\) a Borel algebra on \(X\). Non-negative spectral measure \(M\) is regular if the spectral measures \(M\rho\) are regular for every hermitian projection \(P \in (A_1)_p\), i.e., complex measures

\[
(M_P)_{k_1,k_2} : \text{Bor}(X) \to \mathbb{C}, \quad (M_P)_{k_1,k_2}(\Delta) := (M_P(\Delta))_{k_1,k_2}
\]

are regular for every \(k_1,k_2 \in \mathcal{K}\) and every \(P \in (A_1)_p\). \(M\) is normalized if \(M(X)(\text{Id}_H) = \text{Id}_K\), where \(\text{Id}_H, \text{Id}_K\) denote the identity operators on \(H, K\) respectively.

Let \(\mathcal{A}\) be a unital algebra with an involution \(*\). A \(*\)-representation \(\rho : \mathcal{A} \to \mathcal{W}_2\) is an algebra homomorphism, such that \(\rho(a^*) = \rho(a)^*\) for every \(a \in \mathcal{A}\). \(\rho\) is unital, if \(\rho(e) = \text{Id}_{\mathcal{K}}\), where \(e\) is a unit element of \(\mathcal{A}\). Non-negative spectral measures were introduced in [8] to prove the following result (see [8, Theorem 1.1.]).

THEOREM 2.3. Let \(X\) be a compact Hausdorff space, \(\text{Bor}(X)\) a Borel \(\sigma\)-algebra on \(X\), \(\mathcal{A}_1\) a \(C^\ast\) algebra, \(\mathcal{W}_2\) a von Neumann algebra and \(\rho : C(X, \mathcal{A}_1) \to \mathcal{W}_2\) a map. The following statements are equivalent.

1. \(\rho : C(X, \mathcal{A}_1) \to \mathcal{W}_2\) is a unital \(*\)-representation.
2. There exists a unique regular normalized non-negative spectral measure \(M : \text{Bor}(X) \to B(\mathcal{A}_1, \mathcal{W}_2)\) such that

\[
\rho(F) = \int_X F \, dM
\]

for every \(F \in C(X, \mathcal{A}_1)\).

REMARK 2.4. [8] Theorem 1.1.] deals with a map of the form \(\rho : C(X, \mathcal{W}_1) \to \mathcal{W}_2\), where \(\mathcal{W}_1, \mathcal{W}_2\) are von Neumann algebras. As in Theorem [24] (see Remark 2.2), \(\mathcal{W}_1\) can be replaced by a \(C^\ast\)-algebra \(A_1\). However, \(\mathcal{W}_2\) cannot be replaced by a \(C^\ast\)-algebra \(A_2\) (the representing measure \(M\) would then map into the von Neumann algebra generated by \(A_2\)).

2.2. Spectral Integral on a Hilbert Space. Let \(\Omega\) be a set, \(\text{Bor}(X)\) a \(\sigma\)-algebra on \(X\) and \(E : \text{Bor}(X) \to B(\mathcal{H})\) a spectral measure, where \(\mathcal{H}\) is a Hilbert space. Let \(\mathcal{U}\) denote the set of all \(\text{Bor}(X)\)-measurable functions \(f : \Omega \to \mathbb{C}\). A sequence \((\Delta_n)_{n \in \mathbb{N}}\) of sets \(\Delta_n \in \text{Bor}(X)\) is a bounding sequence for a subset \(\mathcal{F}\) of \(\mathcal{U}\) if each function \(f \in \mathcal{F}\) is bounded on \(\Delta_n\), \(\Delta_n \subseteq \Delta_{n+1}\) for \(n \in \mathbb{N}\), and \(E(\bigcup_{n=1}^\infty \Delta_n) = \text{Id}_\mathcal{H}\).

If \((\Delta_n)_{n \in \mathbb{N}}\) is any bounding sequence, then by the properties of the spectral measure,

\[
E(\Delta_n) \preceq E(\Delta_{n+1})\quad \text{for } n \in \mathbb{N}, \quad \lim_{n \to \infty} E(\Delta_n)x = x\quad \text{for } x \in \mathcal{H},
\]

and the set \(\bigcup_{n=1}^\infty E(\Delta_n)\mathcal{H}\) is dense in \(\mathcal{H}\). Each finite set of element \(f_1, f_2, \ldots, f_r \in \mathcal{U}\) has a bounding sequence

\[
\Delta_n := \{t \in X : |f_j(t)| \leq n \text{ for } j = 1, 2, \ldots, r\}.
\]

The spectral integral \(I(f) := \int_\Omega f \, dE\) of a function \(f \in \mathcal{U}\) is given by the following theorem (see [31] Theorem 4.13).


Theorem 2.5. Suppose that \( f \in \mathcal{U} \) and define
\[
\mathcal{D}(\mathbb{I}(f)) := \left\{ x \in \mathcal{H} : \int_X |f(t)|^2 \ dE(t)x, x < \infty \right\}.
\]

Let \((\Delta_n)_{n \in \mathbb{N}}\) be a bounding sequence for \( f \). Then we have:
(i) A vector \( x \in \mathcal{H} \) is in \( \mathcal{D}(\mathbb{I}(f)) \) iff the sequence \( (\mathbb{I}(f\chi_{\Delta_n}))_{n \in \mathbb{N}} \) converges in \( \mathcal{H} \), or equivalently, if \( \sup_{n \in \mathbb{N}} \|\mathbb{I}(f\chi_{\Delta_n})x\| < \infty \).
(ii) For \( x \in \mathcal{D}(\mathbb{I}(f)) \), the limit of the sequence \( (\mathbb{I}(f\chi_{\Delta_n})) \) does not depend on the bounding sequence \((\Delta_n)\). There is a linear operator \( \mathbb{I}(f) \) on \( \mathcal{D}(\mathbb{I}(f)) \) defined by
\[
\mathbb{I}(f)x = \lim_{n \to \infty} \mathbb{I}(f\chi_{\Delta_n})x \quad \text{for } x \in \mathcal{D}(\mathbb{I}(f)).
\]
(iii) \( \bigcup_{n=1}^\infty E(\Delta_n)\mathcal{K} \) is contained in \( \mathcal{D}(\mathbb{I}(f)) \) and is a core for \( \mathbb{I}(f) \). Further, \( E(\Delta_n)\mathbb{I}(f) \subseteq \mathbb{I}(f)E(\Delta_n) = \mathbb{I}(f\chi_{\Delta_n}). \)
\( \mathcal{D}(\mathbb{I}(f)) \) from Theorem 2.5 is the domain of the operator \( \mathbb{I}(f) \).

The main algebraic properties of the map \( f \to \mathbb{I}(f) \) are given in the following theorem (see [7, Theorem 4.16]).

Theorem 2.6. For \( f, g \in \mathcal{U} \) and \( \alpha, \beta \in \mathbb{C} \) we have:
(i) \( \mathbb{I}(f^*) = \mathbb{I}(f)^* \),
(ii) \( \mathbb{I}(\alpha f + \beta g) = \alpha \mathbb{I}(f) + \beta \mathbb{I}(g) \),
(iii) \( \mathbb{I}(fg) = \mathbb{I}(f)\mathbb{I}(g) \),
(iv) \( \mathbb{I}(f) \) is a closed normal operator on \( \mathcal{K} \), and
\( (f^*)^*\mathbb{I}(f) = \mathbb{I}(f^*) = \mathbb{I}(f)\mathbb{I}(f)^* \),
(v) \( \mathcal{D}(\mathbb{I}(f)\mathbb{I}(g)) = \mathcal{D}(\mathbb{I}(g)) \cap \mathcal{D}(\mathbb{I}(fg)) \).

Remark 2.7. To emphasize with respect to which spectral measure we integrate, we will denote the integral \( \mathbb{I}(f) \) with respect to \( E \) by \( \mathbb{I}_E(f) \).

2.3. \( \ast \)-Representations of Commutative Semigroups with an Involution.
Let \( S \) be a commutative semigroup with a unit element \( e \) and an involution \( \ast \) (i.e., \((s^*)^* = s \) and \((st)^* = s^*t^* \) for all \( s, t \in S \)). A function \( \nu : S \to \mathbb{C} \) which satisfies \( \nu(e) = 1 \) and \( \nu(st^*) = \nu(s)\nu(t) \) for all \( s, t \in S \) is called a character of \( S \). By \( S^* \) we denote the set of all characters of \( S \). A function \( \alpha : S \to [0, \infty) \) is an absolute value if \( \alpha \) is symmetric (i.e., \( \alpha(s^*) = \alpha(s) \)), \( \alpha(e) = 1 \) and \( \alpha(st) \leq \alpha(s)\alpha(t) \) for all \( s, t \in S \). The family of all absolute values is denoted by \( \mathcal{A}(S) \). For a map \( \rho : S \to N(\mathcal{H}) \), where \( N(\mathcal{H}) \) is a vector space of all normal (not necessarily bounded) operators on a Hilbert space \( \mathcal{H} \) and \( \alpha \in S \), define
\[
D_\alpha := \left\{ x \in \bigcap_{s \in S} \mathcal{D}(\rho(s)) : \|\rho(s)x\| \leq \alpha(s) \|x\| \text{ for all } s \in S \right\},
\]
where \( \mathcal{D}(\rho(s)) \) denotes the domain of \( \rho(s) \). By [6, Definition 1.1.], a map \( \rho : S \to N(\mathcal{H}) \) is called a \( \ast \)-representation, if:
(i) \( \rho(e) = \operatorname{Id}_\mathcal{H} \), where \( \operatorname{Id}_\mathcal{H} \) is the identity operator on \( \mathcal{H} \).
(ii) \( \rho(s^*) = \rho(s)^* \), \( s \in S \).
(iii) \( \rho(t)\rho(s) \subseteq \rho(st) \) with \( \mathcal{D}(\rho(t)\rho(s)) = \mathcal{D}(\rho(st)) \cap \mathcal{D}(\rho(s)) \), \( s, t \in S \).
(iv) \( \rho(t)\rho(s) = \rho(st) \), \( s, t \in S \).
(v) $D_c := \bigcup_{\alpha \in \mathcal{A}(S)} D_\alpha$ is dense in $\mathcal{H}$.

The main result of [6] is the following theorem on the integral representation of the $*$-representation $\rho : S \to N(\mathcal{H})$ (see [6 Theorem 1.2]).

**Theorem 2.8.** Let $S, \rho, N(\mathcal{H})$ and $S^*$ be as above. Let $\text{Bor}(S^*)$ be a Borel $\sigma$-algebra on $S^*$. Then there exists a unique regular normalized spectral measure $F : \text{Bor}(S^*) \to B(\mathcal{H})$ such that

$$\rho(s)x = \left(\int_{S^*} \hat{s}(\nu) \, dE(\nu)\right)x$$

for every $s \in S$ and every $x \in \mathcal{D}(\rho(s))$, where $\hat{s} : S^* \to \mathbb{C}$ is defined by $\hat{s}(\nu) = \nu(s)$.

3. **Integrals of Unbounded Measurable Functions with Respect to Non-negative Spectral Measures**

Let $(X, \text{Bor}(X), \mathcal{A}_1 \subseteq B(\mathcal{H}), B(\mathcal{K}), M)$ be a space with a non-negative spectral measure $M$, where $X$ is a $\sigma$-compact topological space, $\text{Bor}(X)$ a Borel $\sigma$-algebra on $X$, $\mathcal{A}_1 \subseteq B(\mathcal{H})$ a $C^*$-algebra, $\mathcal{W}_2 \subseteq B(\mathcal{K})$ a von Neumann algebra and $B(\mathcal{H}), B(\mathcal{K})$ the bounded linear operators on Hilbert spaces $\mathcal{H}, \mathcal{K}$. The set $D_0 \subseteq \mathcal{K}$ is defined by

$$\mathcal{D}_0 := \bigcup_{K \text{ compact}} M(K)(\text{Id}_{\mathcal{H}})\mathcal{K}.$$  

**Proposition 3.1.** $D_0$ is a linear subspace in $\mathcal{K}$.

**Proof.** Let $x_1, x_2 \in \mathcal{D}_0$ and $\lambda_1, \lambda_2 \in \mathbb{C}$. Then it holds that $x_1 = M(K_1)(\text{Id}_{\mathcal{H}})k_1$, $x_2 = M(K_2)(\text{Id}_{\mathcal{H}})k_2$ for some compact sets $K_1, K_2$ and some vectors $k_1, k_2 \in \mathcal{K}$. Since $M(\Delta)(\text{Id}_{\mathcal{H}})$ is a hermitian projection for every $\Delta \in \text{Bor}(X)$ and the inequality $M(\Delta_1)(\text{Id}_{\mathcal{H}}) \leq M(\Delta_2)(\text{Id}_{\mathcal{H}})$ is true if $\Delta_1 \subseteq \Delta_2$, we have $\lambda_1 x_1 + \lambda_2 x_2 = M(K_1 \cup K_2)(\text{Id}_{\mathcal{H}})(\lambda_1 x_1 + \lambda_2 x_2)$ and $K_1 \cup K_2$ is a compact set. Hence $\lambda_1 x_1 + \lambda_2 x_2 \in \mathcal{D}_0$ and $\mathcal{D}_0$ is a linear subspace in $\mathcal{K}$. \qed

Let $\mathcal{U}$ denote the set of all $\text{Bor}(X)$-measurable functions $f : X \to \mathbb{C}$. The aim of this section is to extend the integration of bounded maps of the form $\sum_{i=1}^{n} f_i \otimes A_i$ with respect to the non-negative spectral measure $M$, where $f_i \in \mathcal{U}$ is bounded and $A_i \in \mathcal{A}_1$ for every $i = 1, 2, \ldots, n$, to all maps of the form $\sum_{i=1}^{n} f_i \otimes A_i$, where $f_i \in \mathcal{U}$ and $A_i \in \mathcal{A}_1$ for every $i = 1, 2, \ldots, n$.

For a map $f \otimes P$, where $f \in \mathcal{U}$ and $P \in (\mathcal{A}_1)_p$ is a hermitian projection, we define the non-negative spectral integral $\mathcal{I}(f \otimes P)$ by

$$\mathcal{I}(f \otimes P) := \int_{\Omega} f \, dM_P = \|M_P(f)\|.$$  

**Lemma 3.2.** $\mathcal{D}_0$ is contained in the domain $\mathcal{D}(\|M_P(f)\|)$ and is a core for $\|M_P(f)\|$. Also,

$$M(K)(\text{Id}_{\mathcal{H}})\|M_P(f)\| M(K)(\text{Id}_{\mathcal{H}}) = \|M_P(f \chi_K)\|$$

for every compact set $K$.

**Proof.** Let $K$ be a compact set and $x \in \mathcal{K}$. By the $\sigma$-compactness of $X$ we can find a sequence $K_n$ of compact sets, such that $K_1 := K, K_n \subseteq K_{n+1}$ for $n \in \mathbb{N}$ and...
\[ \bigcup_{n \in \mathbb{N}} K_n = X. \] By the boundedness of the function \( f \chi_K, \chi_K \), and by [8, Proposition 7.2], we have
\[
\left( \int_X f \chi_K \otimes P \, dM \right) x = \left( \int_X f \chi_K \chi_K \otimes P \, dM \right) x \\
= \left( \int_X f \chi_K \otimes P \, dM \right) M(K_1)(\text{Id}_H)x \\
= M(K_1)(\text{Id}_H) \left( \int_X f \chi_K \otimes P \, dM \right) x.
\]
Hence, \( \sup_{n \in \mathbb{N}} \|\mathbb{I}_{M^p}(f \chi_K) M(K_1)(\text{Id}_H)x\| < \infty \) and by Theorem 2.5 (i),
\[ M(K)(\text{Id}_H)x = M(K_1)(\text{Id}_H)x \in \mathcal{D}(\mathbb{I}_{M^p}(f)). \]
That is, \( D_0 \subseteq \mathcal{D}(\mathbb{I}_{M^p}(f)) \).

Let \( x \in K \) be arbitrary. By the same reasoning as above we have
\[
\left( \int_X f \chi_K \otimes P \, dM \right) x = \left( \int_X f \otimes P \, dM \right) M(K_n)(\text{Id}_H)x \\
= \left( \int_X f \chi_K \otimes P \, dM \right) M(K_n)(\text{Id}_H) \left( \int_X f \otimes P \, dM \right) x.
\]
where \( m \geq n \). Letting \( m \to \infty \), we get
\[ \left( \int_X f \chi_K \otimes P \, dM \right) x = \left( \int_X f \otimes P \, dM \right) M(K_n)(\text{Id}_H)x. \]
For \( x \in \mathcal{D}(\mathbb{I}_{M^p}(f)) \), letting \( m \to \infty \), we get
\[ \left( \int_X f \otimes P \, dM \right) M(K_n)(\text{Id}_H)x = M(K_n)(\text{Id}_H) \left( \int_X f \otimes P \, dM \right) x. \]
Since \( M(K_n)(\text{Id}_H)x \to x \) and
\[ \mathbb{I}_{M^p}(f) M(K_n)(\text{Id}_H)x = M(K_n)(\text{Id}_H) \mathbb{I}_{M^p}(f)x \to \mathbb{I}_{M^p}(f)x \]
for \( x \in \mathcal{D}(\mathbb{I}_{M^p}(f)) \), the linear subspace \( \bigcup_{n=1}^{\infty} M(K_n)(\text{Id}_H)K \subseteq D_0 \) is a core for \( \mathbb{I}_{M^p}(f) \). Hence \( D_0 \) is a core for \( \mathbb{I}_{M^p}(f) \). \( \square \)

**Lemma 3.3.** For \( f \in \mathcal{U} \) and \( P, Q \in (A_1)_p \) orthogonal hermitian projections it is true that:

(i) \( \mathcal{I}(f \otimes (P + Q)) = \mathcal{I}(f \otimes P) + \mathcal{I}(f \otimes Q) \).

(ii) \( \text{Im} \left( \mathcal{I}(f \otimes P) \right) \perp \text{Im} \left( \mathcal{I}(f \otimes Q) \right) \),

where \( \text{Im}(T) \) denotes the image of the operator \( T \) on \( K \).

**Proof.** Since \( P, Q \) are orthogonal hermitian projections, \( P + Q \) is also a hermitian projection. Since \( M \) is a non-negative spectral measure, \( \text{Im}(M_P) \) and \( \text{Im}(M_Q) \) are orthogonal (Here \( \text{Im}(M_P), \text{Im}(M_Q) \) denote the images of \( M_P, M_Q \), i.e., \( \text{Im}(M_P) := \bigcup_{\Delta \in \text{Bor}(X)} M_P(\Delta)K \) and analogously for \( M_Q \)). Therefore by the definition of \( \mathcal{D}(\mathbb{I}(f)) \) (see Theorem 2.5),
\[ \mathcal{D}(\mathbb{I}_{M_P+Q}(f)) = \mathcal{D}(\mathbb{I}_{M_P}(f)) \cap \mathcal{D}(\mathbb{I}_{M_Q}(f)). \]
Let $K_n$ be an increasing sequence of compact sets, such that $X = \cup_{n \in \mathbb{N}} K_n$. Since $f\chi_{K_n}$ is a bounded measurable function, by [8 Proposition 3.5],
\[
\|M_{P+Q}(f\chi_{K_n})\| = \|M_P(f\chi_{K_n})\| + \|M_Q(f\chi_{K_n})\|.
\]
Hence by
\[
\|M_{P+Q}(f)x\| = \lim_{n \to \infty} \|M_{P+Q}(f\chi_{K_n})x\| = \lim_{n \to \infty} (\|M_P(f\chi_{K_n})x\| + \|M_Q(f\chi_{K_n})x\|)
\]
\[
= \lim_{n \to \infty} \|M_P(f\chi_{K_n})x\| + \lim_{n \to \infty} \|M_Q(f\chi_{K_n})x\|
\]
\[
= \|M_P(f)x\| + \|M_Q(f)x\|
\]
for every $x \in D(M_{P+Q}(f))$, it follows that
\[
\|M_{P+Q}(f)\| = \|M_P(f)\| + \|M_Q(f)\|
\]
and by the definition of $I$ also
\[
I(f \otimes (P + Q)) = I(f \otimes P) + I(f \otimes Q).
\]
Since $M$ is a non-negative spectral measure, $M_P(\Delta)M_Q(\Delta') = 0$ for every $\Delta, \Delta' \in \text{Bor}(X)$ such that $\Delta \cap \Delta' = \emptyset$, and hence also $\text{Im} (I(f \otimes P) \perp \text{Im} (I(f \otimes Q))$.

In what follows we extend the integral $I$ to all maps of the form $U \otimes A_1$. But first we introduce the preintegral $\psi$, which is needed to assure the well-definedness. For $f \in U$, $\lambda, \mu \in \mathbb{C}$, $P, Q$ orthogonal hermitian projections (i.e., $PQ = 0$) and $x \in D_0$, we define
\[
\psi(f, \lambda P + \mu Q)x := \lambda I(f \otimes P)x + \mu I(f \otimes Q)x.
\]
Before extending $\psi$ to the pairs $(f, A)$, where $f \in U$ is a function and $A \in A_1$ an arbitrary operator, we state the following lemma, which will be used often in what follows.

**Lemma 3.4.** Let $A \in A_+^+$ be a positive operator in a $C^*$-algebra $A \subseteq B(H)$, where $H$ is a Hilbert space. Then there is a sequence $S_t(A)$ of the form
\[
S_t(A) = \sum_{k=1}^{n_t} \zeta_{k,t} P_{k,t},
\]
converging to $A$ in norm, where $\zeta_{k,t} \geq 0$ are non-negative and $P_{k,t}$ are orthogonal hermitian projections.

**Proof.** For the sequence $S_t(A)$ we can take a Riemann sum given by the resolution of unity belonging to $A$, where $S_{t+1}(A)$ is a refinement of $S_t(A)$ (see [8 equality (7.1)] and [7 p. 63-64] for details). 

We will now extend $\psi$ to the pairs $(f, A)$, where $f \in U$ is a function and $A \in (A_1)_+$ is a positive operator. Let $x \in D_0$. We separate two possibilities:

(i) If $A$ has a finite spectral decomposition $\sum_{k=1}^{n} \lambda_k P_k$, where $\lambda_k \geq 0$ are non-negative and $P_k$ are mutually orthogonal hermitian projections (i.e., $P_iP_j = 0$ for every $i \neq j$), then we define
\[
\psi(f, A)x := \sum_{k=1}^{n} \lambda_k I(f \otimes P_k)x.
\]
If $A$ does not have a finite spectral decomposition, then we define

$$\psi(f, A) x := \lim_{\ell \to \infty} \psi(f, S_\ell(A)) x,$$

where $S_\ell(A)$ is a sequence from Lemma 3.4.

**Proposition 3.5.** For $f \in \mathcal{U}$ and a positive operator $A \in (A_1)_+$ without a finite spectral decomposition, the definition of $\psi(f, A)$:

(i) is well-defined,

(ii) does not depend on the choice of the sequence $S_\ell(A)$.

**Proof.** Let us first prove that $(\psi(f, S_\ell(A)) x)_{\ell \in \mathbb{N}}$ is a Cauchy sequence. For $\ell' > \ell$, we have $S_\ell(A) - S_{\ell'}(A) = \sum_{i=1}^{m_{\ell'}} \lambda_i P_i$ for some $\lambda_i \in \mathbb{R}$, mutually orthogonal hermitian projections $P_i$ and $m_{\ell'} \in \mathbb{N}$. Given $\epsilon > 0$ and choosing $\ell$ great enough we can achieve $|\lambda_i| < \epsilon$ for every $i = 1, \ldots, m$. Since $\text{Id}_H = P + (\text{Id}_H - P)$, where $P$, $\text{Id}_H - P$ are mutually orthogonal hermitian projections, it follows that $\|\psi(f, P)x\| \leq \|\psi(f, \text{Id}_H)x\|$ for every $x \in D_0$. We have

$$\left\| \sum_{i=1}^{m} \lambda_i \psi(f, P_i)x \right\| \leq \max_{i} |\lambda_i| \left\| \sum_{i=1}^{m} \psi(f, P_i)x \right\| \leq \max_{i} |\lambda_i| \|\psi(f, \text{Id}_H)x\| \leq \epsilon \|\psi(f, \text{Id}_H)x\|,$$

where the first inequality follows by the fact that $\text{Im}(\psi(f, P_i)) \perp \text{Im}(\psi(f, P_j))$ for $i \neq j$ (see Lemma 3.3(ii)) and the second by the fact that $\sum_{i=1}^{m} P_i$ is a hermitian projection. Since $\epsilon > 0$ was arbitrary, $(\psi(f, S_\ell(A)) x)_{\ell \in \mathbb{N}}$ is a Cauchy sequence and hence $\lim_{\ell \to \infty} \psi(f, S_\ell(A)) x$ exists. This proves (i).

Now we will prove the independence from the sequence $S_\ell(A)$. Let $S'_\ell(A) := \sum_{k=1}^{m} \zeta'_{k,\ell} P'_{k,\ell}$ be another sequence converging to $A$ in norm, where $\zeta'_{k,\ell} \geq 0$ are non-negative and $P'_{k,\ell}$ are mutually orthogonal hermitian projections. We will prove that

$$(\psi(f, S_\ell(A)) x - \psi(f, S'_\ell(A)) x)_{\ell \in \mathbb{N}}$$

converges to 0. We have

$$S_\ell(A) - S'_\ell(A) = \sum_{i=1}^{p} \mu_i Q_i =: h_\ell(x),$$

where $\mu_i \in \mathbb{R}$ and $Q_i$ are mutually orthogonal hermitian projections. Therefore

$$\psi(f, S_\ell(A)) x - \psi(f, S'_\ell(A)) x = \sum_{i=1}^{p} \mu_i \psi(f, Q_i)x.$$

Given $\epsilon > 0$ and choosing $\ell$ great enough we can achieve $|\mu_i| < \epsilon$ for every $i = 1, \ldots, p$. As for part (i) we estimate

$$\left\| \sum_{i=1}^{p} \mu_i \psi(f, Q_i)x \right\| \leq \epsilon \|\psi(f, \text{Id}_H)x\|.$$

Therefore $\psi(f, S_\ell(A)) x - \psi(f, S'_\ell(A)) x$ converges to 0 which proves (ii). \hfill \Box

**Lemma 3.6.** For a function $f \in \mathcal{U}$, a positive operator $A \in (A_1)_+$ and $x \in D_0$, we have

$$\psi(f, A) x = \lim_{n \to \infty} \psi(f \chi_{\Delta_n}, A)x,$$
where \((\Delta_n)_{n \in \mathbb{N}}\) is a bounding sequence of \(f\) with respect to the spectral measure \(M(\cdot)(\operatorname{Id}_H)\) (see Subsection 2.2).

**Proof.** For \(f \in \mathcal{U}\), a positive operator \(A \in \mathcal{A}_1\) and \(x \in \mathcal{D}_0\), it holds by the definition that \(\psi(f, A)x = \lim_{n \to \infty} \psi(f, S_{\lambda}(A))x\), where \((S_{\lambda}(A))_{\lambda \in \mathbb{N}}\) is a sequence from Lemma 3.4. By the equality (11), \(\psi(f, P)x = \lim_{n \to \infty} \psi(f \Delta_n, P)x\) holds for every hermitian projection \(P \in (\mathcal{A}_1)_p\), every bounding sequence \((\Delta_n)_n\) for \(f\) with respect to the spectral measure \(M(\cdot)(\operatorname{Id}_H)\) and every \(x \in \mathcal{D}_0\). Using Lemma 3.3 it is also true that for every hermitian projection \(P \in (\mathcal{A}_1)_p\),

\[
\|\psi(f, P)x - \psi(f \Delta_n, P)x\| \leq \|\psi(f, \operatorname{Id}_H)x - \psi(f \Delta_n, \operatorname{Id}_H)x\|,
\]

and hence by an analogous estimate as in the proof of Proposition 3.5

\[
\|\psi(f, S_{\lambda}(A))x - \psi(f \Delta_n, S_{\lambda}(A))x\| \leq \|A\| \|\psi(f, \operatorname{Id}_H)x - \psi(f \Delta_n, \operatorname{Id}_H)x\|.
\]

Therefore

\[
\psi(f, A)x = \lim_{\ell \to \infty} \psi(f, S_{\lambda}(A))x = \lim_{\ell \to \infty} \lim_{n \to \infty} \psi(f \Delta_n, S_{\lambda}(A))x = \lim_{n \to \infty} \lim_{\ell \to \infty} \psi(f \Delta_n, S_{\lambda}(A))x,
\]

where we used 11 in the third equality. \(\square\)

Let now \(A \in \mathcal{A}_1\) be arbitrary operator, \(f \in \mathcal{U}\) and \(x \in \mathcal{D}_0\). We define \(\psi(f, A)x\) by

\[
\psi(f, \operatorname{Re}(A)_+)x - \psi(f, \operatorname{Re}(A)_-)x + i \cdot \psi(f, \operatorname{Im}(A)_+)x - i \cdot \psi(f, \operatorname{Im}(A)_-)x,
\]

where \(\operatorname{Re}(A), \operatorname{Im}(A)\) denote the real and the imaginary part of the operator \(A\) and \(A_+, A_-\) the positive and the negative part of the hermitian operator \(A\).

**Proposition 3.7.** For a function \(f \in \mathcal{U}\), an operator \(A \in \mathcal{A}_1\) and \(x \in \mathcal{D}_0\), we have

\[
\psi(f, A)x = \lim_{n \to \infty} \psi(f \Delta_n, A)x,
\]

where \((\Delta_n)_{n \in \mathbb{N}}\) is a bounding sequence of \(f\) with respect to the spectral measure \(M(\cdot)(\operatorname{Id}_H)\).

**Proof.** By the definition of \(\psi(f, A)\) for an arbitrary operator \(A \in \mathcal{A}_1\), it is enough to prove the proposition for a positive operator \(A\). But this is just the statement of Lemma 3.6 \(\square\)

We have constructed the map

\[
\psi: \mathcal{U} \times \mathcal{A}_1 \to L(\mathcal{D}_0, \mathcal{K}),
\]

where \(L(\mathcal{D}_0, \mathcal{K})\) denotes the vector space of linear operators mapping \(\mathcal{D}_0\) into \(\mathcal{K}\).

**Proposition 3.8.** The map \(\psi\) is bilinear.

**Proof.** We will first prove the linearity in the first factor. For \(f, g \in \mathcal{U}, \lambda, \mu \in \mathbb{C}\) and \(A \in \mathcal{A}_1\), we have to show that \(\psi(\lambda f + \mu g, A) = \lambda \psi(f, A) + \mu \psi(g, A)\). We may
assume \( A \) is positive. Let \( S_\ell(A) \) be a sequence as in Lemma 3.4. We have

\[
\psi(\lambda f + \mu g, A) = \lim_{\ell \to \infty} \psi(\lambda f + \mu g, S_\ell(A)) = \lim_{\ell \to \infty} \sum_{k=1}^{n_\ell} \zeta_{k,\ell} \psi(\lambda f + \mu g, P_{k,\ell}) = \lambda \lim_{\ell \to \infty} \sum_{k=1}^{n_\ell} \zeta_{k,\ell} P_{k,\ell}
\]

where in the third equality we used the linearity of the integration with respect to the spectral measure \( M \) (see Theorem 2.6.(ii)).

Now we will prove the linearity in the second factor. For \( f \in \mathcal{U} \), \( \lambda, \mu \in \mathbb{C} \) and \( A, B \in A_1 \), we have to show that

\[
\psi(f, \lambda A + \mu B, A) = \lambda \psi(f, A, A) + \mu \psi(f, B, A)
\]

By the usual decomposition of \( \lambda, \mu, A, B \) into the linear combination of four positive parts and since the domain \( D(\psi(f \otimes A)) \) is dense in \( K \), we may assume, by [7, Proposition 3.5.(3.1)] that \( \lambda = \mu = 1 \). For \( x \in D_0 \) and a bounding sequence \( \Delta_n \) for \( f \) with respect to the spectral measure \( M \), we have

\[
\psi(f, A + B) x = \lim_{n \to \infty} \psi(f \chi_{\Delta_n}, A + B) x = \lim_{n \to \infty} \left( \psi(f \chi_{\Delta_n}, A) + \psi(f \chi_{\Delta_n}, B) \right) x = \psi(f, A) x + \psi(f, B) x,
\]

where in the first equality we used Lemma 3.6 and in the second equality we used the linearity of integration of bounded functions with respect to the non-negative spectral measures (see [8, Proposition 3.5.(3.1)]).

By the universal property of tensor products, the bilinear form \( \psi \) extends to the linear map

\[
\overline{\psi} : \mathcal{U} \otimes A_1 \to L(D_0, K), \quad \overline{\psi} \left( \sum_{i=1}^{n} f_i \otimes A_i \right) = \sum_{i=1}^{n} \psi(f_i, A_i).
\]

Now we extend \( \overline{\psi} \) to the integral \( \mathcal{I} \) defined by

\[
\mathcal{I} : \mathcal{U} \otimes A_1 \to L(K), \quad \mathcal{I} \left( \sum_{i=1}^{n} f_i \otimes A_i \right) = \overline{\psi} \left( \sum_{i=1}^{n} f_i \otimes A_i \right),
\]

where \( L(K) \) denotes the vector space of all linear operators on \( K \) (Here we do not demand that \( T \in L(K) \) is defined on all \( K \).) and \( \mathcal{T} \) denotes the closure of a densely defined operator \( T \in L(K) \).

**Lemma 3.9.** For \( f \in \mathcal{U} \) and \( A \in A_1 \), we have \( D_0 \subseteq \mathcal{D} \left( \overline{\psi}(f \otimes A)^* \right) \) and

\[
\overline{\psi}(f \otimes A)^* x = \overline{\psi}(f \otimes A^*) x
\]

for every \( x \in D_0 \).

**Proof.** By the decomposition of \( A \) into the linear combination of four positive parts and since the domain \( D_0 \) of \( \mathcal{D}(\overline{\psi}(f \otimes A)) \) is dense in \( K \), we may assume, by \[7\]
Proposition 1.6(vi)], that $A$ is a positive operator. For $x, y \in \mathcal{D}_0$ and a sequence $S_{\ell}(A)$ as in Lemma 3.4 we have
\[
\langle \overline{v}(f \otimes A)x, y \rangle = \left\langle \lim_{\ell \to \infty} \overline{v}(f \otimes S_{\ell}(A))x, y \right\rangle = \lim_{\ell \to \infty} \langle \overline{v}(f \otimes S_{\ell}(A))x, y \rangle = \lim_{\ell \to \infty} \langle x, \overline{v}(f \otimes S_{\ell}(A))y \rangle = \langle x, \overline{v}(f \otimes A)y \rangle,
\]
where we used Theorem 2.6(i) and [7 Proposition 1.6(vi)] in the fourth equality.

(12) $\mathcal{A}$

Therefore $y \in \mathcal{D}(\overline{v}(f \otimes A))^*$ and $\overline{v}(f \otimes A)^*y = \overline{v}(f \otimes A)y$.

\[\square\]

Proposition 3.10. The map $\mathcal{I}$ is well-defined.

Proof. For $\mathcal{I}$ to be well-defined $\overline{v}(\sum_{i=1}^n f_i \otimes A_i)$ must be closable for every functions $f_1, \ldots, f_n \in \mathcal{U}$ and every $A_1, \ldots, A_n \in A_1$. By [7 Theorem 1.8(i)], it suffices to show that the domain $\mathcal{D} \left( \overline{v}(\sum_{i=1}^n f_i \otimes A_i) \right)$ is dense in $\mathcal{K}$. Since the domain $\mathcal{D} \left( \overline{v}(\sum_{i=1}^n f_i \otimes A_i) \right)$ is dense, by [7 Proposition 1.6(vi)], $\overline{v}(\sum_{i=1}^n f_i \otimes A_i)$ is dense. Therefore it suffices to show that $\sum_{i=1}^n \overline{v}(f_i \otimes A_i)^*$ is densely defined. Further on, it suffices to prove that every operator $\overline{v}(f \otimes A)^*$, where $f \in \mathcal{U}$ and $A \in A_1$, is defined on $\mathcal{D}_0$. But this is the statement of Lemma 3.9, which concludes the proof.

Remark 3.11. In the equality (12) we introduced the integral of $f \otimes P$ for $f \in \mathcal{U}$ and a hermitian projection $P$ by $\mathcal{I}(f \otimes P) = \mathbb{I}_{M_P}(f)$. [7] is another definition of the integral $\mathcal{I}(f \otimes P)$. Since $\mathcal{D}_0$ is a core for $\mathbb{I}_{M_P}(f)$ (by Lemma 3.2), both definitions coincide.

Let $\mathcal{D}(T)$ denotes the domain of the linear operator $T$ on $\mathcal{K}$. The following proposition summarizes some properties of the integral $\mathcal{I}$.

Proposition 3.12. For functions $f, f_1, f_2, \ldots, f_n \in \mathcal{U}$, operators $A_1, A_2, \ldots, A_n \in A_1$, a hermitian operator $P \in (A_1)_+$, a positive operator $A \in (A_1)_+$, $S_{\ell}(A)$ a sequence as in Lemma 3.4, and a map $F \in \mathcal{U} \otimes A_1$, we have:

(i) $\mathcal{D}_0 := \bigcup_{K \text{ compact}} \mathbb{M}(K)(\mathbb{I}_{\mathcal{H}})K \subseteq \mathcal{D} \left( \mathcal{I} \left( \sum_{i=1}^n f_i \otimes A_i \right) \right)$.

(ii) $\mathcal{I}(f \otimes P) = \mathbb{I}_{M_P}(f)$.

(iii) If $(\Delta_m)_m$ is a bounding sequence for every $f_i$ in $\sum_{i=1}^n f_i \otimes A_i$ with respect to the spectral measure $\mathbb{M}(\cdot)(\mathbb{I}_{\mathcal{H}})$, then

\[
\mathcal{I} \left( \sum_{i=1}^n f_i \otimes P_i \right) x = \lim_{m \to \infty} \mathcal{I} \left( \sum_{i=1}^n f_i \chi_{\Delta_m} \otimes P_i \right) x
\]

for every $x \in \bigcap_{i=1}^n \mathcal{D}(\mathbb{I}_{M_P}(f_i))$.

(iv) For every vector $x \in \mathcal{D}_0$ the following holds

\[
\mathcal{I}(f \otimes A)x = \lim_{\ell \to \infty} \mathcal{I}(f \otimes S_{\ell}(A))x = \lim_{m \to \infty} \lim_{\ell \to \infty} \mathcal{I}(f \chi_{\Delta_m} \otimes S_{\ell}(A))x.
\]
(v) $\mathcal{I}(F)$ is closed.

**Proof.** Part (i), the first equality of part (iv) and part (v) are true by the construction of the integral $\mathcal{I}$. Part (ii) is true by Remark 3.11. Since $D_0$ is a core for $l_{M^P}(f)$ for every hermitian projection $P$ and every $f \in \mathcal{U}$, we have $\cap_{i=1}^n D(l_{M^P}(f_i)) \subseteq \mathcal{D}(\mathcal{I}(\sum_{i=1}^n f_i \otimes P_i))$. Hence by Theorem 2.5(ii), part (iii) is true. The second equality of part (iv) is the statement of Lemma 3.6. □

**Lemma 3.13.** For every map $F \in \mathcal{U} \otimes A_1$ and every compact $K$ we have

$$M(K)(\text{Id}_H)\overline{\psi}(F) \subseteq \overline{\psi}(F)M(K)(\text{Id}_H) = \overline{\psi}(F \chi_K).$$

**Proof.** Let $K$ be a compact set. By the linearity of $\overline{\psi}$ and the boundedness of $M(K)(\text{Id}_H)$, we may assume $F = f \otimes A$, where $f \in \mathcal{U}$, $A \in A_1$. With the same argument we can further assume that $A$ is positive operator. For $x \in D_0$ and $S_\ell(A)$ a sequence as in Lemma 3.4, we have

$$M(K)(\text{Id}_H)\overline{\psi}(f \otimes A)x = \lim_{\ell \to \infty} M(K)(\text{Id}_H)\overline{\psi}(f \otimes S_\ell(A))x$$

$$= \lim_{\ell \to \infty} M(K)(\text{Id}_H) \sum_{k=1}^{n_\ell} \zeta_{k,\ell} \overline{\psi}(f \otimes P_{k,\ell})x$$

$$= \lim_{\ell \to \infty} \sum_{k=1}^{n_\ell} \zeta_{k,\ell} \overline{\psi}(f \otimes P_{k,\ell})M(K)(\text{Id}_H)x$$

$$= \lim_{\ell \to \infty} \overline{\psi}(f \otimes S_\ell(A))M(K)(\text{Id}_H)x = \overline{\psi}(f \otimes A)M(K)(\text{Id}_H)x,$$

where the first and the fifth equality follow by the construction of the map $\overline{\psi}$, the second and the fourth by the linearity of the map $\overline{\psi}$ and the third equality follows by Lemma 3.2. This proves the inclusion part of the lemma. Since by Lemma 3.2

$$\lim_{\ell \to \infty} \sum_{k=1}^{n_\ell} \zeta_{k,\ell} \overline{\psi}(f \otimes P_{k,\ell})M(K)(\text{Id}_H)x = \lim_{\ell \to \infty} \sum_{k=1}^{n_\ell} \zeta_{k,\ell} \overline{\psi}(f \chi_K \otimes P_{k,\ell})x$$

$$= \overline{\psi}(f \chi_K \otimes A)x,$$

the equality part of the lemma also holds. □

**Lemma 3.14.** For every $F, G \in \mathcal{U} \otimes A_1$, $D_0$ is contained in the domain of the operator $\mathcal{I}(F)\mathcal{I}(G)$.

**Proof.** Let $K$ be a compact set. We have

$$\mathcal{I}(F) \circ M(K)(\text{Id}_H) \circ \mathcal{I}(G) \circ M(K)(\text{Id}_H)x$$

$$= \overline{\psi}(F) \circ M(K)(\text{Id}_H) \circ \overline{\psi}(G) \circ M(K)(\text{Id}_H)x$$

$$= \overline{\psi}(F) \circ \overline{\psi}(G) \circ (M(K)(\text{Id}_H))^2x$$

$$= \overline{\psi}(F) \circ \overline{\psi}(G) \circ M(K)(\text{Id}_H)x$$

$$= \mathcal{I}(F)\mathcal{I}(G) \circ M(K)(\text{Id}_H)x,$$

where we used Proposition 3.12(i) for the well-definedness of the first line, the definition of $\overline{\psi}$ and $\mathcal{I}$ in the first and the forth equality, Lemma 3.13 in the second equality and the fact that $M(K)(\text{Id}_H)$ is a hermitian projection in the third equality. Hence $D_0 = \bigcup_{K \text{ compact}} M(K)(\text{Id}_H)K$ is contained in the domain of $\mathcal{I}(F)\mathcal{I}(G)$. □
Lemma 3.15. For $F, G \in \mathcal{U} \otimes A_1$ we have
\[ \overline{\psi}(FG) = \overline{\psi}(F)\overline{\psi}(G). \]

Proof. By the linearity of $\overline{\psi}$, we may assume $F = f \otimes A, G = g \otimes B$, where $f, g \in \mathcal{U}$ and $A, B \in A_1$. Let $K$ be a compact set. By the $\sigma$-compactness of $K_n$ of compact sets, such that $K_1 = K, K_n \subseteq K_{n+1}$ for every $n \in \mathbb{N}$ and $X = \bigcup_{n \in \mathbb{N}} K_n$. For $y = M(K)(\text{Id}_H)x, x \in \mathcal{K}$ and a fixed $n \in \mathbb{N}$, we have
\[ \overline{\psi}(fg \otimes AB)y = \lim_{m \to \infty} \overline{\psi}(fg\chi_{K_m} \otimes AB)y \]
\[ = \lim_{m \to \infty} \overline{\psi}(f\chi_{K_m} \otimes A) \circ \overline{\psi}(g\chi_{K_m} \otimes B)y \]
\[ = \lim_{m \to \infty} \overline{\psi}(f\chi_{K_m} \otimes A) \circ \overline{\psi}(g\chi_{K_m} \otimes B) \circ M(K_n)(\text{Id}_H) \circ M(K)(\text{Id}_H)x \]
\[ = \lim_{m \to \infty} \overline{\psi}(f\chi_{K_m} \otimes A) \circ \overline{\psi}(g\chi_{K_m} \chi_{K_n} \otimes A) \circ M(K)(\text{Id}_H)x \]
\[ = \overline{\psi}(f \otimes A) \circ \overline{\psi}(g\chi_{K_n} \otimes B) \circ M(K)(\text{Id}_H)x \]
where we used Proposition 3.7 in the first and the fifth equality, [Proposition 7.2.] in the second equality, the fact that $M$ is a non-negative spectral measure in the third inequality (and $K_n \supseteq K$) and the equality part of Lemma 3.15 in the fourth equality. As $n \to \infty$, we get (by the use of Proposition 3.7)
\[ \overline{\psi}(fg \otimes AB)y = \overline{\psi}(f \otimes A)\overline{\psi}(g \otimes B)y. \]

This concludes the proof. \(\square\)

The main algebraic properties of the integral $\mathcal{I}$ are collected in the following theorem.

Theorem 3.16. For $F, G \in \mathcal{U} \otimes A_1, \alpha, \beta \in \mathbb{C}, f, g \in \mathcal{U}$ and a hermitian projection $P \in (A_1)_p$, we have:

(i) $\mathcal{I}(F^*) \subseteq \mathcal{I}(F)^*$,
(ii) $\mathcal{I}(\alpha F + \beta G) \subseteq \alpha \mathcal{I}(F) + \beta \mathcal{I}(G)$,
(iii) $\mathcal{I}(FG) \subseteq \mathcal{I}(F)\mathcal{I}(G)$,
(iv) $\mathcal{D}(\mathcal{I}(g \otimes P)) \cap \mathcal{D}(\mathcal{I}(f \otimes P)) \subseteq \mathcal{D}(\mathcal{I}(f \otimes P)\mathcal{I}(g \otimes P))$.

Proof. For $x \in \mathcal{D}_0$ and $F := \sum_{i=1}^{n} f_i \otimes A_i$, where $f_1, \ldots, f_n \in \mathcal{U}, A_1, \ldots, A_n \in A_1$, we have
\[ \mathcal{I}(F^*)x = \overline{\psi}(F^*)x = \sum_{i=1}^{n} \overline{\psi}(f_i \otimes A_i^*)x = \sum_{i=1}^{n} \overline{\psi}(f_i \otimes A_i)^*x \]
\[ = \left( \sum_{i=1}^{n} \overline{\psi}(f_i \otimes A_i) \right)^*x = \overline{\psi}(F^*)x = \mathcal{I}(F)^*x, \]
where the first and the sixth equality follow by the definition of $\overline{\psi}$ and $\mathcal{I}$, the second and the fifth by the linearity of $\overline{\psi}$, the third equality follows by Lemma 3.9 and in the forth equality we used [Proposition 1.6(vi)] $(\mathcal{D}_0 = \mathcal{D}(\sum_{i=1}^{n} \overline{\psi}(f_i \otimes A_i))$ is dense in $\mathcal{K}$). Since $\mathcal{I}(F)^*$ is the closed extension of the operator $\overline{\psi}(F^*)$, and $\mathcal{I}(F^*)$ is its closure, part (i) is true.

It suffices to prove that $\mathcal{I}(F + G) \subseteq \mathcal{I}(F) + \mathcal{I}(G)$. Using part (i), we have
\[ \mathcal{D}(\mathcal{I}(F + G)) \supseteq \mathcal{D}(\mathcal{I}(F)^* + \mathcal{I}(G)^*) \supseteq \mathcal{D}(\mathcal{I}(F^*) + \mathcal{I}(G^*)) \supseteq \mathcal{D}_0. \]
Since $D_0$ is dense, $\mathcal{I}(F) + \mathcal{I}(G)$ is closable (by [7] Proposition 1.8(i)). Since $\mathcal{I}(F+G)$ is the closure of $\mathcal{I}(F+G)|_{D_0}$ and $\mathcal{I}(F) + \mathcal{I}(G)$ is the closed extension of $\mathcal{I}(F+G)|_{D_0}$, part (ii) is true.

By Lemma 5.14, $D_0$ is contained in the domain of $(\mathcal{I}(F)\mathcal{I}(G))^*$. Since we have $(\mathcal{I}(F)\mathcal{I}(G))^* \supseteq (\mathcal{I}(G))^*\mathcal{I}(F)^*$ (by [7] Proposition 1.7(ii)) and $(\mathcal{I}(G))^*\mathcal{I}(F)^* \supseteq \mathcal{I}(G)^*\mathcal{I}(F)^*$ by part (i), the operator $\mathcal{I}(F)\mathcal{I}(G)$ is closable (by [7] Proposition 1.8(i)). For $x \in D_0$, $\mathcal{I}(FG)x = \mathcal{I}(F)\mathcal{I}(G)x$ (by Lemma 5.15). Since $\mathcal{I}(FG)$ is the closure for $\overline{\mathcal{I}}(FG)$ and $\overline{\mathcal{I}(F)\mathcal{I}(G)}$ is the closed extension for $\overline{\mathcal{I}}(FG)$, part (iii) follows.

Since $\mathcal{I}(f \otimes P) = \mathbb{1}_{\mathcal{M}_p}(f)$ for every $f \in \mathcal{U}$ and every hermitian projection $P \in (\mathcal{A}_1)_p$, part (iv) follows by Theorem 2.3(v).

4. Extension of Theorem 2.3 to Unbounded $^*$-Representations

Let $(X,\text{Bor}(X),\mathcal{A}_1 \subseteq B(\mathcal{H}),B(K),M)$ be a space with a non-negative spectral measure $M : \text{Bor}(X) \rightarrow B(\mathcal{A}_1,B(K))$, where $\text{Bor}(X)$ is a Borel $\sigma$-algebra on a $\sigma$-compact topological space $X$, $\mathcal{A}_1 \subseteq B(\mathcal{H})$ is a $C^*$-algebra, $\mathcal{W}_2 \subseteq B(\mathcal{K})$ is a von Neumann algebra, and $\mathcal{H},\mathcal{K}$ Hilbert spaces.

The aim of this Section is to extend Theorem 2.3 from the bounded $^*$-representations $\rho : C(X,\mathcal{A}_1) \rightarrow B(K)$ to the unbounded $^*$-representations $\rho : C(X,\mathcal{C}) \otimes \mathcal{A}_1 \rightarrow L(\mathcal{K})$, where $L(\mathcal{K})$ is the algebra of all linear operators of $\mathcal{K}$ (Here we do not demand that a linear operator is defined on all $\mathcal{K}$.)

Let $D_0$ be a dense subspace in $\mathcal{K}$ and $\mathcal{V}$ a subalgebra of the algebra $\mathcal{U}$ of all measurable functions. For a map $\rho : \mathcal{V} \otimes \mathcal{A}_1 \rightarrow L(\mathcal{K})$, $\alpha \in \mathcal{A}(\mathcal{V})$ (see Subsection 2.3 for the definition of $\alpha$ and $\mathcal{A}$) and a hermitian projection $P \in (\mathcal{A}_1)_p$, we define the set

$$D_{\alpha,P} := \left\{ x \in \bigcap_{f \in \mathcal{V}} D(\rho(f \otimes P)) : \|\rho(f \otimes P)x\| \leq \alpha(f)\|x\| \text{ for all } f \in \mathcal{V} \right\}.$$ 

The map $\rho$ is a $^*$-representation on a subspace $D_0$, if for every $\alpha,\beta \in \mathcal{C}$, every $F,G \in \mathcal{V} \otimes \mathcal{A}_1$, every $f,g \in \mathcal{V}$ and every hermitian projection $P \in (\mathcal{A}_1)_p$, we have

\begin{enumerate}
  \item $D_0$ is contained in the domain $D(\rho(F))$ and is a core for $\rho(F)$,
  \item $\rho(1 \otimes \text{Id}_\mathcal{K}) = \text{Id}_\mathcal{K}$,
  \item $\rho(F^*) \subseteq \rho(F)^*$, \hspace{1cm} (i)
  \item $\rho(\alpha F + \beta G) \subseteq \alpha \rho(F) + \beta \rho(G)$, \hspace{1cm} (ii)
  \item $\rho(FG) \subseteq \rho(F)\rho(G)$, \hspace{1cm} (iv)
  \item $\rho(f \otimes P)^* = \rho(f \otimes P)$, \hspace{1cm} (v)
  \item $\rho(f \otimes P)\rho(g \otimes P) \subseteq \rho(fg \otimes P)$ and
  \begin{equation*}
  D(\rho(f \otimes P)\rho(g \otimes P)) = D(\rho(g \otimes P)) \cap D(\rho(fg \otimes P)),
  \end{equation*}
  \item $\rho(f \otimes P)\rho(g \otimes P) = \rho(fg \otimes P)$, \hspace{1cm} (vii)
  \item $D_{\alpha,\text{Id}_\mathcal{K}} := \cup_{\alpha \in \mathcal{A}(\mathcal{V})} D_{\alpha,\text{Id}_\mathcal{K}}$ is dense in $\mathcal{K}$.
\end{enumerate}

**Remark 4.1.** From the condition (ix) it follows that

\begin{enumerate}
  \item $D_{\alpha,P} := \cup_{\alpha \in \mathcal{A}(\mathcal{V})} D_{\alpha,P}$ is dense in $\mathcal{K}$ for every hermitian projection $P \in (\mathcal{A}_1)_p$.
\end{enumerate}
Indeed, let \( f, g \in \mathcal{V} \) be arbitrary functions and \( P \in (A_1)_p \) a hermitian projection. By (vii) we have
\[
\rho(\overline{f} \otimes P)\rho(g \otimes (\text{Id}_\mathcal{H} - P)) \subseteq \rho((\overline{f} \otimes P)(g \otimes (\text{Id}_\mathcal{H} - P))) = \rho(0) = 0.
\]
Hence
\[
\text{Im}(\rho(g \otimes (\text{Id}_\mathcal{H} - P))) \subseteq \text{Ker}(\rho(\overline{f} \otimes P)) = \text{Ker}(\rho(f \otimes P)^*),
\]
where we used (vi) for the equality. By [7, Proposition 1.6(ii)], \( \text{Ker}(\rho(f \otimes P)^*) = \text{Im}(\rho(f \otimes P))^\perp \). Therefore \( \text{Im}(\rho(g \otimes (\text{Id}_\mathcal{H} - P))) \perp \text{Im}(\rho(f \otimes P)) \). Hence the inequality \( \|\rho(f \otimes P)x\| \leq \|\rho(f \otimes \text{Id}_\mathcal{H})x\| \) holds and \( D_{\text{e,Id}_\mathcal{H}} \subseteq D_{\text{e,P}} \).

Let \( \mathcal{U} \) be the algebra of measurable complex functions \( f : X \to \mathbb{C} \). The non-negative spectral integral is a \(*\)-representation on the space \( \mathcal{U} \otimes A_1 \) by the following theorem.

**Theorem 4.2.** The non-negative spectral integral
\[
\mathcal{I} : \mathcal{U} \otimes A_1 \to L(\mathcal{K}),
\]
defined by [4], is a \(*\)-representation on the subspace
\[
\mathcal{D}_0 = \bigcup_{K \text{ compact}} M(K)(\text{Id}_\mathcal{H})\mathcal{K}.
\]

**Proof.** We will check that all parts in the definition of the \(*\)-representation on \( \mathcal{D}_0 = \bigcup_{K \text{ compact}} M(K)(\text{Id}_\mathcal{H})\mathcal{K} \) hold for the non-negative spectral integral \( \mathcal{I} \). Part (i) follows by the construction of the integral \( \mathcal{I} \). Parts (ii)-(v) follow by Theorem 3.10. Parts (vi)-(viii) follow by the equality \( \mathcal{I}(f \otimes P) = \mathbb{1}_{M_P}(f) \) and Theorem 2.6 where \( f \in \mathcal{U} \) is a measurable function and \( P \in (A_1)_p \) is a hermitian projection. Let now \( K \) be a compact set. By the \( \sigma\)-compactness of \( X \), there is an increasing sequence \( K_n \) of compact sets, such that \( K_1 = K \) and \( X = \bigcup_{n \in \mathbb{N}} K_n \). To prove part (ix) we first note that for \( x = M(K)(\text{Id}_\mathcal{H})y, \) where \( y \in \mathcal{K} \), following holds:
\[
\mathcal{I}(f \otimes \text{Id}_\mathcal{H})x = \left( \int_X f \ dM_{\text{Id}_\mathcal{H}} \right) y = \left( \int_X f \ dM_{\text{Id}_\mathcal{H}} \right) M(K)(\text{Id}_\mathcal{H})^2 y = \left( \int_X f \chi_K \ dM_{\text{Id}_\mathcal{H}} \right) y = \left( \int_X f \chi_K \ dM_{\text{Id}_\mathcal{H}} \right) x,
\]
where we used Lemma 3.13 for the third equality. Therefore, using the fact that \( M_{\text{Id}_\mathcal{H}} : \text{Bor}(X) \to B(\mathcal{K}) \) is a spectral measure and [7 Proposition 4.12(iv)], we have
\[
\|\mathcal{I}(f \otimes \text{Id}_\mathcal{H})x\| = \left( \int_X f \chi_K \ dM_{\text{Id}_\mathcal{H}} \right) x \leq \|f\|_{\infty, \mathcal{K}} \|x\| = \alpha_K(f)\|x\|,
\]
where \( \|f\|_{\infty, \mathcal{K}} := \sup_{t \in K} \{|f(t)|\} \) and
\[
\alpha_K : C(X, \mathbb{C}) \to [0, \infty), \quad \alpha_K(f) := \|f\|_{\infty, \mathcal{D}_n}
\]
is an absolute value. Hence \( x \in D_{\text{K,Id}_\mathcal{H}} \) and so
\[
D_{\text{e,Id}_\mathcal{H}} \supseteq \bigcup_{K \text{ compact}} D_{\text{K,Id}_\mathcal{H}} \supseteq D_0.
\]
Since \( D_0 \) is dense in \( \mathcal{K} \), part (ix) holds.
Conversely, for realcompact and \(\sigma\)-compact spaces, equipped with a Borel \(\sigma\)-algebra, \(*\)-representations of the algebra \(C(X, \mathbb{C}) \otimes A_1\) have integral representations on \(X\) by the following theorem.

**Theorem 4.3.** Let \(X\) be a realcompact and \(\sigma\)-compact space, \(\text{Bor}(X)\) be a Borel \(\sigma\)-algebra on \(X\), \(A_1\) a \(C^*\)-algebra, \(L(K)\) a vector space of all linear operators on a Hilbert space \(K\) (not necessarily everywhere defined), \(D_0\) a dense subspace in \(K\), \(L(D_0, K)\) a vector space of all linear operators of \(D_0\) into \(K\) and

\[
\rho : C(X, \mathbb{C}) \otimes A_1 \rightarrow L(K)
\]
a \(*\)-representation on a subspace \(D_0\) of a Hilbert space \(K\), such that for every function \(f \in C(X, \mathbb{C})\) the map

\[
\rho_f : A_1 \rightarrow L(D_0, K), \quad \rho_f(A) = \rho(f \otimes A)
\]
is continuous relative to the operator topology on \(A_1\) and the strong operator topology on \(L(D_0, K)\). Then there exists a unique regular normalized non-negative spectral measure

\[
M : \text{Bor}(X) \rightarrow B(\mathcal{B}(A_1, B(K))
\]
such that

\[
\rho(F)x = \left(\int_X F \, dM\right)x
\]
holds for every \(x \in \mathcal{D}(\rho(f))\) and every \(F \in C(X, \mathbb{C}) \otimes A_1\).

In the proof we will use the following result, which is a special case of Theorem \ref{thm:representations}.

**Proposition 4.4.** Let \(X\) be a realcompact space, \(\text{Bor}(X)\) a Borel \(\sigma\)-algebra on \(X\), \(N(K)\) a vector space of all (not necessarily bounded and everywhere defined) normal operators on \(K\) and \(\rho : C(X, \mathbb{C}) \rightarrow N(K)\) a map. The following statements are equivalent.

1. \(\rho : C(X, \mathbb{C}) \rightarrow N(K)\) is a \(*\)-representation.
2. There exists a unique regular normalized spectral measure \(F : \text{Bor}(X) \rightarrow B(K)\) such that

\[
\rho(f)x = \left(\int_X f \, dF\right)x
\]
holds for every \(x \in \mathcal{D}(\rho(f))\) and every \(f \in C(X, \mathbb{C})\).

**Proof.** First we will prove the direction 2. \(\Rightarrow\) 1. Parts (i)-(iv) in the definition of a \(*\)-representation \(\rho : C(X, \mathbb{C}) \rightarrow N(K)\) (see Subsection \ref{subsection:representations}) follow by Theorem \ref{thm:representations}. Let \(x \in F(K_n)K\), where \(X = \cup_{n \in \mathbb{N}} K_n\), \(K_n \subseteq K_{n+1}\) and \(K_n\) is compact for every \(n \in \mathbb{N}\). Hence

\[
\left(\int_X f \, dF\right)x = \left(\int_X f \, dF\right)F(K_n)x = \left(\int_X f_{XK_n} \, dF\right)x,
\]
where the first equality follows by \(F(K_n)\) being a hermitian projection and the second equality follows by Theorem \ref{thm:representations} (iii). So

\[
\left\|\left(\int_X f \, dF\right)x\right\| = \left\|\left(\int_X f \, dF\right)F(K_n)x\right\| \leq \|f\|_{\infty, K_n}\|x\|
\]
where \(\|f\|_{\infty, K_n} := \sup_{t \in K_n} \{|f(t)|\}, \quad \alpha_n : C(X, \mathbb{C}) \rightarrow [0, \infty), \quad \alpha_n(f) := \|f\|_{\infty, K_n}\).
is an absolute value and the second inequality follows by [7] Lemma 4.11. Hence $x \in D_{\alpha_n}$ and so

$$D_c \supseteq \bigcup_{n \in \mathbb{N}} D_{\alpha_n} \supseteq \bigcup_{n \in \mathbb{N}} F(K_n)\mathcal{K}.$$ 

Since $\bigcup_{n \in \mathbb{N}} F(K_n)\mathcal{K}$ is dense in $\mathcal{K}$ (see [4]), part (v) in the definition of a $*$-representation $\rho : C(X, \mathbb{C}) \rightarrow \mathcal{N}(\mathcal{K})$ holds.

Now we will prove the direction $1. \Rightarrow 2.$ In particular, $C(X, \mathbb{C})$ is a commutative, unital semigroup, having a conjugation as an involution. By Theorem 2.8 there exists a unique regular normalized spectral measure $\tilde{F} : \text{Bor}(C(X, \mathbb{C})^*) \rightarrow B(\mathcal{K})$ such that

$$\rho(f)x = \left(\int_{C(X, \mathbb{C})^*} \nu(f) \, d\tilde{F}(\nu)\right)x,$$

for every $f \in C(X, \mathbb{C})$ and every $x \in D(\rho(f))$. Since $C(X, \mathbb{C})$ is also an algebra, $\tilde{F}$-almost every $\nu \in C(X, \mathbb{C})^*$ is linear (see [5] Proof of Theorem 1, p. 2953 and [6] Proof of Theorem 1.2, p. 230]). Since $X$ is a realcompact space, by [4] Section 3.11, all the linear characters on $C(X, \mathbb{C})$ are evaluations in points. Therefore the topological subspace of all linear characters in $C(X, \mathbb{C})$ is homeomorphic to $X$ and hence $\rho(f) = \int_X f \, dF$ for every $f \in C(X, \mathbb{C})$, where $F : \text{Bor}(X) \rightarrow B(\mathcal{K})$ is a unique regular normalized spectral measure representing $\rho$. 

\begin{proof}[Proof of Theorem 4.3] The proof is analogous to the proof of [8] Theorem 9.1, with some modifications. The idea is to construct a family $\{F_p\}_{p \in (A_1)_p}$ of spectral measures $F_p : \text{Bor}(X) \rightarrow B(\mathcal{K})$ which satisfies the conditions of Theorem 2.1 to obtain a non-negative spectral measure $M$ representing $\rho$. The proofs of the conditions (1) and (3) are the same as in [8] Theorem 9.1, for the bounded Borel sets, but for the arbitrary Borel sets we have to use the countable additivity of the spectral measures and the $\sigma$-compactness of $X$ (every realcompact space is $\sigma$-compact). For $M$ to be the representing measure of $\rho$ we have to use the continuity of the maps $\rho_f$. Precisely, the proof is the following.

Since $\rho$ is a $*$-representation on $D_0$, the maps $\rho_p : C(X, \mathbb{C}) \rightarrow B(\mathcal{K})$, $\rho_p(f) := \rho(f \otimes P)$ are $*$-representations for every hermitian projection $P \in (A_1)_p$. By Proposition 4.1 there exist unique spectral measures $F_p : \text{Bor}(X) \rightarrow B(\mathcal{K})$ such that $\rho_p(f) := \rho(f \otimes P) = \int_X f \, dF_p$ holds for every $f \in C(X, \mathbb{C})$ and every hermitian projection $P \in (A_1)_p$. We will show that the family $\{F_p\}_{p \in (A_1)_p}$ satisfies the conditions of Theorem 2.1.

The family $\{F_p\}_{p \in (A_1)_p}$ satisfies the condition (1) of Theorem 2.1: Let $P_i, Q_j \in (A_1)_p$ be hermitian projections and $\lambda_i, \mu_j \in \mathbb{R}$ real numbers, such that $\sum_{i=1}^{n} \lambda_i P_i = \sum_{j=1}^{m} \mu_j Q_j$. We have to show that for every set $\Delta \in \text{Bor}(X)$, the following equality $\sum_{i=1}^{n} \lambda_i F_{P_i}(\Delta) = \sum_{j=1}^{m} \mu_j F_{Q_j}(\Delta)$ holds. For bounded sets $\Delta \in \text{Bor}(X)$ the proof is the same as in the proof of [8] Theorem 9.1. Let $\Delta \in \text{Bor}(X)$ be arbitrary. By the $\sigma$-compactness of $X$, there exists a sequence $\Delta_n$ of bounded Borel sets, such that $\Delta = \bigcup_n \Delta_n$. By the above, $\sum_{i=1}^{n} \lambda_i F_{P_i}(\Delta_n) = \sum_{j=1}^{m} \mu_j F_{Q_j}(\Delta_n)$ holds for every $n \in \mathbb{N}$. By the countable additivity of the spectral measures $F_{P_i}, F_{Q_j}$, we conclude that

$$\sum_{i=1}^{n} \lambda_i F_{P_i}(\Delta) = \sum_{j=1}^{m} \mu_j F_{Q_j}(\Delta).$$
The family \(\{F_p\}_{p \in (A_1)_p}\) satisfies the condition (2) of Theorem 2.1. Let \(P \in (A_1)_p\) be a hermitian projection and \(\Delta \in \text{Bor}(X)\) a Borel set. We have to find a constant \(k_\Delta \in \mathbb{R}^+\) such that \(\|F_p(\Delta)\| \leq k_\Delta\). We know that

\[
\|F_p(X)\| = \left\| \int_X 1 \, dF_p \right\| = \|\rho_p(1)\| = \|\rho(1 \otimes P)\|.
\]

By the finite additivity of \(F_p\), it follows that \(\|F_p(\Delta)\| \leq \|\rho(1 \otimes P)\|\) for every \(\Delta \in \text{Bor}(X)\).

The family \(\{F_p\}_{p \in (A_1)_p}\) satisfies the condition (3) of Theorem 2.1. Let \(P, Q \in (A_1)_p\) be hermitian projections and \(\Delta_1, \Delta_2 \in \text{Bor}(X)\) Borel sets. It is equivalent to show that \(M_P(\Delta_1)M_Q(\Delta_2) = M_{PQ}(\Delta_1 \cap \Delta_2)\). For bounded sets \(\Delta_1, \Delta_2 \in \text{Bor}(X)\) the proof is the same as in the proof of [8, Theorem 9.1.]. Let \(\Delta_1 \in \text{Bor}(X)\) be a bounded fixed set. There exists a sequence \(\Delta_{1,n}\) of bounded Borel sets, such that \(\Delta_1 = \bigcup_n \Delta_{1,n}\). By the above, \(M_P(\Delta_{1,n})M_Q(\Delta_2) = M_{PQ}(\Delta_{1,n} \cap \Delta_2)\) holds for every \(n \in \mathbb{N}\). By the countable additivity of the spectral measures \(M_P, M_Q\), we conclude that \(M_P(\Delta_1)M_Q(\Delta_2) = M_{PQ}(\Delta_1 \cap \Delta_2)\). Let now \(\Delta_2 \in \text{Bor}(X)\) be arbitrary and \(\Delta_1 \in \text{Bor}(X)\) a bounded fixed set. By the compactness of \(X\), there exists a sequence \(\Delta_{2,n}\) of bounded Borel sets, such that \(\Delta_2 = \bigcup_n \Delta_{2,n}\). By the above, \(M_P(\Delta_1)M_Q(\Delta_{2,n}) = M_{PQ}(\Delta_1 \cap \Delta_{2,n})\) holds for every \(n \in \mathbb{N}\).

By the countable additivity of the spectral measures \(M_P, M_Q\), we conclude that \(M_P(\Delta_1)M_Q(\Delta_2) = M_{PQ}(\Delta_1 \cap \Delta_2)\).

\(M\) is the representing measure of \(\rho\): We first consider the elements \(F \in C(X, \mathbb{C}) \otimes A_1\) of the form \(f \otimes P\), where \(f \in C(X, \mathbb{C})\) and \(P\) is a hermitian projection. By the construction of the measures \(F_p\), we have \(\rho(f \otimes P) = \int_X f \, dF_p = \int_X (f \otimes P) \, dM\). Let now \(f \in C(X, \mathbb{C})\) and \(A \in (A_1)_+\) be a positive operator. Since \(\rho(f \otimes P) = \int_X (f \otimes P) \, dM\) holds for every \(P \in (A_1)_p\), we have \(\rho(f \otimes S_t(A)) = \int_X (f \otimes S_t(A)) \, dM\), where \(S_t(A)\) is a sequence as in Lemma 3.1. By the continuity of the maps \(\rho_f\) relative to the operator topology on \(A_1\) and the strong operator topology on \(L(D_0, K)\), we have

\[
\rho(f \otimes A)x = \rho_f(A)x = \lim_{t \to \infty} \rho_f(S_t(A))x = \lim_{t \to \infty} \rho(f \otimes S_t(A))x,
\]

for every \(x \in D_0\). By the construction of the non-negative spectral integral,

\[
\left(\int_X (f \otimes A) \, dM\right)x = \lim_{t \to \infty} \left(\int_X (f \otimes S_t(A)) \, dM\right)x
\]

for every \(x \in D_0\). Hence

\[
\rho(f \otimes A)x = \left(\int_X (f \otimes A) \, dM\right)x
\]

for every \(x \in D_0\). Since \(D_0\) is a core for \(\rho(f \otimes A)\) and \(\int_X (f \otimes A) \, dM\) is closed, we conclude that

\[
\rho(f \otimes A)x = \left(\int_X (f \otimes A) \, dM\right)x
\]

holds for every \(f \in C(X, \mathbb{C})\), every \(A \in (A_1)_+\) and every \(x \in D(\rho(f \otimes A))\).

By the usual decompositions of \(A\) into the linear combination of four positive parts, we conclude that \(\rho(f \otimes A)x = \left(\int_X (f \otimes A) \, dM\right)x\) holds for every \(f \in C(X, \mathbb{C})\), every \(A \in A_1\) and every \(x \in D(\rho(f \otimes A))\). By the linearity of \(\rho\) and \(f\) on \(D_0, D_0\)
being a core for \( \rho(F) \) and \( \int_X F \, dM \) being closed for every \( F \in C(X, \mathbb{C}) \otimes A_1 \), we finally get
\[
\rho(F)x = \left( \int_X F \, dM \right)x
\]
for every \( x \in D(\rho(F)) \).

\( M \) is unique, regular and normalized: This follows from the uniqueness and the regularity of each \( F_P \) and the unitality of \( \rho \). \qed

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