Finitistic Dimension Conjectures via Gorenstein Projective Dimension

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Abstract

It is a well-known result of Auslander and Reiten that contravariant finiteness of the class $\mathcal{P}_\infty^{\text{fin}}$ (of finitely generated modules of finite projective dimension) over an Artin algebra is a sufficient condition for validity of finitistic dimension conjectures. Motivated by the fact that finitistic dimensions of an algebra can alternatively be computed by Gorenstein projective dimension, in this work we examine the Gorenstein counterpart of Auslander–Reiten condition, namely contravariant finiteness of the class $\mathcal{GP}_\infty^{\text{fin}}$ (of finitely generated modules of finite Gorenstein projective dimension), and its relation to validity of finitistic dimension conjectures. It is proved that contravariant finiteness of the class $\mathcal{GP}_\infty^{\text{fin}}$ implies validity of the second finitistic dimension conjecture over left artinian rings. In the more special setting of Artin algebras, however, it is proved that the Auslander–Reiten sufficient condition and its Gorenstein counterpart are virtually equivalent in the sense that contravariant finiteness of the class $\mathcal{GP}_\infty^{\text{fin}}$ implies contravariant finiteness of the class $\mathcal{P}_\infty^{\text{fin}}$ over any Artin algebra, and the converse holds for Artin algebras over which the class $\mathcal{GP}_0^{\text{fin}}$ (of finitely generated Gorenstein projective modules) is contravariantly finite.

Keywords: Contravariant finiteness; Cotorsion pair; Finitistic dimensions; Gorenstein projective dimension; Tilting module.
**Introduction**

A key problem in homological theory of rings and modules is to understand the range of homological dimensions associated to rings and their modules. In particular, many problems and (homological) conjectures about rings and modules relate in one way or another to the question of how large can the projective dimension of modules over a ring be? To address this question two classes of modules, namely the class $\mathcal{P}_\infty$ of modules of finite projective dimension and the class $\mathcal{P}^\text{fin}_\infty$ of finitely generated modules of finite projective dimension, come naturally to the fore. In order to measure the range of projective dimension of modules in the two classes, the so-called finitistic dimensions are natural invariants to consider. Recall that for any ring $\Lambda$, the big finitistic dimension of $\Lambda$ is defined as

$$\text{FPD}(\Lambda) := \sup \{ \text{pd}_\Lambda(M) \mid M \in \mathcal{P}_\infty \},$$

and the little finitistic dimension of $\Lambda$ is defined as

$$\text{fpd}(\Lambda) := \sup \{ \text{pd}_\Lambda(M) \mid M \in \mathcal{P}^\text{fin}_\infty \}.$$

The importance of these invariants was emphasized in the 1960s by Bass [10], where the following problems due to Rosenberg and Zelinsky, later known as finitistic dimension conjectures, were advertised [10, page 487]:

**First Finitistic Dimension Conjecture:** $\text{FPD}(\Lambda) = \text{fpd}(\Lambda)$.

**Second Finitistic Dimension Conjecture:** $\text{fpd}(\Lambda) < +\infty$.

Over the years, the conjectures have been studied mainly in two different but closely related areas, namely commutative algebra and representation theory of Artin algebras. In commutative algebra, the conjectures are quite well-understood and they both fail in general: it is known through the works of Auslander and Buchsbaum [5], Bass [11], and Gruson and Raynaud [32] that if $\Lambda$ is a commutative noetherian local ring, then $\text{fpd}(\Lambda) < +\infty$ and the equality $\text{FPD}(\Lambda) = \text{fpd}(\Lambda)$ holds if and only if $\Lambda$ is Cohen-Macaulay. Furthermore, there are examples of commutative noetherian rings $\Lambda$ with $\text{fpd}(\Lambda) = +\infty$; see e.g. [31, page 276]. In representation theory of Artin algebras, the finitistic dimension conjectures are not as much well-understood as in the commutative setting: It is proved by Huisgen-Zimmermann [26] that the first finitistic dimension conjecture fails in general for a monomial-relation algebra and examples due to Smalø [34] show that the difference between the first and the second finitistic dimension can indeed be arbitrarily large. However, it is still of particular interest to know for which classes of algebras the equality $\text{FPD}(\Lambda) = \text{fpd}(\Lambda)$ holds. The second finitistic dimension conjecture is still open in general, but the conjecture is verified for many classes of algebras including algebras of finite representation type, monomial-relation algebras, radical square/cube zero algebras; see [27] for more information in this regard.

It is well-known that understanding the structure of modules in the classes $\mathcal{P}^\text{fin}_\infty$ and $\mathcal{P}_\infty$ provides insight to finitistic dimension conjectures, and for this purpose “approximation theory” and “tilting theory” turn out to be invaluable tools; see e.g. (1.17). The first result via this approach was obtained by Auslander and Reiten [6], who proved that “contravariant finiteness
of the class $P_{\infty}^{\text{fin}}$, referred to as the *Auslander–Reiten condition* from now on, is a sufficient condition for validity of the second finitistic dimension conjecture over an Artin algebra. This result was further strengthened or generalized later:

(I). It was proved by Huisgen-Zimmermann and Smalø [28] that contravariant finiteness of the class $P_{\infty}^{\text{fin}}$ actually implies that any module in $P_{\infty}$ is a direct limit of modules in $P_{\infty}^{\text{fin}}$, and as a result both finitistic dimension conjectures hold in this case, i.e. $\text{FPD}(\Lambda) = \text{fpd}(\Lambda) < +\infty$. Thus, Auslander–Reiten condition is actually a sufficient condition for validity of both finitistic dimension conjectures. The condition is not, however, necessary as examples due to Igusa, Smaø and Todorov [30] show.

(II). Trlifaj [35] proved, using tools of approximation theory of modules, that contravariant finiteness of the class $P_{\infty}^{\text{fin}}$ is still sufficient for validity of the second finitistic dimension conjecture over left artinian rings.

(III). In [4], Angeleri-Hügel and Trlifaj presented a somewhat more conceptual proof of the implication

$$P_{\infty}^{\text{fin}} \text{ is contravariantly finite} \implies \text{FPD}(\Lambda) = \text{fpd}(\Lambda) < +\infty$$

mentioned in (I) using tilting theory, by showing that the class $P_{\infty}^{\text{fin}}$ is contravariantly finite over an Artin algebra if and only if the cotorsion pair generated by $P_{\infty}^{\text{fin}}$ is induced by a finitely generated tilting module which renders the equality $\text{FPD}(\Lambda) = \text{fpd}(\Lambda) < +\infty$; see (1.17).

The above-mentioned results can be recapitulated in the following diagram:

The point of departure in the present work is that in studying finitistic dimensions, it is sometimes more convenient to look at some alternative classes of modules, other than the obvious classes $P_{\infty}^{\text{fin}}$ and $P_{\infty}$, and two such alternative classes are:

- the class $GP_{\infty}$ of modules of finite Gorenstein projective dimension, and
- the class $GP_{\infty}^{\text{fin}}$ of finitely generated modules of finite Gorenstein projective dimension.

These classes of modules are usually regarded as “Gorenstein counterparts” of $P_{\infty}$ and $P_{\infty}^{\text{fin}}$ in the so-called “Gorenstein homological algebra”, a branch of homological algebra where the focus is on studying “Gorenstein modules” and their respective dimensions; cf. [23], [18], [36]
and [29]. By a well-known result of Holm [25, Theorem 2.28], finitistic dimensions of a ring can be computed by modules in $\mathcal{GP}_\infty$ and $\mathcal{GP}^{\text{fin}}_\infty$. More precisely, for any ring $\Lambda$,

$$\text{FPD}(\Lambda) = \sup \left\{ \text{Gpd}_\Lambda(M) \mid M \in \mathcal{GP}_\infty \right\}$$

(2)

and if $\Lambda$ is left notherian, then we also have

$$\text{fpd}(\Lambda) = \sup \left\{ \text{Gpd}_\Lambda(M) \mid M \in \mathcal{GP}^{\text{fin}}_\infty \right\}.$$ 

(3)

To illustrate the point that using $\mathcal{GP}_\infty$ and $\mathcal{GP}^{\text{fin}}_\infty$ in place of the obvious classes $\mathcal{P}_\infty$ and $\mathcal{P}^{\text{fin}}_\infty$ is sometimes more convenient, note that validity of finitistic dimension conjectures for Iwanaga-Gorenstein rings follows almost immediately from (2) and (3), because Iwanaga-Gorenstein rings can be characterized in terms of global finiteness of Gorenstein projective dimension of modules [23, Corollary 12.3.2]; also cf. [2] and [12, Corollary 5.2].

**Main Problem and Summary of Results.** In view of the considerations above, it is natural to ask how contravariant finiteness of $\mathcal{GP}^{\text{fin}}_\infty$, regarded as the “Gorenstein counterpart” of the Auslander–Reiten condition, fits in with the implications mentioned in (1). More precisely, our goal in this paper is to investigate the relation between:

(a) contravariant finiteness of $\mathcal{GP}^{\text{fin}}_\infty$,

(b) contravariant finiteness of $\mathcal{P}^{\text{fin}}_\infty$, and

(c) validity of finitistic dimension conjectures.

It is easy to see in the first place that the condition (a) above implies validity of the second finitistic dimension conjecture over artin algebras, using Eq. (3) in conjunction with a well-known result of Auslander and Reiten [6, Proposition 3.8]; see also [38, Proposition 4.8]. Nevertheless, we provide a more complete picture by showing that “contravariant finiteness of $\mathcal{GP}^{\text{fin}}_\infty$” fits in with the implications in diagram (1) as follows:

\[
\begin{array}{c}
\text{FPD}(\Lambda) = \text{fpd}(\Lambda) < +\infty \\
\end{array}
\]

The above implications are proved in Section 2 after the preliminary Section 1. The implication

$$\mathcal{GP}^{\text{fin}}_\infty \text{ is contravariantly finite} \implies \mathcal{P}^{\text{fin}}_\infty \text{ is contravariantly finite}$$

is proved in Theorem (2.1) using tilting theory. Indeed, it is shown using Proposition (1.21) that when $\mathcal{GP}^{\text{fin}}_\infty$ is contravariantly finite, the cotorsion pair generated by $\mathcal{GP}^{\text{fin}}_\infty$ has a “tilting-like structure” in the sense of Definition (1.18), and the underlying tilting module can be taken
finitely generated which renders \( \mathcal{P}_\infty^{\text{fin}} \) to be contravariantly finite. The reverse implication, namely
\[
\mathcal{P}_\infty^{\text{fin}} \text{ is contravariantly finite } \implies \mathcal{G}\mathcal{P}_\infty^{\text{fin}} \text{ is contravariantly finite ,}
\]
it proved in Theorem (2.4) for Artin algebras over which the class \( \mathcal{G}\mathcal{P}_0^{\text{fin}} \) is contravariantly finite. Typical examples of such Artin algebras are CM-finite and virtually Gorenstein Artin algebras; cf. Remark (2.5). Thus, one can say that Auslander–Reiten condition and its Gorenstein counterpart are “almost equivalent”; cf. [38, Proposition 4.8]. In the end, we shift our focus from Artin algebras to the slightly more general setting of left artinian rings and we prove in Theorem (2.6) that contravariant finiteness of \( \mathcal{G}\mathcal{P}_\infty^{\text{fin}} \) still implies validity of the second finitistic dimension conjecture for left artinian rings. This can be regarded as the “Gorenstein counterpart” of the main result of [35].

1 Preliminaries

(1.1) General Notations, Notions and Conventions. Throughout the paper, by a “ring” we mean an arbitrary non-trivial unital ring. Such a ring is denoted by \( \Lambda \). We often assume that \( \Lambda \) is an Artin algebra which means that \( \Lambda \) is an algebra over a commutative artinian ring \( R \) and \( \Lambda \) is finitely generated as a module over \( R \). By a “module over \( \Lambda \)” or a “\( \Lambda \)-module” we always mean a left \( \Lambda \)-module. If \( P_* = \cdots \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \xrightarrow{o} \) is a projective resolution of a \( \Lambda \)-module \( M \), then for any \( i \geq 0 \) the module \( \text{Im}(f_i) \) is called the \( i \)-th syzygy module of \( M \) in the projective resolution \( P_* \). The class of all \( \Lambda \)-modules is denoted by \( \text{Mod}(\Lambda) \) and the class of strongly finitely presented modules, i.e. modules with a degreewise finitely generated projective resolution, is denoted by \( \text{mod}(\Lambda) \). Furthermore, given a class \( C \) of \( \Lambda \)-modules we let \( C^{\text{fin}} := C \cap \text{mod}(\Lambda) \).

For any integer \( n \geq 0 \), the class of \( \Lambda \)-modules of projective dimension at most \( n \) is denoted by \( \mathcal{P}_n \), and the class of all \( \Lambda \)-modules of finite projective dimension is denoted by \( \mathcal{P}_\infty \). These are typical examples of resolving classes of modules, that is by definition extension closed classes of modules containing \( \mathcal{P}_0 \) which are closed under kernels of epimorphisms. Dually, a class \( C \) of \( \Lambda \)-modules is called coresolving if it contains all injective modules, and it is closed under extensions and cokernels of monomorphisms. Resolving classes are always syzygy-closed in the sense that they contain syzygies of projective resolutions of their elements. The notions of a “(co)resolving class” and a “syzygy-closed class” can be defined within \( \text{mod}(\Lambda) \) with the obvious modifications.

Ext-perpendicular classes will be of frequent use in the sequel. For any class \( C \) of \( \Lambda \)-modules let
\[
\begin{align*}
C^\perp &:= \{ M \in \text{Mod}(\Lambda) \mid \text{Ext}_\Lambda^1(C, M) = o \text{ for all } C \in C \}, \\
\perp C &:= \{ M \in \text{Mod}(\Lambda) \mid \text{Ext}_\Lambda^1(M, C) = o \text{ for all } C \in C \}, \\
C^{\perp \infty} &:= \{ M \in \text{Mod}(\Lambda) \mid \text{Ext}_\Lambda^{\geq 1}(C, M) = o \text{ for all } C \in C \}, \\
\perp \infty C &:= \{ M \in \text{Mod}(\Lambda) \mid \text{Ext}_\Lambda^{\geq 1}(M, C) = o \text{ for all } C \in C \}.
\end{align*}
\]
For any module $M$ we let $M^{\perp} := \{M\}^{\perp}$ for simplicity, and a similar notation is adopted for other Ext-perpendicular classes of a singleton. It is easy to see that the class $C^{\perp} = C^{\perp \infty}$ holds provided that $C$ is syzygy-closed.

(1.2) Filtrations. A family $\{M_\alpha\}_{\alpha \leq \sigma}$ (indexed by an ordinal $\sigma$) is called a continuous chain if the inclusion $M_\alpha \subseteq M_{\alpha + 1}$ holds for any ordinal $\alpha < \sigma$, and the equality $M_\alpha = \bigcup_{\beta < \alpha} M_\beta$ holds for any limit ordinal $\alpha \leq \sigma$.

Given a class $C$ of $\Lambda$-modules, a $\Lambda$-module $M$ is called $C$-filtered if there is a continuous chain $\{M_\alpha\}_{\alpha \leq \sigma}$ of $\Lambda$-modules such that $M_0 = \emptyset$, $M_\sigma = M$, and the successive factors $M_{\alpha + 1}/M_\alpha$ are isomorphic to an element in $C$ for any $\alpha < \sigma$. In this case the family $\{M_\alpha\}_{\alpha \leq \sigma}$ is called a $C$-filtration of $M$ of the length $\sigma$, and a $C$-filtration is said to be finite if $\sigma$ is a finite ordinal, i.e. a natural number. In this case, the module $M$ is said to be finitely $C$-filtered.

Notation. Given a class $C$ of $\Lambda$-modules, the class of all $C$-filtered modules is denoted by $\text{Filt}(C)$, and the class of all finitely $C$-filtered modules is denoted by $\text{filt}(C)$.

The following lemma about the length of filtrations of finitely generated modules will be useful later.

(1.3) Lemma. Let $C$ be a class consisting of finitely presented $\Lambda$-modules. If $\{M_\alpha\}_{\alpha \leq \sigma}$ is a strict $C$-filtration (i.e. $M_\alpha \subsetneq M_{\alpha + 1}$ for every $\alpha < \sigma$) of a finitely generated $\Lambda$-module $M$, then $\sigma$ is finite.

Proof. For the sake of contradiction assume that $\sigma$ is infinite. Then we can write $\sigma = \tau + n$ where $\tau$ is a limit ordinal and $n \geq 0$ is a natural number. Since $M_\sigma = M$ is finitely generated and $C$ consists of finitely presented modules, one can deduce inductively that $M_\tau = \bigcup_{\alpha < \tau} M_\alpha$ is finitely generated. Thus there exists $\alpha < \tau$ such that $M_\alpha$ contains a finite generating set of $M_\tau$, i.e. $M_\tau \subseteq M_\alpha$ which is not possible as the $C$-filtration is strict.

As it was already mentioned in the introduction, the pivotal class of modules in this paper are modules of “finite Gorenstein projective dimension” which are modules finitely resolved by “Gorenstein projective modules”.

(1.4) Gorenstein Projective Modules. A $\Lambda$-module $M$ is called Gorenstein projective if it admits a complete projective resolution, that is an exact sequence

$$P_* = \cdots \longrightarrow P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} P_{-1} \longrightarrow \cdots$$

consisting of projective modules such that $M \cong \text{Ker}(f_0)$ and that the sequence remains exact under $\text{Hom}_\Lambda(-, P)$ for every $P \in P_0$. The class of Gorenstein projective $\Lambda$-modules is denoted by $\text{GP}$. Note that by the symmetry in the definition, all the modules $\text{Ker}(f_0)$ in the complete projective resolution $P_*$ are also Gorenstein projective. In the special case where all $f_i$ and $P_i$ are equal in $P_*$, the Gorenstein projective module $M$ is called strongly Gorenstein projective.

The importance of strongly Gorenstein projective modules lies in the following construction from [15]; see also [36, Theorem 11.1.12].
(1.5) **Construction.** Let $M$ be a Gorenstein projective $\Lambda$-module and

$$P_\bullet = \cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} P_{-1} \xrightarrow{d_{-1}} \cdots$$

be a complete projective resolution of $M$. Let $P := \bigoplus_{n \in \mathbb{Z}} P_n$ and $\partial : P \rightarrow P$ be the $\Lambda$-homomorphism induced by $d_i$, i.e. $\partial |_{P_i} = d_i$ for every $i \in \mathbb{Z}$. It is then easy to see (see [15, Theorem 2.7] or [36, Theorem 11.1.12]) that $S := \text{Ker}(\partial)$ is a strongly Gorenstein projective module containing $M$ as a direct summand.

(1.6) **Gorenstein Projective Dimension.** It is well-known that the class $\mathcal{GP}$ is resolving [25, Theorem 2.5] and hence one can define *Gorenstein projective dimension* of modules by resolving modules by the class $\mathcal{GP}$; cf. [25] and [23] for more information. The Gorenstein projective dimension of $\Lambda$-modules is denoted by $\text{Gpd}_\Lambda(-)$. For any integer $n \geq 0$ the class of $\Lambda$-modules of Gorenstein projective dimension at most $n$ is denoted by $\mathcal{GP}_n$. We also let $\mathcal{GP}_\infty$ be the class of $\Lambda$-modules of finite Gorenstein projective dimension.

(1.7) **Remark.** It is well-known that $\mathcal{GP}_n$ has two-of-three property [36, Section 11.3] and it is closed under filtrations [22, Theorem 3.4].

Modules of finite projective dimension are important partly because they behave more or less similarly to modules over rings of finite global dimension. On the other hand, the “Gorenstein version” of these modules, namely modules of finite Gorenstein projective dimension, have historically been studied as the class of modules which behave more or less similarly to modules over Gorenstein rings; cf. [19], [18] and [29].

The following classical result [36, Theorem 11.3] shows that Gorenstein projective dimension of modules in $\mathcal{GP}_\infty$ can be measured via vanishing of $\text{Ext}$-functors.

(1.8) **Theorem.** Let $\Lambda$ be a ring and $n \geq 0$ be an integer. The following statements are equivalent for any $M \in \mathcal{GP}_\infty$:

(i). $\text{Gpd}_\Lambda(M) \leq n$;

(ii). $\text{Ext}^{n+i}_\Lambda(M, N)$ for all $i \geq 1$ and $N \in \mathcal{P}_\infty$;

(iii). $\text{Ext}^{n+1}_\Lambda(M, N)$ for all $N \in \mathcal{P}_\infty$.

The following lemma will be used later in the proofs of a couple of results.

(1.9) **Lemma.** For any integer $n \geq 0$ and $M \in \mathcal{GP}_n^{\text{fin}}$, there exists a short exact sequence

$$0 \longrightarrow M \longrightarrow P \longrightarrow G \longrightarrow 0$$

where $P \in \mathcal{P}_n^{\text{fin}}$ and $G \in \mathcal{GP}_0^{\text{fin}}$.

**Proof.** We proceed by induction on $n$. For $n = 0$ there exists—essentially by definition of a Gorenstein projective module—a short exact sequence

$$0 \longrightarrow M \xrightarrow{u} F \xrightarrow{v} C \longrightarrow 0$$
of $\Lambda$-modules where $F$ is projective and $C$ is Gorenstein projective. Since $F$ is a direct summand of a free module, we may add a suitable projective summand to $F \xrightarrow{\nu} C$ so that we can assume $F$ is free. Now since $M$ is finitely generated, $u : M \to F$ factors through some finitely generated free $\Lambda$-module $\Lambda^n$ and thence we obtain a short exact sequence

$$0 \to M \xrightarrow{f} \Lambda^n \xrightarrow{g} N \to 0$$

wherein $N \in \mathcal{GP}_\infty$ by Remark (1.7), and $\text{Hom}_\Lambda(f, P)$ is surjective for each projective module $P$ because $\text{Hom}_\Lambda(u, P)$ was such. Consequently $\text{Ext}_\Lambda^1(N, P) = 0$ for every $P \in \mathcal{P}_0$ and then Theorem (1.8) yields $N \in \mathcal{GP}_0$. Assume now that $n \geq 1$ and the assertion holds for all modules in $\mathcal{GP}_{n-1}$. Consider a short exact sequence

$$0 \to M \to F \to N \to 0$$

where $F$ is free, and note that $M \in \mathcal{GP}_{n-1}$. Thus, by the inductive hypothesis, we have a short exact sequence

$$0 \to M \to Q \to H \to 0$$

with $Q \in \mathcal{P}_{n-1}^\text{fin}$ and $H \in \mathcal{GP}_{0}^\text{fin}$. Forming the pushout of the last two short exact sequences, we obtain a short exact sequence

$$0 \to Q \to U \to N \to 0$$

where $U \in \mathcal{GP}_{0}^\text{fin}$. Finally, we form one more pushout

where we use the already proved “case $n = 0$” to obtain the middle column with $G \in \mathcal{GP}_{0}^\text{fin}$. The short exact sequence on the right-hand side column is obviously the desired one. □

Next, we review some definitions and facts from approximation theory of modules; we refer to [24] for more information and proofs of the standard facts mentioned below.
(1.10) Approximations. Given a $\Lambda$-module $M$ and a class $C$ of $\Lambda$-modules, a $\Lambda$-homomorphism $f : C \rightarrow M$ with $C \in C$ is said to be a $C$-precover (of $M$) if any $\Lambda$-homomorphism $g : C' \rightarrow M$ with $C' \in C$ factors through $f$.

The class $C$ is called precovering if any $\Lambda$-module has a $C$-precover. The dual of the notion “precover” is called a preenvelope and subsequently we may speak of a preenveloping class of $\Lambda$-modules; cf. [24].

Approximation theory of modules, also known as the “theory of covers and envelopes”, originates in the work of Enochs [21] on torsion-free and flat covers of modules, and also earlier work of Auslander and Smalø [8, 9] in the realm of representation theory of Artin algebras. In the latter setting, the related notions are often confined to the class of finitely generated modules over Artin algebras, and in this setting the term contravariantly finite is often synonymously used instead of “precovering in $\text{mod}(\Lambda)$”.

A typical situation where a $\Lambda$-homomorphism $f : C \rightarrow M$ with $C \in C$ happens to be a $C$-precover is when $f$ is surjective and $\text{Ker}(f) \in C^\perp$. In this case, $f : C \rightarrow M$ is called a special $C$-precover (of $M$). Dually, a $\Lambda$-homomorphism $f : M \rightarrow C$ with $C \in C$ is called a special $C$-preenvelope (of $M$) if $f$ is injective and $\text{Coker}(f) \in C^\perp$. Accordingly, the class $C$ is called special precovering (respectively, special preenveloping) if any $\Lambda$-module has a special $C$-precover (respectively, special $C$-preenvelope).

As the following proposition shows, in the setting of finitely generated modules over Artin algebras, contravariantly finite classes with suitable closure properties actually provide for special precovers.

(1.11) Proposition. Let $\Lambda$ be an Artin algebra and $C$ be a resolving class in $\text{mod}(\Lambda)$. If $C$ is contravariantly finite, then $C$ is special precovering in $\text{mod}(\Lambda)$.

Proof. By the hypothesis each finitely generated $\Lambda$-module $M$ has a $C$-precover $f : C \rightarrow M$ in $\text{mod}(\Lambda)$ which is surjective because $\Lambda \in C$. Furthermore, $f$ can be taken to be “left minimal” in the sense that $f$ cannot factor through a $\Lambda$-homomorphism $\alpha : C \rightarrow C'$ unless $\alpha$ is an isomorphism; cf. [24, Corollary 5.10] or [7, Theorem 2.4]. Now Wakamatsu Lemma [24, Lemma 5.13] in $\text{mod}(\Lambda)$—see also [6, Lemma 1.3]—implies that such a left minimal $C$-precover in $\text{mod}(\Lambda)$ is a special $C$-precover in $\text{mod}(\Lambda)$.

A useful machinery to systematically detect or construct classes of modules which are special precovering or special preenveloping is the notion of a “cotorsion pair”.

(1.12) Cotorsion Pairs. A pair $(\mathcal{L}, \mathcal{R})$ of two classes of $\Lambda$-modules is said to be a cotorsion pair if $\mathcal{L} = \perp \mathcal{R}$ and $\mathcal{R} = \perp \mathcal{L}$. Given a class $C$ of $\Lambda$-modules, it is easily seen that $(\perp \mathcal{C}^\perp, \mathcal{C}^\perp)$ is a cotorsion pair called the cotorsion pair generated by $C$.

The components of a cotorsion pair are known to have some dual properties: A result known as the Rozas Lemma [24, Lemma 5.24] states that in a cotorsion pair $\mathcal{C} := (\mathcal{L}, \mathcal{R})$ the left-hand
side class $\mathcal{L}$ is resolving if and only if the right-hand side class $\mathcal{R}$ is coresolving. In this case
the cotorsion pair $\mathcal{C}$ is said to be hereditary. It is easy to see that every cotorsion pair generated
by a syzygy-closed class of modules is hereditary. Another duality result in cotorsion pairs is
Salce Lemma [24, Lemma 5.20] which states that in the cotorsion pair $\mathcal{C}$ the left-hand side class
$\mathcal{L}$ is special precovering if and only if the right-hand side class $\mathcal{R}$ is special preenveloping. In
this case the cotorsion pair $\mathcal{C}$ is said to be complete. The following key result due to Eklof and
Trlifaj [20] shows abundance of complete cotorsion pairs; cf. [24, Theorem 6.11].

(1.13) Theorem. If $\mathcal{S}$ is a set of $\Lambda$-modules, then for any $\Lambda$-module $M$ there exists a short exact
sequence
\[ o \rightarrow M \overset{f}{\rightarrow} N \overset{g}{\rightarrow} L \overset{h}{\rightarrow} o \]
where $N \in \mathcal{S}^\perp$ and $L$ is $\mathcal{S}$-filtered. In particular, $f$ is a special $\mathcal{S}^\perp$-preenvelope and the cotor-
sion pair generated by $\mathcal{S}$ is complete.

Theorem (1.13) provides in particular a relatively concrete description of modules in the
double-perpendicular class $\perp(\mathcal{S}^\perp)$ when $\mathcal{S}$ is a set.

Notation. For a class $\mathcal{C}$ of modules, $\text{add}_\Lambda(\mathcal{C})$ denotes the class of all direct summands of finite
direct sums of modules in $\mathcal{C}$.

(1.14) Corollary. Let $\mathcal{S}$ be a set of $\Lambda$-modules.

(i). the class $\perp(\mathcal{S}^\perp)$ consists precisely of direct summands of modules filtered by $\mathcal{S} \cup \{\Lambda\}$.

(ii). If $\mathcal{S}$ consists of finitely presented $\Lambda$-modules and $\Lambda \in \mathcal{S}$, then $\perp(\mathcal{S}^\perp) \cap \text{mod}(\Lambda) = \text{add}_\Lambda(\text{filt}(\mathcal{S}))$.

PROOF. Part (i) is a classical result, see e.g. [24, Corollary 6.13]. In order to prove part (ii),
notice first that the inclusion $\text{add}_\Lambda(\text{filt}(\mathcal{S})) \subseteq \perp(\mathcal{S}^\perp) \cap \text{mod}(\Lambda)$ can be proved readily—either
by a straightforward argument or using part (i)—and so it remains to prove the reverse inclusion.
If $M \in \perp(\mathcal{S}^\perp) \cap \text{mod}(\Lambda)$, then $M$ is a direct summand of some $\mathcal{S}$-filtered module $N$ by
part (i). Then it can be proved, say by Hill Lemma [24, Theorem 7.10], that we can replace
$N$ with a finitely presented module, and so we can take $N$ finitely $\mathcal{S}$-filtered by Lemma (1.3).
Consequently, $M \in \text{add}_\Lambda(\text{filt}(\mathcal{S}))$, and this finishes the proof. ■

The double perpendicular class $\perp(\mathcal{S}^\perp)$ assumes a simpler description over left artinian rings
as it turns out that in this case special precovers of simple modules are enough to determine the
structure of all the modules in $\perp(\mathcal{S}^\perp)$.

(1.15) Proposition. Let $\Lambda$ be a left artinian ring and $\mathcal{S}$ be a set $\Lambda$-modules. Let $\{S_1, \ldots, S_n\}$ be
a complete set of simple $\Lambda$-modules, and for every $1 \leq i \leq n$ choose a special $\perp(\mathcal{S}^\perp)$-precover
$A_i \rightarrow S_i$. Then any (finitely generated) $\Lambda$-module $M$ has special $\perp(\mathcal{S}^\perp)$-precover $A \rightarrow M$
where $A$ is (finitely) filtered by $\mathcal{C} = \{A_1, \ldots, A_n\}$. In particular, the class $\perp(\mathcal{S}^\perp)$ coincides with
the class of direct summands of $\mathcal{C}$-filtered modules.

PROOF. See [24, Corollary 17.19]. ■
The above results indicate the use of approximation theory in decoding structure of modules. Another useful tool serving this purpose is the notion of a “tilting module”.

**1.16 Tilting Modules.** Let $n \geq 0$ be an integer. A $\Lambda$-module $T$ is said to be an $n$-tilting module if it satisfies the following conditions:

1. $\text{pd}_\Lambda(T) \leq n$.
2. $\text{Ext}^n_\Lambda(T, T^{(\kappa)}) = 0$ for any integer $n \geq 1$ and any cardinal $\kappa$. Here $T^{(\kappa)}$ denotes the direct sum of $\kappa$ copies of $T$.
3. There exists an exact sequence

$$0 \rightarrow \Lambda \rightarrow T_0 \rightarrow \cdots \rightarrow T_m \rightarrow 0$$

where $T_i \in \text{Add}_\Lambda(T)$ for all $1 \leq i \leq m$. Here $\text{Add}_\Lambda(T)$ denotes the class of modules isomorphic to a direct summands of $T^{(\kappa)}$ for some cardinal $\kappa$.

In this case, the class $T^{\perp_\infty}$ is called the tilting class associated to $T$, and the complete and hereditary cotorsion pair $(\perp(T^{\perp_\infty}), T^{\perp_\infty})$ is called the tilting cotorsion pair induced by $T$. A cotorsion pair is said to be an $n$-tilting cotorsion pair if it is induced by an $n$-tilting module.

Tilting modules are the main objects of study in tilting theory with myriads of applications in representation theory of algebras [1] and structure theory of modules [24, Part III]. Applications of tilting modules to finitistic dimension conjectures were first observed in [4], where the authors prove, among other things, the following results; cf. [24, Chapter 17].

**1.17 Theorem.** Let $\mathcal{P}$ be the cotorsion pair generated by $\mathcal{P}_{\text{fin}}^{\infty}$.

(i). If $\Lambda$ is left noetherian, then $\text{fpd}(\Lambda) < +\infty$ if and only if the cotorsion pair $\mathcal{P}$ is tilting.

(ii). If $\Lambda$ is an Artin algebra, the class $\mathcal{P}_{\text{fin}}^{\infty}$ is contravariantly finite if and only if $\mathcal{P}$ is a tilting cotorsion pair induced by a finitely generated tilting $\Lambda$-module $T$. In this case, $\mathcal{P}_{\infty} = \perp(T^{\perp_\infty})$ and $\text{FPD}(\Lambda) = \text{fpd}(\Lambda) < +\infty$.

Coming back to our main problem, namely the relation between finitistic dimension conjectures and contravariant finiteness of the class $\mathcal{GP}_{\infty}^{\text{fin}}$, it is natural to consider the cotorsion pair $\mathcal{G}$ generated by the class $\mathcal{GP}_{\infty}^{\text{fin}}$ and employ ideas parallel to [4], in particular Theorem (1.17) mentioned above, to gain insight. This approach, however, does not work directly as the cotorsion pair $\mathcal{G}$ cannot be tilting except when $\mathcal{GP}_{0}^{\text{fin}} = \mathcal{P}_{0}^{\text{fin}}$—Artin algebras with this property are known as CM-free in the literature; see e.g. [17]. The remedy to this obstacle is the observation that although the cotorsion pair $\mathcal{G}$ is almost never tilting, it still has a “tilting-like structure” in the sense defined below; see Theorem (2.1).

**1.18 Definition.** A cotorsion pair $(\mathcal{L}, \mathcal{R})$ is said to be $n$-tilting-like (for some integer $n \geq 0$) if $\mathcal{R} = (T \oplus S)^{\perp_\infty}$, where $T$ is a tilting $\Lambda$-module and $S$ is a strongly Gorenstein projective module. Needless to say, for $S = 0$ we recover the tilting cotorsion pair induced by $T$.  

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The following proposition says that in a tilting-like cotorsion pair \((\mathcal{L}, \mathcal{R})\) with the underlying tilting module \(T\), the class \(\perp(T^{\perp\infty})\) coincides with the subclass of \(\mathcal{L}\) consisting of modules of finite projective dimension.

(1.19) Proposition. Let \(U := T \oplus S\) where \(T\) is an \(n\)-tilting \(\Lambda\)-module and \(S\) is a strongly Gorenstein projective module. If \((\mathcal{L}_U, \mathcal{R}_U)\) is the cotorsion pair with \(\mathcal{R}_U = U^{\perp\infty}\), then \(\mathcal{L}_U \cap \mathcal{P}_n = \perp(T^{\perp\infty})\).

Proof. The inclusion \(\perp(T^{\perp\infty}) \subseteq \mathcal{L}_U \cap \mathcal{P}_\infty\) holds because \(T \in \mathcal{L}_U \cap \mathcal{P}_n\). As for the reverse inclusion, let \(M \in \mathcal{L}_U \cap \mathcal{P}_n\) and \(A \in T^{\perp\infty}\). By [24, Proposition 13.13] the module \(A\) has an \(\text{Add}_\Lambda(T)\)-resolution

\[
\cdots \to T_1 \xrightarrow{d_1} T_0 \xrightarrow{d_0} A \to 0
\]

Using condition (T2) in (1.16) and Theorem (1.8) it is easy to see that the modules \(T_i\) in the above sequence belong to \(\mathcal{R}_U\) and so \(\text{Ext}_\Lambda^n(M, T_i) = 0\) for every \(i \geq 0\). Now since \(M \in \mathcal{P}_n\), it follows from dimension shifting that

\[
\text{Ext}_\Lambda^1(M, A) \cong \text{Ext}_\Lambda^{n+1}(M, \text{Im}(d_n)) = 0.
\]

Consequently, \(M \in \perp(T^{\perp\infty})\) and this finishes the proof. ■

Tilting-like cotorsion pairs have recently been studied in [37], where the authors prove the following characterization theorem.

(1.20) Theorem ([37, Theorem 1.1]). Let \(\mathcal{C} := (\mathcal{L}, \mathcal{R})\) be a hereditary cotorsion pair generated by a set \(S\) of \(\Lambda\)-modules. The following statements are equivalent for the cotorsion pair \(\mathcal{C}\):

(i). \(\mathcal{C}\) is \(n\)-tilting like.

(ii). There is an \(n\)-tilting module such that \(\mathcal{L} \cap \mathcal{R} = \text{Add}_\Lambda(T)\).

(iii). \(\mathcal{L} \cap \mathcal{R}\) is closed under direct sums and there exists a strongly Gorenstein projective module \(S \in \mathcal{L}\) that contains some \(n\)-th syzygy module of every module in \(S\) as a direct summand.

(iv). \(\mathcal{L} \subseteq \mathcal{GP}_n\), \(\mathcal{L} \cap \mathcal{R} \subseteq \mathcal{P}_n\) and \(\mathcal{L} \cap \mathcal{R}\) is closed under direct sums.

We will use the following instance of Theorem (1.20) in the next section to prove one of our main theorems, namely Theorem (2.1).

(1.21) Proposition. Let \(\Lambda\) be a ring and \((\mathcal{L}, \mathcal{R})\) be the cotorsion pair generated by a syzygy-closed subclass \(\mathcal{C}\) of \(\text{mod}(\Lambda)\). If \(\mathcal{GP}^{\text{fin}}_n \subseteq \mathcal{C} \subseteq \mathcal{GP}^{\text{fin}}_0\) for some integer \(n \geq 0\), then there exists an \(n\)-tilting module \(T\) and a strongly Gorenstein projective module \(S\) such that \(\mathcal{R} = (T \oplus S)^{\perp\infty}\). If furthermore \(\Lambda\) is an Artin algebra and \(\mathcal{C}^{\text{fin}}\) is contravariantly finite, then the tilting module \(T\) can be taken in \(\text{mod}(\Lambda)\).

Proof. Replacing \(\mathcal{C}\) with a representative set of its element we may assume from the outset that \(\mathcal{C}\) is a set. Since \(\mathcal{C}\) consists of strongly finitely presented modules, the class \(\mathcal{R} = \mathcal{C}^{\perp}\) and hence \(\mathcal{L} \cap \mathcal{R}\) is closed under direct sums. Furthermore, for every \(M \in \mathcal{C}\) it follows from Lemma (1.9)
that any $n$-th syzygy module of $M$ has a complete projective resolution whose cycles lie in $GP_{0}^{\infty} \subseteq L$. Thus it follows from Construction (1.5) that there exists a strongly Gorenstein projective module $S_{M} \in L$ which contains an $n$-th syzygy module of $M$ as a direct summand. Therefore, it follows from Theorem (1.20) that $R = (T \oplus S)^{\perp_{\infty}}$ for some $n$-tilting module $T$ and strongly Gorenstein projective module $S$.

As for the second part of the assertion, note that the tilting module $T$ is constructed in general as follows (cf. proof of part (2) $\Rightarrow$ (3) in [37, Theorem 1.1]): By considering iterated special $R$-preenvelopes of $\Lambda$ one can construct an exact sequence

$$0 \rightarrow \Lambda \rightarrow T_{0} \rightarrow \cdots \rightarrow T_{n} \rightarrow 0$$

where $T_{i} \in L \cap R$, and it then can be proved that $T := \bigoplus_{i=0}^{n} T_{i}$ is the desired tilting module. Now if $L^{\text{fin}}$ is contravariantly finite, then it follows from [3, Theorem 5.3] that $(L^{\text{fin}}, R^{\text{fin}})$ is a complete cotorsion pair in $\text{mod}(\Lambda)$. So one can repeat the construction process of $T$ using iterated special $R^{\text{fin}}$-preenvelopes of $\Lambda$ inside $\text{mod}(\Lambda)$ and thereby $T$ can be taken finitely generated.

\section{Contravariant finiteness of $GP_{\infty}^{\text{fin}}$}

In this section we prove our main results about the relation between contravariant finiteness of $GP_{\infty}^{\text{fin}}$ and finitistic dimension conjectures, advertised earlier in the introduction. We start with the following theorem which states that the Gorenstein version of Auslander–Reiten condition actually implies the usual Auslander–Reiten condition; compare [38, Proposition 4.8].

\textbf{(2.1) Theorem.} Let $\Lambda$ be an Artin algebra and consider the following statements about the cotorsion pair $(L, R)$ generated by a syzygy-closed class $S \subseteq \text{mod}(\Lambda)$:

(i). $L^{\text{fin}}$ is contravariantly finite;

(ii). $R = (T \oplus S)^{\perp_{\infty}}$ for some finitely generated $n$-tilting $\Lambda$-module $T$ and some strongly Gorenstein projective module $S$.

(iii). $L^{\text{fin}} \cap P_{\infty}$ is contravariantly finite.

If $GP_{0}^{\infty} \subseteq S \subseteq GP_{\infty}^{\text{fin}}$, then (i) $\implies$ (ii) $\implies$ (iii). In particular, contravariant finiteness of $GP_{\infty}^{\text{fin}}$ implies contravariant finiteness of $P_{\infty}^{\text{fin}}$.

\textbf{Proof.} Let $\{S_{1}, \ldots, S_{m}\}$ be a complete set of simple $\Lambda$-modules and for every $1 \leq i \leq m$ let $A_{i} \rightarrow S_{i}$ be a special $(\perp)$-precover of $S_{i}$. Let $n := \sup \{\text{Gpd}_{\Lambda}(A_{i}) \mid 1 \leq i \leq m\}$. Since the class $GP_{n}$ is closed under filtrations (1.7), it follows from Proposition (1.15) that $S \subseteq L \subseteq GP_{n}$. Now the implication (i) $\implies$ (ii) follows from Proposition (1.21). As for the implication (ii) $\implies$ (iii), it follows from Proposition (1.19) that $L^{\text{fin}} \cap P_{\infty} = (T^{\perp_{\infty}}) \cap \text{mod}(\Lambda)$, and it is well-known from classical tilting theory that this class is contravariantly finite; cf. [24, Lemma 17.26].
Next we are going to prove in Theorem (2.4) the converse of the implication
\[ \mathcal{GP}^\text{fin}_\infty \text{ is contravariantly finite} \implies \mathcal{P}^\text{fin}_\infty \text{ is contravariantly finite} \]
for Artin algebras over which \( \mathcal{GP}^\text{fin}_0 \) is contravariantly finite; cf. (2.5). The key step in the proof is the observation that modules in the class \( \mathcal{GP}^\text{fin}_n \) are precisely the modules obtained as an extension of a module in \( \mathcal{GP}^\text{fin}_0 \) by a module in \( \mathcal{P}^\text{fin}_n \), and that glueing the two classes by extension preserves contravariant finiteness; see (2.2). In order to precisely state and prove these considerations, we require some preparatory results.

**Notation.** Let \( (\mathcal{U}, \mathcal{V}) \) be a pair of subclasses of \( \text{Mod}(\Lambda) \). Denote by \( \mathcal{U} \star \mathcal{V} \) the class of all modules \( M \) which sit in a short exact sequence of the form
\[ o \longrightarrow U \longrightarrow M \longrightarrow V \longrightarrow o \]
where \( U \in \mathcal{U} \) and \( V \in \mathcal{V} \). In other words, \( \mathcal{U} \star \mathcal{V} \) is the class of all modules which are an extension of a module in \( \mathcal{U} \) by a module in \( \mathcal{V} \).

The importance of the operation “\( \star \)” for our purposes lies in the following fact from [33]; see also [16].

**Remark.** Let \( \Lambda \) be a ring. If \( \mathcal{U} \) and \( \mathcal{V} \) are precovering (resp., preenveloping) classes in \( \text{Mod}(\Lambda) \), then the class \( \mathcal{U} \star \mathcal{V} \) is also precovering (resp., preenveloping) in \( \text{Mod}(\Lambda) \), and this statement remains valid if we replace \( \text{Mod}(\Lambda) \) with \( \text{mod}(\Lambda) \) in case \( \Lambda \) is an Artin algebra.

It is clear from the definition that \( \mathcal{U} \star \mathcal{V} \subseteq \text{filt}(\mathcal{U} \cup \mathcal{V}) \), and the following lemma provides a sufficient condition for the equality.

**Lemma.** Let \( \Lambda \) be a ring and \( (\mathcal{U}, \mathcal{V}) \) be a pair of extension closed subclasses of \( \text{Mod}(\Lambda) \) which contain the zero module. If \( \mathcal{U} \subseteq \mathcal{V} \), then \( \mathcal{U} \star \mathcal{V} = \text{filt}(\mathcal{U} \cup \mathcal{V}) \).

**Proof.** We need to show that if \( M \) is a finitely \( (\mathcal{U} \cup \mathcal{V}) \)-filtered \( \Lambda \)-module, then there exists a short exact sequence
\[ o \longrightarrow U \longrightarrow M \longrightarrow V \longrightarrow o \]
where \( U \in \mathcal{U} \) and \( V \in \mathcal{V} \). By the hypothesis, there exists a finite \( (\mathcal{U} \cup \mathcal{V}) \)-filtration
\[ o = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_k = M. \]

We prove the assertion by induction on \( k \), the length of the filtration: For \( k = 1 \) the assertion holds trivially. Assume that \( k > 1 \) and that the assertion holds for all modules with \( (\mathcal{U} \cup \mathcal{V}) \)-filtration of the length \( k - 1 \). Then we have a short exact sequence
\[ o \longrightarrow L \overset{f}{\longrightarrow} M \longrightarrow N \longrightarrow o \]
where \( L \) has a \( (\mathcal{U} \cup \mathcal{V}) \)-filtration of the length \( k - 1 \) and \( N \in \mathcal{U} \cup \mathcal{V} \). By the induction hypothesis, the module \( L \) sits in a short exact sequence
\[ o \longrightarrow U \overset{g}{\longrightarrow} L \longrightarrow V \longrightarrow o, \]
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where $U \in \mathcal{U}$ and $V \in \mathcal{V}$. Thus we get the short exact sequence

$$0 \to U \xrightarrow{f \circ g} M \xrightarrow{h} W \to 0,$$

wherein $W$ sits in a short exact sequence of the form

$$0 \to V \to W \to N \to 0.$$  \hspace{1cm} (***)

Now if $N \in \mathcal{V}$, then $W \in \mathcal{V}$ and (***) is the desired short exact sequence. Otherwise, $N \in \mathcal{U}$ and so the short exact sequence (**) splits. In this case, we have also the split short exact sequence

$$0 \to N \to W \xrightarrow{\alpha} V \to 0,$$

with which we can form the short exact sequence

$$0 \to K \to M \xrightarrow{\alpha \circ h} V \to 0,$$

wherein $K$ sits in a short exact sequence of the form

$$0 \to U \to K \to N \to 0.$$  \hspace{1cm} (†)

and hence it belong to $\mathcal{U}$. Now since $\mathcal{U}$ is closed under extensions, we have $K \in \mathcal{U}$ and so the short exact sequence (†) shows that $M \in \mathcal{U} \ast \mathcal{V}$. The proof is thus complete. \hfill ■

We are now ready to prove our second main theorem.

(2.4) Theorem. If $\Lambda$ is an Artin algebra, then:

(i). For any integer $n \geq 0$ the equality $\mathcal{GP}_n^{\text{fin}} = \text{add}(\mathcal{GP}_0^{\text{fin}} \ast \mathcal{P}_n^{\text{fin}})$ holds. In particular, if $\mathcal{P}_n^{\text{fin}}$ and $\mathcal{GP}_0^{\text{fin}}$ are contravariantly finite, then $\mathcal{GP}_n^{\text{fin}}$ is contravariantly finite.

(ii). If $\mathcal{GP}_0^{\text{fin}}$ and $\mathcal{P}_\infty$ are contravariantly finite, then $\mathcal{GP}_\infty^{\text{fin}}$ is contravariantly finite.

Proof. Part (i): Since the class $\mathcal{GP}_n^{\text{fin}} \subseteq \text{add}(\mathcal{GP}_0^{\text{fin}} \ast \mathcal{P}_n^{\text{fin}})$ holds by part (ii) of Corollary (1.14). On the other hand, we have the inclusion $\mathcal{GP}_0^{\text{fin}} \subseteq \mathcal{GP}_n^{\text{fin}}$ by Theorem (1.8) and so we obtain the equality $\mathcal{GP}_n^{\text{fin}} \ast \mathcal{P}_n^{\text{fin}} = \text{filt}(\mathcal{GP}_0^{\text{fin}} \cup \mathcal{P}_n^{\text{fin}})$ by Lemma (2.3). Therefore, $\mathcal{GP}_n^{\text{fin}} = \text{add}(\mathcal{GP}_0^{\text{fin}} \ast \mathcal{P}_n^{\text{fin}})$. Furthermore, this equality in conjunction with Proposition (2.2) implies that $\mathcal{GP}_n^{\text{fin}}$ is contravariantly finite provided that $\mathcal{P}_n^{\text{fin}}$ and $\mathcal{GP}_0^{\text{fin}}$ are contravariantly finite.

Part (ii): It is well-known that contravariant finiteness of $\mathcal{P}_\infty^{\text{fin}}$ implies $n := \text{fpd}(\Lambda) < +\infty$; cf. (1.15) or [6]. Thus, $\mathcal{P}_\infty^{\text{fin}} = \mathcal{P}_n^{\text{fin}}$ and $\mathcal{GP}_\infty^{\text{fin}} = \mathcal{GP}_n^{\text{fin}}$ by Eq. (3). Now by part (i) of the theorem we have the equality $\mathcal{GP}_\infty^{\text{fin}} = \text{add}(\mathcal{GP}_0^{\text{fin}} \ast \mathcal{P}_\infty^{\text{fin}})$, which in conjunction with Proposition (2.2) implies contravariant finiteness of $\mathcal{GP}_\infty^{\text{fin}}$. \hfill ■

(2.5) Remark. It follows from part (ii) of Theorem (2.4) that for Artin algebras over which the class $\mathcal{GP}_0^{\text{fin}}$ is contravariantly finite, the following conditions are equivalent:
• contravariant finiteness of $\mathcal{P}_\infty^{\text{fin}}$;
• contravariant finiteness of $\mathcal{GP}_\infty^{\text{fin}}$.

Typical examples of Artin algebras over which the class $\mathcal{GP}_0^{\text{fin}}$ is contravariantly finite are the so-called CM-finite algebras and virtually Gorenstein Artin algebras [13, 14]. By [13] the latter class of algebras include algebras which are derived equivalent, or stably equivalent of Morita type, with Artin algebras of finite representation type or Gorenstein Artin algebras.

Finally, we shift our focus from Artin algebras to the slightly more general setting of left artinian rings and prove in the following theorem that in parallel to [35], contravariant finiteness of $\mathcal{GP}_\infty^{\text{fin}}$ is still a sufficient condition for validity of the second finitistic dimension conjecture.

(2.6) Theorem. Let $\Lambda$ be a left artinian ring and $(A, B)$ be the cotorsion pair generated by $\mathcal{GP}_\infty^{\text{fin}}$. Let $\{S_1, \ldots, S_n\}$ be a complete set of simple $\Lambda$-modules and for any $1 \leq i \leq n$ pick a special $A$-precover $A_i \to S_i$. Then:

(i). $\text{fpd}(\Lambda) = \sup \{\text{Gpd}_{\Lambda}(A_i) \mid 1 \leq i \leq n\}$.

(ii). $\mathcal{GP}_\infty^{\text{fin}}$ is contravariantly finite if and only if the modules $A_i$ can be taken finitely generated for every $1 \leq i \leq n$. In this case, $\text{fpd}(\Lambda) < +\infty$.

Proof. Part (i): If $m := \sup \{\text{Gpd}_{\Lambda}(A_i) \mid 1 \leq i \leq n\} < +\infty$, then $\mathcal{GP}_\infty^{\text{fin}} \subseteq \mathcal{A} \subseteq \mathcal{GP}_m$ by Proposition (1.15) and the fact that $\mathcal{GP}_m$ is closed under filtrations (1.7). Therefore, $\text{fpd}(\Lambda) \leq m$ by Eq.(3). If, on the other hand, $m := \text{fpd}(\Lambda) < +\infty$, then $\mathcal{GP}_\infty^{\text{fin}} = \mathcal{GP}_m$ by Eq. (3) and since $\mathcal{GP}_m$ is closed under filtrations (1.7) it follows that $\mathcal{A} \subseteq \mathcal{GP}_m$. Therefore, $\sup \{\text{Gpd}_{\Lambda}(A_i) \mid 1 \leq i \leq n\} \leq m$.

Part (ii): If $\mathcal{GP}_\infty^{\text{fin}}$ is contravariantly finite, then each $S_i$ has special $\mathcal{GP}_\infty^{\text{fin}}$-precover $f : A_i \to S_i$ in $\text{mod}(\Lambda)$ by (1.11) and so $\text{Ker}(f) \in (\mathcal{GP}_\infty^{\text{fin}})^{\perp} \cap \text{mod}(\Lambda) \subseteq B$. Therefore, $f : A_i \to S_i$ is a special $A$-precover with $A_i$ finitely generated. Conversely, if each $A_i$ is finitely generated, then it follows from Corollary (1.14) that $\mathcal{A}^{\text{fin}} = \mathcal{GP}_\infty^{\text{fin}}$. In particular each $A_i$ belongs to $\mathcal{GP}_\infty^{\text{fin}}$ and then it follows from Proposition (1.15) that $\mathcal{GP}_\infty^{\text{fin}}$ is contravariantly finite.

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References

[1] L. Angeleri-Hügel, D. Happel, and H. Krause. Handbook of Tilting Theory, volume 13. Cambridge University Press, 2007.
[2] L. Angeleri-Hügel, D. Herbera, and J. Trlifaj. Tilting modules and Gorenstein rings. In *Forum Mathematicum*, volume 18, pages 211–229. Walter de Gruyter, 2006.

[3] L. Angeleri-Hügel, J. Šaroch, and J. Trlifaj. On the telescope conjecture for module categories. *Journal of Pure and Applied Algebra*, 212(2):297–310, 2008.

[4] L. Angeleri-Hügel and J. Trlifaj. Tilting theory and the finitistic dimension conjectures. *Transactions of the American Mathematical Society*, 354(11):4345–4358, 2002.

[5] M. Auslander and D. A. Buchsbaum. Homological dimension in local rings. *Transactions of the American Mathematical Society*, 85(2):390–405, 1957.

[6] M. Auslander and I. Reiten. Applications of contravariantly finite subcategories. *Advances in Mathematics*, 86(1):111–152, 1991.

[7] M. Auslander, I. Reiten, and S. O. Smalø. *Representation theory of Artin algebras*, volume 36. Cambridge University Press, 1997.

[8] M. Auslander and S. O. Smalø. Preprojective modules over Artin algebras. *Journal of algebra*, 66(1):61–122, 1980.

[9] M. Auslander and S. O. Smalø. Almost split sequences in subcategories. *Journal of Algebra*, 69(2):426–454, 1981.

[10] H. Bass. Finitistic dimension and a homological generalization of semi-primary rings. *Transactions of the American Mathematical Society*, 95(3):466–488, 1960.

[11] H. Bass. Injective dimension over noetherian rings. *Transactions of the American Mathematical Society*, 102(1):18–29, Jan. 1962.

[12] S. Bazzoni, P. C. Eklof, and J. Trlifaj. Tilting cotorsion pairs. *Bulletin of the London Mathematical Society*, 37(5):683–696, 2005.

[13] A. Beligiannis. Cohen-Macaulay modules,(co)torsion pairs and virtually Gorenstein algebras. *Journal of Algebra*, 288(1):137–211, 2005.

[14] A. Beligiannis and I. Reiten. *Homological and homotopical aspects of torsion theories*. Mem. Amer. Math. Soc., 2007.

[15] D. Bennis and N. Mahdou. Strongly Gorenstein projective, injective, and flat modules. *Journal of Pure and Applied Algebra*, 210(2):437–445, 2007.

[16] X.-W. Chen. Extensions of covariantly finite subcategories. *Archiv der Mathematik*, 93(1):29–35, jun 2009.

[17] X.-W. Chen. Gorenstein homological algebra of Artin algebras. *arXiv preprint arXiv:1712.04587*, 2017.

[18] L. W. Christensen. *Gorenstein dimensions*. Number 1747. Springer Science & Business Media, 2000.
[19] L. W. Christensen, H.-B. Foxby, and H. Holm. Beyond totally reflexive modules and back. In *Commutative Algebra*, pages 101–143. Springer, 2011.

[20] P. C. Eklof and J. Trlifaj. How to make Ext vanish. *Bulletin of the London Mathematical Society*, 33(1):41–51, 2001.

[21] E. E. Enochs. Injective and flat covers, envelopes and resolvents. *Israel Journal of Mathematics*, 39(3):189–209, 1981.

[22] E. E. Enochs, A. Iacob, and O. M. G. Jenda. Closure under transfinite extensions. *Illinois Journal of Mathematics*, 51(2):561–569, 2007.

[23] E. E. Enochs and O. M. G. Jenda. *Relative Homological Algebra*, volume I of *De Gruyter Expositions in Mathematics*. De Gruyter, 2011.

[24] R. Göbel and J. Trlifaj. *Approximations and Endomorphism Algebras of Modules: Volume 1–Approximations/Volume 2–Predictions*, volume 41. Walter de Gruyter, 2012.

[25] H. Holm. Gorenstein homological dimensions. *Journal of pure and applied algebra*, 189(1):167–193, 2004.

[26] B. Huisgen-Zimmermann. Homological domino effects and the first finitistic dimension conjecture. *Inventiones mathematicae*, 108(1):369–383, 1992.

[27] B. Huisgen-Zimmermann. *The Finitistic Dimension Conjectures—A Tale of 3.5 Decades*, pages 501–517. Springer Netherlands, Dordrecht, 1995.

[28] B. Huisgen-Zimmermann and S. Smalø. A homological bridge between finite and infinite-dimensional representations of algebras. *Algebras and Representation Theory*, 1(2):169–188, 1998.

[29] A. Iacob. *Gorenstein Homological Algebra*. CRC Press, 2018.

[30] K. Igusa, S. O. Smalø, and G. Todorov. Finite projectivity and contravariant finiteness. *Proceedings of the American Mathematical Society*, 109(4):937–941, 1990.

[31] J. McConnell, J. Robson, and L. Small. *Noncommutative Noetherian Rings*. Graduate studies in mathematics. American Mathematical Society, 2001.

[32] M. Raynaud and L. Gruson. Critères de platitude et de projectivité. *Inventiones mathematicae*, 13(1-2):1–89, 1971.

[33] S. A. Sikko and S. O. Smalø. Extensions of homologically finite subcategories. *Archiv der Mathematik*, 60(6):517–526, 1993.

[34] S. O. Smalø. Homological differences between finite and infinite dimensional representations of algebras. In *Infinite length modules*, pages 425–439. Springer, 2000.

[35] J. Trlifaj. Approximations and the little finitistic dimension of artinian rings. *Journal of Algebra*, 246(1):343–355, 2001.
[36] F. Wang and H. Kim. *Foundations of Commutative Rings and Their Modules*. Algebra and Applications. Springer Singapore, 2017.

[37] J. Wang, Y. Li, and J. Hu. When the kernel of a complete hereditary cotorsion pair is the additive closure of a tilting module. *Journal of Algebra*, 530:94–113, 2019.

[38] C. Xi. On the finitistic dimension conjecture, III: Related to the pair $eAe \subseteq A$. *Journal of Algebra*, 319(9):3666 – 3688, 2008.