A CONTRACTIBLE LEVI-FLAT HYPERSURFACE WHICH IS A DETERMINING SET FOR PLURIHARMONIC FUNCTIONS

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Abstract. We find a real analytic Levi-flat hypersurface in $\mathbb{C}^2$ containing a bounded contractible domain which is a determining set for pluriharmonic functions.

1. The main result

A real hypersurface $M$ in an $n$-dimensional complex manifold is Levi-flat if it is foliated by complex manifolds of dimension $n-1$; this Levi foliation is as smooth as $M$ itself according to Barrett and Fornæss [2]. If $M$ is real analytic, it is locally near every point defined by a pluriharmonic function $v$: $dd^c v = 2i\partial\bar{\partial} v = 0$. One might expect that an oriented, real analytic, Levi flat hypersurface admits a pluriharmonic defining function on any topologically simple relatively compact domain, perhaps under an additional analytic assumption such as the existence of a fundamental system of Stein neighborhoods (see e.g. Theorem 2 in [10], p. 298). Here we show that, on the contrary, even a most simple domain in a real analytic Levi flat hypersurface may be a determining set for pluriharmonic functions.

Theorem 1.1. There exist an ellipsoid $B \subset \mathbb{C}^2$ and a real analytic, Levi-flat hypersurface $M \subset \mathbb{C}^2$ intersecting the boundary $bB$ transversely such that the Levi foliation of $M$ has trivial holonomy and $A = M \cap B$ satisfies the following:

(i) $A$ is diffeomorphic to the three-ball and admits a Stein neighborhood basis.
(ii) Any real analytic function on $A$ which is constant on Levi leaves is constant.
(iii) Any pluriharmonic function in a connected open neighborhood of $A$ in $\mathbb{C}^2$ which vanishes on $A$ is identically zero.

The Levi foliation of $M$ in our proof is a simple foliation ([5], p. 79) whose leaves are complex discs. Likely one can also obtain a similar example in the ball of $\mathbb{C}^2$. On the other hand, for any compact subset $A$ in a real analytic, simply connected Levi-flat hypersurface $M$ there is a smooth defining function $v$ for $M$ whose pluricomplex Laplacian $dd^c v$ is flat on $A$; this suffices for the construction of Stein neighborhood basis of certain compact subsets of $M$.

We mention that D. Barrett gave an example of a compact real analytic Levi-flat hypersurface with trivial holonomy and without a global pluriharmonic defining function (Theorem 3 in [11]). His example is the quotient of $S^1 \times \mathbb{C}^*$ by

\[dd^c v = 2i\partial\bar{\partial} v = 0.\]
$(\theta, z) \to (\phi(\theta), 2z)$ where $\phi$ is a real analytic diffeomorphism of the circle $S^1$ which is topologically but not diffeomorphically conjugate to a rotation.

2. A REAL ANALYTIC FOLIATION OF $\mathbb{R}^2$ WITHOUT ANALYTIC FIRST INTEGRALS

Our construction of the hypersurface $M$ in theorem 1.1 is based on the following.

**Proposition 2.1.** Let $D$ be the open unit disc in $\mathbb{R}^2$. There exists a real analytic foliation $\mathcal{F}$ of $\mathbb{R}^2$ by closed lines such that any real analytic function on $D$ which is constant on every leaf of the restricted foliation $\mathcal{F}|_D$ is constant.

**Remark 2.2.** While we cannot exclude the possibility that an example of this kind is contained in the vast literature on the subject, we could not find a precise reference in some of the standard sources concerning foliations of the plane ([S], [7], [5], [6], [3]). It is known that every smooth foliation of $\mathbb{R}^2$ by lines has a global continuous first integral but in general not one of class $C^1$, not even in the analytic case (Wazewsky [11]); however, there exists a smooth first integral without critical points on any relatively compact subset (Kamke [9]).

**Proof.** Let $(x, y)$ be coordinates on $\mathbb{R}^2$. Define subsets $E_1, E_2 \subset \mathbb{R}^2$ by

$$E_1 = \{(x, y) \in \mathbb{R}^2 : x < -1 \text{ or } y > 0\}, \quad E_2 = \{(x, y) \in \mathbb{R}^2 : x > 1 \text{ or } y > 0\}.$$ 

Let $\mathcal{F}_j$ denote the restriction of the foliation $\{y = c\}_{c \in \mathbb{R}}$ to $E_j$ ($j = 1, 2$). Let $\psi$ be a real analytic orientation preserving diffeomorphism of the half line $(0, +\infty)$, so $\lim_{t \to 0} \psi(t) = 0$. (We do not require that $\psi$ extends analytically to a neighborhood of 0.) Then $\phi(x, y) = (x, \psi(y))$ is a real analytic diffeomorphism of the upper half plane $E_{1,2} = E_1 \cap E_2 = \{(x, y) \in \mathbb{R}^2 : y > 0\}$ onto itself which maps every leaf of $\mathcal{F}_1|_{E_{1,2}}$ to a leaf of $\mathcal{F}_2|_{E_{1,2}}$. Let $E$ be the quotient of the topological (disjoint) sum $E_1 \cup E_2$ obtained by identifying a point $(x, y) \in E_1$ belonging to $E_{1,2}$ with the point $\phi(x, y) \in E_2$. The foliations $\mathcal{F}_j$ ($j = 1, 2$) amalgamate into a real analytic foliation $\mathcal{F}$ on $E$.

By construction $E$ is a real analytic manifold homeomorphic to $\mathbb{R}^2$, and hence there exists a real analytic diffeomorphism of $E$ onto $\mathbb{R}^2$. (This follows in particular from the classification theorem for simply connected Riemann surfaces.) We identify $E$ with $\mathbb{R}^2$ and denote the resulting real analytic foliation of $\mathbb{R}^2$ by $F = F_\psi$. Let $\pi : \mathbb{R}^2 \to Q = \mathbb{R}^2/\mathcal{F}$ denote the projection onto the space of leaves. $Q$ admits the structure of a non-Hausdorff real analytic manifold such that $\pi$ is a real analytic submersion. (The real analytic structure on $Q$ is obtained by declaring the restriction of $\pi$ to any local analytic transversal $\ell$ to $\mathcal{F}$ to be a diffeomorphism of $\ell$ onto the open set $\pi(\ell) \subset Q$. For the details see [7], [8].) In our case $Q$ is the quotient of the topological sum $\mathbb{R}_1 \sqcup \mathbb{R}_2$ of two copies of the real axis obtained by identifying a point $t > 0$ in $\mathbb{R}_1$ with the point $\psi(t) \in \mathbb{R}_2$ (no identifications are made for points $t \leq 0$). The only pair of branch points in $Q$ (i.e., points without a pair of disjoint neighborhoods) are those corresponding to $0 \in \mathbb{R}_1$ and $0 \in \mathbb{R}_2$.

**Lemma 2.3.** If $\psi$ is flat at origin ($\lim_{t \to 0} \psi^{(k)}(t) = 0$ for $k \in \mathbb{N}$) then every real analytic function on $\mathbb{R}^2$ which is constant on every leaf of $\mathcal{F}_\psi$ is constant.

**Proof.** A real analytic function $f$ on $\mathbb{R}^2$ which is constant on the leaves of the foliation $\mathcal{F}_\psi$ is of the form $f = h \circ \pi$ for some real analytic function $h : Q \to \mathbb{R}$, where $Q$ is the space of leaves. From our construction of the foliation it follows
that \( h \) is given by a pair of real analytic functions \( h_j: \mathbb{R} \to \mathbb{R} \) \((j = 1, 2)\) satisfying \( h_j(t) = h_2(\psi(t)) \) for \( t > 0 \). As \( t \downarrow 0 \), the flatness of \( \psi \) at 0 implies that the derivative \( h_1' \) is flat at 0. Hence \( h_1 \), and therefore also \( h_2 \), are constant. \( \square \)

Fix \( \psi \) and consider the following pair of subsets of \( E \): \( E_1 \) resp. \( E_2 \):

\[
D_1 = \{(x,y) \in \mathbb{R}^2: -3 < x < -2, -1 < y < +2\}, \quad D_2 = \{(x,y) \in \mathbb{R}^2: 2 < x < 3, -1 < y < \psi(2)\} \cup \\
\quad \cup \{(x,y) \in \mathbb{R}^2: -3 < x < 3, \psi(1) < y < \psi(2)\}.
\]

Let \( D \) be the quotient of the disjoint sum \( D_1 \cup D_2 \) obtained by identifying any point \((x,y) \in D_1\) such that \( 1 < y < 2 \) with the point \( \phi(x,y) = (x,\psi(y)) \in D_2 \). Clearly \( D \) is a simply connected domain with compact closure in \( E \simeq \mathbb{R}^2 \), and the space of leaves \( Q_D = D/\mathcal{F} \) is a non-Hausdorff manifold with a simple branch at \( t = 1 \in \mathbb{R} \) resp. \( \psi(1) \in \mathbb{R} \).

**Lemma 2.4.** If \( \psi \) is flat at the origin then every real analytic function \( f \) on \( D \) which is constant on every leaf of \( \mathcal{F}_\psi|_D \) is constant.

**Proof.** As in lemma 2.3 such \( f \) is of the form \( f = h \circ \pi \) for some real analytic function \( h \) on \( Q_D = D/\mathcal{F}_\psi \). Such \( h \) is given by a pair of real analytic functions \( h_1: (-1,2) \to \mathbb{R} \), \( h_2: (-1,\psi(2)) \to \mathbb{R} \) satisfying \( h_1(t) = h_2(\psi(t)) \) for \( 1 < t < 2 \). By analyticity this relation persists on the largest interval on which both sides are defined, which is \((0,2)\). By flatness of \( \psi \) at 0 we conclude as in lemma 2.3 that \( h_1 \) and \( h_2 \) must be constant. \( \square \)

Let \( \mathcal{F} = \mathcal{F}_\psi \) be the foliation of \( \mathbb{R}^2 \) constructed above with the diffeomorphism \( \psi(t) = te^{-1/t} \) of \((0, +\infty)\) (which is flat at 0). Let \( D \subset \mathbb{R}^2 \) satisfy the conclusion of lemma 2.4. Choose a disc containing \( D \); clearly lemma 2.4 still holds for this disc, and by an affine change of coordinates on \( \mathbb{R}^2 \) we may assume this to be the unit disc. This completes the proof of proposition 2.1. \( \square \)

**Remark 2.5.** Proposition 2.1 holds for any foliation \( \mathcal{F}_\psi \) constructed above for which the diffeomorphism \( \psi \) of \((0, +\infty)\) is such that \( h \circ \psi \) does not extend as a real analytic function to a neighborhood of 0 for any real analytic function \( h \) near 0. An example is \( t^\alpha \) for an irrational \( \alpha > 0 \). The foliation of \( \mathbb{R}^2 \) determined by the algebraic 1-form \( \omega = \alpha - x)(1+x)dy - xdx \) has the space of leaves \( C^1 \)-diffeomorphic to the ‘simple branch’ \( Q \) determined by \( \psi(t) = t^\alpha \) (\( \mathbb{R} \), p. 120); hence it might be possible to find a disc \( D \subset \mathbb{R}^2 \) satisfying the proposition 2.1 for this foliation. These examples indicate that a real analytic foliation of \( \mathbb{R}^2 \) only rarely admits real analytic first integrals on large compact subsets.

### 3. Proof of theorem 1.1

Let \( \mathcal{F} \) be a real analytic foliation of \( \mathbb{R}^2 \) furnished by the proposition 2.1 such that any real analytic function on \( D = \{x^2 + x^2 < 1\} \subset \mathbb{R}^2 \) which is constant on the leaves of \( \mathcal{F}|_D \) is constant. Denote by \((x_1 + iy_1, x_2 + iy_2)\) the coordinates on \( \mathbb{C}^2 \) and identify \( \mathbb{R}^2 \) with the plane \( \{y_1 = 0, y_2 = 0\} \subset \mathbb{C}^2 \). Complexifying the leaves of \( \mathcal{F} \) we obtain the Levi foliation of a closed, real analytic, Levi-flat hypersurface \( M \) in an open tubular neighborhood \( \Omega = \mathbb{C}^2 \setminus \mathbb{R}^2 \). Set \( B = \{x^2 + x_2^2 + c(y_1^2 + y_2^2) < 1\} \) where \( c > 0 \) is chosen sufficiently large such that \( \overline{B} \subset \Omega \). Note that \( B \cap \mathbb{R}^2 = D \). A
generic choice of $c$ insures that $M$ intersects $bB$ transversely (since transversality holds along $bD \cap M$). Set $A = M \cap B \subset M$. If $B$ is sufficiently thin (which is the case if $c$ is sufficiently large) then clearly $A$ is diffeomorphic to the closed ball in $\mathbb{R}^3$. If a real analytic function $u \in \mathcal{C}^\omega(A)$ is constant on every Levi leaf of $A$ then $u|_D$ is constant on every leaf of $\mathcal{F}|_D$ and hence is constant. Thus $A$ satisfies property (ii) in theorem 1.1.

The foliation $\mathcal{F}$ of $\mathbb{R}^2$ is transversely orientable and hence admits a transverse analytic vector field $\nu$. Its complexification is a holomorphic vector field $w$ in a neighborhood of $\mathbb{R}^2$ in $\mathbb{C}^2$ such that $iw$ is transverse to $M$ in a neighborhood of $\overline{B}$, provided that $B$ is chosen sufficiently thin. Moving $M$ off itself to either side by a short time flow of $iw$ in a neighborhood of $\overline{B}$ we obtain thin neighborhoods of $\overline{A}$ with two Levi-flat boundary components; intersecting these with $rB$ for $r > 1$ close to 1 gives a fundamental system of Stein neighborhoods of $\overline{A}$.

Suppose that $v$ is a real pluriharmonic function in a connected open neighborhood of $A$ such that $v|_A = 0$. For every point $x \in A$ there is an open connected neighborhood $U_x \subset B$ and a pluriharmonic function $u_x$ on $U_x$, determined up to a real constant, such that $u_x + iv$ is holomorphic on $U_x$. Since $A$ is contractible, $H^1(A, \mathbb{R}) = 0$ and hence the collection $\{u_x\}_{x \in A}$ can be assembled into a pluriharmonic function $u$ in a neighborhood of $A$ such that $u + iv$ is holomorphic. Since $v|_A = 0$, $u$ is constant on every Levi leaf on $A$ and hence constant by property (ii) of $A$. Thus $v$ is constant and hence identically zero. This proves theorem 1.1.

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