Higher-order Abel equations: Lagrangian formalism, first integrals and Darboux polynomials

José F Cariñena\textsuperscript{1}, Partha Guha\textsuperscript{2,3} and Manuel F Rañada\textsuperscript{1}

\textsuperscript{1} Departamento de Física Teórica and IUMA, Facultad de Ciencias Universidad de Zaragoza, 50009 Zaragoza, Spain
\textsuperscript{2} Max Planck Institute for Mathematics in the Sciences Inselstrasse 22, D-04103 Leipzig, Germany
\textsuperscript{3} S.N. Bose National Centre for Basic Sciences, JD Block Sector-3, Salt Lake, Calcutta 700098, India

E-mail: jfc@unizar.es, partha@bose.res.in and mfran@unizar.es

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Abstract

A geometric approach is used to study a family of higher-order nonlinear Abel equations. The inverse problem of the Lagrangian dynamics is studied in the particular case of the second-order Abel equation and the existence of two alternative Lagrangian formulations is proved, both Lagrangians being of a non-natural class (neither potential nor kinetic term). These higher-order Abel equations are studied by means of their Darboux polynomials and Jacobi multipliers. In all the cases a family of constants of the motion is explicitly obtained. The general \( n \)-dimensional case is also studied.

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1. Introduction

The first-order Riccati equation

\[ y' = P(x)y^2 + Q(x)y + R(x) \]

is important mainly because it is a nonlinear one but directly related to the general linear differential equation of second-order via a Cole–Hopf transformation. It is usually considered as the first instance in the study of nonlinear equations \([1]\) and is endowed with many interesting properties. For example, it is a Lie system admitting a nonlinear superposition principle and it
is the only nonlinear equation of the form \( y' = f(x, y) \), where \( f(x, y) \) is a rational function of the variable \( y \) with coefficients analytic in \( x \), that possesses the Painlevé property (nevertheless, the Lie–Scheffers theory or the Painlevé approach will not be considered in this paper).

The (first-order) Riccati equation is therefore a nonlinear equation that has been intensively studied by many authors. The important point is that it has been proved that it admits higher-order generalizations which are also studied by making use of several different approaches [2–4] (according to Davis these higher-order equations were first considered by Vessiot in 1895). All the higher-order Riccati equations can be linearized via a Cole–Hopf transformation to linear differential equations. It is known that the higher-order Riccati equations play the role of Bäcklund transformations for integrable partial differential equations (PDEs) of higher-order than the KdV equation. The Riccati chain without potential is naturally associated with Faà di Bruno polynomials. The Faà di Bruno polynomials appear in several branches of mathematics and physics and can be introduced in several ways.

In fact, higher-order Riccati equations are related to the existence of symmetries [5, 6], Darboux polynomials [7–9] and Jacobi multipliers [10–12]. We also mention that the second-order Riccati equation has been studied in [13] from a geometric perspective and it has been proved to admit two alternative Lagrangian formulations, both Lagrangians being of a non-natural class (neither potential nor kinetic term). An analysis of the higher-order Riccati equations and all these properties (Lagrangians, symmetries, Darboux polynomials and Jacobi multipliers) is presented in [14].

The Abel differential equation can be considered as the simplest nonlinear extension of the Riccati equation [15–17]. The Abel equation of the first kind [18–23] is given by

\[
y' = f_0(x) + f_1(x)y + f_2(x)y^2 + f_3(x)y^3.
\]

There is also another related equation, called the Abel equation of the second kind, given by

\[
[g_0(x) + g_1(x)y]y' = f_0(x) + f_1(x)y + f_2(x)y^2 + f_3(x)y^3,
\]

which is reducible to the previous one [24, 25] and is not going to be considered in this paper. On the one hand, it has striking similarities with the Riccati equation but on the other hand, as the non-linearity is of higher degree, the properties are different (and, in fact, more difficult to study). The objective of this paper is to study a chain of higher-order Abel equations using as an approach the analysis of the differential geometric properties, the Lagrangian formalism and the theory of Darboux polynomials and Jacobi multipliers.

The plan of the paper is as follows: in section 2 we review the hierarchy of higher-order Riccati equations and then introduce in a similar way the hierarchy of higher-order Abel equations. Section 3 is devoted to a particular case of the second-order Abel equation. We study the existence of a Lagrangian formulation, obtain some constants of the motion and establish the relationship of this equation with the theory of Darboux polynomials and Jacobi multipliers. Section 4 is devoted to the third- and fourth-order equations and in section 5 we consider the general \( n \)-dimensional case. Finally in section 6 we make some brief comments.

2. Riccati and Abel equations

Let us start with the definition of higher-order Riccati equations. It is known that these equations can be obtained by reduction from the matrix Riccati equation. The matrix Riccati equation plays an important part in the theory of linear Hamiltonian systems, the calculus of variations and other related topics.
2.1. Hierarchy of higher-order Riccati equations

Let us denote by $D_R$ the following differential operator, depending on a real parameter $k \in \mathbb{R}$, that will be called ‘differential operator of Riccati’:

$$D_R = \frac{d}{dt} + k x(t),$$

in such a way that the action of $D_R$ leads to the following family of differential expressions:

$$D_R^0 x = x,$$

$$D_R^1 x = \left( \frac{d}{dt} + k x \right) x = \ddot{x} + k \dot{x},$$

$$D_R^2 x = \left( \frac{d}{dt} + k x \right)^2 x = \dddot{x} + 3k \ddot{x} + k^2 x,$$

$$D_R^3 x = \left( \frac{d}{dt} + k x \right)^3 x = \ddddot{x} + 6k \dddot{x} + 3k^2 \ddot{x} + k^3 x,$$

$$D_R^4 x = \left( \frac{d}{dt} + k x \right)^4 x = x^{(v)} + 15k \dddot{x} + 10k^2 \ddot{x} + 10k^3 \dot{x} + 10k^4 x.$$  

The Riccati equation of order $m$ of the higher-order Riccati hierarchy (or chain) is given by

$$D_R^m x = 0, \quad m = 0, 1, 2, \ldots$$

In fact, the most general form of a Riccati equation of order $m$ is just a superposition of all the previous equations (linear combination of the different members of the hierarchy)

$$(p_0 D_R^m + p_1 D_R^{m-1} + \cdots + p_{n-1} D_R + p_n)x + p_{n+1} = 0,$$

where each $p_i$ is a function of $t$.

These equations have certain properties that make them interesting from both physical and mathematical points of view. Next we point out some of them.

(1) The higher-order Riccati equation of order $m$, member of the Riccati hierarchy, admits the maximal number of Lie point symmetries that can admit an equation of order $m$.

(2) The higher-order Riccati equation of order $m$ can be linearized and presented as a linear equation of order $m + 1$.

(3) The dimensional reduction of a linear equation of order $m + 1$ leads to the Riccati equation of order $m$.

2.2. Hierarchy of higher-order Abel equations

The most natural generalization of the Riccati equation is

$$\dot{x} = f(t, x),$$

where $f(t, x)$ is a polynomial in the variable $x$ (with coefficients depending on $t$). The particular case of $f(t, x)$ being a cubic polynomial

$$f(t, x) = A_0(t) + A_1(t)x + A_2(t)x^2 + A_3(t)x^3$$

is called an Abel equation. Such an equation can be considered as the simplest nonlinear extension of the Riccati equation.
Let us denote by $\mathbf{D}_A$ the following differential operator, depending on a real parameter $k \in \mathbb{R}$, to be called an ‘Abel differential operator’,

$$\mathbf{D}_A = \frac{d}{dt} + k x^2(t),$$

in such a way that the action of $\mathbf{D}_A$ leads to a family of $k$-dependent differential equations whose first members are given by

$$\begin{align*}
\mathbf{D}_A^0 x &= x,
\mathbf{D}_A x &= \left(\frac{d}{dt} + k x^2\right) x = \dot{x} + k x^3,
\mathbf{D}_A^2 x &= \left(\frac{d}{dt} + k x^2\right)^2 x = \ddot{x} + 4 k x^2 \dot{x} + k^2 x^5,
\mathbf{D}_A^3 x &= \left(\frac{d}{dt} + k x^2\right)^3 x = \dddot{x} + 5 k x^2 \ddot{x} + 8 k x^4 \dot{x} + 9 k^2 x^4 \dot{x} + k^3 x^7,
\mathbf{D}_A^4 x &= \left(\frac{d}{dt} + k x^2\right)^4 x = \ddddot{x} + 2 k (4 \dddot{x}^3 + 13 x \dddot{x} \ddot{x} + 3 x^2 \dddot{x}) + 2 k^2 \dddot{x}^3 (22 \dot{x}^2 + 7 x \ddot{x})
+ 16 k^3 x^6 \ddot{x} + k^4 x^9.
\end{align*}$$

We call this family the hierarchy of higher-order Abel equations. The Abel equation of order $m$, written in the so-called simplified form, is given by

$$\mathbf{D}_A^m x = 0,$$  
$m = 0, 1, 2, \ldots$.

Actually, the most general form of the Abel equation of order $m$ is just a superposition of all the previous equations (linear combination of the different members of the hierarchy with functions $p_i(t)$ as coefficients)

$$(p_0 \mathbf{D}_A^0 + p_1 \mathbf{D}_A^{-1} + \cdots + p_{n-1} \mathbf{D}_A + p_n) x + p_{n+1} = 0.$$  

An important point is that the Abel equation cannot be obtained, as the Riccati equation, by a reduction procedure from a linear equation.

3. Abel equation of second order

In this section we will analyse the particular case of the second-order Abel equation. In particular, we describe the Lagrangian formulation of the second-order Abel equation.

3.1. Lagrangian formalism

The action of $\mathbf{D}_A^2$ on the function $x(t)$ leads to the nonlinear equation

$$\frac{d^3 x}{dt^3} + 4 k x^2 \left(\frac{dx}{dt}\right) + k^2 x^5 = 0,$$  
(1)

which represents the Abel equation of second order. It can be presented as a system of two first-order equations

$$\begin{align*}
\frac{dx}{dt} &= v,
\frac{dv}{dt} &= -4 k x^2 v - k^2 x^5
\end{align*}$$

J F Cariñena et al
which determines a dynamical system that, in differential geometric terms, is represented by
the following vector field:
\[ \Gamma^{(2)} = v \frac{\partial}{\partial x} + F_{A2} \frac{\partial}{\partial v}, \quad F_{A2} = -4kx^2v - k^2x^5 \] (2)
defined on the phase space \( \mathbb{R}^2 \) with coordinates \((x, v)\).

It has been proved in [13] that the second-order Riccati equation
\[ \frac{d^2x}{dt^2} + 3kx \left( \frac{dx}{dt} \right) + k^2x^3 = 0 \]
can be considered as the Lagrange equation determined by the following Lagrangian:
\[ L_R = \frac{1}{v + kx^2}. \]

**Proposition 1.** The nonlinear Abel equation of second order (1) admits a Lagrangian formulation with a non-polynomial Lagrangian.

**Proof.** There are two different ways of obtaining a Lagrangian function for the nonlinear Abel equation; the Helmholtz approach and the generalization of the method used for the corresponding Riccati case.

The Helmholtz conditions are a set of conditions that a multiplier matrix \( g_{ij}(x, \dot{x}, t) \) must satisfy in order for a given system of second-order equations
\[ \ddot{x}_j = f_j(x, \dot{x}, t), \quad j = 1, 2, \ldots, n, \]
when written in the form
\[ g_{ij} \ddot{x}_j = g_{ij}f_j(x, \dot{x}, t), \quad i, j = 1, 2, \ldots, n, \]
to be the set of Euler–Lagrange equations for a certain Lagrangian \( L \) [26–29] (the summation convention on repeated indices is assumed). If a matrix solution \( g_{ij} \) is obtained then it can be identified with the Hessian matrix of \( L \), that is \( g_{ij} = \partial^2 L/\partial v_i \partial v_j \), and a Lagrangian \( L \) can be obtained by direct integration of the \( g_{ij} \) functions. The two first conditions just impose regularity and symmetry of the matrix \( g_{ij} \); the two other ones are equations introducing relations among the derivatives of \( g_{ij} \) and the derivatives of the functions \( f_j \). Here we only write the fourth set of conditions that determine the time evolution of the \( g_{ij} \)
\[ \Gamma(g_{ij}) = g_{ik}A_{kj} + g_{jk}A_{ki}, \quad A_{ab} = -\frac{1}{2} \frac{\partial f_a}{\partial v_b}. \]

When the system is one-dimensional we have \( i = j = k = 1 \) and then the first three sets of conditions become trivial and the fourth one reduces to one single first-order PDE:
\[ \Gamma(g) + \left( \frac{\partial f}{\partial v} \right) g \equiv v \left( \frac{\partial g}{\partial x} \right) + f \left( \frac{\partial g}{\partial v} \right) + \left( \frac{\partial f}{\partial v} \right) g = 0, \] (3)
which in the case of the Abel equation becomes
\[ v \left( \frac{\partial g}{\partial x} \right) - (4kx^2 + k^2x^5) \left( \frac{\partial g}{\partial v} \right) - 4kx^2g = 0. \] (4)

So, the problem reduces to find the function \( g \) as a solution of this equation. Once a solution \( g \) is known a Lagrangian \( L \) is obtained by integrating two times the function \( g \). The function \( L \) obtained from \( g \) is unique up to addition of a gauge term.

Next we consider the second method that is specific for this particular nonlinear problem. The starting point is the idea that, since the Abel equation is very closely related to the Riccati
equation, it seems natural to assume that the Abel Lagrangian must be a non-polynomial function similar to that of the second-order Riccati equation.

Let us begin by considering the following one degree of freedom Lagrangian:

\[ L = \frac{1}{(v + kU(x, t))^m}. \]  (5)

From such a Lagrangian we arrive at the following second-order nonlinear equation:

\[ \ddot{x} + \left( \frac{2 + m}{1 + m} \right) kU_x \dot{x} + \left( \frac{1}{1 + m} \right) k^2 U_{xx} + kU_t = 0. \]  (6)

Hence, in the particular case of \( U \) and \( m \) being given by

\[ U(x, t) = x^3, \quad m = 2, \]

the Lagrangian (5) leads to (1). Thus, the second-order Abel equation (1) turns out to be the Euler–Lagrange equation of the Lagrangian function

\[ L_A = \frac{1}{(v + kx^3)^2}. \]  (7)

Finally, as a by-product of this approach, we have also obtained the Lagrangians for the whole family of nonlinear equations (6) depending on a function \( U \). □

As a corollary of this proposition we can state that when the function \( U \) is time independent the nonlinear equation (6) has a first integral that can be interpreted as a preserved energy. That is, if we restrict the study to nonlinear equations arising from a time-independent Lagrangian of the form

\[ L = \frac{1}{(v + kU(x))^m} \]

then we can define an associated Lagrangian energy \( E_L \) by the usual procedure

\[ E_L = \Delta(L) - L, \quad \Delta = v \frac{\partial}{\partial v}, \]

and we arrive at

\[ E_L = \frac{-(1 + m)(v + kU(x))}{(v + kU(x))^{m+1}}, \quad \frac{d}{dt} E_L = 0. \]

In the particular case of the Abel Lagrangian \( L_A \) we have

\[ E_{L_A} = \frac{(3v + kx^3)}{(v + kx^3)^3}, \quad \frac{d}{dt} E_{L_A} = 0. \]  (8)

Note that \( L_A \) is non-natural and, as there is neither kinetic term \( T \) nor potential function \( V \), the energy cannot be of the standard form \( E_L = T + V \). But, in spite of its rather peculiar form, \( E_{L_A} \) is a conserved function for the Abel equation.

An important property of the Lagrangian formalism is that for one degree of freedom systems if an equation admits a Lagrangian formulation then the Lagrangian is not unique [37, 38]. This property can be proved in two different ways. First, the Helmholtz equation (4) is a linear equation in partial derivatives and thus it admits many different particular solutions. Moreover it is clear from the form of equation (3) that if \( g_1 \) is a particular solution then \( g_2 = f g_1 \) with \( \Gamma(f) = 0 \) is also a solution. A second method is related to the properties of the symplectic formalism. In a two-dimensional manifold all the symplectic forms must be proportional. Hence if \( \omega_L \) is known then any other symplectic form \( \omega_2 \) must be proportional to \( \omega_L \), that is \( \omega_2 = f \omega_L \). Then

\[ i(\Gamma_L)\omega_2 = f i(\Gamma_L)\omega_L = f dE_L. \]
The right-hand side is an exact one-form if, and only if, $df \wedge dE_L = 0$, which shows that $f$ must be a function of $E_L$. In this case it can be proved that the new symplectic form $\omega_2$ is derivable from an alternative Lagrangian $L_2 \neq L$ for $L_E$.

In the particular case of the Abel system $\Gamma^{(2)}$, several alternative Lagrangians can be obtained that, in most cases, are of non-algebraic character (with logarithmic terms). Nevertheless, in the particular case of $f$ given by $f = (-1/E_L)^{3/2}$, we have obtained the following algebraic function:

$$L_A = (3v + k x^3)^{2/3}$$

as a new alternative Lagrangian for the Abel equation (1). This new Lagrangian is equivalent to $L_A$ in the sense that both determine the same dynamics. It determines a new energy $E_{L_A}$ that is a constant of the motion for the Abel equation; nevertheless, it must not be considered as a new fundamental constant since it is a function of the original energy $E_{L_A}$.

### 3.2. Constants of the motion and geometric formalism

A function $T$ that satisfies the following property:

$$\frac{d}{dt} T \neq 0, \ldots, \frac{d^m}{dt^m} T \neq 0, \quad \frac{d^{m+1}}{dt^{m+1}} T = 0,$$

is called a generator of integrals of motion of degree $m$. Note that this means that the function $T$ is a non-constant function generating a constant of motion by successive time derivations.

Let us denote by $T^{(2)}_1$ the following function:

$$T^{(2)}_1 = \frac{x}{v + k x^3}.$$

Then, we have that under the evolution given by Abel’s equation (1)

$$\frac{d}{dt} T^{(2)}_1 = T^{(2)}_2 = 1, \quad \frac{d}{dt} T^{(2)}_2 = 0.$$

Thus, the function $J_{t1}$ defined by

$$J_{t1} = T^{(2)}_1 - t$$

is a time-dependent constant of the motion for the Abel equation.

This means that we have obtained two constants of the motion (of quite different nature) for the Abel equation of second order: the energy $E_{L_A}$ and the time-dependent function $J_{t1}$.

In differential geometric terms a time-independent Lagrangian function $L$ determines an exact two-form $\omega_L$ defined as

$$\theta_L = \left( \frac{\partial L}{\partial v_x} \right) dx, \quad \omega_L = -d\theta_L,$$

and $L$ is said to be regular when the 2-form $\omega_L$ is symplectic. In the particular case of $L$ given by (7) $\omega_{L_A}$ is given by

$$\omega_{L_A} = \left( \frac{6}{(v + k x^3)^4} \right) dx \wedge dv,$$

and the dynamical vector field $\Gamma^{(2)}$ is the solution of the equation

$$i(\Gamma^{(2)}) \omega_{L_A} = dE_{L_A}.$$

Next we consider two interesting classes of symmetries: ‘master symmetries’ and ‘non-Cartan symmetries’. The idea is that, in differential geometric terms, constants of motion that
depend on time but in a polynomial way are related to the existence of master symmetries [30–33] and in some very particular cases to non-Cartan symmetries.

Given a dynamics represented by a certain vector field $\Gamma_1$, then a vector field $Z$ satisfying

$$[Z, \Gamma_1] = \tilde{Z} \neq 0, \quad [\tilde{Z}, \Gamma_1] = 0$$

is called a ‘master symmetry’ of degree $m = 1$ for $\Gamma_1$. When $Z$ is such that

$$[Z, \Gamma_1] = \tilde{Z} \neq 0, \quad [\tilde{Z}, \Gamma_1] \neq 0 \quad \text{and} \quad [[\tilde{Z}, \Gamma_1], \Gamma_1] = 0,$$

then $Z$ is called a ‘master symmetry’ of degree $m = 2$. The generalization to higher values of $m$ is straightforward:

$$(\text{ad}(\Gamma_1))^{m+1}(Z) = 0, \quad \text{but} \quad (\text{ad}(\Gamma_1))^m(Z) \neq 0.$$ 

It is well known that symmetries are important because they give rise to constants of the motion and reduction procedures. Master symmetries, which are a rather peculiar class of symmetries, determine time-dependent constants of motion (the system is time-independent but the constant is however time-dependent). This can be seen as follows: if $Z$ is a master symmetry of degree one, the time-dependent vector field $\gamma Z$ determined by $Z$ as follows [33]:

$$\gamma Z = Z + t [Z, \Gamma_1] + \left( \frac{1}{2} \right) t^2 [Z, [\Gamma_1, \Gamma_1]]$$

is a time-dependent symmetry of $\Gamma_1$, $\gamma = \partial/\partial t + \Gamma_1$, which is the suspension of the vector field $\Gamma_1$ [34]. This symmetry determines a time-dependent constant of motion $J_\gamma = T - t \gamma (T)$ that depends linearly on $t$ (for $m = 2$ the corresponding constant $J_\gamma$ will be quadratic in $t$ and for $m = 3$ will be cubic).

Let $Z_1$ be the Hamiltonian vector field of the function $T_1^{(2)}$, that is, the unique solution of the equation

$$i(Z_1) \omega_L = dT_1^{(2)},$$

which is given by

$$Z_1 = - \left( \frac{1}{6} \right) P_{A1}^2 \left( x \frac{\partial}{\partial x} + (v - 2k x^3) \frac{\partial}{\partial v} \right), \quad P_{A1} = v + k x^3.$$ 

Then $Z_1$ is a symplectic symmetry (that is, $\mathcal{L}_{Z_1} \omega_L = 0$) because it is the Hamiltonian vector field of $T_1^{(2)}$, and moreover it is a dynamical symmetry because

$$i([Z_1, \Gamma_1^{(2)}] \omega_L = i(Z_1)(\mathcal{L}_{\Gamma_1^{(2)}} \omega_L) - \mathcal{L}_{\Gamma_1^{(2)}}(i(Z_1) \omega_L) = - \mathcal{L}_{\Gamma_1^{(2)}}(dT_1^{(2)}) = 0,$$

and therefore, as $\omega_L$ is non-degenerate, $[Z_1, \Gamma_1^{(2)}] = 0$.

Note, however, that $Z_1$ is not a symmetry of the energy since $Z_1(E_{L_1}) \neq 0$.

Thus, $Z_1$ is a dynamical but non-Cartan symmetry of the Lagrangian system [35, 36]. These symmetries are rather peculiar and only appear in some very particular cases. In particular, it was proved in [36] that if the Hamiltonian vector field $X_F$ with the function $F$ as Hamiltonian in a symplectic manifold $(M, \omega)$ is a dynamical but non-Cartan symmetry, then $X_F(H)$ must be a numerical constant $X_F(H) = \alpha \neq 0$. In this case we are considering, $F = T_1^{(2)}$ and we have $Z_1(E_{L_1}) = \alpha = -1$.

We close this section by recalling that the Riccati equation was endowed with similar properties but the function $T_1^{(2)}$ was the Lagrangian $L_R$ itself [14].

### 3.3. Darboux polynomial and Jacobi multiplier approach

The existence of constants of the motion and the Lagrangian inverse problem for polynomial vector fields are two questions related to two important ideas: Jacobi multipliers and Darboux polynomials.
Let $U$ be an open subset of $\mathbb{R}^n$. We say that a polynomial function $D : U \to \mathbb{R}$ is a Darboux polynomial for a polynomial vector field $X$ if there is a polynomial function $f$ defined in $U$ such that $XD = f D$ [7–9, 14]. The function $f$ is said to be the cofactor corresponding to such a Darboux polynomial and the pair $(f, D)$ is called a Darboux pair.

When $f = 0$, then the Darboux polynomial is a first integral. We say that $D$ is a proper Darboux polynomial if $f \neq 0$. If $D_1$ and $D_2$ are Darboux polynomials with the same cofactor, the quotient $D_1/D_2$ is a first integral.

On the other side, given a vector field $X$ in an oriented manifold $(M, \Omega)$, a function $R$ such that $Ri(X)\Omega$ is closed is said to be a Jacobi multiplier (JM) for $X$. Recall that the divergence of the vector field $X$ (with respect to the volume form $\Omega$) is defined by the relation $L_X\Omega = (\text{div}X)\Omega$.

This means that $R$ is a multiplier if and only if $RX$ is a divergenceless vector field and then $L_{RX}\Omega = (\text{div} RX)\Omega = [X(R) + R \text{div} X]\Omega = 0$, and therefore we see that $R$ is a last multiplier for $X$ if and only if

$$X(R) + R \text{div} X = 0.$$ (10)

Note that if $R$ is a never vanishing Jacobi multiplier, then $f R$ is a Jacobi multiplier too if and only if $f$ is a constant of motion. We also note that equation (4) can now be considered as a particular case of equation (10).

The remarkable point is that if $D_1, \ldots, D_k$, are Darboux polynomials with corresponding cofactors $f_i$, $i = 1, \ldots, k$, one can look for multiplier factors of the form

$$R = \prod_{i=1}^{k} D_i^{v_i}$$ (11)

and then

$$\frac{X(R)}{R} = \sum_{i=1}^{k} v_i \frac{X(D_i)}{D_i} = \sum_{i=1}^{k} v_i f_i,$$

and therefore, if the coefficients $v_i$ can be chosen such that

$$\sum_{i=1}^{k} v_i f_i = -\text{div} X$$ (12)

holds, then we arrive at

$$\frac{X(R)}{R} = \sum_{i=1}^{k} v_i f_i = -\text{div} X,$$

and consequently $R$ is a Jacobi last multiplier for $X$.

Finally, if $R$ is a Jacobi multiplier for a vector field which corresponds to a second-order differential equation, there is an essentially unique Lagrangian $L$ (up to addition of a gauge term) such that $R = \partial^2 L/\partial v^2$ [10–12].

From these general concepts we can return to the Abel equation. In this case the polynomial $D_1$ defined by

$$D_1(x, v) = v + kx^3$$

is a Darboux polynomial for $\Gamma^{(2)}$ with cofactor $-kx^2$ since

$$\left( v \frac{\partial}{\partial x} + F_{x2} \frac{\partial}{\partial v} \right) (v + kx^3) = -kx^2(v + kx^3).$$
The divergence of the vector field $\Gamma^{(2)}$ is $-4kx^2$, and then, according to (12), we see that there is a multiplier of the form
\[ R = D_1^{\nu_1}, \]
with $\nu_1 = -4$. Consequently, the Abel equation admits a Lagrangian description by means of a function $L_1$ such that
\[ \frac{\partial^2 L_1}{\partial v^2} = (v + kx^3)^{-4}, \]
from where we obtain the Lagrangian $L_1 = L_A$ given by (7).

But the polynomial $D_2$ defined by
\[ D_2(x, v) = 3v + kx^3 \]
is a Darboux polynomial for $\Gamma^{(2)}$ with cofactor $-3kx^2$, because
\[ \left( v \frac{\partial}{\partial x} + F_{A2} \frac{\partial}{\partial v} \right) (3v + kx^3) = 3kx^2v - 3(4kx^2v + k^2x^5) = -3kx^2(3v + kx^3), \]
and then, using equation (12) we can find another Jacobi multiplier of the form $D_2^{\nu_2}$ with $\nu_2 = -4/3$. The Abel equation admits a Lagrangian description by means of a function $L_2$ such that
\[ \frac{\partial^2 L_2}{\partial v^2} = (3v + kx^3)^{-4/3}, \]
from where we obtain the Lagrangian $L_2 = \tilde{L}_A$ given by (9).

Note that, as indicated above, if $P$ and $Q$ are two Darboux polynomials with the same cofactor then $P/Q$ is a constant of the motion. This is just what happens with the energy $E_{L_A}$ obtained in (8) which is given by $D_2/D_1^3$ (up to the sign).

4. Abel equations of third and fourth order

In the following all the functions that appear as constants of the motion will depend on the time $t$. So we better consider to describe them as first integrals that depend polynomially on time.

4.1. Abel equation of third order

The action of the operator $\mathbb{D}_A$ three times on the function $x(t)$ leads to the following nonlinear equation:
\[ \frac{d^3x}{dt^3} + 5kx^2 \left( \frac{d^2x}{dt^2} \right) + 8kx \left( \frac{dx}{dt} \right)^2 + 9k^2x^4 \left( \frac{dx}{dt} \right) + k^3x^7 = 0, \tag{13} \]
which represents the third-order element of the Abel equation chain. It can be presented as a system of three first-order equations
\[ \begin{align*}
\frac{dx}{dt} &= v, \\
\frac{dv}{dt} &= a, \\
\frac{da}{dt} &= -5kx^2a - 8kxv^2 - 9k^2x^4v - k^3x^7, 
\end{align*} \tag{14} \]
which represents a dynamical system that, in differential geometric terms, is represented by the following vector field in the phase space $\mathbb{R}^3$, with coordinates $(x, v, a)$:

$$
\Gamma^{(3)} = \frac{\partial}{\partial x} + a \frac{\partial}{\partial v} + F_{A1} \frac{\partial}{\partial a}, \quad F_{A3} = -5k x^2 a - 8k x v^2 - 9k^2 x^4 v - k^3 x^7. \quad (15)
$$

In what follows we make use of the following polynomials:

$$
P_{A0} = x, \quad P_{A1} = v - F_{A1} = v + k x^3, \quad P_{A2} = a - F_{A2} = a + 4k x^2 v + k^2 x^5,
$$

defined on the phase space and obtained by making use of the substitution $\dot{x} \mapsto v$ and $\dot{a} \mapsto a$. Then we have

$$
\Gamma^{(3)}(P_{A0}) + k x^2 P_{A0} = v + k x^3,
$$

$$
\Gamma^{(3)}(P_{A1}) + k x^2 P_{A1} = a + 4k x^2 v + k^2 x^5,
$$

$$
\Gamma^{(3)}(P_{A2}) + k x^2 P_{A2} = F_{A3} + 5k x^2 a + 8k x v^2 + 9k^2 x^4 v + k^3 x^7,
$$

which can be rewritten as follows:

$$
\Gamma^{(3)}(P_{A0}) + k x^2 P_{A0} = P_{A1},
$$

$$
\Gamma^{(3)}(P_{A1}) + k x^2 P_{A1} = P_{A2},
$$

$$
\Gamma^{(3)}(P_{A2}) + k x^2 P_{A2} = 0.
$$

Note that according to these properties $P_{A2}$ is a Darboux polynomial with $f = -k x^2$ as cofactor. The divergence of the vector field $\Gamma^{(3)}$ is $-5k x^2$, and using relation (12) we see that $R = (P_{A2})^{\mu_2}$ with $\mu_2 = -5$ is a Jacobi multiplier.

Next let $T_i^{(3)}$ be the following function:

$$
T_i^{(3)} = \frac{x}{P_{A2}},
$$

and then we have

$$
\Gamma^{(3)}(T_i^{(3)}) = T_i^{(3)} = \frac{v + k x^3}{P_{A2}}, \quad \Gamma^{(3)}(T_2^{(3)}) = T_3^{(3)} = 1, \quad \Gamma^{(3)}(T_3^{(3)}) = T_4^{(3)} = 0.
$$

This means that $T_1^{(3)}$ and $T_2^{(3)}$ are generators of constants of motion for the third-order element of the Abel equation chain represented by the dynamical vector field $\Gamma^{(3)}$. Thus, we can state the following proposition.

**Proposition 2.** The two functions $J_1$ and $J_2$ defined as

$$
J_1 = T_1^{(3)} - t, \quad J_2 = T_1^{(3)} - t T_2^{(3)} + \left(\frac{1}{2}\right) t^2
$$

are first integrals, that depend polynomially on time, for the Abel equation of third order.

Note that $J_1$ is linear in the time $t$ and $J_2$ is quadratic. So these expressions are similar to the constants of the motion determined by master symmetries; nevertheless, in this third-order case we have not made use of any symplectic structure and we have obtained these functions without relating them to symmetries of a symplectic structure. This is an interesting situation deserving an additional analysis in the next sections.

Note also that both $J_1$ and $J_2$ can be written as quotients of polynomials; so if we consider the system as a time-dependent system then the dynamics is geometrically represented by the vector field $\Gamma_i^{(3)} = \Gamma^{(3)} + \partial/\partial t$ and the following polynomials

$$
D_2 = P_{A1} - t P_{A2}, \quad D_3 = P_{A0} - t P_{A1} + \left(\frac{1}{2}\right) t^2 P_{A2}
$$

are two Darboux polynomials with the same cofactor as $P_{A2}$

$$
\Gamma_i^{(3)}(D_i) = \left(\Gamma^{(3)} + \frac{\partial}{\partial t}\right)(D_i) = -k x^2 D_i, \quad i = 2, 3.
$$
4.2. Abel equation of fourth order

The action of $I D_{4}$ four times on the function $x(t)$ leads to the following nonlinear equation:

$$x^{(4)} + 2k(4x^3 + 13x^2 \dot{x} + 3x^2 \ddot{x}) + 2k^2 x^3 (22x^2 + 7x \dot{x}) + 16k^3 x^6 \dot{x} + k^4 x^9 = 0,$$

which represents the fourth-order element of the Abel equation chain. This equation determines a dynamical system that, in geometric terms, can be represented by the following vector field on $\mathbb{R}^4$ as phase space, with coordinates $(x, v, a, w)$:

$$\Gamma^{(4)} = v \frac{\partial}{\partial x} + a \frac{\partial}{\partial v} + w \frac{\partial}{\partial a} + F_{A4} \frac{\partial}{\partial w},$$

where

$$F_{A4} = -2k(4v^3 + 13va + 3v^2 w) - 2k^2 x^3 (22v^2 + 7va) - 16k^3 x^6 v - k^4 x^9.$$

Now we introduce the polynomial $P_{A3}$

$$P_{A3} = w - F_{A3} = w + 5kx^2 a + 8kx v^2 + 9k^2 x^4 v + k^3 x^7,$$

obtained from the expression of $I D_{3}^4 x$ with the substitution $\dot{x} \mapsto v, \ddot{x} \mapsto a$ and $\dddot{x} \mapsto w$. Then we have

$$\Gamma^{(4)}(P_{A0}) + kx^2 P_{A0} = v + kx^3,$$

$$\Gamma^{(4)}(P_{A1}) + kx^2 P_{A1} = a + 4kx^2 v + k^2 x^5,$$

$$\Gamma^{(4)}(P_{A2}) + kx^2 P_{A2} = w + 5kx^2 a + 8kx v^2 + 9k^2 x^4 v + k^3 x^7,$$

$$\Gamma^{(4)}(P_{A3}) + kx^2 P_{A3} = F_{A4} + k(8v^3 + 26xva + 6x^2 w) + k^2 (44x^3 v^2 + 14x^5 a)$$

$$+ 16k^3 x^6 v + k^4 x^9,$$

which can be rewritten as follows:

$$\Gamma^{(4)}(P_{A0}) + kx^2 P_{A0} = P_{A1},$$

$$\Gamma^{(4)}(P_{A1}) + kx^2 P_{A1} = P_{A2},$$

$$\Gamma^{(4)}(P_{A2}) + kx^2 P_{A2} = P_{A3},$$

$$\Gamma^{(4)}(P_{A3}) + kx^2 P_{A3} = 0.$$

Let now $T_{1}^{(4)}$ be the following function:

$$T_{1}^{(4)} = \frac{x}{P_{A3}},$$

and then we have

$$\Gamma^{(4)}(T_{1}^{(4)}) = T_{2}^{(4)}, \quad \Gamma^{(4)}(T_{2}^{(4)}) = T_{3}^{(4)}, \quad \Gamma^{(4)}(T_{3}^{(4)}) = T_{4}^{(4)}, \quad \Gamma^{(4)}(T_{4}^{(4)}) = 0,$$

with $T_{2}^{(4)}, T_{3}^{(4)}$ and $T_{4}^{(4)}$ given by

$$T_{2}^{(4)} = \frac{v + kx^3}{P_{A3}}, \quad T_{3}^{(4)} = \frac{a + 4kx^2 v + k^2 x^5}{P_{A3}}, \quad T_{4}^{(4)} = \frac{w + \cdots + k^3 x^7}{P_{A3}} = 1.$$

**Proposition 3.** The three functions $J_{1}, J_{2}$ and $J_{3}$ defined as

$$J_{1} = T_{3} - t,$$

$$J_{2} = t T_{3} + \left(\frac{1}{4}\right) t^2,$$

$$J_{3} = T_{1} - t T_{2} + \left(\frac{1}{4}\right) t^2 T_{3} - \left(\frac{1}{4}\right) t^3$$

are first integrals, that depend polynomially on time, for the fourth-order element of the Abel equation chain.

The situation is similar to the $n = 3$ case and the functions $J_{r}, r = 1, 2, 3,$ are polynomials of order $r$ in the variable $t$. 
5. Equation of Abel of order $n$

We have seen that the second-order element of the Abel equation chain is endowed with some specific properties (e.g., it admits a Lagrangian description) but the third- and fourth-order elements of the chain also enjoy very similar properties. Now in this section we study the equation of order $n$ and prove that these properties characterize all the equations of the family in an independent of the order way.

The equation of Abel of order $n$ can be obtained as the equation arising from the action of the operator $\mathbf{D}_A$ on the equation of order $n-1$

$$\mathbf{D}_A (\mathbf{D}_A^{n-1} x) = \mathbf{D}_A^n x = 0.$$  

This equation determines a dynamical system that, in geometric terms, can be represented by the following vector field defined on the phase space $\mathbb{R}^n$, with coordinates $(x_1, x_2, x_3, \ldots, x_n)$:

$$\Gamma^{(n)} = x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3} + \cdots + F_{Ar} \frac{\partial}{\partial x_{n}},$$  

where $F_{Ar}$ is obtained from the expression for $\mathbf{D}_A^n x$ with the substitution $x \mapsto x_1, \dot{x} \mapsto x_2, \ddot{x} \mapsto x_3, \ldots$.

In the previous sections we have made use of the polynomials $P_{A0}, P_{A1}, P_{A2}$ and $P_{A3}$ defined in the phase space and whose explicit expressions, when written in the notation of the coordinates $x_1, x_2, x_3, \ldots, x_n$, were given by

$$P_{A0} = x_1,$$

$$P_{A1} = x_2 - F_{A1} = x_2 + k x_1^3,$$

$$P_{A2} = x_3 - F_{A2} = x_3 + 4k x_1^2 x_2 + k^2 x_1^5,$$

$$P_{A3} = x_4 - F_{A3} = x_4 + 5k x_1^3 x_3 + 8k x_1 x_3^2 + 9k^2 x_1^4 x_2 + k^3 x_1^7.$$  

In the general case, we have $P_{An-1} = x_n - F_{An-1}$ that leads to an expression of the form

$$P_{An-1} = x_n - F_{An-1} = x_n + (n+1)k x_1 x_{n-1} + \cdots + k^{n-1} x_1^{2n-1}.$$  

**Proposition 4.** The action of the dynamical vector field $\Gamma^{(n)}$ on the $n$ polynomials $P_{Ar}$, $r = 0, 1, 2, \ldots, n-1$, is given by

(i) $\Gamma^{(n)}(P_{Ar}) + k x_2 P_{Ar} = P_{Ar+1}, \quad r = 0, 1, 2, \ldots, n-2,$

(ii) $\Gamma^{(n)}(P_{An-1}) + k x_2 P_{An-1} = 0.$

The property (i) can be proved by induction. Property (ii) follows by direct calculus. 

**Proposition 5.** The time evolution of the functions $P_{Ar}/P_{An-1}$ is given by

$$\Gamma^{(n)} \left( \frac{P_{Ar}}{P_{An-1}} \right) = \frac{P_{Ar+1}}{P_{An-1}}, \quad r = 0, 1, 2, \ldots, n-2.$$  

By direct calculus we have

$$\Gamma^{(n)} \left( \frac{P_{Ar}}{P_{An-1}} \right) = \frac{\Gamma^{(n)}(P_{Ar}) P_{An-1} - P_{Ar} \Gamma^{(n)}(P_{An-1})}{(P_{An-1})^2}.$$

Then, making use of properties (i) and (ii) of proposition (4), we arrive at

$$\Gamma^{(n)} \left( \frac{P_{Ar}}{P_{An-1}} \right) = \frac{[P_{Ar+1} - k x_2 P_{Ar} P_{An-1} - P_{Ar} [-k x_2 P_{An-1}]]}{(P_{An-1})^2} = \frac{P_{Ar+1}}{P_{An-1}}.$$  

□
Note that the first and the last derivatives in this series, corresponding to \( r = 0 \) and \( r = n - 2 \), become

\[
\frac{d}{dr} \left( \frac{x}{P_{A_{n-1}}} \right) = \frac{P_{A_1}}{P_{A_{n-1}}} \quad \text{and} \quad \frac{d}{dr} \left( \frac{P_{A_{n-2}}}{P_{A_{n-1}}} \right) = 1.
\]

Now let \( T_1^{(n)} \) be the following function defined by

\[
T_1^{(n)} = \frac{x}{P_{A_{n-1}}},
\]

and then, making use of the two preceding propositions, we can obtain the values of the sequence of time derivatives of the functions \( T_1^{(n)} \), which are given by

\[
\Gamma^{(n)}(T_1^{(n)}) = T_2^{(n)} = \frac{x_2 + k x^3}{P_{A_{n-1}}} = \frac{P_{A_1}}{P_{A_{n-1}}},
\]

\[
\Gamma^{(n)}(T_2^{(n)}) = T_3^{(n)} = \frac{x_3 + 4k x^2 x_2 + k^2 x^5}{P_{A_{n-1}}} = \frac{P_{A_2}}{P_{A_{n-1}}},
\]

\[
\Gamma^{(n)}(T_3^{(n)}) = T_4^{(n)} = \frac{x_4 + 5k x^2 x_3 + 8k^2 x^4 x_2 + k^3 x^7}{P_{A_{n-1}}} = \frac{P_{A_3}}{P_{A_{n-1}}},
\]

\[
\vdots \quad \text{...........................................}
\]

\[
\Gamma^{(n)}(T_{n-1}^{(n)}) = T_n^{(n)} = 1,
\]

\[
\Gamma^{(n)}(T_n^{(n)}) = 0.
\]

From here we can state the existence of a family of \( n - 1 \) first integrals depending polynomially on time.

**Proposition 6.** The \((n - 1)\) functions \( J_r, r = 1, 2, \ldots, n - 1 \), defined as the following polynomials of order \( r \) in the variable \( t \):

\[
J_1 = T_1^{(n)} - t,
\]

\[
J_2 = T_2^{(n)} - tT_1^{(n)} + \left( \frac{1}{2} \right) t^2,
\]

\[
J_3 = T_3^{(n)} - tT_2^{(n)} + \left( \frac{1}{2} \right) t^2 T_1^{(n)} - \left( \frac{1}{6} \right) t^3
\]

\[
\vdots \quad \text{...........................................}
\]

\[
J_{n-1} = T_1^{(n)} - tT_2^{(n)} + \left( \frac{1}{2} \right) t^2 T_3^{(n)} - \ldots - (-1)^n \left( \frac{1}{n!} \right) t^n
\]

are \( n - 1 \) functionally independent first integrals, that depend polynomially on time, for the Abel equation of order \( n \).

An alternative form of proving the existence of all these constants of the motion is as follows. The \( n \) polynomials \( D_a, a = 1, 2, \ldots, n \), defined in the extended phase space \( \mathbb{R}^n \times \mathbb{R} \) as

\[
D_1 = P_{A_{n-1}},
\]

\[
D_2 = P_{A_{n-2}} - t P_{A_{n-1}},
\]

\[
D_3 = P_{A_{n-3}} - t P_{A_{n-2}} + \left( \frac{1}{2} \right) t^2 P_{A_{n-1}},
\]

\[
D_4 = P_{A_{n-4}} - t P_{A_{n-3}} + \left( \frac{1}{2} \right) t^2 P_{A_{n-2}} - \left( \frac{1}{6} \right) t^3 P_{A_{n-1}};
\]

\[
\vdots \quad \text{...........................................}
\]

\[
D_n = P_{A_0} - t P_{A_1} + \cdots + (-1)^n \left( \frac{1}{n!} \right) t^n P_{A_{n-1}},
\]
are \( n \) Darboux polynomials with the same cofactor

\[
\Gamma^{(n)}(D_a) = \left( \Gamma^{(n)} + \frac{\partial}{\partial t} \right) D_a = -k x^2 D_a, \quad a = 1, 2, \ldots, n.
\]

Hence the functions

\[
J_{ab} = \frac{D_a}{D_b}, \quad a, b = 1, 2, \ldots, n,
\]

are constants of the motion. In fact, we can arrange all these functions as the entries of an \( n \)-dimensional matrix \( [J_{ab}] \) that becomes a matrix formed by constants of the motion (the diagonal elements are just ones) with the fundamental set of functions \( J_{tk} \) placed in the first row.

Finally, the divergence of the vector field \( \Gamma^{(n)} \) is given by \( \text{div} \Gamma^{(n)} = -(n + 2)k x^2 \). Thus, using relation (12), we obtain the following Jacobi multipliers for the Abel equation of order \( n \) (or for the dynamical vector field \( \Gamma^{(n)} \)):

\[
R_a = (D_a)^{\mu_n}, \quad \mu_n = -(n + 2), \quad a = 1, 2, \ldots, n.
\]

We note that all these \( n \) Jacobi multipliers are different, that is \( R_b \neq R_a, b \neq a \), but they are proportional by a function that is an integral of the motion.

6. Final comments

We have studied a chain of higher-order nonlinear Abel equations using, as starting point, the idea that they have many similarities with the higher-order nonlinear Riccati equations. We have made use of the Lagrangian formalism (inverse problem, non-polynomial Lagrangians, nonstandard symmetries) in the case of the second-order equation and of other mathematical tools (Darboux polynomials and Jacobi multipliers) in the case of higher-order nonlinearities. All these questions seem to be really interesting and we think they deserve a deeper study.

Finally, we mention that all these equations have (for any order of the equation) a family of first integrals \( J_{tk} \) that depend on the time as a polynomial in \( t \). In the symplectic case functions of such a class are associated with master symmetries of the (Lagrangian or Hamiltonian) system, but in the general Abel case we have proved the existence of such constants without referring to any symplectic structure. This is, in fact, a very interesting fact that is to be studied.

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