Multiple scalar particle decays and perturbation generation

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Abstract. We study the evolution of the universe which contains multiple non-relativistic scalar fields decaying into both radiation and pressureless matter. We present a powerful analytic formalism for calculating the matter and radiation curvature perturbations, and find that our analytic estimates agree with full numerical results within an error of less than one per cent. Also we discuss the isocurvature perturbation of matter and radiation components, which may be detected by near future cosmological observations, and point out that it crucially depends on the branching ratio of the decay rate of the scalar fields and that it is hard to make any model independent predictions.

Keywords: cosmological perturbation theory, physics of the early universe
1. Introduction

Nowadays it is widely accepted that primordial density perturbations are the origin of the temperature anisotropy in the cosmic microwave background (CMB) and large scale structure in the present observable universe. Various cosmological observations indicate that these perturbations are adiabatic and Gaussian, with an almost scale invariant spectrum [1]. Interestingly, these observational facts are consistent with an earlier inflationary era [2]: during inflation, quantum fluctuations of a slowly rolling scalar field which dominates the energy density, the inflaton, are stretched and become classical perturbations due to the quasi-exponential expansion of the universe. A particularly convenient quantity to study as regards these perturbations is the curvature perturbation $\zeta$ on uniform density hypersurfaces, developed in [3], or $R_c$ on comoving hypersurfaces, which is equivalent to $\zeta$ on large scales. For single-field inflation cases $\zeta$ is known to be conserved on large scales since perturbations are purely adiabatic, and one can obtain the power spectrum of these perturbations with good enough accuracy [4]. Note that, in multi-field inflationary models, in contrast, there exists in general a non-adiabatic pressure perturbation and this makes $\zeta$ no longer conserved on large scales$^3$ [6].

In conventional inflationary models, the inflaton field is assumed to play two roles at the same time: it dominates the energy density during inflation and makes the universe expand enough for solving many cosmological problems such as the homogeneity, isotropy and flatness of the observable universe. Also, its vacuum fluctuations are relevant for the

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$^3$ This is why the power spectrum of primordial perturbations is evaluated only after the possible trajectories of the inflaton fields coalesce in the so-called $\delta N$ formalism [5].
curvature perturbation $\zeta$ and thus responsible for the primordial density perturbations. Generally, the latter requirement introduces extra fine-tuning into the model: for example, in the simplest chaotic inflation model with $V(\phi) = m^2\phi^2/2$, the inflaton mass $m$ can be as large as $\mathcal{O}(m_{\text{Pl}})$, where $m_{\text{Pl}} = (8\pi G)^{-1/2} \approx 2.4 \times 10^{18}$ GeV is the reduced Planck mass, when we do not consider perturbations and try to solve other problems. However, to match the observed amplitude of density perturbations on large scales, we need $m \sim \mathcal{O}(10^{-5}) m_{\text{Pl}}$, i.e. we need a relative fine-tuning of one part in $10^5$ [7]. However, during inflation, any scalar fields with their masses smaller than the Hubble scale acquire almost scale invariant fluctuations. Such fields, depending on the post-inflationary evolution of the universe, may later generate primordial density perturbations by transferring their almost scale invariant isocurvature perturbations to the curvature perturbation.

If this is the case, i.e. in the so-called curvaton scenario [9], such a field, dubbed the ‘curvaton’, should satisfy several requirements: firstly, its effective mass must be light, i.e. less than the Hubble parameter during inflation, to produce an almost flat spectrum of fluctuations and to remain sub-dominant during inflation. It should also couple very weakly to other fields so that its potential in the early universe is not modified appreciably. It is also demanded that it keeps some level of non-zero value [10] and has not yet relaxed to its vacuum expectation value. This is necessary to generate the appropriate amplitude of perturbations. These conditions are basically what the conventional inflaton field, which is assumed to be responsible for the primordial density perturbations, should satisfy, as well as there being enough expansion of the universe. Thus, the curvaton scenario may find its natural accommodation in the context of multi-field inflation [11]: for example, in a recently proposed scenario [12] where a number of string axion fields drive inflation, it is known [13] that there are a number of fields which have not yet relaxed to their minima of the effective potential, with their mass being very small relative to the Hubble parameter during inflation due to the assisted inflation mechanism [14].

Therefore, it is natural to consider the case where multiple curvaton fields are responsible for the generation of the curvature perturbation after inflation. The fluctuations of these curvaton fields are non-adiabatic in nature and thus, as mentioned above, the curvature perturbation $\zeta$ does not remain constant but evolves according to the energy transfer between different components which constitute the universe. In this paper we study this general curvaton model. This paper is outlined as follows. In section 2, we introduce the coupled equations which determine the evolution of the universe. In section 3 we solve these equations analytically in the so-called sudden decay approximation, using a novel and model independent method. In section 4 we apply our results of the previous sections to several examples and compare the analytic estimates with numerical calculations. Finally in section 5 we summarize and present our conclusions.

### 2. Background equations and perturbations

In this section we will summarize the evolution of the background quantities in a flat universe and show the evolution equations for the curvature perturbations of the components in the system of multiple curvatons decaying into radiation and matter. We assume that the universe is initially dominated by radiation due to the decay of the inflaton field(s) after inflation.

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4 There have been some studies on similar scenarios using the decay of neutrino dark matter particles [8].
We assume that the curvatons ($\sigma_i$) decay into both radiation ($\gamma$) and non-relativistic matter (m) with constant decay rates $\Gamma^{(i)}_{\gamma}$ and $\Gamma^{(i)}_{m}$ respectively, which are fixed by underlying physics. The equations for energy transfer between components are given by:

\[ Q_i = -\left(\Gamma^{(i)}_{\gamma} + \Gamma^{(i)}_{m}\right) \rho_i \equiv -\Gamma^{(i)} \rho_i, \quad (2.1) \]

\[ Q_{\gamma} = \sum_i \Gamma^{(i)}_{\gamma} \rho_i \equiv \sum_i Q_{\gamma i}, \quad (2.2) \]

\[ Q_m = \sum_i \Gamma^{(i)}_{m} \rho_i \equiv \sum_i Q_{mi}, \quad (2.3) \]

where we have introduced the total decay width of $\sigma_i$, $\Gamma^{(i)} \equiv \Gamma^{(i)}_{\gamma} + \Gamma^{(i)}_{m}$, and the energy transfer to radiation (matter) by the decay of $\sigma_i$, $Q_{\gamma i}$ ($Q_{mi}$). Note that they obey the constraint of energy conservation

\[ \sum_i Q_i + Q_{\gamma} + Q_m = 0. \quad (2.4) \]

Thus from the general continuity equation of each component including energy transfer [16],

\[ \dot{\rho}_\alpha = -3H(\rho_\alpha + p_\alpha) + Q_\alpha, \quad (2.5) \]

we find that for each component

\[ \dot{\rho}_i = -(3H + \Gamma^{(i)}) \rho_i, \quad (2.6) \]

\[ \dot{\rho}_{\gamma} = -4H \rho_{\gamma} + \sum_i \Gamma^{(i)}_{\gamma} \rho_i, \quad (2.7) \]

\[ \dot{\rho}_m = -3H \rho_m + \sum_i \Gamma^{(i)}_{m} \rho_i. \quad (2.8) \]

Note that we can obtain the continuity equation of the total energy density by summing over that of each component,

\[ \dot{\rho} = -3H(\rho + p) \]

\[ = -H \left( 4\rho_{\gamma} + 3\rho_m + 3 \sum_i \rho_i \right), \quad (2.9) \]

where the total density $\rho$ and pressure $p$ are given by

\[ \rho = \rho_{\gamma} + \rho_m + \sum_i \rho_i, \quad (2.10) \]

\[ p = p_{\gamma} + p_m + \sum_i p_i, \quad (2.11) \]

\[ ^{5} \text{One can add the effect of dark matter freeze-out and annihilation [15], but the qualitative evolution is not too different.} \]
respectively. In the above we take \( p_\gamma = \rho_\gamma / 3 \) and \( p_m = p_i = 0 \), i.e. the equations of state of the curvaton fields are effectively equivalent to that of pressureless matter.

By adopting the density parameters \( \Omega_\gamma, \Omega_m \) and \( \Omega_i \), we can rewrite equations (2.6), (2.7) and (2.8) in more convenient dimensionless forms for numerical calculation. From the Friedmann equation
\[
H^2 = \frac{1}{3m_P^2} \rho, 
\]
the density parameters satisfy the relation
\[
\Omega_\gamma + \Omega_m + \sum_i \Omega_i = 1. 
\]
Then, equations (2.6), (2.7) and (2.8) can be written as
\[
\begin{align*}
\Omega_i' &= \Omega_i \left( \Omega_\gamma - H^{-1} \Gamma^{(i)} \right), \\
\Omega_\gamma' &= H^{-1} \sum_i \Omega_i \Gamma^{(i)} - \Omega_\gamma (1 - \Omega_\gamma), \\
\Omega_m' &= H^{-1} \sum_i \Omega_i \Gamma^{(i)} + \Omega_\gamma \Omega_m,
\end{align*}
\]
and equation (2.12) as
\[
H' = -\frac{3 + \Omega_\gamma}{2} H, 
\]
where a prime denotes a derivative with respect to the number of e-folds,
\[
N \equiv \int H \, dt. 
\]
The total curvature perturbation on uniform curvature hypersurfaces is given by
\[
\zeta = -H \frac{\delta \rho}{\rho}, 
\]
which can be written as a weighted sum of the curvature perturbation of the component \( \alpha \) on the corresponding uniform density hypersurfaces \( \zeta_\alpha [6] \),
\[
\zeta = \sum_\alpha \frac{\dot{\rho}_\alpha}{\rho} \zeta_\alpha, 
\]
where
\[
\zeta_\alpha = -H \delta \rho_\alpha \frac{\rho}{\bar{\rho}_\alpha}. 
\]
The difference between any two components gives an isocurvature perturbation [17]
\[
S_{\alpha\beta} = 3(\zeta_\alpha - \zeta_\beta). 
\]
The total curvature perturbation on large scales evolves as [6]
\[
\dot{\zeta} = -\frac{H}{\rho + p} \delta p_{nad}. 
\]
where the non-adiabatic pressure perturbation is given by
\[
\delta p_{\text{nad}} \equiv \delta p - \frac{\dot{\rho}}{\rho} \delta \rho.
\] (2.24)

Therefore, as mentioned before, \(\zeta\) remains constant on large scales when the perturbations are purely adiabatic. From equations (2.21) and (2.23), and using the perturbed continuity equations of each component \([17, 18]\), we can find that the curvature perturbations of the components evolve on large scales as
\[
\zeta'_i = H^{-1} \frac{\Gamma^{(i)}(3 + \Omega_\gamma)}{2(3 + H^{-1} \Gamma^{(i)})} (\zeta - \zeta_i),
\] (2.25)
\[
\zeta'_\gamma = \left(4 \Omega_\gamma - H^{-1} \sum_i \Gamma^{(i)} \Omega_i\right)^{-1} \left[ \sum_j \Omega_j (3 + H^{-1} \Gamma^{(j)}) H^{-1} \Gamma^{(j)} (\zeta_i - \zeta_\gamma) \right.
\]
\[
- \frac{H^{-1} \sum_k \Gamma^{(k)} \Omega_k}{2} (3 + \Omega_\gamma) (\zeta - \zeta_\gamma),
\] (2.26)
\[
\zeta' = \frac{4 \Omega_\gamma - H^{-1} \sum_i \Gamma^{(i)} \Omega_i}{3 + \Omega_\gamma} (\zeta - \zeta_\gamma).
\] (2.27)

Here we do not solve the evolution of \(\zeta_m\) directly, though it is straightforward to write the evolution equation of \(\zeta_m\): rather, from equation (2.20), \(\zeta_m\) is calculated as
\[
\zeta_m = \frac{\dot{\rho} \zeta - \dot{\rho}_\gamma \zeta_\gamma - \sum_i \dot{\rho}_i \zeta_i}{\dot{\rho}_m}.
\] (2.28)

The reason is the existence of singularity in \(\zeta_m\), because there exists some moment \(\dot{\rho}_m = 0\) when the dilution of matter due to the expansion of the universe is balanced with the creation of matter due to the curvaton decay\(^6\) \([17, 18]\).

We may solve equations (2.14)–(2.17) and (2.25)–(2.27) numerically, which would be the simplest way to study the evolution of the curvature perturbation. However, we can obtain further insights by implementing analytic analysis. In the following section we will find the final curvature perturbations under the so-called sudden decay approximation \([19]\).

3. Analytic approximation

In this section, we study the curvature perturbations under the assumption that there is no interaction between components until the curvaton fields decay and that the decay of each curvaton is instantaneous. Under this ‘sudden decay approximation’, we can derive analytic estimates for the curvature perturbations associated with matter and radiation after all the curvaton fields decay, as we will see in this section. Note that from equations (2.1) to (2.3), after all the curvatos decay, there is no energy transfer between matter and radiation and hence \(\zeta_m\) and \(\zeta_\gamma\) are constant, though \(\zeta\) will still evolve on large scales. In this sense, we will call these \(\zeta_m\) and \(\zeta_\gamma\) after the decay of the curvatons ‘final’ curvature perturbations, and denote them by the superscript (out).

\(^6\) In fact this is the same for the radiation component. However, as long as we assume that the density of radiation is initially high so that the universe is radiation dominated, \(\dot{\rho}_\gamma < 0\) always.
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For our purpose in this section, we decompose the radiation and matter density according to the source of generation,

\[ \rho_\gamma = \rho_{\gamma 0} + \sum_i \rho_{\gamma i}, \]
\[ \rho_m = \rho_{m 0} + \sum_i \rho_{mi}, \]

(3.1)

where \( \rho_{\gamma 0} (\rho_{m 0}) \) is the energy density of radiation (matter) which is due to the decay of the inflaton field(s) and independent of the curvaton decay, and \( \rho_{\gamma i} (\rho_{mi}) \) is the radiation (matter) density generated from the decay of \( \sigma_i \) [20]. Then, equation (2.10) can be written as

\[ \rho = \rho_{\gamma 0} + \rho_{m 0} + \sum_i \left( \rho_{\gamma i} + \frac{\Gamma_{\gamma i}}{\Gamma_{\gamma i}} \rho_i \right) + \sum_i \left( \rho_{mi} + \frac{\Gamma_{mi}}{\Gamma_{mi}} \rho_i \right) \]
\[ \equiv \rho_{\gamma 0} + \rho_{m 0} + \sum_i \tilde{\rho}_{\gamma i} + \sum_i \tilde{\rho}_{mi}, \]

(3.2)

where we have introduced two composite densities \( \tilde{\rho}_{\gamma i} \) and \( \tilde{\rho}_{mi} \) which will play the central role in the discussions below.

3.1. Matter curvature perturbation

From equations (2.1), (2.3) and (3.2), we can see that for the composite density \( \tilde{\rho}_{mi} \),

\[ \tilde{Q}_{mi} = Q_{mi} + \frac{\Gamma_{mi}}{\Gamma_{mi}} Q_i = 0, \]

(3.3)

i.e. the energy transfer is zero. Moreover, since the corresponding equation of state is that of pressureless matter, we can write

\[ \dot{\tilde{\rho}}_{mi} = -3H \tilde{\rho}_{mi}, \]

(3.4)

and therefore the associated curvature perturbation [18],

\[ \tilde{\zeta}_{mi} = -H \frac{\delta \tilde{\rho}_{mi}}{\tilde{\rho}_{mi}} = \frac{\delta \tilde{\rho}_{mi}}{3 \tilde{\rho}_{mi}}, \]

(3.5)

is conserved on large scales. Well before the curvaton \( \sigma_i \) decays \( \rho_{mi} = \delta \rho_{mi} = 0 \) so \( \tilde{\zeta}_{mi} = \zeta_{mi}^{(\text{in})} \); meanwhile after \( \sigma_i \) decays \( \rho_i \) is negligible and thus \( \tilde{\zeta}_{mi} = \zeta_{mi}^{(\text{out})} \). Therefore we have

\[ \zeta_{mi}^{(\text{out})} = \zeta_{mi}^{(\text{in})}. \]

(3.6)

Thus, from equations (2.21), (3.1) and (3.6), we find that the final matter curvature perturbation after all the curvaton decays is given by

\[ \zeta_{m}^{(\text{out})} = \left( \rho_{m 0}^{(\text{in})} \zeta_{m 0}^{(\text{in})} + \sum_i \left( \frac{\Gamma_{mi}}{\Gamma_{mi}} \rho_i^{(\text{in})} \zeta_{mi}^{(\text{in})} \right) \right) \left[ \rho_{m 0}^{(\text{in})} + \sum_j \left( \frac{\Gamma_{mj}}{\Gamma_{mj}} \rho_j^{(\text{in})} \right) \right]^{-1} \]
\[ \equiv \sum_{i=0}^{n} \delta_i \zeta_i^{(\text{in})}, \]

(3.7)
where \( \zeta_0^{\text{(in)}} \equiv \zeta_{m_0}^{\text{(in)}} \). The transfer coefficient \( s_i \) we have introduced above is given by

\[
\begin{align*}
    s_0 &= \Omega_m^{\text{(in)}} \left[ \Omega_m^{\text{(in)}} + \sum_j \frac{\Gamma_m^{(j)}}{\Gamma_j^{(j)}} \Omega_j^{\text{(in)}} \right]^{-1}, \\
    s_i &= \frac{\Gamma_i^{(j)}}{\Gamma_i^{(j)}} \left[ \Omega_i^{\text{(in)}} + \sum_j \frac{\Gamma_m^{(j)}}{\Gamma_j^{(j)}} \Omega_j^{\text{(in)}} \right]^{-1} (i = 1, 2, \ldots, n).
\end{align*}
\]

(3.8)

So we can see that the final matter curvature perturbation is completely determined by the decay rate and the initial energy density \( \rho_i^{\text{(in)}} \), or equivalently, the initial density parameter \( \Omega_i^{\text{(in)}} \) of each curvaton field and that of pre-existing matter as shown above.

### 3.2. Radiation curvature perturbation

In the previous section, we could use the conservation of the curvature perturbation of each composite density \( \tilde{\rho}_{m_i} \) to find the final matter curvature perturbation. This is possible since every \( \tilde{\rho}_{m_i} \) has no energy transfer and in addition a unique equation of state. One may hope that a similar argument is applicable to the other composite density that we have introduced in equation (3.2), \( \tilde{\rho}_{\gamma_1} \), but this is not the case. Nevertheless, \( \tilde{\rho}_{\gamma_1} \) turns out to be a useful quantity to use in calculating the final radiation curvature perturbation as we will see shortly. In this section, we assume that the decay rates of the curvaton fields are different so that they do not decay at the same time: rather, they decay successively due to different decay rates. Without loss of generality we put the order of curvatons by \( \Gamma_i^{(i)} > \Gamma_i^{(i+1)} \).

First we consider a limited time interval around the decay of the curvaton field \( \sigma_1 \), which is assumed to have the largest decay rate. We write a combined density of radiation and the curvaton field:

\[
\rho_{\gamma}^{(1)} \equiv \rho_{\gamma 0} + \tilde{\rho}_{\gamma 1} = \rho_{\gamma 0} + \rho_{\gamma 1} + \frac{\Gamma_{\gamma}^{(1)}}{\Gamma_{(1)}^{(1)}} \rho_1.
\]

(3.9)

Note that although the energy transfer of \( \rho_{\gamma}^{(1)} \) is zero, its equation of state is not unique and thus the corresponding curvature perturbation \( \zeta_{\gamma}^{(1)} \) evolves on large scales. Therefore, as mentioned above, unlike for \( \tilde{\rho}_{m_i} \) we cannot simply connect the initial curvature perturbations in the curvaton fields to the final one in radiation, but rather we have to get through the moments of decay. Now we assume that until \( \sigma_1 \) decays instantaneously there is no energy transfer between the curvaton and radiation. Then, the forms of \( \rho_{\gamma}^{(1)} \) before and after \( \sigma_1 \) decays are written respectively as

\[
\rho_{\gamma}^{(1)} \big|_{\text{before}} = \rho_{\gamma 0} < 1 + \frac{\Gamma_{\gamma}^{(1)}}{\Gamma_{(1)}^{(1)}} \rho_1 < 1,
\]

(3.10)

\[
\rho_{\gamma}^{(1)} \big|_{\text{after}} = \rho_{\gamma 0} > 1 + \rho_{\gamma_1} > 1,
\]

(3.11)

where the superscript \(<1\> (>1)\) means that the expression is evaluated before (after) \( \sigma_1 \) decays, and these densities have the same value at the moment of decay. Since \( \rho_{\gamma_1} \) is
generated only after \( \sigma_1 \) decays, the value of \( \rho_{\gamma_1} \) at the moment of decay corresponds to its initial value and thus

\[
\rho_{\gamma_1}^{\text{(dec)}} = \frac{\Gamma^{(1)}_{\gamma_1}}{\Gamma^{(1)}_{\gamma_1}} \rho_1^{\text{(dec)}}. \tag{3.12}
\]

Using the fact that both \( \rho_{\gamma_0} \) and \( \rho_{\gamma_1} \) scale as \( a^{-4} \), we can write the ratio \( \rho_{\gamma_1}/\rho_{\gamma_0} \) at late times, which is constant after \( \sigma_1 \) decays, as

\[
\frac{\rho_{\gamma_1}}{\rho_{\gamma_0}} = \frac{\rho_{\gamma_1}^{\text{(dec)}}(a^{\text{(dec)}}/a)^4}{\rho_{\gamma_0}^{\text{(dec)}}(a^{\text{(dec)}}/a)^4} = \frac{\Gamma^{(1)}_{\gamma_1}}{\Gamma^{(1)}_{\gamma_1}} \rho_1^{\text{(dec)}}. \tag{3.13}
\]

The individual curvature perturbations \( \zeta_{\gamma_0} \) and \( \zeta_1 \) remain constant on large scales before \( \sigma_1 \) decays. Then, the combined curvature perturbation \( \zeta^{(1)}_{\gamma} \) corresponding to \( \rho^{(1)}_{\gamma} \) is written as

\[
\zeta^{(1)}_{\gamma} \approx (1 - f_1) \zeta_{\gamma_0} + f_1 \zeta_1, \tag{3.14}
\]

where

\[
f_1 = \frac{3(\Gamma^{(1)}_{\gamma_1}/\Gamma^{(1)}_{\gamma_1})\rho_1}{4\rho_{\gamma_0} + 3(\Gamma^{(1)}_{\gamma_1}/\Gamma^{(1)}_{\gamma_1})\rho_1}. \tag{3.15}
\]

Here \( f_1 \), the weight of \( \zeta_1 \), solely describes the evolution of \( \zeta^{(1)}_{\gamma} \) on large scales. After the curvaton \( \sigma_1 \) decays, the energy density \( \rho^{(1)}_{\gamma} \) is identical to \( \rho_\gamma \) at that time and has a unique equation of state. Hence after the decay of \( \sigma_1 \), \( \zeta^{(1)}_{\gamma} \) becomes constant on large scales until the curvaton with the next largest decay width begins to decay, i.e. [17]

\[
\zeta^{(1)}_{\gamma} \approx (1 - f^{(\text{dec})}_1) \zeta_{\gamma_0}^{\text{(in)}} + f^{(\text{dec})}_1 \zeta_1^{\text{(in)}}, \tag{3.16}
\]

where, using equation (3.13), \( f^{(\text{dec})}_1 \) is given by

\[
f^{(\text{dec})}_1 = \frac{3\rho_{\gamma_1}/\rho_0}{4 + 3\rho_{\gamma_1}/\rho_0}. \tag{3.17}
\]

We can take the same step for the successive curvaton decays: e.g. for \( \sigma_2 \) which has the next largest decay width, we just make the replacement

\[
\rho^{(1)}_{\gamma} \big|_{\text{after}} \Rightarrow \rho_{\gamma_0}\!/(\text{new}), \quad \zeta^{(1)}_{\gamma} \Rightarrow \zeta_{\gamma_0}\!/(\text{new}), \tag{3.18}
\]

and so on. In general, after \( i \)th curvaton \( \sigma_i \) decays, the curvature perturbation in the radiation component is constant until the decay of the \((i+1)\)th curvaton, and is written as

\[
\zeta^{(i)}_{\gamma} \approx (1 - f^{(\text{dec})}_i) \zeta^{(i-1)}_{\gamma} + f^{(\text{dec})}_i \zeta^{\text{(in)}}_i, \tag{3.19}
\]

where

\[
f^{(\text{dec})}_i = \frac{3\rho_{\gamma_i}/\rho_0}{4\sum_{k=0}^{i-1}\rho_{\gamma_k}/\rho_0 + 3\rho_{\gamma_i}/\rho_0}. \tag{3.20}
\]
Therefore, after all the $n$ curvatons decay, we find the final curvature perturbations in radiation as

\[
\zeta_\gamma^{(\text{out})} \approx (1 - f_n^{(\text{dec})}) \zeta_\gamma^{(\text{in})} + f_n^{(\text{dec})} \zeta_n^{(\text{in})} \\
= (1 - f_n^{(\text{dec})}) \left( 1 - f_{n-1}^{(\text{dec})} \right) \zeta_\gamma^{(\text{out})} + (1 - f_n^{(\text{dec})}) f_{n-1}^{(\text{dec})} \zeta_{n-1}^{(\text{in})} + f_n^{(\text{dec})} \zeta_n^{(\text{in})} \\
= \ldots \\
= \sum_{i=0}^n r_i \zeta_i^{(\text{in})},
\]

where $\zeta_0^{(\text{in})} \equiv \zeta_0^{(\text{in})}$ and $f_0^{(\text{dec})} = 1$. The transfer coefficient $r_i$ is given by

\[
\begin{align*}
& r_i = \prod_{k=i+1}^n \left( 1 - f_k^{(\text{dec})} \right) f_i^{(\text{dec})} = \left( 1 - \sum_{k=i+1}^n r_k \right) f_i^{(\text{dec})}, & (i = 0, 1, \ldots, n - 1) \\
& r_n = f_n^{(\text{dec})},
\end{align*}
\]

and is completely determined once we find the ratio $\rho_{\gamma i}/\rho_{\gamma 0}$.

### 3.3. Ratio of radiation after curvaton decay

We found in the previous section that the final radiation curvature perturbation depends on the ratio of the radiation generated from curvaton decay with respect to the original radiation component. In this section, we present a general and simple way to calculate this ratio analytically.

From equations (2.6)–(2.8), we can write the continuity equations of the components used in equation (3.2) as

\[
\begin{align*}
\dot{\rho}_0 &= -4H \rho_0, \\
\dot{\rho}_m &= -3H \rho_m, \\
\dot{\rho}_i &= -4H \rho_i - 3H \frac{\Gamma_i}{\Gamma(i)} \rho_i, \\
\dot{\rho}_m &= -3H \rho_m - 3H \frac{\Gamma_m}{\Gamma(i)} \rho_i.
\end{align*}
\]

We can solve these equation analytically and the solutions are given by

\[
\begin{align*}
\rho_0 &= \rho_0^{(\text{in})} \left( \frac{a^{(\text{in})}}{a} \right)^4, \\
\rho_m &= \rho_m^{(\text{in})} \left( \frac{a^{(\text{in})}}{a} \right)^3, \\
\rho_i &= \rho_i^{(\text{in})} \left( \frac{a^{(\text{in})}}{a} \right)^3 \exp \left[ -\Gamma(i)(t - t_0) \right],
\end{align*}
\]

\[
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\]

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\[ \rho_{\gamma i} = \Gamma_\gamma^{(i)} \rho_i^{(\text{in})} \left( \frac{a^{(\text{in})}}{a} \right)^4 \int_{t_0}^{t} \frac{a}{a^{(\text{in})}} \exp \left[ -\Gamma^{(i)} t' \right] \, dt', \] (3.30)

\[ \rho_{m i} = \frac{\Gamma^{(i)} m_i^{(\text{in})}}{\Gamma^{(i)}} \rho_i^{(\text{in})} \left( \frac{a^{(\text{in})}}{a} \right)^3 \left\{ 1 - \exp \left[ -\Gamma^{(i)} (t - t_0) \right] \right\}, \] (3.31)

where we have set the initial time to be \( t_0 \). Now, introducing \([21]\)

\[ z \equiv \frac{a}{a^{(\text{in})}}, \quad x_i \equiv \Gamma^{(i)} t, \] (3.32)

and using equation (3.2), the Friedmann equation,

\[ H^2 = \frac{1}{3m_{\text{pl}}^2} \left[ \rho_{\gamma 0} + \rho_{m 0} + \sum_i (\rho_{\gamma i} + \rho_{m i} + \rho_i) \right], \] (3.33)

becomes

\[ \left( \frac{z'}{z} \right)^2 = x_H^{-2} \left\{ z^{-4} + \frac{m_0^{(\text{in})}}{\Omega_0^{(\text{in})}} z^{-3} + \sum_i \left[ \frac{\Gamma^{(i)} \Omega_i^{(\text{in})}}{\Gamma^{(i)} \Omega_0^{(\text{in})}} z^{-3} e^{-x_i} + \frac{\Gamma^{(i)} m_i^{(\text{in})}}{\Gamma^{(i)} \Omega_0^{(\text{in})}} z^{-3} e^{-x_i} \right] \right\}, \] (3.34)

where

\[ x_H \equiv \frac{\Omega_0^{(\text{in})}^{-1/2}}{H^{(\text{in})}}, \] (3.35)

\[ u_i \equiv \Gamma^{(i)} t' \] and a prime denotes a derivative with respect to \( x_1 \). We choose \( x_1 \) for convenience since the dependence on this particular choice of \( x_1 \) is absorbed into the definition of \( x_H \) as shown above. Finally, introducing a new variable

\[ y \equiv x_H^{1/2} z, \] (3.36)

we finally obtain the dimensionless Friedmann equation

\[ \left( \frac{y'}{y} \right)^2 = y^{-4} + \beta_0 y^{-3} + \sum_i \left[ \alpha_i y^{-3} e^{-x_i} + \beta_i y^{-3} + \alpha_i y^{-4} \int_{x_0}^{x_i} ye^{-u_i} \, du_i \right], \] (3.37)

where the coefficients \( \alpha_i, \beta_i \) and \( \beta_0 \) are defined by

\[ \alpha_i \equiv \frac{\Gamma^{(i)} \Omega_i^{(\text{in})}}{\Gamma^{(i)} \Omega_0^{(\text{in})}} x_H^{-1/2}, \quad \beta_i \equiv \frac{\Gamma^{(i)} \Omega_i^{(\text{in})}}{\Gamma^{(i)} \Omega_0^{(\text{in})}} x_H^{-1/2}, \quad \beta_0 \equiv \frac{\Omega_{m0}^{(\text{in})}}{\Omega_0^{(\text{in})}} x_H^{-1/2}, \] (3.38)

respectively. Then the ratio of radiations which determines the transfer coefficient \( r_i \) is given by

\[ \frac{\rho_{\gamma i}}{\rho_{\gamma 0}} = \alpha_i \int_{x_0}^{x_i} ye^{-u_i} \, du_i, \] (3.39)

where the integrand is suppressed exponentially after the curvaton \( \sigma_i \) decays, and the integral becomes almost constant. In figure 1, we plot this integral as a function of \( x \equiv \Gamma t \). A large change occurs only around the decay time \((x \sim 1)\) and soon becomes constant. We can see that the most significant contribution of this integral comes from the epoch around the moment of decay.
3.4. Final curvature and isocurvature perturbations

After all the curvaton fields decay, i.e. $\Omega_i = 0$, we are left with the overall curvature perturbation given by, from equation (2.20),

$$\zeta = \frac{\dot{\rho}_\gamma}{\dot{\rho}} \zeta^{(\text{out})}_\gamma + \frac{\dot{\rho}_m}{\dot{\rho}} \zeta^{(\text{out})}_m = \frac{4\Omega_\gamma}{4\Omega_\gamma + 3\Omega_m} \zeta^{(\text{out})}_\gamma + \frac{3\Omega_m}{4\Omega_\gamma + 3\Omega_m} \zeta^{(\text{out})}_m. \quad (3.40)$$

The final matter and radiation curvature perturbations are constant on large scales and given by equations (3.7) and (3.21), respectively. Their transfer coefficients are determined by equations (3.8), (3.22) and (3.39). Thus, the isocurvature perturbation of matter and radiation components $S_{m\gamma} = 3(\zeta_m - \zeta_\gamma)$, which is fixed after all the curvaton fields decay so that $\zeta_\gamma$ and $\zeta_m$ become constants, is written as

$$S_{m\gamma}^{(\text{out})} = 3 \left( \zeta_m^{(\text{out})} - \zeta_\gamma^{(\text{out})} \right) = 3 \sum_i (s_i - r_i) \zeta_i^{(\text{in})}. \quad (3.41)$$

A particularly simple case is when all the decay rates are the same: then, from equation (3.8), the transfer coefficient of matter curvature perturbation becomes simply

$$s_i = \frac{\Omega_i^{(\text{in})}}{\sum_j \Omega_j^{(\text{in})}} = \frac{\Omega_i^{(\text{in})}}{1 - \Omega_\gamma^{(\text{in})}}. \quad (3.42)$$
where we have assumed that initially there is no matter component. As can be seen clearly, the most significant contribution to the final matter curvature perturbation comes from the curvaton field which initially occupies the largest energy density among the curvatons. For \( r_i \), we only need to consider a single moment of decay since the curvaton fields decay at the same time. Thus, from equations (3.13) and (3.15), we simply have

\[
 f_i^{(\text{dec})} = \frac{3\rho_{\gamma i}/\rho_{\gamma 0}}{4 + 3\sum_j \rho_{\gamma j}/\rho_{\gamma 0}}, \tag{3.43}
\]

and the final radiation curvature perturbation becomes, from equation (3.16),

\[
 \zeta_\gamma^{(\text{out})} = \left(1 - \sum_i f_i^{(\text{dec})}\right)\zeta_\gamma^{(\text{in})} + \sum_i f_i^{(\text{dec})}\zeta_i^{(\text{in})}. \tag{3.44}
\]

Now, from equations (3.38) and (3.39), we can see that the ratio \( \rho_{\gamma i}/\rho_{\gamma 0} \) is proportional to \( \alpha_i \), which is again proportional to \( \Omega_i^{(\text{in})} \), since \( \text{[integral]} \equiv \int_{x_0}^{x} y \exp(-u_i) \, du_i \) will have the same value as discussed in the previous section. Hence,

\[
 r_i = f_i^{(\text{dec})} = \frac{3\text{[integral]}\alpha_i}{4 + 3\text{[integral]}\sum_j \alpha_j} = \frac{3C\text{[integral]}\Omega_i^{(\text{in})}}{4 + 3C\text{[integral]}\sum_j \Omega_j^{(\text{in})}}, \tag{3.45}
\]

where \( C = \Gamma H^{(\text{in})1/2}/(\Gamma^{3/2}\Omega^{(\text{in})3/4}) \) is the common coefficient of proportionality of \( \alpha_i \) to \( \Omega_i^{(\text{in})} \). Thus, with one further assumption that the initial radiation curvature perturbation is negligible, i.e. \( \zeta_\gamma^{(\text{in})} \approx 0 \), the final isocurvature perturbation is, from equation (3.41),

\[
 S^{(\text{out})}_{\text{m}\gamma} \approx 3 \sum_i \left[ \frac{1}{1 - \Omega_i^{(\text{in})}} - \frac{3C\text{[integral]}}{4 + 3C\text{[integral]}\left(1 - \Omega_i^{(\text{in})}\right)} \right] \Omega_i^{(\text{in})}\zeta_i^{(\text{in})}, \tag{3.46}
\]

and thus the transfer from the initial curvature perturbation \( \zeta_i^{(\text{in})} \) is proportional to the corresponding initial density fraction \( \Omega_i^{(\text{in})} \).

4. Applications

In this section, we apply our analytic estimates obtained in the previous section to several examples and compare with numerical results.

4.1. Single curvaton

First we consider a simple example where a single curvaton field decays into radiation and matter with decay rates \( \Gamma^{(1)}_{\gamma} \) and \( \Gamma^{(1)}_{\text{m}} \), respectively. If we assume that the initial curvature perturbation in radiation is negligible, which is usually taken as the initial condition for the curvaton scenario, the radiation curvature perturbation after curvaton decay is purely due to the decay of the curvaton field and from equations (3.21) is given by

\[
 \zeta_\gamma^{(\text{out})} \approx f_1^{(\text{dec})}\zeta_1^{(\text{in})}, \tag{4.1}
\]
where

$$f^{(\text{dec})}_1 = \frac{3\rho_1/\rho_0}{4 + 3\rho_1/\rho_0}.$$  \hfill (4.2)

As discussed in the previous sections this is constant after the curvaton decay, and is completely determined once we find the ratio $\rho_1/\rho_0$. This ratio is given by equation (3.39) as

$$\frac{\rho_1}{\rho_0} = \alpha_1 \int_{x_0}^{x_1} y(u_1)e^{-u_1} du_1,$$  \hfill (4.3)

and depends only on $x_H$ and $\alpha_1$ given by equations (3.35) and (3.38), respectively.

If initially radiation dominates, i.e. $\Omega_{\gamma}^{(\text{in})} \approx 1$, we find that

$$x_H \approx \frac{\Gamma(1)}{H(\text{in})},$$  \hfill (4.4)

$$\alpha_1 \approx \Omega_{\gamma}^{(\text{in})} \frac{\Gamma(1)}{\Gamma(1)} \left( \frac{\Gamma(1)}{H(\text{in})} \right)^{-1/2},$$  \hfill (4.5)

where $\alpha_1$ becomes identical to $p$ of [18] in the limit $\Gamma(1) \gg \Gamma_m(1)$. In this case since the universe is dominated by the radiation component, $a \propto t^{1/2}$ and $H = (2t)^{-1}$. That is, for small $x_1$ the solution of equation (3.37) is given by

$$y(x_1) = (2x_1)^{1/2},$$  \hfill (4.6)

with $x_1^{(\text{in})} = x_H/2$, and we can see that $y(x_1)$ is independent of $x_H$ [21]. Thus the curvature perturbation depends only on $\alpha_1$, which is shown by using the phase space plot in [17, 18].

Furthermore, in the case where the curvaton does not dominate the density during the evolution, we can further approximate equation (4.3) analytically. From the sudden decay approximation, we can see that $\alpha_1 + \beta_1 \approx \Omega_{\gamma}^{(\text{in})}(\Gamma(1)/H(\text{in}))^{-1/2} \ll 1$ guarantees the radiation domination [17]; thus using $y(t) \simeq x_H^{1/2}(t/t_0)^{1/2}$, we obtain

$$\frac{\rho_1}{\rho_0} \approx \alpha_1 x_H^{1/2} \int_{x_0}^{x_1} \left( \frac{t}{t_0} \right)^{1/2} e^{-u_1} du_1$$

$$= \frac{\alpha_1 x_H^{1/2}}{\sqrt{70} \Gamma(1)} \int_{x_0}^{x_1} u_1^{1/2} e^{-u_1} du_1$$

$$\approx \sqrt{2} \alpha_1 \int_{x_0}^{x_1} u_1^{1/2} e^{-u_1} du_1,$$  \hfill (4.7)

where we have used equations (4.4) and (4.5) and $t_0 = 1/(2H(\text{in}))$ for the last equality.

Now let us take a look at the integral: it is integrated from the initial time to some later time after the curvaton field decays. Since we are free to choose the initial time and the integrand is suppressed at later times after the curvaton decay, we can take the range of integration from zero to infinity without loss of generality. Then the integral becomes just $\Gamma(3/2) = \sqrt{\pi}/2$. Hence,

$$\frac{\rho_1}{\rho_0} = \sqrt{\frac{\pi}{2}} \alpha_1 \approx 1.25331 \alpha_1.$$  \hfill (4.8)
Therefore from equations (4.1), (4.2), and (4.8), the final curvature perturbation \( \zeta_{\gamma}^{(\text{out})} \) after the curvaton decays is

\[
\zeta_{\gamma}^{(\text{out})} \approx \frac{3}{4} \sqrt{\frac{\pi}{2}} \alpha_1 \zeta_{\gamma}^{(\text{in})} \approx 0.939986 \alpha_1 \zeta_1^{(\text{in})},
\]

which is in good agreement with [18].

In the opposite limit \((\alpha_1 + \beta_1 \gg 1)\), i.e. where the curvaton field completely dominates the energy density of the universe before it decays, mostly the region of integration is the matter dominated epoch, and thus \(a \propto t^{2/3}\). We can find the time of the transition from the radiation dominated to the curvaton dominated era \((\Omega_{\gamma 0} = \Omega_1)\) from the sudden decay approximation:

\[
t_{\text{tr}} \approx \frac{1}{2H^{(\text{in})}} \left( \frac{\Omega_{\gamma 0}^{(\text{in})}}{\Omega_1^{(\text{in})}} \right)^2.
\]

Using this, the integral becomes

\[
\frac{\rho_{\gamma 1}}{\rho_{\gamma 0}} \approx \alpha_1 x_H^{1/2} \left[ \int_{t_0}^{t_{\text{tr}}} \left( \frac{t}{t_0} \right)^{1/2} e^{-u_1} du_1 + \left( \frac{t_{\text{tr}}}{t_0} \right)^{1/2} \int_{t_{\text{tr}}}^{t_1} \left( \frac{t}{t_{\text{tr}}} \right)^{2/3} e^{-u_1} du_1 \right],
\]

where \(x_{\text{tr}} = \Gamma t_{\text{tr}}\). Ignoring the contribution from the transient radiation dominated era, we find

\[
\frac{\rho_{\gamma 1}}{\rho_{\gamma 0}} \approx \frac{2^{2/3} \alpha_1}{\Omega_1^{(\text{in})}} \left[ \frac{\Gamma^{(1)}}{H^{(\text{in})}} \left( \frac{\Omega_{\gamma 0}^{(\text{in})}}{\Omega_1^{(\text{in})}} \right) \right]^{-1/6} \int_0^{\infty} u_1^{2/3} e^{-u_1} du_1
\]

\[
\approx 2^{2/3} \frac{\Gamma^{(1)}}{H^{(\text{in})}} \left( \frac{5}{3} \right) (\alpha_1 + \beta_1)^{1/3} \alpha_1
\]

\[
\approx 1.43302 (\alpha_1 + \beta_1)^{1/3} \alpha_1,
\]

where we assume that initially radiation dominates the universe so that \(x_H \approx \Gamma^{(1)}/H^{(\text{in})}\) and \(t_0 \approx 1/(2H^{(\text{in})})\).

For the final matter curvature perturbation, assuming that initially there is no matter component, it is independent of the curvaton domination and is simply given from equations (3.7) and (3.8) by

\[
\zeta_{\text{m}}^{(\text{out})} = \zeta_1^{(\text{in})},
\]

i.e. it is just the same as the initial curvature perturbation in the curvaton, as shown in [18].
4.2. Two curvaton

In this section we consider the next simplest case where there are two curvaton fields decaying into both radiation and matter. If we assume again that the initial curvature perturbation in radiation is negligible, the final curvature perturbation in radiation is, from equation (3.21),

$$\zeta_{\gamma}^{(\text{out})} = r_1 \zeta_{\gamma}^{(\text{in})} + r_2 \zeta_{\gamma}^{(\text{in})}$$

$$= \left(1 - f_2^{\text{(dec)}}\right) f_1^{\text{(dec)}} \zeta_{\gamma}^{(\text{in})} + f_2^{\text{(dec)}} \zeta_{\gamma}^{(\text{in})}, \quad (4.14)$$

where

$$f_1^{\text{(dec)}} = \frac{3 \rho_{\gamma 1} / \rho_{\gamma 0}}{4 + 3 \rho_{\gamma 1} / \rho_{\gamma 0}},$$

$$f_2^{\text{(dec)}} = \frac{3 \rho_{\gamma 2} / \rho_{\gamma 0}}{4 (1 + \rho_{\gamma 1} / \rho_{\gamma 0}) + 3 \rho_{\gamma 2} / \rho_{\gamma 0}}. \quad (4.15)$$

The final curvature perturbation in matter is given by, from equation (3.7),

$$\zeta_{m}^{(\text{out})} = s_1 \zeta_{m}^{(\text{in})} + s_2 \zeta_{m}^{(\text{in})}$$

$$= \frac{\beta_1}{\beta_1 + \beta_2} \zeta_{m}^{(\text{in})} + \frac{\beta_2}{\beta_1 + \beta_2} \zeta_{m}^{(\text{in})}. \quad (4.16)$$

Therefore the final isocurvature perturbation of radiation and matter is now completely determined from equation (3.41). In figure 2, we show some examples where two curvaton fields decay into radiation and matter.

**Figure 2.** The evolution of density parameters (upper row) and curvature perturbations (lower row) for three cases of two curvaton decays: as shown, the energy densities of the two curvaton fields are sub-dominant (left panel), dominant (right panel) and comparable (middle panel) to the radiation energy density. The details are given in table 1.
Table 1. The analytic and numerical results for the cases shown in figure 2. In
the upper half of the table, we give the initial values used in the calculation and
in the lower half we compare the analytic estimates with the numerical results.
For the analytic approximation we first solved equation (3.37) to find the density
ratio equation (3.39) and then used equations (3.20), (3.21) and (3.22). For the
analytic limit we used equations (4.8) and (4.12) to find the radiation ratio in
both limits where the curvaton fields remain sub-dominant/dominant. For the
final matter curvature perturbation we have used equations (3.7) and (3.8) for
analytic estimation. Note that in the middle panel we did not use any of the
analytic limits, since in this case the curvaton fields occupy an amount of energy
density comparable to that of the radiation and this does not correspond to any
of the limiting cases.

|                | Left panel | Middle panel | Right panel |
|----------------|------------|--------------|-------------|
| $\zeta_2^{\text{in}}/\zeta_1^{\text{in}}$ | 0.65       | 0.8          | 0.75        |
| $\Gamma_1^{(1)}/H^{(\text{in})}$ | $10^{-6}$  | $10^{-6}$    | $10^{-8}$   |
| $\Gamma_1^{(2)}/H^{(\text{in})}$ | $10^{-6}$  | $10^{-6}$    | $10^{-8}$   |
| $\Gamma_1^{(1)}/H^{(\text{out})}$ | $10^{-8}$  | $10^{-8}$    | $10^{-10}$  |
| $\Gamma_1^{(2)}/H^{(\text{out})}$ | $10^{-8}$  | $10^{-8}$    | $10^{-10}$  |
| $\Omega_1^{(\text{in})}$ | $10^{-3.7}$ | $10^{-2.5}$  | $10^{-2.0}$ |
| $\Omega_2^{(\text{in})}$ | $10^{-4.0}$ | $10^{-2.8}$  | $10^{-2.1}$ |
| $r_1$          | Analytic approx. | 0.151 008    | 0.582546    | 0.556639    |
|                | Analytic limit      | 0.144 650    | —           | 0.575 484   |
| $r_2$          | Analytic approx. | 0.075 6833   | 0.291 964   | 0.442 155   |
|                | Analytic limit      | 0.072 4969   | —           | 0.423 350   |
| $\zeta_2^{\text{(out)}}/\zeta_1^{\text{(in)}}$ | Analytic approx. | 0.200 202    | 0.816 117   | 0.888 255   |
|                | Analytic limit      | 0.191 773    | —           | 0.892 997   |
|                | Numerical           | 0.195 615    | 0.792 049   | 0.887 648   |
| $s_1$          | Analytic            | 0.666 139    | 0.666 139   | 0.557 312   |
| $s_2$          | Analytic            | 0.333 861    | 0.333 861   | 0.442 688   |
| $\zeta_m^{\text{(out)}}/\zeta_1^{\text{(in)}}$ | Analytic           | 0.883 149    | 0.933 227   | 0.889 327   |
|                | Numerical           | 0.883 149    | 0.933 228   | 0.889 328   |

If the energy density of the curvaton fields remains sub-dominant throughout the
evolution of the universe, which would be guaranteed by the conditions
\[
\alpha_1 + \beta_1 \ll 1, \quad \alpha_2 + \beta_2 \ll 1,
\]
then
\[
\begin{align*}
f_1^{\text{(dec)}} & \approx c_R \alpha_1, \\
f_2^{\text{(dec)}} & \approx c_R \alpha_2,
\end{align*}
\]
where \(c_R = 3\sqrt{\pi/2}/4 \approx 0.939 986\).

Now it is clear that the transfer coefficients are proportional to the initial density
parameter of the corresponding curvaton fields. Thus with sub-dominant curvaton the
isocurvature perturbation is
\begin{equation}
S_{\eta}^{(out)} = 3 \sum_i (s_i - r_i) \zeta_i^{(in)} \\
\approx 3 \left[ \left( \frac{\beta_1}{\beta_1 + \beta_2} - c_R \alpha_1 \right) \zeta_1^{(in)} + \left( \frac{\beta_2}{\beta_1 + \beta_2} - c_R \alpha_1 \right) \zeta_2^{(in)} \right].
\end{equation} (4.19)

For the curvaton dominated case, before they decay, we can take similar steps to in the radiation dominated case. For example in the case of the right panel of figure 2, the transfer coefficients of the matter curvature perturbation \( s_i \) are
\begin{equation}
s_1(2) = \frac{\alpha_1(2)}{\alpha_1 + \alpha_2},
\end{equation} (4.20)
where we have used \( \Gamma^{(1)}_m/\Gamma^{(1)}_m = \Gamma^{(2)}_m/\Gamma^{(2)}_m \). Since the two curvatons dominate at the same epoch, we can use the same normalization for the \( y \)-function; thus \( f_1 \) and \( f_2 \) are easily approximated as
\begin{equation}
f_1 \approx 1, \quad f_2 \approx \frac{\alpha_1}{\alpha_1 + \alpha_2} = \frac{\Omega_1}{\Omega_1 + \Omega_2},
\end{equation} (4.21)
where in the last equality we have used \( \Gamma^{(1)}_\gamma/\Gamma^{(1)}(1) = \Gamma^{(2)}_\gamma/\Gamma^{(2)}(2) \). The transfer coefficients of the radiation curvature perturbation \( r_i \) are
\begin{equation}
r_1(2) = \frac{\alpha_1(2)}{\alpha_1 + \alpha_2}.
\end{equation} (4.22)
The isocurvature perturbation hence almost vanishes, which is shown in the right panel of figure 2.

4.3. Multiple curvatons

Now we consider more general case where there exist a number of curvaton fields decaying into radiation and matter. It is straightforward to extract the final curvature perturbations either numerically by solving equations \( (2.14)-(2.17) \) and \( (2.25)-(2.27) \), or analytically by using equation \( (3.40) \) with \( (3.7) \) and \( (3.21) \). Indeed, as shown in table 2, analytic estimates give good approximations to the full numerical result within an error of 0.7% (5%) with the analytic approximation (analytic limit). However the evolution of each perturbation could be quite non-trivial, as shown in figure 3 where we have plotted several cases with five curvaton fields. We can read the following:

- The evolution of the total curvature perturbation \( \zeta \) depends, not surprisingly, on which component dominates the energy density of the universe. While the curvaton field dominates the energy density, before they decay, \( \zeta \) is the average of the \( \zeta_i \) and is constantly evolving during this epoch, since the curvatons are decaying into radiation and matter. This is clearly seen in the right panel of figure 3. After all the curvatons decay, \( \zeta \) follows \( \zeta_\gamma \) when radiation dominates before matter begins to dominate, and \( \zeta = \zeta_m \) afterwards, as shown in equation \( (3.40) \).
Figure 3. The same as figure 2 but with five curvaton fields. The details of the parameters are given in table 2.

Table 2. The analytic and numerical results of figure 3. As in table 1, we show the initial parameters in the upper half.

|                  | Left panel | Middle panel | Right panel |
|------------------|------------|--------------|-------------|
| $\zeta_i^{(in)}/\zeta_1^{(in)}$ | (1.0, 0.95, 0.9, 0.85, 0.8) | (1.0, 0.95, 0.9, 0.85, 0.8) | (0.6, 0.7, 0.8, 0.9, 1.0) |
| $\log_{10}\left(\Gamma_i^{(i)}/H^{(in)}\right)$ | (-4, -4.5, -5, -5.5, -6) | (-6, -6.5, -7, -7.5, -8) | (-8, -8.5, -9, -9.5, -10) |
| $\log_{10}\left(\Gamma_m^{(i)}/H^{(in)}\right)$ | (-6, -6.5, -7, -7.5, -8) | (-8, -8.5, -9, -9.5, -10) | (-10, -10.5, -11, -11.5, -12) |
| $\log_{10}\left(\Omega_i^{(in)}\right)$ | (-3, -3.25, -3.5, -3.75, -4) | (-3, -3.25, -3.5, -3.75, -4) | (-3, -2.75, -2.5, -2.25, -2) |
| $r_1$            | Analytic approx. 0.065 6799 | 0.187 597 | 0.009 008 59 |
|                  | Analytic limit 0.064 2904 | — | 0.004 060 52 |
| $r_2$            | Analytic approx. 0.065 0727 | 0.172 502 | 0.025 6607 |
|                  | Analytic limit 0.062 5240 | — | 0.014 2846 |
| $r_3$            | Analytic approx. 0.064 6559 | 0.166 930 | 0.076 8897 |
|                  | Analytic limit 0.060 9761 | — | 0.052 7442 |
| $r_4$            | Analytic approx. 0.064 2456 | 0.163 447 | 0.228 450 |
|                  | Analytic limit 0.059 6027 | — | 0.196 372 |
| $r_5$            | Analytic approx. 0.063 7326 | 0.160 106 | 0.659 821 |
|                  | Analytic limit 0.058 3711 | — | 0.732 362 |
| $s_\gamma^{(out)}/s_\gamma^{(in)}$ | Analytic approx. 0.291 284 | 0.768 726 | 0.950 306 |
|                  | Analytic limit 0.275 926 | — | 0.963 727 |
|                  | Numerical 0.291 515 | 0.765 150 | 0.956 406 |
| $s_m^{(out)}/s_1^{(in)}$ | Analyticapprox. 0.950 652 | 0.950 652 | 0.901 304 |
|                  | Numerical 0.950 670 | 0.950 652 | 0.901 304 |
Multiple scalar particle decays and perturbation generation

Figure 4. All the parameters are the same as for the right panel of figure 3 except the ratios of branching to matter: here, $\Gamma_i^{(m)}/H^{(in)}$ is given by $10^{-19}$, $10^{-17.5}$, $10^{-15}$, $10^{-13.5}$ and $10^{-11}$ for each curvaton. In this case, we have $\zeta_\gamma^{(out)}/\zeta_1^{(in)} = 0.956524$ ($\zeta_\gamma^{(out)}/\zeta_1^{(in)} = 0.950257$ with the analytic approximation) and $\zeta_m^{(out)}/\zeta_1^{(in)} = 0.602384$, making $\zeta_m^{(out)}$ not negligible.

- $\zeta_\gamma$ and $\zeta_m$ evolve only during the curvaton field decay and remain constant after the curvaton field decay since, as mentioned before, there is no energy transfer between radiation and matter. In particular, since matter is assumed to be produced purely due to the decay of the curvaton, $\zeta_m$ is greatly affected no matter whether the curvaton fields dominate the energy density before decay or not, e.g. in the left panel of figure 3 where the curvaton never contribute significantly, their impact on $\zeta_m$ is large: when $\rho_{m0} = 0$, $\zeta_m$ is just a weighted sum of the initial curvature perturbations of the curvaton and the weight $s_i$ is basically the ratio of the corresponding curvaton energy density multiplied by the branching ratio to matter to the total curvaton energy density responsible for matter density, as shown in equation (3.8). For $\zeta_\gamma$, it is noticeable that $\zeta_\gamma$ becomes significant only when the curvaton fields occupy a significant fraction of the total energy density before they decay, as can be seen by comparing the different columns of figure 3. This is because practically the radiation is completely generated by the decay of curvaton fields, making the pre-existing radiation irrelevant.

- From the discussion above, one may tempted to conclude that there will be negligible isocurvature perturbation of matter and radiation if the curvaton dominate before they decay, because they are both generated due to the decay of the curvaton fields. This is not true when there are a number of curvaton fields: the final isocurvature perturbation is dependent on the background parameters such as curvaton densities and decay rates. For example, in figure 4, the ratio of branching to matter of the
curvaton \( \sigma_5 \) which has the largest energy density is extremely small, i.e.

\[
\frac{\Gamma_{m}^{(5)}}{\sqrt{\Gamma_{m}^{(5)}}} \approx 10^{-16}.
\] (4.23)

Thus, although \( \zeta_m \) receives a contribution from the decay product of the curvaton with large energy density and this gives a rise of \( \zeta_m \), this rise is never enough to catch up \( \zeta_i \) to make \( \zeta_m^{(\text{out})} \) vanishing if the branching ratio is very small, as in this case. This is reminiscent of multi-field inflation: in multi-field inflation, there is no unique prediction on the isocurvature perturbation produced during inflation. The detail depends on the inflaton trajectory in the field space. Likewise, generally we can hardly make any definite prediction on the isocurvature perturbation without the detail.

5. Conclusions

In this paper, we have studied the evolution of the universe which contains a number of non-interacting scalar particles (the ‘curvatons’ \( \sigma_i \)) decaying into radiation (\( \gamma \)) and pressureless matter (\( m \)) after inflation. We first wrote the evolution equations for the background densities of the components \( \rho_i, \rho_\gamma \) and \( \rho_m \) which compose the universe and of the curvature perturbations of the corresponding components \( \zeta_i, \zeta_\gamma \) and \( \zeta_m \) on flat hypersurfaces, equations (2.14)–(2.17) and (2.25)–(2.27). These equations can be numerically solved and give the resulting curvature perturbations of the components, as well as the total curvature perturbation \( \zeta \) given by equation (2.20).

Using the sudden decay approximation, we have obtained analytic estimates of the final radiation and matter curvature perturbations \( \zeta_\gamma^{(\text{out})} \) and \( \zeta_m^{(\text{out})} \) which are in good agreement with full numerical results. With the composite densities \( \tilde{\rho}_i \) and \( \tilde{\rho}_m \) given by equation (3.2), we can relate \( \zeta_\gamma^{(\text{out})} \) and \( \zeta_m^{(\text{out})} \) to the initial curvature perturbations associated with the curvatons \( \zeta_i^{(\text{in})} \): the curvature perturbation \( \tilde{\zeta}_mi \) is conserved and hence the final matter curvature perturbation \( \zeta_m^{(\text{out})} \) has a very simple relation to \( \zeta_i^{(\text{in})} \) equation (3.7) with the transfer coefficient \( s_i \) given by equation (3.8). Meanwhile, \( \tilde{\zeta}_\gamma^i \) is not constant on large scales since the equation of state of \( \tilde{\rho}_\gamma \) is not unique. Nevertheless, we can find \( \zeta_\gamma^{(\text{out})} \) written in terms of \( \zeta_i^{(\text{in})} \) as equation (3.21), with the transfer coefficient \( r_i \) given by equation (3.22). \( r_i \) is determined once the ratio \( \rho_\gamma / \rho_\gamma^0 \) is found, and we have found a general and model independent result: equation (3.39). This might also be useful for investigating non-Gaussianity of the primordial curvature perturbation in the multi-curvaton scenario [22].

We have applied our results to several different cases. The analytic estimates give good enough fits to the full numerical results, within an error of \( O(0.1) \) %. When the curvatons dominate the energy density of the universe before they decay, the final radiation curvature perturbation \( \zeta_\gamma^{(\text{out})} \) is significantly affected by the curvature perturbations of the curvatons \( \zeta_i \), since practically radiation is generated by the decay of the curvaton fields and the pre-existing radiation is irrelevant. More importantly, the isocurvature perturbation of matter and radiation given by equation (3.41) depends on the detailed decay rate of the curvatons: for example, in the right panel of figure 2, \( \zeta_\gamma^{(\text{out})} \) and \( \zeta_m^{(\text{out})} \) are of almost
the same amplitudes and thus isocurvature perturbation is highly suppressed. However, as shown in figure 4, when the ratios of branching to matter are different for different curvaton, we may have significant isocurvature perturbation depending on the initial values of the background quantities. We can determine $S_{m\gamma}^{(out)}$ which may be detected in the CMB observations only when we have detailed information on the curvaton fields.

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