Langevin equation with Coulomb friction

Hisao Hayakawa

Department of Physics, Yoshida-South Campus, Kyoto University, Kyoto
606-8501, Japan

Abstract

We propose a Langevin model with Coulomb friction. Through the analysis of the corresponding Fokker-Planck equation, we have obtained the steady velocity distribution function under the influence of the external field.

Key words: Langevin equation, Coulomb friction, Velocity distribution function

1 Introduction

The Langevin equation is widely used to describe the motion of stochastic particles. In a typical situation a Brownian particle is driven by the random force originated from collisions of molecules in the environment and the friction force proportional to the velocity of the particle.

The friction force is not always proportional to the velocity of particles. A well-known counter example can be found in the dry friction of macroscopic materials, so called Coulomb friction in which the dynamical friction is independent of the speed of a material but depends on the direction of the motion. In general, the motion of such the material obeying Coulomb friction is not affected by the thermal fluctuations, because the material is too large to be fluctuated by the thermal force. However, if the material we consider is enough small like a nanomaterial or the thermal fluctuation is replaced by the mechanical fluctuation force, Coulomb friction force with fluctuating random force may be relevant. In a recent paper, in fact, Kawarada and Hayakawa suggest that Langevin equation associated with Coulomb friction can be an effective equation in a vibrating system of granular particles through their simulation of granular particles. They demonstrate that (i) granular particles in a quasi-two-dimensional container obey an exponential velocity distribution function (VDF), (ii) the exponential VDF is produced by Coulomb slip between particles and fixed scatters because their
simulation without Coulomb slip produces a Gaussian-like VDF, (iii) Langevin equation with Coulomb friction produces an exponential VDF, and (iv) the relation between the diffusion coefficient \( D_x \) and the granular temperature \( T \) in their simulation obeys \( D_x \propto \sqrt{T} \) as is expected from Langevin equation with Coulomb friction. Their result may encourage us to look for relevancy of such Langevin equation in physical systems, and investigate mathematical properties of the equation.

In this paper, we investigate mathematical properties of Langevin equation with Coulomb friction. The organization of this paper is as follows. In the next section, we introduce our model. In section 3, we obtain the steady VDF under the influence of the steady external field. In section 4 we discuss our result through the comparison of our result with the simulation of driven granular particles in a quasi-two-dimensional box. In the final section, we conclude our results.

2 Langevin model

Let us consider the motion of a particle whose velocity is \( \mathbf{v} \) subjected to the dynamical Coulomb friction force \(-\zeta m \mathbf{v}/|\mathbf{v}|\) with the mass \( m \). When we assume that the motion is affected by the external force \( \mathbf{F}_{ex} \) and the uncorrelated fast agitation by the environment, the equation of motion is given by

\[
\frac{d\mathbf{v}}{dt} = -\zeta \frac{\mathbf{v}}{|\mathbf{v}|} + \frac{\mathbf{F}_{ex}}{m} + \mathbf{\xi} \tag{1}
\]

where \( \mathbf{\xi} \) is the random force whose \( \alpha \) component is assumed to satisfy

\[
<\xi_\alpha(t)> = 0, \quad <\xi_\alpha(t)\xi_\beta(t')> = 2D_{\alpha,\beta}\delta(t-t'). \tag{2}
\]

Here \(<\cdots>\) represents the average over the distribution of the random variable \( \mathbf{\xi} \) and the diffusion constant \( D \) in the velocity space satisfies the fluctuation-dissipation relation \( D = \zeta \sqrt{T/(3m)} \), where \( T \) is the temperature defined from the second moment of VDF as \( T(t) \equiv \frac{1}{2m} \int d\mathbf{v} \mathbf{v}^2 P(\mathbf{v},t) \). Since we assume that higher order correlations of \( \mathbf{\xi} \) vanish, Langevin equation (1) can be converted into Fokker-Planck equation for the probability function \( P(\mathbf{v},t) \):

\[
\frac{\partial P(\mathbf{v},t)}{\partial t} + \frac{\mathbf{F}_{ex}}{m} \cdot \frac{\partial}{\partial \mathbf{v}} P(\mathbf{v},t) = \frac{\partial}{\partial \mathbf{v}} \left( \frac{\zeta}{|\mathbf{v}|} P(\mathbf{v},t) + D \frac{\partial}{\partial \mathbf{v}} P(\mathbf{v},t) \right). \tag{3}
\]
Here $P(v, t)$ satisfies the normalization condition

$$\int dvP(v, t) = 1. \quad (4)$$

Kawarada and Hayakawa (Kawarada04) indicate that eq.(3) without $F_{ex}$ has the steady solution obeying an exponential VDF. Namely, the VDF approaches

$$P(v, t) \to \varphi_0(v) \equiv \frac{\eta^2}{2\pi} \exp[-\eta v] \quad (5)$$

as time goes on, where $\eta = \zeta/D$. Thus, our system has completely different properties from those in the conventional Langevin model.

It is convenient to use dimensionless distribution function. For this purpose, we may introduce

$$P(v, t) = v_0(t)^{-d} \tilde{P}(c, t), \quad c = v/v_0(t) \quad (6)$$

with the normalization

$$\int dc c^2 \tilde{P}_0(c, t) = 1 \quad (7)$$

where $d$ is the dimension, and $\tilde{P}_0$ is the scaled VDF without $F_{ex}$. We adopt $\tilde{P}_0$ instead of $\tilde{P}$ in eq.(7), because the determination of the scaled factor is difficult if we use $\tilde{P}$. In fact, $T$ is the function of $F_{ex}$ and cannot be determined without the complete form of VDF.

### 3 Steady state distribution function

Let us consider the situation that a particle is moving under the influence of $F_{ex} = mg \hat{z}$ with the unit vector $\hat{z}$ parallel to the direction of the external force and Coulomb friction force associated with the random force. We assume that the system is a two-dimensional one, because the particle is typically located on a substrate when Coulomb friction is important. We are interested in the statistical steady state in the balance between the friction and the external force.

Let $\theta$ be the angle between the direction we consider and $\hat{z}$, the steady equation becomes

$$g[\cos \theta \frac{\partial P}{\partial v} + \frac{\sin^2 \theta}{v} \frac{\partial P}{\partial \cos \theta}] = \zeta \{ \frac{P}{v} + \frac{\partial P}{\partial v} \} + D[\frac{\partial^2 P}{\partial v^2} + \frac{1}{v} \frac{\partial P}{\partial v} + \frac{1}{v^2} \frac{\partial^2 P}{\partial \theta^2}] \quad (8)$$
where $v = |\mathbf{v}|$.

Now we assume the expansion

$$P(v, \theta) = \sum_{n=0}^{\infty} P_n(v) \cos n \theta = P_0(v) + P_1(v) \cos \theta.$$  \hspace{1cm} (9)

Here the terms with $n \geq 2$ are irrelevant, because the contribution of $n \geq 2$ is orthogonal to those of $n = 0, 1$, and the effect of gravity appears in the term of $n = 1$. Thus, the normalization (4) is reduced to

$$2\pi \int_0^\infty dv v P_0(v) = 1,$$  \hspace{1cm} (10)

and eq.(8) becomes

$$g \left[ \cos \theta P_0' + \cos^2 \theta P_1' + \frac{\sin^2 \theta}{v} P_1 \right] = \zeta \left[ \frac{P_0}{v} + P_0' + \left( \frac{P_1}{v} + P_1' \right) \cos \theta \right]$$
$$+ D \left[ P_0'' + \frac{P_0'}{v} + \left( P_1'' + \frac{P_1'}{v} \right) \cos \theta - \frac{P_1}{v^2} \cos \theta \right],$$  \hspace{1cm} (11)

where $P_n'$ and $P_n''$ represent $dP_n/dv$ and $d^2P_n/dv^2$, respectively. From the integrations of eq.(11) multiplied by $\cos n \theta$ with $n = 0, 1$ over $(0, 2\pi)$, we obtain

$$g \left( P_1' + \frac{P_1}{v} \right) = \zeta \left( \frac{P_0}{v} + P_0' \right) + D \left( \frac{P_0''}{v} + \frac{P_0'}{v} \right),$$  \hspace{1cm} (12)
$$g P_0' = \zeta \left( \frac{P_1}{v} + P_1' \right) + D \left( \frac{P_1''}{v} + \frac{P_1'}{v} - \frac{P_1}{v^2} \right).$$  \hspace{1cm} (13)

Equation (12) can be integrated easily as

$$P_1 = \frac{\zeta}{g} P_0 + \frac{D}{g} P_0'$$  \hspace{1cm} (14)

for $g \neq 0$. We can check that $P_0 \propto e^{-\eta v}$ and $P_1 = 0$ are the solution of (12) and (13) which correspond to the steady solution (5) at $g = 0$.

To solve eq.(13) we adopt the form

$$P_0(v) = f(v) e^{-\eta v}$$  \hspace{1cm} (15)
and substitute this into (14), eq.(14) is reduced to
\[ P_1(v) = \frac{D}{g} f'(v) e^{-\eta v}. \]  
(16)

With the aid of eqs.(15) and (16), eq.(13) can be rewritten as
\[ f'''' + \left( \frac{1}{v} - \eta \right) f'''' - \left( \frac{1}{v^2} + \epsilon \eta^2 \right) f' + \epsilon \eta^3 f = 0, \]  
(17)
where \( \epsilon = \left( \frac{g}{\zeta} \right)^2 \).

Since we do not know the general procedure to obtain the solution of the third order ordinary differential equation (17) and it may not be easy to get the numerical solution for eq.(17) around the singular point \( v = 0 \), we rewrite it as a set of the second order differential equations under the assumption of small \( \epsilon \). For this purpose, we expand
\[ f(v) = \frac{\eta^2}{2\pi} [1 + \epsilon f^{(1)} + \epsilon^2 f^{(2)} + \cdots]. \]  
(18)

Here the prefactor is determined by the normalization (10). Thus, we obtain the series of equations:
\[ h_1'' + \left( \frac{1}{v} - \eta \right) h_1' - \frac{1}{v^2} h_1 = -\eta^3 \]  
(19)
\[ h_n'' + \left( \frac{1}{v} - \eta \right) h_n' - \frac{1}{v^2} h_n = -\eta^3 f^{(n-1)} + \eta^2 h_{n-1} \text{ for } n \geq 2 \]  
(20)
with \( h_n(v) \equiv df^{(n)}(v)/dv \). We also assume
\[ \int_0^\infty dv v f^{(n)}(v) e^{-\eta v} = 0 \text{ for } n \geq 1, \]  
(21)
which ensures that the normalization condition (10) is satisfied for any \( \epsilon \).

Equations (19) and (20) have a common homogeneous part supplemented by the inhomogeneous parts in the right hand sides. Thus, at first, let us obtain the solution of the homogeneous equation:
\[ w'' + \left( \frac{1}{v} - \eta \right) w' - \frac{1}{v^2} w = 0. \]  
(22)

The solution of eq.(22) can be obtained by the method of a series-expansion around the singular point \( v = 0 \). The result is
\[ w_1(v) = 2 \sum_{n=0}^{\infty} \frac{\eta^n}{(n+2)!} v^{n+1} = \frac{2}{\eta^2 v} (e^{\eta v} - 1 - \eta v) \]  
\[ w_2(v) = \frac{\eta^2}{2} w_1(v) \ln v - \sum_{k=1}^{\infty} \frac{\eta^k}{k!} \{ \sum_{r=1}^{k} \frac{1}{r} \} v^{k-1} \]
\[ = \frac{\eta^2}{2} w_1(v) \ln v - \frac{e^{\eta v}}{v} \{ \gamma + \ln(\eta v) - Ei(-\eta v) \} \]  
\[ (24) \]

where \( \gamma \) is Euler’s constant \( \gamma = \lim_{n \to \infty} \left( \sum_{r=1}^{\infty} \frac{1}{r} - \ln n \right) = 0.57721 \cdots \) and

\[ Ei(-x) = - \int_{x}^{\infty} dt \frac{e^{-t}}{t} \]  
\[ (25) \]

is the exponential integral function. Here we use Frobenius’ method and Bessel’s formula\( [Arfken95]\)

\[ \sum_{k=1}^{\infty} \sum_{r=1}^{k} \frac{1}{r} \frac{x^k}{k!} = e^x \{ \gamma + \ln x - Ei(-x) \}. \]  
\[ (26) \]

The general solutions of inhomogeneous equation (19) or (20) is given by the sum of a special solution of the inhomogeneous equation and the linear combination of the homogeneous solutions (23) and (24). For \( O(\epsilon) \) it is easy to confirm that the special solution \( h_{1,s} \) is given by

\[ h_{1,s}(v) = \eta^2 v. \]  
\[ (27) \]

On the other hand, the leading order contribution \( e^{\eta v} / v \) should be canceled in the linear combination of \( w_1 \) and \( w_2 \) to obtain the finite result for \( \int_{0}^{\infty} dv v f^{(1)}(v) e^{-\eta v} \). Thus, we obtain the solution as

\[ h_1(v) = c_0 \{ w_2(v) + \frac{\eta^2}{2} (\gamma + \ln \eta) w_1(v) \} + \eta^2 v \]
\[ = c_0 \left\{ \frac{Ei(-\eta v)}{v} e^{\eta v} - \frac{1 + \eta v}{v} (\gamma + \ln(\eta v)) \right\} + \eta^2 v \]
\[ (28) \]

where \( c_0 \) is a constant. We should note that \( h_1(v) \) approaches \( h_1(v) \to -c_0 \eta \) in the limit of \( v \to 0 \).

From the integration we obtain the first order solution as

\[ f^{(1)}(v) = c_0 \left\{ G_{31}^{31} \left[ \eta v | \begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right] - \eta v (\gamma - 1) + (\gamma + \eta v) \ln \eta v - \frac{1}{2} (\ln \eta v)^2 \right\} \]
\[ + \frac{\eta^2}{2} v^2 + c_1 \]  
\[ (29) \]
where $c_1$ is a constant and

$$G_{p,q}^{m,n} \left( z \left| \begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{array} \right. \right) \equiv (2\pi i)^{-1} \int ds z^s \prod_{j=1}^m \Gamma(b_j - s) \prod_{j=1}^n \Gamma(1 - a_j + s)$$

$$\times \left( \prod_{j=m+1}^q \Gamma(1 - b_j + s) \prod_{j=n+1}^p \Gamma(a_j - s) \right)^{-1}$$

(30)

is Meijer’s G function which satisfies

$$\frac{d}{dz} G_{23}^{31} \left( z \left| \begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right. \right) = \frac{e^z}{z} Ei(-z).$$

(31)

From the normalization condition (21) constants $c_0$ and $c_1$ in eq.(29) become

$$c_0 = 1, \quad \text{and} \quad c_1 = 4 - \delta_1 + \frac{\pi^2}{12} - \frac{\gamma^2}{2} \simeq -2.98961,$$

(32)

respectively. Here we use \(\delta_1 = G_{23}^{32} \left( 1 \left| \begin{array}{ccc} -1 & 0 & 1 \\ 0 & 0 & 0 \end{array} \right. \right) \simeq 1.64549\). from the formulae \(\text{from Wille86}\)

$$\int_0^\infty dv e^{-\eta v} G_{23}^{31} \left( \eta v \left| \begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right. \right) = \eta^{-(1+\alpha)} G_{23}^{32} \left( 1 \left| \begin{array}{ccc} -\alpha & 0 & 1 \\ 0 & 0 & 0 \end{array} \right. \right).$$

(33)

Using this distribution function, the mobility $\mu$ defined through

$$\bar{v} \equiv \int dv \int_0^{2\pi} d\theta v \cos \theta P(v, \theta) = \mu F_{ex}$$

(34)

is given by

$$\mu = \frac{\sqrt{T}}{4\sqrt{3m^3}\zeta},$$

(35)

where we use $D = \zeta \sqrt{T/3m}$ \(\text{from Kawarada04}\).

Now, we comment on the dimensionless forms of our result obtained here. With the aid of (6) and (7), it is easy to check $\eta v$ becomes $\sqrt{6} c$ in the scaling form. In addition, the prefactor $D/g$ in (16) becomes $D/(gv_0) = 1/\sqrt{6}c$. Figure 1 plots the scaled $P_0(c)$ and $P_1(c)$ as functions of $c = v/v_0$. 

7
Fig. 1. The scaled $P_0(c)$ (solid line) and $P_1(c)$ (dashed line) as functions of $c = v/v_0$, where $P_n(c)$ denotes $P_0(c)$ or $P_1(c)$ in the figure. For $P_1(c)$ we plot $f^{(1)'}(c)e^{-\sqrt{6}c}$ to remove the effect of the expansion parameter $\epsilon$.

4 Discussion

Let us discuss our result. First, we can obtain higher order terms such as $f^{(2)}(v)$ and $f^{(3)}(v)$, because the homogeneous solutions in eqs.(23) and (24) can be used in any order. However, these solutions may have complicated forms and we had better use numerical integrations to represent inhomogeneous terms. Because of the limitation of the length of this paper, we have omitted to give the explicit forms of higher order terms here.

Second, Kawarada and Hayakawa\cite{Kawarada04} assume that the diffusion coefficient in the real space is proportional to the diffusion constant in the velocity space, but this assumption may not be true for our case. Let us briefly evaluate the diffusion constant in the real space as follows. The basic equation is

$$\partial_t P(r, v, t) = \Gamma P = (\Gamma_0 + \Gamma_1)P$$  \hspace{1cm} (36)

where

$$\Gamma_0 = D\partial_v(\eta \frac{v}{v} + \partial_v) \quad \text{and} \quad \Gamma_1 = -v \cdot \nabla.$$  \hspace{1cm} (37)
Now, we introduce the projection operator

$$\hat{P}h(\mathbf{r}, \mathbf{v}, t) = \varphi_0(\mathbf{v}) \int d\mathbf{v} h(\mathbf{r}, \mathbf{v}, t).$$

(38)

With the aid of Mori-Zwanzig projection method (Zwanzig61, Mori65), in general, the distribution function obeys an identity

$$\partial_t \hat{P}P = \hat{P} \hat{\Gamma} \hat{P}P + \hat{P} \hat{\Gamma} \int_0^t d\tau e^{(t-\tau)\hat{Q}\hat{\Gamma}} \hat{Q} \hat{P}P(\tau)$$

$$+ \hat{P} \hat{\Gamma} e^{\tau\hat{Q}\hat{\Gamma}} \hat{Q} \hat{P}P(t_0)$$

(39)

where $\hat{Q} = 1 - \hat{P}$. It is easy to show that the first and the last terms of right hand side of (39) are zero if the initial distribution is proportional to $\varphi_0(\mathbf{v})$. We also assume that $\Gamma_1$ can be regarded as the perturbation of $\Gamma_0$ because the distribution function is unchanged within the mean-free path of particles. Thus, we may replace $e^{(t-\tau)\hat{Q}\hat{\Gamma}}$ by $e^{(t-\tau)\Gamma_0}$ in the second term of right hand side of (39). Noting

$$\hat{Q} \Gamma_1 \varphi_0(\mathbf{v}) \bar{F}(\mathbf{r}, t) = -\mathbf{v} \cdot \nabla \varphi_0(\mathbf{v}) \bar{F}(\mathbf{r}, t)$$

$$\Gamma_0 \mathbf{v} \varphi_0(\mathbf{v}) = -\zeta \mathbf{v} \varphi_0(\mathbf{v})$$

(40)

(41)

with $\bar{F} \equiv \hat{P}P$, eq.(39) may be reduced to

$$\partial_t \bar{F} = -\int d\mathbf{v} \Gamma \int_0^t d\tau e^{(t-\tau)\Gamma_0} \mathbf{v} \varphi_0(\mathbf{v}) \nabla \bar{F}$$

$$\simeq \nabla \int d\mathbf{v} \int_0^t d\tau e^{-\zeta(\tau-t)} v^2 \varphi_0 \nabla \bar{F}$$

$$\simeq \frac{1}{\zeta} \int d\mathbf{v} v^3 \varphi_0(\mathbf{v}) \nabla^2 \bar{F} = D_x \nabla^2 \bar{F}$$

(42)

in the long time limit $t \gg v_0/\zeta$. Thus, the diffusion coefficient in the real space may be $D_x \propto D^3 / \zeta^4 \propto T^{3/2}$. We should note that this procedure is not exact because $t \gg v_0/\zeta$ does not ensure the approximation $\int_0^t d\tau e^{-\zeta t/v} \simeq v/\zeta$ for large $v \gg v_0$. However, the treatment may be plausible and is contradicted with the previous result of simulation of granular particles.

We also note that the behavior of VDF under the influence of $g$ in granular particles does not coincide with the theoretical prediction presented here quantitatively. As shown in Fig.1, $P_0$ and $P_1$ have peaks around $c = 1$ and
become negative near $c = 0$. The behavior of the simulation for granular particles is qualitatively similar, but the peak position is located near $c = 0.5$ and the negativity in the vicinity of $c = 0$ is not large in the simulation.

Thus, we will have to check whether the connection between Langevin equation with Coulomb friction and the vibrated granular particles confined in a quasi-two-dimensional box is superficial. The detailed quantitative comparisons will be reported elsewhere.

5 Conclusion

In conclusion, we have developed the theory of Langevin equation with Coulomb friction and obtained the steady solution of VDF under the influence of a steady external field. We also discuss the relation between our system with the vibrated granular systems confined in a quasi-two-dimensional box.

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