Algebra of chiral currents on the physical surface

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Abstract
Using a particular structure for the Lagrangian action in a one-dimensional Thirring model and performing the Dirac’s procedure, we are able to obtain the algebra for chiral currents which is entirely defined on the constraint surface in the corresponding hamiltonian description of the theory.
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1 Introduction

The Thirring model is a well known model in quantum field theory because it is exactly solvable in (1+1) dimensions [1]. If this model is treated in a Hamiltonian way in terms of currents, with interaction between them, it can be easily solved by performing a Bogoliubov transformation. This is, concerning to find the fermionic correlators using a bosonization procedure [2].

An extensive investigation of its current algebra has been performed. Dell’Antonio et. al. [3], using an exact expansion in bilocal operators products, have solved the model entirely in terms of currents. They defined the so-called Schwinger term as function of a c-number and then determined their form requiring that a spinor transformation law was fulfilled. With a current regularization, Takahashi and Ogura [4] also obtained a very similar Schwinger term in their formulation of Thirring model.

A particularly interesting description of the current algebra of this model is given by Gomes et. al. [5]. Following a Dirac’s procedure, they are able to calculate the current algebra that lives in the constraint surface. However, the

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inclusion in that surface of the Hamiltonian which governs the dynamics and participates in the algebra it is not assured in their procedure.

The purpose of this paper was, taking advantage of the special Lagrangian action introduced by Floreanini and Jackiw for self-dual fields \cite{1, 2}, we shall to analyze the current algebra in a massless Thirring model where the proposed Lagrangian action for the currents generates a Hamiltonian that is defined also in the constraint surface. In this way, the current algebra obtained by the Dirac’s procedure \cite{9} entirely lives in the same physical surface.

The plan of this paper is the following. In section 2, we review the known current algebra in the Dirac formalism \cite{5}, and also obtain the corresponding Hamiltonian on the constraint surface, which coincides with the usual form for the free theory. In section 3, we repeat the procedure that was successful in the free case, obtaining a complete interacting current algebra defined on the physical surface. Finally in section 4, we examine a formal extension of this procedure for the case of \( N \) fermion currents with interactions, and in order to illustrate, we solve the complete algebra for the \( N = 2 \) case, which exhibits all the interesting features claimed above.

2 Non-interacting Kac-Moody \( U(1) \) algebra of chiral currents on the physical surface

Based on the treatment of fermion fields in terms of currents explored by Dashen and Sharp \cite{10}, Sugawara \cite{11} and Sommerfield \cite{12}, let us start with the following Lagrangian for the right (\( J_R \)) and left (\( J_L \)) currents of the model

\[
L = \frac{1}{2} \int dx dy \left[ J_R(x) \varepsilon(x-y) \partial_x J_R(y) - J_R(x) \varepsilon(x-y) \partial_x J_L(y) - J_L(x) \varepsilon(x-y) \partial_x J_R(y) + J_L(x) \varepsilon(x-y) \partial_x J_L(y) \right],
\]

where \( J_{R,L} \) are the right and left currents and \( \varepsilon \) is the Heaviside function (\( \varepsilon(0) \equiv 0 \) is assumed). This Lagrangian is similar to the one used in \cite{6} and \cite{1}, but here the canonical Hamiltonian is zero. Later a new one will be obtained on the constraint surface, which satisfies the same motion equations.

The constraints for this system are

\[
\chi_1(x) = \Pi_R(x) - \frac{1}{2} \int dy J_R(y) \varepsilon(y - x),
\]

\[
\chi_2(x) = \Pi_L(x) + \frac{1}{2} \int dy J_L(y) \varepsilon(y - x).
\]

These are second class constraints which are automatically conserved because the canonical Hamiltonian vanishes.
Following the Dirac’s procedure we calculate the Dirac’s brackets defined as

\[ \{ A, B \}_D = \{ A, B \} - \int dz d\omega \{ A, \chi_i(z) \} C^{-1}_{i,j}(z, \omega) \{ \chi_j(\omega), B \}, \]

where \( C^{-1}_{i,j} \) is the constraint Poisson bracket matrix

\[ C_{i,j}(z, \omega) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \epsilon(z - \omega), \quad (4) \]

with

\[ C^{-1}_{i,j}(z, \omega) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \delta'(z - \omega), \quad (5) \]

and \( \chi_i \) denotes second class constraints; \( \{ \cdot, \cdot \} \) sets for usual Poisson’s brackets. After a simple calculation we obtain the algebra of currents in this formalism

\[ \{ J_L(x), J_L(y) \}_D = \delta'(x - y), \quad (6) \]

\[ \{ J_R(x), J_R(y) \}_D = -\delta'(x - y), \quad (7) \]

\[ \{ J_L(x), J_R(y) \}_D = 0. \quad (8) \]

This is the result given by Gomes et. al. in [5]. Now we want to obtain a similar Hamiltonian, but assuring by construction [9, 13], that it entirely lives on the constraint surface, and that it governs the dynamics of currents involved in the algebra (6)-(8).

To obtain the Hamiltonian we introduce the following generalized vector

\[ Z^\alpha = \{ \Pi_L; \Pi_R; J_L; J_R \}, \quad (9) \]

on the constraint surface. As constraints themselves are strong identities, we can parameterize the momenta in terms of currents. Furthermore, in order to retrieve Hamiltonian relations between our new variables, we must define a new Hamiltonian on that surface according to

\[ K = H_0 + F, \quad (10) \]
where \( H_0 \) is the initial canonical Hamiltonian and \( F \) obeys the relationship

\[
\frac{\delta F(x)}{\delta u^i} = \int dw dy \left( \frac{\partial Z^\alpha(x)}{\partial u^i(w)} W_{\alpha \beta}(x, y) \frac{\partial Z^\beta(y)}{\partial w} \right),
\]

where \( W_{\alpha \beta} \) is the initial symplectic matrix, \( u^i \) are \( J_L \) or \( J_R \), and \( Z^\alpha \) is a component of \( Z \) defined in (9). It looks like

\[
W_{\alpha \beta}(x - y) = \begin{pmatrix}
0 & 0 & \delta(x - y) & 0 \\
0 & 0 & 0 & \delta(x - y) \\
-\delta(x - y) & 0 & 0 & 0 \\
0 & -\delta(x - y) & 0 & 0
\end{pmatrix}.
\]

The consistency of this procedure is ensured because \( W \) is time independent. In order to obtain \( F \), we must observe its variations along \( J_L \) and \( J_R \)

\[
\int dx \delta F(x) = \int dx dz \left\{ \{ F(x), J_L(z) \} \delta J_L(z) + \{ F(x), J_R(z) \} \delta J_R(z) \right\}
\]

\[
= \int dx dz dw \left( \frac{\partial F(x)}{\partial J_L(w)} \{ J_L(w), J_L(z) \} \delta J_L(z) + \frac{\partial F(x)}{\partial J_R(w)}(z) \{ J_R(w), J_R(z) \} \delta J_R(z) \right)
\]

\[
= -\int dx dz \partial_z \left( \frac{\partial F(x)}{\partial J_R(z)} \delta J_R(z) - \frac{\partial F(x)}{\partial J_L(z)} \delta J_L(z) \right)
\]

\[
= \int dx \left( \frac{\partial F(x)}{\partial J_L(z)} \delta J_L(z) - \frac{\partial F(x)}{\partial J_R(z)} \delta J_R(z) \right).
\]

Note that from (10) and (12) is easy to calculate \( \partial F/\partial J_L \) and \( \partial F/\partial J_R \), then

\[
\int dx \delta F(x) = 2 \int dx [J_L(x) \delta J_L(x) - J_R(x) \delta J_R(x)].
\]

Therefore, due to the vanishing of the canonical Hamiltonian, the modified Hamiltonian is

\[
K = \int dx (J_L^2(x) + J_R^2(x)).
\]
Finally,

\[
\{ K, J_{L,R}(x) \}_D = \pm 2 J'_{L,R}(x).
\] (16)

The advantage of this procedure is that the Hamiltonian that participates in the Kac-Moody algebra is defined on the constraint surface, i.e. on the physical surface, which guarantees that the solution of motion equations is in complete consistency with the algebra. On the other hand, this is very important because it assures that in the quantization procedure of the theory, only gauge degrees of freedom are counted.

3 Chiral current algebra on the physical surface for Thirring model

Now consider the Lagrangian

\[
L = \frac{1}{2} \int dx dy \left[ (J_R(x) - g J_L(x)) \varepsilon(x - y) \partial_+ J_R(y) - (J_L(x) + g J_R(x)) \varepsilon(x - y) \partial_+ J_L(y) \right],
\] (17)

\( g \) being the coupling constant. This one admit the same considerations made in the previous section.

Now, our second-class conserved constraints are

\[
\chi_1 = \Pi_R(x) - \frac{1}{2} \int dy (J_R(y) - g J_L(y)) \varepsilon(y - x), \quad \chi_2 = \Pi_L(x) + \frac{1}{2} \int dy (J_L(y) + g J_R(y)) \varepsilon(y - x). \]
(18)

These are again second class constraints and, as in the non-interacting case, they are conserved (in this sense, it is a closed system).

Now we can follow the Dirac’s procedure and calculate the Dirac’s brackets on the constraint surface. To do this, it is necessary to build \( W \) (that is, the matrix that resumes the former bracket relations). We have

\[
C(z) = \begin{pmatrix} 1 & -g \\ -g & -1 \end{pmatrix} \varepsilon(z).
\] (20)
Note that this matrix is symmetric. In order to calculate Dirac’s brackets, we first need to calculate the inverse of $C$. We obtain

$$C^{-1}(z) = \frac{1}{1 + g^2} \begin{pmatrix} 1 & -g \\ -g & -1 \end{pmatrix} \partial_z \delta(z). \quad (21)$$

The Dirac’s brackets between currents on the constraint surface turn out to be

$$\{J_L(x), J_L(y)\}_D = \frac{1}{1 + g^2} \delta'(x - y), \quad (22)$$
$$\{J_R(x), J_R(y)\}_D = \frac{-1}{1 + g^2} \delta'(x - y), \quad (23)$$
$$\{J_L(x), J_R(y)\}_D = \frac{-g}{1 + g^2} \delta'(x - y), \quad (24)$$
$$\{J_R(x), J_L(y)\}_D = \frac{-g}{1 + g^2} \delta'(x - y). \quad (25)$$

Now, if we calculate the new Hamiltonian on the physical surface by the same way as before, we arrive to

$$K = \int dx [J_L^2(x) + J_R^2(x) + 2gJ_L(x)J_R(x)]. \quad (26)$$

So,

$$\{K, J_{L,R}(x)\}_D = \pm \frac{2}{1 + g^2} \left[ J'_{L,R}(x) + gJ'_{R,L}(x) \right]. \quad (27)$$

We have obtained a modification of the Schwinger term as in [3, 4]. It is remarkable that this result involves the last section, which is reached directly vanishing the coupling constant $g$.

4 An extension of the model

The previous results can be extended straightforwardly to the case of $N$ currents with interactions among all them. Let us to consider the following Lagrangian
\[ L = \frac{1}{2} \int dxdy \sum_{i=1}^{N} \left[ J_i^L(x) - \sum_{j=1}^{N} g_{ij} J_j^R(x) \right] \varepsilon(x-y) \partial_x J_i^L(y) \]
\[ -\frac{1}{2} \int dxdy \sum_{i=1}^{N} \left[ J_i^R(x) + \sum_{j=1}^{N} g_{ij} J_j^L(x) \right] \varepsilon(x-y) \partial_x J_i^R(y), \tag{28} \]

where \( g_{ii} \equiv g \) and \( g_{ij} \equiv h \) \((i \neq j)\). The currents are defined by
\[ J_{i}^{L,R}(x) = \psi_{i}^{L,R}(x) \] \( \psi_{L,R}^{*}(x) \).
\[ \tag{29} \]

and the new constraint set is
\[ \chi_k^L(z) = \Pi_k^L(z) - \frac{1}{2} \int dx \left[ J_i^L(x) - \sum_{j=1}^{N} g_{kj} J_j^R(x) \right] \varepsilon(x-z), \tag{30} \]
\[ \chi_k^R(z) = \Pi_k^R(z) + \frac{1}{2} \int dx \left[ J_i^R(x) + \sum_{j=1}^{N} g_{kj} J_j^L(x) \right] \varepsilon(x-z), \tag{31} \]

where \( J \) and \( \Pi \) are canonical fields, subject to
\[ \{ J_i^{L,R}(x), \Pi_k^{L,R}(y) \} = \delta(x-y) \delta_{ik}. \tag{32} \]

For this case, the whole Kac-Moody algebra in Dirac’s formalism is
\[ \{ J_k^\alpha(x), J_m^\beta(y) \}_D = \{ J_k^\alpha(x), J_m^\beta(y) \}
- \int dzd\omega \{ J_k^\alpha(x), \chi_\mu(z) \} C^{\mu\nu}(z-\omega) \{ \chi_\nu(\omega), J_m^\beta(y) \}, \tag{33} \]

where \( C^{\mu\nu} = \| \{ \chi_\mu, \chi_\nu \} \| \) and \( \int dzC_{\mu}^{\nu}(x-z)C_{\tau}^{\nu}(z-y) = \delta(x-y) \delta_{\mu}^{\nu}. \)

The current Hamiltonian on the current surface of this model is
\[ H = \int dx \left\{ \sum_{i=1}^{N} \left[ (J_i^L(x))^2 + (J_i^R(x))^2 \right] + \sum_{i,j=1}^{N} g_{ij} J_i^L(x) J_j^R(x) \right\}, \tag{34} \]
Give a compact expression for the inverse of a \((N \times N)\) matrix in the more general case is a very complicated task, and it is not very illustrative to our main objective. Thus, we will consider the \(N = 2\) case, which allows to illustrate all desired features of this extension.

In this case we have
\[
C_{\alpha\beta}(z) = \begin{pmatrix}
1 & -g & 0 & -h \\
-g & -1 & -h & 0 \\
0 & -h & 1 & -g \\
-h & 0 & -g & -1
\end{pmatrix} \varepsilon(z), \quad (35)
\]
and, after some calculations, we obtain
\[
\{J^L_k(x), J^L_m(y)\}_D = f\delta'(x - y) \left[ (1 + g^2 + h^2) \left( \delta_{k1}\delta_{1m} + \delta_{k2}\delta_{2m} \right) \\
-2gh \left( \delta_{k1}\delta_{2m} + \delta_{k2}\delta_{1m} \right) \right],
\quad (36)
\]
\[
\{J^R_k(x), J^R_m(y)\}_D = f\delta'(x - y) \left[ (1 + g^2 + h^2) \left( \delta_{k1}\delta_{1m} + \delta_{k2}\delta_{2m} \right) \\
-2gh \left( \delta_{k1}\delta_{2m} + \delta_{k2}\delta_{1m} \right) \right],
\quad (37)
\]
\[
\{J^L_k(x), J^R_m(y)\}_D = -f\delta'(x - y) \left[ (g + g^3 - gh^2) \left( \delta_{k1}\delta_{1m} + \delta_{k2}\delta_{2m} \right) \\
+ (h + h^3 - g^2h) \left( \delta_{k1}\delta_{2m} + \delta_{k2}\delta_{1m} \right) \right],
\quad (38)
\]
\[
\{J^R_k(x), J^L_m(y)\}_D = -f\delta'(x - y) \left[ (g + g^3 - gh^2) \left( \delta_{k1}\delta_{1m} + \delta_{k2}\delta_{2m} \right) \\
+ (h + h^3 - g^2h) \left( \delta_{k1}\delta_{2m} + \delta_{k2}\delta_{1m} \right) \right],
\quad (39)
\]
where
\[
f = \frac{1}{[1 + (g - h)^2][1 + (g + h)^2]}.
\quad (40)
\]
The Hamiltonian on the constraint surface is
\[
K = \int dx \left\{ \sum_{i=1}^{2} \left[ (J^L_i(x))^2 + (J^R_i(x))^2 \right] + \sum_{i,j=1}^{2} g_{ij} J^L_i(x) J^R_j(x) \right\},
\quad (41)
\]
\[
\{K, J_m^{L,R}(x)\}_D = \pm 2f \left\{ (1 + g^2 + h^2) \left[ J_1^{L,R}(x) + g J_1^{R,L}(x) + h J_2^{R,L}(x) \right] \\
-2gh \left[ J_2^{L,R}(x) + g J_2^{R,L}(x) + h J_1^{R,L}(x) \right] \\
\mp (g + g^3 - gh^2)h \left[ J_1^{R,L}(x) + g J_1^{L,R}(x) + h J_2^{L,R}(x) \right] \\
\mp (g + g^3 - g^2 h)h \left[ J_2^{R,L}(x) + g J_2^{L,R}(x) + h J_1^{L,R}(x) \right] \right\} \delta_{1m} \\
\pm 2f \left\{ (1 + g^2 + h^2) \left[ J_2^{L,R}(x) + g J_2^{R,L}(x) + h J_1^{R,L}(x) \right] \\
-2gh \left[ J_1^{L,R}(x) + g J_1^{R,L}(x) + h J_2^{R,L}(x) \right] \\
\mp (g + g^3 - gh^2)h \left[ J_2^{R,L}(x) + g J_2^{L,R}(x) + h J_1^{L,R}(x) \right] \\
\mp (g + g^3 - g^2 h)h \left[ J_1^{R,L}(x) + g J_1^{L,R}(x) + h J_2^{L,R}(x) \right] \right\} \delta_{2m}
\]

These are all the relationship for current algebra for the \( N = 2 \) case. Also, it is direct to obtain the previous results by sequentially turning off the constants \( h \) and \( g \) and setting \( m = 1 \) (sections 3 and 2, respectively). So, in this sense, the consistency of this extension is fulfilled.

## 5 Conclusions

In this contribution, a Lagrangian formulation was proposed for the Thirring chiral current model. It allows to give a treatment for the current algebra entirely on the physical surface, and this algebra has the usual features reported by other authors. We also give a first generalization of our results for \( N \) currents with interaction between them. The remarkable advantage of this procedure is that the Hamiltonian as well as the others elements that participate in the Kac-Moody algebra are defined on the constraint surface. The fact that the Hamiltonian lives on the constraint surface ensures that the equation of motion of the theory has a full consistency with the Kac-Moody algebra that currents satisfy. We consider this result very important because the corresponding quantized theory ("la Dirac", for instance) is a gauge theory. The complete theory that lives on the physical surface ensures that the gauge symmetry will be warranted.

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