Discretization and Continuum Limit
of Quantum Gravity
on a Four-Dimensional Space-Time Lattice

Elmar Bittner\textsuperscript{1,2}, Wolfhard Janke\textsuperscript{1} and Harald Markum\textsuperscript{2}

\textsuperscript{1}Universität Leipzig, Institut für Theoretische Physik, Augustusplatz 10/11, D-04109 Leipzig, Germany
\textsuperscript{2}Vienna University of Technology, Atominstutut, Wiedner Hauptstraße 8-10, A-1040 Vienna, Austria
Abstract

The Regge Calculus is a powerful method to approximate a continuous manifold by a simplicial lattice, keeping the connectivities of the underlying lattice fixed and taking the edge lengths as degrees of freedom. The Discrete Regge Model limits the choice of the link lengths to a finite number. We examine the phase structure of Standard Regge Calculus in four dimensions and compare our Monte Carlo results with those of the \( \mathbb{Z}_2 \)-Regge Model as well as with another formulation of lattice gravity derived from group theoretical considerations. Within all of the three models of quantum gravity we find an extension of the well-defined phase to negative gravitational couplings and a new phase transition. We calculate two-point functions between geometrical quantities at the corresponding critical point and estimate the masses of the respective interaction particles. A main concern in lattice field theories is the existence of a continuum limit which requires the existence of a continuous phase transition. The recently conjectured second-order transition of the four-dimensional Regge skeleton at negative gravity coupling could be such a candidate. We examine this regime with Monte Carlo simulations and critically discuss its behavior.
0.1 Introduction

The construction of a non-trivial quantum theory of gravitation represents one of the major open problems in theoretical physics. Approaches based on perturbative methods failed due to the non-renormalizability of the underlying theory in dimensions greater than two. One possibility to deal with the quantization problem consists in discretization of space-time and quantization via the path integral, in close analogy to quantum field theory on a flat background geometry. There are two related schools attempting at a lattice theory of quantum gravity, dynamical triangulation and Regge quantum gravity. We concentrate on the latter approach and refer to [1] for an excellent review of the former framework. Regge calculus is the only discretization scheme reproducing the Bianchi identities of classical general relativity and is formulated exclusively in terms of the edge lengths of the lattice, independent of any coordinate system. Taking advantage of the freedom in how to choose the lattice structure as well as the action, we investigate in this paper different formulations, all of them based on the original work by Regge [2], namely Standard Regge Calculus (SRC) [3, 4], a Group Theoretical Model (GTM) [5], and the $Z_2$-Regge Model ($Z_2$RM) [6, 7]. Different to ordinary lattice gauge theory, quantum fluctuations are represented by fluctuations in the edge lengths. In particular the Monte Carlo method is a convenient tool for evaluating the discrete Euclidean functional integral in a non-perturbative way. It was shown that within a certain range of the gravitational coupling the path integral converges to make up a well-defined phase of lattice quantum gravity. We present an extension of the phase diagram of the models in four dimensions mentioned above. Due to the fact that a Wick rotation from the Lorentzian to the Euclidean sector of quantum gravity is not feasible, in general the sign of the action in the path integral is not fixed a priori. Therefore we investigate the continuation of the well-defined phase into the region of negative gravitational coupling. It turns out to be terminated by a pronounced transition-like cross-over to an ill-defined phase suggesting to look for a continuum limit. In particular the $Z_2$RM permits accurate investigations of the negative coupling region. Further, we calculate two-point functions to probe the existence of massless quanta of the gravitational field. A candidate for a realistic quantum theory of gravity should reproduce the expected long-range interaction behavior observed in nature.

0.2 Lattice quantum gravity

Any smooth $d$-manifold can be approximated by appropriately gluing together pieces of flat space, called $d$-simplices, ending up with a simplicial lattice. We take the edge lengths as the dynamical degrees of freedom and leave the triangulation of the lattice fixed. Adopting the Euclidean path-integral approach we may write down the partition function

$$Z = \int D[q]e^{-I(q)}, \quad (1)$$
with the gravitational action $I$ introduced in more detail below. The functional integration extends over the squared edge lengths $q_l$ of the links $l$ of the simplicial lattice.

One of the problems with (1) is the ambiguity in performing the link-lengths integration. Commonly the measure is written as

$$D[q] = \prod_l dq_l q_l^{\sigma-1} \mathcal{F}(q),$$

with $\mathcal{F}$ a function of the quadratic edge lengths being equal to one if the Euclidean triangle inequalities are fulfilled and zero otherwise. The question remains whether such a local measure is sufficient at all and how the power $\sigma$ might be chosen. In an effort to clarify the role of the measure the conventional definition of diffeomorphisms has been employed in the two-dimensional case, assuming that a piecewise linear space, i.e. a Regge surface, is exactly invariant under the action of the full diffeomorphism group [8]. After a conformal gauge fixing was performed in the continuum formalism with the DeWitt measure, it was shown that the evaluation of the non-local Faddeev-Popov determinant by using such a Regge regularization leads to the usual Liouville field theory results in the continuum limit. All that is based on a description of piecewise linear manifolds with deficit angles, not edge lengths, and is mostly taken as an argument that the correct measure of Standard Regge Calculus has to be non-local. However, to our knowledge it is not obvious that this argument carries over to a discretized Lagrangian, which is formulated in terms of fluctuating edge lengths, obeying triangle inequalities, and which is not invariant under the diffeomorphism group due to the presence of curvature defects: different assignments of edge lengths correspond to different physical geometries, and as a consequence there are no gauge degrees of freedom in Standard Regge Calculus, apart from special geometries (like flat space) [9, 10]. Furthermore, the generalization of this procedure to higher dimensions and its numerical implementation are technically demanding and some of the proposed non-local measures do not agree with their continuum counterparts in the weak field limit, which is a necessary condition for an acceptable discrete measure [11]. But this property is fulfilled for the standard simplicial measure (2).

Working in Euclidean space, i.e. with positive definite metric, the conformal mode renders the four-dimensional continuum Einstein-Hilbert action unbounded from below. This unpleasant feature persists in the discretized Regge-Einstein action but need not necessarily lead to an ill-defined path integral [3]. Indeed, numerical simulations reveal the existence of a well-defined phase with finite expectation values within a certain range of the bare gravitational coupling. A lattice action for gravitation is given by

$$I(q) = -\beta \sum_t R_t(A_t, \delta_t) + \lambda \sum_s V_s,$$

where the first sum runs over all triangles $t$ with areas $A_t(q)$ and the corresponding deficit angles $\delta_t(q)$ yield the curvature elements $R_t$. The second term extends over the volumes $V_s(q)$ of all four-simplices $s$ and allows together with the cosmological constant $\lambda$ to set an overall scale in the action. Next we define our three different models for subsequent numerical treatment.
0.2.1 Standard Regge Calculus (SRC)

Here we employ the Regge-Einstein action with $R_t = A_t \delta_t$, set the cosmological constant $\lambda = 1$ and choose the gravitational measure to be uniform, $\sigma = 1$ [3, 4]. In the classical continuum limit the Regge action $2 \sum_t R_t$ is equivalent to the Einstein-Hilbert action $\int d^4x \sqrt{g} R$ if the so-called fatness $\phi_s$ of a 4-simplex obeys [12]

$$\phi_s \sim \frac{V_s^2}{\max_{\ell \in \partial} (q^4_\ell)} \geq f = \text{const} > 0.$$  \hspace{1cm} (4)

A lower limit $f = 10^{-4}$ restricts the configuration space and thus facilitates numerical simulations [13].

0.2.2 Group Theoretical Model (GTM)

Constructing the dual of a simplicial lattice, Poincaré transformations can be assigned to its links to yield an action in which the $\sin$ of the deficit angle enters, $R_t = A_t \sin \delta_t$, and that reduces to the Regge action in the small curvature limit [5]. While in the classical continuum limit the deficit angles become small, this is in general not the case in the path-integral quantization on a finite lattice, where one has to sum over all possible discrete configurations. A priori it is hence not clear whether both SRC and the GTM result in the same quantum continuum theory. For comparison we again use the uniform measure with $\sigma = 1$, set the cosmological constant $\lambda = 1$, and choose a lower limit on the fatness, $f = 10^{-4}$.

0.2.3 $Z_2$-Regge Model ($Z_2$RM)

This model was invented in an attempt to reformulate SRC as the partition function of a spin system [6, 7]. It is defined by restricting the squared link lengths to take on only two values

$$q_t = b_t (1 + \epsilon \sigma_t), \quad \sigma_t \in Z_2.$$  \hspace{1cm} (5)

By setting $b_t = 1, 2, 3, 4$ for edges, face diagonals, body diagonals and the hyper-body diagonal of a hypercube, respectively, the link lengths are allowed to fluctuate around their flat-space values. The Euclidean triangle inequalities are automatically fulfilled as long as $\epsilon \leq \epsilon_{\text{max}} \in \mathbb{R}_+$ and therefore $\mathcal{F} = 1$ in any case. Consequently, the uniform measure in the quantum-gravity path-integral becomes identical to unity for all possible link configurations. The action can be rewritten in terms of complicated local “spin-spin” interactions. Nevertheless, numerical simulations of the $Z_2$RM become extremely efficient by implementing look-up tables and a heat-bath algorithm. Computations have been performed with the parameter $\epsilon = 0.0875$ and the cosmological constant $\lambda = 0$ because (5) already fixes the average lattice volume.
0.3 Phase structure

Monte Carlo simulations with at least 50k sweeps for each value of the coupling $\beta$ have been performed on regularly triangulated hypercubic lattices with toroidal topology and $N_0 = 4^4$ vertices. We measured expectation values of the average link length $\langle q \rangle = \langle N_1^{-1} \sum_l q_l \rangle$ ($N_1$ denotes the total number of links) and the average curvature

$$\langle R \rangle = \left\langle \frac{2 \sum_t R_t}{\sum_s V_s} q \right\rangle,$$

within the models described above.

Let us first concentrate on SRC, cf. figures 1a and 2a. The expectation values of the average link length change only slightly, increase near $\beta - c \approx -0.16$ and $\beta + c \approx 0.116$, and become arbitrarily large for $\beta < \beta_c$ and $\beta > \beta_c$ indicating the emergence of an ill-defined phase with spike-like structures. The expectation value of the average curvature is negative in the well-defined phase, but $|\langle R \rangle|$ becomes very large in the ill-defined phase where simplices collapse into degenerate configurations with long links and small volumes. The coupling of SU(2)-gauge theory to SRC was examined in [14] and turns out to have little influence on the gravitational phase diagram.

The GTM behaves very similar, see figures 1b and 2b. Again expectation values are finite and well-defined only in a certain coupling region between $\beta - c \approx -0.115$ and $\beta + c \approx 0.14$. In contrast to the Regge model the curvature $\langle R \rangle$ is mostly positive in this coupling interval. However, as long as we deal with non-renormalized entities it is not clear whether the observed differences in the phase structure are physically significant. Although SRC and GTM have the same classical continuum limit, it is necessary to demonstrate that they coincide in their critical properties in order to have the same quantum continuum limit.

The $Z_2$RM is always well-defined because (5) limits the link lengths a priori, cf. figures 1c and 2c. The lattice freezes, forming characteristic configurations where the systems with continuously varying edge lengths become ill-defined. Such a model is of course better suited for investigations of the phase boundaries which here occur at $\beta_c^- \approx -4.665$ and $\beta_c^+ \approx 22.3$. While the transition at $\beta_c^+$ is quite clearly of first order, the behavior at $\beta_c^-$ is more subtle and will be discussed in some detail below in Section 0.5. Different to SRC and the GTM the four kinds of links in the $Z_2$RM are not equivalent by definition (5). This is reflected in the corresponding average link lengths $q$ in a single configuration, see figure 3. Thus we can examine their respective influence on the phase transitions. For example the hyperbody diagonal shows a considerable discontinuity at $\beta_c^+$ and exactly this diagonal was identified from weak-field calculations of Regge gravity to belong to the (five) spurious metric degrees of freedom [15]. It would be interesting to clarify whether physical and spurious degrees of freedom decouple as predicted from weak-field Regge theory. A simulation of the $Z_2$RM with the hyperbody diagonal fixed results in unchanged order of the transition but in slightly shifted values of the critical couplings. An intriguing question remains: to what extent are the (ten) physical degrees of freedom responsible for (the order of) the phase transitions?

A related issue was investigated in [16] using the SRC on general, non-regular tri-
angulations of the four-torus. Also there, even with additional higher-order terms in the action, the phase transition is influenced by the local coordination numbers.

0.4 Correlation functions

One important feature of a physically relevant theory of quantum gravity should be the existence of a massless graviton. The gravitational weak-field propagator can be cast into a spin-zero and a spin-two part [17], with the latter written in terms of connected curvature-curvature correlations,

\[ G_R(d) = \langle \sum_{t \supset v_0} R_t \sum_{t' \supset v_d} R_{t'} \rangle_c . \]

The local operators should be measured at two vertices \( v_0 \) and \( v_d \) separated by the geodesic distance \( d \), which we take to be equal to the index distance along the main axes of the skeleton. This seems a reasonable approximation in the well-defined phase with its small average curvature. In general one expects for (7) at large distances the functional form

\[ G_R(d) \sim \frac{e^{-md}}{d^a} . \]

A power law with \( a = 2 \) and a vanishing effective mass \( m = 0 \) would hint at Newtonian gravity with massless gravitons. Indeed this was found in [17] for SRC with higher-order curvature terms at some positive critical coupling. In contrast to that, fast decaying exponentials were reported for (7) and also for volume correlations in the case of SRC at \( \beta^+ \) [18].

We now compare SRC and GTM (on \( 4^3 \times 8 \) lattices) with \( \mathbb{Z}_2 \)RM (on \( 8^4 \) lattice) [19]. The gravitational couplings were chosen close to the negative critical coupling \( \beta^-_c \) where there is a chance for a continuous phase transition and thus a continuum limit. Figure 4 displays our Monte Carlo data for the two-point functions (7). We took only distances \( d \geq 2 \) into account because \( G_R(d = 1) \) is plagued by lattice artifacts due to contact terms. In order to test whether (7) obeys a power law we fixed \( a = 2 \) and fitted the effective masses \( m \), cf. table 1. For SRC \( m \) remains rather constant towards the critical coupling whereas in the GTM the mass decreases for \( \beta \to \beta^-_c \). However for all fits the uncertainties in the mass parameters are large allowing even for \( m = 0 \). For both the \( \mathbb{Z}_2 \)RM and the GTM, power-law fits to (8) are compatible with vanishing effective mass \( m \). Also for SRC an algebraic decay is compatible but, due to the large uncertainties, obviously no definite conclusion can be drawn here.
0.5 Continuum limit at negative gravitational coupling

The Discrete Regge model $Z_2$RM – like full Regge theory – exhibits two phase transitions [20]. One is located at a negative value and the other one at a positive value of the bare gravitational coupling. Although earlier work concentrated on the latter transition [21], there is no reason for favoring a positive value of $\beta$ over a negative one due to the Wick rotation problem mentioned in the Introduction. In the vicinity of the transition at positive $\beta$, histograms, e.g. of $A_t\delta_t$ [20], clearly show a two-peak structure, see figure 5. The two phases coexist and tunnelling from one phase to the other and back takes place. The system also exhibits a hysteresis; the transition occurs at a larger value of $\beta$ if the simulation is started from a configuration in the well-defined phase than it does if the calculation is started from a “frozen” configuration. Given all these pieces of evidence, we conjecture that the transition at $\beta > 0$ is of first order.

To determine the order of the transition at negative $\beta$ we employed in Ref. [20] histogram techniques and used the Binder–Challa–Landau (BCL) cumulant criterion [22]. The BCL cumulant is defined as

$$B_L := 1 - \frac{\langle E^4 \rangle}{3 \langle E^2 \rangle^2}$$

with $E$ being the action of the system under consideration. It was evaluated for $A_t\delta_t$ on different lattice sizes with $L = 3$ to 10 vertices per direction, simulating the system at several values of the bare coupling $\beta$ with high statistics (using always $\epsilon = 0.0875$ and $\lambda = 0$). For the BCL cumulant a trend towards $2/3$ was observed and all histograms showed a clear one-peak structure, cf. Ref. [20]. In our recent Monte Carlo simulations [23] we typically generated $200\,000 – 500\,000$ iterations, and recorded for every run the time series of the energy density $e = E/N_0$ and the magnetization density $m = \sum_l \sigma_l/N_0$, where $N_0 = L^4$ is the lattice size. To obtain results for the various observables $O$ at values of the bare gravitational coupling $\beta$ in an interval around the simulation point $\beta_0$, the reweighting method [24] was applied. By this means we can compute the specific heat,

$$C(\beta) = \beta^2 N_0 (\langle e^2 \rangle - \langle e \rangle^2),$$

and the (finite lattice) susceptibility,

$$\chi(\beta) = N_0 (\langle m^2 \rangle - \langle |m| \rangle^2),$$

in a certain $\beta$-range around the simulation point. Figure 6 shows the finite-size scaling (FSS) of the maxima of the specific heat $C_{\text{max}}$ and the susceptibility $\chi_{\text{max}},$

| $\beta$  | SRC | GTM | $Z_2$RM |
|---------|-----|-----|---------|
| $-0.155$| $-0.157$ | $-0.159$ | $-0.110$ | $-0.111$ | $-0.112$ | $-4.668$ |
| $m$     | 2(9) | 3(24) | 3(17)   | 1.0(21) | 0.9(8)  | 0.5(7)   | 0.5(9)   |

Table 1: Effective masses $m$ of the curvature two-point function $G_R(d)$ for gravitational couplings close to $\beta_+^\text{c}$. 
respectively. While the behavior of $C_{\text{max}}$ could still be explained as critical scaling with a negative exponent $\alpha$, the flattening of $\chi_{\text{max}}$ as $L$ increases is quite unusual for a second-order transition and should rather be taken as an indication for a cross-over regime.

Another feature of the system can be seen in figure 7 depicting the critical gravitational coupling, as determined from the specific-heat maxima $C_{\text{max}}$ and the susceptibility maxima $\chi_{\text{max}}$. In the case of a second-order transition the extrapolations of all pseudo-transition points lead to one infinite-volume critical value. This seems to be violated in the four-dimensional Discrete Regge model and thus again favors the interpretation as a cross-over phenomenon over a true, thermodynamically defined phase transition.

## 0.6 Summary and conclusion

We explored the complete phase structure of three different formulations of lattice quantum gravity in four dimensions including the region of negative gravitational couplings [20]. The qualitative resemblance of the results from simulations of the $Z_2$-Regge Model to those with continuously varying edge lengths is particularly remarkable. We computed two-point functions close to the critical bare gravitational coupling in the negative coupling regime. In the case of the Group Theoretical Model as well as for the $Z_2$-Regge Model we found some evidence for long-range correlations corresponding to massless gravitons. Eventually, if there is universality between SRC, GTM, and $Z_2$RM, the same results for physical quantities are to be expected in the continuum limit.

In the present analysis of the scaling of the maxima of the specific heat and susceptibility we found evidence for a cross-over regime in the four-dimensional Discrete Regge model of quantum gravity in the negative coupling region [23]. If this can be substantiated by further investigations and also for the Regge theory with continuous link lengths, the existence of a continuum limit at negative bare gravitational coupling is questionable. This is of major concern for a continuum theory of quantum gravity with matter fields [25].

## 0.7 Acknowledgments

E.B. and W.J. were supported by the EU-Network HPRN-CT-1999-000161 “Discrete Random Geometries: From Solid State Physics to Quantum Gravity”.

Figure 1: Expectation values of the average link lengths as a function of the gravitational coupling for (a) SRC, (b) GTM, and (c) $Z_2$RM.
Figure 2: Expectation values of the average curvature as a function of the gravitational coupling for (a) SRC, (b) GTM, and (c) $Z_2$RM. All models exhibit a related phase structure and could lie in the same universality class.
Figure 3: Lattice average of the squared link length of type $i$ as a function of the gravitational coupling for (a) SRC, (b) GTM, and (c) $Z_2$RM. The symbol $\square$ denotes the edges, $\circ$ the face diagonals, $\triangle$ the body diagonals, and $\times$ the hyperbody diagonal of a triangulated hypercube. The dashed lines are to guide the eyes.
Figure 4: Curvature correlation functions for (a) SRC, (b) GTM, and (c) $Z_2$RM in the vicinity of $\beta_c^-$. The curves show fits to the Yukawa ansatz (8) with fixed parameter $a = 2$. The resulting masses $m$ are compiled in table 1.
Figure 5: Histograms of the Regge action $R_t = A_t \delta_t$ from simulations of the $Z_2$-Regge Model. The distributions indicate a first-order transition around $\beta_c^+ = 22.3$ (upper plot) and a continuous transition around $\beta_c^- = -4.665$ (lower plot).
Figure 6: FSS of (a) the specific-heat maxima $C_{\text{max}}$ and (b) the susceptibility maxima $\chi_{\text{max}}$ close to $\beta_c$ as a function of the lattice size $L$. In particular the behavior of $\chi_{\text{max}}$ is indicative for a cross-over rather than a true (continuous) phase transition.

Figure 7: Volume dependence of the (pseudo-)critical gravitational couplings $\beta(C_{\text{max}})$ and $\beta(\chi_{\text{max}})$, as determined from the locations of $C_{\text{max}}$ and $\chi_{\text{max}}$ shown in figure 6.
Bibliography

[1] D.A. Johnston, Nucl. Phys. B (Proc. Suppl.) 53 (1997) 43.
[2] T. Regge, Nuovo Cimento 19 (1961) 558.
[3] B.A. Berg, Phys. Rev. Lett. 55 (1985) 904; Phys. Lett. B176 (1986) 39.
[4] H.W. Hamber, Phys. Rev. D45 (1992) 507; Nucl. Phys. B400 (1993) 347.
[5] M. Caselle, A. D’Adda and L. Magnea, Phys. Lett. B232 (1989) 457.
[6] W. Beirl, H. Markum and J. Riedler, Int. J. Mod. Phys. C5 (1994) 359; W. Beirl, P. Homolka, B. Krishnan, H. Markum and J. Riedler, Nucl. Phys. B (Proc. Suppl.) 42 (1995) 710; W. Beirl, A. Hauke, P. Homolka, B. Krishnan, H. Kröger, H. Markum and J. Riedler, Nucl. Phys. B (Proc. Suppl.) 47 (1996) 625.
[7] T. Fleming, M. Gross and R. Renken, Phys. Rev. D50 (1994) 7363.
[8] P. Menotti and P. Peirano, Phys. Lett. B353 (1995) 444; Nucl. Phys. B473 (1996) 426; B488 (1997) 719.
[9] J. Hartle, J. Math. Phys. 26 (1985) 804; 27 (1986) 287; 30 (1989) 452.
[10] C. Holm and W. Janke, Nucl. Phys. B (Proc. Suppl.) 42 (1995) 722; Nucl. Phys. B477 (1996) 465.
[11] H.W. Hamber and R.M. Williams, Nucl. Phys. B487 (1997) 345; Phys. Rev. D59 (1999) 064014.
[12] J. Cheeger, W. Müller and R. Schrader, Commun. Math. Phys. 92 (1984) 405.
[13] W. Beirl, E. Gerstenmayer, H. Markum and J. Riedler, Phys. Rev. D49 (1994) 5231.
[14] W. Beirl, B.A. Berg, B. Krishnan, H. Markum and J. Riedler, Phys. Rev. D54 (1996) 7421.
[15] M. Roček and R.M. Williams, Z. Phys. C21 (1984) 371.
[16] W. Beirl, H. Markum and J. Riedler, Phys. Lett. B341 (1994) 12.
[17] H.W. Hamber, Phys. Rev. D50 (1994) 3932.
[18] W. Beirl, H. Markum and J. Riedler, Nucl. Phys. B (Proc. Suppl.) 34 (1994) 736.

[19] W. Beirl, A. Hauke, P. Homolka, H. Markum and J. Riedler, Nucl. Phys. B (Proc. Suppl.) 53 (1997) 735; P. Homolka, W. Beirl, H. Markum, J. Riedler and H. Kröger, Nucl. Phys. B (Proc. Suppl.) 53 (1997) 769.

[20] J. Riedler, W. Beirl, E. Bittner, A. Hauke, P. Homolka and H. Markum, Class. Quant. Grav. 16 (1999) 1163.

[21] H.W. Hamber, in: Proceedings of the 1984 Les Houches Summer School, Session XLIII, edited by K. Osterwalder and R. Stora (North Holland, Amsterdam, 1986); H.W. Hamber and R.M. Williams, Phys. Lett. B157 (1985) 368; H.W. Hamber, Phys. Rev. D61 (2000) 124008.

[22] K. Binder and D.P. Landau, Phys. Rev. B30 (1984) 1477; M.S.S. Challa, D.P. Landau and K. Binder, Phys. Rev. B34 (1986) 1841.

[23] E. Bittner, W. Janke and H. Markum, Nucl. Phys. B (Proc. Suppl.) 119 (2003) 924.

[24] A.M. Ferrenberg and R.H. Swendsen, Phys. Rev. Lett. 61 (1988) 2635; 63 (1989) 1195, 1658(E).

[25] E. Bittner, W. Janke and H. Markum, Phys. Rev. D66 (2002) 024008; Acta Physica Slovaca 52 (2002) 241.