Solitary waves of the two-dimensional Camassa–Holm—nonlinear Schrödinger equation

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Abstract
In this work, we study solitary waves in a (2 + 1)-dimensional variant of the defocusing nonlinear Schrödinger (NLS) equation, the so-called Camassa–Holm-NLS (CH-NLS) equation. We use asymptotic multiscale expansion methods to reduce this model to a Kadomtsev–Petviashvili (KP) equation. The KP model includes both the KP-I and KP-II versions, which possess line and lump soliton solutions. Using KP solitons, we construct approximate solitary wave solutions on top of the stable continuous-wave solution of the original CH-NLS model, which are found to be of both the dark and antidark type. We also use direct numerical simulations to investigate the validity of the approximate solutions, study their evolution, as well as their head-on collisions.

Keywords: Camassa–Holm-NLS, multiscale expansion, dark solitons, antidark solitons

(Some figures may appear in colour only in the online journal)
1. Introduction

It is well known that numerous physically relevant nonlinear evolution equations, are completely integrable via the inverse scattering transform, and possess soliton solutions [1, 2]. In particular, universal equations like the nonlinear Schrödinger (NLS), the Korteweg–de Vries (KdV)—as well as its two-dimensional generalization, the Kadomtsev–Petviashvili (KP) equation—, the sine-Gordon equation and others, are ubiquitous in numerous physical settings, such as water waves, plasmas, mechanical systems, nonlinear optics, Bose–Einstein condensates, and so on [2–5]. Interestingly, these soliton equations can be connected—i.e. reduced one to the other—via multiscale expansion methods [6]. Such connections have proved to be extremely useful in the study of non-integrable models: for instance, in the context of optics, the reduction of certain perturbed defocusing NLS equations to the KdV model allowed not only for the derivation of approximate dark (gray) soliton solutions, but also the prediction of novel structures, such as the anti-dark solitons having the form of humps (instead of dips) on top of a continuous-wave background [7–10]. Importantly, relevant studies have also been performed in two-dimensional (2D) settings, leading to the prediction of structures such as ring solitons [11–16], as well as line solitons and lumps satisfying effective KP equations [17–21].

On the other hand, recently, there has been an interest on the study of deformations of integrable equations, which can give rise to novel interesting models in their own right. For instance, the deformation of the KdV equation leads to the Camassa–Holm (CH) equation (which is also an integrable model originally derived in the context of water waves [22]), while the deformation of the NLS equation results in a variant that has been referred to as the Camassa–Holm-nonlinear Schrödinger (CH-NLS) equation [23, 24]. The focusing version of the CH-NLS model was studied in [24], both analytically and numerically, and its bright soliton solutions, as well as their dynamics and interactions were explored. Notice that hereafter the term soliton will be used even when the model is not necessarily integrable, referring to solitary wave structures in such a case. Furthermore, in the recent work [25] similar investigations, but for the defocusing CH-NLS model, were performed; in fact, the methodology used in [25] was relying on the reduction of the CH-NLS model to the KdV equation, which allowed for the derivation of dark and anti-dark soliton solutions. Nevertheless, apart from the above mentioned works, to the best of our knowledge, there exist no studies devoted to the 2D version of the CH-NLS equation.

It is the purpose of this work to contribute to this direction by studying the (2 + 1)-dimensional defocusing CH-NLS equation. Here, we will adopt multiscale expansion methods to eventually reduce the considered model to a KP equation; this allows us to construct approximate soliton solutions, having the form of line solitons or lumps, being of the dark or of the anti-dark type. A brief description of our findings, as well as the outline of the presentation, is as follows. In section 2, we present the model, and study both the linear and the nonlinear regime. We present results of multiscale expansion methods that are used for the derivation of asymptotic reductions of the CH-NLS equation. In particular, at an intermediate stage of the asymptotic analysis, we obtain a 2D Boussinesq-type equation, and also obtain its far-field, namely a pair of KP equations (that are either of the KP-I or KP-II type) for right- and left-going waves. In section 3, we use solutions of the KP-I and KP-II models to construct approximate soliton solutions of the original CH-NLS equation; the derived solutions have the form of dark or anti-dark line solitons and lumps. We also present results of direct numerical simulations concerning the dynamics and interactions between the various approximate soliton solutions. We find that, for sufficiently small amplitudes, all the derived solutions persist and
can undergo quasi-elastic head-on collisions. Finally, in section 4, we summarize our findings and present our conclusions, as well as suggest a number of directions for future study.

2. Model and its analytical consideration

2.1. Presentation of the model and some of its properties

As indicated above, our aim is to study the 2D CH-NLS equation, which is a generalization of the 1D model derived in [23, 24], when developing a theory of a deformation of hierarchies of integrable systems. The 2D CH-NLS equation is expressed as follows:

\[ \text{i} \sigma m \Delta u + 2 \sigma m (|u|^2 - a^2 \nabla |u|^2) = 0, \quad m = u - a^2 \Delta u, \]  

(1)

where \( u(x,y,t) \) and \( m(x,y,t) \) are complex fields, \( \Delta = \nabla^2 \) is the Laplacian in 2D, and \( \nabla = (\partial_x, \partial_y) \) is the gradient operator. In addition, \( \sigma = \pm 1 \) pertains, respectively, to focusing or defocusing nonlinearity, while the constant \( a \) arises within the Helmholtz operator; note that for \( a = 0 \) the above model reduces to the standard, 2D NLS equation. I.e. this parameter measures the ‘size’ of the departure from the original NLS limit.

In terms of the complex field \( u \), the CH-NLS equation can be expressed as:

\[ \text{i} u_t + \Delta u + 2 \sigma u |u|^2 - 2 \sigma a^2 u \nabla |u|^2 - 2 \sigma a^2 \Delta u |u|^2 + 2 \sigma a^4 \Delta u |\nabla u|^2 = 0. \]  

(2)

The simplest nontrivial solution of equation (2) is the continuous-wave (cw):

\[ u = u_0 \exp(2\sigma |u_0|^2 t), \]  

(3)

where \( u_0 \) is an arbitrary complex constant. Since below we will seek nonlinear excitations (e.g. solitary waves) which propagate on top of this cw background, it is relevant to investigate if this solution is subject to modulational instability (MI): naturally, nonlinear excitations corresponding to an unstable background will be less physically relevant. The stability of the cw solution (3) can be investigated upon using the ansatz \( u = u_0(1 + \tilde{u}) \exp(2i\sigma |u_0|^2 t + i\theta) \), where the small perturbations \( \tilde{u} \) and \( \theta \) are assumed to be \( \propto \exp(i(k \cdot r - \omega t)) \), with \( r = (x, y) \), while \( k = (k_x, k_y) \) and \( \omega \) denote the perturbation wave-vector and frequency, respectively. Then, it can readily be found that \( \omega \) and \( k \equiv |k| \) obey the following the dispersion relation:

\[ \omega^2 = \frac{k^2 (-4\sigma |u_0|^2 + k^2)}{(1 + a^2 |k|^2)^2}. \]  

(4)

It is observed that in the case of the defocusing nonlinearity, i.e. for \( \sigma = -1 \), the cw solution is always modulationaly stable, i.e. \( \omega \in \mathbb{R} \forall k \in \mathbb{R} \). On the other hand, for a focusing nonlinearity, \( \sigma = +1 \), the cw solution is unstable for \( k^2 > 4|u_0|^2 \): in this case, perturbations grow exponentially, with the instability growth rate given by \( \text{Im}(k) \). Note that, for \( a = 0 \), and in the 1D case (e.g. \( k_y = 0 \)), equation (4) reduces to the well-known [26] result for the modulational (in)stability of the NLS equation: \( \omega^2 = k^2 (-4\sigma |u_0|^2 + k^2) \). Clearly (and as was also found in the 1D case [24, 25]), the MI band is shared between NLS and CH-NLS and, in both cases, the cw (3) is modulationaly stable in the defocusing realm of \( \sigma = -1 \). For this case, it is also relevant to mention that the long-wavelength limit \( (k \to 0) \) of equation (4) provides the (squared) ‘sound velocity’

\[ C^2 = 4|u_0|^2, \]  

(5)

namely the velocity of small-amplitude linear excitations propagating on top of the cw background.
An important aside regarding equation (1) concerns its potential conservation laws. First of all, it has been verified (both analytically and numerically) that the mass

\[ M = \int_{\mathbb{R}^2} |m|^2 \, dx \, dy \]  

remains a conserved quantity. On the other hand, in the 1D context, the Hamiltonian of the system is a momentum-like quantity of the form:

\[ P_{1D} = \frac{i}{2} \int_{\mathbb{R}} (\bar{m}_x - m_x) \, dx \]  

(7)

(where overbar denotes complex conjugation), while the Hamiltonian structure is non-canonical as discussed, e.g. in [24]. Identifying a generalization of this non-canonical Hamiltonian formulation to the higher dimensional case is an intriguing open problem.

2.2. Multiscale expansions and reduced models

We now consider small-amplitude slowly-varying modulations of the cw solution, and look for solutions of equation (2) in the form of the following asymptotic expansions:

\[ u = u_0 \rho \exp \left[ -2i |u_0|^2 t + i \epsilon^{1/2} \Phi(X, Y, T) \right], \]  

(8)

\[ \rho = 1 + \sum_{j=1}^{\infty} \epsilon^j \rho_j(X, Y, T), \]  

(9)

where the phase \( \Phi \) and densities \( \rho_j \) are unknown real functions of the slow variables

\[ X = \epsilon^{1/2} x, \quad Y = \epsilon^{1/2} y, \quad T = \epsilon^{1/2} t, \]  

(10)

while \( 0 < \epsilon \ll 1 \) is a formal small parameter. Substituting the expansions (8) and (9) into equation (2), and separating real and imaginary parts, we obtain the following results. First, the real part of equation (2) leads, at orders \( O(\epsilon) \) and \( O(\epsilon^2) \), to the following equations:

\[ \Phi_T + C^2 \rho_1 = 0, \]  

(11)

\[ a^2 \Delta \Phi_T - 2 |u_0|^2 (2 \rho_2 + 3 \rho_1^2 - a^2 |\nabla \Phi|^2) + \Delta \rho_1 - \rho_1 \Phi_T - |\nabla \Phi|^2 = 0, \]  

(12)

where \( \Delta \equiv \partial_x^2 + \partial_y^2 \) and \( \nabla \equiv (\partial_x, \partial_y) \). Second, the imaginary part of equation (2), at orders \( O(\epsilon^{1/2}) \) and \( O(\epsilon^{3/2}) \) yields:

\[ \rho_{1T} + \Delta \Phi = 0, \]  

(13)

\[ \rho_{2T} + (1 + a^2 C^2) \rho_1 \Delta \Phi + a^2 \Phi_T \Delta \Phi - a^2 \Delta \rho_{1T} + 2 \nabla \Phi \cdot (a^2 \nabla \Phi_T + \nabla \rho_1) = 0. \]  

(14)

Then, eliminating the functions \( \rho_1 \) and \( \rho_2 \) from the system of equations (11)–(14), we derive the following equation for \( \Phi(X, Y, T) \):

\[ \Phi_{TT} - C^2 \Delta \Phi + \epsilon \left\{ 2a^2 \Delta \Phi_{TT} - 4(1 - 3a^2 |u_0|^2)(\nabla \Phi_T \cdot \nabla \Phi) - 2 \Phi_T \Delta \Phi - \Delta^2 \Phi \right\} = O(\epsilon^2). \]  

(15)

At the leading-order, equation (15) is a second-order linear wave equation, while at order \( O(\epsilon) \) it incorporates fourth-order dispersion and quadratic nonlinear terms, similar to the
Boussinesq and the Benney–Luke [27] equations. These models have originally been used to describe bidirectional shallow water waves [2], but also ion-acoustic waves in plasmas [3], as well as mechanical lattices and electrical transmission lines [4]. Note that the present analysis generalizes the results of [25] (where the 1D case was studied) to the 2D setting; in that regard, it is worth mentioning that similar 2D Boussinesq equations were derived from 2D NLS equations with either a local [28, 29] or a nonlocal [16, 21] defocusing nonlinearity.

Next, using a multiscale expansion method similar to the one employed in the water wave problem [2], we will derive the far-field of the Boussinesq equation, namely a pair of two KP equations for right- and left-going waves. These models will be obtained under the additional assumption of unidirectional propagation. We thus seek solutions of equation (15) in the form of the asymptotic expansion:

\[ \Phi = \Phi_0 + \epsilon \Phi_1 + \cdots, \]  

(16)

where the unknown functions \( \Phi_j \) (\( j = 1, 2, \ldots \)) depend on the variables:

\[ \chi = X - CT, \quad \tilde{\chi} = X + CT, \quad \sqrt{\epsilon}Y, \quad T = \epsilon T. \]  

(17)

Substituting equation (16) into equation (15), and using equation (17), we obtain the following results. First, at the leading order, \( \mathcal{O}(1) \), we obtain the wave equation:

\[ -4C^2 \Phi_{0\chi\tilde{\chi}} = 0, \]  

(18)

which implies that \( \Phi_0 \) can be expressed as a superposition of a right-going wave, \( \Phi_0^{(R)} \), depending on \( \chi \), and a left-going wave, \( \Phi_0^{(L)} \), depending on \( \tilde{\chi} \), namely:

\[ \Phi_0 = \Phi_0^{(R)} + \Phi_0^{(L)}. \]  

(19)

Second, at \( \mathcal{O}(\epsilon) \), we obtain the equation:

\[ 4C^2 \Phi_{1\chi\tilde{\chi}} = C(3\alpha^2 C^2 - 2)\Phi_{0\chi\tilde{\chi}}^{(R)} \Phi_{0\chi\tilde{\chi}}^{(L)} + \left\{ 2C\Phi_{0\chi}^{(L)} - \frac{3C}{2}(2 - \alpha^2 C^2)\Phi_{0\chi}^{(L)^2} - (1 - 2\alpha^2 C^2)\Phi_{0\chi\tilde{\chi}}^{(L)} \right\} \tilde{\chi} - C^2 \Phi_{1\chi\tilde{\chi}}^{(L)} \]  

\[ - \left\{ 2C\Phi_{0\chi}^{(R)} - \frac{3C}{2}(2 - \alpha^2 C^2)\Phi_{0\chi}^{(R)^2} + (1 - 2\alpha^2 C^2)\Phi_{0\chi\tilde{\chi}}^{(R)} \right\} \chi + C^2 \Phi_{1\chi\tilde{\chi}}^{(R)}. \]  

(20)

When integrating equation (20) with respect to \( \chi \) or \( \tilde{\chi} \), it is observed that secular terms arise from the curly brackets in the right-hand side of this equation, which are functions of \( \chi \) or \( \tilde{\chi} \) alone, not both. Hence, these secular terms are set to zero, which leads to two uncoupled non-linear evolution equations for \( \Phi_0^{(R)} \) and \( \Phi_0^{(L)} \). Next, employing equation (11), it is found that the amplitude \( \rho_1 \) can also be decomposed to a left- and a right-going wave, i.e. \( \rho_1 = \rho_1^{(R)} + \rho_1^{(L)} \), with

\[ \Phi_{0\chi}^{(R)} = C\rho_1^{(R)}, \quad \Phi_{0\chi}^{(L)} = -C\rho_1^{(L)}. \]  

(21)

Using these and the above equations for \( \Phi_0^{(R)} \) and \( \Phi_0^{(L)} \) yields the following equations for \( \rho_1^{(R)} \) and \( \rho_1^{(L)} \):

\[ 2C\rho_1^{(R)} - 3C^2(2 - \alpha^2 C^2)\rho_1^{(R)} + (1 - 2\alpha^2 C^2)\rho_1^{(R)} \chi + C^2 \rho_1^{(R)} \tilde{\chi} = 0. \]  

(22)
\[
\left[2C\rho_1^{(L)} + 3C^2(2 - a^2C^2)\rho_1^{(L)} - (1 - 2a^2C^2)\rho_1^{(L)} \right] - C^2\rho_1^{(L)} = 0.
\]

(23)

The result of this analysis is the emergence of two KP equations for left- and right-going waves. Without loss of generality, below we focus on the right-going wave. To proceed further, and express the KP of equation (22) in its standard form, we introduce the rescaling:

\[
\hat{Y} = \sqrt{\frac{3|1 - 2a^2C^2|}{C^2}} Y, \quad \hat{T} = \left(\frac{1 - 2a^2C^2}{2C}\right) T, \quad \rho_1^{(R)} = qU,
\]

(24)

where the parameter \(q\) is given by:

\[
q = \frac{2}{C^2} \left(\frac{1 - 2a^2C^2}{a^2C^2 - 2}\right)
\]

(25)

Then, equation (22) is reduced to the form:

\[
\left(U_\hat{T} + 6UU_\chi + U_{\chi\chi}\right)\chi + 3\delta U\hat{Y}\hat{Y} = 0,
\]

(26)

where \(\delta = -\text{sgn}(1 - 2a^2C^2) = \pm 1\). The case \(\delta = -1\), or \(1 - 2a^2C^2 > 0\), corresponds to the KP-I equation, which models waves in liquid thin films with large surface tension. On the other hand, \(\delta = 1\), or \(1 - 2a^2C^2 < 0\), corresponds to the KP-II equation, arising in the description of shallow water waves characterized by small surface tension (see, e.g. [2]).

3. Line solitons and lumps

Below, soliton solutions of the KP-I and KP-II models will be used for the construction of approximate soliton solutions of the original 2D CH-NLS equation. Furthermore, the dynamics of these structures will be studied by means of direct numerical simulations in the framework of the CH-NLS. For the numerical exploration below, we note the following. For the numerical integration of the original 2D CH-NLS, we use the exponential time-differencing 4th-order Runge–Kutta (ETDRK4) scheme of [30, 31]; for details related to implementation, see [25]. The parameters used in the simulations can be found in the individual figure captions. If a figure shows a collision between two solitons, and only one set of parameters is given, then that set was used for both solitons. Lastly, the background amplitude has been set to unity for all simulations.

3.1. Approximate soliton solutions of the CH-NLS equation

We start with the case of line soliton solutions, which are supported by both KP-I and KP-II equations, given their quasi-1D nature, (see equation (26)) and are of the form [1, 2]:

\[
U = \frac{1}{2}\gamma^2 \text{sech}^2\left[\frac{\gamma}{2}\left(\chi - \beta\hat{Y} - \eta\hat{T} + \chi_0\right)\right].
\]

(27)

where \(\eta, \gamma\) and \(\beta\) are constants with \(\eta = \gamma^2 + 3\delta\gamma^2\), and \(\delta = \pm 1\) for KP-I or KP-II, respectively. The solution (27) is characterized by the parameter \(\gamma\) associated with the soliton...
amplitude and the soliton direction $\beta$ in the $xy$-plane. Using equation (27), and reverting transformations back to the original variables, we find the following approximate (valid up to order $O(\epsilon)$) solution of the CH-NLS, equation (1):

$$u \approx u_0 \left[ 1 + \frac{\epsilon \gamma^2 q}{2} \text{sech}^2(\xi) \right] \exp \left[ -2i|u_0|^2 t + i\sqrt{\epsilon}C \gamma q \tanh(\xi) \right],$$

(28)

where

$$\xi = \frac{\sqrt{\epsilon} \gamma}{2} \left[ x - \sqrt{\epsilon} \beta \sqrt{\frac{3(1 - 2a^2C^2)}{C^2}} y - \left( C + \frac{\epsilon \eta(1 - 2a^2C^2)}{2\gamma C} \right) t + x_0 \right].$$

(29)

Here, it is important to mention that the approximate soliton solution (28) describes two types of solitons: if $q < 0$ the solitons are dark, having the form of density dips on top of the cw background of amplitude $u_0$; if $q > 0$, the solitons are anti-dark, having the form of density humps on top of the cw background. The sign of parameter $q$ depends on the range of values of a single parameter $p \equiv a^2C^2 = 4a^2|u_0|^2$: indeed, if $1/2 < p < 2$ the solitons are antidark, else they are dark.

Using direct numerical simulations (results not shown here) we have found that for sufficiently small $\epsilon$, of the order $O(10^{-2})$, solitons of both types do exist and propagate undistorted, without emitting significant radiation. On the other hand, solitons of relatively large amplitudes feature a different behavior, because—as expected—the results of the asymptotic analysis become less accurate. Indeed, this is shown in figure 1, where the evolution of a dark line soliton (see initial condition, at $t = 0$, in panel (a)) is depicted. As is observed in panel (b), the dark soliton ‘ejects’ an anti-dark line soliton, and a radiation tail forms. Nevertheless, it should be pointed out that even for such relatively large amplitudes, the approximate soliton solutions are supported by the system and can even undergo almost elastic collisions with each other. An example pertaining to the case of a pair of dark line solitons is shown in panels (c) and (d) of figure 1. Here, the collision is deemed as almost elastic, in the sense that the observed dynamics for each soliton is identical to the one depicted in panel (b), i.e. the ejection of the anti-dark soliton and the emission of radiation would occur even if each of the solitons evolved by itself (i.e. in the absence of the other one).

In figure 2, we explore further the relative elasticity of the collisions for different parameter values. In particular, in the figure we show a typical (for different values of $a$ and $C$) result representing the elasticity of the collisions as a function of $\epsilon$, i.e. the parameter characterizing the amplitude of the solitary wave. More specifically, we compare the amplitude of the solitonic depression (and that of the background) for a single soliton (dashed line) and for two solitons (solid line) as a result of their collision. It can be seen that in the case of two solitons the amplitudes remain identical to that of one soliton (implying that the collisions are essentially elastic) for values of $\epsilon$ up to around 0.07. For higher $\epsilon$ values it is clear that the amplitude of the background increases further, while the solitonic depression is also modified as a result of the collision. These features are a byproduct of the partial inelasticity of the associated collisions. We have found this result to be similar to other cases of $a$ and $C$, providing in this way a rough guideline about the deviation of soliton collisions from elasticity.
3.2. Lump solitons

Next, we consider lump solitons, which are weakly localized (exponentially decaying) two-dimensional soliton solutions of the KP-I equation. The lump soliton is of the form:

\[ U = 4 \left\{ \frac{\left[ x + \gamma \beta + 3(\gamma^2 - \beta^2) \beta \right]^2}{\left[ x + \gamma \beta + 3(\gamma^2 - \beta^2) \beta \right]^2 + \beta^2(\beta + 6 \gamma \beta)^2 + 1/\beta^2} \right\}^{1/2}, \]

where \( \gamma \) and \( \beta \) are real parameters. As in the case of the line soliton, we use this solution and revert transformations back to the original variables, and find the following approximate (valid up to order \( O(\epsilon) \)) solution of equation (1):

\[ u \approx u_0 (1 + \epsilon q w_1) \exp[-2i|u_0|^2 t + i\sqrt{\epsilon} Cqw_2], \]

where

Figure 1. Top panels: contour plots showing the evolution of the density of one line dark soliton of relatively large amplitude, with \( \epsilon = 0.1 \), at \( t = 0 \) (panel (a)) and \( t = 100 \) (panel (b)). As expected, it is observed that the soliton emits a small amount of radiation. Bottom panels: collision between two line dark solitons. Panel (c) shows the initial condition, at \( t = 0 \), and panel (d) shows the outcome of the head-on collision, at \( t = 100 \). Here, the leftmost soliton appears as the rightmost one, and vice versa. Parameter values: \( a = 1 \), \( \gamma = 1.3 \), and \( \beta = 1 \).
In the case of small-amplitude lumps, e.g. for $\epsilon = 0.01$, direct numerical simulations are in very good agreement with the analytical findings. Indeed, in figure 3, which depicts the evolution of an anti-dark lump, shown is the initial condition, at $t = 0$ (panel (a)), and a snapshot, at $t = 100$ (panel (b)), as found in the framework of the CH-NLS equation. It is observed that, up to this time, the lump soliton propagates undistorted and the radiation emitted is practically non observable. Furthermore, in the bottom panels of figure 3, shown is the result of a collision between two identical dark lumps—one traveling to the left and one to the right. In panel (c), we depict the initial condition ($t = 0$), and in panel (d) the outcome of the collision ($at t = 100$), where the leftmost lump appears at the rightmost place, and vice versa. It is seen that, for such small-amplitudes, the two lump solitons remain unscathed after the collision while, in this case too, no radiation is observable. We note in passing that the validity of our analytical approximations was also checked in other cases (not shown here), e.g. for individual anti-dark line solitons and dark lump solitons, as well as for collisions between such structures, and—for sufficiently small amplitudes—a very good agreement with the numerical results was found as well.
Next, we consider lumps of larger amplitudes. Figures 4 and 5 depict the evolution and collisions of anti-dark and dark lump solitons, respectively. Regarding their evolution (top panels of figures 4 and 5), it is observed that although both structures remain localized, they spread and bend radially, emitting also radial radiation. Indeed, as seen in the more pronounced case of the dark lump of figure 5, the emitted radiation has the form of nearly concentric circles. The collision between anti-dark or dark lumps (bottom panels of figures 4 and 5, respectively) appears to be elastic; nevertheless, post collision dynamics again features the formation of (nearly) concentric segments of circular rings, which appear to be more pronounced in the case of dark lumps.

We have also performed simulations to study collisions between line and lump solitons. Pertinent results are depicted in figure 6 for solitons of both types, dark (top panels) and anti-dark (bottom panels). In this case too, nearly elastic collision occurs in both cases, with the deformation of the lumps along the radial direction persisting as in the previously studied cases. We note in passing that collisions between line dark solitons and anti-dark lumps (and

Figure 3. Top panels: contour plots showing the evolution of the density of one anti-dark lump soliton of relatively small amplitude, with $\epsilon = 0.01$, at $t = 0$ (panel (a)) and $t = 100$ (panel (b)). It is observed that the lump evolves undistorted and no emission of radiation is observed. Parameter values: $a = \frac{1}{2}$, $\gamma = 1$, $\beta = 0.8$. Bottom panels: collision between two dark lumps. Panel (c) shows the initial condition, at $t = 0$, and panel (d) shows the outcome of the head-on collision, at $t = 100$. Here, the leftmost lump appears as the rightmost one, and vice versa; again, the radiation is barely discernible at $t = 100$. Parameter values: $a = \frac{1}{2}$, $\epsilon = 0.005$, $\gamma = 1$, $\beta = 0.85$. 
Figure 4. Large-amplitude anti-dark lump solitons. Top panels show the evolution of this waveform for $a = \frac{1}{2}$, $\epsilon = 0.01$, $\gamma = 1$, and $\beta = 3$; it is observed that this structure evolves into a bent shape. Bottom panels depict the collision between two identical anti-dark lumps, for $a = 0.6$, $\epsilon = 0.08$, $\gamma = 0$, and $\beta = \frac{1}{2}$; the collision appears to be almost elastic although radiation develops in each lump after it.

Figure 5. Similar to figure 4, but for large-amplitude dark lump solitons. The observed behavior is similar to that of the anti-dark lump case. For the top panels, the parameter values are $a = 1$, $\epsilon = 0.01$, $\gamma = 1$, and $\beta = 1.8$, while for the bottom panels $a = 1$, $\epsilon = 0.08$, $\gamma = 0$, and $\beta = \frac{1}{2}$. 
vice versa) are not possible because solitons of the dark and the anti-dark type and of different dimensionality do not coexist for the same parameter values; such collisions may become possible only in the presence of higher-order effects that may facilitate the coexistence of such structures (see, e.g. [18] where third-order dispersion supports solitons of different types and different dimensionality, which can undergo quasi-elastic head-on collisions).

Lastly, we use generic Gaussian initial data on top of the background to investigate whether the resulting dynamics can share some qualitative features with the one corresponding to the approximate line and lump soliton solutions. To be exact, for the initial condition we place the Gaussian

\[ u(x, y, 0) = 1 + A \exp \left[ -\left( \frac{x}{\sigma_1 L_x} \right)^2 - \left( \frac{y}{\sigma_2 L_y} \right)^2 \right] \]  

(34)

on top of the background, where \( A = -0.2 \), \( L_x \), \( L_y \) are the bounds of the computational domain and \( \sigma_1, \sigma_2 \) control the length and width. We have fixed \( L_x = L_y = 500 \) and \( \sigma_1 = 0.02 \). Pertinent results are shown in figure 7, for a Gaussian very stretched in the vertical direction (top panels, \( \sigma_2 = 0.8 \)), for a slightly stretched Gaussian (middle panels, \( \sigma_2 = 0.07 \)), and for a completely symmetric Gaussian (bottom panels, \( \sigma_2 = 0.02 \)).

It is observed that, in the first case, the extended Gaussian splits into two symmetric waveforms reminiscent of a pair of dark line solitons moving in opposite directions. Here, the initial stripe-like structure, is similar to a line dark soliton but without a phase jump (see equation (28)). To attain the correct phase profile suggested by the energetically preferable approximate line dark soliton, and still preserve the phase structure at infinity, the initial Gaussian has to split to two oppositely moving stripes.

On the other hand, in the case where the initial condition has the form of a slightly extended Gaussian (see middle panels of figure 7), the form of the initial data is closer to that of a dark
lump soliton (rather than that of a line soliton). In this case too, due to not having the proper phase structure—and decay at infinity which is now exponential rather than algebraic—the initial data reorganizes itself into a structure which is reminiscent of a superposition of two bent lumps. Here, one should observe the resemblance of this dynamics with the outcome of the collision between two dark lumps of figure 5. Finally, when the initial Gaussian is completely symmetric, the resulting structure evolves towards an almost perfect ring-like structure resembling an expanding ring soliton.

4. Conclusions and future challenges

In this work, we have studied the defocusing Camassa–Holm–Nonlinear Schrödinger (CH-NLS) equation. We have shown that this model possesses a stable continuous-wave (cw) solution, on top of which small-amplitude soliton solutions can be supported. Our analytical approach was based on asymptotic multiscale expansion methods, which allowed us to reduce the CH-NLS model to a Kadomtsev–Petviashvili (KP) equation. Both versions, namely the KP-I and KP-II, were found to be possible, depending on the sign of a characteristic parameter.
The reduction to the KP model allowed us to construct approximate soliton solutions, both line solitons and lumps, and either of the dark or of the anti-dark type, of the original CH-NLS model. Domains of existence of all these structures, as well as their dynamics by means of direct numerical simulations, were investigated. We found that line and lump solitons do persist in the original model, but as their amplitude is increased, they undergo deformations, i.e. bending and a radial expansion, which may form other, ring-shaped, structures. We also studied head-on collisions between line solitons, between lumps, as well as between line solitons and lumps, and found that they are almost elastic (although less so as the amplitude of the structures increases). In our simulations, we have also used generic Gaussian initial data, the dynamics of which were found to follow qualitative features of the approximate soliton solutions’ dynamics.

There are many interesting topics for future studies. First of all, it would be interesting to study the transverse dynamics of large-amplitude dark solitons and investigate their instability, and the concomitant generation of vortices (similarly to the traditional 2D defocusing NLS model [28, 29]). The studies of other quasi-2D or purely 2D structures, such as the ring dark or anti-dark solitons (which were already identified in our simulations) and vortices, respectively, constitute still other themes of particular interest, as also highlighted by select ones among our numerical computations (e.g. the case of the radially symmetric Gaussian initial condition). Relevant studies are in progress and pertinent results will be reported in future publications.

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References

[1] Ablowitz M J and Clarkson P A 1991 Solitons, Nonlinear Evolution Equations and Inverse Scattering (Cambridge: Cambridge University Press)
[2] Ablowitz M J 2011 Nonlinear Dispersive Waves: Asymptotic Analysis and Solitons (Cambridge: Cambridge University Press)
[3] Infeld E and Rowlands G 1990 Nonlinear Waves, Solitons and Chaos (Cambridge: Cambridge University Press)
[4] Remoissenet M 1999 Waves Called Solitons (Berlin: Springer)
[5] Kevrekidis P G, Frantzeskakis D J and Carretero-González R 2008 Emergent Nonlinear Phenomena in Bose–Einstein Condensates: Theory and Experiment (Heidelberg: Springer)
[6] Zakharov V E and Kuznetsov E A 1986 Physica D 18 455
[7] Kivshar Yu S 1990 Phys. Rev. A 42 1757
[8] Kivshar Yu S and Afanasjev V V 1991 Phys. Rev. A 44 R1446
[9] Bass F G, Konotop V V and Puzenko S A 1992 Phys. Rev. A 46 4185
[10] Frantzeskakis D J 1996 J. Phys. A: Math. Gen. 29 3631
[11] Kivshar Yu S and Yang X 1994 Phys. Rev. E 50 R40
[12] Frantzeskakis D J and Malomed B A 1999 Phys. Lett. A 264 179
[13] Nistazakis H E, Frantzeskakis D J, Malomed B A and Kevrekidis P G 2001 Phys. Lett. A 285 157
[14] Dreischuh A, Neshev D, Paulus G G, Grasbon F and Walther H 2002 Phys. Rev. E 66 066611
[15] Horikis T P and Frantzeskakis D J 2016 *Opt. Lett.* **41** 583
[16] Horikis T P and Frantzeskakis D J 2016 *J. Phys. A: Math. Theor.* **49** 205202
[17] Frantzeskakis D J, Hizanidis K, Malomed B A and Polymilis C 1998 *Phys. Lett.* A **248** 203
[18] Nistazakis H E, Frantzeskakis D J and Malomed B A 2001 *Phys. Rev. E* **64** 026604
[19] Baronio F, Wabnitz S and Kodama Y 2016 *Phys. Rev. Lett.* **116** 173901
[20] Biondini G 2007 *Phys. Rev. Lett.* **99** 064103
[21] Horikis T P and Frantzeskakis D J 2017 *Phys. Rev. Lett.* **118** 243903
[22] Camassa R and Holm D D 1993 *Phys. Rev. Lett.* **71** 1661
[23] Arnaudon A 2016 *J. Nonlinear Sci.* **26** 1133
[24] Arnaudon A 2016 *J. Phys. A: Math. Theor.* **49** 125202
[25] Mylonas I K, Ward C B, Kevrekidis P G, Rothos V M and Frantzeskakis D J 2017 *Phys. Lett.* A **381** 3965
[26] Kivshar Yu S and Agrawal G P 2003 *Optical Solitons: from Fibers to Photonic Crystals* (San Diego, CA: Academic)
[27] Benney D J and Luke J C 1964 *J. Math. Phys.* **43** 309
[28] Pelinovsky D E, Stepanyants Yu A and Kivshar Yu S 1995 *Phys. Rev. E* **51** 5016
[29] Kivshar Yu S and Pelinovsky D E 2000 *Phys. Rep.* **331** 117
[30] Cox S and Matthews P 2002 *J. Comput. Phys.* **176** 430
[31] Kassam A and Trefethen L 2005 *SIAM J. Sci. Comput.* **26** 1214