ON THE DYNAMICS OF A QUADRATIC SCALAR FIELD POTENTIAL

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We review the attractor properties of the simplest chaotic model of inflation, in which a minimally coupled scalar field is endowed with a quadratic scalar potential. The equations of motion in a flat Friedmann-Robertson-Walker universe are written as an autonomous system of equations, and the solutions of physical interest appear as critical points. This new formalism is then applied to the study of inflation dynamics, in which we can go beyond the known slow-roll approximation.

Keywords: Cosmology; scalar fields; inflation.

1. Introduction

One of the most studied issues in Cosmology is the dynamics of cosmological scalar fields, mostly because of their usefulness in providing models for different needed processes in the evolution of the universe. It is not an easy task at all to present a full bibliography on scalar fields, but a nice and comprehensive review was recently presented by Copeland, Sami and Tsujikawa in Ref.\(^1\).

A full plethora of methods exist in the specialized literature to study the dynamics of cosmological scalar fields, but only a few can be told to be extreme useful and of general applicability, in a field where exact solutions are rarely found.

One of these methods is the writing of the scalar and gravitational evolution equations in the form of a dynamical system. This is an idea that has largely pervaded the specialized literature on cosmological scalar fields\(^2\)\(^\text{,}3\)\(^\text{,}4\). The reason is that the theory of dynamical systems can show the existence of (fixed) stationary points that may represent important cosmological solutions; many times these solutions are attractors the system evolves to independently of the initial conditions. This is of wide interest in the case of inflationary and dark energy models\(^5\). However, the dynamical system approach is not always completely suitable for the study of scalar field dynamics.

A typical example is a scalar field model endowed with a quadratic potential. The first attempt for this case was made by Belinsky et al\(^2\) (see also\(^5\)), in which the dynamical variables were just the normalized values of the scalar field and grav-
itational variables (the scalar field value $\phi$, its time derivative $\dot{\phi}$, and the Hubble parameter $H$). Their analysis did not reveal inflationary attractor solutions, but rather asymptotic inflationary behavior at times $t \to -\infty$, for which the corresponding fixed points were all unstable.

In a later paper, de la Macorra and Piccinelli used the same variables suggested in Ref. 3 for an exponential potential; for the latter it is possible to write the equations of motion as an autonomous plane system. On the contrary, the system of equations for a quadratic potential case cannot be written in such a simple form, and one has to introduce extra variables, which are not directly related with the original dynamical variables, to have a closed system of equations. In any case, de la Macorra and Piccinelli were able to obtain asymptotic solutions, and provided a general classification for the behavior of arbitrary scalar potentials.

Our own proposal to study the dynamics of a quadratic potential is somehow in between the last two discussed above. We take the same variables used in 3, 6, and define a third new variable related to the Hubble parameter; the resulting dynamical system is a 3-dimensional autonomous one. However, the nature of the fixed points of physical interest is clearly revealed if one studies a reduced 2-dimensional dynamical system, and the third variable is just taken as a control parameter.

The critical points that will be considered are not fixed in the strict sense, as their location on the 2-dimensional phase space depend upon the values of the control variable. Nevertheless, we shall show that the critical points of the reduced dynamical system carry useful physical information, and that their stability can be established by standard methods.

A similar approach was presented in a recent paper in 7, in which the dynamics of quintessence models is under the control of a so-called roll parameter $\lambda \propto V'/V$. The location of the fixed points, which are actually those found for an exponential potential, is determined by the value of $\lambda$ at any instant of time. Such a property provides a powerful method to study general quintessence models.

Even though a quadratic potential is the simplest and the most well known of the inflationary models, we will revise some inflationary solutions from the point of view of the results on attractor solutions obtained from the dynamical system approach.

A summary of the paper is as follows. In Sec. 2 we describe the mathematical definitions of the new dynamical system, and make a detailed study of its critical points. In Sec. 3 we apply our results to inflationary models in the simplest chaotic scenario. Finally, conclusions are discussed in Sec. 4.

2. Mathematical background

The model is defined by the following action in the scalar field $\phi$,

$$S_\phi = - \int dx^4 \sqrt{-g} \left[ \frac{1}{2} \partial\mu \phi \partial_\mu \phi + \frac{1}{2} m^2 \phi^2 \right],$$  (1)
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where \( m \) indicates the scalar field mass. The evolution equations in a spatially flat FRW model, with a metric given by \( g_{\mu\nu} = \text{diag}(-1, a^2(t), a^2(t), a^2(t)) \), are

\[
\begin{align*}
\ddot{\phi} &= -3H\dot{\phi} - m^2\phi, \\
H &= -4\pi G\dot{\phi}^2,
\end{align*}
\]

(2a, b)
together with the Friedmann constraint

\[
H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{4\pi G}{3} \left(\dot{\phi}^2 + m^2\phi^2\right).
\]

(3)

In all the above equations, a dot means derivative with respect to the cosmic time \( t \), \( a(t) \) is the scale factor, \( H \) is the Hubble parameter, \( G \) is Newton’s gravitational constant, and we use units in which \( c = 1 \).

2.1. Dynamical system

To study the dynamics of the system (2), it is convenient to define new dimensionless variables as follows,

\[
x \equiv \sqrt{\frac{4\pi G}{3H}}\phi, \quad y \equiv \sqrt{\frac{4\pi G}{3H}}m\phi, \quad z \equiv \frac{m}{H}.
\]

(4)
The evolution equations (2) are then replaced by the equations

\[
\begin{align*}
x' &= -3(1 - x^2)x - yz, \\
y' &= 3x^2y + zx, \\
z' &= 3x^2z.
\end{align*}
\]

(5a, b, c)

where a prime denotes derivative with respect to the e-fold number \( N \equiv \ln a \), and we have used Eq. (3) in the form \(-\dot{H}/H^2 = 3x^2\); on the other hand, the Friedmann constraint now reads

\[
x^2 + y^2 = 1.
\]

(6)

Eqs. (5) are written in the form of a dynamical system, which helps to clarify the properties of the original equations of motion.

Unlike the very well known case of an exponential potential, \( \exp(\phi) \), it is not possible to take the Friedmann parameter \( H \) out of the evolution equations, and we are forced to take it into account through the definition of the new variable \( z \); notice that this new variable is, by definition, a monotonic growing function (see [12, 13] for a similar situation in a cosmological dynamical system).

The trajectories in the 3-dim phase space are located on the surface of an infinite cylinder of unitary radius, and, strictly speaking, the only critical point is the origin of coordinates. It is then evident that the 3-dimensional system does not provide enough relevant information about the dynamics of the original physical system.

However, one is usually interested in the evolution of variables \( x \) and \( y \), whose quadratic values represent the contributions of the kinetic and potential energies,
respectively, to the total energy contents of the universe. Because of this, the Friedmann constraint involves the values of \( x \) and \( y \) only, see Eq. (3), and then all important physical features of the original system of equations (2) are more tractable if we restrict ourselves to the \( xy \) plane of the phase space.

Thus, we are to study the reduced 2-dim dynamical system provided by Eqs. (5a) and (5b) only; and variable \( z \) will be considered a control parameter all the results presented below will depend upon. In what follows, and for purposes of generality, we will study the unconstrained system (5a) and (5b), and we shall impose the Friedmann constraint (6) whenever is necessary to clarify the nature of the critical points.

### 2.2. Critical points

The critical points of the 2-dimensional system correspond to those for which \( x' = y' = 0 \). The results can be summarized as follows.

**Null scalar field.** This point corresponds to the origin of coordinates \( x = y = 0 \). This point exists for any value of \( z \); but it should be stressed out that this point does not accomplish with the Friedmann constraint (6), i.e., it is inaccessible to the original physical system.

**Scalar field domination.** There are four critical points with coordinates given by

\[
x^2_0 = \frac{1}{2} \pm \frac{1}{2} \sqrt{1 - \frac{4}{9} z^2}, \quad y^2_0 = \frac{1}{2} \mp \frac{1}{2} \sqrt{1 - \frac{4}{9} z^2}.
\]

(7)

Notice that these critical points cannot be considered fixed in the strict sense, as their values depend upon variable \( z \); but it is remarkable that they all lie, whenever their existence is allowed, on the unitary circumference \( x^2_0 + y^2_0 = 1 \).

For \( z = 0 \), the four critical points are located at \( \{\pm 1, 0\} \) and \( \{0, \pm 1\} \); but as \( z \) evolves to larger values, the points move one to each other in the unitary circumference until they merge into two points located at \( \{\pm 1/\sqrt{2}, \mp 1/\sqrt{2}\} \) once \( z = 3/2 \). The critical points cease to exist for \( z > 3/2 \), see also Fig. 1.

### 2.3. Stability analysis

We now proceed to the stability analysis under the assumption that variable \( z \) is, as mentioned before, only a control parameter. Following standard methods, we make small perturbations around each one of the critical points of the form \( \mathbf{x} = \mathbf{x}_0 + \mathbf{u} \), where \( \mathbf{x} = (x, y) \), \( \mathbf{u} = (u_x, u_y) \) are the corresponding perturbations of the dynamical variables, and a subscript ‘0’ denotes the critical points.

The dynamical system in Eqs. (5a) and (5b) is of the form \( \mathbf{x}' = \mathbf{f} (\mathbf{x}, z) \), and then the evolution equations for the perturbations are \( \mathbf{u}' = \mathcal{M} \mathbf{u} \), where the stability matrix \( \mathcal{M} \) has components

\[
\mathcal{M}_{ij} = \left. \frac{\partial f_i}{\partial x_j} \right|_{\mathbf{x}_0} = \begin{pmatrix} -3 + 9x_0^2 & -z \\ 6x_0y_0 + z & 3x_0^2 \end{pmatrix}.
\]

(8)
We only need to calculate the eigenvalues of the stability matrix by taking the values of $x_0$ and $y_0$ at each one of the critical points; the results are summarized below.

**Null scalar field.** The eigenvalues of the stability matrix in this case are

$$\omega_{1,2} = -\frac{3}{2} \pm \frac{3}{2} \sqrt{1 - \frac{4}{9} z^2}. \quad (9)$$

For any value $z > 0$, this critical point is stable; for values $0 < z < 3/2$, the point is a node, whereas for values $3/2 < z$ the point is a converging focus. Unfortunately, we have already mentioned that this critical point does not accomplish the Friedmann constraint (6), and then it is not of physical interest.

**Scalar field domination.** For the rest of the critical points (7), the eigenvalues are given by

$$\omega_1 = 6x_0^2, \quad \omega_2 = 3(2x_0^2 - 1). \quad (10)$$

It can be seen that all the critical points are unstable under small perturbations whenever they exist, as the first eigenvalue is always positive, $\omega_1 \geq 0$. However, the second eigenvalue is negative (positive) for $x_0^2 < 1/2$ ($x_0^2 \geq 1/2$). This change of sign does not make any difference for the stability of the critical points, except in the case of trajectories subjected to the Friedmann constraint (6), as we are to show now.

The general solution to the perturbations is of the form

$$u = C_1 \left( \begin{array}{c} x_0 \\ y_0 \end{array} \right) e^{\omega_1 t} + C_2 \left( \begin{array}{c} y_0 \\ -x_0 \end{array} \right) e^{\omega_2 t}, \quad (11)$$

where the terms in brackets are the normalized eigenvectors of the stability matrix, and the $C$'s are arbitrary constants which are determined from initial conditions. The Friedmann constraint (6), to first order in perturbations, reads

$$x_0 u_x + y_0 u_y = C_1 e^{\omega_1 t} = 0,$$

and then $C_1 = 0$ at all times. In other words, the Friedmann constraint does not allow the existence of radial perturbations.

Therefore, the nature of the critical points for any trajectory constrained to the unitary circumference only depends on the value of $\omega_2$. In consequence, there are two stable (attractor) points at

$$x_a^2 = \frac{1}{2} + \frac{1}{2} \sqrt{1 - \frac{4}{9} z^2}, \quad y_a^2 = \frac{1}{2} - \frac{1}{2} \sqrt{1 - \frac{4}{9} z^2}, \quad (13)$$

and there are two unstable points at

$$x_u^2 = \frac{1}{2} - \frac{1}{2} \sqrt{1 - \frac{4}{9} z^2}, \quad y_u^2 = \frac{1}{2} + \frac{1}{2} \sqrt{1 - \frac{4}{9} z^2}. \quad (14)$$

Fig. 1 show some sketches of the phase portrait, as depicted in the mathematical package MAPLE, of our dynamical system on the $xy$ plane for different values of $z$; the figures speak by themselves and illustrate well the description of the critical points made in the text above.
Fig. 1. 2-dimensional phase portraits for the set of Eqs. (5a) and (5b), in which variable $z$ is taken as a free parameter. The arrows show the normalized velocity field with components $v = (x', y')$, and the unitary circumference represents the Friedmann constraint (6). As expected, the origin of coordinates is a convergent focus in the four figures, and there are other four critical points given by Eq. (7), which are all unstable, located on the unitary circumference for $0 < z < 3/2$; the latter merge into two points at the special value $z = 3/2$. See the text for more details.

After the attractor points has disappeared, the trajectories move in the counterclockwise direction with increasing angular frequency. To leading order for large values of $z$, one can show that variables $x$ and $y$ obey a harmonic oscillator equation where variable $z$ plays the role of the angular frequency. This fact makes the numerical evolution very difficult to follow for times corresponding to $z > 2/3$. 
2.4. 'Fixed point' (FP) approximation

The method that was developed in the previous sections is going to be called the 'fixed point' (FP) approximation. The reason is that the 2-dim system (5a) and (5b) is not an autonomous one, and then the values \((x, y)\) for which the terms on their r.h.s. vanish are not, in the strict sense, solutions of the 2-dim dynamical system.

However, the 2-dim velocity field vanishes at the critical points, see Fig. 1, which indicates that they should have attractor properties. In this respect, the standard perturbation analysis gives correct information about the stability of the critical points. It should be stressed out that the stability analysis is valid only on constant-\(z\) slices of the 3-dim phase space; in other words, we have to neglect the time evolution of variable \(z\). In consequence, the FP is expected to work well only in the cases for which the value of \(z\)', see Eq. (5c), is small.

An example is shown in Fig. 2 to illustrate that the FP approximation is very good at early times, and that it differs from the true evolution of the system at late times when \(z\)' cannot be neglected. In any case, in inflationary studies one is interested in the early evolution of a given model where the FP approximation is particularly useful.

To finish this section, we would like to compare our method with that used in Ref. 2, where inflationary solutions were shown to be characteristic of the equations of motion (2). Using other dynamical variables different to ours in Eqs. (4), the inflationary solutions were identified with two horizontal separatrices on a 2-dim phase space, see Fig. 1 in Ref. 2. We show in our Fig. 2 that those inflationary separatrices are well described by our stable critical points (13). From the comparison above, we see that the FP approximation may provide more semi-analytical results than other methods.

3. Inflationary dynamics

One of the issues cosmologists are most interested in are the inflationary solutions provided by scalar field models. The minimally-coupled scalar field endowed with a quadratic potential has a long tradition as an inflationary model, and its predictions are very well known, see for instance 8,9,10,14; besides, it seems to be the best model to fit the available observational data 15,16, see also 17. In this section, we will rewrite part of the inflation formalism in terms of our new variables, and discuss the role played by the critical points studied in the previous section.

3.1. Attractor inflationary solutions

The condition for an accelerated expansion is

\[
\frac{\ddot{a}}{a} = \dot{H} + H^2 > 0 \iff 3x^2 = -\frac{\dot{H}}{H^2} < 1.
\]  

(15)

The description of the dynamics of a scalar field with a quadratic potential we provided in Sec. 2 confirms the existence of attractor points that provides an accel-
Fig. 2. (Top) Comparison between the numerical solutions of the full 3-dim equations of motion (red solid lines) and the fixed point (FP) (blue dotted line) and slow-roll (SR) (green dashed line) approximations. The numerical solutions were given initial conditions according to the Friedmann constraint at $z_i = 0.08$. The trajectories for the FP approximation were obtained from the attractor solutions, whereas those corresponding to the SR approximation are given by Eqs. (A.2) in Appendix A. Initially, the numerical solution quickly reaches the attractor solution given by the FP approximation, but they differ at late times. However, the separation from the SR approximation is much larger: see Appendix A for more details. (Bottom) 2-dim phase space with the same variables used in Fig. 1 of Ref. 2. The red solid lines represent different solutions of the equations of motion, whereas the green dashed lines are the trajectories obtained from the stable critical points. Notice that the latter agree well with the inflationary separatrices described in Ref. 2. There is a small discrepancy close to the origin, which shows the break-down of the FP approximation at the late stages of inflation; see the text for more details.
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erated expansion; but that happens only as long as the attractor points accomplish with the condition (13); that is, if $3x_s^2 < 1$.

As the position of the critical points depends upon $z$, we see that the attractor points (13) become non-inflationary for $z > z_{\text{end}} = \sqrt{2}$, where the subscript $\text{end}$ means the time the attractor solution leaves the accelerated regime; the corresponding values for the other variables are $x_{c,\text{end}}^2 = 1/3$ and $y_{c,\text{end}}^2 = 2/3$.

In fact, our calculations show that the scalar field value once the attractor is no longer inflationary is

$$\phi_{\text{end}} = \sqrt{\frac{3}{4\pi}} \frac{y_{\text{end}}}{z_{\text{end}}} = \frac{m_{\text{Pl}}}{2\sqrt{\pi}},$$

where $m_{\text{Pl}} \equiv G^{-1/2}$ is the Planck mass. This value coincides with that provided by the slow-roll formalism of inflation (25).

3.2. Number of e-folds during inflation

Another point we investigated refers to the number of e-folds during which inflation occurs, but for that we required the help of numerical solutions.

First of all, we have to set initial conditions for $\phi_i$, $\dot{\phi}_i$ and $H_i$, which should be translated into the initial conditions $x_i$, $y_i$ and $z_i$. The different physical constraints to be imposed upon them are discussed in the following list.

- The first constraint to be taken into account is $\phi_i/m_{\text{Pl}} \sim O(1)$. In order to obtain results as general as possible, we took the inequality

$$\frac{\phi_i}{m_{\text{Pl}}} = \sqrt{\frac{3}{4\pi}} \frac{y_i}{z_i} < 10.$$  

- The second constraint arises from the requirement that $\rho_{\phi,i} < m_{\text{Pl}}^4$, a constraint that is called the quantum boundary in Ref. 2. With the help of the Friedmann constraint (3), one finds that such constraint corresponds to the lower bound $z_i > m_{\text{Pl}}/m_{\text{Pl}}$, which is in turn equivalent to impose $H_i < m_{\text{Pl}}$.
- The initial value $\dot{\phi}_i$ is found in terms of the Friedmann constraint $x_i^2 = 1 - y_i^2$, once $y_i$ is found from the above constraints.
- Last constraint refers to inflationary solutions, and then $z_i < z_{\text{end}}$.

We noticed, in the diverse numerical experiments we performed, that the number of inflationary e-folds mainly depends upon the initial value $z_i$. This is because the dynamical system reaches the attractor points very quickly, and then its subsequent evolution can be well described by the motion of the attractor points themselves.

Thus, a formula for the number of inflationary e-folds is given by a combination of Eqs. (5c) and (13), namely,

$$N(z; z_i) \equiv \ln \left( \frac{a}{a_i} \right) \simeq \frac{2}{3} \int_{z_i}^z \frac{d\ln \tilde{z}}{1 - \sqrt{1 - 4\tilde{z}^2/9}},$$

where
Needless to say, we have found very good agreement between our numerical results and the analytic ones provided by Eq. (18). The total number of e-folds in a given inflationary stage, defined through $N_{\text{total}} \equiv N(z_{\text{end}}; z_i)$ in Eq. (18), is uniquely determined by the initial value $z_i$, or equivalently, by the initial value of the Hubble parameter $H_i$. If the total number of e-folds should at least be $N_{\text{total}} > 60$, then the initial condition should be $z_i < 0.15$; this guarantees that there is sufficient inflation whatever the initial conditions on the scalar field are, as long as they are in agreement with the Friedmann constraint.

The attractor solution indicates that the end of the accelerating stage happens once the Hubble parameter is $H_{\text{end}} = m/\sqrt{2}$, and the oscillations begin a bit later, when $H_{\text{osc}} = 2m/3$, once the attractor point disappears. These two events are almost instantaneous, as the number of e-folds in between, as calculated again from Eq. (18), is $\Delta N = 0.05$.

### 3.3. Inflationary quantities

We give here some instances about how we can write different inflationary quantities of physical interest in terms of the attractor values of our dynamical variables.

The first one is the amplitude of primordial quantum perturbations of the inflaton field given by

$$\delta_H^2(k) \equiv \frac{4}{25} \left( \frac{H}{\dot{\phi}} \right)^2 \left( \frac{H}{2\pi} \right)^2 = \frac{4}{75\pi m^2_{\text{Pl}}} \frac{1}{x^2_s(z)z^2},$$

(19)

where it is implicitly assumed, as usual, that all quantities on the r.h.s. are evaluated at the time the corresponding wave number $k$ leaves the horizon, i.e. $k = aH$. Also, notice that the last term on the r.h.s. should be calculated using the attractor solutions in Eq. (13).

The last statement is not trivial, as the standard procedure is to calculate the values of $H$ and $\dot{\phi}$ in the slow-roll regime. This is not our case, as we are giving the aforementioned dynamical variables their values at the attractor points, and the latter are not subjected to any order approximation. The same remark applies to other quantities shown below.

The wave number $k$ in Eq. (19) can be given in terms of the smallest scale that leaves the horizon at the end of inflation, which we call $k_{\text{end}}$. In terms of our variables, it is easy to show that

$$\ln(k/k_{\text{end}}) = N(z; z_i) - N_{\text{total}} + \ln(z_{\text{end}}/z).$$

(20)

That given, we can calculate the spectral index $n(k)$ with the help of Eqs. (19) and (20), and then we obtain

$$n(k) - 1 = \frac{d \ln \delta_H^2(k)}{d \ln k} = -\frac{2 \left[ z^2 - 3x_s^2(z) \right]}{[1 - 3x_s^2(z)][1 - 2x_s^2(z)]},$$

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where, as before, the last term on the r.h.s. is the final expression found from the attractor solution in terms of our dynamical variables.

As said before, our calculations rely on exact inflationary attractor solutions, so that we can proceed further and calculate higher order derivatives of the scalar power spectrum. For instance, the running of the spectral index $\alpha$ can be obtained just by one more parametric derivation of Eq. (21),

$$
\alpha \equiv \frac{dn_s}{d \ln k} = \frac{4x_s^2(z)z^2 \left[ z^2 + 9x_s^2(z) - 6 \right]}{3 \left[ 1 - 3x_s^2(z) \right]^3 \left[ 1 - 2x_s^2(z) \right]^3}.
$$

As in other results shown before, this formula is evaluated at the attractor solution. By contrast, the slow-roll formalism needs of a second order calculation in the slow-roll parameters to get a precise result.

For completeness, we give below the values of the aforementioned inflationary quantities at a time corresponding to 60 e-folds before the end of inflation. In terms of our control parameter, that time corresponds to $z \simeq 0.15$, as $N(\sqrt{2}; 0.15) = 60$, see Eq. (18). The resulting values are

$$
\delta_{H}^{(60)} \simeq 3.0105 \times 10^{-2} \frac{m^2}{m_{Pl}^2},
$$

$$
n^{(60)} \simeq 0.969659,
$$

$$
\alpha^{(60)} = -4.65 \times 10^{-4}.
$$

The above values can be compared to those inferred from observations of the Cosmic Microwave Background. For example, from the measured amplitude $\delta_{H} \simeq 1.9 \times 10^{-5}$ one obtains the known result for the scalar field mass, which is $m \simeq 10^{-6}m_{Pl}$.

4. Conclusions

We have shown, with the help of the theory of dynamical systems, the existence of attractor solutions in a scalar field theory minimally coupled to gravity and endowed with a quadratic potential, the simplest and most known of the monomial potentials in inflationary theory.

The attractor solutions and other subsequent results were given explicitly in terms of the auxiliary variable $z = m/H$, which acted as a free parameter and controlled the behavior of the critical points. We called this method the fixed point (FP) approximation. In particular, we applied the formalism to the study of inflationary dynamics with a quadratic potential. Our results are explicitly based on attractor solutions, and then we were able to calculate diverse inflationary quantities without the need of order approximations.

Also, we would like to stress out that the conditions for an inflationary stage are largely determined by the value of our control variable $z$, as it alone determines

\[\text{See Ref.}^{19} \text{and table of parameters in http://lambda.gsfc.nasa.gov.}\]
the existence of an inflationary attractor solution. This confirms the results of previous studies about the feasibility of inflationary solutions from arbitrary initial conditions, see also the discussion in Refs. 20, 21.

We believe that the FP approximation presented in this paper can be used further to fully exploit the predictions of the simplest chaotic inflationary model, and to have a more precise comparison with cosmological observations. The FP approximation can be easily generalized to other inflationary models; this is work in progress that will be published elsewhere.

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Appendix A. Comparison with slow-roll

In this appendix, we are going to make the comparison of the 'fixed point' (FP) approximation to the slow-roll (SR) one. The SR approximation for the equations of motion of a scalar field in a FRW metric and endowed with a quadratic potential is (see also Refs. 2, 5 for an earlier study of the SR equations)

\[ \dot{\phi} \simeq -\frac{m^2 \phi}{3H}, \quad H^2 \simeq \frac{8\pi G}{6}m^2 \phi^2. \] (A.1)

In terms of our auxiliary variables \( \xi \), equivalent expressions for the SR equations are

\[ x_{SR} \simeq -\frac{z^3}{3}, \quad y_{SR}^2 \simeq 1. \] (A.2)

From Eqs. (13), it can be seen that the SR solution is a first order approximation to the FP one. Actually, in the limit \( z \ll 1 \) we get

\[ x_{s} \simeq \mp \frac{z}{3}, \quad y_{s} \simeq \pm \left(1 - \frac{z^2}{18}\right). \] (A.3)

The second equation shows that the correction to the usual SR assumption \( y_{SR} \simeq 1 \) is of the second order. Likewise, the values of the inflationary quantities in the limit \( z \ll 1 \) are, from Eqs. (19), (21), and (22),

\[ d_H^2 \simeq \frac{36}{75\pi} \frac{m^2}{m_{P1}^2} z^{-4}, \quad n(k) - 1 \simeq -\frac{4}{3} z^2, \quad \alpha \simeq -\frac{8}{9} z^4. \] (A.4)

\[^b\text{During the revision process of this work, we discovered that the FP approximation was successfully applied to a quartic monomial potential in Ref. 24, where the authors also make a detailed comparison with the results obtained from slow-roll.}\]
The above formulas can be written in terms of the number of e-folds given by Eq. (18); to first order we find \(N (\sqrt{2}, z_s) \simeq (3/2) z_s^{-2}\) in the limit \(z_s \ll 1\). If we use this last result in Eqs. (A.3), we find the usual expressions of the SR approximation for the different inflationary quantities in terms of the number of e-folds, see also [21].

Some remarks are in turn. First one is that the Friedmann constraint is not fully accomplished in the SR approximation, as we have seen that the deviations are of second order. Second, it is manifest that SR equations (A.2) are not solutions of the original equations of motion. Moreover, the SR points are not even critical points of the dynamical system [1], and so it is difficult to study their attractor properties using standard techniques of dynamical systems.

The SR and the FP trajectories coincide in the very early universe, in the limit \(z \ll 1\). But even for moderate small values of \(z\), the deviations of SR from FP start to be significant, see Fig. [2] and then one has to include correction terms in the SR approximation at different levels [9,10].

\[ \text{References} \]

1. Copeland, Edmund J. and Sami, M. and Tsujikawa, Shinji, Int. J. Mod. Phys. D15 (2006) 1753-1936.
2. Belinsky, V. A. and Khalatnikov, I. M. and Grishchuk, L. P. and Zeldovich, Ya. B., Phys. Lett. B155 (1985) 232-236, hep-th/0603057.
3. Copeland, Edmund J. and Liddle, Andrew R. and Wands, David, Phys. Rev. D57 (1998) 4686-4690, gr-qc/9711068.
4. Coley, Alan A. gr-qc/9910074.
5. Starobinsky, Alexei A., Pis'ma Astron Zh. 4 (1978) 155-159.
6. de la Macorra, A. and Piccinelli, G., Phys. Rev. D61 (2000) 123503, hep-ph/9909459.
7. Chongchitnan, Sirichai and Efthathiou, George, Phys. Rev. D76 (2007) 043508, arXiv:0705.1955.
8. Linde, Andrei D., Phys. Lett. B129 (1983) 177-181.
9. Liddle, A. R. and Lyth, D. H., Cosmological inflation and large-scale structure (Cambridge, UK: Univ. Pr., 2000).
10. Bassett, Bruce A. and Tsujikawa, Shinji and Wands, David, Rev. Mod. Phys. 78 (2006) 537-589, astro-ph/0507632.
11. Liddle, Andrew R. and Mazumdar, Anupam and Schunck, Franz, Phys. Rev. D58 (1998) 061301, astro-ph/9804177.
12. Boehmer, Christian G. and Caldera-Cabral, Gabriela and Lazkoz, Ruth and Maartens, Roy (2008), arXiv:0801.1564 [gr-qc].
13. Matos, Tonatiuh and Vazquez, J. Alberto and Magana, Juan (2008), arXiv:0806.0683 [astro-ph].
14. Alabidi, Laila and Lyth, David H., JCAP 0608 (2006) 013, astro-ph/0603539.
15. Komatsu, E. and others (WMAP) (2008), arXiv:0808.0547 [astro-ph].
16. Spergel, D. N. and others (WMAP), Astrophys. J. Suppl. 170 (2007) 377, astro-ph/0603449.
17. Peiris, H. V. and others (WMAP), Astrophys. J. Suppl. 148 (2003) 213, astro-ph/0302225.
18. Liddle, Andrew R. and Leach, Samuel M., Phys. Rev. D68 (2003) 103503.
19. Liddle, Andrew R. and Parkinson, David and Leach, Samuel M., Phys. Rev. D74 (2006) 083512, astro-ph/0607275.
20. Hollands, Stefan and Wald, Robert M., *Gen. Rel. Grav.* **35** (2002) 2043-2055, [gr-qc/0205058](http://arxiv.org/gr-qc/0205058).

21. Kofman, Lev and Linde, Andrei and Mukhanov, Viatcheslav F., *JHEP* **10** (2002) 057, [hep-th/0206088](http://arxiv.org/hep-th/0206088).

22. Cortes, Marina and Liddle, Andrew R. and Mukherjee, Pia, *Phys. Rev.* **D75** (2007) 083520, [astro-ph/0702170](http://arxiv.org/astro-ph/0702170).

23. Pahud, Cedric and Liddle, Andrew R. and Mukherjee, Pia and Parkinson, David (2007), [astro-ph/0701481](http://arxiv.org/astro-ph/0701481).

24. Kiselev, V. V. and Timofeev, S. A. (2008), [arXiv:0801.2453](http://arxiv.org/0801.2453) [gr-qc].