(Dis)assembling Special Lagrangians

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Abstract: We explain microscopically why split attractor flows, known to underlie certain stationary BPS solutions of four dimensional $\mathcal{N} = 2$ supergravity, are the relevant data to describe wrapped D-branes in Calabi-Yau compactifications of type II string theory. We work entirely in the context of the classical geometry of A-branes, i.e. special Lagrangian submanifolds, avoiding both the use of homological algebra and explicit constructions of special Lagrangians. Our results provide a way to disassemble and assemble arbitrary special Lagrangians to and from more simple building blocks, giving a concrete way to determine for example marginal stability walls and deformation moduli spaces.
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1. Introduction

There are many good reasons to study D-branes on Calabi-Yau compactifications: they provide a nontrivial testing ground for virtually all ideas in D-brane physics, combine a plethora of beautiful results in mathematics and physics, and promise new key insights in the dynamics of $\mathcal{N}=1$ gauge theories (for a nice short review, see [1]).

One distinguishes two types of D-branes in type II Calabi-Yau compactifications, B-branes, wrapping holomorphic cycles and carrying holomorphic bundles, and A-branes, wrapping special Lagrangian submanifolds and carrying flat connections. A- and B-branes are allegedly interchanged by mirror symmetry.

Though A-branes have a much more obvious geometrical framework to be analyzed in than B-branes, the main focus of research [2, 3, 4, 1, 5] has been on B-branes, in part because it is virtually impossible to construct generic special Lagrangians explicitly, and in part because some quantum corrections are better in control for B-branes. However, studying B-branes properly requires the introduction of some arcane — and for most of us even scary — mathematics. The state of the art can be found in [4], briefly summarized in [1].

In this paper on the other hand, we will exclusively consider A-branes, without quantum or stringy corrections, that is, the geometry of special Lagrangian submanifolds. Other work dealing directly with special Lagrangians includes [6, 7, 8, 9, 10, 11, 12]. As it turns out, we will only need some pretty basic mathematics, enhanced with relatively well-known features of special geometry. This hopefully will take away some of the barriers complicating access to this field of research.

The main direct motivation for this work was the observation in [13, 14] that split attractor flows on the vector multiplet moduli space provide a remarkably powerful way to analyze existence and stability of BPS states in type II compactifications. Such flows appear naturally as the basic data describing BPS solutions of the corresponding four dimensional $\mathcal{N}=2$ supergravity theory, but turn out to provide accurate predictions even beyond the regime where supergravity can be expected to be valid. Also the exact same structures, in a completely different guise, had turned up before in the context of $\mathcal{N}=2$ field theories (decoupled from gravity), as the appropriate data to describe BPS states [15]. The obvious question all this prompted was whether the appearance of split flows could also be understood from an entirely microscopic D-brane perspective. In view of the many known examples of identical basic data describing BPS objects in widely different regimes, a positive answer was certainly not unlikely.

We will provide that missing link in what follows. The main idea is that split attractor flows can be associated to certain deformations of special Lagrangians (SLGs), providing a way to assemble and disassemble these from and to more simple building blocks. This in turn gives substantial insight in the structure of arbitrary
SLGs, yielding for instance their domain of stability and in favorable circumstances a parametrization of their (uncorrected) moduli spaces. Our scheme does not require explicit construction of generic SLGs, which, in view of the effective impossibility of such constructions, is obviously rather good news.

The outline of the paper is as follows. In section 2, we recall some well-known and less well-known features of type IIB Calabi-Yau compactifications, including a few useful facts about special Lagrangians and their deformations. We make the latter quite explicit, thus preparing ourselves for section 4. In section 3, we briefly review (split) attractor flows. Though originally introduced in the context of supergravity, we will reformulate things in a way which no longer refers directly to spacetime structures. Section 4 forms the core of this paper. It explains how changing the complex structure along an attractor flow induces a certain deformation of any associated special Lagrangian $L$, determined by requiring preservation of the SLG condition. Such deformations can cause $L$ to split. We investigate in detail the different possible degenerations of this kind, with particular emphasis on determining the “direction” of decay, and link them in a natural way to the different kinds of splits observed in the attractor flow picture. In section 5, we use all this to construct a procedure for splitting up arbitrary special Lagrangians into simpler ones, by letting them flow, according to the rules of section 4, along attractor flow trees. Reversing this procedure gives in turn a way to assemble special Lagrangians of arbitrary complexity out of simple building blocks. Explicit construction of these special Lagrangians is not required; the flow trees themselves already encode a great deal of very useful information (e.g. stability). In this manner we obtain an efficient classification scheme for special Lagrangians. We end the section with a comparison with the Π-stability conjecture. Finally, in section 6, we illustrate some of the general results with a couple of simple examples. The reader might find it useful to have a look at these already while reading the previous sections.

Some of the results derived in this paper have been obtained before by mathematicians. The simplest SLG splittings in section 4 were first studied by Joyce [6]. However, the methods we use, based on infinitesimal deformations of arbitrary compact SLGs (and a useful physical analogy with steady heat flow) rather than local models, are more concrete and perhaps more general, at least from the point of view of a physicist. Secondly, the general concept of studying Lagrangians through Hamiltonian deformations, the importance of various connect sums in this context, and the notion of a Lagrangian decomposition appeared already in the recent work of Thomas [10] and Thomas and Yau [11]. The immediate scope of their work is quite different from ours though. Roughly, in physical terms, they consider (among other things) D-branes wrapped around arbitrary Lagrangians, and investigate whether these will “decay”, at fixed complex structure, to one or more branes wrapped around special Lagrangians, where the decay process is taken to be given by the mean curvature Hamiltonian flow acting on the Lagrangian till it splits, and then further on its
decay products. In the present paper on the other hand, we consider only special Lagrangians, and propose a way to (dis)assemble those by letting the complex structure vary along an attractor flow tree. Physically, the disassembling process can be thought of as adiabatically moving the special Lagrangian into its corresponding large $N$ supergravity solution. So the two setups are inherently different, though (at least in simple cases) some obvious relations exist.

2. Some (less) well-known features of type IIB Calabi-Yau compactifications

For concreteness, we will work in the framework of type IIB string theory compactified on a Calabi-Yau 3-fold $X$. This theory has $\mathcal{N} = 2$ supersymmetry in four dimensions, with $n_v = h^{1,2}(X)$ massless abelian vector multiplets and $n_h = h^{1,1}(X) + 1$ massless hypermultiplets. The vector multiplet scalars are given by the complex structure moduli of $X$, and the lattice of electric and magnetic charges is identified with $H^3(X, \mathbb{Z})$, the lattice of integral harmonic 3-forms on $X$: after a choice of symplectic basis $\alpha^I, \beta_I$ of $H^3(X, \mathbb{Z})$, a D3-brane wrapped around a cycle Poincaré dual to $\Gamma \in H^3(X, \mathbb{Z})$ has electric and magnetic charges equal to its components with respect to this basis.

2.1 Special Geometry of the complex structure moduli space

The geometry of the complex structure moduli space $\mathcal{M}_c$, parametrized by $n_v$ coordinates $z^a$, is special Kähler \cite{16}. The (positive definite) metric

$$g_{ab} = \partial_a \bar{\partial}_b \mathcal{K}$$

(2.1)

is derived from the Kähler potential

$$\mathcal{K} = - \ln \left( i \int_X \Omega_0 \wedge \overline{\Omega_0} \right),$$

(2.2)

where $\Omega_0$ is the holomorphic 3-form on $X$, depending holomorphically on the complex structure moduli. It is convenient to introduce also the normalized 3-form\footnote{In \cite{14}, the holomorphic 3-form was denoted as $\Omega$, and the normalized one as $\tilde{\Omega}$.}

$$\Omega \equiv e^{\mathcal{K}/2} \Omega_0.$$

(2.3)

Then the “central charge” of $\Gamma \in H^3(X, \mathbb{Z})$ is given by

$$Z(\Gamma) \equiv \int_X \Gamma \wedge \Omega \equiv \int_\Gamma \Omega,$$

(2.4)

where we denoted, by slight abuse of notation, the homology class Poincaré dual to $\Gamma$ by the same symbol $\Gamma$. Note that $Z(\Gamma)$ has a nonholomorphic dependence on the
moduli through the Kähler potential. It can be shown \cite{17, 18} that for any three (real) dimensional submanifold $L$ of $X$, $\text{Vol}(L) \geq k|Z(L)|$, where $k$ is a constant independent of the complex structure moduli. More precisely, $k = \sqrt{8\text{Vol}(X)}$. Equality is obtained if $L$ is special Lagrangian (see below). The mass of the corresponding wrapped BPS 3-brane is $M = |Z(L)|/\sqrt{G_N}$, with $G_N$ the four dimensional Newton constant.

Central in what follows will be the (antisymmetric, topological, moduli independent) intersection product, defined as:

$$\langle \Gamma_1, \Gamma_2 \rangle = \int_X \Gamma_1 \wedge \Gamma_2 = \int_{\Gamma_1} \Gamma_2 = \#(\Gamma_1 \cap \Gamma_2),$$

(2.5)

where the intersection points are counted with signs. With this notation, we have for a symplectic basis $\{\alpha^I, \beta_I\}$ by definition $\langle \alpha^I, \beta_J \rangle = \delta^I_J$, so for $\Gamma_i = q_i^I \beta_I - p_i^I \alpha^I$, we have $\langle \Gamma_1, \Gamma_2 \rangle = q_1^I p_{2,I} - p_1^I q_2^I$. This is nothing but the Dirac-Schwinger-Zwanziger symplectic inner product on the electric/magnetic charges. Integrality of this product is equivalent with Dirac charge quantization.

Every harmonic 3-form $\Gamma$ on $X$ can be decomposed according to $H^3(X, \mathbb{C}) = H^{3,0}(X) \oplus H^{2,1}(X) \oplus H^{1,2}(X) \oplus H^{0,3}(X)$ as (for real $\Gamma$):

$$\Gamma = i\bar{Z}(\Gamma) \Omega - ig^{ab} \bar{D}_a \bar{Z}(\Gamma) D_b \Omega + \text{c.c.},$$

(2.6)

where $D_a \equiv (\partial_a + \frac{1}{2} \partial_a \mathcal{K})$. This decomposition is orthogonal with respect to the intersection product (2.5).

### 2.2 Special Lagrangian submanifolds and their deformations

At large volume and zero string coupling, the condition for a single wrapped D3-brane to be supersymmetric is that it is embedded (or, to be precise, immersed) in the Calabi-Yau manifold $X$ as a special Lagrangian (SLG) submanifold, and that the $U(1)$ gauge field on its worldvolume is flat \cite{17}. A three real dimensional submanifold $L$ of $X$ is called special Lagrangian with phase $\alpha$ if

$$\omega|_L = 0$$

(2.7)

$$\text{Im}(e^{-i\alpha} \Omega)|_L = 0$$

(2.8)

and its orientation is such that $\int_L e^{-i\alpha} \Omega > 0$. Here $|_L$ denotes the pull-back to $L$, and $\omega$ is the Kähler form on $X$. Then $\int_L e^{-i\alpha} \Omega|_L$ is up to a constant factor equal to the volume form on $L$, and as stated earlier, the volume of $L$ saturates the BPS bound:

$$\text{Vol}(L) = k|Z(L)|,$$

(2.9)

with $k = \sqrt{8\text{Vol}(X)}$.

It can be shown \cite{4, 8} that the moduli space of deformations of $L$ has real dimension $b^1(L) = \text{dim } H^1(L, \mathbb{R})$, its tangent space at $L$ being isomorphic to $H^1(L, \mathbb{R})$, 5
the space of real harmonic 1-forms on $L$. On the other hand, there are also $b^1(L)$ moduli corresponding to Wilson lines of the flat $U(1)$ gauge field. The deformations pair up with these moduli to form the $b^1(L)$ complex dimensional D-brane moduli space.

The correspondence between SLG deformations and harmonic 1-forms can be made explicit as follows.\(^2\) Let $L$ be a smooth SLG, $I$ some open interval containing 0 and $f_t : L \to X$, $t \in I$, a one parameter family of arbitrary smooth deformations of $L$, with $f_0(L) = L$. Define the map $F : I \times L \to X$ by $F(t, x) \equiv f_t(x)$. Then we can write

\[
F^\ast \omega = \theta^{(1)} \wedge dt + \sigma^{(2)},
\]

\[
F^\ast \text{Im}(e^{-i\alpha \Omega}) = \eta^{(2)} \wedge dt + \chi^{(3)},
\]

where the various Greek letters denote various $t$-dependent differential forms on $L$. Note that the deformations preserve the SLG condition, that is, $f_t^\ast \omega = 0$ and $f_t^\ast \text{Im}(e^{-i\alpha \Omega}) = 0$ for all $t$, if and only if $\sigma^{(2)} = 0$ and $\chi^{(3)} = 0$.

We would like to find an equivalent condition for SLG preservation, but now on the forms $\theta^{(1)}$ and $\eta^{(2)}$. This goes as follows. Since $d F^\ast \omega = F^\ast d \omega = 0$ and $d F^\ast \text{Im}(e^{-i\alpha \Omega}) = F^\ast \text{Im}(e^{-i\alpha}d \Omega) = 0$, equations (2.10)-(2.11) imply:

\[
0 = d \theta^{(1)} \wedge dt + \partial_t \sigma^{(2)} \wedge dt + d \sigma^{(2)},
\]

\[
0 = d \eta^{(2)} \wedge dt - \partial_t \chi^{(3)} \wedge dt + d \chi^{(3)},
\]

where $d$ denotes the exterior derivative on $L$. Therefore, the SLG condition is preserved by the deformations if and only if

\[
d \theta^{(1)} = 0 = d \eta^{(2)}.\]

Writing out components, one sees that the definition (2.10)-(2.11) for these forms is equivalent (for arbitrary deformations) with the following explicit expressions, in obvious notation:

\[
\theta^{(1)} = 2 \text{Im}[g_{mn} \partial_i F^m \partial_j \bar{F}^n] \, dx^i
\]

\[
\eta^{(2)} = 3 \text{Im}[e^{-i\alpha} \Omega_{mnr} \partial_i F^m \partial_j \bar{F}^n \partial_k \bar{F}^r] \, dx^i \wedge dx^j .
\]

($i, j, \ldots$ are real coordinate indices on $L$ and $m, n, r, \ldots$ holomorphic coordinate indices on $X$, with $g_{mn}$ the Ricci-flat metric.) Putting $t = 0$ and decomposing the deformation vector field $\partial_t F \big|_{t=0}$ on $L$ in a tangential\(^3\) ($w$) and a normal ($v$) part as

\[
\partial_t F^m \big|_{t=0} \equiv (w^j + i v^j) \partial_j F^m \big|_{t=0},
\]

\(^2\)The calculation given here follows the reasoning outlined in [7, 8]. We are somewhat more explicit here, preparing for section 4.

\(^3\)The longitudinal part $w$ can always be put to zero; it is simply the gauge degree of freedom for diffeomorphisms of $L$.\]
and using the SLG properties of $L$, this can be rewritten as

\[
\theta^{(1)} = 2h_{ij} v^i dx^j \quad \text{(2.18)}
\]

\[
\eta^{(2)} = \frac{1}{2k} \sqrt{h} \varepsilon_{ijk} v^i dx^j \wedge dx^k , \quad \text{(2.19)}
\]

where $h_{ij}$ is the induced metric on $L$ and, as before, $k = \sqrt{8\text{Vol}(X)}$ (the constant $k$ appears here because we used equation (2.9)). Thus we get a one-to-one correspondence between (arbitrary) infinitesimal deformations and 1- or 2-forms on $L$. It is furthermore straightforward now to check that

\[\ast \theta^{(1)} = 2k \eta^{(2)}.\] (2.20)

Combining this with the condition (2.14) for SLG deformations, we see that $\theta^{(1)}$ corresponds to an infinitesimal SLG deformation if and only if it is harmonic, yielding the isomorphism between the tangent space to the moduli space of SLG manifolds at $L$ and $H^1(L, \mathbb{R})$, as announced.

### 3. Attractor flows and their (not so) basic properties

#### 3.1 Definition

Attractor flows have their origin in the description of black hole solutions of four dimensional $\mathcal{N} = 2$ supergravity theories [19, 20, 21, 13]. Here we will simply define them by a certain flow equation in moduli space.\(^4\) The data specifying a flow are a cohomology class (or charge) $\Gamma \in H^3(X, Z)$ and an initial point $z_0$ in the complex structure moduli space $\mathcal{M}_c$. The corresponding attractor flow is an oriented trajectory in $\mathcal{M}_c$ given by the solution of

\[\mu \frac{dz^a}{d\mu} = g^{ab} \partial_b \ln |Z|^2, \quad \text{(3.1)}\]

where $\mu$ is always positive and runs down starting from 1, $z_{\mu=1} = z_0$, $Z = Z(\Gamma)$ as in (2.4) and $g_{ab}$ is the special Kähler metric (2.1) on $\mathcal{M}_c$.\(^5\) Thus attractor flows are simply gradient lines of the potential $\ln |Z|^2$. In particular, $|Z|$ decreases with decreasing $\mu$, hence the flow will tend to minimal $|Z|$. Generically, this local minimum of $|Z|$ is isolated, so the endpoint of the flow, the so-called attractor point, is invariant under small variations of the starting point $z_0$. It can be shown [21] that all critical points of $\ln |Z|^2$ are in fact local minima.

If the relevant local minimum of $|Z|$ is nonzero, the flow is smooth and $\mu$ runs all the way down to 0. If on the other hand the minimum is zero, this zero will

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\(^4\)In the 4d supergravity context, these flow equation usually involve also the space-dependence of the metric. However, this is easily eliminated [21], leaving pure flow equations in moduli space.

\(^5\)In the 4d black hole context, the physical meaning of $\mu$ is the metric redshift factor: $ds^2 = -\mu^2 dt^2 + \mu^{-2} dx^2$. The spatial dependence of $\mu$ is then given by $\partial_\tau \mu = -\mu^2 |Z|$, with $\tau = 1/|x|$. 

generically be hit before $\mu = 0$, and the flow stops there. We will see later on that there is a further natural distinction between regular zeros of $Z$ and zeros at singularities in $\mathcal{M}_c$ where the cycle $\Gamma$ collapses.$^6$

It will be crucial in what follows to know how $\text{Im}(e^{-i\alpha \Omega})$, with $\alpha \equiv \text{arg } Z$, varies along the attractor flow. A short calculation shows that for an infinitesimal change in complex structure $\delta z^a$ along the attractor flow, we have $\delta \alpha = -\text{Im}(\delta z^a \partial_a \mathcal{K})$, so

$$\delta \text{Im}(e^{-i\alpha \Omega}) = \text{Im}(e^{-i\alpha} \delta z^a D_a \Omega). \quad (3.2)$$

Combining this with (3.1) and the identity $\partial_b \ln |Z|^2 = D_b Z / Z$ gives

$$\mu \frac{d}{d\mu} \text{Im}(e^{-i\alpha \Omega}) = \frac{1}{|Z|} \text{Im}(g^{ab} \bar{D}_b \bar{D}_a \Omega), \quad (3.3)$$

which, using (2.6), can be elegantly rewritten as

$$\mu \frac{d}{d\mu} \text{Im}(e^{-i\alpha \Omega}) - \text{Im}(e^{-i\alpha \Omega}) = \frac{\Gamma}{2|Z|}. \quad (3.4)$$

Since $\Gamma$ is constant in cohomology, this equation can be integrated if we consider it as an equation in $H^3(\mathcal{X}, \mathbb{C})$:

$$2\mu^{-1} \text{Im}(e^{-i\alpha \Omega}) = -\Gamma \tau + 2 \text{Im}(e^{-i\alpha \Omega})_0. \quad (3.5)$$

where $\tau(\mu)$ is defined$^7$ by $|Z|d\tau = -d\mu/\mu^2$, $\tau_{\mu=1} = 0$, and where, as in the remainder of the paper, the index “0” refers to the initial point $z_0$. Note that when $\mu$ runs down from 1 to 0, $\tau$ runs up from 0 to infinity.

Finally, for completeness, we recall that attractor flows can also be considered as geodesic strings$^{14}$ with respect to the action

$$S = |Z_*| + \int 2 \sqrt{g^{ab} \partial_a |Z| \bar{\partial}_b |Z|} \ ds, \quad (3.6)$$

where the starting point of the string is kept fixed at $z_0$, $Z_*$ is $Z(\Gamma)$ evaluated at the free endpoint of the string, and $ds$ is the line element on $\mathcal{M}_c$: $ds^2 = g_{a\bar{b}} dz^a d\bar{z}^\bar{b}$. Requiring $\delta S = 0$ for variations of the free endpoint fixes the latter to be located at the attractor point of $\Gamma$. The action $S$ reaches its minimal value, equal to $|Z|_0$, when evaluated along an attractor flow.

For practical techniques to solve attractor flow equations in this formalism, we refer to$^{14}$.

$^6$Attractor points with nonzero minimal $|Z|$ correspond to spherically symmetric black holes with finite horizon area, regular zeros of $Z$ correspond to charges that don’t admit a spherically symmetric BPS solution, and zeros at singularities can either correspond to singular black holes with zero horizon area, or horizonless enhançon-like “empty holes”$^{21, 13, 22, 23}$.

$^7$In the supergravity picture, $\tau$ appears naturally, as in footnote$^5$. 
3.2 Marginal Stability

A central concept in the discussion of stability of BPS states (and therefore of SLGs) is *marginal stability* (MS). Generically, a BPS state of charge $\Gamma$ is stable against the decay $\Gamma \to \Gamma_1 + \Gamma_2$ by conservation of energy and the triangle inequality: $M = |Z| = |Z_1 + Z_2| \leq |Z_1| + |Z_2| \leq M_1 + M_2$. However, if the decay products $\Gamma_1$ and $\Gamma_2$ are BPS, and $\arg Z_1 = \arg Z_2 (= \arg Z)$, the state is only *marginally* stable against this decay, i.e. the decay is no longer forbidden by conservation of energy. A typical locus where $\arg Z_1 = \arg Z_2$ is of codimension one, and is called a wall (or hypersurface) of $(\Gamma_1, \Gamma_2)$ marginal stability.

Upon crossing such a wall, it is possible (though not necessary) that certain one particle BPS states with charge $\Gamma$ are forced to decay into certain two particle states that are no longer BPS (consisting of the charges $\Gamma_1$ and $\Gamma_2$). The analog of this for SLGs is the Joyce transition, where a single SLG splits into two SLGs with different phases \[8, 9\] (see section 4).

It will be important for us to know whether or not an attractor flow with charge $\Gamma = \Gamma_1 + \Gamma_2$ can cross or reach a wall of marginal stability for the decay $\Gamma \to \Gamma_1 + \Gamma_2$. This is easily answered by taking the intersection product of equation (3.5) with $\Gamma_1$.

This gives, using $\langle \Gamma_1, \Gamma \rangle = \langle \Gamma_1, \Gamma_2 \rangle$:

$$2 \text{Im}[e^{-i\alpha} Z_1] = -\langle \Gamma_1, \Gamma_2 \rangle \mu \tau + 2\mu \text{Im}[e^{-i\alpha} Z_1]_0.$$  \hspace{1cm} (3.7)

By definition, at marginal stability, the left hand side is zero. Now there are two cases to distinguish, as illustrated in fig. 1. The first (right hand side in the figure) is

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8Also often called a *line* of marginal stability, in analogy with the most famous example \[24\], where this locus is indeed a line (and moreover unique, another rather atypical feature).
when $\Gamma_1$ and $\Gamma_2$ have zero intersection product. When the right hand side of (3.7) is equal to zero then gives $\mu \Im[e^{-i\alpha Z}] = 0$, that is, either we are already at $(\Gamma_1, \Gamma_2)$ marginal stability from the beginning (at $z_0$), and then the flow stays inside the MS wall, or we start away from the wall, and then we can only reach it at $\mu = 0$, the attractor point. So, in particular, the flow can never intersect the wall transversally. Note that we do not necessarily reach $(\Gamma_1, \Gamma_2)$ marginal stability at the attractor point: even though, for a flow converging to a nonzero minimal $|Z|$, we will always have $\mu = 0$ at the attractor point and therefore $\Im[Z_1 Z_2] = 0$, it is still possible (and in fact more generic) to have $\arg Z_1 = \arg Z_2 + \pi$ rather than equal phases.

The second case is $\langle \Gamma_1, \Gamma_2 \rangle \neq 0$. Taking the intersection product of $\Gamma_1$ with equation (3.4) then implies that, at an attractor point with $\mu = 0$, we can never be at a $(\Gamma_1, \Gamma_2)$ MS wall. However, again by intersecting (3.3) with $\Gamma_1$, one sees that now the flow can intersect the wall transversely, namely at $\tau_{ms} = 2 \Im[e^{-i\alpha Z}] / \langle \Gamma_1, \Gamma_2 \rangle$, or written more symmetrically, at

$$\tau_{ms} = \frac{2 \Im(Z_1 Z_2)}{|Z| \langle \Gamma_1, \Gamma_2 \rangle} |^0 .$$

A necessary condition for the intersection point to exist is of course $\tau_{ms} > 0$, that is,

$$\langle \Gamma_1, \Gamma_2 \rangle \Im(Z_1 Z_2)|^0 > 0$$

(3.9)

This is also indicated in fig. 1. Note that this condition is not sufficient to have intersection with the marginal stability wall: the flow could instead cross an “anti-MS” wall, i.e. where $\arg Z_1 = \arg Z_2 \pm \pi$, or it could hit a zero (also necessarily on an anti-MS wall) before reaching $\tau = \tau_{ms}$.

### 3.3 Split flows

Split attractor flows arise naturally in the description of non-spherically symmetric stationary BPS supergravity solutions [13]. Again we will take a pragmatic approach here and simply state a definition, without reference to supergravity equations of motion. A split flow is obtained by letting a flow of charge $\Gamma = \Gamma_1 + \Gamma_2$ split, at a $(\Gamma_1, \Gamma_2)$ MS wall, into a flow of charge $\Gamma_1$ and a flow of charge $\Gamma_2$. In view of the previous discussion, there are two cases to consider: splits at a MS wall with $\langle \Gamma_1, \Gamma_2 \rangle \neq 0$ (type 1) and splits at a MS wall with $\langle \Gamma_1, \Gamma_2 \rangle = 0$ (type 2). From the conclusions of the previous discussion, it follows that type 1 splits can happen anywhere on the MS line (and the position will in general be dependent on the starting point of the flow), whereas type 2 splits necessarily occur at an attractor.

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9In the four dimensional theory, this corresponds to mutually local charges, i.e. charges that (possibly after an electromagnetic duality rotation) can simultaneously be considered to be electric, without magnetic components.
Figure 2: Left: type 1 split flows. Right: type 2 split flows. In both cases a set of initial points is indicated by $z^{(i)}_0$, and the two final attractor points by $z_1$ and $z_2$. For the type 2 split, the “intermediate” $\Gamma_1 + \Gamma_2$-attractor point is denoted by $z^{(1+2)}_0$.

point (independent of starting point). Because of that, for generic initial $z_0$, type 1 splits involve just two branches, whereas type 2 splits might generically involve more than two. Furthermore, branches coming out of a type 2 split necessarily stay on their MS line, while those coming out of a type 1 split necessarily leave that line.

The two types of splits are illustrated in fig. 2. The splitting can be repeated on the separate branches, resulting in flow trees of arbitrary complexity, as shown in fig. 3. These split flow trees can still be considered to be geodesic strings with respect to the action (3.6), where the condition for the splits to occur only at MS is moreover an automatic consequence of the minimization of the action.

3.4 Stability of splits

Type 2 splits, being fixed at an attractor point, are insensitive to the initial moduli $z_0$, and are therefore stable for variations of $z_0$ all over Teichmüller space $\tilde{\mathcal{M}}_c$ (the covering space of moduli space $\mathcal{M}_c$).

A type 1 split on the other hand is fragile. If we let $z_0$ cross the MS wall on which the split point is located, from the side where (3.9) is satisfied to the side where it is not, the split ceases to exist.

A stability condition for type 1 splits in terms of phases that makes sense for arbitrary points $z_0$ in Teichmüller space $\tilde{\mathcal{M}}_c$ can be formulated as follows. Consider

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10In the four dimensional supergravity, type 1 splits are related to multicentered configurations at the macroscopic level, whereas type 2 splits are related to multicentered configurations in the near horizon region [14].
The homology classes corresponding to the different branches (and thus the branches themselves) are recursively labeled as follows. If a branch $\Gamma_x$ splits into two new branches through a type 1 split, the new branches are labeled $\Gamma_x^+$ and $\Gamma_x^-$, where the $+/-$ is attributed such that $\langle \Gamma_x, \Gamma_x^+ \rangle > 0$ (so $\langle \Gamma_x, \Gamma_x^- \rangle < 0$). When the split is of type 2, the corresponding branches are labeled $\Gamma_x^a, \Gamma_x^b, \Gamma_x^c$, and so on (in no particular order).

the type 1 split $\Gamma \to \Gamma_+ + \Gamma_-$, with incoming branch starting at $z = z_0$, and take $\langle \Gamma_-, \Gamma_+ \rangle > 0$. Let $\alpha_- = \arg Z(\Gamma_-) \mod 2\pi$ and similarly for $\alpha_+$ and $\alpha$. The relative $2n\pi$-ambiguity between the phases is fixed by requiring $\alpha_- = \alpha_+ = \alpha$ at the MS wall and taking the $\alpha_x$ to be continuous on Teichmüller space. Now, since there can be at most one solution to (3.8), we certainly need that $-\pi < (\alpha_+ - \alpha_-)_0 < \pi$ in order for the split to exist. Combining this with (3.9), this yields the condition

$$0 < (\alpha_- - \alpha_+)_0 < \pi$$

(and consequently also $0 < (\alpha - \alpha_0) < \pi$ and $0 < (\alpha_+ - \alpha)_0 < \pi$). This is a necessary condition. Close to the MS wall it is also sufficient, but when moving further away from the wall, it could fail to be so, as the split flow might cease to exist even without negating (3.10). This will be the case if the flow is “dragged” through a part of the discriminant locus of $\mathcal{M}_c$ and the 3-cycle vanishing there has nonzero intersection with the 3-cycle corresponding to the flow [13, 14]. In general this does not mean that the split decays like it does when the MS line is crossed. As discussed in [13, 14], in such situations, the original flow tree can morph into a new one through the so-called branch creation mechanism: a new branch, corresponding to the vanishing cycle and ending on the discriminant locus, is “pulled” into existence, thus saving the flow tree from collapse.
4. Attractor flows as Hamiltonian (de)formations of SLGs

4.1 SLG deformations along attractor flows

We now turn to the stability of SLG manifolds under small deformations of the complex structure of $X$ induced by moving along an attractor flow. From theorem 2.14 in [6] and the fact that the Kähler class remains constant, one indeed expects the existence of a special Lagrangian near the original one, at least if the SLG does not degenerate during the deformation process. In [10], it was furthermore shown that such deformations can be taken to be Hamiltonian.

Let us make this explicit, along the lines of section 2.2. The only difference with that section is the fact that the factor $\text{Im}(e^{-i\alpha \Omega})$ appearing in equation (2.11) will now also have an explicit $t$-dependence, due to the variation of the complex structure on $X$ as given by (3.4). To make this precise, we have to specify a relation between $\mu$ and $t$. A convenient choice away from the attractor point is $dt \equiv -\frac{\sqrt{|Z|^2 \mu}}{\mu - 1} d\mu$. The $t$-dependence of $\text{Im}(e^{-i\alpha \Omega})$ is then obtained from (3.4), where $\Gamma$ should be understood as the harmonic representative in the cohomology class Poincaré dual to $[L]$, with $L$ the SLG under consideration. Because of this extra $t$-dependence, the left hand side of (2.13) is no longer zero, but equal to $F^* [dt \wedge \partial_t \text{Im}(e^{-i\alpha \Omega})]|_{t=0}$, which by (3.4) and the SLG condition equals $-dt \wedge F^* \Gamma |_{t=0} = \Gamma |_L \wedge dt$. Therefore the condition (2.14) for preservation of the SLG condition is now

$$d\theta^{(1)} = 0$$
$$d\eta^{(2)} = \Gamma |_L.$$ (4.1)

All other identities remain the same. In particular the relation $*\theta^{(1)} = 2k \eta^{(2)}$ is unchanged. A particular solution is therefore obtained by putting $\theta^{(1)} = 2k dH$, $\eta^{(2)} = *dH$, with $H$ the up to a constant unique solution of

$$\Delta H = \Gamma |_L,$$ (4.3)

with $\Delta \equiv d * d$. Such a deformation is Hamiltonian, with Hamiltonian function $H$. Finding $H$ is equivalent to finding the electrostatic potential on $L$ for a given charge density $\Gamma |_L$. As a consistency check, observe that the integrability condition for this problem is trivially satisfied, as the total “charge” $\int_L \Gamma |_L = \langle \Gamma, \Gamma \rangle = 0$. Another physical interpretation, which is particularly useful to keep in mind for intuition in what follows, is viewing $H$ as the equilibrium temperature on an ideal heat conductor $L$ with heating/cooling sources given by $\Gamma |_L$.

Note that because of the SLG condition and equation (3.4), at a regular attractor point, $\Gamma |_L = 0$ and therefore the solution of (4.3) is trivially a constant and the corresponding first order deformation (in $t$) of $L$ zero. This is also required by consistency, since all attractor flows stop at that point.
4.2 Simple degeneration

Now let us see what we get when the SLG submanifold $L$ happens to be degenerate, equal to the union of two (smooth) SLG submanifolds $L_1$ and $L_2$ (necessarily with equal phases). We first consider the case where $L_1$ and $L_2$ have a single, transversal intersection point (so we are at a point in moduli space where the attractor flow crosses a line of $(L_1, L_2)$ marginal stability, i.e. where a flow split of type 1 can occur). Without loss of generality, we can assume that

$$\langle L_2, L_1 \rangle = +1. \quad (4.4)$$

Equation (4.3) does not have a solution on $L_1$ and $L_2$ separately, since $\int_{L_1} \Gamma |_{L_1} = \langle L_1, L \rangle = \langle L_1, L_2 \rangle = -1 \neq 0$. To get a solution, we have to “connect” $L_1$ and $L_2$ in a certain way through their intersection point. This goes as follows. Let $S_1$ ($S_2$) be an infinitesimally small sphere in $L_1$ ($L_2$), centered around the intersection point. We can then consider $L = L_1 \cup L_2$ to be the variety obtained by deleting the insides of the infinitesimal spheres in $L_1$ and $L_2$ and “gluing” the two remainders together along the spheres. With suitable orientations for $S_1$ and $S_2$, we can write $S_1 = \partial L_2$ and $S_2 = \partial L_1$ (see fig. 4). In more mathematical language this procedure is referred to as taking the connect sum of $L_1$ and $L_2$.

After connecting $L_1$ and $L_2$ in this way, an observer sitting in $L_1$ will see the effect on (4.3) of the presence of $L_2$ as a $\delta$-function source at the intersection point:

$$\int_{S_1} \eta_1^{(2)} = \int_{\partial L_2} * dH = \int_{L_2} \Delta H = \int_{L_2} \Gamma |_{L} = \langle L_2, L \rangle = 1. \quad (4.5)$$

As in other places in this paper, when confusion is not possible, we take the liberty to use the same notation for the SLGs and their corresponding homology classes and their Poincaré dual cohomology classes.
Figure 5: SLG deformation of $L_1 \cup L_2$ to $L'$ corresponding to a change in complex structure in the direction of the attractor flow, assuming this deformation exist. Here the intersection curves with the $z^m$ coordinate plane are shown, near the intersection point $z = 0$. The deformation vectors are given by the thin red arrows. Note that, since a small but finite deformation is shown here, the expressions (4.7)-(4.8) for infinitesimal deformations are only valid up to a small but nonzero distance from the intersection point.

Therefore, choosing suitable spherical coordinates $r, \theta, \phi$ in $L_1$, centered at the intersection point, we can write, for small $r$:

$$\eta^{(2)} \approx \frac{1}{4\pi} \sin \theta \, d\theta \wedge d\phi.$$  \hfill (4.6)

Comparison with (2.13) then gives $v^\theta \approx 0 \approx v^\phi$ and $v^r \approx \frac{k}{4\pi r^2}$. Hence, from the definition of $v$, we get for the deformation vector field on $L_1$ (putting $w \equiv 0$):

$$\partial_t F^m_1 \approx i \frac{k}{4\pi r^2} \partial_r F^m_1.$$  \hfill (4.7)

The above reasoning can be repeated for the deformation vector field on $L_2$, giving only a sign difference in the final result (because the $\eta^{(2)}$-flux through $S_2$ is opposite to the flux through $S_1$).

$$\partial_t F^m_2 \approx -i \frac{k}{4\pi r^2} \partial_r F^m_2.$$  \hfill (4.8)

For a deformation corresponding to a change of complex structure induced by following the attractor flow in the opposite direction of the flow, these two equations each acquire an extra minus sign. From the physics of decay at marginal stability and the work of Joyce [6], one expects that following the flow in one direction will make $L_1$ and $L_2$ join into one smooth SLG, whereas following it in the other direction will produce a decay (into two separate SLGs). In other words, we only expect a consistent fusion of the two SLGs for one direction.

Figuring out which one is a rather subtle business. Fig. 5 shows the intersection of a $z^m$ coordinate plane with $L_1$ and $L_2$, tracking out certain rays (out of the
intersection point \( z = 0 \) in \( L_1 \) and \( L_2 \), together with their deformations according to (4.7)-(4.8), assuming they do indeed fuse into a smooth SLG \( L' \). As explained in detail in appendix A, we get equally oriented bases of the tangent spaces to \( L_1 \) and \( L_2 \) at \( z = 0 \) by orienting the deformed smooth curves in each coordinate plane \( m = 1, 2, 3 \), constructing the tangent vectors to the asymptotes to these curves in \( L_1 \) and \( L_2 \), and translating those vectors along the original rays in \( L_1 \) and \( L_2 \) to the intersection point, as indicated in the figure. The orientation of \( L_1 \) and \( L_2 \) determine in turn the sign of the intersection product \( \langle L_1, L_2 \rangle \), which, using the rules outlined in appendix B, can be read off from the picture (repeated for \( m = 1, 2, 3 \)): \( \langle L_1, L_2 \rangle = -(-1)^3 = +1 \). But this is in contradiction with equation (4.4). Therefore, in the direction of the attractor flow, \( L_1 \) and \( L_2 \) can not fuse into a new SLG \( L \) — in physical terms, the brane configuration ceases to be realizable as a BPS state.

If on the other hand we follow the attractor flow in the opposite direction, we get the situation of fig. 6, yielding \( \langle L_1, L_2 \rangle = -(+1)^3 = -1 \), which is consistent. We thus arrive at the important conclusion that, for the degeneration at hand, the decay happens in the direction of the attractor flow. This also implies that for such a degeneration, the stable side of the line of \((L_1, L_2)\) marginal stability is the one that satisfies (3.10). We thus reproduce (and extend) the SLG stability criterion as obtained by Joyce in [6] from an explicit local model of the SLG degeneration.

Finally, note that \( b^1(L) = b^1(L_1) + b^1(L_2) \), so the dimension of the D3-brane moduli space of the single brane \( L \) equals the sum of the dimensions of the moduli spaces of the constituent branes \( L_1 \) and \( L_2 \).

4.3 Degeneration with more than one transversal intersection point

We now turn to the case where \( L_1 \) and \( L_2 \) have multiple transversal intersection
points, say \( n \), as in fig. [3] where \( n = 3 \). We can again assume that \( \kappa \equiv \langle L_2, L_1 \rangle > 0 \). Of the \( n \) intersection points, \( n_+ \) will contribute positively to \( \langle L_1, L_2 \rangle \), and \( n_- \) will contribute negatively. So \( n = n_+ + n_- \) and \( \kappa = n_+ - n_- \).

We can repeat the previous analysis, but we have a little more freedom here: a priori we can choose along which of the intersection points we try to glue together \( L_1 \) and \( L_2 \), and which intersection points remain just (self-)intersection points. We can implement this freedom by allowing but not forcing the intersection points to appear as delta-function sources. More concretely, the difference with the \( n = 1 \) case is that we now have infinitesimal spheres \( S_{(s)}^1 \) and \( S_{(s)}^2 \) for each intersection point \( P_s, s = 1, \ldots, n \), leading to \( \partial L_2 = S_{(1)}^1 \cup \cdots \cup S_{(n)}^1 \) and similarly for \( L_1 \). Therefore, (4.5) now becomes

\[
\sum_{s=1}^{n} \int_{S_{(s)}^1} \eta^{(2)} = \kappa. \tag{4.9}
\]

An observer sitting in \( L_1 \) will still see the points \( P_s \) that get connected to \( L_2 \) as \( \delta \)-function sources, but we face an ambiguity now: only the sum of the corresponding charges \( Q_s \) is constrained by (4.3). We find for the analog of (4.7), close to intersection point \( P_s \):

\[
\partial_t F_1^m \approx \sigma i Q_s \frac{k}{4\pi r^2} \partial_r F_1^m, \tag{4.10}
\]

with

\[
Q_s \equiv \int_{S_{(s)}^1} \eta^{(2)} ; \quad \sum_{s=1}^{n} Q_s = \kappa, \tag{4.11}
\]

and \( \sigma = +1 \) in the direction of the flow, \( \sigma = -1 \) in the opposite direction. Points \( P_s \) through which \( L_1 \) and \( L_2 \) do not get connected have \( Q_s = 0 \). For the deformation vector field on \( L_2 \), we get the same formulas with the opposite sign.

There are additional consistency constraints on the \( Q_s \). First, we will argue that in order to produce a smooth SLG \( L \) from \( L_1 \cup L_2 \), all nonzero \( Q_s \) must have the same
Figure 8: Degeneration of $L$ near $P_s$ for $\sigma Q_s < 0$. The arrows show the “electric field” (or “heat flow”) $\nabla H$, which near degeneration (assuming the “charge density” $\Gamma|_L$ is nonsingular near $P_s$) has approximately equal flux through small spheres slicing the throat. The graphs on the lower row show how the corresponding difference in $H$ between the $L_1$ side and the $L_2$ side diverges when the degeneration is approached.

To prove this, let us consider the opposite process, namely letting a smooth SLG $L$ degenerate to $L_1 \cup L_2$. As long as $L$ is smooth, the function $H$ will be a well defined, finite function on $L$. However, when we approach the degeneration, near an intersection point $P_s$, $H$ will approach $-\sigma Q_s/4\pi r + \text{const.}$ on the $L_1$ side and $\sigma Q_s/4\pi r + \text{const.'}$ on the $L_2$ side. So the difference in $H$ between the $L_1$ side and the $L_2$ side of the “throat” diverges to $\sigma Q_s \infty$, as shown in fig. 8. On the other hand, $H$ stays regular away from the degeneration points (assuming $L_1$ and $L_2$ are regular), so to be consistent, the difference in $H$ between the $L_1$ side and the $L_2$ side must diverge in the same way over all throats, i.e. all to $+\infty$, or all to $-\infty$. This is the case if and only if all nonzero $Q_s$ have the same sign, proving the first assertion. (All this is intuitively quite clear if one recalls the equilibrium temperature interpretation of $H$: $L_1$ and $L_2$ are basically net cooled versus net heated parts of $L$, and the heat flow is squeezed through the necks when approaching degeneration.)

Since $\kappa > 0$, the second equation in (4.11) thus implies that all $Q_s$ must be positive. Therefore, for all “active” intersection points $P_s$ (i.e. those with nonzero $Q_s$), we have again the situation of fig. 7 for $\sigma = +1$ and fig. 8 for $\sigma = -1$, but the resulting consistency constraint is more subtle now: the contribution to the intersection product $\kappa = \langle L_2, L_1 \rangle$ of an active point $P_s$ must be negative for $\sigma = 1$ and positive for $\sigma = -1$. In other words, if we go upstream along the attractor flow, in order to obtain a smooth $L$, we have to connect $L_1$ and $L_2$ along the intersections...
with sign equal to the sign of the total intersection, whereas if we go downstream, we need to connect along the intersection with the opposite sign. Note that the first kind of intersections will always be present, while the second might not (in which case \( L \) necessarily decays downstream).

So from the second equation in (4.11), we see that the connection process comes with \( n_+ - 1 \) degrees of freedom if we move upstream, and \( n_- - 1 \) extra degrees of freedom (if any at all) if we move downstream. It is easy to understand what the origin is of these degrees of freedom: the manifold \( L \) obtained by gluing together \( L_1 \) and \( L_2 \) with \( m \) (\( m = n_+ \) or \( m = n_- \)) infinitesimal spheres cut out, has precisely \( m - 1 \) extra nontrivial 2-spheres as compared to those of \( L_1 \) and \( L_2 \) separately, i.e. \( b_2(L, \mathbb{R}) = b_2(L_1) + b_2(L_2) + m - 1 \), or equivalently \( b^1(L) = b^1(L_1) + b^1(L_2) + m - 1 \).

By the results of section 2.2, this means that we have \( m - 1 \) more degrees of freedom to deform \( L \) as an SLG submanifold than we have to deform \( L_1 \) and \( L_2 \) separately. One can think of these degrees of freedom as the volume of small balls filling up the connecting spheres \( S^1(s) \) in \( X \), parametrizing the scale of the throats.

The values of \( Q_s \) then simply corresponds to the rate of change of the size of these balls when we move in moduli space to or from the degeneration point.

Note that since \( n_+ = n_- + \kappa > n_- \), a transition \( L \rightarrow L_1 \cup L_2 \rightarrow L' \) downstream the attractor flow will always lower the dimension of the deformation moduli space of the SLG (i.e. \( b^1(L') < b^1(L) \)).

Finally, the same kind of reasoning as above on the behavior of \( H \) yields that these degenerations must occur either at all throats at the same time, degenerating \( L \) to \( L_1 \cup L_2 \), or not at all (because if one throat stays open, \( H \) will stay finite everywhere — in intuitive heat flow terms: there can be no temperature divergence if the two parts stay smoothly connected). This implies that the Hamiltonian deformations along attractor flows we are studying here preserve the topology of \( L \) unless \( L \) degenerates to a union \( L_1 \cup L_2 \).

Degenerations involving more than two decay products can be analyzed in a similar way. However, if several identical copies of the same constituent special Lagrangian are involved, additional subtleties might arise, putting more constraints on the set of possible active intersection points. We will not go into those issues here, nor into the analysis of more complicated degenerations, involving for example non-transversal intersections.

### 4.4 Type 2 degenerations

The degenerations we studied up to now can be called type 1 degenerations: they involve splitting into two SLGs with nonzero intersection product and, by the discussion of section 3, occur at a marginal stability point in moduli space where a type 1 split can occur.

\[12\text{This can be made precise along the lines of [3].}\]
Type 2 splits on the other hand occur at attractor points and involve splitting into charges with zero intersection product (recall that all charges having zero intersection product with the total charge have identical or opposite phases at the attractor point, so in general one can expect quite some candidate decay channels). Having zero intersection product does not mean that the decay products are disjoint as SLG submanifolds: they can intersect transversally with as many positive as negative intersections, or they can intersect non-transversally, for example along a circle. An example of the latter on a $T^6$ with coordinates $x^1, \ldots, x^6$ is the D3-brane system $x^1 = x^3 = x^5 = 0$ plus $x^1 = x^4 = x^6 = 0$. The corresponding splitting of a smooth SLG $L$ into two SLGs $L_1$ and $L_2$ will be called a type 2 degeneration.

The solution to (3.4) for initial moduli $z_0 = z_* + \delta z$ very close to the attractor point $z_*$ is

$$\text{Im}(e^{-i\alpha \Omega}) \approx \text{Im}(e^{-i\alpha \Omega})_* + \delta \text{Im}(e^{-i\alpha \Omega}) \mu ,$$

(4.12)

where $\delta \text{Im}(e^{-i\alpha \Omega}) \equiv \text{Im}(e^{-i\alpha \Omega}_0) - \text{Im}(e^{-i\alpha \Omega})_*$. Identifying now $t$ with $\mu$, we thus obtain for the analog of equation (4.3), describing deformations induced by moving away from the attractor point along the attractor flow specified by $z_0 = z_* + \delta z$:

$$\Delta H = -\delta \text{Im}(e^{-i\alpha \Omega})|_L = -\text{Im}(e^{-i\alpha \delta z^a D_a \Omega})_*|_L .$$

(4.13)

For the last equality, we used (3.2).

We will not attempt to classify all possible type 2 degenerations in this paper. Instead let us consider the case where $L = L_1 \cup L_2$ is a trivial Lagrangian 2-fold fibration over a circle, with the fibers of $L_1$ and $L_2$ intersecting transversally over a finite number of points, as in the $T^6$ example given above.

The situation is then similar to the type 1 case, except that we have to look at pictures like fig. 5 and fig. 6 for two-dimensional Lagrangians, fibered over a circle. However, there is one important difference. In the three-dimensional case, going from fig. 5 to fig. 6 (that is, changing the side of the MS wall we move to) results in a jump in the required intersection product of $L_1$ and $L_2$, implying nonexistence of the SLG deformation at one side of the MS line (since, of course, only one intersection product can match). In the two-dimensional case, the corresponding required intersection product (here of the 2d fibers) stays the same, namely +1, so there is no longer reason to expect a decay when crossing the MS line (which in this case means going through the attractor point). This fits nicely with the physical expectation (from supergravity [13]) that if the BPS state exists, it should exist in the entire neighborhood of the attractor point.

The number of deformation moduli of $L$ is also easily computed. Denoting the fiber of $L$ by $l$, we have $b_2(L) = b_2(l) + 1$, and similarly for $L_1$ and $L_2$. If the fibers $l_1$ and $l_2$ of $L_1$ and $L_2$ have $n$ intersection points, we furthermore have $b_1(l) = b_1(l_1) + b_1(l_2) + 2(n - 1)$. Combining this, we find $b_2(L) = b_1(l) + 1 = b_1(l_1) + b_1(l_2) + 2(n - 1) + 1 = b_2(L_1) + b_2(L_2) + 2n - 3$. As in the type 1 case,
the new degrees of freedom can again be viewed as the volumes of certain minimal 3-manifolds with boundary on L, only now their topology will no longer be that of a ball, but rather that of a circle times a disc, i.e. a solid 2-torus.

Many more kinds of degenerations could occur, but it is at this point not clear to us how to proceed with a systematic analysis, so we will leave it at this.

5. (Dis)assembling Special Lagrangians

In this section we will explain how these results can be used to get insight in the structure of arbitrary SLGs, and to obtain a parametrization of their moduli spaces in favorable circumstances.

5.1 Disassembling

Let \( L \) be a smooth special Lagrangian in the Calabi-Yau manifold \( X \) with complex structure given by \( z = z_0 \). If we vary the complex structure along the attractor flow corresponding to the homology class of \( L \), we can keep \( L \) special Lagrangian by deforming it as detailed in section 4.1, at least as long as \( L \) does not split.

When a type 1 split occurs along the flow, splitting \( L \) into two equal phase SLGs \( L_1 \) and \( L_2 \), the procedure can be repeated on the two decay products separately. Correspondingly, the attractor flow will bifurcate in a type 1 split, as described in section 3.3. By iterating this procedure, we end up with a flow tree containing only type 1 splits, with branches ending in a number of attractor points, each associated to one of the “constituent” SLGs. The attractor points will be either at a nonzero minimum of the corresponding central charge modulus |\( Z \)|, or at a singular point\(^{13} \) with the corresponding 3-cycle being a vanishing cycle (i.e. zero volume with respect to \( \sqrt{\text{Vol}(X)} \)). In particular, it is not possible to end up at a regular point with vanishing \( Z \), since for special Lagrangians, equation (2.9) holds, so if \( Z \) vanishes, the SLG must be a vanishing cycle and we are by definition at a singular point of moduli space.

If at an attractor point with nonzero \( Z \), a type 2 degeneration occurs, we can again split the flow, as a type 2 split, and continue to deform the constituents along the new branches, and so on. The whole procedure ends with a flow tree like fig. 3, with branches ending on a set of connected special Lagrangians, at their respective attractor points, that cannot further be decomposed. We will call such SLGs simple. Clearly, any vanishing cycle at its vanishing point is simple (the opposite is probably not true). Note that one can expect the procedure to end after a finite number of splits, because the volumes |\( Z \)| of the SLGs decrease monotonically along the

\(^{13} \)We define singular points of moduli space as points where the volume of a 3-cycle vanishes with respect to the square root of the volume of the entire Calabi-Yau. Physically, those are the points where the corresponding particle masses vanish in 4d Planck units.
attractor flows, and splits always result in constituents lighter than the original SLG. For the same reason, one expects more complicated trees to appear for higher volume SLGs (relative to the CY volume).

In this way, we have constructed a well defined decomposition of any special Lagrangian at a given point in moduli space into simple special Lagrangians. As formulated, the decomposition is unique. However, other and equally natural decompositions, based on the same kind of Hamiltonian deformations, can be possible, namely in those cases where at a type 1 split, we also have the option to continue to deform along the original flow. As explained in section 4.3, this can occur for SLGs transversally intersecting in a number points with both positive and negative contributions to the intersection product. Thus one SLG $L$ can correspond to several different natural flow trees.

Even if we fix this ambiguity, e.g. by only splitting when no other options are open, two SLGs in the same homology class do not necessarily give rise to the same flow tree (examples of different flow trees within one homology class and at fixed $z_0$ can be found in [14]). However, generically, one can expect the decomposition to be stable under small variations of the complex structure moduli of $X$ and the deformation moduli of $L$, where “stable” means that the flow tree and the final constituents are at most a bit (continuously) deformed. For bigger variations, this is not necessary the case. First, there are some “mild” changes possible in the topology of the tree, which can still be regarded as continuous, like changes in connections between the different branches, or, in the presence of a discriminant locus, creation/annihilation of branches ending on that locus. Details and examples of these phenomena can be found in [14].

A more drastic change occurs when the flow tree begins with a type 1 split and $z_0$ passes through the corresponding MS line, causing the split and hence the entire flow tree to decay (cf. section 3.3). If no alternative continuation of $L$ exists at this split point (that is, if all intersection points contribute with the same sign to the intersection product, see section 4.3), this also implies the decay of the associated SLG. Otherwise, a new flow tree should be constructed, and the dimension of the moduli space of the SLG decreases.

The map from the deformation moduli space of $L$ (at fixed $z_0$) to the product of the deformation moduli spaces of the constituent SLGs (at their attractor points) will in general not be injective. Two different SLGs $L$ can end up having exactly the same decomposition, because for instance at a type 1 split with intersection product greater than one, some deformation degrees of freedom disappear, corresponding to the choice of $Q_s$ when going in the opposite direction, as explained in section 4.3. If we add at every split the relevant data resolving this ambiguity (i.e. the $Q_s$ for type 1 splits) to the flow tree data, the map to this “dressed” set of flow trees will be injective. The question whether the map is also surjective will be addressed in the next section.
The fact that every SLG has this kind of decomposition again fits beautifully with supergravity results: the existence of such a flow tree, with none of the branches ending on a regular zero of $Z$, is precisely what is needed for a BPS solution to exist in the 4d supergravity theory, as explained in [13]. And if the existence of a special Lagrangian is equivalent to the existence of a corresponding BPS state in the full string theory, this is clearly a requirement for the consistency of the theory. It is therefore quite pleasing to see this result appearing here.

5.2 Assembling

Given a certain flow tree with none of the branches ending in a regular zero, we can try to reverse the above process to end up with a special Lagrangian $L$ at $z_0$. To do so, we first have to pick SLGs in the homology classes specified by the flow branches terminating in attractor points. This, of course, requires such SLGs to exist, which is already one place where our attempt can fail. Next we deform those SLGs upstream along the attractor flows, as described in section 2.2. At the split points, we have to fuse our SLGs together. This not always possible, as the SLGs have to be “connectable”. For instance if the SLGs are disjoint, there is obviously no way to glue them together (a simple example where this is not possible is $X = T^6$ with $L_1$ given by $x^1 = x^3 = x^5 = 0$ and $x^1 = a, x^2 = x^4 = 0$ when $a \neq 0$). Furthermore, the subtleties that can arise in case the splits involve multiple copies of the same SLG constituents, briefly mentioned at the end of section 4.3, can also produce obstructions to successful gluing. So there will be restrictions on the set of SLGs we start with in order to be able to complete the job. In particular, this implies that the map between SLGs $L$ and candidate constituents for a given flow tree will not be surjective in general. Figuring out these restrictions could be a difficult task in general, though it seems doable for simple models, like $X = T^6$.

In conclusion, if the issue of restrictions on the constituents can be dealt with, this (de)composition of SLGs provides a classification scheme for SLGs in Calabi-Yau threefolds, and in favorable circumstances a parametrization of their moduli spaces: if the moduli spaces of the constituents are known, the moduli space of the assembled Lagrangian is obtained by combining the constituent moduli spaces, adding the extra “$Q_s$” degrees of freedom at the split points, and making some identifications if necessary. A big advantage of this setup is that it is not necessary to construct the SLGs explicitly.

5.3 Comparison with II-stability

Though our basic stability criterion for SLGs assembled in this way is simply that

\footnote{Recall that, as shown in section 4.1, deformations “upstream” the attractor flow do never lead to forced decays.}

\footnote{This might very well be the key to the resolution in this context of the “s-rule problem” discussed in [14, 23].}
the corresponding flow tree must exist for the given \( z_0 \), it is useful to compare this with the (extended) \( \Pi \)-stability conjecture of Douglas et al. [3, 4, 1]. This criterion roughly states that a “topological” “object” \( C \) is physically stable if for any two stable objects \( A \) and \( B \) of which \( C \) can be considered to be “made of”, the “triangle” \( A \to C \to B \) is stable, the latter meaning that the “morphism grades” between two subsequent objects involved all lie between 0 and 1. For a precise definition of the words between quotation marks, we refer to [4, 1]. Essentially the objects here are graded special Lagrangians, the grade distinguishing between different possible \( \mathbb{R} \)-valued phases of the SLG. The morphism grade between two subsequent entries in the triangle given above is simply their (moduli dependent) phase difference divided by \( \pi \). Thus, given the set of stable objects at one point in moduli space, this stability criterion yields in principle the stable objects at all other points.

A similar but not identical statement holds here, essentially because of the stability condition (3.10) for type 1 splits.\(^{16}\) If we formally associate to a type 1 split \( L \to L^+ + L^- \) (in the notation of fig. 3) the triangle \( L^+ \to L \to L^- \), we get indeed from (3.10) that in order for this split to exist, we need the grades \( (\alpha - \alpha_+)_0/\pi \) and \( (\alpha_+ - \alpha)_0/\pi \) to be between 0 and 1, or in the terminology of [1, 4], that the triangle \( L^+ \to L \to L^- \) is stable (at \( z_0 \)).

Obviously, though the setup is quite different, this is formally similar to the \( \Pi \)-stability criterion. However, even after making natural identifications between the two setups, there are differences. First, in the \( \Pi \)-stability criterion, to verify the stability of \( C \), the objects \( A \) and \( B \) in the triangles \( A \to C \to B \) to be checked must be stable at \( z_0 \). This is not necessarily the case in our setup: we need that \( A \) and \( B \) are stable at the split point \( z_s \) (to verify their stability there, in the case that those branches again have type 1 splits, the above analysis has to repeated, but with \( z_0 \) replaced by \( z_s \), and so on if more type 1 splits occur). If \( z_0 \) is not too far from \( z_s \), this will be equivalent, but not necessarily in general. On the other hand, to check \( \Pi \)-stability, one might have to verify several potentially destabilizing triangles involving \( C \), all at \( z_0 \) (presumably corresponding to the possible occurrence of more than two constituents in our flow trees). In our case, there is only one triangle to check at \( z_0 \), the one corresponding to the first split (assumed to be of type 1). However, in general, there are other, “lower level”, triangles to check, namely those corresponding to the subsequent splits, but a priori they should be verified at subsequent split points, not at \( z_0 \).

Since the differences are of the apples versus oranges kind, it is not inconceivable that they “annihilate” each other up to a certain extent, and that in many or even all cases the two criteria are actually equivalent. To settle this, one should try to find specific examples separating the two (or prove equivalence, of course), but we

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\(^{16}\)Recall that type 2 splits cannot be destabilized by variations of \( z_0 \), so as far as the moduli dependence of stability is concerned, we need only to consider type 1 splits.
\begin{figure}[h]
\centering
\includegraphics[width=0.7\textwidth]{diagonal_torus.png}
\caption{The diagonal torus with modulus $\tau$. The 3-brane mirror to the $D0$ is $\alpha_1 \times \alpha_2 \times \alpha_3$, the one mirror to the $D6$ is $-\beta_1 \times \beta_2 \times \beta_3$.}
\end{figure}

will leave this for future work.

6. Some (simple) examples

In this section we will illustrate our results with some simple examples, mainly on $T^6$. A much more elaborate study of this case will appear elsewhere [25].

6.1 Type 1: the mirror of $D0$-$D6$ on the diagonal $T^6$

Let $X_\tau$ be the diagonal $T^6$ with modulus $\tau$, that is, $X_\tau = E_\tau \times E_\tau \times E_\tau$, with $E_\tau$ the 2-torus with standard complex structure parameter $\tau$ (valued in the upper half plane), as shown in fig. 9. For $D3$-brane charges respecting the permutation symmetry of the three 2-tori, this subfamily $\{X_\tau\}_\tau$ of 6-tori is closed under the attractor flow equations, making this a particularly simple example.

Type IIB string theory on $X_\tau$ is mirror (or T-dual\footnote{By T-dualizing along the three $\alpha_i$-cycles}) to IIA on $Y = E_\tau' \times E_\tau' \times E_\tau'$, with $E_\tau'$ the 2-torus with area $\text{Im} \, \tau/2$ and $B$-field flux $\text{Re} \, \tau/2$ (which together determine the complexified Kähler class of $Y$). The IIA $D0$- and $D6$-brane are mirror to respectively the $\alpha_1 \times \alpha_2 \times \alpha_3$ $D3$-brane and the $-\beta_1 \times \beta_2 \times \beta_3$ $D3$-brane (see fig. 9).\footnote{Our orientation conventions, and hence what we call a brane and what an anti-brane, are such that the period $Z(D0) \sim 1$, where “~” means positively proportional, $Z(D2) \sim \tau$, $Z(D4) \sim -\tau^2$ (i.e. such that $D4$ and $D0$ have equal phases at $\text{Re} \, \tau = 0$) and $Z(D6) \sim -\tau^3$ (such that $D6$ and $D2$ have equal phases at $\text{Re} \, \tau = 0$).} It is convenient to label also the IIB $D3$-branes by their IIA names and we will do so henceforth, hoping not to cause confusion.

The Kähler form on $X_\tau$ can be taken to be $\omega = dz^1 \wedge dz^1 + dz^2 \wedge dz^2 + dz^3 \wedge dz^3$, and the normalized holomorphic 3-form $\Omega = (2 \, \text{Im} \, \tau)^{-3/2} d\bar{z}^1 \wedge d\bar{z}^2 \wedge d\bar{z}^3$. The trivial flat embeddings of the $D3$-branes under consideration are special Lagrangian, giving a 3-parameter moduli space (with torus topology) for each of them (complex if Wilson lines are included).
Figure 10: The shaded region $\text{Im} \tau < \sqrt{3} \text{Re} \tau$ shows the stable domain of the composite (or “bound state”) D6-D0 SLG. A typical split attractor flow corresponding to this SLG is sketched in green. The $D0$ and $D6$ branches have attractor points at the “large complex structure” singularities $\tau = i\infty$ and $\tau = 0$ respectively. The dashed straight line $\text{Im} \tau = -\sqrt{3} \text{Re} \tau$ is the line of $(D0, D6)$ anti-MS. It contains a zero for the period $Z(D6 + D0)$, at $\tau = e^{2i\pi/3}$. The would-be single flow for $D6 + D0$ (the dotted red curve) crashes at this zero.

The D6 and the D0 have one transversal intersection point. Starting at a D6-D0 marginal stability line and moving into the stable domain, they will therefore fuse into a single SLG, as explained in section 4.2. The quotient of the D6-period and the D0-period is $-\tau^3$, so D6-D0 marginal stability occurs at the line $\text{Im} \tau = \sqrt{3} \text{Re} \tau$, as shown in fig. 11. The intersection product is easily computed using the rules of appendix B: $\langle D0, D6 \rangle = +1$. The stability condition (3.10) thus becomes $0 < \alpha_{D0} - \alpha_{D6} < \pi$, corresponding to the shaded area $\text{Im} \tau < \sqrt{3} \text{Re} \tau$ in the figure. In the white region, the SLG ceases to exist: it splits in a pure D6 and a pure D0 SLG, with different phases; physically this means the state is no longer BPS. Going to the IIA interpretation, this agrees with the results of [27] (at marginal stability) and [26] (on the full moduli space), where it was shown that a supersymmetric D6-D0 bound state on $T^6$ exists, but only for a certain range of B-field fluxes.

The number of moduli of these composite (“bound state”) D6-D0 SLGs is 6, as is easily computed using the formula at the end of section 4.2. The map between the composite SLGs and their constituents, discussed in section 5, is furthermore one to one here, so the deformation moduli space will simply be the product of the moduli spaces of D0 and D6.

The deformation equation (4.3) and its solution can also be made more explicit here, at least on the MS line. Writing $dz^m = du^m + \tau dv^m$ for $m = 1, 2, 3$, we have $\Gamma = du^1 \wedge du^2 \wedge du^3 + dv^1 \wedge dv^2 \wedge dv^3$ and we can take $L_1 = D6$, $L_2 = D0$, $L = L_1 \cup L_2$. Then on the D0 part of $L$, (4.3) becomes

$$\Delta H = \sigma [1 - \delta^3(u - u_0)] du^1 \wedge du^2 \wedge du^3 = \sigma [1 - \delta^3(u)] dV , \quad (6.1)$$
Figure 11: $\tau$-plane with sketch of a typical (type 2 split) attractor flow for the D4-D0 composite SLG (which is everywhere stable). The intermediate attractor point is at $\tau = i$. Also included is a picture of a connectable D4-D0 system in $T^6$ at this attractor point, where D4 and D0 have equal phases and hence can supersymmetrically coexist.

and on the $D_6$ part

$$\Delta H = \sigma \left[ 1 - \delta^3(v - v_0) \right] dv^1 \wedge dv^2 \wedge dv^3 = -\sigma \left[ 1 - \delta^3(v) \right] dV,$$

where $dV$ is the volume element, $u_0$ and $v_0$ are the coordinates of the intersection point, and $\sigma = 1$ for a change in $\tau$ in the direction of the attractor flow, $\sigma = -1$ in the opposite direction. One could write down explicit solutions to these equations, but we won’t do so here.

It is possible to generalize all this to less trivial examples, like for example replacing the $\beta_m$-cycles by $n\beta_m + \alpha_m$, producing an intersection product equal to $n^3$. Instead of doing that, let us consider an example involving a type 2 split.

### 6.2 Type 2: the mirror of D0-D4 on the diagonal $T^6$

The D3-brane corresponding to the type IIA D0 is still $D0 \equiv \alpha_1 \times \alpha_2 \times \alpha_3$. The D4 is a bit more complicated, because we require the symmetry between the three $E_\tau$ tori to be respected. The natural IIB D3-brane system to consider is then $D4 \equiv -(\alpha_1 \times \beta_2 \times \beta_3 + \beta_1 \times \alpha_2 \times \beta_3 + \beta_1 \times \beta_2 \times \alpha_3)$, which has 3 units of D4-brane charge on the IIA side, and period $Z(D4) = -3(2\text{Im}\, \tau)^{-3/2} \tau^2$. Its intersection product with D0 is zero.

The homology class $D0 + D4$ has period $Z(D4 + D0) = (2\text{Im}\, \tau)^{-3/2} (1 - 3\tau^2)$, which has a nonzero minimal norm (= regular attractor point) at $\tau = i$, where $Z = Z_\star = \sqrt{2}$. At the attractor point, D4 and D0 have the same phase $\alpha = 0$, so the D4-D0 system is supersymmetric there. The same holds for all points on the imaginary $\tau$ axis.
Deformations away from $\tau = i$, fusing the separate branes into one SLG, are governed by equation (4.13). For this equation to make sense (and fusing to be possible), the branes must have common points, along which they can get glued together. This is the case if and only if of each of the three D4 components, the $\alpha$ cycle coincides with the corresponding $\alpha$ cycle of the D0, as sketched in fig. [14]. Then the D0 brane intersects each of the D4 components in a circle. Thus, any D4 component put together with the D0 brane can be seen as a fibration over a circle, with transversally intersecting 2-dimensional SLG fibers (with intersection +1), and we are essentially in the situation considered in section 4.4.

To make (4.13) explicit, denote the position of the D4 component $\alpha_1 \times \beta_2 \times \beta_3$ by $(v^1(1), u^2(1), u^3(1))$ and similarly for the other three components. The requirement of coincidence then implies that the position of the D0 is given by $(v^1(1), v^2(2), v^3(3))$. A simple calculation shows that

$$D_\tau \Omega = i(2 \Im \tau)^{-5/2}(dz^1 \land dz^2 \land dz^3 + dz^1 \land d\bar{z}^2 \land dz^3 + dz^1 \land dz^2 \land d\bar{z}^3),$$

(6.3)

so (4.13), for an infinitesimal deviation $\delta \tau$ from $\tau = i$, becomes on the D0 part:

$$\Delta H = -2^{-5/2} \Re \delta \tau \left[ 3 - \delta(u^2 - u^2_{(1)})\delta(u^3 - u^3_{(1)}) - \delta(u^1 - u^1_{(2)})\delta(u^3 - u^3_{(2)}) - \delta(u^1 - u^1_{(3)})\delta(u^2 - u^2_{(3)}) \right] dV,$$

(6.4)

on the D4 component $\alpha_1 \times \beta_2 \times \beta_3$:

$$\Delta H = 2^{-5/2} \Re \delta \tau \left[ 1 - \delta(v^2 - v^2_{(2)})\delta(v^3 - v^3_{(2)}) \right] dV,$$

(6.5)

and similarly for the other two D4 components. The delta-function sources, localized on the intersecting circles, will produce the required fusions, yielding an SLG with 9 deformation moduli (basically the positions of the four constituents minus 3 because of the coincidence constraint). Note that if $\Re \delta \tau = 0$, we get a constant $H$, so the branes stay at their original $(u^m, v^m)$ positions on the $T^6$. Indeed, on the imaginary axis, the D4-D0 system stays special Lagrangian without deformation.

Again one could consider more complicated (higher charge) configurations and try to deduce the topology of their moduli spaces, which upon quantization should provide for example a microscopic computation of the corresponding black hole entropy.

6.3 The (no longer) mysterious $|10000\rangle_B$ brane on the Quintic

In [3], using CFT techniques, a number of BPS states was established to exist in IIA string theory at the Gepner point on the the Quintic. One of those states, labeled $|10000\rangle_B$, caused some confusion for a while, as it doesn’t correspond to a regular single attractor flow, which at the time was thought to imply that it doesn’t have a corresponding BPS supergravity solution. The issue got cleared up in [4].
where it was noted that it gives rise instead to a (type 1) split flow, which in turn is associated to a multicentered BPS supergravity solution. The two branches of this split flow correspond to SLGs vanishing at two different copies of the conifold point in Teichmüller space, with mutual intersection product equal to 5. Thus, the results of this paper tell us that there exists indeed an SLG in the required homology class at the Gepner point, formed by fusing these two SLGs together. In [14] strong numerical evidence was presented showing that beyond the split point, no consistent flow trees exist for this homology class, so no SLGs either. Such forced decays occur when all intersection points have the same sign (section 4.3), so we can assume the total number of intersection points to be 5. Therefore, since the vanishing cycles are 3-spheres, which have $b^1 = 0$ and therefore no deformation moduli, the number of deformation moduli of the resulting SLG is, according to the formulas at the end of section 4.3, equal to $5 - 1 = 4$. This agrees with the number of moduli found in [4].

7. Conclusions

We have constructed a (de)composition/classification scheme for arbitrary special Lagrangian submanifolds in a Calabi-Yau 3-fold, based on attractor flow trees. In favorable circumstances, this allows one to extract nontrivial information about existence, stability domains and deformation moduli spaces of special Lagrangians without having to construct them explicitly. Considering the virtual impossibility to construct generic SLGs in compact manifolds, this is an important simplification, similar to the way dealing with Calabi-Yau spaces becomes possible without explicit construction by invoking Yau's theorem and other results in algebraic geometry.

Clearly this construction may have many useful applications in string theory, and through string theory in $\mathcal{N} = 1$ field theories, as these are (at large volume) the world-volume theories of D-branes filling the four-dimensional noncompact space and wrapping an SLG. It should be noted though that stringy corrections are expected to the dynamics and moduli spaces of such D-branes as compared to their classical geometric counterparts, certainly away from large volume. However, the role of the attractor flow trees themselves will conceivably remain unchanged also after corrections to the BPS condition on the brane embedding. This is because the decomposition along a flow tree has the physical interpretation of moving the BPS “particle”, no matter how it is represented, adiabatically from spatial infinity into the interior of the corresponding (large N) supergravity solution, letting it decay wherever it is forced to, and repeating all this on the decay products (along different paths) and so on, all the way to the respective black hole cores. Because the IIB complex structure moduli space is exact at tree level, this picture should remain intact after corrections. In fact, it is possible that in general (away from large volume, or perhaps even already at large volume) the Hamiltonian flows on SLGs as defined
in section 4 are not well defined, leading to nasty singularities and the like, and
that to have a well defined deformation flow avoiding singularities, such that the
microscopic interpretation of the flow tree picture makes sense, stringy corrections
must be taken into account.

Unavoidably for a paper of limited size in this context, there are quite some
loose ends. We hope we have given the reader at least an idea of how to proceed
in principle. A first step to get more insight would be to construct a number of
explicit examples, for instance on $T^6$. From the mathematical side, an open problem
is (to my knowledge) under what conditions the existence of an SLG at a regular
attractor point (nonzero $|Z|$) is guaranteed. (This is related, at large volume / large
complex structure, to the “positive discriminant” conditions for vector bundles.)
Getting a better grip on this would increase the usefulness of attractor flow trees
for establishing existence of SLGs. From the physics side, apart from applications
to $\mathcal{N} = 1$ gauge theories, it would be interesting to see what quantization of SLG
moduli spaces obtained through our construction (if feasible) could teach us about
black hole entropy and other four dimensional space-time properties, especially in
the light of the rather special (multi-centered) black hole solutions appearing in the
corresponding low energy four-dimensional supergravity theories.

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A. Connect sums, trajectories and orientations

In section 4.2, we introduced the connect sum of two 3-folds $L_1$ and $L_2$ with one
transversal intersection point $P$ as the singular variety $L$ obtained by cutting out
infinitesimal spheres around the intersection point in $L_1$ and $L_2$ and gluing the two
manifolds together along the spheres. In this appendix, we will develop a practical
way to determine a basis of tangent vectors at $P$ in $L_1$ and $L_2$ which have both
positive (or both negative) orientation, given the way $L_1$ and $L_2$ are connected. This
is needed to compute the intersection product of $L_1$ and $L_2$ (as explained in appendix
B).

For our purposes here, we can locally model $L_1$ and $L_2$ as two positively oriented
copies of $\mathbb{R}^3$, and $L$ as the manifold obtained by removing spheres of radius $\epsilon \to 0$
around the origin and gluing the remainders together along those spheres. More
precisely, picking spherical coordinates $(r_1, \theta_1, \phi_1)$ resp. $(r_2, \theta_2, \phi_2)$ in the two copies.

19I thank R. Thomas for pointing this out to me.
of $\mathbb{R}^3$, we make the identifications

$$ r_1 = \epsilon^2 / r_2 $$  \hspace{1cm} (A.1)

$$ \theta_1 = \theta_2 $$  \hspace{1cm} (A.2)

$$ \phi_1 = \pi - \phi_2. $$  \hspace{1cm} (A.3)

The identifications are chosen such that $dr_2 \wedge d\theta_2 \wedge d\phi_2$ is positively proportional to $dr_1 \wedge d\theta_1 \wedge d\phi_1$, making the orientation of $L$ well defined.

Now imagine three particles in our local model of $L$, flying along the negative $x, y$ and $z$ axes, coming from infinity in the part of $L$ identified with $L_2$, and moving towards the sphere $r_2 = \epsilon$. In other words, the first particle comes from $\theta_2 = \pi/2, \phi_2 = \pi$ (and large $r_2$), the second from $\theta_2 = \pi/2, \phi_2 = -\pi/2$, and the third from $\theta_2 = \pi$. At any time, the velocity vectors of these particles, when further translated to the origin, form a positively oriented basis of $\mathbb{R}^3$ at the origin, thus giving a positively oriented basis of tangent vectors of $L_1$ at $P$. Now when the particles pass through the sphere $r_2 = \epsilon$, moving outwards, according to (A.2)-(A.3) respectively at $\theta_1 = \pi/2, \phi_1 = 0$, at $\theta_1 = \pi/2, \phi_1 = 3\pi/2$ and at $\theta_1 = \pi$. When translated to the origin of the copy of $\mathbb{R}^3$ associated to $L_1$, their velocity vectors are easily seen to be again a positively oriented basis of $\mathbb{R}^3$. (Basically the last two basis vectors flip sign with respect to the basis obtained in the other copy of $\mathbb{R}^3$.)

The upshot of all this is that if we imagine a set of three particle trajectories through $L$ going from asymptotic $L_2$ to asymptotic $L_1$, and we produce bases of the tangent spaces to $L_1$ and $L_2$ at $P$ associated to the asymptotic trajectories as outlined above, we get two identically oriented bases.

Note that this result is not as self-evident as it might seem: in the case of two-dimensional special Lagrangians, the opposite is true, as can be checked by repeating the above reasoning for two copies of $\mathbb{R}^2$. The bases associated to asymptotic trajectories now have opposite orientations. Essentially, this is because only one basis vector flips sign upon passing through the circle connecting the $L_1$ and $L_2$ parts of $L$, resulting in a basis of opposite orientation.

In general, for odd (even) dimensional SLGs, the bases associated to the two trajectory asymptotes will have equal (opposite) orientations.

**B. Computing intersection products**

We want to compute the intersection product of two $d$ real dimensional submanifolds $A$ and $B$ in a $d$ complex dimensional manifold, with all intersections between $A$ and $B$ transversal, and $A$ and $B$ intersecting each complex complex coordinate plane in a curve, as shown in fig. 12. That is, near an intersection point the manifolds can be represented locally as $A_1 \times A_2 \times \cdots A_d$ resp. $B_1 \times B_2 \times \cdots B_d$, with the $A_m$ and $B_m$
Figure 12: Contribution of a transversal intersection in the $z^m$ coordinate plane. Left: $\int C \hat{A}_m \wedge \hat{B}_m = +1$. Right: $\int C \hat{A}_m \wedge \hat{B}_m = -1$.

oriented curves as in fig. 12. Denoting the local Poincaré dual to $A_m$ by $\hat{A}_m$ (that is, locally, $\hat{A}_m = \delta(n)dn$, with $n$ any coordinate normal to $A_m$) and similarly for $B_m$, we find for the contribution of such an intersection point to the intersection product

$$\int_{C^d} \hat{A}_1 \wedge \cdots \wedge \hat{A}_d \wedge \hat{B}_1 \wedge \cdots \wedge \hat{B}_d = (-1)^{d(d-1)/2} \int_{C^d} \hat{A}_1 \wedge \hat{B}_1 \wedge \cdots \wedge \hat{A}_d \wedge \hat{B}_d$$

$$= (-1)^{d(d-1)/2} \int_{C} \hat{A}_1 \wedge \hat{B}_1 \cdots \int_{C} \hat{A}_d \wedge \hat{B}_d.$$

Where the separate integrals $\int \hat{A}_m \wedge \hat{B}_m$ evaluate to $+1$ or $-1$ according to the orientation conventions shown in fig. 12. Finally, the total intersection product is obtained by summing the contributions of all intersection points.

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