**ABSTRACT**

Graph algorithms enormously contribute to the domains such as blockchains, social networks, biological networks, telecommunication networks, and several others. The ever-increasing demand of data-volume as well as speed of such applications have essentially transported these applications from their comfort zone: static setting, to a challenging territory of dynamic updates. At the same time, mainstreaming of multi-core processors have entailed that the dynamic applications should be able to exploit concurrency as soon as parallelization gets inhibited. Thus, the design and implementation of efficient concurrent dynamic graph algorithms have become significant.

This paper reports a novel library of concurrent shared-memory algorithms for breadth-first search (BFS), single-source shortest-path (SSSP), and betweenness centrality (BC) in a dynamic graph. The presented algorithms are provably non-blocking and linearizable. We extensively evaluate C++ implementations of the algorithms through several micro-benchmarks. The experimental results demonstrate the scalability with the number of threads. Our experiments also highlight the limitations of static graph analytics methods in dynamic setting.

**keywords:** concurrent data structure, lock-free, obstruction-free, directed graph, breadth-first-search, single-source-shortest-path, betweenness centrality.

1 Introduction

A graph represents the pairwise relationships between objects or entities that underlie the complex frameworks such as blockchains [2], social networks [9], semantic-web [5], biological networks [17], and many others. Often these applications are implemented on dynamic graphs: they undergo changes like addition and removal of vertices and/or edges [18] over time. For example, consider the computation of shortest path or centrality between nodes in a real-time dynamically changing social network as highlighted in [33]. Such settings are challenging and approaches such as incremental computation [33] or streaming framework, where a graph operation is performed over a static temporal snapshot of the data structure, e.g. Kineograph [14], GraphTau [31], are currently adopted. However, application of concurrency in dynamic graph algorithms is largely unexplored where dynamic dataset-updates severely hinder parallel operation-processing designed for static graphs.

With the rise of multi-core computers around a decade back, concurrent data structures have become popular, for they are able to harness the power of multiple cores effectively. Several concurrent data structures have been developed: stacks [25], queues [3, 26, 36, 44], linked-lists [13, 23, 24, 45], hash tables [37, 38], binary search trees [8]
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1. BFS: Given a query vertex \( v \in V \), output each vertex \( u \in V - v \) reachable from \( v \). The collection of vertices happens in a BFS order: those at a distance \( d_1 \) from \( v \) is collected before those at a distance \( d_2 > d_1 \).

2. SSSP: Given a vertex \( v \in V \), find a shortest path with respect to total edge-weight from \( v \) to every other vertex \( u \in V - v \). Note that, given a pair of nodes \( u,v \in V \), the shortest path between \( u \) and \( v \) may not be unique.

3. BC: Given a vertex \( v \in V \), compute \( BC(v) = \sum_{s,t \in V} \frac{\sigma(s,t|v)}{\sigma(s,t)} \), where \( \sigma(s,t) \) is the number of shortest paths between vertices \( s,t \in V \) and \( \sigma(s,t|v) \) is that passing through \( v \). \( BC(v) \) indicates the prominence of \( v \) in \( V \) and finds several applications where influence of an entity in a network is to be measured.

Overview

In a nutshell, we implement a concurrent non-blocking dynamic directed graph data structure as a composition of lock-free sets: a lock-free hash-table and multiple lock-free BSTs. The set of outgoing edges \( E_v \) from a vertex \( v \in V \) is implemented by a BST, whereas, \( v \) itself is a node of the hash-table. Addition/removal of a vertex amounts to the same of a node in the lock-free hash-table, whereas, addition/removal of an edge translates to the same operation in a lock-free BST. Although lock-free progress is composable \( [16] \), thereby ensuring lock-free updates in the graph, however, optimizing these operations are non-trivial. The operations – BFS, SSSP, BC – are implemented by specialized partial snapshots of the composite data structure. In a dynamic concurrent non-blocking setting, we apply multi-scan/validate \( [1] \) to ensure the linearizability of a partial snapshot. We prove that these operations are non-blocking. The empirical results show the effectiveness of our algorithms.

\( ^{\dagger} \)Panugraham is the Sanskrit translation of Marriage, which undoubtedly is a prominent phenomenon in our lives resulting in networks represented by graphs.

\( ^{\ddagger} \)In this paper we confine the scope of discussion to directed graphs only.
Contributions and paper summary

• First, we introduce the ADT and describe the non-blocking design of directed graph data structure as a composition of lock-free sets. (Section 2)

• After that, we describe our novel framework as an interface operation with its correctness and progress guarantee (Section 3) followed by the detailed concurrent implementation of BFS, SSSP and BC. (Section 4)

• We present an experimental evaluation of the data structure. A novel feature of our experiments is comparison of the concurrent data structure against a parallel graph operations library Ligra [42]. Our experimental observations demonstrate the power of concurrency for dynamic updates in an application. Utilizing the parallel compute resources – 56 threads – in a standard multi-core machine, our implementation performs up to 10x better than Ligra for BFS, SSSP and BC algorithms in some cases. (Section 5)

Related work

The libraries of parallel implementation of graph operations are abundant in literature. A relevant survey can be found in [4]. To mention a few well-known ones: PowerGraph [22] Galois [35], Ligra [42], and the extensions thereof such as Ligra+ [43]. However, they primarily focus on static queries and natively do not allow updates to the data structure, let alone concurrency. The implementations on GPUs [29] adapt the existing parallel algorithms to the enhanced available parallelism therein, whereas, on the streaming frameworks the same algorithms are applied on static snapshots [14, 31]. A couple of exceptions though: Stinger [19] and Congra [40] support concurrent updates, however, they steer away from discussing the main challenges accompanying concurrency – progress guarantee and correctness. To our knowledge, the present work is the first in this direction.

In the field of concurrent data structures, only a couple of previous attempts are known. Kallimanis et al. [32] presented wait-free graph with dynamic traversals. Their design is based on adjacency matrix, can not support an unbounded graph and has no known implementation. Chatterjee et al. [12] presented a lock-free graph designed on lock-free linked-lists that support reachability queries. They have limited implementation results. Thus, in the field of concurrent data structures, this is the first attempt to implement SSSP and BC, while ensuring linearizability and non-blocking progress.

Remark. Before moving to technical details, we pause to emphasize our aim/technique in contrast to existing methods.

• We target more applied “non-local” operations in graph that are used in the domains such as analytics. In contrast, local operations, for instance, vertex/edge coloring, which are highly interesting in their own merits, are not in the scope of this work.

• In general, non-blocking data structures attempt to ensure lock-freedom which packs benefits from both worlds: performance and progress guarantee, however, ensuring lock-freedom in snapshot-like operations, such as ours, is particularly costly, and would easily lose out to highly optimized parallel static graph queries. Therefore, in the spirit of exploring advantages of concurrency in dynamic settings, we worked with obstruction-freedom, which favors performance while still ensuring practical progress guarantee.

• Though we compare against a high performance graph analytics framework, it is important to mention that the individual operations in our work are not even “inline” parallelized. Essentially, our experiments display the collective strength of “sequential” operations, which satisfy linearizability while interleaving concurrently, in a dynamic setting of large recurrence.

2 Non-blocking Graph Data Structure

Our discussion uses a standard shared-memory model [12] that supports atomic read, write, fetch-and-add (FAA) and compare-and-swap (CAS) instructions.

The Abstract Data Type (ADT)

Consider a weighted directed graph \( G = (V, E) \) as defined before. A vertex \( v \in V \) has an immutable unique key drawn from a totally ordered universe. For brevity, we denote a vertex with key \( v: v(v) \) by \( v \) itself. Extending on the notations used in Section 1, we denote a directed edge with weight \( w \) from the vertex \( v_1 \) to \( v_2 \) as \((v_1, v_2|w) \in E\). We consider an ADT \( \mathcal{A} \) as a set of the operations \( \mathcal{A} \):

A precondition for \((v_1, v_2|w) \in E\) is \( v_1, v_2 \in V \).

1. A \text{PUT} V(v)\ updates \( V \to V \cup v \) and returns \text{true} if \( v \notin V \), otherwise it returns \text{false} without any update.
2. A \textsc{RemV}(v) updates \( V \) to \( V - v \) and returns \( \text{true} \) if \( v \in V \), otherwise it returns \( \text{false} \) without any update.

3. A \textsc{GetV}(v) returns \( \text{true} \) if \( v \in V \), and \( \text{false} \) if \( v \notin V \).

4. A \textsc{PutE}(v_1, v_2|w)
   (a) updates \( E \) to \( E \cup (v_1, v_2|w) \) and returns \( \text{true,} \infty \) if \( v_1 \in V \) \& \( v_2 \in V \land (v_1, v_2|\cdot) \notin E \),
   (b) updates \( E \) to \( E - (v_1, v_2|\cdot) \cup (v_1, v_2|w) \); returns \( \text{true,} z \) if \( (v_1, v_2|\cdot) \in E \),
   (c) returns \( \text{false,} w \) if \( (v_1, v_2|w) \in E \) without updates,
   (d) returns \( \text{false,} \infty \) if \( v_1 \notin V \lor v_2 \notin V \) without updates.

5. A \textsc{RemE}(v_1, v_2|w) updates \( E \) to \( E - (v_1, v_2|w) \) and returns \( \text{true,} w \) if \( (v_1, v_2|w) \in E \), otherwise it returns \( \text{false,} \infty \) without any update.

6. A \textsc{GetE}(v_1, v_2|w) returns \( \text{true,} w \) if \( (v_1, v_2|w) \in E \), otherwise it returns \( \text{false,} \infty \).

7. A \textsc{BFS}(v), if \( v \in V \), returns a sequence of vertices reachable from \( v \) arranged in a BFS order as defined before. If \( v \notin V \lor \nexists v' \in V \) s.t. \( (v, v'|\cdot) \in E \), it returns \text{NULL}.

8. An \textsc{SSSP}(v), if \( v \in V \), returns a set \( S(v) = \{ d(v_i) \}_{v_i \in V} \), where \( d(v_i) \) is the summation of the weights of the edge on the shortest-path between \( v \) and \( v_i \). If \( v_i \not\leftrightarrow v \), and \( d(v_i) = \infty \), \( v_i \not\leftrightarrow v \). Note that \( d(v) = 0 \). There can be multiple paths between \( v \) and \( v_i \) with the same sum of edge-weights. If \( v \notin V \), it returns \text{NULL}.

9. A \textsc{BC}(v) returns the betweenness centrality of \( v \) as defined before, if \( v \in V \). It returns \text{NULL} if \( v \notin V \).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figures.png}
\caption{\( \text{a) and b) Data structure components, c) A sample directed graph, d) Our implementation of (c) as a composition of a lock-free hash-table and lock-free BSTs.} \)
\end{figure}

\textbf{Design Requirements}

Firstly, because we intend to implement efficient dynamic modifications in an unbounded graph, we choose its adjacency list representation. An adjacency list of a graph, essentially, translates to a composition of dictionaries: one \textit{vertex-list} and as many \textit{edge-lists} as the number of vertices. The members of the vertex-list correspond to \( v \in V \), wherein each of them has an associated edge-list corresponding to \( E_v \). Implementing the ADT requires the vertex-list and edge-lists offer membership queries along with addition/removal of keys. Significantly, the addition/removal need to be non-blocking.

Next, in essence, the operations BFS, SSSP, and BC, also termed as \textit{queries}, are special partial snapshots of the data structure. Naturally, they scan (almost) the entire graph. This requires supporting the scan while keeping track of the visited nodes. This is more important for traversals to happen non-recursively.

\textbf{Data Structure Components}

Keeping the above requirements in view, we build the data structure based on a composition of a lock-free hash-table implementing the vertex-list, and lock-free BSTs implementing the edge-lists. On a skeleton of this composition, we include the design components for efficient traversals and (partial) snapshots. This is a more efficient design\footnote{We limit our discussion to positive edge-weights only.}.
as compared to Chatterjee et al.’s approach \[12\] where the component dictionaries are implemented using lock-free linked-lists only.

More specifically, the nodes of the vertex-list are instances of the class VNode, see Figure 1(a). A VNode contains the key of the corresponding vertex along with a pointer to a BST implementing its edge-list. The most important member of a VNode is a pointer to an instance of the class OpItem, which serves the purpose of anchoring the traversals as described above.

The OpItem class, see Figure 1(b), encapsulates an array VisA of the size equal to the number of threads in the system, a counter ecnt and other algorithm specific indicators, which we describe in Section 4 while specifying the queries. An element of VisA simply keeps a count of the number of times the node is visited by a query performed by the corresponding thread. The counter ecnt is incremented every time an outgoing edge is added or removed at the vertex. This serves an important purpose of notifying a thread if the same edge is removed and added since the last visit.

The class ENode, see Figure 1(b), structures the nodes of an edge-list. It encapsulates a key, the edge-weight, the left- and right-child pointers and a pointer to the associated VNode where the edge terminates; the key in the ENode is that of the VNode; thus each ENode delegates a directed edge.

The VNodes are bagged in a linked-list being referred to by a pointer from the buckets, see Figure 1(a). A resizable hash-table is constructed of the arrays of these buckets, wherein arrays are linked in the form of a linked-list of HNodes.

At the bare-bone level, our dynamically resizable lock-free hash-table derives from Liu et al. \[37\], whereas the BSTs, implementing the edge-lists, are based on lock-free internal BST of Howley et al. \[30\]. We introduce the OpItem fields in hash-table nodes. To facilitate non-recursive traversal in the lock-free BST, we use stacks. As we will explain later, aligning the operations of the hash-table to the state of OpItem therein brings in nontrivial challenges.

The last but a significant component of our design is the class VisA, see Figure 1(b). It encapsulates the information which we use to validate a scan of the graph to output a consistent specialized partial snapshot. More specifically, it packs the pointers to VNodes visited during a scan along with two pointers nxt and p to keep track of the order of their visit. The field ecnt records the ecnt counter of the corresponding visited VNode, which enables checking if the visited VNode has had any addition or removal of an edge since the last visit.

Non-blocking Data Structure Construction

Having these components in place, we construct a non-blocking graph data structure in a modular fashion. Refer to Figure 1(d) depicting a partial implementation of a small directed graph shown in Figure 1(c). The ENodes, shown as circles in Figure 1(d), with their children and parent pointers make lock-free internal BSTs corresponding to the edge-lists. For simplicity we have only shown the outgoing edges of vertex 5 in Figure 1(d) while the edges of other vertices are represented by small triangles. Thus, whenever a vertex has outgoing edges, the corresponding VNode, shown as small rectangles therein, has a non-null pointer pointing to the root of a BST of ENodes. The VNodes themselves make sorted lock-free linked-lists connected to the buckets of a hash-table. The buckets are cells of a bucket-array implementing the lock-free hash-table. When required, we add/remove bucket-arrays for an unbounded resizable dynamic design. The lock-free VNode-lists have two sentinel VNodes: vh and vt initialized with keys -\(\infty\) and \(\infty\), respectively.

We adopt the well-known technique of pointer marking – using a single-word CAS– via bit-stealing \[30,37\] to perform lazy non-blocking removal of nodes. More pointedly, on a common x86-64 architecture, memory has a 64-bit boundary and the last three least significant bits are unused; this allows us to use the last one significant bit of a pointer to indicate first a logical removal of a node and thereafter cleaning it from the data structure. Specifically, an HNode, a VNode, and an ENode is logically removed by marking its pred, vnxt, and el1 pointer, respectively. We call a node alive which is not logically removed.

Data Structure Invariants

To prove the correctness of the presented algorithm, we fix the invariants corresponding to a consistent state of the composite data structure:

a) each edge-list maintains a BST order based on the ENode’s key e, and alive ENodes are reachable from enxt of the corresponding VNode,

b) a VNode that holds a pointer to a BST containing any alive ENodes it is itself alive,
c) each alive VNode is reachable from vh and vertex-lists connected to buckets are sorted based on the VNode’s keys v, and

d) an HNode which contains a bucket holding a pointer to an alive VNode is itself alive and an alive HNode is always connected to the linked-list of HNodes.

Correctness and Progress Guarantee

To prove linearizability [28], we describe the execution generated by the data structure as a collection of method invocation and response events. We assign an atomic step between the invocation and response as the linearization point (LP) of a method call (operation). Ordering the operations by their LPs provide a sequential history of the execution. We prove the correctness of the data structure by assigning a sequential history to an arbitrary execution which is valid i.e. it maintains the invariants. Furthermore, we argue that the data structure is non-blocking by showing that the queries would return in finite number of steps if no update operation happens and in an arbitrary execution at least one update operation returns in finite number of steps.

The progress properties specify when a thread invoking operations on the shared memory objects completes in the presence of other concurrent threads. In this context, we provide the graph implementation with methods that satisfy wait-freedom, based on the definitions in Herlihy and Shavit [27]. A method of a concurrent data-structure is wait-free if it completes in finite number of steps. A data structure implementation is wait-free if all its methods are wait-free. This ensures per-thread progress and is the most reliable non-blocking progress guarantee in a concurrent system. A data structure is lock-free if its methods get invoked by multiple concurrent threads, then one of them will complete in finite number of steps.

3 PANIGRAMAH Framework

In this section, we describe a non-blocking algorithm that implements the ADT $\mathcal{A}$. The operations $\mathcal{M} := \{\text{PutV}, \text{RemV}, \text{GetV}, \text{Pute}, \text{RemE}, \text{GetE}\} \subset \mathcal{A}$ use the interface of the hash-table and BST with interesting
non-trivial adaptation to our purpose. In the permitted space we describe the execution, correctness and progress property of the operations $\mathcal{O} := \{\text{BFS, SSSP, BC}\} \subset \mathcal{A}$. To de-clutter the presentation, we encapsulate the three queries in a unified framework. The framework comes with an interface operation $O$. $O$ is specialized to the requirements of the three queries. The functionality of $O$ and its specializations are presented in pseudo-code in Figures 2, 4 and 5.

**Pseudo-code convention:** We use $p.x$ to denote the member field $x$ of a class object pointer $p$. To indicate multiple return objects from an operation we use $(x_1, x_2, \ldots, x_n)$. To represent pointer-marking, we define three procedures: (a) $\text{isMRK}(p)$ returns $\text{true}$ if the last significant bit of the pointer $p$ is set to 1, else, it returns $\text{false}$, (b) $\text{MRK}(p)$ sets last significant bit of $p$ to 1, and (c) $\text{UNMRK}(p)$ sets the same to 0. An invocation of $\text{CVNODE}(v)$, $\text{CENODE}(e)$ and $\text{CTNODE}(v)$, creates a new $\text{VNode}$ with key $v$, a new $\text{ENode}$ with key $e$ and a new $\text{SNode}$ with a $\text{VNode}$ $v(v)$ respectively. For a newly created $\text{VNode}$, $\text{ENode}$ and $\text{SNode}$ the pointer fields are initialized with NULL value.

The execution pipeline of $O$ is presented at lines 1 to 12 in Fig. 2. $O$ intakes a query vertex $v$. It starts with checking if $v$ is alive at Line 40. In the case $v$ was not alive, it returns NULL. For this execution case, which results in $O$ returning NULL, the LP is at the atomic step (a) where $O$ is invoked in case $v$ was not in the data structre at that point, and (b) where $v$ was logically removed using a CAS in case it was alive at the invocation of $O$.

Now, if the query vertex $v$ is alive, it proceeds to perform the method $\text{SCAN}$, Line 13 to 23. $\text{SCAN}$, essentially, repetitively, performs (specialized partial) snapshot collection of the data structure along with comparing every consecutive pair of scans, stopping when a consecutive pair of collected snapshots are found identical. Snapshot collection is structured in the method $\text{TREECOLLECT}$, Line 24 to 81 whereas, comparison of collected snapshots in performed by the method $\text{CMPTREE}$, Line 24 to 39.

Method $\text{TREECOLLECT}$ performs a BFS traversal in the data structure to collect pointers to the traversed $\text{VNodes}$, thereby forming a tree. A cell of $\text{VisA}$ corresponding to thread-id is marked on visiting it; notice that it is adaptation of the well-known use of node-dirty-bit for BFS [15]. The traversal over $\text{VNodes}$ is facilitated by a queue: Line 47 whereas, exploring the outgoing edges at each $\text{VNode}$, equivalently, traversing over the BST corresponding to its edge-list uses a stack: Line 57. The snapshot collection for the queries BFS and BC are identical. For SSSP, where edge-weights are to be considered, the snapshot collection is optimized in each consecutive scan based on the last collection; we detail them in the next section. At the core, the collected snapshot is a list of $\text{SNodes}$, where each $\text{SNode}$ contains a pointer to a $\text{VNode}$, pointers to the next and previous $\text{VNodes}$ and the value of the $\text{ecnt}$ field of the $\text{OpItem}$ of the $\text{VNode}$.

Method $\text{CMPTREE}$ essentially compares two snapshots in three aspects: whether the collected $\text{SNodes}$ contain (a) pointers to the same $\text{VNodes}$ (b) have the same $\text{SNode}$ being pointed by previous and next, and (c) have the same $\text{ecnt}$. The three checks ensure that a consistent snapshot is the one which has its collection lifetime not concurrent to (a) a vertex either added to or removed from it, (b) a path change by way of addition or removal of an edge, and, (c) an edge removed and then added back to the same position, respectively. Thus, at the completion of these checks, if two consecutive snapshots match, it is guaranteed to be unchanged during the time of the last two $\text{TREECOLLECT}$ operations. Clearly, we can put a linearization point just after the atomic step where the last check is done: Line 30 or 35 where it returns $\text{CMPTREE}$.

Now, it is clear that any $Q \in \mathcal{O}$ does not engage in helping any other operation. Furthermore, an $M \in \mathcal{M}$ does not help a $Q \in \mathcal{O}$. Thus, given an execution $E$ as a collection of arbitrary $O \in \mathcal{A}$, by the fact that the data-structures hash-table and BST are lock-free, and whenever no $\text{PUTV}$, $\text{PUTE}$, $\text{REMV}$, and $\text{REME}$ happen, a $Q \in \mathcal{O}$ returns, we infer that the presented algorithm is non-blocking.

## 4 The Graph Query Operations

Having described the execution pipeline of $O$, here we specialize it to the queries breadth-first search, single-source shortest-path, and betweenness centrality. As mentioned before, the $\text{OpItem}$ class is inducted with query specific fields for $O \in \mathcal{O} := \{\text{BFS, SSSP, BC}\}$, which sits at the core of its implementation.

### 4.1 Breadth-first search

Given an unweighted graph $G = (V, E)$, BFS($v$) starts from the source vertex $v$ to traverse the vertices $u \in V$, which are reachable from $v$, in BFS order (see section 1) to them.

Implementation of BFS $\in \mathcal{O}$ is shown in Figure 3. The $\text{OpItem}$ class has only two fields: $\text{ecnt}$ and $\text{VisA}$, whose functionality we have already described earlier. A BFS($v$) operation, at Lines 82 to 92 begins by validating the presence of the $v$ in the hash table and is unmarked. If the validation fails, it returns NULL. Once the validation...
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4.2 Single Source Shortest Paths

SSSP \(v \in V\) builds on BFS and the implementation depicted in Figure 4. The OpItem class here contains three fields: 

- ecnt, VisA and an array DistA of size equal to the number of threads. For a SSSP\((v)\), the distance to a visited vertex \(u\) from \(v\) is stored at a cell of \(u\).ol.DistA corresponding the traversing thread. We assume that the edge weights can be a real number: \(w \in \mathbb{R}\), thus, the graph \(G\) can possibly contain negative edge cycles [15].

Essentially, our methodology is an adaptation of the Bellman-Ford algorithm [13] to our framework. Similar to BFS, a SSSP\((v)\) operation, Lines 143 to 154, starts by validating the presence of the \(v\) in the hash table and it is unmarked. After successful validation, it invokes the method SPScan, Lines 155 to 170, works similar to SCAN described before: repeatedly collect SP-trees using the method SPTCLt, return the last one on a match of a consecutive pair of collects, see Line 160. A SP-tree is a BFS-tree with some modifications described below. However, as we consider the presence of negative edge cycles, it can return NULL if a consecutive pair of SPTCLt discover negative edge cycle, see Line 163. The method SPTCLt extends TREECOLLECT as the following:

VERTEX DISTANCE TRACKING: Every time a new vertex is collected, check its distance from source and update the current recorded distance if required using method RLXD, see Lines 205 to 212 and Line 241. Formally, A
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Figure 4: The SSSP query.
We formally define the BC as the following. Given a directed graph \( G \) through an intermediate vertex vertex node in a graph [21]. Here we consider unweighted graphs \( G \) We defined Betweenness centrality (BC) in Section 1. BC is an index measure based on the relative significance of a

4.3 Betweenness Centrality

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value, then we discard the older tree and restart the SPTC

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cells correspond to the threads. In essence, we adapt the algorithm of Brandes [7] to our setting using these arrays to

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algorithm, it computes the distances and shortest path counts from

In our setting, the operation BC ∈ SPC(see Line 160), then SPC returns ttrue and spt to SSSP operation. However, if two consecutive SPTC returns same status value false and both the trees are equal (see Line 163) then SPC returns false to SSSP operation, which says graph has neg-cycle. If returns of two consecutive SPTC do not match in the SPC method or different status value, then we discard the older tree and restart the SPTC.

4.3 Betweenness Centrality

We defined Betweenness centrality (BC) in Section 1. BC is an index measure based on the relative significance of a vertex node in a graph [21]. Here we consider unweighted graphs \( G = (V, E) \): \( u_e = 1 \forall v \in E \).

We formally define the BC as the following. Given a directed graph \( G = (V, E) \) and some \( s, t \in V \), let \( \sigma(s, t) \) be the number of shortest paths between vertex \( s \) and \( t \) and \( \sigma(s, t|v) \) be the number of shortest paths between \( s \) and \( t \) that pass through an intermediate vertex \( v \). Then the pair-dependency of \( s, t \) on \( v \) is defined as \( \delta(s, t|v) = \frac{\sigma(s, t|v)}{\sigma(s, t)} \), if \( s = t \), then \( \delta(s, t|v) = 1 \) and if \( v \in s, t \), then \( \delta(s, t|v) = 0 \). \( \sigma(s, t) \) can be compute recursively as \( \sigma(s, t) = \sum_{u \in P dest(t)} \sigma(s, u) \), where \( P dest(t) = \{ u : (u, t) \in E, d(s, t) = d(s, u) + 1 \} \) (predecessors of \( t \) on shortest path from \( s \)), and \( d(s, u) \) is the distance between vertex node \( s \) and \( u \). With that, the BC of \( v \) is defined as \( C_B(v) = \sum_{s,t \in V} \delta(s, t|v) \). To compute \( \delta(s, t|v) \) one can run the BFS algorithm with each vertex node as source \( s \) and then sum the pair-dependencies for each \( v \in V \).

U. Brandes [7] proposed an algorithm for unweighted graphs by defining an one-sided dependencies equation as \( \delta(s|v) = \sum_{(v, w) \in E, w \in \delta(s, v)} \frac{\sigma(s, v)}{\sigma(v, w)} \times (1 + \delta(s|w)) \), and the BC be \( C_B(v) = \sum_{s \in V} \delta(s|v) \). Brande’s algorithm is as follows: it iterates over the vertices \( v \in V \) and then computes \( \delta(s|v) \) for every \( v \in V \) in two phases, (1) using the BFS algorithm, it computes the distances and shortest path counts from \( s \), and it also keeps tracking of all the vertices onto a stack as they are visited, (2) it visits all the vertices by popping them off from the stack in reverse order and aggregate the dependencies according to the one-sided dependencies equation.

In our setting, the operation BC ∈ SPC (see Line 272) builds on BFS. In this case, the OpItem class, in addition to ecnt and VisA, contains five extra array fields: DistA, sigmaA, deltaA, PredlistA and CBA, whose cells correspond to the threads. In essence, we adapt the algorithm of Brandes [7] to our setting using these arrays to allow concurrent threads compute the measure of BC anchored at their corresponding array-cells.

The termination criterion of the method BCScan is similar to Scan: matching a consecutive pair of specialized partial snapshots, called BC-tree. Besides that, the snapshot method BCTC extends TreeCollect as the following:

DISTANCE tracking: On visiting a vertex, set its distance from \( v \) at the corresponding DistA cell, Line 337.

SHORTEST path counting: If a visited vertex is on the shortest path, adjust the number of shortest paths recorded at the corresponding sigmaA cell, Line 344.

PREDECESSOR list maintenance: For each of the visited vertices record and update the list of predecessors from \( v \) by the shortest path, Line 347.

\[RLXD(tid, u, v, w(u, v))\], at Line 205 to 212 method works similar as [15], it checks for each edge \( (u, v) \) if \( v.oDist[tid] > u.oDist[tid] + w(u, v) \); if so, it sets the \( v.oDist[tid] \) to \( u.oDist[tid] + w(u, v) \).

PROCESS BFS-TREE: The method SPTC (in Lines 213 to 260) collects SP-tree (of \( SNode \)). First, it initializes the shortest paths DistA (Distance Array) to all vertices to \( \infty \), by invoking the method INIT (Line 216) except for the source vertex \( v \) which is initialized to 0 (Line 219). Then it traverse all unmarked reachable VNodes through unmarked ENodes, and also a RLXD (relaxed) (Line 241) method repeatedly called. In the process of traversal it keeps adding all first time visited VNodes in the spt, at Line 245, and if it encounters already visited VNodes which are further relaxed, then it updates the DistA and its parent pointer, by invoking an UPDATE_SPTREE method at 251. At each of the SNode update to make a shortest path from source to its vertex.

An UPDATE_SPTREE (tid, head, par, spn) (Lines 261 to 271), starts from the head node and iterate all nodes in the SP-tree until it reaches the corresponding SNode, and then it updates the shortest path distance, at Line 265 and its parent reference pointer, at Line 266.

NEGATIVE CYCLE CHECKING: After exploring all reachable VNodes and Before returning the collected tree, the SPTC invokes CHECK_NEG CYCLE (Check for Negative Cycle) method (at Line 259). A CHECK_NEG CYCLE (tid, spt), at Lines 171 to 192 starts processing all SNodes in the spt. It uses stack S (Line 175), to process all ENodes a RLXD (relaxed) (Line 185) method called to test whether any further relaxation is possible for any ENodes, if so, it returns false, otherwise, returns true.

At the end the SPTC method terminates by returning spt and a boolean status file(presence of neg-cycle) to the SPScan method. After each two consecutive SPTC, SPC and SPTC is invoked to compare the two trees are equal or not. If two consecutive SPTC returns same status value ttrue and both the trees are equal (see Line 160), then SPC returns ttrue and spt to SSSP operation. However, if two consecutive SPTC returns same status value false and both the trees are equal (see Line 163) then SPC returns false to SSSP operation, which says graph has neg-cycle. If returns of two consecutive SPTC do not match in the SPC or different status value, then we discard the older tree and restart the SPTC.
Dynamic Graph Operations: A Consistent Non-blocking Approach

5 Experiments

Evaluating the performance of a new concurrent data structure vis-a-vis the existing ones implementing the same ADT, in general, limits to comparing the throughput or latency of a combined execution of all of the proposed ADT operations. Moreover, in general, they use non-standard synthetic micro-benchmarks. In contrast, the high performance static graph-query libraries are evaluated for the average latency of a parallelized execution of a single graph operation. Furthermore, the datasets used in the evaluation of graph-query libraries are now standard in the literature. As we aimed to present this work as a library of graph operations in a dynamic setting, keeping the aforementioned points in view, we compared the experimental performance of our non-blocking graph algorithms against a well-known graph-query library Ligra [42]. The dynamic updates for Ligra are simulated by intermittent sequential addition and removal in the dataset. We use a standard synthetic graph dataset – R-MAT graphs \[10\] – with power-law distribution of degrees.

As we discussed before, we needed the repeated snapshot collection and matching methodology to guarantee linearizability of graph queries. However, if the consistency requirement is not as strong as linearizability, we can still have non-blocking progress even if we collect the snapshots once. At the cost of theoretical consistency, we gain a lot in terms of throughput, which is the primary demand of the analytics applications, who often go for approximate queries.
Thus, we have the following execution cases: (1) **PG-Cn**: Linearizable PANIGRAHAM, (2) **PG-Icn**: Inconsistent PANIGRAHAM, and (3) **Ligra**: execution on Ligra [42].

![Figure 6: Latency of the executions containing OP: BFS. The surfaces indicate the total time for an end-to-end run of $10^4$ operations uniformly distributed by a distribution mentioned below the plots. x-axis refers to the number of threads, whereas, y-axis refers to the ratio of number of vertices and edges for a graph instance.](image)

**R-MAT Graph Generation:** R-MAT graph was described by Chakrabarti, et al. [10]. It is a recursive matrix (R-MAT) model that can quickly generate realistic graphs using very few input parameters. It utilizes power-law/DGX [6] degree distributions, which match the characteristics of real-world graphs (degree exponents, diameters, etc.). The R-MAT model generates the graphs in $O(\log^2 E \log N)$ time (Explanations can be seen in [10]). R-MAT model uses adjacency matrix $A$ of a graph of $N$ vertices and an $N \times N$ matrix, with entry $A(i,j) = 1$ if the edge $(i,j)$ exists, and 0 otherwise. It recursively subdivides the adjacency matrix into four equal-sized partitions and distributes edges within these partitions with unequal probabilities $a$, $b$, $c$, and $d$ following a certain distribution. For our case, the default values are $a = 0.5$, $b = 0.1$, $c = 0.1$ and $d = 0.3$, such that $a + b + c + d = 1$, and these values can be changed based on the required application.

To generate a graph, the input parameters required at the running time are the number of vertices and the output file name. The default number of edges is set to 10 times the number of vertices, and that is changed depending on the required application by setting the flag followed by the number of edges. Table 1 shows the different graphs used in the paper for running the BFS, SSSP, and BC algorithms.

To generate a weighted graph, we added random integer weights in the range $[1, \ldots, \log_2(N)]$ to an unweighted graph in adjacency graph format.

**Experimental Setup:** We conducted our experiments on a system with Intel(R) Xeon(R) E5-2690 v4 CPU packing 56 cores running at 2.60GHz. There are 2 logical threads for each core and each having private cache memory L1-64K and L2-256K. The L3-35840K cache is shared across the cores. The system has 32GB of RAM and 1TB of hard disk. It runs on a 64-bit Linux operating system. All the implementations are written in C++ without garbage collection. We used Posix threads for multi-threaded implementation.

**Running Strategy:** The experiments start with a R-MAT graph instance populating the data structure. At the execution initialization, we spawn a fixed set of threads (7, 14, 28 and 56) and each thread randomly performs a set of operations chosen by a certain random workload distribution. The metric for evaluation is the total time taken to complete the set of operations, after a fixed warm-up: $\delta$% of the total number of operations. Each experiment runs for $\delta$ iterations and then we take the median of all iterations.
Figure 7: Latency of the executions containing OP: SSSP.

Figure 8: Latency of the executions containing OP: BC.
Table 1: Initial RMAT graphs used to load before start running BFS, SSSP, and BC algorithms and then perform $10^4$ operations with different workload distributions.

| Vertices | Edges |
|----------|-------|
| 1024     | 10000 |
| 2048     | 20000 |
| 4096     | 30000, 40000 |
| 8192     | 50000, 60000, 70000, 80000 |
| 16384    | 90000, 100000, ..., 160000 |
| 32768    | 170000, 180000, ..., 320000 |
| 65536    | 330000, 340000, ..., 650000 |
| 131072   | 660000, 670000, ..., 1000000 |

Workload Distribution: To evaluate the performance over a number of micro-benchmarks, we used a range of distributions over an ordered (family of) set of operations: \{Update:=\{PUTV, REMV, PUTE, REME\}, Search:=\{GETV, GETE\}, OP\}. In each case, first we load a RMAT graph instance, perform warm-up operations, followed by an end-to-end run of $10^4$ operations in total, assigned in a uniform random order to the concurrent threads. In the plots a label, say, 40/10/50 refers to a distribution \{\{10\%, 10\%, 10\%, 10\%\}, \{5\%, 5\%\}, 50\%\} (i.e. 40\% first four operations distributed equally, 10\% next two distributed equally, and 50\% OP queries) of the aforementioned operations in their order.
Observations and Discussion

Firstly, the surface plots of latency, in Figures 6, 7, and 8, show that for any combination of a graph size and parallelism, as in the number of threads, the inconsistent non-blocking implementation outperforms highly parallel Ligra by an order of magnitude. It depicts the advantage of concurrency in the dynamic settings. The surfaces also show that for BFS, where the non-blocking methods need to maintain only a small amount of information in vertices, the linearizable non-blocking concurrent executions outperform Ligra up to a reasonable graph size. In case of SSSP and BC, where a lot more info is needed for linearizable snapshots, Ligra tends to work better than consistent non-blocking implementation, still under-performing the one that relaxes consistency.

Secondly, for a fixed number of threads, in this instance 56, as shown in Figures 9, 10, and 11, for the smaller graph sizes, non-blocking implementations handsomely outperform Ligra, however, as graph size increases, Ligra starts getting advantage of the parallel implementation. Note that, the non-blocking implementations do not have even an inline parallelization.

To explore the effect of concurrent dynamic updates on linearizable query performance of PG-Cn, we plot the average number of snapshot collection method calls in Figure 12 (a), (b), and (c), and the average number of interrupting updates during the lifetime of a query in Figure 13 (a), (b), and (c). We observe that the average number of snapshot collection grows in a linear ratio to the number of threads, which is on the expected lines. However, comparing the average number of snapshot collections and the corresponding average number of interrupting updates for the respective queries, we observe that the former does not grow in the same ratio as the latter when the ratio of updates in a distribution increases. This shows that a smaller number of queries in a distribution results in a lesser amount of interaction between a query and the concurrent updates. This observation infers that consistent non-blocking implementation of dynamic graph queries is a much better option than sequential highly parallel implementation, such as Ligra, where by design a query has to be stopped for an update.
The experimental observations infer that a design which can take advantage of both concurrency and parallelization will substantially benefit the dynamic graph queries. We plan to work on this in future.
Figure 13: Subfigs. (a), (b), and (c) show the average number of interrupting update operations during the lifetime of the respective queries.

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Appendix

A The Non-blocking Graph Algorithm

```plaintext
1: Operation PUTV(v)
  2: return HASHADD(v);
  3: end Operation

4: Operation REMV(v)
  5: return HASHREM(v);
  6: end Operation

7: Operation GETV(v)
  8: ⟨st, e⟩ ← HASHCON(v);
  9: if (st = true) then
 10: return (true, e);
 11: else
 12: return (false, NULL);
 13: end if
 14: end Operation

15: Operation GETE(v1, v2)
  16: ⟨u, v, st⟩ ← CONVPLUS(v1, v2);
  17: if (st = false) then
  18: return (false, ∞);
  19: end if
 20: ⟨st, e⟩ ← BSTCON(v2, v.enxt);
 21: if (st = FOUND ∧ HASHCON(v1) ∧ HashCON(v2)) then
 22: z ← e.w;
 23: return (true, z);
 24: else return (false, ∞);
 25: end if
 26: end Operation

27: Method CONVPLUS(v1, v2)
  28: ⟨stl,u⟩ ← HASHCON(v1); //modified GETV, returns status along with ref
  29: ⟨st2,v⟩ ← HASHCON(v2);
 30: if (st1 = true ∧ st2 = true) then
 31: return (u, v, true);
 32: else return (u, v, false);
 33: end if
 34: end Method

35: Operation PUTE(v1, v2 [w])
  36: ⟨u, v, st⟩ ← CONVPLUS(v1, v2);
  37: if (st = false) then return (false, ∞);
  38: end if
 39: while (true) do
 40: if (isMKD(u) ∨ ISMKD(v)) then
 41: return (false, ∞);
 42: end if
 43: if (st = FOUND) then
 44: return (false, ∞);
 45: end if
 46: if (st = FOUND) then
 47: if (op = WRITE) then return (false, w);
 48: else
 49: z ← ce.w;
 50: CAS(ce.w, z, w);
 51: u.ecnt.FastAdd(1);
 52: return (true, z);
 53: end if
 54: end if
 55: end if
 56: end if
 57: end while
 58: end Operation
```

Figure 14: Pseudocodes of PUTV, REMV, GETV, PUTE, REME, GETE and CONVPLUS

In this section, we present a detailed implementation of our non-blocking directed graph algorithm. The non-blocking graph composits on the basic structures of the dynamic non-blocking hash table [37] and non-blocking internal binary search tree [30]. For a self-contained reading, we present the algorithms of non-blocking hash table and BST. Because it derives and builds on the earlier works [37] and [30], many keywords in our presentation are identical to theirs. One key difference between our non-blocking BST design from [30] is that we maintain a mutable edge-weight in each BST node, thereby not only the implementation requires extra steps but also we need to discuss extra cases in order to argue the correctness of our design. Furthermore, we also perform non-recursive traversals in the BST for snapshot collections, which were already discussed as part of the graph queries. The pseudo-codes pertaining to the non-blocking hash-table are presented in Figure 15 whereas those for the non-blocking BST are presented in Figures 16 and 17.
A.1 Structures

The declarations of the object structures that we use to build the data structure are listed in Figure 15 and 16. The structures FSet, FSetOp, and HNode are used to build the vertex-list, whereas Node, RelocateOp, and ChildCASOp are the component-objects of the edge-list. The structure FSet, a freezable set of VNodes that serves as a building block of the non-blocking hash table. An FSet object builds a VNode set with PUTV, REMV and GETV operations, and in addition, provides a FREEZE method that makes the object immutable. The changes of an FSet object can be either addition or removal of a VNode. For simplicity, we encode PUTV and REMV operation as FSetOp objects. The FSetOp has a state optype (PUTV or REMV), the key value, done a boolean field that shows the operation was applied or not, and a boolean field that holds the return value.

The vertex-list is a dynamically resizable non-blocking hash table constructed with the instances of VNodes, and it is a linked-list of HNode (Hash Table Node). The HNode consists of an array of buckets of FSet objects, the size field stores the array length and the predecessor HNode is pointed to by the pred pointer. The head of the HNode is pointed to by a shared head pointer.

For clarity, we assume that a RESIZE method grows (doubles) or shrinks (halves) the size of the HNode which amount to modifying the length of the bucket array. The hash function uses modular arithmetic for indexing in the hash table, e.g. index = key mod size.

Based on the boolean parameter taken by RESIZE method, it decides the hash table either to grow or shrink. The INITBKT method ensures all VNodes are physically present in the buckets. It relocates the VNodes to the hash table which are in the predecessor’s list.

The $i^{th}$ bucket of a given HNode $h$ is initialized by INITBKT method, by splitting or merging the buckets of $h$’s predecessor HNode $s$, if $s$ exists. The sizes of $h$ and $s$ are compared and then this method decides whether $h$ is shrinking or expanding with reference to $s$. Then it freezes the respective bucket(s) of $s$ before copying the VNodes. If $h$ halves the size of $s$, then $i^{th}$ and $(i + h$.size$)^{th}$ buckets of $s$ are merged together to form the $i^{th}$ bucket of $h$. Otherwise, $h$ doubles the size of $s$, then approximately half of the VNodes in the $(i \mod h$.size$)^{th}$ bucket of $s$ relocate to the $i^{th}$ bucket of $h$. To avoid any races with the other helping threads while splitting or merging of buckets a CAS is used (Line 182).

The ENode structure is similar to that of a lock-free BST [30] with an additional edge weight $v$ and a pointer field pV which points to the corresponding VNode. This helps direct access to its VNode while doing a BFS traversal and also helps in deletion of the incoming edges. The operation op field stores if any changes are being made, which affects the ENode. To avoid the overhead of another field in the node structure, we use bit-manipulation: last significant bits of a pointer $p$, which are unused because of the memory-alignment of the shared-memory system, are used to store information about the state of the pointer shared by concurrent threads and executing an operation that would potentially update the pointee of the pointer. More specifically, in case of an x86-64 bit architecture, memory has a 64-bit boundary and the last three least significant bits are unused. So, we use the last two significant bits, which are enough for our purpose, of the pointer to store auxiliary data. We define four different methods to change an ENode pointer: ISNULL($p$) returns true if the last two significant bits of $p$ make 00, which indicates no ongoing operation, otherwise, it returns false; ISMREQD($p$) returns true if the last two significant bits of $p$ are set to 01, else it returns false, which indicates the node is no longer in the tree and it should be physically deleted; ISCHILD($p$) returns true if last two bits of $p$ are set to 10, which indicates one of the child nodes is being modified, else it returns false; ISRELOCATE($p$) returns true if the last two bits of $p$ make 11, which indicates that the ENode is undergoing a node relocation operation.

A ChildCASOp object holds sufficient information for another thread to finish an operation that made changes to one of the child — right or left — pointers of a node. A node’s op field holds a flag indicating an active ChildCASOp operation. Similarly, a RelocateOp object holds sufficient information for another thread to finish an operation that removes the key of a node with both the children and replaces it with the next largest key. To replace the next largest key, we need the pointer to the node whose key is to be removed, the data stored in the node’s value, and a boolean field that holds the return value. We define four different methods to change an ENode: true if last two bits of $p$ make 00, which indicates no ongoing operation, otherwise, it returns false; ISMREQD($p$) returns true if the last two significant bits of $p$ are set to 01, else it returns false, which indicates the node is no longer in the tree and it should be physically deleted; ISCHILD($p$) returns true if last two bits of $p$ are set to 10, which indicates one of the child nodes is being modified, else it returns false; ISRELOCATE($p$) returns true if the last two bits of $p$ make 11, which indicates that the ENode is undergoing a node relocation operation.

A.2 The Vertex Operations

The working of the non-blocking vertex operations PUTV, REMV, and GETV are presented in Figure 14. A PUTV($v$) operation, at Lines [1] to [3], invokes HASHADD($v$) to perform an insertion of a VNode $v$ in the hash table. A REMV($v$) operation at lines [4] to [6], invokes HASHREM($v$) to perform a deletion of VNode $v$ from the hash table. The method APPLY, which tries to modify the corresponding buckets, is called by both HASHADD and HASHREM, see Line 146.
It first creates a new FSetOp object consisting of the modification request, and then constantly tries to apply the request to the respective bucket \( b \), see Lines [185] to [197]. Before applying the changes to the bucket it checks whether \( b \) is NULL; if it is, INITBKT method is invoked to initialize the bucket (Line 191). At the end, the return value is stored in the resp field.

The algorithm and the resizing hash table are orthogonal to each other, so we used heuristic policies to resize the hash table. As a classical heuristic we use a HASHADD operation that checks for the size of the hash table with some threshold value, if it exceeds the threshold the size of the table is doubled. Similarly, a HASHREM checks the threshold value, if it falls below threshold, it shrinks the hash table size to halves.

A GETV() operation, at Lines 7 to 14, invokes HASHCON() to search a VNode \( v \) in the hash table. It starts by searching the given key \( v \) in the bucket \( b \). If \( b \) is NULL, it reads \( v \)'s predecessor (Line 163) \( s \) and then starts searching on it. At this point it could return an incorrect result as HASHCON is concurrently running with resizing of \( s \). So, a double check at Line 164 is required to test whether \( s \) is NULL between Lines 161 and 163. Then, we re-read that bucket of \( t \) (Line 165 or 167), which must be initialized before \( s \) becomes NULL, and then we perform the search in that bucket. If \( b \) is not NULL, then we simply return the presence of the corresponding VNode in the bucket \( b \). Note that, at any point in time there are at most two VNodes: only one when no resizing happens and another to support resizing – halving or doubling – of the hash table.

A3 The Edge Operations

The non-blocking graph edge operations – PUTE, REME, and GETE – are presented in Figure [14]. Before describing these operations, we detail the implementation of FIND method, which is used by them. It is shown in Figure [16]. The method FIND, at Lines 268 to 318, tries to locate the position of the key by traversing down the edge-list of a VNode. It returns the position in \( pe \) and \( ce \), and their corresponding op values in \( peOp \) and \( ceOp \) respectively. The result of the method FIND can be one of the four values: (1) FOUND: if the key is present in the tree, (2) NOTFOUND_L: if the key is not in the tree but might have been placed at the left child of \( ce \) if it was added by some other threads, (3) NOTFOUND_R: similar to NOTFOUND_L but for the right child of \( ce \), and (4) ABORT: if the search in a subtree is unable to return a usable result.

A PUTE(\( v_1, v_2 | w \)) operation, at Lines 55 to 68, begins by validating the presence of \( v_1 \) and \( v_2 \) in the vertex-list. If the validation fails, it returns \( \langle \text{false}, \infty \rangle \) (Line 57). Once the validation succeeds, PUTE operation invokes FIND method in the edge-list of the vertex with key \( v_1 \) to locate the position of the key \( v_2 \). The position is returned in the variables \( pe \) and \( ce \), and their corresponding op values are stored in the \( peOp \) and \( ceOp \) respectively. On that, PUTE checks whether an ENode with the key \( v_2 \) is present. If it is present containing the same edge weight value \( w \), it implies that an edge with the exact same weight is already present, therefore PUTE returns \( \langle \text{false}, \infty \rangle \) (Line 48). However, if it is present with a different edge weight, say \( z \), PUTE updates \( ce \)'s old weight \( z \) to the new weight \( w \) and returns \( \langle \text{true}, z \rangle \) (Line 50). We update the edge-weight using a CAS to ensure the correct return in case there were multiple concurrent PUTE operations trying to update the same edge. Notice that, here we are not freezing the ENode in anyway while updating its weight. The linearizability is still ensured, which we discuss in the next section.

If the key \( v_2 \) is not present in the tree, a new ENode and a Ch1dCASOp object are created. Then using CAS the object is inserted logically into \( ce \)'s op field (Line 61). If the CAS succeeds, it implies that \( ce \)'s op field hadn’t been modified since the first read. Which in turn indicates that all other fields of \( ce \) were also not changed by any other concurrent thread. Hence, the CAS on one of the \( ce \)'s child pointer should not fail. Thereafter, using a call to HELPCHILDCAS method the new ENode ne is physically added to the tree. This can be done by any thread that sees the ongoing operation in \( ce \)'s op field.

A REME(\( v_1, v_2 \)) operation, at Lines 69 to 105, similarly begins by validating the presence of \( v_1 \) and \( v_2 \) in the vertex-list. If the validation fails, it returns \( \langle \text{false}, \infty \rangle \). Once the validation succeeds, it invokes FIND method in the edge-list of the vertex having key \( v_1 \) to locate the position of the key \( v_2 \). If the key is not present it returns \( \langle \text{false}, \infty \rangle \). If the key is present, one of the two paths is followed. The first path at Lines 83 to 87 is followed if the node has less than two children. In case the node has both its children present a second path at Lines 90 to 99 is followed. The first path is relatively simpler to handle, as single CAS instruction is used to mark the node from the state NONE to MARKED at this point the node is considered as logically deleted from the tree. After a successful CAS, a HELPMARKED method is invoked to perform the physical deletion. It uses a Ch1dCASOp to replace pe's child pointer to ce's with either a pointer to ce's only child pointer, or a NULL pointer if ce is a leaf node.

The second path is more difficult to handle, as the node has both the children. Firstly, FIND method only locates the children but an extra FIND (Line 90) method is invoked to locate the node with the next largest key. If the FIND method returns ABORT, which indicates that ce's op field was modified after the first search, so the entire REME operation is restarted. After a successful search, a RelocateOp object replace is created (Line 94) to replace ce's key v2 with...
the node returned. This operation added to replace’s op field safeguards it against a concurrent deletion while the REME operation is running by virtue of the use of a CAS (Line 95). Then HELPRELOCATE method is invoked to insert RelocateOp into the node with v2’s op field. This is done using a CAS (Line 327), after a successful CAS the node is considered as logically removed from the tree. Until the result of the operation is known the initial state is set to ONGOING. If any other thread either sees that the operation is completed by way of performing all the required CAS executions or takes steps to perform those CAS operations itself, it will set the operation state from ONGOING to SUCCESSFUL (Line 329), using a CAS. If it has seen other value, it sets the operation state from ONGOING to FAILED (Line 332). After the successful state change, a CAS is used to update the key to new value and a second CAS is used to delete the ongoing RelocateOp from the same node. Then next part of the HELPRELOCATE method performs cleanup on replace by either marking it if the relocation was successful or clearing its op field if it has failed. If the operation is successful and ce is marked, HELPMARKED method is invoked to excise ce from the tree. At the end REME returns ⟨true, ce, w⟩.

Similar to PUTE and REME, a GETE(v1, v2) operation, at Lines [15] to [20] begins by validating the presence of v1 and v2 in the vertex-list. If the validation fail, it returns ⟨false, ∞⟩. Once the validation succeeds, it invokes FIND method in the edge-list of the vertex with key v1 to locate the position of the key v2. If it finds v2, it checks if both the vertices are not marked and also the ceOp not marked; on ensuring that it returns ⟨true, ce, w⟩, otherwise, it returns ⟨false, ∞⟩.

B Proof of Correctness and Progress Guarantee

We argue that a linearizable [28] implementation maintains the data structure invariant. To prove linearizability, we specify the atomic events corresponding to the linearization points (LP) inside the execution interval of each of the operations.

B.1 Linearizability

Each operation implemented by the data structure are represented by their invocation and return steps. We show that it is possible to assign an atomic step as LP inside the execution interval of each operation. The vertex operations have their LPs along the similar lines as that discussed in [37]. However, the edge operations include updating the weights of ENodes in addition to their addition and removal. Accordingly, we have more execution cases compared to a set implemented by a non-blocking BST as implemented in [30]. The specification of LPs of BFS, SSSP, and BC are closer to that of GETPATH operation of [12].

Theorem 0. The ADT operations implemented by the non-blocking graph algorithm are linearizable.

Proof. Based on the return values of the operations we discuss the LPs.

1. PUTV(v): We have two cases:
   (a) true: The key v was not present earlier, then the LP is at Line [193] the INVOKE method that returns true where it sets op.done to true.
   (b) false: The key v was already present, then the LP is at Line [193] the INVOKE method that returns true where it sets op.done to true.

2. REMV(v): We have two cases:
   (a) true: If the key v was already present, then the LP is at Line [193] REMARKED method that sets false.
   (b) false: If the key v was not present earlier, then the LP is at Line [193]

3. GETV(v): We have two cases:
   (a) true: The LP is at Line [165] or [167] second read of the bucket of t or s, respectively.
   (b) false: The LP is at Line [138] because b must have been made immutable by some FREEZE method that sets b.ok to false.

4. PUTE(v1, v2, w): We have four cases:
   (a) ⟨true, ∞⟩: New edge has been added
        i. No concurrent REMV(v1) or REMV(v2): The LP is the successful CAS execution at the Line 61
        ii. With concurrent REMV(v1) or REMV(v2): The LP is just before the first remove’s LP.
struct FSetNode {
    int set; // Set of integer
    bool ok; // check for the set is mutable or not
}

struct FSet {
    FSetNode node;
}

struct FSetOp {
    int optype; // operation type (ADD or REMOVE)
    int key; // key to insert or delete
    bool resp; // holds the return value
}

struct ILNode {
    FSetNode buckets; // an array(or list) of FSet
    int size; // array(or list) length
    ILNode pred; // pointer that points to the predecessor ILNode
}

106. Method GetResponse(op)
107. return op.resp;
108. end Method

109. Method HasMember(b, k)
110. a ← b.node; // local copy of b
111. return k ∈ a.set;
112. end Method

113. Method Invoke(b, op)
114. a ← b.node; // local copy of b
115. while (a.ok)
116. if (op.optype = ADD) then
117.    resp ← op.key ∈ a.set;
118.    set ← a.set ∪ {op.key};
119. else
120.    if (op.optype = REMOVE) then
121.        resp ← op.key ∈ a.set;
122.        set ← a.set \ {op.key};
123.    end if
124. end if
125. n ← new FSetNode(set, true);
126. if (CAS(b.node, n)) then
127.     op.resp ← resp;
128.     return true;
129. end if
130. a ← b.node;
131. end while
132. return false;
133. end Method

134. Method Freeze(b)
135. a ← b.node; // local copy of b
136. while (a.ok)
137. n ← new FSetNode(a.set, false);
138. if (CAS(b.node, n)) then
139.     break;
140. end if
141. a ← b.node;
142. end while
143. return a.set;
144. end Method

145. Operation HASHADD(key)
146. resp ← APPLY(ADD, key);
147. if (heuristic-policy) then
148.     RESIZE(true);
149. end if
150. return resp;
151. end Operation

152. Operation HASHREM(key)
153. resp ← APPLY(REMOVE, key);
154. if (heuristic-policy) then
155.     Resize(false);
156. end if
157. return resp;
158. end Operation

159. Operation HASHCON(key)
160. t ← Head;
161. b ← t.buckets[key mod t.size];
162. if (b = NULL) then
163.     s ← t.pred;
164.     if (s ≠ NULL) then
165.         b ← s.buckets[key mod s.size];
166. else
167.         b ← t.buckets(key mod t.size);
168. end if
169. end if
170. return HasMember(b, key);
171. end Operation

172. Method Resize(grow)
173. t ← Head;
174. if (t.size > 1 ∨ grow = true) then
175.     for (i = 0 to t.size-1) do
176.         INITBKT(t);
177.     end for
178.     t−pred ← NULL;
179.     size ← grow ? t.size + 2 : t.size/2;
180.     buckets ← new FSet[size];
181.     t ← new ILNode(buckets, size, t);
182.     CAS(t, Head, t, t);
183.     end if
184. end Method

185. Method Apply(optype, key)
186. op ← new FSetOp(optype, key, false, false);
187. while (true) do
188.     t ← Head;
189.     b ← t.buckets[key mod t.size];
190.     if (b = NULL) then
191.         b ← INITBKT(t, key, mod t.size);
192.     end if
193.     if (INVoke(b, op)) then
194.         return GetResponse(op);
195.     end if
196.     end while
197. end Method

198. Method INITBKT(f, key)
199. b ← t.buckets[key];
200.     s ← t.pred;
201.     if (b = NULL ∧ s ≠ NULL) then
202.         if (t.size = s.size) then
203.             m ← s.buckets[mod s.size];
204.             set ← FREEZE(m) \ {x | x mod t.size = i};
205.         else
206.             m ← s.buckets[;]
207.             m' ← s.buckets[; + s.size];
208.             set ← FREEZE(m) \ FREEZE(m');
209.         end if
210.     end if
211.     b ← new FSet(set, true);
212.     CAS(t.buckets[;], NULL, b);
213. end if
214. end Method

Figure 15: Structure of FSet, FSetOp and ILNode. Pseudocodes of Invoke, Freeze, Add, Remove, Contains, Resize, Apply and INITBKT methods based on dynamic sized non-blocking hash table [17].

(b) (true, z): Edge has been updated
i. No concurrent REMV(v1) or REMV(v2) or REME(v1, v2): The LP is the atomic update of the edge-weight to u using a successful CAS, at Line 51
ii. With concurrent REMV(v1) or REMV(v2) or REME(v1, v2): The LP is just before the first remove’s LP.

(c) (false, u): Edge already present
i. No concurrent REMV(v1) or REMV(v2) or REME(v1, v2): The LP is the atomic read of the ENode e(v2) has occurred at Line 293 in the FIND method.
ii. With concurrent \texttt{REMV(v_1)} or \texttt{REMV(v_2)} or \texttt{REME(v_1, v_2)}: The LP is just before the first remove’s LP.

(d) (false, \infty): Either \(v_1\) or \(v_2\) or both are not present

i. No concurrent \texttt{REMV(v_1)} or \texttt{REMV(v_2)} or \texttt{REME(v_1, v_2)}: The LP is the end of the search path reached by reading a NULL pointer at Line 298 or 302 in the Find method.

ii. At the time of invocation of \texttt{PUTE(v_1, v_2}|w\rangle) if both vertices \(v_1\) and \(v_2\) were in the vertex-list and a concurrent \texttt{REMV} removed \(v_1\) or \(v_2\) or both then the LP is the just after the LP of the earlier \texttt{REMV}.

iii. At the time of invocation of \texttt{PUTE(v_1, v_2}|w\rangle) if both vertices \(v_1\) and \(v_2\) were not present in the vertex-list, then the LP is the invocation point itself.
5. REME($v_1$, $v_2$): We have two cases:

(a) (\texttt{true}, $u$): Edge already present

i. No concurrent REMV($v_1$) or REMV($v_2$) or REME($v_1$, $v_2$): We have two cases

A. With less than two children: The LP is at Line 83 when the ENode is set as marked.

B. With two children: It needs two CASs to succeed, on one the node having the key to be deleted, and second which has the next largest key. So a successful REME’s LP is at Line 327 when the relocateOp is installed by the HELPRELOCATE method.

ii. With concurrent REMV($v_1$) or REMV($v_2$) or REME($v_1$, $v_2$): The LP is just before the first remove’s LP.

(b) (\texttt{false}, $\infty$): Either $v_1$ or $v_2$ or both are not present

i. No concurrent REMV($v_1$) or REMV($v_2$) or REME($v_1$, $v_2$): The LP is the end of the search path reached by reading a NULL pointer at Line 298 or 302 in the FIND method.

ii. At the time of invocation of REME($v_1$, $v_2$) if both vertices $v_1$ and $v_2$ were in the vertex-list and a concurrent REMV removed $v_1$ or $v_2$ or both then the LP is the just after the LP of the earlier REMV.

iii. At the time of invocation of REME($v_1$, $v_2$) if both vertices $v_1$ and $v_2$ were not present in the vertex-list, then the LP is the invocation point itself.

6. GETE($v_1$, $v_2$): We have two cases:

(a) (\texttt{true}, $u$): Edge present

i. No concurrent REMV($v_1$) or REMV($v_2$) or REME($v_1$, $v_2$): The LP is the atomic read of the key that occurs at Line 295 in the method FIND.

ii. With concurrent REMV($v_1$) or REMV($v_2$) or REME($v_1$, $v_2$): The LP is just before the first remove’s LP.

(b) (\texttt{false}, $\infty$): Either $v_1$ or $v_2$ or both or $e$($v_2$) are not present

i. No concurrent REMV($v_1$) or REMV($v_2$) or REME($v_1$, $v_2$): The LP is the end of the searching path reached by reading a NULL pointer at Line 298 or 302 in the FIND method.

ii. At the time of invocation of GETE($v_1$, $v_2$) if both vertices $v_1$ and $v_2$ were in the vertex-list and a concurrent REMV removes $v_1$ or $v_2$ or both then the LP is just after the LP of the earlier REMV.

iii. At the time of invocation of GETE($v_1$, $v_2$) if both vertices $v_1$ and $v_2$ were not present in the vertex-list, then the LP is the invocation point itself.

7. BFS($v$): Here, there are two cases:
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(a) BFS invoke the BFSSCAN method: Assuming that BFSSCAN invokes \( m \) (greater than equal to 2) BFSTCLT procedures, it is the last atomic read step of the \((m - 1)\)st BFSTCLT call.

(b) BFS does not invoke the BFSSCAN method: If a concurrent REMV operation \( op \) removed \( v \). Then just after the LP of \( op \). If \( v \) did not exist in the vertex-list before the invocation then at the invocation of BFS(\( v \)).

8. SSSP(\( v \)) : Similarly, there are two cases here as well:

(a) SSSP invoke the SPSCAN method: Assuming that SPSCAN invokes \( m \) (greater than equal to 2) SPTCLT procedures, it is the last atomic read step of the \((m - 1)\)st SPTCLT call.

(b) SSSP does not invoke the SPSCAN method: The LP is the same as the case [b] of BFS operation.

9. BC(\( v \)) : Here, there are two cases:

(a) BC invoke the BCSCAN method: Assuming that BCSCAN invokes \( m \) (greater than equal to 2) BCTCLT procedures. Then it is the last atomic read step of the \((m - 1)\)st BCTCLT call.

(b) BC does not invoke the BCSCAN method: The LP is the same as the case [b] of BFS operation.

From the above description, one can notice that the LPs of each of the operations lie in the interval between their invocation and the return steps. We can observe that in any invocation of a PUTV or a REMV operation, with key value \( v \), there is always a unique FSet object for an operation to be applied. If two threads \( T_1 \) and \( T_2 \) try to add or delete the same key \( v \), after indexing in the hash table using modular arithmetic and both call the INVOKE method. Thus, both either hash to the same bucket \( b \), or at least one of them gets mapped to an immutable bucket. This prevents multiple INVOKE methods parallely add or remove the same key at different buckets, and thereby a possible event invalidating linearizability is avoided. The PUTE and REME operations are similar to [20] except the case when the edge-weight is updated. However, update of edge-weight does not interfere with the arrangement of nodes in the BST corresponding to the edge-list. The non-update operations, GETV, GETE, BFS, SSSP, and BC do not modify the data structure. Thus, following from [27] and [30] we conclude that all non-blocking graph operations maintain the invariant of the data structure across the LPs. This completes the correctness proof.

B.2 Non-blocking Progress Guarantee

**Theorem 0.** The presented concurrent graph operations

(i) If the set of keys is finite, the operations GETV and GETE are wait-free.

(ii) The operation BFS, SSSP, and BC are obstruction-free.

(iii) The operations PUTV, REMV, GETV, PUTE, REME, and GETE are lock-free.

**Proof.** If the set of keys is finite, the size of the concurrent graph has a fixed upper bound. This implies that there are only a finite number of VNodes in each bucket. A search for a given VNode having key \( v \) is either in the bucket \( b \), or if \( b \) is NULL, in the predecessor’s buckets, so it terminates in a finite number of steps. Similarly, a GETE operation invokes the FIND method and it terminates by traversing the tree until the key is found or a null node is reached. A BFS operation or a CMPTREE method never returns true with concurrent update operations, which impose the While loop (Line 96) in BFSSCAN method to not terminate. So, unless a non-faulty thread has taken the steps in isolation a BFS operation will never return. A similar argument can be brought for a SSSP, and a BC operation. This shows the [i]Whenever an insertion or a deletion operation is blocked by a concurrent delete operation by the way of a marked pointer, then that blocked operations is helped to make a safe return. Generally, insertion and lookup operations do not need help by a concurrent operation. So, if any random concurrent execution consists of any concurrent data structure operation, then at least one operation finishes its execution in a finite number of steps taken be a non-faulty thread. Therefore, the concurrent graph operations PUTV, REMV, GETV, PUTE, REME, and GETE are lock-free. This shows the [ii]