Almost-bipartite distance-regular graphs with the $Q$-polynomial property \(^*\)

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Abstract

Let $\Gamma$ denote a $Q$-polynomial distance-regular graph with diameter $D \geq 4$. Assume that the intersection numbers of $\Gamma$ satisfy $a_i = 0$ for $0 \leq i \leq D - 1$ and $a_D \neq 0$. We show that $\Gamma$ is a polygon, a folded cube, or an Odd graph.

1 Introduction

In this article we prove the following theorem.

**Theorem 1.1** Let $\Gamma$ denote a distance-regular graph with diameter $D \geq 3$. Assume that the intersection numbers of $\Gamma$ satisfy $a_i = 0$ for $0 \leq i \leq D - 1$ and $a_D \neq 0$. Then $\Gamma$ is $Q$-polynomial if and only if at least one of (i)–(iv) holds below.

(i) $\Gamma$ is the $(2D + 1)$-gon.

(ii) $\Gamma$ is the folded $(2D + 1)$-cube.

(iii) $\Gamma$ is the Odd graph on a set of cardinality $2D + 1$.

(iv) $D = 3$ and there exist complex scalars $\beta$ and $\mu$ such that the intersection numbers of $\Gamma$ satisfy

\[
egin{align*}
k &= 1 + (\beta^2 - 1)(\beta(\beta + 2) - (\beta + 1)\mu), \\
c_2 &= \mu, \\
c_3 &= -(\beta + 1)(\beta^2 + \beta - 1 - (\beta + 1)\mu).
\end{align*}
\]

The following remarks refer to Theorem 1.1

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**Remark 1.2** Suppose that (iv) holds. Then \( \theta_0, \theta_1, \theta_2, \theta_3 \) is a \( Q \)-polynomial ordering of the eigenvalues of \( \Gamma \), where

\[
\theta_0 = 1 + (\beta^2 - 1)\left(\beta(\beta + 2) - (\beta + 1)\mu\right),
\]

\[
\theta_1 = (\beta + 1)(\beta^2 + \beta - 1 - \beta\mu),
\]

\[
\theta_2 = \beta^2 - \beta - 1 - (\beta + 1)\mu,
\]

\[
\theta_3 = 1 - \beta - \beta^2.
\]

**Remark 1.3** \( \Gamma \) is the 7-gon if and only if (iv) holds with \( \mu = 1 \) and \( \beta \in \{\omega + \omega^{-1}, \omega^2 + \omega^{-2}, \omega^3 + \omega^{-3}\} \), where \( \omega \) is a primitive 7th root of unity. \( \Gamma \) is the folded 7-cube if and only if (iv) holds with \( \mu = 2 \) and \( \beta \in \{-2, 2\} \). \( \Gamma \) is the Odd graph on a set of cardinality 7 if and only if (iv) holds with \( \mu = 1 \) and \( \beta = -2 \).

**Remark 1.4** Suppose that (iv) holds but none of (i)–(iii) do. Then \( \beta \) is unique, integral and less than \(-2\). We know of no graph for which this occurs.

## 2 Preliminaries

Let \( \Gamma = (X, R) \) denote a finite, undirected, connected graph, without loops or multiple edges, with vertex set \( X \), edge set \( R \), path-length distance function \( \partial \), and diameter \( D := \max\{\partial(x, y) : x, y \in X\} \). Let \( k \) denote a nonnegative integer. We say \( \Gamma \) is regular with valency \( k \) whenever for all \( x \in X \), \( |\{z \in X : \partial(x, z) = 1\}| = k \). We say \( \Gamma \) is distance-regular whenever for all integers \( h, i, j \) \((0 \leq h, i, j \leq D) \) and all \( x, y \in X \) with \( \partial(x, y) = h \), the scalar \( p_{ij}^h := |\{z \in X : \partial(x, z) = i, \partial(y, z) = j\}| \) is independent of \( x \) and \( y \). For notational convenience, set \( c_i := p_{ii}^1 \) \((1 \leq i \leq D) \), \( a_i := p_{ii}^0 \) \((0 \leq i \leq D) \), \( b_i := p_{ii}^{i+1} \) \((0 \leq i \leq D - 1) \), \( c_0 := 0 \) and \( b_D := 0 \). For the rest of this section, suppose that \( \Gamma \) is distance-regular. We observe that \( \Gamma \) is regular with valency \( k = b_0 \). Further, we observe \( c_i + a_i + b_i = k \) for \( 0 \leq i \leq D \).

We recall the Bose-Mesner algebra. Let \( \mathbb{R} \) denote the field of real numbers. By \( \text{Mat}_X(\mathbb{R}) \) we mean the \( \mathbb{R} \)-algebra consisting of all matrices whose entries are in \( \mathbb{R} \) and whose rows and columns are indexed by \( X \). For each integer \( i \) \((0 \leq i \leq D) \), let \( A_i \) denote the matrix in \( \text{Mat}_X(\mathbb{R}) \) with \( x, y \) entry

\[
(A_i)_{xy} = \begin{cases} 
1 & \text{if } \partial(x, y) = i, \\
0 & \text{otherwise}
\end{cases} \quad (x, y \in X).
\]

Note that \( A_0 = I \), the identity matrix. Abbreviate \( A := A_1 \). We call \( A \) the adjacency matrix of \( \Gamma \). Let \( M \) denote the subalgebra of \( \text{Mat}_X(\mathbb{R}) \) generated by \( A \). By [2, Theorem 20.7], \( A_0, A_1, \ldots, A_D \) is a basis for \( M \). We call \( M \) the Bose-Mesner algebra of \( \Gamma \).

By [3, Theorem 2.6.1], \( M \) has a second basis \( E_0, E_1, \ldots, E_D \) such that \( E_iE_j = \delta_{ij}E_i \) \((0 \leq i, j \leq D) \). We call \( E_0, E_1, \ldots, E_D \) the primitive idempotents of \( \Gamma \). Observe that there exists a sequence of scalars \( \theta_0, \theta_1, \ldots, \theta_D \) taken from \( \mathbb{R} \) such that \( A = \sum_{i=0}^{D} \theta_iE_i \). We call \( \theta_i \) the eigenvalue of \( \Gamma \) associated with \( E_i \) \((0 \leq i \leq D) \). Note that \( \theta_0, \theta_1, \ldots, \theta_D \) are distinct since \( A \) generates \( M \).
We recall the $Q$-polynomial property. Let $\theta_0, \theta_1, ..., \theta_D$ denote an ordering of the eigenvalues of $\Gamma$. We say this ordering is $Q$-polynomial whenever there exists a sequence of real scalars $\sigma_0, \sigma_1, ..., \sigma_D$ and a sequence of polynomials $q_0, q_1, ..., q_D$ with real coefficients such that $q_j$ has degree $j$ and $E_j = \sum_{i=0}^{D} q_j(\sigma_i)A_i$ for $0 \leq j \leq D$. In this case, $\theta_0 = k$ [3, Theorem 8.1.1]; we call $\theta_1$ a $Q$-polynomial eigenvalue of $\Gamma$. We say that $\Gamma$ is $Q$-polynomial whenever there exists a $Q$-polynomial ordering of its eigenvalues.

We recall what it means for $\Gamma$ to be bipartite or almost-bipartite. We say $\Gamma$ is bipartite whenever $a_i = 0$ for $0 \leq i \leq D$. We say $\Gamma$ is almost-bipartite whenever $a_i = 0$ for $0 \leq i \leq D - 1$ but $a_D \neq 0$. (In the literature, such a $\Gamma$ is also called a generalized Odd graph or a regular thin near $(2D+1)$-gon.) For the rest of this section, assume that $\Gamma$ is almost-bipartite.

We recall the bipartite double $2\Gamma$. This graph has vertex set $\{x^+: x \in X\} \cup \{y^-: y \in X\}$. For $x, y \in X$ and $\gamma, \delta \in \{+, -\}$, vertices $x^\gamma$ and $y^\delta$ are adjacent in $2\Gamma$ whenever $x$ and $y$ are adjacent in $\Gamma$ and $\gamma \neq \delta$. The graph $2\Gamma$ is bipartite and distance-regular with diameter $2D+1$. Moreover, $2\Gamma$ is an antipodal 2-cover of $\Gamma$ [3, Theorem 1.11.1(i),(vi)]. The intersection numbers $k$ and $c_2$ are the same in $2\Gamma$ as in $\Gamma$ [3, Proposition 4.2.2(ii)]. The set of eigenvalues for $2\Gamma$ consists of the eigenvalues of $\Gamma$ together with their opposites [3, Theorem 1.11.1(v)]. The concept of an AO eigenvalue was introduced in [6]. A scalar $\theta$ is an AO eigenvalue of $2\Gamma$ if and only if $\theta$ is a $Q$-polynomial eigenvalue of $\Gamma$ [6, Theorem 10.4].

3 Setup

Our strategy for proving Theorem 1.1 is to assume that (i)–(iii) do not hold and then prove that (iv) must.

Lemma 3.1 Let $\Gamma$ denote an almost-bipartite distance-regular graph with diameter $D \geq 3$. Suppose that $\Gamma$ is $Q$-polynomial but not as in Theorem 1.1(i)–(iii). Then $\Gamma$ has a unique $Q$-polynomial eigenvalue.

Proof: Since $\Gamma$ is not a polygon, it has valency $k \geq 3$. Observe that $2\Gamma$ has diameter at least 7. Suppose that $\Gamma$ has at least two $Q$-polynomial eigenvalues. Then $2\Gamma$ has at least two AO eigenvalues. Applying [6, Theorem 16.2], we find that $2\Gamma$ is the $(2D+1)$-cube. Thus $\Gamma$ is the folded $(2D+1)$-cube, contradicting the assumption that Theorem 1.1(iii) does not hold. □

For the rest of this article, we use the following notation.

Notation 3.2 Let $\Gamma = (X, R)$ denote an almost-bipartite distance-regular graph with diameter $D \geq 3$. Assume that $\Gamma$ is $Q$-polynomial but not as in Theorem 1.1(i)–(iii). Let $\theta_0, \theta_1, ..., \theta_D$ denote the eigenvalues of $\Gamma$ in their $Q$-polynomial order. Set

$$\beta := \frac{\theta_0 - \theta_3}{\theta_1 - \theta_2} - 1.$$ (8)

4 Parameters

In this section, we recall some formulae for the intersection numbers and eigenvalues of $\Gamma$. 3
Lemma 4.1 [4, Lemma 15.2, Corollaries 15.4, 15.7, 15.8, Theorem 15.5] With reference to Notation 3.2, there exist complex scalars \( q \) and \( s \) such that the intersection numbers and eigenvalues of \( \Gamma \) satisfy

\[
\begin{align*}
    k &= h(1 + sq), \\
    c_i &= \frac{h(1 - q^i)(1 + sq^{2D+2-i})}{q^i(q^{2D-2i+1} - 1)} \quad (1 \leq i \leq D), \\
    \theta_i &= hq^{-i}(1 + sq^{2i+1}) \quad (0 \leq i \leq D),
\end{align*}
\]

where

\[
h = \frac{q - q^{2D}}{(q - 1)(1 + sq^{2D+1})}.
\]

Moreover,

\[
\begin{align*}
    q &\neq 0, \\
    q^i &\neq 1 \quad (1 \leq i \leq 2D), \\
    sq^i &\neq 1 \quad (2 \leq i \leq 2D), \\
    sq^i &\neq -1 \quad (1 \leq i \leq 2D + 1).
\end{align*}
\]

Corollary 4.2 With reference to Lemma 4.1, we have

\[
\beta = q + q^{-1}
\]

and

\[
\theta_D = \frac{q^{1-D} - q^D}{q - 1}.
\]

5 Restrictions

Throughout this section we refer to Notation 3.2 and Lemma 4.1. We obtain restrictions on the parameters \( q \) and \( s \). Let \( \mathbb{Z} \) denote the ring of integers; let \( \mathbb{Q} \) denote the field of rational numbers.

Lemma 5.1 We have \( \theta_i \in \mathbb{Z} \) for \( 0 \leq i \leq D \).

Proof: Suppose that there exists an integer \( i \) \((0 \leq i \leq D)\) such that \( \theta_i \not\in \mathbb{Z} \). Then \( \Gamma \) has a second \( \mathbb{Q} \)-polynomial eigenvalue by [1, p. 360]. This contradicts Lemma 3.1. \( \square \)

Lemma 5.2 We have \( q^i + q^{-i} \in \mathbb{Z} \) for each positive integer \( i \). In particular, \( \beta \in \mathbb{Z} \).

Proof: Define polynomials \( T_0, T_1, \ldots \) in a variable \( x \) by \( T_0 = 2, T_1 = x, T_{i+1} = xT_i - T_{i-1} \) \((i \geq 1)\). We routinely find that for \( i \geq 1 \),

\[
T_i \in \mathbb{Z}[x], \quad T_i \text{ is monic, and } q^i + q^{-i} \in \mathbb{Z} \quad \text{and} \quad q^i + q^{-i} = T_i(\beta).
\]
To finish the proof it suffices to show \( \beta \in \mathbb{Z} \). To do this we show \( \beta \in \mathbb{Q} \) and \( \beta \) is an algebraic integer.

By (8) and Lemma 5.1 we find \( \beta \in \mathbb{Q} \).

We now show that \( \beta \) is an algebraic integer. The right-hand side of (18) is equal to

\[
\sum_{i=1}^{D-1} q^i.
\]

By this and (19), we find that \( \beta \) is a root of a monic polynomial with coefficients in \( \mathbb{Z} \). It follows that \( \beta \) is an algebraic integer.

We have now shown \( \beta \in \mathbb{Q} \) and \( \beta \) is an algebraic integer, so \( \beta \in \mathbb{Z} \). The result follows. □

**Lemma 5.3** We have \( |\beta| > 2 \). Moreover, \( q \in \mathbb{R} \).

*Proof:* Suppose \( |\beta| \leq 2 \). Since \( \beta \in \mathbb{Z} \) by Lemma 5.2 we find that \( |\beta| \) is 0, 1 or 2. We now use (17). If \( |\beta| = 0 \) then \( q^4 = 1 \). If \( |\beta| = 1 \) then \( q^6 = 1 \). If \( |\beta| = 2 \) then \( q^2 = 1 \). Each of these contradicts (14), so the result follows. □

**Lemma 5.4** We may assume \( q^2 > 1 \).

*Proof:* By (13), we have \( q^2 \neq 0 \). By (14), we have \( q^2 \neq 1 \). We now consider two cases.

First suppose \( s = 0 \). If \(-1 < q < 0 \) then using (19) we find \( c_2 < 0 \). If \( 0 < q < 1 \) then using (2) we find \( k < 0 \). Each of these is a contradiction, so \( q^2 > 1 \) as desired.

Now suppose \( s \neq 0 \). If \( q^2 < 1 \), replace \( q \) by \( q^{-1} \) and \( s \) by \( s^{-1} \). In light of (12), these substitutions leave (9)–(11) unchanged. Moreover, \( q^2 > 1 \) as desired. □

Consider the quantity

\[
\eta := -\frac{(q^2 + 1)(q^{2D} - q^3)}{q^{2D} - q^5}.
\]

We show \( \eta \) to be an integer. To do this, we use the fact \( s^2 q^{2D+3} \neq 1 \). We obtain this fact using the following two lemmas.

**Lemma 5.5** For \( 1 \leq i \leq D \) we have \( (c_2 - 1)\theta_i^2 \neq (k - c_2)(k - 2) \).

*Proof:* Suppose that there exists an integer \( i \) (\( 1 \leq i \leq D \)) such that \( (c_2 - 1)\theta_i^2 = (k - c_2)(k - 2) \). We mentioned earlier that \( \theta_i \) is an eigenvalue of \( 2\Gamma \) and that the intersection numbers \( k \) and \( c_2 \) are the same in \( 2\Gamma \) as in \( \Gamma \). Now by [5, Theorem 25], we find that \( 2\Gamma \) is 2-homogeneous in the sense of Nomura [7]. By assumption, \( 2\Gamma \) is not a cube. Now by [3, Theorem 1.2], the diameter of \( 2\Gamma \) is at most 5. Since this diameter is \( 2D + 1 \), we find \( D \leq 2 \), which is a contradiction. □

**Lemma 5.6** We have

\[
(c_2 - 1)\theta_D^2 - (k - c_2)(k - 2) = \frac{(q^{2D} - 1)(q^{2D} - q^2)(q^{2D} - q)^2(s^2 q^{2D+3} - 1)}{q^{2D}(q - 1)^2(q^{2D} - q^3)(1 + sq^{2D+1})^2}.
\]

*Proof:* Use Lemma 4.1 □
Corollary 5.7  We have $s^2q^{2D+3} \neq 1$.

Proof: Combine Lemmas 5.9 and 5.10

Before proceeding, we recall the local graph $\Gamma_2^2$.

Definition 5.8  Fix a vertex $x \in X$. The corresponding local graph $\Gamma_2^2$ is the graph with vertex set $\{y \in X : \partial(x, y) = 2\}$, where vertices $y$ and $z$ are adjacent in $\Gamma_2^2$ whenever $\partial(y, z) = 2$ in $\Gamma$.

Lemma 5.9  Fix a vertex $x \in X$ and let $\Gamma_2^2$ denote the corresponding local graph from Definition 5.8. Then the scalar $\eta$ from (20) is an eigenvalue of $\Gamma_2^2$. Moreover, $\eta$ is an algebraic integer.

Proof: Our argument uses the subconstituent algebra of $\Gamma$. This object is introduced in [9]. We refer the reader to that paper and its continuations [10] and [11] for background and definitions.

Let $T = T(x)$ denote the subconstituent algebra of $\Gamma$ with respect to $x$. By [4, Theorem 14.1] we find that, up to isomorphism, there exists at most one irreducible $T$-module with endpoint 2, dual endpoint 2 and diameter $D-2$. By [4, Example 16.9(iv)] the multiplicity with which this module appears in the standard module is

$$
\frac{(q^{2D} - 1)(q^{2D} - q^2)(1 + sq)(1 + sq^4)(s^2q^{2D+3} - 1)}{q(q+1)(q-1)^2(s^2q^{2D+4} - 1)(1 + sq^{2D})(1 + sq^{2D+1})}.
$$

This number is nonzero by [13], [14], [15] and Corollary 5.7. Therefore, this module exists.

Let $W$ denote an irreducible $T$-module with endpoint 2, dual endpoint 2 and diameter $D-2$. The dimension of $E_2^*W$ is 1 for $2 \leq i \leq D$ [4, (18), Lemma 10.3]. By construction, $E_2^*W$ is an eigenspace for $E_2^*A_2E_2^*$. By [4, Definition 8.2] and using $A_2 = (A^2 - kI)/c_2$, we find that the corresponding eigenvalue is

$$
c_1(W)b_0(W) - k
$$

where $c_1(W)$ and $b_0(W)$ are intersection numbers of $W$. Evaluating (21) using [4, Theorem 15.5], we find that it is equal to $\eta$. We conclude that $\eta$ is an eigenvalue of $\Gamma_2^2$.

We show $\eta \in \mathbb{Z}$. To do this we use the following result.

Lemma 5.10  We have

$$
\eta + \beta^2 - 1 = \frac{q^{2D} - q^9}{q^{2D+2} - q^4}.
$$

Proof: Use (17) and (20).
Lemma 5.11 We have $\eta \in \mathbb{Z}$.

Proof: First assume $D = 3$. Then $\eta = -\beta(\beta + 1)$ by (17) and (20). Thus $\eta \in \mathbb{Z}$ by Lemma 5.2. Now assume $D \geq 4$. Observe by Lemma 5.9 that $\eta$ is an algebraic integer. We show $\eta \in \mathbb{Q}$. Observe that the right-hand side of (22) is equal to $-(\beta + 1)^{-1}$ for $D = 4$ and

$$\frac{\sum_{i=5-D}^{D-5} q^i}{\sum_{i=3-D}^{D-3} q^i}$$

for $D \geq 5$. By this and Lemma 5.2 we find that the right-hand side of (22) is in $\mathbb{Q}$. By this and since $\beta \in \mathbb{Z}$ we find $\eta \in \mathbb{Q}$. Now $\eta \in \mathbb{Q}$ and $\eta$ is an algebraic integer so $\eta \in \mathbb{Z}$. □

6 Proof

In this section we prove Theorem 1.1 and the associated remarks.

Proof of Theorem 1.1: Assume that $\Gamma$ is $\mathbb{Q}$-polynomial but none of (i)–(iii) hold. We show that $\Gamma$ satisfies (iv).

We first show $D = 3$. On the contrary, suppose $D \geq 4$. For notational convenience, abbreviate $\xi := \eta + \beta^2 - 1$. Recall $\beta \in \mathbb{Z}$ by Lemma 5.2 and $\eta \in \mathbb{Z}$ by Lemma 5.11, so $\xi \in \mathbb{Z}$. By (13), (14) and (22) we find $\xi \neq 0$. Thus $|\xi| \geq 1$. Evaluating $\xi^2 - 1$ using (22) and simplifying, we find $(q^4 - 1)(q^{14} - q^{4D}) \geq 0$. But since $q^2 > 1$ by Lemma 5.4, we find $(q^4 - 1)(q^{14} - q^{4D}) < 0$, for a contradiction. We have now shown $D = 3$.

Evaluating (9)–(12) using $D = 3$ and $\beta = q + q^{-1}$, we routinely obtain (1)–(3) and (4)–(7).

We have proved the theorem in one direction. We now show the converse. First assume that $\Gamma$ satisfies one of (i)–(iii). That $\Gamma$ is $\mathbb{Q}$-polynomial is well known [3, Corollary 8.5.3]. Now assume that $\Gamma$ satisfies (iv). We routinely find that the eigenvalues of $\Gamma$, in a $\mathbb{Q}$-polynomial order, are given by (4)–(7).

Remarks 1.2 and 1.3 are verified routinely.

Proof of Remark 1.4: The $\mathbb{Q}$-polynomial ordering of the eigenvalues is unique by Lemma 3.1 so $\beta$ is unique by (8). It is an integer by Lemma 5.2.

We show $\beta < -2$. Recall that $\theta_1$ is a $\mathbb{Q}$-polynomial eigenvalue of $\Gamma$ and thus is an AO eigenvalue of the bipartite double $2.\Gamma$. Observe that $2.\Gamma$ has diameter 7. Since $2.\Gamma$ is bipartite, we see by [3, p. 82] that half of the eigenvalues of $2.\Gamma$ are positive and half are negative. By [3, Lemma 13.5], we find that $\theta_1$ is the fifth- or seventh-largest of the eight eigenvalues of $2.\Gamma$. Thus $\theta_1 < 0$.

Recall $|\beta| > 2$ by Lemma 5.3, thus $\beta^2 + \beta - 1 > 0$. Observe $b_2 = (\beta^2 + \beta - 1)(\beta^2 + \beta - 1 - \beta\mu)$ by (11). By this and since $b_2 > 0$, we find $\beta^2 + \beta - 1 - \beta\mu > 0$. Combining this with (13), we find $\beta + 1 < 0$. In particular, $\beta < 0$. Now $\beta < -2$ in view of Lemma 5.3. □

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