BIASED PARTITIONS OF $\mathbb{Z}^n$

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Abstract. Given a function $f$ on the vertex set of some graph $G$, a scenery, let a simple random walk run over the graph and produce a sequence of values. Is it possible to, with high probability, reconstruct the scenery $f$ from this random sequence?

To show this is impossible for some graphs, Gross and Grupel, in [3], call a function $f : V \rightarrow \{0,1\}$ on the vertex set of a graph $G = (V,E)$ $p$-biased if for each vertex $v$ the fraction of neighbours on which $f$ is 1 is exactly $p$. Clearly, two $p$-biased functions are indistinguishable based on their sceneries. Gross and Grupel construct $p$-biased functions on the hypercube $\{0,1\}^n$ and ask for what $p \in [0,1]$ there exist $p$-biased functions on $\mathbb{Z}^n$ and additionally how many there are. We fully answer this question by giving a complete characterization of these values of $p$. We show that $p$-biased functions exist for all $p = c/2n$ with $c \in \{0,\ldots,2n\}$ and, in fact, there are uncountably many of them for every $c \in \{1,\ldots,2n-1\}$. To this end, we construct uncountably many partitions of $\mathbb{Z}^n$ into $2^n$ parts such that every element of $\mathbb{Z}^n$ has exactly one neighbour in each part. This additionally shows that not all sceneries on $\mathbb{Z}^n$ can be reconstructed from a sequence of values on attained on a simple random walk.

1. Background

For a graph $G = (V,E)$, we call a function $f$ on $V$ a scenery. Let $\tilde{X} = (X_n)_{n=0}^\infty$ be a simple random walk on $G$. We associate with $\tilde{X}$ the sequence $(f(X_n))_{n=0}^\infty$ of values attained by the scenery. Is it possible to, with high probability, reconstruct the scenery $f$ from this random sequence? This question has an extensive history, in particular, the case where $G = \mathbb{Z}$ has been studied in many papers, see e.g. [1, 2, 4, 6, 7]. In [4], Howard showed that periodic sceneries on $\mathbb{Z}$ or equivalently sceneries on finite cycles can always be reconstructed. Matzinger and Lember [7] extended this result on periodic sceneries to random walks on $\mathbb{Z}$ which include steps of different sizes. In turn, Finucane, Tamuz and Yaar [2] refined this idea and extended it to more general Cayley graphs of finite abelian groups.
Focusing on the question for \( G = \mathbb{Z} \) (and \( G = \mathbb{Z}^2 \)), Benjamini and Kesten\cite{1} showed that almost all sceneries can in fact be distinguished, in the sense that for any given scenery on \( G \), it can be distinguished with probability 1 from a scenery chosen randomly in the product measure on all sceneries. Benjamini and, independently, Keane and Den Hollander conjectured that in fact all pairs of sceneries on \( \mathbb{Z} \) are distinguishable (unpublished)\cite{5}. This, however, was soon disproved by Lindenstrauss\cite{6} who constructed a collection of uncountably many distinct yet indistinguishable sceneries.

In a recent paper, Gross and Grupel\cite{3} showed that for 0-1 functions on the hypercube, i.e. functions of the form \( f : \{0, 1\}^n \to \{0, 1\} \), this reconstruction is not possible in general. To this end, they defined a \( p \)-biased function to be a function which takes the value 1 on exactly a fraction \( p \) of the neighbours of each vertex in the graph. Note that if there exist two non-isomorphic \( p \)-biased functions on a graph, they will be indistinguishable as the sequence of values for both will look like a sequence of independent Bernoulli random variables with success probability \( p \).

Gross and Grupel\cite{3} extended their construction from the hypercube to \( \mathbb{Z}^n \) to find \( p \)-biased functions for \( p = c/2^k \) where \( 2^k \mid n \). They asked for what \( p \in [0, 1] \) such \( p \)-biased functions exist and how many there are for each \( p \). Our main aim in this paper is to give a complete characterization of all the values \( p \in [0, 1] \) for which a \( p \)-biased function on \( \mathbb{Z}^n \) exists and the exact number for each of those \( p \).

As every element of \( \mathbb{Z}^n \) has \( 2n \) neighbours, if there is a \( p \)-biased function, then \( p \) is of the form \( p = \frac{c}{2n} \) for some integer \( c \). In fact, we will find that for all \( c \in \{0, \ldots, 2n\} \), there exists a \( p \)-biased function with \( p = \frac{c}{2n} \). Note that the sum of a \( p \)-biased function and a \( q \)-biased function on disjoint supports is a \( p + q \)-biased function. Hence, to show that these \( \frac{c}{2n} \)-biased functions exist, it suffices to construct \( 2n \frac{1}{2n} \)-biased functions the supports of which partition \( \mathbb{Z}^n \).

We proceed by showing that for all \( n > 1 \) there are uncountably many non-isomorphic such partitions and uncountably many indistinguishable \( p \)-biased functions, answering the second question from \cite{3}. Our result extends the result of Lindenstrauss\cite{6} that there exist uncountably many indistinguishable sceneries on \( \mathbb{Z} \).

2. Definitions

As usual, we write \([n] = \{1, \ldots, n\}\) and \( e_i \) for the \( i \)th basis vector in the canonical basis of \( \mathbb{Z}^n \). We consider \( \mathbb{Z}^n \) to be the graph on that set with edge set \( E = \{\{x, x + e_i\} : x \in \mathbb{Z}^n, i \in [n]\} \). Given vertex \( x \), \( \Gamma(x) \) denotes the neighbourhood, in particular for \( x \in \mathbb{Z}^n \);

\[
\Gamma(x) = \{y : y = x \pm e_i, \text{ for some } i \in [n]\}.
\]

We identify a 0-1 function with its support, so that we can talk of sets rather than functions.

A set \( X \subset \mathbb{Z}^n \) is \( p \)-biased if for all \( x \in \mathbb{Z}^n \), we have \( |\Gamma(x) \cap X| = 2pn \). A partition \( \{X_i\}_{i \in [2n]} \)
of \(\mathbb{Z}^n\) is biased if each of its elements is \(\frac{1}{2n}\)-biased. Hence, as noted in the introduction, if there is a biased partition of \(\mathbb{Z}^n\), then there are \(p\)-biased function on \(\mathbb{Z}^n\) for all \(p = c/2n\).

To stress that a certain union consists of disjoint sets, we use \(\sqcup\) rather than the more common \(\cup\).

3. There is a biased partition of \(\mathbb{Z}^n\) for every \(n \in \mathbb{N}\)

**Theorem 3.1.** For every \(n \in \mathbb{N}\), there is a biased partition of \(\mathbb{Z}^n\).

The proof of this result will consist of a recursive construction. Accordingly, we start by observing that \(\mathbb{Z} = \{m : m \equiv 0, 1 \mod 4\} \sqcup \{m : m \equiv 2, 3 \mod 4\}\) is a biased partition. For the recursive construction of the biased partitions, we define a closely related notion.

Let \(m, n \in \mathbb{N}\) such that \(m\) is a multiple of \(n\) and let \(\{X^j_i\}_{i \in \mathbb{N}, j \in [2n]}\) be a family of subsets of \(\mathbb{Z}^m\). We write \(X^i = \bigcup_j X^i_j\). We say this family of subsets of \(\mathbb{Z}^m\) is \((m, n)\)-filling if the family partitions \(\mathbb{Z}^m\) and if, for each \(i \in \mathbb{N}\) and \(j \in [2n]\), we have that if \(x \in \mathbb{Z}^m \setminus X^i\), then \(|\Gamma(x) \cap X^i_j| = 1\) and if \(x \in X^i\), then \(\Gamma(x) \cap X^i = \emptyset\).

The following lemma is the basis for our recursive construction.

**Lemma 3.2.** If there is a biased partition of \(\mathbb{Z}^n\) and there exists an \((m, n)\)-filling family of subsets of \(\mathbb{Z}^m\), then there is a biased partition of \(\mathbb{Z}^{m+n}\).

Proof. Let \(\{X^i_j\}_{i \in [m+n], j \in [2n]}\) be an \((m, n)\)-filling family of subsets of \(\mathbb{Z}^m\) and let \(\{Y^i_j\}_{i \in [2n]}\) be a biased partition of \(\mathbb{Z}^n\) that witnesses \(n\) allowing a biased partition. Let, for \(i \in [m+n]\) and \(l \in [2n]\);

\[
Z^i_l = \bigcup_j X^i_{j+l} \times Y^i_j
\]

we claim this is set is \(\frac{1}{2(m+n)}\)-biased in \(\mathbb{Z}^{m+n}\). For notational convenience, we take \(i = 1\) and \(l = 0\).

We claim that every \(z = (x, y) \in \mathbb{Z}^m \times \mathbb{Z}^n\) has a unique neighbour in \(Z = Z^1_0\). If \(x \in X^1_j\) for some \(j\), then \(z\) cannot have a neighbour in \(Z\) in the first \(m\) coordinates, as \(\Gamma(x) \cap X^1 = \emptyset\). Fortunately, by definition of \(Y^1_j\), \(y\) has a unique neighbour in \(Y^1_j\), say \(y'\), which gives \(z\) a unique neighbour in the last \(n\) coordinates, i.e. \((x, y')\).

If, on the other hand, \(x \notin X^1 = \bigcup_j X^1_j\), then \(z\) cannot have a neighbour in \(Z\) in the last \(n\) coordinates. Since the sets \(Y^1_j\) partition \(\mathbb{Z}^n\), we find a unique \(j\) such that \(y \in Y^1_j\). By construction, we know that exactly one neighbour of \(x\) is \(X^1_j\), say \(x'\), so that neighbour gives a unique neighbour of \(z\) in \(Z\), i.e. \((x', y')\).
Analogously, $Z_i^l$ is a $\frac{1}{(m+n)}$-biased set for each $i \in [\frac{m+n}{n}], l \in [2n]$. Moreover, these sets partition $\mathbb{Z}^{n+n}$, showing that there is a biased partition of $\mathbb{Z}^{m+n}$.

For the proof of Theorem 3.1, it remains to find suitable $(m, n)$-filling families.

**Lemma 3.3.** Given $n \in \mathbb{N}$, define for $l \in [2], j \in [2n]$ the following subsets of $\mathbb{Z}^n$:

$$X_j^l = \begin{cases} \{ x \in \mathbb{Z}^n : \sum x_i \equiv l \mod 4, \sum ix_i \equiv j \mod n \} & \text{if } j \leq n \\ \{ x \in \mathbb{Z}^n : \sum x_i \equiv l + 2 \mod 4, \sum ix_i \equiv j \mod n \} & \text{if } j > n \end{cases}$$

Then the family $\{X_j^l\}_{l \in [2], j \in [2n]}$ is an $(n, n)$-filling family.

**Proof.** Note that $X_j^l = \bigcup_j X_j^l = \{ x : \sum x_i \equiv l \mod 2 \}$, so the sets $X_j^l$ partition $\mathbb{Z}^n$. Let $x \in \mathbb{Z}^n$ and $l \in [2]$ be such that $x \in X_j^l$. Then $x \pm e_i$, the neighbours of $x$, are in $X_j^{l+1}$ for all $i \in [n]$. It remains to show that these neighbours are all in distinct $X_j^{l+1}$. We find that the neighbour $y = x + e_j$ is such that $\sum y_i \equiv 1 + \sum x_i \mod 4$ and $\sum iy_i \equiv j + \sum ix_i \mod n$. For distinct $j \in [n]$ the second sums are clearly distinct modulo $n$. If we compare $y$ to $z = x - e_k$ for some $k \in [n]$, we find that $\sum z_i \equiv -1 + \sum x_i \mod 4$, so $y$ and $z$ belong to different parts of the partition.

These lemmas imply that if there is a biased partition of $\mathbb{Z}^n$, then there also is a biased partition of $\mathbb{Z}^{2n}$. Hence, there is a biased partition of $\mathbb{Z}^{2k}$ for all $k \in \mathbb{N}$. To extend this to the natural numbers with odd prime divisors, we use the following construction.

**Lemma 3.4.** There exists a $(2mn, n)$-filling family of subsets of $\mathbb{Z}^{2mn}$ for all $m, n \in \mathbb{N}$.

**Proof.** Define for $l \in [2m + 1]$ and $k \in [2n]$:

$$X_j^l = \{ x \in \mathbb{Z}^{2mn} : \sum_{j=1}^{m} \sum_{i=2(j-1)n+1}^{2nj} jx_i \equiv l \mod 2m + 1 \} ,$$

$$X_k^l = \{ x \in \mathbb{Z}^{2mn} : x \in X_j^l, \sum_{i=1}^{2mn} ix_i \equiv k \mod 2n \}$$

We will show that these sets $X_k^l$ form a $(2mn, n)$-filling family. To this end, note that the sets $X_j^l$ partition $\mathbb{Z}^{2mn}$ and that $\{X_k^l\}_{k \in [2n]}$ is a partition of $X_j^l$ into $2n$ parts. It remains to check that each element of $X_j^l$ has exactly one neighbour in each of the parts $X_k^l$ for $l' \neq l$ and no neighbour in $X_j^l$. 

□
Fix some \( l \in [2m + 1] \). Consider \( y \in \mathbb{Z}^{2mn} \). If \( \sum_{j=0}^{m-1} \sum_{i=2jn+1}^{2n(j+1)} jx_i \equiv l \mod 2m + 1 \), we have \( y \in X^l \), and any neighbour of \( y \) is not in \( X^l \) as changing any coordinate would change this sum.

If \( \sum_{j=0}^{m-1} \sum_{i=2jn+1}^{2n(j+1)} jx_i \equiv l \mod 2m + 1 \), for some \( j \in [2m] \), there are two options.

Finally, if \( j \in \{m + 1, \ldots, 2m\} \), then let \( h = 2m + 1 - j \). Then \( y - e_i \in X^l \) for every \( i \in \{2hn+1, \ldots, 2(h+1)n\} \). Again, each of these \( 2n \) vectors is in a different set \( X^l_j \). Note that in both cases, these are the only neighbours of \( y \) in \( X^l \). □

This is the last ingredient needed for the proof of Theorem 3.1.

Proof. As noted, there is a biased partition of \( \mathbb{Z} \). In combination with Lemma 3.3 and Lemma 3.2, this implies that there is a biased partition of \( \mathbb{Z}^k \) for every \( k \in \mathbb{Z}_{\geq 0} \). To extend this to all of \( \mathbb{N} \), let \( n = 2^k(2m + 1) \) for some \( k, m \in \mathbb{Z}_{\geq 0} \). By Lemma 3.4, we can find a \((2m2^k, 2^k)\)-filling family, which by Lemma 3.2 implies that there is a biased partition of \( \mathbb{Z}^{2^k+2m2^k} = \mathbb{Z}^n \). □

4. Counting biased partitions

In [3], Gross and Grupel ask, besides a characterisation of \( p \) values for which \( p \)-biased functions exist, for a count of the number of non-isomorphic such \( p \)-biased functions. In fact, they provide some finite lower bounds on this number for \( p = \frac{1}{n} \) and \( p = \frac{1}{2} \), based on an extension of their construction on the hypercube. We find the following complete characterization.

**Theorem 4.1.** For all \( p = c/2n \) with \( n > 1 \) and \( c \in \{1, \ldots, 2n - 1\} \), there are \( 2^{8n} \) non-isomorphic \( p \)-biased functions in \( \mathbb{Z}^n \).

We say two biased partitions \( \{X_i\}_{i \in [2n]} \) and \( \{Y_i\}_{i \in [2n]} \) of \( \mathbb{Z}^n \) are isomorphic if there exist a graph isomorphism \( \phi : \mathbb{Z}^n \to \mathbb{Z}^n \) and permutation \( \sigma : [2n] \to [2n] \) such that \( \phi(X_i) = Y_{\sigma(i)} \) for all \( i \in [2n] \).

**Theorem 4.2.** For all \( n > 1 \), there are \( 2^{8n} \) non-isomorphic biased partitions of \( \mathbb{Z}^n \).

As there are only \( 2^{8n} \) subsets of \( \mathbb{Z}^n \), the upper bound for so for both Theorems 4.1 and 4.2 are immediate.
In fact, our construction proving Theorems 4.1 and 4.2 will be almost identical to the one in the previous section. By introducing a degree of freedom, we produce an uncountable collection of distinct biased partitions. An automorphism \( \phi : \mathbb{Z}^n \to \mathbb{Z}^n \) consists of three components; a permutation of the coordinates \( (n!) \), a reflection of some of the coordinates \( (2^n) \) and a translation \( (|\mathbb{Z}^n| = \aleph_0) \). There are only countably many combinations of these operations and hence there are only countably many automorphisms \( \phi : \mathbb{Z}^n \to \mathbb{Z}^n \).

To use the previous construction, we need to have a sense of isomorphism for \((m, n)\)-filling families. We say two \((m, n)\)-filling families \( \{ A^i_j \}_{i \in [m+n], j \in [2n]} \) and \( \{ B^i_j \}_{i \in [m+n], j \in [2n]} \) are isomorphic, if there exist a graph isomorphism \( \phi : \mathbb{Z}^m \to \mathbb{Z}^m \), a permutation \( \sigma : [m+n] \to [m+n] \) and a family of permutations \( \tau_j : [2n] \to [2n] \) for all \( i \in [m+n] \), such that \( \phi(A^i_j) = B^\sigma(i) \). Note that by the above observation, each \((m, n)\)-filling family is isomorphic to at most countably many other \((m, n)\)-filling families.

It is interesting to note the following. Given an \((m, n)\)-filling family, we can define a function, taking any element of \( \mathbb{Z}^m \) to the the part of the partition that element is in. Two functions arising in such a way from two non-isomorphic \((m, n)\)-filling families will be indistinguishable by a simple random walk.

**Lemma 4.3.** If there is a collection of pairwise non-isomorphic \((m, n)\)-filling families of size \( 2^{\aleph_0} \) and a biased partition of \( \mathbb{Z}^n \), then there are \( 2^{\aleph_0} \) pairwise non-isomorphic biased partitions of \( \mathbb{Z}^{m+n} \).

Proof. In fact, the construction in Lemma 3.2 produces such a collection. Let \( \{ X^i_{j,x} \}_{i \in [m+n], j \in [2n]} \) be an uncountable family of pairwise non-isomorphic \((m, n)\)-filling families indexed by some \( x \in \mathbb{R} \) and let \( \{ Y_i \}_{i \in [2n]} \) be a biased partition of \( \mathbb{Z}^n \). Let \( \{ Z^i_{l,x} \}_{i \in [m+n], l \in [2n]} \) be the biased partitions as defined in equation 1.

As only countably many of the \( \{ Z^i_{l,x} \}_{i \in [m+n], l \in [2n]} \) are isomorphic, it suffices to prove there are \( 2^{\aleph_0} \) distinct biased partitions.

We claim that all \((m, n)\)-filling families \( \{ X^i_{j,x} \}_{i \in [m+n], j \in [2n]} \) produce distinct biased partitions. For a contradiction, suppose \( \{ Z^i_{l,x} \}_{i \in [m+n], l \in [2n]} = \{ Z^i_{l,y} \}_{i \in [m+n], l \in [2n]} \) for some \( x \neq y \). Then we find that for any \( i \in [m+n] \) and \( l \in [2n] \), we can find some \( i' \in [m+n] \) and \( l' \in [2n] \) such that: \( \bigcup_j X^i_{j,l,x} \times Y_j = \bigcup_j X^{i'}_{j,l',y} \times Y_j \). As we know the \( Y_j \), we can then find for all \( j \in [2n] \); \( X^i_{j,l,x} \times Y_j = X^{i'}_{j,l',y} \times Y_j \) and \( \{ X^i_{j,x} \}_{i \in [m+n], j \in [2n]} = \{ X^i_{j,y} \}_{i \in [m+n], j \in [2n]} \). However, we assumed that the \((m, n)\)-filling families were distinct; this contradiction proves the lemma. \( \square \)

The construction of Lemma 3.2 works in such a way that for any element not in \( X^i \) all \( 2n \) neighbours in \( X^i \) lie in the same hyperplane in \( \mathbb{Z}^{m+n} \) defined by \( \sum_{j=0}^{m-1} \sum_{i=2j+1}^{2n} j x_i = \)
Lemma 4.5. Similarly we can extend the construction from Lemma 3.3. □

Lemma 4.4. For any \( m, n \in \mathbb{N} \), there are \( 2^{\aleph_0} (2mn, n) \)-filling families.

Proof. Consider the following construction for \( l \)

\[
X_{k,f}^l = \left\{ x \in \mathbb{Z}^{2mn} : \exists h \in \mathbb{Z}; \sum_{j=0}^{m-1} \sum_{i=2jn+1}^i j x_i = l + h(2m + 1) \text{ and } \sum_{i=1}^{2mn} ix_i \equiv k + f(h) \mod 2n \right\}
\]

where \( f : \mathbb{Z} \to [2n] \) is any function. This family is \((2mn,n)\)-filling by the proof of Lemma 3.4. There are \(|[2n]^Z| = 2^{\aleph_0}\) such functions, and thus such biased partitions. As each of these biased partitions is isomorphic to at most countably many others, we must have \( 2^{\aleph_0}\) non-isomorphic biased partitions among these.

Similarly we can extend the construction from Lemma 3.3.

Lemma 4.5. For \( n > 1 \), there are \( 2^{\aleph_0} (n, n) \)-filling families.

Proof. Consider the following construction for \( l \in [2], p \in \{0, 1\}, q \in [n] \) and \( f : \mathbb{Z} \to [n] \):

\[
X_{p,q,f}^l = \left\{ x \in \mathbb{Z}^n : \exists h \in \mathbb{Z}; \sum_{i=1}^n x_i = l + 2p + 4h \text{ and } \sum_{i=1}^n ix_i \equiv q + f(h) \mod n \right\}
\]

Writing it in proper form, let \( X_{k,f}^l = X_{p,q,f}^l \) with \( p = \begin{cases} 0 & \text{if } k \leq n \\ 1 & \text{if } k > n \end{cases} \), and \( q \in [n] \) with \( q \equiv k \mod n \). To check that this is in fact a biased partition, note that \( X_f^l = \bigcup_k X_{k,f}^l = \{ x : \sum x_i \equiv l \mod 2 \} \), so the \( X_{k,f}^l \) form a partition of \( \mathbb{Z}^n \) and \( \Gamma(X_f^l) \cap X_f^l = \emptyset \). We proceed to check that each element of \( \mathbb{Z}^n \setminus X_f^l \) has a unique neighbour in \( X_f^l \).

Let \( x \in \mathbb{Z}^n \setminus X_f^l \), i.e. \( \sum x_i \equiv l + 1 \mod 2 \). Write \( X_{p,f}^l = \bigcup_{q} X_{p,q,f}^l \), then \( x - e_j \in X_{p,f}^l \) and \( x + e_j \in X_{1-p,f}^l \) for all \( j \in [n] \) for some \( p \in \{0, 1\} \). Let \( y = x + e_j \), then we have \( \sum_i y_i = 1 + \sum_i x_i = l + 2(1 - p) + 4h \) for some \( h \) not dependent on \( j \). Thus, for different \( j \in [n] \), we find \( \sum_i iy_i = j + \sum_{i=1}^n ix_i \equiv q + f(h) \mod n \) with distinct \( q \). Hence, \( y = x + e_j \in X_{1-p,q,f}^l \) with distinct \( q \) for distinct \( j \).

Analogously \( x - e_j \in X_{p,q,f}^l \) with distinct \( q \) for distinct \( j \).

As in the proof of Lemma 4.4, we find that for \( n > 1 \), there are \( 2^{\aleph_0} \) functions \( f : \mathbb{Z} \to [n] \) and thus sets \( X_{p,q,f}^l \). At most \( \aleph_0 \) of those can be pairwise isomorphic, so there must be \( 2^{\aleph_0} \)
pairwise non-isomorphic \((n,n)\)-filling families.

What remains is to count the number of non-isomorphic biased partitions of \(\mathbb{Z}^2\).

**Lemma 4.6.** There are \(2^{\aleph_0}\) non-isomorphic biased partitions of \(\mathbb{Z}^2\).

**Proof.** Consider the following set \(S = \{x \in \mathbb{Z}^2 : x_1 + x_2 = 0 \text{ and } x_1 \equiv 0 \text{ mod } 2\}\). Note that every element of the set \(\{x \in \mathbb{Z}^2 : x_1 + x_2 \in \{-1, 0, 1, 2\}\}\), i.e. four downward diagonals around the origin, has exactly one neighbour in the set \(S + \{0, e_i\}\) for both \(i = 1\) and \(i = 2\).

Given some function \(f : \mathbb{Z} \rightarrow [2]\), let \(X_f = S + \{(2n, 2n), (2n, 2n) + e_{f(n)} : n \in \mathbb{Z}\}\). Now consider the following biased partition:

\[
X_f^k = \begin{cases} 
X_f & \text{if } k = 1 \\
X_f + (1, -1) & \text{if } k = 2 \\
X_f + (1, 1) & \text{if } k = 3 \\
x_f + (2, 0) & \text{if } k = 4
\end{cases}
\]

These 4 sets partition \(\mathbb{Z}^2\). Each of the sets is \(\frac{1}{4}\)-biased, by the note above. Finally this produces \(2^{\aleph_0}\) non-isomorphic biased partitions by the same argument as for the previous two lemmas. □

**Proof (of Theorem 4.2).**

By Lemma 4.6, we find uncountably many non-isomorphic biased partitions of \(\mathbb{Z}^2\), and by Lemmas 4.4 and 4.5 combined with Lemma 4.3, we find uncountably many non-isomorphic biased partitions of \(\mathbb{Z}^n\) for \(n > 2\). □

Extending the theorem on biased partitions to \(p\)-biased functions is not immediate as different biased partitions might give rise to the same \(p\)-biased functions. Consider for instance the \(\frac{1}{2}\)-biased functions on \(\mathbb{Z}^2\) which has support on \(X_1^1 \cup X_3^2\) from the proof of Lemma 4.6, this function is the same for all choices of \(f\). We will see that for dimensions bigger than two this is not a problem. We consider the case for \(\mathbb{Z}^2\) seperately.

**Lemma 4.7.** For all \(p \in \{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\}\), there exist infinitely many non-isomorphic \(p\)-biased functions in \(\mathbb{Z}^2\).

**Proof.** For \(p = \frac{1}{4}\) and \(p = \frac{3}{4}\), this follows immediately from Lemma 4.6 so consider \(p = \frac{1}{2}\). Let \(f : \mathbb{Z} \rightarrow [2]\) be any function, consider set:

\[
X_f = \{x \in \mathbb{Z}^2 : x_1 \equiv f(x_1 + x_2) \text{ mod } 2\}
\]
Note that for any \( x \in \mathbb{Z}^2 \) either \( x + e_1 \) or \( x + e_2 \) is in \( X_f \), and similarly either \( x - e_1 \) or \( x - e_2 \) is in \( X_f \). Hence, \( X_f \) is \( \frac{1}{2} \)-biased. As there are \( 2^{\aleph_0} \) different functions \( f : \mathbb{Z} \to [2] \), there are as many different \( \frac{1}{2} \)-biased functions, at most countably many of which are isomorphic. □

All that remains is to prove Theorem 4.1

Proof. Note that for \( p = \frac{1}{2n} \) and \( p = \frac{2n-1}{2n} \) this follows immediately from Theorem 4.2. The case \( n = 2 \) follows from Lemma 4.7. Fix some \( n > 2 \) and \( p = c/2n \). Use the constructions of equations 2 and 3 to produce biased partitions indexed by some function \( f \) to feed into the construction in equation 1. This produces the sets \( Z^l_{k,f} \) with \( l \in [r] \) and \( k \in [q] \) with \( r, q \) integers depending on the last step of the recursive construction of biased partitions. That is; \( r = 2 \) and \( q = n \) if \( n \) is some power of 2 and \( r = m + 1 \) and \( q = 2^{k+1} \) if \( n = 2^k(2m + 1) \) for some \( k \in \mathbb{Z}_{\geq 0} \) and \( m \in \mathbb{N} \).

Let \( I \subset [r] \times [q] \) be such that \( |I| = c \) and there is an \( l_0 \in [r] \) such that \( |\{k \in [q] : (l_0,k) \in I\}| \) is either 1 or \( q - 1 \). Let

\[
S_f = \bigsqcup_{(l,k) \in I} Z^l_{k,f}
\]

We claim that \( \{S_f : f \in [r]^2\} \) is a family of uncountably many non-isomorphic \( p \)-biased sets. It suffices to show that \( f \mapsto S_f \) is injective. Let \( S_f = S_g \). Equation 4 implies:

\[
\bigsqcup_{(l,k) \in I} X^{l}_{j+k,f} \times Y_j = \bigsqcup_{(l,k) \in I} X^{l}_{j+k,g} \times Y_j
\]

Consider \( S_f \cap X^{l_0} \times Y_q = S_g \cap X^{l_0} \times Y_q \). Taking complements if \( |\{k \in [q] : (l_0,k) \in I\}| = q - 1 \), this is equal to \( X^{l_0}_{k,f} \times Y_q = X^{l_0}_{k,g} \times Y_q \) for some \( k \), by the construction of \( l_0 \). Thus, \( X^{l_0}_{k,f} = X^{l_0}_{k,g} \) and \( f = g \). Hence, \( f \mapsto S_f \) is injective.

There are uncountably many sets \( S_f \) and only countably many of those can be pairwise isomorphic, so there are uncountably many non-isomorphic \( p \)-biased sets and functions. □

5. Open Problems

The concept of a biased partition introduced in this paper raises the question what graphs contain them. Evidently a graph needs to \( d \)-regular for some \( d \). A biased partition then consists of \( d \) parts, which in turn consist of pairs of neighbors. Hence, we need \( 2d \) to divide the order of the graph.

**Question 5.1.** What conditions on a graph imply the existence of a biased partition?
A more modest question towards a full characterization would be to identify a larger class of graphs allowing biased partitions. This paper shows their existence in $\mathbb{Z}^n$ and the paper by Gross and Grupel [3] implies the existence of biased a partition of the hypercube $\{0, 1\}^n$ exactly if $n = 2^k$ for some $k$.

**Question 5.2.** What discrete tori $\mathbb{Z}_m^n$ allow a biased partition?

Or more generally;

**Question 5.3.** What Cayley graphs allow a biased partition?

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