Reducible Fermi Surfaces for Non-symmetric Bilayer Quantum-Graph Operators

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Abstract. This work constructs a class of non-symmetric periodic Schrödinger operators on metric graphs (quantum graphs) whose Fermi, or Floquet, surface is reducible. The Floquet surface at an energy level is an algebraic set that describes all complex wave vectors admissible by the periodic operator at the given energy. The graphs in this study are obtained by coupling two identical copies of a periodic quantum graph by edges to form a bilayer graph. Reducibility of the Floquet surface for all energies ensues when the coupling edges have potentials belonging to the same asymmetry class. The notion of asymmetry class is defined in this article through the introduction of an entire spectral A-function \( a(\lambda) \) associated with a potential—two potentials belong to the same asymmetry class if their A-functions are identical. Symmetric potentials correspond to \( a(\lambda) = 0 \). If the potentials of the connecting edges belong to different asymmetry classes, then typically the Floquet surface is not reducible. An exception occurs when two copies of certain bipartite graphs are coupled; the Floquet surface in this case is always reducible. This includes AA-stacked bilayer graphene.

Key words: quantum graph, graph operator, periodic operator, bound state, embedded eigenvalue, reducible Fermi surface, local perturbation, Floquet transform, bilayer graphene

MSC: 47A75, 47B25, 39A70, 39A14, 47B39, 47B40, 39A12

1 Introduction

The Fermi surface for a periodic Schrödinger operator \(-\nabla^2 + q(x) (x \in \mathbb{R}^n)\) is the analytic set of complex wavevectors \((k_1, \ldots, k_n)\) for which the operator admits a (non-square-integrable) state at a fixed energy \(\lambda\). For periodic Schrödinger operators on metric graphs, known as quantum graphs, the Fermi surface is an algebraic set in the variables \((z_1, \ldots, z_n) = (e^{ik_1}, \ldots, e^{ik_n})\)—that is, the zero set of a polynomial in several variables. The reducibility of the Fermi surface into the union of two algebraic sets is important because it is intimately related to the existence of embedded eigenvalues induced by a local perturbation of the operator. It is proved by Kuchment and Vainberg [14] that reducibility is required for a local perturbation to engender a square-integrable eigenfunction with unbounded support at an energy that is embedded in the continuous spectrum.

The type of Fermi surface considered in this article is the zero set of a single Laurent polynomial in \((z_1, \ldots, z_n)\), and its reducibility is equivalent to the nontrivial factorability of this polynomial into a product of two Laurent polynomials. That a polynomial in several variables generically cannot be factored nontrivially indicates that a quantum graph must possess special features in order that a local defect be able to support an embedded eigenvalue with eigenfunction having unbounded support. The typical feature is symmetry. For a class of operators possessing reflectional symmetry, it is proved in [20 §3] that such eigenfunctions are possible due to the decomposition of the operator on even and odd states, which, in turn, effects a canonical reduction of the Fermi surface. The present work addresses the reducibility of the Fermi surface for a certain class of quantum graphs that are not decomposable by symmetry. Although the underlying metric graphs are reflectionally symmetric, the Schrödinger operators on them are not. The construction of embedded eigenvalues is not investigated here.