The relaxed linear micromorphic continuum: Existence, uniqueness and continuous dependence in dynamics

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Received 3 September 2013; accepted 18 November 2013

Abstract
We study well-posedness for the relaxed linear elastic micromorphic continuum model with symmetric Cauchy force-stresses and curvature contribution depending only on the micro-dislocation tensor. In contrast to classical micromorphic models our free energy is not uniformly pointwise positive definite in the control of the independent constitutive variables. Another interesting feature concerns the prescription of boundary values for the micro-distortion field: only tangential traces may be determined which are weaker than the usual strong anchoring boundary condition. There, decisive use is made of new coercive inequalities recently proved by Neff, Pauly and Witsch, and by Bauer, Neff, Pauly.

Dedicated to Prof. Antonio Di Carlo in recognition of his academic activity.

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and Starke. The new relaxed micromorphic formulation can be related to dislocation dynamics, gradient plasticity and seismic processes of earthquakes.

**Keywords**
Micromorphic elasticity, symmetric Cauchy stresses, dynamic problem, dislocation dynamics, gradient plasticity, dislocation energy, generalized continua, microstructure, micro-elasticity, non-smooth solutions, well-posedness, Cosserat couple modulus, wave propagation

### 1. Introduction

In this paper we show the well-posedness of a recently introduced new variant of the micromorphic model [1]. Micromorphic elasticity [2–7] is a generalized continuum formulation which tries to incorporate microstructure into the formulation of elasticity problems. This is necessary if one wants to describe size-effects (smaller is relatively stiffer), dispersion of waves phenomena etc. One of the best known such extension is the Cosserat model [8–12]. Our micromorphic model equations are linear and the question is permitted as to what new kind of model there can be after the general framework has been introduced by Mindlin and Eringen [2, 13, 14]. Indeed, our new relaxed micromorphic model is a subclass of the classical model which, however, violates pointwise uniform definiteness of the energy: the new energy is positive semi-definite only.

The relaxed micromorphic model [1] preserves full kinematical freedom (12 degrees of freedom) by reducing the model in order to obtain symmetric Cauchy force-stresses. In fact, beginning in mid-1950, Kröner tried to link the theory of static dialocations to the Cosserat model with asymmetric force stresses. However, since 1964 it became clear to Kröner [15, 16] that the force stress $\sigma$ in such a theory is always symmetric. The relaxed micromorphic model [1] reconciles Kröner’s rejection of antisymmetric force stresses in dislocation theory with the dislocation model of Eringen and Claus [17–19], and it is able to fully describe rotations of the microstructure and to fit a huge class of mechanical behaviors of materials with microstructure. As far as purely mechanical models are considered in the framework of linear elasticity, the need for introducing asymmetric stresses becomes rarer; see the discussions in Neff et al. [1]. The model of Eringen and Claus [17–19] contains the linear Cosserat model [9–12] with asymmetric force stresses upon suitable restriction.

The size effects involved in a natural way in the micromorphic models have recently received much attention in conjunction with nano-devices and foam-like structures. Also other microstructured materials, as granular assemblies are considered to be good candidates for the exploitation of continuum micromorphic theories. Indeed, even if in the literature the averaged models for granular assemblies are often looked for in the framework of classical Cauchy theory (see e.g. [20–22]), it becomes clearer that generalized continuum models are necessary to correctly describe the mechanical behavior of such physical systems (see e.g. [23, 24]. A geometrically nonlinear generalized continuum of micromorphic type in the sense of Eringen for the phenomenological description of metallic foams is given by Neff and Forest [25]. Moreover, in their 2007 paper [25] the authors proved the existence of minimizers and they identified the relevant effective material parameters. The modelling of growth phenomena is also a major challenge to mechanical and mathematical modeling. The question of growth in continuum growth models is examined from a rigorous mathematical approach by DiCarlo and Quiligotti [26].

The mathematical analysis of general micromorphic solids is well-established for infinitesimal, linear elastic models (see e.g. [27–30]). The only known existence results for the static geometrically nonlinear formulation are due to Neff [31] and to Mariano and Modica [5]. In fact, Mariano and Modica [5] treat general microstructures described by manifold-valued variables, even if they discuss essentially what is called by Neff macro-stability [31] (two other cases are treated in this work, one leads to fractures—a situation excluded by Mariano and Modica [5]—the other is left open). When the energy analyzed by Mariano and Modica is reduced to micromorphic materials in the splitted version considered by Neff [5], their coercivity assumptions result as more stringent than Neff’s (the blow up of the determinant of $\det F$ apart), so they restrict the material response. However, the direct comparison of the two existence results is not completely straightforward. As for the numerical implementation, see Mariano and Stazi [4] and the development by Klawonn et al. [32]. In this work [32] the original problem is decoupled into two separate problems. Corresponding domain-decomposition techniques for the subproblem related to balance of forces are investigated by Klawonn et al. [32]. On the other hand, in the classical theory of Mindlin–Eringen micromorphic elasticity, existence and uniqueness results were already
established by Sóos [27], by Hlaváček [28], by Ieşan and Nappa [29] and by Ieşan [30] assuming that the free
energy is a pointwise positive definite quadratic form. Ieşan [30] also gave a uniqueness result for the dynamic
problem without assuming that the free energy is a positive definite quadratic form. Moreover, in order to study
the existence of solution of the resulting system, Hlaváček [28], Ieşan and Nappa [29] and Ieşan [30] considered
the strong anchoring boundary condition. In contrast with the models considered until now, our free energy of
the relaxed model is not uniformly pointwise positive definite in the control of the constitutive variables. To be
more precise, let us recall that the elastic free energy from the Mindlin–Eringen micromorphic elasticity model
can be written as (see Sections 2 and 3 for notation and for the physical significations of the quantities)
\[
2\mathcal{E}(e, e_p, \gamma) = \left\langle \mathbb{C}, (\nabla u - P), (\nabla u - P) \right\rangle + \left\langle \mathbb{G}, \text{sym} P, \text{sym} P \right\rangle + \left\langle \mathbb{L}, \nabla P, \nabla P \right\rangle
+ 2\left\langle \mathbb{F}, \text{sym} P, (\nabla u - P) \right\rangle + 2\left\langle \mathbb{G}, \nabla P, (\nabla u - P) \right\rangle + 2\left\langle \mathbb{C}, \nabla P, \text{sym} P \right\rangle,
\]
where \( u \) is the displacement and \( P \) is the micro-distortion, \( \langle \cdot, \cdot \rangle \) is the standard Euclidean scalar product on \( \mathbb{R}^{3 \times 3} \), the constitutive coefficients are such that
\[
\mathbb{C} : \mathbb{R}^{3 \times 3} \to \mathbb{R}^{3 \times 3}, \quad \mathbb{G} : \text{Sym}(3) \to \text{Sym}(3), \quad \mathbb{F}, \mathbb{G} : \mathbb{R}^{3 \times 3} \to \text{Sym}(3), \quad \mathbb{L} : \mathbb{R}^{3 \times 3} \times \mathbb{R}^{3 \times 3} \to \mathbb{R}^{3 \times 3} \times \mathbb{R}^{3 \times 3},
\]
and the constitutive variables are
\[
e := \nabla u - P, \quad e_p := \text{sym} P, \quad \gamma := \nabla P.
\]
The elastic free energy of our relaxed model is given by
\[
2\mathcal{E}(e, e_p, \alpha) = \left\langle \mathbb{C}, \text{sym} (\nabla u - P), \text{sym} (\nabla u - P) \right\rangle + \left\langle \mathbb{G}, \text{sym} P, \text{sym} P \right\rangle + \left\langle \mathbb{L}_c, \text{Curl} P, \text{Curl} P \right\rangle,
\]
where
\[
\mathbb{C} : \text{Sym}(3) \to \text{Sym}(3), \quad \mathbb{G} : \text{Sym}(3) \to \text{Sym}(3), \quad \mathbb{L}_c : \mathbb{R}^{3 \times 3} \to \mathbb{R}^{3 \times 3},
\]
and the new set of constitutive variables is
\[
\varepsilon := \text{sym} (\nabla u - P), \quad \varepsilon_p = \text{sym} P, \quad \alpha = - \text{Curl} P.
\]
The comparison of the relaxed model with the classical Mindlin–Eringen [2] free energy is then achieved
through observing that
\[
\left\langle \mathbb{C}, X, X \right\rangle_{\mathbb{R}^{3 \times 3}} := \left\langle \mathbb{C}, \text{sym} X, \text{sym} X \right\rangle_{\mathbb{R}^{3 \times 3}},
\]
\[
\left\langle \mathbb{L}, \nabla P, \nabla P \right\rangle_{\mathbb{R}^{3 \times 3} \times \mathbb{R}^{3 \times 3}} := \left\langle \mathbb{L}_c, \text{Curl} P, \text{Curl} P \right\rangle_{\mathbb{R}^{3 \times 3}}.
\]
define only positive semi-definite tensors \( \mathbb{C} \) and \( \mathbb{L}_c \) in terms of positive definite tensors \( \mathbb{C} \) and \( \mathbb{L}_c \) acting on linear subspaces of \( \mathfrak{gl}(3) \cong \mathbb{R}^{3 \times 3} \).

We prove that the new micromorphic relaxed model [1] is still well-posed, i.e. we study the continuous
dependence of solution with respect to the initial data and supply terms and existence and uniqueness of the
solution. These results were announced previously by Neff et al. [1]. All the results are obtained for a standard
set of tangential boundary conditions for the micro-distortion, i.e. \( P, \tau = 0 \) \( (P \times n = 0) \) on \( \partial \Omega \), and not the
usual strong anchoring condition \( P = 0 \) on \( \partial \Omega \). The solution space for the elastic distortion and micro-distortion
is only \( H(\text{Curl}; \Omega) \) and for the macroscopic displacement \( u \in H(\Omega) \). For non-smooth external data we expect
slip lines. Using a fundamental identity which characterizes the conservation of the total energy associated
with the solution of the dynamical problem of the relaxed micromorphic model we prove the uniqueness and the
continuous dependence of the solution with respect to the initial data. These results show that the considered
model is in concordance with physical reality. Then, we transform the initial boundary value problem in an
abstract evolution equation in an appropriate Hilbert space and we use the results of the semigroups theory
of linear operators [33, 34] in order to obtain the existence results. The main point in establishing the desired
estimates is represented by the new coercive inequalities recently proved by Neff et al. [35–37] and by Bauer
et al. [38–40] (see also [41]). The results established in our paper can be easily extended to theories which include electromagnetic and thermal interactions [42–45].

In the paper [46] we investigate the salient features of the new relaxed model with respect to wave-propagation phenomena compared with the classical Mindlin–Eringen micromorphic model [2, 13, 14]. In particular, we show that the considered relaxed model is able to account for the description of frequency band-gaps which are observed in particular microstructured materials as phononic crystals and lattice structures. In particular, such materials can inhibit wave propagation in particular frequency ranges (band-gaps) and could be used as an alternative to piezoelectric materials which are used today for vibration control and which are for this reason extensively studied in the literature (see e.g. [47–52]). Moreover, in a forthcoming paper we will deal with the static model and consider the elliptic regularity question. The numerical treatment of our new model needs FEM-discretisations in \( H(\text{curl}; \Omega) \). This will be left for future work.

2. Notation

For \( a, b \in \mathbb{R}^3 \) we let \( \langle a, b \rangle_{\mathbb{R}^3} \) denote the scalar product on \( \mathbb{R}^3 \) with associated vector norm \( \| a \|_{\mathbb{R}^3}^2 = \langle a, a \rangle_{\mathbb{R}^3} \). We denote by \( \mathbb{R}^{3\times3} \) the set of real \( 3 \times 3 \) second order tensors, written with capital letters. The standard Euclidean scalar product on \( \mathbb{R}^{3\times3} \) is given by \( \langle X, Y \rangle_{\mathbb{R}^{3\times3}} = \text{tr}(XY^T) \), and thus the Frobenius tensor norm is \( \| X \|_F^2 = \langle X, X \rangle_{\mathbb{R}^{3\times3}} \). In the following we omit the index \( \mathbb{R}^{3\times3} \). The identity tensor on \( \mathbb{R}^{3\times3} \) will be denoted by \( \mathbb{1} \), so that \( \text{tr}(X) = \langle X, \mathbb{1} \rangle \). We let \( \text{Sym}(3) \) denote the set of symmetric tensors. We adopt the usual abbreviations of Lie-algebra theory, i.e. \( \text{so}(3) := \{ X \in \mathbb{R}^{3\times3} \mid X^T = -X \} \) is the Lie-algebra of skew symmetric tensors and \( \text{sl}(3) := \{ X \in \mathbb{R}^{3\times3} \mid \text{tr}(X) = 0 \} \) is the Lie-algebra of traceless tensors. For all \( X \in \mathbb{R}^{3\times3} \) we set \( \text{sym} X = \frac{1}{2}(X + X^T) \in \text{Sym} \), \( \text{skew} X = \frac{1}{2}(X - X^T) \in \text{so}(3) \) and the deviatoric part \( \text{dev} X = X - \frac{1}{3} \text{tr}(X) \mathbb{1} \in \text{sl}(3) \), and we have the orthogonal Cartan-decomposition of the Lie-algebra \( \text{gl}(3) \)

\[
\text{gl}(3) = \{ \text{sl}(3) \cap \text{Sym}(3) \} \oplus \text{so}(3) \oplus \mathbb{R} \cdot \mathbb{1},
\]

\[
X = \text{dev} \text{sym} X + \text{skew} X + \frac{1}{3} \text{tr}(X) \mathbb{1}.
\]  

By \( C^\infty_0(\Omega) \) we denote infinitely differentiable functions with compact support in \( \Omega \). We employ the standard notation of Sobolev spaces, i.e. \( L^2(\Omega), H^{1,2}(\Omega), H^{1,1}(\Omega) \), which we use indifferently for scalar-valued functions as well as for vector-valued and tensor-valued functions. Throughout this paper (when we do not specify else) Latin subscripts take the values 1, 2, 3. Typical conventions for differential operations are implied such as comma followed by a subscript to denote the partial derivative with respect to the corresponding cartesian coordinate, while \( t \) after a comma denotes the partial derivative with respect to the time. The usual Lebesgue spaces of square integrable functions, vector or tensor fields on \( \Omega \) with values in \( \mathbb{R}, \mathbb{R}^3 \) or \( \mathbb{R}^{3\times3} \), respectively will be denoted by \( L^2(\Omega) \). Moreover, we introduce the standard Sobolev spaces [53–55]

\[
H^1(\Omega) = \{ u \in L^2(\Omega) \mid \text{grad} u \in L^2(\Omega) \}, \quad \text{grad} = \nabla,
\]

\[
\| u \|_{H^1(\Omega)}^2 := \| u \|_{L^2(\Omega)}^2 + \| \text{grad} u \|_{L^2(\Omega)}^2,
\]

\[
H(\text{curl}; \Omega) = \{ v \in L^2(\Omega) \mid \text{curl} v \in L^2(\Omega) \}, \quad \text{curl} = \nabla \times,
\]

\[
\| v \|_{H(\text{curl}; \Omega)}^2 := \| v \|_{L^2(\Omega)}^2 + \| \text{curl} v \|_{L^2(\Omega)}^2,
\]  

of functions \( u \) or vector fields \( v \), respectively.

Furthermore, we introduce their closed subspaces \( H^1_0(\Omega) \), and \( H_0(\text{curl}; \Omega) \) as completion under the respective graph norms of the scalar valued space \( C^\infty_0(\Omega) \), the set of smooth functions with compact support in \( \Omega \). Roughly speaking, \( H^1_0(\Omega) \) is the subspace of functions \( u \in H^1(\Omega) \) which are zero on \( \partial \Omega \), while \( H_0(\text{curl}; \Omega) \) is the subspace of vectors \( v \in H(\text{curl}; \Omega) \) which are normal at \( \partial \Omega \) (see [35–37, 54, 55]). For vector fields \( v \) with components in \( H^1(\Omega) \) and tensor fields \( P \) with rows in \( H(\text{curl}; \Omega) \), i.e.

\[
v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}, \quad v_i \in H^1(\Omega), \quad P = \begin{pmatrix} p_1^1 \\ p_1^2 \\ p_1^3 \\ p_2^1 \\ p_2^2 \\ p_2^3 \end{pmatrix}, \quad P_i \in H(\text{curl}; \Omega)
\]
we define

\[
\text{Grad} \, v = \begin{pmatrix}
\text{grad}^T v_1 \\
\text{grad}^T v_2 \\
\text{grad}^T v_3
\end{pmatrix}, \quad
\text{Curl} \, P = \begin{pmatrix}
\text{curl}^T P_1 \\
\text{curl}^T P_2 \\
\text{curl}^T P_3
\end{pmatrix}, \quad
\text{Grad} \, P = (\text{Grad} \, P_1, \text{Grad} \, P_2, \text{Grad} \, P_3).
\] (4)

We note that \(v\) is a vector field, whereas \(P\), \(\text{Curl} \, P\) and \(\text{Grad} \, v\) are second order tensor fields. The corresponding Sobolev spaces will be denoted by

\[
H(\text{Grad} \, \Omega) \quad \text{and} \quad H(\text{Curl} \, \Omega).
\] (5)

We recall that if \(C\) is a fourth order tensor and \(X \in \mathbb{R}^{3 \times 3}\), then \(C \cdot X \in \mathbb{R}^{3 \times 3 \times 3 \times 3}\) with the components

\[
(C \cdot X)_{ij} = \sum_{k=1}^{3} \sum_{l=1}^{3} C_{ijkl} X_{kl}.
\] (6)

3. Formulation of the relaxed micromorphic continuum model

We consider a micromorphic continuum which occupies a bounded domain \(\Omega\) and is bounded by the piecewise smooth surface \(\partial \Omega\). Let \(T > 0\) be a given time. The motion of the body is referred to a fixed system of rectangular Cartesian axes \(Ox_i\) \((i = 1, 2, 3)\). The micro-distortion (plastic distortion) \(P = (P_{ij}) : \Omega \times [0, T] \to \mathbb{R}^{3 \times 3}\) describes the substructure of the material which can rotate, stretch, shear and shrink, while \(u = (u_i) : \Omega \times [0, T] \to \mathbb{R}^3\) is the displacement of the macroscopic material points.

The quantities involved in our new relaxed micromorphic continuum model have the following physical signification:

- \((u, P)\) are the kinematical variables.
- \(u\) is the displacement vector (translational degrees of freedom).
- \(P\) is the micro-distortion tensor (plastic distortion, second order, non-symmetric).
- \(\sigma\) is the Cauchy stresses (second order, symmetric).
- \(s\) is the microstress tensor (second order, symmetric).
- \(m\) is the moment stress tensor (micro-hyperstress tensor, third order, in general non-symmetric).
- \(f\) is the body force.
- \(M\) is the body moment tensor (second order, non-symmetric).
- \(e := \nabla u - P\) is the elastic distortion (relative distortion, second order, non-symmetric).
- \(\varepsilon_e := \text{sym} \, e = \text{sym}(\nabla u - P)\) is the elastic strain tensor (second order, symmetric).
- \(\varepsilon_p := \text{sym} \, P\) is the micro-strain tensor (plastic strain, second order, symmetric).
- \(\alpha := \text{Curl} \, e = - \text{Curl} \, P\) the micro-dislocation tensor (second order).

We consider here a relaxed version of the classical micromorphic model with \(\sigma\) symmetric and drastically reduced numbers of constitutive coefficients. More precisely, our model is a subset of the classical micromorphic model in which we allow the usual micromorphic tensors [2] to become positive-semidefinite only [1]. The proof of the well-posedness of this model necessitates the application of new mathematical tools [35–41]. The curvature dependence is reduced to a dependence only on the micro-dislocation tensor \(\alpha := \text{Curl} \, e = - \text{Curl} \, P \in \mathbb{R}^{3 \times 3}\) instead of \(\gamma = \nabla P \in \mathbb{R}^{27} = \mathbb{R}^{3 \times 3 \times 3 \times 3}\), and the local response is reduced to a dependence on the symmetric part of the elastic distortion (relative distortion) \(\varepsilon_e = \text{sym} \, e = \text{sym}(\nabla u - P)\), while the full kinematical degrees of freedom for \(u\) and \(P\) are kept, notably rotation of the microstructure remains possible.
Our new set of independent constitutive variables for the relaxed micromorphic model is thus
\[ \varepsilon_c = \text{sym}(\nabla u - P), \quad \varepsilon_p = \text{sym} P, \quad \alpha = -\text{Curl} P. \] (7)

The system of partial differential equations which corresponds to this special linear anisotropic micromorphic continuum is derived from the following free energy
\[ 2 \mathcal{E}(\varepsilon_c, \varepsilon_p, \alpha) = (\mathbb{C}, \varepsilon_c, \varepsilon_c) + (\mathbb{H}, \varepsilon_p, \varepsilon_p) + (\mathbb{L}_c, \alpha, \alpha) \]
\[ = (\mathbb{C}, \varepsilon_c, \varepsilon_c) + (\mathbb{H}, \text{sym}(\nabla u - P), \text{sym}(\nabla u - P)) + (\mathbb{L}_c, \text{sym} P, \text{sym} P) + (\mathbb{L}_c, \text{Curl} P, \text{Curl} P), \] (8)

where \( \mathbb{C} : \Omega \to L(\mathbb{R}^{3 \times 3}, \mathbb{R}^{3 \times 3}), \mathbb{H} : \Omega \to L(\mathbb{R}^{3 \times 3}, \mathbb{R}^{3 \times 3}) \) and \( \mathbb{L} : \mathbb{R}^{3 \times 3} \to \mathbb{R}^{3 \times 3} \) are fourth order elasticity tensors, positive definite and functions of class \( C^1(\Omega) \).

Further we assume, without loss of generality, that \( \mathbb{C} \), \( \mathbb{L}_c \) and \( \mathbb{H} \) are positive definite. Then, there are positive numbers \( c_M, c_m \) (the maximum and minimum elastic moduli for \( \mathbb{C} \)), \( (L_c)_M, (L_c)_m \) (the maximum and minimum moduli for \( \mathbb{L}_c \)) and \( h_M, h_m \) (the maximum and minimum moduli for \( \mathbb{H} \)) such that
\[ c_m \|X\|^2 \leq (\mathbb{C} X, X) \leq c_M \|X\|^2 \quad \text{for all } X \in \text{Sym}(3), \]
\[ (L_c)_m \|X\|^2 \leq (\mathbb{L}_c X, X) \leq (L_c)_M \|X\|^2 \quad \text{for all } X \in \mathbb{R}^{3 \times 3}, \]
\[ h_m \|X\|^2 \leq (\mathbb{H} X, X) \leq h_M \|X\|^2 \quad \text{for all } X \in \text{Sym}(3). \] (11)

Further we assume, without loss of generality, that \( c_M, c_m, (L_c)_M, h_M, h_m \) and \( (L_c)_m \) are constants.

We introduce the action functional of the considered system to be defined as
\[ \Lambda = \int_0^T \int_{\Omega} (\mathcal{K} + \sigma \dot{x} - \mathcal{E}) \, dx \, dt, \quad \mathcal{K} = \frac{1}{2} \|u_d\|^2 + \frac{1}{2} \|P_d\|^2, \quad \sigma = \langle f, u_d \rangle + \langle M, P_d \rangle, \] (12)

where \( \mathcal{K} \) and \( \sigma \) are the kinetic energy and the work done by external loads on the body, respectively, \( f : \Omega \times [0, T] \to \mathbb{R}^3 \) describes the body force and \( M : \Omega \times [0, T] \to \mathbb{R}^{3 \times 3} \) describes the external body moment.

We consider the weaker boundary conditions
\[ u(x, t) = 0, \quad \text{and the tangential condition } \quad P_i(x, t) \times n(x) = 0, \quad i = 1, 2, 3, \quad \langle x, t \rangle \in \partial \Omega \times [0, T], \] (13)

where \( \times \) denotes the vector product, \( n \) is the unit outward normal vector at the surface \( \partial \Omega \), \( P_i, i = 1, 2, 3 \) are the rows of \( P \). The model is driven by nonzero initial conditions
\[ u(x, 0) = u_0(x), \quad \dot{u}(x, 0) = \dot{u}_0(x), \quad P(x, 0) = P_0(x), \quad \dot{P}(x, 0) = \dot{P}_0(x), \quad x \in \overline{\Omega}, \] (14)

where \( u_0, \dot{u}_0, P_0 \) and \( \dot{P}_0 \) are prescribed functions.

**Remark 1** Since \( P \) is determined in \( \mathbb{H}(\text{Curl} ; \Omega) \), in our relaxed model the only possible description of boundary value is in terms of tangential traces, i.e. \( P \tau = 0 \) for all tangential vectors \( \tau \) at \( \partial \Omega \). This follows from the standard theory of the \( \mathbb{H}(\text{Curl} ; \Omega) \)-space.
Imposing the first variation of the action functional to be zero (Hamilton–Kirchhoff principle), integrating by parts a suitable number of times and considering arbitrary variations $\delta u$ and $\delta P$ of the basic kinematic fields, we obtain that the system of partial differential equations of our relaxed micromorphic continuum model is

\[
\begin{align*}
    u_{,\mu} = \text{Div}[\mathbb{C} \text{ sym}(\nabla u - P)] + f, & \quad \text{balance of forces,} \\
    P_{,\mu} = -\text{Curl}[\mathbb{L}_c \text{Curl} P] + \mathbb{C} \text{ sym}(\nabla u - P) - \mathbb{H} \text{ sym} P + M, & \quad \text{balance of moment stresses}
\end{align*}
\]

in $\Omega \times [0, T]$. For simplicity, the system (15) is considered in a normalized form.

Our new approach, in marked contrast to classical asymmetric micromorphic models, features a symmetric Cauchy stress tensor $\sigma = \mathbb{C} \text{ sym}(\nabla u - P)$. Therefore, the linear Cosserat approach ([9]: $\mu_c > 0$) is excluded here.

The relaxed formulation considered in the present paper still shows size effects and smaller samples are relatively stiffer. It is clear to us that for this reduced model of relaxed micromorphic elasticity in microscopic scales we obtain that the system of partial differential equations of our relaxed micromorphic continuum model is

\[
\begin{align*}
    u_{,\mu} = \text{Div} \sigma + f, & \quad \text{balance of forces,} \\
    P_{,\mu} = -\text{Curl} m + \sigma - s + M & \quad \text{balance of moment stresses}
\end{align*}
\]

where

\[
\begin{align*}
    \sigma &= 2\mu_c \text{ sym}(\nabla u - P) + \lambda_c \text{tr}(\nabla u - P)\cdot I, \\
    m &= \alpha_1 \text{ dev} \text{ sym} \text{ Curl} P + \alpha_2 \text{ skew} \text{ Curl} P + \alpha_3 \text{ tr}(\text{Curl} P)\cdot I, \\
    s &= 2\mu_h \text{ sym} P + \lambda_h \text{tr}(P)\cdot I.
\end{align*}
\]

Thus, we obtain the complete system of linear partial differential equations in terms of the kinematical unknowns $u$ and $P$

\[
\begin{align*}
    u_{,\mu} = \text{Div}[2\mu_c \text{ sym}(\nabla u - P) + \lambda_c \text{tr}(\nabla u - P)\cdot I] + f, & \quad \text{balance of forces,} \\
    P_{,\mu} = -\text{Curl}[\alpha_1 \text{ dev} \text{ sym} \text{ Curl} P + \alpha_2 \text{ skew} \text{ Curl} P + \alpha_3 \text{ tr}(\text{Curl} P)\cdot I] 
    + 2\mu_c \text{ sym}(\nabla u - P) + \lambda_c \text{tr}(\nabla u - P)\cdot I - 2\mu_h \text{ sym} P - \lambda_h \text{tr}(P)\cdot I + M & \quad \text{balance of moment stresses}
\end{align*}
\]

In this model, the asymmetric parts of $P$ are entirely due only to moment stresses and applied body moments! In this sense, the macroscopic and microscopic scales are neatly separated.

The positive definiteness required for the tensors $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{L}_c$ implies for isotropic materials the following restriction upon the parameters $\mu_c, \lambda_c, \mu_h, \lambda_h, \alpha_1, \alpha_2$ and $\alpha_3$

\[
\mu_c > 0, \quad 2\mu_c + 3\lambda_c > 0, \quad \mu_h > 0, \quad 2\mu_h + 3\lambda_h > 0, \quad \alpha_1 > 0, \quad \alpha_2 > 0, \quad \alpha_3 > 0.
\]

Therefore, positive definiteness for our isotropic model does not involve extra nonlinear side conditions [2, 57].

If, by abuse of our notation by neglect of our guiding assumption, we add the anti-symmetric term $2\mu_c \text{ skew}(\nabla u - P)$ in the expression of the Cauchy stress tensor $\sigma$, where $\mu_c \geq 0$ is the Cosserat couple modulus, then our analysis works for $\mu_c \geq 0$. The model in which $\mu_c > 0$ is the isotropic Eringen–Claus model for dislocation dynamics [17–19] and it is derived from the following free energy

\[
\mathcal{E}(e, e_P, \alpha) = \mu_c \text{ sym}(\nabla u - P)^2 + \mu_c \text{ skew}(\nabla u - P)^2 + \frac{\lambda_c}{2} \text{tr}(\nabla u - P)^2 + \mu_h \text{ sym} P^2 + \frac{\lambda_h}{2} \text{tr}(P)^2
\]

\[
+ \frac{\alpha_1}{2} \text{ dev} \text{ sym} \text{ Curl} P^2 + \frac{\alpha_2}{2} \text{ skew} \text{ Curl} P^2 + \frac{\alpha_3}{2} \text{tr}(\text{Curl} P)^2.
\]

For $\mu_c > 0$ and if the other inequalities (19) are satisfied, the existence and uniqueness follow along the classical lines. There is no need for any new integral inequalities. For the sake of simplicity, we only present in the present
paper well-posedness results for the relaxed model. These results still hold for the complete model and can be generalized with some additional calculations.

For the mathematical treatment of the linear relaxed model there arises the need for new integral type inequalities which we present in the next section. Using the new results established by Neff et al. [35–37] and by Bauer et al. [38–40] we are now able to manage also energies depending on the dislocation energy and having symmetric Cauchy stresses.

4. New Poincaré and Korn type estimates

In potential theory use is made of Poincaré’s inequality, that is

\[ \|u\|_{L^2(\Omega)} \leq c_p \| \text{grad } u\|_{L^2(\Omega)}, \] (21)

for all functions \( u \in H^1_0(\Omega) \) with some constants \( c_p > 0 \), to bound a scalar potential in terms of its gradient.

In linearized elasticity theory Korn’s inequality is used, that is

\[ \|\text{grad } u\|_{L^2(\Omega)} \leq c_k \| \text{sym } \text{grad } u\|_{L^2(\Omega)}, \] (22)

for all functions \( u \in H^1_0(\Omega) \) with some constants \( c_k > 0 \), for bounding the deformation of an elastic medium in terms of the symmetric strains.

In electro-magnetic theory the Maxwell inequality, that is

\[ \|u\|_{L^2(\Omega)} \leq c_m (\|\text{curl } u\|_{L^2(\Omega)} + \|\text{div } u\|_{L^2(\Omega)}), \] (23)

for all functions \( u \in H^0(\text{curl}; \Omega) \cap H(\text{div}; \Omega) \) with some constants \( c_m > 0 \), is used to bound the electric and magnetic field in terms of the electric charge and current density, respectively.

In works by Neff et al. [35–37], for tensor fields \( P \in H(\text{curl}; \Omega) \) the following seminorm \( ||| \cdot ||| \) is defined

\[ |||P|||^2 := \| \text{sym } P\|_{L^2(\Omega)}^2 + \|\text{Curl } P\|_{L^2(\Omega)}^2. \] (24)

From the same works [35–37] we have the following result:

**Theorem 4.1** There exists a constant \( \hat{c} \) such that

\[ \|P\|_{L^2(\Omega)} \leq \hat{c} |||P|||, \] (25)

for all \( P \in H(\text{curl}; \Omega) \) with vanishing restricted tangential trace on \( \partial \Omega \), i.e. \( P \tau = 0 \) on \( \partial \Omega \).

Moreover, we have

**Theorem 4.2** On \( H_0(\text{curl}; \Omega) \) the norms \( \| \cdot \|_{H(\text{curl}; \Omega)} \) and \( ||| \cdot ||| \) are equivalent. In particular, \( ||| \cdot ||| \) is a norm on \( H_0(\text{curl}; \Omega) \), and there exists a positive constant \( c \), such that

\[ c \|P\|_{H(\text{curl}; \Omega)} \leq |||P|||. \] (26)

for all \( P \in H_0(\text{curl}; \Omega) \).

Moreover, in a forthcoming paper [39] (see also [35, 38]) the following results are proved:

**Theorem 4.3** There exists a positive constant \( C_{DD} \), only depending on \( \Omega \), such that for all \( P \in H_0(\text{curl}; \Omega) \) the following estimate holds:

\[ \|\text{Curl } P\|_{L^2(\Omega)} \leq C_{DD} \| \text{dev } \text{Curl } P\|_{L^2(\Omega)}. \] (27)

**Theorem 4.4** There exists a positive constant \( C_{DSDC} \), only depending on \( \Omega \), such that for all \( P \in H_0(\text{curl}; \Omega) \) the following estimate holds:

\[ \|P\|_{L^2(\Omega)} \leq C_{DSDC} (\| \text{dev } \text{sym } P\|_{L^2(\Omega)}^2 + \| \text{dev } \text{Curl } P\|_{L^2(\Omega)}^2). \] (28)
Corollary 4.5 For all $P \in H_0(\text{Curl} ; \Omega)$ the following estimate holds:

$$\|P\|_{L^2(\Omega)} + \|\text{Curl} \, P\|_{L^2(\Omega)} \leq (C_{DSC} + C_{DD})(\|\text{dev sym} \, P\|_{L^2(\Omega)}^2 + \|\text{dev Curl} \, P\|_{L^2(\Omega)}^2). \tag{29}$$

We have to remark that the above corollary proves that on $H_0(\text{Curl} ; \Omega)$ the norms $\|\cdot\|_{H(\text{Curl}, \Omega)}$ and $||| \cdot |||$ are equivalent.

Theorem 4.6 There exists a positive constant $C_{DSG}$, only depending on $\Omega$, such that for all $u \in H_1^0(\Omega)$ the following estimate holds:

$$\|\nabla u\|_{L^2(\Omega)} \leq C_{DSG} \|\text{dev sym} \, \nabla u\|_{L^2(\Omega)}. \tag{30}$$

The estimates given by the above theorems will be essential in the study of our relaxed linear micromorphic model.

5. Conservation law, uniqueness, continuous dependence and existence

5.1. Energy conservation

In this subsection we establish a fundamental identity which characterizes the conservation of the total energy associated with the solution of the dynamic problem $(P)$ defined by the equations (15), the boundary conditions (13) and the initial conditions (14). Let us consider a solution $\{u, P\}$ of the problem $(P)$ corresponding to the given data $I = \{f, M, u_0, \dot{u}_0, P_0, \dot{P}_0\}$.

We define the total energy

$$2E(t) = \int_{\Omega} \left( \|u_t\|^2 + \|P_t\|^2 + \langle \text{C. sym}(\nabla u - P), \text{sym}(\nabla u - P) \rangle 
+ \langle \text{H. sym} \, P, \text{sym} \, P \rangle + \langle \text{L. Curl} \, P, \text{Curl} \, P \rangle \right) \, dv, \tag{31}$$

and the power function

$$\Pi(t) = \int_{\Omega} (\langle f, u_t \rangle + \langle M, P_t \rangle) \, dv. \tag{32}$$

Lemma 5.1 (Conservation law) Let $\{u, P\}$ be a solution of the problem $(P)$ corresponding to the loads $I = \{f, M, u_0, \dot{u}_0, P_0, \dot{P}_0\}$. Then, for every time $t \in [0, T]$, we have

$$E(t) = E(0) + \int_0^t \Pi(s) \, ds. \tag{33}$$

Proof. First of all, let us recall the identities

$$\text{div}(\psi A) = \langle A, \text{grad} \, \psi \rangle + \psi \text{ div} \, A, \tag{34}$$
$$\text{div} \, (A \times B) = \langle B, \text{curl} \, A \rangle - \langle A, \text{curl} \, B \rangle,$$

for all $C^1$-functions $\psi : \Omega \to \mathbb{R}$ and $A, B : \Omega \to \mathbb{R}^3$, where $\times$ is the cross product. Hence

$$\text{div}(\varphi_i Q_i) = \langle Q_i, \nabla \varphi_i \rangle + \varphi_i \text{ div} \, Q_i \quad \text{not summed}, \tag{35}$$
$$\text{div} \, (R_i \times S_i) = \langle S_i, \text{curl} \, R_i \rangle - \langle R_i, \text{curl} \, S_i \rangle \quad \text{not summed},$$

for all $C^1$-functions $\varphi_i : \Omega \to \mathbb{R}$ and $Q_i, P_i, S_i : \Omega \to \mathbb{R}^3$, where $\varphi_i$ are the components of the vector $\varphi$ and $Q_i, P_i, S_i$ are the rows of the matrix $Q, P$ and $S$, respectively. We choose

$$\varphi = u_t, \quad Q = \text{C. sym}(\nabla u - P), \tag{36}$$
and we obtain
\[ \text{div}(u_{i,t}[\mathbb{C} \text{ sym}(\nabla u - P)]_i) = \langle [\mathbb{C} \text{ sym}(\nabla u - P)], \nabla u_{i,j} \rangle + u_{i,t} \text{ div } [\mathbb{C} \text{ sym}(\nabla u - P)]_i \text{ not summed.} \] (37)

This leads to
\[ \sum_{i=1}^{3} u_{i,t} \text{ div } [\mathbb{C} \text{ sym}(\nabla u - P)]_i = \sum_{i=1}^{3} \text{div}(u_{i,t}[\mathbb{C} \text{ sym}(\nabla u - P)]_i) - \sum_{i=1}^{3} \langle [\mathbb{C} \text{ sym}(\nabla u - P)], \nabla u_{i,j} \rangle. \] (38)

Thus
\[ \langle \text{Div } [\mathbb{C} \text{ sym}(\nabla u - P)], u_{i,j} \rangle = \sum_{i=1}^{3} \text{div}(u_{i,t}[\mathbb{C} \text{ sym}(\nabla u - P)]_i) - \langle \mathbb{C} \text{ sym}(\nabla u - P), \text{sym } \nabla u_{i,j} \rangle. \] (39)

If we take in (35)
\[ R_i = [\mathbb{L}_c \text{ Curl } P]_i, \quad S_i = P_i, \] (40)

we have
\[ \sum_{i=1}^{3} \text{div } ([\mathbb{L}_c \text{ curl } P]_i \times P_{i,t}) = \sum_{i=1}^{3} \langle P_{i,t}, \text{curl } [\mathbb{L}_c \text{ Curl } P]_i \rangle - \sum_{i=1}^{3} \langle [\mathbb{L}_c \text{ Curl } P], \text{curl } P_{i,t} \rangle. \]

Hence, we obtain
\[ \langle P_{i,t}, \text{Curl } (\mathbb{L}_c \text{ (Curl } P)) \rangle = \sum_{i=1}^{3} \text{div } ([\mathbb{L}_c \text{ Curl } P]_i \times P_{i,t}) + \langle [\mathbb{L}_c \text{ Curl } P, \text{Curl } P]_i \rangle. \] (41)

Using (15), (39) and (41) we have
\[ \langle u_{i,t}, u_{i,j} \rangle + \langle P_{i,t}, P_{i,j} \rangle = \langle \text{Div } (\mathbb{C} \text{ sym}(\nabla u - P)), u_{i,j} \rangle + \langle f, u_{i,t} \rangle \] (42)
\[ - \langle \text{Curl } (\mathbb{L}_c \text{ Curl } (P)), P_{i,j} \rangle + \langle \mathbb{C} \text{ sym}(\nabla u - P), P_{i,j} \rangle - \langle \text{sym } P, P_{i,j} \rangle + \langle M, P_{i,j} \rangle \]
\[ = \sum_{i=1}^{3} \text{div}(u_{i,t}[\mathbb{C} \text{ sym}(\nabla u - P)]_i) - \langle \mathbb{C} \text{ sym}(\nabla u - P), \text{sym } \nabla u_{i,j} \rangle + \langle f, u_{i,t} \rangle \]
\[ - \sum_{i=1}^{3} \text{div } ([\mathbb{L}_c \text{ Curl } P]_i \times P_{i,t}) - \langle [\mathbb{L}_c \text{ Curl } P, \text{Curl } P]_i \rangle \]
\[ + \langle \mathbb{C} \text{ sym}(\nabla u - P), P_{i,j} \rangle - \langle \text{sym } P, P_{i,j} \rangle + \langle M, P_{i,j} \rangle \]
\[ = - \langle \mathbb{C} \text{ sym}(\nabla u - P), \text{sym } (\nabla u_{i,j} - P_{i,j}) \rangle - \langle [\mathbb{L}_c \text{ Curl } P, \text{Curl } P]_i \rangle - \langle \text{sym } P, \text{sym } P_{i,j} \rangle \]
\[ + \sum_{i=1}^{3} \text{div}(u_{i,t}[\mathbb{C} \text{ sym}(\nabla u - P)]_i) + \sum_{i=1}^{3} \text{div } (P_{i,t} \times [\mathbb{L}_c \text{ Curl } P]_i) \]
\[ + \langle f, u_{i,j} \rangle + \langle M, P_{i,j} \rangle. \]

Hence, using the symmetries (10), we have
\[ \frac{1}{2} \frac{\partial}{\partial t} \left( \|u_{i,t}\|^2 + \|P_{i,t}\|^2 + \langle \mathbb{C} \text{ sym}(\nabla u - P), \text{sym}(\nabla u - P) \rangle \right) \]
\[ + \langle \text{sym } P, \text{sym } P \rangle + \langle [\mathbb{L}_c \text{ Curl } P, \text{Curl } P] \rangle \] (43)
\[ = \sum_{i=1}^{3} \text{div}(u_{i,t}[\mathbb{C} \text{ sym}(\nabla u - P)]_i) + \sum_{i=1}^{3} \text{div } (P_{i,t} \times [\mathbb{L}_c \text{ Curl } P]_i) \]
\[ + \langle f, u_{i,j} \rangle + \langle M, P_{i,j} \rangle. \]
Therefore, using the divergence theorem, it follows that
\[
\frac{d}{dt} E(t) = \int_{\partial \Omega} \left( \sum_{i=1}^{3} \langle [\mathbb{C}. \text{sym} (\nabla u - P)], u_{i}, n \rangle + \sum_{i=1}^{3} \langle P_{i}, [\mathbb{L}. \text{Curl} P]_{i}, n \rangle \right) da \tag{44}
\]
\[+ \int_{\Omega} \langle f, u \rangle + \langle M, P \rangle \rangle dv
\]
\[= \int_{\partial \Omega} \left( \sum_{i=1}^{3} \langle [\mathbb{C}. \text{sym} (\nabla u - P)], u_{i}, n \rangle + \sum_{i=1}^{3} \langle [\mathbb{L}. \text{Curl} P]_{i}, n \times P_{i} \rangle \right) da
\]
\[+ \int_{\Omega} \langle f, u \rangle + \langle M, P \rangle \rangle dv,
\]
so that, in view of the boundary conditions \(u = 0, P \cdot \tau = 0\) on \(\partial \Omega\) and by integration over \([0, t]\), the proof is complete. \[\square\]

5.2. Continuous dependence of solution and uniqueness

Throughout this section we study the continuous dependence of solution of the problem \((P)\) with respect to the initial and the body loads. To this aim, let us first prove the following lemma:

**Lemma 5.2** There exists a constant \(a_1\) such that
\[
a_1 \left( \| \text{sym} \nabla u \|^2 + \| \text{sym} P \|^2 \right) \leq \langle \mathbb{C}. \text{sym} (\nabla u - P), \text{sym} (\nabla u - P) \rangle + \langle \mathbb{H}. \text{sym} P, \text{sym} P \rangle \tag{45}
\]
for all \(u \in H^1(\Omega)\) and \(P \in H(\text{Curl}; \Omega)\).

**Proof.** We start the proof with the remark that the arithmetic–geometric inequality and the positivity of \(\mathbb{C}\) imply
\[
\langle \mathbb{C}. \text{sym} (\nabla u - P), \text{sym} (\nabla u - P) \rangle \geq c_m \| \text{sym} \nabla u \|^2 = c_m \| \text{sym} (\nabla u - P), \text{sym} (\nabla u - P) \|
\geq c_m \left( \| \nabla u \|^2 + \| \text{sym} P \|^2 - 2 \langle \text{sym} \nabla u, \text{sym} P \rangle \right)
\geq c_m \left[ (1 - \delta) \| \nabla u \|^2 + \left( 1 - \frac{1}{\delta} \right) \| \text{sym} P \|^2 \right], \tag{46}
\]
for all \(\delta > 0\). Moreover, we have
\[
\langle \mathbb{H}. \text{sym} P, \text{sym} P \rangle \geq h_m \| \text{sym} P \|^2. \tag{47}
\]
Hence, we deduce that
\[
c_m (1 - \delta) \| \nabla u \|^2 + \left( c_m + h_m - \frac{c_m}{\delta} \right) \| \text{sym} P \|^2 \leq \langle \mathbb{C}. \text{sym} (\nabla u - P), \text{sym} (\nabla u - P) \rangle + \langle \mathbb{H}. \text{sym} P, \text{sym} P \rangle, \tag{48}
\]
and the proof is complete. \[\square\]

To establish an estimate describing the continuous dependence upon the initial data we shall assume that \(\{u, P\}\) \(\)is solution of the problem \((P)\) with null boundary data and null body loads. For this type of external data system, using Lemma 5.1, we deduce the following result.
Theorem 5.3 (Continuous dependence upon initial data) Let \( \{u, P\} \) be a solution of the problem \((P)\) with the external data system \( \mathcal{I} = \{0, u_0, u_0, P_0, P_0\} \). Then, there is a positive constant \( a \) so that
\[
a\left(\|u_0\|^2_{L^2(\Omega)} + \|P_0\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)} + \|P\|^2_{L^2(\Omega)} + \|\text{Curl}\, P\|^2_{L^2(\Omega)}\right) \leq E(0), \quad \text{for all } t \in [0, T].
\]

**Proof.** A direct consequence of the conservation law is
\[
\|u_0\|^2 + \|P_0\|^2 + (\mathbb{C} \text{ sym}(\nabla u - P), \text{ sym}(\nabla u - P)) \\
+ (\mathbb{H} \text{ sym} P, \text{ sym} P) + (\mathbb{L}_c, \text{Curl } P, \text{Curl } P) = 2 E(0), \quad \text{for all } t \in [0, T].
\]
Using Lemma 5.2 and the inequality
\[
(\mathbb{L}_c, \text{Curl } P, \text{Curl } P) \geq (L_c)_m \|\text{Curl } P\|^2,
\]
we have that there is a positive constant \( a_2 \) so that
\[
a_2\left(\|u_0\|^2_{L^2(\Omega)} + \|P_0\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)} + \|\text{ sym}\, \nabla u\|^2_{L^2(\Omega)} + \|\text{ sym}\, P\|^2_{L^2(\Omega)} + \|\text{Curl}\, P\|^2_{L^2(\Omega)}\right) \leq E(0). \tag{53}
\]
Because \( P \in H_0(\text{Curl } \Omega) \) and \( u \in H^1(\Omega) \), in view of (22) and (26), there are the positive constants \( c \) and \( c_k \) \([35–37]\), such that
\[
c\|P\|_{H(\text{Curl } \Omega)} \leq |||P|||, \tag{54}
\]
and
\[
\|\nabla u\|_{L^2(\Omega)} \leq c_k \|\text{ sym}\, \nabla u\|_{L^2(\Omega)}. \tag{55}
\]
Hence, we can find a positive constant \( a \) so that
\[
a\left(\|u_0\|^2_{L^2(\Omega)} + \|P_0\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)} + \|P\|^2_{L^2(\Omega)} + \|\text{Curl}\, P\|^2_{L^2(\Omega)}\right) \leq E(0), \quad \text{for all } t \in [0, T]. \tag{50}
\]

**Corollary 5.4** (Uniqueness) Any two solutions of the problem \((P)\) are equal. \qed

Now we study the continuous data dependence of the solution upon the supply terms \( \{f, M\} \).

**Theorem 5.5** (Continuous dependence upon the supply terms) Let \( \{u, P\} \) be a solution of the problem \((P)\) corresponding to external data system \( \mathcal{I} = \{f, M, 0, 0, 0, 0, 0\} \). Then, for all \( t \in I \) we have
\[
\sqrt{a}\left(\|u_0\|^2_{L^2(\Omega)} + \|P_0\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)} + \|P\|^2_{L^2(\Omega)} + \|\text{Curl}\, P\|^2_{L^2(\Omega)}\right) \leq \frac{1}{2} \int_0^t g(s)ds, \tag{56}
\]
where
\[
g(s) = \left\{ \int_{\Omega} \left(\|f(s)\|^2 + \|M(s)\|^2\right)dv\right\}^{\frac{1}{2}}. \tag{57}
\]

**Proof.** Under the hypothesis of the theorem, Lemma 5.1 implies
\[
E(t) = \int_0^t \int_{\Omega} (f, u_j) + (M, P_j)dvds, \quad \forall \ t \geq 0. \tag{58}
\]
By means of the Cauchy–Schwarz inequality we obtain
\[
E(t) \leq \int_0^t \left\{ \int_{\Omega} \left(\|u_j\|^2 + \|P_j\|^2\right)dv\right\}^{\frac{1}{2}} g(s)ds, \tag{59}
\]
for all \( t \in [0, T] \). We define the function \( \mathcal{Y} : [0, T] \to \mathbb{R}_+ \) \( \mathcal{Y}(t) = [E(t)]^{\frac{1}{2}} \). This function is well defined because \( E(\cdot) \) is a positive function. The inequality (59) becomes

\[
\mathcal{Y}^2(t) \leq \int_0^t \mathcal{Y}(s)g(s)ds, \quad \forall \ t \geq 0.
\]

(60)

By the Brezis lemma given in Appendix A (see \([58], p. 47, \text{Lemma } 1.5.3\) we deduce the inequality

\[
\mathcal{Y}(t) \leq \frac{1}{2} \int_0^t g(s)ds, \quad \forall \ t \geq 0,
\]

and the proof is complete. \( \square \)

5.3. Existence of the solution

In this subsection, in order to establish an existence theorem for the solution of the problem (P) we use the results of the semigroup theory of linear operators. First, we will rewrite the initial boundary value problem (P) as an abstract Cauchy problem in a Hilbert space \([33, 34]\). Let us define the space

\[
\mathcal{X} = \left\{ w = (u, v, P, K) \mid u \in H^1_0(\Omega), \quad v \in L^2(\Omega), \quad P \in H_0(\text{Curl}; \Omega), \quad K \in L^2(\Omega) \right\}.
\]

(61)

On \( \mathcal{X} \) we define the following bilinear form

\[
(w_1, w_2) = \int_\Omega \left( (v_1, v_2) + (K_1, K_2) + (\text{C. sym}(\nabla u_1 - P_1), \text{sym}(\nabla u_2 - P_2)) \\
+ (\text{sym} P_1, \text{sym} P_2) + (\text{Curl} P_1, \text{Curl} P_2) \right) dv,
\]

(62)

where \( w_1 = (u_1, v_1, P_1, K_1) \) and \( w_2 = (u_2, v_2, P_2, K_2) \). Using Lemma 5.2 and the same method as in the proof of Theorem 5.3, we observe that there is a positive constant \( a_m \) such that

\[
a_m \left( \|v\|^2_{L^2(\Omega)} + \|K\|^2_{L^2(\Omega)} + \|
abla u\|^2_{L^2(\Omega)} + \|P\|^2_{L^2(\Omega)} + \|\text{Curl} P\|^2_{L^2(\Omega)} \right) \leq (w, w),
\]

(63)

where \( w = (u, v, K, P) \in \mathcal{X} \).

Hence, according with the symmetries (10), we can conclude that the above bilinear form is an inner product on \( \mathcal{X} \).

Remark 2 As in the proof of Theorem 5.3 we have used that \( h_m \neq 0 \) in order to prove the above inequality. Hence, the above bilinear form \((\cdot, \cdot)\) is an inner product on \( \mathcal{X} \) if \( h_m \neq 0 \).

Obviously, in view of (11) and of the following inequalities

\[
\|\text{sym}(\nabla u - P)\|^2 \leq 2 \left( \|\text{sym} \nabla u\|^2 + \|\text{sym} P\|^2 \right),
\]

\[
\|\text{sym} \nabla u\|^2 \leq \|
abla u\|^2, \quad \|\text{sym} P\|^2 \leq \|P\|^2,
\]

(64)

we observe that there is also a positive constant \( a_M \) such that

\[
(w, w) \leq a_M \left( \|v\|^2_{L^2(\Omega)} + \|K\|^2_{L^2(\Omega)} + \|
abla u\|^2_{L^2(\Omega)} + \|P\|^2_{L^2(\Omega)} + \|\text{Curl} P\|^2_{L^2(\Omega)} \right).
\]

(65)

A direct consequence of the above inequalities is the fact that the norm induced by \((\cdot, \cdot)\) is equivalent to the usual norm on \( \mathcal{X} \). Further, we introduce the operators

\[
A_1 w = v, \\
A_2 w = \text{Div}[\text{C. sym}(\nabla u - P)], \\
A_3 w = K, \\
A_4 w = -\text{Curl}[\text{Curl} P] + \text{C. sym}(\nabla u - P) - \text{sym} P,
\]

(66)
where all the derivatives of the functions are understood in the sense of distributions. Let $\mathcal{A}$ be the operator

$$\mathcal{A} = (A_1, A_2, A_3, A_4)$$

with domain

$$\mathcal{D}(\mathcal{A}) = \{ w = (u, v, P, K) \in \mathcal{X} \mid \mathcal{A}w \in \mathcal{X} \}. \quad (68)$$

We note that $C_0^\infty(\Omega) \times C_0^\infty(\Omega) \times C_0^\infty(\Omega) \times C_0^\infty(\Omega)$ is a dense subset of $\mathcal{X}$ which is contained in $\mathcal{D}(\mathcal{A})$. Hence, $\mathcal{D}(\mathcal{A})$ is a dense subset of $\mathcal{X}$.

With the above definitions, the problem $(P)$ can be transformed into the following abstract problem in the Hilbert space $\mathcal{X}$

$$\frac{dw}{dt}(t) = \mathcal{A}w(t) + \mathcal{F}(t), \quad w(0) = w_0, \quad (69)$$

where

$$\mathcal{F}(t) = (0, f, 0, M)$$

and

$$w_0 = (u_0, \dot{u}_0, P_0, \dot{P}_0). \quad (71)$$

**Lemma 5.6** The operator $\mathcal{A}$ is dissipative, i.e.

$$(\mathcal{A}w, w) \leq 0, \text{ for all } w \in \mathcal{D}(\mathcal{A}) \text{ in the inner product } (\cdot, \cdot) \text{ defined in } (62). \quad (72)$$

**Proof.** Using the relations (35) we find that

$$\begin{align*}
(\mathcal{A}w, w) &= \int_\Omega \left( (\text{Div}(\text{C. sym}(\nabla u - P)), v) - (\text{Curl}(\mathbb{L}_c \text{Curl } P), K) \\
&\quad + (\text{C. sym}(\nabla u - P), K) - (\text{H. sym } P, K) \\
&\quad + (\text{C. sym}(\nabla v - K), \text{sym}(\nabla u - P)) \\
&\quad + (\text{H. sym } K, \text{sym } P) + (\mathbb{L}_c \text{ Curl } K, \text{Curl } P) \right) dv \\
&= \int_\Omega \left( \sum_{i=1}^3 \text{div}(v_i[\text{C. sym}(\nabla u - P)]_i) - (\text{C. sym}(\nabla u - P), \text{sym } \nabla v) \\
&\quad - \sum_{i=1}^3 \text{div } [(\mathbb{L}_c \text{ curl } P)_i \times K_i] - (\mathbb{L}_c \text{ Curl } P, \text{Curl } K) \\
&\quad + (\text{C. sym}(\nabla u - P), K) - (\text{H. sym } P, K) \\
&\quad + (\text{C. sym}(\nabla v - K), \text{sym}(\nabla u - P)) \\
&\quad + (\text{H. sym } K, \text{sym } P) + (\mathbb{L}_c \text{ Curl } K, \text{Curl } P) \right) dv.
\end{align*}$$

Hence, using the divergence theorem and the boundary conditions $u = 0$ and $P \times n = 0$, we deduce

$$\begin{align*}
(\mathcal{A}w, w) &= \int_\Omega \left( - (\text{C. sym}(\nabla u - P), \text{sym}(\nabla v - K)) - (\mathbb{L}_c \text{ Curl } P, \text{Curl } K) \\
&\quad - (\text{H. sym } P, \text{sym } K) + (\text{C. sym}(\nabla v - K), \text{sym}(\nabla u - P)) \\
&\quad + (\text{H. sym } K, \text{sym } P) + (\mathbb{L}_c \text{ Curl } K, \text{Curl } P) \right) dv.
\end{align*} \quad (74)$$

The symmetries (10) ensure that

$$(\mathcal{A}w, w) = 0, \text{ for all } w \in \mathcal{D}(\mathcal{A}), \quad (75)$$

and the proof is complete.
Remark 3 The dissipative condition \((Aw, w) \leq 0, \text{ for all } w \in \mathcal{D}(A)\) is already true for \(h_m = 0\), but this alone does not imply that \(A\) is dissipative, since for \(h_m = 0\) the bilinear form \((\cdot, \cdot)\) is not an inner product.

Lemma 5.7 The operator \(A\) satisfies the range condition, i.e.

\[ R(I - A) = X. \]  

Proof. Let us consider \(w^* = (u^*, v^*, P^*, K^*) \in X\). We must show that the system

\[
\begin{align*}
  u - A_1w &= u^*, \\
  v - A_2w &= v^*, \\
  P - A_3w &= P^*, \\
  K - A_4w &= K^*
\end{align*}
\]  

(77)

has a solution in \(\mathcal{D}(A)\).

By eliminating the functions \(v\) and \(K\), we obtain for the determination of the functions \(u\) and \(P\) the following system of equations

\[
\begin{align*}
  L_1y &= u - \text{Div}[\mathbb{C} \text{ sym}(\nabla u - P)] = g_1, \\
  L_2y &= P + \text{Curl}(\mathbb{L}_c \text{Curl } P) - \mathbb{C} \text{ sym}(\nabla u - P) + \mathbb{H} \text{ sym } P = g_2,
\end{align*}
\]  

(78)

where \(y = (u, v, P, K)\),

\[
g_1 = v^* + u^*, \quad g_2 = K^* + P^*,
\]  

(79)

and all the derivatives of the functions are understood in the sense of distributions.

We study this system in the following Hilbert space

\[ Z = H_0^1(\Omega) \times H_0(\text{Curl}; \Omega). \]  

(80)

We introduce the bilinear form \(B : Z \times Z \rightarrow \mathbb{R}\)

\[ B(y, \tilde{y}) = \langle (L_1y, L_2y), (\tilde{u}, \tilde{P}) \rangle_{L^2(\Omega) \times L^2(\Omega)}. \]  

(81)

In view of relations (35) and of the boundary conditions \(u = 0\) and \(P \times n = 0\), we have

\[
B(y, \tilde{y}) = \int_\Omega \left( (u, \tilde{u}) - (\text{Div}(\mathbb{C} \text{ sym}(\nabla u - P)), \tilde{u}) \\
+ (P, \tilde{P}) + (\mathbb{H} \text{ sym } P, \tilde{P}) + \langle \text{Curl}(\mathbb{L}_c \text{Curl } P) \rangle_{L^2(\Omega)} - (\mathbb{C} \text{ sym}(\nabla u - P), \tilde{P}) \right) dv
\]  

(82)

\[
= \int_\Omega \left( (u, \tilde{u}) + (P, \tilde{P}) - \sum_{i=1}^3 \text{div}((\mathbb{H}_i \text{ sym } P)(\nabla u - P)) + (\mathbb{C} \text{ sym}(\nabla u - P), \text{sym } \nabla \tilde{u}) \\
+ (\mathbb{H} \text{ sym } P, \tilde{P}) - \sum_{i=1}^3 \text{div}((\mathbb{L}_c \text{Curl } P)_i \times \tilde{P}) + (\mathbb{L}_c \text{Curl } P, \text{Curl } \tilde{P}) - (\mathbb{C} \text{ sym}(\nabla u - P), \tilde{P}) \right) dv
\]  

(83)

\[
= \int_\Omega \left( (u, \tilde{u}) + (P, \tilde{P}) + (\mathbb{C} \text{ sym}(\nabla u - P), \text{sym}(\nabla \tilde{u} - \tilde{P})) \\
+ (\mathbb{H} \text{ sym } P, \text{sym } \tilde{P}) + (\mathbb{L}_c \text{Curl } P, \text{Curl } \tilde{P}) \right) dv,
\]  

where \(\tilde{y} = (\tilde{u}, \tilde{P})\). Let us define the linear operator \(I : Z \rightarrow \mathbb{R}\)

\[ I(\tilde{y}) = \langle (g_1, g_2), (\tilde{u}, \tilde{P}) \rangle_{L^2(\Omega) \times L^2(\Omega)}. \]  

(84)

From the Cauchy–Schwarz inequality and the Poincaré inequality we see that the linear operator \(I\) is bounded, i.e. there exists a positive constant \(C\) such that

\[ I(\tilde{y}) \leq C ||\tilde{y}||_Z. \]  

(85)
Moreover, the Cauchy–Schwarz inequality leads us to

\[
B(y, \tilde{y}) \leq \left[ \int_{\Omega} \left( \|u\|^2 + \|P\|^2 + \langle \text{sym}(\nabla u - P), \text{sym}(\nabla u - P) \rangle \right. \\
+ \langle \mathbb{H}. \text{sym} P, \text{sym} P \rangle + \langle \mathbb{L}_c. \text{Curl} P, \text{Curl} P \rangle \right) dv \right]^{\frac{1}{2}}
\times \left[ \int_{\Omega} \left( \|\tilde{u}\|^2 + \|\tilde{P}\|^2 + \langle \text{sym}(\nabla \tilde{u} - \tilde{P}), \text{sym}(\nabla \tilde{u} - \tilde{P}) \rangle \\
+ \langle \mathbb{H}. \text{sym} \tilde{P}, \text{sym} \tilde{P} \rangle + \langle \mathbb{L}_c. \text{Curl} \tilde{P}, \text{Curl} \tilde{P} \rangle \right) dv \right]^{\frac{1}{2}}.
\]

(85)

In view of (11) we obtain that

\[
B(y, \tilde{y}) \leq C \left[ \int_{\Omega} \left( \|u\|^2 + \|P\|^2 + \|\text{sym}(\nabla u - P)\|^2 + \|\text{sym} P\| + \|\text{Curl} P\|^2 \right) dv \right]^{\frac{1}{2}}
\times \left[ \int_{\Omega} \left( \|\tilde{u}\|^2 + \|\tilde{P}\|^2 + \|\text{sym}(\nabla \tilde{u} - \tilde{P})\|^2 + \|\text{sym} \tilde{P}\| + \|\text{Curl} \tilde{P}\|^2 \right) dv \right]^{\frac{1}{2}},
\]

where \( C \) is a positive constant.

Hence, using the inequalities (64) and the Poincaré inequality, we can find a positive constant \( C \) such that

\[
B(y, \tilde{y}) \leq C \left[ \int_{\Omega} \left( \|u\|^2 + \|
abla u\|^2 + \|P\|^2 + \|\text{Curl} P\|^2 \right) dv \right]^{\frac{1}{2}}
\times \left[ \int_{\Omega} \left( \|\tilde{u}\|^2 + \|\nabla \tilde{u}\|^2 + \|\tilde{P}\|^2 + \|\text{Curl} \tilde{P}\|^2 \right) dv \right]^{\frac{1}{2}} \leq C \|y\| \|\tilde{y}\|,
\]

(86)

which means that \( B \) is bounded. On the other hand, we have

\[
B(y, y) = \int_{\Omega} \left( \|u\|^2 + \|P\|^2 + \langle \text{sym}(\nabla u - P), \text{sym}(\nabla u - P) \rangle \\
+ \langle \mathbb{H}. \text{sym} P, \text{sym} P \rangle + \langle \mathbb{L}_c. \text{Curl} P, \text{Curl} P \rangle \right) dv
\]

for all \( y = (u, P) \in Z \). The inequality (46) shows that

\[
\langle \text{sym}(\nabla u - P), \text{sym}(\nabla u - P) \rangle \geq c_m \left[ (1 - \delta) \|\nabla u\|^2 + \left( 1 - \frac{1}{\delta} \right) \|\text{sym} P\|^2 \right],
\]

(87)

for all \( \delta > 0 \). Hence, we deduce that

\[
c_m (1 - \delta) \|\nabla u\|^2 + \left( c_m + 1 - \frac{c_m}{\delta} \right) \|\text{sym} P\|^2 \leq \langle \text{sym}(\nabla u - P), \text{sym}(\nabla u - P) \rangle + \|\text{sym} P\|^2 \\
\leq \langle \text{sym}(\nabla u - P), \text{sym}(\nabla u - P) \rangle + \|P\|^2 \\
\leq c_M \|\nabla (\nabla u - P)\|^2 + \|P\|^2.
\]

(90)
If we choose $\delta$ so that

$$\frac{c_m}{c_m + 1} < \delta < 1$$

we obtain a positive constant $C_1$ so that

$$C_1 (\| \text{sym} \nabla u \|^2 + \| \text{sym} P \|^2) \leq \| \text{sym}(\nabla u - P) \|^2 + \| P \|^2. \quad (91)$$

Moreover, as a consequence of the assumptions (11) we have

$$\mathcal{B}(y, y) \geq \int_{\Omega} \left( \| u \|^2 + \| P \|^2 + c_m \| \text{sym}(\nabla u - P) \|^2 + h_m \| \text{sym} P \|^2 + (L_c)_m \| \text{Curl} P \|^2 \right) dv \quad (92)$$

$$\geq \int_{\Omega} \left( \| u \|^2 + \min\{c_m, 1\}(\| P \|^2 + \| \text{sym}(\nabla u - P) \|^2) + h_m \| \text{sym} P \|^2 + (L_c)_m \| \text{Curl} P \|^2 \right) dv \quad (93)$$

Using (22) and (26) we deduce

$$\mathcal{B}(y, y) \geq \int_{\Omega} \left( C(\| u \|^2 + \| \nabla u \|^2 + \| P \|^2 + \| \text{Curl} P \|^2) + h_m \| \text{sym} P \|^2 \right) dv \geq C\| y \|^2_{\mathcal{Z}},$$

where $C$ is a positive constant. Hence $\mathcal{B}(\cdot, \cdot)$ is coercive.

Using the Lax–Milgram theorem we prove the existence of a solution of the system (78) in $\mathcal{Z}$, i.e.

$$u \in H^1_0(\Omega), \quad P \in H_0(\text{Curl}; \Omega). \quad (44)$$

Hence, $v$ and $K$ will be given by

$$v = u - u^*, \quad K = P - P^*. \quad (95)$$

Moreover, because $w^* \in \mathcal{X}$ it follows that $u^* \in H^1_0(\Omega)$ and $P^* \in H_0(\text{Curl}; \Omega)$. Thus

$$v \in H^1_0(\Omega) \subset L^2(\Omega), \quad K \in H_0(\text{Curl}; \Omega) \subset L^2(\Omega). \quad (96)$$

Let us remark that

$$\mathcal{D}(A) \subset H^1_0(\Omega) \times H^1_0(\Omega) \times H_0(\text{Curl}; \Omega) \times H_0(\text{Curl}; \Omega) \subset \mathcal{X}, \quad (97)$$

and that, until now, we have proved that the equation

$$w - \mathcal{A} w = w^* \quad (98)$$

has a solution

$$w = (u, v, P, K) \in H^1_0(\Omega) \times H^1_0(\Omega) \times H_0(\text{Curl}; \Omega) \times H_0(\text{Curl}; \Omega) \subset \mathcal{X} \quad (99)$$

for all $w^* \in \mathcal{X}$. Hence, $-\mathcal{A} w = w^* - w \in \mathcal{X}$, which shows that $w \in \mathcal{D}(A)$, and $w - \mathcal{A} w = w^*$. This implies that we have the desired solution of system (77) and the proof is complete.

**Remark 4** In the proof of the range condition from the previous lemma, we have not used that $h_m > 0$. The result holds true if $\mathbb{H} = 0$. However, the bilinear form $(\cdot, \cdot)$ is an inner product if $h_m > 0$ and we can not prove this fact if $\mathbb{H} = 0$. Our existence result needs that $\mathcal{A}$ is dissipative in the inner product $(\cdot, \cdot)$. Hence, the existence result below holds true only for $h_m > 0$.

**Theorem 5.8** The operator $\mathcal{A}$ defined by (67) generates a $C_0$-contractive semigroups in $\mathcal{X}$.
Proof. The proof follows using the previous lemmas and the Lumer–Phillips corollary to the Hille–Yosida theorem [33] given in Appendix A.

**Theorem 5.9** Assume that \( f, M \in C^1([0, T); L^2(\Omega)) \), \( w_0 \in D(A) \) and the fourth order elasticity tensors \( C, Lc \) and \( H \) are positive definite and satisfy the symmetries (10). Then, there exists a unique solution

\[
w \in C^1((0, T); \mathcal{X}) \cap C^0([0, T); D(A))
\]

of the Cauchy problem (69).

**Proof.** The proof follows from the results concerning the abstract Cauchy problem from Appendix A (see [33, 34]).

**Corollary 5.10** In the hypothesis of Theorem 5.9 we have the following estimate

\[
\|w(t)\|_{\mathcal{X}} \leq \|w_0(t)\|_{\mathcal{X}} + C \int_0^T \left( \|f(s)\|_{L^2(\Omega)} + \|M(s)\|_{L^2(\Omega)} \right) ds, \quad \text{for all} \quad t \in [0, T],
\]

where \( C \) is a positive constant.

**Proof.** For the proof of this Corollary we use the fact that the semigroup generated by \( A \) is contractive and apply the Duhamel Principle (see Appendix A).

**6. Another further relaxed problem**

In this section, we weaken our energy expression further in the following model, where the corresponding elastic energy depends now only on the set of independent constitutive variables

\[
\varepsilon_e = \text{sym}(\nabla u - P), \quad \text{dev} \varepsilon_p = \text{dev sym} P, \quad \text{dev} \alpha = -\text{dev Curl} P.
\]

In this model, it is neither implied that \( P \) remains symmetric, nor that \( P \) is trace-free, but only the trace free symmetric part of the micro-distortion \( P \) and the trace-free part of the micro-dislocation tensor \( \alpha \) contribute to the stored energy.

**6.1. Formulation of the problem**

The model in its general anisotropic form is:

\[
\begin{align*}
\dot{u} &= \text{Div}[C \cdot \text{sym}(\nabla u - P)] + f, \\
\dot{P} &= -\text{Curl}[\text{dev}Lc \cdot \text{dev Curl} P] + C \cdot \text{sym}(\nabla u - P) - H \cdot \text{dev sym} P + M \quad \text{in} \quad \Omega \times [0, T].
\end{align*}
\]

In the isotropic case the model becomes

\[
\begin{align*}
\dot{u} &= \text{Div}[2\mu_e \text{sym}(\nabla u - P) + \lambda_e \text{tr}(\nabla u - P) \cdot \mathbb{I}] + f, \\
\dot{P} &= -\text{Curl}[\alpha_1 \text{dev sym Curl} P + \alpha_2 \text{skew Curl} P] \\
&\quad + 2\mu_e \text{sym}(\nabla u - P) + \lambda_e \text{tr}(\nabla u - P) \cdot \mathbb{I} - 2\mu_h \text{dev sym} P + M \quad \text{in} \quad \Omega \times [0, T].
\end{align*}
\]

To the system of partial differential equations of this model we adjoin the weaker boundary conditions

\[
\begin{align*}
u(x, t) &= 0, \quad P_i(x, t) \times n(x) = 0, \quad i = 1, 2, 3, \quad (x, t) \in \partial \Omega \times [0, T],
\end{align*}
\]

and the nonzero initial conditions

\[
\begin{align*}
u(x, 0) &= u_0(x), \quad \dot{u}(x, 0) = \dot{u}_0(x), \quad P(x, 0) = P_0(x), \quad \dot{P}(x, 0) = \dot{P}_0(x), \quad x \in \bar{\Omega},
\end{align*}
\]

where \( u_0, \dot{u}_0, P_0 \) and \( \dot{P}_0 \) are prescribed functions.

We remark again that \( P \) is not trace-free in this formulation and no projection is performed. We denote the new problem defined by the above equations, the boundary conditions (104) and the initial conditions (105) by \((\tilde{P})\).
6.2. Energy conservation

To a solution \( \{u, P\} \) of the problem \( \tilde{P} \) we associate the total energy

\[
2 \tilde{E}(t) = \int_{\Omega} \left( \|u_{\tau}\|^2 + \|P_{\tau}\|^2 + \langle \mathbb{C} \text{ sym} (\nabla u - P), \text{sym} (\nabla u - P) \rangle \\
+ \langle \mathbb{H} \text{ dev sym} P, \text{dev sym} P \rangle + \langle \mathbb{L}_{\text{c}} \text{ dev Curl} P, \text{dev Curl} P \rangle \right) \, dv .
\] (106)

We observe that since \( \mathbb{H} \) is positive definite on \( \text{Sym}(3) \), in view of (11) we also have the estimate

\[
h_m \| \text{dev sym} P \|^2 \leq \langle \mathbb{H} \text{ dev sym} P, \text{dev sym} P \rangle \leq h_M \| \text{dev sym} P \|^2 \text{ for all } P \in \mathbb{R}^{3 \times 3} .
\] (107)

It is easy to see that taking in (35)

\[
R_i = [\text{dev} (\mathbb{L}_{\text{c}} \text{ dev Curl } P)]_i, \quad S_i = P_i ,
\] (108)

we have

\[
\sum_{i=1}^3 \text{div} ([\text{dev} (\mathbb{L}_{\text{c}} \text{ dev Curl } P)]_i \times P_{\tau i}) = \sum_{i=1}^3 \langle P_{\tau i}, \text{curl} [\text{dev} (\mathbb{L}_{\text{c}} \text{ dev Curl } P)]_i \rangle \\
- \sum_{i=1}^3 \langle [\text{dev} (\mathbb{L}_{\text{c}} \text{ dev Curl } P)]_i, \text{curl} P_{\tau i} \rangle.
\] (109)

Thus, in terms of tensors, we have the following equality

\[
\langle P_{\tau}, \text{Curl} [\text{dev} (\mathbb{L}_{\text{c}} \text{ dev Curl } P)] \rangle = \langle \text{dev} (\mathbb{L}_{\text{c}} \text{ dev Curl } P), \text{Curl } P_{\tau} \rangle \\
+ \sum_{i=1}^3 \text{div} ([\text{dev} (\mathbb{L}_{\text{c}} \text{ dev Curl } P)]_i \times P_{\tau i}) .
\] (110)

Moreover, using that

\[
\langle \text{dev} A, B \rangle = \langle A, \text{dev} B \rangle, \text{ for all } A, B \in \mathbb{R}^{3 \times 3} ,
\] (111)

we have

\[
\langle P_{\tau}, \text{Curl} [\text{dev} (\mathbb{L}_{\text{c}} \text{ dev Curl } P)] \rangle = \langle \mathbb{L}_{\text{c}} \text{ dev Curl } P, \text{dev Curl } P_{\tau} \rangle \\
+ \sum_{i=1}^3 \text{div} ([\text{dev} (\mathbb{L}_{\text{c}} \text{ dev Curl } P)]_i \times P_{\tau i}) .
\] (112)

In view of (102) and using the symmetries (10), we get

\[
\frac{1}{2} \frac{\partial}{\partial t} \left( \|u_{\tau}\|^2 + \|P_{\tau}\|^2 + \langle \mathbb{C} \text{ sym} (\nabla u - P), \text{sym} (\nabla u - P) \rangle \\
+ \langle \mathbb{H} \text{ dev sym} P, \text{dev sym} P \rangle + \langle \mathbb{L}_{\text{c}} \text{ dev Curl} P, \text{dev Curl} P \rangle \right) \\
= \sum_{i=1}^3 \langle \text{div} ([\mathbb{C} \text{ sym} (\nabla u - P)]_i u_{\tau i} \rangle + \sum_{i=1}^3 \text{div} (P_{\tau i} \times [\text{dev} (\mathbb{L}_{\text{c}} \text{ dev Curl } P)]_i \rangle + \langle f, u_{\tau} \rangle + \langle M, P_{\tau} \rangle .
\] (113)

Using the divergence theorem and the boundary conditions \( u = 0 \) and \( P \times n = 0 \), from the above identity we can conclude:
Remark 5 (Energy conservation for the problem (P)) If \( \{u, P\} \) is a solution of the problem (P) corresponding to the loads \( \mathcal{I} = \{f, M, u_0, u_0, P_0, P_0\} \), then we have
\[
\tilde{E}(t) = \tilde{E}(0) + \int_0^t \Pi(s)ds,
\]
for every time \( t \in [0, T] \).

### 6.3. Uniqueness and continuous dependence of the solution

In order to prove the uniqueness and the continuous dependence of the solution with respect to given data, we will use the estimate given by the following lemma:

**Lemma 6.1** For all \( u \in H^1(\Omega) \) and \( P \in H(\text{Curl}; \Omega) \), the following estimate holds true
\[
a\left(\|\nabla u\|_{L^2(\Omega)}^2 + \|\text{dev sym } P\|_{L^2(\Omega)}^2 \right) \leq \int_\Omega \left( \langle \text{C. sym}(\nabla u - P), \text{sym}(\nabla u - P) \rangle + \langle \text{H. dev sym } P, \text{dev sym } P \rangle \right)dv,
\]
where \( a \) is a positive constant.

**Proof:** First, we note that
\[
\|X\|^2 = \|\text{dev } X\|^2 + \frac{1}{3}(\text{tr} X)^2,
\]
for all \( X \in \mathbb{R}^{3 \times 3} \). Using this identity and the properties (11) we deduce
\[
\langle \text{C. sym}(\nabla u - P), \text{sym}(\nabla u - P) \rangle + \langle \text{H. dev sym } P, \text{dev sym } P \rangle \geq c_m \|\text{sym}(\nabla u - P)\|^2 + h_m \|\text{dev sym } P\|^2
\]
\[
\geq c_m \|\text{dev sym}(\nabla u - P)\|^2 + h_m \|\text{dev sym } P\|^2
\]
\[
\geq c_m (1 - \delta) \|\text{dev sym } \nabla u\|^2 + \left( c_m - \frac{c_m}{\delta} + h_m \right) \|\text{dev sym } P\|^2
\]
for all \( \delta > 0 \). If we choose \( \delta \) so that
\[
\frac{c_m}{c_m + h_m} < \delta < 1
\]
we have that there is a positive constant \( a_1 \) so that
\[
\langle \text{C. sym}(\nabla u - P), \text{sym}(\nabla u - P) \rangle + \langle \text{H. dev sym } P, \text{dev sym } P \rangle \geq a_1 \left( \|\text{dev sym } \nabla u\|^2 + \|\text{dev sym } P\|^2 \right).
\]
The estimate (115) follows from Theorem 4.6.

**Theorem 6.2** (Continuous dependence upon initial data) Let \( \{u, P\} \) be a solution of the problem (P) with the external data system \( \mathcal{I} = \{0, 0, u_0, u_0, P_0, P_0\} \). Then, there is a positive constant \( a \) so that
\[
a(\|u, P\|_{L^2(\Omega)}^2 + \|P, P\|_{L^2(\Omega)}^2 + \|\nabla u\|_{L^2(\Omega)}^2 + \|P\|_{L^2(\Omega)}^2 + \|\text{Curl } P\|_{L^2(\Omega)}^2) \leq E(0), \quad \text{for all } t \in [0, T].
\]

**Proof:** Using the conservation law (114) and the estimate (115) we have that there is a positive constant \( a \) so that
\[
a(\|u, P\|_{L^2(\Omega)}^2 + \|P, P\|_{L^2(\Omega)}^2 + \|\nabla u\|_{L^2(\Omega)}^2 + \|\text{dev sym } P\|_{L^2(\Omega)}^2 + \|\text{dev Cull } P\|_{L^2(\Omega)}^2) \leq E(0), \quad \text{for all } t \in [0, T].
\]
Because \( P \in H_0(\text{Curl} ; \Omega) \) and \( u \in H^1_0(\Omega) \), in view of (29), there are the positive constants \( C_{DSDC} \) and \( C_{DD} [35–39] \), such that

\[
\|P\|_{L^2(\Omega)} + \|\text{Curl} P\|_{L^2(\Omega)} \leq (C_{DSDC} + C_{DD})(\|\text{dev sym} P\|_{L^2(\Omega)} + \|\text{dev Curl} P\|_{L^2(\Omega)}) \tag{120}
\]

Hence, we can find a positive constant \( a \) so that the inequality (119) is satisfied.

**Corollary 6.3** (Uniqueness) Any two solutions of the problem \((\tilde{P})\) are equal.

For the modified problem \((\tilde{P})\) we also have the continuous data dependence of solution upon the supply terms \([f, M]\).

**Remark 6** (Continuous dependence upon the supply terms) Let \([u, P]\) be a solution of the problem \((\tilde{P})\) corresponding to the external data system \( I = [f, M, 0, 0, 0] \). Then, for all \( t \in I \) we have

\[
\sqrt{a}(\|u, t\|_{L^2(\Omega)}^2 + \|P, t\|_{L^2(\Omega)}^2 + \|\nabla u\|_{L^2(\Omega)}^2 + \|P\|_{L^2(\Omega)}^2 + \|\text{Curl} P\|_{L^2(\Omega)}^2)^{\frac{1}{2}} \leq \frac{1}{2} \int_0^t g(s)ds. \tag{121}
\]

**Proof.** The proof of this remark is similar to the proof of Theorem 5.5. \( \square \)

### 6.4. Existence of the solution

In this subsection, we study the existence of the solution of the problem \((\tilde{P})\). Because the method is similar with that used in Section 5.3 we only point out the differences which arise for our modified problem.

We consider the same Hilbert space \( \mathcal{X} \) as defined in Section 5.3 and we define the following bilinear form

\[
((w_1, w_2)) = \int_\Omega \left( \langle v_1, v_2 \rangle + \langle K_1, K_2 \rangle + \langle \text{C. sym}(\nabla u_1 - P_1), \text{sym}(\nabla u_2 - P_2) \rangle \right.
\]

\[
\left. + \langle \text{dev sym} P_1, \text{dev sym} P_2 \rangle + \langle \text{sym} \text{Curl} P_1, \text{sym} \text{Curl} P_2 \rangle \right) \, dv,
\]

where \( w_1 = (u_1, v_1, P_1, K_1) \) and \( w_2 = (u_2, v_2, P_2, K_2) \). Let us remark that in view of (115), there is a positive constant \( a \) such that

\[
a(\|v\|_{L^2(\Omega)}^2 + \|K\|_{L^2(\Omega)}^2 + \|\nabla u\|_{L^2(\Omega)}^2 + \|\text{dev sym} P\|_{L^2(\Omega)}^2 + \|\text{dev Curl} P\|_{L^2(\Omega)}^2) \leq ((w, w)), \tag{122}
\]

where \( w = (u, v, K, P) \in \mathcal{X} \). In other words, using Corollary 4.5 we have

\[
a(\|v\|_{L^2(\Omega)}^2 + \|K\|_{L^2(\Omega)}^2 + \|\nabla u\|_{L^2(\Omega)}^2 + \|P\|_{L^2(\Omega)}^2 + \|\text{Curl} P\|_{L^2(\Omega)}^2) \leq ((w, w)), \tag{123}
\]

which implies that the bilinear form \((\cdot, \cdot)\) is an inner product on \( \mathcal{X} \). We also have that there is a positive constant \( C \) such that

\[
((w, w)) \leq C(\|v\|_{L^2(\Omega)}^2 + \|K\|_{L^2(\Omega)}^2 + \|\nabla u\|_{L^2(\Omega)}^2 + \|P\|_{L^2(\Omega)}^2 + \|\text{dev sym} P\|_{L^2(\Omega)}^2 + \|\text{dev Curl} P\|_{L^2(\Omega)}^2). \tag{124}
\]

Hence, using Theorem 4.4 we obtain

\[
((w, w)) \leq C(\|v\|_{L^2(\Omega)}^2 + \|K\|_{L^2(\Omega)}^2 + \|\nabla u\|_{L^2(\Omega)}^2 + \|P\|_{L^2(\Omega)}^2 + \|\text{Curl} P\|_{L^2(\Omega)}^2). \tag{124}
\]

Thus, the norm induced by \((\cdot, \cdot)\) is equivalent with the usual norm on \( \mathcal{X} \). We consider the operators

\[
\tilde{A}_1 w = v, \quad \tilde{A}_2 w = \text{Div} [\text{C. sym} (\nabla u - P)], \quad \tilde{A}_3 w = K, \quad \tilde{A}_4 w = -\text{Curl} [\text{dev} \text{sym} (\text{Curl} P)] + \text{C. sym} (\nabla u - P) - \text{dev sym} P, \tag{125}
\]
where all the derivatives of the functions are understood in the sense of distributions, and the operator \( \tilde{A} \)

\[
\tilde{A} = (\tilde{A}_1, \tilde{A}_2, \tilde{A}_3, \tilde{A}_4)
\]  

(126)

with the domain

\[
D(\tilde{A}) = \{ w = (u, v, P, K) \in \mathcal{X} \mid \tilde{A}w \in \mathcal{X} \}.
\]  

(127)

A similar method with that considered in Section 5.3 gives us the following existence result:

**Theorem 6.4** Assume that \( f, M \in C^1([0, T); L^2(\Omega)) \), \( w_0 \in D(\tilde{A}) \) and the fourth order elasticity tensors \( C, L_c \) and \( H \) are positive definite and satisfy the symmetries (10). Then, there exists a unique solution

\[
w \in C^1((0, T); \mathcal{X}) \cap C^0([0, T); D(\tilde{A}))
\]

of the following Cauchy problem

\[
\frac{dw}{dt}(t) = \tilde{A}w(t) + F(t), \quad w(0) = w_0,
\]  

(128)

where

\[
F(t) = (0, f, 0, M)
\]  

(129)

and

\[
w_0 = (u_0, \dot{u}_0, P_0, \dot{P}_0).
\]  

(130)

Moreover, we have the estimate

\[
\|w(t)\|_{\mathcal{X}} \leq \|w_0(t)\|_{\mathcal{X}} + C \int_0^T \left( \|f(s)\|_{L^2(\Omega)} + \|M(s)\|_{L^2(\Omega)} \right) ds, \quad \text{for all} \quad t \in [0, T],
\]  

(131)

where \( C \) is a positive constant.

**Proof.** It is easy to prove that the operator \( \tilde{A} \) is dissipative. In the following we prove that the operator \( \tilde{A} \) satisfies the range condition

\[
\text{R}(I - \tilde{A}) = \mathcal{X}.
\]  

(132)

Let us define the operator \( \tilde{L}_2 : \mathcal{X} \to \mathcal{X} \) by

\[
\tilde{L}_2 y = P + \text{Curl}[\text{dev}[L_c \cdot \text{Curl} P]] + C \cdot \text{sym}(\nabla u - P) - H \cdot \text{dev sym} P,
\]  

(133)

where \( y = (u, v, P, K) \in \mathcal{X} \) and all the derivatives of the functions are understood in the sense of distributions. We consider the Hilbert space

\[
\mathcal{Z} = H^1_0(\Omega) \times H_0(\text{Curl}; \Omega).
\]  

(134)
Figure 1. Relation between possible relaxed micromorphic models. The non-well-posedness follows from the results in work by Bauer et al. [38–40].

On $\mathcal{Z}$ we consider the bilinear form $\tilde{B} : \mathcal{Z} \times \mathcal{Z} \to \mathbb{R}$

$$B(y, \tilde{y}) = \langle (L_1 y, \tilde{L}_2 y), (\tilde{u}, \tilde{P}) \rangle_{L^2(\Omega) \times L^2(\Omega)},$$  \hspace{1cm} (135)
where $L_1$ is given by (78), and the linear bounded operator $I : Z \to \mathbb{R}$ is given by (83). In view of the boundary conditions we have

$$
\tilde{B}(y, \tilde{y}) = \int_{\Omega} \left( (u, \bar{u}) + (P, \bar{P}) + \langle \text{C. sym}(\nabla u - P), \text{sym}(\nabla \bar{u} - \bar{P}) \rangle 
\right)
$$

$$
+ \langle \mathbb{H}. \text{dev sym } P, \text{dev sym } \bar{P} \rangle + \langle \mathbb{L}_{\text{C. dev Curl } P, \text{dev Curl } \bar{P}} \rangle dv,
$$

where $\tilde{y} = (\tilde{u}, \tilde{P})$. The Cauchy–Schwarz inequality, the Poincaré inequality and the relations (11), (64) and (116) lead us to the estimate

$$
\tilde{B}(y, \tilde{y}) \leq C \left[ \int_{\Omega} \left( \|u\|^2 + \|\nabla u\|^2 + \|P\|^2 + \|\text{dev } P\|^2 + \|\text{dev Curl } P\|^2 \right) dv \right]^{\frac{1}{2}}
$$

$$
	imes \left[ \int_{\Omega} \left( \|\bar{u}\|^2 + \|\nabla \bar{u}\|^2 + \|\bar{P}\|^2 + \|\text{dev } \bar{P}\|^2 + \|\text{dev Curl } \bar{P}\|^2 \right) dv \right]^{\frac{1}{2}}
$$

$$
\leq C \left[ \int_{\Omega} \left( \|u\|^2 + \|\nabla u\|^2 + \|P\|^2 + \|\text{dev sym } P\|^2 + \|\text{dev Curl } P\|^2 \right) dv \right]^{\frac{1}{2}}
$$

$$
\leq C \|y\|_Z \|\tilde{y}\|_Z,
$$

where $C$ is a positive constant. This means that $\tilde{B}$ is bounded. On the other hand, we have

$$
\tilde{B}(y, y) = \int_{\Omega} \left( \|u\|^2 + \|P\|^2 + \langle \text{C. sym}(\nabla u - P), \text{sym}(\nabla u - P) \rangle 
\right)
$$

$$
+ \langle \mathbb{H}. \text{dev sym } P, \text{dev sym } P \rangle + \langle \mathbb{L}_{\text{C. dev Curl } P, \text{dev Curl } P} \rangle dv
$$

for all $y = (u, P) \in Z$. Moreover, as a consequence of the assumptions (11) and of estimate (115), we can find a new positive constant $C$ so that

$$
\mathcal{B}(y, y) \geq C \int_{\Omega} \left( \|u\|^2 + \|\nabla u\|^2 + \|\text{dev sym } P\|^2 + \|\text{dev Curl } P\|^2 \right) dv.
$$

Now, using the Corollary 4.5 we deduce that $\mathcal{B}(\cdot, \cdot)$ is coercive. Using the Lax–Milgram theorem and similar arguments as in the proof of Lemma 5.7 it follows that the operator $\tilde{A}$ satisfies the range condition. Hence, the hypothesis of the general existence theorem for the abstract Cauchy problem are satisfied and the proof is complete.

7. Final remarks

In the present paper we have mathematically studied a large class of evolution equations which describe the behaviour of micromorphic or generalized continua (see e.g. [59–60]). The mathematical existence, uniqueness and continuous dependence theorems which we have obtained here are the logical basis of the studies which will be developed in further investigations, where the manifold variety of propagating mechanical waves which may exist in micromorphic continua may unfold unexpected applications in the design of particularly tailored metamaterials [62, 63], showing very useful and up-to-now unimagined features. We remark that the theorems obtained in the present paper can also be used to give a better grounded basis to many results which are already available in the literature (see e.g. [64, 65]).

The diagram from Figure 1 gives some new possible relaxed micromorphic models and, in view of the status of the mathematical background, we indicate the well-posedness of the dynamic and static problem.
Acknowledgements

ID Ghiba acknowledges support from the Romanian National Authority for Scientific Research (CNCS-UEFISCDI), Project No. PN-II-ID-PCE-2011-3-0521. ID Ghiba would like to thank P Neff for his kind hospitality during his visit at the Faculty of Mathematics, Universität Duisburg-Essen, Campus Essen. P Neff is grateful to F Dell’Isola for making his visit to CISTERNA di LATINA (M&MoCS), in spring 2013, a wonderful scientific experience. A Madeo thanks INSA-Lyon for the financial support assigned to the project BQR 2013-0054 “Matériaux Méso et Micro-Hétérogènes: Optimisation par Modèles de Second Gradient et Applications en Ingénierie”.

Note

1. In fact $(\mathcal{A}w, w) = 0$, for all $w \in \mathcal{D}(\mathcal{A})$.

References

[1] Neff, P, Ghiba, ID, Madeo, A, Placidi, L, and Rosi, G. A unifying perspective: The relaxed linear micromorphic continua. *Cont Mech Therm* 2013; doi:10.1007/s00161-013-0322-9.

[2] Eringen, AC. *Microcontinuum Field Theories*. Heidelberg, Germany: Springer-Verlag, 1999.

[3] Neff, P. On material constants for micromorphic continua. In: Wang Y and Hutter K (eds) *Trends in Applications of Mathematics to Mechanics* (STAMM Proceedings, Seeheim, Germany, 2004). Aachen, Germany: Shaker Verlag, 2005, pp.337–348.

[4] Mariano, PM, and Stazi, FL. Computational aspects of the mechanics of complex materials. *Arch Comput Meth Eng* 2005; 12: 392–478.

[5] Mariano, PM, and Modica, G. Ground states in complex bodies. *ESAIM: COCV* 2009; 15(2): 377–402.

[6] Mariano, PM. Representation of material elements and geometry of substructural interaction. *Quaderni di Matematica* 2007; 20: 80–100.

[7] Steigmann, DJ. Theory of elastic solids reinforced with fibers resistant to extension, flexure and twist. *Int J Non-Linear Mech* 2012; 47: 734–742.

[8] Cosserat, E, and Cosserat, F. *Théorie des corps déformables*. Librairie Scientifique A Hermann et Fils, 1909; Paris: Hermann Librairie Scientifique, 2009, http://www.uni-duje.de/hm0014/Cosserat_files/Cosserat09_eng.pdf.

[9] Neff, P. The Cosserat couple modulus for continuous solids is zero viz the linearized Cauchy-stress tensor is symmetric. *Z Angew Math Mech* 2006; 86: 892–912.

[10] Jeong, J, and Neff, P. Existence, uniqueness and stability in linear Cosserat elasticity for weakest curvature conditions. *Math Mech Solids* 2010; 15(1): 78–95.

[11] Neff, P, and Jeong, J. A new paradigm: The linear isotropic Cosserat model with conformally invariant curvature energy. *Z Angew Math Mech* 2009; 89(2): 107–122.

[12] Neff, P, Jeong J, Münch I, et al. Linear Cosserat elasticity, conformal curvature and bounded stiffness. In: Maugin GA and Metrikine VA (eds) *Mechanics of Generalized Continua. One Hundred Years After the Cosserats*. Advances in Mechanics and Mathematics, Vol. 21. Berlin: Springer-Verlag, 2010, pp.55–63.

[13] Mindlin, RD. Micro-structure in linear elasticity. *Arch Rat Mech Anal* 1964; 16: 51–77.

[14] Eringen, AC, and Suhubi, ES. Nonlinear theory of simple micro-elastic solids. *I. Int J Eng Sci* 1964; 2: 189–203.

[15] Kröner, E. Discussion on papers by AC Eringen and WD Claus, Jr, and N Fox. In: Simmons JA, de Wit R, and Bullough R (eds) *Fundamental Aspects of Dislocation Theory*. Special publication, Vol 1. Washington DC: National Bureau of Standards (US), 1970, pp. 1054–1059.

[16] Kröner, E. Das physikalische Problem der antisymmetrischen Spannungen und der sogenannten Momentenspannungen. In: Görtler H (ed) *Proceedings of 11th international congress of applied mechanics*, Munich, 1964, pp. 143–158.

[17] Claus, WD, and Eringen, AC. Three dislocation concepts and micromorphic mechanics. In: *Proceedings of the 12th Midwestern mechanics conference on developments in mechanics*, Notre Dame, IN, 16–18 August 1971, Vol. 6, pp.349–358.

[18] Eringen, AC, and Claus, WD. A micromorphic approach to dislocation theory and its relation to several existing theories. In: Simmons JA, de Wit R, and Bullough R (eds) *Fundamental Aspects of Dislocation Theory*. Special publication, Vol 1. Washington DC: National Bureau of Standards (US), 1970, pp.1023–1040.

[19] Claus, WD, and Eringen, AC. Dislocation dispersion of elastic waves. *Int J Eng Sci* 1971; 9: 605–610.

[20] Chang, CS, and Misra, A. Packing structure and mechanical properties of granulates. *J Eng Mech—ASCE* 1990; 116(5): 1077–1093.

[21] Misra, A, and Chang, CS. Effective elastic moduli of heterogeneous granular solids. *Int J Solids Struct* 1993; 30(18): 2547–2566.

[22] Misra, A, and Yang, Y. Micromechanical model for cohesive materials based upon pseudo-granular structure. *Int J Solids Struct* 2010; 47(21): 2970–2981.

[23] Yang, Y, and Misra, A. Micromechanics based second gradient continuum theory for shear band modeling in cohesive granular materials following damage elasticity. *Int J Solids Struct* 2012; 49(18): 2500–2514.

[24] Merkel, A, and Tournat, V. Experimental evidence of rotational elastic waves in granular phononic crystals. *Physical Review Letters* 2011; 107: 225502.
[25] Neff, P, and Forest, S. A geometrically exact micromorphic model for elastic metallic foams accounting for affine microstructure. Modelling existence of minimizers, identification of moduli and computational results. *J Elasticity* 2007; 87: 239–276.

[26] DiCarlo, A, and Quiligiotti, S. Growth and balance. *Mech Res Commun* 2002; 29(6): 449–456.

[27] Soós, E. Uniqueness theorems for homogeneous, isotropic, simple elastic and thermoelastic materials having a micro-structure. *Int J Eng Sci* 1969; 7: 257–268.

[28] Hlaváček, I, and Hlaváček, M. On the existence and uniqueness of solutions and some variational principles in linear theories of elasticity with couple-stresses. I: Cosserat continuum. II: Mindlin’s elasticity with micro-structure and the first strain gradient. *J Appl Mat* 1969; 14: 387–426.

[29] Ieşan, D. Extremum principle and existence results in micromorphic elasticity. *Int J Eng Sci* 2001; 39: 2051–2070.

[30] Ieşan, D. On the micromorphic thermoelectricity. *Int J Eng Sci* 2002; 40: 549–567.

[31] Neff, P. Existence of minimizers for a finite-strain micromorphic elastic solid. *Proc Roy Soc Edinb A* 2006; 136: 997–1012.

[32] Klawonn, A, Neff, P, Rheinbach, O, et al. FETI-DP domain decomposition methods for elasticity with structural changes: P-elasticity. *ESAIM: Math Mod Num Anal* 2011; 45: 563–602.

[33] Pazy, A. *Semigroups of Linear Operators and Applications to Partial Differential Equations*. New York; Berlin; Heidelberg; Germany: Tokyo: Springer-Verlag. 1983.

[34] Vrabie, I. *C₀-Semigroups and applications* (North-Holland Mathematics Studies, Vol. 191). Amsterdam: Elsevier, 2003.

[35] Neff, P, Pauly, D, and Witsch, KJ. Poincaré meets Korn via Maxwell: Extending Korn’s first inequality to incompatible tensor fields. arXiv:1203.2744 submitted.

[36] Neff, P, Pauly, D, and Witsch, KJ. A canonical extension of Korn’s first inequality to H(Curl) motivated by gradient plasticity with plastic spin. *C R Acad Sci Paris Ser I* 2011; 349: 1251–1254.

[37] Neff, P, Pauly, D, and Witsch, KJ. Maxwell meets Korn: A new coercive inequality for tensor fields in $\mathbb{R}^{N\times N}$ with square-integrable exterior derivative. *Math Methods Appl Sci* 2012; 35: 65–71.

[38] Bauer, S, Neff P, Pauly, D, et al. Some Poincaré type inequalities for quadratic matrix fields. *Proc Appl Math Mech* 2013; forthcoming.

[39] Bauer, S, Neff P, Pauly, D, et al. Dev-Div and DevSym-DevCurl inequalities for incompatible square tensor fields with mixed boundary conditions. 2013, submitted.

[40] Bauer, S, Neff, P, Pauly, D, et al. New Poincaré type inequalities. *C R Acad Sci Paris Ser I* 2013; submitted.

[41] Lankeit J, Neff P and Pauly D. Uniqueness of integrable solutions to integrable exterior derivative. *Math Methods Appl Sci* 2013; 36: 1241–1249.

[42] Galeş, C, Ghiba, ID, and Ignatescu, I. Asymptotic partition of energy in micromorphic thermopiezoelectricity. *J Therm Stresses* 2011; 34: 1241–1249.

[43] Galeş, C. Some results in micromorphic piezoelectricity. *Eur J Mech—A/Solids* 2012; 31: 37–46.

[44] Grekova, EF, and Maugin, GA. Modelling of complex elastic crystals by means of multi-spin micromorphic media. *Int J Eng Sci* 2005; 43: 494–519.

[45] Maugin, GA. Electromagnetism and generalized continua. In: Altenbach H and Eremeyev V (eds) *Generalized Continua from the Theory to Engineering Applications* (CISM International Centre for Mechanical Sciences, Vol. 541). New York: Springer-Verlag, 2013, pp.301–360.

[46] Madeo, A, Neff, P, Ghiba, ID, et al. Wave propagation in relaxed linear micromorphic continua: Modelling metamaterials with frequency band-gaps. *Cont Mech Therm* 2013; doi: 10.1007/s00161-013-0329-2

[47] Andreas, U, Dell’Isola, F, and Porfiri, M. Piezoelectric passive distributed controllers for beam flexural vibrations. *J Vib Control* 2004; 10(5): 625–659.

[48] Dell’Isola, F, and Vidoli, S. Continuum modelling of piezoelectromechanical truss beams: An application to vibration damping. *Arch Appl Mech* 1998; 68(1): 1–19.

[49] Maurini, C, Pouget, J, and Dell’Isola, F. Extension of the Euler–Bernoulli model of piezoelectric laminates to include 3D effects via a mixed approach. *Comput Struct* 2006; 84(22–23): 1438–1458.

[50] Maurini, C, Dell’Isola, F, and Pouget, J. On models of layered piezoelectric beams for passive vibration control. *J Phys* 2004; 115: 307–316.

[51] Porfiri, M, Dell’Isola, F, and Santini, E. Modeling and design of passive electric networks interconnecting piezoelectric transducers for distributed vibration control. *Int J Appl Electrom* 2005; 21(2): 69–87.

[52] Vidoli, S, and Dell’Isola, F Vibration control in plates by uniformly distributed PZT actuators interconnected via electric networks. *Arch Appl Mech—A/Solids* 2001; 20(3): 435–456.

[53] Adams, RA. *Sobolev Spaces* (Pure and Applied Mathematics, Vol. 65). 1st edition. London: Academic Press, 1975.

[54] Girault, V, and Raviart, PA. *Finite Element Approximation of the Navier–Stokes Equations* (Lecture Notes in Mathematics, Vol. 749). Heidelberg, Germany: Springer-Verlag, 1979.

[55] Leis, R. *Initial Boundary Value problems in Mathematical Physics*. Stuttgart, Germany: Teubner, 1986.

[56] Eringen, AC, and Suhubi, ES. Nonlinear theory of simple microelastic solids: II. *Int J Eng Sci* 1964; 2: 389–404.

[57] Smith, AC. Inequalities between the constants of a linear micro-elastic solid. *Int J Eng Sci* 1968; 6: 65–74.

[58] Vrabie, I. *Differential equations. An introduction to basic concepts*. London: World Scientific, 2004.

[59] Dell’Isola, F, Madeo, A and Placidi, L. Linear plane wave propagation and normal transmission and reflection at discontinuity surfaces in second gradient 3D continua. *Z Angew Math Mech* 2012; 92(1): 52–71.
In particular, each classical solution of the problem satisfies the variation of constants formula

\[ u(t) = S(t-a)\xi + \int_0^t S(t-s)f(s)ds. \quad \text{(VCF)} \]

In particular, each classical solution of the problem (PC) is given by (VCF).