One dimensional SU(3) bosons with δ function interaction

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Abstract

In this paper we solve one dimensional SU(3) bosons with repulsive δ-function interaction by means of Bethe ansatz method. The features of ground state and low-lying excited states are studied by both numerical and analytic methods. We show that the ground state is a SU(3) color ferromagnetic state. The configurations of quantum numbers for the ground state are given explicitly. For finite N system the spectra of low-lying excitations and the dispersion relations of four possible elementary particles (holon, antiholon, σ-coloron and ω-coloron) are obtained by solving Bethe-ansatz equation numerically. The thermodynamic equilibrium of the system at finite temperature is studied by using the strategy of thermodynamic Bethe ansatz, a revised Gaudin-Takahashi equation which is useful for numerical method are given. The thermodynamic quantities, such as specific heat, are obtain for some special cases. We find that the magnetic property of the model in high temperature regime is dominated by Curie’s law: $\chi \propto 1/T$ and the system has Fermi-liquid like specific heat in the strong coupling limit at low temperature.
I. INTRODUCTION

One of the main goals of theoretical physics during the past 40 years is to understand quantum systems involving many particles. Once the interaction between those particles were taken into account, the problem becomes complicated. Meanwhile, as long as their interaction is not sufficient weak, the perturbative methods that were powerful in many quantum mechanics text book become unreliable. In one dimension, various non-perturbative methods were proposed, among which the impact of exactly solvable theoretical models are undeniable. Particles with $\delta$-function interaction is a simple but interesting model. Lieb and Liniger [1] first solved a Bose system under periodic boundary condition in the case of spin-0 or in the absence of internal degree of freedom. The method they used is nowadays referred as coordinate Bethe ansatz. The extension of periodic boundary condition to the boundary condition of potential well of infinite depth were made by Gaudin [2] and Woynarivich [3]. The first attempt to develop the method applied in [1] to deal with spin-1/2 fermions was made by McGuire [4] who can, however, deal with the case of only one spin-down and the other spins keep up. Further step of considering two spin-down and others spin-up was made by Flicker and Lieb [5]. Gaudin [6] and Yang [7] successfully solved the problem for arbitrary number of spins in the state of spin-down. In Yang’s paper [7], the first non-trivial case of Yang-Baxter equation was introduced. Actually, the strategy for general multi-component systems was proposed in [7] though the explicit solution was give only for spin-1/2 particles (it is Fermi system then).

The literature in [7] was later extended by Sutherland [8] to the called any irreducible representation of permutation group. Actually, both Yang and Sutherland adopted antisymmetric wave function under permutating indistinguishable particles. Thus Yang solved the problem of 2-component fermions and Sutherland solved N-component fermions by means of coordinate Bethe ansatz. As 2-component system is mostly associated with “spin-1/2” system which is conventionally referred to Fermi system, the coordinate Bethe ansatz has not been employed to 2-component Bose systems till recently [10].
Along with the developments of quantum inverse scattering method, Kulish [11] discussed multi-component nonlinear Schrödinger equation in terms of quantum inverse scattering method (QISM) in order to re-derive the Bethe-ansatz equations of Yang [7] and Sutherland [8]. The author explicitly formulated 2-component case and conjectured that Sutherland’s results would be obtained by repeating his procedure $n - 2$ times. Actually, it is not possible because the author [11] adopted commutation (instead of anti-commutation) relations but the system both Yang and Sutherland considered is Fermi system. It is now clear that the first quantization form of the system which Kulish considered ought to be a system of SU($n$) bosons with $\delta$-function interaction. The QISM was also employed to nonlinear Schrödinger equation for graded matrix but it breaks the Yang-Baxter relation at first [12] that was noticed and overcome later on [13].

Although the Bethe-ansatz equations for 2-component bosons was early formulated by Kulish, the nature of ground state and the property of low-lying excitations has never been exposed till the paper of Li et. al. [10]. We know that not only the 2-component Bose gas can be formed in magnetically trapped $^{87}\text{Rb}$ [14], but also a 3-component Bose gas can be produced in an optically trapped $^{23}\text{Na}$ [15], it will be valuable to study the model of 3-component Bose system. In present paper, we study a system of three-component bosons with SU(3) symmetry in one dimension. On the basis of Bethe-ansatz equations, we discuss the ground state, low-lying excited states and the thermodynamics of the system at finite temperature, and try to obtain thermal coefficients for some special cases. Our paper is organized as follows: In the following section we introduce the model and the corresponding Bethe-ansatz equations for charge rapidity and color rapidities. In Sec. III, we explicitly show that the ground state is a color ferromagnetic state and how the quantum numbers in Bethe-ansatz equations should be taken for the ground state. In Sec. IV, We study the low-lying excited states extensively by analyzing the possible variations in the sequence of quantum numbers. Numerical results of energy momentum spectra for each excitation are given. Furthermore, the dispersion relation of four possible elementary particles are obtained. In Sec. V we discuss the general thermodynamics of the system with the strategy
of thermodynamic Bethe ansatz (TBA) which was proposed by C.N. Yang and C.P. Yang when they study Bose gas with $\delta$-function interaction in one-dimension [20]. In Sec. VI we discuss the system for some special cases and obtain some analytical results. In Sec. VII a brief summary is given.

II. THE MODEL AND ITS BETHE-ANSATZ SOLUTION

We consider interacting SU(3) Bose field in a one dimensional ring of length $L$. The model Hamiltonian of the system reads,

$$H_0 = \int dx \left[ \sum_a \partial_x \psi_a^* \partial_x \psi_a + \frac{c}{2} \sum_{a,b} \psi_a^* \psi_a \psi_b^* \psi_b \right],$$

where natural unit is adopted for simplicity. Here $c$ is the coupling constant and $a,b = 1,2,3$ (we call colors hereafter) denotes the three states that carry out the fundamental representation of SU(3) group. The fields obey the following commutation relations,

$$[\psi_a^*(x), \psi_b(y)] = \sum_n \delta_{ab} \delta(x - y - nL).$$

In the terminology of group theory, the three states $|1\rangle$, $|2\rangle$ and $|3\rangle$ are labeled by weight vectors, $(1/2,0)$, $(-1/2,1/2)$ and $(0,-1/2)$ respectively. The two $su(2)$ subalgebra in the $su(3)$ Lie algebra are $[T^+, T^-] = 2T^z$ and $[U^+, U^-] = 2U^z$. With the help of those “flipping” operators, $U^\pm$ and $T^\pm$, we can generate the three states from the highest weight state $|1\rangle$, i.e.,

$$T^-|1\rangle = |2\rangle, \quad U^-|2\rangle = |3\rangle.$$

With additional commutation relations defined by

$$V^+ = [T^+, U^+], \quad V^- = [T^-, U^-],$$

the Chevalley bases of the $su(3)$ Lie algebra consists of eight generators $\{T^\pm, U^\pm, V^\pm, T^z, U^z\}$. 
In the domain with \( x_i \neq x_j \), the Hamiltonian (1) reduces to the one for free bosons and its eigenfunction is therefore just the superposition of plane waves. When two particles collide with each other, a scattering process occurs. The coordinate Bethe ansatz embodies that this process is purely elastic, i.e., exchange of their momenta. So for a given momentum \( k = (k_1, k_2, \ldots, k_N) \), the scattering momenta include all permutations of the components of \( k \). Thus for the case of \( N \) bosons, due to the Hamiltonian is invariant under the action of the permutation group \( S_N \), one can adopt the following Bethe-ansatz wave function,

\[
\Psi_a(x) = \sum_{P \in S_N} A_a(P, Q)e^{i(Pk|Qx)}, \quad x \in C(Q),
\]

where \( a = (a_1, a_2, \ldots, a_N) \), \( a_j \) denotes the color label of the \( j^{th} \) particle; \( Pk \) stands for the image of a given \( k := (k_1, k_2, \ldots, k_N) \) by a mapping \( P \in S_N \) and the coefficients \( A(P, Q) \) are functionals of \( P \) and \( Q \) where the \( Q \) denotes a permutation of the coordinates which defines a region with \( 0 < x_{Q_1} < x_{Q_2} < \ldots < x_{Q_N} < L \). For a Bose system, the wave function is supposed to be symmetric under any permutation of both coordinates and color indices, i.e.

\[
(\Pi^i\Psi)_a(x) = \Psi_a(x),
\]

where \( \Pi^i : \{a_1, \ldots, a_j, a_{j+1}, \ldots\} \mapsto \{a_1, \ldots, a_{j+1}, a_j, \ldots\} \) and \( (\Pi^i\Psi)_a \) is well defined by \( \Psi_{\Pi^i\Pi^j}(\Pi^i x) \). Furthermore, using the identity \((Pk|\Pi^i Qx) = (\Pi^i Pk|Qx)\) and rearrangement theorem of group theory, we have the following consequence from (4):

\[
A_a(P, \Pi^i Q) = A_{\Pi^i\Pi_a}(P, Q),
\]

The \( \delta \)-function term in the Hamiltonian (1) contributes a boundary condition across the hyper-plane \( x_{Q_j} = x_{Q_{j+1}} \),

\[
i((P_k)_{j} - (P_k)_{j+1})[A_a(P, \Pi^i Q) - A_a(\Pi^i P, \Pi^i Q) - A_a(P, Q) + A_a(\Pi^i P, Q)] = 2c[A_a(P, Q) + A_a(\Pi^i P, Q)],
\]

By making use of the relations (5) and (6) together with the continuity condition, we can obtain the following relation.
\( A_a(\Pi P, Q) = \frac{i[(Pk)_{j} - (Pk)_{j+1}]P^j + c}{i[(kP)_{j} - (Pk)_{j+1}] - c} A_a(P, Q), \) \hspace{1cm} (7)

where \( P^j \) permutes the color labels of bosons located at \( x_Q, j \) and \( x_Q, j+1 \).

Applying the periodic boundary condition \( \Psi(\cdots, x_Q, \cdots) = \Psi(\cdots, x_Q + L, \cdots) \) and making use of the standard procedure of quantum inverse scattering method [8,11,16], one can obtain the Bethe-ansatz equations

\[
e^{ik_jL} = -\prod_{l=1}^{N} \frac{k_j - k_l + ic}{k_j - k_l - ic} \prod_{\nu=1}^{M} \frac{k_j - \lambda_{\nu} - ic/2}{k_j - \lambda_{\nu} + ic/2},
\]

\[
1 = -\prod_{l=1}^{N} \frac{\lambda_{\gamma} - k_l - ic/2}{\lambda_{\gamma} - k_l + ic/2} \prod_{\nu=1}^{M} \frac{\lambda_{\gamma} - \lambda_{\nu} + ic}{\lambda_{\gamma} - \lambda_{\nu} - ic} \times \prod_{\alpha=1}^{M'} \frac{\lambda_{\gamma} - \mu_{\alpha} - ic/2}{\lambda_{\gamma} - \mu_{\alpha} + ic/2},
\]

\[
1 = -\prod_{\nu=1}^{M} \frac{\mu_{\beta} - \lambda_{\nu} - ic/2}{\mu_{\beta} - \lambda_{\nu} + ic/2} \prod_{\alpha=1}^{M'} \frac{\mu_{\beta} - \mu_{\alpha} + ic}{\mu_{\beta} - \mu_{\alpha} - ic}, \hspace{1cm} (8)
\]

The \( \lambda \) and \( \mu \) are SU(3) color rapidities. There are \( M - M' \) particles in the state \(|2\rangle\), \( M' \) in \(|3\rangle\) and \( N - M \) in \(|1\rangle\). We would like to mention here that the state obtained above is the highest weight state among the multiplet of SU(3) representation labeled by \( (N/2 + M'/2 - M, M/2 - M') \). The other states in the multiplets can be generated by iterate application of the flipping operators \( T^- \) and \( U^- \).

Taking logarithm of Eqs. (8) we have secular equations,

\[
k_jL = 2\pi I_j + \sum_{l=1}^{N} \Theta_1(k_j - k_l) + \sum_{\nu=1}^{M} \Theta_{-1/2}(k_j - \lambda_{\nu}),
\]

\[
2\pi J_{\gamma} = \sum_{l=1}^{N} \Theta_{-1/2}(\lambda_{\gamma} - k_l) + \sum_{\nu=1}^{M} \Theta_1(\lambda_{\gamma} - \lambda_{\nu}) + \sum_{\alpha=1}^{M'} \Theta_{-1/2}(\lambda_{\gamma} - \mu_{\alpha}),
\]

\[
2\pi J'_{\beta} = \sum_{\nu=1}^{M} \Theta_{-1/2}(\mu_{\beta} - \lambda_{\nu}) + \sum_{\alpha=1}^{M'} \Theta_1(\mu_{\beta} - \mu_{\alpha}), \hspace{1cm} (9)
\]

where \( \Theta_n(x) = -2 \tan^{-1}(x/nc) \). The quantum number \( I_j \) for charge rapidity \( k_j \) takes integer or half-integer depending on whether \( N - M \) is odd or even. The quantum number \( J_{\gamma} \) and \( J'_{\beta} \) for SU(3) color rapidities \( \lambda_{\gamma} \) and \( \mu_{\beta} \) take integer or half-integer depending on whether
\(N - M - M'\) and \(M - M'\) is odd or even respectively. Once all roots \(\{k_j, \lambda_\gamma, \mu_\beta\}\) are solved from the above equations (9) for a given set of quantum numbers \(\{I_j, J_\gamma, J'_\beta\}\), the energy and momentum will be calculated by

\[
E = \sum_{j=1}^{N} k_j^2, \quad p = \frac{2\pi}{L} \left[ \sum_{j=1}^{N} I_j - \sum_{\gamma=1}^{M} J_\gamma - \sum_{\beta=1}^{M'} J'_\beta \right],
\]

(10)

where the second equation of Eqs. (10) is obtained from Eqs. (9) directly.

For a state with real roots \((k, \lambda, \mu)\), we may define the distribution densities \(\rho(k), \sigma(\lambda)\) and \(\omega(\mu)\)

\[
\rho(k_j) = \frac{1}{L(k_{j+1} - k_j)}, \quad \sigma(\lambda_\gamma) = \frac{1}{L(\lambda_{\gamma+1} - \lambda_\gamma)}, \quad \omega(\mu_\beta) = \frac{1}{L(\mu_{\beta+1} - \mu_\beta)}.
\]

(11)

In terms of those densities the energy and momentum become

\[
E/L = \int k^2 \rho(k) dk, \quad p/L = \int k \rho(k) dk.
\]

(12)

While \(N, M\) and \(M'\) are determined by

\[
N/L = \int \rho(k) dk, \quad M/L = \int \sigma(\lambda) d\lambda, \quad M'/L = \int \omega(\mu) d\mu.
\]

(13)

As the SU(3) “magnetic” field is characterized by two parameters \(H_1\) and \(H_2\), the Zeeman term is given by

\[
\mathcal{H}_{zee} = H_1(N - 2M + M')/2 + H_2(M - 2M')/2
\]

\[
= \frac{H_1 L}{2} \int \rho(k) dk + \frac{(H_2 - 2H_1)L}{2} \int \sigma(\lambda) d\lambda
\]

\[
+ \frac{(H_1 - 2H_2)L}{2} \int \omega(\mu) d\mu.
\]

(14)
III. THE GROUND STATE

It is easy to show that the first equation of Eqs. (9) is a monotonously increasing function of $k_j$, that is if $I_i < I_j$ we have $k_i < k_j$. So the configuration of $\{I_j\}$ for the ground state is given by successive integers or half integers that is symmetrically arranged around zero, i.e., $I_{j+1} - I_j = 1$. In order to observe the properties of $\{J_\gamma, J'_\beta\}$, it is useful to investigate the Eqs. (9) in the weak coupling limit $c \to 0$. Due to $\Theta_{\pm n}(x) \to \mp \pi \text{sgn}(x)$, Eqs. (9) become

$$2I_j = k_j L/\pi + \sum_{l=1}^{N} \text{sgn}(k_j - k_l) - \sum_{\nu=1}^{M} \text{sgn}(k_j - \lambda_\nu),$$

$$2J_\gamma = \sum_{l=1}^{N} \text{sgn}(\lambda_\gamma - k_l) - \sum_{\nu=1}^{M} \text{sgn}(\lambda_\gamma - \lambda_\nu) + \sum_{\alpha=1}^{M'} \text{sgn}(\lambda_\gamma - \mu_\alpha),$$

$$2J'_\beta = \sum_{\nu=1}^{M} \text{sgn}(\mu_\beta - \lambda_\nu) - \sum_{\alpha=1}^{M'} \text{sgn}(\mu_\beta - \mu_\alpha).$$

(15)

We can choose the subscripts of the rapidities $k_j, \lambda_\gamma, \mu_\beta$ in such a way that $I_j, J_\gamma, J'_\beta$ are all ranged in an increasing order. Then we have

$$2(I_{j+1} - I_j - 1) = \frac{L}{\pi}(k_{j+1} - k_j) - \sum_{\nu=1}^{M} [\text{sgn}(k_{j+1} - \lambda_\nu) - \text{sgn}(k_j - \lambda_\nu)],$$

$$2(J_{\gamma+1} - J_\gamma + 1) = \sum_{l=1}^{N} [\text{sgn}(\lambda_{\gamma+1} - k_l) - \text{sgn}(\lambda_\gamma - k_l)] + \sum_{\alpha=1}^{M'} [\text{sgn}(\lambda_{\gamma+1} - \mu_\alpha) - \text{sgn}(\lambda_\gamma - \mu_\alpha)],$$

$$2(J'_{\beta+1} - J'_\beta + 1) = \sum_{\nu=1}^{M} [\text{sgn}(\mu_{\beta+1} - \lambda_\nu) - \text{sgn}(\mu_\beta - \lambda_\nu)].$$

(16)

Therefore, if $J'_{\beta+1} - J'_\beta = m$, there must exist $m + 1$ solutions of $\lambda_\nu$ satisfying $\mu_\beta < \lambda_\nu < \mu_{\beta+1}$; and if $J_{\gamma+1} - J_\gamma = n$, there must be $n + 1$ solutions of $k_l$ and $\mu_\alpha$ satisfying $\lambda_\gamma < k_l, \mu_\alpha < \lambda_{\gamma+1}$. So the existence of a $\lambda_\nu$ between two $\mu$’s has a positive contribution to the density of $\mu$ (11), and vice versa for $\mu$ to $\lambda$. However, from the first equation of Eqs. (16), for $I_{j+1} - I_j = n$, there will be $k_{j+1} - k_j = 2n\pi/L$ if there is a $\lambda_\gamma$ such that $k_j < \lambda_\gamma < k_{j+1}$, otherwise $k_{j+1} - k_j = 2(n - 1)\pi/L$. So a rapidity of $\lambda_\gamma$ always repels the $k$ rapidity away from that value. As a result, an existing $\lambda_\gamma$ will suppress the density of state in $k$-space at the point $k = \lambda_\gamma$. The weaker the coupling, the more magnificent the effect
will be (see Fig.2). And for a given set \( \{I\} \), the more \( \lambda \) rapidities there are, the higher the energy is.

It is also useful to observe Eq. (16) in the strong coupling limit. We consider two cases: \( M = 0 \) and \( M = 1 \). For \( M = 0 \), the secular equation becomes

\[
k_j L = 2\pi I_j + \sum_{l=1}^{N} \Theta_1(k_j - k_l).
\]

(17)

and for \( M = 1 \) we have

\[
k'_j L = 2\pi I'_j + \sum_{l=1}^{N} \Theta_1(k'_j - k'_l) + \Theta_{-1/2}(k'_j - \lambda_1).
\]

(18)

Here \( I_j - I'_j = 1/2 \) due to \( M \) changing from zero to one. As \( c \rightarrow \infty \), we have \( \tan^{-1}(x/c) \sim x/c \). So the above two equations become

\[
(k_{j+1} - k_j) L \left[ 1 + \frac{2N}{Lc} \right] = 2\pi,
\]

\[
(k'_{j+1} - k'_j) L \left[ 1 + \frac{2(N-1)}{Lc} \right] = 2\pi.
\]

(19)

Whence the distribution is almost a histogram. Referring to Eq. (11) the value of the density distribution for \( M = 0 \) is larger than that for \( M = 1 \), which makes the Fermi momentum for later case to be larger than the former case so that to keep the total number of particles being the same. Therefore the state of \( M = 0 \) has lower energy.

Differing from the SU(3) fermionic model [8] and a toy model of quark cluster [9], the ground state of SU(3) bosonic model is no more a color singlet but a color ferromagnetic state. The difference is due to the distinct permutation symmetries. For \( N \) particles, the ground state is characterized by a one-row \( N \)-column Young tableau \([N]\) whose quantum-number configurations are

\[
\{I_j^0\} = \{-(N-1)/2, \cdots, (N-1)/2\}
\]

\[
M = M' = 0.
\]

(20)

The density of states for the ground state is plotted in Fig. 1 for various coupling with \( L = N = 41 \).
In the thermodynamic limit, the density corresponding to the configuration of quantum numbers of the ground state satisfies the integral equation

$$\rho_0(k) = \frac{1}{2\pi} + \int_{-k_F}^{k_F} K_2(k - k')\rho_0(k')dk'. \tag{21}$$

Here $\rho_0(k), k_F$ are respectively the density and integration limit for the ground state, and

$$K_n(x) = \frac{1}{\pi} \frac{nc/2}{n^2c^2/4 + x^2},$$

The concentration is given by

$$D = N/L = \int_{-k_F}^{k_F} \rho_0(k)dk. \tag{22}$$

From Eq. (21) and Eq. (22), we can determine $\rho_0(k)$ and $k_F$. Here $k_F$ is a quasi-Fermi momentum because the wave function vanishes for any $k_j = k_l$ ($j \neq l$) as long as $c \neq 0$ even in Bose system which can be seen from Eq.(7). The energy can be calculated by

$$E_0/L = \int_{-k_F}^{k_F} k^2\rho_0(k)dk. \tag{23}$$

which is explicitly $\frac{1}{3}\pi^2 D^3(1 - \frac{4}{cD})$ in the strong coupling limit. In the general case one needs to solve the equations numerically. We show the ground state energy for particle densities $D = 1.0, 0.75, 0.5$ in Fig. 3.

**IV. LOW-LYING EXCITED STATES**

The low-lying excited states are obtained by varying the configuration $\{I_j, J_{\gamma}, J'_{\beta}\}$ from that of the ground state.

**Holon-antiholon excitation.** The simplest case is to remove one of $I$ from the configuration of the ground state and add a new one outside the original sequence, i.e.,

$$\{I_j\} = \{-(N - 1)/2, ..., n_1 - 1, n_1 + 1, ..., (N - 1)/2, I_n\},$$

where $|I_n| > (N - 1)/2$ and $M = M' = 0$. We call this holon-antiholon excitation which consists of a “holon” created under Fermi surface and an “antiholon” created outside it.
In Fig. 4, we plot the numerical results of energy-momentum spectrum for a system with \( L = N = 41 \) (the other part is just the mirror image of the plotted part corresponding to the state with \( p \rightarrow -p \) coming from the negative \( I_n \)). From the figure, we notice that there is a minimum in the excitation energy at \( p = 2\pi \) due to the fact that both \( I^0_0 \) replaced by \( I^0_0 = (N + 1)/2 \) and \( I^0_N \) share the same energy, their momenta difference, however, is \( 2\pi \). The overall structure of the spectrum is not changed obviously between \( c = 1 \) and \( c = 10 \). For a system of finite size, the gap of holon-antiholon excitations opens. In the thermodynamic limit, however, it vanishes.

In the configuration of quantum numbers for the ground state \((2, 0)\), replacing \( I^0_N = (N - 1)/2 + n, n = 1, 2, \cdots \) and keeping the others unchanged, we obtain the dispersion relation of antiholon (Fig. 7) by solving the Bethe ansatz equations (9) numerically. In a similar way, replacing \( I^0_n, n = 1, \cdots, N \) of \( \{I^0_j\} \) in turn by \((N + 1)/2\), we have the dispersion relation of holon which is shown in Fig. 8.

In the thermodynamic limit, it is plausible to calculate the excitation energy by making \( \rho(k) = \rho_0(k) + \rho_1(k)/L \) where \( \rho_0(k) \) is the density of the ground state. By creating a hole inside the quasi Fermi sea \( \bar{k} \in [-k_F, k_F] \) and an additional \( k_p > k_F \) outside it, we have the

\[
\rho_1(k) + \delta(k - \bar{k}) = \int_{-k_F}^{k_F} dk' \rho_1(k') K_2(k - k') + K_2(k - k_p). \tag{24}
\]

The excitation energy consists of two terms \( \Delta E = \int k^2 \rho_1(k)dk + k_p^2 = \varepsilon_h(\bar{k}) + \varepsilon_a(k_p) \).

The holon energy \( \varepsilon_h \) and antiholon energy \( \varepsilon_a(k_p) = -\varepsilon_h(k_p) \) are given by,

\[
\varepsilon_h(\bar{k}) = -\bar{k}^2 + \int_{-k_F}^{k_F} k^2 \rho^h_1(k, \bar{k}) dk,
\]

\[
\rho^h_1(k, \bar{k}) = -K_2(k - \bar{k}) + \int_{-k_F}^{k_F} K_2(k - k') \rho^h_1(k', \bar{k}) dk'. \tag{25}
\]

**Holon-coloron excitation:** Excitations related to the color sector is characterized by adding \( \lambda \) and \( \mu \) rapidities into the system. The simplest excitation of this type is obtained by considering \( M = 1 \) which is labeled by \((N/2 - 1, 1/2)\). Comparing to the ground state,
the quantum number changes from half-integer to integer or vice versa. We call this type
excitation $\sigma$-coloron which is regarded as an elementary quasi-particle of the present model.
It’s quantum number takes

$$I_1 = -N/2 + \delta_{1,j_1} \quad (1 \leq j_1 \leq N + 1),$$

$$I_j = I_{j-1} + 1 + \delta_{j,j_1} \quad (j = 2, \ldots, N),$$

while $J_1 = I_1 + m \quad (m = 1, 2, \ldots, N - 1)$ so that $I_1 < J_1 < I_N$. This produces a $N - 1$
multiplets. The excitation spectrum are plotted in Fig. 5 for a system of $N = L = 21$ with $c = 10, 1$ respectively.

Adding an additional $\lambda$ rapidity to the color ferromagnetic ground state brings about a
hole in the $k$-sector. Now we have two-parameter excitation $\Delta E = \int k^2 \rho_1(k)dk$ where the
$\rho_1(k)$ solves

$$\rho_1(k) + \delta(k - \bar{k}) = \int_{-k_F}^{k_F} K_2(k - k') \rho_1(k')dk'$$

$$-K_1(k - \lambda). \quad (26)$$

The energy of the holon-coloron excitation consists of two terms $\Delta E = \varepsilon_h(\bar{k}) + \varepsilon_c(\lambda)$. The $\varepsilon_h$ is determined by Eqs. (25) and the $\varepsilon_c$ defined by $\varepsilon_c(\lambda) = \int k^2 \rho_1^c(k, \lambda)$ with

$$\rho_1^c(k, \lambda) = -K_1(k - \lambda)$$

$$+ \int_{-k_F}^{k_F} K_2(k - k') \rho_1^c(k', \lambda)dk'. \quad (27)$$

The $\varepsilon_h(\bar{k})$ and $\varepsilon_c(\lambda)$ are energies of holon and $\sigma$-coloron whose dispersons are shown in Fig. 8 and Fig. 9 respectively.

Furthermore, the overall structure for the case of $c = 1$ in Fig. 5 differs from the case
of $c = 10$. We interpret the phenomenon as being due to the fact that the dependence of
the dispersion relation of holon and $\sigma$-coloron on the coupling constant are different. This
feature can be concluded from Fig. 8 and Fig. 9. When $c$ decreases, the $\varepsilon_h(p)$ decreases
while $\varepsilon_c(p)$ increases.
The **σ-type coloron-coloron excitation**: Leaving the configuration of the ground state \( \{I^0\} \) unchanged and changing \( M \) from zero to \( M = 2 \), which corresponds to \( (N/2 - 2, 1) \), a two parameters excitation in \( \lambda \)-sector is characterized by

\[-(N - 1)/2 < J_1 < J_2 < (N - 1)/2.\]

There are totally \( N(N - 3)/2 \) possible choices for such type of excitation. In Fig. 6, we plot the numerical result of energy-momentum spectrum for a system with \( N = L = 21 \).

The excitation energy of coloron(\( σ \) type)-coloron (\( σ \) type) can be calculated by \( \Delta E = \int k^2 \rho_1^c(k, \lambda_1, \lambda_2) dk \), where \( \rho_1^c(k, \lambda_1, \lambda_2) \) is determined by

\[
\rho_1^c(k, \lambda_1, \lambda_2) = -K_1(k - \lambda_1) - K_1(k - \lambda_2) + \int_{-k_F}^{k_F} K_2(k - k') \rho_1^c(k', \lambda_1, \lambda_2) dk'.
\]

It consists of two terms \( \Delta E = \varepsilon_c(\lambda_1) + \varepsilon_c(\lambda_2) \), where the coloron energy \( \varepsilon_c(\lambda) \) has been give in the passage before Eq. (27).

**Dispersion relation of \( ω \)-coloron**: The fourth possible excitation involves both the additional quantum number \( J \) and \( J' \). For \( M = 2 \) and \( M' = 1 \) there is no range for \( J' \) varying, but for large \( M \) the excitation is no more low-lying excitations. So we only show its dispersion relation which is described by the following configuration

\[
\{I_j\} = \{- (N - 1)/2, \cdots, (N - 1)/2\},
\]

\[
\{J_\gamma\} = \{- M/2, \cdots, (M - 2)/2\},
\]

\[
J'_1 = -M/2 + 1, \cdots, M/2 - 1,
\]

for a given \( M \). We plotted the dispersion relation of \( ω \)-coloron in Fig. 10 by varying \( J'_1 \) for a system with \( L = N = 40 \). The figure has a minimum around \( p = \pi \) when \( M = N/2 \).

Up to now, we have discussed three low-lying excitation energies and the dispersion relations of four possible elementary particles, holon, antiholon, \( σ \)-coloron and \( ω \)-coloron. We found those low-lying excitations are gapless in thermodynamic limit.
V. THERMODYNAMICS AT FINITE TEMPERATURE

For the ground state (i.e. at zero temperature), the charge rapidity $k_s$ are real roots of the Bethe-ansatz equations (9). For the excited state, however, the $\lambda$ and $\mu$ rapidity can be complex roots [17,18] which always form a “bound state” with several other $\lambda$s. This arises from the consistency of both hand sides of the Bethe-ansatz equations [19] in the limit $L \to \infty$, $N \to \infty$. The n-string rapidity is defined by

$$\Lambda^n_{aj} = \lambda^n_a + (n + 1 - 2j)i\nu + O(\exp(-\delta N)),$$
$$U^n_{aj} = \mu^n_a + (n + 1 - 2j)i\nu + O(\exp(-\delta N))$$

(30)

where $u = c/2$, $j = 1, 2 \cdots n$. The total number of $\lambda$ and $\mu$ are determined by

$$M = \sum_{n=1}^{\infty} nM_n; \quad M' = \sum_{n=1}^{\infty} nM'_n.$$

(31)

where $M_n$ and $M'_n$ denote the number of $\lambda$ n-strings and $\mu$ n-string respectively. The Eqs. (9) become

$$k_j L = 2\pi I_j + \sum_l \Theta_1(k_j - k_l) + \sum_{an} \Theta_{-n/2}(k_j - \lambda^n_a),$$
$$2\pi J^n_a = \sum_l \Theta_{-n/2}(\lambda^n_a - k_l) + \sum_{bl,t \neq 0} A_{nlt} \Theta_{t/2}(\lambda^n_a - \lambda^n_b) + \sum_{ct} B_{nlt} \Theta_{-t/2}(\lambda^n_a - \mu^n_c),$$
$$2\pi J'^n_a = \sum_{bl} B_{nlt} \Theta_{t/2}(\mu^n_a - \lambda^n_b) + \sum_{cl,t \neq 0} A_{nlt} \Theta_{-t/2}(\mu^n_a - \mu^n_c).$$

(32)

where

$$A_{nlt} = \begin{cases} 
1, & \text{for } t = n + l, |n - l|, \\
2, & \text{for } t = n + l - 2, \cdots, |n - l| + 2, \\
0, & \text{otherwise.}
\end{cases}$$

and

$$B_{nlt} = \begin{cases} 
1, & \text{for } t = n + l - 1, n + l - 3, \cdots, |n - l| + 1, \\
0, & \text{otherwise.}
\end{cases}$$

and the quantum numbers $\{I_j, J^n_a, J'^n_a\}$ label the state which is no more the ground state.

Replacing $k_j, \lambda^n_a, \mu^n_a$ in Eqs. (32) by continuous variables $k, \lambda, \mu$ but keeping the summation
still over the solutions of these roots, we can consider the quantum number $I_j, J^a_n, J^m_a$ as functions $I(k), J^n(\lambda)$ and $J^m(\mu)$ given by Eqs. (32). Take $I(\lambda)$ as an example, when $I(k)$ passes through one of the quantum number $I_j$, the corresponding $k$ is equal to one of the roots $k_j$, so is for $J^n(\lambda)$ and $J^m(\mu)$. However, there may exist some integers or half-integers for which the corresponding $k(\lambda, \mu)$ is not in the set of roots. Such a situation is conventionally referred as a “hole”. In the thermodynamic limit, we may introduce the densities of real $k$, $\lambda$ n-string and $\mu$ n-string

$$
\rho(k) + \rho^h(k) = (1/L)dI(k)/dk,
$$

$$
\sigma_n(\lambda) + \sigma_n^h(\lambda) = (1/L)dJ^n(\lambda)/d\lambda,
$$

$$
\omega_n(\mu) + \omega_n^h(\mu) = (1/L)dJ^m(\mu)/d\mu.
$$

(33)

Then Eqs. (32) give rise to the following coupled integral equations,

$$
\rho + \rho^h = \frac{1}{2\pi} + \int K_2(k - k')\rho(k')dk' - \sum_n \int K_n(k - \lambda)\sigma_n(\lambda)d\lambda,
$$

$$
\sigma_n^h = \int K_n(\lambda - k)\rho(k)dk - \sum_{lt} A_{nt} \int K_t(\lambda - \lambda')\sigma_t(\lambda')d\lambda' + \sum_{lt} B_{nt} \int K_t(\lambda - \mu)\omega_t(\mu)d\mu,
$$

$$
\omega_n^h = \sum_{lt} B_{nt} \int K_t(\mu - \lambda)\sigma_t(\lambda)d\lambda - \sum_{lt} A_{nt} \int K_t(\mu - \mu')\omega_t(\mu')d\mu'.
$$

(34)

The $\sigma_n$ and $\omega_n$ arising from the definition (34) that occur in the left hand side have been moved to right hand side by including the $t = 0$ term in the summation. In terms of densities defined above, the total numbers of $\lambda$ and $\mu$ are given by

$$
M/L = \sum_n n \int \sigma_n(\lambda)d\lambda,
$$

$$
M'/L = \sum_n n \int \omega_n(\mu)d\mu.
$$

(35)
In the presence of SU(3) magnetic field $H_1$ and $H_2$, we can define two type of “magnetization” whose $z$-components are

$$T^z/L = \frac{1}{2} \int \rho(k) dk - \sum_n n \int \sigma_n(\lambda) d\lambda$$

$$+ \frac{1}{2} \sum_n n \int \omega_n(\mu) d\mu,$$

$$U^z/L = \frac{1}{2} \sum_n n \int \sigma_n(\lambda) d\lambda - \sum_n n \int \omega_n(\mu) d\mu. \quad (36)$$

Hence the energy contributed by the Zeemann term (14) is

$$E_{Zee} = H_1 T^z + H_2 U^z. \quad (37)$$

For given $\rho(k)$, $\rho^h(k)$, $\sigma_n(\lambda)$, $\sigma^h_n(\lambda)$, $\omega_n(\mu)$ and $\omega^h_n(\mu)$ the entropy has the form [20]

$$S/L = \int [(\rho + \rho^h) \ln(\rho + \rho^h) - \rho \ln \rho - \rho^h \ln \rho^h] dk$$

$$+ \sum_n \int [(\sigma_n + \sigma^h_n) \ln(\sigma_n + \sigma^h_n) - \sigma_n \ln \sigma_n - \sigma^h_n \ln \sigma^h_n] d\lambda$$

$$+ \sum_n \int [(\omega_n + \omega^h_n) \ln(\omega_n + \omega^h_n) - \omega_n \ln \omega_n - \omega^h_n \ln \omega^h_n] d\mu. \quad (38)$$

where the Boltzmann constant is set to unit.

At finite temperature, the thermal equilibrium is obtained by minimizing the free energy

$$F = E - E_{Zee} - TS - \mu N$$

where $\mu$ is the chemical potential and $S$ is the entropy of the system. Making use of the relations derived from Eqs. (34)

$$\delta \rho^h = -\delta \rho + \int K_2(k-k') \delta \rho dk' - \sum_n \int K_n(k-\lambda) \delta \sigma_n d\lambda,$$

$$\delta \sigma^h_n = \int K_n(\lambda - k) \delta \rho dk - \sum_{lt} A_{nl} \int K_\ell(\lambda - \lambda') \delta \sigma_\ell(\lambda') d\lambda' + \sum_{lt} B_{nlt} \int K_\ell(\lambda - \mu) \delta \omega_\ell(\mu) d\mu,$$

$$\delta \omega^h_n = \sum_{lt} B_{nlt} \int K_\ell(\mu - \lambda) \delta \sigma_\ell(\lambda) d\lambda - \sum_{lt} A_{nl} \int K_\ell(\mu - \mu') \delta \omega_\ell(\mu') d\mu', \quad (39)$$

and define

$$\frac{\rho^h(k)}{\rho(k)} = \kappa(k) = e^{e(k)/T}, \quad \frac{\sigma^h_n(\lambda)}{\sigma_n(\lambda)} = \eta_n(\lambda) = e^{\xi_n(\lambda)/T}, \quad \frac{\omega^h_n(\mu)}{\omega_n(\mu)} = \Delta_n(\mu) = e^{\xi_n(\mu)/T}. \quad (40)$$

We obtain the following conditions from the minimum condition $\delta F = 0$, namely
\[ \epsilon(k) = k^2 - \mu - H_1/2 - T \int K_2(k - k') \ln(1 + e^{-\epsilon(k')/T})dk' - T \sum_n \int K_n(k - \lambda) \ln[1 + e^{-\zeta_n(\lambda)/T}]d\lambda \]

\[ \zeta_n(\lambda) = n(2H_1 - H_2)/2 + T \int K_n(\lambda - k) \ln[1 + e^{-\epsilon(k)/T}]dk + T \sum_{l,t\neq0} A_{nlit} \int K_t(\lambda - \lambda') \ln[1 + e^{-\xi_l(\lambda')/T}]d\lambda' \]

\[ -T \sum_{l,t} B_{nlit} \int K_t(\lambda - \mu) \ln[1 + e^{-\xi_l(\mu)}/T]d\mu, \]

\[ \xi_n(\mu) = n(2H_2 - H_1)/2 - T \sum_{l,t} B_{nlit} \int K_t(\mu - \lambda) \ln[1 + e^{-\zeta_n(\lambda)}/T]d\lambda \]

\[ +T \sum_{l,t\neq0} A_{nlit} \int K_t(\mu - \mu') \ln[1 + e^{-\zeta_n(\mu')/T}]d\mu'. \]

(41)

A more useful version of Eqs. (41) is the recursive scheme which is a revised version of Gaudin-Takahashi equations, as obtained by Fourier transform.

\[ T \ln \kappa = k^2 - \mu - H_1/2 - TK_2(k) * \ln[1 + \kappa^{-1}] - T \sum_n K_n(k) * \ln[1 + \eta_n^{-1}], \]

\[ \ln \eta_1 = \frac{1}{4u} \text{sech}(\pi \lambda/2u) * \ln[(1 + \kappa^{-1})(1 + \eta_2)/(1 + \Delta_1^{-1})], \]

\[ \ln \eta_n = \frac{1}{4u} \text{sech}(\pi \lambda/2u) * \ln[(1 + \eta_{n-1})(1 + \eta_{n+1})/(1 + \Delta_n^{-1})], \]

\[ \ln \Delta_1 = \frac{1}{4u} \text{sech}(\pi \lambda/2u) * \ln[(1 + \Delta_2)/(1 + \eta_1^{-1})], \]

\[ \ln \Delta_n = \frac{1}{4u} \text{sech}(\pi \lambda/2u) * \ln[(1 + \Delta_{n-1})(1 + \Delta_{n+1})/(1 + \eta_n^{-1})]. \]

(42)

where * denotes a convolution. And these equations are complete by the asymptotic conditions

\[ \lim_{n\to\infty} [\ln \eta_n/n] = (2H_1 - H_2)/2T, \]

\[ \lim_{n\to\infty} [\ln \Delta_n/n] = (2H_2 - H_1)/2T. \]

(43)

Finally, we obtain the Helmholtz free energy \( F = E - TS \):

\[ F = \mu N - \frac{LT}{2\pi} \int \ln[1 + e^{-\epsilon}]dk, \]

(44)

and the pressure of the system

\[ P = -\frac{\partial F}{\partial L} = \frac{T}{2\pi} \int \ln[1 + e^{-\epsilon}]dk, \]

(45)

which is formally the same as Yang and Yang’s expression but the equation which \( \epsilon \) fulfills is different.
VI. SPECIAL CASES

In general, the free energy can be calculated by using formula (44), where \( \epsilon(k) \) and \( \zeta_n(\lambda) \) are determined from Eqs. (41) which can be solved by iteration. In the following we will consider some special cases respectively as explicit results are obtainable in those cases.

A. Zero temperature limit

The state at zero temperature is the ground state. When \( T \to 0 \), the first equation of Eqs. (41) becomes

\[
\epsilon(k) = k^2 - \mu - H_1/2 + \int K_2(k - k')\epsilon(k')dk' + \sum_n \int K_n(k - \lambda)\zeta_n(\lambda)d\lambda
\]

Then the Fermi surface is determined by \( \epsilon(k_F) = 0 \). Since there is no hole under Fermi surface, we can take the ratio \( \kappa = \rho^h/\rho \) as zero when \( k \in [-k_F, k_F] \). As a result, it is easy to see from Eqs. (42) that \( \eta_n = \Delta_n \to \infty \). That is \( M = M' = 0 \), the state is color ferromagnetic state. This is consistent with the conclusion obtained in Sec. IV. Then Eq. (46) can be rewritten as

\[
\epsilon_0(k) = k^2 - \mu - H_1/2 + \int_{-k_F}^{k_F} K_2(k - k')\epsilon_0(k')dk',
\]

which gives the solution of dressed energy [21], and the ground-state energy can be given in terms of \( \epsilon_0 \)

\[
E_0/L = \frac{1}{2\pi} \int_{-k_F}^{k_F} \epsilon_0(k)dk.
\]

whose dependence on the coupling constant is shown in Fig. (10).

B. High temperature limit

In the high temperature limit \( T \to \infty \), we can assume that all functions \( \eta_n(\lambda) \) and \( \Delta_n(\mu) \) are independent of their corresponding parameter. Due to \( \lim_{u \to 0} \frac{1}{2u} \text{sech}(\frac{\lambda}{2u}) = \delta(\lambda) \), Eqs. (42) can be written as follows,
\[ \eta_1^2 = (1 + \kappa^{-1})(1 + \eta_2)/(1 + \Delta_1^{-1}), \]
\[ \eta_n^2 = (1 + \eta_{n-1})(1 + \eta_{n+1})/(1 + \Delta_n^{-1}), \]
\[ \Delta_1^2 = (1 + \Delta_2)/(1 + \eta_1^{-1}), \]
\[ \Delta_n^2 = (1 + \Delta_{n-1})(1 + \Delta_{n+1})/(1 + \eta_n^{-1}). \quad (49) \]

with the asymptotic conditions Eqs. (43).

Performing the Fourier transform to Eqs. (34), we get the solution of the densities of \( \lambda \) n-strings,
\[ \sigma_1 + \sigma_1^h = \frac{1}{4u} \text{sech}[\pi \lambda/2u] \ast [\rho + \sigma_2^h + \omega_1], \]
\[ \sigma_n + \sigma_n^h = \frac{1}{4u} \text{sech}[\pi \lambda/2u] \ast [\sigma_{n-1}^h + \sigma_{n+1}^h + \omega_n], \]
\[ \omega_1 + \omega_1^h = \frac{1}{4u} \text{sech}[\pi \lambda/2u] \ast [\omega_2^h + \sigma_1] \]
\[ \omega_n + \omega_n^h = \frac{1}{4u} \text{sech}[\pi \lambda/2u] \ast [\omega_{n-1}^h + \omega_{n+1}^h + \sigma_n]. \quad (50) \]

If we assume that \( \sigma_n, \sigma_n^h \) and \( \omega_n, \omega_n^h \) are independent of \( \lambda \) and \( \mu \) respectively, or let \( u = 0 \), we have the following relation
\[ \sum_n n \sigma_n = \frac{\rho}{2} + \frac{1}{2} \sum_n n \omega_n + -\frac{n_m+1}{2} \sigma_{nm} e^{n_m \Omega_1/T}, \]
\[ \sum_n n \omega_n = \frac{1}{2} \sum_n n \sigma_n - \frac{n_{m'}+1}{2} \sigma_{nm'} e^{n_{m'} \Omega_2/T} \quad (51) \]

where \( n_m \) and \( n_{m'} \) are the maximal length of \( \lambda \) string and \( \mu \) string respectively, and \( \Omega_1 = 2H_1 - H_2, \Omega_2 = 2H_2 - H_1 \). In the absence of external field \( H_1 \) and \( H_2 \), it is easy to obtain \( M' = M/2 = N/3 \) which means there are \( N/3 \) particles in each internal state. Then we can also infer that the contribution of the internal degree of freedom to the entropy per site must be \( S = \ln 3 \) which follows from the fact that the internal degree of freedom per particle is three.

If the external field is small, expanding Eq. (51) for small field and integrating the equation over \( \lambda \) and \( \mu \), we get the SU(3) magnetization of the model,
\[ \frac{T^z}{L} = \frac{M_m}{2L} \left[ 1 + \frac{n_m \Omega_1}{T} + \frac{1}{2} \left( \frac{n_m \Omega_1}{T} \right)^2 + \ldots \right] \]
\[
\frac{U^z}{L} = \frac{M'_m}{2L} \left[ 1 + \frac{n_{m'}\Omega_2}{T} + \frac{1}{2} \left( \frac{n_{m'}\Omega_2}{T} \right)^2 + \cdots \right].
\]  

(52)

where \(M_m\) and \(M_{m'}\) are the total number of rapidities in \(\lambda n_m\)-strings and \(\mu n_{m'}\)-strings respectively. The first term in the parentheses of both equations arises from self-magnetization, while the others are contributed by external field. Eq. (52) indicates that the magnetic property of the model in high temperature regime is dominated by Curie's law \(\chi \propto 1/T\).

C. The strong coupling limit

For \(u \to \infty\), \(K_n(k)\) goes to zero, from Eqs. (41) we have

\[
\epsilon = k^2 - \mu. 
\]  

(53)

where the external field is set to unit. The \(k\)-sector are completely decoupled with \(\lambda\) and \(\mu\)-sectors. At arbitrary temperature, the solutions for the \(\eta_n, \Delta_n\) are independent of parameters \(\lambda\) and \(\mu\) respectively, which gives rise to Eq. (49) The free energy of the system defined by Eq. (44) can be solved by integration by part,

\[
\frac{F}{L} = \mu D - \frac{2}{\pi} \left[ \frac{1}{3} \frac{\mu^3}{2} + \frac{T^2 \pi^2}{24 \mu^{1/2}} \right]
\]  

(54)

where the external field is set to zero.

We are not able to deduce the specific heat directly from the free energy obtained above because the chemical potential is a function of temperature. From Eqs. (34), the density of charge rapidity has the form

\[
\rho = \frac{1}{2\pi} \frac{1}{1 + e^{(k^2 - \mu)/T}} 
\]  

(55)

Integrating the charge density over \(k\) space with the condition Eq. (13), we have an explicit expression of the chemical potential,

\[
\mu = \mu_0 \left[ 1 - \frac{\pi^2 T^2}{24 \mu_0^2} \right]^{-2}
\]  

(56)

where \(\mu_0 = \pi^2 D^2\) which denotes \(\mu\) at zero temperature. Then the free energy becomes
The free energy for the SU(3) invariant spin chain also has a $T^2$ dependence [22].

Since in thermodynamics $S = -\partial F/\partial T$ and $C_v = T\partial S/\partial T$, we find the specific heat at low temperature is Fermi-liquid like,

$$S = C_v = \frac{T}{6D}. \quad (58)$$

It is the same as the result of one-component case, since for the strong coupling limit the color degree of freedom and the charge degree of freedom are decoupled completely, the contribution of color degree of freedom to the free energy vanishes.

**VII. CONCLUSIONS**

In this paper, we have solved one dimensional SU(3) bosons with $\delta$-function interaction by means of coordinate Bethe-ansatz method. On the basis of Bethe-ansatz equations we first discussed the ground state of the Bose system and found that the ground state is a color ferromagnetic state which differs from the SU(3) Fermi system greatly. The configuration of quantum numbers for the ground state was given explicitly. The low-lying excitations were discussed extensively by both analytical and numerical methods. The energy-momentum spectra for three type excitations: holon-antiholon, holon-coloron($\sigma$ type) and coloron($\sigma$ type)-coloron($\sigma$ type) were plotted for $c = 10$ and $c = 1$. We also discussed the dispersion relations of four elementary quasi-particles.

The thermodynamics of the system were studied by using the strategy of TBA. A revised version of Gaudin-Takahashi equations were obtained by minimizing the free energy at finite temperature. We found the magnetic property of the system at high temperature regime is dominated by Curie’s law, and for the case of strong coupling the system possess the properties of Fermi-liquid like and its specific heat is a linear function of $T$ at low temperature.
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FIG. 1. The density of state in $k$-space for the ground state. The distribution changes from a histogram to a narrow peak gradually for the coupling from strong to weak. The figure is plotted for $N = L = 41$ and $c = 10, 1, 0.1, 0.01$.

FIG. 2. The density of state in $k$-space for the ground state in the presence of one color rapidity by choosing $J_1 = 0$. The distribution changes from a histogram to a narrow peak gradually for the coupling from strong to weak. The figure is plotted for $N = L = 100$ and $c = 10, 1, 0.1, 0.01$.
FIG. 3. The ground state energy $E/L$ versus the coupling constant $\ln c$ for different densities $D = 1.0, 0.75, 0.5, 0.25$.

FIG. 4. The holon-antiholon excitation spectrum calculated for $N = L = 41$ and $c = 10$ (left), $c = 1$ (right).

FIG. 5. The holon-coloron($\sigma$ type) excitation spectrum calculated for $N = L = 21$ and $c = 10$ (left), $c = 1$ (right).

FIG. 6. The coloron($\sigma$ type)-coloron($\sigma$ type) excitation spectrum calculated for $N = L = 21$ and $c = 10$ (left), $c = 1$ (right).
FIG. 7. The dispersion relation of antiholon excitation for different coupling constants where the curves from bottom to top correspond to $c = 1, 10, 20, 40, 80$ respectively. Here $N = L = 40$.

FIG. 8. The dispersion relation of holon for different coupling constants where the curves from bottom to top correspond to $c = 1, 10, 20, 40, 80$ respectively. Here $N = L = 40$.

FIG. 9. The dispersion relation of $\sigma$-coloron for different coupling constants where the curves from top to bottom correspond to $c = 1, 10, 20, 40, 80$ respectively. Here $N = L = 41$. 
FIG. 10. The dispersion relation of ω-coloron for different coupling constants where the curves from top to bottom correspond to $c = 1, 10, 20, 40, 80$ respectively. Here $N = L = 40$. Zero energy corresponds to $M = N/2$ ground state.