Effective loop quantized theories

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Abstract. We introduce notions of effective gauge field theory and coarse graining that do not use a background metric and are compatible with loop quantization.

1. Introduction
The notions of effective theory, coarse graining and Wilson’s renormalization group (RG) are an integral part of quantum field theory and are tightly related to the concept of scale. Thus, physical phenomena that do not involve a preferred background metric like gravity or any kind of matter coupled to gravity are not covered by the principles of ordinary quantum field theory.

In the context of gauge theories there is an implementation of the RG using lattices (LGT). The lattices themselves play the role of scale because one can write $L_1 \leq L_2$ in the sense that one lattice is contained in the other and also in the sense that all the observables associated to the coarser lattice can be recovered using observables of the finer lattice. It is customary to work in a torus of size $l$ using a family of nested regular lattices $L_n$ with lattice spacing $a_n = \frac{l}{2^n}$. If the RG has an attractive fixed point in the limit $a \rightarrow 0$ the theory has a continuum limit whose dynamics is selected by the RG.

General relativity can be formulated as a gauge theory, but if one wishes to represent the diffeomorphism symmetry it is impossible to use a single embedded lattice to host the theory. A solution to this problem is given by the kinematics of loop quantization “which is a lattice gauge theory for a lattice that is infinitely refined” (for a precise statement see [1]). However, there was no notion of effective theory that let one implement Wilson’s RG and that in the limit of infinite refinement selected a dynamics for the loop quantized theory.

The pioneering work of Markopoulou and Smolin sparked the first articles in search for an implementation of the RG the context of spin foam models [2]. Recently we gave a proposal for effective theory and coarse graining that is consistent with loop quantization both in its canonical (frozen time) and spin foam (spacetime) formulations [3]. It is steel not clear if it leads to the correct physics. Some modifications may be necessary, but its structure is promising. There has been a joined effort to study the proposal and its properties in several fronts.

In this talk I describe the foundations and structure of the proposal. A geometric study of the structures involved was carried out by Martinez and Meneses [4]. The plan to implement Wilson’s RG and a simple example is contained in the work of Manrique [5]. A first study of $U(1)$ Yang-Mills at weak coupling is also part of this collaboration [6]. All the references cited above can be found in this proceedings.
2. Heuristics

In a laboratory we set a detector array $C$ which measures some aspects of the configuration and momentum of a field. Our limited set of available observables captures a minimal portion of the degrees of freedom present in the field.

**What are the measurements from the $C$ array of detectors good for?**

If we are interested in phenomena in the “scale” of $C$ or coarser we may be able to formulate a **$C$-effective theory** that (at least approximately) describes that phenomena. If the details of the microscopic state of the system that we ignore are not relevant for the subsequent behavior of the system at macroscopic scale, such $C$-effective theory exists. We will assume it for the moment.

¿From the nature of our detectors, it should be clear what degrees of freedom our effective theory will have.

**But what would the dynamics be?**

We will provide the dynamics from a regularization of the dynamics of the continuum. Our procedure is the following:

- We will define our effective theory in terms of $C$-regular configurations that belong to the continuum set of configurations.
- This $C$-regular continuum configuration will be selected from the measurements of our $C$-array of detectors and $C$- regularity assumptions.
- Hence we can assign an action, Lagrangian or Hamiltonian (as needed) to our $C$-effective configuration. There should still be some unspecified coupling constants that will let us fine tune our $C$-effective theory to match experiments.
- Upgrading the laboratory setup to refine our detector array from $C_1$ to $C_2$, $C_1 \leq C_2$, is a change of measurement “scale” increasing the resolution giving us a more detailed knowledge of the system. Hence one needs weaker regularity assumptions which define a finer effective theory. Configurations of the finer effective theory include those of a coarser effective theory as specially regular configurations. Once in a finer effective theory one can coarse grain to infer a state of a coarser effective theory.
- Also the dynamics found for the $C_2$ effective theory should be able to predict the fate of any of the $C_1$ observables. That is, the fine tuning of the $C_1$ effective theory should not be independent of that of the $C_2$ effective theory. The two sets of coupling constants should be related by a renormalization prescription.

For concreteness consider a model situation in which we are interested to measure the electromagnetic field. At our disposal we have only a finite set of $E$-flux detectors and devices that measure the vector potential circulation (and hence the $B$-flux). If we arrange the $E$-flux detectors in our laboratory in a cellular configuration we may be able to draw interesting consequences like the total charge inside a cell whose walls are some of our detectors. Thus we fix our $E$-flux detectors in a cellular configuration $C$. The circulation of vector potential is measured along paths that are transverse to the flux detectors and arranged in a lattice dual to $C$.

Given our interest in canonical loop quantization we have special interest in the measurement of the $E$-flux detectors. They are our available set of commuting momentum observables. If our set of available observables was a complete set of commuting observables we would have all the information we could hope for and we would know the quantum state of the system. This state would be a charge network in $H_{\Sigma} = L^2(A_{\Sigma}/\delta_{\Sigma},d\mu_{AL})$.

**However, our set of detectors arranged in the configuration $C$ does not give us a complete set of measurements.** We will define a $C$-effective theory that will give us an irreducible representation for our available set of measurements. A set of measurements of our detectors will lead to a
quantum state in $\mathcal{H}_C = \mathcal{H}_S / \sim_C$, where $\sim_C$ identifies all the charge network states that are not resolved by our array of $E$-flux detectors.

In the next section we will prove that this $C$ effective theory can be constructed with a set of regularity assumptions on the vector potential. A measurement of our array of devises measuring the vector potential circulation characterizes the filed configuration once the mentioned regularity is assumed. Thus, the effective theory is actually a lattice gauge theory with Hilbert space $\mathcal{H}_C = L^2(\mathcal{A}_{C,\text{flat}}/\mathcal{G}_C, d\mu_C)$.

Before closing this section we stress the following conceptual points: Our construction of effective theories depends on an extension of the concept of scale. A ”scale” is given by a set of available measurements and a complementary set of regularity assumptions. In this talk we introduce them as taylored to a cellular decomposition of the manifold. Both gauge invariance and diffeomorphism invariance are broken by the regularity assumptions used in this talk. There is a remanent gauge symmetry in each effective theory which is much like a gauge group of lattice gauge theory. Thus the definition of effective theory that we use in this paper involves a gauge choice. In the following section we mention a result that implies that all the gauge choices lead to equivalent effective theories. The situation for diffeomorphism invariance is similar. There is an equivalent framework which works directly with gauge and diffeomorphism invariant effective theories, but we leave this for a separate publication.

3. $C$-Flat connections and coarse graining: definitions and some properties

In this section our definitions will be more precise than in the previous section. We begin with the definition of a cellular decomposition.

A cellular decomposition $C$ of a manifold is a presentation of it as a union of disjoint cells

$$M = \cup_{c\in C} c$$

where the cells are the image of open disks of dimensions from zero to $\text{dim } M$. A typical example is a triangulation of the sphere with four triangles, six edges and four vertices.

Two directed curves $\gamma_1, \gamma_2$ are defined to be $C$ equivalent, $\gamma_1 \sim_C \gamma_2$, if the sequence of cells that they induce coincide.

As compared to the previous section one must pay special attention to “non generic curves”; that is, we have to consider also curves passing through cells of co-dimension bigger than one.

Now we define the space of $C$-flat connections, $\mathcal{A}_{C,\text{flat}}$. $A \in \mathcal{A}_{C,\text{flat}}$ if and only if $h_{\gamma_1}(A) = h_{\gamma_2}(A)$ whenever $\gamma_1 \sim_C \gamma_2$.

Notice first that the “regularity” assumptions yield distributiuonal connections, $\mathcal{A}_{C,\text{flat}} \subset \tilde{A}_M$. Second, the lattice dual to $C$ contains representatives of “most” curves, but not all. Hence the space of connections on a lattice dual of $C$ can almost parametrize $\mathcal{A}_{C,\text{flat}}$, but not really. Then we complete the dual lattice to include curves $C$ equivalent to curves passing through cells of co-dimension bigger than one; call the resulting lattice $L(C)$. In the case of simplicial cellular decompositions $L(C)$ is the one skeleton of the baricentric subdivision, $L(C) = \text{Sd}(C)^1$.

Then we have described the isomorphism $\mathcal{A}_{C,\text{flat}} \approx \mathcal{A}_{L(C)}$ that will be used to parametrize the space of $C$ flat connections and to endow it with the Haar measure.

Now we will describe refining and coarse graining.

We say that a cellular decomposition $C_2$ is finer than cellular decomposition $C_1$ if each cell $c_\alpha \in C_1$ is the union of $C_2$ cells $c_\alpha = \cup c_\beta$ for some cells $c_\beta \in C_2$. We denote it by $C_1 \leq C_2$.

It should be clear that if $C_1 \leq C_2$ then $\mathcal{A}_{C_1,\text{flat}} \subset \mathcal{A}_{C_2,\text{flat}}$. The regularity assumptions that define the finer $C_2$ effective configurations are weaker than those assumed for the coarser $C_1$ effective configurations. Taking a configuration of an effective theory to a configuration space of a finer theory is called refining, and in our framework it is just an inclusion map.
To coarse grain we need to choose an embedding $\text{Emb}_C : L(C) \to M$. With this map we can pull back connections from $M$ to $L(C)$, and using the isomorphism we can define $\pi_C : \bar{A}_M \to \mathcal{A}_{C\text{-flat}}$.

Finally the coarse graining map

$$\pi_{C_2 \to C_1} : \mathcal{A}_{C_2\text{-flat}} \to \mathcal{A}_{C_1\text{-flat}}$$

is defined as the restriction of $\pi_{C_1}$ to $\mathcal{A}_{C_2\text{-flat}} \subset \bar{A}_M$.

Now we will mention a few properties of our construction.

Given three cellular decompositions $C_1 \leq C_2 \leq C_3$ we can construct three projection maps $\pi_{C_3 \to C_2}$, $\pi_{C_3 \to C_1}$ and $\pi_{C_2 \to C_1}$ after choosing corresponding embedding maps. For most choices $\pi_{C_3 \to C_1} \neq \pi_{C_2 \to C_1} \circ \pi_{C_3 \to C_2}$. However, there are embeddings that make the triangular diagram corresponding to the last expression commute.

Since the $C$ effective theory is isomorphic to a lattice gauge theory, it is clear that configuration observables correspond to holonomies and momentum observables correspond to left invariant vector fields. The set of these observables is an algebra under a Poisson bracket product and is called the holonomy flux algebra, $H-F(C)$.

The pull back of the coarse graining takes holomies from $H-F(C_1)$ to $H-F(C_2)$ and the pull back of the refining can be used to bring fluxes from $H-F(C_1)$ to $H-F(C_2)$. We call these map $\tilde{\pi}_{C_2 \to C_1}$. It turns out that

**Theorem 1** If $C_1 \leq C_2$

$$\tilde{\pi}_{C_2 \to C_1} : H-F(C_1) \to H-F(C_2)$$

is a $*$-algebra embedding.

For each $C$ effective theory we have a measure $d\mu_C$ and it is clear that if $C_1 \leq C_2$ then $\pi_{C_2 \to C_1} * d\mu_{C_2}$ is a faithful measure on $\mathcal{A}_{C_2\text{-flat}}$ and $\mathcal{A}_{C_2\text{-flat}}$ has measure zero inside $\mathcal{A}_{C_2\text{-flat}}$. However, there are sequences with increasingly finer cellular decompositions that carry significant information. Consider for example any triangulation $\Delta$ of the manifold and the sequence of cellular decompositions obtained by repeatedly performing baricentric subdivisions $C_i = Sd^i(\Delta)$ For any such sequence we have the following characterization of measures on the space of generalized connections:

**Theorem 2** Any measure on $\bar{A}_M$ is characterized by a sequence of compatible $(\pi_{C_{i+1} \to C_i} * d\mu_{C_{i+1}} = d\mu_{C_i})$ effective measures, $d\mu_{C_i}$ on $\mathcal{A}_{C_i\text{-flat}}$.

Another important property is the behavior of these spaces of effective connections under fiber bundle maps. The result says:

**Theorem 3** $\mathcal{A}_{C_2\text{-flat}} \subset \bar{A}_M$ is not left invariant by the pull back of fiber bundle maps. Instead

$$f^*(\mathcal{A}_{C_2\text{-flat}, A_0}) = \mathcal{A}_{f^{-1}C_2\text{-flat}, f^*A_0}$$

where $\tilde{f}$ is the restriction of $f$ to the base space and $A_0$ is the auxiliary flat connection implicit in our constructions.

For a cleaner presentation of this point see [4].
4. Quantum side and peculiarities

In the previous section we defined spaces of effective gauge fields $A_{C_{\text{flat}}}$ and coarse graining maps $\pi_C : A_{\Sigma} \rightarrow A_{C_{\text{flat}}}$. In this section we give the quantum counterparts and point out some of the peculiar features of our proposal.

Once we have defined a set of $C$ effective configurations that is parametrized by holonomies along edges of a lattice, it is clear that the relevant quantum states are square integrable functions of the lattice connection with respect to the Haar measure. These wave functions can be written in terms of the character expansion; or in other words, written in terms of a spin network basis.

Our coarse graining puts in correspondence degrees of freedom of an effective theory with “sharp degrees of freedom from the continuum”. The picture in the quantum theory is clear when one writes the action of the pull back of the coarse graining on the effective theory’s spin networks

$$\pi_C^*(j) = \text{Emb}_C(j).$$

If we see spin networks as quantum flux lines, the embedding chooses which sharp flux line in the continuum is responsible for a given flux line of the effective theory.

On the one hand coarse graining comes from giving effective theory significance to sharp configurations, and on the other hand there is a lot of freedom in the choice of a coarse graining map. First we note that it may be possible to consider a ensemble of these choices to cure both problems. Second we point out that all the mentioned choices are related by a choice of embedding and hence are equivalent in a diffeomorphism invariant setting. In that setting also the “sharpness” of the field configuration is a subtle issue.

The relationship between the $C$ effective theories and the continuum is not trivial. We may want to assign a significance to the degrees of freedom of the continuum in terms of where do they go when coarse grained. This is problematic because of the general non commutativity of coarse graining. That is, we can choose to arrive to a given effective theory though a sequence of coarse grainings, but if another physicist chooses a different sequence to arrive to the same effective theory our results could be different.

Of course this phenomena is also related to the non uniqueness of coarse graining, but it is also tied to diffeomorphism invariance. One solution to this problem is to choose a single preferred sequence of effective theories that are increasingly fine. As shown in [7] the sequences of lattices of the type used in Theorem 2 contain representatives of all the knot classes present in the continuum.

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5. References

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