Some inequalities for interval-valued functions on time scales

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Abstract We introduce the interval Darboux delta integral (shortly, the ID $\Delta$-integral) and the interval Riemann delta integral (shortly, the IR $\Delta$-integral) for interval-valued functions on time scales. Fundamental properties of ID and IR $\Delta$-integrals and examples are given. Finally, we prove Jensen’s, Hölder’s and Minkowski’s inequalities for the IR $\Delta$-integral. Also, some examples are given to illustrate our theorems.

Keywords interval-valued functions · time scales · Jensen’s inequality · Hölder’s inequality · Minkowski’s inequality

1 Introduction

Interval analysis was initiated by Moore for providing reliable computations [27]. Since then, interval analysis and interval-valued functions have been extensively studied both in mathematics and its applications: see, e.g., [2,11,12,13,15,21,23,25,28,29,30,34,38,42]. Recently, several classical integral inequalities have been extended to the context of interval-valued functions by Chalco-Cano et al. [9,10], Costa [13], Costa and Román-Flores [16], Flores-Franulić et al. [20], Román-Flores et al. [32,33].

Motivated by [8,13,32], we introduce the ID and IR $\Delta$-integrals, and present some integral inequalities on time scales. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$ with the subspace topology inherited from the standard topology of $\mathbb{R}$. The theory of time scales was born in 1988 with the Ph.D. thesis of Hilger [22]. The aim is to unify various definitions and results from the theories of discrete and continuous dynamical systems, and to extend them to more general classes of dynamical systems. It has undergone tremendous expansion and development on various aspects by several authors over the past three decades: see, e.g., [3,4,5,6,18,19,35,37,39,40,41].

In 2013, Lupulescu introduced the Riemann $\Delta$-integral for interval-valued functions on time scales and presented some of its basic properties [24]. Nonetheless, to our best knowledge, there is no systematic theory of integration for interval-valued functions on time scales. In this work, in order to complete the theory of IR $\Delta$-integration and improve recent results given in [8,13,32], we introduce the ID $\Delta$-integral and the IR $\Delta$-integral on time scales. We show that the ID $\Delta$-integral is a generalization of the IR $\Delta$-integral. Also, some basic properties for the ID and IR $\Delta$-integrals, and some examples, are given. Finally, we present Jensen’s inequality, Hölder’s inequality and Minkowski’s inequality for the IR $\Delta$-integral. Some celebrated inequalities are derived as consequences of our results.

The paper is organized as follows. After a Section 2 of preliminaries, in Section 3 the ID and IR $\Delta$-integrals
for interval-valued functions are introduced. Moreover, some basic properties and examples are given. In Section 4, we prove Jensen's, Hölder's and Minkowski's inequalities for the general IR ∆-integral. We end with Section 5 of conclusions.

2 Preliminaries

In this section, we recall some basic definitions, notations, properties and results on interval analysis and the time scale calculus, which are used throughout the paper. A real interval [u] is the bounded, closed subset of R defined by

\[ [u] = [\underline{u}, \overline{u}] = \{ x \in R : \underline{u} \leq x \leq \overline{u} \}, \]

where \( \underline{u}, \overline{u} \in R \) and \( \underline{u} \leq \overline{u} \). The numbers \( \underline{u} \) and \( \overline{u} \) are called the left and the right endpoints of \([u]\), respectively. When \( \underline{u} \) and \( \overline{u} \) are equal, the interval \([u]\) is said to be degenerate. In this paper, the term interval will mean a nonempty interval. We call \([u]\) positive if \( \underline{u} > 0 \) or negative if \( \overline{u} < 0 \). The partial order “≤” is defined by

\[ \underline{u}, \overline{u} \subseteq \underline{u}', \overline{u}' \iff \underline{u} \leq \underline{u}' \land \overline{u} \leq \overline{u}'. \]

The inclusion “⊆” is defined by

\[ \underline{u}, \overline{u} \subseteq \underline{u}', \overline{u}' \iff \underline{u} \leq \underline{u}' \land \overline{u} \leq \overline{u}'. \]

For an arbitrary real number λ and \([u]\), the interval \( \lambda [u] \) is given by

\[ \lambda [u] = [\lambda \underline{u}, \lambda \overline{u}] = \begin{cases} [\lambda \underline{u}, \lambda \overline{u}] & \text{if } \lambda > 0, \\ \{0\} & \text{if } \lambda = 0, \\ [\lambda \overline{u}, \lambda \underline{u}] & \text{if } \lambda < 0. \end{cases} \]

For \( [u] = [\underline{u}, \overline{u}] \) and \([v] = [\underline{v}, \overline{v}] \), the four arithmetic operators \((+, · , -, /)\) are defined by

\[ [u] + [v] = [\underline{u} + \underline{v}, \overline{u} + \overline{v}], \]
\[ [u] - [v] = [\underline{u} - \overline{v}, \overline{u} - \underline{v}], \]
\[ [u] \cdot [v] = [\min\{\underline{u} \overline{v}, \underline{v} \overline{u}, \underline{v} \overline{u}, \underline{u} \overline{v}\}, \max\{\underline{u} \overline{v}, \underline{v} \overline{u}, \underline{v} \overline{u}, \underline{u} \overline{v}\}], \]
\[ [u]/[v] = \left[ \frac{\min\{\underline{u} \overline{v}, \underline{v} \overline{u}, \underline{v} \overline{u}, \underline{u} \overline{v}\}}{\max\{\underline{u} \overline{v}, \underline{v} \overline{u}, \underline{v} \overline{u}, \underline{u} \overline{v}\}}, \frac{\max\{\underline{u} \overline{v}, \underline{v} \overline{u}, \underline{v} \overline{u}, \underline{u} \overline{v}\}}{\min\{\underline{u} \overline{v}, \underline{v} \overline{u}, \underline{v} \overline{u}, \underline{u} \overline{v}\}} \right], \quad \text{where } 0 \notin [u, v]. \]

We denote by \( R_I \) the set of all intervals of \( R \), and by \( R^+_I \) and \( R^-_I \) the set of all positive intervals and negative intervals of \( R \), respectively. The Hausdorff–Pompeiu distance between intervals \([\underline{u}, \overline{u}]\) and \([\underline{v}, \overline{v}]\) is defined by

\[ d([\underline{u}, \overline{u}], [\underline{v}, \overline{v}]) = \max\{|\underline{u} - \underline{v}|, |\overline{u} - \overline{v}|\}. \]

It is well known that \((R_I, d)\) is a complete metric space.

Let \( T \) be a time scale. We define the half-open interval \([a, b)^T\) by

\[ [a, b)^T = \{ t \in T : a \leq t < b \}. \]

The open and closed intervals are defined similarly. For \( t \in T \), we denote by \( \sigma \) the forward jump operator, i.e., \( \sigma(t) := \inf\{s > t : s \in T\} \), and by \( \rho \) the backward jump operator, i.e., \( \rho(t) := \sup\{s < t : s \in T\} \). Here, we put \( \sigma(\sup T) = \sup T \) and \( \rho(\inf T) = \inf T \), where \( \sup T \) and \( \inf T \) are finite. In this situation, \( T^\kappa := T\setminus\{\sup T\} \) and \( T_\kappa := T\setminus\{\inf T\} \), otherwise, \( T^\kappa := T \) and \( T_\kappa := T \). If \( \sigma(t) > t \), then we say that \( t \) is right-scattered, while if \( \rho(t) < t \), then we say that \( t \) is left-scattered. If \( \sigma(t) = t \) and \( t < \sup T \), then \( t \) is called right-dense, and if \( \rho(t) = t \) and \( t > \inf T \), then \( t \) is left-dense. The graininess functions \( \mu \) and \( \eta \) are defined by \( \mu(t) := \sigma(t) - t \) and \( \eta(t) := t - \rho(t) \), respectively.

A function \( f : [a, b]^T \to R \) is called right-dense continuous (rd-continuous) if it is right continuous at each right-dense point and there exists a finite left limit at all left-dense points. The set of rd-continuous function \( f : [a, b]^T \to R \) is denoted by \( C_{rd}(a, b]^T, R) \).

A function \( f \) is said to be an interval function of \( t \) on \([a, b]^T\) if it assigns a nonempty interval

\[ f(t) = [f(t), \bar{f}(t)] \]

to each \( t \in [a, b]^T \). We say that \( f : [a, b]^T \to R \) is continuous at \( t_0 \in [a, b]^T \) if for each \( \epsilon > 0 \) there exists a \( \delta > 0 \) such that

\[ d(f(t), f(t_0)) < \epsilon \]

whenever \( |t - t_0| < \delta \). The set of continuous function \( f : [a, b]^T \to R \) is denoted by \( C(a, b]^T, R) \). It is clear that \( f \) is continuous at \( t_0 \) if and only if \( f \) and \( \bar{f} \) are continuous at \( t_0 \).

A division of \([a, b]^T\) is any finite ordered subset \( D \) having the form

\[ D = \{a = t_0 < t_1 < \cdots < t_n = b\}. \]

We denote the set of all divisions of \([a, b]^T\) by \( D(a, b]^T) \).

**Lemma 1 (Bohner and Peterson)** For every \( \delta > 0 \) there exists some division \( D \in D(a, b]^T \) given by

\[ a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b \]

such that for each \( i \in \{1, 2, \ldots, n\} \) either \( t_i - t_{i-1} \leq \delta \) or

\[ t_i - t_{i-1} > \delta \] and \( \rho(t_i) = t_{i-1} \).
Let $\mathcal{D}(\delta, [a, b]_\Delta)$ be the set of all $D \in \mathcal{D}([a, b]_\Delta)$ that possess the property indicated in Lemma 1. In each interval $[t_{i-1}, t_i]_\Delta$, where $1 \leq i \leq n$, choose an arbitrary point $\xi_i$ and form the sum

$$S(f, D, \delta) = \sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1}),$$

where $f : [a, b]_\Delta \to \mathbb{R}$ (or $\mathbb{R}_\sigma$). We call $S(f, D, \delta)$ a Riemann $\Delta$-sum of $f$ corresponding to $D \in \mathcal{D}(\delta, [a, b]_\Delta)$.

**Definition 1** (Bohner and Peterson [6]) A function $f : [a, b]_\Delta \to \mathbb{R}$ is called Riemann $\Delta$-integrable on $[a, b]_\Delta$ if there exists an $A \in \mathbb{R}$ such that for each $\epsilon > 0$ there exists a $\delta > 0$ for which

$$|S(f, D, \delta), A| < \epsilon$$

for all $D \in \mathcal{D}(\delta, [a, b]_\Delta)$. In this case, $A$ is called the Riemann $\Delta$-integral of $f$ on $[a, b]_\Delta$ and is denoted by $A = \frac{\Delta}{\Delta} \int_a^b f(t) \Delta t$ or $A = \frac{\Delta}{\Delta} \int_a^b f(t) \Delta t$. The family of all Riemann $\Delta$-integrable functions on $[a, b]_\Delta$ is denoted by $\mathcal{R}(\Delta, [a, b]_\Delta)$.

### 3 The Interval Darboux and Riemann delta integrals

Let $f : [a, b]_\Delta \to \mathbb{R}_\sigma$ be such that $f(t) = \big[ f(t), \overline{f}(t) \big]$ for all $t \in [a, b]_\Delta$. We denote

$$M = \sup \{ \overline{f}(t) : t \in [a, b]_\Delta \}, \quad m = \inf \{ f(t) : t \in [a, b]_\Delta \},$$

and for $1 \leq i \leq n$,

$$M_i = \sup \{ \overline{f}(t) : t \in [t_{i-1}, t_i]_\Delta \}, \quad m_i = \inf \{ f(t) : t \in [t_{i-1}, t_i]_\Delta \}.$$

The lower Darboux $\Delta$-sum $L(f, D)$ of $f$ with respect to $D \in \mathcal{D}([a, b]_\Delta)$ is the sum

$$L(f, D) = \sum_{i=1}^{n} m_i (t_i - t_{i-1}),$$

and the upper Darboux $\Delta$-sum $U(f, D)$ of $f$ is

$$U(f, D) = \sum_{i=1}^{n} M_i (t_i - t_{i-1}).$$

**Definition 2** (The Interval Darboux delta integral) Let $I = [a, b]_\Delta$, where $a, b \in \mathbb{T}$. The lower Darboux $\Delta$-integral of $f$ on $[a, b]_\Delta$ is defined by

$$(D) \int_a^b f(t) \Delta t = \sup_{D \in \mathcal{D}([a, b]_\Delta)} \left\{ L(f, D) \right\}$$

and the upper Darboux $\Delta$-integral of $f$ on $[a, b]_\Delta$ is defined by

$$(D) \int_a^b \overline{f}(t) \Delta t = \inf_{D \in \mathcal{D}([a, b]_\Delta)} \left\{ U(f, D) \right\}.$$
Example 1 Suppose that $[a, b]_{\lambda} = [0, 1]$, $Q$ is the set of rational numbers in $[0, 1]$, and $f : [a, b]_{\lambda} \rightarrow \mathbb{R}$ is defined by
\[
f(t) = \begin{cases} [-1, 0], & \text{if } t \in Q, \\ [1, 2], & \text{if } t \in [0, 1]\setminus Q. \end{cases}
\]

Then,
\[
(ID) \int_0^1 f(t) \Delta t = \left[ (D) \int_0^1 f(t) \Delta t \right] = [-1, 2].
\]

Example 2 Suppose that $[a, b]_{\lambda} = \left\{ 0, \frac{1}{3}, \frac{2}{3}, 1 \right\}$ and $f : [a, b]_{\lambda} \rightarrow \mathbb{R}$ is defined by
\[
f(t) = \begin{cases} [-1, 0], & \text{if } t = 0, \\ [-\frac{1}{3}, \frac{1}{3}], & \text{if } t = \frac{2}{3}, \\ [-\frac{1}{2}, \frac{1}{2}], & \text{if } t = \frac{1}{2}, \\ [1, 2], & \text{if } t = 1. \end{cases}
\]

Then,
\[
(ID) \int_0^1 f(t) \Delta t = \int \frac{1}{3} \Delta t + \frac{1}{6} \Delta t + \frac{1}{2} \Delta t = \frac{11}{36},
\]

and therefore
\[
(ID) \int_0^1 f(t) \Delta t = \left[ \frac{11}{36} \right].
\]

Theorem 2 Let $f, g \in \mathcal{TD}_{\Delta, [a, b]_{\lambda}}$, and $\lambda$ be an arbitrary real number. Then,
1. $\lambda f \in \mathcal{TD}_{\Delta, [a, b]_{\lambda}}$ and
   \[
   (ID) \int_a^b \lambda f(t) \Delta t = \lambda (ID) \int_a^b f(t) \Delta t;
   \]
2. $f + g \in \mathcal{TD}_{\Delta, [a, b]_{\lambda}}$ and
   \[
   (ID) \int_a^b (f(t) + g(t)) \Delta t 
   \leq (ID) \int_a^b f(t) \Delta t + (ID) \int_a^b g(t) \Delta t;
   \]
3. for $c \in [a, b]_{\lambda}$ and $a < c < b$,
   \[
   (ID) \int_a^c f(t) \Delta t + (ID) \int_c^b f(t) \Delta t = (ID) \int_a^b f(t) \Delta t;
   \]
4. if $f \leq g$ on $[a, b]_{\lambda}$, then
   \[
   (ID) \int_a^b f(t) \Delta t \leq (ID) \int_a^b g(t) \Delta t.
   \]

Proof We only prove that part (2) of Theorem 2 holds. The other relations are obvious. Suppose that
\[
f(t) = \left[ f(t), \overline{f}(t) \right], \quad g(t) = \left[ g(t), \overline{g}(t) \right].
\]

Select any division $D \in \mathcal{D}_{[a, b]_{\lambda}}$ having the form
\[
D = \{ a = t_0 < t_1 < \cdots < t_n = b \}.
\]

Then,
\[
\inf_{\ell \in \{t_{\ell-1}, t_{\ell}\}} \{ f(t) \} + \inf_{\ell \in \{t_{\ell-1}, t_{\ell}\}} \{ g(t) \}
\leq \sup_{\ell \in \{t_{\ell-1}, t_{\ell}\}} \{ \overline{f}(t) \} + \sup_{\ell \in \{t_{\ell-1}, t_{\ell}\}} \{ \overline{g}(t) \}
\]
and it follows that
\[
L(f, D) + L(g, D) \leq L(f + g, D),
\]
\[
U(f, D) + U(g, D) \geq U(f + g, D).
\]

The intended result follows.

Example 3 Suppose that $[a, b]_{\lambda} = [0, 1]$, $Q$ is the set of rational numbers in $[0, 1]$, and $f, g : [a, b]_{\lambda} \rightarrow \mathbb{R}$ are defined by
\[
f(t) = \begin{cases} [-1, 0], & \text{if } t \in Q, \\ [1, 2], & \text{if } t \in [0, 1]\setminus Q, \end{cases}
\]
\[
g(t) = \begin{cases} [0, 1], & \text{if } t \in Q, \\ [-2, -1], & \text{if } t \in [0, 1]\setminus Q. \end{cases}
\]

Then
\[
f(t) + g(t) = [-1, 1]
\]
for all $t \in [0, 1]$. It follows that
\[
(ID) \int_0^1 f(t) \Delta t + (ID) \int_0^1 g(t) \Delta t = [-1, 2] + [-2, 1] = [-3, 3],
\]
\[
(ID) \int_0^1 (f(t) + g(t)) \Delta t = [-1, 1].
\]

Therefore, we have
\[
(ID) \int_a^b (f(t) + g(t)) \Delta t \leq (ID) \int_a^b f(t) \Delta t + (ID) \int_a^b g(t) \Delta t.
\]

We now give Riemann’s definition of integrability, which is equivalent to the Riemann $\Delta$-integral given in [24 Definition 13].
Some inequalities for interval-valued functions on time scales

Definition 3 (The Interval Riemann delta integral) A function $f : [a, b]_T \to \mathbb{R}$ is called $IR \Delta$-integrable on $[a, b]_T$ if there exists an $A \in \mathbb{R}_T$ such that for each $\epsilon > 0$ there exists a $\delta > 0$ for which

$$d(S(f, D, \delta), A) < \epsilon$$

for all $D \in D(\delta, [a, b]_T)$. In this case, $A$ is called the $IR \Delta$-integral of $f$ on $[a, b]_T$ and is denoted by $A = (IR) \int_a^b f(t) \Delta t$. The family of all $IR \Delta$-integrable functions on $[a, b]_T$ is denoted by $\mathcal{IR} \Delta [a, b]_T$.

Remark 2 Definitions 2 and 3 are not equivalent. If $f \in \mathcal{IR} \Delta [a, b]_T$, then $f \in \mathcal{ID} \Delta [a, b]_T$. However, the converse is not always true (see Example 1). It is clear that $f \in \mathcal{ID} \Delta [a, b]_T$, but $f \notin \mathcal{IR} \Delta [a, b]_T$. In fact, all bounded interval functions are $ID \Delta$-integrable, but boundedness of $f$ is not a sufficient condition for $IR \Delta$-integrability. If $f$ is a continuous function, then $f \in \mathcal{IR} \Delta [a, b]_T$ if and only if $f \in \mathcal{ID} \Delta [a, b]_T$, in which case the value of the integrals agree.

The following two theorems can be easily verified and so the proofs are omitted.

Theorem 3 If $f, g \in C([a, b]_T, \mathbb{R}_T)$, then $f \in \mathcal{IR} \Delta [a, b]_T$ and

$$(IR) \int_a^b f(t) \Delta t = \left[ \int_a^b f(t) \Delta t, \int_a^b g(t) \Delta t \right].$$

Theorem 4 Let $f, g \in \mathcal{IR} \Delta [a, b]_T$, and $\lambda$ be an arbitrary real number. Then,

1. $\lambda f \in \mathcal{IR} \Delta [a, b]_T$ and

$$(IR) \int_a^b \lambda f(t) \Delta t = \lambda (IR) \int_a^b f(t) \Delta t;$$

2. $f + g \in \mathcal{IR} \Delta [a, b]_T$ and

$$(IR) \int_a^b (f(t) + g(t)) \Delta t = (IR) \int_a^b f(t) \Delta t + (IR) \int_a^b g(t) \Delta t;$$

3. for $c \in [a, b]_T$ and $a < c < b$,

$$(IR) \int_a^c f(t) \Delta t + (IR) \int_c^b f(t) \Delta t = (IR) \int_a^b f(t) \Delta t;$$

4. if $f \leq g$ on $[a, b]_T$, then

$$(IR) \int_a^b f(t) \Delta t \leq (IR) \int_a^b g(t) \Delta t.$$

Example 4 Suppose that $T = [-1, 0]_{3\mathbb{N}_0}$, where $[-1, 0]$ is a real-valued interval and $\mathbb{N}_0$ is the set of nonnegative integers. Let $f : [a, b]_T \to \mathbb{R}_T$ be defined by

$$f(t) = \begin{cases} [t, t + 1], & \text{if } t \in [-1, 0), \\ [t, t^2 + 1], & \text{if } t \in 3\mathbb{N}_0. \end{cases}$$

If $[a, b]_T = [-1, 3]_T$, then

$$(IR) \int_{-1}^{3} f(t) \Delta t = \left[ \int_{-1}^{3} f(t) \Delta t, \int_{-1}^{3} g(t) \Delta t \right]$$

$$= \left[ \int_{-1}^{0} \left( \int_0^1 f(t) \Delta t + \int_1^3 t \Delta t, \int_0^3 \Delta t + \int_1^3 t \Delta t, \int_{-1}^0 (t + 1) \Delta t + \int_0^1 2 \Delta t + \int_1^3 (t^2 + 1) \Delta t \right) \right]$$

$$= \left[ \frac{1}{2} \left[ \int_{-1}^0 t \Delta t + 1 + 2t^2 \right]_1, \frac{1}{2} \left[ \int_{-1}^0 (t^2 + 1) \Delta t + 2 + 2t(t^2 + 1) \right]_1 \right]$$

$$= \left[ \frac{1}{2} \cdot \frac{1}{2}, \frac{1}{2} \right].$$

4 Some inequalities for the interval Riemann delta integral

We begin by recalling the notions of convexity on time scales.

Definition 4 (Dimu [17]) We say that $f : [a, b]_T \to \mathbb{R}$ is a convex function if for all $x, y \in [a, b]_T$ and $\alpha \in [0, 1]$ we have

$$f(\alpha x + (1 - \alpha) y) \leq \alpha f(x) + (1 - \alpha) f(y)$$

for which $\alpha x + (1 - \alpha) y \in [a, b]_T$. If inequality (1) is reversed, then $f$ is said to be concave. If $f$ is both convex and concave, then $f$ is said to be affine. The set of all convex, concave and affine interval-valued functions are denoted by $SX([a, b]_T, \mathbb{R})$, $SV([a, b]_T, \mathbb{R})$, and $SA([a, b]_T, \mathbb{R})$, respectively.

We can now introduce the concept of interval-valued convexity.

Definition 5 We say that $f : [a, b]_T \to \mathbb{R}_T$ is a convex interval-valued function if for all $x, y \in [a, b]_T$ and $\alpha \in (0, 1)$ we have

$$\alpha f(x) + (1 - \alpha) f(y) \subseteq f(\alpha x + (1 - \alpha) y)$$

(2)
for which $\alpha x + (1 - \alpha)y \in [a, b]_{\mathbb{T}}$. If the set inclusion (2) is reversed, then $f$ is said to be concave. If $f$ is both convex and concave, then $f$ is said to be affine. The set of all convex, concave and affine interval-valued functions are denoted by $SX([a, b]_{\mathbb{T}}, \mathbb{R}_I)$, $SV([a, b]_{\mathbb{T}}, \mathbb{R}_I)$ and $SA([a, b]_{\mathbb{T}}, \mathbb{R}_I)$, respectively.

**Remark 3** It is clear that if $\mathbb{T} = \mathbb{R}$, then Definition 5 implies the definition of convexity introduced by Breckner [7].

**Theorem 5** Let $f : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}_I$ be such that

$$f(t) = [f(t), \overline{f}(t)]$$

for all $t \in [a, b]_{\mathbb{T}}$. Then,

1. $f \in SX([a, b]_{\mathbb{T}}, \mathbb{R}_I)$ if and only if $\underline{f} \in SX([a, b]_{\mathbb{T}}, \mathbb{R})$ and $\overline{f} \in SV([a, b]_{\mathbb{T}}, \mathbb{R})$,

2. $f \in SV([a, b]_{\mathbb{T}}, \mathbb{R}_I)$ if and only if $\underline{f} \in SV([a, b]_{\mathbb{T}}, \mathbb{R})$ and $\overline{f} \in SX([a, b]_{\mathbb{T}}, \mathbb{R})$,

3. $f \in SA([a, b]_{\mathbb{T}}, \mathbb{R}_I)$ if and only if $\underline{f}, \overline{f} \in SA([a, b]_{\mathbb{T}}, \mathbb{R})$.

**Proof** We only prove that part (1) of Theorem 5 holds. Suppose that $f \in SX([a, b]_{\mathbb{T}}, \mathbb{R}_I)$ and consider $x, y \in [a, b]_{\mathbb{T}}$, $\alpha \in [0, 1]$. Then,

$$\alpha f(x) + (1 - \alpha)f(y) \subseteq f(\alpha x + (1 - \alpha)y),$$

that is,

$$\begin{align*}
[\alpha f(x) + (1 - \alpha)f(y), \alpha \overline{f}(x) + (1 - \alpha)\overline{f}(y)] &\subseteq [f(\alpha x + (1 - \alpha)y), \overline{f}(\alpha x + (1 - \alpha)y)].
\end{align*}$$

(3)

It follows that

$$\alpha \underline{f}(x) + (1 - \alpha)\underline{f}(y) \leq \underline{f}(\alpha x + (1 - \alpha)y)$$

and

$$\alpha \overline{f}(x) + (1 - \alpha)\overline{f}(y) \leq \overline{f}(\alpha x + (1 - \alpha)y).$$

This shows that $\underline{f} \in SX([a, b]_{\mathbb{T}}, \mathbb{R})$ and $\overline{f} \in SV([a, b]_{\mathbb{T}}, \mathbb{R})$.

Conversely, if $\underline{f} \in SX([a, b]_{\mathbb{T}}, \mathbb{R})$ and $\overline{f} \in SV([a, b]_{\mathbb{T}}, \mathbb{R})$, by Definition 5 and the set inclusion (3), we have $f \in SX([a, b]_{\mathbb{T}}, \mathbb{R}_I)$.

**Theorem 6 (Dinu [17])** A convex function on $[a, b]_{\mathbb{T}}$ is continuous on $(a, b)_{\mathbb{T}}$.

**Theorem 7** Let $f : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}_I$ be such that

$$f(t) = [f(t), \overline{f}(t)]$$

for all $t \in [a, b]_{\mathbb{T}}$. If $f \in SX([a, b]_{\mathbb{T}}, \mathbb{R}_I) \cup SV([a, b]_{\mathbb{T}}, \mathbb{R}_I) \cup SA([a, b]_{\mathbb{T}}, \mathbb{R}_I)$, then $f \in IT_{\Delta, [a,b]_I}$.

**Proof** Suppose that

$$f \in SV([a, b]_{\mathbb{T}}, \mathbb{R}_I) \cup SV([a, b]_{\mathbb{T}}, \mathbb{R}_I) \cup SA([a, b]_{\mathbb{T}}, \mathbb{R}_I).$$

Due to Theorems 5 and 3 it follows that $f$ and $\overline{f}$ are continuous. Then, from Theorem 5.19 of [8], we have that

$$\overline{f}(t), \underline{f}(t) \in \mathcal{R}_{\Delta, [a,b]_I}.$$

Hence, $f \in IT_{\Delta, [a,b]_I}$.

**Theorem 8 (Wong et al. [36])** Let $a, b \in [a, b]_{\mathbb{T}}$ and $c, d \in \mathbb{R}$. Suppose that $g \in C_{rd}([a, b]_{\mathbb{T}}, (c, d))$ and $h \in C_{rd}([a, b]_{\mathbb{T}}, \mathbb{R})$ with

$$\int_a^b |h(s)| \Delta s > 0.$$

If $f \in C((c, d), \mathbb{R})$ is convex, then

$$f \left( \int_a^b |h(s)|g(s)\Delta s \right) \leq \int_a^b |h(s)|f(g(s))\Delta s.$$

(4)

If $f$ is concave, then inequality (4) is reversed.

**Theorem 9 (Jensen’s inequality)** Let $g \in C_{rd}([a, b]_{\mathbb{T}}, (c, d))$ and $h \in C_{rd}([a, b]_{\mathbb{T}}, \mathbb{R})$ with

$$\int_a^b |h(s)| \Delta s > 0.$$

If $f \in C((c, d), \mathbb{R}^+_I)$ is a convex function, then

$$(IR) \int_a^b |h(s)|f(g(s))\Delta s \leq f \left( \int_a^b |h(s)|g(s)\Delta s \right).$$

**Proof** By hypothesis, we have

$$|h|f(g), |h|f(g) \in \mathcal{R}_{\Delta, [a,b]_I}.$$ 

Hence, $|h|f(g) \in IT_{\Delta, [a,b]_I}$ and

$$(IR) \int_a^b |h(s)|f(g(s))\Delta s = \left[ \int_a^b |h(s)f(g(s))\Delta s, \int_a^b |h(s)f(g(s))\Delta s \right].$$

From Theorem 8 it follows that

$$\int_a^b |h(s)|g(s)\Delta s \leq \int_a^b |h(s)|f(g(s))\Delta s$$

and

$$\int_a^b |h(s)|f(g(s))\Delta s \geq \int_a^b |h(s)|f(g(s))\Delta s.$$
which implies
\[
\left\lfloor \int_a^b [h(s)|f(g(s))| \Delta s, \int_a^b |h(s)|f(g(s)| \Delta s] \right\rfloor \leq \left\lfloor \frac{\int_a^b h(s)|f(g(s)| \Delta s}{\int_a^b |h(s)| \Delta s} \right\rfloor.
\]
that is,
\[
\left\lfloor \int_a^b h(s)|f(g(s)| \Delta s, \int_a^b |h(s)|f(g(s)| \Delta s] \right\rfloor \leq \left\lfloor \frac{\int_a^b h(s)|f(g(s)| \Delta s}{\int_a^b |h(s)| \Delta s} \right\rfloor.
\]
Finally, we obtain
\[
\left\lfloor \frac{\int_a^b h(s)|f(g(s)| \Delta s}{\int_a^b |h(s)| \Delta s} \right\rfloor \leq \left\lfloor \frac{\int_a^b h(s)|f(g(s)| \Delta s}{\int_a^b |h(s)| \Delta s} \right\rfloor.
\]
The proof is complete.

Example 5 Suppose that \([a, b] = [0, 1] \cup \left\{ \frac{1}{2} \right\}, \) where \([0, 1] \) is a real-valued interval. Let \(g(s) = s^2, h(s) = e^s, \) and \(f(s) = [s^2, 4\sqrt{s}] \). Then
\[
\left\lfloor \frac{\int_a^b h(s)|f(g(s)| \Delta s}{\int_a^b |h(s)| \Delta s} \right\rfloor = \left\lfloor \frac{\int_a^b [s^2e^s \Delta s, \int_a^b 4se^s \Delta s] \right\rfloor \leq \left\lfloor \frac{\int_a^b s^2e^s \Delta s, \int_a^b 4se^s \Delta s]}{\int_a^b |h(s)| \Delta s} \right\rfloor = \left\lfloor \frac{\int_a^b s^2e^s ds + \int_a^b 4se^s ds, \int_a^b 4se^s ds + \int_a^b 4se^s \Delta s]}{\int_a^b e^s ds + \int_a^b e^s \Delta s} \right\rfloor = \left\lfloor \frac{9\frac{1}{2}e - 24, 4 + 2e]}{\frac{3}{2}e - 1} \right\rfloor = \left\lfloor \frac{19e - 48, 8 + 4e]}{3e - 2 \cdot 3e - 2} \right\rfloor,
\]
and
\[
f \left( \frac{\int_a^b h(s)|f(g(s)| \Delta s}{\int_a^b |h(s)| \Delta s} \right) = f \left( \frac{\int_a^b s^2e^s \Delta s}{\int_a^b e^s \Delta s} \right) = f \left( \frac{\int_a^b s^2e^s ds + \int_a^b 4se^s ds + \int_a^b 4se^s \Delta s]}{\int_a^b e^s ds + \int_a^b e^s \Delta s} \right) = \left\lfloor \frac{3e - 4, 8 + 4e]}{3e - 2 \cdot 3e - 2} \right\rfloor.
\]
It follows that
\[
\left\lfloor \frac{19e - 48, 8 + 4e]}{3e - 2 \cdot 3e - 2} \right\rfloor = \left\lfloor \frac{3e - 4, 8 + 4e]}{3e - 2 \cdot 3e - 2} \right\rfloor.
\]
It is clear that if \([a, b] = [0, 1] \) and \(h(s) \equiv 1, \) then we get a similar result given in [13 Theorem 3.5] by T. M. Costa. Similarly, we can get the following results that generalize [13 Theorem 3.4] and [13 Corollary 3.3].

**Theorem 10** Let \(g \in C_{rd}([a, b] \cap \mathbb{R}^+) \) and \(h \in C_{rd}([a, b] \cap \mathbb{R}^+) \) with
\[
\int_a^b |h(s)| \Delta s > 0.
\]
If \(f \in C((c, d), \mathbb{R}^+) \) is a concave function, then
\[
\left( \frac{\int_a^b h(s)|f(g(s)| \Delta s}{\int_a^b |h(s)| \Delta s} \right) \geq f \left( \frac{\int_a^b h(s)|f(g(s)| \Delta s}{\int_a^b |h(s)| \Delta s} \right).
\]

**Theorem 11** Let \(g \in C_{rd}([a, b] \cap \mathbb{R}^+) \) and \(h \in C_{rd}([a, b] \cap \mathbb{R}^+) \) with
\[
\int_a^b |h(s)| \Delta s > 0.
\]
If \(f \in C((c, d), \mathbb{R}^+) \) is an affine function, then
\[
\left( \frac{\int_a^b h(s)|f(g(s)| \Delta s}{\int_a^b |h(s)| \Delta s} \right) = f \left( \frac{\int_a^b h(s)|f(g(s)| \Delta s}{\int_a^b |h(s)| \Delta s} \right).
\]

**Theorem 12** (Agarwal et al. [11]) Let \(f, g, h \in C_{rd}([a, b] \cap (0, \infty)) \). If \(\frac{1}{p} + \frac{1}{q} = 1, \) with \(p > 1, \) then
\[
\int_a^b h(s)f(s)g(s) \Delta s \leq \left( \int_a^b h(s)f^p(s) \Delta s \right)^{\frac{1}{p}} \left( \int_a^b h(s)g^q(s) \Delta s \right)^{\frac{1}{q}}.
\]
Next we present a Hölder type inequality for interval-valued functions on time scales.

**Theorem 13 (Hölder’s inequality)**

Let \( h \in C_{rd}([a, b], (0, \infty)), f, g \in C_{rd}([a, b], \mathbb{R}_T^+) \).

If \( \frac{1}{p} + \frac{1}{q} = 1 \), with \( p > 1 \), then

\[
\int_a^b h(s)f(s)g(s)\Delta s \\
\leq \left( \int_a^b h(s)f^p(s)\Delta s \right)^{\frac{1}{p}} \left( \int_a^b h(s)g^q(s)\Delta s \right)^{\frac{1}{q}}.
\]

**Proof** By hypothesis, we have

\[
\int_a^b h(s)f(s)g(s)\Delta s \\
= \int_a^b h(s)\left[\int f(s)g(s)\Delta s, \int \overline{g}(s)\Delta s\right]\Delta s \\
= \left[ \int_a^b h(s)f^p(s)\Delta s, \int_a^b h(s)g^q(s)\Delta s \right] \\
\leq \left( \int_a^b h(s)f^p(s)\Delta s \right)^{\frac{1}{p}} \left( \int_a^b h(s)g^q(s)\Delta s \right)^{\frac{1}{q}} \\
= \left( \int_a^b h(s)f^p(s)\Delta s \right)^{\frac{1}{p}} \left( \int_a^b h(s)g^q(s)\Delta s \right)^{\frac{1}{q}}.
\]

This concludes the proof.

For the particular case \( p = q = 2 \) in Theorem 13, we obtain the following Cauchy–Schwarz inequality.

**Theorem 14 (Cauchy–Schwarz inequality)**

Let \( h \in C_{rd}([a, b], (0, \infty)), f, g \in C_{rd}([a, b], \mathbb{R}_T^+) \). Then,

\[
\int_a^b h(s)f(s)g(s)\Delta s \\
\leq \sqrt{\left( \int_a^b h(s)f^2(s)\Delta s \right) \left( \int_a^b h(s)g^2(s)\Delta s \right)}.
\]

**Example 6** Suppose that \([a, b] = [0, \frac{x}{2}]\). Let \( h(s) = s \), \( f(s) = [s, s+1] \), and \( g(s) = [\sin s, s] \) for \( s \in [0, \frac{x}{2}] \). Then

\[
\int_a^b h(s)f(s)g(s)\Delta s \\
= \int_0^{\frac{x}{2}} [s^2 \sin s, s^3 + s^2] \Delta s \\
= \left[ \int_0^{\frac{x}{2}} s^2 \sin s \Delta s, \int_0^{\frac{x}{2}} (s^3 + s^2) \Delta s \right] \\
= \left[ \pi - 2, \frac{\pi}{64} \cdot 3\pi^3 \cdot \frac{24}{24} \right],
\]

and

\[
\int_0^{\frac{x}{2}} h(s)f(s)g(s)\Delta s \\
= \int_0^{\frac{x}{2}} \left( \int_0^1 [s^3, s^3 + 2s^2 + s] \Delta s \right) \left( \int_0^1 [s^2 \sin s, s^3] \Delta s \right) \\
= \left[ \int_0^{\frac{x}{2}} s^3 \Delta s, \int_0^{\frac{x}{2}} (s^3 + 2s^2 + s) \Delta s \right] \cdot \left[ \int_0^{\frac{x}{2}} s^2 \sin s \Delta s, \int_0^{\frac{x}{2}} s^3 \Delta s \right] \\
= \left[ \frac{\pi^4}{64} \cdot \frac{\pi^4}{12} + \frac{\pi^2}{8} \cdot \frac{\pi^2}{16} + \frac{\pi^4}{4} \cdot \frac{\pi^4}{64} \right] \\
= \left[ \frac{\pi^6}{1024} + \frac{\pi^4}{256} \cdot \frac{\pi^8}{4096} + \frac{\pi^6}{768} + \frac{\pi^6}{512} \right].
\]

Consequently, we obtain

\[
\left[ \pi - 2, \frac{\pi^4}{64} + \frac{\pi^3}{24} \right] \leq \left[ \frac{\pi^6}{1024} + \frac{\pi^4}{256} \cdot \frac{\pi^8}{4096} + \frac{\pi^6}{768} + \frac{\pi^6}{512} \right].
\]
Then
\[
\int_a^b h(s)f(s)g(s)\Delta s
\]
\[
= \int_0^b \left[ \frac{3}{2} s^3 + s^2 \right] \Delta s
\]
\[
= \int_0^b \left[ \frac{3}{2} s^3 + s^2 \Delta s \right]
\]
\[
= \left[ \frac{9}{2}, 14 \right],
\]
and
\[
\sqrt{\left( \int_a^b h(s)f^2(s)\Delta s \right) \left( \int_a^b h(s)g^2(s)\Delta s \right)}
\]
\[
= \sqrt{\left( \int_0^b \left[ s^3, s^3 + 2s^2 + s \right] \Delta s \right) \left( \int_0^b \left[ \frac{3}{4} s^3, s^3 \right] \Delta s \right)}
\]
\[
= \sqrt{\left[ 9, 22 \right] \cdot \left[ \frac{9}{4} \right]}
\]
\[
= \left[ \frac{9}{2}, 3\sqrt{22} \right].
\]
Consequently, we obtain
\[
\left[ \frac{9}{2}, 14 \right] \leq \left[ \frac{9}{2}, 3\sqrt{22} \right].
\]

**Theorem 15 (Agarwal et al. [11]; Wong et al. [37])**
Let \( f, g, h \in C_{rd}(a, b; T, \mathbb{R}) \) and \( p > 1 \). Then,
\[
\left( \int_a^b |h(s)|(f(s) + g(s))^p\Delta s \right)^\frac{1}{p}
\]
\[
\leq \left( \int_a^b |h(s)||f(s)|^p\Delta s \right)^\frac{1}{p} + \left( \int_a^b |h(s)||g(s)|^p\Delta s \right)^\frac{1}{p}.
\]

By the same technique used in the proof of Theorem 4 in [32], we get a more general result.

**Theorem 16 (Minkowski’s inequality)**
Let \( h \in C_{rd}(a, b; T, \mathbb{R}) \), \( f, g \in C([a, b; \mathbb{R}, \mathbb{R}_+]) \) and \( p > 1 \). Then,
\[
\left( \int_a^b |h(s)|(f(s) + g(s))^p\Delta s \right)^\frac{1}{p}
\]
\[
\leq \left( \int_a^b |h(s)|f^p(s)\Delta s \right)^\frac{1}{p} + \left( \int_a^b |h(s)|g^p(s)\Delta s \right)^\frac{1}{p}.
\]

**Proof** By hypothesis, we have
\[
\left( \int_a^b |h(s)|(f(s) + g(s))^p\Delta s \right)^\frac{1}{p}
\]
\[
= \left( \int_a^b |h(s)||f(s) + g(s)|\Delta s \right)^\frac{1}{p},
\]
\[
= \left( \int_a^b |h(s)|f^p(s)\Delta s \right)^\frac{1}{p} + \left( \int_a^b |h(s)|g^p(s)\Delta s \right)^\frac{1}{p},
\]
\[
\leq \left( \int_a^b |h(s)||f(s)|^p\Delta s \right)^\frac{1}{p} + \left( \int_a^b |h(s)||g(s)|^p\Delta s \right)^\frac{1}{p},
\]
\[
= \left( \int_a^b |h(s)||f^p(s)\Delta s \right)^\frac{1}{p} + \left( \int_a^b |h(s)||g^p(s)\Delta s \right)^\frac{1}{p}.\]

The proof is complete.

**Example 8** Suppose that \([a, b] = [0, 1] \cup \{2\}\). Let \( h(s) = s \), \( f(s) = [s, 2s] \), \( g(s) = [s, e^s] \) and \( p = 2 \). Then,
\[
\left( \int_a^b |h(s)|(f(s) + g(s))^p\Delta s \right)^\frac{1}{p}
\]
\[
= \sqrt{\int_0^2 \left[ 4s^3, 2se^{2s} + 4s^2e^s + 4s^3 \right] \Delta s}
\]
\[
= \sqrt{\left[ \int_0^2 4s^3 \Delta s, \int_0^2 2se^{2s} + 4s^2e^s + 4s^3 \Delta s \right]}
\]
\[
= \sqrt{\frac{5e^2 + 32e - 11}{2}}.
\]
and
\[
\left( \int_a^b |h(s)|f^p(s)\Delta s \right)^{\frac{1}{p}} \leq \left( \int_a^b |h(s)|g^p(s)\Delta s \right)^{\frac{1}{p}} + \left( \int_a^b |h(s)|g^p(s)\Delta s \right)^{\frac{1}{p}}
\]
\[
= \sqrt{\int_0^2 [s^3, 4s^3]} \Delta s + \sqrt{\int_0^2 [s^3, 6s^2]} \Delta s
\]
\[
= \left[ \frac{\sqrt{5}}{2}, \sqrt{5} \right] + \left[ \frac{\sqrt{5}}{2}, \frac{\sqrt{5}c^2 + 1}{2} \right]
\]
\[
= \left[ \sqrt{5}, \sqrt{5} + \frac{\sqrt{5c^2 + 1}}{2} \right].
\]
Consequently, we obtain
\[
\left[ \sqrt{5}, \frac{\sqrt{5c^2 + 32c - 11}}{2} \right] \leq \left[ \sqrt{5}, \sqrt{5} + \frac{\sqrt{5c^2 + 1}}{2} \right].
\]
The next results follow directly from Theorems 13 and 16 respectively.

**Corollary 1** Let \( h \in C_{rd}(a, b; \mathbb{T}, (0, \infty)) \), and \( f, g \in C_{rd}(a, b; \mathbb{T} \mathbb{R}^2) \). If \( \frac{1}{p} + \frac{1}{q} = 1 \), with \( p > 1 \), then
\[
\int_a^b h(s)f(s)g(s)\Delta s \leq \left( \int_a^b h(s)(-f)^p(s)\Delta s \right)^{\frac{1}{p}} \left( \int_a^b h(s)(-g)^q(s)\Delta s \right)^{\frac{1}{q}}.
\]

**Corollary 2** Let \( h \in C_{rd}(a, b; \mathbb{T}, \mathbb{R}) \), \( f, g \in C([a, b], \mathbb{R}^2) \) and \( p \in \mathbb{R}_+ \). Then,
\[
\left( \int_a^b |h(s)|(f(s) + g(s))^p\Delta s \right)^{\frac{1}{p}} \leq \left( \int_a^b |h(s)|f^p(s)\Delta s \right)^{\frac{1}{p}} + \left( \int_a^b |h(s)|g^p(s)\Delta s \right)^{\frac{1}{p}}.
\]

**5 Conclusion**

We investigated Darboux and Riemann interval delta integrals for interval-valued functions on time scales. Inequalities for interval-valued functions were proved. Our results generalize previous inequalities presented by Costa [13] Corollary 3.3, Theorem 3.4, Theorem 3.5 and Román-Flores [32] Theorem 4.

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**Compliance with ethical standards**

**Conflicts of interest** The authors declare that they have no conflict of interest.

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