Self Focussing as a Coherence Propagation Phenomena. An Application to Calculate the Coherence Length for an Atom Laser

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A theoretical description in terms of the coherence propagation is given for self-focussing. The concept of coherence length is defined in terms of free, self-focussing propagation giving results in accordance with well known experimental criteria for the laser. Extension of the method is given for an Atom Laser showing good results in agreement with recent numerical results of Trippenbach et al. [1. Phys. B:At. Mol. Opt. Phys. 33 47-54 (2000)].

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I. INTRODUCTION

A propagating beam of particles or waves are focused in some point when they sum up constructively, and oscillate in rigorous phase at that point. The focussing of these particles or waves, is experimentally obtained by using some optical arrangement. However, the self-focussing does not need of any experimental arrangement and raises as an unique effect no yet well explained. This became clear since the first experimental observation of self-imaging, today known as the Talbot’s effect. In this report we formulate self-focussing as a coherence propagation phenomena and then use it as a criterion for the definition of coherence length in both cases: interacting (Atom Laser) and non interacting (Photon Laser) particles beam propagation.

This paper is organized as follows: In section II we describe self-focussing as a phenomena of coherence propagation and obtain a master equation in Eq.(13); in section III we review the atom laser formalism and introduce the wave function \( \psi_0 \) for the condensed untrapped atoms as proposed by Gerbier et al., in section IV we calculate the general expression for the coherence length and finally in section V we give some relevant results and conclusions.

II. SELF-FOCUSING OF NON-INTERACTING PARTICLES

The general vector state to describe a highly collimated non interacting beam of neutral atoms propagating in a non-conducting medium is given by

\[
\psi(\vec{r},t) = \int_{-\infty}^{+\infty} A(\vec{k})e^{i(\vec{k} \cdot \vec{r} - \omega t)}d\vec{k}. \tag{1}
\]

The equation (1) indicates a vector sum, where \( \vec{k} \) corresponds to the vector associated to the plane wave solution for free particles. On the other hand, the integral stems from the possibility of the particle beam of taking any (continuous) value of momentum. In case of discrete changes in momentum it is customary to write down:

\[
\psi(\vec{r},t) = \sum_k A_k(\vec{k})e^{i(\vec{k} \cdot \vec{r} - \omega t)}. \tag{2}
\]

Going from Eq.(1) to Eq.(2) it is straightforward, but the former it’s not so easy to deal with. For instance, the current density,

\[
\vec{J} = \frac{\hbar}{2mi}(\psi^* \nabla \psi - \psi \nabla \psi^*), \tag{3}
\]

can be written using Eq.(2) (we drop here the time dependence)

\[
\vec{J} = \frac{\hbar}{2mi}[2i \sum_k \vec{k}|A_k(\vec{k})|^2 + \sum_{k' \neq k} ik\vec{k}' A_k^* A_{k'} e^{i(\vec{k}' - \vec{k}) \cdot \vec{r}} + A_k^* A_{k'} e^{-i(\vec{k}' - \vec{k}) \cdot \vec{r}}], \tag{4}
\]

In the integral form it reads

\[
\vec{J} = \frac{\hbar}{2m}[2 \int \vec{k}|A(\vec{k})|^2 d\vec{k} + \int \vec{k} \Gamma(\vec{k}) d\vec{k'}], \tag{5}
\]

where

\[
\Gamma(\vec{k}') = \int_{k \neq k'} \vec{k}' A(\vec{k}') A^*(\vec{k}) e^{-i(\vec{k}' - \vec{k}) \cdot \vec{r}} + A^*(\vec{k}') A(\vec{k}) e^{i(\vec{k}' - \vec{k}) \cdot \vec{r})d\vec{k}'. \tag{6}
\]

The physical meaning of Eq.(5) is made visible when we set

\[
\vec{r} = \vec{k} - \vec{k'}, \quad d\vec{r} = d\vec{k}, \tag{7}
\]

for a fixed \( \vec{k}' \). Then the first term at the right-hand side of Eq.(6) (the remaining is a complex conjugate), will be

\[
\gamma = \int \vec{k} A(\vec{k}') A^*(\vec{k}) e^{-i(\vec{k}' - \vec{k}) \cdot \vec{r}}d\vec{k},
\]

\[
= \int [A(\vec{k} - \vec{r}) e^{i(\vec{k} - \vec{r}) \cdot \vec{r}}][A^*(\vec{k}) e^{-i(\vec{k} - \vec{r})}]d\vec{k}. \tag{8}
\]
Equation (8) does not quite satisfy the correlation (classical) definition for a pair of functions $g(\xi)$ and $h(\xi)$:

$$\gamma = g(\xi) \otimes h(\xi),$$

$$= \int g(\xi) h^*(\xi - \tau) d\xi,$$  \hspace{0.5cm} (9)

because in this expression $\tau$, is a fixed correlation parameter and in our treatment (see Eq.(7)), $\tau$ is changing continuously together with $\vec{k}$. However, we still can say that Eq.(8) has the meaning of a correlation, since the integrand can be understood as a correlation between particles of momentum $\vec{k}$ and $(\vec{k} - \tau)$ for a given $\vec{k}'$. Keeping this in mind we see that equation (6) makes account of all these correlations for different values of $\vec{k}'$.

Then we write:

$$\Gamma(\vec{k}') = \int [\gamma(\vec{k}) + \gamma^*(\vec{k})] d\vec{k},$$  \hspace{0.5cm} (10)

now, a particle beam can be experimentally prepared in such a way that $A(\vec{k})$ may be a real number. This only means that for initial condition $(t = 0, r = 0)$ $A(\vec{k})$ has already some definite value (including zero). With this argument, Eq.(5) now reads:

$$\vec{J} = \frac{\hbar}{2m}[2\int \vec{k}|A(\vec{k})|^2 d\vec{k} +$$

$$2 \int d\vec{k}' \int Re[A(\vec{k} - \tau)e^{i(\vec{k} - \tau) \cdot \vec{r}}][\vec{k}A^*(\vec{k})e^{-i\vec{k} \cdot \vec{r}}]d\vec{k}]$$  \hspace{0.5cm} (11)

$$\vec{J} = \frac{\hbar}{2m}[2\int \vec{k}|A(\vec{k})|^2 d\vec{k} +$$

$$2 \int \int A(\vec{k}')A^*(\vec{k})\vec{k}\cos[(\vec{k}' - \vec{k}) \cdot \vec{r}]d\vec{k}d\vec{k}']$$  \hspace{0.5cm} (12)

This equation represents the more general expression for describing a non interacting particle beam propagation. We see from here that the second term (associated with coherence through out correlation), describes a focussing phenomena both directional (i.e, certain directions contain a bigger amount of particles than other directions) and longitudinal (along a particular direction, where some focus will exist). In this paper we deal with the longitudinal focussing.

1. Longitudinal focussing phenomena

From the second term on the right hand side of Eq.(12) we note that for a pair of atoms travelling in the same direction and with a slight difference in $\vec{k}$, the corresponding focussing along $z$ will correspond to those points where:

$$(\vec{k}' - \vec{k}) \cdot \vec{z} = 2n\pi;$$  \hspace{0.5cm} (13)

\begin{align*}
&\quad \text{for } n = 1, 2, \ldots
\end{align*}

For De Broglie particles with $\vec{p} = \hbar \vec{k} = m\vec{v}$, we have

$$z \cong \frac{2n\pi\hbar}{\Delta p} = \frac{2n\pi\hbar}{m\Delta v},$$  \hspace{0.5cm} (14)

where $\Delta v = v' - v$ is the atomic velocity difference between atoms, $m$ the atomic mass and $\hbar$ is the Planck constant. If the coordinate $z$ is measured from the beam origin, Eq.(14) can be put in the form

$$\Delta z \Delta p \cong 2n\pi\hbar,$$  \hspace{0.5cm} (15)

to understand Eq.(15), is straightforward from the quantum mechanical point of view. In fact we identify this expression as the uncertainty principle since $\Delta p = 0$ means that we can not localize any focus on $z$-axis. We are in the presence of a perfect monochromatic plane wave (i.e all every particle in the beam has exactly the same energy).

On the other hand, if we choose any two particles into the beam with a momentum difference $\Delta p$, and we track them in time, there is a certain possibility, different from zero, that it will focus in some point $z$ given by Eq.(15). This probability will be smaller as $\Delta p$ increases. It comes out from this argument that, in geometrical terms, the existence of any focus will mean some degree of coherence and therefore focus at infinity it means perfect or total degree of coherence (for given $n$), on the contrary focussing near the origin without any further focus, will mean a lower degree of coherence since the beam will spread along the propagation axis.

When we are dealing with light, we better put equation (15) as:

$$k'z = 2n\pi,$$

$$\frac{1}{c}(\Delta w)z = 2n\pi,$$

$$\frac{2\pi\Delta\nu}{c}z \cong 2n\pi,$$

$$\Delta z = \frac{n}{\Delta \nu} c,$$  \hspace{0.5cm} (16)

For $n = 1$ we obtain the so called coherence length in optics $\Delta z = \frac{\lambda}{\Delta \nu}$. If we know the band width of a laser, we will know the distance at which, the field will oscillate in rigorous phase. So it is interesting that requiring focussing in Eq.(13) as a coherence condition, we arrive at the well known formula of coherence length.
III. SELF-FOCUSSING OF INTERACTING PARTICLES. THE ATOM LASER

Since the obtention of the first Bose Einstein Condensate (BEC) in 1995, the laser of atoms became feasible. This object is defined as a device producing an intense well collimated coherent beam of atoms involving a process of coherent matter-wave amplification. Since the atoms have masses, and they interact while travelling, the coherence of the beam presents an additional spreading, which, as variant of the Photon Laser has to be considered. We do this by replacing in Eq.(1) the outgoing wave function of the condensate $\Psi_0$ and then we perform the same procedure as in section II.

In what follows we describe briefly the obtention of $\Psi_0$. A more detailed and rigorous deduction can be found in the paper by Gerbier et al.

These authors consider a BEC of $^{87}$Rb in the hyperfine level with $F = 1$. This condensate could be in any of the sublevels $m = -1,0,1$. Atoms in a state with $m = -1$ are confined in a magnetic potential of the form $V_{\text{trap}} = \frac{1}{2} M (w_x^2 x^2 + w_y^2 y^2 + w_z^2 z^2)$; atoms in a state with $m = 0$ are the untrapped ones, and those with $m = 1$ are rejected out of the trap. These are the three components of the spinorial wave function of the condensate $\Psi = [\psi_m]_{m=-1,0,1}$ and they obey a set of Schrödinger coupled equations. When the limit of weak coupling is considered, the populations $N_m$ satisfy $N_1 << N_0 << N_{-1}$ and only the states with $m = -1$, and $m = 0$ are considered.

The condensate atoms are transferred from the state $m = -1$ (trapped) to the state $m = 0$ (untrapped) by using a rf pulse

$$\tilde{B}_{rf} = B_{rf} \cos(\omega_{rf} t) \hat{e}_x,$$  

(17)

Thus the components of BEC $\psi_m = \psi_m e^{-i m \omega_{rf} t}$ satisfy the two coupled equations (after R.W.A)

$$i \hbar \frac{\partial \psi_{-1}}{\partial t} = \left[ \hbar \delta_{rf} + \frac{\hbar^2}{2M} + V_{\text{trap}} + U |\psi_{-1}|^2 \right] \psi_{-1} + \frac{\hbar \Omega_{rf}}{2} \psi_0,$$

(18)

$$i \hbar \frac{\partial \psi_0}{\partial t} = \left[ \frac{\hbar^2}{2M} - Mgz + U |\psi_{-1}|^2 \right] \psi_0 + \frac{\hbar \Omega_{rf}}{2} \psi_{-1},$$

(19)

The intensity of the interaction is given by $U = \frac{4 \pi \hbar^2 a N}{M}$, where $N$ is the initial number of trapped atoms, $M$ is the atomic mass, and $a$ is the diffusion length for the interatomic collision process which, for the $^{87}$Rb is $5nm$.

The uncoupling intensity between states $m = -1$, and $m = 0$ is given by the Rabi flopping frequency

$$\hbar \Omega_{rf} = \mu B_{rf} / 2 \sqrt{2},$$

(20)

the detuning $\delta_{rf}$ is

$$\hbar \delta_{rf} = V_{\text{eff}} - \hbar \omega_{rf}$$

(21)

and

$$V_{\text{eff}} = \mu B_0 / 2 + K z^2 / 2$$

(22)

$B_0$ is the background magnetic field due to the coils of the trap.

Equations (18) and (19) are uncoupled in the framework of the meanfield theory and the weak coupling limit, obtaining for $\psi_0$,

$$\psi_0(\mathbf{r},t) \simeq A(\Omega_{rf},F) \frac{e^{i \frac{\pi}{2} |\zeta_{-1}|^2 \cdots - \frac{F}{1 - t} \sqrt{|\zeta_{-1}|^2}}}{\sqrt{|\zeta_{-1}|^2}}$$

(23)

where

$$A(\Omega_{rf, F}) = -\sqrt{\pi \hbar \Omega_{rf} \frac{\phi_{-1}}{M g l} (x,y,z),}$$

(24)

here $F$ describes the finite extension of an atom laser beam due to the finite coupling time (e.g. the time of rf irradiation) and is constant for each particular laser, the adimensional parameter $\zeta = (z - z_r) / l$ provides a scale to the size of the trap, $z_r = \eta z_0 / 2$ is the extraction point from the trap, and

$$\phi_{-1}(x,y,z) = \left( \frac{\mu}{17} \right)^{1/2} \left[ 1 - (x/x_0)^2 - (y/y_0)^2 - (z/z_0)^2 \right]^{1/2}$$

(25)

The quantity $| \phi_{-1}(x,y,z)|^2$, corresponds to the trapped atomic population in the output point $z_r$. In what follows we define some important figures such as $l, x_0, y_0$ and $\eta$

$$l = \left( \frac{\hbar^2}{2 M^2 g^2} \right)^{1/3},$$

$$x_0^2 = 2 \mu / M w_x^2,$$

$$y_0^2 = 2 \mu / M w_y^2,$$

$$z_0^2 = 2 \mu / M w_z^2,$$

$$\eta = (2 \hbar \delta_{rf} + 4 \mu / l) \frac{1}{2 \mu g z_0}$$

(26)

where $\mu$ is the chemical potential, and is understood as the necessary energy to either add or remove a condensed atom in the trap ensemble. The chemical potential $\mu$ is defined

$$\mu = \left( \frac{\hbar \omega}{2} \right) \left( \frac{15 a N_{-1}}{\sigma} \right)^{2/5}$$

(27)

with $\omega = (\omega_2 \omega_3)^{1/3}$ and the harmonic oscillator length defined as $\sigma = (\hbar / M \omega)^{1/2}$, note that $l << x_0, y_0, z_0$ and for $^{87}$Rb it is known that $l \simeq 0.28 \mu m$.
IV. COHERENCE LENGTH FOR AN ATOM LASER

In the spirit of section II, we now use Eq.(13) in order to find the coherence length. To do this we need to know the propagation vector $\vec{k}$, which is calculated using Eq.(23) and following the standard procedures (see for example Flügge). Along these lines we then calculate $\vec{J}$, and find $\vec{v} = \vec{J}/\rho$ with $\rho = |\psi_0|^2$.

\[
\vec{J} = \frac{\hbar}{2mi}(\psi_0^\ast \vec{\nabla} \psi_0 - \psi_0 \vec{\nabla} \psi_0^\ast)
\]
\[
v = \frac{\vec{J}}{\rho} = \frac{\hbar}{ml} \sqrt{\zeta_r}
\]
\[
v = \frac{\hbar}{m} \frac{\sqrt{z + z_r}}{l^{3/2}}
\]  

(28)

here $\zeta_r = \vec{p} + \vec{p}_r$. We now use the De Broglie relation $\vec{p} = \hbar \vec{k} = m \vec{v}$ to obtain the propagation vector $\vec{k}$, we obtain for its magnitude

\[
k = \frac{\sqrt{z + z_r}}{l^{3/2}}
\]  

(29)

We then replace Eq.(29) into Eq.(13)

\[
((z + z_r)^{1/2} - (z' + z_r)^{1/2})z = 2n\pi l^{3/2},
\]  

(30)

It is clear from section II and Eq(13) that $z - z'$ is the correlation length for the interacting atom laser beam between two different points of the beam. Since we are interested in the coherence length measured from the extraction point $z_r$, we make $z' = z_r$, therefore

\[
((z + z_r)^{1/2} - (2z_r)^{1/2})z = 2n\pi l^{3/2},
\]  

(31)

By solving this equation for $n = 1$, we obtain the coherence length $z$ for the atom laser.

V. NUMERICAL RESULTS AND CONCLUSIONS

We use the mathematica program to solve numerically Eq.(31), below we list the results for all possible atoms laser with alkalin species

\[
^{23}Na = 2.4622 \mu m,
\]
\[
^{87}Rb = 1.0299 \mu m,
\]
\[
^7Li = 5.4461 \mu m.
\]

In particular, our treatment leads to a quantitative agreement with the experimental results of Marek Trippenbach et al. for atoms of $^{23}Na$. In their experiment they obtain different results for different $w_{rf}$, this results running between 2.0 and 5.0 $\mu m$.

In conclusion, we have proposed a very simple form for the calculation of the coherence length for an Atom Laser, and shown the validity for the method by comparison with the numerical results of Trippenbach et al. As a final comment we note that the coherence length is not depending on the irradiation time.