On the Riemann-Lie algebras and Riemann-Poisson Lie groups

M. Boucetta

Abstract. A Riemann-Lie algebra is a Lie algebra \( G \) such that its dual \( G^* \) carries a Riemannian metric compatible (in the sense introduced by the author in [1]) with the canonical linear Poisson structure of \( G^* \). The notion of Riemann-Lie algebra has its origins in the study, by the author, of Riemann-Poisson manifolds (see [2]).

In this paper, we show that, for a Lie group \( G \), its Lie algebra \( G \) carries a structure of Riemann-Lie algebra iff \( G \) carries a flat left-invariant Riemannian metric. We use this characterization to construct a huge number of Riemann-Poisson Lie groups (a Riemann-Poisson Lie group is a Poisson Lie group endowed with a left-invariant Riemannian metric compatible with the Poisson structure).

1 Introduction

Riemann-Lie algebras first arose in the study by the author of Riemann-Poisson manifolds (see [2]). A Riemann-Poisson manifold is a Poisson manifold \((P, \pi)\) endowed with a Riemannian metric \(<,>\) such that the couple \((\pi, <,>)\) is compatible in the sense introduced by the author in [1]. The notion of Riemann-Lie algebra appeared when we looked for a Riemannian metric compatible with the canonical Poisson structure on the dual of a Lie algebra. We pointed out (see [2]) that the dual space \( G^* \) of a Lie algebra \( G \) carries a Riemannian metric compatible with the linear Poisson structure iff the Lie algebra \( G \) carries a structure which we called Riemann-Lie algebra. Moreover, the isotropy Lie algebra at a point on a Riemann-Poisson manifold is a Riemann-Lie algebra. In particular, the dual Lie algebra of a Riemann-Poisson Lie group is a Riemann-Lie algebra (a Riemann-Poisson Lie group is a Poisson Lie group endowed with a left-invariant Riemannian metric compatible with the Poisson structure). In this paper, we will show
that a Lie algebra $G$ carries a structure of Riemann-Lie algebra iff $G$ is a semi-direct product of two abelian Lie algebras. Hence, according to a well-known result of Milnor [5], we deduce that, for a Lie group $G$, its Lie algebra carries a structure of Riemann-Lie algebra iff $G$ carries a flat left-invariant Riemannian metric. As application, we give a method to construct a huge number of Riemann-Poisson Lie groups. In particular, we give many examples of bialgebras $(G, [], G^*, [ , ]^*)$ such that both $(G, [])$ and $(G^*, [ , ]^*)$ are Riemann-Lie algebras.

2 Definitions and main results

2.1 Notations

Let $G$ be a connected Lie group and $(G, [ , ])$ its Lie algebra. For any $u \in G$, we denote by $u^l$ (resp. $u^r$) the left-invariant (resp. right-invariant) vector field of $G$ corresponding to $u$. We denote by $\theta$ the right-invariant Maurer-Cartan form on $G$ given by

$$\theta(u^r) = -u, \quad u \in G.$$  \hfill (1)

Let $<;>$ be a scalar product on $G$ i.e. a bilinear, symmetric, non-degenerate and positive definite form on $G$.

Let us enumerate some mathematical objects which are naturally associated with $<;>$:

1. an isomorphism $\# : G^* \rightarrow G$;
2. a scalar product $<, >^*$ on the dual space $G^*$ by
$$< \alpha, \beta >^* = < \#(\alpha), \#(\beta) >, \quad \alpha, \beta \in G^*;$$  \hfill (2)
3. a left-invariant Riemannian metric $<, >^l$ on $G$ by
$$< u^l, v^l >^l = < u, v >, \quad u, v \in G;$$  \hfill (3)
4. a left-invariant contravariant Riemannian metric $<, >^{*l}$ on $G$ by
$$< \alpha, \beta >^{*l} = < T^*_g L_g(\alpha), T^*_g L_g(\beta) >^*$$  \hfill (4)

where $\alpha, \beta \in \Omega^1(G)$ and $L_g$ is the left translation of $G$ by $g$. The infinitesimal Levi-Civita connection associated with $(G, [ , ], <, >)$ is the bilinear map $A : G \times G \rightarrow G$ given by
$$2 < A_\alpha v, w > = < [u, v], w > + < [w, u], v > + < [w, v], u >, \quad u, v, w \in G.$$  \hfill (5)
Note that $A$ is the unique bilinear map from $G \times G$ to $G$ which verifies:
1. $A_u v - A_v u = [u, v]$;
2. for any $u \in G$, $A_u : G \to G$ is skew-adjoint i.e.
   \[ < A_u v, w > + < v, A_u w > = 0, \quad v, w \in G. \]

The terminology used here can be justified by the fact that the Levi-Civita connection $\nabla$ associated with $(G, <, >)$ is given by
\[ \nabla^u v^l = (A_u v)^l, \quad u, v \in G. \]  \hfill (6)

We will introduce now a Lie subalgebra of $G$ which will play a crucial role in this paper.
For any $u \in G$, we denote by $\text{ad}_u : G \to G$ the endomorphism given by $\text{ad}_u(v) = [u, v]$, and by $\text{ad}_u^t : G \to G$ its adjoint given by
\[ < \text{ad}_u^t(v), w > = < v, \text{ad}_u(w) >, \quad v, w \in G. \]

The space
\[ S_{<, >} = \{ u \in G; \text{ad}_u + \text{ad}_u^t = 0 \} \]  \hfill (7)
is obviously a subalgebra of $G$. We call $S_{<, >}$ the orthogonal subalgebra associated with $(G, [ , ], <, >)$.

**Remark 2.1** The scalar product $<, >$ is bi-invariant if and only if $G = S_{<, >}$. In this case $G$ is the product of an abelian Lie algebra and a semi-simple and compact Lie algebra (see [5]). The general case where $<, >$ is not positive definite has been studied by A. Medina and P. Revoy in [4] and they called a Lie algebra $G$ with an inner product $<, >$ such that $G = S_{<, >}$ an orthogonal Lie algebra which justifies the terminology used here.\hfill \Box

Let $(G, [ , ], <, >)$ be a Lie algebra endowed with a scalar product. The triple $(G, [ , ], <, >)$ is called a Riemann-Lie algebra if
\[ [A_u v, w] + [u, A_u v] = 0 \]  \hfill (8)
for all $u, v, w \in G$ and where $A$ is the infinitesimal Levi-Civita connection associated to $(G, [ , ], <, >)$.

From the relation $A_u v - A_v u = [u, v]$ and the Jacobi identity, we deduce that (8) is equivalent to
\[ [u, [v, w]] = [A_u v, w] + [v, A_u w] \]  \hfill (9)
for any \( u, v, w \in \mathcal{G} \). We refer the reader to [2] for the origins of this definition. Briefly, if \((\mathcal{G}, [\ , \ ], <, >)\) is a Lie algebra endowed with a scalar product. The scalar product \(<, >\) defines naturally a contravariant Riemannian metric on the Poisson manifold \( \mathcal{G}^* \) which we denote also by \(<, >\). If we denote by \( \pi^d \) the canonical Poisson tensor on \( \mathcal{G}^* \), \((\mathcal{G}^*, \pi^d, <, >)\) is a Riemann-Poisson manifold iff the triple \((\mathcal{G}, [\ , \ ], <, >)\) is a Riemann-Lie algebra.

### 2.2 Characterization of Riemann-Lie algebras

With the notations and the definitions above, we can state now the main result of this paper.

**Theorem 2.1** Let \( G \) be a Lie group, \((\mathcal{G}, [\ , \ ])\) its Lie algebra and \(<, >\) a scalar product on \( \mathcal{G} \). Then, the following assertions are equivalent:

1) \((\mathcal{G}, [\ , \ ], <, >)\) is a Riemann-Lie algebra.

2) \((\mathcal{G}^*, \pi^d, <, >)\) is a Riemann-Poisson manifold (\( \pi^d \) is the canonical Poisson tensor on \( \mathcal{G}^* \) and \(<, >\) is considered as a contravariant metric on \( \mathcal{G}^* \)).

3) The 2-form \( d\theta \in \Omega^2(G, \mathcal{G}) \) is parallel with respect the Levi-Civita connection \( \nabla \) i.e. \( \nabla d\theta = 0 \).

4) \((G, <, >^d)\) is a flat Riemannian manifold.

5) The orthogonal subalgebra \( S_{<,>} \) of \((\mathcal{G}, [\ , \ ], <, >)\) is abelian and \( \mathcal{G} \) split as an orthogonal direct sum \( S_{<,>} \oplus \mathcal{U} \) where \( \mathcal{U} \) is a commutative ideal.

The equivalence “1) \( \iff \) 2)” of this theorem was proven in [2] and the equivalence “4) \( \iff \) 5)” was proven by Milnor in [5]. We will prove the equivalence “1) \( \iff \) 3)” and the equivalence “1) \( \iff \) 5)”.

The equivalence “1) \( \iff \) 3)” is a direct consequence of the following formula which it is easy to verify:

\[
\nabla d\theta(u^l, v^l, w^l)_g = Ad_g ([u, [v, w]] - [A_u v, w] - [v, A_u w]), \quad u, v, w \in \mathcal{G}, g \in G.
\]

(10)

If \( G \) is compact, connected and non-abelian, the condition \( \nabla d\theta = 0 \) implies that \( d\theta \) is harmonic and defines, according to Hodge Theorem, a non-vanishing class in \( H^2(G, \mathbb{R}) \). If \( G \) is compact, connected and semi-simple Lie group, then \( H^2(G, \mathbb{R}) = 0 \) and hence we get the following lemma which will be used in the proof of the equivalence “1) \( \iff \) 5)” in Section 3.

**Lemma 2.1** Let \( G \) be a compact, connected and semi-simple Lie group. Then the Lie algebra of \( G \) does not admit any structure of Riemann-Lie algebra.

A proof of the equivalence “1) \( \iff \) 5)” will be given in Section 3.
2.3 Examples of Riemann-Poisson Lie groups

This subsection is devoted to the construction, using Theorem 2.1, of many structures of Riemann-Poisson Lie groups. A Riemann-Poisson Lie group is a Poisson Lie group with a left-invariant Riemannian metric compatible with the Poisson structure (see [2]).

We refer the reader to [6] for background material on Poisson Lie groups. Let $\mathcal{G}$ be a Poisson Lie group with Lie algebra $\mathcal{G}$ and $\pi$ the Poisson tensor on $\mathcal{G}$. Pulling $\pi$ back to the identity element $e$ of $\mathcal{G}$ by left translations, we get a map $\pi_l : \mathcal{G} \rightarrow \mathcal{G} \land \mathcal{G}$ defined by $\pi_l(g) = (L_g)_* \pi(g)$ where $(L_g)_*$ denotes the tangent map of the left translation of $\mathcal{G}$ by $g$. Let $d_e \pi : \mathcal{G} \rightarrow \mathcal{G} \times \mathcal{G}$ be the intrinsic derivative of $\pi$ at $e$ given by

$$v \mapsto L_X \pi(e),$$

where $X$ can be any vector field on $\mathcal{G}$ with $X(e) = v$.

The dual map of $d_e \pi$

$$[ , ]_e : \mathcal{G}^* \times \mathcal{G}^* \rightarrow \mathcal{G}^*$$

is exactly the Lie bracket on $\mathcal{G}^*$ obtained by linearizing the Poisson structure at $e$. The Lie algebra $(\mathcal{G}^*, [ , ]_e)$ is called the dual Lie algebra associated with the Poisson Lie group $(\mathcal{G}, \pi)$.

We consider now a scalar product $< , >^l$ on $\mathcal{G}^*$. We denote by $< , >^l$ the left-invariant contravariant Riemannian metric on $\mathcal{G}$ given by (4).

We have shown (cf. [2] Lemma 5.1) that $(\mathcal{G}, \pi, < , >^l)$ is a Riemann-Poisson Lie group iff, for any $\alpha, \beta, \gamma \in \mathcal{G}^*$ and for any $g \in \mathcal{G}$,

$$[\text{Ad}^*_g \left( A^*_\alpha \gamma + a d^*_{\pi_l(g)(\alpha)} \gamma \right), \text{Ad}^*_g (\beta)] + [\text{Ad}^*_g (\alpha), \text{Ad}^*_g \left( A^*_\beta \gamma + a d^*_{\pi_l(g)(\beta)} \gamma \right)] = 0,$$

(11)

where $A^* : \mathcal{G}^* \times \mathcal{G}^* \rightarrow \mathcal{G}^*$ is the infinitesimal Levi-Civita connection associated to $(\mathcal{G}^*, [ , ]_e, < , >^l)$ and where $\pi_l(g)$ also denotes the linear map from $\mathcal{G}$ to $\mathcal{G}$ induced by $\pi_l(g) \in \mathcal{G} \land \mathcal{G}$.

This complicated equation can be simplified enormously in the case where the Poisson tensor arises from a solution of the classical Yang-Baxter equation. However, we need to work more in order to give this simplification.

Let $\mathcal{G}$ be a Lie group and let $r \in \mathcal{G} \land \mathcal{G}$. We will also denote by $r : \mathcal{G}^* \rightarrow \mathcal{G}$ the linear map induced by $r$. Define a bivector $\pi$ on $\mathcal{G}$ by

$$\pi(g) = (L_g)_* r - (R_g)_* r \quad g \in \mathcal{G}.$$
\((G, \pi)\) is a Poisson Lie group if and only if the element \([r, r] \in G \wedge G \wedge G\) defined by

\[
[r, r](\alpha, \beta, \gamma) = \alpha([r(\beta), r(\gamma)]) + \beta([r(\gamma), r(\alpha)]) + \gamma([r(\alpha), r(\beta)])
\]

(12) is \(ad\)-invariant. In particular, when \(r\) satisfies the Yang-Baxter equation

\[
[r, r] = 0
\]

\((Y - B)\) it defines a Poisson Lie group structure on \(G\) and, in this case, the bracket of the dual Lie algebra \(G^*\) is given by

\[
[\alpha, \beta]_e = \text{ad}^*_{r(\beta)}(\alpha) - \text{ad}^*_{r(\alpha)}(\beta), \quad \alpha, \beta \in G^*.
\]

(13)

We will denote by \([ \ ]_r\) this bracket.

We will give now another description of the solutions of the Yang-Baxter equation which will be useful latter.

We observe that to give \(r \in G \wedge G\) is equivalent to give a vectorial subspace \(S_r \subset G\) and a non-degenerate 2-form \(\omega_r \in \wedge^2 S_r^*\).

Indeed, for \(r \in G \wedge G\), we put \(S_r = \text{Im}r\) and \(\omega_r(u, v) = r(r^{-1}(u), r^{-1}(v))\) where \(u, v \in S_r\) and \(r^{-1}(u)\) is any antecedent of \(u\) by \(r\). Conversely, let \((S, \omega)\) be a vectorial subspace of \(G\) with a non-degenerate 2-form. The 2-form \(\omega\) defines an isomorphism \(\omega^b : S \rightarrow S^*\) by \(\omega^b(u) = \omega(u, \cdot)\), we denote by \# : \(S^* \rightarrow S\) its inverse and we put

\[
r = \# \circ i^*
\]

where \(i^* : G^* \rightarrow S^*\) is the dual of the inclusion \(i : S \hookrightarrow G\).

With this observation in mind, the following proposition gives another description of the solutions of the Yang-Baxter equation.

**Proposition 2.1** Let \(r \in G \wedge G\) and \((S_r, \omega_r)\) its associated subspace. The following assertions are equivalent:

1) \([r, r] = 0\).

2) \(r([\alpha, \beta], r(\beta)) = [r(\alpha), r(\beta)]\). (\([\ ]_r\) is the bracket given by (13)).

3) \(S_r\) is a subalgebra of \(G\) and

\[
\delta \omega_r(u, v, w) := \omega_r(u, [v, w]) + \omega_r(v, [w, u]) + \omega_r(w, [u, v]) = 0
\]

for any \(u, v, w \in S_r\).

Moreover, if 1), 2) or 3) holds then \((G^*, [\ ]_r)\) is a Lie algebra and \(r : G^* \rightarrow G\) is a morphism of Lie algebras.
Proof: The proposition follows from the following formulas:
\[
\gamma(r([\alpha, \beta], r) - [r(\alpha), r(\beta)]) = -[r, r]([\alpha, \beta, \gamma], \alpha, \beta, \gamma, \gamma, \in \mathcal{G}^* \quad (14)
\]
and, if \( S \) is a subalgebra,
\[
[r, r]([\alpha, \beta, \gamma]) = -\delta \omega_r(r(\alpha), r(\beta), r(\gamma)). \quad (15)
\]
\[\square\]
This proposition shows that to give a solution of the Yang-Baxter equation is equivalent to give a symplectic subalgebra of \( \mathcal{G} \). We recall that a symplectic algebra (see [3]) is a Lie algebra \( S \) endowed with a non-degenerate 2-form \( \omega \) such that \( \delta \omega = 0 \).

Remark 2.2 Let \( G \) be a Lie group, \( \mathcal{G} \) its Lie algebra and \( S \) an even dimensional abelian subalgebra of \( \mathcal{G} \). Any non-degenerate 2-form \( \omega \) on \( S \) verifies the assertion 3) in Proposition 2.1 and hence \( (S, \omega) \) defines a solution of the Yang-Baxter equation and then a structure of Poisson Lie group on \( G \).\[\square\]

The following proposition will be crucial in the simplification of the equation (11).

**Proposition 2.2** Let \((\mathcal{G}, [\ , \ ], <, >)\) be a Lie algebra endowed with a scalar product, \( r \in \mathcal{G} \wedge \mathcal{G} \) a solution of \((Y - B)\) and \((S_r, \omega_r)\) its associated symplectic Lie algebra. Then, \( S_r \subset S_{<,>} \) iff the infinitesimal Levi-Civita connection \( A^*_r \) associated with \((\mathcal{G}^*, [\ , \ ], r, <, >^*)\) is given by
\[
A^*_r\alpha \beta = -ad^*_r(\alpha, \beta), \quad \alpha, \beta \in \mathcal{G}^*. \quad (16)
\]
Moreover, if \( S_r \subset S_{<,>} \), the curvature of \( A^*_r \) vanishes and hence \((\mathcal{G}^*, [\ , \ ], r, <, >^*)\) is a Riemann-Lie algebra.

**Proof:** \( A^*_r \) is the unique bilinear map from \( \mathcal{G}^* \times \mathcal{G}^* \) to \( \mathcal{G}^* \) such that:
1) \( A^*_r\alpha \beta - A^*_r\beta \alpha = [\alpha, \beta, r] \) for any \( \alpha, \beta \in \mathcal{G}^* \);
2) the endomorphism \( A^*_r : \mathcal{G}^* \rightarrow \mathcal{G}^* \) is skew-adjoint with respect to \( <, >^* \). The bilinear map \((\alpha, \beta) \mapsto -ad^*_r(\alpha, \beta)\) verifies 1) obviously and verifies 2) iff \( S_r \subset S_{<,>} \).
If \( A^*_r\alpha \beta = -ad^*_r(\alpha, \beta) \), the curvature of \( A^*_r \) is given by
\[
R(\alpha, \beta) \gamma = A^*_r[\alpha, \beta, r] \gamma - \left( A^*_r A^*_r \gamma - A^*_r A^*_r \gamma \right) = ad^*_r((\alpha, \beta, r) - [\alpha, r(\beta)] \gamma = 0
\]
from Proposition 2.1 2). We conclude by Theorem 2.1.\[\square\]
Proposition 2.3 Let $(\mathcal{G}, [,], <,>)$ be a Lie algebra with a scalar product. Let $r \in \mathcal{G} \wedge \mathcal{G}$ be a solution of $(Y - B)$ such that $S_r$ is a subalgebra of the orthogonal subalgebra $S_{<,>}$. Then $S_r$ is abelian.

Proof: $S_r$ is unimodular and symplectic and then resolvable (see [3]). Also $S_r$ carries a bi-invariant scalar product so $S_r$ must be abelian (see [5]).

We can now simplify the equation (11) and give the construction of Riemann-Poisson Lie groups announced before. Let $G$ be a Lie group, $(\mathcal{G}, [,])$ its Lie algebra and $<,>$ a scalar product on $\mathcal{G}$. We assume that the orthogonal subalgebra $S_{<,>}$ contains an abelian even dimensional subalgebra $S$ endowed with a non-degenerate 2-form $\omega$.

As in Remark 2.2, $(\mathcal{S}, \omega)$ defines a solution $r$ of $(Y - B)$ and then a Poisson Lie tensor $\pi$ on $G$. We can verify obviously that, for any $g \in G$,

$$\pi^l(g) = r - Ad_g(r).$$

This implies that (11) can be rewritten

$$[Ad_g^* (A^*_\alpha \gamma + ad^*_r(\alpha) \gamma), Ad_g^*(\beta)]_r + [Ad_g^*(\alpha), Ad_g^*(A^*_\beta \gamma + ad^*_r(\beta) \gamma)]_r = [Ad_g^*(ad_{Ad_g(r)(\alpha)} \gamma), Ad_g^*(\beta)]_r + [Ad_g^*(\alpha), Ad_g^*(ad_{Ad_g(r)(\beta)} \gamma)]_r.$$ 

Now, since $S \subset S_{<,>}$, we have by Proposition 2.2

$$A^*_\alpha \gamma + ad^*_r(\alpha) \gamma = 0$$

for any $\alpha, \gamma \in G^*$. On other hand, the following formula is easy to get

$$Ad_g^*[ad^*_r(\alpha) \beta] = ad^*_r(\alpha)(Ad_g^{-1}r)(Ad_g^*\alpha)(Ad_g^*\beta), \quad g \in G, \alpha, \beta \in G^*.$$ 

Finally, $(G, \pi, <,>)$ is a Riemann-Poisson Lie group if

$$[ad^*_r(\alpha) \gamma, \beta]_r + [\alpha, ad^*_r(\beta) \gamma]_r = 0, \quad \alpha, \beta, \gamma \in G^*.$$ 

But, also since $A^*_\alpha \gamma + ad^*_r(\alpha) \gamma = 0$, this condition is equivalent to $(G^*, [ ], <, >^s)$ is a Riemann-Lie algebra which is true by Proposition 2.2. So, we have shown:

Theorem 2.2 Let $G$ be a Lie group, $(\mathcal{G}, [,])$ its Lie algebra and $<,>$ a scalar product on $\mathcal{G}$. Let $S$ be an even dimensional abelian subalgebra of the orthogonal subalgebra $S_{<,>}$ and $\omega$ a non-degenerate 2-form on $S$. Then, the solution of the Yang-Baxter equation associated with $(S, \omega)$ defines a structure of Poisson Lie group $(G, \pi)$ and $(G, \pi, <, >^s)$ is a Riemann-Poisson Lie group.
Let us enumerate some important cases where this theorem can be used.

1) Let $G$ be a compact Lie group and $\mathfrak{g}$ its Lie algebra. For any bi-invariant scalar product $\langle, \rangle$ on the Lie algebra $\mathfrak{g}$, $S_{\langle, \rangle} = \mathfrak{g}$. By Theorem 2.2, we can associate to any even dimensional subalgebra of $\mathfrak{g}$ a Riemann-Poisson Lie group structure on $G$.

2) Let $(\mathfrak{g}, [\ , \ ], \langle , , \rangle)$ be a Riemann-Lie algebra. By Theorem 2.1, the orthogonal subalgebra $S_{\langle, \rangle}$ is abelian and any even dimensional subalgebra of $S_{\langle, \rangle}$ gives rise to a structure of a Riemann-Poisson Lie group on any Lie group whose the Lie algebra is $\mathfrak{g}$. Moreover, we get a structure of bialgebra $(\mathfrak{g}, [\ , \ ], \mathfrak{g}^*, [\ , \ ]_r)$ where both $\mathfrak{g}$ and $\mathfrak{g}^*$ are Riemann-Lie algebras.

Finally, we observe that the Riemann-Lie groups constructed above inherit the properties of Riemann-Poisson manifolds (see [2]). Namely, the symplectic leaves of these Poisson Lie groups are Kählerian and their Poisson structures are unimodular.

3 Proof of the equivalence “1) $\iff$ 5)” in Theorem 2.1

In this section we will give a proof of the equivalence “1) $\iff$ 5)” in Theorem 2.1. The proof is a sequence of lemmas. Namely, we will show that, for a Riemann-Lie algebra $(\mathfrak{g}, [\ , \ ], \langle , , \rangle)$, the orthogonal subalgebra $S_{\langle, \rangle}$ is abelian. Moreover, $S_{\langle, \rangle}$ is the $\langle , , \rangle$-orthogonal of the ideal $[\mathfrak{g}, \mathfrak{g}]$. This result will be the key of the proof.

We begin by a characterization of Riemann-Lie subalgebras.

Proposition 3.1 Let $(\mathfrak{g}, [\ , \ ], \langle , , \rangle)$ be a Riemann-Lie algebra and $\mathcal{H}$ a subalgebra of $\mathfrak{g}$. For any $u, v \in \mathcal{H}$, we put $A^0_{uv} = A^0_{vu} + A^0_{wv}$, where $A^0_{uv} \in \mathcal{H}$ and $A^1_{uv} \in \mathcal{H}^\perp$. Then, $(\mathcal{H}, [\ , \ ], \langle , , \rangle)$ is a Riemann-Lie algebra if and only if, for any $u, v, w \in \mathcal{H}$, $[A^0_{uv}, w] + [v, A^0_{wv}] + [A^1_{uv}, w] \in \mathcal{H}^\perp$.

Proof: We have, from (9), that for any $u, v, w \in \mathcal{H}$

$$[u, [v, w]] = [A^0_{uv}, w] + [v, A^0_{wv}] + [A^1_{uv}, w] + [v, A^1_{wv}].$$

Now $A^0 : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ is the infinitesimal Levi-Civita connection associated with the restriction of $\langle, \rangle$ to $\mathcal{H}$ and the proposition follows. $\square$

We will introduce now some mathematical objects which will be useful latter. Let $(\mathfrak{g}, [\ , \ ], \langle , , \rangle)$ a Lie algebra endowed with a scalar product.
From (5), we deduce obviously that the infinitesimal Levi-Civita connection $A$ associated to $\langle , \rangle$ is given by

$$A_{uv} = \frac{1}{2} [u, v] - \frac{1}{2} \left( \text{ad}_u^t v + \text{ad}_v^t u \right) \quad u, v \in \mathcal{G}. \quad (17)$$

On other hand, the orthogonal with respect to $\langle , \rangle$ of the ideal $[\mathcal{G}, \mathcal{G}]$ is given by

$$[\mathcal{G}, \mathcal{G}]^\perp = \bigcap_{u \in \mathcal{G}} \text{Kerad}^t_u. \quad (18)$$

Let us introduce, for any $u \in \mathcal{G}$, the endomorphism

$$D_u = \text{ad}_u - A_u. \quad (19)$$

We have, by a straightforward calculation, the relations

$$D_u(v) = \frac{1}{2} [u, v] + \frac{1}{2} \left( \text{ad}_u^t v + \text{ad}_v^t u \right),$$

$$D_t^t(v) = \frac{1}{2} [u, v] + \frac{1}{2} \left( \text{ad}_u^t v - \text{ad}_v^t u \right).$$

From these relations, we remark that, for any $u, v \in \mathcal{G}$, $D_t^t(v) = -D_t^t(u)$ and then

$$\forall u \in \mathcal{G}, \; D_t^t(u) = 0. \quad (20)$$

We remark also that

$$D_t^t = D_u \iff \forall v \in \mathcal{G}, \text{ad}_v^t u = 0.$$ 

So, by (18), we get

$$[\mathcal{G}, \mathcal{G}]^\perp = \left\{ u \in \mathcal{G}; \; D_t^t = D_u \right\}. \quad (21)$$

We begin now to state and to prove a sequence of results which will give a proof of the equivalence “1) $\iff$ 5)” in Theorem 2.1.

**Proposition 3.2** Let $(\mathcal{G}, [ , ], \langle , \rangle)$ be a Riemann-Lie algebra. Then $Z(\mathcal{G})^\perp$ (where $Z(\mathcal{G})$ is the center of $\mathcal{G}$) is an ideal of $\mathcal{G}$ which contains the ideal $[\mathcal{G}, \mathcal{G}]$. In particular,

$$\mathcal{G} = Z(\mathcal{G}) \oplus Z(\mathcal{G})^\perp.$$
Proof: For any $u \in Z(G)$ and $v \in G$, from (17) and the fact that $A_u$ is skew-adjoint, $A_u v = -\frac{1}{2} \text{ad}_u^t v \in Z(G)^\perp$. By (8), for any $w \in G$

$$[A_u v, w] = [A_u v, u] = 0,$$

so $A_u v \in Z(G)$ and then $A_u v = -\frac{1}{2} \text{ad}_u^t v = 0$ which shows that $u \in [G,G]^\perp$. So $Z(G) \subset [G,G]^\perp$ and the proposition follows. 

From this proposition and the fact that for a nilpotent Lie algebra $G$, $Z(G) \cap [G,G] \neq \{0\}$, we get the following lemma.

**Lemma 3.1** A nilpotent Lie algebra $G$ carries a structure of Riemann-Lie algebra if and only if $G$ is abelian.

We can now get the following crucial result.

**Lemma 3.2** Let $(G,[ , ],<,>)$ be a Riemann-Lie algebra. Then the orthogonal Lie subalgebra $S_{<,>}$ is abelian.

Proof: By (17), $A_u v = \frac{1}{2}[u,v]$ for any $u,v \in S_{<,>}$. So, by Proposition 3.1, $S_{<,>}$ is a Riemann-subalgebra. By (9), we have, for any $u,v,w \in S_{<,>}$,

$$[u,[v,w]] = [A_u v, w] + [v, A_u w]$$

$$= \frac{1}{2}[[u,v],w] + \frac{1}{2}[v,[u,w]]$$

$$= \frac{1}{2}[u,[v,w]]$$

and then $[S_{<,>},[S_{<,>},S_{<,>}]] = 0$ i.e. $S_{<,>}$ is a nilpotent Lie algebra and then abelian by Lemma 3.1. 

**Lemma 3.3** Let $(G,[ , ],<,>)$ be a Riemann-Lie algebra. Then $[G,G]^\perp = \{u \in G; D_u = 0\}$.

Proof: Firstly, we notice that, by (21), $[G,G]^\perp \supset \{u \in G; D_u = 0\}$. On other hand, remark that the relation (8) can be rewritten

$$[D_u(v),w] + [v, D_u(w)] = 0$$

for any $u,v,w \in G$. So, we can deduce immediately that $[\text{Ker}D_u, \text{Im}D_u] = 0$ for any $u \in G$. 

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Now we observe that, for any $u \in [\mathcal{G}, \mathcal{G}]^\perp$, the endomorphism $D_u$ is auto-
adjoint and then diagonalizable on $\mathbb{R}$. Let $u \in [\mathcal{G}, \mathcal{G}]^\perp$, $\lambda \in \mathbb{R}$ be an
eigenvalue of $D_u$ and $v \in \mathcal{G}$ an eigenvector associated with $\lambda$. We have

$$< D_u(v), v > = \lambda < v, v > \overset{(\alpha)}{=} - < A_u v, v > \overset{(\beta)}{=} - <[v, u], v > \overset{(\gamma)}{=} 0.$$ 

So $\lambda = 0$ and we obtain that $D_u$ vanishes identically. Hence the lemma follows.

The equality $(\alpha)$ is a consequence of the definition of $D_u$, and the equality $(\beta)$
follows from the definition of $A$. We observe that $v \in Im D_u$ and $u \in Ker D_u$
since $D_u(u) = D_u^t(u) = 0$ (see (20)) and the equality $(\gamma)$ follows from the
remark above. $\square$

Lemma 3.4 Let $(\mathcal{G}, [ , ], <, >)$ be a Riemann-Lie algebra. Then

$$S_{<, >} = [\mathcal{G}, \mathcal{G}]^\perp.$$ 

Proof: From Lemma 3.3, for any $u \in [\mathcal{G}, \mathcal{G}]^\perp$, $A_u = \text{ad}_u$ and then $\text{ad}_u$ is
 skew-adjoint. So $[\mathcal{G}, \mathcal{G}]^\perp \subset S_{<, >}$. To prove the second inclusion we need to
work harder than the first one.

Firstly, remark that one can suppose that $Z(\mathcal{G}) = \{0\}$. Indeed, $\mathcal{G} = Z(\mathcal{G}) \oplus Z(\mathcal{G})^\perp$ (see Proposition 3.2), $Z(\mathcal{G})^\perp$ is a Riemann-Lie algebra (see
Proposition 3.1), $[\mathcal{G}, \mathcal{G}] = [Z(\mathcal{G})^\perp, Z(\mathcal{G})^\perp]$ and $S_{<, >} = Z(\mathcal{G}) \oplus S'_{<, >}$ where
$S'_{<, >}$ is the orthogonal subalgebra associated to $(Z(\mathcal{G})^\perp, <, >)$.

We suppose now that $(\mathcal{G}, [ , ], <, >)$ is a Riemann-Lie algebra such that
$Z(\mathcal{G}) = \{0\}$ and we want to prove the inclusion $[\mathcal{G}, \mathcal{G}]^\perp \subset S_{<, >}$. Notice that
it suffices to show that, for any $u \in S_{<, >}$, $A_u = \text{ad}_u$.

The proof requires some preparation. Let us introduce the subalgebra $K$
given by

$$K = \bigcap_{u \in S_{<, >}} \ker \text{ad}_u.$$ 

Firstly, we notice that $K$ contains $S_{<, >}$ because $S_{<, >}$ is abelian (see Lemma
3.2).

On other hand, we remark that, for any $u \in S_{<, >}$, the endomorphism $A_u$
leaves invariant $K$ and $K^\perp$. Indeed, for any $v \in K$ and any $w \in S_{<, >}$, we have

$$[w, A_u v] \overset{(\alpha)}{=} [w, A_u u] \overset{(\beta)}{=} -[A_w u, v] \overset{(\gamma)}{=} 0$$

and then $A_u v \in K$, this shows that $A_u$ leaves invariant $K$. Furthermore, $A_u$
being skew-adjoint, we have $A_u(K^\perp) \subset K^\perp$. 

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The equality \((\alpha)\) follows from the relation \(A_u v = A_v u + [u, v] = A_u v\),
the equality \((\beta)\) follows from (8) and \((\gamma)\) follows from the relation \(A_w u = \frac{1}{2} [w, u] = 0\).
With this observation in mind, we consider now the representation \(\rho: S_{<,>} \to so(K^\perp)\) given by

\[
\rho(u) = ad_u|K^\perp \quad u \in S_{<,>}.
\]

It is clear that

\[
\bigcap_{u \in S_{<,>}} \ker \rho(u) = \{0\}.
\]

This relation and the fact that \(S_{<,>}\) is abelian imply that \(\dim K^\perp\) is even and that there is an orthonormal basis \((e_1, f_1, \ldots, e_p, f_p)\) of \(K^\perp\) such that

\[
\forall i \in \{1, \ldots, p\}, \forall u \in S_{<,>}, \quad ad_u e_i = \lambda^i(u) f_i \quad \text{and} \quad ad_u f_i = -\lambda^i(u) e_i,
\]

where \(\lambda^i \in S^*_{<,>}\).

Now, for any \(u \in S_{<,>}\), since \(A_u\) leaves \(K^\perp\) invariant, we can write

\[
A_u e_i = \sum_{j=1}^{p} \left( < A_u e_i, e_j > e_j + < A_u e_i, f_j > f_j \right),
\]

\[
A_u f_i = \sum_{j=1}^{p} \left( < A_u f_i, e_j > e_j + < A_u f_i, f_j > f_j \right).
\]

From (9), we have for any \(v \in S_{<,>}\) and for any \(i \in \{1, \ldots, p\},\)

\[
[u, [v, e_i]] = [A_u v, e_i] + [v, A_u e_i],
\]

\[
[u, [v, f_i]] = [A_u v, f_i] + [v, A_u f_i].
\]

Using the the equality \(A_u v = 0\) and \((**\)\) and substituting we get

\[
-\lambda^i(u) \lambda^j(v) e_i = \sum_{j=1}^{p} \lambda^j(v) < A_u e_i, e_j > f_j - \sum_{j=1}^{p} \lambda^j(v) < A_u e_i, f_j > e_j,
\]

\[
-\lambda^i(u) \lambda^j(v) f_i = \sum_{j=1}^{p} \lambda^j(v) < A_u f_i, e_j > f_j - \sum_{j=1}^{p} \lambda^j(v) < A_u f_i, f_j > e_j.
\]

Now, it is clear from \((*)\) that, for any \(i \in \{1, \ldots, p\},\) there exists \(v \in S_{<,>}\) such that \(\lambda^i(v) \neq 0\). Using this fact and the relations above, we get

\[
A_u e_i = \lambda^i(u) f_i \quad \text{and} \quad A_u f_i = -\lambda^i(u) e_i.
\]
So we have shown that, for any \( u \in S_{<,>} \),

\[
A_u|K^\perp = ad_u|K^\perp.
\]

Now, for any \( u \in S_{<,>} \) and for any \( k \in K \), \( ad_u(k) = 0 \). So, to complete the proof of the lemma, we will show that, for any \( u \in S_{<,>} \) and for any \( k \in K \), \( A_u k = 0 \). This will be done by showing that \( A_u k \in Z(G) \) and conclude by using the assumption \( Z(G) = \{0\} \).

Indeed, for any \( h \in K \), by (8)

\[
[A_u k, h] = [A_h k, u].
\]

Since \( A_u(K) \subset K \) and since \( K \) is a subalgebra, \( [A_u k, h] \subset K \). Now, \( K \subset \ker ad_u \) and \( ad_u \) is skew-adjoint so \( [A_u k, u] \subset \text{Im}ad_u \subset K^\perp \). So \( [A_u k, h] = 0 \).

On other hand, for any \( f \in K^\perp \), we have, also from (8),

\[
[A_u k, f] = [A_k u, f] = [A_f u, k] = 0
\]

since \( A_f u = [f, u] + A_u f = [f, u] + [u, f] = 0 \).

We deduce that \( A_u k \in Z(G) \) and then \( A_u k = 0 \). The proof of the lemma is complete. □

**Lemma 3.5** Let \((G, [\ , \], <, >)\) be a Riemann-Lie algebra such that \( Z(G) = 0 \). Then

\[ G \neq [G, G]. \]

**Proof:** Let \((G, [\ , \], <, >)\) be a Riemann-Lie algebra such that \( Z(G) = 0 \). We will show that the assumption \( G = [G, G] \) implies that the Killing form of \( G \) is strictly negative definite and then \( G \) is semi-simple and compact which is in contradiction with lemma 2.1.

Let \( u \in G \) fixed. Since \( A_u \) is skew-adjoint, there is an orthonormal basis \((a_1, b_1, \ldots, a_r, b_r, c_1, \ldots, c_l)\) of \( G \) and \((\mu_1, \ldots, \mu_r) \in \mathbb{R}^r \) such that, for any \( i \in \{1, \ldots, r\} \) and any \( j \in \{1, \ldots, l\} \),

\[
A_u a_i = \mu_i b_i, \quad A_u b_i = -\mu_i a_i \quad \text{and} \quad A_u c_j = 0.
\]

Moreover, \( \mu_i > 0 \) for any \( i \in \{1, \ldots, r\} \).

By applying (9), we can deduce, for any \( i, j \in \{1, \ldots, r\} \) and for any \( k, h \in \{1, \ldots, l\} \), the relations:

\[
\begin{align*}
[u, [a_i, a_j]] &= \mu_i [b_i, a_j] + \mu_j [a_i, b_j], \quad [u, [b_i, b_j]] = -\mu_j [b_i, a_j] - \mu_i [a_i, b_j], \\
[u, [a_i, b_j]] &= -\mu_j [a_i, a_j] + \mu_i [b_i, b_j], \quad [u, [b_i, a_j]] = -\mu_i [a_i, a_j] + \mu_j [b_i, b_j], \\
[u, [c_k, a_j]] &= \mu_j [c_k, b_j], \quad [u, [c_k, b_j]] = -\mu_j [c_k, a_j], \quad [u, [c_k, c_h]] = 0.
\end{align*}
\]

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From these relations we deduce

\[
\begin{align*}
ad_u \circ ad_u ([a_i, a_j]) &= -(\mu_i^2 + \mu_j^2)[a_i, a_j] + 2\mu_i\mu_j[b_i, b_j], \\
ad_u \circ ad_u ([b_i, b_j]) &= 2\mu_i\mu_j[a_i, a_j] - (\mu_i^2 + \mu_j^2)[b_i, b_j], \\
ad_u \circ ad_u ([b_i, a_j]) &= -(\mu_i^2 + \mu_j^2)[b_i, a_j] - 2\mu_i\mu_j[a_i, b_j], \\
ad_u \circ ad_u ([a_i, b_j]) &= -2\mu_i\mu_j[b_i, a_j] - (\mu_i^2 + \mu_j^2)[a_i, b_j], \\
ad_u \circ ad_u ([c_k, a_j]) &= -\mu_j^2[c_k, a_j], \\
ad_u \circ ad_u ([c_k, b_j]) &= -\mu_j^2[c_k, b_j], \\
ad_u \circ ad_u ([c_k, c_h]) &= 0.
\end{align*}
\]

By an obvious transformation we obtain

\[
\begin{align*}
ad_u \circ ad_u ([a_i, a_j] + [b_i, b_j]) &= -(\mu_i - \mu_j)^2([a_i, a_j] + [b_i, b_j]), \\
ad_u \circ ad_u ([a_i, a_j] - [b_i, b_j]) &= -(\mu_i + \mu_j)^2([a_i, a_j] - [b_i, b_j]), \\
ad_u \circ ad_u ([b_i, a_j] + [a_i, b_j]) &= -(\mu_i + \mu_j)^2([b_i, a_j] + [a_i, b_j]), \\
ad_u \circ ad_u ([b_i, a_j] - [a_i, b_j]) &= -(\mu_i - \mu_j)^2([b_i, a_j] - [a_i, b_j]), \\
ad_u \circ ad_u ([c_k, a_j]) &= -\mu_j^2[c_k, a_j], \\
ad_u \circ ad_u ([c_k, b_j]) &= -\mu_j^2[c_k, b_j], \\
ad_u \circ ad_u ([c_k, c_h]) &= 0.
\end{align*}
\]

Suppose now \( \mathcal{G} = [\mathcal{G}, \mathcal{G}] \). Then the family of vectors
\[
\{[a_i, a_j] + [b_i, b_j], [a_i, a_j] - [b_i, b_j], [b_i, a_j] + [a_i, b_j],
[b_i, a_j] - [a_i, b_j], [c_k, a_i], [c_k, b_j], [c_k, c_h]; i, j \in \{1, \ldots, r\}, h, k \in \{1, \ldots, l\}\}
\]
spans \( \mathcal{G} \) and then \( ad_u \circ ad_u \) is diagonalizable and all its eigenvalues are non-positive. Now its easy to deduce that \( ad_u \circ ad_u = 0 \) if and only if \( ad_u = 0 \). Since \( Z(\mathcal{G}) = 0 \) we have shown that, for any \( u \in \mathcal{G} \setminus \{0\} \), \( Tr(ad_u \circ ad_u) < 0 \) and then the Killing form of \( \mathcal{G} \) is strictly negative definite and then \( \mathcal{G} \) is semi-simple compact. We can conclude with the Lemma 2.1.\( \square \)

**Proof of the equivalence “1) \( \Leftrightarrow \) 5)” in Theorem 2.1.**

It is an obvious and straightforward calculation to show that 5) \( \Rightarrow \) 1).

Conversely, let \( (\mathcal{G}, [, , ], <, >) \) be a Riemann-Lie algebra. By Proposition 3.2, we can suppose that \( Z(\mathcal{G}) = \{0\} \).

We have, from Lemma 3.5 and Lemma 3.4, \( \mathcal{G} \neq [\mathcal{G}, \mathcal{G}] \) which implies \( S_{<, >} \neq 0 \) and \( \mathcal{G} = S_{<, >} \frac{1}{\Theta} [\mathcal{G}, \mathcal{G}] \). Moreover, \([\mathcal{G}, \mathcal{G}] \) is a Riemann-Lie algebra (see Proposition 3.1) and we can repeat the argument above to deduce that finally \( \mathcal{G} \) is solvable which implies that \([\mathcal{G}, \mathcal{G}] \) is nilpotent and then abelian by Lemma 3.1 and the implication follows.\( \square \)
Remark 3.1  The pseudo-Riemann-Lie algebras are completely different from the Riemann-Lie algebras. Indeed, the 3-dimensional Heisenberg Lie algebra which is nilpotent carries a Lorentzian-Lie algebra structure. On other hand, the non trivial 2-dimensional Lie algebra carries a Lorentzian inner product whose curvature vanishes and don’t carries any structure of pseudo-Riemann-Lie algebra.

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Mohamed BOUCETTA
Faculté des sciences et techniques, BP 549, Marrakech, Maroc
Email: boucetta@fstm-marrakech.ac.ma

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