The Pairwise-Markov Bernoulli Filter

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ABSTRACT The Bernoulli filter is a general, Bayes-optimal solution for tracking a single disappearing and reappearing target, using a sensor whose observations are corrupted by missed detections and a general, known clutter process. Like virtually all target-tracking algorithms it presumes restrictive independence assumptions, namely a hidden Markov model (HMM) structure on the sensor and target. That is, the current state of the target depends only on its previous state, and the measurement collected from it depends only on its current state. Pieczynski’s pairwise Markov model (PMM) relaxes these restrictions. In it, the current target state can additionally depend on the previous measurement; and the current measurement can additionally depend on the previous measurement and previous target state. In this paper we show how to correctly generalize the PMM to the multitarget (MPMM) case; and use the MPMM to derive a “PMM Bernoulli filter” that obeys PMM rather than restrictive HMM sensor/target statistics.

INDEX TERMS Target tracking, random finite set, finite-set statistics, recursive Bayes filter, Bernoulli filter, hidden Markov model, pairwise Markov model.

I. INTRODUCTION

The Bernoulli filter was independently and contemporaneously devised by Vo [1] and Mahler [2, Sec. 14.7]. It is a general and Bayes-optimal solution for tracking a single disappearing and reappearing target, using a sensor whose observations are corrupted by missed detections and a general, known clutter process. It propagates a probability hypothesis density (PHD) $D$ via time-update and measurement-update steps $D(x_k|Z_{1:k-1}) \rightarrow D(x_k|Z_{1:k-1})$ and $D(x_k|Z_{1:k-1}) \rightarrow D(x_k|Z_{1:k})$, where $Z_{1:k} : Z_1, \ldots, Z_k$ is the time-sequence of collected measurement-sets. See Section VI-A for more detail.

Like virtually all target-tracking algorithms, the Bernoulli filter presumes restrictive independence assumptions, namely a hidden Markov model (HMM) structure on the sensor and target. That is, at time $t_k$ the target’s state $x_k$ depends only on its previous state $x_{k-1}$ with Markov transition density $f(x_k|x_{k-1})$; and the measurement $y_k$ that the sensor collects from it depends only on $x_k$ with measurement density $f(y_k|x_k)$. Pieczynski’s pairwise Markov model (PMM) [3]–[7] relaxes these restrictions.

A. THE PAIRWISE MARKOV MODEL (PMM)

The PMM generalizes the HMM by treating the target and sensor as a joint dynamical system with joint state $(x_k, y_k)$, which is governed by a Markov transition density

$$f(x_k, y_k| x_{k-1}, y_{k-1}) = f(x_k|x_{k-1}, y_{k-1}) \cdot f(y_k|x_k, x_{k-1}, y_{k-1})$$

where the factorization on the right is due to Bayes’ rule. In the PMM, the current target state can additionally depend on the previous measurement (as described by $f(x_k|x_{k-1}, y_{k-1})$ i.e., the target can be non-Markovian); and in that the current measurement can additionally depend on the previous measurement and the previous target state as described by $f(y_k|x_k, x_{k-1}, y_{k-1})$ (and thus measurement noise can be colored or correlated with plant noise [7, p. 4487]). See Section III for more detail.

Pieczynski and Desbouvries [6] have described practical Kalman filter-based implementations of PMMs to single-target tracking. Petetin and Desbouvries [7] proposed a PMM generalization of the probability hypothesis density (PHD) filter of [2, Sec. 16.3]; described concrete practical applications and implementations; and demonstrated that their PMM-PHD filter has better tracking performance than the classical HMM-PHD filter under non-HMM conditions. This work has been extended to nonlinear models [8].

B. THE MULTITARGET PMM (MPMM)

Let $X_k$ be the state-set of a multitarget system at time $t_k$ and $Y_k$ the multitarget measurement-set generated by both targets and clutter. In [9] Mahler generalized the PMM to

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the multitarget PMM (MPMM); and also proposed a concrete formula—see (51,53) below—for the MPMM transition density \( f(X_k, Y_k | X_{k-1}, Y_{k-1}) \), based on the “standard” multitarget Markov density \( f(X_k | X_{k-1}) \) [2, Eq. 14.273], [10, Eq. 5.94]; and the “standard” multitarget measurement density \( f(Y_k | X_k) \) [2, Eq. 14.290], [10, Eq. 5.104].

Remark 1: The MPMM transition model (51,53) turns out to be erroneous—see Section V-C. It will be replaced by the corrected, theoretically rigorous model in (60,61).

We shall see that the evolution \( f(X_{k-1}, Y_{k-1} | Z_{1:k-2}) \rightarrow f(X_k, Y_k | Z_{1:k-1}) \) of an MPMM is described in terms of “MPMM densities” \( f(X_k, Y_k | Z_{1:k-1}) \), which describe not only \( X_k \) and \( Y_k \) but also the statistical correlation between them (Section IV). In this paper we will consider the evolution of “Bernoulli MPMM’s” \( (X_k, Y_k) \)—i.e., those such that \( |X_k| \leq 1 \) for all \( k \geq 1 \) (where \( |X| \) denotes the number of elements in \( X \)). In such an MPMM the following dynamical transitions are possible: \((\{x_{k-1}\}, Y_{k-1}) \rightarrow (\{x_k\}, Y_k)\) (target survives); \( (\{x_{k-1}\}, Y_{k-1}) \rightarrow (Ø, Y_k) \) (target disappears); or \((Ø, Y_{k-1}) \rightarrow (\{x_k\}, Y_k) \) (target appears or reappears).

Since \( f(X_k, Y_k | Z_{1:k-1}) = 0 \) identically if \( |X_k| > 1 \), the state of a Bernoulli MPMM at time \( t_k \) is completely described by \( f(Ø, Y_{k-1} | Z_{1:k-2}) \) and \( f(\{x_k\}, Y_k | Z_{1:k-1}) \); and its evolution from time \( t_k \) to time \( t_{k+1} \) is described by the update \( f(Ø, Y_{k-1} | Z_{1:k-2}) \), \( f(\{x_k\}, Y_k | Z_{1:k-1}) \). The ultimate result is a Bayes-optimal “PMMM Bernoulli filter” in which the sensor can have correlated-noise statistics and the target can have non-Markovian dynamics.

C. SUMMARY OF MAIN RESULTS

These are as follows:

1. The corrected MPMM transition model, (60,61).
2. Evolution models for the “elementary” MPMM pairs \( (X_k, Y_k) \)—i.e., those with \( |X_k|, |Y_k| \leq 1 \) (Sections V-D through V-H).
3. The “Bernoulli MPMM filter,” which recursively propagates Bernoulli MPMM densities \( f(X_k, Y_k | Z_{1:k-1}) \) with \( |X_k| \leq 1 \) (Section VI-D).
4. The “PMMM Bernoulli filter,” which, like the usual HMM Bernoulli filter, recursively propagates PHD’s \( D(x_k | Z_{1:k}) \) (see (10)).

The PMMM Bernoulli filter can be summarized as follows. Let us be given: (i) \( \kappa_k(Y_k) \) (the multi-object probability density function of the clutter process); (ii) \( \rho_k(x_{k-1}) \) (the probability that the target will not disappear at time \( t_{k-1} \)); (iii) \( \delta_k^0 \) (the probability that the target will reappear at time \( t_{k-1} \) after having disappeared); (iv) \( \delta_k^R \) (the target’s spatial density after reappearance); (v) \( p_D(x_k) \) (the target’s probability of detection); (vi) \( f(x_k, y_k | x_{k-1}, y_{k-1}) \) (the PMMM transition density); (vii) \( f(x_k | x_{k-1}, y_{k-1}) \) (the marginal of \( f(x_k, y_k | x_{k-1}, y_{k-1}) \)); (viii) \( M_k(x_{k-1}) = f(x_k | x_{k-1}) \) (Markov density associated with transition \((x_{k-1}), Ø \) → \((x_k), Ø \), see (76)); and (ix) \( L_k(y_k) = f(y_k | x_k) \) (measurement density associated with \((Ø, Ø) \rightarrow ((x_k), (y_k)), \) see (89)). Define

\[
\ell_{Y_k}(y_k | x_k) = 1 - p_D(x_k) + p_D(x_k) \sum_{y_k \in Y_k} L_k(y_k) \frac{\kappa_k(Y_k - \{y_k\})}{\kappa_k(Y_k)}
\]

where by convention the summation vanishes if \( Y_k = Ø \). Also, if \( f(x_k | x_{k-1}, y_{k-1}) \) is the marginal of \( f(x_k, y_k | x_{k-1}, y_{k-1}) \),

\[
\ell_{Y_k}(y_k | x_k) = \frac{1 - p_D(x_k)}{|Y_k| - 1} \sum_{y_k \in Y_k} f(x_k, y_k | x_{k-1}, y_{k-1}) \frac{\kappa_k(Y_k - \{y_k\})}{\kappa_k(Y_k)}
\]

if \( Y_k = Ø \) (and by convention the second summation vanishes if \( Y_k = Ø \)); whereas if \( Y_k = Ø \),

\[
\ell_{Y_k}(y_k | x_k) = \ell_{Y_k}(x_k) \cdot f(x_k | x_{k-1}) = \ell_{Y_k}(x_k) \cdot M_k(x_k)
\]

Abbreviate

\[
D_k(x_{k-1}) = D(x_k | Z_{1:k-1}), D_k(x_k) = D(x_k | Z_{1:k})
\]

Define

\[
\ell_{Y_{k-1}, Y_{k-1}}(x_k) = \ell_{Y_{k-1}}(x_k) \cdot f(x_k | x_{k-1}) = \ell_{Y_k}(x_k) \cdot M_k(x_k)
\]

Also, if \( f(x) \) is a density function and \( 0 \leq h(x) \leq 1 \) a unitless function, define

\[
f[h] = \int h(x) \cdot f(x) dx
\]

Given this, the PMMM Bernoulli filter is given by the following single-step recursive update, (10), as shown at the bottom of the next page. This equation is derived in Appendix B.

If the PMMM is actually an HMM, then

\[
f(x_k, y_k | x_{k-1}, y_{k-1}) = f(y_k | x_k) \cdot f(x_k | x_{k-1})
\]

(see (36)) from which it follows that

\[
\ell_{Y_{k-1}, y_{k-1}}(x_k) = \ell_{Y_k}(x_k) \cdot f(x_k | x_{k-1})
\]

in which case, as will be shown in (200), (10) reduces to the single-step HMM Bernoulli filter as given in (111).
D. ORGANIZATION OF THE PAPER
The remainder of the paper is organized as follows: A brief summary of the mathematical theory required to understand the paper (Section II); the PMM (Section III); the MPMM (Section IV); the corrected MPMM transition density (Section V); and the Bernoulli MPMM filter (Section VI). Conclusions can be found in Section VII, and the Bernoulli MPMM and PMM Bernoulli filters are derived in Appendices A and B, respectively. The following notation will be employed hereafter: $A := B$ means “$A$ is defined to be $B$”; and $A !:= B$ means “$A$ is an abbreviation of $B$.”

II. OVERVIEW OF FINITE-SET STATISTICS (FISST)
This section summarizes the theory necessary to understand the remainder of the paper. Greater detail can be found in books [2], [10]–[12], tutorials [13]–[15], and a short survey of advances c. 2015 [16]. Also, systematic investigations of FISST vs. “point processes” can be found in [17], [18] and of FISST vs. measurement-to-track approaches in [19].

Significant recent advances can be found in [20], [21]. Specifically, [20] describes an implementation of the generalized labeled multi-Bernoulli (GLMB) filter that is capable of simultaneously tracking over a million 2D targets in significant clutter in real time using off-the-shelf computing equipment, as well as a theoretically rigorous, large-scale track quality measure, “OSP(A)”; and [21] describes a multiscan extension of the GLMB filter.

The section is organized as follows: random finite sets (Section II-A); multitarget calculus (Section II-B); Bernoulli RFSs (Section II-C); and the multitarget recursive Bayes filter (Section II-D).

A. RANDOM FINITE SETS (RFSs)
Let $\mathcal{Z}$ be a single-target-state space with $\mathbf{x} \in \mathcal{Z}$, and let $\mathcal{N}$ be the sensor measurement space with $\mathbf{z} \in \mathcal{N}$. Then the state of a multitarget system is represented as a finite subset $X = \{x_1, \ldots, x_n\} \subseteq \mathcal{Z}$ with $X = \emptyset$ for $n = 0$. The number of elements in $X$ is denoted as $|X|$. In a Bayesian approach, unknown states are random variables. Thus an unknown multitarget state is a random finite set (RFS) $\Xi \subseteq \mathcal{Z}$.

B. MULTITARGET CALCULUS
A multitarget density function is a function $f(X)$ of the finite-set variable $X \subseteq \mathcal{Z}$ such that the unit of measurement of $f(X)$ is $\ell_{|X|}$, where $\ell_{|X|}$ is the unit of measurement of $\mathcal{Z}$. The set integral of $f(X)$ is

$$\int f(X)\,\delta X = f(\emptyset) + \sum_{n \geq 1} \int f_n(x_1, \ldots, x_n)\,dx_1 \cdots dx_n$$

where $f_n(x_1, \ldots, x_n) := f((x_1, \ldots, x_n)/n!$ for distinct $x_1, \ldots, x_n$ and $f_n(x_1, \ldots, x_n) := 0$ otherwise. Every random finite state-set $\Xi$ has a multitarget probability distribution $f_\Xi(X)$ with $f f_\Xi(X)\,\delta X = 1$.

An MPMM density function is a function $f(X, Y) \geq 0$ of the finite-set variables $X \subseteq \mathcal{Z}$, $Y \subseteq \mathcal{N}$ such that the unit of measurement of $f(X, Y)$ is $\ell_{|X|}^-\ell_{|Y|}^-$ where $\ell_{|\cdot|}$ is the unit of measurement of $\mathcal{N}$. An MPMM density function is a joint probability density if $f_f(X, Y)\,\delta X\,\delta Y = 1$. If $\Xi \subseteq \mathcal{Z}$ and $\Sigma \subseteq \mathcal{N}$ are RFSs then $\Xi, \Sigma$ have an MPMM probability density $f_{\Xi,\Sigma}(X, Y)$ that describes the statistical correlation between them.

The probability generating functional (p.g.f.) of $\Xi$ is, for unitless “test functions” $0 \leq h(x) \leq 1$,

$$G_\Xi[h] := \int h^X \cdot f_\Xi(X)\,\delta X$$

where $h^X = 1$ if $X = \emptyset$ and $h^X = \prod_{x \in X} h(x)$ otherwise. The simplest nontrivial p.g.f.’s are

$$s[h] = \int h(x) \cdot s(x)\,dx$$

where $s(x) \geq 0$ is a probability density function on $\mathcal{Z}$. If $0 \leq g(z) \leq 1$ for $z \in \mathcal{N}$ then the joint p.g.f. of $\Xi, \Sigma$ is

$$G_{\Xi,\Sigma}[h, g] := \int h^X \cdot g^Y \cdot f_{\Xi,\Sigma}(X, Y)\,\delta X\,\delta Y.$$

The intuitive definition of the Volterra functional derivative of $G_\Xi[h]$ is:

$$\frac{\delta G_\Xi[h]}{\delta h} := \lim_{\varepsilon \to 0^+} \frac{G_\Xi[h + \varepsilon \cdot \delta h] - G_\Xi[h]}{\varepsilon}$$

where $\delta h(x)$ is the Dirac delta function concentrated at $x$. (For a rigorous definition see [14].) If $X = \{x_1, \ldots, x_n\}$ with $|X| = n$ then the iterated functional derivative is

$$\frac{\delta G_\Xi[h]}{\delta x} := \frac{\delta^a G_\Xi[h]}{\delta x_1 \cdots \delta x_n} := \frac{\delta^a G_\Xi[h]}{\delta x_n \delta x_1 \cdots \delta x_{n-1}}$$

if $|X| \geq 1$ and $= G_\Xi[h]$ if otherwise. There is an extensive “toolbox” of “turn-the-crank” rules for set integrals and functional derivatives [2, pp. 383-389], [10, pp. 69-80].

The joint functional derivatives of $G_{\Xi,\Sigma}[h, g]$ are:

$$\frac{\delta G_{\Xi,\Sigma}[h, g]}{\delta X} := \frac{\delta^a G_{\Xi,\Sigma}[h, g]}{\delta X^a}$$

where the “$\star$” notation indicates that $\delta \delta X$ is taken with respect to the variable $h$ and $\delta \delta Y$ with respect to the variable $g$. When $Y = \emptyset$ or $X = \emptyset$ we have:

$$\frac{\delta G_{\Xi,\Sigma}[h, g]}{\delta X^a} := \frac{\delta G_{\Xi,\Sigma}[h, g]}{\delta X}$$

$$\frac{\delta G_{\Xi,\Sigma}[h, g]}{\delta Y^a} := \frac{\delta G_{\Xi,\Sigma}[h, g]}{\delta Y}.$$
The p.g.f.l. and distribution of an RFS are related by:

$$f_{\Xi}(X) = \delta G_{\Xi}[0].$$  \hfill (21)

Likewise, the bivariate p.g.f.l. and bivariate multitarget distribution of RFSs $\Xi$, $\Sigma$ are related by:

$$f_{\Xi,\Sigma}(X, Y) = \delta G_{\Xi,\Sigma}[0, 0].$$  \hfill (22)

C. THE BERNOULLI RFS

An RFS of special importance for this paper, the Bernoulli RFS, is most easily described using its p.g.f.l.: $G_{\Xi}[h] = 1 - q + q \cdot s[h]$ where $0 \leq q \leq 1$ and the probability density $s(x)$ are, respectively, the existence probability and spatial distribution of a single target.

D. MULTITARGET RECURSIVE BAYES FILTER

Given a time-sequence $Z_{1:k} | Z_{1:k-1}, \ldots, Z_k$ of collected measurement-sets from a sensor, this is:

$$\ldots \rightarrow f(X_{k-1} | Z_{1:k-1}) \rightarrow f(X_k | Z_{1:k-1}) \rightarrow f(X_k | Z_{1:k}) \rightarrow \ldots$$

where

$$f(X_k | Z_{1:k-1}) = \int f(X_k | X_{k-1}, Z_{1:k-1}) \cdot f(X_{k-1} | Z_{1:k-1}) \, dX_{k-1}$$  \hfill (23)

$$f(X_k | Z_{1:k}) \propto f(Z_k | X_k, Z_{1:k}) \cdot f(X_k | Z_{1:k});$$  \hfill (24)

and where $f(X_k | X_{k-1}, Z_{1:k-1})$ is the multitarget state-transition density and $f(Z_k | X_k, Z_{1:k-1})$ is the sensor multitarget measurement density. It is assumed that $f(X_k | X_{k-1}, Z_{1:k-1}) = f(X_k | Z_{k-1})$ (Markov assumption) and $f(Z_k | X_k, Z_{1:k-1}) = f(Z_k | X_k)$. In this paper we will be concerned with $f(X_k | X_{1:k-1})$ and $f(Z_k | X_k)$ for only the “standard” multitarget motion and measurement models, respectively—see Section V-A.

III. THE PAIRWISE MARKOV MODEL (PMM)

The section is organized as follows: single-target recursive Bayes filter (Section III-A); the PMM (Section III-B); and single-target tracking using PMMs (Section III-C).

A. SINGLE-TARGET RECURSIVE BAYES FILTER

The PMM concept is most easily explained via the single-target recursive Bayes filter. Let $x \in \mathbb{R}$ denote a single-target state and $\mathbb{R}$ a single-target measurement. In the Bayesian approach, the unknown state at time $t_k$ is a random variable $X_{k|k} \in \mathbb{R}$ and the measurement process at time $t_k$ is a random variable $Z_k \in \mathbb{R}$. Let us be given:

1. the distribution $f(x_0)$ of the initial state $X_{0|0}$;
2. the sequence $z_{1:k} | z_1, \ldots, z_k$ of measurements collected from the target at times $t_1, \ldots, t_k$;
3. the transition density $f(x_k | x_{k-1}, z_{1:k-1})$, which describes the evolution of $X_{k-1|k-1}$ at time $t_{k-1}$ to $X_{k|k-1}$ at time $t_k$; and
4. the measurement density $f(z_k | x_k, z_{1:k-1})$ at time $t_k$, which characterizes the statistics of $Z_k$ if $x_k$ is a realization of $X_{k|k-1}$.

Given this, the recursive Bayes filter is defined by the time- and measurement-update equations

$$f(x_k | z_{1:k-1}) = \int f(x_k | x_{k-1}, z_{1:k-1}) \cdot f(x_{k-1} | z_{1:k-1}) \, dx_{k-1}$$  \hfill (25)

$$f(x_k | z_{1:k}) = \frac{f(z_k | x_k, z_{1:k-1}) \cdot f(x_k | z_{1:k-1})}{f(z_k | z_{1:k-1})}$$  \hfill (26)

$$f(z_k | z_{1:k-1}) = \int f(z_k | x_k, z_{1:k-1}) \cdot f(x_k | z_{1:k-1}) \, dx_k$$  \hfill (27)

It is usually assumed that $f(x_k | x_{k-1}, z_{1:k-1}) = f(x_k | x_{k-1})$ and $f(z_k | x_k, z_{1:k-1}) = f(z_k | x_k)$.

B. THE PAIRWISE MARKOV MODEL

Now let the state space be the Cartesian product $\mathbb{R} \times \mathbb{R}$ rather than $\mathbb{R}$. In this case, the unknown quantity at time $t_k$ is the joint state of the joint target-measurement system, and is represented as the random pair $(X_{k|k}, Y_k) \in \mathbb{R} \times \mathbb{R}$. What is unknown is not only $X_{k|k}$ and $Y_k$ but also their statistical correlation, as described by the posterior PMM density $f(x_k, y_k | z_{1:k-1})$. Let us be given

1. a PMM transition density

$$f(x_k, y_k | x_{k-1}, y_{k-1}) = f(x_k | x_{k-1}, y_{k-1})$$

$$f(y_k | x_k, x_{k-1}, y_{k-1})$$  \hfill (28)

that describes the evolution of the PMM system;
2. the measurement density $f(z_k | x_k, y_k) = \delta_{z_k}(z_k)$ of the PMM system, in which case it follows that the measurement equation is $z_k = \eta(x_k, y_k)$ with measurement function $\eta(x, y) = y$—i.e., if the joint system has state $(x, y)$ then $y$ is the only measurement that can be collected from it.

Given this, $f(x_k, y_k | z_{1:k-1})$ can be recursively derived from $f(x_k-1, y_k-1 | z_{1:k-2})$ as (29)–(31), shown at the bottom of the next page. Here, (29) is the Bayes’ filter time-update step for the PMM; (30) incorporates the Bayes’ filter measurement-update step for the PMM; and (31) follows from the fact that $f(z_k | x_k, y_k) = \delta_{y_k}(z_k)$. The initial PMM distribution for the recursion is $f(x_1, y_1) = f(y_1 | x_1) \cdot f(x_1)$ where $f(x_1)$ is an initial target distribution and $f(y_1 | x_1)$ is an initial measurement density.

Remark 2: Since $Z_k = Y_k$, in the PMM literature the distinction between $z_k$ (a collected measurement) and $y_k$ (a realization of the unknown random variable $Y_k$) is notionally suppressed:

$$f(x_k, y_k | y_{1:k-1})$$

$$= \frac{\int f(x_k, y_k | x_{k-1}, y_{k-1}) \cdot f(x_{k-1} | y_{1:k-1}) \, dx_{k-1}}{\int f(x_{k-1}, y_{k-1} | y_{1:k-2}) \, dx_{k-1}}$$  \hfill (32)
The estimated measurement density at time $t_k$ can be determined from $f(x_k, y_k | z_{1:k-1})$ via

$$f(y_k | x_k, y_{1:k-1}) = \frac{f(x_k, y_k | y_{1:k-1})}{f(x_k | y_{1:k-1})} = \frac{f(x_k, y_k | y_{1:k-1})}{\int f(x_k, y_k | y_{1:k-1}) dy_k}.$$ (33)

Likewise, if $f(x_k | x_{k-1}, y_{k-1})$ is the marginal of $f(x_k, y_k | x_{k-1}, y_{k-1})$ then the estimated Markov density at time $t_k$ is

$$f(x_k | x_{k-1}, y_{1:k-2}) = \int f(x_k | x_{k-1}, y_{1:k-2}) dy_k = \frac{\int f(x_k | x_{k-1}, y_{1:k-2}) dy_k}{\int f(x_k | y_{1:k-2}) dy_k}.$$ (34)

A PMM reduces to an HMM if, for $k > 1$,

$$f(y_k | x_k, x_{k-1}, y_{k-1}) = f(y_k | x_k), f(x_k | x_{k-1}, y_{k-1}) = f(x_k | x_{k-1})$$ (35)

in which case (28) reduces to:

$$f(x_k, y_k | x_{k-1}, y_{k-1}) = f(x_k | x_{k-1}) \cdot f(y_k | x_k).$$ (36)

Thus PMMs significantly weaken HMM’s to encompass non-Markov targets and correlated sensor noise [7, p. 4487].

### C. SINGLE-TARGET TRACKING USING PMMs

The single-target posterior distribution $f(x_k | y_{1:k})$ can be recursively propagated as follows [7, Eq. 12]:

$$f(x_k | y_{1:k}) = \frac{\int f(x_k, y_k | x_{1:k-1}, y_{1:k-1}) \cdot f(x_{1:k-1} | y_{1:k-1}) dx_{1:k-1}}{\int f(x_k, y_k | x_{1:k-1}, y_{1:k-1}) \cdot f(x_{1:k-1} | y_{1:k-1}) dx_{1:k-1} dx_k}.$$ (37)

Note that $f(x_k | y_{1:k})$ is related to $f(x_k, y_k | y_{1:k-1})$ as follows:

$$f(x_k | y_{1:k}) = \frac{f(x_k, y_k | y_{1:k-1})}{f(y_k | y_{1:k-1})} = \frac{f(x_k, y_k | y_{1:k-1})}{\int f(x_k, y_k | y_{1:k-1}) dx_k},$$ (38)

so that

$$\hat{x}_k = \arg \sup_{x_k \in \mathbb{X}} f(x_k, y_k | y_{1:k-1})$$ (39)

is the MAP estimate of the target state given $y_{1:k}$. The predicted target distribution is

$$f(x_k | y_{1:k-1}) = \int f(x_k, y_k | y_{1:k-1}) dy_k.$$ (40)

### IV. MULTITARGET PMM (MPMM)

This is a direct generalization of the single-target PMM. Let $Fin(\mathcal{S})$ denote the set of multitarget states (i.e., the finite subsets $X$ of $\mathcal{S}$) and let $Fin(\mathcal{Y})$ denote the set of multitarget measurements (i.e., the finite subsets $Z$ of $\mathcal{Y}$). Then the unknown multitarget state at time $t_k$ is an RFS $\Xi_{t_k} \subseteq \mathcal{S}$ and the multitarget measurement process is an RFS $\Xi_{t_k} \subseteq \mathcal{Y}$.

Now let the state space be $Fin(\mathcal{S}) \times Fin(\mathcal{Y})$. Then the unknown state at time $t_k$ is that of the joint multitarget, multi-measurement system, as represented by the random pair $(\Xi_{t_k}, \Sigma_k) \in Fin(\mathcal{S}) \times Fin(\mathcal{Y})$. What is unknown is not only $\Xi_{t_k}$ and $\Sigma_k$ but also their statistical correlation, as described by the posterior MPMM density $f(X_k, Y_k | Y_{1:k-1})$. Let us be given:

1. an MPMM transition density

$$f(X_k, Y_k | X_{k-1}, Y_{k-1}) = f(X_k | X_{k-1}, Y_{k-1}) f(Y_k | X_k, X_{k-1}, Y_{k-1})$$ (41)

2. describing the evolution of the MPMM system;

3. the MPMM measurement density $f(Z_k | X_k, Y_k) = \delta_{Y_k}(Z_k)$ of the MPMM system.

Then as with the single-target case, the recursions for the MPMM density $f(X_k, Y_k | Y_{1:k-1})$ and the multitarget posterior $f(X_k | Y_{1:k})$ are, respectively,

$$f(X_k, Y_k | Y_{1:k-1}) = \frac{\int f(X_k, Y_k | X_{k-1}, Y_{k-1}) f(X_{k-1}, Y_{k-1} | Y_{1:k-2}) \delta X_{k-1}}{\int f(X_{k-1}, Y_{k-1} | Y_{1:k-2}) \delta X_{k-1}},$$ (42)

$$f(X_k | Y_{1:k}) = \int f(X_k, Y_k | X_{k-1}, Y_{k-1}) f(X_{k-1}, Y_{k-1} | Y_{1:k-1}) \delta X_{k-1} \delta X_k.$$ (43)

From (42) we see that the p.g.fl. of $f(X_k, Y_k | Y_{1:k-1})$ is

$$G[h_k, g_k | Y_{1:k-1}] = \int h_k \cdot g_k \cdot f(X_k, Y_k | Y_{1:k-1}) \delta X_k \delta Y_k$$ (44)

$$= \int G[h_k, g_k | X_{k-1}, Y_{k-1}] f(X_{k-1}, Y_{k-1} | Y_{1:k-2}) \delta X_{k-1} \delta X_k$$ (45)
where the p.g.fl. of $f(X_k, Y_k|X_{k-1}, Y_{k-1})$ is

$$G[h_k, g_k|X_{k-1}, Y_{k-1}] = \int h_k^{X_k} g_k^{Y_k} \cdot f(X_k, Y_k|X_{k-1}, Y_{k-1}) \delta X_k \delta Y_k. \quad (46)$$

Thus $f(X_k, Y_k|X_{k-1}, Y_{k-1})$ can be specified by providing a formula for $G[h_k, g_k|X_{k-1}, Y_{k-1}]$ as in the next section.

V. MPMM TRANSITION DENSITIES

The section is organized as follows: the “standard” multitarget motion and measurement models (Section V-A); the original MPMM transition model (Section V-B); the corrected MPMM transition model (Section V-C); and the evolution of the MPMM pair ($\{x_{k-1}\}, \{y_{k-1}\}$) according to this model (Section V-D). The remaining subsections address extensions of this basic evolution model: the general evolution of ($\{x_{k-1}\}, \{y_{k-1}\}$) (Section V-E); the evolution of ($\{x_{k-1}\}, \emptyset$) (Section V-F); the evolution of ($\emptyset, \emptyset$) (Section V-G); and the evolution of ($\emptyset, \{y_{k-1}\}$) (Section V-H).

A. THE “STANDARD” MULTITARGET MOTION AND MEASUREMENT MODELS

What is $f(X_k, Y_k|X_{k-1}, Y_{k-1})$? This question was addressed in [9] by endeavoring to infer the form of its p.g.fl. $G[h_k, g_k|X_{k-1}, Y_{k-1}]$ from the p.g.fl.’s $G[h_k|X_{k-1}]$ and $G[g_k|X_{k-1}]$ of, respectively, the “standard” multitarget motion and measurement models.

The “standard” multitarget motion model [2, Eq. 14.273], [10, Eq. 5.94], presumes that: (a) individual target motions are statistically independent; (b) the probability that a target with state $x_{k-1}$ at time $t_{k-1}$ will survive to time $t_k$ is $p_S(x_{k-1})$; (c) if so, then $f(x_k|x_{k-1})$ is the probability (density) that it will transition to a target with state $x_k$; and (d) $f^B(x_k)$ is the multitarget density of newly-appearing targets, with corresponding p.g.fl. $G^B[h_k]$. This motion model is used to construct the “standard” multitarget Markov density $f(X_k|X_{k-1})$, with corresponding p.g.fl.

$$G[h_k|X_{k-1}] = G^B[h_k] \cdot (1 - p_S + p_SM_{h_k})^{X_{k-1}},$$

$$M_{h_k}(x_{k-1}) := \int h_k(x_k) \cdot f(x_k|X_{k-1}) \, dx_k. \quad (47)$$

The “standard” multitarget measurement model [2, Eq. 14.290], [10, Eq. 5.104] presumes that: (e) all measurements are generated independently of each other; (f) the probability that a target with state $x_k$ at time $t_k$ will generate a measurement is $p_D(x_k)$; (g) if so, then $f(y_k|x_k)$ is the probability (density) that the measurement is $y_k$; and (h) $f^*(Z_k)$ is the multi-object density of the clutter process, with corresponding p.g.fl. $G^*(g_k)$. This model is used to construct the standard multitarget measurement density $f(Y_k|X_k)$, with corresponding p.g.fl.

$$G[g_k|X_k] = G^*(g_k) \cdot (1 - p_D + p_DL_{g_k})^{X_k},$$

$$L_{g_k}(x_k) := \int g_k(y_k) \cdot f(y_k|x_k) \, dy_k. \quad (48)$$

B. THE ORIGINAL MPMM TRANSITION MODEL

The multitarget analog of (36) is

$$f(X_k, Y_k|X_{k-1}, Y_{k-1}) = f(X_k|X_{k-1}) \cdot f(Y_k|X_k). \quad (49)$$

Given this, it was shown in [9, Sec. 2.4] via substitution that the p.g.fl. of $f(X_k, Y_k|X_{k-1}, Y_{k-1})$ is

$$G[h_k, g_k|X_{k-1}, Y_{k-1}] = \int h_k^{X_k} g_k^{Y_k} \cdot f(X_k, Y_k|X_{k-1}, Y_{k-1}) \delta X_k \delta Y_k \quad (50)$$

where $G^*[g_k]$ characterizes the clutter process; $G^B[h_k]$ characterizes the target-appearance process; and where the evolution model is

$$G^B[h_k] = (1 - p_S + p_S M_{h_k})^{X_{k-1}}. \quad (51)$$

From this, in [9, Sec. 2.4] it was proposed that if $X_k \neq \emptyset$ and $Y_k \neq \emptyset$ then a plausible generalization of $G^B[h_k]$ to $G^E[h_k, g_k|X_{k-1}, Y_{k-1}]$ is

$$G^E[h_k, g_k|X_{k-1}, Y_{k-1}] = \prod_{(x_{k-1}, y_{k-1}) \in X_{k-1} \times Y_{k-1}} G^E[h_k, g_k|\{x_{k-1}\}, \{y_{k-1}\}] \quad (53)$$

where the evolution of the PMM pair ($\{x_{k-1}\}, \{y_{k-1}\}$) is described by the p.g.fl.

$$G^E[h_k, g_k|\{x_{k-1}\}, \{y_{k-1}\}] = \tilde{M}_1 - p_S + p_S M_{h_k}(1 - p_D + p_DL_{g_k}) (x_{k-1}, 1) \quad (54)$$

$$= \int \left( (1 - p_S(x_{k-1}) + p_S(x_{k-1}) \cdot h_k(x_k) ) \right) \times f(x_k, y_k|x_{k-1}, y_{k-1}) \, dx_k \, dy_k. \quad (55)$$

Equation (53) thus presumes that the elementary pairs ($\{x_{k-1}\}, \{y_{k-1}\}$) evolve independently of each other.

C. THE CORRECTED MPMM TRANSITION MODEL

In retrospect, (53) cannot be correct for at least two reasons. First, when the underlying PMM is an HMM—i.e., when $f(x_k, y_k|x_{k-1}, y_{k-1}) = f(x_k|x_{k-1}) \cdot f(y_k|x_k)$—then

$$\tilde{M}_1 - p_S + p_S M_{h_k}(1 - p_D + p_DL_{g_k}) (x_{k-1}, 1) \quad (56)$$

Thus from (52-54) we see that

$$G^E[h_k, g_k|X_{k-1}, Y_{k-1}] = \left( (1 - p_S + p_SM_{1 - p_D + p_DL_{g_k}}) X_{k-1} \right) |Y_{k-1}|$$

$$= G^E[h_k, g_k|X_{k-1}] |Y_{k-1}| \quad (57)$$

rather than, as should be the case, $G^E[h_k, g_k|X_{k-1}, Y_{k-1}] = 1$.

Second, suppose that the scenario contains at most a single target obscured by clutter. Then the multi-object system will

\[1\] The MPMM-CPHD filter proposed in [9] therefore cannot be correct.
always be described by an MPMM pair \((X_k, Y_k)\) with \(|X_k| \leq 1\). In this case the evolution of the system must consist of transitions \(\ldots \to (X_{k-1}, Y_{k-1}) \to (X_k, Y_k) \to \ldots\) where \(|X_{k-1}|, |X_k| \leq 1\). Such an evolution is impossible if it is given as (53) since
\[
G^E[h_k, g_k | \{x_{k-1}\}, Y_{k-1}] = \prod_{y_{k-1} \in Y_{k-1}} G^E[h_k, g_k | \{x_{k-1}\}, \{y_{k-1}\}]
\]
(58)

describes a system that has as many as \(|Y_{k-1}|\) targets.

Accordingly, a corrected model is required, as follows. If \(|Y_{k-1}| > 0\) or \(|Y_{k-1}| = 0\) define, respectively,
\[
\tilde{M}_{1-p_x + p_y h_k(1-p_D+p_D L)}(x_{k-1}, Y_{k-1}) = \frac{1}{|Y_{k-1}|} \sum_{y_{k-1} \in Y_{k-1}} \tilde{M}_{1-p_x + p_y h_k(1-p_D+p_D L)}(x_{k-1}, y_{k-1}) = M_{1-p_x + p_y h_k(1-p_D+p_D L)}(x_{k-1}, \emptyset)
\]
(59)

Then replace (53) with
\[
\bar{G}^E[h_k, g_k | x_{k-1}, Y_{k-1}] = \prod_{x_{k-1} \in X_{k-1}} \tilde{M}_{1-p_x + p_y h_k(1-p_D+p_D L)}(x_{k-1}, Y_{k-1}).
\]
(60)

In this case
\[
G[h_k, g_k | x_{k-1}, Y_{k-1}] = G^* [g_k] \cdot G^E[h_k(1-p_D+p_D L)] \cdot \bar{G}^E[h_k, g_k | x_{k-1}, Y_{k-1}]
\]
(61)
does properly reduce to (51). Thus the incorrect (58) is replaced with
\[
\bar{G}^E[h_k, g_k | \{x_{k-1}\}, Y_{k-1}] = \tilde{M}_{1-p_x + p_y h_k(1-p_D+p_D L)}(x_{k-1}, Y_{k-1}).
\]
(62)

Equation (62) has a physical interpretation. Suppose that \(p_y = 1\) (the target does not disappear) and \(p_D = 1\) (the target is detected) and \(|Y_{k-1}| > 0\). Then (62) reduces to
\[
\bar{G}^E[h_k, g_k | \{x_{k-1}\}, Y_{k-1}] = \frac{1}{|Y_{k-1}|} \sum_{y_{k-1} \in Y_{k-1}} \int h_k(x_k, y_k) \cdot g_k(y_k) f(x_k, y_k | \{x_{k-1}\}, \{y_{k-1}\}) \, dx_k \, dy_k.
\]
(63)
The corresponding distribution is: \(f^E(X_k, Y_k | \{x_{k-1}\}, Y_{k-1}) = 0\) unless \(|X_k| = |Y_k| = 1\), in which case
\[
f^E(\{x_k\}, \{y_k\} | \{x_{k-1}\}, Y_{k-1}) = \frac{1}{|Y_{k-1}|} \sum_{y_{k-1} \in Y_{k-1}} f(x_k, y_k | \{x_{k-1\}, \{y_{k-1}\}).
\]
(64)
That is, the transition from \(\{x_{k-1}\}, Y_{k-1}\) to \(\{x_k\}, \{y_k\}\) is the average transition from \(\{x_{k-1}\}, \{y_{k-1}\}\) to \(\{x_k\}, \{y_k\}\), taken over all \(y_{k-1} \in Y_{k-1}\).

D. Evolution of the MPMM Pair \((x_{k-1}, y_{k-1})\)

It was shown in [9, Sec. 2.4] that \(G^E[h_k, g_k | \{x_{k-1}\}, \{y_{k-1}\}]\) in (54) is the p.g.fl. of the bivariate multitarget probability distribution \(f(X_k, Y_k | \{x_{k-1}\}, \{y_{k-1}\})\) that characterizes the following intuitive dynamics model for \(|X_k|, |Y_k| \leq 1\):

1. If \(x_{k-1}\) evolves to \(x_k\) and measurement \(y_k\) is collected from \(x_k\) then the probability (density) that this event will occur is:
\[
f(\{x_k\}, \{y_k\} | \{x_{k-1}\}, \{y_{k-1}\}) = p_S(x_k) \cdot p_D(x_k) \cdot f(x_k, y_k | x_{k-1}, y_{k-1}).
\]
(65)

2. If \(x_{k-1}\) evolves to \(x_k\) but \(x_k\) is not detected, then the probability (density) that this event will occur is, if \(f(x_k | x_{k-1}, y_{k-1})\) is the marginal of the PMM density \(f(x_k, y_k | x_{k-1}, y_{k-1})\):
\[
f(\{x_k\}, \emptyset | \{x_{k-1}\}, \{y_{k-1}\}) = p_S(x_k) \cdot (1 - p_D(x_k)) \cdot f(x_k | x_{k-1}, y_{k-1})
\]
(66)

3. If \(x_{k-1}\) does not survive to time \(t_k\) then no (nonempty) measurement can be collected from it and so the probability that this event will occur is:
\[
f(\emptyset, \emptyset | \{x_{k-1}\}, \{y_{k-1}\}) = 1 - p_S(x_k).
\]
(67)

4. If \(x_{k-1}\) does not survive to time \(t_k\) and yet measurement \(y_k\) is collected from it, this is an impossibility and so the probability that this event will occur is:
\[
f(\emptyset, \{y_k\} | \{x_{k-1}\}, \{y_{k-1}\}) = 0.
\]
(68)

Subsections V-E through V-H will address generalizations and extensions of this basic evolution model.

E. General Evolution of \((x_{k-1}, y_{k-1})\)

The dynamics model \(f(X_k, Y_k | \{x_{k-1}\}, \{y_{k-1}\})\) of the previous section is a special case of a more general model, deduced from an arbitrary MPMM density \(f(X_k, Y_k | \{x_{k-1}\}, \{y_{k-1}\})\) assuming only that \(|X_k|, |Y_k| \leq 1\). Since a nonexistent target cannot generate a nonempty measurement, we may assume that \(f(\emptyset, \{y_k\} | \{x_{k-1}\}, \{y_{k-1}\}) = 0\) identically. Define
\[
p_S(x_{k-1}, y_{k-1}) = \int f(x_k, y_k | \{x_{k-1}\}, \{y_{k-1}\}) \, dx_k \, dy_k
\]
(69)
\[
p_D(x_k | \{x_{k-1}\}, y_{k-1}) = \int f(x_k, y_k | \{x_{k-1}\}, \{y_{k-1}\}) \, dy_k
\]
(70)
\[
f(x_k, y_k | x_{k-1}, y_{k-1}) = \frac{f(x_k, y_k | \{x_{k-1}\}, \{y_{k-1}\}) \int f(|x_k|, y_k | \{x_{k-1}\}, \{y_{k-1}\}) \, dy_k}{\int f(x_k, y_k | \{x_{k-1}\}, \{y_{k-1}\}) \, dy_k}
\]
(71)
Abbreviate $\tilde{p}_S(x_{k-1}) = p_S(x_{k-1}, y_{k-1})$ and $\tilde{p}_D(x_k) = p_D(x_k|y_{k-1})$. Then it is easily shown that

$$
\begin{align*}
    f(\{x_k\}, \{y_k\}|[x_{k-1}], [y_{k-1}]) &= \tilde{p}_S(x_{k-1}) \cdot \tilde{p}_D(x_k) \cdot f(x_k, y_k|x_{k-1}, y_{k-1}) \\
    f(\{x_k\}, \emptyset|[x_{k-1}], [y_{k-1}]) &= \tilde{p}_S(x_{k-1}) \cdot (1 - \tilde{p}_D(x_k)) \cdot f(x_k, y_k|x_{k-1}, y_{k-1}) \\
    f(\emptyset, \emptyset|[x_{k-1}], [y_{k-1}]) &= 1 - \tilde{p}_S(x_{k-1}) .
\end{align*}
$$

(72)  

(73)  

(74)

This reduces to the model of Section V-D if $p_S(x_{k-1}, y_{k-1}) = p_S(x_{k-1})$ and $p_D(x_k|y_{k-1}, y_{k-1}) = p_D(x_k)$.

**Remark 3**: To simplify notation, this is what will be assumed later in Section VI-C (though this assumption is not a necessity).

**F. EVOLUTION OF $([x_k], \emptyset)$**

Now consider $f(X_k, Y_k|[x_{k-1}], \emptyset)$. It can be presumed that $f(\emptyset, [y_k]|[x_{k-1}], \emptyset) = 0$. Define:

$$
\begin{align*}
    p_S(x_{k-1}, \emptyset) &= \int f(x_k, y_k|[x_{k-1}], \emptyset) dy_k dx_k \\
    f(x_k|[x_{k-1}], \emptyset) &= \int f(x_k, y_k|[x_{k-1}], \emptyset) dy_k dx_k \\
    p_D(x_k|[x_{k-1}], \emptyset) &= \int f(x_k, y_k|[x_{k-1}], \emptyset) dy_k dx_k \\
    f(y_k|x_k, \emptyset, [x_{k-1}], \emptyset) &= \int f(x_k, y_k|[x_{k-1}], \emptyset) dy_k dx_k
\end{align*}
$$

(75)  

(76)  

(77)  

(78)

where we abbreviate $\tilde{p}_S(x_{k-1}) = p_S(x_{k-1}, \emptyset)$ and $\tilde{f}(x_k|[x_{k-1}], \emptyset) = f(x_k|x_{k-1}, \emptyset)$ and $\tilde{p}_D(x_k) = p_D(x_k|\emptyset, x_{k-1}, \emptyset)$ and $\tilde{f}(y_k|x_k, \emptyset) = f(y_k|x_k, \emptyset, \emptyset)$.

**Remark 4**: To simplify notation, it will be assumed in Section VI-C that $p_D(x_k|\emptyset, x_{k-1}, \emptyset)$ does not depend on $x_{k-1}$ and that $f(y_k|x_k, \emptyset, \emptyset)$ does not depend on $x_{k-1}$ or $\emptyset$; in which case $p_D(x_k|\emptyset, x_{k-1}, \emptyset) = p_D(x_k)$, $f(y_k|x_k, \emptyset, [x_{k-1}], \emptyset) = f(y_k|x_k)$.

Then it is easily shown that

$$
\begin{align*}
    f(\{x_k\}, \{y_k\}|[x_{k-1}], [y_{k-1}]) &= \tilde{p}_S(x_{k-1}) \cdot \tilde{p}_D(x_k) \cdot \tilde{f}(x_k|x_{k-1}) \cdot \tilde{f}(y_k|x_k) \\
    f(\{x_k\}, \emptyset|[x_{k-1}], [y_{k-1}]) &= \tilde{p}_S(x_{k-1}) \cdot (1 - \tilde{p}_D(x_k)) \cdot \tilde{f}(x_k|x_{k-1}) \\
    f(\emptyset, \emptyset|[x_{k-1}], [y_{k-1}]) &= 1 - \tilde{p}_S(x_{k-1}) .
\end{align*}
$$

(79)  

(80)  

(81)

For future reference, the p.g.f. of $f(X_k, Y_k|[x_{k-1}], \emptyset)$ is

$$
\begin{align*}
    G^\emptyset[h_k, g_k|[x_{k-1}], \emptyset] &= 1 - \tilde{p}_S(x_{k-1}) + \tilde{p}_S(x_{k-1}) \cdot \tilde{M}_h(1 - \tilde{p}_D + \tilde{p}_D\tilde{L}_h)(x_{k-1}) \\
    &= 1 - \tilde{p}_S(x_{k-1}) + \tilde{p}_S(x_{k-1}) \cdot \tilde{M}_h(1 - \tilde{p}_D + \tilde{p}_D\tilde{L}_h)(x_{k-1}).
\end{align*}
$$

(82)

where $\tilde{M}_h(x_{k-1}) := \int h_k(x_k) \cdot f(x_k|x_{k-1}) dx_k$ and $\tilde{L}_h(x_{k-1}) := \int g_k(y_k) \cdot f(y_k|x_k) dy_k$. For,

$$
\begin{align*}
    G^\emptyset[h_k, g_k|[x_{k-1}], \emptyset] &= \int h_k(x_k) \cdot g_k(y_k) \cdot f(x_k, y_k|[x_{k-1}], \emptyset) dy_k dx_k \\
    &= \int f(x_k, y_k|[x_{k-1}], \emptyset) dy_k dx_k \\
    &= \int f(x_k, y_k|[x_{k-1}], \emptyset) dy_k dx_k \\
    &= \int \int f(x_k, y_k|[x_{k-1}], \emptyset) dy_k dx_k \\
    &= \int \int f(x_k, y_k|[x_{k-1}], \emptyset) dy_k dx_k \\
    &= \int \int f(x_k, y_k|[x_{k-1}], \emptyset) dy_k dx_k .
\end{align*}
$$

(83)  

(84)  

(85)

**G. EVOLUTION OF $([\emptyset], \emptyset)$**

Consider the MPMM transition $f(X_k, Y_k|[\emptyset], \emptyset)$ where, as usual, $f(\emptyset, [y_k]|[\emptyset], \emptyset) = 0$. The evolutions $([\emptyset], [y_k]) \rightarrow ([x_k], [y_k])$ or $([\emptyset], \emptyset) \rightarrow ([x_k], \emptyset)$ describe the target’s first appearance or its subsequent reappearance. Define

$$
\begin{align*}
    q^B_{\emptyset}(\emptyset, \emptyset) &= \int f([x_k], [y_k]|[\emptyset], \emptyset) dy_k dx_k \\
    s^B_{\emptyset}(\emptyset, \emptyset) &= \int f([x_k], [y_k]|[\emptyset], \emptyset) dy_k dx_k \\
    p_D(x_k|\emptyset, \emptyset) &= \int f([x_k], [y_k]|[\emptyset], \emptyset) dy_k dx_k \\
    f(y_k|x_k, \emptyset, \emptyset) &= \int f([x_k], [y_k]|[\emptyset], \emptyset) dy_k dx_k
\end{align*}
$$

(86)  

(87)  

(88)  

(89)

where we abbreviate $\tilde{q}^B_{\emptyset} = q^B_{\emptyset}(\emptyset, \emptyset)$ and $\tilde{s}^B_{\emptyset}(\emptyset, \emptyset) = s^B_{\emptyset}(\emptyset, \emptyset)$ and $\tilde{p}_D(x_k) = p_D(x_k|\emptyset, \emptyset)$ and $\tilde{f}(y_k|x_k) = f(y_k|x_k, \emptyset, \emptyset)$.

**Remark 5**: To simplify notation, it will be assumed in Section VI-C that $q^B_{\emptyset}(\emptyset, \emptyset) = q^B_{\emptyset}(\emptyset, \emptyset)$ and $s^B_{\emptyset}(\emptyset, \emptyset) = s^B_{\emptyset}(\emptyset, \emptyset)$ and $p_D(x_k|\emptyset, \emptyset) = p_D(x_k)$ and $f(y_k|x_k, \emptyset, \emptyset) = f(y_k|x_k)$.

Then it is easily shown that

$$
\begin{align*}
    f(\{x_k\}, \{y_k\}|[\emptyset], [y_{k-1}]) &= \tilde{q}^B_{\emptyset}(x_{k-1}) \cdot \tilde{p}_D(x_k) \cdot \tilde{f}(y_k|x_k) \\
    f(\{x_k\}, \emptyset|[\emptyset], [y_{k-1}]) &= \tilde{q}^B_{\emptyset}(x_{k-1}) \cdot (1 - \tilde{p}_D(x_k)) \cdot \tilde{f}(y_k|x_k) \\
    f(\emptyset, \emptyset|[\emptyset], [y_{k-1}]) &= 1 - \tilde{q}^B_{\emptyset}. \end{align*}
$$

(90)  

(91)  

(92)

Here, $\tilde{q}^B_{\emptyset}$ is the “birth” probability—i.e., the nonexistent target $\emptyset$ at time $t_{k-1}$ transitions to a target with state $x_k$ at time $t_k$—and $\tilde{q}^B_{\emptyset}(x_k)$ is its spatial distribution.

**Remark 6**: The obvious choice for $q^B_{\emptyset}(x_k)$ is the multiobject version of (40):

$$
q^B_{\emptyset}(x_k) := f([x_k]|[y_{1:k-1}] = \int f([x_k], [y_{1:k-1}] \delta y_k .
$$

(93)
This might seem theoretically questionable since from (86,87) it would seem to imply that \( f((x_k), Y_k) \) depends on \( Y_{k-1} \). However, the choice of \( q_k^B x_k^B (x_k) \) is arbitrary and so we can choose (93) as we please.

For future reference, the p.g.fl. of \( f(X_k, Y_k) | \{x_k \} \) is
\[
\tilde{G}^E[h_k, g_k | \{x_k \}, \mathcal{D}] = 1 - q_k^B + q_k^B s_k^B [h_k (1 - \tilde{p}_D + \tilde{p}_D L_{R_k})]
\] 
(94)

where \( L_{R_k} = \int g_k(y_k) \cdot f(y_k | x_k) dy_k \). For,
\[
\tilde{G}^E[h_k, g_k | \mathcal{D}] = \int h_k^Y \cdot s_k^Y \cdot f(X_k, Y_k) | \mathcal{D} \delta X_k \delta Y_k
\]
(95)
\[
= f(\mathcal{D}| \mathcal{D}) + f(x_k) | \mathcal{D}
\] 
(96)
\[+ \int h_k(x_k) \cdot f(\{x_k \}, Y_k) | \mathcal{D} dx_k
+ \int g_k(y_k) \cdot f(\mathcal{D}, \{y_k \}) | \mathcal{D} dy_k
\]
\[+ \int h_k(x_k) \cdot g_k(y_k) \cdot f(\{x_k \}, \{y_k \}) | \mathcal{D} dx_k dy_k
\]
\[= 1 - q_k^B + q_k^B s_k^B [h_k (1 - \tilde{p}_D) + q_k^B \tilde{p}_D L_{R_k}]
\] 
(97)
\[= 1 - q_k^B + q_k^B s_k^B [h_k (1 - \tilde{p}_D + \tilde{p}_D L_{R_k})].
\] 
(98)

VI. THE BERNOULLI MPMM FILTER

The section is organized as follows: the Bernoulli filter (Section VI-A); the Bernoulli MPMM filter (Section VI-B); transition p.g.fl.'s for the Bernoulli MPMM filter (Section VI-C); summary of the Bernoulli MPMM filter (Section VI-D); derivation of the Bernoulli MPMM filter update when \( Y_{k-1} = \emptyset \) (Section VI-E); and derivation of the Bernoulli MPMM filter update when \( Y_{k-1} \neq \emptyset \) (Section VI-F).

A. THE BERNOULLI FILTER

The Bernoulli filter [1], [2, Sec. 14.7] is the special case of the multitarget Bayes filter
\[
\ldots \rightarrow f(X_{k-1} | Z_{1:k-1}) \rightarrow f(X_k | Z_{1:k}) \rightarrow f(X_k | Z_{1:k}) \rightarrow \ldots
\]
when at most a single target is present—i.e., when \(|X_{k-1}|, |X_k| \leq 1\) for all \( k \geq 1 \). Since
\[
f(\mathcal{D} | Z_{1:k-1}) = 1 - \int f(\{x_k \} | Z_{1:k-1}) dx_k,
\]
\[
f(\mathcal{D} | Z_{1:k}) = 1 - \int f(\{x_k \} | Z_{1:k}) dx_k,
\]
(108)
the Bernoulli filter is mathematically equivalent to a filter that propagates the PHD's \( D_{k|k-1}(x_k) \) !: \( \tilde{D}(x_k | Z_{1:k}) \) !: \( f(\{x_k \} | Z_{1:k-1}) \) and \( D_{k|k}(x_k) \) !: \( \tilde{D}(x_k | Z_{1:k}) \) !: \( f(\{x_k \} | Z_{1:k}) \). The time-update equation and measurement-update equation are, respectively,
\[
D_{k|k-1}(x_k) = q_k^B s_k^B (x_k) \cdot (1 - D_{k-1|k-1}[1]) + D_{k-1|k-1}[p_k M_{x_k}]
\] 
(109)
\[
D_{k|k}(x_k) = \frac{\ell_{Z_k}(x_k) \cdot D_{k|k-1}(x_k)}{1 - D_{k|k-1}[1] + D_{k|k-1}[\ell_{Z_k}]},
\] 
(110)
where \( \ell_{Z_k} \) was defined in (2).

Note that, as presented in [2, Sec. 14.7], the Bernoulli filter propagates two items, not one: the probability of target existence \( p_{k|x_k} = f(\{x_k \} | Z_{1:k}) dx_k \) and the target spatial distribution \( f_{k|x_k}(x_k) = f(\{x_k \} | Z_{1:k}) p_{k|x_k} \). But it is clear that the filter in (109,110) differs from that in [2, Sec. 14.7] only by a change of notation (although the former is significantly simpler in form). A tutorial on the (original) Bernoulli filter can be found in [22].

State estimation using \( D_{k|x_k} \) is as in [2]. The target exists if \( p_{k|x_k} > \tau \) for some threshold \( \tau > 1/2 \); and if it exists, its state is the MAP estimate \( \text{argsup}_{x_k} D_{k|x_k}(x_k) \).

Eqs. (109,110) can be consolidated by substitution into the single-step update equation, (111), as shown at the bottom of the next page, where \( M_{x_k}(x_k) \) was defined in (8).
B. THE BERNOLLI MPMM FILTER

This is a special case of the MPMM filter

\[ \ldots \rightarrow f(X_{k-1}, Y_{k-1}|Y_{1:k-2}) \rightarrow f(X_k, Y_k|Z_{1:k-1}) \rightarrow \ldots \]

when \( |X_{k-1}|, |X_k| \leq 1 \) for all \( k \geq 1 \). Section V-G described a simplified target-appearance model. This model will allow us to avoid the factor \( G^B \) in (61) by assuming that \( G^B[h_k] = 1 \) identically.

Remark 8: Note that this simplified target-appearance model would not be acceptable in the multitarget case, since then the number of targets could never increase.

Given that \( G^B[h_k] = 1 \), the p.g.fl. (61) of the MPMM transition density reduces to:

\[
G[h_k, g_k|X_{k-1}, Y_{k-1}] = G^A[g_k] \cdot G^E[h_k, g_k|X_{k-1}, Y_{k-1}] 
\]

(112)

where either \( X_{k-1} = \emptyset \) or \( X_{k-1} = \{x_{k-1}\} \) for all \( k \geq 1 \). From (44), the p.g.fl. update for the Bernoulli MPMM filter is

\[
G[h_k, g_k|Y_{1:k-1}] = \frac{\int G[h_k, g_k|X_{k-1}, Y_{k-1}] \cdot f(X_{k-1}, Y_{k-1}|Y_{1:k-2})dX_{k-1}}{\int f(X_{k-1}, Y_{k-1}|Y_{1:k-2})dX_{k-1}}
\]

(113)

where the numerator is

\[
\int G[h_k, g_k|X_{k-1}, Y_{k-1}] \cdot f(X_{k-1}, Y_{k-1}|Y_{1:k-2})dX_{k-1} = G^A[g_k] \cdot \tilde{G}^E[h_k, g_k|\emptyset, Y_{k-1}] \cdot f(\emptyset, Y_{k-1}|Y_{1:k-2}) + G^A[g_k] \int \tilde{G}^E[h_k, g_k|\{x_{k-1}\}, Y_{k-1}] \cdot f(\{x_{k-1}\}, Y_{k-1}|Y_{1:k-2})dY_{k-1}
\]

(114)

and the denominator is

\[
\int f(X_{k-1}, Y_{k-1}|Y_{1:k-2})dX_{k-1} = f(\emptyset, Y_{k-1}|Y_{1:k-2}) + \int f(\{x_{k-1}\}, Y_{k-1}|Y_{1:k-2})dY_{k-1}.
\]

(115)

C. STATE-TRANSITION P.G.FL’S FOR THE BERNOLLI MPMM FILTER

We therefore need formulas for \( \tilde{G}^E[h_k, g_k|X_{k-1}, Y_{k-1}] \) in the following four cases:

1. \( X_{k-1} = \emptyset \) and \( Y_{k-1} = \emptyset \): By (94),

\[
\tilde{G}^E[h_k, g_k|\emptyset, \emptyset] = 1 - q_k^B + 2q_k^B p_k^B h_k \left(1 - \tilde{p}_D + \tilde{p}_D L_{q_k}\right)
\]

(116)

where (see (86-89)): \( \tilde{p}_D(x_k) \neq p_D(x_k|\emptyset, \emptyset) \) and \( \tilde{f}(y_k|x_k) = f(y_k|x_k, \emptyset, \emptyset) \) and \( \tilde{L}_{q_k}(x_k) = \int g_k(y_k) \cdot \tilde{f}(y_k|x_k)dy_k \). To simplify notation, in what follows we will:

a. further abbreviate \( p_D(x_k) \neq p_D(x_k|\emptyset, \emptyset) \) and \( f(x_k|x_{k-1}) \neq f(x_k|x_{k-1}, \emptyset, \emptyset) \)

2. \( X_{k-1} = \{x_{k-1}\} \) and \( Y_{k-1} = \emptyset \): By (82),

\[
\tilde{G}^E[h_k, g_k|\emptyset, Y_{k-1}] = 1 - \tilde{p}_S(x_{k-1}) + \tilde{p}_S(x_{k-1}) \cdot \bar{M}_k(1 - \tilde{p}_D + \tilde{p}_D L_{q_k})(x_{k-1})
\]

(117)

where (see (75-78): \( \tilde{p}_S(x_{k-1}) = p_S(x_{k-1}|\emptyset) \) and \( f(y_{k|x_k}) = f(y_k|x_k, \emptyset, \emptyset) \), \( \tilde{p}_D(x_k) \neq p_D(x_k|\emptyset, \emptyset) \) and \( f(y_k|x_k, x_{k-1}, \emptyset) = f(y_k|x_k, x_{k-1}) \cdot \tilde{f}(y_k|x_{k-1})dy_k \), \( \bar{L}_{g_k}(x_k) = \int g_k(y_k) \cdot \tilde{f}(y_k|x_k)dy_k \). To simplify notation, in what follows we will:

a. further abbreviate \( p_S(x_{k-1}) \neq p_S(x_{k-1}|\emptyset) \) and \( f(x_k|x_{k-1}) \neq f(x_k|x_{k-1}, \emptyset, \emptyset) \)

b. assume that \( p_D(x_k|\emptyset, \emptyset) = p_D(x_k) \) and \( f(y_k|x_k, x_{k-1}, \emptyset) = f(x_k|x_{k-1}, \emptyset, \emptyset) = f(x_k|x_{k-1}) \).

3. \( X_{k-1} = \emptyset \) and \( Y_{k-1} = \{y_{k-1}\} \): By (107),

\[
\tilde{G}^E[h_k, g_k|\emptyset, Y_{k-1}] = 1 - q_k^B + 2q_k^B p_k^B h_k (1 - \tilde{p}_D + \tilde{p}_D L_{q_k})
\]

(118)

4. \( X_{k-1} = \{x_{k-1}\} \) and \( Y_{k-1} = \{y_{k-1}\} \): By (62),

\[
\tilde{G}^E[h_k, g_k|x_{k-1}, Y_{k-1}] = M_{1-ps+pshk(1-pD+pDgk)}(x_{k-1}, Y_{k-1})
\]

(119)

where (see (69-71): \( \tilde{p}_S(x_{k-1}) = p_S(x_{k-1}, y_{k-1}) \) and \( \tilde{p}_D(x_k) \neq p_D(x_k|\emptyset, y_{k-1}) \) and where \( M_{1-ps+pshk(1-pD+pDgk)}(x_{k-1}, y_{k-1}) \) was defined in (59). To simplify notation we will:

a. assume that \( p_S(x_{k-1}, y_{k-1}) = p_S(x_{k-1}, \emptyset) = p_S(x_{k-1}) \) and \( p_D(x_k|x_{k-1}, y_{k-1}) = p_D(x_k|x_{k-1}, \emptyset) = p_D(x_k|\emptyset, \emptyset) = p_D(x_k) \).

D. SUMMARY OF THE BERNOLLI MPMM FILTER

We are given \( f(\emptyset, Y_{k-1}|Z_{1:k-2}) \) and \( f(\{x_k\}, Y_{k-1}|Z_{1:k-2}) \) and that

\[
K_{Y_{k-1}} = f(\emptyset, Y_{k-1}|Z_{1:k-2}) + \int f(\{x_{k-1}\}, Y_{k-1}|Z_{1:k-2})dY_{k-1}.
\]

(120)

Then the updates \( f(\emptyset, Y_{k}|Z_{1:k-2}) \) and \( f(\{x_k\}, Y_{k}|Z_{1:k-2}) \) are given by the following two recursive formulas:

1. If \( Z_{k-1} = \emptyset \) is collected then:

\[
f(\emptyset, Y_{k}|Z_{1:k-1}) = A_{Y_k}(\emptyset) \cdot f(\emptyset, \emptyset|Z_{1:k-2}) + \int B_{Y_k}(x_k) \cdot f(\{x_k\}, \emptyset|Z_{1:k-2})dY_{k-1}
\]

(121)
\[f(x_k), Y_k | Z_{1:k}\]

\[= A_{Y_k}(x_k) \cdot f(\emptyset, \emptyset | Z_{1:k}) + \int B_{Y_k}(x_k | x_{k-1}) \cdot f(\{x_{k-1}\}, \emptyset | Z_{1:k}) d\mathbf{x}_{k-1}\]

\[(122)\]

where

\[A_{Y_k}(\emptyset) = K_{\emptyset}{(1)} \cdot (1 - q^B_k) \cdot f(\emptyset, Z_{k-1} | Y_{1:k-2}) + K_{Z_{k-1}} \cdot \ell_{Y_k}(x_k) \cdot \int (1 - p_S(x_{k-1})) \cdot f(\{x_{k-1}\}, Z_{k-1} | Z_{1:k-2}) d\mathbf{x}_{k-1}\]

\[(123)\]

\[B_{Y_k}(x_k | x_{k-1}) = K_{\emptyset}{(1)} \cdot \ell_{Y_k}(x_k) \cdot q^B_k \cdot x_k \cdot f(\emptyset, Z_{k-1} | Y_{1:k-2}) + K_{Z_{k-1}} \cdot \ell_{Y_k}(x_k) \cdot p_S(x_{k-1}) \cdot f(x_k | x_{k-1}) \]

\[(124)\]

\[\ell_{Y_k}(x_k) = 1 - p_D(x_k) + p_D(x_k) \cdot \sum_{x_1 \in \mathbb{Y}_k} L_{Y_k}(x_k) \cdot \frac{K}{K_k(Y_k) - |\{x_1\}|}{K_k(Y_k)}.

\[(127)\]

2. If \(Z_{k-1} \neq \emptyset\) is collected then:

\[f(\emptyset, Y_k | Z_{1:k-1}) = K_{\emptyset}{(1)} \cdot K_{Z_{k-1}} \cdot \ell_{Y_k}(x_k) \cdot \int (1 - p_S(x_{k-1})) \cdot f(\{x_{k-1}\}, Z_{k-1} | Z_{1:k-2}) d\mathbf{x}_{k-1}\]

\[(128)\]

\[f(\{x_k\}, Y_k | Z_{1:k}) = K_{\emptyset}{(1)} \cdot K_{Z_{k-1}} \cdot \ell_{Y_k}(x_k) \cdot q^B_k \cdot x_k \cdot f(\emptyset, Z_{k-1} | Y_{1:k-2}) + K_{Z_{k-1}} \cdot \ell_{Y_k}(x_k) \cdot p_S(x_{k-1}) \cdot f(x_k | x_{k-1}) \]

\[(129)\]

These equations are derived in Appendix A.

**Remark 9:** In regard to (129), consider the following special case: \(K_k(Y_k) = 0\) identically (no clutter); \(p_S(x_{k-1}) = 1\) (target never disappears); and \(p_D(x_k) = 1\) (perfect detection); in which case \(|X_k| = |Y_k| = 1\) for all \(k \geq 1\). Then Eq. (129) should reduce to (31)—which is indeed the case.

**VII. CONCLUSION**

The Bernoulli filter is a general solution for tracking a single disappearing and reappearing target, using a sensor whose observations are corrupted by missed detections and a general, known clutter process. The Bernoulli filter presumes restrictive independence assumptions, namely a hidden Markov model (HMM) structure. That is, the current target state depends only on the previous target state; and the measurement that it generates depends only on its current state.

Pieczynski’s pairwise Markov model (PMM) relaxes these restrictions. In it, the current target state can additionally depend on the previous measurement; and the current measurement can additionally depend on the previous measurement and the previous target state.

In this paper we: (i) generalized PMMs to the multitarget case (MPMM); (ii) devised a theoretically rigorous formula for the “standard” MPMM transition density (see (60,61)); (iii) derived transition models for the elementary MPMM pairs \((X_k, Y_k)\) with \(|X_k|, |Y_k| \leq 1\) (Sections V-D through V-H); (iv) used them to derive the Bernoulli MPMM filter (an MPMM generalization of the Bernoulli filter, Section VI); and then used it to derive the PMM Bernoulli filter (a generalization of the Bernoulli filter that obeys PMM rather than HMM sensor and target statistics).

Future work will be devoted to generalization of the PMM Bernoulli filter to multiple correlated sensors.

**APPENDIX A**

**DERIVATION OF THE BERNOULLI MPMM FILTER**

The derivation has two parts: when \(Y_{k-1} = \emptyset\) (Appendix A.1) and when \(Y_{k-1} \neq \emptyset\) (Appendix A.2).

1) DERIVATION OF THE BERNOULLI MPMM FILTER

**UPDATE WHEN \(Y_{k-1} = \emptyset\)**

Let us turn to the derivation of the Bernoulli MPMM filter update when \(Y_{k-1} = \emptyset\). Let

\[K! = \int f(X_{k-1}, \emptyset | Y_{1:k-2}) d\mathbf{x}_{k-1}\]

\[(130)\]

Then from (114), (116), and (117),

\[K \cdot G[h_k, g_k | Y_{1:k-1}] = G^* [g_k] \cdot G^E[h_k, g_k | \emptyset, \emptyset] \cdot f(\emptyset, \emptyset | Y_{1:k-2}) + G^* [g_k] \int G^E[h_k, g_k | \{x_k\}, \emptyset] \cdot f(\{x_k\}, \emptyset | Y_{1:k-2}) d\mathbf{x}_{k-1}\]

\[(131)\]

For fixed \(h_k\), abbreviate

\[L[g_k]! = \left(1 - q^B_k + q^B_k \cdot \int [h_k(1 - p_D + p_DL_{g_k})] f(\emptyset, \emptyset | Y_{1:k-2}) \right.\]

\[(132)\]

in which case

\[K \cdot G[h_k, g_k | Y_{1:k-1}] = G^* [g_k] \cdot L[g_k].\]

(134)
For $W \subseteq Y_k$, note that
\[
\frac{\delta L}{\delta W}[g_k] = \begin{cases} L[g_k], & \text{if } W = \emptyset \\ l(y), & \text{if } W = \{y\} \\ 0, & \text{if } |W| > 1 \end{cases} \tag{135}
\]
where
\[
l(y) = q^B_k s_k^y [h_k pD_L y] \cdot f(\emptyset, \emptyset|Y_{1:k-2}) + \int_{\emptyset|Y_{1:k-2}} p_S(x_{k-1}) \cdot M_{h_k pD_L y}(x_{k-1}) \cdot f([x_{k-1}], \emptyset|Y_{1:k-2}) dx_{k-1} \tag{136}
\]
and where $L_S(x) := f(y|x)$. Thus from the product rule for functional derivatives [2, p. 389],
\[
K \cdot \frac{\delta G}{\delta Y_k}[h_k, g_k|Y_{1:k-1}] = \sum_{W \subseteq Y_k} \left( \frac{\delta}{\delta (Y_k - W)} G^x[g_k] \right) \cdot \frac{\delta L}{\delta W}[g_k] \tag{137}
\]
\[
= \sum_{W \subseteq Y_k} \left( \frac{\delta G^x}{\delta (Y_k - W)}[g_k] \right) \cdot \frac{\delta L}{\delta W}[g_k] \tag{138}
\]
\[
= \sum_{W \subseteq Y_k: |W| \leq 1} \frac{\delta G^x}{\delta (Y_k - W)}[g_k] \cdot \frac{\delta L}{\delta W}[g_k] \tag{139}
\]
\[
= \frac{\delta G^x}{\delta Y_k}[g_k] \cdot L[g_k] + \sum_{y_k \in Y_k} \frac{\delta G^x}{\delta (Y_k - y_k)}[g_k] \cdot l(y_k) \tag{140}
\]
\[
= \frac{\delta G^x}{\delta Y_k}[g_k] \cdot \left( L[g_k] + \sum_{y_k \in Y_k} \frac{\delta G^x}{\delta (Y_k - y_k)}[g_k] \cdot l(y_k) \right) \tag{141}
\]
and so substituting $g_k = 0$ and using the fact that
\[
\frac{\delta G^x}{\delta Y_k}[0] = \kappa_k(Y_k), \quad \frac{\delta G^x}{\delta (Y_k - y_k)}[0] = \kappa_k(Y_k - y_k) \tag{142}
\]
we get
\[
K \cdot \frac{\delta G}{\delta Y_k}[h_k, 0|Y_{1:k-1}] = \kappa_k(Y_k) \cdot \left( L[0] + \sum_{y_k \in Y_k} l(y_k) \cdot \frac{\kappa_k(Y_k - y_k)}{\kappa_k(Y_k)} \right) \tag{143}
\]
where
\[
L[0] = \left( 1 - q^B_k + q^B_k q^B_k p_k (1 - p_D) \right) \cdot f(\emptyset, \emptyset|Y_{1:k-2}) + \int_{\emptyset|Y_{1:k-2}} \left( 1 - p_S(x_{k-1}) + p_S(x_{k-1}) \cdot M_{h_k (1 - p_D)}(x_{k-1}) \right) \cdot f([x_{k-1}], \emptyset|Y_{1:k-2}) dx_{k-1} \tag{144}
\]
and
\[
l(y) = q^B_k q^B_k [h_k pD_L y] \cdot f(\emptyset, \emptyset|Y_{1:k-2}) + \int_{\emptyset|Y_{1:k-2}} p_S(x_{k-1}) \cdot M_{h_k pD_L y}(x_{k-1}) \cdot f([x_{k-1}], \emptyset|Y_{1:k-2}) dx_{k-1} \tag{145}
\]
Thus after substitution and collection of like terms we get:
\[
\frac{\delta G}{\delta Y_k}[h_k, 0|Y_{1:k-1}] = A_{h_k} \cdot f(\emptyset, \emptyset|Y_{1:k-2}) + \int B_{h_k}(x_{k-1}) \cdot f([x_{k-1}], \emptyset|Y_{1:k-2}) dx_{k-1} \tag{146}
\]
where
\[
A_{h_k} = K^{-1} \kappa_k(Y_k) \left( 1 - q^B_k + q^B_k q^B_k p_k (1 - p_D) \right) \cdot \left( \sum_{y_k \in Y_k} q^B_k q^B_k [h_k pD_L y] \cdot \frac{\kappa_k(Y_k - y_k)}{\kappa_k(Y_k)} \right) \tag{147}
\]
\[
B_{h_k}(x_{k-1}) = K^{-1} \kappa_k(Y_k) \left( 1 - p_S(x_{k-1}) + p_S(x_{k-1}) \cdot M_{h_k (1 - p_D)}(x_{k-1}) + p_S(x_{k-1}) \sum_{y_k \in Y_k} M_{h_k pD_L y}(x_{k-1}) \cdot \frac{\kappa_k(Y_k - y_k)}{\kappa_k(Y_k)} \right) \tag{148}
\]
Consequently, and as claimed,
\[
f(\emptyset, Y_k|Z_{1:k-1}) = A_{Y_k}(\emptyset) \cdot f(\emptyset, \emptyset|Z_{1:k-2}) + \int_{\emptyset|Z_{1:k-2}} B_{Y_k}(x_{k-1}) \cdot f([x_{k-1}], \emptyset|Z_{1:k-2}) dx_{k-1} \tag{149}
\]
\[
f([x_k], Y_k|Z_{1:k-1}) = A_{Y_k}(x_k) \cdot f(\emptyset, \emptyset|Z_{1:k-2}) + \int_{\emptyset|Z_{1:k-2}} B_{Y_k}(x_{k-1}) \cdot f([x_{k-1}], \emptyset|Z_{1:k-2}) dx_{k-1} \tag{150}
\]
where
\[
A_{Y_k}(\emptyset) = K^{-1} \kappa_k(Y_k) \cdot (1 - q^B_k) \tag{151}
\]
\[
B_{Y_k}(x_{k-1}) = K^{-1} \kappa_k(Y_k) \cdot (1 - p_S(x_{k-1}) + p_S(x_{k-1}) \cdot M_{h_k (1 - p_D)}(x_{k-1}) + p_S(x_{k-1}) \sum_{y_k \in Y_k} M_{h_k pD_L y}(x_{k-1}) \cdot \frac{\kappa_k(Y_k - y_k)}{\kappa_k(Y_k)} \right) \tag{152}
\]
\[
B_{Y_k}(x_{k-1}) = K^{-1} \kappa_k(Y_k) \cdot \ell_{Y_k}(x_k) \cdot p_S(x_{k-1}) \cdot M_{h_k}(x_{k-1}) \tag{153}
\]
and
\[
\ell_{Y_k}(x_k) = 1 - p_D(x_k) + p_D(x_k) \sum_{y_k \in Y_k} L_{y_k}(x_k) \cdot \frac{\kappa_k(Y_k - y_k)}{\kappa_k(Y_k)} \tag{154}
\]
\[
(2) \text{ DERIVATION OF THE BERNOULLI MPM\textsc{M} FILTER UPDATE WHEN } Y_{k-1} \neq \emptyset \text{ }
\]
Now turn to the derivation of the Bernoulli MPM filter update when $Y_{k-1} \neq \emptyset$. From (114), (119), and (59) we have the following:
\[
K \cdot G[h_k, g_k|Y_{1:k-1}] = G^x[g_k] \cdot G^x[h_k, g_k|Y_{1:k-1}] \cdot f(\emptyset, Y_{k-1}|Y_{1:k-2}) + K \cdot G^x[h_k] \tag{155}
\]
\[
\times \int_{\emptyset|Y_{1:k-2}} \left( \sum_{y_k \in Y_k} M_{1-p_D + p_D} h_k (1-p_D X_L y)\right)(x_{k-1}, y_{k-1}) \cdot f([x_{k-1}], Y_{k-1}|Y_{1:k-2}) dx_{k-1} \tag{156}
\]
with normalization factor
\[
K = \int f(X_{k-1}, Y_{k-1} | Y_{1:k-2}) dx_{k-1} = f(\emptyset, Y_{k-1} | Y_{1:k-2})
\]
\[
+ \int f(\{x_{k-1}\}, Y_{k-1} | Y_{1:k-2}) dx_{k-1}
\]
(157)

and where, by (118),
\[
G^F[h_k, g_k | Y_{k-1}] = 1 - q_k^B + q_k^B \delta_k [h_k(1 - p_D + p_D L_{g_k})].
\]
Thus
\[
K \cdot G[h_k, g_k | Y_{k-1}]
\]
\[
= G^F[g_k] \cdot (1 - q_k^B + q_k^B \delta_k [h_k(1 - p_D + p_D L_{g_k})])
\]
\[
\int f(\emptyset, Y_{k-1} | Y_{1:k-2})
\]
\[
+ G^F[g_k] \int (\sum \tilde{M}_{1-p_S}(x_{k-1}, y_{k-1}))
\]
\[
f(\{x_{k-1}\}, Y_{k-1} | Y_{1:k-2}) dx_{k-1}
\]
(158)

First note that, setting \( h_k = 0 \),
\[
K \cdot G[0, g_k | Y_{k-1}]
\]
\[
= G^F[g_k] \cdot (1 - q_k^B) \cdot f(\emptyset, Y_{k-1} | Y_{1:k-2})
\]
\[
+ G^F[g_k] \int \left( \sum \tilde{M}_{1-p_S}(x_{k-1}, y_{k-1}) \right)
\]
\[
f(\{x_{k-1}\}, Y_{k-1} | Y_{1:k-2}) dx_{k-1}
\]
(159)

where
\[
\tilde{M}_{1-p_S}(x_{k-1}, y_{k-1})
\]
\[
= \int (1 - p_S(x_{k-1})) f(x_k, y_k | x_{k-1}, y_{k-1}) dx_k dy_k
\]
\[
= 1 - p_S(x_{k-1})
\]
(160)

and so
\[
K \cdot G[0, g_k | Y_{k-1}]
\]
\[
= G^F[g_k] \cdot (1 - q_k^B) \cdot f(\emptyset, Y_{k-1} | Y_{1:k-2})
\]
\[
+ G^F[g_k] \int (1 - p_S(x_{k-1})) \cdot f(\{x_{k-1}\}, Y_{k-1} | Y_{1:k-2}) dx_{k-1}.
\]
(161)

Taking \( \delta/\delta Y_k \) of both sides with respect to \( g_k \) and then setting \( g_k = 0 \) we get:
\[
K \cdot f(\emptyset, Y_k | Y_{k-1})
\]
\[
= \kappa_k(Y_k)
\]
\[
\cdot \left( (1 - q_k^B) \cdot f(\emptyset, Y_{k-1} | Y_{1:k-2})
\right.
\]
\[
\left. + \int f(1 - p_S(x_{k-1})) \cdot f(\{x_{k-1}\}, Y_{k-1} | Y_{1:k-2}) dx_{k-1} \right)
\]
(162)

Now note that
\[
K \cdot \frac{\delta G}{\delta x_k}[h_k, g_k | Y_{k-1}]
\]
\[
= G^F[g_k] \cdot q_k^B \delta_k [\delta_{x_k}(1 - p_D + p_D L_{g_k})] \cdot f(\emptyset, Y_{k-1} | Y_{1:k-2})
\]
\[
+ G^F[g_k] \int \left( \sum \tilde{M}_{1-p_S}(x_{k-1}, y_{k-1}) \right)
\]
\[
\cdot \delta(\{x_{k-1}\}, Y_{k-1} | Y_{1:k-2}) dx_{k-1}
\]
(163)

\[
f(\{x_{k-1}\}, Y_{k-1} | Y_{1:k-2}) dx_{k-1}
\]
\[
= G^F[g_k] \int \left( \sum \tilde{M}_{p_S\delta_{x_k}(1 - p_D + p_D L_{g_k})}(x_{k-1}, y_{k-1}) \right)
\]
\[
f(\{x_{k-1}\}, Y_{k-1} | Y_{1:k-2}) dx_{k-1}
\]
(164)

and so
\[
K \cdot \frac{\delta G}{\delta x_k}[0, g_k | Y_{k-1}]
\]
\[
= G^F[g_k] \cdot q_k^B \delta_k [\delta_{x_k}(1 - p_D + p_D L_{g_k})] \cdot f(\emptyset, Y_{k-1} | Y_{1:k-2})
\]
\[
+ G^F[g_k] \int \left( \sum \tilde{M}_{p_S\delta_{x_k}(1 - p_D + p_D L_{g_k})}(x_{k-1}, y_{k-1}) \right)
\]
\[
f(\{x_{k-1}\}, Y_{k-1} | Y_{1:k-2}) dx_{k-1}.
\]
(165)

Thus
\[
K \cdot \frac{\delta G}{\delta x_k}[0, g_k | Y_{k-1}]
\]
\[
= \left( \frac{\delta}{\delta Y_k} \left( G^F[g_k] \cdot q_k^B \delta_k [\delta_{x_k}(1 - p_D + p_D L_{g_k})]\right) \right)
\]
\[
f(\emptyset, Y_{k-1} | Y_{1:k-2})
\]
\[
+ \frac{1}{|Y_{k-1}|} \sum_{y_{k-1} \in Y_{k-1}} \int \delta
\]
\[
\left. \times \left( G^F[g_k] \cdot \tilde{M}_{p_S\delta_{x_k}(1 - p_D + p_D L_{g_k})}(x_{k-1}, y_{k-1}) \right) \right)
\]
\[
f(\{x_{k-1}\}, Y_{k-1} | Y_{1:k-2}) dx_{k-1}.
\]
(166)

For the first term in this sum, note that
\[
\frac{\delta}{\delta Y_k} \left( G^F[g_k] \cdot q_k^B \delta_k [\delta_{x_k}(1 - p_D + p_D L_{g_k})]\right)
\]
\[
= \sum_{w \subseteq Y_k} \left( \frac{\delta G^F}{\delta (\delta_{Y_k} - W)}[g_k] \right)
\]
\[
\times \left( \frac{\delta}{\delta Y_k} q_k^B \delta_k [\delta_{x_k}(1 - p_D + p_D L_{g_k})]\right)
\]
\[
+ \frac{1}{|Y_{k-1}|} \sum_{y_{k-1} \in Y_{k-1}} \int \delta
\]
\[
\left. \times \left( G^F[g_k] \cdot \tilde{M}_{p_S\delta_{x_k}(1 - p_D + p_D L_{g_k})}(x_{k-1}, y_{k-1}) \right) \right)
\]
\[
f(\{x_{k-1}\}, Y_{k-1} | Y_{1:k-2}) dx_{k-1}.
\]
(167)

where, for \( W \subseteq Y_k \),
\[
\frac{\delta}{\delta W} q_k^B \delta_k [\delta_{x_k}(1 - p_D + p_D L_{g_k})]
\]
\[
= \begin{cases} q_k^B \delta_k [\delta_{x_k}(1 - p_D + p_D L_{g_k})], & \text{if } W = \emptyset \\ q_k^B \delta_k [\delta_{x_k} p_D L_{y_k}], & \text{if } W = \{y_k\} \\ 0, & \text{otherwise} \end{cases}
\]
(168)
and so
\[
\left[ \frac{\delta}{\delta y_k} \left( G^*[g_k] \cdot q_k^Y_k \cdot q_k^B_k \cdot \delta y_k (1 - pD + pD L_{r_k}) \right) \right]_{y_l=0} = G^*[g_k] \cdot q_k^Y_k \cdot q_k^B_k \cdot \delta y_k (1 - pD) + \sum_{y_j \in Y_k} \kappa_k(y_k - \{y_j\}) \cdot q_k^Y_k \cdot q_k^B_k \cdot \delta y_k pD L_{y_k} \tag{169}
\]
where
\[
q_k^Y_k \cdot q_k^B_k \cdot \delta y_k (1 - pD) = (1 - pD(x_k)) \cdot q_k^B_k(x_k) \tag{170}
\]
\[
q_k^B_k \cdot \delta y_k pD L_{y_k} = pD(x_k) \cdot L_{y_k}(x_k) \cdot q_k^B_k(x_k) \tag{171}
\]
and so
\[
\left[ \frac{\delta}{\delta y_k} \left( G^*[g_k] \cdot q_k^Y_k \cdot q_k^B_k \cdot \delta y_k (1 - pD + pD L_{r_k}) \right) \right]_{y_l=0} = \kappa_k(y_k)
\]
\[
\cdot \left( 1 - pD(x_k) + pD(x_k) \cdot \sum_{y_j \in Y_k} L_{y_k}(x_k) \cdot \frac{\kappa_k(y_k - \{y_j\})}{\kappa_k(y_k)} \right) \tag{172}
\]
\[
q_k^B_k(x_k) = \kappa_k(y_k) \cdot \ell y_k(x_k) \cdot q_k^B_k(x_k). \tag{173}
\]
For the second term in (166), note that
\[
\frac{\delta}{\delta y_k} \left( G^*[g_k] \cdot \tilde{M}_{ps \delta_k (1-pD+pDGL)}(x_{k-1}, y_{k-1}) \right) = \sum_{W \subseteq Y_k} \left( \frac{\delta G^*}{\delta (y_k - W)} [g_k] \right) \times \left( \frac{\delta}{\delta y_k} \tilde{M}_{ps \delta_k (1-pD+pDGL)}(x_{k-1}, y_{k-1}) \right) \tag{174}
\]
and where for \( W \subseteq Y_k \),
\[
\frac{\delta}{\delta W} \tilde{M}_{ps \delta_k (1-pD+pDGL)}(x_{k-1}, y_{k-1}) = \begin{cases} 
\tilde{M}_{ps \delta_k (1-pD+pDGL)}(x_{k-1}, y_{k-1}), & \text{if } W = \emptyset \\
\tilde{M}_{ps \delta_k pD \delta_{y_k}}(x_{k-1}, y_{k-1}), & \text{if } W = \{y_k\} \\
0, & \text{if otherwise.}
\end{cases} \tag{175}
\]
Thus
\[
\frac{\delta}{\delta y_k} \left( G^*[g_k] \cdot \tilde{M}_{ps \delta_k (1-pD+pDGL)}(x_{k-1}, y_{k-1}) \right) = \frac{\delta G^*}{\delta y_k} [g_k] \cdot \tilde{M}_{ps \delta_k (1-pD+pDGL)}(x_{k-1}, y_{k-1}) + \sum_{y_j \in Y_k} \frac{\delta G^*}{\delta (y_k - \{y_j\})} [g_k] \cdot \tilde{M}_{ps \delta_k pD \delta_{y_k}}(x_{k-1}, y_{k-1}) \tag{176}
\]
where
\[
\tilde{M}_{ps \delta_k (1-pD)}(x_{k-1}, y_{k-1}) = \int p_S(x_{k-1}) \cdot \delta y_k(x_k) \cdot (1 - pD(x_k)) \cdot f(u_k, v_k|x_{k-1}, y_{k-1}) \cdot d\mu_k \tag{177}
\]
\[
f([x_{k}], Y_k|Y_{1:k-1}) = K^{-1} \kappa_k(Y_k) \int p_S(x_{k-1}) \cdot \delta y_k(x_k) \cdot q_k^B_k(x_k) \cdot f(\emptyset, Y_k-1|Y_{1:k-2}) + \frac{K^{-1} \kappa_k(Y_k)}{|Y_{k-1}|} \tag{183}
\]
\[
f([x_{k-1}], Y_{k-1}|Y_{1:k-2}) d\mu_{x_{k-1}} = K^{-1} \kappa_k(Y_k) \int p_S(x_{k-1}) \cdot \frac{1 - pD(x_k)}{|Y_{k-1}|} \sum_{y_l \in Y_k} f(x_k|Y_{k-1}, y_{k-1}) \cdot \frac{pD(x_k) \cdot pD(x_k) \cdot \kappa_k(y_k - \{y_j\})}{\kappa_k(y_k)} \tag{184}
\]
\[
f([x_{k-1}], Y_{k-1}|Y_{1:k-2}) d\mu_{x_{k-1}} = K^{-1} \kappa_k(Y_k) \int p_S(x_{k-1}) \cdot \delta y_k(x_k) \cdot q_k^B_k(x_k) \cdot f(\emptyset, Y_k-1|Y_{1:k-2}) + f([x_{k-1}], Y_{k-1}|Y_{1:k-2}) d\mu_{x_{k-1}} \tag{185}
\]
\[ \tilde{M}_{P_{h_k} P_{d_k} P_{y_k}}(x_{k-1}, y_{k-1}) = \int p_{s}(x_{k-1}) \cdot \delta_{x_k}(u_k) \cdot p_{d}(u_k) \cdot \delta_{y_k}(v_k) \cdot f(u_k, v_k | x_{k-1}, y_{k-1})du_kdv_k \]

\[ = p_{S}(x_{k-1}) \cdot \cdot f(x_k, y_{k} | x_{k-1}, y_{k-1}) \quad (180) \]

Thus setting \( \gamma_k = 0 \) in (166) and substituting (173) and (182), we get (129), (183)-(185), as shown at the bottom of the previous page, where

\[ \epsilon_{Y_k} \cdot f(x_k | x_{k-1}) \]

\[ = \frac{1}{|Y_{k-1}|} \sum_{y_{k-1} \in Y_{k-1}} f(x_k | x_{k-1}, y_{k-1}) \]

\[ = \frac{p_{d}(x_k)}{|Y_{k-1}|} \sum_{y_{k-1} \in Y_{k-1}} \sum_{y_{k-1} \in Y_{k}} f(x_k, y_k | x_{k-1}, y_{k-1}) \]

\[ \cdot \frac{\kappa_k(Y_k - \{y_{k}\})}{\kappa_k(Y_k)} \quad (186) \]

**APPENDIX B**

**DERIVATION OF THE PMM BERNOULLI FILTER**

The derivation has two parts: when \( Z_{k-1} = \emptyset \) (Appendix B.1) and when \( Z_{k-1} = \emptyset \) (Appendix B.2).

1) **DERIVATION OF THE PMM BERNOULLI FILTER WHEN \( Y_k = \emptyset \)**

We are to verify (10) assuming that \( Z_{k-1} = \emptyset \). The multtarget version of (38) is

\[ f(x_k | Z_{1:k}) = \frac{f(x_k, Z_k | Z_{1:k-1})}{\int f(x_k, Z_k | Z_{1:k-1})dx_k} = \frac{f(x_k, Z_k | Y_{1:k-1})}{\int f(Z_k | Z_{1:k-1})dx_k} \quad (187) \]

For a Bernoulli filter, \( |X_k| \leq 1 \) and so the updated PHD is

\[ D_{k|k}(x_k) = f((x_k) | Z_{1:k}) \]

\[ = \frac{f((x_k), Z_{1:k})}{f((x_k), Z_{1:k}) + f((x_k), Z_{1:k} | Z_{1:k-1})dx_k} \quad (188) \]

Recall from (120) that

\[ K_{Z_{k-1}} = \int f((x_{k-1}), Z_{k-1} | Z_{1:k-2})dx_{k-1} \]

\[ + \int f((x_{k-1}), Z_{k-1} | Z_{1:k-2})dx_{k-1} \quad (189) \]

Thus on the one hand, (121) can be written as:

\[ K_0 \cdot f((x_k), Z_{1:k}) \]

\[ = \kappa_k(Y_k) \left( \left(1 - q_k^B\right) \cdot f((x_k), Z_{1:k}) + \int f((x_{k-1}), Z_{1:k})dx_{k-1} \right) \]

\[ \cdot \frac{\lambda_k(Y_k, Z_{k-1})}{\lambda_k(Y_k)} \quad (190) \]

On the other hand, (122) can be written as:

\[ K_0 \cdot f((x_k), Z_{1:k}) \]

\[ = \kappa_k(Y_k) \left( \left(1 - q_k^B\right) \cdot f((x_k), Z_{1:k}) + \int f((x_{k-1}), Z_{1:k})dx_{k-1} \right) \]

\[ \cdot \frac{\lambda_k(Y_k, Z_{k-1})}{\lambda_k(Y_k)} \quad (191) \]

\[ = \kappa_k(Y_k) \cdot \left( \left(1 - q_k^B\right) \cdot f((x_k), Z_{1:k}) + \int f((x_{k-1}), Z_{1:k})dx_{k-1} \right) \]

\[ \cdot \frac{\lambda_k(Y_k, Z_{k-1})}{\lambda_k(Y_k)} \quad (192) \]

Note that

\[ K_0 \int f((x_k), Z_{1:k})dx_k \]

\[ = \kappa_k(Y_k) \cdot \left( \left(1 - q_k^B\right) \cdot f((x_k), Z_{1:k}) + \int f((x_{k-1}), Z_{1:k})dx_{k-1} \right) \]

\[ \cdot \frac{\lambda_k(Y_k, Z_{k-1})}{\lambda_k(Y_k)} \quad (193) \]

Thus adding (193) and (195) we get:

\[ K_0 \cdot f((x_k), Z_{1:k}) \]

\[ = \kappa_k(Y_k) \cdot \left( \left(1 - q_k^B\right) \cdot f((x_k), Z_{1:k}) + \int f((x_{k-1}), Z_{1:k})dx_{k-1} \right) \]

\[ \cdot \frac{\lambda_k(Y_k, Z_{k-1})}{\lambda_k(Y_k)} \quad (194) \]

Consequently, the single-step PHD update is, (199), as shown at the bottom of the page, which is identical to (111).
At the top of the page, and so

\[ K_{z_{k-1}} \cdot f([x_k], Z_{k}|Z_{1:k-1}) = \kappa_k(Y_k) \cdot \left( \ell_{z_k}(x_k) \cdot q_k^{B_k}(x_k) \cdot f(\emptyset, Z_{k-1}|Y_{1:k-2}) + f(p_S(x_{k-1})) \cdot \ell_{z_k, Z_{k-1}}(x_k|x_{k-1}) \cdot f([x_{k-1}], Z_{k-1}|Z_{1:k-2}) \right) \]  \hfill (204)

\[ = \kappa_k(Y_k) \cdot \left( \ell_{z_k}(x_k) \cdot q_k^{B_k}(x_k) \cdot f(\emptyset|Z_{1:k-1}) \right) \]  \hfill (205)

\[ = \kappa_k(Y_k) \cdot \left( \ell_{z_k}(x_k) \cdot q_k^{B_k}(x_k) \cdot (1 - D_{k-1|k-1}[1]) \right) \]  \hfill (206)

\[ D_{k|Z_k}(x_k) = \ell_{z_k}(x_k) \cdot \left( q_k^{B_k}(x_k) \cdot (1 - D_{k-1|k-1}[1]) + D_{k-1|k-1}[p_S\ell_{z_k, Z_{k-1}}] \right) \]  \hfill (207)

Thus adding (203) and (207) we get:

\[ f(Z_k|Z_{k-1}) = \kappa_k(Y_k) \cdot \left( (1 - q_k^B) \cdot (1 - D_{k-1|k-1}[1]) + D_{k-1|k-1}[p_S\ell_{z_k, Z_{k-1}}] \right) \]  \hfill (208)

\[ = \kappa_k(Y_k) \cdot \left( (1 - q_k^B) + q_k^B\ell_{z_k} \cdot (1 - D_{k-1|k-1}[1]) + D_{k-1|k-1}[1 - p_S] \ell_{z_k, Z_{k-1}} \right) \]  \hfill (209)

Thus the single-step PHD update does result in (10), (210), as shown at the top of the page.

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