\(A_4 \times SU(5)\) SUSY GUT of Flavour with Trimaximal Neutrino Mixing

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Abstract

Recent T2K, MINOS and Double CHOOZ results, together with global fits of mixing parameters, indicate a sizeable reactor angle \(\theta_{13} \sim 8^\circ\) which, if confirmed, would rule out tri-bimaximal (TB) lepton mixing. Recently two of us studied the vacuum alignment of the Altarelli-Feruglio \(A_4\) family symmetry model including additional flavons in the \(1'\) and \(1''\) representations, leading to so-called “trimaximal” neutrino mixing and allowing a potentially large reactor angle. Here we show how such a model may arise from a Supersymmetric (SUSY) Grand Unified Theory (GUT) based on \(SU(5)\), leading to sum rule bounds \(|s| \leq \frac{\theta_C}{\pi}, |a| \leq \frac{1}{2}(r + \frac{\theta_C}{\pi}) \cos \delta|\), where \(s, a, r\) parameterise the solar, atmospheric and reactor angle deviations from their TB mixing values, \(\delta\) is the CP violating oscillation phase, and \(\theta_C\) is the Cabibbo angle.

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1 Introduction

Recently T2K have published evidence for a large non-zero reactor angle \[1\]. Combining this with data from MINOS \[2\] and other experiments in a global fit yields \[3\] \[\theta_{13} = 6.5^\circ \pm 1.5^\circ\], assuming a normal neutrino mass hierarchy. Here the errors indicate the 1\(\sigma\) range, although the statistical significance of a non-zero reactor angle is about 3\(\sigma\). Other global fits indicate a larger reactor angle \[\theta_{13} = 9.1^\circ \pm 1.3^\circ\] \[4\]. The first results from Double CHOOZ also show indications of a non-zero reactor angle \[5\]. If confirmed, a sizable reactor angle \[\theta_{13} \sim 8^\circ\], consistent with both global fits, would rule out the hypothesis of exact tri-bimaximal (TB) mixing \[7\].

In the framework of Supersymmetric (SUSY) Grand Unified Theories (GUTs) of Flavour \[8\] (i.e. with a Family Symmetry \[9\] implemented) it is already known that TB mixing cannot be exact. The reason is that, even if TB mixing is realised exactly in the neutrino sector, observable lepton mixing is subject to charged lepton (CL) corrections (due to the fact that \[U_{\text{PMNS}} = U_e U_\nu^\dagger\]) and renormalisation group (RG) corrections, not to mention other corrections due to canonical normalisation (CN) (for a unified discussion of all three corrections see e.g. \[10\] and references therein). Therefore, in the framework of SUSY GUTs of Flavour, the question of whether TB mixing may be maintained in the neutrino sector is a quantitative one: can the above CL, RG and CN corrections be sufficiently large to account for the observed reactor angle? The answer is yes in some cases (see e.g. \[11\]), but no in many other cases. For example, in models based on the Georgi-Jarlskog mechanism \[12\], where the CL corrections are less than or about 3\(^\circ\), and where the RG and CN corrections are less than or about 1\(^\circ\) (which is the case for hierarchical neutrinos), it would be difficult to account for a reactor angle \[\theta_{13} \sim 8^\circ\]. For this reason, there is a good motivation to consider other patterns of neutrino mixing beyond TB mixing, and many alternative proposals \[13\] have indeed been put forward to account for a non-zero \[\theta_{13}\]. On the other hand, since the solar and atmospheric mixing angles remain consistent with TB mixing, there is also a good motivation to maintain these successful predictions of TB mixing.

In going beyond TB mixing, it is useful to consider a general parameterisation of the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) mixing matrix in terms of deviations from TB mixing \[14\],

\[
U_{\text{PMNS}} \approx \begin{pmatrix}
\frac{2}{\sqrt{3}}(1 - \frac{1}{2}s) & \frac{1}{\sqrt{3}}(1 + s) & \frac{1}{\sqrt{2}}r e^{-i\delta} \\
-\frac{1}{\sqrt{6}}(1 + s - a + r e^{i\delta}) & \frac{1}{\sqrt{3}}(1 - \frac{1}{2}s - a - \frac{1}{2}r e^{i\delta}) & \frac{1}{\sqrt{2}}(1 + a) \\
\frac{1}{\sqrt{6}}(1 + s + a - r e^{i\delta}) & -\frac{1}{\sqrt{3}}(1 - \frac{1}{2}s + a + \frac{1}{2}r e^{i\delta}) & \frac{1}{\sqrt{2}}(1 - a)
\end{pmatrix} P ,
\]

where the deviation parameters \(s, a, r\) are defined as \[14\],

\[
\sin \theta_{12} = \frac{2}{\sqrt{3}}(1 + s) , \quad \sin \theta_{23} = \frac{1}{\sqrt{2}}(1 + a) , \quad \sin \theta_{13} = \frac{r}{\sqrt{2}} .
\]

\[1\] In both cases, the quoted ranges are calculated using the new reactor anti-neutrino fluxes \[5\].
For example, the global fit of the conventional mixing angles in [3] can be translated into the $1\sigma$ ranges:

\[ 0.12 < r < 0.20, \quad -0.06 < s < -0.006, \quad -0.06 < a < 0.08. \] (3)

In a SUSY GUT of Flavour, the Family Symmetry is responsible for determining the neutrino mixing pattern, which then gets corrected by CL, RG and CN contributions to yield the observed lepton mixing angles. The question is what is the underlying neutrino mixing pattern? If we want to go beyond TB neutrino mixing, there are many possibilities. One simple scheme is the trimaximal (TM) mixing pattern [15]:

\[
U_{\nu}^{TM} = P' \begin{pmatrix}
\frac{2}{\sqrt{6}} \cos \vartheta & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \sin e^{i\varphi} \\
\frac{1}{\sqrt{6}} \cos \vartheta - \frac{1}{\sqrt{2}} \sin \vartheta e^{-i\psi} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \cos \vartheta - \frac{1}{\sqrt{6}} \sin \vartheta e^{i\psi} \\
\frac{1}{\sqrt{6}} \cos \vartheta + \frac{1}{\sqrt{2}} \sin \vartheta e^{-i\psi} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \cos \vartheta - \frac{1}{\sqrt{6}} \sin \vartheta e^{i\psi}
\end{pmatrix} P,
\] (4)

where $\frac{2}{\sqrt{6}} \sin \vartheta = \sin \theta_{13}^{\nu}, P'$ is a diagonal phase matrix required to put $U_{\nu}^{PMNS} = U^{e}U_{\nu}^{TM}$ into the PDG convention [16], and $P = \text{diag}(1, e^{i\alpha_{2}/2}, e^{i\alpha_{3}/2})$ contains the usual Majorana phases. In particular TM mixing approximately predicts TB neutrino mixing for the solar neutrino mixing angle $\theta_{12}^{\nu} \approx 35^\circ$ as the correction due to a non-zero but relatively small reactor angle is of second order. However we emphasise again that, in a SUSY GUT of Flavour, TM mixing refers to the neutrino mixing angles only, and the physical lepton mixing angles will involve additional CL, RG and CN corrections. Nevertheless, TM neutrino mixing could provide a better starting point than TB neutrino mixing, if the reactor angle proves to be $\theta_{13} \sim 8^\circ$, and this provides the motivation for the approach followed in this paper.

Recently, an $A_{4}$ model of TM neutrino mixing was discussed in [17]. In the original $A_{4}$ models of TB mixing Higgs fields [18] or flavon fields [19] transforming under $A_{4}$ as 3 and 1 but not 1' or 1'' were used to break the family symmetry and to lead to TB mixing. However there is no good reason not to include flavons transforming as 1' or 1'', and once included they will lead to deviations from TB mixing [20], in particular it was noted that they lead to TM mixing [21]. In [17] the vacuum alignment of the Altarelli-Feruglio $A_{4}$ family symmetry model [19], including additional flavons in the 1' and/or 1'' representations, was studied and it was shown that it leads to TM neutrino mixing. In this paper we shall show how such a model may arise from a SUSY GUT based on $SU(5)$, leading to the sum rule bounds $|s| \leq \frac{\theta_{45}^{\nu}}{3}$ and $|a| \leq \frac{1}{2}(r + \frac{\theta_{45}^{\nu}}{3})|\cos \delta|$, up to RG and CN corrections, where $r, s, a$ are the above tri-bimaximal deviation parameters, $\delta$ is the CP violating oscillation phase, and $\theta_{C}$ is the Cabibbo angle. However we shall not be interested in the details of the GUT breaking Higgs potential, which is model dependent and may not even be describable by renormalisable 4d field theory, as is the case where GUT breaking is due to orbifold constructions [22]. Instead we shall take the GUT breaking as “given” and formulate the theory just below the GUT scale, although
Table 1: Matter and Higgs chiral superfields in the model.

The transformation properties of the SU(5) matter and Higgs multiplets are shown in
Table 1. N denotes the right-handed neutrino fields, F the 5 of SU(5), containing
the lepton doublet and the right-handed down-type quark, and T labels the 10 which
includes the quark doublet as well as the right-handed up-type quark and charged lepton.
N and F furnish the triplet representation of A4, thus unifying the three families of
leptons, while the three families of the Ti transform in the three distinct one-dimensional
representations of A4. In the Higgs sector, we have introduced the $H_{3\bar{5}}$ in order to
implement the Georgi-Jarlskog mechanism \(^{12}\). \(^{3}\)

The full set of flavon fields is shown in Table 2. The fields $\varphi_S$ and $\xi_i$ are responsible
for the flavour structure of the neutrino sector, while the flavons $\varphi_T$ and $\theta_i$ control
the quark and charged lepton sector. The vacuum structure is obtained via the standard

\(^2\)The standard MSSM $\mu$-term $\mu H_u H_d$ (where $H_u$ is the SU(2)$_L$ doublet of $H_5$; and $H_d$ is a linear
combination of the SU(2)$_L$ doublets in $H_\pi$ and $H_{3\bar{5}}$) is forbidden by the $A_4$, $U(1)$, $Z_3$ and $Z_5$ symmetries
as well as $U(1)_R$, allowing for a natural solution to the $\mu$-problem of the MSSM using a GUT singlet
from the hidden sector of Supergravity theories \(^{23}\).
Table 2: Flavon chiral superfields in the model.

| Field | \( \varphi_S \) | \( \xi \) | \( \xi' \) | \( \xi'' \) | \( \varphi_T \) | \( \theta \) | \( \theta' \) | \( \theta'' \) | \( \vartheta \) | \( \sigma \) | \( \varphi_0 \) | \( \varphi_S^0 \) | \( \xi^0 \) |
|-------|----------------|----------------|----------------|----------------|----------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| \( SU(5) \) | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| \( A_4 \) | 3 | 1 | 1' | 1'' | 3 | 1 | 1' | 1'' | 1 | 3 | 3 | 1 | 1 |
| \( U(1)_R \) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 2 | 2 | 2 | 2 |
| \( U(1) \) | -2 | -2 | -2 | -2 | 2 | -1 | -1 | -1 | -5 | 2 | -4 | 4 | 4 | 4 |
| \( Z_2 \) | + | + | + | + | + | + | + | + | + | + | + | + | + |
| \( Z_3 \) | \( \omega \) | \( \omega \) | \( \omega \) | \( \omega \) | 1 | \( \omega \) | \( \omega^2 \) | \( \omega^2 \) | 1 | 1 | \( \omega \) | \( \omega \) | \( \omega \) |
| \( Z_5 \) | \( \rho^5 \) | \( \rho^5 \) | \( \rho^5 \) | \( \rho^5 \) | 1 | 1 | 1 | 1 | 1 | 1 | \( \rho^3 \) | \( \rho^3 \) | \( \rho^3 \) |

The auxiliary flavon field \( \sigma \) is introduced for the purpose of achieving the alignment of the \( U(1) \) charged flavon field \( \varphi_T \).
with \( y, y_1, y_2, y'_2, y''_2 \) being dimensionless couplings.

The leading order superpotential terms of the down quark and charged lepton sector are given as follows

\[
W_d \sim \left( \frac{\theta^2 \theta''}{\Lambda_d^4} (F_{\varphi T})' + \frac{\theta^2 \theta'}{\Lambda_d^4} (F_{\varphi T})'' \right) H_{\bar{5}} T_1 + \frac{\sigma \theta \theta' (\theta'')^2}{\Lambda_d^6} (F_{\varphi T}) H_{\bar{5}} T_1 \\
+ \frac{(\theta')^2 \theta''}{\Lambda_d^4} (F_{\varphi T}) H_{\bar{5}} T_2 + \left( \frac{\theta \theta''}{\Lambda_d^3} (F_{\varphi T})' + \frac{\theta \theta'}{\Lambda_d^3} (F_{\varphi T})'' \right) H_{\bar{5}} T_2 \\
+ \left( \frac{\sigma^2 \theta^2 (\theta')^2}{\Lambda_d^2} (F_{\varphi T})' + \frac{1}{\Lambda_d^2} ((F_{\varphi T})'') \right) H_{\bar{5}} T_3 + \left( \frac{\sigma \theta^3}{\Lambda_d^5} (F_{\varphi T})' \right) H_{\bar{5}} T_3,
\]

(7)

where \( \Lambda_d \) is the relevant messenger mass. Note that for some entries of the down quark Yukawa matrix, there are several different operators of the same order; here we simply choose an example for illustrative purposes. The flavons \( \theta^i \) play a role similar to a Froggatt-Nielsen field [24].

Finally the leading order up quark sector Yukawa superpotential terms take the form

\[
W_u \sim \frac{\theta^4 (\theta')^2}{\Lambda_u^6} T_1 T_1 H_5 + \left( \frac{\theta^2 (\theta')^2 (\theta'')^2}{\Lambda_u^6} + \frac{\sigma \theta (\theta')^2 \theta'}{\Lambda_u^5} \right) (T_1 T_2 + T_2 T_1) H_5 \\
+ \frac{\theta^2 \theta'}{\Lambda_u^3} (T_2 T_3 + T_3 T_1) H_5 + \frac{\theta \theta'}{\Lambda_u^2} T_2 T_2 H_5 + \frac{\theta' (\theta'')^2}{\Lambda_u^3} (T_2 T_3 + T_3 T_2) H_5 + T_3 T_3 H_5.
\]

(8)

It should be mentioned that the messenger mass in this sector, \( \Lambda_u \), may in principle be different from that in the down quark sector. The field \( \theta' \) is introduced specifically to generate the \( T_2 T_2 \) term to the required order.

Examples of the many subleading higher order operators allowed by the symmetries of the model are listed in Appendix [B]. As their contribution to the mass matrices is negligible, they do not induce physically relevant modifications of the LO picture.

3 Fermion mass matrices

After spontaneous breakdown of the \( A_4 \) family symmetry by the flavon VEVs, the superpotential terms of Eqs. (6)-(8) predict mass matrices for the respective sectors. In the following, order one coefficients in the quark and charged lepton sectors are omitted (including flavon VEV normalisation factors). Regarding the scale of the flavon VEVs we define

\[
\eta_i = \frac{\langle |\varphi_i| \rangle}{\Lambda},
\]

(9)

where \( \varphi_i = \varphi_T, \theta^i \) or \( \sigma \). In order to get the hierarchical structure of the quark and charged lepton mass matrices we assume

\[
\eta_\varphi = \epsilon^2 \quad \text{and} \quad \eta_{\text{others}} = \epsilon,
\]

(10)
where the numerical values for $\epsilon$ depend on the messenger scale of the relevant sector. We present LO operators for each entry in the mass matrices; NLO operators can be found in Appendix B.

In the Higgs sector, it is not the $H_5$, $H_5^-$ or $H_{45}^-$ which get VEVs but their $SU(2)_L$ doublet components. These are the two MSSM doublets $H_u$ (corresponding to $H_5$) and $H_d$ (corresponding to a linear combination of $H_-^5$ and $H_{45}^-$); they originate below the GUT scale and remain massless down to the electroweak scale. The non-MSSM states all acquire GUT scale masses, including the linear combination of $H_5^-$ and $H_{45}^-$ orthogonal to $H_d$. Electroweak symmetry is broken after the light MSSM doublets $H_u,d$ acquire VEVs $v_u,d$ and they then generate the fermion masses.

In the following all quark and charged lepton mass matrices are given in the L-R convention, i.e. the mass term for a field $\psi$ is given in the order $\psi_L M_{LR} \psi_R$.

### 3.1 Neutrino sector

Eq. (6) gives Dirac and Majorana mass matrices

$$m_D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} y v_u, \quad (11)$$

and

$$M_R = \begin{bmatrix} \alpha \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} & + \beta \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + \gamma' \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} + \gamma'' \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{bmatrix}, \quad (12)$$

with $\alpha = y_1 \langle \varphi_S \rangle$, $\beta = y_2 \langle \xi \rangle$, $\gamma' = y'_2 \langle \xi' \rangle$ and $\gamma'' = y''_2 \langle \xi'' \rangle$. As shown in [17], the standard type I seesaw formula then yields a light neutrino mass matrix of trimaximal structure, and hence a neutrino mixing matrix of the form as given in Eq. (4).

### 3.2 Down quark and charged lepton sector

In the down quark and charged lepton sector, the superpotential of Eq. (7) predicts a mass matrix of the form (with messenger mass $\Lambda_d$ in $\eta_i$)

$$\begin{pmatrix} k_f \eta_0 \eta_0 \eta_0 \eta_0 \eta_0 \eta_0 & \eta_0 \eta_0 \eta_0 \eta_0 \eta_0 \eta_0 & \eta_0 \eta_0 \eta_0 \eta_0 \eta_0 \eta_0 \\ \eta_0 \eta_0 \eta_0 \eta_0 \eta_0 \eta_0 & k_f \eta_0 \eta_0 \eta_0 \eta_0 \eta_0 \eta_0 & \eta_0 \eta_0 \eta_0 \eta_0 \eta_0 \eta_0 \\ \eta_0 \eta_0 \eta_0 \eta_0 \eta_0 \eta_0 & \eta_0 \eta_0 \eta_0 \eta_0 \eta_0 \eta_0 & k_f \eta_0 \eta_0 \eta_0 \eta_0 \eta_0 \eta_0 \end{pmatrix} v_d, \quad (13)$$

where this matrix has to be transposed for the charged leptons. $k_f$ is the Georgi-Jarlskog factor which takes the values

$$k_f = \begin{cases} 1 & \text{for } f = d, \\ -3 & \text{for } f = e. \end{cases}$$
Inserting the $\epsilon$ suppressions of the flavon VEVs from Eq. (10) the down quark mass matrix becomes

$$M_d \sim \begin{pmatrix} e^5 & e^3 & e^3 \\ e^3 & e^2 & e^2 \\ e^6 & e^5 & 1 \end{pmatrix} \epsilon v_d,$$

whilst the charged lepton mass matrix reads

$$M_e \sim \begin{pmatrix} -3e^5 & e^3 & e^6 \\ e^3 & -3e^2 & -3e^5 \\ e^3 & -3e^2 & 1 \end{pmatrix} \epsilon v_d.$$

Here we assume the numerical value $\epsilon \sim 0.15$. Upon diagonalisation, these give mass ratios of $e^4 : e^2 : 1$ for the down-type quarks and $e^4 : 3e^2 : 1$ for the charged leptons. These ratios are in good agreement with quark and lepton data and also predict the Georgi-Jarlskog GUT scale mass relations of $m_u \sim \frac{m_d}{3}$, $m_\mu \sim 3m_s$ and $m_\tau \sim m_b$ as desired. In the low quark angle approximation, the left-handed down quark mixing angles $\theta^d_{12} \sim \epsilon$, $\theta^d_{13} \sim e^3$ and $\theta^d_{23} \sim e^2$ are also predicted in agreement with data (assuming an approximately diagonal up quark sector which we obtain in the next subsection). The corresponding charged lepton mixing angles are $\theta^e_{12} \sim \frac{\epsilon}{3}$, $\theta^e_{13} \sim e^6$ and $\theta^e_{23} \sim 3e^5$. Therefore, the only significant charged lepton correction to the TM mixing of the neutrino sector originates from $\theta^\nu_{12} \sim \frac{\epsilon_\nu}{3}$, where $\theta^\nu_C$ denotes the Cabibbo angle.

### 3.3 Up quark sector

Eq. (5) may be expanded after $A_4$ symmetry breaking and is responsible for up-type quark masses

$$\begin{pmatrix} \eta^2 \eta^2 & \eta^2 \eta^2 & \eta^2 \eta^2 \\ \eta^2 \eta^2 & \eta^2 \eta^2 & \eta^2 \eta^2 \\ \eta^2 \eta^2 & \eta^2 \eta^2 & \eta^2 \eta^2 \end{pmatrix} \frac{\epsilon v}{\sqrt{2}},$$

Taking the VEV hierarchy as in Eq. (10), but now adopting the messenger scale $\Lambda_u \approx \frac{3}{2} \Lambda_d$, we obtain a mass matrix with an expansion parameter $\overline{\epsilon} \sim 0.1$,

$$M_u \sim \begin{pmatrix} \overline{\epsilon}^6 & \overline{\epsilon}^3 & \overline{\epsilon}^3 \\ \overline{\epsilon}^6 & \overline{\epsilon}^3 & \overline{\epsilon}^3 \\ \overline{\epsilon}^6 & \overline{\epsilon}^3 & 1 \end{pmatrix} \epsilon v_u,$$

and an up-type quark mass hierarchy $\overline{\epsilon}^6 : \overline{\epsilon}^3 : 1$. This matrix gives mixing angles of $\theta^u_{12} \sim \theta^u_{13} \sim \theta^u_{23} \sim \overline{\epsilon}^3$. This means that the Cabibbo-Kobayashi-Maskawa (CKM) mixing matrix is dominated by down quark mixing, except that there may be a contribution to $\theta^\nu_{23}^{\text{CKM}}$ from the up quark sector which is almost as significant as the contribution coming from the down-type quarks. The Cabibbo angle is still approximately $\theta^\nu_C \sim \theta^d_{12} \sim \epsilon$. 

8
4 Charged lepton corrections to lepton mixing

We have presented mixing angles which rotate the charged leptons and neutrino fields between the mass and flavour bases, however these individual rotations are not what experiments observe. It is the combination of the two mixing matrices that appears in the electroweak coupling to the $W$ boson, giving the physical mixing matrix

$$U_{\text{PMNS}} = U_{eL} U_{\nu L}^\dagger.$$  \hspace{1cm} (18)

Here it is understood that for Lagrangians in the left right convention, $U_{eL}$ acts as $U_{eL} m_e m_e^\dagger U_{eL}^\dagger = (m_e^{\text{diag}})^2$ and $U_{\nu L}$ as $U_{\nu L} m_\nu U_{\nu L}^T = m_\nu^{\text{diag}}$. While the neutrino sector predicts exact TM mixing, the effect of the charged lepton corrections generates an experimentally detectable deviation from this in the physical parameters. In this section we ignore RG and CN corrections and focus only on the CL corrections.

There are (at least) two popular ways to parameterise the PMNS matrix; firstly one can write $U_{\text{PMNS}} = U_{12}U_{13}U_{23}$ with

$$U_{12} = \begin{pmatrix} c_{12} & s_{12} \exp(-i\delta_{12}) & 0 \\ -s_{12} \exp(i\delta_{12}) & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$  \hspace{1cm} (19)

$$U_{13} = \begin{pmatrix} c_{13} & 0 & s_{13} \exp(-i\delta_{13}) \\ 0 & 1 & 0 \\ -s_{13} \exp(i\delta_{13}) & 0 & c_{13} \end{pmatrix},$$  \hspace{1cm} (20)

$$U_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \exp(-i\delta_{23}) \\ 0 & -s_{23} \exp(i\delta_{23}) & c_{23} \end{pmatrix}.$$  \hspace{1cm} (21)

Here, $c_{ij}$ and $s_{ij}$ stand for $\cos \theta_{ij}$ and $\sin \theta_{ij}$ respectively and the 3 remaining unphysical phases have been rotated away, see e.g. [25]. Individual rotation matrices $U_{eL}$ and $U_{\nu L}$ are parameterised in the same way with relevant superscripts. The second parameterisation is that used by the PDG [16], with a Dirac phase $\delta$ and Majorana phases $\alpha_2$ and $\alpha_3$; this is constructed as $U_{\text{PMNS}}^{\text{PDG}} = R_{23} U_{13}^{\text{PDG}} R_{12} P$ where the $R_{ij}$ are standard orthogonal rotations, $U_{13}^{\text{PDG}} = U_{13} (\delta_{13} = \delta)$ and $P = \text{diag}(1, e^{i\alpha_2}, e^{i\alpha_3})$. A comparison of the two parameterisations, after performing a global phase redefinition to absorb remaining unphysical phases and obtain consistency with the convention stated in the introduction, shows that [25]

$$\delta = \delta_{13} - \delta_{23} - \delta_{12},$$  \hspace{1cm} (22)

$$\alpha_2 = -2\delta_{12},$$  \hspace{1cm} (23)

$$\alpha_3 = -2(\delta_{12} + \delta_{23}).$$  \hspace{1cm} (24)
We can now write the parameters of $U_{PMNS}$ in terms of the neutrino mixing parameters, with perturbative corrections from the charged lepton sector as follows \[25\] (neglecting $\theta_{e13}$ and $\theta_{e23}$ as they are small),

\[
\begin{align*}
\sin \theta_{23} \exp \left( -i \delta_{23} \right) & \approx \sin \nu_{23} \exp \left( -i \delta_{\nu_{23}} \right), \\
\sin \theta_{13} \exp \left( -i \delta_{13} \right) & \approx \theta_{13} \sin \nu_{23} \exp \left( -i \left( \delta_{\nu_{23}} + \delta_{e13} \right) \right), \\
\sin \theta_{12} \exp \left( -i \delta_{12} \right) & \approx \theta_{12} \sin \nu_{23} \exp \left( -i \delta_{\nu_{23}} \right).
\end{align*}
\]

The dominance of the first term in Eq. (27) allows us to approximate $\delta_{12} \approx \delta_{\nu_{12}}$, while Eq. (25) gives directly $\delta_{23} \approx \delta_{\nu_{23}}$. The phase $\delta_{13}$ requires a more careful treatment, since the first term of Eq. (26) is larger but not dominant enough to drop the second term. It turns out to be possible to write

\[
\delta_{13} \approx \delta_{\nu_{13}} - \theta_{e12} \sin \left( \delta_{\nu_{23}} - \delta_{\nu_{13}} + \delta_{e12} \right),
\]

assuming that $\theta_{e12} \sin \nu_{23}$ is small.\[5\] Then the physical Dirac oscillation phase can be approximated by

\[
\delta \approx \delta_{\nu_{13}} - \delta_{\nu_{23}} - \delta_{\nu_{12}} - \theta_{e12} \sin \left( \delta_{\nu_{23}} - \delta_{\nu_{13}} + \delta_{e12} \right).
\]

Turning to the resulting mixing angles, we first observe that the TM mixing of the neutrino sector must necessarily be a small deviation from TB mixing. We can therefore express our results using the neutrino TB deviation parameters \[14\],

\[
\sin \theta_{12}^\nu = \frac{1}{\sqrt{3}} (1 + s^\nu), \quad \sin \theta_{23}^\nu = \frac{1}{\sqrt{2}} (1 + a^\nu), \quad \sin \theta_{13}^\nu = \frac{\nu^\nu}{/\sqrt{2}},
\]

where here these parameters refer only to the neutrino sector. In terms of angles and phases, using Eqs. (25)-(27) (see, e.g. \[10\] for a discussion of this procedure), we can then write the TB deviation parameters for the complete lepton mixing in terms of the TB deviations parameters in the neutrino sector and the charged lepton corrections as,

\[
\begin{align*}
a & \approx a^\nu, \\
r & \approx |\nu^\nu \exp \left( -i \delta_{\nu_{13}} \right) - \theta_{e12}^\nu \exp \left( -i \left( \delta_{\nu_{23}} + \delta_{e12} \right) \right)|, \\
s & \approx s^\nu - \theta_{e12}^\nu \cos \left( \delta_{\nu_{12}} - \delta_{\nu_{13}} \right).
\end{align*}
\]

\[4\]We note that in order to derive these equations consistently to first order, the Majorana phases from Eqs. (22)-(24) must be redefined by a correction of order $\theta_{13}^\nu$; this is however only a subtlety in the derivation and therefore we do not explicitly demonstrate this redefinition, merely point it out to the reader.

\[5\]With $\theta_{e12} \approx \theta_{e13} \approx 0.15$ we obtain a numerical value of $\frac{\theta_{\nu_{13}}}{\theta_{\nu_{13}}} \approx \frac{1}{3}$.  

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With the neutrino mixing being of TM form as given in Eq. (4), the deviation parameters of the neutrino sector can be shown to satisfy, see [14,17,26],
\[ s^\nu = 0 \text{ and } a^\nu \approx -\frac{r^\nu}{2} \cos \delta^\nu. \]
Using this and the fact that \( \theta_{e12}^\nu \sim \theta_C^3 \) and Eq. (29), the above equations for the tri-bimaximal deviation parameters may be further simplified to first order as
\[ a \approx -\frac{r^\nu}{2} \cos \delta, \tag{34} \]
\[ r \approx r^\nu - \frac{\theta_C}{3} \cos (\delta^\nu_{23} - \delta^\nu_{13} + \delta^\nu_{12}), \tag{35} \]
\[ s \approx -\frac{\theta_C}{3} \cos (\delta^\nu_{12} - \delta^\nu_{12}), \tag{36} \]
again assuming that \( \frac{\theta_{e12}^\nu}{\theta_{13}^\nu} \sim \frac{\theta_C}{3} \) is small. In the limit that charged lepton corrections are switched off, the above results reduce to the usual TM sum rules [14,17,26], \( s \approx 0 \) and \( a \approx -\frac{r}{2} \cos \delta \). In the limit that the neutrino mixing angle \( \theta_{13}^\nu \) is switched off the above results reduce to the usual TB sum rules [27], \( s \approx r \cos \delta \) where \( r \approx \theta_C/3 \) and \( \delta \approx \delta_{12}^\nu - \delta_{12}^e \).

The results in Eqs. (34)-(36) imply the relatively simple sum rule bounds:
\[ |s| \leq \frac{\theta_C}{3}, \tag{37} \]
\[ |a| \leq \frac{1}{2} \left( r + \frac{\theta_C}{3} \right) \cos \delta, \tag{38} \]
where, again, \( r, s, a \) are the tri-bimaximal deviation parameters, in particular \( r \approx \sqrt{2} \theta_{13} \), \( \delta \) is the CP violating oscillation phase, and \( \theta_C \) is the Cabibbo angle. We emphasise that these bounds do not include RG and CN corrections, which however are expected to be rather small for the case of hierarchical neutrino masses. For example, assuming \( \theta_{13} \sim 8^\circ \) we find \( r \approx 0.2 \), and using \( \theta_C/3 \approx 0.075 \) these bounds become \( |s| \leq 0.075 \) and \( |a| < 0.14 \cos \delta \). The present approximate limits from the global fit \( |a| < 0.08, -0.06 < s < 0 \) quoted in Eq. (3) are nicely consistent with these sum rule bounds.

5 Conclusions

Recent T2K, MINOS and Double CHOOZ results, together with global fits of mixing parameters, indicate a sizeable reactor angle \( \theta_{13} \sim 8^\circ \) which, if confirmed, would rule out TB lepton mixing. On the other hand, the TB predictions \( \sin \theta_{23} = 1/\sqrt{2} \) and \( \sin \theta_{12} = 1/\sqrt{3} \) remain in agreement with global fits and continue to provide tantalising hints for an underlying Family Symmetry. For example, an \( A_4 \) family symmetry model including additional flavons in the \( 1' \) and \( 1'' \) representations leads to TM neutrino mixing which maintains the prediction \( \sin \theta_{12} \approx 1/\sqrt{3} \), at least approximately, while allowing
an arbitrarily large reactor angle. Indeed, as discussed in a previous paper by two of us, the problem in this model is in explaining why the reactor angle should be smaller than the atmospheric or solar angles, which follows from the fact that the additional flavons would be expected to have VEVs of the same order as the other TB flavon VEVs, with all undetermined coefficients being of order unity. However, apart from this drawback, such a model provides a simple example of a Family Symmetry model with a non-zero reactor angle.

In this paper we have proposed a SUSY GUT of Flavour with a non-zero $\theta_{13}$ based on $A_4$ Family Symmetry with additional flavons in the $1'$ and $1''$ representations, and an $SU(5)$ GUT group. The model involves an additional continuous $U(1)$ family symmetry as well as three discrete symmetries designed to control the operator structure of the model. All flavon representations of $A_4$ are populated, and the main flavon content of the quark sector is copied from the neutrino sector. The vacuum alignment is obtained using the conventional $F$-term mechanism. NLO terms to the mass matrices are negligible, demonstrating the stability of the LO matrix textures. The resulting model exhibits TM mixing in the neutrino sector, with the physical lepton mixing involving charged lepton corrections, which in turn are related to quark mixing angles. In particular, the model involves a Georgi-Jarlskog relation, leading to bounds on the TB deviation parameters $|s| \leq \frac{\Delta \theta}{3}$, $|a| \leq \frac{1}{2}(r + \frac{\Delta \theta}{3})|\cos \delta|$, up to RG and CN corrections, which are in good agreement with current global fits. The considered model shows that it is possible to accommodate $\theta_{13} \sim 8^\circ$, within a SUSY GUT of Favour which relates quark and lepton masses and mixing angles, while continuing to provide an explanation for the TB nature of the solar and atmospheric lepton mixing angles.

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**Appendix A: Vacuum alignment**

In order that the flavon fields obtain the alignment presented in Eq. (5), their potential must be minimised in the correct way. We follow the method of [17] very closely. The leading order contributions to the driving superpotential are:

$$W_0 = \varphi_T^0 (g_1 \langle \sigma \rangle \varphi_T + g_2 \varphi_T \varphi_T) + \varphi_S^0 (g_4 \varphi_S \varphi_S + g_5 \varphi_s \xi + g_6 \varphi_s \xi' + g_7 \varphi_s \xi'') + \xi^0 (g_8 \varphi_s \varphi_S + g_9 \xi + g_{10} \xi') .$$

Here, $g_1 \langle \sigma \rangle = M$ which appears in the vacuum alignment of [17]; this is required since $\varphi_T$ is charged under the auxiliary symmetries and so the original structure $\varphi_T^0 (M \varphi_T + \varphi_T \varphi_T)$
that drives the $\varphi_T$ alignment cannot be used. Minimising with respect to $\varphi_T$ gives

$$\langle \varphi_T \rangle = v_T \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad v_T = -\frac{g_1 \langle \sigma \rangle}{2g_2}.$$  (40)

The conditions from $\varphi_S^0$ and $\xi^0$ are

$$2g_3 \begin{pmatrix} s_1^2 - s_2 s_3 \\ s_2^2 - s_3 s_1 \\ s_3^2 - s_1 s_2 \end{pmatrix} + g_4 u \begin{pmatrix} s_1 \\ s_3 \\ s_2 \end{pmatrix} + g_4' u' \begin{pmatrix} s_3 \\ s_1 \\ s_2 \end{pmatrix} + g_4'' u'' \begin{pmatrix} s_2 \\ s_1 \\ s_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$  (41)

$$g_5 (s_1^2 + 2s_2 s_3) + g_6 u^2 + g_7 u' u'' = 0.$$  (42)

Here, $\langle \varphi_S \rangle = s_i$, $\langle \xi \rangle = u$, $\langle \xi' \rangle = u'$ and $\langle \xi'' \rangle = u''$. The solutions to these equations are

$$\langle \varphi_S \rangle = v_S \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad v_S^2 = -\frac{g_6 u^2 + g_7 u' u''}{3g_5}, \quad u = -\frac{g_4' u' + g_4'' u''}{g_4}.$$  (43)

As in [19], we assume the undetermined singlets obtain their VEVs as a result of their soft mass parameters $m_s^2$ (where $s$ stands for singlet) being driven negative in some portion of parameter space.

**Appendix B: Higher order operators**

There are many higher order corrections to the mass matrices presented in Section 3 of this paper; most of these give negligible contributions to masses and mixings. In Tables 3 and 4 we give suppressions and example terms of the NLO operators for each sector; it can be seen that none of these will change the LO results significantly.
Table 3: NLO corrections in the model. The first column shows each basic term that exists in the neutrino, down quark (and charged lepton) Yukawa superpotential, as specified in the second column. A bunch of flavons is appended to these basic terms to obtain the complete term invariant under the symmetries. In the third column we give an example of such a bunch of flavons at NLO and the order of its contribution, to be compared to the LO contribution given in the final column. Note that in the terms contributing to $M_d$, there is a flavon $\varphi_T$ already present in the basic term. It is furthermore not specified whether the LO term comes from an $H_5$ or an $H_{45}$; the reader may refer back to Eq. (7) if required.
Table 4: NLO corrections in the model. The first column shows each basic term that exists in the up quark Yukawa and vacuum alignment sectors, as specified in the second column. A bunch of flavons is appended to these basic terms to obtain the complete term invariant under the symmetries. In the third column we give an example of such a bunch of flavons at NLO and the order of its contribution, to be compared to the LO contribution given in the final column. The notation $\tilde{\epsilon}$ is simply used to denote that we are in a different sector to $\epsilon$ or $\overline{\epsilon}$.

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