The Non-perturbative Canonical Quantization of the N = 1 Supergravity

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ABSTRACT

The non-perturbative canonical quantization of the N = 1 supergravity with the non-zero cosmological constant is studied using the Ahtekar formalism. A semi-classical wave function is obtained and it has the form of the exponential of the N = 1 supersymmetric extension of the Chern-Simons functional. The N = 1 supergravity in the Robertson-Walker universe is also examined and some analytic solutions are obtained.

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1 Introduction

Recently several attempts have been made at constructing quantum theory of gravity and proved successful in the case of low dimensional gravity theories. In particular, a major progress has been achieved in the 2-dim ensional quantum gravity based on the methods of the conformal eld theory. In the 3-dim ensional case, we also have various approaches to the quantum gravity such as the Chern-Simons gauge theory and the Turaev-Viro theory, and so on. The 4-dim ensional quantum gravity has, however, proved too di cult so far to be constructed completely. The main di culties are that it is unrenormalizable and highly non-linear. While the 3-dim gravity is also unrenormalizable, it has no dynamical degrees of freedom and we can formulate it as a topological eld theory. On the other hand, the 4-dim gravity has the gravitons and can’t be described by some topological eld theory straightforwardly.

There are several approaches to overcome these difficulties of the 4-dim quantum gravity. One of them is to nd the renormalizable theory, which contains the Einstein gravity in a suitable limit. The typical one of this approach is the superstring theory. The stand point of this approach is to modify the Einstein gravity such that it has more tractable perturbative behaviour. On the other hand, we can take the position that the quantum gravity should be de ned non-perturbatively. There are two well-known attempts to construct the non-perturbative quantum gravity. One is the lattice gravity and the other is the canonical quantization by the ADM formalism [1].

The lattice gravity has been successful in 2 or 3 dimension by means of the random triangulation method. The 4-dim ensional lattice gravity is now in progress. But it has not been ascertained in 4 dimension whether we can take a continuum limit. The ADM canonical formalism has its own di culty. As is well-known, the Hamiltonian of the Einstein-Hilbert action reduces to the constraints in the ADM formalism. These constraints are the complicated non-polynomial s of the canonical variables. So the canonical quantization can be carried out only in the cases in which there are few degrees of freedom.

Recently Ashtekar has presented a new formalism of the Einstein gravity [2][3]. In this formalism, all the constraints of the gravity are simple polynomials of the canonical variables, so we would expect to solve the non-perturbative quantum gravity. In fact some wave functions and the physical states of the gravity are derived [4][5]. The Ashtekar formalism can be extended to the N=1,2 supergravities and the constraints are again polynomials of the canonical variables [6][8].

The aim of this paper is to consider the non-perturbative canonical quantization of the N=1 supergravity by the Ashtekar formalism. While the ADM formalism of the N=1 supergravity has been investigated in Ref.[7], it is extremely more complicated than that of the
ordinary gravity, so it seems unsuitable for quantization. As we will see soon, the Ashtekar formalism gives us a powerful aid for our aim. Especially we give attention to the case that the theory has the non-zero cosmological constant. The case of the Einstein gravity in the same condition have been considered by Kodama [9].

This paper is organized as follows. In section 2, we give the brief review of the Ashtekar formalism of the N=1 supergravity. After the 3+1 decomposition of the action, we get the constraints of the supergravity. We solve these constraints semi-classically and obtain the holomorphic wave function of the N=1 supergravity. We consider the special case that the metric is given by the Robertson-Walker metric in section 3. We solve the Wheeler-De Witt equation and again obtain the semi-classical wave function of the universe. In section 4, we derive the equations which determine the classical limit of the quantum Robertson-Walker universe and obtain the several analytic solutions. We examine these equations by the numerical simulation. Section 5 is devoted to the discussion. The notations and the formulas used in this paper are given in the appendix.

2 The Ashtekar Formalism and the WKB Wave Function of N = 1 Supergravity

In this section we present the Ashtekar formalism of the N=1 supergravity and solve the constraints. The Ashtekar formalism of the N=1 supergravity has been given first by Jacobson [6] and reformulated in more elegant form in ref [1].

From now on we use the method of the 2-form gravity [3]. We represent the left- and the right-spinor indices as $A;B;C;\ldots$ and $A^0;C^0;\ldots$, respectively. $e^A_A$ and $A^0$ express the vierbein, the left- and the right-component of the gravitino, respectively. We define the 2-form fields $A^B$ and $A^A$ as

\begin{align}
A^B &= e^{A^0}_A \wedge e^{A^B}_A; \\
A &= e^{A^0}_A \wedge A^0.
\end{align}

Now the chiral Lagrangian of the N=1 supergravity is given as [3]:

\begin{align}
\mathcal{L} &= \mathcal{L}_0 + \mathcal{L}_{\text{cosm}}; \\
\mathcal{L}_0 &= A^B \wedge R^B_{AB} + A^A \wedge D_A \frac{1}{2} A_{BCD} A^B \wedge C_{AB} C^B_{A}; \\
\mathcal{L}_{\text{cosm}} &= \frac{g^2}{6} A^B \wedge A^B + \frac{1}{2} g A^B \wedge A^B \wedge B \frac{g}{6} A^B \wedge A_A^A;
\end{align}

where $R^B_{AB}$ is the curvature of the anti-self-dual part of SO (3;1) connection $!_{AB}$, $D$ is the covariant derivative with respect to $!_{AB}$, and $g$ and $g$ are the real constants. When we
add cosmological term \( \frac{\mu}{6} A^A B^A \) to Lagrangian, we must add other terms appearing in \( L_{\text{cosm}} \). can be regarded as the gravitational constant. Cosmological constant is given by \( g^2 \). \( A_{BCD} \) and \( A_{ABC} \) are the Lagrange multipliers by which we require the algebraic constraints

\[
\begin{align*}
(AB \wedge CD) &= 0; \quad (2.4) \\
(AB \wedge C) &= 0; \quad (2.5)
\end{align*}
\]

where the indices between \( A' \) and \( A'' \) are completely symmetrized, and these equations guarantee the decomposition (2.1) and (2.3).

In this paper we consider \( N=1 \) supergravity with non-zero cosmological term.

The Lagrangian (2.3) has the left- and the right local supersymmetries. The left supersymmetry transformation is given by:

\[
\begin{align*}
L_{AB} &= (A \wedge B); \\
L_{AB}^! &= g (A \wedge B); \\
L_A &= D_A; \\
L_A^! &= g (A \wedge B); \\
L_{ABC} &= ABCD; \\
L_{ABCD} &= 2 g (ABC D); \quad (2.6)
\end{align*}
\]

where \( A \) is the fermionic 0-form parameter. The right transformation has the peculiar form:

\[
\begin{align*}
R_{AB} &= (A \wedge B); \\
R_{AB}^! &= (ABC)^!; \\
R_A &= \frac{g}{3} A; \\
R_A^! &= D_A; \\
R_{ABC} &= 0; \\
R_{ABCD} &= 0; \quad (2.7)
\end{align*}
\]

where the parameter \( A \) is the fermionic 1-form parameter which satisfies the algebraic constraint

\[
(AB \wedge C) = 0; \quad (2.8)
\]

which can be solved on shell as

\[
A = e_A^0 a_0: \quad (2.9)
\]

Now we rewrite the Lagrangian (2.3) in the canonical form. First we define the variables \( ^{ij}A^B \) and \( \sim A \) as

\[
^{ij}A^B = \frac{1}{2} ijk A^B \quad (2.10)
\]
The algebraic constraints (2.4), (2.5) can be written as

\[ \begin{align} 
\left\langle A B \right| \sim_{CD} \rangle + \left\langle C \right| \sim_{AB} \rangle = 0; \\
\left\langle A B \right| \sim_{C} \rangle + \left\langle i_{AB} \right| C \rangle = 0; 
\end{align} \tag{2.12} \tag{2.13} \]

As is shown in the appendix, these equations can be solved as

\[ \begin{align} 
\left\langle A B \right| 0 \rangle = \frac{1}{2} \left\langle ij_{k} \right| \sim_{C} \rangle \sim^{jA}_{kB} + 2N^{j}_{kAB} ; \\
\left\langle A \right| 0 \rangle = \left\langle ij_{k} \right| \sim_{C} \rangle \sim^{jA}_{kB} + N^{j}_{kA} + i_{jk} \left\langle A B \right| \sim^{kB}_{C} \rangle M_{C} ; 
\end{align} \tag{2.14} \tag{2.15} \]

where \( M_{A} \) is the fermionic field of the weight -1, and \( N \) and \( N^{i} \) correspond to the lapse function and the shift vector in the ADM formalism, respectively.

The Lagrangian rewritten in the canonical form is

\[ \begin{align*} 
iL & = \sim^{jA}_{iJAB} + \sim^{jA}_{iA} \\
& \quad + \left\langle 0AB \right| G^{AB} \left( \left\langle 0 \right| A \right) + \frac{1}{2} \left\langle 0 \right| N^{i} \rangle H^{i} \rangle. 
\end{align*} \tag{2.16} \]

The coefficients \( 0_{AB} \), \( 0_{iA} \), \( M_{A} \), \( N \), and \( N^{i} \) are the Lagrange multipliers and the constraints are given by

\[ \begin{align} 
G^{AB} & = D_{i} \sim^{jA}_{iJAB} \left\langle 0_{AB} \rangle i; \\
L^{A} & = D_{i} \sim^{jA}_{iJAB} \left\langle 0_{AB} \rangle i; \\
R^{A} & = \sim^{jA}_{C} \sim^{jC}_{B} \left\langle 0_{B} \rangle iJ + \frac{g}{3} \sim^{i}_{jk} \sim^{k}_{B} ; \\
H & = \sim^{jA}_{C} \sim^{jC}_{B} R_{iJAB} \left( \frac{g^{2}}{3} \sim^{i}_{jk} \sim^{k}_{AB} + \left\langle 0_{iA} \right| B \rangle lj \\
& \quad + 2 \sim^{jA}_{B} \sim^{jB}_{B} \left\langle 0_{A} \rangle iJ + \frac{g}{3} \sim^{i}_{jk} \sim^{k}_{A} ; \\
H_{i} & = \sim^{jA}_{B} R_{iJAB} \left( \frac{g^{2}}{3} \sim^{i}_{jk} \sim^{k}_{AB} + \left\langle 0_{iA} \right| B \rangle lj \\
& \quad + \sim^{jA}_{B} \left\langle 0_{A} \rangle iJ + \frac{g}{3} \sim^{i}_{jk} \sim^{k}_{A} 
\end{align} \tag{2.17} \tag{2.18} \tag{2.19} \tag{2.20} \tag{2.21} \]

The Poisson brackets between the canonical variables\(^1\) are

\[ \begin{align} 
\delta^{n}_{\sim_{AB}} \left( x; t \right) & = \sim^{jC}_{D} \left( y; t \right) = i_{i}^{jC}_{j} \left( \sim^{jA}_{i} \left( \sim^{jB}_{i} \right) \left( x, y \right) \right) ; \\
\delta^{n}_{\sim_{A}} \left( x; t \right) & = \sim^{jB}_{j} \left( y; t \right) = i_{i}^{jB}_{j} \left( \sim^{jA}_{i} \left( x, y \right) \right). 
\end{align} \tag{2.22} \tag{2.23} \]

\(^{1}\)We use the left derivatives for the fermionic fields.
\( G^{AB}, L^A, R^A, H, \) and \( H \) are the generators of the local Lorentz transformation, the left- and the right supersymmetry transformations, the time evolution, and the 3-dimensional duality, respectively. These constraints are written in the polynomial of the canonical variables and form the closed Poisson algebra under (2.22) and (2.23) [3].

Now we start the quantization. The (anti-)commutation relations are

\[
\begin{align*}
\{ i_{AB} (x; t) ; i^{CD}_n (y; t) \} & = \frac{\hbar}{n} \left( \frac{3}{i_{AB}} \right)_{(3)} (x - y); \\
\{ i_A (x; t) ; i^B (y; t) \} & = \frac{\hbar}{i_A} \left( \frac{3}{i_A} \right)_{(3)} (x - y);
\end{align*}
\]

(2.24)

(2.25)

We choose the representation in which the variables \( i_{AB} \) and \( i_A \) are diagonalized.

\[
\begin{align*}
t_{AB} & = \frac{1}{i_{AB}}; \\
t_A & = \frac{1}{i_A}.
\end{align*}
\]

(2.26)

(2.27)

How we should order the operators is the serious problem in the quantum gravity. While there is some discussion to select some special ordering, we have no precise answer to this problem a priori [4]. So now we avoid this problem and simply \( x \) the operator ordering as in (2.17) - (2.21).

We define two 1-form fields \( i_{AB} \) and \( i_A \) as

\[
\begin{align*}
i_{AB} & = i_{AB} dx^i; \\
i_A & = i_A dx^i;
\end{align*}
\]

(2.28)

(2.29)

where the index \( i \) is the space index, \( i = 1, 2, 3 \). Then as we can easily see, the semi-classical solution for the constraints (2.17) - (2.21) are given by

\[
[ i_{AB} ; i_A ] = \exp \left( \frac{3}{\hbar} \sum_i i_{AB} \wedge d i_{AB} + \frac{2}{3} i_{AC} \wedge i^C_B \wedge i_{AB} g^A \wedge D_A \right): (2.30)
\]

We call this the holomorphic wave function of the \( N = 1 \) supergravity. In the Einstein gravity, this type of the wave function is given in Ref[3], and has the form of the exponential of the Chern-Simons functional. In the case of the \( N = 1 \) supergravity, the part of the Chern-Simons functional is replaced by its supersymmetric extension; in fact, the functional

\[
W = \sum_i i_{AB} \wedge d i_{AB} + \frac{2}{3} i_{AC} \wedge i^C_B \wedge i_{AB} g^A \wedge D_A
\]

(2.31)

is invariant under the local supersymmetry transformation

\[
\begin{align*}
i_{AB} & = g_{(A} B); \\
i_A & = D_A;
\end{align*}
\]

(2.32)

(2.33)

where the covariant derivative \( D \) is that corresponding to the connection [2.23].
3 The Robertson–Walker Universe

In this section we consider the special case that the space-time metric is given by the Robertson-Walker metric and re-examine the discussion of the last section.

The Robertson-Walker metric is given by

\[
\text{d}s^2 = N^2 \text{d}t^2 + \frac{1}{8} e^{2A} \text{d}A^B \text{d}B^A;
\] (3.1)

where the 1-form \( A_B \) on the 3-dim space (see appendix) satisfies the structure equation

\[
\text{d} A_B = A^C \wedge C_B;
\] (3.2)

and \( N = N(t) \) and \( \text{d}A = \text{d}t \) depend only on the time. In this metric the 3-dim space has the topology of \( S^3 \).

Now we suppose that the universe is homogeneous and isotropic, and decompose the variables appearing in the theory into the parts which depend only on the time and which depend only on the space coordinates through \( A_B \):

\[
!_{AB} = i!_{AB};
\]

\[
\sim_{AB} = \frac{1}{24V} j j_{AB}^B + \frac{i}{8V} A_B^C B_C^B;
\]

\[
_{A} = \frac{1}{24V} i A_B^B + \frac{1}{8V} A_B^C C_B^C;
\]

\[
\sim_{A} = \frac{1}{6V} j j_{AB}^B + \frac{1}{8V} A_B^C C_B^B;
\] (3.3)

where \(!, \sim, A, A_B, A_C, A\) and \( A_B C_A \) are the variables depending only on the time. We define the dual basis \( i_{AB} \) by \( \text{d}A_B^B = 8 i_{AB} \), and \( j j \) is equal to \( \det (i) \), where \( i_{AB} = \frac{1}{i} \). Here we normalize the volume of \( S^3 \) as \( V = \int d^3x j j = \frac{2}{9} \). The variable \( A \) is related to \( !_{AB} \) as

\[
= 12V e^2
\] (3.4)

We assume that all the Lagrange multipliers \( 0_{AB}; 0_A; M_A; N \); and \( N \) (where \( N = \frac{1}{8V} \) and \( M_A = \frac{1}{8V} M_A \). See appendix, ) depend only on the time. Then the general solution (2.30) is rewritten as

\[
[!; A_A, A_B C_A ] = e^{i S};
\] (3.5)

\[
S = \frac{12}{g^2} \frac{1}{3} \frac{i}{2} \frac{1}{4} g^0(1 - i) A + \frac{1}{4} g^0(1 + 2i) A_B C_A B_C;
\] (3.6)

After integrating out the spatial coordinates, the Lagrangian has the following form:

\[
L = \frac{1}{2} + \frac{A_A}{A} + \frac{A_B C_A B_C}{A};
\] (3.7)
The explicit form of the Hamiltonian $H$ will be given in the next section after some discussion. The sets of the canonical variables are $f!; g; A^A; \ldots$ and $ABC; ABC^O$. The Poisson brackets between the canonical variables are

\begin{align}
[f!; g] & = 1; \\
[A^B; B] & = B^A; \\
[ABC; DEF^O] & = D^E F^{AB C};
\end{align}

In the quantization, we replace the Poisson brackets (3.8)–(3.10) with the canonical (anti-)commutation relations:

\begin{align}
[f!; ] & = i; \\
[A^B; ] & = i_A^B; \\
[ABC; DEF^O] & = i^{D E F}_{ABC};
\end{align}

We choose the representation in which the variables $f!, A$, and $ABC$ are diagonalized:

\begin{align}
[!; A] & = e^{iS}; \\
S & = \frac{12}{g^2} \frac{1}{3} \frac{1}{2} 2(1 - 2i) + \frac{1}{4} g^0(0 - i) A^A; \tag{3.19}
\end{align}

Using these variables, the constraint $L^A$ is rewritten as

\begin{align}
L^A & = \frac{6}{6V_1} j j(2! A^A 0 A^A); \tag{3.17}
\end{align}

The function (3.5) doesn't satisfy the constraint (3.17) in general. To make (3.5) have the left supersymmetry, we must set $ABC = 0$ and $ABC^O = 0$. Then the function

\begin{align}
[!; A] & = e^{iS}; \\
S & = \frac{12}{g^2} \frac{1}{3} \frac{1}{2} 2(1 - 2i) + \frac{1}{4} g^0(0 - i) A^A; \tag{3.19}
\end{align}

satisfies the constraint (3.17) and all the remaining constraints:

\begin{align}
G^{AB} & = \frac{i}{6V} j j(A^B + B^A); \tag{3.20} \\
R^A & = \frac{1}{12V^2} \frac{1}{6V} j j 2(1 - !) A^A + \frac{g^0}{3} A^A; \tag{3.21} \\
H_{1} & = \frac{2}{3V} j j (2(1 - !) A^A + \frac{g^0}{3} A^A); \tag{3.22} \\
H & = \frac{2}{3V^2} j j (1 - !)^2 + \frac{g^0}{12} \frac{1}{4} g^0 A^A + A^A 2(1 - !) A^A + \frac{g^0}{3} A^A \tag{3.23}
\end{align}
Thus we obtain the semi-classical wave function of the $N=1$ supergravity in the Robertson-Walker universe.

4 The Classical Limit of the Quantum Universe

Next we consider what classical universe is involved in the semi-classical wave function $\Psi$. Note that (3.18) has the form of the WKB wave function. In the WKB approximation, $S$ is the classical principal function of the dynamical system. So all the informations about the classical universe involved in (3.18) will be derived from (3.19). Since $S$ is the principal function, it must satisfy the Hamilton-Jacobi equations:

$$\frac{\partial S}{\partial t} + H (!; ; A; ^A) = 0; \quad (4.1)$$

$$\frac{\partial S}{\partial !} = \frac{12}{g^2} !^2 i! + \frac{1}{4} g^0 A ^A; \quad (4.2)$$

$$A = \frac{\partial S}{\partial A} = \frac{6}{g^0} (! i) ^A; \quad (4.3)$$

The Hamiltonian $H$ vanishes under (4.2) and (4.3) because all the constraints vanish under these two relations. Therefore (4.3) is derived from (4.2)-(4.3) and the fact that $S$ doesn’t depend on the time explicitly. So we obtain the result that the classical universe contained in (3.18) obeys the equations (4.2) and (4.3).

The constraints contained in the $N=1$ supergravity are all first class constraints and the Hamiltonian $H$ is the linear combination of these constraints. Therefore all the multipliers are left undetermined. To fix these multipliers and introduce the time evolution, we must choose a gauge. Here we take the following gauge:

$$!_{0A} = 0; \quad 0A = 0; \quad M_A = 0; \quad N = 1; \quad N^i = 0; \quad (4.4)$$

In this gauge the Hamiltonian is

$$H = 4 \frac{p}{12V} \frac{1}{12} + \frac{g^0}{12} \frac{1}{12} + \frac{1}{4} g^0 A ^A + 4 \frac{p}{12V} \frac{1}{12} 2(i! i) ^A + \frac{g^0}{3} A ^A; \quad (4.5)$$

Of course the classical solutions must obey the Hamilton equations:

$$\frac{d!}{dt} = f!; \quad H = \frac{g^2}{3} \frac{p}{12V} \frac{1}{12}; \quad (4.6)$$

$$\frac{dA}{dt} = f; \quad H = g \frac{p}{12V} \frac{1}{12} (\cdot i) ^A; \quad (4.7)$$

$$\frac{d}{dt} = f; \quad H = 4 \frac{p}{12V} \frac{1}{2} (2! i) + 2 \frac{1}{2} A ^A; \quad (4.8)$$

$$\frac{d^A}{dt} = f; \quad 0 A^o; \quad H = 4 \frac{p}{12V} \frac{1}{2} \frac{1}{2} g^0 A + 2(i! i) ^A; \quad (4.9)$$
where we use the relations (4.3)–(4.7). The equations (4.3) and (4.7) can be derived from the equations (4.2), (4.3), (4.6) and (4.7).

The reason why we should set \( A_B C = A_B C = 0 \) can be understood from another point of view. We can derive more general equations corresponding to (4.2), (4.3), (4.6) and (4.7) from (2.12), (2.22), (2.32) and (4.4):

\[
\begin{align*}
0 = \mp \frac{g^2}{3} i_j k^{k}_A & ; \\
\mp R_{AB} i_j & = g^k A_B^{k} + g i_A B_{ij} = 0 ; \\
\frac{\partial t}{\partial t} = \frac{ig^2}{6g^2} j^{k}_A k^{k}_C & ; \\
\frac{\theta}{\partial t} = \frac{i}{3g^2} i_A j^{k}_B & .
\end{align*}
\]

All the remaining Euler-Lagrange equations are derived from the above equations. In the case of the Robertson-Walker universe, (4.11) gives the following three equations by the spinorial decomposition:

\[
\begin{align*}
0 = \frac{1}{3} F_{A B} E_{C D} + (A_B C D) & ; \\
0 = (A_B C D) & ; \\
0 = \frac{1}{12} i ! (g^2)^{0} A_B^{A} + \frac{1}{4} g^{0} A_B^{A} + \frac{1}{4} g^{0} A_B^{A} & .
\end{align*}
\]

The first and the second algebraic constraints can be derived from neither the Euler-Lagrange equations of (3.7) nor the Hamilton-Jacobi equations of (3.8). So when we consider the mini-superspace of the Robertson-Walker universe, these extra constraints must not appear and we must set \( A_B C = A_B C = 0 \). Of course \( A_B C \) and \( A_B C \) may be non-zero in the general case.

In our gauge the line element is

\[
12V ds^2 = d^2 + \frac{1}{8} d^2 ;
\]

where \( d^2 \) is \( A_B A_B \) and \( d = \sqrt{12V} dt \) must be real when (4.17) represents the Lorentzian or the Euclidean universe. Then there exist four cases:

\[
\begin{align*}
\text{case 1} & : \text{real}, \quad < 0 ; \quad \text{Euclidean} , \\
\text{case 2} & : \text{imaginary}, \quad > 0 ; \\
\text{case 3} & : \text{real}, \quad > 0 ; \\
\text{case 4} & : \text{imaginary}, \quad < 0 ; \\
\end{align*}
\]

We examine these four cases respectively. Using (4.3) and (4.3), the equation (4.8) is rewritten as

\[
\frac{d^2}{d} = 4 \frac{1}{i} (6! 5i) \frac{48}{g^2} \frac{1}{i} (6! 5i)^2 ;
\]

(4.19)
This equation and (4.4) contain only the bosonic parameters. When we search for the classical solutions, we first solve these two equations and obtain $!$ and $\ell$. Then we use the results to solve (4.7) and (4.13).

Case 1: We set $\ell = r$. From (4.4), $!$ has the form of

$$! = c + i f; \quad (4.20)$$

where $c$ is the constant to be determined and $f$ is the function depending only on $r$. Then (4.2) and (4.13) give the following equations:

$$\frac{df}{d} = \frac{g_2^2}{3} r \frac{i}{2}; \quad (4.21)$$
$$\frac{dr}{d} = 4 r \frac{i}{2} (6f^5 g_2^2 r \frac{i}{2} - 2c^2 (f + 1) + c^2 f (f + 1)^2); \quad (4.22)$$
$$c \cdot r + \frac{8}{g_2^2} c^2 (f + 1) (3f + 1) = 0; \quad (4.23)$$

We must set $c = 0$ by the requirement of the consistency among the above equations. There exists only the trivial solution when $c$ takes the non-zero value. The equations satisfied by the classical solutions in this case are:

$$\frac{df}{d} = \frac{g_2^2}{3} r \frac{i}{2}; \quad (4.24)$$
$$\frac{dr}{d} = 4 r \frac{i}{2} (6f^5 g_2^2 r \frac{i}{2} f (f + 1)^2); \quad (4.25)$$
$$\frac{d \lambda}{d} = 8r \frac{i}{2} (f + 1) \lambda; \quad (4.26)$$
$$\lambda = \frac{6i^0}{g_2^6} (f + 1) \lambda; \quad (4.27)$$

It seems very difficult to solve these equations analytically. We have been able to find only two analytic solutions. Let us consider the case in which $f^2 f \frac{g_2^2}{12} r = 0$. This condition means that the gravity can be solved analytically by itself and the gravitino exists in the background of the gravity. Then the equations to be solved are

$$\frac{df}{d} = \frac{g_2^2}{3} r \frac{i}{2}; \quad (4.28)$$
$$r = \frac{12}{g_2^2} f \frac{i}{2} (f + 1) \frac{i}{4}; \quad (4.29)$$
$$\frac{d \lambda}{d} = 8r \frac{i}{2} (f + 1) \lambda; \quad (4.30)$$
$$\lambda = \frac{6i^0}{g_2^6} (f + 1) \lambda; \quad (4.31)$$
There exist two cases:

(a) \( f = \frac{1}{2}(1 + \cosh) \);

(b) \( f = \frac{1}{2}(1 - \cosh) \); \hspace{1cm} (4.32)

where \( f \) is the function of to be determined. In either case, \( r \) is given by

\[ r = \frac{3}{g^2} \sinh^2 \] \hspace{1cm} (4.33)

First we consider case (a). From (4.28), we have

\[ \frac{2}{3} j^0 \] + const; \hspace{1cm} (4.34)

We set \( \frac{2}{3} j^0 \). Then the inequality \( 0 \) must be satisfied. The solution is

\[ 12V \, ds^2 = \frac{3}{4g^2} \, \text{d}^2 + \frac{1}{2} \sinh^2 \, \text{d} \] \hspace{1cm} (4.35)

\[ f = \frac{1}{2}(1 + \cosh) \] \hspace{1cm} (4.36)

\[ \lambda = \cosh^4 \frac{1}{2} \, a_0 \] \hspace{1cm} (4.37)

\[ \lambda = \frac{6i}{g^0} \sinh^4 \frac{1}{2} \cosh^4 - \frac{\lambda}{2} \lambda^0 \] \hspace{1cm} (4.38)

where the grassmannian constant \( a_0 \) satisfies \( a_0 \lambda^0 = 0 \). This solution covers the region \( 0 \) and has the topology of the hyperbolic universe \( H^4 \).

The case (b) gives another analytic solution which covers the region \( 0 \). By the similar discussion, we obtain

\[ \frac{2}{3} j^0 \] \hspace{1cm} (4.39)

\[ 12V \, ds^2 = \frac{3}{4g^2} \, \text{d}^2 + \frac{1}{2} \sinh^2 \, \text{d} \] \hspace{1cm} (4.40)

\[ f = \frac{1}{2}(1 - \cosh) \] \hspace{1cm} (4.41)

\[ \lambda = \sinh^4 \frac{1}{2} \, a_0 \] \hspace{1cm} (4.42)

\[ \lambda = \frac{6i}{g^0} \cosh^2 - \frac{1}{2} \sinh^4 - \frac{\lambda}{2} \lambda^0 \] \hspace{1cm} (4.43)

where \( a_0 \lambda^0 = 0 \).
Case 2: We set \( i \), where \( i \) is a real parameter. \( i \) is written as \( i = c + \text{if}(\) \). Then we have

\[
\frac{df}{d} = \frac{g_0^2}{3} \frac{1}{i}; \quad (4.44)
\]

\[
\frac{d^2}{d^2} = 4 \frac{1}{i} (5 - 6f) + \frac{48}{g_0^2} \frac{1}{i} f f (f 1)^2 \quad ; \quad (4.45)
\]

\[
c = \frac{8}{g_0^2} c^2 (f 1) (3f 1) = 0; \quad (4.46)
\]

By the requirement of the consistency among these equations, we must set \( c = 0 \). Otherwise we obtain only the trivial solution. The equations to be solved are:

\[
\frac{df}{d} = \frac{g_0^2}{3} \frac{1}{i}; \quad (4.47)
\]

\[
\frac{d^2}{d^2} = 4 \frac{1}{i} (5 - 6f) \frac{48}{g_0^2} \frac{1}{i} f f (f 1)^2 \quad ; \quad (4.48)
\]

\[
\frac{d \lambda}{d} = 8 \frac{1}{i} (f 1) \lambda; \quad (4.49)
\]

\[
\lambda = \frac{6 i}{g_0} (f 1) \lambda; \quad (4.50)
\]

As is in the case 1, we can obtain the analytic solution when the condition \( f^2 f + \frac{d^2}{ds^2} = 0 \) is satisfied:

\[
\lambda = \frac{24 g_0}{3}; \quad 0 \quad ; \quad (4.51)
\]

\[
12V ds^2 = \frac{3}{4 g_0^2} d^2 + \frac{1}{2} \sin^2 d^2 \quad ; \quad (4.52)
\]

\[
f = \frac{1}{2} (1 \cos ); \quad (4.53)
\]

\[
\lambda = \frac{\sin^4}{2} \quad \lambda; \quad (4.54)
\]

\[
\lambda = \frac{6 i}{g_0} \cos^2 \frac{1}{2} \sin^4 \frac{1}{2} \lambda; \quad (4.55)
\]

where \( 0 \lambda = 0 \), and the universe has the topology of the sphere \( S^4 \).

Case 3: In this case we set \( ! = f(\) + ic. Then we have

\[
\frac{df}{d} = \frac{g_0^2}{3} \frac{1}{i}; \quad (4.56)
\]

\[
\frac{d^2}{d^2} = 24 f \frac{1}{i} \frac{8}{g_0^2} \frac{1}{i} f f (c 1) (3c 1) \quad ; \quad (4.57)
\]

\[
= \frac{48}{g_0^2} 6c \frac{1}{i} f f (3c 2) \quad c(c 1)^2 \quad \lambda; \quad (4.58)
\]
By the consistency, we obtain $c = \frac{1}{2}$. The equation (4.58) gives

$$c = \frac{12}{g^2} f^2 + \frac{1}{4};$$

(4.59)

Because of this relation, there exists only one solution in this case:

$$c = \frac{2g_0^2}{3};$$

(4.60)

$$12V \, ds^2 = \frac{3}{4g_0^2} \, d^2 + \frac{1}{2} \cosh^2 \, df;$$

(4.61)

$$f = \frac{1}{2} \sinh;$$

(4.62)

$$\lambda = \cosh^2 \, \exp \left[ 2i \arctan \sinh \right]_0 \lambda;$$

(4.63)

$$\Lambda = \frac{3}{g_0} \sinh \, i \cosh^2 \, \exp \left[ 2i \arctan \sinh \right]_0 \Lambda;$$

(4.64)

This universe has the topology of the de Sitter universe $dS^4$.

Case 4: We set $= i$, $r = r(r = 0)$, and $! = f(\lambda) + ic$. Then we have

$$\frac{df}{dr} = \frac{g_0^2}{3};$$

(4.65)

$$\frac{dr}{d\lambda} = 4f \cdot 6r \cdot 1 + \frac{48}{g_0^2} \cdot r \cdot \lambda^2 \cdot f^2 \cdot (c \cdot 1) \cdot (3c \cdot 1);$$

(4.66)

$$r = \frac{48}{g_0^2} \cdot \frac{1}{6c} \cdot \frac{h}{5} \cdot (3c \cdot 2) f^2 \cdot c(c \cdot 1)^2;$$

(4.67)

By the consistency we get $c = \frac{1}{2}$. Using this value (4.67) is rewritten as

$$r = \frac{12}{g_0^2} \cdot f^2 + \frac{1}{4} < 0;$$

(4.68)

which is inconsistent with $r = 0$. Therefore there exists no solution in case 4.

Now we examine the differential equations obtained above. If we suitably change the normalization of $f$, $\lambda$, and $\Lambda$, we can absorb $g_0$ and $\lambda$. So we calculate under the condition that $g_0 = 1$ and $\lambda = 1$. Since $f$ is the monotone increasing function of $\lambda$ (or $\lambda$), we may consider $f$ as the time parameter. Since $\lambda$ and $\Lambda$ can be written as

$$\lambda = F \cdot \lambda;$$

(4.69)

$$\Lambda = F \cdot \lambda;$$

(4.70)

where $F$ and $F$ are the c-number functions depending on $f$, and $\lambda$ and $\lambda$ are the grassmannian constants, we calculate and plot the bosonic parameters $F$ and $F$ under some suitable normalizations.
In the case 3, we have the analytic solution, and in the case 4, there exists no solution. So we pay attention mainly to the case 1 and the case 2. In these cases it seems difficult to solve the differential equations analytically in general, though we can find the analytic solutions in some special conditions. Therefore we resort to the numerical calculation to get the solutions, and study the nature of these solutions.

In the case 1 and case 2, using the Runge-Kutta method, we calculated the solution curves numerically. Since we set \( c = 1; h = 1; \) and \( G = 1, \) all quantities appearing in graphs are dimensionless. The \( g[1] \) is the graph of \( r \) (case 1). Corresponding to some of these solutions (labeled by \( A, B, C, D, E \) and \( F \)), the graphs of \( F' \) and \( F'' \) are given by the \( g[2] \) and the \( g[3]. \)

The \( g[4] \) is the graph of \( \Delta \) (case 1), and the graphs of \( F' \) and \( F'' \) are given in the \( g[5-8] \) (for \( G, H, I, \) and \( J \)). The solutions \( B, F \) and \( J \) are the analytic solutions. In the case 3 we draw the graph of the analytic solution \( g[9-10] \).

In these results we can see some new properties which come from the existence of the gravitino. In the case 1 we have some solutions that \( r \not= 1 \) as \( f \not= 1 \) (for example \( B, C \ldots, F \)). We also have some solutions which seem to correspond to the compact universes (A and G), though the pure gravity solution in the case 1 has the topology of the hyperbolic non-compact universe. From \( g[11-14] \) we can see that the very rapid increase of \( F' \) and \( F'' \) may be the causes of these compactification of the universes.

There are the universes which seem to have the singularities in the first and/or the end. The scalar curvature of the Robertson-Walker universe whose metric is given by (3.3) with the lapse function \( N = 1 \) is calculated as

\[
R = 6!4e^2 + \frac{d^2}{dt^2} + 2 \frac{d^2}{dt^2} A : \quad (4.71)
\]

Using the relations \( 12V e^2 = \), \( V = 2^2 = 4 \), and \( d = \frac{P}{3} \), we have

\[
R = 9^2 8 + \frac{d^2}{dt^2} : \quad (4.72)
\]

The graphs of the scalar curvature of the solutions \( A, B, C, D, E \) and \( F \) are given in the \( g[10] \), and that of \( G, H, I, \) and \( K \) in the \( g[11] \).

How we should take the direction of the physical time is the serious problem. In the analytic solutions in the case 1, we have two candidates of the time; and \( f \). We can take as the time because the universe expands with the increase of \( N \) (note that the cosmological constant of the \( N = 1 \) supergravity is positive). But we have the solutions which diverge in the limit \( f \) (or \( 1 \)) in the case 1, and can't decide the direction in the case 2 because all the universes in this case are compact. So we can't decide the direction of the time naively.
5 Discussion

In this paper we considered the non-perturbative canonical quantization of the N=1 supergravity with the cosmological terms by the Ashtekar formalism. We obtained the holomorphic wave function of the universe which is given by the exponential of the N=1 supersymmetric extension of the Chern-Simons functional. We applied this wave function to the Robertson-Walker metric and found that there exist several types of the universe which contain the gravitinos. Furthermore we obtained four exact classical solutions. The gravitinos in the numerical and the analytic solutions can't be deleted by the supersymmetry transformation in general.

The N=1 supergravity in the Ashtekar formalism is the complex theory and we must consider the reality conditions (see [6]) to take out the real solutions. In this paper we have considered the only one condition that the dreibein fields must be real. When we consider the Lorentzian (Euclidean) universe, the action must be real (pure imaginary). The reality condition about the SL(2; C) connection in the Lorentzian universe is the torsion condition, which is derived from the reality of the action:

\[ D e_{\alpha A} = A^\alpha_A; \]  

(5.1)

where \( D \) is the covariant derivative which acts on both left and right spinor indices. But as can be seen easily, the action is real for the classical solution of the case 3 and pure imaginary for that of the case 2 or 3. Therefore we may think that the reality condition on the connection is satisfied. The reality condition on the gravitino is the Majorana condition. The classical solutions in this paper don't satisfy this condition in general. To obtain the real solution, we must transform the solution by the transformations corresponding to the symmetries (local Lorentz, left and right supersymmetries, 3-dim diffeomorphism, and time-reparametrization) in the theory. Since it seems very difficult to determine the parameters of the transformations explicitly and ascertain whether there exist non-trivial solutions, we leave the settlement of the problem about the reality conditions and will consider it on another occasion.

In this paper we used the self-dual representation. We know another approach to the quantum gravity, which is called the loop space representation [4]. Whether there exists the corresponding representation in the N=1 supergravity is the interesting and challenging problem. In the Einstein gravity, the physical states in the loop space representation are related to the invariants of knots. It seems natural that we expect some invariants corresponding to the physical states of the N=1 supergravity.

Recently the Ashtekar formalism of the N=2 supergravity is derived [8]. We can obtain the holomorphic wave function of the N=2 supergravity as well as that of the N=1 supergravity. The study in this case is now in progress.
A cknow ledgem ent

The authors thank Professor T. Eguchi and Dr. H. Kunitomo for helpful discussions and useful comments on the manuscript.

A ppendix

Here we give the notations and formulas used in this paper. Our space-time signature is \((+;+;+;+)\). We represent the 4-dim space-time indices and the local Lorentz indices by \(; ; ;\); and \(a; b; c;\); and the 3-dim space indices and the at space indices by \(i; j; k;\); and \(I; J; K;\); respectively. We take the basis of \(\text{SL}(2; C)\) and \(\text{SU}(2)\) spinors as

\[
0 \overset{0\ A\ A}{\overset{1\ 0\ 0\ 1}{\begin{array}{c} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{array}}} = \frac{1}{2} 0 1 A ; \quad 1 \overset{1\ A\ A}{\overset{0\ 0\ 1\ 1}{\begin{array}{c} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{array}}} = \frac{1}{2} 0 1 A ;
\]

\[
2 \overset{2\ A\ A}{\overset{0\ 1\ 0\ 1}{\begin{array}{c} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{array}}} = \frac{1}{2} 0 1 A ; \quad 3 \overset{3\ A\ A}{\overset{0\ 0\ 1\ 1}{\begin{array}{c} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{array}}} = \frac{1}{2} 0 1 A ;
\]

and

\[
0 \overset{0\ B\ B}{\overset{1\ 0\ 0\ 1}{\begin{array}{c} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{array}}} = 2 i \theta 0 1 A ; \quad 1 \overset{1\ B\ B}{\overset{1\ 0\ 1}{\begin{array}{c} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{array}}} = 2 i \theta 0 1 A ; \quad 2 \overset{2\ B\ B}{\overset{0\ 1\ 0}{\begin{array}{c} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{array}}} = 2 i \theta 0 1 A ; \quad 3 \overset{3\ B\ B}{\overset{0\ 0\ 1}{\begin{array}{c} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{array}}} = 2 i \theta 0 1 A ;
\]

respectively. By these basis, we can transform the \(\text{SO}(3; 1)\) vector \(v_a\) into \(\text{SL}(2; C)\) spinor as \(v_A^a = v_a^a A_a^A\), and the \(\text{SO}(3)\) vector \(u_I\) into \(\text{SU}(2)\) spinor as \(u^A_{AB} = u_I^I I^A_{AB}\). We define the anti-symmetric spinors by

\[
\overset{1\ A\ B}{\overset{0\ 1\ 0\ 1}{\begin{array}{c} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{array}}} = AB = A^B = A^B = 0 1 A ;
\]

The spinor indices can be raised and lowered according to the conventions

\[
A = AB B ; \quad A = B BA ;
\]

Taking an adequate gauge, we x the form of the vierbein eld as follows:

\[
e^a = \begin{pmatrix} 0 & N & N e^i_j & 1 \\ 0 & 0 & e^i_j & \end{pmatrix}
\]

where \(N\) and \(N^i\) are the lapse function and the shift vector, respectively. Defining the 3-dim space metric \(q_{ij}\) by \(q_{ij} = e^i_i e_j^j\), the line-element of space-time is given by

\[
ds^2 = N^2 dt^2 + q_{ij} (N^i dt + dx^i) (N^j dt + dx^j)
\]
Introducing the dual basis $e^i$ by $e^i e^j = \delta^i_j$, we have

$$\sim_{AB} = \frac{1}{2} P - \delta^i_{i AB}$$  \quad (5.8)$$

by the straightforward calculation, where $q = \text{det} q_{ij}$. We also have

$$\frac{A_B}{e_{01}} = \frac{1}{2} N I e_{IAB} + \frac{1}{2} i j k N j^l_{IAB}$$

$$= N i^{-1}_{AB} i j k N j^l_{kAB};$$  \quad (5.9)$$

where $N = \frac{p}{q}$. By the formula $\sim_{AB} = \frac{1}{2} i j k \sim_{AC} \sim_{CB}$, finally we obtain

$$\frac{A_B}{e_{01}} = \frac{1}{2} i j k i N \sim_{AC} \sim_{CB} + 2 N j^l_{kAB};$$  \quad (5.10)$$

We can ascertain that (2.13) is the solution of (2.14) under (2.14). By counting of the degrees of freedom, (2.13) is just the general solution.

Now we list some formulas in the case of the Robertson-Walker metric (3.1) and (3.2). The space-time metric is given by

$$g = \begin{pmatrix}
0 & N^2 & 0 & 1 & 0 & N^2 & 0 & 1 \\
0 & 1 & 0 & N^2 & 0 & 1 & 0 & e^2 \frac{i}{i j} A \\
0 & 1 & 0 & N^2 & 0 & 1 & 0 & e^2 \frac{i}{i j} A
\end{pmatrix};$$  \quad (5.12)$$

and the vierbein is given by

$$e^a = \begin{pmatrix}
0 & N & 0 & 1 \\
0 & e^i & i j & A
\end{pmatrix};$$  \quad (5.13)$$

Moreover we obtain

$$e = \text{det} e^a = N e^3 \frac{j}{j}$$

$$p_{q} = e^3 \frac{j}{j};$$  \quad (5.14)$$

$$\sim_{AB} = \frac{1}{2} j^2 \frac{k}{k AB};$$  \quad (5.15)$$

In this paper we use the Maurer-Cartan form of $SU(2)$ as $\frac{A_B}{e_{01}}$;

$$U^A_{AB} = \begin{pmatrix}
0 & e^{\frac{j}{j}} \cos \frac{1}{2} & e^{-\frac{i}{i}} \sin \frac{1}{2} \\
e^{\frac{i}{i}} \sin \frac{1}{2} & e^{\frac{j}{j}} \cos \frac{1}{2}
\end{pmatrix};$$  \quad (5.16)$$

$$A^A_{B} = U_{AC}^A d U^C_B;$$  \quad (5.17)$$
where $\theta$, $\phi$, and $\varphi$ are the Euler angles. The explicit form of $i^\perp$ is

$$
\begin{bmatrix}
0 & \frac{1}{2} \sin \theta \cos \phi & \frac{1}{2} \sin \theta \sin \phi & \frac{1}{2} \cos \theta \\
\frac{1}{2} \cos \varphi & 0 & \frac{1}{2} \sin \varphi & 0 \\
\frac{1}{2} \sin \varphi & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
$$

and then we have

$$
j j = \frac{1}{64} \sin \theta ;
$$

$$
\Theta_1(j j_{\perp AB}) = 0.
$$

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Figure 1: The graphs of $r$ in the case 1.

Figure 2: The graphs of the real part of $F$ in the case 1.

Figure 3: The graphs of the imaginary part of $F$ in the case 1.

Figure 4: The graphs of in the case 2.

Figure 5: The graphs of the real part of $F$ in the case 2.

Figure 6: The graphs of the imaginary part of $F$ in the case 2.

Figure 7: The graphs of in the case 3.

Figure 8: The graphs of $F$ in the case 3.

Figure 9: The graphs of $F$ in the case 3.

Figure 10: The graphs of the scalar curvature $R$ in the case 1.

Figure 11: The graphs of the scalar curvature $R$ in the case 2.