SUMMATION FORMULAE FOR QUADRICS

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Abstract. We prove a Poisson summation formula for the zero locus of a quadratic form in an even number of variables with no assumption on the support of the functions involved. The key novelty in the formula is that all “boundary terms” are given either by constants or sums over smaller quadrics related to the original quadric. We also discuss the link with the classical problem of estimating the number of solutions of a quadratic form in an even number of variables. To prove the summation formula we compute (the Arthur truncated) theta lift of the trivial representation of SL₂(AF). As previously observed by Ginzburg, Rallis, and Soudry, this is an analogue for orthogonal groups on vector spaces of even dimension of the global Schrödinger representation of the metaplectic group.

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1. Introduction

Let $V_0$ be an even dimensional affine space over a number field $F$ and let $Q_0$ be a non-degenerate anisotropic quadratic form on $V_0$. We allow the degenerate special case where $V_0 = \{0\}$, equipped with the trivial quadratic form. For $i \geq 0$ let

$$V_i := V_0 \oplus \mathbb{G}_\alpha^{2i}$$

equipped with the quadratic form

$$Q_i(x) := \frac{x^t J_i x}{2}.$$ 

Here

$$J_i = \begin{pmatrix} J_0 & J & \cdots & J \\ J & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & J \\ J & \cdots & J & \end{pmatrix}$$

where $J_0$ is the matrix of $Q_0$ and $J := (1 \ 1)$. Here there are $i$ copies of $J$. For $i > i' \geq 0$ we identify $V_i$ with the subspace $V_i \oplus \{0\}^{2i-2i'} < V_i$.

For $F$-algebras $R$ let

$$X_i(R) := \{u \in V_i(R) : Q_i(u) = 0\}$$

and let $X_i^\circ := X_i \setminus \{0\}$. Now fix $\ell \in \mathbb{Z}_{>0}$. In this paper we prove a summation formula for $X_\ell$ analogous to the Poisson summation formula. It will involve the whole family of spaces $X_i$ for $\ell \geq i \geq 0$. If $V_0 = \{0\}$ we will always assume that $\ell > 1$.

1.1. A summation formula. Let $\mathbb{A}_F$ be the adeles of $F$. Fix an additive character $\psi : F \setminus \mathbb{A}_F \to \mathbb{C}^\times$. We can then define the Weil representation

$$\rho_i := \rho_{Q_i, \psi} : SL_2(\mathbb{A}_F) \times S(V_i(\mathbb{A}_F)) \to S(V_i(\mathbb{A}_F))$$

as usual. Here

$$S(V_i(\mathbb{A}_F)) := S(V_i(F_\infty)) \otimes C_c^\infty(V_i(\mathbb{A}_{F_\infty}^\infty))$$

is the usual Schwartz space, where $\infty$ is the set of infinite places of $F$. We have suppressed the orthogonal group in the Weil representation because it plays no role at the moment.

We have an action

$$L^\vee : SL_2(\mathbb{A}_F) \times S(\mathbb{A}_F^2) \to S(\mathbb{A}_F^2)$$

and thus an action

$$r_i := \rho_i \otimes L^\vee : SL_2(\mathbb{A}_F) \times S(V_i(\mathbb{A}_F)) \otimes S(\mathbb{A}_F^2) \to S(V_i(\mathbb{A}_F)) \otimes S(\mathbb{A}_F^2)$$
which extends to
\begin{equation}
\tag{1.7}
\rho_i : \text{SL}_2(\mathbb{A}_F) \times S(V_i(\mathbb{A}_F) \oplus \mathbb{A}_F^2) \rightarrow S(V_i(\mathbb{A}_F) \oplus \mathbb{A}_F^2).
\end{equation}

Let $\infty$ denote the set of infinite places of $F$. For finite places $v$ of $F$ let $O_v$ denote the ring of integers of $F_v$.

**Definition 1.1.** The Schwartz space of $X_i(\mathbb{A}_F)$ is the space of coinvariants
\[ S(V_i(\mathbb{A}_F) \oplus \mathbb{A}_F^2)_{\rho_v(\text{SL}_2(\mathbb{A}_F))} = S(V_i(F_\infty) \oplus F_\infty^2)_{\rho_v(\text{SL}_2(F_\infty))} \otimes \bigotimes_{v \neq \infty} S(V_i(F_v) \oplus F_v^2)_{\rho_v(\text{SL}_2(F_v))}. \]
Here the restricted direct product is taken with respect to the image of $1_{V_i(\mathcal{O}_v) \oplus \mathcal{O}_v^2}$ for all finite $v$. The space
\[ S(V_i(F_\infty) \oplus F_\infty^2)_{\rho_v(\text{SL}_2(F_\infty))} \]
denotes $S(V_i(F_\infty) \oplus F_\infty^2)$ modulo the closure (in the usual Fréchet topology) of the space spanned by vectors of the form $f - \rho_v(g)f$ for $(g, f) \in \text{SL}_2(F_\infty) \times S(V_i(F_\infty) \oplus F_\infty^2)$. The local Schwartz spaces are defined analogously (see (4.2)).

It may seem odd to define the Schwartz space of $X_i(\mathbb{A}_F)$ as a space of coinvariants. However, this algebraic definition is quite convenient. To obtain functions on $X_i(\mathbb{A}_F)$ from elements of $S(X_i(\mathbb{A}_F))$ we define, for $i > 0$, the operator
\begin{equation}
\tag{1.8}
I : S(V_i(\mathbb{A}_F) \oplus \mathbb{A}_F^2) \rightarrow C^\infty(X_i(\mathbb{A}_F))
\end{equation}
\[ f \mapsto \left( \xi \mapsto \int_{N(\mathbb{A}_F) \backslash \text{SL}_2(\mathbb{A}_F)} \rho_v(g) f(\xi, 0, 1) \, dg \right). \]

Here $N \leq \text{SL}_2$ is the unipotent radical of the Borel subgroup $B$ of upper triangular matrices, that is, for $F$-algebras $R$,
\begin{equation}
\tag{1.9}
N(R) := \{(t^1) : t \in R\}.
\end{equation}
The integral $I(f)$ is absolutely convergent and defines a function in $C^\infty(X_i(\mathbb{A}_F))$ by lemmas 4.1 and 4.8 below. We observe that $I$ factors through the canonical map $S(V_i(\mathbb{A}_F) \oplus \mathbb{A}_F^2) \rightarrow S(X_i(\mathbb{A}_F))$ and thus allows us to produce a function on $X_i(\mathbb{A}_F)$ from each element of $S(X_i(\mathbb{A}_F))$ as mentioned above. In Lemma 4.7 we prove that $I(S(X_i(F_v)))$ contains the restrictions of any Schwartz function on $V_i(F_v)$ to $X_i(\mathbb{A}_F)$ for all $v$.

For $s \in \mathbb{C}$ we also define a family of functionals
\begin{equation}
\tag{1.10}
Z_{\rho_v}(\cdot, s) : S(V_i(\mathbb{A}_F) \oplus \mathbb{A}_F^2) \rightarrow \mathbb{C}
\end{equation}
\[ f \mapsto \int_{N(\mathbb{A}_F) \backslash \text{SL}_2(\mathbb{A}_F)} e^{H_B(g)(2 - \dim V_i - s)} \rho_v(g) f(0, 0, 1) \, dg \]
where $H_B$ is the usual Harish-Chandra map (see (3.7)). This is a Tate integral (see Lemma 4.5). For $i > 0$ we let
\begin{equation}
\tag{1.11}
c_i : S(V_i(\mathbb{A}_F) \oplus \mathbb{A}_F^2) \rightarrow \mathbb{C}
\end{equation}
be defined by
\[ c_i(f) := \begin{cases} Z_r(f, 2 - \frac{\dim V_i}{2}) & \text{if } Z_{r_i}(f, s) \text{ is holomorphic at } 2 - \frac{\dim V_i}{2} \\ \lim_{s \to 0} \frac{d}{ds}(sZ_{r_i}(f, s + 2 - \frac{\dim V_i}{2})) & \text{if } Z_{r_i}(f, s) \text{ has a pole at } 2 - \frac{\dim V_i}{2}. \end{cases} \]

One might think of \( c_i(f) \) as the regularized value of \( I(f) \) at \( 0 \in X_i(F) \). We observe that \( Z_{r_i}(f, s) \) is holomorphic except possibly at \( s \in \{0, 1\} \). In fact, it is also holomorphic at \( s \in \{0, 1\} \) when the \( \chi \) is nontrivial, where \( \chi \) is the usual character attached to the family of quadratic spaces \( V_i \) (see (4.9)). We discuss the \( \text{SL}_2(\mathbb{A}_F) \)-invariance of the functionals \( c_i \) in §8 below. In particular \( c_i \) is \( \text{SL}_2(\mathbb{A}_F) \)-invariant provided that \( \dim V_i \not\in \{2, 4\} \).

For each \( i \) we define \( \text{SL}_2(\mathbb{A}_F) \)-intertwining maps
\[ d_i : \mathcal{S}(V_i(\mathbb{A}_F) \oplus \mathbb{A}_F^2) \to \mathcal{S}(V_{i-1}(\mathbb{A}_F) \oplus \mathbb{A}_F^2) \quad (1.12) \]
\[ f \mapsto \mathcal{F}_2(\xi \mapsto f(\xi, 0, 0)) \]
where \( \mathcal{F}_2 : \mathcal{S}(V_i(\mathbb{A}_F)) \to \mathcal{S}(V_{i-1}(\mathbb{A}_F) \oplus \mathbb{A}_F^2) \) is the partial Fourier transform of (4.6) below. For \( i > i' \geq 0 \) we let
\[ d_{i,i'} = d_{i'+1} \circ \cdots \circ d_{i-1} \circ d_i : \mathcal{S}(V_i(\mathbb{A}_F) \oplus \mathbb{A}_F^2) \to \mathcal{S}(V_{i'}(\mathbb{A}_F) \oplus \mathbb{A}_F^2). \]
By convention, \( d_{i,i} \) is the identity. We let
\[ \mathcal{F}_\lambda : \mathcal{S}(\mathbb{A}_F^2) \to \mathcal{S}(\mathbb{A}_F^2) \]
be the usual \( \text{SL}_2(\mathbb{A}_F) \)-equivariant Fourier transform (see (4.13)). We extend it to \( \mathcal{S}(V_i(\mathbb{A}_F) \oplus \mathbb{A}_F^2) \) by setting \( \mathcal{F}_{X_i} := 1_{\mathcal{S}(V_i(\mathbb{A}_F))} \otimes \mathcal{F}_\lambda \). Since this map is \( r_i(\text{SL}_2(\mathbb{A}_F)) \)-equivariant it induces a map
\[ \mathcal{F}_{X_i} : \mathcal{S}(X_i(\mathbb{A}_F)) \to \mathcal{S}(X_i(\mathbb{A}_F)). \]
This is the **Fourier transform for** \( X_i \).

**Remark.** We expect that the image of the map \( I : \mathcal{S}(X(\mathbb{A}_F)) \to C^\infty(X^o(\mathbb{A}_F)) \) is essentially the Schwartz space of \( X(\mathbb{A}_F) \) in either the sense of [BK02, GK19] or the sense of [GHL21] and the map
\[ \mathcal{F}_{X_i} : \mathcal{S}(X_i(\mathbb{A}_F)) \to \mathcal{S}(X_i(\mathbb{A}_F)) \]
descends to the Fourier transform on the Schwartz space of \( X_i(\mathbb{A}_F) \) defined in these references. We say “essentially” because there may be some subtleties when \( i \) is small and with the archimedean theory.

Our main theorem is the following summation formula:

**Theorem 1.2.** For \( \ell \in \mathbb{Z}_{\geq 0} \) and \( f \in \mathcal{S}(V_\ell(\mathbb{A}_F) \oplus \mathbb{A}_F^2) \) one has that
\[
\sum_{i=1}^{\ell} \left( c_i(d_{\ell,i}(f)) + \sum_{\xi \in X'_i(F)} I(d_{\ell,i}(f))(\xi) \right) + \kappa d_{\ell,0}(f)(0_{V_0}, 0, 0)
\]
\[
= \sum_{i=1}^{\ell} \left( c_i(d_{\ell,i}(\mathcal{F}_{X\ell}(f))) + \sum_{\xi \in X_\ell(F)} I(d_{\ell,i}(\mathcal{F}_{X\ell}(f)))(\xi) \right) + \kappa d_{\ell,0}(\mathcal{F}_{X\ell}(f))(0_{V_0}, 0, 0).
\]

Here
\[
\kappa := \begin{cases} 
\text{meas}([\text{SL}_2]) & \text{if } \dim V_0 = 0 \\
0 & \text{otherwise.}
\end{cases}
\]

The sums over \( \xi \) are in general infinite, but they are easily seen to be absolutely convergent by lemmas 4.1 and 4.8 below. In general, the \( i \)th summand on the left and right are not equal.

Let us call the terms other than the sum over \( X_\ell(F) \) in Theorem 1.2 boundary terms. The key novelty in Theorem 1.2 is that it is valid under no restrictions on the test functions involved and the boundary terms are explicit and evidently geometric in nature.

If \( V_{\ell} \) is a split quadratic space and we place additional assumptions on \( f \) then the theorem is a consequence of more general work of Braverman and Kazhdan [BK02]. Their work was generalized to arbitrary test functions \( f \) in [CG21], but the boundary terms were given inexplicitly in terms of residues of Eisenstein series. Related formulae are also established for arbitrary quadratic spaces in [Get18] and in a special case related to Rankin-Selberg products on \( \text{GL}_2 \) in [Get20].

The interest in general test functions and boundary terms is not academic. Restricting test functions restricts the information one can extract from automorphic representations. In more detail, it is expected, and can be verified in some cases, that the boundary terms correspond to poles of appropriate \( L \)-functions. Hence choosing test functions that eliminate these contributions hides information about the poles of \( L \)-functions.

Theorem 1.2 is very closely related to the circle method for quadratic forms and we feel that it will be useful for questions in analytic number theory. To make the relationship transparent we discuss the special case where \( F = \mathbb{Q} \) in §2 below. Interestingly, in analytic number theory it is often only the most degenerate boundary terms that are studied.

1.2. Sketch of the proof of Theorem 1.2. In order to prove Theorem 1.2 for \( f \in \mathcal{S}(V_{\ell}(\mathbb{A}_F) \oplus \mathbb{A}_F^2) \) we compute

\[
\int_{[\text{SL}_2]} \Theta_f^T(g) dg
\]

where \( \Theta_f \) is the usual \( \Theta \)-function and the superscript \( T \) denotes the usual truncation operator employed by Arthur. Here \( [\text{SL}_2] := \text{SL}_2(F) \backslash \text{SL}_2(\mathbb{A}_F) \).
Theorem 1.3. The integral (1.15) is a polynomial in $e^T$ and $T + o_f(1)$ as $T \to \infty$. The constant term of the polynomial is

$$
(1.16) \quad \sum_{i=1}^{\ell} \left( c_i(d_{\ell,i}(\mathcal{F}_2(f))) + \sum_{\xi \in X_0^2(F)} I_i(d_{\ell,i}(\mathcal{F}_2(f)))(\xi) \right) + \kappa d_{\ell,0}(\mathcal{F}_2(f))(0_0, 0, 0).
$$

Theorem 1.2 then follows upon observing that $\Theta_{\mathcal{F}_2^{-1}(\mathcal{F}_X(f))}(g) = \Theta_{\mathcal{F}_2(f)}(g)$ (see §7 for more details and the proof of Theorem 1.3).

We now explain a representation-theoretic interpretation of Theorem 1.3. For each $i$ let $O_{V_i}$ (resp. $GO_{V_i}$) be the orthogonal (resp. orthogonal similitude) group of $V_i$. The partial Fourier transform $\mathcal{F}_2 : \mathcal{S}(V_i(\mathbb{A}_F) \oplus \mathbb{A}_F^2) \to \mathcal{S}(V_i(\mathbb{A}_F) \oplus \mathbb{A}_F^2)$ of (4.6) intertwines the action of $\rho_{i+1}$ and $r_i$ (see Lemma 4.2). Hence it yields a $\mathbb{C}$-linear isomorphism

$$
\mathcal{F}_2 : \mathcal{S}(V_i(\mathbb{A}_F) \oplus \mathbb{A}_F^2)_{\rho_{i+1}(SL_2(\mathbb{A}_F))} \rightarrow \mathcal{S}(V_i(\mathbb{A}_F) \oplus \mathbb{A}_F^2)_{r_i(SL_2(\mathbb{A}_F))}
$$

that is equivariant with respect to the action of $O_{V_i}(\mathbb{A}_F)$ (embedded in $O_{V_{i+1}}(\mathbb{A}_F)$) in the obvious manner. The left hand side admits an obvious action of $GO_{V_{i+1}}(\mathbb{A}_F)$ and we can define a representation

$$
\sigma_i : GO_{V_{i+1}}(\mathbb{A}_F) \times \mathcal{S}(V_i(\mathbb{A}_F) \oplus \mathbb{A}_F^2)_{r_i(SL_2(\mathbb{A}_F))} \rightarrow \mathcal{S}(V_i(\mathbb{A}_F) \oplus \mathbb{A}_F^2)_{r_i(SL_2(\mathbb{A}_F))}
$$

by transport of structure. We will make the action explicit in Proposition 4.3 below. Thus $\sigma_i$ can be considered the big theta lift of the trivial representation of $SL_2(\mathbb{A}_F)$.

Theorem 1.3 amounts to an explicit automorphic realization of $\sigma_i$. The realization is clearly similar in spirit to the Schrödinger model of the metaplectic representation of the two-fold cover of a symplectic group. In the archimedean case this analogy is discussed in detail in [KM11]. We hope that having such an explicit model will aid in applications analogous to the many uses of the Schrödinger model. In particular the explicit geometric description of the model may aid in its use in unfolding arguments that are so crucial in the theory of integral representations of $L$-functions.

We hasten to point out that it is already known that $\sigma_i$ is automorphic. In the case where $V_0 = \{0\}$ it is discussed in detail in [GRS97], which contains a wealth of information about the representation and analogues of it for exceptional groups. In loc. cit., the theta lift is realized as a residue of an Eisenstein series. It is then related to the theta lift of the trivial representation of $SL_2(\mathbb{A}_F)$ to $O_{V_i}(\mathbb{A}_F)$ in the special case where $F$ is totally real using Kudla and Rallis’ regularized Siegel-Weil identity [KR94]. However, it is unclear to the author whether it is possible to use this approach to obtain a formula similar to that in Theorem 1.3 (and hence Theorem 1.2) valid for all test functions $f$. This is one of the main features of the current paper. For the purpose of comparing our work with [GRS97] it is also worth mentioning that we make no use of the theory of Eisenstein series on groups of absolute rank bigger than 1. There is another closely related automorphic realization that was studied by Kazhdan and Polischchuk in [KP04]. However the automorphic realization of the minimal
representation was not completely determined. In particular it is not clear how to extract the explicit formulae of Theorem 1.2 and Theorem 1.3 from [KP04].

We now outline the contents of this paper. We pause to indicate the relationship of our results with classical questions related to the circle method in §2. We set notional and measure conventions in §3. In §4 we define the local integrals that play a role in the statement of our main theorem, define the local Schwartz space, and prove several useful properties of it. The proof of Theorem 1.3 is an induction on \( i \). In §5 the computation of \( \int_{[\text{SL}_2]} \Theta_f^T(g) dg \) is inductively reduced to the case where \( V_i = V_0 \), that is, to the anisotropic case. This case requires one additional idea and is treated in §6. We put the results of §5 and §6 together in §7 to prove Theorem 1.3. We then deduce Theorem 1.2. To aid the reader we have appended an index of notation.

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2. Application to summation over points on a quadric

Just as the Poisson summation formula gives a canonical method for estimating the number of points in a vector space of a given size, the summation formula of Theorem 1.2 gives a canonical method for estimating the number of points in a quadric of a given size. Given the ubiquity of the Poisson summation formula in analytic number theory (and more generally in analysis) we can imagine many applications of Theorem 1.2.

Of course the question of analyzing the number of points on a quadric has a long history, with theta functions and the circle method playing a key role. We point out in particular the author’s work in [Get18] based on earlier work of Heath-Brown [HB96] and Duke, Friedlander and Iwaniec [DFI93]. To make the relationship between the circle method and Theorem 1.2 more transparent we explain Theorem 1.2 in a special case. Let \( F = \mathbb{Q} \) and let \( Q_\ell \) be a quadratic form with matrix \( J_\ell \) defined as in (1.3). We assume that \( J_\ell \in \text{GL}_{V_\ell}(\mathbb{Z}) \) and \( \det J_\ell = (-1)^\ell \). Assume moreover that \( \ell > 0 \) (which is to say that \( V_\ell \) is isotropic) and that \( \dim V_\ell > 4 \). The case \( \dim V_\ell = 4 \) can also be treated, but the formula is slightly more complicated (see [Get18, HB96]). We choose

\[
f = f_\infty 1_{V_{\ell+1}(\mathbb{Z})} \in \mathcal{S}(V_\ell(\mathbb{A}_Q) \oplus \mathbb{A}_Q^2)\
\]
and assume that 

$$d_\ell(f_\infty) = c_\ell(f_\infty) = 0.$$ 

This is easy enough to arrange. Then for any $$B \in \mathbb{R}_{>0} \leq \mathbb{A}_Q^\times$$

$$\sum_{\xi \in X_\ell^0(\mathbb{Q})} I(f)(\frac{\xi}{B}) = \sum_{n=0}^{\infty} \sum_{\xi \in nV_\ell(\mathbb{Z}) - \{0\}} n^{\dim V_\ell/2-2} I(f_\infty)(\frac{\xi}{B}).$$

Here we have used Lemma 4.8. Thus one side of our summation formula is a weighted version of a sum familiar from analytic number theory. It is a smoothed count of the number of integral zeros of $$Q_\ell$$ of size at most $$B$$.

By Corollary 7.2 the other side is

$$B^{\dim V_i-2} c_\ell(F_{X_\ell}(f)) + B^{\dim V_i-2} \sum_{n=0}^{\infty} \sum_{\xi \in nV_\ell(\mathbb{Z}) - \{0\}} n^{\dim V_\ell/2-2} I(F_{X_\ell}(f_\infty))(B\xi)$$

$$+ B^{\dim V_i/2} \sum_{i=1}^{\ell-1} c_i(d_{\ell,i}(F_{X_\ell}(f))) + \sum_{n=0}^{\infty} \sum_{\xi \in nV_\ell(\mathbb{Z}) - \{0\}} n^{\dim V_\ell/2-2} I(d_{\ell,i}(F_{X_\ell}(f_\infty)))(\xi)$$

$$+ B^{\dim V_i/2} \kappa d_{\ell,0}(F_{X_\ell}(f))(0_{V_i},0,0).$$

Thus Theorem 1.2 gives a complete asymptotic expansion of (2.1) as a function of $$B$$. The term $$c_\ell(F_{X_\ell}(f))$$ is essentially the familiar singular series. The flexibility of choosing other test functions in $$\mathcal{S}(X_\ell(\mathbb{A}_Q))$$ allows one to impose congruence conditions on the sum. It is important that one is allowed to choose arbitrary test functions at infinity for classical applications of this formula. More specifically, if one imposed the conditions of [BK02], for example, then the term $$c_\ell(F_{X_\ell}(f))$$ would be zero, which would make the formula useless for the classical application of counting the number of points of size at most $$B$$ on $$X_\ell(\mathbb{Z})$$.

3. Preliminaries

3.1. Groups. As in the introduction GO$$V_i$$ is the similitude group of $$(V_i, Q_i)$$. Let

$$\lambda : \text{GO}_V \longrightarrow \mathbb{G}_m$$

be the similitude norm. We identify GO$$V_i$$ as a subgroup of GO$$V_{i+1}$$ via the embedding given on points in an $$F$$-algebra $$R$$ by

$$\text{GO}_V(R) \longrightarrow \text{GO}_{V_{i+1}}(R)$$

$$h \longmapsto \left( \begin{array}{c} h \\ \lambda(h) \end{array} \right).$$

For $$x \in R^{\dim V_i}$$ (viewed as a column vector) we let

$$u(x) := \left( \begin{array}{c} I_{V_i} \\ J_x \\ 1 \\ -x^t \\ -Q_i(x) \end{array} \right)$$

(3.2)
and set
\[ N_{i+1}(R) := \{ u(x) : x \in R^{\dim V_i} \}. \]

This is the unipotent radical of a maximal parabolic subgroup of (the neutral component of) \( \text{GO}_{V_i+1} \).

### 3.2. Normalization of measures and the Harish-Chandra map.

Fix a nontrivial character \( \psi : F \backslash A_F \to \mathbb{C}^\times \). For each place \( v \) of \( F \) we normalize the Haar measure on \( F_v \) so that the Fourier transform with respect to \( \psi_v \) is self-dual. Then the induced measure on \( A_F \) gives \( F \backslash A_F \) measure 1 \([Wei74, \S VII.2]\). For each place \( v \) of \( F \) we give \( F_v \) the measure \( d^v \times x_v := \zeta_v(1)|x|^{-1}d x_v \).

If \( F_v \) is unramified over its prime field and \( \psi_v \) is unramified then \( d x_v(O_v) = 1 \), where \( O_v \) is the ring of integers of \( F_v \). As usual, let
\[ A_{G_m} \leq F^\times \]
be the diagonal copy of \( \mathbb{R}_{>0} \) and let
\[ (A_f^\times)^1 := \{ x \in A_f^\times : |x| = 1 \}. \]
We choose the Haar measure on \( A_{G_m} \) so that the isomorphism \( | \cdot | : A_{G_m} \to \mathbb{R}_{>0} \) is measure preserving and then endow \( (A_f^\times)^1 \) with the unique Haar measure such that the canonical isomorphism
\[ A_{G_m} \times (A_f^\times)^1 \to A_f^\times \]
is measure-preserving.

Let \( T_2 \leq B \leq \text{SL}_2 \) be the maximal torus of diagonal matrices and the Borel of upper triangular matrices, respectively. Let \( N \) be the unipotent radical of \( B \). Fix a maximal compact subgroup \( K \leq \text{SL}_2(\mathbb{A}_F) \) such that the Iwasawa decomposition
\[ \text{SL}_2(\mathbb{A}_F) = N(\mathbb{A}_F)T_2(\mathbb{A}_F)K \]
holds. For each place \( v \) of \( F \) we give \( \text{SL}_2(F_v) \) the Haar measure
\[ d((\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}) (\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}) k) = \frac{d^v a}{|a|^2} d k \]
where \((t,a,k) \in F_v \times F_v^\times \times K_v \) and \( d k \) gives \( K_v \) measure 1. This induces a measure on \( \text{SL}_2(\mathbb{A}_F) \).

For \( g \in \text{SL}_2(\mathbb{A}_F) \) write \( g = n (\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}) k \) with \((n,a,k) \in N(\mathbb{A}_F) \times \mathbb{A}_F^\times \times K \), where \( K \leq \text{SL}_2(\mathbb{A}_F) \) is the standard maximal compact subgroup, and define
\[ H_B(nak) := \log |a| \in \mathbb{R}. \]
We also use the obvious local analogue of this notation.
If $G$ is an algebraic group over $F$ we let

$$[G] := G(F) \backslash G(\mathbb{A}_F).$$

4. Local integrals and the Schwartz space

In this section we fix a place $v$ of $F$ and omit it from notation, writing $F := F_v$. Thus $F$ can be any local field of characteristic not equal to 2. Let $f \in S(V_i(F) \oplus F^2)$. For $\xi \in X_i^0(F)$ let

$$(4.1) \quad I(f)(\xi) := \int_{N(F) \backslash SL_2(F)} r_i(g) f(\xi, 0, 1) d\hat{g}.$$  

Here, as above, $N \leq SL_2$ denotes the unipotent radical of the Borel subgroup of upper triangular matrices.

**Lemma 4.1.** The integral

$$\int_{N(F) \backslash SL_2(F)} |r_i(g) f(\xi, 0, 1)| d\hat{g}$$

is convergent for all $\xi \in V_i(F)$. For any $A \in \mathbb{Z}_{\geq 0}$ it is bounded by a constant depending on $f$ and $F$ (and $\epsilon > 0$ if $\dim V_i = 4$) times

$$\max(|\xi|, 1)^{2 - \frac{\dim V_i}{2}} \begin{cases} 
\min(|\xi|, 1)^{2 - \frac{\dim V_i}{2}} & \text{if } \dim V_i > 4 \\
\min(|\xi|, 1)^{-\epsilon} & \text{if } \dim V_i = 4 \\
1 & \text{if } \dim V_i = 2.
\end{cases}$$

In the nonarchimedean case, this can be strengthened to the assertion that the support of the integral is contained in the intersection of a compact subset of $V(F)$ with $V^0(F)$.

**Proof.** Decomposing the Haar measure on $SL_2(F)$ using the Iwasawa decomposition we obtain

$$\int_{F^\times \times K} |r_i((a^{-1} a^{-1}) k) f (\xi, 0, 1)| \frac{d^\times adk}{|a|^2}$$

$$= \int_{F^\times \times K} |r_i(k) f (a \xi, 0, a^{-1})| \frac{|a|^{\dim V_i/2} d^\times adk}{|a|^2}$$

$$= \int_{F^\times \times K} |r_i(k) f (a^{-1} \xi, 0, a)| |a|^{2 - \dim V_i/2} d^\times adk.$$

In the nonarchimedean case it is easy to check the assertion of the lemma from this expression. In the archimedean case it requires a small computation which is contained in [GL19, Lemma 8.1].

We define

$$(4.2) \quad S(X_i(F)) := S(V_i(F) \oplus F^2)_{r_i(SL_2(F))}.$$
Temporarily let
\[ W := \langle f - r_i(g) f : (g, f) \in \text{SL}_2(F) \times S(V_i(F) \oplus F^2) \rangle \]
where the brackets denote the \( \mathbb{C} \)-span. In the nonarchimedean case
\[ S(V_i(F) \oplus F^2)_{r_i(\text{SL}_2(F))} := S(V_i(F) \oplus F^2)/W \tag{4.3} \]
and in the archimedean case
\[ S(V_i(F) \oplus F^2)_{r_i(\text{SL}_2(F))} := S(V_i(F) \oplus F^2)/\overline{W} \tag{4.4} \]
where \( \overline{W} \) is the closure of \( W \) in the usual Frechet space topology on \( S(V_i(F) \oplus F^2) \). The map \( I \) defined in (4.1) induces a morphism
\[ I : S(V_i(F) \oplus F^2) \rightarrow C^\infty(X_i^o(F)) \]
that factors through \( S(X_i(F)) \).

Already in the introduction we made use of two representations of \( \text{SL}_2(F) \) on \( S(V_i(F) \oplus F^2) \), namely \( \rho_{i+1}(g) \) and \( r_i(g) \). We now relate these two actions. Define a transform
\[ F_2 : S(F^2) \rightarrow S(F^2) \tag{4.5} \]
\[ f \mapsto \left( (u_1, u_2) \mapsto \int_F f(u_1, x) \psi(u_2 x) dx \right) \]
where \((u_1, u_2) \in F^2\). The subscript 2 is a reminder that this is the Fourier transform in the second variable. Identifying \( V_{i+1}(F) = V_i(F) \oplus F^2 \) this extends to
\[ F_2 := 1_{S(V_i(\mathbb{A}_F))} \otimes F_2 : S(V_{i+1}(F)) \rightarrow S(V_i(F) \oplus F^2). \tag{4.6} \]
It is obvious that \( F_2 \) is \( \text{GO}_{V_i}(F) \)-equivariant. We also have the following equivariance property:

**Lemma 4.2.** For \( g \in \text{SL}_2(F) \), one has that
\[ F_2 \circ \rho_{i+1}(g) = r_i(g) \circ F_2. \]

Before proving the lemma, we set some notation for the Weil representation \( \rho_i \) on \( S(V_i(F)) \). We let
\[ \langle v_1, v_2 \rangle_i := v_1^t J_i v_2 \tag{4.7} \]
be the pairing attached to the quadratic form \( Q_i \), we let
\[ \gamma := \gamma(Q_i) \tag{4.8} \]
be the Weil index (it is independent of \( i \)) and we let
\[ \chi(a) := (a, (-1)^{\dim V_i/2} \det J_i) \tag{4.9} \]
be the usual character attached to the quadratic form (it is independent of \( i \)). Here the right hand side of (4.9) is the usual Hilbert symbol.
Proof of Lemma 4.2. Let $f \in S(V_i(F) \oplus F^2)$. By the Bruhat decomposition, it suffices to check the identity for $g \in T_2(F)$, $g \in N(F)$ and $g = (-1 \, 1)$.  

For $a \in F^\times$, one has
\[
\mathcal{F}_2 \circ \rho_{i+1}(a^{-1}) f(\xi, u_1, u_2) = \chi(a) |a|^{\dim V_i/2+1} \int_{F} f(a \xi, au_1, ax) \psi(u_2x) dx \\
= r_i(a^{-1}) \mathcal{F}_2(f)(\xi, u_1, u_2).
\]

For $t \in F$ one has
\[
\mathcal{F}_2 \circ \rho_{i+1}(1) f(\xi, u_1, u_2) = \mathcal{F}_2(\psi(tQ_{i+1}(\cdot))f)(\xi, u_1, u_2) \\
= \int_{F} f(\xi, u_1, x) \psi(tQ_i(\xi) + tu_1x) \psi(u_2x) dx \\
= \psi(tQ_i(\xi)) \mathcal{F}_2(f)(\xi, u_1, u_2 + tu_1) \\
= r_i(1) \mathcal{F}_2(f)(\xi, u_1, u_2).
\]

Moreover
\[
\mathcal{F}_2 \circ \rho_{i+1}(-1) f(\xi, u_1, u_2) \\
= \gamma \int_{F} \psi(u_2x) \left( \int_{V_i(F) \times F^2} f(w, w_1, w_2) \psi(\langle \xi, w \rangle_i + u_1w_2 + xw_1) dw_1 dw_2 dx \right) dx \\
= \gamma \int_{F \times V_i(F)} f(w, -u_2, w_2) \psi(\langle \xi, w \rangle_i + u_1w_2) dw_2 dx \\
= r_i(-1) \mathcal{F}_2(f)(\xi, u_1, u_2).
\]

This transform plays key role in the proof of our global theorem, see Lemma 5.3 below. It is also important because it is what allows us to view $S(X_i(F))$ as a representation of $GO_{V_{i+1}}(F)$, not just $GO_{V_i}(F)$ as we now explain. By Lemma 4.2 the isomorphism (4.6) yields an isomorphism
\[
\mathcal{F}_2 : S(V_{i+1}(F))_{\rho_{i+1}(SL_2(F))} \longrightarrow S(V_i(F) \oplus F^2)_{r_i(SL_2(F))} =: S(X_i(F)).
\]

We have an action
\[
L : GO_{V_{i+1}}(F) \times S(V_{i+1}(F))_{\rho_{i+1}(SL_2(F))} \longrightarrow S(V_{i+1}(F))_{\rho_{i+1}(SL_2(F))} \quad (h, f) \mapsto (x \mapsto f(h^{-1}x)).
\]

(4.10)

Thus using $\mathcal{F}_2$ we obtain an action $GO_{V_{i+1}}(F)$ on $S(V_i(F) \oplus F^2)$ defined by transport of structure along $\mathcal{F}_2$:
\[
(4.11) \quad \sigma_i(h) := \mathcal{F}_2 \circ L(h) \circ \mathcal{F}_2^{-1}.
\]

Define the semidirect product $SL_2(F) \rtimes GO_{V_i}(F)$ via
\[
(g \rtimes h)(g' \rtimes h') := g \left( \frac{1}{\lambda(h)} \right) g' \left( \frac{1}{\lambda(h)^{-1}} \right) \rtimes hh'.
\]
The group $\text{SL}_2(F) \rtimes \text{GO}_{V_{i+1}}(F)$ acts on $\mathcal{S}(V_{i+1}(A_F))$ via

$$\rho_{i+1}(g \rtimes h)f := \rho_{i+1}(g)(L(h)f)$$

(see [GL19, §3.1]). Thus $r_i$, originally defined as a representation of $\text{SL}_2(F)$, extends to an action

$$r_i : \text{SL}_2(F) \rtimes \text{GO}_{V_{i+1}}(F) : \mathcal{S}(V_i(F) \oplus F^2) \rightarrow \mathcal{S}(V_i(F) \oplus F^2)$$

given by

$$r_i(g \rtimes h)f := r_i(g)(\sigma_i(h)f).$$

Thus $\sigma_i$ descends to an action

$$\sigma_i : \text{GO}_{V_{i+1}}(F) \times \mathcal{S}(X_i(F)) \rightarrow \mathcal{S}(X_i(F)).$$

For $f \in \mathcal{S}(F^2)$ let

$$\mathcal{F}_\lambda(f)(v) := \int_{F^2} f(w)\psi(w \wedge v)dw$$

$$= \int_{F^2} f \left( \frac{w_1}{w_2} \right) \psi(w_1v_2 - w_2v_1)dw_1dw_2.$$  

(4.13)

Thus $\mathcal{F}_\lambda$ is an $\text{SL}_2(F)$-equivariant Fourier transform:

$$\mathcal{F}_\lambda : \mathcal{S}(F^2) \rightarrow \mathcal{S}(F^2).$$

(4.14)

We extend it to $\mathcal{S}(V_{i+1}(F))$ by setting $\mathcal{F}_\lambda := 1_{\mathcal{S}(V_i(F))} \otimes \mathcal{F}_\lambda$. It clearly descends to a linear isomorphism

$$\mathcal{F}_X : \mathcal{S}(X_i(F)) \rightarrow \mathcal{S}(X_i(F)).$$

(4.15)

It is useful to explicitly compute how the action of $\text{GO}_{V_{i+1}}(F)$ interacts with the operator $I$. We use the notation on groups from §3.1.

**Proposition 4.3.** Let $f \in \mathcal{S}(X_i(F))$ and $\xi \in X_i^*(F)$. For $a \in F^\times$, $h \in \text{GO}_{V_i}(F)$ and $x \in F^{\dim V_i}$ one has that

$$I (\sigma_i(h)f)(\xi) = |\lambda(h)|I(f)(h^{-1}\xi),$$

(4.16)

$$I \left( \sigma_i \left( \begin{array}{cc} 1 & iv_i \\ a & a^{-1} \end{array} \right) f \right)(\xi) = \chi(a)|a|^{1-\dim V_i/2}I(f)(a^{-1}\xi),$$

(4.17)

$$I (\sigma_i(u(x))f)(\xi) = \overline{\psi(x^t\xi)}I(f)(\xi),$$

(4.18)

$$\sigma_i \left( \begin{array}{cc} 1 & iv_i \\ 1 & 1 \end{array} \right) f = \mathcal{F}_X(f).$$

(4.19)

The similarity between these formulae and the formulae defining the Schrödinger model of the Weil representation is apparent. Similar models were investigated in [Kaz90, KP04, GK19] and in [KM11] when $F = \mathbb{R}$. 
Proof. Changing variables $g \mapsto \left(1_{\lambda(h)} \right) g \left(1_{\lambda(h)}^{-1} \right)$ we have
\[
I(\sigma_i(h)f)(\xi) = \int_{N(F) \backslash \operatorname{SL}_2(F)} r_i(g)\sigma_i(h)f(\xi, 0, 1)dg
\]
\[
= |\lambda(h)| \int_{N(F) \backslash \operatorname{SL}_2(F)} r_i(1 \rtimes h)r_i(g)f(\xi, 0, 1)dg
\]
\[
= |\lambda(h)|I(f)(h^{-1}\xi).
\]
For $a \in F^\times$ one has
\[
\sigma_i\left( I_{V_i}^a a_{a^{-1}}^{-1} \right) f(\xi, \xi', \xi')
\]
\[
= \int_F \left( \int_F f(\xi, a^{-1}\xi', x)\bar{\psi}(ayx)dx \right) \psi(\xi'y)dy
\]
\[
= |a|^{-1}f(\xi, a^{-1}\xi_1', a^{-1}\xi_2')
\]
\[
= \chi(a)|a|^{-\dim V_i/2-1}r_i\left( a_{a^{-1}}^{-1} \right) \sigma_i\left( a_{I_{V_i}}^a 1 \right) f(\xi, \xi', \xi').
\]
Thus
\[
I \left( \sigma_i\left( I_{V_i}^a a_{a^{-1}}^{-1} \right) f \right)(\xi)
\]
\[
= \chi(a)|a|^{-\dim V_i/2-1} \int_{N(F) \backslash \operatorname{SL}_2(F)} r_i(g \rtimes 1)r_i\left( a_{a^{-1}}^{-1} \right) \sigma_i\left( a_{I_{V_i}}^a 1 \right) f(\xi, 0, 1)dg
\]
\[
= \chi(a)|a|^{-\dim V_i/2-1} \int_{N(F) \backslash \operatorname{SL}_2(F)} r_i(g)\sigma_i\left( a_{I_{V_i}}^a 1 \right) f(\xi, 0, 1)dg
\]
\[
= \chi(a)|a|^{-\dim V_i/2-1}I \left( \sigma_i\left( a_{I_{V_i}}^a 1 \right) f \right)(\xi).
\]
Here we have changed variables $g \mapsto g \left( a_{a^{-1}}^{-1} \right)^{-1}$. We now apply (4.16) to see that the above is
\[
\chi(a)|a|^{-\dim V_i/2+1}I(f)(a^{-1}\xi).
\]
For $f \in \mathcal{S}(V_i(F) \oplus F^2)$ one has
\[
L(u(x))f(x, \xi', \xi') = \int_F f(\xi - Jx\xi', \xi', v)\overline{\psi((x^t\xi - Q_i(x)\xi' + \xi')v)dv}.
\]
Thus
\[
\sigma_i(u(x))f(\xi, \xi', \xi') = \int_F \left( \int_F f(\xi - Jx\xi', \xi', v)\overline{\psi((x^t\xi - Q_i(x)\xi' + u)v)dv} \right) \psi(\xi'u)du
\]
\[
= \int_F \left( \int_F f(\xi - Jx\xi', \xi', v)\overline{\psi(uv)dv} \right) \psi(\xi'(x^t\xi + Q_i(x)\xi'))du
\]
\[
= f(\xi - Jx\xi', \xi', \xi')\psi(\xi'(x^t\xi + Q_i(x)\xi')).
\]
Since $\sigma_i(O_{V_i+1}(F))$ commutes with $r_i(\operatorname{SL}_2(F))$ we deduce (4.18).
We now prove (4.19). Temporarily let \( f \in \mathcal{S}(F^2) \). Letting \( L(h) f(\xi) = f(h^{-1} \xi) \) as usual one has

\[
L(1^1_1) F_2^{-1}(f)(\frac{y}{y}) = L(1^1_1) \left( \frac{y}{y} \right) \mapsto \int_{F} f\left( \frac{y}{y} \right) \psi(-yu)dv = \int_{F} f\left( \frac{y}{y} \right) \psi(-yu)dv.
\]

Applying \( F_2 \) yields

\[
(4.22) \quad \int_{F^2} f\left( \frac{y}{y} \right) \psi(-yu)dv = F_\lambda(f) \left( \frac{y}{y} \right)
\]

and (4.19) follows. \( \square \)

**Corollary 4.4.** For \( h \in GO_{V_i}(F) \) one has

\[
I(F_{X_i} \circ \sigma_i(h)f)(\xi) = |\lambda(h)|^{dim V_i/2} I(F_{X_i}(f)) (\lambda(h)h^{-1} \xi).
\]

**Proof.** One has that

\[
\begin{pmatrix} I_{V_i} \\ 1_1 \end{pmatrix} \begin{pmatrix} h \\ \lambda(h) \end{pmatrix} = \begin{pmatrix} \lambda(h)^{-1}h \\ \lambda(h)^{-1} \lambda(h)^{-1} \end{pmatrix} \lambda(h) I_{V_{i+1}} \begin{pmatrix} I_{V_i} \\ 1_1 \end{pmatrix}
\]

so by (4.16) and (4.19) we have

\[
I(F_{X_i}(\sigma_i(h)f))(\xi) = |\lambda(h)|^{-1} I(\sigma_i(\lambda(h) I_{V_{i+1}}) F_{X_i}(f)) (\lambda(h)h^{-1} \xi).
\]

Now

\[
\lambda(h) I_{V_{i+1}} = \begin{pmatrix} I_{V_i} \\ \lambda(h)^{-1} \lambda(h) \\ \lambda(h)^2 \end{pmatrix}
\]

so by (4.16) and (4.17) we have

\[
|\lambda(h)|^{-1} I(\sigma_i(\lambda(h) I_{V_{i+1}}) F_{X_i}(f)) (\lambda(h)h^{-1} \xi)
\]

\[
= \chi(\lambda(h)) |\lambda(h)|^{dim V_i/2 - 2} I \left( \sigma_i \left( \begin{pmatrix} \lambda(h) \\ \lambda(h)^2 \end{pmatrix} \right) F_{X_i}(f) \right) (\lambda(h)^2 h^{-1} \xi)
\]

\[
= \chi(\lambda(h)) |\lambda(h)|^{dim V_i/2} I(F_{X_i}(f)) (\lambda(h)h^{-1} \xi).
\]

Since \( \chi \) is trivial on the image of the similitude norm [GL19, Lemma 3.2] we deduce the corollary. \( \square \)

We will require another family of operators on \( \mathcal{S}(V_i(F) \oplus F^2) \). For \( f \in \mathcal{S}(V_i(F) \oplus F^2) \) and \( s \in \mathbb{C} \) define

\[
(4.23) \quad Z_{r_i}(f, s) := \int_{N(F) \backslash \text{SL}_2(F)} e^{H_B(g)(2 - dim V_i/2 - s)} r_i(g) f(0_{V_i}, 0, 1) d\hat{g},
\]

\[
(4.24) \quad Z_{\rho_{i+1}}(f, s) := \int_{N(F) \backslash \text{SL}_2(F)} e^{H_B(g)(1 - dim V_i/2 + s)} \rho_{i+1}(g) f(0_{V_i}, 0, 1) d\hat{g}.
\]
Lemma 4.5. The integral $Z_{r_i}(f, s)$ is a Tate integral in the following sense: If

$$
\Psi_f(x) := \int_K r_i(k)f(0, 0, x) \, dk
$$

then

$$
Z_{r_i}(f, s) = Z(\Psi_f, \chi \cdot |^s) := \int_{F^\times} \Psi_f(a)\chi(a)|^s d^\times a.
$$

In view of the lemma, the integral $Z_{r_i}(f, s)$ is meromorphic as a function of $s$ and is a well-defined function of $f \in S(X_i(F))$.

Proof. By the Iwasawa decomposition one has

$$
Z_{r_i}(f, s) = \int_{F^\times \times K} \chi(a)|^s(r_i(k)f)(0_{V_i}, 0, a^{-1}) \, dk d^\times a
$$

$$
= \int_{F^\times \times K} \chi(a)|^s(r_i(k)f)(0_{V_i}, 0, a) \, dk d^\times a.
$$

Lemma 4.6. Let $f \in S(V_i(F) \oplus F^2)$. For $a \in F^\times$, $h \in O_{V_i}(F)$ and $x \in F^{\dim V_i}$ one has

$$
Z_{r_i}(\sigma_i(h)f, s) = Z_{r_i}(f, s),
$$

$$
Z_{r_i}(\sigma_i \left( \begin{array}{cc} I_{V_i} & a \\ 0 & a^{-1} \end{array} \right) f, s) = \chi(a)|^s \chi(a)Z_{r_i}(f, s),
$$

$$
Z_{r_i}(\sigma_i(u(x))f, s) = Z(f, s).
$$

Proof. The first assertion is clear. Similarly we compute

$$
Z \left( \sigma_i \left( \begin{array}{cc} I_{V_i} & a \\ 0 & a^{-1} \end{array} \right) f, s \right) = \int_{F^\times \times K} \chi(b)|^s \left( \sigma_i \left( \begin{array}{cc} I_{V_i} & a \\ 0 & a^{-1} \end{array} \right) r_i(k)f \right) (0, 0, b) d^\times b.
$$

Using (4.20) this is

$$
\int_{F^\times \times K} |b|^s \chi(b)|a|^{-1} (r_i(k)f)(0, 0, a^{-1}b) d^\times b
$$

$$
= |a|^{s-1} \chi(a) \int_{F^\times \times K} \chi(b)|^s (r_i(k)f)(0, 0, b) d^\times b.
$$

The proof of the last formula is the same as the proof of (4.18) in Proposition 4.3.

We now prove some useful properties of the Schwartz space. By a minor variant of the argument proving [GH20, Lemma 5.7] one obtains the following lemma:

Lemma 4.7. One has

$$
S(V_i(F))|_{X_i^\times(F)} < I(S(X_i(F))).
$$
For nonarchimedean $F$ with ring of integers $\mathcal{O}$ we say that the image of $1_{V_i(\mathcal{O})\oplus\mathcal{O}^2}$ in $\mathcal{S}(X_i(F))$ is the basic function. Let $q$ be the order of the residue field of $\mathcal{O}$. Computing as in Lemma 4.1 one obtains the following:

**Lemma 4.8.** Let $F$ be a nonarchimedean local field of characteristic 0. Assume that $\psi$ is unramified, that $F$ is unramified over its prime field, and that $J_i \subseteq \text{GL}_{V_i}(\mathcal{O})$. For $\xi \in X_i^0(F)$ one has that

$$I(1_{V_i(\mathcal{O})\oplus\mathcal{O}^2})(\xi) := \sum_{k=0}^{\infty} \chi(\omega^k)q^{k(dim V_i/2-2)}1_{V_i(\mathcal{O})}(\frac{\xi}{\omega^k}).$$

Moreover

$$\int_{N(F)\setminus \text{SL}_2(F)}|r_i(g)1_{V_i+1(\mathcal{O})}(\xi, 0, 1)|dg$$

is convergent for all $\xi \in V_i^0(F)$. It is supported in $V_i(\mathcal{O}) \cap V_i^0(F)$ and is bounded by

$$\begin{cases}
|\xi|^{2-dim V_i/2} \log_q |\xi| & \text{if } \dim V_i > 4 \\
\log_q |\xi| & \text{if } \dim V_i = 4 \text{ or } 2.
\end{cases}$$

If $F$ is nonarchimedean, unramified over its prime field, $\psi$ is unramified, and $J_i \subseteq \text{GL}_{V_i}(\mathcal{O})$ then it is clear that $\mathcal{F}_{X_i}$ preserves the basic function.

5. **Integrals of truncated isotropic theta functions**

For $f \in \mathcal{S}(V_{i+1}(\mathbb{A}_F))$ with $i \geq 0$ and $g \in \text{SL}_2(\mathbb{A}_F)$ let

$$(5.1) \quad \Theta_f(g) := \sum_{\xi \in V_{i+1}(F)} \rho_{i+1}(g)f(\xi).$$

We refer to this as an isotropic theta function because $Q_{i+1}$ is isotropic.

For $T \in \mathbb{R}_{>0}$ and suitable functions $\varphi$ on $[\text{SL}_2]$ we then define

$$(5.2) \quad \varphi^T(g) := \Lambda^T \varphi(g) := \varphi(g) - \sum_{\gamma \in B(F)\setminus \text{SL}_2(F)} 1_{>T}(H_B(\gamma g)) \int_{[N]} \varphi(n\gamma g)dn.$$ 

Here $1_{>T}$ is the characteristic function of $\mathbb{R}_{>T}$. This is the usual truncation in the special case of $\text{SL}_2$ (in Arthur’s notation, $1_{>0}$ is $7_B$). For fixed $g$ the sum over $\gamma$ in this expression is finite.

For $f \in \mathcal{S}(V_i(\mathbb{A}_F) \oplus \mathbb{A}_F^2) = \mathcal{S}(V_{i+1}(\mathbb{A}_F))$ and $s \in \mathbb{C}$ define

$$(5.3) \quad Z_{r_i}(f, s) := \int_{N(\mathbb{A}_F)\setminus \text{SL}_2(\mathbb{A}_F)} e^{H_B(g)(2-dim V_i/2-s)}r_i(g)f(0, 0, 1) dg,$$

$$Z_{\rho_{i+1}}(f, s) := \int_{N(\mathbb{A}_F)\setminus \text{SL}_2(\mathbb{A}_F)} e^{H_B(g)(1-dim V_i/2+s)}\rho_{i+1}(g)f(0, 0, 1) dg.$$ 

This is the global version of (4.23). The integral $Z_{r_i}(f, s)$ is a Tate integral by Lemma 4.5.
The main theorem of this section is the following:

**Theorem 5.1.** Let $f \in \mathcal{S}(V_{i+1}(\mathbb{A}_F))$. As $T \to \infty$
\begin{align*}
\int_{[\text{SL}_2]} \Theta_T^f(g) dg &= \sum_{\xi \in X_i^\circ(F)} I(\mathcal{F}_2(f)) (\xi) + \int_{[\text{SL}_2]} \Theta_T^{\mathcal{F}_2(f)(\cdot,0,0)}(g) dg \\
&\quad + \left( \sum_{s_i \in \{ \frac{\dim V_i}{2} - 1, \frac{\dim V_i}{2} - 2, 0 \}} \text{Res}_{s=\bar{s}} e^{Ts} Z_{r_i}(\mathcal{F}_2(f), s + 2 - \frac{\dim V_i}{2}) \right) + o_f(1).
\end{align*}

Here
\[ \Theta_{\mathcal{F}_2(f)(\cdot,0,0)}(g) := \sum_{\xi \in V_i(F)} \rho_i(g) \mathcal{F}_2(f)(\xi,0,0) \]

where $\rho_i$ acts via its action on $\mathcal{S}(V_i(\mathbb{A}_F))$. We give the proof at the end of this section. By induction on $i$ this reduces the study of $\int_{[\text{SL}_2]} \Theta_T^f(g) dg$ to a special case to be treated in §6.

**Lemma 5.2.** One has
\[ Z_{r_i}(f,s) = Z_{\rho_{i+1}}(\mathcal{F}_2^{-1}(f), 1-s) \]
as meromorphic functions in $s$.

**Proof.** Let
\[ \Psi_f(x) := \int_K r_i(k) f(0,0,x) dk. \]

One has
\begin{align*}
Z_{r_i}(f,s) &= \int_{\mathbb{A}_F^\times K} \chi(a) |a|^{-s} r_i(k) f(0,0,a^{-1}) dk d^\times a \\
&= \int_{\mathbb{A}_F^\times K} \chi(a) |a|^s r_i(k) f(0,0,a) dk d^\times a \\
&= Z(\Psi_f, \chi \cdot |^s).
\end{align*}

Let $\hat{\Psi}_f(x) := \int_{\mathbb{A}_F} \Psi_f(y) \overline{\psi}(xy) dy$. Then we have
\[ Z(\Psi_f, \chi \cdot |^s) = Z(\hat{\Psi}_f, \chi \cdot |^{1-s}) \]
by Tate’s functional equation. Let us expand $Z(\hat{\Psi}_f, \chi \cdot |^s)$ for $\text{Re}(s)$ sufficiently large. It is equal to
\begin{align*}
Z(\hat{\Psi}_f, \chi \cdot |^s) &= \int_{\mathbb{A}_F^\times K} \chi(a) |a|^s \mathcal{F}_2^{-1}(r_i(k)f)(0,0,a) dk d^\times a \\
&= \int_{\mathbb{A}_F^\times K} \chi(a) |a|^{-1-\dim V/2+s} \rho_{i+1} (a^{-1}) \mathcal{F}_2^{-1}(r_i(k)f)(0,0,1) dk d^\times a.
\end{align*}
By Lemma 4.2 $\mathcal{F}_2^{-1} \circ r_i = \rho_{i+1} \circ \mathcal{F}_2^{-1}$. Thus the above is

$$\int_{\mathbb{A}_F^\times \times K} \chi(a)|a|^{-1-\dim V/2+s} \rho_{i+1} \left( (\begin{smallmatrix} 1 & a^{-1} \\ 0 & 1 \end{smallmatrix}) k \right) \mathcal{F}_2^{-1}(f)(0,0,1) \, dkd^\times a$$

$$= \int_{N(\mathbb{A}_F) \backslash SL_2(\mathbb{A}_F)} e^{H_B(g)(1-\dim V/2+s)} \rho_{i+1}(g) \mathcal{F}_2^{-1}(f)(0,0,1) \, dg$$

$$= Z_{\rho_{i+1}}(\mathcal{F}_2^{-1}(f), s).$$

□

**Lemma 5.3.** The integral $\int_{[SL_2]} \Theta_f^T(g) \, dg$ is equal to

$$\int_{[SL_2]} \Theta_{\mathcal{F}_2(f)(0,0)}^T(g) \, dg + \int_{N(\mathbb{A}_F) \backslash SL_2(\mathbb{A}_F)} \left( 1_{\leq T}(H_B(g)) \sum_{\xi \in X_i(F)} r_i(n g) \mathcal{F}_2(f)(\xi, 0, 1) \right.$$

$$\left. - 1_{> T}(H_B(g)) \sum_{\xi \in V_i(F)} \int_{N(\mathbb{A}_F)} r_i(n g) \mathcal{F}_2(f)(\xi, 1, 0) \, dn \right) \, dg.$$  

Here is where we make key use of the transform $\mathcal{F}_2$. It converts the integral of the truncated $\Theta$-function into an object that can be unfolded. Before beginning the proof we make some remarks on convergence issues. The lemma reduces the absolute convergence of $\int_{[SL_2]} \Theta_f^T(g) \, dg$ to the absolute convergence of $\int_{[SL_2]} \Theta_{\mathcal{F}_2(f)(0,0)}^T(g) \, dg$ together with the absolute convergence of the other summand. The absolute convergence of the other summand is checked in propositions 5.4, 5.5, and 5.6. So by induction we are reduced to the absolute convergence statement in Lemma 6.1. This justifies the manipulations below.

**Proof.** By Lemma 4.2 and Poisson summation we obtain

$$(5.4) \quad \Theta_f(g) = \sum_{\xi \in V_{i+1}(F)} r_i(g) \mathcal{F}_2(f)(\xi).$$

Thus

$$\int_{[SL_2]} \Theta_f^T(g) \, dg = \int_{[SL_2]} \left( \sum_{(\xi, \xi', \xi'') \in V_i(F) \oplus F^2} r_i(g) \mathcal{F}_2(f)(\xi, \xi', \xi'') \right.$$  

$$\left. - \sum_{\gamma \in B(F) \backslash SL_2(F)} 1_{\leq T}(H_B(\gamma g)) \int_{[N]} \sum_{(\xi, \xi', \xi'') \in V(F) \oplus F^2} r_i(n \gamma g) \mathcal{F}_2(f)(\xi, \xi', \xi'') \, dn \right) \, dg.$$  

We separate the contributions of $(\xi', \xi'') = (0,0)$ and $(\xi', \xi'') \neq (0,0)$ to write this as the sum of

$$\int_{[SL_2]} \left( \sum_{\xi \in V_i(F)} r_i(g) \mathcal{F}_2(f)(\xi, 0, 0) - \sum_{\gamma \in B(F) \backslash SL_2(F)} 1_{> T}(H_B(\gamma g)) \int_{[N]} \sum_{\xi \in V_i(F)} r_i(n \gamma g) \mathcal{F}_2(f)(\xi, 0, 0) \right) \, dg$$

$$= \int_{[SL_2]} \Theta_{\mathcal{F}_2(f)(0,0)}^T(g) \, dg$$
Thus, the expression is equal to

\[ \sum_{(\xi, \xi_1, \xi_2) \in V_i(F) \oplus F^2} r_i(g)F_2(f)(\xi, \xi_1, \xi_2) - \sum_{\gamma \in B(F) \setminus SL_2(F)} 1 > T(H_B(\gamma g)) \int_{[N]} \sum_{(\xi, \xi_1, \xi_2)} r_i(n \gamma g)F_2(f)(\xi, \xi_1, \xi_2) dn \] \]

where the sums are over \((\xi, \xi_1, \xi_2) \in V_i(F) \oplus F^2\) such that \((\xi_1', \xi_2') \neq (0, 0)\). The latter expression is equal to

\[ \int_{B(F) \setminus SL_2(\mathbb{A}_F)} \left( \sum_{(\xi, \alpha) \in V_i(F) \times F^\times} r_i(g)F_2(f)(\xi, 0, \alpha) - 1 > T(H_B(g)) \int_{[N]} \sum_{(\xi, \xi_1', \xi_2')} r_i(n g)F_2(f)(\xi, \xi_1', \xi_2') dn \right) dg \]

\[ = \int_{N(F) \setminus SL_2(\mathbb{A}_F)} \left( \sum_{\xi \in V_i(F)} r_i(g)F_2(f)(\xi, 0, 1) - 1 > T(H_B(g)) \int_{[N]} \sum_{\xi \in V_i(F)} \sum_{(\xi_1', \xi_2') \sim} r_i(n g)F_2(f)(\xi, \xi_1', \xi_2') dn \right) dg. \]

The last sum is over \((\xi_1', \xi_2') \in F^2 - \{(0, 0)\}\) up to equivalence, where \((\xi_1', \xi_2')\) is equivalent to \((\alpha \xi_1', \alpha^{-1} \xi_2')\) for all \(\alpha \in F^\times\). Using the definition of the Weil representation this becomes

\[ \int_{N(\mathbb{A}_F) \setminus SL_2(\mathbb{A}_F)} \left( \sum_{\xi \in X_i(F)} r_i(g)F_2(f)(\xi, 0, 1) \right. \]

\[ \left. - 1 > T(H_B(g)) \int_{[N]} \sum_{\xi \in V_i(F)} \sum_{(\xi_1', \xi_2') \sim} r_i(n g)F_2(f)(\xi, \xi_1', \xi_2') dn \right) dg. \]

(5.5)

For each \(\xi \in V_i(F)\) one has

\[ \int_{[N]} \sum_{(\xi_1', \xi_2') \sim} r_i(n g)F_2(f)(\xi, \xi_1', \xi_2') dn \]

\[ = \int_{[N]} r_i(n g)F_2(f)(\xi, 0, 1) dn + \int_{[N]} \sum_{\xi \in F} r_i(n g)F_2(f)(\xi, 1, \alpha) dn. \]

The left summand vanishes unless \(Q_i(\xi) = 0\), in which case it is equal to \(r_i(g)F_2(f)(\xi, 0, 1)\).

Thus (5.5) is equal to

\[ \int_{N(\mathbb{A}_F) \setminus SL_2(\mathbb{A}_F)} \left( 1 \leq T(H_B(g)) \sum_{\xi \in X_i(F)} r_i(g)F_2(f)(\xi, 0, 1) \right. \]

\[ \left. - 1 > T(H_B(g)) \int_{[N]} \sum_{\xi \in V_i(F)} \int_{N(\mathbb{A}_F)} r_i(n g)F_2(f)(\xi, 1, 0) dn \right) dg. \]

(5.6)

We now break the second summand in Lemma 5.3 into three pieces that we compute in the following three propositions:

**Proposition 5.4.** The expression

\[ \int_{N(\mathbb{A}_F) \setminus SL_2(\mathbb{A}_F)} 1 \leq T(H_B(g)) \sum_{\xi \in X_i(F)} |r_i(g)F_2(f)(\xi, 0, 1)| dg \]
converges and one has
\[
\lim_{T \to \infty} \int_{N(\mathbb{A}_F) \setminus SL_2(\mathbb{A}_F)} \mathbbm{1}_{\leq T}(H_B(g)) \sum_{\xi \in X^0_i(F)} r_i(g) F_2(f)(\xi, 0, 1) d\dot{g} = \sum_{\xi \in X^0_i(F)} I(F_2(f))(\xi).
\]

**Proof.** It is easy to see from Lemma 4.1 and Lemma 4.8 that
\[
\sum_{\xi \in V^0_i(F)} \int_{N(\mathbb{A}_F) \setminus SL_2(\mathbb{A}_F)} |r_i(g) F_2(f)|((\xi, 0, 1)) d\dot{g}
\]
is absolutely convergent. The proposition follows. 

**Proposition 5.5.** The integral
\[
\int_{N(\mathbb{A}_F) \setminus SL_2(\mathbb{A}_F)} \mathbbm{1}_{\leq T}(H_B(g)) r_i(g) F_2(f)(0_{V_i}, 0, 1) d\dot{g}
\]
converges absolutely and is equal to
\[
\sum_{s_i \in \left\{ \frac{\dim V_i}{2} - 1, \frac{\dim V_i}{2} - 2, 0 \right\}} \text{Res}_{s=s_i} \frac{e^{Ts} Z_{r_i}(F_2(f), s+2 - \frac{\dim V_i}{2})}{s} + o_f(1)
\]
as \(T \to \infty\).

**Proof.** The absolute convergence statement is a trivial consequence of the Iwasawa decomposition and well-known facts on Tate integrals.

We now prove the asymptotic formula. We have
\[
\int_{N(\mathbb{A}_F) \setminus SL_2(\mathbb{A}_F)} \mathbbm{1}_{\leq T}(H_B(g)) r_i(g) F_2(f)(0_{V_i}, 0, 1) d\dot{g}
\]
\[
= \int_{\mathbb{A}_F^\times K} 1_{\leq T} |\log|a|| \chi(a) |a|^{\frac{\dim V_i}{2} r_i(k) F_2(f)(0_{V_i}, 0, a^{-1})} \frac{dk d^\times a}{|a|^2}
\]
(5.7)

We wish to apply Mellin inversion to this expression. For Re\((s) > 0\) we have
\[
\int_{\mathbb{A}_{cm}} 1_{\leq T} |\log|a|| |a|^s d^\times a = \frac{e^{Ts}}{s}
\]
(5.8)

and for Re\((s)\) sufficiently small we have
\[
\int_{\mathbb{A}_F^\times K} \chi(a) |a|^{\frac{\dim V_i}{2} + r_i(k) F_2(f)(0_{V_i}, 0, a^{-1})} \frac{dk d^\times a}{|a|^2} = Z_{r_i}(F_2(f), 2 - \frac{\dim V_i}{2} - s).
\]
(5.9)

Hence (5.7) is equal to
\[
\frac{1}{2\pi i} \int_{i\mathbb{R}+\sigma} e^{Ts} Z_{r_i}(F_2(f), s + 2 - \frac{\dim V_i}{2}) \frac{ds}{s}
\]
for \(\sigma\) sufficiently large.
We now shift the contour to \(\sigma\) very small to see that the integral in the proposition is equal to

\[
\sum_{s_i \in \left\{ \frac{\dim V_i}{2} - 1, \frac{\dim V_i}{2} - 2, 0 \right\}} \text{Res}_{s=s_i} e^{Ts} Z_{r_i}(\mathcal{F}_2(f), s + 2 - \frac{\dim V_i}{2}) + o_f(1).
\]

\[\square\]

**Proposition 5.6.** As \(T \to \infty\)

\[
\int_{N(\mathbb{A}_F) \setminus \text{SL}_2(\mathbb{A}_F)} 1_{>T}(H_B(g)) \int_{N(\mathbb{A}_F)} \sum_{\xi \in V_i(F)} |r_i(n g)\mathcal{F}_2(f)(\xi, 1, 0)| d\dot{g} = o_f(1).
\]

**Proof.** The integral in the proposition is equal to

\[
\int_{\mathbb{A}_F^\times \times K} \left( 1_{>T}(|a|) \sum_{\xi \in V_i(F)} |a|^{\dim V_i/2 - 2} |r_i(k)\mathcal{F}_2(f)| (a\xi, a^{-1} x) dx d^\times d\dot{k}
\]

\[
= \int_{\mathbb{A}_F^\times \times K} \left( 1_{>T}(|a|) \sum_{\xi \in V_i(F)} |a|^{\dim V_i/2 - 1} |r_i(k)\mathcal{F}_2(f)| (a\xi, a, x) dx \right) d^\times d\dot{k}.
\]

The integral over \(a\) is a truncated Tate integral, and by well-known properties of Tate integrals it converges absolutely for all \(T\). It becomes smaller as \(T\) becomes larger since the integrand is Schwartz as a function of \(\xi\). \(\square\)

**Proof of Theorem 5.1.** This is immediate from Lemma 5.3 and propositions 5.4, 5.5, and 5.6. \(\square\)

### 6. Integrals of Truncated Anisotropic Theta Functions

Assume for this section that \(V_i := V_0\) is an even dimensional vector space equipped with a nondegenerate anisotropic quadratic form \(Q_0\). Let \(f \in \mathcal{S}(V_0(\mathbb{A}_F))\). We refer to \(\Theta_f(g)\) as an **anisotropic theta function**. We allow the special case where \(V_0 = \{0\}\). In this case we define \(\mathcal{S}(V_0(\mathbb{A}_F)) := \mathbb{C}\) and the Weil representation is taken to be the trivial representation of \(\text{SL}_2(\mathbb{A}_F)\).

Our aim is to compute \(\int_{\text{SL}_2} \Theta_f^T(g) dg\). Since \(Q_0\) is anisotropic we cannot reduce this computation to a smaller quadratic space as we did above. Instead, we apply a variant of the classical Rankin-Selberg method. Let

\[
\Phi_s(g) := \int_{\mathbb{A}_F^\times} \Phi((0 \ i) \ g) |t|^{2s} d^\times t.
\]

Moreover let

\[
E(g, \Phi_s) := \sum_{\xi' \in F^2 - \{0\}} \Phi_s(\xi' g).
\]
Then, as is well-known, \( E(g, \Phi_s) \) converges absolutely for \( \Re(s) \) large enough and admits a meromorphic continuation to the plane. Its residue at \( s = 1 \) is

\[
\hat{\Phi}(0) := \frac{1}{2} \int_{A^2} \Phi(x, y)dx dy.
\]

In particular the residue is independent of \( g \). For all of this we refer the reader to [JZ87, §1]. Define

\[
(6.3) \ 
\kappa := \begin{cases} \meas([\text{SL}_2]) & \text{if } \dim V_0 = 0 \\ 0 & \text{otherwise}. \end{cases}
\]

**Lemma 6.1.** The integral \( \int_{[\text{SL}_2]} \Theta^T_f(g) dg \) converges absolutely. If \( \dim V_0 = 0 \)

\[
\lim_{T \to \infty} \int_{[\text{SL}_2]} \Theta^T_f(g) dg = \kappa f(0).
\]

If \( \dim V_0 \geq 4 \), then \( \int_{[\text{SL}_2]} \Theta^T_f(g) dg \) is a polynomial in \( e^T \) and if \( 0 < \dim V_0 < 4 \), it is a polynomial and \( e^{-T} \). If \( \chi \neq 1 \) then this polynomial vanishes identically. If \( \chi = 1 \) then the constant term of the polynomial is 0.

**Proof.** The lemma is easy to check when \( \dim V_0 = 0 \). Thus for the remainder of the proof we assume \( \dim V_0 > 0 \). Assume that \( \Phi \in S(\mathbb{A}^2_F) \) and \( \Phi(\xi'k) = \Phi(\xi') \) for all \( (\xi', k) \in F^\times \setminus \{0\} \times K \). Assume moreover that \( \hat{\Phi}(0) \neq 0 \). Then by the comments before the statement of Lemma 6.1 we have

\[
\frac{\hat{\Phi}(0)}{2} \int_{[\text{SL}_2]} \Theta^T_f(g) dg = \int_{[\text{SL}_2]} \text{Res}_{s=1} E(g, \Phi_s) \Theta^T_f(g) dg.
\]

The function \( \Theta^T_f(g) \) is rapidly decreasing on \( [\text{SL}_2] \) (see [Art80]) and hence using [JS81, Lemma 4.2] we deduce that the above is equal to

\[
(6.4) \ 
\text{Res}_{s=1} \int_{[\text{SL}_2]} E(g, \Phi_s) \Theta^T_f(g) dg.
\]

We also deduce that \( \int_{[\text{SL}_2]} E(g, \Phi_s) \Theta^T_f(g) dg \) has at most a simple pole at \( s = 1 \). One has

\[
\Theta^T_f(g) = \Theta_f(g) - \sum_{\gamma \in B(F) \setminus [\text{SL}_2]} \mathbb{1}_{>T}(H_B(\gamma g)) \int_{[N]} \Theta_f(n\gamma g) dn
\]

\[
= \Theta_f(g) - \sum_{\gamma \in B(F) \setminus [\text{SL}_2]} \mathbb{1}_{>T}(H_B(\gamma g)) \rho_0(\gamma g)f(0).
\]

Thus for \( \Re(s) \) sufficiently large,

\[
\int_{[\text{SL}_2]} E(g, \Phi_s) \Theta^T_f(g) dg
\]

\[
= \int_{[\text{SL}_2]} \sum_{\gamma' \in B(F) \setminus [\text{SL}_2]} \Phi_s(\gamma'g) \left( \Theta_f(g) - \sum_{\gamma \in B(F) \setminus [\text{SL}_2]} \mathbb{1}_{>T}(H_B(\gamma g)) \rho_0(\gamma g)f(0) \right) dg
\]
\[ = \int_{B(F)\backslash SL_2(\mathbb{A}_F)} \Phi_s(g) \left( \Theta_f(g) - \sum_{\gamma \in B(F)\backslash SL_2(F)} 1_{>T}(H_B(\gamma g)) \rho_0(\gamma g) f(0) \right) dg \]

\[ = \int_{T_2(F)N(\mathbb{A}_F)\backslash SL_2(\mathbb{A}_F)} \Phi_s(g) \int_{[N]} \left( \Theta_f/ng - \sum_{\gamma \in B(F)\backslash SL_2(F)} 1_{>T}(H_B(\gamma ng)\rho_0(\gamma ng) f(0)) \right) dngd\gamma. \]

This is equal to

\[ \int_{T_2(F)N(\mathbb{A}_F)\backslash SL_2(\mathbb{A}_F)} \Phi_s(g) \left( \rho_0(g) f(0) 1_{\leq T}(H_B(g)) \right. \]

\[ \left. - \int_{N(\mathbb{A}_F)\backslash SL_2(\mathbb{A}_F)} 1_{>T}(H_B((-1)^{-1} ng)\rho_0((-1)^{-1} ng) f(0)) dngd\gamma. \right. \]

We break (6.5) into two summands. The first is

\[ \int_{T_2(F)N(\mathbb{A}_F)\backslash SL_2(\mathbb{A}_F)} 1_{\leq T}(H_B(g)) \Phi_s(g) \rho_0(g) f(0) d\gamma \]

\[ = \Phi_s(I_2) \int_{[G_m] \times K} 1_{\leq T}(\log |a|)|a|^{2s+\dim V_0/2} \chi(a) \rho_0(k) f(0) d^s a \frac{dk}{|a|^2}. \]

This vanishes unless \( \chi = 1 \). Assuming \( \chi = 1 \) and \( \text{Re}(s) \) is sufficiently large we see that it is equal to

\[ \Phi_s(I_2) \text{meas}(F\backslash (\mathbb{A}_F)^1) \int_{K} \rho_0(k) f(0) dk \int_{0}^{e^T} r^{2s+\dim V_0/2-3} dr \]

\[ = \Phi_s(I_2) \text{meas}(F\backslash (\mathbb{A}_F)^1) \int_{K} \rho_0(k) f(0) dk e^{T(2s+\dim V_0/2-2)} 2s + \frac{\dim V_0}{2} - 2. \]

The residue of this expression at \( s = 1 \) is zero for \( \dim V_0 > 0 \).

The second summand of (6.5) is

\[ - \int_{T_2(F)N(\mathbb{A}_F)\backslash SL_2(\mathbb{A}_F)} \Phi_s(g) \rho_0((-1)^{-1} ng) f(0) 1_{>T}(H_B((-1)^{-1} ng)) dngd\gamma. \]

\[ - \int_{T_2(F)\backslash SL_2(\mathbb{A}_F)} \Phi_s(g) \rho_0((-1)^{-1} g) f(0) 1_{>T}(H_B((-1)^{-1} g)) dg \]

\[ - \int_{T_2(F)\backslash SL_2(\mathbb{A}_F)} \Phi_s((-1)^{-1} g) \rho_0(g) f(0) 1_{>T}(H_B(g)) dg \]

\[ - \int_{K} \int_{[G_m] \times K} \int_{\mathbb{A}_F} \Phi(ta^{-1}) |t|^{2s+\dim V_0/2} \chi(a) \rho_0(k) f(0) 1_{>T}(\log |a|) d^s a \frac{dk}{|a|^2}. \]

We change variables \( t \mapsto ta^{-1} \) and then \( x \mapsto t^{-1} x a^2 \) to see that this is

\[ - \int_{K} \int_{[G_m] \times \mathbb{A}_F} \int_{\mathbb{A}_F} \Phi(t x) |t|^{2s-1} d^s x |a|^{\dim V_0/2-2s} \chi(a) \rho_0(k) f(0) 1_{>T}(\log |a|) d^s x \frac{dk}{|a|^2}. \]
This vanishes unless $\chi$ is trivial, in which case it is equal to
\[
- \int_K \rho_0(k) f(0) dk \text{meas}(F^x \setminus (A_F^x)^1) \int_{e^T}^{\infty} t^{\dim V_0 / 2 - 2s - 1} dt \int_{A_F} \int_{A_F} \Phi(t x) |t|^{2s - 1} d^x t dx
\]
(6.6)
\[
= \int_K \rho_0(k) f(0) dk \text{meas}(F^x \setminus (A_F^x)^1) \frac{e^{T(\dim V_0 / 2 - 2s)}}{2s - \dim V_0 / 2} \int_{A_F} \int_{A_F} \Phi(t x) |t|^{2s - 1} d^x t dx.
\]
Assume for the moment that $\dim V_0 = 4$. If $\int_K \rho_0(k) f(0) dk = 0$ then this expression vanishes. If $\int_K \rho_0(k) f(0) dk \neq 0$ then this expression has a pole of order 2 at $s = 1$ for suitably chosen $\Phi$. This contradicts the fact that $\int_{[SL_2]} E(g, \Phi_s) \Theta_f^T(g) dg$ has at most a simple pole at $s = 1$. Thus if $\dim V_0 = 4$ and $\chi$ is nontrivial we are done.

Assume $\dim V_0 \neq 4$. Then the pole of (6.6) at $s = 1$ is simple with residue equal to
\[
\text{meas}(F^x \setminus (A_F^x)^1) \int_K \rho_0(k) f(0) dk \frac{\hat{\Phi}(0) e^{T(\dim V_0 / 2 - 2)}}{2(2 - \dim V_0 / 2)}.
\]
This is a polynomial in $e^T$ when $\dim V_0 > 4$ and a polynomial in $e^{-T}$ when $0 < \dim V_0 < 4$. In either case the constant term is term 0.

\section{7. Proof of Theorem 1.2}

For $i > 0$ we defined a linear form
\[
(7.1) \quad c_i : S(X_i(A_F)) \rightarrow \mathbb{C}
\]
in (1.11).

\textbf{Lemma 7.1.} As $T \rightarrow \infty$
\[
\sum_{s_i \in \left\{ \frac{\dim V_i}{2} - 1, \frac{\dim V_i}{2} - 2, 0 \right\}} \operatorname{Res}_{s = s_i} e^{Ts} Z_{rs_i}(f, s + 2 - \frac{\dim V_i}{2}) \frac{e^{T(s + 2 - \frac{\dim V_i}{2})}}{s}
\]
is a polynomial in $T$ and $e^T$ plus $o_f(1)$. The constant term of the polynomial is $c_i(f)$.

\textbf{Proof.} The lemma follows immediately from Lemma 4.5 and well-known facts about Tate zeta functions. \hfill \Box

\textbf{Proof of Theorem 1.3.} By Theorem 5.1 we have
\[
\int_{[SL_2]} \Theta_f^T(g) dg = \sum_{\xi \in X_f^0(F)} I(F_2(f))(\xi) + \int_{[SL_2]} \Theta_f^{T_{F_2(f)}(0,0)}(g) dg
\]
\[
+ \left( \sum_{s_i \in \left\{ \frac{\dim V_i}{2} - 1, \frac{\dim V_i}{2} - 2, 0 \right\}} \operatorname{Res}_{s = s_i} e^{Ts} Z_{rs_i}(F_2(f), s + \frac{\dim V_i}{2} - 2) \right) + o_f(1).
\]
By induction we obtain
\[
\int_{[SL_2]} \Theta^T_f(g) dg = \int_{[SL_2]} \Theta^T_{d_{\ell,0}}(f_2(f))(g) dg + \sum_{\ell=1}^\ell \left( \sum_{\xi \in X_2^0(F)} I(d_i(F_2(f)))(\xi) + \sum_{s_i \in \{\dim V_i - 1, \dim V_i - 2, 0\}} \text{Res}_{s=s_i} e^{Ts} Z_{s_i}(F_2(f), s + 2 - \frac{\dim V_i}{2}) \right) + o_f(1).
\]

We now conclude using Lemma 6.1 and Lemma 7.1. \qed

**Proof of Theorem 1.2.** By Poisson summation and the fact that $F_{X_i}$ is SL2($A_F$)-invariant we have
\[
\sum_{\xi \in V_i(F) \oplus F^2} r_i(g)f(\xi) = \sum_{\xi \in V_i(F) \oplus F^2} r_i(g)F_{X_i}(f)(\xi).
\]
By (5.4) this implies we have
\[
\Theta_{F_{X_i}}^{-1}(f)(g) = \Theta_{F_{X_i}}^{-1}(F_{X_i}(f))(g).
\]
Thus
\[
\int_{[SL_2]} \Theta^T_{F_{X_i}}^{-1}(f)(g) dg = \int_{[SL_2]} \Theta^T_{F_{X_i}}^{-1}(F_{X_i}(f))(g) dg.
\]
We conclude using Theorem 1.3. \qed

In applications of Poisson summation the behavior of the functions involved under scaling plays a key role. Since this takes some thought to work out we make it explicit:

**Corollary 7.2.** Assume $\ell > 0$, and that either $\dim V_\ell > 4$ or $\chi \neq 1$. For $a \in A^-_F$ and $f \in S(X_\ell(\mathbb{A}_F))$ one has
\[
|a|^{1-\dim V_\ell/2} \chi(a) c_\ell(f) + \chi(a)|a|^{1-\dim V_\ell/2} \sum_{\xi \in X_\ell^0(F)} I(f)(a^{-1}\xi)
\]
\[
+ |a|^{-1} \sum_{i=1}^{\ell-1} \left( c_i(d_{\ell,i}(f)) + \sum_{\xi \in X_\ell^0(F)} I(d_{\ell,i}(f))(\xi) \right) + |a|^{-1} \kappa d_{\ell,0}(f)(0_{V_0}, 0, 0)
\]
\[
= |a|^{\dim V_\ell/2-1} \chi(a) c_\ell(F_{X_i}(f)) + \chi(a)|a|^{\dim V_\ell/2-1} \sum_{\xi \in X_\ell^0(F)} I(F_{X_i}(f))(a\xi)
\]
\[
+ |a| \sum_{i=1}^{\ell-1} \left( c_i(d_{\ell,i}(F_{X_i}(f))) + \sum_{\xi \in X_\ell^0(F)} I(d_{\ell,i}(F_{X_i}(f)))(\xi) \right) + |a| \kappa d_{\ell,0}(F_{X_i}(f))(0_{V_0}, 0, 0).
\]

**Proof.** Let $f \in S(V_\ell(\mathbb{A}_F) \oplus \mathbb{A}_F^2)$. By (4.17) of Proposition 4.3 we have
\[
I(\sigma_\ell(\frac{I\nu_\ell a}{a^{-1}}) f)(\xi) = \chi(a)|a|^{1-\dim V_\ell/2} I(f)(a^{-1}\xi)
\]

(7.2)
and by (4.17) and (4.19) of Proposition 4.3 we have

\[ I \left( F_{X_{\ell}} \left( \sigma_{\ell} \left( I_{V_{\ell}} a^{-1} \right) \right) f \right) (\xi) = I \left( \sigma_{\ell} \left( \left( I_{V_{\ell}} a^{-1} \right) \left( I_{V_{\ell}} a^{-1} \right) \right) \right) f (\xi) 
\]

\[ = I \left( \sigma_{\ell} \left( \left( \sum_{i=1}^{\dim V_{\ell}} \chi_{i} \right) \right) \right) f (\xi) 
\]

\[ = I \left( \sigma_{\ell} \left( I_{V_{\ell}} a^{-1} \right) F_{X_{\ell}}(f) \right) (\xi) 
\]

\[ = \chi(a) |a|^{\dim V_{\ell}/2} I (F_{X_{\ell}}(f)) (a \xi). \]

Similarly by Lemma 4.6

\[ c_{\ell} \left( \sigma_{\ell} \left( I_{V_{\ell}} a^{-1} \right) f \right) = |a|^{1-\dim V_{\ell}/2} \chi(a) c_{\ell}(f), \]

\[ c_{\ell} \left( F_{X_{\ell}} \left( \sigma_{\ell} \left( I_{V_{\ell}} a^{-1} \right) \right) f \right) = |a|^{\dim V_{\ell}/2-1} \chi(a) c_{\ell}(F_{X_{\ell}}(f)). \]

On the other hand

\[ d_{\ell} \left( \sigma_{\ell} \left( I_{V_{\ell}} a^{-1} \right) f \right) = |a|^{-1} d_{\ell}(f), \quad d_{\ell} \left( F_{X_{\ell}} \left( \sigma_{\ell} \left( I_{V_{\ell}} a^{-1} \right) \right) f \right) = |a| d_{\ell}(F_{X_{\ell}}(f)). \]

Thus applying Theorem 1.2 to the function \( \sigma_{\ell} \left( I_{V_{\ell}} a^{-1} \right) f \) we arrive at the asserted identity.

There is an analogue of Corollary 7.2 that is valid in the case \( \dim V_{\ell} = 4 \) and \( \chi = 1 \) as well. We omit it because it is slightly messier to state.

8. Invariance

We would like to view the identity of our summation formula Theorem 1.2 as an identity of linear functionals on the space of \( \text{SL}_{2}(\mathbb{A}_{F}) \)-coinvariants \( \mathcal{S}(X_{\ell}(\mathbb{A}_{F})) \), not just \( \mathcal{S}(V_{\ell}(\mathbb{A}_{F}) \oplus \mathbb{A}_{F}^{2}) \).

Provided that \( \dim V_{\ell} \geq 4 \) as we have assumed throughout the paper we prove that this is possible in the current section.

To prove that a given functional present in the identity of Theorem 1.2 descends to \( \mathcal{S}(X_{\ell}(\mathbb{A}_{F})) \) it is sufficient to check that it is invariant under \( \text{SL}_{2}(\mathbb{A}_{F}) \), or more briefly invariant. It is obvious that the linear functional \( I \) is invariant. Moreover, when \( V_{0} = \{0\} \) it is obviously true that the functional \( f \mapsto f(0_{V_{0}}, 0, 0) \) is invariant.

For this section we take the convention that \( c_{i} = 0 \) if \( i \leq 0 \). To complete our discussion of invariance it suffices to prove the following theorem:

**Theorem 8.1.** Assuming \( \dim V_{i} \not\in \{2, 4\} \), the functionals \( c_{i} \) are \( r_{i}(\text{SL}_{2}(\mathbb{A}_{F})) \)-invariant. If \( \dim V_{i} = 4 \) then

\[ \mathcal{S}(V_{i}(\mathbb{A}_{F}) \oplus \mathbb{A}_{F}^{2}) \longrightarrow \mathbb{C} 
\]

\[ f \mapsto c_{i}(f) + c_{i-1}(d_{i}(f)) \]

is \( r_{i}(\text{SL}_{2}(\mathbb{A}_{F})) \)-invariant.
The theorem implies that if dim \( V_i \geq 4 \) (as we have assumed throughout this paper) all of the linear functionals in the summation formula of Theorem 1.2 are invariant, at least after possibly grouping two of the linear functionals together.

We first prove a special case of Theorem 8.1:

**Lemma 8.2.** If dim \( V_i \notin \{2, 4\} \) then \( c_i \) is \( r_i(\text{SL}_2(\mathbb{A}_F)) \)-invariant.

**Proof.** By Lemma 4.2 and Lemma 5.2 it suffices to show that
\[
\mathcal{S}(V_{i+1}(\mathbb{A}_F)) \rightarrow \mathbb{C}
\]
\[
f \mapsto Z_{\rho_{i+1}}(f, \frac{\dim V_i}{2} - 1)
\]
is invariant under the action of \( \rho_{i+1}(\text{SL}_2(\mathbb{A}_F)) \). But this is clear. \( \square \)

**Lemma 8.3.** Assume dim \( V_i = 4 \) and that \( f \in \mathcal{S}(V_{i+1}(\mathbb{A}_F) \oplus \mathbb{A}_F^2) \). The difference
\[
c_i(d_{i+1}(f)) + c_{i-1}(d_{i+1,i-1}(f))
- (c_i(d_{i+1}f_{X_{i+1}}(f)) + c_{i-1}(d_{i+1,i-1}f_{X_{i+1}}(f)))
\]
is invariant under \( f \mapsto r_{i+1}(h)f \) for \( h \in \text{SL}_2(\mathbb{A}_F) \).

**Proof.** By Theorem 1.2 the quantity in the statement of the lemma is equal to
\[
- \sum_{j=1}^{i+1} \sum_{\xi \in X_0^j(F)} I(d_{i+1,j}(f))((\xi) + \sum_{j=1}^{i+1} \sum_{\xi \in X_0^j(F)} I(d_{i+1,j}f_{X_{i+1}}(f))((\xi)
- \kappa d_{i+1,0}(f)(0, 0, 0) + \kappa d_{i+1,0}f_{X_{i+1}}(f)(0, 0, 0)
- c_{i+1}(f) + c_{i+1}f_{X_{i+1}}(f).
\]
By our comments at the beginning of this section and Lemma 8.2 each of these terms is invariant under \( f \mapsto r_{i+1}(h)f \) for \( h \in \text{SL}_2(\mathbb{A}_F) \). \( \square \)

**Proof of Theorem 8.1.** If dim \( V_i \notin \{2, 4\} \) then the theorem follows from Lemma 8.2. Thus we assume that dim \( V_i = 4 \). We must show that \( f \mapsto c_i(f) + c_{i-1}(d_i(f)) \) is \( r_i(\text{SL}_2(\mathbb{A}_F)) \)-invariant. Let \( f \in \mathcal{S}(V_i(\mathbb{A}_F) \oplus \mathbb{A}_F^2) \). Pick \( \Phi \in \mathcal{S}(\mathbb{A}_F^2) \) such that \( \Phi(0, 0) = 1 \) and \( f_\lambda(\Phi)(0, 0) = 0 \). Then
\[
F_2^{-1}(f) \otimes \Phi \in \mathcal{S}(V_{i+1}(\mathbb{A}_F) \oplus \mathbb{A}_F^2),
\]
and by Lemma 4.2 and Lemma 8.3
\[
c_i(d_{i+1}(F_2^{-1}(f) \otimes \Phi)) + c_{i-1}(d_{i+1,i-1}(F_2^{-1}(f) \otimes \Phi))
- (c_i(d_{i+1}(F_2^{-1}(f) \otimes F_\lambda(\Phi))) + c_{i-1}(d_{i+1,i-1}(F_2^{-1}(f) \otimes F_\lambda(\Phi))))
\]
is invariant under \( f \otimes \Phi \mapsto r_i(h) \otimes L^\vee(h)(f \otimes \Phi) \) for \( h \in \text{SL}_2(\mathbb{A}_F) \). But the above is
\[
c_i(f)(0, 0) + c_{i-1}(d_{i,i-1}(f))(0, 0)
- c_i(f)(0, 0) + c_{i-1}(d_{i,i-1}(f))(0, 0)
= c_i(f) + c_{i-1}(d_{i,i-1}(f))
\]
and we deduce the proposition. \( \square \)
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