Polarization phase gate with a tripod atomic system

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We analyze the nonlinear optical response of a four-level atomic system driven into a tripod configuration. The large cross-Kerr nonlinearities that occur in such a system are shown to produce nonlinear phase shifts of order $\pi$. Such a substantial shift may be observed in a cold atomic gas in a magneto-optical trap where it could be feasibly exploited towards the realization of a polarization quantum phase-gate. The experimental feasibility of such a gate is here examined in detail.

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I. INTRODUCTION

A great effort has recently gone into the search for practical architecture for quantum information processing systems. While most attention has been devoted toward theoretical issues, several strategies have also been proposed for experimental investigations. However, the laboratory demand for building quantum information devices is quite severe, requiring strong coupling between qubits, the quantum carriers of information, in an environment with minimal dissipation. For this reason experimental progress has so far lagged behind the remarkable development that quantum information theory now witnesses 1.

Here we focus on optical implementations of quantum information processing systems. Travelling optical pulses are the natural candidates for the realization of quantum communication schemes and many experimental demonstrations of quantum key distribution 2–8 and quantum teleportation schemes 9–15 have been already performed. Optical systems have also been proposed for the implementation of quantum computing, even though the absence of significant photon-photon interactions is an obstacle for the realization of efficient two-qubit quantum gates, which are needed for implementing universal quantum computation 16. Various schemes have been proposed to circumvent this problem. One is linear optics quantum computation 17, which is a probabilistic scheme based on passive linear optical devices, efficient single photon sources and detectors and which implicitly exploits the nonlinearity hidden in the photodetection process (see 18–20 for some preliminary demonstrations of this scheme). Other schemes explicitly exploit optical nonlinearities for quantum gate implementations. Typical optical nonlinearities are too small to provide a substantial photon-photon interaction, hence limiting the usefulness of an all-optical quantum gate. However there seems to be a way to overcome the problem. Quantum interference effects associated with electromagnetically induced transparency (EIT) 21–23 have quite recently been shown to enhance these nonlinearities by as much as 10 orders of magnitude 24. This enhancement is commonly exhibited by a weak probe beam in the presence of another strong coupling beam when both impinge off-resonance on a three-level atomic sample at very low temperatures.

The off-resonance condition is rather crucial to the observation of the enhancement and one can, in general, identify two ways for attaining that. One is to introduce an additional laser beam whose detuning from a fourth level is larger than the level linewidth 24. In this “N” configuration one of the ground levels undergoes an ac-Stark shift which disturbs the EIT resonance condition and induces an effective Kerr nonlinearity while keeping absorption negligible. Improvements by many orders of magnitude with respect to conventional nonlinearities have indeed been observed in this way 25. In addition, strong cross-phase modulation 26 and photon blockade (i.e. strong self-phase modulation) have also been predicted 27–29.

Another and related way to obtain large nonlinearities consists in disturbing the exact two-photon resonance condition in a Λ configuration. This can be achieved by slightly mismatching the probe and coupling field frequencies yet within the EIT transparency window making the dispersion of the probe field not exactly zero. In this case enhanced Kerr nonlinearities have been observed in the Λ configuration 30–32 and predicted in the so-called chain-Λ configurations 33–37. By using this second approach Ottaviani et al. 38 have shown that large cross-phase modulations that occur in an “M” configuration may lead to an all-optical two-qubit quantum phase gate (QPG) 16, where one qubit gets a phase shift dependent on the state of the other qubit. Here, the key element enabling large cross-phase modulation is the possibility of group velocity matching. Large cross-phase modulations occur when two optical pulses, a probe and a trigger, interact for a sufficiently long time.

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This happens when their group velocities are both small and comparable \( \delta_1 \simeq 16^\circ \) and there exists several ways by which this can be done \( 24, 31, 33 \).

This paper proposes an alternative scheme for phase gating that can greatly reduce, when compared with other schemes, the experimental effort for its realizability. The mechanism relies on an enhanced cross-phase modulation effect which occurs in a relatively simple and robust four atomic level tripod configuration. Our scheme only requires good control over frequencies and intensities of the laser beams. As in Ref. \( 27 \), we consider a QPG for qubits in which binary information is encoded in the polarization of an optical field.

Optical QPG have been already experimentally studied. A conditional phase shift \( \phi \simeq 8^\circ \) between two frequency-distinct high-Q cavity modes, due to the effective cross modulation mediated by a beam of Cs atoms, has been measured in Ref. \( 34 \). However, the complete truth table of the gate has not been determined in this experiment. A conditional phase shift \( \phi \simeq 8^\circ \) has been instead obtained between weak coherent pulses, using a second-order nonlinear crystal \( 35 \). However, this experiment did not demonstrate a bona fide QPG because \( \phi \) depends on the input states, and the gate can be defined only for a restricted class of inputs (weak coherent states).

The four level tripod configuration that we adopt here has been extensively studied in the past few years. For example, Unanyan et al. \( 36 \) used a tripod configuration to achieve stimulated Raman adiabatic passage (STIRAP) for creating an arbitrary coherent superposition of two atomic states in a controlled way. Paspalakis et al. \( 37, 38, 39 \), in particular, developed the interesting possibility of using a tripod scheme for efficient nonlinear frequency generation. Moreover, it was shown that the group velocity of a probe pulse may be significantly reduced, as in conventional A system \( 37 \). The work of Malakyan \( 40 \) was the first to hint that the tripod scheme may be used to entangle a pair of very weak optical fields in an atomic sample. This work has been recently extended to the case of quantum probe and trigger fields in \( 41 \), where an adiabatic treatment similar to that of \( 31 \) is adopted.

The purpose of this paper is thus twofold. First, we adopt a standard density matrix approach, including spontaneous emission and dephasing, to analyse the nonlinear optical response of a four-level tripod configuration. In particular, we examine the conditions under which large cross-Kerr nonlinearities may occur in a cold atomic sample. Second, we study the possibility of employing such an enhanced cross-phase modulation to devise a polarization phase-gating mechanism which turns out to be rather robust and apt to actual experimental investigations.

The paper is organized as follows. In Sec. \( \text{III} \), dressed states of the atomic tripod are analyzed and their significance emphasized. In Sec. \( \text{III} \), we solve the set of Bloch equations and derive expressions for linear and nonlinear susceptibilities. In Sec. \( \text{IV} \), group velocity matching is discussed in detail, while Sec. \( \text{V} \) discusses the operation of a polarization phase gate. We summarize our results in Sec. \( \text{VI} \).

\section*{II. DRESSED STATES OF THE TRIPOD SYSTEM}

The energy level scheme of a tripod system is given in Fig. \( 1 \). Transitions \( |1\rangle \to |0\rangle \) and \( |3\rangle \to |0\rangle \) are driven by a probe and trigger fields have Rabi frequencies \( \Omega_p \) and \( \Omega_T \) and polarizations \( \sigma_+ \) and \( \sigma_- \). The pump Rabi frequency is \( \Omega \) while \( \delta_j = \omega_0 - \omega_j - \omega_j^{(L)} \) denote the laser (frequency \( \omega_j^{(L)} \)) detunings from the respective transitions \( |j\rangle \leftrightarrow |0\rangle \).

FIG. 1: Energy level scheme for a tripod. Probe and trigger fields have Rabi frequencies \( \Omega_p \) and \( \Omega_T \) and polarizations \( \sigma_+ \) and \( \sigma_- \). The pump Rabi frequency is \( \Omega \) while \( \delta_j = \omega_0 - \omega_j - \omega_j^{(L)} \) denote the laser (frequency \( \omega_j^{(L)} \)) detunings from the respective transitions \( |j\rangle \leftrightarrow |0\rangle \).

There are four eigenstates of Hamiltonian \( 41, 36 \). When the three detunings are equal, \( \delta_i = \delta, i = 1, 2, 3 \), two of them are degenerate with energy equal to \( \delta \) and assume the following form:

\[
|e_1\rangle = \frac{\Omega_T|1\rangle + \Omega_p|3\rangle}{\sqrt{\Omega_p^2 + \Omega_T^2}}, \quad (2a)
\]

\[
|e_2\rangle = \frac{\Omega_T\Omega_p|1\rangle + \Omega_T\Omega_p|3\rangle - (\Omega_p^2 + \Omega_T^2)|2\rangle}{\sqrt{(\Omega_p^2 + \Omega_T^2)(\Omega_p^2 + \Omega_T^2)}}. \quad (2b)
\]

Since these states do not contain any contribution of the excited state \( |0\rangle \), they belong to the class of dark states.
states. The other two eigenstates have energies \( \delta \pm \sqrt{\Omega_P^2 + \Omega_T^2 + \Omega_T^2} \) and are
\[
|e_{\pm}\rangle = \frac{\Omega_P|1\rangle \pm |0\rangle + \Omega_T|3\rangle + \Omega|2\rangle}{\sqrt{\Omega_P^2 + \Omega_T^2 + \Omega_T^2}}. \tag{3}
\]
In the case of different detunings, the expression of the eigenstates becomes more complicated, and the degeneracy of the two dark states is removed because their energies shift from \( \delta \) to \( \delta_i \) and \( \delta_j \) respectively.

The scope of the present paper is to show that the tripod configuration of Fig. 1 allows to achieve a giant cross-Kerr phase shift can be formulated as follows: (i) probe and trigger fields must be tuned to dark states, (ii) the transparency frequency window for each of these dark states has to be narrow and with a steep dispersion to enable significant group velocity reduction, and (iii) there must be a degree of symmetry between the two transparency windows so that trigger and probe group velocities can be made to be equal. These conditions can be satisfied by taking all three detunings nearly equal. When the three detunings are equal the two dark states are degenerate, giving a common transparency window for both fields. However, it can be seen that for perfectly equal detunings the tripod system is linear, i.e., the dispersive nonlinearity vanishes [see Eqs. (9)]. Hence, the exact resonance of symmetry between the two transparency windows so that coherences and to obtain the steady state solution for the latter, yielding the probe and trigger susceptibilities according to
\[
\chi_P = -\lim_{t \to \infty} \frac{N|\mu_P|^2}{\hbar \epsilon_0} \times \frac{\rho_{10}(t)}{\Omega_P}, \tag{5a}
\]
\[
\chi_T = -\lim_{t \to \infty} \frac{N|\mu_T|^2}{\hbar \epsilon_0} \times \frac{\rho_{30}(t)}{\Omega_T}. \tag{5b}
\]
where \( N \) is the atomic density and \( \mu_{P,T} \) the electric dipole matrix elements for probe and trigger transitions respectively. Rabi frequencies are defined in terms of electric field amplitudes \( E_{P,T} \), as \( \Omega_{P,T} = - (\mu_{P,T} \cdot \mathbf{E}_{P,T})/\hbar \), with \( \mathbf{E}_{P,T} \) being the linear polarization unit vector of probe and trigger beams. The resulting general expression for the steady-state (ss) probe and trigger susceptibilities are obtained from
\[
\begin{align*}
\langle \rho_{10} \rangle_{ss} & = \frac{1}{\Omega_P} \left( 1 + \frac{1}{4} \frac{(\Delta_{12}\Delta_{23}/\Delta_{13}^2)|\mu_P|^2|\Omega_T|^2}{(\Omega_{10}\Delta_{23} - |\Omega|^2)} \right) \left( 1 - \frac{1}{2} \frac{\Delta_{12}\Delta_{13}}{\Delta_{12}\Delta_{13} - \Delta_{13}^2|\Omega|^2 - \Delta_{12}|\Omega_T|^2} \right), \\
\langle \rho_{30} \rangle_{ss} & = \frac{1}{\Omega_T} \left( 1 + \frac{1}{4} \frac{(\Delta_{23}\Delta_{12}/\Delta_{13}^2)|\mu_P|^2|\Omega_T|^2}{(\Omega_{30}\Delta_{23} - |\Omega|^2)} \right) \left( 1 - \frac{1}{2} \frac{\Delta_{23}\Delta_{13}}{\Delta_{23}\Delta_{13} - \Delta_{13}^2|\Omega|^2 - \Delta_{23}|\Omega_T|^2} \right).
\end{align*}
\]
We are interested in the cross-phase modulation between the probe and trigger fields. Therefore, we keep the two lowest order contributions in trigger and probe: linear and third-order nonlinear susceptibilities, while neglecting the higher orders in the expansion. This yields
\[
\begin{align*}
\chi_P & = \chi_P^{(1)} + \chi_P^{(3)} |E_T|^2, \\
\chi_T & = \chi_T^{(1)} + \chi_T^{(3)} |E_P|^2.
\end{align*}
\]
that is, each susceptibility has a linear and a cross-Kerr nonlinear term, while self-phase modulation terms are of higher order. Both susceptibilities have a linear contribution because of the nonzero stationary population in

III. BLOCH EQUATIONS AND SUSCEPTIBILITIES

The Bloch equations for the density matrix elements (including atomic spontaneous emission and dephasing) are
\[
\begin{align*}
i\dot{\rho}_{00} & = -i(\gamma_{11} + \gamma_{33})\rho_{00} + \Omega_P\rho_{10} - \Omega_P\rho_{01} + \Omega^*\rho_{20} - \Omega_T\rho_{30} - \Omega_T\rho_{03}, \tag{4a}
i\dot{\rho}_{11} & = i\gamma_{11}\rho_{00} + i\gamma_{12}\rho_{22} + i\gamma_{13}\rho_{33} + \Omega_P\rho_{01} - \Omega_P^*\rho_{10}, \tag{4b}
i\dot{\rho}_{22} & = i\gamma_{22}\rho_{00} - i\gamma_{12}\rho_{11} + i\gamma_{23}\rho_{33} + \Omega\rho_{02} - \Omega^*\rho_{20}, \tag{4c}
i\dot{\rho}_{33} & = i\gamma_{33}\rho_{00} - i(\gamma_{13} + \gamma_{23})\rho_{11} + \Omega_T\rho_{30} - \Omega_T^*\rho_{03}, \tag{4d}
i\dot{\rho}_{10} & = -\Delta_{10}\rho_{10} + \Omega_P\rho_{00} - \Omega_P\rho_{11} - \Omega_P\rho_{12} - \Omega_T\rho_{13}, \tag{4e}
i\dot{\rho}_{20} & = -\Delta_{20}\rho_{20} + \Omega_P\rho_{00} - \Omega_P\rho_{22} - \Omega_P\rho_{21} - \Omega_T\rho_{23}, \tag{4f}
i\dot{\rho}_{30} & = -\Delta_{30}\rho_{30} + \Omega_T\rho_{00} - \Omega_P\rho_{33} - \Omega_P\rho_{31} - \Omega_T\rho_{32}, \tag{4g}
i\dot{\rho}_{12} & = -\Delta_{12}\rho_{12} + \Omega_P\rho_{02} - \Omega_T\rho_{10}, \tag{4h}
i\dot{\rho}_{13} & = -\Delta_{13}\rho_{13} + \Omega_P\rho_{03} - \Omega_T\rho_{10}, \tag{4i}
i\dot{\rho}_{23} & = -\Delta_{23}\rho_{23} + \Omega_P\rho_{03} - \Omega_T\rho_{20}. \tag{4j}
\end{align*}
\]
where \( \rho_{ij} = \text{Tr}[\sigma_i \rho]\) = \( \langle i|\rho|j\rangle \). Decay rates \( \gamma_{ij} \) describe decay of populations and coherences, \( \Delta_{ij} = \delta_j - \delta_i - i\gamma_{ij} \), with \( i, j = 1, 2, 3 \).
levels 1 and 3. Linear susceptibilities are given by

\[
\chi^{(1)}_P = \frac{N \mu_P^2}{\hbar c} \times \frac{\Delta_{12}}{2 \Delta_{10} \Delta_{12} - |\Omega|^2}, \quad (8a)
\]

\[
\chi^{(1)}_T = \frac{N \mu_T^2}{\hbar c} \times \frac{\Delta_{23}^*}{2 \Delta_{30} \Delta_{23} - |\Omega|^2}, \quad (8b)
\]

where the factor 1/2 in each equation comes from the symmetric steady state population distribution. The cross-Kerr susceptibilities are instead given by

\[
\chi^{(3)}_P = \frac{N \mu_P^2 \mu_T^2}{\hbar^3 c} \times \frac{\Delta_{12}/\Delta_{13}}{2 \Delta_{10} \Delta_{12} - |\Omega|^2}
\]

\[
\times \left( \frac{\Delta_{12}}{\Delta_{10} \Delta_{12} - |\Omega|^2} + \frac{\Delta_{23}}{\Delta_{30} \Delta_{23} - |\Omega|^2} \right), \quad (9a)
\]

\[
\chi^{(3)}_T = \frac{N \mu_T^2 \mu_P^2}{\hbar^3 c} \times \frac{\Delta_{12}/\Delta_{13}}{2 \Delta_{10} \Delta_{12} - |\Omega|^2}
\]

\[
\times \left( \frac{\Delta_{12}}{\Delta_{10} \Delta_{12} - |\Omega|^2} + \frac{\Delta_{23}}{\Delta_{30} \Delta_{23} - |\Omega|^2} \right). \quad (9b)
\]

Note that Eqs. (5) and also Eqs. (9) are completely symmetric with respect to the 1 ↔ 3 exchange. This exchange symmetry is ensured by the complex conjugate terms in (8a) and (8b) and it is expected because of the symmetry of the population distribution. Note also that in the absence of dephasing, the nonlinear susceptibility has a singularity at \( \delta_1 = \delta_3 \). The necessary regularization is provided by a nonzero dephasing term \( i \delta_1 \).

Paspalakis and Knight [37] have recently analyzed the properties of the tripod system in a somewhat different setup. It is nevertheless instructive to compare the results of this Section with theirs. In the scheme of [37], population is assumed to be initially in the ground state |1\>. Provided that \( |\Omega_P|^2 \ll |\Omega|^2, |\Omega_T|^2 \) population remains in |1\> in the steady state. Paspalakis and Knight calculate the expression for probe susceptibility to the first order in \( \Omega_P \). It is easy to see that their expression is consistent (up to a factor 1/2 determined by the different population distribution) to our result in Eq. (5): considering only terms to the first order in \( \Omega_P \) leaves only the first term in the curly brackets of (5). Additional terms in Eqs. (9) arise because we are looking for a cross-Kerr nonlinearity in both probe and trigger, so that all the terms of third order have to be included.

**IV. GROUP VELOCITY MATCHING**

The linear and nonlinear susceptibilities of Eqs. (5) and (9) have all the properties required for a large cross-phase modulation. In fact, our tripod system can be seen as formed by two adjacent \( \Lambda \) systems, one involving the probe field and one involving the trigger field, sharing the same control field. Therefore both fields exhibit EIT, which here manifests itself through the presence of two generally distinct transparency windows, corresponding to the two dark states of Eq. (2). Perfect EIT for both fields takes place when the two transparency windows coincide, i.e., when the two dark states are degenerate, which is achieved when the three detunings \( \delta_i \) are all equal. In this case, all physical effects related to standard EIT are present and in particular the steep dispersion responsible for the reduction of the group velocity which is at the basis of the giant cross-Kerr nonlinearity (see Fig. 2). The condition of equal detunings (exact double EIT-resonance condition) is important also for another reason. In fact, together with the symmetry of Eqs. (5) and (9) with respect to the 1 ↔ 3 exchange, it also guarantees identical dispersive properties for probe and trigger and therefore the same group velocity. As first underlined by Lukin and Imamoglu [31], group velocity matching is another fundamental condition for achieving a large nonlinear mutual phase shift because only in this way the two optical pulses interact in a transparent nonlinear medium for a sufficiently long time.

The group velocity of a light pulse is given in general by \( v_g = c/(1+n_g) \), where \( c \) is the speed of light in vacuum.
and

\[ n_g = \frac{1}{2} \text{Re}[\chi] + \frac{\omega_0}{2} \left( \frac{\partial \text{Re}[\chi]}{\partial \omega} \right) \omega_0 \]  

(10)

is the group index, \( \omega_0 \) being the laser frequency. The group index of Eq. (10) is essentially determined by the linear dispersion gradient. One has \( \text{Re}[\chi] \) vanishes, and the group velocity is reduced due to a large transparency window for each field, where \( \text{Re}[\chi] \) vanishes, and the group velocity is reduced due to a large dispersion gradient. One has

\[
(v_g)_P \approx \frac{4\hbar c_0}{\omega_P N |\mu_P|^2} (|\Omega|^2 + |\Omega_T|^2), \tag{11a}
\]

\[
(v_g)_T \approx \frac{4\hbar c_0}{\omega_T N |\mu_T|^2} (|\Omega_P|^2 + |\Omega|^2), \tag{11b}
\]

so that, as expected from the 1 \( \leftrightarrow \) 3 symmetry, group velocity matching is achieved for \( |\Omega_P| = |\Omega_T| \).

Unfortunately, it is possible to check from Eqs. (9) that when \( \delta_i = \delta, \forall i \) exactly, the system becomes linear, i.e., the real part of the nonlinear susceptibilities vanish and there is no cross-phase modulation. This means that we have to “disturb” the exact EIT resonance conditions, by taking slightly different detunings. This is a general conclusion, valid for any atomic level scheme resembling multiple A systems. \( \text{If the double EIT-resonance condition is disturbed by a small amount, one remains within the common transparency window and the absorption is still negligible. Moreover, the two group velocities can be matched also in the non-resonant case. In fact, from the symmetry of Eqs. (9), one has that the gradients - and hence the group velocities - can be kept symmetric and all the conclusions for the exact resonance remain valid in the vicinity of resonance as well.} \)

V. PHASE GATE OPERATION

A medium able to realize a significant cross-phase modulation is the key ingredient for the implementation of a quantum gate between two optical qubits. Such a gate requires the existence of conditional quantum dynamics, which is realized in the cross-Kerr effect where an optical field acquires a phase shift conditional to the state of another optical field. Using cross phase modulation one can implement a QPG, defined by the following input-output relations \( |i_1\rangle|j_2\rangle \to \exp \{i\phi_{ij}\} |i_1\rangle|j_2\rangle \), with \( i, j = 0, 1 \) denoting the logical qubit bases. This gate is a universal two-qubit gate, that is, it is able to entangle two initially factorized qubits, when the conditional phase shift \( \phi = \phi_{11} + \phi_{00} - \phi_{10} - \phi_{01} \neq 0 \).

A natural choice for encoding binary information in optical beams is to use the polarization degree of freedom, where the two logical basis states \( |0\rangle \) and \( |1\rangle \) correspond to two orthogonal polarizations. In this case one can implement a universal QPG if a nontrivial cross-phase modulation between probe and trigger fields arise for only one of the four possible input configurations of their polarization.

A possible experimental configuration employing the giant Kerr nonlinear phase shift achievable in the tripod scheme discussed above is provided by a \( ^{87}\text{Rb} \) atoms confined in a magneto-optical trap (MOT), in which states \( |1\rangle, |2\rangle \) and \( |3\rangle \) correspond to the ground state Zeeman sublevels \( |5\Sigma_{1/2}, F = 1, m = \{-1, 0, 1\} \), and state \( |0\rangle \) corresponds to the excited state \( |5\Sigma_{3/2}, F = 0\rangle \). One realizes the tripod scheme of Fig. 1 (and therefore a significant nonlinear phase shift) only when the probe has \( \sigma_+ \) polarization and the trigger has \( \sigma_- \) polarization. When either the probe or the trigger polarizations (or both) are changed, the phase shifts acquired by the two pulses do not involve the nonlinear susceptibilities and are different, so that the resulting conditional phase shift is nonzero. In fact, when they have the “wrong” polarization (probe \( \sigma_- \) polarized or trigger \( \sigma_+ \) polarized) there is no sufficiently close level which the atoms can be driven to and the fields acquire the trivial vacuum phase shift \( \phi_0^0 = k_0 l \), \( j = P, T \), where \( l \) in the length of the medium. Instead, when only one of them has the right polarization, it acquires a linear phase shift \( \phi_{lin}^j \), \( j = P, T \), where

\[
\phi_{lin}^j = k_0 l \left( 1 + 2\pi \chi^{(1)}_j \right). \tag{12}
\]

Denoting \( \phi_{lin}^{P,T} \) the corresponding probes and trigger nonlinear phase shift when the tripod configuration is realized, we arrive at the following truth table for the polarization QPG

\[
\begin{align*}
|\sigma_-\rangle_P |\sigma_-\rangle_T & \to e^{-i\phi_{lin}^P} |\sigma_-\rangle_P |\sigma_-\rangle_T, \tag{13a} \\
|\sigma_-\rangle_P |\sigma_+\rangle_T & \to e^{-i\phi_{lin}^P} |\sigma_-\rangle_P |\sigma_+\rangle_T, \tag{13b} \\
|\sigma_+\rangle_P |\sigma_+\rangle_T & \to e^{-i\phi_{lin}^P} |\sigma_+\rangle_P |\sigma_+\rangle_T, \tag{13c} \\
|\sigma_+\rangle_P |\sigma_-\rangle_T & \to e^{-i\phi_{lin}^P} |\sigma_+\rangle_P |\sigma_-\rangle_T. \tag{13d}
\end{align*}
\]

with the conditional phase shift being

\[
\phi = \phi_+^P + \phi_+^T - \phi_{lin}^P - \phi_{lin}^T, \tag{14}
\]

with \( \phi_+^P = \phi_{lin}^P + \phi_{lin}^T \) and \( \phi_T^T = \phi_{lin}^P + \phi_{lin}^T \). Notice that only the nonlinear part contributes to the conditional phase shift. The truth table of Eqs. (13) differs from that of Ottaviani et al. \( \text{[25]} \), only in the presence of an additional linear phase shift for the trigger field, which is a consequence of the fact that also level 3 is populated by one half of the atoms.

For a Gaussian trigger pulse of time duration \( \tau_T \), whose peak Rabi frequency is \( \Omega_T \), moving with group velocity \( \mu_T \) through the atomic sample, the nonlinear probe phase shift can be written as

\[
\phi_{lin}^P = k_0 l \left( \frac{3}{2} \hbar |\Omega|^2 \right) \frac{\text{erf}[\sqrt{\zeta_P}] - \text{erf}[\sqrt{\zeta_P}]}{4|\mu|^2} |\chi^{(3)}_P|, \tag{15a}
\]
where $\zeta = (1 - v_P^d/v_P^p)\sqrt{2}/v_P^p \tau_P$. The trigger phase shift is simply obtained by changing $P \leftrightarrow T$ in the equation above

$$\phi_{\text{lin}}^{\text{T}} = k_T \frac{3}{2} \frac{\Omega_T^2 |\mu_P|^2}{|\Omega_T|^2} \frac{\text{erf}[\zeta_T]}{\zeta_T} \text{Re}[\chi_T^{(3)}],$$

with the same appropriate changes in the definition of $\zeta_T$.

In the $^{87}$Rb level configuration chosen above, the decay rates are equal $\gamma_0 = \gamma$, and we choose equal dephasing rates $\gamma_d = \gamma$ for simplicity. For $\Omega_P \approx \Omega_T \approx 0.1 \gamma$, $\gamma = \delta_1$, and detunings $\delta_0 = 20.01 \gamma$, $\delta_2 = 20 \gamma$, $\delta_3 = 20.02 \gamma$, by assuming a low dephasing rate $\gamma_d = 10^{-2} \gamma$, we obtain a conditional phase shift of $\pi$ radians, over the interaction length $l = 1.6$ mm, density $N = 3 \times 10^{13}$ cm$^{-3}$. With these parameters, group velocities are virtually the same, giving $\text{erf}[\zeta_P]/\zeta_P = \text{erf}[\zeta_T]/\zeta_T \approx 2/\sqrt{\pi}$. These parameters correspond to a case where a polarization qubits are encoded into a single photon wave packets, a desired setup for the implementation of a QPG operation. As discussed in Ref. [29], the proposed QPG can also be demonstrated by using post selection of single photon coherent pulses instead of single photon wave packets [44].

Strong cross-phase modulation can also be achieved with classical fields, and we propose here alternative set of parameters that can be used to achieve this. For (classical) Rabi frequencies $\Omega_P \approx \Omega_T = \gamma$, $\Omega = 4.5 \gamma$, and detunings $\delta_1 = 10.01 \gamma$, $\delta_2 = 10 \gamma$, $\delta_3 = 10.02 \gamma$, a conditional phase shift of $\pi$ radians, over the interaction length $l = 0.7$ cm, density $N = 3 \times 10^{12}$ cm$^{-3}$ is obtained. Again, with these parameters, group velocities are the same. Probe and trigger susceptibilities corresponding to these parameter values are shown in Fig. 3. The above parameters are chosen to correspond to those obtained with cold atoms in a MOT. Alternatively, a gas cell of standard length between 2.5 cm and 10 cm can be considered, but the increase in length is then compensated with a lower density. This shows that a demonstration of a deterministic polarization QPG can be made using present technologies.

As discussed above, we had to move from the exact double EIT-resonance condition in order to have a nonzero nonlinearity and in such a condition the linear susceptibilities do not vanish. Actually, the linear contribution is predominant. In fact, the ratios of nonlinear to linear phase shifts are given by

$$\frac{\phi_{\text{lin}}^{\text{nlin}}}{\phi_{\text{lin}}^{\text{lin}}} = \frac{|\Omega_T|^2}{4} \left[ \frac{1}{\Delta_{12}} \left( \frac{\Delta_{12}}{\Delta_{10} \Delta_{12} - |\Omega_T|^2} + \frac{\Delta_{23}}{\Delta_{30} \Delta_{23} - |\Omega_T|^2} \right) \right] \text{(6a)}$$

$$\frac{\phi_{\text{lin}}^{\text{nlin}}}{\phi_{\text{lin}}^{\text{lin}}} = \frac{|\Omega_P|^2}{4} \left[ \frac{1}{\Delta_{13}} \left( \frac{\Delta_{12}}{\Delta_{10} \Delta_{12} - |\Omega_T|^2} + \frac{\Delta_{23}}{\Delta_{30} \Delta_{23} - |\Omega_T|^2} \right) \right] \text{(6b)}$$

and for the above choice of parameters, they are of order $\sim 1/43$ for the first (quantum) set of parameters and $\sim 1/64$ for the second (semiclassical) set of parameters. This means that under the optimal conditions corresponding to a $\pi$ conditional phase shift, the total phase shift in each input–output transformation is very large, of the order of $45 \pi$ and $65 \pi$, respectively. The experimental demonstration of the QPG requires the measurement of the conditional phase shift, i.e., of a phase difference and therefore it is important to keep the errors in the phase measurements small. These errors are mainly due to the fluctuations of the laser intensities and of the detunings. In particular, intensity fluctuations of $1\%$ yield an error of about $4\%$ in the phase measurement. It is more important to minimize the effects of relative detuning fluctuations but this can be achieved by taking all lasers tightly phase locked to each other.

Another important limitation is that due to dephasing of the ground state coherences, whose main effect is to increase absorption. When the polarization qubits are carried by classical pulses one has only to be sure that absorption is not too large, i.e., that it does not dominate over the nonlinear dispersion. Absorption is instead
a more crucial issue in the case of single photon polarization qubits. In fact a non negligible absorption implies a nonzero gate failure probability (one or both qubits missing at the output), making therefore the present QPG, which is deterministic in principle, a probabilistic gate. In our scheme, it can be checked that, if the dephasings do not become very large, i.e., $\gamma_d = 2\pi \times 10 \text{ kHz}$, or $\gamma_d \sim 10^{-2}\gamma$, this increase of absorption is negligible, as shown in Fig. 4.

It should be mentioned that the conclusion above holds for strong control field strengths of order $\Omega \sim \gamma$. If a weaker control is used, the dephasing must also be lower in order to keep absorption negligible.

![Graph showing absorption as a function of scaled frequency](image)

**FIG. 4:** Probe absorption (scaled) at the center of probe transparency window, plotted against the dephasing rate, for $\Omega_p = \Omega = \gamma$, $\Omega = 4.5\gamma$, $\delta_j = 0$.

### VI. Conclusion

In this paper we have studied the nonlinear response of a four-level atomic sample in a tripod configuration to an incident probe and trigger field. The resulting large cross-Kerr modulation between probe and trigger enables one to implement a phase gate with a conditional phase shift of the order of $\pi$. The main advantage of our proposal lies in its experimental feasibility which has been assessed through a detailed study of the requirements needed to observe such a large shift in a cold atomic gas of $^{87}$Rb atoms in a MOT.

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[1] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2000).
[2] N. Gisin, G. Ribordy, W. Tittel, and H. Zbinden, Rev. Mod. Phys. 74, 145 (2002).
[3] F. Grosshans, G. V. Assche, J. Wenger, R. Brouil, N. J. Cerf, and P. Grangier, Nature 421, 238 (2003).
[4] D. Bouwmeester, J.-W. Pan, K. Mattle, M. Eibl, H. Weinfurter, and A. Zeilinger, Nature 390, 575 (1997).
[5] D. Boschi, S. Braica, F. DeMartini, L. Hardy, and S. Popescu, Phys. Rev. Lett. 80, 1121 (1998).
[6] A. Furusawa, J. L. Srensen, S. L. Braunstein, C. A. Fuchs, H. J. Kimble, and E. S. Polzik, Science 282, 706 (1998).
[7] W. P. Bowen, N. Treps, R. Schnabel, and P. K. Lam, Phys. Rev. Lett. 89, 253601 (2002).
[8] T. C. Zhang, K. W. Goh, C. W. Chou, P. Lodahl, and H. J. Kimble, Phys. Rev. A 67, 033802 (2003).
[9] E. Knill, R. Laflamme, and G. J. Milburn, Nature 397, 594 (1999).
[10] A. Imamoğlu, H. Schmidt, G. Woods, and M. Deutsch, Phys. Rev. Lett. 79, 1467 (1997).
[11] P. Grangier, D. F. Walls, and K. M. Gheri, Phys. Rev. Lett. 81, 2833 (1998).
[12] S. Rebić, S. M. Tan, A. S. Parkins, and D. F. Walls, J. Opt. B: Quant. Semiclass. Opt. 1, 490 (1999).
[13] K. M. Gheri, W. Alge, and P. Grangier, Phys. Rev. A 60, R2673 (1999).
[14] A. D. Greentree, J. A. Vaccaro, S. R. de Echaniz, A. V. Duran, and J. P. Marangos, J. Opt. B: Quant. Semiclass. Opt. 2, 252 (2000).
[15] J.-F. Roch, K. Vigneron, P. Grelu, A. Sinatra, J.-P. Poizat, and P. Grangier, Phys. Rev. Lett. 87, 634 (1997).
[16] M. S. Zubairy, A. B. Matsko, and M. O. Scully, Phys. Rev. A 65, 043804 (2002).
[17] A. B. Matsko, I. Novikova, G. R. Welch, and M. S. Zubairy, Opt. Lett. 28, 96 (2002).
[27] A. B. Matsko, I. Novikova, M. S. Zubairy, and G. R. Welch, Phys. Rev. A 67, 043805 (2003).
[28] A. D. Greentree, D. Richards, J. A. Vaccaro, A. V. Dur- rant, S. R. de Echaniz, D. M. Segal, and J. P. Marangos, Phys. Rev. A 67, 023818 (2003).
[29] C. Ottaviani, D. Vitali, M. Artoni, F. Cataliotti, and P. Tombesi, Phys. Rev. Lett. 90, 197902 (2003).
[30] S. Lloyd, Phys. Rev. Lett. 75, 346 (1995).
[31] M. D. Lukin and A. Imamoglu, Phys. Rev. Lett. 84, 1419 (2000).
[32] M. D. Lukin and A. Imamoglu, Nature 413, 273 (2001).
[33] D. Petrosyan and G. Kurizki, Phys. Rev. A 65, 033833 (2002).
[34] Q. A. Turchette, C. J. Hood, W. Lange, H. Mabuchi, and H. J. Kimble, Phys. Rev. Lett. 75, 4710 (1995).
[35] K. J. Resch, J. S. Lundeen, and A. M. Steinberg, Phys. Rev. Lett. 89, 037904 (2002).
[36] R. Unanyan, M. Fleischhauer, B. W. Shore, and K. Bergmann, Opt. Comm. 155, 144 (1998).
[37] E. Paspalakis and P. L. Knight, J. Opt. B: Quantum Semiclass. Opt. 4, S372 (2002).
[38] E. Paspalakis, N. J. Kylstra, and P. L. Knight, Phys. Rev. A 65, 053808 (2002).
[39] E. Paspalakis and P. L. Knight, J. Mod. Opt. 49, 87 (2002).
[40] Y. P. Malakyan, e-print, quant-ph/0112058.
[41] D. Petrosyan and Y. P. Malakyan, e-print, quant-ph/0402070.
[42] D. F. Walls and G. J. Milburn, Quantum Optics (Springer, Berlin, 1994).
[43] The full symmetry also requires $|\mu_T|^2=|\mu_F|^2$, which is fulfilled for the proposed $^{87}$Rb scheme, see Sec. V.
[44] Realization of QPG operation for a single photons relies on an assumption of negligible transfer of fluctuations from a classical pump to probe and trigger single photon pulses/wave packets. Evaluation of the validity of this assumption will be given in our forthcoming publication.