ON THE LAST DIGITS OF CONSECUTIVE PRIMES

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Abstract. Recently Oliver and Soundararajan made conjectures based on computational enumerations about the frequency of occurrence of pairs of last digits for consecutive primes. By studying Eratosthenes sieve, we have identified discrete dynamic systems that exactly model the populations of gaps across stages of Eratosthenes sieve. Our models provide some insight into the observed biases in the occurrences of last digits in consecutive primes, and the models suggest that the biases will ultimately be reversed for large enough primes.

The exact model for populations of gaps across stages of Eratosthenes sieve provides a constructive complement to the probabilistic models rooted in the work of Hardy and Littlewood.

1. Introduction

Recently Oliver and Soundararajan [6, 5] computed the distribution of the last digits of consecutive primes for the first $10^8$ prime numbers. Their calculations revealed a bias: the pairs $(1, 1), (3, 3), (7, 7)$ and $(9, 9)$ occur about a third less often than other ordered pairs of last digits of consecutive primes. Their calculations are shown in Table 1.

For the past several years we have been studying the cycle of gaps $G(p\#)$ that arises at each stage of Eratosthenes sieve. Our work to this point is summarized in [4].

We have identified a population model that describes the growth of the populations of any gap $g$ in the cycle of gaps $G(p\#)$, across the stages of Eratosthenes sieve. The recursion from one cycle of gaps, $G(p_{k-1}\#)$, to the next, $G(p_k\#)$, leads to a discrete dynamic model that provides exact populations for a gap $g$ in the cycle $G(p\#)$, provided that $g < 2p$. The model provides precise asymptotics for the ratio of the population of the gap $g$ to the population of the gap 2 once the prime $p$ is larger than any prime factor of $g$. This discrete dynamic system is deterministic, not probabilistic.

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The discrete dynamic model provides some insight into the phenomenon that Oliver and Soundararajan have observed [6, 5].

i) We look at the asymptotic ratios of the populations of small gaps to the gap $g = 2$. These asymptotic ratios suggest that the reported biases will erode away for samples of much larger primes.

ii) We look at additional terms in the model, to understand rates of convergence to the asymptotic values. To first order this explains some of the biases exhibited in Table 1.

iii) We initially work in base 10, so we then examine the results for a few different bases, to see how the biases depend on the base.

These observations apply to the stages of Eratosthenes sieve as the sieve proceeds. All gaps between prime numbers arise in a cycle of gaps. To connect our results to the desired results on gaps between primes, we would need to better understand how gaps survive later stages of the sieve, to be affirmed as gaps between primes. Until the models for survival have a higher accuracy, the results based on the exact models for $G(p\#)$ can only be approximately applied to gaps between prime numbers.
We offer the exact model on populations of gaps in $G(p\#)$ as a constructive complement to the approaches working from the probabilistic models pioneered by Hardy and Littlewood [2].

1.1. Notes on version 3. Since posting the first version of this work, we extended our analysis of the polynomial approximations in section [4]. We had initially worked with six terms of the polynomial expansion, and we have extended this to twelve terms. This has helped us refine our claims about the convergence for the populations of some of the gaps. Two examples of the progressive approximations, for the gaps $g = 30$ and $g = 420$, are shown in Figure [3].

We have also introduced Mertens’ Third Theorem, which ties the system parameter $\lambda = a_k^2$ in Equation [1] to the magnitude of the prime $p$ in the cycle $G(p\#)$.

2. The model for populations of gaps in $G(p\#)$

Here we restate only a select few results from [4] that are relevant to studying the last digits of consecutive primes.

We do not study the gaps between primes directly. Instead, we study the cycle of gaps $G(p\#)$ at each stage of Eratosthenes sieve. Here, $p\#$ is the primorial of $p$, which is the product of all primes from 2 up to and including $p$. $G(p\#)$ is the cycle of gaps among the generators of $\mathbb{Z} \mod p\#$. These generators and their images through the counting numbers are the candidate primes after Eratosthenes sieve has run through the stages from 2 to $p$. All of the remaining primes are among these candidates.
The cycle of gaps $G(p^\#)$ consists of $\phi(p^\#)$ gaps that sum to $p^\#$. For example, we have $G(5^\#) = 64242462$, and

$$G(7^\#) = 10, 2424626424626468424248642462664246264242, 10, 2.$$ 

There is a substantial amount of structure preserved in the cycle of gaps from one stage of Eratosthenes sieve to the next, from $G(p_k^\#)$ to $G(p_{k+1}^\#)$. This structure is sufficient to enable us to give exact counts for gaps and for sufficiently short constellations in $G(p^\#)$ across all stages of the sieve.

As the sieve proceeds from one prime $p_{k-1}$ to the next $p_k$, the recursive construction leads to a discrete dynamic system that provides exact counts of a gap and its driving terms. The driving terms for a gap $g$ are constellations $s$ in $G(p^\#)$ that have the same sum $g$. For example, the driving terms for $g = 6$ are the gaps $g = 6$ themselves and the constellations of length 2 that sum to 6: e.g., $s = 24$ and $s = 42$. Under the closures in the recursion – which correspond to eliminating a candidate from later stages of the sieve – these constellations will produce more gaps $g = 6$.

These raw counts for populations of gaps grow superexponentially by factors of $(p_k - 2)$, and so to better understand their behavior we take the ratio of a raw count $n_{g,j}(p^\#)$ of the driving terms of length $j$ for the gap $g$ in $G(p^\#)$ to the number of gaps $g = 2$ at this stage of the sieve.

$$w_{g,j}(p^\#) = \frac{n_{g,j}(p^\#)}{n_{2,1}(p^\#)}$$

For a gap $g$ that has driving terms of lengths up to $j$, we take any $J \geq j$, and we form a vector of initial values $\vec{w}_g|_{p_0^\#}$, whose $i^{th}$ entry is the ratio $w_{g,i}$. These values evolve according to the dynamic system:

$$\vec{w}_g(p^\#) = \begin{vmatrix} M_{1:J} |_{p_k} \cdot \vec{w}_g(p_{k-1}^\#) \\
M_{1:J} |_{p_k} \cdot M_{1:J} |_{p_{k-1}} \cdot \cdots \cdot M_{1:J} |_{p_1} \cdot \vec{w}_g(p_0^\#) = M_{1:J}^k \cdot \vec{w}_g(p_0^\#) \end{vmatrix}$$

Here $M(p)$ is a banded matrix with diagonal entries $a_j(p) = (p - j - 1)/(p - 2)$ and superdiagonal entries $b_j(p) = j/(p - 2)$.

We use the notation $M^k$ and $a^k_j$ to indicate the products over the range of primes from $p_1$ to $p_k$, relative to some starting value $p_0$.

$$M^k = M |_{p_k} \cdot M |_{p_{k-1}} \cdots \cdot M |_{p_1}$$

$$a^k_j = \prod_{p=p_j}^{p_k} \frac{p - j - 1}{p - 2}$$
2.1. **Eigenstructure of the system matrix** $M_{1,J}$. The system matrix $M_{1,J}(p)$ is diagonalizable with a particularly nice eigenstructure.

\[ M_{1,J}(p) = R \cdot \Lambda \cdot L \]

with $LR = I$.

The right eigenvectors are the columns of $R$, and $R$ is an upper triangular Pascal matrix of alternating sign:

\[
R_{ij} = \begin{cases} 
(-1)^i j \binom{j-1}{i-1} & \text{if } i \leq j \\
0 & \text{if } i > j 
\end{cases}
\]

The left eigenvectors are the rows of $L$, and $L$ is an upper triangular Pascal matrix:

\[
L_{ij} = \begin{cases} 
\binom{j-1}{i-1} & \text{if } i \leq j \\
0 & \text{if } i > j 
\end{cases}
\]

The eigenvalues are the system coefficients $a_j = (p - j - 1)/(p - 2)$:

\[
\Lambda = \text{diag}(1, a_2, \ldots, a_J).
\]

While the eigenvalues depend on the prime $p$, the eigenvectors do not. We thereby get a similarly nice eigenstructure for $M^k_{1,J}$.

\[ M^k_{1,J} = R \cdot \Lambda^k \cdot L \]

in which the eigenvalues $\lambda_j^k = a_j^k = \prod_{p=p_1}^{p_k} (p - j - 1)/(p - 2)$.

2.2. **Implications of the discrete dynamic system.** We borrow a few results from [4] that have direct bearing on the distributions of last digits between consecutive primes.

Although the asymptotic growth of all gaps is equal, the initial conditions and driving terms are important. Brent [1] made analogous observations. His Table 2 indicates the importance of the lower-order effects in estimating relative occurrences of certain gaps.

**Lemma 2.1.** For a gap $g$, let $p_0$ be any prime greater than the greatest prime factor of $g$, and let $J$ be at least as large as the longest driving term for $g$ in $\mathcal{G}(p_0\#)$. Then in $\mathcal{G}(p_k\#)$,

\[
w_{g,1}(p_k\#) = (L_1 \tilde{w}_g|p_0\#) - a_2^k(L_2 \tilde{w}_g|p_0\#) \\
+ a_3^k(L_3 \tilde{w}_g|p_0\#) \cdots + (-1)^{J+1} a_j^k(L_J \tilde{w}_g|p_0\#) \\
(1) \approx (L_1 \tilde{w}_g|p_0\#) - a_2^k(L_2 \tilde{w}_g|p_0\#) \\
+ (a_2^k)^2(L_3 \tilde{w}_g|p_0\#) \cdots + (-1)^{J+1}(a_2^k)^{J-1}(L_J \tilde{w}_g|p_0\#)\]
This is Equations (6 & 7) in [4]. From this expansion we can compute the asymptotic values of the ratios \( w_{g,1}(p^\#) \), and we can analyze the rate of convergence to the asymptotic value. To obtain the asymptotic ratio \( w_{g,1}(\infty) \), since \( L_1 = \langle 1 \rangle \) we simply add together the initial ratios of all driving terms. As quickly as \( a_k^2 \to 0 \), the ratios \( w_{g,1}(p_k^\#) \) converge to the asymptotic ratio \( w_{g,1}(\infty) \).

For these asymptotic ratios, we restate Corollary 5.4 and Theorem 5.5 of [4] here.

**Theorem 2.2.** For any \( g = 2n \), the gap \( g \) eventually occurs in Eratosthenes sieve. Let \( \bar{q} \) be the largest prime factor of \( g \). Then for \( p \geq \bar{q} \),

\[
    w_{g,1}(\infty) = L_1 \bar{w}_g|_{p^\#} = \sum_{q \in \mathbb{Q}, q|g} w_{g,1}(p^\#) = \prod_{q \geq 2, q | g} \frac{q - 1}{q - 2}
\]

This theorem establishes an analogue of Polignac’s conjecture for Eratosthenes sieve [3], that for any number \( 2n \) the gap \( g = 2n \) does occur infinitely often in the sieve, and further that the ratio of occurrences of this gap to the gap 2 approaches the ratio implied by Hardy & Littlewood’s Conjecture B [2].

2.3. **Estimating the rate of convergence of** \( a_k^2 \to 0 \). We have calculated \( a_k^2 \) for primes into the range of \( 10^{15} \), at which \( a_k^2 \approx 0.105 \) (with \( p_0 = 37 \)). Even into this range, gaps of sizes in the low hundreds are still appearing in ratios far below their asymptotic values. We need a way to estimate \( a_k^2 \) for much larger primes.

Mertens’ Third Theorem provides these estimates. The theorem is that

\[
    \prod_{p \leq q} \left( \frac{p - 1}{p} \right) = e^{-\gamma} + o(1) \ln q.
\]

We define the constant \( c_0 = \prod_{q \leq p_0} q/(q - 1) \). Then we can establish both a lower and an upper bound for \( a_k^2 \) for large primes \( p_k \):

\[
    \frac{p_0 - 1}{p_0} \cdot c_0 \cdot \frac{e^{-\gamma} + o(1)}{\ln p_{k-1}} < a_k^2 < c_0 \cdot \frac{e^{-\gamma} + o(1)}{\ln p_k}
\]

For the calculations in this paper, we are using \( p_0 = 37 \). We used these bounds to extend Figure 1 across the range \( p_k \in [10^{15}, 10^{20}] \).

3. **Ultimate distributions of last digits of consecutive primes**

Consider an ordered pair \( (a, b) \) of last digits of consecutive primes [6], with \( a, b \in \{1, 3, 7, 9\} \). We are interested in the size of the set of indices \( k \), such that \( p_k = a \mod 10 \) and \( p_{k+1} = b \mod 10 \).
By way of example, suppose $b - a = 0 \mod 10$. Then $p_{k+1} - p_k = 10$, or $p_{k+1} - p_k = 20$, or in general $p_{k+1} - p_k$ is some multiple of 10. So how often do these gaps $g = 10, 20, 30, \ldots$ arise?

The ordered pairs $(a, b)$ of last digits correspond to specific gaps as follows:

$$
\begin{array}{c|c}
(a, b)’s & g’s \\
\hline
(1, 1), (3, 3), (7, 7), (9, 9) & 10, 20, 30, 40, \ldots \\
(1, 3), (7, 9), (9, 1) & 2, 12, 22, 32, 42, \ldots \\
(3, 7), (7, 1), (9, 3) & 4, 14, 24, 34, 44, \ldots \\
(1, 7), (3, 9), (7, 3) & 6, 16, 26, 36, 46, \ldots \\
(1, 9), (3, 1), (9, 7) & 8, 18, 28, 38, 48, \ldots \\
\end{array}
$$

This table already provides us with a couple of insights into the problem. The class $b - a = 0 \mod 10$ has four ordered pairs and the other classes have three. So in order for these ordered pairs to occur equally often, the gaps in $g = 0 \mod 10$ must occur $4/3$ as often as the gaps in the other classes. We also note that within any of the five classes, if the distribution of a single gap across the corresponding ordered pairs is not uniform, then this would lead to a biased distribution within this class.

3.1. **Tracking the relative growth of the classes of gaps.** The raw populations of gaps within the cycles of gaps $G(p^\#)$ grow by factors of $(p-2)$. This led us to look at the ratios $w_{g,j}(p^\#) = n_{g,j}(p^\#)/\prod (p - 2)$.

Within each residue class modulo 10 we will be adding up the ratios of an infinite number of gaps. To compare these in a practical manner, we first note that due to the recursive construction of $G(p_k^\#)$ from $G(p_{k-1}^\#)$, the closures within driving terms occur methodically. As a result we see larger gaps introduced generally in order as the sieve progresses. In Table 2 the column for $j = 1$ illustrates this introduction of larger gaps.

As a first comparison of the distributions across the residue classes, we consider the average $\mu_k(w_g)$ as the size of the gaps within each class increases.

$$
\begin{align*}
\mu_0(w_g(\infty))|_N &= \frac{1}{N} [w_{10,1}(\infty) + w_{20,1}(\infty) + \cdots + w_{10N,1}(\infty)] \\
\mu_2(w_g(\infty))|_N &= \frac{1}{N} [w_{2,1}(\infty) + w_{12,1}(\infty) + \cdots + w_{10N-8,1}(\infty)] \\
\mu_4(w_g(\infty))|_N &= \frac{1}{N} [w_{4,1}(\infty) + w_{14,1}(\infty) + \cdots + w_{10N-6,1}(\infty)] \\
\mu_6(w_g(\infty))|_N &= \frac{1}{N} [w_{6,1}(\infty) + w_{16,1}(\infty) + \cdots + w_{10N-4,1}(\infty)] \\
\mu_8(w_g(\infty))|_N &= \frac{1}{N} [w_{8,1}(\infty) + w_{18,1}(\infty) + \cdots + w_{10N-2,1}(\infty)]
\end{align*}
$$
In Table 3 we list the gaps $g < 100$ in their respective residue classes, along with each gap’s asymptotic ratio and the average ratio for the class up to this gap.

Table 3 helps us make a few observations about the effect of Theorem 2.2 on the average asymptotic ratios. Gaps that are divisible by 3 have a factor of 2 in their asymptotic ratio, and gaps that are divisible by 5 have a factor of 4/3 in theirs. These are as large as these factors get. If a gap is divisible by a larger prime $p$, the asymptotic ratio $w_g(\infty)$ has a factor of $(p - 1)/(p - 2)$.

| gap  | $n_{g, j}(37^\#)$: driving terms of length $j$ in $\mathcal{G}(37^\#)$ | $w_{g, 1}(37^\#)$ | $w_{g, 1}(\infty)$ |
|------|---------------------------------------------------------------------|--------------------|---------------------|
| 2, 4 | 217929555875                                                       | 1                  | 1                   |
| 6    | 293928522950                                                      | 1.348698           | 2                   |
| 8    | 915894444450                                                      | 0.420271           | 1                   |
| 10   | 108665865050                                                      | 0.499527           | 4/3                 |
| 12   | 83462164156                                                       | 0.382978           | 2                   |
| 14   | 83462164156                                                       | 0.159695           | 6/5                 |
| 16   | 169960708688                                                     | 0.071989           | 1                   |
| 18   | 21218333416                                                       | 0.097363           | 2                   |
| 20   | 4814320320                                                       | 0.022991           | 4/3                 |
| 22   | 5454179550                                                       | 0.025027           | 10/9                |
| 24   | 4079565144                                                       | 0.018694           | 2                   |
| 26   | 9106945454                                                       | 0.004213           | 12/11               |
| 28   | 857901000                                                        | 0.003937           | 6/5                 |
| 30   | 535673924                                                        | 0.002458           | 1                   |
| 32   | 58664256                                                        | 0.000269           | 10                   |
| 34   | 69104898                                                        | 0.000181           | 16/15               |
| 36   | 46346428                                                        | 0.000213           | 2                   |
| 38   | 7381190                                                         | 0.000034           | 18/17               |
| 40   | 10176048                                                         | 0.000047           | 4/3                 |
| 42   | 4153336                                                         | 0.000199           | 12/5                |
| 44   | 5265966                                                         | 0.000092           | 10/9                |
| 46   | 291342                                                          | 0.000001           | 22/21               |
| 48   | 239760                                                          | 0.000001           | 2                   |
| 50   | 91392                                                           | 4.2E−7             | 4/3                 |
| 52   | 8912                                                            | 4.1E−8             | 12/11               |
| 54   | 25320                                                           | 1.2E−2             | 2                   |
| 56   | 2952                                                            | 1.4E−8             | 6/5                 |
| 58   | 1654                                                            | 7.6E−9             | 28/27               |
| 60   | 452                                                              | 2.1E−9             | 8/3                 |
| 62   | 48                                                              | 1.2E−10            | 30/29               |
| 64   | 24                                                              | 2.2E−10            | 1                   |
| 66   | 24                                                              | 1.1E−10            | 20/9                |

Table 2. For the gaps that actually occur in $\mathcal{G}(37^\#)$, this table lists the number of gaps and driving terms of length $j \leq 4$. The gaps $g \geq 16$ have longer driving terms as well; the gap $g = 66$ has driving terms up to length 16. Also tabulated are the current ratio $w_{g, 1}(37^\#)$ and the asymptotic value for this ratio.
Working in base 10, the gaps divisible by 3 rotate through the classes in this order: \( h = 6, 2, 8, 4, 0 \). When the number \( N \) of gaps in each class is small, we can see the impact of a mod-3 gap on the average. In Table 3 look at \( \mu_4 \) jump at the gaps \( g = 24 \) and \( g = 54 \). To make a fair comparison across classes, we should pick \( N \) to include a complete rotation of 3 through the classes; e.g. stopping when the gap \( g \) is a multiple of 30.

We also observe that the class \( g = 0 \mod 10 \) will have all of the gaps divisible by 5, giving the average for this class a consistent boost. Interestingly, the corresponding factor for \( w_g(\infty) \) is \( 4/3 \), which is the factor needed to compensate for this class having four ordered pairs \((a,b)\) as compared to the other classes having only three ordered pairs of last digits.

The class \( g = 0 \mod 10 \) will also contain all of the primorial gaps \( g = p# \) for \( p \geq 5 \) and all of their multiples. By Theorem 2.2 the primorial gaps mark new maxima for \( w_g(\infty) \). For the gap \( g = 30 \), the asymptotic ratio is

| \( g \) | \( w_g(\infty) \) | \( \mu_0(w_g(\infty)) \) | \( g \) | \( w_g(\infty) \) | \( \mu_2(w_g(\infty)) \) | \( g \) | \( w_g(\infty) \) | \( \mu_4(w_g(\infty)) \) |
|---|---|---|---|---|---|---|---|---|
| 10 | 4/3 | 1.333 | 2 | 1 | 1.000 | 14 | 6/5 | 1.100 |
| 20 | 4/3 | 1.333 | 22 | 10/9 | 1.370 | 24 | 2 | 1.400 |
| 30 | 8/3 | 1.777 | 32 | 1 | 1.277 | 34 | 16/15 | 1.316 |
| 40 | 4/3 | 1.666 | 42 | 12/5 | 1.502 | 44 | 10/9 | 1.275 |
| 50 | 4/3 | 1.600 | 52 | 12/11 | 1.433 | 54 | 2 | 1.396 |
| 60 | 8/3 | 1.777 | 62 | 30/29 | 1.376 | 64 | 1 | 1.339 |
| 70 | 8/5 | 1.752 | 72 | 2 | 1.454 | 74 | 36/35 | 1.300 |
| 80 | 4/3 | 1.700 | 82 | 40/39 | 1.406 | 84 | 12/5 | 1.422 |
| 90 | 8/3 | 1.807 | 92 | 22/21 | 1.370 | 94 | 46/45 | 1.382 |

Table 3. The distribution of gaps \( g < 100 \) that maintain pairs of last digits modulo 10.
Based on these asymptotic distributions, we expect that the biases observed by Oliver and Soundararajan will disappear among large primes. For the cycles of gaps \( G(p^\#) \), the evolution of the dynamic system plays out on massive scales. Compared to the scales on which the primes evolve, the computations by Oliver and Soundararajan [6] are very early. For example, for their computations for distributions in base 3 they considered the first million primes; these occur in the first twelfth of \( G(23^\#) \) (that is, within the first two copies of \( G(19^\#) \)). Their computations for distributions in base 10 use the first one hundred million primes; these occur in the first third of \( G(29^\#) \). We observe in Table 2 that even for the small gaps the ratios \( w_{30,1}(q^\#) \) at this stage are far from their asymptotic values.

### 4. Rate of convergence to the ultimate distributions

From our observations above about the asymptotic ratios for small gaps, we believe that Oliver and Soundararajan are observing transient phenomena. In this section we demonstrate that these transient biases will persist for any computationally tractable primes.

The asymptotic values described in the previous section evolve very slowly, even for small gaps. For example, the gap \( g = 30 \) ultimately occurs \( 4/3 \) times as often as the gap \( g = 6 \) and \( 8/3 \) times as often as the gap \( g = 2 \); but the gap \( g = 30 \) is not even more numerous than the gap \( g = 2 \) until \( G(q^\#) \) with \( q \approx 2E6 \).

From Equation (1) we can determine more specifically the rate at which the ratio \( w_{g,1}(p^\#) \) converges to the asymptotic value \( w_{g,1}(\infty) \). For computational simplicity, we use the approximation \( a_k^j \approx (a_2^k)^{j-1} \) to express the expansion as a polynomial in \( \lambda = a_2^k \).

\[
w_{g,1}(p_k^\#) \approx L_1 \cdot w_g(p_0^\#) - (L_2 \cdot w_g(p_0^\#)) \lambda + (L_3 \cdot w_g(p_0^\#)) \lambda^2 - (L_4 \cdot w_g(p_0^\#)) \lambda^3 + \cdots
\]

This system converges to the asymptotic value \( w_{g,1}(\infty) = L_1 w_g(p_0^\#) \) as quickly as \( \lambda = a_2^k \to 0 \). This decay of \( a_2^k \) is extraordinarily slow. The bounds in Equation 2 show that this decay is proportionate to \( 1/\ln p_k \).

To understand how far we are from convergence for a prime \( q \approx 3E15 \), let’s again consider the gap \( g = 30 \). Ultimately the gap \( g = 30 \) will have ratio \( w_{30}(\infty) = 8/3 \) compared to the gap \( g = 2 \). For \( q \approx 3E15 \), in \( G(q^\#) \) the ratio of gaps \( g = 30 \) to \( g = 2 \) is \( w_{30,1}(q^\#) \approx 1.976 \).

For the next primorial gap \( g = 210 \), the asymptotic ratio is \( w_{210}(\infty) = 48/15 \), but in \( G(q^\#) \) we only have \( w_{210,1}(q^\#) \approx 0.265 \).
For setting the vectors of initial values \( w_g(p_0^\#) \) we have an incentive to pick \( p_0 \) as large as possible, since the dynamic system only holds exactly for those gaps \( g \) all of whose prime factors are less than or equal to \( p_0 \). Gaps \( g \) with larger prime factors will be underrepresented by factors of \( (p−1)/(p−2) \); this is an application of Corollary 5.7 in [4].

**Example.** To apply this polynomial approximation to the distribution of last digits of consecutive primes, we use \( G(37^\#) \) to obtain initial conditions for the gaps \( g = 2, \ldots, 420 \). This will give accurate representations for all of the gaps that are not divisible by the larger primes \( p = 41, 43, \ldots, 199 \).

We take the first twelve terms of the polynomial approximation and apply this model to the ratios \( w_g,p\cdot w_g(p_0^\#) \) for the gaps \( g = 2, \ldots, 420 \) sorted into their residue classes modulo 10. For each gap \( g \), we define the coefficients 
\[
l_i = L_i \cdot w_g(37^\#).
\]
Then we have the degree-11 model
\[
w_{g,1}(p_k^\#) \approx l_1 - l_2 \lambda + l_3 \lambda^2 - \cdots + l_{11} \lambda^{10} - l_{12} \lambda^{11}
\]
Graphs of these models for the gaps \( g = 30 \) and \( g = 420 \) from degree 1 to degree 11 are plotted in Figure 3. These graphs provide a sense of the range of values of \( \lambda = a_2^k \) over which we can rely on the accuracy of these approximations.

Turning our attention back to the residue classes mod10, we consider the aggregate model for the gaps in each residue class. These five aggregate models are depicted in Figure 1. As \( p_k \to \infty \), the parameter
\[
\lambda = a_2^k = \prod_{p_1} \frac{p - 3}{p - 2} \to 0.
\]
Figure 1 depicts the evolution of the populations for each residue class versus \( \log p_k \). For each value of \( p_k \), we normalize the total population of gaps in class \( h \mod 10 \) by the population of gaps with residue \( 2 \mod 10 \):
\[
W_h(p_k^\#) = \frac{\sum_{g=h \mod 10} w_{g,1}(p_k^\#)}{\sum_{g=2 \mod 10} w_{g,1}(p_k^\#)}
\]
To give a sense of the initial conditions underlying this slow evolution of the populations of gap, in \( G(37^\#) \) the gap \( g = 420 \) has a total of 697373938800 driving terms. These range from 2 driving terms of length \( j = 47 \) through 304 driving terms of length \( j = 76 \). Under the twelve-term approximation, the gap \( g = 420 \) will not reach 10\% of its asymptotic ratio of \( w_{420,1}(\infty) = 3.2 \) until \( a_2^k < 0.0365 \); that is until \( G(p_k^\#) \) with \( p_k \approx 1.12E45 \). With this model, we don’t expect the gap \( g = 420 \) to be more numerous than the gap \( g = 2 \) until \( a_2^k < 0.01415 \), that is until \( p_k \approx 3.57E87 \).

In Figure 1 we can see how the biases observed by Oliver and Soundararajan will be corrected for large primes. The distributions calculated by Oliver
and Soundararajan correspond to the dashed line toward the left of Figure. Summing up their computed values by residue class we calculate the following ratios for the first $10^8$ prime numbers:

| $h$ | $(a,b)$ | $\sum_g n_{g,1}$ | $W_h$ |
|-----|---------|-----------------|------|
| 2   | (1,3), (7,9), (9,1) | 22852739 | 1    |
| 4   | (3,7), (7,1), (9,3) | 19790617 | 0.866006 |
| 6   | (1,7), (3,9), (7,3) | 21762703 | 0.952302 |
| 8   | (1,9), (3,1), (9,7) | 17466066 | 0.764288 |
| 0   | (1,1), (3,3), (7,7), (9,9) | 18127875 | 0.793247 |

The calculated ratios $W_h$ for the first $10^8$ primes are consistent with the distributions of gaps in $G(p\#)$ for very small primes $p$. Intuitively, we see that for small primes $p_k$ the residue classes $h = 2, 4, 6$ all start with significant populations of the gaps $g = 2, 4, 6$ respectively, while the classes $h = 8, 0$ have to manufacture representative gaps as the recursion on the cycle of gaps $G(p_k\#)$ proceeds.

We see in Figure that these biases will change for much larger primes. Ultimately, for the sample of gaps $2 \leq g \leq 420$, these ratios will converge to:

$$W_2(\infty) = 1 \quad W_5(\infty) = 1.0026$$
$$W_4(\infty) = 1.0007 \quad W_6(\infty) = 1.3192$$
$$W_6(\infty) = 1.0029$$

For this sample of gaps, the residue class $h = 0 \mod 10$ has about 98.9% of the size necessary to compensate for having four corresponding ordered pairs $(a,b)$ as compared to the three for each of the other residue classes.

5. DISTRIBUTIONS IN OTHER BASES

The work above has addressed the residue classes of primes in base 10. For base 10 the dynamic system that models populations of gaps across stages of Eratosthenes sieve indicates that the biases calculated by Oliver and Soundararajan are transient phenomena. These biases will gradually fade for very large primes. How is this analysis affected by the choice of base?

Our work in base 10 consisted of three components: identifying the ordered pairs of last digits $(a,b)$ and gaps $g$ that correspond to each residue class; comparing the asymptotic ratios $w_{g,1}(\infty)$ for the gaps within each residue class; and looking at the initial conditions and rates of convergence for the gaps within each residue class. As we consider other bases, we look at the effect that a new basis has on each of these components.
Figure 2. The initial conditions for the gaps that occur in \( G(37#) \). The relative frequencies \( w_{g,j}(37#) \) are shown for the gaps and for their driving terms of length 2. We see the early leads that the gaps 2, 4, 6 enjoy. Other small gaps, especially those divisible by 6 have short driving terms that will quickly boost their populations as well.

Since the asymptotic ratios tend toward uniformity across the ordered pairs, the choice of base will simply shift the distributions around the residue classes. The asymptotic ratios will be proportional to the number of ordered pairs \((a,b)\) corresponding to that class.

By setting a base, we set the assignment of gaps, especially the small gaps, to the respective residue classes. These residue classes inherit the initial biases and rates of convergence associated with the assigned gaps.

Note that the bias is dominated by small gaps, especially the gaps 2, 4 and 6. Figure 2 illustrates the components of the bias introduced by small gaps. We can see that the gaps \( g = 12, 10, 18 \) will quickly make significant contributions as well, and that for powers of 2, e.g. \( g = 8, 16, 32 \), the populations will lag compared to gaps of similar size. The biases in the initial populations for small gaps, for example as illustrated in Table 2 for \( G(37#) \), will be inherited by the residue classes to which the small gaps belong.

For the bases 3 and 6, all gaps that are multiples of 6 fall into a single residue class. Similarly for the bases 5 and 10, all gaps that are multiples of 10 fall into a single residue class. The base 30 separates the multiples of 6 and 10 into a small set of residue classes. In contrast, powers of 2, like the base 8, distribute the multiples of 3 and 5 (and any odd prime) across residue classes.

We illustrate this assignment of the initial bias with the gaps \( 2 \leq g \leq 420 \) under the bases 3, 8, and 30.
5.1. **Distributions in bases 3 or 6.** Oliver and Soundararajan calculated the distributions of select pairs \((a,b)\) modulo 3 up through 1012, and they compare these favorably to a conjectured model derived from the Hardy and Littlewood’s [2] work on the \(k\)-tuple conjecture.

In our approach through \(G(p\#)\), for each residue class \(h\) mod 3 we identify the associated gaps \(g\) and the ordered pairs with \(h = b - a\) [6]. We then calculate both the initial estimate \(W_h(1993\#)\) based on the sample of gaps \(g = 2, \ldots, 420\) and initial conditions from \(G(37\#)\); and the asymptotic ratio \(W_h(\infty)\) for this sample of gaps. For the ratios, we normalize by the values for the class \(h = 2\) mod 3.

| \(h\) mod 3 | \(g\)'s | \((a,b)\) | \(W_h(1993\#)\) | \(W_h(\infty)\) |
|---------------|----------|-----------|----------------|----------------|
| 2             | 2, 8, 14, \ldots | (2, 1)    | 1              | 1              |
| 1             | 4, 10, 16, \ldots | (1, 2)    | 1.0009         | 1.0010         |
| 0             | 6, 12, 18, \ldots | (1, 1), (2, 2) | 1.6358         | 1.9868         |

We see that the ratios \(W_1(1993\#)\) and \(W_0(1993\#)\) are consistent with Oliver and Soundararajan’s tabulations for base 3. Additionally, the asymptotic ratios for the sample of gaps indicates that the initial bias will again disappear.

We note that these results for the primes modulo 3 (or in base 3) can be translated directly into results for the primes modulo 6 (or in base 6). Indeed, for any odd base \(B\), there is a direct translation of the residue classes, gaps, and ordered pairs into the base \(2B\). For example, Oliver and Soundararajan’s computations for base 5 could be combined with their initial computations base 10.

One interesting aspect of working in base 3 or 6 is that all multiples of 6 will fall in the class \(h = 0\) mod 6, and thus all of the primorials \(g = p\#\) will fall within this class. In \(G(11\#)\) the gap \(g = 6\) is the most frequent gap, and it grows more quickly than other gaps for many more stages of the sieve.

5.2. **Distributions in base 8.** In base 8 the small gaps are distributed more evenly across the residue classes. We observe that the residue class \(h = 0\) mod 8 starts slowly, and for this sample of gaps \(2 \leq g \leq 420\) this residue class still lags in its asymptotic value.

| \(h\) mod 8 | \(g\)'s | \((a,b)\) | \(W_h(1993\#)\) | \(W_h(\infty)\) |
|---------------|----------|-----------|----------------|----------------|
| 2             | 2, 10, 18, \ldots | (1,3), (3,5), (5,7), (7,1), | 1              | 1              |
| 4             | 4, 12, 20, \ldots | (1,5), (5,1), (3,7), (7,3) | 0.9695         | 1.0185         |
| 6             | 6, 14, 22, \ldots | (1,7), (7,5), (5,3), (3,1) | 1.0086         | 1.0003         |
| 0             | 8, 16, 24, \ldots | (1,1), (3,3), (5,5), (7,7) | 0.7081         | 0.9676         |

5.3. **Distributions in base 30.** The next primorial base is \(30 = 5\#\). This base is big enough that the small gaps are well separated, and the multiples
of \( g = 6 \) and \( g = 10 \) fall into a few distinct classes. The early bias toward small gaps \( g = 2, 4, 6, 10, 12 \) and even \( g = 14, 18 \) fall into separate residue classes. We see the early biases in \( W_h(1993^\#) \) for base 30 in Table 4.

### Table 4. The table for the distribution in base 30.

| \( h \mod 30 \) | \( g's \) | \( (a,b) \) | \( W_h(1993^\#) \) | \( W_h(\infty) \) |
|------------------|---------|---------|-----------------|---------|
| 2                | 2,32,... | (29,1),(11,13),(17,19) | 1      | 1      |
| 4                | 4,34,... | (7,11),(13,17),(19,23) | 1.0180 | 1.0019 |
| 6                | 6,36,... | (1,7),(7,13),(13,19)   | 1.7771 | 2.0021 |
|                  |         | (11,17),(17,23),(23,29) |       |        |
| 8                | 8,38,... | (11,19),(23,1),(29,7)  | 0.8154 | 1.0000 |
| 10               | 10,40,...| (1,11),(7,17),(13,23),(19,29) | 1.0421 | 1.3245 |
| 12               | 12,42,...| (1,13),(7,19),(11,23)  | 1.4228 | 1.9918 |
|                  |         | (17,29),(19,1),(29,11) |       |        |
| 14               | 14,44,...| (17,1),(23,7),(29,13)  | 0.7501 | 1.0028 |
| 16               | 16,46,...| (1,17),(7,23),(13,29)  | 0.5890 | 1.0015 |
| 18               | 18,48,...| (1,19),(11,29),(13,1)  | 1.0775 | 1.9956 |
|                  |         | (19,7),(23,11),(29,17)  |       |        |
| 20               | 20,50,...| (11,1),(17,7),(23,13),(29,19) | 0.6116 | 1.3287 |
| 22               | 22,52,...| (1,23),(7,29),(19,11)  | 0.5109 | 1.0020 |
| 24               | 24,54,...| (7,1),(13,7),(19,13)   | 0.8031 | 1.9920 |
|                  |         | (17,11),(23,17),(29,23) |       |        |
| 26               | 26,56,...| (11,7),(17,13),(23,19)  | 0.3920 | 1.0019 |
| 28               | 28,58,...| (1,29),(13,11),(19,17)  | 0.4122 | 1.0086 |
| 0                | 30,60,...| (1,1),(7,7),(11,11),(13,13), (17,17),(19,19),(23,23), (29,29) | 0.7578 | 2.6153 |

6. CONCLUSION

By identifying structure among the gaps in each stage of Eratosthenes sieve, we have been able to develop an exact model for the populations of gaps and their driving terms across stages of the sieve. We have identified a model for a discrete dynamic system that takes the initial populations of a gap \( g \) and all its driving terms in a cycle of gaps \( G(p_0^\#) \) such that \( g < 2p_1 \), and thereafter provides the exact populations of this gap and its driving terms through all subsequent cycles of gaps.

All of the gaps between primes are generated out of these cycles of gaps, with the gaps at the front of the cycle surviving subsequent closures. The trends across the stages of Eratosthenes sieve indicate probable trends for gaps between primes. We are not yet able to translate the precision of the model for populations of gaps in \( G(p^\#) \) into a robust analogue for gaps between primes.
For the first $10^8$ primes, Oliver and Soundararajan \cite{6,5} calculated how often the possible pairs $(a, b)$ of last digits of consecutive primes occurred, and they observed biases. Regarding their calculations they raised two questions: Does the observed bias persist? Is the observed bias dependent upon the base? We have addressed both of these questions by using the dynamic system that exactly models the populations of gaps across stages of Eratosthenes sieve.

The observed biases are transient phenomena. The biases persist through the range of computationally tractable primes. The asymptotics of the dynamic system play out on superhuman scales – for example, continuing Eratosthenes sieve at least through all 16-digit primes. To put this in perspective, the cycle $G(199\#)$ has more gaps than there are particles in the known universe; yet in $G(p\#)$ for a 16-digit prime $p$, small gaps like $g = 30$ will still be appearing in frequencies well below their ultimate ratios. Gaps the size of $g = 210$ will just be emerging, relative to the prevailing populations of small gaps at this stage.

Our work on the relative frequency of gaps modulo 10 for $G(p\#)$ has addressed the bias between the residue classes. The observed biases are due to the quick appearance of small gaps and the slow evolution of the dynamic system. While we have addressed the inter-class bias, we have said nothing about the intra-class bias, that is, unequal distributions across the ordered pairs $(a, b)$ within a given residue class modulo 10. Our initial calculations here indicate that this bias should also disappear eventually, but this exploration needs to be more thorough. The model developed by Oliver and Soundararajan also depends only on the residue class $h = b - a$.

Our calculations use a sample of gaps $g = 2, \ldots, 420$. To improve the precision of our calculations of the asymptotic ratios $w_h(\infty)$ across residue classes, it would be useful to find a normalization that makes working with all gaps $g = 2n$ manageable.

Once we understand the model for gaps, then any choice of base reassigns the gaps across the residue classes for this base. The number of ordered pairs corresponding to a residue class $h$ is proportional to the asymptotic relative frequency $W_h(\infty)$. The initial biases and more rapid convergence that favor the small gaps can be observed, over any computationally tractable range, for the residue classes to which these small gaps are assigned.

**References**

1. R. Brent, The distribution of small gaps between successive prime numbers, Math. Comp. 28 (1974), 315–324.
2. G.H. Hardy and J.E. Littlewood, Some problems in ‘partitio numerorum’ iii: On the expression of a number as a sum of primes, G.H. Hardy Collected Papers, vol. 1, Clarendon Press, 1966, pp. 561–630.
3. F.B. Holt and H. Rudd, On Polignac’s conjecture, \texttt{arXiv:1402.1970v2}, 15 Feb 2014.
Figure 3. Two examples of the polynomial approximations in Equation 1. The approximations differ from the exact discrete model by substituting $\lambda = (a^2_j)^{j}$ for $a^k_j + 1$. The gap $g = 30$ has driving terms up to length $j = 8$, so the approximations of degree 8 and higher coincide with that of degree 7.

4. F.B. Holt with H. Rudd, Combinatorics of the gaps between primes, arXiv:1510.00743v2, 8 Oct 2015.
5. Mathematicians discover prime conspiracy, Quanta Magazine, 13 March 2016.
6. R.J.L. Oliver and K. Soundararajan, Unexpected biases in the distribution of consecutive primes, arXiv:1603.03720 version 2, 15 March 2016.

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