Is Lipschitz Continuity Preserved under Sampled-Data Discretization?

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Abstract

Usually, given a continuous-time nonlinear model, a closed form solution for an exact discretization can not be found explicitly, originating the need of approximating discrete-time models. This note studies the preservation of the Lipschitz continuity under approximate discretizations.

1 Introduction

The theory of nonlinear sampled data systems has still a long way ahead to be well developed. An interesting and important path for further research is to extend previous results using Euler discretization higher order approximations. In control systems, the discretization is often under the Zero-Order Hold assumption, i.e. the control input is assumed to be constant during the sampling intervals \([kT, (k+1)T]\), where \(T\) is the sampling time. Under this assumption, the system becomes autonomous within the sampling interval and thus it is amenable to a rigorous mathematical treatment rooted in the well-known Taylor-Lie series theory for nonlinear autonomous ODEs widely used in this context. In particular, the analytical solution of the systems is expandable in a uniformly convergent Taylor series within the sampling interval and the resulting coefficients can be easily obtained by taking successive partial derivatives of the right-hand side of systems model. On the other hand, there is a large body of literature for control and estimation of nonlinear systems satisfying a Lipschitz continuity condition. See for example [1-23] and the references therein, for details of the approach and application to control and filtering and different classes of nonlinear systems. The significance of this condition is that it guarantees the existence and untidiness of the solution of the nonlinear systems. Also, it provides a mathematically tractable framework to apply Lyapunov stability theory and establish the stability and performance conditions in the form of Riccati equations or LMIs.

This note studies whether Lipschitz continuity is preserved under approximate discretizations of nonlinear system using a zero order hold, for both standard two-sided Lipschitz condition and its extended one-sided version. Analytical expressions are given relating the Lipschitz constants of discretized systems into their continuous-time values.

2 Problem Statement

We consider the following continuous-time system

\[ \dot{x} = Ax + f(x, u) \tag{1} \]

\[ y = Cx \tag{2} \]
where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$. The control input is assumed to be constant during the sampling intervals $[kT, (k+1)T)$ (zero-order hold assumption). The family of exact discretization is:

$$
x_{k+1} = A^e_0 x_k + F^e_T(x_k, u_k)
\quad y_k = C_d x_k.
$$

Index $T$ means the discretizations is dependent to the sampling time. However, it is realistic to assume that a family of approximate discrete-time models is available

$$
x_{k+1}^a = A^a_0 x_k^a + F^a_T(x_k^a, u_k)
\quad y_k = C_d x_k^a.
$$

Before stating the problem, we need to refer to the following two definitions.

**Definition 1.** \[13\] The system (1)-(2) is said to be *locally Lipschitz* in a region $\mathcal{D}$ including the origin with respect to $x$, uniformly in $u$, if there exist a constant $l > 0$ satisfying:

$$
\|f(x_1, u^*) - f(x_2, u^*)\| \leq \gamma_c \|x_1 - x_2\| \quad \forall x_1, x_2 \in \mathcal{D},
$$

where $u^*$ is any admissible control signal. The smallest constant $\gamma_c > 0$ satisfying (12) is known as the *Lipschitz constant*. The region $\mathcal{D}$ is the *operational region* or our *region of interest*. If the condition (12) is valid everywhere in $\mathbb{R}^n$, then the function is said to be globally Lipschitz.

An extension of this class are the so-called *one-sided* Lipschitz systems. The following definition introduces one-sided Lipschitz functions.

**Definition 2.** \[13\] The system (1)-(2) is said to be *one-sided Lipschitz* if there exist $\rho \in \mathbb{R}$ such that $\forall x_1, x_2 \in \mathcal{D}$

$$
\langle f(x_1, u^*) - f(x_2, u^*), x_1 - x_2 \rangle \leq \rho_c \|x_1 - x_2\|^2,
$$

where $\rho_c \in \mathbb{R}$ is called the *one-sided Lipschitz constant*. As in the case of Lipschitz functions, the smallest $\rho$ satisfying (6) is considered as the one-sided Lipschitz constant. Similar to the Lipschitz property, the one-sided Lipschitz property might be local or global.

Note that while the Lipschitz constant must be positive, the one-sided Lipschitz constant can be positive, zero or even negative \[13\]. For any function $f(x, u)$, we have:

$$
| \langle f(x_1, u^*) - f(x_2, u^*), x_1 - x_2 \rangle | \leq \|f(x_1, u^*) - f(x_2, u^*)\| \|x_1 - x_2\|$

and if $f(x, u)$ is Lipschitz, then:

$$
\leq \gamma_c \|x_1 - x_2\|^2.
$$

Therefore, any Lipschitz function is also one-sided Lipschitz. The converse, however, is not true. For Lipschitz functions, we have

$$
-\gamma_c \|x_1 - x_2\|^2 \leq \langle f(x_1, u^*) - f(x_2, u^*), x_1 - x_2 \rangle \leq \gamma_c \|x_1 - x_2\|^2,
$$

which is a *two-sided* inequality v.s. the *one-sided* inequality in (16). The Lipschitz constants can be bounded by the norms of the Jacobian \[24\] or computed through numerical optimization \[25\]. See \[13\] for further details.

Assuming the continuous-time system is Lipschitz or one-sided Lipschitz, the propose of this note is to study the conditions under which these properties are preserved under zero order hold discretization of the nonlinear system.
3 Lipschitz Conditions under ZOH Discretization

Under the ZOH assumption, similar to the approach given in [26], we have:

\[ x(k+1) = x(k) + \sum_{l=1}^{\infty} \frac{T}{l!} \frac{d^l x}{d t^l} |_{t=k} \]

\[ = x(k) + \sum_{l=1}^{\infty} \frac{T}{l!} d^l-1 [Ax + f(x, u)] |_{t=k} \]

\[ = x(k) + \sum_{l=1}^{\infty} \frac{T}{l!} [A d^l-1 x + \frac{d^l-1}{d t^{l-1}} f(x, u)] |_{t=k}. \]

where

\[ \left\{ \begin{array}{l}
\frac{d}{dt} f(x, u) = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial u} \cdot \frac{du}{dt} \\
\frac{d^n}{dt^n} f(x, u) = \frac{d}{dt} \left( \frac{d^{n-1}}{dt^{n-1}} f(x, u) \right), \quad n \geq 2
\end{array} \right. \]

(8)

Under the ZOH assumption, \( \frac{du}{dt} = 0 \) in each sampling interval and thus:

\[ x(k+1) = x(k) + \sum_{l=1}^{\infty} \frac{T}{l!} [A d^l-1 x + \frac{d^l-1}{d t^{l-1}} f(x, u)] |_{t=k} \]

\[ \frac{d}{dt} f(x, u) = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} \quad \frac{d^n}{dt^n} f(x, u) = \frac{d}{dt} \left( \frac{d^{n-1}}{dt^{n-1}} f(x, u) \right), \quad n \geq 2. \]

(9)

The first order approximation, \( (l = 1) \) leads to the well-known Euler approximate model.

3.1 First Order Discrete Approximation (the Euler Method)

We first analyse Lipschitz conditions under the Euler discretization. This is a trivial case, in which Lipschitz continuity of the discretized system is established following the properties of the inner-product spaces. Yet this is a very important case, since it has significant practical applications due to its computational simplicity. For the Euler approximation we have

\[ x(k+1) = x(k) + T [Ax(k) + f(x_k, u_k)], \]

\[ A_d^0 = I + AT, \quad C_d = C, \]

\[ F_d^0(x_k^0, u_k) = T f(x_k^0, u_k). \]

3.1.1 Lipschitz Continuity

As seen, under the Euler discretization, the structure of the nonlinear function is preserved and is just scaled by the sampling time. Therefore,

\[ \|F_d^0(x_1, u^*) - F_d^0(x_2, u^*)\| = T \|f(x_1, u^*) - f(x_2, u^*)\| \leq T \gamma_c \|x_1 - x_2\| \quad \forall x_1, x_2 \in \mathcal{D}, \]

(11)

where, \( x_1 \) and \( x_2 \) are any two points in the operating space. It is clear that the Lipschitz continuity is preserved under Euler approximation. The Lipschitz constant of the Euler approximate model is \( \gamma_d = T \gamma_c. \)
3.1.2 One-Sided Lipschitz Continuity

A similar argument can be made for the one-sided Lipschitz property.

\[
\langle F_T(x_1, u^*) - F_T(x_2, u^*), x_1 - x_2 \rangle = T \langle f(x_1, u^*) - f(x_2, u^*), x_1 - x_2 \rangle 
\leq T \rho_c \|x_1 - x_2\|^2, \quad \forall x_1, x_2 \in D. \tag{12}
\]

This means that the Euler approximate model in one-sided Lipschitz with one-sided Lipschitz contact \( \rho_d = T \rho_c \).

So, in summary, using the Euler discretization, both two-sided and one-sided Lipschitz constant are just linearly scaled by the sampling time.

3.2 Second Order Discrete Approximation

The Taylor expansion of the second order discrete model gives:

\[
x(k+1) = \left( I + AT + \frac{T^2}{2} A^2 \right) x(k) + \left( T + \frac{T^2}{2} A \right) f(x, u) + \frac{T^2}{2} \left( \frac{\partial f}{\partial x} A x + \frac{\partial f}{\partial x} f(x, u) \right) \tag{13}
\]

3.2.1 Lipschitz Continuity

Based on the above:

\[
\|F_T(x_1, u^*) - F_T(x_2, u^*)\| = \left\| \left( T + \frac{T^2}{2} A \right) f(x_1, u^*) + \frac{T^2}{2} \left( \frac{\partial f}{\partial x} |_{x=x_1} A x_1 + \frac{\partial f}{\partial x} |_{x=x_1} f(x_1, u^*) \right) 
- \left( T + \frac{T^2}{2} A \right) f(x_2, u^*) - \frac{T^2}{2} \left( \frac{\partial f}{\partial x} |_{x=x_2} A x_2 + \frac{\partial f}{\partial x} |_{x=x_2} f(x_2, u^*) \right) \right\|
\leq T \|f(x_1, u^*) - f(x_2, u^*)\| + \frac{T^2}{2} \sigma(A) \alpha \|x_1 - x_2\| + \frac{T^2}{2} \left( \sigma(A) + \alpha \right) \|f(x_1, u^*) - f(x_2, u^*)\|,
\]

where \( \sigma(A) \) is the maximum singular value or the induced 2-norm of \( A \) and \( \alpha \) is the supremum of the norm of the Jacobian of \( f(x, u) \) over the operating region. Using the Lipschitz continuity condition we get:

\[
\|F_T(x_1, u^*) - F_T(x_2, u^*)\| < \left( T \gamma_c + \frac{T^2}{2} \sigma(A) (\alpha + \gamma_c) + \frac{T^2}{2} \alpha \gamma_c \right) \|x_1 - x_2\|, \quad \forall x_1, x_2 \in D, \tag{15}
\]

which shows the satisfaction of the Lipschitz continuity in discrete-time domain, using Cauchy-Schwartz inequalities for normed spaces. On the other hand, \( \alpha \) itself by definition is the Lipschitz constant in continuous-time \[24\].

\[
\alpha = \sup_{x \in D} \left\| \frac{\partial f}{\partial x} \right\|. \tag{16}
\]

Simplifying the above yields to:

\[
\gamma_d = T \gamma_c + T^2 \left( \sigma(A) \gamma_c + \frac{\gamma_c^2}{2} \right). \tag{17}
\]
3.2.2 One-Sided Lipschitz Continuity

Similar to the Lipschitz continuity, we can construct the one-sided Lipschitz continuity condition for the second order approximate model, using the properties of the inner-product spaces. After doing some linear algebra, we get:

\[
\langle F^a_T(x_1, u^*) - F^a_T(x_2, u^*), x_1 - x_2 \rangle < \left( T + \frac{T^2}{2} \bar{\sigma}(A) \right) \rho_c + \frac{T^2}{2} \bar{\sigma}(A) \gamma_c \rho_c \| x_1 - x_2 \|^2, \quad \forall x_1, x_2 \in D,
\]

which leads to the following expression for the discrete-time one-sided Lipschitz constant:

\[
\rho_d = T \rho_c + \frac{T^2}{2} \bar{\sigma}(A)(\rho_c + \gamma_c + \rho_c \gamma_c).
\]

It is interesting to see that in this case, the discrete one-sided Lipschitz constant is not only a function of the continuous one-sided Lipschitz contact, but also a function of the two-sided continuous Lipschitz constant.

3.3 Higher Order Approximations

In most practical applications, first or second order discretization should be enough, specially since the sampling time can be selected small enough to ensure desired bounds on the approximation error. Furthermore, the expressions involving higher-order approximate models rapidly become very complicated. In particular, higher-order partial derivatives require tensor analysis of higher-orders. In this section, we briefly discuss the third-order approximate model, and derive the analytical expression for the two-sided Lipschitz constant, which also serves as a hint to those of higher-order approximate models. For the third order approximate model, under the ZOH assumption we get:

\[
x(k + 1) = \left( I + AT + \frac{T^2}{2} A^2 + \frac{T^3}{6} A^3 \right) x(k) + Tf(x, u) + \frac{T^2}{2} \left( A f(x, u) + \frac{\partial f}{\partial x} A x + \frac{\partial f}{\partial x} f(x, u) \right) + \frac{T^3}{6} \left( 2A^2 \frac{\partial f}{\partial x} x + 2A \frac{\partial^2 f}{\partial x^2} x + A \frac{\partial^2 f}{\partial x^2} f(x, u) + 2A \frac{\partial f}{\partial x} f(x, u) + 2 \frac{\partial^2 f}{\partial x^2} f(x, u) \right) + \left( \frac{\partial f}{\partial x} \right)^2 f(x, u).
\]

Note that the second derivative of the vector field \( f \) with respect to the state vector \( x \) is not a Hessian matrix, but a tensor of order three. For the two-sided Lipschitz continuity condition, after some tedious manipulations, the discrete Lipschitz constant is achieved as:

\[
\gamma_d = T \gamma_c + \frac{T^2}{2} \left[ \bar{\sigma}(A) \gamma_c + \left\| \frac{\partial f}{\partial x} \right\| \gamma_c + \left\| \frac{\partial f}{\partial x} \right\| \bar{\sigma}(A) \right] + \frac{T^3}{6} \left[ \left\| \frac{\partial^2 f}{\partial x^2} \right\| \bar{\sigma}(A) \gamma_c + 2 \left\| \frac{\partial f}{\partial x} \right\| \bar{\sigma}(A) \gamma_c \right. + \left. 2 \left\| \frac{\partial f}{\partial x} \right\| \gamma_c + 2 \left\| \frac{\partial f}{\partial x} \right\| \bar{\sigma}(A) + 2 \left\| \frac{\partial^2 f}{\partial x^2} \right\| \bar{\sigma}(A) \right].
\]
Defining
\[
\gamma_c = \sup_{x \in D} \left\| \frac{\partial f}{\partial x} \right\|, \tag{22}
\]
\[
\beta = \sup_{x \in D} \left\| \frac{\partial^2 f}{\partial x^2} \right\|,
\]
\[
M = \sup_{x \in D} \| f(x, u) \|,
\]
the above is simplified to
\[
\gamma_d = T \gamma_c + T^2 \left( \sigma(A) \gamma_c + \frac{\gamma_c^2}{2} \right) + \frac{T^3}{6} \left[ 2\beta\sigma(A) + \left( \beta\sigma(A) + 2\beta M + 2\sigma^2(A) \right) \gamma_c + 2\sigma(A) \gamma_c^2 + 2\gamma_c^3 \right]. \tag{23}
\]

4 Conclusions

The preservation of the (one-sided) Lipschitz continuity condition was studied for nonlinear systems undergoing a ZOH based approximate discretization. It was shown that the condition can still be established for approximate discretized model, with Lipschitz constants as a function of the continuous Lipschitz constant, and sampling time and induced norms of the system matrices and its Jacobian.

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