Linear embeddings of contractible and collapsible complexes

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Abstract

(1) We show that if a presentation of the trivial group is “hard to trivialize”, in the sense that lots of Tietze moves are necessary to transform it into the trivial presentation, then the associated presentation complex (which is a contractible 2-dimensional cell complex) is “hard to embed in \( \mathbb{R}^3 \), in the sense that lots of linear subdivisions are necessary.

(2) For any \( d \), we show that all collapsible \( d \)-complexes with \( n \) facets linearly embed in \( \mathbb{R}^2 \) after less than \( n \) barycentric subdivisions. This is best possible, as cones over non-planar graphs do not topologically embed in \( \mathbb{R}^3 \).

Introduction

Every finite \( d \)-dimensional simplicial complex can be geometrically realized in \( \mathbb{R}^{2d+1} \) by placing its vertices in generic points. If we try to decrease the ambient dimension by one, then not all \( d \)-complexes can be realized in \( \mathbb{R}^{2d} \). For example, when \( d = 1 \), not all graphs are planar. The class of planar graphs, completely characterized by Kuratowski’s theorem, is invariant under subdivisions.

In higher dimensions, the situation is much more complicated. When \( d \geq 2 \), realizing a complex or just a subdivision of it are no longer equivalent tasks. We say a complex linearly embeds (resp. PL-embeds) in \( \mathbb{R}^k \), if the complex (resp. some subdivision of it) can be realized in \( \mathbb{R}^k \). We say that a complex topologically embeds in \( \mathbb{R}^k \), if it is homeomorphic to some subcomplex of \( \mathbb{R}^k \). Clearly, all linear embeddings are PL embeddings, and all PL embeddings are topological; but the converse inclusions are false. By taking cones over non-planar graphs, one can easily produce examples of \( d \)-complexes that do not embed in \( \mathbb{R}^{2d} \), neither linearly, nor PL, nor topologically.

Whether a specific complex embeds in some \( \mathbb{R}^k \) or not is a delicate matter; see e.g. [MTW11] for a recent algorithmic approach. The conjecture that every contractible 2-complex PL-embeds in \( \mathbb{R}^4 \) is wide open and related to the 4-dimensional smooth Poincaré Conjecture [Cur62]. (Weber proved that every acyclic \( d \)-complex PL-embeds in \( \mathbb{R}^{2d} \) if \( d \geq 3 \) [Web67], but the argument does not extend to \( d = 2 \).) In contrast, Horvatić and Kranjc showed that all acyclic \( d \)-complexes topologically embed in \( \mathbb{R}^{2d} \) [Hor71, Kra88].

In the present paper, we prove the following result. Let us denote by \( \Delta(a) \) a tower of exponentials of base 2 of length \( |a| \). In particular, \( \Delta(0) = 2 \), \( \Delta(1) = 2^2 \), and recursively \( \Delta(n+1) = 2^{\Delta(n)} \). We say a number is mild in \( n \), if it is bounded by a tower of exponentials in \( n \) of bounded length.

Main Theorem A. If a presentation of length \( n \) of the trivial group needs at least \( \Delta(\Omega(n)) \) Tietze moves to trivialize, then the associated presentation complex does not embed linearly into \( \mathbb{R}^3 \) after a mild number of subdivisions.
Here, \( f = \Omega(g) \) is Knuth’s notation, i.e., \( f \) is asymptotically bounded from below by \( g \). That presentations with the properties assumed in Main Theorem A actually exist, is nontrivial; they were constructed by Bridson [Bri15+] and Lishak-Nabutovsky [LN16+]. Main Theorem A reflects the difference between dimension two and higher dimensions. In fact, in any dimension \( d \geq 3 \), the Freedman–Krushkal theorem claims that “every topological embedding of a \( d \)-complex with \( n \) facets in \( \mathbb{R}^{2d} \) is isotopic to a linear embedding with \( O(e^{n^{d+\epsilon}}) \) facets” [FK14], a number which of course is much smaller than a tower of exponentials.

In the paper [FK14] mentioned above, Freedman and Krushkal also show that for each \( d \geq 2 \) one can find \( d \)-complexes with \( n \) facets that need a singly-exponential number of subdivisions to embed. We complement this result as follows:

**Main Theorem B.** For any \( d \in \mathbb{N} \), the \((n − 1)\)-st barycentric subdivision of any collapsible \( d \)-complex with \( n \) facets embeds linearly in \( \mathbb{R}^{2d} \).

**Notation**

**Functions and their growth.** Given a function \( f : \mathbb{N} \to \mathbb{N} \), we say that \( f \) is mild if it grows at most like a tower of exponentials of bounded length. For example,

\[
g(n) = 2^{3^{3^{3^n}}}
\]

is mild, because it grows like a tower of exponentials of length three. By \( f = \Omega(g) \) we mean that \( f \) is asymptotically bounded from below by \( g \). By \( \Delta(u) \) we denote a tower of exponentials of base 2 of length \( \lfloor u \rfloor \).

**Polytopal complexes.** By \( \mathbb{R}^d \) and \( S^d \) we denote the Euclidean \( d \)-space and the unit sphere in \( \mathbb{R}^{d+1} \) with the standard (intrinsic) metric, respectively. A (Euclidean) polytope in \( \mathbb{R}^d \) is the convex hull of finitely many points in \( \mathbb{R}^d \). A spherical polytope in \( S^d \) is the convex hull of a finite number of points that all belong to some open hemisphere of \( S^d \). Spherical polytopes are in natural one-to-one correspondence with Euclidean polytopes, by taking radial projections. A geometric polytopal complex in \( \mathbb{R}^d \) (resp. in \( S^d \)) is a finite collection of polytopes in \( \mathbb{R}^d \) (resp. \( S^d \)) such that the intersection of any two polytopes is a face of both. For details, see e.g. the definition section in [AB17].

We denote by \( sdC \) the barycentric subdivision of \( C \). Recursively, \( sd^n(C) = sd(sd^{n−1}C) \). For the definitions of shellable, line shelling, etc. see e.g. Ziegler [Zie95]. A free face \( \sigma \) is a face strictly contained in only one other face of \( C \). An elementary collapse is the deletion of a free face \( \sigma \) from a polytopal complex \( C \). We say that the complex \( C \) collapses to a subcomplex \( D \), and write \( C \searrow D \), if \( C \) can be reduced to \( D \) by a sequence of elementary collapses. A collapsible complex is a complex that collapses onto a single vertex. Non-evasiveness can be defined by induction on the dimension, as follows: A simplicial complex of dimension \( d \) is called non-evasive if either \( d = 0 \) and the complex consists of a single point, or if \( d > 0 \) and the complex can be reduced to a single vertex by recursively deleting a vertex whose link is \((d−1)\)-dimensional and non-evasive.

**Group presentations.** A presentation with generators \( g_i \) and relators \( r_j \) of a group \( G \), usually denoted by

\[
G = \langle g_1, \ldots, g_s : r_1, \ldots, r_t \rangle,
\]

is called balanced if \( s = t \). Any balanced presentation \( \varphi \) can be associated with a cell complex \( \mathcal{C} = \mathcal{C}(\varphi) \), called presentation complex, so that the 1-cells of \( \mathcal{C} \) (resp. the 2-cells of \( \mathcal{C} \)) are in
one-to-one correspondence with the generators (resp. the relators) of the presentation. If \( G \) is the trivial group, then \( C(\varphi) \) is contractible; for details, see e.g. [HMS93, Chapter I]. The \( \text{presentation length} \ell(\varphi) \) is the sum of all relator lengths in the presentation plus the number of generators. \( \text{Tietze moves} \) are the following four ways to change a presentation \( \varphi \) of a group \( G \) into another presentation of the same group \( G \):

I. Add a relation between generators that is a consequence of the existing ones.
   For example, \( <x : x^3 = 1 > \) can be changed to \( <x : x^3 = 1, x^6 = 1 > \).

II. Delete a relation between generators that is a consequence of the other ones.
   For example, \( <x : x^3 = 1, x^6 = 1 > \) can be changed to \( <x : x^3 = 1 > \).

III. Add a new generator that is expressed as a word in the other generators.
   For example, \( <x : x^3 = 1 > \) can be changed to \( <x, y : x^3 = 1, y = x^2 > \).

IV. If a relation can be formed where one of the generators is a word in the other generators,
   then remove that generator, replacing all occurrences of it with the equivalent word.
   For example, \( <x, y : xy = 1, y = x^2 > \) can be changed to \( <x : x^3 = 1 > \).

The inverse of a type I move is a type II move; same for III and IV.

1 Hard embeddings from hard presentations

In this section we show that if \( \varphi \) is a complicated presentation of the trivial group, then some contractible 2-complex \( X(\varphi) \) associated to it cannot embed in \( \mathbb{R}^3 \) in low complexity. Our first step is to reduce ourselves to working with simplicial complexes:

**Lemma 1.** Let \( \varphi \) be any finite presentation of a group \( G \). Let \( \ell \) be the length of \( \varphi \). Then the presentation complex \( \mathcal{C}(\varphi) \) has a triangulation \( C = C(\varphi) \) with at most \( 24\ell \) faces.

**Proof.** For each \( i \), we realize the 2-cell \( \sigma_i \) corresponding to \( r_i \) as a polygon with \( \ell_i \) edges. This way we obtain a regular cell complex with \( 24\ell \) faces. Then we perform a barycentric subdivision.

The second step is a revisitation of a classical construction due to Bing:

**Lemma 2** (essentially Bing, cf. [Bin59, Lemma 6] and [Bin83, Theorem I.2A]). Let \( X \) be any \( k \)-complex with \( m \) faces that is geometrically realized in \( \mathbb{R}^d \) (respectively, in \( S^d \)). Then \( X \) can be completed to a triangulation of a convex ball \( B \subset \mathbb{R}^d \) (respectively, to a triangulation of \( S^d \)) using \( O(m^k) \) faces, for fixed \( d \).

**Proof.** We follow Bing’s proof, and in particular, we use the terminology in [Bin83, Theorem I.2A]. We proceed by induction on \( k \). When \( k = 1 \) the claim is clear. Let \( f_k(X) \) be the number of \( i \)-faces of \( X \). We first “shield off” each \( k \)-face by taking the join with the boundary of a (suitably small) \((d - k)\)-simplex. This introduces \( f_k \) new \( d \)-dimensional simplices. Of course, \( f_k \leq m \). Once all \( k \)-faces are shielded off, we turn our attention to the link of each \((k - 1)\)-face.

Such link is \( k \)-dimensional in \( S^{d-1} \), but all “exposed faces” are of dimension \((k - 1)\). By the inductive assumption, we can complete the triangulation of the link to a triangulation of \( S^{d-1} \). This introduces \( O(m^{k-1}) \) faces for each \( k \)-face, for a total of \( O(m^k) \) faces.

**Corollary 3.** Let \( X \) be any \( k \)-complex with \( m \) faces that is geometrically realized in \( \mathbb{R}^d \) (or \( S^d \)). There is a realization of the regular neighborhood \( N_X \) of \( X \) in \( \mathbb{R}^d \) that uses at most \( O(m^k) \) faces and collapses onto \( N_X \).

**Proof.** As in Lemma 2 we complete \( X \) to a triangulation \( B \) of a convex ball, take two barycentric subdivisions, and look at the subcomplex of \( B \) induced by all facets intersecting \( X \). This gives the desired realization of the regular neighborhood.
Our third, crucial step shows how to extend a triangulation of a \((d - 1)\)-sphere in \(\mathbb{R}^d\) to a collapsible triangulation of its inside, using a mild number of faces.

**Proposition 4.** Let \(d \leq 4\). Let \(S\) be a triangulated \((d - 1)\)-sphere realized in \(\mathbb{R}^d\) on \(m\) faces. Let \(B\) be the “inside”, i.e., the topological closure of the bounded connected component of the complement of \(S\). There exist a triangulation of the \(d\)-ball \(B\) such that
1. restricted to its boundary, \(B\) coincides with a (mild) subdivision \(S'\) of \(S\);
2. \(B\) has a mild number of faces;
3. \(B\) is collapsible.

**Proof.** Every triangulated sphere of dimension \(\leq 2\) is polytopal. As for triangulated 3-spheres, by a theorem of King [Kin04] polytopality can be achieved with a number of subdivisions estimated by \(e^{O(m^2)}\). So without loss of generality we can assume that \(S\) is polytopal. (Caveat: This does not mean that the inside of \(S\) is convex.)

![Figure 1: Subdividing the geometric cone \(C_1\) to “make it fit” inside \(S\).](image)

Consider now a line shelling of \(S\) that ends by removing the star of a vertex \(v\). We want to triangulate the (abstract) cone over \(S\) with apex \(v\) in such a way that it forms a triangulation of \(S\) in \(\mathbb{R}^d\). For this, consider a first facet \(F_1\) in the shelling order we selected. Let \(C_1\) be the geometric cone over \(F_1\) with apex \(v\) in \(\mathbb{R}^d\). Since the inside of \(S\) is not necessarily convex, we do not know whether all facets of \(S\) are visible from \(v\). In particular, the geometric cone that we have just constructed might intersect \(S\) in faces other than \(F_1\) and \(v\). To fix it, we want to move this cone to the inside of \(B\).

To achieve this, we may use a triangulation parallel to \(S\) whenever the cone encounters the sphere \(S\), using stellar subdivisions of \(C_1\) at boundary faces (cf. Figure 1 above.) The stellar subdivision of the cone is still collapsible, and we have introduced at most \(e^{O(m^2)}\) new faces.

Consider now the cone \(C_2\) over the next facet \(F_2\) with apex \(v\). Again, we move \(C_2\) to the interior by subdividing \((\partial C_2) \setminus C_1\) relative to \(C_1\). Since the latter has \(e^{O(m^2)}\) faces, this yields \(e^{O(m^4)}\) new faces. This is repeated until the triangulation of the disk bounded by \(S\) is completed, and gives us a total of at most
\[
e^{m^eO(m^2)}
\]
new faces. This number is mild in \(m\), as desired.

**Remark 5.** For polyhedra in \(\mathbb{R}^3\) the bound above can be significantly improved, especially if one is willing to renounce to the collapsibility conclusion: See for instance [CP90].

**Remark 6.** The proof of Proposition 4 breaks down if \(d \geq 5\): A triangulated 4-sphere has a polytopal subdivision if and only if it is PL, and whether all 4-spheres are PL is an important open problem, equivalent to the smooth Poincaré conjecture. But even if we restrict ourselves
to PL spheres, making a PL 4-sphere shellable may require a much larger number of barycentric subdivisions. In fact, let \( F(n) \) be the smallest integer \( k \) such that every PL 4-sphere with \( n \) facets becomes shellable after \( k \) consecutive barycentric subdivisions. We know that \( f(n) \) must be larger than a tower of exponentials, by the work of Lishak and Nabutovski [LN16+] [LN15+]. Were \( f(n) \) bounded above by a computable function \( h = h(n) \), then a shellability test for the \( h(n) \)-th barycentric subdivision would yield an algorithm to recognize PL 4-spheres. It is conjectured that no such algorithm exists. If such conjecture holds true, then \( F(n) \) must be larger than any computable function of \( n \).

**Lemma 7.** Let \( D_1, D_2 \) denote two geometric triangulations of the same \( d \)-manifold \( D \) in \( \mathbb{R}^d \), or in \( S^d \). Let \( n_i \) be the number of faces of \( D_i \) (\( i = 1, 2 \)). Then the number of bistellar flips and stellar/inverse stellar subdivisions needed to connect \( D_1 \) and \( D_2 \) is mild in \( n_1 \) and in \( n_2 \).

**Proof.** Consider a regular subdivision \( R \) of the convex hull of \( D \) with no interior vertices. The number of faces of this triangulation is polynomial in the number \( b \) of boundary vertices. In fact, by the Upper Bound Theorem [Sta75], it is bounded above by \( O(b^{\frac{d+1}{2}}) \). Let \( r \) be the number of faces of \( R \).

To find a regular stellar subdivision \( D'_1 \) of \( D_1 \) and \( R \), we perform stellar subdivisions on \( R \) for every transversal intersection of faces of \( D_1 \) and \( R \), resulting in some regular subdivision \( D''_1 \) of \( D_1 \). This requires at most \( n_1 \) subdivisions. As a result, \( D''_1 \) has at most \( r(d+1)^{n_1} \) many faces, cf. [AI15] Lemma 3]. This \( D''_1 \) is not necessarily stellar. However, any stellar subdivision of \( D_1 \) that refines \( D''_1 \) is necessarily regular: Compare the proof of [AI15] Theorem 1]. This latter step requires at most \( r(d+1)^{n_1} \) subdivisions by [AI15] Lemma 3], and brings the count of faces to at most

\[
n_1^{(d+1)r(d+1)^{n_1}}
\]

which is large but mild. Similarly, we find a common regular stellar subdivision \( D'_2 \) of \( D_2 \) and \( R \) after a mild number of subdivisions. The conclusion follows then from the fact that any two regular subdivisions of the same \( d \)-dimensional disk are connected by at most polynomially many bistellar flips in the number of faces, cf. [DRS10] Corollary 5.3.11].

**Lemma 8** (Lishak–Nabutovski [LN16+]). Let \( D_1 \) and \( D_2 \) denote two geometric triangulations of the same convex subset \( D \) in \( \mathbb{R}^d \) or in \( S^d \). If \( D_1 \) and \( D_2 \) are connected by a single bistellar move, the induced presentations of their fundamental groups are at most \( C_d \) Tietze moves apart, where \( C_d \) depends only on \( d \).

**Remark 9.** Any stellar subdivision can be achieved by polynomially many bistellar moves (in terms the number of the faces incident to the subdivided face).

**Proof of Main Theorem** [A] Consider a family of presentations of length \( \ell \) that requires \( \Delta(\Omega(\ell)) \) Tietze moves to trivialize. By Lemma [1] in terms of \( \ell \), the complex \( C(\varphi) \) has linearly many faces. Moreover, since \( \varphi \) presents the trivial group, the complex \( C(\varphi) \) is contractible. Now, suppose there is a piecewise linear embedding \( \varphi \); let us call \( X \) the image of the complex \( C(\varphi) \) under \( \varphi \). Assume by contradiction that the number of faces of \( X \) is mild. Then there exists a mild triangulation of the regular neighborhood, which is a PL disk in \( \mathbb{R}^3 \) (being a contractible compact 3-manifold). As observed, there exists a mild triangulation of that ball that is collapsible. Combining with the previous lemma, we obtain that the number of Tietze moves required to show triviality is mild; a contradiction.

**Remark 10.** By Bridson [Bri15+] and Lishak-Nabutovski [LN16+], there exists a family of presentations of the trivial group that require \( \Delta(\Omega(\ell)) \) Tietze moves to trivialize. So as direct
corollary of Theorem \[A\] there are contractible 2-complexes \(X\) with \(n\) facets that do not linearly embed into \(\mathbb{R}^3\) with a number of subdivisions smaller than a tower of exponentials of length \(n\). This contrasts a recent result by Freedman and Krushkal \[FK14\], who proved a much smaller upper bound for the number of subdivisions necessary to PL embed complexes of dimension \(d \geq 3\).

Of course, if \(Y\) is the cone over any non-planar graph, then \(Y\) does not topologically embed in \(\mathbb{R}^3\). The contractible complexes we build above though have a non-trivial, numerical obstruction to PL embeddability, in the sense that a priori, some might PL embed.

More generally, if \(Y\) is the cone over any \((d-1)\)-complex that does not topologically embed in \(\mathbb{R}^{2d-2}\) (such as the \((d-1)\)-skeleton of the 2d-simplex), then \(Y\) is a collapsible \(d\)-complex that does not topologically embed in \(\mathbb{R}^{2d-1}\). In the next section, we complement this result by showing that if we try an ambient space of one dimension higher, then every collapsible complex linearly embeds after few subdivisions.

\section{All collapsible complexes embed after few subdivisions}

In this section we prove Main Theorem \[B\].

\begin{lemma}
Let \(d, n\) be positive integers. Let \(\Sigma\) be a \(d\)-simplex and let \(\sigma\) be any of its facets. Let \(A_{n-1} = \text{sd}^{n-1}(\partial \Sigma - \sigma)\). Then \(\text{sd}^n \Sigma\) admits a facewise linear map \(\varphi_n\) to \(A_{n-1}\) that restricts to the identity on \(\text{sd} A_{n-1}\).
\end{lemma}

\begin{proof}
We claim that it is enough to prove this for \(n = 1\); in this case, \(\varphi_1\) is defined by sending every face of \(A_0 = \partial \Sigma - \sigma\) to itself, and every remaining vertex in \(\text{sd} \Sigma\) to the unique vertex of \(\Sigma\) not in \(\sigma\). To recursively construct \(\varphi_{n+1}\) when given \(\varphi_n\), it suffices to observe that the barycentric subdivision of \(A_{n-1}\) can be lifted to \(\Sigma\) along the fibers of \(\varphi_n\). See Figure 2 below.
\end{proof}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2}
\caption{How to construct \(\varphi_0\) (left) and \(\varphi_1\) (right).}
\end{figure}

\begin{corollary}
Let \(d, n, k\) be positive integers. Let \(\varepsilon > 0\) be a real number. Let \(\Sigma\) be a \(d\)-simplex and let \(\sigma\) be one of its facets. Let \(A_{n-1} = \text{sd}^{n-1}(\partial \Sigma - \sigma)\). Then any facewise linear embedding \(\phi : A_{n-1} \to \mathbb{R}^k\) can be extended to a facewise linear map \(\phi : \text{sd}^n \Sigma \to \mathbb{R}^k\) so that the image of \(A_{n-1}\) and \(\text{sd}^n \Sigma\) differ by at most \(\varepsilon\) in Hausdorff distance.
\end{corollary}

\begin{proof}
Using the map \(\varphi_n\) of Lemma \[11\] we can deform \(\Sigma\) until it gets close to \(A_{n-1}\).
\end{proof}

\begin{proof}[Proof of Main Theorem \[B\]]
We do not need to worry about the lower-dimensional skeleton, since every \(k\)-complex generically embeds in \(\mathbb{R}^{2d}\) when \(k \leq d - 1\). Also, by genericity, we do not
need to worry about intersections of $d$-faces with faces of dimension lower than $d$. Hence, we proceed by induction on the number $n$ of $d$-dimensional faces.

The case $n = 1$ consists of a collapsible $d$-complex $C$ (not necessarily pure) with only one $d$-simplex. This certainly embeds in $\mathbb{R}^{2d}$ simply by generically embedding into $\mathbb{R}^{2d}$ the $(d-1)$-skeleton of $C$ and then by inserting the unique $d$-simplex. No barycentric subdivision is required.

Now, let $C$ be a collapsible $d$-dimensional simplicial complex with $n + 1$ $d$-faces. Fix a collapse of $C$ and let $\sigma$ be the first free face removed in the sequence. Let $\Sigma$ be the unique $d$-face of $C$ containing $\sigma$. Set $C' = C - \sigma$. Since $C'$ is collapsible and has $n$ $d$-faces, by the inductive assumption we can find a geometric realization of $s\Delta^{n-1}C'$ in $\mathbb{R}^{2d}$. So all we need to do is to figure out how to position the vertices of the $n$-th barycentric subdivision of the simplex $\Sigma$ “conveniently close” to the geometric realization of $s\Delta^{n-1}C'$, so that self-intersections outside of stars of faces are avoided. This is precisely the task carried out by Corollary 12.

Remark 13. A 2-dimensional complex $C$ is called 3-thickenable if there exists a triangulated 3-manifold with boundary $M$ that collapses onto a subdivision of $C$. See Skopenkov [Sko95] for the precise definition; we thank Uli Wagner for introducing us to this notion. Being 3-thickenable is stronger than the planarity of all vertex links planar [Sko95]. If $C$ is a 2-dimensional complex that is 3-thickenable, then reasoning exactly as in the proof of Main Theorem B above, one can conclude by induction on the number $n$ of facets that $C$ embeds linearly into $\mathbb{R}^3$.

Applying the same reasoning to cubical complexes, we obtain an analogous result:

Proposition 14. Every collapsible $d$-dimensional cubical complex with $n$ facets embeds linearly in $\mathbb{R}^{2d}$ after less than $n$ barycentric subdivisions.

Corollary 15. Every $d$-dimensional (finite) CAT(0) cube complex with $n$ facets embeds linearly in $\mathbb{R}^{2d}$ after less than $n$ barycentric subdivisions.

Proof. By [AB17, Corollary II], every $d$-dimensional CAT(0) cube complex is collapsible.

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References

[AB17] K. A. Adiprasito and B. Benedetti, Collapsibility of CAT(0) spaces, to appear in Geometriae Dedicata.

[AI15] K. A. Adiprasito and I. Izmestiev, Derived subdivisions make every PL sphere polytopal, Israel Journal of Mathematics 208 (2015), 443–450.

[Bin59] R. H. Bing, An alternate proof that 3-manifolds can be triangulated, Annals of Math. 69 (1959), 37–65.
[Bin83] R. H. Bing, The geometric topology of 3-manifolds. Providence, American Mathematical Society, 1983.
[Bri15+] M. R. Bridson, The complexity of balanced presentations and the Andrews–Curtis conjecture, preprint at arXiv:1504.04187.
[CP90] B. Chazelle and L. Palios, Triangulating a Nonconvex Polytope, Discrete and Computational Geometry 5 (1990), 505–526.
[Cur62] M. L. Curtis, On 2-complexes in 4-space, in Topology of 3-Manifolds and Related Topics (Proc. Univ. of Georgia Institute, 1961), Prentice–Hall, Englewood Cliffs, N.J. (1962), 204–207.
[DRS10] J. De Loera, J. Rambau and F. Santos, Triangulations. Structures for Algorithms and Applications, Springer, 2010.
[Fre76] B. M. Freed, Embedding contractible 2-complexes in $\mathbb{E}^4$, Proc. Amer. Math. Soc. 54 (1976), 423–430.
[Fre82] M. Freedman, The topology of four-dimensional manifolds, J. Diff. Geom. 17 (1982), 357–453.
[FK14] M. Freedman, V. Krushkal, Geometric complexity of embeddings in $\mathbb{R}^d$, Geom. Funct. Anal. 24 (2014), 1406–1430.
[HMS93] C. Hog–Angeloni, W. Metzler, A. J. Sieradski (eds.), Two-dimensional homotopy and combinatorial group theory. London Mathematical Society Lecture Note Series, Volume 197. Cambridge University Press, Cambridge (1993).
[Hor71] K. Horvatić, On embedding polyhedra and manifolds, Trans. Amer. Math. Soc. 157 (1971), 417–436.
[Kin04] S. King, How to make a triangulation of $S^3$ polytopal, Trans. Am. Math. Soc. 356 (2004), 4519–4542.
[Kra88] M. Kranjc, Embedding 2-complexes in $\mathbb{R}^4$, Pac. J. Math. 133 (1988), no. 2, 301–313.
[LN16+] B. Lishak and A. Nabutovsky, Sizes of spaces of triangulations of 4-manifolds and balanced presentations of the trivial group, preprint at arXiv:1610.06130.
[LN15+] B. Lishak and A. Nabutovsky, Complexity of Unknotting of Trivial 2-knots, preprint at arXiv:1510.02773.
[MTW11] J. Matoušek, M. Tancer, and U. Wagner, Hardness of embedding simplicial complexes in $\mathbb{R}^d$, J. Europ. Math. Society 13(2) (2011), 259–295.
[RS99] D. Repovš and A. B. Skopenkov, New results on embeddings of polyhedra and manifolds in Euclidean spaces, Russ. Math. Surv. 54 (1999), 1149–1196.
[Sko95] A. B. Skopenkov, A generalization of Neuwirth’s theorem on thickening 2-dimensional polyhedra. Mathematical Notes 58 (1995), 1244–1247.
[Sta75] R. P. Stanley, The upper bound conjecture and Cohen-Macaulay rings, Stud. Appl. Math. 54 (1975), 135–142.
[Web67] C. Weber, Plongements de polyhèdres dans le domaine métastable, Comment. Math. Helv. 42 (1967), 1–27.
[Zie95] G. M. Ziegler, Lectures on polytopes, Graduate Texts in Mathematics, vol. 152, Springer-Verlag, New York, 1995.