Remarks on the zero-divisor graph of a commutative ring

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Abstract
In 1988, I. Beck showed that the chromatic number of $G(Z_n)$ is equal to its clique number. In 2004, S. Akbari and A. Mohammadian proved that the edge chromatic number of $G(Z_n)$ is equal to its maximum degree. In 2008, J. Skowronek-Kaziow give formulas calculating the clique number and the maximum degree of $G(Z_n)$, but he have a error about clique number of $G(Z_n)$. We consider the zero-divisor graph $G(Z_n)$ of the ring $Z_n$, we give formulas calculating the clique number of $G(Z_n)$. We present a constructed method to calculate the clique number.

Keywords: Digraph, Group theory, Zero-divisor graph, Chinese Remainder Theorem

1.

The concept of zero-divisor graphs of a commutative rings was introduced by I. Beck in 1988[1]. In 1999 [2], Anderson and Livingston introduced and studied the Zero-divisor graph whose vertices are the non-zero zero-divisors. This graph turns out to best exhibit the properties of the set of zero-divisors of a commutative ring. The zero-divisor graph helps us to study the algebraic properties of rings using graph theoretical tools. We can translate some algebraic properties of a ring to graph theory language and then the geometric properties of graphs help us to explore some interesting results in the algebraic structures of rings. The zero-divisor graph of a commutative ring has been studied extensively by Anderson, Frazier, Lauve, Levy, Livingston and Shapiro, see[2, 3, 4, 10]. The zero-divisor graph concept has recently been extended to non-commutative rings, see[3]. A clique in a graph $G$ is a complete subgraph of $G$, the order of the largest clique in a graph $G$ is its clique number[6]. A subgraph $K_m$ with $m$ vertices is called a clique of size $m$ if any two distinct vertices in it are adjacent. The minimum number of colors that can be used to color the edges of $G$ is called the edge chromatic number and is denoted by $\chi_1(G)$. The vertex chromatic number $\chi(G)$ of a graph $G$ is the minimum $k$ for which $G$ has a $k$-vertex coloring. The zero-divisor graph of the rings $Z_n$, denoted by $G(Z_n)$, is a graph with vertex set in $Z_n - \{0\}$, in which two vertices $x$ and $y$ are adjacent if and only if $x \neq y$ and $x \cdot y \equiv 0 (mod n)$.

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For example, $n = 420 = 2^2 \cdot 3 \cdot 5 \cdot 7, G(Z_{20}) = \{30, 42, 70, 210\}$, the clique number is $4, n = 108 = 2^2 \cdot 3^3, G(Z_{108}) = \{6, 18, 36, 54, 72, 90\}$, the clique number is $6$. In this paper we give formulas calculating the clique number of $G(Z_n)$. Let $n = p_1^{a_1}p_2^{a_2} \cdots p_s^{a_s}$ be the prime power factorization of $n$, where $p_1 < p_2 \cdots < p_s$ are distinct primes and $a_i \geq 1, s \geq 1$.

2. Results

In this section, we show formulas calculating the clique number of $G(Z_n)$ at the same time, we give some examples.

1. Vizing’s Theorem [6, p281] For every nonempty graph $G$, then either $\chi_1(G) = \Delta(G)$ or $\chi_1(G) = \Delta(G) + 1$.

2. Theorem 1. The maximal degree in $G(Z_n)$ has the vertex $n/p_1$ and the maximum degree is equal to $n/p_1 - 1$.

Proof: This is proved in [8].

3. Theorem 2. If $n$ is square-free, then the clique number of the graph $G(Z_n)$ is $s$ if $\alpha_i$ are even numbers for all $1 \leq i \leq s$, then the clique number is $p_1^{a_1/2}p_2^{a_2/2} \cdots p_s^{a_s/2} - 1$, otherwise, the clique number is:

$$p_1^{a_i/2} \cdots p_r^{a_r/2}q_1^{\beta_1-1/2} \cdots q_t^{\beta_t-1/2} + t - 1,$$

where $\delta_i$ is even, $i = 1, \ldots, r$, $\beta_i$ is odd, $i = 1, \ldots, t$.

Proof: We consider the three cases.

1. If $n$ is square-free, let $n = p_1p_2 \cdots p_s$, where $p_i$ are distinct primes, $1 \leq i \leq s$; we consider the set $S = \{n/p_1, n/p_2, \ldots, n/p_s\}$, the product of every pair elements of the set is a multiple of $n$, i.e., the elements of $S$ is in the vertices set of $\delta_i$.

2. If all $\alpha_i$ are even, then the element $m = p_1^{a_1/2}p_2^{a_2/2} \cdots p_s^{a_s/2}$ and element $2m, 3m, 4m, \ldots, (m - 1)m$ form a clique number of $G(Z_n)$, the element $t$ is the smallest number such that the multiple $(m - 1)m$ is possibly the greatest number belonging to $Z_n$ and the clique number in this case is equal to $m - 1$.

3. If $\alpha_i$ are even and odd numbers, let $n = p_1^{a_1/2}p_2^{a_2/2}q_1^{\beta_1-1/2}q_2^{\beta_2-1/2} \cdots q_t^{\beta_t-1/2} + t - 1$, let $i = 1, \ldots, j, p_1 < p_2 < \ldots < p_r, q_1 < q_2 < \ldots < q_j, \sum_{i=1}^{r} \beta_i = \sum_{i=1}^{j} \alpha_i$. We present a constructed method to calculate the clique number.

If $j = 1$, let $k = 1$, let $k = p_1^{a_1/2}p_2^{a_2/2}q_1^{\beta_1-1/2}q_2^{\beta_2-1/2} \cdots q_t^{\beta_t-1/2} + t - 1$, the product of every pair elements of the set $A$ is a multiple of $n$ i.e. the elements of $A$ are in the vertices set of $G(Z_n)$, the element $k$ is the smallest number such that the multiple $(k - 1)k$ is possibly the greatest number belonging to $Z_n$, and the clique number in this case is equal to $k + 1 - 1 = k$.

If $j = 2$, let $c = q_1q_2 \cdots q_t, n = p_1^{a_1/2}p_2^{a_2/2}q_1^{\beta_1-1/2}q_2^{\beta_2-1/2} \cdots q_t^{\beta_t-1/2} + c$.

The proof is complete.
3. Example

(1) If \( n = 60 = 2^2 \cdot 3 \cdot 5 \), by the theorem, the clique number of \( G(Z_{60}) \) is equal to 3. The vertices set of \( G(Z_{60}) \) is \( G(Z_{60}) = \{12, 20, 30\} \).

(2) If \( n = 2^3 \cdot 5^3 \cdot 7^2 \), then, by the theorem, the clique number of \( G(Z_{196000}) \) is equal to 141.

(3) If \( n = 3^3 \cdot 5^2 \cdot 7^3 \), then, by the theorem, the clique number of \( G(Z_{231525}) \) is equal to 106.

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