A NEW PROGRESS ON WEAK DIRAC CONJECTURE

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Abstract. In 2014, Payne-Wood proved that every non-collinear set \( P \) of \( n \) points in the Euclidean plane contains a point in at least \( \frac{n}{37} \) lines determined by \( P \). This is a remarkable answer for the conjecture, which was proposed by Erdős, that every non-collinear set \( P \) of \( n \) points contains a point in at least \( \frac{n}{c_1} \) lines determined by \( P \), for some constant \( c_1 \). In this article, we refine the result of Payne-Wood to give that every non-collinear set \( P \) of \( n \) points contains a point in at least \( \frac{n}{26} + 2 \) lines determined by \( P \). Moreover, we also discuss some relations on theorem Beck that every set \( P \) of \( n \) points with at most \( l \) collinear determines at least \( \frac{1}{61} n(n - l) \) lines.

1. Introduction

Let \( P \) be a set of points in the Euclidean plane. A line that contains at least two points in \( P \) is said to be determined by \( P \).

In 1951, G. Dirac ([4]) made the following conjecture, which remains unsolved:

Conjecture 1 (Strong Dirac Conjecture). Every non-collinear set \( P \) of \( n \) points in the plane contains a point in at least \( \frac{n}{2} - c_0 \) of the lines determined by \( P \), for some constant \( c_0 \).

In 2011, J. Akiyama, H. Ito, M. Kobayashi, and G. Nakamura ([2]) gave some examples to show that the \( \frac{n}{2} \) bound would be tight. We note that if \( P \) is non-collinear and contains \( \frac{n}{2} \) or more collinear points, then Dirac’s Conjecture holds. Thus we may assume that \( P \) does not contain \( \frac{n}{2} \) collinear points, and \( n \geq 5 \).

In 1961, P. Erdős ([5]) proposed the following weakened conjecture.

Conjecture 2 (Weak Dirac Conjecture). Every non-collinear set \( P \) of \( n \) points contains a point in at least \( \frac{n}{c_1} \) lines determined by \( P \), for some constant \( c_1 \).

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In 1983, Beck ([3]) and Szemerédi-Trotter ([18]) proved the Weak Dirac Conjecture for the case $c_1$ but it is unspecified or very large. In 2014, Payne-Wood ([15]) proved the following theorem:

**Theorem 1.** Every non-collinear set $P$ of $n$ points contains a point in at least $\frac{n}{37}$ lines determined by $P$.

For the first purpose of this article, we would like to give a new progress for the Weak Dirac conjecture. In particular, we prove the following:

**Main theorem 1.** Every non-collinear set $P$ of $n$ points contains a point in at least $\frac{n}{26} + 2$ lines determined by $P$.

Moreover, relate to work on the Weak Dirac Conjecture, Beck gave the number of lines determined by $P$. He proved the following theorem.

**Theorem 2.** ([3]) Every set $P$ of $n$ points with at most $l$ collinear determines at least $c_2 n(n - l)$ lines, for some constant $c_2$.

In 2014, Payne-Wood also gave a remarkable improvement of Beck’s theorem by proving the following.

**Theorem 3.** ([15]) Every set $P$ of $n$ points with at most $l$ collinear determines at least $\frac{1}{98} n(n - l)$ lines.

We note that the number 98 can be instead by 93. The details can be found in [14].

For the final purpose, we would like to give some results for the number of lines with few points from $n$ points in plane. Then, we also give the following theorems.

**Main theorem 2.** Every set $P$ of $n$ points with at most $l$ collinear determines at least $\frac{1}{61} n(n - l)$ lines.

**Main theorem 3.** Every set $P$ of $n$ points with at most $l$ collinear determines at least $\frac{1}{122} n(n - l)$ lines with at most 3 points.

2. **Auxiliary Results**

We list here some known results which are very helpful for the proofs of the main theorems.

The crossing number of a graph $G$, denoted by $cr(G)$, is the minimum number of crossings
Lemma 4. *(Crossing lemma [13]).* For every graph with \( n \) vertices and \( m \geq \frac{103}{16}n \) edges, then
\[
\text{cr}(G) \geq \frac{1024m^3}{31827n^2}.
\]

We set \( E(H) \) to be the set of all edges of a graph \( H \). The visibility graph \( G \) of a point set \( P \) has vertex set \( P \), where \( vw \in E(G) \) whenever the line segment \( vw \) contains no other point in \( P \) (that is, \( v \) and \( w \) are consecutive on a line determined by \( P \)). For \( i \geq 2 \), an \( i \)-line is a line containing exactly \( i \) points in \( P \). Let \( s_i \) be the number of \( i \)-lines. Let \( G_i \) be the spanning subgraph of the visibility graph of \( P \) consisting of all edges in \( j \)-lines where \( j \geq i \). Note that since each \( i \)-line contributes \( i-1 \) edges, \( |E(G_i)| = \sum_{j \geq i} (j-1)s_j \).

We introduce some useful results:

**Theorem 5.** *(Hirzebruch’s Inequality [10]).* Let \( P \) be a set of \( n \) points with at most \( n-3 \) collinear. Then
\[
s_2 + \frac{3}{4}s_3 \geq n + \sum_{i \geq 5} (2i - 9)s_i.
\]

**Theorem 6.** *(Szemerédi-Trotter [18]).* Let \( \alpha \) and \( \beta \) be positive constants such that every graph \( H \) with \( n \) vertices and \( m \geq \alpha n \) edges satisfies
\[
\text{cr}(H) \geq \frac{m^3}{\beta n^2}.
\]

Let \( P \) be a set of \( n \) points in the plane. Then
\[
\text{a)} \quad |E(G_i)| = \sum_{j \geq i} (j-1)s_j \leq \max \{ \alpha n, \frac{\beta n^2}{2(i-1)^2} \},
\]
\[
\text{b)} \quad \sum_{j \geq i} s_j \leq \max \{ \frac{\alpha n}{i-1}, \frac{\beta n^2}{2(i-1)^3} \}.
\]

3. A new progress on Weak Dirac’s conjecture

In order to get the main theorem 1, we refine the method of Payne-Wood to find the largest number \( \varepsilon \) such that every set \( P \) of \( n \) non-collinear points in the plane at most \( \varepsilon n + 2 \) collinear points, the arrangement of \( P \) has at least \( \varepsilon n^2 + 2n \) point-line incidents. We start by the following result.
Theorem 7. Let $\alpha$ and $\beta$ be positive constants such that every graph $G$ with $n$ vertices and $m \geq \alpha n$ edges satisfies $cr(G) \geq \frac{m^3}{\beta n^2}$.

Fix two integers $c \geq 8, 0 \leq q \leq 3$ and a real number $\epsilon \in (0; \frac{1}{2})$, $\epsilon n \geq 2$. Let $h := \frac{c(c-2)}{5c-18}$. Then for every set $P$ of $n$ points in the plane with at most $\epsilon n + q$ collinear points, the arrangement of $P$ has at least $\delta n^2 + r n$ point-line incident, where

$$
\delta = \frac{1}{h+1} \left( 1 - \epsilon \alpha - \frac{\beta}{2} \left( -\frac{18(c-2)}{c^3(5c-18)} + \sum_{i \geq c} \frac{i+1}{i^3} \right) \right),
$$

$$
r = \frac{2h-1 + \alpha}{h+1}.
$$

Proof. Let $J = \{2; 3; \ldots; \lfloor \epsilon n \rfloor + q \}$ and assume that $\epsilon n \geq 2$. Considering the visibility graph $G$ of $P$ and its subgraphs $G_i$ as defined previously. Let $k$ be the minimum integer such that $|E(G_k)| \leq \alpha n$. If there is no such $k$ then let $k = \lfloor \epsilon n \rfloor + q + 1$. An integer $i \in J$ is large if $i \geq k$, and is small if $i \leq c$. An integer in $J$ that is neither small nor large is medium.

Recall that an $i$-line is a line containing exactly $i$ points in $P$. An $i$-pair is a pair of points in an $i$-line. A small pair is an $i$-pair for some small $i$. Define large pair, medium pair analogously. Let $P_S, P_M$ and $P_L$ denote the number of small, medium and large pairs respectively. An $i$-incidence is an incidence between a point of $P$ and an $i$-line. A small incidence is an $i$-incidence for some small $i$, and define medium, large incidences analogously. Let $I_S, I_M$ and $I_L$ denote the number of small, medium and large incidences respectively and let $I$ denote the total number of incidences. Since every $s_i$ has $i$ points incidence with its, then

$$I = \sum_{i \in J} i s_i = I_S + I_M + I_L$$

Because $P$ has no more than $\frac{n}{2}$ collinear points and $n \geq 5$, thus $\lfloor \epsilon n \rfloor + q \leq \frac{n}{2} \leq n - 3$. Therefore, for $n$ points of $P$ has no more than $n - 3$ collinear points. Applying the Hirzebruch’s Inequality (Theorem 5), we have

$$s_2 + \frac{3}{4} s_3 \geq n + \sum_{i \geq 5} (2i - 9) s_i.$$
Since $h > 0$ then,

$$hs_2 + \frac{3}{4}hs_3 - hn - h\sum_{i \geq 5} (2i - 9)s_i \geq 0.$$ 

$$P_S = \sum_{i=2}^{c} \left( \frac{i}{2} \right) s_i$$

$$= s_2 + 3s_3 + 6s_4 + \sum_{i=5}^{c} \left( \frac{i}{2} \right) s_i$$

$$\leq (h + 1)s_2 + \left( \frac{3h}{4} + 3 \right)s_3 + 6s_4 + \sum_{i=5}^{c} \left( \frac{i}{2} \right) s_i - hn - h\sum_{i \geq 5} (2i - 9)s_i$$

$$= \frac{h + 1}{2}s_2 + \frac{h + 4}{4}s_3 + \frac{3}{2}4h_4 + \sum_{i=5}^{c} \left( \frac{i - 1}{2} - 2h + \frac{9h}{i} \right) is_i$$

$$- h\sum_{i=c+1}^{k-1} (2i - 9)s_i - h\sum_{i \geq k} (2i - 9)s_i - hn$$

$$\leq \frac{h + 1}{2}s_2 + \frac{h + 4}{4}s_3 + \frac{3}{2}4h_4 + \sum_{i=5}^{c} \left( \frac{i - 1}{2} - 2h + \frac{9h}{i} \right) is_i$$

$$- h\sum_{i=c+1}^{k-1} (2 - \frac{9}{c + 1}) is_i - h\sum_{i \geq k} (2 - \frac{7}{c})(i - 1)s_i - hn.$$ 

Setting $X := \max\{\frac{h + 1}{2}; \frac{h + 4}{4}; \frac{3}{2}; \max_{5 \leq i \leq c} (\frac{i - 1}{2} - 2h + \frac{9h}{i})\}$ implies that,

$$P_S \leq XI_S - h\sum_{i=c+1}^{k-1} (2 - \frac{9}{c + 1}) is_i - h\sum_{i \geq k} (2 - \frac{7}{c})(i - 1)s_i - hn. \quad (3.1)$$

Let $\gamma(h, i) = \frac{i - 1}{2} - 2h + \frac{9h}{i}$ for $5 \leq i \leq c$.

We have: $\gamma_i'' \geq 0 \forall i \in (5, c) \Rightarrow \gamma(h, i)_{\text{max}} = \gamma(h, 5) = 2 - \frac{h}{5}$

or $\gamma(h, i)_{\text{max}} = \gamma(h, c) = \frac{c - 1}{2} - 2h + \frac{9h}{c}$ for $c \geq 8$. 
Clearly, \( h(c) = \frac{c(c - 2)}{5c - 18} \) has minimum value \( \frac{24}{11} \) when \( c = 8 \). Hence,

\[
\begin{align*}
\frac{h + 1}{2} & \geq \frac{3}{2} \\
\frac{h + 1}{2} & \geq \frac{h + 4}{4} \\
\frac{h + 1}{2} & \geq 2 - \frac{h}{5} \\
\frac{h + 1}{2} & = \frac{c - 1}{2} - 2h + \frac{9h}{c}.
\end{align*}
\]

Thus, \( X = \frac{h + 1}{2} \).

On the other hand, if \( i \in J \) is medium \( (c < i < k) \) then \( i \) is not large. Therefore, \( \sum_{j \geq i} (j - 1)s_j > \alpha n \). Because if \( \sum_{j \geq i} (j - 1)s_j \leq \alpha n \) then \( |E(G_i)| \leq \alpha n \), contradict with minimum property of \( k \). Using part (a) and (b) of the Szemerdi-Trotter theorem 6,

\[
\sum_{j \geq i} js_j = \sum_{j \geq i} (j - 1)s_j + \sum_{j \geq i} s_j \leq \frac{\beta n^2}{2(i - 1)^2} + \frac{\beta n^2}{2(i - 1)^3} = \frac{\beta n^2i}{2(i - 1)^3}.
\]  

(3.2)

Given \( X \) as above, we have

\[
P_M - XI_M = \left( \sum_{i = c + 1}^{k-1} \left( \begin{array}{c} i \\ 2 \end{array} \right) s_i \right) - X \left( \sum_{i = c + 1}^{k-1} is_i \right)
= \frac{1}{2} \sum_{i = c + 1}^{k-1} (i - 1 - 2X)is_i.
\]

Combining with (3.1), we get

\[
P_S + P_M \leq XIS - hn + XI_M + \frac{1}{2} \sum_{i = c + 1}^{k-1} \left( i - 1 - 2X - 4h + \frac{18h}{c + 1} \right) is_i - h(2 - \frac{7}{c})|E(G_k)|.
\]

(3.3)

We define

\[
Y = c - 5h - 2 + \frac{18h}{c + 1}
= c - 2 - 5\frac{c(c - 2)}{5c - 18} + \frac{18c(c - 2)}{(c + 1)(5c - 18)}
= -18\frac{(c - 2)}{(c + 1)(5c - 18)}.
\]
This implies $-1 < Y < 0$ with $c \geq 8$. Thus we have,

$$T = \frac{1}{2} \sum_{i=c+1}^{k-1} \left( i - 1 - 2X - 4h + \frac{18h}{c+1} \right) is_i = \frac{1}{2} \sum_{i=c+1}^{k-1} (i - c + Y) is_i$$

$$= \frac{1}{2} \left( \sum_{i=c+1}^{k-1} \sum_{j=1}^{k-1} js_j \right) + \frac{Y}{2} \left( \sum_{i=c+1}^{k-1} is_i \right)$$

$$\leq \frac{1}{2} \left( \sum_{i=c+1}^{k-1} \sum_{j=i}^{k-1} js_j \right) + \frac{Y}{2} \left( \sum_{i=c+1}^{k-1} is_i \right).$$

Applying (3.2) and $Y + 1 > 0$, this yields

$$T \leq \frac{1}{2} \sum_{i \geq c+1}^{k-1} \beta n^2 i \frac{2}{2(i-1)^2} + \frac{Y}{2} \beta n^2 (c+1) = \frac{\beta n^2}{4} \left( Y \frac{c+1}{c^3} + \sum_{i \geq c} \frac{i+1}{i^3} \right). \quad (3.4)$$

Finally, we have

$$P_L - XI_L = \sum_{i=k}^{\lceil \epsilon n \rceil + q} \frac{i}{2} \left( s_i - \sum_{i \geq k} \sum_{i \geq k} (i-1)s_i - X \sum_{i \geq k} (i-1)s_i = \left( \frac{\epsilon n + q}{2} - X \right)|E(G_k)|. \quad (3.5)$$

Combining (3.3), (3.4), (3.5), we get

$$P_S + P_M + P_L \leq X(I_S + I_M + I_L) - hn$$

$$+ \frac{\beta n^2}{4} \left( Y \frac{c+1}{c^3} + \sum_{i \geq c} \frac{i+1}{i^3} \right) + \frac{1}{2} (\epsilon n + q - 2X - 4h + \frac{7h}{c})|E(G_k)|$$

$$\leq XI - hn + \frac{\beta n^2}{4} \left( Y \frac{c+1}{c^3} + \sum_{i \geq c} \frac{i+1}{i^3} \right) + \frac{1}{2} (\epsilon n - 2)|E(G_k)| \quad (by 1 \leq q \leq 3, c \geq 8)$$

$$\leq XI - hn + \frac{\beta n^2}{4} \left( Y \frac{c+1}{c^3} + \sum_{i \geq c} \frac{i+1}{i^3} \right) + \frac{1}{2} (\epsilon n - 2)\alpha n.$$ 

On the other hand, we have $P_S + P_M + P_L = \left( \frac{n}{2} \right) = \frac{1}{2}(n^2 - n)$.

Thus, we get

$$\frac{1}{2}(n^2 - n) \leq XI - hn + \frac{\beta n^2}{4} \left( Y \frac{c+1}{c^3} + \sum_{i \geq c} \frac{i+1}{i^3} \right) + \frac{\epsilon \alpha n^2}{2} - \alpha n.$$

$$\Rightarrow I \geq \frac{1}{2X} \left( 1 - \epsilon \alpha - \frac{\beta}{2} \left( Y \frac{c+1}{c^3} + \sum_{i \geq c} \frac{i+1}{i^3} \right) \right) n^2 + \frac{2h - 1 + \alpha}{2X} n.$$
Since $X = \frac{h + 1}{2}$ and $Y = \frac{-18(c - 2)}{c^3(5c - 18)}$ then,

$$I \geq \frac{1}{h + 1} \left( 1 - \epsilon \alpha - \frac{\beta}{2} \left( \frac{-18(c - 2)}{c^3(5c - 18)} + \sum_{i \geq c} \frac{i + 1}{i^3} \right) \right) n^2 + \frac{2h - 1 + \alpha n}{h + 1}$$

$$= \delta n^2 + rn.$$

□

**Theorem 8.** Every set $P$ of $n$ non-collinear points in the plane with at most $\frac{n}{26} + 2$ collinear points, the arrangement of $P$ has at least $\frac{n^2}{26} + 2n$ point-line incidents.

**Proof.** Case 1. If $0 < \frac{n}{26} < 1$, then the arrangement of $P$ is $n^2 - n > \frac{n^2}{26} + 2n$ by $n \geq 5$.

Case 2. If $1 \leq \frac{n}{26} < 2$, then $I = 2s_2 + 3s_3 \geq s_2 + 3s_3 = \frac{n^2 - n}{2} > \frac{n^2}{26} + 2n$ by $n \geq 26$.

Case 3. If $\frac{n}{26} \geq 2$, then the assumptions of Theorem 7 satisfy with $\epsilon = \frac{1}{26}, c = 46, q = 2$.

We have

$$I \geq \delta n^2 + rn \geq \frac{n^2}{26} + 2n.$$

The proof of Theorem 8 is completed.

□

So we now can give the proof of Main theorem 1.

**Proof.** Let $P$ be a set of $n$ non-collinear points in the plane. If $P$ contains at least $\frac{n}{26} + 2$ collinear points, then every other point is in at least $\frac{n}{26} + 2$ lines $P$ (one through each of the collinear points). Otherwise, by Theorem 8, the arrangement of $P$ has at least $\frac{n^2}{26} + 2n$ incidences, and so some point is incident with at least $\frac{n}{26} + 2$ lines determined by $P$. Main theorem 1 is proved.

□

We note that the number $\epsilon = \frac{1}{26}$ is best possible in this technic. Indeed, for our purpose, we need $\delta \geq \epsilon$ to get a constant $\epsilon$ in Theorem 7. Using equivalent transformation,

$$\epsilon \leq \frac{1 - \frac{\beta}{2} \left( \frac{-18(c - 2)}{c^3(5c - 18)} + \sum_{i \geq c} \frac{i + 1}{i^3} \right)}{h + 1 + \alpha}$$

$$= \frac{1 - \frac{\beta}{2} \left( \frac{-18(c - 2)}{c^3(5c - 18)} + \sum_{i \geq c} \frac{i + 1}{i^3} \right)}{\frac{c(c - 2)}{5c - 18} + 1 + \alpha}.$$

We have $0 < \epsilon < 1$, and so $1 - \epsilon \alpha - \sum_{i \geq c} \frac{i + 1}{i^3} \geq 0$, we have

$$\epsilon \leq \frac{1 - \frac{\beta}{2} \left( \frac{-18(c - 2)}{c^3(5c - 18)} + \sum_{i \geq c} \frac{i + 1}{i^3} \right)}{\frac{c(c - 2)}{5c - 18} + 1 + \alpha}.$$
In order to having maximum value $\varepsilon$ we need to optimal value $c$. We define

$$f(c) = \frac{1 - \frac{\beta}{2} \left( \frac{-18(c - 2)}{c^3(5c - 18)} + \sum_{i \geq c} \frac{i + 1}{i^3} \right)}{\frac{c(c - 2)}{5c - 18} + 1 + \alpha},$$

for defined constant $\alpha, \beta$ in Crossing lemma 4. Using Maple application we have that the maximum value of $f(c)$ is at $c = 46$. Hence, we can choose $\varepsilon > \frac{1}{26}$. This shows that $\frac{1}{26}$ is the best constant.

4. THE LINES WITH FEW POINTS

**Theorem 9.** Let $\alpha, \beta$ be positive constants such that every graph $H$ with $n$ vertices and $m \geq \alpha n$ edges satisfies

$$cr(H) \geq \frac{m^3}{\beta n^2}.$$

Fix an integer $c \geq 29$. Then for every set $P$ of $n$ points in the plane with at most $l$ collinear points, the arrangement of $P$ has at least

$$s_2 + \frac{3h}{4}s_3 - hn - h \sum_{i \geq 5} (2i - 9)s_i \geq 0.$$
Now we have,

\[ P_S = \sum_{i=2}^{c} \binom{i}{2} s_i \]

\[ = s_2 + 3s_3 + 6s_4 + \sum_{i=5}^{c} \binom{i}{2} s_i \]

\[ \leq (h+1)s_2 + \left( \frac{3h}{4} + 3 \right) s_3 + 6s_4 + \sum_{i=5}^{c} \binom{i}{2} s_i - hn - h \sum_{i \geq 5}^{c} (2i-9)s_i \]

\[ \leq (h+1)s_2 + \frac{3}{4}(h+4)s_3 + 6s_4 + \sum_{i=5}^{c} \left( \frac{i(i-1)}{2} - h(2i-9) \right) s_i - hn - h \sum_{i \geq c+1}^{c} (2i-9)s_i. \]

By \( c \geq 29 \), it is easy to see that

\[ X := h + 1 = \max\{h + 1; \frac{3}{4}(h + 4); 6; \max_{5 \leq i \leq c} \left( \frac{i(i-1)}{2} - h(2i-9) \right) \}, \]

and thus we get

\[ P_S \leq XL_S - hn - h \sum_{i \geq c+1}^{c} (2i-9)s_i. \]

For the medium index \( i \), we use the Crossing Lemma 4 and part (a) of Theorem 6 to imply that

\[ \sum_{j \geq i} (j-1)s_j \leq \frac{\beta n^2}{2(i-1)^2}, \]
thus we have

\[
P_S + P_M - XLS \leq -hn - h \sum_{i \geq c+1} (2i - 9)s_i + \sum_{i = c+1}^{k-1} \left( \binom{i}{2} \right) \frac{i(i - 1)}{2} - h(2i - 9)s_i
\]

\[
= -hn - h \sum_{i \geq k} (2i - 9)s_i + \sum_{i = c+1}^{k-1} \left( \binom{i}{2} \right) \frac{i(i - 1)}{2} - h(2i - 9)s_i
\]

\[
= -hn - h \sum_{i \geq k} (2i - 9)s_i + \frac{1}{2} \left( \sum_{i = c+1}^{k-1} \left( c - \frac{4hi - 18h}{i - 1} \right)(i - 1)s_i + \sum_{i = c+1}^{k-1} \sum_{j = i}^{k-1} (j - 1)s_j \right)
\]

\[
\leq -hn - h \sum_{i \geq k} (2i - 9)s_i + \frac{1}{2} \left( \sum_{i = c+1}^{k-1} \left( c - \frac{4h + 14h}{i - 1} \right)(i - 1)s_i + \sum_{i = c+1}^{k-1} \sum_{j = i}^{k-1} (j - 1)s_j \right)
\]

\[
= -hn - h \sum_{i \geq k} (2i - 9)s_i + \frac{1}{2} \left( \sum_{i = c+1}^{k-1} \left( c - \frac{4h + 14h}{i - 1} \right)(i - 1)s_i + \sum_{i = c+1}^{k-1} \sum_{j = i}^{k-1} (j - 1)s_j \right)
\]

On the other hand, we have

\[
c - 4h + \frac{14h}{c} = c - 4\frac{c^2 - c - 2}{4c - 16} + \frac{7c^2 - 7c - 14}{c(2c - 8)} = \frac{c^2 - 3c - 14}{2c(c - 4)}
\]

\[
\Rightarrow c - 4h + \frac{14h}{c} > 0 \text{ (by } c \geq 29)\]

So we get

\[
P_S + P_M - XLS \leq -hn - h \sum_{i \geq k} (2i - 9)s_i + \frac{1}{2} \left( \sum_{i = c+1}^{k-1} \left( c^2 - 3c - 14 \right) \frac{2c^3 - 4c^2}{2c(c + 4)} (i - 1)s_i + \sum_{j = c+1}^{k-1} \sum_{i \geq j} (i - 1)s_i \right)
\]

\[
\leq -hn - h \sum_{i \geq k} (2i - 9)s_i + \left( \sum_{i = c+1}^{k-1} \left( c^2 - 3c - 14 \right) \frac{2c^3 - 4c^2}{2c^3(c - 4)} + \sum_{i \geq c} \frac{1}{i^2} \right) \frac{\beta n^2}{4}.
\]
Thus, we now have

\[
\binom{n}{2} - XL_S = P_S + PM + PL - XL_S
\]

\[
\leq -hn - h \sum_{i \geq k} (2i - 9) s_i + \left( \frac{c^2 - 3c - 14}{2c^3(c - 4)} + \sum_{i \geq e} \frac{1}{i^2} \right) \frac{\beta n^2}{4} + \sum_{i = k}^{l} \binom{i}{2} s_i
\]

\[
= -hn + \left( \frac{c^2 - 3c - 14}{2c^3(c - 4)} + \sum_{i \geq e} \frac{1}{i^2} \right) \frac{\beta n^2}{4} + \sum_{i \geq k} \left( \frac{i(i - 1)}{2} - 2hi + 9h \right) s_i
\]

\[
= -hn + \left( \frac{c^2 - 3c - 14}{2c^3(c - 4)} + \sum_{i \geq e} \frac{1}{i^2} \right) \frac{\beta n^2}{4} + \sum_{i \geq k} \left( \frac{i}{2} - 2h + \frac{7h}{i - 1} \right) (i - 1) s_i
\]

\[
\leq -hn + \left( \frac{c^2 - 3c - 14}{2c^3(c - 4)} + \sum_{i \geq e} \frac{1}{i^2} \right) \frac{\beta n^2}{4} + \frac{l}{2} E(G_k) |\]

\[
\leq -hn + \left( \frac{c^2 - 3c - 14}{2c^3(c - 4)} + \sum_{i \geq e} \frac{1}{i^2} \right) \frac{\beta n^2}{4} + \frac{l}{2} \alpha n.
\]

So we get

\[
L_S \geq \left( 1 - \beta \frac{1}{4} \left( \frac{c^2 - 3c - 14}{2c^3(c - 4)} + \sum_{i \geq e} \frac{1}{i^2} \right) \right) n^2 \frac{X}{X} + \left( h - \frac{1}{2} - \frac{l}{2} \alpha \right) n
\]

\[
\geq \left( 1 - \frac{1}{2} - \frac{\beta}{4} \left( \frac{c^2 - 3c - 14}{2c^3(c - 4)} + \sum_{i \geq e} \frac{1}{i^2} \right) \right) n^2 \frac{X}{X} - \frac{l}{2} \alpha n.
\]

On the other hand, \( X = h + 1 = \frac{c^2 + 3c - 18}{4c - 16} \), we thus get

\[
L_S \geq \left( 1 - \frac{\beta}{2} \left( \frac{c^2 - 3c - 14}{2c^3(c - 4)} + \sum_{i \geq e} \frac{1}{i^2} \right) \right) \frac{2c - 8}{c^2 + 3c - 18} n^2 - \frac{(2c - 8) \alpha}{c^2 + 3c - 18} ln.
\]

Theorem 9 is proved.

For the case \( c = 36 \), we get the following.

**Corollary 10.** Every set \( P \) of \( n \) points with at most \( l \) collinear determines at least \( \frac{1}{39} n^2 - \frac{1}{3} ln \) lines with at most 36 points.

We now apply Theorem 9 to give the proof of Main theorem 2.
Proof. We may assume that \( l \) is the size of the longest line. For some integer \( c \geq 29 \), then by Theorem 9 we have \( L \geq L_s \geq A(c)n^2 - B(c)nl \) for some \( A(c) \) and \( B(c) \) evident in the theorem. Observe that,

\[
A(c) = \left( 1 - \frac{\beta}{2} \left( \frac{c^2 - 3c - 14}{2c^3(c - 4)} + \sum_{i \geq c} \frac{1}{i^2} \right) \right) \frac{2c - 8}{c^2 + 3c - 18}
\]

\[
B(c) = \frac{(2c - 8)\alpha}{c^2 + 3c - 18}.
\]

We note that,

\[
\frac{2A}{1 + 2B} \geq \epsilon
\]

\[
\Rightarrow A \geq \frac{\epsilon}{2} + B\epsilon - \frac{\epsilon^2}{2}
\]

\[
\Rightarrow An \geq \frac{en}{2} + (B - \frac{\epsilon}{2})en
\]

\[
\Rightarrow An \geq \frac{en}{2} + (B - \epsilon)nl
\]

\[
\Rightarrow An^2 - Bnl \geq \frac{en(n - l)}{2}.
\]

So we can find the maximum of \( \frac{2A(c)}{1 + 2B(c)} \) to get a largest number \( \epsilon \). Now, set \( c = 44 \) we get \( \epsilon \leq \frac{1}{30.2} \). So we choose \( \epsilon = \frac{1}{30.5} \) to complete Main theorem 2. \( \square \)

We now get Main theorem 3 by using Main theorem 2 and the following observation:

**Theorem 11.** ([15]) Let \( P \) be a set of \( n \) non-collinear points in a plane. Then at least half the lines determined by \( P \) contain at most 3 points.

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