DEFINING AN $m$-CLUSTER CATEGORY

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September 2007

Abstract. We show that a certain orbit category considered by Keller encodes the combinatorics of the $m$-clusters of Fomin and Reading in a fashion similar to the way the cluster category of Buan, Marsh, Reineke, Reiten, and Todorov encodes the combinatorics of the clusters of Fomin and Zelevinsky. This allows us to give type-uniform proofs of certain results of Fomin and Reading in the simply laced cases.

For $\Phi$ any root system, Fomin and Zelevinsky [FZ] define a cluster complex $\Delta(\Phi)$, a simplicial complex on $\Phi_{\geq -1}$, the almost positive roots of $\Phi$. Its facets (maximal faces) are called clusters. In [BM+], starting in the more general context of a finite dimensional hereditary algebra $H$ over a field $K$, Buan et al. define a cluster category $\mathcal{C}(H) = \mathcal{D}^b(H)/\tau^{-1}[1]$. ($\mathcal{D}^b(H)$ is the bounded derived category of representations of $H$; more will be said below about it, its shift functor [1], and its Auslander-Reiten translate $\tau$.) The cluster category $\mathcal{C}(H)$ is a triangulated Krull-Schmidt category. We will be mainly interested in the case where $H$ is a path algebra associated to the simply laced root system $\Phi$, in which case we write $\mathcal{C}(\Phi)$ for $\mathcal{C}(H)$. There is a bijection $V$ taking $\Phi_{\geq -1}$ to the indecomposables of $\mathcal{C}(\Phi)$. A (cluster)-tilting set in $\mathcal{C}(\Phi)$ is a maximal set $S$ of indecomposables such that $\text{Ext}^1_{\mathcal{C}(\Phi)}(X,Y) = 0$ for all $X,Y \in S$. $\mathcal{C}(\Phi)$ encodes the combinatorics of $\Delta(\Phi)$ in the sense that the clusters of $\Phi$ correspond bijectively to the tilting sets of $\mathcal{C}(\Phi)$ under the map $V$.

Tilting sets in $\mathcal{C}(\Phi)$ always have cardinality $n$, the rank of $\Phi$. An almost complete tilting set is a set $T$ of $n-1$ indecomposables such that $\text{Ext}^1_{\mathcal{C}(\Phi)}(X,Y) = 0$ for $X,Y \in T$. A complement for $T$ is an indecomposable $M$ such that $T \cup \{M\}$ is a tilting set. A tilting set always has exactly two complements. (This was shown from the cluster perspective in [FZ] and from the representation theoretic perspective in [BM+].)

In [FR], Fomin and Reading introduced a generalization of clusters known as $m$-clusters, for $m \in \mathbb{N}$. When $m = 1$, the classical clusters are recovered. The $m$-cluster complex $\Delta_m(\Phi)$ is a simplicial complex on a set of coloured roots $\Phi_{\geq -1}^m$. It has been studied further in [AT1, T, AT2]. The facets of $\Delta_m(\Phi)$ are known as $m$-clusters. The goal of this paper is to show that the category $\mathcal{C}_m(\Phi) = \mathcal{D}^b(\Phi)/\tau^{-1}[m]$, which we will call the $m$-cluster category, plays a similar role to the cluster category but with respect to the combinatorics of $m$-clusters. We define a bijection $W : \Phi_{\geq -1}^m \to \text{ind}\mathcal{C}_m(\Phi)$. We define an $m$-tilting set in $\mathcal{C}_m(\Phi)$ to be a maximal set of indecomposables $S$ satisfying $\text{Ext}^i_{\mathcal{C}_m(\Phi)}(X,Y) = 0$ for all $X,Y \in S$ and $i = 1 \ldots m$. Then we show:
Theorem 1. The map $W$ induces a bijection from $m$-clusters of $\Phi$ to $m$-tilting sets of $C_m(\Phi)$.

We then prove two facts about $C_m(\Phi)$. First:

Theorem 2. The $m$-tilting sets of $C_m(\Phi)$ have cardinality $n$.

Second, we make the natural definition of an almost complete $m$-tilting set, namely, that it is a collection $\mathcal{T}$ of $n-1$ indecomposables of $C_m(\Phi)$ such that $\text{Ext}^i(X,Y) = 0$ for all $X,Y \in \mathcal{T}$ and $i = 1 \ldots m$, and we show:

Theorem 3. An almost complete $m$-tilting set $\mathcal{T}$ has $m+1$ complements.

Via Theorem 1, Theorems 2 and 3 are equivalent to facts proved about the $m$-cluster complex by Fomin and Reading ([FR, Theorem 2.9 and Proposition 2.10]). The proofs of these results in [FR] depend on case-by-case arguments and a computer check for the exceptional types. Using Theorem 1, our proofs of Theorems 2 and 3 therefore provide a type-uniform and computer-free proof of these results from [FR] in the simply laced cases.

After this paper was completed, we received a copy of Zhu’s paper [Z] which proves Theorems 1 and 2 without the simply laced assumption, by drawing on some sophisticated results reported in Iyama [I], and presented in detail in the recent preprint of Iyama and Yoshino [IY]. Our approach is different and more elementary.

Clusters

We begin with a quick introduction to the combinatorics of clusters. Our presentation is based on [FZ] and [FR].

Let $\Phi$ be a crystallographic root system of rank $n$. (In fact, the assumption that $\Phi$ is crystallographic is not essential [FR], but since we will shortly be assuming that $\Phi$ is not merely crystallographic but also simply laced, there is no advantage to considering the slightly more general situation.) For convenience, we will also assume that $\Phi$ is irreducible; the analysis extends easily to the reducible case.

Label the vertices of the Dynkin diagram for $\Phi$ by the numbers from 1 to $n$. Let $W$ be the Weyl group corresponding to $\Phi$. Let the simple roots of $\Phi$ be $\Pi = \{\alpha_1, \ldots, \alpha_n\}$, and let $s_i$ be the reflection in $W$ corresponding to $\alpha_i$.

The ground set for the cluster complex $\Delta(\Phi)$ is the set of almost positive roots, $\Phi_{\geq -1}$, which are, by definition, the positive roots $\Phi^+$ together with the negative simple roots $-\Pi$.

Since the Dynkin diagram for $\Phi$ is a tree, it is a bipartite graph. Let $I_+, I_-$ be a decomposition of $[n]$ corresponding to the bipartition. ($I_+$ and $I_-$ are determined up to interchanging + and −. We fix this choice once and for all.)

For $\epsilon \in \{+, -\}$, define the bijection $\tau_\epsilon : \Phi_{\geq -1} \to \Phi_{\geq -1}$ by

$$\tau_\epsilon(\beta) = \begin{cases} \beta & \text{if } \beta = -\alpha_i \text{ for some } i \in I_\epsilon \\
(\prod_{i \in I_\epsilon} s_i) \beta & \text{otherwise} \end{cases}$$

Now set $R = \tau_+ \tau_-$. $R$ is in some sense a deformation of a Coxeter element of $W$. (We will give a more representation-theoretic interpretation for $R$ in Lemma 1 below.)
The crucial fact about \( R \) is that every root in \( \Phi \geq -1 \) has at least one negative simple root in its \( R \)-orbit. For that reason, the following suffices to define a relation called compatibility.

1. \(-\alpha_i \) is compatible with \( \beta \) iff \( \alpha_i \) does not appear when we write \( \beta \) as a sum of simple roots. (This is called the simple root expansion for \( \beta \).)

2. \( \alpha \) and \( \beta \) are compatible iff \( R(\alpha) \) and \( R(\beta) \) are compatible.

This relation is well-defined (not \textit{a priori} obvious, since a root may have two negative simples in its \( R \)-orbit) and symmetric \([FZ, \S\S 3.1-2]\).

In fact, there is more information associated to a pair of almost positive roots than mere compatibility or incompatibility. The compatibility degree \((\alpha \parallel \beta)\) can be defined by saying that:

1. \((\beta \parallel -\alpha_i)\) is the coefficient of \( \alpha_i \) in the expansion of \( \beta \) if \( \beta \) is positive and 0 if \( \beta \) is negative.

2. \((R(\beta) \parallel R(\alpha)) = (\beta \parallel \alpha)\).

Compatibility degree is well-defined, and, if \( \Phi \) is simply laced, it is also symmetric \([FZ, \S 3.1]\). Two roots are compatible iff their compatibility degree is zero.

The \textit{cluster complex} \( \Delta(\Phi) \) is defined to be the simplicial complex whose faces are the sets of almost positive roots which are pairwise compatible. The facets (maximal faces) of \( \Delta(\Phi) \) are all of the same cardinality, \( n \), the rank of \( \Phi \). They are called \textit{clusters}.

**Derived Category**

Fix \( Q \) an orientation of the Dynkin diagram for \( \Phi \). The representations of \( Q \) are denoted \( \mathcal{L}(Q) \). The bounded derived category \( \mathcal{D}^b(Q) \) is a triangulated category, and it comes with a \( \mathbb{Z} \) grading and a shift functor \([1]\) which takes \( \mathcal{D}^b(Q)_i \) to \( \mathcal{D}^b(Q)_{i-1} \). \( \mathcal{D}^b(Q)_1 \) is just a copy of \( \mathcal{L}(Q) \). We refer to this grading as the \textit{coarse grading} on \( \mathcal{D}^b(Q) \), and denote the degree function with respect to this grading by \( d_C \).

To give a more concrete description of \( \mathcal{D}^b(Q) \), we will define an infinite quiver \( ZQ^{op} \). Its vertex set consists of \([n] \times \mathbb{Z} \). For each edge from \( v_i \) to \( v_j \) in \( Q \), \( ZQ^{op} \) has an edge from \( (j,p) \) to \( (i,p) \) and one from \( (i,p) \) to \( (j,p-1) \), for all \( p \in \mathbb{Z} \). This means that one way of thinking of \( ZQ^{op} \) is as \( \mathbb{Z} \) many copies of \( Q^{op} \) (\( Q \) with its orientation reversed) with some edges added connecting copy \( i \) to copy \( i-1 \).

It turns out that \( ZQ^{op} \) is the Auslander-Reiten quiver for \( \mathcal{D}^b(Q) \), so in particular the indecomposables of \( \mathcal{D}^b(Q) \) can be identified with the vertices of \( ZQ^{op} \). See \([H]\) for further details and proofs.

If \( Q \) and \( Q' \) are two different orientations of the Dynkin diagram for \( \Phi \), then \( \mathcal{D}^b(Q) \) and \( \mathcal{D}^b(Q') \) are isomorphic as triangulated categories, but their coarse gradings disagree. We will generally therefore forget the orientation (and the grading it induces), and write \( \mathcal{D}^b(\Phi) \).

Since \( \mathcal{D}^b(\Phi) \) does not depend on the choice of an orientation, we may fix a convenient orientation if we like. Let \( Q_{bip} \) denote the bipartite orientation of the Dynkin diagram of \( \Phi \) in which the arrows go from roots in \( I_+ \) and towards roots in \( I_- \). We want to fix a grading on the vertices of \( ZQ^{op}_{bip} \), which we shall call the \textit{fine grading}, and denote it \( d_F \). Vertices in \( ZQ^{op}_{bip} \) are indexed by \((i,k)\) with \( i \in [n] \) and \( k \in \mathbb{Z} \). We say that a vertex \((i,k)\) is in fine degree \( 2k \) if \( i \in I_- \) and \( 2k-1 \) if \( i \in I_+ \).

It follows that all the arrows in \( ZQ^{op}_{bip} \) diminish fine degree by 1.

The coarse and fine gradings of \( \mathcal{D}^b(Q_{bip}) \) are related: \( d_C(M) = [d_F(M)/h] \), where \( h \) is the Coxeter number for \( \Phi \). (The Coxeter number is the order of any
Coxeter element, and can be computed from the fact that $|\Phi| = nh$.

The copy of the indecomposables of $\mathcal{L}(Q_{bip})$ which sits in coarse degree 0, consists of the vertices of $ZQ_{bip}^{op}$ in fine degree between 0 and $-h + 1$. The indecomposables of $\mathcal{L}(Q_{bip})$ which are projective are exactly those in fine degree 0 and $-1$.

Here is an example for $A_3$. Here $h = 4$, and $I_+$ consists of the two outside nodes while $I_-$ is the middle node.

\[
\begin{array}{ccccccc}
1 & 0 & -1 & \text{coarse degree} \\
\hline
& & & & & & \\
\hline
2 & 1 & 0 & -1 & -2 & -3 & -4 & -5 & \text{fine degree} \\
\end{array}
\]

We can define an automorphism $\tau$ of $ZQ^{op}$ which takes $(i, p)$ to $(i, p + 1)$ for all $i \in \{1, \ldots, n\}$ and $p \in \mathbb{Z}$. This automorphism corresponds to an auto-equivalence of $\mathcal{D}^b(\Phi)$, also denoted $\tau$, which is the Auslander-Reiten translate for $\mathcal{D}^b(\Phi)$. The functor $\tau$ respects the fine degree, increasing it by 2. The shift functor $[1]$ also respects the fine degree, decreasing it by $h$.

\section*{Factor categories of the derived category}

Let $\mathcal{D}^b(H)$ be the bounded derived category of modules over a hereditary algebra $H$, finite dimensional over a field $K$. We quote some general results from [BM+] and [K] about the factor of $\mathcal{D}^b(H)$ by a suitable automorphism.

Let $G$ be an automorphism of $\mathcal{D}^b(H)$, satisfying conditions (g1) and (g2) of [BM+]:

(g1): For each $U \in \text{ind } \mathcal{D}^b(H)$, only a finite number of $G^nU$ lie in $\text{ind } H$ for $n \in \mathbb{Z}$.

(g2): There is some $N \in \mathbb{N}$ such that $\{U[n] \mid U \in \text{ind } H, n \in [-N, N]\}$ contains a system of representatives of the orbits of $G$ on $\mathcal{D}^b(H)$.

$\mathcal{D}^b(H)/G$ denotes the corresponding factor category: the objects of $\mathcal{D}^b(H)/G$ are by definition $G$-orbits in $\mathcal{D}^b(H)$. Let $X$ and $Y$ be objects of $\mathcal{D}^b(H)$, and let $\tilde{X}$ and $\tilde{Y}$ be the corresponding objects in $\mathcal{D}^b(H)/G$. Then the morphisms in $\mathcal{D}^b(H)/G$ are given by:

$$\text{Hom}_{\mathcal{D}^b(H)/G}(\tilde{X}, \tilde{Y}) = \prod_{i \in \mathbb{Z}} \text{Hom}_{\mathcal{D}^b(H)}(G^iX, Y).$$

From [K] we know that $\mathcal{D}^b(H)/G$ is a triangulated category, and the canonical map from $\mathcal{D}^b(H)$ to $\mathcal{D}^b(H)/G$ is a triangle functor. It is shown in [BM+, Proposition 1.2] that $\mathcal{D}^b(H)/G$ is a Krull-Schmidt category.

$G$ defines an automorphism $\phi$ of the AR quiver $\Gamma(\mathcal{D}^b(H))$: the factor category $\mathcal{D}^b(H)/G$ has almost split triangles, and the AR quiver $\Gamma(\mathcal{D}^b(H)/G)$ is $\Gamma(\mathcal{D}^b(H))/\phi$ [BM+, Proposition 1.3]
The shift $[1]$ on $\mathcal{D}^b(H)$ passes to $\mathcal{D}^b(H)/G$; we use the same notation. Define $\text{Ext}^1_{\mathcal{D}^b(H)/G}(X,Y) = \text{Hom}_{\mathcal{D}^b(H)/G}(X,Y[i])$.

It is also straightforward to show [BM+, Proposition 1.4] that Serre duality in $\mathcal{D}^b(H)$ passes to $\mathcal{D}^b(H)/G$, so $\text{Ext}^1_{\mathcal{D}^b(H)/G}(X,Y)$ is dual to $\text{Hom}_{\mathcal{D}^b(H)/G}(Y,\tau X)$.

### Cluster category

The cluster category is defined by $\mathcal{C}(H) = \mathcal{D}^b(H)/\tau^{-1}[1]$. Since $\tau^{-1}[1]$ is an automorphism satisfying conditions (g1) and (g2), $\mathcal{C}(H)$ is a triangulated Krull-Schmidt category.

We will mainly be concerned in this section with $\mathcal{C}(\Phi) = \mathcal{D}^b(\Phi)/\tau^{-1}[1]$. In [BM+], the connection between $\mathcal{C}(\Phi)$ and $\mathcal{D}(\Phi)$ is made via a reformulation of clusters in terms of decorated representations due to Marsh, Reineke, and Zelevinsky [MRZ]. We proceed in a different fashion, the basis of which is our Lemma 1 below.

Fix $Q = Q_{\text{bip}}$. Identify the indecomposables of $\mathcal{D}^b(\Phi)$ with the vertices of $\mathbb{Z}Q^{\text{op}}$. It is clear that the vertices of $\mathbb{Z}Q^{\text{op}}$ satisfying $2 \geq d_F(M) \geq -h+1$ are a fundamental domain for $\tau^{-1}[1]$. The representations of $Q$ in coarse degree 0 correspond to indecomposables with fine degree between 0 and $-h+1$, so the fundamental domain we have identified for $\tau^{-1}[1]$ consists of the indecomposable representations of $Q$ in coarse degree zero together with $n$ extra indecomposables which correspond to the injective representations in coarse degree 1.

We wish to put these indecomposables in bijection with $\Phi_{\Phi,-1}$. Given a representation $V$ of $Q$, its dimension is by definition $\dim(V) = \sum_i \dim_K(V_i)\alpha_i$. By Gabriel’s Theorem, $\dim$ is a bijection from indecomposable representations of $Q$ to $\Phi^+$. We write $V(\beta)$ for the indecomposable representation in coarse degree zero whose dimension is $\beta$. We write $P_i$ for the projective representation corresponding to vertex $v_i$, and we write $I_i$ for the injective representation corresponding to vertex $v_i$. Observe that $\tau P_i = I_i[-1]$. We define $V(-\alpha_i)$ to be $I_i[-1]$.

**Lemma 1.** $V(R(\alpha)) = V(\alpha)[1]$.

**Proof.** On the representations of $Q$ which do not lie in fine degree 0 or $-1$ (i.e. the indecomposable representations of $Q$ which are not projective), $\tau_+\tau_-$ acts like a product of the corresponding reflection functors, and the product of the reflection functors coincides with $\tau$ [BB]. So $V(\tau_+\tau_-(\beta)) = \tau(V(\beta)) = V(\beta)[1]$, as desired.

Now consider the case that $V(\alpha)$ is projective. If $V(\alpha) = P_i$ is simple projective, then $i \in I_-$ and $\alpha = \alpha_i$, so $\tau_+\tau_-(\alpha) = -\alpha_i$. If $V(\alpha) = P_i$ is non-simple projective, then $i \in I_+$ and $\alpha = \alpha_i + (\text{the sum of the adjacent roots})$. Thus, again, $\tau_+\tau_-(\alpha) = -\alpha_i$. In both these cases, $V(R(\alpha)) = V(-\alpha_i) = I_i[-1] = \tau P_i = \tau V(\alpha) = V(\alpha)[1]$, as desired.

Finally we consider the case where $\alpha = -\alpha_i$. For $i \in I_-$, we know that $V(-\alpha_i)$ sits in fine degree 2. Now $\tau_+\tau_-(\alpha_i) = \alpha_i + (\text{the sum of the roots adjacent to } \alpha_i)$. We recognize this as $\dim I_i$; in other words, $V(R(\alpha)) = I_i = V(\alpha)[1]$, as desired. For $i \in I_+$, the object $V(-\alpha_i)$ sits in fine degree 1. In this case $\tau_+\tau_-(\alpha_i) = \alpha_i$. Now $V(\alpha_i) = I_i$, so again $V(R(\alpha)) = \tau V(\alpha) = V(\alpha)[1]$. This completes the proof.

The connection between representation theory and clusters now appears strongly:

**Proposition 1 [BM+].** $\dim_K \text{Ext}^1_{\mathcal{C}(\Phi)}(V(\beta), V(\alpha)) = (\beta \parallel \alpha)$
Proof. We check the two defining properties of compatibility degree given above.

\[(d1) \quad \dim_K \Ext^1(C(\Phi))(V(\beta), V(-\alpha_i)) = \dim_K \Ext^1(C(\Phi))(V(\beta), \mathcal{I}_i[-1]) = \dim_K \Hom_{C(\Phi)}(\mathcal{I}_i[-1], \tau V(\beta)) = \dim_K \Hom_{C(\Phi)}(\tau^{-1} \mathcal{I}_i[-1], V(\beta)) = \dim_K \Hom_{L(\Phi)}(P_i, V(\beta))\]

(The first equality is because \(V(-\alpha_i) = \mathcal{I}_i[-1]\). The second equality is by Serre duality. The third follows because \(\tau\) is an auto-equivalence; the fourth from the fact that \(\tau^{-1} \mathcal{I}_i[-1] = P_i\).) Now \(\dim_K \Hom_{L(\Phi)}(Q, P_i, V(\beta))\) is the coefficient of \(\alpha_i\) in the simple root expansion of \(\beta\), proving condition (i).

\[(d2) \quad \text{The invariance under } R \text{ follows by Lemma 1 from fact that } [1] \text{ is an auto-equivalence of } C(\Phi).\]

Thus, the roots in a cluster correspond to a maximal collection of irreducible modules in \(C(\Phi)\) such that all the \(\Ext^1(\Phi)\)'s between them vanish. This is exactly the definition of a \((\text{cluster-})\text{tilting set}\) for \(\Phi\), so we have seen that tilting sets for \(\Phi\) are in one-one correspondence with clusters for \(\Phi\).

\(m\)-Clusters

The \(m\)-clusters are a simplicial complex whose ground set, denoted \(\Phi_{\geq -1}^m\), consists of the negative simple roots \(-\Pi\) together with \(m\) copies of \(\Phi^+\). These \(m\) copies are referred to as having \(m\) different “colours” 1 through \(m\). To keep track of the roots of different colour, we use superscripts. So \(\beta^i\) is the root \(\beta\) with colour \(i\). Negative simple roots are considered to have colour 1.

Fomin and Reading define an \(m\)-ified rotation on \(\Phi_{\geq -1}^m\):

\[R_m(\alpha^k) = \begin{cases} \alpha^{k+1} & \text{if } \alpha \in \Phi^+ \text{ and } k < m \\ \tau(\alpha) & \text{otherwise} \end{cases} \]

Again, the crucial fact (which follows from the \(m = 1\) case) is that every root has at least one negative simple in its \(R_m\)-orbit.

We now follow [FR] in defining a relation called compatibility. (Strictly speaking, perhaps, we should call this \(m\)-compatibility, but no ambiguity will result, because this is a relation on \(\Phi_{\geq -1}^m\), not \(\Phi_{\geq -1}\).)

\(m1\) \(-\alpha_i\) is compatible with all negative simple roots and any positive root (of whatever colour) that does not use \(\alpha_i\) in its simple root expansion.

\(m2\) \(\alpha\) and \(\beta\) are compatible iff \(R_m(\alpha)\) and \(R_m(\beta)\) are compatible.

Because of the crucial fact mentioned above, this is sufficient to define compatibility, but not to prove that such a relation exists. This is verified in [FR], where the relation is also shown to be symmetric. We shall give our own proof that there is a compatibility relation on \(\Phi_{\geq -1}^m\) satisfying \((m1)\) and \((m2)\), below (Proposition 2).

The \(m\)-cluster complex \(\Delta_m(\Phi)\) is the simplicial complex on \(\Phi_{\geq -1}^m\) whose faces are sets of pairwise compatible roots. The facets of the complex are called \(m\)-clusters.

\(m\)-Cluster Category

We define the \(m\)-cluster category to be \(\mathcal{C}_m(\Phi) = D^b(\Phi)/\tau^{-1}[m]\). This category is discussed in [K, Section 8.3], where it is shown to be triangulated, and in [KR].
It is also being studied at present by A. Wraalsen [W]. The type $A$ case has been considered in detail in [BM].

We now identify the indecomposables of $C_m(\Phi)$ with $\Phi_{\geq -1}^m$, as follows. For $\beta^j$ a positive root in $\Phi_{\leq -1}^m$, let $W(\beta^j) = V(\beta[j-1])$. Let $W(-\alpha_i) = I_i[-1]$. Observe that the set of $W(\beta^k)$ which we have identified are a fundamental domain with respect to $F = \tau^{-1}[m]$, and therefore they correspond in a 1-1 fashion to the indecomposables of $C_m(\Phi)$.

We now prove the $m$-ified analogue of Lemma 1.

**Lemma 2.** $W(R_m(\beta^k)) = W(\beta^k)[1]$.

**Proof.** There are three cases to consider: firstly when $\beta^k = -\alpha_i$, secondly when $\beta$ is a positive root and $k < m$, and thirdly when $\beta$ is a positive root and $k = m$.

In the first case, $\beta^k = -\alpha_i$, and $R_m(-\alpha_i) = R(-\alpha_i)^1$. In this case $W(-\alpha_i) = I_i[-1]$, and by the proof of Lemma 1, $W(R(-\alpha_i)^1) = I_i$, which proves the claim in this case.

In the second case ($\beta$ a positive root and $k < m$), $R_m(\beta^k) = \beta^{k+1}$, and the desired result follows by the definition of $W$.

In the third case, $R_m(\beta^m) = R(\beta)^1$. By the proof of Lemma 1, $W(R_m(\beta^m)) = \tau(V(\beta)) = V(\beta)[m] = W(\beta^m)[1]$, as desired.

We now prove the following analogue of Proposition 1.7(b) of [BM+]:

**Lemma 3.** If $X, Y \in C_m(\Phi)$, then $\dim_K Ext^i_{C_m(\Phi)}(X, Y) = \dim_K Ext^{m+1-i}_{C_m(\Phi)}(Y, X)$.

**Proof.** This is essentially (a slightly naive version of) the Calabi-Yau condition of dimension $m + 1$, proved for $C_m(\Phi)$ by Keller in [K, Section 8.3]. Observe that $\dim_K Ext^i_{C_m(\Phi)}(X, Y) = \dim_K Ext^i_{C_m(\Phi)}(X, Y[i-1]) = \dim_K Hom_{C_m(\Phi)}(\tau^{-1}Y[i-1], X) = \dim_K Hom_{C_m(\Phi)}(Y[i-1-m], X) = \dim_K Ext^{m+1-i}_{C_m(\Phi)}(Y, X)$. The second equality follows by Serre duality, and the third because we are in $C(m/\Phi)/\tau^{-1}[m]$.

We now prove an $m$-ified analogue of Proposition 1. Here, we consider only compatibility, not compatibility degree, as [FR] does not define compatibility degree in the $m$-cluster context (though it is not difficult to do so).

**Proposition 2.** There is a well-defined and symmetric relation (called compatibility) on $\Phi_{\leq -1}^m$ satisfying (m1) and (m2). A pair of coloured roots $\beta^j$ and $\gamma^k$ are compatible in $\Phi_{\leq -1}^m$ if $Ext^i_{C_m(\Phi)}(W(\beta^j), W(\gamma^k)) = 0$ for $i = 1 \ldots m$.

**Proof.** The proof is analogous to the proof of Proposition 1: since (m1) and (m2) define a unique relation, it suffices to check that the relation given in the statement of the proposition is symmetric and satisfies (m1) and (m2). Symmetry follows from Lemma 3. We now proceed to check (m1) and (m2).

(m1) $\dim_K Ext^j_{C_m(\Phi)}(W(\beta^k), W(-\alpha_i)) = \dim_K Ext^j_{C_m(\Phi)}(W(\beta^k), I_i[-1])$

$= \dim_K Ext^j_{C_m(\Phi)}(W(\beta^k), I_i[j-2])$

$= \dim_K Hom_{C_m(\Phi)}(I_i[j-2], W(\beta^k))$

$= \dim_K Hom_{C_m(\Phi)}(\tau^{-1}I_i[j-2], W(\beta^k))$

$= \dim_K Hom_{C_m(\Phi)}(P_i[j-1], W(\beta^k))$. 

(The second equality is from the definition of $\operatorname{Ext}^i$; the third is from Serre duality; the fourth follows because $\tau$ is an auto-equivalence.) If $k \neq j$, then $\operatorname{Hom}_{C_m(\Phi)}(P_1[j-1], W(\beta^k))$ is zero. If $k = j$, it is the coefficient of $\alpha_i$ in the root expansion of $\beta$. This implies that $\beta^k$ is incompatible with $-\alpha_i$ if and only if $\alpha_i$ appears with positive coefficient in the simple root expansion of $\beta$.

**(m2)** This follows from Lemma 2 and the fact that $[1]$ is an auto-equivalence of $C_m(\Phi)$.

We define an $m$-tilting set in $C_m(\Phi)$ to be a maximal set of indecomposables $S$ satisfying $\operatorname{Ext}^i(X, Y) = 0$ for all $X, Y \in S$ and $i = 1 \ldots m$. The following theorem is an immediate consequence of Proposition 2:

**Theorem 1.** The map $W$ induces a bijection from $m$-clusters of $\Phi$ to $m$-tilting sets of $C(\Phi)$.

**Combinatorics of $m$-clusters**

In this section, we prove Theorems 2 and 3. These are representation-theoretic reformulations of the following results from [FR]:

**Theorem 2' [FR, Theorem 2.9].** All the facets of $\Delta_m(\Phi)$ are of size $n$.

**Theorem 3' [FR, Proposition 2.10].** Given a set $T$ of $n - 1$ pairwise compatible roots from $\Phi^{m}_{\geq -1}$, there are exactly $m + 1$ roots not in $T$ which are compatible with all the roots of $T$. (In other words, every codimension 1 face of $\Delta_m(\Phi)$ is contained in exactly $m + 1$ facets.)

The proofs of these results in [FR] rely on the following theorem, which is proved on a type-by-type basis, with a computer check for the exceptional. We will give a type-free proof.

**Theorem 4 [FR, Theorem 2.7].** If $\alpha$ and $\beta$ are roots of $\Phi$ contained in a parabolic root system $\Psi$ within $\Phi$, then $\alpha(i)$ and $\beta(j)$ are compatible in $\Phi^m_{\geq -1}$ iff they are compatible in $\Psi^m_{\geq -1}$.

**Proof.** Let $\tilde{X}$ and $\tilde{Y}$ be indecomposables of $C_m(\Phi)$ corresponding to $\alpha(i)$ and $\beta(j)$ respectively. Let $X$ and $Y$ be corresponding indecomposables of $\mathcal{D}^b(\Phi)$, chosen so that $2 \geq d_F(X), d_F(Y) \geq -mh + 1$. Without loss of generality, we may assume that $d_F(X) \geq d_F(Y)$.

Suppose that $\alpha(i)$ and $\beta(j)$ are not compatible in $\Phi^m_{\geq -1}$. So there is some $1 \leq k \leq m$ with $\operatorname{Ext}^k_{C_m(\Phi)}(\tilde{X}, \tilde{Y}) \neq 0$. This asserts that there is a non-zero morphism in $\mathcal{D}^b(\Phi)$ from some $G^pX$ to $Y[k]$, where $G = \tau^{-1}[m]$ and $p$ is some integer. Since $d_F(X) \geq d_F(Y)$, so $d_F(X) - d_F(Y[k]) \geq h$, it follows that $p$ must be strictly positive. On the other hand, $d_F(Y[k]) > d_F(G^pX)$ for $p \geq 2$. So $p = 1$, and $\operatorname{Hom}_{\mathcal{D}^b(\Phi)}(\tau^{-1}X[m], Y[k]) \neq 0$. By Serre duality, $\operatorname{Ext}^1_{\mathcal{D}^b(\Phi)}(Y[k], X[m]) \neq 0$, so $\operatorname{Ext}^m_{\mathcal{D}^b(\Phi)}(Y, X) \neq 0$. The crucial point here is that we know this statement on the level of the derived category, rather than just the $m$-cluster category.

Let $Q'$ be the subquiver of $Q$ corresponding to $\Psi$. There is a natural inclusion of $\mathcal{L}(Q')$ into $\mathcal{L}(Q)$ as a full subcategory, which extends to an inclusion of $\mathcal{D}^b(Q')$ into $\mathcal{D}^b(Q)$ as a full triangulated subcategory, where the inclusion respects the coarse grading. $X$ and $Y$ represent $\alpha(i)$ and $\beta(j)$ respectively in both $C_m(\Phi)$ and $C_m(\Psi)$. 

Thus, the non-vanishing Ext that we have shown exists in $D^b(\Phi)$ also exists in $D^b(\Psi)$, and testifies that $\alpha^{(i)}$ and $\beta^{(j)}$ are not compatible in $\Psi_{m-1}$ either.

The converse is proved similarly.

Now that we have established Theorem 4, the proofs of Theorems 2 and 3 go through exactly as in [FR]. We include the proofs for completeness.

Proof of Theorem 2. The proof is by induction on $n$. The statement is clear when $n = 1$. Let $S$ be an $m$-tilting set in $C_m(\Phi)$. Pick $X$ an indecomposable in $S$. Applying $\tau$ if necessary, we may assume that $X$ is of the form $I_i[-1]$ for some $i$. Let $Q'$ be the quiver $Q$ with the vertex $v_i$ removed, and let $\Psi$ be the associated root subsystem. For each indecomposable $Y \in S \setminus \{X\}$, choose a representative $\hat{Y}$ in $D^b(Q)$ with fine degree between 2 and $-hm + 1$ (in other words, $\hat{Y}$ is either of the form $I_j[-1]$ for $j \neq i$ or in $L(Q)[k]$ for some $0 \leq k \leq m - 1$.

Since $Y$ is compatible with $X$, $\text{Ext}_C^i_{C_m(\Phi)}(P_i[-1], Y) = 0$ for all $1 \leq j \leq m$. By Serre duality, this is equivalent to the condition that $\text{Hom}_{C_m(\Phi)}(P_i[-1], Y) = 0$ for all $1 \leq j \leq m$, or, in other words, that, if $\check{Y} \in L(Q)[k]$, that in fact $\check{Y} \in L(Q')[k]$. Thus, by Theorem 4, the images of the $\check{Y}$ form an $m$-tilting set in $C_m(\Psi)$, so $S \setminus \{X\}$ contains $n - 1$ indecomposables by induction, and thus $S$ contains $n$ indecomposables.

Proof of Theorem 3. The proof is, again, by induction on $n$. The base case, when $n = 1$, is clear. For the induction step, let $T$ be an almost complete tilting set. As before, we choose an indecomposable $X$ in $T$, which we may assume is of the form $I_i[-1]$, and then we observe that $T \setminus \{X\}$ consists of an almost complete tilting set for a root system of rank $n - 1$, and the $m + 1$ complements for that almost complete $m$-tilting set are precisely the complements of $T$ in $C_m(\Phi)$.

Acknowledgements

We would like to thank Colin Ingalls for sharing his knowledge of quiver representations, and Drew Armstrong, Sergei Fomin, Osamu Iyama, Bernhard Keller, Nathan Reading, Idun Reiten, Ralf Schiffler, David Speyer, Andrei Zelevinsky, and an anonymous referee for helpful comments.

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