Deformation of fillable $CR$ structures

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Abstract

We study the fillability (or embeddability) of $CR$ structures under the gauge-fixed Cartan flow. We prove that if the initial $CR$ structure is fillable with nowhere vanishing Tanaka-Webster curvature and free torsion, then it keeps having the same property after a short time. In the Appendix, we show the uniqueness of the solution to the gauge-fixed Cartan flow.

Key Words: Cartan flow, $CR$ structure, fillable, embeddable, pseudohermitian structure, torsion, Tanaka-Webster curvature.

1. Introduction

In [CL1], we study an evolution equation for $CR$ structures $J(t)$ on a closed (compact with no boundary) contact 3-manifold $(M, \xi)$ according to their Cartan (curvature) tensor $Q_{J(t)}$ (see also §2):

\[ \frac{\partial J(t)}{\partial t} = 2Q_{J(t)}. \]

We will often call this evolution equation (1.1) the Cartan flow. Since the equation (1.1) is invariant under a big symmetry group, namely, the contact diffeomorphisms, we add a gauge-fixing term on the right-hand side to break the symmetry. The gauge-fixed (called "regularized" in [CL1]) Cartan flow reads as follows:

\[ \frac{\partial J(t)}{\partial t} = 2Q_{J(t)} - \frac{1}{6} D_{J(t)} F_{J(t)} K \]

(see [CL1] or §2 for the meaning of notations). In this paper, we investigate the fillability of $CR$ structures under the gauge-fixed Cartan flow (1.2). A closed $CR$ manifold $M$ is fillable if $M$ bounds a complex manifold in the smooth $(C^\infty)$ sense (i.e.
there exists a complex manifold with smooth boundary $M$, and the complex structure restricts to the $CR$ structure on $M$). The notion of fillability is weaker than that of embeddability. Recall that a $CR$ manifold is embeddable if it can be embedded in $C^N$ for large $N$ with the $CR$ structure being the one induced from the complex structure of $C^N$. The embeddability is a special property for 3-dimensional $CR$ manifolds since any closed $CR$ manifold of dimension $\geq 5$ is embeddable ([BdM]). It is easy to see that a closed embeddable (strongly pseudoconvex) $CR$ 3-manifold is fillable by some well-known results (see the argument on page 543 in [Ko]). Conversely, if there exists a smooth strictly plurisubharmonic function defined in a neighborhood of a fillable $M$, then $M$ is embeddable ([Ko], Theorem 5.3; in fact, any compact complex surface with nonempty strongly pseudoconvex boundary can be made Stein by deforming it and blowing down any exceptional curves according to [Bo]). Now it is natural to ask the following question:

**Is the embeddability (or fillability) preserved under the (gauge-fixed) Cartan flow (1.1) (or (1.2))?**

An affirmative answer to the above question has an application in determining the topology of the space of all fillable $CR$ structures. For instance, one can apply such a result plus the convergence of the long time solution to (1.2) (expected for $S^3$) to prove that the space of all fillable $CR$ structures on $S^3$ is contractible (cf. Remark 4.3 in [El]). For other topological applications of solving (1.2), we refer the reader to [Ch].

Now by choosing a contact form, we can talk about the torsion of the associated (positive) pseudohermitian structure (see §2). Our first observation is the following result.

**Theorem A.** Suppose there is a contact form $\theta$ such that the torsion $A_{J(0),\theta}$ vanishes. Then under the gauge-fixed Cartan flow (1.2) (assuming smooth solution) with $K = J(0)$, $A_{J(t),\theta}$ stays vanishing.

Let $T$ denote the Reeb vector field associated with the contact form $\theta$. The vanishing of the torsion is equivalent to saying that $T$ is an infinitesimal $CR$ diffeomorphism (see (2.3)). We say a $CR$ manifold has transverse symmetry if the infinitesimal generator of a one-parameter group of $CR$ diffeomorphisms is everywhere transverse to $\xi$. Such an infinitesimal generator can be realized as the Reeb vector field for a certain contact form $\hat{\theta}$ ([L2]). It follows from Theorem A that

**Corollary B.** The $CR$ structures $J(t)$ stay having the same transverse symmetry as $J(0)$ does under the gauge-fixed Cartan flow (1.2) with $K = J(0)$ and $\theta = \hat{\theta}$.
Let $W_{J,\theta}$ denote the Tanaka-Webster curvature of a pseudohermitian structure $(J, \theta)$ (see §2 for the definition). We have the following result.

**Theorem C.** Suppose $J_{(0)}$ is fillable with $A_{J_{(0)},\theta} = 0$ and $W_{J_{(0)},\theta} > 0$ (or $< 0$, respectively). Then the solution $J(t)$ to (1.2) with $K = J_{(0)}$ stays fillable for a short time.

Our proof of Theorem C is a direct construction of an integrable almost complex structure $\tilde{J}$ on $M \times (0, \tau)$ for a small $\tau$ so that $\tilde{J}|_{\xi} = J(t)$ at $M \times \{t\}$ (see §3 for details). Then we glue this complex structure $\tilde{J}$ with the one induced by the complex surface that $(M, J_{(0)})$ bounds along $M \times \{0\}$ (identified with $M$). After we obtained the above result, László Lempert pointed out to the author that the existence of a CR vector field $T$ is sufficient to imply the embeddability of the CR structure. (see [Lem]) So by Theorem A the condition in Theorem C on the Tanaka-Webster curvature can be removed according to [Lem]. We speculate that the embeddability (or fillability) is preserved under the (gauge-fixed) Cartan flow without any conditions.

On the other hand, the zero torsion condition reduces the complexity of our flow a lot. It seems to be a good starting point. We are in a situation analogous to Hamilton’s Ricci flow. Namely, given a closed contact 3-manifold $(M, \xi)$, suppose there is a (positive) pseudohermitian structure $(J, \theta)$ with vanishing torsion and positive Tanaka-Webster curvature. Then can we deform $(J, \theta)$ according to the (gauge-fixed) Cartan flow to a limit CR structure (together with the fixed contact form $\theta$) that has the positive constant Tanaka-Webster curvature? It follows that this limit CR structure has the vanishing Cartan tensor (recall that the torsion stays vanishing for all time). Therefore it is spherical.

In the Appendix (§5), we show the uniqueness of the solution to (1.2) with given smooth initial data.

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2. **Review in CR and pseudohermitian geometry** For most of basic material we refer the reader to [We], [Ta] or [L1]. Throughout the paper, our base space $M$ is a closed (compact with no boundary) contact 3-manifold with the oriented contact structure $\xi$. A CR structure $J$ (compatible with $\xi$) is an endomorphism on $\xi$ with $J^2 = -\text{identity}$. By choosing a (global) contact form $\theta$ (exists if the normal bundle
of $\xi$ in $TM$ is orientable), we can talk about pseudohermitian geometry. The Reeb vector field $T$ is uniquely determined by $\theta(T) = 1$ and $T[d\theta] = 0$. We choose a (local) complex vector field $Z_1$, an eigenvector of $J$ with eigenvalue $i$, and a (local) complex 1-form $\theta^1$ such that $\{\theta, \theta^1, \bar{\theta}^1\}$ is dual to $\{T, Z_1, Z_1\}$ (here $\theta^1$ and $Z_1$ mean the complex conjugates of $\theta^1$ and $Z_1$ respectively). It follows that $d\theta = ih_{1\bar{1}}\theta^1 \wedge \bar{\theta}^1$ for some nonzero real function $h_{1\bar{1}}$. If $h_{1\bar{1}}$ is positive, we call such a pseudohermitian structure $(J, \theta)$ positive, and we can choose a $Z_1$ (hence $\theta^1$) such that $h_{1\bar{1}} = 1$. That is to say

\begin{equation}
(2.1) \quad d\theta = i\theta^1 \wedge \bar{\theta}^1.
\end{equation}

We’ll always assume our pseudohermitian structure $(J, \theta)$ is positive (by changing the sign of $\theta$ if negative) and $h_{1\bar{1}} = 1$ throughout the paper. The pseudohermitian connection of $(J, \theta)$ is the connection $\nabla^{\psi,h}$ on $TM \otimes C$ (and extended to tensors) given by

\[
\nabla^{\psi,h} Z_1 = \omega^1 \otimes Z_1, \quad \nabla^{\psi,h} Z_1 = \omega^1 \otimes Z_1, \quad \nabla^{\psi,h} T = 0
\]

in which the connection 1-form $\omega^1$ is uniquely determined by the following equation and associated normalization condition:

\begin{equation}
(2.2) \quad \begin{cases}
    d\theta^1 = \theta^1 \wedge \omega^1 + A^1 \theta \wedge \theta^1, \\
    \omega^1 + \omega^1 = 0.
\end{cases}
\end{equation}

The coefficient $A^1_1$ in (2.2) and its complex conjugate $A^\bar{1}_1$ are components of the torsion (tensor) $A_{J,\theta} = i A^1 \theta^1 \otimes Z_1 - i A^1 \theta^1 \otimes Z_1$. Since $h_{1\bar{1}} = 1$, $A_{1\bar{1}} = A_{1\bar{1}} A^1_1 = A^1_1$. Further $A_{1\bar{1}}$ is just the complex conjugate of $A_{1\bar{1}}$. Write $J = i \theta^1 \otimes Z_1 - i \theta^1 \otimes Z_1$. It is not hard to see from (2.1) and (2.2) that

\begin{equation}
(2.3) \quad L_T J = 2A_{J,\theta}
\end{equation}

where $L_T$ denotes the Lie differentiation in the direction $T$ (this is the case when $f = -1$ in Lemma 3.5 of [CL1]). So the vanishing torsion is equivalent to $T$ being an infinitesimal $CR$ diffeomorphism. We can define the covariant differentiations with respect to the pseudohermitian connection. For instance, $f_1 = Z_1 f$, $f_{1\bar{1}} = Z_1 Z_1 f - \omega^1(Z_1) Z_1 f$ for a (smooth) function $f$ (see, e.g., §4 in [L1]). Now Differentiating $\omega^1$ gives

\begin{equation}
(2.4) \quad d\omega^1 = W\theta^1 \wedge \bar{\theta}^1 + 2iIm(A_{1\bar{1}} \theta^1 \wedge \theta)
\end{equation}

where $W$ or $W_{J,\theta}$ (to emphasize the dependence of the pseudohermitian structure) is called the (scalar) Tanaka-Webster curvature. There are distinguished $CR$ structures $J$, called spherical, if $(M, \xi, J)$ is locally $CR$ equivalent to the standard 3-sphere.
$(S^3, \hat{\xi}, \hat{J})$, or equivalently if there are contact coordinate maps into open sets of $(S^3, \hat{\xi})$ so that the transition contact maps can be extended to holomorphic transformations of open sets in $C^2$. In 1930’s, Elie Cartan ([Ca], [CL1]) obtained a geometric quantity, denoted as $Q_J$, by solving the local equivalence problem for $CR$ structure so that the vanishing of $Q_J$ characterizes $J$ to be spherical. We will call $Q_J$ the Cartan (curvature) tensor. Note that $Q_J$ depends on a choice of contact form $\theta$. It is $CR$-covariant in the sense that if $\tilde{\theta} = e^{2f} \theta$ is another contact form and $\tilde{Q}_J$ is the corresponding Cartan tensor, then $\tilde{Q}_J = e^{-4f} Q_J$. We can express $Q_J$ in terms of pseudohermitian invariants. Write $Q_J = iQ_{11} \theta^1 \otimes Z_1 - iQ_{11} \theta^1 \otimes Z_1$ (note that $Q_{11}^1 = Q_{11}$ and $Q_{11}^1 = Q_{11}$ since we always assume $h_{11} = 1$). We have the following formula ([CL1], Lemma 2.2):

$$Q_{11} = \frac{1}{6} W_{11} + \frac{1}{2} WA_{11} - A_{11,0} - \frac{2}{3} A_{11,11}. \quad (2.5)$$

In terms of local coframe fields we can express the Cartan flow (1.1) as follows:

$$\dot{\theta}^1 = -Q_{11} \theta^1 \quad (2.6)$$

(cf. (2.16) in [CL1] with $E_{11}$ replaced by $-iQ_{11}$). The torsion evolves under the Cartan flow as shown in the follow formula:

$$\dot{A}_{11} = -Q_{11,0} \quad (2.7)$$

(this is the complex conjugate of (2.18) in [CL1] with $E_{11}$ replaced by $-iQ_{11}$). Since the Cartan flow is invariant under the pullback action of contact diffeomorphisms (cf. the argument in the proof of Proposition 3.6 in [CL1]), we need to add a gauge-fixing term to the right-hand side of (1.1) to get the subellipticity of its linearized operator. Let us recall what this term is. First we define a quadratic differential operator $F_J$ from endomorphism fields to functions by ([CL1], p.236 and note that $h_{11} = 1$ here)

$$F_J E = (iE_{11} E_{11,11} + iE_{11} E_{11,11}) + \text{conjugate}. \quad (2.8)$$

Also we define a linear differential operator $D_J$ from functions to endomorphism fields and its formal adjoint $D_J^*$ by

$$D_J f = (f_{11} + iA_{11} f) \theta^1 \otimes Z_1 + (f_{11} - iA_{11} f) \theta^1 \otimes Z_1, \quad D_J^* E = E_{11,11} + E_{11,11} - iA_{11} E_{11} + iA_{11} E_{11}. \quad (2.9)$$

(note that we have used the notations $D_J$, $D^*_J$ instead of $B'_J$, $B_J$ in [CL1], respectively.) Now let $K$ be a fixed $CR$ structure. The Cartan flow with a gauge-fixing term reads as follows: (this is (1.2))

$$\frac{\partial J_{(\omega)}}{\partial t} = 2Q_{J_{(\omega)}} - \frac{1}{6} D_{J_{(\omega)}} F_{J_{(\omega)}} K. \quad (2.10)$$
We also need the following commutation relations often:

\begin{align}
C_{I,01} - C_{I,10} & = C_{I,\bar{1}}A_{1\bar{1}} - kC_I A_{11,\bar{1}} \\
C_{I,01} - C_{I,10} & = C_{I,\bar{1}}A_{1\bar{1}} - mC_I A_{\bar{1}1,1} \\
C_{I,11} - C_{I,1\bar{1}} & = iC_{I,0} + kC_I W.
\end{align}

Here $C_I$ denotes a coefficient of a tensor with multi-index $I$ consisting of 1 and $\bar{1}$, and $k$ is the number of 1 in $I$ while $m$ is the number of $\bar{1}$ in $I$ ([L2]).

3. **Proof of Theorem A** We’ll compute the evolution of the torsion under the flow (1.2) (with $K$ being the initial CR structure $J_{(0)}$). First, instead of (2.7), we have

\begin{equation}
\dot{A}_{1\bar{1}} = -Q_{11,0} - \frac{i}{12}(D_J F_J K)_{11,0}.
\end{equation}

From the formula (2.5) for $Q_{11}$, we compute $Q_{11,0}$. Using the commutation relations (2.10) and the Bianchi identity: $W_{0} = A_{11,\bar{1}1} + A_{\bar{1}1,11}$ ([L2]), we can express $Q_{11,0}$ only in terms of $A_{11,\bar{1}1}$ and their covariant derivatives as follows:

\begin{equation}
Q_{11,0} = \frac{1}{6}(A_{11,\bar{1}111} + A_{\bar{1}1,1111}) - A_{11,00} - \frac{2k}{3}A_{11,110} + \text{l.w.t.}
\end{equation}

where l.w.t. means a lower weight term in $A_{11}$ and $A_{\bar{1}1}$. We count covariant derivatives in 1 or $\bar{1}$ direction (0 direction, resp.) as weight 1 (weight 2, resp.) and we call a term of weight $m$ if its total weight of covariant derivatives is $m$. For instance, $A_{11,\bar{1}111}$, $A_{11,00}$ and $A_{11,110}$ are all of weight 4. So more precisely each single term in l.w.t. must contain terms of weight $\leq 3$ in $A_{11}$ or $A_{\bar{1}1}$ in particular, if $A_{11} = 0$, then l.w.t. = 0. Note that $A_{11,1111}$ is a ”bad” term in the sense that we need a gauge-fixing term to cancel it and obtain a fourth order subelliptic operator in $A_{11}$. Now by (2.9) the gauge-fixing term in (3.1) (up to a constant) reads as

\begin{equation}
(D_J F_J K)_{11,0} = (F_J K)_{110} + i[A_{11}(F_J K)]_0 \\
= (F_J K)_{110} + \text{l.w.t.} \quad \text{(in } A_{11})
\end{equation}

(we have used the commutation relations (2.10) for the last equality). Write $K = K_{11}\theta^1 \otimes Z_1 + K_{\bar{1}1} \theta^1 \otimes Z_{\bar{1}} + K_{1\bar{1}} \theta^1 \otimes Z_1 + K_{\bar{1}1} \theta^1 \otimes Z_{\bar{1}}$ where $K_{11}, K_{\bar{1}1}$ are the complex conjugates of $K_{11}$, $K_{\bar{1}1}$, respectively. We compute

\begin{equation}
K_{11,0} = T[\theta^1(K Z_1)] - 2\omega_1^1(T)K_{11} \\
= (L_T \theta^1)(K Z_1) + \theta^1[(L_T K)Z_1] + \theta^1[K(L_T Z_1)] - 2\omega_1^1(T)K_{11}.
\end{equation}
It is easy to compute the first term using the (complex conjugate of) structure equation (2.2) and the third term using the formula \([T, Z_1] = -A_{11}Z_1 + \omega_1(T)Z_1\) ([L1]). For the second term, if we take \(K\) to be the initial CR structure \(J_{(0)}\), then

\[
L_T K = L_T J_{(0)} = 2A_{J_{(0)}, \theta} = 0
\]

by (2.3) and the assumption. So altogether we obtain

\[
K_{11,0} = A_{11}(K_{11} - K_{11}) = 2A_{11}K_{11}.
\]

Note that \(K^2 = -I\) implies that \(K_{11} = \pm i(1 + |K_{11}|^2)^{\frac{1}{2}}\) and \(K_{11} = -K_{11}\). It follows that

\[
K_{11,0} = -A_{11}K_{11} + A_{11}K_{11}.
\]

Here the point is that both \(K_{11,0}\) and \(K_{11,0}\) are linear in \(A_{11}\) and \(A_{11}\) with coefficients being ”0th-order” in a (co)frame. Using (3.6), (3.7), we can express \((F_j K)_{011}\) as follows:

\[
(F_j K)_{011} = iK_{11}K_{11,11011} + iK_{11}K_{11,11011} - iK_{11}K_{11,11011} - iK_{11}K_{11,11011} + l.w.t..
\]

Here and hereafter \(l.w.t.\) will mean a lower weight term in \(A_{11}, A_{11}\) up to weight 3 with coefficients in \(K_{11}, K_{11}, K_{11}, K_{11}\) and their covariant derivatives up to weight 5. Note that \(A_{11}, A_{11}\) are of weight 2 in \(K_{11}, K_{11}, K_{11}, K_{11}\). The first four terms on the right-hand side of (3.8) contain the highest weight terms of weight 4 in \(A_{11}, A_{11}\) in view of the commutation relations (2.10) and (3.6), (3.7) as will be shown below. Using (2.10) repeatedly and (3.6), we compute

\[
K_{11,1011} = K_{11,0111} + l.w.t. = K_{11,0111} = 2K_{11}A_{11,1111} + l.w.t.
\]

\[
= 2K_{11}A_{11,1111} - 2K_{11}A_{11,1111} + l.w.t.
\]

Similarly we obtain

\[
K_{11,11011} = 2K_{11}A_{11,1111} - 2K_{11}A_{11,1111} + l.w.t.
\]

\[
K_{11,11011} = 2K_{11}A_{11,1111} - 2K_{11}A_{11,1111} + l.w.t.
\]

Substituting (3.9), (3.10) in (3.8), we get, in view of (3.3),

\[
(D_j F_j K)_{11,0} = -2iA_{11,1111} + 2iA_{11,1111} + l.w.t..
\]

Now substituting (3.2) and (3.11) in (3.1) gives
Let \( \tilde{A}_{11} = -\frac{1}{2}A_{11,111} + A_{11,00} + \frac{2i}{3}A_{11,110} + l.w.t. \)

(note that the "bad" terms cancel). Define \( L_\alpha A_{11} = -A_{11,11} - A_{11,11} + i\alpha A_{11,0} \) for a complex number \( \alpha \). Let \( L^*_\alpha \) be the formal adjoint of \( L_\alpha \). It is a direct computation (cf. p1257 in [CL2]) that

\[
L^*_\alpha L_\alpha A_{11} = 2(A_{11,1111} + A_{11,1111}) - i(3 + \alpha + \bar{\alpha} - |\alpha|^2)A_{11,110} + i(3 - \alpha - \bar{\alpha} - |\alpha|^2)A_{11,110} + l.w.t..
\]

Using the commutation relations (2.10), we can easily obtain

\[
A_{11,1111} = A_{11,1111} + 2iA_{11,110} + 2iA_{11,110} + l.w.t.
\]

\[
A_{11,00} = -iA_{11,110} + iA_{11,110} + l.w.t.
\]

In view of (3.14) and (3.13), we can rewrite (3.12) as follows:

\[
\tilde{A}_{11} = -\frac{1}{12}L^*_\alpha L_\alpha A_{11} + l.w.t.
\]

for \( \alpha = 4 + i\sqrt{3} \). Since \( \alpha \) is not an odd integer, \( L_\alpha \) and hence \( L^*_\alpha L_\alpha \) (note \( L^*_\alpha = L_\alpha \)) are subelliptic (e.g. [CL1]). Taking the complex conjugate of (3.15) gives a similar equation for \( A_{11} \) only with \( \alpha \) replaced by \(-\bar{\alpha}\). On the other hand, we observe that \( A_{11} = 0, A_{1\bar{1}} = 0 \) for all (valid) time is a solution to (3.15) and its conjugate equation (note that \( l.w.t. \) vanishes if \( A_{11} \) and \( A_{1\bar{1}} \) vanish as remarked previously). Therefore by the uniqueness of the solution to a (or system of) subparabolic equation(s) (in the Appendix §5), we give a proof of the uniqueness of the solution to (1.2). An analogous argument works for a general subparabolic equation on a closed manifold), we conclude that \( A_{11} \) stays vanishing under the flow (1.2).

### 4. Proof of Theorem C

Let \( J(t) \) be a solution to (1.2) for \( 0 \leq t < \tau \) with given initial \( J(0) \) having the property stated in Theorem C. We are going to construct an almost complex structure \( \tilde{J} \) on \( M \times [0, \tau] \), integrable on \( M \times (0, \tau) \).

There is a canonical choice of the (unitary) frame \( Z_{1}(t) \) with respect to \( J(t) \) ([CL1]). Write \( Z_{1}(t) = \frac{1}{2}(e_{1}(t) - ie_{2}(t)) \) where \( e_{1}(t), e_{2}(t) \in \xi \) and \( J(t)e_{1}(t) = e_{2}(t) \). Let \( \{\theta, e_{1}^{1}(t), e_{2}^{2}(t)\} \) be a coframe dual to \( \{T, e_{1}(t), e_{2}(t)\} \) on \( M \). We'll identify \( M \times \{t\} \) with \( M \) (hence \( T(M \times \{t\}) \) with \( TM \)). Now we define an almost complex structure \( \tilde{J} \) at each point in \( M \times \{t\} \) as follows:

\[
\tilde{J} |_\xi = J(t), \tilde{J}T = -a \frac{\partial}{\partial t} + bT + a(\alpha e_{1}(t) + \beta e_{2}(t)).
\]

Here \( a, b, \alpha, \beta \) are some real (smooth) functions of space variable and \( t \), and \( a \neq 0 \) (so \( \tilde{J} \frac{\partial}{\partial t} \) is completely determined from the above formulas and \( \tilde{J}^{2} = -\text{identity} \)).
Strictly speaking, $\alpha, \beta$ depend on the choice of frame while $a, b$ are global. It is easy to see that the coframe dual to $\{e_1(t), e_2(t), \frac{\partial}{\partial t} - \frac{b}{a}T - \alpha e_1(t) - \beta e_2(t), (1/a)T\}$ is $\{e_1^1 + adt, e_2^2 + \beta dt, dt, a\theta + b dt\}$. So the following complex 1-forms:

\begin{align}
\Theta^1 &= (e_1^1 + adt) + i(e_2^2 + \beta dt) = \theta^1_t + \gamma^1_t dt, \\
\eta &= (a\theta + b dt) - idt = a\theta + (b - i) dt
\end{align}

are type (1,0) forms with respect to $\tilde{\alpha}$. Here $\gamma^1 = \alpha + i \beta$ is really the $Z_{1(t)}$ coefficient of the vector field $\alpha e_1(t) + \beta e_2(t)$. Let $\Lambda^{p,q}$ denote the space of type $(p,q)$ forms. The integrability of $\tilde{\alpha}$ is equivalent to $d\Lambda^{1,0} \subset \Lambda^{2,0} + \Lambda^{1,1}$ or $\Lambda^{2,0} \land d\Lambda^{1,0} = 0$. In terms of $\Theta^1, \eta$, the integrability conditions read as follows:

\begin{align}
\eta \land \Theta^1 \land d\eta &= 0, \\
\eta \land \Theta^1 \land d\Theta^1 &= 0.
\end{align}

Substituting (4.1), (4.2) in (4.3) and making use of $d\theta = d_M\theta = i\theta^1_t \land \theta^1_t$ (here $d_M$ denotes the exterior differentiation on $M$ and $d = d_M + dt\frac{\partial}{\partial t}$ on $M \times (0, \tau)$), we obtain

$$0 = \eta \land \Theta^1 \land d\eta = [ab_\perp - (b - i)a_\perp + ia^2\gamma^1] t \land \theta^1_t \land \theta^1_t \land dt.$$ 

Here $b_\perp = Z_{1(t)} b, a_\perp = Z_{1(t)} a$. Therefore (4.3) is equivalent to the relation between $a, b$ and $\gamma^1$ as below:

$$\gamma^1 = ia^{-1}b_\perp - ia^{-2}(b - i)a_\perp.$$ 

Next note that $d\theta^1_t = d_M\theta^1_t + dt \land \dot{\theta}^1_t$ and

$$\dot{\theta}^1_t = \{-Q_{11(t)} + \frac{i}{12}[(F_{J(t)} J(0)),_{11(t)} - iA_{11(t)}(F_{J(t)} J(0))]\} \theta^1_t.$$ 

($Q_{11(t)}$ is the $\overline{\Pi}$—component of the Cartan tensor with respect to $J_{(t)}$) So substituting (4.1),(4.2) in (4.4) and making use of (2.2) for $\theta^1_t$, we obtain

$$0 = \eta \land \Theta^1 \land d\Theta^1 = (a\{Q_{11(t)} - \frac{i}{12}[(F_{J(t)} J(0)),_{11(t)} - iA_{11(t)}(F_{J(t)} J(0))]\} + (b - i)A_{11(t)} + a\gamma^1_{\perp} \theta \land \theta^1_t \land dt.$$ 

Here $A_{11(t)}$ is the $\overline{\Pi}$—component of the torsion tensor with respect to $J(t)$ and $\gamma^1_{\perp} = Z_{1(t)} \gamma^1 + \omega^1_{\perp(t)}(Z_{1(t)}) \gamma^1$ where $\omega^1_{\perp(t)}$ is just the pseudohermitian connection form with respect to $\theta^1_t$. Therefore (4.4) is equivalent to the following relation between $a, b$ and $\gamma^1_{\perp}$:
(4.6) \[ \gamma_1^1 = -Q_{11(t)} + \frac{i}{12}[(F_{J(t)}J(0))_{11(t)} - iA_{11(t)}(F_{J(t)}J(0))] - a^{-1}(b - i)A_{11(t)}. \]

Substituting (4.5) in (4.6) and letting \( f = a^{-1}, g = ba^{-1}, u = f + ig \), we obtain an equation for a complex valued function \( u \):

(4.7) \[ u_{11} - iuA_{11(t)} = -Q_{11(t)} + \frac{i}{12}[(F_{J(t)}J(0))_{11(t)} - iA_{11(t)}(F_{J(t)}J(0))]. \]

In view of (2.9), we can express (4.7) in an intrinsic form:

(4.8) \[ J_t \circ D_{J_t}f - D_{J_t}g = Q_{J_t} - \frac{1}{12}D_{J_t}F_{J(t)}J(0). \]

Recall that \( D_{J(t)}f = \frac{1}{2}L_{X_f}J(t) \) ([CL1]) in which \( X_f = -fT + i(Z_{1(t)}f)Z_{1(t)} \)

\[-i(Z_{1(t)}f)Z_{1(t)} \text{ is the infinitesimal contact diffeomorphism induced by } f. \]

So the image of \( D_{J(t)} \) describes the tangent space of the orbit of the symmetry group acting on \( J(t) \) by the pullback (in this case, the contact diffeomorphisms are our symmetries). Now (4.8) means that if \( Q_{J(t)} \) sits in the “complexification” of the infinitesimal symmetry group orbit for all \( t \in (0, \tau) \), then \( \tilde{J} \) is integrable on \( M \times (0, \tau) \).

Now by Theorem A we have \( A_{11(t)} = 0 \) for \( 0 \leq t < \tau \). So in view of (2.5), we get

(4.9) \[ Q_{11(t)} = \frac{1}{6}W_{11(t)}. \]

Here \( W_{11(t)} = (Z_{1(t)})^2W(t) - \omega_{1(t)}(Z_{1(t)})Z_{1(t)}W(t) \) and \( W(t) \) is the Tanaka-Webster curvature with respect to \( J(t) \) (and fixed \( \theta \)). Therefore \( a = -6W_{11}^{-1} \) and \( b = -\frac{1}{2}(F_{J(t)}J(0))W_{11}^{-1} \), hence \( u = \frac{1}{6}W(t) + \frac{i}{12}F_{J(t)}J(0) \) is a solution to (4.7) by (4.9) for \( 0 \leq t < \tau \) with \( \tau \) small so that \( W(t) > 0 \) or \( W(t) < 0 \). Thus for such a choice of \( a, b, \) and \( \tau \), our \( \tilde{J} \) is integrable on \( M \times (0, \tau) \).

On the other hand, \((M, J(0)) \) bounds a complex surface \( N \) by our assumption. So we have another almost complex structure \( \tilde{J} \) on \( M \times (-\delta, 0) \) induced from \( N \), integrable on \( M \times (-\delta, 0) \), and restricting to \( J(0) \) on \((M, \xi)\). Up to a diffeomorphism from \( M \times (-\delta_1, 0] \) to \( M \times (-\delta_2, 0] \), identity on \( M \times \{0\} \) for \( \delta_1, \delta_2 \) perhaps smaller than \( \delta \), we can assume that \( \tilde{J} \) and \( \tilde{J} \) coincide at \( M \times \{0\} \) where they may not coincide up to \( C^k \) for \( k \geq 1 \), however. We want to find a local diffeomorphism \( \Phi \) from a neighborhood \( U \) of a point in \( M \times (-\delta_1, 0) \) to a similar set so that \( \Phi \) is an identity on \( U \times \{0\} \), and \( \Phi \circ \tilde{J} \) coincides with \( \tilde{J} \) up to \( C^k \) for some large integer \( k \) at \( U \times \{0\} \). Let \( x^i, 0 \leq i \leq 3 \) denote the coordinates of \( U \times (-\delta_1, 0] \) with \( x^0 \) being the time variable for \((-\delta_0, 0] \). Let \( y^i, 0 \leq i \leq 3 \) denote the corresponding coordinates of the image of \( \Phi \) with \( y^0 \) being the time variable. If we express \( \tilde{J}_1 = \Phi^* \tilde{J} = \Phi_{*}^{-1} (\tilde{J} \circ \Phi) \Phi_{*} \) in coordinates, we usually write

\[ (\tilde{J}_1)_m = \tilde{J}_i^j \frac{\partial y^j}{\partial x^m} \frac{\partial x^i}{\partial y^j}. \]
for \( \dot{J}_1 = (\dot{J}_1)^i_m dx^m \otimes \frac{\partial}{\partial x^i} \) and \( \dot{J} = \dot{J}_i^j dy^j \otimes \frac{\partial}{\partial y^i} \). Let \( \eta = \Phi^{-1} \). Then \( \eta^{-1} \) has the expression \( \frac{\partial \eta}{\partial x^m} \), the Jacobian matrix of \( \Phi \), in coordinates. We require \( \eta^{-1} = \text{identity} \) at each point with \( x^0 = 0 \) where \( \dot{J} \) coincides with \( \dot{J} \). Differentiating \( \dot{J}_1 = \Phi^* \dot{J} = \Phi^{-1}(J \circ \Phi)\dot{\Phi} = \eta(J \circ \Phi)\eta^{-1}(\text{considered as a matrix equation with respect to the above-mentioned bases}) \) in \( x^0 \) at \( x^0 = 0 \), we obtain

\[
(4.10) \quad \dot{J}_1 - \dot{J}' = \eta' \dot{J} - \dot{J} \eta'.
\]

Here the prime of \( \dot{J}' \) means the \( y^0 \)-derivative at \( y^0 = 0 \) while the prime of \( \dot{J}_1' \) and \( \eta' \) means the \( x^0 \)-derivative at \( x^0 = 0 \). Finding \( \Phi \) such that \( \dot{J}_1 = \Phi^* \dot{J} \) coincides with \( \dot{J} \) up to \( C^1 \) at \( U \times \{0\} \) is reduced to solving the above equation (4.10) for \( \eta' \) with \( \dot{J}_1' = \dot{J}' \). Here the prime of \( \dot{J}' \) means the \( t \)-derivative at \( t = 0 \). And this can be done by simple linear algebra as follows. First note that \( C = \dot{J}' - \dot{J}' \) satisfies \( JC + C \dot{J} = 0 \) since \( \dot{J} = \dot{J} \) at \( U \times \{0\} \) and both \( \dot{J}' \) and \( \dot{J}' \) satisfies the same relation as \( C \) does.

With respect to a suitable basis, \( \dot{J} \) has a canonical matrix representation:

\[
\begin{pmatrix}
  0 & -1 & 0 & 0 \\
  1 & 0 & 0 & 0 \\
  0 & 0 & 0 & -1 \\
  0 & 0 & 1 & 0 
\end{pmatrix}
\]

Then \( C \) has the matrix form \( \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \) where each \( C_{ij} \) is a \( 2 \times 2 \) matrix \( \begin{pmatrix} a_{ij} & b_{ij} \\ b_{ij} & -a_{ij} \end{pmatrix} \).

Now the solution \( \eta' \) to (4.10) has the matrix form \( \begin{pmatrix} \eta_{11}' & \eta_{12}' \\ \eta_{21}' & \eta_{22}' \end{pmatrix} \) where each \( \eta_{ij}' \) is a \( 2 \times 2 \) matrix \( \begin{pmatrix} u_{ij} & v_{ij} \\ w_{ij} & s_{ij} \end{pmatrix} \) satisfying the relations: \( v_{ij} + w_{ij} = -a_{ij}, u_{ij} - s_{ij} = b_{ij} \). Once \( \eta' \) is determined by the equation (4.10), it is easy to construct the "local" diffeomorphism \( \Phi_1 \) such that the inverse Jacobian and its \( x^0 \)-derivative at \( x^0 = 0 \) of \( \Phi_1 \) is \( \eta \) =the identity and \( \eta' \), respectively (we may need to shrink the time interval \((-\delta_1, 0])\). So if we start with \( \dot{J}_1 = \Phi_1^* \dot{J} \) instead of \( \dot{J} \) and repeat the above procedure looking for \( \Phi_2 \) so that \( \dot{J}_2 = \Phi_2^* \dot{J}_1 \) coincides with \( \dot{J} \) at \( U \times \{0\} \) up to \( C^2 \), we differentiate \( \dot{J}_2 = \eta_1 \dot{J}_1 \eta_1^{-1} \) twice with respect to \( x^0 \) at \( x^0 = 0 \). Here \( \eta_1 \) denotes the inverse Jacobian matrix of \( \Phi_2 \) (to be determined). Requiring \( \dot{J}_2' = \dot{J}_1' \) and \( \eta_1 = \text{identity} \) (at \( x^0 = 0 \)) implies \( \eta_1' = 0 \). It then follows that \( \eta_2'' \) the second derivative of \( \eta_1 \) in \( x^0 \) at \( x^0 = 0 \), satisfies a similar equation as in (4.10):

\[
(4.11) \quad \eta_2'' \dot{J}_1 - \dot{J}_1 \eta_2'' = \dot{J}_2'' - \dot{J}_2''.
\]

Now we can verify that the right-hand side anti-commutes with \( \dot{J}_1 \) as follows:

\[
(\dot{J}_2'' - \dot{J}_1'') \dot{J}_1 + \dot{J}_1 (\dot{J}_2'' - \dot{J}_1'') = (\dot{J}_2'' \dot{J}_1 + \dot{J}_1 \dot{J}_2'') - (\dot{J}_1'' \dot{J}_1 + \dot{J}_1 \dot{J}_1'') = -2(\dot{J}_2'^2 + 2(\dot{J}_1')^2 = 0
\]
(here we have used $\dot{J}_2 = \dot{J}_1$, $\dot{J}_2' = \dot{J}_1'$ and $J''J + 2(J')^2 + JJ'' = 0$ for any almost complex structure $J$ by differentiating $J^2 = -I$ twice. So we can solve (4.11) for $\eta^i_1$ with $J''_2 = \dot{J}_2''$ and hence find a $\Phi_2$ with the required properties as before. In general, suppose we have found $\Phi_{n-1}$ such that $\tilde{J}_{n-1} = \Phi_{n-1}^*\hat{J}_{n-2} = \eta_{n-2}\hat{J}_{n-2}\eta_{n-1}^{-1}$ coincides with $\tilde{J}$ up to $C^{n-1}$ at $x^0 = 0$. Then by the similar procedure we can find $\Phi_n$ such that $\tilde{J}_n = \Phi_n^*\hat{J}_{n-1} = \eta_{n-1}\hat{J}_{n-1}\eta_{n-1}^{-1}$ coincides with $\tilde{J}$ up to $C^n$ at $x_0 = 0$, and the $x^0$-derivatives of $\eta_{n-1}$ vanish up to the order $n - 1$. Furthermore the $n$-th $x^0$-derivative $\eta_{n-1}^{(n)}$ satisfies a similar equation as in (4.10) or (4.11):

\begin{equation}
(4.12) \quad \eta_{n-1}^{(n)}\tilde{J}_{n-1} - \dot{J}_{n-1}\eta_{n-1}^{(n)} = \dot{\eta}^{(n)} - \dot{\eta}_{n-1}^{(n)}.
\end{equation}

Here $\tilde{J}^{(n)}$ denotes the $n$-th $t$-derivative of $\tilde{J}$ at $t = 0$ while $\dot{\tilde{J}}_{n-1}^{(n)}$ means the $n$-th $x^0$-derivative of $\tilde{J}_{n-1}$ at $x^0 = 0$.

Now $\tilde{J}_n$ defined on $U \times (-\delta_n, 0]$ and $\dot{J}$ defined on $U \times [0, \delta_n)$ for a small $\delta_n > 0$ together form a $C^m$ integrable almost complex structure on $U \times (-\delta_n, \delta_n)$. Therefore $U \times (-\delta_n, \delta_n)$ is a complex manifold for $n \geq 4$ by a theorem of Newlander-Nirenberg ([NN]). Since $M$ is compact, we can cover it by a finite number of $U$'s and have corresponding $\delta_n$'s. For each point in the overlap of two $U$'s considered in $U \times \{0\}$, we can find local coordinate maps from an open neighborhood $V$ contained in the intersection of two associated $U \times (-\delta_n, \delta_n)$'s into $C^2$ so that the transition map $\psi$ on the ”concave” part corresponding to positive ”time variable” is holomorphic (note that our $(M, J_{(0)})$ is a strongly pseudoconvex boundary of $N$. And on $V \cap \{M \times [0, \tau]\}$, we have the ”same” integrable almost complex structure $\tilde{J}$ while on the intersection of $V$ and $U \times (-\delta_n, 0)$'s, we may have ”different” ($\tilde{J}_n$)'s. We then extend $\psi$ to the pseudoconvex part holomorphically, and denote the extension map by $\tilde{\psi}$. Now glue $V \cap \{U \times (-\delta_n, 0)\}$ (complex structure $\tilde{J}_n$) with $V \cap \{another copy of U \times (-\delta_n, 0)\}$ (perhaps different $\tilde{J}_n$) through $\tilde{\psi}$. In this way we can manage to extend the complex structure $\tilde{J}$ across $M \times \{0\}$ to $M \times (-\delta, 0)$ (globally) for some small $\delta > 0$. Finally the identity (a $CR$ diffeomorphism) on $(M, J_{(0)})$ extends to a biholomorphism $\rho$ between $M \times (-\delta, 0)$ (perhaps smaller $\delta$) and an open set in $N$ near $M$ (recall that $N$ is a complex surface that $M$ bounds). Glue $M \times (-\delta, t)$ ($t < \tau$) and $N$ via $\rho$ to form a complex surface $N_t$ that $(M \times \{t\}, J_{(t)})$ bounds. We have shown that $J_{(t)}$ is fillable for $0 < t < \tau$.

5. Appendix: uniqueness of the solution to (1.2) In this section, we’ll show that the short-time solution ([CL1]) to the gauge-fixed Cartan flow (1.2) with given initial data is actually unique. As mentioned in Introduction, the idea of proof was suggested by Jack Lee.

First we refer the reader to [CL1] for the definitions of various notations, e.g., the Folland-Stein space $S_k$, some time-dependent space $E_{k,\tau}$, the vector bundle $E_I$,
the operators $L_\alpha, \Lambda$, etc.. We define the space $\tilde{A}_{k+4,\tau}$ to consist of all the elements $u$ in $E_{k+4,\tau}$ with the initial value $u(0)$ in $S_{k+2}$. Let $\Xi_{k,\tau} = \{(u(0), u) \in S_{k+2} \times \tilde{A}_{k+4,\tau} \mid u \in \tilde{A}_{k+4,\tau}\}$. Let $P(t)$ be a time-dependent linear operator on sections of $E_J$ over $M$, involving only spatial derivatives of weight $\leq 4$ and depending smoothly on $t \in [0, 1]$, such that $P(0) = cL_\alpha^*L_\alpha + S$, where $c$ is a positive constant, $\alpha$ is admissible and $S$ is an operator of weight $\leq 3$ (here $L_\alpha^*$ instead of $L_\alpha$ in [CL1] means the adjoint operator of $L_\alpha$). The following theorem extends Theorem 4.6 in [CL1] to the case of nonvanishing initial data $u(0)$.

**Theorem 5.1.** For any integer $k \geq 0$, there exists $0 < \tau \leq 1$ such that the map
\[
\Psi : \Xi_{2k,\tau} \to S_{2k+2} \times E_{2k,\tau}
\]
defined by
\[
\Psi((u(0), u)) = (u(0), (\partial_t + P(t))u)
\]
is a bounded isomorphism.

**Proof.** We will follow the treatment given in the proof of Theorem 4.6 in [CL1]. It is clear that $\Psi$ is linear and bounded. $\Psi$ being surjective is equivalent to solving the following initial-value problem:

\[
(5.1) \quad (\partial_t + P(t))u(t) = f(t),
\]

\[
(5.2) \quad u(0) = g
\]

for $u \in \tilde{A}_{2k+4,\tau}$ with given $(g, f) \in S_{2k+2} \times E_{2k,\tau}$. Recall ([CL1], p.244) that $B_{k,\tau} = \{u \in E_{k,\tau} : u(\tau) = 0\}$. Define $\tilde{B}_{k,\tau} = \{u \in B_{k,\tau} : u(0) \in S_{k-2}\}$. Let $\Sigma_{k,\tau} = \{(v(0), v) \in S_{k-2} \times \tilde{B}_{k,\tau} \mid v \in B_{k,\tau}\}$. Define a hermitian bilinear form $\Omega : \Sigma_{2k+4,\tau} \times (S_{2k+2} \times S_{2k+4,\tau}) \to C$ by

\[
(5.3) \quad \Omega((v(0), v), (h, u)) = A(v, u).
\]

Here $A(v, u)$ is just the hermitian bilinear form that we used in [CL1] (see (4.9) on page 245). Observe that

1. For any $(v(0), v) \in \Sigma_{2k+4,\tau}$, the linear functional $(h, u) \to \Omega((v(0), v), (h, u))$ is bounded on $S_{2k+2} \times S_{2k+4,\tau}$ since $|A(v, u)| \leq C||u||_{2k+4,\tau} \leq C(||h||_{2k+2} + ||u||_{2k+4,\tau})$.

2. For some positive constant $C$, $C(||v(0)||_{2k+2}^2 + ||v||_{2k+4,\tau}^2) \leq \Omega((v(0), v), (v(0), v))$ for all $(v(0), v) \in \Sigma_{2k+4,\tau}$.

To verify this, we review the argument at the bottom of page 245 and the top of page 246 in [CL1] and conclude that

\[
|\Omega((v(0), v), (v(0), v))| = |A(v, v)| \geq Re A(v, v)
\]

\[
\geq C''||v||_{2k+4,\tau}^2 + \frac{1}{2}||A^{k+1}v(0)||_0^2
\]

\[
\geq C(||v||_{2k+4,\tau}^2 + ||v(0)||_{2k+2}^2).
\]

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Here $C'$ and $C$ are some positive constants, and the last inequality follows from Corollary 4.3 in [CL1]. Under conditions (1), (2), we can apply a generalized Lax-Milgram lemma due to J. L. Lions ([Tr], lemma 41.2) to assert that for any continuous linear functional $G$ on $S_{2k+2} \times S_{2k+4,\tau}$, there exists $(\tilde{h}, \tilde{u}) \in S_{2k+2} \times S_{2k+4,\tau}$ with

\[(5.4) \quad ||\tilde{h}||_{2k+2} + ||\tilde{u}||_{2k+4,\tau} \leq C||G|| \quad \text{(operator norm)}\]

such that

\[(5.5) \quad \Omega((v(0), v), (\tilde{h}, \tilde{u})) = G((v(0), v)) \quad \text{for all } (v(0), v) \in \Sigma_{2k+4,\tau}.\]

Now given $(g, f) \in S_{2k+2} \times E_{2k,\tau}$, we take $G : S_{2k+2} \times S_{2k+4,\tau} \rightarrow C$ to be the functional

\[(5.6) \quad G((h, v)) = F(v) + (\Lambda^{k+1}h, \Lambda^{k+1}g)_J\]

in which $F : S_{2k+4,\tau} \rightarrow C$ is given by ([CL1], p.246)

\[F(v) = \int_0^\tau (\Lambda^{k+2}v(t), \Lambda^k(e^{-\kappa t}f(t)))dt.\]

It is easy to see that $|G((h, v))| \leq C_1(||h||_{2k+2} + ||v||_{2k+4,\tau})$ since $|F(v)| \leq C_2||v||_{2k+4,\tau}$. Here $C_1, C_2$ are some positive constants. Thus there exists $(\tilde{h}, \tilde{u}) \in S_{2k+2} \times S_{2k+4,\tau}$ satisfying (5.4) so that (5.5) holds. By taking $v \in C^\infty_0((0, \tau) \times M)$ (smooth with compact support)(which implies $v(0) = 0$), we are reducing the equation (5.5) to the equation (4.12) in [CL1]. So an argument there on p.246 shows that $u(\tau) = e^{\kappa \tau} \tilde{u}(\tau)$ satisfies (5.1). Furthermore, we can show that $u \in E_{2k+4,\tau}$ by (5.4) and a similar argument as in [CL1], p.247. Now for $v \in B_{2k+4,\tau}$, we have

\[(5.7) \quad 0 = A(v, \tilde{u}) - G((v(0), v)) \quad \text{(by (5.5), (5.3))}
\]

\[= (A(v, \tilde{u}) - F(v)) - (\Lambda^{k+1}v(0), \Lambda^{k+1}g)_J \quad \text{(by (5.6))}
\]

\[= (\Lambda^{k+1}v(0), \Lambda^{k+1}\tilde{u}(0)) - (\Lambda^{k+1}v(0), \Lambda^{k+1}g)_J
\]

(see the formula in the middle of p.247 in [CL1])

Since, for any $w \in C^\infty(M)$, we can find $v \in B_{2k+4,\tau}$ such that $v(0) = w$, we obtain from (5.7) that $\Lambda^{2k+2}(\tilde{u}(0) - g) = 0$ in the distribution sense, and hence $u(0) = \tilde{u}(0) = g$.

We have shown that $u \in \bar{A}_{2k+4,\tau}$ and is a solution to (5.1) – (5.2). Therefore $\Psi$ is surjective. By the same proof for the uniqueness as in [CL1], p.247 (the last paragraph of the proof for Theorem 4.6), we conclude that $\Psi$ is injective.

Q.E.D.

We remark that Theorem 5.1 may still be valid for any positive half-integer $k$ if we can make sense of $\Lambda^k$, say, in terms of pseudodifferential operators. In the elliptic
case, the (complex) power of an elliptic operator (acting on sections of a vector bundle over a closed manifold) can be well defined (see, e.g., [See]). For our case, one expects to define the power of a subelliptic operator like $\Lambda$ along the same line of ideas as in [See] together with the symbol calculus for so-called $V$-operators ([BG]). On the other hand, Theorem 5.1 is sufficient for our purpose of proving the uniqueness of the solution to (1.2).

**Theorem 5.2.** Let $\hat{J}$ be any smooth (i.e., $C^\infty$) oriented CR structure on $M$. For a large enough integer $m$, say $m \geq 14$, suppose $J^1_{(t)}$, $J^2_{(t)}$ are two $C^m$ solutions to (1.2) on some time interval $[0, \varepsilon]$ with $J^1_{(0)} = J^2_{(0)} = \hat{J}$. Then $J^1_{(t)} = J^2_{(t)}$ on $[0, d]$ for a small positive $d < \varepsilon$.

**Proof.** Recall ([CL1]) that we can parametrize $J_{(t)}$ by a section $u_{(t)}$ of $\mathcal{E}_j$ and describe the nonlinear operator $P(J) = -2Q_J + \frac{1}{6}D_JF_JK$ by $\hat{P}$ from sections of $\mathcal{E}_j$ to sections of $\mathcal{E}_j$. The equation (1.2) together with the initial condition $J_{(0)} = \hat{J}$ is equivalent to

\begin{align}
(5.8) & \quad \partial_t u_{(t)} + \hat{P}(u_{(t)}) = 0, \\
(5.9) & \quad u_{(0)} = 0
\end{align}

([CL1], (5.1),(5.2)). Let $\tilde{u}_{(t)}$ be an infinite-order formal solution to (5.8) – (5.9) so that $\tilde{f} = (\partial_t + \hat{P})\tilde{u}$ vanishes to infinite order at $t = 0$. Let $\hat{P}$ denote the linearization of $\hat{P} : \hat{A}_{2k+4,\tau} \to E_{2k,\tau}$ about $\tilde{u}$. Observe that $P_{(0)}$ satisfies the required property, and $\hat{P}$ is $C^1$ for $k$ large enough, say, $k \geq 5$ ([CL1], pp.250-251). So in view of Theorem 5.1, we can apply the inverse function theorem to conclude that

**Lemma 5.3.** For $k \geq 5$, there exists $0 < \tau \leq 1$ such that the map $Id \times (\partial_t + \hat{P}) : \Xi_{2k,\tau} \to S_{2k+2} \times E_{2k,\tau}$ defined by

\[(u_{(0)}, u) \to (u_{(0)}, (\partial_t + \hat{P})u)\]

has a $C^1$ inverse on some neighborhood of $(0, \tilde{u})$ in $\Xi_{2k,\tau}$.

Note that the smooth section $f_{\varepsilon}$ given by $f_{\varepsilon(t)} = 0$ for $0 \leq t \leq \varepsilon$, and $f_{\varepsilon(t)} = \tilde{f}_{(t-\varepsilon)}$ for $\varepsilon \leq t \leq \tau$ is arbitrarily close to $\tilde{f}$ in $E_{2k,\tau}$ for small $\varepsilon > 0$. So by Lemma 5.3 there exists $u \in A_{2k+4,\tau}$ ($u_{(0)} = 0$) satisfying $(\partial_t + \hat{P})u = f_{\varepsilon}$. For $t \in [0, \varepsilon]$, $u$ is a solution to (5.8) – (5.9).

Now suppose $v \in A_{2k+4,\tau}$ is another solution to (5.8) – (5.9) also for $t \in [0, \varepsilon]$. Observe that $f_{\varepsilon-d}$ is arbitrarily close to $f_{\varepsilon}$ in $E_{2k,\tau}$ for small $d > 0$. Moreover, $(v_{(d)}, f_{\varepsilon-d})$ is in the neighborhood of $(0, \hat{f})$ where we can apply Lemma 5.3 if $d$ is small enough. So by Lemma 5.3 there exists $\tilde{v}$ close to $\tilde{u}$ in $A_{2k+4,\tau}$ such that
\[(5.10) \quad (\partial_t + \hat{P})\tilde{v} = f_{\varepsilon - d},\]

\[(5.11) \quad \tilde{v}(0) = v(d).\]

Let \(w(t) = v(t)\) for \(0 \leq t \leq d\) and \(w(t) = \tilde{v}(t-d)\) for \(d \leq t \leq \tau\). The equation insures that the derivatives of \(v\) and \(\tilde{v}\) match up at \(t = d\). Compute \((\partial_t + \hat{P})w = 0\) for \(0 \leq t \leq d\) and \((\partial_t + \hat{P})w = f_{\varepsilon}\) for \(d \leq t \leq \tau\). (note that \(f_{\varepsilon - d(t-d)} = f_{\varepsilon(t)}\)) Since \(d < \varepsilon\), we actually have \((\partial_t + \hat{P})w = f_{\varepsilon}\). Also, for \(d\) small, \(w\) is close to \(\tilde{u}\) since \(v(0) = 0\) and \(\tilde{v}\) is close to \(\tilde{u}\). Therefore \(w = u\). It follows that \(v(t) = u(t)\) for \(t \in [0, d]\).

Q.E.D.

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