Efficient Quantum Algorithm for Hidden Quadratic and Cubic Polynomial Function Graphs

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Abstract

We introduce the Hidden Polynomial Function Graph Problem as a natural generalization of an abelian Hidden Subgroup Problem (HSP) where the subgroups and their cosets correspond to graphs of linear functions over the finite field $\mathbb{F}_p$. For the Hidden Polynomial Function Graph Problem the functions are not restricted to be linear but also can be multivariate polynomial functions of higher degree.

For a fixed number of indeterminates and bounded total degree the Hidden Polynomial Function Graph Problem is hard on a classical computer as its black box query complexity is polynomial in $p$. In contrast, this problem can be reduced to a quantum state identification problem so that the resulting quantum query complexity does not depend on $p$. For univariate polynomials we construct a von Neumann measurement for distinguishing the states. We relate the success probability and the implementation of this measurement to certain classical problems involving polynomial equations. We present an efficient algorithm for hidden quadratic and cubic function graphs by establishing that the success probability of the measurement is lower bounded by a constant and that it can be implemented efficiently.

1 Introduction

Shor’s algorithm for factoring integers and calculating discrete logarithms [18] is one of the most important and well known examples of quantum computational speedups. This algorithm as well as other fast quantum algorithms for number-theoretic problems [9,10,17] essentially rely on the efficient solution of an abelian hidden subgroup problem (HSP) [4]. This has naturally raised the questions of what interesting problems can be reduced to the nonabelian HSP and of whether the general nonabelian HSP can also be solved efficiently on a quantum computer.

It is known that an efficient quantum algorithm for the dihedral HSP would give rise to efficient quantum algorithms for certain lattice problems [16], and that an efficient

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quantum algorithm for the symmetric group would give rise to an efficient quantum algorithm for the graph isomorphism problem [6]. Despite the fact that efficient algorithms have been developed for several nonabelian HSPs (see, for example, [14] and the references therein), the HSP over the dihedral group and the symmetric group have withstood all attempts so far. Moreover, there is evidence that the nonabelian HSP might be hard for some groups such as the symmetric group [11].

Another idea for the generalization of the abelian HSP is to consider Hidden Shift Problems [3,7] or problems with hidden non-linear structures [5]. In the latter context, we define a new black-box problem, called the Hidden Polynomial Function Graph Problem, and present efficient quantum algorithms for special cases. More specific, the Hidden Polynomial Function Graph Problem is a natural generalization of the abelian HSP over groups of the special form $G := \mathbb{F}_p^{m+1}$, where the hidden subgroups are generated by the $m$ generators $(0, \ldots, 1, \ldots, 0, q_i) \in \mathbb{F}_p^{m+1}$ with $q_i \in \mathbb{F}_p$ and the 1 is in the $i$th component. Therefore, the hidden subgroups $H_Q$ and their cosets $H_{Q,z}$ are given by

$$H_Q := \{(x, Q(x)) : x \in \mathbb{F}_p^m\} \quad \text{and} \quad H_{Q,z} := \{(x, Q(x) + z) : x \in \mathbb{F}_p^m\},$$

where $z \in \mathbb{F}_p$ and $Q$ runs over all polynomials $Q(X_1, \ldots, X_m) = q_1 X_1 + \ldots + q_m X_m$. In the Hidden Polynomial Function Graph Problem the polynomials are no longer restricted to be linear but can also be of degree $n \geq 2$. The subgroups and their cosets are generalized to graphs of polynomial multivariate functions going through the origin and to translated function graphs, respectively.

Our approach to solve this problem on a quantum computer is to generalize standard techniques for the HSP. First, we reduce the problem to a quantum state identification problem and show that the resulting quantum query complexity does not depend on $p$. Second, we design a measurement scheme for distinguishing the quantum states in the univariate case. Third, we relate the success probability and implementation of the measurement to certain classical problems involving polynomial equations.

The paper is organized as follows: In Section 2 we define the Hidden Polynomial Function Graph Problem and compare it to the Hidden Polynomial Problem studied in Ref. [5]. In Section 3 we show that the standard approach for HSPs can be used to reduce the new problem to a state distinguishing problem. In Section 4 we derive upper and lower bounds for the query complexity for this approach. In Section 5 we discuss the properties of the states for univariate polynomials and construct measurements to distinguish these states. In Sections 6 and 7 we discuss the cases of quadratic and cubic univariate functions thoroughly and show that an efficient solution for these special cases exists. In Section 8 we conclude and discuss possible objectives for further research.

### 2 Hidden Polynomial Function Graph Problem

**Definition (Hidden Polynomial Function Graph Problem):**

Let $Q(X_1, X_2, \ldots, X_m) \in \mathbb{F}_p[X_1, X_2, \ldots, X_m]$ be an arbitrary $m$-variate polynomial of total degree at most $n$ whose constant term is equal to zero. Let $B : \mathbb{F}_p^{m+1} \to \mathbb{F}_p$ be a
black-box function hiding the polynomial $Q$ in the following sense:

$$B(r_1, r_2, \ldots, r_m, s) = B(\bar{r}_1, \bar{r}_2, \ldots, \bar{r}_m, \bar{s})$$

iff there is an element $z \in \mathbb{F}_p$ such that

$$s = Q(r_1, r_2, \ldots, r_m) + z \text{ and } \bar{s} = Q(\bar{r}_1, \bar{r}_2, \ldots, \bar{r}_m) + z,$$

i.e., the function $B$ is constant on the subsets

$$H_{Q,z} := \{(r_1, r_2, \ldots, r_m, Q(r_1, r_2, \ldots, r_m) + z) : r_1, r_2, \ldots, r_m \in \mathbb{F}_p\}$$

of $\mathbb{F}_p^{m+1}$ and distinct for different values of $z$.

The Hidden Polynomial Function Graph Problem is to identify the polynomial $Q$ if only the black-box function $B$ is given. An algorithm for $m$-variate polynomials with total degree less or equal to $n$ (where $n$ and $m$ are both constant) is efficient if its running time is polylogarithmic in $p$.

An alternative definition of the function $B$ is

$$B(r_1, r_2, \ldots, r_m, s) := \pi(s - Q(r_1, r_2, \ldots, r_m))$$

where $\pi$ is an unknown and irrelevant bijection $\pi : \mathbb{F}_p \to \mathbb{F}_p$ which permutes the elements of $\mathbb{F}_p$ arbitrarily.

The classical query complexity of the Hidden Polynomial Function Graph Problem is polynomial in $p$. This is because for univariate polynomials (i.e., $m = 1$) at least $n$ different points

$$(r^{(1)}, s^{(1)}), \ldots, (r^{(n)}, s^{(n)}) \text{ with } B(r^{(1)}, s^{(1)}) = \ldots = B(r^{(n)}, s^{(n)})$$

are required in order to determine the hidden polynomial $Q$ of degree $n$. The probability of obtaining such an $n$-fold collision is smaller than the probability of obtaining a 2-fold collision. The probability of the latter is $1/p$.

The Hidden Polynomial Function Graph Problem is related to the Hidden Polynomial Problem defined in [5] which can be equivalently reformulated as follows. The black-box function $h : \mathbb{F}_p^m \to \mathbb{F}_p$ is given by $h(r_1, \ldots, r_m) := \sigma(Q(r_1, \ldots, r_m))$, where $\sigma$ is an arbitrary permutation of $\mathbb{F}_p$ and $Q(X_1, \ldots, X_m)$ is the hidden polynomial. It is readily seen that the black-boxes $h$ can be obtained from the black-boxes $B$ by querying $B$ only at points of the form $(r_1, \ldots, r_m, 0)$. For this reason the black-boxes $B$ offer more flexibility in designing quantum algorithms. We are able to design an efficient quantum algorithm for the black-boxes $B$ hiding univariate quadratic and cubic polynomials, whereas no algorithms are known for the black-boxes $h$.

### 3 Standard Approach

Most quantum algorithms for HSPs are based on the standard approach which reduces black box problems to state distinguishing problems. We apply this approach to the Hidden Polynomial Function Graph Problem in the following.
• Evaluate the black-box function on an equally weighted superposition of all 
\( (r_1, r_2, \ldots, r_m, s) \in F_p^{m+1} \). The resulting state is
\[
\frac{1}{\sqrt{p^{m+1}}} \sum_{r_1, r_2, \ldots, r_m, s \in F_p} |r_1, r_2, \ldots, r_m\rangle \otimes |s\rangle \otimes |F(r_1, r_2, \ldots, r_m, s)\rangle
\]

• Measure and discard the third register. Assume we have obtained the result \( \pi(z) \). Then the state on the first and second register is \( \rho_{Q,z} := |\phi_{Q,z}\rangle \langle \phi_{Q,z}| \) where
\[
|\phi_{Q,z}\rangle := \frac{1}{\sqrt{p^m}} \sum_{r_1, r_2, \ldots, r_m \in F_p} |r_1, r_2, \ldots, r_m\rangle \otimes |Q(r_1, r_2, \ldots, r_m, s) + z\rangle
\]
with the unknown polynomial \( Q \) hidden by \( B \), and \( z \) is uniformly at random. The corresponding density matrix is
\[
\rho_Q := \frac{1}{p} \sum_{z \in F_p} |\phi_{Q,z}\rangle \langle \phi_{Q,z}|. \tag{1}
\]

We refer to the states \( \rho_Q \) as polynomial function states. We have to distinguish these states in order to solve the black box problem.

## 4 Quantum Query Complexity

We show that the quantum query complexity of the Hidden Polynomial Function Graph Problem is independent of \( p \). To prove this result we make use of the upper and lower bounds of Ref. [12] on the number of copies required for state discrimination. The former is expressed in terms of fidelity which can be bounded by the following technical lemma.

**Lemma 1:** Let \( \rho \) and \( \sigma \) be two quantum states with corresponding spectral decompositions \( \rho = \sum_i \lambda_i |\psi_i\rangle \langle \psi_i| \) and \( \sigma = \sum_j \mu_j |\phi_j\rangle \langle \phi_j| \). Assume that \( \max_{i,j} |\langle \psi_i| \phi_j\rangle| \leq \alpha \) for some value \( \alpha \). Then we have
\[
F(\rho, \sigma) \leq \alpha \cdot \min \left\{ \sum_i \sqrt{\lambda_i}, \sum_j \sqrt{\mu_j} \right\}
\]
where \( F(\rho, \sigma) := \|\sqrt{\rho} \sqrt{\sigma}\|_1 \) is the fidelity of \( \rho \) and \( \sigma \).

**Proof.** We have
\[
\|\sqrt{\rho} |\phi_i\rangle\|_1 \leq \alpha \tag{2}
\]
for all \( i \). This is derived by observing that \( \|\sqrt{\rho} |\phi_i\rangle\|_1 = \|\sqrt{\rho} |\psi_i\rangle\| \| |\phi_i\rangle\| \) and
\[
\|\sqrt{\rho} |\phi_j\rangle\|^2 \leq \sum_i \lambda_i |\langle \psi_i| \phi_j\rangle|^2 \leq \alpha^2.
\]
Using first the triangle inequality and then Eq. (2) we obtain

\[
\|\sqrt{\rho} \|_1 = \|\sqrt{\rho} \sum_j \sqrt{\mu_j} |\phi_j\rangle\langle \phi_j| \|_1 \leq \sum_j \sqrt{\mu_j} \|\sqrt{\rho}|\phi_j\rangle\langle \phi_j| \|_1 = \alpha \cdot \left( \sum_j \sqrt{\mu_j} \right) . \tag{3}
\]

The same arguments apply if we use the spectral decomposition of \(\rho\) instead. This completes the proof. \(\square\)

**Corollary 1:** We have \(F(\rho_Q, \rho_{\tilde{Q}}) \leq n/\sqrt{p}\), where \(\rho_Q\) and \(\rho_{\tilde{Q}}\) are two different polynomial states and their total degree is at most \(n\).

This corollary follows by observing that

\[
|\langle \phi_Q,z|\phi_{\tilde{Q}},\tilde{z}\rangle| = \frac{1}{p^m} \sum_{r_1,\ldots,r_m \in \mathbb{F}_p} \langle Q(r_1,\ldots,r_m) + z|\tilde{Q}(r_1,\ldots,r_m) + \tilde{z}\rangle \\
= \frac{1}{p^m} \left| \{(r_1,\ldots,r_m) \in \mathbb{F}_p^m : Q(r_1,\ldots,r_m) + z = \tilde{Q}(r_1,\ldots,r_m) + \tilde{z}\} \right| \\
\leq \frac{1}{p^m} n p^{m-1} = \frac{n}{p}.
\]

The last inequality follows from the Schwartz-Zippel theorem saying that two different \(m\)-variate polynomials of total degree less or equal to \(n\) can intersect in at most \(np^{m-1}\) points \([15]\).

**Theorem 1:** The query complexity of the Hidden Function Graph Problem is at most \(4 \binom{n+m}{m}\).

**Proof.** The results in \([12]\) imply that there is a POVM \(\{E_Q\}\) acting on \(k\) copies of a polynomial function state such that

\[
P_{\text{success}} := \min_Q \operatorname{Tr}(\rho_Q^{\otimes k} E_Q) \geq 1 - \epsilon
\]

provided that \(k \geq 2(\log N - \log \epsilon)/(-\log F)\), where \(N := p^{(n+m)-1}\) is the number of different polynomial function states and \(F\) is the maximal fidelity over all pairs of different polynomial function states. This bound and the lower bound on the fidelity \(F \leq n/\sqrt{p}\) imply that the success probability \(P_{\text{success}}\) is at least 1/2 for \(k = 4 \binom{n+m}{m}\) (provided that \(p\) is sufficiently large). \(\square\)

The lower bound presented in \([12]\) implies that at least \(\binom{n+m}{m}/m - 1\) copies are required to have \(P_{\text{success}} \geq 1/2\).

5 Distinguishing Polynomial Function States

In the remainder of the article we consider only the univariate case, i.e., \(m = 1\).
Structure of Polynomial Function States  The states $\rho_{Q,z}$ can be written as

$$
\rho_{Q,z} = \frac{1}{p} \sum_{b,c \in \mathbb{F}_p} |b\rangle \langle c| \otimes |Q(b) + z\rangle \langle Q(c) + z|.
$$

The density matrix $\rho_Q$ of Eq. (1) is the average of these states over $z$. To obtain a compact notation we introduce the cyclic shift $S_p|x\rangle := |x + 1 \text{ mod } p\rangle$ for which we have the identity

$$
\sum_{z \in \mathbb{F}_p} |b + z\rangle \langle c + z| = S_p^{b-c}.
$$

This directly leads to

$$
\rho_Q = \frac{1}{p^2} \sum_{b,c \in \mathbb{F}_p} |b\rangle \langle c| \otimes S_p^{Q(b)-Q(c)}.
$$

Now we use the fact that the shift operator and its powers can be diagonalized simultaneously with the Fourier matrix $F_p := \sqrt{1/p} \sum_{k,\ell \in \mathbb{F}_p} \omega_p^{k\ell} |k\rangle \langle \ell|$, i.e., we have

$$
F_p S_p F_p^\dagger = \sum_{u \in \mathbb{F}_p} \omega_p^u |u\rangle \langle u|,
$$

where $\omega_p := e^{2\pi i/p}$ is a $p$th root of unity. Hence, the density matrices have the block diagonal form

$$
\tilde{\rho}_Q := (I_p \otimes F_p) \rho_Q (I_p \otimes F_p^\dagger) = \frac{1}{p^2} \sum_{b,c,x \in \mathbb{F}_p} \omega_p^{[Q(b)-Q(c)]x} |b\rangle \langle c| \otimes |x\rangle \langle x|
$$

in the Fourier basis where $I_p$ denotes the identity matrix of size $p$.

By repeating the standard approach $k$ times for the same black-box function $B$, we obtain the density matrix $\tilde{\rho}_Q^\otimes k$. After rearranging the registers we can write

$$
\tilde{\rho}_Q^\otimes k = \frac{1}{p^{2k}} \sum_{b,c,x \in \mathbb{F}_p^k} \omega_p^{\sum_{j=1}^k [Q(b_j)-Q(c_j)]x_j} |b\rangle \langle c| \otimes |x\rangle \langle x|
$$

where $|q| := (q_1, q_2, \ldots, q_n) \in \mathbb{F}_p^{1 \times n}$ is the row vector whose entries are the coefficients of the hidden polynomial $Q(X) = \sum_{i=1}^n q_i X^i$. 

- $\langle q|$, $\Phi^{(n)}(b)$, $\Phi^{(n)}(c)$, and $|x\rangle$ are defined as follows:

- $\langle q| := (q_1, q_2, \ldots, q_n) \in \mathbb{F}_p^{1 \times n}$
• $\Phi^{(n)}(b)$ is the $n \times k$ matrix

$$
\Phi^{(n)}(b) := \sum_{i=1}^{n} \sum_{j=1}^{k} b_j^i |i\rangle \langle j| = \begin{pmatrix}
    b_1 & b_2 & \cdots & b_k \\
b_1^2 & b_2^2 & \cdots & b_k^2 \\
\vdots & \vdots & \ddots & \vdots \\
b_1^n & b_2^n & \cdots & b_k^n
\end{pmatrix},
$$

• $|x\rangle := (x_1, \ldots, x_k)^T \in \mathbb{F}_p^k$ is the column vector whose entries are $x_i$.

Algebraic-geometric problem We simplify the techniques of [1, 2, 3] and use them to construct a von Neumann measurement for distinguishing the states $\tilde{\rho}_Q^\otimes k$. Let $w := (w_1, \ldots, w_n) \in \mathbb{F}_p^n$ and $|w\rangle \in \mathbb{F}_p^n$ be the corresponding column vector. Consider the algebraic-geometric problem to determine all $b \in \mathbb{F}_p^k$ for given $x \in \mathbb{F}_p^k$ and $w \in \mathbb{F}_p^n$ such that $\Phi^{(n)}(b)|x\rangle = |w\rangle$, i.e.,

$$
\begin{pmatrix}
    b_1 & b_2 & \cdots & b_k \\
b_1^2 & b_2^2 & \cdots & b_k^2 \\
\vdots & \vdots & \ddots & \vdots \\
b_1^n & b_2^n & \cdots & b_k^n
\end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}.
$$

We denote the set of solutions to these polynomial equations and its cardinality by

$$
S^x_w := \{ b \in \mathbb{F}_p^k : \Phi^{(n)}(b)|x\rangle = |w\rangle \}
$$

and

$$
\eta^x_w := |S^x_w|,
$$

respectively. We also define the quantum states $|S^x_w\rangle$ to be the equally weighted superposition of all solutions

$$
|S^x_w\rangle := \frac{1}{\sqrt{\eta^x_w}} \sum_{b \in S^x_w} |b\rangle
$$

if $\eta^x_w > 0$ and $|S^x_w\rangle$ to be the zero vector otherwise. Using this notation, we can express the state $\tilde{\rho}_Q^\otimes k$ as

$$
\tilde{\rho}_Q^\otimes k := \frac{1}{p^{2k}} \sum_{x \in \mathbb{F}_p^k} \sum_{w \in \mathbb{F}_p^n} \omega_p^{(q|w\rangle - (q|v\rangle)} \sqrt{\eta^x_w \eta^v_v} |S^x_w\rangle \langle S^v_v| \otimes |x\rangle \langle x|.
$$

Measurement for distinguishing the polynomial states The block structure of the states $\tilde{\rho}_Q^\otimes k$ implies that we can measure the second register in the computational basis without any loss of information. The probability of obtaining a particular $x$ is

$$
\text{Tr} \left( \tilde{\rho}_Q^\otimes k (I_{p^k} \otimes |x\rangle \langle x|) \right) = \frac{1}{p^{2k}} \sum_{w \in \mathbb{F}_p^n} \eta^x_w = \frac{1}{p^k}
$$

and the resulting reduced state is

$$
\tilde{\rho}_Q^x := \frac{1}{p^k} \sum_{w, v \in \mathbb{F}_p^n} \omega_p^{(q|w\rangle - (q|v\rangle)} \sqrt{\eta^x_w \eta^v_v} |S^x_w\rangle \langle S^v_v|.
$$
In the following we assume that for a result \( x \) and all \( w \) the cardinality \( \eta^x_w \) is at most polylogarithmic in \( p \) and that the elements of the sets \( S^x_w \) can be computed efficiently. In this case we have an efficiently computable bijection between \( S^x_w \) and the set \( \{(w,j) : j = \{0, \ldots, \eta^x_w - 1\}\} \). This bijection is obtained by sorting the elements of \( S^x_w \) according to the lexicographic order on \( \mathbb{F}_p^k \) and associating to each \( b \in S^x_w \) the unique \( j \in \{0, \ldots, \eta^x_w - 1\} \) corresponding to its position in \( S^x_w \). We rely on this bijection to implement a transformation \( U_x \) satisfying

\[
U_x |S^x_w\rangle = |w\rangle
\]

for all \((x, w)\) with \( \eta^x_w > 0 \). This is done as follows.

- Implement a unitary with

\[
\sum_{b \in S^x_w} |b\rangle \otimes |0\rangle \otimes |0\rangle \mapsto \frac{1}{\sqrt{\eta^x_w}} |w\rangle \otimes \sum_{j=1}^{\eta^x_w} |j\rangle \otimes |\eta^x_w\rangle.
\]

Note that \( b \) and \( x \) determine \( j \) and \( w \) uniquely and vice versa. Furthermore, we can compute \( w \) and \( j \) efficiently since \( \eta^x_w \) is at most polylogarithmic in \( p \).

- Apply the unitary

\[
\sum_{\ell=0}^{\eta^x_w-1} (F_{\ell+1} \oplus I_{p^k-\ell-1}) \otimes |\ell\rangle \langle \ell| + \sum_{\ell=\eta^x_w}^{p^k-1} I_{p^k} \otimes |\ell\rangle \langle \ell|
\]

on the second and third register. This implements the embedded Fourier transform \( F_\ell \) of size \( \ell \) controlled by the second register in order to map the superposition of all \( |j\rangle \) with \( j \in \{0, \ldots, \ell - 1\} \) to \( |0\rangle \). The resulting state is \( |w\rangle \otimes |0\rangle \otimes |\eta^x_w\rangle \).

- Uncompute \( |\eta^x_w\rangle \) in the third register with the help of \( w \) and \( x \). This leads to the state \( |w\rangle \otimes |0\rangle \otimes |0\rangle \)

We apply \( U_x \) to the state of Eq. (4) and obtain

\[
U_x \rho^x_Q U_x^\dagger = \frac{1}{p^k} \sum_{w, v \in \mathbb{F}_p^n} \omega_p^{(q|w) - (q|v)} \sqrt{\eta^x_w \eta^x_v} |w\rangle \langle v|.
\]

We now measure in the Fourier basis, i.e., we carry out the von Neumann measurement with respect to the states

\[
|\psi_Q\rangle := \frac{1}{\sqrt{p^n}} \sum_{w \in \mathbb{F}_p^n} \omega_p^{(q|w)} |w\rangle.
\]

Simple computations show that the probability for the correct detection of the state \( \tilde{\rho}^x_Q \) is

\[
\langle \psi_Q | \tilde{\rho}^x_Q | \psi_Q \rangle = \frac{1}{p^{k+n}} \left( \sum_{w \in \mathbb{F}_p^n} \sqrt{\eta^x_w} \right)^2.
\]

The probability to identify \( Q \) correctly is obtained by summing the probabilities in Eq. (6) over all \( x \) for which we can implement the transformation \( U_x \) and multiplying the sum by \( 1/p^{k} \).
6 Hidden Quadratic Polynomials

For a single copy of the polynomial function state $\rho_Q$ it turns out that the pretty good measurement [13] is the optimal measurement for distinguishing the states. However, the resulting success probability is only in the order of $1/p$. In contrast, the success probability of our measurement scheme for two copies is lower bounded by a constant. This strongly resembles the situation for the Heisenberg-Weyl HSP, where a single copy is also not sufficient but the pretty good measurement of two copies leads to an efficient quantum algorithm [2].

For quadratic polynomials we have to consider the sets

$$ S^{(x_1,x_2)}_{(w_1,w_2)} = \left\{ \left( \begin{array}{c} b_1 \\ b_2 \end{array} \right) \in \mathbb{F}_p^2 : \left( \begin{array}{ccc} b_1 & b_2 & b_1 b_2^2 \\ b_1 b_2 & b_2^3 & b_1^2 b_2 \\ b_1^3 b_2 & b_2^3 & b_1^2 b_2^3 \end{array} \right) \cdot \left( \begin{array}{c} x_1 \\ x_2 \end{array} \right) = \left( \begin{array}{c} w_1 \\ w_2 \end{array} \right) \right\} . $$

We set $b = b_1, c = b_2, x = x_1, y = x_2, v = w_1,$ and $w = w_2$ to avoid too many indices. Therefore, we have to find the set of solutions of the equations

$$ bx + cy = v \quad \text{and} \quad b^2x + c^2y = w. \quad (7) $$

Depending on $x$ and $y$ which are determined by the orthogonal measurement in the first stage as well as by $v$ and $w$ the set of solutions can encompass 0, 1, 2, $p$ or $p^2$ solutions. To derive a lower bound on the success probability it suffices to consider the $p^2 - 3p + 2$ cases where $x, y \neq 0$ and $x \neq -y$. In these cases the Eqs. (7) have the solutions $(b_j, c_j)$ with

$$ c_{1/2} := \frac{v}{x+y} \pm \frac{1}{x+y} \sqrt{D} \quad \text{and} \quad b_{1/2} = \frac{v}{x} - \frac{y}{x}c_{1/2} $$

provided that

$$ D := \frac{x}{y}w((x+y) - v^2) $$

is a square in $\mathbb{F}_p$. For each pair $(x, y)$ there are $p(p+1)/2$ pairs $(v, w)$ such that the resulting $D$ is a square. In this case, there are one or two solutions. Therefore, we have the following lower bound on the success probability

$$ \frac{1}{p^6} \sum_{(x,y)} \left( \sum_{(v,w)} \sqrt{\eta_{(v,w)}} \right)^2 \geq \frac{1}{p^6} \left( p^2 - 3p + 2 \right) \left( \frac{p(p+1)}{2} \right)^2 = \frac{1}{4} - O \left( \frac{1}{p} \right). $$

We now argue that the measurement can be implemented efficiently. Following the discussion of Sec. 5 we only have to show that we can implement the transform of (5) efficiently, i.e., given $x, y, b$ and $c$ we must find the index of the solution $(b, c)$ to Eqs. (7) efficiently. This is possible since the solutions of the $p^2 - 3p + 2$ considered cases can be computed with $O(\log(p))$ operations on a classical computer (see Cor. 14.16 in [8]).

7 Hidden Cubic Polynomials

For cubic polynomials we obtain the sets

$$ S^{(x_1,x_2,x_3)}_{(w_1,w_2,w_3)} = \left\{ \left( \begin{array}{c} b_1 \\ b_2 \\ b_3 \end{array} \right) \in \mathbb{F}_p^3 : \left( \begin{array}{ccc} b_1 & b_2 & b_3 \\ b_2 & b_2 & b_2 \\ b_3 & b_3 & b_3 \end{array} \right) \cdot \left( \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} \right) = \left( \begin{array}{c} w_1 \\ w_2 \\ w_3 \end{array} \right) \right\}. \quad (8) $$
To simplify the following computations we assume that \( x_1 \neq 0 \). Therefore, the set of Eq. (8) can be written as

\[
S^\kappa_\lambda = \left\{ \left( \begin{array}{c} b \\ c \\ d \end{array} \right) \in \mathbb{F}_p^3 : \left( \begin{array}{ccc} b & c & d \\ b^2 & c^2 & d^2 \\ b^3 & c^3 & d^3 \end{array} \right) \cdot \left( \begin{array}{c} 1 \\ x \\ y \end{array} \right) = \left( \begin{array}{c} u \\ v \\ w \end{array} \right) \right\}
\]  
(9)

with \( \kappa := (1, x, y) \), \( \lambda := (u, v, w) \), and the coefficients

\[
x := \frac{x_2}{x_1}, \quad y := \frac{x_3}{x_1}, \quad u := \frac{w_1}{x_1}, \quad v := \frac{w_2}{x_1}, \quad \text{and} \quad w := \frac{w_3}{x_1}.
\]

In the appendix we show that for \( x \neq 0, \pm 1 \) and \( y \neq 0, -1, -x, \pm (x + 1) \)

and for all \( u, v, w \) the inequality

\[
\eta^\kappa_\lambda \leq 10 \quad (11)
\]

holds for the size \( \eta^\kappa_\lambda \) of the sets of Eq. (9). This bound now implies that for all pairs \((x, y)\) there are at least \( p^3/10 \) tuples \((u, v, w)\) with \( \eta^\kappa_\lambda \geq 1 \) because of the equality

\[
\sum_{\lambda \in \mathbb{F}_p^3} \eta^\kappa_\lambda = p^3.
\]

We obtain a lower bound on the success probability \( P_{\text{success}} \) for our measurement scheme as follows. First, we discard all tuples \((x_1, x_2, x_3, w_1, w_2, w_3)\) with \( x_1 = 0 \). This leads to

\[
P_{\text{success}} = \frac{1}{p^9} \sum_{x_1, x_2, x_3 \in \mathbb{F}_p} \left( \sum_{w_1, w_2, w_3 \in \mathbb{F}_p} \sqrt{\eta^{(x_1, x_2, x_3)}_{(w_1, w_2, w_3)}} \right)^2 \\
\geq \frac{p - 1}{p^9} \sum_{x, y \in \mathbb{F}_p} \left( \sum_{u, v, w \in \mathbb{F}_p} \sqrt{\eta^{(1, x, y)}_{(u, v, w)}} \right)^2.
\]

Second, we take the disequalities (10) into account and obtain

\[
P_{\text{success}} \geq \frac{(p - 1)(p^2 - 8p + 16)}{p^6} \left( \sum_{u, v, w \in \mathbb{F}_p} \sqrt{\eta^{(1, x, y)}_{(u, v, w)}} \right)^2
\]

because there are \((p - 3)(p - 5) + 1\) pairs \((x, y)\) which satisfy these disequalities. Third, we lower bound the sum by \( p^3/10 \) and obtain

\[
P_{\text{success}} \geq \frac{(p - 1)(p^2 - 8p + 16)}{p^9} \left( \frac{p^3}{10} \right)^2 = \frac{1}{100} - O \left( \frac{1}{p} \right).
\]

Therefore, the success probability can be lower bounded by a constant for sufficiently large \( p \). Furthermore, the computations in the appendix show that we find the solutions of the polynomial system (9) by solving univariate polynomials of degree six or less. This leads to an efficient quantum algorithm because the roots of these polynomials can be computed with a polylogarithmic number of operations.
8 Conclusion and Outlook

We have introduced the Hidden Polynomial Function Graph Problem as a generalization of a particular abelian Hidden Subgroup Problem. We have shown that the standard approach for HSPs can be successfully applied to this problem and leads to an efficient quantum algorithm for quadratic and cubic polynomials over prime fields. A generalization of all the methods to non-prime fields $\mathbb{F}_d$ is straightforward. The Fourier transform over $\mathbb{F}_p$ has to be replaced by the Fourier transform over $\mathbb{F}_d$ which can be implemented efficiently \cite{7}.

The central points of interest for future research are the generalization to polynomials over rings (admitting a Fourier transform), polynomials of higher degree, multivariate polynomials, and a broader class of functions. Moreover, it would be important to find real-life problems which could be reduced to our black-box problem and the problems defined in \cite{5}.

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A Analysis of the Cubic Case

In this appendix we use Buchberger’s algorithm\footnote{We use the lexicographical order of monomials.} to show that the ideal that is generated by the polynomials of Eq. (9) contains the elements

$$
\begin{align*}
    b + xc + yd - u &= 0 \\
    c + g_1d^5 + g_2d^4 + g_3d^3 + g_4d^2 + g_5d + g_6 &= 0 \\
    d^6 + h_1d^5 + h_2d^4 + h_3d^3 + h_4d^2 + h_5d + h_6 &= 0
\end{align*}
$$

for a subset of the tuples $(x, y, u, v, w)$ which we refer to as regular cases. From these equations inequality (11) follows directly because there are at most six solutions for $d$ and each value of $d$ determines $b$ and $c$ uniquely. Additionally, we consider non-regular cases in order to establish the inequality for all $(x, y, u, v, w)$ with certain $x$ and $y$. In the latter cases we obtain at most ten solutions since there are at most five possible values for $d$ and for each of those values there are at most two pairs $(b, c)$ which lead to a solution of the system.

A.1 Buchberger’s Algorithm in Regular Cases

Before computing S-polynomials following Buchberger’s algorithm we reduce the polynomials of Eq. (9) with the linear polynomial, i.e., we eliminate $b$ in the second and third polynomial equation with the substitution

$$
b = u - xc - yd.
$$
This leads to the equations

\[ c^2 + c_1cd + c_2c + c_3d^2 + c_4d + c_5 = 0 \quad (12) \]
\[ c^3 + d_1c^2d + d_2c^2 + d_3cd^2 + d_4cd + d_5c + d_6d^2 + d_7d^2 + d_8d + d_9 = 0 \quad (13) \]

with the coefficients

\[
\begin{align*}
c_1 &:= \frac{2y}{x(x+1)} & c_2 &:= \frac{-2u}{x(x+1)} & c_3 &:= \frac{y(x+1)}{x(x+1)} & c_4 &:= \frac{-2u y}{x(x+1)} \\
c_5 &:= \frac{u^2-w}{x(x+1)} & d_1 &:= \frac{-3xy}{1-x^2} & d_2 &:= \frac{3ux}{1-x^2} & d_3 &:= \frac{-3y^2}{1-x^2} \\
d_4 &:= \frac{6uy}{1-x^2} & d_5 &:= \frac{-3uw}{1-x^2} & d_6 &:= \frac{y(1-y^2)}{x(1-x^2)} & d_7 &:= \frac{-3y^2}{x(1-x^2)} \\
d_8 &:= \frac{-3u^2y}{x(1-x^2)} & d_9 &:= \frac{u^2-w}{x(1-x^2)}
\end{align*}
\]

Here and in the remainder of this section we assume that all occurring denominators are unequal to zero. We reduce Eq. (13) with Eq. (12) and obtain the polynomial

\[ cd^2 + e_1cd + e_2c + e_3d^3 + e_4d^2 + e_5d + e_6 \quad (14) \]

where we have

\[
\begin{align*}
e_1 &:= \frac{d_6-c_1d_4-c_5d_1-c_2+2c_1c_4}{d_4-c_1d_1-c_3+c_1^2} & e_2 &:= \frac{d_5-c_2d_2-c_5+c_2^2}{d_4-c_1d_1-c_3+c_1^2} \\
e_3 &:= \frac{d_6-c_1d_4-c_5d_1-c_2}{d_4-c_1d_1-c_3+c_1^2} & e_4 &:= \frac{d_5-c_2d_2-c_5d_1-c_4+c_1c_1+c_3}{d_4-c_1d_1-c_3+c_1^2} \\
e_5 &:= \frac{d_6-c_4d_2-c_5d_1+c_1c_1+c_4}{d_4-c_1d_1-c_3+c_1^2} & e_6 &:= \frac{d_9-c_5d_2+c_2c_5}{d_4-c_1d_1-c_3+c_1^2}
\end{align*}
\]

After these reductions we compute the reduced S-polynomial

\[ cd + f_1c + f_2d^4 + f_3d^3 + f_4d^2 + f_5d + f_6 \quad (15) \]

of the polynomials (12) and (14). We have the coefficients

\[
\begin{align*}
f_1 &:= \frac{e_6 - e_2e_4 + e_1e_2e_3}{e_5 - e_1e_4 - e_2e_3 + e_1^2e_3} \\
f_2 &:= -\frac{e_1^2 - c_1e_3 + e_3}{e_5 - e_1e_4 - e_2e_3 + e_1^2e_3} \\
f_3 &:= -\frac{2e_3e_4 - c_1e_4 - e_1e_3^2 - e_2e_3 + c_3e_1 + c_4}{e_5 - e_1e_4 - e_2e_3 + e_1^2e_3} \\
f_4 &:= -\frac{e_3e_5 - c_1e_5 + e_3^2 - e_1e_3e_4 - c_2e_4 + c_3e_2 + c_4e_1 + c_5}{e_5 - e_1e_4 - e_2e_3 + e_1^2e_3} \\
f_5 &:= -\frac{e_3e_6 - c_1e_6 + e_4e_5 - e_1e_3e_5 - c_2e_5 + c_4e_2 + e_5e_1}{e_5 - e_1e_4 - e_2e_3 + e_1^2e_3} \\
f_6 &:= -\frac{e_4e_6 - e_1e_3e_6 - c_2e_6 + c_5e_2}{e_5 - e_1e_4 - e_2e_3 + e_1^2e_3}
\end{align*}
\]

The reduced S-polynomial of the polynomials (14) and (15) is

\[ c + g_1d^5 + g_2d^4 + g_3d^3 + g_4d^2 + g_5d + g_6 \quad (16) \]
with the coefficients

\[
\begin{align*}
g_1 &:= -\frac{f_2}{f_1^2 - e_1 f_1 + e_2} & g_2 &:= -\frac{f_3 - f_1 f_2 + e_1 f_2}{f_1^2 - e_1 f_1 + e_2} & g_3 &:= -\frac{f_4 - f_1 f_3 + e_1 f_3 - e_3}{f_1^2 - e_1 f_1 + e_2} \\
g_4 &:= -\frac{f_2}{f_2^2 - e_1 f_1 + e_2} & g_5 &:= -\frac{f_3 - f_1 f_5 + e_1 f_5 - e_3}{f_1^2 - e_1 f_1 + e_2} & g_6 &:= -\frac{f_6 - f_1 f_6 + e_1 f_6}{f_1^2 - e_1 f_1 + e_2}
\end{align*}
\]

The reduced S-polynomial of the polynomials in Eq. (15) and (16) is

\[
d^6 + h_1 d^5 + h_2 d^4 + h_3 d^3 + h_4 d^2 + h_5 d + h_6
\]

with the coefficients

\[
\begin{align*}
h_1 &:= \frac{g_2 + h_1 g_1}{g_1} & h_2 &:= \frac{g_3 + h_1 g_2 - f_2}{g_1} & h_3 &:= \frac{g_4 + h_1 g_3 - f_3}{g_1} \\
h_4 &:= \frac{g_2 + h_4 g_1 - f_1}{g_1} & h_5 &:= \frac{g_3 + h_4 g_2 - f_3}{g_1} & h_6 &:= \frac{g_4 + h_4 g_3}{g_1}
\end{align*}
\]

After this step we stop Buchberger’s algorithm since the S-polynomials (16) and (17) are the polynomials we are looking for. This computation is only possible for regular tuples \((x, y, u, v, w)\), i.e., all denominators are non-vanishing.

### A.2 Characterization of Regular Cases

In the preceding section, all denominators are unequal to zero if the disequalities

\[
\begin{align*}
0 &\neq x(x + 1) & 0 &\neq x(1 - x^2) \\
0 &\neq d_3 - c_1 d_1 - c_3 + c_1^2 & 0 &\neq e_5 - e_1 e_4 - e_2 e_3 + e_1^2 e_3 \\
0 &\neq f_1^2 - e_1 f_1 + e_2 & 0 &\neq f_1
\end{align*}
\]

are satisfied. The substitution of \(c_j, d_j, e_j, f_1,\) and \(g_1\) with their expressions in \(x, y, u, v, w\) leads to the disequalities

\[
\begin{align*}
0 &\neq x(x + 1) & 0 &\neq x(1 - x^2) \\
0 &\neq y(y + x + 1) & 0 &\neq (y - x - 1)(vy + vx + v - u^2) \\
0 &\neq (r_3 v^3 + r_2 v^2 + r_1 v + r_0)(x + 1) & 0 &\neq (y + 1)(y + x)
\end{align*}
\]

where we have the coefficients

\[
\begin{align*}
r_0 &:= w^2 x y^3 + w^2 y^3 + 2 w^2 x^2 y^2 + 4 w^2 x y^2 + 2 w^2 y^2 + w^2 x^3 y + 3 w^2 x y + 3 w^2 x y + 4 u^3 w x y + w^2 y + 4 u^3 w y + u^6 \\
r_1 &:= -3 u(y + x + 1)(2 w x y + 2 w y + u^3) \\
r_2 &:= 3 u^2(y^2 + x y + y + x^2 + 2 x + 1) \\
r_3 &:= -(y - x - 1)^2(y + x + 1)
\end{align*}
\]

For the following analysis we separate the factors of the disequalities into two sets: The first set contains all factors which only depend on \(x\) and \(y\) and the second set contains all factors which also depend on \(u, v,\) or \(w.\)
A.3 Analysis of Non-Regular Cases

In this section we discuss the polynomial system for non-regular tuples \((x, y, u, v, w)\), i.e., one or more of the denominators of Sec. A.1 vanish. We assume that all factors of the denominators which solely depend on \(x\) and \(y\) are unequal to zero since the other cases can be discarded in the analysis of Sec. 7. The remaining disequalities which depend on the \(u\), \(v\), and \(w\) are

\[
0 \neq v(y + x + 1) - u^2 \quad \text{and} \quad 0 \neq r_3v^3 + r_2v^2 + r_1v + r_0.
\]

First, we assume that \(v(y + x + 1) - u^2 = 0\). Then Buchberger’s algorithm leads to the polynomial system

\[
\begin{align*}
    b + xc + yd - u &= 0 \\
    cd^2 + e_1cd + e_2c + e_3d^3 + e_4d^2 + e_5d + e_6 &= 0 \\
    \tilde{f}_1c + \tilde{f}_2d^4 + \tilde{f}_3d^3 + \tilde{f}_4d^2 + \tilde{f}_5d + \tilde{f}_6 &= 0
\end{align*}
\]

where we have

\[
\tilde{f}_j := (e_5 - e_1e_4 - e_2e_3 + e_1^2e_3)f_j
\]

with the \(f_j\) of Section A.1. If \(\tilde{f}_1 \neq 0\) then we substitute \(c\) in the second equation with

\[
c = -\frac{1}{\tilde{f}_1}\left(\tilde{f}_2d^4 + \tilde{f}_3d^3 + \tilde{f}_4d^2 + \tilde{f}_5d + \tilde{f}_6\right).
\]

This substitution leads to a polynomial in \(d\) which always has degree six since

\[
\tilde{f}_2 = -\frac{(y+1)(y+x)}{(x-1)^2x}
\]

is always non-zero for the \(x\) and \(y\) we consider. Hence, there are at most six solutions for \(d\) and inequality (11) also holds in this non-regular case because \(b\) and \(c\) are uniquely defined by the value of \(d\). For \(\tilde{f}_1 = 0\) we have the system

\[
\begin{align*}
    b + xc + yd - u &= 0 \\
    cd^2 + e_1cd + e_2c + e_3d^3 + e_4d^2 + e_5d + e_6 &= 0 \\
    \tilde{f}_2d^4 + \tilde{f}_3d^3 + \tilde{f}_4d^2 + \tilde{f}_5d + \tilde{f}_6 &= 0
\end{align*}
\]

where again \(\tilde{f}_2 \neq 0\) is always true. In this case there are at most four solutions for \(d\). Furthermore, it follows from Eq. (12) that for each value of \(d\) there are at most two solutions for \(c\). Since \(b\) is uniquely defined by \(c\) and \(d\) there are at most eight solutions.

For \(v(y + x + 1) - u^2 \neq 0\) we consider the case \(r_3v^3 + r_2v^2 + r_1v + r_0 = 0\). We obtain the reduced S-polynomial

\[
\tilde{g}_1d^5 + \tilde{g}_2d^4 + \tilde{g}_3d^3 + \tilde{g}_4d^2 + \tilde{g}_5d + \tilde{g}_6
\]

(18)

where we have the coefficients

\[
\tilde{g}_j := (f_1^2 - e_1f_1 + e_2)g_j
\]
with the \( g_j \) of Section A.1. There are always at most five solutions for \( d \) since

\[
\tilde{g}_1 = \frac{-y(y + 1)(y + x)(y + x + 1)^2}{x(x - 1)(y - x - 1)(vy + vx + v - u^2)}
\]

shows that polynomial (18) is not the zero polynomial. Therefore, the system has at most ten solutions as the discussion of the preceding case shows.

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