HIGHER DIMENSIONAL STEADY RICCI SOLITONS
WITH LINEAR CURVATURE DECAY

YUXING DENG* AND XIAO HUA ZHU**

Abstract. We prove that any noncompact \( \kappa \)-noncollapsed steady (gradient) Ricci soliton with nonnegative curvature operator must be rotationally symmetric if it has a linear curvature decay.

1. Introduction

As one of singular model solutions of Ricci flow, it is important to classify steady (gradient) Ricci solitons under a suitable curvature condition [15, 21, 18], etc. In his celebrated paper [21], Perelman conjectured that any 3-dimensional \( \kappa \)-noncollapsed steady (gradient) Ricci soliton must be rotationally symmetric. The conjecture has been solved by Brendle in 2012 [2]. It is known by a result of Chen that any 3-dimensional ancient solution has nonnegative sectional curvature [6]. As a natural generalization of Perelman’s conjecture in higher dimensions, we have

Conjecture 1.1. Any \( n \)-dimensional \((n \geq 4)\) \( \kappa \)-noncollapsed steady (gradient) Ricci soliton with positive curvature operator must be rotationally symmetric.

An essential progress to Conjecture 1.1 has been made by Brendle in [3]. In fact, he proved that any steady (gradient) Ricci soliton with positive sectional curvature must be rotationally symmetric if it is asymptotically cylindrical. For \( \kappa \)-noncollapsed steady Kähler-Ricci solitons with nonnegative bisectional curvature, the authors recently proved that they must be flat [8, 11].

Definition 1.2. An \( n \)-dimensional steady gradient Ricci soliton \((M, g, f)\) is called asymptotically cylindrical if the following holds:

(i) Scalar curvature \( R(x) \) of \( g \) satisfies

\[
\frac{C_1}{\rho(x)} \leq R(x) \leq \frac{C_2}{\rho(x)}, \quad \forall\ \rho(x) \geq r_0,
\]

2000 Mathematics Subject Classification. Primary: 53C25; Secondary: 53C55, 58J05.
Key words and phrases. Ricci flow, Ricci soliton, \( \kappa \)-solution, Perelman’s conjecture.
*Partially supported by the NSFC Grants 11701030, ** by the NSFC Grants 11331001 and 11771019.
Higher dimensional steady Ricci solitons

where $C_1, C_2$ are two positive constants and $\rho(x)$ denotes the distance from a fixed point $x_0$.

(ii) Let $p_m$ be an arbitrary sequence of marked points going to infinity. Consider rescaled metrics $g_m(t) = r_m^{-1} \phi_t^* g$, where $r_m R(p_m) = \frac{n-1}{2} + o(1)$ and $\phi_t$ is a one-parameter subgroup generated by $X = -\nabla f$. As $m \to \infty$, flows $(M, g_m(t), p_m)$ converge in the Cheeger-Gromov sense to a family of shrinking cylinders $(\mathbb{R} \times S^{n-1}(1), \tilde{g}(t)), t \in (0, 1)$. The metric $\tilde{g}(t)$ is given by

$$\tilde{g}(t) = dr^2 + (n-2)(2-2t)g_{S^{n-1}(1)},$$

where $S^{n-1}(1)$ is the unit sphere in Euclidean space.

The property (ii) above means that for any $p_i \to +\infty$, rescaled flows $(M, R(p_i)g(R^{-1}(p_i)t), p_i)$ converge subsequently to $(\mathbb{R} \times S^{n-1}, ds^2 + g_{S^{n-1}(t)})$ ( $t \in (-\infty, 0]$), where $g_{S^{n-1}(t)}$ is a family of shrinking round spheres. In this paper, we verify the properties (i) and (ii) in Definition 1.2 to show that Conjecture 1.1 is true in addition that the scalar curvature of steady Ricci soliton has a linear curvature decay. More precisely, we have

**Theorem 1.3.** Let $(M, g)$ be a noncompact $\kappa$-noncollapsed steady (gradient) Ricci soliton with nonnegative curvature operator. Then, it is rotationally symmetric if its scalar curvature $R(x)$ satisfies

$$R(x) \leq \frac{C}{\rho(x)}.$$  \hspace{1cm} (1.1)

In Theorem 1.3, we do not assume that $(M, g)$ has positive Ricci curvature. In fact, we can prove that $(M, g)$ is an Euclidean space if the Ricci curvature is not strictly positive (cf. Section 5). Since the Euclidean space is rotationally symmetric, we may assume that $(M, g)$ is not a flat space. Then we show that the condition (1.1) in Theorem 1.3 also implies

$$R(x) \geq \frac{c_0}{\rho(x)}, \quad \forall \, \rho(x) \geq r_0 > 0,$$  \hspace{1cm} (1.2)

where $c_0 > 0$ is a constant (cf. Corollary 5.3). (1.2) has been proved for the steady Ricci soliton with nonnegative curvature operator and positive Ricci curvature [8] (also see Theorem 2.5).

Theorem 1.3 is reduced to prove

**Theorem 1.4.** Let $(M, g)$ be a $\kappa$-noncollapsed steady (gradient) Ricci soliton with nonnegative sectional curvature. Suppose that $(M, g)$ has an exactly linear curvature decay, i.e.

$$\frac{C_0^{-1}}{\rho(x)} \leq R(x) \leq \frac{C_0}{\rho(x)}.$$  \hspace{1cm} (1.3)
Higher dimensional steady Ricci solitons

for some constant \( C_0 > 0 \). Let \( g(t) = \phi_t^* g_0 \). Then for any \( p_i \to +\infty \), rescaled flows \((M, R(p_i)g(R^{-1}(p_i)t), p_i)\) converge subsequently to \((\mathbb{R} \times S^{n-1}, ds^2 + g_{S^{n-1}}(t))\) (\( t \in (-\infty, 0) \)) in the Cheeger-Gromov topology, where \((S^{n-1}, g_{S^{n-1}}(t))\) is a \( \kappa \)-noncollapsed ancient Ricci flow with nonnegative sectional curvature. Moreover, scalar curvature \( R_{S^{n-1}}(x, t)\) of \((S^{n-1}, g_{S^{n-1}}(t))\) satisfies

\[
R_{S^{n-1}}(x, t) \leq \frac{C}{|t|}, \quad \forall \ x \in S^{n-1},
\]

where \( C \) is a uniform constant.

Theorem 1.4 is about steady Ricci solitons with nonnegative sectional curvature, which is a weaker condition than nonnegative curvature operator. In fact, applying Theorem 1.4 to 4-dimensional steady Ricci solitons, we further prove

**Theorem 1.5.** Any 4-dimensional noncompact \( \kappa \)-noncollapsed steady (gradient) Ricci soliton with nonnegative sectional curvature must be rotationally symmetric if it has a linear curvature decay \( \text{(1.3)} \).

We conjecture that Theorem 1.5 holds for all dimensions. It is closely related to the classification of shrinking solitons on \( S^{n-1} \) with positive sectional curvature (see Section 6.1 for details). This will improve Brendle’s result without the condition (ii) in Definition 1.2 [3].

The main step in the proof of Theorem 1.4 is to estimate the diameter of level sets of steady Ricci soliton (cf. Section 3). We study a metric flow of level sets. This flow is very similar to the Ricci flow. Then we can use Perelman’s argument to estimate the distance functions in level sets [21]. In Section 4, we prove Theorem 1.4 for steady Ricci solitons with positive Ricci curvature by constructing a parallel vector field in a limit space as in [8, 10, 9]. The proof of Theorem 1.4 will be completed in Section 5. The proofs of Theorem 1.3 and 1.5 are given in Section 6.

2. Previous results

In this section, we recall some results for steady Ricci solitons in [8, 10]. \((M, g, f)\) is called a steady gradient Ricci soliton if Ricci curvature \( \text{Ric}(g) \) of \( g \) on \( M \) satisfies

\[
\text{Ric}(g) = \text{Hess} f,
\]

where \( f \) is a smooth function on \( M \). We always assume that \( \text{Ric}(g) > 0 \) and there is an equilibrium point \( o \) in \( M \) from this section to section 4. The
Higher dimensional steady Ricci solitons

latter means that $\nabla f(o) = 0$. By the identity

$$|\nabla f|^2 + R \equiv \text{const.},$$

we see that $R(o) = R_{\text{max}}$ and

$$|\nabla f|^2 + R = R_{\text{max}}.$$  

(2.2)

Under (1.1), the equilibrium point always exists (cf. [10, Corollary 2.2]). Moreover, it is unique since $\text{Ric}(g) > 0$. By Morse lemma, we have (cf. [10]).

Lemma 2.1. The level set $\Sigma_r = \{ f(x) = r \}$ is a closed manifold for any $r > f(o)$, which is diffeomorphic to $S^{n-1}$.

The following lemma is due to [10, Lemma 3.1]

Lemma 2.2. Let $o \in M$ be an equilibrium point of steady Ricci soliton $(M, g, f)$ with positive Ricci curvature. Then for any $p \in M$ and number $k > 0$ with $f(p) - \frac{k}{\sqrt{R(p)}} > f(o)$, it holds

$$B(p, \frac{k}{\sqrt{R_{\text{max}}}}; R(p)g) \subset M_{p,k},$$

(2.3)

where the set $M_{p,k}$ is defined by

$$M_{p,k} = \{ x \in M \mid f(p) - \frac{k}{\sqrt{R(p)}} \leq f(x) \leq f(p) + \frac{k}{\sqrt{R(p)}} \}.$$

By Lemma 2.2, we prove

Lemma 2.3. Let $(M, g)$ be a steady (gradient) Ricci soliton with nonnegative sectional curvature and positive Ricci curvature. Suppose that (1.3) holds. Then there exists a constant $C$ such that

$$\frac{|\nabla R(p)|}{R^{3/2}(p)} \leq C, \forall p \in M.$$  

(2.4)

Proof. Fix any $p \in M$ with $f(p) \geq r_0 >> 1$. Then

$$|f(x) - f(p)| \leq \frac{1}{\sqrt{R(p)}}, \forall x \in M_{p,1}.$$  

It is known by [4],

$$c_1 \rho(x) \leq f(x) \leq c_2 \rho(x), \forall \rho(x) \geq r_0.$$  

(2.5)

Thus by the curvature decay, we get

$$c_2 \rho(x) \geq f(p) - \frac{1}{\sqrt{R(p)}} \geq c_1 \rho(p) - \sqrt{C_0 \rho(p)}.$$  

It follows that

$$\frac{R(x)}{R(p)} \leq C_0^2 \frac{\rho(p)}{\rho(x)} \leq \frac{2c_2C_0^2}{c_1}, \forall x \in M_{p,1}.$$  

4
Higher dimensional steady Ricci solitons

On the other hand, by (2.3), we have

\[ B(p, \frac{1}{\sqrt{R_{\text{max}}}}; g_p) \subseteq M_{p,1}. \]

Hence

\[ R(x) \leq C'R(p), \quad \forall \ x \in B(p, \frac{1}{\sqrt{R_{\text{max}}}}; g_p). \]

Let \( \phi_t \) be generated by \( -\nabla f \). Then \( g(t) = \phi_t^* g \) satisfies the Ricci flow,

\[ \frac{\partial g(t)}{\partial t} = -2 \text{Ric}(g(t)). \]

Also rescaled flow \( g_p(t) = R(p)g(R^{-1}(p)t) \) satisfies (2.7). Since the Ricci curvature is positive,

\[ B(p, \frac{1}{\sqrt{R_{\text{max}}}}; g_p(t)) \subseteq B(p, \frac{1}{\sqrt{R_{\text{max}}}}; g_p(0)), \quad t \in [-1,0]. \]

and

\[ \frac{\partial}{\partial t} R = 2 \text{Ric}(\nabla f, \nabla f) \geq 0. \]

Combining the above two relations with (2.6), we get

\[ R_{g_p(t)}(x) \leq C', \quad \forall \ x \in B(p, \frac{1}{\sqrt{R_{\text{max}}}}; g_p(0)), \quad t \in [-1,0]. \]

Thus, by Shi’s higher order estimates [22], we obtain

\[ |\nabla (g_p(t)) R_{g_p(t)}|(x) \leq C_1', \quad \forall \ x \in B(p, \frac{1}{2\sqrt{R_{\text{max}}}}; g_p(-1)), \quad t \in [-\frac{1}{2},0]. \]

It follows that

\[ |\nabla R|(x) \leq C'_1 R^{3/2}(p), \quad \forall \ x \in B(p, \frac{1}{2\sqrt{R_{\text{max}}}}; g_p(-1)). \]

In particular, we have

\[ |\nabla R|(p) \leq C'_1 R^{3/2}(p), \quad \text{as } \rho(p) \geq r_0. \]

The lemma is proved.

**Remark 2.4.** From the argument in the proof of Lemma 2.3, we can further prove that there exists a constant \( C(k) \) for each \( k \in \mathbb{N} \) such that

\[ \frac{|\nabla^k Rm|(p)}{R^{\frac{k+2}{2}}(p)} \leq C(k), \quad \forall \ p \in M. \]

In Proposition 4.3 of [8], the authors have obtained a lower decay estimate for steady Kähler-Ricci solitons with nonnegative bisectional curvature and positive Ricci curvature. Our proof essentially depends on the Harnack inequality and the existence of equilibrium point\(^2\). Thus the argument there

\(^2\)The existence of equilibrium points is proved for steady Kähler-Ricci solitons in [7].
Higher dimensional steady Ricci solitons

still works for steady Ricci solitons with nonnegative curvature operator and positive Ricci curvature if the soliton admits an equilibrium point. Namely, we have

**Theorem 2.5.** Let \((M, g)\) be a \(\kappa\)-noncollapsed steady Ricci soliton with nonnegative curvature operator and positive Ricci curvature. Suppose that \((M, g)\) has an equilibrium point. Then scalar curvature of \(g\) satisfies (1.2).

3. DIAMETER ESTIMATE OF LEVEL SETS

By Lemma 2.1 there is a one parameter group of diffeomorphisms \(F_r : \mathbb{S}^{n-1} \to \Sigma_r \subseteq M \ (r \geq r_0)\), which is generated by flow

\[
\frac{\partial F_r}{\partial r} = \frac{\nabla f}{|\nabla f|^2}.
\]

Let \(h_r = F_r^*(g)\) and \(e_i = F_r^*(\bar{e}_i), e_j = F_r^*(\bar{e}_j)\), where \(\bar{e}_i, \bar{e}_j \in T\mathbb{S}^{n-1}\). Then,

\[
\frac{\partial h_r}{\partial r} (\bar{e}_i, \bar{e}_j) = L \frac{\nabla f}{|\nabla f|^2} g(e_i, e_j)
\]

\[
= \langle \nabla_{\bar{e}_i} \left( \frac{\nabla f}{|\nabla f|^2} \right), e_j \rangle + \langle \nabla_{\bar{e}_j} \left( \frac{\nabla f}{|\nabla f|^2} \right), e_i \rangle
\]

\[
= \langle \frac{\nabla e_i \nabla f}{|\nabla f|^2}, e_j \rangle + \langle \nabla f, e_j \rangle \nabla e_i \left( \frac{1}{|\nabla f|^2} \right)
\]

\[
+ \langle \frac{\nabla e_j \nabla f}{|\nabla f|^2}, e_i \rangle + \langle \nabla f, e_i \rangle \nabla e_j \left( \frac{1}{|\nabla f|^2} \right)
\]

\[
= 2 \frac{\nabla f}{|\nabla f|^2} \text{Ric}(e_i, e_j).
\]

(3.1) is like the Ricci flow for metrics \(h_r\) since \(|\nabla f|^2(x)\) goes to a constant as \(\rho(x) \to \infty\) under the condition (1.3). In this section, we use the argument in [21] Lemma 8.3 (b) to study the distance functions of \(h_r\) in order to get an estimate of the diameter of \((\mathbb{S}^{n-1}, h_r)\), i.e., the diameter of \((\Sigma_r, g)\). First, we prove

**Lemma 3.1.** Let \((M, g, f)\) be a steady Ricci soliton as in Theorem 1.4 with positive Ricci curvature. Then for \(x_1, x_2 \in \mathbb{S}^{n-1}\) with \(d_r(x_1, x_2) \geq 2\tau_0\), we have

\[
\frac{d}{dr} d_r(x_1, x_2) \leq C \left( \frac{\tau_0}{r} + \frac{1}{\tau_0} + \frac{d_r(x_1, x_2)}{r^{3/2}} \right),
\]

where \(d_r(\cdot, \cdot)\) is the distance function of \((\mathbb{S}^{n-1}, h_r)\).

**Proof.** Let \(\gamma\) be a normalized minimal geodesic from \(x_1\) to \(x_2\) with velocity field \(X(s) = \frac{d\gamma}{ds}\) and \(V\) any piecewise smooth normal vector field along \(\gamma\)
Higher dimensional steady Ricci solitons

which vanishes at the endpoints. By the second variation formula, we have

\begin{equation}
\int_0^{d_r(x_1,x_2)} \left( |\nabla_X V|^2 + \langle \bar{R}(V,V)X \rangle \right) ds \geq 0.
\end{equation}

(3.3)

Let \( \{e_i(s)\}_{i=1}^{n-1} \) be a parallel orthonormal frame along \( \gamma \) that is perpendicular to \( X \). Put \( V_i(s) = f(s)e_i(s) \), where \( f(s) \) is defined as

\[
\begin{align*}
    f(s) &= \frac{s}{\tau_0}, \text{ if } 0 \leq s \leq \tau_0; \\
    f(s) &= 1, \text{ if } \tau_0 \leq s \leq d_r(x_1, x_2) - \tau_0; \\
    f(s) &= \frac{d_r(x_1, x_2) - s}{\tau_0}, \text{ if } d_r(x_1, x_2) - \tau_0 \leq s \leq d_r(x_1, x_2).
\end{align*}
\]

Then \( |\nabla_X V_i| = |f'(s)| \) and

\[
\int_0^{d_r(x_1,x_2)} |\nabla_X V_i|^2 ds = 2 \int_0^{\tau_0} \frac{1}{\tau_0} ds = \frac{2}{\tau_0}.
\]

Moreover

\[
\begin{align*}
    \int_0^{d_r(x_1,x_2)} & \langle \bar{R}(V_i,V_i)X \rangle ds \\
    &= \int_0^{\tau_0} s^2 \frac{1}{\tau_0} \langle \bar{R}(e_i, X)e_i, X \rangle ds + \int_0^{d_r(x_1,x_2) - \tau_0} \langle \bar{R}(e_i, X)e_i, X \rangle ds \\
    &+ \int_{d_r(x_1,x_2) - \tau_0}^{d_r(x_1,x_2)} \frac{(d_r(x_1, x_2) - s)^2}{\tau_0^2} \langle \bar{R}(e_i, X)e_i, X \rangle ds.
\end{align*}
\]

Thus

\[
0 \leq \sum_{i=1}^{n-1} \int_0^{d_r(x_1,x_2)} \left( |\nabla_X V_i|^2 + \langle \bar{R}(V_i,V_i)X \rangle \right) ds
\]

\[
= \frac{2(n-1)}{r_0} - \int_0^{d_r(x_1,x_2)} \text{Ric}(X,X) ds + \int_0^{\tau_0} \left( 1 - \frac{s^2}{\tau_0^2} \right) \text{Ric}(X,X) ds
\]

\[
+ \int_{d_r(x_1,x_2) - \tau_0}^{d_r(x_1,x_2)} \left( 1 - \frac{(d_r(x_1, x_2) - s)^2}{\tau_0^2} \right) \text{Ric}(X,X) ds.
\]

(3.4)

We claim

\[
\text{Ric}(X,X) \leq C \frac{\bar{g}(X,X)}{r}, \forall x \in \Sigma_r.
\]

(3.5)

By Gauss formula, we have

\[
\text{Rm}(X,Y,Z,W) = \text{Rm}(X,Y,Z,W) \\
+ \langle B(X,Z), B(Y,W) \rangle - \langle B(X,W), B(Y,Z) \rangle,
\]

where \( B(X,Y,Z,W) = \text{Rm}(X,Y,Z,W) \).
Higher dimensional steady Ricci solitons

where $X, Y, Z, W \in T \Sigma_r$ and $B(X, Y) = (\nabla_X Y)^\perp$. Since

$$B(X, Y) = \nabla_X Y \cdot \frac{\nabla f}{|\nabla f|^2} = \left[\nabla_X (Y, \nabla f) - (Y, \nabla_X \nabla f)\right] \cdot \frac{\nabla f}{|\nabla f|^2} = -\text{Ric}(X, Y) \cdot \frac{\nabla f}{|\nabla f|^2},$$

we get

$$Rm(X, Y, Z, W) = Rm(X, Y, Z, W) + \frac{1}{|\nabla f|^2} (\text{Ric}(X, Z)\text{Ric}(Y, W) - \text{Ric}(X, W)\text{Ric}(Y, Z))$$

(3.6)

and

$$R_{ij} = \overline{R}_{ij} + R \left( \frac{\nabla f}{|\nabla f|}, e_i, e_j, \frac{\nabla f}{|\nabla f|} \right) - \frac{1}{|\nabla f|^2} \sum_k (R_{i,j,k,k} - R_{i,k,k,j}),$$

where indices $i, j, k$ are corresponding to vector fields on $T \Sigma_r$. Thus for a unit vector $Y$, we derive

$$(\text{Ric} - \overline{\text{Ric}})(Y, Y) = R \left( \frac{\nabla f}{|\nabla f|}, Y, Y, \frac{\nabla f}{|\nabla f|} \right) - \frac{1}{|\nabla f|^2} \sum_{i=1}^{n-1} \left[ \text{Ric}(Y, Y)\text{Ric}(e_i, e_i) - \text{Ric}^2(Y, e_i) \right].$$

Note that

$$R \left( \frac{\nabla f}{|\nabla f|}, Y, Y, \frac{\nabla f}{|\nabla f|} \right) \leq \text{Ric} \left( \frac{\nabla f}{|\nabla f|}, \frac{\nabla f}{|\nabla f|} \right) = \frac{|(\nabla R, \nabla f)|}{|\nabla f|^2} \leq \frac{|\nabla R|}{|\nabla f|} \leq \frac{C}{r^{3/2}},$$

and

$$\frac{1}{|\nabla f|^2} \sum_{i=1}^{n-1} |\text{Ric}(Y, Y)\text{Ric}(e_i, e_i) - \text{Ric}^2(Y, e_i)| \leq \frac{R^2 + |\text{Ric}|^2}{|\nabla f|^2} \leq \frac{C}{r^2}.$$

Hence, we obtain

$$|(\text{Ric} - \overline{\text{Ric}})(Y, Y)| \leq \frac{C}{r^{3/2}}, \quad r \geq r_0.$$  

(3.7)

In particular,

$$\overline{\text{Ric}}(Y, Y) \leq \frac{C_1}{r}, \quad r \geq r_0.$$  

This proves (3.5).
By (3.4) and (3.5), it is easy to see
\[
\int_0^{d_r(x_1,x_2)} \text{Ric}(Y, Y) \, ds \leq \frac{2(n-1)}{\tau_0} + \frac{4C\tau_0}{3r}.
\]
Also by (3.7), we see
\[
\int_0^{d_r(x_1,x_2)} (\text{Ric} - \text{Ric})(Y, Y) \, ds \leq \frac{C}{r^{3/2}} d_r(x_1,x_2).
\]
On the other hand, if we let \(Y(s) = (F_r)_*(X(s))\) with \(|Y(s)|_{(\Sigma_r, \bar{g})} \equiv 1\), then by (3.1), we have
\[
\frac{d}{dr} d_r(x_1, x_2) = \frac{1}{2} \int_0^{d_r(x_1,x_2)} (L \nabla f, g)(Y,Y) \, ds
\]
\[
= \frac{1}{|\nabla f|^2} \int_0^{d_r(x_1,x_2)} \text{Ric}(Y, Y) \, ds + \frac{1}{|\nabla f|^2} \int_0^{d_r(x_1,x_2)} (\text{Ric} - \text{Ric})(Y, Y) \, ds.
\]
Thus inserting (3.8) and (3.9) into the above relation, we obtain
\[
\frac{d}{dr} d_r(x_1, x_2) \leq C\left(\frac{\tau_0}{r} + \frac{1}{\tau_0} + \frac{d_r(x_1, x_2)}{r^{3/2}}\right).
\]

\[\square\]

**Corollary 3.2.** Let \(d_r(\cdot, \cdot)\) be the distance function as in Lemma 3.1. Then for any \(x_1, x_2 \in S^{n-1}\), we have
\[
\frac{d}{dr} d_r(x_1, x_2) \leq C\left(\frac{2}{\sqrt{r}} + \frac{d_r(x_1, x_2)}{r^{3/2}}\right), \quad r \geq r_0.
\]

**Proof.** If \(d_r(x_1, x_2) \geq 2\sqrt{r}\), the corollary follows from Lemma 3.1 by taking \(\tau_0 = \sqrt{r}\). If \(d_r(x_1, x_2) < 2\sqrt{r}\), we have
\[
\frac{d}{dr} d_r(x_1, x_2) = \frac{1}{2} \int_0^{d_r(x_1,x_2)} (L \nabla f, g)(Y,Y) \, ds
\]
\[
= \frac{1}{|\nabla f|^2} \int_0^{d_r(x_1,x_2)} \text{Ric}(Y, Y) \, ds
\]
\[
\leq \frac{C}{r} d_r(x_1, x_2) \leq \frac{2C}{\sqrt{r}}.
\]
The corollary is proved. \[\square\]

By Corollary 3.2, we get the following diameter estimate for \((\Sigma_r, \bar{g})\).

**Proposition 3.3.** Let \((M, g, f)\) be a steady Ricci soliton as in Theorem 1.4 with positive Ricci curvature. Then there exists a constant \(C\) independent of \(r\) such that
\[
\text{diam}(\Sigma_r, g) \leq C\sqrt{r}, \quad \forall \ r \geq r_0.
\]
Higher dimensional steady Ricci solitons

Proof. For any fixed $x_1, x_2 \in S^{n-1}$, by Corollary 3.2 we have

$$d_r(x_1, x_2) \leq e^{\int_0^r \tau^{-3/2} d\tau} (d_{r_0}(x_1, x_2) + \int_0^r \frac{2C}{\sqrt{\tau}} e^{-\int_0^\tau s^{-3/2} ds} d\tau)$$

$$\leq 4C(\sqrt{r} - \sqrt{r_0}) + \frac{2}{\sqrt{r_0}} \cdot \text{diam}(S^{n-1}, h_{r_0}).$$

Thus

$$\text{diam}(\Sigma_r, g) = \text{diam}(S^{n-1}, h_r) \leq 4C' \sqrt{r}, \quad r \geq r_0.$$

□

As a corollary of Proposition 3.3, we get

Corollary 3.4. Let $(M, g, f)$ be a steady Ricci soliton as in Theorem 1.4 with positive Ricci curvature. Then there exists a uniform constant $C_0 > 0$ such that the following is true: for any $k \in \mathbb{N}$, there exists $\bar{r}_0 = \bar{r}_0(k)$ such that

$$(3.12) \quad M_{p,k} \subset B(p, C_0 + 2k \sqrt{R_{\max}}; R(p)g), \quad \forall \rho(p) \geq \bar{r}_0.$$

Proof. By Proposition 3.3 and (1.3), it is easy to see that

$$\Sigma_{f(p)} \subset B(p, C_0; R(p)g)$$

for some uniform constant $C_0$ as long as $f(p)$ is large enough. Note that by (2.2),

$$\frac{R_{\max}}{2} \leq |\nabla f|^2(x) \leq R_{\max}, \quad \forall x \in M_{p,k}, \quad \rho(p) \geq r_0.$$

Since there exists $q' \in \Sigma_{f(p)}$ for any $q \in M_{p,k}$ such that $\phi_s(q) = q'$ for some $s \in \mathbb{R}$, we derive

$$d(p, q) \leq d(p, q') + d(q', q)$$

$$\leq \text{diam}(\Sigma_{f(p)}, g) + \mathcal{L}(\phi_r|_{[0,s]})$$

$$\leq C_0 R^{-\frac{1}{2}}(p) + \int_0^s |\frac{d\phi_r(q)}{d\tau}| d\tau$$

$$= C_0 R^{-\frac{1}{2}}(p) + \int_0^s |\sqrt{\frac{\nabla f(\phi_r(q))}{R_{\max}}}| d\tau$$

$$\leq C_0 R^{-\frac{1}{2}}(p) + \int_0^s |\nabla f(\phi_r(q))|^2 \cdot \frac{2}{\sqrt{R_{\max}}} d\tau$$

$$= C_0 R^{-\frac{1}{2}}(p) + \int_0^s \frac{d(f(\phi_r(q)))}{d\tau} \cdot \frac{2}{\sqrt{R_{\max}}} d\tau.$$
Higher dimensional steady Ricci solitons

\[ \leq C_0 R^{-\frac{1}{2}}(p) + |f(q) - f(p)| \cdot \frac{2}{\sqrt{R_{\text{max}}}} \]

\[ \leq \left( C_0 + \frac{2k}{\sqrt{R_{\text{max}}}} \right) \cdot \frac{1}{\sqrt{R(p)}}. \]

This implies

\[ M_{p,k} \subset B(p, C_0 + \frac{2k}{\sqrt{R_{\text{max}}}}, R(p)g). \]

□

Corollary 3.4 will be used in the next Section.

4. Proof of Theorem 1.4-II

In this section, we prove Theorem 1.4 for steady Ricci solitons with positive Ricci curvature as follows.

**Theorem 4.1.** Let \((M, g, f)\) be a \(\kappa\)-noncollapsed steady Ricci soliton with nonnegative sectional curvature and positive Ricci curvature. Suppose that (1.3) is satisfied. Then for any \(p_i \to +\infty\), rescaled flows \((M, R(p_i)g(R^{-1}(p_i)t), p_i)\) converge subsequently to \((\mathbb{R} \times S^{n-1}, ds^2 + g_{S^{n-1}}(t)) (t \in (-\infty, 0])\) in the Cheeger-Gromov topology, where \((S^{n-1}, g_{S^{n-1}}(t))\) is a \(\kappa\)-noncollapsed ancient Ricci flow with nonnegative sectional curvature. Moreover, scalar curvature of \(g_{S^{n-1}}(t)\) satisfies (1.4).

We need several lemmas to prepare for the proof of Theorem 4.1. First we give a volume comparison for level sets \(\Sigma_r\).

**Lemma 4.2.** Under the condition of Theorem 4.1, for any small \(\varepsilon > 0\), there is a \(r_0 > 0\) such that for any \(s \in [-1, 1]\),

\[ 1 - \varepsilon \leq \frac{\text{vol}(\Sigma_{r+s\sqrt{r}}, g)}{\text{vol}(\Sigma_r, g)} \leq 1 + \varepsilon, \quad r \geq r_0. \]

**Proof.** Let \(V \in TS^{n-1}\) be any fixed nonzero vector. By (3.1), we have

\[ |\frac{\partial}{\partial r} h_r(V, V)| = \frac{2}{|V|^2} |\text{Ric}((F_r)_* V, (F_r)_* V)| \leq \frac{C}{r} h_r(V, V). \]

Thus

\[ \frac{r_1^C}{r_2^C} \leq \frac{h_{r_2}(V, V)}{h_{r_1}(V, V)} \leq \frac{r_2^C}{r_1^C}, \quad \forall 0 \leq r_1 \leq r_2. \]

It follows

\[ \frac{h_{r+s\sqrt{r}}(V, V)}{h_r(V, V)} \to 1, \quad \text{as} \quad r \to \infty. \]
Higher dimensional steady Ricci solitons

and

\[
\frac{\det(h + s\sqrt{r})}{\det(h)} \to 1, \quad \text{as } r \to \infty.
\]

Hence (4.1) follows. \qed

Let \( g_p = R(p)g \) be a rescaled metric of \((M, g)\). Then

**Lemma 4.3.** Under the condition of Theorem 4.1, there exists a constant \( C(\kappa) \) such that

\[
(4.2) \quad \text{vol}(\Sigma_{f(p)}, g_p) \geq C(\kappa), \quad \text{if } f(p) \geq r_0.
\]

**Proof.** We define a set \( M_r(s) \) by

\[
M_r(s) = \{ x \in M | r - s\sqrt{r} \leq f(x) \leq r + s\sqrt{r} \}.
\]

By the uniform curvature decay of \( R(x) \), it is easy to see that there is a constant \( c > 0 \) such that

\[
M_{p,cs} \subseteq M_{f(p)}(s), \quad \forall s \in [-1, 1],
\]

as long as \( f(p) \) is large enough. Then by Lemma 2.2, we see

\[
B(p, R_{\max}^{-\frac{1}{2}}; g_p) \subseteq M_{p,1} \subseteq M_{f(p)}(c^{-1}).
\]

Note that

\[
R_{g_p}(x) = \frac{R(x)}{R(p)} \leq C, \quad \forall x \in M_{f(p)}(c^{-1}).
\]

Thus

\[
R_{g_p}(x) \leq C, \quad \text{in } B(p, R_{\max}^{-\frac{1}{2}}; g_p).
\]

Since \((M, g_p)\) is \( \kappa \)-noncollapsed, we get

\[
\text{vol}(M_{f(p)}(c^{-1}), g_p) \geq \text{vol}(B(p, R_{\max}^{-\frac{1}{2}}; g_p)) \geq c(\kappa).
\]

On the other hand, by the co-area formula and (4.1), we have

\[
\text{vol}(M_{f(p)}(c^{-1}), g_p) = R^\frac{\alpha}{2}(p)\text{vol}(M_{f(p)}(c^{-1}), g)
\]

\[
= R^\frac{\alpha}{2}(p) \int_{f(p) - c^{-1}\sqrt{f(p)}}^{f(p) + c^{-1}\sqrt{f(p)}} \frac{1}{|\nabla f|} \text{vol}(S^{n-1}, h_r)dr
\]

\[
\leq R^\frac{\alpha}{2}(p) \frac{4}{c\sqrt{R_{\max}}} \sqrt{f(p)} \text{vol}(S^{n-1}, h_{f(p)})
\]

\[
\leq C \cdot \text{vol}(\Sigma_{f(p)}, g_p).
\]

Hence, (4.2) follows from the above inequalities. \qed
Lemma 4.4. Let $\bar{D} \in \otimes_{i=1}^{k} T^*\Sigma_r$ be a $k$-multiple tensor on $\Sigma_r$ and $D \in \otimes_{i=1}^{k} T^* M$ a $k$-multiple tensor on $M$, respectively. Then, under the condition of Theorem 4.1, we have

\begin{equation}
(\bar{\nabla} \bar{D})(e_{i_0}, e_{i_1}, \cdots, e_{i_k}) = (\nabla D)(e_{i_0}, e_{i_1}, \cdots, e_{i_k}) \\
+ \sum_{s=1}^{k} D(e_{i_1}, \cdots, e_{i_{s-1}}, \bar{\nabla}_{e_{i_0}} e_{i_s}, e_{i_{s+1}}, \cdots, e_{i_k}) \text{Ric}(e_{i_0}, e_{i_s}).
\end{equation}

Proof. Let $e_{i_1}, e_{i_2}, \cdots, e_{i_k}$ be unit vector fields which are tangent to $\Sigma_r$. Let $\nabla$ and $\bar{\nabla}$ be the Levi-Civita connections $M$ and $\Sigma_r$, respectively. Then

\begin{align*}
(\bar{\nabla} \bar{D})(e_{i_0}, \cdots, e_{i_k}) &= e_{i_0} [\bar{D}(e_{i_1}, \cdots, e_{i_k})] \\
&\quad + \sum_{s=1}^{k} \bar{D}(e_{i_1}, \cdots, e_{i_{s-1}}, \bar{\nabla}_{e_{i_0}} e_{i_s}, e_{i_{s+1}}, \cdots, e_{i_k}).
\end{align*}

and

\begin{align*}
(\nabla D)(e_{i_0}, \cdots, e_{i_k}) &= e_{i_0} [D(e_{i_1}, \cdots, e_{i_k})] \\
&\quad + \sum_{s=1}^{k} D(e_{i_1}, \cdots, e_{i_{s-1}}, \nabla_{e_{i_0}} e_{i_s}, e_{i_{s+1}}, \cdots, e_{i_k}).
\end{align*}

Note

\[
\nabla_{e_{i_0}} e_{i_s} - \bar{\nabla}_{e_{i_0}} e_{i_s} = \langle \nabla_{e_{i_0}} e_{i_s}, \frac{\nabla f}{|\nabla f|} \rangle \frac{\nabla f}{|\nabla f|}
\]

\[
= -\langle e_{i_s}, \nabla_{e_{i_0}} \nabla f \rangle \frac{\nabla f}{|\nabla f|^2}
\]

\[
= -\text{Ric}(e_{i_0}, e_{i_s}) \frac{\nabla f}{|\nabla f|^2}.
\]

Combining the identities above, we get \(4.5\). \hfill \Box

Proposition 4.5. Under the condition of Theorem 4.1, for any $p_i \to \infty$, $(\Sigma_{f(p_i)}, \bar{g}_{p_i}, p_i)$ converges subsequently to $(S_{\infty}, h_{\infty}, p_{\infty})$ in Cheeger-Gromov sense as $i \to \infty$. Here $\bar{g}_{p_i} = R(p_i)\bar{g}$ and $(\Sigma_{f(p_i)}, \bar{g})$ is as a hypersurface of $(M, g)$ with induced metric $\bar{g}$. Moreover, $S_{\infty}$ is diffeomorphic to $S^{n-1}$.

Proof. By (3.6), we have

\[
Rm(X, Y, Z, W) = \bar{Rm}(X, Y, Z, W)
\]

\[
+ \frac{1}{R_{\text{max}} - R} (\text{Ric}(X, Z)\text{Ric}(Y, W) - \text{Ric}(X, W)\text{Ric}(Y, Z)).
\]

Let

\[D^{(0)} = Rm - \frac{1}{R_{\text{max}} - R} \text{Ric} \wedge \text{Ric}.\]
be a $(0,4)$-type tensor on $M$. Then $D^{(0)}|_{\Sigma_f(p_i)} = Rm$. Note that by Remark 2.4 we have

$$|\nabla^k Rm|(x) = \frac{1}{R^\frac{k+2}{2}(p_i)} \cdot \frac{R^\frac{k+2}{2}(x)}{R^\frac{k+2}{2}(p_i)} \leq C(k), \quad \forall \ x \in \Sigma_f(p_i).$$

Since

$$\nabla(\frac{1}{R_{\max} - R}) = \frac{\nabla R}{(R - R_{\max})^2},$$

by induction on $m$, we get

$$(4.6) \quad |\nabla^m D^{(0)}|(x) \leq C(m) R^\frac{m+2}{2}(p_i), \quad \forall \ x \in \Sigma_f(p_i).$$

Let

$$D^{(k)} = \nabla D^{(k-1)}$$

$$+ \sum_{s=1}^{k+4} D^{(k-1)}(e_{i_1}, \ldots, e_{i_{s-1}}, e_{i_{s+1}}, \ldots, e_{i_{k+4}}) \frac{\nabla f}{|\nabla f|^2} \cdot \text{Ric}(e_{i_0}, e_{i_s}).$$

Then by Lemma 4.4 we have

$$\nabla^k Rm = D^{(k)}|_{\Sigma_f(p_i)}.$$

On the other hand, by induction on $k$ with the help of $(4.6)$, we get

$$|\nabla^m D^{(k)}|(x) \leq C(m, k) R^\frac{m+k+2}{2}(p_i), \quad \forall \ x \in \Sigma_f(p_i).$$

In particular,

$$|D^{(k)}|(x) \leq C(k) R^\frac{k+2}{2}(p_i), \quad \forall \ x \in \Sigma_f(p_i).$$

Thus

$$(4.7) \quad |\nabla^k Rm|_{\tilde{g}_{p_i}}(x) \leq \frac{|D^{(k)}|(x)}{R^\frac{k+2}{2}(p_i)} \leq C(k), \quad \forall \ x \in \Sigma_f(p_i).$$

By Lemma 4.3 and Theorem 3.3, respectively, we have

$$\text{vol}(\Sigma_f(p_i), \tilde{g}_{p_i}) \geq C(\kappa)$$

and

$$\text{diam}(\Sigma_f(p_i), \tilde{g}_{p_i}) \leq C.$$

Then by Cheeger-Gromov compactness theorem together with $(4.7)$, we see that $(\Sigma_f(p_i), \tilde{g}_{p_i}, p_i)$ converges subsequently to $(S_\infty, h_\infty, p_\infty)$. Note that $\Sigma_f(p_i)$ are all diffeomorphic to $S^{n-1}$. Therefore, $S_\infty$ is also diffeomorphic to $S^{n-1}$. \qed
4.1. Proof of Theorem 4.1. We are now in a position to prove Theorem 4.1. The proof consists of the following three lemmas. First, by the arguments in [8, 10], we prove

**Lemma 4.6.** Under the condition of Theorem 4.1, let \( p_i \to \infty \). Then by taking a subsequence of \( p_i \) if necessary, we have

\[
(M, g_{p_i}(t), p_i) \to (\mathbb{R} \times N, g_\infty(t); p_\infty), \quad \text{for } t \in (-\infty, 0],
\]

where \( g_{p_i}(t) = R(p_i)g(R^{-1}(p_i)t), \) \( g_\infty(t) = ds \otimes ds + g_N(t) \) and \( (N, g_N(t)) \) is an ancient solution of Ricci flow on \( N \).

**Proof.** Fix \( \bar{r} > 0 \). By (1.3), it is easy to see that there exists a uniform \( C_1 \) independent of \( \bar{r} \) such that

\[
R(x) \leq C_1 R(p_i), \quad \forall \, x \in M, \quad \forall \, x \in B(p_i, \bar{r} \sqrt{R_{\max}})
\]
as long as \( i \) is large enough. Then by Lemma 2.3, we have

\[
R_{g_{p_i}}(x) \leq C_1, \quad \forall \, x \in B(p_i, \bar{r}; g_{p_i}),
\]

where \( g_{p_i} = g_{p_i}(0) \). Since the scalar curvature is increasing along the flow (cf. (2.8)) and the sectional curvature is nonnegative, for any \( t \in (-\infty, 0] \), we get

\[
|Rm_{g_{p_i}(t)}(x)|_{g_{p_i}(t)} \leq C(n) R_{g_{p_i}(t)}(x) \leq C(n) R_{g_{p_i}}(x) \leq C(n) C_1, \quad \forall \, x \in B(p_i, \bar{r}; g_{p_i}).
\]

Note that \((M, g(t))\) is \( \kappa \)-noncollapsed. Hence \( g_{p_i}(t) \) converges subsequently to a limit flow \( (M_\infty, g_\infty(t); p_\infty) \) for \( t \in (-\infty, 0] \). Moreover, the limit flow has uniformly bounded curvature. It remains to prove the splitting property.

Let \( X(i) = R(p_i)^{-\frac{1}{2}} \nabla f \). Then

\[
\sup_{B(p_i, \bar{r}; g_{p_i})} |\nabla(g_{p_i}) X(i)|_{g_{p_i}} = \sup_{B(p_i, \bar{r}; g_{p_i})} \frac{|\nabla|}{\sqrt{R(p_i)}} \leq C \sqrt{R(p_i)} \to 0.
\]

By Remark 2.3, it follows that

\[
\sup_{B(p_i, \bar{r}; g_{p_i})} |\nabla^m(g_{p_i}) X(i)|_{g_{p_i}} \leq C(n) \sup_{B(p_i, \bar{r}; g_{p_i})} |\nabla^{m-1}(g_{p_i}) \text{Ric}(g_{p_i})|_{g_{p_i}} \leq C_1.
\]

Thus \( X(i) \) converges subsequently to a parallel vector field \( X(\infty) \) on \( (M_\infty, g_\infty(0)) \). Moreover,

\[
|X(i)|_{g_{p_i}}(x) = |\nabla f|(p_i) = \sqrt{R_{\max}} + o(1) > 0, \quad \forall \, x \in B(p_i, \bar{r}; g_{p_i}),
\]

Higher dimensional steady Ricci solitons
Higher dimensional steady Ricci solitons

as long as \( f(p_i) \) is large enough. This implies that \( X(\infty) \) is non-trivial. Hence, \((M_\infty, g_\infty(t))\) locally splits off a piece of line along \( X(\infty) \). In the following, we show that \( X(\infty) \) generates a line through \( p_\infty \).

By Corollary 3.4,

\[
M_{p_i, k} \subset B(p_i, C_0 + \frac{2k}{\sqrt{R_{\max}}}; g_{p_i}(0)), \ \forall \ p_i \to \infty.
\]

Let \( \gamma_{i,k}(s), s \in (-D_{i,k}, E_{i,k}) \) be an integral curve generated by \( X(i) \) through \( p_i \), which restricted in \( M_{p,k} \), i.e., \( \gamma_{i,k}(s) \) satisfies \( f(\gamma_{i,k}(-D_{i,k})) = f(p_i) - \frac{k}{\sqrt{R(p_i)}} \) and \( f(\gamma_{i,k}(E_{i,k})) = f(p_i) + \frac{k}{\sqrt{R(p_i)}} \). Then \( \gamma_{i,k}(s) \) converges to a geodesic \( \gamma_\infty(s) \) generated by \( X(\infty) \) through \( p_\infty \), which restricted in \( B(p_\infty, 2\pi \sqrt{B} + \frac{2k}{\sqrt{R_{\max}}}; g_\infty(0)) \). If we let \( L_{i,k} \) be lengths of \( \gamma_{i,k}(s) \) and \( L_{\infty,k} \) length of \( \gamma_\infty(s) \), respectively,

\[
L_{i,k} = \int_{-D_{i,k}}^{E_{i,k}} |X(i)|_{g_{p_i}(0)} ds
\]

\[
= \int f(p_i) + \frac{k}{\sqrt{R(p_i)}} |X(i)|_{g_{p_i}(0)} df \frac{\langle \nabla f, X(i) \rangle}{\sqrt{R(p_i)}}
\]

\[
= \int f(p_i) + \frac{k}{\sqrt{R(p_i)}} \sqrt{R(p_i)} \frac{\langle \nabla f \rangle}{\sqrt{R(p_i)}}
\]

\[
\geq 2R_{\max}^{-\frac{1}{2}} k,
\]

and so,

\[
L_{\infty,k} \geq \frac{1}{2} L_{i,k} \geq R_{\max}^{-\frac{1}{2}} k.
\]

Thus \( X(\infty) \) generates a line \( \gamma_\infty(s) \) through \( p_\infty \) as \( k \to \infty \). As a consequence, \((M_\infty, g_\infty(0))\) splits off a line and so does the flow \((M_\infty, g_\infty(t); p_\infty)\). The lemma is proved. 

\]
Higher dimensional steady Ricci solitons

and \((\Phi_i^{-1})(B(p_{ik}, k; R(p_{ik})g))\) exhausts \(M_\infty\) as \(k \to \infty\). For any \(q \in B(p_{ik}, k; R(p_{ik})g)\), there exists a minimal geodesic \(\gamma(s) : [0, l] \to M\) such that \(\gamma(0) = p_{ik}, \gamma(l) = q\). Note \(\gamma|_{[0, l]} \subseteq B(p_{ik}, k; R(p_{ik})g)\). It follows that \((\Phi_i^{-1})(B(p_{ik}, k; R(p_{ik})g))\) is connected for each \(k\). Therefore, \(M_\infty\) is connected, and so is \(N\).

By Proposition 4.5 and Lemma 4.6, we may assume that \((\Sigma_{f(p_i)}, g_{p_1}, p_i)\) converge to a limit \((S^{n-1}, h_\infty, p_\infty)\) and \(S^{n-1} \subseteq M_\infty = N \times \mathbb{R}\). Then by the above claim, it suffices to prove that \(S^{n-1} \subseteq N \times \{p_\infty\}\). Let \(X_\infty\) and \(X_i\) be the vector fields defined as in the proof of Lemma 4.6. Let \(V \in TS^{n-1}\) with \(|V|_{g_\infty} = 1\). Thus by the convergence in Proposition 4.5, we see that there are \(V_{(i)} \in T\Sigma_{f(p_i)}\) such that \(R(p_i)^{-\frac{1}{2}}V_{(i)} \to V\). It follows

\[
g_\infty(V, X_\infty) = \lim_{i \to \infty} R(p_i)g(R(p_i)^{-\frac{1}{2}}V_{(i)}, X_{(i)}) = \lim_{i \to \infty} g(V_{(i)}, \nabla f) = 0.
\]

This shows that \(V\) is vertical to \(X_\infty\) for any \(V \in TS^{n-1}\). Hence, \(S^{n-1} \subseteq N \times \{p_\infty\}\). Note that \(\text{dim}N = n - 1\). We complete the proof.

\[\square\]

Finally, we verify the condition (1.4). We prove

**Lemma 4.8.** Let \((M_\infty = N \times \mathbb{R}, g_\infty(t))\) be the limit manifold in Lemma 4.6. Then, scalar curvature \(R_\infty(x, t)\) of \(g_\infty(t)\) satisfies

\[
R_\infty(x, t) \leq \frac{C}{|t|} \quad \forall \ t < 0, \ x \in M_\infty.
\]

**Proof.** Let \(\phi_t\) be generated by \(-\nabla f\). Then,

\[
\frac{d|f(\phi_t(p))|}{dt} = -|\nabla f|^2(\phi_t(p))
\]

and

\[
\frac{d|\nabla f|^2(\phi_t(p))}{dt} = -2\Ric(\nabla f, \nabla f)(\phi_t(p)) \leq 0.
\]

It follows that

\[
|\nabla f|^2(\phi_\tau(p)) \leq |\nabla f|^2(\phi_t(p)), \ \forall \ \tau \geq t.
\]

Hence,

\[
f(\phi_t(p)) \geq f(\phi_\tau(p)) + |t - \tau| |\nabla f|^2(\phi_\tau(p)).
\]

By taking \(\tau = 0\), for any \(p \in \{q \in M | f(q) \geq 1\}\), we get

\[
f(\phi_t(p)) \geq f(p) + |t||\nabla f|^2(p) \geq 1 + c|t|,
\]

where \(c = \min_{p \in \Sigma_1} |\nabla f|^2\). On the other hand, by (1.3) and (2.5), we have

\[
R(p) \leq \frac{C}{f(p)}, \ \forall \ p \in \{q \in M | f(q) \geq 1\}.
\]

17
Higher dimensional steady Ricci solitons

Hence,
\[
R(p, t) \leq \frac{C}{f(\phi_t(p))} \leq \frac{C}{1 + c|t|}, \quad \forall p \in \{ q \in M \mid f(q) \geq 1 \}.
\]

(4.10)

Let \( x \in M_\infty \) and \( d_{g_\infty}(0)(x, p_\infty) = \tau \). By the convergence in Lemma 4.6, there are \( x_i \in B(p_i, 2\tau; g_{p_i}(0)) \) such that \( x_i \to x \) as \( i \to \infty \). Moreover,
\[
\lim_{i \to \infty} R_{g_{p_i}(t)}(x_i) = R_\infty(x, t).
\]

Note that \( x_i \in B(p_i, 2\tau; g_{p_i}(0)) \subseteq M_{p_i, 2\tau \sqrt{R_{\text{max}}}} \). It means that
\[
f(x_i) \geq f(p_i) - 2\tau \sqrt{\frac{R_{\text{max}}}{R(p_i)}} \gg 1, \quad \text{as } i \to \infty.
\]

Hence, by (4.10), we derive
\[
R_{g_{p_i}(t)}(x_i) = \frac{R(x_i, R^{-1}(p_i)t)}{R(p_i)} \leq \frac{C}{R(p_i) + c|t|} \leq \frac{2C}{c|t|}, \quad \text{as } i \to \infty.
\]

Let \( i \to \infty \), we get (4.9). \( \square \)

By Lemma 4.6 and Lemma 4.7, (4.9) in Lemma 4.8 implies (1.4). The proof of Theorem 4.1 is completed.

5. PROOF OF THEOREM 1.4-II

In this section, we complete the proof of Theorem 1.4. We need to describe the structure of level set \( \Sigma_r \) of \((M, g, f)\) without assumption of positive Ricci curvature.

Lemma 5.1. Let \((M, g, f)\) be a non-flat steady Ricci soliton with nonnegative sectional curvature. Let \( S = \{ p \in M \mid \nabla f(p) = 0 \} \) be set of equilibrium points of \((M, g, f)\). Suppose that scalar curvature \( R \) of \( g \) decays uniformly. Then the following statements are true.

1. \((S, g_S)\) is a compact flat manifold, where \( g_S \) is an induced metric \( g \).
2. Let \( o \in S \). Then level set \( \Sigma_r = \{ x \in M \mid f(x) = r \} \) is a closed hypersurface of \( M \). Moreover, each \( \Sigma_r \) is diffeomorphic to each other whenever \( r > f(o) \).
3. \( M_r = \{ x \in M \mid f(x) \leq r \} \) is compact for any \( r > f(o) \).
4. \( f \) satisfies (2.5).

Proof. (1) Let \((\tilde{M}, \tilde{g})\) be the universal cover of \((M, g, f)\) with the covering map \( \pi \). Let \( \tilde{f} = f \circ \pi \). It is clear that \((\tilde{M}, \tilde{g}, \tilde{f})\) is also a steady gradient Ricci soliton. Then by [12, Theorem 1.1], there is an \((n - k)\)-dimensional steady gradient Ricci soliton \((N, h, f_N)\) with nonnegative sectional curvature and positive Ricci curvature such that \((\tilde{M}, \tilde{g}) = (N, h) \times \mathbb{R}^k \) \((k \geq 0)\). Let
Higher dimensional steady Ricci solitons

\((q, y')\) be a coordinate system on \(\tilde{M} = N \times \mathbb{R}^k\), where \(q \in N\) and \(y' = (y_1, \cdots, y_k) \in \mathbb{R}^k\). We claim

\[
\frac{\partial \tilde{f}}{\partial y_j} = 0, \quad \forall \ 1 \leq j \leq k.
\]

Fix \(q \in N\) and \(y_i\) with \(i \neq j\). Let

\[
\tilde{f}_j(y) = \tilde{f}(q, y_1, \cdots, y_{j-1}, y, y_{j+1}, \cdots, y_k), \quad \forall \ y \in \mathbb{R}.
\]

By the Ricci soliton equation, we have

\[
\frac{\partial^2 \tilde{f}_j}{\partial y^2} = \tilde{\text{Ric}}\left(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}\right) = 0.
\]

It follows

\[
\tilde{f}_j(y) = c_1 y + c_2,
\]

where \(c_1\) and \(c_2\) are constants. Thus

\[
|\tilde{f}_j(y) - \tilde{f}_j(-y)| = 2|c_1 y|, \quad \forall \ y \in \mathbb{R}.
\]

We define a set

\[
E = \{p \in M \mid p = \pi(q, y_1, \cdots, y_{j-1}, y, y_{j+1}, \cdots, y_k), \ y \in \mathbb{R}\}.
\]

Then for any \(p \in E\), we have

\[
R(p) = \bar{R}(q, y_1, \cdots, y_{j-1}, y, y_{j+1}, \cdots, y_k) = R_h(q) > 0.
\]

Thus

\[
E \subseteq \{p \in M \mid R(p) \leq R_h(q)\} = E'.
\]

Since \(R\) decays uniformly, the set \(E'\) is bounded and so is \(E\). Hence

\[
diam(E, g) = D < +\infty.
\]

Note that \(|\nabla f| \leq \sqrt{R_{\max}}\). Therefore, for any \(p_1, p_2 \in E\), we integrate from \(p_2\) to \(p_1\) along a minimal geodesic to get

\[
f(p_1) - f(p_2) \leq \sqrt{R_{\max}}d(p_1, p_2) \leq D\sqrt{R_{\max}}.
\]

Choose \(p_1 = \pi(q, y_1, \cdots, y_{j-1}, m, y_{j+1}, \cdots, y_k)\) and \(p_2 = \pi(q, y_1, \cdots, y_{j-1}, -m, y_{j+1}, \cdots, y_k)\). By \([5.3]\), we derive

\[
2|c_1 m| = |\tilde{f}_j(m) - \tilde{f}_j(-m)| = |f(p_1) - f(p_2)| \leq D\sqrt{R_{\max}},
\]

As a consequence, \(c_1 = 0\) by taking \(m \to \infty\). This implies \([5.1]\) by \([5.2]\).

By \([5.1]\), we may assume that \(f_N(q) = \tilde{f}(q, \cdot)\). Since \(R(p)\) attains its maximum in \(M\), \(R_h(q)\) attains its maximum at some point \(o_N \in N\). Note
Higher dimensional steady Ricci solitons

that \( \text{Ric}(h) \) is positive. Then by an argument in [10] Corollary 2.2], we see
\( \nabla_h f_N(o_N) = 0. \) Moreover, such a \( o_N \) is unique. Thus
(5.4)
\[
\pi^{-1}(S) = \{ o_N \} \times \mathbb{R}^k.
\]

Since \( S = \{ p \in M \mid R(p) = R_{\text{max}} \} \) is compact by the curvature decay, \((S, g_S)\)
is a compact flat manifold.

(2) Let \( o \in S \). Then by (5.4), we have \( f(S) \equiv f(o) \). By (2.2), it follows
that for any \( r > f(o) \),
\[
|\nabla f|^2(p) = R_{\text{max}} - R(p) > 0, \quad \forall \ p \in \Sigma_r.
\]
Thus \( \Sigma_r \) is a hypersurface of \( M \). In the following, we show that it is bounded.

Choose \( q \in N \) such that \( \pi(q, y) = p \in \Sigma_r \). Then \( f_N(q) = r \). Let \( (\phi_N)_t \) be
a one-parameter diffeomorphisms generated by \( -\nabla_h f_N \). Thus
(5.5)
\[
d_h((\phi_N)_t(q), o_N) \to 0, \quad \text{as } t \to \infty.
\]
Moreover, the above convergence is uniform for all \( q \in \{ x \in N \mid f_N(x) = r \} \).
Similarly, we have
(5.6)
\[
\bar{d}(\bar{\phi}_t(q, y), (o_N, y)) \to 0, \quad \text{as } t \to \infty,
\]
where \( \bar{\phi}_t \) is a one-parameter diffeomorphisms generated by \( -\nabla \bar{f} \) and the convergence of (5.6)
is uniform on \( \{ \bar{x} \in \bar{M} \mid \bar{f}(\bar{x}) = r \} \). Note that \( \pi(\bar{\phi}_t(q, y)) = \phi_t(p) \). Thus
\[
d(\phi_t(p), \pi(o_N, y)) \leq \bar{d}(\bar{\phi}_t(q, y), (o_N, y)) \to 0, \quad \text{as } t \to \infty.
\]
It follows that
(5.7)
\[
d(\phi_t(p), S) \to 0, \quad \text{as } t \to \infty.
\]
Moreover, the convergence is uniform on \( \Sigma_r \).

By (5.7), there is a sufficiently large \( t_0 \) such that
(5.8)
\[
d(\phi_t(\Sigma_r), S) \leq 1, \quad \forall \ t \geq t_0.
\]
Let \( \gamma_p(s) = \phi_s(p), s \in [0, t_0] \). Then,
(5.9)
\[
d(p, \phi_{t_0}(p)) \leq \text{Length}(\gamma_p, g) = \int_0^{t_0} |\nabla f|(\phi_t(p))ds \leq t_0 \sqrt{R_{\text{max}}}.
\]
It follows that
\[
d(\Sigma_r, S) \leq d(\Sigma_r, \phi_{t_0}(\Sigma_r)) + d(\phi_{t_0}(\Sigma_r), S) \leq t_0 \sqrt{R_{\text{max}}} + 1.
\]
Hence \( \Sigma_r \) is bounded since \( S \) is compact. \( \Sigma_{r_1} \) and \( \Sigma_{r_2} \) are diffeomorphic to
each other for all \( r_1, r_2 > f(o) \) by the fact \( |\nabla f|(x) > 0 \) for all \( f(x) > f(o) \).

(3) Since \( \Sigma_r \) is a closed set, it suffices to show that the set \( M_r' = \{ f(o) <
\]
\( f(x) < r \} \) is bounded by the above properties (1) and (2). For any \( x \in M_r' \),
choose a point \( x' \in \Sigma_r \) and a number \( t_x > 0 \) such that \( \phi_{t_x}(x') = x \). If \( t_x \geq t_0, \)

Higher dimensional steady Ricci solitons

then $d(x, S) \leq 1$ by (5.8). If $t_x < t_0$, then $d(x, \Sigma_r) \leq d(x, x') \leq t_0 \sqrt{R_{\text{max}}}$ by (5.9). Thus

\begin{equation}
(5.10) \quad d(x, S) \leq d(x, \Sigma_r) + d(\Sigma_r, S) \leq 2(t_0 \sqrt{R_{\text{max}}} + 1).
\end{equation}

Hence, $M'_r$ is bounded.

(4) Note that $(N, h, f_N)$ has positive Ricci curvature. Then similar to (2.5), we have

$f_N(q) \geq C d_h(q, o_N), \quad \forall \ f_N(q) \geq r_0.$

It follows that

$f(\tilde{x}) \geq C d(\tilde{x}, \{o_N\} \times \mathbb{R}^k), \quad \tilde{f}(\tilde{x}) \geq r_0,$

where $\pi(\tilde{x}) = x$, Thus

$f(x) = \tilde{f}(\tilde{x}) \geq C d(\tilde{x}, \{o_N\} \times \mathbb{R}^k) \geq C d(x, S), \quad \forall \ f(x) \geq r_0,$

As a consequence, we get

$f(x) \geq C \rho(x) - CC_S, \quad \forall \ f(x) \geq r_0,$

where $C_S = \text{diam}(S, g)$. Since \( \{x \in M| \rho(x) \leq k\} \}_{k \in \mathbb{N}} \) exhaust $M$ as $k \to \infty$ and $M_{r_0}$ is compact by the above property (3), there exists a constant $r'_0$ ($\gg r_0$) such that

$M_{r_0} \subset \{x \in M| \rho(x) < r'_0\}.$

Namely,

$\{x \in M| \rho(x) \geq r'_0\} \subset \{x \in M| f(x) \geq r_0\}$

Hence

$f(x) \geq C \rho(x) - CC_S, \quad \forall \ \rho(x) \geq r'_0.$

Therefore, we get the left side of (2.5). The right side follows from the fact

$f(x) \leq f(o) + \sqrt{R_{\text{max}}} \rho(x), \quad \forall \ x \in M.$

The lemma is proved.

\[ \square \]

**Remark 5.2.**
(1) It is possible that $\Sigma$ is empty for steady Ricci solitons with nonnegative sectional curvature. For example, $(\mathbb{R}^n, g_{\text{Euclid}}, f = \Sigma_{i=1}^n x_i)$ is a steady Ricci soliton with $|\nabla f|^2 \equiv n$.

(2) The estimate $\tilde{f}(\tilde{x}) \geq C \rho(\tilde{x}), \ \tilde{f}(\tilde{x}) \geq r_0$ fails on the universal cover $(\tilde{M}, \tilde{g})$ of $(M, g)$ in Lemma 5.1 since $\tilde{R}(x)$ doesn’t decay uniformly.

**Corollary 5.3.** Let $(M, g, f)$ be a non-flat $\kappa$-noncollapsed steady Ricci soliton with nonnegative curvature operator and uniform curvature decay. Then scalar curvature of $g$ satisfies (1.2).
Higher dimensional steady Ricci solitons

Proof. Let \((\overline{M}, \overline{g}, \overline{f})\) be the covering steady Ricci soliton of \((M, g, f)\) as in Lemma 5.1. Then \(\overline{M} = N \times \mathbb{R}^k\) is also \(\kappa\)-noncollapsed, and \((N, h, f_N)\) is a \(\kappa\)-noncollapsed steady gradient Ricci soliton with nonnegative curvature operator and positive Ricci curvature, where \(f_N(q) = \overline{f}(q, \cdot)\). Moreover, \((N, h, f_N)\) admits a unique equilibrium point \(o_N\). Thus by Theorem 2.5, we have

\[ R_N(q)f_N(q) \geq C_0, \quad \forall f_N(q) \geq r_0, \quad q \in N. \]

It follows that

\[ \overline{R}(x)\overline{f}(x) \geq C_0, \quad \forall \widehat{f}(\hat{x}) \geq r_0, \quad \hat{x} \in \overline{M}, \]

and

\[ R(x)f(x) \geq C_0, \quad \forall f(x) \geq r_0, \quad x \in M. \]

Combining the above with (2.5), we get (1.2) immediately. \(\square\)

With the help of (2)-(4) in Lemma 5.1, we can extend the arguments in Section 2-4 to prove a weak version of Theorem 1.4.

Theorem 5.4. Let \((M, g)\) be a \(\kappa\)-noncollapsed steady Ricci soliton with nonnegative sectional curvature. Suppose that \((M, g)\) satisfies (1.3). Then, for any \(p_i \to \infty\), rescaled flows \((M, R(p_i)g(R^{-1}(p_i)t), p_i)\) converge subsequently to \((\mathbb{R} \times \Sigma, ds^2 + g_{\Sigma}(t)) \ (t \in (-\infty, 0])\) in the Cheeger-Gromov topology, where \(\Sigma\) is diffeomorphic to a level set \(\Sigma_{r_0}\) in \((M, g)\) and \((\Sigma, g_{\Sigma}(t))\) is a \(\kappa\)-noncollapsed ancient Ricci flow with nonnegative sectional curvature. Moreover, scalar curvature \(R_{\Sigma}(x, t)\) of \((\Sigma, g_{\Sigma}(t))\) satisfies

\[ R_{\Sigma}(x, t) \leq \frac{C}{|t|}, \quad \forall x \in \Sigma, \]

where \(C\) is a uniform constant.

The proof of Theorem 5.4 is almost the same as one of Theorem 1.1 by replacing \(\mathbb{S}^{n-1}\) with \(\Sigma\). We leave it to the readers.

Proof of Theorem 1.4. It suffices to show that \((M, g)\) has positive Ricci curvature. Otherwise, we assume that the Ricci curvature is not strictly positive. Let \((\overline{M}, \overline{g}, \overline{f})\) be the covering steady Ricci soliton of \((M, g, f)\) as in Lemma 5.1. Then \(\overline{M}\) splits off \(\mathbb{R}^k\) \((k \geq 1)\) as \(N \times \mathbb{R}^k\), where \((N, h)\) has positive Ricci curvature. Let \(\overline{V} \in T\overline{M}\) be a vector field parallel to \(\mathbb{R}^k\) and \(|\overline{V}|g \equiv 1\). Then \(\overline{g}(\overline{V}, \nabla \overline{f}) = 0\) since \(\overline{f}|_{\mathbb{R}^k} \equiv \text{const}\). Let \(\{p_i\}\) be a sequence of points in \(M\) with \(f(p_i) \to \infty\), and \(\{\overline{p}_i\}\) in \(\overline{M}\) with \(\pi(\overline{p}_i) = p_i\). Let \(W_{(i)} = R(p_i)^{-1/2}\pi_*(V(\overline{p}_i)) \in T_{\overline{p}_i}\overline{M}\) and \(X_{(i)} = R(p_i)^{-1/2}\nabla f\). Then

\[ \text{Ric}_{g(t)}(W_{(i)}, W_{(i)}) = 0, \quad |W_{(i)}|_{R(p_i)g} \equiv 1, \quad \langle W_{(i)}, X_{(i)}(p_i) \rangle = 0, \quad \forall i, \quad t \leq 0. \]
Higher dimensional steady Ricci solitons

Since \((M, R(p_i)g(R^{-1}(p_i)t), p_i)\) converges subsequently to \((\mathbb{R} \times \Sigma, ds^2 + g_\Sigma(t), p_\infty)\) by Theorem 5.3, as in the proof of Lemma 4.6 we get \(X(i) \to X(\infty)\), where the limit vector field \(X(\infty)\) is parallel to \(\mathbb{R}\) in \(\mathbb{R} \times \Sigma\). Moreover

\[
W(i) \to W(\infty), \quad \langle W(\infty), X(\infty)(p_\infty) \rangle = 0,
\]
and

\[
|W(\infty)|_{g_\Sigma(0) + ds^2} = 1, \quad \text{Ric}_{g_\Sigma(t) + ds^2}(W(\infty), W(\infty)) = 0, \quad \forall \ t \leq 0.
\]

Thus, \((\Sigma, g_\Sigma(t))\) satisfies \((1.4)\), by Theorem 3.1 in \([20]\), we see that \((\Sigma, \tau_i g_\Sigma(\tau_i^{-1}t), q)\) subsequently converges to a shrinking Ricci soliton \((\Sigma_\infty, g_{\Sigma_\infty}(t), q_\infty)\) for any fixed \(q \in \Sigma\) and any sequence \(\{\tau_i\} \to \infty\). On the other hand, by a result in \([19]\), there exists a constant \(C_1\) such that

\[
\text{diam}(\Sigma, g_\Sigma(t)) \leq C_1 \sqrt{|t|}.
\]

In particular,

\[
|\tau_i^{-1}g_\Sigma(\tau_i) - g_\Sigma(t)| \leq C_1.
\]

Hence, \((\Sigma_\infty, g_{\Sigma_\infty}(t))\) is a compact shrinking Ricci soliton with nonnegative sectional curvature, but not strictly positive Ricci curvature. By \([12\text{ Theorem 1.1}]\), the universal cover of \((\Sigma_\infty, g_{\Sigma_\infty}(t))\) must split off a flat factor \(\mathbb{R}^l\) \((l \geq 1)\). On the other hand, since the fundamental group of any compact shrinking Ricci soliton is finite \([16\text{ Theorem 1}]\) (also see \([17, 5]\)), the universal cover of \(\Sigma_\infty\) should be compact. Therefore, we get a contradiction. The proof is completed.

\[\square\]

6. Proofs of Theorem 1.3 and Theorem 1.5

Proof of Theorem 1.3. We may assume that the steady (gradient) Ricci soliton \((M, g)\) is not flat. By the condition \((1.1)\) in Theorem 1.3 together with Corollary 5.3 we see that Theorem 1.4 is true for a \(\kappa\)-noncollapsed \((M, g)\) with nonnegative curvature operator if \((M, g)\) satisfies \((1.1)\). Then the ancient solution \(g_{\Sigma^{-1}}(t)\) in Theorem 1.4 is in fact a compact \(\kappa\)-solution which satisfies \((1.4)\). By a result of Ni \([19]\), \(g_{\Sigma^{-1}}(t)\) must be a flow of shrinking
Higher dimensional steady Ricci solitons round spheres. This means that \((M, g)\) is asymptotically cylindrical. Hence by Brendle’s result \([3]\), it is rotationally symmetric.

\[\square\]

**Proof of Theorem 1.5.** We note that the ancient solution \(g_{S^{n-1}}(t)\) in Theorem \([4]\) satisfies the condition \([1.4]\). Then, as in the proof of Theorem \([1.4]\), \((S^{n-1}, \tau^{-1}_{i} g_{S^{n-1}}(\tau_{i}t), x)\) subsequently converges to a compact gradient shrinking Ricci soliton \((N', g'(t), x')\) for any fixed \(x \in S^{n-1}\) and any sequence \(\{\tau_{i}\} \to \infty\). Moreover, \(N'\) is diffeomorphic to \(S^{n-1}\). Note that \(n = 4\). By \([13]\) or \([1]\), any shrinking Ricci soliton with nonnegative sectional curvature on \(S^{3}\) must be a round sphere. Hence, \((N', g'(t), x')\) is a flow of shrinking round spheres. This implies that \((S^{3}, \tau^{-1}_{i} g_{S^{3}}(\tau_{i}t_{0}))\) has a strictly positive curvature operator as long as \(\tau_{i}\) is sufficiently large. As a consequence, \((S^{3}, g_{S^{3}}(\tau_{i}t_{0}))\) has positive curvature operator. Since the positivity of curvature operator is preserved under Ricci flow, \(g_{S^{3}}(t)\) has positive curvature operator for all \(t \in (-\infty, 0)\). Therefore, we see that the ancient solution \(g_{S^{3}}(t)\) is a compact \(\kappa\)-solution which satisfies \([1.4]\). As in the proof of Theorem \([1.3]\) \((M, g)\) is asymptotically cylindrical, and so it is rotationally symmetric by Brendle’s result \([3]\).

\[\square\]

6.1. **Further remarks.** The rigidity of nonnegatively curved \(n\)-dimensional \(\kappa\)-noncollapsed steady (gradient) Ricci solitons is closely related to the classification of positively curved shrinking Ricci soliton on \(S^{n-1}\). To the authors’ knowledge, it is still unknown whether the shrinking soliton with positive sectional curvature on \(S^{n}\) is unique when \(n \geq 4\). If it is true, then the argument in proof of Theorem \([1.5]\) can be generalized to any dimension. Namely, we may prove that any \(n\)-dimensional steady (gradient) Ricci soliton with nonnegative sectional curvature, positive Ricci curvature and exactly linear curvature decay must be rotationally symmetric.

**References**

[1] Böhm, C. and Wilking, B., *Manifolds with positive curvature operators are space forms*, Ann. of Math., 167 (2008), 1079-1097.

[2] Brendle, S., *Rotational symmetry of self-similar solutions to the Ricci flow*, Invent. Math. , 194 No.3 (2013), 731-764.

[3] Brendle, S., *Rotational symmetry of Ricci solitons in higher dimensions*, J. Diff. Geom., 97 (2014), no. 2, 191-214.

[4] Cao, H.D. and Chen, Q., *On locally conformally flat gradient steady Ricci solitons*, Trans. Amer. Math. Soc., 364 (2012), 2377-2391.
Higher dimensional steady Ricci solitons

[5] Cao, H., Tian, G. and Zhu, X.H., Kähler-Ricci solitons on compact complex manifolds with $c_1(M) > 0$, Geom. Funct. and Anal., 15 (2005), 697-719.

[6] Chen, B.L., Strong uniqueness of the Ricci flow, J. Diff. Geom. 82 (2009), 363-382.

[7] Deng, Y.X. and Zhu, X.H., Complete non-compact gradient Ricci solitons with nonnegative Ricci curvature, Math. Z., 279 (2015), no. 1-2, 211-226.

[8] Deng, Y.X. and Zhu, X.H., Asymptotic behavior of positively curved steady Ricci solitons, arXiv:math/1507.04802, to appear in Trans. Amer. Math. Soc..

[9] Deng, Y.X. and Zhu, X.H., Steady Ricci solitons with horizontally $\epsilon$-pinched Ricci curvature, arXiv:math/1601.02111.

[10] Deng, Y.X.; Zhu, X.H., 3D steady gradient Ricci solitons with linear curvature decay, arXiv:math/1612.05713, to appear in IMRN.

[11] Deng, Y.X. and Zhu, X.H., Asymptotic behavior of positively curved steady Ricci solitons, II, arXiv:math/1604.00142.

[12] Guan, P.F., Lu, P. and Xu, Y.Y., A rigidity theorem for codimension one shrinking gradient Ricci solitons in $\mathbb{R}^{n+1}$ Calc. Var. Partial Differential Equations 54 (2015), no. 4, 4019-4036.

[13] Hamilton, R.S., Three manifolds with positive Ricci curvature, J. Diff. Geom., 17 (1982), 255-306.

[14] Hamilton, R.S., A compactness property for solution of the Ricci flow, Amer. J. Math., 117 (1995), 545-572.

[15] Hamilton, R.S., Formation of singularities in the Ricci flow, Surveys in Diff. Geom., 2 (1995), 7-136.

[16] Lott, J., Some geometric properties of the Bakry-Émery-Ricci tensor. Comment. Math. Helv. 78 (2003), no. 4, 865-883.

[17] Mabuchi, T., Heat kernel estimates and the Green functions on Multiplier Hermitian manifolds. Tohoku Math. J. 54 (2002), 259-275.

[18] Morgan, J. and Tian, G., Ricci flow and the Poincaré conjecture, Clay Math. Mono., 3. Amer. Math. Soc., Providence, RI; Clay Mathematics Institute, Cambridge, MA, 2007, xlii+521 pp. ISBN: 978-0-8218-4328-4.

[19] Ni, L., Closed type-I Ancient solutions to Ricci flow, Recent Advances in Geometric Analysis, ALM, vol. 11 (2009), 147-150.

[20] Naber, A., Noncompact shrinking four solitons with nonnegative curvature, J. Reine Angew Math., 645 (2010), 125-153.

[21] Perelman, G., The entropy formula for the Ricci flow and its geometric applications, arXiv:math/0211159.

[22] Shi, W.X., Ricci deformation of the metric on complete noncompact Riemannian manifolds, J. Diff. Geom., 30 (1989), 223-301.

YUXING DENG, SCHOOL OF MATHEMATICAL SCIENCES, BEIJING NORMAL UNIVERSITY, BEIJING, 100875, CHINA, dengyuxing@mail.bnu.edu.cn

XIAOHUA ZHU, SCHOOL OF MATHEMATICAL SCIENCES AND BICMR, PEKING UNIVERSITY, BEIJING, 100871, CHINA, xhzhu@math.pku.edu.cn