Stability of a Vortex in Spinor Bose-Einstein Condensate

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We propose a method to create a vortex in BEC utilizing the spin degree of freedom. We consider the optical plug at the center of the vortex, which makes the vortex-creation process stable. We also investigate the instability of the halfway state to complete vortex state without the optical plug.

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1. INTRODUCTION

The vortex formation in Bose-Einstein condensate of atom gases has been achieved in several groups \cite{1,2,3,4,5,6,7} in variety of methods. The first method is called phase imprinting \cite{1} and an another method rotate the potential which confine the condensate. \cite{2} We proposed another method in Ref. \cite{8}, which utilized the turning magnetic field. This process is made stable using pinning potential with dipole force at the center of the condensate. This makes the condensate ring shaped and the refined properties of vortices around the core is hidden. So it is more desirable to form the vortex without the pinning potential. In this paper, we compare the stability of condensate with and without the pinning potential. We analyze the excitation spectrum to obtain the insight to the stability of the process.

The next section introduces Bogoliubov equations. Section 3 is devoted to the explanation of method to create the vortex state. In Section 4, we discuss the instability of vortex creation process. Section 5 is devoted to summary.
In order to treat an interacting Boson system, we use the formulation based on Bogoliubov theory, which is extended to treat the spin degrees of freedom. The wavefunction of condensate is written with a set of 3 wavefunctions $\phi_i(r)(i = 0, \pm 1)$. These are given by the Gross-Pitaevskii (GP) equation

$$\left[-\frac{\hbar^2 \nabla^2}{2m} + V(r) + g_n \sum_k |\phi_k|^2 \delta_{ij} - B_{ij} \right. \\
\left. + g_s \sum_\alpha \sum_{kl} (F_\alpha)_{ij} (F_\alpha)_{kl} \phi_k^* \phi_l \right] \phi_j = 0, \quad (1)$$

where $\mu$ is the chemical potential. The coefficient $g_n$ and $g_s$ are interaction parameters. The numbers $B_{ij}$ are matrix elements of $B$-matrix which shows the magnetic field. The form of this matrix is given below. The subscripts are $\alpha = (x, y, z)$ and $i, j, k, l = (0, \pm 1)$. $V(r)$ is the spin-independent potential which is given in Eqs. (12) and (13).

The excitation levels $\varepsilon_q$ and corresponding wavefunctions $u_q, v_q$ are given by Bogoliubov equations

$$\sum_j \{ A_{ij} u_q(r, j) - B_{ij} v_q(r, j) \} = \varepsilon_q u_q(r, i), \quad (2)$$

$$\sum_j \{ B_{ij}^* u_q(r, j) - A_{ij}^* v_q(r, j) \} = \varepsilon_q v_q(r, i) \quad (3)$$

where

$$A_{ij} = \left( -\frac{\hbar^2 \nabla^2}{2m} - \mu + V(r) + g_n \sum_k |\phi_k|^2 \right) \delta_{ij} - B_{ij} \\
+ g_n \phi_i^* \phi_j + g_s \sum_\alpha [(F_\alpha)_{ij} (F_\alpha)_{kl} \phi_k^* \phi_l + (F_\alpha)_{ji} (F_\alpha)_{kl} \phi_k \phi_l], \quad (4)$$

$$B_{ij} = g_n \phi_i \phi_j + g_s \sum_\alpha \sum_{kl} (F_\alpha)_{ik} \phi_k (F_\alpha)_{jl} \phi_l. \quad (5)$$

Most of $\varepsilon_q$ are positive. When the lowest $\varepsilon_q$ is negative, it shows the local instability of condensate.

3. PROCESS TOWARD THE VORTEX

We consider a two dimensional system of BEC with rotational symmetry around the $z$ axis. We introduce the cylindrical coordinates $r = (r, \theta, z)$.
Suppose that an Ioffe-Pritchard field

\[ \mathbf{B} = (B_{\perp} \cos \theta, -B_{\perp} \sin \theta, B_z) \]  

is applied to the system. The coefficients \( B_{\perp} \) and \( B_z \) are amplitude of magnetic field measured by Zeeman energy. This magnetic field is rewritten in matrix form as

\[
\mathbf{B} = \begin{pmatrix}
    B_z & B_{\perp} e^{i\theta} & 0 \\
    B_{\perp} e^{-i\theta} & 0 & B_{\perp} e^{i\theta} \\
    0 & B_{\perp} e^{-i\theta} & -B_z
\end{pmatrix}.
\]

The eigenvectors of \( \mathbf{B} \) show the arrangement of spin components when the field is strong enough. The eigenvalues of \( \mathbf{B} \) are 0 and \( \pm B \) where \( B = \sqrt{B_{\perp}^2 + B_z^2} \). The eigenvectors corresponding to \( \pm B \) are

\[
\begin{pmatrix}
    \phi_{+1} \\
    \phi_0 \\
    \phi_{-1}
\end{pmatrix} = \frac{1}{2B} \begin{pmatrix}
    (B \pm B_z) e^{i\theta} \\
    \pm \sqrt{2} B_{\perp} \\
    (B \mp B_z) e^{-i\theta}
\end{pmatrix}.
\]

The winding number (coefficient of \( \theta \) in exponential) becomes 1, 0, and -1 for \( \phi_1, \phi_0 \) and \( \phi_{-1} \) respectively. When the coefficients \( B_{\perp} \) and \( B_z \) vary as

\[
\begin{align*}
    B_z &= B_0 \cos[\pi (1 - t/T)] \\
    B_{\perp} &= B_{\perp}^t \sin[\pi (1 - t/T)]
\end{align*}
\]

where \( t \) is time, the spin of condensate shifts following Eq. (8). If the process begins with condensate of -1 component without vortex, the initial arrangement of condensate is Eq. (8) multiplied by \( e^{i\theta} \). The coefficient of \( \theta \) in exponential of \( \phi_{-1} \) is zero and the condensate takes the form

\[
\begin{pmatrix}
    \phi_{+1} \\
    \phi_0 \\
    \phi_{-1}
\end{pmatrix} \propto \begin{pmatrix}
    (B + B_z) e^{2i\theta} \\
    \sqrt{2} B_{\perp} e^{i\theta} \\
    (B - B_z)
\end{pmatrix}.
\]

This process ends with condensate of +1 component with winding 2 at \( t = T \). This means that the vortex is nucleated.

The explanation above treat only the magnetic field. We can take into account other terms with GP equation Eq. (1). This does not change the arrangement of winding numbers as long as the condition of cylindrical symmetry is kept.

When the condensate follows Eq. (1), the winding numbers of excitation levels are written as \( q_\theta + 2, q_\theta + 1, q_\theta \) for \( u_{+1}, u_0, u_{-1} \) and \( q_\theta - 2, q_\theta - 1, q_\theta \) for \( v_{+1}, v_0, v_{-1} \) respectively. When the pinning potential does not exist, the modes with \( q_\theta = \pm 1 \) and \( \pm 2 \) can localize at the core (and have negative energy) because they have winding number 0 in \( u_i \) or \( v_i \).
4. ANALYSIS OF STABILITY

The stability of the condensate can be evaluated with sign of the lowest excitation level, which is given by Bogoliubov equations Eqs. (2) and (3). We decide the distribution of condensate with GP equation Eq. (1) at each time $t$, and the excitation levels with Eqs. (2) and (3). The whole condensate is confined with spin-independent harmonic potential, and the magnetic field which is described in Eqs. (9) and (10) handle the spin of condensate. The parameters are $m \approx 1.44 \times 10^{-25} \text{ kg}$, $a_0 = 5.5 \times 10^{-9} \text{ m}$, $g_n = \frac{4\pi\hbar^2}{m} \frac{2a_0 + a_2}{3}$, $g_s = \frac{4\pi\hbar^2}{m} \frac{a_2 - a_0}{3}$. The scattering length $a_2$ is 0.75$a_0$ or 1.375$a_0$, which means $g_s = -0.1g_n$ and $g_s = +0.1g_n$ respectively.

The case when the pinning potential at the center of the system is introduced is discussed in next subsection, and the case without the pinning potential is discussed in subsection 4.2. The time $T$ is supposed to be enough longer than the Larmor frequency, namely the adiabatic condition is satisfied. When the time $T$ is too short, the spin of atoms will not have enough time to follow the perpendicular magnetic field. Whether the vortex nucleates or not is uncertain in this case.

4.1. WITH PINNING POTENTIAL

The whole condensate is confined with spin-independent harmonic potential radially, and the pinning potential at the $z$-axis of the system is also given. This stabilizes the vortex creation process. These two spin-independent potentials are written as

$$V(r) = \frac{m(2\pi \nu)^2}{2} r^2 + U \exp \left( -\frac{r^2}{2r_0^2} \right)$$

where $\nu = 200\text{ Hz}$, $U = 2 \times 10^3 \text{ J}$ and $r_0 = 1\mu\text{m}$.

The magnetic field which is given by Eq. (9) and (10) are applied. The wavefunctions of the condensate and the excitation spectrum are calculated at each $t$, at each pairs of magnetic field coefficient $B_r$ and $B_\perp$. Fig. (a) shows the density distribution of the condensate at several $t$. Fig. (b) shows the lowest excitation levels at $q_\theta = \pm 1, \pm 2$. The change of total magnetization $M$ along the $z$-axis accompanies the change of $t$, and the horizontal axis of Fig. (b) shows $M/N$, which is $f(|\phi_1|^2 - |\phi_{-1}|^2) dr / \sum_i \int |\phi_i|^2 dr$. It is found that the lowest $\varepsilon$ never becomes negative. Because the negative excitation level does not appear, the condensate stays locally stable.
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Fig. 1. In (a) and (c), the solid line shows the density profile of condensate $\sum |\phi_i|^2$ and the various dashed lines show the density of three spin components $|\phi_i|^2$ ($i = 0, \pm 1$). In (b), (d) and (e), the lowest eigenenergy $\varepsilon$ at winding number index $q_\theta = \pm 1, \pm 2$ are plotted. The unit of $\varepsilon$ is $200hJ$, which is the trap unit when the pinning potential is not exist. The horizontal axis show magnetization along z-axis which varies from -1 to +1 during the process. (a) and (b) show shift of density distribution of the condensate and the lowest eigenenergies when the pinning potential exists. The eigenenergies never become negative. (c), (d) and (e) show those when the pinning potential does not exist. Some of eigenenergies become negative and even complex as the magnetization increases. In (a), (b), (c) and (d) the interaction parameter $g_s = -0.1$, which means ferromagnetic interaction. $a_2$ is $0.75a_0$. In (e), the interaction parameter $g_s = +0.1$, which means antiferromagnetic interaction and $a_2$ is $1.375a_0$. The density distributions (a) and (b) hardly change when the sign of interaction parameter $g_s$ is turned to antiferromagnetic.
4.2. WITHOUT PINNING POTENTIAL

In this subsection, we consider the case when the optical pinning does not exist. The spin-independent trapping potential written in Eqs. (1) and (2) is

\[ V(r) = \frac{m(2\pi \nu)^2}{2} r^2, \quad \text{(13)} \]

and the time-dependent magnetic field which is given in Eqs. (9) and (10) is also applied. The halfway states are obtained by the static form of Gross-Pitaevskii equation. Figure 1(c) shows some of the density distributions of condensate. Figure 1(d) and 1(e) show the excitation levels obtained through Bogoliubov equations Eqs. (2) and (3). The filled dot shows that there is a pair of eigenstate with complex eigenenergy. The real parts of them are plotted. The complex eigenenergy appears where the \( q_\theta = 2 \) mode and the \( q_\theta = -2 \) mode crossed, which result is consistent with Ref. 9. This fact means the spiraling out or the splitting of the vortex.

5. CONCLUSION

In this paper, we analyzed the stability of vortex creation process. The distributions of condensate and excitation levels are derived using Bogoliubov equations. When the pinning potential is absent, negative and complex eigenenergies are found and the vortex forming become unstable halfway.

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