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A new scheme for the running coupling constant in gauge theories using Wilson loops

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Abstract

We propose a new renormalization scheme of the running coupling constant in general gauge theories defined by using the Wilson loops. The renormalized coupling constant is obtained from the Cretz ratio in lattice simulations and the corresponding perturbative coefficient at the leading order. The latter calculation is performed by adopting the zeta-function resummation techniques. We make a benchmark test of our scheme in quenched QCD with the plaquette gauge action. The running of the coupling constant is determined by applying the step scaling procedure. Using several methods to improve the statistical accuracy, we show that the running coupling constant can be determined in a wide range of energy scales with relatively small number of gauge configurations.
§1. Introduction

One of key subjects upon which recent attention has been focusing is the flavor dependence of a $SU(N)$ Yang-Mills theory. In particular, given the number of flavors $N_f$, the question is whether the theory has an infrared fixed point, or, in other words, an approximate conformal behavior at long distances. This question is triggered by an effort to construct an alternative mechanism of electroweak symmetry breaking assuming the existence of a new, strongly interacting sector beyond the electroweak scale. The earliest model of this sort, the so-called technicolor, gives rise to a dynamical electroweak symmetry breaking by introducing a QCD-like sector scaled up to some TeV. While theoretically appealing, the simplest form of the technicolor model and its variants with QCD-like dynamics are ruled out or disfavored with constraint by electroweak precision measurements. However, the possibility of such mechanism with non-QCD-like theory is still open, and may provide observable signatures at LHC. It is thus challenging task to investigate the low-energy landscape of spontaneously broken, strongly interacting gauge theories.

Among theoretical tools in hand, numerical approach to lattice gauge theories has made possible to gain quantitative information about strong dynamics of gauge theories. The current understanding can be summarized as follows. A vector-like field theory like QCD is known to exhibit confinement and dynamical chiral symmetry breaking for small number of massless fermions, $N_f$. When $N_f$ is just below the value at which asymptotic freedom sets in, $N_f$, the theory is conformal (no confinement, unbroken chiral symmetry) in the infrared. The theory is believed to remain conformal down to some critical value $N_f$, where the coupling becomes strong enough and the transition to the confined chirally broken phase occurs. The range $N_f < N_f < N_f$ is called as conformal window.

It is thus demanded to investigate strongly interacting gauge theories in a wide range of parameters, such as the number of colors, the number of flavors, and the fermion representations. While previous approaches, such as the Schrödinger functional scheme, have been useful to explore the infrared nature of the gauge theories, it is important to study the running coupling constant in different schemes, in order to convince the fixed point is not an artifact due to particular scheme but a physical one. For this purpose, we propose a new renormalization scheme which uses as key ingredient the Wilson loops, and, as such, it can be used a renormalization scheme in general gauge theories, as long as the Wilson loops can be defined. In particular this scheme provides an efficient computational method for lattice gauge theories. Specifically, a renormalized amplitude is defined as the ratio among the Wilson loops, namely the Creutz ratio, and its perturbative counterpart. The former can be evaluated non-perturbatively by Monte Carlo simulation, while the latter calculated
analytically once the underlying theory is specified. By properly defining the nonperturbatively renormalized coupling constant, its scale dependence is extracted using the step scaling procedure, i.e., from the volume dependence of the coupling.

Our scheme will provide not only an independent check on the extent of the conformal window, but also several computational advantages. Since the Creutz ratio is free from $O(a)$ discretization error, our scheme is in principle free from any $O(a)$ systematic error. Furthermore, this scheme uses simple gluonic observables, and does not introduce any particular kinematical setup which may deteriorate the discretization error and may break the exact chiral symmetry, even when retained in the action. Therefore it can be easily applied to the case of dynamical fermions of any type, without restriction to $N_f$. For these features, this scheme may be an attractive alternative to the Schrödinger functional scheme or the twisted Polyakov loop scheme.16)–18)

Before challenging the gauge theories with dynamical fermions, as a benchmark test, we apply this new scheme to computation of the running coupling constant in quenched lattice QCD. The numerical calculation is performed using the plaquette gauge action with the periodic boundary condition. Although this boundary condition is adopted just for simplicity, as a drawback, we have to be careful about the contribution from degenerate vacua known as “toron”.21) Our scheme can, however, be applied in principle to any choice of the boundary condition, such as the twisted boundary condition, which ensures no unwanted zero-mode contribution by inducing non-trivial background configurations. Adopting several methods to improve the statistical accuracy, we can determine the running of the coupling constant in a wide range of the energy scale with relatively small number of gauge configurations.

Another essential ingredient of our scheme is the perturbative calculation of the renormalization constant. This is performed analytically using zeta function resummation techniques, which prove to be quite convenient. First of all, zeta function techniques offer a natural method to study the analyticity (and regularity) properties of the perturbative counterpart of the Creutz ratio. In addition, some algebraic rearrangements of zeta functions, originally due to Chowla and Selberg, allow us to recast the expressions in terms of analytic functions accompanied by some exponentially converging series, whose evaluation is almost trivial and requires little computer power. The zeta function methods we apply can be easily extended to any boundary conditions and to the case of the Polyakov lines.20)

This paper is organized as follows. In the next section we give the definition of the new scheme. The perturbative calculation is illustrated in Sec. 3. Sec. 4 is devoted to the details of our numerical simulations, after brief introduction to the step-scaling procedure. Sec. 5 contains discussion on the numerical results and comparison with other results in the literature. Finally, Sec. 6 summarizes our conclusions. The paper is equipped with
three appendices where some technical details and tables of the simulation parameters are reported. Preliminary results of this work have been reported in Ref.¹

§2. Wilson Loop Scheme

In this section, we define a new renormalization scheme that we term as ‘Wilson loop scheme’. Let us consider an amplitude $A$ whose tree-level contribution is

$$A^{\text{tree}} = kg_0^2,$$

(2.1)

where $g_0$ is the bare coupling constant, and $k$ is a coefficient of proportionality that does not depend on $g_0$ and can be explicitly calculated for a given underlying theory. With a non-perturbatively calculated amplitude $A^{\text{NP}}$ at the scale $\mu$, the renormalization constant $Z(\mu) \equiv A^{\text{NP}}(\mu)/A^{\text{tree}}$ relates the renormalized coupling constant, $g(\mu)$, to the bare one, leading to the relation

$$g^2(\mu) = \frac{A^{\text{NP}}(\mu)}{k}.$$

(2.2)

Although in particle physics an $S$-matrix element, i.e. a scattering amplitude, is usually adopted as $A$, to define $g(\mu)$ one can equivalently use any physical quantity that can be perturbatively expanded and is proportional to $g_0^2$ at the tree-level.

We define the Wilson loop scheme by taking the ‘amplitude’ to be

$$A_W(R, T; L_0, T_0; g_0) \equiv -R^2 \frac{\partial^2}{\partial R \partial T} \ln \langle W(R, T; L_0, T_0) \rangle_{T=0},$$

(2.3)

where $W(R, T)$ is the Wilson loop with the temporal and spatial sizes $T$ and $R$, respectively. We also include as arguments of $W$ the physical box sizes of the lattice, $T_0$ and $L_0$, which will later be regarded as the renormalization scale. On a finite lattice, $W$, and thus $A_W$, also depend on the lattice spacing $a$ which is determined by the bare coupling $g_0$. The dependence of $A_W$ on $a$ is removed by taking the continuum limit, $a \to 0$. A pictorial definition of the Wilson loop is shown in Fig. 1. It is easy to see, by using lattice perturbation theory (see Fig. 2), that at the lowest order in the coupling constant $g_0$, $A_W$ is proportional to $g_0^2$:

$$A_W(R, T; L_0, T_0; g_0) = -g_0^2 R^2 \frac{\partial^2}{\partial R \partial T} \ln \langle (W(R, T; L_0))^{\text{tree}} \rangle_{T=R} + 0(g_0^4).$$

(2.4)

Thus, once the value of $k$ is calculated, relation (2.2) gives after taking the continuum limit a straightforward prescription to get the renormalized coupling:

$$g^2 \left( \frac{L_0}{L_0}, \frac{R}{L_0} \right) = -\frac{R^2}{k(R/L_0)} \frac{\partial^2}{\partial R \partial T} \ln \langle W(R, T; L_0) \rangle^{\text{NP}}_{T=R}.$$

(2.5)

1
Fig. 1. Wilson loop defined on the latticized space-time box. \( T_0, L_0 \) and \( T, R \) represent the size of the box and the Wilson loop in the temporal and spatial directions, respectively; \( a \) is the lattice spacing.

\[
W(R, T; L_0, T_0; a; g_0) = \begin{array}{c}
\quad
\end{array} + \begin{array}{c}
\begin{array}{c}
g_0
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
g_0
\end{array}
\end{array} + \cdots
\]

Fig. 2. Schematic illustration of the perturbative expansion of the Wilson loop.

In the above expression we have made explicit that in the continuum limit \( k \) is a regular function of \( R/L_0 \) only. This will be proved in the next section. The remaining factor on the right hand side of Eq. (2.5) can be evaluated on the lattice as the Creutz ratio,

\[
\chi(\hat{R} + 1/2; \hat{T} + 1/2; L_0/a) = -\ln \left( \frac{W(\hat{R} + 1, \hat{T} + 1; L_0/a) \cdot W(\hat{R}, \hat{T}; L_0/a)}{W(\hat{R} + 1, \hat{T}; L_0/a) \cdot W(\hat{R}, \hat{T} + 1; L_0/a)} \right),
\]

where \( \hat{T} \equiv T/a \) and \( \hat{R} \equiv R/a \). The value of \( \chi \) is evaluated by a Monte Carlo simulation.

The renormalized coupling constant in the Wilson loop scheme can be written as

\[
g_w^2 \left( L_0, \frac{R + a/2}{L_0}, \frac{a}{L_0} \right) = (\hat{R} + 1/2)^2 \cdot \chi(\hat{R} + 1/2; L_0/a)/k,
\]

where we used the shorthand notation \( \chi(\hat{R} + 1/2, \hat{T} + 1/2; L_0/a)\big|_{\hat{T}=\hat{T}} \equiv \chi(\hat{R} + 1/2; L_0/a) \).

The quantity \( g_w^2 \) depends on three different scales, \( L_0, R, \) and \( a \); by taking the ratio to \( L_0 \), we use as the independent parameters \( r \equiv (R + a/2)/L_0, a/L_0, \) and \( L_0 \). Fixing \( r \) to a specific value means fixing the renormalization scheme. The ratio \( a/L_0 \) specifies the discretization of the box, and can be removed by taking the continuum limit, \( a/L_0 \rightarrow 0 \). After fixing
two dimensionless parameters $r$ and $a/L_0$, $g_w^2$ becomes a function of single scale, $L_0$. In our scheme, following the step scaling procedure, $L_0$ is identified as the scale at which the renormalized coupling is defined.

There are several advantages in using the Wilson loop scheme. An evident one is that our scheme does not have any $O(a)$ systematic error as long as we use actions without $O(a)$ errors. This is because the Creutz ratio is free from $O(a)$ discretization error, since the heavy quark propagator is automatically $O(a)$-improved after the redefinition of the mass and the wavefunction. This is in contrast to the case of the Schrödinger functional scheme, in which the boundary counter term gives rise to an additional $O(a)$ systematic error. Such a particular kinematical setup also spoils the exact chiral symmetry even when retained in the action. Another good point is that this scheme uses simple gluonic observables and can be easily applied to the case with any type of dynamical fermions without restriction to the number of flavors. This is an advantage over the twisted Polyakov loop scheme, in which one has to introduce the 'smell' degrees of freedom to the fermion, so that the number of flavors is restricted to the multiple of the color degrees of freedom, as $N_f = kN_c$, with $k$ an integer.

§3. Computation of $k$

One of the indispensable steps of the scheme presented in the previous section is the calculation of the coefficient of proportionality $k$ in Eq. (2.1). It can be generically split into two terms:

$$k = k_0 + k_1 ,$$

(3.1)

where $k_0$ represents the zero-mode contribution, whereas $k_1$ can be expressed as

$$k_1 = -2 R^2 C_F \frac{\partial^2}{\partial R \partial T} \left[ \frac{4}{(2\pi)^4} \sum_n \left( \frac{\sin \frac{n\pi T}{L_0}}{n_0} \right)^2 e^{i \frac{2\pi T R}{L_0}} \right]_{T=R} .$$

(3.2)

In this expression, the summation is taken over integer values of $n_i$ ($i=0, \ldots, 3$) except for the case $n_0 = n_1 = n_2 = n_3 = 0$ (indicated by the prime in the sum), and $n^2 = n_0^2 + n_1^2 + n_2^2 + n_3^2$. The zero mode contribution depends on the boundary conditions. In the following, we will concentrate on the case of periodic boundary conditions. In this case, $k_0$ was initially calculated in Ref.\textsuperscript{21}) For SU(3) gauge group, $k_0$ is given by:

$$k_0 = \frac{2}{3} C_F \left( \frac{R}{L_0} \right)^4 .$$

(3.3)
The scope of this section is to present a method to compute the quantity $k$. The method we develop will be illustrated for the case of periodic boundary conditions, but it can be applied, with minor changes, to the case of twisted or mixed boundary conditions. As we have seen, the contribution from the zero mode, $k_0$, can be separated from the contribution $k_1$. From relation (3.3), the regularity of $k_0$ is obvious. Thus to compute (and prove the regularity of) $k$, we only need to consider $k_1$. Our starting point is the quantity:

$$S(T/L_0, R/L_0) = \sum_{n_0=-\infty}^{\infty} \frac{\sin \frac{2\pi T n_0}{L_0}}{n_0} \left[ 2 \sum_{n_1, n_2} \sum_{n_3=1}^{\infty} \frac{\cos \frac{2\pi R n_3}{L_0}}{n_0^2 + n_1^2 + n_2^2 + n_3^2} + \sum_{n_1, n_2} \frac{1}{n_0^2 + n_1^2 + n_2^2} \right] \quad (3.4)$$

$k_1$ can be obtained from $S(T/L_0, R/L_0)$ by taking an additional derivative with respect to $R$:

$$k_1 = \frac{R^2 C_F}{2\pi^3 L_0} \frac{\partial S}{\partial R}(T/L_0, R/L_0).$$

Although it is not possible to find a closed form for $S(T/L_0, R/L_0)$ in terms of elementary functions, the use of zeta function resummation techniques and basic analytic continuation allow us to recast $S(T/L_0, R/L_0)$ in the form of a practically computable quantity, and also to explicitly prove the regularity of $k$. The computation is carried out in a few basic steps. The first one consists in evaluating the sum over the direction $n_3$ by using the Poisson formula. Then, the summation over the other indices, $n_1$ and $n_2$, is expressed in terms of Epstein zeta functions. After these simple steps, formulas become compact, but, without further rearrangements, they are of little practical use. To this aim, it is convenient to rewrite the Epstein zeta functions using the Chowla-Selberg formula, that re-expresses the zeta functions as a sum of some elementary analytic functions plus some rapidly converging series. The subsequent step is to perform analytically the integration, and finally perform the remaining summations numerically. Although the above procedure may seem involved, the actual steps are rather simple. The method has also the bonus of providing a proof of the regularity of the Creutz ratio, as we will explicitly show in the following.

The first step of our procedure is to employ the Poisson summation formula:

$$\sum_{n_3=1}^{\infty} f(n_3) = -\frac{1}{2} f(0) + \int_{0}^{\infty} dt f(t) + 2 \sum_{n=1}^{\infty} \int_{0}^{\infty} f(t) \cos(2\pi nt) dt \quad . \quad (3.5)$$

A straightforward application of the above relation to the function $S(T/L_0, R/L_0)$ gives:

$$S(T/L_0, R/L_0) = 2 \sum_{n_0=-\infty}^{\infty} \frac{\sin \frac{2\pi T n_0}{L_0}}{n_0} \sum_{m=-\infty}^{\infty} \int_{0}^{\infty} \cos(2\pi (m + R/L_0)t) \zeta_\epsilon(s; n_0) dt \quad , \quad (3.6)$$

where we have used the standard definition of the generalized Epstein zeta function:

$$\zeta_\epsilon(s; n_0) \equiv \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \frac{1}{(n_0^2 + n_1^2 + n_2^2 + t^2)^{-s}} \quad . \quad (3.7)$$
The parameter $s$ is a regulator, introduced to perform the necessary analytical continuations. The limit $s \to 1$ will be taken at the end. It is interesting that the function $S(T/L_0, R/L_0)$ can be entirely expressed in terms of the integral function

$$Z(\Omega) = \int_0^\infty \cos (2\pi \Omega t) \zeta_t(s; n_0) dt . \quad (3.8)$$

Although compact, the result Eq. (3.6) requires further manipulation. A useful way to handle these functions is to make use of the Chowla-Selberg formula. Refs. (27), (28) develop the appropriate formalism that allows us to express $\zeta_t(s; n_0)$ as the sum of analytic functions plus a rapidly converging series:

$$\zeta_t(s; n_0) = \frac{\Gamma(s-1)}{\Gamma(s)} n_0^2 + t^2 (1-s) + 2\pi \frac{\Gamma(s)}{\Gamma(s)} \sum_{p,q=-\infty}^{\infty} \left[ \pi^2 (p^2 + q^2) \right]^{-(1-s)/2} \times (n_0^2 + t^2)^{(1-s)/2} K_{1-s} \left( \frac{2\pi \sqrt{n_0^2 + t^2 \sqrt{p^2 + q^2}}}{2\pi \sqrt{p^2 + q^2}} \right) . \quad (3.9)$$

The other tool is the following integral formula (see (29)):

$$\int_0^\infty \cos (2\pi \Omega t) \left( t^2 + n_0^2 \right)^{(1-s)/2} K_{1-s} \left( 2\pi (p^2 + q^2)^{1/2} \left( t^2 + n_0^2 \right)^{1/2} \right) dt = \sqrt{\pi} \left( 2\pi \sqrt{p^2 + q^2} \right)^{1-s} n_0^{1/2+1/2} \left( 4\pi^2 \left( \Omega^2 + p^2 + q^2 \right) \right)^{s-1/4} K_{(s-1)-1/2} \left( 2\pi n_0 \sqrt{(p^2 + q^2) + \Omega^2} \right) . \quad (3.10)$$

The procedure is now straightforward and consists in using the relations (3.9) and (3.10) in (3.6). Some computations lead to

$$Z(\Omega) = \sqrt{\pi} \frac{\Gamma(2-s)}{\Gamma(s)} \frac{\Gamma(s-1)}{\Gamma(s)} \cos (\pi((1-s) + 1/2)) \left( \frac{n_0}{\pi \Omega} \right)^{(1-s)+1/2} K_{(s-1)-1/2} \left( 2\pi \Omega n_0 \right) \right) + (2\pi)^{3/2} \frac{2^{-s}}{\Gamma(s)} \sum_{p,q=-\infty}^{\infty} n_0^{1/2+(1-s)} \left[ \left( 4\pi^2 \left( p^2 + q^2 + \Omega^2 \right) \right)^{(s-1)/2} \right]^{1/2} \times K_{(s-1)-1/2} \left[ 2\pi n_0 \left( p^2 + q^2 + \Omega^2 \right) \right]^{1/2} . \quad (3.11)$$

It can be easily checked that, in the above expression, the limit $s \to 1$ can be taken safely giving

$$Z(\Omega) = \frac{\pi}{2|\Omega|} e^{-2\pi |\Omega| n_0} + \frac{\pi}{2} \sum_{p,q=-\infty}^{\infty} \left( p^2 + q^2 + \Omega^2 \right)^{-1/2} e^{-2\pi n_0 \sqrt{p^2 + q^2 + \Omega^2}} . \quad (3.12)$$

From the above expressions, it is evident that the quantity $Z(\Omega)$ is regular. We can now plug (3.12) into (3.4), and rearrange it as

$$S(T/L_0, R/L_0) = \frac{4\pi^2 T}{L_0} \left[ S_0(R/L_0) + S_1(T/L_0, R/L_0) \right] , \quad (3.13)$$
where we have separated the contribution coming from the \( n_0 = 0 \) contribution from the remaining part which is exponentially suppressed. We have defined

\[
S_0(R/L_0) = A_1(R/L_0) + A_2(R/L_0) + A_3(R/L_0),
\]

\[
S_1(T/L_0, R/L_0) = B_1(T/L_0, R/L_0) + B_2(T/L_0, R/L_0) + B_3(T/L_0, R/L_0),
\]

where

\[
A_1(R/L_0) = \sum_{m=-\infty}^{\infty} \frac{1}{2|m + (R/L_0)|},
\]

\[
A_2(R/L_0) = 2 \sum_{m=-\infty}^{\infty} \sum_{p,q=1}^{\infty} \frac{1}{(p^2 + q^2 + |m + (R/L_0)|)^{1/2}},
\]

\[
A_3(R/L_0) = 2 \sum_{m=-\infty}^{\infty} \sum_{p,q=1}^{\infty} \frac{1}{(p^2 + |m + (R/L_0)|)^{1/2}},
\]

\[
B_1(T/L_0, R/L_0) = \sum_{n_0=1}^{\infty} \frac{\sin(2\pi T_{L_0}/n_0)}{\pi n_0 T/L_0} \sum_{m=-\infty}^{\infty} \frac{e^{-2\pi|m+(R/L_0)| n_0}}{2|m+(R/L_0)|},
\]

\[
B_2(T/L_0, R/L_0) = 2 \sum_{n_0=1}^{\infty} \frac{\sin(2\pi T_{L_0}/n_0)}{\pi n_0 T/L_0} \sum_{m=-\infty}^{\infty} \sum_{p,q=1}^{\infty} \frac{e^{-2\pi n_0 \sqrt{p^2+q^2+|m+(R/L_0)|^2}}}{(p^2 + q^2 + |m + (R/L_0)|)^{1/2}},
\]

\[
B_3(T/L_0, R/L_0) = 2 \sum_{n_0=1}^{\infty} \frac{\sin(2\pi T_{L_0}/n_0)}{\pi n_0 T/L_0} \sum_{m=-\infty}^{\infty} \sum_{p,q=1}^{\infty} \frac{e^{-2\pi n_0 \sqrt{p^2+|m+(R/L_0)|^2}}}{(p^2 + |m + (R/L_0)|)^{1/2}}.
\]

Due to the exponential suppression, the terms \( B_1, B_2, \) and \( B_3, \) and thus \( S_1(T/L_0, R/L_0), \) are clearly regular. Therefore, to prove the regularity of \( k, \) we only have to show that \( S_0(R/L_0) \) is also regular. To show that the terms (3.16), (3.17), and (3.18) also lead to a regular expression for \( S_0(R/L_0) \) (and to compute them), it requires further manipulations.

The first term, (3.16), can be computed analytically:

\[
A_1(R/L_0) = \frac{1}{2} \left[ -\frac{L_0}{R} - \psi(R/L_0) + \psi(-R/L_0) \right],
\]

where \( \psi(x) \) is the Euler psi function. The remaining sums, instead, can be rearranged by performing first the summation over \( m, \) and then by using the Chowla-Selberg formula. We give some details in appendix to the paper, and here present the results:

\[
A_2(R/L_0) = 8 \sum_{j=1}^{\infty} \cos(2j\pi R/L_0) K_0 \left( 2j\pi \sqrt{p^2 + q^2} \right) \text{ (R-independent terms)},
\]

\[
A_3(R/L_0) = 8 \sum_{j=1}^{\infty} \cos(2j\pi R/L_0) K_0 \left( 2j\pi q \right) \text{ (R-independent terms).}
\]
Written as above, it is a trivial matter to see that, after deriving $A_1$, $A_2$, and $A_3$ with respect to $R$, $S_0(R/L_0)$, and thus $k$, is nicely behaved due to the exponential fall-off of the Kelvin functions $K_\nu(z)$.

The last step of our procedure consists in evaluating the above expressions. However, the numerical evaluation of (3-19), (3-20), (3-21), (3-23), and (3-24), does not present any problem and, due to the exponential suppression of the sums, can be performed easily. Differentiating with respect to $R$, substituting $R/L_0 = T/L_0$, and combining the results according to (3-2), (3-3), (3-4) is all that is left to arrive at the desired result for $k$. Fig. (3) shows the dependence of the function $k$ with respect to $R/L_0$. Table 3 provides some indicative values for $k$.

![Fig. 3. The figure shows the dependence of $k$ as a function of $R/L_0$. The solid line represents $k$, in the continuum limit, according to the analytical expressions given in the text. The squares result from a naive lattice calculation. As shown explicitly in Sec. 3, the (continuum) limit of $k$ for $L_0/a \to \infty$ exists and is finite. Also, the convergence of the lattice computation to the continuum value is faster for larger values of $R/L_0$.](image_url)

| $R/L_0$ | $k$  | $R/L_0$ | $k$  |
|---------|------|---------|------|
| 0.02    | 0.17365 | 0.30    | 0.16608 |
| 0.10    | 0.17360 | 0.35    | 0.15694 |
| 0.15    | 0.17336 | 0.40    | 0.13970 |
| 0.20    | 0.17259 | 0.45    | 0.10885 |
| 0.25    | 0.17058 | 0.50    | 0.05556 |

Table I. Values for $k$ from the continuum calculation.
§4. Numerical Simulation

In this section, we will describe the details of our numerical simulations. For later use, we define the coupling-squared, $\tilde{g}_w^2$, by removing the factor $1/k$ from $g_w^2$:

$$\tilde{g}_w^2(\beta, r, \frac{L_0}{a}) \equiv kg_w^2,$$

(4·1)

where $r \equiv (R + a/2)/L_0$. Note that we defined $\tilde{g}_w^2$ as a function of $\beta$, $r$, and $a/L_0$ instead of $L_0$, $r$ and $L_0/a$. The above redefinition is chosen for convenience, since $\beta$, $r$, and $a/L_0$ are the actual input parameters for the simulations.

4.1. Step Scaling

We begin by briefly reviewing the step-scaling procedure (see Refs.23-25 for details), that we use to evaluate the evolution of the running coupling for a wide range of energy scale on the lattice.

The first step is to fix a value for $r$, and find a set of parameters, $(\beta, L_0/a)$, which reproduce $\tilde{g}_w^2$ for several different values of $L_0/a$:

$$\left\{ \left( \beta_1^{(1)}, \frac{(L_0/a)^{(1)}}{a} \right) , \left( \beta_2^{(1)}, \frac{(L_0/a)^{(1)}}{a} \right) , \ldots \right\}.$$  

(4·2)

What we are doing here is tuning the value of $\beta$ in such a way that the physical volume $L_0$ is fixed for different values of $L_0/a$. Let us call this fixed physical volume for the starting point of the step scaling procedure as $L_0$.

The next step is to vary the physical volume from $L_0$ to $\sigma L_0$, which gives the evolution of the running coupling from the energy scale $L_0^{-1}$ to $(\sigma L_0)^{-1}$, where $\sigma$ is the scaling factor. This step can be achieved by changing the lattice size from $(L_0/a)^{(1)}$ to $\sigma(L_0/a)^{(1)}$, leaving each value of $\beta^{(1)}$ unchanged. Values of $g_w^2$ calculated with these new parameter sets should be considered as the coupling at the energy scale $(\sigma L_0)^{-1}$ up to discretization error, so the extrapolation to the continuum limit can be taken by using those data.

$$g_w^2 \left( \frac{1}{\sigma L_0} \right) \equiv \lim_{a \to 0} \left[ Z \left( \frac{1}{\sigma L_0}, \frac{a}{L_0} \right) g_w^2(a) \right].$$

(4·3)

The resultant value of coupling, $g_w^2$, should be considered as the renormalized coupling at the energy scale $(\sigma L_0)^{-1}$. This is the way to obtain a single discrete step of evolution of the running coupling with scaling factor $\sigma$.

Next, we find a new parameter set for $(\beta^{(2)}, (L_0/a)^{(2)})$, which reproduce the value of $g_w^2(\sigma L_0)$ obtained in the previous step. Here, we chose the parameter set in such a way that the new lattice size $(L_0/a)^{(2)}$ is equal to the original one, $(L_0/a)^{(1)}$. From here, we
can repeat exactly the same procedure described so far: we calculate $g'_w$ with the parameter set $(\beta^{(2)}, \sigma(L_0/a)^{(1)})$. By iterating this procedure $n$ times, we obtain the evolution of the running coupling from the energy scale $1/L_0$ to $(\sigma^n L_0)^{-1}$.

4.2. Simulation Parameters

We use the standard Wilson plaquette gauge action defined on a four-dimensional Euclidean lattice with finite volume $L_0^4$. As for the boundary condition on the lattice, in the present analysis, we adopt the untwisted periodic boundary condition, however, it is straightforward to use other boundary condition (e.g., twisted boundary condition) when we need them. Gauge configurations are generated by using the pseudo-heatbath algorithm with over-relaxation, mixed in the ratio of 1:5. In the remainder of this paper, we use the word "a sweep" to refer to the combination of one pseudo-heatbath update sweep followed by five over-relaxation sweeps. In order to eliminate the influence of autocorrelation, we either take large enough number of sweeps between measurements, or adopt the method of binning with a large enough size of bin to estimate the correct size of statistical error. More details are given in Appendix A. We perform the numerical simulations based on the step scaling procedure explained in the previous section for a fixed value $r = 0.3$. (The reason for this choice will be given in the next subsection.) We take the scaling parameter $\sigma = 1.5$ with five different starting lattice sizes being $L_0/a = 10, 12, 14, 16$ and $18$, which means lattice sizes after the scaling at each step are $L_0/a = 15, 18, 21, 24$ and $27$, respectively. We take $g'_w = 0.2871$ (which corresponds to $g^2_w = \frac{g'_w}{k(r=0.3)} \simeq 1.728$) as a starting value of the first step of the step-scaling procedure. Tunings of the values of $\beta$ (namely, finding values of $\beta$ which satisfies $g'_w (\beta, r = 0.3, \frac{L_0}{a}) = 0.2871$ for each $\frac{L_0}{a} = 10, 12, 14, 16$ and $18$ in the case of the first step, for example) are carried out by interpolating the data obtained from simulations for different values of $L_0/a$ and $\beta$ shown in Fig. 4. Each data in the figure is calculated from 200 gauge configurations with 1000-sweep separation between configurations. (It will be shown, in Appendix A, that 1000 sweeps are large enough to remove the autocorrelation among configurations.) Once we obtain values of $\beta$ which reproduce $g'_w (\beta, r = 0.3, \frac{L_0}{a}) = 0.2871$ for $\frac{L_0}{a} = 10, 12, 14, 16$ and $18$, we carry out the simulation for $\sigma = 1.5$ step scaling, namely simulations for $\frac{L_0}{a} = 15, 18, 21, 24$ and $27$ with fixed values of beta we tuned. Those results are used to take the continuum limit, then the resultant value of $g'_w$ becomes a starting value for the next step. We iterate this procedure seven times. The combination of $L_0/a$ and $\beta$ used for simulations are shown in Table II.
4.3. Simulation Details

There are several practical steps to calculate the quantity \( \tilde{g}_w^2(\beta, r, L_0/a) \) from numerical simulations. Here, we explain various technical details of our computations. Let us start from the explanation of the smearing procedure for the reduction of the statistical error. We use the APE smearing\(^{26}\) of link variables defined by the following equation;

\[
U^{(n+1)}_{x,\mu} = \text{Proj}_{SU(3)} \left[ U^{(n)}_{x,\mu} + \frac{1}{c} \sum_{\mu \neq \rho} U^{(n)}_{x,\rho} U^{(n)}_{x+\mu,\rho} U^{(n)}_{x+\rho,\mu} U^{(n)}_{x+\mu+\rho,\nu} \right],
\]

where \( n \) and \( c \) denote a smearing level and a smearing parameter, respectively. The result does not depend on the value of \( c \) significantly, and we take \( c = 2.3 \) in the present study. Here, we need to find an optimal set of a value of \( r \equiv \frac{R+a/2}{L_0} \) and a smearing level \( n \). We find those by considering the following requirements. For better control of discretization error, it is preferable to choose a larger value of \( r \). Meanwhile, for the purpose of reducing the statistical error, it is better to take a smaller value of \( r \) and higher number of \( n \). Fig. 5 shows the smearing-level dependence of \( \tilde{g}_w^2 \) in the case of \( \beta = 8.25 \) and \( L_0/a = 18 \) as an example. From this figure, we find the statistical error is notably reduced even at the smearing level one. In order to avoid over-smearing, \( n \) should be smaller than \( \hat{R}/2 \). This condition gives
Table II. Parameter sets, $L_0/a$ and $\beta$, used for the simulation.

| Step 1 | Step 2 | Step 3 | Step 4 |
|--------|--------|--------|--------|
| $L_0/a$ | $\beta$ | $L_0/a$ | $\beta$ | $L_0/a$ | $\beta$ | $L_0/a$ | $\beta$ |
| 15     | 8.31   | 15     | 7.80   | 15     | 7.44   | 15     | 6.968  |
| 18     | 8.25   | 18     | 7.83   | 18     | 7.45   | 18     | 7.040  |
| 21     | 8.27   | 21     | 7.86   | 21     | 7.49   | 21     | 7.076  |
| 24     | 8.32   | 24     | 7.91   | 24     | 7.55   | 24     | 7.156  |
| 27     | 8.40   | 27     | 7.97   | 27     | 7.61   | 27     | 7.243  |

the lower bound, $L_0/a > (4n + 1)/2$. We summarize the bound from this requirement in Table III. We actually see, for example in the case of $L_0/a = 18$ (see Fig. 5), that the data of $(\hat{R} + 1/2) = 1.5, 2.5$ in higher smearing level are not reliable because of over-smearing. By considering all the above requirements, we find that $(r, n) = (0.3, 1)$ is the optimal choice.

| Step 5 | Step 6 | Step 7 |
|--------|--------|--------|
| $L_0/a$ | $\beta$ | $L_0/a$ | $\beta$ | $L_0/a$ | $\beta$ |
| 15     | 6.571  | 15     | 6.207  | 15     | 5.907  |
| 18     | 6.656  | 18     | 6.303  | 18     | 6.000  |
| 21     | 6.734  | 21     | 6.377  | 21     | 6.087  |
| 24     | 6.797  | 24     | 6.463  | 24     | 6.170  |
| 27     | 6.871  | 27     | 6.546  | 27     | 6.229  |

Table III. The lower bound on $L_0/a$ to avoid over smearing.

Once we fix the value of $r$ ($r = 0.3$ in the case of our study), we need to estimate the value of $g_\alpha^2$ for noninteger $\hat{R}$. We interpolate the value of $g_\alpha^2$ using a quadratic fit function:

$$f(\hat{R} + 1/2) = c_0 + c_1(\hat{R} + 1/2) + c_2(\hat{R} + 1/2)^2,$$

with fit ranges for each lattice size listed in Table IV. We confirmed that the data can be well fitted by our fit function with these fit ranges for all parameter sets. An example of
The values of $\tilde{g}_w^2$ with statistical error for several values of $(\hat{R} + 1/2)$ in the case of $\beta = 8.25$ and $L_0/a = 18$. Data connected by solid, dotted and dashed lines denote the data with 0, 1 and 2 smearing levels, respectively.

Interpolated data is shown in Fig. 5 at $(\hat{R} + 1/2) = 5.4$, which corresponds to the interpolation to $r = 0.3$ in the case of $L_0/a = 18$.

![Fig. 5](image)

Table IV. Fit ranges used to interpolate the value of $\tilde{g}_w^2$. The column "$\hat{R} + 1/2$" is the value that corresponds to $r = 0.3$.

| $L_0/a$ | $\hat{R} + 1/2$ | $\hat{R}_{\text{min}}$ | $\hat{R}_{\text{max}}$ | $L_0/a$ | $\hat{R} + 1/2$ | $\hat{R}_{\text{min}}$ | $\hat{R}_{\text{max}}$ |
|---------|----------------|----------------|----------------|---------|----------------|----------------|----------------|
| 10      | 3.0            | 2              | 4              | 18      | 5.4            | 4              | 6              |
| 12      | 3.6            | 2              | 4              | 21      | 6.3            | 5              | 7              |
| 14      | 4.2            | 2              | 5              | 24      | 7.2            | 5              | 7              |
| 16      | 4.8            | 3              | 5              | 27      | 8.1            | 6              | 8              |

The last step of calculation is to take the continuum limit of $\tilde{g}_w^2$ from data obtained for different combinations of $\beta$ and $L_0/a$ listed in each column of Table II. We show two example plots in Fig. 6, which show the continuum limit for Step 1 and Step 7 listed in Table II. Since our Wilson loop scheme does not have $O(a)$ systematic error as explained in section 2, for the extrapolation of data to the continuum limit, we use a fit function which is linear in terms of $(a/L_0)^2$. Four points ($L_0/a = 27, 24, 21$ and $18$) of data are used to extrapolate to $(a/L_0)^2 = 0$ (shown as red lines in Fig. 6), and the resultant value is adopted as the central value of $\tilde{g}_w^2$ in the continuum limit. We also take the continuum limit by using a fit function which is quadratic in terms of $(a/L_0)^2$ with five points ($L_0/a = 27, 24, 21, 18$ and
15) of data (indicated by pink curves in Fig. 6), and the difference between central values of two continuum limits are adopted as a (upper bound) of the systematic error coming from possible higher order discretization errors. In Fig. 6, we have also plotted extrapolation by linear function with five points of data just for comparison. In the figure, resultant values of continuum limit obtained from different fit functions are plotted at \((a/L_0)^2 = 0\). (For better visibility, we slightly displaced the data obtained from 5-point quadratic and 5-point linear extrapolations.) Error bars indicate statistical errors of the values at the continuum limit.

### 4.4. Numerical Results

Here, we show the results of our simulations which were done for combinations of parameters shown in Table II with procedures explained in the previous section. Details of parameter choice and numerical results are summarized in Appendix B.

In Fig. 7, we plot the resultant values of \(\tilde{g}_w^2\) and their statistical errors for \(L_0/a = 18, 21, 24\) and 27 for Step 1 ~ 7. Continuum limit were taken in the way explained in the previous section, and both statistical and systematic errors were estimated. In Fig. 7 the values of \(\tilde{g}_w^2\) in the continuum limit are shown with total error which is defined by the square root of the sum of the squares of the statistical and systematic errors of each data.

The running coupling constant \(\tilde{g}_w^2\) is obtained by dividing \(\tilde{g}_w^2\) by \(k(r = 0.3) = 0.1661\). The evolution of the running coupling constant is obtained by connecting the resultant values for Step 1 ~ 7 by assigning appropriate scales to each data. We plot the results in Fig. 8. We define the starting energy scale of Step 1 as \(1/\tilde{L}_0\), and the evolution of the running coupling constant is plotted as a function of energy in the unit of \(1/\tilde{L}_0\). The total error
Fig. 7. Results of simulations and continuum limit of $g_w^2$ in Step 1 ~ 7 (from bottom to top).

of the data is also indicated in the figure. Here, errors are accumulated with evolution of the running coupling, i.e., relative error of $g_w^2$ at a certain energy scale is defined as the square root of the sum of the squares of the relative error at that scale and all the relative errors of previous steps. For comparison, we also plotted scheme-independent perturbative running couplings with one-loop and two-loop approximation, as well as that with three-loop approximation in MS-bar scheme (from bottom to top). In the high energy region, where the perturbative computation is reliable, the Wilson loop scheme is consistent with the perturbative calculations. The results also shows that the coupling calculated using the Wilson loop scheme deviate from the two-loop order running below a certain energy scale. This deviation arises due to effects that are not captured by the two-loop approximation.

§5. Discussion

So far, we have concentrated on how the coupling constant runs under the relative change of the scale without referring to its absolute value. To determine the absolute value, we employ the Sommer scale, $r_0$, defined by

$$r^2 F(r) \bigg|_{r=r_0} = 1.65. \quad (5-1)$$
Fig. 8. Evolution of the running coupling constant in Wilson loop scheme, \( g_w^2 \), obtained from step scaling procedure. Horizontal axis shows the energy scale in the unit of \( 1/\tilde{L}_0 \). Three curves, from bottom to top, show scheme-independent perturbative running couplings with one-loop and two-loop approximation, as well as that with three-loop approximation in MS-bar scheme.

The formula relating \( \beta \) and \( r_0 \) is given in Ref.\(^{32} \) which is

\[
\ln(a/r_0) = -1.6805 - 1.7139(\beta - 6) + 0.8155(\beta - 6)^2 - 0.6667(\beta - 6)^3, \tag{5.2}
\]

which is valid in the region \( 5.7 \leq \beta \leq 6.57 \). For our lattices, \( \beta \) values in Step 7 in Table II are suitable. We summarize in Table V \( r_0/a \) and corresponding values of \( r_0/\tilde{L}_0 \). In Ref.\(^{32} \) the values of \( r_0 \) in this range of \( \beta \) are determined within 0.3% accuracy. Therefore, we assign this size of error to the values. By extrapolation to the continuum limit, we obtain

\[
\frac{r_0}{\tilde{L}_0} = 5.1(?) \quad \text{(to be replaced with precise value)} \tag{5.3}
\]

We can now estimate the \( \Lambda \) scale in units of \( r_0 \). Since it is obvious from Fig. 8 that \( g_w^2 \) is well approximated by two-loop perturbative running coupling at high energy region, it is reasonable to estimate the scale \( \Lambda \) by using the value of \( g_w^2(\tilde{L}_0) \) from the following two-loop
Table V. The Sommer scale at each $\beta$ of the Step 7. The value of $r_0$ at each $\beta$ is from Ref. 32.

| $L_0/a$ | $\beta$ | $r_0/a$ | $r_0/L_0$ |
|--------|--------|--------|-----------|
| 15     | 5.907  | ?      | ?         |
| 18     | 6.00   | 5.368(22) | 5.09(?)  |
| 21     | 6.087  | ?      | ?         |
| 24     | 6.170  | ?      | ?         |
| 27     | 6.229  | ?      | ?         |

The Sommer scale at each $\beta$ of the Step 7 is from Ref. 32.

relation$^*)$ between $\bar{L}_0 A$ and $g_w^2(\bar{L}_0)$,

\[
\bar{L}_0 A^{2\text{-loop}}_{\text{WL}} = e^{-\frac{r_0 g_w^2(\bar{L}_0)}{b_0 g_w^2(\bar{L}_0)}} \left( \frac{b_0 g_w^2(\bar{L}_0)}{b_0 + b_1 g_w^2(\bar{L}_0)} \right)^{-\frac{b_1}{2b_0}},
\]

where $b_0 = 11/(4\pi)^2$ and $b_1 = 102/(4\pi)^4$ are the one-loop and the two-loop coefficients of the $\beta$ function of quenched QCD, respectively. By substituting the value $g_w^2(\bar{L}_0) = 1.728$, we found

\[
\bar{L}_0 A^{2\text{-loop}}_{\text{WL}} \simeq 0.0399.
\]

Combining this result with the value of $r_0/\bar{L}_0$ in Eq. (?), we obtain the value of $A^{2\text{-loop}}_{\text{WL}}$ in the unit of $r_0$ as

\[
r_0 A^{2\text{-loop}}_{\text{WL}} = 0.188(?) \pm 0.008.
\]

We also estimated, in the similar way as above, the value of $r_0 A^{2\text{-loop}}$ in the case of SF scheme by using the data reported in 31), and found the following result:

\[
r_0 A^{2\text{-loop}}_{\text{SF}} = 0.301 \pm 0.025.
\]

By fixing $r_0$ as a reference scale, we obtain the following ratio of $A^{2\text{-loop}}_{\text{SF}}$ to $A^{2\text{-loop}}_{\text{WL}}$:

\[
\frac{A^{2\text{-loop}}_{\text{SF}}}{A^{2\text{-loop}}_{\text{WL}}} = 1.60(?) \pm 0.15.
\]

We also estimated the value of $A^{2\text{-loop}}_{\text{SF}}/A^{2\text{-loop}}_{\text{WL}}$ without relying on the measurement of any low-energy physical quantity. This is possible, if we carry out the simulations with exactly the same combinations of values for $\beta$ and $L_0/a$ in the two schemes, corresponding to fixing the physical box size at each $L_0/a$ as a reference scale. Here, we again take a set of data found in 31). In Table VI, we list the values of the coupling-squared in the SF scheme, $g^2_{\text{SF}}$.
Table VI. Values for $g^2$ and $L_0 \Lambda^2$-loop in the SF and the WL schemes for several sets of $L_0/a$ and $\beta$. Values of $g^2_{SF}$ were taken from$^{31}$). The values in parentheses represent statistical errors in units of the last digits.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|c|}
\hline
$L_0/a$ & $\beta$ & $g^2_{SF}$ & $L_0 \Lambda^2$-loop & $g^2_{WL}$ & $L_0 \Lambda^2$-loop & $\Lambda^2_{SF}/\Lambda^2_{WL}$ \\
\hline
10 & 7.8538 & 1.8776(93) & 0.0539(9) & 1.9427(68) & 0.06043(71) & 0.891(18) \\
12 & 7.9993 & 1.8811(38) & 0.0542(4) & 1.8620(66) & 0.05232(65) & 1.036(15) \\
14 & 8.1380 & 1.884(11) & 0.0545(11) & 1.8028(73) & 0.04667(68) & 1.168(29) \\
16 & 8.2500 & 1.864(10) & 0.0525(10) & 1.7662(79) & 0.04331(72) & 1.213(30) \\
\hline
\end{tabular}
\end{table}

(which is denoted as $\bar{g}^2$ in$^{31}$), and values of $\Lambda^2$-loop estimated from them by using Eq. (5·4) for a given sets of parameters ($L_0/a, \beta$).

The values of $g^2_{SF}$ (and $\Lambda^2$-loop estimated from those values) are also listed in Table VI. These values result from an interpolation procedure using the data shown in Fig. 4. From these data, we estimated the ratio of $\Lambda^2_{SF}/\Lambda^2_{WL}$ at each $L_0/a$, and the resulting values are listed in Table VI. A linear fit to these data, gives the following value in the continuum limit:

$$\frac{\Lambda^2_{SF}}{\Lambda^2_{WL}} = 1.78 \pm 0.07 \text{ (stat.)} \pm 0.04 \text{ (sys.)} \quad (5·9)$$

Here, systematic error was estimated by the difference between values in the continuum limit with linear and quadratic extrapolations. We could have expected even smaller systematic error if we could use our data with larger lattice size, i.e., $L_0/a = 15, 18, 21, 24$, however, we couldn’t do that simply because of the lack of data to which we can compare in the SF scheme.

A comparison between the results of Eqs. (5·8) and (5·9) shows the "universality" of the estimate of $\Lambda^2_{SF}/\Lambda^2_{WL}$, namely, the two different estimates are consistent with each other.$^{*}$ From the theoretical point of view, this "universality" might be a trivial result since QCD has only one scale in the theory. However, from the numerical computation point of view, it is a rather non-trivial consistency check since one estimate involves the measurement of a low-energy physical quantity while the other is estimated completely from high-energy physics.

$^{*}$ From the difference between two-loop and three-loop $\beta$ function in the SF scheme, we see the correction to $\Lambda$ estimation due to the higher order effect is about 3%, which is reasonably small compared to the numerical ambiguity of the current analysis.

$^{*}$ Three-loop perturbative calculation (though quite challenging to carry out it in the WL scheme, while it’s already done in the SF scheme), together with the improvement of systematic and statistical errors, would enable us to do even more precise test of this "universality" by defining $\Lambda$ scale with three-loop $\beta$ function.
§6. Summary

We proposed a new scheme for the determination of the running coupling on the lattice. Our method is based on the measurement of the finite volume dependence of the Wilson loop. Unlike the SF scheme, our method does not have any $O(a)$ discretization error, therefore the systematic error arising from the extrapolation to the continuum limit is expected to be quite small. We showed results of numerical study for the quenched QCD as a feasibility test of our scheme, and confirmed that the method actually reproduced the step scaling of the coupling which is consistent with the perturbative running coupling at high energy. We also showed that the coupling calculated by this newly proposed scheme deviates from that with two-loop approximation below a certain energy scale. This deviation arises from the effects that are not captured by the two-loop approximation. We have confirmed that our scheme works well for the calculation of the running coupling with relatively small number of gauge configurations, via demonstrating that the statistical error is under control by properly choosing the smearing level and $r$. We expect that this new method is quite useful also for the calculation of the running coupling in other gauge theories, including the $SU(N)$ gauge theory with a large number of dynamical fermions, which will be studied in our future work.

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Appendix A

--- Thermalization and Decorrelation ---

In this appendix, we discuss the effect of beginning with a cold gauge configuration and autocorrelation among gauge configurations, and estimate required number of sweeps for thermalization and decorrelation.

The left panel of Fig. 9 shows the dependence of the value of $\bar{g}_w^2$ on the number of sweeps discarded after beginning with a cold gauge configurations in the case of for $(\beta, L_0/a) =$
(7.243, 27). Here, we took measurement at every sweep, and collected data from 13500 sweeps. Cold gauge configuration is used for beginning simulations for every 500 sweeps. The result in Fig. 9 shows that at least 30 sweeps should be discarded to remove the influence of the cold gauge configurations. The right panel in Fig. 9 shows the bin-size dependence of the estimated statistical error of the value of $\tilde{g}_w^2$ when we used the binning procedure for the analysis. In the figure, result for $(\beta, L_0/a) = (8.4, 10)$ is shown as an example. Here, we took measurement at every sweep again, and statistical errors were estimated by using the binning procedure, and the magnitude of the estimated errors are plotted for different number of bin sizes. Analysis with binning method underestimate the statistical error when the bin size is not large enough compared to the length of the autocorrelation between configurations. The result in the figure indicates that binning with bin size $50 \sim 100$ (or larger) correctly estimate the magnitude of statistical errors in our simulations. We also confirmed that the length needed for thermalization and decorrelation do not change very much for other parameter sets we use in the present analysis. As we will show in Appendix B, we take large enough number of sweeps between measurement, or take large enough bin size when we take measurement at every sweeps to remove the effect of the autocorrelation. We also run large enough sweeps before we take the first measurement after beginning with a cold configurations.

Fig. 9. Left: Dependence of the value $\tilde{g}_w^2$ on the number of sweeps discarded after beginning with a cold gauge configurations (denoted as $N_{\text{cold}}$) in the case of $(\beta, L_0/a) = (7.243, 27)$. Right: Bin-size dependence of the estimated statistical error of $\tilde{g}_w^2$. Result for $(\beta, L_0/a) = (8.4, 10)$ is shown as an example. The average value of $\tilde{g}_w^2$ in this case is $\tilde{g}_w^2 = 0.279$. 
Appendix B

—— Summary of Simulation Parameters and Numerical Results ——

Here, we summarize simulation parameters, and numerical results obtained from those simulations. In the tables below, \( N_{\text{conf}}, N_{\text{sweep}} \) and \( N_{\text{bin size}} \) respectively represent number of gauge configurations from which measurement were taken, number of sweeps between each configurations, and number of bin size when we estimate the statistical error. The values in parentheses in the column of \( \bar{g}_w^2 \) represent statistical errors in units of the last digits. We also listed the values of \( \bar{g}_w^2 \) in the continuum limit with the magnitude of total error at each step.

### Step 1

| \( L_0/a \) | \( \beta \) | \( N_{\text{conf}} \) | \( N_{\text{sweep}} \) | \( N_{\text{bin size}} \) | \( \bar{g}_w^2 \) |
|---|---|---|---|---|---|
| 15 | 8.31 | 200 | 1000 | 1 | 0.2846(24) |
| 18 | 8.25 | 200 | 1000 | 1 | 0.2999(24) |
| 21 | 8.27 | 400 | 1000 | 1 | 0.3100(21) |
| 24 | 8.32 | 400 | 1000 | 1 | 0.3119(27) |
| 27 | 8.40 | 200 | 1000 | 1 | 0.3137(51) |

continuum limit: \( \bar{g}_w^2 = 0.328 \pm 0.010 \)

### Step 2

| \( L_0/a \) | \( \beta \) | \( N_{\text{conf}} \) | \( N_{\text{sweep}} \) | \( N_{\text{bin size}} \) | \( \bar{g}_w^2 \) |
|---|---|---|---|---|---|
| 15 | 7.80 | 200 | 1000 | 1 | 0.3371(26) |
| 18 | 7.83 | 200 | 1000 | 1 | 0.3465(29) |
| 21 | 7.86 | 200 | 1000 | 1 | 0.3482(36) |
| 24 | 7.91 | 200 | 1000 | 1 | 0.3585(48) |
| 27 | 7.97 | 200 | 1000 | 1 | 0.3632(66) |

continuum limit: \( \bar{g}_w^2 = 0.371 \pm 0.012 \)

### Step 3

| \( L_0/a \) | \( \beta \) | \( N_{\text{conf}} \) | \( N_{\text{sweep}} \) | \( N_{\text{bin size}} \) | \( \bar{g}_w^2 \) |
|---|---|---|---|---|---|
| 15 | 7.44 | 200 | 1000 | 1 | 0.3831(31) |
| 18 | 7.45 | 400 | 1000 | 1 | 0.4056(25) |
| 21 | 7.49 | 400 | 1000 | 1 | 0.4125(33) |
| 24 | 7.55 | 400 | 1000 | 1 | 0.4278(50) |
| 27 | 7.61 | 200 | 1000 | 1 | 0.4249(92) |

continuum limit: \( \bar{g}_w^2 = 0.445 \pm 0.009 \)
### Step 4

| $L_0/a$ | $\beta$ | $N_{\text{conf}}$ | $N_{\text{sweep}}$ | $N_{\text{bin.size}}$ | $\tilde{g}_w^2$  |
|---------|---------|------------------|--------------------|----------------------|------------------|
| 15      | 6.968   | 10000            | 1                  | 100                  | 0.4829(9)       |
| 18      | 7.040   | 10000            | 1                  | 100                  | 0.4959(12)      |
| 21      | 7.076   | 10000            | 1                  | 100                  | 0.5146(15)      |
| 24      | 7.156   | 10000            | 1                  | 100                  | 0.5204(20)      |
| 27      | 7.243   | 10000            | 1                  | 100                  | 0.5181(24)      |

Continuum limit: $\tilde{g}_w^2 = 0.547 \pm 0.004$

### Step 5

| $L_0/a$ | $\beta$ | $N_{\text{conf}}$ | $N_{\text{sweep}}$ | $N_{\text{bin.size}}$ | $\tilde{g}_w^2$  |
|---------|---------|------------------|--------------------|----------------------|------------------|
| 15      | 6.571   | 10000            | 1                  | 100                  | 0.6307(14)       |
| 18      | 6.656   | 10000            | 1                  | 100                  | 0.6489(19)       |
| 21      | 6.734   | 10000            | 1                  | 100                  | 0.6606(27)       |
| 24      | 6.797   | 10000            | 1                  | 100                  | 0.6794(33)       |
| 27      | 6.871   | 10000            | 1                  | 100                  | 0.6931(41)       |

Continuum limit: $\tilde{g}_w^2 = 0.719 \pm 0.024$

### Step 6

| $L_0/a$ | $\beta$ | $N_{\text{conf}}$ | $N_{\text{sweep}}$ | $N_{\text{bin.size}}$ | $\tilde{g}_w^2$  |
|---------|---------|------------------|--------------------|----------------------|------------------|
| 15      | 6.207   | 10000            | 1                  | 100                  | 0.978(3)         |
| 18      | 6.303   | 10000            | 1                  | 100                  | 1.016(5)         |
| 21      | 6.377   | 10000            | 1                  | 100                  | 1.069(6)         |
| 24      | 6.463   | 10000            | 1                  | 100                  | 1.075(7)         |
| 27      | 6.546   | 10000            | 1                  | 100                  | 1.074(9)         |

Continuum limit: $\tilde{g}_w^2 = 1.144 \pm 0.037$

### Step 7

| $L_0/a$ | $\beta$ | $N_{\text{conf}}$ | $N_{\text{sweep}}$ | $N_{\text{bin.size}}$ | $\tilde{g}_w^2$  |
|---------|---------|------------------|--------------------|----------------------|------------------|
| 15      | 5.907   | 10000            | 1                  | 100                  | 1.811(5)         |
| 18      | 6.000   | 10000            | 1                  | 100                  | 1.833(7)         |
| 21      | 6.087   | 10000            | 1                  | 100                  | 1.846(12)        |
| 24      | 6.170   | 10000            | 1                  | 100                  | 1.861(13)        |
| 27      | 6.229   | 10000            | 1                  | 100                  | 1.917(27)        |

Continuum limit: $\tilde{g}_w^2 = 1.914 \pm 0.042$
Appendix C

Formulas (3.23), and (3.24)

In the following, we will briefly show how to obtain formulas (3.23), and (3.24). Basically the method we use is a repetition of the generic technique we have adopted in section 3: we first perform the sum over \( m \), and then use the Chowla-Selberg formula, which rearranges the expression as a sum of some analytic function plus a series, suppressed by the presence of the Kelvin functions \( K_\nu(x) \). For the first term one has:

\[
\sum_{m=-\infty}^{+\infty} \sum_{p,q=1}^{\infty} \frac{1}{(p^2 + q^2 + |m + R/L_0|^2)^s} = 
\sum_{p,q=1}^{\infty} \left[ \frac{\sqrt{\pi} \left( p^2 + q^2 \right)^{\frac{s}{2} - \frac{1}{2}} \Gamma \left( s - \frac{1}{2} \right)}{\Gamma(s)} + \frac{4\pi^s}{\Gamma(s)} \left( p^2 + q^2 \right)^{\frac{s}{2} - \frac{1}{2}} \sum_{j=1}^{\infty} j^{s-\frac{1}{2}} \cos(2j\pi R/L_0) K_{s-\frac{1}{2}} \left( 2j\pi \sqrt{p^2 + q^2} \right) \right] \quad (C.1)
\]

In the above formula, the first term disappears upon derivation with respect to \( r \) and thus will not contribute to \( k \). The second term is exponentially suppressed due to the presence of \( K_0(z) \) and, thus, regular. An analogous procedure applies to \( A_3(r) \):

\[
\sum_{m=-\infty}^{+\infty} \sum_{q=1}^{\infty} \frac{1}{(q^2 + |m + R/L_0|^2)^s} = 
\sum_{q=1}^{\infty} \frac{\sqrt{\pi} \Gamma(2s - 1)}{\Gamma \left( \frac{1}{2} \right)} + 4 \sum_{q,j=1}^{\infty} \cos(2j\pi R/L_0) K_0 \left( 2j\pi q \right) . \quad (C.2)
\]

As before, the first term is independent of \( r \) and disappears, when differentiated with respect to \( r \). The second term is exponentially suppressed and regular.

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