MODULI OF CURVES OF GENUS ONE WITH TWISTED FIELDS*

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Abstract. We construct a smooth Artin stack parameterizing the stable weighted curves of genus one with twisted fields and prove that it is isomorphic to the blowup stack of the moduli of genus one weighted curves studied by Hu and Li. This leads to a blowup-free construction of Vakil-Zinger’s desingularization of the moduli of genus one stable maps to projective spaces. This construction provides the cornerstone of the theory of stacks with twisted fields, which is thoroughly studied in [8] and leads to a blowup-free resolution of the stable map moduli of genus two.

Key words. moduli of weighted curves, twisted fields, blowup-free desingularization.

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1. Introduction. Moduli problems are of central importance in algebraic geometry. Many moduli spaces possess arbitrary singularities [12]. Among them, the moduli \( \overline{M}_g(P^n, d) \) of degree \( d \) stable maps from genus \( g \) nodal curves into projective spaces \( P^n \) are particularly important. We aim to resolve singularities of \( \overline{M}_g(P^n, d) \), that is, to construct a new Deligne-Mumford stack \( \tilde{M}_g(P^n, d) \) admitting a proper morphism onto \( \overline{M}_g(P^n, d) \) and having smooth irreducible components with at worst normal crossing singularities such that the primary component of \( \tilde{M}_g(P^n, d) \) dominates the primary component of \( \overline{M}_g(P^n, d) \) birationally, at least, when \( d \) is sufficiently large. The problem of resolution of singularities is arguably among the hardest ones in algebraic geometry [3, 4, 9, 10].

The stable map moduli are smooth if \( g = 0 \) and singular if \( g \geq 1 \) and \( n \geq 2 \). For \( g = 1 \), a resolution was constructed by Vakil and Zinger [13], followed by an algebraic approach of Hu and Li [5]. The latter is achieved by constructing a canonical smooth blowup \( \tilde{M}_1^{\text{wt}} \) of the Artin stack \( M_1^{\text{wt}} \) of stable weighted nodal curves of genus one. The method of [5] was further developed in [7] to finally establish a resolution in the case of \( g = 2 \). The resolution of [7] is achieved by constructing a canonical smooth blowup \( \tilde{P}_2 \) of the relative Picard stack \( P_2 \) of nodal curves of genus two.

In higher genus cases, the construction of a possible resolution of the stable map moduli may seem formidable. The constructions of the explicit resolutions in [13, 5, 7] rely on certain precise knowledge on the singularities of the moduli. For arbitrary genus, it calls for a more abstract and geometric approach. As advocated by the first author, every singular moduli space should admit a resolution which itself is also a moduli. Following this principle, we interpret the blowup stack \( \tilde{M}_1^{\text{wt}} \) of [5] as a smooth algebraic stack of stable weighted nodal curves of genus one with twisted fields, and consequently, the resolution \( \tilde{M}_1(P^n, d) \) of \( M_1(P^n, d) \) as a Deligne-Mumford stack of genus one stable maps with twisted fields. The results in this paper are the first step to tackle the arbitrary genus case.

The main theorem of this paper is the following:

**Theorem 1.1.** There exists a smooth Artin stack \( \mathcal{M}_1^{\text{tf}} \) parameterizing the weighted nodal curves of genus one with twisted fields, along with a universal family \( C^{\text{tf}} \).
The weighted curves. By saying smooth components, each of which is of finite type and indexed by the total weight together using smooth charts in $\mathcal{M}$. It provides a resolution of $\mathcal{M}$ is birational to $\mathcal{M}$, we mean it is birational on each connected component.

The outline of the proof of Theorem 1.1 is as follows. We first construct the strata of $\mathcal{M}$ and the forgetful map $\varpi$ in §2; see (2.14). We then glue the strata of $\mathcal{M}$ together using smooth charts in §3 and conclude that $\mathcal{M}$ is a smooth Artin stack and is birational to $\mathcal{M}$ in Corollary 3.8. The universal family $\mathcal{C}$ is described in Proposition 3.12. We finally show that $\mathcal{M}$ is isomorphic to $\mathcal{M}$ in Proposition 4.4, which implies the properness of $\varpi$. These results together establish Theorem 1.1.

We remark that both $\mathcal{M}$ and $\mathcal{M}$ are the disjoint unions of infinitely many smooth components, each of which is of finite type and indexed by the total weight of the weighted curves. By saying $\mathcal{M}$ is birational, we mean it is birational on each connected component.

We remark that a direct approach to the properness of $\varpi$ (i.e. without the comparison with the blowup $\mathcal{M}$) is provided in the proof of [8, Theorem 2.19(p1)], in a more general setting.

We also point out that there should exist a groupoid, represented by $\mathcal{M}$, that sends any scheme $S$ to the set of the flat families of stable weighted nodal curves of genus 1 with twisted fields over $S$ as in (3.34); see Remark 3.13 for some details.

According to [5], the resolution $\tilde{\mathcal{M}}_1(\mathbb{P}^n, d)$ of $\mathcal{M}_1(\mathbb{P}^n, d)$ is given by

$$\tilde{\mathcal{M}}_1(\mathbb{P}^n, d) = \mathcal{M}_1(\mathbb{P}^n, d) \times_{\mathcal{M}_1} \mathcal{M}_1$$

where

$$\mathcal{M}_1(\mathbb{P}^n, d) \twoheadrightarrow \mathcal{M}$$

and $\mathcal{M}_1 \rightarrow \mathcal{M}$ is the canonical blowup. Analogously, we take

$$\tilde{\mathcal{M}}_1(\mathbb{P}^n, d) := \mathcal{M}_1(\mathbb{P}^n, d) \times_{\mathcal{M}_1} \mathcal{M}_1$$

where $\mathcal{M}_1 \rightarrow \mathcal{M}$ is the forgetful morphism aforementioned. Theorem 1.1 then leads to the following conclusion immediately.

Corollary 1.2. $\tilde{\mathcal{M}}_1(\mathbb{P}^n, d)$ is a proper Deligne-Mumford stack and is isomorphic to $\mathcal{M}_1(\mathbb{P}^n, d)$.

Via the above isomorphism and applying [5], one sees that the stack $\tilde{\mathcal{M}}_1(\mathbb{P}^n, d)$ provides a resolution of $\mathcal{M}_1(\mathbb{P}^n, d)$. Nonetheless, without relating to $\tilde{\mathcal{M}}_1(\mathbb{P}^n, d)$, we can directly prove the resolution property of $\tilde{\mathcal{M}}_1(\mathbb{P}^n, d)$ by investigating the local equations of $\mathcal{M}_1(\mathbb{P}^n, d)$ in [5] and their pullbacks to $\tilde{\mathcal{M}}_1(\mathbb{P}^n, d)$; see Remark 3.9.

The methods and ideas of this paper are essential to the development in [8] and forthcoming works. Based on the construction of $\mathcal{M}_1$, we introduce the theory of stacks with twisted fields (STF) in [8, Theorem 2.19]. To be somewhat more informative, we work on a smooth stack $\mathcal{M}$ that has a stratification indexed by a set $\Gamma$ of graphs similar to (2.10); see [8, Definition 2.15]. The graphs in $\Gamma$ need not to come from the dual graphs as in (2.10), but the stratification of $\mathcal{M}$ should resemble (3.2) locally. Moreover, $\Gamma$ need not to consist of trees, but it should contain necessary information on the notion of the (weighted) level trees in Definition 2.1 so that we can add the twisted fields to the strata of $\mathcal{M}$ parallel to (2.13) and obtain a new stack.
Such \( M_{tf} \) enjoys desirable properties as in Corollary 3.8 and Remark 3.9. As an application of the STF theory, in [8], we construct a smooth Artin stack \( \mathcal{M}_2^{tf} \) of genus 2 nodal curves with line bundles and twisted fields, along with a proper and birational forgetful morphism \( \mathcal{M}_2^{tf} \to \mathcal{P}_2 \), such that

\[
\tilde{M}_2^{tf}(\mathbb{P}^n, d) = \mathcal{M}_2(\mathbb{P}^n, d) \times_{\mathcal{P}_2} \mathcal{M}_2^{tf} \to \mathcal{M}_2(\mathbb{P}^n, d)
\]

provides a resolution. Further, we expect that they can be extended to arbitrary genus, as far as the existence of moduli of nodal curves with twisted fields is concerned. This is the main motivation of the current article.

In a related work [11], D. Ranganathan, K. Santos-Parker, and J. Wise provide a different modular perspective of \( \tilde{M}_1(\mathbb{P}^n, d) \) using logarithmic geometry.

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**Convention.** The subscript "1" of the relevant stacks indicating the genus appears only in §1 and will be omitted starting §2, as we only deal with the genus 1 case in this paper. In particular, we will denote by

\[
\mathcal{M}^{wt} \quad \text{and} \quad \mathcal{M}^{tf}
\]

the aforementioned stacks \( \mathcal{M}_1^{wt} \) and \( \mathcal{M}_1^{tf} \), respectively.

**2. Set-theoretic descriptions.** In §2.1, we discuss the combinatorics of the dual graphs of nodal curves and introduce the notion of the weighted level trees. They will be used to define \( \mathcal{M}^{tf} \) set-theoretically in §2.2.

**2.1. Weighted level trees.** Let \( \gamma \) be a rooted tree, i.e. a connected finite graph that contains no cycles, along with a special vertex \( o \), called the root. The sets of the vertices and the edges of \( \gamma \) are denoted by

\[
\text{Ver}(\gamma) \quad \text{and} \quad \text{Edg}(\gamma),
\]

respectively. The set \( \text{Ver}(\gamma) \) is endowed with a partial order \( >_{\gamma} \), called the tree order, so that \( v >_{\gamma} v' \) if and only if \( v \neq v' \) and \( v \) belongs to a path between \( o \) and \( v' \). When the context is clear, we drop the subscript \( \gamma \) and simply denote by \( > \) the tree order. We write \( v \geq v' \) if either \( v > v' \) or \( v = v' \). The root \( o \) is thus the unique maximal element of \( \text{Ver}(\gamma) \) with respect to the tree order.

For each \( e \in \text{Edg}(\gamma) \), we denote by \( v_e^+ \in \text{Ver}(\gamma) \) the endpoints of \( e \) such that \( v_e^+ > v_e^- \). Then, every vertex \( v \in \text{Ver}(\gamma) \setminus \{o\} \) corresponds to a unique

\[
e_e \in \text{Edg}(\gamma) \quad \text{satisfying} \quad v_e^- = v.
\]

The tree order on \( \text{Ver}(\gamma) \) induces a partial order on \( \text{Edg}(\gamma) \), still called the tree order, so that

\[
e > e' \iff v_e^- \geq v_{e'}^-.
\]

We write \( e \geq e' \) if either \( e > e' \) or \( e = e' \).

We call a pair \( \tau = (\gamma, w) \) consisting of a rooted tree \( \gamma \) and a function

\[
w : \text{Ver}(\gamma) \to \mathbb{Z}_{\geq 0}
\]
a weighted tree. For such \( \tau \), we write \( \text{Ver}(\tau) = \text{Ver}(\gamma) \) and \( \text{Edg}(\tau) = \text{Edg}(\gamma) \). The set of all the weighted trees is denoted by \( \mathcal{T}^{\text{wt}}_R \), which will be used to index the stratification 2.10.

Next, we introduce the extra structures needed for \( \mathcal{M}^{\text{tf}} \). We call a map \( \ell : \text{Ver}(\gamma) \rightarrow \mathbb{R}_{\leq 0} \) satisfying
\[
\ell^{-1}(0) = \{0\} \quad \text{and} \quad \ell(v) > \ell(v') \quad \text{whenever} \quad v > v'
\]
a level map. For each \( i \in \ell(\text{Ver}(\gamma)) \setminus \{0\} \), let
\[
i^2 = \min \{ k \in \ell(\text{Ver}(\gamma)) : k > i \},
\]
i.e. the level \( i^2 \) is right “above” the level \( i \); see Figure 1. We remark that a rooted tree along with a level map is called a level graph with the root as the unique top level vertex in \([1, \S 1.5]\).

**Definition 2.1.** We call the tuple
\[
t = (\gamma, w : \text{Ver}(\gamma) \rightarrow \mathbb{Z}_{\geq 0}, \ell : \text{Ver}(\gamma) \rightarrow \mathbb{R}_{\leq 0})
\]
a weighted level tree if \((\gamma, w) \in \mathcal{T}^{\text{wt}}_R \) and \( \ell \) is a level map.

For every weighted level tree \( t \) as above, we write \( \text{Ver}(t) = \text{Ver}(\gamma) \) and \( \text{Edg}(t) = \text{Edg}(\gamma) \). Set
\[
\begin{align*}
m(t) &= \max \{ \ell(v) : v \in \text{Ver}(t), w(v) > 0 \} \quad (\leq 0), \\
\widehat{\text{Edg}}(t) &= \{ e \in \text{Edg}(t) : \ell(v_e^+) > m \} \quad (\subseteq \text{Edg}(t)).
\end{align*}
\]
For any two levels \( i, j \in \mathbb{R}_{\leq 0} \), we write
\[
[i, j]_t = \ell(\text{Ver}(t)) \cap (i, j), \quad [i, j]_t = \ell(\text{Ver}(t)) \cap [i, j].
\]
For every \( e \in \widehat{\text{Edg}}(t) \), let
\[
\ell(e) = \max \{ \ell(v_e^-), m \} \quad (\in [m, 0]_i).
\]
For each level \( i \in [m, 0]_t \), we set
\[
\mathcal{E}_i = \mathcal{E}_i(t) = \{ e \in \widehat{\text{Edg}}(t) : \ell(e) \leq i < \ell(v_e^+) \}.
\]
Intuitively, \( \mathcal{E}_i \) consists of all the edges crossing the gap between the levels \( i \) and \( i^2 \).

We point out that all the notions in the preceding paragraph depend on the weighted level tree \( t \), and we may hereafter omit \( t \) in any of such notions when the context is clear.

**Remark 2.2.** The weighted level trees will be used to label the strata \( \mathcal{M}^{\text{tf}}_{[t]} \) of \( \mathcal{M}^{\text{tf}} \). Locally speaking, the edges in \( \widehat{\text{Edg}}(t) \) correspond to the normal directions of the blowup centers of \( \mathcal{M}^{\text{wt}} \rightarrow \mathcal{M}^{\text{wt}} \) in [5]; each level \( i \) (\( \geq m \)) corresponds to an exceptional divisor \( \tilde{E}_i' \); the edges \( e \in \mathcal{E}_i \) with \( \ell(v_e^-) = i \) correspond to the non-zero components of \( \tilde{E}_i' \); the level \( m \) indicates when the sequential blowup terminates.
Every weighted level tree $t$ determines a unique index set
\[
\begin{align*}
\mathbb{I}(t) &= \mathbb{I}_+(t) \sqcup \mathbb{I}_m(t) \sqcup \mathbb{I}_-(t), \quad \text{where} \quad \mathbb{I}_+(t) = [m, 0]_t, \\
\mathbb{I}_m(t) &= \{e \in \text{Edg}(t) : \ell(v^-_e) < m\}, \quad \mathbb{I}_-(t) = (\text{Edg}(t) \setminus \text{Edg}(t)).
\end{align*}
\] (2.4)

The set $\mathbb{I}_+(t)$ becomes empty if $m = 0$, i.e. the root $o$ is positively weighted. As mentioned before, we may simply write
\[
\mathbb{I} = \mathbb{I}(t), \quad \mathbb{I}_+ = \mathbb{I}_+(t), \quad \mathbb{I}_m = \mathbb{I}_m(t)
\] when the context is clear. We point out that $\mathbb{I}_+$ consists of levels, whereas $\mathbb{I}_m$ and $\mathbb{I}_-$ consist of edges.

For each $I \subset \mathbb{I}$ (possibly empty), let
\[
I_m = I \cap \mathbb{I}_m, \quad I_+ = I \cap \mathbb{I}_+.
\]
The subset $I$ determines a weighted level tree
\[
t(I) = (\tau(I), \ell(I)) = (\gamma(I), w(I), \ell(I))
\] (2.5)
intuitively as follows: the weighted level tree $t(I)$ is obtained from $t$ by contracting all the edges labeled by $I_-$, then lifting all the vertices $v$ with $e_v \in I_m$ to the level $m$, and finally contracting all the levels in $I_+$; see Figures 2 and 3 for illustration. Such $t(I)$ will be used to describe the local structure of $\mathfrak{M}^G$ in §3.

Below we present the formal definition of $t(I)$:
- the rooted tree $\gamma(I)$ is obtained via the edge contraction
  \[
  \pi(I) : \text{Ver}(\gamma) \to \text{Ver}(\gamma(I))
  \] such that the set of the contracted edges is
  \[
  \text{Edg}(t) \setminus \text{Edg}(t(I)) = \{e \in (\text{Edg}(t) \setminus I_m) \cup I_m : \{\ell(e), \ell(v_e^+)\}_l \subset I_+ \cup I_-\};
  \] (2.6)
- the weight function $w(I)$ is given by
  \[
  w(I) : \text{Ver}(\gamma(I)) \to \mathbb{Z}_{\geq 0}, \quad w(I)(v) = \sum_{v' \in \pi^{-1}(v)} w(v');
  \]
- the level map $\ell(I)$ is such that for any $e \in \text{Edg}(\gamma(I))$, (⊂ Edg(t)),
  \[
  \ell(I)(v^-_e) = \begin{cases} 
  \min\{i \in \mathbb{I}_+ \cup I_+ : i \geq \ell(v) \land \forall v \in \pi^{-1}(v^-_e)\} & \text{if } e \in \text{Edg}(t) \setminus I_m, \\
  \min I_+ & \text{if } e \in I_m, \\
  \max\{\ell(v) : v \in \pi^{-1}(v^-_e)\} & \text{if } e \in (I_m \setminus I_m) \cup I_-.
  \end{cases}
  \]

It is a direct check that $\tau(I)$ is a weighted tree and $\ell(I)$ satisfies the criteria of a level map, hence (2.5) gives a well defined weighted level tree.

The construction of $t(I)$ implies
\[
\begin{align*}
\mathbb{I}_+(t(I)) &= \mathbb{I}_+ \setminus I_+, \quad m(t(I)) = \min((\mathbb{I}_+ \cup \{0\}) \\
\mathbb{I}_m(t(I)) &= \{e \in \mathbb{I}_m \setminus I_m : \ell(v^-_e) > m(t(I))\}, \\
\mathbb{I}_-(t(I)) &= \mathbb{I}_- \cup \{e \in \mathbb{I}_m \setminus I_m : \ell(v^-_e) \leq m(t(I))\}.
\end{align*}
\] (2.7)

**Definition 2.3.** Two weighted level trees $t = (\gamma, w, \ell)$ and $t' = (\gamma', w', \ell')$ are said to be equivalent, written as $t \sim t'$, if
(E1) \((\gamma, w) = (\gamma', w')\) as weighted trees;
(E2) \(\ell^{-1}[m(t), 0]_1 = (\ell')^{-1}[m(t'), 0]_0\);
(E3) there exists a (unique) order preserving bijection \(\alpha: [m(t), 0]_1 \rightarrow [m(t'), 0]_0\) so that \(\alpha \circ \ell = \ell'\) on \(\ell^{-1}[m(t), 0]_1\).

It is a direct check that \(\sim\) is an equivalence relation on the set of weighted level trees. Intuitively, this equivalence relation records the relative positions of the vertices above or in the level \(m(t)\); see Figure 1 for illustration. We do not need to know the relative positions of the vertices below \(m(t)\), because from the blowup point of view, the level \(m(t)\) corresponds to when the blowup \(\mathcal{M}^{wt} \rightarrow \mathfrak{M}^{wt}\) stops locally; c.f. Remark 2.2.

We denote by \(\mathcal{R}^{wt}\) the set of the equivalence classes of the weighted level trees. There is a natural forgetful map

\[
\mathcal{f}: \mathcal{R}^{wt} \rightarrow \mathcal{R}^{rwt}, \quad [\gamma, w, \ell] \mapsto (\gamma, w),
\]

which is well defined by Condition (E1) of Definition 2.3.

### 2.2. Twisted fields.

For every genus 1 nodal curve \(C\), its dual graph \(\gamma_C^*\) has either a unique vertex \(o\) corresponding to the genus 1 irreducible component of \(C\) or a unique loop. In the former case, \(\gamma_C^*\) can be considered as a rooted tree with the root \(o\); in the latter case, we contract the loop to a single vertex \(o\) and obtain a rooted tree with the root \(o\). Such defined rooted tree is denoted by \(\gamma_C\) and called the reduced dual tree of \(C\) (c.f. [5, §3.4]). We call the minimal connected genus 1 subcurve of \(C\) the core and denote it by \(C_o\). Other irreducible components of \(C\) are smooth rational curves and denoted by \(C_v, v \in \text{Ver}(\gamma_C) \setminus \{o\}\). For every incident pair \((v, e)\), let

\[
q_{v,e} \in C_v
\]

be the nodal point corresponding to the edge \(e\).

Let \(\mathfrak{M}^{wt}\) be the Artin stack of genus 1 stable weighted curves introduced in [5, §2.1]. Here the subscript "1" indicating the genus is omitted as per our convention. The stack \(\mathfrak{M}^{wt}\) consists of the pairs \((C, w)\) of genus 1 nodal curves \(C\) with non-negative weights \(w \in H^2(C, \mathbb{Z})\), meaning that \(w(\Sigma) \geq 0\) for all irreducible \(\Sigma \subset C\). Here \((C, w)\) is said to be stable if every rational irreducible component of weight 0 contains at least three nodal points. The weight of the core \(w(C_o)\) is defined as the sum of the weights of all irreducible components of the core.

Every \((C, w) \in \mathfrak{M}^{wt}\) uniquely determines a function

\[
w : \text{Ver}(\gamma_C) \rightarrow \mathbb{Z}_{\geq 0}, \quad v \mapsto w(C_v),
\]
which makes the pair \((\gamma_C, w)\) a weighted tree, called the weighted dual tree. Thus, the stack \(\mathcal{M}^{wt}\) can be stratified as

\[
\mathcal{M}^{wt} = \bigsqcup_{\tau \in \mathcal{F}^{wt}} \mathcal{M}^{wt}_\tau = \bigsqcup_{\tau \in \mathcal{F}^{wt}} \{ (C, w) \in \mathcal{M}^{wt} : (\gamma_C, w) = \tau \}.
\]

(2.10)

If the sum of the weights of all vertices is fixed, the stability condition of \(\mathcal{M}^{wt}\) then guarantees there are only finitely many \(\tau \in \mathcal{F}^{wt}\) so that \(\mathcal{M}^{wt}_\tau\) is non-empty.

Given \(\tau = (\gamma, w) \in \mathcal{F}^{wt}\) and \(e \in \text{Edg}(\tau)\), let

\[
L_e^\pm \longrightarrow \mathcal{M}^{wt}_\tau
\]

be the line bundles whose fibers over a weighted curve \((C, w)\) are the tangent vectors of the irreducible components \(C_{e^\pm}\) at the nodal points \(q_{e^\pm}\), respectively. We take

\[
L_e = L_e^+ \otimes L_e^- \longrightarrow \mathcal{M}^{wt}_\tau, \quad L_e^\geq = \bigotimes_{e' \in \text{Edg}(\tau), e' \geq} L_{e'}, \quad \longrightarrow \mathcal{M}^{wt}_\tau.
\]

(2.11)

The line bundles \(L_e\) describe normal directions of \(\mathcal{M}^{wt}_\tau\); c.f. Lemma 3.1 and its proof.

**Remark 2.4.** The motivation of \(L_e^\geq\) is twofold: one comes from the comparison with the blowup construction \(\hat{\mathcal{M}}^{wt} \longrightarrow \mathcal{M}^{wt}\), while the other from close examination of the local equations of the local equations of \(\hat{M}_1(\mathbb{P}^n, d)\). The former will be elaborated in (4.5) and the paragraph after it; we now explain the latter. Given \((C, u) \in \hat{M}_1(\mathbb{P}^n, d)\), we denote by \((C, w)\) its image in \(\mathcal{M}^{wt}\) under (1.1). By (2.10), there exists a unique \(\tau \in \mathcal{F}^{wt}\) such that \((C, w) \in \mathcal{M}^{wt}_\tau\). W.l.o.g. we assume that \(u|_{C_{\gamma}}\) is constant (i.e. \(w(o) = 0\)), for otherwise \(\hat{M}_1(\mathbb{P}^n, d)\) is already smooth near \((C, u)\). As shown in [5, §5.2], over an affine smooth chart \(\mathcal{V} \longrightarrow \mathcal{M}^{wt}\) containing \((C, w)\), one can identify an open neighborhood of \((C, u) \in \hat{M}_1(\mathbb{P}^n, d)\) with the kernel of a homomorphism of the trivial vector bundles: \((\mathcal{O}_{\mathcal{V}}^\otimes d)^\otimes \longrightarrow \mathcal{O}_{\mathcal{V}}^\otimes n\). Under suitable trivialization, the vector bundle homomorphism can be written as a matrix. Each entry of the matrix is (up to a unit on \(\mathcal{V}\)) in the form \(\prod_{e' \geq e} \zeta_{e'}\), where \(e \in \text{Edg}(\tau)\) satisfies \(w(v^{-}_e) > 0\), and the local parameters \(\zeta_{e'}\) correspond to the smoothing of the nodes (see (3.1) for exact definition).

To desingularize \(\hat{M}_1(\mathbb{P}^n, d)\), one needs to compare the products \(\prod_{e' \geq e} \zeta_{e'}\) for different edges \(e\) near \(\mathcal{M}^{wt}_\tau\) (where all \(\zeta_{e'} = 0\)). Given two edges \(e_1\) and \(e_2\), the limit of the ratio \([(\prod_{e' \geq e_1} \zeta_{e'}) : (\prod_{e' \geq e_2} \zeta_{e'})] \) at \(\mathcal{M}^{wt}_\tau\) depends on the choice of the local parameters \(\zeta_{e'}\) in (3.1) unless \(e_1\) and \(e_2\) are comparable. Nonetheless, with \(\partial_{\zeta_{e'}}\) as before Lemma 3.1, the limit of the ratio

\[
\left[ \otimes_{e' \geq e_1} \zeta_{e'} \partial_{\zeta_{e'}} : \otimes_{e'' \geq e_2} \zeta_{e''} \partial_{\zeta_{e''}} \right]
\]

at \(\mathcal{M}^{wt}_\tau\) does not depend on the choice of \(\zeta_{e'}\); c.f. (3.31). By Lemma 3.1, the restriction of each \(\partial_{\zeta_{e'}}\) to \(\mathcal{M}^{wt}_\tau\) is a nowhere vanishing section of \(L_e/(\mathcal{V} \cap \mathcal{M}^{wt}_\tau)\), which suggests that \(L_e^\geq\) should be a key part of (2.13).

For any direct sum of line bundles \(V = \bigoplus_m L'_m\) (over any base), we write

\[
\widehat{\mathcal{P}}(V) := \{(x, [v_m]) \in \mathcal{P}(V) : v_m \neq 0 \text{ } \forall \text{ } m\}.
\]

(2.12)

For any vector bundle \(V'\) over the same base, by abuse of notation, we identify \(V\) with \(V \oplus 0 \subset V \oplus V'\) and identify \(\widehat{\mathcal{P}}(V)\) with its image in \(\mathcal{P}(V \oplus V')\). For any morphisms \(M_1, \ldots, M_k \longrightarrow S\), we write

\[
\prod_{1 \leq i \leq k} (M_i/S) := M_1 \times_S M_2 \times_S \cdots \times_S M_k.
\]
With notation as above, given \( \tau \in \mathcal{T}_R^{wt} \) and \([t]=\left[\tau, \ell\right] \in \mathcal{T}_L^{wt}\), let

\[\varpi: \mathcal{M}_{[t]}^f = \left( \prod_{i \in I_+ (t)} \left( \mathbb{P} \left( \bigoplus_{e \in \text{Edg}(t), \ell (v_e^-) = i} \mathcal{M}_\tau^{eq} \right) \right) / \mathcal{M}_\tau^{eq} \right) \rightarrow \mathcal{M}_\tau^{wt},\]

\[\mathcal{E}_{[t]} = \left( \prod_{i \in I_+ (t)} \left( \mathbb{P} \left( \bigoplus_{e \in \mathcal{E}_i} \mathcal{M}_\tau^{eq} \right) \right) / \mathcal{M}_\tau^{eq} \right) \rightarrow \mathcal{M}_\tau^{wt},\]

(2.13)

where \( I_+ (t) \), \( L_e^\geq \), and \( \mathcal{E}_i \) are as in (2.4), (2.11), and (2.3), respectively. It is straightforward that both bundles in (2.13) are independent of the choice of the weighted level tree \( t \) representing \([t]\). Since

\[\{ e \in \text{Edg}(t) : \ell (v_e^-) = i \} \subset \mathcal{E}_i \quad \forall \ i \in I_+ (t),\]

we see that \( \mathcal{M}_{[t]}^f \) is a subset of \( \mathcal{E}_{[t]} \). In addition, since each stratum \( \mathcal{M}_\tau^{wt} \) is an algebraic stack, so are \( \mathcal{M}_{[t]}^f \) and \( \mathcal{E}_{[t]} \).

Using (2.13) and (2.10), we define

\[\mathcal{M}^f := \bigsqcup_{[t] \in \mathcal{T}_L^{wt}} \mathcal{M}_{[t]}^f \xrightarrow{\varpi} \mathcal{M}^{wt}.\]

(2.14)

This is the set-theoretic definition of the proposed stack \( \mathcal{M}^f \) as well as the forgetful map in Theorem 1.1. For any \( x \in \mathcal{M}_\tau^{wt} \), the points of the fiber \( \mathcal{M}_{[t]}^f \big|_x \) are called the twisted fields over \( x \).

**Remark 2.5.** By (2.14), \( \tilde{M}_1^f (\mathbb{P}^n, d) \) in Corollary 1.2 consists of the tuples

\[(C, u, [t], \eta),\]

where \( (C, u) \) are stable maps in \( \overline{M}_1 (\mathbb{P}^n, d) \), \([t]\) are the equivalence classes of weighted level trees satisfying \( \bar{f}[t] = (\gamma_C, c_1 (u^* \mathcal{O}_{\mathbb{P}^n} (1))) \), and \( \eta \) are twisted fields over \( (C, c_1 (u^* \mathcal{O}_{\mathbb{P}^n} (1))) \).

### 2.3. Examples on the strata of \( \mathcal{M}^f \)

In this subsection, we use two examples to illustrate the strata of \( \mathcal{M}^f \). These two examples will be revisited in §3.3 to illustrate how the strata are glued together by the twisted charts (3.16).

**Example 2.6.** Let \([t]=\left[\tau, \ell\right] \in \mathcal{T}_L^{wt}\) be given by the leftmost diagram in Figure 2. Then,

\[\text{Edg}(t) = \{a, b, c, d\}, \quad m = -2, \quad I = I_+ = \{-1, -2\}, \quad I_m = I_- = \emptyset.\]
As illustrated in Figure 2, each \( I \subset \mathbb{I} \) determines a weighted level tree \( t(I) \), whose underlying weighted tree is denoted by \( \tau(I) \) as in (2.5). Each \( t(I) \) in turn gives rise to a stratum of \( \mathcal{M}^\text{tf} \) as follows:

\[
\mathcal{M}^\text{tf}_{[t]} = \mathbb{P}(L_b) \times \mathcal{M}^\text{wt} \left( \mathbb{P}(L_a \oplus L_b \otimes L_c \oplus L_b \otimes L_d) \right), \\
\mathcal{M}^\text{tf}_{[t_{(1,-1,2)}]} = \mathbb{P}(L_a \oplus L_b \otimes L_d)/\mathcal{M}^\text{wt}_{\tau_{(1,-1,2)}}, \\
\mathcal{M}^\text{tf}_{[t_{(1,-1,2)}]} = \mathbb{P}(L_a \oplus L_b \otimes L_d)/\mathcal{M}^\text{wt}_{\tau_{(1,-1,2)}}.
\]

In Example 3.10, we will demonstrate how to glue these strata by charts.

**Example 2.7.** Let \( [t] = [\tau, \ell] \in \mathcal{L}^\text{wt} \) be given by the leftmost diagram in the first row of Figure 3. Then,

\[
\text{Edg}(t) = \{a, b, c, d\}, \quad m = -2, \quad \mathbb{I} = \{-1, -2, a\}, \quad \mathbb{I} = \{-1, -2\}, \quad \mathbb{I}_m = \{a\}, \quad \mathbb{I}_e = \emptyset.
\]

As illustrated in Figure 3, each \( I \subset \mathbb{I} \) determines a weighted level tree \( t(I) \), whose underlying weighted tree is denoted by \( \tau(I) \) as in (2.5). Each \( t(I) \) in turn gives rise to a stratum of \( \mathcal{M}^\text{tf} \) as follows:

\[
\mathcal{M}^\text{tf}_{[t]} = \mathbb{P}(L_b) \times \mathcal{M}^\text{wt} \left( \mathbb{P}(L_a \oplus L_b \otimes L_c \oplus L_b \otimes L_d) \right), \\
\mathcal{M}^\text{tf}_{[t_{(1,-2,2)}]} = \mathbb{P}(L_b)/\mathcal{M}^\text{wt}_{\tau_{(1,-2,2)}}, \\
\mathcal{M}^\text{tf}_{[t_{(1,-1,2)}]} = \mathbb{P}(L_a \oplus L_a \otimes L_d)/\mathcal{M}^\text{wt}_{\tau_{(1,-1,2)}}, \\
\mathcal{M}^\text{tf}_{[t_{(1,-1,2)}]} = \mathbb{P}(L_a \oplus L_a \otimes L_d)/\mathcal{M}^\text{wt}_{\tau_{(1,-1,2)}}, \\
\mathcal{M}^\text{tf}_{[t_{(1,-1,2)}]} = \mathbb{P}(L_a \oplus L_a \otimes L_c \oplus L_b \otimes L_c)/\mathcal{M}^\text{wt}_{\tau_{(1,-1,2)}}.
\]

In Example 3.11, we will demonstrate how to glue these strata by charts.

**3. The stack structure of \( \mathcal{M}^\text{tf} \).** In §3, we show \( \mathcal{M}^\text{tf} \) is naturally a smooth Artin stack and describe its universal family. To help understand the combinatorial notation in §3.1 and §3.2, the readers may consult the examples in §3.3.

**3.1. Twisted charts.** We first fix \( [t] = [\gamma, w, \ell] \in \mathcal{L}^\text{wt} \) and \( x \in \mathcal{M}^\text{tf}_{[t]} \), and write \( \tau = f[t] = (\gamma, w) \in \mathcal{L}^\text{wt} \), \( (c, w) = \varpi(x) \in \mathcal{M}^\text{wt}_{\gamma} \).

Since \( \mathcal{M}^\text{wt} \) is smooth, we take an affine smooth chart

\[
\mathcal{V} = \mathcal{V}_{\varpi(x)} \longrightarrow \mathcal{M}^\text{wt}
\]

containing \( (c, w) \).

As in [5, §4.3] and [7, §2.5], there exists a set of regular functions

\[
\zeta_e \in \Gamma(\mathcal{O}_{\mathcal{V}}) \quad \text{with} \quad e \in \text{Edg}(\gamma),
\]

called the **modular parameters**, so that for each \( e \in \text{Edg}(\gamma) \), the locus

\[
\mathcal{Z}_e = \{ \zeta_e = 0 \} \subset \mathcal{V}
\]
is the irreducible smooth Cartier divisor on $\mathcal{V}$ where the node labeled by $e$ is not smoothed. For any $I \subset \mathbb{I} = \mathbb{I}(t)$, let

$$
\mathcal{V}_I^o := \{ \zeta' \neq 0; e' \in (\text{Edg}(\gamma) \setminus \text{Edg}(\gamma(I))) \} \subset \mathcal{V},
$$

$$
\mathcal{V}_{\tau(I)} := \mathcal{V}_I^o \cap \{ \zeta_e = 0; e \in \text{Edg}(\gamma(I)) \} \subset \mathcal{V}_I^o \subset \mathcal{V}.
$$

Then, $\mathcal{V}_{\tau(I)}$ is an open subset of $\mathcal{V}$. Shrinking $\mathcal{V}$ if necessary, we see that

$$
\mathcal{V}_{\tau(I)} \subset \pi_0(\mathcal{M}_{\tau(I)}^{\text{wt}} \cap \mathcal{V}),
$$

where $\pi_0$ denotes the set of the connected components. In particular, $\mathcal{V}_{\tau(I)}$ can be considered as a smooth chart of the stratum $\mathcal{M}_{\tau(I)}^{\text{wt}}$. Rigorously, the sets $\mathbb{I}$ and $I$ depend on the choice of the weighted level tree $t$ representing $[\mathbb{I}(t)]$, however, $\mathcal{V}_{\tau(I)}$ and $\mathcal{V}_{\tau(I)}$ are independent of such choice.

Given a set of the modular parameters as in (3.1), we may extend it to a set of local parameters on $\mathcal{V}$ centered at $(C, \omega)$:

$$
\{ \zeta_e \}_{e \in \text{Edg}(\gamma)} \cup \{ \zeta_j \}_{j \in J} \quad \text{with} \quad (C, \omega) = 0 := (0, \ldots, 0),
$$

where $J$ is a finite set. We do not impose other conditions on $\zeta_j$.

For each $e \in \text{Edg}(\gamma)$, let $\partial_{\zeta_e} \in \Gamma(\mathcal{V}; T\mathcal{M}_{\tau(I)}^{\text{wt}})$ be such that

$$
\langle \partial_{\zeta_e}, d\zeta_e \rangle = 1, \quad \langle \partial_{\zeta_e}, d\zeta_{e'} \rangle = 0 \quad \forall e', e \in \text{Edg}(\gamma) \setminus \{e\}, \quad \langle \partial_{\zeta_e}, d\zeta_j \rangle = 0 \quad \forall j \in J.
$$

For $Z_e = \{ \zeta_e = 0 \}$, we denote by $\overline{\partial}_{\zeta_e}$ the image of $\partial_{\zeta_e}|_{Z_e}$ in the normal bundle of $Z_e$, i.e.

$$
\overline{\partial}_{\zeta_e}|_Z := \partial_{\zeta_e}|_Z + T_Z Z_e \quad \forall Z \in Z_e.
$$

**Lemma 3.1.** For every $I \subset \mathbb{I}$ and every $e \in \text{Edg}(\gamma(I))$, the restriction of $\overline{\partial}_{\zeta_e}$ to $\mathcal{V}_{\tau(I)}$ is a nowhere vanishing section of the restriction of the line bundle $L_e$ in (2.11) to $\mathcal{V}_{\tau(I)}$.
Proof. Since the restriction of $d\zeta_e$ to $Z_e \subset Z_{(1)}$ is identically zero, we observe that $\partial \zeta_e$ is a nowhere vanishing section of the normal bundle of $Z_e$. It is a well-known fact of the moduli of curves that the normal bundle of $Z_e$ is $L_e$; see [2, Proposition 3.31].

For each level $i \in \mathbb{Z}$, we choose a special vertex

$$v_i \in \text{Ver}(\gamma) \quad \text{s.t.} \quad \ell(v_i) = i.$$  \hspace{1cm} (3.4)

Recall that for every $v \in \text{Ver}(\gamma) \setminus \{o\}$, the edge $e_v$ “above” $v$ is given by $e_v \supset v$. We then denote by $e_i$, $e_i^+$, and $v_i^+$ respectively the edges and the vertex satisfying

$$e_i = e_{v_i}, \quad v_i^+ = v_{e_i}, \quad e_i^+ = e_{v_i};$$  \hspace{1cm} (3.5)

see Figure 1. We remark that $e_i^+$ is not defined if $v_i^+ = o$. Each $i \in \mathbb{Z}$ determines a strictly increasing sequence

$$i[0] := i < i[1] := \ell(v_i) < i[2] := \ell(v_{i[1]}) < \cdots.$$  \hspace{1cm} (3.6)

We would like to remark that $i[1]$ and $i^2$ in (2.1) need not to be the same; see Figure 1 for illustration. This sequence is finite, as there is a unique step $h$ satisfying $i[h] = 0$.

Remark 3.2. As mentioned in Remark 2.2, for each level $i \in \mathbb{Z}$, the vertices $v$ with $\ell(v) = i$ (or equivalently the edges $e$ with $\ell(e) = i$) locally correspond to the non-zero components of an exceptional divisor in the blowup $\overline{\mathcal{M}}^{\text{wt}} \to \mathcal{M}^{\text{wt}}$ of [5]. Thus, choosing $v_i$ (or equivalently $e_i$) means selecting a standard chart for the exceptional divisor. Such choices affect the expression (3.14), hence affect the twisted chart (3.16). Nonetheless, Lemma 3.5 guarantees that they do not affect the stack structure of $\mathcal{M}^{\text{tw}}$.

By Lemma 3.1, there exist $\lambda \in \mathbb{A}$ with $e \in \overline{\text{Ed}}(t)$ so that the fixed $x \in \mathcal{M}^{\text{if}} \times (C, w) = 0 \in \mathcal{M}^{\text{if}} \times (C, w)$ can uniquely be written as

$$x = \left(0; \prod_{i \in \mathbb{Z}_+} \left(\lambda_i e_i, (\mathcal{M}^{\text{if}}_i)_{0} : \ell(e) = i\right)\right)$$

$$\in \mathcal{M}^{\text{if}} \times (C, w),$$

where $\lambda_i = 1, \forall i \in \mathbb{Z}_+$, $\lambda_i = 0, \forall i \in \mathbb{Z}$. With the index set $J$ as in (3.3) (which indexes the local parameters $\zeta_j$ on $\mathcal{V}$ other than the modular parameters $\zeta_e$), let

$$\mathcal{M}_x \subset \mathbb{A}^{\mathbb{Z}_+} \times \mathbb{A}^{\overline{\text{Ed}}(t) \setminus \{e_i : i \in \mathbb{Z}_+\}} \times \mathbb{A}^{\mathbb{Z}_-} \times \mathbb{A}^{J}$$

be an open subset containing the point

$$y_x := \left(0, (\lambda_i e_i)_{e \in \overline{\text{Ed}}(t) \setminus \{e_i : i \in \mathbb{Z}_+\}}, 0, 0\right).$$  \hspace{1cm} (3.8)

The coordinates on $\mathcal{M}_x$ are denoted by

$$\left((\xi_i)_{i \in \mathbb{Z}_+}, (u_e)_{e \in \overline{\text{Ed}}(t) \setminus \{e_i : i \in \mathbb{Z}_+\}}, (z_e)_{e \in \mathbb{Z}_-}, (w_j)_{j \in J}\right).$$  \hspace{1cm} (3.9)
For any $I \subset \mathbb{I}$, we set

$$\Upsilon^c_{x; (I)} = \{ \varepsilon_i \neq 0 \, \forall \, i \in I_+ \, ; \, u_e \neq 0 \, \forall \, e \in I_m \, ; \, z_e \neq 0 \, \forall \, e \in I_- \} \subset \Upsilon_x,$$

$$\Upsilon_{x; [t(I)]} = \Upsilon^c_{x; (I)} \cap \{ \varepsilon_i = 0 \, \forall \, i \in \mathbb{I} \setminus I_+ \, ; \, u_e = 0 \, \forall \, e \in I_m \, ; \, z_e = 0 \, \forall \, e \in I_- \}.$$  

This gives rise to a stratification

$$\Upsilon_x = \bigcup_{I \subset \mathbb{I}} \Upsilon_{x; [t(I)]}. \quad (3.10)$$

We remark that neither $\Upsilon_x$ nor its stratification (3.10) depends on the choice of the weighted level tree $t$ representing $[\mathbb{I}]$, even though the sets $\mathbb{I}$ and $I$ depend on such choice. We also notice that $\Upsilon^c_{x; (I)}$ is an open subset of $\Upsilon_x$, but the strata $\Upsilon_{x; [t(I)]}$ are not open unless $I = \mathbb{I}$.

For each $i \in \mathbb{I}_+$, we take

$$u_{e_i} := 1.$$  

By (3.7), shrinking $\Upsilon_x$ if necessary, we have

$$u_e \in \Gamma\left( \mathcal{E}_{\Upsilon_x}^e \right) \quad \forall \, e \in \overline{\text{Edg}(t) \setminus \mathbb{I}_m}. \quad (3.11)$$

With the local parameters $\zeta_e$ and $\varsigma_j$ as in (3.3), we construct a morphism

$$\theta_x : \Upsilon_x \longrightarrow \mathcal{V} \quad (\longrightarrow \mathcal{M}^{\text{wt}})$$

given by

$$\theta^*_x \zeta_e = \frac{u_e \cdot u_{e_{\ell(e)}} \cdot u_{e_{\ell(e)[1]}} \cdots}{u_{e_{v^+_e}} \cdot u_{e_{v^-_{\ell(e)}}} \cdot u_{e_{v^-_{\ell(e)[1]}}}} \prod_{i \in \mathbb{I}} \varepsilon_i \quad \forall \, e \in \overline{\text{Edg}(t)};$$

$$\theta^*_x \varsigma_j = z_e \quad \forall \, e \in \mathbb{I}_- = \overline{\text{Edg}(t) \setminus \mathbb{I}_m}; \quad \theta^*_x \varsigma_j = w_j \quad \forall \, j \in J. \quad (3.12)$$

The numerator and the denominator in the first line of (3.12) are both finite products, because (3.6) is always a finite sequence.

For any $I \subset \mathbb{I}$, it follows from (3.11) and (3.12) that

$$\theta_x (\Upsilon^c_{x; (I)}) \subset \mathcal{V}_{(I)}; \quad \theta_x (\Upsilon_{x; [t(I)]}) \subset \mathcal{V}_{t(I)}, \quad (3.13)$$

where $\mathcal{V}_{(I)}$ and $\mathcal{V}_{t(I)}$ are described before (3.2).

Fix $I \subset \mathbb{I}$ ($I$ may be empty). With

$$[t(I)] = [\tau(I), \ell(I)] \in \mathcal{T}^{\text{wt}}$$

as in (2.5), $\mathcal{M}^{\text{lf}}_{[t(I)]}$ as in (2.14), and the chart $\mathcal{V}_{\tau(I)} \longrightarrow \mathcal{M}^{\text{wt}}_{\tau(I)}$ as in (3.2), let

$$\Phi_{x; (I)} : \Upsilon_{x; [t(I)]} \longrightarrow \mathcal{V}_{\tau(I)} \times \mathcal{M}^{\text{wt}}_{\tau(I)} \mathcal{M}^{\text{lf}}_{[t(I)]}$$

be the morphism so that for any $y \in \Upsilon_{x; [t(I)]},$

$$\Phi_{x; (I)} (y) = \left( \theta_x (y); \prod_{i \in \mathbb{I}_+, I_+} \left( \mu_{e; i; I} (y) \cdot \bigotimes_{e \in e_{\ell(e)[1]}} \zeta_e \right) ; e \in \mathcal{E}_i \right), \quad (3.14)$$
where
\[
\mu_{e;i} = \left( \frac{u_{e_i}^+ \cdot u_{e_i}^+ \cdots}{u_{e_i}^+ \cdot u_{e_i}^+ \cdots} \right) \left( \prod_{e > e_i: [\ell(e), \ell(v_e^+)] \subset I} \epsilon_{h_e}^{ + \zeta_e} \right) \left( \prod_{e' > e; [\ell(e'), \ell(v_{e'}^-)] \subset I} \epsilon_{h_e'}^{ + \zeta_{e'}} \right)
\]
for all \( i \in \mathbb{I}_+ \backslash I_+ \) and \( e \in \mathcal{E}_i \). Similar to (3.12), the products in the first pair of parentheses above are both finite products.

By (3.13), the description of \( \mathcal{V}_I^{\circ} \) above (3.2), and (2.6), we see that
\[
\mu_{e;i} : I \in \Gamma\left( \mathcal{O}_{M_{X(I)}} \right) \quad \forall \, I \subset \mathbb{I}, \, i \in \mathbb{I}_+ \backslash I_+, \, e \in \mathcal{E}_i.
\]
Moreover, by (3.11),
\[
\mu_{e;i} : I |_{\mathcal{U}_{x[t(I)} \left\{ \begin{array}{ll}
0 & \text{if } \ell(I)(v_e^-) < i, \\
\epsilon_{h_e}^{ + \zeta_e} & \text{if } \ell(I)(v_e^-) = i.
\end{array} \right.
\] (3.15)

This, along with (3.13), Lemma 3.1, and (2.13), implies \( \Phi_{x(I)} \) is well-defined.

The morphisms \( \Phi_{x[I]} \), \( I \subset \mathbb{I} \), together determine
\[
\Phi_x : \mathcal{U}_x \rightarrow \mathcal{M}^\text{tf}, \quad \Phi_x(y) = \Phi_{x(I)}(y) \text{ if } y \in \mathcal{U}_{x[t(I)}.
\] (3.16)

We remark that \( \Phi_{x(I)} \) and \( \Phi_x \) are also independent of the choice of the weighted level tree \( t \) representing \( [t] \). Moreover, we observe that
\[
\Phi_x(y_x) = \Phi_{x(\emptyset)}(y_x) = x \quad \forall \, x \in \mathcal{M}^\text{tf},
\] (3.17)

where \( y_x \in \mathcal{U}_x \) is given in (3.8).

A priori \( \Phi_x \) is just a map, for the set-theoretic definition (2.14) of \( \mathcal{M}^\text{tf} \) does not describe its stack structure, although each \( \mathcal{M}^\text{tf}_{[t]} \) is a stack. In §3.2, we will show such \( \Phi_x \) patch together to endow \( \mathcal{M}^\text{tf} \) with a smooth stack structure. Each \( \Phi_x \) will hereafter be called a twisted chart centered at \( x \) (lying over \( \mathcal{V} \rightarrow \mathcal{M}^\text{wt} \)), although rigorously it becomes a chart of \( \mathcal{M}^\text{tf} \) only after Corollary 3.8 is established.

**Lemma 3.3.** For every \( I \subset \mathbb{I} \), \( \Phi_{x(I)} : \mathcal{U}_{x[t(I)} \rightarrow \mathcal{M}^\text{tf}_{[t(I)} \) of (3.14) is an isomorphism to an open subset of \( \mathcal{M}^\text{tf}_{[t(I)} \).

**Proof.** For any \( i \in \mathbb{I}_+(e(I)) = \mathbb{I}_+ \backslash I_+ \), notice that every edge in \( \mathcal{E}_i \) of the weighted level tree \( t \) is not contracted in the construction of \( \mathcal{U}_{x[t(I]} \) (c.f. (2.6)). Thus,
\[
\mathcal{E}_i \subset \text{Edg}(t(I)) \quad \forall \, i \in \mathbb{I}_+(t(I)).
\]

In particular, the edges \( e_i, \, i \in \mathbb{I}_+(t(I)) \), can be used as the special edges of \( t(I) \). For conciseness, let
\[
E[t(I)] := \text{Edg}(t(I)) \backslash \left( \{ e_i : i \in \mathbb{I}_+(t(I)) \} \cup I_m(t(I)) \right)
\]
\[
= \bigsqcup_{i \in \mathbb{I}_+(t(I))} \left\{ e \in \text{Edg}(t(I)) : \ell(I)(v_e^-) = i, \, e \neq e_i \right\} \subset \text{Edg}(t(I)) \subset \text{Edg}(t(I));
\]
see (2.7) for notation.

Let \( \{ \zeta_e \}_{e \in \text{Edg}(t(I))} \cup \{ \zeta_j \}_{j \in J} \) be a set of the local parameters on \( \mathcal{V} \) centered at \( \varpi(x) \) as in (3.3). By the definition of \( \mathcal{V}_{t(I)} \) above (3.2),
\[
\{ \zeta_e \}_{e \in \text{Edg}(t(I)) \text{Edg}(t(I))} \cup \{ \zeta_j \}_{j \in J}
\] (3.18)
is a set of local parameters of $\mathcal{V}_{\tau(I)}$.

Recall that there exist $\lambda_e \in \mathbb{A}^*$, $e \in \widehat{\text{Edg}}(t)$, such that
\[ x = \left( 0 : \prod_{e \in \mathbb{I}_+} \lambda_e \cdot \left( \bigotimes_{e \geq e} \partial_{\zeta^i} |_{\emptyset} : \ell(v_e^-) = i \right) \right) \]
as in (3.7). Let $U_x : \mathbb{E}[t(I)]$ be an open subset of $(\mathbb{A}^*)^\mathbb{E}[t(I)]$ such that
\[ (\lambda_e)_{e \in \mathbb{E}[t(I)]} \in U_x : \mathbb{E}[t(I)] \subset (\mathbb{A}^*)^\mathbb{E}[t(I)]. \]
The coordinates of $U_x : \mathbb{E}[t(I)]$ are denoted by
\[ (\mu_e)_{e \in \mathbb{E}[t(I)]}. \]
In addition, we set
\[ \mu_{e_i} = 1 \quad \forall \ i \in \mathbb{I}_+^+(t(I)), \quad \mu_e = 0 \quad \forall \ e \in \mathbb{I}_m(t(I)). \]
Thus, the function $\mu_e$ is defined for all $e \in \widehat{\text{Edg}}(t(I))$, and is nowhere vanishing on $U_x : \mathbb{E}[t(I)]$ for all $e \in \widehat{\text{Edg}}(t(I)) \backslash \mathbb{I}_m(t(I))$.

The smooth chart $\mathcal{V}_{\tau(I)} \longrightarrow \mathcal{M}_x^{\mathfrak{h}}(t(I))$ in (3.2) induces a smooth chart
\[ U'_x : \mathbb{E}[t(I)] := \mathcal{V}_{\tau(I)} \times U_x : \mathbb{E}[t(I)] \longrightarrow \mathcal{M}_x^{\mathfrak{h}}(t(I)) \]
given by
\[ (\mathfrak{z}, (\mu_e)_{e \in \mathbb{E}[t(I)]}) \mapsto (\mathfrak{z}, \prod_{i \in \mathbb{I}_+^+(t(I))} \mu_e \left( \bigotimes_{e \in \text{Edg}(t(I)), e \geq e} \partial_{\zeta^i} |_{\emptyset} : \ell_e(v_e^-) = i \right)). \]

We will construct a morphism
\[ \Psi_{x : t(I)} : U'_x : \mathbb{E}[t(I)] \longrightarrow \mathfrak{U}_x : \mathbb{E}[t(I)] \tag{3.19} \]
such that $\Phi_{x : t(I)} \circ \Psi_{x : t(I)}$ and $\Psi_{x : t(I)} \circ \Phi_{x : t(I)}$ are both the identity morphisms, which will then establish Lemma 3.3.

Given $(\mathfrak{z}, (\mu_e)_{e \in \mathbb{E}[t(I)]}) \in U'_x : \mathbb{E}[t(I)]$, we denote its image by
\[ y := \Psi_{x : t(I)}(\mathfrak{z}, (\mu_e)_{e \in \mathbb{E}[t(I)]}) \in \mathfrak{U}_x : \mathbb{E}[t(I)], \]
which is to be constructed. With the coordinates on $\mathfrak{U}_x$ as in (3.9), we set
\[ z_e(y) = \zeta_e(\mathfrak{z}) \quad \forall \ e \in \mathbb{I}_-, \quad w_j(y) = \zeta_j(\mathfrak{z}) \quad \forall \ j \in J. \tag{3.20} \]
By (3.2), we see that
\[ z_e(y) = \zeta_e(\mathfrak{z}) = 0 \quad \iff \quad e \in \mathbb{I}_- \backslash \mathbb{I}_- \left( \subset \mathbb{I}_-(t(I)) \right). \tag{3.21} \]
The rest of the coordinates of $y$ are much more complicated; we describe them by induction over the levels in $\mathbb{I}_+ = \mathbb{I}_+(t)$. More precisely, we will show that $\varepsilon_i(y)$ with $i \in \mathbb{I}_+$ and $u_e(y)$ with $e \in \widehat{\text{Edg}}(t) \backslash \{ e_i : i \in \mathbb{I}_+ \}$ are all rational functions in $\zeta_e(\mathfrak{z})$ and $\mu_{e'}$, satisfying
\[ (\varepsilon_i(y) = 0 \iff i \in \mathbb{I}_+ \backslash \mathbb{I}_+) \quad \text{and} \quad (u_e(y) = 0 \iff e \in \mathbb{I}_m \backslash \mathbb{I}_m). \tag{3.22} \]
In particular, (3.22) and (3.21) imply \( y \in \mathfrak{u}_{x_i(t_I)} \), i.e. \( \Psi_{x_i(t)} \) is well-defined.

The base case of the induction is for the level

\[
i_0 := \max \mathbb{I}_+(t).
\]

We take

\[
\epsilon_{i_0}(y) = \zeta_{e_{i_0}}(\mathfrak{z}).
\]

By (3.2), we see that \( \epsilon_{i_0}(y) \) satisfies (3.22). We take

\[
u_{e_{i_0}}(y) = 1.
\]

For any \( e \neq e_{i_0} \) with \( \ell(e) = i_0 \), we set

\[
u_e(y) = \begin{cases} 
\mu_e & \text{if } i_0 \notin I_+ \quad \text{(i.e. } i_0 \notin \mathbb{I}_+(t_I) \text{, } e \in \mathbf{E}(t_I))
\zeta_e(y) & \text{if } i_0 \in I_+ \quad \text{(i.e. } e_{i_0} \in \mathop{\text{Edg}}(t) \setminus \mathop{\text{Edg}}(t_I))
\end{cases}.
\]\n
If \( i_0 \in I_+ \), then by (3.18) and (3.2), we have \( \zeta_e(y) = 0 \) if and only if \( (m = i_0) \) and \( e \in \mathbb{I}_m \setminus \mathbb{I}_m \). If \( i_0 \notin I_+ \), then \( \mu_e = 0 \) if and only if \( (m = i_0) \) and \( e \in \mathbb{I}_m \setminus \mathbb{I}_m \). We thus conclude that \( \nu_e(y) \) satisfies (3.22) for all \( e \in \mathop{\text{Edg}}(t) \) with \( \ell(e) = i_0 \). Moreover, such \( \epsilon_{i_0}(y) \) and \( \nu_e(y) \) are obviously rational functions in \( \zeta_{e'}(\mathfrak{z}) \) and \( \mu_{e''} \). Hence, the base case is complete.

Next, for any \( i \in \mathbb{I}_+ \), assume that all \( \epsilon_k(y) \) with \( k > i \) and all \( \nu_e(y) \) with \( e \in \mathop{\text{Edg}}(t) \) and \( \ell(e) > i \) have been expressed as rational functions in \( \zeta_{e'}(\mathfrak{z}) \) and \( \mu_{e''} \), satisfying (3.22).

For the level \( i \), we first construct \( \epsilon_i(y) \). The construction is subdivided into three cases.

**Case 1.** If \( i \notin I_+ \), then set

\[
\epsilon_i(y) = 0.
\]

Obviously this satisfies (3.22).

**Case 2.** If \( i \in I_+ \) and \( [i, i[1]] \notin I_+ \), then \( e_i \in \mathbf{E}(t_I) \), hence \( \mu_{e_i} \neq 0 \). Let

\[
\hat{i} := \min ([i, i[1]] \setminus I_+) \in \mathbb{I}_+(t_I).
\]

Intuitively, \( \hat{i} \) is the level containing the image of \( v_i \) in \( t_I \). Thus,

\[
\ell_I(e_i) = \hat{i}.
\]

Let \( \epsilon_i(y) \) be given by

\[
\mu_{e_i} = \epsilon_i(y) \cdot \prod_{e_i \neq e_i} \cdot \prod_{e_i \in [i, i[1]} \zeta_{e'}(\mathfrak{z}) \cdot \prod_{h \in [\hat{i}, i]} \epsilon_h(y).
\]\n
The inductive assumption implies that all \( \epsilon_h(y) \) with \( h \in [i, i] \), as well as all \( u_{e_i}^+(\mathfrak{y}) \) and \( u_{e_i(h)}^+ \) for \( h > 0 \), are non-zero and are rational functions in \( \zeta_{e'}(\mathfrak{z}) \) and \( \mu_{e''} \).

By (3.18) and (2.6), we also see that all \( \zeta_{e'}(\mathfrak{z}) \) and \( \zeta_{e''}(\mathfrak{z}) \) in (3.25) are non-zero. Therefore, such defined \( \epsilon_i(y) \) is a rational function in \( \zeta_{e'}(\mathfrak{z}) \) and \( \mu_{e''} \), satisfying (3.22).
Case 3. If $[i, i[1]] \subset I_+$ (hence $i \in I_+$), then we see $e_i \in \text{Edg}(t) \setminus \text{Edg}(t_{(1)}); \text{c.f. (2.6)}$. Intuitively, this means $e_i$ is contracted in the construction of $t_{(1)}$. By the description of $V_{(1)}$ above (3.2), we see that $\zeta_{e_i}(3) \neq 0$. Let $\varepsilon_i(y)$ be given by

$$\zeta_{e_i}(3) = \varepsilon_i(y) \cdot \prod_{h \in \{i, j\}} \varepsilon_h(y). \quad (3.26)$$

Mimicking the argument in Case 2, we conclude that $\varepsilon_i(y)$ is a rational function in $\zeta_{e_i}(3)$ and $\mu_{e''}$, satisfying (3.22).

Next, we construct $u_e(y)$ for $e \in \text{Edg}(t)$ with $\ell(e) = i$. Set

$$u_e(y) = 1.$$

For $e \neq e_i$, the construction is subdivided into two cases.

Case A. If $[i, i[1]] \subset I_+$, then

$$e \in E(t_{(1)}) \cup \Pi_m(t_{(1)}) \left( = \text{Edg}(t_{(1)}) \setminus \{e_i : i \in \Pi_+(t_{(1)})\}\right),$$

hence $\mu_e$ exists, and $\mu_e = 0$ if and only if $e \in \Pi_m(t_{(1)})$. In Case A, since $[i, i[1]] \subset I_+$, (2.7) further implies

$$\mu_e = 0 \iff e \in \Pi_m \setminus I_m. \quad (3.27)$$

Let

$$\kappa = \kappa_e := \min \left(\{i, \ell(v^+_{e_i})\} \setminus I_+ \right) \quad (i \in \Pi_+(t_{(1)})).$$

Since $[i, \kappa] \subset I_+$, we have

$$\ell_{(1)}(e) = \kappa.$$

Let $u_e(y)$ be given by

$$\mu_e = u_e(y) \cdot \frac{u_{e_i}(y) \cdot u_{e_i}(y) \cdot \prod_{e > e_i, [\ell(e), [\ell(v^+_{e_i})] \subset I} \zeta_{e}(3)}{u_{e_i}(y) \cdot u_{e_i}(y) \cdot \prod_{e' > e_i, [\ell(e'), [\ell(v^+_{e_i})] \subset I} \zeta_{e}(3)} \cdot \prod_{h \in \{i, \kappa\}} \varepsilon_h(y). \quad (3.28)$$

We observe that if $i \notin I_+$, then $\kappa = i$ and hence $\prod_{h \in \{i, \kappa\}} \varepsilon_h(y) = 1$; if $i \in I_+$, then the previous construction of $\varepsilon_i(y)$, along with the inductive assumption, guarantees $\prod_{h \in \{i, \kappa\}} \varepsilon_h(y) \neq 0$. Mimicking the argument of Case 2 of the construction of $\varepsilon_i(y)$ and taking (3.27) into account, we see that $u_e(y)$ determined by (3.28) is a rational function in $\zeta_{e}(3)$ and $\mu_{e''}$, and it satisfies (3.22).

Case B. If $[i, i[1]] \subset I_+$, then (2.6) gives

$$e \in \text{Edg}(t_{(1)}) \quad (\text{i.e. } \zeta_{e}(3) = 0) \iff e \in \Pi_m \setminus I_m. \quad (3.29)$$

Let $u_e(y)$ be given by

$$\zeta_{e}(3) = u_e(y) \cdot \frac{u_{e_i}(y) \cdot u_{e_i}(y) \cdot \prod_{e > e_i, [\ell(e), [\ell(v^+_{e_i})] \subset I} \zeta_{e}(3)}{u_{e_i}(y) \cdot u_{e_i}(y) \cdot \prod_{e' > e_i, [\ell(e'), [\ell(v^+_{e_i})] \subset I} \zeta_{e}(3)} \cdot \prod_{h \in \{i, \kappa\}} \varepsilon_h(y). \quad (3.30)$$

Once again, mimicking the argument of Case A of the construction of $u_e(y)$, and taking (3.29) as well as the description of $V_{(1)}$ right before (3.2) into account, we see that $u_e(y)$ determined by (3.30) is a rational function in $\zeta_{e}(3)$ and $\mu_{e''}$, and it satisfies (3.22).

The cases 1-3, A, and B together complete the inductive construction of $\Psi_{x_{(1)}}$.
• (3.20) with the second line of (3.12),
• (3.23), the second case of (3.24), (3.26), and (3.30) with the first line (3.12),
• the first case of (3.24), (3.25), and (3.28) with the expressions of \( \mu_{e;i;I} \) right after (3.14),
we observe that \( \Psi_{x;i(1)} \) is the inverse of \( \Phi_{x;i(1)} \). □

**Corollary 3.4.** \( \Phi_x : \mathcal{U}_x \rightarrow \mathcal{M}^{tf} \) is injective.

**Proof.** This follows from Lemma 3.3 and the stratification (3.10) and (2.14) directly. □

### 3.2. Stack structure.
In this subsection, we will show the twisted charts \( \Phi_x \) patch together to endow \( \mathcal{M}^{tf} \) with a smooth stack structure; c.f. Proposition 3.7 and Corollary 3.8. Note that a priori, \( \Phi_x \) depends on the choices of the special vertices \( \{v_i\}_{i \in \mathbb{I}_+} \) (3.4) and of the local parameters (3.3). Nonetheless, Lemmas 3.5 and 3.6 below will guarantee that such choices do not affect the proposed stack structure of \( \mathcal{M}^{tf} \), hence will make the proof of Proposition 3.7 more concise.

Let \( \{w_i : i \in \mathbb{I}_+\} \) be another set of the special vertices satisfying (3.4), and \( w_i^+, d_i, \) and \( d_i^+ \) be the analogues of \( v_i^+, e_i, \) and \( e_i^+ \) in (3.5), respectively. As in (3.6), each level \( i \in \mathbb{I}_+ \) similarly determines a finite sequence
\[
i(0) = i < i(1) = \ell(w_i^+) < i(2) = \ell(w_i^+) < \cdots .
\]
We take an open subset
\[
\mathcal{U}_x^a \subset \mathbb{A}^{\mathbb{I}_+} \times \mathbb{A}^{\text{Edg}(t) \setminus \{d_i : i \in \mathbb{I}_+\}} \times \mathbb{A}^{\mathbb{I}^-} \times \mathbb{A}^J
\]
with the coordinates
\[
(\bar{\delta}_i)_{i \in \mathbb{I}_+}, (u^a_i)_{e \in \text{Edg}(t) \setminus \{d_i : i \in \mathbb{I}_+\}}, (z^a_e)_{e \in \mathbb{I}^-}, (w^a_j)_{j \in J},
\]
as in (3.9), and then construct
\[
\theta_x^a : \mathcal{U}_x^a \rightarrow \mathcal{V}, \quad \mu_{e;i;I} \in \Gamma(\theta_x, \mathcal{U}_x^{a(1)}), \quad \Phi_x^a : \mathcal{U}_x^a \rightarrow \mathcal{M}^{tf}
\]
parallel to (3.12) and (3.14). Let \( \mathcal{U} = \Phi_x^a(\mathcal{U}_x^a) \cap \Phi_x(\mathcal{U}_x) \).

**Lemma 3.5.** The transition map
\[
(\Phi_x^a)^{-1} \circ \Phi_x : \Phi_x^{-1}(\mathcal{U}) \rightarrow (\Phi_x^a)^{-1}(\mathcal{U})
\]
is an isomorphism.

**Proof.** Let \( g : \Phi_x^{-1}(\mathcal{U}) \rightarrow (\Phi_x^a)^{-1}(\mathcal{U}) \) be the isomorphism given by
\[
g^* \bar{\delta}_i = \bar{\epsilon}_i, \quad g^* u_{e;i} = u_{e;i}^+, \quad g^* u_{e;[1]} = u_{d;[1]}, \quad g^* u_{d;[1]} = u_{d;[1]}^+, \quad g^* u_{e;[\mathbb{I}_+]} = u_{e;[\mathbb{I}_+]}^+, \quad g^* u_{d;[\mathbb{I}_+]} = u_{d;[\mathbb{I}_+]}^+, \quad g^* u_{e;[\mathbb{I}^-]} = u_{e;[\mathbb{I}^-]}^+, \quad g^* u_{d;[\mathbb{I}^-]} = u_{d;[\mathbb{I}^-]}^+, \quad g^* u_{e;J} = u_{e;J}, \quad g^* u_{d;[J]} = u_{d;[J]}, \quad g^* u_{e;\mathbb{I}_+} = u_{e;\mathbb{I}_+}, \quad g^* u_{d;\mathbb{I}_+} = u_{d;\mathbb{I}_+}, \quad g^* u_{e;\mathbb{I}^-} = u_{e;\mathbb{I}^-}, \quad g^* u_{d;\mathbb{I}^-} = u_{d;\mathbb{I}^-}, \quad g^* u_{e;J} = u_{e;J}, \quad g^* u_{d;J} = u_{d;J}.
\]

The fact that \( g \) is an isomorphism can be shown by constructing its inverse explicitly, which is similar to the proof of Lemma 3.3, but is simpler. The key point of the construction is that
\[
u_{d_i} = \frac{1}{g^* u_{e_i}} \quad \forall i \in \mathbb{I}_+, \quad u_{e} = \frac{g^* u_{e}}{g^* u_{e_i}} \quad \forall e \in \text{Edg}(t) \text{ with } \ell(e) = i.
\]
and each $\varepsilon_i$ is a product of $g^*\delta_i$ and a rational function of $u_e$ with $e \in \hat{\text{Edg}}(t)$.

It is a direct check that the isomorphism $g$ satisfies

$$\theta_2^* \circ g = \theta_x \quad \text{and} \quad g^* \mu_{\varepsilon;i;I}^* = \frac{\mu_{\varepsilon;i;I}}{\mu_{d;i;I}} \quad \forall I \subset \mathbb{I}, \ i \in \mathbb{I}_+ \setminus I_+, \ e \in \mathcal{E}_i.$$

Thus, $(\Phi_x^*)^{-1} \circ \Phi_x = g$ and hence is an isomorphism. □

Let

$$\{\hat{\zeta}_e : e \in \text{Edg}(\gamma)\} \sqcup \{\zeta_j : j \in J\}$$

be another set of extended modular parameters centered at $x$ on the same chart $\mathcal{V} \to \mathcal{M}^\text{wt}$; see (3.3). We use this set of local parameters to construct another twisted chart $\hat{\Phi}_x : \hat{\mathcal{U}}_x \to \mathcal{M}^\text{tw}$; in particular, we have $\hat{\theta}_x : \hat{\mathcal{U}}_x \to \mathcal{V}$ and $\hat{\mu}_{e;i;I} \in \Gamma(\mathcal{O}_{\hat{\mathcal{U}}_x}^*)$ as in (3.12) and (3.14), respectively. Parallel to (3.9), the coordinates on $\hat{\mathcal{U}}_x$ are denoted by

$$(\hat{\zeta}_i)_{i \in \mathbb{I}_+}, (\hat{\mu}_e)_{e \in \text{Edg}(t)} \setminus \{e_i : i \in \mathbb{I}_+\}, (\hat{\zeta}_j)_{j \in J}.$$

Let $\mathcal{U} = \Phi_x(\mathcal{U}_x) \cap \hat{\Phi}_x(\hat{\mathcal{U}}_x)$.

**Lemma 3.6.** The transition map

$$(\Phi_x)^{-1} \circ \Phi_x : \Phi_x^{-1}(\mathcal{U}) \to \hat{\Phi}_x^{-1}(\mathcal{U})$$

is an isomorphism.

**Proof.** By Lemma 3.5, it suffices to use the same set of the special vertices $\{\nu_i\}_{i \in \mathbb{I}_+}$ for both $\Phi_x$ and $\hat{\Phi}_x$. For any $e \in \text{Edg}(\gamma)$, the local parameters $\hat{\zeta}_e$ and $\zeta_e$ defines the same locus $Z_e = \{\zeta_e = 0\} = \{\hat{\zeta}_e = 0\}$, hence there exists $f_e \in \Gamma(\mathcal{O}_e^*)$ such that

$$\hat{\zeta}_e = f_e \cdot \zeta_e.$$

Since

$$\partial_{\hat{\zeta}_e} |_{Z_e} \in \Gamma(TZ_e) \quad \forall e \in \text{Edg}(\gamma) \setminus \{e\}, \quad \partial_{\zeta_e} |_{Z_e} \in \Gamma(TZ_e) \quad \forall j \in J$$

(where $\partial_{\hat{\zeta}_e}$ are defined in the same way as $\partial_{\zeta_e}$), we have

$$\partial_{\zeta_e} = \frac{1}{f_e} \partial_{\hat{\zeta}_e} \quad (3.31)$$

as nowhere vanishing sections of the normal bundle $L_e \to Z_e$.

Let $g : \Phi_x^{-1}(\mathcal{U}) \to \hat{\Phi}_x^{-1}(\mathcal{U})$ be the isomorphism given by

$$g^* \hat{\zeta}_i = \varepsilon_i \cdot \theta_x^* \left( \frac{f_{e_i} \cdot f_{e_i(t)} \cdots \cdot f_{e_i(t_{[n_i]}}}{f_{e_i} \cdot f_{e_i(t_{[1]}} \cdots \cdot f_{e_i(t_{[n_i]}}}} \right) \quad \forall i \in \mathbb{I}_+, \quad g^* \hat{\zeta}_e = z_e \cdot (\theta_x^* f_e) \quad \forall e \in \mathbb{I}_-, \quad g^* \hat{\mu}_j = \theta_x^* \zeta_j \quad \forall j \in J,$$

$$g^* \hat{u}_e = u_e \cdot \theta_x^* \left( \frac{\prod_{e' \supset e} f_{e'}}{\prod_{e' \supset e \subset e_i} f_{e'}} \right) \quad \forall e \in \text{Edg}(t) \setminus \{e_i : i \in \mathbb{I}_+\}.$$

Shrinking $\mathcal{U}$ if necessary, we see that such defined $g$ is invertible, because

$$(\hat{\zeta}_i)_{i \in \mathbb{I}_+}, (\hat{u}_e)_{e \in \text{Edg}(t) \setminus \{e_i : i \in \mathbb{I}_+\}}, (z_e)_{e \in \mathbb{I}_-}, (\theta_x^* \zeta_j)_{j \in J}$$
can be considered as coordinates on $\Phi^{-1}_x(U)$.

It is a direct check that $\hat{\theta}_x \circ g = \theta_x$ and

$$\frac{g^* \hat{\mu}_{e;i}; I}{\theta_2^* \left( \prod_{e \in E, \ell(e), \ell(v_e^+) \in I} f_e \right)} = \frac{\mu_{e;i}; I}{\theta_2^* \left( \prod_{e \in E, \ell(e), \ell(v_e^+) \in I} f_e \right)} \quad \forall I \subset \mathbb{I}, \ i \in \mathbb{I}_+ \setminus I_+, \ e \in \mathcal{E}_i.$$

Taking (3.31) into account, we conclude that $(\hat{\Phi}_x)^{-1} \circ \Phi_x = g$ and hence is an isomorphism. \(\Box\)

Given $I \subset \mathbb{I}$ and $x' \in \Phi_x(U_x; t(I))$, let $\mathcal{V}_{\varpi'(x')} \rightarrow \mathcal{M}_{\text{wt}}$ be a chart containing $\varpi(x')$ and $\Phi_{x'} : \mathcal{V}_{x'} \rightarrow \mathcal{M}_{\text{wt}}$ be a twisted chart centered at $x'$ over $\mathcal{V}_{\varpi(x')} \rightarrow \mathcal{M}_{\text{wt}}$. Let $U = \Phi_x(U_x) \cap \Phi_{x'}(U_{x'})$.

**Proposition 3.7.** The transition map

$$\Phi_{x'}^{-1} \circ \Phi_x : \Phi_{x'}^{-1}(U) \rightarrow \Phi_x^{-1}(U)$$

is an isomorphism.

**Proof.** Since $x' \in \Phi_x(U_x; t(I)) \subset \Phi_x(U_x)$, its underlying weighted curve satisfies

$$\varpi(x') \in \mathcal{V}_{\varpi}(I) \quad \left( = \{ \zeta_e \neq 0 : e \in \text{Edg}(t) \setminus \text{Edg}(t(I)) \} \subset \mathcal{V} \right).$$

Thus, replacing $\mathcal{V}_{\varpi(x')}$ by $\mathcal{V}_{\varpi(x')} \cap \mathcal{V}_{\varpi}(I)$ if necessary, we may assume

$$\varpi(x') \in \mathcal{V}_{\varpi(x')} \subset \mathcal{V}_{\varpi}(I).$$

Moreover, the following modular parameters on $V$:

$$\zeta_e, \quad e \in \text{Edg}(t(I)) \subset \text{Edg}(t)$$

also serve as modular parameters on $\mathcal{V}_{\varpi(x')}$. Thus by Lemma 3.6, we may assume $\Phi_{x'}$ is constructed using the local parameters on $\mathcal{V}_{\varpi(x')}$:

$$\{ \zeta_e \}_{e \in \text{Edg}(t(I))} \cup \left( \{ \zeta_j \}_{j \in t(I)} \cup \{ \zeta_e \}_{e \in \text{Edg}(t) \setminus \text{Edg}(t(I))} \right)$$

as the analogue of (3.3).

Let the special vertices $v_i$ and edges $e_i$ of $t$ be respectively as in (3.4) and (3.5). By Lemma 3.5, we may further assume that the special vertices and edges of $t(I)$ are respectively

$$v_i \quad \text{and} \quad e_i, \quad i \in \mathbb{I}_+(t(I)) = \mathbb{I}_+ \setminus I_+.$$

For any $i \in \mathbb{I}_+ \setminus I_+$ and $h \in \mathbb{Z}_{\geq 0}$, let $i^h$, $e_i^h$, and $i(h)$ be the analogues of $i^h$, $e_i^h$, and $i[h]$, respectively, for the weighted level tree $t(I)$ instead of $t$; see (2.1), (3.5), and (3.6) for notation.

Recall that $I_-(t(I)) = I_+ \setminus I \cup \{ e \in \mathbb{I}_m \setminus \mathbb{I}_m : \ell(v_e^+) \leq m(t(I)) \}$. We denote by

$$\left( (\varepsilon_i^h)_{i \in \mathbb{I}_+ \setminus I_+}, (u_j^h)_{e \in \text{Edg}(t(I)) \setminus \{ e_i : i \in I_+ \setminus I_+ \}}, (z_e^h)_{e \in I_-(t(I))}, (w_j^h)_{j \in I \cup (\text{Edg}(t) \setminus \text{Edg}(t(I)))} \right)$$

the coordinates on $\mathcal{U}_{x'}$ parallel to (3.9), and construct

$$\theta_{x'} : \mathcal{U}_{x'} \rightarrow \mathcal{V}_{\varpi(x')} \quad \text{and} \quad \mu_{t(I')} \in \Gamma \left( \Theta_{\mathcal{U}_{x'}'(t(I'))} \right), \quad I' \subset \mathbb{I}, \ i \in \mathbb{I}_+(I_+ \cup I'), \ e \in \mathcal{E}_i$$
parallel to
\[ \theta_x : \mathcal{U}_x \to \mathcal{V} \quad \text{and} \quad \mu_{e;i}^{t} : \Gamma(\mathcal{O}_{\mathcal{U}_x(t)})_i \to \prod_{I} \mathcal{O}_{\mathcal{U}_x(t)}, \quad i \in \mathbb{P} \setminus \mathbb{I}_+, \; e \in \mathcal{E}_i, \]
of (3.12) and (3.14), respectively. In this way, \( \Phi_x : \mathcal{U}_x \to \mathcal{M}^{\mathrm{tf}} \) is constructed analogously to \( \Phi_x \).

Let \( g : \Phi_x^{-1}(\mathcal{U}) \to \Phi_x^{-1}(\mathcal{U}) \) be the isomorphism given by

\[
g^* \varepsilon_i = \varepsilon_i \cdot \left( \prod_{i=0}^{n} \varepsilon_h \right) \cdot \frac{\prod_{e \in \mathcal{E}_i} \mu_{e;i}^{t} : \mu_{e;i}^t \cdot \cdots \cdot \mu_{e;i}^t \cdot \cdots}{\prod_{e \in \mathcal{E}_i} \mu_{e;i}^{t} : \mu_{e;i}^t \cdot \cdots \cdot \mu_{e;i}^t \cdot \cdots}, \quad \forall i \in \mathbb{P} \setminus \mathbb{I}_+ \]

and

\[
g^* u_e = \mu_{e;i}^{t} : \mu_{e;i}^t \cdot \cdots \cdot \mu_{e;i}^t \cdot \cdots, \quad \forall e \in \mathcal{E}_e \setminus \mathcal{E}_i, \]

\[
g^* z_e = u_e \quad \forall e \in \{ e \in \mathcal{E}_e \mid \mathcal{E}_m \cap \mathcal{E}_e \}
\]

\[
g^* w_j = w_j \quad \forall j \in \mathcal{J}, \quad g^* \omega_e = \theta_x : \omega_e \quad \forall e \in \mathcal{E}_e \setminus \mathcal{E}_e \].

To see \( g \) is well defined, notice that \( \mathcal{V}^{\omega(x)} \subseteq \mathcal{V}_x \) implies that

\[ \Phi_x^{-1}(\mathcal{U}) \subseteq \mathcal{M}^{\mathrm{tf}} \].

(3.32)

Thus, every \( \mu_{e;i}^{t} \) above can be considered as a function on \( \Phi_x^{-1}(\mathcal{U}) \). By (3.32) and (3.13), the function \( \theta_x : \omega_e \in \mathcal{E}_e \setminus \mathcal{E}_e \) is nowhere vanishing on \( \Phi_x^{-1}(\mathcal{U}) \) whenever \( \mathcal{E}_m \cap \mathcal{E}_e \cap \mathcal{E}_i \subseteq \mathcal{I} \). Taking (3.11) and (3.15) into account, we conclude that \( g \) is well defined.

Once again, the explicit expression of \( g \) implies it is invertible; see the parallel argument in the proof of Lemma 3.5. Moreover, it is a direct check that \( \theta_x \circ g = \theta_x \) and

\[
g^* \mu_{e;i}^{t} : \mu_{e;i}^t \cdot \cdots \cdot \mu_{e;i}^t \cdot \cdots, \quad \forall i \in \mathcal{I}, \; i \in \mathbb{P} \setminus (\mathbb{I}_+ \cup \mathbb{I}_d), \; e \in \mathcal{E}_i. \]

Thus, \( \Phi_x^{-1} \circ \Phi_x = g \) and hence is an isomorphism. \( \square \)

**Corollary 3.8.** \( \mathcal{M}^{\mathrm{tf}} \) is a smooth Artin stack that is birational to \( \mathcal{M}^{\mathrm{wt}} \), with \( \{ \Phi_x : \mathcal{U}_x \to \mathcal{M}^{\mathrm{tf}} \}_{x \in \mathcal{M}^{\mathrm{wt}}} \) as smooth charts. Moreover, the structure of the stratification (2.14) is locally identical to the one induced by (3.10), and each stratum \( \mathcal{M}^{\mathrm{tf}} \) is of codimension \( \mathcal{H}(t) \). Furthermore, for any \( x \in \mathcal{M}^{\mathrm{tf}} \), any chart \( \mathcal{V} \to \mathcal{M}^{\mathrm{wt}} \) containing \( \mathcal{Z}(x) \in \mathcal{M}^{\mathrm{wt}} \), and any twisted chart \( \Phi_x : \mathcal{U}_x \to \mathcal{M}^{\mathrm{tf}} \) centered at \( x \) lying over \( \mathcal{V} \to \mathcal{M}^{\mathrm{wt}} \), we have the Cartesian diagram

\[
\begin{array}{ccc}
\mathcal{U}_x & \xrightarrow{\Phi_x} & \mathcal{M}^{\mathrm{tf}} \\
\theta_x \downarrow & & \downarrow \mathcal{Z} \\
\mathcal{V} & \to & \mathcal{M}^{\mathrm{wt}}
\end{array}
\]

where \( \mathcal{Z} \) is the forgetful morphism as in (2.14) and \( \theta_x \) is as in (3.12).

**Proof.** The first statement follows from Proposition 3.7, (3.17), and the fact that \( \mathcal{Z} \) restricts to the identity map on the preimage of the open subset

\[ \{(C, \mathcal{W}) \in \mathcal{M}^{\mathrm{wt}} : \mathcal{W}(C_0) > 0 \} \subseteq \mathcal{M}^{\mathrm{wt}}. \]
Lemma 3.3 then implies for every \([t] \in \mathcal{T}_l^{wt}\), the stack structure of \(M_{[t]}^{tf}\) is the same as that induced from the inclusion \(M_{[t]}^{tf} \hookrightarrow \mathcal{M}_l^{tf}\). Besides, the \(I = \emptyset\) case of Lemma 3.3, along with the description of \(\mathcal{U}_{x,[t]}\) before (3.10) and the definition of the index set \(I(t)\) in (2.4), gives rise to the codimension of \(M_{[t]}^{tf}\). The last statement follows from (3.14).

**Remark 3.9.** By (3.12) and Corollary 3.8, one sees that on an arbitrary twisted chart \(\mathcal{U}_x\) of \(M^{tf}\),

\[
\varpi^* \left( \prod_{e' \geq m} \zeta_{e'} \right) = (u_e \cdots u_{m+1}) \prod_{i \in I_m} \varepsilon_i,
\]

\[
\varpi^* \left( \prod_{e' \geq e} \zeta_{e'} \right) = (u_e u_m \cdots u_{m+1}) \prod_{i \in I_m} \varepsilon_i = u_e \cdot \varpi^* \left( \prod_{e' \geq e} \zeta_{e'} \right), \quad \forall e \in E_m.
\]

This, along with the local equations of \(\overline{M}_1(\mathbb{P}^n, d)\) in [5, §5.2], implies that \(\overline{M}_1^{tf}(\mathbb{P}^n, d)\) has smooth irreducible components and contains at worst normal crossing singularities. This observation should be useful for the cases of higher genera.

### 3.3. Examples on the twisted charts

In this subsection, we revisit the two examples of §2.3 to illustrate how the strata of \(M_{[t]}^{tf}\) are glued together by the twisted charts.

**Example 3.10.** We continue with the setup of Example 2.6. Let \(x \in \mathcal{M}_{[t]}^{tf}\) be a weighted curve of genus 1 with twisted fields over \((C, w) \in \mathcal{M}^{wt}\). The core and the nodes of \(C\) are labeled by \(o\) and by \(a, b, c, d\), respectively.

Let \(\mathcal{V} \to \mathcal{M}^{wt}\) be an affine smooth chart containing \((C, w)\), with a set of local parameters

\[
\{\zeta_a, \zeta_b, \zeta_c, \zeta_d\} \sqcup \{\zeta_j\}_{j \in J}
\]

centered at \((C, w)\) as in (3.3), where \(\zeta_a, \ldots, \zeta_d\) correspond to the smoothing of the nodes. There then exist non-zero \(\lambda_c\) and \(\lambda_d\) such that

\[
x = (\emptyset ; [0, \partial \zeta_b |_0], [\partial \zeta_c |_0], \lambda_c \cdot (\partial \zeta_b \otimes \partial \zeta_c) |_0, \lambda_d \cdot (\partial \zeta_b \otimes \partial \zeta_c) |_0) \in \mathcal{V}_x \times_{\mathcal{M}^{wt}} \mathcal{M}_{[t]}^{tf}.
\]

Here we identify \([0, \partial \zeta_b |_0]\) with \([\partial \zeta_b |_0]\), as mentioned below (2.12).

We choose the special edges (3.5) of \(t\) to be \(e_1 = b\) and \(e_2 = a\). Let

\[
\mathcal{U}_x \subset \mathbb{A}^{-1, -2} \times \mathbb{A}^{c, d} \times \mathbb{A}^J
\]

be an open subset containing the point

\[
y_x = (0, 0, \lambda_c, \lambda_d, 0, \ldots, 0).
\]

The coordinates of \(\mathcal{U}_x\) are denoted by

\[
\varepsilon_{-1}, \varepsilon_{-2}, u_c, u_d, \quad \text{and} \quad w_j \text{ with } j \in J,
\]

where \(J\) is the same as the index set for the local parameters \(\zeta_j\) on \(\mathcal{V}\). Since \(\lambda_c, \lambda_d \neq 0\), by shrinking \(\mathcal{U}_x\) if necessary we assume that \(\mathcal{U}_x \subset \{u_c \neq 0, u_d \neq 0\}\).
By Corollary 3.8 and (3.12), the forgetful morphism \( \varpi : \mathcal{M}^\text{tf} \rightarrow \mathcal{M}^\text{wt} \) can locally be written as \( \theta_x : \mathcal{U}_x \rightarrow \mathcal{V} \) such that

\[
\theta_x \left( \varepsilon_{-1}, \varepsilon_{-2}, u_c, u_d, (w_j) \right) = \left( \varepsilon_{-1} \cdot \varepsilon_{-2}, \varepsilon_{-1}, \varepsilon_{-2} \cdot u_c, \varepsilon_{-2} \cdot u_d, (w_j) \right).
\]

Considering all possible subsets \( I \) of \( \mathbb{I} \) in (3.14), we obtain a twisted chart \( \Phi_x : \mathcal{U}_x \rightarrow \mathcal{M}^\text{tf} \) centered at \( x \) over \( \mathcal{V} \rightarrow \mathcal{M}^\text{wt} \) so that for any

\[
y = \left( \varepsilon_{-1}, \varepsilon_{-2}, u_c, u_d, (w_j) \right) \in \mathcal{U}_x,
\]

- if \( \varepsilon_{-1} = \varepsilon_{-2} = 0 \) (i.e. \( y \in \mathcal{U}_{x;[t]} \)), then

\[
\Phi_x(y) = \left( \left( 0, 0, 0, 0, (w_j) \right); \left[ 0, \partial_{\varepsilon_{-1}} \theta_x(y), \partial_{\varepsilon_{-1}} \theta_x(y), u_c(\partial_{\varepsilon_{-2}} \theta_x(y) \otimes \partial_{\varepsilon_{-2}} \theta_x(y)), u_d(\partial_{\varepsilon_{-2}} \theta_x(y) \otimes \partial_{\varepsilon_{-2}} \theta_x(y)) \right] \right)
\in \mathcal{V}_x \times \mathcal{M}^\text{tf}_{[t]};
\]

- if \( \varepsilon_{-1} \neq 0 \) and \( \varepsilon_{-2} = 0 \) (i.e. \( y \in \mathcal{U}_{x;[t_{(-1)}]} \)), then

\[
\Phi_x(y) = \left( \left( 0, \varepsilon_{-1}, 0, 0, (w_j) \right); \left[ 0, \partial_{\varepsilon_{-1}} \theta_x(y), \frac{u_c}{\varepsilon_{-1}} \cdot \partial_{\varepsilon_{-1}} \theta_x(y), \frac{u_d}{\varepsilon_{-1}} \cdot \partial_{\varepsilon_{-1}} \theta_x(y) \right] \right)
\in \mathcal{V}_{t_{(-1)}} \times \mathcal{M}^\text{tf}_{[t_{(-1)}]};
\]

- if \( \varepsilon_{-1} = 0 \) and \( \varepsilon_{-2} \neq 0 \) (i.e. \( y \in \mathcal{U}_{x;[t_{(-2)}]} \)), then

\[
\Phi_x(y) = \left( \left( 0, 0, \varepsilon_{-2} \cdot u_c, \varepsilon_{-2} \cdot u_d, (w_j) \right); \left[ \varepsilon_{-2} \cdot \partial_{\varepsilon_{-2}} \theta_x(y), \frac{\partial_{\varepsilon_{-2}} \theta_x(y)}{\varepsilon_{-2}} \right] \right)
\in \mathcal{V}_{t_{(-2)}} \times \mathcal{M}^\text{tf}_{[t_{(-2)}]};
\]

- if \( \varepsilon_{-1} \neq 0 \) and \( \varepsilon_{-2} \neq 0 \) (i.e. \( y \in \mathcal{U}_{x;[t_{(-1,-2)}]} \)), then

\[
\Phi_x(y) = \theta_x(y) = \left( \varepsilon_{-1} \cdot \varepsilon_{-2}, \varepsilon_{-1}, \varepsilon_{-2} \cdot u_c, \varepsilon_{-2} \cdot u_d, (w_j) \right) \in \mathcal{V}_{t_{(-1,-2)}}.
\]

When a point \( y \in \mathcal{U}_x \) satisfying \( \varepsilon_{-1}(y) \neq 0 \) and \( \varepsilon_{-2}(y) = 0 \) approaches a point \( y' \in \mathcal{U}_x \) satisfying \( \varepsilon_{-1}(y') = \varepsilon_{-2}(y') = 0 \), it may seem that the limit of the expression of \( \Phi_x(y) \) does not match \( \Phi_x(y') \), however, as suggested in Remark 2.4, \( \left[ \partial_{\varepsilon_{-1}} \theta_x(y'), u_c(\partial_{\varepsilon_{-2}} \theta_x(y') \otimes \partial_{\varepsilon_{-2}} \theta_x(y')) \right] \) of \( \Phi_x(y') \) captures the limit of the ratio \( [\zeta_a : \zeta_b \zeta_c : \zeta_b \zeta_d] \), whereas \( \left[ \partial_{\varepsilon_{-1}} \theta_x(y), \frac{u_c}{\varepsilon_{-1}} \cdot \partial_{\varepsilon_{-1}} \theta_x(y), \frac{u_d}{\varepsilon_{-1}} \cdot \partial_{\varepsilon_{-1}} \theta_x(y) \right] \) of \( \Phi_x(y) \) captures the limit of the ratio \( [\zeta_a : \zeta_c : \zeta_d] \). If we multiply the last two components of the twisted field of \( \Phi_x(y) \) by \( \varepsilon_{-1} \) (i.e. \( \zeta_b \)) and then take \( \varepsilon_{-1} \rightarrow 0 \), we can recover \( \Phi_x(y') \).

With the expressions of \( \Phi_x \) as above, it is straightforward to check

\[
\Phi_x(0, 0, \lambda_c, \lambda_d, 0, \ldots, 0) = x,
\]
as well as to verify the statements of Lemmas 3.3, 3.5, 3.6, and Proposition 3.7 in this situation.

**Example 3.11.** We continue with the setup of Example 2.7. Let \( x \in \mathcal{M}^\text{tf}_{[t]} \) be a weighted curve of genus 1 with twisted fields over \( (C, w) \in \mathcal{M}^\text{wt} \). The core and the nodes of \( C \) are still labeled by \( o \) and by \( a, b, c, d \), respectively.
Let $\mathcal{V} \to \mathcal{M}^{wt}$ be an affine smooth chart containing $(C, \mathbf{w})$, with a set of local parameters
\[
\{\zeta_a, \zeta_b, \zeta_c, \zeta_d\} \cup \{\zeta_j\}_{j \in J}
\]
centered at $(C, \mathbf{w})$ as in Example 3.10. There then exists a non-zero $\lambda_d$ such that
\[
x = \left( 0, \begin{bmatrix} 0, \partial_{\zeta_b} \partial_{\zeta_c} \zeta_d \end{bmatrix} \right) \in \mathcal{V} \times \mathcal{M}^{\text{tf}}_{(1)}.
\]
We choose the special edges (3.5) of $\mathcal{M}$ by shrinking $\mathcal{U}_x$ if necessary we assume that $\mathcal{U}_x \subset \{ u_d \neq 0 \}$.

By Corollary 3.8 and (3.12), the forgetful morphism $\varpi : \mathcal{M}^{\text{tf}} \to \mathcal{M}^{\text{wt}}$ can locally be written as $\theta_x : \mathcal{U}_x \to \mathcal{V}$ such that
\[
\theta_x(x_0, x_1, u_a, u_d, (w_j)) = \left( x_0, x_0, x_1, x_1, x_0, x_0, x_0, x_0, \ldots, 0 \right)
\]
be an open subset containing the point
\[
y_x = (0, 0, 0, \lambda_d, 0, \ldots, 0).
\]
The coordinates of $\mathcal{U}_x$ are denoted by
\[
\varepsilon_{-1}, \varepsilon_{-2}, u_a, u_d, \text{ and } w_j \text{ with } j \in J,
\]
where $J$ is the same as the index set for the local parameters $\zeta_j$ on $\mathcal{V}$. Since $\lambda_d \neq 0$, by shrinking $\mathcal{U}_x$ if necessary we assume that $\mathcal{U}_x \subset \{ u_d \neq 0 \}$.

Considering all possible subsets $I$ of $\mathbb{I}$ in (3.14), we obtain a twisted chart $\Phi_x : \mathcal{U}_x \to \mathcal{M}^{\text{tf}}$ centered at $x$ over $\mathcal{V} \to \mathcal{M}^{\text{wt}}$ so that for any
\[
y = \left( \varepsilon_{-1}, \varepsilon_{-2}, u_a, u_d, (w_j) \right) \in \mathcal{U}_x,
\]
- if $\varepsilon_{-1} = \varepsilon_{-2} = u_a = 0$ (i.e. $y \in \mathcal{U}_{x;I_{(1)}}$), then
  \[
  \Phi_x(y) = \left( 0, 0, 0, 0, (w_j) \right) \in \mathcal{V} \times \mathcal{M}^{\text{tf}}_{(1)};
  \]
- if $\varepsilon_{-1} = u_a = 0$ and $\varepsilon_{-2} \neq 0$ (i.e. $y \in \mathcal{U}_{x;I_{(2)}}$), then
  \[
  \Phi_x(y) = \left( 0, 0, \varepsilon_{-2}, u_d, (w_j) \right) \in \mathcal{V} \times \mathcal{M}^{\text{tf}}_{(2)};
  \]
- if $\varepsilon_{-1} \neq 0$ and $\varepsilon_{-2} = u_a = 0$ (i.e. $y \in \mathcal{U}_{x;I_{(1)}}$), then
  \[
  \Phi_x(y) = \left( 0, 0, \varepsilon_{-2}, u_d, (w_j) \right) \in \mathcal{V} \times \mathcal{M}^{\text{tf}}_{(2)};
  \]
- if $\varepsilon_{-1} = \varepsilon_{-2} = 0$ and $u_a \neq 0$ (i.e. $y \in \mathcal{U}_{x;I_{(3)}}$), then
  \[
  \Phi_x(y) = \left( 0, 0, 0, (w_j) \right) \in \mathcal{V} \times \mathcal{M}^{\text{tf}}_{(3)};
  \]
- if $\varepsilon_{-1} = \varepsilon_{-2} = 0$ and $u_a \neq 0$ (i.e. $y \in \mathcal{U}_{x;I_{(3)}}$), then
  \[
  \Phi_x(y) = \left( 0, 0, 0, (w_j) \right) \in \mathcal{V} \times \mathcal{M}^{\text{tf}}_{(3)};
  \]
- if $\varepsilon_{-1} = \varepsilon_{-2} = 0$ and $u_a \neq 0$ (i.e. $y \in \mathcal{U}_{x;I_{(3)}}$), then
  \[
  \Phi_x(y) = \left( 0, 0, 0, (w_j) \right) \in \mathcal{V} \times \mathcal{M}^{\text{tf}}_{(3)};
It is straightforward that

\[ M \cong \text{Im}(\overline{\varphi}_y) \]

as well as to verify the statements of Lemmas 3.3, 3.5, 3.6, and Proposition 3.7 in this situation.

3.4. Universal family. Let \( \pi^w : C^w \to M^w \) be the universal weighted nodal curves of genus 1. The stratification (2.10) gives rise to a stratification

\[ C^w = \bigsqcup_{\tau \in \mathcal{R}^w} C^w_{\tau} \]

satisfying \( \pi^w(C^w_{\tau}) = M^w_{\tau} \) \( \forall \tau \in \mathcal{R}^w \).

Parallel to (2.13) and (2.14), we set

\[ C^w_{[t]} = \left( \prod_{i \in I(t)} \left( \left( \sum_{e \in \text{Edg}(t), \ell(v_e^-) = i} \overline{\varphi}_y(e) \right) \right) \right) \to C^w_{[t]} \]

\[ C^w = \bigsqcup_{[t] \in \mathcal{L}^w} C^w_{[t]} \]

Mimicking the construction of the stack structure of \( M^w \) in \S 3.1 and \S 3.2, we can endow \( C^w \) with a stack structure analogously. Furthermore, the projection \( \pi^w : C^w \to M^w \) induces a unique projection

\[ \pi^w : C^w \to M^w. \]

It is straightforward that

\[ C^w \cong C^w \times_{M^w} M^w \to M^w. \]
For any scheme $S$, a flat family $\mathcal{Z}/S$ of stable weighted nodal curves of genus 1 with twisted fields corresponds to a morphism $f : S \rightarrow \mathcal{M}^\text{tf}$ such that $\mathcal{Z}/S$ is the pullback of (3.33):

$$\mathcal{Z}/S \cong \left( S \times_{\mathcal{M}^\text{wt}} C^\text{tf} \right)/S.$$  

(3.34)

This leads to the following statement.

**Proposition 3.12.** $C^\text{tf} \rightarrow \mathcal{M}^\text{tf}$ in (3.33) gives the universal family of $\mathcal{M}^\text{tf}$.

**Remark 3.13.** One may establish a moduli interpretation of $\mathcal{M}^\text{tf}$ as follows: for any scheme $S$, every flat family $\mathcal{Z}/S$ of stable weighted nodal curves of genus 1 with twisted fields can be constructed directly as follows. A priori, $\mathcal{Z}/S$ is over a flat family $\mathcal{C}_S/S$ of stable weighted curves, thus by the universality of the moduli $\mathcal{M}^\text{wt}$, there exists a morphism

$$\alpha : S \rightarrow \mathcal{M}^\text{wt} = \bigsqcup_{\tau \in \mathcal{T}_R^\text{wt}} \mathcal{M}^\text{wt}_\tau,$$

such that $\mathcal{C}_S/S$ is the pullback of $C^\text{wt}/\mathcal{M}^\text{wt}$ via $\alpha$. This induces a stratification of the scheme $S$:

$$S = \bigsqcup_{\tau \in \mathcal{T}_R^\text{wt}} S_\tau \text{ satisfying } \alpha(S_\tau) \subset \mathcal{M}^\text{wt}_\tau \quad \forall \tau \in \mathcal{T}_R^\text{wt}. \quad (3.35)$$

We take

$$S^\text{tf}[t] = \left( \prod_{i \in L^\text{ct}(t)} \left( \left( \widehat{P} \left( \bigoplus_{e \in \text{Edg}(t), \ell(v^-_e) = i} \alpha^* L^\text{wt}_e \right) \right) / S^\text{tf}[t] \right) \right)^{\pi_S} S^\text{tf}[t] \quad \forall [t] \in \mathcal{T}^\text{ct}_L,$$

$$S^\text{tf} = \bigsqcup_{[t] \in \mathcal{T}^\text{ct}_L} S^\text{tf}[t] \xrightarrow{\pi_S} S.$$  

For any chart $V_S \rightarrow S$, shrinking it if necessary, we see there exists a (smooth) chart $V \rightarrow \mathcal{M}^\text{wt}$ such that

$$\begin{array}{ccc}
V_S & \longrightarrow & V \\
\downarrow & & \downarrow \\
S & \longrightarrow & \mathcal{M}^\text{wt}
\end{array}$$

commutes. The modular parameters $\zeta_e$ on $V$ pull back to regular functions on $V_S$, which are denoted by $\zeta^S_e$. By (3.35), we have

$$S_\tau \cap V_S = \{ \zeta^S_e = 0 : e \in \text{Edg}(\tau) \} \cap \{ \zeta^S_{e'} \neq 0 : e' \notin \text{Edg}(\tau) \}.$$  

Mimicking the construction in §3.1 and §3.2, we can thus endow $S^\text{tf}$ with a scheme structure.

We say $\mathcal{Z}/S$ is a flat family of stable weighted nodal curves of genus 1 with twisted fields if and only if there exists a section $\sigma^\text{tf}$ of $\pi_S : S^\text{tf} \rightarrow S$ such that

$$\mathcal{Z} = \mathcal{C}_S \times_S (\sigma^\text{tf}(S)).$$

This construction is consistent with (3.34). One can check that the groupoid sending any scheme $S$ to the set of all such defined flat families $\mathcal{Z}/S$ is represented by $\mathcal{M}^\text{tf}$.

We would like to remark that a more succinct definition of a flat family of stable weighted nodal curves of genus 1 with twisted fields should be desirable.
forms of the closed substacks said to be both of the weighted level trees.\(\gamma\) of Edg are all smooth, so is \(\gamma\) indicating the genus. In Proposition 4.4, we show that \(\gamma\) is isomorphic to \(\tilde{M}^{wt}\). Lemma 4.3 is rather technical; it is only used in the proof of Proposition 4.4.

We briefly recall the notion of the \textit{locally tree-compatible blowups} described in [7, §3]. Let \(M\) be a smooth stack, \(\gamma\) be a rooted tree, and \(V\) be an affine smooth chart of \(M\). If there exists a set of local parameters on \(V\) so that a subset of which can be written as

\[
\{ z_e \in \Gamma(E_V) : e \in \text{Edg}(\gamma) \},
\]

then the set is called a \(\gamma\)-labeled subset of local parameters on \(V\). For example, if \(M = M^{wt}\) and \(V\) is a chart centered at a weighted curve whose reduced dual tree is \(\gamma\), then the set of the modular parameters \(\{ z_e \}_{e \in \text{Edg}(\gamma)}\) is a \(\gamma\)-labeled subset of local parameters.

Let \(\text{Ver}(\gamma)_{\text{min}}\) be the set of the minimal vertices of \(\gamma\) with respect to the tree order. We call a subset \(E\) of \(\text{Edg}(\gamma)\) a traverse section if for any \(v \in \text{Ver}(\gamma)_{\text{min}}\), the path between \(o\) and \(v\) contains exactly one element of \(E\). For example, the subsets \(E_i\) of \(\text{Edg}(\gamma)\) as in (2.3) are traverse sections. Let \(\Xi(\gamma)\) be the set of the traverse sections. The tree order on \(\text{Edg}(\gamma)\) induces a partial order on \(\Xi(\gamma)\) such that

\[
E > E' \iff (E \neq E') \text{ and } \forall e \in E, \exists e' \in E' \text{ s.t. } e \geq e'.
\]

We remark that the tree order on \(\text{Edg}(\gamma)\) and the induced order on \(\Xi(\gamma)\) in this paper are both \textit{opposite} to those in [7], in order to be consistent with the order of the levels of the weighted level trees.

\textbf{Example 4.1.} Let \(\gamma\) be the rooted tree in Figure 4. The set \(\Xi(\gamma)\) consists of four traverse sections, as listed in Figure 4. The partial order on \(\Xi(\gamma)\) is given by \(E_1 > E_2 > E_4\) and \(E_1 > E_3 > E_4\); the traverse sections \(E_2\) and \(E_3\) are not comparable.

Let \(\tilde{M} \to M\) be the sequential blowup of \(M\) successively along the proper transforms of the closed substacks \(Z_1, Z_2, \ldots\) of \(M\).

\textbf{Definition 4.2 ([7, Definitions 3.2.4 & 3.2.1])}. The blowup \(\tilde{M} \to M\) above is said to be \textit{locally tree-compatible} if there exists an étale cover \(\{ V \}\) of \(M\) such that for

\[
E_1 = \{ a, b \} \quad E(\gamma) = \{ E_1, E_2, E_3, E_4 \}
\]

\[
E_2 = \{ a, e, f \} \quad \Xi = \{ E_1, E_2, E_3, E_4 \}
\]

\[
E_3 = \{ c, d, b \} \quad E_1 > E_2 > E_4
\]

\[
E_4 = \{ c, d, e, f \} \quad E_1 > E_3 > E_4
\]

\textbf{Fig. 4. An example of traverse sections}

4. Comparison with Hu-Li’s blowup stack \(\tilde{M}^{wt}\). Let \(\pi : \tilde{M}^{wt} \to M^{wt}\) be the sequential blowup constructed in [5, §2.2]. More precisely, let \(Z_k \subset M^{wt}\), \(k \in \mathbb{Z}_{>0}\), be the closed locus whose \textit{general} point is obtained by attaching \(k\) smooth positively-weighted rational curves to the smooth 0-weighted elliptic core at pairwise distinct points. Then \(\tilde{M}^{wt}\) is obtained by blowing up \(M^{wt}\) along the proper transforms of \(Z_1, Z_2, \ldots\) sequentially. Since \(M^{wt}\) is a smooth Artin stack and the blowup centers are all smooth, so is \(\tilde{M}^{wt}\). As per the convention of this paper, we omit the subscript indicating the genus. In Proposition 4.4, we show that \(\tilde{M}^{wt}\) is isomorphic to \(\tilde{M}^{wt}\).

Let \(\text{Ver}(\gamma)\) be the rooted tree, and \(E\) is a chart centered at a weighted curve whose reduced dual tree is \(\gamma\), then the set of the traverse sections \(\{ z_e \}_{e \in \text{Edg}(\gamma)}\) is a \(\gamma\)-labeled subset of local parameters.

Let \(\text{Ver}(\gamma)_{\text{min}}\) be the set of the minimal vertices of \(\gamma\) with respect to the tree order. We call a subset \(E\) of \(\text{Edg}(\gamma)\) a traverse section if for any \(v \in \text{Ver}(\gamma)_{\text{min}}\), the path between \(o\) and \(v\) contains exactly one element of \(E\). For example, the subsets \(E_i\) of \(\text{Edg}(\gamma)\) as in (2.3) are traverse sections. Let \(\Xi(\gamma)\) be the set of the traverse sections. The tree order on \(\text{Edg}(\gamma)\) induces a partial order on \(\Xi(\gamma)\) such that

\[
E > E' \iff (E \neq E') \text{ and } \forall e \in E, \exists e' \in E' \text{ s.t. } e \geq e'.
\]

We remark that the tree order on \(\text{Edg}(\gamma)\) and the induced order on \(\Xi(\gamma)\) in this paper are both opposite to those in [7], in order to be consistent with the order of the levels of the weighted level trees.

\textbf{Example 4.1.} Let \(\gamma\) be the rooted tree in Figure 4. The set \(\Xi(\gamma)\) consists of four traverse sections, as listed in Figure 4. The partial order on \(\Xi(\gamma)\) is given by \(E_1 > E_2 > E_4\) and \(E_1 > E_3 > E_4\); the traverse sections \(E_2\) and \(E_3\) are not comparable.

Let \(\tilde{M} \to M\) be the sequential blowup of \(M\) successively along the proper transforms of the closed substacks \(Z_1, Z_2, \ldots\) of \(M\).

\textbf{Definition 4.2 ([7, Definitions 3.2.4 & 3.2.1])}. The blowup \(\tilde{M} \to M\) above is said to be \textit{locally tree-compatible} if there exists an étale cover \(\{ V \}\) of \(M\) such that for
each \( \mathcal{V} \in \{ \mathcal{V} \} \), there exist a rooted tree \( \gamma \), a partition of \( \Xi(\gamma) \):

\[
\Xi(\gamma) = \bigcup_{k \geq 1} \Xi_k(\gamma)
\]

and a \( \gamma \)-labeled subset of local parameters on \( \mathcal{V} \) such that

- for every \( k \geq 1 \),
  \[
  Z_k \cap \mathcal{V} = \bigcup_{\mathcal{E} \in \Xi_k(\gamma)} \{ z_e = 0 : e \in \mathcal{E} \};
  \]

- if \( \mathcal{E}' \in \Xi_{k'}(\gamma) \), \( \mathcal{E}'' \in \Xi_{k''}(\gamma) \), and \( \mathcal{E}' \succ \mathcal{E}'' \), then \( k' < k'' \).

If a sequential blowup \( \overline{\mathcal{M}} \rightarrow \mathcal{M} \) is locally tree-compatible, then the blowup procedure is finite on each \( \mathcal{V} \in \{ \mathcal{V} \} \), because the set \( \Xi(\gamma) \) is finite.

**Lemma 4.3.** If the blowup \( \overline{\mathcal{M}} \rightarrow \mathcal{M} \) successively along the proper transforms of the closed substacks \( Z_1, Z_2, \ldots \) of \( \mathcal{M} \) is locally tree-compatible, then the blowup \( \overline{\mathcal{M}'} \rightarrow \mathcal{M} \) successively along the total transforms of

\[
Y_1 = Z_1, \quad Y_2 = Z_1 \cup Z_2, \quad Y_3 = Z_1 \cup Z_2 \cup Z_3, \quad 
\]
yields the same space, i.e. \( \overline{\mathcal{M}'} = \overline{\mathcal{M}} \).

**Proof.** We prove the statement by induction. For each \( h \geq 1 \), we will show that after the \( h \)-th step, the blowup stacks \( \overline{\mathcal{M}}(h) \) of \( \mathcal{M} \) along the total transforms of \( Y_1, \ldots, Y_h \) is the same as the blowup \( \overline{\mathcal{M}}(h) \) of \( \mathcal{M} \) along the proper transforms of \( Z_1, \ldots, Z_h \).

The base case of the induction is trivial. Suppose the blowup \( \overline{\mathcal{M}}(k) = \overline{\mathcal{M}}(k) \). We will show that for any \( x \in \mathcal{M} \) and any lift \( \tilde{x} \) of \( x \) after the \( k \)-th step, the blowup along the total transform \( \tilde{Y}_{k+1} \) of \( Y_{k+1} \) has the same effect as that along the proper transform \( \tilde{Z}_{k+1} \) of \( Z_{k+1} \) near \( \tilde{x} \). Since \( x \) and \( \tilde{x} \) are arbitrary, this will establish the \((k+1)\)-th step of the induction.

W.l.o.g. we may assume \( x \in \cap_{i=1}^{k+1} Z_k \) (otherwise we simply omit the loci \( Z_i \) not containing \( x \) and change the indices of \( Z_i \) and \( Y_i \) accordingly). The blowup \( \overline{\mathcal{M}} \rightarrow \mathcal{M} \) is locally tree-compatible, hence there exist a rooted tree \( \gamma \), an affine smooth chart \( \mathcal{V} \) containing \( x \), and a \( \gamma \)-labeled subset of local parameters \( z_e, e \in \text{Edg}(\gamma) \) on \( \mathcal{V} \) such that

\[
x \in \{ z_e = 0 : e \in \text{Edg}(\gamma) \}.
\]

As shown in [7, Lemma 3.3.2], there exist traverse sections \( \mathcal{G}_{(k)} \in \Xi_k(\gamma) \) and \( \mathcal{G}_{(k+1)} \in \Xi_{k+1}(\gamma) \) (c.f. Definition 4.2), an affine smooth chart \( \tilde{\mathcal{V}}_\mathcal{E} \), and a subset of local parameters

\[
\tilde{z}_1, \ldots, \tilde{z}_k; \quad \tilde{z}_e \text{ with } e \in \mathcal{G}_{(k+1)} \setminus \mathcal{G}_{(k)}; \quad \tilde{z}_e \text{ with } e \in \mathcal{G}_{(k+1)} \cap \mathcal{G}_{(k)}
\]
on \( \tilde{\mathcal{V}}_\mathcal{E} \) so that \( \tilde{Z}_{k+1} \) is locally given by

\[
\tilde{Z}_{k+1} \cap \tilde{\mathcal{V}}_\mathcal{E} = \{ \tilde{z}_e = 0 : e \in \mathcal{G}_{(k+1)} \setminus \mathcal{G}_{(k)} \}, \quad \tilde{z}_e = 0 : e \in \mathcal{G}_{(k+1)} \cap \mathcal{G}_{(k)} \}
\]

Moreover, by [7, (3.13)], the total transform of each \( Z_i \) with \( 1 \leq i \leq k \) is locally given by \( \{ \tilde{z}_i = 0 \} \). Thus, \( \tilde{Y}_{k+1} \) is locally given by

\[
\tilde{Y}_{k+1} \cap \tilde{\mathcal{V}}_\mathcal{E} = (\tilde{Z}_{k+1} \cap \tilde{\mathcal{V}}_\mathcal{E}) \cup \{ \prod_{1 \leq i \leq k} \tilde{z}_i = 0 \}.
\]
That is, on the chart $\tilde{Y}_{\tilde{z}}$, $\tilde{Z}_{k+1}$ and $\tilde{Y}_{k+1}$ are defined by the ideals

$$\mathcal{I}_{\tilde{Z}_{k+1}} = \langle z_e : e \in S_{(k+1)} \setminus S_{(k)}, \ z_{\tilde{e}} : e \in S_{(k)} \cap S_{(k)} \rangle \quad \text{and} \quad \mathcal{I}_{\tilde{Z}_{k+1}} = \prod_{1 \leq i \leq k} \tilde{e}_i,$$

respectively. Therefore, blowing up along $\tilde{Z}_{k+1}$ has the same effect on $\tilde{Y}_{\tilde{z}}$ as that along $Y_{k+1}$. □

**Proposition 4.4.** $\mathcal{M}^t / \mathcal{M}_{wt}$ is isomorphic to $\widetilde{\mathcal{M}}_{wt} / \mathcal{M}_{wt}$. In particular, $\varpi : \mathcal{M}^t \longrightarrow \mathcal{M}_{wt}$ is proper.

**Proof.** Our goal is to construct two morphisms $\psi_1$ and $\psi_2$ between $\widetilde{\mathcal{M}}_{wt}$ and $\mathcal{M}^t$ so that the following diagram

$$\begin{array}{ccc}
\tilde{M}_{wt} & \xrightarrow{\psi_2} & \mathcal{M}^t \\
\pi \downarrow & & \downarrow \\
\mathcal{M}_{wt} & \xrightarrow{\psi_1} & \tilde{M}_{wt}
\end{array}$$

commutes. Since $\pi$ and $\varpi$ restrict to the identity map on the preimages of the open subset

$$\{(C, w) \in \mathcal{M}_{wt} : w(C_\alpha) > 0\} \subset \mathcal{M}_{wt},$$

respectively, we see that $\psi_1$ and $\psi_2$ are the identity maps over the open subsets, respectively, hence $\psi_2 \circ \psi_1$ and $\psi_1 \circ \psi_2$ are the identity maps. This then implies the former statement of Proposition 4.4. The latter statement follows from the former as well as the properness of the blowup $\widetilde{\mathcal{M}}_{wt} \longrightarrow \mathcal{M}_{wt}$.

We first construct $\psi_1$. For each $k \in \mathbb{Z}_{>0}$, let $Z_k \subset \mathcal{M}_{wt}$ be the closed locus whose general point is obtained by attaching $k$ smooth positively-weighted rational curves to the smooth 0-weighted elliptic core at pairwise distinct points. By Lemma 4.3, the blowup $\pi : \widetilde{\mathcal{M}}_{wt} \longrightarrow \mathcal{M}_{wt}$ successively along the proper transforms of $Z_1, Z_2, \ldots$ can be identified with the blowup of $\mathcal{M}_{wt}$ successively along the total transforms of $\gamma$

$$Y_1 = Z_1, \ Y_2 = Z_1 \cup Z_2, \ Y_3 = Z_1 \cup Z_2 \cup Z_3, \ldots$$

We observe that for each $k \in \mathbb{Z}_{>0}$, the pullback $\varpi^{-1}(Y_k)$ to $\mathcal{M}^t$ is a Cartier divisor. In fact, for any $[t] = [\gamma, w, \ell] \in \mathcal{F}_{wt}$ and $x \in \mathcal{M}^t_{[t]}$, let $\mathcal{U} \longrightarrow \mathcal{M}^t$ be a twisted chart centered at $x$, lying over a chart $V \longrightarrow \mathcal{M}_{wt}$. In [5], the blowup $\pi$ locally on $V$ is proved to be compatible with the weighted tree $(\gamma, \mathcal{W})$ obtained by contracting all the edges $e$ of $\gamma$ as long as there exists $v \geq v_\gamma^+$ satisfying $w(v) > 0$. Let $\{e \in \text{Edg}(\gamma)\}$ be a set of modular parameters on $V$ as in (3.1) and

$$\{e_i\}_{i \in I_+} \cup \{u_e\}_{e \in \text{Edg}(t) \setminus \{e_i : i \in I_+\}} \cup \{z_e\}_{e \in I_-}$$

be the subset of the parameters (3.9) on $\mathcal{U}$. We claim that

$$\varpi^{-1}(Y_k) \cap \mathcal{U} = \{ \prod_{i \in I_+ : |e_i| \leq k} e_i = 0 \}.$$  \hspace{1cm} (4.1)

To show (4.1), we first notice that $\varpi^{-1}(Y_k) \cap \mathcal{U} = \varpi^{-1}(Y_k \cap V)$ by Corollary 3.8. Every irreducible component of $Y_k \cap V$ can be written in the form

$$Y_{k, \mathcal{G}} := \{e_\gamma = 0 : e \in \mathcal{G}\} \quad \text{with} \quad \mathcal{G} \in \Xi(\gamma), \ |\mathcal{G}| \leq k, \ \mathcal{G} \cap (\text{Edg}(t) \setminus \mathcal{I}_{m}) \neq \emptyset.$$
For each irreducible component $Y_{k,\mathcal{O}}$ of $Y_k \cap Y$, the local expression $\theta_x$ of $\varpi$ as in (3.12) implies the pullback $\varpi^{-1}(Y_{k,\mathcal{O}})$ can be written as

$$\{ \prod_{h \in \mathcal{E}(\ell(v^+_e), \ell(v^+_e))} \varepsilon_h = 0 : e \in \mathcal{S} \cap (\mathcal{Edg}(t) \setminus \mathcal{I}_m), \quad (u_e \cdot \prod_{h \in \mathcal{E}(\ell(v^+_e), \ell(v^+_e))} \varepsilon_h) = 0 : e \in \mathcal{S} \cap \mathcal{I}_m, \quad z_e = 0 : e \in \mathcal{S} \cap \mathcal{I}_- \}. \quad (4.2)$$

Since $\mathcal{S} \cap (\mathcal{Edg}(t) \setminus \mathcal{I}_m) \neq \emptyset$ and $|\mathcal{S}| \leq k$, we can always find $e \in \mathcal{S} \cap (\mathcal{Edg}(t) \setminus \mathcal{I}_m)$ such that $|\mathcal{E}_h| \leq |\mathcal{S}| \leq k$ for all $h \in \mathcal{E}(\ell(v^+_e), \ell(v^+_e))$. By (4.1) and (4.2), the pullback $\varpi^{-1}(Y_{k,\mathcal{O}})$ is thus a sub-locus of the right-hand side of (4.1). Moreover, it is a direct check that

$$\varpi^{-1}(Y_{k,\mathcal{O}}) \cap \mathcal{U} = \{ \varepsilon_i = 0 \} \quad \forall i \in \mathcal{I}_+ \text{ with } |\mathcal{E}_i| \leq k.$$ 

Therefore, (4.1) holds.

Since every $\varpi^{-1}(Y_k)$ is a Cartier divisor of $\mathcal{M}^{wt}$, by the universality of the blowup $\pi: \mathcal{M}^{wt} \to \mathcal{M}^{wt}$, we obtain a unique morphism

$$\psi_1: \mathcal{M}^{t} \to \mathcal{M}^{wt}$$

that $\varpi: \mathcal{M}^{t} \to \mathcal{M}^{wt}$ factors through.

We next construct $\psi_2$ explicitly. For any $\bar{x} \in \mathcal{M}^{wt}$, let $(C, \omega)$ be its image in $\mathcal{M}^{wt}$. As shown in [7, §3.3], there exists a unique maximal sequence of exceptional divisors

$$\bar{E}_{i_1}, \ldots, \bar{E}_{i_k} \subset \mathcal{M}^{wt}, \quad 1 \leq i_1 < \cdots < i_k$$

containing $\bar{x}$. Each $\bar{E}_{i_j}$ is obtained from blowing up along the proper transform of $Z_{i_j}$. Note that $k$ is possibly 0, which means $(C, \omega)$ is not in the blowup loci. The weighted dual tree $\tau = (\gamma_C, \omega)$, along with the exceptional divisors $\bar{E}_{i_1}, \ldots, \bar{E}_{i_k}$, uniquely determines a weighted level tree $t_{\bar{x}}$ such that

$$\mathcal{I}_+ = \mathcal{I}_+(t_{\bar{x}}) = \{-i_k, \ldots, -i_1\}.$$ 

In particular, $\mathcal{m} = \mathcal{m}(t_{\bar{x}}) = -i_k$.

With the line bundles $\mathcal{L}_{i} = L_{i} \otimes \bigotimes_{j \in \mathcal{E}(\ell(v^+_e), \ell(v^+_e))} \mathcal{L}_{j}^{\ast}$, as in (2.11), the notation $\langle \cdot, \cdot \rangle_{t_{\bar{x}}}$ and $\llbracket \cdot, \cdot \rrbracket_{t_{\bar{x}}}$ as in (2.2), and the notation $i[\ell]$ as in (3.6), the line bundles

$$\mathcal{L}_i = L_{i} \otimes \bigotimes_{j \in \mathcal{E}(\ell(v^+_e), \ell(v^+_e))} \mathcal{L}_{j}^{\ast} \rightarrow \mathcal{M}^{wt}_{\tau}, \quad i \in \mathcal{I}_+,$$ 

can be constructed inductively over $\mathcal{I}_+$. Then, we take

$$\mathcal{L}_e = L_{e} \otimes \bigotimes_{j \in \mathcal{E}(\ell(v^+_e), \ell(v^+_e))} \mathcal{L}_{j}^{\ast} \rightarrow \mathcal{M}^{wt}_{\tau}, \quad e \in \mathcal{Edg}(t_{\bar{x}}). \quad (4.4)$$

In particular, $\mathcal{L}_e = \mathcal{L}_1$. For each $e \in \mathcal{Edg}(t_{\bar{x}})$, (4.3) and (4.4) imply

$$\mathcal{L}_e \otimes \bigotimes_{e' > e} (\mathcal{L}_{e'} \otimes \mathcal{L}_{e'}^{\ast}) = L_{e}^{\ast} \otimes \bigotimes_{j \in \mathcal{E}(\ell(v^+_e), \ell(v^+_e))} \mathcal{L}_{j}^{\ast}.$$

Hence for each $i \in \mathcal{I}_+$,

$$\hat{\mathcal{P}} \left( \bigoplus_{\ell(v^+_e) = i, e' > e} (\mathcal{L}_e \otimes \bigotimes_{e' > e} (\mathcal{L}_{e'} \otimes \mathcal{L}_{e'}^{\ast})) \right) = \hat{\mathcal{P}} \left( \bigoplus_{\ell(v^+_e) = i} L_{e}^{\ast} \right). \quad (4.5)$$
For \( h \geq 1 \), let \( \tilde{x}_{(h)} \) be the image of \( \tilde{x} \) in the exceptional divisor of the \( h \)-th step. Given \( i \in I_+ \), the proper transform of \( Z_{-i} \) after the first \(-i-1\) steps of the blowup may have several connected components; see [7, Lemma 3.3.2]. The normal bundle of the component containing \( \tilde{x}_{(-i-1)} \) is the pullback \( \pi^*(-i-1) \oplus \mathcal{E}_e \), where \( \pi(h) : \mathcal{M}^{\text{wt}}(h) \to \mathcal{M}^{\text{wt}} \) is the blowup after the \( h \)-th step.

Notice that the non-zero entries of \( \tilde{x}_{(-i)} \) exactly correspond to the edges \( e \in \mathcal{E}_i \) satisfying \( \ell(v^-_e) = i \). Therefore,

\[
\tilde{x}_{(-i)} \in \pi^*(-i-1) \mathcal{P}( \bigoplus_{\ell(v^-_e) = i} \mathcal{E}_e ).
\]

Then, \( \tilde{x}_{(-j)} \) with \( j \in [i, 0]_{\mathbb{T}_h} \) together determine a unique

\[
\eta_j(\tilde{x}) \in \mathcal{P}( \bigoplus_{\ell(v^-_e) = i} (\mathcal{E}_e \otimes \mathcal{L}_{\ell(e')})) = \mathcal{P}( \bigoplus_{\ell(v^-_e) = i} \mathcal{L}_e ).
\]

The last equality above follows from (4.5). We then set

\[
\psi_2(\tilde{x}) = \left( (C, w), [t_{\mathbb{T}_h}], (\eta_j(\tilde{x}) : i \in I_+(t_{\mathbb{T}_h})) \right) \in \mathcal{M}^{\text{tf}}_{[t_{\mathbb{T}_h}]}.
\]

Obviously, this implies \( \varpi \circ \psi_2 = \pi \).

It remains to verify such defined \( \psi_2 \) is a morphism. Let \( \mathcal{V} \to \mathcal{M}^{\text{wt}} \) be a smooth chart containing \( (C, w) \), and \( \{ \zeta_e : e \in \text{Edg}(t_{\mathbb{T}_h}) \} \cup \{ \varsigma_j : j \in J \} \) be a set of local parameters centered at \( (C, w) \) as in (3.3). As shown in [7, §3.1&§3.3], there exists a chart \( \tilde{\mathcal{V}}_{\tilde{x}} \to \tilde{\mathcal{M}}^{\text{wt}} \) containing \( \tilde{x} \) with local parameters

\[
\tilde{z}_i, \quad i \in I_+; \quad \rho_e, \quad e \in \text{Edg}(t_{\mathbb{T}_h}) \setminus (\ell \cup \{ e_i : i \in I_+ \});
\]

\[
\tilde{z}_e, \quad e \in I_{\mathbb{R}}; \quad \tilde{z}_e, \quad e \in I_{-\mathbb{R}}; \quad s_j, \quad j \in J.
\]

All \( \rho_e \) are nowhere vanishing on \( \tilde{\mathcal{V}}_{\tilde{x}} \). Moreover, with \( \pi : \tilde{\mathcal{V}}_{\tilde{x}} \to \mathcal{V} \) denoting the blowup, we have

\[
\pi^* \zeta_e = \prod_{h \in [i, i+1]} \tilde{z}_i \quad \forall i \in I_+; \quad \pi^* \zeta_e = \rho_e \prod_{i \in \ell(e), \ell(v^+_e)} \tilde{z}_i \quad \forall e \in \text{Edg}(t_{\mathbb{T}_h}) \setminus (\ell \cup \{ e_i : i \in I_+ \});
\]

\[
\pi^* \zeta_e = \tilde{z}_e \prod_{i \in \ell(e), \ell(v^+_e)} \tilde{z}_i\quad \forall e \in I_{\mathbb{R}}; \quad \pi^* \zeta_e = \tilde{z}_e \quad \forall e \in I_{-\mathbb{R}}; \quad \pi^* \varsigma_j = s_j \quad \forall j \in J.
\]

For \( e \in \{ e_i \}_{i \in I_+} \), we set \( \rho_e = 1 \). Then,

\[
\rho_e \in \Gamma( \vartheta_{\tilde{\mathcal{V}}_{\tilde{x}}} ) \quad \forall e \in \text{Edg}(t_{\mathbb{T}_h}) \setminus I_{\mathbb{R}}.
\]

Let \( \mathcal{U}_{\psi_2(\tilde{x})} \to \mathcal{M}^{\text{tf}} \) be a twisted chart centered at \( \psi_2(\tilde{x}) \), lying over \( \mathcal{V} \to \mathcal{M}^{\text{wt}} \). The parameters on \( \mathcal{U}_{\psi_2(\tilde{x})} \) are as in (3.3). It is a direct check that the point-wise defined \( \psi_2 \) can locally be written as

\[
\psi_2 : \tilde{\mathcal{V}}_{\tilde{x}} \to \mathcal{U}_{\psi_2(\tilde{x})}
\]

such that

\[
\psi_2^* \tilde{z}_i = \tilde{z}_i \quad \forall i \in I_+; \quad \psi_2^* u_e = \prod_{e \geq e} \frac{\rho_e}{\prod_{e \geq e' \cap e} \rho_{e'}} \quad \forall e \in \text{Edg}(t) \setminus (\ell \cup \{ e_i : i \in I_+ \});
\]

\[
\psi_2^* u_e = \tilde{z}_e \prod_{e \geq e' \cap e} \frac{\rho_e}{\rho_{e'}} \quad \forall e \in I_{\mathbb{R}}; \quad \psi_2^* z = \tilde{z}_e \quad \forall e \in I_{-\mathbb{R}}; \quad \psi_2^* w_j = s_j \quad \forall j \in J.
\]
This shows \( \psi_2: \overline{\mathcal{M}}^\text{wt} \longrightarrow \mathcal{M}^\text{tf} \) is a morphism. \( \square \)

**Remark 4.5.** In [6], another resolution \( \overline{\mathcal{M}}^\text{dr} \longrightarrow \mathcal{M}^\text{wt} \), called the derived resolution of \( \mathcal{M}^\text{wt} \), is constructed for the purpose of diagonalizing certain direct image sheaves. That resolution is “smaller” in the sense that the resolution \( \overline{\mathcal{M}}^\text{wt} \longrightarrow \mathcal{M}^\text{wt} \) of [5] factors through \( \overline{\mathcal{M}}^\text{dr} \longrightarrow \mathcal{M}^\text{wt} \). Mimicking the approach of \( \S 3 \), we may construct a moduli stack

\[
\mathcal{N} = \bigcup_{[t] \in \mathcal{P}^\text{wt}} \mathcal{N}_t, \quad \mathcal{N}_t = \overline{\mathcal{P}} \left( \bigoplus_{e \in \text{Edg}(t), \ell(e) = \mathfrak{m}(t)} L^e \right) \longrightarrow \mathcal{M}^\text{wt}_{j[t]},
\]

This moduli should be isomorphic to \( \overline{\mathcal{M}}^\text{dr} \).

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