THE ASYMPTOTIC FORMULA FOR WARING’S PROBLEM
IN FUNCTION FIELDS

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Abstract. Let $F_q[t]$ be the ring of polynomials over $F_q$, the finite field of $q$ elements, and let $p$ be the characteristic of $F_q$. We denote $\tilde{G}_q(k)$ to be the least integer $t_0$ with the property that for all $s \geq t_0$, one has the expected asymptotic formula in Waring’s problem over $F_q[t]$ concerning sums of $s$ $k$-th powers of polynomials in $F_q[t]$. For each $k$ not divisible by $p$, we derive a minor arc bound from Vinogradov-type estimates, and obtain bounds on $\tilde{G}_q(k)$ that are quadratic in $k$, in fact linear in $k$ in some special cases, in contrast to the bounds that are exponential in $k$ available only when $k < p$. We also obtain estimates related to the slim exceptional sets associated to the asymptotic formula.

1. Introduction

In the early twentieth century, Hardy and Littlewood developed the technique now known as the Hardy-Littlewood circle method in a series of papers on Waring’s problem. Waring’s problem is regarding the representation of a natural number as a sum of integer powers. More precisely, given $n, s, k \in \mathbb{N}$, $k \geq 2$, we let

$$R_{s,k}(n) = \# \{(x_1, \ldots, x_s) \in \mathbb{N}^s : x_1^k + \ldots + x_s^k = n, ~ x_i \leq n^{1/k} ~(1 \leq i \leq s)\},$$

and we consider the smallest number $s$ such that $R_{s,k}(n) > 0$. There are various questions studied related to Waring’s problem, one of which is to find the minimum number of variables required to establish the expected asymptotic formula. This is an important aspect of Waring’s problem as a “brief review of the progress achieved in nearly a century of development of the Hardy-Littlewood (circle) method reveals that a substantial part has originated in work devoted to the challenge of establishing the asymptotic formula in Waring’s problem” [10]. As stated in [10], by a heuristic application of the circle method, one expects that when $k \geq 3$ and $s \geq k + 1$,

$$R_{s,k}(n) = \frac{\Gamma(1 + 1/k)^s}{\Gamma(s/k)} \mathcal{G}_{s,k}(n) n^{s-1/k} + o(n^{s-1/k}),$$

where

$$\mathcal{G}_{s,k}(n) = \sum_{a=1}^{\infty} \sum_{(a,q)=1}^{\infty} \left( \frac{1}{q} \sum_{r=1}^{q} e^{2\pi i (ar^k/q)} \right)^s e^{-2\pi i na/q}.$$ 

We note that subject to modest congruence conditions on $n$, one has $1 \ll \mathcal{G}_{s,k}(n) \ll n^\varepsilon$ [8, Chapter 4]. Let $\tilde{G}(k)$ be the least integer $t_0$ with the property that, for all $s \geq t_0$, and all sufficiently large natural numbers $n$, one has the asymptotic formula (1.1). As a consequence of his recent work concerning Vinogradov’s mean value theorem, Wooley has

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significantly improved estimates on $\tilde{G}(k)$ [9, 10, 11]. In particular, it was proved in [11] that $\tilde{G}(k) \leq 2k^2 - 2k - 8$ ($k \geq 6$).

In this paper, we consider an analogous problem in the setting of $\mathbb{F}_q[t]$, where $\mathbb{F}_q$ is a finite field of $q$ elements. In other words, we consider the asymptotic Waring’s problem over $\mathbb{F}_q[t]$. We later define $\tilde{G}_q(k)$, an analogue of $\tilde{G}(k)$ over $\mathbb{F}_q[t]$, and establish bounds on it. As the function field analogue of Wooley’s work on Vinogradov’s mean value theorem [9] has been established in [6] and its multidimensional version in [2], it is natural to consider its consequences in improving the number of variables required to establish the asymptotic formula in Waring’s problem over $\mathbb{F}_q[t]$. Here we accomplish this task by taking the approach of [10].

Before we can state our main results, we need to introduce notation, some of which we paraphrase from the material in introduction of [5]. We denote the characteristic of $\mathbb{F}_q$, a positive prime number, by $\text{ch}(\mathbb{F}_q) = p$. Unless we specify otherwise, we always assume $p$ to be the characteristic of $\mathbb{F}_q$ even if it is not explicitly stated so. Let $k$ be an integer with $k \geq 2$, let $s \in \mathbb{N}$, and consider a polynomial $n \in \mathbb{F}_q[t]$. We are interested in the representation of $n$ of the form

$$n = x_1^k + x_2^k + \ldots + x_s^k,$$

where $x_i \in \mathbb{F}_q[t]$ ($1 \leq i \leq s$). It is possible that a representation of the shape (1.2) is obstructed for every natural number $s$. For example, if the characteristic $p$ of $\mathbb{F}_q$ divides $k$, then $x_1^k + x_2^k + \ldots + x_s^k = (x_1^{k/p} + x_2^{k/p} + \ldots + x_s^{k/p})^p$, and thus $n$ necessarily fails to admit a representation of the shape (1.2) whenever $n \notin \mathbb{F}_q[t^p]$, no matter how large $s$ may be. In order to accommodate this and other intrinsic obstructions, we define $\mathbb{J}_q^k[t]$ to be the additive closure of the set of $k$-th powers of polynomials in $\mathbb{F}_q[t]$, and we restrict attention to those $n$ lying in the subring $\mathbb{J}_q^k[t]$ of $\mathbb{F}_q[t]$. It is also convenient to define $\mathbb{J}_q^k$ to be the additive closure of the set of $k$-th powers of elements of $\mathbb{F}_q$.

Given $n \in \mathbb{J}_q^k[t]$, we say that $n$ is an exceptional element of $\mathbb{J}_q^k[t]$ when its leading coefficient lies in $\mathbb{F}_q \backslash \mathbb{J}_q^k$, and in addition $k$ divides $\deg n$. As explained in [5], the strongest constraint on the degrees of the variables that might still permit the existence of a representation of the shape (1.2) is plainly $\deg x_i \leq \lceil (\deg n) / k \rceil$ ($1 \leq i \leq s$). When $p < k$, however, it is possible that $\mathbb{J}_q^k$ is not equal to $\mathbb{F}_q$, and then the leading coefficient of $n$ need not be an element of $\mathbb{J}_q^k$. If $k$ divides $\deg n$, so that $n$ is an exceptional polynomial, such circumstances obstruct the existence of a representation (1.2) of $n$ with variables $x_i$ satisfying the above constraint on their degrees. For these reasons, following [5], we define $P = P_k(n)$ by setting

$$P = \begin{cases} \left\lfloor \frac{\deg n}{k} \right\rfloor, & \text{if } n \text{ is not exceptional}, \\ \frac{\deg n}{k} + 1, & \text{if } n \text{ is exceptional}. \end{cases}$$

In particular, when $n$ is not exceptional, then $P$ is the unique integer satisfying $k(P - 1) < \deg n \leq kP$. We say that $n$ admits a strict representation as a sum of $s$ $k$-th powers when for some $x_i \in \mathbb{F}_q[t]$ with $\deg x_i \leq P_k(n)$ ($1 \leq i \leq s$), the equation (1.2) is satisfied.

For notational convenience, let $X = X_k(n) := P_k(n) + 1$, and we define $I_X := \{ x \in \mathbb{F}_q[t] : \deg x < X \}$. For $n$ a polynomial in $\mathbb{F}_q[t]$, we denote $R_{s,k}(n)$ to be the number of strict representations of $n$, in other words

$$R_{s,k}(n) = \# \{(x_1, \ldots, x_s) \in (I_X)^s : x_1^k + \ldots + x_s^k = n \}.$$
Though it is not explicit in the notation, $R_{s,k}(n)$ does depend on $q$. Suppose the leading coefficient of the polynomial $n$ is $c(n)$. We define $b = b(n)$ to be $c(n)$ when $k$ divides $\deg n$ and $n$ is not exceptional, and otherwise we set $b(n)$ to be 0. In addition, we write $J_{\infty}(n) = J_{\infty}(n; q)$ for the number of solutions of the equation $y_1^k + \ldots + y_m^k = b$ with $(y_1, \ldots, y_s) \in \mathbb{F}_q \setminus \{0\}$. Analogously to the case of integers, one expects the following asymptotic formula

\begin{equation}
R_{s,k}(n) = \mathcal{G}_{s,k}(n) J_{\infty}(n) q^{(s-k)P} + o\left(q^{(s-k)P}\right),
\end{equation}

where

$$
\mathcal{G}_{s,k}(n) = \sum_{g \in \mathbb{F}_q[t]} \frac{1}{q^{n(\deg g)}} \sum_{\deg a < \deg g} \left( \sum_{\deg r < \deg g} e(ar^k/g) \right)^s e(-na/g),
$$

to hold whenever $s$ is sufficiently large with respect to $k$. We postpone the definition of the exponential function $e(\cdot)$ to Section 2. By making the circle method applicable over $\mathbb{F}_q[t]$, the following theorem was proved in \cite[Theorem 30]{7}. We note that the theorem stated below is slightly different from the statement of \cite[Theorem 30]{7}. The reason for this difference is explained in the paragraph before Theorem 2.1 on page 7.

**Theorem 1.1** (Theorem 30, \cite{7}). Suppose $3 \leq k < p$ and $s \geq 2^k + 1$. Let $n \in \mathbb{F}_q[t]$. Then there exists $\epsilon > 0$ such that the following asymptotic formula holds,

\begin{equation}
R_{s,k}(n) = \mathcal{G}_{s,k}(n) J_{\infty}(n) q^{(s-k)P} + O\left(q^{(s-k-\epsilon)P}\right),
\end{equation}

where

\begin{equation}
1 \ll \mathcal{G}_{s,k}(n) J_{\infty}(n) \ll 1.
\end{equation}

Note that the implicit constants in the theorem may depend on $k$, $s$, and $q$, where the constant in \eqref{eq:1.4} may also depend on $\epsilon$, but they are independent of $n$ and $P$.

We denote $\tilde{G}_q(k)$ to be the least integer $t_0$ with the property that, for all $s \geq t_0$, and all $n \in \mathbb{F}_q[t]$ with $\deg n$ sufficiently large, one has the above asymptotic formula \eqref{eq:1.3}. Thus, in this language we have the following corollary as an immediate consequence of Theorem 1.1 except for the case $k = 2$. (The estimate on $\tilde{G}_q(2)$ is treated in the paragraph after the proof of Theorem 2.1 on page 7.)

**Corollary 1.2.** Suppose $2 \leq k < p$. Then we have

$$
\tilde{G}_q(k) \leq \begin{cases} 
2^k + 1, & \text{if } k \geq 3, \\
5, & \text{if } k = 2.
\end{cases}
$$

It is worth mentioning that one of the main advantages of using Vinogradov-type estimates established in \cite{2} or \cite{3} is that we can avoid the use of Weyl differencing as the primary tool during the computation of minor arc bounds, which is the source of the restriction $k < p$ in Theorem 1.1 and Corollary 1.2. Thus, via Vinogradov-type estimates we can obtain an estimate for $\tilde{G}_q(k)$ for a larger range of $k$, which is for all $k$ not divisible by $p$.

We are now ready to state our main results. To avoid clutter in the exposition, we present the cases $k > p$ and $k < p$ separately. When $k > p$, as a result of our approach we further consider three cases, $p \nmid (k-1)$, $k = p^b + 1$, and $k = mp^b + 1$, where $b, m \in \mathbb{N}$ and $p \nmid m$. Throughout the paper, whenever we write $k = mp^b + 1$ we are assuming $b, m \in \mathbb{N}$ and $p \nmid m$, even when these conditions are not explicitly stated.
Theorem 1.3. Let \( k \geq 3 \) be an integer, where \( p \nmid k \). Suppose \( k > p \), then we have

\[
\tilde{G}_q(k) \leq \begin{cases} 
2k \left( k - \left\lfloor \frac{k}{p} \right\rfloor \right) - 5 + \left\lfloor \frac{6|k/p| - 4}{k-2} \right\rfloor, & \text{if } p \nmid (k-1), \\
4k + 5, & \text{if } k = p^b + 1, \\
\left( 2 - \frac{2}{p} \right) k^2 - 2(p^b - p^{b-1} - 2)k - c_k, & \text{if } k = mp^b + 1 \text{ and } m > 1,
\end{cases}
\]

where \( c_k = 2 \left( p^b - p^{b-1} - 1 - \frac{1}{p} \right) + \left\lfloor \frac{(m-1)(1-1/p)}{2} \right\rfloor. \]

We note that when \( p \nmid (k-1) \) the above theorem is proved using Lemma 9 in Section 3 which involves an application of the pigeon hole principle. However, when \( k = mp^b + 1 \) this approach is no longer effective. As a result, we have to use analogous results which rely on the large sieve inequality instead when \( m > 1 \), and another separate approach when \( m = 1 \). This explains why we consider the three cases separately.

We also remark that when \( k > p \) our estimates for \( \tilde{G}_q(k) \) given above are sharper than the current available bound of \( \tilde{G}(k) \leq 2k^2 - 2k - 8 \) \((k \geq 6)\) for the integer case [11]. In particular, note that in the special case when \( k = p^b + 1 \) and \( k > 3 \), we obtain a sharp linear bound of \( \tilde{G}_q(k) \leq 4k + 5 \) in contrast to the quadratic bound for \( \tilde{G}(k) \).

We now state the result for the case \( 3 \leq k < p \).

Theorem 1.4. Suppose \( 3 \leq k < p \). Then we have \( \tilde{G}_q(k) \leq 2k^2 - 2[\log k/(\log 2)] \). Furthermore, \( \tilde{G}_q(7) \leq 86 \) and \( \tilde{G}_q(k) \leq 2k^2 - 11 \) when \( k \geq 8 \).

We also study the slim exceptional sets associated to the asymptotic formula (1.3). These sets measure the frequency with which the expected formula (1.3) does not hold. In other words, we estimate the number of polynomials that in a certain sense do not satisfy the asymptotic formula. For \( \psi(z) \) a function of positive variable \( z \), we denote by \( \tilde{E}_{s,k}(N, \psi) \) the set of \( n \in I_N \cap \mathbb{Z}[t] \) for which

\[
(1.6) \quad \left| R_{s,k}(n) - \mathcal{E}_{s,k}(n) J_\infty(n) q^{(s-k)p} \right| > q^{(s-k)p} \psi(q^P)^{-1}.
\]

Note that \( \tilde{E}_{s,k}(N, \psi) \) is dependent on \( q \). We define \( \tilde{G}_q^+(k) \) to be the least positive integer \( s \) for which \( |\tilde{E}_{s,k}(N, \psi)| = o(q^N) \) for some function \( \psi(z) \) increasing to infinity with \( z \). We obtain the following estimates on \( \tilde{G}_q^+(k) \). We first present the case \( k > p \).

Theorem 1.5. Let \( k \geq 3 \) be an integer, where \( p \nmid k \). Suppose \( k > p \), then we have

\[
\tilde{G}_q^+(k) \leq \begin{cases} 
k \left( k - \left\lfloor \frac{k}{p} \right\rfloor \right) - 2 + \left\lfloor \frac{3|k/p| - 2}{k-2} \right\rfloor, & \text{if } p \nmid (k-1), \\
2k + 3, & \text{if } k = p^b + 1, \\
\left( 1 - \frac{1}{p} \right) k^2 - (p^b - p^{b-1} - 2)k - c'_k, & \text{if } k = mp^b + 1 \text{ and } m > 1,
\end{cases}
\]

where \( c'_k = \left( p^b - p^{b-1} - 1 - \frac{1}{p} \right) + \left\lfloor \frac{(m-1)(1-1/p)}{4} \right\rfloor. \)

We now state the result for the case \( 3 \leq k < p \).
Theorem 1.6. Suppose $3 \leq k < p$. Then we have $\tilde{G}_q^+(k) \leq k^2 - \lceil \log k / (\log 2) \rceil$. Furthermore, we have $\tilde{G}_q^+(7) \leq 43$, and $\tilde{G}_q^+(k) \leq k^2 - 5$ when $k \geq 8$.

The organization of the rest of the paper is as follows. In Section 2 we introduce some notation and basic notions required to carry out our discussion in the setting over $\mathbb{F}_q[t]$. In Section 3 we go through technical details to prove an upper bound for $\psi(\alpha, \theta)$, which is defined in (3.1). This estimate is one of the main ingredients to obtain our minor arc estimates, for the cases $p \nmid (k - 1)$ and $k = mp^b + 1$ with $m > 1$, in Section 4. We also obtain minor arc estimates for the case $k = p^b + 1$ in Section 4. We then prove a useful result related to Weyl differencing in Section 5. The content of Sections 6 and 7 are similar; we combine the material from previous sections to obtain a variant of minor arc estimates achieved in Section 4, from which our results follow.

We denote $\mathbf{x} = (x_1, ..., x_{2s})$, where $x_i \in \mathbb{F}_q[t]$ $(1 \leq i \leq 2s)$. We write $N_1 \leq \text{ord} \mathbf{x} \leq N_2$ to denote that $N_1 \leq \text{ord} x_i \leq N_2$ for $1 \leq i \leq 2s$, and given $n_0 \in \mathbb{F}_q[t]$, we write $(\mathbf{x} - n_0)$ to denote the $2s$-tuple $(x_1 - n_0, ..., x_{2s} - n_0)$. Confusion should not arise if the reader interprets analogous statements in a similar manner.

2. Preliminary

While the Hardy-Littlewood circle method for $\mathbb{F}_q[t]$ mirrors the classical version familiar from applications over $\mathbb{Z}$, the substantial differences in detail between these rings demand explanation. Our goal in the present section is to introduce notation and basic notions that are subsequently needed to initiate discussion of key components of this version of the circle method. The material here is taken from various sources including [2], [4], [5], and [7]. Associated with the polynomial ring $\mathbb{F}_q[t]$ defined over the field $\mathbb{F}_q$ is its field of fractions $K = \mathbb{F}_q(t)$. For $f/g \in K$, we define an absolute value $\langle \cdot \rangle : K \to \mathbb{R}$ by $\langle f/g \rangle = q^{\deg f - \deg g}$ (with the convention that $\deg 0 = -\infty$ and $\langle 0 \rangle = 0$). The completion of $K$ with respect to this absolute value is $K_\infty = \mathbb{F}_q((1/t))$, the field of formal Laurent series in $1/t$. In other words, every element $\alpha \in K_\infty$ can be written as $\alpha = \sum_{i=-\infty}^{n} a_i t^i$ for some $n \in \mathbb{Z}$, coefficients $a_i = a_i(\alpha) \in \mathbb{F}_q$ $(i \leq n)$ and $a_n \neq 0$. For each such $\alpha \in K_\infty$, we refer to $a_{-1}(\alpha)$ as the residue of $\alpha$, an element of $\mathbb{F}_q$ that we abbreviate to $\text{res} \alpha$. If $n < -1$, then we let $\text{res} \alpha = 0$. We also define the order of $\alpha$ to be $\text{ord} \alpha = n$. Thus if $f$ is a polynomial in $\mathbb{F}_q[t]$, then $\text{ord} f = \deg f$. Note that the order on $K_\infty$ satisfies the following property: if $\alpha, \beta \in K_\infty$ satisfies $\text{ord} \alpha > \text{ord} \beta$, then

$$\text{ord} (\alpha + \beta) = \text{ord} \alpha.$$  

The field $K_\infty$ is a locally compact field under the topology induced by the absolute value $\langle \cdot \rangle$. Let $T = \{ \alpha \in K_\infty : \text{ord} \alpha < 0 \}$. Every element $\alpha \in K_\infty$ can be written uniquely in the shape $\alpha = [\alpha] + \|\alpha\|$, where the integral part of $\alpha$ is $[\alpha] \in \mathbb{F}_q[t]$ and the fractional part of $\alpha$ is $\|\alpha\| \in T$. Note that $[\cdot]$ and $\|\cdot\|$ are $\mathbb{F}_q$-linear functions on $K_\infty$ [7, pp.12]. Since $T$ is a compact additive subgroup of $K_\infty$, it possesses a unique Haar measure $d\alpha$. We normalise it, so that $\int_T 1 \, d\alpha = 1$. The Haar measure on $T$ extends easily to a product measure on the $D$-fold Cartesian product $T^D$, for any positive integer $D$. For convenience, we will use the notation

$$\int_T d\alpha := \int_T \ldots \int_T d\alpha_1 \ldots d\alpha_D,$$

where the positive integer $D$ should be clear from the context.
We are now equipped to define an analogue of the exponential function. Recall \( ch(\mathbb{F}_q) = p \). There is a non-trivial additive character \( e_q : \mathbb{F}_q \to \mathbb{C}^\times \) defined for each \( a \in \mathbb{F}_q \) by taking 
\[ e_q(a) = \exp(2\pi i \frac{\mathrm{tr}(a)}{p}), \]
where \( \mathrm{tr} : \mathbb{F}_q \to \mathbb{F}_p \) denotes the familiar trace map. This character induces a map \( e : \mathbb{K}_\infty \to \mathbb{C}^\times \) by defining, for each element \( \alpha \in \mathbb{K}_\infty \), the value of \( e(\alpha) \) to be \( e_q(a^{-1}(\alpha)) \). The orthogonality relation underlying the Fourier analysis of \( \mathbb{F}_q[t] \) takes the following shape.

**Lemma 1.** Let \( h \) be a polynomial in \( \mathbb{F}_q[t] \). Then we have
\[
\int_{\mathbb{T}} e(h\alpha) \, d\alpha = \begin{cases} 
0, & \text{if } h \in \mathbb{F}_q[t]\backslash\{0\}, \\
1, & \text{if } h = 0.
\end{cases}
\]

**Proof.** This is [7, Lemma 1 (f)]. □

The following estimate on exponential sums will be useful during the analysis in subsequent sections.

**Lemma 2.** Let \( Y \in \mathbb{N} \). Then we have
\[
\sum_{\ord x \leq Y} e(\beta x) = \begin{cases} 
q^{Y+1}, & \text{if } \ord \|\beta\| < -Y - 1, \\
0, & \text{if } \ord \|\beta\| \geq -Y - 1.
\end{cases}
\]

**Proof.** This is [7, Lemma 7]. □

For each \( k \geq 2 \), we define the following exponential sum
\[
g(\alpha) = \sum_{x \in I_X} e(\alpha x^k).
\]
Then, it is a consequence of the orthogonality relation (2.2) that
\[
R_{s,k}(n) = \int_{\mathbb{T}} g(\alpha)^s e(-n\alpha) \, d\alpha.
\]

We analyse the integral (2.5) via the Hardy-Littlewood circle method, and to this end we define sets of **major** and **minor arcs** corresponding to well and poorly approximable elements of \( \mathbb{T} \). Let \( R_k = (k - 1)X \). Given polynomials \( a \) and \( g \) with \( (a, g) = 1 \) and \( g \) monic, we define the **Farey arcs** \( \mathcal{M}_k(g, a) \) about \( a/g \) (associated to \( k \)) by
\[
\mathcal{M}_k(g, a) = \{ \alpha \in \mathbb{T} : \ord (\alpha - a/g) < -R_k - \ord g \}.
\]
The set of major arcs \( \mathcal{M}_k \) is defined to be the union of the sets \( \mathcal{M}_k(g, a) \) with
\[
a, g \in \mathbb{F}_q[t], \quad g \text{ monic}, \quad 0 \leq \ord a < \ord g \leq X, \quad \text{and } (a, g) = 1.
\]
The set of minor arcs is defined to be \( \mathcal{m}_k = \mathbb{T} \backslash \mathcal{M}_k \). It follows from Dirichlet’s approximation theorem in the setting of \( \mathbb{F}_q[t] \) [7, Lemma 3] that \( \mathcal{m}_k \) is the union of the sets \( \mathcal{M}_k(g, a) \) with
\[
a, g \in \mathbb{F}_q[t], \quad g \text{ monic}, \quad 0 \leq \ord a < \ord g, \quad X < \ord g \leq R_k, \quad \text{and } (a, g) = 1.
\]
Notice \( \mathcal{m}_2 = \emptyset \) and for this reason, we assume \( k \geq 3 \) for results involving minor arcs. We will suppress the subscript \( k \) whenever there is no ambiguity with the choice of \( k \) being used. We can then rewrite (2.5) as
\[
R_{s,k}(n) = \int_{\mathcal{m}} g(\alpha)^s e(-n\alpha) \, d\alpha + \int_{\mathcal{m}} g(\alpha)^s e(-n\alpha) \, d\alpha,
\]
and study the contribution from the major arcs and the minor arcs separately.
We have the following estimate on the major arcs, which is slightly different from what is established in [7]. The difference comes from our choice of $P(n)$ following [5], instead of the approach taken in [7], and this choice allows us to have a cleaner statement of the result. Applying Theorem 2.1 below for the estimate of the major arcs results in the statement of Theorem 1.1 in contrast to that of [7, Theorem 30].

**Theorem 2.1.** Suppose $p \nmid k$ and $s \geq 2k + 1$. Then there exists $\epsilon > 0$ such that given any $n \in \mathbb{J}_q^k$, the following asymptotic formula holds

\[(2.10) \quad \int_{\mathfrak{M}} g(\alpha)^s \, d\alpha = \mathcal{G}_{s,k}(n) J_\infty(n) q^{(s-k)P} + O(q^{(s-k-\epsilon)P}),\]

where

\[1 \ll \mathcal{G}_{s,k}(n) J_\infty(n) \ll 1.\]

Note that the implicit constants in the theorem may depend on $s, q, k$, and $\epsilon$, where the constant in (2.10) may also depend on $\epsilon$, but they are independent of $n$ and $P$.

**Proof.** Let $0 < \epsilon < 1$. Similarly as explained in the proof of [5, Lemma 5.3], by applying Lemma 17 of [7] with $m = X$ and $m' = R_k + \text{ord} \, g$, where $\text{ord} \, g \leq \epsilon X$, we obtain

\[(2.11) \quad \int_{\text{ord} \beta < R_k - \text{ord} \, g} g(\beta)^s e(-n\beta) \, d\beta = J_\infty(n) q^{(s-k)P} + O(1),\]

where the implicit constant may depend on $s, k, q$, and $\epsilon$. We note that when $P$ is sufficiently large in terms of $k$ and $\epsilon$, it is only the cases (a) and (b) of [7, Lemma 17] that are relevant, and in fact we obtain (2.11) without the $O(1)$ term. The $O(1)$ term in (2.11) comes from the small values of $P$ where this does not apply. It is also explained in the proof of [5, Lemma 5.3] that for $s \geq k + 1$, we have $1 \leq J_\infty(n) \ll 1$. By [5, Lemma 5.2], we know that if $n \in \mathbb{J}_q^k$ and $s \geq 2k + 1$, then $1 \ll \mathcal{G}_{s,k}(n) \ll 1$.

The equation (2.10) is a consequence of (2.11) and [5, Lemma 5.2], and it is essentially contained in the proof of [7, Theorem 30], where we replace the use of [7, Theorem 18] with (2.11). We remark that the condition $s \geq 3k + 1$ is imposed in [7, Lemma 23], which is also used in the proof of [7, Theorem 30]. However, as stated in [5, pp.19] this is a result of an oversight and in fact we can relax the condition to $s \geq 2k + 1$ in [7, Lemma 23]. It can easily be verified that the arguments to prove (2.10) within [7, Theorem 30] also remains valid when $s \geq 2k + 1$. \[\square\]

When $k = 2$, we know that $m_2 = \varnothing$. Hence, it follows that

\[(2.12) \quad R_{s,k}(n) = \int_{\mathfrak{M}} g(\alpha)^s \, d\alpha.\]

Therefore, as an immediate consequence of Theorem 2.1 we obtain $\tilde{G}_q(2) \leq 5$.

It was proved in [7, Lemma 28] that $\mathbb{F}_q[t] = \mathbb{J}_q^k[t]$ when $k < p$, which explains the use of $\mathbb{F}_q[t]$ in the statement of Theorem 1.1 instead of $\mathbb{J}_q^k[t]$ as above in Theorem 2.1.

Let $\mathcal{R}$ be a finite subset of $\mathbb{N}$ satisfying the following condition in [2, pp.846] with $d = 1$:

\[(2.13) \quad \text{Condition*} : \text{Given } l \in \mathbb{N}, \text{if there exists } j \in \mathcal{R} \text{ such that } p \nmid \left(\frac{j}{l}\right), \text{ then } l \in \mathcal{R}.\]
Let $J_s(\mathcal{R}; X)$ denote the number of solutions of the system
\begin{equation}
\sum_{i=1}^{s} (u_i^j - v_i^j) = \left( \sum_{i=1}^{s} (u_i^j - v_i^j) \right)^{p^v}.
\end{equation}
Thus, the equations in (2.14) are not always independent. The absence of independence suggests that Vinogradov-type estimates for integers cannot be adapted directly into a function field setting. To regain independence, we instead consider
\begin{equation}
\mathcal{R}' = \{ j \in \mathbb{N} : p \nmid j \text{ and } p^v j \in \mathcal{R} \text{ for some } v \in \mathbb{N} \cup \{0\} \}.
\end{equation}
Then we see that $J_s(\mathcal{R}; X)$ also counts the number of solutions of the system
\begin{equation}
\sum_{i=1}^{s} (u_i^j - v_i^j) = \left( \sum_{i=1}^{s} (u_i^j - v_i^j) \right)^{p^v}.
\end{equation}
with $u_i, v_i \in I_X$ (1 ≤ i ≤ s), or in other words $J_s(\mathcal{R}; X) = J_s(\mathcal{R}'; X)$. We note here that although the equations in (2.16) are independent, the set $\mathcal{R}'$ is not necessarily contained in $\mathcal{R}$. The following theorem was proved in [3] and in [2] Theorem 1.1 with $d = 1$.

**Theorem 2.2** (Theorem 1.1, [2]). Suppose $\mathcal{R}$ satisfies Condition* given in (2.13). Let $r = \text{card } \mathcal{R}'$, $\phi = \max_{j \in \mathcal{R}'} j$, and $\kappa = \sum_{j \in \mathcal{R}'} j$. Suppose $\phi \geq 2$ and $s \geq r\phi + r$. Then for each $\epsilon > 0$, there exists a positive constant $C = C(s; r, \phi, \kappa; q; \epsilon)$ such that
\begin{equation}
J_s(\mathcal{R}; X) \leq C \left( q^X \right)^{2s - \kappa + \epsilon}.
\end{equation}

The following is a useful criterion, which we utilize.

**Lemma 3.** Let $p$ be any prime and $k = a_h p^{b_h} + \ldots + a_1 p + a_0$ with $0 \leq a_i < p$ (0 ≤ i ≤ h) and $a_h \neq 0$. The binomial coefficient $\binom{k}{n}$ is coprime to $p$ if and only if $n = b_h p^{b_h} + \ldots + b_1 p + b_0$, where $0 \leq b_i \leq a_i$ (0 ≤ i ≤ h).

**Proof.** It follows by Lucas’ Criterion [5, pp.33] or apply [12] Lemma A.1 with $d = 1$.

As a consequence of Lemma 3, we have the following lemma.

**Lemma 4.** Let $p$ be any prime. Suppose $k = mp^b + 1$ with $m, b \in \mathbb{N}$ and $p \nmid m$. Then, $(k - p^b)$ is the largest number less than $(k - 1)$ such that $\binom{k}{k-1} \neq 0 \mod p$.

**Proof.** Let $m = c_a p^a + c_{a-1} p^{a-1} + \ldots + c_1 p + c_0$ with $0 \leq c_i < p$ and $0 < c_0$. Thus, we have $k = c_a p^{a+b} + c_{a-1} p^{a-1+b} + \ldots + c_1 p^{b+1} + c_0 p^b + 1$. For $1 \leq j \leq p^b$, write $k - j = c_a p^{a+b} + c_{a-1} p^{a-1+b} + \ldots + c_1 p^{b+1} + d_0 b^0 + d_{b-1} p^{b-1} + \ldots + d_1 p + d_0$ with $0 \leq d_i < p$. Then, by Lemma 3 $\binom{k}{k-j} \neq 0 \mod p$ if and only if $d_0 \leq c_0$, $d_i = 0 (1 \leq i < b)$ and $d_0 \leq 1$. Therefore, it is not too difficult to verify that $\binom{k}{k-j} \neq 0 \mod p$ only when $j = 1$ and $p^b$ in the range $1 \leq j \leq p^b$.

For a prime $p = \text{ch}(\mathbb{F}_q)$ and $k \in \mathbb{N}$ with $p \nmid k$, we define $j_0(k, q) = j_0$ to be
\begin{equation}
(2.17) \quad j_0 := \max_{0 < j < k} \left\{ j : p \nmid j \text{ and } \binom{k}{j} \neq 0 \mod p \right\}.
\end{equation}
If \( p \nmid (k - 1) \), then \( j_0 = k - 1 \). On the other hand, if \( k = mp^b + 1 \) for some \( m, b \in \mathbb{N} \) and \( p \nmid m \), then \( j_0 = k - p^b \) by Lemma 4. We record the values of \( j_0 \) here for reference,

\[
(2.18) \quad j_0 = \begin{cases} 
    k - 1, & \text{if } p \nmid (k - 1), \\
    (m - 1)p^b + 1, & \text{if } k = mp^b + 1.
\end{cases}
\]

With application of Theorem 2.2 in mind, we define the following two sets

\[
(2.19) \quad \mathcal{R} = \{1, 2, \ldots, j_0, k\} \cup \{k - 1\}
\]

and

\[
(2.20) \quad \mathcal{R}' = \{j \in \mathbb{N}: p \nmid j \text{ and } p^j \in \mathcal{R} \text{ for some } v \in \mathbb{N} \cup \{0\}\} 
= \{j : j \in \mathcal{R} \text{ and } p \nmid j\}.
\]

The first equality is the definition of \( \mathcal{R}' \), which comes from (2.15), but the second equality requires a slight justification. If \( p \nmid (k - 1) \), then \( \mathcal{R} = \{1, 2, \ldots, k\} \) and the second equality of (2.20) is immediate. If \( k = mp^b + 1 \), then \( k \in \mathcal{R}' \). We also have \( k - 1 = mp^b \notin \mathcal{R}' \) and \( m \in \mathcal{R}' \). However, since \( j_0 = (m - 1)p^b + 1 > m \) and \( p \nmid m \), it follows that \( \mathcal{R}' = \{j : 1 \leq j \leq j_0 \text{ and } p \nmid j\} \cup \{k\} \) from which we obtain the second equality of (2.20).

We let \( \text{card } \mathcal{R}' = r \) and let \( \mathcal{R}' = \{t_1, \ldots, t_r\} \), where \( t_1 < \ldots < t_r \). Clearly, we have \( t_r = k \) and it follows by our definition of \( j_0 \) and \( \mathcal{R} \) that \( t_{r-1} = j_0 \). We can verify by simple calculation that

\[
(2.21) \quad r = \begin{cases} 
    \lfloor k/p \rfloor, & \text{if } p \nmid (k - 1), \\
    (1 - 1/p)(k - p^b) + (1 + 1/p), & \text{if } k = mp^b + 1.
\end{cases}
\]

In particular, if \( k = p^b + 1 \), then \( r = 2 \). For the remainder of the paper, whenever we refer to \( \mathcal{R} \), \( \mathcal{R}' \) and \( r \), we mean (2.19), (2.20), and (2.21), respectively.

**Lemma 5.** \( \mathcal{R} \) satisfies Condition* given in (2.13).

**Proof.** If \( p \nmid (k - 1) \), then \( \mathcal{R} = \{1, 2, \ldots, k\} \) and it satisfies Condition*. This is easy to see, because suppose for some \( l \in \mathbb{N} \), there exists \( j \in \mathcal{R} \) such that \( p \nmid \binom{l}{j} \). Then we have \( 1 \leq l \leq j \leq k \), and hence \( l \in \mathcal{R} \). On the other hand, if \( k = mp^b + 1 \), then we have \( \mathcal{R} = \{1, 2, \ldots, j_0, k - 1, k\} \). Suppose we are given some \( l \in \mathbb{N} \). It is clear that if \( l > k \), then there does not exist \( j \in \mathcal{R} \) such that \( p \nmid \binom{l}{j} \), because \( \binom{l}{j} = 0 \). Thus, it suffices to show that for \( j_0 < l < (k - 1) \), \( \binom{l}{j_0} \equiv 0 \pmod{p} \) for all \( j \in \mathcal{R} \). Clearly, \( \binom{l}{j} \equiv 0 \pmod{p} \) for \( j \leq j_0 \). Lemma 4 gives us that \( \binom{k - 1}{j} \equiv 0 \pmod{p} \) for \( j_0 < l < (k - 1) \). Therefore, we only need to verify \( \binom{k - 1}{l} \equiv 0 \pmod{p} \) for \( j_0 = (k - p^b) < l < (k - 1) \). Every \( l \) in this range can be written as \( l = (m - 1)p^b + c_{b-1}p^{b-1} + \ldots + c_1p + c_0 \), where \( 0 \leq c_i < p \). Since \( (k - 1) = mp^b \), by Lemma 3 we have \( \binom{k - 1}{l} \not\equiv 0 \pmod{p} \) if and only if \( c_i = 0 \) for \( 0 \leq i < b \), or in other words \( l = (m - 1)p^b = k - p^b - 1 \). Because \( l = k - p^b - 1 \) is not in the range of \( l \) we are considering, it follows that \( \mathcal{R} \) satisfies Condition*. \( \square \)

3. **Technical Lemmas**

We will be applying the following large sieve inequality in this section. Given a set \( \Gamma \subseteq K_{\infty} \), if for any distinct elements \( \gamma_1, \gamma_2 \in \Gamma \) we have \( \text{ord}(\gamma_1 - \gamma_2) > \delta \), then we say the points \( \{\gamma : \gamma \in \Gamma\} \) are spaced at least \( q^\delta \) apart in \( \mathbb{T} \).
Theorem 3.1 (Theorem 2.4, [1]). Given \( A, Z \in \mathbb{Z}^+ \), let \( \Gamma \subseteq \mathbb{K}_\infty \) be a set whose elements are spaced at least \( q^{-A} \) apart in \( \mathbb{T} \). Let \((c_x)_{x \in \mathbb{F}_q[t]}\) be a sequence of complex numbers. For \( \alpha \in \mathbb{K}_\infty \), define
\[
S(\alpha) = \sum_{\text{ord } x \leq Z} c_x e(x\alpha).
\]
Then we have
\[
\sum_{\gamma \in \Gamma} |S(\gamma)|^2 < \max\{q^{Z+1}, q^{A-1}\} \sum_{\text{ord } x \leq Z} |c_x|^2.
\]

Recall \( I_X := \{x \in \mathbb{F}_q[t]: \text{ord } x < X\} \). Let \( k \geq 3, \theta \in \mathfrak{m}_k, 0 \neq c \in \mathbb{F}_q, \alpha \in \mathbb{T}, \) and \( j_0 \) be as defined in Section 2. In this section, we find an upper bound for the following exponential sum,
\[
\psi(\theta, \alpha) = q^{-X} \sum_{y \in I_X} \sum_{\text{ord } h \leq j_0(X-1)} e(-chy^{k-j_0}\theta - \alpha h).
\]
The estimates obtained for \( \psi(\theta, \alpha) \) is one of our main ingredients for computing the minor arc estimates in Section 4. To achieve this goal, the precise value of \( j_0 \) plays an important role. Hence, we consider the following two cases separately: \( k \) and \( k \) do not play an important role. Here, because we apply a different method to bound the minor arcs in this case.

First, we make several observations, which we use throughout this section. Let \( \theta = a/g + \beta \), where \((a, g) = 1\). Let \( x, y \in I_X \) and \( x \neq y \). Then, since \( \| \cdot \| \) is \( \mathbb{F}_q \)-linear, we have
\[
\text{ord}(\|cx^{k-j_0}\theta + \alpha\| - \|cy^{k-j_0}\theta + \alpha\|) = \text{ord}(\|(x^{k-j_0} - y^{k-j_0})\theta\|) = \text{ord}(\|(x^{k-j_0} - y^{k-j_0})a/g\| + \|(x^{k-j_0} - y^{k-j_0})\beta\|).
\]
Since \( \mathbb{F}_q[t] \) is a unique factorization domain, we have \((x^{k-j_0} - y^{k-j_0})a \neq 0 \) as long as \( a \neq 0 \). Note that it is possible to get \( a = 0 \), when \( \text{ord } g = 0 \).

Suppose \((x^{k-j_0} - y^{k-j_0})a/g \notin \mathbb{F}_q[t] \). Then, we have \( \|(x^{k-j_0} - y^{k-j_0})a/g\| = 0 \) and
\[
\text{ord}(\|cx^{k-j_0}\theta + \alpha\| - \|cy^{k-j_0}\theta + \alpha\|) = \text{ord}(\|(x^{k-j_0} - y^{k-j_0})\beta\|).
\]
On the other hand, if \((x^{k-j_0} - y^{k-j_0})a/g \notin \mathbb{F}_q[t] \), write
\[
\frac{a}{g}(x^{k-j_0} - y^{k-j_0}) = s_0 + a_{-j} t^{-j} + a_{-j-1} t^{-j-1} + \ldots
\]
with \( s_0 \in \mathbb{F}_q[t], a_i \in \mathbb{F}_q \) for \( i \leq -j \leq -1 \) and \( a_{-j} \neq 0 \). Here we know such \( a_{-j} \neq 0 \) exists, because \((x^{k-j_0} - y^{k-j_0})a/g \notin \mathbb{F}_q[t] \). Then it follows that
\[
a(x^{k-j_0} - y^{k-j_0}) - gs_0 = g(a_{-j} t^{-j} + a_{-j-1} t^{-j-1} + \ldots).
\]
Since the left hand side is a polynomial, we have \( -j + \text{ord } g \geq 0 \). Consequently, we obtain
\[
\text{ord}(\|(x^{k-j_0} - y^{k-j_0})a/g\|) \geq -\text{ord } g.
\]
3.1. Case \( p \nmid (k - 1) \). Here we have \( j_0 = k - 1 \), or equivalently \( k - j_0 = 1 \). In this situation, we obtain an upper bound for \( \psi(\theta, \alpha) \) in a way analogous to the case for integers in [10]. We have the following lemma.

**Lemma 6.** Suppose \( k \geq 3 \), \( p \nmid k \), and \( p \nmid (k - 1) \). Let \( \theta \in \mathfrak{m}_k \) and \( \alpha \in \mathbb{T} \). Then we have 

\[
\psi(\theta, \alpha) \leq q^{(j_0 - 1)X}.
\]

Proof. Let \( \theta = a/g + \beta \in \mathfrak{M}_k(g,a) \subseteq \mathfrak{m}_k \). Let \( x, y \in I_X \) and \( x \neq y \). Then, we know 

\[
(x^{k-j_0} - y^{k-j_0})a/g \notin \mathbb{F}_q[t],
\]

because \( k - j_0 = 1 \) and \( g > X \). Consequently, we have (3.4).

Recall \( R_k = (k - 1)X \). For simplicity we let \( R = R_k \). Since \( R > (k - j_0)(X - 1) \) and \( \text{ord} \beta < (-R - \text{ord } g) \), we have 

\[
\text{ord} (x^{k-j_0} - y^{k-j_0}) \beta < (k - j_0)(X - 1) - R - \text{ord } g < -\text{ord } g \leq 0.
\]

Thus, we obtain from (2.1) and (3.2)

\[
\text{ord} \left( ||cx^{k-j_0} + \alpha|| - ||cy^{k-j_0} + \alpha|| \right) = \text{ord} \left( (x^{k-j_0} - y^{k-j_0})a/g \right)
\geq -\text{ord } g
\geq -R.
\]

Suppose there exists \( y \in I_X \) such that \( \text{ord} ||cy^{k-j_0} + \alpha|| < (-j_0(X - 1) - 1) \), or equivalently, 

\[
\text{ord} ||cy + \alpha|| < -(k - 1)(X - 1) - 1.
\]

This means the first \(((k - 1)(X - 1) + 1)\) coefficients of \( ||cy + \alpha|| \) are 0. Hence, it takes the form 

\[
||cy + \alpha|| = 0 t^{-1} + 0 t^{-2} + \ldots + 0 t^{-(k-1)(X-1)-1} + a_{-(k-1)(X-1)-1}t^{-(k-1)(X-1)-2} + \ldots + a_{-R}t^{-R} + \ldots.
\]

Note that there are only \( q^{k-2} \) possibilities for the \((k - 2)\)-tuple \((a_{-(k-1)(X-1)-2}, \ldots, a_{-R})\). Thus, if there are more than \( q^{k-2} \) such polynomials \( y \in I_X \) satisfying (3.6), then by the pigeon hole principle there exists a pair \( x \) and \( y \) in \( I_X \) for which the first \( R \) coefficients of \( ||cx + \alpha|| \) and \( ||cy + \alpha|| \) agree. However, this contradicts (3.5). Therefore, it follows by (3.1) and Lemma 2 that 

\[
\psi(\theta, \alpha) \leq q^{-X+k-2+(k-1)(X-1)+1} = q^{(k-2)X}.
\]

\[\square\]

3.2. Case \( k = mp^b + 1 \) with \( m > 1 \). Here we have \( j_0 = k - p^b > \alpha = k - j_0 \). When \( p \nmid (k - 1) \), we had that the difference between \( j_0(X - 1) + 1 \) and \( R_k = (k - 1)X \) was small enough compared to \( X \) - in fact it was constant with respect to \( X \) - which was the reason our application of the pigeon hole principle was effective in Lemma 6. However, when \( k = mp^b + 1 \) this is no longer the case as \( R_k - j_0(X - 1) - 1 = (p^b - 1)X + (k - p^b - 1) \).

It follows from the definition of the major arcs that \( \mathfrak{M}_k \subseteq \mathfrak{M}_{k-j_0+1} \), hence \( \mathfrak{m}_{k-j_0+1} \subseteq \mathfrak{m}_k \). Therefore, given \( \theta \in \mathfrak{m}_k \), we have either \( \theta \in \mathfrak{m}_{k-j_0+1} \) or \( \theta \in \mathfrak{M}_{k-j_0+1} \). We consider these two cases separately in Lemmas 7 and 8. The argument in Lemma 7 is similar to that of Lemma 6. However, in Lemma 8 we use a different approach, which relies on the large sieve inequality given in Theorem 3.1 instead.

**Lemma 7.** Let \( k = mp^b + 1 \) with \( m > 1 \) and \( \theta \in \mathfrak{m}_k \). Suppose \( \theta \in \mathfrak{m}_{k-j_0+1} \).Then we have 

\[
\psi(\theta, \alpha) \ll q^{(j_0-1/p^b)X},
\]

where the implicit constant depends only on \( q \) and \( k \).
Proof. Let \( \theta = a/g + \beta \in M_{k-j_0+1}(g, a) \subseteq M_{k-j_0+1} \), and we know \( R' \geq \text{ord} \, g \geq X \), where \( R' = R_{k-j_0+1} = (k-j_0)X \). Given \( y \in I_X \), it takes the form
\[
(3.7) \quad y = c_{X-1}t^{X-1} + \cdots + c_{\lfloor X/p^b \rfloor}t^{\lfloor X/p^b \rfloor} + \cdots + c_0.
\]
Let \( L = (X - \lfloor X/p^b \rfloor) \). Order the \( L \)-tuples of elements of \( \mathbb{F}_q \) in any way, for example, we may take one bijection between \( \mathbb{F}_q \) and \( \{1, \ldots, q\} \), and use the lexicographic ordering on \( (\mathbb{F}_q)^L \).

We can then split \( I_X \) into \( q^L \) subsets \( T_1, T_2, \ldots, T_{q^L} \), where
\[
T_i = \{ y \in I_X : \text{given} \ y \ \text{in the form} \ (3.7), \ \text{the coefficients} \ (c_{X-1}, \ldots, c_{\lfloor X/p^b \rfloor}) \ \text{is exactly the} \ l\text{-th} \ \text{\( L \)-tuple} \}.
\]

Then, we have for some \( T' = T_i \)
\[
(3.8) \quad \psi(\theta, \alpha) \ll q^{-X+X-X/p^b} \sum_{y \in T'} \sum_{\text{ord} \ h \leq j_0(X-1)} e(-c_y y^{k-j_0} - \alpha h).
\]

Given any distinct \( x, y \in T' \), we have
\[
\text{ord} \, (x^{k-j_0} - y^{k-j_0}) = \text{ord} \, (x - y)^{p^b} \leq X,
\]
and hence, \( (x^{k-j_0} - y^{k-j_0})a/g \notin \mathbb{F}_q[t] \). Thus, by (3.4) we have \( \| (x^{k-j_0} - y^{k-j_0})a/g \| \geq -\text{ord} \, g \). Since \( \text{ord} \, \beta < -R' \) and \( R' = (k-j_0)X > X \), we have \( \text{ord} \, (x^{k-j_0} - y^{k-j_0})\beta < X - R' < \text{ord} \, g < 0 \). Therefore, by (2.1) and (3.2), we obtain
\[
(3.9) \quad \text{ord} \, (\| cx^{k-j_0} + \alpha \| - \| cy^{k-j_0} + \alpha \| ) \geq -\text{ord} \, g \geq -R'.
\]

Suppose there exists \( y \in T' \) such that \( \text{ord} \, \| cy^{k-j_0} + \alpha \| < -j_0(X-1) - 1 \). This means the first \( j_0(X-1) + 1 \) coefficients of \( \| cy^{k-j_0} + \alpha \| \) must be 0, or in other words it takes the form
\[
\| cy^{k-j_0} + \alpha \| = 0 \ t^{-1} + 0 \ t^{-2} + \cdots + 0 \ t^{-j_0(X-1)-1} + a_{-j_0(X-1)-2}t^{-j_0(X-1)-2} + \cdots
\]
If there is another distinct \( x \in T' \), which satisfies the same condition, then the first \( j_0(X-1)+1 \) coefficients of \( \| cx^{k-j_0} + \alpha \| \) agree with that of \( \| cy^{k-j_0} + \alpha \| \). However, this contradicts (3.9) as \( R' = (k-j_0)X < j_0(X-1)+1 \) for \( X \) sufficiently large. Hence, there is at most one such \( y \). Therefore, it follows by (3.8) and Lemma 2 that
\[
\psi(\theta, \alpha) \ll q^{-X+X-X/p^b+j_0(X-1)+1} \ll q^{(j_0-1/p^b)X}.
\]

Lemma 8. Let \( k = mp^b + 1 \) with \( m > 1 \) and \( \theta \in M_k \). Suppose \( \theta \in M_{k-j_0+1} \). Then we have
\[
\psi(\theta, \alpha) \ll q^{(j_0-1/(4p^b))X},
\]
where the implicit constant depends only on \( q \).

Proof. Let \( \theta = a/g + \beta \in M_{k-j_0+1}(g, a) \subseteq M_{k-j_0+1} \). Then, we have \( \text{ord} \, g \leq X \) and
\[
(3.10) \quad -R_k - \text{ord} \, g \leq \text{ord} \, \beta < -R_{k-j_0+1} - \text{ord} \, g,
\]
where \( R_k = (k-1)X \) and \( R_{k-j_0+1} = (k-j_0)X \). For simplicity, we denote \( R = R_k \) and \( R' = R_{k-j_0+1} \). We have the above lower bound, for otherwise it would mean \( \theta \in M_k \).

By the Cauchy-Schwartz inequality, we obtain
\[
(3.11) \quad \psi(\theta, \alpha) \ll q^{-X} q^{X/2} S^{1/2},
\]
where

\[ S = \sum_{y \in I_X} \sum_{\text{ord } h \leq j_0(X-1)} e(-chy^{k-j_0}\theta - \alpha h)^2. \]

Let \( \delta' > 0 \) be sufficiently small, and in particular we make sure \( \delta' \leq 1 \). We consider two cases: \( \text{ord } g > \delta'X \) and \( \text{ord } g \leq \delta'X \).

Case 1: Suppose \( \text{ord } g > \delta'X \). Given \( y \in I_X \), it takes the form

\[ y = c_{X-1}t^{X-1} + \ldots + c_{[\delta'X/p^b]}t^{[\delta'X/p^b]} + \ldots + c_0. \]

Let \( L = X - [\delta'X/p^b] \). Order the \( L \)-tuples of elements of \( \mathbb{F}_q \) in any way. We can then split \( I_X \) into \( q^L \) subsets, \( T_1, T_2, \ldots, T_{q^L} \), where

\[ T_i = \{ y \in I_X : \text{given } y \text{ in the form } (3.12), \text{the coefficients } \( c_{X-1}, \ldots, c_{[\delta'X/p^b]} \) \]

is exactly the \( l \)-th \( L \)-tuple).

Then we have for some \( T' = T_i \)

\[ (3.13) \quad S \ll q^{X-\delta'X/p^b} \sum_{y \in T'} \sum_{\text{ord } h \leq j_0(X-1)} e(-chy^{k-j_0}\theta - \alpha h)^2. \]

Recall \( k - j_0 = p^b \). Given any \( x, y \in T' \), we have

\[ \text{ord } (x^{k-j_0} - y^{k-j_0}) = \text{ord } (x - y)^{p^b} \leq \delta'X, \]

and hence, \( (x^{k-j_0} - y^{k-j_0})a/g \notin \mathbb{F}_q[t] \). Thus, we have \( \| (x^{k-j_0} - y^{k-j_0})a/g \| \geq -\text{ord } g \)

by (3.4). Since \( \text{ord } \beta < (-R' - \text{ord } g) \) and \( R' = X > \delta'X \), we have

\[ \text{ord } (x^{k-j_0} - y^{k-j_0})\beta < \delta'X - R' - \text{ord } g < -\text{ord } g \leq 0. \]

Therefore, by (2.1) and (3.2), we obtain

\[ (3.14) \quad \text{ord } (\| (cx^{k-j_0} + \alpha) \| - \| (cy^{k-j_0} + \alpha) \|) \geq -\text{ord } g \geq -X. \]

Since \( \max \{ X, j_0(X - 1) + 1 \} \leq j_0X \), we have by Theorem 3.1

\[ (3.15) \quad S \ll q^{X-\delta'X/p^b} \sum_{y \in T'} \sum_{\text{ord } h \leq j_0(X-1)} e(-chy^{k-j_0}\theta - \alpha h)^2 \ll q^{X-\delta'X/p^b} q^{2j_0X}. \]

Case 2: Suppose \( \text{ord } g \leq \delta'X \). Let \( \epsilon > 0 \) be sufficiently small. We order the polynomials of degree less than \( L' = \lceil (1-\epsilon)X \rceil \) in any way, and call them \( p_1, p_2, \ldots, p_{q^{L'}} \). We then split \( I_X \) into \( q^{L'} \) subsets, \( T_1, T_2, \ldots, T_{q^{L'}} \), where given any \( x \in T_l \), \( 1 \leq l \leq q^{L'} \), the coefficients of \( x \) for powers less than \( L' \) agree with that of \( p_l \). Thus, we have for some \( T' = T_l \)

\[ (3.16) \quad S \ll q^{(1-\epsilon)X} \sum_{y \in T'} \sum_{\text{ord } h \leq j_0(X-1)} e(-chy^{k-j_0}\theta - \alpha h)^2. \]

Given any \( x, y \in T' \) with \( x^{k-j_0} \neq y^{k-j_0} \) (mod \( g \)), we have \( (x^{k-j_0} - y^{k-j_0})a/g \notin \mathbb{F}_q[t] \). Thus, by (3.4) we have \( \| (x^{k-j_0} - y^{k-j_0})a/g \| \geq -\text{ord } g \). Since \( \text{ord } \beta < -R' - \text{ord } g \) and \( R' = (k-j_0)X > (k-j_0)(X - 1) \), we have

\[ (3.17) \quad \text{ord } (x^{k-j_0} - y^{k-j_0})\beta < (k-j_0)(X - 1) - R' - \text{ord } g < -\text{ord } g \leq 0. \]

Therefore, by (2.1) and (3.2), we obtain

\[ (3.18) \quad \text{ord } (\| (cx^{k-j_0} + \alpha) \| - \| (cy^{k-j_0} + \alpha) \|) \geq -\text{ord } g \geq -\delta'X. \]
On the other hand, suppose we have distinct \( x, y \in T' \), where \( x^{k-j_0} \equiv y^{k-j_0} \pmod g \). Then we have \( (x^{k-j_0} - y^{k-j_0})a/g \in \mathbb{F}_q[t] \) from which (3.3) follows. Also, because \( x, y \in T' \) and \( k - j_0 = p^b \), we obtain

\[
\text{ord} (x^{k-j_0} - y^{k-j_0}) = \text{ord} (x - y)^{p^b} \geq p^b L'.
\]

Therefore, it follows by (3.3), (3.10) and (3.17),

\[
\text{ord} \left( \| (cx^{k-j_0} + \alpha) \| - \| (cy^{k-j_0} + \alpha) \| \right) = \text{ord} \left( x^{k-j_0} - y^{k-j_0} \right) \beta \\
\geq p^b L' - R - \text{ord} g \\
\geq (k - j_0)(1 - \epsilon)X - (k - 1)X - \delta'X \\
= -j_0X + (1 - (k - j_0)\epsilon - \delta')X \\
\geq -j_0X.
\]

Since \( \max\{\delta'X, j_0X, j_0(X - 1) + 1\} \leq j_0X \), we have by Theorem 3.1

\[
S \ll q^{(1-\epsilon)X} \sum_{y \in T'} \left| \sum_{\text{ord} h \leq j_0(X-1)} e(-ch^{k-j_0} \theta - \alpha h) \right|^2 \ll q^{(1-\epsilon)X} q^{2j_0X}.
\]

Note that the only restrictions we had so far for \( \delta' \) and \( \epsilon \) were: \( 0 < \delta' \leq 1 \), \( 0 < \epsilon \), and

\[
0 \leq 1 - (k - j_0)\epsilon - \delta'.
\]

We have by (3.15) and (3.20)

\[
S \ll q^{X-\delta'X/p^b} q^{2j_0X} + q^{(1-\epsilon)X} q^{2j_0X}.
\]

In order to minimize the right hand side of the above inequality, we set \( \epsilon = \delta'/p^b \). Then, since \( k - j_0 = p^b \), (3.21) can be simplified to

\[
2\delta' \leq 1.
\]

By letting \( \delta' = 1/2 \), we obtain by (3.11)

\[
\psi(\theta, \alpha) \ll q^{-X/2} S^{1/2} \ll q^{(j_0-\delta)}X,
\]

where \( \delta = 1/(4p^b) \).

\[ \square \]

4. A bound on the minor arcs

We obtain estimates on the minor arcs in this section. In Section 4.1, we give bounds on the minor arcs when \( p \nmid (k - 1) \) and \( p = mk^b + 1 \), \( m > 1 \). The remaining case when \( k = p^b + 1 \) requires a different approach, and it is treated separately in Section 4.2. The reason we require a different approach is that when \( k = p^b + 1 \), the method in Section 4.1 results in an exponential sum that is more complicated to estimate than \( \psi(\alpha, \theta) \). Thus we take a more basic approach in this case.
4.1. **Cases** \( p \nmid (k-1) \) and \( p = mk^b + 1, \ m > 1 \). Let \( \mathcal{R}' \) be as defined in (2.20). Recall from the paragraph after Lemma 5 that \( \text{card} \mathcal{R}' = r \), and \( t_1, \ldots, t_r \) are the elements of \( \mathcal{R}' \) in increasing order. The main results of this section are the following estimates on the minor arcs.

**Theorem 4.1.** Suppose \( k \geq 3 \) and \( p \nmid k \). Suppose further that either \( p \nmid (k-1) \) or \( p = mk^b + 1, \ m > 1 \). Let \( \kappa = \sum_{j=1}^{r} t_j \), where \( \mathcal{R}' = \{t_1, \ldots, t_r\} \) and \( t_j \leq t_{j+1} \). Let

\[
\delta_0 = \begin{cases} 
1, & \text{if } p \nmid (k-1), \\
\frac{1}{4p}, & \text{if } k = mp^b + 1, m > 1.
\end{cases}
\]

Then we have

\[
\int_{m} |g(\alpha)|^{2s} \ d\alpha \ll q^{(\kappa-k-\delta_0)X} J_s(\mathcal{R}', X),
\]

where the implicit constant depends only on \( q \) and \( k \).

Recall from above that if \( p \nmid (k-1) \), then \( r = k - [k/p] \). On the other hand, if \( k = mp^b + 1 \), then \( r = (1 - 1/p)(k-p^b) + (1 + 1/p) \).

**Corollary 4.2.** Suppose \( k \geq 3, \ p \nmid k \) and \( s \geq (rk+r) \). Suppose further that either \( p \nmid (k-1) \) or \( p = mk^b + 1, \ m > 1 \). Let \( \delta_0 \) be as in the statement of Theorem 4.1. Then for each \( \epsilon > 0 \), we have

\[
\int_{m} |g(\alpha)|^{2s} \ d\alpha \ll q^{(2s-k-\delta_0+\epsilon)X},
\]

where the implicit constant depends only on \( s, q, k, \mathcal{R}', \) and \( \epsilon \).

**Proof.** This is an immediate consequence of applying Theorem 2.2 to Theorem 4.1. \( \square \)

Before we begin with our proof of Theorem 4.1, we set some notation. First we define the following exponential sums:

\[
f(\alpha) = \sum_{x \in I_X} e \left( \sum_{j=1}^{r-1} \alpha_{t_j} x_j + \alpha_k x^k \right),
\]

and

\[
F(\beta, \theta) = \sum_{x \in I_X} e \left( \sum_{j=1}^{r-2} \beta_{t_j} x_j + \theta x^k \right).
\]

We will also use the notation \( f(\alpha, \theta) \) to mean

\[f(\alpha, \theta) = f(\alpha_{t_1}, \alpha_{t_2}, \ldots, \alpha_{t_{r-1}}, \theta).\]

We also define for \( 1 \leq j \leq k \),

\[
\sigma_{s,j}(x) = \sum_{i=1}^{s} (x_i^j - x_{s+i}^j).
\]

Recall \( J_s(\mathcal{R}', X) \) is the number of solutions of the system

\[u_1^j + \ldots + u_s^j = v_1^j + \ldots + v_s^j \quad (j \in \mathcal{R}')\]
with \( u_j, v_j \in I_X \) \((1 \leq j \leq s)\). By the orthogonality relation (2.2), it follows that
\[
\begin{align*}
\left(\int f(\alpha)\right)^{2s} = \prod_{i=1}^{s} \left( \sum_{x_i, x_{s+i} \in I_X} e \left( \sum_{j=1}^{r-2} \beta_{t_j} (x_i^{t_j} - x_{s+i}^{t_j}) + \theta (x_i^k - x_{s+i}^k) \right) \right)
\end{align*}
\]

where
\[
\begin{align*}
\delta(x, t) &= \prod_{j=1}^{r-2} \left( \int e(\beta_{t_j}(x_{s,t_j}(x) - h_{t_j})) d\beta_{t_j} \right).
\end{align*}
\]

Thus, the orthogonality relation (2.2) gives us
\[
\begin{align*}
\int \delta(x, t) = \left\{ \begin{array}{ll}
1, & \text{when } \sigma_{s,t_j}(x) = h_{t_j}, \\
0, & \text{when } \sigma_{s,t_j}(x) \neq h_{t_j}.
\end{array} \right.
\end{align*}
\]

When \( \text{ord } x < X \), we have \( \text{ord } \sigma_{s,t_j}(x) \leq t_j(X - 1) \) for \( 1 \leq j \leq r - 2 \), and so it follows from (4.6) and (4.7) that
\[
\begin{align*}
\sum_{\text{ord } h_{t_1} \leq t_1(X - 1)} \ldots \sum_{\text{ord } h_{t_{r-2}} \leq t_{r-2}(X - 1)} \delta(x, h) = 1.
\end{align*}
\]

Since
\[
|g(\theta)|^{2s} = \sum_{\text{ord } x < X} e(\theta \sigma_{s,k}(x)),
\]

we obtain by (4.5) and (4.8),
\[
\begin{align*}
\int_m \left( \sum_{\text{ord } h_{t_1} \leq t_1(X - 1)} \ldots \sum_{\text{ord } h_{t_{r-2}} \leq t_{r-2}(X - 1)} \int_m \delta(x, \alpha) e(\sum_{j=1}^{r-2} \beta_{t_j} h_{t_j}) d\beta \right) d\theta
&= \int_m \left( \sum_{\text{ord } x < X} \delta(x, h) e(\theta \sigma_{s,k}(x)) \right) d\theta
&= \int_m |g(\theta)|^{2s} d\theta.
\end{align*}
\]
It therefore follows by the triangle inequality,

\begin{equation}
\int_{m} |g(\alpha)|^{2s} \, d\alpha \leq \sum_{\text{ord } h_1 \leq t_1 (X-1)} \ldots \sum_{\text{ord } h_{t_r-2} \leq t_r-2 (X-1)} \int_{m} \int_{m} |F(\beta, \theta)|^{2s} \, d\beta \, d\theta \leq q^{(k-k-t_r-1)X} \int_{m} \int_{m} |F(\beta, \theta)|^{2s} \, d\beta \, d\theta.
\end{equation}

An argument similar to that employed in the last paragraph permits us to relate the mean value of $F(\beta, \theta)$ to a sum of integrals involving $f(\alpha, \theta)$ as follows

\begin{equation}
\int_{m} \oint |F(\beta, \theta)|^{2s} \, d\beta \, d\theta = \sum_{\text{ord } h \leq t_r-1 (X-1)} \int_{m} \oint |f(\alpha, \theta)|^{2s} e(-\alpha_{t_r-1} h) \, d\alpha \, d\theta.
\end{equation}

The advantage of this maneuver is that we can rewrite the integral in the summand with similar expression involving an extra new variable $y \in I_X$. We then take the average of these integrals over $y \in I_X$ to get a sharper upper bound for the left hand side of (4.10), which ultimately gives us the desired result. This task will be achieved during the course of the rest of the proof, but first we prove (4.10). For $h \in \mathbb{F}_q[t]$, let

\begin{equation}
\tilde{\delta}(x, h) = \oint e(\alpha_{t_r-1} (\sigma_{s,t_r-1}(x) - h)) \, d\alpha_{t_r-1}.
\end{equation}

We have by the orthogonality relation (2.2),

\begin{equation}
\tilde{\delta}(x, h) = \begin{cases} 
1, & \text{when } \sigma_{s,t_r-1}(x) = h, \\
0, & \text{when } \sigma_{s,t_r-1}(x) \neq h.
\end{cases}
\end{equation}

Clearly, ord $x < X$ implies ord $\sigma_{s,t_r-1}(x) \leq t_r-1 (X-1)$. Hence we have

\begin{equation}
\sum_{\text{ord } h \leq t_r-1 (X-1)} \tilde{\delta}(x, h) = 1.
\end{equation}

Since $f(\alpha, \theta) = f(-\alpha, -\theta)$, we get

\begin{align*}
|f(\alpha, \theta)|^{2s} &= \prod_{i=1}^{s} \left( \sum_{x_i, x_{i+1} \in I_X} e \left( \sum_{j=1}^{r-1} \alpha_{t_j} (x_i^{t_j} - x_{i+1}^{t_j}) + \theta (x_i^k - x_{i+1}^k) \right) \right) \\
&= \sum_{\text{ord } x < X} e \left( \sum_{j=1}^{r-1} \alpha_{t_j} \sigma_{s,t_j}(x) + \theta \sigma_{s,k}(x) \right).
\end{align*}

Thus, it follows by (4.11) that

\begin{equation}
\int_{m} \oint |f(\alpha, \theta)|^{2s} e(-\alpha_{t_r-1} h) \, d\alpha \, d\theta = \sum_{\text{ord } x < X} \tilde{\delta}(x, h) \int_{m} \oint e \left( \sum_{j=1}^{r-2} \beta_{t_j} \sigma_{s,t_j}(x) + \theta \sigma_{s,k}(x) \right) \, d\beta \, d\theta.
\end{equation}
Therefore, we obtain by (4.12) and (4.13),

$$
\sum_{\text{ord } h \leq t_{r-1}(X-1)} \int_{m} \oint |f(\alpha, \theta)|^{2s} e \left( -\alpha_{t_{r-1}} h \right) \ d\alpha \ d\theta
$$

(4.14)

$$
= \sum_{\text{ord } x < X} \sum_{\text{ord } h \leq t_{r-1}(X-1)} \overline{\delta}(x, h) \int_{m} \oint e \left( \sum_{j=1}^{r-2} \beta_{t_{j}} \sigma_{s, t_{j}}(x) + \theta \sigma_{s, k}(x) \right) \ d\beta \ d\theta
$$

$$
= \int_{m} \oint \sum_{\text{ord } x < X} e \left( \sum_{j=1}^{r-2} \beta_{t_{j}} \sigma_{s, t_{j}}(x) + \theta \sigma_{s, k}(x) \right) \ d\beta \ d\theta
$$

$$
= \int_{m} \oint |F(\beta, \theta)|^{2s} \ d\beta \ d\theta,
$$

which is exactly the equation (4.10) we aimed to prove.

Given \( y \in I_X \), observe that \( I_X \) is invariant under translation by \( y \), or in other words

$$
I_X = \{ x : x \in \mathbb{F}_q[t], \text{ord } x < X \} = \{ x + y : x \in \mathbb{F}_q[t], \text{ord } x < X \}.
$$

Let

$$
\lambda(z; \alpha) = \sum_{j=1}^{r-1} \alpha_{t_{j}} z^{t_{j}} + \alpha_{k} z^{k}.
$$

By the above observation, shifting the variable of summation in \( f(\alpha) \) by \( y \) gives us

$$
f(\alpha) = \sum_{x \in I_X} e(\lambda(x; \alpha)) = \sum_{x \in I_X} e(\lambda(x - y; \alpha)).
$$

(4.15)

Define \( \Delta(\theta, h, y) \) as follows:

$$
\Delta(\theta, h, y) = e(\theta \sigma_{s, k}(x - y)),
$$

when the 2s-tuple \( x \) satisfies

$$
\sum_{i=1}^{s} ((x_{i} - y)^{t_{j}} - (x_{s+i} - y)^{t_{j}}) = 0 \quad (1 \leq j \leq r - 2)
$$

(4.16)

and

$$
\sum_{i=1}^{s} ((x_{i} - y)^{t_{r-1}} - (x_{s+i} - y)^{t_{r-1}}) = h.
$$

(4.17)

Otherwise, we let \( \Delta(\theta, h, y) = 0 \). Substituting the expression (4.15) for \( f(\alpha, \theta) \), we find by the orthogonality relation (2.2),

$$
\oint |f(\alpha, \theta)|^{2s} e(-\alpha_{t_{r-1}} h) \ d\alpha = \sum_{\text{ord } x < X} \Delta(\theta, h, y).
$$

(4.18)

We now simplify the function \( \Delta(\theta, h, y) \) and obtain another expression for the left hand side of (4.18). First, we prove that the 2s-tuple \( x \) satisfies (4.16) and (4.17) if and only if \( x \) satisfies

$$
\sum_{i=1}^{s} (x_{i}^{t_{j}} - x_{s+i}^{t_{j}}) = 0 \quad (1 \leq j \leq r - 2)
$$

(4.19)
and

\[(4.20) \quad \sum_{i=1}^{s} (x_{t_i}^{t-r-1} - x_{s+i}^{t-r-1}) = h.\]

Suppose \(x\) satisfies \((4.16)\) and \((4.17)\). Since \(F_q\) has characteristic \(p\), we have \((x - y)^p = x^p - y^p\). Recall \(t_{r-1} = j_0\). Thus, we can prove by induction and the definition of \(R'\) that \((4.16)\) implies

\[(4.21) \quad \sum_{i=1}^{s} \left( (x_i - y)^j - (x_{s+i} - y)^j \right) = 0 \quad (1 \leq j < t_{r-1}).\]

Note we can verify that \(t_{r-1} > 1\) for the cases we consider here. By applying the binomial theorem, we obtain that whenever a 2s-tuple \(x\) satisfies \((4.17)\) and the system \((4.21)\), then \(x\) satisfies

\[(4.22) \quad \sum_{i=1}^{s} (x^j_i - x^j_{s+i}) = 0 \quad (1 \leq j < t_{r-1})\]

and \((4.20)\). Clearly the system \((4.22)\) implies \((4.19)\). For the converse direction, since \((4.19)\) implies \((4.22)\), we can obtain the desired result in a similar manner as in the forward direction.

Suppose \(x\) satisfies \((4.19)\) and \((4.20)\), and consequently \((4.22)\). If \(p \nmid (k - 1)\), then \(t_{r-1} = j_0 = k - 1\) and we have

\[(4.23) \quad \sigma_{s,k}(x - y) = \sum_{i=1}^{s} ((x_i - y)^k - (x_{s+i} - y)^k) = \sigma_{s,k}(x) - chy^{k-t_{r-1}},\]

where \(c = \binom{k}{t_{r-1}} \equiv 0 \pmod{p}\). If \(k = mp^b + 1\), then we can deduce from \(t_{r-1} = j_0 = (m - 1)p^b + 1 > m\), which we note does not hold if \(m = 1\), and \((4.22)\) that

\[\sum_{i=1}^{s} x_i^{k-1} - x_{s+i}^{k-1} = \left( \sum_{i=1}^{s} x_i^m - x_{s+i}^m \right)^{p^b} = 0.\]

Therefore, by the binomial theorem, the above equation, and the definition of \(j_0\) given in \((2.17)\), we also obtain \((4.23)\) when \(k\) is of the form \(k = mp^b + 1, m > 1\). Thus, we can rewrite the definition of \(\Delta(\theta, h, y)\) as

\[\Delta(\theta, h, y) = e(\theta \sigma_{s,k}(x) - chy^{k-t_{r-1}}),\]

whenever \(x\) satisfies \((4.19)\) and \((4.20)\); otherwise, \(\Delta(\theta, h, y)\) is equal to 0. Thus, we have

\[\oint |f(\alpha, \theta)|^{2s} e(-chy^{k-t_{r-1}} - \alpha_{t_{r-1}} h) d\alpha = \sum_{\text{ord } x < X} \Delta(\theta, h, y),\]

and consequently, it follows from \((4.18)\) that

\[\oint |f(\alpha, \theta)|^{2s} e(-\alpha_{t_{r-1}} h) d\alpha = \oint |f(\alpha, \theta)|^{2s} e(-chy^{k-t_{r-1}} - \alpha_{t_{r-1}} h) d\alpha.\]
From here, we have by (4.10),
\[
\int \phi |F(\beta, \theta)|^{2s} \, d\beta \, d\theta
\]
(4.24)
\[
= \int \phi |f(\alpha, \theta)|^{2s} \sum_{\text{ord } h \leq t_{r-1}(X-1)} e(-c(hy^{k-t_{r-1}} - \alpha_{t_{r-1}})) \, d\alpha \, d\theta.
\]

Since the left hand side of (4.24) is independent of \( y \), we can average the right hand side over \( y \in I_X \) to obtain
\[
\int \phi |F(\beta, \theta)|^{2s} \, d\beta \, d\theta
\]
(4.25)
\[
= q^{-X} \sum_{y \in I_X} \int \phi |f(\alpha, \theta)|^{2s} \sum_{\text{ord } h \leq t_{r-1}(X-1)} e(-c(hy^{k-t_{r-1}} - \alpha_{t_{r-1}})) \, d\alpha \, d\theta.
\]

In the last equality displayed above, we invoked (3.1), the definition of \( \psi(\theta, \alpha) \). We apply the appropriate lemma depending on \( k \) from Section 3, namely Lemmas 6, 7 and 8 to \( \psi(\theta, \alpha) \) and obtain an upper bound for the right hand side of (4.25). We then use the resulting estimate and (4.4) to bound (4.9), from which we obtain
\[
\int \phi |g(\alpha)|^{2s+1} \, d\alpha \ll q^{(1+k-\delta_0)X} J_s(\mathcal{R}', X),
\]
(4.26)
for suitable \( \delta > 0 \).  \( \square \)

4.2. Case \( k = p^b + 1 \). Recall from above that if \( k = p^b + 1 \), then \( r = (1 - 1/p)(k - p^b) + (1 + 1/p) = 2 \). We obtain the following minor arc bound when \( k = p^b + 1 \).

**Theorem 4.3.** Suppose \( k \geq 3 \) and \( k = p^b + 1 \). Let \( \kappa = 1 + k \), \( \mathcal{R}' = \{1, k\} \), and
\[
\delta_0 = \frac{1}{16(p^b + 2)}.
\]

Then we have
\[
\int \phi |g(\alpha)|^{2s+1} \, d\alpha \ll q^{(1+k-\delta_0)X} J_s(\mathcal{R}', X),
\]
where the implicit constant depends only on \( q \) and \( k \).

By applying Theorem 2.2 to Theorem 4.3, we also obtain the following corollary.

**Corollary 4.4.** Suppose \( k \geq 3 \), \( k = p^b + 1 \), and \( s \geq (2k + 2) \). Let
\[
\delta_0 = \frac{1}{16(p^b + 2)}.
\]

Then for each \( \epsilon > 0 \), we have
\[
\int \phi |g(\alpha)|^{2s+1} \, d\alpha \ll q^{(2s+1-k-\delta_0+\epsilon)X},
\]
where the implicit constant depends only on \( s, q, k, \) and \( \epsilon \).
We introduce some notation before we get into the proof of Theorem 4.3. Given \( j, j' \in \mathbb{Z}^+ \), we write \( j \preceq_p j' \) if \( p \nmid \binom{j'}{j} \). By Lucas’ Theorem, this happens precisely when all the digits of \( j \) in base \( p \) are less than or equal to the corresponding digits of \( r \). From this characterization, it is easy to see that the relation \( \preceq_p \) defines a partial order on \( \mathbb{Z}^+ \). If \( j \preceq_p j' \), then we necessarily have \( j \leq j' \). Let \( \mathcal{K} \subseteq \mathbb{Z}^+ \). We say an element \( k \in \mathcal{K} \) is maximal if it is maximal with respect to \( \preceq_p \), that is, for any \( j \in \mathcal{K} \), either \( j \preceq_p k \) or \( j \) and \( k \) are not comparable. Following the notation of [3], we define the shadow of \( \mathcal{K} \), \( \mathcal{S}(\mathcal{K}) \), to be

\[
\mathcal{S}(\mathcal{K}) = \{ j \in \mathbb{Z}^+ : j \preceq_p j' \text{ for some } j' \in \mathcal{K} \}. 
\]

We also define

\[
\mathcal{K}^* = \{ k \in \mathcal{K} : p \nmid k \text{ and } p^v k \notin \mathcal{S}(\mathcal{K}) \text{ for any } v \in \mathbb{Z}^+ \}.
\]

We invoke the following result from [3]. The theorem allows us to estimate certain coefficients of a polynomial \( h(u) \) by an element in \( \mathbb{K} \) when the exponential sum of \( h(u) \) is sufficiently large. We use the result to bound exponential sums over the minor arcs.

**Theorem 4.5 (Theorem 12, [3]).** Let \( \mathcal{K} \subseteq \mathbb{Z}^+ \) and \( h(u) = \sum_{j \in \mathcal{K} \cup \{0\}} \alpha_j u^j \in \mathbb{K}_\infty[u] \), where \( \alpha_j \neq 0 \) (\( j \in \mathcal{K} \)). Suppose that \( k \in \mathcal{K}^* \) is maximal in \( \mathcal{K} \). Then there exist constants \( c, C > 0 \), depending only on \( \mathcal{K} \) and \( q \), such that the following holds: suppose that for some \( 0 < \eta \leq cX \), we have

\[
\left| \sum_{x \in I_X} e(h(x)) \right| \geq q^{X-\eta}.
\]

Then for any \( \epsilon > 0 \) and \( X \) sufficiently large in terms of \( \mathcal{K} \), \( \epsilon \) and \( q \), there exist \( a, g \in \mathbb{F}_q[t] \) such that

\[
\text{ord} (g\alpha_k - a) < -kX + \epsilon X + C\eta \quad \text{and} \quad \text{ord} g \leq \epsilon X + C\eta.
\]

**Proof of Theorem 4.3.** We bound \( \sup_{\theta \in \mathfrak{m}} |g(\theta)| \) using Theorem 4.5. For \( g(\theta) \) with \( k = p^b + 1 \), we have \( \mathcal{K} = \{ k \} \), and thus

\[
\mathcal{S}(\mathcal{K}) = \{ k, p^b, 1 \},
\]

and

\[
\mathcal{K}^* = \{ k \}.
\]

Clearly, \( k \) is maximal in \( \mathcal{K} \). We also have

\[
\mathcal{S}(\mathcal{K})' := \{ i \in \mathbb{N} : p \nmid i \text{ and } p^v i \in \mathcal{S}(\mathcal{K}) \text{ for some } v \in \mathbb{N} \cup \{0\} \}
\]

\[
= \{ k, 1 \}.
\]

It is given at the end of the proof of [3, Theorem 12] that we may take \( c = 1/(8(r_0 \phi + r_0)) \) and \( C = 2(r_0 \phi + r_0) \), where \( r_0 = \# \mathcal{S}(\mathcal{K})' \) and \( \phi = \max_{i \in \mathcal{S}(\mathcal{K})'} i \). Therefore, we can apply Theorem 4.5 with

\[
c = \frac{1}{8(2k + 2)} = \frac{1}{16(p^b + 2)} \quad \text{and} \quad C = 2(2k + 2).
\]

Take any \( \theta \in \mathfrak{m} \). We set \( \epsilon = 1/2 \). Suppose for some \( X \) sufficiently large, with respect to \( \mathcal{K} \) and \( q \), we have

\[
|g(\theta)| \geq q^{X-cX}.
\]

Then, by Theorem 4.5 there exist \( \tilde{g}, \tilde{a} \in \mathbb{F}_q[t] \) such that

\[
\text{ord} (\tilde{g}\theta - \tilde{a}) < -kX + \epsilon X + \frac{1}{4}X \quad \text{and} \quad \text{ord} \tilde{g} \leq \epsilon X + \frac{1}{4}X.
\]
Let \((\tilde{g}, \tilde{a}) = \ell,\) and denote \(\tilde{g} = \ell g_0\) and \(\tilde{a} = \ell a_0.\) We obtain from above inequalities,

\[
\text{ord} (\theta - a_0/g_0) = \text{ord} (\theta - \tilde{a}/\tilde{g}) < -kX + \epsilon X + \frac{1}{4}X - \text{ord} \tilde{g} \leq -(k-1)X - \text{ord} g_0
\]

and

\[
\text{ord} g_0 \leq \text{ord} \tilde{g} \leq \epsilon X + \frac{1}{4}X < X.
\]

By the definition of major arcs (2.6), this implies that \(\theta \in \mathcal{M}_k,\) which is a contradiction. Therefore, we must have

\[
|g(\theta)| < q^{X-cX}
\]

for all \(X\) sufficiently large with respect to \(\mathcal{K}\) and \(q.\) Since the result is independent of the choice of \(\theta \in \mathfrak{m},\) it follows that

\[
(4.27) \quad \sup_{\theta \in \mathfrak{m}_k} |g(\theta)| < q^{X - \frac{1}{16(p^h+2)} X}.
\]

When \(k = p^h + 1,\) we have \(r = 2;\) therefore, we have \(F(\beta, \theta) = g(\theta).\) Thus, we obtain by (4.10) and the triangle inequality that

\[
(4.28) \quad \int_m |g(\theta)|^{2s+1} d\theta \\
\leq \sup_{\theta \in \mathfrak{m}} |g(\theta)| \cdot \int_m |g(\theta)|^{2s} d\theta \\
= \sup_{\theta \in \mathfrak{m}} |g(\theta)| \cdot \sum_{h \in I_X} \int_m \oint |f(\alpha, \theta)|^{2s} e(-\alpha h) \ d\alpha \ d\theta \\
\leq \sup_{\theta \in \mathfrak{m}} |g(\theta)| \cdot q^X \cdot \oint |f(\alpha, \theta)|^{2s} \ d\alpha \ d\theta \\
= \sup_{\theta \in \mathfrak{m}} |g(\theta)| \cdot q^X \cdot J_s(\mathcal{R}', X).
\]

Consequently, substituting (4.27) into the above inequality (4.28) gives us

\[
\int_m |g(\theta)|^{2s+1} d\theta \ll q^{2X - \frac{1}{16(p^h+2)} X} J_s(\mathcal{R}', X).
\]

\[
\square
\]

5. Weyl Differencing

Let \(w_0(u)\) be a polynomial in \(\mathbb{F}_q[t][u].\) Let \(z_1, \ldots, z_h\) be indeterminates. We define the differencing operator \(\Delta_{z_1}\) by

\[
\Delta_{z_1} (w_0)(u) = w_0(u + z) - w_0(u) \in \mathbb{F}_q[t][u, z_1],
\]

where we denote \(\Delta_{z_1} (w_0) = \Delta_{z_1} (w_0)(u).\) We also define recursively

\[
\Delta_{z_h \ldots z_1} (w_0)(u) = \Delta_{z_h \ldots z_1} (w_0)(u + z_h) - \Delta_{z_h \ldots z_1} (w_0)(u) \in \mathbb{F}_q[t][u, z_1, \ldots, z_h],
\]

and we denote \(\Delta_{z_h \ldots z_1} (w_0) = \Delta_{z_h \ldots z_1} (w_0)(u).\)

While in characteristic zero the above differencing process, known as Weyl differencing, decreases the degree (in \(u\)) of the polynomial by one, the situation in positive characteristic is more subtle. With application of Hua’s lemma (Proposition 5.1) in mind, it will be useful to know how many times one can apply Weyl differencing to \(u^k\) in \(\mathbb{F}_q[t][u]\) before it becomes
identically zero. Note that given an indeterminate \( z \) and a monomial \( u^\ell \), we have \( \Delta_z(u^\ell) = 0 \) if and only if \( \ell = 0 \). To see this, suppose we have \( \ell \geq 1 \) and 
\[
0 = \Delta_z(u^\ell) = (u + z)^\ell - u^\ell = \sum_{j=0}^{\ell-1} \binom{\ell}{j} u^j z^{\ell-j}.
\]
Then, in particular it must be that \( \binom{\ell}{0} = 1 \equiv 0 \pmod{p} \), which is a contradiction. Therefore, we have \( \ell = 0 \). The converse direction is trivial. The following lemma is a slight modification of \cite[Lemma 8.1]{5} and we omit the proof here.

**Lemma 9.** Let \( k = c_0 p^r + \ldots + c_0 \) with \( 0 \leq c_i < p \) \((0 \leq i \leq v)\), and let \( h_0 = h_0(k) = c_0 + \ldots + c_0 \). Let \( z_1, \ldots, z_{h_0+1} \) be indeterminates. Then, we have 
\[
0 \neq \Delta_{z_{h_0}} \ldots \Delta_{z_1} u^k \in \mathbb{F}_q[t][u, z_1, \ldots, z_{h_0}]
\]
and
\[
0 = \Delta_{z_{h_0+1}} \ldots \Delta_{z_1} u^k \in \mathbb{F}_q[t][u, z_1, \ldots, z_{h_0+1}].
\]
Combining Lemma 9 and \cite[Proposition 13]{7}, we have the following version of Hua’s lemma.

**Proposition 5.1.** Let \( w_0(u) \) be a polynomial in \( \mathbb{F}_q[t][u] \) of degree \( k \) in \( u \), and let \( w(\alpha) = \sum_{x \in \mathcal{I}} e(w_0(x) \alpha) \). Let \( h_0(k) \) be as defined in the statement of Lemma 9. Suppose \( j \leq h_0(k) \). Then for every \( \epsilon > 0 \), we have 
\[
\oint |w(\alpha)|^{2j} \mathrm{d}\alpha \ll q^{(2j-j+\epsilon)X},
\]
where the implicit constant depends only on \( k, q, \) and \( \epsilon \).

We apply Proposition 5.1 in Sections 6 and 7 with \( w_0(u) = u^k \).

6. Asymptotic Formula and \( \tilde{G}_q(k) \)

We now lower the bound on \( s \) in Corollary 4.2 via combination of Proposition 5.1 and Hölder’s inequality, and obtain Theorems 1.3 and 1.4. First, we consider the case when \( p \nmid (k-1) \) in Proposition 6.1. We then take care of the case \( k = mp^b + 1 \) in Proposition 6.2. Let
\[
s'_0(j) = 2k^2 + 1 - \left\lfloor \frac{2kj - 2j}{k+1-j} \right\rfloor.
\]
If \( k < p \), we set
\[
s_1(k) = \min_{1 \leq j < k, 2j \leq k(2k+1)} s'_0(j).
\]
On the other hand, if \( k > p \) and \( p \nmid (k-1) \), we set
\[
s_1(k) = 2rk + 1 - \left\lfloor \frac{6r-8}{k-2} \right\rfloor.
\]
We choose where the implicit constant depends only on \( s, q, k \), \( j \).

Note that our restriction on Proposition 6.1.

Suppose \( k < p \) and in (6.2) when \( k > p \). If \( s \geq s_1(k) \), then there exists \( \delta_1 > 0 \) such that

\[
\int \left| g(\alpha) \right|^s \, d\alpha \ll q^{(s-k-\delta_1)X},
\]

where the implicit constant depends only on \( s, q, k, R', \) and \( \delta_1 \).

**Proof.** Let \( h_0(k) \) be as in the statement of Lemma 9. We have by Proposition 5.1 if \( j \leq h_0(k) \), then for any \( \epsilon > 0 \),

\[
(6.3) \quad \int \left| g(\alpha) \right|^j \, d\alpha \ll q^{(2j-\epsilon)X}.
\]

We let \( s_0(j) = 2r(k+1)a' + 2j' \), where \( a' + b' = 1 \). Then Hölder’s inequality gives us

\[
(6.4) \quad \int \left| g(\alpha) \right|^{s_0(j)} \, d\alpha \leq \left( \int \left| g(\alpha) \right|^{2r(k+1)} \, d\alpha \right)^{a'} \left( \int \left| g(\alpha) \right|^{2j} \, d\alpha \right)^{b'}.
\]

Recall for the range of \( k \) we are considering, we can take \( \delta_0 = 1 \) in Corollary 4.2. We consider \( j \) in the following range: \( 1 \leq j < k, 2j \leq (2r-1)(k+1) + 1 \) and \( j \leq h_0(k) \). Define

\[
\eta(j) = \frac{2rj}{k-j+1} - \frac{2j}{k-j+1},
\]

and let

\[
\gamma(j) = 1 + \eta(j) - \lfloor \eta(j) \rfloor.
\]

We choose

\[
a' = \frac{k-j}{k-j+1} + \frac{\gamma(j)}{2r(k+1) - 2j}
\]

and

\[
b' = \frac{1}{k-j+1} - \frac{\gamma(j)}{2r(k+1) - 2j}.
\]

Note that our restriction on \( j \) ensures \( b' > 0 \). Also, this choice of \( a' \) and \( b' \) ensures \( a' - (k-j)b' > 0 \). Then, by Corollary 4.2 and (6.3), we have the following bound for (6.4):

\[
\int \left| g(\alpha) \right|^{s_0(j)} \, d\alpha \ll q^{X} q^{a'(2r(k+1)-k-1)X} q^{b'(2j'-j)X} \ll q^{(s_0(j)-k-(a'-(k-j)b') + \epsilon)X}.
\]

By the trivial bound \( |g(\alpha)| \leq q^X \), it follows that for any \( s \geq s_0(j) \) we have

\[
\int \left| g(\alpha) \right|^s \, d\alpha \ll q^{(s-s_0(j))X} \int \left| g(\alpha) \right|^{s_0(j)} \, d\alpha \ll q^{(s-k-(a'-(k-j)b') + \epsilon)X}.
\]

We can simplify \( s_0(j) \) as

\[
s_0(j) = 2r(k+1) \left( \frac{k-j}{k-j+1} + \frac{\gamma(j)}{2r(k+1) - 2j} \right) + 2j \left( \frac{1}{k-j+1} - \frac{\gamma(j)}{2r(k+1) - 2j} \right)
\]

\[
= 2rk - \eta(j) + \gamma(j)
\]

\[
= 2rk + 1 - \lfloor \eta(j) \rfloor.
\]

To establish our result, all we have left is to choose \( j \) within the appropriate range given above such that \( s_0(j) \) is as small as possible. This value of \( s_0(j) \) will be our \( s_1(k) \). We consider the two cases separately.
Case 1: $k > p$. From $p \nmid k$, $p \nmid (k - 1)$, and $k > p$, we can verify that $3 \leq h_0(k)$. Thus we know we can apply Weyl differencing at least three times. Therefore, we set $s_1(k) = s_0(3)$. Since
\begin{equation}
0 < \eta(3) = \frac{6r - 8}{k - 2},
\end{equation}
we obtain
\[ s_1(k) = 2rk + 1 - \left\lfloor \frac{6r - 8}{k - 2} \right\rfloor. \]

Case 2: $k < p$. In this case, we have $h_0(k) = k$. We set
\begin{equation}
\label{eq:case2}
s_1(k) = \min_{2^j \leq (2r-1)(k+1)+1} s_0(j).
\end{equation}
Since $r = k - \lfloor k/p \rfloor = k$, we have $s_0(j) = s'_0(j)$ and $(2r-1)(k+1)+1 = k(2k+1)$. Therefore, we see that $s_1(k)$ given above in (6.6) coincides with (6.1).

Now we consider the case $k = mp^b + 1$. If $m = 1$, we set $s_1(k) = 4k + 5$. If $m > 1$, then we set
\begin{equation}
\label{eq:case3}
s_1(k) = 2rk + 2r - \left\lfloor \frac{(m-1)(1-1/p)}{2} \right\rfloor.
\end{equation}

**Proposition 6.2.** Suppose $k = mp^b + 1$ with $p \nmid m$. Let $s_1(k)$ be $4k + 5$ when $m = 1$ and as in (6.7) when $m > 1$. If $s \geq s_1(k)$, then there exists $\delta_1 > 0$ such that
\[ \int_m |g(\alpha)|^s \, d\alpha \ll q^{(s-k-\delta_1)r}, \]
where the implicit constant depends only on $s, q, k, R'$, and $\delta_1$.

**Proof.** We first deal with the case $m > 1$. Let $h_0(k)$ be as in the statement of Lemma 6.3. If $j \leq h_0(k)$, then for any $\epsilon > 0$ we have (6.3). We let $s_0(j) = 2r(k+1)a' + 2^jb'$, where $a' + b' = 1$, as before in Proposition 6.1. Then by Hölder’s inequality, we have (6.4). We consider $j$ in the following range: $1 \leq j < k$, $2^j \leq (2r - 1)(k+1) + 1$ and $j \leq h_0(k)$. Let $\epsilon(j)$ be a small positive number. We choose
\[ a' = \frac{k - j}{k - j + \delta} + \frac{\epsilon(j)}{2r(k+1) - 2^j}, \]
and
\[ b' = \frac{\delta}{k - j + \delta} - \frac{\epsilon(j)}{2r(k+1) - 2^j}, \]
where we let $\delta = \delta_0 = 1/(4p^b)$ from Corollary 4.2.

Note that we pick $\epsilon(j)$ sufficiently small to make sure $b' > 0$. Also, the range of $j$ we are considering and this choice of $a'$ and $b'$ ensure
\[ \delta a' - (k - j)b' = \frac{(\delta + k - j)\epsilon(j)}{2r(k+1) - 2^j} > 0. \]
By Corollary 4.2 and (6.3), we have the following bound for (6.4):

$$\int |g(\alpha)|^{s_0(j)} \, d\alpha \ll q^X q^{a(2r(k+1)-k-\delta)X} q^{b'(2j-3)X} \ll q^{(s_0(j) - k - (\delta a' - (k-j)b') + \epsilon)X}.$$ 

By the trivial bound $|g(\alpha)| \leq q^X$, it follows that for any $s \geq s_0(j)$ we have

$$\int |g(\alpha)|^s \, d\alpha \ll q^{(s-s_0(j))X} \int |g(\alpha)|^{s_0(j)} \, d\alpha \ll q^{(s-k - (\delta a' - (k-j)b') + \epsilon)X}.$$

We can simplify $s_0(j)$ as

$$s_0(j) = 2r(k+1) \left( \frac{k-j}{k-j+\delta} + \frac{\epsilon(j)}{2r(k+1) - 2j} \right) + 2^j \left( \frac{\delta}{k-j+\delta} - \frac{\epsilon(j)}{2r(k+1)-2j} \right).$$

To establish our result, all we have left is to choose $j$ within the appropriate range given above such that $s_0(j)$ is as small as possible. We would like to maximize the value

$$\delta \frac{2r(k+1) - 2j}{k-j+\delta}$$

in order to minimize $s_0(j)$. We then let the smallest integer greater than the $s_0(j)$ found to be our $s_1(k)$.

Since $m > 1$, we can verify that $h_0(k) \geq 3$. Thus we know we can apply Weyl differencing at least three times. We have

$$r = (1 - 1/p)(k - p^b) + (1 + 1/p) = (m - 1)(p^b - p^{b-1}) + 2.$$ 

Also, recall from above we have set $\delta = \delta_0 = 1/(4p^b)$. Let $j = 3$ and we obtain

$$s_0(3) = 2rk + 2r - \frac{2r(k+1) - 2^3}{4p^b(k-3+\delta)} + \epsilon(3)$$

$$= 2rk + 2r - \frac{2(m-1)(p^b - p^{b-1})(k+1)}{4p^b(k-3+\delta)} - \frac{4(k+1)}{4p^b(k-3+\delta)} + \frac{8}{4p^b(k-3+\delta)} + \epsilon(3)$$

$$= 2rk + 2r - \frac{(m-1)(1-1/p)(k+1)}{2(k-3+\delta)} - \frac{k-1}{p^b(k-3+\delta)} + \epsilon(3)$$

$$\leq 2rk + 2r - \frac{(m-1)(1-1/p)(k+1)}{2(k-3+\delta)}$$

$$\leq 2rk + 2r - \frac{(m-1)(1-1/p)}{2}.$$ 

Therefore, we let $s_1(k) = \left\lfloor 2rk + 2r - \frac{(m-1)(1-1/p)}{2} \right\rfloor = 2rk + 2r - \left\lfloor \frac{(m-1)(1-1/p)}{2} \right\rfloor \geq s_0(3)$.

The case $m = 1$ is an immediate consequence of Corollary 4.4. When $m = 1$, we have $r = 2$ and the saving in the exponent of $\delta_0 = \frac{1}{10(p^b+2)}$ from Corollary 4.4.
these values our approach above is not effective as in the case $m > 1$. Therefore, we let $s_1(k) = 4k + 5$ in this case.

We are now in position to prove Theorems 1.3 and 1.4. By using the bounds on minor arcs from this section, we obtain an estimate for $\tilde{G}_q(k)$.

Proof of Theorems 1.3 and 1.4. The result is an immediate consequence of combining our major arc estimates, Theorem 2.1, and our minor arc estimates, Propositions 6.1 and 6.2, from which we obtain $\tilde{G}_q(k) \leq \max\{s_1(k), 2k + 1\}$. We then simplify $s_1(k)$ from Propositions 6.1 and 6.2 via (2.21) to obtain the estimates given in the statement of Theorem 1.3. When $k < p$, we see that $s_1(k)$ given in (6.1) is identical to that defined for the integer case in [10]. Consequently, our estimates for $\tilde{G}_q(k)$ when $k < p$ are identical to the estimates of $\tilde{G}(k)$ obtained in [10].

7. Slim Exceptional Sets

We carry out a similar calculation here as in Section 6 and obtain Theorems 1.5 and 1.6. Recall from Section 1 that $\tilde{E}_{s,k}(N, \psi)$ is defined to be the set of $n \in I_N \cap \mathbb{F}_q[t]$ which satisfies (1.6). As in [10], we refer to a function $\psi(z)$ as being *sedately increasing* when $\psi(z)$ is a function of positive variable $z$ increasing monotonically to infinity, and satisfying the condition that when $z$ is large, one has $\psi(z) = O(z^\epsilon)$ for a positive number $\epsilon$ sufficiently small in the ambient context. We also prove the following theorem on the estimate of $|\tilde{E}_{s,k}(N, \psi)|$ when $\psi$ is a sedately increasing function. In order to avoid clutter in the exposition, we present the case $k = p^b + 1$ separately from the rest of the cases.

**Theorem 7.1.** Suppose $k \geq 3$ and $p \nmid k$. Suppose further that either $p \nmid (k - 1)$ or $k = mp^b + 1$, $m > 1$. Let $\delta_0$ be as in the statement of Theorem 4.1. If $\psi(z)$ is a sedately increasing function, then for $s \geq rk + r$ we have

$$|\tilde{E}_{s,k}(N, \psi)| \ll q^{(k-\delta_0+\epsilon)P} \psi(q^P)^2,$$

where the implicit constant depends on $s, q, k, \epsilon, R'$, and $\psi$.

**Theorem 7.2.** Suppose $k \geq 3$ and $p \nmid k$. Suppose further that $k = p^b + 1$. Let

$$\delta_0 = \frac{1}{16(p^b + 2)}.$$

If $\psi(z)$ is a sedately increasing function, then for $s \geq 2k + 3$ we have

$$|\tilde{E}_{s,k}(N, \psi)| \ll q^{(k-\delta_0+\epsilon)P} \psi(q^P)^2,$$

where the implicit constant depends on $s, q, k, \epsilon, R'$, and $\psi$.

First, we consider the case when $p \nmid (k - 1)$ in Proposition 7.3. We then take care of the case $k = mp^b + 1$ in Proposition 7.4.

Let

$$u'_0(j) = k^2 + 1 - \left\lfloor \frac{kj - 2^{j-1}}{k + 1 - j} \right\rfloor.$$

If $k < p$, we set

$$u_2(k) = \min_{\frac{1}{2} \leq j < k} u'_0(j).$$

(7.1)
On the other hand, if $k > p$ and $p \nmid (k - 1)$, we set

\begin{equation}
(7.2) \quad u_2(k) = rk + 1 - \left\lfloor \frac{3r - 4}{k - 2} \right\rfloor.
\end{equation}

**Proposition 7.3.** Suppose $k \geq 3$, $p \nmid k$, and $p \nmid (k - 1)$. Let $u_2(k)$ be as given in (7.1) when $k < p$ and in (7.2) when $k > p$. If $s \geq u_2(k)$, then there exists $\delta_2 > 0$ such that

\[ \int |g(\alpha)|^{2s} \, d\alpha \ll q^{(2s-k-\delta_2)}X, \]

where the implicit constant depends only on $s, q, k, R'$, and $\delta_2$.

**Proof.** Since the proof is similar to that of Proposition 6.1, we only give the set up of the proof here. Let $\eta(j) = \frac{rj}{k-j+1} - \frac{2j-1}{k-j+1}$ and let

\[ \gamma(j) = 1 + \eta(j) - \lfloor \eta(j) \rfloor. \]

We choose

\[ a' = \frac{k-j}{k-j+1} + \frac{\gamma(j)}{r(k+1) - 2j-1}, \]

and

\[ b' = \frac{1}{k-j+1} - \frac{\gamma(j)}{r(k+1) - 2j-1}. \]

Note that our restriction on $j$ ensures $b' > 0$. Also, this choice of $a'$ and $b'$ ensures $a' - (k-j)b' > 0$. Then, by Corollary 4.2 and (6.3), we have the following bound for (7.3):

\[ \int \left( |g(\alpha)|^{2u_0(j)} d\alpha \right)^{a'} \left( |g(\alpha)|^{2j} d\alpha \right)^{b'} \ll q^{(2u_0(j) - k - (a' - (k-j)b') + \epsilon)}X. \]

We then obtain the result by proceeding in a similar manner as in the proof of Proposition 6.1. We leave verifying the remaining details of the proof as an exercise for the reader. □

Now we consider the case $k = mp^h + 1$. If $m = 1$, we set $u_2(k) = 2k + 3$. If $m > 1$, then we set

\begin{equation}
(7.4) \quad u_2(k) = rk + r - \left\lfloor \frac{(m-1)(1-1/p)}{4} \right\rfloor.
\end{equation}
Proposition 7.4. Suppose $k = mp^b + 1$ with $p 
mid m$. Let $u_2(k)$ be $2k + 3$ when $m = 1$ and as in (7.4) when $m > 1$. If $s \geq u_2(k)$, then there exists $\delta_2 > 0$ such that

$$\int_m \left| g(\alpha) \right|^{2s} d\alpha \ll q^{(2s-k-\delta_2)X},$$

where the implicit constant depends only on $s, q, k, {\mathcal R}'$, and $\delta_2$.

Proof. Since the proof is similar to that of Proposition 6.2, we only give the set up of the proof here. For the case $m = 1$, by a similar reasoning as in Proposition 6.2, we let $u_2(k) = 2k + 3$, and the result is an immediate consequence of Corollary 4.4. We now deal with the case $m > 1$. Let $h_0(k)$ be as in the statement of Lemma 9. If $j \leq h_0(k)$, then for any $\epsilon > 0$ we have (6.3). We let $2u_0(j) = 2r(k+1)a' + 2j' b'$, where $a' + b' = 1$, as before in Proposition 7.3. Then by Hölder’s inequality, we have (7.3). We consider $j$ in the following range: $1 \leq j < k$, $2^j \leq (2r - 1)(k+1) + 1$ and $j \leq h_0(k)$.

Let $\epsilon(j)$ be a small positive number. We choose

$$a' = \frac{k - j}{k - j + \delta} + \frac{\epsilon(j)}{r(k+1) - 2^{j-1}},$$

and

$$b' = \frac{\delta}{k - j + \delta} - \frac{\epsilon(j)}{r(k+1) - 2^{j-1}},$$

where we let $\delta = \delta_0 = 1/(4p^b)$ from Corollary 4.2.

Note that we pick $\epsilon(j)$ sufficiently small such that $b' > 0$. Also, the range of $j$ we are considering and this choice of $a'$ and $b'$ ensure

$$\delta a' - (k-j) b' = \frac{(\delta + k-j) \epsilon(j)}{r(k+1) - 2^{j-1}} > 0.$$

By Corollary 4.2 and (6.3), we have the following bound for (7.3):

$$\int_m \left| g(\alpha) \right|^{2u_0(j)} d\alpha \ll q^{(a' r(k+1) - k - \delta)X} q^{b'(2^j-j)X} \ll q^{(2u_0(j) - k - (\delta a' - (k-j)b') + \epsilon)X}.$$

We then obtain the result by proceeding in a similar manner as in the proof of Proposition 6.2. We leave verifying the remaining details of the proof as an exercise for the reader. $\square$

For $\psi(z)$ a function of positive variable $z$, recall we denote $\tilde{E}_{s,k}(N, \psi)$ to be the set of $n \in I_N \cap \mathbb{J}_q^k[t]$ for which

$$(7.5) \quad \left| R_{s,k}(n) - \mathcal{S}_{s,k}(n) J_\infty(n) q^{(s-k)P} \right| > q^{(s-k)P} \psi(q^P)^{-1}.$$

By Theorem 2.1 for $s \geq 2k + 1$ and any polynomial $n \in \tilde{E}_{s,k}(N, \psi)$ we have

$$\int_{\mathbb{M}} g(\alpha)^s e(-n\alpha) \, d\alpha = \mathcal{S}_{s,k}(n) J_\infty(n) q^{(s-k)P} + O(q^{(s-k-2\epsilon)P}),$$

for sufficiently small $\epsilon > 0$. Hence, it follows by (2.9) that

$$R_{s,k}(n) = \mathcal{S}_{s,k}(n) J_\infty(n) q^{(s-k)P} + \int_{m} g(\alpha)^s e(-n\alpha) \, d\alpha$$

$$+ \mathcal{O}(q^{(s-k-2\epsilon)P}).$$
By (7.5), (7.7) and the triangle inequality, we see that there exists a constant \( C_1 > 0 \) such that given any \( n \in \tilde{E}_{s,k}(N,\psi) \),

\[
(7.8) \quad \left| \int_m g(\alpha)^s e(-n\alpha) \, d\alpha \right| + C_1 q^{(s-k-2\epsilon)P} > q^{(s-k)P} \psi(qP)^{-1}.
\]

Suppose \( \psi(z) < C_2 z^\epsilon \) for some constant \( C_2 > 0 \). Then it follows that \( C_1 q^{(s-k-2\epsilon)P} < C_3 q^{(s-k-\epsilon)P} \psi(qP)^{-1} \) for some constant \( C_3 > 0 \). Now there exists \( M_0 > 0 \) such that \( C_3 q^{-\epsilon P} < 1/2 \) for all \( P \geq M_0 \). Therefore, for \( P \) sufficiently large we have that given any \( n \in \tilde{E}_{s,k}(N,\psi) \),

\[
(7.9) \quad \left| \int_m g(\alpha)^s e(-n\alpha) \, d\alpha \right| > \frac{1}{2} q^{(s-k)P} \psi(qP)^{-1}.
\]

Let \( E = \left| \tilde{E}_{s,k}(N,\psi) \right| \). Define the complex numbers \( \eta(n) \), depending on \( s \) and \( k \), for \( n \in \tilde{E}_{s,k}(N,\psi) \) by means of the equation

\[
\left| \int_m g(\alpha)^s e(-n\alpha) \, d\alpha \right| = \eta(n) \int_m g(\alpha)^s e(-n\alpha) \, d\alpha.
\]

Clearly, \( |\eta(n)| = 1 \) for all \( n \in \tilde{E}_{s,k}(N,\psi) \). Define the exponential sum \( K(\alpha) \) by

\[
(7.10) \quad K(\alpha) = \sum_{n \in \tilde{E}_{s,k}(N,\psi)} \eta(n) e(n\alpha).
\]

Then, it follows from (7.9) that for \( P \) sufficiently large

\[
(7.11) \quad \frac{1}{2} q^{(s-k)P} \psi(qP)^{-1}E < \sum_{n \in \tilde{E}_{s,k}(N,\psi)} \eta(n) \int_m g(\alpha)^s e(-n\alpha) \, d\alpha = \int_m g(\alpha)^s K(-\alpha) \, d\alpha.
\]

We apply Cauchy-Schwartz inequality to the right hand side of (7.11) to obtain

\[
(7.12) \quad \frac{1}{2} q^{(s-k)P} \psi(qP)^{-1}E < \left( \int_m |g(\alpha)|^{2s} \, d\alpha \right)^{1/2} \left( \int_m |K(-\alpha)|^2 \, d\alpha \right)^{1/2}.
\]

We note that we have established the above inequality (7.12) assuming \( s \geq 2k+1 \) here. The orthogonality relation (2.2) gives us

\[
(7.13) \quad \int \left| K(\alpha) \right|^2 \, d\alpha = \sum_{n \in \tilde{E}_{s,k}(N,\psi)} 1 = E.
\]

With this set up, we are ready to prove Theorems 1.5, 1.6, 7.1, and 7.2.

Proof of Theorems 1.5, 1.6, 7.1, and 7.2. Recall we defined \( X = P + 1 \). By Propositions 7.3 and 7.4 for \( s \geq u_2(k) \) we know there exists \( \delta_2 > 0 \) such that

\[
\left( \int_m |g(\alpha)|^{2s} \, d\alpha \right)^{1/2} \ll q^{(s-k/2-\delta_2/2)P}.
\]
Therefore, we can further bound the right hand side of (7.12) by the above inequality and (7.13), and obtain for \( s \geq \max\{u_2(k), 2k + 1\}, \)
\[
\frac{1}{2} q^{(s-k)P} \psi(q^P)^{-1} E^{1/2} < \left( \int_{m} |g(\alpha)|^{2s} d\alpha \right)^{1/2} \ll q^{(s-k-\delta_2/2)P},
\]
which simplifies to
\[
(7.14) \quad E \ll q^{(k-\delta_2)P} \psi(q^P)^2.
\]
Fix \( \epsilon > 0 \) sufficiently small and let \( \psi(z) \) be such that \( \psi(q^P) \ll q^{\epsilon P/2} \). Then we have by (7.14) that
\[
E \ll q^{(k-\delta_2+\epsilon)P} \ll q^{\ord n-(\delta_2-\epsilon)P} \ll q^{N-(\delta_2-\epsilon)N/2} = o(q^N).
\]
Therefore, we obtain \( \tilde{G}_q^+(k) \leq \max\{u_2(k), 2k + 1\} \). We then simplify \( u_2(k) \) via (2.21) to obtain the estimates given in the statement of Theorem 1.5. When \( k < p \), we see that \( u_2(k) \) given in (7.1) is identical to \( u_1(k) \) defined in [10]. Consequently, our estimates for \( \tilde{G}_q^+(k) \) when \( k < p \) are identical to the estimates of \( \tilde{G}_q^+(k) \) obtained in [10]. We have now completed the proof of Theorems 1.5 and 1.6.

Finally, to prove Theorems 7.1 and 7.2, we substitute (7.13) into (7.12), apply Corollary 4.2 or Corollary 4.4 (depending on \( k \) and \( p \)), and obtain for \( P \) sufficiently large
\[
\frac{1}{2} q^{(s-k)P} \psi(q^P)^{-1} E^{1/2} < \left( \int_{m} |g(\alpha)|^{2s} d\alpha \right)^{1/2} \ll q^{(s-k/2-\delta_0/2+\epsilon/2)P}.
\]
Rearranging the above inequality yields
\[
E \ll q^{(k-\delta_0+\epsilon)P} \psi(q^P)^2,
\]
as desired. \qed

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