Quantum resource theories (QRTs) provide a unified theoretical framework for understanding inherent quantum-mechanical properties that serve as resources in quantum information processing, but resources motivated by physics may possess intractable mathematical structure to analyze, such as non-uniqueness of maximally resourceful states, lack of convexity, and infinite dimension. We investigate state conversion and resource measures in general QRTs under minimal assumptions to figure out universal properties of physically motivated quantum resources that may have such intractable mathematical structure. In the general setting, we prove the existence of maximally resourceful states in one-shot state conversion. Also analyzing asymptotic state conversion, we discover catalytic replication of quantum resources, where a resource state is infinitely replicable by free operations. In QRTs without assuming uniqueness of maximally resourceful states, we formulate the tasks of distillation and formation of quantum resources, and introduce distillable resource and resource cost based on the distillation and the formation, respectively. Furthermore, we introduce consistent resource measures that quantify the amount of quantum resources without contradicting the rate of state conversion even in QRTs with non-unique maximally resourceful states. Progressing beyond the previous work showing a uniqueness theorem for additive resource measures, we prove the corresponding uniqueness inequality for the consistent resource measures; that is, consistent resource measures of a quantum state take values between the distillable resource and the resource cost of the state. These formulations and results establish a foundation of QRTs applicable to mathematically intractable but physically motivated quantum resources in a unified way.

I. INTRODUCTION

Advantages in quantum information processing compared to conventional classical information processing arise from various inherent properties of quantum states. A framework for systematically investigating quantum-mechanical properties is essential for better understandings of quantum mechanics and quantum information processing. Quantum resource theories (QRTs) give such a framework, in which the quantum properties are regarded as resources for overcoming restrictions on operations of quantum systems; especially, manipulation and quantification of resources are integral parts of QRTs. QRTs have covered numerous aspects of quantum properties such as entanglement [2–4], coherence [5–11], atthermality [12–14], magic states [15–16], asymmetry [17], purity [18], non-Gaussianity [19–22], and non-Markovianity [23–24]. Recently, QRTs for a general resource have been studied to figure out common structures shared among known QRTs and to understand the quantum properties systematically [23–28].

However, general QRTs are not necessarily mathematically tractable to analyze, and simply extending the formulation of a known QRT such as bipartite entanglement is insufficient. For example, maximal resources in the QRT of magic states [15] and the QRT of coherence with physically incoherent operations (PIO) [10] are not unique. Gaussian states [29, 30], quantum discord [31], and quantum Markov chain [32] are quantum-mechanical properties emerging in a non-convex quantum state space. Gaussian operations [29, 30] are conventionally defined on a non-convex state space while existing QRTs of non-Gaussianity [19–22] are formulated on convex state spaces. Furthermore, the state spaces of QRTs of non-Gaussianity are infinite-dimensional, and analysis of QRTs on finite-dimensional quantum systems is not necessarily applicable to infinite-dimensional systems. These physically motivated quantum properties are mathematically intractable to analyze.

To analyze general quantum properties including those shown in the previous paragraph, we investigate conversion and quantification of quantum resources in general QRTs that can be mathematically intractable but physically motivated. We do not make mathematical assumptions such as the existence of a unique maximal resource, a convex state space, and a finite-dimensional state space. In this paper, we take a position that free operations determine free states. A free operation is an element in a subset of quantum operations. The set of free operations describes what is freely capable when we operate a quantum system. A quantum state that may not be obtained by free operations is regarded as a resource state, while a quantum state freely obtained by free operations is called a free state. Convertibility of quantum states...
under free operations introduces a mathematical order of the states in terms of resourcefulness. A maximally resourceful state is a special resource state at the top of this ordering, regarded as a unit of the resource. The existence of maximally resourceful states is essential for quantifying quantum resources. Due to the generality of our formulation, the existence of a maximally resourceful state is not obvious, but we prove that a maximally resourceful state always exists by introducing compactness in our framework. Furthermore, we analyze the structure of the set of free states, and clarify a condition where a maximally resourceful state is not free.

To investigate manipulation of a quantum resource in general QRTs, we analyze one-shot and asymptotic state conversion in the general framework of QRTs, rather than specific resources. We discover a type of quantum resources with a counter-intuitive property, which is a resource state that is not free to generate but can be replicated infinitely by free operations using a given copy catalytically. While a catalytic conversion of quantum resources are originally found in the entanglement copy catalytically. While a catalytic conversion of quantum resources, our discovery provides another form of catalytic property of quantum resources. We call this resource state a catalytically replicable state. In addition, we formulate resource conversion tasks in general QRTs, namely, distillation and formation of a resource, and introduce general definitions of the distillable resource and the resource cost through these tasks, which generalize those defined for bipartite entanglement, coherence, and athermality. Formulation of the distillation and the formation of a resource is not straightforward when the QRT has non-unique maximally resourceful states. To overcome this issue, we formulate the distillable resource as how much resource can be extracted from the state in the worst-case scenario, and the resource cost as how much resource is needed to generate the state in the best-case scenario. Under this formulation, we identify a condition of the distillable resource being smaller than the resource cost.

A resource measure is a tool for quantifying resources. In the QRT of bipartite entanglement, it is known that a resource measure satisfying certain axioms given in Ref. [33] is lower-bounded by the distillable resource and upper-bounded by the resource cost, which we call the uniqueness inequality. In this paper, we show that the uniqueness inequality holds for a general QRT under the same axioms even in infinite-dimensional cases, but at the same time show that these axioms applicable to the QRT of bipartite entanglement are too strong to be satisfied in known QRTs such as magic states [15]. Motivated by this issue, we introduce a concept of consistent resource measures, which provide quantification of quantum resources without contradicting the rate of asymptotic state conversion. We prove that the uniqueness inequality also holds for the consistent resource measure and observe that this uniqueness inequality is more widely applicable than the uniqueness inequality previously proved through the axiomatic approach. Moreover, we show that the regularized relative entropy of resource serves as a consistent resource measure, generalizing the existing results in reversible QRTs [20].

These formulations and results establish a framework of general QRTs that are applicable even to mathematically intractable but physically motivated restrictions on quantum operations. Our results clarify the general structures of quantum resources, leading to a theoretical foundation for further understandings of quantum mechanical phenomena through a systematic approach based on QRTs.

The rest of this paper is organized as follows. In Sec. II we recall descriptions of infinite-dimensional quantum mechanics and provide a framework of general QRTs. In Sec. III we investigate maximally resourceful states and free states in general QRTs. In Sec. IV we analyze manipulation of quantum states in general QRTs, especially, asymptotic state conversion. In Sec. V we focus on the distillation of a resource from a quantum state and the formation of a quantum state from a resource, and prove the uniqueness inequality. In Sec. VI we investigate the quantification of a resource, introducing and analyzing a consistent resource measure. Our conclusion is given in Sec. VII.

II. PRELIMINARIES

We provide preliminaries to quantum resource theories (QRTs) that we analyze in this paper. In Sec. II A we present notations on describing quantum mechanics on infinite-dimensional quantum systems that QRTs in this paper cover. In Sec. II B we recall a formulation of QRTs. The readers who are interested in QRTs on finite-dimensional quantum systems and are familiar with finite-dimensional quantum mechanics can skip Sec. II A while we may use notions summarized in Sec. II A to show our result in Sec. III A.

A. Quantum Mechanics on Infinite-Dimensional Quantum Systems

We provide mathematical notations of quantum mechanics that cover infinite-dimensional quantum systems. Notice that some inherent properties of quantum mechanics, such as non-Gaussianity [19–22], are easier to formulate on an infinite-dimensional quantum system than its approximation by a finite-dimensional quantum system. As for proofs of mathematical facts that we use in the following, see, e.g., Refs. [39, 40].

To represent a (finite- and infinite-dimensional) quantum system, we use a complex Hilbert space $\mathcal{H}$, i.e., a complex inner product space that is also a complete metric space with respect to the distance function induced by the inner product. We represent a multipartite system as a tensor product of the Hilbert spaces representing its subsystems. We may write an orthonormal basis (i.e., a
complete orthonormal system) of $\mathcal{H}$ as
\[ B_{\mathcal{H}} := \{ |k\rangle \}_{k}. \] (1)

In cases where $\mathcal{H}$ represents a $D$-dimensional system, $B_{\mathcal{H}}$ is a finite set with cardinality $D$, while $B_{\mathcal{H}}$ can be an uncountable set in this paper.

We use a subset of operators on $\mathcal{H}$, in particular, the von Neumann algebra, to describe quantum mechanics on the system represented by $\mathcal{H}$, since $\mathcal{H}$ can be infinite-dimensional. Let $\mathcal{L}(\mathcal{H})$ denote the set of linear operators on $\mathcal{H}$. Let $B(\mathcal{H}) \subset \mathcal{L}(\mathcal{H})$ denote the set of bounded operators, that is, for any $A \in B(\mathcal{H})$, the operator norm $\|A\|_{\infty} := \sup \{ \|A|v\|_{\mathcal{H}} : |v\rangle \in \mathcal{H}, \|v\|_{\mathcal{H}} \leq 1 \}$ is bounded, where $\|\cdot\|_{\mathcal{H}}$ denotes the norm induced by the inner product of $\mathcal{H}$.

To define trace-class operators, we use the trace defined as $\text{Tr} T := \sum_{|k\rangle \in B_{\mathcal{H}}} \langle k| T |k\rangle$, where $B_{\mathcal{H}}$ is an orthonormal basis of $\mathcal{H}$ defined in (1), each term on the right-hand side denotes the inner product of $T |k\rangle$ and $|k\rangle$ on $\mathcal{H}$, and if $B_{\mathcal{H}}$ is an infinite set, the summation on the right-hand side means the limit of a net, i.e., a generalization of sequence. While a sequence $(a_{n} : n \in \mathbb{N})$ is indexed by a natural number, that is, a countably infinite and totally ordered set, a net $(a_{\alpha})$ is indexed by $\alpha$ in a directed set, a generalization of totally ordered sets, while this directed set can be uncountable. In the definition of the trace, the net is indexed by any finite subset $B'_{\mathcal{H}} \subset B_{\mathcal{H}}$ and defined as $\sum_{|k\rangle \in B'_{\mathcal{H}}} \langle k| T |k\rangle$, which approaches to the limit $\text{Tr} T$ as $B'_{\mathcal{H}}$ gets larger. Note that $\text{Tr} T$ is independent of the choice of $B_{\mathcal{H}}$. Let $T (\mathcal{H}) \subset B(\mathcal{H})$ denote the set of trace-class operators; that is, for any $T \in T (\mathcal{H})$, we have finite and hence well-defined $\text{Tr} |T\rangle$, where $|T\rangle := \sqrt{\text{Tr} T}$, and $T^\dagger$ denotes the conjugation of $T$. For any $T \in T (\mathcal{H})$, the trace norm of $T$ is defined as
\[ \|T\|_{1} := \text{Tr} |T\rangle . \] (2)

To define the von Neumann algebra, we need to discuss convergence of bounded operators mathematically. To discuss convergence of bounded operators, we need a topology defined for $B(\mathcal{H})$, and we use the ultraweak operator topology. The ultraweak operator topology of $B(\mathcal{H})$ is a topology where any sequence $A_{1}, A_{2}, \ldots \in B(\mathcal{H})$, or more generally, any net $A_{\alpha}$, converges to $A$ if and only if $\text{Tr} [T A_{\alpha}]$ converges to $\text{Tr} [T A]$ for any $T \in T (\mathcal{H})$. A von Neumann algebra $\mathcal{M}$ on $B(\mathcal{H})$ is a subset of $B(\mathcal{H})$ (or $B(\mathcal{H})$ itself) that contains the identity operator $\mathds{1}$ on $\mathcal{H}$, is closed under linear combination, product, and conjugation, and is also closed in terms of the ultraweak operator topology.

A noncommutative von Neumann algebra can be used for describing a quantum system, while a commutative von Neumann algebra for a classical system. To describe quantum mechanics on $\mathcal{H}$, we use a set of operators represented as a von Neumann algebra $\mathcal{M}$ on $B(\mathcal{H})$. For example, for any finite-dimensional Hilbert space $\mathcal{H}$, the algebra of all the linear operators $\mathcal{L}(\mathcal{H}) = B(\mathcal{H})$ is a von Neumann algebra, which suffices to describe the finite-dimensional quantum mechanics. More generally, for any $\mathcal{H}$ that can be infinite-dimensional, the algebra of all the bounded operators $B(\mathcal{H})$ is a von Neumann algebra.

In this paper, a system $\mathcal{H}$ is always accompanied with a set of operators $\mathcal{M}$ therein, where we may implicitly consider $\mathcal{M} = B(\mathcal{H})$ unless stated otherwise.

Given that a quantum state associates a measurement of an observable with a probability of a measurement outcome, we introduce a quantum state using a linear functional from an operator to a scalar. In particular, for a system $\mathcal{H}$ with $\mathcal{M}$, a state is defined as a linear functional $f_{\psi} : \mathcal{M} \rightarrow \mathbb{C}$ that is positive semidefinite $f_{\psi} (M^\dagger M) \geq 0$, $\forall M \in \mathcal{M}$, satisfies the normalization condition $f_{\psi} (\mathds{1}) = 1$, and is also normal, i.e., continuous in terms of the ultraweak operator topology. We require this continuity in order for $f_{\psi}$ to be in the predual $\mathcal{M}_{*}$ of $\mathcal{M}$, i.e., the space whose dual $(\mathcal{M}_{*})'$ equals (can be identified with) $\mathcal{M}$. Given this duality and under the condition of $\mathcal{M} = B(\mathcal{H})$, we identify $f_{\psi}$ with the operator $\psi \in T (\mathcal{H})$ that satisfies for any $M \in \mathcal{M}$
\[ \text{Tr} [\psi M] = f_{\psi} (M) . \] (3)

Note that we have one-to-one correspondence between $f_{\psi}$ and $\psi$ if $\mathcal{M} = B(\mathcal{H})$. This operator $\psi$ is the density operator representing the state $f_{\psi}$, and let $D (\mathcal{H}) := \{ \psi \in T (\mathcal{H}) : \psi \geq 0, \text{Tr} \psi = 1 \}$ denote the set of density operators on $\mathcal{H}$ with $\mathcal{M} = B(\mathcal{H})$. For simplicity, we may call $\psi$ a quantum state, rather than $f_{\psi}$.

We introduce a quantum channel on a system $\mathcal{H}$ with $\mathcal{M} = B(\mathcal{H})$ in the Heisenberg picture as a completely positive and unital linear map on $\mathcal{M}$, which correspondingly yields the definition of the channel in the Schrödinger picture as a completely positive and trace-preserving linear map of density operators on $\mathcal{H}$. Given two systems $\mathcal{H}^{(\text{in})}$ with $\mathcal{M}^{(\text{in})}$ and $\mathcal{H}^{(\text{out})}$ with $\mathcal{M}^{(\text{out})}$ representing the spaces of the input and the output respectively, a channel $\tilde{\mathcal{E}} : \mathcal{M}^{(\text{out})} \rightarrow \mathcal{M}^{(\text{in})}$ in the Heisenberg picture is defined as a linear map that is completely positive
\[ \sum_{j,k=0}^{n-1} M_{j}^{(\text{in})} \tilde{\mathcal{E}} \left( M_{j}^{(\text{out})} \right) M_{k}^{(\text{in})} \geq 0, \]
\[ \forall M_{0}^{(\text{in})}, \ldots, M_{n-1}^{(\text{in})} \in \mathcal{M}^{(\text{in})}, \]
\[ \forall M_{0}^{(\text{out})}, \ldots, M_{n-1}^{(\text{out})} \in \mathcal{M}^{(\text{out})}, \]
\[ \forall n \in \mathbb{N}, \] unital $\tilde{\mathcal{E}} (\mathds{1}^{(\text{out})}) = \mathds{1}^{(\text{in})}$, and normal, i.e., continuous in terms of the ultraweak operator topologies of $\mathcal{M}^{(\text{out})}$ and $\mathcal{M}^{(\text{in})}$, where $\mathds{1}^{(\text{out})}$ and $\mathds{1}^{(\text{in})}$ are the identity operators on $\mathcal{H}^{(\text{out})}$ and $\mathcal{H}^{(\text{in})}$, respectively. In the same way as the identification (3) of a functional $f_{\psi}$ of a state with the density operator $\psi$ of the state, under the conditions of $\mathcal{M}^{(\text{in})} = B(\mathcal{H}^{(\text{in})})$ and $\mathcal{M}^{(\text{out})} = B(\mathcal{H}^{(\text{out})})$, we identify a channel $\tilde{\mathcal{E}} : \mathcal{M}^{(\text{out})} \rightarrow \mathcal{M}^{(\text{in})}$ in the Heisenberg picture.
picture with the channel $\mathcal{E} : \mathcal{T}(\mathcal{H}^{(\text{in})}) \to \mathcal{T}(\mathcal{H}^{(\text{out})})$ in the Schrödinger picture that satisfies for any $M^{(\text{out})} \in \mathcal{M}^{(\text{out})}$

$$\text{Tr}\left[\mathcal{E}(\psi)M^{(\text{out})}\right] = \left(f_\psi \circ \tilde{\mathcal{E}}\right)\left(M^{(\text{out})}\right),$$

(5)

where $\psi$ and $f_\psi$ are related as $[\psi]$. Note that $\mathcal{E}$ is a completely positive and trace-preserving (CPTP) linear map by definition, and if $\mathcal{M}^{(\text{in})} = \mathcal{B}(\mathcal{H}^{(\text{in})})$ and $\mathcal{M}^{(\text{out})} = \mathcal{B}(\mathcal{H}^{(\text{out})})$, we have one-to-one correspondence between $\tilde{\mathcal{E}}$ and $\mathcal{E}$.

The set of channels from an input system $\mathcal{H}^{(\text{in})}$ with $\mathcal{M}^{(\text{in})}$ to an output system $\mathcal{H}^{(\text{out})}$ with $\mathcal{M}^{(\text{out})}$ is denoted by $\mathcal{C}(\mathcal{H}^{(\text{in})} \to \mathcal{H}^{(\text{out})})$, which we may write $\mathcal{C}(\mathcal{H})$ if $\mathcal{H} = \mathcal{H}^{(\text{in})} = \mathcal{H}^{(\text{out})}$. In this paper, we use the Schrödinger picture with $\mathcal{M}^{(\text{in})} = \mathcal{B}(\mathcal{H}^{(\text{in})})$ and $\mathcal{M}^{(\text{out})} = \mathcal{B}(\mathcal{H}^{(\text{out})})$; that is, $\mathcal{C}(\mathcal{H}^{(\text{in})} \to \mathcal{H}^{(\text{out})})$ is the set of the CPTP linear maps, while it would be possible to use the Heisenberg picture otherwise. We represent quantum operations as channels, while it is possible to include measurements in our formulation as channels from a quantum input system to a classical output system.

To discuss compactness of a set of states, we need further definitions of topologies for the set of states. A compact set in terms of some topology is a set where for any net that converges in terms of this topology, its limit point is in the set. Note that a compact set in a Hausdorff space is a closed set, which holds in this paper. Several different topologies can be defined for the set of states. The weak operator topology of $\mathcal{H}^{(\text{in})}$ is the strongest topology of $\mathcal{H}^{(\text{in})}$. Several different topologies can be defined for $\mathcal{H}^{(\text{out})}$, a map $\mathcal{S}_{\psi,M^{(\text{out})}} : \mathcal{C}(\mathcal{H}^{(\text{in})} \to \mathcal{H}^{(\text{out})}) \to \mathcal{C}$ given by $\mathcal{S}_{\psi,M^{(\text{out})}}(\mathcal{E}) = \left(f_\psi \circ \tilde{\mathcal{E}}\right)\left(M^{(\text{out})}\right)$ is continuous, where $\mathcal{E}$ and $\tilde{\mathcal{E}}$ are related as $[\psi]$. Note that if $\mathcal{H}^{(\text{in})}$ and $\mathcal{H}^{(\text{out})}$ are finite-dimensional, $\mathcal{C}(\mathcal{H}^{(\text{in})} \to \mathcal{H}^{(\text{out})})$ is compact, while $\mathcal{C}(\mathcal{H}^{(\text{in})} \to \mathcal{H}^{(\text{out})})$ for infinite-dimensional systems is not compact in terms of the BW topology.

**B. Framework of Quantum Resource Theories**

In this section, we provide a formulation of quantum resource theories (QRTs) starting from free operations with physically motivated assumptions. In the definition, we consider a compact set of the free operations. We also present justification of the compactness by examples from the perspective of indistinguishability. To represent the state set of interest in a QRT, e.g., the set of pure states on finite-dimensional $\mathcal{H}$, we consider a compact set of quantum states chosen as desired

$$\mathcal{S}(\mathcal{H}) \subseteq \mathcal{D}(\mathcal{H}).$$

(6)

Note that the quantum system $\mathcal{H}$ can be infinite-dimensional as we have introduced in Sec. II A.

A map contained in $\mathcal{O}(\mathcal{H}^{(\text{in})} \to \mathcal{H}^{(\text{out})})$ is called a free operation from $\mathcal{H}^{(\text{in})}$ to $\mathcal{H}^{(\text{out})}$. If the input space and output space are the same quantum system $\mathcal{H}$, we write the set of free operations from $\mathcal{H}$ to $\mathcal{H}$ as $\mathcal{O}(\mathcal{H}) \subseteq \mathcal{C}(\mathcal{H})$. We consider a compact set because two arbitrarily close CPTP maps are indistinguishable by any protocol in a task of channel discrimination [11], as we will discuss below by examples. We assume that $\mathcal{O}$ satisfies the following axioms of general QRTs:

1. Let $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$ be arbitrary quantum systems. For any $\mathcal{M} \in \mathcal{O}(\mathcal{H}_1 \to \mathcal{H}_2)$ and $\mathcal{N} \in \mathcal{O}(\mathcal{H}_2 \to \mathcal{H}_3)$, it holds that $\mathcal{N} \circ \mathcal{M} \in \mathcal{O}(\mathcal{H}_1 \to \mathcal{H}_3)$, where $\circ$ represents the composition.

2. Let $\mathcal{H}_1^{(\text{in})}$, $\mathcal{H}_1^{(\text{out})}$, $\mathcal{H}_2^{(\text{in})}$ and $\mathcal{H}_2^{(\text{out})}$ be arbitrary quantum systems. For any $\mathcal{M} \in \mathcal{O}(\mathcal{H}_1^{(\text{in})} \to \mathcal{H}_1^{(\text{out})})$ and $\mathcal{N} \in \mathcal{O}(\mathcal{H}_2^{(\text{in})} \to \mathcal{H}_2^{(\text{out})})$, it holds that $\mathcal{M} \otimes \mathcal{N} \in \mathcal{O}(\mathcal{H}_1^{(\text{in})} \otimes \mathcal{H}_2^{(\text{in})} \to \mathcal{H}_1^{(\text{out})} \otimes \mathcal{H}_2^{(\text{out})})$.

3. Let $\mathcal{H}$ be an arbitrary quantum system. Then, it holds that id $\in \mathcal{O}(\mathcal{H})$, where id is an identity map.
4. Let $\mathcal{H}$ be an arbitrary quantum system. Then, it holds that $\text{Tr} \in O(\mathcal{H} \rightarrow \mathbb{C})$, where $\text{Tr}$ is the trace. Note that due to the above conditions, it is necessary that the partial trace is also free.

The meanings of these axioms are as follows:

1. We always have access to free operations and can use free operations as many times as necessary.

2. We can arbitrarily apply free operations to a quantum system regardless of what free operations are applied to another quantum system.

3. Doing nothing is free.

4. Ignorance is free.

Remark 1 (QRTs Not Satisfying the Axioms). There can be classes of operations that do not satisfy the axioms stated above. For example, Ref. [24] considers $\epsilon$-resource non-generating operations. However, the composition of two $\epsilon$-resource non-generating operations is not necessarily an $\epsilon$-resource non-generating operation, which implies the set of $\epsilon$-resource non-generating operations does not satisfy the first axiom. Hence, we do not employ this class of operations as free operations. In addition, Refs. [42] and [43] consider separability preserving (SEPP) operations. However, the set of SEPP operations is not closed under tensor product, and hence does not satisfy the second axiom. We do not use these operations as free operations since they are not free to apply to multiple quantum systems simultaneously.

In the definition above, we use a compact set as the set of free operations. Some classes of operations that are conventionally used as free operations does not satisfy this compactness, such as local quantum operations and classical communication (LOCC) in the QRT of bipartite entanglement [41]. However, in this case, we take a position that the closure of LOCC, i.e., a compact superset of LOCC, can be considered to be free in the sense that any channel in the closure of LOCC is indistinguishable from a channel implementable in the setting of LOCC, as discussed in Example 1. In the same way, Example 2 shows that we conventionally consider any unitary transformation to be implementable by the Clifford+$T$ gate set in the sense that any unitary can be approximated with arbitrary precision by this gate set. Note that the compactness of the set of free operations is essential for guaranteeing the existence of maximally resourceful states as we will see in Sec. IIIA.

Example 1 (LOCC and Closure of LOCC). In the case of the QRT of entanglement, LOCC is conventionally considered to be physically implementable operations, but our formulation of QRTs may use the closure of LOCC in this case as a compact set of free operations instead of LOCC. In particular, let $O_{\text{LOCC}}(\mathcal{H}^{\text{in}} \rightarrow \mathcal{H}^{\text{out}})$ be the set of LOCC from $\mathcal{H}^{\text{in}}$ to $\mathcal{H}^{\text{out}}$. It is known that $O_{\text{LOCC}}(\mathbb{C}^4 \rightarrow \mathbb{C}^4)$ is not closed; that is, $O_{\text{LOCC}}(\mathbb{C}^4 \rightarrow \mathbb{C}^4) \neq O_{\text{LOCC}}(\mathbb{C}^4 \rightarrow \mathbb{C}^4)$ [44]. In this case, we use $O(\mathbb{C}^4 \rightarrow \mathbb{C}^4) = O_{\text{LOCC}}(\mathbb{C}^4 \rightarrow \mathbb{C}^4)$ as the set of free operation because for any CPTP map $\mathcal{N} \in O_{\text{LOCC}}(\mathbb{C}^4 \rightarrow \mathbb{C}^4)$, we can construct a CPTP map $\tilde{\mathcal{N}} \in O_{\text{LOCC}}(\mathbb{C}^4 \rightarrow \mathbb{C}^4)$ that is indistinguishable from $\mathcal{N}$ up to an $\epsilon$ probability by any protocol in a task of channel discrimination [41].

In the next example, we consider a situation of universal quantum computation where for any positive integer $d$, any $d$-depth quantum circuit composed of a universal gate set is implementable as the free operations.

Example 2. For any finite-dimensional quantum systems $\mathcal{H}^{\text{in}}$ and $\mathcal{H}^{\text{out}}$, we define $O'(\mathcal{H}^{\text{in}} \rightarrow \mathcal{H}^{\text{out}})$ as the set of CPTP maps that can be realized by a $d$-depth circuit composed of the identity gate, the partial trace, the Hadamard gate $H$, the controlled-NOT gate, the $\pi/8$ phase gate $T$, appending $\ket{0}$, and the measurement in the computational basis for any positive integer $d > 0$. Conventionally, combination of these operations can be considered to be universal because $O'(\mathcal{H}^{\text{in}} \rightarrow \mathcal{H}^{\text{out}})$ is dense in the set of all the channels $\mathcal{C}(\mathcal{H}^{\text{in}} \rightarrow \mathcal{H}^{\text{out}})$, but $O'(\mathcal{H}^{\text{in}} \rightarrow \mathcal{H}^{\text{out}})$ is different from $\mathcal{C}(\mathcal{H}^{\text{in}} \rightarrow \mathcal{H}^{\text{out}})$ [45]. In this case, our framework may use the closure of the set of operations implemented by all the $d$-depth circuits for any $d$ as the set of the free operations; that is, we take the set of free operations in this case as the set of all the channels $O(\mathcal{H}^{\text{in}} \rightarrow \mathcal{H}^{\text{out}}) = \mathcal{C}(\mathcal{H}^{\text{in}} \rightarrow \mathcal{H}^{\text{out}})$.

We do not assume the convexity of the set of free operations in our framework. Convex QRTs are a class of QRTs where the set of free operations is convex. For instance, the QRT of bipartite entanglement [44], the QRT of coherence [37] and the QRT of magic states [13] are known as convex QRTs. We can achieve a convex combination of operations using classical randomness. In general, randomness is regarded as a resource [46, 47], and randomness generation [48] is indeed a promising application of noisy intermediate quantum devices [49]. Therefore, we also consider non-convex QRTs in our framework, such as the following example.

Example 3 (Non-Convex QRT). The QRT of non-Markovianity [22, 23] is known as a non-convex QRT, where the set of free operations is not convex.

### III. Maximally Resourceful States and Free States

In this section, we analyze properties of maximally resourceful states and free states in general quantum resource theories (QRTs). In Sec. IIIA, we provide a mathematical order introduced by free operations, and prove the existence of maximally resourceful states in QRTs under our formulation in Sec. IIIB. In Sec. IIIIB, we provide the definition of free states, and give a condition under which a maximally resourceful state is not free.
A. Maximally Resourceful States

We analyze a mathematical order of resourcefulness of quantum states introduced by free operations. We also define maximally resourceful states in terms of this order. It is not trivial in general that the set of maximally resourceful states are not empty although it is desired for quantification of the resource. We prove the existence of maximally resourceful states in any QRT that satisfies the four axioms and the compactness given in Sec. II B.

For any quantum system $\mathcal{H}$, free operations introduce a preorder $\succeq$ on $S(\mathcal{H})$, which is a binary relation defined on $S(\mathcal{H})$ satisfying
\begin{align}
\phi \succeq \phi, \\
\phi \succeq \psi, \psi \succeq \sigma \Rightarrow \phi \succeq \sigma
\end{align}
for any states $\phi, \psi, \sigma \in S(\mathcal{H})$. The preorder is introduced in terms of the exact one-shot state conversion under free operations. Given two states $\phi, \psi \in S(\mathcal{H})$, the exact one-shot state conversion from $\phi$ to $\psi$ is a task of transforming a single $\phi$ exactly into a single $\psi$ by a free operation $N \in \mathcal{O}(\mathcal{H})$. Formally, we write
\begin{align}
\phi \succeq \psi
\end{align}
if there exists a free operation $N \in \mathcal{O}(\mathcal{H})$ such that
\begin{align}
N(\phi) = \psi.
\end{align}
With respect to this preorder, two states $\phi, \psi \in S(\mathcal{H})$ are said to be equivalent if both $\phi \succeq \psi$ and $\psi \succeq \phi$ hold. If $\phi$ and $\psi$ are equivalent, we write
\begin{align}
\phi \sim \psi.
\end{align}
The preorder also introduce maximal elements in the set of states. Given a quantum system $\mathcal{H}$, we let $\mathcal{G}(\mathcal{H})$ denote the set of the maximal states of $S(\mathcal{H})$ in terms of the preorder defined as $\mathcal{G}(\mathcal{H}) := \{\phi \in S(\mathcal{H}) : \forall \psi \in S(\mathcal{H}), \psi \succeq \phi \Rightarrow \psi \succeq \phi\}$. The elements of $\mathcal{G}(\mathcal{H})$ are called maximally resourceful states. Note that there may be several non-equivalent maximally resourceful states that are not comparable with each other. Here, we recall two QRTs that have two or more non-equivalent maximally resourceful states. The first one is the QRT of magic for qutrits $[15]$, which has two classes of maximally resourceful states.

Example 4 (QRT of Magic Has Non-Equivalent Maximally Resourceful States). In the QRT of magic for qutrits $[15]$, there exist two non-equivalent maximally resourceful states, which are called the Norrell state and the Strange state.

The second example is the QRT of coherent with physically incoherent operations (PIO) $[10]$, which has infinitely many non-equivalent maximally resourceful states.

Example 5 (QRT of Coherent with PIO Has Infinitely Many Non-Equivalent Maximally Resourceful States). In the QRT of coherent with PIO $[10]$, a free operation cannot change diagonal elements of a quantum state represented in the standard basis. Therefore, there exist infinitely many non-equivalent maximally resourceful states, which have different diagonal elements from each other.

We here prove that a maximally resourceful state always exists in any QRT satisfying axioms and the compactness of the set of states discussed in Sec. II B. Maximally resourceful states are regarded as a unit of the resource $[3, 35, 50]$. For example, in the QRT of bipartite entanglement, the amount of entanglement of the Bell state is defined as one ebit. Therefore, it is crucial for QRTs to have a maximally resourceful state. In general, whether a maximally resourceful state exists is not obvious. For example, a maximally entangled state does not necessarily exist in a QRT of bipartite entanglement for an infinite-dimensional system with a non-compact set of free operations such as LOCC while a unique maximally entangled state exists for a finite-dimensional system. Theorem I shows that for any given state, there exists a maximally resourceful state that is more resourceful than the state, which ensures the existence of maximally resourceful states in our framework.

Theorem 1 (Existence of a Maximally Resourceful States). Let $\mathcal{H}$ is a quantum system. For any state $\psi \in S(\mathcal{H})$, there exists a state $\phi \in \mathcal{G}(\mathcal{H})$ that upper-bounds $\psi$; that is, $\psi \preceq \phi$.

Proof. It is known that a compact space $X$ with a preorder $\succeq$ has a maximal element if the upper closure $U_x := \{y \in X | x \succeq y\}$ is closed for any $x \in X$ $[51]$ (Proposition VI-1.6.(i)). Thus, it suffices to show $U_x := \{\phi \in S(\mathcal{H}) | \psi \succeq \phi\}$ is weakly closed, or equivalently norm closed due to Lemma 27 in Appendix. We take a sequence $(\phi_n)$ in $U_x$ norm convergent to $\psi \in S(\mathcal{H})$ and prove $\psi \in U_x$. By the definition of the preorder $\succeq$, for each $n \in \mathbb{N}$, there exists a free operation $N_n \in \mathcal{O}(\mathcal{H})$ such that $\psi = N_n(\phi_n)$. By the BW-compactness of $\mathcal{O}(\mathcal{H})$, there exists a subnet $(N_{n(i)})_{i \in I}$ BW-convergent to some $N \in \mathcal{O}(\mathcal{H})$. In the following, we show $\psi = N(\phi)$ to prove the theorem.

Take an arbitrary $\epsilon > 0$ and an arbitrary $A \in B(\mathcal{H}) \setminus \{0\}$, which satisfies $\|A\|_{\infty} > 0$. By the norm compactness of $S(\mathcal{H})$, there exists a finite subset $\{\chi_k : k \in \{1, \ldots, N\}\}$ of $S(\mathcal{H})$ such that for any $\chi \in S(\mathcal{H})$
\begin{align}
\min_{k \in \{1, \ldots, N\}} \|\chi - \chi_k\|_1 < \frac{\epsilon}{\|A\|_{\infty}}.
\end{align}
By definition of the BW-convergence $N_{n(i)} \xrightarrow{BW} N$ in terms of $i$, there exists $i_{\epsilon, A}$ in $I$ such that for any $i \geq i_{\epsilon, A}$
\begin{align}
\max_{k \in \{1, \ldots, N\}} |\text{Tr}(N_{n(i)}(\chi_k) A) - \text{Tr}(N(\chi_k) A)| < \epsilon.
\end{align}
Thus, for any $i \geq i_{\epsilon, A}$ and any $\chi \in S(\mathcal{H})$, we have
\[
|\text{Tr} (\mathcal{N}_{n(i)} (\chi) A) - \text{Tr} (\mathcal{N} (\chi) A)|
= |\text{Tr} (\mathcal{N}_{n(i)} (\chi - \chi_k) A) + \text{Tr} (\mathcal{N}_{n(i)} (\chi_k) A) - \text{Tr} (\mathcal{N} (\chi - \chi_k) A) - \text{Tr} (\mathcal{N} (\chi_k) A)|
\leq |\text{Tr} (\mathcal{N}_{n(i)} (\chi - \chi_k) A)| + |\text{Tr} (\mathcal{N} (\chi - \chi_k) A)| + |\text{Tr} (\mathcal{N}_{n(i)} (\chi_k) A) - \text{Tr} (\mathcal{N} (\chi_k) A)| + \epsilon,
\]
(16)
where $\chi_k$ is an element in the finite subset $\{\chi_k\}$ of $S(\mathcal{H})$ in (14) satisfying
\[
\|\chi - \chi_k\|_1 < \frac{\epsilon}{\|A\|_\infty},
\]
(17)
and we use (15) in the last line. With
\[
\|\mathcal{N}_{n(i)}\|_\infty := \sup \{\|\mathcal{N}_{n(i)}(T)\|_1 : T \in \mathcal{T}(\mathcal{H}), \|T\|_1 \leq 1\}
\]
(18)
denoting the operator norm of the linear map $\mathcal{N}_{n(i)} : \mathcal{T}(\mathcal{H}) \rightarrow \mathcal{T}(\mathcal{H})$, we have
\[
|\text{Tr} (\mathcal{N}_{n(i)} (\chi - \chi_k) A)|
\leq \|\mathcal{N}_{n(i)} (\chi - \chi_k)\|_1 \cdot \|A\|_\infty
\leq \|\mathcal{N}_{n(i)}\|_\infty \cdot \|\chi - \chi_k\|_1 \cdot \|A\|_\infty
< 1 \cdot \frac{\epsilon}{\|A\|_\infty} \cdot \|A\|_\infty = \epsilon,
\]
(19)
where the last inequality follows from the fact that any CPTP map $\mathcal{N}_{n(i)}$ satisfies
\[
\|\mathcal{N}_{n(i)}\|_\infty = \sup \{\|\mathcal{N}_{n(i)}(T)\|_1 : \|T\|_1 \leq 1\} \leq 1.
\]
(20)
In the same way as (19) by substituting $\mathcal{N}_{n(i)}$ with $\mathcal{N}$, it holds that
\[
|\text{Tr} (\mathcal{N} (\chi - \chi_k) A)| < \epsilon.
\]
(21)
Therefore, applying (15) and (21) to (16), for any $i \geq i_{\epsilon, A}$ and any $\chi \in S(\mathcal{H})$, we have
\[
|\text{Tr} (\mathcal{N}_{n(i)} (\chi) A) - \text{Tr} (\mathcal{N} (\chi) A)| < \epsilon + \epsilon + \epsilon = 3\epsilon.
\]
(22)
Consequently, for any $i \geq i_{\epsilon, A}$, we obtain
\[
|\text{Tr} ((\psi - \mathcal{N} (\phi)) A)|
= |\text{Tr} (\psi A) - \text{Tr} (\mathcal{N} (\phi) A)|
\leq |\text{Tr} (\mathcal{N}_{n(i)} (\phi_{n(i)}) A) - \text{Tr} (\mathcal{N} (\phi) A)|
\leq |\text{Tr} (\mathcal{N}_{n(i)} (\phi_{n(i)}) A) - \text{Tr} (\mathcal{N} (\phi_{n(i)}) A)|
+ |\text{Tr} (\mathcal{N} (\phi_{n(i)}) A) - \text{Tr} (\mathcal{N} (\phi) A)|
\leq 3\epsilon + \|\mathcal{N}\|_\infty \cdot \|\phi_{n(i)} - \phi\|_1 \cdot \|A\|_\infty \rightarrow 3\epsilon,
\]
(23)
where the last inequality follows from (22) by substituting $\chi$ with $\phi_{n(i)}$ and from the inequality shown in the same way as (19)
\[
|\text{Tr} (\mathcal{N} (\phi_{n(i)}) A) - \text{Tr} (\mathcal{N} (\phi) A)|
\leq \|\mathcal{N}\|_\infty \cdot \|\phi_{n(i)} - \phi\|_1 \cdot \|A\|_\infty
\]
(24)
and the limit in the last line in terms of $i$ yields $\|\phi_{n(i)} - \phi\|_1 \rightarrow 0$. Since $\epsilon > 0$ and $a \in B(\mathcal{H}) \setminus \{0\}$ are arbitrary, this shows $\psi = \mathcal{N}(\phi)$.
Q.E.D.

Remark 2. In a similar manner, we can prove that the set of minimal elements
\[
\{\phi \in S(\mathcal{H}) : \forall \psi \in S(\mathcal{H}), \phi \supseteq \psi \Rightarrow \psi \supseteq \phi\}
\]
(25)
is not empty as well. Note that a state in this set may be different from free states in that it may not be prepared by a free operation. The set of minimal elements is considered as the set of the least resourceful states.

B. Free States

In this section, we analyze properties of free states. A free state is defined as a state that can be generated from any other state by a free operation. Let $F(\mathcal{H})$ denote the set of free states; that is,
\[
F(\mathcal{H}) := \left\{ \psi \in S(\mathcal{H}) : \forall \mathcal{H}', \forall \phi \in S(\mathcal{H}'), \exists \mathcal{N} \in \mathcal{O}(\mathcal{H}' \rightarrow \mathcal{H}) \text{ s.t. } \psi = \mathcal{N}(\phi) \right\}
\]
(26)
A state $\psi \in S(\mathcal{H}) \setminus F(\mathcal{H})$ that is not free is called a resourceful state or a resource state. Since $\text{Tr}$ is a free operation, the set of free states is equal to the set of states that can be generated from the scalar $1 \in S(\mathbb{C})$.

Proposition 2. Let $\mathcal{H}$ be a quantum system. Then, it holds that
\[
F(\mathcal{H}) = \left\{ \psi \in S(\mathcal{H}) : \exists \mathcal{N} \in \mathcal{O}(\mathcal{H} \rightarrow \mathcal{H}) \text{ s.t. } \psi = \mathcal{N}(1) \right\}
\]
(27)
Proof. By the definition (26) of $F(\mathcal{H})$, it trivially holds that
\[
F(\mathcal{H}) \subseteq \left\{ \psi \in S(\mathcal{H}) : \exists \mathcal{N} \in \mathcal{O}(\mathcal{H} \rightarrow \mathcal{H}) \text{ s.t. } \psi = \mathcal{N}(1) \right\}
\]
(28)
To show the converse inclusion, assume that
\[
\psi \in \left\{ \psi \in S(\mathcal{H}) : \exists \mathcal{N} \in \mathcal{O}(\mathcal{H} \rightarrow \mathcal{H}) \text{ s.t. } \psi = \mathcal{N}(1) \right\}
\]
(29)
Let $\mathcal{N} \in \mathcal{O}(\mathcal{H} \rightarrow \mathcal{H})$ be a free operation such that $\psi = \mathcal{N}(1)$. Consider an arbitrary quantum system $\mathcal{H}'$ and an arbitrary state $\phi \in S(\mathcal{H}')$. Since $\text{Tr} \in \mathcal{O}(\mathcal{H}' \rightarrow \mathbb{C})$, it holds that
\[
\psi = \mathcal{N} \circ \text{Tr}(\phi).
\]
(30)
Therefore, $\psi \in F(\mathcal{H})$, which yields the conclusion.
Q.E.D.
The set of free states $\mathcal{F}(\mathcal{H})$ may be empty for some $\mathcal{H}$ while the set of minimal elements defined in (25) is not empty as seen in Remark 2. For example, if the set of free operations $\mathcal{O}(\mathcal{C} \rightarrow \mathcal{H})$ does not contain any operation, then $\mathcal{F}(\mathcal{H}) = \emptyset$ for any $\mathcal{H}$. The following example gives a more concrete scenario, where we take the logical 2-dimensional space of the Gottesman-Kitaev-Preskill (GKP) code [52] as $\mathcal{S}(\mathbb{C}^2)$. In this paper, to investigate constraints and properties of QRTs in as general a setup as possible, we do not make any assumption on whether $\mathcal{F}(\mathcal{H})$ is empty or not.

**Example 6 (QRT of Non-Gaussianity on GKP Code).** The resource theory of non-Gaussianity has applications to analyzing continuous-variable quantum computation using the Gottesman-Kitaev-Preskill (GKP) code [22]. The GKP code encodes a qubit into an infinite-dimensional oscillator of an optical mode, and the logical 2-dimensional space can be defined by dividing the Hilbert space of the bosonic mode into a logical qubit and a gauge mode [53]. Gaussian operations [54] at a physical level suffice to implement logical Clifford gates for the GKP code [52]. Suppose that $\mathcal{S}(\mathbb{C}^2)$ is the set of logical states in the logical 2-dimensional space of the GKP code. Take the quantum operations on $\mathcal{S}(\mathbb{C}^2)$ implementable by the Gaussian operations as the free operations. Any physical state of the GKP code is non-Gaussian, and hence in this case, $\mathcal{F}$ only has the trivial element 1; that is, $\mathcal{F}(\mathcal{H}) = \{1\}$ if $\dim \mathcal{H} = 1$, and $\mathcal{F}(\mathcal{H}) = \emptyset$ otherwise.

The following proposition guarantees that a maximally resourceful state cannot be a free state if a resource state exists.

**Proposition 3.** Let $\mathcal{H}$ be a quantum system. Suppose that the set of resource state is not empty; that is $\mathcal{S}(\mathcal{H}) \setminus \mathcal{F}(\mathcal{H}) \neq \emptyset$. Then, it holds that

$$\mathcal{G}(\mathcal{H}) \cap \mathcal{F}(\mathcal{H}) = \emptyset. \quad (31)$$

**Proof.** The proof is by contradiction. To prove (31), assume that $\phi \in \mathcal{G}(\mathcal{H}) \cap \mathcal{F}(\mathcal{H})$. Take a resource state $\psi \in \mathcal{S}(\mathcal{H}) \setminus \mathcal{F}(\mathcal{H})$. (32)

Since $\phi \in \mathcal{F}(\mathcal{H})$, it holds that $\psi \succeq \phi$. Then, since $\phi \in \mathcal{G}(\mathcal{H})$, it holds that $\phi \succeq \psi$. Therefore, $\psi$ is also a free state; that is, $\psi \in \mathcal{F}(\mathcal{H})$, which contradicts (31).

Q.E.D.

We can observe that some properties of the set of free states $\mathcal{F}(\mathcal{H})$ are inherent in the set of the free operations $\mathcal{O}(\mathcal{H})$. The compactness of $\mathcal{O}(\mathcal{H})$ leads to the closed set of free states $\mathcal{F}(\mathcal{H})$. If $\mathcal{O}(\mathcal{H})$ is convex, $\mathcal{F}(\mathcal{H})$ is also convex.

**IV. ASYMPTOTIC STATE CONVERSION**

In this section, we characterize the asymptotic state conversion in general quantum resource theories (QRTs). Asymptotic state conversion gives a fundamental limit of large-scale quantum information processing exploiting quantum resources, and it has been widely discussed for known QRTs [1]. We provide a general definition of a state conversion rate in Sec. IV.B. In terms of the conversion rate, we find a class of resource that cannot be generated from any free state with any free operation but can be replicated infinitely by free operations. We call this state a catalytically replicable state. We give the definition and an example of catalytically replicable states in Sec. IV.B. In Sec. IV.C, we formulate relations between asymptotic state conversion and one-shot state conversion that hold in general QRTs, which may have catalytically replicable states. In the following, the ceiling function is denoted by $\lceil \cdots \rceil$, and the floor function is denoted by $\lfloor \cdots \rfloor$.

**A. Formulation of State Conversion Rate**

We recall the concept of asymptotic state conversion and provide possible two definitions of asymptotic state conversion rates. We show the equivalence of these two definitions.

For two quantum systems $\mathcal{H}_1$ and $\mathcal{H}_2$, and two quantum states $\phi \in \mathcal{S}(\mathcal{H}_1)$ and $\psi \in \mathcal{S}(\mathcal{H}_2)$, asymptotic state conversion from $\phi$ to $\psi$ is a task of transforming infinitely many copies of $\phi$ into as many copies of $\psi$ as possible by a sequence of free operations $\mathcal{N}_1, \mathcal{N}_2, \ldots$ within a vanishing error. There are two possible ways to define state conversion rates from $\phi$ to $\psi$: how many $\psi$’s can be generated from a single $\phi$, and how many $\phi$’s are necessary to generate a single $\psi$. We write the first conversion rate as $r_{\text{conv}}(\phi \rightarrow \psi)$, and the second conversion rate as $r'_{\text{conv}}(\phi \rightarrow \psi)$. As will be shown in Theorem 4, these two conversion rates are related to each other in such a way that $r'_{\text{conv}}(\phi \rightarrow \psi)$ is the inverse of $r_{\text{conv}}(\phi \rightarrow \psi)$. Therefore, we consider $r_{\text{conv}}(\phi \rightarrow \psi)$ as the asymptotic state conversion rate in this paper.

More formally, $r_{\text{conv}}(\phi \rightarrow \psi)$ is defined as follows. A set of asymptotic achievable rates is defined as

$$\mathcal{R}(\phi \rightarrow \psi) := \left\{ r \geq 0 : \exists \left( \mathcal{N}_n \in \mathcal{O}(\mathcal{H}_1^{\otimes n} \rightarrow \mathcal{H}_2^{\otimes \lceil rn \rceil}) : n \in \mathbb{N} \right) : \liminf_{n \to \infty} \left\| \mathcal{N}_n(\phi^{\otimes n}) - \psi^{\otimes \lceil rn \rceil} \right\|_1 = 0 \right\}, \quad (33)$$

where $\phi^{\otimes 0} := 1$. An asymptotic state conversion rate $r_{\text{conv}}(\phi \rightarrow \psi)$ is defined as

$$r_{\text{conv}}(\phi \rightarrow \psi) := \sup \mathcal{R}(\phi \rightarrow \psi). \quad (34)$$

Similarly, we give the other definition of a state conversion rate $r'_{\text{conv}}(\phi \rightarrow \psi)$. Here, we define another set of
Theorem 4

We choose a fixed positive real number \( r \) in the following theorem. Hereafter, we will use operation \( n \) to define the partial trace of \( n \) systems.

From (40) and (41), it holds that

\[
\lim_{n \to \infty} \left\| N_n^r \left( \phi \otimes [r n] \right) - \psi \otimes [n/r] \right\|_1 = 0.
\]

With respect to this definition of achievable rates, an asymptotic conversion rate \( r'_{\text{conv}}(\phi \to \psi) \) is defined as

\[
r'_{\text{conv}}(\phi \to \psi) := \inf \mathcal{R}'(\phi \to \psi),
\]

where \( r'_{\text{conv}}(\phi \to \psi) \) is infinity if the set on the right-hand side is empty. These two conversion rates \( r_{\text{conv}}(\phi \to \psi) \) and \( r'_{\text{conv}}(\phi \to \psi) \) are related to each other as shown in the following theorem. Hereafter, we will use \( r_{\text{conv}}(\phi \to \psi) \) as the asymptotic states conversion rate rather than \( r'_{\text{conv}}(\phi \to \psi) \).

**Theorem 4 (Relation Between Two Conversion Rates).** Let \( \mathcal{H} \) and \( \mathcal{H}' \) be quantum systems. For any states \( \phi \in \mathcal{S}(\mathcal{H}) \) and \( \psi \in \mathcal{S}(\mathcal{H}') \), it holds that

\[
r_{\text{conv}}(\phi \to \psi) = \frac{1}{r_{\text{conv}}(\phi \to \psi)},
\]

where we regard \( 1/0 = \infty \).

**Proof.** It suffices to show that

\[
r \in \mathcal{R}(\phi \to \psi) \Rightarrow \frac{1}{r} \in \mathcal{R}'(\phi \to \psi),
\]

and that

\[
r \in \mathcal{R}'(\phi \to \psi) \Rightarrow \frac{1}{r} \in \mathcal{R}(\phi \to \psi).
\]

First, assume that \( r \in \mathcal{R}(\phi \to \psi) \) to show (38). Choose a fixed positive real number \( \epsilon > 0 \). Let \( n \) be an arbitrary positive integer such that

\[
\left\| N_n^r \left( \phi \otimes [rn] \right) - \psi \otimes [n/r] \right\|_1 < \epsilon.
\]

Let \( n' = [rn] \). Because \( n \leq [n'/r] \), we can define a free operation \( N_{n'}^r \) as the partial trace of \( [n'/r] - n \) systems so that

\[
\mathcal{M}_{n'} \left( \phi \otimes [n'/r] \right) = \phi \otimes n.
\]

From (40) and (41), it holds that

\[
\left\| N_n \circ N_{n'}^r \left( \phi \otimes [n'/r] \right) - \psi \otimes [n/r] \right\|_1 < \epsilon,
\]

and \( 1/r \in \mathcal{R}'(\phi \to \psi) \) follows.

On the other hand, assume that \( r \in \mathcal{R}'(\phi \to \psi) \) to show (39). Choose a fixed positive real number \( \epsilon > 0 \). Let \( n \) be an arbitrary positive integer such that

\[
\left\| N_n \left( \phi \otimes [rn] \right) - \psi \otimes [n/r] \right\|_1 < \epsilon.
\]

Finally, we recall a useful relation of state conversion rates given in Ref. [52]. Let \( \mathcal{H}_1, \mathcal{H}_2 \) and \( \mathcal{H}_3 \) be quantum systems. Let \( \rho \in \mathcal{S}(\mathcal{H}_1), \sigma \in \mathcal{S}(\mathcal{H}_2) \) and \( \omega \in \mathcal{S}(\mathcal{H}_3) \) be quantum states. Suppose that we first asymptotically generate \( \sigma \) from \( \rho \), then we generate \( \omega \) from \( \sigma \) to achieve conversion from \( \rho \) to \( \omega \). While this protocol generate \( \omega \) from \( \rho \), the protocol is not necessarily optimal. In fact, it is known that

\[
r_{\text{conv}}(\rho \to \omega) \geq r_{\text{conv}}(\rho \to \sigma) r_{\text{conv}}(\sigma \to \omega).
\]
Theorem 5 (Replication of State). Let $\mathcal{H}$ be a quantum system. For any state $\psi \in \mathcal{S}(\mathcal{H})$, $r_{\text{conv}}(\psi \rightarrow \psi)$ is equal to either 1 or $+\infty$.

Proof. It is trivially holds that $r_{\text{conv}}(\psi \rightarrow \psi) \geq 1$ because $id \in \mathcal{O}(\mathcal{H})$.

Assume that $r_{\text{conv}}(\psi \rightarrow \psi) > 1$, that is, there exists $r > 1$ such that $r \in \mathcal{R}(\psi \rightarrow \psi)$. To prove $r_{\text{conv}}(\psi \rightarrow \psi) = \infty$, it suffices to show that

$$2r - 1 \in \mathcal{R}(\psi \rightarrow \psi)$$

(48)

because if $r > 1$ holds, an arbitrarily large rate can be achieved by exploiting (48) repeatedly.

Choose a fixed positive real number $\epsilon$. There exists a sequence of free operations $(\mathcal{N}_n \in \mathcal{O}(\mathcal{H}^{\otimes n} \rightarrow \mathcal{H}^{\otimes [rn]}) : n \in \mathbb{N})$ such that

$$\left\| \mathcal{N}_n(\psi^{\otimes n}) - \psi^{\otimes [rn]} \right\|_1 < \frac{\epsilon}{2}$$

(49)

holds for an infinitely large subset of $\mathbb{N}$.

Define

$$r_0 := \inf_{n} r_n,$$

(50)

where

$$r_n := \max \{ r' \geq 0 : [r'n] = 2[rn] - n \},$$

(51)

and the infimum is taken over $n$ satisfying (49).

For $n$ satisfying (49), it holds that

$$r_n = 2r - 1 + \frac{2\alpha_n}{n},$$

(52)

where $\alpha_n$ is a real number satisfying $0 \leq \alpha_n < 1$ and $[rn] = rn + \alpha_n$. The term $2\alpha_n/n$ approaches to zero as $n$ approaches to infinity. Then, it holds that

$$\inf_{n} r_n = \inf_{n} \left\{ 2r - 1 + \frac{2\alpha_n}{n} \right\} = 2r - 1.$$

(53)

Therefore, it holds that

$$r_0 = 2r - 1.$$ 

(54)

Then, due to (48) and (54), it suffices to show that

$$r_0 \in \mathcal{R}(\psi \rightarrow \psi).$$

(55)

Now, observe that for any $n$,

$$[rn] > n$$

(56)

always holds. This implies that for any $n$,

$$2[rn] - n > [rn]$$

(57)

holds.

For $n$ satisfying (49) and (56), we define a free operation $\mathcal{M}_n$ as the partial trace of $(2[rn] - n) - [rn]$ systems so that we can obtain

$$\mathcal{M}_n(\psi^{\otimes [2rn] - n}) = \psi^{\otimes [rn]}.$$ 

(58)

Therefore, from the triangle inequality, it follows that

$$\left\| \mathcal{M}_n \circ (\mathcal{N}_n \otimes id) \circ \mathcal{N}_n (\psi^{\otimes n}) - \psi^{\otimes [rn]} \right\|_1 \leq \left\| \mathcal{M}_n \circ (\mathcal{N}_n \otimes id) \circ \mathcal{N}_n (\psi^{\otimes n}) \right\|_1$$

$$- \mathcal{M}_n \circ (\mathcal{N}_n \otimes id) \left( \psi^{\otimes [rn]} \right) \right\|_1 + \left\| \mathcal{M}_n \circ (\mathcal{N}_n \otimes id) (\psi^{\otimes [rn]}) - \psi^{\otimes [rn]} \right\|_1,$$

(59)

where id is the identity map on $[rn] - n$ systems. Since the trace distance is non-increasing for quantum operations, it holds that

$$\left\| \mathcal{M}_n \circ (\mathcal{N}_n \otimes id) \circ \mathcal{N}_n (\psi^{\otimes n}) \right\|_1$$

$$- \mathcal{M}_n \circ (\mathcal{N}_n \otimes id) \left( \psi^{\otimes [rn]} \right) \right\|_1 \leq \left\| \mathcal{N}_n (\psi^{\otimes n}) - \psi^{\otimes [rn]} \right\|_1,$$

(60)

and that

$$\left\| \mathcal{M}_n \circ (\mathcal{N}_n \otimes id) \left( \psi^{\otimes [rn]} \right) - \psi^{\otimes [rn]} \right\|_1$$

$$\leq \left\| (\mathcal{N}_n \otimes id) \left( \psi^{\otimes (n + [rn]-n)} \right) - \psi^{\otimes (2[rn]-n)} \right\|_1$$

$$= \left\| \mathcal{N}_n (\psi^{\otimes n}) - \psi^{\otimes [rn]} \right\|_1$$

$$= \left\| \mathcal{N}_n (\psi^{\otimes n}) - \psi^{\otimes [rn]} \right\|_1.$$ 

(61)

Therefore, by (59), (60), and (61), we obtain

$$\left\| \mathcal{M}_n \circ (\mathcal{N}_n \otimes id) \circ \mathcal{N}_n (\psi^{\otimes n}) - \psi^{\otimes [rn]} \right\|_1 \leq \left\| \mathcal{N}_n (\psi^{\otimes n}) - \psi^{\otimes [rn]} \right\|_1,$$

(62)

which implies that $r_0 \in \mathcal{R}(\psi \rightarrow \psi).$ Q.E.D.

Remarkably, we here give an example where $r_{\text{conv}}(\psi \rightarrow \psi) = \infty$, but $\psi$ is not a free state, that is, $\psi \notin \mathcal{F}(\mathcal{H})$. In this paper, we call a state $\psi$ that satisfies $r_{\text{conv}}(\psi \rightarrow \psi) = \infty$ and $\psi \notin \mathcal{F}(\mathcal{H})$ a catalytically replicable state. A catalytically replicable state is regarded as a form of catalytic property of quantum resources, which are similar to catalytic state conversion in the entanglement theory. Any free state $\psi$ is a trivial example of $r_{\text{conv}}(\psi \rightarrow \psi) = \infty$, but the following example shows that this is not the whole story; that is, $r_{\text{conv}}(\psi \rightarrow \psi) = \infty$ implies that $\psi$ is free or catalytically replicable.
Example 7 (Catalytically Replicable Resource). Suppose that $\mathcal{S}(\mathbb{C}^2) = \{0\} \cup \{1\}$. Further suppose that the set of free operations $\mathcal{O}$ consists of operations that are realized by circuits composed of the identity gate, the partial trace, the controlled-NOT gate, the preparation of an auxiliary qubit in $|0\rangle$ state. For any integer $n \geq 0$, the set of free states is

$$\mathcal{F}((\mathbb{C}^2)^{\otimes n}) = \{|0\rangle \otimes \ldots \otimes |0\rangle\}.$$  

(63)

In this case, whereas $|1\rangle \otimes |1\rangle \notin \mathcal{F}(\mathbb{C}^2)$, $r_{\text{conv}}(|1\rangle \langle 1|) = +\infty$ because we can convert $|1\rangle \langle 1|$ into $|1\rangle \langle 1|^{\otimes n}$ for any $n$ by appending an auxiliary system prepared in $|0\rangle$ and applying controlled-NOT repeatedly.

C. Relations Between One-Shot State Conversion and Asymptotic State Conversion

In this section, we analyze relations between the asymptotic state conversion and the exact one-shot state conversion. The asymptotic state conversion from $\phi$ to $\psi$ is a task transforming infinitely many copies of $\phi$ into many copies of $\psi$ with a vanishing error, while the exact one-shot conversion $\phi$ to $\psi$ is a task transforming a single $\phi$ into a single $\psi$ exactly.

We prove two propositions both of which give relations between asymptotic state conversion and one-shot state conversion. The first proposition provides the relation that holds for inequivalent states. On the other hand, the second proposition characterizes the asymptotic conversion rate between two equivalent states. Firstly, the following proposition shows that the more resourceful a state is, the harder it is to distill the state and the easier it is to form another state from the state.

Proposition 6. Let $\mathcal{H}_1, \mathcal{H}_2$ be quantum systems. Let $\phi, \psi \in \mathcal{S}(\mathcal{H}_1)$ and $\rho \in \mathcal{S}(\mathcal{H}_2)$ be quantum states. If $\phi \succeq \psi$, then it holds that

$$r_{\text{conv}}(\rho \to \phi) \leq r_{\text{conv}}(\rho \to \psi).$$  

(64)

$$r_{\text{conv}}(\psi \to \rho) \leq r_{\text{conv}}(\phi \to \rho).$$  

(65)

Proof. To prove (64), it suffices to show that $\mathcal{R}(\rho \to \phi) \subseteq \mathcal{R}(\rho \to \psi)$. Suppose that $r \in \mathcal{R}(\rho \to \phi)$. Then, there exists a sequence of free operations $\{N_n \in \mathcal{O}(\mathcal{H}_2^{\otimes n} \to \mathcal{H}_1^{\otimes [rn]}): n \in \mathbb{N}\}$ such that for arbitrary $\epsilon > 0$,

$$\|N_n(\rho^{\otimes n}) - \phi^{\otimes [rn]}\|_1 < \epsilon$$  

(66)

holds for an infinitely large subset of $\mathbb{N}$. As $\phi \succeq \psi$, there exists a free operation $\mathcal{N}$ such that

$$\mathcal{N}(\phi) = \psi.$$  

(67)

Define a sequence of free operations $\{M_n \in \mathcal{O}(\mathcal{H}_2^{\otimes n} \to \mathcal{H}_1^{\otimes [rn]}): n \in \mathbb{N}\}$ as

$$M_n := N_{\otimes [rn]} \circ N_n.$$  

(68)

Then, for any $n$ satisfying (66),

$$\|M_n(\rho^{\otimes n}) - \psi^{\otimes [rn]}\|_1 \leq \|N_{\otimes [rn]}(\rho^{\otimes n}) - \phi^{\otimes [rn]}\|_1 < \epsilon$$  

(69)

holds, and this implies that $r \in \mathcal{R}(\rho \to \psi)$.

Next, we investigate the other relation between the asymptotic conversion and the exact one-shot conversion, i.e., the asymptotic state conversion between two equivalent states. Asymptotically, we may achieve conversion between states that are not convertible to each other in one-shot state conversion. One may wonder whether we can achieve a better asymptotic conversion rate between states that are equivalent under one-shot conversion. The following proposition shows that the asymptotic conversion rate for two equivalent states is equal to 1 in a QRT without catalytically replicable states, which implies that we cannot do better asymptotically than in one-shot conversion for any state $\psi$ except catalytically replicable states and free states; that is, $r_{\text{conv}}(\psi \to \psi) = 1$ as shown in Theorem 5.

Next, we prove (65). Note that (65) is equivalent to $r_{\text{conv}}(\phi \to \rho) \leq r_{\text{conv}}(\psi \to \rho)$ because of Theorem 1. It suffices to show that $\mathcal{R}'(\psi \to \rho) \subseteq \mathcal{R}'(\phi \to \rho)$. Suppose that $r \in \mathcal{R}'(\psi \to \rho)$. Then, there exists a sequence of free operations $\{N'_n \in \mathcal{O}(\mathcal{H}_1^{\otimes n} \to \mathcal{H}_2^{\otimes [rn]}): n \in \mathbb{N}\}$ such that for arbitrary $\epsilon > 0$,

$$\|N'_n(\psi^{\otimes [rn]}) - \rho^{\otimes n}\|_1 < \epsilon$$  

(70)

holds for an infinitely large subset of $\mathbb{N}$. As $\phi \succeq \psi$, there exists an free operation $\mathcal{N}'$ such that

$$\mathcal{N}'(\phi) = \psi.$$  

(71)

Define a sequence of free operations $\{M'_n \in \mathcal{O}(\mathcal{H}_1^{\otimes n} \to \mathcal{H}_2^{\otimes [rn]}): n \in \mathbb{N}\}$ as

$$M'_n := N'_n \circ N_n.$$  

(72)

Then, for any $n$ satisfying (70),

$$\|M'_n(\phi^{\otimes [rn]}) - \rho^{\otimes n}\|_1 \leq \|N'_n(\psi^{\otimes [rn]}) - \rho^{\otimes n}\|_1 < \epsilon$$  

(73)

holds and this implies that $r \in \mathcal{R}'(\phi \to \rho)$.

Q.E.D.
Proposition 7. Let \( \mathcal{H} \) be a quantum system. Let \( \psi, \phi \in \mathcal{S}(\mathcal{H}) \) be quantum states such that \( \psi \sim \phi \). Suppose that \( r_{\text{conv}}(\psi \rightarrow \phi) = 1 \). Then, it holds that
\[
 r_{\text{conv}}(\psi \rightarrow \phi) = r_{\text{conv}}(\phi \rightarrow \psi) = 1. \tag{74}
\]

Proof. Since \( r_{\text{conv}}(\psi \rightarrow \psi) = 1 \), it follows that
\[
 r_{\text{conv}}(\psi \rightarrow \phi) r_{\text{conv}}(\phi \rightarrow \psi) \leq r_{\text{conv}}(\psi \rightarrow \psi) = 1 \tag{75}
\]
from (10). On the other hand, since \( \psi \sim \phi \), there exist free operations \( \mathcal{M} \in \mathcal{O}(\mathcal{H}) \) and \( \mathcal{N} \in \mathcal{O}(\mathcal{H}) \) such that
\[
 \mathcal{M}(\psi) = \phi \tag{76}
\]
\[
 \mathcal{N}(\phi) = \psi, \tag{77}
\]
which implies that
\[
 r_{\text{conv}}(\psi \rightarrow \phi) \geq 1 \tag{78}
\]
\[
 r_{\text{conv}}(\phi \rightarrow \psi) \geq 1. \tag{79}
\]
Therefore, both \( r_{\text{conv}}(\psi \rightarrow \phi) \) and \( r_{\text{conv}}(\phi \rightarrow \psi) \) must be equal to 1. Q.E.D.

V. DISTILLABLE RESOURCE AND RESOURCE COST

In this section, we analyze properties of the distillable resource \( R_D \) and the resource cost \( R_C \), which represent how much resource can be extracted from a state and how much resource is needed to generate a state respectively. As noted in the Sec. IIIA, maximally resourceful states are not necessarily unique in general quantum resource theories (QRTs). In Sec. V A we define the distillable resource \( R_D \) as how many resourceful states can be generated from a state in the worst-case scenario, and we define the resource cost \( R_C \) as how many resourceful states are needed to generate a state in the best-case scenario. Our definition formulates distillation and formation of a resource even in a case where maximally resource states are not unique. In Sec. V B we analyze distillation and formation of catalytically replicable states. In Sec. V C we prove weak subadditivity of the distillable resource and the resource cost. In Sec. V D we further investigate the resource cost, and prove that an upper bound of the resource cost is achievable by a maximally resourceful state if the number of non-equivalent maximally resourceful states is finite. In Sec. V E generalizing the fact that the distillable entanglement is always smaller than the entanglement cost \( E \), we prove that the same inequality holds in general if there is no catalytically replicable state in the QRT, which is applicable to any QRT formulated in Sec. III.

A. Definitions of Resource Cost and Distillable Resource

In this section, we provide a formulation of distillation and formation of a resource, and give the definitions of the distillable resource and the resource cost, generalizing those in known QRTs such as bipartite entanglement \(35,36,37\), coherence \(37\), and thermality \(13\). In contrast with the definition in these previous works, our definitions are applicable to general QRTs, where maximally resourceful states are not necessarily unique. Our definition of the distillable resource represents how much resource can be generated in the worst case, and the definition of the resource cost represents how much resource is needed to form a state in the best case.

Our formulation of distillation and formation of a resource is as follows. For a quantum system \( \mathcal{H} \) and a state \( \psi \in \mathcal{S}(\mathcal{H}) \), distillation from the state \( \psi \) is a task of extracting many copies of state \( \phi \in \mathcal{S}(\mathcal{H}) \) from many copies of \( \psi \), where \( \phi \) is a state that is the most difficult to generate from \( \psi \). More formally, distillation is regarded as state conversion from \( \psi \) to a state \( \phi \) for which \( r_{\text{conv}}(\psi \rightarrow \phi) \) takes a minimum value. Similarly, formation of \( \psi \) is a task of generating many copies of \( \psi \) from many copies of state \( \phi \in \mathcal{S}(\mathcal{H}) \) where \( \phi \) is a state that can the most easily generate \( \psi \). Formation is regarded as state conversion from \( \phi \) to a state \( \psi \), where \( r_{\text{conv}}(\phi \rightarrow \psi) \) takes a maximum value for \( \phi \).

The distillable resource \( R_D \) represents the amount of resource obtained by distillation; the resource cost \( R_C \) represents the amount of resource needed for formation of a state. Formally, the distillable resource of any state \( \psi \in \mathcal{S}(\mathcal{H}) \) is defined as
\[
 R_D(\psi) := \inf_{\phi \in \mathcal{S}(\mathcal{H})} \left\{ r_{\text{conv}}(\psi \rightarrow \phi) R^{(\mathcal{H})}_{\text{max}} \right\}, \tag{80}
\]
where \( R^{(\mathcal{H})}_{\text{max}} \) is a normalization constant, which represents the maximum amount of a resource in \( \mathcal{S}(\mathcal{H}) \). If the dimension of \( \mathcal{H} \) is finite, we typically take \( R^{(\mathcal{H})}_{\text{max}} \) as the required number of qubits for representing the system \( \mathcal{H} \); that is,
\[
 R^{(\mathcal{H})}_{\text{max}} = \log_2(\dim \mathcal{H}), \tag{81}
\]
where \( \dim \mathcal{H} \) denotes the dimension of \( \mathcal{H} \). Similarly, the resource cost of any state \( \psi \in \mathcal{S}(\mathcal{H}) \) is defined as
\[
 R_C(\psi) := \inf_{\phi \in \mathcal{S}(\mathcal{H})} \left\{ \frac{R^{(\mathcal{H})}_{\text{max}}}{r_{\text{conv}}(\phi \rightarrow \psi)} \right\}. \tag{82}
\]

In the QRT of bipartite entanglement, \( R_D \) and \( R_C \) reduce to the distillable entanglement and the entanglement cost, respectively.

We obtain the following proposition for QRTs without catalytically replicable states, while QRTs with catalytically replicable states will be discussed in the next subsection. This proposition provides general bounds of the distillable resource and the resource cost, and we will also analyze achievability of the bound of the resource cost in Sec. V D.

Proposition 8. Let \( \mathcal{H} \) be a quantum system, and let \( \psi \in \mathcal{S}(\mathcal{H}) \) be a state. Suppose that \( \mathcal{S}(\mathcal{H}) \setminus \mathcal{F}(\mathcal{H}) \neq \emptyset \). If
ψ is not a catalytically replicable state, it holds that
\[ 0 \leq R_D (\psi) \leq R^{(H)}_{\text{max}}, \]
\[ 0 \leq R_C (\psi) \leq R^{(H)}_{\text{max}}. \]  
(83)
(84)

Especially, if ψ is a free state, it holds that
\[ R_D (\psi) = 0, \]
\[ R_C (\psi) = 0. \]  
(85)
(86)

Proof. Note that by the definitions [80] of \( R_D \) and \( R_C \), \( 0 \leq R_D (\psi) \) and \( 0 \leq R_C (\psi) \) trivially hold. First, we prove the statement for a free state. Let \( \psi \in \mathcal{F}(\mathcal{H}) \) be a free state. Since the set of free states is closed, for any resource state \( \phi \in \mathcal{F}(\mathcal{H}) \), it holds that
\[ r_{\text{conv}} (\psi \rightarrow \phi) = 0. \]  
(87)

Therefore, it holds that
\[ 0 \leq R_D (\psi) \leq R^{(H)}_{\text{max}} R_{\text{conv}} (\psi \rightarrow \phi) = 0, \]  
(88)
which shows \( R_D (\psi) = 0 \). On the other hand, by Proposition 2 there exists a free operation \( N \in \mathcal{O}(\mathcal{C} \rightarrow \mathcal{H}) \) such that \( N (1) = \psi \). Therefore, it holds that
\[ N^{\otimes n} (1) = \psi^{\otimes n} \]  
(89)
for any positive integer \( n \), which implies that
\[ r_{\text{conv}} (1 \rightarrow \psi) = \infty. \]  
(90)

Therefore, it holds that
\[ 0 \leq R_C (\psi) \leq \frac{R^{(H)}_{\text{max}}}{r_{\text{conv}} (1 \rightarrow \psi)} = 0, \]  
(91)
which shows \( R_C (\psi) = 0 \).

Next, we prove [81] for a resource state \( \psi \in \mathcal{S}(\mathcal{H}) \setminus \mathcal{F}(\mathcal{H}) \). We have \( r_{\text{conv}} (\psi \rightarrow \psi) = 1 \) because \( \psi \) is not a catalytically replicable state. Then, from [80], we obtain
\[ R_D (\psi) \leq R^{(H)}_{\text{max}} r_{\text{conv}} (\psi \rightarrow \phi) = R^{(H)}_{\text{max}}. \]  
(92)
We can show [83] by replacing \( R_D \) with \( R_C \) and \( r_{\text{conv}} (\psi \rightarrow \phi) \) with \( 1/r_{\text{conv}} (\psi \rightarrow \phi) \) respectively in the proof of [80]. Q.E.D.

In fact, from the relation between the preorder introduced by the free operations and the asymptotic conversion rate shown in Proposition 6, we obtain the following theorem, which shows that it is sufficient to take the infimum over the maximally resourceful states in the definitions of \( R_D \) and \( R_C \), rather than the infimum over the whose set of states.

**Theorem 9** (Maximally Resourceful States are Sufficient for Distillable Resource and Resource Cost). Let \( \psi \in \mathcal{S}(\mathcal{H}) \) be an arbitrary state. It is sufficient to consider \( \mathcal{G}(\mathcal{H}) \) instead of \( \mathcal{S}(\mathcal{H}) \) when we take the infimum in the definitions [80] and [82] of \( R_D \) and \( R_C \); that is, it holds that
\[ R_D (\psi) = \inf_{\phi \in \mathcal{G}(\mathcal{H})} \left\{ r_{\text{conv}} (\psi \rightarrow \phi) R^{(H)}_{\text{max}} \right\}, \]  
(93)
\[ R_C (\psi) = \inf_{\phi \in \mathcal{G}(\mathcal{H})} \left\{ \frac{R^{(H)}_{\text{max}}}{r_{\text{conv}} (\phi \rightarrow \psi)} \right\}. \]  
(94)

Proof. To show [91], it suffices to show that
\[ R_D (\psi) = \inf_{\phi \in \mathcal{G}(\mathcal{H})} \left\{ r_{\text{conv}} (\phi \rightarrow \psi) R^{(H)}_{\text{max}} \right\}. \]  
(95)
By Proposition 6 and Theorem 1, for any state \( \rho \in \mathcal{S}(\mathcal{H}) \), there always exists a maximal state \( \phi \in \mathcal{G}(\mathcal{H}) \) such that
\[ r_{\text{conv}} (\phi \rightarrow \psi) R^{(H)}_{\text{max}} \leq r_{\text{conv}} (\rho \rightarrow \psi) R^{(H)}_{\text{max}}. \]  
(96)
Therefore, (95) holds. Equation (93) can be shown by replacing \( R_C \) with \( R_D \) and \( r_{\text{conv}} (\rho \rightarrow \psi) \) with \( r_{\text{conv}} (\rho \rightarrow \psi) \) in [95]. Q.E.D.

**Remark 3.** Due to Theorem 3, the infimum in the definitions of the distillable resource and the resource cost is achieved in the following cases. Let
\[ \mathcal{G}(\mathcal{H}) / \sim := \{ C_\phi : \phi \in \mathcal{G}(\mathcal{H}) \} \]  
(97)
be the set of equivalence classes of the maximally resourceful states, where
\[ C_\phi := \{ \psi \in \mathcal{G}(\mathcal{H}) : \psi \sim \phi \} \]  
(98)
is the equivalence class of \( \phi \). Suppose that the number of non-equivalent maximally resourceful states is finite; that is, \( |\mathcal{G}(\mathcal{H}) / \sim| < \infty \). For example, in the QRT of bipartite entanglement, \( |\mathcal{G}(\mathcal{H}) / \sim| = 1 \); in the QRT of magic states for qutrits, \( |\mathcal{G}(\mathcal{H}) / \sim| = 2 \). In these cases, the infimum is achievable by a maximally resourceful state because of Proposition 7. Thus, for these existing QRTs, we can actually replace the infimum in the definitions of the distillable resource [81] and the resource cost [82] with the minimum, while further research is needed to clarify whether or not we can replace the infimum with the minimum for QRTs with infinitely many non-equivalent maximally resourceful states, i.e., \( |\mathcal{G}(\mathcal{H}) / \sim| = \infty \).

**B. Distillable Resource and Resource Cost of Catalytically Replicable States**

In this section, we analyze the distillable resource and the resource cost of a catalytically replicable state. As the conversion rate between a catalytically replicable state is infinite, we obtain a counter-intuitive result, which shows that an infinitely large number of a resource can be distilled from a catalytically replicable state.
state and that a catalytically replicable state can be generated without any cost.

The following proposition shows that the resource cost needed to form a catalytically replicable state is equal to zero. Moreover, if the distillable resource of a catalytically replicable state is nonzero, an infinite amount of a resource can be distilled from the state.

**Proposition 10.** Let \( \psi \in S(\mathcal{H}) \) be a state satisfying \( r_{\text{conv}}(\psi \to \psi) = \infty \). Then,

\[
R_C(\psi) = 0.
\]

holds. Moreover, if \( R_D(\psi) > 0 \),

\[
R_D(\psi) = \infty
\]

holds.

**Proof.** Note that \( 0 \leq R_C(\psi) \) and \( 0 \leq R_D(\psi) \) hold by the definitions. Since \( r_{\text{conv}}(\psi \to \psi) = \infty \),

\[
R_C(\psi) \leq \frac{R_{\text{max}}(\mathcal{H})}{r_{\text{conv}}(\psi \to \psi)} = 0
\]

holds. Therefore, it holds that \( R_C(\psi) = 0 \).

Recall that for quantum states \( \rho, \sigma \) and \( \omega \), it holds that \( r_{\text{conv}}(\rho \to \omega) \geq r_{\text{conv}}(\rho \to \sigma) r_{\text{conv}}(\sigma \to \omega) \) as shown in \( \ref{convadditivity} \). Take an arbitrary positive number \( \epsilon \). Let \( \phi \in S(\mathcal{H}) \) be a maximally resourceful state such that

\[
R_D(\psi) + \epsilon \geq r_{\text{conv}}(\psi \to \phi) R_{\text{max}}(\mathcal{H}),
\]

Then, it holds that

\[
R_D(\psi) + \epsilon \geq r_{\text{conv}}(\psi \to \phi) R_{\text{max}}(\mathcal{H}) \geq r_{\text{conv}}(\psi \to \psi) r_{\text{conv}}(\psi \to \phi) R_{\text{max}}(\mathcal{H}) = r_{\text{conv}}(\psi \to \psi) R_D(\psi) = \infty.
\]

As we can take an arbitrarily small \( \epsilon \), it holds that \( R_D(\psi) = \infty \). Q.E.D.

In a QRT whose maximally resourceful states are catalytically replicable, there may be a state of which the distillable resource is infinite, and the resource cost is zero, as shown in the Example \( \ref{example3} \).

**Example 8** (Zero Resource Cost and Infinite Distillable Resource). As shown in Proposition \( \ref{example3} \) and Proposition \( \ref{example2} \), the distillable resource of a catalytically replicable state may be infinite while that of a free state is zero. Consider the same setup as Example \( \ref{example1} \). In this case,

\[
R_D(|1\rangle \langle 1|) = \infty,
\]

\[
R_C(|1\rangle \langle 1|) = 0
\]

follows from \( G(\mathbb{C}^2) = \{|1\rangle \langle 1|\} \) and \( r_{\text{conv}}(|1\rangle \langle 1| \to |1\rangle \langle 1|) = \infty \). In contrast, for any free state \( \psi \), Proposition \( \ref{example3} \) shows that

\[
R_D(\psi) = 0,
\]

\[
R_C(\psi) = 0.
\]

C. **Weak Subadditivity of Distillable Resource and Resource Cost**

In this section, we prove that the distillable resource and the resource cost are weakly subadditive if the Hilbert space \( \mathcal{H} \) is finite-dimensional and the normalized constant is set as \( R_{\text{max}}(\mathcal{H}) = \log_2(\dim \mathcal{H}) \). The definitions of additivity and subadditivity are as follows.

**Definition 11** (Additivity and Subadditivity). Let \( f_\mathcal{H} \) be a family of functions from \( S(\mathcal{H}) \) to \( \mathbb{R} \), where \( \mathcal{H} \) is a quantum system. We may omit the subscript of \( f_\mathcal{H} \) to write \( f \) for brevity. Then, \( f \) is said to be fully additive if it holds that

\[
f(\psi \otimes \phi) = f(\psi) + f(\phi)
\]

for any states \( \psi \in S(\mathcal{H}) \) and \( \phi \in S(\mathcal{H}'). \) On the other hand, \( f \) is said to be weakly additive if it holds that

\[
f(\psi^{\otimes n}) = n f(\psi)
\]

for any state \( \psi \in S(\mathcal{H}) \) and for any positive integer \( n \). In this paper, we use the word “additivity” to refer to weak additivity for brevity.

Similarly, \( f \) is said to be fully subadditive if it holds that

\[
f(\psi \otimes \phi) \leq f(\psi) + f(\phi)
\]

for any states \( \psi \in S(\mathcal{H}) \) and \( \phi \in S(\mathcal{H}'). \) On the other hand, \( f \) is said to be weakly subadditive if it holds that

\[
f(\psi^{\otimes n}) \leq n f(\psi)
\]

for any state \( \psi \in S(\mathcal{H}) \) and for any positive integer \( n \).

The proof of the weak subadditivity exploits the following proposition.

**Proposition 12.** Let \( \mathcal{H} \) and \( \mathcal{H}' \) be quantum systems. Let \( \psi \in S(\mathcal{H}) \) and \( \phi \in S(\mathcal{H}') \) be quantum states. Then for any \( n \in \mathbb{N} \), it holds that

\[
r_{\text{conv}}(\psi \to \phi) = r_{\text{conv}}(\psi^{\otimes n} \to \phi^{\otimes n}).
\]

**Proof.** It suffices to show that \( R(\psi \to \phi) = R(\psi^{\otimes n} \to \phi^{\otimes n}) \). First, assume \( r \in R(\psi^{\otimes n} \to \phi^{\otimes n}) \) to show \( R(\psi^{\otimes n} \to \phi^{\otimes n}) \subseteq R(\psi \to \phi) \). Choose a positive number \( \epsilon > 0 \). Then, there exists a sequence of free operations \( \{\mathcal{M}_m^{(n)} \in \mathcal{O}(\mathcal{H}^{\otimes nm} \to \mathcal{H}^{\otimes [rnm]}): m \in \mathbb{N}\} \) such that

\[
\left\| \mathcal{M}_m^{(n)}(\psi^{\otimes n})^{\otimes m} - (\phi^{\otimes n})^{\otimes [rnm]} \right\|_1 < \epsilon
\]

holds for an infinitely large subset of \( \mathbb{N} \). Since \( n [rnm] \geq [rnm] \) holds, we can define \( \mathcal{N}_m^{(n)} \in \mathcal{O}(\mathcal{H}^{\otimes [rnm]} \to \mathcal{H}^{\otimes [rnm]}) \) as the partial
trace of $n \lfloor rm \rfloor - \lfloor rnm \rfloor$ systems. Then, we have $\mathcal{N}^{(n)}_m (\phi \otimes^n [rm]) = \phi \otimes^n [rnm]$.

Therefore, it holds that

$$
\left\| \mathcal{L}^n_m \circ \mathcal{M}^{(n)}_m (\psi \otimes^m n) - \phi \otimes^n [rm] \right\|_1 \leq \left\| \mathcal{M}^{(n)}_m ((\psi \otimes^m n) \otimes^n m) - \mathcal{N}^{(n)}_m (\phi \otimes^n [rm]) \right\|_1
$$

$$
< \epsilon.
$$

(117)

Therefore, for an integer $k = nm$ with a sufficiently large $m$, there exists a free operation $\mathcal{L}_k := \mathcal{L}^n_m \circ \mathcal{M}^{(n)}_m$ such that

$$
\left\| \mathcal{L}_k (\psi \otimes^k n) - \phi \otimes [rk] \right\|_1 < \epsilon.
$$

(118)

Therefore, $r \in \mathcal{R} (\psi \rightarrow \phi)$, which implies $\mathcal{R} (\psi \otimes^n n \rightarrow \phi \otimes^n n) \subseteq \mathcal{R} (\psi \rightarrow \phi)$.

On the other hand, to show $\mathcal{R} (\psi \rightarrow \phi) \subseteq \mathcal{R} (\psi \otimes^n n \rightarrow \phi \otimes^n n)$, assume $r \in \mathcal{R} (\psi \rightarrow \phi)$. Choose a positive number $\epsilon$. Then, there exists a sequence of free operations $(\mathcal{M}_m \in \mathcal{O} (\mathcal{H} \otimes m \rightarrow \mathcal{H} \otimes^n m) : m \in \mathbb{N})$ such that for a fixed positive integer $m$, holds for an infinitely large subset of $\mathcal{N}$. Therefore, for any $n$ satisfying (119), it holds that

$$
\left\| \mathcal{M}_m (\psi \otimes^m n) - \phi \otimes^n [rm] \right\|_1 < \frac{\epsilon}{n}
$$

(119)

which implies that $\mathcal{R} (\psi \rightarrow \phi) \subseteq \mathcal{R} (\psi \otimes^n n \rightarrow \phi \otimes^n n)$.

Q.E.D.

Using Proposition 12, we show Theorem 13. In the statement of Theorem 13 (121) means that for a maximally resourceful state $\phi \in \mathcal{G} (\mathcal{H})$, there may be $\phi_n \in \mathcal{G} (\mathcal{H} \otimes^n n)$ that is harder to distill than $\phi \otimes^n n$. On the other hand, (122) means that for a maximally resourceful state $\phi \in \mathcal{G} (\mathcal{H})$, there may be a more resourceful state $\phi_n \in \mathcal{G} (\mathcal{H} \otimes^n n)$ in resource formation than $\phi \otimes^n n$ for $n \geq 2$.

Theorem 13 (Weak Subadditivity of Distillable Resource and Resource Cost). Let $\mathcal{H}$ be an arbitrary finite-dimensional system. Set the normalization constant as $R^{(\mathcal{H})}_{\text{max}} = \log_2 (\dim \mathcal{H})$ as shown in (11). For any $n \in \mathbb{N}$ and for any state $\psi \in \mathcal{S} (\mathcal{H})$,

$$
R_D (\psi \otimes^n n) \leq n R_D (\psi),
$$

(121)

$$
R_C (\psi \otimes^n n) \leq n R_C (\psi).
$$

(122)

Proof. First, we prove (121). Let $n$ be a fixed positive integer, and let $\epsilon$ be an arbitrary positive number. Due to Theorem 9, we can take a maximally resourceful state $\phi \in \mathcal{G} (\mathcal{H})$ such that

$$
R_D (\psi) + \epsilon \geq R_{\text{max}} (\psi \rightarrow \phi).
$$

(123)

Since $\phi \otimes^n \in \mathcal{S} (\mathcal{H} \otimes^n n)$,

$$
n R_D (\psi) + \epsilon = n R_{\text{max}} (\psi \rightarrow \phi)
$$

(124)

$$
= R^{(\mathcal{H})}_{\text{max}} (\psi \otimes^n n \rightarrow \phi \otimes^n n)
$$

(125)

$$
\geq R_D (\psi \otimes^n n)
$$

(126)

holds where (125) follows from Proposition 12. As we can take an arbitrarily small $\epsilon$, (121) holds. We can show (121) in a similar way by replacing $R_D$ with $R_C$ and $r_{\text{conv}} (\psi \rightarrow \phi)$ with $1/r_{\text{conv}} (\phi \rightarrow \psi)$.

Q.E.D.

D. Maximally Resourceful State Maximizing Resource Cost

In this section, we prove that the upper bound $R^{(\mathcal{H})}_{\text{max}}$ of the resource cost $R_C$ shown in Proposition 8 is indeed achievable by a maximally resourceful state if the number of equivalence classes of the maximally resourceful states is finite and if there is no catalytically replicable state. Note that this property holds even in infinite-dimensional cases; that is, $R^{(\mathcal{H})}_{\text{max}} = \log_2 (\dim \mathcal{H})$ for finite-dimensional $\mathcal{H}$ is not assumed in this section. First, Proposition 12 also leads to the following proposition, which shows that the cost needed to form a state is always upper-bounded by the resource cost of a maximally resourceful state.

Proposition 14. For any state $\psi \in \mathcal{S} (\mathcal{H})$, there exists $\phi \in \mathcal{G} (\mathcal{H})$ such that

$$
R_C (\psi) \leq R_C (\phi)
$$

(127)

holds.

Proof. Given any $\psi$, due to Theorem 11, we take $\phi \in \mathcal{G} (\mathcal{H})$ satisfying $\phi \succeq \psi$. Let $\epsilon$ be an arbitrary positive number. Due to Theorem 12, we can take a maximally resourceful state $\phi \in \mathcal{G} (\mathcal{H})$ such that

$$
R_C (\phi) + \epsilon \geq \frac{R^{(\mathcal{H})}_{\text{max}}}{r_{\text{conv}} (\rho \rightarrow \phi)}.
$$

(128)

Then, by Proposition 12,

$$
R_C (\psi) \leq \inf_{\sigma \in \mathcal{S} (\mathcal{H})} \left\{ \frac{R^{(\mathcal{H})}_{\text{max}}}{r_{\text{conv}} (\sigma \rightarrow \psi)} \right\}
$$

(129)

$$
\leq \frac{R^{(\mathcal{H})}_{\text{max}}}{r_{\text{conv}} (\phi \rightarrow \psi)}
$$

(130)

$$
\leq \frac{R^{(\mathcal{H})}_{\text{max}}}{r_{\text{conv}} (\phi \rightarrow \phi)}
$$

(131)

$$
\leq R_C (\phi) + \epsilon
$$

(132)

holds. As we can take an arbitrarily small $\epsilon$, $R_C (\psi) \leq R_C (\phi)$ holds.

Q.E.D.

Using Proposition 7, we prove Theorem 13. Recall the set of equivalence classes of the maximally resourceful...
states \( \mathcal{G} (\mathcal{H}) / \sim \) defined in [67]. Consider a QRT where the number of maximally resourceful states is finite up to the equivalence with regard to the preorder, that is,

\[ |\mathcal{G} (\mathcal{H}) / \sim| < \infty. \tag{133} \]

In this case, the following theorem shows that the upper bound \( R_{\text{max}}^{(\mathcal{H})} \) of the resource cost given in Proposition 8 is actually achievable by a maximally resourceful state.

**Theorem 15 (Maximally Resourceful State that Maximizes Resource Cost).** Suppose that there is no catalytically replicable state. Suppose further that the set of resource states is not empty; that is, \( \mathcal{S} (\mathcal{H}) \setminus \mathcal{F} (\mathcal{H}) \neq \emptyset \). If \( |\mathcal{G} (\mathcal{H}) / \sim| < \infty \) where \( \mathcal{G} (\mathcal{H}) / \sim \) is defined in [67], then there exists a maximal state \( \phi \in \mathcal{G} (\mathcal{H}) \) such that

\[ R_C (\phi) = R_{\text{max}}^{(\mathcal{H})}. \tag{134} \]

**Proof.** Assume that for any state \( \phi \in \mathcal{G} (\mathcal{H}) \), \( R_C (\phi) < R_{\text{max}}^{(\mathcal{H})} \). Because of Theorem 9, this assumption implies that for any state \( \phi \in \mathcal{G} (\mathcal{H}) \), there exists a maximally resourceful state \( \rho \in \mathcal{G} (\mathcal{H}) \) such that \( r_{\text{conv}} (\rho \rightarrow \phi) > 1 \). From Proposition 7, \( \rho \) must be in a different equivalence class from \( C_\phi \). Write this relation as \( \phi \rightarrow \rho \); that is, \( \phi \in \mathcal{G} (\mathcal{H}) / \sim \) and \( \rho \in \mathcal{G} (\mathcal{H}) / \sim \) are written as \( \phi \rightarrow \rho \) if \( r_{\text{conv}} (\rho \rightarrow \phi) > 1 \). Since \( |\mathcal{G} (\mathcal{H}) / \sim| < \infty \), there must exist a loop of elements in \( \mathcal{G} (\mathcal{H}) / \sim \)

\[ \rho_0 \rightarrow \rho_1 \rightarrow \cdots \rightarrow \rho_n \rightarrow \rho_0. \tag{135} \]

Therefore, Theorem 9 shows that

\[ r_{\text{conv}} (\rho_0 \rightarrow \rho_1) \times r_{\text{conv}} (\rho_1 \rightarrow \rho_2) \times \cdots \times r_{\text{conv}} (\rho_n \rightarrow \rho_0) > 1. \]

On the other hand, note that for any maximally resourceful state \( \rho \in \mathcal{G} (\mathcal{H}) \), it holds that \( r_{\text{conv}} (\rho \rightarrow \rho) = 1 \) because there is no catalytically replicable state and because \( \rho \notin \mathcal{F} (\mathcal{H}) \) due to Theorem 9. From [40], it follows that

\[ r_{\text{conv}} (\rho_0 \rightarrow \rho_1) \times r_{\text{conv}} (\rho_1 \rightarrow \rho_0) \leq r_{\text{conv}} (\rho_0 \rightarrow \rho_0) = 1, \tag{136} \]

which contradicts (136). Therefore, there exists a maximal state \( \phi \in \mathcal{G} (\mathcal{H}) \) such that \( R_C (\phi) = R_{\text{max}}^{(\mathcal{H})} \). Q.E.D.

### VI. RESOURCE MEASURES

In this section, we investigate a formulation of resource measures in general QRTs and clarify general properties of the resource measures. The resource measures quantify the amount of quantum resources, which is a central interest in QRTs [1]. In Sec. VI A we provide the definition of a resource measure. In Sec. VI B progressing beyond the existing result on bipartite entanglement [38], we show in the general setting that a resource measure is upper-bounded by the resource cost and lower-bounded by the distillable resource if it satisfies the same axioms as those given in Ref. [38]. At the same time, we show that the QRT of magic for qutrits [57], which has several non-equivalent maximally resourceful states, have no resource
measure satisfying the axioms. To overcome this problem in the axiomatic approach based on Ref. [38], we here introduce a concept of consistency of a resource measure in Sec. VI C. In contrast with the previous approach, a consistent resource measure exists in the case where multiple non-equivalent maximally resourceful states exist. Furthermore, we prove a similar uniqueness inequality to the previous approach; that is, the consistent resource measure is bounded by the distillable resource and the resource cost if it is normalized. In Sec. VI D, we provide the definition of the relative entropy of resource, and show that the regularized relative entropy of resource serves as a consistent resource measure.

A. Axioms on Resource Measures

In this section, we provide a definition of resource measure as discussed in previous works on studying resource measures in a wide class of QRTs, such as Refs. [24, 58–61], and recall axioms on a resource measure some of which are also discussed in Ref. [1]. A resource measure \( R \) quantifies the amount of the resource of a state. It takes a state as an input and outputs a real number that represents the amount of the resource. To quantify the resource consistently with the fact that free operations cannot generate resources by themselves, a resource measure must satisfy a property called monotonicity; i.e., the amount of the resource quantified by a resource measure does not increase through application of free operations. Formally, the monotonicity is defined as follows. For quantum systems \( \mathcal{H}^{(\text{in})} \) and \( \mathcal{H}^{(\text{out})} \), any state \( \psi \in \mathcal{S} (\mathcal{H}^{(\text{in})}) \), and any free operation \( \mathcal{N} \in \mathcal{O} (\mathcal{H}^{(\text{in})} \rightarrow \mathcal{H}^{(\text{out})}) \), it holds that

\[
R_{\mathcal{H}^{(\text{in})}}(\psi) \geq R_{\mathcal{H}^{(\text{out})}}(\mathcal{N}(\psi)).
\]  

(143)

Here, we recall the definition of a resource measure.

Definition 17 (Resource Measure). A resource measure \( R_{\mathcal{H}} \) is a family of real functions from \( \mathcal{S} (\mathcal{H}) \) for a quantum system \( \mathcal{H} \) to \( \mathbb{R} \) satisfying the monotonicity. We may omit the subscript of \( R_{\mathcal{H}} \) to write \( R \) for brevity.

By the monotonicity, a resource measure \( R_{\mathcal{H}} \) for a quantum system \( \mathcal{H} \) quantifies the resource consistently with the preorder introduced by free operations. For two states satisfying

\[
\phi \succeq \psi,
\]

(144)

it holds that

\[
R_{\mathcal{H}}(\phi) \geq R_{\mathcal{H}}(\psi).
\]

(145)

Note that if two states satisfy

\[
\phi \sim \psi
\]

(146)

then we have

\[
R_{\mathcal{H}}(\phi) = R_{\mathcal{H}}(\psi).
\]

(147)

Furthermore, using a resource measure, we can uniformly evaluate the resource amounts of two different states that cannot be compared in terms of the preorder introduced by free operations.

Now, we recall several axioms on a resource measure. Additivity: Strong superadditivity refers to

\[
R (\psi^{AB}) \geq R (\psi^{A}) + R (\psi^{B}),
\]

(148)

and full superadditivity refers to

\[
R (\psi \otimes \phi) \geq R (\psi) + R (\phi).
\]

(149)

Full subadditivity refers to

\[
R (\psi \otimes \phi) \leq R (\psi) + R (\phi).
\]

(150)

Full additivity refers to

\[
R (\psi \otimes \phi) = R (\psi) + R (\phi),
\]

(151)

while additivity refers to

\[
R (\psi^{\otimes n}) = n R (\psi).
\]

(152)

Regularization of \( R \) provides a measure that is additive for tensor product of the same states

\[
R^\infty (\psi) := \lim_{n \to \infty} \frac{R (\psi^{\otimes n})}{n},
\]

(153)

as long as the right-hand side exists. The following proposition shows that the additivity of a resource measure implies that free states have zero resource, which is a generalization of the statement shown for entanglement in Ref. [38] to general QRTs.

Proposition 18. If a resource measure \( R \) is additive, \( R (\phi) = 0 \) for any free state \( \phi \).

Proof. Suppose that \( \psi \) is a free state. Then, there exists a free operation \( \mathcal{M} \) such that \( \mathcal{M} (1) = \psi \). Therefore, for any \( n \in \mathbb{N} \), \( \mathcal{M}^{\otimes n} (1) = \psi^{\otimes n} \) holds, which implies \( \psi^{\otimes n} \) is also a free state. Then, there exists free operations \( \mathcal{N}_1 \) and \( \mathcal{N}_2 \) such that

\[
\mathcal{N}_1 (\psi^{\otimes n}) = \psi,
\]

(154)

\[
\mathcal{N}_2 (\psi) = \psi^{\otimes n}
\]

(155)

hold. Therefore, it holds that \( R (\psi) = R (\psi^{\otimes n}) \). Since \( R \) is additive, \( R (\psi) = n R (\psi) \) for any \( n \), which implies \( R (\psi) = 0 \).

Q.E.D.

One conventional way of normalizing resource measures such as that in the entanglement theory is as follows, which we call conventional normalization:

- For any free state \( \sigma \),

\[
R (\sigma) = 0.
\]

(155)

- For any maximal state \( \phi \in \mathcal{G} (\mathcal{H}) \),

\[
R (\phi) = R_{\text{max}}^{(\mathcal{H})}.
\]

(156)
Because of the monotonicity of a resource measure, a resource measure takes the least value for free states. From Proposition 18, the least value is automatically set to zero for an additive measure. We can assume this normalization also for non-additive measures. Furthermore, we can set the greatest value of a resource measure to \( R_{\mathrm {max}}^H \) in the same way that we normalize the distillable resource \( S_1^H \) and the resource cost \( S_2^H \) with the normalized constant \( R_{\mathrm {max}} \). In a finite-dimensional case with \( R_{\mathrm {max}}^H = \log_2 (\dim H) \) shown in (51), (158) provides the normalization generalizing that of entanglement measures in the entanglement theory, but our definition is applicable to infinite-dimensional cases. We here remark that general QRTs do not necessarily have resource measures satisfying this conventional normalization, as we will prove in the next subsection.

**Asymptotic continuity:** For any quantum system \( H \), \( R_H \) is asymptotically continuous if for any sequence of positive integers \( (n_i)_{i \in \mathbb N} \), and any sequences of states \( (\phi_{n_i} \in S(\mathcal H^{\otimes n_i}))_i \) and \( (\psi_{n_i} \in S(\mathcal H^{\otimes n_i}))_i \) satisfying \( \lim_{n_i \to \infty} \| \phi_{n_i} - \psi_{n_i} \|_1 = 0 \), it holds that

\[
\lim_{n_i \to \infty} \frac{|R_{\mathcal H^{\otimes n_i}} (\phi_{n_i}) - R_{\mathcal H^{\otimes n_i}} (\psi_{n_i})|}{n_i} = 0. \tag{157}
\]

Our definition of asymptotic continuity is applicable to an infinite-dimensional system. If we take \( R_{\mathrm {max}}^H = \log_2 (\dim H) \) as shown in (51) for a finite-dimensional system \( H \), our definition (157) corresponds to Condition (E3) in Ref. 38. Note that our definition includes the asymptotic continuity discussed in (52) (Definition 1) as a tighter bound applicable to a finite-dimensional system.

**Remark 4 (Continuity and Asymptotic Continuity).** Since a resource measure is a family of functions each of which may be defined for different quantum systems, we employ the asymptotic continuity of a family of functions as an axiom on resource measures rather than the continuity of a single function \( R_H \) for a fixed quantum system \( H \) defined as follows. A function \( R_H : S(H) \to \mathbb R \) is **continuous** if for any sequences of states \( (\phi_n \in S(H))_n \) and \( (\psi_n \in S(H))_n \) satisfying \( \lim_{n \to \infty} \| \phi_n - \psi_n \|_1 = 0 \), it holds that

\[
\lim_{n \to \infty} |R_H (\phi_n) - R_H (\psi_n)| = 0. \tag{158}
\]

Our definition of asymptotic continuity implies continuity as special cases.

**B. Generalization of Uniqueness Inequality**

In this section, we show that we have the inequality \( R_D \leq R \leq R_C \) for a resource measure \( R \) if \( R \) satisfies conventional normalization, asymptotic continuity, and additivity. We call this inequality the uniqueness inequality. The uniqueness inequality are originally proved in the QRT of bipartite entanglement in finite-dimensional cases 38. We show that the proof of this uniqueness inequality can be generalized to all the QRTs in our framework that covers infinite-dimensional cases.

At the same time, we also show a QRT in which no resource measure satisfies these axioms; that is, the set of states satisfying the uniqueness inequality becomes empty. First, we prove the uniqueness inequality for a general QRT in our framework.

**Proposition 19 (Uniqueness Inequality).** Let \( H \) be a quantum system. Suppose that there is no catalytically replicable state; that is, \( R_{\mathrm {conv}} (\phi \to \phi) = 1 \) for any resource state \( \phi \in S(\mathcal H) \setminus \mathcal F (\mathcal H) \). If a resource measure \( R_H \) satisfies the conventional normalization, the asymptotic continuity, and the additivity, then for any state \( \psi \in S(H) \), \( R_H \) satisfies

\[
R_D (\psi) \leq R_H (\psi) \leq R_C (\psi). \tag{159}
\]

**Proof.** First, we prove \( R_D (\psi) \leq R_H (\psi) \) for any state \( \psi \in S(H) \). Let \( \delta \) be an arbitrary positive number. Due to Theorem 19, we take a maximally resourceful state \( \phi \in \mathcal G (\mathcal H) \) such that

\[
R_D (\psi) + \delta \leq R_{\mathrm {conv}} (\psi \to \phi) R_{\max}^H. \tag{160}
\]

Let \( r := R_{\mathrm {conv}} (\psi \to \phi) \). For any positive integer \( n \), it holds that

\[
r R_{\max}^H \leq \frac{[rn]}{n} R_{\max}^H. \tag{161}
\]

Then, by the conventional normalization and the additivity, it holds that

\[
\frac{[rn]}{n} R_{\max}^H = \frac{[rn]}{n} R_H (\phi)
= \frac{R_{H^{\otimes [rn]}} (\phi^{\otimes [rn]})}{n}. \tag{162}
\]

By the definition of \( R_{\mathrm {conv}} (\psi \to \phi) \) shown in (34), there exists free operations \( (N_n \in \mathcal O (\mathcal H^{\otimes n} \to H^{\otimes [rn]}) \) such that for any \( \epsilon \), it holds that

\[
\left\| N_n (\psi^{\otimes n}) - \phi^{\otimes [rn]} \right\|_1 < \epsilon, \tag{163}
\]

for an infinitely large subset of \( \mathbb N \). Because of the asymptotic continuity, for any \( \epsilon > 0 \), there exists sufficiently large \( n \) such that

\[
\frac{R_{H^{\otimes [rn]}} (\phi^{\otimes [rn]})}{n} \leq \frac{R_{H^{\otimes [rn]}} (M_n (\psi^{\otimes n}))}{n} + \epsilon. \tag{164}
\]

By the monotonicity and the additivity, it holds that

\[
\frac{R_{H^{\otimes [rn]}} (M_n (\psi^{\otimes n}))}{n} \leq \frac{R_{H^{\otimes [rn]}} (\psi^{\otimes n})}{n} = n R_H (\psi) \leq R_H (\psi). \tag{165}
\]
Therefore, by (160), (161), (162), (164), and (165), it holds that

\[ R_D (\psi) + \delta \leq R_H (\psi) + \epsilon. \]  

(166)

Since we can take arbitrarily small \( \epsilon \) and \( \delta \), it holds that \( R_D (\psi) \leq R_H (\psi) \).

Next, we prove \( R_C (\psi) \geq R_H (\psi) \) for any state \( \psi \in \mathcal{S} (\mathcal{H}) \). Let \( \delta \) be an arbitrary positive number. Due to Theorem 14, we take a maximally resourceful state \( \phi \in \mathcal{G} (\mathcal{H}) \) such that

\[ R_C (\psi) + \delta \geq \frac{R_H^{(H)}}{r_{\text{conv}} (\phi \rightarrow \psi)} = r'_{\text{conv}} (\phi \rightarrow \psi) R_H^{(H)}. \]  

(167)

Let \( r := r'_{\text{conv}} (\phi \rightarrow \psi) \). For any positive integer \( n \), it holds that

\[ r R_H^{(H)} \geq \frac{\lfloor rn \rfloor}{n} R_H^{(H)}. \]  

(168)

By the conventional normalization, the additivity, and the monotonicity, it holds that

\[ \frac{\lfloor rn \rfloor}{n} R_H^{(H)} = \frac{\lfloor rn \rfloor}{n} R_H (\phi) = \frac{R_H (\phi^{|\lfloor rn \rfloor|})}{n} \geq \frac{R_H (\phi^{|\lfloor rn \rfloor|})}{n}. \]  

(169)

By the definition of \( r'_{\text{conv}} (\phi \rightarrow \psi) \) shown in (172), there exists free operations \( (\mathcal{N}_n \in \mathcal{O} (\mathcal{H}^{|\lfloor rn \rfloor|} \rightarrow \mathcal{H}^{|\lfloor rn \rfloor|})) \) such that for any \( \epsilon \), it holds that

\[ \left\| \mathcal{N}_n (\phi^{|\lfloor rn \rfloor|}) - \phi^{|\lfloor rn \rfloor|} \right\|_1 < \epsilon, \]  

(170)

for an infinitely large subset of \( \mathbb{N} \). Because of the asymptotic continuity, for any \( \epsilon > 0 \), there exists sufficiently large \( n \) such that

\[ R_H^{(\phi^{|\lfloor rn \rfloor|})} = R_H^{(\phi^{|\lfloor rn \rfloor|})} - \epsilon. \]  

(171)

Therefore, by (167), (168), (169), and (171), and the additivity, it holds that

\[ R_C (\psi) + \delta \geq \frac{R_H^{(\phi^{|\lfloor rn \rfloor|})}}{n} - \epsilon \]

\[ = \frac{n R_H (\psi)}{n} - \epsilon \]

\[ = R_H (\psi) - \epsilon. \]  

(172)

Since we can take arbitrarily small \( \epsilon \) and \( \delta \), it holds that

\[ R_C (\psi) \geq R_H (\psi). \]  

(173)

Q.E.D.

Despite the general uniqueness inequality shown in Proposition 19, we show a condition of QRTs where no resource measure satisfies the conventional normalization, the asymptotic continuity, and the additivity simultaneously. In these QRTs, Proposition 19 is not applicable. Note that because of Proposition 2 the condition (175) in the following theorem is not satisfied for QRTs with a unique maximally resourceful state, but may hold for QRTs with two or more different equivalence classes of maximally resourceful states; that is,

\[ |\mathcal{G} (\mathcal{H}) / \sim| \geq 2. \]  

(174)

**Theorem 20 (Inconsistency of Axioms).** Suppose that there exist maximally resourceful elements \( \phi_0, \phi_1 \in \mathcal{G} (\mathcal{H}) \) such that

\[ r_{\text{conv}} (\phi_0 \rightarrow \phi_1) > 1. \]  

(175)

If \( R_H^{(\mathcal{H})} > 0 \), then there exists no resource measure satisfying the conventional normalization, the asymptotic continuity, and the additivity simultaneously.

**Proof.** The proof is by contradiction. Assume that there exists a resource measure \( R \) that satisfies all of the conventional normalization, the asymptotic continuity, and the additivity. Let \( \phi_0, \phi_1 \in \mathcal{G} (\mathcal{H}) \) be maximally resourceful states such that

\[ r := r_{\text{conv}} (\phi_0 \rightarrow \phi_1) > 1. \]  

(176)

By the definition of \( r_{\text{conv}} (\phi_0 \rightarrow \phi_1) \) shown in (177), there exists free operations \( (\mathcal{N}_n \in \mathcal{O} (\mathcal{H}^{|\lfloor rn \rfloor|} \rightarrow \mathcal{H}^{|\lfloor rn \rfloor|})) \) such that for any \( \epsilon \), it holds that

\[ \left\| \mathcal{N}_n (\phi^{|\lfloor rn \rfloor|}) - \phi^{|\lfloor rn \rfloor|} \right\|_1 < \epsilon, \]  

(177)

for an infinitely large subset of \( \mathbb{N} \). Then, by the asymptotic continuity, for a fixed \( \epsilon > 0 \), there exists sufficiently large \( n \) such that

\[ R_H^{(\phi^{|\lfloor rn \rfloor|})} (\mathcal{N}_n (\phi^{|\lfloor rn \rfloor|})) \geq \frac{R_H^{(\phi^{|\lfloor rn \rfloor|})}}{\mathcal{N}_n (\phi^{|\lfloor rn \rfloor|})} (\phi^{|\lfloor rn \rfloor|}) - \epsilon. \]  

(178)

By the monotonicity, it holds that

\[ \frac{R_H^{(\phi^{|\lfloor rn \rfloor|})}}{\mathcal{N}_n (\phi^{|\lfloor rn \rfloor|})} (\phi^{|\lfloor rn \rfloor|}) \geq \frac{R_H^{(\phi^{|\lfloor rn \rfloor|})}}{\mathcal{N}_n (\phi^{|\lfloor rn \rfloor|})} (\phi^{|\lfloor rn \rfloor|}) - \epsilon. \]  

(179)

Then, from the additivity, it follows that

\[ R_H (\phi_0) \geq \frac{[\lfloor rn \rfloor]}{n} R_H (\phi_1) - \epsilon. \]  

(180)

Therefore, due to the conventional normalization, it is necessary that for any \( \epsilon \) and \( n \), it holds that

\[ R_H (\phi_0) \geq \frac{[\lfloor rn \rfloor]}{n} R_H^{(\mathcal{H})} \geq -\epsilon. \]  

(181)
Since $R_{\text{max}}^{(H)} > 0$, we have
\begin{equation}
1 - \frac{[rn]}{n} \geq -\frac{\epsilon}{R_{\text{max}}^{(H)}}. \quad (182)
\end{equation}
Since $r > 1$, this inequality does not hold for sufficiently small $\epsilon$ or sufficiently large $n$, which implies that there is no such $R$.

The following example shows that the QRT of magic has no measure with the conventional normalization, the asymptotic continuity, and the additivity.

**Example 9** (QRT without Measure with Conventional Normalization, Asymptotic Continuity, and Additivity). The QRT of magic for qutrits $\mathbb{T}$ shown in Example 8 does not have any conventionally normalized, asymptotic continuous and additive measure. It has been proved that the asymptotic conversion rate from Theorem 20, it follows that no measure can satisfy the conventional normalization, the asymptotic continuity and the additivity simultaneously in this QRT.

### C. Consistency of Resource Measures

In this section, we here introduce consistent resource measures in place of resource measures in the previous axiomatic approach in Sec. VI A and VI B. As Theorem 20 and Example 9 suggest, the conventional normalization, the asymptotic continuity and the additivity do not necessarily hold simultaneously with monotonicity in general QRTs. On the other hand, a consistent resource measure is compatible with the state conversion rate. Thus, we prove that the uniqueness inequality (159) also holds for a consistent resource measure that is appropriately normalized. Note that this normalization respects the state conversion rate, and hence can be different from the conventional normalization given by (155) and (156).

First, we introduce a definition of a consistent resource measure. A consistent resource measure quantifies the amount of a resource without contradicting monotonicity under the free operations.

**Definition 21** (Consistent Resource Measure). For quantum systems $\mathcal{H}$ and $\mathcal{H}'$, a resource measure $R$ is called a consistent resource measure if for any states $\psi \in \mathcal{S}(\mathcal{H})$ and $\phi \in \mathcal{S}(\mathcal{H}')$, it holds that
\begin{equation}
R_{\mathcal{H}}(\psi) r_{\text{conv}}(\phi \rightarrow \psi) \leq R_{\mathcal{H}'}(\phi). \quad (183)
\end{equation}

The following proposition shows that a consistent resource measure must be additive for any non-catalytically replicable resource states.

**Proposition 22.** Let $R$ be a consistent resource measure. Then, for any resource state $\psi \in \mathcal{S}(\mathcal{H}) \setminus \mathcal{F}(\mathcal{H})$ that is not catalytically replicable and for any positive integer $n$, it holds that
\begin{equation}
R(\psi^\otimes n) = nR(\psi). \quad (184)
\end{equation}

**Proof.** By (16), it holds that
\begin{align}
r_{\text{conv}}(\psi \rightarrow \psi^\otimes n) &\leq r_{\text{conv}}(\psi^\otimes n \rightarrow \psi) \\
&\leq r_{\text{conv}}(\psi \rightarrow \psi) \\
&= 1.
\end{align}

Since the identity map is a free operation, we have
\begin{align}
r_{\text{conv}}(\psi \rightarrow \psi^\otimes n) &\geq n, \quad (186) \\
r_{\text{conv}}(\psi^\otimes n \rightarrow \psi) &\geq \frac{1}{n}. \quad (187)
\end{align}

Thus, we have
\begin{align}
r_{\text{conv}}(\psi \rightarrow \psi^\otimes n) &= \frac{1}{n}, \quad (188) \\
r_{\text{conv}}(\psi^\otimes n \rightarrow \psi) &= n. \quad (189)
\end{align}

By the definition of a consistent resource measure combined with the equations above, it holds that
\begin{equation}
nR(\psi) \leq R(\psi^\otimes n), \quad (190)
\end{equation}
\begin{equation}
\frac{1}{n} R(\psi^\otimes n) \leq R(\psi). \quad (191)
\end{equation}

Therefore, it holds that
\begin{equation}
R(\psi^\otimes n) = nR(\psi). \quad (192)
\end{equation}

Q.E.D.

We prove that the uniqueness inequality holds for a consistent resource measure that satisfies normalizations in the following propositions.

**Proposition 23.** Let $R_{\mathcal{H}}$ be a consistent resource measure. Suppose that $0 \leq R_{\mathcal{H}}(\psi) \leq R_{\text{max}}^{(H)}$ for any state $\psi \in \mathcal{S}(\mathcal{H})$. Then, $R_{\mathcal{H}}$ satisfies
\begin{equation}
R_{\mathcal{H}}(\psi) \leq R_{\mathcal{C}}(\psi). \quad (193)
\end{equation}

**Proof.** Let $\epsilon$ be an arbitrary positive number. Due to Theorem 20, we take a maximally resourceful state $\phi \in \mathcal{G}(\mathcal{H})$ such that
\begin{equation}
R_{\mathcal{C}}(\psi) + \epsilon \geq \frac{R_{\text{max}}^{(H)}}{r_{\text{conv}}(\phi \rightarrow \psi)}. \quad (194)
\end{equation}

Then, it holds that
\begin{align}
R_{\mathcal{C}}(\psi) + \epsilon &\geq \frac{R_{\text{max}}^{(H)}}{r_{\text{conv}}(\phi \rightarrow \psi)} \\
&\geq \frac{R_{\mathcal{H}}(\phi)}{r_{\text{conv}}(\phi \rightarrow \psi)} \quad (195) \\
&\geq R_{\mathcal{H}}(\psi), \quad (197)
\end{align}

where the second inequality follows from the definition of consistent resource measures. As we can take an arbitrarily small $\epsilon$, $R_{\mathcal{H}}(\psi) \leq R_{\mathcal{C}}(\psi)$ holds.

Q.E.D.
Proposition 24. Let $\mathcal{R}_\mathcal{H}$ be a consistent resource measure. Suppose that there exists a maximally resourceful state $\phi \in \mathcal{G}(\mathcal{H})$ such that $\mathcal{R}_\mathcal{H}(\phi) = R_{\mathcal{H}}^{(\max)}$. Then, $\mathcal{R}_\mathcal{H}$ satisfies

$$R_{\mathcal{H}}(\psi) \geq R_D(\psi)$$

for any state $\psi \in \mathcal{S}(\mathcal{H})$.

Proof. By the definition of the distillable resource, it holds that

$$R_D(\psi) \leq r_{\text{conv}}(\psi \rightarrow \sigma) R_{\mathcal{H}}^{(\max)}.$$  \hspace{1cm} (198)

Then, it holds that

$$R_D(\psi) \leq r_{\text{conv}}(\psi \rightarrow \sigma) R_{\mathcal{H}}^{(\max)} = r_{\text{conv}}(\psi \rightarrow \sigma) R_{\mathcal{H}}(\sigma) \leq R_{\mathcal{H}}(\psi).$$  \hspace{1cm} (199)

Therefore, $R_D(\psi) \leq R_{\mathcal{H}}(\psi)$ holds. Q.E.D.

From Proposition 23 and Proposition 24, we obtain the following corollary.

Corollary 25. Let $\mathcal{R}_\mathcal{H}$ be a consistent resource measure. Suppose that $\mathcal{R}_\mathcal{H}$ satisfies the following assumptions:

- For any state $\psi \in \mathcal{S}(\mathcal{H})$, $0 \leq R_{\mathcal{H}}(\psi) \leq R_{\mathcal{H}}^{(\max)}$;
- There exists $\phi \in \mathcal{G}(\mathcal{H})$ such that $R_{\mathcal{H}}(\phi) = R_{\mathcal{H}}^{(\max)}$.

Then, it holds that

$$R_D(\psi) \leq R_{\mathcal{H}}(\psi) \leq R_C(\psi),$$

for any state $\psi \in \mathcal{S}(\mathcal{H})$.

Remark 5. The second condition of Corollary 25 can be replaced by the existence of a state $\rho \in \mathcal{S}(\mathcal{H})$ (which is not necessarily maximally resourceful) such that $R_{\mathcal{H}}(\rho) = R_{\mathcal{H}}^{(\max)}$. Suppose a resource measure $R_{\mathcal{H}}$ satisfies the following conditions:

- The measure $R_{\mathcal{H}}$ is normalized in such a way that there exists a positive real number $C$ such that $0 \leq R_{\mathcal{H}}(\psi) \leq C$ for any state $\psi \in \mathcal{S}(\mathcal{H})$;
- The upper-bound of $R_{\mathcal{H}}$ is achieved by some state; that is, there exists $\rho \in \mathcal{S}(\mathcal{H})$ such that $R_{\mathcal{H}}(\rho) = C$.

Then, there exists a maximally resourceful state $\phi \in \mathcal{G}(\mathcal{H})$ such that $R_{\mathcal{H}}(\phi) = C$ by monotonicity of the resource measure.

To observe whether a consistent resource measure satisfies the asymptotic continuity, take any two states $\phi, \psi \in \mathcal{S}(\mathcal{H})$ such that $r_{\text{conv}}(\phi \rightarrow \psi) \leq 1$. Consider a consistent resource measure $\mathcal{R}$ satisfying $0 \leq \mathcal{R}_{\mathcal{H}}(\psi) \leq R_{\mathcal{H}}^{(\max)}$ for any $\psi \in \mathcal{S}(\mathcal{H})$. By the definition of a consistent resource measure, it holds that

$$\frac{R_{\mathcal{H}}(\psi) - R_{\mathcal{H}}(\phi)}{R_{\mathcal{H}}^{(\max)}} \leq 1 - r_{\text{conv}}(\phi \rightarrow \psi).$$  \hspace{1cm} (204)

Assume that the quantum system $\mathcal{H}$ is finite-dimensional, and take $R_{\mathcal{H}}^{(\max)} = \log_2(\dim \mathcal{H})$ as shown in [31]. Inequality [24] suggests that a consistent resource measure has the asymptotic continuity if $1 - r_{\text{conv}}(\phi_n \rightarrow \psi_n)$ converges to zero for any sequence of positive integers $(n_i)_{i \in \mathbb{N}}$ and any sequences of states $(\phi_n \in \mathcal{S}(\mathcal{H}^{\otimes n_i}))$ and $(\psi_n \in \mathcal{S}(\mathcal{H}^{\otimes n_i}))$ satisfying $\lim_{i \to \infty} \|\phi_n - \psi_n\|_1 = 0$. However, in general, $1 - r_{\text{conv}}(\phi_n \rightarrow \psi_n)$ is not necessarily small even if $\|\phi_n - \psi_n\|_1$ is small, because the convertibility of two states under free operations is not related to the distance between the two states. Therefore, a consistent resource measure is not necessarily asymptotically continuous. More generally, the difference of the resource amounts between two resource states is not necessarily related to the distance between the states due to the irrelevance between the preorder and the distance. Thus, we do not assume the asymptotic continuity in the definition of a consistent resource measure, while further research is needed to explicitly construct an example of consistent resource measures that are not asymptotically continuous.

D. Example of Consistent Resource Measures

In this section, we show an example of the consistent resource measure, which is known as the regularized relative entropy of resource and widely used in known QRTs such as bipartite entanglement [62], coherence [8] and magic states [13]. We give the definition of the relative entropy of resource $R_{\mathcal{R}}$ in our framework.

Definition 26 (Relative Entropy of Resource). The relative entropy of resource $R_{\mathcal{R}}$ is defined as

$$R_{\mathcal{R}}(\psi) := \inf_{\phi \in \mathcal{F}(\mathcal{H})} D(\psi || \phi),$$

where $D(\cdot || \cdot)$ is the quantum relative entropy defined as

$$D(\psi || \phi) = \text{Tr} \psi \log_2 \psi - \text{Tr} \psi \log_2 \phi.$$

The relative entropy of resource $R_{\mathcal{R}}$ is fully subadditive since the set of free operations is closed under tensor product. Therefore, by the subadditivity of $R_{\mathcal{R}}$, the regularized relative entropy of resource defined as

$$R_{\mathcal{R}}^\infty(\rho) := \lim_{n \to \infty} \frac{R_{\mathcal{R}}(\rho^{\otimes n})}{n}$$

exists [53].

We show that the regularized relative entropy of resource serves as a consistent resource measure for a finite-dimensional convex QRT in which $\mathcal{F}(\mathcal{H})$ for each $\mathcal{H}$ contains at least one full-rank state. Consider a convex QRT
that is defined for finite-dimensional systems and satisfies the axioms of QRTs given in Sec. III. It has been shown that if the set of free states $F(\mathcal{H})$ for each $\mathcal{H}$ contains at least one full-rank state, the relative entropy of resource is asymptotically continuous $[62,63]$. Therefore, using a bound on the asymptotic conversion rate given in Refs. [1,65], it is shown that the regularized relative entropy of resource $R^\infty$ is a consistent resource measure for a convex QRT that has a full-rank free state in each dimension. The QRT of bipartite entanglement $[44]$, coherence $[37]$ and magic $[15]$ are known as convex QRTs with full-rank free states. In these QRTs, the regularized relative entropy of resource works as a consistent resource measure.

We remark that this proof of the existence of a consistent measure is not applicable to non-convex QRTs because the relative entropy of resource for a non-convex set of free states can be discontinuous $[66]$. There may be a consistent resource measure even in QRTs that are not convex, not finite-dimensional, or does not contains full-rank free states while further research is needed to explicitly construct a consistent resource measures in these QRTs.

VII. CONCLUSION

We have formulated and investigated quantum state conversion and resource measures in a framework of general QRTs to figure out general properties of quantum resources. Our framework is based on physically motivated assumptions that are not necessarily mathematically tractable, and hence covers a broad range of QRTs including those with non-unique maximally resourceful states, non-convexity, and infinite dimension. In our general framework, the existence of maximally resourceful states is no longer trivial, but we proved that there always exists a maximally resourceful state in the general QRTs.

To clarify general properties of resource manipulation, we investigated one-shot and asymptotic state conversions, which are central tasks in QRTs. We discovered a catalytically replicable state, which is a resource that is infinitely replicable by free operations. In addition, we introduced the distillable resource and the resource cost in our framework without assuming uniqueness of maximally resourceful states. We showed that the distillable resource and the resource cost are weakly subadditive. Furthermore, we showed that the distillable resource is always smaller or equal to the resource cost if there is no catalytically replicable state.

As for quantification of quantum resources, we proved that the conventional normalization, the asymptotic continuity, and the additivity are incompatible with each other in general QRTs with non-unique maximally resourceful states. Motivated by this incompatibility, we introduced a consistent resource measure, which is consistent with the asymptotic state conversion rate. Moreover, we proved a normalized consistent resource measure is bounded by the distillable resource and the resource cost, generalizing the previous work on the uniqueness inequality in the entanglement theory to general QRTs.

Owing to the generality, our formulations and results broaden potential applications of QRTs in the following future research directions. Since we formulated a framework of QRTs applicable to non-convex QRTs where randomness can be regarded as a resource, it would be interesting to find further applications of non-convex QRTs, such as analyses of random-number generation $[67]$ and quantum $t$-designs $[68]$. In addition, since our framework forms a basis of QRTs on infinite-dimensional quantum systems, our results provide a foundation for applying QRTs to quantum field theory. Since we discovered a counter-intuitive phenomenon of catalytically replicable resources, it is interesting to find physically motivated situations where catalytically replicable states arise. Furthermore, while we showed that the regularized relative entropy serves as a consistent resource measure in convex finite-dimensional QRTs that have full-rank free states, construction of a consistent resource measure for all the QRTs in our framework including non-convex or infinite-dimensional QRTs is still open. Finally, extension of our framework to dynamic resource $[1]$ would also be an interesting future direction.

We established general and fruitful structures of QRTs disclosing universal properties of quantum resources. Owing to the broad applicability of our formulations, our results open a way to quantitative understandings of complicated and sometimes mathematically intractable quantum-mechanical phenomena through a unified approach using our general formulation of QRTs.

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Appendix: Equivalence of Compactness in Weak Operator Topology and Trace Norm Topology

In this section, we prove the following lemma, which shows that compactness in the weak operator topology is equivalent to that in the trace norm topology on a set of density operators. We exploit this lemma in the proof of Theorem I in Sec. IIIA.

Lemma 27. For any set of density operators $K \subseteq D(\mathcal{H})$, $K$ is weakly compact if and only if $K$ is compact in the trace norm topology.
Proof. Since the trace norm topology is stronger than the weak operator topology, the “if” part is obvious. Assume that \( K \) is weakly compact to show the “only if” part. To show the compactness in the trace norm topology, take an arbitrary sequence \( (\psi_n)_{n \in \mathbb{N}} \) in \( K \). According to the Eberlein-Šmulian theorem (e.g. [92], Theorem V.6.1), in the weak operator topology, the condition of the compactness coincides with that of the sequential compactness. Therefore, there exists a subsequence \( (\psi_{n(k)})_{k \in \mathbb{N}} \) weakly converging to some \( \psi \in K \subset \mathcal{D} (\mathcal{H}) \). Moreover, according to [70] (Lemma 2), a sequence in \( \mathcal{D} (\mathcal{H}) \) weakly convergent to a density operator is in fact norm convergent to that density operator. Thus, \( (\psi_{n(k)})_{k \in \mathbb{N}} \) is norm convergent to \( \psi \). Therefore \( K \) is sequentially compact, hence compact, in the trace norm topology. Q.E.D.

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