Mode coupling on a geometrodynamical quantization of an inflationary universe

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Abstract. A geometrodynamical quantization of an inflationary universe is considered in order to estimate quantum-gravity effects for the primordial perturbations. Contrary to previous studies in the literature, the back-reaction produced by all the modes of the system is included in our computations. Even if at a classical level the assumption that every mode evolves independently provides a good estimate for the dynamics, our results explicitly show that this is not the case when considering quantum-gravity effects. More precisely, both the self-interaction, as well as the back-reaction from other modes, provide a correction of the same order of magnitude to the usual power spectrum as computed in the approximation of quantum field theory on classical backgrounds. In particular, these quantum-gravity effects introduce certain characteristic scale-dependence on the expression of the power spectrum.

1 Introduction

The unification between gravity and the quantum theory has been one of the great challenges of theoretical physics during several decades. Apart from the many conceptual issues that this construction has to face, the lack of experimental evidence makes this search even more difficult. Indeed, quantum-gravity effects might only become measurable at energies of the order of the Planck scale, which is completely out of reach for the current particle accelerators. Nonetheless, it has been argued that the highly energetic inflationary phase of the primordial universe might be an adequate place to look for such effects.

In order to study the evolution of the fluctuations, as a first approximation, one usually considers quantum field theory (QFT) on classical backgrounds. In this context perturbations are treated as quantum variables, while the background follows its classical behavior. The goal of our approach is to consider also the quantum nature of the background, and compute corrections to the dynamics derived within the QFT approximation. In particular, we will be especially interested on the form of the power spectrum, for both scalar and tensor modes, since it entails observable effects.

More precisely, we will consider the geometrodynamical framework for quantum gravity that leads to the Wheeler–DeWitt equation. Therefore, starting from this equation, the natural question is how to recover the equations of QFT on classical backgrounds plus certain correction terms, which would encode the quantum-gravity effects. An approach based on a Born–Oppenheimer type of decomposition of the wavefunction has been widely used in the literature for inflationary scenarios [1]–[12]. This approximation is based on a factorization of the wavefunction: one part only depending on the slow degrees of freedom (background) and the other part encoding the information about the fast degrees of freedom (perturbations) for a given configuration of the background.

Another alternative is to perform a decomposition of the wavefunction into its infinite set of moments and follow the prescription for constrained systems presented in [13,14]. This formalism has been used

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to introduce a concept of time for semiclassical states in quantum cosmological models [15]–[19], and it has also already been applied to the analysis of the inflationary universe in [20]. In fact, the present paper will generalize the analysis performed in [20] in several important aspects. In particular, in that reference a truncation of the system at second-order in moments was implemented. The corrections to the power spectrum were obtained to be quadratic in second-order moments which, formally, would correspond to fourth-order terms. Therefore, in this paper we will explore the third- and fourth-order truncation schemes in detail. In addition, it was assumed that at the onset of inflation the system was on its fundamental state. Nevertheless, this might not be the case [21]–[28]. Hence, as already performed in the context of the Born–Oppenheimer approximation [29], here generically excited initial states will be considered. Finally, the most important generalization is that we will take into account the back-reaction produced by other modes on the evolution of the mode under consideration. Up to our knowledge, this effect has been neglected in all the previous studies in the literature, but our results will point out that it enters in the leading-order correction term to the power spectrum and it is thus not negligible.

The rest of the paper is organized as follows. In Sec. 2 the quantization of the system is performed. In Sec. 3 the quantum moments are defined and the general method is explained. Sec. 4 considers a truncation of third-order in moments and shows that, at this order, the power spectrum does get any correction term. In Sec. 5, after imposing a fourth-order truncation in the moments, the power spectrum for initial excited states and under the presence of infinite modes is obtained. Finally, our conclusions are presented in Sec. 6.

2 Canonical quantization of an inflationary model

We will consider an inflationary model given by a spatially flat Friedmann-Lemaitre-Robertson-Walker metric minimally coupled to an inflaton field \( \phi \) with potential \( V(\phi) \). As it is well known, by performing a harmonic decomposition on the homogeneous spatial sections, the linear perturbations of this model can be classified into three different types: scalar, vector and tensor perturbations. At linear level different sectors evolve independently and thus can be treated separately.

By construction, tensor perturbations are invariant under gauge transformations, while scalar and vector perturbations are not. Nonetheless, one can perform suitable canonical transformations and factor out all the gauge and constrained degrees of freedom, in such a way that the complete physical information of the problem is contained in three variables: on the one hand, the Mukhanov-Sasaki variable \( v_s \), which encodes the gauge-invariant perturbations of the inflaton field \( \phi \). And, on the other hand, \( v^+ \) and \( v^\times \), which describe the two polarizations of the gravitational wave. Let us, for compactness, denote all these variables as \( v^\sigma \), with \( \sigma = s, +, \times \).

The smeared Hamiltonian constraint that describes the full (both background and perturbative) dynamics of this system takes the following form,

\[
\int d\eta H := \int d\eta \int dx N C = \int d\eta \int dx \left[ -\frac{G}{2} \pi_a^2 + \frac{\pi_\phi^2}{2a^2} + a^4 V(\phi) \right] + \int d\eta H_{\text{pert}},
\]

where the lapse has been chosen equal to the scale factor \( N = a \). In addition, the reduced gravitational constant \( G \) has been defined as \( G := \frac{4\pi G_N}{3} \), with \( G_N \) being Newton’s gravitational constant. The variables \( \pi_a := -a'/G \) and \( \pi_\phi := a^2 \phi' \) are, respectively, the conjugate momentum of the scale factor \( a \) and of the inflaton field \( \phi \), where the prime stands for derivative with respect to the conformal time \( \eta \).

Since we are assuming flat spatial slices, in order to write down the perturbative Hamiltonian \( H_{\text{pert}} \), it is convenient to perform a Fourier transformation of the variables \( v^\sigma \), and their corresponding conjugate
momenta $\pi^\sigma$, as follows,

$$v^\sigma(\eta, x) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} dk \, u_k^\sigma(\eta) e^{ik \cdot x},$$  \hspace{1cm} (2)

$$\pi^\sigma(\eta, x) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} dk \, p_k^\sigma(\eta) e^{ik \cdot x}.$$  \hspace{1cm} (3)

These complex Fourier modes obey the reality condition $u_{-k}^\sigma = (u_k^\sigma)^*$ and $p_{-k}^\sigma = (p_k^\sigma)^*$, and the conjugate momentum of $u_k^\sigma$ is $p_{-k}^\sigma$, that is, $\{u_k^\sigma, p_q^\sigma\} = \delta(k + q)$. Nonetheless, in order to work with just real modes, it is possible to perform a subsequent transformation by simply decomposing each mode on its real and imaginary parts:

$$u_k^\sigma = \frac{1}{\sqrt{2}} (u_k^{\sigma,R} + i u_k^{\sigma,I}),$$  \hspace{1cm} (4)

$$p_k^\sigma = \frac{1}{\sqrt{2}} (p_k^{\sigma,R} + i p_k^{\sigma,I}),$$  \hspace{1cm} (5)

in such a way that $\{u_k^{\sigma,R}, p_q^{\sigma,R}\} = \delta(k - q)$ and $\{u_k^{\sigma,I}, p_q^{\sigma,I}\} = \delta(k - q)$. In terms of these real modes, the contribution to the Hamiltonian constraint from the perturbative variables can be written in the following compact form,

$$H_{\text{pert}} = \frac{1}{2} \sum_{r=R,I} \sum_{\sigma=s,+,-,\times} \int d^3k \left[ (p_k^{\sigma,r})^2 + \omega_{k,\sigma}^2 (u_k^{\sigma,r})^2 \right],$$  \hspace{1cm} (6)

where the integration is taken over half Fourier space, that is, $\mathbb{R}^2 \times \mathbb{R}^+$. Note that, in this expression there are two sums: one in $r$ over the real ($R$) and imaginary ($I$) part of the modes, and another one in $\sigma$ over the three different degrees of freedom (the scalar sector and the two polarizations of the tensor sector). Therefore, for each wavevector $k$, this Hamiltonian is the linear combination of six parametric oscillators. In the classical treatment, as well as in the approximation given by QFT on classical backgrounds, these six oscillators evolve independently.

Concerning the frequency $\omega_{k,\sigma}$, it depends on the conformal time $\eta$, on the wavenumber $k$, and on the sector: for the scalar modes it is given by $\omega_{k,s}^2 := k^2 - a''/a$, whereas for tensor modes it reads as $\omega_{k,+}^2 = \omega_{k,\times}^2 := k^2 - z''/z$, with $z := (2a'^2 - aa'')^{1/2}a'/a' = a(1 - (a'/a')^\gamma/(a'/a)^2)^{1/2}$. In consequence, since they obey the same equations of motion, the two oscillators corresponding to the scalar sector will follow an identical evolution, as long as one chooses the same initial state for both. This argument certainly also applies to the four oscillators of the tensor sector. Even if in our analysis the full Hamiltonian constraint (1) will be quantized and all these oscillators will couple, we will also consider the same initial state for the oscillators of a given sector, as it is done in QFT. Hence, one can perform the following transformation,

$$v_k^s := \sqrt{2} u_k^{s,R} = \sqrt{2} u_k^{s,I},$$  \hspace{1cm} (7)

$$\pi_k^s := \sqrt{2} p_k^{s,R} = \sqrt{2} p_k^{s,I},$$  \hspace{1cm} (8)

$$v_k^t := 2 u_k^{+,R} = 2 u_k^{+,I} = 2 u_k^{X,R} = 2 u_k^{X,I},$$  \hspace{1cm} (9)

$$\pi_k^t := 2 p_k^{+,R} = 2 p_k^{+,I} = 2 p_k^{X,R} = 2 p_k^{X,I},$$  \hspace{1cm} (10)

that leaves just two harmonic oscillators per wavevector: one for the scalar sector $(v_k^s, \pi_k^s)$ and one for the tensor sector $(v_k^t, \pi_k^t)$, with canonical brackets $\{v_k^\lambda, \pi_k^\lambda\} = \delta(k - q)\delta_{\sigma\lambda}$. These are the variables that will be considered for the rest of the analysis.
Let us now comment on the explicit spatial integral on the right-hand side of expression (1). Note that, due to homogeneity, none of the variables inside this integral have any spatial dependence. Therefore, it is immediate to perform the integral, which provides the spatial volume $L^3$ as a global factor.

This volume can be absorbed by performing the following rescaling of the scale factor $a$ and of the conformal time $\eta$:

$$a \rightarrow a/L, \quad \eta \rightarrow \eta L.$$  \hfill (11)

The rescaling of the time $\eta$ implies that the momenta are being rescaled as $\pi_a \rightarrow \pi_a/L^2$ and $\pi_\phi \rightarrow \pi_\phi/L^3$. If the volume $L^3$ is finite, all the above integrals in the Fourier space will be replaced by discrete sums.

In particular, if one assumes periodic boundary conditions, in Cartesian coordinates, the wavevector would be given by $k = 2\pi/L(n_1, n_2, n_3)$, and the integral $\int d^3k$ would be replaced by the sum

$$\left(\frac{2\pi}{L}\right)^3 \sum_{n_1, n_2, n_3},$$

running over half the Fourier space, which is given by the following set of integers: $\{\forall n_1, n_2, n_3 > 0\} \cup \{n_1, n_2 > 0, n_3 = 0\} \cup \{n_1 > 0, n_2 = 0, n_3 = 0\}$. Furthermore, the discretization factor that appears in front of the sum can be absorbed in the perturbative variables by a rescaling $v_k \rightarrow L^2/(2\pi)^3/2 v_k$ and $k \rightarrow k/L$. This rescaling, in combination with the above rescaling for the time variable, implies a transformation $\pi_k \rightarrow L/(2\pi)^{3/2} \pi_k$ for the momenta. Finally, the smeared Hamiltonian constraint takes the following form,

$$\mathcal{H} = -\frac{G}{2 \pi_a^2} + \frac{\pi_\phi^2}{2a^2} + a^4 V(\phi) + \frac{1}{2} \sum_{\sigma=S,T} \sum_{n_1, n_2, n_3} (\pi^\sigma_k)^2 + \omega^2_{k,\sigma} (v^\sigma_k)^2,$$  \hfill (12)

with canonical Poisson brackets $\{a, \pi_a\} = \{\phi, \pi_\phi\} = 1$ and $\{v^\sigma_k, \pi^\lambda_q\} = \delta_{kq} \delta_{\sigma\lambda}$, and $k$ being given in terms of the integers $n_1$, $n_2$ and $n_3$, as commented above. From this point on, and in order to alleviate the notation, the sum that appears in the last expression will be compactly denoted as $\sum_{k,\sigma}$. In addition, since it will be possible to treat all the sectors on the same footing, the $\sigma$ superindex will also be removed from the explicit notation of the different variables. In this way, we will write the above Hamiltonian as,

$$\mathcal{H} = -\frac{G}{2 \pi_a^2} + \frac{\pi_\phi^2}{2a^2} + a^4 V(\phi) + \frac{1}{2} \sum_{k} (\pi_k^2 + \omega_k^2 v_k^2).$$  \hfill (13)

The canonical quantization of the system is then performed by promoting different variables to operators, and Poisson brackets to commutators: $\{\cdot, \cdot\} \rightarrow i\hbar [\cdot, \cdot]$. Physical states are defined as those annihilated by the Hamiltonian operator, which leads to the Wheeler–DeWitt equation,

$$\hat{\mathcal{H}} \Psi = 0.$$  \hfill (14)

The goal of this paper is to solve this equation by performing a decomposition of the wavefunction in its infinite set of quantum moments. In particular, we will be interested on the evolution of the fluctuation of the perturbative variable $v_k$, that defines the power spectrum. More precisely, we will define the dimensionless quantity,

$$P_k := \frac{G k^3}{a^2} \langle \pi^2_k \rangle,$$  \hfill (15)

which, up to global numerical factors, provides the power spectrum for each sector.
3 Moment decomposition

Let us define the central moments of the wavefunction as follows,

$$
\Delta (a_{1}^{i_{1}}a_{2}^{i_{2}}\hat{\phi}^{i_{3}}\hat{\pi}_{k_{1}}^{m_{1}}\cdots \hat{\pi}_{k_{n}}^{m_{n}}\pi_{q_{1}}^{n_{1}}\cdots \pi_{q_{r}}^{n_{r}}) := \langle (a - a)^{i_{1}}(\hat{\pi}_{a} - \pi_{a})^{i_{2}}(\hat{\phi} - \phi)^{i_{3}}(\hat{\pi}_{\phi} - \pi_{\phi})^{i_{4}}(\hat{v}_{k_{1}} - v_{k_{1}})^{m_{1}}\cdots (\hat{v}_{k_{n}} - v_{k_{n}})^{m_{n}}(\pi_{q_{1}} - \pi_{q_{1}})^{n_{1}}\cdots (\pi_{q_{r}} - \pi_{q_{r}})^{n_{r}} \rangle_{Weyl},
$$

where the subscript Weyl stands for totally symmetric ordering, and the symbols without hat stand for the expectation value of their corresponding operator, that is, \( X := \langle \hat{X} \rangle \). The sum of the different powers on the above definition \((i_{1} + i_{2} + i_{3} + m_{1} + \cdots + m_{n} + n_{1} + \cdots + n_{r})\) will be referred as the order of the quantum moment.

These moments form an infinite set of variables and encode the complete physical information of the wavefunction. They are purely quantum variables and thus, in the classical limit, all of them are vanishing. On the one hand, in this formalism, instead of solving the corresponding equation for the wavefunction, which for the present model would be the Wheeler–DeWitt equation \((14)\), one deals directly with the equations of motion for the moments. In addition, the coupling between the moments and the classical equations of motion provide a direct measure of the quantum back-reaction. On the other hand, the main disadvantage of this formalism is that one deals with infinite variables. Therefore, in practice, a reduction of the system is usually necessary, which can be achieved by simply dropping moments of an order higher than a given truncation order. This kind of truncation is valid for peaked semiclassical states, since for such states a moment of order \(n\) has a value of the order of \(O(\hbar^{n/2})\).

In a previous study \([20]\), this system was analyzed with a truncation at second order. Here, we will study the third and fourth-order truncations, in order to check how these higher-order moments affect the physical results. Nonetheless, as already commented above, this is not the only generalization of the mentioned study. We are also considering the presence of infinite perturbative modes, which means that, even with a truncation at a given order in moments, we will still be dealing with an infinite number of variables.

In order to convert the Wheeler–DeWitt equation \((14)\) on a system of equations for the moments, we will follow the method presented in \([13]\). The main idea of this method is that, since the action of the Hamiltonian \(\hat{H}\) on any physical wavefunction must be zero, the expectation value of the Hamiltonian \(\langle \hat{H} \rangle\), as well as the expectation value of the Hamiltonian multiplied from the left by any operator \(\langle \hat{O}\hat{H} \rangle\), should also be vanishing. By adequately choosing the operator \(\hat{O}\) as the symmetrically ordered product of basic fluctuations,

$$
\hat{O} = (a - a)^{i_{1}}(\hat{\pi}_{a} - \pi_{a})^{i_{2}}(\hat{\phi} - \phi)^{i_{3}}(\hat{\pi}_{\phi} - \pi_{\phi})^{i_{4}}(\hat{v}_{k_{1}} - v_{k_{1}})^{m_{1}}\cdots (\hat{v}_{k_{n}} - v_{k_{n}})^{m_{n}}(\pi_{q_{1}} - \pi_{q_{1}})^{n_{1}}\cdots (\pi_{q_{r}} - \pi_{q_{r}})^{n_{r}}\rangle_{Weyl},
$$

and writing the mentioned expectation values in terms of the moments, one obtains a system of first-class relations that constraint the moments. These constraints encode a gauge freedom of the system, and one can then fix the gauge following the usual prescription for constrained systems. In the present context, this gauge freedom is related to the choice of a time variable. Classically there is just one constraint \((\mathcal{H} = 0)\), which allows one to choose one time variable and deparametrize the Hamiltonian. The new quantum constraints that appear at every order in moments are related to the choice of fluctuations and higher-order moments of the time variable.

In order to choose an internal time variable, the natural gauge-fixing conditions would be that all the moments related to that variable should be vanishing; so that it can be understood as a parameter, instead of a physical degree of freedom. In our case, as performed in the previous analysis \([20]\), the scale
factor $a$ will be chosen as the internal time variable, which is a monotonically increasing function. In particular, this is a well-behaved time variable in the case of a constant potential that will be discussed later; unlike the matter variables $(\phi, \pi, \omega)$, which are constant in that case. The system of constraints will be solved for its conjugate momentum $\pi_a$, which will then become the physical Hamiltonian of the deparametrized system.

4 Third-order truncation

In this section we will consider a third-order truncation in moments. In the first subsection we will assume a generic potential $V(\phi)$ for the inflaton field. The full system of first-class constraints will be obtained and, by choosing the scale factor $a$ as the time parameter, it will be solved in order to obtain the physical Hamiltonian of the system $\mathcal{H}_a := -\pi_a$. In the second subsection the dynamics for the constant-potential case will be analyzed in detail.

4.1 Deparametrization of the model for a general potential

At third order the relevant constraint equations for the moments will be given by the following expectation values:

$$\langle \hat{H} \rangle = 0, \quad (18)$$

$$\langle (\hat{X} - X)\hat{H} \rangle = 0, \quad (19)$$

$$\frac{1}{2} \left[ (\hat{X} - X)(\hat{Y} - Y) + (\hat{Y} - Y)(\hat{X} - X) \right] \hat{H} = 0, \quad (20)$$

where $X$ and $Y$ stand for all our basic variables: $(a, \pi, \phi, \pi, v_k, \pi_k)$. By performing an expansion around the expectation values, equation (18) can be written in terms of the moments as follows,

$$\langle \hat{H} \rangle = \sum \frac{\partial^N \mathcal{H}}{\partial a^{i_1} \partial \pi^{i_2} \partial a^i \partial \pi^{i_4} \partial a^{i_1} \partial \pi^{i_2} \partial a^{i_3} \partial \pi^{m_1} \partial \pi^{n_1} \partial \pi^{n_2}} \Delta(a^{i_1} \pi^{i_2} \phi^{i_3} \pi^{m_4} \omega_k^{m_1} \pi^{n_1} \pi^{n_2} v_k^{m_1} \pi^{n_1} \pi^{n_2}) = 0,$$

where $\mathcal{H}$ is the classical Hamiltonian (13), $N$ is the order of the derivative $(N := i_1 + i_2 + i_3 + i_4 + m_1 + \cdots + m_n + n_1 + \cdots + n_r)$ and the sum runs over all nonnegative integer values for all the indices $i, m, n$ and $j$. In addition, in this expression $\Delta(1) = 1$ and $\partial^0 \mathcal{H} = \mathcal{H}$ should be understood, and $\Delta(X) = 0$ by definition. If one truncates this expression at third order in moments, the expectation value of the Hamiltonian takes the following explicit form,

$$\langle \hat{H} \rangle = -\frac{G^2}{2\pi^2} + \frac{\pi^2}{2a^2} + a^4 V(\phi) + \frac{1}{2} \sum_{k,\sigma} \left( \frac{\pi_k^2}{\omega_k^2} \right) - \frac{G^2}{2\pi^2} \Delta(\pi^2) + \frac{3\pi^2 \Delta(a^2)}{2a^4} + \frac{3\pi \Delta(a^2 \pi)}{a^4}$$

$$- \frac{2\pi \Delta(\pi \phi)}{a^3} - \frac{2\pi \Delta(a^3)}{a^5} + \frac{\Delta(\pi \phi)}{2a^2} - \frac{\Delta(a \pi \phi)}{a^2} + 2a \left[ 3a \Delta(a^2) V(\phi) + 2\Delta(a^3) \right] V(\phi)$$

$$+ 2a^2 \left[ 2a \Delta(a \phi) + 3\Delta(a^2 \phi) \right] V'(\phi) + \frac{a^3}{2} \left[ a \Delta(\phi^2) + 4a^3 \Delta(a \phi^2) \right] V''(\phi) + \frac{a^4}{6} \Delta(\phi^3) V^{(3)}(\phi)$$

$$+ \frac{1}{2} \sum_{k,\sigma} \left( \Delta(\pi_k^2) + \omega_k^2 \Delta(\pi_k^2) \right) = 0. \quad (21)$$

The first four terms correspond to the classical Hamiltonian (13), whereas the rest are terms linear in moments that encode the quantum back-reaction of the state. In particular, it is clear from this expression that, upon quantization, the classical constraint is no longer fulfilled, as all the moments can not be vanishing at the same time.
Concerning the other two types of constraints that we need to consider, \([19] – [20]\), it can be shown that at third order they take the following form,

\[
(\langle \dot{X} - X \rangle \hat{H}) = \frac{i \hbar}{2} \frac{\partial \hat{H}}{\partial p_X} + \sum \frac{\partial^N \mathcal{H}}{\partial p_X} \Delta(X a_{i_1} \pi_{a_{i_2}} \phi_{j_1} \phi_{j_2} \phi_{j_3} \pi_{a_{j_4}} v_{k_1} \ldots v_{k_n} \pi_{q_1} \ldots \pi_{q_r}) \frac{1}{i_1! i_2! i_3! i_4! m_1! \ldots m_n! n_1! \ldots n_r!},
\]

with \(p_X\) being the conjugate momentum of the variable \(X\) under consideration, and

\[
\frac{1}{2} \langle (\hat{X} - X)(\dot{Y} - Y) + (\dot{Y} - Y)(\dot{X} - X) \rangle \hat{H} = \sum \frac{\partial^N \mathcal{H}}{\partial p_X} \Delta(XY a_{i_1} \pi_{a_{i_2}} \phi_{j_1} \phi_{j_2} \phi_{j_4} \phi_{j_3} \pi_{a_{j_4}} v_{k_1} \ldots v_{k_n} \pi_{q_1} \ldots \pi_{q_r}) \frac{1}{i_1! i_2! i_3! i_4! m_1! \ldots m_n! n_1! \ldots n_r!}.
\]

The \(X\) (and \(Y\)) inside the \(\Delta\) that defines the moment means that the order of the variable \(X\) (and \(Y\)) under consideration should be increased by one. For instance, for the case \(X = a\) and \(Y = \phi\),

\[
\Delta(XY a_{i_1} \pi_{a_{i_2}} \phi_{j_1} \phi_{j_2} \phi_{j_3} \phi_{j_4}) := \Delta(a_{i_1+1} \pi_{a_{i_2+1}} \phi_{j_1+1} \phi_{j_2+1} \phi_{j_3+1} \phi_{j_4+1}).
\]

In principle, in these last two expressions the sum should run over all nonnegative integers \(i_j, m_j\) and \(n_j\). But, since we are considering a truncation at third order in moments, the order of the derivative \(N\) (defined again as \(N := i_1 + i_2 + i_3 + i_4 + m_1 + \ldots + m_n + n_1 + \ldots + n_r\)) would just take the values \(N = 0, 1, 2\) in \([22]\), whereas in the expression \([23]\) it has only two possible values \(N = 0, 1\). For illustration purposes, the constraints \([22]\) are given in the App. \(A\) for each basic variable \(X\), but the constraints \([23]\) are quite lengthy and we will refrain from displaying them explicitly.

Note that, as one would expect, the constraints \([22]\) are complex. The imaginary term comes from the reordering of the operators and, at this order, is given by the derivative of the classical Hamiltonian constraint \(\mathcal{H}\) with respect to the conjugate momentum of the variable under consideration. Nonetheless, the constraints \([23]\) are real. In this latter case, the reordering terms are of order \(\hbar^2\), which corresponds to fourth order in moments, and thus have been dropped for the present third-order approximation.

As already commented above, these constraint equations are related to a gauge freedom. Hence, following the standard procedure, one needs to fix a gauge condition which, in this context, amounts to the choice of a time variable. In our case, the most natural time variable is the scale factor, as it is a monotonically increasing function during an inflationary evolution. Other expectation values, like \(\pi_a, \pi_q, v_k\) and \(\pi_k\), are vanishing (or very small), whereas the scalar field \(\phi\) is (approximately) constant during the slow-roll evolution. Choosing \(a\) as the time parameter implies that all the moments related to \(a\), that is, of the form \(\Delta(Xa)\) and \(\Delta(XYa)\), for all \(X, Y \neq \pi_a\), should be chosen to be vanishing. Thus, these will be our gauge-fixing conditions. The constraints \([21] – [23]\) should then be understood as equations to be solved for the moments related to \(\pi_a\) (that is, \(\Delta(X\pi_a)\) and \(\Delta(XY\pi_a)\) for all \(X, Y\)), as well as \(\pi_a\) itself. This procedure will deparametrize the system and define \(\mathcal{H}_a := -\pi_a\) as our physical Hamiltonian.

At this point, it is very instructive to count degrees of freedom and constraint equations. For definiteness let us, just for this purpose, assume the presence of only two different perturbative modes: \((v_k, \pi_k)\) and \((v_q, \pi_q)\). In this way, one would have eight independent basic variables \((a, \pi_a, \phi, \pi_{\phi}, v_k, \pi_k, v_q, \pi_q)\), and thus one would get eight constraints of the form \([22]\). Concerning constraints of the form \([24]\), there would be 36 of them, one for each independent couple of the basic variables. Even if large, the system of equations \([22]\), \([23]\), is linear in moments. Following the commented gauge-fixing procedure, these 44 equations should be solved for the 44 moments related to \(\pi_a\); that is, the 8 second-order moments of the form \(\Delta(X\pi_a)\) and the 36 third-order moments of the form \(\Delta(XY\pi_a)\), for all \(X, Y\). The last step would then be to replace this solution in the Hamiltonian constraint \([21]\), and solve it for
\( \pi_a \). In this way, and requesting \( \Delta(Xa) = 0 = \Delta(XY a) \), for all \( X, Y \neq \pi_a \), one would get the form of the physical Hamiltonian \( \mathcal{H}_a := -\pi_a \) in terms of the time \( a \), the physical variables \( (\phi, \pi_\phi, v_k, \pi_k, v_q, \pi_q) \), and their moments \( \Delta(XY) \) and \( \Delta(XYZ) \), with \( X, Y, Z \neq a, \pi_a \).

Note that the solution to the linear system (22)–(23) gives the moments \( \Delta(X\pi_a) \) and \( \Delta(XY \pi_a) \) in terms of moments unrelated to \( \pi_a \) with coefficients that depend on expectation values, and in particular on powers of \( \pi_a \). In general, the higher the truncation under consideration, the higher the powers of \( \pi_a \) that would appear in those coefficients as one considers expectation values of higher powers of operators. Therefore, when applying the commented procedure under the presence of infinite modes, and using the solution to the linear system (22)–(23) to remove moments of the form \( \Delta(\pi_a) \) and \( \Delta(X\pi_a) \), the Hamiltonian constraint (21) turns into the following sixth-order polynomial equation for \( \pi_a \):

\[
G^3\pi_a^6 - G^2 \left( Gp_a^2 + A_4 \right) \pi_a^4 + iG^2 A_3 \pi_a^3 + GA_2 \pi_a^2 + A_0 = 0,
\]

where \( p_a \) is defined as the classical value of \( \pi_a \),

\[
p_a := -\frac{1}{\sqrt{G}} \left[ \frac{\pi_\phi^2}{a^2} + 2a^4V(\phi) + \sum_{k,\sigma} \pi_k^2 + \omega_k^2 v_k^2 \right]^{1/2},
\]

and the coefficients \( A_i \) are real functions of the expectation values and moments unrelated to \( \pi_a \). In fact, the coefficients \( A_3 \) and \( A_4 \) have a quite simple form,

\[
A_3 = \frac{\pi_\phi^2}{a^2} - 4a^3V(\phi),
\]

\[
A_4 = \frac{19}{16} \Delta(\pi_\phi^2) + \frac{19}{16} a^4 \Delta(\phi^2) V''(\phi) + \frac{1}{3} a^4 \Delta(\phi^3) V^{(3)}(\phi) + \frac{19}{16} \sum_{k,\sigma} (\omega_k^2 \Delta(v_k^2) + \Delta(\pi_k^2)),
\]

whereas \( A_0 \) and \( A_2 \) are much lengthier and are given in App. B. Note that the only imaginary term that appears in equation (24) is multiplied by the coefficient \( A_3 \), which is equal to the time derivative of the Hamiltonian \( \partial \mathcal{H} / \partial a \).

We are interested in the (slow-roll) inflationary evolution of the system. And, even if we are considering quantum corrections, this formalism is valid as long as the evolution is close to a classical trajectory. Hence, let us study how the different coefficients behave during such an evolution. In particular, the scale factor goes as \( a \approx e^{Ht} \), with \( t \) being the cosmological time and \( H \) the (approximately) constant Hubble factor. The rest of the variables behave as \( \pi_a \approx e^{2Ht}, \pi_\phi = 0, V(\phi) \approx H^2, v_r \approx 0 \), and \( \pi_r \approx 0 \). Using these dependencies, we find that all the terms in the equation (24), except the imaginary term, increase as follows

\[
\pi_a^6 \approx A_4 \pi_a^4 \approx A_2 \pi_a^2 \approx A_0 \approx e^{12Ht}.
\]

The imaginary term also grows with time but much slower, concretely as \( A_3 \pi_a^3 \approx e^{9Ht} \), and thus can be neglected in relation with the rest. Therefore, under this assumption, equation (24) is simplified to the following form

\[
G^3\pi_a^6 - G^2 \left( Gp_a^2 + A_4 \right) \pi_a^4 + GA_2 \pi_a^2 + A_0 = 0.
\]

This is a third-order polynomial equation for \( \pi_a^2 \), with three independent solutions. Out of these three solutions, two happen to be complex. Therefore our physical Hamiltonian will be given by the unique real solution. Even if it has a quite complicate form, taking into account that \( A_0, A_2 \) and \( A_4 \) are linear in second- and third-order moments, we can make a Taylor expansion of the solution in the order of the moments, and write the physical Hamiltonian as

\[
\mathcal{H}_a := -\pi_a = -p_a - \frac{A_0}{2G^3 p_a^6} + \frac{A_2}{2G^2 p_a^3} - \frac{A_4}{2Gp_a}.
\]
From this expression one can clearly see that in the classical limit one would recover \( \pi_a = p_a \), since all the coefficients \( A_0, A_2, A_4 \) would vanish. Note also that the sign of \( p_a \) has been chosen as negative in order to get a positive Hamiltonian.

This last expression is the main result of this subsection, and provides a physical Hamiltonian that rules the quantum evolution of this model for any potential \( V(\phi) \), with the scale factor \( a \) playing the role of the evolution parameter. Due to this choice, there are no quantum moments related to the degree of freedom \((a, \pi_a)\): moments related to \( a \) are directly fixed by the gauge-fixing condition, whereas moments related to its conjugate momentum \( \pi_a \) are determined by solving the constraint equations. Therefore, the physical variables will be the expectation values \((\phi, \pi_\phi, v_k, \pi_k)\), along with its fluctuations and the correlations between them, as well as their third-order moments. The evolution equations for different variables can be obtained by computing their Poisson brackets with this Hamiltonian, and making use of the usual relation that provides the Poisson brackets between expectation values in terms of the commutator between the corresponding operators: \( \{\langle \hat{A} \rangle, \langle \hat{B} \rangle \} = -i\hbar[\hat{A}, \hat{B}] \).

4.2 Constant potential

Let us now analyze the dynamics of the system for the particular, but relevant, case of a constant potential \( V(\phi) = H^2/(2G) \), with \( H \) being the constant Hubble factor. At a classical level this model corresponds to the de Sitter universe, and it provides the dominant contribution to any slow-roll inflationary dynamics. For this case, the frequency of the modes coincides both for the tensor and scalar sectors, and takes the simple form \( \omega_k^2 = k^2 - 2a^2H^2 \).

Classically, if one chooses an initially vanishing value for the momentum \( \pi_\phi \), the inflaton field \( \phi \) is constant and \( \pi_\phi \) vanishes all along evolution. Nonetheless, this does not need to be the case for the quantum evolution, since the back-reaction terms might produce a nonvanishing time derivative of either of these variables. But we will show below that, by choosing an adequate initial (adiabatic) state at the onset of inflation, this will also be obeyed by the quantum evolution.

At the beginning of inflation \((a \to 0)\), the mode is well inside the horizon and the frequency tends to the wavenumber \((\omega_k \to k)\). Therefore, in the context of QFT on classical backgrounds, one usually considers the stationary state of a free mode on a flat Minkowski background. In particular, this implies the vanishing of the expectation value of the perturbative variable and its conjugate momentum: \( v_k = \pi_k = 0 \). Let us then analyze whether it is possible in our model to choose an adiabatic initial state, that evolves coherently and in particular keeps the initial conditions \( v_k = \pi_k = \pi_\phi = 0 \), as well as the value of the inflaton \( \phi \), constant. Under these conditions, the evolution of these four variables would be given by:

\[
\begin{align*}
\frac{dv_k}{da} &= \{v_k, H_a\} = -\frac{G}{2a^6H^3} \left[ \frac{\Delta(\pi_\phi^2 v_k)}{a^2} + \sum_{q,\sigma} \omega_q^2 \Delta(\pi_k^2 v_q^2) + \omega_q^2 \Delta(\pi_k v_q^2) \right], \\
\frac{d\pi_k}{da} &= \{\pi_k, H_a\} = \frac{G\omega_k^2}{2a^6H^3} \left[ \frac{\Delta(\pi_\phi^2 v_k)}{a^2} + \sum_{q,\sigma} \omega_q^2 \Delta(v_k^2 v_q^2) + \Delta(v_k^2) \right], \tag{30}
\end{align*}
\]

\[
\begin{align*}
\frac{d\phi}{da} &= \{\phi, H_a\} = -\frac{G}{2a^6H^3} \left[ \frac{\Delta(\pi_\phi^3)}{a^2} + \sum_{q,\sigma} \omega_q^2 \Delta(\pi_\phi v_q^2) + \Delta(\pi_\phi) \right], \tag{31}
\end{align*}
\]

\[
\begin{align*}
\frac{d\pi_\phi}{da} &= \{\pi_\phi, H_a\} = 0. \tag{32}
\end{align*}
\]
The form of these equations suggest that one should choose an initial state with,
\[
\Delta(\pi_k v_q^2) = \Delta(\pi_k^2 v_q) = \Delta(\pi_k^2 \pi_k) = 0,
\]
\[
\Delta(v_k \pi_q^2) = \Delta(v_k^2 v_q) = \Delta(\pi_k^2 v_k) = 0,
\]
\[
\Delta(\pi_\phi v_k^2) = \Delta(\pi_\phi \pi_k^2) = \Delta(\pi_\phi^3) = 0,
\]
so that the right-hand side of the above equations of motion is vanishing. The next step, in order to construct an adiabatic state, is to compute the evolution of these set of moments and check whether one can choose an initial state so that their time derivative is vanishing. This is in fact the case, under the condition that the following moments are also vanishing,
\[
\Delta(v_k \pi_q \pi_r) = \Delta(v_k \pi_q v_r) = 0,
\]
\[
\Delta(v_k \pi_q \pi_r) = \Delta(\pi_k \pi_q \pi_r) = 0,
\]
\[
\Delta(\pi_\phi v_k v_r) = \Delta(\pi_\phi \pi_k v_q) = 0.
\]
(34)

Under all the above conditions, the time derivative of this last set of moments also vanish, and makes the construction of the initial adiabatic state consistent. Therefore, in summary, if one chooses an initial state with vanishing value of the expectation values \(v_k, \pi_k, \pi_\phi\) and of the third-order moments (34)–(35), all these variables will be vanishing all along the evolution and \(\phi\) will be a constant of motion. (Note that there are certain third-order moments, as those related to the scalar field \(\phi\), which would not be vanishing.) For such a state, the Hamiltonian (29) takes the following simple form:
\[
H_a = a^2 H - \frac{\Delta(\pi_\phi^2)}{2a^2 H} + \frac{1}{2a^2 H} \sum_{k,\sigma} \left( \Delta(\pi_k^2) + \omega_k^2 \Delta(v_k^2) \right).
\]
(36)

We are particularly interested in computing the power spectrum of the perturbations, which is encoded in the fluctuation \(\Delta(v_k^2)\). In the present approximation, the evolution equation of this fluctuation is only explicitly coupled to the fluctuation of the momentum \(\Delta(\pi_k^2)\) and the correlation \(\Delta(v_k \pi_k)\). The equations for these three variables read as follows,
\[
\frac{d}{da} \Delta(v_k \pi_k) = \frac{1}{a^2 H} \left[ \Delta(\pi_k^2) - \omega_k^2 \Delta(v_k^2) \right],
\]
(37)
\[
\frac{d}{da} \Delta(v_k^2) = \frac{2}{a^2 H} \Delta(v_k \pi_k),
\]
(38)
\[
\frac{d}{da} \Delta(\pi_k^2) = -2 \omega_k^2 \Delta(v_k \pi_k).
\]
(39)

As it can be seen, there is no contribution from third-order moments, and in fact these equations are the same as the ones obtained in the approximation of QFT on classical backgrounds. Therefore, we conclude that the third-order terms will not contribute to modify the power spectrum. In the next section, we will then move on to study the fourth-order truncation, which will indeed produce some relevant contributions.

5 Fourth-order truncation

The analysis at fourth order would proceed in the same way as for third order: one has to write the Wheeler–DeWitt equation as a set of constraints for the moments, impose the gauge conditions \(\Delta(Xa) = 0 = \Delta(XY a)\), for all \(X, Y \neq \pi_a\), and solve them for \(\pi_a\) and its moments. This procedure
would define the physical Hamiltonian $\mathcal{H}_a = -\pi_a$, which would then provide the evolution equations for the different variables. Nonetheless, as will be explained below, at this order of truncation, it will not be possible to complete this algorithm analytically and some assumptions will have to be performed.

More precisely, at fourth order, the set of constraints is given by the equations we have already considered at the third-order truncation level, that is, $(18)$–$(20)$, as well as the new set of constraints,

$$\langle(\hat{X} - X)(\hat{Y} - Y)(\hat{Z} - Z)\rangle_{Weyl} = 0. \quad (40)$$

Although this adds a large number of equations to the system, these constraints, as well as the previous ones $(19)$–$(20)$, are linear in moments. Therefore, one can solve all these equations for moments related to $\pi_a$, and replace their form in the Hamiltonian constraint $(18)$, along with the gauge conditions $\Delta(XY a) = \Delta(Xa) = 0$, for all $X, Y \neq \pi_a$. In this way, one ends up with an equation, which does not contain any moment related to the degree of freedom $(a, \pi_a)$, and that should be solved for $\pi_a$ in order to deparametrize the system. But here appears the main technical difficulty: whereas within the third-order truncation this equation was the sixth-order polynomial equation $(24)$, at fourth-order the resulting equation is not polynomial in $\pi_a$. In fact, it contains complicated rational expressions of $\pi_a$ and it is thus not possible to obtain a closed analytic form for its solution. There are two main contributions that produce such complicated expressions. On the one hand, when considering higher-order truncations, one needs to consider expectation values of higher powers of operators which, when expanded in moments, introduce higher powers of the expectation values, and in particular of $\pi_a$, in the coefficients of the moments. On the other hand, the coupling of the equations $(19)$–$(20)$ and $(40)$ is more involved than at previous orders, since a given moment appears in more constraints. Therefore, linear combinations involving a larger number of equations, and with more complicated coefficients of the expectation values, have to be considered to solve the system for each moment.

Hence, in order to analyze the behavior of the system with a fourth-order truncation in moments, we will implement several simplifying assumptions. On the one hand, we will consider from the beginning the case of a constant potential $V = H^2/(2G)$, with the frequencies given by the expression $\omega_k^2 = k^2 - 2a^2 H^2$. On the other hand, we will assume that some properties of the system, that have been shown to be valid up to third order, are still valid at the next level of truncation. In particular, $\nu_k = \pi_k = \pi_\phi = 0$ will be considered to be an exact solution of the system. Since the Poisson brackets between any moment and the expectation values $(\nu_k, \pi_k, \pi_\phi)$ are vanishing, this solution can then be strongly imposed to construct the Hamiltonian for the moments. Finally, we will also assume that different imaginary terms do not contribute to the solution in a relevant way, so that it is safe to neglect them. In this way, the Hamiltonian constraint takes the simple form,

$$\langle \hat{H} \rangle = -\frac{G}{2} \pi_a^2 + \frac{a^4 H^2}{2 G} + \Delta(\pi_\phi^2) + \frac{1}{2} \sum_{k, \sigma} (\Delta(\pi_k^2) + \omega_k^2 \Delta(\nu_k^2)) = 0, \quad (41)$$

where the gauge conditions $\Delta(Xa) = \Delta(XYa) = 0$ have already been imposed. The nontrivial part of this equation is contained in the fluctuation $\Delta(\pi_a^2)$, which must be written in terms of moments unrelated to $\pi_a$ by solving the other constraint equations $(19)$, $(20)$ and $(40)$. The expression that is obtained for $\Delta(\pi_a^2)$ (as a function of $\pi_a$) is quite involve and, even under the present assumptions, makes $(41)$ to be a quite complicate equation for $\pi_a$. Thus, we will perform a Taylor expansion on the order of the moments, and assume that the solution can be written as,

$$\pi_a = p_0 + p_2 + p_3 + p_4, \quad (42)$$

with $p_a$ being the classical value of $\pi_a$ $(25)$, and $p_2$, $p_3$ and $p_4$ being respectively the contribution of second-, third- and fourth-order. One can then solve equation $(41)$ iteratively order by order, which
leads to the following form for the physical Hamiltonian,

\[
\mathcal{H}_a := -\pi_a = \frac{a^2 H}{G} + \frac{\Delta(\pi^2_\phi)}{2a^4 H} - \frac{G}{8a^{10} H^3} \Delta(\pi_\phi^4) + \frac{\hbar^2 G}{8a^6 H^3} \left( 4a^2 H^2 + \sum_k \omega_k^2 \right) \\
+ \frac{1}{2a^2 H} \sum_{k,\sigma} (\Delta(\pi_k^2) + \omega_k^2 \Delta(v_k^2)) - \frac{G}{4a^8 H^3} \sum_{k,\sigma} (\Delta(\pi_k^2 \pi_\sigma^2) + \omega_k^2 \Delta(v_k^2 \pi_\sigma^2)) \\
- \frac{G}{8a^6 H^3} \sum_{k,\sigma} \sum_{q,\sigma'} (\omega_q^2 \Delta(\pi_k^2 v_q^2) + \omega_k^2 \Delta(\pi_q^2 v_k^2) + \Delta(\pi_k^2 \pi_q^2) + \omega_k^2 \omega_q^2 \Delta(v_k^2 v_q^2)).
\]

As can be seen, unlike third-order moments, there are several fourth-order moments present in this Hamiltonian, which in principle will back-react on the evolution of second-order moments.

At this point, in order to obtain the evolution equations for different moments, one just needs to compute the Poisson brackets between each moment and this Hamiltonian. In particular, we are interested in obtaining the power spectrum (15), which is directly related to the fluctuation of the Hamiltonian, which in principle will back-react on the evolution of second-order moments.

Furthermore, by combining equations (44), (45) and (46), it is easy to get a unique third-order equation for the fluctuation of the perturbative variable \(\Delta(v_k^2)\), \(\Delta(\pi_k^2)\), \(\Delta(v_k \pi_k)\) take the form,

\[
\frac{d}{d\xi}(\Delta(v_k^2)) = -\frac{2}{k} \Delta(v_k \pi_k) + S_1, \\
\frac{d}{d\xi}(\Delta(v_k \pi_k)) = -\frac{1}{k} \left[ \Delta(\pi_k^2) - \omega_k^2 \Delta(v_k^2) \right] + S_2, \\
\frac{d}{d\xi}(\Delta(\pi_k^2)) = \frac{2}{k} \omega_k^2 \Delta(v_k \pi_k) + S_3,
\]

where we have introduced the dimensionless time variable \(\xi := k/(aH)\) in order to see more clearly how the dependence on the wavenumber \(k\) enters the different equations. Note that, except the source terms \(S_1\), \(S_2\) and \(S_3\), these equations are the same as one finds within the approximation of QFT on classical backgrounds. Therefore, quantum-gravity effects (that is, the back-reaction produced by other modes) is completely encoded in the sources,

\[
S_1 = \frac{G\xi^6 H^4}{k^7} \Delta(v_k \pi_k \pi_\sigma^2) + \frac{G\xi^4 H^2}{k^5} \sum_{q,\sigma} (\omega_q^2 \Delta(v_k \pi_k v_q^2) + \Delta(v_k \pi_k \pi_q^2)), \\
S_2 = \frac{G\xi^6 H^4}{2k^7} (\Delta(\pi_k^2 \pi_\sigma^2) - \omega_k^2 \Delta(v_k^2 \pi_\sigma^2)) \\
+ \frac{G\xi^4 H^2}{2k^5} \sum_{q,\sigma} (\omega_q^2 \Delta(\pi_k^2 v_q^2) - \omega_k^2 \Delta(\pi_q^2 v_k^2) + \Delta(\pi_k^2 \pi_q^2) - \omega_k^2 \omega_q^2 \Delta(v_k^2 v_q^2)), \\
S_3 = -\omega_k^2 S_1.
\]

Furthermore, by combining equations (14), (15) and (16), it is easy to get a unique third-order equation for the fluctuation of the perturbative variable \(\Delta(v_k^2)\),

\[
\xi^3 \frac{d^3 \Delta(v_k^2)}{d\xi^3} + 4\xi (\xi^2 - 2) \frac{d\Delta(v_k^2)}{d\xi} + 8 \Delta(v_k^2) = S,
\]

where the source \(S\) is defined as a linear combination of the time-derivatives of the above sources,

\[
S := \xi^3 \frac{d^2 S_1}{d\xi^2} - \frac{2\xi^3}{k} \frac{dS_2}{d\xi}.
\]
Our goal then is to solve equation (50). But this equation is coupled, through the source term $S$, to other second- and fourth-order moments. Hence, in principle one should deal with the complete system of evolution equations for the moments. Nonetheless, since we are interested in slight modifications to the evolution of the modes predicted by the approximation of QFT on classical backgrounds, it is natural to consider that the coupling between different degrees of freedom is weak, so that they can be treated as independent variables, and thus the moments are factorized; that is, $\Delta(v^n_k \pi^m_k \pi^2_\phi) = \Delta(v^n_k \pi^m_k) \Delta(\pi^2_\phi)$ and $\Delta(v^n_k \pi^m_k v^l_q \pi^r_q) = \Delta(v^n_k \pi^m_k) \Delta(v^l_q \pi^r_q)$. In addition, the moments that define the source term $S$ will be assumed to follow the evolution given by the approximation of QFT on classical backgrounds. These assumptions will allow us to obtain an analytic solution to equation (50), which will provide us with insight about the specific corrections that quantum-gravity effects produce on the power spectrum.

In particular, the factorization of the moments leads to the following form for the sources,

$$S_1 = \frac{G \xi^4 H^2}{k^8} \left[ \Delta(v_k \pi_k) \left( \frac{\Delta(\pi^2_k)}{a^2} + 2E_k \right) + (\Delta(v_k \pi^3_k) + \omega_k^2 \Delta(v_k^3 \pi_k)) \right],$$

(52)

$$S_2 = \frac{G \xi^4 H^2}{2k^5} \left[ \left( \Delta(\pi_k)^2 - \omega_k^2 \Delta(v_k)^2 \right) \left( \frac{\Delta(\pi^2_k)}{a^2} + 2E_k \right) + (\Delta(\pi^4_k) - \omega_k^4 \Delta(v_k^4)) \right].$$

(53)

In these expressions, the contribution from modes with a wavevector (and/or a sector) different from $k$ has been completely encoded in the function,

$$E_k := -\frac{1}{2} \left[ \Delta(\pi_k^2) + \omega_k^2 \Delta(v_k^2) \right] + \frac{1}{2} \sum_{q,\sigma} \left[ \Delta(\pi_q^2) + \omega_q^2 \Delta(v_q^2) \right],$$

(54)

which measures the difference between the energy of all modes and the energy of the particular mode under consideration. Note that the sum is for all wavevectors and all sectors $\sigma = s, +, \times$. Considering now the QFT evolution for the different moments, as given in App. C this energy function takes the explicit form,

$$E_k = \frac{\hbar}{2} k/\beta_k \left( 1 - \frac{1}{\xi^2} - \frac{1}{2\xi^4} \right) + \sum_{q,\sigma} \frac{\hbar}{2} q \beta_q \left( 1 - \frac{k^2}{\xi^2 q^2} - \frac{k^4}{2\xi^4 q^4} \right),$$

(55)

where the factor $\beta_k := (2N_k + 1)$ parametrizes the possibly excited state of the mode $k$ at the onset of inflation ($\xi \to \infty$). The source $S$ takes then the simple form,

$$S = -\frac{4\hbar \bar{H}^2 \beta_k}{k^4 \xi^2} (3 + 2\xi^2) + \alpha_1 k \xi^2 + 9\alpha_2 k^3),$$

(56)

where the dimensionless Hubble factor $\bar{H} := H/\sqrt{G\hbar}$ and the positive constants,

$$\alpha_1 := \sum_{q,\sigma} \frac{\beta_q}{q},$$

(57)

$$\alpha_2 := \sum_{q,\sigma} \frac{\beta_q}{6q^3},$$

(58)

have been defined. As a side remark, we note that the sums in the definitions of the coefficients $\alpha_1$ and $\alpha_2$ might be divergent and, in order to obtain their value, one would need to consider a regularization. For instance, it would make sense to consider a cut-off so that these sums only run over modes that are actually coupled to the mode under consideration. As we have considered a fiducial volume $L^3$ of the spatial sections, the natural infrared cut-off would be given by the inverse of this characteristic
background length scale $1/L$. Regarding the ultraviolet limit, one could consider as cut-off the inverse of the quantum-gravity scale $1/\sqrt{\hbar G}$. In any case, for the forthcoming discussion, $\alpha_1$ and $\alpha_2$ will be considered to be two positive constants that parametrize the quantum back-reaction on the dynamics of a particular mode.

Let us also comment that, as can be seen in (56), under the assumption of the QFT evolution for the moments that define the source $S$, the fluctuation of the momentum of the inflaton field $\Delta(\pi_\phi^2)$, which is constant through evolution, has completely disappeared from that expression. This is due to an exact cancellation of the coefficients that multiply this term and erases all the contributions from the moments of the background matter sector $(\phi, \pi_\phi)$ to the evolution of the fluctuation $\Delta(v_k^2)$. Nevertheless, even if this cancellation was not exactly fulfilled, this term would provide a term proportional to $\bar{H}^4$, which would negligible, since $\bar{H}$ is very small during inflation and the source $S$ is of order $\bar{H}^2$.

Now, with the expression (56) for the source, it is possible to obtain the analytic solution to equation (50),

$$\Delta(v_k^2) = \frac{\hbar \beta_k}{2k \xi^2} \left( \xi^2 + 1 \right) + \frac{\hbar \beta_k \bar{H}^2}{6k^4 \xi^2} \left[ 11 \beta_k + 3k \alpha_1 + 15k^3 \alpha_2 
+ 2 \left( \beta_k + 3k^3 \alpha_2 \right) \right.
\left. \left( 2 \cos(2\xi) \left( (\xi^2 - 1) \text{Ci}(2\xi) + 2\xi \text{Si}(2\xi) - \pi \xi \right)
- \sin(2\xi) \left( (\xi^2 - 1) (\pi - 2\text{Si}(2\xi)) + 4\xi \text{Ci}(2\xi) \right) \right) \right],$$  

where $\text{Si}$ and $\text{Ci}$ are, respectively, the sine and cosine integral functions. The integration constants have been fixed by requesting that at the onset of inflation $(\xi \to \infty)$ the fluctuation tends to its stationary form $\Delta(v_k^2) = \beta_k \hbar / (2k)$ for a flat background. This expression shows explicitly that the trajectory of the fluctuation is given by its form in the approximation of QFT on classical backgrounds, as can be seen in App. C plus certain correction terms that are of the order of $\bar{H}^2$.

Considering then the definition (15), and performing an expansion around the super-Hubble limit $(\xi \to 0)$, leads to the following expression for the power spectrum at late times,

$$\mathcal{P}_k = \mathcal{P}_k^{(0)} \left[ 1 + \bar{H}^2 \left( \frac{\beta_k}{3k^3} [11 - 4\gamma_E - 4\ln(2\xi)] + \frac{\alpha_1}{k^2} + \alpha_2 [5 - 4\gamma_E - 4\ln(2\xi)] \right) \right],$$  

with $\mathcal{P}_k^{(0)} := \bar{H}^2 / 2 \beta_k$ being the usual scale-independent spectrum obtained in the QFT approximation, and $\gamma_E \approx 0.577$ the Euler–Mascheroni constant. By evaluating the numerical values, this expression can be rewritten as follows,

$$\mathcal{P}_k \approx \mathcal{P}_k^{(0)} \left[ 1 + \bar{H}^2 \left( \frac{1.97 \beta_k}{k^3} + \frac{\alpha_1}{k^2} - 0.081 \alpha_2 - \frac{4}{3} \left( 3 \alpha_2 + \frac{\beta_k}{k^3} \right) \ln \xi \right) \right],$$  

where the time-dependence is contained in the last logarithmic term. At the horizon crossing $(\xi = 1)$ this term exactly vanishes; whereas, from that point on, for super Hubble scales $(\xi < 1)$, it will produce a positive contribution to the power spectrum.

The expression for the power spectrum (61) is the main result of this paper. As can be seen, the relative correction to the power spectrum obtained in the QFT approximation, is proportional to $\bar{H}^2$, which takes a very small value during inflation. In addition, this correction depends explicitly on the wavenumber $k$ and therefore breaks the scale-invariance of $\mathcal{P}_k^{(0)}$. In fact, concerning their dependence on the wavenumber, there are three different contributions that encode the quantum-gravity effects.

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3Nonetheless, let us mention that, in the super-Hubble limit $(\xi \to 0)$ only one of the three integration constants survives and provides the amplitude of the power spectrum $\mathcal{P}_k^{(0)}$. Therefore, modifying the integration constants would just introduce a global factor in the power spectrum $\mathcal{P}_k^{(0)}$. 

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On the one hand, the factor proportional to $\beta_k/k^3$ comes purely from the back-reaction of the mode with itself. That is, unlike in the QFT approximation, the evolution of the fluctuation $\Delta(v_k^2)$ is coupled to higher-order moments of the same mode, like $\Delta(v_k\pi_k^3)$ and $\Delta(v_k^3\pi_k)$. This coupling leads to the mentioned contribution to the power spectrum, which is inversely proportional to $k^3$. Therefore, this effect is more relevant for small wavenumbers, which encode the information about large scales. Furthermore, and as obtained also in the analysis presented in [29], this correction is linear on $\beta_k$, which parametrizes the energy of the initial state of the mode.

On the other hand, the terms that go with the positive constants $\alpha_1$ and $\alpha_2$ represent the back-reaction of all the modes on the mode under consideration. This includes modes with a different wavenumber from the one under consideration, as well as those from a different sector, but also self-interactions; since the sum in the definition of the positive coefficients $\alpha_1$ and $\alpha_2$ include all of them. The term with $\alpha_2$ provides an overall shift of the power spectrum, which is independent of the wavenumber. Therefore, in principle this effect could be absorbed in the amplitude of the power spectrum. Nevertheless, the term described by $\alpha_1$ is proportional to $1/k^2$ and, like the pure self-interaction term, does include a distinctive scale-dependence on the power spectrum. Up to our knowledge, these two corrective factors have not been obtained on any other previous study since, in one way or another, different computations in the literature assume that the evolution of a given mode is not affected by other modes. Nonetheless, we have shown that other modes provide a correction that is of the same order of magnitude as the one given by the mode itself, and thus their effects are not negligible.

It is interesting to note that, when particularizing the above computation to the presence of an unique mode, the three different terms contribute to provide a correction proportional to $\beta_k/k^3$. More precisely, if one considers a mode initially on its fundamental state ($\beta_k = 1$), and neglects the contribution from other modes ($\alpha_1 = 1/k$, and $\alpha_2 = 1/(6k^3)$), one obtains the power spectrum

$$P_{k_{\text{one mode}}} = P_k^{(0)} \left[1 + \frac{\bar{H}^2}{k^3} \left(\frac{11}{2} - 2\gamma_E - 2\ln(2\xi)\right)\right] \approx P_k^{(0)} \left[1 + \frac{\bar{H}^2}{k^3} (2.96 - 2\ln \xi)\right]. \tag{62}$$

This result coincides with the one presented in [20] and, up to small numerical differences, it also agrees with the form obtained in other approaches; see in particular the most recent value found within the Born–Oppenheimer type of approximation [12]. This particularization to one mode remarks the relevance of considering the contribution from the other modes. Note that just looking at the result from one mode [62], one would infer a correction of the form $1/k^3$ to the power spectrum. Nevertheless, when considering other modes, some of this dependence changes because one sums over all wavenumbers. In particular, as we have shown, this dependence is splitted in three different terms: one that keeps the form $1/k^3$, another one with a dependence of the form $1/k^2$ and the third one completely independent of $k$.

6 Conclusions

In this work we have obtained quantum-gravity corrections to the power spectrum for inflationary scalar and tensor perturbations in the context of a geometrodynamical quantization. Contrary to the usual approximation of QFT on classical backgrounds, the quantum behavior of all the degrees of freedom of the model (both background and perturbative) is considered in this case. More precisely, in this paper

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4In fact, there is a factor 3 of difference, which can be explained by the fact that, in [20], a second-order truncation in moments was considered, although quadratic combinations of second-order moments were kept to see how higher-order contributions might enter. In this way, essentially fourth-order contributions were estimated by quadratic combinations of second-order moments as, for instance, $\Delta(v_k^4) \approx \Delta(v_k^2)^2$. But, as can be seen in App. C this estimate is slightly short, and there is indeed an exact factor 3 in that relation, that is, $\Delta(v_k^4) = 3\Delta(v_k^2)^2$. 

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we have generalized previous results presented in \cite{20} in three different and relevant aspects. On the one hand, the third- and fourth-order truncations in moments have been studied. On the other hand, initially excited states have also been included. Furthermore, and more importantly, the presence of infinite modes have been considered. This last generalization has introduced some new scale-dependence in the power spectrum, and has shown that the back-reaction of other modes is as important as the self-interaction of a specific mode with itself. Therefore, as one would expect from a quantum-gravity theory, all the degrees of freedom are coupled and it is not correct to assume an independent evolution of a given mode, even to compute the leading-order quantum-gravity corrections.

From a methodological perspective, we have chosen the moments (16) as our basic variables in order to describe the system. In this way, the Wheeler–DeWitt equation (14) has been rewritten as an infinite set of constraint equations for these moments. These equations form a first-class system and encode certain gauge freedom, that is related to the choice of time variable. In our case, we have chosen the scale factor as the internal time parameter and, thus, once the system is deparametrized, its conjugate momentum turns out to be the physical Hamiltonian. In order to deal with this infinite system, as already commented, we have considered two different truncations in the order of the moments.

At third order we have been able to obtain the complete expression for the physical Hamiltonian (29) for any potential. Then, we have considered the case of a constant potential and, by constructing an initial adiabatic state, we have concluded that third-order moments do not contribute to the power spectrum of the perturbations.

At fourth-order the analysis is technically much more involve and therefore we have assumed the constant-potential case from the beginning. We have then obtained the evolution equation for the fluctuation of the perturbative variable $\Delta(v_k^2)$ (50), that is directly related to the power spectrum. The main differential part of this equation takes exactly the same form as within the approximation of QFT on classical backgrounds, and the quantum-gravity effects are completely encoded in the source term $S$. In order to solve this equation analytically, we have assumed that the source term follows the evolution given by the approximation of QFT on classical backgrounds. In this way, the exact form for the power spectrum (61) has been computed.

As has been obtained in several other approaches in the literature, the relative correction to the power spectrum due to quantum-gravity effects is of the order of the square of the Hubble factor $H^2$, which is a very small quantity during inflation. In addition, quantum-gravity effects introduce a distinctive scale-dependence on the power spectrum. In particular, there are three different terms. On the one hand, due to the self-interaction of the mode with itself, a dependence of the form $1/k^3$ arises. On the other hand, the back-reaction of all modes on the mode under consideration introduces two different terms: one that behaves as $1/k^2$, and another one that is scale-independent and produces an overall shift of the amplitude of the power spectrum. In previous studies only the dependence $1/k^3$ has been computed since, in one way or another, the back-reaction from other modes has been completely neglected. Nevertheless, we have shown that this produces an effect of the same order of magnitude as the self-interaction, and thus it is important to take it into account.

\section*{Acknowledgments}

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A Constraints of the form \((\hat{X} - X)\hat{H}\)

In this appendix we explicitly display the six independent constraints of the form \((\hat{X} - X)\hat{H}\) up to third order:

\[
\langle (\hat{a} - a)\hat{H} \rangle = -iG\hbar\pi_a + \frac{\pi_\phi}{a^2}\Delta(a\pi_\phi) + \frac{\Delta(a\pi_\phi^2)}{a^2} - 2G\pi_a\Delta(a\pi_a) - G\Delta(a\pi_a^2) - \frac{2\pi_\phi^2}{a^3}\Delta(a^2) + \frac{4\pi_\phi}{a^3}\Delta(a^2a) \\
+ \frac{2\pi_\phi^2}{a^4}\Delta(a^3) + 8a^3\Delta(a^2)\phi + 2a^2\Delta(a^3)\phi + 2a^4\Delta(a\phi^2)V'(\phi) + 8a^3\Delta(a^2\phi)V'(\phi) \\
+ a^4\Delta(a\phi^2)\phi''(\phi) + \sum_{k,\sigma} (2\pi_k\Delta(a\pi_k) + \Delta(a\pi_k^2) + 2\omega_k^2\pi_k\Delta(a\pi_k^2)) ,
\]

\[
\langle (\pi_a - a\pi_a)\hat{H} \rangle = \frac{i\hbar}{a^3}(\pi^2_\phi - 4a^6V(\phi)) + \frac{2\pi_\phi}{a^2}\Delta(a\pi_\phi) + \frac{\Delta(a\pi_\phi^2)}{a^2} - G(2\pi_a\Delta(a\pi_a) + \Delta(a\pi_a^2) + 3\pi_\phi^2a^3\Delta(a^2a) \\
- \frac{2\pi_\phi^2}{a^2}\Delta(a\pi_a) + \frac{4\pi_\phi}{a^3}\Delta(a\pi_a^2) + (2a\Delta(a\pi_a) + 3\Delta(a^2\pi_a) + 2a^2\pi_\phi)2a^2\phi V(\phi) + a^4\Delta(a\phi^2)\phi''(\phi) \\
+ (a\Delta(a\pi_a) + 4\Delta(a\pi_a^2))2a^2\phi V'(\phi) \\
+ \sum_{k,\sigma} (2\pi_k\Delta(a\pi_k) + \Delta(a\pi_k^2) + \omega_k^2(2v_k\Delta(a\pi_k) + \Delta(a\pi_k^2))) ,
\]

\[
\langle (\phi - \phi)\hat{H} \rangle = \frac{i\hbar\pi_\phi}{a^2} + \frac{2\pi_\phi}{a^2}\Delta(\phi_\pi_\phi) + \frac{\Delta(\phi_\pi_\phi^2)}{a^2} - 2G\pi_a\Delta(a\phi_\pi_\phi) - G\Delta(a\pi_\phi) - \frac{2\pi_\phi^2}{a^3}\Delta(a^2\phi_\pi_\phi) \\
+ \frac{2\pi_\phi^2}{a^4}\Delta(a^2\phi_\phi) + 2a^4\Delta(a^2\phi)\phi''(\phi) + 8a^3\Delta(a\phi^2)\phi''(\phi) + a^4\Delta(a^2\phi_\phi)\phi''(\phi) + 8a^3\Delta(a\phi^2)\phi''(\phi) \\
+ 12a^2\Delta(a^2\phi)\phi'(\phi) + \sum_{k,\sigma} (2\pi_k\Delta(\phi_\pi_k) + \Delta(\phi_\pi_k^2) + 2\omega_k^2\pi_k\Delta(\phi_\pi_k^2) ,
\]

\[
\langle (\pi_\pi - \pi_a\pi_\phi)\hat{H} \rangle = -iha^4V'(\phi) + \frac{2\pi_\phi}{a^2}\Delta(\pi_\phi^2) + \frac{\Delta(\pi_\phi^2)}{a^2} - 2G\pi_a\Delta(\pi_\phi^2) - G\Delta(\pi_\phi^2) - \frac{2\pi_\phi^2}{a^3}\Delta(a\pi_\phi) \\
- \frac{4\pi_\phi^2}{a^3}\Delta(a\pi_\phi^2) + \frac{4\pi_\phi^2}{a^4}\Delta(a^2\pi_\phi) + 3\Delta(a\phi^2)\phi V'(\phi) \\
+ (2a\Delta(a\pi_\phi) + 3\Delta(a^2\pi_\phi))2a^2\phi V(\phi) \\
+ a^4\Delta(\phi_\phi^2)\phi''(\phi) + \sum_{k,\sigma} (2\pi_k\Delta(\phi_\pi_k^2) + \Delta(\phi_\pi_k^2) + 2\omega_k^2\pi_k\Delta(\phi_\pi_k^2) ,
\]

\[
\langle (\pi_\pi - \pi_\pi)\hat{H} \rangle = -ih\omega_\pi\pi_\pi + \frac{2\pi_\phi^2}{a^2}\Delta(\pi_\phi^2) + \frac{\Delta(\pi_\pi^2)}{a^2} - 2G\pi_a\Delta(\pi_\pi^2) - G\Delta(\pi_\phi^2) - \frac{2\pi_\phi^2}{a^3}\Delta(a\pi_\phi) \\
- \frac{4\pi_\phi^2}{a^3}\Delta(a\pi_\phi^2) + \frac{3\pi_\phi^2}{a^4}\Delta(a^2\pi_\phi) + (2a\Delta(a\phi_\pi) + 2\Delta(a\phi_\pi))2a^3V'(\phi) \\
+ a^4\Delta(\phi^2\pi_\phi)\phi''(\phi) + (2a\Delta(a\pi_\phi) + 3\Delta(a^2\pi_\phi))2a^2\phi V(\phi) \\
+ \sum_{k,\sigma} (2v_k\Delta(\pi_k\pi_k) + \Delta(\pi_k^2\pi_k) + 2\omega_k^2\pi_k\Delta(\pi_k^2) ,
\]
\[
\langle (\hat{v}_q - v_q) \hat{H} \rangle = i \hbar \pi \frac{2\pi^2}{a^2} \Delta(\pi, v_q) + \frac{\Delta(\pi, v_q)}{a^2} - 2G_{\pi a} \Delta(\pi_a v_q) \quad G_{\pi a}(\pi_a v_q) - G_{\pi a} \Delta(\pi_a v_q) - \frac{2\pi^2}{a^3} \Delta(\pi, v_q)
\]

\[
- 4\frac{\pi^2}{a^3} \Delta(\pi, v_q) + \frac{3\pi^2}{a^4} \Delta(\pi, v_q) - (a \Delta(\phi, v_q)) + 4a' \Delta(\phi, v_q) + 2a^3 V''(\phi) + a^4 \Delta(\phi^2, v_q) V''(\phi) + 8a^3 \Delta(a v_q) V(\phi) + 12a^2 \Delta(a^2 v_q) V(\phi) + \sum_{k, \sigma} (2\pi_k \Delta(v_k v_q) + \Delta(v_k, v_q^2) + 2\omega_k v_k \Delta(v_k v_q) + \omega_k v_k \Delta(v_k^2 v_q))
\]

B Coefficients $A_0$ and $A_2$

In this appendix we provide the coefficients $A_0$ and $A_2$ that appear in the physical Hamiltonian (24) within the third-order truncation in moments:

\[
A_0 = \frac{1}{4} a^6 \left( V(\phi)^2 \left( a^6 \Delta(\phi^2) V''(\phi) + \Delta(\pi, \phi) \right) - 4a^6 \Delta(\phi^3) V'(\phi)^3 - 6a^6 \Delta(\phi^2) V(\phi) V'(\phi)^2 \right)
\]

\[
\left( 11\pi_0 \Delta(\pi^2, \phi) + 16 \Delta(\pi, \phi) \right) + \frac{\pi^2}{a^6} \left( \pi_0 \Delta(\pi, \phi) + 4 \Delta(\phi, \pi, \phi) \right) \quad V'(\phi)^2 a^6 + \frac{15}{a^6} V(\phi) \Delta(\pi, \phi)
\]

\[
\frac{1}{4} a^6 \left( V'(\phi)^2 \Delta(\pi, \phi) + \Delta(\pi, \phi) \right) + \frac{1}{16 a^4} \pi_0 \left( \Delta(\pi, \phi) \right) \pi_0 + \frac{1}{a^2} V'(\phi) \Delta(\pi, \phi)
\]

\[
- 24 \left( v_k \Delta(\pi, v_k) \omega_k^2 + \pi_k \Delta(\pi, \pi_k) \right) \pi_0 - 2\omega_k v_k (5v_k \Delta(\pi, \phi) - 24 \pi_k \Delta(\pi, \phi)) + 10 \pi_k \Delta(\pi, \phi))
\]

\[
- 3 \left( V(\phi) \pi_0 \left( \pi_0 \left( \Delta(\pi, \phi) \omega_k^2 + \Delta(\pi, \phi) \right) \right) - 12 \left( v_k \Delta(\pi, v_k) \omega_k^2 + \pi_k \Delta(\pi, \pi_k) \right) \right)
\]

\[
- 3 \left( \pi_0 \left( \pi_0 \left( \Delta(\pi, v_k) \omega_k^2 + \Delta(\pi, v_k) \right) \right) - 12 \left( \pi_0 \Delta(\pi, v_k) \omega_k^2 + \pi_k \Delta(\pi, \pi_k) \right) \right)
\]

\[
- 3 \left( \pi_0 \left( \pi_0 \left( \Delta(\pi, v_k) \omega_k^2 + \Delta(\pi, v_k) \right) \right) - 12 \left( \pi_0 \Delta(\pi, v_k) \omega_k^2 + \pi_k \Delta(\pi, \pi_k) \right) \right)
\]

\[
+ 18 \left( V(\phi) \pi_0 \Delta(\pi, v_k) \omega_k^2 + \pi_k \Delta(\pi, v_k) \omega_k^2 + \pi_k \Delta(\pi, v_k) \omega_k^2 + \pi_k \Delta(\pi, v_k) \omega_k^2 \right)
\]

\[
+ 3 \left( V(\phi) \left( \pi_0 \Delta(\pi, v_k) \omega_k^2 + \pi_k \Delta(\pi, v_k) \omega_k^2 + \pi_k \Delta(\pi, v_k) \omega_k^2 + \pi_k \Delta(\pi, v_k) \omega_k^2 \right) \right)
\]

\[
+ \sum_{k, \sigma} \sum_{q, \sigma'} \left\{ - \frac{3}{4 a^4} \omega_k v_k (v_q \Delta(\phi^2) + 4 \Delta(\phi, v_q) + \omega_q \Delta(\phi^2) + 4 \pi_q \Delta(\phi, v_q)) \right\}
\]

\[
+ \frac{18}{a^4} V'(\phi) \pi_0 \Delta(\phi, v_q) + v_q \Delta(\phi, v_q) + 4 \Delta(\phi, v_q) \right) - \frac{3 v_q \Delta(\pi, v_q) \omega_q^2 + 2 \Delta(\pi, v_q)^3}{2}
\]

\[
+ \frac{3}{a^2} \pi_0 \left( 12 \pi_v q \Delta(\phi, v_q) \omega_q^2 + \Delta(\phi, v_q) \omega_q^2 + 5 \omega_v q \Delta(\pi, v_q) - \pi_q \Delta(\pi, v_q) \omega_q^2 \right) \right)
\]

\[
+ \frac{1}{8 a^2} \pi_0 \omega_v q \Delta(\pi, v_q) \omega_q^2 + 12 \omega_v q \Delta(\pi, v_q) + 2 \Delta(\pi, v_q) \right) - \frac{2 \pi_0 \pi_v q \Delta(\pi, v_q) \omega_q^2 + 2 \Delta(\pi, v_q) \omega_q^2 \right) \right)
\]

\[
- \frac{1}{16} \pi_0 \left( \pi_0 \left( \omega_v q \Delta(\phi, v_q) + 2 \Delta(\phi, v_q) \right) + \pi_v q \Delta(\phi, v_q) + 2 \Delta(\phi, v_q) \right)
\]

\[
+ \frac{\pi_0 \left( \pi_0 \left( \omega_v q \Delta(\phi, v_q) + 2 \Delta(\phi, v_q) \right) + \pi_v q \Delta(\phi, v_q) + 2 \Delta(\phi, v_q) \right) \right)
\]
The evolution of different moments in the approximation of QFT value of the Hamiltonian operator, which, by performing an expansion around the expectation values motion for each variable just by computing the Poisson brackets with it. In particular, since it is

This Hamiltonian encodes the dynamics of the different modes, and one can obtain the equations of

\[ A_2 = \frac{1}{8a^4} \left[ 2a^2V'(\phi)V''(\phi)\Delta(\phi^2) + a^6V'(\phi) (8\Delta(\phi^2) + a^6(8V''(\phi)\Delta(\phi^3) + 14V'(\phi)\Delta(\phi^2))) \\
+ 36a^6\pi_0V'(\phi) + V''(\phi)a^6\pi_0\Delta(\phi^2) + 8\Delta(\phi^2\pi_0) + 2a^6V(\phi)\Delta(\pi_0^2) + 15\pi_0^2\Delta(\pi_0^2) \\
+ \sum_{k,\sigma} \left\{ 8a^4\pi_0\Delta(\pi_0^2) + 15a^4\pi_0^2\Delta(\pi_0^2) + 2a^6\pi_0\Delta(\pi_0^2 + \omega_k\Delta(\pi_0^2)) \\
+ 4a^8V'(\phi)(2\Delta(\phi^2) + 7\pi_0\Delta(\phi\pi_0)) + 2a^6\pi_0^2\Delta(\phi^2) + 7\pi_0\omega_k\Delta(\phi\pi_0)) + a^4\pi_0^2\Delta(\pi_0^2) \\
+ a^2\pi_0^2\Delta(\pi_0^2) + 28a^2\pi_0\omega_k\Delta(\pi_0^2) + 8a^6V'(\phi)\pi_0\omega_k\Delta(\phi^2) + 8a^4\pi_0\omega_k^2\Delta(v_k^2) + 28a^4\pi_0\omega_k^2\Delta(v_k^2) \\
+ a^2\pi_0^2\Delta(\pi_0^2) + 28a^2\pi_0\omega_k\Delta(\pi_0^2) + 8a^6V'(\phi)\pi_0\omega_k\Delta(\phi^2) + 8a^4\pi_0\omega_k^2\Delta(v_k^2) + 28a^4\pi_0\omega_k^2\Delta(v_k^2) \\
+ a^2\pi_0^2\Delta(\pi_0^2) + 28a^2\pi_0\omega_k\Delta(\pi_0^2) + 8a^6V'(\phi)\pi_0\omega_k\Delta(\phi^2) + 8a^4\pi_0\omega_k^2\Delta(v_k^2) + 28a^4\pi_0\omega_k^2\Delta(v_k^2) \right\} \\
+ \sum_{k,\sigma} \sum_{q,\sigma'} \left\{ 8a^4\pi_0\Delta(\pi_0^2v_q^2) + a^4\pi_0^2\Delta(\pi_0^2v_q^2) + 28a^4\pi_0^2\Delta(\pi_0^2v_q^2) + 28a^4\pi_0^2\Delta(\pi_0^2v_q^2) \\
+ a^4\pi_0^2\Delta(\pi_0^2v_q^2) + \omega_k^2\Delta(v_k^2) \right\}.
\]

C The evolution of different moments in the approximation of QFT on classical backgrounds

In this appendix we derive the evolution of the moments under the approximation of QFT on classical backgrounds. In this case, for each mode, one has a physical Hamiltonian and the quantum dynamics can be written as a Schrödinger functional equation,

\[ \hat{H}_k\Psi_k := \frac{1}{2}(\hat{\pi}_k^2 + \omega_k^2\hat{v}_k^2)\Psi_k(\eta, v_k) = i\hbar\frac{\partial\Psi_k(\eta, v_k)}{\partial\eta}, \]

with the conformal time \( \eta \). Equivalently, one can define an effective Hamiltonian as the expectation value of the Hamiltonian operator, which, by performing an expansion around the expectation values \( v_k := \langle \hat{v}_k \rangle \) and \( \pi_k := \langle \hat{\pi}_k \rangle \), takes the form

\[ \langle \hat{H} \rangle := \frac{1}{2}(\pi_k^2 + \omega_k^2v_k^2 + \Delta(v_k^2) + \Delta(\pi_k^2)). \]

This Hamiltonian encodes the dynamics of the different modes, and one can obtain the equations of motion for each variable just by computing the Poisson brackets with it. In particular, since it is quadratic, moments of different orders completely decouple. For the expectation values, second- and fourth-order moments one obtains the evolution equations,

\[ \begin{align*}
\hat{v}_k &= \pi_k, \\
\hat{\pi}_k &= -\omega_k^2\hat{v}_k, \\
\hat{v}_k &= 2\Delta(\pi_k^2), \\
\hat{\pi}_k &= 2\Delta(\pi_k^2) - \omega_k^2\Delta(v_k^2), \\
\Delta(v_k^2) &= -2\omega_k^2\Delta(v_k^2), \\
\Delta(v_k^2) &= 4\Delta(v_k^2), \\
\Delta(v_k^2) &= 3\Delta(v_k^2) - \omega_k^2\Delta(v_k^2), \\
\Delta(v_k^2) &= 2\Delta(v_k^2) - 2\omega_k^2\Delta(v_k^2), \\
\Delta(v_k^2) &= \Delta(\pi_k^2) - 3\omega_k^2\Delta(v_k^2), \\
\Delta(v_k^2) &= -4\omega_k^2\Delta(v_k^2),
\end{align*} \]
where the prime stands for the derivative with respect to the conformal time $\eta$.

Taking into account the form of the frequency for the constant-potential case, $\omega_k^2 = k^2 - \frac{2}{\eta^2}$, it is immediate to solve these equations. In addition, in order to impose the integration constants, one considers an adiabatic state at the onset of inflation ($k\eta \to -\infty$). In particular, this leads to a vanishing value of the expectation values, $v_k = 0$ and $\pi_k = 0$, all along evolution, and to the following form for the moments,

\[
\Delta(v_k^2) = \frac{h\beta_k}{2k\xi^2} (\xi^2 + 1),
\quad \Delta(\pi_k^2) = \frac{hk\beta_k}{2\xi^4} (\xi^4 - \xi^2 + 1),
\]

\[
\Delta(v_k\pi_k) = \frac{h\beta_k}{2\xi^3},
\quad \Delta(v_k^3) = \frac{3h^2\beta_k^2}{4k\xi^5} (\xi^2 + 1),
\]

\[
\Delta(\pi_k^2) = \frac{3k^2\beta_k^2}{4\xi^6} (\xi^4 - \xi^6 + 1),
\quad \Delta(v_k\pi_k^2) = \frac{h^2\beta_k^2}{2\xi^6} (\xi^6 + 3),
\]

\[
\Delta(v_k^3) = \frac{3h^2\beta_k^2}{4k\xi^7} (\xi^2 + 1),
\quad \Delta(\pi_k^3) = \frac{3h^2\beta_k^2}{4k\xi^7} (\xi^4 - \xi^6 + 1).
\]

These expressions have been given for the dimensionless time variable $\xi = -k\eta$. Furthermore, the factor $\beta_k := (2N_k + 1)$ parametrizes the initial state of each mode. Finally, we are not interested in third-order moments, as we do not need them for the computations performed in the main body of the article, but it is easy to show that they are vanishing for the initial state under consideration.

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