Revised Conditions for MRI due to Isorotation Theorem

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Abstract

We re-analyze the physical conditions for Magneto-rotational Instability (MRI) within a steady axisymmetric stratified disk of plasma, in order to account for the so-called isorotation theory (the spatial profile of differential angular velocity depends on the magnetic flux surface). We develop the study of linear stability around an astrophysical background configuration, following the original derivation in [15], but implementing the isorotation condition as the orthogonality between the background magnetic field and the angular velocity gradient. We demonstrate that a dependence on the background magnetic field direction is restored in the dispersion relation and, hence, the emergence of MRI is also influenced by field orientation.

Keywords: Magnetohydronamics (MHD)

1. Introduction and basic formalism

The existence of an Alfvénic instability in weakly magnetized rotating plasmas (the plasma parameter $\beta$ is much greater than unity) was discovered in 1959 by E. P. Velikhov [1] and implemented on an astrophysical setting by S. Chandrasekar in 1960 [2] and it is well known as the Magneto-Rotational Instability (MRI).

The intrinsic nature of such an unstable mode consists of the coupling between Alfvén waves and the plasma inhomogeneity, due to differential rotation (indeed MRI is suppressed in uniformly rotating systems). The most intriguing feature of MRI is, however, that it vanishes when the magnetic field has sufficiently high amplitude, because of the effect of the restoring force due to magnetic tension, which is able to stabilize the plasma displacements.

Despite being an instability driven by magnetic tension which does not transport matter, nonetheless MRI is very relevant for astrophysical systems and, in particular, for accretion structures.

The Shakura idea for accretion [3, 4] relies on the existence of an effective viscosity, emerging from the plasma turbulence and the differential rotation of the accreting matter. Actually MRI turns out to be the most appropriate (if not the only reliable) mechanism to promote such a turbulent regime in astrophysical accreting systems [5, 6] (see also [7], where MRI was re-analyzed, after the original derivation). The great impact of this issue can be easily understood when realizing that most space plasmas are weakly magnetized, especially stellar disk configurations [8].

This scenario has led many authors, over the years, to investigate MRI in some different contexts and system morphologies, see for instance [9, 10, 11, 12, 13, 14]. A rather general result has been achieved in [15], where a stratified and differentially rotating disk has been addressed in the context of MRI as a general property of axially symmetric profiles. The very surprising feature derived in [15] consists of the...
independence of the stability conditions on the intensity and direction of the background magnetic field. The aim of the present analysis is to demonstrate how the direction of the magnetic field is actually involved in the stability of the stratified disk, as soon as the so-called isorotation theorem (also known as corotation theorem) is taken into account [16]. This theorem states that, for a steady, axially symmetric and purely rotating background (exactly the adopted hypotheses in [15]), the disk angular velocity depends on the magnetic flux function only. In what follows, this aspect will be equivalently formulated as the orthogonality character of the angular velocity gradient with respect to the background magnetic field.

Here, we perform the linear perturbation analysis by accounting for the isorotation profile of the background plasma and then we demonstrate that the obtained dispersion relation takes the same form as the one in [15], but after the orthogonality constraint between the background magnetic field and the angular velocity gradient is imposed. Then we investigate how such a revised dispersion relation affects the general stability properties of the stratified disk. The main merit of the present study consists of re-introducing a dependence of the stable profile on the angle that the background magnetic field forms with the entropy gradient, i.e. the extent of the stability region depends on the direction of the stationary magnetic configuration.

A specific example is considered of a stratified disk around a compact massive astrophysical object, showing how a relevant stability region exists in the inner thick part of the disk.

The basic theoretical framework of the present discussion is the ideal magnetohydrodynamic (MHD) theory, as viewed in axial symmetry, i.e. reducing the dynamical problem to a two-dimensional one. In particular, in what follows, we will implement the momentum conservation equation:

\[ \rho \left( \frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} \right) = -\nabla p + \frac{1}{4\pi} \left( \nabla \times \vec{B} \right) \times \vec{B} . \]  

(1)

and the induction equation:

\[ \frac{\partial \vec{B}}{\partial t} = \nabla \times \left( \vec{v} \times \vec{B} \right) , \]  

(2)

where \( \vec{v} \) and \( \vec{B} \) denote the velocity and magnetic fields, respectively, while \( \rho \) is the plasma mass density and \( p \) its total pressure.

The ideal (hydrogen-like) plasma thermodynamics is regulated by the mass and entropy conservation equations, which read respectively

\[ \frac{\partial \rho}{\partial t} + \vec{v} \cdot \left( \rho \vec{v} \right) = 0 \]  

(3)

\[ \left( \frac{\partial \rho}{\partial t} + \vec{v} \cdot \nabla \right) \ln \left( \frac{p}{\rho} \right)^{-5/3} = 0 . \]  

(4)

Appropriately for the axial symmetry, in what follows we adopt standard cylindrical coordinates \((r,\phi,z)\) and we take both the background (denoted by a subscript \(0\)) and the perturbed (characterized by a corresponding subscript \(1\)) quantities as independent of the angle \(\phi\), while the former are also stationary (independent of the time variable \(t\)). More specifically, the background equilibrium is guaranteed by the simultaneous validity of the gravitostatic equilibrium and the Lorentz force-free condition, i.e.

\[ \nabla p_0 = \rho_0 \left( \omega_0^2 r \hat{r} - \omega_K^2 \hat{r}_p \right) \]  

(5)

\[ \left( \nabla \times \vec{B}_0 \right) \times \vec{B}_0 = 0 , \]  

(6)

where \( \omega_0 \) is the background disk angular velocity, \( \omega_K \) the Keplerian angular velocity and we adopt a compact notation for the poloidal vector radius \((i.e. \ \hat{r}_p = (r,0,z))\). We consider separately the force-free condition, because we regard the background field \( \vec{B}_0 \) as the radiation one due to the central star, i.e. we assume, according to standard literature, that, on the background level, the plasma does not provide a back-reaction field. Furthermore, the basic isorotation condition must hold \[16\]

\[ \vec{B}_0 \cdot \nabla \omega_0 = 0 . \]  

(7)
The linear perturbation theory assumes the local approximation limit, in which the wavelength of the disturbances is much less than the typical scale of variation characterizing the background material, i.e. we will deal with the condition $| \vec{k} \parallel \vec{r}_p | \gg 1$ ($\vec{k}$ being the wave vector used below) . Such assumption allows to treat the background variables as parametric quantities and to use Fourier analysis (i.e. both in the radial and vertical directions the perturbations have short wave length compared to background profile variations).

It is worth noting that the implications of the isorotation theorem are relevant for a stratified thick disk only, since, in the case of a thin disk it holds automatically. In fact, for a thin disk, well confined near the equatorial plane, the magnetic field is essentially vertical, the angular velocity gradient is essentially radial, $\omega_0$ depending on the radial coordinate only, and the vertical shear does not enter the problem. Thus the present analysis, as that one in [15], is physically relevant if the angular velocity changes rapidly enough along the vertical direction in comparison to the perturbation wavelengths (for discussion of perturbation analyses in the presence of vertical gradients see [17] [18]). The isorotation condition, stating the orthogonality between the background magnetic field and the angular velocity gradient, implies the relation

$$\partial_r \omega_0 - \frac{B_0}{B_0} \partial_r \omega_0. \tag{8}$$

Thus the vertical shear is comparable to the radial one only in thick disk configurations (see for instance the dipole-like case of Section 3).

Finally, we will adopt also the so-called Boussinesq approximation [5], which reduces the mass conservation equation to the local incompressibility of the perturbations, i.e. equation (3) becomes the constraint $\vec{k} \cdot \vec{v}_{p1} = 0$, where $\vec{v}_{p1}$ denotes the poloidal perturbed velocity. In a weakly magnetized plasma ($\beta \gg 1$), the sound speed $v_s$ is much larger than the Alfvén speed, since $\beta \sim v_A^2/v_s^2$. The limit of an arbitrarily large $v_s$ is equivalent to a diverging value of the polytropic index $\gamma$, associated to the equation of state $p_0 \propto \rho_0^\gamma$. In fact, we easily have $v_s^2 = \gamma p_0/\rho_0$ and the limit $\gamma \to \infty$ corresponds to an incompressible plasma (magnetized liquid), since a huge variation of the pressure corresponds to an infinitesimally small change in the mass density. Thus, for $\beta \gg 1$ the incompressibility condition, proper of the Boussinesq approximation, is adequate to describe the dominant features of the considered plasma system.

The study of the perturbation dynamics does not require, as we shall see, the specification of the equation of state, relating $p_1$ and $\rho_1$, since the perturbed pressure will be provided by requiring the validity of the local incompressibility during the plasma evolution. The resulting expression for $p_1$, for sufficiently large wavenumbers, results to be negligible in equation (4), which de facto becomes a link between $\rho_1$ and $\vec{v}_{p1}$.

2. Linear approximation

Let us consider the case of axisymmetric perturbations of the form $(\ldots)(t, \vec{r}_p) \propto e^{i(k \cdot \vec{r}_p - \Omega t)}$, with a poloidal wave-vector $\vec{k} = (k_r, 0, k_z)$ and frequency $\Omega$, according to the analysis in [15] that we are here revising (for a discussion on non axisymmetric perturbations allowed by the validity of the isorotation theorem see [14]). The basic equations from which we start are the perturbed poloidal and azimuthal components of the momentum conservation equation (1) and of the induction equation (2) in the linear approximation, which read

$$\Omega \vec{v}_{1r} - 2i\omega_0 \vec{v}_{p0} \vec{e}_r - \frac{\vec{k}}{\rho_0} p_1 - i \frac{\vec{e}_r p_0}{\rho_0^2} - \frac{1}{4 \pi \rho_0} (\vec{B}_0 \cdot \vec{B}_1 - \vec{k} \cdot \vec{B}_0 \vec{B}_{1p}) = 0 \tag{9}$$

$$\Omega \vec{v}_{1o} + 2i\omega_0 \vec{v}_{1pr} + \frac{\vec{k} \cdot \vec{B}_0}{4 \pi \rho_0} B_{1o} = 0 \tag{10}$$

and

$$\Omega \vec{B}_{1p} + \vec{k} \cdot \vec{B}_0 \vec{v}_{1p} = 0 \tag{11}$$

$$\Omega B_{1o} - i r \vec{B}_1 \cdot \vec{v}_{1p} + \vec{k} \cdot \vec{B}_0 \vec{v}_{1o} = 0, \tag{12}$$
respectively.

From (11) and (12) the components \( \vec{B}_1 \) and \( \vec{B}_0 \) of the magnetic field perturbation can be written in terms of other variables and the resulting expressions can be inserted into (9) (10). An expression for the perturbed pressure \( p_1 \) can be given by taking the scalar product of (9) with \( \vec{k} \) and using the mass conservation equation in the Boussinesq approximation, namely \( \vec{k} \cdot \vec{p}_1 = 0 \), which must be preserved during the dynamics. Eventually, the perturbed mass density can be given from the entropy conservation equation (4), which written for adiabatic perturbations, in the limit when \( p_1/p_0 \ll \rho_1/p_0 \) (which is consistent with the large value of \( k \) in the local approximation scheme, since \( p_1 \) is suppressed by a factor \( \sim 1/k^2 \)), reduces to the form

\[
\frac{d\Omega}{\Omega} \frac{\rho_1}{\rho_0} + \dot{v}_1 \cdot \vec{C}_0 = 0, \tag{13}
\]

where \( \vec{C}_0 = \vec{\nabla} \ln \left( p_0 \rho_0^{-5/3} \right) \).

The dispersion relation can be obtained by combining the conditions one gets from (9) and (10) after all these manipulations and by considering the following decomposition for the wave vector: since, by virtue of the isorotation theorem [7], the two vectors \( \vec{B}_0 \) and \( \vec{\nabla} \omega_0 \) are orthogonal and the wavevector \( \vec{k} \) has zero azimuthal component (the perturbations preserve axial symmetry), we can write

\[
\vec{k} = k_0 \frac{\vec{B}_0}{|\vec{B}_0|} + k_\perp \frac{\vec{\nabla} \omega_0}{|\vec{\nabla} \omega_0|}, \tag{14}
\]

or more explicitly

\[
k_r = k_0 \cos \iota + k_\perp \sin \iota, \quad k_z = -k_0 \sin \iota + k_\perp \cos \iota, \tag{15}
\]

with \( 0 \leq \iota < 2\pi \) being the angle between the background magnetic field and the equatorial plane, such that the background field reads \( \vec{B}_0 = |\vec{B}_0| \cos \iota, 0, -\sin \iota \). The axisymmetric configuration has a “north-south” symmetry, corresponding to the transformation \( r \to r \) and \( z \to -z \), under which the radial components of the magnetic field and the vertical components of vector fields (as the entropy gradient) change sign, while other components are unchanged. As a consequence, \( \iota \to \pi - \iota \), \( k_\parallel \to -k_\parallel \) and \( k_\perp \to k_\perp \). In what follows, one can verify that the north-south symmetry is realized, since all the formulas contain invariant terms (as \( k_\perp \sin \iota, k_\parallel k_\perp \cos \iota, \ldots \)).

The final dispersion relation is

\[
\tilde{\Omega}^6 + A \tilde{\Omega}^4 + B \tilde{\Omega}^2 + C = 0, \tag{16}
\]

with

\[
A = \frac{3}{5} \left( \vec{\nabla} p_0 \cdot \vec{C}_0 \right) + \left( \vec{\nabla} (\rho_0 \vec{k}) \cdot (\vec{k} \cdot \vec{C}_0) \right) - \frac{k_\perp^2}{k^2} \sin \iota + \frac{k_\parallel}{k^2} \frac{k_\perp}{k^2} \cos \iota - 4 \omega_0^2 \frac{k_\perp^2}{k^2}, \tag{17}
\]

\[
B = \frac{3}{5} \left( \frac{k_\parallel^2}{k^2} \sin \iota - \frac{k_\parallel}{k^2} \frac{k_\perp}{k^2} \cos \iota + 4 \omega_0^2 \frac{k_\perp^2}{k^2} \right) \left( \vec{\nabla} p_0 \cdot \vec{C}_0 \right) + \left( \vec{\nabla} (\rho_0 \vec{k}) \cdot (\vec{k} \cdot \vec{C}_0) \right) - 4 \omega_0^2 \omega_0^2 \frac{k_\perp^2}{k^2}, \tag{18}
\]

\[
C = \frac{12}{5} \omega_0^2 \left( \frac{k_\perp^2}{k^2} \vec{\nabla} p_0 \cdot \vec{C}_0 \right) - \left( \vec{\nabla} (\rho_0 \vec{k}) \cdot (\vec{k} \cdot \vec{C}_0) \right) + \left( \frac{C_0 k_\perp^2}{k^2} - C_\omega k_\parallel k_\perp \frac{\omega_0^2}{k^2} \right) \left( \frac{\omega_0^2}{\rho_0} - \frac{\omega_0^2}{\rho_0} \right) \frac{k_\perp^2}{k^2}. \tag{19}
\]

Above, we defined the quantities \( \omega_\Lambda^2 = \frac{\left( \vec{k} \cdot \vec{B}_0 \right)^2}{4 \pi \rho_0}, \quad \Omega^2 = \omega_\Lambda^2 - \omega_\Omega^2 \) and \( y_m = 2 \omega_\Omega r |\vec{\nabla} \omega_0| \).

If as in [15] one introduces

\[
\mathcal{D} = \frac{k_r}{k_\perp} \dot{\omega}_\Omega - \dot{\omega}_\iota, \tag{20}
\]
it can be shown that
\[ C \frac{k_0^2}{k^2} - C_0 \frac{k_0 k_z}{k^2} = - \frac{k_0^2}{k^2} D \ln \left( p_0 \rho_0^{-5/3} \right) \] (21)
\[ \frac{\vec{v}_{p_0} \cdot \vec{C}_0}{\rho_0} = \left( \frac{\vec{v}_{p_0} \cdot \vec{k}}{k^2 \rho_0} \right) \left( \vec{k} \cdot \vec{C}_0 \right) = \frac{k_z^2}{k^2} \frac{\rho_0}{\rho_0} D \ln \left( p_0 \rho_0^{-5/3} \right) \] (22)
\[ \frac{\partial_x p_0}{\rho_0} \frac{k_z^2}{k^2} - \frac{\partial_z p_0 k_0 k_z}{\rho_0} \frac{k_z^2}{k^2} = \frac{k_z^2}{k^2} \frac{\rho_0}{\rho_0} D \ln \left( p_0 \rho_0^{-5/3} \right) \] (23)

from which it follows that \( C = 0 \).

Moreover, the following relation holds for the second term on the right-hand side of (17)
\[ - \frac{k_z^2}{k^2} \frac{\sin \iota}{\partial_\iota} + \frac{y_m k_0 k_z}{k^2} \cos \iota - 4 \omega_0^2 \frac{k_z^2}{k^2} = \frac{k_z^2}{k^2} \frac{1}{r^3} D \left( \omega_0^2 r^4 \right). \] (24)

Similarly, using (21), (22) and (24) also the expression (18) for \( B \) can be simplified. Collecting all these results, we end up with the following expressions
\[ A = \left[ \frac{3 k_0^2}{k^2} D \ln \left( p_0 \rho_0^{-5/3} \right) \right] \frac{\rho_0}{\rho_0} + \frac{k_z^2}{k^2} \frac{1}{r^3} D \left( \omega_0^2 r^4 \right) \] (25)
\[ B = \left[ -4 \omega_0^2 \frac{k_z^2}{k^2} \right] \frac{\rho_0}{\rho_0} \] (26)
\[ C = 0, \] (27)

where \( \rho_0 \) means the imposition of the isorotation theorem, i.e. the decomposition (14) for \( \vec{k} \). The resulting dispersion relation formally coincides with that given by Balbus in [15], but for the implementation of the isorotation theorem. By other words, we are demonstrating that the isorotation theorem can be also formally coincides with that given by Balbus in [15], but for the implementation of the conditions can be easily shown to be equivalent

\[ x^2 \xi + \xi \left[ \frac{3}{5 p_0} \left( \frac{\partial_x p_0}{\rho_0} \partial_\iota \ln \left( p_0 \rho_0^{-5/3} \right) + \partial_x p_0 \partial_\iota \ln \left( p_0 \rho_0^{-5/3} \right) \right) - r \partial_x \omega_0^2 \right] + \frac{1}{r} \partial_r \partial_\iota \omega_0^2 > 0, \] (28)

where
\[ x = \frac{k_0}{k_z}, \quad N_2^2 = \frac{3}{5 p_0} \partial_x p_0 \partial_\iota \ln \left( p_0 \rho_0^{-5/3} \right). \] (29)

The inequality above is always verified if (we note that in [15] the first stability condition is different, but the whole set of conditions can be easily shown to be equivalent)
\[ N_2^2 > 0 \] (30)
\[ -\partial_x p_0 \left[ \partial_\iota \omega_0^2 \partial_\iota \ln \left( p_0 \rho_0^{-5/3} \right) - \partial_x \omega_0^2 \partial_\iota \ln \left( p_0 \rho_0^{-5/3} \right) \right] > 0. \] (31)

The imposition of the isorotation theorem here means that the derivatives of \( \omega_0 \) must be written according to the following relations
\[ r \partial_r \omega_0^2 = y_m \sin \iota \quad r \partial_\iota \omega_0^2 = y_m \cos \iota, \] (32)

which come from the decomposition of the radial and vertical directions as in (14). Therefore, the stability condition (31) becomes
\[ - \frac{y_m}{y_m} \partial_x p_0 \left[ \sin \iota \partial_\iota \ln \left( p_0 \rho_0^{-5/3} \right) - \cos \iota \partial_\iota \ln \left( p_0 \rho_0^{-5/3} \right) \right] > 0. \] (33)
which can be rewritten as follows

$$- y_m \partial_z p_0 \left( \mathbf{B}_0 \cdot \mathbf{C}_0 \right) > 0.$$  

(34)

This relation outlines how for a given matter configuration, i.e. given the spatial profiles of \( \rho_0 \) and \( p_0 \), the stability of the system depends on the direction of the magnetic field. It is worth noting how the north-south symmetry is still realized, since for \( z \rightarrow -z \) both \( \partial_z p_0 \) and \( \mathbf{B}_0 \cdot \mathbf{C}_0 \) change sign (\( \mathbf{B}_0 \) and \( \mathbf{C}_0 \) are pseudo-vector and vector fields, respectively), such that (34) is unchanged.

Summarizing, we demonstrated that deriving the dispersion relation when the isorotation theorem holds is equivalent to directly impose such a restriction on the spectrum presented in [15]. By other words, we revised the analysis of Balbus, by recognizing that the disk background must satisfy the isorotation condition and hence the dependence of the dispersion relation on the background magnetic field direction is revealed, under the assumption of isorotation.

It is worth noting that a similar conclusion can be inferred also from the global analysis of [19]. The dependence on the magnetic field direction is revealed, since for \( z \rightarrow -z \) both \( \partial_z p_0 \) and \( \mathbf{B}_0 \cdot \mathbf{C}_0 \) change sign (\( \mathbf{B}_0 \) and \( \mathbf{C}_0 \) are pseudo-vector and vector fields, respectively), such that (34) is unchanged.

3. Relevant application

Let us consider as an application of the previous analysis the case of a specific background morphology, describing a stratified (thick) accretion disk around a compact astrophysical object. According to equations [5], the background magnetic field is essentially that generated by the central object and it is here modeled by a dipole-like configuration (a reliable approximation for many concrete astrophysical systems), i.e. it is determined by the function \( \psi = \frac{N r^2}{(r^2 + z^2)^{5/2}} \), where the constant \( N \) fixes the magnetic field amplitude, and it leads to the following magnetic fields components

$$\mathbf{B} = \hat{\nabla} \wedge \left( \frac{\psi}{r} \right) \hat{e}_\phi = \frac{N}{(r^2 + z^2)^{5/2}} (3 rz, 0, 2 z^2 - r^2).$$  

(39)

The gradients of the pressure can be inferred from [4], by assuming \( \omega_0^2 = GM r^3 / (r^2 + z^2)^{9/2} \), \( M \) being the mass of the central object. This assumption comes from implementing the isorotation theorem and requiring that on the equatorial plane the angular velocity be Keplerian [20]. We investigate the equations for the
pressure gradients [5] and implement the consistency condition for which a solution exists (namely, that the mixed derivatives of the pressure coincide). This condition provides a restriction on the admissible form of the energy density: it turns out that the matter density is the product of \((r^2 + z^2)^{9/2}\) times an arbitrary function of \(r^8 - (r^2 + z^2)^4\), while the pressure can be explicitly computed and it equals the primitive of such arbitrary function modulo a constant. We stress that this form of the energy density and pressure is a mathematical requirement for the existence of a pressure profile for which (5) holds. The choice of the arbitrary function of \(r^8 - (r^2 + z^2)^4\) determines the physical properties of the considered matter distribution, but any choice is admissible mathematically. We choose an exponential function, leading to the following particular configuration

\[
\rho_0 = K \frac{r^8 - (r^2 + z^2)^4}{(r^2 + z^2)^{9/2}} e^{r_0^8 - (r^2 + z^2)^4 \ell}, \quad p_0 = K \ell^8 e^{r_0^8 - (r^2 + z^2)^4 \ell},
\]

\((40)\)

\(K\) and \(\ell\) being two integration constants. It is easy to check that these expressions solve (5) by direct substitution. The corresponding profile is characterized by an exponential decay along \(z\), with a decreasing decay length along the radial direction (see figure 1).

\[\text{Figure 1. The stable region of the system, denoted by the crossed area, is compared with its decay length (defined as the distance where the mass density is reduced by a factor } e \text{ with respect to the equatorial value), represented by the solid line. It is worth noting how the stable region comprises some areas in which the matter density is not negligible in the thick region of the disk.}\]

The stability condition (30) can be shown to be already contained into (33), such that the stable region is solely determined by the direction of the magnetic field according to (34) (it is drawn in figure 1). This is clarified in figure 2: in the stable region the angle between the magnetic field and the entropy gradient is acute, while in the unstable region it is obtuse.

This analysis outlines how the relative orientation of the magnetic field with respect to the entropy gradient is crucial for the stability of the whole system. In particular, for the considered configuration MRI is suppressed close to the central object in the inner part of the stratified disk (in correspondence to the lobes in figure 1) and therefore no turbulent transport can easily emerge there.

We observe how, far enough from the symmetry centre, the disk configuration is essentially thin and the stability region would take place in the outer part of the disk, i.e. at a bigger vertical distance than the characteristic decay length of the configuration (defined as the distance at which the density is reduced by a factor \(e\) with respect to its value at the equator). This means that the stability region is of little relevance far from the centre.

The relevant modification of the spectrum with respect to [15] takes place close to the centre of the disk configuration, where the stability domain overlaps the region in which the vertical distance is smaller than the decay length. This means that the stability condition holds in a domain where the disk mass density is significant.

We conclude this section observing how the provided example describes an accretion disk very far from a thin disk model: in the inner region the configuration has a spherical profile and its mass density rapidly
decays outward. Such a thick and stratified configuration is however predicted by the equilibrium and, in this respect, it does not constitute a limiting case. In Nature, plasma disks of this sort are, for instance, the so-called Advective Dominated Accretion Flows [21], or tori surrounding Active Galactic Nuclei.

Indeed, the dependence we traced above of the stability on the magnetic field direction better emerges in configurations allowing a significant range of variation for the ratio of the poloidal magnetic components.

4. Concluding remarks

We developed a perturbation analysis which accounts for the validity of the isorotation theorem on the background disk configuration, here translated as the orthogonality of the background magnetic field and the angular velocity gradients. The effect of such a restriction on the dispersion relation corresponds to restore a dependence on the background magnetic field direction into the mode spectrum (absent in [15]). Such an effect is relevant only for a stratified plasma disk since, when the $z$-dependence is weak or removed (like in a thin disk morphology), the isorotation constraint is automatically satisfied: the magnetic field is mainly vertical and the angular velocity gradient is essentially radially directed.

We have seen how, in a specific sample, the restored magnetic field dependence leads to a significant modification of the MRI morphology across the disk, since a threshold angle emerges. In fact, MRI can develop only when the angle between the magnetic field and the entropy gradient of the background configuration is bigger than $\pi/2$.

From an astrophysical point of view, the obtained result could affect the transport properties of thick stratified disks, since in some regions of the plasma profile the turbulent flow due to MRI is clearly suppressed (as elucidated in the previous section), together with the associated effective viscosity. This feature is particularly enhanced far from the equatorial plane (when also the dipole magnetic field is significantly inclined) and for stiff radial pressure gradients, i.e. when the disk rotation deviates from the Keplerian behavior [22].

Independently from the astrophysical range of applicability for the isorotation theorem (for a discussion see [14]), the present study removes the surprising and, a bit unnatural, feature discussed in [15], about the independence of MRI spectrum on the magnetic configuration, i.e. on its own trigger mechanism: actually the magnetic field direction enters the spectrum and also a forbidden spatial region for MRI emerges.
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