Relaxation limit in larger Besov spaces for compressible Euler equations

Jiang Xu∗
Department of Mathematics,
Nanjing University of Aeronautics and Astronautics,
Nanjing 211106, P.R.China

Shuichi Kawashima†
Graduate School of Mathematics,
Kyushu University, Fukuoka 812-8581, Japan

Abstract
The work is devoted to the relaxation limit in larger Besov spaces for compressible Euler equations, which contains previous results in Sobolev spaces and Besov spaces with critical regularity. Such an extension depends on a revision of commutator estimates and an elementary fact which indicates the connection between homogeneous and inhomogeneous Chemin-Lerner spaces.

Keywords. compressible Euler equations, relaxation limit, Chemin-Lerner spaces

AMS subject classification: 35L25, 35L45, 76N15

1 Introduction
In this paper, we consider the following nondimensional compressible Euler equations

\[
\begin{aligned}
\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) &= 0, \\
\partial_t (\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v}) + \nabla p(\rho) &= -\rho \mathbf{v}/\tau
\end{aligned}
\]  

(1.1)

for \((t, x) \in [0, +\infty) \times \mathbb{R}^N\) with \(N \geq 1\). Here \(\rho = \rho(t, x)\) is the fluid density function; \(\mathbf{v} = (v^1, v^2, \ldots, v^d)^\top\) (\(\top\) represents the transpose) denotes the fluid velocity. The pressure \(p(\rho)\) satisfies the classical assumption

\[p'(\rho) > 0, \quad \text{for any } \rho > 0.\]

*E-mail: jiangxu_79@yahoo.com.cn
†E-mail: kawashim@math.kyushu-u.ac.jp
An usual simplicity \( p(\rho) := \rho^\gamma (\gamma \geq 1) \), where the adiabatic exponent \( \gamma > 1 \) corresponds to the isentropic flow and \( \gamma = 1 \) corresponds to the isothermal flow. The nondimensional number \( 0 < \tau \leq 1 \) is a (small) relaxation time. The notation \( \nabla, \otimes \) are the gradient operator (in \( x \)) and the symbol for the tensor products of two vectors, respectively.

System (1.1) is complemented by the initial conditions

\[
(\rho, \mathbf{v})(0, x) = (\rho_0, \mathbf{v}_0).
\]  

(1.2)

For fixed \( \tau > 0 \), as we all know, the relaxation term which plays the role of damping can prevent the finite time blow-up and the Cauchy problem (1.1)-(1.2) admits a unique global classical solution, provided the initial data is small under certain norms. In this direction, such problem was widely studied by many authors, see e.g. [8, 9, 10, 15, 21, 22, 25, 26] and references therein. In addition, it is proved that the solutions in [25] has the \( L^\infty \) convergence rate \((1 + t)^{-3/2} (N = 3)\) to the constant background state and the optimal \( L^p(1 < p \leq \infty) \) convergence rate \((1 + t)^{-N/2/(1 - 1/p)}\) in general several dimensions [26], respectively. In one space dimension in Lagrangian coordinates, Nishida [21] obtained the global classical solutions with small data, and the solutions following Darcy's law asymptotically as time tends to infinity was shown by Hsiao and Liu [8]. For the large-time behavior of solutions with vacuum, the reader is referred to [9, 10]. Nishihara and Yang [22] studied the boundary effect on the asymptotic behavior of the solutions to the one-dimensional initial-boundary value problem. Later, Liu and Wang [15] considered the 2-D initial-boundary value problem in the wedge-space, and proved the global existence of classical solutions.

Another interesting line of research is to justify the singular limit as \( \tau \to 0 \) in (1.1). To do this, we introduce the time variable by considering an \( \mathcal{O}(1/\tau) \) time scale

\[
(\rho^\tau, \mathbf{v}^\tau)(s, x) = \left( \rho^\tau(s/\tau), \mathbf{v}^\tau(s/\tau), x \right).
\]  

(1.3)

Then

\[
\begin{aligned}
\partial_s \rho^\tau + \nabla \cdot \left( \frac{\rho^\tau}{\tau} \mathbf{v}^\tau \right) &= 0, \\
\tau^2 \partial_s \left( \frac{\rho^\tau}{\tau} \mathbf{v}^\tau \right) + \tau^2 \nabla \cdot \left( \frac{\rho^\tau}{\tau^2} \mathbf{v}^\tau \otimes \mathbf{v}^\tau \right) + \frac{\rho^\tau}{\tau} = -\nabla p(\rho^\tau)
\end{aligned}
\]  

(1.4)

with the initial data

\[
(\rho^\tau, \mathbf{v}^\tau)(x, 0) = (\rho_0, \mathbf{v}_0).
\]  

(1.5)

At the formal level, at least, if we can assume that \( \frac{\rho^\tau}{\tau} \) is uniformly bounded, it will be shown that the limit \( \mathcal{N} \) of \( \rho^\tau \) as \( \tau \to 0 \) satisfies the classical porous medium equation

\[
\begin{aligned}
\partial_t \mathcal{N} - \Delta p(\mathcal{N}) &= 0, \\
\mathcal{N}(x, 0) &= \rho_0,
\end{aligned}
\]  

(1.6)

which is a parabolic equation since \( p(\mathcal{N}) \) is strictly increasing.

This singular limit from hyperbolic relaxation to parabolic equations have attracted much attention, see [5, 14, 13, 17, 18, 20, 27] and therein references. The relaxation results mentioned for smooth solutions fell in the framework of the classical existence theory of Kato and Majda[11, 16]. The regularity index of Sobolev spaces (in \( x \)) \( H^\sigma(\mathbb{R}^N) \) is assumed to be high (\( \sigma > 1 + N/2 \) with integer). Recently, the first author and Wang [29] studied the limit case of regularity index (\( \sigma = 1 + N/2 \)) where the classical theory fails. They developed a new commutator estimate to
overcome the technique difficulty and constructed global classical solutions in the critical Besov spaces $B^{1+N/2}_{2,1}(\mathbb{R}^N)$. Furthermore, based on the Aubin-Lions compactness lemma in [23], it was justified that the (scaled) density converges to the solution of the porous medium equation.

The main purpose of this paper is to generalize the relaxation limits in Sobolev spaces with higher regularity and Besov spaces with critical regularity. Due to the elementary fact that Sobolev spaces $H^\sigma(\mathbb{R}^N) := B^\sigma_{2,2}(\mathbb{R}^N)$, a natural question is whether those results hold in larger Besov spaces $B^\sigma_{2,r}(\mathbb{R}^N)$ or not, whose indices satisfy the following condition:

$$\sigma > 1 + N/2, \ 1 \leq r \leq 2 \text{ or } \sigma = 1 + N/2, \ r = 1.$$  \hfill (1.7)

In the present paper, we shall answer the question. The main difficulty lies in the a priori nonlinear estimates, in particular, the commutator estimates. To overcome it, we need a more general version of commutator estimates in Proposition 2.3 (also see [12]), which relaxes the restriction on the couple $(s, r)$. For completeness and the reader convenience, we present the proof in Appendix with the aid of the Bony’s decomposition. Besides, this extension also heavily depends on an elementary fact developed for general hyperbolic system of balance laws (see Lemma 1.1 or [28]), which indicates the connection between homogeneous and inhomogeneous Chemin-Lerner spaces. Precisely, our results are stated as follows.

**Theorem 1.1.** Let the couple $(\sigma, r)$ satisfy the condition (1.7) and let $\bar{\rho} > 0$ be a constant reference density. There exists a positive constant $\delta_0$ independent of $\tau$ such that if

$$\|(\rho_0 - \bar{\rho}, m_0)\|_{B^\sigma_{2,r}(\mathbb{R}^N)} \leq \delta_0$$

with $m_0 := \rho_0 v_0$, then the Cauchy problem (1.1)-(1.2) has a unique global classical solution $(\rho, m) \in C^1(\mathbb{R}^+ \times \mathbb{R}^N)$ satisfying $(\rho - \bar{\rho}, m) \in \tilde{C}(B^\sigma_{2,r}(\mathbb{R}^N)) \cap \tilde{C}^1(B^{\sigma-1}_{2,r}(\mathbb{R}^N))$. Furthermore, the global solutions satisfy the inequality

$$\|(\rho - \bar{\rho}, m)\|_{\tilde{L}^\infty(B^\sigma_{2,r}(\mathbb{R}^N))} + \mu_0 \left(\|\frac{m}{\sqrt{\tau}}\|_{\tilde{L}^2(B^\sigma_{2,r}(\mathbb{R}^N))} + \|\sqrt{\tau} (\nabla \rho, \nabla m)\|_{\tilde{L}^2(B^{\sigma-1}_{2,r}(\mathbb{R}^N))}\right) \leq C_0\|(\rho_0 - \bar{\rho}, m_0)\|_{B^\sigma_{2,r}(\mathbb{R}^N)}, \hfill (1.8)$$

where $m := \rho v$, $\mu_0$ and $C_0$ are some uniform positive constants independent of $\tau(0 < \tau \leq 1)$.

**Remark 1.1.** It should be pointed out Theorem 1.1 contains previous results (see [5, 14, 25, 26, 29] and the references therein) for compressible Euler equations. The proof depends on the Fourier localization methods and Shizuta-Kawashima algebraic condition which has been well developed by the second author et al. [12, 30] for generally hyperbolic systems of balance laws. In particular, a concrete entropy for current compressible Euler equations is available, which has been verified in, e.g., [14, 28]. In addition, we track the singular parameter $\tau$ in the uniform energy inequality (1.8), which plays a key role in the analysis of relaxation limit problem.

As a direct consequence, using the standard weak convergence method and Aubin-Lions compactness lemma (23), we further have the analogue relaxation convergence as in [29].

**Theorem 1.2.** Let $(\rho, m)$ be the global solution of Theorem 1.1. Then it yields

$$\rho^\tau - \bar{\rho} \text{ is uniformly bounded in } C(\mathbb{R}^+, B^\sigma_{2,r}(\mathbb{R}^N));$$
\( \frac{\rho^\tau V^\tau}{\tau} \) is uniformly bounded in \( L^2(\mathbb{R}^+, B^2_{2,2}(\mathbb{R}^N)) \).

Furthermore, there exists some function \( N \in C(\mathbb{R}^+; \bar{n} + B^2_{2,2}(\mathbb{R}^N)) \) which is a unique solution of (1.6). For any \( 0 < T, R < \infty \), \( \{\rho^\tau(s,x)\} \) strongly converges to \( N(s,x) \) in \( C([0,T], (B^2_{2,2})^{-\delta}(B_r)) \) as \( \tau \to 0 \), where \( \delta \in (0,1) \) and \( B_r \) denotes the ball of radius \( r \) in \( \mathbb{R}^N \). In addition, it holds that

\[
\| (N(s,\cdot) - \bar{\rho})_{B^2_{2,2}(\mathbb{R}^N)} \leq C_0 \| (\rho_0 - \bar{\rho}, m_0)_{B^2_{2,2}(\mathbb{R}^N)}, s \geq 0, \tag{1.9}
\]

where \( C_0 > 0 \) is a uniform constant independent of \( \tau \).

Remark 1.2. Theorem 1.2 gives a rigorous description in larger spaces that the porous medium equation is usually regarded as an appropriate model for compressible inviscid fluids in small amplitude regime of relaxation time \( \tau \). In comparison with our recent results in [29], Theorem 1.1-1.2 also hold in the cases of general pressure and arbitrary space dimensions except for the regularity consideration.

Remark 1.3. From the symmetrization in Sect.3 we see that the dependence of the matrices \( A^j (j = 0,1,2 \cdots, N) \) with respect to the total variable \( W \) rather than only \( W_2 \). The concrete context of matrices enables us to obtain the uniform frequency-localization estimates and the relaxation limit. However, to the best of our knowledge, it is unknown for generally hyperbolic systems to get corresponding results. Therefore, this paper can be regarded as the effort to the open question in [3] (Remark 15, p.225).

The rest of this paper unfolds as follows. In Sect. 2 we briefly review the Littlewood-Paley decomposition and properties of Besov spaces and Chemin-Lerner spaces. In Sect. 3 we reformulate the equations (1.1) as a symmetric hyperbolic form in terms of entropy variables, and the local existence and blow-up criterion of classical solutions in critical spaces are presented. Sect. 4 is devoted to deduce the a priori estimates in Chemin-Lerner spaces by using Fourier-localization arguments, which are used to achieve the global existence of classical solutions.

2 Preliminary

Throughout the paper, \( f \lesssim g \) denotes \( f \leq C g \), where \( C > 0 \) is a generic constant. \( f \approx g \) means \( f \lesssim g \) and \( g \lesssim f \). Denote by \( C([0,T], X) \) (resp., \( C^1([0,T], X) \)) the space of continuous (resp., continuously differentiable) functions on \( [0,T] \) with values in a Banach space \( X \). Also, \( \|(f,g,h)\|_X \) means \( \|f\|_X + \|g\|_X \), where \( f,g \in X \). \( (f,g) \) denotes the inner product of two functions \( f,g \) in \( L^2(\mathbb{R}^N) \).

In this section, we briefly review the Littlewood-Paley decomposition and some properties of Besov spaces. The reader is also referred to, e.g., [1] for more details. We omit the space dependence, since all functional spaces (in \( x \)) are considered in \( \mathbb{R}^N \).

Let us start with the Fourier transform. The Fourier transform \( \hat{f} \) of a \( L^1 \)-function \( f \) is given by

\[
\mathcal{F} f = \int_{\mathbb{R}^N} f(x)e^{-2\pi i x \cdot \xi} \, dx.
\]

More generally, the Fourier transform of any \( f \in S' \), the space of tempered distributions, is given by

\[
(\mathcal{F} f, g) = (f, \mathcal{F} g)
\]
for any \( g \in \mathcal{S} \), the Schwartz class.

First, we fix some notation.

\[
\mathcal{S}_0 = \left\{ \phi \in \mathcal{S}, \partial^\alpha \mathcal{F} f(0) = 0, \forall \alpha \in \mathbb{N}^N \text{ multi-index} \right\}.
\]

Its dual is given by

\[
\mathcal{S}_0' = \mathcal{S}' / \mathcal{P},
\]

where \( \mathcal{P} \) is the space of polynomials.

We now introduce a dyadic partition of \( \mathbb{R}^N \). We choose \( \phi_0 \in \mathcal{S} \) such that \( \phi_0 \) is even,

\[
\text{supp} \phi_0 := A_0 = \left\{ \xi \in \mathbb{R}^N : \frac{3}{4} \leq |\xi| \leq \frac{8}{3} \right\}, \quad \text{and} \quad \phi_0 > 0 \text{ on } A_0.
\]

Set \( A_q = 2^q A_0 \) for \( q \in \mathbb{Z} \). Furthermore, we define

\[
\phi_q(\xi) = \phi_0(2^{-q} \xi)
\]

and define \( \Phi_q \in \mathcal{S} \) by

\[
\mathcal{F} \Phi_q(\xi) = \frac{\phi_q(\xi)}{\sum_{q \in \mathbb{Z}} \phi_q(\xi)}.
\]

It follows that both \( \mathcal{F} \Phi_q(\xi) \) and \( \Phi_q \) are even and satisfy the following properties:

\[
\mathcal{F} \Phi_q(\xi) = \mathcal{F} \Phi_0(2^{-q} \xi), \quad \text{supp } \mathcal{F} \Phi_q(\xi) \subset A_q, \quad \Phi_q(x) = 2^{Nq} \phi_0(2^q x)
\]

and

\[
\sum_{q=-\infty}^{\infty} \mathcal{F} \Phi_q(\xi) = \begin{cases} 1, & \text{if } \xi \in \mathbb{R}^N \setminus \{0\}, \\ 0, & \text{if } \xi = 0. \end{cases}
\]

As a consequence, for any \( f \in \mathcal{S}_0' \), we have

\[
\sum_{q=-\infty}^{\infty} \Phi_q * f = f.
\]

To define the homogeneous Besov spaces, we set

\[
\hat{\Delta}_q f = \Phi_q * f, \quad q = 0, \pm 1, \pm 2, ...
\]

**Definition 2.1.** For \( s \in \mathbb{R} \) and \( 1 \leq p, r \leq \infty \), the homogeneous Besov spaces \( \dot{B}^s_{p,r} \) is defined by

\[
\dot{B}^s_{p,r} = \left\{ f \in \mathcal{S}_0' : \|f\|_{\dot{B}^s_{p,r}} < \infty \right\},
\]

where

\[
\|f\|_{\dot{B}^s_{p,r}} = \begin{cases} \left( \sum_{q \in \mathbb{Z}} \|2^{qs} \|\hat{\Delta}_q f\|_{L^p}\right)^{1/r}, & r < \infty, \\ \sup_{q \in \mathbb{Z}} 2^{qs} \|\hat{\Delta}_q f\|_{L^p}, & r = \infty. \end{cases}
\]
To define the inhomogeneous Besov spaces, we set $\Psi \in C^\infty_0(\mathbb{R}^N)$ be even and satisfy

$$\mathcal{F}\Psi(\xi) = 1 - \sum_{q=0}^\infty \mathcal{F}\Phi_q(\xi).$$

It is clear that for any $f \in S'$, yields

$$\Psi * f + \sum_{q=0}^\infty \Phi_q * f = f.$$

We further set

$$\Delta_q f = \begin{cases} 0, & q \leq -2, \\
\Psi * f, & q = -1, \\
\Phi_q * f, & q = 0, 1, 2, \ldots \end{cases}$$

**Definition 2.2.** For $s \in \mathbb{R}$ and $1 \leq p, r \leq \infty$, the inhomogeneous Besov spaces $\overset{\text{s}}{B}_{p,r}$ is defined by

$$\overset{\text{s}}{B}_{p,r} = \{ f \in S' : \| f \|_{\overset{\text{s}}{B}_{p,r}} < \infty \},$$

where

$$\| f \|_{\overset{\text{s}}{B}_{p,r}} = \left\{ \left( \sum_{q=-1}^\infty (2^{qs} \| \Delta_q f \|_{L^p})^r \right)^{1/r}, \quad r < \infty, \right.$$

$$\left. \sup_{q \geq -1} 2^{qs} \| \Delta_q f \|_{L^p}, \quad r = \infty. \right.$$

Next we turn to Bernstein inequalities.

**Lemma 2.1.** Let $k \in \mathbb{N}$ and $0 < R_1 < R_2$. There exists a constant $C$, depending only on $R_1, R_2$ and $N$, such that for all $1 \leq a \leq b \leq \infty$ and $f \in L^a$,

$$\text{Supp} \mathcal{F} f \subset \{ \xi \in \mathbb{R}^N : |\xi| \leq R_1 \lambda \} \Rightarrow \sup_{|\alpha|=k} \| \partial^\alpha f \|_{L^b} \leq C^{k+1} \lambda^{k+d(\frac{1}{a} - \frac{1}{b})} \| f \|_{L^a};$$

$$\text{Supp} \mathcal{F} f \subset \{ \xi \in \mathbb{R}^N : R_1 \lambda \leq |\xi| \leq R_2 \lambda \} \Rightarrow C^{-k-1} \lambda^k \| f \|_{L^a} \leq \sup_{|\alpha|=k} \| \partial^\alpha f \|_{L^a} \leq C^{k+1} \lambda^k \| f \|_{L^a}.$$

As a direct corollary of the above inequalities, it holds that

**Remark 2.1.**

$$\frac{1}{C} \| f \|_{\overset{\text{k}}{B}_{p,r}^{+|\alpha|}} \leq \| \partial^\alpha f \|_{\overset{\text{s}}{B}_{p,r}^{+|\alpha|}} \leq C \| f \|_{\overset{\text{s}}{B}_{p,r}^{+|\alpha|}};$$

$$\| \partial^\alpha f \|_{\overset{\text{s}}{B}_{p,r}^+} \leq C \| f \|_{\overset{\text{s}}{B}_{p,r}^{+|\alpha|}},$$

for all multi-index $\alpha$.

The Besov spaces defined above obey various inclusion relations. In particular, we have

**Lemma 2.2.** Let $s \in \mathbb{R}$ and $1 \leq p, r \leq \infty$, then

1. If $s > 0$, then $\overset{\text{s}}{B}_{p,r}^s = L^p \cap \overset{\text{s}}{B}_{p,r}^s$;

2. If $\overset{\text{s}}{s} \leq s$, then $\overset{\text{s}}{S}_{p,r}^s \hookrightarrow \overset{\text{s}}{B}_{p,r}^s$.
(3) \( \dot{B}_{p,1}^{N/p} \hookrightarrow C_0, \quad B_{p,1}^{N/p} \hookrightarrow C_0(1 \leq p < \infty); \)

where \( C_0 \) is the space of continuous bounded functions which decay at infinity.

On the other hand, we also present the definition of Chemin-Lerner space-time spaces initialed by J.-Y. Chemin and N. Lerner [4], which are the refinement of the spaces \( L_T^0(\dot{B}_{p,r}^s) \) or \( L_T^0(B_{p,r}^s) \).

**Definition 2.3.** For \( T > 0, s \in \mathbb{R}, 1 \leq r, \theta \leq \infty \), the homogeneous mixed time-space Besov spaces \( \tilde{L}_T^0(\dot{B}_{p,r}^s) \) is defined by

\[
\tilde{L}_T^0(\dot{B}_{p,r}^s) := \{ f \in L^\theta(0,T;\mathcal{S}_0') : \|f\|_{\tilde{L}_T^0(\dot{B}_{p,r}^s)} < +\infty \},
\]

where

\[
\|f\|_{\tilde{L}_T^0(\dot{B}_{p,r}^s)} := \left( \sum_{q \in \mathbb{Z}} (2^{qs}\|\Delta_q f\|_{L_T^r(L^p)})^r \right)^{\frac{1}{r}}
\]

with the usual convention if \( r = \infty \).

**Definition 2.4.** For \( T > 0, s \in \mathbb{R}, 1 \leq r, \theta \leq \infty \), the inhomogeneous mixed time-space Besov spaces \( \tilde{L}_T^0(B_{p,r}^s) \) is defined by

\[
\tilde{L}_T^0(B_{p,r}^s) := \{ f \in L^\theta(0,T;\mathcal{S}') : \|f\|_{\tilde{L}_T^0(B_{p,r}^s)} < +\infty \},
\]

where

\[
\|f\|_{\tilde{L}_T^0(B_{p,r}^s)} := \left( \sum_{q \geq -1} (2^{qs}\|\Delta_q f\|_{L_T^r(L^p)})^r \right)^{\frac{1}{r}}
\]

with the usual convention if \( r = \infty \).

We only state some basic properties on the inhomogeneous Chemin-Lerner spaces, since the similar ones follow in the homogeneous Chemin-Lerner spaces.

The first one is that \( \tilde{L}_T^0(B_{p,r}^s) \) may be linked with the classical spaces \( L_T^0(B_{p,r}^s) \) via the Minkowski’s inequality:

**Remark 2.2.** It holds that

\[
\|f\|_{\tilde{L}_T^0(B_{p,r}^s)} \leq \|f\|_{L_T^0(B_{p,r}^s)} \quad \text{if} \quad r \geq \theta; \quad \|f\|_{\tilde{L}_T^0(B_{p,r}^s)} \geq \|f\|_{L_T^0(B_{p,r}^s)} \quad \text{if} \quad r \leq \theta.
\]

Let us also recall the property of continuity for product in Chemin-Lerner spaces \( \tilde{L}_T^0(B_{p,r}^s) \).

**Proposition 2.1.** The following inequality holds:

\[
\|fg\|_{\tilde{L}_T^0(B_{p,r}^s)} \leq C(\|f\|_{L_T^{\theta_1}(L^\infty)}\|g\|_{\tilde{L}_T^{\theta_2}(B_{p,r}^s)} + \|g\|_{\tilde{L}_T^{\theta_3}(L^\infty)}\|f\|_{\tilde{L}_T^{\theta_4}(B_{p,r}^s)})
\]

whenever \( s > 0, 1 \leq p \leq \infty, 1 \leq \theta, \theta_1, \theta_2, \theta_3, \theta_4 \leq \infty \) and

\[
\frac{1}{\theta} = \frac{1}{\theta_1} + \frac{1}{\theta_2} = \frac{1}{\theta_3} + \frac{1}{\theta_4}.
\]

As a direct corollary, one has

\[
\|fg\|_{\tilde{L}_T^0(B_{p,r}^s)} \leq C\|f\|_{\tilde{L}_T^{\theta_1}(B_{p,r}^s)}\|g\|_{\tilde{L}_T^{\theta_2}(B_{p,r}^s)}
\]

whenever \( s \geq d/p, \frac{1}{\theta} = \frac{1}{\theta_1} + \frac{1}{\theta_2} \).
In the next symmetrization, we meet with some composition functions. The following continuity result for compositions is used to estimate them.

**Proposition 2.2.** Let \( s > 0, 1 \leq p, r, \theta \leq \infty, F' \in W_{loc}^{[s]+1,\infty}(I;\mathbb{R}) \) with \( F(0) = 0, T \in (0,\infty] \) and \( v \in \tilde{L}_T^p(B_{p,r}^s) \cap L_T^\infty(L^\infty) \). Then

\[
\|F(f)\|_{\tilde{L}_T^\theta(B_{s,p}^r)} \leq C(1 + \|f\|_{L_\infty^p(L^\infty)})^{[s]+1} \|F'\|_{W_\infty^s(B_{p,r}^0)} \|f\|_{\tilde{L}_T^\theta(B_{s,p}^r)}.
\]

In addition, we present some estimates of commutators in homogeneous and inhomogeneous Chemin-Lerner spaces to bound commutators.

**Proposition 2.3.** Let \( 1 < p < \infty, 1 \leq \theta \leq \infty \) and \( s \in (-\frac{N}{p} - 1, \frac{N}{p}] \). Then there exists a generic constant \( C > 0 \) depending only on \( s, N \) such that

\[
\|[f, \Delta_q]g\|_{L^p} \leq C c_q 2^{-q(s+1)} \|f\|_{W_{B_{p,1}^s}^\theta} \|g\|_{\tilde{L}_T^\theta(B_{s,p}^r)},
\]

and

\[
\|[f, \Delta_q]g\|_{L^p(T^*)} \leq C c_q 2^{-q(s+1)} \|f\|_{L_{B_{p,1}^s}^\theta} \|g\|_{\tilde{L}_T^\theta(B_{s,p}^r)},
\]

with \( 1/\theta = 1/\theta_1 + 1/\theta_2 \), where the commutator \([\cdot, \cdot]\) is defined by \([f, g] = fg - gf\) and \(\{c_q\}\) denotes a sequence such that \(\|(c_q)\|_{l_1} \leq 1\).

## 3 Entropy and Local well-posedness

First, let us introduce the energy function which is just an entropy in the sense of Definition 2.1 of [12]:

\[
\eta(\rho, m) := \frac{|m|^2}{2\rho} + h(\rho) \quad \text{with} \quad m = \rho v \quad \text{and} \quad h'(\rho) = \int_1^\rho \frac{p'(s)}{s} ds.
\]

For this rigorous verification, see, e.g., [28]. Furthermore, the associated entropy flux is

\[
q(\rho, m) = \left(\frac{|m|^2}{2\rho^2} + \rho h'(\rho)\right) \frac{m}{\rho}.
\]

Define

\[
W = \left(\begin{array}{c} W_1 \\ W_2 \end{array}\right) := \nabla \eta(\rho, m) = \left(\begin{array}{c} -\frac{|m|^2}{2\rho^2} + h'(\rho) \\ m/\rho \end{array}\right).
\]

Clearly, the mapping \( U \to W \) is a diffeomorphism from \( \mathcal{O}_{(\rho, m)} := \mathbb{R}^+ \times \mathbb{R}^d \) onto its range \( \mathcal{O}_W \), and for classical solutions \((\rho, v)\) away from vacuum, (1.1) is equivalent to the symmetric system

\[
A^0(W)W_t + \sum_{j=1}^d A^j(W)W_{x_j} = H(W)
\]

with

\[
A^0(W) = \left(\begin{array}{cc} 1 & W_2^T \\ W_2 & W_2 \otimes W_2 + p'(\rho)I_d \end{array}\right),
\]

and

\[
A^j(W) = \left(\begin{array}{cc} 0 & W_2^T \\ W_2 & 0 \end{array}\right).
\]
\[
A^j(W) = \begin{pmatrix}
W_{2j} & W_2^\top W_{2j} + p'(\rho) e_j^\top \\
W_2 W_{2j} + p'(\rho) e_j & W_2 (W_2 \otimes W_2 + p'(\rho) I_d) + p'(\rho) (W_2 \otimes e_j + e_j \otimes W_2)
\end{pmatrix},
\]

where \( I_d \) stands for the \( d \times d \) unit matrix, and \( e_j \) is \( d \)-dimensional vector where the \( j \)th component is one, others are zero. It follows from the definition of entropy variable \( W \) that \( h'(\rho) = W_1 + |W_2|^2/2 \), so \( p'(\rho) \) can be viewed as a function of \( W \), since \( \rho \) is the function of \( W_1 + |W_2|^2/2 \), i.e. of \( W \).

The corresponding initial data become into

\[
W(0, x) := W_0 = \left( -\frac{|v_0|^2}{2} + h'(\rho_0), v_0 \right).
\] (3.2)

In [28], we have recently established a local well-posedness theory for generally symmetric hyperbolic systems in the framework of critical spaces, which is regarded as the generalization of the classical existence theory of Kato and Majda [11, 16]. Actually, the theory is also true in larger Besov spaces, whose the regularity indices satisfy the condition (1.7). Hence, we can have the following local existence result for the concrete problem (3.1)-(3.2).

**Proposition 3.1.** For any fixed relaxation time \( \tau > 0 \), assume that the initial data \( W_0 \) satisfy \( W_0 - \bar{W} \in \tilde{C}_{T_0} B_{2,r}^\sigma, \) \((W := (h'(\bar{\rho}), 0))\) and take values in a compact subset of \( \mathcal{O}_W \), then, there exists a time \( T_0 > 0 \) such that

(i) **Existence:** the Cauchy problem (3.1)-(3.2) has a unique classical solution \( W \in C^1([0, T_0] \times \mathbb{R}^d) \) satisfying

\[
W - \bar{W} \in \tilde{C}_{T_0} B_{2,r}^\sigma, \cap \tilde{C}^1_{T_0} B_{2,r}^{\sigma-1}.
\]

(ii) **Blow-up criterion:** there exists a constant \( C_0 > 0 \) such that the maximal time \( T^* \) of existence of such a solution can be bounded from below by \( T^* \geq \frac{C_0}{\|W_0 - \bar{W}\|_{B_{2,r}^\sigma}} \). Moreover, if \( T^* \) is finite, then

\[
\limsup_{t \to T^*} \|W - \bar{W}\|_{B_{2,r}^\sigma} = \infty
\]

if and only if

\[
\int_0^{T^*} \|\nabla W\|_{L^\infty} dt = \infty.
\]

4 **Global well-posedness**

This Section is devoted to the global existence result in Theorem [11]. To show that the solutions of (3.1)-(3.2), are globally defined, we need further a priori estimates.

To do this, for any time \( T > 0 \) and for any solution \( W - \bar{W} \in \tilde{C}_T B_{2,r}^\sigma, \cap \tilde{C}^1_T B_{2,r}^{\sigma-1} \), we define by \( E(T) \) the energy functional and by \( D_T(T) \) the corresponding dissipation functional:

\[
E(T) := \|W - \bar{W}\|_{\tilde{L}^\infty_T B_{2,r}^\sigma},
\]

\[
D_T(T) := \int_0^T \|\nabla W\|_{L^\infty} dt.
\]
\[ D_\tau(T) := \frac{1}{\sqrt{T}}\|W_2\|_{\tilde{L}_2^2(B_{2,r}^\tau)} + \sqrt{T}\|\nabla W\|_{\tilde{L}_2^2(B_{2,r}^\tau)}, \]

and \( E(0) := \|W_0 - \bar{W}\|_{B_{2,r}^\tau}. \) We also define
\[ S(T) := \|W\|_{L^\infty([0,T] \times \mathbb{R}^d)} + \|\nabla W\|_{L^\infty([0,T] \times \mathbb{R}^d)}. \]

Note that the imbedding in Lemma 2.2 and Remark 2.2, we have \( S(T) \leq CE(T) \) for some generic constant \( C > 0. \)

The next central task is to construct the desired a priori estimate, which is included in the following proposition.

**Proposition 4.1.** Let \( W \) be the solution of (3.1)-(3.2) satisfying \( W - \bar{W} \in \tilde{C}_T(B_{2,r}^\tau) \cap \tilde{C}_{1,T}(B_{2,r}^{\tau-1}). \) If \( W(t,x) \) takes values in a neighborhood of \( \bar{W}, \) then there exists a non-decreasing continuous function \( C: \mathbb{R}^+ \to \mathbb{R}^+ \) which is independent of \( \tau, \) such that the following nonlinear inequality holds:
\[ E(T) + D_\tau(T) \leq C(S(T)) \left( E(0) + E(T)^{1/2}D_\tau(T) + E(T)D_\tau(T) \right). \]

(4.1)

Furthermore, there exist some positive constants \( \delta_1, \mu_1 \) and \( C_1 \) independent of \( \tau, \) if \( E(T) \leq \delta_1, \) then
\[ E(T) + \mu_1D_\tau(T) \leq C_1E(0). \]

(4.2)

The proof of Proposition 4.1 in fact, is to capture the dissipation rates of \( W = (W_1, W_2) \) in turn by using the Fourier localization arguments. First, we give an elementary fact that indicates the connection between the homogeneous and inhomogeneous Chemin-Lerner spaces, which have been established in the recent work [28]. Precisely, it reads as follows:

**Lemma 4.1.** Let \( s > 0, 1 \leq \theta, p, r \leq +\infty. \) When \( \theta \geq r, \) it holds that
\[ L_{T}^{\theta}(L^{p}) \cap \tilde{L}_{T}^{\theta}(\tilde{B}_{p,r}^{\tau}) = \tilde{L}_{T}^{\theta}(B_{p,r}^{\tau}) \]

(4.3)

for any \( T > 0. \)

Next we begin to prove the Proposition 4.1. For clarity, we divide it into several lemmas.

**Lemma 4.2.** Under the assumptions stated in Proposition 4.1, there exists a non-decreasing continuous function \( C: \mathbb{R}^+ \to \mathbb{R}^+ \) which is independent of \( \tau, \) such that the following estimate holds:
\[ \|W - \bar{W}\|_{\tilde{L}_\infty^\infty(B_{2,r}^\tau)} + \frac{1}{\sqrt{T}}\|W_2\|_{\tilde{L}_2^2(B_{2,r}^\tau)} \leq C(S(T))(E(0) + E(T)^{1/2}D_\tau(T)). \]

(4.4)

**Proof.** The proof is divided into three steps.

Step 1. The \( \tilde{L}_\infty^\infty(B_{2,r}^\tau) \) estimate of \( W \) and the \( \tilde{L}_2^2(B_{2,r}^\tau) \) one of \( W_2 \)
Applying the homogeneous localization operator $\hat{\Delta}_q(q \in \mathbb{Z})$ to (3.1), and taking the inner product with $\hat{\Delta}_q$, we obtain

$$A^0(W)\hat{\Delta}_qW_t + \sum_{j=1}^d A^j(W)\hat{\Delta}_qW_{x_j} + L\hat{\Delta}_qW$$

with

$$L\hat{\Delta}_qW := \frac{1}{\tau}(p'(\bar{\rho})\hat{\Delta}_qW_2)$$

and

$$r(W) := \frac{1}{\tau}(\hat{\Delta}_q(p'(\rho) - p'(\bar{\rho})W_2),$$

where the commutator $[\cdot, \cdot]$ is defined by $[f, g] := fg - gf$.

Take the $L^2$-inner product of (4.5) with $\hat{\Delta}_qW$ to get

$$\frac{1}{2}\langle A^0(W)\hat{\Delta}_qW, \hat{\Delta}_qW\rangle_t + \frac{p'(\bar{\rho})}{\tau}\|\hat{\Delta}_qW_2\|^2_{L^2} = \sum_{i=1}^5 I_i(t),$$

where

$$I_1 := \frac{1}{2}(\partial_t A^0(W)\hat{\Delta}_qW, \hat{\Delta}_qW), \quad I_2 := \frac{1}{2}\sum_{j=1}^d(\partial_{x_j} A^j(W)\hat{\Delta}_qW, \hat{\Delta}_qW),$$

$$I_3 := -\langle[\hat{\Delta}_q, A^0(W)]\partial_t W, \hat{\Delta}_qW\rangle, \quad I_4 := -\sum_{j=1}^d\langle[\hat{\Delta}_q, A^j(W)]\partial_{x_j} W, \hat{\Delta}_qW\rangle, \quad I_5 := \langle r(W), \hat{\Delta}_qW\rangle.$$

In what follows, we begin to bound these nonlinear energy terms. Firstly, remark that

$$A^0(W) = \left(\begin{array}{cc} 1 & W_2^T \\
W_2 & W_2 \otimes W_2 \end{array}\right) + \left(\begin{array}{cc} 0 & 0 \\
0 & p'(\rho)I_d \end{array}\right)$$

$$:= A^0_{I}(W_2) + A^0_{II}(W),$$

we further get

$$\partial_t A^0_{I}(W_2) = \left(\begin{array}{cc} 0 & \partial_t W_2^T \\
\partial_t W_2 & \partial_t W_2 \otimes W_2 + W_2 \otimes \partial_t W_2 \end{array}\right)$$

and

$$\partial_t A^0_{II}(W) = \left(\begin{array}{cc} 0 & 0 \\
0 & p''(\rho)(D_W \rho(W)\partial_t W)I_d \end{array}\right),$$

where $D_W$ stands for the (row) gradient operator with respect to $W$. From the decomposition in (4.7) of $A^0(W)$, $I_1$ can be written as the sum

$$I_1(t) = \frac{1}{2}\langle \partial_t A^0_{I}(W_2)\hat{\Delta}_qW, \hat{\Delta}_qW\rangle + \frac{1}{2}\langle \partial_t A^0_{II}(W)\hat{\Delta}_qW, \hat{\Delta}_qW\rangle =: I_{11}(t) + I_{12}(t).$$
With the aid of the embedding properties in Lemma 2.2 and Remark 2.1, we obtain

\[
\left| \int_0^T I_{11}(t) dt \right| = \left| \int_0^T \int_{\mathbb{R}^d} (\partial_t W_2 \cdot \hat{\Delta}_q W_2 \hat{\Delta}_q W_1 + \frac{1}{2} |\hat{\Delta}_q W_2|^2 W_2 \cdot \partial_t W_2) \right|
\leq C(S(T)) \left( \int_0^T \|\partial_t W_2\|_{L_\infty} \|\hat{\Delta}_q W_2\|_{L^2} \|\hat{\Delta}_q W\|_{L^2} dt \right)
\leq C(S(T)) \left( \int_0^T \left( \|\nabla W\|_{L_\infty} + \frac{1}{r} \|W_2\|_{L_\infty} \right) \|\hat{\Delta}_q W_2\|_{L^2} \|\hat{\Delta}_q W\|_{L^2} dt \right)
\leq C(S(T)) \left( \|\nabla W\|_{L^\infty(L^\infty)} \|\hat{\Delta}_q W_2\|_{L^2(L^2)} \|\hat{\Delta}_q W\|_{L^2(L^2)} + \frac{1}{r} \|\hat{\Delta}_q W\|_{L^\infty(L^\infty)} \|W_2\|_{L^2(L^2)} \|\hat{\Delta}_q W_2\|_{L^2(L^2)} \right).
\tag{4.11}
\]

Furthermore, multiplying the factor \(2^{2q_\sigma}\) on both sides of (4.11) gives

\[
2^{2q_\sigma} \left| \int_0^T I_{11}(t) dt \right| \leq C(S(T)) 2^{q_\sigma} \left( \left( \int_0^T \int_{\mathbb{R}^d} |\partial_t W| |\hat{\Delta}_q W_2|^2 dx dt \right) \right.
+ \left. \frac{1}{r} \|W_2\|_{L^2_\sigma(B^\sigma_{2,r})} \|W_2\|_{L^2_\sigma(B^\sigma_{2,r})} \right).
\tag{4.12}
\]

Here and below, \(\{c_q\}\) denotes some sequence which satisfies \(\|\{c_q\}\|_{l^r} \leq 1\) although each \(\{c_q\}\) is possibly different in (4.12). Allow us to abuse the notation for simplicity. Similarly, we have

\[
2^{2q_\sigma} \left| \int_0^T I_{12}(t) dt \right| \leq C(S(T)) 2^{q_\sigma} \left( \left( \int_0^T \int_{\mathbb{R}^d} |\partial_t W| |\hat{\Delta}_q W_2|^2 dx dt \right) \right.
+ \left. \frac{1}{r} \|W_2\|_{L^2_\sigma(B^\sigma_{2,r})} \|W_2\|_{L^2_\sigma(B^\sigma_{2,r})} \right).
\tag{4.13}
\]

where we used the smallness of \(\tau(0 < \tau \leq 1)\).

Combining (4.12)–(4.13), it follows from the basic fact in Lemma 4.1 and Young’s inequality that

\[
2^{q_\sigma} \left| \int_0^T I_1(t) dt \right| \leq C(S(T)) c^2_\sigma E(T) D_\tau(T)^2. \tag{4.14}
\]

Secondly, we turn to estimate \(I_2\). For this purpose, we write

\[
A^j(W) = \left( \begin{array}{cc}
W_{2j} & W_2^T W_{2j} \\
W_2 W_{2j} & W_{2j} (W_2 \otimes W_2)
\end{array} \right)
+ \left( \begin{array}{cc}
0 & p'(\rho) e_j^T \\
p'(\rho) e_j & W_{2j} p'(\rho) I_2 + p'(\rho) (W_2 \otimes e_j + e_j \otimes W_2)
\end{array} \right)
:= A^j_1(W_2) + A^j_{11}(W).
\tag{4.15}
\]

Therefore, it is not difficult to get

\[
2^{2q_\sigma} \left| \int_0^T \frac{1}{2} \sum_{j=1}^d (\partial_{y_j} A^j_1(W_2) \hat{\Delta}_q W, \hat{\Delta}_q W) dt \right|
\]
\begin{align*}
&\leq C(S(T))2^{2\sigma} \|\nabla W_2\|_{L^2_{q}(L^\infty)} \|\hat{\Delta}_q W\|_{L^2_{q}(L^2)} \|\hat{\Delta}_q W\|_{L^2_{q}(L^2)} \\
&\leq C(S(T))2^q \|W - \hat{W}\|_{L^\infty_{q}(B_{q,r}^+)\|W_2\|_{L^2_{q}(B_{q,r}^+)} \|\nabla W\|_{L^2_{q}(B_{q,r}^+)}} \\
&\leq C(S(T))2^q \|W - \hat{W}\|_{L^\infty_{q}(B_{q,r}^+)} \left(\frac{1}{\tau} \|W_2\|_{L^2_{q}(B_{q,r}^+)} + \tau \|\nabla W\|_{L^2_{q}(B_{q,r}^+)}}\right) \\
&\leq C(S(T))2^q E(T)D_\tau(T)^2. \tag{4.16}
\end{align*}

Thanks to the null block located at the first line and the first column of $A^j_{II}(W)$, we are led to the estimate

\[
2^{2\sigma} \left| \int_0^T \frac{1}{2} \sum_{j=1}^d \langle \partial_{x_j}, A^j_{II}(W_2) \hat{\Delta}_q W, \hat{\Delta}_q W \rangle dt \right| \leq C(S(T))2^{2\sigma} \left| \int_0^T \|\nabla W\|_{L^\infty} \|\hat{\Delta}_q W\|_{L^2} \|\hat{\Delta}_q W\|_{L^2} dt \right|
\]

\[
\leq C(S(T))2^q \|W - \hat{W}\|_{L^\infty_{q}(B_{q,r}^+)} \left(\frac{1}{\tau} \|W_2\|_{L^2_{q}(B_{q,r}^+)} + \tau \|\nabla W\|_{L^2_{q}(B_{q,r}^+)}}\right)
\]

\[
\leq C(S(T))2^q E(T)D_\tau(T)^2. \tag{4.17}
\]

Hence, together with (4.16)--(4.17), we arrive at

\[
2^{2\sigma} \left| \int_0^T I_2 dt \right| \leq C(S(T))2^q E(T)D_\tau(T)^2. \tag{4.18}
\]

Thirdly, we also use the decomposition (4.17) of $A^0(W)$ in order to estimate the commutator occurring in $I_3$. Using the commutator estimate (5.2) in Proposition 5.1, we get

\[
2^{2\sigma} \left| \int_0^T -\langle [\hat{\Delta}_q, A^0_{II}(W_2)] \partial_\ell W, \hat{\Delta}_q W \rangle dt \right| \leq 2^{2\sigma} \left| \langle [\hat{\Delta}_q, A^0_{II}(W_2)] \partial_\ell W, \hat{\Delta}_q W \rangle \right|_{L^2_{q}(L^2)} \\
\leq C2^{2\sigma} c_q \left( \|\nabla A^0_{II}(W_2)\|_{L^2_{q}(L^\infty)} \|\partial_\ell W\|_{L^2_{q}(L^2)} + \|\partial_\ell W\|_{L^2_{q}(L^2)} \|A^0_{II}(W_2)\|_{L^2_{q}(L^2)} \right) \|\hat{\Delta}_q W\|_{L^2_{q}(L^2)} \\
\leq C2^{2\sigma} c_q \left( \|A^0_{II}(W_2)\|_{L^2_{q}(L^\infty)} \|\partial_\ell W\|_{L^2_{q}(L^2)} \right) \|\hat{\Delta}_q W\|_{L^2_{q}(L^2)} \\
\leq C(S(T))2^q E(T)D_\tau(T)^2, \tag{4.19}
\]

where $(\sigma, r)$ satisfies the condition (4.17) and we take $\theta_1 = \theta_2 = \theta_3 = \theta_4 = 2$ in (5.2).

Recall that $p'(\rho)$ can be viewed as a function of $W$ and the nice construction of $A^0_{II}(W)$, we gather that

\[
2^{2\sigma} \left| \int_0^T -\langle [\hat{\Delta}_q, A^0_{II}(W)] \partial_\ell W, \hat{\Delta}_q W \rangle dt \right| \leq C(S(T))2^q E(T)D_\tau(T)^2.
\]
\[ 2^{2q_\sigma} \int_0^T \frac{\|\partial_t W_2\|^2}{\|L^2\|} dt \leq 2^{2q_\sigma} \int_0^T \frac{\|\hat{\Delta}_q, P'(\rho) I_4\|}{\|L^2\|} dt \]
\[ \leq C^{2q_\sigma} c_q \left( \frac{\|\nabla P'(\rho)\|^2}{\|L^2\|} \right) + \frac{\|\hat{\Delta}_q, p'(\rho)\|^2}{\|L^2\|} \]
\[ \leq C^{2q_\sigma} c_q \|P'(\rho)\| \frac{\|\hat{\Delta}_q, p'(\rho)\|^2}{\|L^2\|} \]
\[ \leq C(S(T)) c_q^2 E(T) D_r(T)^2, \] (4.20)
where we used the commutator estimates in Proposition 5.1 and the homogeneous case of Proposition 2.2.

Therefore, we finally get
\[ 2^{2q_\sigma} \left| \int_0^T I_3 dt \right| \leq C(S(T)) c_q^2 E(T) D_r(T)^2. \] (4.21)

The estimates of commutators arising in \( I_4 \) are actually simpler than (4.19)-(4.20), since they only involve spatial derivatives. Let us give the simplified steps only:

\[ 2^{2q_\sigma} \left| \int_0^T - \sum_{j=1}^d (\langle \hat{\Delta}_q, A_j^I(W) \rangle) \partial_x_j W \hat{\Delta}_q W \right| dt \]
\[ \leq C(S(T)) c_q^2 \|W - \hat{W}\| \|\hat{\Delta}_q W\| \|\nabla W\| \|\hat{\Delta}_q W\| \]
\[ \leq C(S(T)) c_q^2 E(T) D_r(T)^2, \] (4.22)

and

\[ 2^{2q_\sigma} \left| \int_0^T - \sum_{j=1}^d (\langle \hat{\Delta}_q, A_{II}^j(W) \rangle) \partial_x_j W \hat{\Delta}_q W \right| dt \]
\[ \leq C^{2q_\sigma} \int_0^T \left( \|\hat{\Delta}_q, A_{II}^j(W)\| \|\hat{\Delta}_q W\| \|\nabla W\| \|\hat{\Delta}_q W\| \right) dt \]
\[ \leq C(S(T)) c_q^2 \|W - \hat{W}\| \|\hat{\Delta}_q W\| \|\nabla W\| \|\hat{\Delta}_q W\| \]
\[ \leq C(S(T)) c_q^2 E(T) D_r(T)^2, \] (4.23)

where \( A_{II}^j(W) \) and \( A_{II}^j(W) \) can be viewed as the smooth functions of \( W \).

Hence, from (4.22)-(4.23), we deduce that
\[ 2^{2q_\sigma} \left| \int_0^T I_4 dt \right| \leq C(S(T)) c_q^2 E(T) D_r(T)^2. \] (4.24)

Finally, due to the homogeneous cases of Proposition 2.1, 2.2, \( I_5 \) can be estimated as
\[ 2^{2q_\sigma} \left| \int_0^T I_5 dt \right| \leq \frac{2^{2q_\sigma}}{\tau} \int_0^T \|\hat{\Delta}_q, (p'(\hat{\rho}) - p'(\rho)) W_2\| \|\hat{\Delta}_q W_2\| \]
\[ \leq \frac{c_q^2}{\tau} \|\hat{\Delta}_q, (p'(\hat{\rho}) - p'(\rho)) W_2\| \|\hat{\Delta}_q W_2\| \]
\[ \leq C(S(T)) \frac{c_q^2}{\tau} \|W - \hat{W}\| \|\hat{\Delta}_q W_2\| \|\hat{\Delta}_q W_2\| \]
\[ \leq C(S(T)) c_q^2 E(T) D_r(T)^2. \] (4.25)
In addition, we define
\[\|f\|_{L^2_{\lambda_0}} := \langle A^0(W)f, f \rangle^{1/2}.\]
Clearly, the norm \(\|f\|_{L^2_{\lambda_0}} \approx \|f\|_{L^2}\), since \(W(t, x)\) takes values in a neighborhood of \(\bar{W}\). Consequently, according to inequalities (4.14), (4.18), (4.21) and (4.24)-(4.25), we conclude that
\[
2^{2q\sigma} \|\dot{\Delta}_q(W(t) - \bar{W})\|_{L^2_{\tau}}^2 + \frac{2^{2q\sigma}}{\tau} \|\dot{\Delta}_q W_2\|_{L^2_{\tau}(L^2)}^2 \\
\leq C(S(T))2^{2q\sigma} \|\dot{\Delta}_q(W_0 - \bar{W})\|_{L^2_{\tau}}^2 + C(S(T))c_q E(T)D_\tau(T)^2. \tag{4.26}
\]
Then it follows from the classical Young’s inequality that
\[
2^{q\sigma} \|\dot{\Delta}_q(W - \bar{W})\|_{L^2_{\tau}(L^2)} + \frac{2^q}{\sqrt{\tau}} \|\dot{\Delta}_q W_2\|_{L^2_{\tau}(L^2)}^2 \\
\leq C(S(T))2^{q\sigma} \|\dot{\Delta}_q(W_0 - \bar{W})\|_{L^2_{\tau}}^2 + C(S(T))c_q E(T)^{1/2}D_\tau(T). \tag{4.27}
\]
Hence, taking the \(\ell^r\)-norm on \(q \in \mathbb{Z}\) on both sides of (4.27) gives immediately
\[
\|W - \bar{W}\|_{L^\infty_{\tau}(L^2_{\lambda_2})} + \frac{1}{\sqrt{\tau}} \|W_2\|_{L^2_{\tau}(L^2_{\lambda_2})} \leq C(S(T))(E(0) + E(T)^{1/2}D_\tau(T)). \tag{4.28}
\]

**Step 2.** The \(L^\infty_{\tau}(L^2)\) estimate of \(W\) and the \(L^2_{\tau}(L^2)\) one of \(W_2\)

It is convenient to introduce the relative entropy \(\tilde{\eta}(\rho, m)\) by
\[
\tilde{\eta}(\rho, m) := \eta(\rho, m) - \eta(\bar{\rho}, 0) - D_\rho \eta(\bar{\rho}, 0)(\rho - \bar{\rho}). \tag{4.29}
\]

Then the strictly convex quantity \(\tilde{\eta}(\rho, m)\) satisfies
\[
\tilde{\eta}(\rho, m) \geq 0, \quad \tilde{\eta}(\bar{\rho}, 0) = 0, \quad D_\rho \tilde{\eta}(\bar{\rho}, 0) = 0. \tag{4.30}
\]
Furthermore, \(\tilde{\eta}(U)\) is equivalent to the quadratic function \(|\rho - \bar{\rho}|^2 + |m|^2\) and hence to \(|W - \bar{W}|^2\), since \(|\rho - \bar{\rho}| + |m|\) lies in a bounded set. Accordingly, the entropy flux \(q_j(\rho, m)(j = 1, 2, \ldots, d)\) are modified as follows
\[
\tilde{q}_j(\rho, m) = q_j(\rho, m) - D_\rho \eta(\bar{\rho}, 0)m_j, \tag{4.31}
\]
and we get the entropy-entropy flux equation
\[
\partial_t \tilde{\eta}(\rho, m) + \sum_{j=1}^d \partial_{x_j} \tilde{q}_j(\rho, m) = -\frac{1}{\tau} \rho |v|^2. \tag{4.32}
\]
Integrating (4.32) over \([0, T] \times \mathbb{R}^N\) implies immediately
\[
\|W - \bar{W}\|_{L^\infty_{\tau}(L^2)} + \frac{1}{\sqrt{\tau}} \|W_2\|_{L^2_{\tau}(L^2)} \leq C(S(T))\|W_0 - \bar{W}\|_{L^2} \\
\leq C(S(T))(E(0) + E(T)^{1/2}D_\tau(T)). \tag{4.33}
\]

**Step 3.** Combining the above analysis.

Note that the fact (4.3), (4.4) is followed by (4.28) and (4.33). □
The next step consists in deriving the $\tilde{L}^2(B^s_{2,r})$ estimate of $\nabla W$. To this end, we use an important stability condition ("Shizuta-Kawashima" condition), which has been first formulated in [24] by the second author, and well developed in [12, 28, 30]. As shown by the recent work [28], the compressible Euler equations with relaxation satisfies the stability condition at the constant state naturally. Furthermore, we obtain the concrete content of the compensating matrix $K(\xi)$, which has been pointed out by [5, 14]. Precisely,

**Lemma 4.3** (Shizuta-Kawashima). For all $\xi \in \mathbb{R}^N$, $\xi \neq 0$, there exists a $(N+1) \times (N+1)$ matrix $K(\xi)$ depending smooth on the unit sphere $S^{N-1}$:

$$K(\xi) = \left(\begin{array}{cc} 0 & -\frac{1}{|\xi|} \frac{\xi^T}{|\xi|} \\ -\frac{\xi}{|\xi|} & 0 \end{array}\right),$$

such that $K(\xi)A^0(W)$ is skew-symmetric and

$$K(\xi) \sum_{j=1}^N \xi_j A^j(W) = \left(\begin{array}{cc} |\xi| & 0 \\ 0 & -p'(\bar{\rho}) \xi \otimes \xi |\xi| \end{array}\right), \quad (4.34)$$

where $A^0$ and $A^j$ are the matrices appearing in the symmetric system (3.1).

Next we shall use the skew-symmetry of the compensating matrix $K(\xi)$ in the Fourier spaces to establish the estimate of $\nabla W$.

**Lemma 4.4.** Under the assumptions stated in Proposition 4.1, there exists a non-decreasing continuous function $C : \mathbb{R}^+ \to \mathbb{R}^+$ which is independent of $\tau$, such that the following estimate holds:

$$\sqrt{\tau} \|\nabla W\|_{L^2(B^s_{2,r})} \leq C(S(T)) \left( E(0) + E(T)^{1/2} D_\tau(T) + E(T)D_\tau(T) \right). \quad (4.35)$$

**Proof.** Firstly, we linearize (3.1) around the constant state $\bar{W}$:

$$A^0(\bar{W}) \partial_t \bar{W} + \sum_{j=1}^d A^j(\bar{W}) \partial_{x_j} \bar{W} = -LW + \mathcal{G} \quad (4.36)$$

with $\bar{W} := W - \bar{W}$, where

$$LW := \frac{1}{\tau} \left(\begin{array}{c} 0 \\ p'(\bar{\rho}) W_2 \end{array}\right)$$

and

$$\mathcal{G} := - \sum_{j=1}^d A^0(\bar{W})[A^0(\bar{W})^{-1} A^j(\bar{W}) - A^0(\bar{W})^{-1} A^j(\bar{W})] \partial_{x_j} W$$

$$- \frac{1}{\tau} A^0(\bar{W})[p'(\rho) A^0(\bar{W})^{-1} - p'(\bar{\rho}) A^0(\bar{W})^{-1}] \left(\begin{array}{c} 0 \\ W_2 \end{array}\right). \quad (4.37)$$
Applying the inhomogeneous localization operator \( \Delta_q(q \geq -1) \) to (4.36) implies
\[
A^0(\bar{W}) \partial_t \Delta_q \bar{W} + \sum_{j=1}^{d} A^j(\bar{W}) \partial_{x_j} \Delta_q \bar{W} = -L \Delta_q W + \Delta_q G. \tag{4.38}
\]

Take the Fourier transform (in the space variable \( x \)), multiply by \(-i \tau(\Delta_q \bar{W})^* K(\xi)(^* \text{ transposed conjugate})\), and compute the real part of each term in the resulting equality to get
\[
\tau \text{Im} \left( (\Delta_q \bar{W})^*(K(\xi) A^0(\bar{W})) \frac{d}{dt} \Delta_q \bar{W} \right) + \tau (\Delta_q \bar{W})^* K(\xi) \left( \sum_{j=1}^{d} \xi_j A_j(\bar{W}) \right) \Delta_q \bar{W} = -p'(\rho) \text{Im} \left( (\Delta_q \bar{W})^* \xi^\top \Delta_q W_2 \right) + \tau \text{Im} \left( (\Delta_q \bar{W})^* K(\xi) \Delta_q G \right). \tag{4.39}
\]

Using the skew-symmetry of \( K(\xi) A^0(\bar{W}) \), we have
\[
\text{Im} \left( (\Delta_q \bar{W})^* (K(\xi) A^0(\bar{W})) \frac{d}{dt} \Delta_q \bar{W} \right) = \frac{1}{2 \tau} \text{Im} \left( (\Delta_q \bar{W})^* (K(\xi) A^0(\bar{W})) \Delta_q \bar{W} \right). \tag{4.40}
\]
Then, with the help of (4.34), the left-hand of (4.39) is bounded from below by
\[
\frac{\tau}{2} \frac{d}{dt} \text{Im} \left( (\Delta_q \bar{W})^* (K(\xi) A^0(\bar{W})) \Delta_q \bar{W} \right) + \frac{1}{\tau} p'(\rho) |\xi| |\Delta_q \bar{W}|^2 - C \tau |\xi| |\Delta_q W_2|^2, \tag{4.41}
\]
where \( C > 0 \) is a generic constant independent of \( \tau \). By Young’s inequality, the right-hand of (4.39) is dominated by
\[
\frac{1}{2} \frac{d}{dt} p'(\rho) |\xi| |\Delta_q \bar{W}|^2 + \frac{C}{\tau |\xi|} |\Delta_q W_2|^2 + \frac{C \tau}{|\xi|} |\Delta_q G|^2. \tag{4.42}
\]

Hence, combine the inequalities (4.41), (4.42) to deduce that
\[
\frac{1}{2} \frac{d}{dt} p'(\rho) |\xi| |\Delta_q \bar{W}|^2 \leq \frac{1}{\tau} \left( |\xi| + \frac{1}{|\xi|} \right) |\Delta_q W_2|^2 + \frac{C \tau}{|\xi|} |\Delta_q G|^2 - \frac{1}{\tau} \frac{d}{dt} \text{Im} \left( (\Delta_q \bar{W})^* (K(\xi) A^0(\bar{W})) \Delta_q \bar{W} \right). \tag{4.43}
\]

Multiplying (4.43) by \( |\xi| \) and integrating it over \([0, T] \times \mathbb{R}^N\), then by Plancherel’s theorem, we arrive at
\[
\tau \|\Delta_q \nabla W\|_{L^2_t(L^2)}^2 \leq (\|\Delta_q \bar{W}_0\|_{L^2}^2 + \|\Delta_q \nabla \bar{W}_0\|_{L^2}^2) + (\|\Delta_q \bar{W}\|_{L^2_t(L^2)}^2 + \|\Delta_q \nabla \bar{W}\|_{L^2_t(L^2)}^2) + \frac{1}{\tau} \left( \|\Delta_q W_2\|_{L^2_t(L^2)}^2 + \|\Delta_q \nabla W_2\|_{L^2_t(L^2)}^2 \right) + \tau \|\Delta_q G\|_{L^2_t(L^2)}^2. \tag{4.44}
\]

where we used the uniform boundedness of the matrix \( K(\xi) A^0(\bar{W})(\xi \neq 0) \) and the smallness of \( \tau(0 < \tau \leq 1) \).
By multiplying the factor $2^{2q(s-1)}$ and taking the $\ell^r$-norm on $q \geq -1$ on both sides of \((4.44)\), then extracting the square root of the resulting inequality, we are led to
\[
\sqrt{\tau} \| \nabla W \|_{L^2(B_{2r}^{-1})} \lesssim \| W_0 - \bar{W} \|_{B^s_{2r}} + \| W - \bar{W} \|_{L^\infty(B_{2r}^s)} + \frac{1}{\sqrt{\tau}} \| W_2 \|_{L^2(B_{2r}^s)} + \sqrt{\tau} \| \mathcal{G} \|_{L^2(B_{2r}^{-1})}.
\]  \(4.45\)

Recalling the definition in \((4.33)\) of $\mathcal{G}$, with the aid of Propositions 2.1, 2.2, we obtain
\[
\| \mathcal{G} \|_{L^2(B_{2r}^{-1})} \leq C(S(T)) \| W - \bar{W} \|_{L^\infty(B_{2r}^s)} \left( \frac{1}{\tau} \| W_2 \|_{L^2(B_{2r}^s)} + \| \nabla W \|_{L^2(B_{2r}^{-1})} \right). \quad (4.46)
\]

Therefore, we get ultimately
\[
\sqrt{\tau} \| \nabla W \|_{L^2(B_{2r}^{-1})} \leq C(S(T)) \left\{ \| W_0 - \bar{W} \|_{B^s_{2r}} + \| W - \bar{W} \|_{L^\infty(B_{2r}^s)} + \frac{1}{\sqrt{\tau}} \| W_2 \|_{L^2(B_{2r}^s)} + \right. \nonumber
\]
\[
\left. + \| W - \bar{W} \|_{L^\infty(B_{2r}^s)} \left( \frac{1}{\tau} \| W_2 \|_{L^2(B_{2r}^s)} + \sqrt{\tau} \| \nabla W \|_{L^2(B_{2r}^{-1})} \right) \right\} \nonumber
\]
\[
\leq C(S(T)) \left( E(0) + E(T)^{1/2} D_{\tau}(T) + E(T) D_{\tau}(T) \right), \quad (4.47)
\]

which completes the proof of Lemma 4.4. \(\square\)

**Proof of Proposition 4.1.** By combining \((4.4)\) and \((4.35)\), we conclude the nonlinear inequality \((4.1)\). In addition, the inequality \((4.2)\) follows from \((4.1)\) and the a priori assumption $E(T) \leq \delta_1$ readily. Hence the proof of Proposition 4.1 is finished. \(\square\)

Thanks to the standard boot-strap argument, for instance, see [19], Theorem 7.1 or the outline given in [29], we extend the local-in-time solutions in Proposition 3.1 to the global-in-time classical solutions of \((4.1)\). Furthermore, we arrive at Theorem 1.1.

The proof of Theorem 1.2 is just the same as that in [29], we thus feel free to skip the details, the interested readers is referred to [29].

## 5 Appendix

In this section, we present a revision of commutator estimates in Proposition 2.3 and relax the restriction on the couple $(s, r)$ in comparison with the commutator estimate in Proposition 2.3 which enables us to construct the a priori estimates in more general Besov spaces.

**Proposition 5.1.** For $s > -1$, $1 \leq p \leq \infty$ and $1 \leq r \leq \infty$, there is a constant $C > 0$ such that
\[
\| [f, \hat{A}_q]g \|_{L^p} \leq C c_q 2^{-q(s+1)} \left( \| \nabla f \|_{L^\infty} \| g \|_{B^s_{p,r}} + \| g \|_{L^p_{s+1}} \| f \|_{B^s_{p,2r}} \right) \quad (5.1)
\]

and
\[
\| [f, \hat{A}_q]g \|_{L^p_{s,t}(L^p)} \leq C c_q 2^{-q(s+1)} \left( \| \nabla f \|_{L^{\theta_1}_{s,t}(L^\infty)} \| g \|_{\tilde{B}^{s+1}_{p,r}} + \| g \|_{L^{\theta_3}_{s,t}(L^\infty)} \| f \|_{\tilde{B}^{s+1}_{p,2r}} \right), \quad (5.2)
\]
where $1/p = 1/p_1 + 1/p_2$, $1/\theta = 1/\theta_1 + 1/\theta_2 = 1/\theta_3 + 1/\theta_4$ and $c_q$ denotes a sequence such that $\| (c_q) \|_{\ell^r} \leq 1$. 

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Proof. We decompose

\[ [f, \dot{\Delta}_q]g := K_1 + K_2 + K_3 + K_4 + K_5 \]  

with

\[
\begin{align*}
K_1 &= [Tf, \dot{\Delta}_q]g, \quad K_2 = R(f, \dot{\Delta}_q g) \quad K_3 = -\dot{\Delta}_q R(f, g), \\
K_4 &= T_{\dot{\Delta}_q} g, \quad K_5 = -\dot{\Delta}_q T_g f,
\end{align*}
\]

where \( T \) and \( R \) stand for the paraproduct and remainder operators (see [2] by J.-M. Bony), which are given by \( Tfg = \sum_{q' \leq q-2} \dot{\Delta}_q f \dot{\Delta}_q' g \) and \( R(f, g) = \sum_{|q-q'| \leq 1} \dot{\Delta}_q f \dot{\Delta}_q' g \). (5.3) is obtained after noticing that

\[ fg = Tfg + Tg f + R(f, g). \]

From the definition of \( \dot{\Delta}_q \), we have the almost orthogonal properties:

\[
\begin{align*}
\dot{\Delta}_p \dot{\Delta}_q f &\equiv 0 \quad \text{if} \quad |p-q| \geq 2, \\
\dot{\Delta}_q(S_{q-1} f \dot{\Delta}_p g) &\equiv 0 \quad \text{if} \quad |p-q| \geq 5.
\end{align*}
\]

Now, we go back to the proof of inequality (5.1). For \( K_1 \), it follows from the Taloy’s formula of first order and Young’s inequality that

\[
\|K_1\|_{L^p} \leq C \sum_{|q-k| \leq 4} 2^{-q} \|S_{k-1} \nabla f\|_{L^\infty} \|\dot{\Delta}_k g\|_{L^p}
\]

\[
\leq Cc_q 2^{-q(s+1)} \|\nabla f\|_{L^\infty} \|g\|_{\dot{B}^s_{p, r}}. \quad (5.4)
\]

For \( K_2 \), we have

\[
K_2 = \sum_{k \in \mathbb{Z}} \sum_{|k-k'| \leq 1} (\dot{\Delta}_k f) (\dot{\Delta}_{k'} \dot{\Delta}_q g)
\]

\[
= \sum_{|k-q| \leq 2} \sum_{|k-k'| \leq 1} (\dot{\Delta}_k f) (\dot{\Delta}_{k'} \dot{\Delta}_q g).
\]

Then by Hölder inequalities, we obtain

\[
\|K_2\|_{L^p} \leq C \sum_{|k-q| \leq 2} \sum_{|k-k'| \leq 1} \|\dot{\Delta}_k f\|_{L^{p_2}} \|\dot{\Delta}_{k'} \dot{\Delta}_q g\|_{L^{p_1}}
\]

\[
\leq C\|g\|_{L^{p_1}} \sum_{|k-q| \leq 2} \|\dot{\Delta}_k f\|_{L^{p_2}}
\]

\[
\leq Cc_q 2^{-q(s+1)} \|f\|_{\dot{B}_{p_2}^{s+1}} \|g\|_{L^{p_1}}, \quad (5.5)
\]

where \( 1/p = 1/p_1 + 1/p_2 \).

For \( K_3 \), we have

\[
K_3 = -\dot{\Delta}_q \left( \sum_{k \in \mathbb{Z}} \sum_{|k-k'| \leq 1} \dot{\Delta}_k f \dot{\Delta}_{k'} g \right)
\]

\[
= -\sum_{\max(k,k') \geq q-2} \sum_{|k-k'| \leq 1} \dot{\Delta}_q (\dot{\Delta}_k f \dot{\Delta}_{k'} g).
\]

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Using the Hölder inequality and Young’s inequality for sequences, we proceed $K_3$ as follows:

\[
\|K_3\|_{L^p} \leq C \sum_{\max(k,k') \geq q-2} \sum_{|k-k'| \leq 1} \|\Delta_k f\|_{L^{p_2}} \|\Delta_{k'} g\|_{L^{p_1}} \\
\leq C \|g\|_{L^{p_1}} \sum_{k \geq q-3} \|\Delta_k f\|_{L^{p_2}} \\
\leq C c_q 2^{-q(s+1)} \|f\|_{\dot{B}^{s+1}_{p_2,r}} \|g\|_{L^{p_1}},
\]

(5.6)

where $s + 1 > 0$ is required.

For $K_4$, we have

\[
K_4 = - \sum_{k \geq q+1} S_{k-1} \Delta_q g \Delta_k f.
\]

In a similar manner as $K_3$, we are led to the estimate

\[
\|K_4\|_{L^p} \leq C \sum_{k \geq q+1} \|\Delta_k f\|_{L^{p_2}} \|g\|_{L^{p_1}} \\
\leq C c_q 2^{-q(s+1)} \|f\|_{\dot{B}^{s+1}_{p_2,r}} \|g\|_{L^{p_1}}.
\]

(5.7)

For the estimate of $K_5$, we recall that

\[
K_5 = - \sum_{|q-k| \leq 4} \Delta_q (S_{k-1} g \Delta_k f),
\]

furthermore, we get

\[
\|K_5\|_{L^p} \leq C \sum_{|q-k| \leq 4} \|\Delta_k f\|_{L^{p_2}} \|g\|_{L^{p_1}} \\
\leq C c_q 2^{-q(s+1)} \|f\|_{\dot{B}^{s+1}_{p_2,r}} \|g\|_{L^{p_1}}.
\]

(5.8)

Together with (5.3)-(5.8), the inequality (5.1) follows immediately.

In addition, the similar process enables us to obtain (5.2), whereas the time exponent $\theta$ behaves according to the Hölder inequality.

Following from the above proof, it is not difficult to obtain the commutator estimates in the inhomogeneous case.

**Corollary 5.1.** For $s > -1$, $1 \leq p \leq \infty$ and $1 \leq r \leq \infty$, there is a constant $C > 0$ such that

\[
\|[f, \Delta_q]g\|_{L^p} \leq C c_q 2^{-q(s+1)} \left(\|\nabla f\|_{L^\infty} \|g\|_{\dot{B}^s_{p,r}} + \|g\|_{L^1} \|f\|_{\dot{B}^{s+1}_{p_2,r}}\right)
\]

(5.9)

and

\[
\|[f, \Delta_q]g\|_{L^p((L^p)^*)} \leq C c_q 2^{-q(s+1)} \left(\|\nabla f\|_{L^{p_1}(L^\infty)} \|g\|_{\dot{B}^s_{r,p}} + \|g\|_{L^{p_1}(L^1)} \|f\|_{\dot{B}^{s+1}_{p_2,r}}\right),
\]

(5.10)

where $1/p = 1/p_1 + 1/p_2$, $1/\theta = 1/\theta_1 + 1/\theta_2 = 1/\theta_3 + 1/\theta_4$ and $c_q$ denotes a sequence such that $\|(c_q)\|_{\ell^r} \leq 1$. 

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