Some Results on Triangle Partitions

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Abstract. We show that there exist efficient algorithms for the triangle packing problem in colored permutation graphs, complete multipartite graphs, distance-hereditary graphs, k-modular permutation graphs and complements of k-partite graphs (when k is fixed). We show that there is an efficient algorithm for $C_4$-packing on bipartite permutation graphs and we show that $C_4$-packing on bipartite graphs is NP-complete. We characterize the cobipartite graphs that have a triangle partition.

1 Introduction

A triangle packing in a graph $G$ is a collection of vertex-disjoint triangles. The triangle packing problem asks for a triangle packing of maximal cardinality. The triangle partition problem asks whether the vertices of a graph can be partitioned into triangles. We refer to Appendix A for an overview of known results on triangle packing problems.

Our objective is the study of the triangle partition problem on permutation graphs. We establish polynomial-time algorithms for several classes of graphs that are related to cographs and permutation graphs. We show that there exist polynomial-time algorithm for triangle packings of colored permutations, complete multipartite graphs, k-modular permutation graphs, distance-hereditary graphs and complements of k-partite graphs. We show that the $C_4$-packing problem can be solved on bipartite permutation graphs and that this problem becomes NP-complete on the class of bipartite graphs. We also characterize the cobipartite graphs that admit a triangle partition.

Since a lot of research on triangle packing centers on interval graphs, we start our discussion with this class of graphs.

2 Interval graphs

A graph is an interval graph if it is the intersection graph of a collection of intervals on the real line [29].

A consecutive clique arrangement of a graph $G$ is a linear arrangement of its maximal cliques such that for each vertex, the cliques that contain it are

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consecutive. Let $G$ be an interval graph with $n$ vertices. Then it has at most $n$ maximal cliques. The following theorem was proved in [21, Theorem 7.1].

**Theorem 1 ([21]).** A graph $G$ is an interval graph if and only if it has a consecutive clique arrangement.

Consider the following problem, called the ‘partition into bounded cliques’ problem in [5].

Let $G = (V, E)$ be a graph and let $r$ and $s$ be integers. Can $V$ be partitioned into $s$ cliques each of cardinality at most $r$?

This problem can be solved in linear time on interval graphs [5, 54].

In the following we denote by $\tau(G)$ the maximal number of vertex disjoint triangles in $G$. The following lemma is easy to check.

**Lemma 1.** Let $G$ be an interval graph and let $[C_1, \ldots, C_t]$ be a consecutive clique arrangement.

(i) If there is a maximal clique with only one vertex $x$ then $x$ is an isolated vertex. In that case $\tau(G) = \tau(G - x)$.

(ii) If there is a maximal clique with exactly two vertices $x$ and $y$ then $(x, y)$ is a bridge. Let $G'$ be the graph obtained from $G$ by deleting the edge $(x, y)$ but not its endvertices. Then $G'$ is an interval graph and $\tau(G) = \tau(G')$.

Henceforth we may assume that every maximal clique in the consecutive clique arrangement has at least three vertices.

**Lemma 2.** Let $G$ be an interval graph and let $[C_1, \ldots, C_t]$ be a consecutive arrangement of its maximal cliques. Consider an ordering of the vertices in $C_1$ by increasing degree or, equivalently, by increasing right endpoints of the corresponding intervals. There is a triangle packing of $G$ with maximal cardinality such that all vertices of $C_1$ are covered except, possibly the smallest or, the two smallest vertices.

**Proof.** If there are at least three vertices in $C_1$ not covered by a triangle then we can add a triangle to the packing.

Let $\alpha$ and $\beta$ be two vertices in $C_1$ and assume that $\alpha < \beta$ in the right endpoint ordering. Assume that $\alpha$ is in a triangle of a triangle packing $\mathcal{P}$ and that $\beta$ is not. Then we can switch $\alpha$ and $\beta$ in the triangle and obtain an alternative packing $\mathcal{P}'$ with the same number of triangles such that $\beta$ is covered. The claim follows by induction by recursively switching any vertex that is not covered with the smallest vertex in $C_1$ that is covered. \qed

**Lemma 3.** Let $G$ be an interval graph and let $[C_1, \ldots, C_t]$ be a consecutive arrangement of its maximal cliques. Let $\alpha$, $\beta$ and $\gamma$ be the three smallest vertices of $C_1$ in the ordering by right endpoints. If $\alpha$ is covered by a triangle in a triangle packing $\mathcal{P}$ then there exists a triangle packing $\mathcal{P}'$ of the same cardinality as $\mathcal{P}$ such that $\{\alpha, \beta, \gamma\}$ is a triangle of $\mathcal{P}'$. 2
Proof. Since $C_1$ is a maximal clique it has a vertex that is not contained in $C_2$. Consequently, $\alpha$ is contained only in $C_1$. Let $p$ and $q$ are in $C_1$. Let $p$ be the smallest of the two. Assume that $\beta \neq p$. If $\beta$ is not covered by any triangle of $P$ then we can replace $p$ with $\beta$. Assume that $\beta$ is in a triangle $\{\beta, r, s\}$ of $P$. Let $C_1$ be the first clique that contains all three vertices $\beta$, $r$ and $s$. Then $C_1$ also contains $p$, since $p$ is larger than $\beta$. Replace the two triangles $\{\alpha, p, q\}$ and $\{\beta, r, s\}$ with $\{\alpha, \beta, q\}$ and $\{p, r, s\}$. A similar argument shows that, if $q \neq \gamma$ then we can replace $q$ with $\gamma$ and obtain an alternative packing $P'$ with the same number of triangles. 

**Theorem 2.** The triangle partition problem can be solved in linear time on interval graphs. The triangle packing problem can be solved by an exponential algorithm which runs in $O^*(1.47^n)$ time.

**Proof.** The first claim follows from Lemma 3. The second claim follows from the recurrence

$$T(n) = T(n - 1) + T(n - 3).$$

To see that this recurrence holds, observe that the minimal element of $C_1$ is either not in any triangle or it is in a triangle together with the next two smallest elements of $C_1$. 

An $O(n \log n)$ algorithm for maximum matching in interval graphs is presented in [39, 48]. For the class of strongly chordal graphs, which includes the class of interval graphs, there exists a linear-time algorithm for maximum matching when a strong elimination ordering of the graph is part of the input [11]. Dahlhaus et al., extend the greedy algorithm for a $K_r$-partition for general $r$ on interval graphs to the class of strongly chordal graphs [11].

Concerning packings of vertex-disjoint maximal cliques we have the following theorem.

**Theorem 3.** Let $G$ be an interval graph. There exists a linear-time algorithm that finds the maximal number of vertex-disjoint maximal cliques.

**Proof.** This can be seen as follows. First recall that the maximal number of vertex-disjoint maximal cliques in an interval graph is equal to the minimal number of vertices that represent all maximal cliques [21]. A vertex $x$ represents a clique $C$ if $x \in C$. A set of vertices that together represent all maximal cliques is called a clique-transversal.

Let $[C_1, \ldots, C_t]$ be a consecutive clique ordering of $G$. We use the following trick that we learned from [27] to reduce the problem to a domination problem. Add one vertex to each maximal clique $C_i$. Then the new graph $H$ has $n + t$ vertices and $H$ is an interval graph. It is easy to check that the minimal cardinality of a clique transversal in $H$ is equal to the minimal cardinality of a dominating set in $H$. Here, a dominating set in a graph is a set $S$ of vertices such that every vertex not in $S$ has a neighbor in $S$. There exists a linear-time algorithm that finds a dominating set of minimal cardinality in an interval graph [36]. This proves the claim. 

\[\blacksquare\]
3 Triangle partition on colored permutation graphs

We refer to Appendix B for a brief overview on permutation graphs.

Notice that the triangle partition problem for permutation graphs is equivalent to the following problem.

Let \( \pi \) be a permutation of \( V = \{1, \ldots, n\} \). Decide if there exists a partition \( P \) of \( \{1, \ldots, n\} \) into triples such that for each element \( \{i, j, k\} \in P \) with \( i < j < k \),

\[ \pi(i) < \pi(j) < \pi(k). \]

As far as we know, both the triangle partition - and the triangle packing problem for permutation graphs are open. In this section we consider a variation of the partition problem.

**Theorem 4.** Let \( \pi \in S_n \) and let

\[ c : \{1, \ldots, n\} \rightarrow \{1, 2, 3\}. \]

There exists a polynomial-time algorithm that decides whether there exists a partition \( P \) of \( \{1, \ldots, n\} \) into triples such that for each \( \{i, j, k\} \in P \) with \( i < j < k \):

\[ \pi(i) < \pi(j) < \pi(k) \quad \text{and} \quad c(\pi(i)) = 1, \quad c(\pi(2)) = 2 \quad \text{and} \quad c(\pi(3)) = 3. \]

**Proof.** Construct two bipartite graphs as follows. The first bipartite graph has vertices that are the elements with colors 1 and 2. A vertex \( i \) with color 1 is adjacent to a vertex \( j \) with color 2 if

\[ i < j \quad \text{and} \quad \pi(i) < \pi(j). \]

The second bipartite graph has vertices with colors 2 and colors 3. A vertex \( p \) with color 2 is adjacent to a vertex \( q \) with color 3 if

\[ p < q \quad \text{and} \quad \pi(p) < \pi(q). \]

It is easy to check that there exists a partition into triangles if and only if both bipartite graphs have a perfect matching.

One can find a perfect matching in \( O(n^\omega) \) time [52], where \( \omega \) is the exponent of a matrix multiplication algorithm.

\( \square \)

It is easy to check that this result generalizes to the \( K_r \)-partitioning problem when an \( r \)-coloring is a part of the input.

4 Triangle partition on complete multipartite graphs

Complete \( t \)-partite graphs form a subclass of the permutation graphs. In this section we show that the triangle partition problem can be solved in polynomial time for complete \( t \)-partite graphs.

Let \( a_1, \ldots, a_t \) be positive natural numbers. We use \( K(a_1, \ldots, a_t) \) to denote the complete \( t \)-partite graph with color classes \( A_1, \ldots, A_t \) such that \( |A_i| = a_i \) for all \( i \in \{1, \ldots, t\} \).
Lemma 4. Let $G$ be a $t$-partite graph. If there exists a triangle partition $\mathcal{P}$ of $G$ then there exists a triangle partition $\mathcal{P}'$ which contains a triangle with vertices in three of the largest color classes.

Proof. Let $A_1, \ldots, A_t$ be the color classes of $G = (V, E)$ and let $a_i = |A_i|$ such that
$$0 < a_1 \leq a_2 \leq \ldots \leq a_t.$$  
We prove the claim by induction on the number of vertices. If there are only three vertices then the claim is obviously true. Consider a triangle $P \in \mathcal{P}$ and consider the subgraph $G'$ of $G$ induced by $V - P$. Then $G'$ has a triangle partition $\mathcal{P}' - P$. By induction there exists a triangle partition $Q$ of $G'$ with a triangle $Q$ contained in three of the maximal color classes of $G'$. Note that color classes with the same cardinality are interchangeable. Therefore, we may assume that the three maximal classes of $G'$ that contain the vertices of $Q$ are also maximal classes of $G$. $\square$

Theorem 5. There exists a linear-time algorithm that solves the triangle partition problem on complete $t$-partite graphs.

Proof. By Lemma 4 a greedy algorithm which chooses recursively triangles from three of the largest color classes produces a triangle partition if it exists. It is easy to see that this algorithm can be implemented to run in linear time. $\square$

Remark 1. We have not found an easy condition on the numbers $a_1, \ldots, a_t$ which characterizes the complete $t$-partite graphs that have a triangle partition.

Remark 2. A similar greedy algorithm solves the triangle packing problem on complete $t$-partite graphs.

5 C₄-Packing on bipartite permutation graphs

Let $G = (X, Y, E)$ be a bipartite permutation graph and consider the left-to-right orderings of the vertices of $X$ and $Y$ on the topline of a permutation diagram. It is easy to check that this is a strong ordering, which is defined as follows.

Definition 1. Let $G = (X, Y, E)$ be a bipartite graph. A strong ordering is a pair of linear orderings $<_1$ and $<_2$ on $X$ and $Y$ such that for all $x_1, x_2 \in X$ and $y_1, y_2 \in Y$ with $x_1 <_1 x_2$ and $y_1 <_2 y_2$
$$((x_1, y_2) \in E \text{ and } (x_2, y_1) \in E) \Rightarrow ((x_1, y_1) \in E \text{ and } (x_2, y_2) \in E).$$

Spinrad, et al., obtained the following characterization [60].

Theorem 6 ([60]). Let $G = (X, Y, E)$ be a bipartite graph. Then $G$ is a bipartite permutation graph if and only if there is a strong ordering on $X$ and $Y$. 

5
In this section we show that there is a greedy algorithm that computes a $C_4$-packing on bipartite permutation graphs. Consider a diagram for a bipartite permutation graph $G = (X, Y, E)$ and let $<$ be the left-to-right ordering of the points on the topline. Denote by $<_1$ and $<_2$ the sub-orderings of $<$ induced by the vertices of $X$ and $Y$.

**Lemma 5.** Let $\mathcal{P}$ be a $C_4$-packing on $G$. Assume that $a, c \in X$ with $a <_1 c$ and assume that $a$ and $c$ are in a square $C \in \mathcal{P}$. Let $a <_1 b <_1 c$ and assume that $b$ is the smallest element $>_1 a$. Then there is a packing $\mathcal{P}'$ of the same cardinality as $\mathcal{P}$ such that $a$ and $b$ are in a square $C' \in \mathcal{P}'$.

**Proof.** Let $C = \{a, c, p, q\}$. First notice that for each vertex $y \in Y$, its neighborhood $N(y)$ forms an interval in $(X, <_1)$ [60]. Then

$$p, q \in N(a) \cap N(c) \quad \text{and} \quad a <_1 b <_1 c \quad \Rightarrow \quad p, q \in N(a) \cap N(b).$$

Thus $\{a, b, p, q\}$ is a square. If $b$ is not in any square of $\mathcal{P}$ then we can replace $C$ with $C' = \{a, b, p, q\}$. Assume that $b$ is in a square $C_2 = \{b, d, r, s\} \in \mathcal{P}$. We consider the following cases.

First assume that $a <_1 b <_1 d <_1 c$. Then $p$ and $q$ are adjacent to $a$, $b$, $c$ and $d$. Since $r$ and $s$ intersect the line segments of $b$ and $d$ but not the line segments of $p$ and $q$ each of $r$ and $s$ intersects at least one of $a$ and $c$. If $r$ and $s$ both intersect $a$ then we can replace $C$ and $C_2$ with $\{a, b, r, s\}$ and $\{c, d, p, q\}$. Similarly, of both $r$ and $s$ intersect $c$ we can replace $C$ and $C_2$ with $\{a, b, p, q\}$ and $\{c, d, r, s\}$. Assume that $r$ and $s$ intersect $a$ and that $r$ intersects $c$. Then replace $C$ and $C_2$ with $\{a, b, p, r\}$ and $\{c, d, q, s\}$.

Now assume that $a <_1 b <_1 c <_1 d$. Then

$$b <_1 c <_1 d \quad \text{and} \quad r, s \in N(b) \cap N(d) \quad \Rightarrow \quad r, s \in N(c) \cap N(d).$$

Thus $\{c, d, r, s\}$ is a square. Replace $C$ and $C_2$ with $\{a, b, p, q\}$ and $\{c, d, r, s\}$.

This proves the lemma. \hfill $\square$

By Lemma 5, there exists a maximum $C_4$-packing $\mathcal{P}$ of $G$ such that the pairs of vertices of $X$ that are contained in a square of $\mathcal{P}$ are consecutive in $<_1$ and the pairs of vertices of $Y$ that are contained in a square of $\mathcal{P}$ are consecutive in $<_2$.

**Lemma 6.** Consider two squares $C_1 = \{a, b, r, s\}$ and $C_2 = \{c, d, p, q\}$ in a $C_4$-packing $\mathcal{P}$. Assume that $a < b < c < d$ and that $p < q < r < s$. There exists a packing $\mathcal{P}'$ of the same cardinality as $\mathcal{P}$ such that $\{a, b, p, q\}$ and $\{c, d, r, s\}$ are squares in $\mathcal{P}'$.

**Proof.** Assume that $d < p$. Since the line segments of $p$ and $q$ intersect the line segments of $c$ and $d$, and $r$ and $s$ intersect $a$ and $b$, and since the line segments of $p$, $q$, $r$ and $s$ are parallel, by the ordering the line segments of $p$, $q$, $r$ and $s$ intersect all line segments of $a$, $b$, $c$ and $d$. Thus $\{c, d, r, s\}$ and $\{a, b, p, q\}$ are squares. The only other possible case is where $s < a$. This case is similar.

This proves the lemma. \hfill $\square$
Theorem 7. There exists a linear-time algorithm which computes a maximum $C_4$-packing in a bipartite permutation graph.

Proof. Let $G = (X,Y,E)$ be a bipartite permutation graph and let $<$ the the ordering of the vertices on the topline of a diagram for $G$. We prove that there exists a maximum packing $P$ such that the first four vertices in the ordering $<$ that form a square are in $P$. Consider the first four vertices $C = \{x_1,x_2,y_1,y_2\}$ that form a square. We may assume that $x_1 < x_2 < y_1 < y_2$. Assume that $C \notin P$. Consider the square $C_1 = \{x'_1,x'_2,y'_1,y'_2\}$ in $P$ with the smallest vertices in $X$. If $C \cap C_1 = \emptyset$ then $C$ is disjoint from all squares in $P$ which contradicts the maximality of $P$. In all other cases we can replace $C_1$ with $C$.

This proves the correctness of the following algorithm. Remove vertices that are smallest in the $<$-ordering that are not in any square of $G$. Assume next that the smallest element is $x_1 \in X$. Thus $x_1$ is in a square. Take the first element $x_2 \in X$ that is in a square with $x_1$. Then take the first two elements $y_1, y_2 \in Y$ such that $C = \{x_1,x_2,y_1,y_2\}$ induces a square in $G$. Put $C$ in $P$. Remove the vertices $C$ from $G$ and recurse. It is easy to see that, with some care this algorithm can be implemented to run in linear time. \hfill \Box

6  NP-Completeness for $C_4$-packing on bipartite graphs

In the previous section we proved that the $C_4$-packing problem on bipartite permutation graphs can be solved in linear time. In this section we show that the $C_4$-packing problem is NP-complete for general bipartite graphs.

Theorem 8. $C_4$-Packing on bipartite graphs is NP-complete.

Proof. It is easy to see that the problem is in NP. To show the NP-hardness for our problem we use a reduction from the 3-dimensional matching problem (3DM) which is described as follows. Suppose we are given three sets $X,Y$ and $Z$ such that $|X| = |Y| = |Z| = q$, and a set $M \subseteq X \times Y \times Z$ of triples $(x,y,z)$. The 3DM problem asks for a subset $M'$ of $M$ such that each element of $X,Y$ and $Z$ is contained in exactly one triple in $M'$.

We apply local replacements to the input instance of 3DM. For each pair $x$ and $y$ that appear in some triple $(x,y,z) \in M$, we create a vertex $v_{xy}$ and a path of length two $[x,v_{xy},y]$ as shown in Figure 1.

For each triple $\tau = (x,y,z) \in M$, we create four local vertices $a_\tau[i]$ $1 \leq i \leq 4$

and eight edges as shown in Figure 1. This finishes the description of the construction of the bipartite graph $G$. The bipartite graph $G$ is obtained in linear time.

Suppose that $M'$ is a solution of the 3DM problem. We obtain a $C_4$-packing of size $p = q + |M|$ as follows. The packing is constructed by taking 4-cycles

\[
\begin{align*}
\{x,v_{xy},y,a_\tau[1]\} \quad & \text{if} \quad \tau = (x,y,z) \in M' \\
\{a_\tau[2],a_\tau[3],a_\tau[4],z\} \quad & \text{if} \quad \tau = (x,y,z) \in M' \\
\{a_\tau[1],a_\tau[2],a_\tau[3],a_\tau[4]\} \quad & \text{if} \quad \tau \notin M'.
\end{align*}
\]
This ensures that each element in $X \cup Y \cup Z$ is included in exactly one 4-cycle in the packing.

Now assume that there is a $C_4$-packing of size $p$, where $p = q + |M|$. In the constructed figure for any triple $\tau = (x, y, z) \in M$, there are only two possible ways to pack the 4-cycles: one way contains the 4-cycle induced by

$$\{a_\tau[1], a_\tau[2], a_\tau[3], a_\tau[4]\}$$

and the other way contains the two 4-cycles induced by

$$\{x, v_{xy}, y, a_\tau[1]\}$$ and $$\{a_\tau[2], a_\tau[3], a_\tau[4], z\}.$$

The first choice contains none of vertices from $X \cup Y \cup Z$ and the second contains exactly one vertex from each of $X, Y$ and $Z$. Our packing is of size $p = q + |M|$ so there must be at least $q$ triples that use the second kind of $C_4$’s in the packing. Since none of these $q$ triples have a common element, they cover all $3q$ elements in $X \cup Y \cup Z$. Thus these $q$ triples forms a 3-dimensional matching. 

\section{Triangle partition on cobipartite graphs}

A cobipartite graph is the complement of a bipartite graph. We denote a bipartite graph $G$ with color classes $A$ and $B$ by $G = (A, B, E)$. We use the same notation for a cobipartite graph where $A$ and $B$ are the color classes of the complement. A star is a bipartite graph $G = (A, B, E)$ with $|A| = 1$. The single vertex in $A$ is called the center of the star.

In this section we show that there is a good characterization of the cobipartite graphs that have a triangle partition.

\textbf{Theorem 9.} Let $G = (A, B, E)$ be cobipartite. Then $G$ can be partitioned into triangles if and only if one of the following holds true.

\begin{enumerate}
  \item $|A| \mod 3 = |B| \mod 3 = 0$, or
  \item $|A| \mod 3 = 1$ and $|B| \mod 3 = 2$ and $G$ has a triangle with one vertex in $A$ and two in $B$, or
  \item similar as above with the role of $A$ and $B$ interchanged.
\end{enumerate}
Proof. If $|A| \mod 3 = |B| \mod 3 = 0$ then $G$ can be partitioned into triangles. Assume that $|A| \mod 3 = 1$ and that $|B| \mod 3 = 2$. If there exists a triangle with one vertex in $A$ and two in $B$ then $G$ can be partitioned into triangles. For the converse, assume that there is no such triangle. Let $G'$ be the bipartite graph obtained from $G$ by deleting edges between vertices that are contained in the same color class. Then $G'$ does not contain a $P_3$, i.e., a path with three vertices, with its midpoint in $A$. Then $G'$ has no $P_4$ and no $C_4$ and so $G'$ is trivially perfect [63]. It follows that $G'$ is a disjoint collection of isolated vertices in $A$ and stars with their centers in $B$. Then each triangle of $G$ that is not contained in $A$ nor in $B$ has one vertex in $B$ and two in $A$. If $G$ has a triangle partition then $|A| \mod 3 = 2(|B| \mod 3)$, which is a contradiction. \hfill \qed

Theorem 10. There exists a linear-time algorithm which check if a cobipartite graph has a triangle partition.

Proof. Assume that $|A| \mod 3 = 1$ and that $|B| \mod 3 = 2$. It is easy to check in linear time whether $G'$ is disjoint collection of isolated vertices and stars with their midpoints in $B$. By Theorem 9 the graph $G$ has a partition into triangles if and only if $G'$ is not a disjoint collection of isolated vertices in $A$ and stars with midpoints in $B$. \hfill \qed

8 Complements of multipartite graphs

A $k$-partite graph is a graph $G = (V, E)$ of which the vertices can be partitioned into $k$ independent sets. Notice that the recognition of 3-partite graphs is NP-complete since it is an instance of the 3-coloring problem. Henceforth, we assume that the partition into color classes of a $k$-partite graph is a part of the input.

In this section we show that the triangle partition problem on the complements of $k$-partite graphs can be solved in polynomial time. We start with the case where $k = 3$.

Lemma 7. Let $G$ be the complement of a 3-partite graph with color classes $A_1$, $A_2$ and $A_3$. There exists a collection of colored graphs $H_1, \ldots, H_t$, each with at most 42 vertices, such that $G$ has a partition into triangles if and only if one of the graphs $H_i$ is an induced subgraph of $G$.

Proof. Consider a partition of $G$ into triangles. Each triangle is either contained in one of the color classes or, it has one vertex in each color class or, it has one vertex in one color class and two vertices in another color class. Assume that 3 vertices in $A_1$, $A_2$ and $A_3$ are mutually connected by triangles. Then we can change the triangle partition such that it contains the triangles on the three vertices in each $A_i$ instead. Assume that there are 3 vertices in $A_1$ that are in triangles with 3 edges in $A_2$. Then we may replace those triangles by one triangle in $A_1$ and two in $A_2$. In this way we obtain a partition into triangles such that there are at most 14 vertices in each color class $A_i$ that are in triangles that are
not completely contained in $A_i$. Consider all possible colored graphs $H_i$ with at most 42 vertices. Then there is a partition of $G$ into triangles if and only if there is a colored induced subgraph $H_i$, which can be partitioned into triangles, such that the number of remaining vertices in each class $A_i$ is $0 \mod 3$. □

**Corollary 1.** There exists a polynomial-time algorithm that checks whether the vertices of the complement of a 3-partite graph can be partitioned into triangles.

**Remark 3.** It is folklore that the triangle partition problem is NP-complete on 3-partite graphs, see, e.g., [51].

**Theorem 11.** Let $k$ be a natural number. There exists a polynomial-time algorithm that solves the triangle partition problem on complements of $k$-partite graphs.

**Proof.** Let $G = (V, E)$ be the complement of a $k$-partite graph with color classes $A_1, \ldots, A_k$. Assume that $V$ can be partitioned into triangles. First we show that there exists a triangle partition of $G$ with only a constant number of triangles of which the vertices are not monochromatic. Consider three color classes $A_1$, $A_2$ and $A_3$. By Lemma 7 we may assume that at most 14 triangles that are not monochromatic are contained in $A_1 + A_2 + A_3$. Since this holds for any three color classes, we find that there is a partition with at most $14k^3$ non-monochromatic triangles. □

### 9 Triangle packing on distance-hereditary graphs

A graph $G$ is distance hereditary if for every component in every induced subgraph the distance between two vertices is the same as their distance in $G$ [33]. In this section we show that the triangle packing problem can be solved in polynomial time on distance-hereditary graphs.

A decomposition tree for a graph $G = (V, E)$ is a pair $(T, f)$ where $T$ is a ternary tree and where $f$ is a 1-1 map from the vertices in $G$ to the leaves of $T$. A line in $T$ induces a partition of $V$ into two sets, say $A$ and $B$. The twinset of $A$ is the subset of vertices in $A$ that have neighbors in $B$. The graph $G$ is distance hereditary if and only if it has a decomposition tree $(T, f)$ such that for every partition $\{A, B\}$ induced by a line in $T$ every pair of vertices in the twinset of $A$ have the same neighbors in $B$ [53]. If $G$ is distance hereditary, such a decomposition tree for $G$ can be found in linear time [10].

**Theorem 12.** There exists a polynomial-time algorithm that solves the triangle packing problem on distance-hereditary graphs.

**Proof.** Our method resembles the one used in [28] used to solve the triangle packing problem on cographs.

Let $G = (V, E)$ be distance hereditary and let $(T, f)$ be a decomposition tree for $G$ which satisfies the properties mentioned above. The algorithm performs dynamic programming on branches of $T$ of increasing size. Consider a branch $B$ rooted at some line $e$ of $T$. Suppose that $B$ decomposes into two smaller branches
Let $S_1$ and $S_2$ be the twinsets of the vertices mapped to leaves of $B_1$ and $B_2$. Then every vertex of $S_1$ is adjacent to every vertex of $S_2$ or no vertex of $S_1$ is adjacent to a vertex of $S_2$. Moreover, the twinset for $B$ is either $S_1 + S_2$ or, it is one of the two or it is empty.

The dynamic programming keeps track of the maximum cardinality of a triangle packing in a branch $B$ that avoids a specified number of 'free' vertices and a specified number of 'free' edges in the twinset. As an invariant, the free edges do not contain any of the free vertices and they form a matching. Free vertices and edges can be used to form triangles with vertices outside the branch. It is easy to update the table for a branch $B$ from the tables of $B_1$ and $B_2$. Details for the updating procedure can be found in [28]. Since the number of entries of each table is bounded by $O(n^2)$, it follows that the algorithm can be implemented to run in polynomial time [28].

\[ \square \]

Remark 4. It is fairly easy to see that the algorithm above can be extended so that it works for graphs of bounded rankwidth [53]. We have not been able to formulate the triangle partition problem in monadic second-order logic. Gurusswami, et al., describe an algorithm for the $K_r$-packing problem on cographs that runs in polynomial time for each fixed $r$. The problem whether this problem can be solved by a fixed-parameter algorithm with respect to $r$ remains an open problem.

## 10 Modular permutations

Let $\pi$ be a permutation of $\{1, \ldots, n\}$. A module in $\pi$ is a consecutive subsequence of $\pi$ that is a permutation of a consecutive subsequence of $[1, \ldots, n]$.

Let $G$ be a permutation graph. Consider a diagram for $G$ with the labels $[1, \ldots, n]$ in order on the topline and with the labels $[\pi(1), \ldots, \pi(n)]$ in order on the bottom line. Then a module in the permutation corresponds with a subset $M$ of vertices such that every vertex outside $M$ is adjacent to all vertices of $M$ or to no vertex of $M$.

A subset $M$ of vertices in a graph $G$ with this property is called a module of $G$. A module $M$ is trivial if it contains zero, one or all the vertices of the graph. A module is strong if it does not overlap with other modules. A graph is prime if it contains only trivial modules. If $M \neq V$ is a strong module then there exists a unique strong module $M'$ of minimal size that properly contains $M$. This defines a parent relation in a modular decomposition tree for $G$.

A modular decomposition tree is a rooted tree $T$ with a 1-1 map from the leaves to a set $V$ of vertices. Each internal node of $T$ is labeled as a join node, as a union node, or as a prime node. A modular decomposition tree $T$ defines a graph $G$ with vertex set $V$ as follows. A join node $j$ stands for the operation which adds an edge between every pair of vertices that are mapped to leaves in different subtrees of $j$. A union node $u$ stands for the operation that unites the subgraphs represented by the children of $u$. Each prime node $p$ is labeled with a graph $H_p$. Each vertex in $H_p$ corresponds with one child of $p$. If two vertices
In $H_p$, are connected by an edge then every pair of vertices in the two graphs represented by the two corresponding children is connected by an edge.

Given a graph $G$, a tree that decomposes $G$ recursively into strong modules can be constructed in linear time [61].

**Definition 2.** A graph is $k$-modular if it has a modular decomposition tree such that the graph $H_p$ of every prime node $p$ has at most $k$ vertices.

If the modular decomposition tree has no prime nodes then the graph is decomposable by unions and joins. We call these graphs 0-modular. The class of 0-modular graphs is exactly the class of cographs.

**Theorem 13.** The graphs that are $k$-modular are characterized by a finite collection of forbidden induced subgraphs.

**Proof.** This follows from Kruskal’s theorem [38]. □

Consider the class of graphs obtained from paths by replacing the endvertices by false twins, i.e., modules consisting of two nonadjacent vertices. This class is contained in the class of permutation graphs and it is not well-quasi-ordered by the induced subgraph relation. This proves the following corollary. Another way to see that is by showing that paths are prime.

**Corollary 2.** There exists a function $f(k)$ such that $k$-modular permutation graphs have no induced paths of length more than $f(k)$.

We omit the easy proof of the following theorem.

**Theorem 14.** For each natural number $k$ there exists a polynomial-time algorithm which computes a triangle packing in $k$-modular graphs.

11 Concluding remark

The main question that we leave open in this paper is whether there exists a polynomial-time algorithm that checks if a permutation of $\{1, \ldots, 3n\}$ can be partitioned into increasing subsequences of length three.

12 Acknowledgement

We thank Klaas Zwartenkot for doing some calculations on the number of permutations that can be partitioned into triangles.

An obvious lowerbound for the number of permutations of $\{1, \ldots, 3n\}$ that can be partitioned into triangles is $\binom{3n}{n}$. We have not been able to determine an exact formula for the number of permutations that can be partitioned into triangles nor have we been able to determine the asymptotics for those permutations.
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A Preliminaries on triangle packings

Despite great interest in the cycle packing – and cycle cover problem there is relatively little theoretical progress on the first of the two problems.

The two kinds of problems are related by the Erdős and Pósa theorem which states that there is a function \( f(k) = O(k \log k) \) such that any graph contains either \( k \) vertex-disjoint cycles or a set with \( f(k) \) vertices which intersect every cycle [15].

A graph \( H \) is topologically contained in a graph \( G \) if \( G \) has a subgraph \( H' \) which is a subdivision of \( H \). It is well-known that the graphs that have no \( k \) disjoint cycles are well-quasi ordered by topological containment [44]. See also [23, Theorem 5.6]. It follows that there exists a finite set \( \mathcal{F}_k \) of graphs, each containing a maximal number of \( k \) disjoint cycles, such that a graph \( G \) has \( k \) disjoint cycles if and only if some element of \( \mathcal{F}_k \) is topologically contained in it. This can also be seen as follows. Notice that a graph \( G \) does not have \( k \) vertex-disjoint cycles if and only if \( G \) does not contain the graph \( H \) that consists of \( k \)
disjoint triangles as a minor. This implies the previous observation [23, Proposition 5.16]. It follows also that the class of graphs without \( k \) disjoint cycles is minor closed. Furthermore, this class does not contain all planar graphs, so the class has a uniform bound on the treewidth [57]. Bodlaender subsequently showed that the elements of this class can be recognized in \( O(n) \) time [4]. Likewise, for any natural number \( k \) one can check in \( O(n) \) time whether a graph has a cycle cover with at most \( k \) vertices.

Another research area that is related to the topic of this paper is that of finding vertex colorings of graphs that are restricted in some way. There are too many variations to cover in any limited survey, even when restricted to permutation graphs. We mention some of the results that seem closely related to our research.

On partially ordered sets, one of the major contributions is the result of Greene and Kleitman and of Frank [19, 24, 25]. Greene and Kleitman generalize Dilworth's theorem [12] and Frank describes an efficient algorithm that finds an optimal solution. If \( P = (V, \leq) \) is a partially ordered set then one can find in polynomial time a collection of \( t \) antichains \( \{A_1, \ldots, A_t\} \) that maximizes \(| \bigcup_i A_i | \). It follows that one can find in polynomial time an induced subgraph of a permutation graph with a maximal number of vertices that has chromatic number at most \( t \). Likewise, one can find a collection of \( t \) cliques \( \{C_1, \ldots, C_t\} \) that maximizes \(| \bigcup_i C_i | \).

When one bounds the number of vertices in the color classes the picture changes drastically. The following problem has been investigated in great detail due to its applications in various scheduling problems. Suppose we wish to find a vertex coloring with a minimal number of colors, such that each color class contains at most \( q \) vertices [30, 32, 35, 42, 49, 50]. This problem is NP-complete on permutation graphs for each \( q \geq 6 \) [34].

Suppose one wishes to color the vertices of a graph with a minimal number of colors such that each color class induces a clique or an independent set. We call this a homogeneous coloring. The Erdős-Hajnal conjecture states that for every graph \( H \) there exists a \( \delta < 1 \) such that every graph \( G \) that does not contain \( H \) as an induced subgraph can be homogeneously colored with at most \( n^\delta \log n \) colors [1, 14]. Considerable progress towards proving this conjecture is reported in [18]. The conjecture is known to be true for perfect graphs with \( \delta = \frac{1}{2} \) and recently it was proved for bull-free graphs with \( \delta = \frac{5}{4} \) [6]. One of the smallest graphs \( H \) for which the conjecture is still open is \( C_5 \). Wagner showed that finding the minimum number of colors in a coloring of this type is NP-complete, even for permutation graphs [62]. On the other hand, for any pair of nonnegative numbers \( r \) and \( s \), the class of permutation graphs that have a homogeneous coloring with \( r \) cliques and \( s \) independent sets (possibly empty) is characterized by a finite collection of forbidden induced subgraphs [37].

Lonc mentions the following open problem in [41]. Given a sequence of \( 3n \) distinct positive integers. Find a partition of the sequence into \( n \) increasing subsequences, each of 3 terms. This is equivalent to finding a partition of the vertices of a permutation graph into triangles.
Packing triangles in a graph is NP-complete [2, 32], even when restricted to chordal graphs, planar graphs and linegraphs [28]. Interestingly, the question whether the vertices of a chordal graph can be partitioned into triangles can be solved in polynomial time [11]. Packing triangles in splitgraphs, unit interval graphs and cographs can be solved in polynomial time [11, 28, 45]. For \( r \geq 4 \) the \( K_r \)-packing problem is NP-complete for splitgraphs [28].

When one allows besides triangles also edges in the packing and one wishes to maximize the number of vertices that are covered, then this packing problem becomes polynomial [9, 31, 32, 40].

B Preliminaries on permutation graphs

A permutation diagram is obtained as follows. Let \( L_1 \) and \( L_2 \) be two horizontal lines in the plane, one above the other. Label \( n \) points on the topline and on the bottom line by 1, 2, \ldots, \( n \). Connect each point on the topline by a straight line segment with the point with the identical label on the bottom line. A graph is a permutation graph if it is the intersection graph of the line segments of a permutation diagram [55].

If \( G \) is a permutation graph then its complement \( \bar{G} \) is also a permutation graph. This is easy to see; simply reverse the ordering of the points on one of the two horizontal lines. Also notice this: for any independent set the corresponding line segments are noncrossing. So they can be ordered left to right, which is of course a transitive ordering. This shows that \( \bar{G} \) is a comparability graph and so also \( G \) is a comparability graph. The converse holds as well since transitive orderings of the vertices of \( G \) and of \( \bar{G} \) provides the ordering of the points on the top– and bottom line. This can be seen as follows. Let \( F_1 \) and \( F_2 \) be transitive orientations of \( G \) and \( \bar{G} \). We claim that \( F_1 + F_2 \) is an acyclic orientation of the complete graph. Otherwise there is a directed triangle, and so two edges in the triangle are directed according to one of \( F_1 \) and \( F_2 \) and the third is directed according to the other one of \( F_1 \) and \( F_2 \). But this contradicts the transitivity of \( F_1 \) or the transitivity of \( F_2 \). Likewise, \( F_1^{-1} + F_2 \) is acyclic. Order the vertices on the topline according to \( F_1 + F_2 \) and the vertices on the bottom line according to \( F_1^{-1} + F_2 \). It is easy to check that this yields the permutation diagram.

**Theorem 15 ([13]).** A graph \( G \) is a permutation graph if and only if \( G \) and \( \bar{G} \) are comparability graphs.

The following characterization of permutation graphs illustrates the relation of this class of graphs to the class of interval graphs. Consider a collection of intervals on the real line. Construct a graph of which the vertices are the intervals and make two vertices adjacent if one of the two intervals contains the other. Such a graph is called an interval containment graph.

Consider a diagram of a permutation graph. When one moves the bottom line to the right of the topline then the line segments in the diagram transform into intervals. It is easy to check that two line segments intersect if and only if one of the intervals is contained in the other one. This proves that permutation
graphs are interval containment graphs. Now consider an interval containment graph. Construct a permutation diagram as follows. Put the left endpoints of the intervals in order on the topline and the right endpoints in order on the bottom line of the diagram. Then one interval is contained in another interval if and only if the two line segments intersect. This proves that every interval containment graph is a permutation graph.

**Theorem 16 ([13]).** A graph is a permutation graph if and only if it is an interval containment graph.

Permutation graphs can be recognized in linear time [46, 61]. Notice that permutation graphs are perfect since they have no induced cycles of length more than four [7]. A graph is perfect if for every induced subgraph the clique number is the same as the chromatic number. If the clique – and chromatic number of a graph are the same then these numbers can be computed in polynomial time [26]. However, in a permutation graph computing a largest clique corresponds to finding a longest increasing subsequence and this can be computed very efficiently [20]. (Notice that a permutation graph has a clique or an independent set with at least $\sqrt{n}$ vertices [16].)