A NOTE ON THE CHERN CONJECTURE IN DIMENSION FOUR

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Abstract. Let $M^4$ be a closed immersed minimal hypersurface with constant squared length of the second fundamental form $S$ and constant 3-mean curvature $H_3$ in $S^5$. If $H_2^3 \leq \frac{1}{2}$ and Gauss-Kronecker curvature $K_M$ satisfies $K_M \leq 1$ (or $K_M \leq \frac{e^2}{4\pi^2}$), then $M^4$ is isoparametric.

1. Introduction

More than 50 years ago, S. S. Chern [5, 6] proposed the following famous and original conjecture:

Conjecture 1.1. Let $M^n$ be a closed immersed minimal hypersurface of the unit sphere $S^{n+1}$ with constant scalar curvature $R_M$. Then for each $n$, the set of all possible values for $R_M$ is discrete.

With the development of the study, mathematicians realized the importance of Conjecture 1.1 and proposed the following stronger version. Up to now, it is so far from a complete solution of this problem and S. T. Yau raised it again as the 105th problem in his Problem Section [38]. Please see the excellent and detailed surveys on this topic by Scherfner-Weiss [28], Scherfner-Weiss-Yau [29] and Ge-Tang [18].

Conjecture 1.2. (Chern Conjecture) Let $M^n$ be a closed immersed minimal hypersurface of the unit sphere $S^{n+1}$ with constant scalar curvature. Then $M^n$ is isoparametric.

The problem of classification for isoparametric hypersurfaces in spheres began in 1930 by Cartan and was finished by many mathematicians until 2020 (cf. Cecil-Chi-Jenson [1], Chi [8, 9, 10], Dorfmeister-Neher [15], Immervoll [19] and Miyaoka [22, 23], etc.), please see the elegant book [2] and survey [7] for more details. In 1968, J. Simons [30] showed the following theorem:

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Theorem 1.3. (Simons inequality) Let $M^n$ be a closed immersed minimal hypersurface of the unit sphere $\mathbb{S}^{n+1}$ with the squared length of the second fundamental form $S$. Then
\[
\int_M S(S-n) \geq 0.
\]
In particular, if $0 \leq S \leq n$, one has either $S \equiv 0$ or $S \equiv n$ on $M^n$.

The classification of $S \equiv n$ in Theorem 1.3 was characterized by Chern-do Carmo-Kobayashi [6] and Lawson [20] independently: The Clifford tori are the only closed minimal hypersurfaces in $\mathbb{S}^{n+1}$ with $S \equiv n$, i.e., $S^k(\sqrt{\frac{k}{n}}) \times S^{n-k}(\sqrt{\frac{n-k}{n}})$, $1 \leq k \leq n-1$.

For a closed immersed minimal hypersurface in $\mathbb{S}^{n+1}$, notice that $S = n(n-1) - R_M$, by the Gauss and Codazzi equations. Hence, Simons inequality gave the first pinching gap of Conjecture 1.1.

In 1983, Peng and Terng [26, 27] made the first breakthrough towards Chern Conjecture 1.1, they proved: If $S > n$, then $S > n + \frac{1}{12n}$. Moreover, for $n = 3$, $S \geq 6$ if $S > 3$. In 1993, Chang [3] completed the proof of Chern Conjecture 1.2 for $n = 3$. Next, Yang-Cheng [37] and Suh-Yang [31] improved the second pinching constant from $\frac{1}{12n}$ to $\frac{3}{4n}$. However, it is still an open problem for higher dimensional case that if $S \equiv \text{Constant} > n$, then $S \geq 2n$?

If $S \not\equiv \text{Constant}$, then the problem becomes more difficult. Peng and Terng [26, 27] obtained that there exists a positive constant $\delta(n)$ depending only on $n$, such that if $n \leq S \leq n + \delta(n), n \leq 5$, then $S \equiv n$. Later, Cheng and Ishikawa [4] improved the previous pinching constant for $n \leq 5$, Wei-Xu [35] extended the result to $n = 6, 7$ and Zhang [39] promoted it to $n \leq 8$. Finally, Ding-Xin [14] proved all the dimensions, in particular, they showed that if the dimension is $n \geq 6$, then the pinching constant $\delta(n) = \frac{2\sqrt{n}}{n}$. After that, Xu-Xu [36] improved it to $\delta(n) = \frac{2\sqrt{n}}{n}$ and Li-Xu-Xu [21] showed $\delta(n) = \frac{2\sqrt{n}}{n}$. Actually, due to some counterexamples of Otsuki [25], the condition $S \geq n$ is essential in the pinching results above. Very recently, using the height functions of the normal vector field (cf. [17, 24]), Ge-Li [16] proved that there is a positive constant $\delta(n) > 0$ depending only on $n$ such that on any closed embedded, non-totally geodesic, minimal hypersurface $M^n$ in $\mathbb{S}^{n+1}$, $\int_M S \geq \delta(n)\text{Vol}(M^n)$.

Lately, de Almeida-Brito-Scherfner-Weiss [12] showed that $M^n (n \geq 4)$ is isoparametric if it is a closed, minimally immersed hypersurface of $\mathbb{S}^{n+1}$ with constant Gauss-Kronecker curvature and it has three pairwise distinct principal curvatures everywhere. For the case that $n = 4$, Tang and Yang [32] proved that, if $R_M \geq 0$, $H_3$ and the number of distinct principal curvatures $g$ are constant, then $M^4$ is isoparametric. Deng-Gu-Wei [13] proved that if $M^4$ is a closed Willmore minimal hypersurfaces with
constant scalar curvature in $\mathbb{S}^5$, then it is isoparametric. In other words, they dropped the non-negativity assumption of the scalar curvature under the condition $H_3 \equiv 0$.

A recent great progress of Tang-Wei-Yan [33] and Tang-Yan [34] generalized the theorem of de Almeida and Brito [11] for $n = 3$ to any dimension $n$, strongly supporting Chern Conjecture 1.2. Note that the scalar curvature $R_M \geq 0$ for all isoparametric hypersurfaces and it can be found in [34].

**Theorem 1.4.** (Tang and Yan [34]) Let $M^n (n \geq 4)$ be a closed immersed hypersurface in $\mathbb{S}^{n+1}$. If the following conditions are satisfied:

(i) $\sum_{i=1}^{n} \lambda_i^k (k = 1, \cdots, n-1)$ are constants for principal curvatures $\lambda_1, \lambda_2, \cdots, \lambda_n$;

(ii) $R_M \geq 0$;

then $M^n$ is isoparametric. Moreover, if $M^n$ has $n$ distinct principal curvatures somewhere, then $R_M \equiv 0$.

As an application of Theorem 1.4 in dimension four, we remove the condition of the scalar curvature $R_M \geq 0$, but we have some requirements for the Gauss-Kronecker curvature $K_M$ and 3-mean curvature $H_3$.

**Theorem 1.5.** Let $M^4$ be a closed immersed minimal hypersurface with constant scalar curvature $R_M$ and constant 3-mean curvature $H_3$ in $\mathbb{S}^5$. If $H_3^2 \leq \frac{1}{2}$ and Gauss-Kronecker curvature $K_M$ satisfies $K_M \leq 1$ (or $K_M \leq \frac{s^2}{144}$), then $M^4$ is isoparametric.

2. **Proof of Theorem 1.5**

In this section, we will prove Theorem 1.5. Let $M^n$ be a closed immersed minimal hypersurface in the unit sphere $\mathbb{S}^{n+1}$ and denote by $h$ the second fundamental form of hypersurface with respect to the unit normal vector field $\nu$. If $\{\omega_1, \omega_2, \omega_3, \omega_4\}$ is a smooth orthonormal coframe field, then $h$ can be written as

$$h = \sum_{i,j} h_{ij} \omega_i \otimes \omega_j.$$

The covariant derivative $\nabla h$ with components $h_{ijk}$ is given by

$$\sum_k h_{ijk} \omega_k = dh_{ij} + \sum_k h_{kj} \omega_{ik} + \sum_k h_{ik} \omega_{jk},$$

and $\{\omega_{ij}\}$ is the connection forms of $M^4$ with respect to $\{\omega_1, \omega_2, \omega_3, \omega_4\}$, which satisfy the following structure equations:

$$d\omega_i = -\sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0,$$

$$d\omega_{ij} = -\sum_k \omega_{ijk} \wedge \omega_{kj} + \frac{1}{2} \sum_{k,l} R_{ijkl} \omega_k \wedge \omega_l,$$
where \( R_{ijkl} \) is the coefficients of the Riemannian curvature tensor on \( M^4 \). We have the Gauss and Codazzi equations:

\[
R_{ijkl} = \delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk} + h_{ik} h_{jl} - h_{il} h_{jk},
\]

and

\[
h_{ijk} = h_{ikj}.
\]

It is a well-known fact that the dual (1, 1) tensor \( A \) (shape operator) of \( h \) is a self-adjoint linear operator in each tangent plane \( T_p M \) and its eigenvalues \( \lambda_1(p), \lambda_2(p), \ldots, \lambda_n(p) \) are the principal curvatures. Associated to the shape operator \( A \) there are \( n \) algebraic invariants given by

\[
\sigma_r(p) = \sigma_r(\lambda_1(p), \lambda_2(p), \ldots, \lambda_n(p)), \quad 1 \leq r \leq n,
\]

where \( \sigma_r : \mathbb{R}^n \to \mathbb{R} \) is the elementary symmetric functions in \( \mathbb{R} \) given by

\[
\sigma_r(x_1, \ldots, x_n) = \sum_{i_1 < i_2 < \cdots < i_r} x_{i_1} x_{i_2} \cdots x_{i_r}.
\]

Observe that the characteristic polynomial of \( A \) can be written as

\[
det(\lambda I_n - A) = \sum_{r=0}^{n} (-1)^r \sigma_r \lambda^{n-r}.
\]

The \( r \)-mean curvature \( H_r \) of the hypersurface is then defined by

\[
(\binom{n}{r}) H_r = \sigma_r.
\]

Suppose

\[
f_k = \text{Tr}(A^k),
\]

by \( n = 4, H_1 = 0, \sigma_4 = K_M \) and \( f_2 = \text{Tr}(A^2) = ||h||^2 = S \) we have

\[
\begin{align*}
 f_1 &= \sigma_1 = nH_1 = 0 \\
 f_2 &= \sigma_2^2 - 2\sigma_2 = S \\
 f_3 &= \sigma_3 - 3\sigma_1\sigma_2 + 3\sigma_3 = 3\sigma_3 \\
 f_4 &= \sigma_4 - 4\sigma_1^2\sigma_2 + 4\sigma_1\sigma_3 + 2\sigma_2^2 = 4\sigma_4 = \frac{S^2}{2} - 4K_M.
\end{align*}
\]

**Lemma 2.1.** Let \( M^4 \) be a closed immersed minimal hypersurface in \( S^5 \) with constant scalar curvature \( R_M \neq 6 \) and constant 3-mean curvature \( H_3 \) (or equivalently \( f_3 \) is constant). If there are 4 distinct principal curvatures at the minimum point and maximum point of \( K_M \), then Gauss-Kronecker curvature \( K_M \) satisfies

\[
\sup_{x \in M^4} K_M(x) \geq \frac{S^2(S - 10) + 6f_3^2}{48(S - 6)} \geq \inf_{x \in M^4} K_M(x).
\]
A NOTE ON THE CHERN CONJECTURE IN DIMENSION FOUR 5

Proof. Set \( x_{\text{max}} \in M^4 \) and \( x_{\text{min}} \in M^4 \) such that
\[
K_M(x_{\text{max}}) = \sup_{x \in M^4} K_M(x), \quad K_M(x_{\text{min}}) = \inf_{x \in M^4} K_M(x).
\]
At point \( p \) (\( p = x_{\text{max}} \) or \( p = x_{\text{min}} \)), we can take orthonormal frames such that \( h_{ij} = \lambda_i \delta_{ij} \) for all \( i, j \). Thus at this point, we have
\[
\begin{cases}
\sum_{i=1}^{4} h_{ikk} = 0 \\
\sum_{i=1}^{4} \lambda_i h_{iik} = 0 \\
\sum_{i=1}^{4} \lambda_i^2 h_{iik} = 0 \\
\sum_{i=1}^{4} \lambda_i^3 h_{iik} = 0.
\end{cases}
\]

The first, second and third equations hold because \( f_1, f_2 \) and \( f_3 \) are constant. The fourth one comes from the fact that \( p \) is an extreme point of \( K_M \). Then \( h_{iik} = 0 \) by \( \lambda_i \neq \lambda_j \) \((i \neq j)\) at \( p \). Since \( f_3 \) is constant and due to Peng and Terng \cite{26, 27}, one has

\[
(2.4) \quad \mathcal{A} - 2\mathcal{B} = S f_4 - f_3^2 - S^2,
\]
and

\[
(2.5) \quad \frac{1}{4} \Delta f_4 = (4 - S)f_4 + 2\mathcal{A} + \mathcal{B},
\]
where
\[
\mathcal{A} = \sum_{i,j,k} h_{ijk} \lambda_i^2, \quad \mathcal{B} = \sum_{i,j,k} h_{ijk} \lambda_i \lambda_j.
\]
In addition, due to \( S \) is constant and by Simons’ identity \cite{30} we obtain

\[
(2.6) \quad 0 = \frac{1}{2} \Delta S = |\nabla h|^2 + S(4 - S),
\]
where \( |\nabla h|^2 = \sum_{i,j,k} h_{ijk}^2 \). Since \( h_{iik} = 0 \) for all \( i, k \) at \( p \) and let
\[
C = \lambda_1^2 h_{234}^2 + \lambda_2^2 h_{134}^2 + \lambda_3^2 h_{124}^2 + \lambda_4^2 h_{123}^2,
\]
we can directly calculate

\[
(2.7) \quad 3(\mathcal{A} - 2\mathcal{B}) = \sum_{i,j,k} h_{ijk}^2 \left( (\lambda_i^2 + \lambda_j^2 + \lambda_k^2) - 2\lambda_i \lambda_j - 2\lambda_i \lambda_k - 2\lambda_j \lambda_k \right)
\]
\[
= \sum_{i,j,k} h_{ijk}^2 \left( 2 (\lambda_i^2 + \lambda_j^2 + \lambda_k^2) - (\lambda_i + \lambda_j + \lambda_k)^2 \right)
\]
\[
= 6 \left( h_{234}^2(2S - 3\lambda_1^2) + h_{134}^2(2S - 3\lambda_2^2) + h_{124}^2(2S - 3\lambda_3^2) + h_{234}^2(2S - 3\lambda_4^2) \right)
\]
\[
= 2S|\nabla h|^2 - 18C
\]
\[
= 2S^2(S - 4) - 18C.
\]
Similarly
\[ A = \frac{1}{3} \sum_{i,j,k} h_{ijk}^2 (\lambda_i^2 + \lambda_j^2 + \lambda_k^2) \]
(2.8)
\[ = 2 \left( h_{123}^2 (S - \lambda_1^2) + h_{124}^2 (S - \lambda_2^2) + h_{134}^2 (S - \lambda_3^2) + h_{234}^2 (S - \lambda_4^2) \right) \]
\[ = \frac{1}{3} S |\nabla h|^2 - 2C \]
\[ = \frac{1}{3} S^2 (S - 4) - 2C. \]

By (2.7) and (2.8), we have
\[ B = -\frac{1}{6} S^2 (S - 4) + 2C. \]

Due to \( f_4 = S^2 - 4K_M \) by (2.2) and (2.4)-(2.9), we have
\[ 18 \Delta K_M(p) + 48K_M(p)(S - 6) = S^2 (S - 10) + 6f_3^2. \]

The maximum principle implies that
\[ \Delta K_M(x_{\text{max}}) \leq 0, \quad \Delta K_M(x_{\text{min}}) \geq 0. \]
Hence
\[ 48K_M(x_{\text{max}})(S - 6) \geq S^2 (S - 10) + 6f_3^2 \geq 48K_M(x_{\text{min}})(S - 6). \]
Specially, if \( 0 \leq S < 6 \), then \( K_M(x_{\text{max}}) \leq K_M(x_{\text{min}}) \), we have \( K_M \) is constant and \( M^4 \) is isoparametric, i.e., \( S = 0 \) or \( S = 4 \). The proof is complete. \( \square \)

**Lemma 2.2.** Let \( M^4 \) be a closed immersed minimal hypersurface in \( S^5 \) with constant scalar curvature \( R_M \) and constant 3-mean curvature \( H_3 \) (or equivalently \( f_3 \) is constant). If there exists a point \( p \in M^4 \) with three distinct principal curvatures, then Gauss-Kronecker curvature \( K_M \) satisfies
\[ -\Delta K_M(p) = 4(S - 4)K_M(p) - 2C_1 + 2(6\lambda^2 - S) \left( h_{111}^2 + h_{112}^2 \right) , \]
where \( C_1 = \lambda_1^2 h_{234}^2 + \lambda_2^2 h_{341}^2 + \lambda_3^2 (h_{124}^2 + h_{114}^2) + \lambda_4^2 (h_{123}^2 + h_{134}^2) \), \( \lambda_1(p) = \lambda_2(p) = \lambda \), \( \lambda_3(p) = \mu - \lambda \) and \( \lambda_4(p) = -\mu - \lambda \).

**Proof.** At point \( p \in M^4 \), we can take orthonormal frames such that \( h_{ij} = \lambda_i \delta_{ij} \) for all \( i, j \). Thus at this point, we have
\[
\begin{aligned}
& \sum_{i=1}^4 h_{iik} = 0 \\
& \sum_{i=1}^4 \lambda_i h_{iik} = 0 \\
& \sum_{i=1}^4 \lambda_i^2 h_{iik} = 0 .
\end{aligned}
\]

The first, second and third equations hold because \( f_1, f_2 \) and \( f_3 \) are constant. Then for all \( 1 \leq k \leq 4 \), one has
\[ h_{11k} = -h_{22k}, \quad h_{33k} = h_{44k} = 0, \]
(2.11)
By \( \lambda_i \neq \lambda_j \) (\( 2 \leq i \neq j \leq 4 \)) at \( p \). Let \( C_2 = h_{234}^2 + h_{123}^2 + h_{124}^2 + h_{134}^2 + h_{114}^2 + h_{113}^2 \), by (2.11) we have

\[
3(\mathcal{A} - 2\mathcal{B}) = \sum_{i,j,k} h_{ijk}^2 \left( 2(\lambda_i^2 + \lambda_j^2 + \lambda_k^2) - (\lambda_i + \lambda_j + \lambda_k)^2 \right)
\]

(2.12) By (2.12) and (2.13) we have

\[
B = 6(h_{234}^2 - 3\lambda_4^2) + h_{124}^2 S - 3\lambda_3^2) + h_{134}^2 (2S - 3\lambda_2^2) + h_{234}^2 (2S - 3\lambda_1^2)) +
\]

\[
\lambda_2 h_{111}^2 + h_{112}^2 (S - \lambda_1^2) + h_{113}^2 (S - \lambda_2^2) +
\]

\[
\lambda_3 h_{222}^2 + 3\lambda_2 h_{224}^2 + h_{223} (S - \lambda_1) + h_{224} (S - \lambda_2)
\]

(2.15) By (2.12) and (2.13), we have

\[
\mathcal{A} = \frac{1}{3} S^2 (S - 4) - 2C_1 + 4\lambda^2 (h_{111}^2 + h_{112}^2) .
\]

(2.16) By (2.5) and (2.11) we obtain

\[
|\nabla h|^2 = S(S - 4) = \sum_{i,j,k} h_{ijk}^2
\]

(2.17) Due to (2.2), (2.5), (2.16) and (2.17), we have

\[
-\Delta K_M(p) = 4(S - 4)K_M(p) - 2C_1 + 2(6\lambda^2 - S) (h_{111}^2 + h_{112}^2) .
\]
The proof is complete. □

**Proof of Theorem 1.5.** Suppose \( K_M(p) = \sup_{x \in M^4} K_M(x) \in M^4 \), we have just five possibilities for the principal curvatures \( \lambda_1(p), \ldots, \lambda_4(p) \) at point \( p \in M^n \):

1. \( \lambda_i(p) = \lambda_j(p) \) for all \( 1 \leq i, j \leq 4 \).
2. \( \lambda_1(p) = \lambda_2(p) = \lambda \) and \( \lambda_3(p) = \lambda_4(p) = -\lambda \),
   \[
   A(p) = \begin{pmatrix}
   \lambda & 0 & 0 & 0 \\
   0 & \lambda & 0 & 0 \\
   0 & 0 & -\lambda & 0 \\
   0 & 0 & 0 & -\lambda 
   \end{pmatrix}.
   \]
3. \( \lambda_1(p) = \lambda_2(p) = \lambda_3(p) = \lambda \) and \( \lambda_4(p) = -3\lambda \),
   \[
   A(p) = \begin{pmatrix}
   \lambda & 0 & 0 & 0 \\
   0 & \lambda & 0 & 0 \\
   0 & 0 & \lambda & 0 \\
   0 & 0 & 0 & -3\lambda 
   \end{pmatrix}.
   \]
4. \( \lambda_1(p) = \lambda_2(p) = \lambda \), \( \lambda_3(p) = \mu - \lambda \) and \( \lambda_4(p) = -\mu - \lambda \),
   \[
   A(p) = \begin{pmatrix}
   \lambda & 0 & 0 & 0 \\
   0 & \lambda & 0 & 0 \\
   0 & 0 & \mu - \lambda & 0 \\
   0 & 0 & 0 & -\mu - \lambda 
   \end{pmatrix}.
   \]
5. \( \lambda_i(p) \neq \lambda_j(p) \) for all \( 1 \leq i \neq j \leq 4 \).

Due to Theorem 1.4, \( M^4 \) is isoparametric if \( S \leq 12 \). Hence, we just need to prove \( S \leq 12 \) at \( p \), since \( S \) is constant on \( M^4 \).

In the case (1), \( S \equiv 0 \).

In the case (2), due to \( H_3 \) is constant, also \( f_3 = 12H_3 \) is constant by (2.1) and (2.2), then \( f_3 = 0 \), \( M^4 \) is isoparametric (see Deng-Gu-Wei [13]). In fact, \( K_M \leq 1 \) (or \( K_M \leq \frac{S^2}{144} \)) implies that

\[
K_M = \lambda^4 \leq 1 \quad \text{(or } K_M = \lambda^4 \leq \frac{S^2}{144}, \text{)}
\]

and \( S = 4\lambda^2 \leq 4 \) (or \( \lambda^4 \leq \frac{S^2}{144} = \frac{16\lambda^4}{144} \)). Hence, \( S \leq 4 \) in this case.

In the case (3), if \( f_3^2 = 576\lambda^6 \leq 576 \), then \( \lambda^6 \leq 1 \) and \( S = 12\lambda^2 \leq 12 \).

In the case (4), some direct calculations show

\[
\begin{cases}
S & = 4\lambda^2 + 2\mu^2 \\
f_3 & = -6\mu^2\lambda \\
K_M & = \lambda^2(\lambda^2 - \mu^2) 
\end{cases}
\]

(2.18)
By (2.18), one has

\( (2.19) \quad \lambda^2 = \frac{\mu^2 + \sqrt{\mu^4 + 4K_M}}{2} \quad \text{or} \quad \lambda^2 = \frac{\mu^2 - \sqrt{\mu^4 + 4K_M}}{2}. \)

If \( K_M(p) = \lambda^2(\lambda^2 - \mu^2) < 0 \), we have \( 0 < \lambda^2 < \mu^2 \). The maximum principle implies

\( \Delta K_M(p) \leq 0. \)

Due to (2.10) by Lemma 2.2 and \( S > 4 \), one has

\[ 0 \leq -\Delta K_M(p) = 4(S - 4)K_M(p) - 2C_1 + 2(6\lambda^2 - S) \left(h_{111}^2 + h_{112}^2\right), \]

and

\[ 0 \leq C_1 < (6\lambda^2 - S) \left(h_{111}^2 + h_{112}^2\right) = 2(\lambda^2 - \mu^2) \left(h_{111}^2 + h_{112}^2\right) \leq 0. \]

This creates a contradiction. Thus, \( K_M(p) \geq 0 \). By (2.19), we obtain

\( f_{3}^{2} = 36\mu^4\lambda^2 = 18\mu^4 \left(\mu^2 + \sqrt{\mu^4 + 4K_M}\right) \geq 36\mu^6. \)

If \( f_{3}^{2} \leq \frac{576\sqrt{16}}{25} \) and \( K_M \leq 1 \) (or \( K_M \leq \frac{S^2}{144} \)), then

\[ S = 4\lambda^2 + 2\mu^2 = 4\mu^2 + 2\sqrt{\mu^4 + 4K_M} \leq 2\sqrt{10(\mu^4 + 2K_M)} \]

\[ \leq 2\sqrt{10}\left(\frac{f_{3}^{2}}{36}\right)^{\frac{2}{3}} + 2 \]

\[ = 12, \]

(or \( S \leq 2\sqrt{10}\left(\frac{f_{3}^{2}}{36}\right)^{\frac{2}{3}} + \frac{S^2}{144} \leq \sqrt{64 + \frac{5S^2}{36}} \) shows that \( S \leq 12 \)).

In the case (5), by (2.3) in Lemma 2.1 and \( K_M \leq 1 \) (or \( K_M \leq \frac{S^2}{144} \)), we have

\[ \left(\text{or} \quad \frac{S^2}{144} \geq 1 \right) \geq K_M(p) \geq \frac{S^2(S - 10) + 6f_{3}^{2}}{48(S - 6)} \geq \frac{S^2(S - 10)}{48(S - 6)}, \]

and it implies that \( S \leq 12 \) if \( S \neq 6 \).

To sum up, all the cases show that \( S \leq 12 \) if \( f_{3}^{2} \leq \min\{576, \frac{576\sqrt{16}}{25}\} = \frac{576\sqrt{16}}{25} \). By (2.1) and (2.2), we have

\[ H_{3}^{2} = \frac{f_{3}^{2}}{16} = \frac{f_{3}^{2}}{144} \leq \frac{4\sqrt{10}}{25} \approx 0.5059. \]

This completes the proof by \( H_{3}^{2} \leq 0.5 < 0.5059. \)

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