NONEXISTENCE OF EXTREMALS FOR THE ADJOINT RESTRICTION INEQUALITY ON THE HYPERBOLOID

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Abstract. We study the problem of existence of extremizers for the $L^2$ to $L^p$ adjoint Fourier restriction inequalities for the hyperboloid in dimensions 3 and 4 in the case $p$ is an even integer. We use the method developed by Foschi in [5] to show that extremizers do not exist.

1. Introduction

For $d \geq 1$, let $\mathbb{H}^d$ denote the hyperboloid in $\mathbb{R}^{d+1}$, $\mathbb{H}^d = \{(y, \sqrt{1+|y|^2}) : y \in \mathbb{R}^d\}$, equipped with the measure

$$\sigma(y,y') = \delta(y' - \sqrt{1+|y|^2}) \frac{dydy'}{\sqrt{1+|y|^2}}$$

defined by duality as

$$\int_{\mathbb{H}^d} g(y,y')d\sigma(y,y') = \int_{\mathbb{R}^d} g(y, \sqrt{1+|y|^2}) \frac{dy}{\sqrt{1+|y|^2}},$$

for all $g \in C_0(\mathbb{R}^{d+1})$.

A function $f : \mathbb{H}^d \to \mathbb{R}$ can be identified with a function from $\mathbb{R}^d$ to $\mathbb{R}$, and in what follows, we do so. We denote the $L^p(\mathbb{H}^d,\sigma)$-norm of a function $f$ by $\|f\|_{L^p(\mathbb{H}^d,\sigma)}$.

The extension or adjoint Fourier restriction operator for $\mathbb{H}^d$ is given by

$$Tf(x,t) = \int_{\mathbb{R}^d} e^{ix \cdot y} e^{it \sqrt{1+|y|^2}} f(y) (1 + |y|^2)^{-1/2} dy,$$ (1.1)

where $(x,t) \in \mathbb{R}^d \times \mathbb{R}$ and $f \in S(\mathbb{R}^d)$. With the Fourier transform in $\mathbb{R}^{d+1}$ defined to be $\hat{g}(\xi) = \int_{\mathbb{R}^{d+1}} e^{-ix \cdot \xi} g(x) dx$, we see that $Tf(x,t) = \hat{f}(-x,-t)$.

It is known [10] that there exists $C_{d,p} < \infty$ such that for all $f \in L^2(\mathbb{H}^d)$, the estimate for $Tf$

$$\|Tf\|_{L^p(\mathbb{R}^{d+1})} \leq C_{d,p} \|f\|_{L^2(\mathbb{H}^d)}$$ (1.2)

holds provided that

$$\frac{2(d+2)}{d} \leq p \leq \frac{2(d+1)}{d-1} \quad \text{if } d > 1,$$

$$6 \leq p < \infty \quad \text{if } d = 1.$$ (1.3)

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For $p$ satisfying (1.3), we denote by $H_{d,p}$ the best constant in (1.2),
\[ H_{d,p} = \sup_{0 \neq f \in L^2(\mathbb{H}^d)} \frac{\|Tf\|_{L^p(\mathbb{R}^{d+1})}}{\|f\|_{L^2(\mathbb{H}^d)}}. \]

We also consider the two-sheeted hyperboloid
\[ \mathbb{H}^d = \{ (y, y') \in \mathbb{R}^d \times \mathbb{R} : y^2 = 1 + |y|^2 \}. \]
and endow it with the measure $\tilde{\sigma} = \sigma^+ + \sigma^-$, where
\[ \sigma^+(y, y') = \delta \left( y' - \sqrt{1 + |y|^2} \right) \frac{dy dy'}{\sqrt{1 + |y|^2}}, \]
\[ \sigma^-(y, y') = \delta \left( y' + \sqrt{1 + |y|^2} \right) \frac{dy dy'}{\sqrt{1 + |y|^2}}. \]

The corresponding adjoint Fourier restriction operator is $\tilde{T}f = \hat{f} \sigma^+ + \hat{f} \sigma^-$. If $(d, p)$ satisfies (1.3), then
\[ \|\tilde{T}f\|_{L^p(\mathbb{R}^{d+1})} \leq \tilde{H}_{d,p} \|f\|_{L^2(\mathbb{H}^d)}, \] (1.4)
where
\[ \tilde{H}_{d,p} = \sup_{0 \neq f \in L^2(\mathbb{H}^d)} \frac{\|\tilde{T}f\|_{L^p(\mathbb{R}^{d+1})}}{\|f\|_{L^2(\mathbb{H}^d)}} \] (1.5)
is finite.

**Definition 1.1.** An extremizing sequence for inequality (1.2) is a sequence $\{f_n\}_{n \in \mathbb{N}}$ of functions in $L^2(\mathbb{H}^d)$ satisfying $\|f_n\|_{L^2(\mathbb{H}^d)} \leq 1$ such that
\[ \|Tf_n\|_{L^p(\mathbb{R}^{d+1})} \to H_{d,p} \text{ as } n \to \infty. \]

An extremizer for inequality (1.2) is a function $f \neq 0$ which satisfies $\|Tf\|_{L^p(\mathbb{R}^{d+1})} = H_{d,p} \|f\|_{L^2(\mathbb{H}^d)}$. These terms are defined analogously for inequality (1.4).

We are interested in the following pairs $(2, 4)$, $(2, 6)$ and $(3, 4)$ of $(d, p)$, which are the only cases for $d > 1$ where $p$ is an even integer. The main result of this paper is the following theorem.

**Theorem 1.2.** The values of the best constants are $H_{2,4} = \frac{3}{4} \pi$, $H_{2,6} = (2\pi)^{5/6}$ and $H_{3,4} = (2\pi)^{5/4}$. Moreover, extremizers for inequality (1.2) do not exist in each of the three cases of $(d, p)$.

The best constants for the two-sheeted hyperboloid are $\tilde{H}_{2,4} = (3/2)^{1/4}H_{2,4}$, and $\tilde{H}_{3,4} = (3/2)^{1/4}H_{3,4}$, and extremizers for inequality (1.4) do not exist.

When $(d, p) = (2, 6)$ we only prove an inequality for $\tilde{H}_{2,6}$ as recorded in the next remark.

**Remark 1.3.** In Proposition 7.5 we show that for each $f \in L^2(\mathbb{H}^2)$, $f \neq 0$
\[ \|\tilde{T}f\|_{L^6(\mathbb{R}^2)} \|f\|_{L^6(\mathbb{H}^2)} < \frac{25}{4} \tilde{H}_{2,6}^6. \]
and therefore $\hat{H}_{2,6} \leq (5/2)^{1/3}H_{2,6}$. Moreover, a refinement of the argument shows that there is a strict inequality $H_{2,6} < (5/2)^{1/3}H_{2,6}$.

We normalize the Fourier transform in $\mathbb{R}^d$ as $\hat{g}(\xi) = \int_{\mathbb{R}^d} e^{-ix\cdot\xi} g(x) dx$. With this normalization, the convolution and $L^2(\mathbb{R}^d)$ norm satisfy

$$\widehat{f \ast g} = \hat{f} \hat{g} \quad \text{and} \quad \|f\|_{L^2(\mathbb{R}^d)} = (2\pi)^{d/2}\|\hat{f}\|_{L^2(\mathbb{R}^d)},$$

respectively. If $p = 2k$ is an even integer, we can write (1.2) in “convolution form”

$$\|Tf\|_{L^{2k}(\mathbb{R}^{d+1})} = \|\hat{(T\sigma)}^k\|_{L^2(\mathbb{R}^{d+1})} = \|(f\sigma \ast \cdots \ast f\sigma)\|_{L^2(\mathbb{R}^{d+1})},$$

where $f\sigma \ast \cdots \ast f\sigma$ denotes the $k$-fold convolution of $f\sigma$ with itself. Therefore, for $p$ an even integer, (1.2) is equivalent to

$$\|f\sigma \ast \cdots \ast f\sigma\|_{L^p(\mathbb{R}^{d+1})} \leq (2\pi)^{-(d+1)/2k}C_{d,2k}\|f\|_{L^2(\mathbb{R}^d)},$$

for all $f \in S(\mathbb{R}^{d+1})$.

For reference, we write here the best constants in convolution form:

$$\sup_{0 \neq f \in L^2(\mathbb{H}^d)} \|f\sigma \ast f\sigma\|_{L^2(\mathbb{R}^d)}^{1/2} \|f\|_{L^2(\mathbb{H}^d)}^{-1} = \pi^{1/4},$$

(1.7)

$$\sup_{0 \neq f \in L^2(\mathbb{H}^d)} \|f\sigma \ast f\sigma \ast f\sigma\|_{L^2(\mathbb{R}^d)}^{1/3} \|f\|_{L^2(\mathbb{H}^d)}^{-1} = (2\pi)^{1/3},$$

(1.8)

$$\sup_{0 \neq f \in L^2(\mathbb{H}^d)} \|f\sigma \ast f\sigma \ast f\sigma\|_{L^2(\mathbb{R}^d)}^{1/2} \|f\|_{L^2(\mathbb{H}^d)}^{-1} = (2\pi)^{1/4}.$$  

(1.9)

It would be interesting to analyze the case $d = 1$ for even integers greater than or equal to 6. Our argument relies on the explicit computation of the $n$-fold convolution of the measure $\sigma$ with itself, and this seems to be computationally complicated if $n \geq 3$.

Interpolation shows that $H_{2,p} \leq H_{2,6}^{1/4}H_{1,6}^{1/4}$ for $d = 2$ and $p \in [4,6]$, where $\frac{4}{p} = \frac{1}{p} + \frac{1-\frac{d}{2}}{6}$. We do not know whether extremizers exist for $p \in (4,6)$, as our method only applies when $p$ is an even integer.

We consider, for $s > 0$, the hyperboloid $H^d_s = \{(y, \sqrt{s^2 + |y|^2}) : y \in \mathbb{R}^d\}$, equipped with the measure

$$\sigma_s(y, y') = \delta(y' - \sqrt{s^2 + |y|^2}) \frac{dydy'}{\sqrt{s^2 + |y|^2}}.$$  

(1.10)

As we mention in Section 3, this measure is natural, since up to multiplication by scalar, it is the only Lorentz invariant measure on $H^d_s$. Let $T_s f(x, t) = \int_{H^d_s} f(x, t') \sigma_s(x, t') dt'$. For $(d, p)$ satisfying (1.3),

$$\|T_s f\|_{L^p(\mathbb{R}^{d+1})} \leq H_{d,p,s} \|f\|_{L^2(\mathbb{H}^d)},$$

(1.11)

where

$$H_{d,p,s} = \sup_{0 \neq f \in L^2(\mathbb{H}^d)} \frac{\|T_s f\|_{L^p(\mathbb{R}^{d+1})}}{\|f\|_{L^2(\mathbb{H}^d)}},$$

(1.12)

is a finite constant.
As shown in Appendix 1, simple scaling relates \( H_{d,p,s} \) and \( H_{d,p} \) by
\[
H_{d,p,s} = s^{(d-1)/2-(d+1)/p} H_{d,p}.
\]
Moreover, \( \{f_n\}_{n \in \mathbb{N}} \) is a extremizing sequence for inequality (1.12) if and only if the sequence \( \{s^{-1/2} f_n(s^{-1}.\} \}_{n \in \mathbb{N}} \) is extremizing for inequality (1.11). Thus, for the problem of extremizers and properties of extremizing sequences, it is enough to study the case \( s = 1 \).

For each \( \rho \in (0, \infty) \), we consider the truncated hyperboloid
\[
H^d_{s,\rho} = \left\{ (y, \sqrt{s^2 + |y|^2}) : y \in \mathbb{R}^d, |y| \leq \rho \right\},
\]
endowed with the measure which is the restriction of \( \sigma_s \) to \( H^d_{s,\rho} \). For \( f \in L^2(H^d_{s,\rho}) \), let \( T_{s,\rho} f = T_s f \) denote the corresponding adjoint Fourier restriction operator. Since \( \|T_{s,\rho} f\|_{L^\infty(\mathbb{R}^d)} \leq C \|f\|_{L^2(\mathcal{H}^d_{s,\rho})} \), it follows that for \( d \geq 1 \),
\[
\|T_{s,\rho} f\|_{L^p(\mathbb{R}^{d+1})} \leq C \|f\|_{L^2(\mathcal{H}^d_{s,\rho})}
\]
for \( p \geq 2(d+2)/d \) and some constant \( C = C(d, p, s, \rho) < \infty \).

The main theorem of Fanelli, Vega and Visciglia in [3, Theorem 1.1] implies that if \( d \geq 1 \) and \( p > 2(d+2)/d \), complex-valued extremizers for (1.14) exist. There exist nonnegative extremizers if \( p \) is an even integer, as can be seen from the equivalent “convolution form” of (1.14). This shows that for \( (d, p) = (2, 6) \) and \( (d, p) = (3, 4) \), there exist extremizers for (1.14). The case \( (d, p) = (2, 4) \) does not follow from the result in [3], since it is the endpoint. In Proposition 5.6, we prove that in this case, extremizers do not exist and that the best constant in (1.14) is independent of \( \rho \) and equals the best constant for the full hyperboloid \( H^d_s \).

2. Some related results

In this section, we discuss the results in [4] and their connection to the case of the adjoint Fourier restriction inequalities for the hyperboloid analyzed in this paper.

For \( r \in \mathbb{R} \), the \textbf{(nonhomogeneous) Sobolev space} \( H^r(\mathbb{R}^d) \) consists of tempered distributions \( g \) over \( \mathbb{R}^d \) such that \( \hat{g} \in L^1_{\text{loc}}(\mathbb{R}^d) \) and the norm
\[
\|g\|_{H^r(\mathbb{R}^d)}^2 := \int_{\mathbb{R}^d} |\hat{g}(y)|^2 (1 + |y|^2)^r dy
\]
is finite. The \textbf{homogeneous Sobolev space} \( \dot{H}^r(\mathbb{R}^d) \) is the space of tempered distributions \( g \) over \( \mathbb{R}^d \) such that \( \hat{g} \in L^1_{\text{loc}}(\mathbb{R}^d) \) and the norm
\[
\|g\|_{\dot{H}^r(\mathbb{R}^d)}^2 := \int_{\mathbb{R}^d} |\hat{g}(y)|^2 |y|^{2r} dy
\]
is finite. Note that the \( \dot{H}^r(\mathbb{R}^d) \)-norm satisfies the scaling property \( \|g(\lambda \cdot)\|_{\dot{H}^r(\mathbb{R}^d)} = \lambda^{r-d/2} \|g\|_{\dot{H}^r(\mathbb{R}^d)} \).

Let us introduce the notation used in [3]. For a function \( h : \mathbb{R}^d \to \mathbb{R} \), the operator \( e^{ih(D)} \) is defined by
\[
e^{ih(D)} g(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix \cdot y} e^{ih(y)} \hat{g}(y) dy \quad \text{for } g \in \mathcal{S}(\mathbb{R}^d),
\]
(2.1)
and for a function \( \eta : \mathbb{R} \to \mathbb{R} \) we define \( e^{it\sqrt{-\Delta}} = e^{it\eta(D)} \).

Note that \( e^{it\sqrt{-\Delta + \lambda^2}} g(x) = (2\pi)^{-d} T f(x, t) \), where \( \hat{g}(y) = f(y)(s^2 + |y|^2)^{-1/2} \); therefore, \( \Box \) is equivalent to the estimate

\[
\|e^{it\sqrt{-\Delta + \lambda^2}} g\|_{L^p_t L^r_x(\mathbb{R}^{d+1})} \leq C_{d,p,s} \|g\|_{L^q_x L^1_t(\mathbb{R}^d)}
\]  

(2.2)

for a constant \( C_{d,p,s} < \infty \) and \( p \) as in \( \Box \). For \( s > 0 \), the operator \( e^{it\sqrt{-\Delta + \lambda^2}} \) satisfies more general mixed-norm Strichartz estimates, namely,

\[
\|e^{it\sqrt{-\Delta + \lambda^2}} g\|_{L^p_t L^r_x(\mathbb{R}^{d+1})} \leq C \|g\|_{L^q_x L^{1+s\Delta t}_t(\mathbb{R}^d)}
\]  

(2.3)

where \( p \in [2, \infty] \), \( q \in [2, 2d/(d - 2)] \) \( (q \in [2, \infty] \) if \( d = 1, 2 \), and \( \frac{2}{p} + \frac{d - 1 + \theta}{q} = \frac{d - 1 + \theta}{2} \), \( (p, q) \neq (2, \infty) \).

Here, \( \theta \in [0, 1] \). We refer the reader to \([6]\) and the references therein for these estimates.

Using (2.3), the Sobolev Embedding Theorem, and interpolation, we obtain that for \( d \geq 2 \),

\[
\|e^{it\sqrt{-\Delta + \lambda^2}} g\|_{L^p_t L^r_x(\mathbb{R}^{d+1})} \leq C \|g\|_{L^r_x \cap H^s(\mathbb{R}^d)}
\]  

(2.4)

for all \( p \) and \( r \) satisfying

\[
\frac{1}{2} \leq r < \frac{d}{2}, \quad \frac{2(d + 2)(d - 1)}{d(d - 2r)} \leq p \leq \frac{2(d + 1)}{d - 2r}.
\]  

(2.5)

An equivalent way to look at the adjoint Fourier restriction inequalities for the hyperboloid \( \mathbb{H}^d \) is through Strichartz estimates for the Klein-Gordon equation

\[
\partial_t^2 u = \Delta u - s^2 u \quad \text{in} \quad \mathbb{R}^{d+1}
\]

\[
u(0, x) = u_0(x), \quad \partial_t u(x, x) = u_1(x).
\]

(2.6)

Writing the solution of (2.6) as

\[
u(t, \cdot) = \cos(t \sqrt{-\Delta + s^2}) u_0(\cdot) + \sin(t \sqrt{-\Delta + s^2}) \frac{u_1(\cdot)}{\sqrt{-\Delta + s^2}},
\]

or equivalently as

\[
u(t, \cdot) = \frac{1}{2} \left( e^{it \sqrt{-\Delta + s^2}} u_0(\cdot) + \frac{1}{i} e^{it \sqrt{-\Delta + s^2}} u_1(\cdot) \right)
\]

\[
+ \frac{1}{2} \left( e^{-it \sqrt{-\Delta + s^2}} u_0(\cdot) - \frac{1}{i} e^{-it \sqrt{-\Delta + s^2}} u_1(\cdot) \right),
\]

we see that (2.4) is equivalent to the Strichartz estimate for \( \nu \)

\[
\|\nu\|_{L^p_t L^r_x(\mathbb{R}^{d+1})} \leq C \|(u_0, u_1)\|_{H^r(\mathbb{R}^d) \times H^{r-1}(\mathbb{R}^d)},
\]  

(2.7)

where \( \|(u_0, u_1)\|_{H^r(\mathbb{R}^d) \times H^{r-1}(\mathbb{R}^d)} \) := \( \|u_0\|_{H^r(\mathbb{R}^d)} + \|u_1\|_{H^{r-1}(\mathbb{R}^d)} \), \( p \) and \( r \) are as in (2.5), and \( C < \infty \) is a constant depending only on \( d, p, r \) and \( s \).

In the context of this paper, it is natural to ask whether inequalities (2.4) and (2.7) admit extremizers \( g \in H^r(\mathbb{R}^d) \) and \( (u_0, u_1) \in H^r(\mathbb{R}^d) \times H^{r-1}(\mathbb{R}^d) \), respectively,
and whether extremizing sequences are precompact, after the possible application of symmetries. Here, extremizers and extremizing sequences are defined similarly as for inequality (1.2) in Definition 1.1.

In [4], the existence of extremals and precompactness of extremizing sequences is studied for an inequality of the form

\[ \| e^{ith(D)} g \|_{L^p_{t,x} (\mathbb{R}^{d+1})} \leq C \| g \|_{\dot{H}^r(\mathbb{R}^d)}, \]  

(2.8)

for operators \( e^{ith(D)} \) that satisfy mixed-norm estimates

\[ \| e^{ith(D)} g \|_{L^p_{t,y} L^q_{-\gamma} (\mathbb{R}^{d+1})} \leq C \| g \|_{\dot{H}^r(\mathbb{R}^d)} \]

for some \( 0 < r < d/2 \) and \( p \) and \( q \) satisfying \( 2 \leq p < q \leq \infty \) where the function \( h(\xi) \) is homogenous of some degree \( k > 0 \), meaning that \( h(\lambda \xi) = \lambda^k h(\xi) \) for all \( \lambda > 0 \) and \( \xi \in \mathbb{R}^d \).

The argument in [4] uses the homogeneous Sobolev spaces \( \dot{H}^r(\mathbb{R}^d) \) and that the function \( h(\xi) \) is homogenous. Indeed, it is the scaling property of the \( \dot{H}^r(\mathbb{R}^d) \)-norm and the homogeneity of the function \( h \) that imply that (2.8) is invariant under scaling, and therefore the sequence defined in [4] Equation 2.15] is still an extremizing sequence for the same inequality.

For the hyperboloid, the function \( h(\xi) = \sqrt{s^2 + |\xi|^2} \) is not homogeneous if \( s \neq 0 \). Therefore, in this case, the question of existence of extremizers in \( H^r(\mathbb{R}^d) \), \( 1/2 < r < d/2 \), for inequality (2.4) is not answered in [4], although information can be obtained from arguments therein, which we record in Proposition 2.1. We can contrast this situation with the case of the cone \( \Gamma^d \) from arguments therein, which we record in Proposition 2.1. We can contrast this situation with the case of the cone \( \Gamma^d \) from arguments therein, which we record in Proposition 2.1. We can contrast this situation with the case of the cone \( \Gamma^d \). Let \( \sigma_0 = \delta(y' - |y|) |y|^{-1} dydy' \). This cone can be seen as the limiting case of the hyperboloid \( (H^s_\sigma, \sigma_s) \) as \( s \to 0 \).

Let \( T_c \) denote the adjoint Fourier restriction operator on the cone \( (\Gamma^d, \sigma_0) \):

\[ T_c f(x, t) := \int_{\mathbb{R}^d} e^{ix \cdot \hat{y} - |y|} f(y) |y|^{-1} dy, \quad f \in S(\mathbb{R}^d). \]  

(2.9)

The operator \( e^{it\sqrt{\Delta}} \) is related to \( T_c \) by \( e^{it\sqrt{\Delta}} g(x) = (2\pi)^{-d} T_c f(x, t) \), where \( \hat{g}(y) = f(y) |y|^{-1} \). For \( d \geq 2 \), the operator \( e^{it\sqrt{\Delta}} \) satisfies

\[ \| e^{it\sqrt{-\Delta}} g \|_{L^{2(d+1)}_{t,x} (\mathbb{R}^{d+1})} \leq C \| g \|_{H^r(\mathbb{R}^d)}, \quad \frac{1}{2} \leq r < \frac{d}{2}. \]  

(2.10)

The main result of [4], Theorem 1.1, implies that for \( d \geq 2 \), extremizers exist for inequality (2.10) for every \( 1/2 < r < d/2 \) and, moreover, extremizing sequences are precompact after the application of symmetries.

For the case \( r = 1/2 \), the existence of extremizers was proved by Carneiro [1] in the cases \( d = 2 \) and \( d = 3 \); he also found the exact form of the extremizers. The precompactness of extremizing sequences after the application of symmetries, and thus the existence of extremizers, was proved in [10] for \( d = 2 \) and by Ramos [11] for \( d \geq 2 \).
The limiting case of the Klein-Gordon equation (2.6) as $s \to 0$ is the wave equation
\begin{align}
\partial_t^2 u &= \Delta u \quad \text{in } \mathbb{R}^{d+1}, \\
u(0, x) &= u_0(x), \quad \partial_t u(x, 0) = u_1(x).
\end{align}

Its solution can be written as
\begin{align}
u(t, \cdot) &= \frac{1}{2} \left( e^{it\sqrt{-\Delta}} u_0(\cdot) + e^{it\sqrt{-\Delta}} u_1(\cdot) \right) + \frac{1}{2} \left( e^{-it\sqrt{-\Delta}} u_0(\cdot) - e^{-it\sqrt{-\Delta}} u_1(\cdot) \right)
\end{align}
and satisfies, for $d \geq 2$ and $1/2 < r < d/2$,
\begin{align}
\|u\|_{L^{d/(d+1)}_{t,x} (\mathbb{R}^{d+1})} \leq C \|u_0, u_1\|_{\dot{H}^r(\mathbb{R}^d) \times \dot{H}^{r-1}(\mathbb{R}^d)}.
\end{align}

Just as for the case of the adjoint Fourier restriction inequality for the cone, there are results concerning the existence of extremizers for inequality (2.12), $(u_0, u_1) \in H^r(\mathbb{R}^d) \times H^{r-1}(\mathbb{R}^d)$. Foschi [5] studied the case $r = 1/2$ for $d = 2$ and $d = 3$, proved the existence of extremizers, and found their exact form. The existence of extremizers for (2.12) when $d \geq 2$ and $1/2 < r < d/2$ was proved in [4], while the case $d \geq 2$ and $r = 1/2$ was proved by Ramos [11]. See also the discussion at the end of [4, Example 1.4] for complementary results.

We note that the argument in [4] does not apply to inequality (2.7) for the same reasons stated before for inequality (2.2).

Let us return to inequality (2.4), where we consider the nonendpoint case, that is, the case $p$ and $r$ satisfy
\begin{align}
1/2 \leq r < \frac{d}{2} \quad \frac{2(d+2)(d-1)}{d(d-2r)} < p \leq \frac{2(d+1)}{d-2r},
\end{align}
that is, (2.3) with the endpoint $p = 2(d+2)(d-1)/(d-2r)$ removed.

In the next proposition, we show that the only obstruction to the convergence of extremizing sequences for inequality (2.4), after the applications of symmetries, is "concentration at infinity" of the Fourier transform.

**Proposition 2.1.** Suppose that $p$ and $r$ satisfy (2.13). Let $\{g_n\}_{n \in \mathbb{N}}$ be an extremizing sequence for inequality (2.4). Then one of the following two possibilities holds.

(i) For all $R \in (0, \infty)$, $\lim_{n \to \infty} \int_{|y| \leq R} |g_n(y)|^2 (1 + |y|^2)^r dy = 0.$

(ii) There exist a subsequence $\{g_{n_k}\}_{k \in \mathbb{N}}$ and a sequence $\{(y_k, t_k)\}_{k \in \mathbb{N}} \subset \mathbb{R}^d \times \mathbb{R}$ such that $e^{it_k\sqrt{-\Delta+s^2}} g_{n_k}(y-y_k) \to 0$ in $L^p(\mathbb{R}^d)$.

Moreover, if (i) holds, then there exist a subsequence $\{g_{n_k}\}_{k \in \mathbb{N}}$, a sequence of positive real numbers $\{\lambda_k\}_{k \in \mathbb{N}}$, $\lambda_k \to 0$ as $k \to \infty$, a sequence $\{(y_k, t_k)\}_{k \in \mathbb{N}} \subset \mathbb{R}^d \times \mathbb{R}$, and $0 \neq v \in H^r(\mathbb{R}^d)$ such that
\begin{align}
\lambda_k^{d/2-r} e^{it_k\sqrt{-\Delta+s^2}} g_{n_k}(\lambda_k(y-y_k)) \to v \quad \text{as } k \to \infty
\end{align}
weakly in the homogeneous Sobolev space $\dot{H}^r(\mathbb{R}^d)$.
In the dual formulation, in “physical space” instead of “frequency space”, that is, via the equality \( \hat{y}(y) = f(y)(s^2 + |y|^2)^{-1/2} \), inequality (2.14) becomes the weighted estimate

\[
\|T_s f\|_{L^p(R^d)} \leq C\|f\|_{L^2(\mu_s)},
\]

where the measure \( \mu_s(y, y') = (1 + |y|^2)^{(s^2 + |y|^2)^{-1/2}} \) is supported on \( \mathbb{H}_s^d \).

The two possibilities in the previous proposition, when written for (2.15), are as follows.

(i) The sequence \( \{f_n\}_{n \in \mathbb{N}} \) concentrates at spatial infinity, that is, for all \( R \in (0, \infty) \),

\[
\lim_{n \to \infty} \int_{|y| \leq R} |f_n(y)|^2(s^2 + |y|^2)^{r-1} dy = 0.
\]

(ii) There exist a subsequence \( \{f_{n_k}\}_{k \in \mathbb{N}} \) and a sequence \( \{(y_k, t_k)\}_{k \in \mathbb{N}} \subset \mathbb{R}^d \times \mathbb{R} \) such that \( \{e^{i\sqrt{s^2 + |y|^2} f_{n_k}(y)}\}_{k \in \mathbb{N}} \) converges in \( L^2(\mu_s) \).

For a set \( A \subset \mathbb{R}^d \) we denote \( \chi_A \) the characteristic function of the set \( A \).

**Sketch of the proof of Proposition 2.7.** If condition (i) is not satisfied, then there exist \( R \in (0, \infty) \) and a subsequence \( \{g_n\}_{n \in \mathbb{N}}, \) which we also call \( \{g_n\}_{n \in \mathbb{N}}, \) satisfying

\[
\inf_{n \in \mathbb{N}} \int_{|y| \leq R} |g_n(y)|^2(1 + |y|^2)^r dy =: \varepsilon^2 > 0.
\]

We define \( g_{n,1} \) and \( g_{n,2} \) by their Fourier transforms, \( \hat{g}_{n,1}(y) = \hat{g}_n(y)\chi_{\{|y| \leq R\}}, \hat{g}_{n,2}(y) = \hat{g}_n(y)\chi_{\{|y| > R\}}. \) Then \( g_n = g_{n,1} + g_{n,2} \), and for all large enough \( n \), we have \( \|g_{n,1}\|_{H^r(\mathbb{R}^d)} \geq \varepsilon/2 \) and \( \|\hat{g}_{n,1}\|_{L^p(\mathbb{R}^d)} \geq c > 0 \) for a certain constant \( c \) independent of \( n \).

Under the assumptions on \( p \) and \( r \), we can apply the “first step” in the proof of [3] Theorem 1.1 to the sequence \( \{g_{n,1}\}_{n \in \mathbb{N}} \) to show that there exist a subsequence, which we also call \( \{g_{n,1}\}_{n \in \mathbb{N}} \), and a sequence \( \{(y_n, t_n)\}_{n \in \mathbb{N}} \subset \mathbb{R}^d \times \mathbb{R} \) such that the functions \( u \mapsto (e^{i\sqrt{s^2 + |y|^2} g_{n,1}}(y - y_n)) \) have a nonzero uniform limit in \( \{y \in \mathbb{R}^d : |y| \leq R\} \). This implies that weak limits of the sequence \( \{e^{i\sqrt{s^2 + |y|^2} g_{n,1}}(\cdot - y_n)\}_{n \in \mathbb{N}} \) in \( H^r(\mathbb{R}^d) \) are nonzero. Using the argument given in the proof of [4] Theorem 1.1 or an argument similar to that in [10] Proposition 8.3, we see that all the hypotheses of [3] Proposition 1.1 are satisfied by the sequence \( \{e^{i\sqrt{s^2 + |y|^2} g_{n,1}}(\cdot - y_n)\}_{n \in \mathbb{N}} \), which is extremizing. Therefore the latter sequence is precompact in \( H^r(\mathbb{R}^d) \), and (ii) is satisfied.

Let us now suppose that (i) is satisfied. The existence of the subsequence \( \{g_{n_k}\}_{k \in \mathbb{N}} \), the sequences \( \{\lambda_k\}_{k \in \mathbb{N}} \), and \( \{(y_k, t_k)\}_{k \in \mathbb{N}} \subset \mathbb{R}^d \times \mathbb{R} \), and the function \( v \in H^r(\mathbb{R}^d) \), \( v \neq 0 \), satisfying (2.14) follows as in [4] Proof of Theorem 1.1. That \( \lambda_k \to 0 \) as \( k \to \infty \) follows from (i). Indeed, if there exists a subsequence \( \{\lambda_{k_l}\}_{l \in \mathbb{N}} \) with \( \inf_{l \in \mathbb{N}} \lambda_{k_l} > 0 \), then the functions \( h_{k_l}(y) := \lambda_{k_l}^{d/2 - r} e^{i\sqrt{s^2 + |y|^2} g_{n_{k_l}}(\lambda_{k_l}(y - y_{k_l}))} \) satisfy

\[
\int_{|y| \leq R} |\hat{h}_{k_l}(y)|^2(1 + |y|^2)^r dy = \int_{|y| \leq R/\lambda_{k_l}} |\hat{g}_{n_{k_l}}(y)|^2(1/\lambda_{k_l}^2 + |y|^2)^r dy \to 0 \quad \text{as} \quad l \to \infty,
\]

for every \( R < \infty \), which is impossible since \( v \neq 0 \).
NONEXISTENCE OF EXTREMALS FOR THE HYPERBOLOID

3. THE LORENTZ INVARiance

The Lorentz group is defined as the group of invertible linear transformations in \( \mathbb{R}^{d+1} \) preserving the bilinear form \( (x, y) \in \mathbb{R}^{d+1} \times \mathbb{R}^{d+1} \mapsto x \cdot Jy, \) where \( J = \text{diag}(-1, \ldots, -1, 1). \)

Let us denote by \( \mathcal{L}^+ \) the subgroup of Lorentz transformations in \( \mathbb{R}^{d+1} \) that preserve \( \mathbb{H}^d \). It is known that \( \sigma_s \) is invariant under the action of \( \mathcal{L}^+ \) and moreover is, up to multiplication by scalar, the unique measure on \( \mathbb{H}^d \) invariant under such Lorentz transformations; see [12] where the case \( d = 3 \) is considered. The same argument can be adapted to \( d \geq 2. \)

For \( t \in (-1, 1), \) we define the linear map \( L^t : \mathbb{R}^{d+1} \rightarrow \mathbb{R}^{d+1} \) by

\[
L^t(\xi_1, \ldots, \xi_d, \tau) = \left( \frac{\xi_1 + t \tau}{\sqrt{1 - t^2}}, \xi_2, \ldots, \xi_d, \frac{\tau + t \xi_1}{\sqrt{1 - t^2}} \right).
\]

Then \( \{L^t\}_{t \in (-1,1)} \) is a one parameter subgroup of Lorentz transformations contained in \( \mathcal{L}^+. \)

For \( i, j \in \{1, \ldots, d\}, \) let \( P_{i,j} \) be the linear transformation that swaps the \( i^{\text{th}} \) and \( j^{\text{th}} \) components of every vector in \( \mathbb{R}^{d+1}. \) More precisely, for \( (\xi_1, \ldots, \xi_d, \tau) \in \mathbb{R}^{d+1}, \)

\[
P_{i,j}(\xi_1, \ldots, \xi_d, \tau) = (\xi_{\omega_{i,j}(1)}, \ldots, \xi_{\omega_{i,j}(d)}, \tau),
\]

where \( \omega_{i,j} \) is the permutation of \( \{1, \ldots, d\} \) defined by

\[
\omega_{i,j}(k) = \begin{cases} j & \text{if } k = i, \\ i & \text{if } k = j, \\ k & \text{otherwise}. \end{cases}
\]

For every orthogonal matrix \( A \in O(d, \mathbb{R}), \) the transformation \( (\xi, \tau) \mapsto R_A(\xi, \tau) = (A\xi, \tau) \) belongs to \( \mathcal{L}^+. \)

Composing the transformations \( P_{i,j} \) and \( L^t \) for suitable \( i, j \)'s and \( t \)'s, we easily see that if \( (\xi, \tau) \in \mathbb{R}^{d+1} \) satisfies \( \tau > |\xi|, \) then there exists \( L \in \mathcal{L}^+ \) such that

\[
L(\xi, \tau) = (0, \sqrt{\tau^2 - \xi^2}).
\]

Alternatively, this can be seen using the transformations \( R_A \) and \( L^t. \) We first find \( A \in O(d, \mathbb{R}) \) such that \( A\xi = (|\xi|, 0, \ldots, 0). \) We take \( t = -|\xi|^{\tau^{-1}} \) and note that \( L^t(R_A(\xi, \tau)) = L^t(|\xi|, 0, \ldots, 0, \tau) = (0, \sqrt{\tau^2 - |\xi|^2}). \)

For \( p \in [1, \infty], \) \( L \in \mathcal{L}^+, \) and \( f \in L^p(\mathbb{H}^d) \) we define \( L^* f = f \circ L; \) here “\( \circ \)” denotes composition. The invariance of the measure \( \sigma_s \) under the action of \( \mathcal{L}^+ \) implies that \( \|f\|_{L^p(\mathbb{H}^d)} = \|L^* f\|_{L^p(\mathbb{H}^d)} \) for all \( p \in [1, \infty], \) equality holding for \( p = \infty \) since Lorentz transformations are invertible. It is also straightforward to check that \( \|T_s(L^* f)\|_{L^p(\mathbb{H}^d)} = \|T_s f\|_{L^p(\mathbb{H}^d)} \) for \( p \in [1, \infty]. \) Therefore, if \( \{f_n\}_{n \in \mathbb{N}} \) is an extremizing sequence for (1.11) and \( \{L_n\}_{n \in \mathbb{N}} \subset \mathcal{L}^+ \), then \( \{L_n^* f_n\}_{n \in \mathbb{N}} \) is also an extremizing sequence for (1.11).

We use the Lorentz transformations \( P_{i,j}, R_A \) and \( L^t. \) The invariance of \( \sigma_s \) with respect to these transformations can be seen directly from an examination of the Jacobians in the change of variables formula.

4. ON FOSCHI’S ARGUMENT

For ease of notation, let \( \psi_s(x) = \sqrt{s^2 + x^2} \) for \( s, x \in \mathbb{R} \) and set \( \psi := \psi_1. \) We also write \( \psi_s(y) \) to mean \( \psi_s(|y|) \) for \( y \in \mathbb{R}^d. \) We define the convolution of measures \( \mu, \nu \)
The region $\mathcal{P}$ for all $x$ a point $(\xi, \tau)$ where $\Phi(\xi, \tau)$ is well-defined on $\mathcal{P}$ is supported in the closure of the region $\mathcal{P}_{d,n} = \{ (\xi, \tau) : \tau > \sqrt{(ns)^2 + |\xi|^2} \}$. For each fixed $(\xi, \tau) \in \mathcal{P}_{d,n}$, we define the measure on $(\mathbb{R}^d)^n$

$$\mu(\xi, \tau) = \delta \left( \tau - \psi_s(x_1) - \cdots - \psi_s(x_n) \right) \frac{d\mu(x_1) \cdots d\mu(x_n)}{\xi - x_1 - \cdots - x_n}.$$

Recall that the Dirac delta measure $\delta_0$ on $\mathbb{R}^d \times \mathbb{R}$, is defined by

$$\langle \delta_0, f \rangle = f(0), \quad \text{for all } f \in S(\mathbb{R}^d \times \mathbb{R}).$$

The measure $\mu(\xi, \tau)$ is the pullback of $\delta_0$ on $\mathbb{R}^d \times \mathbb{R}$ by $\Phi(\xi, \tau) : (\mathbb{R}^d)^n \rightarrow \mathbb{R}^d \times \mathbb{R}$ given by

$$\Phi(\xi, \tau)(x_1, \ldots, x_n) = (\xi - x_1 - \cdots - x_n, \tau - \psi_s(x_1) - \cdots - \psi_s(x_n)).$$

As discussed in [5], the pullback is well-defined as long as the differential of $\Phi(\xi, \tau)$ is surjective at the points where $\Phi(\xi, \tau)$ vanishes. The differential of $\Phi(\xi, \tau)$ is surjective at a point $(x_1, \ldots, x_n)$ if and only if $x_1, \ldots, x_n$ are not all equal. Now $\Phi(\xi, \tau)(x, \ldots, x) = 0$ if and only if $\tau^2 = (ns)^2 + |\xi|^2$, that is, at the boundary of $\mathcal{P}_{d,n}$. Thus, the pullback is well-defined on $\mathcal{P}_{d,n}$.

For each $(\xi, \tau) \in \mathcal{P}_{d,n}$, we define the inner product $\langle \cdot, \cdot \rangle_{(\xi, \tau)}$ and norm $\| \cdot \|_{(\xi, \tau)}$ associated to $\mu(\xi, \tau)$ as

$$\langle F, G \rangle_{(\xi, \tau)} = \int_{(\mathbb{R}^d)^n} F(x_1, \ldots, x_n) \overline{G(x_1, \ldots, x_n)} d\mu(\xi, \tau)(x_1, \ldots, x_n),$$

$$\| F \|^2_{(\xi, \tau)} = \int_{(\mathbb{R}^d)^n} |F(x_1, \ldots, x_n)|^2 d\mu(\xi, \tau)(x_1, \ldots, x_n).$$

What connects this inner product with inequality (1.2) is the following identity. For $f_1, \ldots, f_n \in L^2(\mathbb{H}_s^d)$,

$$f_1 \sigma_s \cdots f_n \sigma_s = \int_{(\mathbb{R}^d)^n} \frac{f_1(x_1) \cdots f_n(x_n)}{\psi_s(x_1) \cdots \psi_s(x_n)} \delta(\xi - x_1 - \cdots - x_n)$$

$$\cdot \delta(\tau - \psi_s(x_1) - \cdots - \psi_s(x_n)) d\mu(x_1) \cdots d\mu(x_n)$$

$$= \int_{(\mathbb{R}^d)^n} \frac{f_1(x_1) \cdots f_n(x_n)}{\psi_s(x_1) \cdots \psi_s(x_n)} d\mu(\xi, \tau)(x_1, \ldots, x_n)$$

$$= \langle F, G \rangle_{(\xi, \tau)},$$

where

$$F(x_1, \ldots, x_n) = \frac{f_1(x_1) \cdots f_n(x_n)}{\psi_s(x_1)^{1/2} \cdots \psi_s(x_n)^{1/2}}$$
and 
\[ G(x_1, \ldots, x_n) = \frac{1}{\psi_s(x_1)^{1/2} \cdots \psi_s(x_n)^{1/2}}. \]

**Lemma 4.1.** If \( f \in S(\mathbb{R}^d) \), then 
\[ \| (f \sigma_s)^{(sn)} \|_{L^2(\mathbb{R}^d)} \leq \| \sigma_s^{(sn)} \|_{L^\infty(\mathbb{R}^d)} \| f \|_{L^2(\mathbb{H}^d)}. \] (4.1)

Moreover, for \( f \neq 0 \), equality holds in (4.1) only if \( \sigma_s^{(sn)}(\xi, \tau) = \| \sigma_s^{(sn)} \|_{L^\infty(\mathbb{R}^d)} \) for a.e. \((\xi, \tau)\) in the support of \((f \sigma_s)^{(sn)}\).

**Proof.** Let \( g \in S(\mathbb{R}^{d+1}) \). By definition of the convolution,
\[
\langle g, (f \sigma_s)^{(sn)} \rangle = \int_{(\mathbb{R}^d)^n} g(x_1 + \cdots + x_n, \psi_s(x_1) + \cdots + \psi_s(x_n)) \cdot f(x_1) \cdots f(x_n) \psi_s(x_1)^{1/2} \cdots \psi_s(x_n)^{1/2} dx_1 \cdots dx_n \\
= \int_{(\mathbb{R}^d)^n} \frac{g(x_1 + \cdots + x_n, \psi_s(x_1) + \cdots + \psi_s(x_n))}{\psi_s(x_1)^{1/2} \cdots \psi_s(x_n)^{1/2}} f(x_1) \cdots f(x_n) dx_1 \cdots dx_n \\
\leq \left( \int_{(\mathbb{R}^d)^n} \frac{g^2(x_1 + \cdots + x_n, \psi_s(x_1) + \cdots + \psi_s(x_n))}{\psi_s(x_1)^{1/2} \cdots \psi_s(x_n)^{1/2}} f(x_1) \cdots f(x_n) dx_1 \cdots dx_n \right)^{1/2} \\
\leq \left( \int_{(\mathbb{R}^d)^n} \frac{g^2}{\psi_s(x_1)^{1/2} \cdots \psi_s(x_n)^{1/2}} f(x_1) \cdots f(x_n) dx_1 \cdots dx_n \right)^{1/2} = \langle g^2, \sigma_s^{(sn)} \rangle^{1/2} \| f \|_{L^2(\mathbb{H}^d)}^{n} \]
\[ \leq \| g \|_{L^2(\mathbb{R}^d)} \| \sigma_s^{(sn)} \|_{L^\infty(\mathbb{R}^d)}^{1/2} \| f \|_{L^2(\mathbb{H}^d)}^{n}. \] (4.2)

Taking the supremum over \( g \in L^2(\mathbb{R}^{d+1}) \) proves (4.1).

Now if \( \| (f \sigma_s)^{(sn)} \|_{L^2(\mathbb{R}^d)} = \| \sigma_s^{(sn)} \|_{L^\infty(\mathbb{R}^d)} \| f \|_{L^2(\mathbb{H}^d)} \), then taking \( g = (f \sigma_s)^{(sn)} \), we must have equality in (4.3), i.e.,
\[
\langle ((f \sigma_s)^{(sn)})^2, \sigma_s^{(sn)} \rangle = \| (f \sigma_s)^{(sn)} \|_{L^2(\mathbb{H}^d)}^2 \| \sigma_s^{(sn)} \|_{L^\infty(\mathbb{R}^d)}^2,
\]
which occurs if and only if \( \sigma_s^{(sn)}(\xi, \tau) = \| \sigma_s^{(sn)} \|_{L^\infty(\mathbb{R}^d)} \) for a.e. \((\xi, \tau)\) in the support of \((f \sigma_s)^{(sn)}\).

From Lemma 4.1 and (1.6), we obtain the following result.

**Corollary 4.2.** Let \((d, p)\) satisfy (1.3) and suppose \( p = 2k \) is an even integer. Then 
\[ \| T_s f \|_{L^p(\mathbb{R}^{d+1})} \leq (2\pi)^{(d+1)/p} \| \sigma_s^{(sk)} \|_{L^\infty(\mathbb{R}^{d+1})}^{1/p} \| f \|_{L^2(\mathbb{H}^d)}, \] (4.4)
and thus 
\[ H_{d,p,s} \leq (2\pi)^{(d+1)/p} \| \sigma_s^{(sk)} \|_{L^\infty(\mathbb{R}^{d+1})}^{1/p}. \] (4.5)
Lemma 4.4. If $(d,p)$ that interest us in this paper, \((4.5)\) gives

\[
\begin{align*}
    &H_{2,4,s} \leq (2\pi)^{3/4}||\sigma_s * \sigma_s||_{L^\infty(\mathbb{R}^3)}^{1/4}, \\
    &H_{2,6,s} \leq (2\pi)^{1/2}||\sigma_s * \sigma_s * \sigma_s||_{L^\infty(\mathbb{R}^3)}^{1/6}, \text{ and} \\
    &H_{3,4,s} \leq 2\pi||\sigma_s * \sigma_s||_{L^\infty(\mathbb{R}^4)}^{1/4}.
\end{align*}
\]

To prove the nonexistence of extremizers, we use the following result.

Corollary 4.3. Let $(d,p)$ satisfy \((1.3)\) and suppose $p = 2k$ is an even integer. Suppose that $H_{d,p} = (2\pi)^{(d+1)/p}||\sigma_s||_{L^\infty(\mathbb{R}^{d+1})}^{1/p}$ and that $\sigma^{(s,k)}(\tau,\xi) < ||\sigma^{(s,k)}||_{L^\infty(\mathbb{R}^{d+1})}$ for a.e. $(\xi,\tau)$ in the support of $\sigma^{(s,k)}$. Then extremizers for inequality \((1.2)\) do not exist for the pair $(d,p)$.

Proof. This is direct consequence of the last assertion in Lemma \[4.1\] \qed

Lemma 4.4. If $f \in \mathcal{S}(\mathbb{R}^d)$, then

\[
\|f \sigma_s^{(m)}\|_{L^2(\mathbb{R}^d)}^2 \leq \int_{(\mathbb{R}^d)^n} \frac{f(x_1) \cdots f(x_n)}{\psi_s(x_1) \cdots \psi_s(x_n)} \cdot \sigma_s^{(m)}(x_1 + \cdots + x_n, \psi_s(x_1) + \cdots + \psi_s(x_n)) \, dx_1 \cdots dx_n. \tag{4.6}
\]

Proof. We prove the Lemma only for the case $n = 2$; the proof for the general case is similar and only requires more notation. Following Foschi’s argument, we write

\[
f \sigma_s * f \sigma_s(\xi,\tau) = \int_{(\mathbb{R}^d)^2} \frac{f(x) f(y)}{\psi_s(x) \psi_s(y)} \delta(\xi - x - y) \delta(\tau - \psi_s(x) - \psi_s(y)) \, dx \, dy \tag{4.7}
\]

\[
= \int_{(\mathbb{R}^d)^2} \frac{f(x) f(y)}{\psi_s(x) \psi_s(y)} d\mu(\tau,\xi)(x,y).
\]

From the Cauchy-Schwarz inequality, for $(\xi,\tau) \in \mathcal{P}_{d,2}$,

\[
|f \sigma_s * f \sigma_s(\tau,\xi)| \leq \left\| \frac{f(x) f(y)}{\psi_s(x)^{1/2} \psi_s(y)^{1/2}} \right\|_{(\tau,\xi)} \left\| \frac{1}{\psi_s(x)^{1/2} \psi_s(y)^{1/2}} \right\|_{(\tau,\xi)} . \tag{4.8}
\]

Now

\[
\left\| \frac{1}{\psi_s(x)^{1/2} \psi_s(y)^{1/2}} \right\|_{(\tau,\xi)}^2 = \sigma_s * \sigma_s(\xi,\tau) \tag{4.9}
\]
as can be seen from (1.7) by taking \( f \equiv 1 \). Then

\[
\|f \sigma_s * f \sigma_s\|_2^2 \leq \int_{\mathbb{R}^d} \left\| \frac{f(x) f(y)}{\psi_s(x)^{1/2} \psi_s(y)^{1/2}} \right\|_{(\tau, \xi)}^2 \sigma_s \sigma_s(\xi, \tau) d\tau \, d\xi
\]

\[
= \int_{\mathbb{R}^d} \left( \frac{f^2(x) f^2(y)}{\psi_s(x)^2} \delta(\tau - \psi_s(x) - \psi_s(y)) \sigma_s \sigma_s(\xi, \tau) \right) dx \, dy \, d\tau \, d\xi
\]

\[
= \int_{\mathbb{R}^d} \frac{f^2(x) f^2(y)}{\psi_s(x)^2} \delta(\tau - \psi_s(x) - \psi_s(y)) \sigma_s \sigma_s(\xi, \tau) \, d\tau \, d\xi \, dx \, dy
\]

\[
= \int_{\mathbb{R}^d} \frac{f^2(x) f^2(y)}{\psi_s(x)^2} \sigma_s \sigma_s(x + y, \psi_s(x) + \psi_s(y)) \, dx \, dy.
\]

\[\square\]

5. Nonexistence of Extremizers

In this section, we prove the first part of Theorem 1.2 related to the best constants and nonexistence of extremizers for the adjoint Fourier restriction inequality for \( \mathbb{H}^d \). We start with the computation of the double and triple convolution of \( \sigma_s \) with itself.

**Lemma 5.1.** Let \( d = 2, s > 0 \), and let \( \sigma_s \) be the measure on \( \mathbb{H}^d \) given in (1.10). Then for \((\xi, \tau) \in \mathbb{R}^2 \times \mathbb{R}\),

\[
\sigma_s \ast \sigma_s(\xi, \tau) = \frac{2\pi}{\sqrt{\tau^2 - |\xi|^2}} \chi_{\{\tau \geq \sqrt{(2s)^2 + |\xi|^2}\}},
\]

\[
\sigma_s \ast \sigma_s \ast \sigma_s(\xi, \tau) = (2\pi)^2 \left( 1 - \frac{3s}{\sqrt{\tau^2 - |\xi|^2}} \right) \chi_{\{\tau \geq \sqrt{(3s)^2 + |\xi|^2}\}}.
\]

In particular, \( \|\sigma_s \ast \sigma_s\|_{L^\infty(\mathbb{R}^3)} = \pi/s \), and \( \sigma_s \ast \sigma_s(\xi, \tau) < \|\sigma_s \ast \sigma_s\|_{L^\infty(\mathbb{R}^3)} \) for all \((\xi, \tau) \) in the interior of the support of \( \sigma_s \ast \sigma_s \). Also, \( \|\sigma_s \ast \sigma_s \ast \sigma_s\|_{L^\infty(\mathbb{R}^3)} = (2\pi)^2 \), and for all \((\xi, \tau) \in \mathbb{R}^{d+1}, \sigma_s \ast \sigma_s \ast \sigma_s(\xi, \tau) < \|\sigma_s \ast \sigma_s \ast \sigma_s\|_{L^\infty(\mathbb{R}^3)} \).

**Proof.** It is easy to compute the convolution

\[
\sigma_s \ast \sigma_s(0, \tau) = \int_{\mathbb{R}^2} \delta(\tau - 2\sqrt{s^2 + |y|^2}) \frac{dy}{s^2 + |y|^2} = 2\pi \int_0^\infty \delta(\tau - 2\sqrt{s^2 + r^2}) \frac{r \, dr}{s^2 + r^2}.
\]

Let \( u = 2\sqrt{s^2 + r^2} \). Then

\[
\sigma_s \ast \sigma_s(0, \tau) = 2\pi \int_2s^2 \delta(\tau - u) \frac{du}{u} = \frac{2\pi}{\tau} \chi_{\{\tau \geq 2s\}}.
\]

By Lorentz invariance, we obtain

\[
\sigma_s \ast \sigma_s(\xi, \tau) = \frac{2\pi}{\sqrt{\tau^2 - |\xi|^2}} \chi_{\{\tau \geq \sqrt{(2s)^2 + |\xi|^2}\}}.
\]
Lemma 5.2. Let $u = \sqrt{s^2 + r^2}$. Then
\[
\sigma_s * \sigma_s * \sigma_s(0, \tau) = (2\pi)^2 \int_{\mathbb{R}^2} \sigma \ast \sigma(-y, \tau - \sqrt{s^2 + |y|^2}) \frac{dy}{\sqrt{s^2 + |y|^2}} = (2\pi)^2 \int_0^\infty \frac{\chi_{\{r - \sqrt{s^2 + r^2} \geq \sqrt{(2s)^2 + r^2}\}} rdr}{\sqrt{(\tau - \sqrt{s^2 + r^2})^2 - r^2 \sqrt{s^2 + r^2}}}
\]

By Lorentz invariance,
\[
\sigma_s * \sigma_s * \sigma_s(x, \tau) = (2\pi)^2 \left(1 - \frac{3s}{\sqrt{\tau^2 - |\xi|^2}}\right) \chi_{\{r \geq \sqrt{(2s)^2 + |\xi|^2}\}}.
\]

Lemma 5.2. Let $d = 3$ and $s > 0$. Then for $(\xi, \tau) \in \mathbb{R}^3 \times \mathbb{R}$,
\[
\sigma_s * \sigma_s(\xi, \tau) = 2\pi \left(1 - \frac{4s^2}{\tau^2 - |\xi|^2}\right)^{1/2} \chi_{\{r \geq \sqrt{(2s)^2 + |\xi|^2}\}}.
\]

In particular, $\|\sigma_s * \sigma_s\|_{L^\infty(\mathbb{R}^d)} = 2\pi$, and for all $(\xi, \tau) \in \mathbb{R}^d$, $\sigma_s * \sigma_s(\xi, \tau) < \|\sigma_s * \sigma_s\|_{L^\infty(\mathbb{R}^d)}$.

Proof.
\[
\sigma_s * \sigma_s(0, \tau) = \int_{\mathbb{R}^3} \frac{\delta(\tau - 2\sqrt{s^2 + |y|^2}) dy}{s^2 + |y|^2} = 4\pi \int_0^\infty \frac{\delta(\tau - 2\sqrt{s^2 + r^2}) r^2 dr}{s^2 + r^2}.
\]

Let $u = 2\sqrt{s^2 + r^2}$. Then
\[
\sigma_s * \sigma_s(0, \tau) = 2\pi \int_{2s}^{\infty} \delta(\tau - u) \frac{\sqrt{u^2 - 4s^2}}{u} du = 2\pi \frac{\sqrt{\tau^2 - 4s^2}}{\tau} \chi_{\{r \geq 2s\}}.
\]

Therefore, by the Lorentz invariance,
\[
\sigma_s * \sigma_s(\xi, \tau) = 2\pi \left(1 - \frac{4s^2}{\tau^2 - |\xi|^2}\right)^{1/2} \chi_{\{r \geq \sqrt{(2s)^2 + |\xi|^2}\}}.
\]

From Corollary 4.2 and Lemmas 5.1 and 5.2, we obtain the following result.
Corollary 5.3.

\[ H_{2,4} \leq 2^{3/4} \pi, \quad H_{2,6} \leq (2\pi)^{5/6}, \quad \text{and} \quad H_{3,4} \leq (2\pi)^{5/4}. \]

To obtain the lower bound for the best constants, we exhibit explicit extremizing sequences.

Lemma 5.4. Let \( d = 2 \) and \( s > 0 \). For \( a > 0 \), let \( f_a(y) = e^{-a\sqrt{x^2+y^2}}, y \in \mathbb{R}^2 \). Then

\[
\lim_{a \to \infty} \|T_s f_a\|_{L^4(\mathbb{R}^3)} \|f_a\|_{L^2(\mathbb{H}_s^2)}^{-1} = \frac{2^{3/4} \pi}{s^{1/4}},
\]

(5.4)

\[
\lim_{a \to 0^+} \|T_s f_a\|_{L^6(\mathbb{R}^3)} \|f_a\|_{L^2(\mathbb{H}_s^2)}^{-1} = (2\pi)^{5/6}.
\]

(5.5)

The proof of this is given in Appendix 2. For the case \( d = 3 \), we have an analogous result.

Lemma 5.5. Let \( d = 3 \) and \( s > 0 \). For \( a > 0 \), let \( f_a(y) = e^{-a\sqrt{x^2+y^2}}, y \in \mathbb{R}^3 \). Then

\[
\lim_{a \to 0^+} \|T_s f_a\|_{L^4(\mathbb{R}^4)} \|f_a\|_{L^2(\mathbb{H}_s^3)}^{-1} = (2\pi)^{5/4}.
\]

The proof of Lemma 5.5 is in Appendix 3.

Note that Corollary 5.3 and Lemmas 5.4 and 5.5 imply that for \((d, p) = (2, 4)\), the family \( \{f_a/\|f_a\|_{L^2(\sigma_s)}\}_{a > 0} \) is an extremizing family as \( a \to \infty \), while for \((d, p) = (2, 3)\), \( \{f_a/\|f_a\|_{L^2(\sigma_s)}\}_{a > 0} \) is an extremizing family as \( a \to 0^+ \), and for \((d, p) = (3, 6)\), \( \{f_a/\|f_a\|_{L^2(\sigma_s)}\}_{a > 0} \) is an extremizing family as \( a \to 0^+ \). An extremizing family \( \{f_a\}_{a > 0} \) is defined as in Definition 1.1 where we replace the limit in \( n \) by a limit in \( a \).

We now give the proof of the part of Theorem 1.2 related to the best constants and extremizers for the adjoint Fourier restriction inequality on \( \mathbb{H}_s^d \); the proof of the second part, related to the two-sheeted hyperboloid \( \mathbb{H}_s^d \), is deferred to Section 7.

Proof of the first part of Theorem 1.2. Combining Corollary 5.3 and Lemmas 5.4 and 5.5, we obtain the first part of Theorem 1.2 namely

\[ H_{2,4} = 2^{3/4} \pi, \quad H_{2,6} = (2\pi)^{5/6}, \quad \text{and} \quad H_{3,4} = (2\pi)^{5/4}. \]

That extremizers do not exist is a consequence of Corollary 4.3 and the last assertions about the infinity norm of the double and triple convolutions of \( \sigma \) with itself, contained in Lemmas 5.1 and 5.2.

We now prove the assertion given in the Introduction about extremizers for the truncated operator \( T_\rho \) for \( d = 2 \) and \( p = 4 \).

Proposition 5.6. Let \((d, p) = (2, 4)\) and \( s > 0 \). For any \( \rho > 0 \), the best constant in inequality (1.14) equals \( 2^{3/4} \pi / s^{1/4} \), and there are no extremizers for inequality (1.14).

Proof. The nonexistence of extremizers for inequality (1.14) follows from the nonexistence of extremizers for inequality (1.2) once we prove that the best constant for the truncated hyperboloid equals the best constant for the entire hyperboloid, \( H_{2,4,s} \). For this, we need a lower bound.
Since the extremizing family \( \{ f_n \}_{n \geq 0} \) given in Lemma 5.4 concentrates at \( y = 0 \) as \( a \to \infty \), one easily sees that
\[
T_\rho \left( \frac{f_n \chi_{\{|y| \leq \rho\}}}{{\|f_n \chi_{\{|y| \leq \rho\}}\|}_{L^2(\sigma_s)}} \right) \to 2^{3/4} \pi s^{1/4}, \quad a \to \infty
\]
for the family \( \{ f_n \chi_{\{|y| \leq \rho\}} \}_{n \geq 0} \). This gives the desired lower bound.

\[\square\]

6. ON EXTREMIZING SEQUENCES

In this section, we obtain some general properties concerning concentration of extremizing sequences for inequality (1.2) for the cases \((d, p) = (2, 4), (2, 6)\) and \((3, 4)\).

The Lorentz invariance of \( \sigma_s \) implies that given an extremizing sequence \( \{ f_n \}_{n \in \mathbb{N}} \) for inequality (1.1) and a sequence of Lorentz transformations \( \{ L_n \}_{n \in \mathbb{N}} \) preserving \( \mathbb{H}^d \), \( \{ f_n \circ L_n \}_{n \in \mathbb{N}} \) is also an extremizing sequence. In this section, it is only in the case \((d, p) = (2, 4)\) that the Lorentz group is used explicitly, but an equivalent result can be written without it, as discussed before the statement of Proposition 6.3.

Consider first the case \( d = 2 \) and \( p = 6 \). From Lemma 5.3 it follows that the family of functions \( \{ f_n \}_{n \geq 0} \) is an extremizing family as \( a \to 0^+ \). This particular extremizing family concentrates at spatial infinity, that is, for every \( \varepsilon, R > 0 \), there exists \( a_0 > 0 \) such that for all \( 0 < a < a_0 \), \( \| f_n \|_{L^2(B(0, R))} < \varepsilon \), where \( B(0, R) = \{ y \in \mathbb{R}^2 : |y| < R \} \). Next we show that this is the case for every extremizing sequence.

**Proposition 6.1.** Let \( \{ f_n \}_{n \in \mathbb{N}} \) be an extremizing sequence for inequality (1.1) in the case \((d, p) = (2, 6)\). Then for any \( \varepsilon, R > 0 \), there exists \( N \in \mathbb{N} \) such that for \( n \geq N \),
\[
\| f_n \|_{L^2(B(0, R))} < \varepsilon,
\]
that is, the sequence concentrates at spatial infinity.

**Proof.** Let \( \varepsilon, R > 0 \) be given. From the proof of Lemma 4.4 and from Lemma 5.1 we have for the inequality in convolution form
\[
\| f_n \sigma_s \ast f_n \sigma_s \ast f_n \sigma_s \|_{L^2(\mathbb{R}^3)}^2 \\
\leq \int_{\mathbb{R}^3} \left\| \frac{f_n(x) f_n(y) f_n(z)}{\psi_s(x)^{1/2} \psi_s(y)^{1/2} \psi_s(z)^{1/2}} \right\|_{(\tau, \xi)}^2 \sigma_s \ast \sigma_s \ast \sigma_s(\tau, \xi) d\tau d\xi
\leq (2\pi)^2 \int_{\mathbb{R}^3} \left\| \frac{f_n(x) f_n(y) f_n(z)}{\psi_s(x)^{1/2} \psi_s(y)^{1/2} \psi_s(z)^{1/2}} \right\|_{(\tau, \xi)}^2 \left( 1 - \frac{3s}{\sqrt{\tau^2 - |\xi|^2}} \right) d\tau d\xi
= (2\pi)^2 \| f_n \|_{L^2(\sigma_s)}^6 - (2\pi)^2 \int_{\mathbb{R}^3} \left\| \frac{f_n(x) f_n(y) f_n(z)}{\psi_s(x)^{1/2} \psi_s(y)^{1/2} \psi_s(z)^{1/2}} \right\|_{(\tau, \xi)}^2 \frac{3s d\tau d\xi}{\sqrt{\tau^2 - |\xi|^2}}.
\]

Since \( \| f_n \sigma_s \ast f_n \sigma_s \ast f_n \sigma_s \|_{L^2(\mathbb{R}^3)}^2 \to (2\pi)^2 \) as \( n \to \infty \), we obtain
\[
\int_{\mathbb{R}^3} \left\| \frac{f_n(x) f_n(y) f_n(z)}{\psi_s(x)^{1/2} \psi_s(y)^{1/2} \psi_s(z)^{1/2}} \right\|_{(\tau, \xi)}^2 \frac{d\tau d\xi}{\sqrt{\tau^2 - |\xi|^2}} \to 0 \quad \text{as} \quad n \to \infty; \quad (6.1)
\]
and thus there exists \( N \in \mathbb{N} \) such that for all \( n \geq N \),
\[
\int_{\mathbb{R}^3} \left\| \frac{f_n(x)f_n(y)f_n(z)}{\psi_s(x)^{1/2}\psi_s(y)^{1/2}\psi_s(z)^{1/2}} \right\|^2 \frac{d\tau d\xi}{\sqrt{\tau^2 - |\xi|^2}} < \frac{\varepsilon}{3\sqrt{s^2 + R^2}}.
\]

By Lemma 4.3, the expression in the left hand side can be written as
\[
\int_{\mathbb{R}^3} \left\| \frac{f_n(x)f_n(y)f_n(z)}{\psi_s(x)^{1/2}\psi_s(y)^{1/2}\psi_s(z)^{1/2}} \right\|^2 \frac{d\tau d\xi}{\sqrt{\tau^2 - |\xi|^2}} = \int_{(B(0,R))^3} \frac{f_n^2(x)f_n^2(y)f_n^2(z)}{\psi_s(x)^{1/2}\psi_s(y)^{1/2}\psi_s(z)^{1/2}} \delta \left( \tau - \psi_s(x) - \psi_s(y) - \psi_s(z) \right) \frac{d\tau d\xi}{\sqrt{\tau^2 - |\xi|^2}} dx dy dz
\]
\[
= \int_{(B(0,R))^3} \frac{f_n^2(x)f_n^2(y)f_n^2(z)}{\psi_s(x)^{1/2}\psi_s(y)^{1/2}\psi_s(z)^{1/2}} \psi_s(x) + \psi_s(y) + \psi_s(z) \right) \frac{d\tau d\xi}{\sqrt{\tau^2 - |\xi|^2}} dx dy dz
\]
\[
\geq \frac{3}{3\sqrt{s^2 + R^2}} \int_{B(0,R)} \left\| f_n \right\|_{L^2(B(0,R))}^6
\]
If \( x, y, z \in B(0, R) \), then \( 3s < \psi_s(x) + \psi_s(y) + \psi_s(z) \leq 3\psi_s(R) = 3\sqrt{s^2 + R^2} \). Therefore, for all \( n \geq N \),
\[
\frac{\varepsilon}{3\sqrt{s^2 + R^2}} > \int_{\mathbb{R}^3} \left\| \frac{f_n(x)f_n(y)f_n(z)}{\psi_s(x)^{1/2}\psi_s(y)^{1/2}\psi_s(z)^{1/2}} \right\|^2 \frac{d\tau d\xi}{\sqrt{\tau^2 - |\xi|^2}} \geq \frac{1}{3\sqrt{s^2 + R^2}} \left\| f_n \right\|_{L^2(B(0,R))}^6
\]
and so, \( \sup_{n \geq N} \left\| f_n \right\|_{L^2(B(0,R))} < \varepsilon \) as desired. \( \square \)

We now turn to the case \( d = 3 \) and \( p = 4 \). Here we can also prove the analog of Proposition 6.1, namely, that extremizing sequences must concentrate at spatial infinity.

**Proposition 6.2.** Let \( \{f_n\}_{n \in \mathbb{N}} \) be an extremizing sequence for inequality (1.11) in the case \((d, p) = (3, 4)\). Then for any \( \varepsilon, R > 0 \), there exists \( N \in \mathbb{N} \) such that for all \( n \geq N \),
\[
\left\| f_n \right\|_{L^2(B(0,R))} < \varepsilon,
\]
that is, the sequence concentrates at spatial infinity.

**Proof.** The proof follows the same lines as that of Proposition 6.1. Using the convolution form of the inequality, we obtain the analog of equation (6.1),
\[
\int_{\mathbb{R}^3} \left\| \frac{f_n(x)f_n(y)f_n(z)}{\psi_s(x)^{1/2}\psi_s(y)^{1/2}\psi_s(z)^{1/2}} \right\|^2 \left( 1 - \left( 1 - \frac{4s^2}{\tau^2 - |\xi|^2} \right)^{1/2} \right) d\tau d\xi \to 0 \text{ as } n \to \infty.
\]
If we use the bound
\[
1 - \left( 1 - \frac{4s^2}{\tau^2 - |\xi|^2} \right)^{1/2} \geq 1 - \left( 1 - \frac{4s^2}{\tau^2} \right)^{1/2}
\]
and the fact that \( 0 < \psi_s(x) + \psi_s(y) \leq 2\psi_s(R) \) whenever \(|x|, |y| \leq R\), we obtain
\[
\int_{\mathcal{P}_{3,2}} \left| \frac{f_n(x)f_n(y)f_n(z)}{\psi_s(x)^{1/2}\psi_s(y)^{1/2}\psi_s(z)^{1/2}} \right|^2 \left( 1 - \left( 1 - \frac{4s^2}{\tau^2 - |\xi|^2} \right)^{1/2} \right) d\tau d\xi \\
\geq \left( 1 - \left( \frac{R}{\tau^2 + s} \right)^{1/2} \right) \|f_n\|_{L^2(B(0,R))}^2.
\]

The conclusion follows as in the proof of Proposition 6.1. \(\square\)

We now analyze the last case \((d,p) = (2,4)\). Since \(\sigma_s * \sigma_s(\xi,\tau) = \|\sigma_s * \sigma_s\|_{L^\infty(\mathbb{R}^3)}\) whenever \(\tau = \sqrt{(2s)^2 + |\xi|^2}\), that is, at the boundary of the support of \(\sigma_s * \sigma_s\), it is not hard to see that there are extremizing sequences that concentrate at any given point in \(\mathbb{H}^2_s\). For the example of an extremizing sequence given in Lemma 5.4, the concentration occurs at the vertex of the hyperboloid \((\xi,\tau) = (0,s) =: P\). We want to show that all extremizing sequences concentrate.

Since every point in \(\mathbb{H}^2_s\) has an extremizing sequence concentrating at it, we can construct an extremizing sequence that concentrates along any given sequence in \(\mathbb{H}^2_s\) in the sense that given a sequence \(\{y_n\}_{n \in \mathbb{N}} \subset \mathbb{H}^2_s\), there exists an extremizing sequence \(\{f_n\}_{n \in \mathbb{N}} \subset L^2(\mathbb{H}^2_s)\) with the property that for every \(\varepsilon > 0\) and \(r > 0\), there exists \(N \in \mathbb{N}\) such that
\[
\int_{|y-y_n|>r} |f_n(y)|^2 d\sigma_s(y) \leq \varepsilon
\]
for all \(n \geq N\). Equivalently, taking a Lorentz transformation \(L_n \in \mathcal{L}^+\) with \(L_n^{-1}(y_n) = (0,s) = P\) and using the Lorentz invariance of the measure \(\sigma_s\), we can write (6.2) as
\[
\int_{L_n^{-1}(\{y:|y|>r\})+P} |L_n^*f_n(y)|^2 d\sigma_s(y) \leq \varepsilon,
\]
where \(L_n^*f_n(y) = f_n(L_ny)\). We show next that this is the case for every extremizing sequence.

**Proposition 6.3.** Let \(\{f_n\}_{n \in \mathbb{N}}\) be an extremizing sequence for inequality (1.11) in the case \((d,p) = (2,4)\). There exists a sequence \(\{L_n\}_{n \in \mathbb{N}} \subset \mathcal{L}^+\) with the property that for all \(\varepsilon, r > 0\), there exists \(N \in \mathbb{N}\) such that
\[
\int_{|y-P|>r} |L_n^*f_n(y)|^2 d\sigma_s(y) \leq \varepsilon,
\]
for all \(n \geq N\), where \(P = (0,s)\) is the vertex of the hyperboloid \(\mathbb{H}^2_s\).

To prove this proposition, we introduce the function \(d_s: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}\) given by the formula
\[
d_s(x,y) = \frac{1}{2s}((\psi_s(x) + \psi_s(y))^2 - |x + y|^2)^{1/2} - 1.
\]
Elementary properties of \(d_s\) are described in the next lemma, whose proof is left to the reader.

**Lemma 6.4.** (i) For all \(x, y \in \mathbb{R}^2\), \(d_s(x,y) = d_s(y,x) \geq 0\), and \(d_s(x,y) = 0\) if and only if \(x = y\).
(ii) For all \( x \in \mathbb{R}^2 \), \( \lim_{|y| \to \infty} d_s(x, y) = \infty \).

(iii) For every \( R > 0 \), there exist \( 0 < C_1(R), C_2(R) < \infty \) such that

\[
C_1(R)|x - y|^2 \leq d_s(x, y) \leq C_2(R)|x - y|,
\]

for all \( x, y \) with \( |x|, |y| \leq R \).

Property (ii) implies that for given \( y \in \mathbb{R}^2 \), the \( d_s \)-ball of radius \( R > 0 \) and center \( y \), \( B_d(y, R) := \{ x \in \mathbb{R}^2 : d_s(x, y) \leq R \} \), is a bounded set. Property (iii) relates the \( d_s \)-ball to the Euclidean ball; namely, it implies that for \( y \) with \( |y| \leq R \) and \( r > 0 \)

\[
B(y, cr) \subset B_d(y, r) \subset B(y, c'R), \tag{6.4}
\]

for some constants \( c, c' \) depending only on \( R \) and \( r \).

**Proof of Proposition 6.3** The first task is to find a sequence \( \{ y_n \}_{n \in \mathbb{N}} \subset \mathbb{H}^2 \) such that an analog of (6.2) is satisfied. It is convenient, for notational purposes only, to identify functions from \( \mathbb{H}^2 \) to \( \mathbb{R} \) with functions from \( \mathbb{R}^2 \) to \( \mathbb{R} \), and points in \( \mathbb{H}^2 \) with points in \( \mathbb{R}^2 \). This is done via the projection of \( \mathbb{H}^2 \) onto \( \mathbb{R}^2 \times \{ 0 \} \).

From Lemmas 4.4 and 5.1, for the inequality in convolution form, we have

\[
\| f_n \sigma_s * f_n \sigma_s \|_{L^2(\mathbb{R}^3)} \leq \int_{P_{2,2}} \left\| \frac{f_n(x)f_n(y)}{\psi_s(x)^{1/2}\psi_s(y)^{1/2}} \right\|_{(\tau, \xi)}^2 \sigma_s * \sigma_s(\tau, \xi) d\tau d\xi
\]

\[
= \frac{\pi}{s} \int_{P_{2,2}} \left\| \frac{f_n(x)f_n(y)}{\psi_s(x)^{1/2}\psi_s(y)^{1/2}} \right\|_{(\tau, \xi)}^2 \frac{2s}{\sqrt{\tau^2 - |\xi|^2}} d\tau d\xi
\]

\[
\leq \frac{\pi}{s} \| f_n \|_{L^2}. \tag{6.5}
\]

Since \( \| f_n \sigma_s * f_n \sigma_s \|_{L^2(\mathbb{R}^3)} \to \pi/s \) as \( n \to \infty \), we obtain

\[
\int_{P_{2,2}} \left\| \frac{f_n(x)f_n(y)}{\psi_s(x)^{1/2}\psi_s(y)^{1/2}} \right\|_{(\tau, \xi)}^2 \frac{2s}{\sqrt{\tau^2 - |\xi|^2}} d\tau d\xi \to 1 \text{ as } n \to \infty. \tag{6.5}
\]

As in the proof of Lemma 4.4, the expression on the left hand side can be written as

\[
\int_{P_{2,2}} \left\| \frac{f_n(x)f_n(y)}{\psi_s(x)^{1/2}\psi_s(y)^{1/2}} \right\|_{(\tau, \xi)}^2 \frac{2s}{\sqrt{\tau^2 - |\xi|^2}} d\tau d\xi
\]

\[
= \int_{(\mathbb{R}^2)^2} \frac{f_n^2(x)}{\psi_s(x)} \frac{f_n^2(y)}{\psi_s(y)} \int_{P_{2,2}} \delta \left( \frac{\tau - \psi_s(x) - \psi_s(y)}{\xi - x - y} \right) \frac{2s}{\sqrt{\tau^2 - |\xi|^2}} d\tau d\xi \ dx \ dy
\]

\[
= \int_{(\mathbb{R}^2)^2} \frac{f_n^2(x)}{\psi_s(x)} \frac{f_n^2(y)}{\psi_s(y)} \left( \frac{\psi_s(x) + \psi_s(y)}{2} \right)^2 \frac{2s}{(x + y)^{1/2}} d\tau d\xi \ dx \ dy
\]

\[
= \int_{(\mathbb{R}^2)^2} \frac{f_n^2(x)}{\psi_s(x)} \frac{f_n^2(y)}{\psi_s(y)} \frac{2s}{\psi_s(x)} K_s(x, y) d\tau d\xi \ dx \ dy.
\]

Observe that

\[
\int_{(\mathbb{R}^2)^2} \frac{f_n^2(x)}{\psi_s(x)} \frac{f_n^2(y)}{\psi_s(y)} d\tau d\xi = \| f_n \|_{L^2(\mathbb{H}^2)}^2 \to 1 \text{ as } n \to \infty.
\]
Using Fubini's Theorem, we can write

\[ K_s(x, y) := \frac{2s}{((\psi_s(x) + \psi_s(y))^2 - |x + y|^2)^{1/2}} = \frac{1}{d_s(x, y) + 1} \leq 1 \]

for all \( x, y \in \mathbb{R}^2 \). Equation (6.5) implies that there exists \( N \) such that \( \int_{\mathbb{R}^2} K_s(x, y)dx \ dy \to 1 \) as \( n \to \infty \).

Let \( h_n(y) = f_n(y)^2/\psi_s(y) \), so that \( \lim_{n \to \infty} \int_{\mathbb{R}^2} h_n(y)dy = 1 \). For \( \varepsilon > 0 \),

\[ \int_{\mathbb{R}^2} h_n(x)h_n(y)K_s(x, y)dx \ dy = \int_{d_s(x,y) \leq \varepsilon} h_n(x)h_n(y)K_s(x, y)dx \ dy + \int_{d_s(x,y) > \varepsilon} h_n(x)h_n(y)K_s(x, y)dx \ dy \leq \|h_n\|_{L^1(\mathbb{R}^2)} \leq \left( 1 - \frac{1}{\varepsilon + 1} \right) \int_{d_s(x,y) > \varepsilon} h_n(x)h_n(y)dy \ dx. \]

Since the left hand side tends to 1 as \( n \to \infty \), we conclude that

\[ \lim_{n \to \infty} \int_{d_s(x,y) \leq \varepsilon} h_n(x)h_n(y)dx \ dy = 1. \]

Using Fubini's Theorem, we can write

\[ \int_{d_s(x,y) \leq \varepsilon} h_n(x)h_n(y)dx \ dy = \int_{\mathbb{R}^2} h_n(y) \int_{d_s(x,y) \leq \varepsilon} h_n(x)dx dy \leq \|h\|_{L^1(\mathbb{R}^2)} \sup_{y \in \mathbb{R}^2} \int_{d_s(x,y) \leq \varepsilon} h_n(x)dx. \]

Then

\[ \lim_{n \to \infty} \sup_{y \in \mathbb{R}^2} \int_{B_{d_s(y,\varepsilon)}} h_n(x)dx = 1. \tag{6.6} \]

Equation (6.6) implies that there exists \( N(\varepsilon) \in \mathbb{N} \) such that for all \( n \geq N(\varepsilon) \),

\[ \sup_{y \in \mathbb{R}^2} \int_{B_{d_s(y,\varepsilon)}} h_n(y)dy \geq 1 - \frac{\varepsilon}{2}, \]

and hence there exists \( \{y_n^\varepsilon\}_{n \geq N(\varepsilon)} \subset \mathbb{R}^2 \) such that

\[ \int_{B_{d_s(y_n^\varepsilon,\varepsilon)}} h_n(y)dy \geq 1 - \varepsilon. \]

Applying (6.6) in this way, we obtain, for each \( \varepsilon > 0 \), a number \( N(\varepsilon) \) and a sequence \( \{y_n^\varepsilon\}_{n \geq N(\varepsilon)} \).

The construction of the sequence \( \{y_n\}_{n \in \mathbb{N}} \) is obtained by a diagonal process. We take a strictly decreasing sequence \( \{\varepsilon_k\}_{k \in \mathbb{N}} \) such that \( \varepsilon_k \to 0 \) as \( k \to \infty \). This gives sequences \( \{N(k)\}_{k \in \mathbb{N}} \) and \( \{y_n^k\}_{n \geq N(k), k \geq 0} \). We can take the sequence \( \{N(k)\}_{k \in \mathbb{N}} \)
strictly increasing. For each \(n \geq N(1)\), we let \(l(n) = \sup\{k \in \mathbb{N} : N(k) \leq n\}\). Next, define \(\{y_n\}_{n \in \mathbb{N}}\) by

\[
y_n = \begin{cases} 
y_n^{(l(n))} & \text{if } n \geq N(1), 
y_0 & \text{if } n < N(1),
\end{cases}
\]

where \(y_0 \in \mathbb{R}^2\) is arbitrary, but fixed.

Now let \(\varepsilon, r > 0\) be given. Take \(k\) such that \(\varepsilon_k < \min\{\varepsilon, r\}\). For \(n \geq N(k)\), \(l(n) \geq k\), so \(\varepsilon_{l(n)} \leq \varepsilon_k < \min\{\varepsilon, r\}\) and \(\int_{B_{d_1}(y_n^{(l(n))}, \varepsilon_{l(n)})} h_n(y) dy \geq 1 - \varepsilon_{l(n)}\); hence

\[
\int_{B_{d_1}(y_n, r)} h_n(y) dy \geq \int_{B_{d_1}(y_n^{(l(n))}, \varepsilon_{l(n)})} h_n(y) dy \geq 1 - \varepsilon_{l(n)} \geq 1 - \varepsilon.
\]

Therefore, for every \(\varepsilon, r > 0\), there exists \(N \in \mathbb{N}\) such that

\[
\int_{B_{d_1}(y_n, r)} |f_n(y)|^2 \frac{dy}{\sqrt{s^2 + |y|^2}} \geq 1 - \varepsilon.
\]  

(6.7)

for all \(n \geq N\).

To finish the proof we use the Lorentz invariance. This is better done without identifying \(\mathbb{H}^2\) with \(\mathbb{R}^2\). So now we lift everything to \(\mathbb{H}^2\). Let \(D_s : \{(\xi, \tau) \in \mathbb{R}^2 \times \mathbb{R} : \tau > |\xi|^2\} \rightarrow \mathbb{R}\) be defined by

\[
D_s((\xi_1, \tau_1), (\xi_2, \tau_2)) = \frac{1}{2s}((\tau_1 + \tau_2)^2 - |\xi_1 + \xi_2|^2)^{1/2} - 1.
\]

Observe that for every \(L \in \mathcal{L}^+, D_s(L(\xi_1, \tau_1), L(\xi_2, \tau_2)) = D_s((\xi_1, \tau_1), (\xi_2, \tau_2))\).

Let \(z_n = (y_n, \psi(y_n)) \in \mathbb{H}^2\). We can write (6.7) as

\[
\int_{D_s(z, z_n) \leq r} |f_n(z)|^2 d\sigma_s(z) \geq 1 - \varepsilon.
\]

By the Lorentz invariance of \(D_s\) and \(\sigma_s\), we have that for \(L_n \in \mathcal{L}^+\) for which \(L_n^{-1}(z_n) = (0, s) = P\) and for every \(\varepsilon, r > 0\), there exists \(N \in \mathbb{N}\) such that

\[
\int_{D_s(z, P) \leq r} |L_n^* f_n(z)|^2 d\sigma_s(z) \geq 1 - \varepsilon \text{ for all } n \geq N.
\]  

(6.8)

Property (iii) in Lemma 6.4 and (6.8) imply that for every \(\varepsilon, r > 0\), there exists \(N \in \mathbb{N}\) such that

\[
\int_{|z - P| \leq r} |L_n^* f(z)|^2 d\sigma_s(z) \geq 1 - \varepsilon
\]

for all \(n \geq N\). \(\square\)

7. The Two-sheeted Hyperboloid

In this section, we consider the two-sheeted hyperboloid

\[
\mathbb{H}^d = \{(y, y') \in \mathbb{R} \times \mathbb{R}^d : y^2 = s^2 + |y'|^2\}
\]

with measure

\[
\tilde{\sigma}_s(y, y') = \delta(y' - \sqrt{s^2 + |y|^2}) \frac{dy dy'}{\sqrt{s^2 + |y|^2}} + \delta(y' + \sqrt{s^2 + |y|^2}) \frac{dy dy'}{\sqrt{s^2 + |y|^2}}
\]
and the adjoint Fourier restriction operator defined by \( \hat{T}_s f = \hat{f} \sigma_s \), for \( f \in S(\mathbb{R}^{d+1}) \).

\( \mathbb{H}_s^d \) is the union of the two sheets

\[
\mathbb{H}_s^{d,\pm} = \{(y, y') \in \mathbb{R}^d \times \mathbb{R} : y = \pm \sqrt{s^2 + |y|^2}\}.
\]

What in this section we are calling \( \mathbb{H}_s^{d,\pm} \) is what before we denoted by \( \mathbb{H}_s^d \). In the previous section, we proved that for \( \mathbb{H}_s^{d,+} \) (and thus also for \( \mathbb{H}_s^{d,-} \)), extremizers do not exist for the cases \( (d, p) = (2, 4), (2, 6) \) and \( (3, 4) \). Here, we show that extremizers for \( \mathbb{H}_s^d \) do not exist either and compute the best constants.

The adjoint Fourier restriction operator on \( \mathbb{H}_s^{d,+} \) is denoted by \( T_s \), and the adjoint Fourier restriction operator on \( \mathbb{H}_s^{d,-} \) by \( T_s^{-} \). For \( s = 1 \), we drop the subscript \( s \). For \( A, B \subseteq \mathbb{R}^d \), we set \( A + B = \{a + b : a \in A, b \in B\} \) and \( -A = \{-a : a \in A\} \).

**Lemma 7.1.** For \( d \geq 1 \),

\[
\begin{align*}
\mathbb{H}_s^{d,+} + \mathbb{H}_s^{d,+} & \subseteq \{\xi, \tau \in \mathbb{R}^d \times \tau \geq \sqrt{(2s)^2 + |\xi|^2}\}, && (7.1) \\
\mathbb{H}_s^{d,+} + \mathbb{H}_s^{d,-} & \subseteq \{\xi, \tau \in \mathbb{R}^d \times |\tau| \leq |\xi|\}, && (7.2) \\
\mathbb{H}_s^{d,-} + \mathbb{H}_s^{d,-} & \subseteq \{\xi, \tau \in \mathbb{R}^d \times \tau \leq -\sqrt{(2s)^2 + |\xi|^2}\}. && (7.3)
\end{align*}
\]

**Proof.** To establish (7.1), let \( \xi = x + y \) and \( \tau = \psi_s(x) + \psi_s(y) \). Thus

\[
\tau^2 = (\psi_s(x) + \psi_s(y))^2 = 2s^2 + |x|^2 + |y|^2 + 2(s^2 + |x|^2)^{1/2}(s^2 + |y|^2)^{1/2},
\]

while \( |\xi|^2 = |x + y|^2 = |x|^2 + |y|^2 + 2x \cdot y \). Then (7.1) is equivalent to the inequality

\[
(s^2 + |x|^2)^{1/2}(s^2 + |y|^2)^{1/2} \geq s^2 + x \cdot y
\]

for all \( x, y \in \mathbb{R}^d \). Using \( x \cdot y = |x||y|\cos \theta \), where \( \theta \) is the angle between \( x \) and \( y \), we see that (7.4) is equivalent to

\[
(s^2 + a^2)^{1/2}(s^2 + b^2)^{1/2} \geq s^2 + ab
\]

for all \( a, b, s \geq 0 \), which is easily shown to hold by squaring both sides.

We proceed in a similar way for (7.2). Let \( \xi = x + y \) and \( \tau = \psi_s(x) - \psi_s(y) \). Then \( \tau^2 = 2s^2 + |x|^2 + |y|^2 - 2(s^2 + |x|^2)^{1/2}(s^2 + |y|^2)^{1/2} \). As before, (7.2) is equivalent to the inequality

\[
-(s^2 + |x|^2)^{1/2}(s^2 + |y|^2)^{1/2} \leq -s^2 + x \cdot y
\]

for all \( x, y \in \mathbb{R}^d \), which in turn is equivalent to

\[
(s^2 + a^2)^{1/2}(s^2 + b^2)^{1/2} \geq s^2 + ab,
\]

which holds for all real numbers \( a, b, s \geq 0 \).

As for (7.3), it follows from (7.1) observing that \( \mathbb{H}_s^{d,-} = -\mathbb{H}_s^{d,+} \). \( \square \)

**Lemma 7.2.** Let \( d \geq 1 \),

\[
\begin{align*}
\mathbb{H}_s^{d,+} + \mathbb{H}_s^{d,+} + \mathbb{H}_s^{d,+} & \subseteq \{\xi, \tau \in \mathbb{R}^d \times \tau \geq \sqrt{(3s)^2 + |\xi|^2}\}, && (7.5) \\
\mathbb{H}_s^{d,-} + \mathbb{H}_s^{d,-} + \mathbb{H}_s^{d,-} & \subseteq \{\xi, \tau \in \mathbb{R}^d \times \tau \leq -\sqrt{(3s)^2 + |\xi|^2}\}, && (7.6) \\
\mathbb{H}_s^{d,+} + \mathbb{H}_s^{d,+} + \mathbb{H}_s^{d,-} & \subseteq \{\xi, \tau \in \mathbb{R}^d \times \tau \geq -\sqrt{s^2 + |\xi|^2}\}, && (7.7) \\
\mathbb{H}_s^{d,+} + \mathbb{H}_s^{d,-} + \mathbb{H}_s^{d,-} & \subseteq \{\xi, \tau \in \mathbb{R}^d \times \tau \leq \sqrt{s^2 + |\xi|^2}\}. && (7.8)
\end{align*}
\]
Proof. We know from Lemma 7.1 that
\[ \mathbb{H}_s^{d,+} + \mathbb{H}_s^{d,-} \subseteq \{ (\xi, \tau) : \tau \geq \sqrt{(2s^2 + |\xi|^2)} \}. \]
We start with (7.5). Setting \( \xi = x + y \) and \( \tau \geq \psi_{2s}(x) + \psi_s(y) > 0 \) and squaring the latter inequality for \( \tau \) gives
\[ \tau^2 \geq 5s^2 + |x|^2 + |y|^2 + 2(4s^2 + |x|^2)^{1/2}(s^2 + |y|^2)^{1/2}. \]
Then (7.5) follows from the inequality
\[ (4s^2 + |x|^2)^{1/2}(s^2 + |y|^2)^{1/2} \geq 2s^2 + x \cdot y \text{ for all } x, y \in \mathbb{R}^d, \]
which is equivalent to
\[ (4s^2 + a^2)^{1/2}(s^2 + b^2)^{1/2} \geq 2s^2 + ab, \]
which is easy to verify for all \( a, b, s \geq 0 \).

We now establish (7.7). Let \( \xi = x + y \) and \( \tau \geq \psi_{2s}(x) - \psi_s(y) \). If \( \tau \geq 0 \), we are done. So, we suppose that \( 0 \geq \tau \geq \psi_{2s}(x) - \psi_s(y) \). Then
\[ \tau^2 \leq 5s^2 + |x|^2 + |y|^2 - 2(4s^2 + |x|^2)^{1/2}(s^2 + |y|^2)^{1/2}, \]
and (7.7) follows from the inequality
\[ -(4s^2 + |x|^2)^{1/2}(s^2 + |y|^2)^{1/2} \leq -2s^2 + x \cdot y \text{ for all } x, y \in \mathbb{R}^d, \]
which is equivalent to
\[ (4s^2 + a^2)^{1/2}(s^2 + b^2)^{1/2} \geq 2s^2 + ab, \]
which holds for all \( a, b, s \geq 0 \).

Both (7.6) and (7.8) can be proved similarly or obtained from (7.5) and (7.7) using that \( \mathbb{H}_s^{d,-} = -\mathbb{H}_s^{d,+} \).

For a function \( f \in L^2(\mathbb{H}_s^d) \), we write \( f = f_+ + f_- \), where \( f_+ \) is supported on \( \mathbb{H}_s^{d,+} \) and \( f_- \) is supported on \( \mathbb{H}_s^{d,-} \). Then
\[ ||f||_{L^2(\mathbb{H}_s^d)}^2 = ||f_+||_{L^2(\mathbb{H}_s^{d,+})}^2 + ||f_-||_{L^2(\mathbb{H}_s^{d,-})}^2. \]

Proposition 7.3. Let \( d \in \{2, 3\} \) and \( f \in L^2(\mathbb{H}_s^d), f \neq 0 \). Then
\[ ||T_s f||_{L^4(\mathbb{R}^{d+1})}^4 ||f||_{L^2(\mathbb{H}_s^d)}^{-4} \leq \frac{3}{2} H_{d,4,s}^4. \] (7.9)
If equality holds in (7.9),
\[ ||T_s f_+||_{L^4(\mathbb{R}^{d+1})} = H_{d,4,s} ||f_+||_{L^2(\mathbb{H}_s^{d,+})} \quad \text{and} \quad ||T_s f_-||_{L^4(\mathbb{R}^{d+1})} = H_{d,4,s} ||f_-||_{L^2(\mathbb{H}_s^{d,-})}. \]

Moreover, if \( \{ f_n \}_{n \in \mathbb{N}} \) is an extremizing sequence for \( T_s \), then \( \{ f_n+/\|f_n+\|_2 \}_{n \in \mathbb{N}} \) and \( \{ f_n-/\|f_n-\|_2 \}_{n \in \mathbb{N}} \) are extremizing sequences for \( T_s \) in \( \mathbb{H}_s^{d,+} \) and \( T_s^- \) in \( \mathbb{H}_s^{d,-} \), respectively.
Proof. The proof of (7.9) is analogous to the argument in [5, pp. 754-755]. We restrict attention to the case \( s = 1 \), but the other cases follow in the same way or by the use of scaling.

Observe that
\[
\| T f \|_{L^4}^4 = \| T f_+ + T^- f_- \|_{L^4}^4 = \| (T f_+ + T^- f_-)^2 \|_{L^2}^2 \\
= \| (T f_+)^2 + (T^- f_-)^2 + 2(T f_+)(T^- f_-) \|_{L^2}^2.
\]
Using the fact that product transforms into convolution under the Fourier transform, we see that the Fourier transforms of \((T f_+)^2\), \((T^- f_-)^2\) and \((T f_+)(T^- f_-)\) are supported on \( \mathbb{H}^{d,+} + \mathbb{H}^{d,+} \), \( \mathbb{H}^{d,-} + \mathbb{H}^{d,-} \), and \( \mathbb{H}^{d,+} + \mathbb{H}^{d,-} \), respectively. By Lemma 7.1, the pairwise intersections of these three sets have measure zero. Therefore,
\[
\| T f \|_{L^4}^4 = \| T f_+ \|_{L^4}^4 + \| T^- f_- \|_{L^4}^4 + 4\| (T f_+)(T^- f_-) \|_{L^2}^2 \quad (7.10)
\]
\[
\leq H_{d,4}^2(\| f_+ \|_{L^2}^2 + \| f_- \|_{L^2}^2 + 4\| f_+ \|_{L^2}^2 \| f_- \|_{L^2}^2) \\
\leq \frac{3}{2} H_{d,4}^2(\| f_+ \|_{L^2}^2 + \| f_- \|_{L^2}^2)^2 \\
= \frac{3}{2} H_{d,4}^2 \| f \|_{L^2}^4, \quad (7.12)
\]
where we have used the sharp inequality (as in [5])
\[
X^2 + Y^2 + 4XY \leq \frac{3}{2} (X + Y)^2, \quad X, Y \geq 0,
\]
where equality holds if and only if \( X = Y \). Thus,
\[
\| T f \|_{L^4}^4 \leq \frac{3}{2} H_{d,4}^2. \quad (7.15)
\]
For \( f \neq 0 \), equality holds in (7.15) if and only if it holds in (7.11) and (7.12). Equality holds in (7.11) if and only if \( \| T f_+ \|_{L^4} = H_{d,4} \| f_+ \|_{L^2(\mathbb{H}^{d,+)}} \), \( \| T^- f_- \|_{L^4} = H_{d,4} \| f_- \|_{L^2(\mathbb{H}^{d,-})} \), and \( \| T f_+ \| = \lambda T^- f_- \) a.e. in \( \mathbb{R}^{d+1} \) for some \( \lambda \geq 0 \), and in (7.12) if and only if \( \| f_+ \|_{L^2} = \| f_- \|_{L^2} \). Note that equality in (7.12) implies that \( \lambda = 1 \).

Let \( \{ f_n \} \) be an extremizing sequence for \( \bar{\mathcal{T}} \), so that \( \lim_{n \to \infty} \| T f_n \|_{L^4(\mathbb{R}^d)} = \bar{\mathcal{H}}_{d,4} \) and \( \| f_n \|_{L^2} \leq 1 \). For the decomposition \( f_n = f_{n,+} + f_{n,-} \), we see that
\[
\lim_{n \to \infty} (\| f_{n,+} \|_{L^2}^4 + \| f_{n,-} \|_{L^2}^4 + 4\| f_{n,+} \|_{L^2}^2 \| f_{n,-} \|_{L^2}^2) = \frac{3}{2}.
\]
This implies that if \( \lim_{n \to \infty} \| f_{n,+} \|_{L^2} \) and \( \lim_{n \to \infty} \| f_{n,-} \|_{L^2} \) exist, then they must be equal, and thus equal to \( 1/\sqrt{2} \). Therefore, any subsequence has a convergent subsequence with limit \( 1/\sqrt{2} \). This implies the existence of both limits and
\[
\lim_{n \to \infty} \| f_{n,+} \|_{L^2} = \lim_{n \to \infty} \| f_{n,-} \|_{L^2} = \frac{1}{\sqrt{2}}.
\]
If we write
\[
\| T f_{n,+} \|_{L^4} = a_n H_{d,4} \| f_{n,+} \|_{L^2(\mathbb{H}^{d,+)}} \) and \( \| T^- f_{n,-} \|_{L^4} = b_n H_{d,4} \| f_{n,-} \|_{L^2(\mathbb{H}^{d,-})}, \)

then, as before, \( \lim_{n \to \infty} a_n \|f_{n+}\|_2 = 1/\sqrt{2} \), and so \( \lim_{n \to \infty} a_n = 1 \); similarly, \( \lim_{n \to \infty} b_n = 1 \). Hence, \( \{f_{n+}/\|f_{n+}\|_2\}_{n \in \mathbb{N}} \) and \( \{f_{n-}/\|f_{n-}\|_2\}_{n \in \mathbb{N}} \) are extremizing sequences for \( T \) and \( T^- \) in \( \mathbb{H}^{d,+} \) and \( \mathbb{H}^{d,-} \), respectively. \( \square \)

**Corollary 7.4.** For \( d \in \{2,3\} \), \( p = 4 \) and \( s > 0 \), \( \mathbb{H}_{d,4,s} = (3/2)^{1/4} \mathbb{H}_{d,4,s} \). Moreover, extremizers for the adjoint Fourier restriction inequality for \( \mathbb{H}_{d}^s \) do not exist.

**Proof.** The only part missing is the lower bound for the value of the best constant. For this, take \( \{f_{n+}\}_{n \in \mathbb{N}} \) to be an extremizing sequence for \( T_s \), then, identifying a function on \( \mathbb{H}^{d,\pm}_s \) with a function from \( \mathbb{R}^d \) to \( \mathbb{R} \), set \( f_{n-}(y) = \overline{f_{n+}}(-y) \), (the complex conjugate of \( f_{n+} \) evaluated at \(-y) \), \( y \in \mathbb{R}^d \). Then \( \{f_n\}_{n \in \mathbb{N}} = \{(f_{n+} + f_{n-})/\sqrt{2}\}_{n \in \mathbb{N}} \) is an extremizing sequence for \( T_s \) in \( \mathbb{H}^d \), since inequalities (7.11) and (7.12) become equalities in the limit \( n \to \infty \). \( \square \)

**Proposition 7.5.** Let \( d \in \{1,2\} \), \( s > 0 \) and \( f \in L^2(\mathbb{H}_s^d) \), \( f \neq 0 \). Then

\[
\|T_s f\|_{L^6(\mathbb{R}^{d+1})} \|f\|_{L^6(\mathbb{H}_s^d)}^6 < \frac{25}{4} \mathbb{H}_{d,6,s}^6.
\]

In particular,

\[
\mathbb{H}_{d,6,s} \leq \left( \frac{5}{2} \right)^{1/3} \mathbb{H}_{d,6,s}.
\]

When \( d = 2 \) we have the refinement

\[
\mathbb{H}_{2,6,s} \leq \left( \frac{5}{8}(4 + 3\sqrt{2}) \right)^{1/6} \mathbb{H}_{2,6,s}.
\]

**Proof.** The proof follows the same lines as [5, pp. 758-760], and Proposition 7.3 using Lemma 7.2. Since we want to highlight that (7.16) is a strict inequality and that a refinement is possible we provide the details. Let us take \( s = 1 \) as other values of \( s \) follow by scaling. We start by writing \( \tilde{T} f = Tf_+ + T^- f_- \), so that

\[
\|\tilde{T} f\|_{L^6} = \|Tf_+ + T^- f_-\|_{L^6}^6 = \|(Tf_+ + T^- f_-)^3\|_{L^2}^2 = \|(Tf_+)^3 + 3(Tf_+)^2(T^- f_-) + 3(Tf_+)(T^- f_-)^2 + (T^- f_-)^3\|_{L^2}^2.
\]

The Fourier transform of the functions \( (Tf_+)^3 \), \( (Tf_+)^2(T^- f_-) \), \( (Tf_+)(T^- f_-)^2 \) and \( (T^- f_-)^3 \) are supported on \( \mathbb{H}^{d,+} + \mathbb{H}^{d,+} + \mathbb{H}^{d,-} + \mathbb{H}^{d,+} + \mathbb{H}^{d,+} + \mathbb{H}^{d,-} + \mathbb{H}^{d,-} + \mathbb{H}^{d,-} \) and \( \mathbb{H}^{d,+} + \mathbb{H}^{d,-} + \mathbb{H}^{d,-} \), respectively. Therefore, using Lemma 7.2 we obtain

\[
\|\tilde{T} f\|_{L^6}^6 = \|(Tf_+)^6_{L^6} + ||T^- f_-\|_{L^6}^6 + 9|(Tf_+)^2(T^- f_-)\|_{L^2}^2 + 9|((Tf_+)(T^- f_-)^2)\|_{L^2}^2 + 6|(Tf_+)^3\|_{L^2}^3 + (Tf_+)^2(T^- f_-))\|_{L^2}^2 + 6|((Tf_+)(T^- f_-)^2, (T^- f_-)^3)\|_{L^2}^2 + 18|(Tf_+)(T^- f_-)\|_{L^2}^3.
\]

Using the Cauchy-Schwarz and Hölder’s inequalities together with the sharp inequality for \( T \) and \( T^- \) we obtain

\[
\|\tilde{T} f\|_{L^6}^6 \leq \mathbb{H}_{d,6}^6 \|f_+\|_{L^2}^6 + \|f_-\|_{L^2}^6 + 9\|f_+\|_{L^2}^3 \|f_-\|_{L^2}^3 + 9\|f_+\|_{L^2}^2 \|f_-\|_{L^2}^2 + 9\|f_+\|_{L^2}^2 \|f_-\|_{L^2}^2 + 18\|f_+\|_{L^2}^3 \|f_-\|_{L^2}^3.
\]
We now use the numerical inequality from [5, Lemma 6.6], namely, for \( X, Y \geq 0 \)
\[
X^6 + Y^6 + 9X^4Y^2 + 9X^2Y^4 + 6X^5Y + 6XY^5 + 18X^3Y^3 \leq \frac{25}{4}(X^2 + Y^2)^3,
\]
with equality if and only if \( X = Y \). In this way we obtain
\[
\|Tf\|^6_{L^6} \leq \frac{25}{4} H_{d,6}^6 \|f_+\|_{L^2}^6 + \|f_-\|_{L^2}^6 = \frac{25}{4} H_{d,6}^6 \|f\|_{L^2}^6. \tag{7.19}
\]
From the first part of Theorem 1.2 we have the inequalities \( \|Tf_+\|^6_{L^6} \leq H_{d,6}^6 \|f_+\|^6_{L^2} \)
and \( \|T^-f_-\|^6_{L^6} \leq H_{d,6}^6 \|f_-\|^6_{L^2} \), which are strict whenever \( f_+ \neq 0 \) and \( f_- \neq 0 \), so that if \( f \neq 0 \) then (7.19) is a strict inequality. More importantly, the inequalities
\[
\langle (Tf_+)^3, (Tf_+)^2(T^-f_-) \rangle \leq \langle (Tf_+)^3 \|L^2 \| (Tf_+)^2(T^-f_-) \|L^2 \rangle \tag{7.20}
\]
\[
\langle (Tf_+)(T^-f_-)^2, (T^-f_-)^3 \rangle \leq \langle (Tf_+)(T^-f_-)^2 \|L^2 \| (T^-f_-)^3 \|L^2 \rangle \tag{7.21}
\]
\[
\langle (Tf_+)^2(T^-f_-), (Tf_+)(T^-f_-)^2 \rangle \leq \langle (Tf_+)(T^-f_-)^2 \|L^2 \| (T^-f_-)^3 \|L^2 \rangle \tag{7.22}
\]
are strict, whenever \( f_+, f_- \neq 0 \). Indeed, equality in the Cauchy-Schwarz inequality
(7.20) forces \( (Tf_+)^3 = \lambda (Tf_+)^2(T^-f_-) \) for some \( \lambda \in \mathbb{C}, \lambda \neq 0 \), which by the use of the Fourier transform implies that \( f_+ \sigma^+ \ast f_+ \sigma^+ \ast f_+ \sigma^+ = \lambda f_+ \sigma^+ \ast f_+ \sigma^+ \ast f_- \sigma^- \), so that the support of \( f_+ \sigma^+ \ast f_+ \sigma^+ \ast f_- \sigma^- \) is contained in \( \mathbb{H}^4 + \mathbb{H}^4 + \mathbb{H}^4 \), which is impossible if \( f_+, f_- \neq 0 \). A similar argument shows that (7.21) and (7.22) are strict inequalities when \( f_+, f_- \neq 0 \).

It was observed by D. Foschi in a related argument that it is possible to sharpen an inequality such as (7.20), (7.21) and (7.22) which then can be used to obtain a better bound for the best constant \( H_{d,6} \). In what follows we adapt the argument to the hyperboloid in the case \( d = 2 \).

Let us write \( \tilde{f}_-(y) = \overline{f}_-(\overline{y}) \), where the overline denotes complex conjugation. Then
\[
\langle (Tf_+)^3, (Tf_+)^2(T^-f_-) \rangle = \langle (Tf_+)^3(T^-f_-), (Tf_+)^2 \rangle = \langle (Tf_+)^3(Tf_+), (Tf_+)^2 \rangle = (2\pi)^3 \langle f_+ \sigma \ast f_+ \sigma \ast f_+ \sigma \ast f_- \sigma, f_+ \sigma \ast f_+ \sigma \rangle \leq (2\pi)^3 \| \sigma(\pm) \|_{L^\infty}^{1/2} \| f_+ \|_{L^2} \| f_- \|_{L^2},
\]
where in the last line we used an argument as in Lemma 4.1. From Lemma 5.1 we know
\[
\sigma \ast \sigma(\xi, \tau) = \frac{2\pi}{\sqrt{\tau^2 - |\xi|^2}} \chi_{\{\tau \geq \sqrt{2^2 + |\xi|^2}\}},
\]
while the fourth convolution can be calculated in a similar way
\[
\sigma^{(4)}(\xi, \tau) = 4\pi^3 \frac{\sqrt{\tau^2 - |\xi|^2} - 4^2}{\sqrt{\tau^2} - |\xi|^2} \chi_{\{\tau \geq 4^2 + |\xi|^2\}}.
\]
Then
\[
\| \sigma(\pm) \|_{L^\infty(\mathbb{R}^3)} = \left\| 8\pi^4 \frac{\sqrt{\tau^2 - |\xi|^2} - 4^2}{\tau^2 - |\xi|^2} \chi_{\{\sqrt{\tau^2} - |\xi|^2 \geq 4\}} \right\|_{L^\infty(\mathbb{R}^3)} = 8\pi^4.
\]
We obtain the inequality
\[ \langle Tf \rangle^2, (Tf)^2 (T^{-} f) \rangle \leq 16\sqrt{2} \pi^6 \|f_+\|_{L^2}^5 \|f_-\|_{L^2}, \] (7.23)
and a similar method gives improved inequalities for (7.21) and (7.22) with the same constant on the right hand side. Note that $16\sqrt{2} \pi^6 < H_{2,6}^6 = (2\pi)^5$, so there is an improvement over using the Cauchy-Schwarz and Hölder’s inequality together with the sharp bound for $T$ and $T^-$. Using (7.17) and (7.23) we can obtain the analog of (7.18),
\[ \|T f\|_{L^6}^6 \leq H_{d,6}^6 (\|f_+\|_{L^2}^6 + \|f_-\|_{L^2}^6 + 9\|f_+\|_{L^2}^4 \|f_-\|_{L^2}^2 + 9\|f_+\|_{L^2}^2 \|f_-\|_{L^2}^4 + 3\sqrt{2} \|f_+\|_{L^2} \|f_-\|_{L^2} + 3\sqrt{2} \|f_+\|_{L^2} \|f_-\|_{L^2} + 9\sqrt{2} \|f_+\|_{L^2}^3 \|f_-\|_{L^2}^3). \] (7.24)

There is the sharp numerical bound
\[ X^6 + Y^6 + 9X^4 Y^2 + 9X^2 Y^4 + 3\sqrt{2} X^3 Y + 3\sqrt{2} X Y^5 + 9\sqrt{2} X^3 Y^3 \leq \frac{5}{8} (4 + 3\sqrt{2}) (X^2 + Y^2)^3, \]
for all $X, Y \geq 0$, with equality if and only if $X = Y$. It implies
\[ \|T f\|_{L^6}^6 \leq \frac{5}{8} (4 + 3\sqrt{2}) H_{d,6}^6 \|f\|_{L^2}^6, \]
which is the desired improvement over (7.19).

Proposition 7.3 gives the proof of the second part of Theorem 1.2 while Proposition 7.5 explains the comment in Remark 1.3.

8. Scaling

Here, we record the scaling for the family of operators $\{T_s\}_{s>0}$. Recall from the Introduction that for $s > 0$, $H_{s}^d := \{(y, \sqrt{s^2 + |y|^2}) : y \in \mathbb{R}^d\}$ is equipped with the measure $\sigma_s(y, y') = \delta(y' - \sqrt{s^2 + |y|^2}) dy' / \sqrt{s^2 + |y|^2}$. The operator $T_s$ is defined on $\mathcal{S}(\mathbb{R}^d)$ by
\[ T_s f(x, t) = \overline{f} \sigma_s(-x, -t) = \int_{\mathbb{R}^d} e^{ix \cdot y} e^{it \sqrt{s^2 + |y|^2}} f(y) \frac{dy}{\sqrt{s^2 + |y|^2}}. \]

We want to show that $H_{d,p,s}$ defined in (1.12) satisfies (1.13). With the change of variables $v = sy$ in (1.1), we have
\[ Tf(x, t) = \int_{\mathbb{R}^d} e^{ix \cdot y} e^{it \sqrt{1 + |y|^2}} f(y) \frac{dy}{\sqrt{1 + |y|^2}} = \int_{\mathbb{R}^d} e^{is^{-1} x \cdot y} e^{is^{-1} t \sqrt{1 + s^{-2} |y|^2}} f(s^{-1} y) \frac{dy}{\sqrt{1 + s^{-2} |y|^2}} = s^{-d+3/2} \int_{\mathbb{R}^d} e^{is^{-1} x \cdot y} e^{is^{-1} t \sqrt{s^{2} + |y|^2}} s^{-1/2} f(s^{-1} y) \frac{dy}{\sqrt{s^2 + |y|^2}}, \]
from which it follows that $s^{d-3/2} Tf(sx, st) = T_s(s^{-1/2} f(s^{-1}))(x, t)$ and
\[ s^{d-3/2-(d+1)/p} \|Tf\|_{L^p(\mathbb{R}^{d+1})} = \|T_s s^{-1/2} f(s^{-1})\|_{L^p(\mathbb{R}^{d+1})}. \]
Proof.
We first compute the $g$-form. Let $x > 0$, where the principal value is taken for $s$. The formulas in (9.2) and (9.3) are easier to compute in their equivalent convolution form:

$$s^{-d+2} \int_{\mathbb{R}^d} |s^{-1/2}f(s^{-1}y)|^2 \frac{dy}{\sqrt{s^2 + |y|^2}},$$

that is, $\|f\|_{L^2(\sigma)} = s^{-(d-2)/2} \|s^{-1/2}f(s^{-1})\|_{L^2(\sigma)}$. Thus

$$s^{(d-1)/2-(d+1)/p} \|Tf\|_{L^p(\mathbb{R}^{d+1})} \|f\|_{L^2(\sigma)}^{-1} = \|T_s s^{-1/2}f(s^{-1} \cdot)\|_{L^p(\mathbb{R}^{d+1})} \|s^{-1/2}f(s^{-1} \cdot)\|_{L^2(\sigma)},$$

and it follows that for all $s > 0$,

$$H_{d,p,s} = s^{(d-1)/2-(d+1)/p}H_{d,p}.$$

(8.1)

9. SOME EXPLICIT CALCULATIONS FOR THE CASE $d = 2$

The exponential integral function $\text{Ei}(x)$, $x \neq 0$, is defined by

$$\text{Ei}(x) = - \int_{-x}^{\infty} \frac{e^{-t}}{t} dt = \int_{-\infty}^{x} \frac{e^{t}}{t} dt \quad (9.1)$$

where the principal value is taken for $x > 0$.

**Lemma 9.1.** Let $a > 0$ and $f_a(y) = e^{-a\sqrt{x^2+|y|^2}}$, $y \in \mathbb{R}^2$. Then

$$\|T_s f_a\|_{L^6(\mathbb{R}^3)} \|f_a\|_{L^6(\sigma)}^{-6} = (2\pi)^3(1 - 6as - 36a^2s^2e^{6as} \text{Ei}(-6as)), \quad \text{and}$$

$$\|T_s f_a\|_{L^4(\mathbb{R}^3)} \|f_a\|_{L^4(\sigma)}^{-4} = \frac{2\pi^4}{s}(4ase^{4as} \text{Ei}(-4as)). \quad (9.3)$$

**Proof.** We first compute the $L^2(\sigma)$-norm of $f_a$:

$$\|f_a\|_{L^2(\sigma)}^2 = \int_{\mathbb{R}^2} e^{-2a\sqrt{x^2+|y|^2}} \frac{dy}{\sqrt{x^2+|y|^2}} = 2\pi \int_0^\infty e^{-2a\sqrt{r^2+y^2}} \frac{r}{\sqrt{r^2+y^2}} dr$$

$$= 2\pi \int_0^{\sqrt{2a}} e^{-2ar} dr = \frac{\pi}{a} e^{-2as}. \quad (9.2)$$

The formulas in (9.2) and (9.3) are easier to compute in their equivalent convolution form. Let $g_a(x, \tau) = e^{-a\tau}$ and observe that $f_{a\sigma} \ast f_{a\sigma} = g_{a\sigma} \ast g_{a\sigma}$ and $f_{a\sigma} \ast f_{a\sigma} \ast f_{a\sigma} \ast f_{a\sigma} = g_{a\sigma} \ast g_{a\sigma} \ast g_{a\sigma} \ast g_{a\sigma}$. Then, because $g_a$ is the exponential of a linear function, $g_{a\sigma} \ast g_{a\sigma}(x, \tau) = g_a(x, \tau) \sigma_s \ast \sigma_s(x, \tau)$ and $g_{a\sigma} \ast g_{a\sigma} \ast g_{a\sigma}(x, \tau) = g_a(x, \tau) \sigma_s \ast \sigma_s \ast \sigma_s(x, \tau)$.
$\sigma_s(\xi, \tau)$. Therefore,

$$\| f_a \sigma_s * f_a \sigma_s * f_a \sigma_s \|^2_{L^2(\mathbb{R}^3)}$$

$$= \int_{\mathbb{R}^2} e^{-2at} (2\pi)^{\frac{1}{2}} \left(1 - \frac{3s}{\sqrt{\tau^2 - |\xi|^2}}\right)^2 \chi_{\tau \geq \sqrt{(3s)^2 + |\xi|^2}} d\tau d\xi$$

$$= (2\pi)^5 \int_{3s}^\infty \int_0^{\sqrt{\tau^2 - (3s)^2}} e^{-2at} \left(1 - \frac{3s}{\sqrt{\tau^2 - r^2}}\right)^2 r dr d\tau$$

$$= (2\pi)^5 \int_{3s}^\infty \int_0^{\sqrt{\tau^2 - (3s)^2}} e^{-2at} \left(r + (3s)^2 \frac{r}{\tau^2 - r^2} - 6s \frac{r}{\sqrt{\tau^2 - r^2}}\right) dr d\tau$$

$$= (2\pi)^5 \left(\frac{1}{2} \int_{3s}^\infty e^{-2at} \tau^2 d\tau + (3s)^2 \int_{3s}^\infty e^{-2at} \log \tau d\tauight.$$\n
$$- 6se^{-6as} \int_0^{\infty} e^{-2at} \tau dr d\tau - \frac{9}{2} s^2 + (3s)^2 \log(3s)\right)(e^{-6as} \int_{3s}^\infty e^{-2at} d\tau)$$

$$= (2\pi)^5 \left(\frac{1}{8a^3} - (3s)^2 \frac{Ei(-6as)e^{6as}}{2a} - 6s \frac{e^{-6as}}{8a^2}\right).$$

Rearranging terms, we have

$$\| f_a \sigma_s * f_a \sigma_s * f_a \sigma_s \|^2_{L^2(\mathbb{R}^3)} = (2\pi)^5 e^{-6as} \left(\frac{1}{8a^3} - (3s)^2 \frac{Ei(-6as)e^{6as}}{2a} - 6s \frac{e^{-6as}}{8a^2}\right).$$

Thus

$$\| f_a \sigma_s * f_a \sigma_s * f_a \sigma_s \|^2_{L^2(\mathbb{R}^3)}/|f_a|_{L^2(\sigma_s)}^6 = (2\pi)^5 \pi^{-3} a^3 \left(\frac{1}{8a^3} - (3s)^2 \frac{Ei(-6as)e^{6as}}{2a} - 6s \frac{e^{-6as}}{8a^2}\right)$$

$$= (2\pi)^2 (1 - 6as - 36a^2 s^2 e^{6as} Ei(-6as)).$$

For the case of $L^4(\mathbb{R}^3)$,

$$\| f_a \sigma_s * f_a \sigma_s \|^2_{L^2(\mathbb{R}^3)} = \int_{\mathbb{R}^2} e^{-2at} \frac{(2\pi)^2}{\tau^2 - |\xi|^2} \chi_{\tau \geq \sqrt{(2s)^2 + |\xi|^2}} d\tau d\xi$$

$$= (2\pi)^3 \int_{2s}^\infty \int_0^{\sqrt{\tau^2 - (2s)^2}} e^{-2at} \frac{r}{\tau^2 - r^2} dr d\tau$$

$$= (2\pi)^3 \left(e^{-4as} \log(2s) - Ei(-4as) - \log(2s) e^{-4as}\right)$$

$$= -(2\pi)^3 \frac{Ei(-4as)}{2a}.$$
Thus
\[ \| f_a \sigma_s * f_a \sigma_s \|_{L^2(\mathbb{R}^3)}^2 \| f_a \|_{L^2(\sigma_s)}^{-4} = -(2\pi)^3 a \frac{\text{Ei}(-4as)}{2\pi^2} e^{4as} \]
\[ = \frac{\pi}{8} (-4ase^{4as} \text{Ei}(-4as)). \]
\[ \square \]

**Proof of Lemma 5.4.** Using the expressions in Lemma 9.1, we obtain
\[ \lim_{a \to 0^+} \| T_s f_a \|_{L^6(\mathbb{R}^3)}^6 \| f_a \|_{L^6(\sigma_s)}^{-6} = \lim_{a \to 0^+} (2\pi)^5 (1 - 6as - 36a^2s^2 e^{6as} \text{Ei}(-6as)) = (2\pi)^5 \]
and
\[ \lim_{a \to \infty} \| T_s f_a \|_{L^4(\mathbb{R}^3)}^4 \| f_a \|_{L^4(\sigma_s)}^{-4} = \lim_{a \to \infty} 2^3 \frac{\pi^4}{s} (-4ase^{4as} \text{Ei}(-4as)) = \frac{2^3\pi^4}{s}. \]
\[ \square \]

**Remark 9.2.**
1. It is not hard to see that the function \( a \mapsto 1 - a + a^2 e^a \text{Ei}(-a) \) is strictly decreasing for \( a \in [0, \infty) \) and tends to 0 as \( a \to \infty \) and to 1 as \( a \to 0^+ \). Thus \( \| T_s f_a \|_{L^6(\mathbb{R}^3)}^6 \| f_a \|_{L^6(\sigma_s)}^{-6} \) is a strictly decreasing function of \( a \), for each fixed \( s \).
2. The function \( a \mapsto -ae^a \text{Ei}(-a) \) is strictly increasing for \( a \in [0, \infty) \) and tends to 0 as \( a \to 0^+ \) and to 1 as \( a \to \infty \). Thus \( \| T_s f_a \|_{L^4(\mathbb{R}^3)}^4 \| f_a \|_{L^4(\sigma_s)}^{-4} \) is a strictly increasing function of \( a \), for each fixed \( s \).

10. Some explicit calculations for the case \( d = 3 \)

**Proof of Lemma 5.5.** For the \( L^2(\sigma_s) \)-norm of \( f_a \), we have
\[ \| f_a \|_{L^2(\sigma_s)}^2 = \int_{\mathbb{R}^3} e^{-2a\sqrt{s^2 + |y|^2}} \frac{dy}{\sqrt{s^2 + |y|^2}} = 4\pi \int_0^\infty e^{-2a\sqrt{r^2 + s^2}} \frac{r^2 dr}{\sqrt{s^2 + r^2}} \]
\[ = 4\pi \int_s^\infty e^{-2aur} \sqrt{u^2 - s^2} du = \frac{4\pi}{a^2} \int_{as}^\infty e^{-2\pi \sqrt{x^2 - (as)^2}} dx. \]

Then
\[ \lim_{a \to 0^+} a^2 \frac{\pi}{\pi} \| f_a \|_{L^2(\sigma_s)}^2 = 1. \]

Using the convolution form of the inequality, our goal is to show
\[ \lim_{a \to 0^+} a^4 \| f_a \sigma_s * f_a \sigma_s \|_{L^2(\mathbb{R}^3)}^2 = 2\pi^3. \]
As in the proof of Lemma 9.1,
\[
\|f_a \sigma_s \ast f_a \sigma_s\|^2_{L^2(\mathbb{R}^4)} = \int_{\mathbb{R} \times \mathbb{R}^3} e^{-2a\tau}(2\pi)^2 \left(1 - \frac{4s^2}{\tau^2 - |\xi|^2}\right) \chi_{\{\tau \geq \sqrt{(2s)^2 + |\xi|^2}\}} d\tau d\xi
\]
\[
= (2\pi)^2 4\pi \int_2^\infty \int_0^{\sqrt{\tau^2 - (2s)^2}} e^{-2a\tau} \left(1 - \frac{4s^2}{\tau^2 - \tau^2}\right) r^2 dr d\tau
\]
\[
= 16\pi^3 \int_2^\infty e^{-2a\tau} \left(\frac{1}{3}(\tau^2 - (2s)^2)^{3/2} + 4s^2((\tau^2 - (2s)^2)^{1/2} - \tau \log \left(\frac{\tau + \sqrt{\tau^2 - (2s)^2}}{2s}\right)\right) d\tau
\]
\[
= \frac{16\pi^3}{a} \int_{2as}^\infty e^{-2\tau} \left(\frac{1}{3a^3}(\tau^2 - (2as)^2)^{3/2} + \frac{4s^2}{a}((\tau^2 - (2as)^2)^{1/2} - \tau \log \left(\frac{\tau + \sqrt{\tau^2 - (2as)^2}}{2as}\right)\right) d\tau.
\]
Multiplying by $a^4$ and taking the limit as $a \to 0^+$ gives
\[
\lim_{a \to 0^+} a^4 \|f_a \sigma_s \ast f_a \sigma_s\|^2_{L^2(\mathbb{R}^4)} = \frac{16\pi^3}{3} \int_0^{\infty} e^{-2\tau^3} d\tau = 2\pi^3. \quad \Box
\]

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