Schwarzschild Black Hole Quantum Statistics, Droplet Nucleation and DLCQ Matrix Theory

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Abstract

Generalizing previous quantum gravity results for Schwarzschild black holes from 4 to $D \geq 4$ spacetime dimensions yields an energy spectrum

$$E_n = n^{1-1/(D-2)} \sigma E_{P,D}, \quad n = 1, 2, \ldots, \sigma = O(1).$$

Assuming the degeneracies $d_n$ of these levels to be given by $d_n = g^n, g > 1$, leads to a partition function which is the same as that of the primitive droplet nucleation model for 1st-order phase transitions in $D$-2 spatial dimensions. Exploiting the well-known properties of the so-called critical droplets of this model immediately leads to the Hawking temperature and the Bekenstein-Hawking entropy of Schwarzschild black holes. Thus, the ”holographic principle” of ’t Hooft and Susskind is naturally realised. The values of temperature and entropy appear closely related to the imaginary part of the partition function which describes metastable states.

Finally some striking conceptual similarities (”correspondence point” etc.) between the droplet nucleation picture and the very recent approach to the quantum statistics of Schwarzschild black holes in the framework of the DLCQ Matrix theory are pointed out.

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1 Introduction

In two recent papers [1, 2] I discussed the quantum statistics of the energy spectrum

\[ E_n = \sigma \sqrt{n} E_P, \quad n = 1, 2, \ldots, \quad E_P = c^2 \sqrt{c \hbar / G}, \quad \sigma = O(1), \quad (1) \]

with degeneracies

\[ d_n = g^n, \quad g > 1. \quad (2) \]

Many authors have proposed that Schwarzschild black holes in 4-dimensional spacetimes have such a spectrum (see the list in Ref. [1]).

The canonical partition function of the above spectrum not only leads to the Hawking temperature and the associated Bekenstein-Hawking entropy, but, very amazingly, it is formally the same as the classical grand canonical potential of the primitive droplet nucleation model in the context of first-order phase transitions in two space dimensions. Thus the so-called ”holographic principle” [3, 4], namely that the essential physics of black holes should be associated with the 2-dimensional horizon, is very evident here and comes out as a result!

Furthermore, as the canonical partition function of the spectrum (1) becomes complex for the degeneracies (2), because \( g > 1 \), one leaves the well-established framework of equilibrium thermodynamics (KMS states) and moves on the perhaps more slippery ground of metastable states and nonequilibrium thermodynamics [5].

The paper is organized as follows: I first generalize the spectrum (1) to \( D \geq 4 \), spacetime dimensions [6]. Then it will be shown that the resulting quantum canonical partition function is the same as the classical grand canonical potential of the (nonrelativistic) primitive droplet nucleation model in \( D - 2 \) spatial dimensions, the essential features of which for our purpose are discussed in chapter 3.

Using known results from the droplet nucleation model - especially the notion of a critical droplet - in chapter 4 the Hawking temperature and the Bekenstein-Hawking entropy are derived, up to a normalization factor which is discussed separately. In chapter 5 it is shown how an effective Hamiltonian of the Born-Infeld type can be used to describe the spectrum plus the degeneracy factor. Its classical mechanical counterpart has some amusing properties, too. Finally, very preliminary and very sketchy, in chapter
6 some surprising similarities to the very recent approach of understanding the quantum statistics of Schwarzschild black holes in the "Discretized Light-Cone Quantization" (DLCQ) version of "M(atrix)-theory" are pointed out which are perhaps not be accidental!

2 The quantum Schwarzschild BH spectrum in $D$-dimensional spacetime

The spectrum (1) may be derived as follows: A canonical Dirac-type treatment of spherically symmetric pure Einstein gravity leads to a reduced 2-dimensional phase space having only the ADM mass $M$ and a canonically conjugate time functional $T$ as (observable) pair of variables [7]. An observer at spatially (flat) infinity will only have the mass $M$ and his own proper time $\tau$ available in order to describe the system. His very simple Schrödinger equation for it is

$$i\hbar \partial_{\tau} \phi(\tau) = Mc^2 \phi(\tau) ,$$

which has the plane wave solutions

$$\phi(M, \tau) = \chi(M)e^{-\frac{i}{\hbar}Mc^2\tau} ,$$

where $M \geq 0$ is assumed. If the system with mass $M$ stays there forever, then $M$ is to be considered as a continuous quantity. However – just like the momenta of plane waves in a spatial box with finite extension –, if the above system, represented by the plane wave (4), has only a finite duration $\Delta$ then this property may be (crudely) implemented by imposing periodic boundary conditions on the plane wave (4), implying the relation [8]

$$c^2M\Delta = 2\pi\hbar n , \ n = 1, 2, \ldots .$$

The question now is how to choose $\Delta$! As the only intrinsic quantity available at spatial infinity to characterize a time interval is $M$ itself, or a function of it, namely the Schwarzschild radius $R_S(M)$, we assume [8]

$$\Delta = \gamma R_S(M)/c ,$$
where $\gamma$ is a dimensionless number of order 1. Inserting (6) into (5) gives the mass quantization condition

$$\gamma c M_n R_S(M_n) = 2\pi \hbar n \ , \ n = 1, 2, \ldots . \quad (7)$$

For $D = 4$ we have $R_S = 2MG/c^2$ and the spectrum (1) results. However, I assume (7) to be valid in any dimension $D \geq 4$, because the Schrödinger Eq. (3) has to hold in any such dimension!

Before I discuss the consequences let me point out that (7) is also the appropriate generalization of a Bohr-Sommerfeld type quantization of the 2-dimensional horizon as suggested very early by Bekenstein [9], Mukhanov [10] (see also Bekenstein’s recent review [11]) and - in the context of string theory - by Kogan [12].

If we interpret $cM$ as canonical momentum and $R_S$ as canonical coordinate, then (7) is, qualitatively, nothing else but the old-fashioned quantum counting of phase space cells.

A relation like (7) appears also in the very recent discussion of the Schwarzschild black hole in the DLCQ-version of ”M theory” [13] (see ch. 6 below).

In the following it is convenient to use these notations: We put $D = 1 + d = 2 + \tilde{d}$: $d$ gives the number of space dimensions and $\tilde{d}$ the spatial dimensions of the black hole horizon.

In $D$ dimensions the Schwarzschild radius is given by [14]

$$R_S(M) = \left( \frac{16\pi G_D M}{c^2 \omega_{\tilde{d}} \tilde{d}} \right)^{1/(\tilde{d}-1)}, \quad (8)$$

where $G_D$ is the gravitational constant in $D$-dimensional spacetime and $\omega_{\tilde{d}} = 2\pi^{(\tilde{d}+1)/2}/\Gamma((\tilde{d}+1)/2)$ is the volume of $S^{\tilde{d}}$. Inserting this $R_S$ into the relation (7) gives the mass spectrum

$$M_n = \sigma_D n^{1-\eta} m_{P,D}, \quad \eta = 1/\tilde{d}, \quad (9)$$

$$\sigma_D = \left( \frac{(2\pi)^{\tilde{d}-2} \omega_{\tilde{d}} \tilde{d}}{8 \gamma^{\tilde{d}-1}} \right)^{\eta},$$

where

$$m_{P,D} = \left( \frac{\hbar^{D-3} c^{5-D}}{G_D} \right)^{1/(D-2)}, \quad l_{P,D} = \left( \frac{\hbar G_D}{c^3} \right)^{1/(D-2)} \quad (10)$$
are the corresponding Planck mass and Planck length in D spacetime dimensions, respectively.

As to the degeneracies I again assume (2) to hold. I would like to stress that this assumption is very important for the thermodynamics involved! Bekenstein [8, 11] and Mukhanov [10] have presented convincing intuitive physical arguments for it, but it does not follow directly within the framework of Refs. [7] where the Schrödinger Eq. (3) comes from. Perhaps loop quantum gravity [13] or ”Matrix-theory” (see below) can be of help here. Mukhanov uses information theoretical reasoning to put \(g = 2\). Bekenstein recently [11] has summerized the corresponding arguments. They are closely related to those of the ”stretched horizon” approach by Zurek and Thorne [16]. Others [4, 17] argue similarly.

All those arguments are strongly supported by the results above: Combining the Eqs. (7) and (8) yields the quantized \(\tilde{d}\)-dimensional horizon

\[
A_{\tilde{d}}(n) = (R_S(M_n))^{\tilde{d}} \omega_{\tilde{d}} = \frac{32 \pi^2}{\gamma \tilde{d}} n \tilde{l}_{P,D}^{\tilde{d}}, \quad n = 1, 2, \ldots.
\]

Thus, the horizon is ”paved” additively with Planck-sized area elements \(\tilde{l}_{P,D}^{\tilde{d}}\). Such a pavement (parquet) can be assembled in many different ways, the total number of possibilities being \(2^{n-1}\). The ansatz (2) is a mathematically convenient generalization of this!

Another point to comment on is the free parameter \(\gamma\). We shall see below how the Gibbons-Hawking geometrical approach [18] to the partition function may be used to restrict it, but it is worthwhile to point out that such a free parameter also occurs in loop quantum gravity [13] and in the discussion of the Schwarzschild black hole thermodynamics of Matrix theory [13].

The canonical partition function can now be written as

\[
Z_D(t, x) = \sum_{n=0}^{\infty} e^{nt - n^{1-\eta} x},
\]

\[
t = \ln g, \quad \eta = 1/\tilde{d} \equiv 1/(D - 2),
\]

\[
x = \beta \sigma_D E_{P,D}, \quad \beta = \frac{1}{k_B T}, \quad E_{P,D} = m_{P,D} c^2.
\]

\(Z_D(t, x)\) obeys the linear PDE

\[
\partial_t^{\tilde{d}-1}Z_D = (-1)^{\tilde{d}} \partial_x^{\tilde{d}}Z_D,
\]
which can be used to determine $Z_D$ in closed form. This has been done \[19\] for $d = 2$ and 3.

### 3 Primitive droplet nucleation model in $D − 2$ space dimensions

The model assumption that 1st-order phase transitions are initiated by the formation (homogeneous nucleation) of expanding droplets of the new phase within the old phase is a popular and important one (see the reviews \[20\] with their references to the original literature). In its most primitive form the droplets are assumed to be spherical and to consist of $n$ constituents (e.g. droplets of Ising spins on a lattice or liquid droplets of molecules etc.), the "excess" energy $\epsilon_n$ of which is given by a "bulk" term proportional to the volume $n$ and a term proportional to the surface $n^{1-\eta}$, $\eta = 1/\tilde{d}$, where $\tilde{d} \geq 2$ is the spatial dimension of the system:

$$\epsilon_n = -\hat{\mu} n + \phi n^{1-\eta} , \quad \eta = 1/\tilde{d} . \quad (14)$$

In the case of negative Ising spin droplets, formed in a background of positive spins by turning an external magnetic field $H$ slowly negative - below the critical temperature -, the coefficient $\hat{\mu}$ in Eq. (14) takes the form $\hat{\mu} = -2H$ and for liquid droplets of $n$ molecules condensing from a supersaturated vapour one has $\hat{\mu} = \mu - \mu_c$, where $\mu$ is the chemical potential and $\mu_c$ its critical value at condensation point. $\phi$ is the constant surface energy.

Assuming the average number $\bar{\nu}(n)$ of droplets with $n$ constituents to be proportional to a Boltzmann factor,

$$\bar{\nu}(n) \propto e^{-\beta \epsilon_n} , \quad \beta = \frac{1}{k_B T} , \quad (15)$$

and that the droplets form a noninteracting dilute gas leads to the grand canonical potential $\psi_{\tilde{d}}$ per spin or per volume

$$\psi_{\tilde{d}}(\beta, t = \beta \hat{\mu}) = \ln Z_G = p\beta = \sum_{n=0}^{\infty} e^{tn} - xn^{1-\eta} , \quad (16)$$

$$t = \beta \hat{\mu} , \quad x = \beta \phi ; \quad p : \text{pressure} ,$$

$$d\psi_{\tilde{d}} = -U d\beta + \bar{n} dt . \quad (17)$$
(For physical reasons the sum (16) may not start at \( n = 0 \) but at some finite \( n_0 > 0 \). This can easily be taken care of. It is mathematically convenient to start at \( n = 0 \).)

Obviously the sum (16) is the same as in (12)!

The interpretation of the sum (16) as a grand canonical potential comes about as follows \[21, 22\]: One starts with a canonical partition function \( Z_c \), where the same terms as above are summed up. However, then one has the thermodynamics of single \( n \)-droplets, but there may be \( N \) of them. If one assumes these to be noninteracting and indistinguishable, then the grand canonical partition function is

\[
Z_G = \sum_{N} \frac{Z_c^N}{N!} = e^{Z_c}, \tag{18}
\]

which leads to the grand canonical potential \( \tilde{\psi}_d \) of Eq. (16).

It will be important, however, that we interpret \( Z_D \) of (12) as a canonical partition function, where \( g = e^t \) describes the fixed temperature – independent degeneracies of the corresponding quantum levels, whereas in the droplet nucleation model \( z = e^t, t = \mu \beta \), is the temperature – dependent fugacity of a classical Boltzmann gas!

Notice that \( \tilde{\psi}_d \) contains no explicit information about properties of the phases before and after the phase transition. Consider, e.g., a vapour \( \rightarrow \) fluid phase transition. Then the properties of the vapour are only very indirectly present in \( \tilde{\psi}_d \), namely in form of the surface energy \( \phi \) of the droplets emerged in the vapour. The model here merely is supposed (for more details see the Refs. \[20\]) to describe that (metastable!) part of the Van der Waals isotherm in the \((V, p)\)-plane which starts where, with decreasing volume, the (theoretically!) strict equilibrium line of the Maxwell construction branches off to the left, till the (local) maximum of the Van der Waals ”loop”, the ”spinodal” point, is reached.

The series (16) converges for \( t \leq 0 \) only. This follows, e.g., from the Maclaurin-Cauchy integral criterium \[23\]. In applications to metastable systems, however, one is interested in the behaviour of \( \tilde{\psi}_d(t, x) \) for \( t \geq 0 \). This calls for an analytic continuation in \( t \) or in the fugacity \( z = e^t \) which reveals a branch cut of \( \tilde{\psi}_d \) from \( z = 1 \) to \( z = \infty \) \[24\].

Qualitatively the following happens: For \( t < 0 \) (i.e. positive magnetic field) \( \epsilon_n \) increases monotonically with \( n \), making the corresponding terms in \( \tilde{\psi}_d \)
decrease monotonically. The small droplets are favoured and no phase transition occurs.
If, however, $t > 0$, then $\epsilon_n$ has a maximum for

$$n^* = \left(\frac{(1 - \eta)\phi}{\mu}\right)^{\hat{d}} = \left(\frac{(1 - \eta)x}{t}\right)^{\hat{d}}, \quad x = \phi\beta,$$

(19)

with

$$\epsilon^* \equiv \epsilon_{n^*} = a\eta(1 - \eta)^{\hat{d} - 1}, \quad a = \frac{\phi^{\hat{d}}}{\mu^{\hat{d} - 1}} = \frac{x^{\hat{d}}}{\beta t^{\hat{d} - 1}},$$

(20)

after which $\epsilon_n$ becomes increasingly negative with increasing $n$ and the series (16) explodes!

The physical interpretation is the following: If, by an appropriate fluctuation, a "critical droplet" of "size" $n > n^*$ has appeared, it is energetically favoured to grow. Such an overcritical droplet – and others of a similarly large size – will destabilize the original phase and will send the system to the phase for which it has served as a nucleus!

The energy $\epsilon^*$ may be interpreted as a measure for the critical barrier of the free energy the system has to "climb" over in order to leave the metastable state for a more stable one.
Furthermore, the rate $\Gamma$ for the transition of the metastable state to the more stable one is proportional to $\exp(-\beta\epsilon^*)$. However, calculating the rate is no longer a problem of equilibrium thermodynamics. One has to deal with tools of nonequilibrium processes like the Fokker-Planck equation etc. For the droplet model this was essentially done by Becker and Döring [25]. They assumed a stationary situation, where a steady flow of small, but in size increasing, droplets leave the metastable state and all overcritical droplets which have passed the barrier are removed from the system.

Their approach was considerably improved by Langer [26] who related the transition rate $\Gamma$ to the imaginary part of $\psi_{\hat{d}}$. This can be seen roughly as follows: If one turns the sum $\psi_{\hat{d}}$ in (16) into an integral by interpreting $n$ as a continuous variable:

$$\tilde{\psi}_{\hat{d}} = \int_0^\infty dne^{\beta(\tilde{\mu}n - \phi n^{1-\eta})},$$

(21)
then a saddle point approximation [21,19] for large $\beta$ gives the asymptotic expansion

$$\tilde{\psi}_d \sim (1 - \eta)^{\hat{d}/2} \left[ \frac{\pi \hat{d}}{2\mu \beta} \left( \frac{\phi}{\mu} \right)^{\hat{d}/2} \right] e^{-\beta \eta (1 - \eta)^{\hat{d}-1}} (i + O(1/\beta)) .$$  \hspace{1cm} (22)

Here the path in the complex $n$-plane goes from $n = 0$ to $n^*$ and then parallel to the imaginary axis to $+i\infty$ [19]. (Thus, only half of the associated Gaussian integral along the steepest descents contributes!)

The crucial point is that the saddle point is given by $n^*$ and the associated $\epsilon^*$ of Eqs. (19) and (20), that is to say, by the critical droplet!

The leading term in the saddle point approximation (22) is purely imaginary. Performing a Fokker-Planck type analysis, Langer found [26] that the transition rate $\Gamma$ is essentially proportional to the imaginary part $\Im(\tilde{\psi}_d)$, the other factor being a "dynamical" one, genuinely related to nonequilibrium properties.

An essential point for us here is the result that the imaginary part of $\psi_d$ can be interpreted, at least intuitively, in terms of equilibrium concepts although it is related to nonequilibrium properties which are, however, near to stationary situations. For further discussions see the reviews mentioned in Ref. [20].

4 Hawking temperature and Bekenstein-Hawking entropy

We are now ready to apply the droplet nucleation model results to the Schwarzschild black hole: If we denote the "critical" term in the series (12) by $Z_D^*$, we have

$$Z_D^* = e^{-\left[ \eta(1 - \eta)^{\hat{d}-1} x^\hat{d} \right] / t^{\hat{d}-1}} .$$  \hspace{1cm} (23)

The essential point now is that $t = \ln g$ here is no longer a temperature dependent quantity – as in the droplet model – but a fixed number. Therefore the equation for the associated internal energy,

$$U^* = -\frac{\partial \ln Z_D^*}{\partial \beta} = (1 - \eta)^{\hat{d}-1} \left( \frac{x}{\hat{d}} \right)^{\hat{d}-1} \sigma_D E_{P,D} = \hat{d} \epsilon^*, \hspace{1cm} (24)$$
can be used to determine the (inverse) temperature $\beta^*$ needed for a (potential) heat bath, if the rest energy $U^*$ is given! Solving Eq. (24) for $x$ and using the relation (8) between Schwarzschild radius $R_S$ and mass $M = U^*/c^2$ we obtain

$$\beta^* = \lambda \left( \frac{4\pi R_S^*}{(\tilde{d} - 1)\hbar c} \right), \quad \lambda \equiv \frac{t \tilde{d} \gamma}{8\pi^2},$$

(25)

where $R_S^* = R_S(M = U^*/c^2)$ (Eq. (8)).

The expression in the bracket of Eq. (25) is exactly the inverse Hawking temperature in $D$-dimensional spacetime [14], if we identify $U^* = Mc^2$, where $M$ is the macroscopic rest mass of the black hole! Thus, up to a numerical factor $\lambda$ of order 1, we obtain the Hawking temperature in this way.

For the entropy $S_D^* = \beta^* U^* + \ln Z_D^*$ we get

$$S_D^*/k_B = (1 - \eta)^{\tilde{d}} (x^*)^{\tilde{d}}/t^{\tilde{d}-1} = (1 - \eta)\beta^* U^*$$

$$= t n^* = \ln (g n^*), \quad x^* = \beta^* \sigma_D E_{P,D},$$

(26)

where $n^*$ is the same as in (19).

If we express $S_D^*$ in terms of the $\tilde{d}$-dimensional surface $A_{\tilde{d}} = \omega_{\tilde{d}} (R_S)^{\tilde{d}}$, we have

$$S_D^*/k_B = \lambda \frac{A_{\tilde{d}}}{4\ell_{P,D}^{\tilde{d}}} = \lambda \frac{c^3 A_{\tilde{d}}}{4\hbar G_D}.$$  

(27)

So, up to the same numerical factor $\lambda$ we already encountered in connection with the inverse temperature, we obtain the Bekenstein-Hawking entropy!

The mean square fluctuations of the energy

$$\langle \Delta E \rangle^2 = \partial^2_{\beta^*} (\ln Z_D^*) = - \frac{(1 - \eta)^{\tilde{d}-1}(\tilde{d} - 1)}{t^{\tilde{d}-1}} x^{\tilde{d}-2} (\sigma_D E_{P,D})^2$$

(28)

are negative (negative specific heat!), but relatively small for large masses because

$$\frac{\langle \Delta E \rangle^2}{U^*} = - \frac{\tilde{d} - 1}{\beta^*},$$

(29)

which appears to be quite an universal relation: the r.h.s. of Eq. (29) depends only on $\tilde{d}$ and $\beta^*$!

As the factor $\gamma$, up to now, is a free parameter, we can possibly choose it in
such a way that the above prefactor $\lambda$ equals one:

$$\gamma = \frac{8\pi^2}{td}.$$  \hspace{1cm} (30)

A suitable argument for such a normalization $\lambda = 1$ comes from the practically classical geometrical result for $S$ of Gibbons and Hawking [18], derived from the euclidean section of the Schwarzschild solution. In view of the surprising approximate equalities of the classical and the quantum theoretical values for $S$, one may use the classical result for normalizing the quantum one.

There is a corresponding analogue in QED, where the physical value of the electric charge $e$ in the quantum theory (to all orders) is normalized via the universal classical total Thomson cross section $\sigma_{\text{tot}} = (8/3)\pi r_0^2$, $r_0 = e^2/(mc^2)$, for Compton scattering in the limit of vanishing photon energy [27].

One has to be careful here, however, because only $Z^*_D$ has the same simple exponential form as one finds in the Gibbons-Hawking approach in lowest order. Fluctuations lead to prefactors with powers of $x$ in front of $Z^*_D$ as can already be seen if we take the imaginary part $Z_{i,D}$ of the saddle point approximation (22) as a slightly more sophisticated "pseudo" partition function, or, if we take the imaginary part of the purely imaginary partition function for the (euclidean) Schwarzschild black hole if one includes "quadratic" quantum fluctuations around the classical solution [28, 30, 2]. Such additional powers of $x$ lead to corrections to $\beta$ and to logarithmic corrections of the entropy (27) [1].

As to the connection between the statistically defined entropy and the one obtained geometrically and as to possible quantum corrections see the recent review by Frolov und Fursaev [31].

The corresponding normalization problem in loop quantum gravity has been discussed in Refs. [15, 32].

I said "pseudo" partition function because they imply negative mean square fluctuations, see Eq. (28) and Ref. [1], which appears to be associated with the metastability of the system. More theoretical work seems to be necessary in order to understand these partially surprising features better beyond the realm of strict equilibrium thermodynamics.

Altogether we see the following picture emerging: If the quantum system in $D$ spacetime dimensions of (whatever) total mass $M$ and represented by
the plane wave (4) collapses after a finite time, then the theoretical imple-
mentation of this collapse through periodic boundary conditions enforces the
mass quantisation (7), which may also be obtained heuristically through the
old-fashioned Bohr-Sommerfeld rules.

If, in addition, the degeneracies (2) are assumed, the associated quantum
statistics is formally the same as that of the classical primitive droplet nu-
cleation model in \( D - 2 \) space dimensions, represented as a classical grand
canonical ensemble. This shows how the holographic principle is indeed im-
plemented. It further indicates how the quantum background is hidden be-
hind a classically appearing facade, very probably formed by the thermal
physics of the horizon, which is being built up during the nucleation of the
black hole.

The ”blurring” of the quantum properties is also indicated by the fact that
the traces of the Bose statistics of the quanta (1) are rather hidden, contrary
to, e.g., the canonical partition function of the simple harmonic oscilla-
tor. If we look at the exact expression for \( Z_4 \) in closed form, derived in Ref. [1],

\[
Z_4(t, x) = \int_0^\infty d\tau \hat{K}(\tau, x) \frac{1}{1 - e^{(t - \tau)}},
\]

\[
\hat{K}(\tau, x) = \frac{x}{2\sqrt{\pi \tau^3}} e^{-x^2/(4\tau)},
\]

\[
\Re[Z_4(t, x)] = \text{p.v.} \int_0^\infty d\tau \hat{K}(\tau, x) \frac{1}{1 - e^{t-\tau}}, \quad \Im[Z_4(t, x)] = \pi \hat{K}(t, x),
\]

only the factor \( 1/(e^{t-\tau} - 1) \) in the principal value integral for the real part
of \( Z_4 \) indicates Bose statistics, whereas the, for our discussion above crucial,
imaginary part does not show such traces (see also the next chapter). Here
may lie a key to the information loss problem [29].

The role of the external magnetic field (or a corresponding chemical poten-
tial) in the droplet case finds its correspondence in the degeneracy factor \( g \)
which intuitively represents the gravitational pull leading to the formation
of the black hole by nucleation.

What is really new, compared to the droplet model, is that the free energy
barrier \( e^* \) determines its own temperature, namely \( T_H \), and the associated
entropy (26), as a function of the total internal energy \( U^* = Mc^2 \). This is a
genuinely quantum mechanical effect, because \( t = \ln g \) is a fixed number, not
a temperature-dependent quantity as in the droplet nucleation case. Here
lies the real difference. This temperature, after nucleation, then becomes
that of the black hole itself. Any thermal radiation emitted from the horizon carries the imprint of this temperature.

In the above physical interpretation of the nucleation process concerning the black hole I followed the droplet nucleation picture and have assumed that the black hole is the result of the decay of metastable states to a more stable one (the black hole) which then Hawking radiates with the corresponding temperature.

Another interpretation is that the black hole itself is the metastable state which slowly decays, due to its interaction with the heat bath consisting of Hawking radiation. A decision between these two alternative pictures probably needs the explicit introduction of matter fields coupled to the quantum black hole in order to see what really happens.

The nucleation of black holes has been discussed quite early by Gross, Perry and Yaffe in terms of the euclidean Schwarzschild instanton [30, 2].

One can arrive at similar results for the Hawking temperature and the Bekenstein-Hawking entropy as above if one performs a microcanonical counting of states [33], however, then one loses the very inspiring connection to the droplet nucleation picture.

5 Effective quantum and classical Hamiltonians

The partition function $Z_D$ of Eq. (12) may be rewritten as

$$Z_D = \text{tr}(e^{-\beta \hat{H}}), \quad \hat{H} = -\mu a^+a + \epsilon(a^+a)^{1-\eta}, \quad \mu = t/\beta, \quad \epsilon = \sigma_D E_{P,D},$$

where $a$ and $a^+$ are the annihilation and creation operators of the harmonic oscillator. If $u_n$ is an eigenfunction of the harmonic oscillator, then $a^+a u_n = n u_n$ and the assertion (32) follows immediately. As the trace is independent of the basis one uses for its calculation, one might also use another one, e.g. the coherent states $|z>$, which are eigenstates of $a$ with complex eigenvalues $z$. Doing so [34] for $\bar{d} = 2$ leads to the same exact result as in Ref. [1].

It is amusing to look [35] briefly at the corresponding classical effective sys-
Let us define the classical quantity

$$\tilde{N} = \frac{1}{\hbar\omega_0} \left( \frac{1}{2m} p^2 + \frac{m}{2} \omega_0^2 q^2 \right).$$  (33)

After an appropriate rescaling of $q$ and $p$ we have the effective Hamiltonian

$$\tilde{H} = -\mu \tilde{N} + \epsilon \tilde{N}^{1-\eta}, \quad \tilde{N} = \frac{1}{2} (p^2 + q^2),$$  (34)

leading to

$$\dot{p} = -\frac{\partial \tilde{H}}{\partial q} = -\tilde{\omega} q, \quad \tilde{\omega}(\tilde{N}) = -\mu + (1 - \eta) \epsilon \tilde{N}^{-\eta},$$  (35)

$$\dot{q} = \frac{\partial \tilde{H}}{\partial p} = \tilde{\omega} p.$$  (36)

It follows that $\tilde{N}$ is a constant of motion $\tilde{N}_0$ for the associated fictitious point particle and that this particle moves on a circle with radius $\sqrt{2\tilde{N}_0}$ in phase space with frequency $\tilde{\omega}$ which is a function of $\tilde{N}_0$ (this is a new feature compared to the usual harmonic oscillator).

The critical value $\tilde{\omega} = 0$ results if $\tilde{N} = \tilde{N}_c = ((1 - \eta)\epsilon/\mu)^{\frac{1}{2}}$ which is just the same as $n^*$ from Eq. (19) above and for which $\tilde{H}$ has its maximum. For $\tilde{N} < \tilde{N}_c$ the frequency $\tilde{\omega}$ is positive and for $\tilde{N} > \tilde{N}_c$ it is negative.

As

$$\dot{q} = ((1 - \eta)\epsilon \tilde{N}^{-\eta} - \mu) p,$$  (37)

it is in general not at all trivial to calculate the Lagrange function $L(q, \dot{q})$, because one has to solve an algebraic equation if one wants $p(\dot{q})$ from (37). Already for $d = 2$ this equation is of order 4. If $\mu = 0$ we get in this case

$$L(q, \dot{q}) = -q \left( \frac{1}{2} \epsilon^2 - q^2 \right)^{1/2},$$  (38)

which may be interpreted as a simple example of a Born-Infeld type Lagrangean [32].

Finally I mention how the classical partition function $Z_{cl}$ associated with the Hamiltonian (34) looks like:

$$Z_{cl} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dp dq}{2\pi \hbar} \exp \left\{-\beta \left[ -\frac{\mu}{2\hbar} (p^2 + q^2) + \frac{\epsilon}{\sqrt{2\hbar}} (p^2 + q^2)^{1/2} \right] \right\}. \quad (39)$$

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Introducing polar coordinates in the \((q, p)\)-plane and making appropriate substitutions gives

\[
Z_{cl} = 2 \int_{0}^{\infty} du u e^{tu^2 - xu}, \quad t = \beta \mu, \quad x = \epsilon \beta.
\]  
(40)

The integral exists for \(t < 0\) and gives

\[
Z_{cl} = -e^{-x^2/(8t)} D_{-2}(\frac{x}{\sqrt{-2t}}),
\]  
(41)

where \(D_p(z)\) is the parabolic cylinder function of order \(p\). Continuing now from negative to positive \(t\) finally yields

\[
Z_{cl} = -\frac{1}{t} \Phi(1, 1/2; -\frac{x^2}{4t}) + i\sqrt{\pi x} \frac{e^{-x^2/(4t)}}{2t^{3/2}}
\]  
(42)

where \(\Phi(a, c; z)\) is the confluent hypergeometric function with \(\Phi(a, c; z = 0) = 1\).

What is remarkable here is that the imaginary part of the classical partition function \(Z_{cl}\) is the same as that of the quantum theoretical one \((31)\). The real parts are different (put \(x = 0\)), however, reflecting the difference between classical and quantum mechanics.

The "invariance" of the imaginary part under quantization reminds one of the Rutherford scattering cross section for charged particles which is the same in mechanics and quantum mechanics, due to the long range of the Coulomb forces.

6 Remarks on possible relationships to DLCQ Matrix Theory

The following remarks are very preliminary and very qualitative, but I hope they are nevertheless useful and not too much beside the point! They are made under the assumption that the correspondence between the quantum statistics of Schwarzschild black holes and the classical droplet nucleation model is not accidental.

Let me start by slightly changing the language: Instead of a "droplet" in \(\tilde{d}\) space dimension we may speak of a spherical "\(\tilde{d} - brane\)" with an associated
“(\(d - 1\))-brane” as its boundary. For \(D = 4\) the corresponding objects in 2 dimensions are compact membranes (“2-branes”) with closed strings as their boundary. Recall that closed strings are essential for having gravity in an effective low energy string limit \[39\]. If the \(d\)-branes are spherical they are in their ground states.

For \(n < n^*\) the \(d\)-branes are (meta)stable, but for \(n > n^*\) they are unstable, nucleating the new phase, the black hole.

This role of the critical droplet size \(n^*\) finds its analogy in the ”correspondence principle for black hole and strings” \[10, 11\] which says that a highly excited string state becomes a black hole when its length scale \(l_S\) shrinks to less than the Schwarzschild radius \(R_S\) of the associated black hole of the same mass. At this ”correspondence point”, which can be reached by adjusting the string coupling constant \(g_s\), the entropies of the two systems become comparable if their masses and charges are. This then opens the possibility to calculate the entropy of black holes by counting the states of an associated string system. For Schwarzschild black holes this program has very recently been discussed in the framework of the ”DLCQ”-version of ”Matrix theory” (as to this see the reviews \[42\]) which structurally - not in detail, of course - shows some surprising similarities with the droplet nucleation picture which are, perhaps, not accidental:

In the DLCQ approach (in the following natural units are being used) one compactifies the lightlike coordinate \(x^- = x^0 - x^{10}\) of the 11-dimensional theory on a circle of radius \(R\), thus making the related momentum discrete, \(P_- = N/R, N = 1, 2, \ldots,\) where \(N\) labels the different ”parton” sectors in the infinite momentum frame. The number of states belonging to a given \(N\) depends on the supersymmetric Galilei invariant part of the system in the nine "transverse" directions several of which may be compactified.

The number \(N\) associated with a lightlike compactification corresponds to the number \(n\) of Eqs. (5) or (7) coming from a timelike compactification. This correspondence becomes evident as follows: in the DLCQ framework one asks what is the minimal \(N = N_{\text{min}}\) such that an appropriate Lorentz boost makes \(R\) large enough for a black hole of mass \(M\) just to fit in. The answer is

\[N_{\text{min}} \approx MR_S \approx S,\]

where \(S\) is the entropy. Thus, it is \(N_{\text{min}}\) which characterizes the correspondence point \[13, 13, 14\]! Obviously \(N_{\text{min}}\) plays quite a similar role as the
critical $n^*$ in the droplet model above (Eqs. (7) and (26)).
The counting of states which yield the Bekenstein-Hawking entropy and the
Hawking temperature are, of course, very different: in our case the essential
input is (2), whereas in the DLCQ approach one either counts the states from
the SUSY YM fields etc. just below the correspondence point [13, 43, 45] or
the states of the nonrelativistic D0-brane gas slightly above that point [13, 40].
In the latter case one needs Boltzmann counting which one obviously has in
the droplet nucleation model, too.
A very essential difference between the DLCQ approach and the droplet nu-
cleation model is, of course, that the latter is not supersymmetric.
Questions are: Can Matrix theory (or string theory) determine the normal-
ization factor $\lambda$ of Eq. (27) for neutral Schwarzschild black holes (see Refs.
[45]), can it provide the spectrum (9) (very likely in view of Eq. (43)) and,
most importantly, can it explain the degeneracies (2) and the value of $g$?

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