Path space forms and surface holonomy

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Abstract. We develop parallel transport on path spaces from a differential geometric approach, whose integral version connects with the category theoretic approach. In the framework of 2-connections, our approach leads to further development of higher gauge theory, where end points of the path need not be fixed.1

Keywords: Path space, Lie 2-group, Category theory

1. INTRODUCTION

The issue of parallel transport along surfaces has been the subject of a growing body of literature ([4, 1, 3, 8], to mention a few). The importance of surface parallel transport lies in the fact that to describe string like objects, it is the natural framework. Ordinary gauge theory describes the interactions between particles, where the gauge group is the relevant symmetry group of the particles; if instead of particles we have string-like objects, a higher gauge theoretic structure becomes more natural, and category theoretic argument show that, unless the group is Abelian, a single group is not sufficient for that purpose. In this article we describe a bridge between the differential geometric and category theoretic approaches to the problem of surface parallel transport. We start with path-space forms to build up the necessary frame work. Then starting from a differential geometric approach we develop the category theoretic structure of the parallel transport on path space. We work in a principal fiber bundle \((\Pi, P, M)\), with the gauge group \(G\). We consider the \(A\)-horizontal path space \(\mathcal{P}_A P\), where \(A\) is a \(LG\) valued connection on \(P\). Keeping in mind that a single group is insufficient to describe surface parallel transport, we introduce another group \(H\) related to \(G\) and construct a connection \(\mathcal{A}\) for the path space from another \(LG\) valued connection \(A\) and a \(LH\) valued 2-form \(B\). With the help of this connection we develop parallel transport on the path space, which leads to a natural construction of an integrated picture or category structure for surface parallel transport.

Path space forms

We start with the construction of 'path-space forms' on a manifold \(M\). Let us first define the path space \(\mathcal{P}M\) of a manifold \(M\) as the space of all smooth paths in \(M\), i.e. if

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1 Talk delivered by S. Chatterjee at XXVIII WGMP, 28th June-4th July, 2009. Bialowieza, Poland.
\( \gamma \in \mathcal{P}M, \) then \( \gamma(t) \in M \) and
\[
\gamma : I \rightarrow M \quad I = [0, 1]
\]
is smooth. We define the tangent space of the path space as follows: for a \( \gamma \in \mathcal{P}M, \) a vector \( X \in T_{\gamma(t)}(\mathcal{P}M) \) is given by a vector field \( X(t) \in T_{\gamma(t)}(M) \) [4]. Note here that \( \mathcal{P}M \) is infinite dimensional, which is consistent with the fact that to define a ‘single’ vector on a point \( \gamma \) of \( \mathcal{P}M, \) we need to define a vector field along a path \( \gamma(t) \in M. \)

Let \( ev \) be the general evaluation map, i.e.
\[
ev : \mathcal{P}M \times I \longrightarrow M, \quad ev_t \overset{\text{def}}{=} ev(\cdot, t)
\]
Then \( ev_t : \mathcal{P}M \rightarrow M \) defines \( \gamma \mapsto \gamma(t), \) and given a \( p-\)form field \( \alpha_p \in \Omega^p M \) we can construct a \( p-\)form \( ev^*_t \alpha_p \) on the \( \mathcal{P}M \) by pulling it back. The above definition of the path space tangent vector leads to the following contraction formula
\[
ev^*_t(\alpha_p)(X_1, X_2, \ldots, X_p) = (\alpha_p)(X_1(t), X_2(t), \ldots, X_p(t))
\]
(2)

Another construction of path space forms follows the method of K.T. Chen[6], [7], known as the ‘Chen integral’. We will not discuss the Chen integral in detail, only a first order Chen integral is sufficient for our present purpose. Higher Chen integrals can be defined by the method of iteration. Given \( \alpha_{p+1} \in \Omega^{p+1} M, \) a first order Chen integral is defined as
\[
\int_{\text{chen}} \alpha_{p+1} \overset{\text{def}}{=} \int_I ev^* \alpha_{p+1} = \int_0^1 \alpha_{p+1}(\gamma(t), \ldots) dt \in \Omega^p \mathcal{P}M
\]
(3)

Here \( \gamma \) is a path on \( M \) defined on the interval \( I = [0, 1]. \) More explicitly, the contraction formula for a first order Chen integral reads as
\[
\int_{\text{chen}} \alpha_{p+1} |_{\gamma} (X_1, X_2, \ldots, X_p) = \int_0^1 \alpha_{p+1}(\gamma(t), X_1(t), X_2(t), \ldots, X_p(t)) dt
\]
(4)

For simplicity, we will often denote \( \int_I ev^* \alpha_p \) as \( \int \alpha_p, \) which should not be confused with the ordinary integral of \( \alpha_p \) on a manifold.

\[\text{\text{\(\bar{\alpha}\)-horizontal path space}}\]

Let us consider a principal \( G \)-bundle \((\Pi, P, M)\)
\[
\Pi : P \rightarrow M
\]
with the usual right action of the Lie group \( G \) on \( P \)
\[
P \times G \rightarrow P : (p, g) \rightarrow pg
\]
If \( \bar{\alpha} \) is a connection on this bundle, we consider the space of \( \bar{\alpha} \)-horizontal paths in \( P. \) An \( \bar{\alpha} \)-horizontal lift \( \tilde{\gamma} \) of a path \( \gamma : I \rightarrow M \) satisfies
\[
\Pi(\tilde{\gamma}(t)) = \gamma(t)
\]
This $\bar{A}$ horizontal path space $\mathcal{P}_{\bar{A}}P$ can be viewed as a principal $G$-bundle over $\mathcal{P}M$, for details see [4]. It can be shown (Proposition 2.1 in [5]) that if $\tilde{\Gamma} : [0, 1] \times [0, 1] \to P : (t, s) \mapsto \tilde{\Gamma}(t, s) = \tilde{\Gamma}_s(t)$ is a smooth map and $\tilde{X}_s(t) = \partial_t \tilde{\Gamma}(t, s)$, then each transverse path $\tilde{\Gamma}_s : [0, 1] \to P$ is $\bar{A}$-horizontal implies that the initial path $\tilde{\Gamma}_0$ is $A$-horizontal, and the tangency condition

$$\frac{\partial \bar{A}(\tilde{X}_s(t))}{\partial t} = F^\bar{A} (\partial_t \tilde{\Gamma}(t, s), \tilde{X}_s(t))$$

holds. In integral form this is:

$$ev^*_t A - ev^*_0 A = \int_0^T F^\bar{A}$$

(6)

The right hand side is a Chen integral over the interval $[0, T]$. Now we can define the tangent space $T_{\tilde{\gamma}}\mathcal{P}_{\bar{A}}P$ at a point $\tilde{\gamma}$ of $\mathcal{P}_{\bar{A}}P$ as the space of all vector fields $t \to \tilde{X}(t) \in T_{\tilde{\gamma}(t)}P$ along $\tilde{\gamma}$ for which (5) holds, i.e.

$$\frac{\partial \bar{A}(\tilde{X}(t))}{\partial t} = F^\bar{A} (\tilde{\gamma}'(t), \tilde{X}(t))$$

(7)

for all $t \in [0, 1]$. The vertical subspace of $T_{\tilde{\gamma}}\mathcal{P}_{\bar{A}}P$ is the linear space of all vectors $\tilde{X}$ for which $\tilde{X}(t)$ is vertical (a more detailed discussion can be found in [5]).

**Parallel transport on path space**

A description of parallel transport on path space by naively using a connection, with values in the Lie algebra $L\mathcal{G}$, on the path space (or horizontal path space) leads to a serious inconsistency with some natural requirements. It is natural to require that 'vertical' and 'horizontal' composition of surface parallel transports satisfy a consistency condition:

$$(H' \times H) \times (H'' \times H') = (H' \times H') \times (H \times H'')$$

(8)

here $H, H', H'', H'''$ are ‘surface parallel transport’ operators in Figure 1 and $\times$ and $\bullet$ denote vertical and horizontal composition for surfaces respectively.

>From this it is clear that if we take the surface parallel transport operator to be a group element and assign the same composition law (the group product) for both horizontal and vertical compositions, the group must be Abelian. Thus there is a ‘no-go theorem’ (see [1] for much more on this). This problem is avoided by using two groups $G$ and $H$ to describe surface parallel transport, and defining different composition laws for the ‘horizontal’ and ‘vertical’ compositions. The proper framework here is provided by the notion of a Lie 2-group [2, 5], which we discuss below.

A Lie 2-group is described by two Lie groups $G$ and $H$, along with a smooth homomorphism $\tau : H \to G$ and a smooth map for $g \in G$ and $h, h' \in H$

$$G \times H \to H : (g, h) \mapsto \alpha(g)h$$

where $\alpha(g)$ is an automorphism of $H$, and the following identities hold

$$\tau(\alpha(g)h) = g\tau(h)g^{-1}, \quad \alpha(\tau(h))h' = hh'h^{-1}$$

(9)
(There are fancier, category-theoretic formulations of the notion of Lie 2-group.) For simplicity we will denote the derivative mappings $\alpha'(e) : LG \to LH$ and $\tau'(e) : LH \to LG$ as $\alpha$ and $\tau$ respectively, here $LG$ and $LH$ are Lie algebras of $G$ and $H$ respectively.

**Connection form on path space**

Suppose we have a connection $A$ on the bundle $P$ and an $LH$ valued $\alpha$-equivariant (under the right action of $G$) 2–form $B$ on $P$, which vanishes on vertical vectors. i.e.

$$B(X, Y) = 0, \quad \text{if } X \text{ or } Y \text{ is vertical}$$

$$R_g^*B = \alpha(g^{-1})B \quad \text{for all } g \in G$$

here $R_g : P \to P : p \mapsto pg$ and according to our convention $\alpha(g^{-1})B = d\alpha(g^{-1})|eB$.

Keeping the ‘no-go’ theorem in mind, we define our connection as

$$\mathcal{A} = ev_1^*A + \tau \int_0^1 B$$

(10)

The integration on the right hand side is a first order Chen integral. For a proof that the right hand side of the (10) is a connection see [5]. At the infinitesimal level, the parallel transport of a path by the connection $\mathcal{A}$ is equivalent to lifting a given vector field $X : I \to T M$, along $\gamma \in \mathcal{P} M$, to a vector field $\tilde{X}$ along $\tilde{\gamma}$ such that it is $\mathcal{A}$ horizontal and satisfies the condition (7):

$$A(\tilde{X}) + \tau \int_0^1 B(\gamma'(t), \tilde{X}(t)) dt = 0$$

$$\frac{\partial A(\tilde{X}(t))}{\partial t} = F^A(\gamma'(t), \tilde{X}(t))$$

(11)
Now decomposing a lifted vector $\bar{X}(t) = \bar{X}_A^h(t) + \bar{X}_V(t)$ into horizontal and vertical parts with respect to the connection $\bar{A}$ and noting that $B$ is zero on the vertical vectors, it can be shown [5] that we can find a vector field $\bar{X}(t)$ which satisfies (11). The basic idea in our construction is that the equations (11) specify ‘parallel transport’ of the ‘right endpoint’ $\bar{\gamma}(1)$ and then (11) specifies the parallel transport of the entire path $\bar{\gamma}$.

\begin{equation}
\tau(h) = a^{-1}b^{-1}cd \tag{12}
\end{equation}

The above equation leads to the 2-categorical picture, where the set of objects for both of the categories is the group $G$ and the set of morphisms is $G^4 \times H$, in the above figure the morphism is $(a,b,c,d;h)$, for the category $\text{Vert}$ (vertical category) and $\text{Horz}$ (horizontal category), the source and targets are as follows

$s_{\text{Vert}}(a,b,c,d;h) = a$

$t_{\text{Vert}}(a,b,c,d;h) = c$

$s_{\text{Horz}}(a,b,c,d;h) = d$

$t_{\text{Horz}}(a,b,c,d;h) = b$

Keeping the condition (12) in mind, the composition law for $\text{Vert}$ is given by

$$ (a,b,c,d;h) \times (c,b',d,d';h') = (a,b'b,d,d'd;h(\alpha(d)h')) \tag{13} $$

and that of $\text{Horz}$ is given by

$$ (a,b,c,d;h) \bullet (a',f,c',b;h') = (a'a,f,c',d;\alpha(d^{-1})h'h) \tag{14} $$

It is easy to check that the identity morphism $a \to a$ for $\text{Vert}$ is $(a,e,a,e;e)$ and inverse of $(a,b,c,d;h)$ is $(c,b^{-1},a,d^{-1};\alpha(d)h^{-1})$, on the other hand for the $\text{Horz}$ the identity morphism $d \to d$ is $(e,d,e,d;e)$ and the inverse of $(a,b,c,d;h)$ is $(a^{-1},d,c^{-1},b;\alpha(a)h^{-1})$, here $e$ denotes the identity element for both $G$ and $H$. The category axioms can be readily verified.

In ordinary gauge theory a parallel transport operator $U(\gamma,0,1)$ between $\gamma(0)$ and $\gamma(1)$ along the path $\gamma$ transforms homogeneously as $U(\gamma(1))U(\gamma(0))^{-1}$, here $U(\gamma(0))$ and $U(\gamma(1))$ are two elements of the gauge group associated with the end
points of the path and \( H(\gamma, 0, 1) \) is also an element of the same group. Now consider a plaquette as in Figure 2. Here instead of a group-valued parallel transport operator we have a morphism like \((a, b, c, d; h)\) and have two end paths rather than two end points. So in the same spirit we define gauge transformation of a surface parallel transport operator as

\[
(a, b, c, d; h) \overset{\text{def}}{=} (c, \tilde{U}(1), c, \tilde{U}(0); \tilde{W}) \times (a, b, c, d; h) \times (a, U(1), a, U(0); W)^{-1} \tag{15}
\]

Here \( U(0), U(1) \in G \) are group elements associated with the left and right end points of the initial path in Figure 2 respectively, \( \tilde{U}(0), \tilde{U}(1) \in G \) are those of the final path, and \( W, \tilde{W} \in H \) are path ordered exponentials of some \( LH \)-valued one form \( \lambda \) over the initial and the final path respectively. As we have already defined the vertical composition in (13), from (15) we have following transformations

\[
\begin{align*}
\bar{a} &= U(1) \cdot a \cdot \tau(W) \cdot U(0)^{-1} \\
\bar{b} &= \tilde{U}(0) \cdot b \cdot U(0)^{-1} \\
\bar{c} &= \tilde{U}(1) \cdot c \cdot \tau(\tilde{W}) \cdot \tilde{U}(0)^{-1} \\
\bar{d} &= \tilde{U}(1) \cdot b \cdot U(1)^{-1} \\
\bar{h} &= (\alpha(U(0)))(W^{-1} \cdot h \cdot (\alpha(d^{-1})\tilde{W})))
\end{align*}
\]

To conclude, we summarize the main points: (i) we have described how connections \( A, \tilde{A} \) on a bundle, and a 2-form \( B \) taking values in a different Lie algebra, give rise to a connection over path-spaces, (ii) we described a pair of categories which arise from considerations of parallel-transport along paths and surfaces, (iii) we outlined ideas on the effect of gauge-transformations on the categorical/parallel-transport structures.

**Acknowledgments** ANS acknowledges research supported from US NSF grant DMS-0601141. AL acknowledges research support from Department of Science and Technology, India under Project No. SR/S2/HEP-0006/2008.

**REFERENCES**

1. J. Baez, *Higher Yang-Mills Theory*, arXiv:math/0206130
2. J. Baez and U. Schreiber, *Higher Gauge Theory*, arXiv:hep-th/0511710v2
3. J. Baez and U. Schreiber, *Higher Gauge Theory II: 2-connections on 2-bundles*, arXiv:hep-th/0412325
4. A. S. Cattaneo, P. Cotta-Ramusino, M. Rinaldi, *Loop and Path Spaces and Four-Dimensional BF Theories: Connections, Holonomies and Observables*, Commun. Math. Phys. **204** (1999) 493-524
5. S. Chatterjee, A. Lahiri, A. N. Sengupta *Parallel transport over path spaces*, arXiv:0906.1864
6. Kuo-Tsai Chen, *Algebras of Iterated Path Integrals and Fundamental Groups*, Transactions of the American Mathematical Society, Vol. 156, May, 1971 (May, 1971), pp. 359-379.
7. Kuo-Tsai Chen, *Iterated Integrals of Differential Forms and Loop Space Homology*, The Annals of Mathematics, 2nd Ser., Vol. 97, No. 2 (Mar., 1973), pp. 217-246.
8. F. Girelli and H. Pfeiffer, *Higher gauge theory - differential versus integral formulation*, J. Math. Phys. **45** (2004) 3949-3971.