MORIMOTO’S CONJECTURE FOR M-SMALL KNOTS

TSUYOSHI KOBAYASHI AND YO’AV RIECK

Abstract. Let $X$ be the exterior of connected sum of knots and $X_i$ the exteriors of the individual knots. In $[10]$ Morimoto conjectured (originally for $n = 2$) that $g(X) < \Sigma_{i=1}^n g(X_i)$ if and only if there exists a so-called primitive meridian in the exterior of the connected sum of a proper subset of the knots. For m-small knots we prove this conjecture and bound the possible degeneration of the Heegaard genus (this bound was previously achieved by Morimoto under a weak assumption $[11]$):

$$\Sigma_{i=1}^n g(X_i) - (n - 1) \leq g(X) \leq \Sigma_{i=1}^n g(X_i).$$

1. Introduction

This proceeding article is based on a talk given in Waseda University on the 18 December 2002, about Morimoto’s Conjecture which is concerned with the behavior of Heegaard genus of knot exteriors under connected sum. For a knot exterior, we consider two equivalent decomposition: the first is given by tunnel system, which is a collection of embedded arcs and the other a Heegaard splitting, given by an embedded surface. (See next section for standard definition.) The complexity of a tunnel system is the number of arcs and the complexity of a Heegaard surface is its genus. The complexity of a knot $K \subset M$ is the minimal number of tunnels required for a tunnel system (called the tunnel number and denoted $t(K)$) or the genus of the minimal genus Heegaard surface (called the Heegaard genus and denoted $g(X)$, here $X = M \setminus N(K)$). It is immediate from definitions that $g(X) = t(K) + 1$. Let $K_1 \subset M_1$ and $K_2 \subset M_2$ be two knots, and

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let \( K(= K_1 \# K_2) \subset M(= M_1 \# M_2) \) be the connected sum. In Section 2 we recall the easy fact: \( t(K) \leq t(K_1) + t(K_2) + 1 \). (More generally, \( t(K) \leq \sum_{i=1}^{n} t(K_i) + (n - 1) \).) Translating this into the language of Heegaard genus, we get \( g(X) \leq g(x_1) + g(x_2) \). (More generally, \( g(X) \leq \sum_{i=1}^{n} g(X_i) \).) In this paper we use the notation of Heegaard genus which seems simpler than tunnel number. Note that a knot exterior \( X \) (resp. \( X_i \)) has a distinguished slope, the meridian, denoted \( \mu \) (resp. \( \mu_i \)). Since we are interested in the knot exterior, instead of studying the connected sum of knots, we consider what happens to the exteriors. It is easy to see that \( X = X_1 \cup_A X_2 \), where \( A \subset \partial X_i \) is an annulus that is a neighborhood of \( \mu_i \) in \( \partial X_i \) \((i = 1, 2)\). We denote that operation \( X_1 \star X_2 \) but emphasize that \( X_1 \star X_2 \) depends not only on \( X_i \) but on \( \mu_i \) as well. It should be noted that the operation \( \star \) is more closely related to torus decomposition than connected sum.

Y. Moriah and J.H. Rubinstein \( ^3 \) showed that there exist knots \( K_1 \) and \( K_2 \) for which \( g(X) = g(x_1) + g(x_2) \). K. Morimoto \( ^{10} \) constructed examples for which \( g(X) = g(x_1) + g(x_2) - 2 \) and T. Kobayashi \( ^{11} \) generalized them to examples were \( g(X) = g(x_1) + g(x_2) - n \) for arbitrarily large \( n \). These and all other known examples have one thing in common: at least one of the two knot exteriors has a minimal genus Heegaard surface with a primitive meridian.

**Definition 1.1.** A minimal genus Heegaard splitting for a knot exterior \( X \) has a **primitive meridian** if there exists a compressing disk for the compression body that intersect a spanning annulus with a meridian slope exactly once.

**Remark.** A non-minimal genus Heegaard surface, even if it is irreducible, is never said to have a primitive meridian. Otherwise we need to modify Assertion 1.2.

Morimoto further observes that the existence of a primitive meridian guarantees a degeneration of the genus:

**Assertion 1.2.** Let \( X_1 \) and \( X_2 \) be two knot exteriors and let \( X = X_1 \star X_2 \) be the exterior of the connect sum.
If $X_1$ or $X_2$ contains a primitive meridian then $g(X) \leq g(x_1) + g(x_2) - 1$.

Proof. Recall that $X_1 \ast X_2$ is obtained from $X_1$, $X_2$ by identifying meridional annuli.

Let $\Sigma_1 \subset X_1$ and $\Sigma_2 \subset X_2$ be minimal genus Heegaard surfaces, say $\Sigma_2$ contains a primitive meridian. After gluing $X_1$ to $X_2$ we surge the Heegaard surfaces together along an annulus that runs across $A$. The component of $X_1 \ast X_2$ containing the boundary is easily seen to be a compression body: it is simply a neighborhood of the boundary union neighborhoods of tunnels for both knots. The other component is obtained by gluing the handlebody components of $X_i$ cut open along $\Sigma_i$ to each other along an annulus, which is general does not yield a handlebody. However, since $X_2$ contained a primitive meridian the annulus is longitudinal there, and a handlebody is obtained. \qed

Before stating our results we define:

Definitions 1.3. (1) A surface with non empty boundary, properly embedded in a manifold is called essential if it is incompressible, boundary incompressible and not boundary parallel.

(2) A knot exterior is called meridionally small if there is no essential surface with non empty boundary whose boundary forms parallel copies of the meridian; in other words the meridian is not a boundary slope.

We prove two results, the first numerical:

Theorem 1.4 (The Numerical Theorem). Let $\{X_i\}_{i=1}^n$ be exteriors of $m$-small knots and $X = \ast_{i=1}^n X_i$ be the exterior of the connected sum. Then:

$$\Sigma_{i=1}^n g(X_i) - (n - 1) \leq g(X) \leq \Sigma_{i=1}^n g(X_i).$$

Remarks. The assumption required for this bound is in fact weaker than m-smallness: we just need a minimal genus Heegaard surface to weakly reduce to a swallow follow torus. (see Theorems 1.6 and 1.1).
Thus, in contrast to the examples of Morimoto and of Kobayashi mentioned above, the degeneration of Heegaard genus under connected sum is bounded. This result, with a small assumption (that none of the ambient manifolds has a lens space component) was obtained by Morimoto [11].

In this theorem and throughout the paper we are considering knot exteriors. The statements can be rephrased in the language of manifolds with boundary torus. Let \((X_i, \mu_i)\) be the manifold \(X_i\) with a choice of meridian \(\mu_i\) \((i = 1, \ldots, n)\). By Hatcher’s Theorem [2] \((X_i, \mu_i)\) is \(\mu_i\)-small for all but finitely many choices of \(\mu_i\). Therefore for all other values of \(\mu_i\) the bound above is valid; we get the following theorem (other theorems in this paper can be modified similarly).

**Theorem** (The Numerical Theorem (1.4) version 2). Given \(X_i\) \((i = 1, \ldots, n)\) manifolds with boundary torus, after excluding a finite set of slopes from each \(X_i\) we get the following inequality for all remaining slopes:

\[
\sum_{i=1}^n g(X_i) - (n - 1) \leq \star_{i=1}^n (X_i, \mu_i) \leq \sum_{i=1}^n g(X_i).
\]

Our second result is geometric; it generalizes Morimoto’s Conjecture for the connected sum of \(n\) knots (we return to the language of knot exteriors):

**Theorem 1.5** (Morimoto’s Conjecture). Let \(\{X_i\}_{i=1}^n\) be exteriors of \(m\)-small knots. Then:

If \(g(X) < \sum_{i=1}^n g(X_i)\) there exists some \(I \subset \{1, \ldots, n\}\) a proper subset so that \(X = \star_{i \in I} X_i\) contains a primitive meridian.

We remark that by saying “\(I\) is a proper subset” we mean that \(I \neq \{1, \ldots, n\}\) and \(I \neq \emptyset\).

This theorem too has a second version, again using Hatcher’s Theorem:

**Theorem** (Morimoto’s Conjecture (1.5) version 2). Given \(X_i\) \((i = 1, \ldots, n)\) manifolds with boundary torus, after excluding a finite set of slopes from each \(X_i\) for all remaining slopes we get:
If \( g(\star_{i=1}^{n}(X_i,\mu_i)) < \sum_{i=1}^{n}g(X_i) \) there exists some \( I \subset \{1,...,n\} \) a proper subset so that \( X = \star_{i \in I}X_i \) contains a primitive meridian.

The main tool for proving both these results is:

**Theorem 1.6** (The Swallow Follow Theorem). Let \( X_i \) be a collection of exteriors of meridionally small knots.

Then any Heegaard splitting of \( X \) (the exterior of the connected sum) weakly reduces to a swallow follow torus.

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**2. Background**

We review some standard definitions (see also [3] or [1]): let \( M \) be a closed 3-manifold. A knot \( k \subset M \) is a smooth embedding of \( S^1 \) into \( M \). The knot exterior \( X \) (which is a manifold with boundary torus) is \( M \setminus N(k) \), where \( N(\cdot) \) is an open normal neighborhood. A **compression body** is a manifold \( C \) with distinguished boundary component (denoted \( \partial_+ C \), and \( \partial_- C = \partial C \setminus \partial_+ C \)) so that after compressing \( \partial_+ \) maximally a collection of balls and components of the form \( (\partial_- C \times [0,1]) \) are obtained. A compressing disk for \( \partial_+ C \) is called a meridian disk. A **handlebody** (denoted \( H \)) is the special case of compression body where only balls are obtained (equivalently if \( \partial_- H = \emptyset \)). A **tunnel system** is a collection of arcs properly embedded in the knot exterior whose exterior is a handlebody. It is well known (and an easy consequence of Morse theory) that every knot has a tunnel system. The tunnel number \( t(K) \) is the least number of arcs required for a tunnel system. If \( T \) is a tunnel system, \( \partial(N(\partial X \cup T)) \) is a surface that decomposes the knot exterior into a compression body \( C \) (with \( \partial_- C = \partial X \)) and a handlebody \( H \). A closed surface that decomposes \( X \) into a handlebody and a compression body is called a **Heegaard surface** (denoted \( \Sigma(X) = \Sigma \)). It is well known
that any Heegaard surface is given as the neighborhood of some tunnel system. Note that the genus of the $\Sigma$ (denoted $g(\Sigma)$) is exactly one more than the number of tunnels.

A Heegaard splitting is called stabilized if there are meridional disks on opposite sides intersecting exactly once, non-stabilized otherwise. If a Heegaard splitting is stabilized, by cutting along one of the disks a Heegaard splitting of lower genus is obtained (it is a consequence of Scharlemann and Thompson [17] that the converse is also true). We will be mostly concerned with minimal genus Heegaard splittings which are therefore always non-stabilized.

An extremely important concept for the study of Heegaard splittings is strong irreducibility (see [1]). A weak reduction for $\Sigma$ is a pair meridional disks on opposite sides of $\Sigma$ that are disjoint. $\Sigma$ is called strongly irreducible if there does not exist a weak reduction, weakly reducible otherwise. If after compressing $\Sigma$ to both sides an essential surface $F$ is obtained we say that $F$ was obtained from $\Sigma$ by weak reduction. Casson and Gordon [1] showed that if a non stabilized Heegaard surface weakly reduces, some weak reduction yields an essential surface. This was the first time a Heegaard surface was used to produce an essential surface, but by no means the last.

A connected sum of knots $K_i \subset M_i (i = 1, 2)$ is defined (much like the case $M_i \cong S^3$) by removing a small ball around each knot and gluing the spheres obtained so that the endpoint of the arcs match up. After removing the intersection of $N(K)$ with the sphere an essential annulus is obtained, called a decomposing annulus. The connected sum is denoted $K_1 \# K_2$ and is naturally a knot in $M_1 \# M_2$. It is now straightforward to see that $t(K) \leq t(K_1) + t(K_2) + 1$ by taking the union of tunnel system for $K_1$ and $K_2$ and a tunnel that is an essential arc on the decomposing annulus. As noted above the exterior of $K_1 \# K_2$ is obtained from the exteriors of $K_1$ and $K_2$ by gluing a meridional annulus. We denote the exterior of $K_1 \# K_2$ by $X_1 \times X_2$ (or just $X$), where $X_i$ is the exterior of $K_i$. Connected sum of $n$ knots is defined by induction, and similar notation is used.

Scharlemann and Thompson [18] refined Casson and Gordon’s weak reduction [1] in a construction called untelescoping (Kobayashi [1] showed
that untelescoping is strictly finer than weak reduction). We briefly describe untelescoping: by Casson and Gordon any non-stabilized Heegaard splitting can be untelescoped to give a collection of closed, essential surfaces (if the Heegaard surface was strongly irreducible this collection is empty and we are done). Each submanifold obtained by cutting the manifold open along these surfaces inherit a Heegaard splitting (called the induced Heegaard splitting). If the induced Heegaard splitting weakly reduces, apply Casson and Gordon’s result again (see Sedgwick [20] for the case with boundary) to obtain a larger collection of essential surfaces. We repeat this process that eventually terminates. Scharlemann and Thompson show that the induced Heegaard splitting after this final stage is strongly irreducible. We will be using these facts in an essential way in the proof presented in Section 5 refer the reader to [18] and [5] for a more detailed account of untelescoping. Since getting Σ back from the induced Heegaard splittings is the converse of untelescoping, called amalgamating.

3. BASIC CONSTRUCTIONS

The construction behind all our work is based on the following theorem:

**Theorem (1.6).** Let $X_i$ be a collection of exteriors of meridionally small knots. Then any Heegaard splitting of $X$ (the exterior of the connected sum) weakly reduces to a swallow follow torus.

**Sketch of proof.** Let $Σ$ be a Heegaard surface for $X$. Let $A = \{A_j\}_{j=1}^{n-1}$ be some collection of essential annuli so that $X$ cut open along $A$ is $\bigsqcup_{i=1}^{n} X_i$ ($\bigsqcup$ denotes disjoint union). We consider three cases.

**Case One: assume $Σ$ is strongly irreducible:** We show that this leads to a contradiction. If $Σ$ is strongly irreducible we can then isotope it to intersect $A$ in curves that are all essential in both $A$ and $Σ$ (this is a standard application of strong irreducibility and we omit the details). For $i = 1, 2$ let $Σ_i$ be the collection of surfaces obtained by cutting $Σ$ along $A_i$ with $Σ_i \subset X_i$ ($Σ_i$ need not be connected). Since $Σ$ is a Heegaard
surface for $X$ (and $X$ is not a compression body) $\Sigma$ compresses into both sides. Let $D_+$ and $D_-$ be compressing disks for $\Sigma$ on opposite sides of $\Sigma$. By minimizing the intersection of $D_+$ and $D_-$ with $A$, and using innermost and outermost disk arguments, we obtain a compression or a boundary compression from some component of $\Sigma_i$, and since $\partial X_i$ is a torus, boundary compression implies compression (this point requires a little care about components that are boundary parallel annuli, but we ignore that here). We choose a normal direction for $\Sigma$. Each component of $\Sigma_1$ and $\Sigma_2$ inherits a normal direction. Some component of some $\Sigma_i$ compresses to one side and some component of some $\Sigma_j$ (possibly $i = j$) compresses to the other side. A compression for $\Sigma_i$ is also a compression for $\Sigma$ (recall that all curves of intersection are essential in both). By strong irreducibility the compressing disks must intersect, and therefore both compress the same component of the same $\Sigma_i$. Furthermore strong irreducibility implies that all other components of $\Sigma_1$ and $\Sigma_2$ are essential. This violates $m$-smallness of the knot exteriors $X_i$. (Compare this with [14] and [15]: there too the authors produce bounded essential surfaces using Heegaard surfaces.)

Thus we may assume that $\Sigma$ weakly reduces, and we can maximally untelscope it ([13], recall Section 3) obtaining a collection of closed essential surfaces denoted $S_j$, with $S_j$ denoting a component of $S$. Minimize the intersection $S$ with $A$. Any component of $S$ that actually intersects $A$ is broken up into pieces that are incompressible, and if some such component is also of negative Euler characteristic, it must be essential, a contradiction similarly to above. (Note that no such component is a disk.) But this would contradict $m$-smallness. We conclude that each component $S_j$ of $S$ is exactly one of the following:

1. $S_j \cap A = \emptyset$, or—
2. each component of $S_j$ cut open along $A$ is an annulus (these annuli must be boundary parallel).

We are now ready for the remaining two cases:
**Case Two:** \((\forall j)S_j \cap A = \emptyset\). Cutting \(X\) open along \(S\) we get a collection of manifolds, one of them containing \(A\). By the maximality of the untelescoping the Heegaard splitting for this component is strongly irreducible, but in Case One of this proof we showed that if an annular manifold contains a strongly irreducible Heegaard surface then it contains an essential surface with boundary on \(A\). (In our main paper we show that that proof is valid here as well.) This surface remains essential in \(X\), and can be used to contradict \(m\)-smallness.

**Case Three:** \((\exists j)S_j \cap A \neq \emptyset\). We following an argument of Morimoto [11]. \(S_j\) is cut up to annuli, since any component of negative Euler characteristic would contradict \(m\)-smallness. Thus \(S_j\) is a torus. To conclude this sketch we show that a torus that cannot be disjoint from \(A\) is a swallow follow torus. After arranging the intersection of \(S_j\) and \(A\) to be essential and minimal, an outermost subannulus of \(A\) cut open along \(S_j\) is an annulus with one boundary on \(S_j\) and the other a meridian. Surgering the torus \(S_j\) along this subannulus we obtain a meridional annulus that must be decomposing. Thus \(S_j\) was obtained from some decomposing annulus by tubing along the boundary and is therefore a swallow follow torus by definition. This completes the proof.

\(\square\)

We obtain the following corollary, see [11] for a detailed description. This corollary is a restatement of Theorem 1.6 in an easy to use manner.

**Definition 3.1.** Let \(Y\) be a knot exterior. Then \(\hat{Y}\) denotes the manifold obtained from \(Y\) by drilling out a curve parallel to a meridian on \(\partial Y\).

**Corollary 3.2.** Let \(T\) be the swallow follow torus obtained in Theorem 1.6.

Suppose \(T\) follows \(X_I = \ast_{i \in I}X_i\) for some \(I \subset \{1, \ldots, n\}\) and swallows \(X_J = \ast_{i \in J}X_i\) (here \(J = \{1, \ldots, n\} \setminus I\) and of course \(I \neq \emptyset\) and \(J \neq \emptyset\)). Then \(X \cong X_I \cup_T \hat{X}_J\).
Note in the corollary above \( \hat{X}_J \) has two boundary components and \( X_I \) only one. As we shall see, the primitive meridian we are trying to find is in \( \hat{X}_J \) and not necessarily in \( X_I \), so there is no symmetry between the roles of \( X_I \) and \( X_J \). More about that later.

4. NUMERICAL BOUNDS

Corollary 3.2 holds whenever a minimal genus Heegaard surface weakly reduces to a swallow follow torus. We demonstrate its usefulness. In the following theorem we are not assuming that \( X_1 \) and \( X_2 \) are m-small (or even prime).

**Theorem 4.1.** Suppose a minimal genus Heegaard surface of \( X_1 \star X_2 \) weakly reduces to a swallow follow torus that follows one and swallows the other.

Then \( g(X_1) + g(X_2) - 1 \leq g(X_1 \star X_2) \leq g(X_1) + g(X_2) \).

**Remark.** Recall that the right hand side inequality is easy and always holds.

Before sketching the proof of Theorem 4.1 we bring two corollaries. The first is obtained as follows: let \( X \) be the exterior of a connected sum of m-small knots. Then \( X_I \) and \( X_J \) described in Corollary 3.2 are exteriors of connected sum of m-small knots as well (perhaps just one summand). By induction we get:

**Theorem (1.4).** Let \( \{X_i\}_{i=1}^n \) be exteriors of m-small knots and \( X = \star_{i=1}^n X_i \) be the exterior of the connected sum. Then:

\[
\Sigma_{i=1}^n g(X_i) - (n - 1) \leq g(X) \leq \Sigma_{i=1}^n g(X_i).
\]

Another corollary is about Morimoto’s examples [11]. Morimoto has examples of knot exteriors so that \( g(X_1) = 2, g(X_2) = 3 \) and \( g(X_1 \star X_2) = 3 \). We get:

**Corollary 4.2.** Let \( K_1 \) and \( K_2 \) be knots with irreducible and a-toroidal exteriors.

If \( g(X_1) + g(X_2) \geq 5 \) and \( g(X_1 \star X_2) = 3 \) then any minimal genus Heegaard surface for \( X_1 \star X_2 \) is strongly irreducible.
The knots $K_1$ and $K_2$ in Morimoto’s example are hyperbolic knots in $S^3$ and therefore fulfil the assumptions of the corollary above, thus we get an example of a connected sum that has only strongly irreducible minimal genus Heegaard surfaces (this was obtained independently by Y. Moriah [8] using different techniques).

Proof of Corollary 4.2. Assume for contradiction that a minimal genus Heegaard surface weakly reduces. Then by Casson and Gordon [1] some weak reduction yields an essential surface $F$. The genus of $F$ is at most $g(X_1 \times X_2) - 2$, hence at most 1. Our assumptions imply that $X_1 \times X_2$ contains no essential sphere, hence $F$ must be a torus. If $F$ can be isotoped to be disjoint from the decomposing annulus it must be parallel to $\partial X_1$ or $\partial X_2$, thus a swallow follow torus. On the other hand if $F$ cannot be made disjoint from the decomposing annulus, we can arrange the intersection of $F$ and the annulus to consist of a non-empty collection of essential curves. As we saw above, surgering $F$ along an outermost subannulus yields an essential annulus and therefore $F$ is a swallow follow torus. In both cases we conclude that the minimal genus Heegaard surface weakly reduces to a swallow follow torus. Hence by Theorem 4.1 the genus reduces by one at most, contradiction. □

Sketch of proof of Theorem 4.1. Corollary 3.2 gives the decomposition $X = X_1 \cup \hat{X}_2$ or $X = \hat{X}_1 \cup X_2$ (say the former). Recall that in Theorem 4.1 we assumed the Heegaard surface for $X$ is minimal genus. Consider the induced Heegaard surfaces on $X_1$ and $\hat{X}_2$. If the induced Heegaard surfaces on $X_1$ and $\hat{X}_2$ are not minimal genus, replacing them with a lower genus Heegaard surface and amalgamating we get a lower genus Heegaard surface for $X$, contradiction. Using these Heegaard surfaces we get (e.g. [13]):

$$(1) \quad g(X) = g(X_1) + g(\hat{X}_2) - g(T) = g(X_1) + g(\hat{X}_2) - 1$$

By Definition 3.1 $X_2$ is obtained from $\hat{X}_2$ by Dehn filling, and moreover, the core of the attached solid torus is parallel into a meridian curve on $\partial X_2$. It is therefore parallel into every Heegaard surface of
In [13] this type of filling is called *good* and it was shown there that for good fillings:

\begin{equation}
\hat{g}(X) - 1 \leq g(X_2) \leq g(\hat{X})
\end{equation}

Equations (1) and (2) complete the proof of Theorem 4.1. □

5. Morimoto’s Conjecture for $n = 2$

**Theorem 5.1.** Suppose $X_1$ and $X_2$ are m-small knots and that $g(X) < g(X_1) + g(X_2)$. Then at least one of $X_1$ or $X_2$ contains a primitive meridian.

**Remarks 5.2.**

1. Morimoto proved Theorem 5.1 for m-small knots in $S^3$ (Theorem 1.6 of [10]).

2. We again emphasize that there is no symmetry between $X_1$ and $X_2$. The the swallow follow torus found in Theorem 1.6 follows one (say $K_1$) and swallows the other ($K_2$), and as we shall see in the proof, in this case $X_2$ contains a primitive meridian. By Moriah and Rubinstein [9] and Morimoto, Sakuma and Yokota [12] there exist knots without a primitive meridian. Taking such knot as $K_1$ and connecting it with $K_2$ a knot that does have a primitive meridian (for example any 2-bridge knot, by Kobayashi [7]) we always get $X = X_1 \cup T\hat{X}_2$ and never $X = \hat{X}_1 \cup T X_2$. This is used in Section 6.

3. Note that the proof in this section holds in greater generality than stated: all we need is that $X_2$ is the exterior of a prime knot, but $X_1$ may be the exterior of a connected sum of many components. This too is used in Section 6 (as the induction hypothesis).

**Sketch of proof of Theorem 5.1.** Let $X = X_1 \ast X_2$ and let $\Sigma$ be a minimal genus Heegaard splitting for $X$. By Theorem 1.6 $\Sigma$ weakly reduces to a swallow follow torus (say $T$). Thus $X = X_1 \cup T\hat{X}_2$ or $X = \hat{X}_1 \cup T X_2$, say the former. Our goal is showing that $X_2$ contains a primitive meridian. This follows from the following three statements. In all three we assume that $X$ is the exterior of a connected sum of m-small knots; the
assumption that $X_2$ is prime is used only in Theorem 5.4 (we do not assume that $X_1$ is prime).

**Lemma 5.3.** If $g(X) < g(X_1) + g(X_2)$ then $g(X_2) = g(\hat{X}_2)$.

**Theorem 5.4.** Let $A$ be the essential annulus in $\hat{X}_2$ with one boundary component a meridian of $X_2$ and the other a longitude of $\partial\hat{X}_2 \setminus \partial X_2$.

Then there is a minimal genus Heegaard surface for $\hat{X}_2$ that intersects $A$ in exactly one essential simple closed curve.

**Remark.** Note that Theorem 5.4 implies that $\hat{X}_2$ has a minimal genus Heegaard surface that separates the boundary components. We do not know if this is always the case.

By Definition 3.1 $X_2$ is obtained from $\hat{X}_2$ by Dehn filling; therefore any Heegaard surface for $\hat{X}_2$ is a Heegaard surface for $X_2$.

**Lemma 5.5.** The Heegaard surface found in Theorem 5.4 (when considered as a Heegaard surface for $X_2$) contains a primitive meridian.

Theorem 5.1 clearly follows since by Lemma 5.3 the surface found in Theorem 5.4 is minimal genus in $X_2$. We conclude this section by indicating the proofs of these statements in the order in which they appeared:

**Proof of Lemma 5.3.** From the previous section, recall Equations (1): $g(X) = g(X_1) + g(\hat{X}_2) - 1$, and (2): $g(\hat{X}_2) - 1 \leq g(X_2) \leq g(\hat{X}_2)$. The lemma follows. □

**Sketch of proof for Theorem 5.4.** This is the most difficult part of the proof and accordingly our sketch is very rough. Let $\hat{\Sigma}_2 \subset \hat{X}_2$ be a minimal genus Heegaard surface. The proof has three cases not unlike the proof of Theorem 1.6.

**Case one:** assume $\hat{\Sigma}_2$ is strongly irreducible: Since $A$ is essential and we assumed $\Sigma$ to be strongly irreducible we can isotope $\Sigma$ to intersect $A$ is essential curves only. Denote the subannuli of $A$ cut open along $\Sigma$ by $\{A_j\}_{j=0}^k$, with $A_0$ adjacent to $\partial X_2$ and $A_k$ adjacent to $\partial \hat{X}_2 \setminus \partial X_2$. Next we fill $\partial \hat{X}_2 \setminus \partial X_2$ to obtain $X_2$, and denote the core of the attached solid torus by
\( \gamma \). Note that we can still consider \( A \) as an embedded annulus (although not properly) with one curve of \( \partial A \) a meridian of \( X_2 \) and the other \( \gamma \). We show that the equator of \( A_1 \) is a core of a 1-handle (of one of the compression bodies obtained by cutting \( X_2 \) open along \( \hat{\Sigma}_2 \)) by compressing \( \hat{\Sigma}_2 \) maximally into the side containing \( A_1 \) but only along disks disjoint from \( A_1 \). By applying Scharlemann’s No Nesting Lemma [16], we see that either the surface we obtain is a torus bounding a solid torus with the core curve of \( A_1 \) as its core, or the components of this surface cut open along \( A \) are essential. Since by cutting \( \hat{X}_2 \) along \( A \) we obtain a manifold homeomorphic to \( X_2 \), the latter contradicts \( m \)-smallness. The former proves the theorem by isotoping \( \gamma \) along \( A \) to the core of \( A_1 \) and drilling it out again.

From now on we assume \( \hat{\Sigma}_2 \) weakly reduces. We untelescope (see Section 2) \( \hat{\Sigma}_2 \) to obtain a collection of essential surfaces \( S_i \) and consider two more cases:

**Case Two:** \((\forall j) S_j \cap A = \emptyset\): let \( M \) be the component of \( \hat{X}_2 \) cut open along \( S = \bigcup_i S_i \) containing \( A \). Since the induced Heegaard splitting is strongly irreducible [18] as in Case One we can arrange for it to intersect \( A \) in a single essential curve. By amalgamating, we retrieve \( \hat{\Sigma}_2 \). We complete the proof by using an explicit description of amalgamation to verify that this can be done without introducing curves of intersection with \( A \).

**Case Three:** \((\exists j) F_j \cap A \neq \emptyset\): (this is similar to an argument of Morimoto [11]) each component of \( F_j \) cut open along \( A \) is incompressible. It is therefore either essential or a boundary compressible annulus (since for surfaces of negative Euler characteristic boundary compression implies compression). Thus \( F_j \) is constructed by pasting annuli together, and is a torus. By cutting \( \hat{X}_2 \) and \( F_j \) along \( A \) we obtain meridional incompressible annuli in \( X_2 \), that must be boundary parallel by our assumption that \( X_2 \) is a prime knot. This implies that \( F_j \) is a boundary parallel torus in \( \hat{X}_2 \). However, a surface obtained by untelescoping
a minimal genus Heegaard surface is never boundary parallel.

With contradiction we see that Case Three never occurs.

This completes our sketch the proof of Theorem 5.4. □

Proof of Lemma 5.5. This is immediate. After filling \( \hat{X}_2 \) to get \( X_2 \) the meridian of the attached solid torus extends to a meridian of the handlebody that intersects the meridional annulus \( A \) exactly once. □

This completes our sketch of the proof of Theorem 5.1. □

6. Proof of Morimoto’s Conjecture for m-small knots

The Swallow Follow Torus Theorem (1.6) and the Numerical Theorem (1.4) were proved for the connected sum of any number of m-small knots. We conclude this paper by the inductive step of the proof of Morimoto’s Conjecture:

**Theorem (1.5).** Let \( \{X_i\}_{i=1}^n \) be exteriors of m-small knots. Then:

If \( g(X) < \sum_{i=1}^n g(X_i) \) there exists some \( I \subset \{1,\ldots,n\} \) a proper subset so that \( X = \star_{i \in I} X_i \) contains a primitive meridian.

**Sketch of proof.** Let \( \Sigma \) be a minimal genus Heegaard surface for \( X \). By Theorem 1.6 \( \Sigma \) weakly reduces to a swallow follow torus \( T \) yielding the decomposition \( X_I \cup_T \hat{X}_J \) (with \( I \sqcup J = \{1,\ldots,n\}, I, J \neq \emptyset \)). We assign a complexity to a knot exterior \( Y \): the number of prime summands, denoted \( n(Y) \). To the decomposition \( X = X_I \cup_T \hat{X}_J \) we assign the complexity \( (n(X), n(X_J)) \) (ordered lexicographically). Suppose \( X = X_I \cup_T \hat{X}_J \) is a minimal complexity counterexample to Theorem 1.5.

If \( g(X) = g(X_I) + g(X_J) \) then, since \( g(X) < \sum_{i=1}^n g(X_i) \), either \( g(X_I) < \sum_{i \in I} g(X_i) \) or \( g(X_J) < \sum_{i \in J} g(X_i) \). By the minimality assumption, in the former case some subcollection of \( I \) has a primitive meridian and in the latter some subcollection of \( J \). In either case \( X \) is not a counterexample. This contradiction together with Theorem 4.1 shows:

\[
g(X) = g(X_I) + g(X_J) - 1.
\]

If \( n(X_J) = 1 \) by (3) of Remark 5.2, \( X_J \) contains a primitive meridian. So we may assume that \( n(X_J) > 1 \). By Theorem 1.6 (applied to \( X_J \))
the minimal genus Heegaard splitting of $X_J$ induced by the minimal genus Heegaard splitting of $X$ weakly reduces to a swallow follow torus in $X_J$, say $X_J = X_{J_1} \cup T_{J_2} \hat{X}_{J_2}$. If $g(X_J) < g(X_{J_1}) + g(X_{J_2})$ then (by the minimality of $(n(X), n(X_J))$ some subcollection of $X_J$ has a primitive meridian and $X$ is not a counterexample, a contradiction. We get:

$$g(X_J) = g(X_{J_1}) + g(X_{J_2}).$$

Next we consider $X_I \cup T X_{J_1}$. If $g(X_I \cup T X_{J_1}) < g(X_I) + g(X_{J_1})$ then some subcollection of $\{X_i\}_{i \in I \cup J_1}$ contains a primitive meridian (minimality again) so $X$ is not a counterexample, a contradiction. We get:

$$g(X_I \cup T X_{J_1}) = g(X_I) + g(X_{J_1}).$$

Since the Heegaard splitting induced by $\Sigma$ on $X_J$ weakly reduces to $T_J$, $\Sigma$ also weakly reduces to $T_J$. This gives $(X_I \cup T \hat{X}_{J_1}) \cup T_{J_2} \hat{X}_{J_2}$. The equations above imply that $g(X) = g(X_I \cup T \hat{X}_{J_1}) + g(X_{J_2}) - 1$. By the minimality, $(X, X_{J_2})$ is not a counterexample, and we conclude that $X$ itself is not a counterexample at all. With this contradiction we complete the proof. □

**References**

[1] A. J. Casson and C. McA. Gordon, *Reducing Heegaard splittings*, Topology Appl. 27 (1987), no. 3, 275–283. MR 89c:57020

[2] A. E. Hatcher, *On the boundary curves of incompressible surfaces*, Pacific J. Math. 99 (1982), no. 2, 373–377. MR 83h:57016

[3] John Hempel, *3-Manifolds*, Princeton University Press, Princeton, N. J., 1976, Ann. of Math. Studies, No. 86. MR 54 #3702

[4] William Jaco, *Lectures on three-manifold topology*, CBMS Regional Conference Series in Mathematics, vol. 43, American Mathematical Society, Providence, R.I., 1980. MR 81k:57009

[5] Tsuyoshi Kobayashi, *Scharlemann-Thompson untelescoping of Heegaard splittings is finer than Casson-Gordon’s*, electronic preprint [arXiv:math.GT/0205121].

[6] ______, *A construction of arbitrarily high degeneration of tunnel numbers of knots under connected sum*, J. Knot Theory Ramifications 3 (1994), no. 2, 179–186. MR 95g:57011
[7] ______, Heegaard splittings of exteriors of two bridge knots, Geom. Topol. 5 (2001), 609–650 (electronic). MR 2002k:57013
[8] Yoav Moriah, Connected sums of knots and weakly reducible heegaard splittings, electronic preprint [math.GT/9912171], 1999.
[9] Yoav Moriah and Hyam Rubinstein, Heegaard structures of negatively curved 3-manifolds, Comm. Anal. Geom. 5 (1997), no. 3, 375–412. MR 98j:57029
[10] Kanji Morimoto, On the super additivity of tunnel number of knots, Math. Ann. 317 (2000), no. 3, 489–508. MR 2001g:57016
[11] ______, Tunnel number, connected sum and meridional essential surfaces, Topology 39 (2000), no. 3, 469–485. MR 2001a:57015
[12] Kanji Morimoto, Makoto Sakuma, and Yoshiyuki Yokota, Examples of tunnel number one knots which have the property “1+1 = 3,” Math. Proc. Cambridge Philos. Soc. 119 (1996), no. 1, 113–118. MR 96i:57007
[13] Yo’av Rieck, Heegaard structures of manifolds in the Dehn filling space, Topology 39 (2000), no. 3, 619–641. MR 2001b:57037
[14] Yo’av Rieck and Eric Sedgwick, Finiteness results for Heegaard surfaces in surgered manifolds, Comm. Anal. Geom. 9 (2001), no. 2, 351–367. MR 2002j:57040
[15] ______, Persistence of Heegaard structures under Dehn filling, Topology Appl. 109 (2001), no. 1, 41–53. MR 2001k:57021
[16] Martin Scharlemann, Local detection of strongly irreducible Heegaard splittings, Topology Appl. 90 (1998), no. 1-3, 135–147. MR 99h:57040
[17] Martin Scharlemann and Abigail Thompson, Heegaard splittings of (surface) × I are standard, Math. Ann. 295 (1993), no. 3, 549–564. MR 94b:57020
[18] ______, Thin position for 3-manifolds, Geometric topology (Haifa, 1992), Contemp. Math., vol. 164, Amer. Math. Soc., Providence, RI, 1994, pp. 231–238. MR 95e:57032
[19] Jennifer Schultens, Heegaard splittings of Seifert fibered spaces with boundary, Trans. Amer. Math. Soc. 347 (1995), no. 7, 2533–2552. MR 95j:57019
[20] Eric Sedgwick, Genus two 3-manifolds are built from handle number one pieces, Algebr. Geom. Topol. 1 (2001), 763–790 (electronic). MR 2002k:57051

DEPARTMENT OF MATHEMATICS, NARA WOMEN’S UNIVERSITY KITAUOYA NISHIMACHI, NARA 630-8506, JAPAN

DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF ARKANSAS, FAYETTEVILLE, AR 72701

E-mail address: tsuyoshi@cc.nara-wu.ac.jp
E-mail address: yoav@uark.edu