Phenomenology of a Higgs triplet model at future $e^+e^-$ colliders

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In this work, we investigate the prospects of future $e^+e^-$ colliders in testing a Higgs triplet model with a scalar triplet and a scalar singlet under $SU(2)$. The parameters of the model are fixed so that the lightest $CP$-even state corresponds to the Higgs particle observed at the LHC at around 125 GeV. This study investigates if the second heaviest $CP$-even, the heaviest $CP$-odd and the singly charged states can be observed at existing and future colliders by computing their accessible production and decay channels. In general, the LHC is not well equipped to produce a Higgs boson which is not mainly doublet-like, so we turn our focus to lepton colliders. We find distinctive features of this model in cases when the second heaviest $CP$-even Higgs is triplet-like, singlet-like or a mixture. These features could distinguish the model from other scenarios at future $e^+e^-$ colliders.

I. INTRODUCTION

The discovery of the Higgs boson at the LHC$^{1,2}$ confirms the particle content of the Standard Model (SM) of particle physics. Still one of the main beyond the SM puzzles remains neutrino mass generation. Several extensions to the SM Higgs sector that give a mass term to neutrinos involve the spontaneous violation of lepton number via the vacuum expectation value of an $SU(2)$ singlet (for a review, see Ref. $^3$). A common feature of these models is the presence of a massless goldstone boson, the Majoron $J$.

We investigate the phenomenology of a Higgs triplet model (HTM) of the kind mentioned above that has a scalar singlet and a scalar triplet under $SU(2)$, in addition to a $SU(2)$ scalar doublet. The model was originally proposed in $^4$, where the authors defined it as the “123” HTM. Once the triplet field acquires a vacuum expectation value (vev), a neutrino mass term is generated. The parameters in the neutrino sector include the vev of the triplet and the Yukawa couplings between the two-component fermion $SU(2)$ doublet, including charged leptons and majorana neutrinos, and the triplet field. In this work, we study the collider phenomenology of the “123” model, which is almost decoupled from its neutrino sector $^5$. This is why we don’t discuss experimental constrains on neutrino masses and mixing angles, which are beyond the scope of this paper and which we leave for a future work. Models in which neutrino masses arise from the interaction with a triplet field have also been discussed extensively in the literature $^6$.$^{10}$

The phenomenology of “123” models was studied before in $^{11,12}$, paying particular attention to the consistency of the presence of the Majoron with experimental data. The Majoron is mainly singlet in this model, so its interaction with gauge bosons such as the $Z$ is negligible, making its existence fully consistent with collider data. This is in contrast to what happens in models with spontaneous violation of lepton number without the singlet field $^8$, which are excluded.

A characteristic signature of models with Higgs triplets is the existence of a doubly charged scalar ($\Delta^{±±}$), in addition to the existence of a tree-level $H^{±}W^{±}Z$ vertex, where $H^{±}$ is a singly charged Higgs $^6$. The LHC collider phenomenology of a doubly charged scalar in Higgs triplet models (in particular the “23” HTM, without the singlet field) has been discussed in $^8$.$^{14}$. Production of doubly charged scalars at $e^+e^-$ colliders has also been studied in the literature as probes of Higgs triplet models $^{13}$, the Georgi–Machacek model $^{16}$ and left-right symmetric models $^{17}$, which have a similar phenomenology.

The phenomenology of the neutral scalar sector in Higgs triplet models has been less studied than the charged sector. Production and decays of the neutral Higgs bosons in the “23” HTM, was studied in $^{18,19}$. Associated production of the charged and neutral Higgs at the ILC was studied in $^{20,21}$. In particular for the “123” HTM of interest in this paper, only discovery prospects at colliders were discussed in $^{11}$ and a fermiophobic Higgs was studied in $^{12}$.

The collider phenomenology of neutral and singly charged Higgs bosons in the HTM has received much less attention in the literature than the doubly charged Higgs. In addition, the phenomenology of the doubly charged Higgs depends directly on neutrino physics we are not evaluating at this time (as noticed earlier), so we focus on the neutral sector and singly charged Higgs of the “123” HTM.

In this paper, we study the production and decay of the next to heaviest neutral $CP$-even Higgs $h_2$, the $CP$-odd Higgs $A$ and the singly charged Higgs $H^{±}$ of the “123” HTM. We extend the work in Refs. $^{11,12}$ by identifying the lightest state in the $CP$-even neutral sector, $h_1$, as the SM-like Higgs discovered at the LHC. This rules out the fermiophobic SM-like Higgs boson sce-
nario described in [11]. Constrains are imposed on the parameter space of the model in order to retain the SM-like Higgs properties. In particular, we define $h_1$ to be mainly doublet and fix its mass to be $m_{h_1} \approx 125$ GeV. We also identify the necessary constrains on the parameters of the scalar potential to suppress its decays to Majorons, so that its invisible decay width is negligible.

We identify three characteristic benchmarks of the model related to the composition of $h_2$. $h_2$ can be mainly singlet, mainly triplet or a mixture. Note that $h_2$ can not be mainly a doublet since this is reserved for the SM like Higgs boson. We compute production cross-sections and decays in these three benchmarks. We find that the main 2-body production mode for $h_2$ is associated production with a $CP$–odd state $A$ and note that cross-sections are in general larger when $A$ is produced on-shell. Production of $A$ may be observable at CLIC when produced in association with an $h_2$ or $h_3$ (the heaviest $CP$–even Higgs), depending on the benchmark. The singly charged Higgs boson $H^\mp$ is potentially observable at CLIC when produced in association with another $H^\pm$. Decay rates of $h_2$ to fermions are suppressed. Invisible decays of $h_2$ to Majorons can be very important, depending on the benchmark. Decays of $A \to h_i Z$, with $i = 1, 2$ or $A \to h_1 \bar{t} t_\ell$ dominate, depending on the benchmark. The decays of $H^\pm \to h_1 W^\pm$ dominate in all three benchmarks.

The paper is organized as follows. In Section II we introduce the model under study. Section III describes our restrictions and scan over the parameter space. In Section IV we comment on the low production cross-section of the new heavy Higgs of this model at the LHC. Section V describes production of $h_2$, $A$ and $H^\pm$ at future $e^+e^-$ colliders, while in Section VI we comment on the decay phenomenology of the model. We briefly comment on the most promising channels for discovery in Section VII. After a summary and conclusions in Section VIII we define the relevant Feynman rules in Appendix B for easy reference by the reader.

\[ M^2_\chi = \begin{bmatrix} 2\beta_1 v^2 + \frac{1}{2}\kappa v^2 \omega \sigma & \beta_2 v \phi v_\sigma - \kappa v \phi v_\Delta & \beta_3 v \phi v_\sigma - \frac{1}{2}\kappa v^2 \phi \sigma \\ \beta_2 v \phi v_\sigma - \kappa v \phi v_\Delta & 2\lambda_1 v^2 \phi & 2\lambda_1 v^2 \phi + \frac{1}{2} \kappa v^2 \phi \sigma \\ \beta_3 v \phi v_\sigma - \frac{1}{2}\kappa v^2 \phi \sigma & (\lambda_2 + \lambda_5) v_\phi v_\Delta - \kappa v_\phi v_\sigma & 2(\lambda_2 + \lambda_4) v_\phi v_\Delta + \frac{1}{2} \kappa v^2 \phi \sigma \end{bmatrix}. \]  

By diagonalizing this matrix with $O_\chi M^2_\chi O_\chi^T = \text{diag}(m^2_{h_1}, m^2_{h_2}, m^2_{h_3})$, one obtains the masses of the neutral scalar fields $h_1$, $h_2$, and $h_3$. The fields are such that $O_\chi [\chi_\sigma, \chi_\phi, \chi_\Delta]^T = [h_1, h_2, h_3]^T$. We assume that the lightest of them is the Higgs boson discovered in 2012 [1, 2], with mass $m_{h_1} \approx 125$ GeV [22]. In the present article we concentrate on the phenomenology of the second $CP$–even Higgs boson $h_2$, the massive $CP$–odd Higgs boson $A$, and the charged Higgs boson $H^\pm$, in consistency with the SM-like higgs found at the LHC being $h_1$ in the “123” model.

The pseudoscalar fields $\varphi_\sigma$, $\varphi_\phi$, and $\varphi_\Delta$ mix due to the mass matrix $M^2_\varphi$. The term in the Lagrangian has the form

\[ V(\varphi, \Delta, \phi) = \mu_3^2 \sigma^2 + \mu_2^2 \phi^2 + \mu_1^2 \Delta^2 + \lambda_1 \phi^2 + \lambda_2 \sigma^2 + \lambda_3 \Delta^2 + \lambda_4 \sigma \Delta \phi + \lambda_5 \phi \Delta \phi + \nabla^2 (\sigma^2 + \Delta^2) + \nabla^2 (\phi^2 + \Delta^2). \]
form \( \frac{1}{2}[\varphi^c, \varphi, \varphi]\) \(M^2_φ [\varphi^c, \varphi, \varphi]\) with

\[
M^2_φ = \begin{bmatrix}
\frac{1}{2} \kappa v^2 \phi + \kappa v^2 \Delta & \kappa v^2 \Delta & \frac{1}{2} \kappa v^2 \\
\kappa v^2 \Delta & 2 \kappa v^2 \Delta & \kappa v^2 \Delta \\
\kappa v^2 \Delta & \kappa v^2 \Delta & \kappa v^2 \Delta
\end{bmatrix}.
\] (4)

By inspection, we know that there are two null eigenvalues, since two rows are linearly dependent of the third. The mass matrix is diagonalized by another rotation given by \(O_φ M^2_φ O^T_φ = \text{diag}(m^2_φ, m^2_J, m^2_A)\), where \(G^0\) is the massless nonphysical neutral Goldstone boson and \(J\) is the massless physical Majoron. \(A\) is the massive pseudoscalar, and \(O_φ [\varphi, \varphi, \varphi \Delta] = [G^0, J, A]^T\) is satisfied.

\[
M^2_+ = \begin{bmatrix}
-\frac{1}{2} \lambda_5 v^2_\Delta + \kappa v^2 \Delta & \kappa v^2 \Delta & \frac{1}{2} \kappa v^2 \\
\kappa v^2 \Delta & \frac{1}{2} \lambda_5 v^2_\Delta & \kappa v^2 \Delta \\
\kappa v^2 \Delta & \kappa v^2 \Delta & \frac{1}{2} \lambda_5 v^2_\Delta
\end{bmatrix}
\] (6)

which is diagonalized by a rotation given by \(O_+ M^2_+ O^T_+ = \text{diag}(m^2_{H^+}, m^2_{H^+})\). As in the previous case, by inspection this mass matrix has a null eigenvalue corresponding to the charged Goldstone boson. The mass eigenstate fields satisfy \(O_+ [\varphi^-, \Delta^+] = [G^+, H^+]^T\). The charged Higgs mass is,

\[
m^2_{H^\pm} = \frac{1}{2} \left( \frac{\kappa v^2_\Delta}{v^2 - \frac{1}{2} \lambda_5} \right) \left( v^2_\Delta + 2 v^2 \right).
\] (7)

Finally, the doubly charged boson \(\Delta^{++}\) mass is given by

\[
m^2_{\Delta^{++}} = -\lambda_4 v^2_\Delta - \frac{1}{2} \lambda_5 v^2_\Delta + \frac{1}{2} \kappa v^2 \Delta.
\] (8)

since it does not mix (it is purely triplet).

### III. Restrictions on the Parameter Space

In this Section we explain our restrictions on the model parameters. We first comment that the invisible decay width of the \(Z\) gauge boson in our model is suppressed since the Majoron \(J\) is mostly singlet \((O^2 \approx 1)\). We define \(\Gamma^{12}_{\text{inv}}\) as the decay width of the \(Z\) into undetected particles excluding the decay into neutrinos, \(Z \to \nu\nu\). Experimentally, \(\Gamma^{12}_{\text{inv}} < 2\text{ MeV at 95\% CL}\). and in our model there could be a contribution from the mode \(Z \to JZ^* \to J\nu\nu\). This contribution is automatically suppressed because the Majoron is mainly singlet (see Appendix A).

The pseudoscalar \(A\) has a mass,

\[
m^2_A = \frac{1}{2} \kappa \left( \frac{v^2_\Delta}{v^2_\sigma} + \frac{v_\Delta v^2_\sigma}{v^2_\sigma} + 4 v_\sigma v^2_\Delta \right).
\] (5)

A value of \(\kappa\) different from zero is necessary to have a massive pseudoscalar \(A\). For experimental reasons, we would like to take the massless Majoron as mainly singlet in order to comply with the well measured \(Z\) boson invisible width \([23, 24]\). Nevertheless, in the “123” model imposing this is unnecessary because the Majoron results mostly singlet as long as the triplet vev is small (see Appendix A). The Majoron can acquire a small mass via different possible mechanisms \([23]\). In cases where this particle has a small mass, it can be a candidate for Dark Matter \([26]\).

We mention also the electrically charged scalars. The singly charged bosons \(\varphi^-\) and \(\Delta^+\) mix to form the term in the Lagrangian \([\varphi^-, \Delta^+]M^2_+ [\varphi^-, \Delta^+]^T\), with

Also, this model includes three \(CP\)-even Higgs bosons. We assume that the lightest of them is SM-like, and therefore fits with the experimental results. That is, we assume its mass is near 125 GeV, that it is mainly doublet \((O_{\chi}^{12} \approx 1)\), and that its invisible decay width is negligible \([27]\). This last condition is obtained if we suppress the \(h_1\) coupling to Majorons taking \(|\beta_2| < 0.05\).

The constraints we implement are:

- \(|O^{21}_φ| \geq 0.95\ (J\text{ mainly singlet})
- The \(\rho\) parameter is also very well measured: \(\rho = 1.00037 \pm 0.00023\). In this model it is

\[
\rho = 1 - \frac{2 v^2_\Delta}{v^2 + 4 v^2_\Delta}.
\] (9)

This restricts the value of \(v_\Delta\) to be smaller than a few GeV. Nevertheless, we consider \(v_\Delta < 0.35\) GeV as in Ref. \([11]\) in order to satisfy astrophysics bounds.

- \(m_{h_1} = 125.09 \pm 0.24\ \text{GeV}\)
- \(|O^{12} \chi| \geq 0.95\ (h_1\text{ mainly doublet})
- \(|\beta_2| \leq 0.05\ (\text{small} \ h_1\text{ invisible decay})
- \(m_{H^\pm} > 80\ \text{GeV}\)

We make a general scan where we vary all the independent parameters. We generate their values randomly from uniform distributions. We do our scan with positive values of \(\lambda_1\), \(\beta_1\) and \(\kappa\), as negative values of these
parameters typically result in negative eigenvalues of the mass matrix in eq. (3). The window for \( v_2 \) is reduced because of its dependency with the masses of the \( W \) and \( Z \) bosons [12]. Considering the range of \( v_2 \) and \( v_3 \), the scanned range for \( \lambda_1 \) is mostly fixed due to its strong dependency with \( m_{h_1} \approx 125 \text{ GeV} \), and also because of the small effects of the mixings with other \( CP^- \)even scalars (see eq. (3)). Terms outside of the mass matrix diagonal are generally much smaller than those on the diagonal, making the terms in the diagonal leading almost directly to the masses of \( h_1, h_2 \) and \( h_3 \). The scanned range for \( \beta_2 \) is forced to be small to avoid a large \( h_1 \) invisible decay (see Section VII A).

After imposing our constraints we note a clear hierarchy where \( v_\sigma \gg v_\phi \gg v_\Delta \) that we have partially imposed: \( v_\Delta \) is small in order to account for the measured \( p \) parameter, and \( v_\phi \approx 246 \text{ GeV} \) to account for the Higgs mass. With that, a large value for \( v_\sigma \) comes naturally.

We find a small effect from our filters in \( \lambda_2, \lambda_3, \lambda_4, \lambda_5 \) and \( \beta_3 \). We note that the value of \( \kappa \) cannot be zero because in that case the \( CP^- \)odd Higgs \( A \) would be massless, and since it is mostly triplet that would contradict the measurements for the invisible decay of the \( Z \) boson. Its value cannot be large neither because mixing in the \( CP^- \)even sector would move \( h_1 \) away from the mostly doublet-like scenario (a SM-like Higgs boson). After the scan and imposing the filters we can see the distribution of the physical masses in our model. This is shown in Fig. [H] where the thick black line shows the distribution before cuts to appreciate their effect. The most distinctive feature is that we impose the lightest scalar mass to be \( m_{h_1} \approx 125 \text{ GeV} \). All the other masses are free. The model allows for heavier scalars considering that we still have room for large parameters.

We highlight that the Majoron is massless in this model and is naturally mainly singlet, as can be inferred from eq. (A.5), which is related to the exact diagonalization of the \( CP^- \)odd mass matrix shown in Appendix A. Also notice that the new scalar states have the tendency to be heavy, with extreme values for the masses obtained for high values of the parameters. The shape of the distributions in Fig. [H] of course depends on using a linear generation of random values, which highlights large masses. Anyhow, we consider this to be an argument against colliders with small values for the centre of mass (CM) energy.

There is also an ambiguity related to the composition of the \( h_2 \) field: it can be mainly singlet, mainly triplet, or anything in between, as long as it is not mainly doublet, which is reserved for \( h_1 \), our SM-like Higgs boson. If \( h_2 \) is mainly triplet its mass tends to be similar to the masses of \( A, H^\pm \), and \( \Delta^{++} \) (all these fields are mainly triplet). If \( h_2 \) is mainly singlet, the mass of \( h_3 \) tends to be equal to the masses of \( A, H^\pm \), and \( \Delta^{++} \), and in this case, a mainly-singlet \( h_2 \) can be lighter. The masses of \( h_2 \) and \( h_3 \) are strongly correlated with the values of \( (M_\lambda)_{11} \) and \( (M_\lambda)_{33} \) depending on which is mainly singlet or triplet. Obtaining a scenario where \( h_2 \) and \( h_3 \) are not purely singlet or triplet requires \( (M_\lambda)_{11}^2 \) numerically very close to \( (M_\lambda)_{33}^2 \), making that scenario highly fine-tuned.

The splitting between the mainly triplet fields is controlled by \( |\lambda_5| \). This can be algebraically understood starting from the hierarchy \( v_\Delta \ll v_\phi, v_\sigma \) and approximating eq. (5):

\[
m_\lambda^2 \approx \frac{1}{2} \kappa v_\sigma v_\phi^2 \left( \frac{v_\Delta}{v_\phi} \right) \quad \text{(10)}
\]

Using the same approximation in eqs. (7) and (8), we get for the singly and doubly charged Higgs masses,

\[
m_{H^\pm}^2 \approx m_\lambda^2 - \frac{1}{4} \lambda_5 v_\phi^2 \\

m_{\Delta^{++}}^2 \approx m_\lambda^2 - \frac{1}{2} \lambda_5 v_\phi^2 \approx m_{H^\pm}^2 - \frac{1}{4} \lambda_5 v_\phi^2. \quad \text{(11)}
\]

Thus, \( H^\pm, \Delta^{++} \) and \( A \) can differ appreciably in mass as long as \( |\lambda_5| \) is large.

The previous considerations motivate us to define three benchmarks, characterized by the composition of \( h_2 \) in Table I. The parameters for each benchmark are defined in Table [H]. Note that these are chosen thinking of \( e^+e^- \) colliders, given the masses below 1 TeV.

We stress the fact that there is an ambiguity in the composition of \( h_2 \). By definition \( h_1 \) is mainly doublet. The \( H^\pm \) and \( \Delta^{++} \) fields are always mainly triplet. The \( A \) field is also always mainly triplet because \( J \) is mainly singlet. The composition of \( h_3 \) is complementary to the composition of \( h_2 \).

Table [H] shows the physical masses obtained for the three benchmarks. In B1 \( h_2 \) is mainly triplet, thus it has a mass similar to \( A, H^\pm \), and \( \Delta^{++} \) masses, with \( h_3 \) heavier. In B2 \( h_2 \) is mainly singlet, thus it is \( h_3 \) that has a mass similar to the masses of \( A, H^\pm \), and \( \Delta^{++} \), with \( h_2 \) lighter.

### IV. PRODUCTION AT THE LHC

Here we briefly comment on the production cross-section at the LHC for the scalars \( h_2, A \) and \( H^\pm \) for our model benchmarks (which we choose thinking of \( e^+e^- \) colliders). We implement the “123” HTM in FeynRules [28] and interface the output to the MadGraph5 [29] event generator to compute production cross-sections.

When thinking of a SM-like Higgs boson (such as \( h_1 \) in our model), the main production mode at the LHC is gluon-gluon fusion (ggF).
FIG. 1. Distribution of the physical masses in the general scan. Parameters are varied as in Table III.

| Parameter  | Scanned Range | B1 | B2 | B3 | Units |
|------------|---------------|----|----|----|-------|
| $v_\sigma$ | $[0, 5000]$   | 1500 | 3300 | 2500 | GeV |
| $v_\phi$   | $[245, 247]$  | 246 | 246 | 246 | GeV |
| $v_\Delta$ | $[0, 0.35]$   | 0.2 | 0.2 | 0.3 | GeV |
| $\lambda_1$ | $[0.127, 0.15]$ | 0.13 | 0.13 | 0.13 | - |
| $\lambda_2$ | $[-4, 4]$     | 0.1 | 0.1 | 0.1 | - |
| $\lambda_3$ | $[-4, 4]$     | 0.1 | 0.1 | 0.1 | - |
| $\lambda_4$ | $[-4, 4]$     | 0.1 | 0.1 | 0.1 | - |
| $\lambda_5$ | $[-4, 4]$     | 1.0 | 0.5 | 0.8 | - |
| $\beta_1$  | $[0, 4]$      | 0.3 | 0.02 | 0.008 | - |
| $\beta_2$  | $[-0.05, 0.05]$ | 0.02 | 0.005 | 0 | - |
| $\beta_3$  | $[-4, 4]$     | 0.1 | 0.5 | 0.6 | - |
| $\kappa$   | $[0, 1]$      | 0.001 | 0.0015 | 0.0004 | - |

| Parameter  | | | | |
|------------| | | | |

TABLE III. Physical masses in GeV for the different benchmarks.

| Parameter  | B1 | B2 | B3 |
|------------|----|----|----|
| $m_{h_1}$  | 125 | 125 | 125 |
| $m_{h_2}$  | 476 | 660 | 316 |
| $m_{h_3}$  | 1162 | 865 | 318 |
| $m_{H^+}$  | 443 | 857 | 277 |
| $m_{\Delta^{++}}$ | 460 | 861 | 298 |
This process dominates SM-like Higgs production not only because the $ht$ coupling is large, but also because the parton distribution functions indicate that it is easier to find a gluon inside the proton than a heavy quark or an electroweak gauge boson.

Nevertheless, this mechanism is not be efficient for a not-mainly-doublet Higgs boson (which is the case for $h_2$ and $A$ in our model benchmarks), because that Higgs couples to quarks very weakly. In the model studied here, the ratio of production cross-sections in the gluon-gluon fusion mode for $h_1$ and $h_2$ is,

$$\frac{\sigma(ggF, h_2)}{\sigma(ggF, h_1, m_{h_1} = m_{h_2})} = \left(\frac{O_X^{22}}{O_X^{12}}\right)^2 \approx (O_X^{22})^2. \quad (12)$$

The last approximation is valid because we have $h_1$ mainly doublet (SM-like). The production cross-section at $\sqrt{s} = 14$ TeV for $h_2$ reaches $5.7 \times 10^{-6}$ pb in B1, $5.7 \times 10^{-5}$ pb in B2 and $3.9 \times 10^{-6}$ pb in B3. For $A$ production, the above ratio is proportional to $(O_X^{22})^2$ and we get similar numbers. The cross-section at $\sqrt{s} = 14$ TeV reaches $6.8 \times 10^{-6}$ pb in B1, $4.0 \times 10^{-7}$ pb in B2 and is somewhat higher in B3, reaching $2.5 \times 10^{-5}$ pb. So we conclude that the above ratio is around $10^{-4}$ at most. This is why, if the model is correct, we may have not seen $h_2$ (nor $A$) at the LHC via $ggF$, as is not a dominant production mode since $h_2$ does not behave like a SM-like Higgs.

Other production mechanisms that can be relevant at the LHC are electroweak modes, for example vector boson fusion (VBF), but they also produce small cross-sections for our given benchmarks. When considering the sum over all VBF processes like the diagram below, the highest cross-section at $\sqrt{s} = 14$ TeV we get is $2.5 \times 10^{-5}$ pb for the charged Higgs production,

$$q \xrightarrow{Z} q, \quad q \xrightarrow{W^+} q, \quad q \xrightarrow{W^+} h_2$$

in B3. Production processes via quark anti-quark annihilation can also be relevant. In the case of $h_2$ production, the highest contribution comes from the diagram for B1 and B3. The cross-section at $\sqrt{s} = 14$ TeV for B1 is $4.5 \times 10^{-4}$ pb. Production of $A$ at $\sqrt{s} = 14$ TeV dominates in B1 when in the above diagram we replace $h_2$ with $A$, $W^+$ with a $Z$, $h_1$ also with a $Z$ and $H^+$ with $h_2$, leading to the AZZ final state. This gives a cross-section of $3.7 \times 10^{-4}$ pb. It can go higher in B3 in the $AJJ$ final state, with a cross-section reaching $2.3 \times 10^{-3}$ pb. Charged Higgs production at $\sqrt{s} = 14$ TeV can reach $4.3 \times 10^{-3}$ pb in B3 in the $H^+W^-W^-$ final state (replacing $W^+$ with $h_1$ with $W^-$, $H^+$ with $\Delta$ and $h_2$ with $H^+$ in the above diagram).

The highest cross-section found in our model benchmarks for each characteristic production mechanism at the LHC is summarized in Table IV for comparison.

**TABLE IV.** Highest LHC production cross-section (in units of pb) found in our benchmarks for $h_2$, $A$ and $H^\pm$ at $\sqrt{s} = 14$ TeV via the three characteristic production mechanisms: $ggF$, $VBF$ and $q\bar{q}$ annihilation.

| $\sigma$ | $h_2$ | $A$ | $H^\pm$ |
|--------|-------|-----|--------|
| $ggF$  | $5.7 \times 10^{-6}$ (B2) | $2.5 \times 10^{-5}$ (B3) | - |
| $VBF$  | $4.4 \times 10^{-6}$ (B3) | $2.2 \times 10^{-5}$ (B1) | $2.5 \times 10^{-5}$ (B3) |
| $q\bar{q}$ | $4.5 \times 10^{-4}$ (B1) | $2.3 \times 10^{-3}$ (B3) | $4.3 \times 10^{-3}$ (B3) |

To finish, not even the HL-LHC will help, because it is expected to have a factor of 10 increase in luminosity, and it will not compensate the smallness of the production cross-section.

In summary, it seems that hadron colliders are not well equipped to produce the new states $h_2$, $A$ and $H^\pm$. Production for $h_2$ and $A$ via $ggF$ at the LHC is not efficient since these Higgs bosons are not-mainly doublet. Productions for $h_2$ and $A$ and $H^\pm$ via VBF can be only as large as $\sim 10^{-5}$ pb for our benchmarks. Electroweak production via quark anti-quark annihilation can be as high as $\sim 10^{-3}$ pb. Given that our benchmarks are not likely to be observed at the LHC (a dedicated analysis is needed to confirm this), the large hadronic background at the LHC and the advantage of a cleaner collider environment at lepton colliders, we focus on the production for these states at future electron-positron colliders.
V. PRODUCTION AT $e^+e^-$ COLLIDERS

In order to assess the discovery potential of the model, we implement it in FeynRules [38], so we can extract relevant parameters and Feynman rules. We then interface the output to the MadGraph5 [23] event generator in order to compute production cross-sections, as we did in the previous section.

The FCC-ee machine is a hypothetical circular $e^+e^-$ collider at CERN with a high luminosity but low energy, designed to study with precision the Higgs boson [31]. We consider its highest projected energy 350 GeV with a luminosity of 2.6 ab$^{-1}$, which was calculated by taking the 0.13 ab$^{-1}$ quoted in [31] and assuming 4 interaction points and 5 years of running of the experiment.

The canonical program for the ILC [32] includes three CM energies given by 250 GeV, 500 GeV, and 1000 GeV, with integrated luminosities 250 fb$^{-1}$, 500 fb$^{-1}$ and 1000 fb$^{-1}$, respectively. CLIC [33] has three operating CM energies: $\sqrt{s} = 350$ GeV, 1.4 TeV and 3 TeV, with estimated luminosities 500 fb$^{-1}$, 1.5 ab$^{-1}$ and 2 ab$^{-1}$, respectively. Based on this, we compute $e^+e^-$ production cross-sections for $h_2$, $A$ and $H^+$ for all 3 benchmarks. In B1 (left frame) this particle is potentially observed at CLIC only when the $A$ scalar is also on-shell. Thus, the main 2-body production mode is the so-called associated production,

\begin{equation}
\gamma_2 \rightarrow h_2X, \quad \text{where } X \text{ is a particle that does not decay.}
\end{equation}

The production cross-sections shown in Table V are dominated by the 2-body production process (or mode) $e^+e^- \rightarrow h_2A$ and by 3-body production processes as follows. In B1 the process $e^+e^- \rightarrow h_2\bar{t}\bar{t}$ is the most important one. In B2 the dominating process is $e^+e^- \rightarrow h_2Ah_1$. In B3 the process $e^+e^- \rightarrow h_2Zh_3$ is the dominant one. All of them are enhanced when a second heavy particle is also on-shell. We show in Fig. 2 the main $h_2$ production modes for all 3 benchmarks. In B1 (left frame) this particle is potentially observed at CLIC only when the $A$ scalar is also on-shell. Thus, the main 2-body production mode is the so-called associated production,

\begin{equation}
\gamma_2 \rightarrow h_2X, \quad \text{where } X \text{ is a particle that does not decay.}
\end{equation}

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TABLE V. Production cross-section (in units of ab) for $h_2$ at an $e^+e^-$ collider for projected energies in the 3 benchmarks. Estimated luminosities are also given in units of ab$^{-1}$.

| $\sqrt{s}$ [TeV] | $\mathcal{L}_{FCCee}$ | $\mathcal{L}_{ILC}$ | $\mathcal{L}_{CLIC}$ | B1: $\sigma$ | B2: $\sigma$ | B3: $\sigma$ |
|------------------|-----------------------|---------------------|---------------------|--------------|--------------|--------------|
| 0.250            | -                      | 0.25                | -                   | 0            | 0            | 0            |
| 0.350            | 2.6                    | -                   | 0.5                | $3.1 \times 10^{-6}$ | 0            | $1.7 \times 10^{-5}$ |
| 0.500            | -                      | 0.5                 | -                   | $1.4 \times 10^3$ | 0.9          | $2.5 \times 10^{-2}$ |
| 1.0              | -                      | 1                   | -                   | $1.1 \times 10^4$ | 3.6          | $3.7 \times 10^3$ |
| 1.4              | -                      | -                   | 1.5                | $6.1 \times 10^3$ | $3.5 \times 10^{-2}$ | $4.1 \times 10^3$ |
| 3                | -                      | -                   | 2                  | $8.2 \times 10^3$ | $2.0 \times 10^3$ |

FIG. 2. Production modes for $h_2$ at an $e^+e^-$ collider in the 3 benchmarks. The legend shows the final state after the $e^+e^-$ collision.

plus Feynman diagrams where in the last sub-process we replace $(A, J)$ by $Z$ and/or interchange $h_1$ with $h_2$. This mode is enhanced when $h_3$ is on-shell, since in B2 $h_3$ is mainly triplet and the coupling $ZAh_3$ is large resulting in $e^+e^- \rightarrow h_3A \rightarrow h_2h_1A$.

B3 is an intermediate situation. Even in this case, $h_2$...
production cross-sections are potentially observable when $A$ is also on-shell. The production cross-section $e^+e^-\rightarrow h_2A$ is smaller than in B1, but still large. The main 3-body production mode in this case is $e^+e^-\rightarrow h_2Zh_1$, with sub-processes given by, a SM-like Higgs boson in an $e^+e^-$ collider, known collectively as vector boson fusion, $e^+e^\rightarrow h_2e^+e^-$ (fusion of two $Z$ bosons) or $e^+e^-\rightarrow h_3\nu_e\bar{\nu}_e$ (fusion of two $W$ bosons) do not work in our case because the $h_2$ couplings to vector bosons are suppressed by the triplet vev $v_3$. In addition, most of the charged leptons go through the beam pipe, thus $\sigma(e^+e^-\rightarrow h_2e^+e^-)$ is further penalized when a cut on the charged lepton pseudo-rapidity is imposed. We use MADGRAPH5 default cuts, which impose that the absolute value of the charged lepton pseudo-rapidity is smaller than 2.5.

#### B. $A$ Production

Table VII shows $A$ production at $e^+e^-$ colliders, prospected luminosities and CM energies for the FCC-ee, ILC and CLIC colliders. The cross-sections are calculated in the same manner explained before. In B1 and B2 the dominating process is $e^+e^-\rightarrow AZZ$, and in B3 the dominating process is $e^+e^-\rightarrow AJJ$, and all of them are enhanced when a second heavy particle is also on-shell.

Fig. 4 shows the production cross-sections for an $A$ boson. In B1 (left frame) $A$ is potentially observable at CLIC when produced in association with an $h_2$. In this case the mode $e^+e^-\rightarrow Ah_1$ is suppressed because $O^{ij}_{e}$ and $O^{ij}_{W}$ are both small (see Feynman rule in Appendix A), thus the coupling $h_1AZ$ itself is suppressed with respect to $g$. Three body production modes are also in Fig. 4. The dominant 3-body production mode in B1 is $e^+e^-\rightarrow AZZ$, represented by the Feynman diagrams,

It is enhanced when $h_2$ is on-shell, with a branching fraction $B(h_2 \rightarrow ZZ) = 0.6$, as indicated in Table VIII. As

where $i = 1, 2, 3$, and missing are a graph with the $CP$-odd scalar replaced by a $Z$ and one formed with a $ZZh_1h_2$ quartic coupling. This production mode is enhanced when the $A$ boson is on-shell, $e^+e^-\rightarrow h_2A\rightarrow h_2h_1Z$, with a branching fraction $B(A \rightarrow h_1Z) = 0.9$ as shown in Table IX.

Fig. 3 shows a scan for the production mode $e^+e^-\rightarrow h_2\ell\bar{\ell}$ (left frame) and $e^+e^-\rightarrow h_2h_1A$ (right frame), two of the important 3-body $h_2$ production modes. In the case of $e^+e^-\rightarrow h_2\ell\bar{\ell}$, the production cross-section reaches up to 0.01 pb. The largest cross-sections are seen when $h_2$ is mainly triplet (black triangular points), with a typical value between 0.001 and 0.01 pb. B1 is shown as a black solid curve. The value of the cross-section drops when $h_2$ is mainly singlet (orange star points), with values typically smaller than $10^{-4}$ pb. This is because a singlet does not couple to the $Z$ gauge boson. The chosen B2 lies within the cloud of points. The case where $h_2$ is mixed is much more rare and no point has been generated in this scenario due to its fine-tuned character.

The case of $e^+e^-\rightarrow h_2Ah_1$ is shown in the right frame of Fig. 3. This is the main process in B2, where $h_2$ is mainly singlet (orange star points). In this case, cross-sections can reach up to $10^{-3}$ pb, but can also be as low as $10^{-14}$ pb, depending on whether $h_3$ is on-shell or not. In the case where $h_2$ is mainly triplet (black triangular points) the cross-section is more restricted. It can vary between $10^{-3}$ and $10^{-8}$ pb and B1 is a very typical case. Cross-sections are larger when an intermediate heavy scalar is also on-shell.

Notice that the popular modes for the production of the important 3-body production mode in this case is $e^+e^-\rightarrow h_2Zh_1$, with sub-processes given by,
explained later in the decay Section, the coupling $h_2ZZ$ is large if $h_2$ is mainly triplet (B1).

In B2 the $CP$-even Higgs boson created in association with $A$ is no longer $h_2$ but $h_3$. If $h_2$ is mainly singlet, $h_3$ is mainly triplet, and the coupling $ZAh_3$ is not suppressed. This is confirmed in the central frame of Fig. 4 where we have B2. The most important 2-body production mode is precisely $e^+e^- \rightarrow Ah_3$, represented by the Feynman diagram

Also in the central frame of Fig. 4 we see the main 3-body $A$ production modes. The most important one is again $e^+e^- \rightarrow AZZ$, and it is enhanced when $h_3$ is on-shell.

B3 is an intermediate case, and we can see in the right

| $\sqrt{s}$ [TeV] | $\mathcal{L}_{FCCee}$ | $\mathcal{L}_{ILC}$ | $\mathcal{L}_{CLIC}$ | B1: $\sigma$ | B2: $\sigma$ | B3: $\sigma$ |
|------------------|---------------------|---------------------|---------------------|-------------|-------------|-------------|
| 0.250            | -                   | -                   | -                   | 0           | 0           | 1.4 x 10^{-10}|
| 0.350            | 2.6                 | 0.5                 | 0                   | 0           | 0           | 1.5 x 10^{-2}  |
| 0.500            | -                   | 0.5                 | -                   | 1.5 x 10^{-12} | 0           | 1.5 x 10^{-2}  |
| 1.0              | -                   | 1                   | -                   | 1.4 x 10^{3} | 2.2 x 10^{-5} | 2.5 x 10^{4}  |
| 1.4              | -                   | -                   | 1.5                 | 1.1 x 10^{4} | 3.5 x 10^{-3} | 2.1 x 10^{4}  |
| 3                | -                   | -                   | 2                   | 6.2 x 10^{3} | 3.6 x 10^{3} | 7.5 x 10^{3}  |
FIG. 4. Production modes for $A$ at an $e^+e^-$ collider in all 3 benchmarks. The legend shows the final state after the $e^+e^-$ collision.

and it is enhanced when $h_2$ and $h_3$ are on-shell.

Fig. 5 shows scans for the process $e^+e^- \rightarrow AZZ$ (left frame), important for B1 and B2, and the process $e^+e^- \rightarrow AJJ$ (right frame), important in B3. In the first case, the production cross-section is increased when $h_2$ is also on-shell, as explained before. The cross-section is not larger than 0.01 pb, and B1 is not far below from that value. In the last process a triple scalar coupling is important, and the exact values of the parameters in the potential are crucial. In this case, B3 is characterized by a large value of $\beta_3$ which increases the coupling $h_3JJ$. As before, in Fig. 5 we include the curves corresponding to each benchmark to facilitate comparisons.

C. $H^+$ Production

Table VII shows $H^+$ production cross-sections at $e^+e^-$ colliders, prospected luminosities and CM energies for the FCC-ee, ILC and CLIC colliders. Besides the 2-body production cross-section for $e^+e^- \rightarrow H^+H^-$, in B1 and B2 the 3-body process $e^+e^- \rightarrow H^+h_1W^-$ dominates. In B3 the process $e^+e^- \rightarrow H^+W^+\Delta^-\bar{\nu}$ dominates. The last case presents a high interest, as the doubly charged Higgs boson gives us an independent window to study neutrinos.

Fig. 6 shows the 2-body and 3-body production of an $H^+$ boson. The charged Higgs boson is potentially ob-
FIG. 5. Production modes $e^+e^- \rightarrow AZZ$ and $e^+e^- \rightarrow AJJ$.

TABLE VII. Production cross-section (in units of ab) for $H^+$ at an $e^+e^-$ collider for projected energies in the 3 benchmarks. Estimated luminosities are also given in units of ab$^{-1}$.

| $\sqrt{s}$ [TeV] | $\mathcal{L}_{\text{FCCee}}$ | $\mathcal{L}_{\text{ILC}}$ | $\mathcal{L}_{\text{CLIC}}$ | B1: $\sigma$ | B2: $\sigma$ | B3: $\sigma$ |
|------------------|------------------|------------------|------------------|-----------|-----------|-----------|
| 0.250            | -                | 0.25             | -                | 0         | 0         | 0         |
| 0.350            | 2.6              | -                | 0.5              | $0.35 \times 10^{-3}$ | 0         | 0         | $5.8 \times 10^{-3}$ |
| 0.500            | -                | 0.5              | -                | $1.9 \times 10^{-4}$ | 0         | 0         | 0         |
| 1.0              | -                | 1                | -                | $1.6 \times 10^{3}$ | $4.1 \times 10^{-3}$ | 0         | $1.7 \times 10^{4}$ |
| 1.4              | -                | -                | 1.5              | $7.0 \times 10^{3}$ | $3.5 \times 10^{-2}$ | 0         | $1.5 \times 10^{4}$ |
| 3                | -                | -                | 2                | $5.0 \times 10^{3}$ | $2.4 \times 10^{3}$ | 0         | $6.6 \times 10^{3}$ |

servable at CLIC when produced in association with another $H^-$, represented by the graph,

The couplings $H^+H^-\gamma$ and $H^+H^-Z$ are both of the order of electroweak couplings, as can be seen in Appendix B. Among the 3-body modes, in B1 and B2 the main production mode is $e^+e^- \rightarrow H^+h_1W^-$, represented by the sub-processes,
FIG. 6. Production modes for $H^+$ at an $e^+e^-$ collider in all 3 benchmarks. The legend shows the final state after the $e^+e^-$ collision.

plus a graph where the intermediate charged Higgs is replaced by a $W$ and removing the intermediate photon, graphs where the external charged Higgs and the $W$ are interchanged (also removing the photon), a graph where $(A,J)$ is replaced by a $Z$, graphs that involve quartic couplings, and a graph with a neutrino in the $t$--channel. This mode is dominated by the graph where the charged Higgs is on-shell. Note that the coupling $ZH^+W^-$ is suppressed by the triplet vev. This mode is enhanced when $H^-$ is also on-shell, corroborated by the fact that $B(H^- \rightarrow h_1W^-) = 0.8$ in B2.

Similarly, in Fig. 6 we see that the mode $e^+e^- \rightarrow H^+W^+\Delta^{--}$ dominates in B3. It is represented by,
plus a graph where the external particles $H^+$ and $\Delta^{--}$ are interchanged and at the same time the intermediate $\Delta^{++}$ is replaced by $H^-$, plus two graphs where the $H^-$ is replaced by a $W^-$ with $Z$ exchanged for a photon, and two graphs with quartic couplings. As it was mentioned before, the production of a $\Delta^{++}$ is important because it could lead to the observation of its decay into two charged leptons, which could probe the mechanism for neutrino masses.

Fig. 4 shows a general scan for the 3-body production modes $e^+e^- \to H^+h_1W^-$ (left frame) and $e^+e^- \to H^+W^+\Delta^{--}$ (right frame). For the case $e^+e^- \to H^+h_1W^-$, the majority of the scenarios give a cross-section between $10^{-2}$ and $10^{-4}$ pb, as long as a second heavy particle is also on-shell. In the case of $e^+e^- \to H^+W^+\Delta^{--}$, the cross-section is of the same order between $10^{-3}$ and $10^{-5}$ pb, also independent of the composition of $h_2$. If neutrinos acquire their mass via a coupling to the triplet, the mechanism can be probed through the production of a double charged Higgs boson.

VI. DECAY BRANCHING RATIOS

In this section, we study the decay modes of the SM-like Higgs boson $h_1$, the next-to- heaviest Higgs $h_2$, the $CP$–odd Higgs $A$, and the charged Higgs $H^+$. For the computation of branching fractions, we consider $B = \Gamma(H \to (XX)_i)/\sum_i \Gamma(H \to (XX)_i)$, with $H = h_1, h_2, A, H^\pm$. For the $CP$–even Higgses we have $XX = \tau\bar{\tau}, bb, WW, ZZ, \gamma\gamma, Z\gamma, gg, JJ, JZ$ for $h_1$ and we include $tt$ and $h_1h_1$ to the previous list for $h_2$. For $A$ we consider $XX = \tau\bar{\tau}, bb, tt, h_1Z, h_1J, \gamma\gamma, Z\gamma, gg$, with $i = 1, 2$. For $H^\pm$, we have $XX = tb, h_1W^\pm, JW^\pm, ZW^\pm$, with $i = 1, 2$.

We define

$$\lambda(a, b, c) = a^2 + b^2 + c^2 - 2ab - 2ac - 2bc. \quad (13)$$

In the special case $b = c$, it is reduced to the function $\beta$,

$$\beta(b/a) = \frac{1}{a} \sqrt[4]{\lambda(b/a)}. \quad (14)$$

A. $h_1$ and $h_2$ Decays

We first mention the decay modes to fermions for $h_i$ ($i = 1, 2$), which include $h_i \to bb$ and $h_i \to \tau\bar{\tau}$. The decay $h_2 \to tt$ is considered for $h_2$, but not for $h_1$. The corresponding Feynman diagram is

\begin{center}
\begin{tikzpicture}
  \node (h1) at (0,0) {$h_1$};
  \node (f) at (2,0) {$f$};
  \node (ff) at (0,-1.5) {$f$};
  \draw (h1) -- (f);
  \draw (h1) -- (ff);
\end{tikzpicture}
\end{center}

with Feynman rule given in Appendix B.

The decay widths are given by

$$\Gamma(h_i \to f\bar{f}) = \frac{N_c m_{h_i}}{8\pi} \alpha^4 (m_f^2/m_{h_i}^2) |\lambda_{h_i ff}|^2, \quad (15)$$

where the number of colors is $N_c = 3$ for quarks and $N_c = 1$ for leptons. We define the coupling $\lambda_{h_i ff} = \alpha_i^2 h_i v_i/\sqrt{2}$, where $v_i$ corresponds to the respective Yukawa coupling in the convention $m_i = h_i v_i/\sqrt{2}$.

Since $h_1$ is always mainly doublet and $h_2$ is not, decay rates of $h_1$ to fermions are consistently larger than decay rates of $h_2$ to fermions. Similarly, since the $h_2$ component to doublet is larger in B2 compared to B1 and B3, the corresponding decay rate is larger too.

Also important are the vector boson decays $h_i \to W^+W^-$, $h_i \to ZZ$, with Feynman diagram

\begin{center}
\begin{tikzpicture}
  \node (h1) at (0,0) {$h_i$};
  \node (Z) at (2,0) {$Z, W$};
  \node (W) at (2,1) {$Z, W$};
  \draw (h1) -- (Z);
  \draw (h1) -- (W);
\end{tikzpicture}
\end{center}

The decay rate where both gauge bosons are on-shell is
with $V = Z, W$, $\delta^' W = 2$ and $\delta^' Z = 1$. The decay rate where one vector boson is off-shell is

$$\Gamma(h_i \rightarrow VV^*) = \frac{3g_h^2 m_h \delta^V V}{512 \pi^3 m_V^4} F(m_V/m_h)|M_{h_i V V}|^2,$$  \hfill (17)
where $A$ is defined as
\[ A = A_W + A_t + A_0^{H^+} + 2A_0^{\Delta^{++}} , \] (22)
with
\[
A_W + A_t = c_W M_{h,WW} A_1(\tau_W, \lambda_W) + \frac{g m_W}{c_W} N_c Q_t (1-4 Q_t s_W^2)\lambda_{h,tt} A_{1/2}(\tau_t, \lambda_t)
\]
\[
A_0^{H^+} = \frac{m_W^2}{g s_W m_W^2} \lambda_{ZH+H^-} M_{h,H+H^-} A_0(\tau_{H^+}, \lambda_{H^-})
\]
\[
A_0^{\Delta^{++}} = \frac{m_W^2}{g s_W m_{\Delta^{++}}^2} \lambda_{Z\Delta^{++}H^-} M_{h,\Delta^{++}H^-} A_0(\tau_{\Delta^{++}}, \lambda_{\Delta^{++}}),
\] (23)
where
\[
\lambda_{ZH+H^-} = -\frac{g}{2c_W} (s_W^2 - 2 s_W^2),
\]
\[
\lambda_{Z\Delta^{++}H^-} = -\frac{g}{c_W} (c_W^2 - s_W^2),
\] (24)
as can be seen from Appendix B. The loop functions are,
\[
A_0(\tau, \lambda) = I_1(\tau, \lambda),
\]
\[
A_1(\tau, \lambda) = 4(3 - \tan^2 \theta_W) I_2(\tau, \lambda) + [(1 + 2/\tau) \tan^2 \theta_W - (5 + 2/\tau)] I_1(\tau, \lambda),
\]
\[
A_{1/2}(\tau, \lambda) = I_1(\tau, \lambda) - I_2(\tau, \lambda),
\] (25)
\]
\[
M_{h,\beta\beta} is defined from the corresponding Feynman rule in Appendix B.
\]
\[
\text{Finally, the decay } h_2 \to h_1 h_1 \text{ is given by}
\]
\[
\Gamma(h_2 \to h_1 h_1) = \frac{\beta (m_{h_2}^2 / m_{h_1}^2)}{32\pi m_{h_2}} |M_{h_2h_1h_1}|^2,
\] (30)
\]
\[
M_{h,\beta\beta} is defined from the corresponding Feynman rule in Appendix B.
\]
\[
\text{In the case of } h_1 \text{ we require that its mass is } \approx 125
\text{GeV and that it is mostly doublet. Besides the usual decay modes for this SM-like Higgs boson, in this model there are two more. These are } h_1 \to JJ \text{ and } h_1 \to JZ.\text{ For the three benchmarks, the branching fractions are } B(h_1 \to JJ) \approx 3 \times 10^{-5} \text{ and } B(h_1 \to JZ) \approx 3 \times 10^{-15}.\text{ We are well within experimental constraints on the Higgs invisible width, as branching fractions bigger than 22% are excluded at 95% CL [27]. These modes are suppressed due to two different reasons. The mode } h_1 \to JZ \text{ is suppressed because the Majoron } J \text{ is mostly singlet. The decay mode } h_1 \to JJ \text{ is suppressed because in addition we require a small value for } \beta_2.\text{ Fig. 5 shows the branching fractions of our light Higgs bosons. In the top frame we scan the parameters without any restriction, varying } \lambda_1 \text{ between } [0, 4], \text{ in order not to constrain the Higgs mass, as we need to make sure the points}
Nevertheless, this fact is obscured in branching fractions because the total decay rate is also very different. Similarly, decay rates to gauge bosons are larger in B2, but not necessarily the same is true at the level of branching fractions. Clearly, looking at branching fractions, decays of \( h_2 \) to two Majorons (invisible decay) dominate in B2 and B3 because \( h_2 \) has a large singlet component in those two benchmarks.

Fig. 8 shows the branching fractions as a function of the scalar mass \( m_{h_2} \), evolving from our three benchmarks, while Fig. 10 shows a scan of the \( h_2 \) decays, with all the constrains from Section 111 implemented.

The curves shown in Fig. 9 confirms the previous observations. These curves are found by keeping the values of the independent parameters as in the 3 different benchmarks, and varying the value of \( \kappa \) in order to keep \( m_{h_2} \) free. Since due to mixing this procedure will also vary the value of \( m_{h_2} \approx 125 \text{ GeV} \), we keep \( \lambda_1 \) also free to compensate, as in Table 111. We show also as a vertical solid line the value of \( m_{h_2} \) in the corresponding benchmark. In the case of B2, near the vertical line \( h_2 \) is mainly singlet, and \( \kappa \) affects very little to \( m_{h_2} \). If \( \kappa \) is sufficiently different from its starting value in B2 \( h_2 \) becomes mostly triplet. The value for \( m_{h_2} \) cannot be larger than its value in the benchmark because by then \( h_2 \) is mostly singlet and \( \kappa \) has little effect. Something similar happens with B3. In all cases \( h_2 \rightarrow ZZ \) and \( h_2 \rightarrow WW \) are important. Decays

\[
\begin{array}{|c|c|c|c|}
\hline
\text{Branching Fraction} & \text{B1} & \text{B2} & \text{B3} \\
\hline
B(h_2 \rightarrow \ell \ell) & 0.3 & 7.9 \times 10^{-3} & - \\
B(h_2 \rightarrow t \bar{t}) & 6.0 \times 10^{-4} & 9.5 \times 10^{-6} & 3.4 \times 10^{-7} \\
B(h_2 \rightarrow \tau \tau) & 3.0 \times 10^{-5} & 4.5 \times 10^{-7} & 1.6 \times 10^{-8} \\
B(h_2 \rightarrow WW) & 7.0 \times 10^{-3} & 3.0 \times 10^{-2} & 3.6 \times 10^{-6} \\
B(h_2 \rightarrow ZZ) & 0.6 & 1.0 \times 10^{-2} & 1.3 \times 10^{-4} \\
B(h_2 \rightarrow gg) & 7.2 \times 10^{-3} & 1.3 \times 10^{-4} & 1.0 \times 10^{-6} \\
B(h_2 \rightarrow \gamma \gamma) & 7.7 \times 10^{-6} & 2.9 \times 10^{-5} & 1.8 \times 10^{-3} \\
B(h_2 \rightarrow Z \gamma) & 1.6 \times 10^{-6} & 1.6 \times 10^{-7} & 1.9 \times 10^{-7} \\
B(h_2 \rightarrow JJ) & 1.2 \times 10^{-4} & 0.9 & 0.9 \\
B(h_2 \rightarrow JZ) & 3.0 \times 10^{-2} & 3.6 \times 10^{-12} & 2.5 \times 10^{-6} \\
B(h_2 \rightarrow h_1 h_1) & 0.1 & 1.7 \times 10^{-2} & 1.0 \times 10^{-6} \\
\hline
\end{array}
\]

in the plot are consistent with a SM-like Higgs. Also is useful to keep the mass free to observe the effect of the constraints and to facilitate the comparison with \( h_2 \). On the top frame \( \beta_2 \) is not constrained and varies between [−4, 4] so we can clearly see the suppression in the Majoron decays once we constrain its value in the bottom frame. The bottom frame includes all constrains from Section 111. The branching fractions in our three benchmarks for \( h_2 \) are given in Table VIII. We mention first that \( h_2 \) has a larger doublet component in B2, and for that reason decay rates to fermions are larger in that benchmark.

FIG. 8. Branching fractions for the \( h_1 \) scalar with (bottom) and without (top) restrictions, as explained in the text.
The decay of $h_2$ to fermions depend strongly on the (small) $h_2$ component to doublet. In the scan in Fig. 10 we plot $h_2$ branching fractions while all the parameters are varied according to Table III. We see that the values of the branching fractions separates in two regions, that we plot separately in the two column plot. These two sectors corresponds to a mainly triplet (left column) or mainly singlet (right column) $h_2$. The scan shows that if $h_2$ is mainly triplet (as in B1) decay modes $h_2 \to ZZ$ and $h_2 \to h_1 h_1$ can dominate, with $h_3 \to JZ$ sometimes also important. On the contrary, if $h_2$ is mainly singlet (as in B2) the decay mode $h_2 \to JJ$ dominates by far, with $h_2 \to WW$ and $h_2 \to ZZ$ following in importance. The $h_2 \to t\bar{t}$ branching fractions can be large as long as the other decay rates are also small.

B. $A$ Decays

Now we study the decays of the $CP$–odd Higgs boson $A$. The relevant decays at tree-level are to third generation fermions, $A \to t\bar{t}$, $A \to b\bar{b}$, $A \to \tau\tau$, to $CP$–even Higgs bosons and a Majoron, $A \to h_2 h_1$, and to $CP$–even Higgs bosons and a $Z$ gauge boson, $A \to h_i Z$. We also consider the 1-loop decays to $\gamma\gamma$, $Z\gamma$ and $gg$ for completeness.

The decay of $A$ to fermions, represented by the Feynman diagram,

$$\Gamma(A \to f\bar{f}) = \frac{N_c m_A}{8\pi} \left[ 1 - 4\frac{m_f^2}{m_A^2} \right] \frac{1}{2} |\lambda_{Aff}|^2, \quad (31)$$

with a coupling

$$\lambda_{Aff} = \frac{1}{\sqrt{2}} O^{32}_{f} h_f, \quad (32)$$

as seen in Appendix B. $h_f$ is the Yukawa coupling of the fermion. Since $A$ is always mainly triplet, $O^{32}_{f}$ is always small. The decay $A \to f\bar{f}$ proceeds just because the $A$ eigenfunction has a small component of doublet, as indicated in eq. (A.3).

The $A$ boson can also decay into a $CP$–even Higgs and a $Z$ boson. The corresponding Feynman diagram is,
The decay rate is given by the formula,

\[ \Gamma(A \rightarrow h_iZ) = \frac{\lambda_{Ah_iZ}^2 m_A^3}{16\pi} \frac{m_A^3}{m_Z^3} \lambda^{3/2}(1, m^2_{h_i}/m^2_A, m^2_Z/m^2_A), \] (33)

with a coupling

\[ \lambda_{Ah_iZ} = \frac{g^2}{2c_W^2} (O^2_{\alpha\beta} O^3_{\gamma\delta} - 2O^2_{\alpha\beta} O^3_{\gamma\delta}), \] (34)

as seen in Appendix B. The \( \lambda \) function is defined in eq. (13). In the case \( A \rightarrow h_2Z \), since \( A \) is always mainly triplet, there is no phase space in B1, where \( h_2 \) is also a triplet and has a mass almost equal to the mass of \( A \). In the case \( A \rightarrow h_1Z \), since the couplings are more or less similar for B1 and B2, the difference is due to the value of \( m_A \).

The decay to a \( CP \)-even Higgs boson and a Majoron is represented by the following Feynman diagram,

The decay rate is

\[ \Gamma(A \rightarrow h_iJ) = \frac{M_{h_i}^2(s) s^2}{16\pi m_A} \lambda^{1/2}(1, m^2_{h_i}/m^2_A, m^2_J/m^2_A), \] (35)

with the coupling \( M_{h_i} \) (with units of mass) given in Appendix B.

The decay to \( \gamma\gamma \) is given by (34)

\[ \Gamma(A \rightarrow \gamma\gamma) = \frac{g^2 m_A^3}{1024\pi^4 m_W^4} \frac{4\sqrt{2}}{3h_t} F_{1/2}(\tau_t) \lambda_{AAU} \] (36)

with \( \tau_t = 4m_t^2/m_A^2 \) and the \( F_{1/2} \) function for a pseudoscalar is defined in Appendix C of Ref. [34].
The Feynman diagram, is very important. 

\[ \Gamma(A \to Z\gamma) = \frac{\alpha g^2}{2048 \pi^4 m_W^2} |A_i|^2 m_A^3 (1 - \frac{m_{\gamma}^2}{m_A^2})^3, \]  

where \( A_i \) is defined in equation 23 (replacing \( h \) with \( A \)). Finally, the decay to two gluons is \( 34 \)

\[ \Gamma(A \to gg) = \frac{\alpha^2 g^2 m_A^3}{128 \pi^4 m_W^2} \left| \frac{4\sqrt{2}}{3} h \right|^2 \left( F_{1/2}(\tau_1) \right) |\lambda_{AB}|^2. \]  

Branching fractions for the decay of \( A \) for our three benchmarks are given in Table IX. Table IX. Branching fractions for \( A \) in our three different benchmarks.

| Branching Fraction | B1 | B2 | B3 |
|--------------------|----|----|----|
| \( B(A \to h\bar{h}) \) | 0.5 | 0.2 | 0.2 |
| \( B(A \to bb) \) | 5.5 \times 10^{-4} | 1.5 \times 10^{-4} | 6.0 \times 10^{-3} |
| \( B(A \to \pi\pi) \) | 2.6 \times 10^{-5} | 7.0 \times 10^{-6} | 2.8 \times 10^{-4} |
| \( B(A \to h_1 Z) \) | 0.5 | 0.8 | 0.9 |
| \( B(A \to h_1 J) \) | 1.7 \times 10^{-2} | 4.4 \times 10^{-3} | 2.0 \times 10^{-2} |
| \( B(A \to h_2 Z) \) | - | 5.0 \times 10^{-2} | - |
| \( B(A \to h_2 J) \) | - | 1.1 \times 10^{-4} | - |
| \( B(A \to gg) \) | 1.4 \times 10^{-2} | 2.7 \times 10^{-3} | 6.2 \times 10^{-2} |
| \( B(A \to J\bar{\gamma}) \) | 1.7 \times 10^{-5} | 3.4 \times 10^{-6} | 7.7 \times 10^{-5} |
| \( B(A \to Z\gamma) \) | 8.2 \times 10^{-7} | 2.6 \times 10^{-7} | 2.0 \times 10^{-6} |

Table IX. Branching fractions for \( A \) in our three different benchmarks.

The decay to \( Z\gamma \) is given by 34.

\[ \Gamma(A \to Z\gamma) = \frac{\alpha g^2}{2048 \pi^4 m_W^2} |A_i|^2 m_A^3 (1 - \frac{m_{\gamma}^2}{m_A^2})^3, \]  

where \( A_i \) is defined in equation 23 (replacing \( h \) with \( A \)). Finally, the decay to two gluons is 34.

\[ \Gamma(A \to gg) = \frac{\alpha^2 g^2 m_A^3}{128 \pi^4 m_W^2} \left| \frac{4\sqrt{2}}{3} h \right|^2 \left( F_{1/2}(\tau_1) \right) |\lambda_{AB}|^2. \]  

Branching fractions for the decay of \( A \) for our three benchmarks are given in Table IX. Table IX. Branching fractions for \( A \) in our three different benchmarks.

| Branching Fraction | B1 | B2 | B3 |
|--------------------|----|----|----|
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| \( B(A \to bb) \) | 5.5 \times 10^{-4} | 1.5 \times 10^{-4} | 6.0 \times 10^{-3} |
| \( B(A \to \pi\pi) \) | 2.6 \times 10^{-5} | 7.0 \times 10^{-6} | 2.8 \times 10^{-4} |
| \( B(A \to h_1 Z) \) | 0.5 | 0.8 | 0.9 |
| \( B(A \to h_1 J) \) | 1.7 \times 10^{-2} | 4.4 \times 10^{-3} | 2.0 \times 10^{-2} |
| \( B(A \to h_2 Z) \) | - | 5.0 \times 10^{-2} | - |
| \( B(A \to h_2 J) \) | - | 1.1 \times 10^{-4} | - |
| \( B(A \to gg) \) | 1.4 \times 10^{-2} | 2.7 \times 10^{-3} | 6.2 \times 10^{-2} |
| \( B(A \to J\bar{\gamma}) \) | 1.7 \times 10^{-5} | 3.4 \times 10^{-6} | 7.7 \times 10^{-5} |
| \( B(A \to Z\gamma) \) | 8.2 \times 10^{-7} | 2.6 \times 10^{-7} | 2.0 \times 10^{-6} |

This leads to larger decay rates to fermions in B2. Since the total decay rate is also different, this is not observed for branching fractions and in fact, the opposite happens. Note that in B1 and B3 the decays of \( A \) to \( h_2 \) and a \( J \) or a \( Z \) are not kinematically allowed. The same happens in B3 for the decay to top quarks. In B2, \( A \) can be much heavier than \( h_2 \) thus, the decay \( A \to h_2 Z \) is open.

Fig. 11 shows the branching fractions of \( A \) as a function of its mass. The curves are obtained starting from each of the 3 benchmarks and vary \( \kappa \) to change \( m_A \). Since this procedure will also change \( m_{h_1} \), which we want fixed to 125 GeV, we change also the value of \( \lambda_1 \) to recover \( m_{h_1} \approx 125 \text{ GeV} \), as in Table I. In all cases, the modes \( A \to h_1 Z \) and \( A \to t\bar{t} \) dominate. In B3 the decay mode \( A \to h_2 Z \) is open and can be relevant too.

Fig. 12 shows a general scan where all the parameters are varied according to Table I. It shows that the decay mode \( A \to h_1 Z \) dominates. If the channel is open, when \( h_2 \) is mainly singlet, the decay channel \( A \to h_2 Z \) is also very important.

C. \( H^\pm \) Decays

In this Section we study tree-level decays of the singly charged Higgs boson. The decay to \( tb \), represented by the Feynman diagram,
FIG. 11. $CP$–odd Higgs $A$ branching fractions in the three benchmarks as a function of $m_A$. The parameter $\kappa$ is varied to move $m_A$, as explained in the text. The vertical solid line in each frame corresponds to our benchmark point. The plot includes all constrains from Section III.

and has the following decay rate

$$\Gamma(H^\pm \to ZW^\pm) = \frac{g^2 |M_{H^\pm ZW^\pm}|^2}{256\pi m_W^4 m_{H^\pm}^2} \left[ m_{H^\pm}^4 + m_Z^4 + 10 m_Z^2 m_W^2 + m_W^4 - 2 m_{H^\pm}^2 (m_W^2 + m_Z^2) \right] \lambda^{1/2}(m_{H^\pm}^2, m_Z^2, m_W^2), \quad (44)$$

with

$$M_{H^\pm ZW^\pm} = O_+^{21} s_W v_\phi - \sqrt{2} O_+^{22} (1 + s_W^2) v_\Delta. \quad (45)$$

In Table X we show the singly charged Higgs branching fractions in our three benchmarks. Note that the decay $H^\pm \to h_2 W^\pm$ is not kinematically allowed in B1 and B3. Branching fractions of $H^\pm \to h_1 W^\pm$ are dominant in the three benchmarks.
Fig. 12. Branching fractions for the $A$ scalar as a function of $m_A$. The left column shows points where $h_2$ is triplet-like (i.e. $|O_2^2| > 0.95$). The right column shows points where $h_2$ is singlet-like (i.e. $|O_1^2| > 0.95$). Parameters are varied according to Table III. The scan includes all constraints from Section III.

VII. PROMISING CHANNELS FOR $h_2$, $A$ AND $H^\pm$

We now briefly comment on the most promising channels for discovery of $h_2$, $A$ and $H^\pm$ at future $e^+e^-$ colliders.

A promising channel for the discovery of $h_2$, given its large cross-section as discussed in Section V, is $e^+e^- \rightarrow h_2 t\bar{t}$. Thinking of B1, the largest decays fractions for $h_2$ are to $ZZ$ as shown in Table VIII. Considering leptonic decays of the $W$ and $Z$, the signal is

$$e^+e^- \rightarrow ZZ t\bar{t} \rightarrow l^+l^-l'^+l'^-\nu\bar{\nu}b\bar{b}$$

with $l = e, \mu$. The signal contains 2 $b$-jets + 6 leptons + $p_T^{miss}$ (missing transverse momenta). For B1 at $\sqrt{s} = 1$ TeV, the cross-section is estimated as

$$\sigma_{266lp^{miss}} \approx \sigma(e^+e^- \rightarrow h_2 t\bar{t}) \times B(h_2 \rightarrow ZZ) \times B(Z \rightarrow l^+l^-)^2 \times B(W^\pm \rightarrow l^\pm\nu)^2 \approx 3 \times 10^{-5} \text{ fb}$$

resulting in less than one event to be discoverable with $\mathcal{L} = 1000 \text{ fb}^{-1}$, so too little to be observed unfortunately. Possible SM backgrounds to this signature include $e^+e^- \rightarrow ZZZ$ and $e^+e^- \rightarrow ZZ t\bar{t}$. Multi-lepton
signatures in the “23” HTM were studied in the context of the LHC in Refs. [19, 38], where it was shown that after requiring kinematic cuts in the transverse momenta of the leptons, signatures with 6 leptons have no background, even though the signal is also scarce. Therefore, multilepton signatures are relevant for higher integrated luminosities. We could require similar leptonic kinematic cuts in the case of $e^+e^-$, in addition of requiring 2 $b$-tagged jets and small $p_T^{miss}$ due to the two neutrinos.

For B2 the decay $b_2 \rightarrow JJ$ dominates. If one W boson decays hadronically and the other leptonically, then we will have a 4 $b$-jets + $p_T^{miss}$ signature, assuming the lepton escapes undetected. This channel was studied in detail in Ref. [11] for our “123” model, where it was shown that with appropriate cuts in $p_T^{miss}$, number of jets and invariant mass distributions the background is removed while keeping high signal efficiency.

In the case of the $CP$–odd Higgs $A$, there are two relevant processes, $e^+e^- \rightarrow AZZ$ has the highest cross-section for B1 and B2. In the case where $A \rightarrow t\bar{t}$ we have the same signature as before for $b_2$. The decay $A \rightarrow h_1Z$ also dominates in our benchmarks. The dominant decay $h_1 \rightarrow b\bar{b}$ follows, leading to topologies with leptons and $b$–jets (with no missing transverse momenta), depending on the decay of the $Z$. The cross-section for,

$$e^+e^- \rightarrow AZZ \rightarrow h_1ZZZ \rightarrow b\bar{b}l^+l^-l^+l^-l^+l^-$$  \hspace{1cm} (48)

leads to a 2$b$–jet+6 leptons signature. The cross-section for B1 at $\sqrt{s} = 1$ TeV is estimated as,

$$\sigma_{2666} \approx \sigma(e^+e^- \rightarrow AZZ) \times B(A \rightarrow h_1Z) \times B(h_1 \rightarrow b\bar{b}) \times B(Z \rightarrow l^+l^-)^3 \approx 1.0 \times 10^{-4} \text{ fb}$$  \hspace{1cm} (49)

resulting in less than one event with $\mathcal{L} = 1000$ fb$^{-1}$. Possible backgrounds are very similar and include the ones in equation (47) so similar cuts can be applied to suppress them.

The associated production $e^+e^- \rightarrow AJJ$ dominates in B3 with $A \rightarrow b\bar{b}$, leading to the topology of 2 $b$–jets + $p_T^{miss}$. This signal was studied for the “23” HTM in [37], with largest background coming from $e^+e^- \rightarrow W^+W^-$ and $e^+e^- \rightarrow ZZ$. The authors concluded that the most efficient way to improve the signal-to-background ratio is to require $b$–tagged jets and large $p_T^{miss}$, in addition to charged multiplicity and an invariant mass cut close to the mass of the visibly decaying particle.

Production for the singly charged Higgs dominates in $e^+e^- \rightarrow H^+H^- \rightarrow H^+h_1W^-$ for most of our benchmarks (see Figure [1]). This is followed by the decay of
can be selected with singly charged Higgs can be reconstructed and the events of Two-Higgs doublet models [38, 39]. The mass of the signature was studied for a charged Higgs in the context cally and the other leptonically, when (see Table X). An optimal discovery channel would be H

\begin{table}
\centering
\begin{tabular}{|c|c|c|c|}
\hline
\textbf{Branching Fractions} & & & \\
\hline
\textbf{\(H^+\)} & \textbf{\(H^+ \to h_1 W^+\)} & \textbf{\(H^+ \to h_2 W^+\)} & \\
\hline
\textbf{\(B(H^+ \to h_1 W^+)\)} & \textbf{\(B(H^+ \to h_2 W^+)\)} & & \\
\hline
\end{tabular}
\end{table}

FIG. 14. Branching fractions for the \(H^+\) scalar as a function of \(m_{H^+}\). The left column shows points where \(h_2\) is triplet-like (i.e. \(O^2_{23} > 0.95\)). The right column shows points where \(h_2\) is singlet-like (i.e. \(O^2_{23} > 0.95\)). Parameters are varied according to Table III. The scan includes all constrains from Section III.

\(H^+ \to h_1 W^+\), which has the highest branching fraction (see Table X). An optimal discovery channel would be when \(h_1 \to b\bar{b}\) and when one \(W\) boson decays hadronically and the other leptonically,

\[ e^+e^- \to H^+ h_1 W^- \to h_1 W^+ h_1 W^- \to b\bar{b} l^+ l^- q\bar{q} \]  

resulting in an event topology of 4b−jets + 2 jets + 1 lepton + \(p_T^{\text{miss}}\), where the lepton \(l = e, \mu\). This distinctive signature was studied for a charged Higgs in the context of Two-Higgs doublet models [38, 59]. The mass of the singly charged Higgs can be reconstructed and the events can be selected with \(b\)−tagging techniques, in addition to requiring one isolated lepton. Also, two jets must have the \(W\) mass.

We can estimate the visible cross-section for this final state. For \(\sqrt{s} = 1\) TeV in B1 we have,

\[ \sigma_{4b+\text{miss}+ljj} \approx \sigma(e^+e^- \to H^+ h_1 W^-) \times B(H^+ \to h_1 W^+) \times B(h_1 \to b\bar{b})^2 \times B(W^\pm \to l^\pm l^- q\bar{q}) \approx 0.04\ \text{fb} \]  

and since the ILC has a yearly integrated luminosity of 1000 fb\(^{-1}\), this results in about 40 potentially discoverable events. A relevant SM background for this signature is the process \(e^+e^- \to t\bar{b}b\). Our estimation yields a visible cross-section of \(\sigma_{4b+\text{miss}+ljj} \approx 0.4\ \text{fb}\), which is quite significant. The signal-to-background ratio can be enhanced by applying the selection cuts above mentioned. It was also shown in Ref. [38] that one can suppress this big irreducible background to a negligible level by using a technique that allows the reconstruc the neutrino four-momentum.

Of course a more detailed simulation study should be done in order to suppress backgrounds further and improve signal efficiency for the channels mentioned. A fully fledge study in this direction, considering also detector efficiencies, goes beyond the scope of this paper and we leave it for a future work.

\textbf{VIII. CONCLUSIONS}

We have studied the Higgs phenomenology of a model with a scalar triplet, a scalar singlet and a scalar doublet under SU(2). In this “123” variant of the Higgs triplet model the singlet acquires a vacuum expectation value, which spontaneously breaks lepton number. The vacuum expectation value generated for the triplet provides a mass term for neutrinos. This feature makes it a well motivated model to look for at particle colliders.

The lightest CP-even Higgs, \(h_1\), has been identified
with the SM-like Higgs boson discovered at the LHC, which constrains the parameters in the scalar potential of the model. We studied the production cross-sections and decay ratios of the second heaviest of the model. We found that production cross-sections at hadron colliders can be very low for these states, so we perform a numerical analysis assessing the discovery potential at future lepton colliders.

We find characteristic features in cases where \( h_2 \) is singlet-like, triplet-like or a mixture. The main 2-body production mode for \( h_2 \) is associated production with a CP-odd state \( A \). We note that cross-sections for states \( A \) and \( H^\pm \) are enhanced when a second heavy particle is also produced on-shell. Invisible decays of \( h_2 \) to Majorons can be very important. Decays of the singly charged Higgs \( H^\pm \to h_1 W^\pm \) dominate. These features lead to promising channels for discovery of \( h_2 \) and \( A \), in particular in the \( 4b-\text{jets} + p_T^{\text{miss}} \) and \( 2b-\text{jets} + p_T^{\text{miss}} \) final states, as shown in Ref. [1] and Ref. [37], respectively, as we estimate the most promising signal with leptons in the final state are too small to be observed. The \( 4b-\text{jets} + 2 \text{jets} + 1 \text{lepton} + p_T^{\text{miss}} \) final state is optimal for the discovery of the singly charged Higgs. These signals provides a test of the “123” HTM at future \( e^+e^- \) colliders.

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Appendix A: Convention for Diagonalization.

The diagonalization in the charged scalar sector is,

\[
\begin{bmatrix}
  h_1^+ \\
  h_2^+
\end{bmatrix}
= O_+ \begin{bmatrix}
  \phi^- \\
  \Delta^+
\end{bmatrix}
= \begin{bmatrix}
  -c_\beta & s_\beta \\
  s_\beta & c_\beta
\end{bmatrix}
\begin{bmatrix}
  \phi^- \\
  \Delta^+
\end{bmatrix}
\]

and the diagonalization in the neutral scalar sector proceeds as,

\[
\begin{bmatrix}
  h_1 \\
  h_2 \\
  h_3
\end{bmatrix}
= O_\chi \begin{bmatrix}
  \chi_{\sigma} \\
  \chi_{\phi} \\
  \chi_{\Delta}
\end{bmatrix},
\begin{bmatrix}
  a_1 \\
  a_2 \\
  a_3
\end{bmatrix}
= \begin{bmatrix}
  G^0 \\
  J \\
  A
\end{bmatrix}
= O_\varphi \begin{bmatrix}
  \varphi_{\sigma} \\
  \varphi_{\phi} \\
  \varphi_{\Delta}
\end{bmatrix},
\]

where \( O_\chi \) and \( O_\varphi \) are \( 3 \times 3 \) matrices.

The mass matrix in eq. [1] is diagonalized by the matrix,

\[
O_\varphi = \begin{bmatrix}
  0 & 1 & 1 \\
  -\frac{1}{N_G} v_\phi & 0 & \frac{v_\phi v_\Delta}{v_\sigma} \\
  -\frac{1}{N_J} v_\phi & \frac{v_\phi v_\Delta}{v_\sigma} & 0
\end{bmatrix},
\]

where

\[
N_G = \sqrt{1 + \frac{v_\phi^2}{v_\sigma^2}},
\]

\[
N_J = \sqrt{N_G^2 + 4 \frac{v_\phi^2}{v_\sigma^2} + \frac{v_\Delta^2}{v_\sigma^2}},
\]

\[
N_A = \sqrt{1 + 4 \frac{v_\phi^2}{v_\sigma^2} + \frac{v_\Delta^2}{v_\sigma^2}}.
\]

The mass eigenstate fields are,

\[
G^0 = \frac{1}{N_G} \varphi_{\phi} - \frac{2}{N_G} \frac{v_\Delta}{v_\phi} \varphi_{\Delta},
\]

\[
J = \frac{N_J}{N_J} \frac{v_\phi}{v_\sigma} - \frac{2}{N_J} \frac{v_\phi}{v_\sigma} \frac{v_{\Delta}}{v_\phi} - \frac{1}{N_J} \frac{v_\Delta}{v_\sigma} \frac{v_{\Delta}}{v_\phi} - \frac{1}{N_A} \frac{v_\phi}{v_\sigma} \frac{v_{\Delta}}{v_\phi},
\]

\[
A = \frac{1}{N_A} \frac{v_\phi}{v_\sigma} \frac{v_{\Delta}}{v_\phi} + \frac{2}{N_A} \frac{v_\phi}{v_\sigma} \frac{v_{\Delta}}{v_\phi} + \frac{1}{N_A} \frac{v_\phi}{v_\sigma} \frac{v_{\Delta}}{v_\phi}.
\]

From here we conclude that the Majoron has the tendency to be mainly singlet and that the neutral Goldstone boson has no singlet component (the singlet does not couple to the Z boson).

Appendix B: Feynman Rules.

1. One scalar and two fermions

\[
\overrightarrow{h_i} \overrightarrow{f} = -i \overline{O} \gamma_5 \gamma^\mu \gamma^\nu \gamma^\tau \frac{h_i}{\sqrt{2}} \overrightarrow{f} = -i \bar{O} \gamma_5 \gamma^\mu \gamma^\nu \gamma^\tau \frac{h_i}{\sqrt{2}}
\]

2. One scalar and two gauge bosons

\[
\overrightarrow{h_i} \overrightarrow{Z_{\mu}} \overrightarrow{Z_{\nu}} = \frac{i}{2} \overline{O} g^2 \gamma_5 \gamma^\mu \gamma^\nu \gamma^\tau \overrightarrow{Z_{\mu}} \overrightarrow{Z_{\nu}} \frac{h_i}{\sqrt{2}}
\]

\[
= \frac{i}{2} (g^2 + g'^2) (O_{\chi}^2 v_\phi + 4O_{\chi}^3 v_\Delta) g^\mu g^\nu
\]

\[
\overrightarrow{Z_{\mu}} \overrightarrow{Z_{\nu}} = \frac{i}{2} \overline{O} g^2 \gamma_5 \gamma^\mu \gamma^\nu \gamma^\tau \overrightarrow{Z_{\mu}} \overrightarrow{Z_{\nu}} \frac{h_i}{\sqrt{2}}
\]

\[
= \frac{i}{2} (g^2 + g'^2) (O_{\chi}^2 v_\phi + 4O_{\chi}^3 v_\Delta) g^\mu g^\nu
\]
3. Two scalars and one gauge boson

\[ h_i \quad \begin{array}{c} \downarrow \quad \text{W}^+ \\ \text{W}_\nu^- \end{array} \quad = \frac{i}{2} \left( O^{i2}_\phi v_\phi + 2O^{i3}_\phi v_\Delta \right) g_{\mu \nu} \]

\[ Z_\mu \quad \begin{array}{c} \downarrow \quad p' \\ p \end{array} \quad a_j \quad \begin{array}{c} \downarrow \quad h_i^- \end{array} \quad = \frac{g}{2c_W} \left( O^{i2}_\phi O^{j2}_\varphi - 2O^{i3}_\phi O^{j3}_\varphi \right) (p + p')_\mu \]

\[ Z_\mu \quad \begin{array}{c} \downarrow \quad p' \\ p \end{array} \quad h_j^+ \quad \begin{array}{c} \downarrow \quad h_i^- \end{array} \quad = \frac{-ig}{2c_W} \left[ O^{i1}_+ O^{j1}_+ (c_W^2 - s_W^2) - 2O^{i2}_+ O^{j2}_+ s_W^2 \right] (p + p')_\mu \]

\[ Z_\mu \quad \begin{array}{c} \downarrow \quad p' \\ p \end{array} \quad \Delta^{++} \quad \begin{array}{c} \downarrow \quad \text{W}^+ \end{array} \quad = \frac{-ie}{2} (c_W^2 - s_W^2) (p + p')_\mu \]

4. Three Scalars

For the case with one \( CP \)-even and two \( CP \)-odd Higgs bosons, the relevant term in the Lagrangian is

\[ \mathcal{L}_{h_{i}a_{j}a_{k}} = M_{h_{i}a_{j}a_{k}} h_{i} a_{j} a_{k}, \quad (B1) \]

where we sum over \( i, j, k \). The coupling \( M_{h_{i}a_{j}a_{k}} \) (with units of mass), after symmetrization in \( j \) and \( k \) is given by the expression
For one \( CP \)-even and two doubly charged Higgs bosons, the relevant term in the Lagrangian is,

\[
\mathcal{L}_{h_i h_j h_k} = M_{h_i h_j h_k} h_i h_j h_k
\]  
(B3)

where we sum over \( i, j, k \). The coupling \( M_{h_i h_j h_k} \) (with units of mass) is given by the expression

\[
M_{h_i h_j h_k} = -2\lambda_1 v_\phi O^2 \chi O^{i2} O^{k2} - 2(\lambda_2 + \lambda_4) v_\Delta O^3 \chi O^{i3} O^{k3} - (\lambda_3 + \frac{1}{2} \lambda_5) v_\sigma O^{2i} O^{2k} + 2(\lambda_3 + 2\lambda_5) v_\Delta O^{i2} O^{k2} - (\lambda_3 + \frac{1}{2} \lambda_5) v_\sigma O^{i3} O^{k3} + \frac{1}{2} \lambda_5 v_\sigma O^{3i} O^{3k}.
\]  
(B4)

and the Feynman rule is,

\[
h_i \quad h_j^+ \quad h_k^- = i M_{h_i h_j h_k} \quad \text{(twice larger if } j = k).\]

For one \( CP \)-even and two charged Higgs bosons, the relevant term in the Lagrangian is

\[
\mathcal{L}_{h_i h_j h_k} = M_{h_i h_j h_k} h_i h_j h_k
\]  
(B5)

with

\[
M_{h_i h_j h_k} = -2\lambda_1 v_\phi O^2 \chi O^{i2} O^{j2} O^{k2} - 2(\lambda_2 + \lambda_4) v_\Delta O^3 \chi O^{i3} O^{j3} O^{k3} - (\lambda_3 + \frac{1}{2} \lambda_5) v_\sigma O^{2i} O^{2j} O^{2k} + 2(\lambda_3 + 2\lambda_5) v_\Delta O^{i2} O^{j2} O^{k2} - (\lambda_3 + \frac{1}{2} \lambda_5) v_\sigma O^{i3} O^{j3} O^{k3} + \frac{1}{2} \lambda_5 v_\sigma O^{3i} O^{3j} O^{3k}.
\]  
(B6)

leading to the following Feynman rule

\[
h_i \quad h_j^+ \quad h_k^- = i M_{h_i h_j h_k} \quad \Delta^{++} \quad \Delta^{--}.
\]  
(B7)

For three \( CP \)-even Higgs bosons, the relevant term in the Lagrangian is

\[
\mathcal{L}_{h_i h_j h_k} = M_{h_i h_j h_k} h_i h_j h_k
\]  
(B7)

where we sum over \( i, j, k \). The coupling \( M_{h_i h_j h_k} \) (with units of mass), after symmetrization in \( j \) and \( k \), is given by
\[ M_{h_2 h_1 h_3} = -6\lambda_1 v_\phi O_{\chi^2}^2 O_{\chi^2}^2 O_{\chi^2}^3 - 6(\lambda_2 + \lambda_3) v_\Delta O_{\chi^3}^2 O_{\chi^3}^2 O_{\chi^3}^3 \\
- (\lambda_3 + \lambda_4) v_\phi \left[ O_{\chi^2}^2 O_{\chi^2}^2 O_{\chi^2}^2 + O_{\chi^2}^2 O_{\chi^3}^2 O_{\chi^3}^3 + O_{\chi^2}^2 O_{\chi^3}^2 O_{\chi^3}^3 \right] \\
- (\lambda_3 + \lambda_5) v_\Delta \left[ O_{\chi^2}^2 O_{\chi^2}^2 O_{\chi^3}^3 + O_{\chi^2}^2 O_{\chi^2}^3 O_{\chi^3}^3 + O_{\chi^2}^2 O_{\chi^3}^2 O_{\chi^3}^3 \right] + 6\beta_3 v_\phi O_{\chi^3}^2 O_{\chi^3}^2 O_{\chi^3}^3 \]

The corresponding Feynman rule is given by

\[ h_2 \rightarrow h_1^2 = i M_{h_2 h_1 h_1}. \]
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