A Note on the Toda Criterion for Interacting Dipole–Quadrupole Vibrations

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Abstract

The Toda criterion of the Gaussian curvature is applied to calculate analytically the transition energy from regular to chaotic motion in a schematic model describing the interaction between collective dipole and quadrupole modes in atomic nuclei.

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In recent years great interest has been shown in nonlinear effects in nuclear\(^1\) and particle physics\(^2\). These phenomena may be considered the precursors of order–chaos transition. In these systems this transition has usually been studied numerically with Lyapunov exponents and Poincaré sections\(^3\). Less attention has been paid to analytical criteria.

In this paper we study the Hamiltonian:

\[
H = \frac{1}{2}(B_1 \dot{q}_1^2 + C_1 q_1^2) + \frac{1}{2}(B_2 \dot{q}_2^2 + C_2 q_2^2) - k_1 q_1^2 q_2 ,
\]

introduced by Bohr and Mottelson\(^4\) in order to describe the coupling between the giant–dipole resonance (GDR) and the quadrupole excitation, later used by the authors of reference\(^5\). The vibration in the \(q_1\) direction is associated with the GDR, while the \(q_2\) motion plays the role of quadrupole deformation of the nuclear surface. Here the parameters of inertia \(B_i\) and of stiffness \(C_i\) \((i = 1, 2)\) are real constants. The perturbing potential \(k_1 q_1^2 q_2\) induces a splitting of the dipole frequency caused by the static quadrupole deformation. Bohr and Mottelson\(^4\) give the estimation \(k_1 = (5/2)\sqrt{3/(2\pi)}C_1\) for the coupling parameter, in order to reproduce the experimental data of photoabsorption in deformed nuclei\(^4\).

The aim of this work is to apply the Toda criterion of the Gaussian Curvature\(^6\) to the Hamiltonian (1) in order to calculate the energy of the onset of chaotic motion, henceforth \(E_c\).

The Toda criterion is based on a local estimation of the rate of separation of neighboring trajectories in the classical phase space of the model\(^6,7\). To obtain the time evolution of a dynamical system with the Hamiltonian

\[
H = \frac{p_1^2}{2B_1} + \frac{p_2^2}{2B_2} + V(q_1, q_2) ,
\]
where \( p_k = B_k \dot{q}_k \), \( k = 1, 2 \), the following equations have to be solved:

\[
\frac{dq}{dt} = \frac{\partial H}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial H}{\partial q},
\]

(3)

where \( q = (q_1, q_2) \) and \( p = (p_1, p_2) \). The linearized equation of motion for the deviations are

\[
\frac{d\delta p}{dt} = M^{-1} \delta p, \quad \frac{d\delta q}{dt} = -S(t) \delta q,
\]

(4)

where \( M_{ij}^{-1} = \delta_{ij} B_i^{-1} \), and

\[
S_{ij}(t) = \left. \frac{\partial^2 V}{\partial \dot{q}_i \dot{q}_j} \right|_{q=q(t)},
\]

(5)

where \( q(t) \) is the solution of (3). The stability of the dynamical system is then determined by the eigenvalues of the \( 4 \times 4 \) stability matrix

\[
\Gamma(q(t)) = \begin{pmatrix} 0 & M^{-1} \\ -S(t) & 0 \end{pmatrix}.
\]

(6)

If at least one of the eigenvalues \( \lambda_i \) of the stability matrix \( \Gamma \) is real, then the separation of the trajectories grows exponentially and the motion is unstable. Imaginary eigenvalues correspond to stable motion.

To diagonalize the matrix \( \Gamma \), we must first solve the equation of motion (3). The problem can be significantly simplified by assuming that the time dependence can be eliminated by replacement of the time–dependent point \( q(t) \) of configuration space by a time–independent coordinate \( q \), i.e. \( \Gamma(q(t)) = \Gamma(q) \). The eigenvalues then are

\[
\lambda = \pm \left[ -b \pm \sqrt{b^2 - 4c} \right]^\frac{1}{2},
\]

(7)
where
\[ b = B_1^{-1}B_2^{-1}\left[ \frac{\partial^2 V}{\partial q_1^2} + \frac{\partial^2 V}{\partial q_2^2} \right], \quad (8) \]
\[ c = B_1^{-1}B_2^{-1}\left[ \frac{\partial^2 V}{\partial q_1^2} \frac{\partial^2 V}{\partial q_2^2} - \left( \frac{\partial^2 V}{\partial q_1 q_2} \right)^2 \right]. \quad (9) \]

Now, if \( b > 0 \) then with \( c \geq 0 \) the eigenvalues are purely imaginary and the motion is stable, while with \( c < 0 \) the pair of eigenvalues becomes real and this leads to exponential separation of neighboring trajectories, i.e. chaotic motion. The parameter \( c \) has the same sign as the Gaussian curvature \( K_G \) of the potential–energy surface:
\[ K_G(q_1, q_2) = \frac{\frac{\partial^2 V}{\partial q_1^2} \frac{\partial^2 V}{\partial q_2^2} - \left( \frac{\partial^2 V}{\partial q_1 q_2} \right)^2}{1 + \left( \frac{\partial^2 V}{\partial q_1^2} \right)^2 + \left( \frac{\partial^2 V}{\partial q_2^2} \right)^2}. \quad (10) \]

For our nuclear model the potential energy is given by
\[ V(q_1, q_2) = \frac{1}{2}(C_1 q_1^2 + C_2 q_2^2) - k_1 q_1^2 q_2, \quad (11) \]
and the \( \Gamma \) matrix reads
\[ \Gamma(q_1, q_2) = \begin{pmatrix}
0 & 0 & B_1^{-1} & 0 \\
0 & 0 & 0 & B_2^{-1} \\
-C_1 + 2k_1 q_2 & 2k_1 q_1 & 0 & 0 \\
2k_1 q_1 & -C_2 & 0 & 0
\end{pmatrix}. \quad (12) \]

The potential has one minimum for \( q_1 = q_2 = 0 \) with \( V = 0 \), and two saddle points for \( q_1 = \pm \sqrt{C_1 C_2 / 2k_1^2} \), \( q_2 = \frac{C_1^2 C_2}{8k_1^2} \). These are the points for which the curvature criterion is exact (they are the fixed points of the Hamilton equations). The origin is a stable elliptic point and the two saddle points are unstable hyperbolic points.
The dissociation energy of this model is \( E_D = \frac{C^2_2 C_2}{8k_1} \) and it occurs at the saddle points of the potential energy surface. Classically, a particle located within the potential well with energy \( 0 < E < E_D \) is always bound.

The Gaussian curvature vanishes at the points that satisfy the equation

\[
C_2(C_1 - 2k_1 q_2) - 4k_1^2 q_1^2 = 0 ,
\]

where

\[
q_1^2 = \frac{C_2}{4k_1^2} (C_1 - 2k_1 q_2) ,
\]

with \( q_2 \leq \frac{C_2}{2k_1} \). At low energies, the motion near the minimum of the potential energy, where the curvature is positive, is periodic or quasi-periodic and is separated from the region of instability by a line of zero curvature; if the energy increases, the system will be, for some initial conditions, in a region of negative curvature where the motion is chaotic. In accordance with this scenario, the energy of order→chaos transition \( E_c \) is equal to the minimum value of the line of zero Gaussian curvature \( K_G \) of the potential–energy surface of the system\(^6\),\(^7\). The energy of the zero–curvature line is determined by the expression

\[
V(K_G = 0, q_2) = C_2 q_2^2 - \frac{C_1 C_2}{2k_1} q_2 + \frac{C_2^2 C_2}{8k_1^2} .
\]

It is easy to show that the minimal energy on the zero–curvature line is given by

\[
E_c = V(K_G = 0, q_2 = \frac{C_1}{4k_1}) = \frac{C_1^2 C_2}{16k_1^2} .
\]

This is the energy of the onset of chaos of the model and we see that \( E_c = E_D/2 \).
With the estimation \( k_1 = (5/2) \sqrt{2/(2\pi)} C_1 \) we have

\[
E_c = \frac{\pi}{150} C_2 .
\]  

(17)

By using the charged liquid drop approximation\(^4\) for the stiffness coefficient \( C_2 \), we obtain for \( E_c \) the following relation

\[
E_c = (b_s \pi A^2 - 3 Z^2 e^2 \pi \frac{1}{150}) ,
\]  

(18)

where \( b_s \approx 17 \text{ MeV} \), \( r_0 \approx 1.2 \text{ fm} \), \( A \) and \( Z \) are the atomic and proton numbers respectively (see Figure 1). With the liquid-drop approximation\(^4\), the Coulomb repulsion counteracts the effect of the surface tension and leads to negative values of \( C_2 \) for \( Z^2/A > (10/3)(b_s r_0/e^2) \approx 49 \).

We can use the experimental quadrupole frequency \( \hbar \omega_2 \) and \( B(E2) \) values to determine\(^4\) the restoring force parameter \( C_2 \). In this case

\[
E_c = \frac{\pi}{60} \hbar \omega_2 \left( \frac{3}{4\pi} Ze R_0^2 \right)^2 B(E2; 0 \rightarrow 2)^{-1} .
\]  

(19)

The experimental values of \( C_2 \) exhibit variations greater by a factor 10 above and below the liquid-drop estimation. This effect can be understood in terms of the approach to instability as particles are added to the closed shells and so \( E_c \) is in the range \( \approx 0.5-5 \text{ MeV} \) depending on the nucleus.

The curvature criterion is able to characterize the local behaviour of the system (for example the local instability) and may give only a signature of the global properties (e.g. the global instability). As is well known, a very useful tool for the study of global properties is provided by the Poincaré sections\(^9\). With this aim the classical trajectories have been calculated by the fourth order Runge–Kutta method. In order to avoid numerical errors
connected to the use of finite temporal intervals, a first-order interpolation has been used\textsuperscript{10}.

The Hamilton equations of the systems are:

\begin{align*}
\dot{q}_1 &= B_1 p_1 , \quad \dot{q}_2 = B_2 p_2 , \\
\dot{p}_1 &= -C_1 q_1 + 2k_1 q_1 q_2 , \quad \dot{p}_2 = -2C_2 q_2 + k_1 q_1^2 .
\end{align*}

(20)

Figure 2 shows the Poincarè sections for different values of the energy. Below the critical energy $E_c$ the system is regular but above $E_c$, the Poincarè sections clearly show an order–chaos transition as the energy increases.

In conclusion, the Poincarè sections show that in our schematic model, which describes the interaction between collective dipole and quadrupole modes in atomic nuclei, the Toda criterion of the Gaussian curvature gives a quite good estimate of the energy $E_c$ of the onset of chaos. This critical energy $E_c$ depends on the stiffness coefficient of the quadrupole vibration, which can be estimated analytically within the liquid-drop approximation or can be experimentally determined by using the quadrupole frequency $\hbar \omega_2$ and the $B(E2)$ transition probabilities.

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Figure Captions

**Figure 1**: The critical energy $E_c$ *versus* the atomic number $A$, with $Z = A/2$.

**Figure 2**: The Poincaré sections for different values of the energy; from the top-left, clockwise: $E = 0.02$, $E = 0.03$, $E = 0.035$ and $E = 0.04$. We chose $B_1 = B_2 = 1$, $C_1 = 100$ and $C_2 = 1$. The critical energy is $E_c = \frac{\pi}{150} C_2 \simeq 0.021$. 