Spherical gravitational collapse and electromagnetic fields in radially homothetic spacetimes

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We consider a spherically symmetric, Petrov-type D, spacetime with hyper-surface orthogonal, radial, homothetic Killing vector. In this work, some general properties of this spacetime for non-singular and non-degenerate data are presented. We also present the source-free electromagnetic fields in this spacetime. We then discuss general astrophysical relevance of the results obtained for this spacetime.

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I. INTRODUCTION

In general relativity, non-gravitational processes are included via the energy-momentum tensor for matter. Various non-gravitational processes determine, apart from other physical characteristics such as radiation, the relation of density and pressure of matter. The temporal evolution of matter is to be determined from such a relation, and from other physical relations, if any.

By physically realizable gravitational collapse, we mean gravitational collapse of matter that leads matter, step by step, through different “physical” stages of evolution, namely, from dust to matter with pressure and radiation, to matter with exothermic nuclear reactions etc.

Then, the spacetime of “physically realizable” collapse of matter must be able to begin with any stage in the chain of evolution of matter under the action of its self-gravity. The temporal evolution from any “initial” data, any “physical” stage in question, is to be obtained from applicable non-gravitational properties of matter. This is as per the principle of causal development of data.

In a recent work [1], we obtained a spherically symmetric spacetime by considering a metric separable in co-moving coordinates and by imposing a relation of pressure $p$ and density $\rho$ of the barotropic form $p = \alpha \rho$ where $\alpha$ is a constant. Such a relation determined only the temporal metric functions of that spacetime.

Therefore, although the temporal metric functions for this spacetime were determined in [1] using the above barotropic equation of state, same metric functions are determinable from any relation of pressure and density. (See later.)

This spacetime admits a hyper-surface orthogonal, radial, Homothetic Killing Vector (HKV). (See later.) Hence, it will be called a radially homothetic spacetime. Since this spacetime describes appropriate “physical” stages of evolution of spherical matter, we have argued in [2] that it is the spacetime of physically realizable spherical collapse of matter. We have also shown in [2] that it is a Petrov-type D spacetime. We note that all general relativistic black hole spacetimes are Petrov-type D spacetimes.

In [3], we studied the shear-free problem for the sake of physical understanding of the issues involved in the study of the spherical gravitational collapse in the spacetime of [1].

In this paper, we consider a radially homothetic spherically symmetric problem for non-singular and non-degenerate matter data in its full “physical” generality, ie, with shear and radiation. Further, we also obtain the source-free electromagnetic fields in this spacetime explicitly.

We describe, in §II, the spacetime under consideration and its properties. Next, we summarize the Hertz-Debye formalism in §III. The electromagnetic fields in a radially homothetic spacetime are obtained in §IV. In §V, we discuss the astrophysical relevance and other implications of the results obtained in these works.

II. SPACETIME METRIC

In general, a HKV captures [1] the notion of the scale-invariance of the spacetime. If, in terms of the chosen coordinates, a homothetic Killing vector $X$ has component only in the direction of one coordinate, the Einstein field equations separate for that coordinate, generating also an arbitrary function of that coordinate. This is the broadest (Lie) sense of the scale-invariance leading not only to the reduction of the field equations as partial differential equations to ordinary differential equa-
A spherically symmetric spacetime has only one spatial scale associated with it - the radial distance scale. Therefore, for a radially homothetic spacetime, the metric admits one arbitrary function of the radial coordinate. We then obtain for a radially homothetic spacetime arbitrary radial characteristics for matter. That is, due to the radial scale-invariance of the spherical spacetime, matter has arbitrary radial properties in a radially homothetic spacetime.

In co-moving coordinates, a radially homothetic, spherically symmetric spacetime admits a spacelike HKV of the form

$$X^a = (0, \frac{y}{y'}, 0, 0)$$  \hspace{1cm} (1)

and the spacetime metric is given by

$$ds^2 = -y'^2 dt^2 + \gamma^2(y')^2 B^2 dr^2 + y^2 Y^2 d\Omega^2$$  \hspace{1cm} (2)

with a prime indicating a derivative with respect to \(r\), \(B \equiv B(t)\), \(Y \equiv Y(t)\) and \(\gamma\) being a constant. (We absorb the temporal function in \(g_{tt}\) by redefinition of the time coordinate.)

The Ricci scalar for (2) is:

$$\mathcal{R} = \frac{4YB}{y'^2 Y B} + \frac{2B}{y'^2 Y} - \frac{6}{y'^2 \gamma^2 B^2}$$

$$+ \frac{2}{y^2 Y^2} + \frac{2Y^2}{y'^2 Y} + \frac{4Y}{y'^2 Y}$$  \hspace{1cm} (3)

The non-vanishing components of the Weyl tensor for (2) are:

$$C_{trtr} = \frac{B^2 \gamma^2 (y')^2}{3} F(t)$$  \hspace{1cm} (4)

$$C_{t\theta t\theta} = -\frac{y^2 Y^2}{6} F(t)$$  \hspace{1cm} (5)

$$C_{t\phi t\phi} = \sin^2 \theta C_{t\theta t\theta}$$  \hspace{1cm} (6)

$$C_{r\theta r\theta} = \frac{B^2 \gamma^2 Y^2 (y')^2}{6} F(t)$$  \hspace{1cm} (7)

$$C_{r\phi r\phi} = \sin^2 \theta C_{r\theta r\theta}$$  \hspace{1cm} (8)

$$C_{\theta \phi \theta \phi} = -\frac{y^2 Y^4 \sin^2 \theta}{3} F(t)$$  \hspace{1cm} (9)

where

$$F(t) = \frac{\ddot{Y}}{Y} - \frac{\dot{Y}^2}{Y^2} - \frac{1}{Y^2} - \frac{\ddot{B}}{B} + \frac{\dot{B} \dot{Y}}{BY}$$  \hspace{1cm} (10)

In what follows, we shall assume, unless stated explicitly, that there are no singular initial-data and that there are no degenerate situations for the metric (3). See later for the singularities and the degeneracies of the metric (2).

It is well-known that spherical spacetimes are either Petrov-type O or Petrov-type D. Type-D spacetimes are not conformally flat. As can be easily verified, the spacetime of (2) is not conformally flat for non-singular and non-degenerate data. That the spacetime of (2) is of Petrov-type D can also be seen differently by verifying that only \(\Psi_2 = -C_{abcd}m^a \tilde{m}^b n^c \tilde{n}^d\) is non-vanishing for the metric (2) where \(\ell, n, m\) and \(\tilde{m}\) are the Newman-Penrose tetrad vectors for it.

### A. Elementary flatness and center

Now, the spacetime of (2) is required, by definition, to be locally flat at all of its points including the center.

In the case of (2), a small circle of coordinate radius \(\epsilon\) with center at the origin has circumference of \(2\pi\epsilon\). On the other hand, the circle has the proper radius \(\gamma y'\epsilon\). Then, requiring that the ratio of the circumference to the proper radius of the circle to be \(2\pi\) in the neighborhood of the origin, we obtain the condition for the center to possess a locally flat neighborhood as

$$y'|_{r=0} \approx 1/\gamma$$  \hspace{1cm} (11)

This condition must be imposed on any \(y(r)\). With this condition, (11), the HKV of metric (2) is, at the center, \(y|_{r=0} \partial/\partial r\).

Now, \(y(r)\) is the “area radius” in (2). When \(y|_{r=0} = 0\), the orbits of the rotation group \(SO(3)\) do not shrink to zero radius at the center for (2). Consequently, the center is not regular for (2) when \(y|_{r=0} = 0\) although the curvature invariants remain finite at the center.

Also, when \(y|_{r=0} = 0\), the center is regular for the spacetime of (2). But, the curvature invariants blow up at the center, then.

It is well-known that the center and the initial data for matter, both, are not simultaneously regular for a spherical spacetime with hyper-surface orthogonal HKV. Therefore, the spacetime of (2) does not possess a regular center and regular matter data, simultaneously.

However, the lack of regularity of the center of (2) for non-singular matter data is understandable since the orbits of the rotation group do not shrink to zero radius for every observer. It is a relative conception and the co-moving observer of (2) is not expected to observe the orbits shrink to zero radius. (See also later.)
B. Singularities of spacetime

Clearly, we may use the function \( y(r) \) in (2) as a new radial coordinate - the area coordinate - as long as \( y' \neq 0 \). However, the situation of \( y' \equiv 0 \) represents a coordinate singularity that is similar to, for example, the one on the surface of a unit sphere where the analogue of \( y \) is \( \sin \theta \). The curvature invariants do not blow up at locations for which \( y' = 0 \).

The genuine spacetime singularities of the strong curvature, shell-focussing type exist when either \( y(r) = 0 \) for some \( r \) or when the temporal functions vanish for some \( t = t_\ast \).

There are, therefore, two types of curvature singularities of the spacetime of (2), namely, the first type for \( B(t_\ast) = 0 \) and, the second type for \( y(r) = 0 \) for some \( r \).

Note that the “physical” radial distance corresponding to the “coordinate” radial distance \( \delta r \) is

\[
\ell = \gamma(y') B \delta r
\]

Then, collapsing matter forms the spacetime singularity in (2) when \( B(t) = 0 \) is reached for it at some \( t = t_\ast \). Therefore, the singularity of first type is a singular hyper-surface for (2).

The singularity of the second type is a singular sphere of coordinate radius \( r \). The singular sphere reduces to a singular point for \( r = 0 \) that is the center of symmetry. For \( y(r) = 0 \) for some range of \( r \), there is a singular thick shell. Singularities of the second type constitute a part of the initial data, singular data, for the evolution.

C. Degeneracies of the metric (2)

The metric (2) has evident degeneracies when \( y(r) = 0 \), \( y(r) \to \infty \) either on a degenerate sphere of coordinate radius \( r \), for some “thick shell” or globally. The degeneracy \( y(r) = 0 \) is equivalent to an infinite density while the degeneracy \( y(r) = \infty \) is equivalent to vacuum. (See later for the expression of density of matter in the spacetime of (2).) Another degeneracy occurs for \( y(r) = \text{constant} \) for some “thick” shell or globally. This degeneracy corresponds to uniform density.

D. Self-similarity Nature

Any vector (1) can always be transformed (3) into

\[
\tilde{X}^a = (T, S, 0, 0)
\]

via a non-singular coordinate transformation

\[
S = l(t) \exp \left( \int F^{-1} dr \right) \\
T = k(t) \exp \left( \int F^{-1} dr \right)
\]

We note that, if we invoke (4) for (2), the resulting metric will not be diagonal. The imposition of diagonality of the metric will require a relationship between \( l(t) \) and \( k(t) \). Such a relation can, of course, always be imposed.

Hence, the metric (2) can always be transformed, under non-singular coordinate transformations (4), to a form which admits a HKV (3) in the transformed coordinates. The transformed metric under consideration is, therefore,

\[
ds^2 = -P^2 dt^2 + Q^2 dS^2 + S^2 Z^2 d\Omega^2
\]

where \( P, Q, Z \) are the metric functions of the self-similarity variable \( T/S \) or \( S/T \). Note that for the transformed metric the radiation or the heat flux is, in general, non-vanishing.

For (13), we are therefore led to consider the spacetime singularity at \( S = 0 \) and \( T = 0 \).

As we noted earlier, a relationship exists between \( l(t) \) and \( k(t) \) of (14). Thus, \( S = 0 \) and \( T = 0 \) for (13) corresponds to \( l(t) = 0 \) for (2) when \( l(t) \propto k(t) \) and \( l(t) = 0 \). There is thus no constraint on the radial function \( y(r) \) in (2) that it should vanish. Consequently, \( y(r) \neq 0 \) at \( r = 0 \) or, for that matter, at any \( r \), is permissible. Then, \( y|_{r=0} \) is arbitrary.

However, \( S = 0 \) and \( T = 0 \) also corresponds to \( y(r) = 0 \) in (13) with \( l(t) \) not proportional to \( k(t) \). However, assuming \( y(r) = 0 \) leads to initially singular density. Such a curvature singularity will then always exist on any hyper-surface \( t = \text{constant} \) in the spacetime of (2).

We note that we may begin with the metric form (13) even when the transformations (4) are singular. The metric (13) is then not reducible to the form (2). For such a spacetime, the Einstein field equations reduce to ordinary differential equations. However, the resultant equations are not entirely separated in terms of the variables \( T \) and \( S \).

E. Mass function

The mass function for the spacetime of (2) can be defined as

\[
m(r,t) = \frac{y}{2} \left( 1 - \frac{Y^2}{\gamma B^2} \right)
\]
where \( m(r, t) \) denotes the effective total energy per unit mass of a fluid element labelled by the co-moving radial coordinate \( r \) at co-moving time \( t \) in the spacetime. Note that it includes the “effective” contribution due to the flux of radiation or heat in the spacetime.

Note that, using (13), we may rewrite the metric (12) as:

\[
ds^2 = -\, dt^2 + \frac{R^2 \, dr^2}{1 - (2m/R)} + R^2 \, d\Omega^2
\]

where \( R \equiv R(\tau, r) = yY, \) \( m \equiv m(\tau, r) \) and \( d\tau = ydt \). Now, the metric (12) is recognizable as a generalization of the Tolman-Bondi dust metric to include pressure and radiation.

Now, we note the following. An asymptotic observer of the Schwarzschild spacetime does not see a sphere, drawn around the central mass point, shrink to a zero radius as a result of the red-shift of the sphere becoming infinite at the gravitational radius. Consequently, the “area radius” of the sphere does not shrink to a zero radius as a result of the red-shift for the co-moving observer of (2), the central Schwarzschild mass. For the asymptotic observer, \( r = 2M \) is then the “center” of the spacetime.

A co-moving observer, as a cosmological observer in (3), is “equivalent” to the asymptotic observer of the Schwarzschild geometry. Hence, for the co-moving observer of (3), the area of any sphere cannot shrink to zero radius at the center since there is always an equivalent mass point at the center for that observer. (See also (11).)

Therefore, the lack of regularity of the center of (3) for non-singular matter data is understandable since the orbits of the rotation group do not shrink to zero radius for the co-moving observer of (3). Note also that the singular hyper-surface of (3) is then the infinite-mass singularity.

F. Field equations

For completeness, we reproduce the arguments of (3) here.

Now, define the co-moving time-derivative

\[
D_t \equiv U^a \frac{\partial}{\partial x^a} = \frac{1}{y} \frac{\partial}{\partial t}
\]

where

\[
U^a = \frac{1}{y} \delta^a_t
\]

is the four-velocity of the co-moving observer. Then, the radial velocity of fluid with respect to the co-moving observer is

\[
V^r = D_t (yY) = \dot{Y}
\]

where an overhead dot has been used to denote a time derivative.

The co-moving observer is accelerating for (3) since

\[
\dot{U}_a = U_{a:}^b U^b = \left( 0, \frac{y'}{y}, 0, 0 \right)
\]

is, in general, non-vanishing for \( y' \neq 0 \). The expansion is

\[
\Theta = \frac{1}{y} \left( \frac{\dot{B}}{B} + \frac{2 \dot{Y}}{Y} \right)
\]

The Einstein tensor for (3) is:

\[
G_{tt} = \frac{\gamma^2 B^2 y^2}{y^2} \left[ -2 \frac{Y'}{Y} + \frac{Y^2}{Y} + \frac{3}{\gamma^2 B^2} \frac{1}{Y^2} \right]
\]

\[
G_{rr} = -\gamma \dot{Y} \dot{Y} - \frac{\dot{B}}{B} Y - \frac{B}{b} \frac{\dot{Y}}{Y} + \frac{Y^2}{\gamma^2 B^2}
\]

\[
G_{\theta\theta} = \sin^2 \theta G_{\theta\theta}
\]

\[
G_{t\tau} = \frac{\dot{B} y'}{B y}
\]

Now, to see, and only to see, that \( \dot{B} \) is related to the flux of radiation in the spacetime of (3), we may consider matter to be described by the energy-momentum tensor

\[
\tilde{T}_{ab} = (p + \rho) U_a U_b + \rho g_{ab}
\]

and consider that it fails to satisfy a local conservation law due to the emission of radiation that escapes radially along the radial null vector \( \ell^a \). For the radiation, we may then assume the “geometrical optics” form

\[
E^{ab} = Q \ell^a \ell^b
\]

with \( Q \) being the energy density of radiation or the energy flux density in the rest frame of the fluid. Then, it is seen that \( Q \propto \dot{B} \). Thus, the radiation-flux depends on \( \dot{B} \).

Now, define the quantity

\[
\sigma \equiv \sigma^1_1 = \sigma^2_2 = -\frac{1}{2} \sigma^3_3 = \frac{1}{3y} \left( \dot{Y} - \frac{\dot{B}}{B} \right)
\]
Here, $\sigma_{ab}$ represents the shear-tensor of the fluid and the shear-scalar is given by $\sqrt{\sigma}$. The spacetime of (2) is shearing when $B(t)$ is not proportional to $\dot{Y}(t)$. Therefore, the spacetime of (2) is, in general, shearing and radiating, both.

Now, we note that Penrose [11] is led to the Weyl hypothesis on the basis of thermodynamical considerations, in particular, those related to the thermodynamic arrow of time. On the basis of these considerations, we may consider the Weyl tensor to be “some” sort of measure of the entropy in the spacetime at any given epoch.

Then, for non-singular and non-degenerate data in (2), the Weyl tensor of (2) blows up at the singularity, but is “constant” at the “initial” hyper-surface since $\dot{Y} = \dot{B} = 0$ for the “initial” hyper-surface (2).

This behavior of the Weyl tensor of (2) is in conformity with Penrose’s Weyl curvature hypothesis [11]. Thus, the spacetime of (2) has the “right” kind of thermodynamic arrow of time in it.

**Stages of collapse**

Now, we turn to steps of collapse of matter in the spacetime of (2) for non-singular and non-degenerate data. For non-singular and non-degenerate data, we then have a “cosmological” situation - continued spherical collapse of matter from the assumed “initial” state.

**Step I - Evolution of dust**

We begin with collision-less and pressureless dust matter with density distribution given such that $y(r) > 1$ everywhere on the initial hyper-surface in the spacetime of (2). Emission of radiation and, hence, radiation itself is not expected in such dust.

Then, self-gravity leads to mass or energy-flux in the radial direction. But, this is not the flux of radiation. Therefore, there is no mass-flux in the rest frame of collapsing dusty matter, but it is present for other observers in the spacetime.

Then, for vanishing flux of radiation, we have from (27), $B = \text{constant} \equiv B_o$

Then, the co-moving density, $\rho$, of dust is

$$\rho = \frac{1}{y^2} \left[ \dot{Y}^2 \frac{\dot{Y}^2}{Y^2} + \frac{1}{Y^2} - \frac{1}{\gamma^2 B_o^2} \right]$$

and the function $Y(t)$ is determined by the condition of vanishing of the isotropic pressure:

$$4Y\ddot{Y} + \dot{Y}^2 + 1 - \zeta Y^2 = 0$$

Here, $\zeta = 5/\gamma^2 B_o^2$, a positive constant.

A solution of this equation is obtainable as

$$\frac{dY}{\sqrt{-1 + \zeta/5Y^2 + c_o Y^{-1/2}}} = t - t_0$$

where $c_o$ is constant. Since $\dot{Y}$ is the radial velocity of matter for the co-moving observer, we require that solution for which $\dot{Y} \rightarrow 0$ for $t \rightarrow -\infty$.

That the dust exists in the spacetime of (2) is not surprising since it is a generalization of the Tolman-Bondi dust metric.

**Step II - Evolution with pressure and radiation**

Next stage of collapse is reached when particles of dust begin to collide with each other. Negligible amount of radiation, but existing nonetheless, is expected from whatever atomic excitations or from whatever free electrons get created in atomic collisions in such matter. Therefore, dusty matter evolves into matter with pressure and radiation, both simultaneously non-vanishing.

The energy-flux can no longer be removed by going to the rest frame of matter.

Now, pressure and radiation, both, get simultaneously switched on in the spacetime of (2) when $B(t) \neq 0$. This is as per the expectation that dusty matter evolves to one with simultaneous occurrence of pressure and radiation, both.

Then, the co-moving density, $\rho$ and isotropic pressure, $p$, are given by

$$\rho = \frac{1}{y^2} \left[ \dot{Y}^2 \frac{\dot{Y}^2}{Y^2} + \frac{2}{Y} \frac{\dot{Y} B}{Y B} + \frac{1}{Y^2} - \frac{1}{\gamma^2 B_o^2} \right]$$

$$p = \frac{1}{y^2} \left[ \frac{5}{3\gamma^2 B_o^2} - \frac{1}{3Y^2} - \frac{4\dot{Y}}{3Y} - \frac{2\dot{B}}{3B} - \frac{Y^2}{3Y^2} - \frac{2\dot{Y}}{3B} \right]$$

From (33) and (34), we obtain

$$\frac{2}{Y} \frac{\ddot{Y}}{Y} + \frac{\dot{B}}{B} = \frac{2}{\gamma^2 B_o^2} \frac{y^2}{2} (\rho + 3p)$$

Then, from (33), the relation of pressure and density of matter is the required additional “physical” information. Also required is other relevant “physical” information to determine the radiation generation in the spacetime of (2). This is a non-trivial task in general relativity just as it is for Newtonian gravity.

Note that the radiation may be “negligible” but what is important to considerations here is its presence in the spacetime.
Step III - Stellar object

At some further stage of evolution, radiation from “central part” of collapsing matter may become non-negligible and the temperatures in the central region may become appropriate for exothermic, thermonuclear reactions. With the onset of exothermic thermonuclear reactions, a “shining” star is born in the spacetime.

The exothermic thermonuclear reactions in the stellar core may support the overlying “stellar layers” and such a stellar object may “appear” gravitationally stable.

But, the spacetime continues to be dynamic since radiation is present in it. The central stellar object may also accrete matter from its surrounding while emitting radiation.

Now, as and when “heating” of the overlying stellar layers decreases due to changes in exothermic thermonuclear processes in the core of the star, the self-gravity of the stellar object leads to its gravitational contraction. These are, in general, very slow and involved processes.

Gravitational contraction leads to generation of pressure by compression and by the occurrence of exothermic thermonuclear reactions involving heavier nuclei. The star may stabilize once more.

This chain, of gravitational contraction of star, followed by pressure increase, followed by subsequent stellar stabilization, continues as long as thermonuclear processes produce enough heat to support the overlying stellar layers.

The theory of the atomic nucleus shows that exothermic nuclear processes do not occur when Iron nucleus forms. With time, the rate of heat generation in iron-dominated-core becomes insufficient to support the overlying stellar layers which may then bounce off the iron-core resulting into a stellar explosion, a supernova.

Then, many, different such, stages of evolution are the results of physical processes that are unrelated to the phenomenon of gravitation. These are, for example, collisions of particles of matter, electromagnetic and other forces between atomic or sub-atomic constituents of matter etc.

As an example, let some non-gravitational process, opposing collapse, result into pressure that does not appreciably rise in response to small contraction of the stellar matter. That is, pressure does not appreciably rise when gravitational field is increased by a small amount. Then, the collapse of a sufficiently massive object would not be halted by that particular non-gravitational process. Therefore, a mass limit is obtained in this situation. For example, electron degeneracy pressure leads to the Chandrasekhar limit.

Clearly, some of the non-gravitational processes determine the gravitational stability of physical objects. This is true in Newtonian gravity as well as in general relativity, both.

If we consider that density alone does not determine pressure completely but that the isotropic pressure is a function of density and temperature, both, then we need another equation, from thermodynamical considerations, perhaps. In this case, the relation of pressure and density can be considered to be a function of time.

But, since the relation of pressure and density is arbitrary for \( T \), a changing pressure-density relationship is allowed for it, we note. To provide for the required physical information is, once again, a non-trivial task.

Evidently, to provide for the required information of “physical” nature is a non-trivial task in general relativity just as it is for Newtonian gravity. The details of these considerations are, of course, very involved and have been left out of the considerations of the present paper.

We note that, in general, the relation of pressure and density at extremely high densities is not known. However, it can be surmised that, in the final stages of collapse, the collapsing matter will be ultra-relativistic and will end up in the singularity as such. Then, the relation of pressure and density of such matter may be expected to remain fairly unchanged in the final stages of the gravitational collapse. That is to say, the spacetime of collapsing matter should describe a relation of pressure and density of ultra-relativistic matter that is not changing in the final collapse stages.

Consequently, it seems reasonable to treat the thermodynamic state of collapsing matter near the spacetime singularity by a relation of the “barotropic” form \( p = \alpha \rho \) where \( \alpha \) is a constant characteristic of the collapsing matter. Therefore, by assuming this equation of state, we may obtain the temporal metric functions in (2) to study the gravitational collapse in its final stages. But, for (3), such a possibility of final stage of collapse is approached only asymptotically, that is for infinite co-moving time. (See §III later.)

But, it is clear that the field equations determine only the temporal functions of (3) from any suitable energy-momentum tensor including that of electromagnetic fields, if any.

The temporal functions \( B(t) \) and \( Y(t) \) are to be obtained from the properties of matter such as a relation of pressure and density, the rate of loss of internal energy to radiation, processes of quantum mechanical nature etc.

Moreover, it is also clear that matter will con-
time to pile up on such a star in a “cosmological setting” and, hence, such a star will always be taken over any mass-limit in operation at any stage of its evolution. The evolution of collapsing matter will, asymptotically, lead to the singular hyper-surface of the spacetime of (3).

The radial dependence of matter properties is “specified” as $1/y^2$ but the field equations of general relativity do not determine the metric function $y(r)$ in (3).

Therefore, the radial distribution of matter is arbitrary in terms of the co-moving radial coordinate $r$. This is the “maximal” physical freedom compatible with the assumption of spherical symmetry, we may note. Note, however, that the physical generality here is not be taken to mean the “geometrical” generality.

G. Absence of null or one-way membrane

A spherical surface $r = \text{constant}$ in the geometry of (3) has a normal vector

$$n_a = (0, 1, 0, 0) \quad n^a n_a = \frac{1}{\gamma^2(y)^2 B^2} \quad (37)$$

Within the range $(0, \infty)$ of the co-moving radial coordinate $r$, the character of $n^a$ does not change from spacelike to null to timelike in the spacetime of (3). Then, the norm $n^a n_a$ does not vanish at any $r$ in non-singular and non-degenerate cases.

To see the same differently, the coordinate speed of light in the spacetime of (3) is

$$\frac{dr}{dt} = \pm \frac{y}{\gamma(y^2)B} \quad (38)$$

This speed cannot vanish for non-singular and non-degenerate cases. At the singular hyper-surface, the coordinate speed of light becomes infinite.

Therefore, in the spacetime of (3), there cannot be a spherical, spatially finite, null membrane or a one-way membrane, i.e., a black hole in the usual sense of the term, in non-singular and non-degenerate cases.

H. Spherical collapse, singular hyper-surface and the infinite red-shift surface

We emphasize that we are using non-singular and non-degenerate data for (3). A co-moving observer in (3) is then "a cosmological observer". The four-velocity of matter fluid with respect to a co-moving observer is:

$$u^a = (u^t, u^r, 0, 0) \quad V^r = \frac{u^r}{u^t} \quad (39)$$

We then obtain from the metric (3):

$$u^a = \frac{1}{y \sqrt{\Delta}} (1, V^r, 0, 0) \quad (40)$$

$$\Delta = 1 - \gamma^2 \left(\frac{y'}{y}\right)^2 B^2 (V^r)^2 \quad (41)$$

Now, if $d\tau_{CM}$ is a small time duration for a co-moving observer and if $d\tau_{RF}$ is the corresponding time duration for the observer in the rest frame of matter, then we have

$$d\tau_{CM} = \frac{d\tau_{RF}}{\sqrt{\Delta}} \quad (42)$$

From (12), we also get the red-shift formula

$$\nu_{CM} = \nu_{RF} \sqrt{\Delta} \quad 1 + z = 1 + \frac{\nu_{RF}}{\nu_{CM}} \quad (43)$$

in the spacetime of (3) where $\nu_{CM}$ is frequency of a photon in the co-moving frame, $\nu_{RF}$ is the frequency in the rest frame and $z$ is the red-shift of the photon. Then, $\Delta = 0$ is the infinite red-shift surface that is, however, not a null membrane. A co-moving observer waits for an infinite period of its time to receive a signal from the rest-frame observer when $\Delta = 0$.

Then, we distinguish regions of the spacetime of (3) as

$$(\Delta > 0) \quad |\gamma(y') BV^r| < y \quad (44)$$

$$(\Delta = 0) \quad |\gamma(y') BV^r| = y \quad (45)$$

$$(\Delta < 0) \quad |\gamma(y') BV^r| > y \quad (46)$$

Now, the geodesic equations of motion for (3) are easily obtainable. The $r$-equation is:

$$\frac{d}{ds} \left(\gamma^2 B^2 y y' \tilde{r} \right) = 2\mathcal{L} \quad (47)$$

where an overhead tilde denotes derivative with respect to the affine parameter $s$. Then,

$$\tilde{r} = \frac{2\mathcal{L} s + k_1}{\gamma^2 B^2 y y'} \quad (48)$$

where $k_1$ is a constant of integration.

For the motion of the particle in the equatorial plane $\theta = \pi/2$, the solution of the $t$-equation of motion, using (18) in the lagrangian of motion, is

$$\tilde{t} = \sqrt{-\frac{2\mathcal{L}}{y^2} + \frac{1}{y^4} \left(\frac{2\mathcal{L}s + k_1)^2}{\gamma^2 B^2}\right)} \quad (49)$$

Then, since $V^r \equiv dr/ydt$, we have

$$V^r V_r = \frac{(k_1 - s)^2}{y^2 \gamma^2 B^2 + (k_1 - s)^2} \quad (2\mathcal{L} = -1) \quad (50)$$

where $s$ is the affine parameter along the geodesic and we have $V^r = 0$ at $s = k_1$. 

1. **Singular hyper-surface**

Clearly, \( V^r V_r = 1 \) for \( B(t_s) = 0 \) at \( t = t_s \), ie, the velocity of the particle with respect to a co-moving observer is the speed of light at the singular hyper-surface of \((2)\) - \( B(t_s) = 0 \) at \( t = t_s \). This is, of course, happening only asymptotically.

But, from \((41)\), \( \Delta = 1 \) for \( B(t_s) = 0 \). A co-moving observer also moves with the speed of light at the singular hyper-surface. After all, matter everywhere should become relativistic as the central mass condensate continues to grow (due to accretion onto it) to influence the entire spacetime to become relativistic everywhere. This too is happening only asymptotically. But, the singular hyper-surface is not the infinite red-shift surface.

2. **Infinite red-shift surface**

A spatially bounded infinite red-shift surface “occurs” in \((2)\) when \( \Delta = 0 \). Since

\[
\Delta = \frac{\gamma^2 y^2 B^2 + (k_1 - s)^2 (1 - \frac{1}{\ell^2})}{\gamma^2 y^2 B^2 + (k_1 - s)^2} \tag{51}
\]

this requires \( y(r) < 1 \).

Thus, the initial “density distribution”, from \( y(r) \), decides whether \((2)\) has an infinite red-shift surface or not, ie, whether \( y^2 = V^r V_r \). This is possible for \( V^r V_r \ll 1 \), but for sufficient mass concentration at the center, ie, \( y < 1 \).

But, a co-moving observer does not see the infinite red-shift surface form. Thus, an initially “small” matter density \( y > 1 \) cannot, for a co-moving observer, become “large” enough that the infinite red-shift surface forms \( y < 1 \).

Matter piles up at the center of the spacetime. But, the density at the center does not become infinite at any finite co-moving time.

Now, let \( y < 1 \), ie, let there be an infinite red-shift surface in \((2)\) at the initial epoch itself. (For \( y|_{r=0} = 0 \), there is an infinite-density singularity at the center.) Then, from \((48)\), it follows that a particle can reach and cross this infinite red-shift surface along its radial geodesic.

But, such a central region - “\( y < 1 \) apparition” - cannot communicate to its exterior since the co-moving time duration is imaginary in this region. Therefore, the interior of an infinite red-shift surface is causally disconnected from the rest of the spacetime at the initial time itself.

Then, matter external to \( y < 1 \) region eventually collapses onto this central “apparition” without a co-moving observer seeing matter enter it. The spacetime of \((2)\) then describes the accretion of matter onto the central “apparition” - the spatially bounded infinite red-shift surface.

We shall refer to the infinite red-shift surface, as described above, as the black hole surface and its interior as an interior of a black hole. This is the conception of a black hole that arises in the spacetime of \((2)\) for non-singular and non-degenerate data. But, here, a black hole is not a null membrane or horizon.

3. **Apparent horizon**

A radially outgoing null vector of \((2)\) is

\[
\ell^\alpha \partial_\alpha = \frac{1}{y} \frac{\partial}{\partial t} + \frac{1}{\gamma y B} \frac{\partial}{\partial r} \tag{52}
\]

Light gets trapped inside a particular radial coordinate \( r \) when the expansion of the above principle null vector vanishes at \( r \). The formation of the outermost light-trapping surface or the apparent horizon is then obtained by setting the expansion of \((52)\) to zero.

The zero-expansion of \((52)\) yields a condition only on the temporal metric functions as

\[
\frac{\dot{B}}{B} + 2 \frac{\dot{Y}}{Y} = -\frac{3}{\gamma B} \tag{53}
\]

The condition \((53)\) implies an “instant of time”.

This is seen as as follows. An outgoing photon moves along the trajectory

\[
\frac{dr}{dt} = \frac{y}{\gamma (y') B}
\]

and crosses a sphere of coordinate radius \( r \) at co-moving time \( t \).

The mass inside this sphere is given by \((16)\). Now, the equation of the light trajectory can be used to express the mass function \( m \) as a function of either \( r \) alone or \( t \) alone. The light trapping mass is then obtained by setting \( 2m = yY \). Then, for every value of \( r \) there is some \( t \) and vice versa for which \( 2m = yY \). Thus, condition \((53)\) implies, in essence, an “instant of light trapping”. Essentially, for non-singular and non-degenerate data in \((2)\), every instant, of the co-moving time, is an instant of light trapping.

Alternatively, let a spherical light front be emitted from the center of symmetry. As it travels radially outwards, it brings in more mass to its interior. When sufficient mass is in the interior, light trapping occurs. For non-singular and non-degenerate data in \((2)\), we can always draw a sphere containing enough mass that can trap light. This is the essence of the statement that “Every co-moving
instant is a ‘Light Trapping Instant’ in this spacetime’. In a sense, every observer is inside some light trapped sphere in (2) for non-singular and non-degenerate data.

I. Shell black hole

Now, an interesting possibility is that of (2) containing many concentric infinite red-shift surfaces. Then, what we have here is the possibility of shells of black holes!

Intuitively, this is the only possibility that can arise in spherical symmetry apart from that of a single spherical black hole as considered earlier. It is also clear that this possibility arises only as “initial data” in the spacetime of (2).

To analyze such a “shell” black hole, we will, of course, be required to use the gaussian coordinate system since \( y' = 0 \) at some radial locations. We recall that \( y' = 0 \) is a coordinate singularity of the metric (2).

That the shell black hole is obtained in the spacetime of (2) is not a coincidence since (2) is the spherically symmetric spacetime with maximal “physical” freedom.

J. When is the singularity of (2) a locally or globally naked singularity?

The radial null geodesic in (2) satisfies (58), ie,

\[
\frac{dt}{dr} = \pm \frac{y'}{y} B
\]

The above differential equation of the radial null geodesic does not possess a singularity for non-singular and non-degenerate data.

For non-degenerate data, the spacetime singularity develops as a result of only the temporal evolution of matter in the spacetime to the future, ie, as \( B(t) \to 0 \).

But, then the above tangent is vanishing at the singularity indicating that there does not exist a future-directed, timelike tangent at the spacetime singularity. Hence, the curvature singularity of (2) is neither globally nor locally visible for the non-singular and non-degenerate data. It is a singular hyper-surface occurring to the future of every observer in the spacetime of (2).

On the other hand, for singular data, ie, when \( y(r) = 0 \) for some \( r \), the equation of radial null geodesic (58) has a singularity at that \( r \).

Then, whether such a singularity or singular sphere is visible to any observer or not depends on the limit of the quantity \( y'/y \) as we approach the singular point for \( B(t) \neq 0 \). For some functions \( y(r) \) such a limit can be positive making the singularity in question a locally naked or a locally visible one.

Therefore, for the spacetime of (2), the visibility of the spacetime singularity is determined principally by whether we assume the existence of a visible singularity at the initial time or not. The temporal evolution of non-singular and non-degenerate data does not lead to a visible singularity in this spacetime, we note.

A short comment on the the possible naked singularity solutions of general relativity will not be out of place here.

For non-singular and non-degenerate data in (2), no naked singularities arise. A black hole, an infinite red-shift surface, can exist in (2) as a part of this initial data.

Now, naked singularities can “arise” for singular and degenerate data in (2). We emphasize that for singular and degenerate data we have to abandon the metric (2) and seek an entirely different solution of the field equations of general relativity. In cases wherein the HKV of these spacetimes of naked singularities is (13) and it is not reducible to (4), the rationale of “pure” radial self-similarity is lost for these spacetimes.

In the absence of “pure” radial scale-invariance, it is then not clear what is the “physical” significance of the existence of the HKV (13). Perhaps, there is none.

III. HERTZ-DEBYE FORMALISM

Having presented the properties of the spacetime of (2), we now turn to the Hertz-Debye formalism before embarking upon the nature of the electromagnetic fields in the spacetime of (2).

A. Hertz-Debye potentials

In flat space, Hertz [12] introduced two vector potentials \( \vec{P}_E \) and \( \vec{P}_M \) related to the standard electromagnetic potentials \( \Phi \) and \( \vec{A} \) as

\[
\Phi = -\nabla \cdot \vec{P}_E \quad \vec{A} = \frac{\partial \vec{P}_E}{\partial t} + \nabla \times \vec{P}_M \quad (54)
\]

and, hence, the bi-vector or the anti-symmetric second-rank tensor potential is related by second derivatives to the physical fields.

The gauge freedom associated with the Hertz potentials is such as to preserve the source-free character of the Maxwell equations while the gauge
terms appear as sources in the equations:

\[ \tilde{Q}_E = \tilde{\nabla} \times \tilde{G} \quad \tilde{Q}_M = -\frac{\partial \tilde{G}}{\partial t} - \tilde{\nabla} g \quad (55) \]

and

\[ \tilde{R}_E = -\frac{\partial \tilde{W}}{\partial t} - \tilde{\nabla} w \quad \tilde{R}_M = -\tilde{\nabla} \times \tilde{W} \quad (56) \]

where \((\tilde{G}, g)\) and \((\tilde{W}, w)\) are arbitrary 4-vectors.

This gauge freedom can be used [13] to reduce the Hertz bi-vector to purely radial vectors of the form \(\tilde{P}_E = P_E \hat{r}\) and \(\tilde{P}_M = P_M \hat{r}\) where \(\hat{r}\) is the unit radial vector. The functions \(P_E\) and \(P_M\) are the Debye potentials [14] and obey, both, a wave equation. It should be noted that only the “monopole” field is missing in this scheme [15].

In essence, the arbitrary, source-free electromagnetic field is specified by two scalar functions which obey a single, separable second-order wave equation. Therefore, a remarkable economy is achieved by the Debye potentials. In [15], this is expressed as: “since a zero rest-mass field possesses two degrees of freedom, no more economical representation of the Maxwell field is possible” than that provided by the Debye potentials.

B. Cohen & Kegeles generalization

Differential forms generalize the flat space Maxwell equations to any curved spacetime in a natural way. Define the Maxwell 2-form as

\[ f = \frac{1}{2} f_{ab} \omega^a \wedge \omega^b \quad (57) \]

where \(f_{ab}\) is the Maxwell tensor and \(\omega^a\) are the basis forms. The Maxwell equations are simply

\[ df = 0 \quad \delta f = 0 \quad (58) \]

where \(d\) is the exterior derivative and \(\delta = \star d \star\) is the co-derivative. Here, \(\star\) is the Hodge dual operation.

The Hertz bi-vector \(P\) (2-form) is related to the electro-magnetic four-potential 1-form \(A\) and the Maxwell 2-form as

\[ A = \delta P \quad f = d\delta P = -\delta dP \quad (59) \]

Then, the equality of the last two expressions in [13] requires that

\[ \triangle P \equiv (d\delta + \delta d)P \equiv (d \star d \star + \star d \star d)P = 0 \quad (60) \]

where \(\triangle\) is the harmonic operator.

The 2-form gauge terms are:

\[ Q = dG \quad R = \star dW \quad (61) \]

where \(G\) and \(W\) are arbitrary 1-forms. Therefore, the wave equation with the gauge terms is

\[ \triangle P = dG + \star dW \quad (62) \]

so that the transformed fields are

\[ f = d\delta P - dG = \star dW - \delta dP \quad (63) \]

The transformed fields still obey the source-free Maxwell equations as a consequence of the important identities:

\[ d^2 \equiv \delta^2 = 0 \quad (64) \]

resulting to \(df = 0\) and \(\delta f = 0\).

Equations (62) and (63) provide an elegant and fully covariant generalization of the Hertz potential formalism to curved spacetimes.

The problem now consists of determining special bi-vector directions in the spacetime so that (62) yields decoupled wave equations for the corresponding components of the potential for some choice of the gauge terms (61).

In a class of spacetimes, the principal directions of the Weyl tensor [5, 16] provide such special bi-vectors [13]. Such special bi-vector directions are defined geometrically and independently of the Maxwell fields to be computed. In essence, one chooses a null tetrad (the Carter tetrad) with one null vector aligned along the repeated principal null direction of the Weyl tensor of an algebraically special spacetime in this scheme.

This completes our overview of the Hertz-Debye formalism or the Cohen-Kegeles formalism [13]. We now turn to the problem of electromagnetic fields in the spacetime of [2].

IV. ELECTROMAGNETIC FIELDS IN THE SPACETIME OF [2]

Choosing an orthonormal tetrad as a basis, the spacetime metric [2] is

\[ ds^2 = -(\omega^0)^2 + (\omega^1)^2 + (\omega^2)^2 + (\omega^3)^2 \quad (65) \]

where

\[ \omega^0 = ydt \quad \omega^1 = \gamma yBdr \quad (66) \]
\[ \omega^2 = yYd\theta \quad \omega^3 = yY \sin \theta d\phi \quad (67) \]

The Hertzian potential 2-form is chosen to be

\[ P = P_E \omega^0 \wedge \omega^1 + P_M \omega^2 \wedge \omega^3 \quad (68) \]
Under the Hertz-Debye formalism, we obtain a wave equation for $P_E$ and $P_M$, each.

The electric components are obtained by setting $P_M = 0$ in \( \text{(68)} \) and solving for $P_E$ while the magnetic components are obtained by setting $P_E = 0$ in \( \text{(68)} \) and solving for $P_M$. Since the resultant wave equation is identical in both these cases, we shall adopt the generic notation $\Psi$ for the Debye potentials $P_E$ and $P_M$, both \( \text{[15]} \).

The physical correspondence with the fields is:

\[
E_i = f_{i0} \quad B_1 = f_{23} \\
B_2 = f_{31} \quad B_3 = f_{12} \quad \text{(69)}
\]

where the index $i$ ranges from 1, 2, 3, $E_i$ are the electric field components and $B_i$ are the magnetic field components.

Choose (See Appendix - A1 for computational details)

\[
\Psi = T_{\ell n}(t)R_n(r)P_\ell(cos \theta)e^{im\phi} \quad \text{(70)}
\]

where $P_\ell(cos \theta)$ is an associated Legendre function, the temporal function $T_{\ell n}(t)$ satisfies \( \text{(A8)} \) and the radial function $R_n(r)$ satisfies \( \text{(A9)} \). As usual, $\ell$ and $m$ are integers: $m$ to ensure single-valued nature of $e^{im\phi}$ and $\ell$ to ensure that the associated Legendre functions do not diverge for $cos \theta = \pm 1$. That is to say, the associated Legendre functions are polynomials.

When $y' \neq 0$, the radial equation \( \text{(A9)} \) can be written as:

\[
\frac{d^2 R_n}{dy^2} + \frac{1}{y} \frac{dR_n}{dy} + \frac{n}{y^2} R_n = 0 \quad \text{(71)}
\]

and it is an Euler equation. The solutions of this Euler equation, for $y > 0$, are:

\[
R_n = \begin{cases} 
  c_1 \cos(\sqrt{n} \ln y) + c_2 \sin(\sqrt{n} \ln y) & \text{for } n > 0 \\
  c_3 y^{\sqrt{-n}} + c_4 y^{-\sqrt{-n}} & \text{for } n < 0 \\
  c_5 \ln y + c_6 & \text{for } n = 0
\end{cases} \quad \text{(72)-(74)}
\]

where $c_1, c_2, c_3, c_4, c_5$ and $c_6$ are constants.

The temporal equation \( \text{(A8)} \) is of the Fuchsian form and is amenable to series solutions as per the theorem of Fuchs. Alternatively, a substitution

\[
T_{\ell n}(t) = \frac{T_{\ell n}(t)}{\sqrt{B}} \quad \text{(75)}
\]

can be used to recast \( \text{(A8)} \) into the form:

\[
\ddot{T}_{\ell n} = -\left(1 \frac{\dot{B}^2}{4 B^2} - \frac{\ell(\ell+1)}{Y^2} - \frac{n}{\gamma^2 B^2} - \frac{1}{2 B} \right) T_{\ell n} \equiv -W(t)T_{\ell n} = -(n_c - n) \frac{T_{\ell n}}{\gamma^2 B^2} \quad \text{(76)}
\]

At any given co-moving time, the temporal function $W(t)$ has an inflexion point $W(t) = 0$ at a critical value denoted by $n_c(\ell, t)$ with

\[
n_c(\ell, t) = \frac{\gamma^2 B^2}{2 Y^2} \left[ \frac{Y^2 \dot{B}^2}{4 B^2} - \frac{Y^2 \dot{B}}{2 B} - \ell(\ell+1) \right] \quad \text{(77)}
\]

For $n_c > n$, the function $T_{\ell n}(t)$ displays an oscillatory behavior. On the other hand, for $n_c < n$, the function $T_{\ell n}(t)$ displays an exponential behavior. At the critical value, $n = n_c$, the function $T_{\ell n} \propto (\zeta + \iota)/\sqrt{B}$ where $\zeta$ and $\iota$ are constants.

In terms of the solutions $T_{\ell n}(t)$, $R_n(r)$, $P_\ell(cos \theta)$ and $e^{im\phi}$, the electric multi-pole fields, from \( \text{(63)} \), are:

\[
E_1 = \frac{\ell(\ell+1)}{y^2 Y^2} T_{\ell n}(t)R_n(r)P_\ell(cos \theta)e^{im\phi} \quad B_1 = 0
\]
The fields components and are related to (78) by inserting an independent solution $\Psi$ to (72) for $P_M$ and performing the duality operation $E_i \rightarrow B_i$ and $B_i \rightarrow -E_i$.

For the sake of further works, we explicitly provide here the magnetic multi-poles. These are:

$$
E_2 = \frac{1}{\gamma' y B y Y} T_{r n}(t) R_n'(r) e^{im\phi} \frac{d}{d\theta} P_t(\cos \theta)
$$

$$
E_3 = \frac{i m \csc \theta}{\gamma' y B y Y} T_{r n}(t) R_n'(r) P_t(\cos \theta) e^{im\phi}
$$

These are the electric multi-poles (except for $\ell = 0$), both static and dynamic. The magnetic multi-poles are obtained similarly from (72) and are related to (78) by inserting an independent solution $\Psi$ to (72) for $P_M$ and performing the duality operation $E_i \rightarrow B_i$ and $B_i \rightarrow -E_i$.

For the sake of further works, we explicitly provide here the magnetic multi-poles. These are:

$$
B_1 = \frac{\ell(\ell + 1)}{y^2 Y^2} T_{r n}(t) R_n(r) P_t(\cos \theta) e^{im\phi}
$$

$$
B_2 = \frac{1}{\gamma' y B y Y} T_{r n}(t) R_n'(r) e^{im\phi} \frac{d}{d\theta} P_t(\cos \theta)
$$

$$
B_3 = \frac{i m \csc \theta}{\gamma' y B y Y} T_{r n}(t) R_n'(r) P_t(\cos \theta) e^{im\phi}
$$

Of particular interest are the “fall-off properties” of the fields. It is noticed that the fields components are proportional to either $R/y^2$ or $R'/yy'$. The fields components

A. Behavior of electromagnetic fields at early times

At early times, we may consider the initial hyper-surface, at time $t = t_i$, to contain sparsely distributed, pressure-less matter collapsing without any radiation. The emission of radiation will be switched on at some suitable time when the particles of matter collide and produce some radiation.

Then, as initial condition, we have $B = \text{constant} \equiv B_0$. Therefore, at $t = t_i$, we have $n_c \approx -\gamma^2 B_0^2 \ell(\ell + 1)/Y^2$.

It is then to be noticed that the source-free electromagnetic fields in this spacetime are to be given as “initial” conditions.

B. Behavior of electromagnetic fields near the singular hyper-surface

From (76) and (77), it is clear that $n_c(\ell, t)$ is an increasing function of $t$ in a collapse situation, i.e., for $\dot{B} < 0$ and $\dot{B} < 0$. Then, any initial mode with $n_c < n$, an initially exponential mode, becomes an oscillatory mode, $n_c > n$, with the progress of gravitational collapse of matter. The frequency of oscillations of the fields then continues to increase with the progress of the collapse. In the limit of the singularity, i.e., for $t \rightarrow t_s$ - the singular hyper-surface, $n_c \rightarrow \infty$ and the frequency of field oscillations becomes infinite.

Thus, all the $\ell$ - modes (electric and magnetic, both), with $\ell > 0$, become oscillatory with the progress of the collapse irrespective of their nature at initial time. In particular, this is the case near the singular hyper-surface $t = t_s$. Note that this is happening only asymptotically with time. The singular hyper-surface is to the infinite future of a co-moving observer who is also the “cosmological observer” for the spacetime of (2).

This is clearly consistent with the result that the co-moving observer too becomes relativistic in the limit $t \rightarrow t_s$. Therefore, the co-moving observer only sees radiation in this limit. That the co-moving observer becomes relativistic in the limit of the singular hyper-surface is irrespective of whether there is any black hole - the infinite redshift surface - in the spacetime of (2). This is also irrespective of whether there are electromagnetic fields in the spacetime of (2) or not.

However, this is not the astrophysically interesting or relevant situation since it is reached for an infinite co-moving time when matter in the entire spacetime has attained relativistic speeds as a result of the continued pile up of matter to the center of the spacetime.
C. Behavior of electromagnetic fields in the presence of a black hole

Perhaps, what is astrophysically relevant is the situation of a black hole existing in the spacetime of (2). However, it must be emphasized that a black hole is only an infinite red-shift surface here and that it must exist in the spacetime of (2) as an “initial” condition for non-singular and non-degenerate matter data. It is therefore likely that the black hole of (2) may only be of academic interest. That is to say, the black hole of (2) may not be astrophysically relevant.

Then, of some, nonetheless questionable, astrophysical interest is the behavior of electromagnetic fields when a black hole exists in (2) as an infinite red-shift surface.

As seen in §1H2, the necessary condition for the presence of a black hole in (2) is \( y(r) < 1 \). Therefore, we need to consider the field solutions of (3) in the range \( y : (y_c, \infty) \) when \( y_c < 1 \). We will then obtain the fields within the black hole region for \( y : (y_c, 1) \) and outside or exterior to the black hole region for \( y : (1, \infty) \).

In this case, the radial behavior is obtained for values \( y_c < y < 1 \) and there is no difference in the temporal behavior of the fields.

V. DISCUSSION

In General Relativity, a continuum of “curved” 4-dimensional spacetime geometry describes the evolution of matter. In its (3+1)-formulation, we can consider some distribution of gravitating matter on an “initial” spacelike hyper-surface, the Cauchy surface, and, from the Einstein equations, can obtain the temporal evolution of matter from that “initial” datum.

We may select a variety of “matter datum” on the initial hyper-surface. For example, we may consider a single particle, ie, a single mass-point, or a blob of matter surrounded by vacuum or a blob of matter that is radiating, and evolve that matter datum using the field equations.

Now, any such “initial” data is replaceable with “that” mass-point or “that” blob of matter being surrounded by “more” matter, this replacement being ad infinitum till the entire initial hyper-surface has matter everywhere. Note also that this “replacement” can be achieved in uncountably many different ways, ie, by distributing “more” matter in uncountably many ways on the initial hyper-surface.

Further, each stage of the evolution is obtainable from the previous stage of matter. This is the principle of causal development.

Now, when the matter datum is specified over all of the initial hyper-surface, we obtain a “cosmological” situation or spacetime.

Different initial data could evolve to distinct four-dimensional spacetime geometries. Therefore, these spacetime geometries; of a mass-point, of a matter blob and the “cosmological” spacetime obtained for an ad infinitum replacement of any of the considered situations; possess different geometric and, hence, physical features.

The question then arises: Which geometric or physical features of these spacetimes are relevant to “real” objects of the observed Universe? To be able to make contact with physical objects embedded in the Universe, a spacetime of the object in question is then needed to be “cosmological”.

Thus, a point of view can be advocated that the geometric features of only cosmological spacetimes are the ones that are relevant to actual physical objects of the real Universe.

The point here is that, in General Relativity, the idealization of an “isolated” object comes with its own pitfalls of the above nature. The issue here is that we could, without changing the Newtonian law of gravitation, add two masses to produce a new mass in the Newtonian theory but that is not permissible with “spacetimes” of arbitrary nature in General Relativity.

Note that the local spacetime geometry is, of course, Minkowskian since the cosmological spacetime is locally flat. Moreover, it may also be that some features of a cosmological spacetime are present with non-cosmological spacetimes.

In the above general spirit, we discussed, in this paper, “physical characteristics” of a radially homothetic spacetime of (4).

We first showed that the requirement of radial homothety, ie, of the existence of a radial homothetic Killing vector for a spherical spacetime, fixes the spacetime metric uniquely to (4). The existence of a radial HKV allows, in accordance with Lie’s theory, an arbitrary function of the co-moving radial coordinate in the metric.

In a radially homothetic spacetime of (4), there is, therefore, the maximal “physical” freedom, but not the geometrical freedom, compatible with the assumption of spherical symmetry. Thus, we concentrated on the non-singular and non-degenerate matter data for (4).

We then discussed the issue of the regularity of the center of a radially homothetic spacetime and discussed the behavior of physical quantities. The lack of regularity of the center of (4) for non-singular and non-degenerate data is compatible with the “physical” expectation that the “cosmological” observer does not witness the formation of
the black hole.

Next, we showed that a black hole of (2) is only an infinite red-shift surface and is a part of the “initial” data of the spacetime.

Further, the curvature singularity of such a spacetime is a singular spacelike hyper-surface at which matter in the spacetime attains the speed of light. This spacetime singularity of (2) occurs to the future of every observer in (2). Hence, the spacetime singularity of (2) is not visible to any observer. That is to say, the singularity of (2) is not naked locally or globally.

These results are in agreement with the strong Cosmic Censorship that demands that the spacetime singularity be not visible to any observer unless and until it is actually encountered [17].

We then discussed the steps of collapse of matter from the initial dusty state. It was argued that “all” the necessary ingredients of the expected physical evolution of matter are obtainable in the radially homothetic spacetime.

Following the Cohen-Kegeles generalization [13] of the flat-space Hertz-Debye potential formalism, we have also obtained the source-free electromagnetic fields, \(\mathcal{E}(\mathcal{D})\), in the spacetime of (2). In the limit of the singular hyper-surface, all the field modes are oscillatory. This behavior of the electromagnetic fields is asymptotically reached in the limit of infinite co-moving time.

In this spacetime, when self-gravity dominates with finality, the unstoppable collapse begins and leads to an eventual spacetime singularity, the singular hyper-surface, only asymptotically. However, such a situation is not astrophysically relevant since “all” the matter in the spacetime becomes relativistic in this situation. This is expected only in the asymptotic future.

The radially homothetic spacetime of (2) therefore provides us the spacetime of astrophysically interesting gravitational collapse problem. In many such collapse situations, matter may trap and carry radiation with it as it collapses.

From the astrophysical point of view, the recent observations appear to point to the existence of a “null hyper-surface” in candidate objects. In particular, [18, 19] have recently pointed out that potential black hole candidates and known neutron stars separate in two categories in \(\log (L_{\text{max}})\) versus \(\log (L_{\text{min}}/L_{\text{max}})\) plots for X-ray luminosity of these sources.

The explanation for this separation is provided on the basis of the Advection-Dominated Accretion Flows (ADAF) that have more core luminosity in the X-rays in the case of Neutron Stars. The accreting matter exhibiting energy-advection encounters the physical surface of the neutron star at which it deposits the stored energy to become X-ray bright. On the other hand, for a black hole, matter is expected to encounter no such physical surface and, hence, matter is not expected to become X-ray bright.

The point is that in these models of ADAF the above is implemented in the form of a boundary condition that essentially implies that the matter exhibiting energy-advection accretes without encountering a physical surface like that of a neutron star or not.

The black hole of (2) - an infinite red-shift surface - is, in the co-moving frame, approached asymptotically by collapsing matter that crosses it in its rest frame. Consequently, matter falling onto the black hole will appear, to a co-moving observer, to be collapsing without encountering a physical surface like that of a neutron star. Hence, matter displaying energy-advection will not deposit the stored energy at any surface. Such an object will, therefore, appear less X-ray bright as compared to an object with a physical surface. This is as per the interpretation of observations in [18, 19].

However, the black hole of (2) is a part of the “initial” data for it. Therefore, unless a black hole is assumed to exist, there is no reason for the non-existence of any hard surface of the collapsing object in the spacetime of (2).

Thus, it seems that the reason(s) for the observations used in [18, 19] are, in all probability, different than have been explored therein. For (2), such reasons can only be explored on the basis of the detailed “physical” considerations leading to solutions to (34) and (35).

In (2), any spherical “object” will collapse asymptotically to the singular hyper-surface. It is an Eternally Collapsing Object. However, collapse to singular hyper-surface of (2) requires the entire spacetime to be “relativistic” and that takes infinite co-moving time. Therefore, in (2), the “Eternally Collapsing Object” is also not relevant to the present astrophysical observations that have been used in [18, 19].

Then, from the results of the present work, it seems most likely that the astrophysically relevant possibility is that of “sufficiently Collapsed Objects” accreting matter in astrophysical environments. We are, of course, considering only spherically symmetric situation in this paper.

Now, from the above, it is then puzzling that no null membrane black holes emerged in the present study. The following issue, therefore, arises.
**Issue of null membranes in the physically realizable collapse**

The standard scenario of gravitational collapse leading to a null membrane black hole is, so far, only an “expectation” based on plausibility arguments. The exact general relativistic spacetime of the gravitational collapse of a “physical” star supporting the standard scenario with a null membrane is not known.

We have shown here that the radially homothetic, spherical spacetime of (2) describes the physically realizable spherical collapse.

Therefore, the standard picture of spherical collapse of matter leading to a null membrane black hole can be verified using (3).

We have then shown that a null membrane does not arise for physically realizable gravitational collapse from any non-singular and non-degenerate data in (2). However, as has been shown here, an infinite red-shift surface can “exist” in (2) as a part of non-singular and non-degenerate data.

The standard picture of the gravitational collapse leading to a null membrane black hole cannot therefore be realized in this spacetime for non-singular and non-degenerate data.

Then, the question is of the existence of another cosmological spacetime describing physically realizable spherical collapse of matter and supporting the standard expectation of the existence of a null membrane black hole.

The question will also be of the difference between such a spacetime and a radially homothetic spacetime considered here. Does general relativity allow two inequivalent such spherical spacetimes? This is a fundamental issue.

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**APPENDIX A: COMPUTATIONAL DETAILS**

1. **Electromagnetic fields**

Some useful Hodge dual operations on basis forms are:

\[
\begin{align*}
\star \omega^0 \wedge \omega^1 \wedge \omega^2 \wedge \omega^3 &= 1 \\
\star \omega^0 \wedge \omega^2 &= -\omega^1 \wedge \omega^3 \\
\star \omega^1 \wedge \omega^3 &= \omega^0 \wedge \omega^2 \\
\star \omega^0 \wedge \omega^2 \wedge \omega^3 &= \omega^1 \\
\star \omega^0 \wedge \omega^1 \wedge \omega^3 &= \omega^2 \\
\star \omega^1 \wedge \omega^2 \wedge \omega^3 &= \omega^2
\end{align*}
\]

Now, in the following we shall set \( P_M = 0 \) in (23), that is to say, we shall evaluate the electric multi-poles for the metric (2). We shall also denote the potential as \( \Psi \). Then, from (28), we obtain

\[
P = \Psi \omega^0 \wedge \omega^1
\]

\[
\star P = \Psi \omega^2 \wedge \omega^3 = \Psi y^2 Y^2 \sin \theta \, d\theta \wedge d\phi
\]

\[
d \star P = \left( \frac{\Psi_y}{y} + 2\Psi \frac{Y'}{Y} \right) \frac{y^2 Y^2}{Y} \sin \theta \, dt \wedge d\theta \wedge d\phi + \left( \frac{\Psi_r}{y} + 2\frac{\Psi_y}{y} \right) \frac{y^2 Y^2}{Y} \sin \theta \, dr \wedge d\theta \wedge d\phi
\]

\[
= \left( \frac{\Psi_y}{y} + 2\Psi \frac{Y'}{Y} \right) \omega^0 \wedge \omega^1 \wedge \omega^3 + \left( \frac{\Psi_r}{\gamma y B} + 2\Psi \frac{1}{\gamma y B} \right) \omega^1 \wedge \omega^2 \wedge \omega^3
\]

\[
(A1)
\]
\[ \Delta P = \left\{ \begin{array}{l} 
\left( B\Psi_{tt} + B\dot{\Psi}_{tt} + 2\Psi_{t} B\dot{Y} + 2\Psi B\dot{Y} - 2\Psi B\dot{Y}^2 + 2\Psi\dot{B} \frac{\dot{Y}}{y} \right) \frac{1}{y^2 B} \\
- \left( \frac{y\Psi_{rr}}{y'} + 3 \Psi_{r} - \frac{yy''\Psi_{r}}{y'^2} \right) \frac{1}{\gamma B} \right \} \omega^0 \wedge \omega^1  
\end{array} \right. 
\]
where $P$ and is amenable to solution by separation of variables. Dividing (A6) by $\Psi$ and setting

$$G = \frac{2\Psi}{\gamma By} \omega^0 + \frac{2\dot{\Psi}}{yY} \omega^1$$

we get

$$dG = \left[ \left( \frac{2\dot{\Psi}}{yY} + \frac{2\ddot{\Psi}}{yY^2} \right) - \frac{1}{By} - \frac{2\dot{\Psi}}{\gamma^2 B^2 y^2} \right] \omega^0 \wedge \omega^1$$

$$- \frac{2\Psi_\theta}{\gamma By^2Y} \omega^0 \wedge \omega^2 - \frac{2\dot{\Psi}}{\gamma By^2Y} \omega^0 \wedge \omega^3 - \frac{2\dot{\Psi}}{\gamma By^2Y} \omega^1 \wedge \omega^2 - \frac{2\dot{\Psi}}{\gamma By^2Y} \omega^1 \wedge \omega^3$$

(A5)

Therefore, substituting (A3) and (A5) in (62), it is seen that the $\omega^0 \wedge \omega^1$ term yields the required wave equation while all other terms lead to identities. The wave equation is:

$$Y^2 \left( \Psi_{tt} + \frac{\dot{B}}{B} \Psi_t \right) - \left( \frac{y \Psi_{rr}}{y'} + \Psi_r - \frac{yy'' \Psi_r}{y'^2} \right) \frac{y}{y'} \frac{Y^2}{\gamma^2 B^2} = \Psi_{,\theta\theta} + \cot \theta \Psi_{,\theta} + \csc \theta \Psi_{,\phi\phi}$$

(A6)

and is amenable to solution by separation of variables. Dividing (A6) by $\Psi$ and setting

$$\Psi (t, r, \theta, \phi) = T_{\ell n} (t) R_n (r) P_\ell (\cos \theta) e^{im\phi}$$

(A7)

where $P_\ell (\cos \theta)$ is an associated Legendre polynomial, we obtain

$$\frac{T_{\ell n}}{T_{\ell n}} + \frac{\dot{B}}{B} \frac{T_{\ell n}}{T_{\ell n}} + \frac{\ell (\ell + 1)}{\gamma^2 B^2} + \frac{n}{Y^2} = 0$$

(A8)

$$\frac{R''_n}{R_n} + \left( \frac{y'}{y} - \frac{y''}{y'} \right) \frac{R'_n}{R_n} = 0$$

(A9)

where $m, \ell$ and $n$ are separation constants. In the above, an overhead dot denotes a time-derivative and an overhead prime denotes a derivative with respect to $r$. 

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