SPECTRAL STABILITY OF BI-FREQUENCY SOLITARY WAVES IN SOLER AND DIRAC–KLEIN–GORDON MODELS

Dedicated to Professor Vladimir Georgiev on the occasion of his sixtieth birthday

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ABSTRACT. We construct bi-frequency solitary waves of the nonlinear Dirac equation with the scalar self-interaction, known as the Soler model (with an arbitrary nonlinearity and in arbitrary dimension) and the Dirac–Klein–Gordon with Yukawa self-interaction. These solitary waves provide a natural implementation of qubit and qudit states in the theory of quantum computing.

We show the relation of ±2ωi eigenvalues of the linearization at a solitary wave, Bogoliubov SU(1, 1) symmetry, and the existence of bi-frequency solitary waves. We show that the spectral stability of these waves reduces to spectral stability of usual (one-frequency) solitary waves.

1. Introduction. The Soler model [18, 29] is the nonlinear Dirac equation with the minimal scalar self-coupling,
\[ i\partial_t \psi = D_m \psi - f(\psi^* \beta \psi) \beta \psi, \quad x \in \mathbb{R}^n, \quad \psi(t, x) \in \mathbb{C}^N, \]
where \( f \) is a continuous real-valued function with \( f(0) = 0 \). Above, \( \bar{\psi} = \psi^* \beta \), with \( \psi^* \) the hermitian conjugate. This is one of the main models of the nonlinear Dirac equation, alongside with its more physically relevant counterpart, the Gross–Neveu model [17, 21], and with the massive Thirring model [30]. Above, the Dirac operator is given by
\[ D_m = -i\alpha \cdot \nabla + m\beta, \quad m > 0, \]
with \( \alpha^j \ (1 \leq j \leq n) \) and \( \beta \) mutually anticommuting self-adjoint matrices such that \( D_m^2 = -\Delta + m^2 \). All these models are hamiltonian, \( U(1) \)-invariant, and relativistically invariant. The classical field \( \psi \) could be quantized (see e.g. [21]).

The Soler model shares the symmetry features with its more physically relevant counterpart, Dirac–Klein–Gordon system (the Dirac equation with the Yukawa self-interaction,
with \( \phi \) produces bi-frequency solitary waves of the form
\[
\tag{2}
\frac{\partial_t \psi}{\partial^2 - \Delta + M^2} \Phi = \psi^* \beta \psi, \quad x \in \mathbb{R}^n,
\]
where \( M > 0 \) is the mass of the scalar field \( \Phi(t, x) \in \mathbb{R} \).

The solitary wave solutions in the Soler model (already constructed in [29]) possess certain stability properties [3, 13, 6]; in particular, small amplitude solitary waves corresponding to the nonrelativistic limit \( \omega \lesssim m \) of the charge-subcritical and charge-critical case \( f(\tau) = |\tau|^k \) with \( k \lesssim 2/n \) are spectrally stable: the linearized equation on the small perturbation of a particular solitary wave has no exponentially growing modes. The opposite situation, the linear instability of small amplitude solitary waves (presence of exponentially growing modes) in the charge-supercritical case \( k > 2/n \) was considered in [11]. Recent results on asymptotic stability of solitary waves in the nonlinear Dirac equation [7, 26, 10] rely on the assumptions on the spectrum of the linearization at solitary waves, although this information is not readily available, especially in dimensions above one. This stimulates the study of the spectra of linearizations at solitary waves. It was shown in [3] that the Soler model in one spatial dimension linearized at a solitary wave \( \phi(x)e^{-i\omega t} \) has eigenvalues \( \pm 2\omega i \). While the zero eigenvalues correspond to symmetries of the system (unitary, translational, etc.), the eigenvalues \( \pm 2\omega i \) are related to the presence of bi-frequency solitary waves and to the Bogoliubov \( SU(1, 1) \) symmetry of the Soler model and Dirac–Klein–Gordon models, first noticed by Galindo in [16]. In the three-dimensional case \((n = 3, N = 4)\) with the standard choice of the Dirac matrices, this symmetry group takes the form
\[
\tag{3}
G_{\text{Bogoliubov}} = \left\{ a - ib\gamma^2 K : a, b \in \mathbb{C}, \ |a|^2 - |b|^2 = 1 \right\} \cong SU(1, 1),
\]
where \( K : \mathbb{C}^N \to \mathbb{C}^N \) is the antilinear operator of complex conjugation; the group isomorphism is given by \( a - ib\gamma^2 K \mapsto \begin{bmatrix} a \\ b \\ \bar{a} \\ \bar{b} \end{bmatrix} \). By the Noether theorem, the continuous symmetry group leads to conservation laws (see Section 2). The Bogoliubov group, when applied to standard solitary waves \( \phi(x)e^{-i\omega t} \) in the form of the Wakano Ansatz [31],
\[
\phi(x) = \begin{bmatrix} v(r, \omega) \xi \\ iu(r, \omega) \bar{\xi} \end{bmatrix}, \quad \xi \in \mathbb{C}^2, \quad |\xi| = 1,
\]
produces bi-frequency solitary waves of the form
\[
\tag{4}
a\phi(x)e^{-i\omega t} + b\phi_C(x)e^{i\omega t}, \quad a, b \in \mathbb{C}, \quad |a|^2 - |b|^2 = 1,
\]
with \( \phi_C = -i\gamma^2 K \phi \) the charge conjugate of \( \phi \); here \(-i\gamma^2 K\) is one of the infinitesimal generators of \( SU(1, 1) \). Above, \( v(r, \omega) \) and \( u(r, \omega) \) are real-valued functions which satisfy
\[
\begin{cases}
\omega v = \partial_r u + \frac{n-1}{r} u + (m-f) v, \\
\omega u = -\partial_r v - (m-f) u,
\end{cases}
\]
\[
\lim_{r \to 0} u(r, \omega) = 0, \quad \lim_{r \to 0} u(r, \omega) = 0,
\]
\[
\tag{6}
\text{with}
\]
\[
\tag{7}
f = f(v^2 - u^2)
\]
in the case of nonlinear Dirac equation (1) (see e.g. [5]) and
\[
\tag{8}
f = (-\Delta + M^2)^{-1}(v^2 - u^2)
\]
in the case of Dirac–Klein–Gordon system (2). We assume that the functions \( u(r, \omega) \) and \( v(r, \omega) \) satisfy
\[
\tag{9}
\sup_{r \geq 0} |u(r, \omega)/v(r, \omega)| < 1,
\]
which is true in particular for small amplitude solitary waves with \( \omega \lesssim m \); see e.g. [5].

We now switch to the general case of a general spatial dimension \( n \geq 1 \) and a general number of spinor components \( N \geq 2 \). By usual arguments (see e.g. [5]), without loss of generality, we may assume that the Dirac matrices have the form

\[
\alpha^j = \begin{bmatrix} 0 & \sigma^j \\ \sigma_j & 0 \end{bmatrix}, \quad 1 \leq j \leq n; \quad \beta = \begin{bmatrix} 1_{N/2} \quad 0 \\ 0 \quad -1_{N/2} \end{bmatrix}.
\]

(10)

Here \( \sigma_j, 1 \leq j \leq n \), are \( \frac{N}{2} \times \frac{N}{2} \) matrices which are the higher-dimensional analogue Pauli matrices:

\[
\{ \sigma_j, \sigma_k^* \} = 2\delta_{jk}1_{N/2}, \quad 1 \leq j, k \leq n.
\]

(11)

In the case \( n = 3, N = 4 \), one takes \( \sigma_j, 1 \leq j \leq 3 \), to be the standard Pauli matrices.

**Remark 1.1.** In general, \( \sigma_j \) are not necessarily self-adjoint; for example, for \( n = 4 \) and \( N = 4 \), one can choose \( \sigma_j \) to be the standard Pauli matrices for \( 1 \leq j \leq 3 \) and set \( \sigma_4 = i1_2 \).

We denote

\[
\sigma_r = \frac{x \cdot \sigma}{r}, \quad x \in \mathbb{R}^n \setminus \{0\}, \quad r = |x|.
\]

(12)

In Section 3, we show that if \( \phi(x) e^{-i\omega t} \), with \( \phi \) from (4), is a solitary wave solution to (1) (or (2)), then there is the following family of exact solutions to (1) (or (2), respectively):

\[
\theta_{\Xi, H}(t, x) = |\Xi|\phi_\xi(x) e^{-i\omega t} + |H|\chi_\eta(x) e^{i\omega t},
\]

(13)

\[
\phi_\xi(x) = \begin{bmatrix} v(r, \omega) \xi \\ iu(r, \omega) \sigma_r \xi \end{bmatrix}, \quad \chi_\eta(x) = \begin{bmatrix} -i u(r, \omega) \sigma_r^\ast \eta \\ v(r, \omega) \eta \end{bmatrix},
\]

with \( \Xi, H \in \mathbb{C}^{N/2} \setminus \{0\} \), \( |\Xi|^2 - |H|^2 = 1 \), \( \xi = \Xi/|\Xi|, \eta = H/|H| \). See Lemma 3.1 below. This shows that in any dimension there is a larger symmetry group, \( SU(N/2, N/2) \), which is present at the level of bi-frequency solitary wave solutions in the models (1) and (2) while being absent at the level of the Lagrangian.

**Remark 1.2.** We note that if \( f \) in (1) is even, then \( \theta_{\Xi, H}(t, x) \) given by (13) with \( \Xi, H \in \mathbb{C}^{N/2} \) such that \( |\Xi|^2 - |H|^2 = -1 \) is also a solitary wave solution.

Two-frequency solitary waves (13) clarify the nature of the eigenvalues ±2\( \omega i \) of the linearization at (one-frequency) solitary waves in the Soler model: these eigenvalues could be interpreted as corresponding to the tangent vectors to the manifold of bi-frequency solitary waves. See Corollary 3.2 below. We point out that the exact knowledge of the presence of ±2\( \omega i \) eigenvalues in the spectrum of the linearization at a solitary wave is important for the proof of the spectral stability: namely, it allows us to conclude that in the nonrelativistic limit \( \omega \lesssim m \) the only eigenvalues that bifurcate from the embedded thresholds at ±2\( mi \) are ±2\( \omega i \); no other eigenvalues can bifurcate from ±2\( mi \), and in particular no eigenvalues with nonzero real part. For details, see [6].

We point out that the asymptotic stability of standard, one-frequency solitary waves can only hold with respect to the whole manifold of solitary wave solutions (13), which includes both one-frequency and bi-frequency solitary waves: if a small perturbation of a one-frequency solitary wave is a bi-frequency solitary wave, which is an exact solution, then convergence to the set of one-frequency solitary waves is out of question. In this regard, we recall that the asymptotic stability results [26, 10] were obtained under certain restrictions on the class of perturbations. It turns out that these restrictions were sufficient to remove not only translations, but also the perturbations in the directions of bi-frequency solitary waves.
waves; this is exactly why the proof of asymptotic stability of the set of one-frequency solitary waves with respect to such class of perturbations was possible at all.

While the stability of one-frequency solitary waves turns out to be related to the existence of bi-frequency solitary waves, one could question the stability of such bi-frequency solutions, too. In Section 4, we show that the bi-frequency solitary waves are spectrally stable as long as so are the corresponding one-frequency solitary waves. While this conclusion may seem natural, we can only give the proof for the case when the number of spinor components satisfies $N \leq 4$ (which restricts the spatial dimension to $n \leq 4$).

Let us mention that the bi-frequency solitary waves (5) may play a role in Quantum Computing. Indeed, such states produce a natural implementation of qubit states $a|0\rangle + b|1\rangle$, $|a|^2 + |b|^2 = 1$, except that now the last relation takes the form $|a|^2 - |b|^2 = 1$. Just like for standard cubits, our bi-frequency states (5) have two extra parameters besides the orbit of the $U(1)$-symmetry group. Below, we are going to show that qubits (5) can be linearly stable. Moreover, the manifold of bi-frequency solitary waves (13) admits a symmetry group $SU(N/2, N/2)$ (which may be absent on the level of the Lagrangian). For these solitary waves, the number of degrees of freedom (after we factor out the action of the unitary group) is $d = 2N - 2$. The states (13) with $N \geq 4$ correspond to higher-dimensional versions of qubits – $d$-level qudits, quantum objects for which the number of possible states is greater than two. These systems could implement quantum computation via compact higher-level quantum structures, leading to novel algorithms in the theory of quantum computing. Bi-frequency solitary waves could also provide a simple stable realization of higher-dimensional quantum entanglement, or hyperentanglement, which is used in cryptography based on quantum key distribution; by [8, 14], using qudits over qubits provides increased coding density for higher security margin and also an increased level of tolerance to noise at a given level of security. Qudits have already been implemented in the system with two electrons [23] as quantum walks of several electrons (just like one qubit could be represented by a quantum walk, a distribution of an electron in a “quantum tunnel” between individual quantum dots, considered as potential wells). Being sensitive to the external noise, the quantum-walk implementation of qudits in [23] was indicated to be highly unstable, requiring excessive cooling and making the practical usage very difficult. In [19], the on-chip implementation of qudit states is achieved by creating photons in a coherent superposition of multiple high-purity frequency modes. We point out that the bi-frequency solitary waves (13) can possess stability properties, as we show below; moreover, the simplicity of the model suggests that such states could be implemented using photonic states in optical fibers without excessive quantum circuit complexity.

We need to mention that several novel nonlinear photonic systems currently explored are modeled by Dirac-like equations (often called coupled mode systems) which are similar to (1). Examples include fiber Bragg gratings [15], dual-core photonic crystal fibers [4], and discrete binary arrays, which refer to systems built as arrays coupled of elements of two types. Earlier experimental work on binary arrays has already shown the formation of discrete gap solitons [24]. Three of the many novel examples that have been recently considered are: a dielectric metallic waveguide array [2, 1]; an array of vertically displaced binary waveguide arrays with longitudinally modulated effective refractive index [22], and arrays of coupled parity-time ($PT$) nanoresonators [20]. We also constructed bi-frequency solutions of the form (5) in the Dirac-type models with the $PT$-symmetry which arise in nonlinear optics in the model describing arrays of optical fibers with gain-loss behavior [12]. This venue of research is pursued for optical implementation of traditional circuits [28, 27] aimed at the energy-efficient computing and at the challenges in reducing the footprint of optics-based devices.
2. **The Bogoliubov SU(1, 1) symmetry and associated charges.** Chadam and Glassey [9] noted an interesting feature of the model (2): under the standard choice of $4 \times 4$ Dirac matrices $\alpha^j$ and $\beta$, as long as the solution $\psi$ is sufficiently regular, there is a conservation of the quantity

$$
\int_{\mathbb{R}^3} (|\psi_1 - \bar{\psi}_4|^2 + |\psi_2 + \bar{\psi}_3|^2) \, dx.
$$

(14)

As a consequence, if (14) is zero at some and hence at all moments of time, then $|\psi_1| = |\psi_4|$ and $|\psi_2| = |\psi_3|$ for almost all $x$ and $t$, hence $\psi^* \beta \psi \equiv 0$ (in the distributional sense), meaning that the self-interaction plays no role in the evolution and that as the matter of fact the solution solves the linear equation (without self-interaction). A similar feature of the Soler model (1) was analyzed in [25]. The relation of the conservation of the quantity (14) in Dirac–Klein–Gordon to the SU(1, 1) symmetry of the corresponding Lagrangian based on combinations of $\bar{\psi} \gamma^0 D_m \psi$ and $\bar{\psi} \psi$ was noticed by Galindo [16].

Let us state the above results in a slightly more general setting. Assume that $B \in \text{End}(\mathbb{C}^N)$ is a matrix which satisfies

$$
\{B K, D_m\} = 0, \quad B K = K B^*, \quad B^* B = 1_N.
$$

(15)

The relations (15) imply that

$$(BK)^2 = KB^* BK = 1_N,$$

$$
(\beta B)^t = K (\beta B)^* K = KB^* \beta K = -\beta KB^* K = -\beta B.
$$

(16)

Above, “$t$” denotes the transpose.

**Remark 2.1.** One can think of $K$ as self-adjoint in the sense that

$$
\text{Re}[(K \psi)^* \bar{\theta}] = \text{Re}[\bar{\psi}^* K \bar{\theta}] = \text{Re}[\psi^* K \theta].
$$

We can summarize [9, 25] as follows:

**Lemma 2.2.** If the solution to (1) or (2) satisfies $BK \psi|_{t=0} = z \psi|_{t=0}$, for some $z \in \mathbb{C}$, $|z| = 1$, then $BK \psi = z \psi$ for all $t \in \mathbb{R}$ and moreover $\psi^* \beta \psi = 0$.

**Proof.** The first statement is an immediate consequence of the fact that $BK$ commutes with the flow of the equation. If $\psi(t, x)$ satisfies $i \psi = D_m \psi - f(\psi^* \beta \psi) \beta \psi$, then

$$
BK \psi = BK (-i(D_m - \beta f)) \psi = (-i(D_m - \beta f)) BK \psi = z(-iD_m + i\beta f) \psi = z \psi.
$$

Finally, since $BK \psi = z \psi$, then

$$
z \psi^* \beta \psi = \psi^* \beta BK \psi = (K \psi)^t \beta BK \psi = -(K \psi)^t \beta BK \psi = 0.
$$

We took into account that $(\beta B)^t = -\beta B$ by (16).

As in [16], the Lagrangians of the Soler model (1) and of the Dirac–Klein–Gordon model (2), with the densities

$$
\mathcal{L}_{\text{Soler}} = \psi^* D_m \psi + F(\bar{\psi} \psi),
$$

$$
\mathcal{L}_{\text{DKG}} = \psi^* D_m \psi + \Phi \bar{\psi} \psi + \frac{1}{2} (|\Phi|^2 + |\nabla \Phi|^2 + M^2 |\Phi|^2),
$$

with $\psi(t, x) \in \mathbb{C}^N$, $\Phi(t, x) \in \mathbb{R}$, are invariant under the action of the continuous symmetry group

$$
g \in G_{\text{Bogoliubov}}, \quad g : \psi \mapsto (a + bB K) \psi, \quad |a|^2 - |b|^2 = 1
$$

(cf. (3)). The Noether theorem leads to the conservation of the standard charge $Q = \int_{\mathbb{R}^3} \psi^* \psi \, dx$ corresponding to the standard charge-current density $\bar{\psi} \gamma^0 \psi$ (note that the unitary group is a subgroup of SU(1, 1)), and the complex-valued Bogoliubov charge...
\[ \Lambda = \int_{\mathbb{R}^n} \psi^* B K \psi \, dx \] which corresponds to the complex-valued four-current density \( \psi^* \gamma^0 \gamma^\mu B K \psi \). Now Galindo’s observation \([16]\) could be stated as follows.

**Lemma 2.3** (The Bogoliubov SU(1, 1) symmetry and the charge conservation).

1. If \( \psi(t, x) \in \mathbb{C}^N \) is a solution to (1) or (2), then so is \( g\psi(t, x) \), for any \( g \in \text{G_{Bogoliubov}} \).
2. The Hamiltonian density of the Soler model (1),
\[
\mathcal{H} = \psi^* D_m \psi - F(\psi^* \beta \psi),
\]
where \( F(\tau) = \int_0^\tau f(t) \, dt, \tau \in \mathbb{R} \), satisfies \( \mathcal{H}(g\psi) = \mathcal{H}(\psi) \), \( \forall g \in \text{G_{Bogoliubov}} \).
3. \( \text{G_{Bogoliubov}} \cong \text{SU}(1, 1) \), with the group isomorphism \( a + b B K \mapsto \begin{bmatrix} a & b \\ b & \bar{a} \end{bmatrix} \in \text{SU}(1, 1) \), where \( a, b \in \mathbb{C} \) satisfy \( |a|^2 - |b|^2 = 1 \).
4. For solutions of the nonlinear Dirac equation (1), the following quantities are (formally) conserved:
\[
Q(\psi) = \langle \psi, \psi \rangle = \int_{\mathbb{R}^n} \psi(t, x)^* \psi(t, x) \, dx,
\]
\[
\Lambda(\psi) = \langle \psi, B K \psi \rangle = \int_{\mathbb{R}^n} B_{jk} \overline{\psi_j(t, x)} \psi_k(t, x) \, dx.
\]

**Proof.** Let \( g = a + b B K, |a|^2 - |b|^2 = 1 \). Since \( B K \) anti-commutes with both \( i \) and with \( D_m \), one has \( i\partial_t (B K \psi) = (D_m - f(\psi^* \beta \psi) \beta (B K \psi) \). Taking the linear combination with (1), we arrive at
\[
i\partial_t (a + b B K) \psi = (D_m - f(\psi^* \beta \psi) \beta ((a + b B K) \psi).
\]

It remains to notice that \( \varphi^* K \rho = (K \varphi)^* \rho, \forall \varphi, \rho \in \mathbb{C}^N \), hence
\[
\text{Re}\{ \varphi^* K \rho \} = \text{Re}\{ (K \varphi)^* \rho \},
\]
resulting in
\[
(g \psi)^* \beta g \psi = \text{Re}\{ \psi^* (a + K B^* \bar{b}) \beta (a + b B K) \psi \}
= \text{Re}\{ \psi^* (a + b B K) (a - b B K) \beta \psi \} = \psi^* \beta \psi.
\]

The invariance of the Hamiltonian density follows from (cf. (18))
\[
(g \psi)^* D_m g \psi = \text{Re}\{ \psi^* (a + b B K) (a - b B K) \psi \} = \psi^* D_m \psi
\]
and from \( F((g \psi)^* \beta g \psi) = F(\psi^* \beta \psi) \).

By the Nöther theorem, the invariance under the action of a continuous group results in the conservation laws. Let us check the (formal) conservation of the complex-valued \( \Lambda \)-charge. Writing \( f = f(\psi^* \beta \psi) \), we have:
\[
\partial_t \Lambda(\psi) = \langle -i(D_m - f \beta) \psi, B K \psi \rangle + \langle \psi, B K (-i(D_m - f \beta) \psi) \rangle
= \langle \psi, i(D_m - f \beta) B K \psi \rangle + \langle \psi, i B K (D_m - f \beta) \psi \rangle = 0.
\]

In the last relation, we took into account the anticommutation relations from (15). We also note that for the densities, we have
\[
\partial_t (\psi^* B K \psi) = - (\alpha^1 \partial_j \psi)^* B K \psi - \psi^* B K \alpha^1 \partial_j \psi = - \partial_{\alpha^1} (\psi^* \alpha^1 \psi B K \psi),
\]
showing that the Minkowski vector of the Bogoliubov charge-current density is given by
\[
S^\mu(t, x) = \psi(t, x)^* \gamma^0 \gamma^\mu B K \psi(t, x).
\]

**Remark 2.4.** Three conserved quantities, one being real and one complex, correspond to \( \text{dim}_\mathbb{R} \text{SU}(1, 1) = 3 \).
Example 2.5. For \( N = 2 \) and \( n \leq 2 \), with \( D_m = -i \sum_{j=1}^{n} \sigma_j \partial_j + \sigma_3 m \), one takes \( B = \sigma_1 \) (\( \sigma_j \) being the standard Pauli matrices);

\[ B K \psi = \sigma_1 K \psi =: \psi_C, \quad \psi \in \mathbb{C}^2. \]

The conserved quantity is

\[ \Lambda = \int_R \psi^* \sigma_1 K \psi \, dx = 2 \int_R \bar{\psi} \psi_2 \, dx. \]

It follows that the charge can be decomposed into

\[ Q = Q_- + Q_+, \]

with both \( Q_\pm := \frac{1}{2} (Q \pm \text{Re} \, \Lambda) = \frac{1}{2} \int_R (|\psi_1|^2 + |\psi_2|^2 \pm 2 \text{Re} \, \bar{\psi}_1 \psi_2) \, dx = \frac{1}{2} \int_R |\psi_1 \pm \bar{\psi}_2|^2 \, dx \)

conserved in time; so, if at \( t = 0 \) one has \( \bar{\psi}_2 = \psi_1 \) (or, similarly, if \( \bar{\psi}_2 = -\psi_1 \)), then this relation persists for all times (hence \( \psi^* \sigma_3 \psi = |\psi_1|^2 - |\psi_2|^2 = 0 \) for all times) due to the conservation of \( Q_\pm \).

Example 2.6. In the case \( n = 3 \), \( N = 4 \), using the standard choice of the Dirac matrices, one can take \( B = -i \gamma^2 \), so that

\[ B K \psi = -i \gamma^2 K \psi =: \psi_C, \quad \psi \in \mathbb{C}^4. \]

Then the quantity

\[ \Lambda(\psi) = \int_{\mathbb{R}^3} \psi^* (\psi^\gamma^2) K \psi \, dx = 2 \int_{\mathbb{R}^3} ( - \bar{\psi}_4 \psi_2 + \bar{\psi}_3 \psi_1 ) \, dx \]

is conserved, hence so are

\[ Q_\pm := \frac{1}{2} (Q \pm \text{Re} \, \Lambda) = \frac{1}{2} \int_{\mathbb{R}^3} ( |\psi_1 + \bar{\psi}_4|^2 + |\psi_2 \pm \bar{\psi}_3|^2 ) \, dx. \]

Thus, if at some moment of time one has \( \bar{\psi}_4 = \psi_1 \) and \( \bar{\psi}_3 = -\psi_2 \) (or, similarly, if \( \bar{\psi}_4 = -\psi_1 \) and \( \bar{\psi}_3 = \psi_2 \)), then this relation persists for all times (hence \( \psi^* \beta \psi = 0 \) due to the conservation of \( Q_\pm \)). Note that \( 2Q_- \) coincides with \( (14) \), so our conclusions are in agreement with \([9, 25]\).

Remark 2.7. For \( n = 4 \) and \( N = 4 \) (cf. Remark 1.1), there is no \( B \) satisfying (15) and thus no \( SU(1, 1) \) symmetry.

Lemma 2.8 (Transformation of the charges under the action of \( SU(1, 1) \)). Let \( a, \ b \in \mathbb{C}, \ |a|^2 - |b|^2 = 1 \), so that \( g = a + b B K \in G_{\text{Bogoliubov}} \). Then \( g \psi = (a + b B K) \psi \) satisfies

\[
Q((a + b B K) \psi) = (|a|^2 + |b|^2) Q(\psi) + 2 \text{Re} \{a b \Lambda(\psi)\}, \quad (19)
\]

\[
\Lambda(a + b B K \psi) = a^2 \Lambda(\psi) + 2a b Q(\psi) + b^2 \bar{\Lambda}(\psi). \quad (20)
\]

The quantity \( Q^2 - |\Lambda|^2 \) is invariant under the action of \( SU(1, 1) \):

\[
Q(g \psi)^2 - |\Lambda(g \psi)|^2 = Q(\psi)^2 - |\Lambda(\psi)|^2. \quad (21)
\]

Proof. For the charge density of \( g \psi = (a + b B K) \psi \), one has

\[
(g \psi)^* g \psi = \text{Re} \{ \psi^* (a + b B K)(a + b B K) \psi \} = \text{Re} \{ \psi^* (|a|^2 + |b|^2 + 2a b B K) \psi \}
\]

\[
= (|a|^2 + |b|^2) \psi^* \psi + 2 \text{Re} \{a b \psi^* B K \psi\}. \]


For the Λ-charge density of \( g\psi \), using the identity \((BKu)^* BKv = \overline{u^* v}, \forall u, v \in \mathbb{C}^N \) which follows from (15), one has
\[
(g\psi)^* BKg\psi = ((a + bBK)\psi)^* BK(a + bBK)\psi
\]
\[
= \overline{\psi}^* (\overline{\psi} B\psi) + b^2 \psi^* (a + bBK)\psi = \overline{\psi}^2 \psi + b\overline{\psi}^* B\psi + b^2 \overline{\psi}^* B\psi.
\]
The integration of the above charge densities leads to (20). The relation (21) is verified by the explicit computation.

3. Bi-frequency solitary waves. We recall that \( \sigma_r = \frac{\omega r}{r} \) (for \( r = |x| > 0 \)); the relations (11) imply that \( \sigma_r \sigma_r^* = I_{N/2} \).

Lemma 3.1. Let \( n \in \mathbb{N}, \omega \in [-m, m] \). If \( v(r), u(r) \) are real-valued functions which solve (6) with \( f \) given by (7) for the nonlinear Dirac equation (or (8) for Dirac–Klein–Gordon system), so that for any \( \xi \in \mathbb{C}^{N/2}, |\xi| = 1, \) the function
\[
\psi(t, x) = \phi_\xi(x) e^{-i\omega t},
\]
with
\[
\phi_\xi(x) = \begin{bmatrix} v(r)\xi \\ iu(r)\sigma_r \xi \end{bmatrix}, \quad r = |x|,
\]
is a solitary wave solution to the nonlinear Dirac equation (1) (or the Dirac–Klein–Gordon system (2)), then for any \( \Xi, H \in \mathbb{C}^{N/2} \setminus \{0\}, |\Xi|^2 - |H|^2 = 1, \) the function
\[
\theta_{\Xi, H}(t, x) = |\Xi|\phi_\xi(x) e^{-i\omega t} + |H|\chi_\eta(x) e^{i\omega t}, \quad \xi = \frac{\Xi}{|\Xi|}, \quad \eta = \frac{H}{|H|},
\]
with
\[
\chi_\eta(x) = \begin{bmatrix} -iu(r)\sigma_r \eta \\ v(r)\eta \end{bmatrix}, \quad r = |x|,
\]
is a solution to (1) (or (2), respectively).

Proof. The lemma is verified by the direct substitution. First one checks that
\[
\theta_{\Xi, H}^* \beta \theta_{\Xi, H} = |\Xi|^2 (v^2 - u^2) + |H|^2 (u^2 - v^2) = v^2 - u^2 = \phi_\xi^* \beta \phi_\xi.
\]
It remains to prove that the relation \( \omega \phi_\xi = D_m \phi_\xi - \beta f \phi_\xi \) implies the relation
\[
-\omega \chi_\eta = D_m \chi_\eta - \beta f \chi_\eta,
\]
with \( \chi_\eta(x) \) defined in (25), for any \( \eta \in \mathbb{C}^{N/2}, |\eta| = 1 \). With \( D_m \) built with the Dirac matrices from (10), the above relations on \( \phi_\xi \) and \( \chi_\eta \) are written explicitly as follows:
\[
\omega \begin{bmatrix} v\xi \\ i\sigma_r u\xi \end{bmatrix} = \begin{bmatrix} 0 & -i\sigma^* \cdot \nabla \\ -i\sigma \cdot \nabla & 0 \end{bmatrix} \begin{bmatrix} v\xi \\ i\sigma_r u\xi \end{bmatrix} + (m - f) \begin{bmatrix} v\xi \\ i\sigma_r u\xi \end{bmatrix},
\]
\[
-\omega \begin{bmatrix} -i\sigma^* u\eta \\ v\eta \end{bmatrix} = \begin{bmatrix} 0 & -i\sigma^* \cdot \nabla \\ -i\sigma \cdot \nabla & 0 \end{bmatrix} \begin{bmatrix} -i\sigma^* u\eta \\ v\eta \end{bmatrix} + (m - f) \begin{bmatrix} -i\sigma^* u\eta \\ v\eta \end{bmatrix};
\]
each of these relations is equivalent to the system (6) (with \( f \) given by (7) for the nonlinear Dirac equation or (8) for Dirac–Klein–Gordon system) once we take into account that \( \sigma \cdot \nabla v = \sigma_r \partial_v, \sigma^* \cdot \nabla v = \sigma_r^* \partial_v, \) and
\[
(\sigma^* \cdot \nabla) \sigma_r u = (\sigma^* \cdot \nabla) \sigma_r \partial_v U = (\sigma^* \cdot \nabla)(\sigma \cdot \nabla) U = \Delta U = \partial_u + \frac{n - 1}{r - u},
\]
where we introduced \( U(r) = \int_0^r u(s) \, ds \) and used the identity \( (\sigma^* \cdot \nabla)(\sigma \cdot \nabla) = 1_{N/2} \Delta \) which follows from (11); similarly, one has \( \sigma \cdot \nabla \sigma_r^* u = \partial_v u + \frac{2}{r - u} \).

Assume that in (1) \( f \in C^1(\mathbb{R} \setminus \{0\}) \cap C(\mathbb{R}), \) so that the linearization at a solitary wave makes sense.
Corollary 3.2. The linearization at a (one-frequency) solitary wave has eigenvalues $\pm 2\omega i$ of geometric multiplicity (at least) $N/2$.

Proof. Since $\psi(t, x) = ((1 + \epsilon^2)^{1/2}\phi(\epsilon^2 x) + \epsilon\chi_\eta(\epsilon x)e^{2i\omega t})e^{-i\omega t}$, for any $0 < \epsilon < 1$ and $\chi, \eta \in \mathbb{C}^{N/2}$, satisfies the nonlinear Dirac equation (1), one concludes that $r(t, x) = \chi_\eta(x)e^{2i\omega t}$ is a solution to the nonlinear Dirac equation linearized at $\phi e^{-i\omega t}$. This shows that $2\omega i$ is an eigenvalue of the linearization. Due to the symmetry of the spectrum with respect to Re $\lambda = 0$ and Im $\lambda = 0$, so is $-2\omega i$. \hfill $\square$

Remark 3.3. The presence of the eigenvalues $\pm 2\omega i$ in the spectrum of the linearization of the Soler model at a solitary wave was noticed in [3] (initially in the one-dimensional case) and eventually led to the conclusion that there exist bi-frequency solitary waves. We note that the existence of such bi-frequency solutions could have already been deduced applying the Bogoliubov transformation (3) from [16] to one-frequency solitary waves $\phi(x)e^{-i\omega t}$ which were constructed in [29].

We define the solitary manifold of one- and bi-frequency solutions of the form (24) corresponding to some value $\omega$ by

$$\mathcal{M}_\omega = \{ \theta_{\Xi, H}(t, x) ; \Xi, H \in \mathbb{C}^{N/2}, |\Xi|^2 - |H|^2 = 1 \} .$$

(26)

In general, the solitary manifold $\mathcal{M}_\omega$ can be larger than the orbit of $\phi e^{-i\omega t}$ under the action of the available symmetry groups: $G_{\text{Bogoliubov}}$ defined in (3) and SO($n$); we denote this orbit by

$$O_\phi = \{ rg(\phi e^{-i\omega t}) ; r \in \text{SO}(n), g \in G_{\text{Bogoliubov}} \} \subset \mathcal{M}_\omega .$$

Remark 3.4. We do not consider translations and Lorentz boosts, thus preserving the spatial location of the solitary wave, just like it is preserved in the definition of $\mathcal{M}_\omega$.

In lower spatial dimensions $n \leq 2$, when $N = 2$, the orbit $O_\phi$ is given by

$$O_\phi = \{ g(\phi(x)e^{-i\omega t}) ; g \in G_{\text{Bogoliubov}} \},$$

with $G_{\text{Bogoliubov}} = \{ a + b\gamma_1 K, a, b \in \mathbb{C}, |a|^2 - |b|^2 = 1 \}$; thus, $O_\phi$ coincides with the solitary manifold

$$\mathcal{M}_\omega = \{ a \begin{pmatrix} v(x) \\ u(x) \end{pmatrix} e^{-i\omega t} + b \begin{pmatrix} u(x) \\ v(x) \end{pmatrix} e^{i\omega t} ; a, b \in \mathbb{C}, |a|^2 - |b|^2 = 1 \} .$$

In three spatial dimensions, $n = 3$ and $N = 4$, the solitary manifold $\mathcal{M}_\omega$ is larger than the orbit $O_\phi$ of $\phi(x)e^{-i\omega t}$ under the action of the available symmetry groups: spatial rotations SO($n$) and the Bogoliubov group $G_{\text{Bogoliubov}}$ given by elements of the form $a + b(-i\gamma^2)K$, where $a, b \in \mathbb{C}, |a|^2 - |b|^2 = 1$. Indeed, one has $O_\phi \subseteq \mathcal{M}_\omega$, since

$$\dim \mathfrak{g} O_\phi = 5 < \dim \mathfrak{g} (SO(3) + \mathfrak{g} G_{\text{Bogoliubov}}) = 3 + 3,$$

$$\dim \mathfrak{m} \mathcal{M}_\omega = \dim \mathfrak{m} \{ (\Xi, H) \in \mathbb{C}^{N/2} \times \mathbb{C}^{N/2} ; |\Xi|^2 - |H|^2 = 1 \} = 2N - 1 = 7 .$$

Note that in the above inequality for $\dim \mathfrak{g} O_\phi$ one has “strictly smaller”, since the generator corresponding to the standard U(1)-invariance which enters the Lie algebra of $G_{\text{Bogoliubov}}$ also coincides with the generator of rotation around z-axis.

In the case $n = 4, N = 4$, the symmetry group simplifies to U(1) since the generator $-i\gamma^2 K$ is no longer available: the Dirac operator now contains $-i\alpha^4 \partial_{\alpha^4} = \beta \gamma^5 \partial_{\alpha^4}$ (with $\alpha^4 := \begin{pmatrix} 0 & -i \gamma_2 \\ i \gamma_2 & 0 \end{pmatrix} = -i \beta \gamma^5$), which breaks the anticommutation of $D_m$ with $-i\gamma^2 K$:

$$\{ -i\alpha^4, i\alpha^2 \beta K \} = \{ \beta \gamma^5, i\alpha^2 \beta K \} = \beta \gamma^5 i\alpha^2 \beta K + i\alpha^2 \beta K \beta \gamma^5 .$$
Again, $O_\phi \subseteq \mathcal{M}_\omega$ since
\[ \dim_{\mathbb{R}} O_\phi \leq \dim_{\mathbb{R}} SO(4) + \dim_{\mathbb{R}} G_{\text{Bogoliubov}} \leq 6 + 1, \quad \dim_{\mathbb{R}} \mathcal{M}_\omega = 2N - 1 = 7, \]
with the Lie algebras of $SO(4)$ and $G_{\text{Bogoliubov}}$ sharing one element (a generator of the standard $U(1)$-symmetry). Moreover, the action of $SO(4)$ (dim$_{\mathbb{R}} = 6$) on $\mathbb{C}^2$ (dim$_{\mathbb{R}} = 4$) could not be faithful: the orbit of an element $\xi \in \mathbb{C}^2$ under the action of $SO(4)$ is only three-dimensional. As a result, in the case $n = 4$, $N = 4$, one has
\[ \dim_{\mathbb{R}} O_\phi = 3, \quad \dim_{\mathbb{R}} \mathcal{M}_\omega = 2N - 1 = 7. \]

**Remark 3.5.** We can rephrase the above situation in the following way. When moving from $n = 3$ to $n = 4$, additional rotations in $\mathbb{R}^4$ do not add to the orbit of $\xi \in \mathbb{C}^2$ which has already been of maximal dimension when $n = 3$ (which equals three: it is the real dimension of the unit sphere in $\mathbb{C}^2$), while the loss of the generator $B_4 K$ from the Bogoliubov group led to the loss of two real dimensions of the orbit $O_\phi$.

**Remark 3.6.** Let us briefly discuss the pseudo-scalar theories in spatial dimension $n = 3$. Instead of the (scalar) Yukawa interaction, given by the term $\phi \bar{\psi} \psi$ in the Lagrangian, one can consider pseudoscalar interaction, introducing the term $\phi \bar{\psi} i \gamma^5 \psi$, which we write as $-\phi \psi^* \alpha^4 \psi$ with
\[ \alpha^4 = -i \beta \gamma^5 = \begin{bmatrix} 0 & -iI_2 \\ iI_2 & 0 \end{bmatrix}. \]

The Bogoliubov symmetry $SU(1, 1)$ is no longer present in a model with such an interaction. For $g = a - ib \gamma^2 K$, $|a|^2 - |b|^2 = 1$, omitting taking the real part in the intermediate computations, one has:
\[
\begin{align*}
(g \psi)^* \alpha^4 (g \psi) &= \text{Re}(g \psi)^* \alpha^4 (g \psi) = \text{Re}((a - ib \gamma^2 K) \psi)^* \alpha^4 (a - ib \gamma^2 K) \psi \\
&= \text{Re} \psi^*(\bar{a} - Kib \gamma^2) \alpha^4 (a - ib \gamma^2 K) \psi = \text{Re} \psi^*(\bar{a} - ib \gamma^2 K) \alpha^4 (a - ib \gamma^2 K) \psi \\
&= \text{Re} \psi^* \alpha^4 (\bar{a} - ib \gamma^2 K) (a - ib \gamma^2 K) \psi \\
&= \text{Re} \psi^* \alpha^4 (|a|^2 + |b|^2 - 2 \bar{a}a \gamma^2 K) \psi = \psi^* \alpha^4 (|a|^2 + |b|^2) \psi,
\end{align*}
\]
which in general is different from $\psi^* \alpha^4 \psi$. Above, in the last equality, we took into account that the matrix $\alpha^4 \gamma^2 = \begin{bmatrix} \sigma_2 & 0 \\
0 & i \sigma_2 \end{bmatrix}$ is antisymmetric, hence $\psi^* \alpha^4 \gamma^2 K \psi = (K \psi)^\dagger \alpha^4 \gamma^2 K \psi = 0$.

4. Spectral stability of bi-frequency solitary waves.

**Definition 4.1.** We say that the bi-frequency solitary wave solution $\theta_{\Xi, H}(t, x)$ (see (24)) to (1) or (2) is linearly unstable if there are nonzero functions $\rho_j \in L^2(\mathbb{R}^n, \mathbb{C}^N)$ and numbers $\Lambda_j \in \mathbb{C}$, $1 \leq j \leq J$, $J \geq 1$, with $\Lambda_i \neq \Lambda_j$ except when $i = j$ and with $\text{Re} \Lambda_j > 0$, such that
\[ \theta_{\Xi, H}(t, x) = c \sum_{j=1}^{J} \rho_j(x) e^{\Lambda_j t} \]
solves (1) or (2) up to $o(\epsilon)$, $0 < \epsilon \ll 1$. Otherwise, we call the bi-frequency solitary wave solution spectrally stable.

**Theorem 4.1.** Let $n \leq 4$, $N = 2$ or $N = 4$. Let (9) be satisfied. Then the bi-frequency solitary wave (24) is spectrally stable as long as the corresponding one-frequency solitary wave solution (22) is spectrally stable.
Proof. Let us first give the proof of Theorem 4.1 in the simple case,
\[ n \leq 2, \quad N = 2, \]
when the argument could be based on reduction with the aid of the Bogoliubov transformation. In this case, by the above considerations, the solitary manifold \( \mathcal{M}_\omega \) coincides with the orbit \( O_\phi \) of a one-frequency solitary wave under the action of the symmetry group of the equation. Therefore, a bi-frequency wave \( \theta(t, x) = a\phi(x)e^{-i\omega t} + b\chi(x)e^{i\omega t} \), \( a, b \in \mathbb{C}, |a|^2 - |b|^2 = 1 \), can be written in the form \( \theta(t, x) = a\phi e^{-i\omega t} + bB(\mathbf{K}\phi)e^{i\omega t} = \left[ \phi(x)e^{-i\omega t} \right] \). The perturbation (27) of the bi-frequency solitary wave can be written in the form
\[
a\phi e^{-i\omega t} + b\chi e^{-i\omega t} + \sum_{j=1}^{J} \rho_j(x)e^{i\lambda_j t} = g \left[ (\phi(x) + \rho(t, x))e^{-i\omega t} \right],
\]
where we take \( \rho(t, x) = e^{i\omega t}g^{-1}\left[ \sum_{j=1}^{J} \rho_j(x)e^{i\lambda_j t} \right] \), with \( g^{-1} = \bar{a} - b\mathbf{B}\mathbf{K} \). The above relation shows that the exponential growth of \( \sum_{j=1}^{J} \rho_j(x)e^{i\lambda_j t} \) is in one-to-one correspondence to the exponential growth of \( \rho(t, x) \). As a result, the spectral stability of the bi-frequency wave \( a\phi(x)e^{-i\omega t} + b\chi(x)e^{i\omega t} \) (cf. Definition 4.1) takes place if and only if the corresponding one-frequency solitary wave \( \phi(x)e^{-i\omega t} \) is spectrally stable. This completes the proof in the case \( n \leq 2, N = 2. \)

Now we assume that
\[ n \leq 4, \quad N = 4. \]
Given \( \Xi, H \in \mathbb{C}^{N/2 \setminus \{ 0 \}} \) such that \( |\Xi|^2 - |H|^2 = 1 \), let
\[
\theta_{\Xi, H}(t, x) = a\phi_\Xi(x)e^{-i\omega t} + b\chi_\Xi(x)e^{i\omega t}, \tag{28}
\]
with \( a = |\Xi|, b = |H|, \xi = \Xi/|\Xi| \) and \( \eta = H/|H| \) (with \( \phi_\Xi \) and \( \chi_\Xi \) from (23) and (25), respectively) be a bi-frequency solitary wave. We will consider the perturbation of this solitary wave in the form
\[
\psi(t, x) = a(\phi_\Xi(x) + \rho(t, x))e^{-i\omega t} + b(\chi_\Xi(x) + \sigma(t, x))e^{i\omega t},
\]
where we will impose the following condition on \( \rho(t, x) \) and \( \sigma(t, x) \):
\[ \bar{a}b\phi_\Xi^*\beta\sigma + \bar{a}b\rho^*\beta\chi_\Xi = 0. \tag{29} \]
If (29) is satisfied, we will say that there is no frequency mixing; in this case, \( \psi^*\beta\psi \) does not contribute terms with factors \( e^{-2i\omega t} \) or \( e^{2i\omega t} \). We will show that indeed there is a way to split the perturbation into \( \rho(t, x) \) and \( \sigma(t, x) \) so that (29) is satisfied (see Proposition 4.3 and Remark 4.6 below).

Let \( (e_j)_{1 \leq j \leq N/2} \) be the standard basis in \( \mathbb{C}^{N/2} \) and let \( R, S \in \mathbf{SU}(N/2) \) be such that \( \xi = Re_1, \eta = Se_1 \). Denote
\[
\phi_j(x) = \begin{bmatrix} v(r)Re_j \\ iu(r)\sigma_SRe_j \end{bmatrix}, \quad \chi_j(x) = \begin{bmatrix} -iu(r)\sigma_S^*Se_j \\ v(r)Se_j \end{bmatrix}, \quad 1 \leq j \leq N/2. \tag{30}
\]
(Above, we do not indicate the dependence of \( v, u \) or \( \omega \).) We consider the perturbation of the bi-frequency solitary wave (28) in the form
\[
\psi(t, x) = a\left( \phi_1(x) + \sum_{j=1}^{N/2} (p_j(t, x)\phi_j(x) + q_j(t, x)\chi_j(x)) \right)e^{-i\omega t}
+ b\left( \chi_1(x) + \sum_{j=1}^{N/2} (r_j(t, x)\phi_j(x) + s_j(t, x)\chi_j(x)) \right)e^{i\omega t}. \tag{31}
\]
Above, $p_j$, $q_j$, $r_j$, $s_j$, $1 \leq j \leq N/2$, are complex scalar-valued functions of $x$ and $t$. The condition (29) of the absence of frequency mixing takes the form

$$ab(\{q_1(t, x) - r_1(t, x)\}) = 0.$$  \hfill (32)

As long as (32) is satisfied, the linearized terms in the expansion of $\psi^* \beta \psi$ do not contain factors $e^{\pm i\omega t}$; the ones that are left are given by

$$2 \text{Re}(\ldots) = 2 \text{Re}(a|\phi|^2_1 + b|\chi|^2_1) = 2 \phi_1^* \beta \phi_1 \text{Re}(|a|^2 - |b|^2).$$  \hfill (33)

The linearized equation will contain two groups of terms, with factors $e^{\pm i\omega t}$; to satisfy the linearized equation, it is enough to equate these groups separately. The terms with the factor $e^{i\omega t}$:

$$i\partial_t p_k \phi_k + i\partial_t q_k \chi_k - 2\omega q_k \chi_k = (D_0 p_k) \phi_k + (D_0 q_k) \chi_k - 2f' \text{Re}(\ldots) \beta \phi_1;$$  \hfill (34)

the terms with the factor $e^{-i\omega t}$:

$$i\partial_t r_k \phi_k + i\partial_t s_k \chi_k - 2\omega r_k \chi_k = (D_0 r_k) \phi_k + (D_0 s_k) \chi_k - 2f' \text{Re}(\ldots) \beta \chi_1.$$  \hfill (35)

**Remark 4.2.** When deriving the above equations, we eliminated terms with the derivatives of $\phi_j$ and $\chi_j$ by using the stationary Dirac equations satisfied by $\phi_j$ and $\chi_j$:

$$\omega \phi_j = (D_m - \beta f) \phi_j, \quad -\omega \chi_j = (D_m - \beta f) \chi_j, \quad 1 \leq j \leq N/2,$$

where $f = f(\tau)$ is evaluated at $\tau = \nu^2 - \omega^2 = \phi_1^* \beta \phi_1$.

**Proposition 4.3.** The system (34), (35) is invariant in the subspace specified by the relations

$$r_j = \bar{q}_j, \quad s_j = \bar{p}_j, \quad 1 \leq j \leq N/2.$$  \hfill (36)

**Proof.** We claim that if $p_j$, $q_j$, $r_j$, and $s_j$, $1 \leq j \leq 2$, satisfy (36), then equations (34) and (35) yield

$$\partial_t r_j = \partial_t \bar{q}_j, \quad \partial_t s_j = \partial_t \bar{p}_j, \quad 1 \leq j \leq N/2.$$  \hfill (37)

Multiplying the relation (34) by $\beta$ and coupling with $\phi_j$ (in the $\mathbb{C}^N$-sense; no integration in $x$):

$$i\partial_t p_k \phi_j^* \beta \phi_k = -i\phi_j \beta \alpha^i \phi_k \partial_t p_k - i\phi_j \beta \alpha^i \chi_k \partial_t q_k - 2f' \text{Re}(\ldots) \phi_j^* \phi_1.$$  \hfill (38)

We took into account that $\phi_j^* \beta \chi_k = 0$ for all $1 \leq j, k \leq N/2$ (cf. Lemma 4.4 below). Multiplying the relation (35) by $\beta$ and coupling with $\chi_j$:

$$i\partial_t s_k \chi_j^* \beta \chi_k = -i\chi_j^* \beta \alpha^i \phi_k \partial_t r_k - i\chi_j^* \beta \alpha^i \chi_k \partial_t s_k - 2f' \text{Re}(\ldots) \chi_j^* \chi_1.$$  \hfill (39)

Multiplying the relation (34) by $\beta$ and coupling with $\chi_j$ one has:

$$i\partial_t q_k \chi_j^* \beta \chi_k - 2\omega q_k \chi_j^* \beta \chi_k$$
$$= -i\chi_j \beta \alpha^i \phi_k \partial_t p_k - i\chi_j \beta \alpha^i \chi_k \partial_t q_k - 2f' \text{Re}(\ldots) \chi_j^* \phi_1.$$  \hfill (40)

Multiplying the relation (35) by $\beta$ and coupling with $\phi_j$:

$$i\partial_t r_k \phi_j^* \beta \phi_k + 2\omega r_k \phi_j^* \beta \phi_k$$
$$= -i\phi_j \beta \alpha^i \phi_k \partial_t r_k - i\phi_j \beta \alpha^i \chi_k \partial_t s_k - 2f' \text{Re}(\ldots) \phi_j^* \chi_1.$$  \hfill (41)

The proof of Proposition 4.3 will follow if we prove that (38) and (39) are complex conjugates of each other, and that so are (40) and (41).
Lemma 4.4. For $1 \leq j, k \leq N/2$,
\begin{align*}
\phi_j^* \phi_k & = \chi_j^\ast \chi_k, & \phi_j^\ast \chi_k & = \chi_j^\ast \phi_k; \tag{42} \\
\phi_j^\ast \beta \phi_k & = -\chi_j^\ast \beta \chi_k, & \phi_j^\ast \beta \chi_k & = 0, & \chi_j^\ast \beta \phi_k & = 0; \tag{43} \\
\phi_j^\ast (-i\beta \alpha^t) \phi_k & = \chi_j^\ast (-i\beta \alpha^t) \chi_k, & \phi_j^\ast (-i\beta \alpha^t) \chi_k & = \chi_j^\ast (-i\beta \alpha^t) \phi_k, & 1 \leq i \leq n. \tag{44}
\end{align*}

Proof. We note that $\sigma_j^\ast \sigma_r = \sigma_r \sigma_j^* = 1$ (cf. (12), (11)). We have:
\begin{align*}
\phi_j^\ast \phi_k & = \begin{bmatrix} vRe_j \\ iu_\sigma_r Re_j \end{bmatrix}^\ast \begin{bmatrix} vRe_k \\ iu_\sigma_r Re_k \end{bmatrix} = v^2 e_j^\ast e_k + u^2 e_j^\ast R^\ast \sigma_r \sigma_s Re_k = (v^2 + u^2) e_j^\ast e_k, \\
\chi_j^\ast \chi_k & = \begin{bmatrix} -iu_\sigma_j S e_j \\ vSe_j \end{bmatrix}^\ast \begin{bmatrix} -iu_\sigma_s S e_k \\ vSe_k \end{bmatrix} = u^2 e_j^\ast S^\ast \sigma_r \sigma_s S e_k + v^2 e_j^\ast e_k = (v^2 + u^2) e_j^\ast e_k, \\
\phi_j^\ast \chi_k & = \begin{bmatrix} vRe_j \\ iu_\sigma_r Re_j \end{bmatrix}^\ast \begin{bmatrix} -iu_\sigma_s S e_k \\ vSe_k \end{bmatrix} = -2iuv e_j^\ast R^\ast \sigma_r \sigma_s S e_k, \\
\chi_j^\ast \phi_k & = \begin{bmatrix} -iu_\sigma_j S e_j \\ vSe_j \end{bmatrix}^\ast \begin{bmatrix} vRe_k \\ iu_\sigma_r Re_k \end{bmatrix} = 2iuv e_j^\ast S^\ast \sigma_r \sigma_s Re_k. \tag{45} \\
\end{align*}

To show that (45) and (46) are complex conjugates of each other, it suffices to mention the identities
\begin{align*}
\overline{e_j^\ast R^\ast \sigma_i^\ast S e_2} & = e_j^\ast S^\ast \sigma_i Re_2, & \overline{e_j^\ast R^\ast \sigma_i^\ast S e_1} & = e_j^\ast S^\ast \sigma_i Re_1, & 1 \leq i \leq n, \tag{47}
\end{align*}
valid for all $R, S \in SU(2)$. (Indeed, $M := R^\ast \sigma_i^\ast S$ satisfies $M \in U(2)$, det $M = -1$; for such matrices, one has $M_{12} = M_{21}$.) This proves relations (42).

Remark 4.5. The relation $M_{ji} = M_{ij}$ for $i \neq j$ is no longer true for $M = R^\ast \sigma_i^\ast S$ with $\sigma_i$ the equivalent of the Pauli matrix of size $N/2$ and with $R, S \in SU(N/2)$ with $N > 4$, seemingly limiting the present approach to four-component spinors.

We continue:
\begin{align*}
\phi_j^\ast \beta \phi_k & = \begin{bmatrix} vRe_j \\ iu_\sigma_r Re_j \end{bmatrix}^\ast \begin{bmatrix} 1_{N/2} & 0 \\ 0 & -1_{N/2} \end{bmatrix} \begin{bmatrix} vRe_k \\ iu_\sigma_r Re_k \end{bmatrix} = v^2 e_j^\ast e_k - u^2 e_j^\ast R^\ast \sigma_r \sigma_s Re_k = (v^2 - u^2) e_j^\ast e_k, \\
\chi_j^\ast \beta \chi_k & = \begin{bmatrix} -iu_\sigma_j S e_j \\ vSe_j \end{bmatrix}^\ast \begin{bmatrix} 1_{N/2} & 0 \\ 0 & -1_{N/2} \end{bmatrix} \begin{bmatrix} -iu_\sigma_s S e_k \\ vSe_k \end{bmatrix} = u^2 e_j^\ast S^\ast \sigma_r \sigma_s S e_k - v^2 e_j^\ast e_k = (v^2 - u^2) e_j^\ast e_k, \\
\phi_j^\ast \beta \chi_k & = \begin{bmatrix} vRe_j \\ iu_\sigma_r Re_j \end{bmatrix}^\ast \begin{bmatrix} 1_{N/2} & 0 \\ 0 & -1_{N/2} \end{bmatrix} \begin{bmatrix} -iu_\sigma_s S e_k \\ vSe_k \end{bmatrix} = 0.
\end{align*}

This proves the relations (43). Finally, we prove (44):
\begin{align*}
\phi_j^\ast \beta \alpha^t \phi_k & = \begin{bmatrix} vRe_j \\ iu_\sigma_r Re_j \end{bmatrix}^\ast \begin{bmatrix} 0 & \sigma_i^\ast \\ -\sigma_i & 0 \end{bmatrix} \begin{bmatrix} vRe_k \\ iu_\sigma_r Re_k \end{bmatrix} = ivue_j^\ast R^\ast \sigma_r \sigma_s Re_k + ivue_j^\ast R^\ast \sigma_r \sigma_s Re_k = 2iuv \frac{d}{dr} e_j^\ast e_k,
\end{align*}
\[ \chi_j^* \beta^i \chi_k = \begin{bmatrix} -iu \sigma_r^* S e_j & 0 & -i u \sigma_r^* S e_k \\ v S e_j & -\sigma_i & 0 \\ v S e_k & 0 & -\sigma_i \end{bmatrix} \]

\[ = i u v e_j^* S \sigma_r^* \sigma_i^* S e_k + i u v e_j^* S^* \sigma_i \sigma_r^* S e_k = 2 i u v \frac{x^i}{r} e_j^* e_k, \]

\[ \phi_j^* \beta^i \chi_k = \begin{bmatrix} v R e_j & 0 & -i u \sigma_r^* S e_k \\ i u \sigma_r R e_j & -\sigma_i & 0 \\ -i u \sigma_r R e_k & 0 & -\sigma_i \end{bmatrix} \]

\[ = v^2 e_j^* R^* \sigma_r^* S e_k + u^2 e_j^* R^* \sigma_i \sigma_r^* S e_k, \]

\[ \chi_j^* \beta^i \phi_k = \begin{bmatrix} -iu \sigma_r^* S e_j & 0 & v R e_k \\ v S e_j & -\sigma_i & i u \sigma_r R e_k \\ i u \sigma_r R e_k & 0 & -\sigma_i \end{bmatrix} \]

\[ = -v^2 e_j^* S^* \sigma_i R e_k - u^2 e_j^* S^* \sigma_r \sigma_i \sigma_r^* R e_k. \] (48)

To argue that the last two lines are anti-complex conjugates, we note that

\[ \sigma_r^* \sigma_i \sigma_r^* = \sigma_r^* \left( 2 \frac{x^i}{r} - \sigma_r \sigma_i^* \right) = 2 \sigma_r^* \frac{x^i}{r} - \sigma_i^*, \quad 1 \leq i \leq n, \]

and then use the same reasoning as above (when showing that (45) and (46) are complex conjugates, basing on the identities (47)).

Lemma 4.4 proves (37), finishing the proof of Proposition 4.3.

Remark 4.6. We note that the relation (32) is satisfied in the invariant subspace described in Proposition 4.3, and thus there is no frequency mixing in this subspace: given \( \psi \) of the form (31) with \( r_j = \tilde{q}_j, s_j = \tilde{p}_j, 1 \leq j \leq N/2 \), the expression \( \psi^* \beta \psi \) does not contain terms with the factors \( e^{\pm 2i \omega t} \).

We introduce the following functions:

\[ \Phi_j(x) = \frac{1}{v(r)} \phi_j(x), \quad X_j(x) = \frac{1}{v(r)} \chi_j(x), \quad 1 \leq j \leq N/2; \] (49)

at each \( x \in \mathbb{R}^n \), these functions form a basis in \( \mathbb{C}^N \).

Lemma 4.7. Let (9) be satisfied. Then \( \Phi_j(x), 1 \leq j \leq N/2, \) and \( X_j(x), 1 \leq j \leq N/2, \) are linearly independent, uniformly in \( x \).

Proof. For any \( 1 \leq j \leq N/2 \) and any \( x \in \mathbb{R}^n \), \( \| \Phi_j(x) \|_{\mathbb{C}^N} + \| X_j(x) \|_{\mathbb{C}^N} \leq C < \infty \) since \( |u(r)/v(r)| \leq c < 1 \) by (9), while \( \det [\Phi_1 \Phi_2 X_1 X_2] \) is given by

\[ \det \begin{bmatrix} e_1 & e_2 \\ i u \sigma_r e_1 & i u \sigma_r e_2 \end{bmatrix} \begin{bmatrix} -i u \sigma_r^* e_1 \\ -i u \sigma_r^* e_2 \end{bmatrix} \begin{bmatrix} -i u \sigma_r^* e_1 \\ -i u \sigma_r^* e_2 \end{bmatrix} \]

\[ = \det \begin{bmatrix} I_2 & -i u \sigma_r^* \\ i u \sigma_r & I_2 \end{bmatrix} = \det \left( I_2 - \frac{u^2}{v^2} \sigma_r \sigma_r^* \right); \]

in the last equality, one can use the Schur complement to compute the determinant of a matrix written in the block form. Using (9) and taking into account that \( \sigma_r \sigma_r^* = I_2 \), one concludes that the right-hand side of the above is separated from zero uniformly in \( x \in \mathbb{R}^n \).
The perturbation of a bi-frequency solitary wave could be rewritten as follows (cf. (31)):

\[
\psi(t, x) = a \left( \phi_1(x) + \sum_{j=1}^{N/2} (P_j(t, x)\Phi_j(x) + Q_j(t, x)X_j(x)) \right) e^{-i\omega t} \\
+ b \left( \chi_1(x) + \sum_{j=1}^{N/2} (\bar{Q}_j(t, x)\Phi_j(x) + \bar{P}_j(t, x)X_j(x)) \right) e^{i\omega t}.
\] (50)

Taking into account (49), we note that \( P_j \) and \( Q_j \) in the above formula differ from \( p_j \) and \( q_j \) in (31) by the factor of \( v(r) \):

\[
P_j(t, x) = v(r)p_j(t, x), \quad Q_j(t, x) = v(r)q_j(t, x), \quad 1 \leq j \leq N/2.
\]

We claim that at the initial moment there is a unique way to decompose the perturbation \( f \in L^2(\mathbb{R}^n, \mathbb{C}) \) over the terms in (50) with factors \( e^{\pm i\omega t} \):

**Lemma 4.8.** Let \( a, b \in \mathbb{C}, |a|^2 - |b|^2 = 1 \), and let (9) be satisfied. Then for any \( \varrho \in L^2(\mathbb{R}^n, \mathbb{C}^N) \), there is a unique choice of scalar functions \( (P_j, Q_j)_{1 \leq j \leq N/2} \in L^2(\mathbb{R}^n, \mathbb{C})^N \) such that

\[
a \sum_{j=1}^{N/2} (P_j \Phi_j + Q_j X_j) + b \sum_{j=1}^{N/2} (\bar{Q}_j \Phi_j + \bar{P}_j X_j) = \varrho.
\] (51)

The map \( L^2(\mathbb{R}^n, \mathbb{C}^N) \to L^2(\mathbb{R}^n, \mathbb{C})^N, \varrho \mapsto (P_j, Q_j)_{1 \leq j \leq N/2} \), is continuous.

**Proof.** Let \( \varrho \in L^2(\mathbb{R}^n, \mathbb{C}^N) \). By Lemma 4.7, there are \( f_j \in L^2(\mathbb{R}^n, \mathbb{C}) \) and \( g_j \in L^2(\mathbb{R}^n, \mathbb{C}) \) be such that \( \varrho = \sum_{j=1}^{N/2} \Phi_j f_j + \sum_{j=1}^{N/2} X_j g_j \), and the map

\[
L^2(\mathbb{R}^n, \mathbb{C}^N) \to L^2(\mathbb{R}^n, \mathbb{C})^N, \quad \varrho \mapsto (f_j, g_j)_{1 \leq j \leq N/2} \in L^2(\mathbb{R}^n, \mathbb{C})^N
\]

is continuous. Equation (51) takes the form

\[
aP_j(x) + b\bar{Q}_j(x) = f_j(x), \quad aQ_j(x) + b\bar{P}_j(x) = g_j(x), \quad 1 \leq j \leq N/2.
\] (52)

Since \( |b/a| < 1 \), for any \( (f_j, g_j)_{1 \leq j \leq N/2} \in L^2(\mathbb{R}^n, \mathbb{C})^N \) the map

\[
(P_j, Q_j) \mapsto \left( \frac{1}{a} (f_j - b\bar{Q}_j), \frac{1}{a} (g_j - b\bar{P}_j) \right), \quad 1 \leq j \leq N/2,
\]

is a contraction in \( L^2(\mathbb{R}^n, \mathbb{C})^N \) and thus has a unique fixed point (a solution to (52)) which continuously depends on \( (f_j, g_j) \). \( \square \)

We can now conclude the proof of Theorem 4.1 in the case \( n \leq 4, N = 4 \). By Proposition 4.3 and Lemma 4.8, any solution to the linearization at the bi-frequency solitary wave

\[
\theta(t, x) = a\phi_1(x)e^{-i\omega t} + b\chi_1(x)e^{i\omega t},
\]

where

\[
\phi_1(x) = \begin{bmatrix} v(r)\xi \\ iu(r)\xi \end{bmatrix}, \quad \chi_1(x) = \begin{bmatrix} -iu(r)\eta \\ v(r)\eta \end{bmatrix},
\]

\( \xi, \eta \in \mathbb{C}^{N/2}, |\xi| = |\eta| = 1, \quad a, b \in \mathbb{C}, \quad |a|^2 - |b|^2 = 1 \)
(cf. (30)) can be written in the form

\[ \psi(t, x) = a \left( \phi_1(x) + \sum_{j=1}^{N/2} \left( p_j(t, x) \phi_j(x) + q_j(t, x) \chi_j(x) \right) \right) e^{-i\omega t} \]

\[ + b \left( \chi_1(x) + \sum_{j=1}^{N/2} \left( \bar{p}_j(t, x) \phi_j(x) + \bar{q}_j(t, x) \chi_j(x) \right) \right) e^{i\omega t}; \]

this expression solves the nonlinear Dirac equation in the zero and first order of the linear perturbation. Taking in this Ansatz \( a = 1 \) and \( b = 0 \), we see that

\[ \psi(t, x) = \left( \phi_1(x) + \sum_{j=1}^{N/2} \left( p_j(t, x) \phi_j(x) + q_j(t, x) \chi_j(x) \right) \right) e^{-i\omega t} \]

also solves the nonlinear Dirac equation in the zero and first order of the perturbation.

We conclude that the bi-frequency solitary wave solution (28) to the nonlinear Dirac equation (1) (or the Dirac–Klein–Gordon system (2)) is linearly unstable if and only if the one-frequency solitary wave \( \phi_\xi(x) e^{-i\omega x} \) is linearly unstable.

**REFERENCES**

[1] A. Aceves, A. Auditore, M. Conforti and C. De Angelis, Discrete localized modes in binary waveguide arrays, in *Nonlinear Photonics (NLP), 2013 IEEE 2nd International Workshop*, 2013, 38–42.

[2] A. Auditore, M. Conforti, C. De Angelis and A. B. Aceves, Dark-antidark solitons in waveguide arrays with alternating positive-negative couplings, *Optics Communications*, 297 (2013), 125–128.

[3] G. Berkolaiko and A. Comech, On spectral stability of solitary waves of nonlinear Dirac equation in 1D, *Math. Model. Nat. Phenom.*, 7 (2012), 13–31.

[4] A. Betlej, S. Suntsov, K. G. Makris, L. N. Christodoulides, G. I. Stegeman, J. Fini, R. T. Bise and D. J. DiGiovanni, All-optical switching and multifrequency generation in a dual-core photonic crystal fiber, *Opt. Lett.*, 31 (2006), 1480–1482.

[5] N. Boussaïd and A. Comech, Nonrelativistic asymptotics of solitary waves in the Dirac equation with Soler-type nonlinearity, *SIAM J. Math. Anal.*, 49 (2017), 2527–2572.

[6] N. Boussaïd and A. Comech, Spectral stability of small amplitude solitary waves of the Dirac equation with the Soler-type nonlinearity, ArXiv e-prints, arXiv:1705.05481.

[7] N. Boussaïd and S. Cuccagna, On stability of standing waves of nonlinear Dirac equations, *Comm. Partial Differential Equations*, 37 (2012), 1001–1056.

[8] N. J. Cerf, M. Bourennane, A. Karlsson and N. Gisin, Security of quantum key distribution using d-level systems, *Physical Review Letters*, 88 (2002), 127902.

[9] J. M. Chadam and R. T. Glassey, On certain global solutions of the Cauchy problem for the (classical) coupled Klein-Gordon-Dirac equations in one and three space dimensions, *Arch. Rational Mech. Anal.*, 54 (1974), 223–237.

[10] A. Comech, T. V. Phan and A. Stefanov, Asymptotic stability of solitary waves in generalized Gross–Neveu model, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 34 (2017), 157–196.

[11] A. Comech, M. Gian and S. Gustafson, On linear instability of solitary waves for the nonlinear Dirac equation, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 31 (2014), 639–654.

[12] J. Cuevas-Maraver, P. G. Kevrekidis, A. Saxena, F. Cooper, A. Khare, A. Comech and C. M. Bender, Solitary waves of a \( PT \)-symmetric nonlinear Dirac equation, *IEEE Journal of Selected Topics in Quantum Electronics*, 22 (2016), 1–9.

[13] J. Cuevas-Maraver, P. G. Kevrekidis, A. Saxena, A. Comech and R. Lan, Stability of solitary waves and vortices in a 2D nonlinear Dirac model, *Phys. Rev. Lett.*, 116 (2016), 214101.

[14] T. Durt, D. Kaszlikowski, J.-L. Chen and L. C. Kwek, Security of quantum key distributions with entangled qudits, *Phys. Rev. A*, 69 (2004), 032313.

[15] B. J. Eggleton, C. M. de Sterke and R. E. Slusher, Nonlinear pulse propagation in Bragg gratings, *J. Opt. Soc. Am. B*, 14 (1997), 2980–2993.

[16] A. Galindo, A remarkable invariance of classical Dirac Lagrangians, *Lett. Nuovo Cimento (2)*, 20 (1977), 210–212.
[17] D. J. Gross and A. Neveu, Dynamical symmetry breaking in asymptotically free field theories, Phys. Rev. D, 10 (1974), 3235–3253.
[18] D. D. Ivanenko, Notes to the theory of interaction via particles, Zh. Éksp. Teor. Fiz., 8 (1938), 260–266.
[19] M. Kues, C. Reimer, P. Roztocki, L. R. Cort, S. Sciara, B. Wetzel, Y. Zhang, A. Cino, S. T. Chu, B. E. Little, D. J. Moss, L. Caspani, J. Aza and R. Morandotti, On-chip generation of high-dimensional entangled quantum states and their coherent control, Nature, 546 (2017), 622–626.
[20] N. Lazarides and G. P. Tsironis, Gain-driven discrete breathers in $\mathcal{PT}$-symmetric nonlinear metamaterials, Phys. Rev. Lett., 110 (2013), 055901.
[21] S. Y. Lee and A. Gavrielides, Quantization of the localized solutions in two-dimensional field theories of massive fermions, Phys. Rev. D, 12 (1975), 3880–3886.
[22] A. Marini, S. Longhi and F. Biancalana, Optical simulation of neutrino oscillations in binary waveguide arrays, Phys. Rev. Lett., 113 (2014), 150401.
[23] A. A. Melnikov and L. E. Fedichkin, Quantum walks of interacting fermions on a cycle graph, Sci. Rep., 6 (2016), 34226.
[24] R. Morandotti, D. Mandelik, Y. Silberberg, J. S. Aitchison, M. Sorel, D. N. Christodoulides, A. A. Sukhorukov and Y. S. Kivshar, Observation of discrete gap solitons in binary waveguide arrays, Opt. Lett., 29 (2004), 2890–2892.
[25] T. Ozawa and K. Yamauchi, Structure of Dirac matrices and invariants for nonlinear Dirac equations, Differential Integral Equations, 17 (2004), 971–982.
[26] D. E. Pelinovsky and A. Stefanov, Asymptotic stability of small gap solitons in nonlinear Dirac equations, J. Math. Phys., 53 (2012), 073705, 27.
[27] J. Schindler, Z. Lin, J. M. Lee, H. Ramezani, F. M. Ellis and T. Kottos, $\mathcal{PT}$-symmetric electronics, Journal of Physics A: Mathematical and Theoretical, 45 (2012), 444029.
[28] J. Schindler, A. Li, M. C. Zheng, F. M. Ellis and T. Kottos, Experimental study of active LRC circuits with $\mathcal{PT}$ symmetries, Phys. Rev. A, 84 (2011), 040101.
[29] M. Soler, Classical, stable, nonlinear spinor field with positive rest energy, Phys. Rev. D, 1 (1970), 2766–2769.
[30] W. E. Thirring, A soluble relativistic field theory, Ann. Physics, 3 (1958), 91–112.
[31] M. Wakano, Intensely localized solutions of the classical Dirac-Maxwell field equations, Progr. Theoret. Phys., 35 (1966), 1117–1141.

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