Confined Brownian Motion Tracked With Motion Blur
Estimating Diffusion Coefficient and Size of Confining Space

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1 CALCULATIONS FOR THE VARIANCE OF A PARTICLE IN A 1D BOX

We insert the expression for $P(x, t|x_0, t_0)$ from Equation (13) in the time-ordered auto-correlation function in Equation (17) and find

$$\langle x(t|x_0)x(t'|x_0) \rangle_{0,t'>t} = \int_0^L \int_0^L xx'P(x, t|x_0, 0)P(x', t'|x, t)dx'dx'$$
$$= \int \int dx'dx \frac{xx'}{L^2} + 2 \sum_{q=1}^\infty \exp \left[ -\left( \frac{q\pi}{2} \right)^2 \frac{t' - t}{\tau} \right] \left[ \int dx \cos \left( \frac{q\pi x}{L} \right) \right]^2$$
$$+ \left[ \text{two terms proportional to } \cos \left( \frac{n\pi x_0}{L} \right) \right] \tag{S1}$$

The integrals are

$$\int \int dx'dx \frac{xx'}{L^2} = \frac{L^2}{4}, \tag{S2}$$

$$\int dx \cos \left( \frac{q\pi x}{L} \right) = \frac{[-1 + (-1)^q]L^2}{\pi^2 q^2}, \text{ i.e. } \frac{-2L^2}{\pi^2 q^2} \text{ for } n \text{ odd, and } 0 \text{ otherwise}, \tag{S3}$$

which we insert in Equation (S1) and get

$$\langle x(t|x_0)x(t'|x_0) \rangle_{0,t'>t} = \frac{L^2}{4} + \frac{8L^2}{\pi^4} \sum_{p=0}^\infty \exp \left[ -\left( \frac{(1 + 2p)\pi}{2} \right)^2 \frac{t' - t}{\tau} \right] \frac{1}{(1 + 2p)^4}$$
$$+ \left[ \text{two terms proportional to } \cos \left( \frac{n\pi x_0}{L} \right) \right]. \tag{S4}$$
Inserting this expression in Equation (16) gives

\[
\langle x_{\text{msr}}(x_0)^2 \rangle_0
= \frac{2}{(\Delta t_{\text{open}})^2} \int_{\Delta t}^{\Delta t + \Delta t_{\text{open}}} dt' \int_{\Delta t}^{\Delta t + \Delta t_{\text{open}}} dt \langle x(t|x_0)x(t'|x_0) \rangle_{0,t' > t}
= \frac{L^2}{4} + \frac{8L^2}{\pi^4} \sum_{p=0}^{\infty} \frac{1}{(1 + 2p)^4} \left\{ \frac{2}{(\Delta t_{\text{open}})^2} \int_{\Delta t}^{\Delta t + \Delta t_{\text{open}}} dt' \int_{\Delta t}^{\Delta t + \Delta t_{\text{open}}} dt \exp \left[ -\left( \frac{(1 + 2p)\pi}{2} \right)^2 \frac{t' - t}{\tau} \right] \right\}
+ \left[ \text{two terms proportional to } \cos \left( \frac{n\pi x_0}{L} \right) \right].
\]

The double-integral in the curly brackets is

\[
\frac{2}{(\Delta t_{\text{open}})^2} \int_{\Delta t}^{\Delta t + \Delta t_{\text{open}}} dt' \int_{\Delta t}^{\Delta t + \Delta t_{\text{open}}} dt \exp \left[ -\left( \frac{(1 + 2p)\pi}{2} \right)^2 \frac{t' - t}{\tau} \right]
= \frac{8}{\pi^2(1 + 2p)^2} \frac{\tau}{\Delta t_{\text{open}}} + \frac{32}{\pi^4(1 + 2p)^4} \left\{ \exp \left[ -\left( \frac{(1 + 2p)\pi}{2} \right)^2 \frac{\Delta t_{\text{open}}}{\tau} \right] - 1 \right\} \left( \frac{\tau}{\Delta t_{\text{open}}} \right)^2,
\]

which is inserted in Equation (S5).

\[
\langle x_{\text{msr}}(x_0)^2 \rangle_0
= \frac{64L^2}{\pi^6} \sum_{p=0}^{\infty} \frac{1}{(1 + 2p)^6} \left\{ \frac{\tau}{\Delta t_{\text{open}}} + \frac{4}{\pi^2(1 + 2p)^2} \left\{ \exp \left[ -\left( \frac{(1 + 2p)\pi}{2} \right)^2 \frac{\Delta t_{\text{open}}}{\tau} \right] - 1 \right\} \left( \frac{\tau}{\Delta t_{\text{open}}} \right)^2 \right\}
+ \frac{L^2}{4} + \left[ \text{two terms proportional to } \cos \left( \frac{n\pi x_0}{L} \right) \right].
\]

When the expression for \( \langle x_{\text{msr}}(x_0)^2 \rangle_0 \) is inserted in Equation (S15) and the integration over initial positions is carried out, we find

\[
\text{var}(x_{\text{msr}})
= \frac{64L^2}{\pi^6} \sum_{p=0}^{\infty} \frac{1}{(1 + 2p)^6} \left\{ \frac{\tau}{\Delta t_{\text{open}}} + \frac{4}{\pi^2(1 + 2p)^2} \left\{ \exp \left[ -\left( \frac{(1 + 2p)\pi}{2} \right)^2 \frac{\Delta t_{\text{open}}}{\tau} \right] - 1 \right\} \left( \frac{\tau}{\Delta t} \right)^2 \right\},
\]

as the integral over \( x_0 \) removes terms proportional to \( \cos \left[ n\pi x_0 / L \right] \) and leaves the other terms unchanged. Using that \( \sum_{p=0}^{\infty} (1 + 2p)^{-6} = \pi^6 / 960 \) \([\text{Bickel (2007)}] \), we arrive at Equation (18).
For $\Delta t_{\text{open}}/\tau \ll 1$, we Taylor expand the terms in the outer curly brackets in Equation (S8) and find

$$
\frac{\tau}{\Delta t_{\text{open}}} + \frac{4}{\pi^2(1+2p)^2} \left\{ \exp \left[ - \left( \frac{(1 + 2p)\pi}{2} \right)^2 \frac{\Delta t_{\text{open}}}{\tau} \right] - 1 \right\} \left( \frac{\tau}{\Delta t_{\text{open}}} \right)^2
$$

$$
= \frac{\tau}{\Delta t_{\text{open}}} + \frac{4}{\pi^2(1+2p)^2} \left\{ - \left( \frac{(1 + 2p)\pi}{2} \right)^2 \frac{\Delta t_{\text{open}}}{\tau} + \frac{1}{2} \left( \frac{(1 + 2p)\pi}{2} \right)^4 \left( \frac{\Delta t_{\text{open}}}{\tau} \right)^2 
\right. 
\left. - \frac{1}{6} \left( \frac{(1 + 2p)\pi}{2} \right)^6 \left( \frac{\Delta t_{\text{open}}}{\tau} \right)^3 + \mathcal{O}[(\Delta t_{\text{open}}/\tau)^4] \right\} \left( \frac{\tau}{\Delta t_{\text{open}}} \right)^2
$$

$$
= \frac{(1 + 2p)^2 \pi^2}{8} \frac{\tau}{\Delta t_{\text{open}}} + \mathcal{O}[(\Delta t_{\text{open}}/\tau)^2].
$$

Inserting this in Equation (S8) gives

$$
\text{var}(x) \simeq \frac{64L^2}{\pi^6} \sum_{p=0}^{\infty} \frac{1}{(1+2p)^6} \left\{ \frac{(1 + 2p)^2 \pi^2}{8} - \frac{(1 + 2p)^4 \pi^4}{96} \frac{\tau}{\Delta t_{\text{open}}} \right\}
$$

$$
= \frac{L^2}{12} \left( 1 - \frac{\Delta t_{\text{open}}}{\tau} \right),
$$

where we used that $\sum_{p=0}^{\infty} (1+2p)^{-2} = \pi^2/8$ and $\sum_{p=0}^{\infty} (1+2p)^{-4} = \pi^4/96$ [Bickel (2007)].

### 2 CALCULATIONS FOR THE MSD FOR A PARTICLE IN A 1D BOX

For the MSD, we have to calculate $\langle x_{\text{msr}}, n x_{\text{msr}}, 0 \rangle$ from Equation (21). The calculation is similar to the one for the variance, except that the time points $t'$ and $t$ in the time-ordered autocorrelation belong to difference time-lapses [see Equation (S1)]. That is, $t \in [0, \Delta t]$ and $t' \in [(n+1)\Delta t, (n+1+1)\Delta t]$. So the spatial integrals are identical to those in Sec. 1 but the time integrals are different.

Carrying out the time integrals gives

$$
\int_{0}^{\Delta t} dt \int_{n\Delta t}^{(n+1)\Delta t} dt' \exp \left[ - \left( \frac{(1 + 2p)\pi}{2} \right)^2 \frac{t' - t}{\tau} \right] s_1(t) s_{n+1}(t')
$$

$$
= \frac{1}{(\Delta t_{\text{open}})^2} \int_{\Delta t - \Delta t_{\text{open}}}^{\Delta t} dt \int_{(n+1)\Delta t - \Delta t_{\text{open}}}^{(n+1)\Delta t} dt' \exp \left[ - \left( \frac{(1 + 2p)\pi}{2} \right)^2 \frac{t' - t}{\tau} \right]
$$

$$
= \left( \frac{\tau}{\Delta t_{\text{open}}} \right)^2 \left( \frac{2}{(1 + 2p)\pi} \right)^4 \exp \left[ - \left( \frac{(1 + 2p)\pi}{2} \right)^2 \frac{n\Delta t + \Delta t_{\text{open}}}{\tau} \right]
$$

$$
\times \left( 1 - e^{\left( \frac{(1+2p)\pi}{2} \right)^2 \Delta t_{\text{open}}/\tau} \right)^2.
$$

This is inserted in the expression for $\langle x_{\text{msr}}, nx_{\text{msr}}, 0 \rangle$ and we arrive at Equation (22) in a similar way as when calculating the variance.
3 CALCULATIONS FOR THE VARIANCE AND MSD FOR A PARTICLE ON A 2D CIRCULAR DISC

The calculation follows the same steps as for the 1D case. All radial integrals below are from 0 to $a$, and all angular integrals are from 0 to $2\pi$.

Due to symmetry it holds that $\langle r_{\text{msr}} \rangle = (0, 0)$, so

$$\text{var}(r_{\text{msr}}) = \langle r_{\text{msr}}^2 \rangle = \frac{1}{\pi a^2} \int d\phi_0 \int dr_0 r_0 \langle r_{\text{msr}}^2(r_0, \phi_0) \rangle_0.$$  \hspace{1cm} (S12)

We insert the definition of the measured position [see Equation (1)] and find

$$\langle r_{\text{msr}}(x_0, y_0)^2 \rangle_0 = \frac{2}{(\Delta t_{\text{open}})^2} \int_{\Delta t - \Delta t_{\text{open}}}^{\Delta t} dt' \int_{\Delta t - \Delta t_{\text{open}}}^{t'} dt \langle r(t|x_0, y_0) \cdot r(t'|x_0, y_0) \rangle_{0, t'>t},$$  \hspace{1cm} (S13)

and then write the time-ordered autocorrelation function as [recall that $r = (x, y) = (r \cos \phi, r \sin \phi)$]

$$\langle r(t|x_0, y_0) \cdot r(t'|x_0, y_0) \rangle_{0, t'>t} = \int r dr \int r' dr' \int d\phi \int d\phi' r r' (\cos \phi \cos \phi' + \sin \phi \sin \phi')$$

$$\times P(r, \phi, t|r_0, \phi_0, t_0) P(r', \phi', t'|r, \phi, t).$$  \hspace{1cm} (S14)

Using Equation (27), the product of propagators are

$$P(r, \phi, t|r_0, \phi_0, t_0) P(r', \phi', t'|r, \phi, t)$$

$$= \left( \frac{1}{\pi a^2} \right)^2 + \left( \frac{1}{\pi a^2} \right)^2 \sum_{n'} \cos[n'(\phi' - \phi)] \sum_{m'} (n') + \left( \frac{1}{\pi a^2} \right)^2 \sum_n \cos[n(\phi - \phi_0)] \sum_m (n),$$  \hspace{1cm} (S15)

$$+ \left( \frac{1}{\pi a^2} \right)^2 \sum_n \sum_{n'} \cos[n(\phi - \phi_0)] \cos[n'(\phi' - \phi)] \sum_m (n) \sum_{m'} (n'),$$

with the short-hand notation

$$\sum_{m} (n) = \sum_{m=1}^{\infty} \frac{\alpha_{nm}^2}{\alpha_{nm}^2 - n^2} \exp \left[-\alpha_{nm}^2 \frac{t}{\tau} \right] \frac{J_n(\alpha_{nm} \frac{r}{a}) J_n(\alpha_{nm} \frac{r_0}{a})}{J_n(\alpha_{nm})^2},$$  \hspace{1cm} (S16)

$$\sum_{m'} (n') = \sum_{m'=1}^{\infty} \frac{\alpha_{n'm'}^2}{\alpha_{n'm'}^2 - n'^2} \exp \left[-\alpha_{n'm'}^2 \frac{t}{\tau} \right] \frac{J_{n'}(\alpha_{n'm'} \frac{r'}{a}) J_{n'}(\alpha_{n'm'} \frac{r_0}{a})}{J_{n'}(\alpha_{n'm'})^2}. $$  \hspace{1cm} (S17)
The integrals over the angles in Equation (S14) are

\[ \int d\phi \int d\phi' \cos[n'(\phi' - \phi)] \cos \phi \cos \phi' = \pi^2 \delta_{n',\pm 1}, \]  
(S18)

\[ \int d\phi \int d\phi' \cos[n(\phi - \phi_0)] \cos \phi \cos \phi' = 0, \]  
(S19)

\[ \int d\phi \int d\phi' \cos[n(\phi - \phi_0)] \cos[n'(\phi' - \phi)] \cos \phi \cos \phi' = \pi^2 \delta_{n,0} \delta_{n',\pm 1}, \]  
(S20)

or \( \frac{\pi^2}{2} \cos[2x_0] \delta_{n,2} \delta_{n',\pm 1}, \)

and

\[ \int d\phi \int d\phi' \cos[n'(\phi' - \phi)] \sin \phi \sin \phi' = \pi^2 \delta_{n',\pm 1}, \]  
(S21)

\[ \int d\phi \int d\phi' \cos[n(\phi - \phi_0)] \sin \phi \sin \phi' = 0, \]  
(S22)

\[ \int d\phi \int d\phi' \cos[n(\phi - \phi_0)] \cos[n'(\phi' - \phi)] \sin \phi \sin \phi' = \pi^2 \delta_{n,0} \delta_{n',\pm 1} \]  
(S23)

or \( -\frac{\pi^2}{2} \cos[2x_0] \delta_{n,2} \delta_{n',\pm 1}. \)

The integrals over \( 1/(\pi a^2) \) vanish for both \( \cos \) and \( \sin \) integrals.

This gives

\[ \int d\phi \int d\phi' \int dr^2 \int dr'^2 \left( \cos \phi \cos \phi' + \sin \phi \sin \phi' \right) \times P(r,\phi,t|r_0,\phi_0,t_0)P(r',\phi',t'|r,\phi,t) \]  
(S24)

\[ = \frac{4}{a^4} \left\{ \int dr \int dr'^2 r'^2 \sum_{n'} (n' = 1) + \int dr^2 \int dr'^2 r'^2 \sum_{n}(n = 0) \sum_{n'} (n' = 1) \right\}. \]

where we used that \( J_n(x) = (-1)^n J_{-n}(x) \), so

\[ J_{-n} \left( \frac{r_{nm}}{a} \right) J_{-n} \left( \frac{r_{nm}}{a} \right) \frac{\alpha_{nm}^2}{\alpha_{nm}^2 - (-n)^2} = J_n \left( \frac{r_{nm}}{a} \right) J_n \left( \frac{r_{nm}}{a} \right) \frac{\alpha_{nm}^2}{\alpha_{nm}^2 - n^2}. \]  
(S25)

We need the following relations for the Bessel functions \( \text{[Riseborough and H"anggli(1982)]} \)

\[ \int_0^a r^q J_n(\lambda r) dr = \left[ \frac{r^q}{\lambda} J_{n+1}(\lambda r) \right]_0^a + \frac{n + 1 - q}{\lambda} \int_0^a r^{q-1} J_{n+1}(\lambda r) dr, \]  
(S26)

\[ J_{n+1}(r) = \frac{n}{r} J_n(r) - J'_n(r), \]  
(S27)

\[ J'_{n+1}(r) = \frac{n + 1}{r} J'_n(r) + \left( 1 - \frac{n(n + 1)}{r^2} \right) J_n(r), \]  
(S28)
to calculate integrals on the form

\[
\int_0^a dr \frac{r^2}{a} J_1 \left( \frac{\alpha_1 m r}{a} \right) = \left[ \frac{r^2}{a} J_2 \left( \frac{\alpha_1 m r}{a} \right) \right]_0^a + \frac{1 + 2}{\alpha_1 m / a} \int_0^a r J_{2+1} (r \alpha_1 m / a) dr
\]

\[
= \frac{a^3}{\alpha_1 m} [J_2 (\alpha_1 m) - J_2 (0)]
\]

\[
= \frac{a^3}{\alpha_1 m} \left[ \frac{1}{\alpha_1 m} J_1 (\alpha_1 m) - J'_1 (\alpha_1 m) \right]
\]

\[
= \frac{a^3}{\alpha_1 m} J_1 (\alpha_1 m),
\]

where we used Equation (S27), \( J_2 (0) = 0 \), and \( J'_1 (\alpha_1 m) = 0 \).

Returning to Equation (S24), we write the last term as

\[
\int dr \int dr' r^2 r'^2 \sum_m (n = 0) \sum_{m'} (n' = 1)
\]

\[
= \int dr \int dr' r^2 r'^2 \left( \sum_{m=1}^{\infty} \sum_{m'=1}^{\infty} \frac{\alpha_0^2}{\alpha_{0m}^2 - 0^2} \exp \left[ -\alpha_{0m}^2 \frac{t - t_0}{\tau} \right] \frac{J_0 (\alpha_{0m} r_0 / a)}{J_0 (\alpha_{0m}^2)} \right)
\]

\[
\times \left( \sum_{m'=1}^{\infty} \frac{\alpha_{1m'}^2}{\alpha_{1m'}^2 - 1^2} \exp \left[ -\alpha_{1m'}^2 \frac{t' - t_0}{\tau} \right] \frac{J_1 (\alpha_{1m'} r'_0 / a)}{J_1 (\alpha_{1m'}^2)} \right)
\]

Expanding the product of the sums, we see that all terms are proportional to \( J_0 (\alpha_{0m} r_0 / a) \). Recall that in Equation (S12) we have an integral of the form \( \int_0^a dr_0 r_0 \ldots \), and as

\[
\int_0^a dr_0 r_0 J_0 \left( \frac{\alpha_{0m} r_0}{a} \right) = \left[ \frac{r a}{\alpha_{0m}} J_1 \left( \frac{\alpha_{0m} r_0}{a} \right) \right]_0^a + \frac{0 + 1 - 1}{\alpha_{0m}} \int_0^a dr_0 J_1 \left( \frac{\alpha_1 m r_0}{a} \right)
\]

\[
= \frac{a^2}{\alpha_{0m}} J_1 (\alpha_{0m})
\]

\[
= \frac{a^2}{\alpha_{0m}} \left[ \frac{0}{\alpha_{0m}} J_0 (\alpha_{0m}) - J'_0 (\alpha_{0m}) \right]
\]

\[
= 0,
\]

we can neglect the last term in Equation (S24). The integral \( \int_0^a dr_0 r_0 \) over the first term in Equation (S24) simply gives \( \pi a^2 \), which cancels the prefactor in Equation (S12).
Putting the pieces together, we find that Equation (S14) is

$$\langle \mathbf{r}(t|x_0,y_0) \cdot \mathbf{r}(t'|x_0,y_0) \rangle_{0,t'>t} = \frac{4}{a^4} \int dr \int dr' r^2 r'^2 \sum_{m=1}^{\infty} \frac{\alpha_{1m}^2}{\alpha_{1m}^2 - 1} \exp \left[ -\alpha_{1m}^2 \frac{t' - t}{\tau} \right] \frac{J_1(\alpha_{1m}^2 \tau)}{J_1(\alpha_{1m}^2 \tau')} + f(r_0)$$

(S32)

where $f(r_0)$ represents terms that depend on $r_0$.

We then insert the expression in Equation (S13) and carry out the time integrals [see, e.g., Equation (S6)], and the result is inserted in Equation (S4). The integral over initial positions removes terms with $r_0$ and leaves the others unchanged. So we finally arrive at the expression for the variance in Equation (28).

If we expand the exponential in Equation (28) in powers of $\Delta t_{\text{open}}/\tau$, we find that the variance becomes

$$\text{var}(r_{\text{msr}}) \simeq \frac{8a^2}{(\Delta t_{\text{open}})^2} \sum_{m=1}^{\infty} \left( \frac{\tau \Delta t_{\text{open}}}{\alpha_{1m}^6 - \alpha_{1m}^4} - \frac{\tau^2}{\alpha_{1m}^8 - \alpha_{1m}^6} \right) \times \left( 1 - \left[ 1 - \alpha_{1m}^2 \frac{\Delta t_{\text{open}}}{\tau} + \frac{1}{2} \alpha_{1m}^4 \frac{(\Delta t_{\text{open}})^2}{\tau^2} - \frac{1}{6} \alpha_{1m}^6 \frac{(\Delta t_{\text{open}})^3}{\tau^3} + \ldots \right] \right)$$

(S33)

$$\simeq 4a^2 \sum_{m=1}^{\infty} \frac{1}{\alpha_{1m}^2 (\alpha_{1m}^2 - 1)} - \frac{4a^2}{3} \sum_{m=1}^{\infty} \frac{1}{\alpha_{1m}^2 (\alpha_{1m}^2 - 1)}$$

$$= \frac{a^2}{2} \left( 1 - \frac{4}{3} \frac{\Delta t_{\text{open}}}{\tau} \right),$$

where the expressions for the sums may be found in [Bickel (2007)].

For $\Delta t_{\text{open}} \gg \tau$, we get

$$\text{var}(r_{\text{msr}}) \simeq \frac{8a^2 \tau}{\Delta t_{\text{open}}} \sum_{m=1}^{\infty} \frac{1}{\alpha_{1m}^4 (\alpha_{1m}^2 - 1)} = \frac{7a^2 \tau}{24 \Delta t_{\text{open}}}. \quad (S34)$$

Here we used that the sum equals $7/192$, which is seen by writing the fraction in the sum as $[x^4(x^2-1)]^{-1} = (x^2 - 1)^{-1} - x^{-4} - x^{-2}$ and using the method described in App. C in Riseborough and Hänggi (1982).

As for the case of a particle in a 1D box, the calculation for the mean-squared displacement for the 2D case is similar to the calculation of the variance, except that the time integrals are different [see Sec. 2].
4 CALCULATIONS FOR THE VARIANCE AND MSD FOR A PARTICLE CONFINED IN A 3D SPHERE

The calculation is similar to the 1D and 2D cases. All radial integrals over \( r \) are from 0 to \( a \), all integrals over \( \theta \) are from 0 to \( \pi \), and all integrals over \( \phi \) are from 0 to \( 2\pi \). Due to symmetry it holds that \( \langle r_{\text{msr}} \rangle = (0, 0, 0) \), so

\[
\text{var}(r_{\text{msr}}) = \langle r_{\text{msr}}^2 \rangle = \frac{3}{4\pi a^3} \int_0^\pi \sin \theta \, d\theta \int_0^{2\pi} \sin \phi \, d\phi \int_0^a dr \, r_0^2 \langle r_{\text{msr}}^2(r_0, \theta_0, \phi_0) \rangle_0.
\]  

(S35)

We insert the definition of the measured position [see Equation (1)] and find

\[
\langle r_{\text{msr}}^2(r_0, \theta_0, \phi_0) \rangle_0 = \frac{2}{(\Delta t_{\text{open}})^2} \int_{\Delta t_{\text{open}}}^\Delta t \int_0^t dt' \int_0^{\Delta t_{\text{open}}} dt \langle r(t|r_0, \theta_0, \phi_0) \cdot r(t'|r_0, \theta_0, \phi_0) \rangle_{0,t'>t},
\]

and then write the time-ordered autocorrelation function as

\[
\langle r(t|r_0, \theta_0, \phi_0) \cdot r(t'|r, \theta, \phi) \rangle_{0,t'>t} = \int dr \, r^2 \int dr' \, r'^2 \int d\theta \sin \theta \int d\theta' \sin \theta' \int d\phi \int d\phi' 
\times \, r \, r' \cos \gamma \, P(r', \theta', \phi', t'|r, \theta, \phi, t) \, P(r, \theta, \phi, t|r_0, \theta_0, \phi_0, t_0).
\]

(S36)

Here we recall that \( \langle r(t|r_0, \theta_0, \phi_0) \cdot r(t'|r_0, \theta_0, \phi_0) \rangle_{0,t'>t} = \langle r(t|r_0, \theta_0, \phi_0) \cdot r(t'|r_0, \theta_0, \phi_0) \rangle_{0,t'>t} \) as the points are on the same trajectory and define \( \gamma = \gamma(\theta', \phi', \theta, \phi) \) to be the angle between \( r(t|r, \theta, \phi) \) and \( r(t'|r', \theta', \phi') \). Following [Bickel (2007)], we express this angle in terms of spherical harmonics,

\[
\cos \gamma = \frac{4\pi}{3} \sum_{m=-1}^{m=1} Y_{1m}^*(\theta', \phi') Y_{1m}(\theta, \phi).
\]  

(S38)

The product of the propagators [see Equation (34)] and the angle is

\[
\cos \gamma \, P(r', \theta', \phi', t'|r, \theta, \phi, t) \, P(r, \theta, \phi, t|r_0, \theta_0, \phi_0, t_0) 
= \left[ \frac{3}{4\pi a^3} + \frac{2}{a^3} \sum_{\ell=0}^\infty \sum_{m=-\ell}^{\ell} Y_{\ell m}(\theta', \phi') Y_{\ell m}^*(\theta, \phi) R_\ell(r', r, t) \right] 
\times \left[ \frac{3}{4\pi a^3} + \frac{2}{a^3} \sum_{\ell'=0}^\infty \sum_{m'=-\ell'}^{\ell'} Y_{\ell m'}(\theta, \phi) Y_{\ell m'}^*(\theta, \phi) R_{\ell'}(r, r_0, t, t_0) \right] 
\times \frac{4\pi}{3} \sum_{m''=-1}^{m''=1} Y_{1m''}^*(\theta', \phi') Y_{1m''}(\theta, \phi),
\]

(S39)

where we introduced the notation for the radial and temporal part of the propagator

\[
R_\ell(r', r, t, t) = \sum_{n=1}^\infty \frac{\beta_{\ell n}^2}{\beta_{\ell n}^2 - \ell(\ell + 1)} \exp \left[ -\beta_{\ell n}^2 \frac{t'}{\tau} \right] \frac{jt \left( \beta_{\ell n} \frac{r'}{a} \right)}{jd \left( \beta_{\ell n} \frac{r}{a} \right)^2}.
\]

(S40)
Due to the orthogonality relation of spherical harmonics,

$$\int_0^\pi d\theta \int_0^{2\pi} d\phi Y_{\ell m}^*(\theta, \phi)Y_{\ell' m'}(\theta, \phi) \sin \theta = \delta_{\ell,\ell'} \delta_{m, m'}, \quad (S41)$$

terms with only $Y_{1m'}^*(\theta, \phi)$ or $Y_{1m''}(\theta', \phi')$ vanish when inserting Equation (S39) in Equation (S37) and integrating over $(\theta, \phi)$ or $(\theta', \phi')$, as the constant term is proportional to $Y_{00}$. So, with the use of Equations (S39) and (S41), Equation (S37) becomes

$$\langle r(t|r_0, \theta_0, \phi_0) \cdot r(t'|r, \theta, \phi) \rangle_{0,t'>t}$$

$$= \int dr r^2 \int dr' r'^2 \int d\theta \sin \theta \int d\theta' \sin \theta' \int d\phi \int d\phi'$$

$$\times \left[ \frac{rr'}{4\pi a^3} \times \frac{2}{a^3} \sum_{\ell=0}^\infty \sum_{m=-\ell}^{\ell} Y_{\ell m}(\theta', \phi')Y_{\ell m}^*(\theta, \phi)R_{\ell}(r', r, t', t) \right]$$

$$\times \frac{4\pi}{3} \sum_{m'=-1}^{1} Y_{1m'}^*(\theta', \phi')Y_{1m''}(\theta, \phi) \right] + \text{[terms with } Y_{\ell m}^*(\theta_0, \phi_0)\text{]} \quad (S42)$$

$$= \frac{6}{a^6} \int_0^a dr' r'^3 \int_0^a dr r^3 \int_0^a dr' r'^3 \int_0^a dr r^3 \sum_{n=1}^\infty \frac{\beta_n^2}{\beta_n^2 - 2} \exp \left[ -\frac{\beta_n^2}{2} \frac{t' - t}{\tau} \right] \frac{j_1(\beta_n \frac{r'}{a})}{j_1(\beta_n \frac{r}{a})} \frac{j_1(\beta_n \frac{r'}{a})}{j_1(\beta_n \frac{r}{a})}$$

$$+ \text{[terms with } Y_{\ell m}^*(\theta_0, \phi_0)\text{]}.$$
We insert this in Equation (S36) and carry out the integrals over time. The result is inserted in Equation (S35), and when integrating over all initial positions, the terms with $Y_{m}^{*} (\theta_0, \phi_0)$ vanish and we get the final result for the variance of the measured positions [Equation (35)].

An expansion to lowest orders in $\Delta t_{\text{open}}/\tau$ gives

$$
\text{var}(r_{\text{msr}}) \approx \frac{12a^2}{(\Delta t_{\text{open}})^2} \sum_{n=1}^{\infty} \left[ \frac{\tau \Delta t_{\text{open}}}{\beta_{1n}^4 (\beta_{1n}^2 - 2)} - \frac{\tau^2}{\beta_{1n}^6 (\beta_{1n}^2 - 2)} \left( \frac{\beta_{1n}^2 \Delta t_{\text{open}}}{\tau} - \frac{1}{2} \left[ \frac{\beta_{1n}^2 \Delta t_{\text{open}}}{\tau} \right]^2 + \frac{1}{6} \left[ \frac{\beta_{1n}^2 \Delta t_{\text{open}}}{\tau} \right]^3 \right) \right]
$$

$$
= \frac{12a^2}{(\Delta t_{\text{open}})^2} \sum_{n=1}^{\infty} \left[ \frac{1}{2} \frac{\Delta t_{\text{open}}^2}{\beta_{1n}^2 (\beta_{1n}^2 - 2)} - \frac{1}{6} \frac{1}{\beta_{1n}^2 (\beta_{1n}^2 - 2)} (\Delta t_{\text{open}})^3 \right]
$$

$$
= \left( 6 \sum_{n=1}^{\infty} \frac{1}{2} \frac{\beta_{1n}^2 (\beta_{1n}^2 - 2)}{\beta_{1n}^2 (\beta_{1n}^2 - 2)} \right) a^2
$$

$$
= \frac{3a^2}{5} \left( 1 - \frac{5 \Delta t_{\text{open}}}{3 \tau} \right)
$$

where we in the last step used that the first and second sums may be found to equal, respectively, $(1 - 3/5)/4 = 1/10$ and $1/2$ [Bickel (2007)].

As for a particle in a 1D box or on a 2D disc, the calculation for the mean-squared displacement for the 3D case is similar to the calculation of the variance, except that the time integrals are different [see Sec. 2].

5 PLOTS OF THE VARIANCE FOR 1D DIFFUSION

Figure S1. (a) The variance of measured positions as function of $\Delta t_{\text{open}}/\tau$ for the exact expression in Equation (18) (blue full line) [same as in Figure 2d], the expression in Equation (18) but keeping only the term with $p = 0$ in the sum (dashed, red line), and the variance obtained as half the value of the expression for the MSD in Equation (25) for $n \to \infty$ (dashed, black line). (b) Relative differences between the approximate and exact results.
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