Lattice field theories with an energy current

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Abstract

We investigate a lattice scalar field theory in the presence of a bias favouring the establishment of an energy current, as a model for stationary nonequilibrium processes at low temperature in a non-integrable system. There is a transition at a finite value of the bias to a gapless modulated phase which carries a classical current; however, unlike in similar, integrable, models, quantum effects also allow for a non-zero current at arbitrarily small bias. The transition is second order in the magnetically disordered phase, but is pre-empted by a first-order transition in the ferromagnetic case, at least at the mean-field level.

1 Introduction

Finding a general framework for the description of nonequilibrium processes has been a long standing problem in statistical physics. In particular, there is a substantial body of literature dealing with the microscopic theory of energy, or heat, conduction \cite{1}\cite{2}\cite{3}\cite{4}. Unfortunately, no clear conclusions have been reached as a result of these investigations. Results are dependent on boundary conditions and on basic assumptions concerning the underlying process responsible for energy conduction.

Recently, Antal, Racz, and Sasvari have proposed studying a related but slightly different problem, namely the microscopic description of energy conduction in quantum systems at zero temperature \cite{5}\cite{6}. Denoting the quantum Hamiltonian by $H^{(0)}$ and the space integral of the energy current by $J$, they introduce a Lagrange multiplier $\lambda$ and study the ground state properties of $H^{(0)} - \lambda J$, for certain exactly solvable one dimensional quantum spin systems. An interesting feature of these models is that the energy current turns on only at a critical, nonvanishing, value of the Lagrange multiplier. At the critical point the system undergoes a second order phase transition to a gapless energy-conducting state, in which the spin correlation function has a characteristic oscillatory behaviour, whose amplitude is modulated by a power law fall-off.

It is of considerable interest to find out to what extent the phenomena found in \cite{5}\cite{6} apply to other systems, in particular whether a phase transition to an energy-conducting state persists in higher dimensions, and whether integrability plays an essential role in the results.

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Having departed from exactly solvable theories, one needs a systematic approximate method to attack the problem. In quantum field theory the effective action \[ \mathcal{A} \], combined with the loop expansion, has been extremely successfully in treating all kinds of problems related to symmetry breaking. Standard applications of the method of effective action are usually restricted to the breaking of global or gauge symmetries \[ \text{[8]} \]. In spontaneous symmetry breaking of continuous symmetries in three or more dimensions a scalar field acquires a constant vacuum expectation value. As this expectation value, or classical field, is constant the effective action reduces to the effective potential, simplifying calculations considerably. An additional bonus of working with the effective action is that the inverse correlation function is obtained as the second functional derivative of the effective action with respect to the classical field. The definiteness of the spectral function of the propagator implies the convexity of the effective action. Just like its analogue in statistical mechanics, the Gibbs free energy, the absolute minimum of the effective action selects the correct phase of the system. Although, as we shall show, the method of the effective action when applied to our problem leads to a coordinate dependent classical field and, thus, spontaneous breaking of translation invariance, we expect that the effective action is still minimised by the correct classical field configuration.

The model we consider in this paper is a continuous spin version, in \( D \) space dimensions, of the transverse Ising model discussed in \[ \text{[5]} \] for \( D = 1 \). In the absence of a current bias, the model has a phase transition from a disordered state to one in which in which the \( Z_2 \) symmetry is spontaneously broken, in the universality class of the \( (D+1) \)-dimensional classical Ising model. The phase diagram in the presence of the current bias \( \lambda \) is shown schematically in Fig. \[ \text{[4]} \]. Qualitatively it is quite similar to that

![Figure 1](https://via.placeholder.com/150)

Figure 1: Schematic phase diagram at \( D > 1 \). \( T \) signifies translation symmetry. The dashed line represents a first order transition.

found in \[ \text{[5]} \]. However, there are important differences. At a finite value of \( \lambda \) there is a
transition to a situation in which, for $D > 1$, the ground state is no longer invariant under lattice translations. In this phase there is a characteristic wave number $p_0$ such that Fourier components of the field which are an integer multiple of this acquire a vacuum expectation value. For $\mu^2 > 0$, only the fundamental frequency is important near the transition, and its amplitude vanishes like $(\lambda - \lambda_c)^{1/2}$. This classical spin density wave supports a non-zero energy current. There are Goldstone modes corresponding to this breaking of translational symmetry, and for $D = 1$ these actually destroy the long range order. The phase transition is then induced by the condensation of vortices, just like in the $D = 1$ planar rotor model. The Goldstone modes lead to a power-law modulation of the amplitude of the oscillatory spin correlation function. Even for $D = 1$, this state with only quasi-long range order is still capable of supporting a classical current.

However, our model differs from those considered in [5],[6] in that, due to quantum effects, the current is also nonvanishing in the region $0 < \lambda < \lambda_c$. This may be simply understood as follows. In [5],[6] the space integral $J$ of the current happens to commute with the hamiltonian $H^{(0)}$. This means, in particular, that the vacuum state of $H^{(0)}$ is also an eigenstate of $H = H^{(0)} - \lambda J$. In the general case where the spectrum of $H^{(0)}$ has a gap, we expect the vacuum state of $H$ to depend analytically on $\lambda$, and therefore to remain unchanged at least for some finite neighbourhood of $\lambda = 0$. In this region, the current will continue to vanish, and, indeed, all the equal-time correlations will be identical to the case when $\lambda = 0$. The onset of the current then occurs at the first nonanalyticity, $\lambda = \lambda_c$. In general, and in our model in particular, there is no a priori reason for $H^{(0)}$ and $J$ to commute, and the structure of the vacuum may change as soon as $\lambda \neq 0$, allowing the current to flow. Interesting enough, we find that this happens due to quantum processes which first arise only at $O(h^3)$, so that such effects might be very small in a real system.

Most of our analysis applies to the case when $\lambda$ is switched on in the disordered, symmetric phase of the continuous spin Ising model. However, a very similar picture seems to hold in the ferromagnetic phase. There is a transition at a finite value of $\lambda$ to a non-translational invariant state (for $D > 1$) which carries a classical energy current. However, in our particular model, in weak coupling, it can be shown that the expected second order transition is preempted by a first order one to a modulated state in which the higher harmonics are important.

The outline of this paper is as follows. In the next section, after setting up an appropriate hamiltonian which captures the physics we want to describe, we calculate the effective action at tree level. Sec. 3. discusses the theory and its quantum corrections in the ferromagnetically disordered phase, first at small values of the bias, then at the critical value, and finally in the modulated phase. In all three cases we evaluate the classical current and its lowest order quantum corrections. In Sec. 4 we pay particular attention to the Goldstone modes which appear in the modulated phase for $D > 1$ and discuss how these dramatically modify the physics for $D = 1$. The next section describes our (limited) results for the ferromagnetically ordered phase, where there appears to be a first-order transition to the modulated phase at mean-field level. Finally, in Sec. 6 we summarise our results, contrast them with those found in the integrable $D = 1$ example.
of Ref. 5, and connect them with those in other different, but mathematically similar, models of modulated phases.

2 A lattice model

Our first aim is to select a suitable field theory hamiltonian $H^{(0)}$ whose long-distance behaviour is expected to be in the same universality class as the transverse Ising model whose $D = 1$ version was discussed in 5. In the absence of any Lagrange multiplier, it is well known that the continuum relativistic $\phi^4$ field theory, with a suitable momentum cut-off, has this property, and that it undergoes a zero-temperature phase transition in the same universality class as the classical Ising model in $D + 1$ space dimensions.

However, it turns out that such a relativistic continuum theory cannot exhibit the physics we are trying to study at non-zero $\lambda$. This is because the energy current is given by the components $T^{0i}$ of the energy-momentum tensor, which, in a relativistic theory, are equal to those of the momentum density $T^{i0}$. The current $J$ is therefore nothing but the total momentum $P$, which commutes with $H^{(0)}$. Any simultaneous eigenstate has, for large $P$, an energy $E \sim P$. From the convexity of the relativistic dispersion relation it is easy to see that the ground state of $H^{(0)} - \lambda P$ is the usual vacuum for $\lambda < 1$, while for $\lambda > 1$ this operator is unbounded from below, so that the theory does not exist.

Clearly it is necessary to modify the dispersion relation, and the natural way to do this is to consider the same field theory on a lattice, since this also provides an ultraviolet regulator. The hamiltonian of the $D$-dimensional lattice model that we consider is thus

$$H^{(0)} = \frac{1}{2} \sum_r \left\{ \Pi^2_r + \mu^2 \phi^2_r + \sum_\alpha^D [\phi_r - \phi_{r-e_\alpha}]^2 + \frac{g}{12} \phi^4_r \right\}, \quad (1)$$

where summation is taken over the vertices of a $D$-dimensional cubic lattice and $e_\alpha$ are unit (lattice) vectors. The lattice constant is taken to be unity. The canonical momentum, $\Pi$, is introduced and the time coordinate is kept continuous to facilitate transformation to the lagrangian formalism later.

Let us calculate the energy current $j^E_{r,\alpha}$ of this hamiltonian. This satisfies

$$i[H^{(0)}, H^{(0)}_r] = \nabla_\alpha j^E_{r,\alpha}, \quad (2)$$

where $\nabla_\alpha$ denotes the lattice divergence. The solution of (2) for the space integral of the energy current, the only quantity of interest for us, is unique.

After a simple calculation we find from (1,2)

$$j^E_{r,\alpha} = \frac{1}{2} \left[ (\Pi_r + \Pi_{r-e_\alpha})(\phi_r - \phi_{r-e_\alpha}) \right]. \quad (3)$$

Note that this is hermitian but is odd under time-reversal and parity.

Now we modify our lattice hamiltonian by the addition of the space integral of a component of the energy current, $\sum_r j^E_{r,1}$, multiplied by a Lagrange multiplier, to get

$$H = \frac{1}{2} \sum_r \left\{ \Pi^2_r + \mu^2 \phi^2_r + \sum_\alpha^D [\phi_r - \phi_{r-e_\alpha}]^2 + \frac{g}{12} \phi^4_r + \frac{\lambda}{2} j^E_{r,1} \right\}, \quad (4)$$
Note that we have selected the first coordinate as the direction of the energy current.

The term $\sum r j^{E}_{r,1}$ does not commute with $H^{(0)}$, due to the presence of the $\phi^4$ interaction term. In fact the commutator is proportional to

$$\sum r \phi^3_r [\phi_{r+e_1} - \phi_{r-e_1}]$$

Note that in the naive continuum limit this is proportional to $\phi^3 \partial_1 \phi$, which is a total derivative, but when higher order derivatives are taken into account it has a non-vanishing effect.

As we emphasised in the Introduction, the main advantage of our field theoretic model is that we are able to utilise the lagrangian formalism. However, it is instructive first to examine the hamiltonian (4) in the limit $g = 0$ when the self-interaction is neglected. In that case $H$ may be diagonalised in terms of momentum-space annihilation and creation operators, with the result

$$H = \int \frac{Dp}{(2\pi)^D} \left( 2 \sum_{\alpha=1}^D (1 - \cos p_\alpha) + \mu^2 \right)^{1/2} - \lambda \sin p_1 \right) a_\lambda^\dagger(p) a(p),$$

where the integral is over the first Brillouin zone. For $\mu^2 > 0$ and small $\lambda$, the effect is only to tilt slightly the usual one-particle dispersion relation (solid line), as illustrated by the dotted line of Fig. 2. The ground state is however still the original vacuum. However,

![Figure 2: Dispersion relations at $\lambda = 0$, $\lambda = 1 < \lambda_c$, $\lambda = 2.2 > \lambda_c$ (solid lines), and for the ‘negative energy branch’ at $\lambda = 2.2$ (dashed line).](image)

there is a critical value of $\lambda > 0$ for which the dispersion curve touches the zero-energy axis at some positive $p_1 = p_1^{(0)}$, and beyond this value there are negative energy single particle states (dashed line of Fig. 2), which will carry a nonzero energy current. Unlike the case considered in [5], however, where the elementary excitations were fermions, the particles will now try to Bose condense into the lowest energy single particle state. Since the number of such quanta is unbounded, the energy will diverge to minus infinity, an unphysical result. Of course, this picture will change in the presence of the repulsive $\phi^4$ interaction, since this will limit the total number of quanta. However, the interaction will also introduce scattering between quasiparticles of different momenta, and in particular
will introduce virtual processes involving the branch of the dispersion relation found by taking the other sign of the square root, indicated by the broken line in Fig. 2. For \( \lambda \) greater than its critical value the energies of some of these states begin to overlap with those of the original single-particle states, indicating a complete breakdown of the quasi-particle picture.

For this reason, among others, we have found it more useful to analyse the interacting theory through the path integral and the effective action formalism. First we need to calculate the lagrangian. We use the canonical equation to find

\[
\dot{\phi}_r = \Pi_r + \frac{\lambda}{2}[\phi_{r+e_1} - \phi_{r-e_1}].
\]

(7)

Then the lagrangian,

\[
L = \sum_r \Pi_r \dot{\phi}_r - H,
\]

(8)

takes the form

\[
L = \frac{1}{2} \left\{ \sum_r [\dot{\phi}_r - \frac{\lambda}{2}(\phi_{r+e_1} - \phi_{r-e_1})]^2 - \mu^2 \phi_r^2 - \sum_\alpha [\phi_r - \phi_{r-e_\alpha}]^2 - \frac{g}{12} \phi_r^4 \right\},
\]

(9)

In the lowest order of the loop expansion the effective action is identical to the classical action. Then, at tree level, the classical field minimises the classical action. As we have argued above, for \( \lambda \) larger than a critical value we expect to find a macroscopic number of particles with momentum \( \approx p_1^{(0)} \) in the ground state. This means that the classical field which minimises the action will be space-dependent. From the point of view of the path integral, there is also no reason to exclude a time-dependence as well. We shall assume that, since we are considering a stationary process, that this is not the case. Further justification for this may be found by considering the euclidean action, whose minimum is guaranteed to give the lowest energy state. This is found from (9) by changing the sign of all the terms and also letting \( \dot{\phi} \rightarrow i \dot{\phi} \). In general, then, euclidean saddle point solutions will be complex, unless they are time-independent. Since we are seeking a ground state with a real field, we should exclude complex solutions.

If we allowed the classical field to depend on time the solution of the extremisation problem would become far from unique. In fact, one could find a solution at all values of the parameters, \( \lambda \) and \( \mu^2 \). Furthermore, on a more practical level, we would completely lose the guiding principle of finding the classical field by minimising the effective action. One can see this problem arising already at the classical level. If the classical field would depend on time, then its Fourier transform would depend on ‘energy’ and the kinetic part of the lagrangian would become indefinite. Looking at this question from a slightly different angle, remember that the second functional derivative of the effective action, or at tree level, classical action, is the inverse propagator. The inverse of this, the propagator has a spectral representation with a positive definite weight function and non-negative threshold. Thus, at any space or timelike four-momentum (or
generalised four momentum for lattices) the propagator is positive. In momentum representation this insures the definiteness of the inverse propagator, as well. However, if the energy component is nonvanishing then the spectral representation does not insure the definiteness of the propagator and there is no minimum principle for the effective action.

Note that, as we shall discuss in detail later, the fact that the classical field is time-independent does not preclude the ground state from carrying a current. This is because, as may be seen from (3,7), if \( \phi \) is space-dependent then \( \dot{\phi} \) may vanish without \( \Pi \), and therefore the classical current, vanishing. From the hamiltonian perspective, this is related to the fact that the ‘time’ appearing in the lagrangian (9) is not in fact the true time, but rather the conjugate quantity to the effective hamiltonian \( H = H^{(0)} - \lambda J \), whose ground state properties we are trying to find. The real time-dependence of the system is still generated by the original hamiltonian \( H^{(0)} \). Thus, in the Heisenberg picture, having a ‘time’-independent classical field which commutes with \( H \) in fact corresponds to a true time dependence \( \dot{\phi} = -i\lambda[J,\phi] \) - thus, as expected, local inhomogeneities in \( \phi \) are transported along by the current in this state.

### 3 Energy current in the disordered phase \( \mu^2 > 0 \).

The minimisation of the classical action\(^3\) on functions independent of time leads to the following equation for classical field \( \phi_r \) that will be henceforth denoted by \( \phi_r \):

\[
i \frac{\delta \Gamma_{\text{eff}}}{\delta \phi_r} = \frac{\lambda^2}{4} \left[ 2\phi_r - \phi_{\mathbf{r}+2\mathbf{e}_1} - \phi_{\mathbf{r}+2\mathbf{e}_1} \right] - \sum^D_\alpha \left[ 2\phi_r - \phi_{\mathbf{r}+\mathbf{e}_\alpha} - \phi_{\mathbf{r}+\mathbf{e}_\alpha} \right] - \mu^2 \phi_r - \frac{g}{6}[\phi_r]^3 = 0. \tag{10}\]

Introduce the Fourier decomposition of the time dependent classical field \( \phi_r \)

\[\phi_r = \frac{1}{(2\pi)^{(D+1)/2}} \int_{-\pi}^{\pi} d^D \mathbf{p} \int_0^\infty dE \left[ \psi(\mathbf{p}, E)e^{iE(t)} + \text{h. c.} \right]. \tag{11}\]

and let \( \phi(\mathbf{p}, E) = \psi(\mathbf{p}, E) + \psi^\dagger(-\mathbf{p}, -E). \)

A time-dependent solution for the classical field then has the form setting \( \phi(\mathbf{p}, E) = 2\pi \phi(\mathbf{p}) \delta(E) \) (10) where \( \phi(\mathbf{p}) \) satisfies

\[
i \frac{\delta \Gamma_{\text{eff}}}{\delta \phi(\mathbf{p})} = 0 = D_0^{-1}(\mathbf{p}, 0) \phi(\mathbf{p}) - \frac{g}{6}[\phi^3(\mathbf{p})], \tag{12}\]

and we have introduced the notation

\[
D_0^{-1}(\mathbf{p}, E) = (E + \lambda \sin p_1)^2 - 2 \sum_\alpha (1 - \cos p_\alpha) - \mu^2 \tag{13}
\]

\(^3\)For convenience, we will use the negative of the effective action that will have a maximum, rather than a minimum at the correct classical field.

\(^4\)This implies the relations \( \phi(\mathbf{p}, E) = \psi(\mathbf{p}, E) \) for \( E > 0 \) and \( \phi(\mathbf{p}, E) = \psi^\dagger(-\mathbf{p}, -E) \) if \( E < 0 \).
and
\[ [\phi^k](p) = \frac{1}{(2\pi)^D} \int d^Dp_1 \cdots d^Dp_k \delta(p - p_1 - \cdots - p_k)\phi(p_1)\cdots\phi(p_k). \] (14)

The tree level expression for the inverse propagator is
\[ i \frac{\delta^2 \Gamma_{\text{eff}}}{\delta\phi(p, E)\delta\phi(-p', -E')} \sim \delta(E - E') \left\{ \delta(p - p')D_0^{-1}(p, E) - \frac{g}{2} [\phi^2](p - p') \right\}. \] (15)

Note that \( D_0^{-1}(p, E) \) vanishes on the single particle dispersion curve discussed earlier, as well as on the 'negative-energy' branch. The coefficient \( D_0^{-1}(p, 0) \) in (12) can be easily analysed. It may have several extrema. Besides the one at \( p = 0 \), provided that \( \lambda > 1 \), there are two at \( p_1 = \pm p_1^{(0)}, p_\alpha = 0 \), if \( \alpha \neq 1 \), where \( \cos p_1^{(0)} = 1/\lambda^2 \). At \( \lambda > 1 \) the latter ones are the only maxima. The value of \( D_0^{-1} \) at these maxima is
\[ D_0^{-1}(p^{(0)}) = (\lambda - 1/\lambda)^2 - \mu^2. \] (16)

Then for every real value of the mass gap, \( \mu \), there is a \( \lambda_c > 1 \), such that the maximum of this coefficient is exactly zero,
\[ \lambda_c = \frac{1}{2} \left[ \mu + \sqrt{\mu^2 + 4} \right] \] (17)

Note that the convexity of the effective action requires that after setting \( E = 0 \) (15) is negative at the appropriate classical field. Indeed, when \( D_0^{-1}(p) > 0 \) one can find a nonvanishing solution of (12). In fact, this nonvanishing solution will provide the minimum of the action in contrast to the symmetric solution, \( \phi = 0 \). We expect that if the vacuum expectation value is independent of time then the propagator (the second derivative of the action) is negative definite. Then the effective action is a convex function of the classical field and the classical field should be chosen to minimise it.

### 3.1 The symmetric phase

Suppose first that \( \lambda < \lambda_c \). Then \( D_0^{-1}(p^{(0)}) < 0 \) and the only real solution of (12) is \( \phi p = 0 \). Consequently, the symmetry is unbroken. The tree level inverse propagator becomes
\[ D_0^{-1}(p, E) = \Delta_0^{-1}(p, E + \lambda \sin p_1), \] (18)
where \( \Delta_0 \) is the propagator in the absence of the perturbation. Then the only change in the perturbed theory is the change of the energy-momentum dispersion relation to
\[ E = -\lambda \sin p_1 + \sqrt{\mu^2 + 2 \sum_{\alpha}(1 - \cos p_\alpha)}. \]

Though this relation dips near \( p = p^{(0)} \), the energy is always positive. In other words, the energy gap is nonvanishing. Perturbative corrections are not expected to change this result, but they may change the location of the critical point.
Although the propagator is modified when $\lambda \neq 0$, in fact the effect on the ground state properties is rather subtle. For example, the equal-time spin correlation function in the absence of loop corrections entails integrating $D_0(p, E)$ over all $E$, and therefore the correction term $\lambda \sin p_1$ is simply shifted away. Does this result survive the addition of loop corrections? This may be examined by making a time-dependent transformation on the Fourier components of the field
\[
\phi(p, t) \rightarrow \phi(p, t) e^{it\lambda \sin p_1},
\]
which shifts away the $\lambda \sin p_1$ term in the bare propagator. The interaction now has the form
\[
\int dt \prod_{n=1}^{4} \left[ d^4p^{(n)} \phi(p^{(n)}, t) \right] e^{it\lambda \sum_{n=1}^{4} \sin p_1^{(n)} \delta(\sum_{n=1}^{4} p^{(n)})}. \tag{20}
\]
From this it can be seen that if we approximate $\sin p_1$ by $p_1$, the $t$-dependence disappears and the result is the same as in the theory with $\lambda = 0$. This is clearly connected with the fact that, to lowest order in the derivative expansion, the current does commute with the interaction.

The subtlety of this modification may be seen, for example, in the computation of the expectation value of the current in this phase. From (3, 7) this is
\[
\langle j_1^E \rangle = \int dE d^4p (E + \lambda \sin p_1) \lambda \sin p_1 \langle \phi(p, E)\phi(-p, -E) \rangle. \tag{21}
\]
If we make the above shift $E \rightarrow E + \lambda \sin p_1$, and we compute the correlator in the absence of any interaction, it is easy to see that this vanishes through the symmetry $E \rightarrow -E$ (or $p_1 \rightarrow -p_1$). Indeed, this will remain true even when the interaction is included, if we can ignore the momentum-dependent phase factors in (20), since these symmetries then continue to hold. But the phase factors couple these two symmetries together, and, since the current is even under the joint reflection of $E$ and $p_1$ it does not necessarily vanish when loop corrections are included. However, it turns out that the first non-zero contribution first occurs at three loops, due to the diagram shown in Fig. 3a. The one- and two-loop diagrams still vanish. For example, at the internal vertex in the diagram shown in Fig. 3b the incoming momenta are $(p, -p, p', -p')$ so the phase $\sum_{n=1}^{4} \sin p_n$ still vanishes. We conclude that the first non-vanishing contribution to the current in this phase occurs at three loops, and so is $O(g^2\hbar^3)$.

### 3.2 The critical theory

At the critical point $\lambda = \lambda_c$, the classical field still vanishes at the minimum of the action. Therefore the tree level propagator has the form
\[
D(p, E) = \frac{1}{(E + \lambda_c \sin p_1)^2 - 2 \sum_{\alpha} (1 - \cos p_\alpha) - \mu^2}. \tag{22}
\]
It is possible to make the same transformation as above and shift $E$ in the bare propagator. However, this obscures the fact that there is now no energy gap, and the theory
may develop infrared singularities close to \( E = 0 \) which modify the large distance behaviour of the correlation functions. This can be seen if we expand the denominator of the propagator around \( p = \pm p^{(0)} \) and \( E = 0 \). The constant term vanishes at the critical point. The term linear in \( p \) also vanishes, as \( \pm p^{(0)} \) were defined as locations of maxima of the inverse propagator. Then, keeping leading order terms in \( E \) and \( \pm p^{(0)} \) only, we obtain

\[
D(p, E) \simeq \frac{1}{\pm 2 E \lambda_c \sin p^{(0)} - \lambda_c^2 \sin^2 p^{(0)}(p_1 \mp p^{(0)})^2 - \sum_{\alpha \neq 1} p_\alpha^2}, \tag{23}
\]

where

\[
p^{(0)}_1 = \cos^{-1} \frac{1}{\lambda_c^2} = \cos^{-1} \left( 1 - \mu \frac{\sqrt{\mu^2 + 4} - \mu}{2} \right) > 0. \tag{24}
\]

The two signs in (23) correspond to poles that are separated from one another by a finite distance. These two poles lead to two propagators of states which, at least in the absence of interactions, do not mix. After rescaling \( E \), as \( 2E \lambda_c |\sin p^{(0)}_1| \to E \), and shifting and rescaling \( p_1 \) as \( \lambda_c \sin p^{(0)}_1(p_1 - p^{(0)}_1) \to p_1 \) we obtain the following expression, which can be regarded as propagators for a ‘particle’ \( A \) and for an ‘antiparticle’ \( \bar{A} \), both, however, non-relativistic:

\[
D^\pm(p, E) \simeq \frac{1}{\pm E - p^2 + i\epsilon}. \tag{25}
\]

When the interactions are included, in terms of the non-relativistic excitations above, the terms allowed by overall momentum conservation are the scattering processes \( AA \to AA, \bar{A}A \to 
\bar{A}\bar{A}, \) and also \( AAA\bar{A} \to 0 \) and its inverse. Without these last two processes, the theory is identical to that of non-relativistic particles with a local 2-body repulsive interaction. Such as theory does have infrared singularities below \( D = 2 \), but they do not affect the propagator, since it is easy to see that there are in fact no loop corrections. For the same reason the quantum corrections to the current all vanish.

This is no longer true in the presence of the last two vacuum processes, which indeed lead, at two loops, to corrections to the current corresponding to those illustrated, for \( \mu^2 > 0 \), in Fig. 3a. However, because of the way the momentum flows through these diagrams, it can be seen that they are not in fact infrared singular. Thus we expect
that the quantum corrections to the current are not singular, leaving the singularity in classical current, to be studied in the next section.

### 3.3 The broken phase

When $\lambda$ is increased beyond its critical value, $\lambda_c$, then the maximum of the inverse propagator becomes positive, $D_0^{-1}(p^{(0)}) > 0$. Then a symmetry breaking solution of (12) exists, along with the symmetric solution, $\phi(p) = 0$. As the nonzero solution minimises the effective action we must choose it over the symmetric one.

It is not a priori clear whether the relevant solution of (12) is a classical field with a sharp spectrum (i.e. delta function in momentum space) or at any given $\lambda$ the spectrum of the classical field contains all the momenta for which $D^{-1}(p) > 0$. Examining the second functional derivative resolves this problem. On one hand, the convexity of the action demands that the right hand side of (15) is negative even in the neighbourhood of $p = p^{(0)}$. On the other hand, at $\lambda > \lambda_c$ the multiplier of the momentum conservation delta function (that is $D^{-1}(p)$) is positive in the neighbourhood of $p^{(0)}$. The negativity of the second variation can only be secured if the last term of (12) also contains a term proportional to $\delta(p - p')$. In turn, that is possible only if $\phi(p)$ itself contains a term proportional to a delta function (sharp spectrum), as well. Suppose that the delta function fixes the momentum at $p = k_0$, or in other words, $\phi \sim \delta(p - k_0)$. Then it is easy to see that the minimum of the action, $S_{\text{min}} \sim -D^{-1}(k_0)^2$. This is minimised, as a function of $k_0$ if $k_0 = \pm p^{(0)}$. In other words, the delta function in the classical field must set the momentum equal to $\pm p^{(0)}$.

Since $\phi(x, t)$ is real at $E = 0$ both “negative” and “positive” frequency terms contribute. Therefore, we use a symmetric ansatz

$$\phi(p) = \phi_0(p) + \Delta \phi(p),$$

where

$$\phi_0(p) = \rho \left[ e^{i\theta_0} \delta(p - p^{(0)}) + e^{-i\theta_0} \delta(p + p^{(0)}) \right].$$

In (27) $\rho$ is a real amplitude, and $\theta_0$ is an arbitrary real phase. As we shall see in the next section the arbitrary phase is related to the presence of a Goldstone boson in the broken phase.

Near the phase transition the term $\Delta \phi(p)$ is much smaller than the leading term, $\phi_0(p)$. The self-consistency of this requirement will become obvious below.

Upon substituting (26) into condition (12) we obtain

$$0 = \left[ -\mu^2 + \left( \lambda - \frac{1}{\lambda} \right)^2 - \frac{g}{2(2\pi)^D} \rho^3 \right] \phi_0(p)$$

$$+ \left[ \lambda^2 \sin^2 p_1 - 2 \sum_\alpha (1 - \cos p_\alpha) - \mu^2 \right] \Delta \phi(p)$$

$$- \frac{g}{6(2\pi)^D} \rho^3 \left[ e^{i\theta_0} \delta(p - 3p^{(0)}) + e^{-i\theta_0} \delta(p + 3p^{(0)}) \right] + O(\rho^2 \Delta \phi(p)).$$

(28)
Consequently, the leading order term \( (O(\lambda - \lambda_c)) \), proportional to \( \delta(p \pm p^{(0)}) \) of (28) vanishes if we choose
\[
\rho^2 = \frac{2(2\pi)^D}{g} \left[ \left( \frac{\lambda}{\lambda_c} - 1 \right)^2 - \mu^2 \right]. \tag{29}
\]
Then near the phase transition the amplitude of the classical field vanishes as \( \rho \sim \sqrt{\lambda - \lambda_c} \). Note that at \( \mu^2 = 0 \), \( \lambda_c = 1 \) and \( \rho \sim \lambda - \lambda_c \).

The leading order third harmonics terms also cancel if we choose
\[
\Delta \phi(p) = \rho_3 \left[ e^{3i\theta_0} \delta(p - 3p^{(0)}) + e^{-3i\theta_0} \delta(p + 3p^{(0)}) \right] + O(\rho^5), \tag{30}
\]
where the amplitude of the third harmonics, \( \rho_3 \), has the form
\[
\rho_3 = -\frac{1}{2(1 - \cos 3p^{(0)}) - \lambda^2 \sin^2 \theta_0} \frac{g}{6(2\pi)^D} \rho^3. \tag{31}
\]
It follows from (31) that \( \Delta \phi(p) = O((\lambda - \lambda_0)^{3/2}) \).

In general, it is easy to see that near the phase transition \( \lambda \simeq \lambda_c \) one can solve equation (12) iteratively, resulting in amplitudes for the higher harmonics that have a following behaviour near \( \lambda = \lambda_c \):
\[
\rho_{2n+1} \sim \left( \lambda - \lambda^{(0)} \right)^{(2n+1)/2}. \tag{32}
\]
Here \( \rho_{2n+1} \) is the coefficient of \( (2n + 1) \)st harmonics of momentum \( p^{(0)} \).

The conclusion is that, at least at the classical level, the system undergoes a second order phase transition to a phase with spontaneously broken translation symmetry.

Finally, we comment on the classical part of the energy current, found by substituting the classical expectation value of the field into (3). Substituting the canonical momentum by its lagrangian expression and using that the classical field is time independent we obtain
\[
\langle \int d^Dxj^E_\alpha(x) \rangle = -\frac{\lambda^2}{2} \int d^Dx \frac{1}{2i} \langle (\phi x + e_1) - \phi x - e_1 \rangle (\phi x + e_0) - \phi x - e_0 \rangle. \tag{33}
\]
This expression is obviously zero unless \( \alpha = 1 \). For \( \alpha = 1 \) we obtain, after substituting \( \phi x = \rho e^{iD^{(0)}x} + \text{h.c.} \)
\[
\langle \int d^Dxj_\alpha^E(x) \rangle \simeq V \rho^2 \lambda^2 \sin^2 p^{(0)}_1 \sim V \rho^2 \sim \frac{1}{g} \begin{cases} \lambda - \lambda_c & \text{if } \mu^2 > 0 \\ (\lambda - \lambda_c)^2 & \text{if } \mu^2 = 0. \end{cases} \tag{34}
\]
As well as this \( O(1/g) \) contribution, there are perturbative quantum corrections just as for \( \lambda < \lambda_c \).

4 Goldstone modes and the case \( D = 1 \).

Up to this point we have investigated the \( D > 1 \) case only. In one dimension, the Mermin-Wagner-Hohenberg theorem asserts that there can be no spontaneous breaking
of the continuous symmetry of translational invariance, due to infrared singularities of the Goldstone modes. In fact, the physics is very similar to that of spin or charge density waves, and of modulated phases such as occur in the ANNNI model \cite{9}, in 1+1 dimensions.

Rather than include all the fluctuations about the classical solution, it should be adequate to derive an effective action for these Goldstone modes only, by writing the field $\phi(x,t)$ in the following form:

$$
\phi(x,t) = \rho \left\{ [1 + \xi(x,t)] e^{i \theta(x,t)} e^{ip(0)x} \right\} + c.c.,
$$

where $\rho$ and $p(0)$ are chosen to minimise the classical action, and $\xi(x,t)$ and $\theta(x,t)$ are real. These functions are chosen so that their Fourier transforms have support only for $|p| < p(0)$: in this way the decomposition in (35) is unique. In principle we should also include higher harmonics, but, as shown in Sec. 3.3, these are suppressed near the transition. We have included possible longitudinal fluctuations through the field $\xi(x,t)$.

Upon substituting (35) into the action and keeping terms quadratic in oscillating fields $\xi(p,E)$ and $\theta(p,E)$ only we obtain

$$
S = \rho^2 \frac{1}{(2\pi)^{D+1/2}} \int d^D p dE \left\{ [D_0^{-1}(p-p(0),E) + D_0^{-1}(p+p(0),E)] - 3g\rho^2 \right\} \xi(p,E) \xi^*(p,E) + [D_0^{-1}(p-p(0),E) + D_0^{-1}(p+p(0),E) - gp^2 \theta(p,E) \theta^*(p,E) + i [D_0^{-1}(p-p(0),E) - D_0^{-1}(p+p(0),E)] [\theta(p,E) \xi^*(p,E) - \theta^*(p,E) \xi(p,E)] \right\},
$$

where the notation $D_0^{-1}(p,E) = (E + \lambda \sin p_1)^2 - 2 \sum_\alpha (1 - \cos p_\alpha) - \mu^2$ has been introduced. Terms containing fields $\xi(p,E)$ or $\theta(p,E)$ taken at a momentum of $O(p(0))$ were omitted from (36). For convenience, we also symmetrised (36) in momentum.

Expression (36) of the action has a few important features:

- $p$ measures the deviation of the momentum from its “critical” values, $\pm p(0)$.

- Radial and angular degrees of freedom are mixed. The term mixing them is proportional to $\lambda$. Thus, no such term appears in the somewhat similar case of spontaneous breaking of $U(1)$ global symmetry, where the radial and phase degrees do not mix at tree level.

- If we omit the mixing term then, as expected, the radial mode is massive, while the angular mode is massless. In other words, the energy gap vanishes at $p = 0$: $D_0^{-1}(p(0),0) + D_0^{-1}(-p(0),0) - gp^2 = 0$.

Thus, due to the mixing of the two modes, which is a peculiar feature of our model, we cannot analyse the zero modes by omitting all degrees of freedom but the phase. The eigenmodes of propagation are found by diagonalising the propagator matrix in the two
dimensional space \((\xi, \theta)\). The eigenvalues of this matrix are

\[
D_{1,2}^{-1} = \frac{\rho^2}{2} \left\{ D_0^{-1}(p - p^{(0)}, E) + D_0^{-1}(p + p^{(0)}, E) - 2g\rho^2 \pm \sqrt{g^2\rho^4 + 16\lambda^2E^2\sin^2 p^{(0)} \cos^2 p} \right\} 
\]

(37)

In the far infrared region, \(E, p^2 \ll g\rho^2\), one finds the mode corresponding to the upper sign tends to the \(\theta\)-mode, (the \(\xi\) admixture is of \(O(E/g\rho^2)\)), while the one corresponding to the lower sign tends to the \(\xi\) mode. Yet the dispersion relation obtained for the upper sign is different from the one we would have obtained, had we omitted the “radial” mode from the outset. This can be seen if we consider the asymptotic form of eigenvalues of the inverse propagator matrix,

\[
D_{1,2}^{-1} = \left( \begin{array}{c}
(\lambda^2 - 1/\lambda^2) \left( \frac{4}{g} E^2 - \rho^2 p_1^2 \right) - \rho^2 p_1^2 \\
[\rho^2 - \frac{4}{g}(\lambda^2 - 1/\lambda^2)] E^2 - (\lambda^2 - 1/\lambda^2)\rho^2 p_1^2 - \rho^2 p_1^2 - 2g\rho^2
\end{array} \right) 
\]

(38)

The first of these modes is massless. The second mode appears to be massive, except near the phase transition, if \(\rho^2 < 4(\lambda^2 - 1/\lambda^2)/g\), the energy squared term has the wrong sign. Having imaginary energy, this is not a propagating mode. We will soon show, however, that at small values of \(\rho\) the classical approach becomes unreliable, at least at \(D = 1\).

Next we shall consider the case \(D = 1\). Near the phase transition the regularised equal time correlation function of the gapless mode becomes

\[
D(x - y) = \frac{1}{i4\pi^2(\lambda^2 - 1/\lambda^2)} \int dE dp - e^{ip(x - y)} - 1 
\]

\[
\frac{4E^2/\rho - \rho^2 p^2}{8\pi\rho(\lambda^2 - 1/\lambda^2)} \approx -\sqrt{\frac{\pi}{4\rho}} \log |x - y|
\]

(39)

The correlation function is proportional to \(1/\rho\), so it diverges as \(1/\sqrt{\lambda - \lambda_c}\) near \(\lambda = \lambda_c\).

Near \(E = 0\) (39) approximates the correlation function for the phase mode of oscillations. Then the equal time correlation function of \(\phi(x, 0)\) is

\[
\langle \phi(x, 0)\phi(y, 0) \rangle \sim \rho^2 \left( e^{i\phi(x, 0) - i\phi(y, 0)} \right) \sim \frac{\rho^2 \cos[p^{(0)}(x - y)]}{|x - y|^{\eta}} 
\]

(40)

where \(\eta = \sqrt{\frac{g}{[8\pi\rho(\lambda^2 - 1/\lambda^2)]}}\), which exhibits an oscillating power-like behaviour with a nonuniversal critical exponent. This means that there is only quasi-long range order in \(D = 1\).

Note that the parameter \(\rho \propto (\lambda - \lambda_0)^{1/2}\) plays a very similar role to \(\beta = 1/T\) in the \(XY\)-model. Just as in this model, there are also vortex-like excitations whose condensation at high enough \(T\) (small \(\rho\)) completely destroys even the quasi-long range order. In our case, these result from the fact that \(\theta(x, t)\) is an angular variable, defined only modulo \(2\pi\). A ‘vortex’ corresponds to a fluctuation in which an extra oscillation of the field \(\phi(x, t)\), localised in \(x\), is either inserted or removed in a large but finite space and time interval. This has an action which grows like the logarithm of the system size. By the Kosterlitz-Thouless criterion [14], they become relevant when \(\eta > 1/4\).

We conclude that the transition to the quasi-modulated phase occurs not at \(\lambda = \lambda_0\), but at a higher value when \(\rho\) has become sufficiently small. At the transition, we should
then have $\eta = 1/4$, with the value of this exponent decreasing as we go further into the modulated phase.

Although this phase exhibits only quasi-long range order, it nevertheless supports a non-zero classical current, as for $D > 1$, since the leading contribution $\sim \lambda \rho^2 \sin^2 p^{(0)}$ is not affected by infrared problems. In fact, this quantity corresponds to the superfluid density in the XY model, and is expected to jump discontinuously at the transition [11].

In the ultraviolet limit of (37) the two roots of the inverse propagator reduce to the two poles of (25). This shows the consistency of our descriptions of the critical and broken phases.

5 Energy current in the ferromagnetically ordered state.

In the regime $\mu^2 \leq 0$ we have not been able to obtain results comparable to those above, especially in the difficult case of $\mu^2 = 0$. We can show however, at least at the classical level, that the second order transition at $\mu^2 > 0$ turns into a first order transition at $\mu^2 < 0$.

When $\mu^2 < 0$ the $Z_2$ symmetry of the theory is broken even at $\lambda \ll 1$, where the translation invariance is certainly not broken. Thus, we must shift the field to minimise the effective potential, but the classical field is a constant, $\phi_0$. When $\lambda$ is increased the classical field will become coordinate dependent above a certain critical value. Writing the classical field as $\phi(p^1) = \phi_0 + \psi(p^1)$ the extremum condition for field $\psi$ will read as:

$$i \frac{\delta \Gamma_{\text{eff}}}{\delta \psi(p)} = 0 = \left[ \lambda^2 \sin^2 p - 2 \sum_{\alpha} (1 - \cos p_{\alpha}) - 2|\mu^2| \right] \psi(p)$$

$$- \frac{g \phi_0}{2(2\pi)^D} \int d^D p_1 d^D p_2 \psi(p_1) \psi(p_2) \delta(p - p_1 - p_2)$$

$$- \frac{g}{6(2\pi)^D} \int d^D p_1 d^D p_2 d^D p_3 \psi(p_1) \psi(p_2) \psi(p_3) \delta(p - p_1 - p_2 - p_3),$$

(41)

where $\phi_0 = \sqrt{6|\mu^2|(2\pi)^D/g}$.

We can rescale (41) by dividing by $|\mu^2|\phi_0$ and introducing the new field $\psi \rightarrow \psi \phi_0$. Then, using the fact that the ground state depends only on $p_1$ (in what follows we will drop the subscript), one obtains the following equation for $\psi$:

$$0 = \left[ 2 - \frac{\lambda^2 \sin^2 p - 2(1 - \cos p)}{|\mu^2|} \right] \psi(p)$$

$$+ 3 \int dp^1 dp^2 \psi(p^1) \psi(p^2) \delta(p - p^1 - p^2)$$

$$+ \int dp^1 dp^2 dp^3 \psi(p^1) \psi(p^2) \psi(p^3) \delta(p - p^1 - p^2 - p^3).$$

(42)

Naively, one would think that just like for $\mu^2 > 0$, at least at the tree level, the phase transition is determined by the linear term in (42) and occurs at $\lambda = \lambda_1$, where
\( \lambda_0 - 1/\lambda_0 = \sqrt{2}\mu. \) (Note the extra factor of \( \sqrt{2} \), reflecting the fact that the bare mass for \( \mu^2 < 0 \) is \( \sqrt{2}|\mu| \).) Contrary to this expectation we will prove that, due to the presence of the term quadratic in \( \psi \) in (42), the system undergoes a first order phase transition, at a critical value of \( \lambda = \lambda_c < \lambda_1 \), to a state with broken translation invariance. In order to show this we merely have to exhibit a solution of (42) which gives a lower action, for \( \lambda_c < \lambda < \lambda_1 \), than the solution \( \psi(p) = 0 \). To this end, consider the ansatz

\[
\psi(p) = \psi_0 \delta(p_1) + \rho_1 \left[ e^{i\theta_0} \delta(p_1 - p_1^{(0)}) + e^{-i\theta_0} \delta(p_1 + p_1^{(0)}) \right]
+ \psi_2 \left[ e^{i2\theta_0} \delta(p_1 - 2p_1^{(0)}) + e^{-i2\theta_0} \delta(p_1 + 2p_1^{(0)}) \right]
\]

(43)

where, as before, \( \cos p^{(0)} = 1/\lambda^2 \). After substituting this ansatz into (41) we minimise the action with respect to \( \psi_0, \rho_1 \), and \( \psi_2 \). We obtain

\[-2\psi_0 = 3(\psi_0^2 + 2\rho_1^2 + 2\psi_2^2) + \psi_0^3 + 3\psi_0(\rho_1^2 + \psi_1^2)\]

\[\epsilon \rho_1 = 6(\psi_0 \rho_1 + \psi_2 \rho_1) + 3\rho_1(\rho_1^2 + \psi_0^2 + 2\psi_2^2)\]

\[-2\psi_2 = 6(\rho_1^2 + \psi_2 \psi_0) + 2\psi_0 \rho_1^2 + \psi_0^2 \psi_2\]

(44)

where we used the relation \( \lambda^2 \sin^2(2p_1^{(0)}) - 2[1 - \cos(2p_1^{(0)})] = 0 \) and the notation \( \left( x - \frac{1}{x} \right)^2 / |\mu^2| - 2 = \epsilon. \)

Note that in terms of the new variables a second order transition is possible only at \( \epsilon \geq 0 \). (44) can however be solved numerically and doing so we find that it has a real solution for all \( \epsilon > \epsilon_0 = -0.3395 \). This value of \( \epsilon \) corresponds to \( \lambda_c \approx 1.834 < \lambda_1 = 1.932 \). Below \( \lambda_c = 1.834 \) there is no real solution of the form (43), but above this value a pair of complex roots become real and the parameters take the values \( \rho_1 \approx 0.130 \), \( \psi_0 \approx -0.0410 \), and \( \psi_2 = -0.0358 \). The true tree level transition is then at a value \( \lambda_c < 1.834 \). The phase transition line is at a fixed value of \( (\lambda - 1/\lambda)^2 / |\mu^2| \). This fixed value is smaller than 1.6605, and certainly smaller than the superficial prediction, 2, that was based on the assumption that the transition is second order.

Of course, this argument does not imply that the ansatz (43) represents the true minimum of the effective potential (now there is no reason to ignore higher harmonics as there was in the second order case) but is does imply that the transition cannot be second order, at least at tree level. Nor does the argument rule out the possibility of a restoration of the second order nature of the transition by the fluctuations, especially in low dimensions (see next section).

6 Summary

Following investigations of similar nature for integrable one dimensional spin chains [3] [4], we have studied the effect of an energy current in field theories, at zero temperature. The energy current was imposed on the system by adding the global energy current operator, multiplied by a Lagrange multiplier, \( \lambda \), to the Hamiltonian. The common feature of these models is the appearance of a phase transition at a critical value of \( \lambda \).
Field theoretic models modified by an energy current can roughly be divided into two groups. In the first, less interesting group of theories the Hamiltonian is destabilised at the critical value of $\lambda$, $\lambda_c$, whereby the energy spectrum has no lower bound at $\lambda > \lambda_c$. A typical example for these models is a relativistic, self-interacting, real, scalar field theory. Theories, in which at $D > 1$ (where $D$ is the number of spatial dimensions) translation invariance breaks spontaneously, are more interesting. A typical example for this second group is a self-interacting real scalar field theory on a lattice.

In the lattice models that we consider the global energy current does not commute with the Hamiltonian and consequently, unlike in spin models $^4$ $^5$, the ground state expectation value of the energy current has a nonvanishing perturbative contribution at all $\lambda \neq 0$. Another, nonperturbative contribution to the energy current appears at the phase transition point.

The system has a few unusual features in the broken phase, at $D > 1$. The first of these is that the field has a nonvanishing, oscillating vacuum expectation value. Correlation functions are also oscillating functions of the coordinate. The oscillations are the consequence of the unusual form of the energy-momentum dispersion relation and the vanishing of the energy gap at a nonvanishing value of the momentum. The second order phase transition to the broken phase is accompanied by the appearance of a Goldstone boson, in spite of the fact that the translation symmetry is a discrete symmetry on the lattice. This happens because near the phase transition the lattice structure of the system becomes irrelevant.

The second interesting feature of the field theoretic model is the apparent spontaneous generation of degrees of freedom and enlargement of the internal symmetry near the phase transition. As the energy gap vanishes at $p = p^0 \neq 0$, it is meaningful to rewrite the real scalar field into the form $\phi(x) = \chi(x)e^{ip^0 x} + \chi^*(x)e^{-ip^0 x}$. At least in the infrared domain ($p << p^0$) $\chi(x)$ is a genuine complex field with $U(1)$ symmetry (planar model) in contrast to the $Z_2$ symmetry of the of the field $\phi(x)$. When the vacuum expectation value of $\chi$ is non-vanishing, its phase degree of freedom becomes a Goldstone mode. The extra near infrared degrees of freedom can of course be related to large momentum modes of the original theory, so in reality the number of degrees of freedom is unchanged. What we really observe is the transmutation of a space-time symmetry (translation invariance) into a broken internal symmetry (phase transformations of field $\chi$).

The analogy with the $U(1)$ model in $D > 1$ spatial dimensions is not complete. Due to the peculiar form of the single particle dispersion relation the propagator takes a nonrelativistic form in the infrared domain. This leads to a critical dynamics different than that of a complex scalar field, and the anomalous dimension vanishes at all $D > 1$.

When $D = 1$ the translation invariance cannot be broken spontaneously. Yet the characteristic momentum $p^0$ and the complexification of the degrees of freedom survive due to large fluctuations, the field $\chi(x)$ must have a vanishing position-dependent vacuum expectation value. Still, the analogy with the $U(1)$ symmetric $XY$ model, based on the decomposition of the scalar field into a superposition of a complex field and of its complex conjugate, is maintained. One obtains damped oscillating correlation functions.
with non-universal anomalous dimensions. The phase transition at \( \lambda = \lambda_c \) is driven by the condensation of vortices of the \( \chi(x) \) field.

Despite some similarities with the exactly solvable cases treated in [5], [6], there are also some important differences in our results. Partly this is due to the fact that in our case the current does not commute with the Hamiltonian, so that the current is non-vanishing even for arbitrarily small bias \( \lambda \). However, the singular part of the current, which we have argued occurs only in its ‘classical’ piece, has a different behaviour, vanishing linearly as \( \lambda \to \lambda_c^+ \), rather than proportional to \( \sqrt{\lambda - \lambda_c} \) as in [5]. In \( D = 1 \), this linear vanishing is modified by the vortices to a discontinuous jump. This behaviour is related to that of the correlation function: we find that in our model it should exhibit oscillations modulated by a power law \( |x|^{-\eta} \), where \( \eta \) is continuously varying throughout the modulated phase, increasing to the universal value \( 1/4 \) at the transition. This should be compared with the constant value \( \eta = 1/2 \) found for the transverse Ising chain in [5].

These differences in detail are not necessarily surprising. When such integrable models as the Ising or XY chains are mapped onto Coulomb gas models, the additional conservation laws in the integrable system may show up in unusual constraints (for example the charge at infinity) which can modify the critical exponents away from their usual values. However it should be clear that our results do not depend so much on details of our model, and therefore should possibly be more generic.

A comment should be made concerning the time-dependence in our model. In most of this paper the parameter \( t \) referred to as ‘time’ is merely an artifice introduced in order to provide a Lagrangian formulation of the problem of finding the ground state properties of the operator \( H^{(0)} - \lambda J \). Thus no meaning should be attached to quantities which are not \( t \)-independent. The true time dependence is still generated by \( H^{(0)} \). In particular, this means that the expectation value of the current, which does not commute with \( H^{(0)} \), should be time-dependent. The constant quantity which we have computed is therefore only its zero-frequency part. It would be interesting, but difficult, to investigate the time-dependent part as well.

In fact, the mathematics of our model bears a very close resemblance to that which appears in the study of other systems which exhibit spatially modulated phases, such as spin or charge density waves, or systems such as the ANNNI model [9] with competing ferromagnetic nearest and antiferromagnetic next-nearest neighbour interactions. In fact, if one neglects the ‘time’-dependence in [9], and takes the limit \( g \to \infty, \mu^2 \to -\infty \) with \( \mu^2/g \) fixed, so that the field at each site is frozen to the values \( \pm \sqrt{3|\mu^2|/g} \), this becomes precisely the energy function for the classical ANNNI model in \( D + 1 \) dimensions. Even in mean field theory the phase diagram for this model is extremely rich. Between the ferromagnetic \( \cdots + + + + + \cdots \) phase and the \( \cdots - - - + + + \cdots \) antiphase lie an infinite number of other modulated phases. In 1+1 dimensions, these can also be separated by Kosterlitz-Thouless incommensurate phases, bounded by Pokrovsky-Talapov transitions[12]. It should be stressed that the limit above is quite different from the regime considered in the body of this paper where it was assumed that the characteristic wave vector \( p^{(0)} \) is much smaller than the reciprocal lattice vector, and that lattice effects are irrelevant. The modulated phases considered there are therefore
incommensurate. The ‘time’-dependent fluctuations in our model are rather different from those in transverse direction of the ANNNI-type models. Nevertheless it is possible that for $D = 1$ the first-order transition predicted in the last section is replaced by one of Pokrovsky-Talapov type.

Besides these correspondences, there remains the important question of the relevance of these kinds of model, whether integrable as in Refs. [3],[5], or non-integrable as in our example, to the study of non-equilibrium processes at finite temperature where dissipation plays a crucial role. It is to be hoped that some of the phenomena discussed in this paper will survive into this regime.

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