ROTA-BAXTER LIE BIALGEBRAS, CLASSICAL YANG-BAXTER EQUATIONS AND SPECIAL L-DENDRIFORM BIALGEBRAS

CHENGMING BAI, LI GUO, GUILAI LIU, AND TIANSHUI MA

Abstract. We establish a bialgebra structure on Rota-Baxter Lie algebras following the Manin triple approach to Lie bialgebras. Explicitly, Rota-Baxter Lie bialgebras are characterized by generalizing matched pairs of Lie algebras and Manin triples of Lie algebras to the context of Rota-Baxter Lie algebras. The coboundary case leads to the introduction of the admissible classical Yang-Baxter equation (CYBE) in Rota-Baxter Lie algebras, for which the antisymmetric solutions give rise to Rota-Baxter Lie bialgebras. The notions of $\mathcal{O}$-operators on Rota-Baxter Lie algebras and Rota-Baxter pre-Lie algebras are introduced to produce antisymmetric solutions of the admissible CYBE. Furthermore, extending the well-known property that a Rota-Baxter Lie algebra of weight zero induces a pre-Lie algebra, the Rota-Baxter Lie bialgebra of weight zero induces a bialgebra structure of independent interest, namely the special L-dendriform bialgebra, which is equivalent to a Lie group with a left-invariant flat pseudo-metric in geometry. This induction is also characterized as the inductions between the corresponding Manin triples and matched pairs. Finally, antisymmetric solutions of the admissible CYBE in a Rota-Baxter Lie algebra of weight zero give special L-dendriform bialgebras and in particular, both Rota-Baxter algebras of weight zero and Rota-Baxter pre-Lie algebras of weight zero can be used to construct special L-dendriform bialgebras.

Contents

1. Introduction
   1.1. Rota-Baxter Lie algebras
   1.2. Bialgebra structures on Rota-Baxter Lie algebras and special L-dendriform algebras
   1.3. Layout of the paper
2. Rota-Baxter Lie algebras, pre-Lie algebras and special L-dendriform algebras
   2.1. Rota-Baxter Lie algebras and their representations
   2.2. Pre-Lie algebras and special L-dendriform algebras
3. Rota-Baxter Lie bialgebras and special L-dendriform bialgebras
   3.1. Rota-Baxter Lie bialgebras
   3.2. Rota-Baxter Lie bialgebras of weight zero and special L-dendriform bialgebras
4. Coboundary Rota-Baxter Lie bialgebras, admissible CYBEs and the induced special L-dendriform bialgebras
   4.1. Coboundary Rota-Baxter Lie bialgebras and the induced special L-dendriform bialgebras
   4.2. Admissible CYBEs, $\mathcal{O}$-operators on Rota-Baxter Lie algebras and Rota-Baxter pre-Lie algebras
5. References
1. Introduction

This paper establishes a bialgebra structure on Rota-Baxter Lie algebras following the approach of Manin triples in the classical work on Lie bialgebras [17] and the recent development on Rota-Baxter antisymmetric infinitesimal bialgebras [6]. The well-known connection of Rota-Baxter Lie algebras with pre-Lie algebras is lifted to the bialgebra level, establishing a relationship of Rota-Baxter Lie bialgebras with special L-dendriform bialgebras [5].

1.1. Rota-Baxter Lie algebras. The importance of Rota-Baxter Lie algebras can be viewed from several perspectives.

1.1.1. Rota-Baxter operators and Rota-Baxter associative algebras. Let $A$ be a vector space equipped with a binary operation $\ast$ and let $\lambda$ be a scalar. A linear operator $R : A \to A$ is called a Rota-Baxter operator of weight $\lambda$ if

$$R(x) \ast R(y) = R(x \ast R(y)) + R(R(x) \ast y) + \lambda R(x \ast y), \quad \forall x, y \in A. \quad (1)$$

Then $(A, \ast, R)$, or simply $(A, R)$, is called a Rota-Baxter algebra for the binary operation $\ast$, most notably, Rota-Baxter associative algebras and Rota-Baxter Lie algebras.

The notion of Rota-Baxter associative algebra originated from the 1960 work [9] of G. Baxter in a probability study, where it was noted that the identity is an abstraction and generalization of the integration by parts formula. Forty years later it reappeared in the fundamental work of Connes and Kreimer [16] on Hopf algebra approach to renormalization of quantum field theory. Motivated by this and other connections in combinatorics, number theory and operads, the study of Rota-Baxter algebras has experienced a great expansion in the recent years. See [21, 30] for introductions and further references.

1.1.2. Rota-Baxter Lie algebras and the classical Yang-Baxter equation. As a remarkable coincidence, Rota-Baxter operators on Lie algebras were discovered independently as the operator form of the classical Yang-Baxter equation (CYBE), named after the physicists.

The CYBE arose from the study of inverse scattering theory in the 1980s and was recognized as the “semi-classical limit” of the quantum Yang-Baxter equation [10, 35]. The study of the CYBE is also related to classical integrable systems and quantum groups [14].

An important approach in the study of the CYBE was through the interpretation of its original tensor form in various operator forms which, along with the well-known method of Belavin and Drinfeld [11], has proved to be effective in providing solutions of the CYBE. Semonov-Tian-Shansky [31] first showed that, if there exists a nondegenerate symmetric invariant bilinear form on a Lie algebra $g$, then a skew-symmetric solution $r$ of the CYBE can be equivalently expressed as a linear operator $R : g \to g$ satisfying the operator identity

$$[R(x), R(y)] = R([R(x), y] + R([x, R(y)]), \quad \forall x, y \in g, \quad (2)$$

which is then regarded as an operator form of the CYBE. Thus this operator form is simply the Rota-Baxter relation (of weight zero) in Eq. (1) for Lie algebras.

This approach was expanded more generally by Kupershmidt [23] by generalizing the notion of Rota-Baxter operators to $\partial$-operators, later also called relative Rota-Baxter operators and generalized Rota-Baxter operators [29, 33].

1.1.3. Rota-Baxter Lie algebras and pre-Lie algebras. Another important role played by Rota-Baxter Lie algebras (of weight zero) is that they produce pre-Lie algebras, defined to be a vector space $A$ with a binary operation $\circ$ satisfying

$$(x \circ y) \circ z - x \circ (y \circ z) = (y \circ x) \circ z - y \circ (x \circ z), \quad \forall x, y, z \in A. \quad (3)$$
Pre-Lie algebras, also called left-symmetric algebras, originated from diverse areas of study, including convex homogeneous cones [34], affine manifolds and affine structures on Lie groups [22], deformation of associative algebras [19], and then appear in many more fields in mathematics and mathematical physics, such as symplectic and Kähler structures on Lie groups [15, 25], vertex algebras [8], quantum field theory [16] and operads [13]. From the operadic viewpoint, the pre-Lie algebra is the splitting (successor) of the Lie algebra [4]. See [12] and the references therein for more details.

By [20], for a Rota-Baxter Lie algebra \((g, [-, -], P)\) of weight zero, in defining
\[
x \circ y = [P(x), y], \quad \forall x, y \in g,
\]
we obtain a pre-Lie algebra \((g, \circ)\), called the **induced pre-Lie algebra** from the Rota-Baxter Lie algebra \((g, [-, -], P)\).

1.1.4. **Rota-Baxter Lie algebras and Lie bialgebras.** The Lie bialgebra is the algebraic structure corresponding to a Poisson-Lie group. It is also the classical structure of a quantized universal enveloping algebra [14, 17]. The great importance of Lie bialgebras is reflected by its close relationship with several other fundamental notions. First Lie bialgebras are characterized by Manin triples and matched pairs of Lie algebras [18]. In fact, there is a one-one correspondence between Lie bialgebras and Manin triples of Lie algebras. The same holds for Lie bialgebras and matched pairs of Lie algebras associated to coadjoint representations.

Furthermore, antisymmetric solutions of the CYBE, or the classical \(r\)-matrices, naturally give rise to coboundary Lie bialgebras [14, 31]. Furthermore, such solutions are provided by \(\mathcal{O}\)-operators which in turn are provided by pre-Lie algebras. In particular, as the \(\mathcal{O}\)-operators associated to adjoint representations, Rota-Baxter operators provide antisymmetric solutions of the CYBE in the bigger Lie algebras and hence give rise to Lie bialgebras. See [2, 3, 12, 23] for more details.

As a summary, the following diagram illustrates the close relationship of Rota-Baxter Lie algebras and \(\mathcal{O}\)-operators with CYBE, pre-Lie algebras and Lie bialgebras. Here the correspondences going both ways are shown by arrows in both directions, and the one-one correspondences are shown by bi-directional double arrows.

![Diagram](attachment:image.png)
be given by a bialgebra structure on Rota-Baxter Lie algebras and shed further light on giving more solutions of the quantum Yang-Baxter equation. While little is known about the (associative) bialgebra structure of Rota-Baxter Lie algebras in the universal enveloping algebras, Rota-Baxter Lie bialgebras constructed here might be regarded as an infinitesimal variation thereof.

Preferably, such a Rota-Baxter operator on Lie bialgebras should be part of a package of Rota-Baxter type actions applied systematically throughout the diagram in (5). This is what we will achieve, as depicted in the following diagram with the corresponding Lie bialgebra boldfaced in both diagrams.

Thus in this paper, we first establish a bialgebra theory for Rota-Baxter Lie algebras following the Manin triple approach of Lie bialgebras [11] and Rota-Baxter ASI bialgebras given in [6]. Most of the results in [6] for Rota-Baxter associative algebras remain valid for Rota-Baxter Lie algebras. Therefore by adding the roles of Rota-Baxter operators in diagram (5), we get the left part of the diagram (6).

For a Lie bialgebra given by a Manin triple \((\mathfrak{g} \oplus \mathfrak{g}^*, \mathfrak{g}, \mathfrak{g}^*)\) together with a Rota-Baxter operator \(P\) on \(\mathfrak{g}\), in building a Rota-Baxter Lie bialgebra on top of this, we allow freedom on a Rota-Baxter operator \(Q^*\) on the Lie algebra \(\mathfrak{g}^*\), as long as \(P\) and \(Q^*\) satisfy certain admissibility condition. This general approach allows us to recover the previous constructions as special cases: the Rota-Baxter Lie bialgebra independently developed in [24] corresponds to the case when \(Q = -P - \text{id}\) and the weight \(\lambda = 1\), while the Rota-Baxter Lie bialgebra introduced in [32] corresponds to the case when \(Q = -P\) and \(\lambda = 0\).

Another interesting phenomenon in our construction is reflected in the induced structures from Rota-Baxter Lie bialgebras. Since a Rota-Baxter Lie algebra induces a pre-Lie algebra by Eq. (4), it is natural to expect that a Rota-Baxter Lie bialgebra induces a pre-Lie bialgebra as defined in [3]. As it turns out, this is not the case. To determine the correct induced structure from a Rota-Baxter Lie bialgebra, we regard the latter as a Lie bialgebra with a Rota-Baxter operator. Since a Lie bialgebra is characterized by a matched pair or a Manin triple of Lie algebras, as shown in the right column in the diagram (5), we can also regard a Rota-Baxter Lie bialgebra as a matched pair or Manin triple of Lie algebras equipped with a Rota-Baxter operator. Since a Rota-Baxter operator on a Lie algebra induces a pre-Lie algebra, a Rota-Baxter operator on a matched pair or Manin triple of Lie algebras should induce a matched pair or Manin triple of pre-Lie algebras. Quite unexpectedly to us, the resulting bialgebra structure is not a pre-Lie bialgebra as introduced in [3], but a special L-dendriform bialgebra [5], leading to the commutative double square diagram in (6).

As it turns out, the class of special L-dendriform bialgebra is quite interesting on its own right as a subclass of L-dendriform bialgebras [28] and has already been studied in some
depth. The operad of L-dendriform algebras [7] is the two-fold splitting (successor) of the operad of Lie algebras (see Remark 2.16). Thus the correspondences in the right column of the diagram in (6) suggest that the splitting of a bialgebra structure is the bialgebra of the two-fold splitting of the structure. Moreover, the special L-dendriform bialgebra has a clear and important geometric interpretation. Since the study of Lie groups with a left-invariant flat pseudo-metric is equivalent to the study of the pre-Lie algebras with a nondegenerate symmetric left-invariant bilinear form [27], the special L-dendriform bialgebras correspond to a class of Lie groups with a left-invariant flat pseudo-metric.

1.3. Layout of the paper. As outlined in the diagram (6), the paper is organized as follows.

In Section 2, we give the notion of representations of Rota-Baxter Lie algebras. An admissibility of a linear operator for a Rota-Baxter Lie algebra is introduced in order to construct a reasonable representation on the dual space. We also observe that an invariant bilinear form on a Rota-Baxter Lie algebra of weight zero is left-invariant on the induced pre-Lie algebra. Moreover, we interpret special L-dendriform algebras in terms of the representations of pre-Lie algebras and hence a Rota-Baxter Lie algebra of weight zero with a linear operator satisfying the aforementioned admissibility condition gives a special L-dendriform algebra which is compatible with the induced pre-Lie algebra.

In Section 3, we introduce the notion of Rota-Baxter Lie bialgebras, equivalently characterized by Manin triples of Rota-Baxter Lie algebras and matched pairs of Rota-Baxter Lie algebras. We establish the explicit relationship between Rota-Baxter Lie bialgebras of weight zero and special L-dendriform bialgebras, previously established in [5], both directly and in their respective equivalent interpretations in terms of the corresponding Manin triples and matched pairs.

In Section 4, we focus on coboundary Rota-Baxter Lie bialgebras, leading to the introduction of the admissible CYBE in Rota-Baxter Lie algebras whose antisymmetric solutions are used to construct Rota-Baxter Lie bialgebras. The notions of O-operators on Rota-Baxter Lie algebras and Rota-Baxter pre-Lie algebras, are introduced to produce antisymmetric solutions of the admissible CYBE and hence Rota-Baxter Lie bialgebras. Furthermore, when the weight is zero, we study the induced special L-dendriform bialgebras from these Rota-Baxter Lie bialgebras, and thus give the construction of special L-dendriform bialgebras from antisymmetric solutions of the admissible CYBE in Rota-Baxter Lie algebras of weight zero. In particular, both Rota-Baxter Lie algebras of weight zero and Rota-Baxter pre-Lie algebras of weight zero give rise to special L-dendriform bialgebras.

Notations. Unless otherwise specified, all the vector spaces and algebras are finite dimensional over a field $\mathbb{K}$ of characteristic zero, although many results and notions, in particular that of a Rota-Baxter Lie bialgebra, remain valid in the infinite-dimensional case.

2. Rota-Baxter Lie algebras, pre-Lie algebras and special L-dendriform algebras

We give the notion of representations of Rota-Baxter Lie algebras. An admissibility of a linear map for a Rota-Baxter Lie algebra is introduced in order to obtain a reasonable representation on the dual space. On the other hand, we observe that an invariant bilinear form on a Rota-Baxter Lie algebra of weight zero is left-invariant on the induced pre-Lie algebra. Moreover, we interpret special L-dendriform algebras in terms of the representations of pre-Lie algebras. Hence a Rota-Baxter Lie algebra of weight zero with a linear operator satisfying the aforementioned admissibility condition gives a special L-dendriform algebra.
2.1. Rota-Baxter Lie algebras and their representations.

We first recall some basic facts on the representations of Lie algebras. For a Lie algebra \( \mathfrak{g} := (\mathfrak{g}, [-, -]) \), a representation of \( \mathfrak{g} \) is a pair \((V, \rho)\) consisting of a vector space \( V \) and a Lie algebra homomorphism
\[
\rho : \mathfrak{g} \to \mathfrak{gl}(V),
\]
for the natural Lie algebra structure on \( \mathfrak{gl}(V) = \text{End}(V) \).

In particular, the linear map
\[
ad : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}), \quad \text{ad}(x)y = [x, y], \quad \forall x, y \in \mathfrak{g},
\]
defines a representation \((\mathfrak{g}, \text{ad})\) of \((\mathfrak{g}, [-, -])\), called the adjoint representation.

For a vector space \( V \) and a linear map \( \rho : \mathfrak{g} \to \mathfrak{gl}(V) \), the pair \((V, \rho)\) is a representation of \((\mathfrak{g}, [-, -])\) if and only if the operation \([-,-]_{\mathfrak{g}\oplus V}\) (often still denoted by \([-,-]\) for simplicity) on \( \mathfrak{g} \oplus V \) defined by
\[
[x + u, y + v]_{\mathfrak{g}\oplus V} := [x, y] + \rho(x)v - \rho(y)u, \quad \forall x, y \in \mathfrak{g}, u, v \in V,
\]
makes \( \mathfrak{g} \oplus V \) into a Lie algebra, called the semi-direct product of \((\mathfrak{g}, [-,-])\) by \( V \), and denoted by \( \mathfrak{g} \ltimes \rho V \).

Let \( A \) and \( V \) be vector spaces. For a linear map \( \rho : A \to \text{End}(V) \), we set \( \rho^* : A \to \text{End}(V^*) \) by
\[
\langle \rho^*(x)v^*, u \rangle = -\langle v^*, \rho(x)u \rangle, \quad \forall x \in A, u \in V, v^* \in V^*.
\]
Here \( \langle , \rangle \) is the usual pairing between \( V \) and \( V^* \). If \((V, \rho)\) is a representation of a Lie algebra \((\mathfrak{g}, [-,-])\), then \((V^*, \rho^*)\) is also a representation of \((\mathfrak{g}, [-,-])\), called the dual representation of \((V, \rho)\). In particular, \((\mathfrak{g}^*, \text{ad}^*)\) is a representation of \((\mathfrak{g}, [-,-])\).

When we extend these notions to Rota-Baxter Lie algebras next, special attention needs to be given to the dual representations.

**Definition 2.1.** A representation of a Rota-Baxter Lie algebra \((\mathfrak{g}, [-,-], P)\) is a triple \((V, \rho, \alpha)\), such that \((V, \rho)\) is a representation of the Lie algebra \((\mathfrak{g}, [-,-])\) and \( \alpha : V \to V \) is a linear map satisfying the following equation:
\[
\rho(P(x))\alpha(v) = \alpha(\rho(P(x))v) + \alpha(\rho(x)\alpha(v)) + \lambda \alpha(\rho(x)v), \quad \forall x \in \mathfrak{g}, v \in V.
\]
Two representations \((V_1, \rho_1, \alpha_1)\) and \((V_2, \rho_2, \alpha_2)\) of a Rota-Baxter Lie algebra \((\mathfrak{g}, [-,-], P)\) are called equivalent if there exists a linear isomorphism \( \varphi : V_1 \to V_2 \) such that
\[
\varphi(\rho_1(x)v) = \rho_2(x)\varphi(v), \quad \varphi(\alpha_1(v)) = \alpha_2(\varphi(v)), \quad \forall x \in \mathfrak{g}, v \in V_1.
\]

**Example 2.2.** Let \((\mathfrak{g}, [-,-], P)\) be a Rota-Baxter Lie algebra of weight \( \lambda \).

(a) \((\mathfrak{g}, \text{ad}, P)\) is a representation of \((\mathfrak{g}, [-,-], P)\), called the adjoint representation of \((\mathfrak{g}, [-,-], P)\).

(b) Let \((V, \rho)\) be a representation of the Lie algebra \((\mathfrak{g}, [-,-])\). Then \((V, \rho, -\lambda \text{id}_V)\) are representations of \((\mathfrak{g}, [-,-], P)\).

For vector spaces \( V_1 \) and \( V_2 \) and linear maps \( \phi_1 : V_1 \to V_1 \) and \( \phi_2 : V_2 \to V_2 \), let \( \phi_1 + \phi_2 \) denote the linear map:
\[
\phi_{V_1 \oplus V_2} : V_1 \oplus V_2 \to V_1 \oplus V_2, \quad \phi_{V_1 \oplus V_2}(v_1 + v_2) = \phi_1(v_1) + \phi_2(v_2), \quad \forall v_1 \in V_1, v_2 \in V_2.
\]

The representations of Rota-Baxter Lie algebras are characterized as follows.
Proposition 2.3. Let \((g, [-, -], P)\) be a Rota-Baxter Lie algebra of weight \(\lambda\) and \(V\) be a vector space. Let \(\rho : g \to \text{End}(V)\) and \(\alpha : V \to V\) be linear maps. Then \((V, \rho, \alpha)\) is a representation of \((g, [-, -], P)\) if and only if, for the Lie algebra \((g \oplus V, [-, -]_{g \oplus V})\) defined in Eq. (7) and the linear map \(P + \alpha\) defined in Eq. (11), the triple \((g \oplus V, [-, -]_{g \oplus V}, P + \alpha)\) is a Rota-Baxter Lie algebra of weight \(\lambda\). In this case, the resulting Rota-Baxter Lie algebra is denoted by \((g \ltimes \rho V, [-, -]_{g \oplus V}, P + \alpha)\) and called the semi-direct product of \((g, [-, -], P)\) by \((V, \rho, \alpha)\).

The proof is omitted since the proposition is a special case of the matched pairs of Rota-Baxter Lie algebras in Proposition 3.5, when \(B = V\) is equipped with the zero operation.

For a representation of a Rota-Baxter Lie algebra, in order to obtain a representation of the Rota-Baxter Lie algebra on the dual space, an additional condition is needed.

Let \(V\) and \(W\) be vector spaces. For a linear map \(T : V \to W\), the transpose map \(T^* : W^* \to V^*\) is characterized by

\[
\langle T^*(w^*), v \rangle = \langle w^*, T(v) \rangle, \quad \forall v \in V, w^* \in W^*.
\]

Lemma 2.4. Let \((g, [-, -], P)\) be a Rota-Baxter Lie algebra of weight \(\lambda\), \((V, \rho)\) be a representation of \((g, [-, -])\), and \(\beta : V \to V\) be a linear map. Then \((V^*, \rho^*, \beta^*)\) is a representation of \((g, [-, -], P)\) if and only if

\[
\beta(\rho(P(x))v) - \rho(P(x))\beta(v) - \beta(\rho(x)\beta(v)) - \lambda \rho(x)\beta(v) = 0, \quad \forall x \in g, v \in V.
\]

In particular, for a linear map \(Q : g \to g\), the triple \((g^*, \text{ad}^*, Q^*)\) is a representation of \((g, [-, -], P)\) if and only if

\[
Q([P(x), y]) - [P(x), Q(y)] - Q([x, Q(y)]) - \lambda [x, Q(y)] = 0, \quad \forall x, y \in g.
\]

Proof. The proof of the first claim is similar to that of [6, Lemma 2.11]. The second claim follows from the first by taking the adjoint representation. 

Definition 2.5. Let \((g, [-, -], P)\) be a Rota-Baxter Lie algebra of weight \(\lambda\), \((V, \rho)\) be a representation of \((g, [-, -])\), and \(\beta : V \to V\) be a linear map. If \((V^*, \rho^*, \beta^*)\) is a representation of \((g, [-, -], P)\) (that is, Eq. (12) holds), then we say that \(\beta\) is admissible to the Rota-Baxter Lie algebra \((g, [-, -], P)\) on \((V, \rho)\), or \((g, [-, -], P)\) is \(\beta\)-admissible on \((V, \rho)\). In particular, if there is a linear map \(Q : g \to g\) satisfying Eq. (13) such that \((g^*, \text{ad}^*, Q^*)\) is a representation of \((g, [-, -], P)\), we simply say that \(Q\) is admissible to \((g, [-, -], P)\) or \((g, [-, -], P)\) is \(Q\)-admissible.

Proposition 2.6. Let \((g, [-, -], P)\) be a Rota-Baxter Lie algebra of weight \(\lambda\) and \((V, \rho, \alpha)\) be a representation of \((g, [-, -], P)\). Then \(-\alpha - \lambda \text{id}_V\) is admissible to \((g, [-, -], P)\) on \((V, \rho)\). In particular, for any representation \((V, \rho)\) of the Lie algebra \((g, [-, -])\), both \(-\lambda \text{id}_V\) and \(0\) are admissible to \((g, [-, -], P)\) on \((V, \rho)\).

Proof. Note that Eq. (12) holds by taking \(\beta = -\alpha - \lambda \text{id}_V\). Hence the first conclusion follows. The other conclusions follow by taking \(\alpha = 0\) and \(\alpha = -\lambda \text{id}_V\) respectively. 

Taking the adjoint representation in Proposition 2.6, we obtain

Corollary 2.7. Let \((g, [-, -], P)\) be a Rota-Baxter Lie algebra of weight \(\lambda\). Then the linear maps \(-P - \lambda \text{id}_g\), \(-\lambda \text{id}_g\) and \(0\) are admissible to \((g, [-, -], P)\).
We next give admissible representations by invariant bilinear forms on Lie algebras. Recall that a bilinear form $\mathcal{B}$ on a Lie algebra $(\mathfrak{g}, [-,-])$ is called **invariant** if
\[
\mathcal{B}([x,y], z) = \mathcal{B}(x, [y,z]), \quad \forall x, y, z \in \mathfrak{g}.
\] (14)

**Proposition 2.8.** Let $(\mathfrak{g}, [-,-], P)$ be a Rota-Baxter Lie algebra and $\mathcal{B}$ be a nondegenerate invariant bilinear form on the Lie algebra $(\mathfrak{g}, [-,-])$. Let $\tilde{P}$ be the adjoint linear map of $P$ with respect to $\mathcal{B}$, that is, $\tilde{P}$ is characterized by
\[
\mathcal{B}(P(x), y) = \mathcal{B}(x, \tilde{P}(y)), \quad \forall x, y \in \mathfrak{g}.
\] (15)
Then $\tilde{P}$ is admissible to $(\mathfrak{g}, [-,-], P)$, or equivalently, $(\mathfrak{g}^*, \text{ad}^*, \tilde{P}^*)$ is a representation of $(\mathfrak{g}, [-,-], P)$. Moreover, $(\mathfrak{g}^*, \text{ad}^*, \tilde{P}^*)$ is equivalent to $(\mathfrak{g}, \text{ad}, P)$ as a representation of $(\mathfrak{g}, [-,-], P)$.

Conversely, let $(\mathfrak{g}, [-,-], P)$ be a Rota-Baxter Lie algebra and $Q : \mathfrak{g} \to \mathfrak{g}$ be a linear map that is admissible to $(\mathfrak{g}, [-,-], P)$. If the resulting representation $(\mathfrak{g}^*, \text{ad}^*, Q^*)$ of $(\mathfrak{g}, [-,-], P)$ is equivalent to $(\mathfrak{g}, \text{ad}, P)$, then there exists a nondegenerate invariant bilinear from $\mathcal{B}$ on $(\mathfrak{g}, [-,-], P)$ such that $\tilde{Q} = P$.

**Proof.** The proof is similar to that of [6, Proposition 3.9]. Note that in the context of Lie algebras, by the antisymmetry of the Lie bracket, the bilinear form $\mathcal{B}$ no longer needs to be symmetric. \hfill $\square$

### 2.2. Pre-Lie algebras and special L-dendriform algebras.

We first recall some facts on pre-Lie algebras [12]. For a pre-Lie algebra $(A, \circ)$ as defined in Eq. (3), the commutator
\[
[x,y] = x \circ y - y \circ x, \quad \forall x, y \in A,
\] (16)
defines a Lie algebra $(\mathfrak{g}(A), [-,-])$, called the **sub-adjacent** Lie algebra of $(A, \circ)$, and $(A, \circ)$ is called a **compatible** pre-Lie algebra structure on the Lie algebra $(\mathfrak{g}(A), [-,-])$.

For a vector space $A$ with a binary operation $\circ : A \otimes A \to A$, define linear maps
\[
L_\circ, R_\circ : A \to \text{End}(A), \quad L_\circ(x)(y) := x \circ y =: R_\circ(y)x, \quad \forall x, y \in A.
\]
Then for a pre-Lie algebra $(A, \circ)$, $(A, L_\circ)$ is a representation of the sub-adjacent Lie algebra $(\mathfrak{g}(A), [-,-])$.

**Definition 2.9.** A bilinear form $\mathcal{B}$ on a pre-Lie algebra $(A, \circ)$ is called **left-invariant** if
\[
\mathcal{B}(x \circ y, z) + \mathcal{B}(y, x \circ z) = 0, \quad \forall x, y, z \in A.
\] (17)

**Remark 2.10.** The references [1, 27] give a natural bijection between the set of the pre-Lie algebras with a nondegenerate symmetric left-invariant bilinear form and the set of the connected and simply-connected Lie groups with a left-invariant flat pseudo-metric. Under correspondence, the sub-adjacent Lie algebra of a pre-Lie algebra in the former set is precisely the Lie algebra of the corresponding Lie group.

**Proposition 2.11.** Let $(\mathfrak{g}, [-,-], P)$ be a Rota-Baxter Lie algebra of weight zero and $(\mathfrak{g}, \circ)$ be the induced pre-Lie algebra. If there is an invariant bilinear form $\mathcal{B}$ on the Lie algebra $(\mathfrak{g}, [-,-])$, then $\mathcal{B}$ is left-invariant on the pre-Lie algebra $(\mathfrak{g}, \circ)$.

**Proof.** For all $x, y, z \in \mathfrak{g}$, we have
\[
\mathcal{B}(x \circ y, z) = \mathcal{B}([P(x), y], z) = -\mathcal{B}(y, [P(x), z]) = -\mathcal{B}(y, x \circ z).
\]
Hence the conclusion holds. \hfill $\square$
**Definition 2.12.** A representation of a pre-Lie algebra \((A, \circ)\) is a triple \((V, l_\circ, r_\circ)\), where \(V\) is a vector space, and \(l_\circ, r_\circ : A \to \text{End}(V)\) are linear maps satisfying

\[
\begin{align*}
l_\circ(x)l_\circ(y)v - l_\circ(x \circ y)v & = l_\circ(y)l_\circ(x)v - l_\circ(y \circ x)v, \quad (18) \\
l_\circ(x)r_\circ(y)v - r_\circ(y)l_\circ(x)v & = r_\circ(y)(x \circ y)v - r_\circ(y)l_\circ(x)v, \quad (19)
\end{align*}
\]

for all \(x, y \in A, v \in V\).

In fact, \((V, l_\circ, r_\circ)\) is a representation of a pre-Lie algebra \((A, \circ)\) if and only if the direct sum \(A \oplus V\) of vector spaces is equipped with a (semi-direct product) pre-Lie algebra structure by the multiplication on \(A \oplus V\) defined by

\[
(x + u) \circ (y + v) := x \circ y + l_\circ(x)v + r_\circ(y)u, \quad \forall x, y \in A, u, v \in V. \quad (20)
\]

We denote the resulting pre-Lie algebra by \(A \ltimes_{l_\circ,r_\circ} V\) or simply \(A \ltimes V\). A representation of a pre-Lie algebra also has a naturally defined dual representation.

**Lemma 2.13.** [3] Let \((V, l_\circ, r_\circ)\) be a representation of a pre-Lie algebra \((A, \circ)\). Then \((V^*, l_\circ^* - r_\circ^*, r_\circ^*)\) is a representation of \((A, \circ)\).

A representation of a Rota-Baxter Lie algebra gives rise to a representation of the induced pre-Lie algebra.

**Proposition 2.14.** Let \((V, \rho, \alpha)\) be a representation of a Rota-Baxter Lie algebra \((g, [-, -], P)\) of weight zero. Define \(l_{\rho,\alpha}, r_{\rho,\alpha} : g \to \text{End}(V)\) by

\[
l_{\rho,\alpha}(x)v = \rho(P(x))v, \quad r_{\rho,\alpha}(x)v = -\rho(x)\alpha(v), \quad \forall x \in g, v \in V. \quad (21)
\]

Then \((V, l_{\rho,\alpha}, r_{\rho,\alpha})\) is a representation of the induced pre-Lie algebra \((g, \circ)\), called the induced representation of the pre-Lie algebra \((g, \circ)\) from \((V, \rho, \alpha)\).

**Proof.** By Proposition 2.3, there is a Rota-Baxter Lie algebra \((g \ltimes_{\rho} V, P + \alpha)\) of weight zero. Hence by Eq. (4), there is an induced pre-Lie algebra structure on \(g \oplus V\), defined by

\[
(x + u) \circ (y + v) = [(P + \alpha)(x + u), y + v] = [P(x), y] + \rho(P(x))v - \rho(y)\alpha(u)
\]

for all \(x, y \in g, u, v \in V\). Thus \((V, l_{\rho,\alpha}, r_{\rho,\alpha})\) is a representation of \((g, \circ)\). \(\square\)

We next recall the notions of dendriform algebras and special dendriform algebras.

**Definition 2.15.** [5, 7] An L-dendriform algebra is a triple \((A, \triangleright, \triangleleft)\), such that \(A\) is a vector space, and \(\triangleright, \triangleleft : A \otimes A \to A\) are linear maps satisfying

\[
\begin{align*}
(x \triangleright y) \triangleright z + (x \triangleleft y) \triangleright z + y \triangleright (x \triangleright z) - (y \triangleleft x) \triangleright z - (y \triangleright x) \triangleright z - x \triangleright (y \triangleright z) & = 0, \quad (22) \\
(x \triangleright y) \triangleleft z + y \triangleleft (x \triangleright z) + y \triangleleft (x \triangleleft z) - (y \triangleleft x) \triangleleft z - x \triangleright (y \triangleleft z) & = 0, \quad (23)
\end{align*}
\]

for all \(x, y, z \in A\). An L-dendriform algebra is called special if \(\triangleleft\) is antisymmetric.

**Remark 2.16.**

(a) The operad of L-dendriform algebras is the successor [4] of the operad pre-Lie of pre-Lie algebras. Thus it is also the Manin black product pre-Lie • pre-Lie [4, Corollary 3.5].

(b) A Rota-Baxter operator \(P\) of weight zero on a pre-Lie algebra \((A, \circ)\) gives rise to an L-dendriform algebra with the multiplications \([7, Corollary 3.9]\)

\[
x \triangleright y := P(x) \circ y, \quad x \triangleleft y := -y \circ P(x), \quad \forall x, y \in A.
\]

In a similar manner, a commuting pair of Rota-Baxter operators of weight zero on a Lie algebra gives rise to an L-dendriform algebra [7, Corollary 3.10].
Furthermore [5, 7], for an L-dendriform algebra \((A, \rhd, \triangleleft)\), there are pre-Lie algebras \((A, \circ)\) and \((A, \star)\) given by
\[
x \circ y = x \rhd y - y \triangleleft x, \quad x \star y = x \rhd y + x \triangleleft y, \quad \forall x, y \in A,
\]
called the horizontal and vertical pre-Lie algebras respectively. Moreover, if (and only if under our assumption of characteristic zero) \((A, \rhd, \triangleleft)\) is special, then the horizontal pre-Lie algebra \((A, \circ)\) and the vertical pre-Lie algebra \((A, \star)\) coincide, that is, \(x \circ y = x \star y, \forall x, y \in A\). In this case, \((A, \circ)\) is called the sub-adjacent pre-Lie algebra of \((A, \rhd, \triangleleft)\), and \((A, \rhd, \triangleleft)\) is called a compatible special L-dendriform algebra of \((A, \circ)\).

We now interpret special L-dendriform algebras in terms of representations of pre-Lie algebras.

**Proposition 2.17.** Let \((A, \circ)\) be a pre-Lie algebra. Suppose that \(\triangleleft : A \otimes A \to A\) is an antisymmetric multiplication on \(A\). Define a multiplication \(\rhd\) on \(A\) by
\[
x \rhd y \eqdef x \circ y - x \triangleleft y, \quad \forall x, y \in A.
\]
Then the following statements are equivalent.

(a) \((A, \rhd, \triangleleft)\) is a special L-dendriform algebra;

(b) The following equation holds:
\[
x \triangleleft (y \triangleleft z) + y \triangleleft (x \circ z) - z \triangleleft (x \circ y) - x \circ (y \triangleleft z) = 0, \quad \forall x, y, z \in A;
\]

(c) \((A, \mathcal{L}_o - \mathcal{L}_o, -\mathcal{L}_o)\) is a representation of \((A, \circ)\);

(d) \((A^*, \mathcal{L}_o^*, -\mathcal{L}_o^*)\) is a representation of \((A, \circ)\).

**Proof.** (a) \(\iff\) (b). Let \(x, y, z \in A\). Then we have
\[
(x \rhd y) \triangleleft z + (x \triangleleft y) \rhd z + y \rhd (x \circ z) - (y \rhd x) \triangleleft z - x \rhd (y \triangleleft z) = (x \circ y) \triangleleft z + y \triangleleft (x \circ z) - x \circ (y \triangleleft z) + x \triangleleft (y \triangleleft z).
\]
Thus Eq. (23) holds if and only if Eq. (26) holds. Moreover, if this is the case, then we have
\[
(x \rhd y) \rhd z + (x \triangleleft y) \rhd z + y \rhd (x \circ z) - (y \rhd x) \rhd z - (y \rhd x) \rhd z - x \rhd (y \rhd z) = (x \circ y) \triangleleft z + (y \circ x) \triangleleft z - x \circ (y \circ z) + y \triangleleft (x \circ z) + y \triangleleft (x \circ z) - x \triangleleft (y \circ z) - x \triangleleft (y \triangleleft z) = 0.
\]
Hence Eq. (22) holds automatically.

(b) \(\iff\) (c). For \(x, y, z \in A\), we have
\[
-(\mathcal{L}_o - \mathcal{L}_o)(x)\mathcal{L}_o(y)z + \mathcal{L}_o(y)(\mathcal{L}_o - \mathcal{L}_o)(x)z + \mathcal{L}_o(x \circ y)z + \mathcal{L}_o(y)\mathcal{L}_o(x)z = -x \circ (y \triangleleft z) + x \triangleleft (y \triangleleft z) + y \triangleleft (x \circ z) + (x \circ y) \triangleleft z.
\]
Thus Eq. (19) holds for the triple \((A, \mathcal{L}_o - \mathcal{L}_o, -\mathcal{L}_o)\) if and only if Eq. (26) holds. Furthermore, in this case, we have
\[
(\mathcal{L}_o - \mathcal{L}_o)(x)(\mathcal{L}_o - \mathcal{L}_o)(y)z - (\mathcal{L}_o - \mathcal{L}_o)(x \circ y)z - (\mathcal{L}_o - \mathcal{L}_o)(y)(\mathcal{L}_o - \mathcal{L}_o)(x)z + (\mathcal{L}_o - \mathcal{L}_o)(y \circ x)z = x \circ (y \triangleleft z) - x \triangleleft (y \triangleleft z) - x \circ (y \triangleleft z) + x \triangleleft (y \triangleleft z) - (x \circ y) \triangleleft z - (x \circ y) \triangleleft z + (x \circ y) \triangleleft z - x \triangleleft (y \triangleleft z) - (y \circ x) \triangleleft z - (y \circ x) \triangleleft z = 0.
\]
Thus Eq. (18) holds for the triple \((A, \mathcal{L}_o - \mathcal{L}_o, -\mathcal{L}_o)\).

(c) \(\iff\) (d). It follows from Lemma 2.13. \(\square\)
Proposition 2.18. Let $(\mathfrak{g}, [-, -], P)$ be a Rota-Baxter Lie algebra of weight zero and $(\mathfrak{g}, \circ)$ be the induced pre-Lie algebra. Let $Q : \mathfrak{g} \to \mathfrak{g}$ be a linear map that is admissible to $(\mathfrak{g}, [-, -], P)$.

(a) The triple $(\mathfrak{g}^*, L_{\ad^*, Q^*}, r_{\ad^*, Q^*})$ is a representation of the pre-Lie algebra $(\mathfrak{g}, \circ)$.

(b) Define an operation $\triangleleft : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ on $\mathfrak{g}$ by

$$x \triangleleft y = -Q([x, y]), \ \forall x, y \in \mathfrak{g}. \quad (27)$$

Then $L_{\ad^*, Q^*} = L^*_\circ, r_{\ad^*, Q^*} = L^*_\circ.$

(c) Define an operation $\triangleright : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ on $\mathfrak{g}$ by Eq. (25). Then $(\mathfrak{g}, \triangleright, \triangleleft)$ is a compatible special L-dendriform algebra of the pre-Lie algebra $(\mathfrak{g}, \circ)$.

Proof. (a) Since $Q$ is admissible to $(\mathfrak{g}, [-, -], P)$, $(\mathfrak{g}^*, \ad^*, Q^*)$ is a representation of the Rota-Baxter Lie algebra $(\mathfrak{g}, [-, -], P)$. Hence by Proposition 2.14, $(\mathfrak{g}^*, L_{\ad^*, Q^*}, r_{\ad^*, Q^*})$ is the induced representation of $(\mathfrak{g}, \circ)$ from $(\ad^*, Q^*; \mathfrak{g}^*)$.

(b) For all $x, y \in \mathfrak{g}, a^* \in \mathfrak{g}^*$, we have

\begin{align*}
\langle l_{\ad^*, Q^*} (x)a^*, y \rangle &= \langle \ad^*(P(x))a^*, y \rangle = -\langle a^*, [P(x), y] \rangle = -\langle a^*, x \circ y \rangle = \langle L^*_\circ (x)a^*, y \rangle, \\
\langle r_{\ad^*, Q^*} (x)a^*, y \rangle &= -\langle \ad^*(x)Q^* (a^*), y \rangle = \langle Q^* (a^*), [x, y] \rangle = \langle a^*, Q([x, y]) \rangle = \langle L^*_\circ (x)a^*, y \rangle.
\end{align*}

Hence $L_{\ad^*, Q^*} = L^*_\circ, r_{\ad^*, Q^*} = L^*_\circ.$

(c) By Items (a) and (b), $(\mathfrak{g}^*, L^*_\circ, L^*_\circ)$ is a representation of $(\mathfrak{g}, \circ)$. Thus $(\mathfrak{g}, \triangleright, \triangleleft)$ is a compatible special L-dendriform algebra of $(\mathfrak{g}, \circ)$ by Proposition 2.17. □

Corollary 2.19. Let $(\mathfrak{g}, [-, -], P)$ be a Rota-Baxter Lie algebra of weight zero and $(\mathfrak{g}, \circ)$ be the induced pre-Lie algebra.

(a) There is a compatible special L-dendriform algebra $(\mathfrak{g}, \triangleright, \triangleleft)$ of the pre-Lie algebra $(\mathfrak{g}, \circ)$ defined by

$$x \triangleleft y = P([x, y]), \quad x \triangleright y = x \circ y - x \triangleleft y = [P(x), y] - P([x, y]), \ \forall x, y \in \mathfrak{g}. \quad (28)$$

(b) Suppose that $B$ is a nondegenerate symmetric invariant bilinear form on $(\mathfrak{g}, [-, -])$. Let $\hat{P} : \mathfrak{g} \to \mathfrak{g}$ be the adjoint linear operator of $P$ with respect to $B$ as defined in Eq. (15). Then there is a compatible special L-dendriform algebra $(\mathfrak{g}, \triangleright, \triangleleft)$ of the pre-Lie algebra $(\mathfrak{g}, \circ)$ defined by

$$x \triangleleft y = -\hat{P}([x, y]), \quad x \triangleright y = x \circ y - x \triangleleft y = [P(x), y] + \hat{P}([x, y]), \ \forall x, y \in \mathfrak{g}. \quad (29)$$

Proof. (a). By Corollary 2.7, $-P$ is admissible to $(\mathfrak{g}, [-, -], P)$. Hence the conclusion follows from Proposition 2.18.

(b). By Proposition 2.8, $\hat{P}$ is admissible to $(\mathfrak{g}, [-, -], P)$. Then the conclusion follows from Proposition 2.18. □

Remark 2.20. By [7], for a Rota-Baxter Lie algebra $(\mathfrak{g}, [-, -], P)$ of weight zero and the induced pre-Lie algebra $(\mathfrak{g}, \circ)$, the operator $P$ is also a Rota-Baxter operator of weight zero on the pre-Lie algebra $(\mathfrak{g}, \circ)$. Further, for the binary operations

$$x \triangleright y = P(x) \circ y = [P^2(x), y], \quad x \triangleleft y = x \circ P(y) = [P(x), P(y)], \ \forall x, y \in \mathfrak{g},$$

the triple $(\mathfrak{g}, \triangleright, \triangleleft)$ is an L-dendriform algebra. Obviously, this L-dendriform algebra is special. In general, it is different from the special L-dendriform algebra defined by Eq. (28) and is not compatible with the pre-Lie algebra $(\mathfrak{g}, \circ)$.

We recall another lemma before the next application.
Lemma 2.21. [5] Let \((A, \circ)\) be a pre-Lie algebra with a nondegenerate symmetric left-invariant bilinear form \(B\). Then there is a compatible special L-dendriform algebra \((A, \triangleright, \triangleleft)\) with the multiplications \(\triangleright\) and \(\triangleleft\) defined by

\[
B(x \triangleleft y, z) = B(x, z \circ y), \quad x \triangleright y = x \circ y - x \triangleleft y, \quad \forall x, y, z \in A.
\]  

(30)

Corollary 2.22. Let \((\mathfrak{g}, [-,-], P)\) be a Rota-Baxter Lie algebra of weight zero and \((\mathfrak{g}, \circ)\) be the induced pre-Lie algebra. Suppose that \(B\) is a nondegenerate symmetric invariant bilinear form on \((\mathfrak{g}, [-,-])\). Let \(\hat{P} : \mathfrak{g} \rightarrow \mathfrak{g}\) be the adjoint linear operator of \(P\) with respect to \(B\). Then the special L-dendriform algebra defined by Eq. (29) coincides with the one defined by Eq. (30).

Proof. On the one hand, by Corollary 2.19 (b), there is a compatible special L-dendriform algebra \((\mathfrak{g}, \triangleright_1, \triangleleft_1)\) of the pre-Lie algebra \((\mathfrak{g}, \circ)\) defined by Eq. (29). On the other hand, by Proposition 2.11, \(B\) is left-invariant on the pre-Lie algebra \((\mathfrak{g}, \circ)\). Hence by Lemma 2.21, there is another compatible special L-dendriform algebra \((\mathfrak{g}, \triangleright_2, \triangleleft_2)\) of the pre-Lie algebra \((\mathfrak{g}, \circ)\) defined by Eq. (30). For all \(x, y, z \in \mathfrak{g}\), we have

\[
B(x \triangleleft_2 y, z) = B(x, z \circ_2 y) = B(x, [P(z), y]) = -B([x, y], P(z)) = -B(\hat{P}([x, y]), z) = B(x \triangleleft_1 y, z).
\]

Thus \(x \triangleleft_2 y = x \triangleleft_1 y\). Moreover

\[
x \triangleright_2 y = x \circ y - x \triangleleft_2 y = x \circ y - x \triangleleft_1 y = x \triangleright_1 y, \quad \forall x, y \in \mathfrak{g}.
\]

Hence the two special L-dendriform algebras \((\mathfrak{g}, \triangleright_1, \triangleleft_1)\) and \((\mathfrak{g}, \triangleright_2, \triangleleft_2)\) coincide. \(\square\)

Example 2.23. Let \((\mathfrak{g}, [-,-])\) be the 3-dimensional simple Lie algebra \(\mathfrak{sl}(2, \mathbb{K})\) with a basis \(\{x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, y = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\}\) and with the multiplication

\[
[h, x] = 2x, \quad [h, y] = -2y, \quad [x, y] = h.
\]

Define a linear operator \(P : \mathfrak{g} \rightarrow \mathfrak{g}\) by

\[
P(x) = x + y, \quad P(h) = 2h + 4y, \quad P(y) = x - 2h - 3y.
\]

Then \(P\) is a Rota-Baxter operator of weight zero on \((\mathfrak{g}, [-,-])\), and the induced pre-Lie algebra \((\mathfrak{g}, \circ)\) from Eq. (4) is given by

\[
x \circ x = -h, \quad x \circ h = -2x + 2y, \quad x \circ y = h, \quad h \circ x = 4x - 4h, \quad h \circ h = 8y, \quad h \circ y = -4y, \quad y \circ x = 3h - 4y, \quad y \circ y = h + 4y, \quad y \circ h = -2x - 6y.
\]

Moreover, there is a nondegenerate symmetric invariant bilinear form \(B\) on \((\mathfrak{g}, [-,-])\) whose nonzero values are

\[
B(x, y) = B(y, x) = 1, \quad B(h, h) = 2.
\]

The adjoint linear operator \(\hat{P}\) of \(P\) with respect to \(B\) is given by

\[
\hat{P}(x) = -3x + 2h + y, \quad \hat{P}(h) = -4x + 2h, \quad \hat{P}(y) = x + y.
\]

Thus a compatible special L-dendriform algebra \((\mathfrak{g}, \triangleright, \triangleleft)\) of the pre-Lie algebra \((\mathfrak{g}, \circ)\) is given by

\[
x \triangleleft h = -\hat{P}([x, h]) = -6x + 4h + 2y, \quad x \triangleleft y = -\hat{P}([x, y]) = 4x - 2h, \quad h \triangleleft y = -\hat{P}([h, y]) = 2x + 2y,
\]

and \(a \triangleright b = a \circ b - a \triangleleft b\) for all \(a, b \in \mathfrak{g}\). Also note that \(\hat{P}\) commutes with \(P\).
### 3. Rota-Baxter Lie Bialgebras and Special L-Dendriform Bialgebras

In this section we introduce the notion of Rota-Baxter Lie bialgebras which comes naturally from the notions of Manin triples of Rota-Baxter Lie algebras and matched pairs of Rota-Baxter Lie algebras. We establish the explicit relationship between Rota-Baxter Lie bialgebras of weight zero and special L-dendriform bialgebras introduced in [5]. Similar relations are also established from the Manin triples (resp. the matched pairs) of Rota-Baxter Lie algebras to those of special L-dendriform algebras.

#### 3.1. Rota-Baxter Lie bialgebras

Following the Manin triple approach to Lie bialgebras [17], we derive the notion of Rota-Baxter Lie bialgebras from Manin triples and matched pairs of Rota-Baxter Lie algebras.

**3.1.1. Manin triples of Rota-Baxter Lie algebras.**

Recall [14] that a (standard) Manin triple of Lie algebras is a triple \((g \oplus g^*, g, g^*)\) of Lie algebras, such that \((g, [-, -]_g)\) and \((g^*, [-, -]_{g^*})\) are Lie subalgebras of the Lie algebra \((g \oplus g^*, [-, -])\), and the natural nondegenerate symmetric bilinear form \(B_d\) on \((g \oplus g^*, [-, -])\) given by

\[
B_d(x + a^*, y + b^*) = \langle x, b^* \rangle + \langle a^*, y \rangle, \quad \forall x, y \in g, a^*, b^* \in g^*
\]

is invariant. We extend this notion to Rota-Baxter Lie algebras.

**Definition 3.1.** A Manin triple of Rota-Baxter Lie algebras is a triple \(((g \oplus g^*, P_{g \oplus g^*}), (g, P), (g^*, Q^*))\) of Rota-Baxter Lie algebras such that \((g \oplus g^*, g, g^*)\) is a Manin triple of Lie algebras and \(P_{g \oplus g^*} = P + Q^*\). Then we denote the Manin triple by \(((g \oplus g^*, P + Q^*), (g, P), (g^*, Q^*))\).

By definition, for a Manin triple of Rota-Baxter Lie algebras \(((g \oplus g^*, P + Q^*), (g, P), (g^*, Q^*))\), the triples \((g, [-, -]_g, P)\) and \((g^*, [-, -]_{g^*}, Q^*)\) are clearly Rota-Baxter Lie subalgebras of \((g \oplus g^*, [-, -], P + Q^*)\). Moreover, we have the following conclusion.

**Lemma 3.2.** Let \(((g \oplus g^*, P + Q^*), (g, P), (g^*, Q^*))\) be a Manin triple of Rota-Baxter Lie algebras.

(a) The adjoint \(P + Q^*\) of \(P + Q^*\) with respect to \(B_d\) is \(Q + P^*\). Further \(Q + P^*\) is admissible to \((g \oplus g^*, [-, -], P + Q^*)\).

(b) \(Q\) is admissible to \((g, [-, -]_g, P)\).

(c) \(P^*\) is admissible to \((g^*, [-, -]_{g^*}, Q^*)\).

**Proof.** The proof is similar to the one of [6, Lemma 3.11].

Then by Corollary 2.19 (b), we obtain

**Corollary 3.3.** Let \(((g \oplus g^*, P + Q^*), (g, P), (g^*, Q^*))\) be a Manin triple of Rota-Baxter Lie algebras of weight zero. There is a special L-dendriform algebra \((g \oplus g^*, \triangleright, \triangleleft)\) defined by

\[
(x + a^*) \triangleleft (y + b^*) = -(Q + P^*)([x + a^*, y + b^*]),
\]

\[
(x + a^*) \triangleright (y + b^*) = (x + a^*) \circ (y + b^*) - (x + a^*) \triangleleft (y + b^*), \quad \forall x, y \in g, a^*, b^* \in g^*,
\]

which contains \((g, \triangleright_g, \triangleleft_g)\) and \((g^*, \triangleright_{g^*}, \triangleleft_{g^*})\) as special L-dendriform subalgebras, where

\[
x \triangleleft_g y = -Q([x, y]_g), \quad x \triangleright_g y = x \circ_g y - x \triangleleft_g y, \quad \forall x, y \in g, \quad (34)
\]

\[
a^* \triangleleft_{g^*} b^* = -P^*([a^*, b^*]_{g^*}), \quad a^* \triangleright_{g^*} b^* = a^* \circ_{g^*} b^* - a^* \triangleleft_{g^*} b^*, \quad \forall a^*, b^* \in g^*. \quad (35)
\]
3.1.2. Matched pairs of Rota-Baxter Lie algebras.

Let \((\mathfrak{g}, [-,-])_G\) and \((\mathfrak{h}, [-,-])_H\) be Lie algebras, with their respective representations \((\mathfrak{h}, \rho_\mathfrak{h})\) and \((\mathfrak{g}, \rho_\mathfrak{g})\). Then \((\mathfrak{g}, \mathfrak{h}, \rho_\mathfrak{g}, \rho_\mathfrak{h})\) is called a matched pair of Lie algebras [26] if

\[
\rho_\mathfrak{g}(x)[a, b]_\mathfrak{h} - [\rho_\mathfrak{g}(x)a, b]_\mathfrak{h} - [a, \rho_\mathfrak{g}(y)b]_\mathfrak{h} + \rho_\mathfrak{g}(\rho_\mathfrak{h}(a)x)b - \rho_\mathfrak{g}(\rho_\mathfrak{h}(b)x)a = 0,
\]

\[
\rho_\mathfrak{h}(a)[x, y]_\mathfrak{g} - [\rho_\mathfrak{h}(a)x, y]_\mathfrak{g} - [x, \rho_\mathfrak{h}(a)y]_\mathfrak{g} + \rho_\mathfrak{h}(\rho_\mathfrak{g}(x)a)y - \rho_\mathfrak{h}(\rho_\mathfrak{g}(y)a)x = 0, \forall x, y \in \mathfrak{g}, a, b \in \mathfrak{h}.
\]

For Lie algebras \((\mathfrak{g}, [-,-])_G\) and \((\mathfrak{h}, [-,-])_H\) with linear maps \(\rho_\mathfrak{g} : \mathfrak{g} \to \text{End}(\mathfrak{h}), \rho_\mathfrak{h} : \mathfrak{h} \to \text{End}(\mathfrak{g})\), there is a Lie algebra structure on the vector space \(\mathfrak{g} \oplus \mathfrak{h}\) given by

\[
[x + a, y + b] = [x, y]_\mathfrak{g} + \rho_\mathfrak{g}(a)y - \rho_\mathfrak{h}(b)x + [a, b]_\mathfrak{h} + \rho_\mathfrak{g}(x)b - \rho_\mathfrak{g}(y)a, \forall x, y \in \mathfrak{g}, a, b \in \mathfrak{h}
\]

if and only if \((\mathfrak{g}, \mathfrak{h}, \rho_\mathfrak{g}, \rho_\mathfrak{h})\) is a matched pair of Lie algebras. We denote the resulting Lie algebra \((\mathfrak{g} \oplus \mathfrak{h}, [-,-])\) by \(\mathfrak{g} \bowtie \mathfrak{h}\) or simply \(\mathfrak{g} \bowtie \mathfrak{h}\).

We extend this construction to Rota-Baxter Lie algebras.

Definition 3.4. A matched pair of Rota-Baxter Lie algebras is a quadruple \(((\mathfrak{g}, P_\mathfrak{g}), (\mathfrak{h}, P_\mathfrak{h}), (\mathfrak{h}, P_\mathfrak{h}), \rho_\mathfrak{g}, \rho_\mathfrak{h})\), such that \((\mathfrak{g}, [-,-])_G\) and \((\mathfrak{h}, [-,-])_H\) are Rota-Baxter Lie algebras, \((\mathfrak{h}, \rho_\mathfrak{h}, P_\mathfrak{h})\) is a representation of \((\mathfrak{h}, [-,-])_H\), \((\mathfrak{g}, \rho_\mathfrak{g}, P_\mathfrak{g})\) is a representation of \((\mathfrak{h}, [-,-])_H\), and \((\mathfrak{g}, \mathfrak{h}, \rho_\mathfrak{g}, \rho_\mathfrak{h})\) is a matched pair of Lie algebras.

Adapting the argument for Theorem 3.4 in [6] from associative algebras to Lie algebras, we obtain

Proposition 3.5. Let \((\mathfrak{g}, [-,-])_G\) and \((\mathfrak{h}, [-,-])_H\) be Rota-Baxter Lie algebras and \((\mathfrak{g}, \mathfrak{h}, \rho_\mathfrak{g}, \rho_\mathfrak{h})\) be a matched pair of Lie algebras. With the resulting Lie algebra \(\mathfrak{g} \bowtie \mathfrak{h}\) given by Eq. (36), \(\mathfrak{g} \bowtie \mathfrak{h}, [-,-], P_\mathfrak{g} + P_\mathfrak{h}\) is a Rota-Baxter Lie algebra if and only if \(((\mathfrak{g}, P_\mathfrak{g}), (\mathfrak{h}, P_\mathfrak{h}), (\mathfrak{h}, P_\mathfrak{h}), \rho_\mathfrak{g}, \rho_\mathfrak{h})\) is a matched pair of the Rota-Baxter Lie algebras \((\mathfrak{g}, [-,-])_G, P_\mathfrak{g}\) and \((\mathfrak{h}, [-,-])_H, P_\mathfrak{h}\).

For Lie algebras \((\mathfrak{g}, [-,-])_G\) and \((\mathfrak{g}^*, [-,-])_G^*\), there is a Manin triple of Lie algebras \((\mathfrak{g} \oplus \mathfrak{g}^*, \mathfrak{g} \bowtie \mathfrak{g}^*)\) if and only if \((\mathfrak{g}, \mathfrak{g}^*, \text{ad}^*_{\mathfrak{g}} , \text{ad}^*_{\mathfrak{g}^*})\) is a matched pair of Lie algebras [14]. This characterization can be extended to Rota-Baxter Lie algebras by the same argument as [6, Theorem 3.12] for Rota-Baxter associative algebras.

Proposition 3.6. Let \((\mathfrak{g}, [-,-])_G, P\) be a Rota-Baxter Lie algebra. Suppose that there is a Rota-Baxter Lie algebra \((\mathfrak{g}^*, [-,-])_G^*, Q^*\) on the dual space \(\mathfrak{g}^*\). Then there is a Manin triple of Rota-Baxter Lie algebras \(((\mathfrak{g} \oplus \mathfrak{g}^*, P + Q^*), (\mathfrak{g}, P), (\mathfrak{g}^*, Q^*))\) if and only if \(((\mathfrak{g}, P), (\mathfrak{g}^*, Q^*), \text{ad}^*_{\mathfrak{g}}, \text{ad}^*_{\mathfrak{g}^*})\) is a matched pair of Rota-Baxter Lie algebras.

3.1.3. Rota-Baxter Lie bialgebras.

A Lie coalgebra is a pair \((\mathfrak{g}, \delta)\) with a vector space \(\mathfrak{g}\) and a linear map \(\delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}\) such that

(a) \(\delta\) is co-antisymmetric, that is, \(\delta = -\tau\delta\) for the flip map \(\tau : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}\), and
(b) the co-Jacobian identity holds:

\[
(id + \sigma + \sigma^2)(id \otimes \delta) \delta = 0,
\]

where \(\sigma(x \otimes y \otimes z) := z \otimes x \otimes y\) for \(x, y, z \in \mathfrak{g}\).

Definition 3.7. [14] A Lie bialgebra is a triple \((\mathfrak{g}, [-,-], \delta)\) consisting of a vector space \(\mathfrak{g}\) and linear maps \([-,-] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}, \delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}\) such that

(a) \((\mathfrak{g}, [-,-])\) is a Lie algebra.
(b) \((\mathfrak{g}, \delta)\) is a Lie coalgebra.
(c) \(\delta\) is a 1-cocycle of \(\mathfrak{g}\) with values in \(\mathfrak{g} \otimes \mathfrak{g}\), that is,
\[
\delta([x, y]) = (\text{ad}(x) \otimes \text{id} + \text{id} \otimes \text{ad}(x))\delta(y) - (\text{ad}(y) \otimes \text{id} + \text{id} \otimes \text{ad}(y))\delta(x), \forall x, y \in \mathfrak{g}. \tag{38}
\]

We now extend the notion of Lie bialgebras to Rota-Baxter Lie bialgebras.

**Definition 3.8.** A Rota-Baxter Lie coalgebra (of weight \(\lambda\)) is a triple \((\mathfrak{g}, \delta, Q)\), where \((\mathfrak{g}, \delta)\) is a Lie coalgebra, and \(Q : \mathfrak{g} \to \mathfrak{g}\) is a linear map such that
\[
(Q \otimes Q)\delta(x) = (Q \otimes \text{id} + \text{id} \otimes Q)\delta(Q(x)) + \lambda\delta(Q(x)), \forall x \in \mathfrak{g}. \tag{39}
\]

Generalizing the well-known duality between Lie coalgebras and Lie algebras, for a finite dimensional vector space \(\mathfrak{g}\), the pair \((\mathfrak{g}^*, [-, -]_{\mathfrak{g}^*}, Q^*)\) is a Rota-Baxter Lie algebra if and only if \((\mathfrak{g}, \delta, Q)\) is a Rota-Baxter Lie coalgebra, where \(\delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}\) is the linear dual of \([-,-]_{\mathfrak{g}^*} : \mathfrak{g}^* \otimes \mathfrak{g}^* \to \mathfrak{g}^*\), that is,
\[
\langle \delta(x), a^* \otimes b^* \rangle = \langle x, [a^*, b^*]_{\mathfrak{g}^*} \rangle, \forall x \in \mathfrak{g}, a^*, b^* \in \mathfrak{g}^*. \tag{40}
\]

In this case, the Lie algebra structure on \(\mathfrak{g}^*\) is also denoted by \(\delta^*\), that is,
\[
\delta^*(a^* \otimes b^*) = [a^*, b^*]_{\mathfrak{g}^*}, \forall a^*, b^* \in \mathfrak{g}^*.
\]
Moreover, for a linear map \(P : \mathfrak{g} \to \mathfrak{g}\), the condition that \(P^*\) is admissible to the Rota-Baxter Lie algebra \((\mathfrak{g}^*, [-, -]_{\mathfrak{g}^*}, Q^*)\), that is, for all \(a^*, b^* \in \mathfrak{g}^*\),
\[
P^*([-Q^*(a^*), b^*]_{\mathfrak{g}^*}) - [Q^*(a^*), P^*(b^*)]_{\mathfrak{g}^*} - P^*([a^*, P^*(b^*)]_{\mathfrak{g}^*}) - \lambda [a^*, P^*(b^*)]_{\mathfrak{g}^*} = 0, \tag{41}
\]
can be rewritten in terms of the comultiplication \(\delta\) as follows:
\[
(P \otimes Q)\delta(x) + (P \otimes \text{id} - \text{id} \otimes Q)\delta(P(x)) + \lambda(P \otimes \text{id})\delta(x) = 0, \forall x \in \mathfrak{g}. \tag{42}
\]

**Definition 3.9.** A Rota-Baxter Lie bialgebra (of weight \(\lambda\)) is a vector space \(\mathfrak{g}\) together with linear maps
\[
[-,-] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}, \quad \delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}, \quad P, Q : \mathfrak{g} \to \mathfrak{g}
\]
such that
(a) the triple \((\mathfrak{g}, [-,-], \delta)\) is a Lie bialgebra.
(b) the triple \((\mathfrak{g}, [-,-], P)\) is a Rota-Baxter Lie algebra.
(c) the triple \((\mathfrak{g}, \delta, Q)\) is a Rota-Baxter Lie coalgebra.
(d) \(Q\) is admissible to \((\mathfrak{g}, [-,-], P)\), that is, Eq. (13) holds.
(e) \(P^*\) is admissible to \((\mathfrak{g}^*, \delta^*, Q^*)\), that is, Eq. (42) holds.

We denote the Rota-Baxter Lie bialgebra by \((\mathfrak{g}, [-,-], P, \delta, Q)\) or simply \((\mathfrak{g}, P, \delta, Q)\).

Let \((\mathfrak{g}, [-,-], \delta)\) be a Lie algebra. Suppose that there is a Lie algebra \((\mathfrak{g}^*, [-,-]_{\mathfrak{g}^*})\) on the dual space \(\mathfrak{g}^*\). Let \(\delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}\) be the linear dual of \([-,-]_{\mathfrak{g}^*} : \mathfrak{g}^* \otimes \mathfrak{g}^* \to \mathfrak{g}^*\). Then \((\mathfrak{g}, [-,-], \delta)\) is a Lie bialgebra if and only if \((\mathfrak{g}, \mathfrak{g}^*, \text{ad}^*_{\mathfrak{g}}, \text{ad}^*_{\mathfrak{g}^*})\) is a matched pair of Lie algebras [14]. Generalizing this fact to the context of Rota-Baxter Lie algebras, we have

**Proposition 3.10.** Let \((\mathfrak{g}, [-,-], \delta)\) be a Rota-Baxter Lie algebra. Suppose that there is a Rota-Baxter Lie algebra \((\mathfrak{g}^*, [-,-]_{\mathfrak{g}^*}, Q^*)\) on the dual space \(\mathfrak{g}^*\). Let \(\delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}\) be the linear dual of \([-,-]_{\mathfrak{g}^*} : \mathfrak{g}^* \otimes \mathfrak{g}^* \to \mathfrak{g}^*\). Then \((\mathfrak{g}, [-,-], P, \delta, Q)\) is a Rota-Baxter Lie bialgebra if and only if \((\mathfrak{g}, P, (\mathfrak{g}^*, Q^*), \text{ad}^*_{\mathfrak{g}}, \text{ad}^*_{\mathfrak{g}^*})\) is a matched pair of Rota-Baxter Lie algebras.

**Proof.** The proof follows the one of [6, Theorem 3.5]. \(\square\)
Combining Theorems 3.6 and 3.10, we obtain

**Theorem 3.11.** Let \((\mathfrak{g}, [-, -]_g, P)\) be a Rota-Baxter Lie algebra. Suppose that there is a Rota-Baxter Lie algebra \((\mathfrak{g}^*[-, -]_{g^*}, Q^*)\) on the dual space \(\mathfrak{g}^*\). Let \(\delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}\) be the linear dual of \([-,-]_{g^*} : \mathfrak{g}^* \otimes \mathfrak{g}^* \to \mathfrak{g}^*\). Then the following conditions are equivalent.

(a) There is a Manin triple \(((\mathfrak{g} \oplus \mathfrak{g}^*, P + Q^*), (\mathfrak{g}, P), (\mathfrak{g}^*, Q^*))\) of Rota-Baxter Lie algebras.
(b) \(((\mathfrak{g}, P), (\mathfrak{g}^*, Q^*), \text{ad}_g, \text{ad}_{g^*})\) is a matched pair of Rota-Baxter Lie algebras.
(c) \((\mathfrak{g}, [-,-]_g, P, \delta, Q)\) is a Rota-Baxter Lie bialgebra.

### 3.2. Rota-Baxter Lie bialgebras of weight zero and special L-dendriform bialgebras

Here we show that the well-known connection between Rota-Baxter Lie algebras of weight zero and pre-Lie algebras has its bialgebra enrichment as a connection between Rota-Baxter Lie bialgebras of weight zero and special L-dendriform bialgebras. Furthermore, as in the case of Lie bialgebras, this connection among the various bialgebra structures is also characterized by suitable Manin triples and matched pairs.

#### 3.2.1. Manin triples of pre-Lie algebras with respect to the nondegenerate symmetric left-invariant bilinear form \(\mathcal{B}_d\).

**Definition 3.12.** [5] Suppose that there are three pre-Lie algebras \((A, \circ_A), (A^*, \circ_{A^*})\) and \((A \oplus A^*, \circ)\) such that \((A, \circ_A)\) and \((A^*, \circ_{A^*})\) are pre-Lie subalgebras of \((A \oplus A^*, \circ)\), and the natural nondegenerate symmetric bilinear form \(\mathcal{B}_d\) on \(A \oplus A^*\) defined by Eq. (31) is left-invariant. Then the triple \(((A \oplus A^*, \circ), (A, \circ_A), (A^*, \circ_{A^*}))\) is called a Manin triple of pre-Lie algebras with respect to the nondegenerate symmetric left-invariant bilinear form \(\mathcal{B}_d\) associated to \((A, \circ_A)\) and \((A^*, \circ_{A^*})\), and is denoted by \((A \circ A^*, A, A^*)\).

**Corollary 3.13.** Let \(((\mathfrak{g} \oplus \mathfrak{g}^*, P + Q^*), (\mathfrak{g}, P), (\mathfrak{g}^*, Q^*))\) be a Manin triple of Rota-Baxter Lie algebras of weight zero. Then the induced pre-Lie algebras (defined by Eq. (4)) from the three Rota-Baxter Lie algebras form a Manin triple \(((\mathfrak{g} \otimes \mathfrak{g}^*, \circ), (\mathfrak{g}, \circ_g), (\mathfrak{g}^*, \circ_{g^*}))\) of pre-Lie algebras with respect to the nondegenerate symmetric left-invariant bilinear form \(\mathcal{B}_d\) associated to \((\mathfrak{g}, \circ_g)\) and \((\mathfrak{g}^*, \circ_{g^*})\).

**Proof.** It follows from the definition that the induced pre-Lie algebra \((\mathfrak{g} \oplus \mathfrak{g}^*, \circ)\) from the Rota-Baxter Lie algebra \((\mathfrak{g} \oplus \mathfrak{g}^*, P + Q^*)\) contains the induced pre-Lie algebras \((\mathfrak{g}, \circ_g)\) and \((\mathfrak{g}^*, \circ_{g^*})\) as pre-Lie subalgebras. The left-invariance of \(\mathcal{B}_d\) follows from Proposition 2.11. □

#### 3.2.2. Matched pairs of pre-Lie algebras.

**Definition 3.14.** [3] Let \((A, \circ_A)\) and \((B, \circ_B)\) be pre-Lie algebras and \((\mathfrak{g}(A), [-, -]_A)\) and \((\mathfrak{g}(B), [-, -]_B)\) be their sub-adjacent Lie algebras respectively. Suppose that there are linear maps \(l_A, r_A : A \to \text{End}(B)\) and \(l_B, r_B : B \to \text{End}(A)\) such that \((B, l_A, r_A)\) is a representation of \((A, \circ_A)\), \((A, l_B, r_B)\) is a representation of \((B, \circ_B)\), and for all \(x, y \in A, a, b \in B\), the following equations hold:

\[
l_A(x)(a \circ_B b) = -l_A(l_B(a)x - r_B(a)x)b + (l_A(x)a) - r_A(a)b + l_A(r_B(b)x)a + a \circ_B (l_A(x)b),
\]

\[
r_A(x)[a, b]_B = r_A(l_B(b)x)a - r_A(l_B(a)x)b + a \circ_B (r_A(x)b) - b \circ_B (r_A(x)a),
\]

\[
l_B(x)(x \circ_A y) = -l_B(l_A(x)a - r_A(x)a)y + (l_B(a)x - r_B(a)x) \circ_A y + r_B(r_A(y)a)x + x \circ_A (l_B(a)y),
\]

\[
r_B(a)[x, y]_A = r_B(l_A(y)a)x - r_B(l_A(x)a)y + x \circ_A (r_B(a)y) - y \circ_A (r_B(a)x).
\]

Then \((A, B, l_A, r_A, l_B, r_B)\) is called a matched pair of pre-Lie algebras.
Let \((A, \circ_A)\) and \((B, \circ_B)\) be pre-Lie algebras, and \(l_A, r_A : A \to \text{End}(B)\), \(l_B, r_B : B \to \text{End}(A)\) be linear maps. Define a multiplication \(\circ\) on \(A \oplus B\) by
\[
(x + a) \circ (y + b) = x \circ_A y + l_B(b)(y) + r_B(b)x + a \circ_B b + l_A(x)b + r_A(y)a, \quad \forall x, y \in A, a, b \in B.
\]
Then \((A \oplus B, \circ)\) is a pre-Lie algebra if and only if \((A, B, l_A, r_A, l_B, r_B)\) is a matched pair of pre-Lie algebras. In this case, we denote the pre-Lie algebra structure on \(A \oplus B\) by \(A \triangleright_A B\) or simply \(A \bowtie B\).

The relationship between Manin triples of pre-Lie algebras with respect to the nondegenerate symmetric left-invariant bilinear form \(B_d\) and matched pairs of pre-Lie algebras is characterized as follows in terms of special L-dendriform algebras.

**Theorem 3.15.** Let \((A, \circ_A)\) and \((A^*, \circ_{A^*})\) be pre-Lie algebras. Suppose that there is a pre-Lie algebra structure \(\circ\) on the direct sum \(A \oplus A^*\) of vector spaces such that \((A \bowtie A^*, A, A^*)\) is a Manin triple of pre-Lie algebras with respect to the nondegenerate symmetric left-invariant bilinear form \(B_d\). Then there is a compatible special L-dendriform algebra \((A \oplus A^*, \triangleright, \triangleleft)\) of the pre-Lie algebra \((A \bowtie A^*, \circ)\) defined by
\[
B_d((x + a^*) \triangleright (y + b^*), z + c^*) = B_d(x + a^*, (z + c^*) \circ (y + b^*)),
\]
\[
(x + a^*) \triangleright (y + b^*) = (x + a^*) \circ (y + b^*) - (x + a^*) \circ (y + b^*), \quad \forall x, y, z \in A, a^*, b^*, c^* \in A^*.
\]
It contains the two compatible special L-dendriform subalgebras \((A, \triangleright_A, \triangleleft_A)\) and \((A^*, \triangleright_{A^*}, \triangleleft_{A^*})\) of the pre-Lie algebras \((A, \circ_A)\) and \((A^*, \circ_{A^*})\) respectively, such that \((A, A^*, L_{\circ_A}, L^*_{\circ_A})\) is a matched pair of pre-Lie algebras.

Conversely, suppose that \((A, \triangleright_A, \triangleleft_A)\) and \((A^*, \triangleright_{A^*}, \triangleleft_{A^*})\) are compatible special L-dendriform algebras of the pre-Lie algebras \((A, \circ_A)\) and \((A^*, \circ_{A^*})\) respectively. If \((A, A^*, L_{\circ_A}, L^*_{\circ_A})\) is a matched pair of pre-Lie algebras, then there is a Manin triple \((A \bowtie A^*, A, A^*)\) of pre-Lie algebras with respect to the nondegenerate symmetric left-invariant bilinear form \(B_d\). Moreover, Eqs. (47) and (48) hold.

The relationship between matched pairs of Rota-Baxter Lie algebras of weight zero and matched pairs of the induced pre-Lie algebras is given as follows.

**Proposition 3.16.** Let \((\mathfrak{g}, [-, -]_g, P)\) and \((\mathfrak{g}^*, [-, -]_{g^*}, Q^*)\) be Rota-Baxter Lie algebras of weight zero such that \(Q\) is admissible to \((\mathfrak{g}, [-, -]_g, P)\) and \(P^*\) is admissible to \((\mathfrak{g}^*, [-, -]_{g^*}, Q^*)\). Let \((\mathfrak{g}, \circ_g)\) and \((\mathfrak{g}^*, \circ_{g^*})\) be the induced pre-Lie algebras defined in Eq. (4), and \((\mathfrak{g}, \triangleright_g, \triangleleft_g)\) and \((\mathfrak{g}^*, \triangleright_{g^*}, \triangleleft_{g^*})\) be the compatible special L-dendriform algebras of the pre-Lie algebras \((\mathfrak{g}, \circ_g)\) and \((\mathfrak{g}^*, \circ_{g^*})\) defined by Eqs. (34) and (35) respectively. If \((\mathfrak{g}, P, \mathfrak{g}^*, Q^*), \text{ad}_{g^*}^\mathfrak{g}, \text{ad}_{g^*}^\mathfrak{g}\) is a matched pair of Rota-Baxter Lie algebras of weight zero, then \((\mathfrak{g}, \mathfrak{g}^*, L_{\circ_g}^*, L_{\circ_{g^*}}^*, L^*_g, L^*_g)\) is a matched pair of pre-Lie algebras.

**Proof.** By Proposition 2.18, \((\mathfrak{g}^*, L_{\circ_{g^*}}^*, L^*_g)\) is a representation of \((\mathfrak{g}, \circ_g)\), and \((\mathfrak{g}, L_{\circ_g}^*, L^*_g)\) is a representation of \((\mathfrak{g}^*, \circ_{g^*})\). Let \(x \in \mathfrak{g}, a^* \in \mathfrak{g}^*\). Then we have
\[
L_{\circ_{g^*}}^*(x)(a^* \circ_{g^*} b^*) = \text{ad}_{g^*}^\mathfrak{g}(P(x))[Q^*(a^*), b^*]_{g^*},
\]
\[
L_{\circ_g}(L_{\circ_{g^*}}^*(a^*) x) - L_{\circ_{g^*}}^*(a^*) x b^* = \text{ad}_{g^*}^\mathfrak{g}(P(\text{ad}_{g^*}^\mathfrak{g}(Q^*(a^*)) + \text{ad}_{g^*}^\mathfrak{g}(a^*)P(x)))b^*.
\]
\[
-(L_{\circ_g}(x)a^* - L_{\circ_{g^*}}(x)a^*) \circ_{g^*} b^* = -[Q^*(\text{ad}_{g^*}^\mathfrak{g}(P(x))a^* + \text{ad}_{g^*}^\mathfrak{g}(x)Q^*(a^*)), b^*]_{g^*},
\]
\[
\text{ad}_{g^*}^\mathfrak{g}(\text{ad}_{g^*}^\mathfrak{g}(Q^*(a^*))P(x))b^*.
\]
- \mathcal{L}_g^*(\mathcal{L}_g^*(b^*)x)a^* = -\text{ad}_g^*[\text{ad}_g^*(b^*)P(x)]Q^*(a^*), \\
- a^* \circ_{\mathcal{L}_g^*} \mathcal{L}_g^*(x)b^* = -[Q^*(a^*), \text{ad}_g^*(P(x))b^*]_{g^*}.

Since \((\mathfrak{g}, \mathfrak{g}^*, \text{ad}_{g^*}, \text{ad}_{g^*}^*\)) is a matched pair of Lie algebras, we have
\[\text{ad}_{g^*}^*[a^*, b^*]_{g^*} - [\text{ad}_{g^*}^*(a^*)b^*]_{g^*} - [a^*, \text{ad}_{g^*}^*(b^*)]_{g^*} + \text{ad}_{g^*}^*[\text{ad}_{g^*}^*(a^*)b^* - \text{ad}_{g^*}^*(a^*)]_{g^*} = 0.\]
Thus Eq. (43) holds by taking

\[(A, \circ_A) = (\mathfrak{g}, \circ_{\mathfrak{g}}), (B, \circ_B) = (\mathfrak{g}^*, \circ_{\mathfrak{g}^*}), l_A = \mathcal{L}_g^*, r_A = \mathcal{L}_{g^*}^*, l_B = \mathcal{L}_{g^*}^*, r_B = \mathcal{L}_{g^*}^*.\]

Similarly, Eqs. (44)-(46) hold. Hence \((\mathfrak{g}, \mathfrak{g}^*, \mathcal{L}_{g^*}^*, \mathcal{L}_{g^*}^*, \mathcal{L}_{g^*}^*)\) is a matched pair of pre-Lie algebras.

3.2.3. Special L-dendriform bialgebras. There is a bialgebra structure for special L-dendriform algebras obtained in \([5]\).

**Definition 3.17.** Let \(A\) be a vector space and \(\Delta, \nabla : A \to A \otimes A\) be linear maps. Suppose that \(\nabla\) is co-antisymmetric. Set \(\diamond := \Delta + \nabla\). If the following two conditions hold:

\[(id \otimes \nabla)\nabla + (\tau \otimes \text{id})(id \otimes \nabla)\nabla + (\nabla \otimes \text{id})\nabla - (\text{id} \otimes \nabla)\nabla = 0, \quad (49)\]

\[(\nabla \otimes \text{id})\nabla - (\text{id} \otimes \nabla)\nabla = (\tau \otimes \text{id})(\nabla \otimes \text{id})\nabla - (\text{id} \otimes \nabla)(\nabla \otimes \text{id})\nabla, \quad (50)\]

then \((A, \Delta, \nabla)\) is called a special L-dendriform coalgebra.

**Proposition 3.18.** Let \(A\) be a finite dimensional vector space, and \(\Delta, \nabla : A \to A \otimes A\) be linear maps. Let \(\triangleright_{A^*}, \triangleleft_{A^*} : A^* \otimes A^* \to A^*\) be the linear duals of \(\Delta\) and \(\nabla\) respectively. Then \((A, \Delta, \nabla)\) is a special L-dendriform coalgebra if and only if \((A^*, \triangleright_{A^*}, \triangleleft_{A^*})\) is a special L-dendriform algebra.

**Proof.** It is obvious that \(\nabla\) is co-antisymmetric if and only if \(\triangleleft_{A^*}\) is antisymmetric. Let \(\circ_{A^*} : A^* \otimes A^* \to A^*\) be a linear operation given by
\[a^* \circ_{A^*} b^* = a^* \triangleright_{A^*} b^* + a^* \triangleleft_{A^*} b^*, \quad \forall a^*, b^* \in A^*.\]
Then for all \(x \in A, a^*, b^*, c^* \in A^*\), we have
\[
\langle (id \otimes \nabla)\nabla(x) + (\tau \otimes \text{id})(id \otimes \nabla)\nabla(x) + (\nabla \otimes \text{id})\nabla(x) - (\text{id} \otimes \nabla)\nabla(x), a^* \otimes b^* \otimes c^* \rangle \\
= \langle x, a^* \triangleleft_{A^*} (b^* \triangleleft_{A^*} c^*) + b^* \triangleleft_{A^*} (a^* \circ_{A^*} c^*) + (a^* \circ_{A^*} b^*) \triangleleft_{A^*} c^* - a^* \circ_{A^*} (b^* \triangleleft_{A^*} c^*) \rangle, \\
\langle (\nabla \otimes \text{id})\nabla(x) - (\text{id} \otimes \nabla)\nabla(x) - (\tau \otimes \text{id})(\nabla \otimes \text{id})\nabla(x) + (\text{id} \otimes \nabla)(\nabla \otimes \text{id})\nabla(x), a^* \otimes b^* \otimes c^* \rangle \\
= \langle x, (a^* \circ_{A^*} b^*) \triangleleft_{A^*} c^* - a^* \circ_{A^*} (b^* \circ_{A^*} c^*) - (b^* \circ_{A^*} a^*) \circ_{A^*} c^* + b^* \circ_{A^*} (a^* \circ_{A^*} c^*) \rangle.
\]
Hence the conclusion follows by Proposition 2.17. \(\square\)

**Definition 3.19.** A special L-dendriform bialgebra \((A, \triangleright, \triangleleft, \Delta, \nabla)\) consists of a special L-dendriform algebra \((A, \triangleright, \triangleleft)\), a special L-dendriform coalgebra \((A, \Delta, \nabla)\) such that the following conditions hold:
\[\diamond(x \circ y) - (\text{id} \otimes \mathcal{R}_o(y))\Delta(x) + (\mathcal{L}_o(y) \otimes \text{id})\nabla(x) - (\mathcal{L}_o(x) \otimes \text{id} + \text{id} \otimes \mathcal{L}_o(x))\diamond(y) = 0, \quad (51)\]

\[(\tau - \text{id})(\text{id} \otimes \mathcal{L}_o(x))\diamond(y) - (\text{id} \otimes \mathcal{L}_o(y))\diamond(x) - \diamond(x \circ y) = 0, \quad (52)\]

\[\nabla([x, y]) + (\mathcal{L}_o(y) \otimes \text{id} + \text{id} \otimes \mathcal{L}_o(y))\nabla(x) - (\mathcal{L}_o(x) \otimes \text{id} + \text{id} \otimes \mathcal{L}_o(x))\nabla(y) = 0, \quad (53)\]
for all \(x, y \in A\), where \(\diamond = \Delta + \nabla\).
Remark 3.20. This notion of a special L-dendriform bialgebra is equivalent to the one introduced in [5] which is given in terms of the operations in the dual space. Indeed, in [5] the compatible conditions are given by four equations in Theorem 3.16 there. The first and second equations coincide with Eqs. (53) and Eq. (51) respectively, while the third and fourth equations are equivalent to Eqs. (52) and (51) respectively.

Theorem 3.21. [5] Let \((A, \triangleright_A, \triangleleft_A)\) be a special L-dendriform algebra. Suppose that there is a special L-dendriform algebra structure \((A^*, \triangleright_{A^*}, \triangleleft_{A^*})\) on the dual space \(A^*\). Let \((A, \circ_A)\) and \((A^*, \circ_{A^*})\) be the sub-adjacent pre-Lie algebras of \((A, \triangleright_A, \triangleleft_A)\) and \((A^*, \triangleright_{A^*}, \triangleleft_{A^*})\) respectively, and \(\Delta, \nabla : A \rightarrow A \otimes A\) be the linear duals of \(\triangleright_{A^*}, \triangleleft_{A^*}\) respectively. Then the sextuple \((A, A^*, \mathcal{L}_{\circ_A}, \mathcal{L}_{\triangleleft_A}, \mathcal{L}_{\circ_{A^*}}, \mathcal{L}_{\triangleleft_{A^*}})\) is a matched pair of pre-Lie algebras if and only if \((A, \triangleright_A, \triangleleft_A, \Delta, \nabla)\) is a special L-dendriform bialgebra.

Combining Theorems 3.15 and 3.21, we obtain the following conclusion.

Corollary 3.22. With the conditions in Theorem 3.21, the following statements are equivalent.

(a) There is a Manin triple \((A \bowtie A^*, A, A^*)\) of pre-Lie algebras with respect to the non-degenerate symmetric left-invariant bilinear form \(B_d\) such that the induced special L-dendriform algebra \((A \oplus A^*, \triangleright, \triangleleft)\) defined by Eqs. (47) and (48) includes \((A, \triangleright_A, \triangleleft_A)\) and \((A^*, \triangleright_{A^*}, \triangleleft_{A^*})\) as subalgebras.
(b) \((A, A^*, \mathcal{L}_{\circ_A}, \mathcal{L}_{\triangleleft_A}, \mathcal{L}_{\circ_{A^*}}, \mathcal{L}_{\triangleleft_{A^*}})\) is a matched pair of pre-Lie algebras.
(c) \((A, \triangleright_A, \triangleleft_A, \Delta, \nabla)\) is a special L-dendriform bialgebra.

Proposition 3.23. Let \((g, [\cdot, \cdot], P)\) and \((g^*, [-, -], Q^*)\) be Rota-Baxter Lie algebras of weight zero such that \(Q\) is admissible to \((g, [\cdot, \cdot], P)\) and \(P^*\) is admissible to \((g^*, [-, -], Q^*)\). Let \((g, \circ_g)\) and \((g^*, \circ_{g^*})\) be the induced pre-Lie algebras defined by Eq. (4), and let \((g, \triangleright_{g^*}, \triangleleft_g)\) and \((g^*, \triangleright_{g^*}, \triangleleft_{g^*})\) be the compatible special L-dendriform algebras of the pre-Lie algebras \((g, \circ_g)\) and \((g^*, \circ_{g^*})\) defined by Eqs. (34) and (35) respectively. Let \(\delta, \Delta, \nabla : g \rightarrow g \otimes g\) be the linear duals of \([-, -]_{g^*}, \triangleright_{g^*}\) and \(\triangleleft_{g^*}\) respectively. Then \((g, \triangleright_{g^*}, \triangleleft_g, \Delta, \nabla)\) is a special L-dendriform bialgebra if and only if the following equations hold:

\[
(Q \otimes \text{id})(\delta([P(x), y]_g) + (\text{id} \otimes \text{ad}_g(y) + \text{ad}_g(y) \otimes \text{id})\delta(P(x)))
\]

\[
- (\text{id} \otimes \text{ad}_g(P(x)) + \text{ad}_g(P(x)) \otimes \text{id})\delta(y) = 0,
\]

\[
(Q \otimes Q)(\delta([x, y]_g + (\text{id} \otimes \text{ad}_g(y) + \text{ad}_g(y) \otimes \text{id})\delta(x) - (\text{id} \otimes \text{ad}_g(x) + \text{ad}_g(x) \otimes \text{id})\delta(y)) = 0,
\]

\[
\delta([P(x), P(y)]_g) = (\text{id} \otimes \text{ad}_g(P(x)) + \text{ad}_g(P(x)) \otimes \text{id})\delta(P(y))
\]

\[
- (\text{id} \otimes \text{ad}_g(P(y)) + \text{ad}_g(P(y)) \otimes \text{id})\delta(P(x)), \tag{56}
\]

for all \(x, y \in g\). In particular, if \((g, [\cdot, \cdot], P, \delta, Q)\) is a Rota-Baxter Lie bialgebra of weight zero, then \((g, \triangleright_{g^*}, \triangleleft_g, \Delta, \nabla)\) is a special L-dendriform bialgebra.

Proof. It is obvious that \((g, \Delta, \nabla)\) is a special L-dendriform coalgebra. Let \(\hat{\diamond}\) be the linear dual of \(\circ_{g^*}\). Let \(x, y \in g\). Then by Eqs. (4), (34) and (35), we have

\[
\hat{\diamond}(x) = (Q \otimes \text{id})\delta(x), \quad \nabla(x) = -\delta(P(x)), \quad \Delta(x) = (Q \otimes \text{id})\delta(x) + \delta(P(x)),
\]

\[
\mathcal{L}_{\circ_g}(x) = \text{ad}_g(P(x)), \mathcal{R}_{\circ_g}(x) = -\text{ad}_g(x)P,
\]

\[
\mathcal{L}_{\triangleleft_g}(x) = -Q\text{ad}_g(x), \mathcal{L}_{\circ_{g^*}}(x) = \text{ad}_g(P(x)) + Q\text{ad}_g(x). \tag{58}
\]

Therefore we have

\[
\hat{\diamond}(x \circ_{g^*} y) = (Q \otimes \text{id})\delta([P(x), y]_g),
\]

\[
-(\text{id} \otimes \mathcal{R}_{\circ_g})\Delta(x) = (\text{id} \otimes \text{ad}_g(y)P)((Q \otimes \text{id})\delta(x) + \delta(P(x)))
\]
there exists an $A$ Rota-Baxter Lie bialgebra (Definition 4.1).

4.1. Coboundary Rota-Baxter Lie bialgebras and the induced special L-dendriform bialgebras.

Recall [14] that a Lie bialgebra $(\mathfrak{g}, [-, -], \delta)$ is called coboundary if there exists an $r \in \mathfrak{g} \otimes \mathfrak{g}$ such that

$$\delta(x) := \delta_r(x) := (\text{id} \otimes \text{ad}_g)(x) + (\text{id} \otimes P)\delta(P(x))$$

Thus Eq. (51) is equivalent to Eq. (54). Similarly, Eq. (52) and Eq. (53) are equivalent to Eq. (55) and Eq. (56) respectively. In particular, if $(\mathfrak{g}, [-, -], P, \delta, Q)$ is a Rota-Baxter Lie bialgebra of weight zero, then Eq. (38) holds. Therefore Eqs. (54)-(56) hold. Thus $(\mathfrak{g}, \triangleright, \triangleleft, \Delta, \nabla)$ is a special L-dendriform bialgebra.

The relations among the various structures in this section can be summarized in the following commutative diagram:

- Manin triples of Rota-Baxter Lie algebras $\xrightarrow{\text{Cor. 3.13}}$ Manin triples of pre-Lie algebras
- Matched pairs of Rota-Baxter Lie algebras $\xrightarrow{\text{Prop. 3.16}}$ Matched pairs of pre-Lie algebras
- Rota-Baxter Lie bialgebras $\xrightarrow{\text{Prop. 3.23}}$ Special L-dendriform bialgebras

4. Coboundary Rota-Baxter Lie bialgebras, admissible CYBEs and the induced special L-dendriform bialgebras

In the last section of the paper, we study the coboundary Rota-Baxter Lie bialgebras, leading to the notion of admissible CYBE in Rota-Baxter Lie algebras, whose antisymmetric solutions can be used to construct Rota-Baxter Lie bialgebras. Furthermore, the notions of $\mathcal{O}$-operators on Rota-Baxter Lie algebras and Rota-Baxter pre-Lie algebras are introduced to produce antisymmetric solutions of the admissible CYBE, and hence Rota-Baxter Lie bialgebras. When the weight of the Rota-Baxter operator is zero, we study the induced special L-dendriform bialgebras from these Rota-Baxter Lie bialgebras and thus give the construction of special L-dendriform bialgebras from antisymmetric solutions of the admissible CYBE in Rota-Baxter Lie algebras of weight zero. In particular, both Rota-Baxter Lie algebras of weight zero and Rota-Baxter pre-Lie algebras of weight zero can be used to construct special L-dendriform bialgebras.
Let \((g, [-, -])\) be a Lie algebra, and \(r = \sum a_i \otimes b_i \in g \otimes g\). Let \(\delta : g \to g \otimes g\) be a linear map defined by Eq. (59). Then \(\delta\) satisfies Eq. (38) automatically. Moreover, by [14], \(\delta\) makes \((g, \delta)\) into a Lie coalgebra such that \((g, [-, -], \delta)\) is a Lie bialgebra if and only if for all \(x \in g\),

\[
(ad(x) \otimes id + id \otimes ad(x))(r + \tau(r)) = 0,
\]

\[
(ad(x) \otimes id \otimes id + id \otimes ad(x) \otimes id + id \otimes id \otimes ad(x))(\left[r_{12}, r_{13}\right] + \left[r_{12}, r_{23}\right] + \left[r_{13}, r_{23}\right]) = 0,
\]

where

\[
\left[r_{12}, r_{13}\right] := \sum_{i,j} [a_i, a_j] \otimes b_i \otimes b_j, \left[r_{12}, r_{23}\right] := \sum_{i,j} a_i \otimes [b_i, a_j] \otimes b_j, \left[r_{13}, r_{23}\right] := \sum_{i,j} a_i \otimes a_j \otimes [b_i, b_j].
\]

Hence in order for \((g, [-, -], P, \delta, Q)\) to be a Rota-Baxter Lie bialgebra, we only need to further require that \((g^*, \delta^*, Q^*)\) is a \(P^*\)-admissible Rota-Baxter Lie algebra, that is, \((g, \delta, Q)\) is a Rota-Baxter Lie coalgebra and Eq. (42) holds.

**Proposition 4.2.** Let \((g, [-, -], P)\) be a \(Q\)-admissible Rota-Baxter Lie algebra of weight \(\lambda\) and \(r \in g \otimes g\). Define a linear map \(\delta : g \to g \otimes g\) by Eq. (59). Suppose that \(\delta^*\) defines a Lie algebra structure on \(g^*\). Then the following conclusions hold.

(a) Eq. (39) holds if and only if for all \(x \in g\),

\[
(id \otimes Q(ad(x)) - id \otimes ad(Q(x)))(Q \otimes id - id \otimes P)(r) + (Q(ad(x)) \otimes id - ad(Q(x)) \otimes id)(P \otimes id - id \otimes Q)(r) = 0.
\]

(b) Eq. (42) holds if and only if for all \(x \in g\),

\[
(id \otimes ad(P(x)) + ad(P(x)) \otimes id + id \otimes Q(ad(x)) - P(ad(x)) \otimes id + \lambda id \otimes ad(x))(P \otimes id - id \otimes Q)(r) = 0.
\]

**Proof.** The conclusion is obtained by following the proof of [6, Theorem 4.3]. \(\Box\)

**Theorem 4.3.** Let \((g, [-, -], P)\) be a \(Q\)-admissible Rota-Baxter Lie algebra and \(r \in g \otimes g\). Define a linear map \(\delta : g \to g \otimes g\) by Eq. (59). Then \((g, [-, -], P, \delta, Q)\) is a Rota-Baxter Lie bialgebra if and only if Eqs. (60)-(63) hold.

**Proof.** By the assumption, \((g, [-, -], P, \delta, Q)\) is a Rota-Baxter Lie bialgebra if and only if \((g^*, \delta^*, Q^*)\) is a \(P^*\)-admissible Rota-Baxter Lie algebra. By Proposition 4.2 and the results before it, the latter holds if and only if Eqs. (60)-(63) hold. \(\Box\)

In particular, we have the following conclusion.

**Corollary 4.4.** Let \((g, [-, -], P)\) be a \(Q\)-admissible Rota-Baxter Lie algebra and \(r \in g \otimes g\). Define a linear map \(\delta : g \to g \otimes g\) by Eq. (59). Then \((g, [-, -], P, \delta, Q)\) is a Rota-Baxter Lie bialgebra if Eq. (60) and the following equations hold:

\[
\left[r_{12}, r_{13}\right] + \left[r_{12}, r_{23}\right] + \left[r_{13}, r_{23}\right] = 0,
\]

\[
(P \otimes id - id \otimes Q)(r) = 0,
\]

\[
(Q \otimes id - id \otimes P)(r) = 0.
\]

Eq. (64) is just the well-known classical Yang-Baxter equation (CYBE) in \(g\) [14]. In view of this, Corollary 4.4 suggests the following variation of the CYBE.
Definition 4.5. Let \((\mathfrak{g}, [-, -], P)\) be a Rota-Baxter Lie algebra. Suppose that \(r \in \mathfrak{g} \otimes \mathfrak{g}\) and \(Q : \mathfrak{g} \to \mathfrak{g}\) is a linear map. Then Eq. (64) with conditions Eq. (65) and Eq. (66) is called the \(Q\text{-admissible classical Yang-Baxter equation}\) in \((\mathfrak{g}, [-, -], P)\) or simply the \(Q\text{-admissible CYBE}\).

Note that if \(r\) is antisymmetric (that is, \(r = -\tau(r)\)), then Eq. (65) holds if and only if Eq. (66) holds.

Proposition 4.6. Let \((\mathfrak{g}, [-, -], P)\) be a \(Q\text{-admissible Rota-Baxter Lie algebra}\) and \(r \in \mathfrak{g} \otimes \mathfrak{g}\) be an antisymmetric solution of the \(Q\text{-admissible CYBE}\) in \((\mathfrak{g}, [-, -], P)\). Then \((\mathfrak{g}, [-, -], P, \delta, Q)\) is a coboundary Rota-Baxter Lie bialgebra, where the linear map \(\delta = \delta_r\) is defined by Eq. (59).

Proof. It follows from Corollary 4.4 immediately. \(\square\)

On the other hand, there is a similar “coboundary” construction of special L-dendriform bialgebras considered in [5].

Proposition 4.7. [5] Let \((A, >, <)\) be a special L-dendriform algebra and \((A, \circ)\) be the sub-adjacent pre-Lie algebra. Let \(r = \sum_i a_i \otimes b_i \in A \otimes A\) be antisymmetric. Define linear maps \(\Delta, \nabla : A \to A \otimes A\) by

\[
\Delta(x) = (\mathcal{L}_r(x) \otimes \text{id} + \text{id} \otimes \text{ad}(x))(r), \quad \nabla(x) = (\mathcal{L}_\circ(x) \otimes \text{id} + \text{id} \otimes \mathcal{L}_\circ(x))(-r), \quad \forall x \in A.
\]

Then \((A, >, <, \Delta, \nabla)\) is a special L-dendriform bialgebra if \(r\) satisfies

\[
(68) \quad r_{12} < r_{13} = r_{12} \circ r_{23} + r_{13} \circ r_{23},
\]

\[
(67) \quad r_{12} < r_{13} = \sum_{i,j} a_i < a_j \otimes b_i \otimes b_j, r_{12} \circ r_{23} = \sum_{i,j} a_i \otimes b_i \circ a_j \otimes b_j, r_{13} \circ r_{23} = \sum_{i,j} a_i \otimes a_j \otimes b_i \circ b_j.
\]

Remark 4.8. In fact, [5] uses \(-r\) instead of \(r\) to define \(\Delta\) and \(\nabla\) in Eq. (67). We change the sign of \(r\) in order to be consistent with another construction given in the following Proposition 4.10.

Proposition 4.9. Let \((\mathfrak{g}, [-, -], P)\) be a Rota-Baxter Lie algebra of weight zero and \(Q : \mathfrak{g} \to \mathfrak{g}\) be a linear map which is admissible to \((\mathfrak{g}, [-, -], P)\). Let \((\mathfrak{g}, \circ)\) be the induced pre-Lie algebra and \((\mathfrak{g}, >, <, \triangleleft)\) be the compatible special L-dendriform algebra of \((\mathfrak{g}, \circ)\), where \(\circ, <\) and \(\triangleleft\) are defined by Eqs. (4), (27) and (25) respectively, that is,

\[
x \circ y = [P(x), y], \quad x < y = -Q([x, y]), \quad x \triangleright y = x \circ y - x < y, \quad \forall x, y \in \mathfrak{g}.
\]

If \(r \in \mathfrak{g} \otimes \mathfrak{g}\) is a solution of the \(Q\text{-admissible CYBE}\) in \((\mathfrak{g}, [-, -], P)\), then \(r\) satisfies Eq. (68).

Proof. Let \(r = \sum_i a_i \otimes b_i \in \mathfrak{g} \otimes \mathfrak{g}\). By Eq. (65), we have

\[
r_{12} \circ r_{23} + r_{13} \circ r_{23} - r_{12} \triangleleft r_{13} = \sum_{i,j} a_i \otimes [P(b_i), a_j] \otimes b_j + a_i \otimes a_j \otimes [P(b_i), b_j] + Q([a_i, a_j]) \otimes b_i \otimes b_j
\]

\[
= \sum_{i,j} Q(a_i) \otimes [b_i, a_j] \otimes b_j + Q(a_i) \otimes a_j \otimes [b_i, b_j] + Q([a_i, a_j]) \otimes b_i \otimes b_j
\]

\[
= (Q \otimes \text{id} \otimes \text{id})([r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}]).
\]

Hence the conclusion follows. \(\square\)
Let \((\mathfrak{g}, [-,-], P)\) be a Rota-Baxter Lie algebra of weight zero and \(Q : \mathfrak{g} \to \mathfrak{g}\) be a linear map which is admissible to \((\mathfrak{g}, [-,-], P)\). An antisymmetric solution \(r \in \mathfrak{g} \otimes \mathfrak{g}\) of the \(Q\)-admissible CYBE in \((\mathfrak{g}, [-,-], P)\) constructs a special L-dendriform bialgebra in two ways. On the one hand, by Proposition 4.6, there is a coboundary Rota-Baxter Lie bialgebra \((\mathfrak{g}, [-,-], P, \delta, Q)\) of weight zero, where the linear map \(\delta = \delta_r\) is defined by Eq. (59). Thus by Proposition 3.23, there is a special L-dendriform bialgebra \((\mathfrak{g}, \triangleright, \triangleleft, \Delta, \nabla)\), where \(\triangleright, \triangleleft, \Delta, \nabla\) are given by Eqs. (34) and (57) respectively. On the other hand, by Proposition 4.9, \(r\) satisfies Eq. (68). Hence by Proposition 4.7, there is a special L-dendriform bialgebra \((\mathfrak{g}, \triangleright, \triangleleft, \Delta', \nabla')\), where \(\triangleright, \triangleleft, \Delta', \nabla'\) are given by Eqs. (34) and (67) respectively. The two constructions turn out to be the same.

**Proposition 4.10.** With the above notations, the special L-dendriform bialgebras \((\mathfrak{g}, \triangleright, \triangleleft, \Delta, \nabla)\) and \((\mathfrak{g}, \triangleright, \triangleleft, \Delta', \nabla')\) coincide.

**Proof.** Let the sub-adjacent Lie algebra of \((\mathfrak{g}, \circ)\) be \((\mathfrak{g}, [-,-]'\)). Let \(r = \sum_i a_i \otimes b_i \in \mathfrak{g} \otimes \mathfrak{g}\) and \(x \in \mathfrak{g}\). Then
\[
\nabla(x) = -\delta(P(x)) = -(\text{ad}(P(x)) \otimes \text{id} + \text{id} \otimes \text{ad}(P(x)))r = -\sum_i ([P(x), a_i] \otimes b_i + a_i \otimes [P(x), b_i])
\]
and similarly, for \(\text{ad}'(x)y = [x, y]'\), \(\forall x, y \in \mathfrak{g}\), we have
\[
\Delta(x) = (Q \otimes \text{id})\delta(x) + \delta(P(x)) = (\mathcal{L}_o(x) \otimes \text{id} + \text{id} \otimes \text{ad}'(x))r = \Delta'(x),
\]
showing that the two special L-dendriform bialgebras coincide. \(\square\)

### 4.2. Admissible CYBEs, \(\circ\)-operators on Rota-Baxter Lie algebras and Rota-Baxter pre-Lie algebras

We first give operator forms of the antisymmetric solutions of the \(Q\)-admissible CYBE. For a vector space \(V\), the isomorphism \(V \otimes V \cong \text{Hom}(V^*, V)\) identifies an \(r \in V \otimes V\) with a linear map \(T_r : V^* \to V\). Thus for \(r = \sum_i u_i \otimes v_i\), the corresponding map \(T_r\) is
\[
T_r : V^* \to V, \quad T_r(u^*) = \sum_i \langle u^*, u_i \rangle v_i, \quad \forall u^* \in V^*.
\]

**Theorem 4.11.** Let \((\mathfrak{g}, [-,-], P)\) be a Rota-Baxter Lie algebra and \(r \in \mathfrak{g} \otimes \mathfrak{g}\) be antisymmetric. Let \(Q : \mathfrak{g} \to \mathfrak{g}\) be a linear map. Then \(r\) is a solution of the \(Q\)-admissible CYBE in \((\mathfrak{g}, [-,-], P)\) if and only if \(T_r\) satisfies
\[
[T_r(a^*), T_r(b^*)] = T_r(\text{ad}^*(T_r(a^*))b^* - \text{ad}^*(T_r(b^*))a^*), \quad \forall a^*, b^* \in \mathfrak{g}^*,
\]
\[
PT_r = T_r Q^*.
\]

**Proof.** The proof follows the same argument as in the proof of [6, Theorem 4.12]. \(\square\)

Then it is natural to introduce the following notion.

**Definition 4.12.** Let \((\mathfrak{g}, [-,-], P)\) be a Rota-Baxter Lie algebra, \((V, \rho)\) be a representation of \((\mathfrak{g}, [-,-])\) and \(\alpha : V \to V\) be a linear map. A linear map \(T : V \to \mathfrak{g}\) is called a weak \(\circ\)-operator associated to \((V, \rho)\) and \(\alpha\) if \(T\) satisfies
\[
[T(u), T(v)] = T(\rho(T(u))v - \rho(T(v))u), \quad \forall u, v \in V,
\]
\[
PT = T\alpha.
\]
If in addition, \((V, \rho, \alpha)\) is a representation of \((\mathfrak{g}, [-], P)\), then \(T\) is called an \(\mathcal{O}\)-operator associated to \((V, \rho, \alpha)\).

Note that for a Lie algebra \((\mathfrak{g}, [\cdot, \cdot])\) and a representation \((V, \rho)\) of \((\mathfrak{g}, [\cdot, \cdot])\), a linear map \(T: V \to \mathfrak{g}\) satisfying Eq. (72) is called an \(\mathcal{O}\)-operator of \((\mathfrak{g}, [\cdot, \cdot])\) associated to \((V, \rho)\) [23]. The terms relative Rota-Baxter operator and generalized Rota-Baxter operator are also used [29, 33].

**Example 4.13.** Let \((\mathfrak{g}, [\cdot, \cdot], P)\) be a Rota-Baxter Lie algebra of weight zero. Then \(P\) is an \(\mathcal{O}\)-operator of \((\mathfrak{g}, [\cdot, \cdot], P)\) associated to the adjoint representation \((\mathfrak{g}, \text{ad}, P)\).

Theorem 4.11 can be rewritten in terms of \(\mathcal{O}\)-operators as follows.

**Corollary 4.14.** Let \((\mathfrak{g}, [\cdot, \cdot], P)\) be a Rota-Baxter Lie algebra and \(r \in \mathfrak{g} \otimes \mathfrak{g}\) be antisymmetric. Let \(Q: \mathfrak{g} \to \mathfrak{g}\) be a linear map. Then \(r\) is a solution of the \(Q\)-admissible CYBE in \((\mathfrak{g}, [\cdot, \cdot], P)\) if and only if \(T_r\) is a weak \(\mathcal{O}\)-operator associated to \((\mathfrak{g}^*, \text{ad}^*)\) and \(Q^*\). If in addition, \((\mathfrak{g}, [\cdot, \cdot], P)\) is a \(Q\)-admissible Rota-Baxter Lie algebra, then \(r\) is a solution of the \(Q\)-admissible CYBE in \((\mathfrak{g}, [\cdot, \cdot], P)\) if and only if \(T_r\) is an \(\mathcal{O}\)-operator associated to the representation \((\mathfrak{g}^*, \text{ad}^*, Q^*)\).

On the other hand, an \(\mathcal{O}\)-operator of a Lie algebra gives rise to a solution of the CYBE in the semi-direct product Lie algebra as follows.

**Lemma 4.15.** [2] Let \((\mathfrak{g}, [\cdot, \cdot])\) be a Lie algebra and \((V, \rho)\) be a representation. Let \(T: V \to \mathfrak{g}\) be a linear map which is identified as an element in \((\mathfrak{g} \ltimes_{\rho^*} V^*) \otimes (\mathfrak{g} \ltimes_{\rho^*} V^*)\) through \(\text{Hom}(V, \mathfrak{g}) \cong \mathfrak{g} \otimes V^* \subseteq (\mathfrak{g} \ltimes_{\rho^*} V^*) \otimes (\mathfrak{g} \ltimes_{\rho^*} V^*)\). Then \(T = T - \tau(T)\) is an antisymmetric solution of the CYBE in the Lie algebra \(\mathfrak{g} \ltimes_{\rho^*} V^*\) if and only if \(T\) is an \(\mathcal{O}\)-operator of \((\mathfrak{g}, [\cdot, \cdot])\) associated to \((V, \rho)\).

In order to extend the above construction to the context of Rota-Baxter Lie algebras, we consider the admissibility of linear maps to the semi-direct products of Rota-Baxter Lie algebras.

**Theorem 4.16.** Let \((\mathfrak{g}, [\cdot, \cdot], P)\) be a Rota-Baxter Lie algebra of weight \(\lambda\), and let \((V, \rho)\) be a representation of \((\mathfrak{g}, [\cdot, \cdot])\). Let \(Q: \mathfrak{g} \to \mathfrak{g}\) and \(\alpha, \beta: V \to V\) be linear maps. Then the following conditions are equivalent.

(a) There is a Rota-Baxter Lie algebra \((\mathfrak{g} \ltimes_{\rho} V, P + \alpha)\) such that the linear map \(Q + \beta\) on \(\mathfrak{g} \oplus V\) is admissible to \((\mathfrak{g} \ltimes_{\rho} V, P + \alpha)\).

(b) There is a Rota-Baxter Lie algebra \((\mathfrak{g} \ltimes_{\rho^*} V^*, P + \beta^*)\) such that the linear map \(Q + \alpha^*\) on \(\mathfrak{g} \oplus V^*\) is admissible to \((\mathfrak{g} \ltimes_{\rho^*} V^*, P + \beta^*)\).

(c) The following conditions are satisfied:

(i) \((V, \rho, \alpha)\) is a representation of \((\mathfrak{g}, [\cdot, \cdot], P)\), that is, Eq. (9) holds;

(ii) \(Q\) is admissible to \((\mathfrak{g}, [\cdot, \cdot], P)\), that is, Eq. (13) holds;

(iii) \(\beta\) is admissible to \((\mathfrak{g}, [\cdot, \cdot], P)\) on \((V, \rho)\), that is, Eq. (12) holds;

(iv) The following equation holds:

\[
\beta(\rho(x)\alpha(v)) = \beta(\rho(Q(x))v) + \rho(Q(x))\alpha(v) + \lambda\rho(Q(x))v, \quad \forall x \in \mathfrak{g}, v \in V.
\]

**Proof.** The proof follows the same argument as in the proof of [6, Theorem 4.20].

**Corollary 4.17.** Let \((\mathfrak{g}, [\cdot, \cdot], P)\) be a Rota-Baxter Lie algebra of weight \(\lambda\) and \((V, \rho)\) be a representation of the Lie algebra \((\mathfrak{g}, [\cdot, \cdot])\). Let \(\alpha: V \to V\) be a linear map. Then the following conditions are equivalent.
(a) \((V, \rho, \alpha)\) is a representation of \((\mathfrak{g}, [-, -], P)\).
(b) \(\alpha^*\) is admissible to \((\mathfrak{g} \ltimes_{\rho^*} V^*, P)\).
(c) \(-\lambda \text{id}_g + \alpha^*\) is admissible to \((\mathfrak{g} \ltimes_{\rho^*} V^*, P - \lambda \text{id}_{V^*})\).
(d) \(-P - \lambda \text{id}_g + \alpha^*\) is admissible to \((\mathfrak{g} \ltimes_{\rho^*} V^*, P - \alpha^* - \lambda \text{id}_{V^*})\).

**Proof.** Suppose that Item (a) holds. Then by Proposition 2.6 and Corollary 2.7, \(Q\) is admissible to \((\mathfrak{g}, [-, -], P)\) and \(\beta\) is admissible to \((\mathfrak{g}, [-, -], P)\) on \((V, \rho)\) in the cases when \(Q = 0, \beta = 0\), or \(Q = -\lambda \text{id}_g, \beta = -\lambda \text{id}_V\), or \(Q = -P - \lambda \text{id}_g, \beta = -\alpha - \lambda \text{id}_V\). Moreover, in these cases, Eq. (74) holds. Hence by Theorem 4.16, Items (b)-(d) follow since they correspond to these cases respectively.

Conversely, suppose that any of Items (b)-(d) holds, then by Theorem 4.16, \((V, \rho, \alpha)\) is a representation of \((\mathfrak{g}, [-, -], P)\), that is, Item (a) holds. \(\square\)

In the following, we apply \(\mathcal{O}\)-operators to the constructions of antisymmetric solutions of the admissible CYBE, and of Rota-Baxter Lie bialgebras.

**Theorem 4.18.** Let \((\mathfrak{g}, [-, -], P)\) be a Rota-Baxter Lie algebra of weight \(\lambda\) and \((V, \rho)\) be a representation of \((\mathfrak{g}, [-, -], \mathfrak{g})\). Let \(\beta : V \to V\) be a linear map which is admissible to \((\mathfrak{g}, [-, -], \mathfrak{g})\) on \((V, \rho)\). Let \(Q : \mathfrak{g} \to \mathfrak{g}, \alpha : V \to V\) and \(T : V \to \mathfrak{g}\) be linear maps.

(a) \(r = T - \tau(T)\) is an antisymmetric solution of the \((Q + \alpha^* )\)-admissible CYBE in the Rota-Baxter Lie algebra \((\mathfrak{g} \ltimes_{\rho^*} V^*, P + \beta^* )\) if and only if \(T\) is a weak \(\mathcal{O}\)-operator associated to \((V, \rho)\) and \(\alpha\), and satisfies \(T\beta = QT\).

(b) Assume that \((V, \rho, \alpha)\) is a representation of \((\mathfrak{g}, [-, -], \mathfrak{g})\). If \(T\) is an \(\mathcal{O}\)-operator associated to \((V, \rho, \alpha)\) and \(T\beta = QT\), then \(r = T - \tau(T)\) is an antisymmetric solution of the \((Q + \alpha^* )\)-admissible CYBE in the Rota-Baxter Lie algebra \((\mathfrak{g} \ltimes_{\rho^*} V^*, P + \beta^* )\). If in addition, \((\mathfrak{g}, [-, -], \mathfrak{g})\) is \(Q\)-admissible and Eq. (74) holds such that the Rota-Baxter algebra \((\mathfrak{g} \ltimes_{\rho^*} V^*, P + \beta^* )\) is \((Q + \alpha^* )\)-admissible, then there is a Rota-Baxter Lie bialgebra \((\mathfrak{g} \ltimes_{\rho^*} V^*, P + \beta^*, \delta, Q + \alpha^* )\) of weight \(\lambda\), where the linear map \(\delta = \delta_r\) is defined by Eq. (59) with \(r = T - \tau(T)\).

**Proof.** (a). It is the same as the proof of [6, Theorem 4.21 (a)].

(b). It follows from Item (a) and Theorem 4.16. \(\square\)

**Corollary 4.19.** Let \((\mathfrak{g}, [-, -], P)\) be a Rota-Baxter Lie algebra of weight \(\lambda\) and \((V, \rho, \alpha)\) be a representation of \((\mathfrak{g}, [-, -], P)\) associated to \((V, \rho, \alpha)\). Then there are the Rota-Baxter Lie bialgebras \((\mathfrak{g} \ltimes_{\rho^*} V^*, P, \delta, \alpha^* )\) \((\mathfrak{g} \ltimes_{\rho^*} V^*, P - \lambda \text{id}_{V^*}, \delta, -\lambda \text{id}_g + \alpha^* )\) and \((\mathfrak{g} \ltimes_{\rho^*} V^*, P - \alpha^* - \lambda \text{id}_{V^*}, \delta, -P - \lambda \text{id}_g + \alpha^* )\), where the linear map \(\delta = \delta_r\) is defined by Eq. (59) with \(r = T - \tau(T)\).

**Proof.** By Corollary 4.17, the facts that the operator \(\alpha^*\) (resp. \(-\lambda \text{id}_g + \alpha^*\), resp. \(-P - \lambda \text{id}_g + \alpha^*\)) is admissible to \((\mathfrak{g} \ltimes_{\rho^*} V^*, P)\) (resp. \((\mathfrak{g} \ltimes_{\rho^*} V^*, P - \lambda \text{id}_{V^*})\), resp. \((\mathfrak{g} \ltimes_{\rho^*} V^*, P - \alpha^* - \lambda \text{id}_{V^*})\)) correspond to the case of \(Q = 0, \beta = 0\) (resp. \(Q = -\lambda \text{id}_g, \beta = -\lambda \text{id}_V\), resp. \(Q = -P - \lambda \text{id}_g, \beta = -\alpha - \lambda \text{id}_V\)) in Theorem 4.16. Note that in each of these cases, \(T\beta = QT\). Hence the conclusion follows from Theorem 4.18 (b). \(\square\)

To illustrate the construction of Rota-Baxter Lie bialgebras by \(\mathcal{O}\)-operators, we focus on the special case when the \(\mathcal{O}\)-operators are associated to the adjoint representation of the Rota-Baxter Lie algebra, as given in Example 4.13.

**Proposition 4.20.** Let \((\mathfrak{g}, [-, -], P)\) be a Rota-Baxter Lie algebra of weight \(\lambda\).
Let $T : \mathfrak{g} \to \mathfrak{g}$ be an $\mathcal{O}$-operator of $(\mathfrak{g}, [-, -], P)$ associated to the adjoint representation $(\mathfrak{g}, \text{ad}, P)$. Suppose that $Q$ is admissible to $(\mathfrak{g}, [-, -], P)$ and $TQ = QT$. Then there is a Rota-Baxter Lie bialgebra $(\mathfrak{g}_{\text{ad}}^* \mathfrak{g}^*, P + Q^*, \delta_r, Q + P^*)$, with the linear map $\delta_r$ defined by Eq. (59) with $r = T - \tau(T)$.

(b) Let $\lambda = 0$. Suppose that $Q : \mathfrak{g} \to \mathfrak{g}$ is a linear map that is admissible to $(\mathfrak{g}, [-, -], P)$ and commutes with $P$. Then there is a Rota-Baxter Lie algebra $(\mathfrak{g}_{\text{ad}}^* \mathfrak{g}^*, P + Q^*, \delta_r, Q + P^*)$ of weight zero, with $\delta_r$ as defined in the last item by letting $T = P$. In particular, there are Rota-Baxter Lie bialgebras $(\mathfrak{g}_{\text{ad}}^* \mathfrak{g}^*, P - P^*, \delta_r, -P + P^*)$ and $(\mathfrak{g}_{\text{ad}}^* \mathfrak{g}^*, P, \delta_r, P^*)$ of weight zero.

Proof. (a). It follows from Theorem 4.18 (b) in the case that $\rho = \text{ad}$, $\alpha = P$, $\beta = Q$.

(b). By Example 4.13, $P$ is an $\mathcal{O}$-operator of $(\mathfrak{g}, [-, -], P)$ associated to $(\mathfrak{g}, \text{ad}, P)$. Then by Item (a), the first conclusion follows by letting $T = P$. Furthermore, note that both $Q = -P$ and $Q = 0$ are admissible to $(\mathfrak{g}, [-, -], P)$ and commute with $P$. Then the second conclusion holds.

Definition 4.21. A Rota-Baxter pre-Lie algebra of weight $\lambda$ is a triple $(A, \circ, P)$, such that $(A, \circ)$ is a pre-Lie algebra, and $P : A \to A$ is a Rota-Baxter operator of weight $\lambda$ on $(A, \circ)$, that is, $P$ satisfies

$$P(x) \circ P(y) = P(P(x) \circ y) + P(x \circ P(y)) + \lambda P(x \circ y), \ \forall x, y \in A. \quad (75)$$

By a direct verification, we obtain

Proposition 4.22. Let $(A, \circ, P)$ be a Rota-Baxter pre-Lie algebra of weight $\lambda$. Then the following conclusions hold.

(a) $(\mathfrak{g}(A), [-, -], P)$ is a Rota-Baxter Lie algebra of weight $\lambda$, which is called the sub-adjacent Rota-Baxter Lie algebra of $(A, \circ, P)$.

(b) $(A, \mathcal{L}_\circ, P)$ is a representation of the Rota-Baxter Lie algebra $(\mathfrak{g}(A), [-, -], P)$.

(c) The identity map $\text{id}_A$ on $A$ is an $\mathcal{O}$-operator on the Rota-Baxter Lie algebra $(\mathfrak{g}(A), [-, -], P)$ associated to $(A, \mathcal{L}_\circ, P)$.

On the other hand, the following conclusion is also easy to check.

Proposition 4.23. Let $(\mathfrak{g}, [-, -], \mathfrak{g})$ be a Rota-Baxter Lie algebra of weight $\lambda$ and $(V, \rho, \alpha)$ be a representation of $(\mathfrak{g}, [-, -], \mathfrak{g})$. Let $T : V \to \mathfrak{g}$ be an $\mathcal{O}$-operator associated to $(V, \rho, \alpha)$. Then there exists a Rota-Baxter pre-Lie algebra structure $(V, \circ, \rho)$ on $V$, with $\circ$ given by

$$u \circ v := \rho(T(u))v, \ \forall u, v \in V. \quad (76)$$

In particular, if $(\mathfrak{g}, [-, -], \mathfrak{g})$ is a Rota-Baxter Lie algebra of weight zero and $(\mathfrak{g}, \circ)$ is the induced pre-Lie algebra, then $P$ is a Rota-Baxter operator of weight zero on $(\mathfrak{g}, \circ)$.

There is a simple construction of Rota-Baxter Lie bialgebras from Rota-Baxter pre-Lie algebras.

Proposition 4.24. Let $(A, \circ, P)$ be a Rota-Baxter pre-Lie algebra of weight $\lambda$, and let the sub-adjacent Rota-Baxter Lie algebra be $(\mathfrak{g}(A), [-, -], P)$. Let $\{e_1, \ldots, e_n\}$ be a basis of $A$, $\{e_1^*, \ldots, e_n^*\}$ be the dual basis and $r = \sum_{i=1}^n e_i \otimes e_i^* - e_i^* \otimes e_i$. Define the linear map $\delta = \delta_r$ by Eq. (59). Then there are Rota-Baxter Lie bialgebras $(\mathfrak{g}(A) \otimes \mathfrak{L}_{\lambda} A^*, P, \delta, P^*)$, $(\mathfrak{g}(A) \otimes \mathfrak{L}_{\lambda} A^*, P - \lambda \text{id}_A^*, \delta, -\lambda \text{id}_A + P^*)$, and $(\mathfrak{g}(A) \otimes \mathfrak{L}_{\lambda} A^*, P - \lambda \text{id}_A^*, -P^*, \delta, -P - \lambda \text{id}_A + P^*)$. 
Proof. By Proposition 4.22 (c), $T = \text{id}_A$ is an $\Theta$-operator of the Rota-Baxter Lie algebra $(g(A), [-, -], P)$ associated to $(A, L_\Theta, P)$. Note that $r = \text{id}_A - \tau(\text{id}_A) = \sum_{i=1}^{n} e_i \otimes e_i^* - e_i^* \otimes e_i$. Then the conclusion follows from Corollary 4.19. □

To complete the paper, we construct special L-dendriform bialgebras from $\Theta$-operators on Rota-Baxter Lie algebras of weight zero and from Rota-Baxter pre-Lie algebras.

**Proposition 4.25.** Let $(g, [-, -], g, P)$ be a Rota-Baxter Lie algebra of weight zero. Let $Q : g \rightarrow g$ be a linear map which is admissible to $(g, [-, -], g, P)$. Let $(V, \rho, \alpha)$ be a representation of $(g, [-, -], g, P)$ and $\beta : V \rightarrow V$ be a linear map which is admissible to $(V, \rho)$ on $(V, \rho)$. Suppose Eq. (74) holds. If $T$ is an $\Theta$-operator associated to $(V, \rho, \alpha)$ and $T \beta = QT$, then there is a special L-dendriform bialgebra $(g \ltimes_{\rho^*} V^*, \triangleright, \triangleleft, \Delta, \nabla)$, where

$$a \triangleleft b = -(Q + \alpha^*)([a, b]), \quad a \triangleright b = [(P + \beta^*)a, b] + (Q + \alpha^*)[a, b],$$

$$\nabla(a) = -\delta((P + \beta^*)a), \quad \Delta(a) = ((Q + \alpha^*) \otimes \text{id})\delta(a) + \delta((P + \beta^*)a),$$

for all $a, b \in g \ltimes_{\rho^*} V^*$. Here $[-, -]$ is defined by the Lie algebra $g \ltimes_{\rho^*} V^*$ and the linear map $\delta = \delta_r$ is defined by Eq. (59) with $r = T - \tau(T)$.

**Proof.** It follows from Theorem 4.18 (b) and Proposition 3.23. □

**Corollary 4.26.** Let $(g, [-, -], g, P)$ be a Rota-Baxter Lie algebra of weight zero. Suppose that $Q : g \rightarrow g$ is a linear map that is admissible to $(g, [-, -], g, P)$ and commutes with $P$. Let $[-, -]$ be the bracket on the Lie algebra $g \ltimes_{\text{ad}^*} g^*$, and let $\delta = \delta_r$ be the linear map defined by Eq. (59) with $r = P - \tau(P)$. Then there is a special L-dendriform bialgebra $(g \ltimes_{\text{ad}^*} g^*, \triangleright, \triangleleft, \Delta, \nabla)$, where

$$a \triangleleft b = -(Q + P^*)([a, b]), \quad a \triangleright b = [(P + Q^*)a, b] + (Q + P^*)[a, b],$$

$$\nabla(a) = -\delta((P + Q^*)a), \quad \Delta(a) = ((Q + Q^*) \otimes \text{id})\delta(a) + \delta((P + Q^*)a),$$

for all $a, b \in g \ltimes_{\text{ad}^*} g^*$. In particular, there are two special L-dendriform bialgebras $(g \ltimes_{\text{ad}^*} g^*, \triangleright_1, \triangleleft_1, \Delta_1, \nabla_1)$ and $(g \ltimes_{\text{ad}^*} g^*, \triangleright_2, \triangleleft_2, \Delta_2, \nabla_2)$, where

$$a \triangleleft_1 b = -((P + P^*)[a, b]), \quad a \triangleright_1 b = [(P + P^*)a, b] + (P + P^*)[a, b],$$

$$\nabla_1(a) = -\delta((P + P^*)a), \quad \Delta_1(a) = ((-P + P^*) \otimes \text{id})\delta(a) + \delta((-P + P^*)a),$$

$$a \triangleleft_2 b = -P^*[a, b], \quad a \triangleright_2 b = [P(a), b] + P^*[a, b],$$

$$\nabla_2(a) = -\delta(P(a)), \quad \Delta_2(a) = (P^* \otimes \text{id})\delta(a) + \delta(P(a)).$$

**Proof.** It follows from Propositions 4.20 (b) and 4.25. □

**Corollary 4.27.** Let $(A, \circ, P)$ be a Rota-Baxter pre-Lie algebra of weight zero with its subadjacent Rota-Baxter Lie algebra $(g(A), [-, -], P)$. Then there are two special L-dendriform bialgebras $(g(A) \ltimes_{\circ} A^*, \triangleright_1, \triangleleft_1, \Delta_1, \nabla_1)$ and $(g(A) \ltimes_{\circ} A^*, \triangleright_2, \triangleleft_2, \Delta_2, \nabla_2)$, where $\triangleright_1, \triangleleft_1, \Delta_1, \nabla_1$ and $\triangleright_2, \triangleleft_2, \Delta_2, \nabla_2$ are defined by Eqs. (79)-(82) respectively. $[-, -]$ is defined by the Lie algebra $g(A) \ltimes_{\circ} A^*$ and the linear map $\delta = \delta_r$ is defined by Eq. (59) with $r = \sum_{i=1}^{n} e_i \otimes e_i^* - e_i^* \otimes e_i$, where $\{e_1, \ldots, e_n\}$ is a basis of $A$ and $\{e_1^*, \ldots, e_n^*\}$ is the dual basis.

**Proof.** It follows from Propositions 4.24 and 3.23. □

We end the paper by showing that, starting with any Rota-Baxter Lie algebra of weight zero, the various Rota-Baxter Lie bialgebras from it allow us to construct a family of special L-dendriform bialgebras.
**Example 4.28.** Let \((g, [-,-]_g, P)\) be a Rota-Baxter Lie algebra of weight zero. Let \((g, \circ)\) be the induced pre-Lie algebra, whose sub-adjacent Lie algebra is denoted by \((g', [-,-]_{g'})\) or simply \(g'\), that is, \([-,-]_g'\) is defined by

\[ [x, y]_g' = [P(x), y]_g + [x, P(y)]_g, \quad \forall x, y \in g. \]

(a) By Corollary 4.26, there are special L-dendriform bialgebras \((g \ltimes_{ad^*} g^*, \triangleright_1, \triangleleft_1, \nabla_1)\) and \((g \ltimes_{ad^*} g^*, \triangleright_2, \triangleleft_2, \nabla_2)\), where \(\triangleright_1, \triangleleft_1, \nabla_1\) and \(\triangleright_2, \triangleleft_2, \nabla_2\) are defined by Eqs. (79)-(82) respectively. \([-,-]\) is defined by the Lie algebra \(g \ltimes_{ad^*} g^*\) and the linear map \(\delta = \delta_r\) is defined by Eq. (59) with \(r = P - \tau(P)\).

(b) By Proposition 4.23, \(P\) is a Rota-Baxter operator of weight zero on \((g, \circ)\). Then by Corollary 4.27, there are two special L-dendriform bialgebras \((g' \ltimes_{L_2} g^*, \triangleright_3, \triangleleft_3, \nabla_3)\) and \((g' \ltimes_{L_2} g^*, \triangleright_4, \triangleleft_4, \nabla_4)\), where \(\triangleright_3, \triangleleft_3, \nabla_3\) and \(\triangleright_4, \triangleleft_4, \nabla_4\) are defined by Eqs. (79)-(82) respectively. \([-,-]\) is defined by the Lie algebra \(g' \ltimes_{L_2} g^*\) and the linear map \(\delta = \delta_r\) is defined by Eq. (59) with \(r = \sum_{i=1}^n e_i \otimes e_i^* - e_i^* \otimes e_i\), where \(\{e_1, \ldots, e_n\}\) is a basis of \(g\) and \(\{e_1^*, \ldots, e_n^*\}\) is the dual basis.

(c) By Proposition 4.22, \(P\) is a Rota-Baxter operator of weight zero on the Lie algebra \((g', [-,-]_{g'}).\) Hence for this Rota-Baxter Lie algebra \((g', [-,-]_{g'}, P)\) of weight zero, we can repeat Items (a) and (b) to get four special L-dendriform bialgebras. Further repeating the same procedure gives rise to a series of special L-dendriform bialgebras.

**Acknowledgments.** This work is supported by National Natural Science Foundation of China (Grant No. 11931009), the Fundamental Research Funds for the Central Universities and Nankai Zhid Foundation.

**References**

1. A. Aubert and A. Medina, Groupes de Lie pseudo-Riemanniens plats, *Tohoku Math. J.* 55 (2003), 487-506.
2. C. Bai, A unified algebraic approach to the classical Yang-Baxter equation, *J. Phys. A: Math. Theor.* 40 (2007), 11073-11082.
3. C. Bai, Left-symmetric bialgebras and an analogue of the classical Yang-Baxter equation, *Commun. Contemp. Math.* 10 (2008), 221-260.
4. C. Bai, O. Bellier, L. Guo and X. Ni, Splitting of operations, Manin products and Rota-Baxter operators, *Int. Math. Res. Not.* 2013 (2013), 485-524.
5. C. Bai, D. Hou and Z. Chen, On a class of Lie groups with a left-invariant flat pseudo-metric, *Monatsh. Math.* 164 (2011), 243-269.
6. C. Bai, L. Guo and T. Ma, Bialgebras, Frobenius algebras and associative Yang-Baxter equations for Rota-Baxter algebras, arXiv:2112.10928.
7. C. Bai, L. Liu and X. Ni, Some results on L-dendriform algebras, *J. Geom. Phys.* 60 (2010), 940-950.
8. B. Bakalov and V. Kac, Field algebras, *Int. Math. Res. Not.* 2003 (2003), 123-159.
9. G. Baxter, An analytic problem whose solution follows from a simple algebraic identity, *Pacific J. Math.* 10 (1960), 731-742.
10. R. J. Baxter, Partition function of the eight-vertex lattice model, *Ann. Phys.* 70 (1972), 193-228.
11. A. A. Belavin and V. G. Drinfeld, Solutions of the classical Yang-Baxter equation for simple Lie algebras, *Funct. Anal. Appl.* 16 (1982), 159-180.
12. D. Burde, Left-symmetric algebras, or pre-Lie algebras in geometry and physics, *Cent. Eur. J. Math.* 4 (2006), 323-357.
13. F. Chapoton and M. Livernet, Pre-Lie algebras and the rooted trees operad, *Int. Math. Res. Not.* 8 (2001), 395-408.
[14] V. Chari and A. Pressley, A Guide to Quantum Groups, Cambridge University Press, Cambridge (1994).
[15] B. Y. Chu, Symplectic homogeneous spaces, Trans. Amer. Math. Soc. 197 (1974), 145-159.
[16] A. Connes and D. Kreimer, Hopf algebras, renormalization and noncommutative geometry, Comm. Math. Phys. 199 (1998), 395-408.
[17] V. Drinfeld, Hamiltonian structure on the Lie groups, Lie bialgebras and the geometric sense of the classical Yang-Baxter equations, Sov. Math. Dokl. 27 (1983), 68-71.
[18] V. Drinfeld, Quantum Groups, Proceedings of the International Congress of Mathematicians (Berkeley 1986), 798-820, Amer. Math. Soc., 1987.
[19] M. Gerstenhaber, The cohomology structure of an associative ring, Ann. Math. 78 (1963), 267-288.
[20] I. Z. Golubitskii and V. V. Sokolov, Generalized operator Yang-Baxter equations, integrable ODES and nonassociative algebras, J. Nonlinear Math. Phys. 7 (2000), 184-197.
[21] L. Guo, An Introduction to Rota-Baxter Algebra, Surveys of Modern Mathematics 4, International Press, Somerville, MA: Higher Education Press, Beijing, 2012.
[22] J.-L. Koszul, Domaines bornés homogènes et orbites de groupes de transformation affines, Bull. Soc. Math. France 89 (1961), 515-533.
[23] B.A. Kupershmidt, What a classical r-matrix really is, J. Nonlinear Math. Phys. 6 (1999), 448-488.
[24] H. Lang and Y. Sheng, Factorizable Lie bialgebras, quadratic Rota-Baxter Lie algebras and Rota-Baxter Lie bialgebras, arXiv:2112.07902.
[25] A. Lichnerowicz and A. Medina, On Lie groups with left-invariant symplectic or Kählerian structures, Lett. Math. Phys. 16 (1988), 225-235.
[26] S. Majid, Matched pairs of Lie groups associated to solutions of the Yang-Baxter equation, Pac. J. Math. 141 (1990), 311-332.
[27] J. Milnor, Curvatures of left invariant metrics on Lie groups, Adv. Math. 21 (1976), 293-329.
[28] X. Ni and C. Bai, Pseudo-Hessian Lie algebras and L-dendriform bialgebras, J. Algebra 400 (2014), 273-289.
[29] J. Pei, C. Bai and L. Guo, Splitting of operads and Rota-Baxter operators on operads, Appl. Cat. Stru. 25 (2017), 505-538.
[30] G. C. Rota, Baxter operators, an introduction, In: Gian-Carlo Rota on Combinatorics: Introductory Papers and Commentaries, Joseph P. S. Kung, Editor, Birkhäuser, Boston, 1995.
[31] M.A. Semonov-Tian-Shansky, What is a classical R-matrix? Funct. Anal. Appl. 17 (1983), 259-272.
[32] M. Shi, Rota-Baxter Lie bialgebras, Master’s thesis, Nankai University, 2014.
[33] K. Uchino, Twisting on associative algebras and Rota-Baxter type operators, J. Noncommut. Geom. 4 (2010), 349-379.
[34] E.B. Vinberg, Convex homogeneous cones, Trans. Moscow Math. Soc. 12 (1963), 340-403.
[35] C. N. Yang, Some exact results for the many-body problem in one dimension with repulsive delta-function interaction, Phys. Rev. Lett. 19 (1967), 1312-1314.

Chern Institute of Mathematics & LPMC, Nankai University, Tianjin 300071, China
Email address: baicm@nankai.edu.cn

Department of Mathematics and Computer Science, Rutgers University, Newark, NJ 07102, USA
Email address: liguo@rutgers.edu

Chern Institute of Mathematics & LPMC, Nankai University, Tianjin 300071, China
Email address: 1120190007@mail.nankai.edu.cn

School of Mathematics and Information Science, Henan Normal University, Xinxiang 453007, China
Email address: matianshui@htu.edu.cn