Abstract—In this paper, we consider the waveform design of a multiple-input multiple-output (MIMO) transmitter which simultaneously functions as a MIMO radar and a base station for downlink multiuser communications. In addition to a power constraint, we require the covariance of the transmit waveform be equal to a given optimal covariance for MIMO radar, to guarantee the radar performance. With this constraint, we formulate and solve the signal-to-interference-plus-noise ratio (SINR) balancing problem for multiuser transmit beamforming via convex optimization. Considering that the interference cannot be completely eliminated with this constraint, we introduce dirty paper coding (DPC) to further cancel the interference, and formulate the SINR balancing and sum rate maximization problem in the DPC regime. Although both of the two problems are non-convex, we show that they can be reformulated to convex optimizations via the Lagrange and downlink-uplink duality. In addition, we propose gradient projection based algorithms to solve the equivalent dual problem of SINR balancing, in both transmit beamforming and DPC regimes. The simulation results demonstrate significant performance improvement of DPC over transmit beamforming, and also indicate that the degrees of freedom for the communication transmitter is restricted by the rank of the covariance.

Index Terms—Joint radar and communications, multiple-input multiple-output (MIMO), transmit beamforming, dirty paper coding (DPC), conic linear programming.

I. INTRODUCTION

JOINT radar and communications on a single platform is an emerging technique which can reduce the cost of the platform, achieve spectrum sharing, and enhance the performance via the cooperation of radar and communications [2]–[4]. Because of these promising advantages, numerous schemes are proposed in recent years to implement joint radar and communications, including multi-functional waveform design [5]–[10], information embedding [11]–[16], joint transmit beamforming [3], [7], [9], [17], [18] and so on.

We focus on the joint transmit beamforming scheme here, which achieves spatial multiplexing of radar and communications by forming multiple transmit beams towards the radar targets and communication receivers. Previous works based on joint transmit beamforming mainly consider the joint waveform design of a multiple-input multiple-output (MIMO) radar and downlink multiuser MIMO communications. In particular, these works consider the optimization of the MIMO radar performance, such as the beam pattern mismatch [3], [17] and Cramér-Rao Bound [18], under individual signal-to-interference-plus-noise ratio (SINR) constraints at the communication receivers. Alternatively, some variants of the design [3], [9] simultaneously optimize the performance of radar and communications in the objective function. However, MIMO radars exhibit performance trade-off with multiuser communications in these works. In other words, to guarantee the SINRs at users, the achievable performance of MIMO radar is worse than the counterpart of a separate MIMO radar without considering communications. In high speed communication scenario, the performance loss of MIMO radar can be significant to achieve high SINRs at users [17].

In our work, we consider a joint MIMO radar and multiuser communication system, in which radar is the primary function and communication is the secondary function. Under this scenario, the efficiency of MIMO radar needs to be first guaranteed and the performance loss of radar is not desired. Literature on MIMO radar reported that the performance of MIMO radars highly depends on the covariance of the transmit waveform [19]–[22]. Therefore, we formulate the waveform design for communications, under the transmit covariance constraint that the covariance of the transmit waveform is equal to the given optimal one for MIMO radar without communication function. The proposed approach in [9] considers a similar constraint, but constrains the instantaneous covariance and needs to optimize the instantaneous transmit waveform with the constraint. Different from [9], we constrain the average covariance, and optimize the precoding matrices as in [3], [17], [18].

To further improve the spectrum efficiency of communications for joint radar and communications, recent works consider the full-duplex multi-user communications [23]–[25], which allow simultaneous transmission of downlink and uplink in the same band. These works consider separate radar and communication systems which share spectrum, while in our work radar and communication share both spectrum and transmit devices.

At the transmitter, linear precoding technique is usually applied to generate the transmit waveform, which performs
transmit beamforming to improve the SINRs at downlink users [26]–[29]. For transmit beamforming, we formulate the SINR balancing [30], [31] problem for multiuser communications, which designs the precoding matrices by maximizing the worst SINR at the users with the transmit covariance constraint. We show that the problem can be reformulated to a linear conic optimization [32] in an innovative way, and further propose a novel iteration algorithm to solve its Lagrange dual [33], which has a low complexity and converges fast.

Despite the low complexity of transmit beamforming, the numerical results show that, the transmit covariance constraint, introduced by the radar function, typically results in low SINRs via transmit beamforming. This still happens even if the signal-to-noise ratio (SNR) is high, because the inter-user interference cannot be canceled under such constraint. To further eliminate the interference, we investigate the application of dirty paper coding (DPC) [34], which reveals that the interference in an additive white Gaussian noise (AWGN) channel does not reduce the capacity if the interference is known at the transmitter, and was applied to the interference canceling in downlink multiuser communications [35]–[42].

We apply DPC for the transmit design of joint radar and communications, and formulate the SINR balancing problem for DPC with the strict radar performance constraint. Considering the optimization is non-convex, we derive its equivalent dual problem from the Lagrange dual of the power minimization problem, which finds the minimal transmit power to achieved given SINRs at users. This duality approach was first proposed in [42], [43] to tackle with the per-antenna power constraint, and here we extend it to the considered transmit covariance constraint. We observe that the dual problem has a convex structure, and propose a novel gradient projection iteration method to solve it.

Meanwhile, we consider to maximize the sum rate of the users, which is still a non-convex optimization problem. To solve this problem, we use the downlink-uplink duality approach, which was first proposed in [42] to tackle with the per-antenna power constraint, and is generalized to the transmit covariance constraint here. Via the downlink-uplink duality, we show that the problem is equivalent to the sum rate maximization for an equivalent uplink channel, which is expressed as a convex-concave saddle point problem. Further, we give a novel proof that the saddle point can be obtained by solving an equivalent linear conic optimization. The simulation results in the DPC regime show that the DPC can significantly improve the obtained SINRs at users compared to transmit beamforming.

Our main contributions towards the joint design of MIMO radar and downlink multiuser communications are summarized here:

- We propose to optimize the communication performance with the transmit covariance for radar, which can better guarantee the radar performance.
- We formulate the SINR balancing problem for transmit beamforming, and solve it via linear conic optimization and a novel iteration algorithm.
- To further eliminate the interference and improve the communication performance, we introduce DPC and formulate the SINR balancing problem in the DPC regime. A novel iteration algorithm is proposed to solve it.
- In addition, we formulate the sum rate maximization in the DPC regime, and propose a novel method to solve it via linear conic optimization.

While the proposed DPC approaches relieve the interference issues for downlink users, we note that the hard constraint on transmit covariance matrix essentially limits the communication performance. In particular, we reveal that the degrees of freedom of the communication transmitter is limited by the rank of the transmit covariance, from the following two observations, with the number of users denoted by $K$ and the rank denoted by $r_o$:

- The balanced SINR for DPC encounters a significant decrease when $K$ exceeds $r_o$;
- Under a high transmit SNR, the maximal sum rate for DPC is asymptotically affine in the transmit SNR in dB. The multiplexing gain [44], (i.e., the rate gain in bits/channeluse for every 3-dB power gain), is $K$ with a power constraint [40], while it reduces to $\min(K, r_o)$ with the transmit covariance constraint.

The rest of the paper is organized as follows. In Sec. II, we give the signal model, introduce the transmit covariance constraint, and derive the obtained SINRs at communication users via both transmit beamforming and DPC. In Sec. III-V, we formulate and solve the SINR balancing for transmit beamforming, SINR balancing for DPC and sum rate maximization for DPC, respectively. The implementation of the numerical algorithms and the complexity analysis are given in Sec. VI. We demonstrate the performance of joint radar and communications, and the convergence property of the proposed iteration algorithms via numerically results in Sec. VII. Sec. VIII draws the conclusion.

### II. Signal Model

Consider a joint transmitter which simultaneously functions as a MIMO radar transmitter and a base station for downlink multiuser communications. In the transmitter, radar and communications share transmit signal, whose expression

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**TABLE I**

| Symbol | Description |
|--------|-------------|
| $a$    | Scalar      |
| $a$    | Vector      |
| $A$    | Matrix      |
| $\text{diag}(a)$ | Diagonal matrix formed from vector $a$ |
| $1_K$  | $K$-dimensional vector whose elements are all 1 |
| $(\cdot)^\dagger$ | Moore-Penrose inverse of matrices |
| $(\cdot)^{1/2}$ | Squared root of matrices |
| $C(\cdot)$ | Column space of matrices |
| $E(\cdot)$ | Expectation of random variables |
| $\preceq$ | Matrix inequality operator ($A \preceq B$ if $A - B$ is positive definite) |
| $\partial(\cdot)$ | Partial differential |
| $\mathbb{R}^N$ | $N$-dimensional Euclidean space |
| $\mathbb{C}^N$ | $N$-dimensional complex Euclidean space |
| $H$    | Communication channel |
| $R$    | Covariance of transmit waveform |
| $W_c, W_r$ | Preceding matrix for communication and radar |
| $F$    | $F = HW_c$ |
is given in Sec. II-A, following [17]. Considering the radar performance, we introduce a transmit covariance constraint to the transmit signal in Sec. II-B. For communications, we first give the obtained SINRs for the downlink users via transmit beamforming in Sec. II-C, and then give the counterparts in the DPC regime in Sec. II-D.

A. Shared Transmit Signal

The transmitter is equipped with a transmit array with $M$ antennas and sends independent communication symbols to $K$ users, where $K \leq M$. The average transmit power is $P$. The transmit signal $x(n)$ for the shared transmit array is generated by the joint linear precoding scheme in [17]. In particular, $x(n)$ is the sum of linear precoded radar waveforms and communication symbols, given by

$$x(n) = W_r s(n) + W_c c(n), \quad n = 0, \ldots, N - 1, \quad (1)$$

where $N$ is the number of samples. Here, $s(n) = [s_1(n), \ldots, s_M(n)]^T$ includes $M$ orthogonal radar waveforms, and the $M \times M$ matrix $W_r$ is the precoding matrix for radar [12]. The orthogonality of radar waveforms means that $(1/N) \sum_{n=0}^{N-1} s(n)s^H(n) = I_M$. The $K$ parallel communication symbols to the users are contained in $c(n) = [c_1(n), \ldots, c_K(n)]^T$, precoded by the $M \times K$ matrix $W_c$.

Following [3], [17], [29], we rely on the following conditions to the communications symbols and radar waveforms:

1) The communication symbols to different users are mutually independent, have zero mean, and are normalized to have unit average power. Therefore, $E(c(n)c^H(n)) = I_M$.
2) The radar waveforms and communication symbols are statistically independent.

Given $\{s(n)\}$ and $\{c(n)\}$, transmit waveform design for joint MIMO radar and communication becomes designing $W_r$ and $W_c$.

B. Transmit Covariance Constraint for Radar

The radar is monostatic so that the communication signals can also be used for target detection because they are completely known at the radar receiver. Unlike phased array radars, MIMO radars transmit independent or partially correlated signals from the array elements. The performance of MIMO radars highly depends on its transmit covariance

$$R = E\left\{ \frac{1}{N} \sum_{n=0}^{N-1} x(n) x^H(n) \right\}. \quad (2)$$

It was shown that the transmit beam pattern [19], the angular estimation accuracy [21], [22] and the detection performance of MIMO radars [22] are determined by $R$. Substituting (1) into (2), $R$ is given by

$$R = W_r W_r^H + W_c W_c^H. \quad (3)$$

To derive (3), we use the conditions that the radar waveform and communication signals for different users are uncorrelated. However, when the signal length $N$ is finite, the signals are not perfectly orthogonal, and the sampled covariance can be slightly different from $R$ in (3). Nevertheless, the numerical result in Sec. VII-G verifies that the covariance mismatch is very small when $N$ is large.

Considering the radar performance, in a separate MIMO radar without communications, the transmit covariance is optimized, yielding $R_o$, under a uniform elemental power constraint

$$R_{m,m} = P/M, \quad m = 1, \ldots, M, \quad (4)$$

as in [19]–[21].

Then, in the joint design, $W_c$ and $W_r$ are constrained so that the obtained $R$ in (3) equals to $R_o$, namely the optimal transmit covariance for radar is achieved. We note that transmitting combined radar and communication waveform is important to the transmit covariance constraint. If $W_r = 0$, namely only communication waveform is transmitted, with single-stream symbol transmission we have $R = W_c W_c^H$ and as a result the rank of $R$ cannot exceed $K$. Then, if the rank of $R_o$ is higher than $K$, the transmit covariance constraint can never be met.

In the following, we design $W_c$ and $W_r$ in the regimes of transmit beamforming and DPC, respectively, to optimize the communication performance with this transmit covariance constraint for radar.

C. Transmit Beamforming for Multiuser Communications

For downlink multiuser communication, transmit beamforming is performed to increase the signal power at intended users and reduce interference to non-intended users [26], [29]. Here, a vector Gaussian broadcast channel (GBC) [45] is considered in which each user is equipped with a single receive antenna. The channel is denoted by a $K \times M$ matrix $H$, where we assume that the channel is non-frequency selective and is perfectly known at the transmitter. The channel output of the GBC is given by [17]

$$r(n) = Hx(n) + v(n) = HW_c c(n) + HW_r s(n) + v(n). \quad (5)$$

Here, the $k$-th elements of $r(n)$ represents the received signal at the $k$-th user, and $v(n)$ is complex AWGN whose covariance is $\sigma^2 I_K$. For convenience, we let $\sigma^2 = 1$ in the sequel.

In (5), each user receives the mixture of its own signals, the interference from other users, the radar signal and the noise. Let $F = HW_c$ and $G = HW_r$. For the $k$-th user, the sum power of received signal, including both desired signal and interference is $\sum_{i=1}^{M} |F_{k,i}|^2 + \sum_{i=1}^{M} |G_{k,i}|^2$, and the power of desired signal is $|F_{k,k}|^2$. Then, in the transmit beamforming regime, the SINR at the $k$-th user is given by [3], [17]

$$\text{SINR}_k = \frac{|F_{k,k}|^2}{\sum_{i\neq k} |F_{k,i}|^2 + \sum_{i=1}^{M} |G_{k,i}|^2 + 1}, \quad (6)$$

for $k = 1, \ldots, K$. In the regime of DPC, the interference is treated differently. Hence, the definition of SINR is different, as will be introduced later in Sec. II-D.

The goal of transmit beamforming is to improve the SINRs by designing $W_c$ and $W_r$. In Sec. III, we formulate and solve the SINR balancing problem for transmit beamforming, which optimize the worst SINR among the users.
D. DPC for Multiuser Communications

The numerical results in Sec. VII-B indicate that transmit beamforming may not be able to achieve high SINRs with the transmit covariance constraint. Therefore, some trade-off designs [3], [9], [17] were proposed to improve the SINR by relaxing the constraint. To further improve the SINRs, one can further perform non-linear precoding techniques, which eliminate the interference by encoding the communication signals to adapt the interference. In particular, we consider DPC [34], which reveals that the capacity of an AWGN channel corrupted by interference equals to the capacity of an interference-free AWGN channel if the interference is known at the transmitter. DPC was applied to downlink multiuser communications for inter-user interference elimination [35]–[38], and was shown to be able to achieve the capacity region of MIMO GBC [39].

We apply DPC to the GBC in (5) by serially encoding the source signal of each user. The encoding operations are conducted in the order \{1, \ldots, K\}. When performing DPC for the k-th user, \{c_1(n)\}, \ldots, \{c_{k-1}(n)\} are already encoded while \{c_k+1(n)\}, \ldots, \{c_K(n)\} are not encoded yet. Thus, the interference from the 1, \ldots, (k − 1)-th user is known while the interference from the k + 1, \ldots, K-th user is unknown at the transmitter. The radar interference is also known at the transmitter. Therefore, the effective SINR at the k-th user in the DPC regime is [36]

\[
\text{SINR}_{k}^{\text{dpc}} = \frac{|F_{k,k}|^2}{\sum_{i>k} |F_{k,i}|^2 + 1},
\]

for \(k = 1, \ldots, K\).

Note the SINRs in (7) is achievable when the channel state information (CSI) is perfectly known at the transmitter. When the CSI is not perfectly known, there exists a difference between the obtained CSI at the transmitter and the actual channel, leading to a residual between the known interference at transmitter and the actual interference, which is unknown and cannot be eliminated by DPC. Therefore, the achievable SINRs may become lower than that in (7). Although the precoding matrix design in the presence of CSI error [46], [47] can be meaningful, we only consider the case that the CSI is perfectly known in this work.

Comparing (6) and (7), it is observed that DPC improves the SINR compared with transmit beamforming by eliminating the interference. Similarly, \(W_c\) and \(W_r\) can be designed to improve the SINRs in the DPC regime. In particular, we consider the two criteria, SINR balancing and sum rate maximization for DPC, in Sec. IV and V, respectively. Prior to these DPC approaches, we first discuss below in Sec. III the SINR balancing in the transmit beamforming regime as a benchmark.

III. SINR BALANCING FOR TRANSMIT BEAMFORMING

In this section, we consider the SINR balancing problem for downlink transmit beamforming with the transmit covariance constraint from radar. The problem is first formulated into an optimization with respect to \(W_c\) and \(W_r\) in Sec. III-A. We then reformulate it into a linear conic optimization that can be effectively solved by optimization solvers, shown in Sec. III-B. Later in Sec. III-C, we further derive the dual problem of the conic optimization, from which a more efficient iteration method can be developed.

A. Problem Formulation

The goal of SINR balancing is to maximize the worst SINR at the users. With the transmit covariance constraint, we let \(\gamma = \min_{1 \leq k \leq K} \text{SINR}_k\) be the worst SINR, yielding the optimization problem:

\[
\begin{aligned}
\max_{W_c, W_r, \gamma} & \quad \gamma, \\
\text{s.t.} & \quad R_o = W_r W_r^H + W_c W_c^H, \\
\text{SINR}_k & \geq \gamma, \quad k = 1, \ldots, K.
\end{aligned}
\]

The solvers for (8) are discussed in the following.

B. Conic Optimization Solution

While the constraints in (8b) are non-convex, we can convert them to a convex constraints, and hence reformulate (8) to a linear conic optimization.

First, with the covariance constraint, the sum power of the desired signal and the interference at the k-th user equals to \([R_h]_{k,k}\), where \(R_h = H R_o H^H\), i.e.

\[
\sum_{i=1}^{K} |F_{k,i}|^2 + \sum_{i=1}^{M} |G_{k,i}|^2 = [R_h]_{k,k}.
\]

Substituting (9) into (6), the SINR constraints in (8b) can be simplified to

\[
|F_{k,k}| \geq \sqrt{\frac{\gamma}{1+\gamma}} s_k, \quad k = 1, \ldots, K,
\]

where \(s_k = ([R_h]_{k,k} + 1)^{1/2}\).

In (10), the reformulated SINR constraint only involves \(F\) but not \(G\). Similarly, the covariance constraint can also be rewritten as a constraint on \(F\), as stated by Theorem 1.

Theorem 1: Given \(F \in \mathbb{C}^{K \times K}\), there exists \(W_c, W_r\) that obey

\[
F = HW_c, \quad R_o = W_r W_r^H + W_c W_c^H,
\]

if and only if \(FF^H \preceq R_h\).

Proof:

If (11) holds, there is

\[
FF^H = HW_c W_c^H H^H \preceq HR_o H^H = R_h,
\]

which proves the necessity.

Now we prove that the condition is sufficient. Given \(F\) that obeys \(FF^H \preceq R_h\), we construct \(W_c, W_r\) by

\[
W_c = R_o^{1/2}(HR_o^{1/2})^{-\dagger} F, \quad W_r = (R_o - W_c W_c^H)^{1/2},
\]

where \((\cdot)^{\dagger}\) is the Moore-Penrose inverse [48]. The sufficiency can be proved by verifying that the constructed \(W_c, W_r\) obey (11). The detailed verification is given in Appendix A. □
We have thus reformulated (8) into the following optimization with respect to $F, \gamma$:

\[
\begin{align*}
\max_{F, \gamma} \gamma, \quad & \text{s.t. } FF^H \preceq R_h, \quad (13a) \\
|F_{k,k}| \geq \sqrt{\frac{\gamma}{1 + \gamma s_k}}, \quad & k = 1, \ldots, K. \quad (13b)
\end{align*}
\]

From Schur complement [49], the constraint $FF^H \preceq R_h$ is equivalent to a convex semidefinite constraint. Note that for a feasible $F$ in (13), multiplying its $k$-th column by a scalar phase factor $e^{j\theta_k}$ does not violate feasibility [29], [42]. Therefore, we only need to consider $F$ with real diagonal elements. Introducing a new variable $t = \sqrt{\gamma/(1 + \gamma)}$, (13) is equivalent to the following:

\[
\begin{align*}
\max_{F, t} t, \quad & \text{s.t. } FF^H \preceq R_h, \quad (14a) \\
\Re\{F_{k,k}\} \geq ts_k, \quad & k = 1, \ldots, K. \quad (14b)
\end{align*}
\]

which is solvable by linear conic programming.

The scale of (14) can be further reduced when $R_h$ is singular. In particular, $R_h$ becomes singular if the rank of $R_o$ is less than $K$ under Rayleigh fading channel. We let $r \leq K$ be the rank of $R_h$, and write the eigen decomposition of $R_h$ as

\[
R_h = U \Sigma_r U^H,
\]

where $\Sigma_r$ is a $r \times r$ diagonal matrix, $U \in \mathbb{C}^{K \times r}$ and $U^H U = I_r$. Since $FF^H \preceq R_h$, there exists $F_u \in \mathbb{C}^{r \times K}$ so that

\[
F = U \Sigma_{r/2} F_u. \quad (20)
\]

In (16), the dimension of $F_u$ reduces to $r \times K$ if $r < K$.

The linear conic optimizations in (14) and (16) can be effectively solved by optimization softwares in polynomial time.

C. Dual Program Solution

Here, we propose a more efficient method to solve (16) based on its Lagrange dual problem.

1) Formulation of the Dual Problem: Define the Lagrange function [33] of (16) by

\[
\mathcal{L}_1(F_u, t, Y, d) = t + \text{tr}\left\{Y(I_r - F_u F_u^H)\right\} + \sum_{k=1}^K d_k (\Re\{u_k^H f_k\} - ts_k),
\]

where $Y \succeq 0$ and $d = [d_1, \ldots, d_K]^T \succeq 0$ are associated dual variables. Then, the dual objective function is

\[
g_1(Y, d) = \max_{F_u, t} \mathcal{L}_1(F_u, t, Y, d), \quad (18)
\]

and the dual problem is

\[
\min_{Y, d} g_1(Y, d), \quad \text{s.t. } d \succeq 0, \quad Y \succeq 0. \quad (19)
\]

Let $s = [s_1, \ldots, s_K]^T$. It can be verified that $g_1(Y, d)$ is not positive infinity if and only if

\[
s^T d = 1, \quad d_k u_k \in \mathcal{C}(Y) \text{ for } k = 1, \ldots, K. \quad (20)
\]

with its expression given by

\[
g_1(Y, d) = \text{tr}\left\{Y + \frac{1}{4} Y^\dagger DD^H\right\}, \quad (21)
\]

where $D = [d_1 u_1, \ldots, d_K u_K]$. Thus, the dual problem of (16) is formulated as

\[
\min_{Y, d} \text{tr}\left\{Y + \frac{1}{4} Y^\dagger DD^H\right\}, \quad \text{s.t. } (20), \quad d \succeq 0, \quad Y \succeq 0. \quad (22)
\]

2) Solving the Dual Problem: To solve the dual problem, we first consider to find the optimal $Y$ with a given $d$. From equation (2.4) in [50], one has

\[
\text{tr}\left\{Y + \frac{1}{4} Y^\dagger DD^H\right\} \geq \text{tr}\left\{(YY^\dagger DD^H)^{1/2}\right\} = \text{tr}\left\{(DD^H)^{1/2}\right\}, \quad (23)
\]

where the inequality holds with equality when $Y = \frac{1}{2}(DD^H)^{1/2}$. In (23), we use the equality that $YY^\dagger D = D$ since the columns of $D$ are in $\mathcal{C}(Y)$ according to (20). Note that (23) gives the optimal value of (22) under a given $d$. Therefore, the dual problem in (22) is equivalent to

\[
\min_{d \succeq 0} \text{tr}\left\{(DD^H)^{1/2}\right\}, \quad \text{s.t. } s^T d = 1. \quad (24)
\]

Generally, solving (24) can be simpler than solving (16) as the variable is only a $K$ dimensional vector. The constraints in (24) are all linear, and the objective function is actually the nuclear norm [51] of $D$, which is convex in $d$. Nuclear norm minimization with linear constraints can be effectively solved by optimization softwares such as CVX [52], [53] in polynomial time.

Since (24) is a constrained convex optimization problem, we also propose a gradient projection based method to solve it. The detailed implementation of the iteration algorithm is described in Sec. VI-A. The numerical results in Sec. VII-D shows that the proposed iteration algorithm can be much more efficient than using CVX.

3) Computing the Solutions of (16): After the dual problem is solved, we in turn compute the solution of the primal problem in (16). The relationship between the primal and dual solutions can be revealed by the Karush-Kuhn-Tucker (KKT) condition [33]:

\[
\frac{\partial \mathcal{L}_1}{\partial F_u} = 2Y F_u - D = 0. \quad (25)
\]

Since $Y = \frac{1}{2}(DD^H)^{1/2}$ at the optimum, the solution of $F_u$ can be computed via

\[
F_u = \frac{1}{2} Y^\dagger D = [DD^H]^{-1/2} D. \quad (26)
\]
IV. SINR BALANCING FOR DPC

In this section, we formulate and solve the SINR balancing problem in the DPC regime. The optimization is formulated in Sec. IV-A. To solve it, we provide a power minimization based method in Sec. IV-B, which is solvable with optimization toolboxes. Later in Sec. IV-C, we give a more efficient method based on its dual problem.

A. Problem Formulation

We maximize the worst SINR among the users with the transmit covariance constraint in the DPC regime. Similar to (8), we define \( \gamma = \min_1 \leq k \leq K \) SINR\(^{\text{dpc}}\), and then the optimization is expressed as

\[
\max_{\mathbf{W}_c, \mathbf{W}_r, \gamma} \gamma, \quad \text{s.t.} \quad \mathbf{R}_o = \mathbf{W}_r \mathbf{W}^H_r + \mathbf{W}_c \mathbf{W}^H_c, \quad (27a)
\]

\[
\text{SINR}^{\text{dpc}} \geq \gamma, \quad k = 1, \ldots, K. \quad (27b)
\]

From the expression of the SINR in (7), the SINR constraints in (27b) is written as

\[
\frac{1}{\gamma} |f_{k,k}|^2 \geq \sum_{i \neq k} |f_{k,i}|^2 + 1, \quad k = 1, \ldots, K. \quad (28)
\]

According to Theorem 1, (27) can also be reformulated into an optimization with respect to \( \mathbf{F}, \gamma \):

\[
\max_{\mathbf{F}, \gamma} \gamma, \quad \text{s.t.} \quad \mathbf{F} \mathbf{F}^H \preceq \mathbf{R}_h, \quad (28).
\]

Similarly, the scale of (29) can be further reduced when \( \mathbf{R}_h \) is singular. With the eigen decomposition of \( \mathbf{R}_o \) in (15), we let \( \mathbf{F} = \mathbf{U} \Sigma^{1/2} \mathbf{F}_u \). Then (30) becomes an optimization with respect to \( \mathbf{F}_u \):

\[
\max_{\mathbf{F}_u, \gamma} \gamma, \quad \text{s.t.} \quad \mathbf{F}_u \mathbf{F}_u^H \preceq \mathbf{I}_r, \quad (30a)
\]

\[
\frac{1}{\gamma} |u^H_k f_k|^2 \geq \sum_{i \neq k} |u^H_k f_i|^2 + 1, \quad k = 1, \ldots, K, \quad (30b)
\]

where \( f_k \) is the \( k \)-th column in \( \mathbf{F}_u \) and \( u^H_k \) is the \( k \)-th row in \( \mathbf{U} \Sigma^{1/2} \).

In the following, we give methods to solve the optimization in (30).

B. Solution via Power Minimization

Unlike the SINR balancing problem for transmit beamforming in (16), (30) is nonconvex since the constraint in (30b) is nonconvex. Despite its non-convexity, (30) can be solved with polynomial time complexity. Firstly, we formulate the corresponding power minimization problem [27], [30], [42], which is a linear conic optimization, solvable with a polynomial time complexity. Based on the results, the optimizer of the original problem is then obtained with bisection search, also solvable with polynomial time complexity.

The power minimization problem is formulated for a given \( \gamma \) in (30b). Adjusting the transmit power by a scaling factor \( \lambda \geq 0 \), the transmit power and covariance becomes \( \lambda \mathbf{P} \) and \( \lambda \mathbf{R}_o \), respectively. As a result, the constraint to \( \mathbf{F}_u \) in (30a) becomes \( \mathbf{F}_u \mathbf{F}_u^H \preceq \lambda \mathbf{I}_r \). Regarding \( \lambda \) as a variable, the power minimization problem seeks for the minimal \( \lambda \) so that there exists \( \mathbf{F}_u \) meeting the SINR constraints in (30b), given by

\[
\min_{\lambda, \mathbf{F}_u} \lambda, \quad \text{s.t.} \quad \mathbf{F}_u \mathbf{F}_u^H \preceq \lambda \mathbf{I}_r, \quad (31a)
\]

\[
\frac{1}{\gamma} |u^H_k f_k|^2 \geq \sum_{i \neq k} |u^H_k f_i|^2 + 1, \quad k = 1, \ldots, K. \quad (31b)
\]

Here, \( \gamma \) is given and we let \( \lambda^*(\gamma) \) be the optimal value of (31).

The minimal transmit power problem is a solvable linear conic optimization. In (31), the constraint in (31a) can be recast to a semi-definite constraint. Similar to (13), we replace \( |u^H_k f_k| \) by \( \Re\{u^H_k f_k\} \) in (31b), and the constraints become second order cone constraints. Therefore, the optimization in (31) can be effectively solved by linear conic solvers with a polynomial time complexity.

The relationship between the original problem (30) and (31) is built from this observation: The SINR \( \gamma \) is achievable if and only if \( \lambda^*(\gamma) \leq 1 \), namely the minimal transmit power to achieve the SINR \( \gamma \) is less than \( \mathbf{P} \). Note that \( \lambda^*(\gamma) \) increases monotonically in \( \gamma \). Therefore, the optimal \( \gamma \) in (30), denoted by \( \gamma^*_o \), satisfies \( \lambda^*(\gamma^*_o) = 1 \), and can be found by a bisection search [30], which can be finished in polynomial time.

Although both linear conic optimization and bisection search are solvable and have polynomial time complexity, below we provide a more efficient approach to solve (30) based on its dual problem.

C. Dual Program Solution

This section propose a more efficient method to solve (30) based on its dual problem. We first derives the Lagrange dual of (31), based on which the dual problem of (30) is later derived. Compared to (30), the dual program can be expressed as a convex optimization and thus can be more effectively solved.

1) The Lagrange Dual of (31): The Lagrange function [33] of (31) is

\[
\mathcal{L}_2(\mathbf{F}_u, \lambda, \mathbf{Y}, d) = \lambda + \text{tr}(\mathbf{Y} (\mathbf{F}_u \mathbf{F}_u^H - \lambda \mathbf{I}_r))
\]

\[
+ \sum_{k=1}^{K} d_k \left\{ \sum_{i \neq k} |u^H_k f_i|^2 + 1 - \frac{1}{\gamma} |u^H_k f_k|^2 \right\}, \quad (32)
\]

where \( d = [d_1, \ldots, d_K]^T \geq 0 \) and \( \mathbf{Y} \succeq 0 \) are dual variables. Then, the dual objective function is

\[
g_2(\mathbf{Y}, d) = \min_{\mathbf{F}_u, \lambda} \mathcal{L}_2(\mathbf{F}_u, \lambda, \mathbf{Y}, d), \quad (33)
\]

and the dual program of (31) is

\[
\max_{\mathbf{Y}, d} g_2(\mathbf{Y}, d), \quad \text{s.t.} \quad d \succeq 0, \quad \mathbf{Y} \succeq 0. \quad (34)
\]

It can be shown that \( g_2(\mathbf{Y}, d) \) is not negative infinity under the following conditions [42]:

\[
\text{tr}(\mathbf{Y}) = 1, \quad (35a)
\]

\[
\mathbf{Y} + \sum_{i \neq k} d_i u_i u_i^H \geq \frac{1}{\gamma} d_k u_k u_k^H, \quad k = 1, \ldots, K. \quad (35b)
\]
We observe that, when the expression given by $g_2(Y, d) = \sum_{k=1}^{K} d_k$. Therefore, the dual problem of (31) is formulated as

$$\max_{Y \geq 0} \max_{d \geq 0} \sum_{k=1}^{K} d_k, \quad \text{s.t.} \quad (35a) \text{ and } (35b).$$

(36)

2) The Dual Problem of (30): The dual problem of (30) is formulated from the duality of (31). It can be proven that (31) has strong duality [42], so the optimal value of (36), denoted by $\lambda^*_\text{dual}(\gamma)$, equals to $\lambda^*(\gamma)$.

Then, the optimal value of (30), $\gamma_0^*$, satisfies $\lambda^*_\text{dual}(\gamma^*_0) = \lambda^*(\gamma^*_0) = 1$. In other words, $\gamma^*_0$ is the minimal $\gamma \geq 0$ so that $\lambda^*_\text{dual}(\gamma) \geq 1$. Note that $\lambda^*_\text{dual}(\gamma) \geq 1$ means there exists a pair of feasible solution $Y$, $d$ in (36) satisfying $\sum_{k=1}^{K} d_k \geq 1$.

Therefore, $\gamma^*_0$ is equal to the optimal value of

$$\min_{Y \geq 0} \min_{d \geq 0, \gamma \geq 0} \sum_{k=1}^{K} d_k \geq 1, \quad (35a) \text{ and } (35b), \quad (37)$$

which is the dual of (30).

Note that (37) is still nonconvex, since the constraints in (35b) are bilinear. Nevertheless, (37) actually has a hidden convex structure. Given a feasible $Y$, we denote the optimal value of the inner minimization in (37) with respect to $d$ and $\gamma$ by

$$\gamma_0(Y) = \min_{d \geq 0, \gamma \geq 0} \gamma, \quad \text{s.t.} \quad \sum_{k=1}^{K} d_k \geq 1 \quad (38).$$

Then (37) is reformulated into an optimization with respect to $Y$:

$$\min_{Y \geq 0} \gamma_0(Y), \quad \text{s.t.} \quad (35a).$$

(39)

We observe that, when $Y \succ 0$, $\gamma_0(Y)$ is a convex and differential function. Therefore, the constrained convex optimization in (39) can be solved via projected gradient methods.

To implement the iteration algorithm, we need to numerically compute the objective function $\gamma_0(Y)$ and the gradient $\nabla \gamma_0(Y)$, which is non-trivial because there is no analytical expression of $\gamma_0(Y)$. The detailed implementation of the iteration algorithm is summarized in Sec. VI-B.

3) The Solution for the Primal Problem: After the dual problem in (37) and (39) is solved, we then compute the solution of the primal problem in (30). The relationship between the primal and dual problems is stated as follow.

Given $\gamma = \gamma_0^*$ in (31) and (36), the optimums of (30) and (37) are also optimal for (31) and (36), respectively. From the KKT conditions of (31) and (36), the optimal solutions of (30) and (37) obey

$$\frac{\partial \bar{L}_2}{\partial f_k} = 2(Y + \sum_{i<k} d_i u_i u_i^H) f_k - \frac{2}{\gamma} d_k u_k u_k^H f_k = 0,$$

(40)

for $k = 1, \ldots, K$.

From (40), we compute $F_u$ by $f_k = \sqrt{b_k} \bar{f}_k$, where

$$\bar{f}_k = (Y + \sum_{i<k} d_i u_i u_i^H)^{\frac{1}{2}} u_k,$$

and $b_k$ is a real factor, for $k = 1, \ldots, K$. The factors $\{b_k\}$ can be determined by solving the following $K$ linear SINR equations:

$$\frac{1}{\gamma_k} \left| u_k^H \tilde{f}_k \right|^2 b_k = \sum_{i>k} \left| u_i^H \tilde{f}_i \right|^2 b_k = 1, \quad k = 1, \ldots, K.$$

(41)

V. SUM RATE MAXIMIZATION FOR DPC

In this section, we consider the sum rate maximization in the DPC regime.

Via the sum rate maximization, we maximize the data throughput at the transmitter and the overall spectral efficiency of multiuser communications. The formulated optimization is non-convex and hard to solve. To solve it, we use the downlink-uplink duality [36], [37], [42], which introduces a dual uplink multiple access channel (MAC) that has the same achievable rate region as the downlink GBC. Then the problem becomes the sum rate maximization for the MAC. Further, we compute the optimal solution of the sum rate maximization for the MAC via an equivalent convex optimization.

A. Problem Formulation

We maximize the achievable sum rate among the users with the transmit covariance constraint in the DPC regime. Defining the variables $\gamma_k \leq \text{SINR}^\text{dpc}_k$, for $k = 1, \ldots, K$, the optimization is expressed as

$$\max_{W_c, W_r, \gamma} \sum_{k=1}^{K} \log(1 + \gamma_k),$$

(42a)

s.t. $R_o = W_r W_r^H + W_c W_c^H$, (42b)

$$\text{SINR}^\text{dpc}_k \geq \gamma_k, \quad k = 1, \ldots, K.$$ (42c)

where $\gamma = [\gamma_1, \ldots, \gamma_K]^T$.

Similar to the reformulations in Sec. IV-A, (42) can be reformulated into optimizations with respect to $F$ and $F_u$. In particular, the equivalent optimization with respect to $F_u$ is

$$\max_{F_u, \gamma} \sum_{k=1}^{K} \log(1 + \gamma_k), \quad \text{s.t.} \quad F_u \preceq F_u^H \succeq I_r,$$ (43a)

$$\frac{1}{\gamma_k} |u_k^H f_k|^2 \geq \sum_{i>k} |u_i^H f_i|^2 + 1, \quad k = 1, \ldots, K.$$ (43b)

It is observed that (43) is non-convex since the SINR constraints in (43b) is non-convex. In the following, we apply the downlink-uplink duality to solve this non-convex optimization.

B. Optimization Reformulation Based on Downlink-Uplink Duality

This section first introduces the signal model of the dual uplink MAC with respect to the original downlink model, given in Sec. V-B.1. Based on the dual uplink MAC, we formulate the sum rate maximization problem in Sec. V-B.2. Then in Sec. V-B.3, we illustrate the downlink-uplink duality by showing that the uplink MAC and the downlink GBC have the same achievable rate region. Based on the duality, Sec. V-B.4 provides the solutions to the original downlink transmit design problem in (43).
1) Dual Uplink MAC Model: Consider a uplink MAC, in which $K$ users simultaneously transmit to a base station with $r$ antennas. Each user is equipped with a single transmit antenna. The channel is $\Sigma^{1/2}U^H \in \mathbb{C}^{r \times K}$. The received signal is

$$y_{ul} = \Sigma^{1/2}U^H x_{ul} + v_{ul},$$

(44)

where $x_{ul} = [x_{ul},1, \ldots, x_{ul},k]^T$ includes the transmit signal of the users, and $v_{ul}$ is additive Gaussian noise that has uncertain covariance $Y$ constrained by $\text{tr}(Y) = 1$, analogy to (35a). The transmit power of the $k$-th user, denoted by $d_k$, for $k = 1, \ldots, K$, will be allocated, under the sum power constraint $\sum_{k=1}^{K} d_k \leq 1$.

For the $k$-th user, the receiver applies a linear filter $\hat{f}_k$, and the output is

$$\hat{f}_k^H y_{ul} = \sum_{i=1}^{K} \hat{f}_k^H u_i x_{ul,i} + \hat{f}_k^H v_{ul}. $$

(45)

Corresponding to the DPC strategy in the downlink regime, we use successive cancellation with the reverse order $\{K, \ldots, 1\}$ [36] so that the signal from the $k+1, \ldots, K$-th user can be subtracted when decoding for the $k$-th user. Therefore, the SINR for the $k$-th user is

$$\text{SINR}^{\text{mac}}_k = \frac{d_k |\hat{f}_k^H u_k|^2}{\sum_{i<k} d_i |\hat{f}_k^H u_i|^2 + |\hat{f}_k^H Y f_k|^2}. $$

(46)

The sum rate is then given by $\sum_{k=1}^{K} \log(1 + \text{SINR}^{\text{mac}}_k)$. Below, we formulate the sum rate maximization for the MAC with respect to the filters $\{\hat{f}_k\}$, the transmit power $d := [d_1, \ldots, d_K]^T$, and the noise covariance $Y$.

2) Sum Rate Maximization Formulation: The maximum sum rate is defined with regard to the worst case of noise: seeking for the noise variance $Y$ that most worsens the sum rate. Therefore, we only need to consider non-singular $Y$, i.e., $Y \succ 0$, because the sum rate can be infinity when $Y$ is singular.

When $Y$ is non-singular, the minimum mean square error (MMSE) filter that maximizes the output SINR for the $k$-th user is given by [42]

$$\hat{f}_k = (Y + \sum_{i<k} d_i u_i u_i^H)^{-1} u_k, \quad k = 1, \ldots, K. $$

(47)

Correspondingly, the achieved SINR for the $k$-th user in (46) becomes

$$\text{SINR}^{\text{mac}}_k = d_k u_k^H (Y + \sum_{i<k} d_i u_i u_i^H)^{-1} u_k, \quad k = 1, \ldots, K,$$

(48)

and further the sum rate is written as

$$\sum_{k=1}^{K} \log(1 + \text{SINR}^{\text{mac}}_k) = \log \|Y + \sum_{k=1}^{K} d_k u_k u_k^H\| - \log \|Y\|.$$

(49)

Now we write the sum rate maximization for the dual MAC as

$$\min_{Y \succeq 0} \max_{d \succeq 0} \log \|Y + \sum_{k=1}^{K} d_k u_k u_k^H\| - \log \|Y\| $$

(50a)

s.t. $\text{tr}(Y) = 1$, $\sum_{k=1}^{K} d_k = 1.$

(50b)

In (50), all the constraints are convex. The objective function, denoted by $g(Y, d)$, is convex in $d$ and is concave in $Y$. Therefore, (50) is a solvable convex-concave saddle point problem. Solutions to this problem will be discussed later in Sec. V-C. As in [42], the sum rate maximization can be generalized to weighted sum rate maximization, which is still a convex-concave saddle point problem.

3) Downlink-Uplink Duality: The downlink-uplink duality is established from the power minimization problem and its dual. In (31), we assign individual SINR thresholds to the users, and the power minimization becomes:

$$\min_{\lambda, F_u} \lambda, \quad \text{s.t. } F_u F_u^H \preceq \lambda I_r,$$

(51a)

$$\frac{1}{\gamma_k} |u_k^H f_k|^2 \geq \sum_{i>k} |u_k^H f_i|^2 + 1, \quad k = 1, \ldots, K,$$

(51b)

where $\gamma_k$ is the given SINR for the $k$-th user, for $k = 1, \ldots, K$. Correspondingly, the Lagrange dual problem becomes:

$$\max_{\lambda \geq 0} \min_{d \succeq 0} \frac{1}{\gamma_k} \sum_{i<k} d_i u_i u_i^H + \frac{1}{\gamma_k} d_k u_k u_k^H, \quad k = 1, \ldots, K.$$ 

(52a)

$$Y + \sum_{i<k} d_i u_i u_i^H \geq \frac{1}{\gamma_k} d_k u_k u_k^H, \quad k = 1, \ldots, K.$$ 

(52b)

We denote the optimal value of (51) and (52) by $\lambda^*(\gamma)$ and $\lambda^*_\text{dual}(\gamma)$, respectively, where $\gamma = [\gamma_1, \ldots, \gamma_K]^T$.

Recall that for the GBC, the SINRs $\gamma_1, \ldots, \gamma_K$ are achievable if and only if $\lambda^*(\gamma) \leq 1$. Meanwhile, we note that the inner maximization in (52) is equivalent to [42]

$$\min_{d \succeq 0} \frac{1}{\gamma_k} \sum_{i<k} d_i u_i u_i^H + \frac{1}{\gamma_k} d_k u_k u_k^H, \quad k = 1, \ldots, K,$$

(53)

which finds the minimal transmit power of the MAC to achieve the SINRs under a given $Y$. Further, the optimal value of the outer maximization in (52) is the worst-case minimal transmit power under all possible $Y$ constrained by (35a). Therefore, $\lambda^*_\text{dual}(\gamma)$ gives the minimal transmit power to achieve the SINRs $\gamma_1, \ldots, \gamma_K$ in the MAC. Since the transmit power cannot exceed 1 in the MAC, the SINRs $\gamma_1, \ldots, \gamma_K$ are achievable in the MAC if and only if $\lambda^*_\text{dual}(\gamma) \leq 1$. From strong duality, $\lambda^*_\text{dual}(\gamma) = \lambda^*(\gamma)$, so the achievable region of the GBC and MAC are the same.

4) Solutions to the Original Downlink Problem: We compute $F_u$ for the original downlink problem in (43) after the saddle point solution for (50) is obtained. First, from the downlink-uplink duality, the obtained SINRs in the MAC given in (48) also give the SINRs in the GBC. Then, similar to
Sec. IV-C3, from the KKT conditions of (51) and (52), we compute $F_w$ by $f_k = \sqrt{b_k} \tilde{f}_k$, where $\tilde{f}_k$ is defined in (47) and $b_k$ is a real factor, for $k = 1, \ldots, K$. The factors $\{b_k\}$ can be determined by solving the following $K$ linear SINR equations:

$$\frac{1}{\text{SINR}_{k_{\text{max}}}} \left[ u_k^H \tilde{f}_k \right]^2 b_k - \sum_{i > k} \left[ u_k^H \tilde{f}_i \right]^2 b_i = 1, \quad k = 1, \ldots, K. \tag{54}$$

C. Solutions to (50)

To our knowledge, there are three types of methods to solve the convex concave saddle point problem in (50): (a)

1) The first type is first order algorithms, such as extra-gradient and optimistic gradient descent ascent [54], which only require the gradient;

2) The second type is interior-point algorithms [33], [42], which solve the KKT equations via Newton method and thus require second derivative;

3) The third one, as stated in [55], is to convert the saddle point problem to equivalent linear conic problems that is acceptable to convex optimization solvers like CVX [52], [53].

The implementation of the interior-point algorithms and first order algorithms is omitted here. Considering the well-structure of the saddle point problem in (50), we show that it is equivalent to the following convex optimization:

$$\min_{Z \succeq 0} \log |I + Z| - \log |Z|, \quad \text{s.t.} \quad u_k^H Zu_k \leq 1, \forall k. \tag{55}$$

Here, the optimal $Z$ is non-singular. The equivalence between (50) and (55) is from the following theorem.

**Theorem 2**: Let $Z^*$ be the optimum of (55). Hence, $Z^*$ obeys

$$Z^{*-1} - (I + Z^*)^{-1} = \sum_{k=1}^{K} \phi_k u_k u_k^H, \quad \phi_k u_k^H Z^* u_k = \phi_k, \tag{56}$$

for $k = 1, \ldots, K$, where $\{\phi_k \geq 0\}$ are the dual variables. Then the saddle point $(Y^*, d^*)$ of (50) can be computed by

$$d^* = \frac{1}{\eta} \phi, \quad Y^* = \frac{1}{\eta} (Z^* + I_r)^{-1}. \tag{57}$$

where $\phi = [\phi_1, \ldots, \phi_K]^T$ and $\eta = 1^T \phi$.

**Proof**: The KKT condition of (55) directly yields equations in (56). To show that $(Y^*, d^*)$ is a saddle point of (50), we only need to verify the KKT conditions; See Appendix E.

The problem in (55) can be reformulated to a linear conic optimization [56], and thus can be solved by optimization softwares in polynomial time. After $Z^*$ is obtained, we can first compute $\{\phi_k\}$ from (56), and then compute $(Y^*, d^*)$ from (57).

D. Discussions

It is worth noting that the maximized sum rate in the DPC regime equals to the sum rate capacity. To see this, we introduce a new variable $Z' = U \Sigma^{1/2} Z \Sigma^{1/2} U^H$, and then (55) becomes

$$\min_{Z' \succeq 0} \log |R_h + Z'| - \log |Z'|, \quad \text{s.t.} \quad Z'_{k,k} = 1, \quad k = 1, \ldots, K, \tag{58}$$

when $R_h$ is non-singular. In (58), the inequality constraints become equality constraints since the equality holds at the optimum. The optimal value of (58), named Sato upper bound [36]–[38], gives the upper bound for the sum rate capacity of the GBC. Note that the maximized sum rate via (42) is equal to the optimal value of (55), and thus equals to the Sato upper bound. Therefore, the sum rate capacity is achieved by DPC, which corresponds with the conclusion in [39] that DPC achieves the capacity region of MIMO GBC.

VI. NUMERICAL ITERATION ALGORITHMS

In this section, we provide the numerical iteration algorithms to solve the optimizations in (24) and (39), respectively. Here, (24) is formulated for the SINR balancing in the transmit beamforming regime, and (39) is formulated for the SINR balancing in the DPC regime.

A. Iteration Algorithm to Solve (24)

We solve the constrained optimization in (24) via gradient projection. The key step is to calculate the descend direction at a point $d$, denoted by $\Delta_d \in \mathbb{R}^K$, under the constraint. For convenience, we let $h(d)$ denote the objective function in (24), which is differential if $d > 0$. The gradient of $h(d)$, denoted by $\nabla h(d)$, is given by

$$\nabla h(d)_k = d_k^{-1} [(D^H D)^{1/2}]_{k,k}. \tag{59}$$

for $k = 1, \ldots, K$.

Considering the constraints, gradient projection computes the descend direction $\Delta_d$ via a projection operation [57], [58]:

$$\Delta_d = \mathcal{P}_{\Omega_1} (d - \rho \nabla h(d)) - d, \tag{60}$$

where the parameter $\rho > 0$ and $\mathcal{P}_{\Omega_1}(\cdot)$ is the orthogonal projection onto the constraint set $\Omega_1 = \{d \mid d \geq 0, s^T d = 1\}$. The projection in (60) does not have a close form expression. Nevertheless, this operation can be finished by at most $K$ loops. The details on the computation of the projection is omitted here, and is given in Appendix B.

The procedure of the iteration algorithm to solve (24) is summarized in Algorithm 1, where the step size $\alpha \in (0, 1)$. The values of $\rho$ and $\alpha$ can be different in each iteration. Algorithm 2.1 in [58] gives an efficient choice of $\rho$ and $\alpha$. The iterations will be stopped if the norm of the descend direction is smaller than a threshold $\varepsilon$ or the number of iterations exceeds $L$.

Here, we give some discussions on the computational complexity of the algorithm, which is evaluated by the number of complex scalar multiplications. In each iteration, we first need to obtain $h(d)$ and $\nabla h(d)$, which can be computed via the
singular value decomposition of $D$ and has a complexity of $O(K \tau(K + r))$. Then we find the descend direction via the projection operation. According to Appendix B, the projection can be obtained by at most $K$ loops, and in each loop we need at most $K$ multiplications. Therefore, the complexity for the projection is $O(K^2)$. Let $\epsilon$ be the acceptable error of the objective function. For gradient descend, the number of iterations needed to find a solution with error $\epsilon$ is $O(\log(1/\epsilon))$. As a summary, the total complexity of the algorithm is

$$O(\log(1/\epsilon)K \tau(K + r)),$$

given by the product of the complexity in each iteration and the number of iterations.

**B. Iteration Algorithm to Solve (39)**

We propose an iteration algorithm to solve (39). Note the objective function $\gamma_0(Y)$ is the optimal value of inner minimization in (37). To introduce the algorithm, we first give a fixed-point method for this inner minimization, then give the method to compute the gradient $\nabla \gamma_0(Y)$, and finally summarize the gradient projection algorithm to solve the constrained optimization in (39).

1) Fixed-Point Iterations for Inner Minimization: For a given $Y$, we compute $\gamma_0(Y)$ by solving (38), namely solving the inner minimization in (37). Define the index set $\mathcal{I}(Y) = \{k \mid u_k \notin C(Y), 1 \leq k \leq K\}$. The following equations hold at the optimum of (38):

$$\sum_{k=1}^{K} d_k = 1, \ d_k = 0 \text{ for } k \in \mathcal{I}(Y), \tag{61a}$$

$$\gamma = d_k u_k^H (Y + \sum_{i<k} d_i u_i u_i^H)^{-1} u_k \text{ for } k \notin \mathcal{I}(Y). \tag{61b}$$

For the case of $k \notin \mathcal{I}(Y)$, according to [30], [42] we compute $\{d_k\}$ and $\gamma$ by the fixed-point iteration method given in Algorithm 2, which converges rapidly in practice. After the algorithm converges, we have $\gamma_0(Y) = \gamma$.

2) Gradient Derivation: We give the expression of the gradient $\nabla \gamma_0(Y)$ below for $Y > 0$. For readability, we leave the derivation in Appendix C. Define a $K \times K$ lower triangular matrix $A$ by

$$A_{k,i} = -d_k |u_i^H \hat{f}_k|^2, \quad A_{k,k} = u_k^H \hat{f}_k, \tag{62}$$

for $1 \leq k \leq K$ and $1 \leq i < k$. Here, the expression of $\hat{f}_k$ is given in (47) and $d$ takes the value of the optimum in (38).

**Algorithm 1 Gradient Projection Algorithm to Solve (24)**

**Input:** $s \in R^K, u_k \in C^r, k = 1, \ldots, K$

1: Initialization: $d^{(0)} > 0, s^T d^{(0)} = 1, \ell \leftarrow 0$

2: repeat

3: $\Delta_d^{(\ell)} \leftarrow \mathcal{P}_{\Omega_1} \left( d^{(\ell)} - \rho \nabla h(d^{(\ell)}) \right) - d^{(\ell)}$

4: $d^{(\ell+1)} \leftarrow d^{(\ell)} + \omega \Delta_d^{(\ell)}$

5: $\ell \leftarrow \ell + 1$

6: until $||\Delta_d^{(\ell-1)}||_2 \leq \epsilon$ or $\ell > L$

**Output:** $d = d^{(\ell-1)}$

Letting $a = (A^{-1})^T 1 K$, we have

$$\nabla \gamma_0(Y) = -\sum_{k=1}^{K} a_k d_k \hat{f}_k \hat{f}_k^H, \tag{63}$$

where $a_k$ is the $k$-th element in $a$.

3) Summary of the Gradient Projection Iterations: Note that $\gamma_0(Y)$ is not differential when $Y$ is singular. To improve the convergence property when $Y$ is nearly singular, we introduce a log-det barrier function [33], and then (39) is reformulated into

$$\min_{Y \geq 0} \gamma_0(Y) - (1/\mu) \log |Y|, \text{ s.t. (35a)}, \tag{64}$$

where $\mu > 0$. The gradient projection iteration algorithm are developed for (64) and $\mu$ will approach $+\infty$ in the iterations. For a given $\mu$ and $Y > 0$, we computes the descend direction via a projection operation [57], [58]:

$$\Delta_y = \mathcal{P}_{\Omega_2} (Y - \rho \nabla \gamma_0(Y) + (\rho/\mu) Y^{-1}) - Y. \tag{65}$$

where the parameter $\rho > 0$ and $\mathcal{P}_{\Omega_2}()$ is the orthogonal projection onto the constraint set $\Omega_2 = \{Y \mid Y \succeq 0, \text{tr}(Y) = 1\}$. The projection does not have a close form expression, but can be computed by at most $K$ loops; See Appendix D.

The procedure of the gradient projection iteration is summarized in Algorithm 3, where $\alpha \in (0, 1), \beta > 1$ and FixPoint() represents the output of Algorithm 2. The values of $\rho$ and $\alpha$ can be different in each iteration. Algorithm 2.1 in [58] gives an efficient choice of $\rho$ and $\alpha$. The initial value of $\mu$ is $\mu_0$. For a given $\mu$, the inner iterations update $Y$ via gradient projection until the norm of the descend direction is smaller than a threshold $\epsilon$ or the number of iterations exceeds $L_2$. Once the inner iterations end, we increase $\mu$ by $\beta \mu$ and continue to run the inner iterations. The whole iterations will be stopped if $\mu$ is large enough, namely larger than a given $\mu_1$.

Here, we give some discussions on the computational complexity of the algorithm, which is evaluated by the number of complex scalar multiplications. In each iteration, the main computational steps include solving (38) via Algorithm 2, computing the gradient, and computing the projection. In Algorithm 2, the matrix inverse operations in step 3 has the highest complexity $O(K^3)$. Note that the complexity can actually be reduced by first computing $Y^{-1}$ with complexity $O(r^3)$ and then serially computing the matrix inverse via matrix inversion lemma with complexity $O(K^2)$.
Algorithm 3 Gradient Projection Algorithm to Solve (39)

Input: $u_k \in \mathbb{C}^r, k = 1, \ldots, K$

1: Initialization: $Y^{(0)} > 0$, $\text{tr}(Y^{(0)}) = 1$, $\ell \leftarrow 0$, $\mu \leftarrow \mu_0$
2: repeat
3: $j \leftarrow 0$
4: repeat
5: $d^{(t)} \leftarrow \text{FixPoint}(Y^{(t)})$
6: $\Delta^{(t)} = P_{	ext{d2}} \left( Y^{(t)} - \rho \nabla \gamma_o(Y^{(t)}) + \frac{2}{n} Y^{(t)} [Y^{(t)}]^{-1} \right) - Y^{(t)}$
7: $Y^{(t+1)} \leftarrow Y^{(t)} + \alpha \Delta^{(t)}$
8: $\ell \leftarrow \ell + 1$, $j \leftarrow j + 1$
9: until $\|\Delta^{(t)}\|_2 \leq \epsilon$ or $j > L_2$
10: $\mu \leftarrow \beta \mu$
11: until $\mu > \mu_1$

Output: $d = d^{(t-1)}$, $Y = Y^{(t-1)}$

The fixed-point method usually converges after 3 to 5 iterations in practice. Theoretically, the number of iterations to obtain a solution with error $\epsilon_1$ is $\log(1/\epsilon_1)$. So the total complexity of Algorithm 2 is $O(r^3 + \log(1/\epsilon_1)K^3r^2)$. In the computation of the gradient, $\alpha_k$ is already obtained in Algorithm 2, the complexity for computing $A$ in (62) is $O(K^2r)$, the complexity for computing $a$ is $O(K^2)$ as $A$ is lower triangular, and the complexity for computing $\nabla \gamma_o(Y)$ in (63) is $O(Kr^2)$. In the computation of the projection, we need to compute the eigen decomposition and solve (78), which has a complexity of $O(r^3 + K^2)$. In total, the complexity for each iteration is $O(r^3 + \log(1/\epsilon_1)Kr^2 + K^2 + r)$. With gradient descend, the number of inner iterations to find a solution with error $\epsilon_2$ is $O(\log(1/\epsilon_2))$, and the number of outer iterations does not exceed $\log(\beta)(\mu_1/\mu_0)$. Therefore, the total complexity of Algorithm 3 is

$$O(\log(\beta)(\mu_1/\mu_0)\log(1/\epsilon_2)(r^3 + \log(1/\epsilon_1)Kr^2 + K^2 + r)).$$

VII. NUMERICAL RESULTS

We performed numerical simulations to demonstrate the performance of multuser communications and radar via the proposed joint waveform design methods. The simulation settings are introduced in Sec. VII-A. In Sec. VII-B, the simulation results for SINR balancing in the transmit beamforming and DPC regimes are compared. The results of sum rate maximization are displayed in Sec. VII-C. The convergence property of the proposed iteration algorithm is displayed in Sec. VII-D. Sec. VII-E shows the radar performance gain of the proposed method compared to existing trade-off designs. It is also shown that the performance trade-off between radar and communication can be achieved in our method by varying $R_o$ in Sec. VII-F. Sec. VII-G verified that the error of the covariance in (3) is very small when $N$ is large.

A. Preliminaries

In the simulations, the transmit array is a uniform linear array with equal antenna spacing. The antenna spacing is half of the wavelength, and the number of transmit antenna is $M = 10$. The optimal covariance for radar $R_o$ is given by $R_o = PS_o$, where $P$ is the transmit power and $S_o$ is the power normalized covariance. For a given $S_o$, we performed numerical experiments with different $P$ to obtain the communication performance versus transmit SNR $P/\sigma^2$. We also compared the communication performance under three different values of $S_o$, which corresponds to three different radar beam patterns. The first value is $S_o = I_M/M$, with which the array transmits orthogonal waveforms and forms an omni-directional beam pattern for radar. The second value is $S_o = (1/M)I_M1_M^H$, which means that the array works in phased-array mode and forms a single beam towards 0°. The third value is obtained via the beam pattern matching design in [19] to form multiple beams towards $-40^\circ$, $0^\circ$, $40^\circ$ with a beam width of 10°. The corresponding transmit beam patterns under the three values of $S_o$ are displayed in Fig. 1.

For communications, the channel $H$ obeys Rayleigh fading, namely the elements in $H$ satisfy independent standard complex normal distributions. The noise power is $\sigma^2 = 1$. To display the communication performance, we run Monte Carlo tests with randomly generated $H$, and computed the average performance of communications.

B. Balanced SINR Versus Transmit SNR

The balanced SINR under different transmit SNR and $S_o$ for $K = 4$ and $K = 6$ is displayed in Fig. 2, in both transmit beamforming and DPC regimes. In the legend of Fig. 2, “TBF” is the abbreviation for transmit beamforming.

From Fig. 2, it is observed that DPC achieves higher balanced SINR than transmit beamforming. The performance improvement of DPC over transmit beamforming is especially impressive under a high transmit SNR. For omni-directional and multi-beam patterns, the balanced SINR via DPC increases linearly with the transmit SNR in dB scale, while the counterpart for transmit beamforming does not increase when the transmit SNR is high. The reason is that the interference cannot be effectively canceled via transmit beamforming with the transmit covariance constraint. To zero-forcing the interference, transmit beamforming requires $R_h = HR_oH^H$ be a diagonal matrix [17], while this condition generally does not.
hold if $H$ is Rayleigh fading. Since the interference cannot be eliminated, the balanced SINR for transmit beamforming keeps constant even if the SNR is high. Conversely, DPC are still able to cancel the interference under the transmit covariance constraint. To zero-forcing the interference in the DPC regime, we can compute a lower triangular $F$ via the Cholesky decomposition of $R_h$, which only requires $R_h$ be non-singular. This condition can be met if the rank of $S_o$, denoted by $r_o$, is not less than $K$ when $H$ obeys Rayleigh fading. We note that the value of $r_o$ are 10, 4 and 1 for the omni-directional, multi-beam and phased-array pattern, respectively. When $K = 4$, the condition holds for omni-directional and multi-beam patterns, and thus the corresponding balanced SINR for DPC is acceptable under a high SNR. For phased-array mode, $r_o$ is less than $K$, and thus its balanced SINR in the DPC regime becomes much lower compared to omni-directional and multi-beam patterns. Nevertheless, DPC is still able to achieve an acceptable balanced SINR for phased-array beam when the transmit SINR is high, while we observe that the counterpart via transmit beamforming is even less than 0-dB.

By comparing the results for $K = 4$ and $K = 6$ in Fig. 2, one observes that the balanced SINR becomes lower when $K$ increases, namely the service quality for each user can worsen when the number of users increases. We also observe that the loss of balanced SINR in the DPC regime is slight for omni-directional beam pattern, but is notable for the multi-beam pattern. We note that when $K = 6$, $r_o$ is larger than $K$ for omni-directional beam pattern, and thus DPC is still able to zero-forcing the interference. However, for the multi-beam pattern, $r_o$ is less than $K = 6$, and thus zero forcing (ZF) DPC is not applicable, leading to an obvious performance degradation. Based on the above facts, we can regard the $r_o$, the rank of $S_o$, as the degrees of freedom of the communication transmitter. In the DPC regime, increasing $K$ generally does not cause serious loss of service quality if $K$ does not exceed the degrees of freedom, while the loss can be more significant when $K$ exceeds it. It is worth noting that when $K$ exceeds $r_o$, simultaneously servicing for $K$ users via transmit beamforming is almost unrealistic, since the balanced SINR is extremely low.

C. Maximized Sum Rate Versus Transmit SNR

The maximized sum rate versus transmit SINR in the DPC regime is given in Fig. 3, for $K = 4, 6$. In Fig. 3, the sum rate is asymptotically affine in the transmit SNR in dB, and the slope of the line determines the multiplexing gain [40], [44] of multiuser communications, which equals to the rate gain in bits/channeluse for every 3-dB transmit power gain. With a power constraint, it is proven in [40] that the multiplexing gain of the GBC is $K$. With the considered transmit covariance constraint, the results is different. For instance, we read from Fig. 3 that the multiplexing gain for the multi-beam pattern does not increase when $K$ increases from 4 to 6. A similar result is observed from the curve for phase-array mode. Nevertheless, for omni-directional beam pattern, the multiplexing gain increases from 4 to 6 when $K$ increases from 4 to 6. In summary, one can find that the multiplexing gain is $\min\{K, r_o\}$ with the transmit covariance constraint. The explanation is from the fact that ZF DPC is asymptotic optimal under a high SNR. When $K \leq r_o$, the GBC can be simplified to $K$ AWGN channels to the $K$ users via ZF DPC, and thus the multiplexing gain is $K$. However, when $K > r_o$, to meet the constraint $F F^H \preceq R_h$, $F$ can have at most $r_o$ non-zero diagonal elements if it is lower triangular. In other words, to zero forcing the interference, only $r_o$ users are active while the others are inactive. Therefore, when $K > r_o$, the multiplexing gain is restricted by $r_o$, and the sum rate gain is not obvious if $K$ exceeds $r_o$.

D. Convergence Performance of the Iteration Algorithms

The convergence performance of Algorithm 1 to solve the SINR balancing problem for transmit beamforming is displayed in Fig. 4 and Fig. 5, where the SINR gap $\gamma^{(t)} - \gamma^*$ versus iteration index $\ell$ and CPU time under 6 channel realizations are demonstrated, for $K = 4$, $S_o = (1/M)I_M$ and $P = 10$. Here, $\gamma^*$ is the balanced SINR, obtained by the linear conic programming in (16), and $\gamma^{(t)}$ is the temporary SINR after the $\ell$-th iteration in Algorithm 1, given by

$$t^{(\ell)} = \min_{i \leq \ell} h(d^{(i)}), \quad \gamma^{(t)} = [t^{(\ell)}]^2/(1 - [t^{(\ell)}]^2),$$
where $\ell^{(e)}$ is the best value of the objective function in the first $\ell$ iterations. The data of CPU time is collected by implementing Algorithm 1 on MATLAB and running the program on a workstation equipped with Intel Xeon Gold 6132 CPU @ 2.60 GHz.

From Fig. 4, we observe that the iteration in Algorithm 1 converges fast. In some experiments, the SINR gap is less than $10^{-6}$ after no more than 10 iterations. In Fig. 5, it is demonstrated that the iteration generally converges in around 1 millisecond when $K = 4$. We found that Algorithm 1 is much more efficient than solving (16) with CVX, which usually consumes about 200 milliseconds. Considering the interference control, the number of users for transmit beamforming needs to be limited. Therefore, the dimension of the variable $d$ can be low in practice, and it is hopeful to implement the algorithm in real time.

The convergence performance of Algorithm 3 to solve the SINR balancing problem for DPC is displayed in Fig. 6 and Fig. 7, which gives the SINR gap $\gamma^{(e)} - \gamma^*$ versus iteration index $\ell$ and CPU time under 6 channel realizations, for $K = 4$, $S_o = (1/M)I_M$ and $P = 10$. Here, the balanced SINR $\gamma^*_o$ is the optimal value of (30) and (37), and $\gamma^{(e)}$ is the best value of $\gamma_o(Y)$ in the first $\ell$ iterations, i.e.

$$\gamma^{(e)} = \min_{i \leq \ell} \gamma_o(Y^{(e)}).$$

To perform the experiments, we first let the balanced SINR be $\gamma^*_o = 10$, next compute the minimal $\lambda$ to achieve the SINR by solving (31) with $R_o = S_o$, and then use Algorithm 3 to solve (37) with the new covariance $R_o = \lambda S_o$. The data of CPU time is collected by implementing Algorithm 3 on MATLAB and running the program on a workstation equipped with Intel Xeon Gold 6132 CPU @ 2.60 GHz.

Compared with Algorithm 1 for transmit beamforming, Algorithm 3 for DPC needs more iterations to achieve a small SINR gap. This is mainly because the optimization in (37) has a more complex structure and a higher dimension than that in (24). In the experiments, we observe that the iteration converges within a moderate number of times when the optimal $Y$ is non-singular, while the SINR gap decreases slower when the optimal $Y$ is singular. As Fig. 7 shows, the iteration in Algorithm 3 generally needs hundreds of milliseconds to converge. Although this algorithm may not be implemented in real time, it is much more efficient than the bisection search method in Sec. IV-B, which generally needs more than 5 seconds to make the SINR gap less than $10^{-6}$.

For practical applications, a trade-off between the accuracy and computation time can be considered. In other words, we can control the number of iterations and obtain an approximate solution. To improve the efficiency and reduce the CPU time of
In particular, we give the relative beam pattern mean square error (MSE) between the beam pattern by the trade-off design and the optimal one for radar in Fig. 8 under two channel realizations, where $K = 6$ and the achieved downlink SINR is 15-dB. The optimal radar pattern in our scheme is also given for comparison.

The motivation of the proposed scheme is to improve the communication performance under the constraint that the trade-off design in [17] by comparing their radar performance and the achieved downlink SINR is 15-dB. Meanwhile, the optimal radar pattern in our scheme is also given for comparison.

The algorithm, one may further consider deep leaning enabled acceleration schemes as in [31].

**E. Comparison With Existing Trade-off Designs**

Here, we compared the proposed joint design scheme with existing trade-off designs, which typically optimize the transmit covariance for radar with individual SINR constraints for downlink communication users. These trade-off designs guarantee the communication performance, but lead to performance loss of MIMO radar. Conversely, our proposed scheme constrains that the transmit covariance is optimal for radar. With this constraint, our schemes achieves better radar performance than the existing trade-off designs, but the price is the reduction of the downlink SINRs. Nevertheless, our scheme employs DPC to further cancel the interference, and may still achieve the same downlink SINRs as the trade-off designs which only use transmit beamforming, without sacrificing the radar performance.

We show the advantage of the proposed scheme over the trade-off design in [17] by comparing their radar performance under the same SINR at communication receivers. Consider the transmit beamforming design for MIMO radar, which aims to form multiple beams towards $-40^\circ, 0^\circ, 40^\circ$ with a beam width of $10^\circ$. The corresponding parameters for radar is given in [17] and the optimal radar beam pattern is the multi-beam pattern in Fig. 1. For the multi-beam pattern, we read from Fig. 2 that the average balanced SINR in the DPC regime is 15-dB when the transmit SNR is 30-dB, i.e. $P = 1000$, for $K = 6$. Then in the trade-off design, we perform a beam pattern match design under the constraints that the achieved SINRs for the $K$ users are higher than 15-dB.

In Fig. 8, we compare the achieved radar transmit beam pattern by the proposed design and the trade-off design under two channel realizations. Here, the proposed design always achieves the optimal beam pattern for radar, while the trade-off design results in a notable beam pattern mismatch. Meanwhile, we observe that the transmit beam pattern by the trade-off design is different under different channel realizations. In particular, we give the relative beam pattern mean square error (MSE) between the beam pattern by the trade-off design and the optimal one for radar in Fig. 9 under 100 channel realizations. Here, the relative beam pattern MSE is defined in eq. (30) of [19] and averaged over the angular interval $[-90^\circ, 90^\circ]$. As shown in Fig. 9, in the trade-off design, the beam pattern mismatch can be significant under some channel realizations, so some relevant radar performance will fluctuate with the fading of the channel and cannot be well guaranteed, which is not desirable for radars.

In addition to the transmit beam pattern, we also compared the proposed scheme with the trade-off design in [17] in terms of angular estimation accuracy. In the simulation, there are three radar targets located at the beam directions $\theta_1 = -40^\circ$, $\theta_2 = 0^\circ$, and $\theta_3 = 40^\circ$, respectively. These targets are in the same range resolution bin and the complex amplitude of the targets are all 1. The number of signal samples is $N = 1024$. The radar reuses the $M$ transmit antennas to receive the targets' reflected signal, which is corrupted by AWGN with power $\sigma_r^2$. The angles of the targets are estimated by finding the peaks in the Capon spatial spectrum [19]. The angle estimation performance is evaluated by the root-mean-square-error (RMSE), defined as

$$\text{RMSE} = \sqrt{\mathbb{E}\left\{\frac{1}{3} \sum_{p=1}^{3} (\theta_p - \hat{\theta}_p)^2\right\}},$$  

(66)

where $\theta_p$ is the real angle and $\hat{\theta}_p$ is the estimated angle for the $p$-th target, for $p = 1, \ldots, 3$. We run the simulations under different $\sigma_r^2$ and show the estimation RMSE versus radar SNR $P/\sigma_r^2$ in Fig. 10. From Fig. 10, we observe that the proposed scheme achieves better estimation accuracy than the trade-off design, since the transmit beamforming pattern of the trade-off design is not stable while the proposed design can achieve the optimal beam pattern.

**F. Performance Trade-off Between Radar and Communications**

The motivation of the proposed scheme is to improve the communication performance under the constraint that the
transmit covariance is optimal for radar, in which the performance loss of radar is not desired. Nevertheless, our method is still able to achieve the performance trade-off between radar and communication by changing the covariance matrix $R_o$. In particular, we consider the following $R_o$ parameterized by $\varrho$, $0 \leq \varrho \leq 1$, according to [20]

$$
[R_o]_{i,j} = \begin{cases} 
\frac{P}{M} & i = j, \\
(P/M)\varrho^{i-j} & i \neq j,
\end{cases}
$$

for $i, j = 1, \ldots, M$.

When $\varrho = 0$, $R_o = (P/M)I_M$ and the array elements transmit orthogonal waveforms. When $\varrho = 1$, $R_o = (P/M)1_M1_M^T$ and the radar works in phase array mode. According to the results in Sec. VII-B and Sec. VII-C, the communication performance is best when $\varrho = 0$. For radar, we consider the target scene where a single target is located in the direction of $0^\circ$. When there is no clutter, the receiver SINR of radar can be maximized when $\varrho = 1$. Therefore, we can observe a performance trade-off between radar and communication when $\varrho$ varies from 0 to 1.

In this simulation, we demonstrate the communication and radar performance under different $\varrho$, for $K = 4$ and $P = 10$. In the setting of radar, the number of signal samples is $N = 1024$ and the received radar signal is corrupted with AWGN whose power is $\sigma_r^2 = 1000$. The target is parameterized by the direction $\theta_0$ and amplitude $\alpha_0$, where $\theta_0 = 0^\circ$ and $E(|\alpha_0|^2) = 1$. The impact of spatial clutters in the radar observation area is also considered. In particular, there are 11 clutter points with their directions $\{\theta_q\}$ and amplitudes $\{\alpha_q\}$ given by

$$
\theta_q = (29 + q)^\circ, \quad E(|\alpha_q|^2) = \sigma_c^2, \quad q = 1, \ldots, 11,
$$

where $\sigma_c^2$ determines the clutter power. At receiver, radar employs the MVDR filter to maximize the signal-to-interference-plus-noise ratio (SCNR) [59]. Fig. 11 gives the radar SCNR under different $\varrho$, for $\sigma_c = 1000$ and $\sigma_c = 0$. Here, $\sigma_c = 0$ means that there is no clutter. It is observed that the SCNR can be improved if $\varrho$ increases. Comparing the results for $\sigma_c = 1000$ and $\sigma_c = 0$, it can be inferred that the spatial clutters only have minor affect to the detection performance of radar if their directions are far away from that of the targets.

As for the communication performance, (12) gives the sum rate as a function of $\varrho$. When $\varrho$ increases, the sum rate of communications becomes lower, but the radar performance becomes better, leading to a performance trade-off between radar and communications.

G. Sampled Transmit Covariance of Finite-Length Signals

In (2) and (3), the transmit covariance is defined and derived in a statistical sense, under the condition that the radar waveform and the communication symbols to different users are statistically uncorrelated. However, when the signal length is finite, the sampled covariance matrix

$$
R_s = \frac{1}{N} \sum_{n=0}^{N-1} x(n)x^H(n)
$$

may be slightly different from the average covariance matrix given in (3), which further leads to slight difference of the transmit beam pattern. Nevertheless, when the signal length $N$ is large, the covariance mismatch can be approximately ignored.

To verify this statement, we compute the relative beam pattern MSE between the transmit beam patterns for the
optimal covariance and sampled covariance, under different signal length \( N \). Here, the optimal covariance corresponds to the multi-beam pattern in Fig. 1. The precoders are obtained by the SINR balancing design for transmit beamforming. The radar signal \( s(n) \) are generated by random phase coding, and the communication symbols are generated by 16-quadrature amplitude modulation (QAM) modulation. The relative beam pattern MSE is defined in eq. (30) of [19] and averaged over the angular interval \([-90^\circ, 90^\circ] \). The beam pattern MSE versus \( N \) is given in Fig. 13, which shows that the beam pattern MSE decreases when \( N \) increases. The beam pattern MSE is very small when \( N \) is large, further indicating that the covariance mismatch can be ignored for a large \( N \).

VIII. CONCLUSION

In this paper, we consider the transmit design of a joint MIMO radar and downlink multiuser communications system, in which the communication performance is optimized under a transmit covariance constraint from radar. In particular, we formulate the SINR balancing problem in both transmit beamforming and DPC regimes, and the sum rate maximization in the DPC regime. Further, we proposed methods to solve these problems via convex optimization. Despite the low complexity of transmit beamforming, the achievable SINR via transmit beamforming may be low even if the transmit SNR is high. As the theoretically optimal scheme for multiuser precoding, DPC has a impressive performance gain over transmit beamforming, with increased complexity for encoding and optimization. In the simulations, it is observed that the degrees of freedom for the communication transmitter is restricted by the rank of the transmit covariance.

It is worth noting that our work relies on some assumptions on the channel. First, we assume that the channel is perfectly known at the transmitter. However, in practical applications this assumption may not hold, e.g. when there exists channel estimation error. In the future, it is meaningful to consider the precoder design in the case that the channel is only partially known at the transmitter, under some prior knowledge on the difference between the actual channel and known channel at transmitter. Second, in our model the channel is assumed to be non-frequency selective. However, in wideband communications, the channel may be frequency selective and MIMO communication transmitters generally transmit OFDM waveform to eliminate the effects of multipath. The OFDM waveform design for joint radar and communications under frequency selective channel can be studied in the future.

APPENDIX A

THE PROOF OF THEOREM 1

To finish the proof, we need to verify that the constructed \( W_c, W_r \) in (12) obeys (11). We first verify that \( HW_c = F_c \). Since \( FF^H \preceq R_h \), one has \( F \in \mathcal{C}(HR_o^{1/2}) \) [49], where \( \mathcal{C}(\cdot) \) represents the column space of a matrix. As a corollary, \( HR_o^{1/2}R_c \) \( \preceq \) \( F \), and thus

\[
HW_c = HR_o^{1/2}R_cF = F.
\]

Then, we show that \( R_o \geq WW_c^H \). Since \( FF^H \preceq R_h \), it follows that

\[
W_cW_c^H = R_o^{1/2}(HR_o^{1/2})^1FF^H(R_o^{1/2}H^H)^1R_o^{1/2} \leq R_o^{1/2}R_o^{1/2} = R_o.
\]

Therefore, it is valid to compute \( W_r \) via (12), and the second equation in (11) holds if \( W_r \) is given by (12), completing the proof.

APPENDIX B

COMPUTATION OF THE PROJECTION ONTO \( \Omega_1 \)

The projection of a point \( d = [d_1, \ldots, d_K]^T \in \mathbb{R}^K \) onto \( \Omega_1 \) is expressed as

\[
x = \arg \min_{x} \frac{1}{2} \|d - \bar{x}\|^2_2, \quad \text{s.t. } \bar{x} \geq 0, \quad s^T \bar{x} = 1,
\]

where \( s = [s_1, \ldots, s_K]^T \geq 0 \). Letting \( \theta = [\theta_1, \ldots, \theta_K]^T \geq 0 \) and \( v \) be the dual variables associated with the constrains \( \bar{x} \geq 0 \) and \( s^T \bar{x} = 1 \), \( x \) is the solution of the KKT system [33]:

\[
x_k = d_k + \theta_k - vs_k = 0, \quad s^T x = 1, \quad \theta_k x_k = 0, \quad x_k \geq 0, \quad \theta_k \geq 0,
\]

(69)

where \( x_k \) is the \( k \)-th element in \( x \), for \( k = 1, \ldots, K \).

To solve (69), we let \( r_k = d_k/s_k \). Without loss of generality, it is assumed that \( r_1 \leq \cdots \leq r_K \). First, we note that \( v \) obeys \( v \leq r_K \). If \( v \geq r_K \), we have \( x_k^2 = x_k \theta_k + x_k(d_k - vs_k) \leq 0 \) for all \( k \). Then \( x = 0 \), which is not feasible. There exists an integer \( 1 \leq j \leq K \) such that \( v \in [r_{j-1}, r_j) \). If \( j = 1 \), the interval is \((\infty, r_1)\). Similarly, we have \( x_k = 0 \) for all \( k < j \). In addition, for \( k \geq j \), we have \( x_k - \theta_k = d_k - vs_k > 0 \), so \( \theta_k = 0 \). Based on these conclusions, we have

\[
x_k = 0, \quad \forall k < j, \quad x_k = d_k - vs_k, \quad \forall k \geq j.
\]

(70)

Since \( s^T x = 1 \), \( v \) is given by

\[
v = \sum_{k=j}^{K} \frac{d_k - 1}{\sum_{k=j}^{K} s_k}.
\]

(71)

The problem is that \( j \) is unknown. Nevertheless, we can search for a \( j \) so that \( v(j) \in [r_{j-1}, r_j) \), where \( v(j) \) is the
value of \( v \) computed by (71). Since \( j \) is an integer in \([1, K]\), the times of searching is not larger than \( K \). Once \( j \) is obtained, the projection \( x \) can be computed via (70) and (71).

**APPENDIX C**

**THE GRADIENT OF THE OBJECTIVE FUNCTION IN (39)**

To derive the gradient, we note that the optimal \( \gamma, d \) for (38) is the solution of the equations in (61). When \( Y > 0 \), the index set \( \mathcal{I}(Y) \) is empty. Computing the differential to the equations in (61), we have

\[
1_k^T d\gamma = 0, \quad d\gamma - dd_k u_k u_k^H f_k = -d_k f_k^H (dY + \sum_{i<k} dd_i u_i u_i^H) f_k, \quad (72a)
\]

\[
\text{for } k = 1, \ldots, K. \quad \text{We rewrite (72b) into a matrix form:}
\]

\[
Add = d\gamma 1_K + b, \quad (73)
\]

where the \( k \)-th element in \( b \) is \( d_k f_k^H dY f_k \) and \( A \) is defined in (62). Combining (72a) and (73), one has

\[
1_k^T A^{-1} 1_K d\gamma + 1_k^T A^{-1} b = 0. \quad (74)
\]

Letting \( a_k \) be the \( k \)-th element in \( a = (A^{-1})^T 1_K \), there is

\[
d\gamma = -\frac{1}{\sum_{k=1}^K a_k} \text{tr}\{ \sum_{k=1}^K a_k d_k f_k f_k^H dY \}, \quad (75)
\]

from which one can obtain the gradient in (63).

**APPENDIX D**

**COMPUTATION OF THE PROJECTION ONTO \( \Omega_2 \)**

The projection of a Hermitian matrix \( Y \in \mathbb{C}^{r \times r} \) onto \( \Omega_2 \) is

\[
\arg \min_X \frac{1}{2} \| Y - X \|_F^2 \quad \text{subject to } \text{tr}(X) = 1, \quad X \succeq 0. \quad (76)
\]

To solve (76), we write the eigen decomposition of \( Y \) as \( Y = V \Sigma_y V^H \), where \( V \) is a \( r \times r \) unitary matrix and \( \Sigma_y \) is diagonal. Letting \( \Sigma_x = V^H X V \), the optimization in (76) is reformulated to

\[
\min_{\Sigma_x} \frac{1}{2} \| \Sigma_y - \Sigma_x \|_F^2 \quad \text{subject to } \text{tr}(\Sigma_x) = 1, \quad \Sigma_x \succeq 0. \quad (77)
\]

It can be observed that \( \Sigma_x \) is diagonal at the optimum. We let \( y_k \) and \( x_k \) be the \((k, k)\)-th elements in \( \Sigma_y \) and \( \Sigma_x \), respectively. Then (77) is equivalent to

\[
\min_x \frac{1}{2} \| y - x \|_2^2 \quad \text{subject to } 1^T x = 1, \quad x \succeq 0. \quad (78)
\]

where \( x = [x_1, \ldots, x_r]^T \) and \( y = [y_1, \ldots, y_r]^T \). Here, (78) has the same form as the optimization in (68), and can be solved with no more than \( K \) loops. Once the optimal \( x \) is obtained, the projection is given by \( V \text{diag}(x) V^H \).

**APPENDIX E**

**PROOF FOR THEOREM 2**

In this proof, we verify that \( Y^* \) and \( d^* \) meet the KKT condition of (50), which is stated as

\[
(Y^* + \sum_{k=1}^K d_k^* u_k u_k^H)^{-1} - [Y^*]^{-1} + \nu I_r = 0, \quad (79a)
\]

\[
\text{tr}(Y^*) = 1, \quad Y^* > 0, \quad (79b)
\]

\[
u_j(Y^* + \sum_{k=1}^K d_k^* u_k u_k^H)^{-1} u_j - \eta + \varphi_j = 0, \quad (79c)
\]

\[
\sum_{k=1}^K d_k^* = 1, \quad d_j^* \geq 0, \quad \varphi_j \geq 0, \quad \varphi_j d_j^* = 0, \quad (79d)
\]

for \( j = 1, \ldots, K \). Here, \( \nu, \eta \) and \( \{\varphi_k\} \) are the dual variables associated with the constraints \( \text{tr}(Y) = 1, \sum_{k=1}^K d_k = 1 \) and \( d \succeq 0 \), respectively, and their values are given by

\[
\eta = 1_k^T \phi, \quad \nu = \varphi, \quad \varphi_k = \eta - u_k^H Z^* u_k, \quad (80)
\]

for \( k = 1, \ldots, K \).

In the following, we check the conditions in (79) one by one. First, we point out an important relationship between \( d^* \), \( Y^* \) and \( Z^* \). From (56), one has

\[
(1/\eta) [Z^*]^{-1} = Y^* + \sum_{k=1}^K d_k^* u_k u_k^H. \quad (81)
\]

Then

\[
(Y^* + \sum_{k=1}^K d_k^* u_k u_k^H)^{-1} - [Y^*]^{-1} = \eta Z^* - [Y^*]^{-1} = -\nu I_r,
\]

i.e. (79a) holds, and

\[
u_j(Y^* + \sum_{k=1}^K d_k^* u_k u_k^H)^{-1} u_j = \eta u_j^H Z^* u_j = \eta - \varphi_j,
\]

i.e. (79c) holds. In (79b), \( Y^* > 0 \) is trivial, and

\[
\text{tr}(Y^*) = \frac{1}{\eta} \text{tr}((Z^* + I_r)^{-1}) = \frac{1}{\eta} \text{tr}((Z^* + I_r)^{-1} Z^*). \quad (82)
\]

Multiply the left and right side of (56) by \( Z^* \) and take the matrix trace, we have

\[
\text{tr}((Z^* + I_r)^{-1} Z^*) = r - \sum_{k=1}^K \phi_k u_k^H Z^* u_k. \quad (83)
\]

Substituting (83) into (82), one has

\[
\text{tr}(Y^*) = \frac{1}{\eta} \sum_{k=1}^K \phi_k u_k^H Z^* u_k = \frac{1}{\eta} \sum_{k=1}^K \phi_k = 1, \quad (84)
\]

so (79b) holds. Finally we show that (79d) holds. The first three conditions in (79d) are trivial. According to (56), it can be shown that

\[
\varphi_j d_j^* = \phi_j (1 - u_j^H Z^* u_j) = 0. \quad (85)
\]

Therefore (79d) holds and the proof is completed.
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