ALMOST $\alpha$-COSYMPLECTIC $(\kappa, \mu, \nu)$-SPACES

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Abstract. Main interest of the present paper is to investigate the almost $\alpha$-cosymplectic manifolds for which the characteristic vector field of the almost $\alpha$-cosymplectic structure satisfies a specific $(\kappa, \mu, \nu)$-nullity condition. This condition is invariant under $D$-homothetic deformation of the almost cosymplectic $(\kappa, \mu, \nu)$-spaces in all dimensions. Also, we prove that for dimensions greater than three, $\kappa, \mu, \nu$ are not necessary constant smooth functions such that $df \wedge \eta = 0$. Then the existence of the three-dimensional case of almost cosymplectic $(\kappa, \mu, \nu)$-spaces are studied. Finally, we construct an appropriate example of such manifolds.

1. Introduction

It is well known that there exist contact metric manifolds $(M^{2n+1}, \phi, \xi, \eta, g)$, for which the curvature tensor $R$ and the direction of the characteristic vector field $\xi$ satisfy $R(X, Y)\xi = 0$, for any vector fields on $M^{2n+1}$. Using a $D$-homothetic deformation to a contact metric manifold with $R(X, Y)\xi = 0$ we get a contact metric manifold satisfying the following special condition

\begin{equation}
R(X, Y)\xi = \eta(Y)(\kappa I + \mu h)X - \eta(X)(\kappa I + \mu h)Y,
\end{equation}

where $\kappa, \mu$ are constants and $h$ is the self-adjoint $(1,1)$-tensor field. This condition is called $(\kappa, \mu)$-nullity on $M^{2n+1}$. Contact metric manifolds with $(\kappa, \mu)$-nullity condition studied for $\kappa, \mu$ = const. in [15], [16].

In [15], the author introduced contact metric manifold whose characteristic vector field belongs to the $(\kappa, \mu)$-nullity condition and proved that non-Sasakian contact metric manifold is completely determined locally by its dimension for the constant values of $\kappa$ and $\mu$.

Koufogiorgos and Tsichlias found a new class of 3-dimensional contact metric manifolds that $\kappa$ and $\mu$ are non-constant smooth functions. They generalized $(\kappa, \mu)$-contact metric manifolds $(M^{2n+1}, \phi, \xi, \eta, g)$ for dimensions greater than three on non-Sasakian manifolds, where the functions $\kappa, \mu$ are constant.

Following these works, P.Dacko and Z.Olszak extensively have studied almost cosymplectic $(\kappa, \mu, \nu)$ manifolds. These almost cosymplectic manifolds whose almost cosymplectic structures $(\phi, \xi, \eta, g)$ satisfy the condition

\begin{equation}
R(X, Y)\xi = \eta(Y)(\kappa I + \mu h + \nu \phi h)X - \eta(X)(\kappa I + \mu h + \nu \phi h)Y,
\end{equation}

for $\kappa, \mu, \nu \in \mathcal{R}_\eta(M^{2n+1})$, where $\mathcal{R}_\eta(M^{2n+1})$ be the subring of the ring of smooth functions $f$ on $M^{2n+1}$ for which $df \wedge \eta = 0$. In the sequel, such manifolds are called almost cosymplectic $(\kappa, \mu, \nu)$-spaces ([17]). The condition (1.2) is invariant with respect to the $D$-homothetic deformations of those structures. The authors showed

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that the integral submanifolds of the distribution $\mathcal{D}$ of such manifolds were locally flat Kaehlerian manifolds and found a new characterization which was established up to a $D$-homothetic deformation of the almost cosymplectic manifolds. In [11], a complete local description of almost cosymplectic $(-1, \mu, 0)$-spaces via “model spaces” is given, depending on the function $\mu$. When $\mu$ is constant, the models are Lie groups with a left-invariant almost cosymplectic structure.

Moreover, Pastore and Dileo are studied the curvature properties of almost Kenmotsu manifolds, with special attention to $(\kappa, \mu)$-nullity condition for $\kappa, \mu = \text{const.}$ and $\nu = 0$ (see [13], [14]). In [13] the authors prove that an almost Kenmotsu manifold $M^{2n+1}$ is locally a warped product of an almost Kaehler manifold and an open interval. If additionally $M^{2n+1}$ is locally symmetric then it is locally isometric to the hyperbolic space $H^{2n+1}$ of constant sectional curvature $c = -1$. It is recall that model spaces for almost cosymplectic case were given in ([17]). But we did not know any example of an almost $\alpha$-Kenmotsu manifold satisfying (1.2) with non-constant smooth functions. The following question sounds that especially interesting. Do there exist almost $\alpha$-cosymplectic manifolds satisfying (1.2) with $\kappa, \mu$ non-constant smooth functions? In this paper, we will try to give an answer to this question for dimension 3.

The existence and invariance of the condition (1.2) have been our motivation in studying almost $\alpha$-cosymplectic manifold.

Section 2 is devoted to preliminaries on almost contact metric structures. In section 3 we give the concept of almost $\alpha$-cosymplectic manifolds, state general curvature properties and derive several important formulas on almost $\alpha$-cosymplectic manifolds. These formulas enable us to find the geometrical properties of almost $\alpha$-cosymplectic manifolds with $\eta$-parallel tensor $h$. In section 4 we study almost $\alpha$-cosymplectic manifolds with $\eta$-parallel tensor field $h$ under some certain conditions and prove that the integral submanifolds of the distribution $\mathcal{D}$ have Kaehlerian structures if $h$ is $\eta$-parallel. In section 5 we introduce the notion of almost $\alpha$-cosymplectic $(\kappa, \mu, \nu)$-spaces in terms of a specific curvature condition. We give a characterization of a $D$-homothetic deformation of the almost co-symplectic $(\kappa, \mu, \nu)$-spaces. We prove that for dimensions greater than three, $\kappa, \mu, \nu$ are not necessary constant smooth functions. Also, we will prove why the functions $\kappa, \mu$ and $\nu$ are element of $\mathcal{R}_\eta(M^{2n+1})$.

Finally, in section 6 we investigate the existence of the three-dimensional case of almost cosymplectic $(\kappa, \mu, \nu)$-spaces. Then we give an example and describe the three-dimensional case.

2. Almost $\alpha$-cosymplectic Manifolds

An almost contact manifold is an odd-dimensional manifold $M^{2n+1}$ which carries a field $\phi$ of endomorphisms of the tangent spaces, a vector field $\xi$, called characteristic or Reeb vector field, and a 1-form $\eta$ satisfying $\phi^2 = -I + \eta \otimes \xi$ and $\eta(\xi) = 1$, where $I : TM^{2n+1} \to TM^{2n+1}$ is the identity mapping. From the definition it follows also that $\phi \xi = 0$, $\eta \circ \phi = 0$ and that the $(1,1)$-tensor field $\phi$ has constant rank $2n$ (see [4]). An almost contact manifold $(M^{2n+1}, \phi, \xi, \eta)$ is said to be normal when the tensor field $N = [\phi, \phi] + 2d\eta \otimes \xi$ vanishes identically, $[\phi, \phi]$ denoting the Nijenhuis tensor of $\phi$. It is known that any almost contact manifold $(M^{2n+1}, \phi, \xi, \eta)$ admits a Riemannian metric $g$ such that

\begin{equation}
    g(\phi X, \phi Y) = g(X, Y) - \eta(X) \eta(Y),
\end{equation}
for any vector fields $X,Y$ on $M^{2n+1}$. This metric $g$ is called a compatible metric and the manifold $M^{2n+1}$ together with the structure $(M^{2n+1}, \phi, \xi, \eta, g)$ is called an almost contact metric manifold. As an immediate consequence of (2.1), one has

$$\Delta \eta = 0$$

for any $\Delta \in D$. The 2-form $\Phi$ of $M^{2n+1}$ defined by $\Phi(X, Y) = g(\phi X, Y)$, is called the fundamental 2-form of the almost contact metric manifold $M^{2n+1}$. Almost contact metric manifolds such that both $\eta$ and $\Phi$ are closed are called almost cosymplectic manifolds and almost contact metric manifolds such that $d\eta = 0$ and $d\Phi = 2\eta \wedge \Phi$ are almost Kenmotsu manifolds. Finally, a normal almost cosymplectic manifold is called a cosymplectic manifold and a normal almost Kenmotsu manifolds is called Kenmotsu manifold.

An almost contact metric manifold $M^{2n+1}$ is said to be almost $\alpha$-Kenmotsu if $d\eta = 0$ and $d\Phi = 2\alpha \eta \wedge \Phi$, $\alpha$ being a non-zero real constant. Geometrical properties and examples of almost $\alpha$-Kenmotsu manifolds are studied in [1], [5], [6] and [7]. Given an almost Kenmotsu metric structure $(\phi, \xi, \eta, g)$, consider the deformed structure

$$\eta' = \frac{1}{\alpha} \eta, \quad \xi' = \alpha \xi, \quad \phi' = \phi, \quad g' = \frac{1}{\alpha^2} g, \quad \alpha \neq 0, \quad \alpha \in \mathbb{R},$$

where $\alpha$ is a non-zero real constant. So we get an almost $\alpha$-Kenmotsu structure $(\phi', \xi', \eta', g')$. This deformation is called a homothetic deformation (see [1], [7]). It is important to note that almost $\alpha$-Kenmotsu structures are related to some special local conformal deformations of almost cosymplectic structures, (see [5]).

If we join these two classes, we obtain a new notion of an almost $\alpha$-cosymplectic manifold, which is defined by the following formula

$$d\eta = 0, \quad d\Phi = 2\alpha \eta \wedge \Phi,$$

for any real number $\alpha$, (see [1]). Obviously, a normal almost $\alpha$-cosymplectic manifold is an $\alpha$-cosymplectic manifold. An $\alpha$-cosymplectic manifold is either cosymplectic under the condition $\alpha = 0$ or $\alpha$-Kenmotsu $\alpha \neq 0$ for $\alpha \in \mathbb{R}$.

We denote by $D$ the distribution orthogonal to $\xi$, that is

$$D = \ker(\eta) = \{ X : \eta(X) = 0 \}$$

and let $M^{2n+1}$ be an almost $\alpha$-cosymplectic manifold with structure $(\phi, \xi, \eta, g)$. Since the 1-form is closed, we have $L_\xi \eta = 0$ and $[X, \xi] \in D$ for any $X \in D$. The Levi-Civita connection satisfies $\nabla_\xi \xi = 0$ and $\nabla_\xi \phi \in D$, which implies that $\nabla_\xi X \in D$ for any $X \in D$.

Now, we set $A = -\nabla \xi$ and $h = \frac{1}{2} L_\xi \phi$. Obviously, $A(\xi) = 0$ and $h(\xi) = 0$. Moreover, the tensor fields $A$ and $h$ are symmetric operators and satisfy the following relations

\begin{align}
(2.2) \quad \nabla_X \xi &= -\alpha \phi^2 X - \phi h X, \\
(2.3) \quad (\phi \circ h) X + (h \circ \phi) X + (A \circ \phi) X &= -2\alpha \phi, \\
(2.4) \quad (\nabla_X \eta) Y &= \alpha [g(X, Y) - \eta(X) \eta(Y)] + g(\phi Y, hX), \\
(2.5) \quad \delta \eta &= -2\alpha n, \quad tr(h) = 0,
\end{align}

for any vector fields $X, Y$ on $M^{2n+1}$. We also remark that

$$h = 0 \iff \nabla \xi = -\alpha \phi^2.$$

From (2.1), we have

$$(\nabla_X \phi) Y + (\nabla_Y \phi) \phi Y = -\alpha \eta(Y) \phi X - 2\alpha g(X, \phi Y) \xi - \eta(Y) h X,$$

for any vector fields $X, Y$ on $M^{2n+1}$. 
Olszak proved that the integral submanifold of the distribution $D$ on an almost cosymplectic manifold has Kaehlerian structures if and only if it satisfies the following condition
\[ (\nabla_X \phi)Y = -g(\phi AX, Y)\xi + \eta(Y)\phi AX, \]
where $A$ is defined by $A = \phi h$. Analogously, we give the following proposition.

**Proposition 1.** Let $M^{2n+1}$ be an almost $\alpha$-cosymplectic manifold. The integral submanifold of the distribution $D$ on an almost $\alpha$-cosymplectic manifold has Kaehlerian structures if and only if it satisfies the condition
\[ (\nabla_X \phi)Y = -g(\phi AX, Y)\xi + \eta(Y)\phi AX, \]
where $A$ is given by $A = \alpha \phi^2 + \phi h$, for any vector fields $X, Y$ on $M^{2n+1}$.

Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost $\alpha$-cosymplectic manifold. We denote the curvature tensor and Ricci tensor of $g$ by $R$ and $S$ respectively. We define a self adjoint operator $l = R(\cdot, \xi)\xi$ (The Jacobi operator with respect to $\xi$). By simple computations, we have the following equations
\[
R(X, Y)\xi = \alpha^2 [\eta(X)Y - \eta(Y)X] - \alpha [\eta(X)\phi h Y - \eta(Y)\phi h X] \\
\quad + (\nabla_Y \phi h) X - (\nabla_X \phi h) Y,
\]
\[
lX = \alpha^2 \phi^2 X + 2\alpha \phi h X - h^2 X + \phi(\nabla_X h) X,
\]
\[
lX - \phi \phi h X = 2 [\alpha^2 \phi^2 X - h^2 X],
\]
\[
(\nabla_X h) X = -\phi h X - \alpha^2 \phi X - 2\alpha h X - \phi h^2 X,
\]
\[
S(X, \xi) = -2\alpha^2 \eta(X) - (\text{div}(\phi h)h)X,
\]
\[
S(\xi, \xi) = -[2\alpha^2 + \text{tr}(h^2)],
\]
where $X, Y$ arbitrary vector fields on $M^{2n+1}$.

3. **Almost $\alpha$-Cosymplectic Manifolds with $\eta$-Parallelism**

Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost $\alpha$-cosymplectic manifold. For any vector field $X$ on $M^{2n+1}$, we can take $X = X^T + \eta(X)\xi$, $X^T$ is tangentially part of $X$ and $\eta(X)\xi$ the normal part of $X$. We say that any symmetric $(1, 1)$-type tensor field $B$ on a $(M^{2n+1}, \phi, \xi, \eta, g)$ is said to be a $\eta$-parallel tensor if it satisfies the equation
\[ g((\nabla_X T^2) B, Y^T, Z^T) = 0, \]
for all tangent vectors $X^T, Y^T, Z^T$ orthogonal to $\xi$.

Dileo and Pastore study almost Kenmotsu manifolds such that $h' = h \circ \phi$ is $\eta$–parallel and prove that this condition is not to equivalent to the characteristic vector field $\xi$ belongs to the $(\kappa, \mu)'$–nullity distribution ([14]) for some constants $\kappa$ and $\mu$, that is the Riemannian curvature satisfies
\[ R(X, Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)h'X - \eta(X)h'Y), \]
for all vector fields $X$ and $Y$. For this, taking into account the spectrum of the operator $h'$, which is of type $\{0, \lambda_1, -\lambda_1, ..., \lambda_r, -\lambda_r\}$, each $\lambda_i$, being positive constant function along $D$, they prove that an integral submanifold of $\tilde{M}$ of $D$ is locally the
Riemannian product $M_0 \times M_\lambda \times M_{-\lambda_1} \times \ldots \times M_\lambda \times M_{-\lambda_r}$, where $M_0$, $M_\lambda$, and $M_{-\lambda_r}$ are integral submanifolds of the distributions of the eigenvectors with eigenvalues $0, \lambda_1$ and $-\lambda_r$ respectively. Moreover, $M_0$ is an almost Kaehlerian manifold and each $M_\lambda \times M_{-\lambda}$ is a bi-Lagrangian Kaehlerian manifold and the structure is CR-integrable if and only if $0$ is a simple eigenvalue or $M_0$ is a Kaehlerian manifold. Also, the authors consider almost Kenmotsu manifolds with $\eta$-parallel and satisfies the additional condition $\nabla_\xi h_\tau = 0$, the almost Kenmotsu manifold is locally a warped product \cite{13}.

In this section, we study almost $\alpha$-cosymplectic manifolds with $\eta$-parallel tensor field $h$ under the condition $\nabla_\xi h = 0$ and the relation between $\eta$-parallelity of the tensor field $h$ and the distribution $\mathcal{D}$. In their work Pastore and Dileo studied $\eta$-parallelity of the tensor field $h\phi$ in almost Kenmotsu manifolds $(\alpha = 1)$ which is a particular case of almost $\alpha$-cosymplectic manifolds (see \cite{13}). In this work, we will be especially focused on providing conditions under which $h$ is $\eta$-parallel in almost $\alpha$-cosymplectic structures. For such manifolds, we also give results about some certain tensor conditions.

The starting point of the investigation of almost $\alpha$-cosymplectic manifolds with $\eta$-parallel tensor $h$ is the following propositions:

**Proposition 2.** Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost $\alpha$-cosymplectic manifold. If the tensor field $h$ is $\eta$-parallel, then we have

\[
(\nabla_X h)Y = -\eta(X) [\phi Y + \alpha^2 \phi Y + 2\alpha h Y + \phi h^2 Y]
-\eta(Y) [-\alpha \phi^2 h X + \phi h^2 X] + g(Y, \alpha h X + \phi h^2 X) \xi,
\]

for all vector fields $X, Y$ on $M^{2n+1}$, where the tensor $l = R(\cdot, \xi) \xi$ is the Jacobi operator with respect to the characteristic vector field $\xi$ and $h$ is a $(1, 1)$-type tensor field.

**Proof.** We suppose that $h$ is $\eta$-parallel. If we denote by $X^T$ the component of $X$ orthogonal to $\xi$, then we get

\[
0 = g((\nabla_X h)^T) Y^T = g((\nabla_X \eta(X) \xi h) (Y - \eta(Y) \xi), Z - \eta(Z) \xi)
= g((\nabla_X h) Y, Z) - \eta(X) g((\nabla_X h) Y, Z) - \eta(Y) g((\nabla_X h) \xi, Z)
-\eta(Z) g((\nabla_X h) \xi, \xi) + \eta((X) g((\nabla_X h) \xi, Z) + \eta(Y) \eta(Z) g((\nabla_X h) \xi, \xi)
+\eta(Z) \eta(X) g((\nabla_X h) Y, \xi) - \eta(X) \eta(Y) \eta(Z) g((\nabla_X h) \xi, \xi),
\]

for all vector fields $X, Y, Z$ on $M^{2n+1}$. If we simplify the above equation, we get

\[
0 = g((\nabla_X h) Y, -\phi^2 Z) - \eta(X) g((\nabla_X h) Y, Z) - \eta(Y) g((\nabla_X h) \xi, Z).
\]

Using $\mathbf{(2.2)}$, $\mathbf{(2.3)}$ and $\mathbf{(2.12)}$, we obtain $\mathbf{(3.1)}$. \hfill \Box

**Proposition 3.** Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost $\alpha$-cosymplectic manifold such that $h$ is $\eta$-parallel. If in addition $\nabla_\xi h = 0$, then the eigenvalues of $h$ are constant on $M^{2n+1}$.

**Proof.** Let $\lambda$ be an eigen function of $h$ and $Y$ be a local unit vector field orthogonal to $\xi$ such that $h(Y) = \lambda Y$. Since $h$ is $\eta$-parallel, using $\mathbf{(3.1)}$ we have

\[
g((\nabla_X h) Y, Y) = \eta(X) \xi(\lambda),
\]

for any vector field $X$. Also, the left-hand side of the equation $\mathbf{(3.3)}$ can be written as

\[
g((\nabla_X h) Y, Y) = X(\lambda),
\]
for any $Y \in \mathcal{D}$. From (3.2) and (3.3), we get $d\lambda = \xi(\lambda) \otimes \eta$, for $X \in \chi(M^{2n+1})$. On the other hand, if $\nabla_{\xi} h = 0$, then $\xi(\lambda) = 0$ for any eigen function $\lambda$. Thus we obtain $d\lambda = 0$ which completes the proof. □

**Theorem 1.** Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost $\alpha$-cosymplectic manifold such that $h$ is $\eta$-parallel and $\nabla_{\xi} h = 0$. If the sectional curvatures of all plane sections $\xi$ are $\neq \alpha^2$ at some point, then $h$ has eigenvalues $\lambda_i \neq 0$ on $\mathcal{D}$.

**Proof.** Assuming the maximal open subset $W$ of $M^{2n+1}$ such that the multiplicities of the eigenvalue functions of $h$ are constant on each connected component of $W$. Let $W^*$ be such a connected component of $W$ and $\lambda_j$ ($j = 1, \ldots, m$) the distinct eigenvalue functions of $h$ restricted to $\mathcal{D}$ on $W^*$. If $X_j$ is a local unit vector field of $h$ such that $h(X_j) = \lambda_j X_j$, then the spectrum $D(\lambda_j)$ can be written

$$D(\lambda_j) = \{ X_j : h(X_j) = \lambda_j X_j \},$$

for $X_j \in \mathcal{D}$. As $h$ anti-commutes with $\phi$, it follows that $h(\phi X_j) = -\lambda_j \phi X_j$. Since $M^{2n+1}$ is connected, the eigenvalues of $h$ are constant on $M^{2n+1}$. Now, we suppose that $hX_j = 0$ for some unit vector fields. Using (3.1) and $\nabla_{\xi} h = 0$, we have

$$(\nabla_{\xi} h) X_j = -\phi R(X_j, \xi)\xi - \alpha^2 \phi X_j - 2\alpha hX_j - \phi h^2 X_j$$

$$g(\phi R(X_j, \xi)\xi, \phi X_j) = -\alpha^2.$$ 

This means $K(\xi, X_j) = -\alpha^2$ everywhere on $M^{2n+1}$ that contradicts our assumption. Thus we obtain $hX_j \neq 0$. Hence, $h$ cannot have an eigenvalue to $0$ on $\mathcal{D}$. Consequently, $h$ is non-degenerate on the distribution $\mathcal{D}$. □

**Remark 1.** If the sectional curvatures of all plane sections $\xi$ are equal to $\alpha^2$ at some point, then we have

$$g(R(X, \xi)\xi, X) + g(R(\phi X, \xi)\xi, \phi X) = 2 \left[ \alpha^2 g(\phi^2 X, X) - g(h^2 X, X) \right],$$

for any unit vector field $X$ on $\mathcal{D}$. The above equation reduces to $g(h^2 X, X) = 0$. This equation yields $\text{trace}(h^2) = 0$ and implies that $h = 0$. So this condition guarantees that $h$ is not equal to $0$.

**Proposition 4.** Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost $\alpha$-cosymplectic manifold and the characteristic vector space of $h$ is completely formed by the direct sum $D(\lambda) \oplus D(-\lambda)$ on $\mathcal{D}$. Then the tensor field $h$ satisfies the following relation

$$h^2 = \lambda^2(I - \eta \otimes \xi),$$

for any vector fields.

**Proof.** If we denote by $X^T$ the component of $X$ orthogonal to $\xi$, then we have $h^2 X^T = \lambda^2 X^T$ for any eigen functions $\lambda$ on $\mathcal{D}$, where $X^T = X - \eta(X)\xi$ for any vector field $X$. So we get $h^2 X = \lambda^2(X - \eta(X)\xi)$. Thus this completes the proof. □

**Proposition 5.** Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost $\alpha$-cosymplectic manifold. Then we have

$$g(R_{\xi X} Y, Z) - g(R_{\xi Y} \phi X, \phi Z) + g(R_{\xi \phi X} Y, \phi Z) + g(R_{\xi \phi X Y} Z)$$

$$= 2(\nabla h \phi)(Y, Z) + 2\alpha^2 \eta(Y)g(X, Z) - 2\alpha^2 \eta(Z)g(X, Y)$$

$$- 2\alpha \eta(Y)g(\phi hX, Z) + 2\alpha \eta(Z)g(\phi hX, Y),$$

for any vector fields $X, Y, Z$ on $M^{2n+1}$. 

Proof. This formula is proved by Pastore for an almost Kenmotsu manifold (see [14]).

**Theorem 2.** Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost $\alpha$-cosymplectic manifold, $\nabla_\xi h = 0$ and the sectional curvatures of all plane sections $\xi$ are $\neq \alpha^2$ at some point. If the tensor field $h$ is $\eta$-parallel, then the integral submanifolds of the distribution $\mathcal{D}$ have Kaehlerian structures.

**Proof.** Let $X^T, Y^T, Z^T$ be an orthogonal vector fields to $\xi$. Using Eqs. (2.9) and (3.1) we obtain

\begin{equation}
(3.6)
g(R(Y^T, Z^T)\xi, X^T) = g((\nabla_{Y^T} \phi)Z^T - (\nabla_{Z^T} \phi)Y^T, hX^T),
\end{equation}

for any vector fields $X, Y, Z$, where $(\nabla_X h)Y = -(\nabla_X h)Y - h(\nabla_X h)Y$. In view of (3.6), we obtain

\begin{equation}
(3.7)
-g(R(\xi, X^T)\phi Y^T, \phi Z^T) = -g((\nabla_{\phi Y^T} \phi)Z^T - (\nabla_{\phi Z^T} \phi)\phi Y^T, hX^T),
\end{equation}

\begin{equation}
(3.8)
g(R(\xi, \phi X^T)Y^T, \phi Z^T) = g((\nabla_{Y^T} \phi)Z^T - (\nabla_{Z^T} \phi)Y^T, h\phi X^T),
\end{equation}

\begin{equation}
(3.9)
g(R(\xi, \phi X^T)Y^T, Z^T) = g((\nabla_{\phi Y^T} \phi)Z^T - (\nabla_{\phi Z^T} \phi)\phi Y^T, h\phi X^T).
\end{equation}

Consider Eqs. (3.7), (3.8), (3.9) and taking sum on both sides of those equalities, respectively, we find

\begin{equation}
(3.10)
g(R(\xi, X^T)Y^T, Z^T) - g(R(\xi, X^T)\phi Y^T, \phi Z^T) + g(R(\xi, \phi X^T)Y^T, \phi Z^T) + g(R(\xi, \phi X^T)Y^T, \phi Z^T)
+ g(R(\xi, \phi X^T)Y^T, \phi Z^T) = g((\nabla_{Y^T} \phi)Z^T, hX^T) - g((\nabla_{Z^T} \phi)Y^T, hX^T)
- g((\nabla_{\phi Y^T} \phi)Z^T, h\phi X^T) + g((\nabla_{\phi Z^T} \phi)\phi Y^T, hX^T) + g((\nabla_{\phi Y^T} \phi)Z^T, h\phi X^T)
- g((\nabla_{\phi Z^T} \phi)\phi Y^T, h\phi X^T).
\end{equation}

We will also need the equality

\begin{equation}
(3.11)
\nabla_{X^T} \phi^2 Y^T = -\nabla_{X^T} Y^T,
\end{equation}

which holds for vector fields $X, Y, Z$ orthogonal to $\xi$. From the above equation, we have

\begin{equation}
(3.12)
g((\nabla_{X^T} \phi)\phi Y^T, Z^T) = g((\nabla_{X^T} \phi)Y^T, \phi Z^T).
\end{equation}

According to Eq. (3.12) we also have

\begin{equation}
(3.13)
g((\nabla_{X^T} \phi)Y^T, Z^T) = -g((\nabla_{\phi X^T} \phi)\phi Y^T, Z^T).
\end{equation}

If we substitute (3.11) and (3.12) into (3.10), then we obtain

\begin{equation}
(3.14)
\begin{aligned}
g(R(\xi, X^T)Y^T, Z^T) &- g(R(\xi, X^T)\phi Y^T, \phi Z^T) \\
+ g(R(\xi, \phi X^T)Y^T, \phi Z^T) + g(R(\xi, \phi X^T)Y^T, \phi Z^T)
= & g((\nabla_{Y^T} \phi)Z^T, hX^T) - g((\nabla_{Z^T} \phi)Y^T, hX^T) - g((\nabla_{\phi Y^T} \phi)Z^T, h\phi X^T)
+ g((\nabla_{\phi Z^T} \phi)\phi Y^T, hX^T) + g((\nabla_{\phi Y^T} \phi)Z^T, h\phi X^T)
- g((\nabla_{\phi Z^T} \phi)\phi Y^T, h\phi X^T).
\end{aligned}
\end{equation}
The left-hand side of Eq. (3.13) using Eq. (3.15) we can be written as follows:

\[ g(R(\xi, X^T)Y^T, Z^T) - g(R(\xi, X^T)\phi Y^T, \phi Z^T) + g(R(\xi, \phi X^T)Y^T, \phi Z^T) + g(R(\xi, \phi X^T)\phi Y^T, Z^T) = 2(\nabla_{hX}\Phi)(Y, Z) = 2g(Y^T, (\nabla_{hX}\phi)Z^T). \]

Combining this equality, we get

\[ g(Y^T, (\nabla_{hX}\phi)Z^T) = 2[g((\nabla_{Y^T}\phi)Z^T, hX^T) - g((\nabla_{Z^T}\phi)Y^T, hX^T)]. \]

In the above equation, if we restrict \( X \) to \( D \) and replace by \( h^{-1}X \), then we have

\[ g(Y^T, (\nabla_{X^T}\phi)Z^T) = 2[g((\nabla_{Y^T}\phi)Z^T, X^T) - g((\nabla_{Z^T}\phi)Y^T, X^T)]. \]

Since the tensor field \( h \) is non-degenerate, its invertible on \( D \). Thus we obtain

\[
\begin{align*}
\phi((\nabla_{X^T}\phi)Y^T, Z^T) + g((\nabla_{Y^T}\phi)Z^T, X^T) + g((\nabla_{Z^T}\phi)X^T, Y^T) &= -2g((\nabla_{Y^T}\phi)Z^T, X^T) - g((\nabla_{Z^T}\phi)Y^T, X^T) \\
&+ g((\nabla_{Z^T}\phi)X^T, Y^T) - g((\nabla_{X^T}\phi)Y^T, Z^T) \\
&+ g((\nabla_{Z^T}\phi)Y^T, X^T) - g((\nabla_{Y^T}\phi)Z^T, X^T) \\
&= 2g((\nabla_{Z^T}\phi)X^T, Y^T) - g((\nabla_{X^T}\phi)Z^T, Y^T)
\end{align*}
\]

The left-hand side of Eq. (3.14) is equal to \( d\Phi(X^T, Y^T, Z^T) \), where \( \Phi \) is the two-form is given by \( \Phi(X, Y) = g(\xi, \phi Y) \). On the other hand, since \( M^{2n+1} \) is an almost \( \alpha \)-cosymplectic manifold, then we have

\[ d\Phi(X^T, Y^T, Z^T) = 2\alpha(\eta(X^T)\Phi(Y^T, Z^T) + \eta(Z^T)\Phi(X^T, Y^T) + \eta(Y^T)\Phi(Z^T, X^T)) = 0 \]

Thus Eq. (3.15) reduces to

\[ g((\nabla_{Z^T}\phi)X^T, Y^T) = g((\nabla_{X^T}\phi)Z^T, Y^T). \]

Moreover, we get

\[ g((\nabla_{Y^T}\phi)Z^T, X^T) = g((\nabla_{Z^T}\phi)Y^T, X^T) = -g((\nabla_{Y^T}\phi)Z^T, X^T), \]

by using (3.16) and anti-commute property of \( \phi \). Hence, we obtain

\[ g((\nabla_{Y^T}\phi)Z^T, X^T) = 0. \]

As this is valid for all vector fields \( X^T, Y^T, Z^T \) orthogonal to \( \xi \), this is equivalent to the equality

\[ (\nabla_Y\phi)Z = [g(Z, hY) - \alpha g(\phi Z, Y)]\xi - \eta(Z)[\alpha \phi Y + hY] \]

\[ = -g(\phi Ay, Z)\xi + \eta(Z)\phi Ay, \]

for all vector fields \( X, Y, Z \) on \( M^{2n+1} \). Therefore, Eq. (3.18) implies that the integral submanifolds of the distribution \( D \) are Kaehlerian. \( \square \)

4. Almost \( \alpha \)-cosymplectic \((\kappa, \mu, \nu)\)-spaces

4.1. \( D \)-homothetic deformations. Let \( M^{2n+1} \) be an almost \( \alpha \)-cosymplectic manifold and \( (\phi, \xi, \eta, g) \) its almost \( \alpha \)-cosymplectic structure. Let \( \mathcal{R}_g(M^{2n+1}) \) be the subring of the ring of smooth functions \( f \) on \( M^{2n+1} \) such that \( df \wedge \eta = 0 \).

Consider a \( D \)-homothetic deformation of \((\phi, \xi, \eta, g) \) into an almost contact metric structure \((\phi', \xi', \eta', g') \) defined as

\[ \phi' = \phi, \xi' = \frac{1}{\beta} \xi, \eta' = \beta \eta, \quad g' = \gamma g + (\beta^2 - \gamma)\eta \otimes \eta, \]

\[ (4.1) \]
Proof. Using Kozsul’s formula we have
\[ d\gamma' = d\beta \wedge \eta + \beta d\eta = 0, \]
and moreover \( d\Phi' = 2(\frac{d\gamma}{\beta})\eta' \wedge \Phi', \) since the fundamental two forms \( \Phi, \Phi' \) of the structures are related by \( \Phi' = \gamma \Phi. \) Taking \( \frac{d\gamma}{\beta} = \beta', \) deformed structure \( (\phi', \xi', \eta', g') \) can be written
\[ \Phi' = \gamma \Phi, \quad d\eta' = 0, \quad d\Phi' = 2\beta' \eta' \wedge \Phi', \]
for \( d\beta = d\beta(\xi)\eta \) and \( \beta' = \frac{d\beta}{\beta} \in \mathcal{R}_\eta(M^{2n+1}). \)

Thus a D-homothetic deformation of an almost \( \alpha \)-cosymplectic structure \( (\phi, \xi, \eta, g) \) gives a new almost \( (\frac{d\beta}{\beta}) \)-cosymplectic structure \( (\phi', \xi', \eta', g') \) on the same manifold.

**Proposition 6.** Let \( (M^{2n+1}, \phi, \xi, \eta, g) \) be an almost \( \alpha \)-cosymplectic manifolds. For D-homothetic deformations of almost \( \alpha \)-cosymplectic structures on \( M^{2n+1}, \) the Levi-Civita connections \( \nabla' \) and \( \nabla \) are related by
\[
\nabla' X = \nabla X - \left( \frac{\beta^2 - \gamma}{\beta^2} \right) g(AX,Y)\xi + \frac{d\beta(\xi)}{\beta} \eta(X)\eta(Y)\xi.
\]

**Proof.** Using Kozsul’s formula we have
\[
2g'\left( \nabla'_X Y, Z \right) = Xg'(Y, Z) + Yg'(X, Z) - Zg'(X, Y) + g'([X, Y], Z) + g'([Z, X], Y) + g'([Z, Y], X),
\]
for any vector fields \( X, Y, Z. \) By applying \( g' = \gamma g + (\beta^2 - \gamma)\eta \otimes \eta \) with all components of Kozsul’s formula, then we find
\[
2g'(\nabla'_X Y, Z) = 2\gamma g(\nabla_X Y, Z) + 2\beta d\beta(\xi)\eta(X)\eta(Y)\eta(Z) + (\beta^2 - \gamma) \left[ 2\eta(\nabla_X Y)\eta(Z) + 2g(Y, \nabla_X \xi)\eta(Z) \right].
\]
Also, since we have
\[
2g'(\nabla'_X Y, Z) = 2\gamma g(\nabla_X Y, Z) + 2(\beta^2 - \gamma)\eta(\nabla_X Y)\eta(Z),
\]
we obtain the formula
\[
\gamma g(\nabla'_X Y, Z) + (\beta^2 - \gamma)\eta(\nabla'_X Y)\eta(Z) = \gamma g(\nabla_X Y, Z) + \beta d\beta(\xi)\eta(X)\eta(Y)\eta(Z) + (\beta^2 - \gamma)\eta(\nabla_X Y)\eta(Z) + g(AX, \nabla_X \xi)\eta(Z),
\]
where
\[
\eta(\nabla'_X Y) = \frac{1}{\beta} d\beta(\xi)\eta(X)\eta(Y) + \eta(\nabla_X Y) + \left( \frac{\beta^2 - \gamma}{\beta} \right) g(Y, \nabla_X \xi).
\]
By using Eq. (1.3) into (1.3) and making use some computations, we get Eq. (1.2) which completes the proof. \( \square \)

**Proposition 7.** For D-homothetic deformations of almost \( \alpha \)-cosymplectic structures, then the following relations are held:
\[
A'X = \frac{1}{\beta} AX, \quad h'X = \frac{1}{\beta} hX,
\]
\[
R'(X, Y)\xi' = \frac{1}{\beta} R(X, Y)\xi + \frac{1}{\beta^2} d\beta(\xi) \eta(X)AY - \eta(Y)AX,
\]
for any vector fields \( X, Y, Z. \)
Proof. By using Eqs. (2.2), (2.3), (4.1) and (4.2), we obtain

\[ A'X = \frac{X(\beta)}{\beta^2} \xi - \frac{1}{\beta^2}\nabla_X \xi - \frac{1}{\beta^2}d\beta(\xi)\eta(X)\xi. \]

By considering the above equation and Eqs. (4.1), Eq. (4.2), then we also have the second equality of (4.5). In order to prove Eq. (4.6), we may also use the Riemannian curvature tensor and Eqs. (4.1), that is, it holds

\[ R'(X,Y)\xi' = \nabla_X \nabla_Y \xi' - \nabla_Y \nabla_X \xi' - \nabla_[X,Y]\xi' \]

\[ = -\frac{X(\beta)}{\beta^2} \nabla_Y \xi - \frac{1}{\beta^2}\nabla_{X,Y} \xi + \frac{1}{\beta}\nabla_X \nabla_Y \xi \]

\[ -\frac{1}{\beta}\nabla^2_Y \nabla_X \xi - \frac{1}{\beta}\nabla_{[X,Y]}\xi. \]

In Eq. (4.7) making use some computations by using the formula

\[ \nabla_X \nabla_Y \xi = \nabla_X \nabla_Y \xi + \left( \frac{\beta^2 - \gamma}{\beta^2} \right) g(Y, A^2 X) \xi, \]

we find

\[ R'(X,Y)\xi' = \frac{X(\beta)}{\beta^2}AY - \frac{Y(\beta)}{\beta^2}AX + \frac{1}{\beta}R(X,Y)\xi, \]

which is a consequence of Eqs. (4.2) and (4.5). \( \square \)

4.2. \((\kappa, \mu, \nu)-spaces.\) In this part, we are especially interested in almost almost -cosymplectic manifolds whose almost \(\alpha\)-cosymplectic structure \((\phi, \xi, \eta, g)\) satisfies the condition (1.2) for \(\kappa, \mu, \nu \in \mathcal{R}_\eta(M^{2n+1}).\) Such manifolds are said to be almost \(\alpha\)-cosymplectic \((\kappa, \mu, \nu)-spaces\) and \((\phi, \xi, \eta, g)\) be called almost \(\alpha\)-cosymplectic \((\kappa, \mu, \nu)-structure.\) We will explain why the functions \(\kappa, \mu, \nu\) are element of \(\mathcal{R}_\eta(M^{2n+1})\) in the latter.

Proposition 8. If \((\phi, \xi, \eta, g)\) is an almost \(\alpha\)-cosymplectic \((\kappa, \mu, \nu)-structure\) for \(D\)-homothetic deformations of almost \(\alpha\)-cosymplectic structures, then \((\phi', \xi', \eta', g')\) is an almost \((\beta')\)-cosymplectic structure with \((\kappa', \mu', \nu') \in \mathcal{R}_{\eta'}(M^{2n+1})\) being related to \(\kappa, \mu, \nu\) by the following equalities

\[ (4.8) \]

\[ \kappa' = \kappa, \mu' = \mu, \nu' = \frac{\beta \nu - d\beta(\xi)}{\beta^2}, \]

which holds

\[ (4.9) \]

\[ R'(X,Y)\xi' = \beta \kappa' [\eta(Y)X - \eta(X)Y] + \mu' [\eta(Y)hX - \eta(X)hY] \]

\[ + \nu' [\eta(Y)\phi hX - \eta(X)\phi hY]. \]

Proof. Applying Eqs. (1.2), (4.1) and (4.5) into (4.6), we get (4.9). By using simple computations, we also obtain

\[ [\eta(Y)X - \eta(X)Y] (\beta \kappa') \]

\[ + [\eta(Y)hX - \eta(X)hY] (\mu') \]

\[ + [\eta(Y)\phi hX - \eta(X)\phi hY] (\nu') = [\eta(Y)X - \eta(X)Y] \left( \frac{\kappa}{\beta} \right) \]

\[ + [\eta(Y)hX - \eta(X)hY] \left( \frac{\mu}{\beta} \right) \]

\[ + [\eta(Y)\phi hX - \eta(X)\phi hY] \left( \frac{\nu}{\beta} - \frac{d\beta(\xi)}{\beta^2} \right), \]

completing the proof. \( \square \)
Theorem 3. An almost $\alpha$-cosymplectic $(\kappa, \mu, \nu)$-structure, $\kappa < -\alpha^2$, can be $D$-homothetically transformed to an almost \((\kappa')\)-cosymplectic \((-1 - \frac{3\alpha^2 + \alpha\nu}{\beta^2}, \mu, \frac{2\alpha}{\beta})\)-structure with $\beta^2 = -(\kappa + \alpha^2)$.

Proof. We suppose that \((\phi, \xi, \eta, g)\) be an almost $\alpha$-cosymplectic $(\kappa, \mu, \nu)$-structure. By making use of $D$-homothetic deformation of the structure \((\phi, \xi, \eta, g)\) with $\kappa < -\alpha^2$ and $\beta^2 = -(\kappa + \alpha^2)$, then we obtain an almost \((\kappa')\)-cosymplectic $(\kappa', \mu', \nu')$-structure \((\phi', \xi', \eta', g')\) with

$$
\kappa' = \frac{\kappa - 2\alpha^2 + \alpha\nu}{\beta^2}, \quad \mu' = \frac{\mu}{\sqrt{-(\kappa + \alpha^2)}}
$$

by the means of the above proposition, where

To prove the theorem, we need the formula $\nu' = \frac{2\alpha}{\beta}$ by using the equation $\nu' = \frac{2\nu'\cdot \delta h(\xi)}{\beta^2}$. Thus \((\kappa - \alpha^2, \frac{\mu}{\beta}, 2\alpha)\)-structure is obtained for the structure \((\phi', \xi', \eta', g')\) with $\beta^2 = -(\kappa + \alpha^2)$. \(\Box\)

Remark 2. The above theorem is proved by Olszak and Dacko for the case $\alpha = 0$ with $\mu' = \frac{\mu}{\sqrt{\kappa}}$ (see \([17]\)).

Proposition 9. The following relations are held on every \((2n + 1)\)-dimensional almost $\alpha$-cosymplectic $(\kappa, \mu, \nu)$-space $(M^{2n+1}, \phi, \xi, \eta, g)$.

\((4.10)\)

$$l = -\kappa\phi^2 + \mu h + \nu\phi h,$$

\((4.11)\)

$$l\phi - \phi l = 2\mu h\phi + 2\nu h,$$

\((4.12)\)

$$h^2 = (\kappa + \alpha^2)\phi^2, \text{ for } \kappa \leq -\alpha^2,$$

\((4.13)\)

$$\nabla_\xi h = -\mu\phi h + (\nu - 2\alpha)h,$$

\((4.14)\)

$$\nabla_\xi h^2 = 2(\nu - 2\alpha)(\kappa + \alpha^2)\phi^2,$$

\((4.15)\)

$$\xi(\kappa) = 2(\nu - 2\alpha)(\kappa + \alpha^2),$$

\((4.16)\)

$$R(\xi, X)Y = \kappa(g(Y, X)\xi - \eta(Y)X) + \mu(g(hY, X)\xi - \eta(Y)hX) + \nu(g(\phi hY, X)\xi - \eta(Y)\phi hX),$$

\((4.17)\)

$$Q\xi = 2\kappa\kappa\xi,$$

\((4.18)\)

$$\nabla_\phi Y = g(\alpha\phi h + \phi h + \nu\phi h),$$

\((4.19)\)

$$\nabla_\phi h - \nabla_\eta hX = -(\kappa + \alpha^2)(\eta(Y)X - \eta(X)Y) - \mu(\eta(Y)hX - \eta(X)hY) + (\alpha - \nu)(\eta(Y)\phi hX - \eta(X)\phi hY),$$

\((4.20)\)

$$\nabla_\phi h - \nabla_\eta hX = (\kappa + \alpha^2)(\eta(Y)\phi hX - \eta(X)\phi hY + 2g(\phi hX, Y)\xi) + \mu(\eta(Y)\phi hX - \eta(X)\phi hY) + (\alpha - \nu)(\eta(Y)hX - \eta(X)hY).$$
for all vector fields \( X, Y \) on \( M^{2n+1} \).

**Proof.** From Eq. (1.2) we get

\[
(4.21) \quad lX = R(X, \xi)\xi = \kappa(X - \eta(X)\xi) + \mu hX + \nu \phi hX.
\]

Replacing \( X \) by \( \phi X \) in Eq. (4.21), it gives

\[
l\phi X = \kappa\phi X + \mu \phi hX + \nu \phi^2 hX.
\]

Also, applying the tensor field \( \phi \) both sides of the last equation we have

\[
\phi lX = -\phi\kappa\phi^2 X + \phi\mu hX + \phi\nu hX.
\]

Then subtracting the last two equations, we obtain

\[
l\phi X - \phi lX = \mu (\phi hX - \phi hX) - 2\nu \phi^2 hX,
\]

completing the proof of Eq. (4.11). By using Eq. (4.21) we deduce

\[
(4.22) \quad \phi l\phi X = \kappa\phi^2 X + \phi\mu hX + \phi\nu hX.
\]

Eqs. (4.21) and (4.22) shows that

\[
lX - \phi l\phi X = -2\kappa \phi^2 X.
\]

Using Eq. (4.21) we have

\[
-2\kappa \phi^2 X = 2(\alpha^2 \phi^2 X - h^2 X),
\]

which gives Eq. (4.12). Moreover, differentiating Eq. (4.12) along \( \xi \) we get

\[
(\nabla \xi h)_X = -\phi lX - \alpha^2 \phi X - 2\alpha hX - \phi h^2 X,
\]

\[
= -\kappa\phi X - \mu \phi hX + \nu hX - \alpha^2 \phi X - 2\alpha hX
\]

\[
+ (\kappa + \alpha^2)\phi X.
\]

Alternately, using (4.12), we obtain

\[
(\nabla \xi h)_X = (\nabla \xi h)h + h(\nabla \xi h) = 2(\nu - 2\alpha)h^2 X.
\]

The proof of Eq. (4.14) is obvious from Eq. (4.13). Then differentiating Eq. (4.12) along \( \xi \) we find

\[
2(\nu - 2\alpha)(\kappa + \alpha^2)\phi^2 X = [\xi(\kappa)] \phi^2 X.
\]

Since \( g(R(\xi, X)Y, Z) = g(R(Y, Z)\xi, X) \), we have

\[
g(R(Y, Z)\xi, X) = \kappa(\eta(Z)g(Y, X) - \eta(X)g(Z, X)) + \mu(\eta(Z)g(hY, X) - \eta(Y)g(hZ, X))
\]

\[
+ \nu(\eta(Z)g(\phi hY, X) - \eta(Y)g(\phi hZ, X)),
\]

by using Eq. (1.2). The last equation completes the proof of Eq. (4.14). Contracting Eq. (4.14) with respect to \( X, Y \) and using the definition of Ricci tensor, we obtain

\[
S(\xi, Z) = \sum_{i=1}^{2n+1} g(R(\xi, E_i)E_i, Z) = 2\kappa \eta(Z),
\]

for any vector field \( Z \). Next, it is clear that Eq. (4.17) is valid. In addition, Eq. (4.17) implies that

\[
g(R_{\xi X} Y, Z) = \kappa [g(Y, X)\eta(Z) - \eta(Y)g(X, Z)] + \mu [g(hY, X)\eta(Z) - \eta(Y)g(hX, Z)]
\]

\[
+ \nu [g(\phi hY, X)\eta(Z) - \eta(Y)g(\phi hX, Z)].
\]

Accordingly, combining the last equation and Eq. (3.3), we deduce that

\[
-2\kappa [\eta(Y)g(X, Z) - \eta(Z)g(X, Y)].
\]
Remark 3.\textit{D}efinition of the distribution sequence of Eq. (4.24). Moreover, Eq. (4.18) shows that the integral submanifold is Kaehlerian for an almost \( \alpha \)-cosymplectic manifold with Kaehlerian integral submanifolds. Then the following relation is valid
\[
M \text{ for all vector fields on } M.
\]

For the curvature transformation of almost \( \alpha \)-cosymplectic manifold with Kaehlerian integral submanifolds, we have
\[
R(X, Y)\phi Z - \phi R(X, Y)Z = g(AX, \phi Z)AY - g(AY, \phi Z)AX - g(AX, Z)\phi AY + g(AY, Z)\phi AX + \eta(Z)\phi((\nabla_X A)Y - (\nabla_Y A)X) + g(\nabla_X A)Y - (\eta^2)(\phi A)X + (\eta^2)(\phi AX) - g(AX, Z)\phi AY + g(AY, Z)\phi AX - \eta(Z)\phi(R(X, Y)\xi) - g(R(X, Y)\xi, \phi Z)\xi.
\]

Proposition 10.\textit{L}et \( (\alpha \)-cosymplectic manifold with \( \alpha \)-cosymplectic \( (\kappa, \mu, \nu) \)-spaces satisfy the Kaehlerian structure condition.}

Now, we need the following formula for the latter usage.

Remark 3. \textit{Eq. (4.18) shows that almost \( \alpha \)-cosymplectic \( (\kappa, \mu, \nu) \)-spaces satisfy the Kaehlerian structure condition.}

Proof. For the curvature transformation of almost \( \alpha \)-cosymplectic manifold with Kaehlerian integral submanifolds, we have
\[
-2\kappa [\eta(Y)g(X, Z) - \eta(Z)g(X, Y)] = 2(\nabla_{hX} \Phi)(Y, Z) + 2\alpha^2 \eta(Y)g(X, Z) - 2\alpha^2 \eta(Z)g(X, Y) - 2\alpha(\eta^2)g(\phi hX, Z) + 2\alpha\eta(Z)g(\phi hX, Y).
\]
In view of the last equation, we obtain
\[
(4.23) \quad -(\nabla_{hX} \Phi)(Y, Z) = (\kappa + \alpha^2) [\eta(Y)g(X, Z) - \eta(Z)g(X, Y)] - \alpha [\eta(Y)g(\phi hX, Z) - \eta(Z)g(\phi hX, Y)].
\]
Substituting \( X = hX \) in Eq. (4.24) and considering the relation \( (\nabla_{X} \Phi)(Y, Z) = g((\nabla_{X} \phi)Z, Y) \), then we get
\[
0 = (\kappa + \alpha^2)g((\nabla_{X} \phi)Z, Y) - \alpha [\eta(Y)g(\phi X, Z) - \eta(Z)g(\phi X, Y)] - [\eta(Y)g(hX, Z) - \eta(Z)g(hX, Y)].
\]
The last equation implies that
\[
(\nabla_{X} \phi)Z = \alpha [g(\phi X, Z)\xi - \eta(\phi X)] + g(hX, Z)\xi - \eta(Z)hX.
\]
Here, replacing \( Z \) by \( Y \) and organizing the last equation, we obtain Eq. \( (4.13) \). On the other hand, Eq. \( (4.13) \) can be written as follows:
\[
(\nabla_{X} \phi)Y = -g(\phi AX, Y)\xi + \eta(Y)\phi AX
\]
by using the tensor field \( A \). Using Eq. \( (2.9) \), we also have
\[
(\nabla_{X} \phi h)Y - (\nabla_{Y} \phi h)X = -R(X, Y)\xi + \alpha^2 [\eta(X)Y - \eta(Y)X] - \alpha [\eta(X)\phi hY - \eta(Y)\phi hX].
\]
The proof of Eq. \( (1.19) \) is obvious from Eq. \( (1.2) \). Eq. \( (4.23) \) is an immediate consequence of Eq. \( (4.24) \). Moreover, Eq. \( (4.18) \) shows that the integral submanifold of the distribution \( D \) is Kaehlerian for an almost \( \alpha \)-cosymplectic \( (\kappa, \mu, \nu) \)-space. □

Remark 3. \textit{Eq. (4.18) shows that almost \( \alpha \)-cosymplectic \( (\kappa, \mu, \nu) \)-spaces satisfy the Kaehlerian structure condition.}

Proposition 10. \textit{Let} \( (M^{2n+1}, \phi, \xi, \eta, g) \) \textit{be an almost} \( \alpha \)-\textit{cosymplectic manifold with} \( \alpha \)-\textit{cosymplectic} \( (\kappa, \mu, \nu) \)-\textit{spaces}. \textit{Then the following relation is valid}
\[
(4.25) \quad Q\phi - \phi Q = l\phi - \phi l + 4\alpha(1 - n)\phi A + 4\alpha^2(1 - n)\phi X + (\eta \circ Q\phi)\xi - \eta \circ (\phi Q\xi),
\]
\textit{for all vector fields on} \( M^{2n+1} \).

Proof. For the curvature transformation of almost \( \alpha \)-cosymplectic manifold \( M^{2n+1} \) with Kaehlerian integral submanifolds, we have
\[
R(X, Y)\phi Z - \phi R(X, Y)Z = g(AX, \phi Z)AY - g(AY, \phi Z)AX - g(AX, Z)\phi AY + g(AY, Z)\phi AX + \eta(Z)\phi((\nabla_X A)Y - (\nabla_Y A)X) + g(\nabla_X A)Y - (\eta^2)(\phi A)X + (\eta^2)(\phi AX) - g(AX, Z)\phi AY + g(AY, Z)\phi AX - \eta(Z)\phi(R(X, Y)\xi) - g(R(X, Y)\xi, \phi Z)\xi.
\]
Substitution of the (4.27) into (4.28) yields immediately

\[ g(\phi R(\phi X, \phi Y)Z, \phi W) = g(\phi R(Z, W)X, \phi Y) + g(AZ, \phi X)g(AW, \phi Y) \]

\[-g(AW, \phi X)g(AZ, \phi Y) - g(AZ, X)g(\phi AW, \phi Y) + g(AW, X)g(\phi AZ, \phi Y) \]

\[-\eta(X)g(\phi R(Z, W)\xi, \phi Y) - \eta(R(\phi X, \phi Y)Z)\eta(W). \]

Putting \( X = \phi X \) and \( Y = \phi Y \) in (4.26) we have,

\[ g(R(\phi X, \phi Y)\phi Z, \phi W) - g(\phi R(\phi X, \phi Y)Z, \phi W) \]

\[ = g(A\phi X, \phi Z)g(A\phi Y, \phi W) - g(A\phi Y, \phi Z)g(A\phi X, \phi W) \]

\[-g(A\phi X, Z)g(\phi A\phi Y, \phi W) + g(A\phi Y, Z)g(\phi A\phi X, \phi W) \]

\[-\eta(Z)g(\phi R(\phi X, \phi Y)\xi, \phi W). \]

Substitution of the (4.27) into (4.28) yields immediately

\[ g(R(\phi X, \phi Y)\phi Z, \phi W) = g(\phi R(Z, W)X, \phi Y) + g(AZ, \phi X)g(AW, \phi Y) \]

\[-g(AW, \phi X)g(AZ, \phi Y) - g(AZ, X)g(\phi AW, \phi Y) \]

\[ + g(AW, X)g(\phi AZ, \phi Y) - \eta(X)g(\phi R(Z, W)\xi, \phi Y) \]

\[-\eta(R(\phi X, \phi Y)Z)\eta(W) + g(A\phi X, \phi Z)g(A\phi Y, \phi W) \]

\[-g(A\phi Y, \phi Z)g(A\phi X, \phi W) - g(A\phi X, Z)g(\phi A\phi Y, \phi W) \]

\[ + g(A\phi Y, Z)g(\phi A\phi X, \phi W) - \eta(Z)g(\phi R(\phi X, \phi Y)\xi, \phi W). \]

By using (2.1), the relation (4.29) can be written as

\[ g(R(\phi X, \phi Y)\phi Z, \phi W) = g(R(Z, W)X, Y) - \eta(R(Z, W)X)\eta(Y) \]

\[-g(AZ, X)g(AW, Y) + g(AW, X)g(AZ, Y) \]

\[-\eta(X)g(R(Z, W)\xi, Y) - \eta(R(\phi X, \phi Y)Z)\eta(W) \]

\[ + g(A\phi X, \phi Z)g(A\phi Y, \phi W) - g(A\phi Y, \phi Z)g(A\phi X, \phi W) \]

\[ -\eta(Z)g(\phi R(\phi X, \phi Y)\xi, \phi W). \]

Substituting \( Y = Z = e_i \) in (4.30), summing over \( i = 1, 2, \ldots, 2n + 1 \), and using the relation \( tr(A) = -2\alpha n \), it is not hard to prove

\[-\phi Q\phi X - QX = -\phi l\phi X - lX + 4\alpha(1 - n)A \]

\[-4\alpha^2(1 - n)\phi^2 X - \eta(X)QX + \sum_{i=1}^{2n+1} \eta(R(\phi X, \phi e_i)e_i)\xi. \]

The rest of the proof follows acting \( \phi \) on the last equation. Also, the last equation reduces to the following formula

\[ Q\phi - \phi Q = l\phi - \phi l + 4\alpha(1 - n)\phi(\alpha\phi^2 + \phi h) + 4\alpha^2(n - 1)\phi X, \]

for all vector fields on \( M^{2n+1} \).

\[ \square \]

**Proposition 11.** Let \((M^{2n+1}, \phi, \xi, \eta, g)\) be an almost \( \alpha \)-cosymplectic \((\kappa, \mu, \nu)\)-space. Then the following relation is true

\[ Q\phi - \phi Q = 2\mu\phi + 2(2\alpha(n - 1) + \nu)h, \]

for all vector fields on \( M^{2n+1} \).
Proof. In view of Eq. (4.31) and (4.11), we find
\[ Q\phi - \phi Q = 2\mu h\phi + 2\nu h - 4\alpha(1 - n)h, \]
which proves the required result. \qed

**Theorem 4.** Let \((M^{2n+1}, \phi, \xi, \eta, g)\) be an almost \(\alpha\)-cosymplectic manifold, \(\nabla_\xi h = 0\), the sectional curvatures of all plane sections \(\xi\) are \(\neq \alpha^2\) at some point and the characteristic vector space of \(h\) is completely formed by the direct sum \(D(\lambda) \oplus D(-\lambda)\) on \(D\). If the tensor field \(h\) is \(\eta\)-parallel, then \((M^{2n+1}, \phi, \xi, \eta, g)\) is a \((\kappa, 0, 2\alpha)\)-space with \(\kappa = -(\alpha^2 + \lambda^2)\).

**Proof.** Using (2.9) we obtain
\[
R(X,Y)\xi = \alpha^2 [\eta(X)Y - \eta(Y)X] - \alpha [\eta(X)\phi h - \eta(Y)\phi h] + (\nabla_X h)\phi Y - (\nabla_Y h)\phi X + h((\nabla_X \phi)Y - (\nabla_Y \phi)X),
\]
by acting the formula \((\nabla_X h)\phi Y = -(\nabla_X \phi)hY - h(\nabla_X \phi)Y\) for any vector fields \(X, Y\). Since \(h\) is \(\eta\)-parallel, substituting \(Y = \phi Y\) in Eq. (3.1) we find
\[
(\nabla_X h)\phi Y = g(\phi Y, \alpha X + \phi^2 Y)\xi.
\]
Then replacing \(X\) by \(Y\) in Eq. (4.34) and by virtue of Eqs. (4.33) and (4.34), we get
\[
(\nabla_X h)\phi Y - (\nabla_Y h)\phi X = 0.
\]
Using (2.18) and combining (4.33) in Eq. (4.33), we also get
\[
R(X,Y)\xi = -\alpha^2 [\eta(Y)X - \eta(X)Y] + 2\alpha [\eta(Y)\phi hX - \eta(X)\phi hY] - [\eta(Y)h^2 X - \eta(X)h^2 Y].
\]
Furthermore, Eq. (3.14) is valid because of the our assumption. At that rate, Eq. (4.36) reduces to
\[
R(X,Y)\xi = -(\alpha^2 + \lambda^2) [\eta(Y)X - \eta(X)Y] + 2\alpha [\eta(Y)\phi hX - \eta(X)\phi hY],
\]
by using Eq. (3.14). The proof is completed. \qed

**Theorem 5.** The following differential equation is valid on every \((2n+1)\)-dimensional almost \(\alpha\)-cosymplectic \((\kappa, \mu, \nu)\)-space \((M^{2n+1}, \phi, \xi, \eta, g)\):
\[
0 = \xi(\kappa)(\eta(Y)X - \eta(X)Y) + \xi(\mu)(\eta(Y)hX - \eta(X)hY) + \xi(\nu)(\eta(Y)\phi hX - \eta(X)\phi hY)
- \eta(X)\phi hY - X(\kappa)\phi^2 Y + X(\mu)hY + X(\nu)\phi hY - Y(\mu)hX - Y(\nu)\phi hX
+ Y(\kappa)\phi^2 X + 2(\kappa + \alpha^2)\mu g(\phi X,Y)\xi + 2\mu g(hX, \phi hY)\xi.
\]
**Proof.** Differentiating the formula (1.22) along an vector field \(Z\) we have
\[
(\nabla_Z R)(X,Y)\xi = Z(\kappa)[\eta(Y)X - \eta(X)Y] + Z(\mu)[\eta(Y)hX - \eta(X)hY]
+ Z(\nu)[\eta(Y)\phi hX - \eta(X)\phi hY] + \kappa [\eta(\nabla_Z Y)X + g(Y, \nabla_Z \xi)X
+ \eta(Y)\nabla_Z X] + \kappa [-\eta(\nabla_Z Y)Y - g(X, \nabla_Z \xi)Y - \eta(X)\nabla_Z Y]
+ \mu [\eta(\nabla_Z Y)hX + g(Y, \nabla_Z \xi)hX]
+ \mu [\eta(Y)\nabla_Z hX - \eta(hX)\nabla_Z Y - g(X, \nabla_Z \xi)hY - \eta(X)\nabla_Z hY]
+ \nu [\eta(\nabla_Z Y)\phi hX + g(Y, \nabla_Z \xi)\phi hX + \eta(Y)\nabla_Z \phi hX]
+ \nu [-\eta(\nabla_Z X)\phi hY - g(X, \nabla_Z \xi)\phi hY - \eta(X)\nabla_Z \phi hY],
\]
by considering the equation

\[(\nabla_Z R)(X, Y)\xi = \nabla_Z R(X, Y)\xi - R(\nabla_Z X, Y)\xi - R(X, \nabla_Z Y)\xi - R(X, Y)\nabla_Z \xi.\]

Then by using (2.2) we also have

\[\begin{align*}
(\nabla_Z R)(X, Y)\xi &= Z(\kappa) [\eta(Y) X - \eta(X) Y] + Z(\mu) [\eta(Y) hX - \eta(X) hY] \\
&\quad + Z(\nu) [\eta(Y) \phi hX - \eta(X) \phi hY] + \kappa [\eta(\nabla_Z Y) X] \\
&\quad + \nu [\alpha g(\phi Y, \phi Z) X - g(Y, \phi h Z) X + g(X, \phi h Z) Y] \\
&\quad + \kappa [\eta(\nabla_Z X) - \eta(\nabla_Z X) Y - \alpha g(\phi X, \phi Z) Y - \eta(X) \nabla_Z Y] \\
&\quad + \mu [\eta(\nabla_Z Y) hX + \alpha g(\phi Y, \phi Z) hX - g(Y, \phi h Z) hX] \\
&\quad + \mu [g(X, \phi h Z) hY - \eta(X) (\nabla_Z Y) Y] + \nu [\alpha g(\phi Y, \phi Z) \phi h X] \\
&\quad + \nu [\eta(\nabla Z Y) \phi h X - \eta(X) (\nabla_Z Y) \phi Y] \\
&\quad + \nu [g(X, \phi h Z) \phi h Y] - \kappa [\eta(Y) \nabla_Z X - \eta(\nabla_Z X) Y] \\
&\quad - \mu [\eta(Y) h \nabla_Z X - \eta(\nabla_Z X) h Y] - \kappa [-\eta(X) \nabla_Z Y + \eta(\nabla_Z Y) X] \\
&\quad - \mu [-\eta(X) h \nabla_Z Y + \eta(\nabla_Z Y) h X] - \nu [-\eta(X) \phi h \nabla_Z Y + \eta(\nabla_Z Y) \phi h X] \\
&\quad - \nu [\eta(Y) \phi h \nabla_Z X - \eta(\nabla_Z X) \phi h Y] + \alpha \kappa \eta(Z) [\eta(Y) X - \eta(X) Y] \\
&\quad + \alpha \mu \eta(Z) [\eta(Y) h X - \eta(X) h Y] + \alpha \nu \eta(Z) [\eta(Y) \phi h X - \eta(X) \phi h Y] \\
&\quad - \alpha R(X, Y) Z + R(X, Y) \phi h Z.
\end{align*}\]

Now, using again Eqs. (2.2) and (1.2) the last equation reduces to

\[(\nabla_Z R)(X, Y)\xi = Z(\kappa) [\eta(Y) X - \eta(X) Y] + Z(\mu) [\eta(Y) h X - \eta(X) h Y] \\
+ Z(\nu) [\eta(Y) \phi h X - \eta(X) \phi h Y] + \kappa [\alpha g(Z, X) Y] \\
+ \nu [\alpha g(Z, X) Y + g(X, \phi h Z) Y - g(Y, \phi h Z) Y] \\
+ \mu [-g(Y, \phi h Z) h X + \eta(Y) (\nabla_Z h) X + \alpha g(Y, Z) h X] \\
+ \mu [-\alpha g(X, Z) Y + g(X, \phi h Z) h Y - \eta(X) (\nabla_Z h) Y] \\
+ \nu [\alpha g(Y, Z) \phi h X - g(Y, \phi h Z) \phi h Y + \eta(Y) (\nabla_Z \phi h) X] \\
+ \nu [-\alpha g(X, Z) \phi h Y + g(X, \phi h Z) \phi h Y - \eta(X) (\nabla_Z \phi h) Y] \\
- \alpha R(X, Y) Z + R(X, Y) \phi h Z.
\]

Next using the last equation and the second Bianchi identity

\[(\nabla_Z R)(X, Y)\xi + (\nabla_X R)(Y, Z)\xi + (\nabla_Y R)(Z, X)\xi = 0,
\]
we obtain

\begin{align*}
0 &= Z(\kappa) [\eta(Y)X - \eta(X)Y] + Z(\mu) [\eta(Y)hX - \eta(X)hY] \\
&+ Z(\nu) [\eta(Y)\phi hX - \eta(X)\phi hY] + X(\kappa) [\eta(Z)Y - \eta(Y)Z] \\
&+ X(\mu) [\eta(Z)hY - \eta(Y)hZ] + X(\nu) [\eta(Z)\phi hY - \eta(Y)\phi hZ] \\
&+ Y(\kappa) [\eta(X)Z - \eta(Z)X] + Y(\mu) [\eta(X)hZ - \eta(Z)hX] \\
&+ Y(\nu) [\eta(X)\phi hZ - \eta(Z)\phi hX] + \mu [\eta(Y)((\nabla hZ)X - (\nabla hX)Z)] \\
&+ \mu [\eta(Z)((\nabla hX)Y - (\nabla hY)X) + \eta(X)((\nabla hY)Z - (\nabla hZ)Y)] \\
&+ \nu [\eta(Y)((\nabla hZ)X - (\nabla hX)Z) + \eta(Z)((\nabla hY)Y - (\nabla hY)X)] \\
&+ \nu [\eta(X)((\nabla hY)Z - (\nabla hZ)Y)] + R(X, Y)\phi hZ + R(Y, Z)\phi hX \\
&- \alpha [R(X, Y)Z + R(Y, Z)X + R(Z, X)Y] + R(Z, X)\phi hY,
\end{align*}

for all vector fields $X, Y, Z$. Putting $\xi$ instead of $Z$ in the above equation, we obtain

\begin{align*}
0 &= \xi(\kappa) [\eta(Y)X - \eta(X)Y] + \xi(\mu) [\eta(Y)hX - \eta(X)hY] \\
&+ \xi(\nu) [\eta(Y)\phi hX - \eta(X)\phi hY] - X(\kappa)\phi^2 Y + X(\mu)hY \\
&+ X(\nu)\phi hY + Y(\kappa)\phi^2 X - Y(\mu)hX - Y(\nu)\phi hX \\
&+ \mu \eta(Y)[-\kappa + (\mu + \alpha)^2]X - \mu \phi hX - (\alpha - \nu)hX] \\
&+ \mu \eta(Y)[\kappa + \alpha + \alpha^2]X - \eta(X)\phi hY + 2\eta(X, Y)\xi] \\
&+ \mu \eta(X)[-\kappa + (\mu + \alpha)^2]X + \mu \phi hX + (\alpha - \nu)hX] \\
&+ \nu \eta(Y)[-\kappa + (\mu + \alpha)^2]X + \mu \phi hX - (\alpha - \nu)\phi hX] \\
&- \nu \eta(Y)[-\kappa + (\mu + \alpha)^2]X - \eta(X)\phi hY + \nu \eta(X)\phi hX + \nu \eta(X)(\kappa + \alpha^2)\phi^2 Y \\
&+ \nu \eta(X)[\mu \phi hY + (\alpha - \nu)\phi hY] - R(\xi, Y)\phi hX + R(\xi, Z)\phi hY.
\end{align*}

Finally, if we substitute (4.16), (4.19), (4.20) in the last equation, then we deduce Eq. (4.37).  

By the means of the proof of [Koufogiorgos et al., Lemma 4.4], we have

**Lemma 1.** Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost $\alpha$-cosymplectic $(\kappa, \mu, \nu)$-space. For every $p \in N$, there exists neighborhood $W$ of $p$ and orthonormal local vector fields $X_i, \phi X_i$ and $\xi$ for $i = 1, \ldots, n$, defined on $W$, such that

\begin{equation}
(4.38) \quad hX_i = \lambda X_i, \quad h\phi X_i = -\lambda X_i, \quad h\xi = 0,
\end{equation}

for $i = 1, \ldots, n$, where $\lambda = \sqrt{-\kappa + \alpha^2}$.

**Proof.** The proof of that lemma is similar to that of [Koufogiorgos et al. 2008, Lemma 4.2].

Now, we will explain why the functions $\kappa, \mu$ and $\nu$ are element of $R_{\eta}(M^{2n+1})$. Using Lemma 1 we will prove the following theorem.

**Theorem 6.** Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost $\alpha$-cosymplectic $(\kappa, \mu, \nu)$-space with the dimension greater than 3. Then the functions $\kappa, \mu$ and $\nu$ are element of $R_{\eta}(M^{2n+1})$.  

Proof. Lemma 1 implies that the existence of a local orthonormal basis \( \{X_i, \phi X_i, \xi\} \) such that
\[
he_i = \lambda e_i, \quad b\phi e_i = -\lambda \phi e_i, \quad h\xi = 0, \quad \lambda = \sqrt{-\left(\kappa + \alpha^2\right)},
\]
on \( W \). Substituting \( X = e_i \) and \( Y = \phi e_i \) in Eq. (4.37), we obtain that
\[
e_i(\kappa)\phi e_i - \lambda e_i(\mu)\phi e_i + \lambda e_i(\nu)e_i - \lambda\phi e_i(\mu)e_i - \lambda\phi e_i(\nu)e_i - \phi e_i(\kappa)e_i = 0.
\]
This means that
\[
[e_i(\kappa) - \lambda e_i(\mu) - \lambda\phi e_i(\nu)]\phi e_i + [\lambda e_i(\nu) - \lambda\phi e_i(\mu) - \phi e_i(\kappa)] = 0.
\]
Since \( \{e_i, eX_i\} \) is linearly independent, we have
\[
e_i(\kappa) - \lambda e_i(\mu) - \lambda\phi e_i(\nu) = 0, \quad \lambda e_i(\nu) - \lambda\phi e_i(\mu) - \phi e_i(\kappa) = 0.
\]
In addition, replacing \( X \) and \( Y \) by \( e_i \) and \( e_j \), respectively, for \( i \neq j \), Eq. (4.37) provides that
\[
e_i(\kappa) + \lambda e_i(\mu) = 0, \quad e_i(\nu) = 0.
\]
Besides, substituting \( X = \phi e_i \) and \( Y = \phi e_j \), for \( i \neq j \), in Eq. (4.37), we get
\[
\phi e_i(\kappa) - \lambda\phi e_i(\mu) = 0, \quad \phi e_i(\nu) = 0.
\]
In view of Eqs. (4.39), (4.40) and (4.41), we deduce
\[
e_i(\kappa) = e_i(\mu) = e_i(\nu) = \phi e_i(\kappa) = \phi e_i(\mu) = \phi e_i(\nu) = 0.
\]
For an arbitrary function \( \kappa \), we obtain that \( d\kappa = \xi(\kappa)\eta \) in the last equation system. In this way, we can write
\[
0 = d^2\kappa = d(d\kappa) = d\xi(\kappa) \land \eta + \xi(\kappa)d\eta.
\]
Since \( d\eta = 0 \), acting the last equation, it gives that \( d\xi(\kappa) \land \eta = 0 \). Namely, the function \( \kappa \) is an element of \( R_\eta(M^{2n+1}) \) on every connected component of \( W \). Analogously, it can be shown that the functions \( \mu \) and \( \nu \) are also non-constant on every connected component of \( W \). \( \square \)

Corollary 1. The functions \( \kappa, \mu \) and \( \nu \) are constants if and only if these functions are constants along the characteristic vector field \( \xi \) for almost \( \alpha \)-cosymplectic \( (\kappa, \mu, \nu) \)-space \( (M^{2n+1}, \phi, \xi, \eta, g) \) with \( n > 1 \).

5. THE EXISTENCE OF NON-CONSTANT \((\kappa, \mu, \nu)\)-SPACES IN DIMENSION 3

In this part, we will show that the existence of almost \( \alpha \)-cosymplectic \( (\kappa, \mu, \nu) \)-space for the non-constant functions \( \kappa, \mu \) and \( \nu \) in dimension 3.

Let \( U \) be the open subset of \( M^3 \) where the tensor field \( h \neq 0 \) and let \( U' \) be the open subset of points \( p \in M^3 \) such that \( h = 0 \) in a neighborhood of \( p \). Thus the association set of \( U \cup U' \) is an open and dense subset of \( M^3 \). For every \( p \in U \) there exists an open neighborhood of \( p \) such that \( he = \lambda e \) and \( h\phi e = -\lambda\phi e \), where \( \lambda \) is a positive non-vanishing smooth function. So every properties satisfying on \( U \cup U' \) is valid on \( M^3 \). Therefore, there exists a local orthonormal basis \( \{e, \phi e, \xi\} \) of smooth eigen functions of \( h \) in a neighborhood of \( p \) for every point \( p \in U \cup U' \). This basis is called \( \phi \)-basis. The following lemma will be useful for the latter case.
Lemma 2. Let \((M^3, \phi, \xi, \eta, g)\) be an almost \(\alpha\)-cosymplectic manifold. Then for the covariant derivative on \(U\) the following equations are valid

\[
\begin{align*}
\nabla_{\xi} e &= -a \phi e, \\
\nabla_{e} \xi &= a e - \lambda \phi e, \\
\nabla_{\phi e} \xi &= -\lambda e + \alpha \phi e, \\
\nabla_{e} e &= b \phi e - \alpha \xi, \\
\nabla_{\phi e} e &= ce - \alpha \xi, \\
\nabla_{\phi e} \phi e &= -b e + \lambda \xi, \\
\nabla_{\phi e} \phi e &= -c \phi e + \lambda \xi,
\end{align*}
\]

where \(a\) is a smooth function, \(b = g(\nabla_{\xi} e, \phi e)\) and \(c = g(\nabla_{\phi e} \phi e, e)\) defined by

\[
b = \frac{1}{2\lambda} \left[ (\phi e)(\lambda) + \sigma(e) \right], \quad \sigma(e) = S(\xi, e) = g(Q\xi, e),
\]

and

\[
c = \frac{1}{2\lambda} \left[ e(\lambda) + \sigma(\phi e) \right], \quad \sigma(\phi e) = S(\xi, \phi e) = g(Q\xi, \phi e),
\]

respectively.

Proof. Replacing \(X\) by \(e\) and \(\phi e\) in Eq. (2.2), respectively, we find

\[
\begin{align*}
\nabla_{\xi} e &= -a \phi^2 e - \phi \phi e, \\
\nabla_{\phi e} \xi &= -a \phi^2 e - \phi \phi e, \\
\n&= a e - \alpha \eta(e) \xi + h \phi e, \\
\nabla_{e} e &= \alpha \phi - \lambda e, \\
\nabla_{\phi e} \phi e &= \alpha \phi - \lambda e,
\end{align*}
\]

for any vector field \(X\). Also, we have

\[
\begin{align*}
\nabla_{\xi} e &= g(\nabla_{\xi} e, e) + g(\nabla_{\xi} e, \phi e) \phi e + g(\nabla_{\xi} e, \xi) \phi e \\
&= -g(e, \nabla_{\xi} \phi e) \phi e,
\end{align*}
\]

where \(a\) is defined by \(a = g(e, \nabla_{\xi} \phi e)\). So \(\nabla_{\xi} e\) is obtained by the formula \(\nabla_{\xi} e = -a \phi e\). Following this procedure, the other covariant derivative equalities can easily find. We recall that the curvature tensor \(3\)-dimensional Riemannian manifold is given by

\[
R(X, Y)Z = -S(X, Z)Y + S(Y, Z)X - g(X, Z)QY + g(Y, Z)QX + \frac{\lambda}{2} [g(X, Z)Y - g(Y, Z)X],
\]

with the dimensional three case, for any vector fields \(X, Y, Z\). Putting \(X = e, Y = \phi e\) and \(Z = \xi\) in the last equation, we obtain

\[
R(e, \phi e)\xi = -g(Qe, \xi) \phi e + g(Q\phi e, \xi) e.
\]

Since \(\sigma(X) = g(Q\xi, X)\), we have

\[
R(e, \phi e)\xi = -\sigma(e) \phi e + \sigma(\phi e) e,
\]

for any vector field \(X\). By using the curvature properties of the Riemannian tensor, we also have

\[
R(e, \phi e)\xi = (\nabla_{\phi e} \phi h) e - (\nabla_{e} \phi h) \phi e,
\]

(5.3)

In this case, combining (5.2) and (5.3), we deduce that

\[
\sigma(e) = 2\lambda b - (\phi e)(\lambda), \quad \sigma(\phi e) = 2\lambda c - e(\lambda).
\]

Hence, the functions \(b\) and \(c\) are obtained by the above relations. \(\square\)
Proposition 12. Let \((M^3, \phi, \xi, \eta, g)\) be an almost \(\alpha\)-cosymplectic manifold. On \(U\) the following relation is true:

\[
\nabla_\xi h = 2ah\phi + \xi(\lambda)s,
\]

where \(s\) is the tensor field of type \((1,1)\) defined by \(s\xi = 0\), \(se = e\) and \(s\phi = -\phi e\).

Proof. First, we check the tensor field \(h\) which differentiating along \(\xi\). In that case, we have

\[
(\nabla_\xi h)e = -2\lambda a\phi + \xi(\lambda)e, \quad (\nabla_\xi h)\phi e = -2\lambda ae - \xi(\lambda)\phi e.
\]

In addition, we also have \((\nabla_\xi h)\xi = 0\). Then we obtain (5.4) with the help of the last equations. It is clear that \(tr(s) = 0\). \(\square\)

Remark 4. Since \(h = 0\) on the open subset \(U'\), we have \(\xi(\lambda)s = \nabla_\xi h = 0\).

Proposition 13. Let \((M^3, \phi, \xi, \eta, g)\) be an almost \(\alpha\)-cosymplectic manifold. Then the integral submanifold of the distribution \(D\) on \(M^3\) has Kaehlerian structures if and only if the following relation is true

\[
(\nabla_X h)Y = g(\alpha\phi X + hX, Y)\xi - \eta(Y)(\alpha\phi X + hX),
\]

for arbitrary vector fields \(X, Y\) on \(M^3\).

Proof. The 3-form \(\eta \wedge \Phi\) is equal to the volume element of \(M^3\). Since every volume element is constant, it gives \(\nabla_X(\eta \wedge \Phi) = 0\), for any vector field \(X\). By the means of this equation, we obtain

\[
0 = (\nabla_X \eta)(Y)\Phi(Z, W) + \eta(Y)(\nabla_X \Phi)(Z, W)
+ (\nabla_X \eta)(Z)\Phi(W, Y) + \eta(Z)(\nabla_X \Phi)(W, Y)
+ (\nabla_X \eta)(W)\Phi(Y, Z) + \eta(W)(\nabla_X \Phi)(Y, Z).
\]

Setting \(\xi\) instead of \(W\) in the last equation, we have

\[
(\nabla_X \Phi)(Z, Y) = -\eta(Z)(\nabla_X \Phi)(Y, \xi) + \eta(Y)(\nabla_X \Phi)(Z, \xi).
\]

In view of Eq. (5.5), we deduce that

\[
g(Z, (\nabla_X \phi)Y) = g(Z, g(\phi\nabla_X \xi, Y)\xi - \eta(Y)\phi\nabla_X \xi),
\]

which yields

\[
(\nabla_X \phi)Y = g(\phi\nabla_X \xi, Y)\xi - \eta(Y)\phi\nabla_X \xi,
\]

for arbitrary vector fields \(X\) and \(Y\). \(\square\)

Proposition 14. Let \((M^3, \phi, \xi, \eta, g)\) be an almost \(\alpha\)-cosymplectic manifold. Then the following relation is satisfied on \(M^3\):

\[
h^2 - \alpha^2\phi^2 = \frac{tr(l)}{2}\phi^2.
\]

Proof. By using (2.14), we get \(tr(l) = -2\left[\alpha^2 + \lambda^2\right]\), for all vector fields on \(M^3\). Besides, we calculate the statement of \(h^2 - \alpha^2\phi^2\) with respect to the basis components, then we obtain that

\[
h^2e - \alpha^2\phi^2e = \frac{tr(l)}{2}\phi^2e, \quad h^2\phi e - \alpha^2\phi^3e = \frac{tr(l)}{2}\phi^2\phi e.
\]

Also, it has been obviously seen that \(h^2\xi - \alpha^2\phi^2\xi = \frac{tr(l)}{2}\phi^2\xi = 0\). These equations completes the proof of Eq. (5.6). \(\square\)
Lemma 3. Let \((M^3, \phi, \xi, \eta, g)\) be an almost \(\alpha\)-cosymplectic manifold. Then the Ricci operator \(Q\) satisfies the relation

\[
Q = \tilde{a}I + \tilde{b}\eta \otimes \xi + 2\alpha \phi h + \phi(\nabla_\xi h) - \sigma(\phi^2) \otimes \xi + \sigma(e)\eta \otimes e + \sigma(\phi e)\eta \otimes \phi e,
\]

where \(\tilde{a}\) and \(\tilde{b}\) are smooth functions defined by \(\tilde{a} = \frac{1}{2} r + \alpha^2 + \lambda^2\) and \(\tilde{b} = -\frac{1}{2} r - 3\alpha^2 - 3\lambda^2\), respectively.

Proof. For 3-dimensional case, we deduce that

\[
lX = tr(l)X - S(X, \xi)\xi + QX - \eta(X)Q\xi - \frac{r}{2} (X - \eta(X)\xi),
\]

for any vector field \(X\). The above equation implies

\[
QX = \alpha^2 \phi^2 X + 2\alpha \phi h X - h^2 X + \phi(\nabla_\xi h)X - tr(l)X - S(X, \xi)\xi + \eta(X)Q\xi + \frac{r}{2} (X - \eta(X)\xi).
\]

Otherwise, since \(S(X, \xi) = -S(\phi^2 X, \xi) + \eta(X)tr(l)\), we have

\[
QX = -\frac{tr(l)}{2} \phi^2 X + 2\alpha \phi h X + \phi(\nabla_\xi h)X - tr(l)X - S(\phi^2 X, \xi)\xi + \eta(X)\sigma(e)\phi e + \eta(X)Q\xi + \frac{r}{2} \phi^2 X.
\]

Thus it is clear that \(Q\xi = \sigma(e)e + \sigma(\phi e)\phi e + tr(l)\xi\). Acting the last equation in Eq. (5.8), we obtain

\[
QX = \left[\frac{1}{2} r + \alpha^2 + \lambda^2\right] X + \left[-\frac{1}{2} r - 3\alpha^2 - 3\lambda^2\right] \eta(X)\xi + 2\alpha \phi h X + \phi(\nabla_\xi h)X - S(\phi^2 X, \xi)\xi + \eta(X)\sigma(e)e + \eta(X)\sigma(\phi e)\phi e,
\]

for arbitrary vector field \(X\). Therefore, we find the functions \(\tilde{a}\) and \(\tilde{b}\) mentioned in Eq. (5.7). \(\square\)

Theorem 7. Let \((M^3, \phi, \xi, \eta, g)\) be an almost \(\alpha\)-cosymplectic manifold. If \(\sigma = 0\), then the \((\kappa, \mu, \nu)\)-structure always exists on every open and dense subset of \(M^3\).

Proof. Substituting \(\sigma = 0\) and \(s = \frac{1}{2} h\) in Eq. (5.7) we deduce

\[
Q = \tilde{a}I + \tilde{b}\eta \otimes \xi + 2ah + (2\alpha + \frac{\xi(\lambda)}{\lambda})\phi h,
\]

which yields

\[
Q\xi = tr(l)\xi,
\]

for any vector fields on \(M^3\). Setting \(\xi\) instead of \(Z\) in Eq. (5.1) we obtain

\[
R(X, Y)\xi = -S(X, \xi)Y + S(Y, \xi)X + \eta(Y)QX - \eta(X)QY - \frac{\xi}{2} \eta(\nabla_\xi h)X - \eta(X)Y\],
\]

and replacing \(X\) by \(\xi\), then we find \(Q\xi = tr(l)\). So the last equation shows that

\[
S(Y, \xi) = tr(l)\eta(Y),
\]
for any vector field $Y$. Thus by virtue of Eqs. (5.9), (5.10) and (5.12), we get
\[
R(X, Y)\xi = -(\alpha^2 + \lambda^2)(\eta(Y)X - \eta(X)Y) + 2\alpha(\eta(Y)hX - \eta(X)hY) + (2\alpha + \xi(\lambda))\lambda(\eta(Y)\phi hX - \eta(X)\phi hY),
\]
where the functions $\kappa, \mu$ and $\nu$ defined by
\[
\kappa = \text{tr}(l)\eta(X), \quad \mu = 2a, \quad \nu = 2\alpha + \xi(\lambda),
\]
respectively and it completes the proof of the Theorem. $\square$

**Corollary 2.** By using Eq. (4.32) for 3-dimensional ($n = 1$), we have
\[
Q\phi - \phi Q = 2\mu h\phi + 2\nu h.
\]

Now, we investigate the inversion of the above corollary on the three dimensional almost $\alpha$-cosymplectic manifolds.

**Theorem 8.** Let $(\mathcal{M}^3, \phi, \xi, \eta, g)$ be an almost $\alpha$-cosymplectic manifold. If the following relation is satisfied
\[
(5.13) \quad Q\phi - \phi Q = f_1 h\phi + f_2 h,
\]
then the manifold $(\mathcal{M}^3, \phi, \xi, \eta, g)$ is an almost $\alpha$-cosymplectic $(\kappa, \mu, \nu)$-space, where the functions $f_1, f_2 \in C^\infty$.

**Proof.** Considering the equations (2.10) and (5.13) we have
\[
\alpha^2 \phi^2 X + 2\alpha \phi hX - h^2 X + \phi(\nabla_\xi h)X = QX - 2\text{tr}(l)\eta(X)\xi + \text{tr}(l)X - \frac{r}{2}(X - \eta(X)\xi) \tag{5.14}
\]
Applying $\phi$ both two sides of Eq. (5.14), we get that
\[
-\alpha^2 \phi X - \phi h^2 X - 2\alpha hX - (\nabla_\xi h)X = \phi QX + \text{tr}(l)\phi X - \frac{r}{2}\phi X \tag{5.15}
\]
On the other hand, replacing $X$ by $\phi X$ in Eq. (5.15), we find
\[
-\alpha^2 \phi X + 2\alpha hX - h^2 \phi X + (\nabla_\xi h)X = Q\phi X + \text{tr}(l)\phi X - \frac{r}{2}\phi X, \tag{5.16}
\]
with the help of $\phi(\nabla_\xi h)\phi X = -\phi^2(\nabla_\xi h)X$. Combining Eqs. (5.15) and (5.16), we deduce
\[
Q\phi X + \phi QX = -2\left[\alpha^2 \phi + \phi h^2\right]X - 2\text{tr}(l)\phi X + r\phi X.
\]
Then substituting Eq. (5.16) in the last equation and using Eq. (5.13), we obtain
\[
Q\phi X + \phi QX = -\text{tr}(l)\phi X + r\phi X.
\]
By virtue of Eqs. (5.15), (5.16) and (5.13), we also obtain that
\[
(\nabla_\xi h)X = \frac{1}{2} f_1 h\phi X + \frac{1}{2}(f_2 - 4\alpha)hX. \tag{5.17}
\]
Using Eq. (5.17) in Eq. (5.17), we have
\[
QX = \tilde{a}X + \tilde{b}\eta(X)\xi + 2\alpha \phi hX + \frac{1}{2} f_1 hX + \frac{1}{2}(f_2 - 4\alpha)\phi hX, \tag{5.18}
\]
for $\sigma \equiv 0$. Finally, substituting Eq. \((5.18)\) in Eq. \((5.11)\), we deduce

$$R(X,Y)\xi = (tr(l) + \tilde{a} - \frac{r}{2})[\eta(Y)X - \eta(X)Y] + \frac{1}{2}f_1[\eta(Y)hX - \eta(X)hY]$$

$$+ \frac{1}{2}f_2[\eta(Y)\phi hX - \eta(X)\phi hY],$$

where $\tilde{a} = \frac{1}{2}r + a^2 + \lambda^2$. Thus this shows that $(M^3, \phi, \xi, \eta, g)$ is a $(\kappa, \mu, \nu)$-space. $\square$

An interesting question is, Do there exist almost $\alpha$-cosymplectic manifolds satisfying \((1.2)\) with $\kappa, \mu$ non-constant smooth functions? Now, we construct an example in order to answer this question for the three dimensional case.

**Example 1.** Consider the three dimensional manifold

$$M^3 = \{(x, y, z) \in \mathbb{R}^3, \quad z \neq 0\},$$

where $(x, y, z)$ are the Cartesian coordinates in $\mathbb{R}^3$. We define three vector fields on $M$ as

$$e = \frac{\partial}{\partial x}, \quad \phi e = \frac{\partial}{\partial y},$$

$$\xi = [\alpha x - y(e^{-2\alpha z} + z)] \frac{\partial}{\partial x} + [x(z - e^{-2\alpha z}) + \alpha y] \frac{\partial}{\partial y} + \frac{\partial}{\partial z},$$

where $\alpha$ is a real number. We easily get

$$[e, \phi e] = 0, \quad [e, \xi] = \alpha e + (z - e^{-2\alpha z})\phi e, \quad [\phi e, \xi] = -(e^{-2\alpha z} + z)e + \alpha \phi e.$$

Moreover, the matrix form of the metric tensor $g$, the tensor fields $\phi$ and $h$ are given by

$$g = \begin{pmatrix}
1 & 0 & -d \\
0 & 1 & -k \\
-d & -k & 1 + d^2 + k^2
\end{pmatrix},$$

and

$$\phi = \begin{pmatrix}
0 & -d & k \\
1 & 0 & -d \\
0 & 0 & -d
\end{pmatrix}, \quad h = \begin{pmatrix}
e^{-2z} & 0 & -de^{-2z} \\
0 & -e^{-2z} & ke^{-2z} \\
0 & 0 & 0
\end{pmatrix},$$

where

$$d = \alpha x - y(e^{-2\alpha z} + z),$$

$$k = x(z - e^{-2\alpha z}) + \alpha y.$$ 

Let $\eta$ be the 1-form defined by $\eta = k_1dx + k_2dy + k_3dz$ for all vector fields on $M^3$. Since $\eta(X) = g(X, \xi)$, we can easily obtain that $\eta(e) = 0, \eta(\phi e) = 0$ and $\eta(\xi) = 1$. By using these equations, we get $\eta = dz$ for all vector fields. Since $d\eta = d(dz) = d^2z$, we obtain $d\eta = 0$. Using Koszul’s formula, we have seen that $d\Phi = 2\alpha \eta \wedge \Phi$. Hence, it has been showed that $M^3$ is an almost $\alpha$-cosymplectic manifold. Thus we obtain

$$R(X,Y)\xi = -(e^{-4\alpha z} + \alpha^2)[\eta(Y)X - \eta(X)Y] + 2z[\eta(Y)hX - \eta(X)hY],$$

where $\kappa = -(e^{-4\alpha z} + \alpha^2)$ and $\mu = 2z.$
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