Affinity driven social networks

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In this work we present a model for evolving networks, where the driven force is related to the social affinity between individuals in a population. In the model, a set of individuals initially arranged on a regular ordered network and thus linked with their closest neighbors are allowed to rearrange their connections according to a dynamics closely related to that of the stable marriage problem. We show that the behavior of some topological properties of the resulting networks follows a non trivial pattern.

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I. INTRODUCTION

The stable marriage problem, introduced by Gale and Shapley in 1962 [1], is a well known example of an optimization problem. In the original formulation, men and women look for a partner to marry. Previously, each agent, man or woman, ranks all the individuals of the opposite sex according to a personal preference. The only motivation that governs the evolution towards a given configuration of marriages is to get married to someone at the top of ones priorities list. But as the individual preference is not necessarily symmetric, the evolution towards a stable situation is not trivial.

In general terms, the stable matching problem is a prototype model in economics and social sciences, where agents act following selfish premises to optimize their own satisfaction, and with underlying mutually conflicting constraints. However, the emergence of global configurations promoted by more collective or collaborative attitudes have also been studied. For example, in [2] Nieuwenhuizen focussed on the properties of globally optimal matchings which are advantageous for the society as a whole, but not necessarily for all individuals.

It is not surprising that besides its practical relevance, the stable marriage problem presents many interesting theoretical features that have attracted researchers from computer and social sciences, mathematics, economics and game theory [3, 4, 5, 6, 7, 8]. The connection between the stable marriage problem and classical disordered systems, established in [9] promoted the interest on the problem within the physics community [2, 10, 11, 12].

Letting the system evolve from its initial configuration, we can claim to have achieved a stable situation when it is not possible to find a man and a woman not married each other that would prefer to join themselves in a new couple, leaving their corresponding partners alone. Such a matching is called stable since no individual has the chance to break it without remaining alone. Previous studies have been mostly concerned with finding algorithms for getting stable pairing configurations [13, 14].

Given the rules according to which marriages may be reconfigured, the main question is whether a stable situation where no benefits can be obtained from ulterior rearrangement can be achieved. In [1] the authors proved that each instance of the marriage problem has at least one stable solution, and they presented an efficient algorithm to find it.

An instance of the stable matching problem is completely specified by the preference or ranking matrix $X$, where all the information about each individual preferences is contained. We can also define the problems in social terms, by considering a marriage tension associated to the mutual attraction. The more mutually affine the individuals in a marriage are, the lower the marriage tension will be. We can thus associate the cost of a marriage with the mutual affinity. It is possible to store in the elements $x_{ij}$ and $x_{ji}$ of $X$ the information about the tension that a marriage between $i$ and $j$ will represent for each one of them respectively. The energy or tension of a configuration of a marriage can be defined as the sum of the individual tensions of the each of the partners in the couple.

In the next sections we present the system under study and the obtained results.

II. THE MODEL

A. Social Affinity

The system consists in $N$ interacting individuals represented by the nodes of an evolving network. Though the number of nodes and links will remain fixed throughout the whole process, networks can evolve by changing their topology through the rewiring of the links. The rewiring is done following the premise that each individual will try to be connected to those individuals who are socially more affine to him. This dynamics is different from the dynamics of the stable marriage problem in that here individuals try to conform groups not necessarily isolated instead of couples. At this point we recall the concept of social affinity between a couple of individuals. We understand that social affinity is the result of the mutual interest that a given pair of individuals arise in each other.

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The affinity is composed then by sum of the personal interest or unilateral affinity of each of the individuals in a given eventual relationship. This last feature does not need to present symmetry properties. In an extreme exercise of abstraction we quantify this “emotional” concept by letting each individual to conform a list of her/his priorities at the moment of choosing a partner or friend and assigning to each of the other individuals a score ranging from zero to one. We define thus the ranking matrix assigning to each of the other individuals a score ranging of abstractness we quantify this “emotional” concept of the others or if the index associated to the node is greater than 1, classified into a certain number of discrete steps. A social interpretation of this can be made in terms of a case where people to whom one would like to be linked, follows no special pattern within the network, and there is no correlation between the mutual affinity and the social or geographical proximity. That means that (initial) vicinity or popularity plays no role on the affinity development. This could be the case if the system under study consist in individuals with no previous knowledge of the others or if the index associated to the node is for identification purposes only. As a mean to quantify the initial range of variability of the affinities we allow the random distributed variable to adopt only a discrete set of values (or steps). The extreme cases correspond to those when a) the affinities can take only two values and b) when the values are uniformly distributed, with the number of steps only limited by the discrete nature of our algorithm.

In the second approach, which will be referred as neighbor correlated, the affinity, on the contrary, is closely related to vicinity. Given a couple of nodes $i$ and $j$, we define the normalized periodical distance between them, $dist(i,j)$, as:

$$dist(i,j) = \frac{2}{N} \left[ |i-j| \mod \frac{N}{2} \right].$$

(4)

Then, we define $x_{ij}$ by the following expression:

$$x_{ij} = dist(i,j)(q + (1-q)r).$$

(5)

where $r$ is a random number between 0 and 1, and $q$ a parameter which we will call slope. In an immediate analysis of the above expression, it can be seen that the value of tension (the inversely proportional to the affinity) as a function of normalized distance is a random number restricted to the interior of two straight lines of slope $\pm 1$ and $\pm q$ respectively. In Fig. 1 we can see an example of the distribution of affinities around node 1 in a system of 1000 individuals and $q = 0.5$. The social correlate of this choice in the distribution of affinity is to privilege closer neighbors when choosing who to be connected with. One can imagine a number of examples where this is the case and clearly, the vicinity or spatial proximity affects the affinity between two individuals.

The choice of the values that conform $X$ is of fundamental importance in the development of the interactions, and therefore in the outcome of the dynamics. Hence, special attention is paid to this feature of the model resulting in two different approaches, distinguished by the characteristics of the (initial) distribution of affinities. The first one, namely random distributed, assigns to the elements of $X$ random values between 0 and 1, classified into a certain number of discrete steps. A social interpretation of this can be made in terms of a case when people to whom one would like to be linked, follows no special pattern within the network, and there is no correlation between the mutual affinity and the physical or geographical proximity. That means that (initial) vicinity or popularity plays no role on the affinity development. This could be the case if the system under study consist in individuals with no previous knowledge of the others or if the index associated to the node is for identification purposes only. As a mean to quantify the initial range of variability of the affinities we allow the random distributed variable to adopt only a discrete set of values (or steps). The extreme cases correspond to those when a) the affinities can take only two values and b) when the values are uniformly distributed, with the number of steps only limited by the discrete nature of our algorithm.

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![FIG. 1: Example of the tension values distribution of node 1, when $N = 1000$ and $q = 0.5$.](image-url)
due to the circular symmetry in the definition of distance. One can infer, that as in the case of the stable marriage problem, the dynamic of the system will be governed by the affinity between individuals, in the sense that an individual will try to be connected to others in the first positions in her/his list. If we consider the case when \( r \) is constant, the final state of the system would necessary be a regular network, in which each node is connected to her/his firsts neighbors. To avoid this trivial case we have introduced a random distribution for \( r \). Furthermore, the parameter \( q \) will be closely related to how close to the trivial case the final configuration will be, reaching a complete matching when \( q = 1 \). On the other hand, choosing \( q \) close to 0 will lead the system to be closer to a random network. In other words, \( q \) (or even better \( 1/q \)) can be considered as an order parameter of the network. All these vague concepts regarding the final state of the system will be formalized after defining the dynamics.

B. Network Dynamics

Once the affinities list has been assigned to each individual we let the social configuration or network evolve. We start with nodes conforming an initially ordered network and let the dynamics associated to the traditional marriage problem lead the social structure to a more stable (less energetic) configuration. The governing dynamics may be chosen to minimize the individual energy, Eq. 2, or the global one, Eq. 3. We will refer to these cases respectively as (global dynamics and individual dynamics). We will show that in all the considered cases the system also achieves a configuration with both lower energies.

It is important to point out that on this first stage we preserve the number of initial links. The process of network rewiring comprises two aspects that must be clearly defined. On one side there is an affinity based probability to break an already established link to create a new one. On the other hand, we may consider that each individual has a minimum number of associated links, thus all the links will have an extreme attached permanently to a given node. These are what we call one side-fixed edges. The possibility of no a priori attached links will be also considered as well.

1. Case 1: One side-fixed edges

We will start by considering the case when each node has \( u \) edges attached to it by one of the extremes, the connectivity of each node is at least \( u \), and the mean connectivity is \( 2u \). Mind that this restriction does not imply that the edges are directed, since once the evolution has finished, the edges are considered indistinguishable. The further rewiring of a given link can be chosen in terms of a global or individual energy minimization rule. Thus we distinguish between the following two cases

a) Global decision dynamics (GDD)

In this case, the changes in the configuration of the network are oriented to minimize the global energy \( E \). Since the energy landscape can be extremely complicated most of the times, the system will get trapped in a configuration associated with a local minimum. The mean individual energies will still be quite high, representing a discomfort situation among the individuals. To avoid this situation we recur to a scheme based on simulated annealing [13, 14], therefore allowed changes will have a non-zero probability of increasing the global energy, which in the end will help to achieve lower value of \( E \). To continue the analogy with the real annealing, we define a temperature \( T \), which will be responsible for the fluctuations in the global energy. The process starts with an ordered network at a certain initial \( T \). At each time step, a node and one of its links are randomly chosen. The evaluation of the rewiring of the chosen links follows, by measuring the change in energy involved in this procedure, namely \( \Delta E \). The change is accepted with probability \( p_a \) with

\[
p_a = \begin{cases} 
1 & \text{if } \Delta E < 0 \\
\frac{1}{2} \exp(-\frac{\Delta E}{T}) & \text{if } \Delta E > 0
\end{cases}
\]  

(6)

After a certain amount of time steps (each one comprising \( N \) computational steps), the threshold value \( T \) is reduced, and we iterate until the chosen minimum temperature is achieved. This temperature usually is chosen to let the system reach a steady state. Since changes in the network configuration are made taken into consideration the global benefit, we can say that the individuals are not “selfish” or that they are not in charge of the evolution. If, for example, we break the link \((i, j)\) and create the link \((i, k)\) the changes in global energy and in \( i \)'s energy are, respectively

\[
\Delta E = \frac{1}{2} (x_{ik} + x_{ki} - x_{ij} - x_{ji})
\]

\[
\Delta E_i = x_{ik} - x_{ij}.
\]

(7)

It is no difficult to imagine a situation in which \( \Delta E_i < 0 \) and \( \Delta E_i > 0 \). In this case, the change is accepted, even though from \( i \)'s point of view is not convenient.

b) Individual decision dynamics 1 (IDD1)

In this case we consider that the individuals make selfish decisions, since only the individual benefit is taken into account at each change. The mechanism here is very much the same as the previous case, the only difference is that instead of using the value \( \Delta E \) as a threshold we take \( \Delta E_i \). Thus, we can not call the process a simulated annealing anymore though the acceptance of changes is made according to the probability \( p_b \) defined as

\[
p_b = \begin{cases} 
1 & \text{if } \Delta E_i < 0 \\
\frac{1}{2} \exp(-\frac{\Delta E_i}{T}) & \text{if } \Delta E_i > 0
\end{cases}
\]

(8)

Even though changes are made in a way to benefit the individual who is making the decision, without regarding what happens with the global energy, it can be seen that \( E \) tends to decrease in this process, as shown if Fig. 2.
2. Case 2: Free edges

c) Individual decision dynamics 2 (IDD2)

The reason for this different approach is that we want to deal with a more realistic individual oriented dynamic. This time, edges are no longer attached to any node. The only restraint in the dynamics is to preserve the number of link and the connectivity of the underlying network. Therefore we impose that no node is left without a link, in other words, we do not accept isolated individuals. Though this case is similar to IDD1 with \( u = 1 \) there are some important differences. Again, we proceed within a simulated annealing frame, but the way we reconnect the network in every time step changes.

The new mechanism goes as follows: we choose a node \( i \) randomly, and look at every node \( i \) is connected to, choosing the one with lower affinity (higher energy value), lets say \( j \). Now we choose another node \( k \) at random not connected to \( i \), and accept to link \( i \) with \( k \) (replacing \( j \)) with a probability given by Eq. (8). The main difference between this dynamics and the previous one, IDD1, is that here we don’t assign \( u \) edges to every node, so when it comes to decide which link to cut, the choice does not restrict to what we previously called assigned edges, but to any edge the node has. Furthermore, as the decision whether to accept a connection or not is taken by only one of the pair of linked nodes, individuals now have the possibility to detach from those undesirable connections. This balance plays a very important role in self organization.

III. NUMERICAL RESULTS

In what follows we will describe the results corresponding to the cases of GDD, IDD1 and IDD2, obtained after extensive numerical simulations of the described model.

Most of the simulations were done with networks of 100 individuals, with mean connectivity equal to 4 ( \( u = 2 \)). When effects that could be associated to size effects appeared, we increased the size of the system to confirm our results.

As said, for GDD we expected the system to achieve a steady state (hopefully the minimum energy state), where no more changes take place. Meanwhile in IDD1 and IDD2, we thought of the system self organizing in a state of stationary global energy, where changes still take place. However, both for random and neighbor correlated distributions of affinity, we found that GDD and IDD1 achieved steady states, with similar values of the final global energy (GDD’s value is slightly smaller). On the other hand, IDD2 reaches, as expected, a stationary situation just as the one described above, see Fig. 2. The fact that GDD reaches a steady state is a direct result of how the dynamics is defined. Once the system reaches an energy minimum and the temperature is not high enough to displace it from there, no more changes can take place. The constraint imposed on one of the extremes of each link also drives the system to a steady situation in the case of IDD1.

The absence of this constraints is what lets the system evolving under the IDD2 scheme not to freeze in a given configuration. Once in a configuration with a minimum energy, the system continues to explore all the available changes leading to other configurations without a change in the energy. It is now completely possible for a change to take place in a way that global energy increases, but the individual energy (of a certain node) decreases, being the latter the one governing the dynamics.

All the three final states present close values of global energy, generally differing in less than 10% (see insent in Fig. 2), indicating that though the dynamics may be associated to global or individual decisions, the network reaches an ordered state.

![Global energies as a function of temperature for the cases GDD (full), IDD1 (dashed), and IDD2 (dotted), with \( q = 0.35 \) and \( N = 1000 \). In the inset we show a detail of the evolution for larger times i.e. smaller T](image)

According to the way how matrix \( X \) was defined, it is apparent that, except in a number of cases that depends on \( q \), individuals will prefer to be connected to their closer neighbors. We wonder what kind of network topology will be the final output of a dynamics dominated by this feature.

We are interested in knowing whether the dynamics of the process can lead to a structure with enhanced clusterization or not, stressing even more the local organization of an ordered network. We study thus, the clustering features of the resulting networks, normalizing the obtained values to the clustering coefficient of an ordered network. Our first observation is that when considering a randomly distributed affinity the final configuration presents a normalized clustering coefficient \( C_q \) considerably lower than 1, indicating that no apparent local structures are being formed. On the contrary, when
produced by but the interplay with a certain degree of disorder intro-
dividuals tend to favor connections to closed neighbors, are being formed. Given the correlated distribution, in-
fluence value reflects that some kind of closed structures
bors, the fact that clustering is greater than the refer-
tering is a measure of connectivity among close neigh-
work with equal number of nodes and links. Since clus-
tering presents a marked non monotonous dependence
higher than 1, meaning that the network clustering is
takes place, the normalized clustering increases to values
large 1
with parameter
in Fig. 3 we find that it is similar to the one displayed by
an algorithm proposed in [18, 19]. The algorithm allows
us to evaluate the community structure in a network,
generating a dendrogram depicting the partition of a net-
into smaller entities. At each step the fusion of two
structures is proposed and accepted only if this fusion
lowers a global quantity called modularity. A high mod-
ularity is an evidence of having obtained a good partition
of the network under analysis. However, the final result
does not only depends on the network itself, but also on
the algorithm used to find it. Examples of the resulting
dendograms are depicted in Figures 4a and b. The algo-
ism is finished when the maximal value of modularity
M is reached. What we observe is that the degree of local
organization is higher when the value of q corresponds to
the one presenting the maximum clusterization, depicted
in Fig. 3. Both the visible structure of the dendogram
and the higher value of modularity reflect this fact.

Another quantity that reflects this behavior is what we
denominate $C_1$ nodes, defined as the number of nodes $c_1$ with absolute clustering equal to 1. A node being
$c_1$ means that all its neighbors are connected between
them. A high mean clustering coefficient might be related with a high number of $c_1$ nodes in the network. When
analyzing the behavior of $C_1$ as a function of $q$ (Inset of
Fig. 3) we find that it is similar to the one displayed by
the clustering. It reaches a maximum at a finite value of
$q$. This reinforces the idea that the networks evolves to
a locally compact configuration, with nodes tending to
conform closed groups. When we say closed groups we
do not mean disconnected subgraphs but a nodes highly
connected among them defined a cluster connected to the
rest of the networks through a few number of links.

Another way to analyze the topology of the resulting
networks is by studying the community structure using
an algorithm proposed in [18, 19]. The algorithm allows
us to evaluate the community structure in a network,
generating a dendrogram depicting the partition of a net-
into smaller entities. At each step the fusion of two
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and the higher value of modularity reflect this fact.

![Diagram](image1)

**FIG. 3:** Normalized clustering as a function of $1/q$, for the cases GDD (squares), IDD1 (circles), IDD2 (triangles). In the inset we show the density of nodes $c_1$ for the same cases. With $N = 100$, $u = 2$

![Diagram](image2)

**FIG. 4:** Examples of the dendograms of the resulting networks for IDD2, $N = 100$ and a) $q = 0.48$, b) $q = 10^{-4}$

**IV. CONCLUSIONS**

In this work we have presented a model for evolving
networks where the dynamics of the architecture of the
links is related to the affinity between individuals. This
aspect associates the model with that of the stable mar-
rriage though in the present case individuals are not form-
ing couples but groups of affine agents.

One of the most interesting feature is the evolution of
the clustering coefficient as a function of the disorder of
the initial condition. If we associate the parameter $q$ with
a degree of disorder, we can see that the clustering has a maximum for an intermediate value of $q$. The clustering is an interesting characteristic of the network since it is related to local efficiency in transmission of information. Previous model of networks only displayed a monotonic behavior of this quantity.

On the other hand it is interesting to stress when the system is driven either by a collective or individual initiative, the results are qualitatively the same. Though the values of the energy $E$ where higher for the cases related to individual dynamics, the overall behavior of the systems is preserved.

We want to end up by saying that the present work is only the first stage towards a model of co evolution of the affinity of agents and the topology of the underlying network. The feedback between these two features in a more general model can lead to an collection of interesting results with social relevance. But first we wanted to isolate those aspects related to the network topology as was done in a previous work [20] where it was the topology of the network that remained unchanged and the affinity among agents evolved.

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