CONVERSE OF SCHUR’S THEOREM - A STATEMENT

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Abstract. Let $G$ be an arbitrary group such that $G/Z(G)$ is finite, where $Z(G)$ denotes the center of the group $G$. Then $\gamma_2(G)$, the commutator subgroup of $G$, is finite. This result is known as Schur’s theorem (the Schur’s theorem). In this short note we provide a quick survey on the converse of Schur’s theorem, generalize known results in this direction and prove the following result (which is perhaps the most suitable statement for converse of the Schur’s theorem): If $G$ is an arbitrary group with finite $\gamma_2(G)$, then $G/Z(G)$ is finite if $Z_2(G) \subseteq Z_2(G)$ is finitely generated, where $Z_2(G)$ denotes the second center of a group $G$. If $G/Z(G)$ is finite, then $\gamma_2(G)$ is also finite and $|G/Z(G)| \leq |\gamma_2(G)|^d$, where $d$ denotes the number of elements in any minimal generating set for $G/Z(G)$. We classify all nilpotent groups $G$ of class $2$ up to isoclinism (in the sense of P. Hall) such that $|G/Z(G)| = |\gamma_2(G)|^d$, and ask some questions in the sequel.

1. Introduction and some history

Let $G$ be an arbitrary group. Let $Z(G)$, $Z_2(G)$, $\gamma_2(G)$ denote the center, the second center and the commutator subgroup of $G$. Let $K(G)$ denote the set of all commutators of $G$ and for $x \in G$, $[x, G]$ denote the set $\{[x, g] | g \in G\}$. Notice that $|[x, G]| = |x^G|$, where $x^G$ denotes the conjugacy class of $x$ in $G$. If $[x, G] \subseteq Z(G)$, then $[x, G]$ becomes a subgroup of $G$. Exponent of a subgroup $H$ of $G$ is denoted by $exp(H)$. For a subgroup $H$ of $G$, $C_G(H)$ denotes the centralizer of $H$ in $G$ and for an element $x \in G$, $C_G(x)$ denotes the centralizer of $x$ in $G$.

Understanding the relationship between $\gamma_2(G)$ and $G/Z(G)$ goes back at least to 1904 when I. Schur [14] proved that the finiteness of $G/Z(G)$ implies the finiteness of $\gamma_2(G)$. This result will be referred to as ‘the Schur’s theorem’ throughout the article. A natural question which arises here is about converse of the Schur’s theorem, i.e., whether the finiteness of $\gamma_2(G)$ implies the finiteness of $G/Z(G)$. Unfortunately the answer is negative as can be seen for infinite extraspecial $p$-group for an odd prime $p$. But there has been attempts to modify the statement and get conclusions.

On one hand, people studied the situation by putting some extra conditions on the group. For example, B. H. Neumann [9] Corollary 5.41] proved that $G/Z(G)$ is finite if $\gamma_2(G)$ is finite and $G$ is finitely generated. Moreover, he proved that if $G$ is generated by $k$ elements, then $|G/Z(G)| \leq |\gamma_2(G)|^k$. This result is recently generalized by P. Niroomand [12] by proving that $G/Z(G)$ is finite if $\gamma_2(G)$ is finite and $G/Z(G)$ is finitely generated. B. H. Neumann [10] Theorem 3.1] proved the following result: For an arbitrary group $G$, the lengths of the conjugacy classes of elements of $G$ are bounded above by a finite natural number if and only if $\gamma_2(G)$ is finite. So it follows that if $G/Z(G)$ is finitely generated and the lengths of the conjugacy classes of elements of $G$ are bounded above by a finite natural number, then $G/Z(G)$ is finite. A special case of this result was re-proved by B. Sury [15] on the way to generalizing Niroomand’s result. The following result generalizes all the statements mentioned in this paragraph. The proof is surprisingly elementary.
Theorem A. Let \( G \) be an arbitrary group such that \( G/Z(G) \) is finitely generated by \( x_1 Z(G), x_2 Z(G), \ldots, x_t Z(G) \) such that \( [x_i, G] \) is finite for \( 1 \leq i \leq t \). Then \( |G/Z(G)| \leq \Pi_i |x_i^G| < \infty \). Moreover, \( \gamma_2(G) \) is finite.

On the other hand, somewhat weaker conclusion were obtained by assuming the finiteness of the commutator subgroup. For example, see the following result of P. Hall [3].

Theorem 1.1. If \( G \) is an arbitrary group such that \( \gamma_2(G) \) is finite, then \( G/Z_2(G) \) is finite.

Explicit bounds on the order of \( G/Z_2(G) \) were first given by I. D. Macdonald [4, Theorem 6.2] and later on improved by K. Podoski and B. Szegedy [13] by showing that if \( \gamma_2(G)/(\gamma_2(G) \cap Z(G)) = n \), then \( |G/Z_2(G)| \leq n^{c \log n} \) with \( c = 2 \). These are really very striking results which suggest to look for the obstruction in the direction of converse of the Schur’s theorem. And it is very surprizing (at least for the people who didn’t know P. Hall’s result) to observe that all these obstructions, which stop \( G/Z(G) \) to be finite, lie between \( Z(G) \) and \( Z_2(G) \). In view of Theorem A, one can easily say that whenever \( \gamma_2(G) \) is finite, \( G/Z(G) \) is finite if and only if \( Z_2(G)/Z(G) \) is finitely generated. The following result shows that something more precise can be said.

Theorem B. Let \( G \) be an arbitrary group such that \( \gamma_2(G) \) is finite. Then \( G/Z(G) \) is finite if and only if \( Z_2(G)/Z(Z_2(G)) \) is finitely generated.

So a statement of converse of the Schur’s theorem is: Let \( G \) be an arbitrary group such that \( \gamma_2(G) \) is finite and \( Z_2(G)/Z(Z_2(G)) \) is finitely generated. Then \( G/Z(G) \) is finite.

Let \( G \) be an arbitrary group such that \( G/Z(G) \) is minimally generated by \( x_1 Z(G), x_2 Z(G), \ldots, x_d Z(G) \) with \( [x_i, G] \) is finite for \( 1 \leq i \leq d \). Then by Theorem A we have \( |G/Z(G)| \leq \Pi_i |x_i^G| \leq \infty \) and \( \gamma_2(G) \) is finite. Since \( [x, G] \subseteq \gamma_2(G) \) for all \( x \in G \), it follows that

\[
(1.2) \quad |G/Z(G)| \leq |\gamma_2(G)|^d.
\]

Notice that equality holds in (1.2) for all abelian groups \( G \). So a natural question which arises here is: What can one say about non-abelian groups \( G \) for which equality holds in (1.2)? In the last section, we mention various classes of such non-abelian groups and classify, upto isoclinism (see Section 3 for definition), all nilpotent groups \( G \) of class 2 such that \( G/Z(G) \) is finite and equality holds in (1.2) for \( G \). We also pose some questions in the sequel.

Using the results of B. H. Neumann [11, Theorem 6], we can also state Theorem B in graph theoretic terminology. For an arbitrary group \( G \), associate a graph \( \Gamma = \Gamma(G) \) as follows: the vertices of \( \Gamma \) are non-central elements of \( G \), and two vertices \( g \) and \( h \) of \( \Gamma \) are joined by an undirected edge if and only if \( g \) and \( h \) do not commute as elements of \( G \). Such a graph \( \Gamma(G) \) is called the non-commuting graph associated with the group \( G \). The following interesting result is due to B. H. Neumann [11, Theorem 6].

Theorem 1.3. Let \( G \) be an arbitrary group. Then \( \Gamma(G) \) contains no infinite complete subgraph if and only if \( G/Z(G) \) is finite.

Now Theorem B can also be stated in the following form.

Theorem B’. Let \( G \) be an arbitrary group such that \( \gamma_2(G) \) is finite. Then \( G/Z(G) \) is finite if and only if \( \Gamma(Z_2(G)) \) contains no infinite complete subgraph.
2. PROOF OF THEOREMS A AND B

We start with the elementary proof of Theorem A.

Proof of Theorem A. Let $G$ be the group as in the statement of Theorem A. Then $G$ is generated by $x_1, x_2, \ldots, x_t$ along with $Z(G)$. Notice that $Z(G) = \bigcap_{i=1}^t C_G(x_i)$. Since $|[x_i, G]| < \infty$, we have $|G : C_G(x_i)| = |[x_i, G]| < \infty$ for $1 \leq i \leq t$. Hence $|G/Z(G)| = |G/\bigcap_{i=1}^t C_G(x_i)| \leq \prod_{i=1}^t |G : C_G(x_i)| = \prod_{i=1}^t |[x_i, G]|$. Since $|[x_i, G]| = |x_i^G|$, we have $|G/Z(G)| \leq \prod_{i=1}^t |x_i^G|$. That $\gamma_2(G)$ is finite, follows from the Schur’s theorem. This completes the proof of the theorem. $\square$

Now we prove the following lemma, which is a modification of a result of B. H. Neumann [11], whose proof is essentially same as the one given by Neumann.

Lemma 2.1. Let $G$ be an arbitrary group having a normal abelian subgroup $A$ such that the index of $C_G(A)$ in $G$ is finite and $G/A$ is finitely generated by $g_1A, g_2A, \ldots, g_rA$, where $|[g_i, G]| < \infty$ for $1 \leq i \leq r$. Then $G/Z(G)$ is finite.

Proof. Let $G/A$ be generated by $g_1A, g_2A, \ldots, g_rA$ for some $g_i \in G$, where $1 \leq i \leq r < \infty$. Let $X := \{g_1, g_2, \ldots, g_r\}$ and $A$ be generated by a set $Y$. Then $G = \langle X \cup Y \rangle$ and $Z(G) = C_G(X) \cap C_G(Y)$. Notice that $C_G(A) = C_G(Y)$. Since $C_G(A)$ is of finite index, $C_G(Y)$ is also of finite index in $G$. Also, since $|[g_i, G]| < \infty$ for $1 \leq i \leq r$, $C_G(X)$ is of finite index in $G$. Hence the index of $Z(G)$ in $G$ is finite and the proof is complete. $\square$

Proposition 2.2. Let $G$ be an arbitrary group such that $\gamma_2(G)$ is finite, but $G/Z(G)$ is not finite. Then $Z_2(G)/Z(Z_2(G))$ cannot be generated by finite number of elements.

Proof. Since $\gamma_2(G)$ is finite, it follows from Theorem [11] that $G/Z_2(G)$ is finite. Thus $Z_2(G)/Z(G)$ can not be finite, otherwise $G/Z(G)$ will be finite, which we are not considering. Assume that $Z_2(G)/Z(Z_2(G))$ is finitely generated. Since $G/Z_2(G)$ is finite, this implies that $G/Z(Z_2(G))$ is finitely generated. Notice that $Z(Z_2(G))$ is a normal abelian subgroup of $G$ such that the index of $C_G(Z(Z_2(G)))$ in $G$ is finite, since $Z_2(G) \leq C_G(Z(Z_2(G)))$. Hence it follows from Lemma 2.1 that $G/Z(G)$ is finite, which is a contradiction to the given hypothesis. Hence $Z_2(G)/Z(Z_2(G))$ can not be generated by finite number of elements. This completes the proof. $\square$

Proof of Theorem B. If $G/Z(G)$ is finite, then obviously $Z_2(G)/Z(Z_2(G))$ is finite. Proof of the converse part follows from Proposition 2.2. $\square$

3. FURTHER RESULTS AND SOME QUESTIONS

Let $x \in Z_2(G)$ for a group $G$. Then, notice that $[x, G]$ is a central subgroup of $G$. We start with the following simple result.

Lemma 3.1. Let $G$ be an arbitrary group such that $Z_2(G)/Z(G)$ is finitely generated by $x_1Z(G), x_2Z(G), \ldots, x_tZ(G)$ such that $\exp([x_i, G])$ is finite for $1 \leq i \leq t$. Then

$$Z_2(G)/Z(G) = \prod_{i=1}^t \exp([x_i, G]).$$
Proof. By the given hypothesis \( \exp([x_i, G]) \) is finite for all \( i \) such that \( 1 \leq i \leq t \). Suppose that \( \exp([x_i, G]) = n_i \). Since \([x_i, G] \subseteq Z(G)\), it follows that \([x_i^{n_i}, G] = [x_i, G]^{n_i} = 1\). Thus \( x_i^{n_i} \in Z(G)\) and no lesser power of \( x_i \) than \( n_i \) can lie in \( Z(G)\), which implies that the order of \( x_i Z(G) \) is \( n_i \). Since \( Z_2(G)/Z(G) \) is abelian, we have \( Z_2(G)/Z(G) = \prod_{i=1}^{t} \exp([x_i, G]) \). \( \square \)

If \(|\gamma_2(G)Z(G)/Z(G)| = n\) is finite for a group \( G\), then it follows from \cite{13} Theorem 1 that \(|G/Z_2(G)| \leq n^{2\log_2 n}\). Using this and the preceding lemma we can also provide an upper bound on the size of \( G/Z(G)\) in terms of \( n\), the rank of \( Z_2(G)/Z(G)\) and exponents of certain sets of commutators (here these sets are really subgroups of \( G\) of coset representatives of generators of \( Z_2(G)/Z(G)\) with the elements of \( G\). This is given in the following theorem.

**Theorem 3.2.** Let \( G\) be an arbitrary group. Let \(|\gamma_2(G)Z(G)/Z(G)| = n\) is finite and \( Z_2(G)/Z(G)\) is finitely generated by \( x_1 Z(G), x_2 Z(G), \ldots, x_t Z(G)\) such that \( \exp([x_i, G]) \) is finite for \( 1 \leq i \leq t\). Then

\[
|G/Z(G)| \leq n^{2\log_2 n} \prod_{i=1}^{t} \exp([x_i, G]).
\]

The following concept is due to P. Hall \cite{2}. Two groups \( K\) and \( H\) are said to be isoclinic if there exists an isomorphism \( \phi \) of the factor group \( \bar{K} = K/Z(K)\) onto \( \bar{H} = H/Z(H)\), and an isomorphism \( \theta \) of the subgroup \( \gamma_2(K)\) onto \( \gamma_2(H)\) such that the following diagram is commutative

\[
\begin{array}{ccc}
\bar{K} \times \bar{K} & \xrightarrow{\alpha_G} & \gamma_2(K) \\
\phi \times \phi \downarrow & & \downarrow \theta \\
\bar{H} \times \bar{H} & \xrightarrow{\alpha_H} & \gamma_2(H)
\end{array}
\]

The resulting pair \((\phi, \theta)\) is called an isoclinism of \( K\) onto \( H\). Notice that isoclinism is an equivalence relation among groups.

The following proposition (also see Macdonald’s result \cite{7} Lemma 2.1) is important for the rest of this section.

**Proposition 3.3.** Let \( G\) be a group such that \( G/Z(G)\) is finite. Then there exists a finite group \( H\) isoclinic to the group \( G\) such that \( Z(H) \leq \gamma_2(H)\). Moreover if \( G\) is a \( p\)-group, then \( H\) is also a \( p\)-group.

Proof. Let \( G\) be the given group. Then by Schur’s theorem \( \gamma_2(G)\) is finite. Now it follows from a result of P. Hall \cite{2} that there exists a group \( H\) which is isoclinic to \( G\) and \( Z(H) \leq \gamma_2(H)\). Since \(|\gamma_2(G)| = |\gamma_2(H)|\) is finite, \( Z(H)\) is finite. Hence, by the definition of isoclinism, \( H\) is finite. Now suppose that \( G\) is a \( p\)-groups. Then it follows that \( H/Z(H)\) as well as \( \gamma_2(H)\) are \( p\)-groups. Since \( Z(H) \leq \gamma_2(H)\), this implies that \( H\) is a \( p\)-group. \( \square \)

Now we mention examples of non-abelian groups \( G\) for which equality holds in \( \cite{12}\)? If there exists such a group, then by Proposition 3.3 it follows that there must also exist such a finite group. So let us first look for finite groups only. Perhaps the simplest examples of such groups are finite extraspecial \( p\)-groups. Other class of examples is the class of 2-generated finite capable nilpotent groups with cyclic commutator subgroup. A group \( G\) is said to be capable if there exists a group \( H\) such that \( G \cong H/Z(H)\). M. Isaacs \cite{5} Theorem 2 proved: Let \( G\) be finite and capable, and suppose that \( \gamma_2(G)\) is cyclic and that all elements of order 4 in \( \gamma_2(G)\) are central in \( G\). Then
$|G/Z(G)| \leq |\gamma_2(G)|^2$, and equality holds if $G$ is nilpotent. So if $G$ is a group as in this statement and $G$ is 2-generated, then equality holds in (1.2). Such examples of groups of nilpotency class 2 are given in [8]. Indeed a complete classification of 2-generated finite capable $G$ and $|m|$ holds in (1.2) can be constructed as follows. For any positive integer $Y$, let $Y$ elements in any minimal generating set for $G$ with any finite group for which equality holds in (1.2). Notice that in all of the above examples, the commutator subgroup is cyclic. Infinite groups for which equality holds in (1.2) can be easily obtained by taking a direct product of an infinite abelian group with any finite group for which equality holds in (1.2).

Motivated by finite extraspecial $p$-groups, a more general class of groups $G$ such that equality holds in (1.2) can be constructed as follows. For any positive integer $m$, let $G_1, G_2, \ldots, G_m$ be 2-generated finite $p$-groups such that $\gamma_2(G_i) = Z(G_i) \cong X$ (say) is cyclic of order $q$ for $1 \leq i \leq m$, where $q$ is some power of $p$. Consider the central product

\[(3.4) \quad Y = G_1 * \times G_2 * \times \cdots * \times G_m\]

of $G_1, G_2, \ldots, G_m$ amalgamated at $X$ (isomorphic to cyclic commutator subgroups $\gamma_2(G_i)$, $1 \leq i \leq m$). Then $|Y| = q^{2m+1}$ and $|Y/Z(Y)| = q^{2m} = |\gamma_2(Y)|^{d(Y)}$, where $d(Y) = 2m$ is the number of elements in any minimal generating set for $Y$. Thus equality holds in (1.2) for the group $Y$. Notice that in all of the above examples, the commutator subgroup is cyclic. Infinite groups for which equality hold in (1.2) can be easily obtained by taking a direct product of an infinite abelian group with any finite group for which equality holds in (1.2).

We now proceed to show that any finite $p$-group $G$ of class 2 for which equality holds in (1.2) is isoclinic to a group $Y$ defined in (3.4). Let $\Phi(X)$ denote the Frattini subgroup of a group $X$.

**Proposition 3.5.** Let $H$ be a finite $p$-group of class 2 such that $Z(H) = \gamma_2(H)$ and equality holds in (1.2). Then

(i) $\gamma_2(H)$ is cyclic;
(ii) $H/Z(H)$ is homocyclic;
(iii) $[x, H] = \gamma_2(H)$ for all $x \in H - \Phi(H)$;
(iv) $H$ is minimally generated by even number of elements.

**Proof.** Let $H$ be the group given in the statement. Let $H$ be minimally generated by $d$ elements $x_1, x_2, \ldots, x_d$ (say). Since $Z(H) = \gamma_2(H)$, it follows that $H/Z(H)$ is minimally generated by $x_1 Z(H)$, $x_2 Z(H), \ldots, x_d Z(H)$. Thus by the identity $|H/Z(H)| = |\gamma_2(H)|^d$, it follows that order of $x_i Z(H)$ is equal to $|\gamma_2(H)|$ for all $1 \leq i \leq d$. Since the exponent of $H/Z(H)$ is equal to the exponent of $\gamma_2(H)$, we have that $\gamma_2(H)$ is cyclic and $H/Z(H)$ is homocyclic. Now by Lemma 3.1, $|\gamma_2(H)|^d = |H/Z(H)| = \prod_{i=1}^d \exp([x_i, H])$. Since $[x_i, H] \subseteq \gamma_2(H)$, this implies that $[x_i, H] = \gamma_2(H)$ for each $i$ such that $1 \leq i \leq d$. Let $x$ be an arbitrary element in $H - \Phi(H)$. Then the set $\{x\}$ can always be extended to a minimal generating set of $H$. Thus it follows that $[x, H] = \gamma_2(H)$ for all $x \in H - \Phi(H)$. This proves first three assertions.

For the proof of (iv), consider the group $\tilde{H} = H/\Phi(\gamma_2(H))$. Notice that both $H$ as well as $\tilde{H}$ are minimally generated by $d$ elements. Since $[x, H] = \gamma_2(H)$ for all $x \in H - \Phi(H)$, it follows that for no $x \in H - \Phi(H)$, $\bar{x} \in Z(\tilde{H})$, where $\bar{x} = x\Phi(\gamma_2(H)) \in \tilde{H}$. Thus it follows that $Z(\tilde{H}) \leq \Phi(\tilde{H})$. Also, since $\gamma_2(H)$ is cyclic, $\gamma_2(\tilde{H})$ is cyclic of order $p$. Thus it follows that $\tilde{H}$ is isoclinic to a finite extraspecial $p$-group, and therefore it is minimally generated by even number of elements. Hence $H$ is also minimally generated by even number of elements. This completes the proof of the proposition.

Using the definition of isoclinism, we have

**Corollary 3.6.** Let $G$ be a finite $p$-group of class 2 for which equality holds in (1.2). Then $\gamma_2(G)$ is cyclic and $G/Z(G)$ is homocyclic.
We need the following important result.

**Theorem 3.7** ([1], Theorem 2.1). *Let $G$ be a finite $p$-group of nilpotency class 2 with cyclic center. Then $G$ is a central product either of two generator subgroups with cyclic center or two generator subgroups with cyclic center and a cyclic subgroup.*

**Theorem 3.8.** *Let $G$ be a $p$-group of class 2 such that $G/Z(G)$ is finite and equality holds in (1.2). Then $G$ is isoclinic to the group $Y$, defined in (3.4), for suitable positive integers $m$ and $n$.***

Proof. Let $G$ be a group as in the statement. Then by Proposition 3.3 there exists a finite $p$-group $H$ isoclinic to $G$ such that $Z(H) = \gamma_2(H)$. Obviously $H$ also satisfies $|H/Z(H)| = |\gamma_2(H)|^d$, where $d$ denotes the number of elements in any minimal generating set of $G/Z(G)$. Then by Proposition 3.3, $\gamma_2(H) = Z(H)$ is cyclic of order $q = p^n$ (say) for some positive integer $n$, and $H/Z(H)$ is homocyclic of exponent $q$ and is of order $q^{2m}$ for some positive integer $m$. Since $Z(H) = \gamma_2(H)$ is cyclic, it follows from Theorem 3.7 that $H$ is a central product of 2-generated groups $H_1, H_2, \ldots, H_m$. It is easy to see that $\gamma_2(H_i) = Z(H_i)$ for $1 \leq i \leq m$ and $|\gamma_2(H)| = q$. This completes the proof of the theorem. □

**Remark 3.9.** *Let $G$ be any nilpotent group of class 2 such that $G/Z(G)$ is finite and equality holds in (1.2). Then it is straightforward to show that $G$ is isoclinic to a finite group which can be written as a direct product of it’s Sylow $p$-subgroups, each of which attains the equality in (1.2). Thus $G$ can be classified (upto isoclinism) using Theorem 3.8.*

We do not know of any example of a finite non-nilpotent group $G$ for which equality holds in (1.2). Rather there exists many important classes of groups which can never be the required examples. First such obvious class is the class of all finite simple groups. Other such class consist of finite groups with trivial Frattini subgroup. This follows from the following result of Herzog, Kaplan and Lev [1 Theorem A] (the same result is also proved independently by Halasi and Podoski in [6 Theorem 1.1]).

**Theorem 3.10.** *Let $G$ be any non-abelian group with trivial Frattini subgroup. Then $|G/Z(G)| < |\gamma_2(G)|^2$.***

In view of the above discussion, we pose the following questions.

**Question 1.** Does there exist a non-nilpotent finite group $G$ such that equality holds in (1.2)?

It is interesting to classify all finite nilpotent groups $G$ for which equality holds in (1.2). But let us pose the following concrete question.

**Question 2.** *Let $G$ be a finite nilpotent group for which equality holds in (1.2). Is it true that $\gamma_2(G)$ is always cyclic?***

Finally let us get back to the situation when $G$ is a group with finite $\gamma_2(G)$ but infinite $G/Z(G)$. The well known examples of such types are infinite extraspecial $p$-groups. Other class of examples can be obtained by taking a central product (amalgamated at their centers) of infinitely many copies of a 2-generated finite $p$-group of class 2 such that $\gamma_2(H) = Z(H)$ is cyclic of order $q$, where $q$ is some power of $p$. Notice that both of these classes consists of groups of nilpotency class 2. Now if we take any finite group $X$ and take its direct product with any group $G$ with finite $\gamma_2(G)$ but infinite
$G/\mathbb{Z}(G)$, then the resulting group $X \times G$ also enjoys the property which is enjoyed by the group $G$. This is suggested by Marcel Herzog and Alireza Abdollahi through group pub forum. We feel that such examples may also be obtained by taking central product of suitable groups. Although we have various examples of groups $G$ with finite $\gamma_2(G)$ but infinite $G/\mathbb{Z}(G)$, still it will be interesting to see answers to the following questions.

**Question 3.** Let $n$ be a natural number $> 2$. Does there exist a nilpotent group $G$ of class $n$, which is not obtained by the process mentioned in the preceding paragraph, such that $\gamma_2(G)$ is finite but $G/\mathbb{Z}(G)$ is infinite?

Regarding non-nilpotent groups, we ask

**Question 4.** Does there exist a non-nilpotent group $G$, which is not obtained by the process mentioned in the paragraph preceding Question 3, such that $\gamma_2(G)$ is finite but $G/\mathbb{Z}(G)$ is infinite?

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