Hamiltonian description of singular Lagrangian systems with spontaneously broken time translation symmetry

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Abstract

Shapere and Wilczek recently found some singular Lagrangian systems which spontaneously breaks time translation symmetry. The common feature of their models is that the energy functions are multivalued in terms of the canonical phase space variables and the symmetry breaking ground states are all located at the branching point singularities. By enlarging the phase space and making use of Dirac’s theory on constrained Hamiltonian systems, we present the Hamiltonian description of some of the models discussed by Shapere and Wilczek and found that both the multivaluedness and the branching point singularities can be avoided, while the spontaneous breaking of time translation becomes more transparent. It is also shown that the breaking of time translation is always accompanied by the breaking of time reversal.

Keywords: spontaneous breaking of symmetry, time translation, time reversal, constrained Hamiltonian systems

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1 Introduction

Conventional wisdom says that every classical conservative mechanical system has a conserved total energy $H$ which generates a continuous time translation symmetry. Until very recently, it is commonly believed that time translation symmetry cannot be spontaneously broken. This picture has changed drastically due to the work [1] of Shapere and Wilczek, who found that in some special Lagrangian systems the time translation symmetry can be spontaneously broken in the lowest energy state (or called ground state). All models considered by Shapere and Wilczek bears the same property
that the energy functions cannot be made singlevalued in terms of the generalized co-
ordinates and their conjugate momenta. Historically, such systems are called singular
Lagrangian systems [2, 3] or multivalued Hamiltonian systems [4], and, even though a
plausible quantization procedure exists [4] in certain special cases, a systematic Hamil-
tonian description for the classical case is still missing.

In this paper, we shall address the Hamiltonian description of singular Lagrangian
systems with spontaneously broken time translation symmetry. Before doing this,
we would like to pay some words clarifying the meaning of spontaneous breaking of
time translation symmetry. First of all, in Hamiltonian mechanics, states of motion
are represented by points in the phase space. With properly chosen coordinates on
the phase space, we can say that a state of motion is just a point in phase space with
definite coordinate values. The physical notion of motion is just the change of the state
in the course of time governed by the classical equations of motion, and the ground state
is the state that minimizes the Hamiltonian function, usually subjects to no motion.
For standard canonical Hamiltonian systems, the most commonly used coordinates
on the phase space are generalized coordinates and their conjugate momenta, which
constitute a set of Darboux coordinates on the phase space regarded as a symplectic
manifold. However, this does not mean that Darboux coordinates are the only viable
coordinates on the phase space. In principle, every set of $2n$ ($n$ being the number
of degrees of freedom of the mechanical system) linearly independent functions of the
Darboux coordinates can be chosen as a coordinate system on the phase space. Let
\{\{X_i, i = 1 \cdots 2n\}\} be any coordinate system on the phase space, under which the
ground state has coordinates \{X_i^G\}. A state is subject to nontrivial motion if there
exist some $i$ such that $\dot{X}_i \equiv \frac{d}{dt}X_i \neq 0$. Recall that under infinitesimal time translation
t \rightarrow t + \epsilon, X_i(t) \rightarrow X_i(t + \epsilon) \simeq X_i(t) + \epsilon \dot{X}_i(t)$. If $\dot{X}_i^G = 0$ for all $i$, then the ground
state is preserved by time translation symmetry. This is the case for most familiar
mechanical systems (e.g. classical harmonic oscillator). If, on the contrary, $\dot{X}_i^G \neq 0$
for some $i$, then the ground state is not preserved by time translation, i.e. the time
translation symmetry is spontaneously broken. We see that spontaneous breaking of
time translation symmetry requires nontrivial motion in the ground state.

One may wonder how a true ground state of classical conservative mechanical sys-
tem can carry through nontrivial motion. This question is beautifully answered by
Shapere and Wilczek by presenting concrete examples. The models they discussed are
all Lagrangian systems depending on higher powers of generalized velocities. For such
systems the conjugate momenta are nonlinear functions in terms of generalized veloc-
ties, and this makes the Hamiltonian multivalued in the conjugate momenta. The trick
is that the ground states always appear at the branching points (they called these the
“turning points”) so that their motion is simultaneously governed by several different
Hamiltonian functions, rendering the result of the motion unpredictable. From the
point of view of Hamiltonian mechanics, the multivaluedness of the Hamiltonian sig-
nifies that the system is ill defined, and the ground states at the branching points of
the multivalued Hamiltonian are in contradiction with the usual smooth minimization
procedure. Below we shall present Hamiltonian descriptions of some of the models con-
sidered by Shapere and Wilczek, avoiding the appearance of multivalued Hamiltonian
and the branching point singularities. The only price needed to be paid is that we have to abandon the use of Darboux coordinates on the phase space.

2 General considerations

Consider the following Lagrangian involving higher powers of generalized velocity,

\[ L = \sum_{k=1}^{n} \frac{1}{2k} f_k(\phi) \dot{\phi}^{2k} - f_0(\phi), \quad (n > 1) \]  

(1)

where \( \phi \) denotes the generalized coordinate on the configuration space and \( f_k(\phi) \) are functions of \( \phi \) with \( f_n(\phi) > 0 \) and at least one of the \( f_k(\phi) \) is negative for \( k = 1, ..., n-1 \). This Lagrangian contains only even powers of \( \dot{\phi} \) because we implicitly imposed a time reversal symmetry.

The conjugate momenta associated with \( \phi \) reads

\[ p = \frac{\delta L}{\delta \dot{\phi}} = \sum_{k=1}^{n} f_k(\phi) \dot{\phi}^{2k-1}. \]

Following the standard procedure, we introduce the Hamiltonian

\[ H = p \dot{\phi} - L = \sum_{k=1}^{n} \frac{2k-1}{2k} f_k(\phi) \dot{\phi}^{2k} + f_0(\phi). \]

The Hamiltonian should in principle be regarded as a function of the phase space variables \((\phi, p)\). However, since \( p \) is a nonlinear polynomial in \( \dot{\phi} \), \( H \) will in general be multivalued in \( p \). So, for a given set of values of \((\phi, p)\), there can be several different values for \( H \). If, as the states tend to some special points \((\phi(B), p(B))\), some of the different values of \( H \) become degenerate, we call these special points the branching points of the Hamiltonian.

The existence of branching points indicates that the original Lagrangian system is singular and possesses some special features. Indeed, though at the first glance, the Lagrangian depends only on \((\phi, \dot{\phi})\) and the standard Euler-Lagrangian equation of motion

\[ \frac{\delta L}{\delta \phi} - \frac{d}{dt} \left( \frac{\delta L}{\delta \dot{\phi}} \right) = 0 \]

should hold without problem, a moment thinking would reveal that things are more involved. Consider the action \( S = \int L dt \) for the above system. After integration by parts, the Lagrangian can be changed into a form containing higher order time derivatives of \( \phi \). In the conventional treatment of such systems, one takes \( \phi, \dot{\phi}, ..., \frac{d^{n-1}}{dt^{n-1}} \phi \) as independent generalized coordinates and define variational derivatives of the corresponding Lagrangian with respect to the time derivatives of these coordinates, i.e.
\( \dot{\phi}, \ddot{\phi}, \ldots, \frac{d^{\ell} \phi}{dt^{\ell}}, \) as generalized momenta (\( \ell \) is the highest order of time derivatives appearing in the Lagrangian). In doing so, one effectively turns the system into a constrained Lagrangian system with a larger number of degrees of freedom, and to get the consistent time evolution of the unconstrained degrees of freedom, Dirac’s theory for constrained Hamiltonian systems [5] is often employed.

Actually, the idea of turning the Lagrangian system into a constrained Hamiltonian system is the key to resolve the problem of presenting a consistent Hamiltonian description for singular Lagrangian systems with spontaneously broken time translation symmetry. To employ Dirac’s method, it is not necessary to carry out the integration by parts in the action. Rather, we can manually introduce novel coordinates \( \gamma, \rho \) and introduce a new Lagrangian

\[
\bar{L} = \sum_{k=1}^{n} \frac{1}{2k} f_k(\phi) \rho^{2k} - f_0(\phi) + \gamma (\rho - \dot{\phi}) \tag{2}
\]

which is equivalent to the original one by solving the \( \gamma \) equation of motion and substituting back in \( \bar{L} \). Since the Lagrangian (2) depends only linearly on \( \dot{\phi} \) and does not depend on \( \dot{\gamma}, \dot{\rho} \), it is clear that this system is a constrained system with a larger phase space containing the phase space of the original system as a subspace. Our task is then turned into finding the consistent Hamiltonian structure on an unconstrained subspace of the enlarged phase space which gives the proper Hamiltonian description of our problem at hand. This last step, however, must be carried out on a case by case basis.

Let us remark that although in the above we considered only systems involving a single generalized coordinate, the generalization to cases with multiple generalized coordinates is straightforward.

### 3 Model analysis

To make our analysis clear and understandable, we shall consider some of the models presented by Shapere and Wilczek, i.e. the \( \dot{\phi}^4 \) model, the \( fgh \) model and the double sombrero model. These models will be analyzed in separate subsections.

#### 3.1 The \( \dot{\phi}^4 \) model

The \( \dot{\phi}^4 \) model is unambiguously defined by its Lagrangian

\[
L(\phi) = -\frac{\kappa}{2} \dot{\phi}^2 + \frac{\lambda}{4} \dot{\phi}^4,
\]

where \( \lambda, \kappa > 0 \) are constant parameters. Clearly, this is the simplest case of \( n = 2 \) in (1) with the choice \( f_2 = \lambda, f_1 = -\kappa \) and \( f_0 = 0 \). Using the idea outlined in the last
section, we introduce the following Lagrangian involving two novel coordinates \( \rho \) and \( \gamma \),
\[
L(\phi, \rho, \gamma) = -\frac{\kappa \rho^2}{2} + \frac{\lambda}{4} \rho^4 + \gamma (\rho - \dot{\phi}).
\]

This system possesses an enlarged phase space spanned by \((\rho, \gamma, \phi, \pi_\rho, \pi_\gamma, \pi_\phi)\), where \((\pi_\rho, \pi_\gamma, \pi_\phi)\) are conjugate momenta of the respective generalized coordinates. This enlarged, 6-dimensional phase space does not possess a well defined symplectic structure, because the definition of the conjugate momenta gives rise to 3 primary constraints,
\[
G_1 = \pi_\rho + \gamma \approx 0,
G_2 = \pi_\gamma \approx 0,
G_3 = \pi_\rho \approx 0.
\]

Using standard Legendre transformation we can obtain the expression for the Hamiltonian \( H \) associated with \( L(\phi, \rho, \gamma) \). Consequently, we are ready to analyze the time evolution of the primary constraints with respect to the total Hamiltonian
\[
H_{\text{total}} \equiv H + \sum_{i=1}^{3} \mu_i G_i
\]
\[
= \frac{\kappa}{2} \rho^2 - \frac{\lambda}{4} \rho^4 - \gamma \rho + \tilde{\mu}_1 G_1 + \tilde{\mu}_2 G_2 + \tilde{\mu}_3 G_3
\]  
(3)

using the canonical Poisson brackets for the phase space variables \((\rho, \gamma, \phi, \pi_\rho, \pi_\gamma, \pi_\phi)\). It turns out that only the time evolution of \( G_3 \) leads to a secondary constraints
\[
G_4 = -\kappa \rho + \lambda \rho^3 + \gamma \approx 0,
\]
and there is no further constraints. Notice that the Lagrangian multipliers \( \mu_i \) and \( \tilde{\mu}_i \) appearing on the first and second lines of (3) are not the same quantities. We have absorbed the unknown \( \dot{\phi}, \dot{\gamma}, \dot{\rho} \) etc into the variables \( \tilde{\mu}_i \).

Next we need to evaluate the matrix Poisson brackets between the constraints, i.e.
\[
M_{\alpha \beta} \equiv \{ G_\alpha, G_\beta \}.
\]
The result reads
\[
M = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & \kappa - 3\lambda \rho^2 & 0 & 0 \\
0 & -1 & -\kappa + 3\lambda \rho^2 & 0 & 0 & 0 \\
\end{pmatrix}.
\]
As long as \( \kappa - 3\lambda \rho^2 \neq 0 \), the above matrix has rank 4 and all the constraints \( G_1 \sim G_4 \) are of the second class. Following the standard procedure we introduce the Dirac Poisson bracket
\[
\{ A, B \}_{DB} = \{ A, B \} - \{ A, G_\alpha \} (M^{-1})^{\alpha \beta} \{ G_\beta, B \}
\]
for any two functions $A, B$ on the phase space. After doing so, all second class constraints can be consistently set equal to strong zeros. The only remaining phase space variables are $(\phi, \rho)$, with

$$\{\phi, \rho\}_{DB} = \frac{1}{3\lambda \rho^2 - \kappa}, \quad \{\rho, \phi\}_{DB} = \{\phi, \phi\}_{DB} = 0.$$  \hfill (4)

The two dimensional subspace spanned by $(\phi, \rho)$ will be henceforth referred to as reduced phase space. The final Hamiltonian on the reduced phase space is given by

$$H_{\text{fin}} = -\frac{1}{2}\kappa \rho^2 + \frac{3}{4}\lambda \rho^4,$$

and the Hamiltonian equations of motion are

$$\dot{\phi} = \{\phi, H_{\text{fin}}\}_{DB} = \rho, \quad \dot{\rho} = \{\rho, H_{\text{fin}}\}_{DB} = 0.$$

The final Hamiltonian is smooth and single valued, its minima obey the usual extremal conditions

$$\frac{\partial H_{\text{fin}}}{\partial \phi} = 0, \quad \frac{\partial H_{\text{fin}}}{\partial \rho} = 0.$$  

The minima occur at

$$\rho^G = \pm \sqrt{\kappa/3\lambda},$$

and $\phi^G$ is not fixed by the minimization condition. Rather, its rate of change in time is given by the equation of motion,

$$\dot{\phi}^G = \dot{\rho}^G = \pm \sqrt{\kappa/3\lambda}.$$

If we take $(\phi^G, \rho^G)$ as parameters specifying the ground states, then the above system will possess two distinct families of infinitely degenerated ground states parametrized by $(\phi^G, \rho^G) = (\sqrt{\kappa/3\lambda} + \phi_0, \sqrt{\kappa/3\lambda})$ and $(\phi^G, \rho^G) = (-\sqrt{\kappa/3\lambda} + \phi_0, -\sqrt{\kappa/3\lambda})$ respectively, where the constant $\phi_0$ is the initial value of $\phi^G$. The explicit dependence of $\phi^G$ on $t$ indicates that the time translation symmetry is spontaneously broken. Moreover, the actual ground state must be chosen from only one of the two families of degenerate ground states. The concrete choice of ground state also breaks the time reversal symmetry.

What happens when $\kappa - 3\lambda \rho^2 = 0$? It appears at first glance that in this case the matrix $M$ becomes uninvertible and the above procedure for constructing Poisson brackets on the reduced phase space breaks down. This is also signified in the final Poisson bracket $\{\phi, \rho\} = \frac{1}{3\lambda \rho^2 - \kappa}$. If the singularity at the right hand side appears at other places in the phase space, this would indeed be a problem. However, notice that at exactly this particular singularity, the final Hamiltonian acquires its critical value. In the standard treatment of Hamiltonian mechanics, the symplectic structure on the phase space is often written as

$$\Omega = dq \wedge dp = dt \wedge dH,$$

where in the second equality the time parameter $t$ and the Hamiltonian $H$ are taken as a pair of action-angle variables. It is clear that the the critical points of $H$, the symplectic structure vanishes and the Poisson bracket is not well defined. Nonetheless, such singularities will not affect the effectiveness of the Poisson bracket in general, as we have witnessed above. In particular, when evaluating the equations of motion using the Poisson brackets (4), all such singularities disappears and the resulting equations hold without problem even at the criticalities of the Hamiltonian.
3.2 The $fgh$ model

The previous model does not allow the inclusion of a potential, which makes it not useful when interaction is involved. To allow the inclusion of interacting potentials, let us look at a slightly more complex model – the $fgh$ model specified by the Lagrangian

$$L(\phi) = f(\phi)\dot{\phi}^4 + g(\phi)\dot{\phi}^2 + h(\phi)$$

with $f(\phi) > 0, g(\phi) < 0$. This model is the special case of (1) with $n = 2$, $f_2 = 4f(\phi), f_1 = 2g(\phi)$ and $f_0 = h(\phi)$. Minimization of the corresponding energy function (here and below we shall omit the arguments of $f, g$ and $h$ and denote the derivatives such as $\frac{df}{d\phi}$ as $f_\phi$ for simplicity)

$$E = 3f\dot{\phi}^4 + g\dot{\phi}^2 - h,$$

$$= 3f \left( \dot{\phi}^2 + \frac{g}{6f} \right)^2 - \frac{g^2}{12f} - h$$

requires both $\dot{\phi}^2 + \frac{g}{6f} = 0$ and $\frac{g^2}{12f} + h = c$, where $c$ is a constant. This is impossible for generic $h$, because the former condition implies nontrivial time dependence of $\phi$, while the latter requires $\phi$ to take some specific constant value(s). To circumvent this contradiction, it is necessary [1] to take $h$ to be the solution of the algebraic equation $\frac{g^2}{12f} + h = c$. Doing so the Lagrangian is rewritten as

$$L(\phi) = f\dot{\phi}^4 + g\dot{\phi}^2 + c - \frac{g^2}{12f}.$$

We proceed as outlined in the last subsection and introduce an equivalent constrained Lagrangian,

$$\bar{L}(\phi, \rho, \gamma) = f\rho^4 + g\rho^2 + c - \frac{g^2}{12f} + \gamma(\rho - \dot{\phi}).$$

After a standard procedure for constructing Dirac Poisson bracket, we are left with a reduced phase space spanned by $(\rho, \phi)$ with the following Poisson bracket and Hamiltonian,

$$\{\phi, \rho\}_{DB} = \frac{1}{2(6f\rho^2 + g)}, \quad \{\phi, \phi\} = \{\rho, \rho\} = 0,$$

$$H_{fin} = 3f\rho^4 + g\rho^2 - c + \frac{g^2}{12f} = 3f \left( \rho^2 + \frac{g}{6f} \right)^2 - c.$$  

The Hamiltonian equations of motion are

$$\dot{\phi} = \{\phi, H_{fin}\}_{DB} = \rho,$$

$$\dot{\rho} = \{\rho, H_{fin}\}_{DB} = \frac{1}{24f^2} (f_\phi g - 2g_\phi f) - \frac{1}{4f} f_\phi \rho^2.$$
Once again, the apparent singularities in the $\{\phi, \rho\}_{DB}$ bracket appears at the critical points go $H_{fin}$, which is not a problem as we have explained in the last subsection.

Eq.(6) shows that the final Hamiltonian is smooth and single valued, and its minima are not isolated points, but rather a pair of curves in the reduced phase space given by the equations $\rho = \pm \sqrt{-g_0(\phi^G)}$. It is easy to check that each of these two curves is consistent with the Hamiltonian equation of motion.

Due to the complicated form of the equations of motion, it is not easy to determine the time dependence of $\phi^G, \rho^G$ in the ground states. However, given any value of $\phi^G$ consistent with the equation of motion, the states $(\phi^G, \rho^G) = (\phi^G, \pm \sqrt{-g_0(\phi^G)})$ constitute two families of infinitely degenerated ground states. The time dependence of each states in these two families indicates that the time translation symmetry is spontaneously broken. Moreover, the actual ground state must be chosen from only one of the two families. The concrete choice of ground state automatically breaks the time reversal symmetry.

### 3.3 The double sombrero model

The double sombrero model is a little bit more complicated than the previous two models because more degrees of freedom are involved. The model Lagrangian is

$$L(\rho, \phi) = \frac{1}{4}(\dot{\rho}^2 + \rho^2 \dot{\phi}^2 - \kappa)^2 + \frac{\mu}{2} \rho^2 - \frac{\lambda}{4} \rho^4,$$

where $\kappa, \mu, \lambda$ are all positive constants. Following the above procedure, we first introduce the first order Lagrangian,

$$\tilde{L}(\rho, \phi, \varphi, \psi, \gamma, \eta) = \frac{1}{4}(\varphi^2 + \rho^2 \psi^2 - \kappa)^2 + \frac{\mu}{2} \rho^2 - \frac{\lambda}{4} \rho^4 + \gamma(\varphi - \dot{\rho}) + \eta(\psi - \dot{\phi}).$$

The construction of Dirac Poisson brackets in this model is much involved comparing to the case of the previous two models because the presence of more constraints. However, with the help of existing algorithm [6], it is not difficult to write down a computer algebra procedure using Maple. It turns out that the reduced phase subspace is spanned by $(\rho, \phi, \varphi, \psi)$, with the following Dirac Poisson brackets:

$$\begin{align*}
\{\rho, \phi\}_{DB} &= 0, \\
\{\rho, \varphi\}_{DB} &= \frac{3\rho^2 \psi^2 + \varphi^2 - \kappa}{(3\rho^2 \psi^2 + 3\varphi^2 - \kappa)(\rho^2 \psi^2 + \varphi^2 - \kappa)}, \\
\{\rho, \psi\}_{DB} &= -\frac{2\psi \varphi}{(3\rho^2 \psi^2 + 3\varphi^2 - \kappa)(\rho^2 \psi^2 + \varphi^2 - \kappa)}, \\
\{\phi, \varphi\}_{DB} &= -\frac{2\psi \varphi}{(3\rho^2 \psi^2 + 3\varphi^2 - \kappa)(\rho^2 \psi^2 + \varphi^2 - \kappa)}, \\
\{\phi, \psi\}_{DB} &= \frac{3\rho^2 \psi^2 + \varphi^2 - \kappa}{(3\rho^2 \psi^2 + 3\varphi^2 - \kappa)(\rho^2 \psi^2 + \varphi^2 - \kappa)},
\end{align*}$$

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\[ \{ \varphi, \psi \}_{DB} = \frac{2\psi}{\rho} \left( \frac{2}{3} \rho^2 \psi^2 + \varphi^2 - \kappa \right) \]

and

\[ \{ \rho, \rho \}_{DB} = \{ \phi, \phi \}_{DB} = \{ \varphi, \varphi \}_{DB} = \{ \psi, \psi \}_{DB} = 0. \]

The final Hamiltonian on the reduced phase space reads

\[ H_{\text{fin}} = \frac{1}{12} \left( \frac{3}{2} \rho^2 \psi^2 + 3 \varphi^2 - \kappa \right)^2 + \frac{1}{4\lambda} \left( \lambda \rho^2 - \mu \right)^2 - \frac{\kappa^2}{3} - \frac{\mu^2}{4\lambda}, \] (8)

which yields the Hamiltonian equations of motion

\[ \dot{\varphi} = \{ \varphi, H_{\text{fin}} \}_{DB} = \rho \psi^2 - \frac{\rho (\lambda \rho^2 - \mu)(3 \rho^2 \psi^2 + \varphi^2 - \kappa)}{(3 \rho^2 \psi^2 + 3 \varphi^2 - \kappa)(\rho^2 \psi^2 + \varphi^2 - \kappa)}, \] (11)

\[ \dot{\psi} = \{ \psi, H_{\text{fin}} \}_{DB} = - \frac{2 \psi \varphi}{\rho} + \frac{2 \psi^2 \rho (\lambda \rho^2 - \mu)}{(3 \rho^2 \psi^2 + 3 \varphi^2 - \kappa)(\rho^2 \psi^2 + \varphi^2 - \kappa)}. \] (12)

Eq. (8) shows that the final Hamiltonian is smooth and single valued. The minima of the Hamiltonian are determined by the conditions

\[ 3 \rho^2 \psi^2 + 3 \varphi^2 - \kappa = 0, \] (13)
\[ \lambda \rho^2 - \mu = 0 \] (14)

together with the equations of motion. Concretely, from (14), we can get

\[ \rho^G = \pm \sqrt{\frac{\mu}{\lambda}}, \]

which are constants. Therefore, from the equation of motion \( \dot{\varphi} = \varphi \), we get

\[ \varphi^G = 0. \]

Inserting these results into (13), we get

\[ \psi^G = \pm \sqrt{\frac{\lambda \kappa}{3 \mu}}, \]

and further, from the equation \( \dot{\phi} = \psi \), we get

\[ \phi^G = \pm \sqrt{\frac{\lambda \kappa}{3 \mu}} t + \phi_0, \]

where \( \phi_0 \) is a constant.
Finally, the ground states are identified with the following curves,

\[
(\phi^G, \rho^G, \psi^G, \varphi^G) = \left( \epsilon_2 \sqrt{\frac{\lambda}{3\mu}}, \epsilon_1 \sqrt{\frac{\mu}{\lambda}}, \epsilon_2 \sqrt{\frac{\lambda}{3\mu}}, 0 \right),
\]

where \( \epsilon_1 = \pm 1, \epsilon_2 = \pm 1 \) and are independent of each other. The number of independent choices of \((\epsilon_1, \epsilon_2)\) is 4. Therefore, there are totally 4 distinct families of infinitely degenerate ground states in this model, each family subjects to nontrivial motion, which spontaneously breaks time translation. The actual ground states must be chosen from one of the 4 families. Any such choice breaks time reversal symmetry.

Remarks:

1. As in the previous two models, the apparent singularities on the right hand sides of \((11)\) and \((12)\) are located exactly at the critical points of \(H_{fm}\), so these do not give rise to any problem as explained earlier.

2. The consistency of Hamiltonian equations of motion with the corresponding Euler-Lagrangian equation of motion for the \(\dot{\phi}^4\) model and \(fgh\) model can be easily checked. For the double sombrero model, the consistency between Hamiltonian equations of motion and the Euler-Lagrangian equations of motion seems more complicated to check. We leave this check in the appendix.

3. The Hamiltonian equations of motion for the double sombrero model, eqs. \((11)\) and \((12)\), seems to be singular at the ground states because the indeterminate 0/0 appears on the right hand side. However, this is completely superficial. Actually, if we first collect a common denominator and then substituting in the ground state values of the phase space variables, the right hand sides of both \((11)\) and \((12)\) vanishes.

4 Discussions

Spontaneous breaking of time translation symmetry is an amazing discovery. The existence of such mechanical systems indicates that there is a hidden piece of classical mechanics about which our understanding is still very poor. Nonetheless, the importance of such models cannot be overestimated. On the one hand, these models provide explicit counter examples to the inverse of Noether’s first theorem (which roughly says that any conserved charge generates a continuous symmetry). On the other hand, similar Lagrangian systems appear widely in modern theoretical physics such as the \(k\)-inflation models in cosmology [7], ghost condensation [8], higher curvature and \(f(R)\) gravity models etc.

The original treatment [1] of Shapere and Wilczek on the spontaneous breaking of time translation symmetry has depended on the use of standard Darboux coordinates on the phase space. Using such variables, the multivaluedness of energy functions and the branching point singularities seem to be inevitable. However, our work shows that both the multivaluedness of energy functions and the branching point singularities can
be avoided, the only sacrifice to be paid is that non Darboux coordinates have to be adopted in parametrizing the phase space. The use of non Darboux coordinates also resolves the puzzling jumping effects at the ground states described by Shapere and Wilczek. Actually, from the detailed model analysis presented in the last section, it is clear that there is no such jumping at the ground states using the non Darboux coordinates. We believe that the motion with momentum jumping in the ground states is just a signature indicating that the momentum, as part of the Darboux coordinates, is not a good coordinate for parametrizing the phase space in such models. As a by-product, we also revealed that the time reversal symmetry is broken by the choice of ground states in such models.

Is it possible to extract Darboux coordinates from our treatment on the Hamiltonian description? The answer is yes, but the original problems will reappear in terms of these coordinates. To be more explicit, let us take the simplest $\dot{\phi}^4$ model as an example. It is easy to see that both $(q_1, p_1) = (\kappa \rho - \lambda \rho^3, \phi)$ and $(q_2, p_2) = (\phi, -\kappa \rho + \lambda \rho^3)$ are viable choices of Darboux coordinates. Using either choices of Darboux coordinates, the Hamiltonian becomes multivalued, either in $q_1$ or in $p_2$. The multivaluedness of the Hamiltonian in terms of Darboux coordinates also indicates that these coordinates are not good coordinates on the phase space. On this point, please also notice that $p_1$ ($p_2$ reps.) is nonlinear in $q_1$ ($q_2$ reps.), therefore $(q_1, p_1)$ (or $(q_2, p_2)$) does not parametrize the cotangent bundle of the configuration space as apposed to the cases of canonical Hamiltonian systems.

For usual Hamiltonian systems, passing from classical to the quantum description is straightforward, one only needs to replace the canonical Poisson bracket $\{q, p\}_{PB} = 1$ by the canonical commutator $[\hat{q}, \hat{p}] = i\hbar$. However, for the models discussed in the present work, the Poisson brackets between the “good coordinates” on the phase space are all nonlinear, hence the canonical quantization process does not work. The systematic quantization of such models remain an open problem\textsuperscript{1}. Perhaps the path integral quantization for the $\dot{\phi}^4$ model [4] can provide some clue for the other models.

Another related issue is whether the spontaneous breaking of time translation symmetry can be dynamical, i.e. can we construct models which initially preserve time translation but eventually evolve into a phase with spontaneous breaking of time translation symmetry? A moment thinking seems to indicate that this is possible. Take the $fg\hbar$ model for example. If we take $f(\phi)$ to be a positive constant, $g(\phi)$ to be a function such that $g(\phi) \geq 0$ for the initial value of $\phi$ but decreasing as $\phi$ evolves in time and eventually becomes negative, then the ground states will transit from the initial symmetry preserving phase $(\phi^G, \rho^G) = (\phi_0, 0)$ to the symmetry breaking phase $(\phi^G, \rho^G) = (\phi^G, \pm \sqrt{\frac{-g(\phi^G)}{6f(\phi^G)}})$. We leave the detailed study of this phenomenon for later works.

\textsuperscript{1}After the first version of this paper appeared in arXiv, we have noticed that Shapere and Wilczek (together with Xiong) have gone further on the issue of quantization [9] [10].
Appendix

In this appendix we check the consistency between Euler-Lagrangian equations and Hamiltonian equations of motion for the double sombrero model.

Variation of the Lagrangian (7) with respect to $\phi$ yields

$$\left( 2\rho\ddot{\rho} + \rho^2 \dot{\phi} (\rho^2 + \rho^2 \dot{\phi}^2 - \kappa) \right) + 2\rho^2 \dot{\phi} (\ddot{\rho} \dot{\phi} + \rho \dot{\phi}^2 + \rho^2 \ddot{\phi}) = 0.$$ 

Similarly, variation of (7) with respect to $\rho$ gives

$$\ddot{\rho} (\rho^2 + \rho^2 \dot{\phi}^2 - \kappa) + 2\dot{\rho} (\dot{\phi} \ddot{\rho} + \rho \dot{\phi}^2 + \rho^2 \ddot{\phi}) + [(\lambda \rho^2 - \mu) \rho - \rho \ddot{\phi} (\dot{\rho}^2 + \rho^2 \dot{\phi}^2 - \kappa)] = 0.$$ 

We rearrange these two equations in the form

$$a_1 \ddot{\phi} + b_1 \dot{\phi} + c_1 = 0, \quad (15)$$
$$a_2 \ddot{\phi} + b_2 \dot{\phi} + c_2 = 0, \quad (16)$$

where

$$a_1 = \rho^2 (3\rho^2 \dot{\phi}^2 + \dot{\phi}^2 - \kappa), \quad b_1 = 2\rho^2 \dot{\phi}, \quad c_1 = 2\rho \ddot{\phi} (2\rho^2 \dot{\phi}^2 + \dot{\phi}^2 - \kappa),$$
$$a_2 = 2\rho^2 \dot{\phi}, \quad b_2 = 3\rho^2 + \rho^2 \dot{\phi}^2 - \kappa, \quad c_2 = (\lambda \rho^2 - \mu) \rho - \rho \ddot{\phi} (\dot{\rho}^2 + \rho^2 \dot{\phi}^2 - \kappa),$$

and take (15) and (16) as a system of linear algebraic equations for $\ddot{\phi}$ and $\ddot{\rho}$. The solution is easily found to be

$$\ddot{\phi} = \frac{a_1 c_2 - a_2 c_1}{a_2 b_2 - a_1 b_2}, \quad (17)$$
$$\ddot{\rho} = -\frac{b_1 c_2 - b_2 c_1}{a_2 b_1 - a_1 b_2}. \quad (18)$$

Inserting the expressions for $a_1, a_2, b_1, b_2, c_1, c_2$ and after some algebraic simplifications, we find that (17) and (18) are exactly the same as the results which we can get by substituting (9) and (10) into (11) and (12).

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References

[1] A. Shapere and F. Wilczek, “Classical Time Crystals,” arXiv: 1202.2537.

[2] V.V. Nesterenko, “Singular Lagrangians with higher derivatives,” J. Phys. A: Math. Gen. 22 (1989)1673.

[3] J.U. Cisneros-Parra, “On singular lagrangians and Dirac’s method,” Revista Mexicana de Fisica 58 (2012) 6168.

[4] M. Henneaux, C. Teitelboim, and J. Zanelli, “Quantum mechanics for multivalued Hamiltonians,” Phys. Rev. A36, No.9 (1987) 4417.

[5] P.A.M. Dirac, Lectures on Quantum Mechanics (Belfer Graduate School of Science, Yeshiva University, 1964).

[6] V. P. Gerdt and S. A. Gogilidze, “Constrained Hamiltonian Systems and Groebner Bases,” Computer Algebra in Scientific Computing, V.G.Ganzha, E.W.Mayr and E.V.Vorozhtsov, eds., Springer-Verlag, Berlin, 1999, pp.139-146 [arXiv: math/9909113].

[7] C. Armendariz-Picon, T. Damour, and V. Mukhanov, “k-Inflation,” Phys. Lett. B458: 209-218, 1999 [arXiv: hep-th/9904075].

[8] N. Arkani-Hamed, H.-C. Cheng, M. A. Luty, and S. Mukhyama, “Ghost Condensation and a Consistent Infrared Modification of Gravity,” JHEP 0405 (2004) 074 [arXiv: hep-th/0312099].

[9] A. Shapere and F. Wilczek, “Branched Quantization,” arXiv: 1207.2677.

[10] A. D. Shapere, F. Wilczek, and Z. Xiong, “Models of Topology Change,” arXiv: 1210.3545.