Two-dimensional algorithm to study electromagnetic waves generating by a relativistic electron beam

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Abstract. In the current work a 3D hybrid numerical model of the interaction of a relativistic electron beam with the stationary plasma to study the generated high-frequency electromagnetic radiation is proposed. Ions are considered in the frame of the hydrodynamics approach and an electron component of plasma is described by the kinetic Vlasov equation. Also, the Maxwell equations are added. At initial time there is the plasma circled by the vacuum in a computational domain. A relativistic electron beam of a small density is injected through a domain boundary and, interacting with the plasma, generates electromagnetic waves. The waves spread out into the vacuum where the wave diagnostics has been carried out. To find amplitudes and directions of generated waves the algorithm to define directions and amplitudes of all electromagnetic waves, presenting in the considered domain at a certain time, has been developed. The diagnostics has been made in a 2D case. Also, the tests showing the implementation of the proposed algorithm have been carried out.

1. Introduction
In the current paper a hybrid scheme to model the generation of the high-frequency electromagnetic radiation on the open system ‘dense plasma – a relativistic electron beam’ is proposed. The study of this problem is very important for the thermonuclear research on open traps as it will permit to move further in understanding of the turbulent processes in this kind of systems and in the development of new schemes of the generation of electromagnetic radiation. The present research is the topical one for such physics problems as the generation of radiation in the solar radio bursts, the fast burning of a target in the inertial thermonuclear synthesis and the forming of collisionless shock waves in astrophysics.

The solving of the problem of the interaction of an electron beam with the plasma for the periodic boundary conditions, which allowed to estimate the level of the peak radiation power only on the dynamic stage of the beam instability evolution, was presented in [1]. To solve the problem, the full two-dimensional kinetic model was developed by the particle-in-cell (PIC) method.

Numerical models, using in solving the plasma physics problems, are divided into three types and deal with the kinetic and hydrodynamics approaches. The magnetohydrodynamics (MHD) models are economical from the computational point of view as they require, in their implementation, the storing of only grid arrays. Unfortunately, it is not possible to use them all the time as there are many assumptions on stationarity of ion and electron distribution functions by velocities were made. The most full description can take place on the base of the kinetic Vlasov equation and the Maxwell equations. These models are developed by the PIC-method requiring large computational resources. When solving
problems, it is necessary to store in the RAM two- or three-dimensional arrays of electric and magnetic field intensities, current and charge densities. Besides, it is required to have enough model particles in each cell. The research based on the hybrid (combined) models [2] allows to reduce the requirements for computational resources and at present time is the most perspective according to computational experiments.

2. Three-dimensional hybrid model

Let us consider a three-dimensional hybrid numerical model of the interaction of a relativistic electron beam with the stationary plasma. The problem is stated as follows.

![Figure 1. Geometry of the problem of the interaction of a relativistic electron beam with the stationary plasma.](image)

There is the stationary plasma, having the cylindrical form of a radius $R$, in a domain of the rectangular parallelepiped form $0 < x < L_x$, $0 < y < L_y$, $0 < z < L_z$ (see figure 1). The cylinder axis is along the axis $x$. A plasma density is $n_0$. The stationary magnetic field $\vec{B}_0$ is along the axis $x$. There is no matter in the rest of the considered domain. At initial time an electron beam with a speed $u_b$ and a density $n_b$ enters the domain through the left boundary ($x = 0$). At initial time a plasma temperature $T_0$ and a beam temperature $T_b$ are given. When plasma flows interact, the electromagnetic waves are generated in the plasma. These waves can expand in different directions, interact with the plasma and can be changed by amplitude and motion direction. It is not possible to study the electromagnetic waves appearing in the modeling in the plasma due to the non-linear character of such interaction. To solve the problem, one should circle the plasma area by the vacuum, and then study there (in the vacuum) electromagnetic waves generated by the plasma. Earlier, in works [3,4], there was studied the electromagnetic radiation generated under interaction of an electron beam with the plasma. As an illustration of the arisen electromagnetic field, there is a figure of one electric field component, from which one can conclude that the radiation moves by a normal to the boundary of a computational domain. However, wave characteristics had not been defined by the authors.

In the present paper an algorithm to define the directions and amplitudes of all electromagnetic waves, presenting in the studied area at a definite time is proposed.

2.1. Initial system of equations. Let us write down the initial equations in our hybrid numerical model. The ion motion is described by the MHD equations

$$\frac{\partial n_i}{\partial t} + \nabla \cdot (n_i \vec{V}_i) = 0,$$

$$m_i n_i \frac{d\vec{V}_i}{dt} = -\nabla p_i + e n_i \left(\vec{E} + \frac{1}{c} \vec{V}_i \times \vec{B}\right),$$

$$n_i \left(\frac{\partial T_i}{\partial t} + (\vec{V}_i \cdot \nabla)T_i\right) + (\gamma - 1)p_i\nabla \cdot \vec{V}_i = (\gamma - 1)(Q_i - \nabla \cdot \vec{q}_i),$$

where $n_i, \vec{V}_i, T_i$ are a density, a velocity and a temperature of ions, $\vec{E}$ and $\vec{B}$ are electric and magnetic field intensities, $p_i = n_iT_i$ is an ion component pressure, $m_i$ is an ion mass, $c$ is the speed of light, $e$ is an electron charge, $\gamma$ is the ratio of specific heats, $Q_i$ is an ion heat, $\vec{q}_i = -\kappa \nabla T_i$ is a heat flow, $\kappa$ is the heat conductivity coefficient.
The electron plasma component is described by the kinetic Vlasov equation for the electron distribution function by velocities $f = f(t, \vec{r}, \vec{v})$

$$\frac{\partial f}{\partial t} + \vec{v} \frac{\partial f}{\partial \vec{r}} - e \left( \vec{E} + \frac{1}{c} \vec{v} \times \vec{B} \right) \frac{\partial f}{\partial \vec{v}} = 0$$

(1)

where $\vec{p} = \frac{m_e \vec{v}}{\sqrt{1 - v^2/c^2}}$ is a relativistic electron pulse, $m_e$ is an electron mass.

Also, to these equations the Maxwell equations are added

$$\nabla \times \vec{B} = \frac{4\pi}{c} j + \frac{1}{c} \frac{\partial \vec{E}}{\partial t}, \quad \nabla \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}$$

where

$$\vec{j} = en_c \vec{V}_i - e \int f \vec{v} \, d\vec{v}.$$ 

To solve the problem a uniform rectangular grid with steps $h_1, h_2, h_3$ by the directions $x, y, z$, accordingly, has been introduced in the computational domain. The problem is solved by time with time steps $\tau$. The finite-difference time-domain (FDTD) scheme is used for solving the Maxwell equations [5]. This scheme is characterized by that the space and time grids to define electric and magnetic field intensities are shifted relatively each other. It permits to achieve the approximation of the Maxwell equations to be of the second order.

2.2. Stages of computation. On the first stage the values of the magnetic field intensities on the time layer $m$ ($t^m = m \tau$) are found by the known values of the electric field intensities on the layer $m$ and the magnetic field intensities on the layer $m - 1/2$

$$\frac{\vec{B}^m - \vec{B}^{m-1/2}}{\tau/2} = -c \nabla_h \times \vec{E}^m.$$ 

Here $\nabla_h \times$ is the difference approximation of the rotor operator on the grid.

On the second stage the velocity $\vec{V}_{i}^{m+1/2}$ and the density $n_{i}^{m+1}$ of ions are defined:

$$m_i n_i^m \left( \frac{\vec{V}_{i}^{m+1/2} - \vec{V}_{i}^{m-1/2}}{\tau} + \vec{V}_{i,x}^{m-1/2} \frac{\Delta_1 \vec{V}_{i}^{m-1/2}}{h_1} + \vec{V}_{i,y}^{m-1/2} \frac{\Delta_2 \vec{V}_{i}^{m-1/2}}{h_2} + \vec{V}_{i,z}^{m-1/2} \frac{\Delta_3 \vec{V}_{i}^{m-1/2}}{h_3} \right) =$$

$$= -\nabla_h p_i^m + e n_i^m \left( \vec{E}^m + \frac{1}{c} \frac{\vec{V}_{i}^{m+1/2} + \vec{V}_{i}^{m-1/2}}{2} \times \vec{B}^m \right),$$

$$\frac{n_{i}^{m+1} - n_i^m}{\tau} + \vec{V}_{i,x}^{m+1/2} \frac{\Delta_1 n_i^m}{h_1} + \vec{V}_{i,y}^{m+1/2} \frac{\Delta_2 n_i^m}{h_2} + \vec{V}_{i,z}^{m+1/2} \frac{\Delta_3 n_i^m}{h_3} + n_i^m \nabla_h \vec{V}_{i}^{m+1/2} = 0$$

where $\nabla_h$ is the difference approximation of the gradient operator on the grid, $\Delta_i$ is one side differences of grid functions depending on a sign of the velocity (there is the difference forward if the velocity is positive or the difference backwards if it is vice versa).

On the next stage, using the PIC-method, the kinetic equation (1) is solved and the electron current density is determined. The motion equations of model particles are the characteristics of the Vlasov equation and have the following form

$$\frac{d\vec{p}}{dt} = -e \left( \vec{E} + \frac{1}{c} \vec{v} \times \vec{B} \right), \quad \frac{d\vec{r}}{dt} = \vec{v}.$$ 

The difference scheme for these equations is

$$\frac{\vec{p}_{i}^{m+1/2} - \vec{p}_{i}^{m-1/2}}{\tau} = -e \left( \vec{E}^m + \frac{1}{c} \vec{v}^{m+1/2} + \vec{v}^{m-1/2} \times \vec{B}^m \right),$$

$$\frac{\vec{r}_{i}^{m+1} - \vec{r}_{i}^{m}}{\tau} = \vec{v}_{i}^{m+1/2}.$$
The electron current density is defined by the formula
\[
j_{e}^{m+1/2} = -q \frac{\sum_k \hat{v}_k^{m+1/2} R \left( \frac{\hat{r}^m - \hat{r}_k^m + \hat{r}_{k+1}^m}{2} \right)}{\sum_k R \left( \frac{\hat{r}^m - \hat{r}_k^m + \hat{r}_{k+1}^m}{2} \right)},
\]
where \( q \) is a charge of a model particle, \( R \) is any kernel of the PIC-method [6].

On the fourth stage the magnetic field intensity on the layer \( m + 1/2 \) and the electric field intensity on the layer \( m + 1 \) are computed
\[
\frac{\vec{B}^{m+1/2} - \vec{B}^{m-1/2}}{\tau} = -c \nabla_h \times \vec{E}^{m},
\]
and
\[
\frac{\vec{E}^{m+1} - \vec{E}^{m}}{\tau} = 4\pi \left( n_e^{m+1} - \frac{1}{2} \left( n_i^{m+1} + n_i^{m} \right) \hat{v}_i^{m+1/2} \right) + c \nabla_h \times \vec{B}^{m+1/2}.
\]

On the last stage the equation for the ion temperature is solved
\[
n_i^{m+1} \left( \frac{T_i^{m+1} - T_i^{m}}{\tau} + V_{i,x}^{m+1/2} \frac{\Delta_1 T_i^{m}}{h_1} + V_{i,y}^{m+1/2} \frac{\Delta_2 T_i^{m}}{h_2} + V_{i,z}^{m+1/2} \frac{\Delta_3 T_i^{m}}{h_3} \right) + (y - 1) p_i^{m} \nabla_h \cdot \hat{V}_i^{m+1} = (y - 1) \left( Q_i^{m+1} - \tilde{q}_i^{m} \right)
\]
where \( \nabla_h \cdot \) is the difference approximation of the divergence operator.

2.3. Test computations. Let us consider some results of the test computations using the code developed for the scheme written above. Figure 2 demonstrates the electric fields in the cross section, parallel to the plane \((x, y)\) and going through the beam axis, at time \( t = 0.5 t_0 \) where \( t_0 \) is some characteristic time. From this figure one can see that the beam passed by the half of the domain, approximately, in the direction of \( x \). The incoming beam caused the growth of the electric field intensity \( E_x \) impeding the beam entry into domain.

3. Algorithm to determine directions and amplitudes of all electromagnetic waves
As it was mentioned before, the vacuum area has been introduced to have the possibility to study the electromagnetic waves generated in the plasma. Let us describe the two-dimensional algorithm to define directions and amplitudes of all electromagnetic waves presenting in the vacuum domain at any moment of time. The Maxwell equations are
\[
\frac{\partial \vec{E}}{\partial t} = \nabla \times \vec{B}, \quad \frac{\partial \vec{B}}{\partial t} = -\nabla \times \vec{E}.
\]
Note that here and below all equations are written in dimensionless variables. The characteristic values are the speed of light \( c \), time \( t_0 \) and a length \( L = c t_0 \). This system of equations breaks apart into two different subsystems (all functions depend on time \( t \) and the coordinates \( x \) and \( y \))

\[
\begin{align}
\frac{\partial B_x}{\partial t} &= -\frac{\partial E_z}{\partial y}, \\
\frac{\partial B_y}{\partial t} &= \frac{\partial E_z}{\partial x}, \\
\frac{\partial E_z}{\partial t} &= \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y};
\end{align}
\]

\[
\begin{align}
\frac{\partial B_x}{\partial t} &= \frac{\partial E_y}{\partial y}, \\
\frac{\partial E_x}{\partial t} &= \frac{\partial B_y}{\partial y}, \\
\frac{\partial E_y}{\partial t} &= -\frac{\partial B_x}{\partial x}.
\end{align}
\]
To demonstrate how the algorithm works it is enough to consider the system (2) only. The procedure for the system (3) is similar. So, approximating the system (2) by the FDTD scheme [5], we get

\[
\begin{align*}
\frac{(B_x)_{m+1/2}^{i-1/2,k} - (B_x)_{m-1/2}^{i-1/2,k}}{\tau} &= \frac{1}{h_y} \frac{E_z}{\tau} (i-1/2,k+1/2) - \frac{E_z}{\tau} (i-1/2,k-1/2), \\
\frac{(B_y)_{m+1/2}^{k-1/2,i} - (B_y)_{m-1/2}^{k-1/2,i}}{\tau} &= \frac{1}{h_x} \frac{E_z}{\tau} (i+1/2,k-1/2) - \frac{E_z}{\tau} (i-1/2,k-1/2), \\
\frac{(E_z)_{m}^{i-1/2,k-1/2} - (E_z)_{m-1}^{i-1/2,k-1/2}}{\tau} &= \frac{1}{h_x} \frac{B_x}{\tau} (i-1/2,k-1/2) - \frac{B_x}{\tau} (i-1/2,k-1/2), \\
\frac{(B_x)_{m-1/2}^{i-1/2,k} - (B_x)_{m+1/2}^{i-1/2,k}}{\tau} &= \frac{1}{h_y} \frac{B_y}{\tau} (i+1/2,k) - \frac{B_y}{\tau} (i+1/2,k).
\end{align*}
\]

An arbitrary wave, satisfying to the considered system, can be written in the following form

\[
\begin{align*}
E_z &= \omega A \sin(-\omega t + k_x x + k_y y + \varphi_0), \\
B_x &= k_x A \sin(-\omega t + k_x x + k_y y + \varphi_0), \\
B_y &= -k_x A \sin(-\omega t + k_x x + k_y y + \varphi_0),
\end{align*}
\]

where \(A\) is a wave amplitude, \(\omega\) is a wave frequency, \(k_x\) and \(k_y\) are components of a wave vector, \(\varphi_0\) is a wave shift by space. The components of the wave vector are bound with the frequency by the following formula

\[\omega^2 = k_x^2 + k_y^2.\]

It allows us to introduce a slope \(\alpha\) of the wave vector to the axis \(x\)

\[k_x = \omega \cos \alpha, \quad k_y = \omega \sin \alpha.\]

As the result of solving the problem, at any instant of time any grid function can be represented in the form of the doubled Fourier series. For example, for \(E_z\) it becomes

\[
E_z = \sum_{n_x=0}^{\infty} \sum_{n_y=0}^{\infty} \left[ R_{1,n_x,n_y} \cos(k_x x) \cos(k_y y) + R_{2,n_x,n_y} \cos(k_x x) \sin(k_y y) + \\
+ R_{3,n_x,n_y} \sin(k_x x) \cos(k_y y) + R_{4,n_x,n_y} \sin(k_x x) \sin(k_y y) \right]
\]
where the coefficients $R$ are determined by the Euler-Fourier formulas [7].

From the Fourier analysis mixes up four different waves having the equal $|k_x|$ and $|k_y|$ and different direction of motion. For instance, for the function $E_z$ the following four waves having different amplitudes and phase shifts will be mixed up

\[
E_z^{(1)} = \omega A_1 \sin(k_x x + k_y y + \varphi_1), \\
E_z^{(2)} = \omega A_2 \sin(k_x x - k_y y + \varphi_2), \\
E_z^{(3)} = \omega A_3 \sin(-k_x x + k_y y + \varphi_3), \\
E_z^{(4)} = \omega A_4 \sin(-k_x x - k_y y + \varphi_4).
\]

As the analysis of functions is carried out at a definite moment of time, here $-\omega t$ has been added to the phase shift $\varphi_0$ (see the formulas (4)).

The problem is that at any time $t$ knowing the values of functions $E_x, E_y, E_z, B_x, B_y, B_z$, it is necessary to find out what waves are present in the considered domain, i.e. for each $k_x$ and $k_y$ one should determine a wave amplitude corresponding to this wave vector. As all waves are linearly independent, let us consider below only the waves having the given $|k_x|$ and $|k_y|$. Consequently, the mixture of four waves, mentioned above, is of the form

\[
E_z = E_z^{(1)} + E_z^{(2)} + E_z^{(3)} + E_z^{(4)}. \tag{5}
\]

The expansions of all functions $E_z^{(i)}$ are

\[
E_z^{(1)} = p_1 \cos(k_x x) \cos(k_y y) + q_1 \cos(k_x x) \sin(k_y y) + \\
+ q_1 \sin(k_x x) \cos(k_y y) - p_1 \sin(k_x x) \sin(k_y y),
\]

\[
E_z^{(2)} = p_2 \cos(k_x x) \cos(k_y y) - q_2 \cos(k_x x) \sin(k_y y) + \\
+ q_2 \sin(k_x x) \cos(k_y y) + p_2 \sin(k_x x) \sin(k_y y),
\]

\[
E_z^{(3)} = p_3 \cos(k_x x) \cos(k_y y) + q_3 \cos(k_x x) \sin(k_y y) - \\
- q_3 \sin(k_x x) \cos(k_y y) + p_3 \sin(k_x x) \sin(k_y y),
\]

\[
E_z^{(4)} = p_4 \cos(k_x x) \cos(k_y y) - q_4 \cos(k_x x) \sin(k_y y) - \\
- q_4 \sin(k_x x) \cos(k_y y) - p_4 \sin(k_x x) \sin(k_y y)
\]

where

\[p_i = \omega A_i \sin \varphi_i, \quad q_i = \omega A_i \cos \varphi_i, \quad i = 1,2,3,4.\]

Substituting these expansions into (5) and equating the coefficients by the same functions, we get

\[
p_1 + p_2 + p_3 + p_4 = R_1, \\
-p_1 + p_2 + p_3 - p_4 = R_4, \\
q_1 - q_2 + q_3 - q_4 = R_2, \\
q_1 + q_2 - q_3 - q_4 = R_3.
\]

The unknowns here are $p_i$ and $q_i$. Solving it we obtain the following equalities

\[
\begin{align*}
p_2 + p_3 &= (R_1 + R_4)/2, \\
p_1 + p_4 &= (R_1 - R_4)/2, \\
q_1 - q_4 &= (R_2 + R_3)/2, \\
q_2 - q_3 &= (R_2 - R_3)/2.
\end{align*}
\]

However, we do not have enough equations in this system to define all coefficients $p_i$ and $q_i$. To do it, we need to use the expansion coefficients for the functions $B_x$ and $B_y$.

So, in a similar way we get the system for $B_x$

\[
\begin{align*}
p_2 - p_3 &= (U_1 + U_4)/(2\sin \alpha), \\
p_1 - p_4 &= (U_1 - U_4)/(2\sin \alpha), \\
q_1 + q_4 &= (U_2 + U_3)/(2\sin \alpha), \\
q_2 + q_3 &= (U_2 - U_3)/(2\sin \alpha).
\end{align*}
\]
Solving this system along with the system (6), we obtain all coefficients $p_i$ and $q_i$:

$$p_1 = \frac{1}{4} \left( R_1 - R_4 + \frac{U_1 - U_4}{\sin \alpha} \right), \quad p_2 = \frac{1}{4} \left( R_1 + R_4 - \frac{U_1 + U_4}{\sin \alpha} \right),$$

$$p_3 = \frac{1}{4} \left( R_1 + R_4 + \frac{U_1 + U_4}{\sin \alpha} \right), \quad p_4 = \frac{1}{4} \left( R_1 - R_4 - \frac{U_1 - U_4}{\sin \alpha} \right),$$

$$q_1 = \frac{1}{4} \left( R_2 + R_3 + \frac{U_2 + U_3}{\sin \alpha} \right), \quad q_2 = \frac{1}{4} \left( R_3 - R_2 + \frac{U_2 - U_3}{\sin \alpha} \right),$$

$$q_3 = \frac{1}{4} \left( R_2 - R_3 + \frac{U_2 - U_3}{\sin \alpha} \right), \quad q_4 = \frac{1}{4} \left( -R_2 - R_3 + \frac{U_2 + U_3}{\sin \alpha} \right).$$

Then, knowing all $p_i$ and $q_i$, one can determine amplitudes of all waves having equal $|k_x|$ and $|k_y|$:

$$A_i = \frac{1}{\omega} \sqrt{p_i^2 + q_i^2}, \quad i = 1, \ldots, A.$$  

The phase shift is found by the formula

$$\tan \varphi_i = \frac{p_i}{q_i}, \quad i = 1, \ldots, A.$$

It is worth mentioning that these equations do not work when $\sin \alpha$ is zero or close to zero. In that case, one can use the system which can be obtained from the expansion of the function $B_y$

$$\begin{align*}
    p_3 - p_2 &= (V_1 + V_4)/(2\cos \alpha), \\
    p_1 - p_4 &= (V_4 - V_1)/(2\cos \alpha), \\
    q_1 + q_4 &= -(V_2 + V_3)/(2\cos \alpha), \\
    q_2 + q_3 &= (V_2 - V_3)/(2\cos \alpha)
\end{align*}$$

which along with the system (6) gives the same solution.

4. Implementation of the algorithm

Let us show the algorithm implementation on a simple example. Using the formula (5) and analogous ones for the functions $B_x$ and $B_y$, let us set the sum of four waves having $n_x = \pm 14$, $n_y = \pm 8$ and the following amplitudes and phase shifts:

$$A_1 = 4, \quad A_2 = 1, \quad A_3 = 2, \quad A_4 = 3; \quad \varphi_1 = 0.1, \quad \varphi_2 = 0.2, \quad \varphi_3 = 0.3, \quad \varphi_4 = 0.4.$$  

The results of the implementation of the proposed algorithm for this example are given on figure 3, where the axes represent the harmonic numbers.

![Figure 3. Amplitudes of electromagnetic waves.](image)
2) \((n_x, n_y) = (14, -8)\) and \((n_x, n_y) = (-14, 8)\).

From the current figure we can see that all waves have been separated as one can expect and the wave amplitudes coincide with the given ones.

**Conclusion**

In the present paper the 3D hybrid numerical model to model the interaction of a relativistic electron beam with the stationary plasma with the aim to study the generated high-frequency electromagnetic radiation has been considered. The provided example shows the efficiency of the algorithm. To study the waves formed by the interaction of a beam with the plasma, it is proposed the methodology allowing to define amplitudes and directions expanded waves by the distribution electric and magnetic fields at some moment of time. It has been considered for a 2D case and permits, using the coefficients of the 2D Fourier transform for all functions (there are 24 coefficients for each wave number), to determine amplitudes and phase shifts of all eight waves, including those of the different polarization, having equal modules of a wave number (there are 16 numbers). It is worth mentioning that the analysis results depend on a domain which will be cut. As it is assumed that all functions are periodic, then results will be different, especially for short-wave harmonics. Nevertheless, the implementation of the algorithm will be useful to interpret computational results.

For a 3D case, as the number of directions is doubled, by 48 coefficients of the Fourier expansion of six functions it will be necessary to find 16 amplitudes and 16 phase shifts.

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