Bound States in Models of Asymptotic Freedom

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Abstract

We study some quantum theories which are divergent but for which renormalization can be performed nonperturbatively and explicitly. The result is a well-defined, finite formulation of these theories, in which neither a cutoff nor a bare coupling constant appears. Such theories describe ‘contact’ interactions between particles which are encoded into the boundary conditions of the wavefunction rather than in the hamiltonian. It is the attempt to describe them in terms of conventional potentials or self-interactions that lead to divergences.

We discover that, after renormalization, the dynamics is described by a new operator (the ‘principal operator’ $\Phi(E)$): the Schrödinger equation for energy levels is replaced by the eigenvalue problem $\Phi(E)\psi = 0$ for this operator. (More generally, the resolvent of the hamiltonian has an explicit

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formula in terms of the inverse of \( \Phi(E) \). Moreover the interactions are not described by the boundary conditions (domain) of the operator, but by the formula for it; i.e., by its action on smooth functions. Even when the theory cannot be explicitly solved, it can be given a finite formulation once this operator is determined. It is then possible to apply standard approximation methods; in particular we can determine the energy of bound states, which would be impossible in perturbation theory.

We propose to call such theories which are apparently divergent, but have a finite formulation in terms of the principal operator, *transfinite quantum theories*. We construct some examples of such transfinite quantum field theories: quantum mechanics with ‘contact’ interactions, three body problem with contact interactions, quantum fields (fermionic and bosonic) interacting with a point source, many body problems with contact interactions, and non-relativistic field theory with polynomial interactions.

As an application we develop a theory of self-interacting Bose fields in two dimensions with an attractive self-interaction. The ground state is a Bose-condensate for which the conventional many body theory breaks down due to divergences. The magnitude of the ground state energy grows *exponentially* with the number of particles, rather than like a power law as for conventional many body systems.

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1 Introduction

There is a fact about classical mechanics that is so basic that we don't even call it a law of mechanics. Perhaps it should be called the zeroth law of mechanics:

All particles behave like free particles for short enough intervals of time.

In the language of modern analysis, the path of any particle is differentiable, so that it can be approximated by a straight line for small enough intervals of time. There are other physical phenomena (such as diffusion) where this law does not hold. Hence it is not a self-evident fact, but rather a law of nature. Newton’s second law states that the deviation of the path from that of a free particle is second order in time (when the path is viewed as a curve in configuration space) and is given by the force divided by the mass.

In quantum mechanics we have a similar zeroth law as well. The propagator (i.e., the integral kernel $h_t(x, y) = <x|e^{-Ht}|y>$, where $H = p^2 + V(x)$ is the hamiltonian operator) is a gaussian upto power law corrections:

$$h_t(x, y) = e^{-\frac{(x-y)^2}{4t}} \left(1 + O(t)\right).$$

(1)

(Of course this statement, just like the previous statement in classical mechanics, is true only for non-singular potentials. The quantum theory is more forgiving of singularities than the classical theory. We will consider some singular situations later.)

Now let us look at quantum field theory. The analogous statement would be that the correlation functions $G(x, y) = <\phi(x)\phi(y)>$ of a quantum field $\phi$ will approach those of a free field theory when the distance $|x - y|$ between points becomes small. This is however not true for realistic quantum field
theories. In Quantum Electrodynamics for example, quite the opposite is true: the smaller the distance between points the larger the deviation of a correlation function from that of the free theory. Asymptotically free theories (such as Quantum ChromoDynamics) are better behaved: their correlators do approach those of the free theory but the deviations are of order $\frac{1}{\log |x-y|}$ rather than $|x-y|$.

Thus if we apply conventional ideas of dynamics to such quantum field theories we will run into divergences. These have to be removed by the unwieldy procedure of renormalization. This leads us to ask if there is another way to formulate such theories: to describe the dynamics of the theory by some operator (instead of the hamiltonian) which is able to describe the logarithmic deviation from the free theory. We must first construct such a new picture by working within the traditional renormalization method applied to simple model problems. Once we learn the basic ideas we might be able to deal with more realistic situations. In this paper we will propose such a reformulation of renormalization by studying a series of examples, starting from quantum mechanics (with singular potentials) and ending with some non-relativistic field theories.

Such a new point of view is necessary, since the most fundamental theory of nature—the standard model of elementary particles—is a divergent quantum field theory. It was discovered in part by the requirement of renormalizability: that the divergences can be removed order by order in perturbation theory by redefining the parameters. The main remaining problems in the standard model are due to the unpleasantness left over from this procedure. For example, the ‘naturalness’ problem in the Higgs sector is an extreme
sensitivity (quadratic dependence) of observables (like the Higgs mass) on microscopic parameters.

There are also more practical reasons to look for a new formulation of renormalization. One of the main obstacles to solving for the dynamics of non-abelian gauge theories is also that its quantum theory is divergent. We only know how to remove these divergences by perturbative renormalization. But perturbation theory cannot describe the formation of bound states: a serious problem since all the observable states of the theory are expected to be bound states (confinement).

There are also motivations that come from outside of particle physics. We don’t yet have a quantum theory of gravity due to the divergences that arise in quantizing general relativity: this time the infinities cannot be removed by renormalization. Perhaps a better understanding of renormalization will lead to a way of quantizing non-renormalizable theories as well: there might be a non-trivial fixed point for the renormalization group.

String theory is a serious candidate for a quantum theory of gravitation and, possibly, even a unified theory of all forces. Moreover it is a finite theory. Thus it may be the ultimate solution for many of the problems noted above. However, it should be possible to account for the spectacular success of renormalizable quantum field theories at current energies without having to resort to a new theory of spacetime at Planck energies. There should be a consistent nonperturbative formulation of quantum field theories whether or not the ultimate theory is a finite string theory. To draw an analogy, it is possible to have a mathematically consistent formulation of continuum (fluid) mechanics even though the ultimate description of fluids is in terms of a finite
(but enormously large) number of particles. Quantum field theory should have a self-contained formulation whether or not it is just an approximation to a more fundamental theory.

We also mention another great success of the idea of renormalization: the divergences that afflicted the theory of second order phase transitions were finally understood by Wilson’s development of the renormalization group. New ideas on renormalization might also deepen our understanding of phase transitions. We will give some examples in this paper.

Thus, at the heart of each of the fundamental problems of modern theoretical physics is a divergent quantum theory. Progress is not possible without a better understanding of these infinities. The situation is analogous to knowing the basic laws of mechanics but without knowing calculus: anything beyond the simplest examples are inaccessible. Indeed it was a deeper understanding of the nature of infinity (Cantor’s work on the transfinite numbers) that made modern analysis possible and by extension the modern theory of dynamical systems. We must look for examples which will help us develop such a general theory of divergent quantum systems: a truly analytical quantum mechanics.

We will start with simple quantum mechanical models with singular potentials [1] and progress to systems with several degrees of freedom, the three-body problem and then to non-relativistic field theories. Eventually we hope to formulate QCD in this way, but even the cases we have studied so far address some physically interesting problems.

To be more precise, we have found a way to reformulate some quantum theories that are divergent in the usual formulation in a new way that is
manifestly finite: there is no need for a regularization and all the parameters are physical, not ‘bare’ coupling constants. We call such systems *transfinite quantum systems*. In these systems, interactions are not specified by potentials or any other simple modification of the hamiltonian operator: indeed on smooth wavefunctions the hamiltonian acts just like the free hamiltonian. It is the attempt to shoehorn these system into a conventional description in terms of a hamiltonian such as $H = -\nabla^2 + V(x)$ that leads us to divergences. The interactions are encoded not into the formula for the hamiltonian but into the boundary conditions on the wavefunctions (i.e., domain of self-adjointness of the hamiltonian). Thus the resolvent of the hamiltonian (which is a sort of Green’s function) contains the complete specification of the system. The resolvent has the complete physical description of the system, in terms of physical and not bare parameters.

Upto this point the ideas are not very new: the resolvent has been worked out in many simple cases, most notably by Krein and his school [2]. Unfortunately in most cases, determining the resolvent is the same as exactly solving the system. We need a way to think of such ‘contact’ interactions which is free from divergences and yet does not require us to solve the whole dynamics first: we just need to solve the ‘singular part’ of the dynamics. Our main discovery is that there is a new operator, which we call the *Principal Operator* $\Phi(E)$, which describes the dynamics of transfinite quantum systems. It is free of divergences, can be determined explicitly (usually as an integral operator) and is quite simple in many cases. There are no subtleties in the definition of its domain: the interaction is described by a term in the formula for $\Phi(E)$. In this way the formulation of transfinite quantum systems is no
more difficult than that of finite quantum systems.

The eigenvalues of energy are given by the solutions to $\Phi(E)|\psi> = 0$; the scattering amplitude is determined by the inverse of $\Phi(E)$. Of course the solution of the eigenvalue equation or the inversion of $\Phi(E)$ is a difficult dynamical problem: as difficult as solving the Schrödinger equations would be in finite quantum systems. But the point is that it is no more complicated than that: we can apply the standard methods such as variational principles or perturbation theory to the principal operator, since now we have a formulation free of divergences. The system need not be exactly solvable for it to have a transfinite formulation. Formulating the system amounts to finding the principal operator $\Phi(E)$ while solving it amounts to inverting $\Phi(E)$.

The principal operator can be thought of as an effective Hamiltonian obtained after ‘integrating out’ (or eliminating) the short distance degrees of freedom. This is reminiscent of Wilson’s program of renormalization. The main new point is that we can get a closed expression for such an operator, rather than give an implicit prescription to find it as in Wilson’s program.

Simple, exactly solvable, examples of such transfinite quantum theories have been known for a long time: the two dimensional delta function potential is a good example. We will in fact start our discussion with such a simple example, the pole model of low energy scattering in quantum mechanics. It has been known for a long time that at momenta small compared to the size of the scatterer, the scattering amplitude of a particle tends to a universal form $f(k) = \frac{1}{\xi - ik}$, the number $\xi$ being the scattering length. For positive scattering length there is a bound state near threshold and for negative scattering length a resonance (‘virtual bound state’). But there is
no finite quantum system that can give this simple model of scattering amplitude: such a system can only be constructed as a limit of hamiltonians. Indeed this is a beautiful example of the idea of renormalization at work. We will construct a transfinite quantum theory of two body interactions that describes exactly this case of low energy scattering.

These ideas are of interest in modern (late 1990’s) atomic physics. Technical advances in the cooling and trapping of atoms have made it possible very recently to study experimentally the interaction of atoms at very low momenta: i.e., wavelengths large compared to the size of the atoms themselves. Scattering lengths for several species have been measured. In the case of Rubidium and Sodium the scattering length is positive while for Lithium it is negative. (e.g., \( \xi = -27.3 \pm 0.8 \) Bohr radii for Lithium.) Interspecies scattering also should give a whole range of values of scattering lengths, both positive and negative. Indeed it is even possible to ‘tune’ the scattering length to any desired value using Feshbach resonances. Thus many of the phenomena associated to low energy scattering have become or will soon become experimentally accessible. Of greatest interest is the formation of condensates of a large number of atoms. This would be described by a non-relativistic field theory (bosonic or fermionic) with a contact interaction. The techniques developed may eventually be useful to study phase transitions in such atomic condensates. So far we have a way to describe condensates in two dimensional systems.

We will then pass to studying some field theoretical models. T. D. Lee [4] introduced a simple model for renormalization which has been studied further by many people. It describes a Bose field interacting with a point-
like source. The source itself has two possible states, which can be thought of as describing the internal states of a heavy particle. In the limit when the size of the source goes to zero, there is an ultra-violet divergence in this model: for example, the energy difference between the two states of the source is infinite. This can be removed by a renormalization procedure: the parameters of the model are made to depend on a cut-off (the size of the source) in such a way that the energies of physical states are finite in the limit as the cut-off is removed.

An even simpler, non-relativistic, limit of the Lee model was studied in ref. [5]. The heavy particle was interpreted as the nucleon, its two states being the neutron and the proton. The light (but non-relativistic) boson was thought of as the charged pion. The energy of the ‘neutron’ (more precisely the neutron-proton mass difference) is infinite. This divergence can be regularized by requiring the source to have a finite size $\Lambda^{-1}$ – an ultra-violet cutoff. Then we require a bare parameter $\mu_\Lambda$ depend on $\Lambda$ in such a way that in the limit we get a finite answer for the neutron energy.

Now we have much better descriptions of the nucleon pion system. The static source model is thought of these days as a historical curiosity. But this kind of model has been very valuable as a proving ground for new approaches to renormalization. The most spectacular example has been the work of Wilson, who also perfected his ideas on a variant of the static source model [6]. A later (one dimensional) version of the static source model, the Kondo model, has become a classic example of renormalization. We will also use the static source model and (later on, the Kondo problem) to test our ideas on renormalization.
The simplest example of such a field theory we will study is a system of (non-relativistic) fermions interacting with a point source. This fermionic variant of the Lee model also has a static source with two states: one of these states is now a fermion and the other a boson. We will show that the ultra-violet divergences can be removed by a nonperturbative renormalization method. The energies are all finite and the theory in fact has a completely finite description, with no cutoffs or bare coupling constants in sight. The main lesson we learn is that renormalizable quantum field theories of this type (‘transfinite’) also have interactions that arise from the boundary conditions of the wave function. Hence the dynamical information is not encoded in the hamiltonian, but in the the principal operator, Φ(E), which we construct. The spectrum of the principal operator determines the energy levels: the Schrödinger equation is replaced by the equation Φ(E)ψ = 0. Although the problem is not exactly solvable, we can apply traditional approximation methods such as the variational principle once we have a finite form of the theory. We won’t be able to get all the energy levels: but we will show that they are all finite and determine the ground state and first excited state energies.

We will then apply our methods to the bosonic static source model (the original Lee model). This case is more subtle. In fact the original analysis of the Lee model was incomplete: the divergences were shown to be removed only in states which contain at most two bosons. The possibility remained that the hamiltonian of the system is unbounded below, when the number of bosons is more than two. This is not a mere technicality: there are systems in which an analogous renormalization model still leaves behind divergent
energies for multi-particle states \[7\]. (We will give such an example ourselves later on in this paper.) We will show that the non-relativistic Lee model has energies bounded below in each sector with a fixed number of bosons (this number is a conserved quantity of this model.) Moreover we will make a variational estimate of the ground state energy of the Lee model in the limit of a large number of bosons. Thus it will be established that the Lee model is indeed free of divergences.

Next, we will apply these ideas to the case of a system of non-relativistic bosons in two dimensions, interacting through a contact interaction: non-relativistic \( \lambda \phi^4 \) theory in \( 2 + 1 \) dimensions. This model has been studied in various guises by other authors as well \[8, 9, 10\], but our approach is somewhat different. We will obtain a closed form for the principal operator after renormalization. We will then show how to solve the many body problem in the mean field approximation. In fact we will obtain a solution for the wavefunction of the bosonic condensate in this approximation, as well as its energy.

2 Scattering at Low Momentum and Renormalization

In the limit of small momentum we should expect that the scattering amplitude of two atoms does not depend on the details of the atomic form-factors: the ‘shape’ of the atoms should not matter at wavelengths much larger than the atoms. This is a situation to which the philosophy of renormalization applies perfectly: the cut-off is the size of the atom: the independence of
the atomic scattering amplitude on the shape of the atom arises from the independence on the regularization scheme of the renormalized system. We will now show in some detail how to perform this renormalization.

Elastic scattering \([11]\) of a particle by a heavy target is described by the scattering amplitude \(f(k, n, n')\). (Of course the case of two-particle scattering can be reduced to this case by passing to the center of mass frame.) It is defined in terms of the asymptotic form of the wavefunction as \(r \to \infty\),

\[
\psi(k, x) \sim e^{ikr n \cdot n'} + f(k, n, n') \frac{e^{ikr}}{r}.
\] (2)

Here, \(k = kn\) is the momentum of the incoming wave and \(n' = \frac{x}{r}\) is the direction of the outgoing wave. Conservation of probability requires the “unitarity condition” on the scattering amplitude:

\[
f(k, n, n') - f^*(k, n', n) = 2ik \int f(k, n, n'')f^*(k, n', n'')\frac{d\Omega_{n''}}{4\pi}.
\] (3)

Let us also recall the formula for the total scattering cross-section:

\[
\sigma(k) = \int |f(k, n, n')|^2 d\Omega_{n'}.
\] (4)

If a wave is scattered by a target whose size is small compared with the wavelength, only the partial wave with zero angular momentum will scatter: only this partial wave has an appreciable probability of interacting with the target. Thus the scattering amplitude will become independent of angles in this limit, *even if the target is not spherically symmetric*. The unitarity condition will then require that

\[
\text{Im}f(k) = k|f(k)|^2.
\] (5)
In other words

\[ \text{Im} \frac{1}{f(k)} = -k \] (6)

Thus, in the limit of vanishing momentum the scattering amplitude approaches a real constant (the unitarity condition requires the imaginary part to be \(O(k)\)):

\[ \lim_{k \to 0} f(k, n, n') = -\xi. \] (7)

The quantity \(\xi\) is called the "scattering length". It can be either positive or negative. It is stated in some textbooks that a positive scattering length corresponds to a generally repulsive interaction and a negative one to an attractive interaction. But this is only true of interactions that are weak enough to be treated in the Born expansion. We will in fact see that a positive scattering length can even lead to a bound state.

Now the imaginary part of \(\frac{1}{f(k)}\) must be \(O(k)\) to satisfy the unitary condition. In the limit of small but nonzero momentum we thus have,

\[ \frac{1}{f(k)} = -ik - \frac{1}{\xi} \] (8)

or,

\[ f(k, n, n') = -\frac{1}{\xi^{-1} + ik}. \] (9)

In fact the scattering amplitude of any target whose size is small compared to the wavelength is described asymptotically by this formula: even if the target has no special property such as spherical symmetry. We will call this the 'simple pole model' for low momentum scattering. (Due to Wigner (1933) and Bethe and Peierls (1935).) The scattering cross-section of the target will
be, in this model,
\[ \sigma(k) = 4\pi\xi^2 \frac{1}{1 + k^2\xi^2}. \] (10)
This gives the geometrical meaning of the scattering length: at low momenta the target will appear to be a hard sphere of radius \( \xi \).

The scattering amplitude has a simple pole at
\[ k = \frac{i}{\xi}. \] (11)
This pole is in the upper half of the complex \( k \)-plane for \( \xi > 0 \); it then corresponds to a bound state of binding energy \( \frac{1}{2m\xi^2} \). If \( \xi \) is negative, the pole does not describe a bound state. Poles of \( f(k) \) in the lower half of the \( k \)-plane correspond to solutions of the Schrödinger equation that grow at infinity: it corresponds to what is called a ‘virtual level’. (See [11] section 133.)

This simple picture for the scattering at low momenta however has an important peculiarity: there is no hamiltonian of the form
\[ H = \frac{p^2}{2m} + V(x) \] (12)
with a potential function \( V(x) \) which can reproduce such a scattering amplitude exactly. If there were such a potential, it would have zero range and infinite height (for positive \( \xi \)) or depth (for negative \( \xi \)). \( V(x) \) cannot be described even by a familiar distribution such as a delta function: the delta function potential in three dimensions either has vanishing scattering amplitude (repulsive case) or has no well-defined ground state (attractive case).

Instead we have to view the hamiltonian as arising from a limit of potentials, with widths tending to zero; the heights have to be carefully adjusted
as a function of the widths in this limit in order to get a non-vanishing scattering amplitude or a finite ground state energy. This process of obtaining a finite scattering amplitude from the limit of a sequence of potentials with finite width is reminiscent of the renormalization program of quantum field theory; the width is the ‘short-distance cut-off’ and the height (or depth) of the potential the ‘bare coupling constant’. The main difference from conventional quantum field theory is that the renormalization has to be carried out non-perturbatively, since we expect to recover a bound state in some cases. The answer will be independent of the details of the limiting process.

We will now show that there is in fact a perfectly well-defined quantum theory with the simple pole model above as the scattering amplitude. Its Hamiltonian is the same as that of the free particle as far its action on smooth wavefunctions in position space is concerned. The interactions are encoded into the boundary conditions of the wavefunction at short distances (or at infinity in momentum space). Thus the Hamiltonian is practically useless as a tool in studying this system. We will instead obtain a formula for the resolvent of the regularized Hamiltonian and take its limit as the cutoff is removed. It will turn out that it is determined in terms of a function \( \phi(E) \) (the ‘principal function’): we get a version of the Krein formula for resolvents. The zeros of this principal function give the point spectrum of the renomalized system.

When we have more than two particles, we can still preform the renormalization as before and get a formula for the resolvent. However, this time the principal function is replaced by an operator. In the case of the three body problem, we can study the spectrum of this operator. If the dimension of
space is two, we show that this problem is well-posed and has a well-defined ground state energy. In the case of three-dimensional three body problem, the ground state energy of the renormalized theory still diverges: there are further renormalizations necessary. Thus even if we can obtain a Krein formula and a principal operator, we still need to show that the spectrum is bounded below in order to have a well-defined theory. This is why we take pains to establish this lower bound in the case of some quantum field theories.

2.1 Renormalized Resolvent

Consider a pair of particles with an attractive short range interaction. After separating out the center of mass variable, we can reduce this to the scattering of a particle of mass \( m \) (equal to the reduced mass of the pair) against an immovable ‘target’ representing the interaction between the particles. We are interested in the limit as the inverse range of the interaction \( \Lambda \) is very large compared to the momentum of the particle.

This can be modelled by the Hamiltonian operator, in momentum space,

\[
H_\Lambda \psi(p) = \frac{p^2}{2m} \psi(p) - g(\Lambda) \rho_\Lambda(p) \int \rho_\Lambda(q) \psi(q) [dq].
\]

Here, \( \rho_\Lambda(p) \) is a function that is equal to one near the origin and falls off rapidly at infinity. For example,

\[
\rho_\Lambda(p) = \theta(|p| < \Lambda)
\]

would be a typical choice. The function \( g(\Lambda) \) will be picked later in such a way that the scattering amplitude has a limit as \( \Lambda \to \infty \).

Here, the interaction is represented by a separable kernel which makes the calculations simple. If instead we choose a potential of range \( \frac{1}{\Lambda} \) we
will get similar answers in the limit $\Lambda \to \infty$ but the calculations are more complicated.

Consider the inhomogenous equation

$$\left(\frac{p^2}{2m} - E\right)\psi_\Lambda(p) - g(\Lambda)\rho_\Lambda(p)\int \rho_\Lambda(q)\psi_\Lambda(q)[dq] = \chi(p).$$  \tag{15}$$

Then

$$\psi_\Lambda(p) = \frac{\chi(p)}{\frac{p^2}{2m} - E} + \frac{A_\Lambda}{\frac{p^2}{2m} - E}\rho_\Lambda(p)$$  \tag{16}$$

where $A_\Lambda$ is

$$A_\Lambda = g(\Lambda)\int \rho_\Lambda(p)\psi_\Lambda(p)[dp].$$  \tag{17}$$

We can put the expression for $\psi_\Lambda(q)$ into this equation for $A_\Lambda$ to get,

$$A_\Lambda\left[g^{-1}(\Lambda) - \int \frac{\rho_\Lambda^2(p)}{\frac{p^2}{2m} - E}\right] = \int \rho_\Lambda(p)\frac{\chi(P)}{\frac{p^2}{2m} - E}[dp]$$  \tag{18}$$

We now choose $g(\Lambda)$ such that

$$g^{-1}(\Lambda) - \int \frac{\rho_\Lambda^2(p)}{\frac{p^2}{2m} - E}[dp]$$  \tag{19}$$

has a limit as $\Lambda \to \infty$:

$$g^{-1}(\Lambda) = \int \rho_\Lambda^2(p)\frac{1}{\frac{p^2}{2m} + \frac{\mu^2}{2m}}[dp]$$  \tag{20}$$

for some real constant $\mu$.

This number $\mu$ (which has the dimension of momentum) is the true physical parameter which describes the strength of the interaction: it remains meaningful even in the limit as $\Lambda$ goes to infinity. If the ‘bare coupling constant’ $g(\Lambda)$ is eliminated in favor of $\mu$, all the divergences will dissappear: this is the essence of renormalization. This kind of replacement of a (divergent)
coupling constant by a momentum scale is quite common in renormalization theory. It is sometimes called ‘dimensional transmutation’.

Then $A_\Lambda$ will have a limit as $\Lambda \to \infty$, and so will the solution $\psi(p) = \lim_{\Lambda \to \infty} \psi_\Lambda(p)$:

$$\psi(p) = \frac{\chi(p)}{p^2 - 2m - E} + \frac{1}{\phi(\mu, E)} \frac{1}{p^2 - 2m - E} \int \frac{\chi(q)}{q^2 - 2m - E} [dq].$$

(21)

Here,

$$\phi(\mu, E) = \int [dp] \left[ \frac{1}{p^2 + \frac{\mu^2}{2m} - E} - \frac{1}{p^2 - 2m - E} \right]$$

$$= 2m \int \frac{[dp]}{p^2} \left[ - \frac{\mu^2}{p^2 + \mu^2} + \frac{(-2mE)}{p^2 - 2mE} \right].$$

(22)

This is a convergent integral. Note that the limiting solution is independent of the choice of $\rho_\Lambda(p)$.

Now, the resolvent kernel is given by the formula,

$$\psi(p) = \int R(E; p, q) \chi(q) [dp].$$

(23)

Thus we get,

$$R(E; p, k) = \frac{(2\pi)^4 \delta(p - k)}{p^2 - 2m - E} + \frac{1}{\phi(\mu, E)} \frac{1}{p^2 - 2m - E} \frac{1}{q^2 - 2m - E}.$$

(24)

This is called the ‘Krein formula’ for the resolvent. We can regard our limiting system as defined by this formula for the resolvent. The hamiltonian from which this follows is, as a differential operator, the same as that of the free particle. The interaction is encoded into the boundary conditions at the origin. The resolvent, being a Green’s function, encodes the information of these boundary conditions as well. The function $\frac{1}{\phi(\mu, E)}$ gives thus a convenient
description of the interaction. We will see that this is (upto a constant) the scattering amplitude.

Thus the interactions of the theory are described by the function \( \phi(\mu, E) \). Apart from it, all the terms in the formula for the resolvent just involve the free theory. We will see that with more than two particles, we still have a similar formula, but the real valued function \( \phi(\mu, E) \) is replaced by an operator valued function, the ‘principal operator’. This principal operator acts on a reduced Hilbert space which for the simple case of this section is one dimensional. That is why we just have a function \( \phi(\mu, E) \) rather than an operator.

### 2.2 The Krein Formula and Boundary Conditions

The resolvent of a differential operator contains the information on its boundary conditions as well. It is interesting to make explicit these boundary conditions implied by the Krein formula for the hamiltonian. These boundary conditions determine the domain of the hamiltonian thought of as a self-adjoint unbounded operator in \( L^2(\mathbb{R}^3) \).

The domain of the hamiltonian is the range of its resolvent: the set of functions in momentum space that can be written as

\[
\int R(E; p, k)\chi(k)[dk]
\]

for square integrable \( \chi(k) \). Thus for the free particle with resolvent

\[
R_0(E; p, k) = \frac{(2\pi)^d\delta(p - k)}{\frac{p^2}{2m} - E}
\]
we have the set
\[
\{ \psi(p) = \frac{\chi(p)}{2m - E} \mid \chi \in L^2(\mathbb{R}^3) \}. \tag{27}
\]
In other words, the domain of the free hamiltonian consists of wave functions \( \psi(p) \) for which both \( \psi \) and \( p^2 \psi \) are square integrable. In position space this means that the domain of the free hamiltonian is the set of wavefunctions that are square integrable and have square integrable second derivatives.

Now let us ask how the form of the resolvent changes if we change the domain of the hamiltonian. The difference of the resolvent from its value for a free particle is (in position space)
\[
\tilde{R}(E; x, y) - \tilde{R}_0(E; x, y)
\]
a homogenous solution of the Schrödinger equation: two different Greens functions for the same differential equation differ by a homogenous solution. The simplest possibility is to change the behavior of the resolvent at one point (say the origin) by putting there a pointlike scatterer. (More complicated modifications are also allowed mathematically, but are not as interesting). Then we must have, for \( d = 3 \),
\[
\tilde{R}(E; x, y) = \tilde{R}_0(E; x, y) + C(E) \frac{e^{i\sqrt{(2mE)}|x|}}{|x|} \frac{e^{-i\sqrt{(2mE)}|y|}}{|y|}
\]
for some constant \( C(E) \). We require that the additional term correspond to outgoing waves at spatial infinity, a physical requirement. The formula for arbitrary \( d \) is similar and involves Hankel functions. This means the wavefunction can blow up at the origin as \( \frac{C(E)}{|x|^{1/d}} \) (or \( \log |x| \) when \( d = 2 \)). Note that such a singularity is still square integrable.
In momentum space this is of the form
\[ R(E; p, k) = \frac{(2\pi)^d \delta(p - k)}{\frac{p^2}{2m} - E} + B(E) \frac{1}{\frac{k^2}{2m} - E} \frac{1}{\frac{k^2}{2m} - E'}. \] (30)

The quantity \( B(E) \) is determined by the condition that this operator be in fact a resolvent:
\[ \frac{R(E; p, k) - R(E'; p, k)}{E - E'} = \int R(E; p, k') R(E', k', k)[dk]. \] (31)

After some calculations we get
\[ \frac{B^{-1}(E) - B^{-1}(E')}{E - E'} = \int [dk] \left\{ \frac{1}{\frac{k^2}{2m} - E} - \frac{1}{\frac{k^2}{2m} - E'} \right\} \] (32)

Or,
\[ B^{-1}(E) - B^{-1}(E') = \int [dk] \left\{ \frac{1}{\frac{k^2}{2m} - E} - \frac{1}{\frac{k^2}{2m} - E'} \right\} \] (33)

But this is precisely what we got from renormalization: our quantity \( B(E) \) is just the inverse of the \( \phi(E) \) we had previously.

Thus we see that the renormalized hamiltonian is just the free hamiltonian with a modified boundary condition on the wavefunctions at the origin of position space.

### 2.3 The Scattering Amplitude

Consider the Schrödinger equation for a free particle:
\[ \frac{1}{2m} \nabla^2 \tilde{\psi}(x) = E \tilde{\psi}(x). \] (34)

Suppose that \( E > 0 \). If we require that the wavefunction be continuous everywhere in space, we have the usual plane wave solution:
\[ Ce^{ik \cdot x} \] (35)
with $E = \frac{k^2}{2m}$. In momentum space this corresponds to a wavefunction

$$(2\pi)^d \delta(p - q).$$

But this is not the only solution if we allow the solution to blow up at one point (say the origin). This would mean that there is a static scatterer of zero size sitting at the origin. We would still require the Schrödinger equation to hold away from the origin. Hence the solution would have to differ from the plane wave by a multiple of a homogenous solution of the differential equation, one that may diverge at the origin. This homogenous solution represents the scattering by the particle at the origin; hence we should require that it become an outgoing wave at infinity:

$$\tilde{\psi}(x) = e^{ik \cdot x} + f(k) e^{ikr}.$$  

Here the constant $f(k)$ has the meaning of the scattering amplitude. In momentum space this becomes

$$\psi(p) = (2\pi)^d \delta(p - k) + f(k) \frac{2\pi}{m} \frac{1}{p^2/2m - k^2/2m - i\epsilon}.$$  

it being understood as usual that $\epsilon \to 0^+$. We can now compare this with what we get from the formula for the resolvent and see that the extra term in the resolvent is precisely of this form:

$$\psi(p) = (2\pi)^d \delta(p - k) + \frac{1}{\phi(\mu, \frac{k^2}{2m} + i\epsilon)} \frac{1}{p^2/2m - k^2/2m - i\epsilon}. $$

This is in line with our argument that the Krein formula for the resolvent describes a boundary condition on the wavefunction at the origin: it is allowed to blow up.
Thus we have a scattering amplitude that is independent of angles:

\[ f(k) = \frac{m}{2\pi \phi(\mu, \frac{k^2}{2m} + i\epsilon)}. \]  

(40)

Evaluating the integral,

\[ \phi(\mu, E) = \frac{m}{2\pi} \left[ \sqrt{(-2mE - i\epsilon) - \mu} \right] \]  

(41)

so that the scattering amplitude is

\[ f(k) = \frac{1}{-\mu - ik}. \]  

(42)

The sign is fixed by the rule \( \lim_{\epsilon \to 0} (-1 - i\epsilon)^{1/2} = -i \).

Thus we find exactly the ‘simple pole model’ for the scattering amplitude for low momenta that we had in the last section, with scattering length \( \xi = \frac{1}{\mu} \).

Later on we will study the many body problem of particles with such contact interactions. We will give a complete theory only in the simpler case of two dimensional space. Problems with the extension to the three dimensional case will be described as well.

3 The Fermions with a Static Source

Since our fermionic variant of the Lee model is simpler we will describe it first.

Let \( \psi(p), \psi^\dagger(p) \) be the creation-annihilation operators for a fermion field in \( d+1 \)-dimensional space-time. They are represented on the Fock space \( \mathcal{F} \) built from the vacuum \( |0> \):

\[ [\psi(p), \psi^\dagger(q)]_+ = (2\pi)^d \delta(p - q), \quad [\psi^\dagger(p), \psi^\dagger(q)]_+ = 0 = [\psi(p), \psi(q)]_+ \]  

(43)
and

\[ \psi(p)|0>= 0. \] (44)

We are mainly interested in the case \( d = 3 \) but our method applies for any \( d < 4 \).

On \( \mathcal{F} \otimes C^2 \), define the hamiltonian

\[ H_\Lambda = H_0 + H_{1\Lambda} \] (45)

where,

\[ H_0 = \int [dp] \psi^\dagger(p) \psi(p) \omega(p) \] (46)

and

\[ H_{1\Lambda} = \mu_\Lambda \frac{1 - \sigma_3}{2} + g \int [dp] [\rho_\Lambda(p) \psi(p) \sigma_- + h.c.]. \] (47)

There is no loss in assuming that the coupling constant \( g > 0 \) since any phase in \( g \) can be absorbed into a redefinition of the field \( \psi(p) \). The quantity \( \mu_\Lambda \) is a 'bare coupling constant' whose dependence on \( \Lambda \) will be determined later by renormalization.

The dispersion relation of the fermion field is chosen to be non-relativistic:

\[ \omega_p = m + \frac{p^2}{2m}. \] (48)

The Pauli matrices \( \sigma_\pm, \sigma_3 \) act on \( C^2 \) in the usual way. There is a \( U(1) \) symmetry (rather like isospin) which leads to the conserved quantity

\[ Q = \frac{1}{2} - \sigma_3 + \int [dp] \psi^\dagger \psi(p). \] (49)

Clearly, \( Q \geq 0 \).

This model describes the interaction of the fermions with some heavy particles sitting at the origin. The function \( \rho_\Lambda(p) \) is a sort of form-factor: it
describes the internal structure of these heavy particles. For us it will serve the purpose of an ultra-violet regulator, e.g. we may choose

\[ \rho_\Lambda(p) = \theta(|p| < \Lambda) \]  

for some momentum cut-off \( \Lambda \). In the limit \( \Lambda \to \infty \) we will have point-like heavy particles. But in this limit there is a UV divergence in the theory. This divergence can be removed by renormalizing the constant \( \mu_\Lambda \). The final answer will not depend on the choice of the form-factor \( \rho_\Lambda \): any function which is equal to one at the origin and falls of faster than any power at infinity will give the same answers. This is part of the universality of the transfinite theory: its independence on the regularization scheme.

Our theory can be thought of as a model for the interaction of the top and charm quarks with the Higgs boson: the heavy boson is the Higgs boson, the heavy fermion the top quark and light fermion the charm quark. Our model is an approximation where \( m_t, m_H \gg m_c = m \) but \( |m_t - m_H| \ll m_c \). In this limit we would have the interaction of non-relativistic b-quarks with either a Higgs boson or a top quark; the parameter \( \mu \) describes the mass difference between the two heavy particles.

It is certainly true that \( m_t \gg m_c \) and it is almost certain that \( m_H \gg m_c \). It would be unusual for the Higgs boson and the top quark to have almost the same mass, as we are assuming. Also, the Higgs-top-charm coupling is very small in the standard model (induced by higher loop effects) but is not so small in some variants of the standard model (for example with supersymmetry). Thus our model describes a somewhat unusual possibility in particle physics: but this has not yet been ruled out experimentally. In any case we can use this system as a physical picture that guides our mathe-
mathematical analysis of this renormalization problem, much like the nucleon-pion system in the Lee model. More realistic cases can be studied later, once the principles are established in this simple case.

In some extensions of the standard model (e.g., the minimal supersymmetric extension) there are charged Higgs particles. In these models, a decay of the top quark into a Higgs boson and a bottom quark is possible. Again, if the Higgs and the top quark happen to be almost degenerate, our model will describe this system.

The hamiltonian can be expressed as a $2 \times 2$ block split according to $C^2$:

$$H_\Lambda = \begin{pmatrix} H_0 & g \int [dp] \rho_\Lambda(p) \psi^\dagger(p) \\ g \int [dp] \rho_\Lambda(p) \psi(p) & H_0 + \mu_\Lambda \end{pmatrix}.$$  (51)

We will now construct the resolvent of this hamiltonian using some identities given in the appendix. Then we will show that the limit as $\Lambda \to \infty$ of the resolvent exists: this is renormalization. After that we will study the spectrum of the renormalized theory.

In the appendix, we work out some formulae for inverting operators split into $2 \times 2$ blocks as above. In the notation used there, the resolvent of the regularized hamiltonian is

$$R_\Lambda(E) = \frac{1}{H_\Lambda - E} = \begin{pmatrix} \alpha & \beta^\dagger \\ \beta & \delta \end{pmatrix}.$$  (52)

where,

$$\alpha = \frac{1}{H_0 - E} + \frac{1}{H_0 - E} b^\dagger \Phi_\Lambda(E)^{-1} b \frac{1}{H_0 - E}, \quad \beta = -\frac{1}{H_0 - E} b^\dagger \Phi_\Lambda(E)^{-1}$$  (53)

and

$$\delta = \Phi_\Lambda(E)^{-1}, \quad b = g \int [dp] \rho_\Lambda(p) \psi(p).$$  (54)
We express everything in terms of \( \Phi_\Lambda(E) \) since we will see that all the divergences are removed once we have a proper definition for \( \Phi_\Lambda(E) \). Once we know \( \Phi \), the resolvent is given by the above explicit formula.

Indeed, we have,

\[
\Phi_\Lambda(E) = H_0 - E + \mu_\Lambda - g^2 \int [dp][dq] \rho_\Lambda(p) \rho_\Lambda(q) \psi(p) \frac{1}{H_0 - E} \psi^\dagger(q).
\]

Now we normal-order the last term; i.e., we reorder the operators so that the creation operators stand to the left of the annihilation operators. This can be done using the identities

\[
H_0 \psi^\dagger(q) = \psi^\dagger(q) [H_0 + \omega(q)], \quad \psi(p) \psi^\dagger(q) = -\psi^\dagger(q) \psi(p) + (2\pi)^d \delta(p-q),
\]

Thus

\[
\Phi_\Lambda(E) = H_0 - E + g^2 \int [dp][dq] \rho_\Lambda(p) \rho_\Lambda(q) \psi^\dagger(p) \frac{1}{H_0 + \omega(p) + \omega(q) - E} \psi(q)
\]

\[
+ \mu_\Lambda - g^2 \int [dp] \rho_\Lambda^2(p) \frac{1}{H_0 + \omega(p) - E} \psi(q)
\]

Suppose we now choose

\[
\mu_\Lambda = \mu + g^2 \int \frac{\rho_\Lambda(p)^2 dp}{\omega(p) - \mu}.
\]

Here \( \mu \) is a parameter independent of \( \Lambda \). Moreover we will require that \( m > \mu \) for simplicity. (This is not essential: it guarantees that both the heavy states are stable against decay.)

The point of this choice of \( \mu_\Lambda \) is that it cancels the divergent part of \( \Phi_\Lambda(E) \):

\[
\Phi_\Lambda(E) = H_0 - E + g^2 \int [dpdq] \psi^\dagger(p) \frac{\rho_\Lambda(p) \rho_\Lambda(q)}{H_0 + \omega(p) + \omega(q) - E} \psi(q)
\]
\[-g^2 \int [dp] \rho_\Lambda(p)^2 \left[ \frac{1}{H_0 + \omega(p) - E} - \frac{1}{\omega(p) - \mu} \right] + \mu \]  

(59)

The integrand of the last term behaves for large \(|p|\) like \(\int [dp] \frac{\omega(p)}{\omega(p)}\): it is convergent if \(d < 4\), since \(\omega = m + \frac{g^2}{2m}\). Thus we can take the limit as \(\Lambda \to \infty\) keeping \(\mu\) fixed at a positive value less than \(m\). Then, the limiting operator

\[
\Phi(E) = H_0 - E + g^2 \int [dpdq] \psi^\dagger(p) \frac{1}{H_0 + \omega(p) + \omega(q) - E} \psi(q) - g^2 \int [dp] \left[ \frac{1}{H_0 + \omega(p) - E} - \frac{1}{\omega(p) - \mu} \right] + \mu
\]

(60)

exists. This is the renormalized form of the principal operator. We can just put this into the earlier formula to get a formula for the resolvent.

### 3.1 The Principal Operator

To get more explicit expressions we specialize to the case \(d = 3\). (In fact the following arguments apply to any \(d < 4\).) Evaluating the integral,

\[
\Phi(E) = H_0 - E + \mu + 4\pi^2 g^2 (2m)^\frac{3}{2} \left[ \sqrt{(H_0 + m - E)} - \sqrt{(m - \mu)} \right] + g^2 \int [dpdq] \psi^\dagger(p) \frac{1}{H_0 + \omega(p) + \omega(q) - E} \psi(q)
\]

(61)

This principal operator can be written as the sum of a ‘kinetic’ (single particle) term

\[
K(E) = H_0 - E + \mu + 4\pi^2 g^2 (2m)^\frac{3}{2} \left[ \sqrt{(H_0 + m - E)} - \sqrt{(m - \mu)} \right]
\]

(62)

and an ‘interaction’ term

\[
U(E) = g^2 \int [dpdq] \psi^\dagger(p) \frac{1}{H_0 + \omega(p) + \omega(q) - E} \psi(q).
\]

(63)

The fermion number operator \(N = \int [dp] \psi^\dagger(p) \psi(p)\) commutes with \(\Phi(E)\).

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The principal operator determines the dynamics of the theory. The resolvent of the Hamiltonian of the renormalized theory is

\[ R(E) = \begin{pmatrix} \alpha(E) & \beta^\dagger(E) \\ \beta(E) & \Phi(E)^{-1} \end{pmatrix} \] (64)

where

\[ \alpha(E) = \frac{1}{H_0 - E} + \frac{g}{H_0 - E} \int [dq] \psi^\dagger(q) \frac{1}{\Phi(E)} \int [dp] \psi(p) \frac{g}{H_0 - E} \] (65)

and

\[ \beta(E) = -\frac{g}{\Phi(E)} \int [dp] \phi(p) \frac{1}{H_0 - E}. \] (66)

This can be thought of as a sort of Krein formula for the resolvent of our field theory.

Indirectly it defines the Hamiltonian \( H \) as the operator for which \( R(E) = \frac{1}{H - E} \) is the resolvent. But if we were to think of this renormalized Hamiltonian directly as an operator on the fermionic Fock space, it would appear to be just the free fermion operator. For example, consider the space of states

\[ |f> = \int f(p_1 \cdots p_n) \psi^\dagger(p_1) \cdots \psi^\dagger(p_n)[dp_1 \cdots dp_n] |0> \] (67)

with smooth \( f(p_1, \cdots p_n) \) decreasing at infinity faster than any polynomial. They correspond to fermionic wavefunctions that are smooth in position space that also fall off faster than any polynomial. On these ‘nice’ states, the Hamiltonian \( H \) of the transfinite theory is the same the free Hamiltonian:

\[ H |f> = \int [\omega(p_1) + \cdots \omega(p_n)] f(p_1 \cdots p_n) \psi^\dagger(p_1) \cdots \psi^\dagger(p_n) |0> [dp_1 \cdots dp_n]. \] (68)

But this formula does not uniquely define \( H \) as a self-adjoint operator. It has many self-adjoint extensions, corresponding to different sets of boundary
conditions as the momenta go to infinity. One of them is the free hamiltonian $H_0$, but there are other extensions too which define interacting theories. The interactions are hidden in the domain of definition of this unbounded operator: in other words in the boundary conditions of the fermion wavefunctions in the limit of large momentum. It is quite awkward to think this way: for example it would be difficult to calculate the spectrum of the renormalized theory.

But the same information is contained in a more explicit form in the above Krein formula for the resolvent. The resolvent is a sort of Green’s function which therefore contains also the information on the boundary conditions. Another way to understand it is that (away from the spectrum) the resolvent is a bounded operator, so there is no need to specify its domain.

Transfinite field theories are thus theories in which the interaction takes place at infinity in momentum space. We are still able to give a sensible description of these theories.

3.2 The Spectrum of the Transfinite Theory

We saw that there is an explicit formula for the resolvent of the hamiltonian in terms of the inverse of the Principal operator. Thus, all the dynamical information about the theory is contained in its principal operator. For example the discrete spectrum of the hamiltonian correspond to the poles of the resolvent. There are no poles in $\frac{1}{H_0 - E}$: its spectrum is purely continuous. Thus the poles must arise from those of $\Phi(E)^{-1}$; i.e., roots of the equation

$$\Phi(E)|u> = 0$$

(69)
This equation now plays the role of the Schrödinger eigenvalue problem in the transfinite theory. The residue of the pole of the resolvent is the projection operator to the corresponding eigenspace of \( H \). Thus the eigenvector of \( H \) corresponding to a root \( E_0 \) of the principal operator is given by

\[
\left( \frac{g}{H_0 - E_0} \int dp \psi_\downarrow(p) \right) \left| u > \right.
\]

This is because the residue of a pole in the resolvent is the projection operator to the eigenspace with that eigenvalue. We can read off this residue and see that it is the projection to the above state, once a root of the equation \( \Phi(E) |u > = 0 \) is found.

An example is the vacuum state:

\[
\Phi(E) |0 > = (-E + \mu + g^2(2m)^{\frac{3}{2}}[\sqrt{(m - E)} - \sqrt{(m - \mu)}]) |0 >.
\]

There is a root when

\[
E = \mu.
\]

Since \( \mu < m \), it is not in the spectrum of \( H_0 \). The corresponding eigenvector of the hamiltonian is

\[
\left( \frac{g}{H_0 - \mu} \int dp \psi_\downarrow(p) \right) |0 >.
\]

It contains a fermion in the first entry, so it is not the vacuum in the whole Hilbert space \( \mathcal{F} \otimes C^2 \). In spite of this, it is the state of lowest energy—the ground state—when \( \mu < 0 \). (Proof is in the next section). When \( m > 0 \),

\[
This root is assumed not be in the spectrum of \( H_0 \). In this case it would be be embedded in the continuous spectrum of \( H \) and should be unstable. We will deal with the continuous spectrum of \( H \) later.
\( \mu > 0 \) the above state is the first excited state. (The ground state is just the ‘vacuum’ state \( |0\rangle \).) When \( \mu > m \), we don’t get an eigenstate of the hamiltonian since this state is then unstable. (The ground state in this case is also \( |0\rangle \).)

More generally, the spectrum of the hamiltonian is the set of values of \( E \) at which the resolvent either does not exist (discrete spectrum) or exists but is unbounded (continuous spectrum). Thus the continuous spectrum will be that of \( H_0 \) plus the values of \( E \) at which \( \Phi(E) \) does not have a bounded inverse.

### 3.3 Proof That The Ground State Energy is Finite

In order to see that we have exorcised all the infinities, we must show not only that the resolvent of the hamiltonian exists in the limit as \( \Lambda \to \infty \) but also that the ground state energy is finite. This is nontrivial to prove since we will see that there are many theories where even after a renormalization there are further divergences which make the spectrum not bounded below.

We will estimate the norm \( ||\Phi(E)^{-1}|| \) for the case \( E < \mu \); our aim is to show that this is finite. Then the ground state energy is either zero (when \( \mu > 0 \)) or is equal to \( \mu \).

It is sufficient to consider the sector with the fermion number held fixed at some value \( n \), since \( \Phi(E) \) preserves this number.

Recall that \( \Phi(E) = K(E) + U(E) \) with

\[
K(E) = H_0 - E + \mu + 4\pi^2 g^2 (2m)^{\frac{3}{2}} \left[ \sqrt{(H_0 + m - E)} - \sqrt{(m - \mu)} \right] \tag{74}
\]
and
\[ U(E) = g^2 \int [dpdq] \psi^\dagger(p) \frac{1}{H_0 + \omega(p) + \omega(q) - E} \psi(q). \] (75)

Moreover,
\[ K(E) \geq nm + (\mu - E), \quad U(E) \geq 0. \] (76)

The inequalities become equalities when the fermion number is zero.

Now write,
\[ \Phi(E) = K(E)^{1/2} [1 + \tilde{U}(E)] K(E)^{1/2}, \quad \tilde{U}(E) = K(E)^{-1/2} U(E) K(E)^{-1/2}. \] (77)

Since \( \tilde{U}(E) \geq 0 \),
\[ ||\Phi(E)^{-1}|| \leq ||K(E)^{-1}|| ||[1 + \tilde{U}(E)]^{-1}|| \]
\[ \leq [nm + (\mu - E)]^{-1} \]
\[ \leq \frac{1}{\mu - E}. \] (78)

This shows that \( \Phi(E) \) has spectrum bounded below by \( \mu \).

This simple method is not sufficient to determine the ground state in the bosonic case. We will have to supplement it with a sort of mean field theory.

4 The Lee Model

The Lee model has a charged bosonic field satisfying:
\[ [\phi(p), \phi^\dagger(q)] = (2\pi)^d \delta(p - q), \quad [\phi^\dagger(p), \phi^\dagger(q)] = 0 = [\phi(p), \phi(q)]. \] (79)

The bosonic Fock space \( B \) is built from the vacuum in the usual way:
\[ \phi(p)|0> = 0. \] (80)
Again, the complete Hilbert space of the system is $B \otimes C^2$.

The Hamiltonian is, again,

$$H_\Lambda = H_0 + H_{1\Lambda} \quad (81)$$

with

$$H_0 = \int [dp] \phi^\dagger(p) \phi(p) \omega_p \quad (82)$$

and

$$H_{1\Lambda} = \mu_\Lambda \frac{1 - \sigma_3}{2} + g \int [dp] [\rho_\Lambda(p) \phi(p) \sigma_- + \text{h.c.}] \quad (83)$$

Some of the methods are the same as in the fermionic case; we will then omit the details. But the bosonic case requires some new analysis as well.

The divergences are removed as before by normal ordering and choosing the bare mass difference to be

$$\mu_\Lambda = \mu + g^2 \int \frac{\rho_\Lambda(p)^2 [dp]}{\omega(p) - \mu} \quad (84)$$

In the limit the resolvent has the form:

$$R(E) = \lim_{\Lambda \to \infty} \frac{1}{H_\Lambda - E} = \begin{pmatrix} \alpha(E) & \beta^\dagger(E) \\ \beta(E) & \Phi(E)^{-1} \end{pmatrix}; \quad (85)$$

$$\alpha(E) = \frac{1}{H_0 - E} + \frac{1}{H_0 - E} \int [dq] \phi^\dagger(q) \frac{1}{\Phi(E)} \int [dp] \phi(p) \frac{1}{H_0 - E}; \quad (86)$$

$$\beta(E) = -\frac{1}{\Phi(E)} \int [dp] \phi(p) \frac{1}{H_0 - E}; \quad (87)$$

and finally

$$\Phi(E) = H_0 - E + \mu + 4\pi^2 g^2 (2m)^3 [\sqrt{(H_0 + m - E)} - \sqrt{(m - \mu)}]$$

$$- g^2 \int [dpdq] \phi^\dagger(p) \frac{1}{H_0 + \omega(p) + \omega(q) - E} \phi(q). \quad (88)$$
Other than the sign of the last term the answer is essentially the same as before. But this sign makes an important difference.

The proton is the state \( |0> \) and it is an eigenstate of the renormalized hamiltonian with eigenvalue zero. The neutron is an eigenstate of the hamiltonian,

\[
\left( \frac{\hbar^2}{\mu_0 - \mu} \int [dp] \phi^\dagger(p)|0> \right|0>
\]

(89)

It corresponds the root of

\[
\Phi(E)|0> = (-E + \mu + g^2 (2m)^{3/2} [\sqrt{(m - E)} - \sqrt{(m - \mu)}])|0>
\]

(90)

with \( E = \mu \). Thus the parameter \( \mu \) is the renomalized value of the neutron-proton mass difference. So it is reasonable to assume \( m > \mu > 0 \).

But is the proton the state of least energy? Could there be states which contain many bosons that have a lower energy? Is there even a ground state? These questions can be answered by studying the principal operator \( \Phi(E) \).

### 4.1 Lower Bound for the Ground State Energy

We will estimate the norm \( \|\Phi(E)^{-1}\| \) for the case \( E < \mu \). It is sufficient to consider the sector with the number of bosons held fixed at \( n \), since \( \Phi(E) \) preserves this number.

Let us define the “kinetic” part of \( \Phi(E) \) to be:

\[
K(E) = H_0 - E + \mu + 4\pi^2 g^2 (2m)^{3/2} [\sqrt{(H_0 + m - E)} - \sqrt{(m - \mu)}].
\]

(91)

Clearly,

\[
K(E) \geq nm + (\mu - E).
\]

(92)

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Define $-U(E)$ to be the “potential” part of $\Phi(E)$:

$$\Phi(E) \equiv K(E) - U(E).$$  \hfill (93)

Notice that $\Phi(E)$ is now the difference of two positive operators rather than the sum as in the fermionic case. This is what makes the bosonic case more complicated.

As before,

$$\Phi(E) = K(E)^{\frac{1}{2}}[1 - \tilde{U}(E)] K(E)^{\frac{1}{2}}$$  \hfill (94)

where

$$\tilde{U}(E) = K(E)^{-\frac{1}{2}}U(E)K(E)^{-\frac{1}{2}}.$$  \hfill (95)

Also,

$$||\Phi(E)^{-1}|| \leq ||K(E)^{-1}|| ||[1 - \tilde{U}(E)]^{-1}||;$$  \hfill (96)

i.e.,

$$||\Phi(E)^{-1}|| \leq [nm + (\mu - E)]^{-1} ||[1 - \tilde{U}(E)]^{-1}||.$$  \hfill (97)

Now, by explicit calculation,

$$\tilde{U}(E) = g^2 \int [dpdq] \phi^\dagger(p) \frac{1}{K(E - \omega_p)^{\frac{1}{2}}(H_0 + \omega_p + \omega_q - E)K(E - \omega_q)^{\frac{1}{2}}} \phi(q).$$  \hfill (98)

Thus, (remembering that inside the square bracket the boson number is $n - 1$),

$$\tilde{U}(E) \leq g^2 \int [dpdq] \phi^\dagger(p) [(n - 1)m + \mu - E + \omega_p]^{-\frac{1}{2}}$$

$$\frac{1}{[(n - 1)m + \omega_p + \omega_q - E]^{-1}}$$

$$\frac{1}{[(n - 1)m + \mu - E + \omega_q]^{-\frac{1}{2}}} \phi(q).$$  \hfill (99)
Also, in the sector with \( n \) bosons,

\[
|| \int \phi^\dagger(p) u(p, q) \phi(q) [dpdq] || \leq n \left[ \int |u(p, q)|^2 [dpdq] \right]^{\frac{1}{2}}. \tag{100}
\]

Combining these, we get

\[
|| \tilde{U}(E) ||^2 \leq g^4 n^2 \int [dpdq] \left[ (n-1)m + \mu - E + \omega_p \right]^{-1} \\
\left[ (n-1)m + \omega_p + \omega_q - E \right]^{-2} \\
\left[ (n-1)m + \mu - E + \omega_q \right]^{-1}. \tag{101}
\]

Now we put in \( \omega(p) = \frac{p^2}{2m} + m \). In the middle factor we can replace an \( m \) by a \( \mu \) since \( m > \mu \):

\[
[(n-1)m + 2m + \frac{p^2}{2m} + \frac{q^2}{2m} - E]^{-2} < [nm + \mu + \frac{p^2}{2m} + \frac{q^2}{2m} - E]^{-2}. \tag{102}
\]

This makes all the constants in the denominators the same so that we can scale them out to get

\[
|| \tilde{U}(E) ||^2 \leq g^4 n^2 \frac{(2m)^3}{[nm + \mu - E]} \int \left[ \frac{[dpdq]}{(1 + p^2)(1 + q^2)(1 + p^2 + q^2)^2} \right]. \tag{103}
\]

The integral is convergent. Working our way back to the beginning,

\[
|| \Phi(E) ||^{-1} \leq \frac{1}{\sqrt{(nm + \mu - E)}} \frac{1}{\sqrt{(nm + \mu - E)} - g^2 n (2m)^{\frac{3}{2}} C} \tag{104}
\]

where

\[
C^2 = \int \left[ \frac{[dpdq]}{(1 + p^2)(1 + q^2)(1 + p^2 + q^2)^2} \right]. \tag{105}
\]

If \( E \) is to be an eigenvalue of energy, it must be big enough to make the denominators on the r.h.s. vanish; otherwise, \( \Phi(E)^{-1} \) would remain bounded.
This means there is a lower bound on all eigenvalues, which really is the same as a lower bound on the ground state energy:

\[ E_{gr} \geq (nm + \mu) - n^2 g^4 (2m)^3 C^2. \] (106)

Thus we see that in each sector with a fixed number of bosons, there is a ground state. However, there is still the possibility that the ground state energy diverges as \( n \) grows to infinity. In this limit we should expect all the bosons to settle into the same state. The ground state of the free theory has all the bosons in the zero momentum state. In the interacting theory, the ground state could in general be something quite different. For example, the bosons might settle into a state which is concentrated at the origin. The difference between that bosonic number density and its free field value is called the ‘boson condensate’. Whether such a non–zero condensate of bosons forms cannot be settled by the present analysis: we need to study the limit as \( n \to \infty \).

### 4.2 The Large \( n \) Limit of the Lee Model

In the limit that the number of bosons becomes large, we should be able to use mean field theory. We would expect all the bosons to occupy the same state \( u(p) \). This state is normalized so that the occupation number is \( n \):

\[ ||u||^2 = \int |u(p)|^2 [dp] = n. \] (107)

In the limit \( n \to \infty \), operators can be approximated by their expectation values in this state: a kind of mean field theory. They will then become
functions on the space of such states, the complex projective space\(^3\) of \(L^2(R^3)\).

Thus our principal operator becomes the principal function

\[
\Phi(E, u) = h_0(u) - E + \mu + 4\pi^2 g^2 (2m)^{\frac{3}{2}} \left[ \sqrt{h_0(u) + m - E} - \sqrt{m - \mu} \right] - g^2 \int [dpdq] \frac{u^*(p)u(q)}{h_0(u) + \omega(p) + \omega(q) - E}.
\]

Here,

\[
h_0(u) = \int \omega(p)|u(p)|^2[dp].
\]

We must solve \(\Phi(E, u) = 0\) to get \(E\) as a function of \(E\). Then we must find the \(u\) that gives the smallest such \(E\), subject to the constraint on the norm \(|u|^2 = n\).

It is convenient to reexpress the problem in terms of some new variables. The principal function depends on \(E\) only through the combination

\[
\lambda = \int \omega(p)|u(p)|^2[dp] - E.
\]

So define,

\[
f(\lambda, u) = \lambda + \mu + (2\pi g)^2 (2m)^{\frac{3}{2}} \left[ \sqrt{\lambda + m} - \sqrt{m - \mu} \right] - g^2 \int [dpdq] \frac{u^*(p)u(q)}{\lambda + \omega(p) + \omega(q)}.
\]

Then \(\lambda\) is determined by the equation \(f(\lambda, u) = 0\) and \(E\) by

\[
E = nm + \frac{1}{2m} \int p^2 |u(p)|^2[dp] - \lambda.
\]

Putting in the explicit form of \(\omega(p)\),

\[
f(\lambda, u) = \lambda + \mu + (2\pi g)^2 (2m)^{\frac{3}{2}} \left[ \sqrt{\lambda + m} - \sqrt{m - \mu} \right]
\]

\(^3\)There is no physical effect if \(u\) is replaced by \(e^{i\theta}u\). The space of vectors of fixed length, modulo a phase, is complex projective space.

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\[-2mg^2 \int [dpdq] \frac{u^*(p)u(q)}{p^2 + q^2 + 2m(\lambda + 2m)}. \tag{113}\]

We now make the change of variables

\[u(p) = \sqrt{n} \ [2m(2m + \lambda)]^{\frac{3}{2}} v(\sqrt{2m(2m + \lambda)} |p|). \tag{114}\]

The change of scale of momentum will simplify the denominator in the integral for \(f(\lambda, u)\). The overall factor of \(\sqrt{n}\) turns the normalization condition into

\[\int |v(p)|^2 [dp] = 1 \tag{115}\]

so that we can separate out the \(n\) dependence.

Now we have

\[E = nm - \lambda + n(\lambda + 2m) \int p^2 |v(p)|^2 [dp], \tag{116}\]

and

\[f(\lambda, u) = \lambda + \mu + (2\pi g)^2 (2m)^{\frac{3}{2}} \{\sqrt{(\lambda + m)} - \sqrt{(m - \mu)}\] 
\[-ng^2 (2m)^{\frac{3}{2}} \sqrt{\lambda + 2m} \int [dpdq] \frac{v^*(p)v(q)}{p^2 + q^2 + 1}. \tag{117}\]

Imposing \(f(\lambda, u) = 0\) will give an equation for \(\lambda\):

\[[\lambda + 2m]^{-\frac{1}{2}} \{\lambda + \mu + (2\pi g)^2 (2m)^{\frac{3}{2}} \{\sqrt{\lambda + m} - \sqrt{(m - \mu)}\}] = 
- \sqrt{\lambda + 2m} \int [dpdq] \frac{v^*(p)v(q)}{p^2 + q^2 + 1}. \tag{118}\]

Then \(\lambda\) determined as a function of \(v\). The l.h.s. can be seen to be a monotonically increasing function of \(\lambda\) by writing it as

\[\sqrt{(\lambda + 2m)} - \frac{m}{\sqrt{(\lambda + 2m)} + \ldots}\]
\[ (2\pi g)^2 (2m)^{\frac{3}{2}} \sqrt{1 - \frac{m}{\lambda + 2m}} + \frac{\mu - m - (2\pi g)^2 (2m)^{\frac{3}{2}} \sqrt{(m - \mu)}}{\sqrt{(\lambda + 2m)}} \]  

(119)

(Recall that \( \mu < m \)). Thus we can solve for \( \lambda \) in terms of the r.h.s. as a monotonically increasing function as well. Its minimum values is \( -\mu \), attained when the r.h.s. is zero. Thus, we have

\[ \lambda = -\mu + f_1(nU) \]  

(120)

where

\[ U = g^2 (2m)^{\frac{3}{2}} \int [dpdq] \frac{v^*(p)v(q)}{p^2 + q^2 + 1} \]  

(121)

and \( f_1 \) is monotonically increasing. (We won’t need an explicit form for \( f_1 \).)

Now put this it into the expression for energy:

\[ E = nm + \mu + (2m - \mu)K + f_1(nU)[nK - 1] \]  

(122)

where

\[ K = \int [dp] |v(p)|^2. \]  

(123)

Now suppose we replace \( v(p) \) by \( v_a(p) = a^{-\frac{3}{2}} v(a^{-1}p) \). Let \( K(v_a) = K_a(v) \), \( E(v_a) = E_a(v) \) etc. Now, \( K_a = a^2 K \) and \( U_a = a^4 U \): as a function of \( a \) both \( K_a \) and \( U_a \) are increasing. Then \( E_a \) is also a monotonically increasing function of \( a \); the minimum for \( E_a \) will occur at \( a = 0 \) and \( K, U = 0 \). Thus we find that the groundstate energy is just

\[ E = nm + \mu. \]  

(124)

All the bosons are in the zero momentum state in the large \( n \) limit: there is no condensate. In other words the ground state of the bosonic field is essentially the same as in the free theory: the interaction is not strong enough to modify the ground state substantially. We will see examples later where the ground state is affected by the interaction.

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5 The $\lambda \phi^4_{2+1NR}$ Model

So far we considered theories where the coupling constants did not need to be renormalized. In fact the UV divergence was removed by a normal ordering of the principal operator. Now we will study a system where the coupling constant needs to be renormalized: in fact one that is asymptotically free. We will discover a trick of introducing fictitious degrees of freedom (‘angels’) which will help us reduce the renormalization again to a normal ordering of the principal operator.

The theory of interest is the nonrelativistic scalar field theory in two space dimensions, with a $\lambda \phi^4$ interaction. This is the many body problem of non-relativistic bosons interacting through a ‘delta function’ potential. It is well-known that, the two body problem has an ultra-violet divergence which can be removed by a coupling constant renormalization. In Ref. [15] the ‘few body problem’ was studied by quantum mechanical renormalization methods. In this section we will renormalize this model by viewing it as a non-relativistic quantum field theory: in other words we will study the many body problem. We will carry out the analysis in part for arbitrary $d$; this will show why our renormalization method is not sufficient for $d = 3$.

Define, on the Bosonic Fock space $\mathcal{B}$, the regularized hamiltonian operator

$$ H_{\Lambda} = H_0 + H_{1\Lambda} $$

where

$$ H_0 = \int \frac{p^2}{2} \phi^\dagger(p)\phi(p) [dp] $$

and

$$ H_{1\Lambda} = -g(\Lambda) \int [dp_1 dp_2 dp_1' dp_2'] \rho_\Lambda(p_1 - p_2) \rho_\Lambda(p_1' - p_2') $$
\[(2\pi)^d \delta(p_1 + p_2 - p_1' - p_2') \phi^\dagger(p_1) \phi^\dagger(p_2) \phi(p_1') \phi(p_2'). \quad (127)\]

Here
\[\rho_\Lambda(p) = \theta(|p| < \Lambda) \quad (128)\]
as before. The dimensionless constant \(g(\Lambda)\) is positive which corresponds to an attractive interaction between the bosons. We have shown elsewhere that there is an ultra-violet divergence as \(\Lambda \to \infty\), which can be removed by renormalizing the coupling constant. Our aim is to get a manifestly finite expression for the resolvent of the renormalized quantum field theory.

Now we will introduce a little trick that simplifies our problem: we will introduce new particles called angels that describe a bound state of a pair of bosons. However, they are created by operators with unusual defining relations: these are necessary to avoid over-counting the number of degrees of freedom.

### 5.1 Angels

Define operators satisfying
\[\chi(p)\chi^\dagger(q) = (2\pi)^d \delta(p - q), \quad \chi(p)\chi(q) = 0 = \chi^\dagger(p)\chi^\dagger(q). \quad (129)\]

Note that it is the product and not the commutator that appears here. These operators can be represented on the Hilbert space \(C \oplus L^2(R^d)\). We can regard \(\chi^\dagger(p)\) as creating an entity (we will give it the somewhat whimsical name ‘angel’) out of the empty state represented by \(C\). There can be at most one angel in any state: that is the meaning of the product of the creation operators being zero. (This is an extreme example of the exclusion statistics considered in some other contexts.)
Now consider an augmentation of the Bosonic Hilbert space, $\tilde{\mathcal{B}} = \mathcal{B} \oplus \mathcal{B} \otimes L^2(R^d)$. On it define the Hamiltonian

$$
\tilde{H}_\Lambda = H_0 \Pi_0 + \int [dp_1 dp_2 dp_3] \rho_\Lambda(p_1 - p_2) \phi^\dagger(p_1) \phi^\dagger(p_2) \chi(p_3)(2\pi)^d \delta(p_1 + p_2 - p_3) + h.c. \right] + \frac{1}{g(\Lambda)} \Pi_1. \quad (130)
$$

Here $\Pi_0$ is the projection operator to the subspace containing no angel:

$$
\Pi_0 = \int [dp] \chi(p) \chi^\dagger(p), \quad (131)
$$

and $\Pi_1$ the projection operator to the subspace with exactly one angel:

$$
\Pi_1 = \int [dp] \chi^\dagger(p) \chi(p). \quad (132)
$$

The point of introducing angels is this: the projection of the resolvent of the $\tilde{H}_\Lambda$, to the $\mathcal{B}$ (the sector with no angels) is just the resolvent of the original bosonic system. But we will get another formula for this resolvent, using angels, which has a finite limit as $\Lambda \to \infty$. Indeed we will see that the infinity is avoided by keeping the energy of the bound state of a pair of bosons fixed in this limit; an angel is essentially such a bound state.

Let us split the Hilbert space according to the angel number, with a corresponding splitting of the operator:

$$
\tilde{H}_\Lambda - E \Pi_0 = \begin{pmatrix} a & b^\dagger \\ b & d \end{pmatrix} \quad (133)
$$

with

$$
a : \mathcal{B} \to \mathcal{B}, \quad b^\dagger : \mathcal{B} \otimes L^2(R^d) \to \mathcal{B}, d : \mathcal{B} \otimes L^2(R^d) \to \mathcal{B} \otimes L^2(R^d). \quad (134)
$$

Define an operator $\tilde{R}_\Lambda(E)$ split in the same way:

$$
\tilde{R}_\Lambda(E) = \frac{1}{H_\Lambda - E \Pi_0} = \begin{pmatrix} \alpha & \beta^\dagger \\ \beta & \delta \end{pmatrix} \quad (135)
$$

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We claim that
\[\alpha = \frac{1}{H_\Lambda - E}; \quad (136)\]
i.e., \(\tilde{R}_\Lambda\) projected to \(\mathcal{B}\) is just the resolvent of \(H_\Lambda\). To see this, we use the formula we obtained earlier:
\[\alpha = [a - b^\dagger d^{-1}b]^{-1}. \quad (137)\]

For us now\(^4\),
\[
\begin{align*}
a &= H_0 - E, \quad d = \frac{1}{g(\Lambda)}, \\
b^\dagger &= \int [dp_1 dp_2] \rho_\Lambda(p_1 - p_2)\phi^\dagger(p_1)\phi^\dagger(p_2)\chi(p_1 + p_2) \quad (138)
\end{align*}
\]
This gives
\[
\begin{align*}
b^\dagger d^{-1}b &= g(\Lambda) \int [dp_1 dp_2 dp_3] \rho_\Lambda(p_1 - p_2)\phi^\dagger(p_1)\phi^\dagger(p_2)\chi(p_3) \\
&\quad (2\pi)^d \delta(p_1 + p_2 - p_3) \\
&\quad \int [dp'_1 dp'_2 dp'_3] \rho_\Lambda(p'_1 - p'_2)\chi^\dagger(p'_3)\phi(p'_1)\phi(p'_2)(2\pi)^d \delta(p'_1 + p'_2 - p'_3)
\end{align*}
(139)
\]
If we use
\[\chi(p)\chi^\dagger(p') = (2\pi)^d \delta(p - p') \quad (140)\]
we will get the required result.

But we have another formula for this resolvent:
\[\alpha = a^{-1} + a^{-1}b^\dagger[d - ba^{-1}b^\dagger]^{-1}a^{-1}. \quad (141)\]
This will give,
\[
\frac{1}{H_\Lambda - E} = a^{-1} + \frac{1}{2}a^{-1}b^\dagger \Phi_\Lambda(E)^{-1}ba^{-1} \quad (142)
\]
\(^4\) We hope there will be no confusion in using the same symbol \(d\) both for an operator and the dimension.
where

\[
\Phi_\Lambda(E) = \frac{1}{g(\Lambda)} - \int [dp_1 dp_2 dp'_1 dp'_2] \rho_\Lambda(p_1 - p_2) \rho_\Lambda(p'_1 - p'_2) \chi^\dagger(p_1 + p_2) \left[ \phi(p_1) \phi(p_2) \frac{1}{H_0 - E} \phi^\dagger(p'_1) \phi^\dagger(p'_2) \right] \chi(p'_1 + p'_2) \tag{143}
\]

We can regard \(b\) as the operator that converts a pair of bosons into an angel. Then \(\Phi_\Lambda(E)\) is a kind of effective hamiltonian in the sector with one angel and two fewer bosons: its zeros are energy levels of the manybody problem.

Note that in this way of writing the resolvent of \(H_\Lambda\), the coupling constant appears additively! Its renormalization can be done by separating out a divergent constant from \(\Phi_\Lambda(E)\). We will do this by normal ordering the operators in \(\Phi_\Lambda(E)\).

Using the canonical commutation relations and

\[
H_0 \phi^\dagger(p) = \phi^\dagger(p) H_0 + \frac{p^2}{2} \tag{144}
\]

we can rewrite the quantity in the square brackets above equation as

\[
\phi^\dagger(p'_1) \phi^\dagger(p'_2) \frac{1}{H_0 + \omega(p'_1) + \omega(p'_2) + \omega(p_1) + \omega(p_2)} \phi(p_1) \phi(p_2)
\]

\[
+ (2\pi)^d \delta(p_1 - p'_1) \phi^\dagger(p'_2) \frac{1}{H_0 + \omega(p'_1) + \omega(p'_2) + \omega(p_2)} \phi(p_2)
\]

\[
+ (2\pi)^d \delta(p_2 - p'_2) \phi^\dagger(p'_1) \frac{1}{H_0 + \omega(p'_1) + \omega(p'_2) + \omega(p_1)} \phi(p_1)
\]

\[
+ (2\pi)^d \delta(p_2 - p'_1) \phi^\dagger(p'_2) \frac{1}{H_0 + \omega(p'_1) + \omega(p'_2) + \omega(p_1) - E} \phi(p_1)
\]

\[
(2\pi)^d \delta(p_1 - p'_1) (2\pi)^d \delta(p_2 - p'_2) \frac{1}{H_0 + \omega(p'_1) + \omega(p'_2) - E}
\]

\[
(2\pi)^d \delta(p_1 - p'_2) (2\pi)^d \delta(p_2 - p'_1) \frac{1}{H_0 + \omega(p'_1) + \omega(p'_2) - E} \tag{145}
\]
This gives,

\[
\Phi_{\Lambda}(E) = g^{-1}(\Lambda) - \int [dp_1 dp_2 dp_1' dp_2'] \rho_{\Lambda}(p_1 - p_2) \rho_{\Lambda}(p_1' - p_2') \\
\phi^+(p_1 + p_2) \left[ \phi^+(p_1') \phi^+(p_2') \frac{1}{H_0 + \omega(p_1') + \omega(p_2') + \omega(p_1) + \omega(p_2) + E - E} \phi(p_1) \phi(p_2) \right. \\
\left. + 4(2\pi)^d \delta(p_1 - p_1') \phi^+(p_2') \frac{1}{H_0 + \omega(p_1') + \omega(p_2') + \omega(p_2) - E} \phi(p_2) \right] \\
\chi(p_1' + p_2'). \tag{146}
\]

Now we can take the limit as \( \Lambda \to \infty \). The only divergent term is the last one. If we choose for \( g(\Lambda) \) the expression from the two-body problem,

\[
g^{-1}(\Lambda) = \frac{4}{2^d} \int \rho_{\Lambda}^2(p) \frac{1}{p^2 + \mu^2} \tag{147}
\]

this divergence will cancel yielding a finite expression for \( \Phi_{\Lambda}(E) \) as \( \Lambda \to \infty \).

In fact, we have,

\[
\lim_{\Lambda \to \infty} \int [dq] \rho_{\Lambda}^2(q) [dq] \left[ \frac{1}{q^2 + \nu^2} - \frac{1}{H_0 + \frac{q^2 + \nu^2}{2} - E} \right] = \xi \left( \frac{\mu^2}{2}, \frac{\nu^2}{2} \right). \tag{148}
\]

Here,

\[
\xi \left( \frac{\mu^2}{2}, \frac{\nu^2}{2} \right) := \frac{4}{2^d} \int [dp] \left[ \frac{1}{p^2 + \mu^2} - \frac{1}{p^2 + \nu^2} \right] = \frac{4}{2^d} \left[ \int [dp] \frac{\nu^2}{p^2(p^2 + \nu^2)} - \int [dp] \frac{\mu^2}{p^2(p^2 + \mu^2)} \right] \\
:= \xi \left( \frac{\nu^2}{2} \right) - \xi \left( \frac{\mu^2}{2} \right). \tag{149}
\]

We will get

\[
\Phi(E) = \int [dp] \chi^+(p) \xi \left( \frac{\mu^2}{2}, \frac{\nu^2}{2} - E \right) \chi(p)
\]
\[ - \int [dp_1 dp_2 dp'_1 dp'_2] \chi^\dagger (p_1 + p_2) \left[ \frac{1}{H_0 + \omega(p'_1) + \omega(p'_2) + \omega(p_1) + \omega(p_2) - E} \phi(p_1) \phi(p_2) \\
+ 4(2\pi)^d \delta(p_1 - p'_1) \phi(p'_2) \frac{1}{H_0 + \omega(p'_1) + \omega(p'_2) + \omega(p_2) - E} \phi(p_2) \right] \chi(p'_1 + p'_2). \]  

(150)

Notice that the dependence on the cut-off of the previous operator is traded for a dependence on the renormalization scale \( \mu \).

The resolvent operator has the finite form

\[
R(E) = \frac{1}{H_0 - E} + \frac{1}{H_0 - E} b^\dagger \Phi(\mu, E) b \frac{1}{H_0 - E} 
\]  

(151)

which is the analogue of the Krein formula in the case of the manybody problem.

### 5.2 The Principal Operator

Now let us understand the spectrum of the theory in the special case \( d = 2 \). We will be mostly interested in eignestates with \( E < 0 \): the bound states of the system. In each sector with a fixed number of particles we expect the ground state to be of this form. ( We will make some remarks later on the case \( d = 3 \) where we will see that the ideas in this section will not work in that case.)

Then,

\[
\xi(\frac{\mu^2}{2}, \frac{\nu^2}{2}) = \frac{1}{4\pi} \ln \frac{\nu^2}{\mu^2}. 
\]  

(152)

We can rescale all the momenta by \( \sqrt{|E|} \) to get

\[
\Phi(\mu, E) = \left[ \frac{1}{2\pi} \ln \frac{|E|}{\mu^2} + W \right], 
\]  

(153)
where

\[
W = \frac{1}{2\pi} \int [dp] \chi\dagger(p) \log \left[ H_0 + \omega(p) + 1 \right] \chi(p)
- \int [dp_1 dp_2 dp'_1 dp'_2] \chi\dagger(p_1 + p_2) \left[ \frac{1}{H_0 + \omega(p'_1) + \omega(p'_2) + \omega(p_1) + \omega(p_2) + 1} \phi(p_1)\phi(p_2)
+ 4(2\pi)^d \delta(p_1 - p'_1)\phi^\dagger(p'_2) \frac{1}{H_0 + \omega(p'_1) + \omega(p'_2) + \omega(p_2) + 1} \phi(p_2) \right] \chi(p'_1 + p'_2). \tag{154}
\]

Thus finding the bound state energy of our manybody system amounts to finding the eigenvalues of \(W\):

\[
W|\psi> = w\psi, \quad E = -\mu^2 e^{-2\pi w} \tag{155}
\]

The special case of the three body problem was studied in previous papers \[15\]. We showed not only that the ground state energy is finite, but that it can be estimated by a simple variational ansatz. Since we have described it elsewhere, we won’t elaborate on this point here.

### 5.3 Bosonic Condensation in Two Dimensions

We will now discuss the case of a large number of particles: how to do transfinite quantum many body theory. We will only discuss the mean field approximation, which should be good in the limit of a large number of particles. In the ground state, we should expect all the bosons to condense to a common state \(u(p)\). Thus the problem is to determine the pair of functions \(u(p), \psi(p)\), where \(\psi(p)\) being the wavefunction of the angel. The wavefunction \(u(p)\) describes a condensate of bosons, which breaks translation invariance.
Of course there is no Bose-Einstein condensation for free bosons in two dimensions. What we will show is that with an attractive interaction of zero range there is in fact such a condensation.

The expectation value of the operator $W$ in the state $|u, \psi>$ becomes, (for large $n$) the ‘principal function’ $U$,

$$U = \frac{1}{2\pi} \int |\psi(p)|^2 \log[nh_0(u) + \omega(p) + 1]$$

$$- \int [dp_1 dp_2 dp'_1 dp'_2] \psi(p_1 + p_2) \psi(p'_1 + p'_2) \left[ \frac{u^*(p'_1) u^*(p'_2) u(p_1) u(p_2) n(n-1)}{nh_0(u) + \omega(p'_1) + \omega(p'_2) + \omega(p_1) + \omega(p_2) + 1} + \frac{4}{nh_0(u) + \omega(p'_1) + \omega(p'_2) + \omega(p_1) + \omega(p_2) + 1} \right]$$

(156)

where

$$h_0(u) = \int |u(p)|^2 \omega(p) [dp].$$

(157)

This $U$ is to be minimized subject to the normalization conditions

$$\int |u(p)|^2 [dp] = \int |\psi(p)|^2 [dp] = 1.$$  

(158)

To get more explicit answers, let us ignore all except the leading terms as $n \to \infty$:

$$U = -n \frac{\int \psi^*(p_1 + p_2) u(p_1) u(p_2) [dp_1 dp_2]^2}{nh_0(u)} + \frac{1}{2\pi} \log n + O(n^0).$$

(159)

Thus the ground state energy $E_n$ of such a system of $n$ particles is given by

$$E_n = -\mu^2 \frac{\xi}{n} [C_1 + O(\frac{1}{n})]$$

(160)

where

$$\xi = \inf_{u, \psi} \frac{\int |\psi(p)|^2 [dp] \int |u(p)|^2 [dp] \int \omega(p) |u(p)|^2 [dp] \int \psi^*(p_1 + p_2) u(p_1) u(p_2) [dp_1 dp_2]^2}{\int \psi^*(p_1 + p_2) u(p_1) u(p_2) [dp_1 dp_2]^2}.$$ 

(161)
We will determine $\xi$ by solving the variational problem. Determination of the $C_1$ takes more work which we will not carry out here. It is not needed to determine the leading large $n$ behavior.

It is more convenient to use the corresponding expressions in position space, with $\tilde{u}(x) = \int u(p)e^{ip\cdot x}[dp]$ etc.

$$
\xi = \inf_{\tilde{u}, \psi} \frac{\int |\tilde{\psi}(x)|^2dx \int |\tilde{u}(x)|^2dx \int \frac{1}{2} |\nabla \tilde{u}(x)|^2dx}{\int |\tilde{\psi}^* (x)\tilde{u}^2(x)dx}.
$$

Eliminating $\tilde{\psi}$ gives

$$
\xi = \inf_{\tilde{u}} I[\tilde{u}]
$$

where,

$$
I[\tilde{u}] = \frac{\int |\tilde{u}(x)|^2dx \int \frac{1}{2} |\nabla \tilde{u}(x)|^2dx}{\int |\tilde{u}(x)|^4dx}.
$$

We can see that this amounts to solving a nonlinear differential equation, which can be derived by minimizing $\log I[\tilde{u}]$:

$$
\nabla^2 \tilde{u} - \beta \tilde{u} + g |\tilde{u}|^2 \tilde{u} = 0.
$$

Here,

$$
\beta = \frac{\int |\nabla \tilde{u}|^2dx}{\int |\tilde{u}|^2dx} , \quad g = \frac{\int |\nabla \tilde{u}|^2dx}{\int |\tilde{u}|^4dx}.
$$

The above partial differential equation (but without the constraints on $\beta$ and $g$) has been studied in [16], Theorem 6.7.25. (We just need the special case of dimension two.) There is a normalizable solution for every positive $\beta$ and $g$. In fact it is enough to find the solution for a particular pair of values of these constants. For example, let $\tilde{u}_1$ be a solution of

$$
\nabla^2 \tilde{u}_1 - \tilde{u}_1 + |\tilde{u}_1|^2 \tilde{u}_1 = 0.
$$
Then, \( u(x) = au_1(bx) \) will solve the general equation with

\[
\beta = b^{-2}, \quad g = a^2b^{-1}.
\]  \hspace{1cm} (168)

The quantity \( I[u] \) is invariant under such scale transformations: \( I[u_1] = I[u] \). So it is enough to solve the special case.

Being the ground state of a many body problem, we should expect the solution to be real and have circular symmetry around a point. Thus we only have to solve an ODE, a kind of nonlinear Bessel’s equation

\[
v''(r) + \frac{1}{4r^2}v(r) + \frac{v^3(r)}{r} = v(r)
\]  \hspace{1cm} (169)

with \( \tilde{u}(x) = \sqrt{rv(r)} \) and \( r = |x| \). That a square integrable solution exists is proven\(^5\) in ref. \cite{16}.

We can get a value for \( \xi \sim 12 \) by numerical solution of the ODE. We also get this way the shape of the ‘soliton’ (or ‘condensate’); i.e., the wavefunction of the bound state of a large number of bosons with pointlike interactions. We plot below the solution \( u(r) \); as shown above, the scales of the \( u \) and \( r \) axes are not physically relevant. The wavefunction is peaked at the origin and decays exponentially at infinity. The ground state thus breaks translation invariance but not rotation invariance.\(^6\)

\(^5\)There is a typo in equation \( \dagger \) of ref. \cite{16}, p. 384; the \( \frac{1}{4r^2}v(r) \) term is missing. It does not appear to affect the rest of the argument.

\(^6\) I thank Govind Krishnaswami for solving the ODE numerically and producing the graph.
One important conclusion of our analysis is that the magnitude of the ground state energy grows exponentially with the number of particles:

\[ E_n \sim -\mu^2 e^{2\pi n}. \]  

(170)

If we had a non-singular pair–wise interaction, we would expect the magnitude of the ground state energy to grow like the number of pairs; i.e., like \( n^2 \).

### 5.4 Vortices in Superconductors

A limiting case of the Landau-Ginzberg theory of superconductivity gives a realization of the example we have been discussing. The lagrangian of this field theory is

\[ L = \int |\nabla \phi|^2 d^3x + \frac{1}{2} \int |\text{curl} A|^2 d^3x + \int \lambda(|\phi|^2 - a^2)^2. \]  

(171)
Here, $\phi$ is a complex valued function on $R^3$, $A$ a covariant vector field describing the gauge potential, and

$$\nabla \phi = \partial \phi - ieA\phi$$ \hfill (172)

the covariant derivative. If $\lambda > 1$ (Type II superconductor) this Lagrangian has a static solution that is cylindrically symmetric around a point in a plane $R^2$ and translationally invariant in the orthogonal direction, carrying a unit of magnetic flux. Along the axis of symmetry, the field $\phi$ vanishes. Two such vortices have a repulsive interaction: the energy decreases as the position of the zeros of $\phi$ move further apart in the plane. (For given flux per unit area, the solution that minimizes energy density is a triangular lattice—the Abrikosov lattice.) As $\lambda \rightarrow 1^+$ the force vanishes and the energy is independent of the position of the zeros. (As $\lambda \rightarrow 1^-$, the force vanishes at finite distances, but leads to a residual attractive contact interaction.) In fact for any integer $n$ there is a solution carrying $n$ units of magnetic flux, with zeros of $\phi$ at any prescribed set of $n$ points on the plane [14].

Thus this vortices behave like identical free particles, located at the position of the zeros of the field $\phi$. We can now consider an approximate quantization of this system where only the modes of the field that carry infinitesimal energy are excited. This is just the quantum mechanics of $n$ identical particles on the plane (The classical thermodynamics of this system has been studied in [20].) In the simplest quantization scheme, we can assume these particles are bosons. There is no potential energy between them, if $\lambda$ has exactly the critical value.

However, as we saw earlier, there might be more subtle interactions that arise from the boundary conditions on the wavefunctions; the boundary being
the region where a pair of particles come together. Such contact interactions
seem to have been ignored in the literature on this subject: they arise if we
approach the limit \( \lambda = 1 \) from below rather than above. These lead to bound
states of vortices, with an energy that is determined by a new short distance
scale \( \mu \) exactly as discussed previously. In fact it is also possible to have a
condensate of a large number \( n \) of vortices with a mean density of vortices
(or magnetic flux) given by the function \(|u(r)|^2\) determined in the last sec-
tion. It is of much interest to search experimentally for such a condensation
of vortices in superconductors which are on the borderline between type I
and type II. The prediction that the energy of such a configuration depends
exponentially on the number of vortices also should be tested experimentally.

6 Scalar Field in Three dimensions: a Cautionary Tale

Now let us see why the above approach cannot work as it stands in three
dimensions. Although a renormalized resolvent and a principal operator can
be constructed, the spectrum is not bounded below for more than two parti-
cles. (However there may be other approaches that give a sensible formulation
of this problem. See e.g., [18].

We get again, a Krein formula:

\[
R(E) = \frac{1}{H_0 - E} + \frac{1}{H_0 - E} b^\dagger \frac{1}{\Phi(\mu, E)} b \frac{1}{H_0 - E}. \tag{173}
\]

The eigenvalues are given by \( \Phi(E)|\psi >= 0 \). Again, we can rescale all the
momenta by $\sqrt{|E|}$ to get

$$
\Phi(\mu, E) = \left[ -\frac{\mu}{16\pi} + \sqrt{|E|} \right] W,
$$

where

$$
W = \frac{1}{8\pi} \int [dp] \chi(p) \left[ H_0 + \omega(p) + 1 \right] \chi(p)
- \int [dp_1 dp_2 dp'_1 dp'_2] \chi(p_1 + p_2) \left[ \phi(p'_1) \phi(p'_2) \frac{1}{H_0 + \omega(p'_1) + \omega(p'_2) + \omega(p_1) + \omega(p_2) + 1} \phi(p_1) \phi(p_2) 
+ 4(2\pi)^4 \delta(p_1 - p'_1) \phi(p'_2) \frac{1}{H_0 + \omega(p'_1) + \omega(p'_2) + \omega(p_2) + 1} \phi(p_2) \right] 
\chi(p'_1 + p'_2).
$$

This time the new eigenvalues of the interacting system are given in terms of the eigenvalues $w_n$ of $W$ by

$$
E_n = -\left( \frac{\mu}{16\pi} \right)^2 \frac{1}{w_n^2}.
$$

The operator $W$ itself depends on no parameters: it determines the spectrum of the interacting theory in terms of the parameter $\mu$. Our system will have a finite ground state energy if and only if $W$ is strictly positive. The eigenvalue closest to zero of $W$ will give the ground state energy by the above formula.

So far everything looks fine: it looks like the same approach as in two dimensions is going to work. But we will now see that in the three body sector the operator $W$ has a zero eigenvalue which means the energy of the ground state is infinite.

In the sector of interest we have just one angel and one boson. The states will have the form

$$
|\psi> = \int [dp][dk] \psi(p|k) \phi(k) \chi(p)|0 >.
$$
Only the first two terms in the Principal Operator will contribute to this sector. The eigenvalue problem for $W$ becomes an integral equation for $\psi$:

$$\frac{1}{8\pi} \sqrt{\frac{p^2}{2}+k^2+1} \psi(p|k) - \int [dk'] \psi(p + k - k'|k') \frac{\psi(p + k - k'|k')}{1 + k^2 + 2k'^2 - 2p \cdot k' + p^2} = \frac{\mu}{16\pi \sqrt{|E|}} \psi(p|k)$$

(178)

The total momentum $p + k$ of the angel and the boson is conserved. By going to the center of mass frame, we impose $p = -k$:

$$[1 + \frac{3}{2} k^2] u(k) - 8\pi \int [dk'] \frac{u(k')}{\frac{1}{2} + k^2 + k'^2 + k \cdot k'} = \frac{\mu}{\sqrt{|E|}} u(k)$$

(179)

Moreover it is reasonable to expect that the ground state will have zero angular momentum: $u(k) = u(k)$. Performing the angular integration gives

$$4\pi \int [dk'] \frac{u(k')}{\frac{1}{2} + k^2 + k'^2 + k \cdot k'} = 4\pi \frac{1}{4\pi} \int_{0}^{\infty} k'^2 u(k') dk' \int_{-1}^{1} d(cos \theta) \frac{1}{\frac{1}{2} + k^2 + k'^2 + kk' \cos \theta} = \frac{1}{\pi} \int_{0}^{\infty} k'^2 u(k') ln \left[ \frac{1}{2} + k^2 + k'^2 + kk' \right] dk'.

$$

Then, putting

$$v(k) = ku(k)$$

(180)

we get

$$\left[ 1 + \frac{3}{2} x^2 \right]^\frac{1}{2} v(x) - 2 \int_{0}^{\infty} v(y) \log \left[ \frac{1}{2} + x^2 + y^2 + xy \right] dy = \frac{\mu}{\sqrt{|E|}} v(x).$$

(181)

We will now ask whether the ground state energy determined by this integral equation is finite; in other words if there are solutions to this equation for large $|E|$. We use the same ideas as the corresponding argument in two dimensions and see why they break down \[17, 13\].

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We rewrite the equation as

\[ v(x) = \int_0^\infty U_E(x, y)v(y)dy \]  
\[ \text{where} \]

\[ U_E(x, y) = \frac{1}{\pi \left[ -\frac{\mu}{\sqrt{|E|}} + \sqrt{(1 + \frac{3}{2}x^2)} \right]} \log \left[ \frac{\frac{1}{2} + x^2 + y^2 + xy}{\frac{1}{2} + x^2 + y^2 - xy} \right] \]

The question is whether this equation has a normalizable solution for large \(|E|\); if such a solutions exists, the three body problem has a divergent ground state energy even after our renormalization.

The idea in two dimension is to show that the Hilbert-Schmidt norm \(||U_E||_2\) of this integral kernel is less than one as \(|E|\) becomes large. Then the equation \(v = K_Ev\) will have no solution for large \(|E|\). But in our case this will diverge. In fact,

\[ ||K_E||_2^2 = \int_0^\infty \int_0^\infty K_E(x, y)^2 dx dy \]

Consider first the integral over \(y\):

\[ g(x) = \int_0^\infty \left\{ \log \left[ \frac{\frac{1}{2} + x^2 + y^2 + xy}{\frac{1}{2} + x^2 + y^2 - xy} \right] \right\}^2 dy. \]

We will first show that there exists a constant \(C\) such that

\[ g(x) \rightarrow Cx \]

for large \(x\). By scaling \(y \rightarrow xy\) this equivalent to showing that

\[ \int_0^\infty \left\{ \log \left[ \frac{\frac{1}{2x^2} + 1 + y^2 + y}{\frac{1}{2x^2} + 1 + y^2 - y} \right] \right\}^2 dy \rightarrow C \]

for large positive \(x\). Now,

\[ \int_0^\infty \left\{ \log \left[ \frac{\frac{1}{2x^2} + 1 + y^2 + y}{\frac{1}{2x^2} + 1 + y^2 - y} \right] \right\}^2 dy \rightarrow \int_0^\infty \left\{ \log \left[ \frac{1 + y^2 + y}{1 + y^2 - y} \right] \right\}^2 dy = C \]
The integral converges (in fact has a value of about $C \sim 3.489$). Thus we see that

$$||K_E||^2 = \int_0^\infty \frac{1}{\pi^2} \left[ \frac{1}{\sqrt{|E|}} + \sqrt{(1 + \frac{3}{2} x^2)} \right]^2 g(x) dx$$  \hspace{1cm} (189)$$

But this diverges logarithmically, since the integrand goes like $\frac{1}{x}$ for large $x$. The corresponding integral in two dimensions converges which makes the problem well-posed there. This is why the proof breaks down here.

In fact we can just put $|E| = \infty$ and see that there is a normalizable solution for $v$. It can be determined by iterating the integral equation starting from the trial function $v_0(x) = (1 + \frac{3}{2} x^2)^{-1}$. We spare the reader the details.

### 7 Appendix: An Elementary Formula for Inverses

Suppose a self-adjoint operator $X : \mathcal{H} \to \mathcal{H}$ can be split into $2 \times 2$ blocks,

$$X = \begin{pmatrix} a & b^\dagger \\ b & d \end{pmatrix} \hspace{1cm} (190)$$

with respect to a splitting $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$

$$a : \mathcal{H}_1 \to \mathcal{H}_1, \hspace{0.5cm} b^\dagger : \mathcal{H}_1 \to \mathcal{H}_2, \hspace{0.5cm} b : \mathcal{H}_2 \to \mathcal{H}_1, \hspace{0.5cm} d : \mathcal{H}_2 \to \mathcal{H}_2. \hspace{1cm} (191)$$

Let the inverse be split similarly:

$$X^{-1} = \begin{pmatrix} \alpha & \beta^\dagger \\ \beta & \delta \end{pmatrix}. \hspace{1cm} (192)$$

Then,

$$a \beta^\dagger + b^\dagger \delta = 0, \hspace{0.5cm} b \alpha + d \beta = 0 \hspace{1cm} (193)$$
and
\[ a\alpha + b^\dagger \beta = 1, \quad b\beta^\dagger + d\delta = 1. \quad (194) \]

Solving the first pair of equations we get two expressions for \( \beta \):
\[ -\delta ba^{-1} = \beta = -d^{-1}b\alpha. \quad (195) \]

Putting either of these into the equation for \( \alpha \) gives two expressions for it:
\[ a^{-1} + a^{-1}b^\dagger \delta ba^{-1} = \alpha = [a - b^\dagger d^{-1}b]^{-1}. \quad (196) \]

Similarly,
\[ d^{-1} + d^{-1}b\alpha b^\dagger d^{-1} = \delta = [d - ba^{-1}b^\dagger]^{-1} \quad (197) \]

By combining the two ways of writing \( \alpha \) and \( \delta \) we get
\[ \alpha = a^{-1} + a^{-1}b^\dagger [d - ba^{-1}b^\dagger]^{-1}ba^{-1}, \quad (198) \]
\[ \delta = d^{-1} + d^{-1}b[a - b^\dagger d^{-1}b]^{-1}b^\dagger d^{-1}. \quad (199) \]

These identities will be used in the text.

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