A (POSSIBLY NEW) STRUCTURE WITHOUT THE CANONICAL BASE PROPERTY

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Abstract. In this short note, we introduce a generalization of the canonical base property, called transfer of internality on quotients. A structural study of groups definable in theories with this property yields as a consequence infinitely many new uncountably categorical additive covers of the complex numbers without the canonical base property.

1. Introduction

Recall that a stable theory is one-based if the canonical base of every strong type \( \text{stp}(a/B) \) is algebraic over the realization \( a \). In particular, the tuple \( a \) is independent from \( B \) over the intersection \( \text{acl}^\text{eq}(a) \cap \text{acl}^\text{eq}(B) \). This characterization of the geometry of forking was later captured by Pillay \[15\], see also Evans \[5\], in terms of the ample hierarchy. In particular, one-basedness coincides with non-1-ampleness for stable theories. Similarly, non-2-ampleness agrees with CM-triviality, a notion introduced by Hrushovski in his \textit{ab initio} construction \[7\].

Hrushovski and Pillay \[9\] showed that one-basedness has strong consequences for definable (or rather interpretable) groups in stable theories. Among other results, they showed that one-based groups are central-by-finite. Later, Pillay proved that CM-trivial groups of finite Morley rank are nilpotent-by-finite.

The canonical base property (in short, CBP) provides a different generalization of one-basedness, by replacing the role of algebraicity with almost internality to a particular \( \emptyset \)-invariant family of types. The study of groups according to this relativization was first considered by Kowalski and Pillay, who showed \[11\] that definable groups in a stable theory with the CBP are now central-by-(almost internal). Blossier, Martin-Pizarro and Wagner \[11\] adapted the ample hierarchy to this relative context, which they called \textit{tightness}, and proved that a 2-tight group of finite Lascar rank is nilpotent-by-(almost internal).

If two types \( \text{stp}(b/A_1) \) and \( \text{stp}(b/A_2) \) in a stable theory are almost internal to a fixed family of types of Lascar rank one, then so is \( \text{stp}(b/\text{acl}^\text{eq}(A_1) \cap \text{acl}^\text{eq}(A_2)) \), whenever \( A_1 \) is independent from \( A_2 \) over \( \text{acl}^\text{eq}(A_1) \cap \text{acl}^\text{eq}(A_2) \). The latter always holds for one-based stable theories. A remarkable result of Chatzidakis \[3\] (and of Moosa \[13\] under the stronger assumption of the UCBP) states that theories of finite rank with the CBP \textit{transfer} \( \Sigma \)-internality \textit{on quotients} with respect to a
fixed (but arbitrary) invariant family $\Sigma$ of types of Lascar rank one, that is, the type $tp(b/acF^q(A_1) \cap acF^q(A_2))$ is almost $\Sigma$-internal, whenever both $stp(b/A_1)$ and $stp(b/A_2)$ are.

\[ b \quad \text{almost } \Sigma\text{-int.} \]

\[ A_1 \quad \text{almost } \Sigma\text{-int.} \]

\[ A_2 \quad \text{almost } \Sigma\text{-int.} \]

\[ acF^q(A_1) \cap acF^q(A_2) \]

In this short note, we will undertake a first structural description of groups definable in stable theories of finite Lascar rank preserving $\Sigma$-internality on quotients. As a consequence, we obtain infinitely many new stable (actually uncountably categorical) structures without the CBP, in terms of additive covers of the field of complex numbers, motivated by the primordial counter-example to the CBP given by Hrushovski, Palacín and Pillay [8].

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## 2. Types, cosets and quotients

We fix a complete stable theory $T$ of finite Lascar rank and work inside a sufficiently saturated ambient model of $T$. Let $\Sigma$ be an $\emptyset$-invariant family of types of finite rank. Motivated by work of Chatzidakis and of Moosa, we will study the following generalization of the canonical base property.

**Definition 2.1.** The theory $T$ transfers $\Sigma$-internality on quotients if for every imaginary element $b$ the type

$$stp(b/acF^q(A_1) \cap acF^q(A_2))$$

is almost $\Sigma$-internal, whenever both $stp(b/A_1)$ and $stp(b/A_2)$ are almost $\Sigma$-internal.

**Remark 2.2.** The above property is preserved under naming and forgetting parameters [12, Corollary 2.9]. We will see before the Proposition 4.5 that imaginary elements $b$ must be considered in the definition of transfer of internality on quotients, in contrast to other notions in geometric stability theory.

Chatzidakis showed that the CBP already implies a seemingly stronger property (called UCBP), which appeared in work of Moosa and Pillay [14] on the model theory of compact complex manifolds. A key step in her argument consisted in showing that the CBP implies transfer of internality on quotients (using our terminology).

**Fact 2.3.** [3, Proposition 2.2 & Lemma 2.3] Every theory of finite Lascar rank with the CBP (with respect to the family of types of rank one) transfers internality on quotients with respect to any $\emptyset$-invariant family of types of rank one.

As pointed out in the introduction, Chatzidakis’ result (Fact 2.3) can be seen as a generalization of a general phenomenon for stable theories.
Remark 2.4. Stable theories transfer internality on independent quotients with respect to any ∅-invariant family Σ′ of types of rank one:

The type \(\text{stp}(b/\text{acl}^\text{eq}(A_1) \cap \text{acl}^\text{eq}(A_2))\) is almost Σ′-internal, whenever the types \(\text{stp}(b/A_1)\) and \(\text{stp}(b/A_2)\) are almost Σ′-internal and

\[
\begin{array}{c}
A_1 \downarrow_{\text{acl}^\text{eq}(A_1) \cap \text{acl}^\text{eq}(A_2)} A_2.
\end{array}
\]

In particular, one-based stable theories transfer Σ′-internality on quotients.

Proof. For simplicity of notation, we assume that \(\text{acl}^\text{eq}(A_1) \cap \text{acl}^\text{eq}(A_2) = \emptyset\). By almost internality, there are sets of parameters \(C_1\) and \(C_2\) with

\[
\begin{align*}
C_1 & \downarrow_{A_1} b, A_2 \quad \text{and} \quad C_2 \downarrow_{A_2} b, C_1
\end{align*}
\]

such that \(b\) is algebraic over \(C_1, e_1\) and over \(C_2, e_2\), where \(e_1\) and \(e_2\) are tuples of realizations of types (each one based over \(C_i\)) in Σ′. Note that \(C_1 \downarrow_{A_1} C_2\). Choosing a maximal subtuple \(e_1′\) of \(e_1\) which is independent from \(C_2\) over \(C_1\), we get that \(e_1\) is algebraic over \(C_1, e_1′, C_2\) and \(C_1, e_1′ \downarrow_{A_1} C_2\). Since the type \(\text{stp}(e_1/C_1, e_1′)\) is almost Σ′-internal, we deduce from [3 Lemma 1.7] that \(e_1\) is algebraic over \((C_1, e_1′, (C_2)_{\emptyset})^{\Sigma′}\), where \((C_2)_{\emptyset}^{\Sigma′}\) is the maximal almost Σ′-internal subset of \(\text{acl}^\text{eq}(C_2)\). Considering a maximal subtuple \(e_2′\) of \(e_2\) with

\[
e_2′ \downarrow_{C_2} C_1, e_1′,
\]

we similarly deduce that \(e_2\) is algebraic over \((C_1, e_1′)^{\Sigma′}, e_2′, C_2\) and the independence

\[
\begin{array}{c}
C_1, e_1 \downarrow_{(C_1, e_1′)^{\Sigma′}, (C_2)_{\emptyset}^{\Sigma′}} C_2, e_2
\end{array}
\]

follows. We conclude that \(b\) is algebraic over \((C_1, e_1′)^{\Sigma′}, (C_2)_{\emptyset}^{\Sigma′}\), so the type \(\text{stp}(b)\) is almost Σ′-internal, as desired. \(\square\)

We will now investigate structural properties of type-definable groups under transfer of Σ-internality on quotients, so for the rest of this section fix a type-definable group \(G\) over \(\emptyset\). Given a type-definable subgroup \(H\), the coset \(aH\) can be seen as the hyperimaginary \([a]_E\), where \(E(x, y)\) is the type-definable equivalence relation given by \(x^{-1}y \in H\). Since stable theories eliminate hyperimaginaries, there is a (possibly infinite) tuple \(\tau aH\) of imaginaries which is interdefinable with the class \([a]_E\). This process is uniform with respect to \(a\), since every type-definable group in a stable theory is an intersection of definable ones.

The generic elements of \(G\) are those having maximal Lascar rank, which we refer to as the Lascar rank of \(G\). For elements \(a\) and \(b\) independent over some set \(C\), we will frequently use the following equivalence

\[
b \cdot a \downarrow_C a \iff U(b \cdot a/C) = U(b/C),
\]

whilst \(U(b \cdot a/C) \supseteq U(b/C)\) always holds, since \(U(b/C) = U(b/C, a)\). In particular, if the element \(a\) is generic over \(b\), so is \(a \cdot b\) generic over \(b\). Ziegler noticed a sort of converse:
Fact 2.5. ([2, Lemma 1.2] & [16, Theorem 1]) If the elements $a, b$ and $a \cdot b$ are pairwise independent over $C$, then each one is generic in an acl$^{eq}(C)$-type-definable coset of the same connected type-definable subgroup

\[ \text{Stab}(a/C) = \text{Stab}(b/C) = \text{Stab}(a \cdot b/C). \]

Motivated by Ziegler’s lemma, we will introduce the following definition.

Definition 2.6. The strong type $stp(a/C)$ of an element $a$ of $G$ is coset-free if there is no proper type-definable connected subgroup $H$ of the connected component $G^0$ such that $a$ is contained in an acl$^{eq}(C)$-type-definable coset of $H$.

Note that every generic type is coset-free. The following example of an additive coset-free type in the field of complex numbers will be used later.

Example 2.7. For a transcendental element $\alpha$ in the field $C$ consider the type

\[ p(x) = stp(\alpha, \alpha^2, \ldots, \alpha^n). \]

Every $Q_{\text{alg}}$-definable coset of a definable additive subgroup is a linear variety over $Q_{\text{alg}}$, i.e. an intersections of subsets defined by

\[ \varepsilon_1 x_1 + \ldots \varepsilon_n x_n = \varepsilon_{n+1} \]

for some algebraic elements $\varepsilon_i$. Since $\alpha$ is transcendental, we deduce that the type $p(x)$ is coset-free in the additive group of the field of complex numbers.

The two following lemmata will be helpful for the study of type-definable groups in theories preserving internality on quotients.

Lemma 2.8. The product of $U(G)$-many independent realizations of any coset-free type $stp(a/C)$ is generic over $C$.

Proof. Let $(a_i)_{i \geq 1}$ be a Morley sequence of $stp(a/C)$. For $m \geq 1$, set $b_m = a_1 \cdots a_m$. We will show that for $n = U(G)$ the element $b_n$ is generic. Since $b_{m+1} = b_m \cdot a_{m+1}$, we have the increasing chain

\[ U(b_1/C) \leq U(b_2/C) \leq \cdots \leq U(b_n/C) \leq n, \]

as remarked after the equivalence ($\clubsuit$), which we will implicitly use constantly in the following. Hence, there is an index $m \leq n$ such that $U(b_m/C) = U(b_{m+1}/C)$. Therefore (by the equivalence ($\clubsuit$)),

\[ b_{m+1} \downarrow_C a_{m+1}, \]

so

\[ a_2 \cdot a_3 \cdots a_{m+2} \downarrow_C a_{m+2}, \]

by indiscernability of the Morley sequence. As

\[ a_1 \downarrow_C a_{m+2}, a_2 \cdot a_3 \cdots a_{m+2}, \]

we deduce that

\[ b_{m+2} \downarrow_C a_{m+2}, \]

and hence,

\[ U(b_{m+2}/C) = U(b_{m+1}/C) = U(b_m/C). \]
Iterating this process, it follows that $U(b_n/C)$ is maximal among the ranks $U(b_j/C)$, which yields the following independences:

$$b_n \downarrow a_{n+1} \cdots a_{2n}, \quad b_n \downarrow b_{2n} \quad \text{and} \quad a_{n+1} \cdots a_{2n} \downarrow b_{2n}.$$

Fact 2.5 implies that the coset $b_n \text{Stab}(b_n/C)$ is type-definable over $\text{acl}^\text{eq}(C)$.

In order to conclude that $b_n$ is generic in $G$, we need only show that the connected type-definable subgroup $H = \text{Stab}(b_n/C)$ equals the connected component $G^0$. Therefore, we use that the type $\text{stp}(a/C)$ is coset-free. Note that $a_n H = a_{n-1} \cdots a_1 b_n H$. Since the sequence $(a_i)_{i \geq 1}$ is independent over $\text{acl}^\text{eq}(C)$, we obtain the independence

$$r_{a_n H^n} \downarrow r_{a_n H^n},$$

so the coset $a_n H$ is $\text{acl}^\text{eq}(C)$-type-definable. We conclude that $H = G^0$, since $\text{stp}(a/C)$ is coset-free.

**Lemma 2.9.** Suppose that the type $\text{stp}(a/C)$ is coset-free and $g$ is generic over $C, a$. The intersection $\text{acl}^\text{eq}(g, C) \cap \text{acl}^\text{eq}(g \cdot a^{-1}, C) = \text{acl}^\text{eq}(C)$.

**Proof.** For simplicity of notation, we assume that $C = \emptyset$. Let $(a_i)_{i \geq 1}$ be a Morley sequence of $\text{stp}(a)$ independent from $g$ with $a_1 = a$. Set now

$$g_1 = g \quad \text{and} \quad g_{i+1} = g_i \cdot a_{2i-1}^{-1} \cdot a_{2i} \quad \text{for} \quad i \geq 1.$$

By stationarity, the independence

$$a_{2i}, a_{2i-1} \downarrow g_i \cdot a_{2i-1}^{-1}$$

implies that

$$a_{2i} \overset{\text{st}}{=} g_i \cdot a_{2i-1}^{-1} \cdot a_{2i-1},$$

i.e. the elements $a_{2i}$ and $a_{2i-1}$ have the same strong type over $g_i \cdot a_{2i-1}^{-1}$. Hence

$$g_{i+1} \overset{\text{st}}{=} g_i \cdot a_{2i-1}^{-1} \cdot g_i,$$

by construction, so

$$\text{acl}^\text{eq}(g_i) \cap \text{acl}^\text{eq}(g_i \cdot a_{2i-1}^{-1}) = \text{acl}^\text{eq}(g_i) \cap \text{acl}^\text{eq}(g_i \cdot a_{2i-1})$$

Again by stationarity $a_{2i}^{-1} \overset{\text{st}}{=} g_{i+1}, a_{2i-1}$ and thus

$$g_i \cdot a_{2i-1}^{-1} = g_{i+1} \cdot a_{2i}^{-1} \overset{\text{st}}{=} g_{i+1} \cdot a_{2i} = g_{i+1} \cdot a_{2i+1}^{-1}.$$

We deduce that the intersection

$$\text{acl}^\text{eq}(g_{i+1}) \cap \text{acl}^\text{eq}(g_i \cdot a_{2i-1}^{-1}) = \text{acl}^\text{eq}(g_{i+1}) \cap \text{acl}^\text{eq}(g_{i+1} \cdot a_{2i+1}^{-1})$$

Thus,

$$\text{acl}^\text{eq}(g_i) \cap \text{acl}^\text{eq}(g_i \cdot a_{2i-1}^{-1}) = \text{acl}^\text{eq}(g_{i+1}) \cap \text{acl}^\text{eq}(g_{i+1} \cdot a_{2i+1}^{-1})$$

for all $i \geq 1$. Hence, we obtain that

$$\text{acl}^\text{eq}(g) \cap \text{acl}^\text{seq}(g \cdot a^{-1}) = \ldots = \text{acl}^\text{eq}(g_n) \cap \text{acl}^\text{eq}(g_n \cdot a_{2n-1}^{-1}) \subseteq \text{acl}^\text{eq}(g) \cap \text{acl}^\text{eq}(g_n).$$

We need only show that the type $\text{stp}(a_1^{-1} \cdot a_2)$ is also coset-free: Indeed, in that case, the Lemma 2.8 implies that the element

$$b = (a_1^{-1} \cdot a_2) \cdots (a_{2i-1}^{-1} \cdot a_{2i}) \cdots (a_{2n-1}^{-1} \cdot a_{2n})$$
is generic over $g$, where $n = U(G)$. Thus, the product $g_n = g \cdot b$ is generic and independent from $g$, so the intersection $acl^{eq}(g) \cap acl^{eq}(g_n) = acl^{eq}(\emptyset)$, as desired.

Suppose that the element $a_1^{-1} \cdot a_2$ is contained in an $acl^{eq}(\emptyset)$-type-definable coset of a connected subgroup $H$. The independence $a_1^{-1} \mid acl^{eq}(\emptyset)$ implies that $acl^{eq}(\emptyset)$-definable. Since $stp(a)$ is coset-free, we deduce that $H = G^s$, as desired. □

We now have all the ingredients in order to show the following property of certain type-definable groups.

**Theorem 2.10.** Suppose that the ambient theory $T$ transfers $\Sigma$-internality on quotients and consider a type-definable subgroup $H$ of $G$ possibly over parameters. If the type $stp(a)$ of the element $a$ of $G$ is coset-free and $stp(\langle aH \rangle)$ is almost $\Sigma$-internal, then $stp(\langle gH \rangle)$ is almost $\Sigma$-internal, whenever $g$ is generic in $G$ over $a$.

Notice that we do not require that the type $stp(a/\langle H \rangle)$ is coset-free. Indeed, if $stp(a/\langle H \rangle)$ is coset-free, the same conclusion holds without imposing that the theory $T$ transfers $\Sigma$-internality on quotients: A finite product of independent realizations of the type $stp(a/\langle H \rangle)$ is generic in $G$, by the Lemma 2.8.

**Proof.** Let $g$ be generic in $G$ over $a$. The type $stp(\langle gH \rangle)$ is almost $\Sigma$-internal, because $H = (aH)^{-1}aH$. Since the coset $gH$ is definable over $g, \langle gH \rangle$, the type

$$stp(\langle gH \rangle/g)$$

is almost $\Sigma$-internal. On the other hand, the identity $gH = (g \cdot a^{-1})aH$ yields that the type

$$stp(\langle gH \rangle/g \cdot a^{-1})$$

is almost $\Sigma$-internal. Preservation of $\Sigma$-internality on quotients implies that

$$stp(\langle gH \rangle/\langle acl^{eq}(g) \cap acl^{eq}(g \cdot a^{-1}) \rangle)$$

is almost $\Sigma$-internal. Since the type $stp(a)$ is coset-free and $g$ is generic over $a$, the intersection

$$acl^{eq}(g) \cap acl^{eq}(g \cdot a^{-1}) = acl^{eq}(\emptyset),$$

by the Lemma 2.9, so the type $stp(\langle gH \rangle)$ is almost $\Sigma$-internal, as desired. □

For a connected type-definable group $G$ in a theory with the CBP relative to the family $\Sigma$, Kowalski and Pillay [11] Theorem 4.3] showed that the quotient $G/Z(G)$ is almost $\Sigma$-internal. We do not know whether the same holds under the weaker assumption of transfer of $\Sigma$-internality on quotients, without assuming the existence of a suitable coset-free type $stp(a)$, as in the previous theorem.
3. A cover of a group

In this section, we will apply the previous results to covers of groups, which will then be needed in Section 4 in order to produce new structures without the CBP.

**Definition 3.1.** Suppose that \((G, \cdot)\) is a stable group of finite Lascar rank (possibly with additional structure). A cover of \(G\) is a structure \(M = (S, P, \pi, \star, \odot, \ldots)\) with two distinguished sorts \(S\) and \(P\) such that the following conditions hold:

- The sort \(P\) equals \(G\) and the induced structure on \(P\) coincides with the full structure of \(G\).
- The map \(\pi : (S, \odot) \to (P, \cdot)\) is a surjective group homomorphism.
- The group action \(\star\) of \(P\) on \(S\) turns each fiber of \(\pi\) into a principal homogeneous space.

The cover is non-degenerated, if the sort \(S\) is not almost \(P\)-internal.

Note in particular that \(P\) is stably embedded in the cover \(M\), that is, every definable subset of \(P^n\) is definable with parameters from \(P\). Hence, every type of a tuple in the cover over \(P\) is definable over a subset of \(P\). Furthermore, the cover \(M\) is again stable of finite rank, since each fiber is definably isomorphic to the sort \(P\).

The example of an uncountably categorical theory without the CBP given by Hrushovski, Palacín and Pillay [8] may be seen as an (additive) cover of the complex numbers.

**Example 3.2.** Consider the (additive) non-degenerated cover \(M_1 = (S, P, \pi, \star, \oplus, \otimes)\), where \(P\) is the field \(\mathbb{C}\) of complex numbers and \(S = \mathbb{C} \times \mathbb{C}\). The projection \(\pi : S \to P\) maps an element of \(S\) onto the first coordinate (seen as a pair of complex numbers) and the group action \(\star\) of \(P\) on \(S\) is given by \(\beta \star (\alpha, a') = (\alpha, a' + \beta)\).

Moreover, the cover \(M_1\) has two distinguished operations:

\[
\oplus : S \times S \to S, \quad ((\alpha, a'), (\beta, b')) \mapsto (\alpha + \beta, a' + b');
\]

and

\[
\otimes : S \times S \to S, \quad ((\alpha, a'), (\beta, b')) \mapsto (\alpha \beta, \alpha b' + \beta a').
\]

which define a ring structure on \(S\).

**Henceforth, we fix a non-degenerated cover** \(M = (S, P, \pi, \star, \odot, \ldots)\) of a group \(G\) of finite Lascar rank.

By \(P\)-internality or internality with respect to \(P\) we mean internality with respect to the family \(\Sigma\) of types in the sort \(P\).

**Proposition 3.3.** Suppose that \(M\) transfers \(P\)-internality on quotients. There exists no tuple \(a = (a_1, \ldots, a_n)\) in \(S^n\) such that \(\text{stp}(\pi(a))\) is coset-free in \((P^n, \cdot)\)

and \(a_n\) is algebraic over \(a_1, \ldots, a_{n-1}, P\).

The idea of the proof is to mimic Theorem 2.10, despite we do not explicitly have a type-definable subgroup of \(S^n\), whose coset should be the set \(H_n\) in the proof.
below, due to the possible lack of compatibility between the group action $\ast$ and the
group law $\odot$.

Proof. Since $P$ is stably embedded, there is a subset $B$ of $P$ such that the type
$p(x) = \text{stp}(a/P)$ is definable over $B$. Choose a formula $\varphi(x, b)$ in $p(x)$ witnessing
that $a_n$ is algebraic over $a_1, \ldots, a_{n-1}, B$. Set

$$H = \{ \gamma \in P^n \mid \forall y \in P^{n}\left( d_p x \varphi(x, y) \leftrightarrow d_p x \varphi(\gamma^{-1} \ast x, y) \right) \}$$

and

$$H_a = \{ x \in S^n \mid x = \gamma \ast a \text{ for some } \gamma \in H \}.$$ 

Note that the canonical parameter of the definable set $\gamma H_a$ is definable over $P$.

Given an automorphism $\sigma$ fixing $P$ pointwise, we have that $a$ and $\sigma(a)$ lie in the
same fiber with respect to the projection $\pi$, so $\sigma(a) = \gamma \ast a$ for some element $\gamma$ in
$P^n$. By the definition of $H$, it is immediate that $\gamma$ belongs to $H$, so $\sigma(a)$ lies in $H_a$.

It now follows that $\sigma$ permutes $H_a$, since the group $H$ is fixed pointwise.

Choose now a generic element $g = (g_1, \ldots, g_n)$ in $S^n$ over $a$. The set

$$g \odot a^{-1} \odot H_a = \{ g \odot a^{-1} \odot x \mid x \in H_a \}$$

is definable over $g \odot a^{-1}, \gamma H_a$. Since every fiber of $\pi$ is $P$-internal, so is the type

$$\text{stp}(\gamma g \odot a^{-1} \odot H_a / \pi(g \odot a^{-1}))$$

We are now led to show that the type

$$\text{stp}(\gamma g \odot a^{-1} \odot H_a)$$

is almost $P$-internal. By transfer of $P$-internality on quotients, it suffices to show
that the type

$$\text{stp}(\gamma g \odot a^{-1} \odot H_a / \pi(g))$$

is $P$-internal, for the intersection

$$\text{acl}^P(\pi(g)) \cap \text{acl}^P(\pi(g) \cdot \pi(a)^{-1}) = \text{acl}^P(\emptyset),$$

by Lemma 2.9.

Notice that

$$\pi(a^{-1} \odot x) = \pi(a)^{-1} \cdot \pi(x) = e_G,$$

where $e_G$ is the neutral element of $G$, for $x$ in $H_a$. Thus, the set $g \odot a^{-1} \odot H_a$ is
contained in the fiber of $g$, which immediately yields that the type

$$\text{stp}(\gamma g \odot a^{-1} \odot H_a / \pi(g))$$

is $P$-internal, as desired.

In order to conclude, we need only show that the type $\text{stp}(g_n/g_1, \ldots, g_{n-1})$ is
almost $P$-internal, since the generic element $g$ of $S^n$ consists of generic independent
coordinates. Note that the set

$$Z = \{ z \in S \mid (a_1, \ldots, a_{n-1}, z) \in H_a \}$$

is finite and contains $a$, witnessed by the formula $\varphi(x, b)$, and thus so is the set

$$\{ u \in S \mid (g_1, \ldots, g_{n-1}, u) \in g \odot a^{-1} \odot H_a \} = g_n \odot a_n^{-1} \odot Z.$$ 

Hence, the element $g_n$ is algebraic over $g_1, \ldots, g_{n-1}, \gamma g \odot a^{-1} \odot H_a$, so the type

$\text{stp}(g_n/g_1, \ldots, g_{n-1})$ is almost $P$-internal, as desired. $\square$
We conclude this section showing that non-degenerated covers of a group of Lascar rank one cannot eliminate finite imaginaries, if the cover transfers $P$-internality on quotients. In particular, every such cover with elimination of imaginaries is a counterexample to the CBP.

**Corollary 3.4.** If the non-degenerated cover $\mathcal{M}$ of a group of Lascar rank one transfers $P$-internality on quotients, then $\mathcal{M}$ does not eliminate finite imaginaries.

**Proof.** Choose two independent realizations $a$ and $b$ of the principal generic type of the group $(S, \circ)$ over $acl^{eq}(\emptyset)$. We will show that the finite set $\{a, b\}$ has no real canonical parameter $(c, \varepsilon)$, where $c$ is a tuple in $S$ and $\varepsilon$ is a tuple in $P$.

**Claim.** The projection $\pi(c)$ is not algebraic over $\pi(a \circ b)$.

**Proof of the Claim.** Since $a$ and $b$ are independent generic elements, their projections $\alpha = \pi(a)$ and $\beta = \pi(b)$ are distinct. Hence, the element $a$ is the unique element of the set $\{a, b\}$ in the fiber of $\alpha$. Thus, the generic $a$ of $S$ is definable over $c, \mathcal{M}$. If the tuple $\gamma = \pi(c)$ were algebraic over $\pi(a \circ b)$, the independence $a \perp a \circ b$ yields that the type $stp(a)$ is almost $P$-internal, contradicting our assumption that the sort $S$ is not almost $P$-internal. □

Choose a coordinate $\gamma_i$ of $\gamma$ of rank one over $\pi(a \circ b) = \alpha \cdot \beta$. By Proposition 3.3 the type $p = stp(\alpha, \beta, \gamma_i)$ is not coset-free. Hence, the tuple $(\alpha, \beta, \gamma_i)$ is contained in an $acl^{eq}(\emptyset)$-definable coset of a proper connected type-definable subgroup $H$ of $G^3$. We first show that $H$ equals the stabilizer of $p$. Note that

$$2 = U(\alpha, \beta, \gamma_i) \leq U((\alpha, \beta, \gamma_i) \cdot H) = U(H) < 3,$$

since $H$ is a proper subgroup of $G^3$. Thus, the type $p$ is the unique generic type of the $acl^{eq}(\emptyset)$-definable coset $(\alpha, \beta, \gamma_i) \cdot H$. Hence, the stabilizer of $p$ contains $H$ and so they are equal, as desired.

Now, stationarity of the principal generic type of $S$ yields that

$$(a, b) \equiv (b, a).$$

Choose an automorphism $\sigma$ fixing $acl^{eq}(\emptyset)$ with $\sigma(a, b) = (b, a)$, so $\sigma(c) = c$. Since

$$(\alpha, \beta, \gamma_i) \cdot H = (\sigma(\alpha), \sigma(\beta), \sigma(\gamma_i)) \cdot \sigma(H) = (\beta, \alpha, \gamma_i) \cdot H,$$

the tuple $(\beta^{-1} \cdot \alpha, \alpha^{-1} \cdot \beta, 1_G)$ is contained in $H$. The two elements $\alpha^{-1} \cdot \beta$ and $\beta$ realize the same type over $acl^{eq}(\emptyset)$ (the principal generic type of $G$), so we deduce that $(\beta, \beta^{-1}, 1_G)$ belongs to $H$. Hence, the tuple $(1_G, \alpha \cdot \beta, \gamma_i)$ is contained in the coset

$$(1_G, \alpha \cdot \beta, \gamma_i) \cdot H = (\beta, \alpha, \gamma_i) \cdot H = (\alpha, \beta, \gamma_i) \cdot H.$$

Since $U(1_G, \alpha \cdot \beta, \gamma_i) = 2$, the tuple $(1_G, \alpha \cdot \beta, \gamma_i)$ realizes $p$, the unique generic type of the coset $(\alpha, \beta, \gamma_i) \cdot H$, which is a blatant contradiction to $\alpha$ being generic. □

4. **Uncountably categorical covers without the CBP**

In this section, we restrict our attention to additive covers of the field of complex numbers, in analogy to the Example 3.2, in order to produce uncountably categorical structures without the CBP, similar to the counter-example of Hrushovski, Palacin and Pillay 8.
Definition 4.1. An additive cover of the complex numbers \( \mathbb{C} \) is a cover of the additive group \((\mathbb{C}, +, \cdot)\) (equipped with the full field structure), where the sort \( S \) equals \( \mathbb{C} \times \mathbb{C} \) and such that:

- The projection \( \pi : S \to P \) maps a pair onto the first coordinate.
- The (lifted) group operation on \( S \) is the map \( \oplus : S \times S \to S \) given by \((\alpha, a'), (\beta, b') \mapsto (\alpha + \beta, a' + b').\)
- The group action \( \star \) of \( P \) on \( S \) is given by \( \beta \star (\alpha, a') = (\alpha, a' + \beta). \)

Note that every additive cover of the complex numbers is an uncountably categorical structure with \( P \) the unique strongly minimal set up to non-orthogonality.

Remark 4.2. Given an additive cover \( M \), there is a canonical embedding \( \text{Aut}(M/P) \hookrightarrow \{ F : \mathbb{C} \to \mathbb{C} \text{ additive} \} \)
uniquely determined by the identity \( \sigma(x) = F_\sigma(\pi(x)) \star x. \)

For the additive cover \( M_1 = (S, P, \pi, \star, \oplus) \) in Example 3.2, the above embedding yields an isomorphism
\[ \text{Aut}(M_1/P) \cong \{ D : \mathbb{C} \to \mathbb{C} \text{ derivation} \}. \]

As every derivation induces an automorphism fixing \( P \) pointwise, the sort \( S \) is not almost \( P \)-internal [8, Corollary 3.3], so the cover \( M_1 \) is non-degenerated.

It was shown in [12, Proposition 6.1] that the cover \( M_1 \) does not transfer almost \( P \)-internality on quotients. We will now sketch an alternative argument, which will serve as a leitmotiv in the sequel: Given a generic element \( a_1 \) in \( S \), the product \( a_2 = a_1 \otimes a_1 \) is algebraic (or actually, definable) over \( a_1 \). The type \( \text{stp}(\pi(a_1), \pi(a_2)) \)
is coset-free in the additive group \( P \times P \), by the Example 2.7. Therefore Proposition 3.3 immediately gives that \( M_1 \) does not transfer \( P \)-internality on quotients.

In the previous short argument, the pair \((a_1, a_2)\) is the generic element of the a binary predicate \( R_1 \), namely the graph of the squaring function. With this in mind, we introduce the relations \( R_n \).

Definition 4.3. Let \( M_n \) be the additive cover of the complex numbers with the additional relation
\[ R_n((\alpha_1, a'_1), \ldots (\alpha_{n+1}, a'_{n+1})) \iff \bigwedge_{i=1}^{n+1} a_i = \alpha_i \]
and \( a'_{n+1} = \sum_{i=1}^{n} \binom{n+1}{i} (-1)^{n-i} \alpha_{n+1-i} \alpha_i \)
on \( S^{n+1} \).

A short calculation yields that the group \( \text{Aut}(M_n/P) \) corresponds to the collection of functions
\[ D_n = \{ F : \mathbb{C} \to \mathbb{C} \text{ additive} \mid F(\alpha^{n+1}) = \sum_{i=1}^{n} \binom{n+1}{i} (-1)^{n-i} \alpha^{n+1-i} F(\alpha_i) \}. \]
It is straightforward to see that $D_1$ is exactly the collection of derivations on $\mathbb{C}$. We will see that the $n^{th}$ iterate of a derivation is contained in $D_n$, which explains why its elements are referred to as higher order derivations [6]. The set $D_n$ is the closure of the additive subgroup generated by the maps which are compositions of at most $n$-many derivations with respect to the product topology on the space of all maps $\mathbb{C} \to \mathbb{C}$, where the image $\mathbb{C}$ carries the discrete topology [10, Theorem 1.1]. In particular, the collection $D_n$ is strictly contained in $D_{n+1}$. For completeness, we will now provide a self-contained proof of the latter, which is a verbatim adaptation of Proposition 4.5 using the translation of Satz.

**Proposition 4.4.** For every natural number $n \geq 1$, the collection $D_n$ is contained in $D_{n+1}$. Given a non-trivial derivation $D$ on $\mathbb{C}$, the composition

$$D^{n+1} = D \circ \cdots \circ D$$

$n+1$ times

belongs to $D_{n+1} \setminus D_n$.

**Proof.** We first show that an additive function $F$ is in $D_n$ if and only if

$$F(x_1 \cdots x_{n+1}) = \sum_{k=1}^{n} (-1)^{k+1} \sum_{1 \leq i_1 < \cdots < i_k \leq n+1} x_{i_1} \cdots x_{i_k} F(x_1 \cdots \hat{x}_{i_j} \cdots x_{i_k} \cdots x_{n+1}),$$

where $\hat{x}_j$ indicates that the variable $x_j$ is omitted. One direction is immediate, setting $x_1 = \ldots = x_{n+1} = \alpha$. For the other direction, let $G_{F,n}(x_1, \ldots, x_{n+1})$ denote the right-hand side of the above equality. For $F$ in $D_n$, we have by definition the identity

$$F((x_1 + \ldots + x_{n+1})^{n+1}) = \sum_{i=1}^{n} \binom{n+1}{i} (-1)^{n-i} (x_1 + \ldots + x_{n+1})^{n+1-i} F((x_1 + \ldots + x_{n+1})^i).$$

Since the function $F$ is additive, expanding the left-hand-side, it is immediate to isolate that the only term which is odd in all the variables $x_i$'s is exactly

$$(n + 1)! \cdot F(x_1 \cdots x_{n+1}).$$

Comparing to the terms on the right-hand-side which are also odd in all the variables, we deduce by renaming $n + 1 - i = k$ and dividing by $(n + 1)!$ that

$$F(x_1 \cdots x_{n+1}) = G_{F,n}(x_1, \ldots, x_{n+1}),$$

as desired.

Using the above equivalence, we now show that the function $F$ in $D_n$ belongs to $D_{n+1}$. Indeed,

$$F(x_1 \cdots x_{n+2}) = F(x_1 \cdots (x_{n+1}x_{n+2})) = G_{F,n}(x_1, \ldots, x_{n+1} \cdot x_{n+2}) =$$

$$G_{F,n+1}(x_1, \ldots, x_{n+1}, x_{n+2}) - x_{n+2}F(x_1 \cdots x_{n+1}) - x_{n+1}F(x_1 \cdots \hat{x}_{n+1} x_{n+2}) +$$

$$x_{n+2}G_{F,n}(x_1, \ldots, x_{n+1}) + x_{n+1}G_{F,n}(x_1, \ldots, \hat{x}_{n+1}, x_{n+2}) =$$

$$G_{F,n+1}(x_1, \ldots, x_{n+1}, x_{n+2}) - x_{n+2}(F(x_1 \cdots x_{n+1}) - G_{F,n}(x_1, \ldots, x_{n+1})) -$$

$$x_{n+1}(F(x_1 \cdots \hat{x}_{n+1} x_{n+2}) - G_{F,n}(x_1, \ldots, \hat{x}_{n+1}, x_{n+2})) =$$

$$G_{F,n+1}(x_1, \ldots, x_{n+1}, x_{n+2}) - x_{n+2} \cdot 0 - x_{n+1} \cdot 0 = G_{F,n+1}(x_1, \ldots, x_{n+1}, x_{n+2}),$$
so $F$ is in $D_{n+1}$.

Fix now some derivation $D$ on the complex numbers. We show inductively on $n$ that the composition $D^{n+1}$ belongs to $D_{n+1}$. Note that

$$D^{n+1}(x^{n+2}) = D(D^n(x^{n+2})) = D\left( \sum_{i=1}^{n+1} \binom{n+2}{i} (-1)^{n+1-i} x^{n+2-i} D^i(x^i) \right),$$

since $D^n$ belongs to $D_n \subseteq D_{n+1}$. Hence,

$$D^{n+1}(x^{n+2}) = \sum_{i=1}^{n+1} \binom{n+2}{i} (-1)^{n+1-i} (D(x^{n+2-i})D^n(x^i) + x^{n+2-i}D^{n+1}(x^i)) = \sum_{i=1}^{n+1} \binom{n+2}{i} (-1)^{n+1-i} x^{n+2-i}D^{n+1}(x^i) + \sum_{i=1}^{n+1} \binom{n+2}{i} (-1)^{n+1-i} D(x^{n+2-i})D^n(x^i).$$

We need only show that the subsum

$$\sum_{i=1}^{n+1} \binom{n+2}{i} (-1)^{n+1-i} D(x^{n+2-i})D^n(x^i) = 0.$$

Indeed,

$$\sum_{i=1}^{n+1} \binom{n+2}{i} (-1)^{n+1-i} D(x^{n+2-i})D^n(x^i) =$$

$$D(x) \sum_{i=1}^{n+1} \binom{n+2}{i} (-1)^{n+1-i} (n + 2 - i)x^{n+1-i}D^n(x^i) =$$

$$D(x) \sum_{i=1}^{n+1} \binom{n+2}{i} (n + 2 - i)(-1)^{n+1-i} x^{n+1-i}D^n(x^i) =$$

$$D(x) \sum_{i=1}^{n+1} \binom{n+1}{i} (n + 2)(-1)^{n+1-i} x^{n+1-i}D^n(x^i) =$$

$$D(x)(n + 2) \left( D^n(x^{n+1}) - \sum_{i=1}^{n} \binom{n+1}{i} (-1)^{n-i} x^{n+1-i}D^n(x^i) \right) = 0,$$

since $D^n$ is in $D_n$.

Assuming now that the composition $D^{n+1}$ belongs to the collection $D_n$, we will deduce that $D$ is the trivial derivation. We show inductively on $k \leq n$ that $D^{n+1-k}$ must be contained in $D_{n-k}$. Since $D_0 = \{0\}$, this yields that $D$ is trivial. The
calculations will be very similar to the above. We have
\[ \sum_{i=1}^{n-k} \binom{n+1-k}{i} (-1)^{n-k-i} x^{n+1-k-i} D^{n+1-k}(x^i) = D^{n+1-k}(x^{n+1-k}) = \]
\[ D(D^{n-k}(x^{n+1-k})) = D \left( \sum_{i=1}^{n-k} \binom{n+1-k}{i} (-1)^{n-k-i} x^{n+1-k-i} D^{n-k}(x^i) \right) = \]
\[ = \sum_{i=1}^{n-k} \binom{n+1-k}{i} (-1)^{n-k-i} D(x^{n+1-k-i}) D^{n-k}(x^i) + \]
\[ + \sum_{i=1}^{n-k} \binom{n+1-k}{i} (-1)^{n-k-i} x^{n+1-k-i} D^{n+1-k}(x^i). \]

Therefore,
\[ 0 = \sum_{i=1}^{n-k} \binom{n+1-k}{i} (-1)^{n-k-i} D(x^{n+1-k-i}) D^{n-k}(x^i) = \]
\[ = D(x)(n+1-k) \sum_{i=1}^{n-k} \binom{n-k}{i} (-1)^{n-k-i} x^{n-k-i} D^{n-k}(x^i). \]

We conclude that (even in case that \( D(x) = 0 \)) the sum
\[ \sum_{i=1}^{n-k} \binom{n-k}{i} (-1)^{n-k-i} x^{n-k-i} D^{n-k}(x^i) = 0, \]
for all \( x \) in \( C \), so \( D^{n-k} \) belongs to \( D^{n-(k+1)} \), as desired. \( \square \)

In particular, the covers \( \mathcal{M}_n \) have pairwise distinct automorphism groups. However, at the moment of writing, we do not know whether these additive covers are bi-interpretable, and more importantly bi-interpretable with \( \mathcal{M}_1 \). A possible way to tackle this question is in terms of transfer of internality on intersections, a property which is already present in Chatzidakis’ proof of Fact 2.3. In [12, Proposition 6.3] it is shown that the additive cover \( \mathcal{M}_1 \) does not transfer \( P \)-internality on intersections.

Before showing that the additive covers \( \mathcal{M}_n \), for \( n \geq 1 \), do not transfer \( P \)-internality on quotients, notice that the presentation of \( \mathcal{M}_1 \) in the Example 3.2 coincides with the Definition 3.1, in the sense that these two structures have the same same definable sets. Indeed, the relation \( R_1(a_1, a_2) \) holds if and only if \( a_2 = a_1 \otimes a_1 \), so the formula
\[ \psi(a, b, z) = \exists z_1 \exists z_2 \exists z_3 \left( R_1(a, z_1) \land R_1(b, z_2) \land R_1(a \oplus b, z_3) \land z = z_3 - z_2 - z_1 \right) \]
defines \( a \otimes b \). Note that each cover \( \mathcal{M}_n \) is a (definitional) reduct of \( \mathcal{M}_1 \):
\( R_n(a_1, \ldots, a_{n+1}) \iff \exists \varepsilon_2, \ldots, \exists \varepsilon_{n+1} \in P \left( \bigwedge_{i=2}^{n+1} a_i = \varepsilon_i \ast \left( a_1 \otimes \cdots \otimes a_1 \right) \right) \land \\
\varepsilon_{n+1} = \sum_{i=2}^{n} \binom{n+1}{i} (-1)^{n-i} \pi(a_1)^{n+1-i} \varepsilon_i \)

We will reproduce now the argument sketched in the Remark 4.2 in order to show that these new covers do not transfer \( P \)-internality on quotients. Note first that there is no real tuple \( b \) in \( \mathcal{M}_n \) such that the types \( \text{stp}(b/A_1) \) and \( \text{stp}(b/A_1) \) are both almost \( P \)-internal for some sets \( A_1 \) and \( A_2 \), but the type \\
\( \text{stp}(b/acl_{\text{eq}}(A_1) \cap acl_{\text{eq}}(A_2)) \)

is not almost \( P \)-internal. This is because the tuple \( \pi(b) \) has to be algebraic over \( A_1 \) and over \( A_2 \), by [8, Corollary 3.3], since every derivation induces an automorphism in \( \text{Aut}(\mathcal{M}_n/P) \).

**Proposition 4.5.** For each \( n \geq 1 \), the additive cover \( \mathcal{M}_n \) does not transfer \( P \)-internality on quotients. In particular, the CBP does not hold in \( \mathcal{M}_n \).

Note that the so-called group version of the CBP [11, Theorem 4.1] yields a more direct argument for the failure of the CBP since the stabilizer of a generic realization of \( R_n \) is trivial.

**Proof.** Note that the sort \( S \) is not almost \( P \)-internal, since \( \mathcal{M}_n \) is a reduct of \( \mathcal{M}_1 \). Choose a realization \( a = (a_1, \ldots, a_n) \) of the relation \( R_n \) with generic projection \( \alpha = \pi(a_1) \). The element \( a_n \) is definable over \( a_1, \ldots, a_{n-1}, P \), by construction, since \( D_n \) is (isomorphic to) \( \text{Aut}(\mathcal{M}_n/P) \). The type \( \text{stp}(\pi(a)) \) is coset-free by the Example 2.7. Hence, the result is a direct application of the Proposition 3.3.

Since the additive cover \( \mathcal{M}_1 \) eliminates finite imaginaries [12, Corollary 4.5], Corollary 3.4 yields another proof for the failure of the transfer of internality on quotients (and thus of the CBP) for \( \mathcal{M}_1 \). We will now conclude this work by showing that none of the additive covers \( \mathcal{M}_n \), with \( n > 1 \), eliminate finite imaginaries. The main step consists in showing that subgroups of \( G^n \) of the form \\
\( G(a/P) = \{ (g_1, \ldots, g_n) \in G^n \mid (g_1 \ast a_1, \ldots, g_n \ast a_n) \equiv_P (a_1, \ldots, a_n) \} \)

for a suitable tuple \( a = (a_1, \ldots, a_n) \) of \( S^n \) will be \( acl_{\text{eq}}(\emptyset) \)-definable. For additive covers, these definable groups are linear. Nonetheless, we will circumvent this description in the first part of the proof in order to provide a general criteria which can be easily adapted to arbitrary non-degenerated covers of a group of finite Morley rank.

**Proposition 4.6.** For \( n > 1 \), the additive cover \( \mathcal{M}_n \) does not eliminate finite imaginaries.

**Proof.** Assume for a contradiction that the finite set \( \{a, b\} \) with \( a \) and \( b \) generic independent elements of \( S \) has a real canonical parameter \( (c, \varepsilon) \), where \( c = (c_i)_{i \leq m} \) is a tuple in \( S \) and \( \varepsilon \) is a tuple in \( P \).
Claim. There is an index \( j \leq m \) such that the group \( G(a, b, c_j/P) \) is not definable over \( \text{acf}^q(\emptyset) \).

Proof of the Claim. Suppose otherwise that all such groups are \( \text{acf}^q(\emptyset) \)-definable and choose an elementary substructure \( N \) of the cover \( \mathcal{M}_n \) independent from \( a, b \).

We first show that for each \( \sigma \) with \( (\sigma(a) = g_1 \ast a, \sigma(b) = g_2 \ast b, \sigma(c) = g_3 \ast c) \). Note that \( (a, b) \equiv_N (b, a) \), so choose an automorphism \( \tau \) fixing \( N \) with \( \tau(a, b) = (b, a) \). The tuple \( c \) is fixed by \( \tau \) and the conjugate \( \tau^{-1} \circ \sigma \circ \tau \) belongs to \( \text{Aut}(\mathcal{M}/P) \). Since

\[
\tau^{-1} \circ \sigma \circ \tau(a) = g_2 \ast a, \quad \tau^{-1} \circ \sigma \circ \tau(b) = g_1 \ast b \quad \text{and} \quad \tau^{-1} \circ \sigma \circ \tau(c) = g_3 \ast c,
\]

the triple \( (g_2, g_1, g_3) \) belongs to \( G(a, b, c_i/P) \), as desired.

As \( N \) was chosen to be an elementary substructure, we deduce that for each \( i \leq m \)

\[
\mathcal{M} \models \forall g_1 \forall g_2 \forall g_3 ((g_1, g_2, g_3) \in G(a, b, c_i/P) \rightarrow (g_2, g_1, g_3) \in G(a, b, c_i/P)).
\]

Given \( (g_1, g_2, g_3) \) in \( G(a, b, c_i/P) \cap N \) there is an automorphism \( \sigma \) in \( \text{Aut}(\mathcal{M}/P) \) with \( \sigma(a) = g_1 \ast a, \sigma(b) = g_2 \ast b \) and \( \sigma(c) = g_3 \ast c \). Set \( \rho(c) = h \ast c \).

Now, the sort \( S \) is not \( P \)-internal, so \( a \) is not definable over \( b, P \). Hence, there is an automorphism \( \rho \) in \( \text{Aut}(\mathcal{M}/P) \) fixing \( b \) with \( \rho(a) = g \ast a \neq a \). Set \( \rho(c) = h \ast c \). By the above, both tuples \( (g, 1_G, h) \) and \( (1_G, g, h) \) belong to each \( G(a, b, c_i/P) \), and hence so does \( (g, g^{-1}, 1_G) \). Therefore, there exists an automorphism \( \rho_1 \) in \( \text{Aut}(\mathcal{M}/P) \) with \( \rho_1(c) = c \) and

\[
\rho_1(a) = g \ast a.
\]

The element \( g \ast a \) does not lie in the fiber of \( \pi(b) \) and differs from \( a \), so \( \rho_1(a) = g \ast a \) is not contained in the set \( \{a, b\} \), which gives the desired contradiction. \( \square \) Claim.

In order to get the final contradiction, we will now use the description of the group \( G(a, b, c_j/P) \) as a definable subgroup of \( (\mathbb{C}^3, +) \). Every such subgroup is given by a system of linear equations

\[
\varepsilon_1 x_1 + \varepsilon_2 x_2 + \varepsilon_3 x_3 = 0
\]

for some complex numbers \( \varepsilon_i \). We will show that the coefficients in each one of these linear equations can be taken in \( \mathbb{Q}^{alg} \), which gives the desired contradiction. Choose hence an arbitrary non-trivial linear equation \( \varepsilon_1 x_1 + \varepsilon_2 x_2 + \varepsilon_3 x_3 = 0 \) which occurs in the above system defining \( G(a, b, c_j/P) \). Note that \( \varepsilon_3 \) cannot be zero, for otherwise (up to permutation of \( a \) and \( b \)) every automorphism fixing \( a \) and \( P \) would fix \( b \), contradicting that the cover \( \mathcal{M}_n \) is non-degenerated. Therefore, we may normalize and assume that \( \varepsilon_3 = 1 \).

Set now \( \alpha = \pi(a), \beta = \pi(b) \) and \( \gamma = \pi(c_j) \). Choose derivations \( D_1 \) and \( D_2 \) with \( D_1(\alpha) = 0 \neq D_2(\beta) \) and the kernel of \( D_2 \) is exactly \( \mathbb{Q}^{alg} \). As noted before Proposition 4.4, both additive maps \( D_1 \) and \( D_2 \) are contained in the image of the canonical embedding

\[
\text{Aut}(\mathcal{M}_n/P) \rightarrow \{F : \mathbb{C} \rightarrow \mathbb{C} \text{ additive}\},
\]

since \( n > 1 \), so there are automorphisms \( \sigma_1 \) and \( \tau \) in \( \text{Aut}(\mathcal{M}_n/P) \) with

\[
\sigma_1(a) = D_1(\alpha) \ast a = a \quad \sigma_1(b) = D_1(\beta) \ast b \neq b \quad \text{and} \quad \sigma_1(c_j) = D_1(\gamma) \ast c
\]
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\[ \tau(a) = D_2(D_1(\alpha)) \star a = a \quad \tau(b) = D_2(D_1(\beta)) \star b \quad \text{and} \quad \tau(c) = D_2(D_1(\gamma)) \star c. \]

The above description of the group \( G(a, b, c; \mathbb{P}) \) yields

\[ \varepsilon_1 D_1(\alpha) + \varepsilon_2 D_1(\beta) + D_1(\gamma) = \varepsilon_2 D_1(\beta) + D_1(\gamma) = 0. \quad (\spadesuit) \]

Similarly,

\[ \varepsilon_1 D_2(D_1(\alpha)) + \varepsilon_2 D_2(D_1(\beta)) + D_2(D_1(\gamma)) = \varepsilon_2 D_2(D_1(\beta)) + D_2(D_1(\gamma)) = 0, \]

Differentiating \((\spadesuit)\) with respect to \( D_2 \), we conclude from the above equation that

\[ 0 = D_2(\varepsilon_2) D_1(\beta), \]

so \( D_2(\varepsilon_2) = 0 \) and thus \( \varepsilon_2 \) lies in \( \mathbb{Q}_{\text{alg}} \). Replacing now \( D_1 \) by a derivation \( \tilde{D}_1 \) with \( \tilde{D}_1(\beta) = 0 \neq \tilde{D}_1(\alpha) \) we conclude that \( \varepsilon_1 \) is an algebraic number as well. \qed

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