On $K$-reflections of Scott spaces

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Abstract
In this paper, for a full subcategory $K$ of the category of all $T_0$ spaces with continuous mappings, we investigate the questions under what conditions the $K$-reflection of a Scott space is still a Scott space and under what conditions the Scott $K$-completion of a poset exists. Some necessary and sufficient conditions for the $K$-reflection of a Scott space to be a Scott space and for the existence of Scott $K$-completion of a poset are established, respectively. It is shown that neither the sobrification nor the well-filtered reflection of the Johnstone space is a Scott space. The $K$-reflections of Alexandroff spaces and the $K$-completions of posets are also discussed.

Keywords: $K$-reflection; Scott $K$-completion; $K$-completion; Scott space; Alexandroff space; Sober space; Well-filtered space; $d$-space

1. Introduction
In domain theory and non-Hausdorff topology, the Scott topology on posets is the most important topology, and the sober spaces, well-filtered spaces and $d$-spaces form three of the most important classes (see [1, 4, 5]). The reflectivities of these spaces in $T_0$ spaces have attracted considerable attention (see [2, 3, 4, 5, 10, 12, 17, 18, 20, 21, 23]).

Let $\text{Top}_0$ be the category of all $T_0$ spaces with continuous mappings and $\text{Sob}$ the full subcategory of $\text{Top}_0$ containing all sober spaces. Denote the category of all $d$-spaces with continuous mappings and that of all well-filtered spaces respectively by $\text{Top}_d$ and $\text{Top}_w$. It is well-known that $\text{Sob}$ is reflective in $\text{Top}_0$ (see [4, 5]). Using $d$-closures, Wyler [18] proved that $\text{Top}_d$ is reflective in $\text{Top}_0$ (see also [2, 3]). In [10], Keimel and Lawson proved that for a full subcategory $K$ of $\text{Top}_0$ containing $\text{Sob}$, if $K$ has certain properties, then $K$ is reflective in $\text{Top}_0$. They showed that $\text{Top}_d$ and some other categories have such properties.

For quite a long time, it was not known whether $\text{Top}_w$ is reflective in $\text{Top}_0$. Recently, this problem has been positively answered by three different methods (see [14, 16, 21, 12]). More generally, for an adequate and full subcategory $K$ of $\text{Top}_0$ containing $\text{Sob}$, a direct and uniform approach to the $K$-reflections of $T_0$ spaces was provided in [20]. It was shown in [20] that $\text{Sob}$, $\text{Top}_d$, $\text{Top}_w$ and Keimel-Lawson categories are all adequate. Therefore, they are all reflective in $\text{Top}_0$.

Directed complete posets (dcpo’s for short) play a fundamental role in domain theory. In [23], using $d$-closures, Zhao and Fan showed that for any poset $P$, the DCPO-completion of $P$ exists. As Keimel and Lawson pointed out in [11], that the DCPO-completion of a poset $P$ is essentially the $d$-reflection of Scott space $\Sigma P$ or, equivalently, the $d$-reflection of Scott space of a poset $P$ is still a Scott space (see also [21, Proposition 5.12]).

For a full subcategory $K$ of $\text{Top}_0$ containing $\text{Sob}$, a natural question arises:

**Question 1.** When is the $K$-reflection of a Scott space still a Scott space?
Let \( P \) be a poset. We call \( P \) a \( K \)-dcpo if the Scott space \( \Sigma P \) is a \( K \)-space. A Scott \( K \)-completion, \( K \)-completion for short, of a poset \( P \) is a pair \( \langle \hat{P}, \eta \rangle \) consisting of a \( K \)-dcpo \( \hat{P} \) and a Scott continuous mapping \( \eta : P \to \hat{P} \), such that for any Scott continuous mapping \( f : P \to Q \) to a \( K \)-dcpo \( Q \), there exists a unique Scott continuous mapping \( \hat{f} : \hat{P} \to Q \) such that \( \hat{f} \circ \eta = f \). For \( K = \text{Top}_w \), the \( K \)-completion is simply called the \( WF \)-completion.

The ordered form of Question 1 is the following:

**Question 2.** When does the \( K \)-completion of a poset exist? In particular, when does the \( \text{Sob}_x \)-completion of a poset exist? When does the \( WF \)-completion of a poset exist?

This paper is mainly devoted to investigating the above two questions. Some necessary and sufficient conditions for the \( K \)-reflection of a Scott space to be still a Scott space and for the existence of \( K \)-completion of a poset are established, respectively. A few related examples and counterexamples are presented. It is shown that for a full subcategory \( K \) of \( \text{Top}_w \) containing \( \text{Sob} \) which is adequate and closed with respect to homeomorphisms, the \( K \)-reflection of the Johnstone space is not a Scott space. In particular, neither the sobrification nor the well-filtered reflection of the Johnstone space is a Scott space. In the final section, the \( K \)-reflections of Alexandroff spaces and the \( K \)-completions of posets are discussed.

2. Preliminary

In this section, we briefly recall some fundamental concepts and basic results that will be used in the paper. For further details, we refer the reader to [4, 5, 20].

For a set \( X \), \(|X|\) will denote the cardinality of \( X \). Let \( \mathbb{N} \) denote the set of all natural numbers with the usual order and \( \omega = |\mathbb{N}| \). The set of all subsets of \( X \) is denoted by \( 2^X \). Let \( X^{(\leq \omega)} = \{ F \subseteq X : F \) is a finite set \} and \( X^{(\leq \omega)} = \{ F \subseteq X : F \) is a countable set \}.

For a poset \( P \) and \( A \subseteq P \), let \( \downarrow A = \{ x \in P : x \leq a \) for some \( a \in A \) \} and \( \uparrow A = \{ x \in P : x \geq a \) for some \( a \in A \} \). For \( x \in P \), we write \( \downarrow x \) for \( \downarrow \{x\} \) and \( \uparrow x \) for \( \uparrow \{x\} \). A subset \( A \) is called a lower set (resp., an upper set) if \( A = \downarrow A \) (resp., \( A = \uparrow A \)). Let \( \text{Fin}P = \{ \uparrow F : F \in P^{(\leq \omega)} \} \) and \( \text{max}(P) = \{ m \in P : m \) is a maximal point of \( P \} \). In the following, when \( \text{Fin}P \) is considered as a poset, the order \( \leq \) on \( \text{Fin}P \) always means the reverse inclusion order \( \supseteq \), that is, for \( \uparrow F_1, \uparrow F_2 \in \text{Fin}P \), \( \uparrow F_1 \leq \uparrow F_2 \) if and only if \( \uparrow F_1 \supseteq \uparrow F_2 \).

A nonempty subset \( D \) of a poset \( P \) is called directed if every two elements in \( D \) have an upper bound in \( D \). The set of all directed sets of \( P \) is denoted by \( \text{D}(P) \). A subset \( I \subseteq P \) is called an ideal of \( P \) if \( I \) is a directed lower subset of \( P \). The poset of all ideals (with the order of set inclusion) of \( P \) is denoted by \( \text{Id}(P) \). \( P \) is called a directed complete poset, or dcpo for short, provided that \( \forall D \) exists in \( P \) for each \( D \in \text{D}(P) \). For \( x, y \in P \), we say \( x \) is way below \( y \), written \( x \ll y \), if for each \( D \in \text{D}(P) \) for which \( D \) exists, \( y \leq \forall D \) implies \( x \leq d \) for some \( d \in D \). Let \( \downarrow x = \{ u \in P : u \ll x \} \). An element \( k \in P \) is called compact if \( k \ll k \). The subset of all compact elements of \( P \) is denoted by \( K(P) \). \( P \) is called a continuous domain, if for each \( x \in P \), \( \downarrow x \) is directed and \( x = \forall \downarrow x \). A continuous domain which is a complete lattice is called a continuous lattice. \( P \) is called an algebraic domain, if for each \( x \in P \), \( \{ k \in K(P) : k \leq x \} \) is directed and \( x = \lor \{ k \in K(P) : k \leq x \} \). An algebraic domain which is a complete lattice is called an algebraic lattice. It is easy to verify that every algebraic domain is a continuous domain (see, for example, [3, Proposition 4.3]).

As in [4], the upper topology on a poset \( P \), generated by the complements of the principal ideals of \( P \), is denoted by \( \nu(P) \). The upper sets of \( P \) form the (upper) Alexandroff topology \( \gamma(P) \). The space \( \Gamma P = (P, \gamma(P)) \) is called the Alexandroff space of \( P \). A subset \( U \) of \( P \) is Scott open if (i) \( U = \uparrow U \), and (ii) for any directed subset \( D \) for which \( \forall D \) exists, \( \forall D \subseteq U \) implies \( D \cap U \neq \emptyset \). All Scott open subsets of \( P \) form a topology. This topology is called the Scott topology on \( P \) and denoted by \( \sigma(P) \). The space \( \Sigma P = (P, \sigma(P)) \) is called the Scott space of \( P \). For the chain \( 2 = \{0, 1\} \) (with the order \( 0 < 1 \)), we have \( \sigma(2) = \{0, \{1\}, \{0, 1\} \} \). The space \( \Sigma 2 \) is well-known under the name of Sierpinski space.

**Lemma 2.1.** ([3, Proposition II-2.1]) Let \( P, Q \) be posets and \( f : P \to Q \). Then the following two conditions are equivalent:

1. \( f \) is Scott continuous, that is, \( f : \Sigma P \to \Sigma Q \) is continuous.
(2) For any \( D \in \mathcal{D}(P) \) for which \( \vee D \) exists, \( f(\vee D) = \vee f(D) \).

For two spaces \( X \) and \( Y \), we use the symbol \( X \cong Y \) to represent that \( X \) and \( Y \) are homeomorphic. Similarly, for two posets \( P \) and \( Q \), the symbol \( P \cong Q \) represents that \( P \) and \( Q \) are isomorphic. Let \( \mathcal{O}(X) \) (resp., \( \mathcal{C}(X) \)) be the set of all open subsets (resp., closed subsets) of \( X \). For \( A \subseteq X \), the closure of \( A \) in \( X \) is denoted by \( \overline{A} \) or simply by \( \overline{A} \) if there is no confusion. If \( X \) is a \( T_0 \) space, we use \( \leq_X \) to denote the specialization order on \( X \): \( x \leq_X y \) iff \( x \in \overline{\{y\}} \). The poset \( X \) with the specialization order is denoted by \( \Omega X \) or simply by \( X \) if there is no confusion. The set \( \mathcal{D}(\Omega X) \) is short denoted by \( \mathcal{D}(X) \). Define \( \mathcal{S}_c(X) = \{ \{x\} : x \in X \} \) and \( \mathcal{D}_c(X) = \{ \overline{D} : D \in \mathcal{D}(X) \} \).

In the following, when a \( T_0 \) space \( X \) is considered as a poset, the partial order always means the specialization order unless otherwise indicated. A subset \( A \) of \( X \) is called saturated if \( A \) equals the intersection of all open sets containing it (equivalently, \( A \) is an upper set in the specialization order).

Lemma 2.2. ([14, Lemma 6.2]) Let \( f : X \rightarrow Y \) be a continuous mapping of \( T_0 \) spaces. If \( D \in \mathcal{D}(X) \) has a supremum to which it converges, then \( f(D) \) is directed and has a supremum in \( Y \) to which it converges, and \( f(\vee D) = \vee f(D) \).

The category of \( T_0 \) spaces and continuous mappings is denoted by \( \text{Top}_0 \). For a full subcategory \( K \) of \( \text{Top}_0 \), the objects of \( K \) will be called \( K \)-spaces. In [10], Keimel and Lawson proposed the following properties:

(K1) Homeomorphic copies of \( K \)-spaces are \( K \)-spaces.

(K2) All sober \( K \)-spaces are \( K \)-spaces or, equivalently, \( \text{Sob} \subseteq K \).

(K3) In a sober space \( S \), the intersection of any family of \( K \)-subspaces is a \( K \)-space.

(K4) Continuous maps \( f : S \rightarrow T \) between sober spaces \( S \) and \( T \) are \( K \)-continuous, that is, for every \( K \)-subspace \( K \) of \( T \), the inverse image \( f^{-1}(K) \) is a \( K \)-subspace of \( S \).

\( K \) is said to be closed with respect to homeomorphisms if \( K \) has (K1). Clearly, \( \text{Sob} \), \( \text{Top}_0 \), and \( \text{Top}_w \) all are closed with respect to homeomorphisms. We call \( K \) a Keimel-Lawson category if it satisfies (K1)-(K4).

In what follows, \( K \) always refers to a full subcategory \( \text{Top}_0 \) containing \( \text{Sob} \) which is closed with respect to homeomorphisms.

Let \( X \) be a \( T_0 \) space. A \( K \)-reflection of \( X \) is a pair \( \langle \tilde{X}, \eta_X \rangle \) consisting of a \( K \)-space \( \tilde{X} \) and a continuous mapping \( \eta_X : X \rightarrow \tilde{X} \) satisfying that for any continuous mapping \( f : X \rightarrow Y \) to a \( K \)-space, there exists a unique continuous mapping \( f^* : \tilde{X} \rightarrow Y \) such that \( f^* \circ \eta_X = f \), that is, the following diagram commutes.

\[
\begin{array}{ccc}
X & \xrightarrow{\eta_X} & \tilde{X} \\
\downarrow f & & \downarrow f^* \\
Y & & \\
\end{array}
\]

\( \text{Sob} \)-reflections and \( \text{Top}_w \)-reflections are exactly sobrifications and well-filtered reflections respectively. \( \text{Top}_d \)-reflections are simply called \( d \)-reflections.

By a standard argument, \( K \)-reflections, if they exist, are unique up to homeomorphism. We shall use \( X^k \) to denote the space of the \( K \)-reflection of \( X \) if it exists.

Definition 2.3. ([20, Definition 3.2]) A subset \( A \) of a \( T_0 \) space \( X \) is called a \( K \)-set, provided for any continuous mapping \( f : X \rightarrow Y \) to a \( K \)-space \( Y \), there exists a unique \( y_A \in Y \) such that \( \overline{\{y_A\}} = f(\overline{A}) \). Denote by \( K(X) \) the set of all closed \( K \)-sets of \( X \).

Obviously, a subset \( A \) of a space \( X \) is a \( K \)-set iff \( \overline{A} \) is a \( K \)-set. For simplicity, let \( d(X) = \text{Top}_d(X) \) and \( \text{WF}(X) = \text{Top}_w(X) \).

Lemma 2.4. ([24, Lemma 3.3, Corollary 3.4 and Proposition 3.8]) Let \( X \) be a \( T_0 \) space. Then

1. \( \text{Sob}(X) = \text{Irr}_c(X) \).
(2) \( S_c(X) \subseteq K(X) \subseteq \operatorname{lr}_c(X) \).
(3) \( S_c(X) \subseteq T_c(X) \subseteq d(X) \subseteq \text{WF}(X) \subseteq \operatorname{lr}_c(X) \).

**Lemma 2.5.** ([24, Lemma 3.11]) Let \( X, Y \) be two \( T_0 \) spaces. If \( f : X \rightarrow Y \) is a continuous mapping and \( A \in K(X) \), then \( f(A) \in K(Y) \).

**Lemma 2.6.** Let \( X \) be a \( T_0 \) space and \( A \) a closed subspace of \( X \). Then \( K(A) \subseteq K(X) \).

**Proof.** For \( B \in K(A) \), we need to show that \( B \in K(X) \). Suppose that \( f : X \rightarrow Y \) is a continuous mapping from \( X \) to a \( K \)-space \( Y \). Then \( f_A : A \rightarrow Y, f_A(a) = f(a) \), is continuous. By \( B \in K(A) \), there exists a unique \( y_B \in Y \) such that \( \text{cl}_X f_a(B) = \{y_A\} \). Since \( A \in C(X) \), we have \( \text{cl}_X f(B) = \text{cl}_X f_a(B) = \{y_A\} \). Thus \( B \in K(X) \). □

3. \( d \)-spaces, well-filtered spaces, sober spaces and \( K \)-spaces

In this section, we give some known and new results about sober spaces, well-filtered spaces, \( d \)-spaces and \( K \)-spaces that will be used in the other sections.

A \( T_0 \) space \( X \) is called a \( d \)-space (or monotone convergence space) if \( X \) (with the specialization order) is a dcpo and \( \mathcal{O}(X) \subseteq \sigma(X) \) (cf. [4, 18]). Let \( \text{Top}_d \) be the full subcategory of \( \text{Top}_0 \) containing all \( d \)-spaces.

**Lemma 3.1.** Let \( P \) be a poset, \( Y \) a \( T_0 \) space and \( f : P \rightarrow Y \). Consider the following two conditions:

(1) \( f : \Sigma P \rightarrow Y \) is continuous.
(2) \( f : \Sigma P \rightarrow \Sigma Y \) is continuous.

Then \((1) \Rightarrow (2) \). Moreover, if \( Y \) is a \( d \)-space, then two conditions are equivalent.

**Proof.** (1) \( \Rightarrow \) (2): By Lemma 2.1 and Lemma 2.2

(2) \( \Rightarrow \) (1): Suppose that \( Y \) is a \( d \)-space. Then \( Y \) is a dcpo (with the specialization order) and \( \mathcal{O}(Y) \subseteq \sigma(Y) \). For each \( U \in \mathcal{O}(Y) \), since \( f : \Sigma P \rightarrow \Sigma Y \) is continuous, we have \( f^{-1}(U) \in \mathcal{O}(X) \). Thus \( f : \Sigma P \rightarrow Y \) is continuous. □

A nonempty subset \( A \) of a \( T_0 \) space \( X \) is said to be irreducible if for any \( \{F_1, F_2\} \subseteq C(X) \), \( A \subseteq F_1 \cup F_2 \) always implies \( A \subseteq F_1 \) or \( A \subseteq F_2 \). Denote by \( \text{lr}_c(X) \) (resp., \( \text{lr}_e(X) \)) the set of all irreducible (resp., irreducible closed) subsets of \( X \). The space \( X \) is called sober, if for any \( A \in \text{lr}_e(X) \), there is a unique point \( x \in X \) such that \( A = \{x\} \). Let \( \text{Sob} \) be the full subcategory of \( \text{Top}_0 \) containing all sober spaces.

The following result is well-known (cf. [4, Corollary II-1.12]).

**Proposition 3.2.** For a continuous domain \( P \), \( \Sigma P \) is sober.

For a \( T_0 \) space \( X \), we shall use \( K(X) \) to denote the set of all nonempty compact saturated subsets of \( X \) and endow it with the Smyth order, that is, for \( K_1, K_2 \in K(X) \), \( K_1 \sqsubseteq K_2 \) iff \( K_2 \sqsubseteq K_1 \). The space \( X \) is called well-filtered if for any filtered family \( K \subseteq K(X) \) and any open set \( U \), \( \bigcap K \subseteq U \) implies \( K \subseteq U \) for some \( K \in K \). Let \( \text{Top}_{wf} \) be the full subcategory of \( \text{Top}_0 \) containing all well-filtered spaces.

We have the following implications (which can not be reversed):

sobriety \( \Rightarrow \) well-filteredness \( \Rightarrow \) \( d \)-space.

In [19, Corollary 3.2] Xi and Lawson gave the following useful result.

**Proposition 3.3.** For a complete lattice \( L \), \( \Sigma L \) is well-filtered.

It is well-known that the Johnstone space \( \Sigma \mathbb{U} \) is a \( d \)-space but not well-filtered (see the proof of Proposition 4.1.1 in Section 5). For the complete lattice \( L \) constructed by Isbell in [8], it is well-known that the Isbell space \( \Sigma L \) is not sober. By Proposition 3.3 \( \Sigma L \) is well-filtered.

For any topological space \( X, \mathcal{G} \subseteq 2^X \) and \( A \subseteq X \), let \( \Diamond \mathcal{G} A = \{G \in \mathcal{G} : G \cap A \neq \emptyset\} \) and \( \Box \mathcal{G} A = \{G \in \mathcal{G} : G \subseteq A\} \). The symbols \( \Diamond \mathcal{G} A \) and \( \Box \mathcal{G} A \) will be simply written as \( \Diamond A \) and \( \Box A \) respectively if there is no
confusion. The lower Vietoris topology on $\mathcal{G}$ is the topology that has $\{\emptyset U : U \in \mathcal{O}(X)\}$ as a subbase, and the resulting space is denoted by $P_H(\mathcal{G})$. If $\mathcal{G} \subseteq \Ir(X)$, then $\{\emptyset U : U \in \mathcal{O}(X)\}$ is a topology on $\mathcal{G}$. The space $P_H(\mathcal{C}(X) \setminus \{\emptyset\})$ is called the Hoare power space or lower space of $X$ and is denoted by $P_H(X)$ for short (cf. [15]). Clearly, $P_H(X) = (\mathcal{C}(X) \setminus \{\emptyset\}, \nu(\mathcal{C}(X) \setminus \{\emptyset\}))$ and hence it is always sober (see [21, Proposition 2.9]).

Remark 3.4. Let $X$ be a $T_0$ space.

(1) If $\mathcal{S}_c(X) \subseteq \mathcal{G}$, then the specialization order on $P_H(\mathcal{G})$ is the order of set inclusion, and the canonical mapping $\eta_X : X \rightarrow P_H(\mathcal{G})$, given by $\eta_X(x) = \{x\}$, is an order and topological embedding (cf. [4, 5, 15]).

(2) The space $X^* = P_H(\Ir_c(X))$ with the canonical mapping $\eta_X : X \rightarrow X^*$ is the sobrification of $X$ (cf. [4, 5]).

A full subcategory $\mathcal{K}$ of $\text{Top}_d$ is said to be adequate if for any $T_0$ space $X$, $P_H(\mathcal{K}(X))$ is a $\mathcal{K}$-space. When $\mathcal{K}$ is adequate, we have the following characterization of $\mathcal{K}$-spaces by $\mathcal{K}$-sets.

Lemma 3.5. ([20, Corollary 4.10]) Let $\mathcal{K}$ be a full subcategory of $\text{Top}_d$ containing $\text{Sob}$ and $X$ a $T_0$ space. Suppose that $\mathcal{K}$ is adequate and closed with respect to homeomorphisms. Then the following two conditions are equivalent:

1. $X$ is a $\mathcal{K}$-space.
2. $\mathcal{K}(X) = \mathcal{S}_c(X)$.

Lemma 3.6. ([20, Proposition 5.1, Theorem 5.4, Theorem 5.14 and Theorem 5.17]) $\mathcal{Sob}$, $\text{Top}_d$ and $\text{Top}_w$ all are adequate. Moreover, every Keimel-Lawson category $\mathcal{K}$ is adequate. Therefore, they all are reflective in $\text{Top}_d$.

By Lemma 3.4, Lemma 3.5 and Lemma 3.6 we get the following two corollaries.

Corollary 3.7. For a $T_0$ space $X$, the following conditions are equivalent:

1. $X$ is a $d$-space.
2. $\mathcal{D}_c(X) = \mathcal{S}_c(X)$.
3. $\mathcal{D}(X) = \mathcal{S}_c(X)$.

Corollary 3.8. For a $T_0$ space $X$, the following two conditions are equivalent:

1. $X$ is well-filtered.
2. $\mathcal{W}_F(X) = \mathcal{S}_c(X)$.

Now we show that if $\mathcal{K}$ is adequate and closed with respect to homeomorphisms, then $\mathcal{K}$ has equalizers and the property of being a $\mathcal{K}$-space is closed-hereditary and saturated-hereditary.

Proposition 3.9. Suppose that $\mathcal{K}$ is adequate and closed with respect to homeomorphisms. Let $X$ be a $\mathcal{K}$-space and $Y$ a $T_0$ space. Then for any pair of continuous mappings $f, g : X \rightarrow Y$, the equalizer $E(f, g) = \{x \in X : f(x) = g(x)\}$ (as a subspace of $X$) is a $\mathcal{K}$-space.

Proof. For $B \in \mathcal{K}(E(f, g))$, we show that $B \in \mathcal{K}(\mathcal{S}_c(E(f, g)))$. By Lemma 3.5, $\text{cl}_XB \in \mathcal{K}(X)$. As $X$ is a $\mathcal{K}$-space, by Lemma 3.3 there is $x \in X$ such that $\text{cl}_XB = \mathcal{C}(\{x\})$. By the continuity of $f$ and $g$, we have that $\{f(x)\} = f(\text{cl}_XB) = f(\mathcal{C}(\{x\})) = g(\text{cl}_XB) = g(\mathcal{C}(\{x\})) = \{g(x)\}$, and consequently, $x \in E(f, g)$ since $Y$ is a $T_0$ space. It follows that $B = (\text{cl}_XB) \cap E(f, g) = (\mathcal{C}(\{x\})) \cap E(f, g) = \mathcal{C}(\{g(x)\})$, and hence $B \in \mathcal{K}(\mathcal{S}_c(E(f, g)))$. By Lemma 3.5, $E(f, g)$ is a $\mathcal{K}$-space.

By Lemma 3.6 and Proposition 3.9 we obtain the following two results.

Corollary 3.10. Suppose that $\mathcal{K} \in \{\text{Sob}, \text{Top}_d, \text{Top}_w\}$ or $\mathcal{K}$ is a Keimel-Lawson category. Then for any pair of continuous mappings $f, g : X \rightarrow Y$, the equalizer $E(f, g) = \{x \in X : f(x) = g(x)\}$ (as a subspace of $X$) is a $\mathcal{K}$-space.
**Proposition 3.11.** Let $K$ be a full subcategory of $\text{Top}_d$ containing $\text{Sob}$ and $X$ a $T_0$ space. Suppose that $K$ is adequate and closed with respect to homeomorphisms. Then the property of being a $K$-space is closed-hereditary and saturated-hereditary.

**Proof.** Suppose $X$ is a $K$-space.

Case 1: $A$ is a closed subspace of $X$.

By Lemma 2.4, Lemma 2.6 and Lemma 3.5, we have that $S_c(A) \subseteq K(A) \subseteq K(X) = S_c(X)$, whence $K(A) = S_c(A)$. By Lemma 3.5, $A$ is a $K$-space.

Case 2: $U$ is a saturated subspace of $X$.

For $B \in K(U)$, we show that $\text{cl}_X B \in K(X)$. Suppose that $f : X \to Y$ is a continuous mapping from $X$ to a $K$-space $Y$. Then $f_U : U \to Y$, $f_U(u) = f(u)$, is continuous. 

By $B \in K(U)$, there exists a unique $y_U \in Y$ such that $f_U(B) = \{y_U\}$, that is, $f(\text{cl}_X B) = f(B) = \{y_U\}$. Thus $\text{cl}_X B \in K(X)$. Since $X$ is a $K$-space, by Lemma 3.5, there is $x \in X$ with $\text{cl}_X B = \text{cl}_X \{x\}$. As $B \subseteq U$ and $U = \uparrow_X U$, we get $x \in U$ and hence $\text{cl}_U B = U \cap \text{cl}_X B = U \cap \text{cl}_X \{x\} = \text{cl}_U \{x\}$. By Lemma 3.5 again, $U$ is a $K$-space.

Therefore, the property of being a $K$-space is closed-hereditary and saturated-hereditary. □

By Lemma 3.6 and Proposition 3.11 we have the following result.

**Corollary 3.12.** Suppose that $K \in \{\text{Sob}, \text{Top}_d, \text{Top}_w\}$ or $K$ is a Keimel-Lawson category. Then the property of being a $K$-space is closed-hereditary and saturated-hereditary.

**Definition 3.13.** For a poset $P$, let $P_\top = P \cup \{\top\}$ ($\top \not\in P$) denote the poset obtained from $P$ by adjoining a largest element $\top$ (whether $P$ has one or not).

Clearly, the order on $P_\top$ is as follows: $x \leq y$ if $x \leq y$ in $P$ or $y = \top$. The element $\top$ is the largest element of $P_\top$ (even if $P$ has a largest element).

It is straightforward to verify the following result.

**Lemma 3.14.** Let $P$ be a poset. Then

1. A poset $P$ is a dcpo iff $P_\top$ is a dcpo. If $P$ is a dcpo, then in $P_\top$ we have $\top \ll \top$, i.e., $\top \in \sigma(P_\top)$.
2. $\zeta : \Sigma P \to \Sigma P_\top$, $x \mapsto x$, is continuous.

**Definition 3.15.** For a topological space $X$, select a point $\top$ such that $\top \not\in X$. Then $C(X) \cup \{X \cup \{\top\}\}$ (as the set of all closed sets) is a topology on $X \cup \{\top\}$. The resulting space is denoted by $X_\top$.

**Remark 3.16.** For a topological space $X$ and a dcpo $P$, we have

(a) $\{\top\}$ is an open set in $X_\top$ and hence $X$ is a closed subspace of $X_\top$.
(b) $\text{cl}_{X_\top} \{\top\} = X_\top$.
(c) $X$ is $T_0$ if $X_\top$ is $T_0$.
(d) $(\Sigma P)_\top = \Sigma P_\top$.

**Proposition 3.17.** For a topological space $X$, $X$ is sober if and only if $X_\top$ is sober.

**Proof.** Clearly, If $X_\top$ is sober, then $X$, as a closed subspace of $X_\top$, is also sober since sobriety is closed-hereditary (see Exercise 0-5.16 or Corollary 3.12 below).

Conversely, if $X$ is sober, then $\text{Irr}_c(X) = \{\text{cl}_X \{x\} : x \in X\}$. Since $C(X_\top) = C(X) \cup \{X_\top\}$ and $X$ is a closed subspace of $X_\top$, we have $\text{Irr}_c(X_\top) = \text{Irr}_c(X) \cup \{X_\top\} = \{\text{cl}_{X_\top} \{x\} : x \in X_\top\}$. Thus $X_\top$ is sober. □

Similarly, we have the following result.

**Proposition 3.18.** For a topological space $X$, $X$ is a well-filtered space (resp., d-space) if and only if $X_\top$ is a well-filtered space (resp., d-space).
Proof. If $X_\tau$ is a well-filtered space (resp., $d$-space), then by Corollary 3.12 below, as a closed subspace of $X_\tau$, $X$ is a well-filtered space (resp., $d$-space).

Conversely, assume that $X$ is a $d$-space. Since $\text{cl}_X \{ \top \} = X_\tau$, $\top$ is the largest element of $X_\tau$ with the specialization order. For $D \in \mathcal{D}(X_\tau)$, if $\top \notin D$, then $D \in \mathcal{D}(X)$. As $X$ is a $d$-space, by Lemma 3.7 there is $x \in X$ such that $\text{cl}_X D = \text{cl}_X \{ x \}$ and hence $\text{cl}_X D = \text{cl}_X \{ x \}$ since $X \in \mathcal{C}(X_\tau)$. If $\top \in D$, then $\text{cl}_X D = \text{cl}_X \{ \top \}$. Thus $X_\tau$ is a $d$-space. Now we assume that $X$ is a well-filtered space. Let $\{ K_i : i \in I \} \subseteq K(X_\tau)$ be a filtered family and $U \in \mathcal{O}(X_\tau)$ such that $\bigcap_{i \in I} K_i \subseteq U$. Note that $\top$ is the largest element in $X$ with respect to the specialization order, so $\top \in \bigcap_{i \in I} K_i \subseteq U$. Let $V = U \setminus \{ \top \} = X \setminus (X_\tau \setminus U)$. Then $V \in \mathcal{O}(X)$ and $U = V \cup \{ \top \}$. For each $i \in I$, let $K_i^* = K_i \setminus \{ \top \}$. One can easily check that $\{ K_i^* : i \in I \} \subseteq K(X)$ is a filtered family and $\bigcap_{i \in I} K_i^* \subseteq V$. Since $X$ is well-filtered, there exists $i_0 \in I$ such that $K_{i_0} \subseteq V$, which implies that $K_{i_0} \subseteq U$. Thus $X_\tau$ is well-filtered. 

By Remark 3.14, Proposition 3.17 and Proposition 3.18, we have the following corollary.

Corollary 3.19. For a dcpo $P$, we have the following conclusions:

1. $\Sigma P$ is a $d$-space iff $\Sigma P_\tau$ is a $d$-space.
2. $\Sigma P$ is well-filtered iff $\Sigma P_\tau$ is well-filtered.
3. $\Sigma P$ is sober iff $\Sigma P_\tau$ is sober.

4. K-reflections of Scott spaces

In this section, we will give some necessary and sufficient conditions for the K-reflection of a $T_0$ space (especially, a Scott space) to be a Scott space. A few related examples and counterexamples are presented.

Lemma 4.1. (24, Lemma 4.3) For a $T_0$ space $X$ and $A \subseteq X$, $\eta_X(A) = \eta_X(\text{cl}_X A) = \overline{\text{K}(X)} A = \text{cl}_X A$ in $X = P_H(\text{K}(X))$.

By Lemma 2.3 $\{ \diamond_{\text{K}(X)} U : U \in \mathcal{O}(X) \}$ is a topology on $\text{K}(X)$. In the following, let $\eta_X : X \to P_H(\text{K}(X))$, $\eta_X(x) = \{ x \}$, be the canonical mapping from $X$ to $P_H(\text{K}(X))$. It is easy to verify that $\eta_X$ is a topological embedding. When $X = \Sigma P$ for some poset $P$, $\eta_X$ is simply denoted by $\eta_P$.

Lemma 4.2. (24, Theorem 4.6) Let $\text{K}$ be a full subcategory of $\text{Top}_0$ and $X$ a $T_0$ space. If $P_H(\text{K}(X))$ is a $\text{K}$-space, then the pair $(X = P_H(\text{K}(X)), \eta_X)$ is a $\text{K}$-reflection of $X$. More precisely, for any continuous mapping $f : X \to Y$ to a $\text{K}$-space $Y$, there exists a unique continuous mapping $f^* : P_H(\text{K}(X)) \to Y$ such that $f^* \circ \eta_X = f$, that is, the following diagram commutes.

$$
\begin{array}{ccc}
X & \xrightarrow{\eta_X} & P_H(\text{K}(X)) \\
\downarrow{f} & & \downarrow{f^*} \\
Y & & 
\end{array}
$$

The unique continuous mapping $f^* : P_H(\text{K}(X)) \to Y$ is defined by $f^*(A) = y_A$, where $y_A$ is the unique point of $Y$ such that $f(A) = \{ y_A \}$.

Theorem 4.3. Let $\text{K}$ be a full subcategory of $\text{Top}_d$ containing $\text{Sob}$ which is adequate and closed with respect to homeomorphisms. For a $T_0$ space $X$, consider the following three conditions:

1. $\Sigma \text{K}(X)$ is a $\text{K}$-space.
2. $\eta_X^* : X \to \Sigma \text{K}(X)$, $\eta_X^*(x) = \{ x \}$, is continuous.
3. The $\text{K}$-reflection $X^k$ of $X$ is a Scott space.

Then (1) $\Rightarrow$ (2), (2) $\Rightarrow$ (3), and (3) $\Rightarrow$ (1). Moreover, when conditions (1) and (2) hold, the Scott space $\Sigma \text{K}(X)$ with the canonical mapping $\eta_X^* : X \to \Sigma \text{K}(X)$, $\eta_X^*(x) = \{ x \}$, is a $\text{K}$-reflection of $X$. 

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Proof. (1)+(2) $\Rightarrow$ (3): Suppose that $X$ satisfies conditions (1) and (2). Then by Lemma 4.2, the pair $(X^k = P_H(K(X)), \eta_X)$ is a K-reflection of $X$, and there is a unique continuous mapping $(\eta_X^*)^*: X^k \rightarrow \Sigma K(X)$ such that $(\eta_X^*)^* \circ \eta_X = \eta_X$, that is, the following diagram commutes.

$$
\begin{array}{ccc}
X & \xrightarrow{\eta_X} & X^k \\
\downarrow{\eta_X^*} & & \downarrow{\eta_X^*} \\
\Sigma K(X) & & \Sigma K(X)
\end{array}
$$

The unique continuous mapping $(\eta_X^*)^*: X^k \rightarrow \Sigma K(X)$ is defined by $(\eta_X^*)^*(A) = B_A$ (for each $A \in K(X)$), where $B_A$ is the unique element of $K(X)$ such that $cl_{\sigma(K(X))}(\eta_X^*(A)) = cl_{\sigma(K(X))}(B_A) = \downarrow_{K(X)} B_A$. It follows from Lemma 2.3 (2) that $A \subseteq B_A$. On the other hand, $X^k$ is a d-space by $K \subseteq \text{Top}_d$, and hence $K(X)$ (with the order of set inclusion) is a dcpo and $\mathcal{O}(X^k) \subseteq \sigma(K(X))$. By Lemma 4.1, we have $\downarrow_{K(X)} B_A = cl_{\sigma(K(X))}(\eta_X^*(A)) \subseteq cl_{\sigma(K(X))}(A)$, and consequently, $B_A \subseteq A$ by Lemma 2.3 (2). Whence $A = B_A$, that is, $(\eta_X^*)^*(A) = A$ for each $A \in K(X)$. By the continuity of $(\eta_X^*)^*$, we have $\sigma(K(X)) \subseteq \mathcal{O}(X^k)$ and hence $\mathcal{O}(X^k) = \sigma(K(X))$, proving that the Scott space $\Sigma K(X)$ with the canonical mapping $\eta_X^*: X \rightarrow \Sigma K(X)$ is a K-reflection $X^k$ of $X$.

(3) $\Rightarrow$ (1): By the adequateness of $K$, $(X^k = P_H(K(X)), \eta_X)$ is a K-reflection of $X$. Suppose that the K-reflection $X^k$ of $X$ is a Scott space. Then there is a poset $P$ and a continuous mapping $\xi_X: X \rightarrow \Sigma P$ such that $\Sigma P$ is a K-space and $(\Sigma P, \xi_X)$ is a K-reflection of $X$. By a standard argument, $X^k$ and $\Sigma P$ are homeomorphic, whence $K(X) (= \Omega P_H(K(X)))$ and $P (= \Omega \Sigma P)$ are isomorphic. It follows that $\Sigma K(X)$ and $\Sigma P$ are homeomorphic. Since $\Sigma P$ is a K-space and $K$ is closed with respect to homeomorphisms, $\Sigma K(X)$ is a K-space.

In particular, we have the following result for the Scott space of a poset.

**Theorem 4.4.** Let $K$ be a full subcategory of $\text{Top}_d$ containing $\text{Sob}$ which is adequate and closed with respect to homeomorphisms. Then for any poset $P$, the following two conditions are equivalent:

1. $\Sigma K(\Sigma P)$ is a K-space.
2. The K-reflection $(\Sigma P)^k$ of $\Sigma P$ is a Scott space.

Moreover, when condition (1) holds, the Scott space $\Sigma K(\Sigma P)$ with the canonical mapping $\eta_P^*: \Sigma P \rightarrow \Sigma K(\Sigma P)$, $\eta_P^*(x) = cl_{\sigma(P)}\{x\} = \downarrow x$, is a K-reflection of $X$.

**Proof.** First, we show that $\eta_P^*: \Sigma P \rightarrow \Sigma K(\Sigma P)$, $\eta_P^*(x) = cl_{\sigma(P)}\{x\} = \downarrow x$, is continuous. Since $K$ is adequate, $P_H(K(\Sigma P))$ is a K-space, and by Lemma 4.2, $(\Sigma P)^k = P_H(K(\Sigma P)), \eta_P)$ is a K-reflection of $X$, where $\eta_P: \Sigma P \rightarrow (\Sigma P)^k$ is defined by $\eta_P(x) = cl_{\sigma(P)}\{x\} = \downarrow x$ for each $x \in P$. Whence by Lemma 3.1, $\eta_P^*: \Sigma P \rightarrow \Sigma K(\Sigma P)$ is continuous.

Then by Theorem 4.4, (1) and (2) are equivalent, and when condition (1) holds, the Scott space $\Sigma K(\Sigma P)$ with the canonical mapping $\eta_P^*: \Sigma P \rightarrow \Sigma K(\Sigma P)$ is a K-reflection of $X$.

When $K = \text{Top}_d$ in Theorem 4.4, we get the following corollary.

**Corollary 4.5.** [74, Lemma 7.2] Let $X$ be a $T_0$ space. If $\eta_X^*: X \rightarrow \Sigma d(X)$, $\eta_X^*(x) = \overline{\{x\}}$, is continuous, then the d-reflection of $X$ is a Scott space. More precisely, the Scott space $\Sigma d(X)$ with the canonical mapping $\eta_X^*: X \rightarrow \Sigma d(X)$ is a d-reflection of $X$.

**Proof.** By Lemma 3.9, $P_H(d(X))$ is a d-space and hence $d(X)$ (with the order of set inclusion) is a dcpo. Therefore, $\Sigma d(X)$ is a d-space. If $\eta_X^*: X \rightarrow \Sigma d(X)$ is continuous, then by Theorem 4.3, the Scott space $\Sigma d(X)$ with the canonical mapping $\eta_X^*: X \rightarrow \Sigma d(X)$ is a d-reflection of $X$.

**Corollary 4.6.** [74, Corollary 5.9] For any poset $P$, $d(\Sigma P)$ is a dcpo and the Scott space $\Sigma d(\Sigma P)$ with the canonical mapping $\eta_P: \Sigma P \rightarrow \Sigma d(\Sigma P)$, $\eta_P(x) = cl_{\sigma(P)}\{x\}$, is a d-reflection of $\Sigma P$.
Proof. By Lemma 4.2 and Lemma 3.6, $P_U(d(\Sigma P))$ with the canonical mapping $\eta_P : \Sigma P \rightarrow P_U(d(\Sigma P))$, $x \mapsto cl_{\sigma(P)}(x)$, is a $d$-reflection of $\Sigma P$. Therefore, $d(\Sigma P)$ is a dcpo and $\eta_P : \Sigma P \rightarrow \Sigma d(X)$ is continuous by Lemma 5.1. By Corollary 4.5, the Scott space $\Sigma d(\Sigma P))$ with the canonical mapping $\eta_P : \Sigma P \rightarrow \Sigma d(\Sigma P)$ is a $d$-reflection of $\Sigma P$.

For a $T_0$ space $X$ with $\text{lr}_e(X) = \{[x] : x \in X\} \cup \{X\}$, Theorem 4.3 can be simplified as the following one.

**Theorem 4.7.** Let $K$ be a full subcategory of $\text{Top}_d$ containing $\text{Sob}$ which is adequate and closed with respect to homeomorphisms. Suppose that $X$ is a $T_0$ space for which $\text{lr}_e(X) = \{[x] : x \in X\} \cup \{X\}$ and $X$ is not a $K$-space. Consider the following three conditions:

1. $\Sigma(\Omega X)_{\top}$ is a $K$-space.
2. $\zeta^X : X \rightarrow \Sigma(\Omega X)_{\top}$, $\zeta^X(x) = x$, is continuous.
3. The $K$-reflection $X^K$ of $X$ is a Scott space.

Then (1) + (2) ⇒ (3), and (3) ⇒ (1). Moreover, when conditions (1) and (2) hold, the Scott space $\Sigma(\Omega X)_{\top}$ with the canonical mapping $\zeta^X : X \rightarrow \Sigma(\Omega X)_{\top}$, $\zeta^X(x) = x$, is a $K$-reflection of $X$.

**Proof.** (1) + (2) ⇒ (3): Since $X$ is not a $K$-space (and hence not a sober space) and $\text{lr}_e(X) = \{[x] : x \in X\} \cup \{X\}$, by Lemma 2.4 and Lemma 3.5, we have $K(X) = \text{lr}_e(X) = \{[x] : x \in X\} \cup \{X\}$ and $X \neq \{y\}$ for every $y \in X$. Define a mapping $\varphi : (\Omega X)_{\top} \rightarrow K(X)$ by

$$\varphi(u) = \begin{cases} \{u\}, & u \in X, \\ X, & u = \top. \end{cases}$$

Since $X$ is a $T_0$ space, $\varphi : (\Omega X)_{\top} \rightarrow K(X)$ is a poset isomorphism, and hence induces a homeomorphism from $\Sigma(\Omega X)_{\top}$ to $\Sigma K(X)$. It follows from condition (2) that the mapping $\eta^X_K = \varphi \circ \zeta^X : X \rightarrow \Sigma K(X)$ is continuous. Since $K$ is closed with respect to homeomorphisms, by condition (1), $\Sigma K(X)$ is a $K$-space. Therefore, by Theorem 4.3, $\Sigma K(X)$ with the canonical mapping $\eta^X_K : X \rightarrow \Sigma K(X)$ is a $K$-reflection of $X$, and hence $\Sigma(\Omega X)_{\top}$ with the continuous mapping $\zeta^X : X \rightarrow \Sigma(\Omega X)_{\top}$ is a $K$-reflection of $X$.

(3) ⇒ (1): By Theorem 4.3, $\Sigma K(X)$ is a $K$-space. It was shown in the proof of the implication (1) + (2) ⇒ (3) that $\varphi : \Sigma(\Omega X)_{\top} \rightarrow \Sigma K(X)$, defined by $\varphi(u) = \{u\}$ for $u \in X$ and $\varphi(\top) = X$, is a homeomorphism. Since $K$ is closed with respect to homeomorphisms, $\Sigma(\Omega X)_{\top}$ is a $K$-space.

By Lemma 3.14 (2) and Theorem 4.7, we get the following corollary.

**Corollary 4.8.** Let $K$ be a full subcategory of $\text{Top}_d$ containing $\text{Sob}$ which is adequate and closed with respect to homeomorphisms. Suppose that $P$ is a poset for which $\text{lr}_e(\Sigma P) = \{[x] : x \in P\} \cup \{P\}$ and $\Sigma P$ is not a $K$-space. Then the following two conditions are equivalent:

1. $\Sigma P_{\top}$ is a $K$-space.
2. The $K$-reflection $(\Sigma P)^K$ of $\Sigma P$ is a Scott space.

Moreover, when condition (1) holds, the Scott space $\Sigma P_{\top}$ with the embedding $i_P : \Sigma P \rightarrow \Sigma P_{\top}$, $i_P(x) = x$, is a $K$-reflection of $X$.

Now we give some examples and counterexamples related to the $K$-reflections of $T_0$ spaces (esp., Scott spaces).

**Example 4.9.** Let $K$ be a full subcategory of $\text{Top}_d$ containing $\text{Sob}$ which is adequate and closed with respect to homeomorphisms. Since $\mathbb{N}$ is not a dcpo, $\Sigma \mathbb{N}$ is not a $d$-space and hence not a $K$-space. Clearly, $\text{lr}_e(\Sigma \mathbb{N}) = \{[n] = \{n\} : n \in \mathbb{N}\} \cup \{[\mathbb{N}]\}$. As $\mathbb{N}_{\top}$ is an algebraic lattice, by Proposition 4.2, $\Sigma \mathbb{N}_{\top}$ is a sober space and hence a $K$-space. By Corollary 4.3, the $K$-reflection of $\Sigma \mathbb{N}$ is a Scott space. More precisely, $\Sigma \mathbb{N}_{\top}$ with the embedding $i_{\mathbb{N}} : \Sigma \mathbb{N} \rightarrow \Sigma \mathbb{N}_{\top}$, $i_{\mathbb{N}}(n) = n$, is a $K$-reflection of $\Sigma \mathbb{N}$.
Example 4.10. Let $K$ be a full subcategory of $\textbf{Top}_d$ containing $\textbf{Sob}$ which is adequate and closed with respect to homeomorphisms and $P = \mathbb{N} \cup \{a, b\}$. Define a partial order $\leq$ on $P$ as follows:

(i) $n < n + 1$ for each $n \in \mathbb{N}$,
(ii) $n < a$ and $n < b$ for all $n \in \mathbb{N}$, and
(iii) $a$ and $b$ are incomparable.

Then $\max(P) = \{a, b\}$ and $P$ is not a dcpo since the chain $\mathbb{N}$ does not have a least upper bound in $P$. So $\Sigma P$ is not a $d$-space and hence not a $K$-space. It is easy to verify that $\text{Irr}(\Sigma P) = \{\{x : x \in P\} \cup \{\mathbb{N}\}\}$. Whence by Lemma 2.3 and Lemma 3.3 $K(\Sigma P) = \text{Irr}_e(\Sigma P) = \{\{x : x \in P\} \cup \{\mathbb{N}\}\}$. Now we show that $\Sigma K(\Sigma P)$ is sober. Let $Q = \mathbb{N} \cup \{a, b, c\}$. Define a partial order $\leq_Q$ on $Q$ as follows:

(a) for $x, y \in P$, $x \leq_Q y$ iff $x \leq_P y$ in $P$,
(b) $n <_Q c$ for all $n \in \mathbb{N}$, and
(c) $c <_Q a$ and $c <_Q b$.

Clearly, $Q$ is an algebraic domain and $K(Q) = \mathbb{N} \cup \{a, b\}$. Define a mapping $\psi : K(\Sigma P) \to Q$ by

$$
\psi(x) = \begin{cases} 
n, & x = \uparrow n \ (n \in \mathbb{N}), 
\end{cases}
$$

$$
\begin{cases}
\psi(x) = \begin{cases} 
c, & x = \mathbb{N}, 
\end{cases}
\end{cases}
$$

$$
\begin{cases} 
a, & x = \{a\}, 
\end{cases}
$$

$$
\begin{cases} 
b, & x = \{b\}. 
\end{cases}
$$

It is straightforward to verify that $\psi$ is a poset isomorphism, and hence induces a homeomorphism from $\Sigma K(\Sigma P)$ to $\Sigma Q$. Clearly, $Q$ is a dcpo, $K(Q) = \mathbb{N} \cup \{a, b\}$ and $c = \lor_Q \mathbb{N}$, whence $Q$ is an algebraic domain.

By Proposition 3.2 $\Sigma Q$ is sober, and consequently, $\Sigma K(\Sigma P)$ is a sober space and hence a $K$-space. It follows from Theorem 1.1 that the $K$-reflection of $\Sigma P$ is a Scott space. More precisely, $\Sigma Q$ with the embedding $i_P = \psi \circ \eta_P : \Sigma P \to \Sigma Q$, $i_P(x) = x$, is a $K$-reflection of $\Sigma P$.

A poset $P$ is said to be $Noetherian$ if it satisfies the ascending chain condition: every ascending chain has a greatest member. Clearly, $P$ is Noetherian iff every directed set of $P$ has a largest element (equivalently, every ideal of $P$ is principal).

The following two examples show that for a $T_0$ space $X$, condition (1) of Theorem 4.3 is only a necessary condition but not a sufficient condition for the $K$-reflection of $X$ to be a Scott space.

Example 4.11. Let $K$ be a full subcategory of $\textbf{Top}_m$ containing $\textbf{Sob}$ which is adequate and closed with respect to homeomorphisms. Let $X$ be a countably infinite set and $X_{\text{cof}}$ the space equipped with the co-finite topology (the empty set and the complements of finite subsets of $X$ are open). Then

(a) $C(X_{\text{cof}}) = \{\emptyset, X\} \cup X^{(<\omega)}$, $X_{\text{cof}}$ is $T_1$ and hence a $d$-space.
(b) $K(X_{\text{cof}}) = 2^X \setminus \{\emptyset\}$.
(c) $X_{\text{cof}}$ is locally compact and first-countable.
(d) $X_{\text{cof}}$ is not well-filtered and hence not a $K$-space.

Let $\mathcal{K} = \{X \setminus F : F \in X^{(<\omega)}\}$. Then $\mathcal{K}$ is a filtered family of saturated compact subsets of $X_{\text{cof}}$ and $\bigcap \mathcal{K} = \emptyset$, but $X \setminus F \neq \emptyset$ for every $F \in X^{(<\omega)}$. Thus $X_{\text{cof}}$ is not well-filtered.

(e) $K(X_{\text{cof}}) = \text{Irr}_e(X_{\text{cof}}) = \{\{x\} : x \in X\} \cup \{X\}$.

It is easy to see that $\text{Irr}_e(X_{\text{cof}}) = \{\{x\} : x \in X\} \cup \{X\}$. By Lemma 2.3 and Lemma 3.3 $K(X_{\text{cof}}) = \text{Irr}_e(X_{\text{cof}}) = \{\{x\} : x \in X\} \cup \{X\}$.

(f) $\Sigma K(X_{\text{cof}})$ is sober and hence a $K$-space.

Clearly, $K(X_{\text{cof}})$ (with the order of set inclusion) is a Noetherian dcpo and hence an algebraic domain.

By Proposition 3.2 $\Sigma K(X_{\text{cof}})$ is sober, whence it is a $K$-space.

(g) $\eta^\circ_{X_{\text{cof}}} : X_{\text{cof}} \to \Sigma \text{Irr}_e(X_{\text{cof}})$, $x \mapsto \{x\}$, is not continuous.

Let $C \notin O(X_{\text{cof}})$. Then $\{\{x\} : x \in C\} \cup \{X\} \in \sigma(\text{Irr}_e(X_{\text{cof}}))$, but $(\eta^\circ_{X_{\text{cof}}})^{-1}(\{\{x\} : x \in C\} \cup \{X\}) = C \notin O(X_{\text{cof}})$, proving that $\eta^\circ_{X_{\text{cof}}} : X_{\text{cof}} \to \Sigma \text{Irr}_e(X_{\text{cof}})$ is not continuous.
(h) The $K$-reflection of $X_{cof}$ is not a Scott space. In particular, the well-filtered reflection of of $X_{cof}$ is not a Scott space and the sobrification of $X_{cof}$ is also not a Scott space.

Assume, on the contrary, that the $K$-reflection of $X_{cof}$ is a Scott space. Then there is a poset $P$ such that $(X_{cof})^k = P_H(K(X_{cof}))$ is homeomorphic to $\Sigma P$, whence by (e), $\mathcal{Irr}(X_{cof}) = K(X_{cof}) (= \Omega D_H(\mathcal{Irr}(X_{cof})))$ and $P (= \Omega \Sigma P)$ are isomorphic. It follows that $\mathcal{Irr}(X_{cof})$ and $\Sigma P$ are homeomorphic, and consequently, $P_H(\mathcal{Irr}(X_{cof})) \cong \mathcal{Irr}(X_{cof})$. Therefore, $O(P_H(\mathcal{Irr}(X_{cof}))) \cong \mathcal{Irr}(X_{cof})$, and hence $2^\omega = |\mathcal{Irr}(X_{cof})| = |O(P_H(\mathcal{Irr}(X_{cof})))| \leq |O(X_{cof})| = |X^{(\omega)}| = \omega$, which is a contradiction by Cantor’s Theorem (see [11], III-2.13 Cantor’s Theorem). So the $K$-reflection of $X_{cof}$ is not a Scott space.

**Example 4.12.** Let $X = 2^\mathbb{N}$ (the set of all subsets of $\mathbb{N}$) and $X_{cof}$ the space equipped with the co-countable topology (the empty set and the complements of countable subsets of $X$ are open). Then

(a) $|X| = 2^\omega$ (where $\omega = |\mathbb{R}|$ and $\mathbb{R}$ is the set of all reals) and $X$ is an uncountably infinite set.

(b) $X_{cof}$ is $T_1$ and $C(X_{cof}) = \{\emptyset, X\} \cup X^{(\omega)}$.

(c) $K(X_{cof}) = X^{(\omega)} \setminus \{\emptyset\}$.

Clearly, every finite subset is compact. Conversely, if $C \subseteq X$ is infinite, then $C$ has an infinite countable subset $\{c_n : n \in \mathbb{N}\}$. Let $C_0 = \{c_n : n \in \mathbb{N}\}$ and $U_m = (X \setminus C_0) \cup \{c_m\}$ for each $m \in \mathbb{N}$. Then $\{U_n : n \in \mathbb{N}\}$ is an open cover of $C$, but has no finite subcover. Whence $C$ is not compact. Thus $K(X_{cof}) = X^{(\omega)} \setminus \{\emptyset\}$.

(d) $X_{cof}$ is well-filtered.

To see this suppose that $\{F_d : d \in D\} \subseteq K(X_{cof})$ is a filtered family and $U \in O(X_{cof})$ with $\bigcap_{d \in D} F_d \subseteq U$. As $\{F_d : d \in D\}$ is filtered and all $F_d$ are finite, $\{F_d : d \in D\}$ has a least element $F_{d_0}$, and hence $F_{d_0} \cap \bigcap_{d \in D} F_d \subseteq U$, proving that $X_{cof}$ is well-filtered.

(e) $\mathcal{Irr}(X_{cof}) = \{\{x\} : x \in X\} \cup \{X\}$.

(f) $\mathcal{Irr}(X_{cof})$ is sober.

Let $P = \{\{x\} : x \in X\} \cup \{X\}$ with the order of set inclusion. It is easy to see that $P$ is a Noetherian dcpo and hence $\Sigma P$ is sober by Proposition 3.2. Clearly, $\sigma(P) = \gamma(P)$ and hence $|\sigma(P)| = |\gamma(P)| = 2^\omega$. By Remark 3.14, $(\mathcal{Irr}(X_{cof}))^{-1}(\{\{x\} : x \in X\}) = \emptyset \notin C(X_{cof})$, proving that $\eta_x : X_{cof} \to \Sigma \mathcal{Irr}(X_{cof})$ is not continuous.

(g) $\eta_x : X_{cof} \to \Sigma \mathcal{Irr}(X_{cof})$, $x \mapsto \{x\}$, is not continuous.

(h) The sobrification of $X_{cof}$ is not a Scott space.

Assume, on the contrary, that the sobrification of $X_{cof}$ is a Scott space. Then there is a poset $P$ such that $(X_{cof})^\omega = P_H(\mathcal{Irr}(X_{cof}))$ is homeomorphic to $\Sigma P$, whence $\mathcal{Irr}(X_{cof}) (= \Omega H(\mathcal{Irr}(X_{cof})))$ and $P (= \Omega \Sigma P)$ are isomorphic. It follows that $\mathcal{Irr}(X_{cof})$ and $\Sigma P$ are homeomorphic, and consequently, $P_H(\mathcal{Irr}(X_{cof})) \cong \mathcal{Irr}(X_{cof})$. Therefore, $O(P_H(\mathcal{Irr}(X_{cof}))) \cong \mathcal{Irr}(X_{cof})$, and hence $2^\omega = |\mathcal{Irr}(X_{cof})| = |O(P_H(\mathcal{Irr}(X_{cof})))| \leq |O(X_{cof})| = |X^{(\omega)}| = \omega$, which is a contradiction by Cantor’s Theorem (see [11], III-3.29 Proposition), which is a contradiction by Cantor’s Theorem. So the sobrification of $X_{cof}$ is not a Scott space.

Let $\mathcal{J} = \mathbb{N} \times (\mathbb{N} \cup \{\omega\})$ with ordering defined by $(j,k) \leq (m,n)$ iff $j = m$ and $k \leq n$, or $n = \omega$ and $k \leq m$. $\mathcal{J}$ is a well-known dcpo constructed by Johnstone in [10].

**Proposition 4.13.** Let $K$ be a full subcategory of Top$_{eq}$ containing Sob which is adequate and closed with respect to homeomorphisms. Then the $K$-reflection of the Johnstone space $\Sigma \mathcal{J}$ is not a Scott space. In particular, neither the sobrification nor the well-filtered reflection of $\Sigma \mathcal{J}$ is a Scott space.

**Proof.** Clearly, $\mathcal{J}_{max} = \{(n,\omega) : n \in \mathbb{N}\}$ is the set of all maximal elements of $\mathcal{J}$. By Remark 3.14, $\mathcal{J}$ is a dcpo, and $\top$ is the largest element of $\mathcal{J}_{\max}$ and $\{\top\} \in \mathcal{S}(\mathcal{J})$. The following three conclusions about $\mathcal{J}$ are known (see, for example, [13], Example 3.1 and [14], Lemma 3.1):

(i) $\mathcal{Irr}(\mathcal{J}) = \{\{c_m(\{j\}) = 1 : x \in \mathcal{J}\} \cup \{\mathcal{J}\}\}$.

(ii) $K(\mathcal{J}) = (2^{\mathcal{J}_{max}} \setminus \{\emptyset\}) \cup \text{Fin} \mathcal{J}$.
(iii) \( \Sigma \) is not well-filtered and hence not a K-space.

Whence we have

(a) \( \text{irr}(\Sigma J \tau) = \{ \cl_{J \tau} \{ x \} = \downarrow_J x : x \in J \tau \} \cup \{ \} \) by (i).
(b) \( K(\Sigma J \tau) = \{ \uparrow G : G \text{ is nonempty and } G \subseteq J_{\text{max}} \cup \{ \top \} \} \cup \text{Fin} J \tau \) by (ii).
(c) \( \Sigma J \tau \) is not well-filtered and hence not a K-space.

Indeed, let \( K = \{ \uparrow_{J \tau}(J_{\text{max}} \setminus F) : F \in (J_{\text{max}})^{(\omega)} \} \). Then by (b), \( K \subseteq K(\Sigma J \tau) \) is a filtered family and
\[
\bigcap K = \bigcap_{F \in (J_{\text{max}})^{(\omega)}} \uparrow_{J \tau}(J_{\text{max}} \setminus F) = \bigcap_{F \in (J_{\text{max}})^{(\omega)}} (J_{\text{max}} \setminus F) \cup \{ \top \} = \{ \top \} \cup (J_{\text{max}} \setminus (J_{\text{max}})^{(\omega)}) = \{ \top \} \in \sigma(J \tau),
\]
but there is no \( F \in (J_{\text{max}})^{(\omega)} \) with \( \uparrow_{J \tau}(J_{\text{max}} \setminus F) \subseteq \{ \top \} \). Therefore, \( \Sigma J \tau \) is not well-filtered. As \( K \) is a full subcategory of \( \text{Top}_w \), \( \Sigma J \tau \) is not a K-space.

(d) The K-reflection \((\Sigma \mathcal{J})^k\) of \( \Sigma \) is not a Scott space.

By (iii), (c) and Corollary 5.3, the K-reflection \((\Sigma \mathcal{J})^k\) of \( \Sigma \) is not a Scott space.

\[\Box\]

5. Scott K-completions of posets

In this section, we give some applications of the results of Section 5 to the Scott K-completions of posets.

The category whose objects are posets and whose morphisms are monotone (i.e., order-preserving) mappings will be denoted by \( \text{Poset} \), and the full subcategory of dcpos by \( \text{DCPO} \). Let \( \text{Poset}_s \) denote the category of all posets with Scott continuous mappings and \( \text{DCPO}_s \) be the full subcategory of dcpos.

**Definition 5.1.** Let \( K \) be a full subcategory of \( \text{Top}_d \) containing \( \text{Sob} \). A poset \( P \) is called a Scott K-dcpo, a K-dcpo for short, if \( \Sigma P \) is a K-space. A poset (even a dcpo) \( Q \) is said to be a non-K poset if \( Q \) is not a K-dcpo. Let \( \text{K-DCPO}_s \) denote the category of all K-dcpos with Scott continuous mappings.

\( \text{K-DCPO}_s \) is a full subcategory of \( \text{DCPO}_s \), and it is a subcategory of \( \text{DCPO} \), but not a full subcategory of \( \text{DCPO} \).

Clearly, a poset \( P \) is a \( \text{Top}_d \)-dcpo (d-dcpo for short) iff \( P \) is a dcpo. For \( K = \text{Top}_w \), the K-dcpos are simply called the \( \text{WF} \)-dcpos and the category \( \text{Top}_w \text{-DCPO}_s \) is simply denoted as \( \text{WF}-\text{DCPO}_s \).

**Definition 5.2.** Let \( K \) be a full subcategory of \( \text{Top}_d \) containing \( \text{Sob} \). A Scott K-completion, \( K_s \)-completion for short, of a poset \( P \) is a pair \( \langle \hat{P}, \eta \rangle \) consisting of a K-dcpo \( \hat{P} \) and a Scott continuous mapping \( \eta : P \rightarrow \hat{P} \), such that for any Scott continuous mapping \( f : P \rightarrow Q \) to a K-dcpo \( Q \), there exists a unique Scott continuous mapping \( \hat{f} : \hat{P} \rightarrow Q \) such that \( \hat{f} \circ \eta = f \), that is, the following diagram commutes.

\[
\begin{array}{ccc}
P & \xrightarrow{\eta} & \hat{P} \\
\downarrow{f} & & \downarrow{\hat{f}} \\
Q & & Q
\end{array}
\]

For \( K = \text{Top}_d \) (resp., \( K = \text{Top}_w \)), the \( K_s \)-completion is simply called the \( D_s \)-completion (resp., \( \text{WF}_s \)-completion).

By a standard argument, \( K_s \)-completions, if they exist, are unique up to isomorphism. We use \( K_s(P) \) to denote the \( K_s \)-completion of \( P \) if it exists. We will use \( D_s(P), \text{WF}_s(P) \) and \( \text{Sob}_s(P) \) to denote the \( D_s \)-completion, \( \text{WF}_s \)-completion and \( \text{Sob}_s \)-completion of \( P \), respectively.

**Definition 5.3.** Let \( K \) be a full subcategory of \( \text{Top}_d \) containing \( \text{Sob} \). A K-completion of a poset \( P \) is a pair \( \langle \hat{P}, \phi \rangle \) consisting of a K-dcpo \( \hat{P} \) and a monotone mapping \( \phi : P \rightarrow \hat{P} \), such that for any monotone mapping \( f : P \rightarrow Q \) to a K-dcpo \( Q \), there exists a unique Scott continuous mapping \( \hat{f} : \hat{P} \rightarrow Q \) such that \( \hat{f} \circ \phi = f \).
For $K = \text{Top}_d$ (resp., $K = \text{Top}_s$), the $K$-completion is simply called the $D$-completion (resp., $\text{WF}$-completion).

Similarly, $K$-completions, if they exist, are unique up to isomorphism. We use $K(P)$ to denote the $K$-completion of $P$ if it exists. We will use $D(P)$, $\text{WF}(P)$ and $\text{Sob}(P)$ to denote the $D$-completion of $P$, $\text{WF}$-completion and $\text{Sob}$-completion of $P$, respectively.

**Remark 5.4.** The $D_s$-completion was called the $D$-completion in [23, Definition 1]. For the sake of distinction, here we call such a completion the $D_s$-completion and give the $D$-completion a little different meaning.

**Definition 5.5.** Let $K$ be a full subcategory of $\text{Top}_d$ containing $\text{Sob}$. A poset $P$ is called a weak $K$-$dcpo$ if there is a $K$-space such that $P$ is isomorphic to $\Omega X$.

Clearly, every weak $K$-$dcpo$ is a dcpo, and a poset $P$ is a dcpo iff $P$ is a $d$-$dcpo$ iff $P$ is a weak $d$-$dcpo$. By Proposition 3.3, every complete lattice is a $\text{WF}$-dcpo and also is a weak $\text{Sob}$-$dcpo$. The Isbell lattice $L$ constructed in [7] as a complete lattice, is a weak $\text{Sob}$-$dcpo$ but not a $\text{Sob}$-$dcpo$.

**Theorem 5.6.** Let $K$ be a full subcategory of $\text{Top}_d$ containing $\text{Sob}$ which is adequate and closed with respect to homeomorphisms. For a poset $P$, if $K(\Sigma P)$ is a $K$-$dcpo$, then $K_s(P) = K(\Sigma P)$ with the canonical mapping $\eta_P : P \rightarrow K(P)$, $\eta_P(x) = cl_{K(P)}(x) = \downarrow x$, is a $K_s$-completion of $P$.

**Proof.** By Theorem 5.4 $\Sigma K(\Sigma P)$ with the canonical mapping $\eta_P : \Sigma P \rightarrow \Sigma K(\Sigma P)$, $\eta_P(x) = cl_{K(P)}(x)$, is the $K$-reflection of $\Sigma P$. Therefore, by Lemma 5.5, $K_s(P) = K(\Sigma P)$ with the canonical mapping $\eta_P : P \rightarrow K_s(P)$, $\eta_P(x) = cl_{K(P)}(x) = \downarrow x$, is a $K_s$-completion of $P$. $\Box$

**Definition 5.7.** Let $K$ be a full subcategory of $\text{Top}_d$. A poset $P$ is called a $S_K$-poset if $\Sigma K(\Sigma P)$ is a $K$-space. Let $S_K$-$\text{Poset}_s$, denote the category of all $S_K$-posets with Scott continuous mappings.

Clearly, $S_K$-$\text{Poset}_s$ is a full subcategory of $\text{Poset}_s$. If $K$ is a full subcategory of $\text{Top}_d$ containing $\text{Sob}$, then by Lemma 5.5 every $K$-$dcpo$ is a $S_K$-poset, and hence $K$-$\text{DCPO}_s$ is a full subcategory of $S_K$-$\text{Poset}_s$.

From Theorem 5.6 we deduce the following result.

**Corollary 5.8.** Let $K$ be a full subcategory of $\text{Top}_d$ containing $\text{Sob}$ which is adequate and closed with respect to homeomorphisms. Then $K$-$\text{DCPO}_s$ is reflective in $S_K$-$\text{Poset}_s$. Therefore, if $K(\Sigma P)$ is a $K$-$dcpo$ for any poset $P$, then $K$-$\text{DCPO}_s$ is reflective in $\text{Poset}_s$.

**Proposition 5.9.** For a poset $P$, $D_s(P) = d(\Sigma P)$ with the canonical mapping $\eta_P : P \rightarrow D(P)$, $\eta_P(x) = cl_{d(P)}(x)$, is a $D_s$-completion of $P$.

**Proof.** Clearly, $\text{Top}_d$ is closed with respect to homeomorphisms. By Lemma 5.6 $\text{Top}_d$ is adequate and. And by Lemma 5.1 and Corollary 4.6, $d(\Sigma P)$ is a dcpo and $D_s(P) = d(\Sigma P)$ with the canonical mapping $\eta_P : P \rightarrow D(P)$, $\eta_P(x) = cl_{d(P)}(x)$, is a $D_s$-completion of $P$. $\Box$

**Corollary 5.10.** (23, Corollary 2) $\text{DCPO}_s$ is reflective in $\text{Poset}_s$.

**Remark 5.11.** In [23], using the $D$-topology defined in [23] (see also [10]), which originates from Wyler [18], Zhao and Fan proved that for any poset $P$, the $D_s$-completion of $P$ exists. Proposition 5.9 (or Corollary 4.6) shows that the $D_s$-completion of a poset $P$ is essentially the $d$-reflection of Scott space $\Sigma P$.

By Lemma 5.6 and Theorem 5.6 we get the following two corollaries.

**Corollary 5.12.** For a poset $P$, if $\text{Irr}_s(\Sigma P)$ is a $\text{Sob}$-$dcpo$, then $\text{Sob}_s(P) = \text{Irr}_s(\Sigma P)$ with the canonical mapping $\eta_P : P \rightarrow \text{Sob}_s(P)$, $\eta_P(x) = cl_{\sigma(P)}(x) = \downarrow x$, is a $\text{Sob}_s$-completion of $P$.

**Corollary 5.13.** For a poset $P$, if $\text{WF}(\Sigma P)$ is a $\text{WF}$-$dcpo$, then $\text{WF}_s(P) = \text{WF}(\Sigma P)$ with the canonical mapping $\eta_P : P \rightarrow \text{WF}_s(P)$, $\eta_P(x) = cl_{\sigma(P)}(x) = \downarrow x$, is a $\text{WF}_s$-completion of $P$. 

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Proposition 5.14. Let $K$ be a full subcategory of $\text{Top}_d$ containing $\text{Sob}$ which is adequate and closed with respect to homeomorphisms. For a non-$K$ poset $P$, if $\text{irr}_s(\Sigma P) = \{\{x\} : x \in P\} \cup \{P\}$ and $P_\tau$ is a $K$-dcpo, then $K_s(P) = P_\tau$ with the canonical mapping $\eta_P : P \to P_\tau$, $\eta_P(x) = x$, is a $K_s$-completion of $P$.

Proof. By Corollary 4.13, $K_s(P) = \Sigma P_\tau$ with the canonical mapping $\eta_P : P \to P_\tau$, $\eta_P(x) = x$, is a $K_s$-completion of $P$.

By Corollary 4.13 and Theorem 5.6, we get the following result.

Corollary 5.15. Let $P$ be a poset $P$. If $\text{irr}_s(\Sigma P) = \{\{x\} : x \in P\} \cup \{P\}$ and $\Sigma P_\tau$ is a sober space, then $K_s(P) = P_\tau$ with the canonical mapping $\eta_P : P \to P_\tau$, $\eta_P(x) = x$, is a $\text{Sob}_s$-completion of $P$.

Corollary 5.16. Let $P$ be a poset $P$. If $\text{irr}_s(\Sigma P) = \{\{x\} : x \in P\} \cup \{P\}$ and $\Sigma P_\tau$ is a well-filtered space, then $\text{WF}_s(P) = P_\tau$ with the canonical mapping $\eta_P : P \to P_\tau$, $\eta_P(x) = x$, is a $\text{WF}_s$-completion of $P$.

6. $K$-reflections of Alexandroff spaces

In the final section, we discuss the $K$-reflections of Alexandroff spaces and the $K$-completions of posets. First, it is easy to verify the following result (cf. [24, Theorem 5.7]).

Proposition 6.1. For any poset $P$, the following conditions are equivalent:

1. $\Gamma P$ is sober.
2. $\Gamma P$ is well-filtered.
3. $\Gamma P$ is an $d$-space.
4. $P$ is Noetherian.
5. $P$ is a dcpo such that every element of $P$ is compact (i.e., $x \ll x$ for all $x \in P$).
6. $P$ is a dcpo such that $\gamma(P) = \sigma(P)$.

It is straightforward to verify the following lemma.

Lemma 6.2. For a poset $P$, a $T_0$ space $Y$ and a mapping $f : \Gamma P \to Y$, the following conditions are equivalent:

1. $f : \Gamma P \to Y$ is continuous.
2. $f : P \to \Omega Y$ is monotone.
3. $f : \Gamma P \to \Gamma \Omega Y$ is continuous.

Lemma 6.3. Let $K$ be a full subcategory of $\text{Top}_d$ containing $\text{Sob}$ and $P$ a poset. Then $K(\Gamma P) = \text{Id} P$.

Proof. It is straightforward to verify that $\text{irr}(\Gamma P) = \mathcal{D}(P)$ and $\text{irr}_s(\Gamma P) = \text{Id} P$ (see, for example, the first paragraph of [24, Section 3.2]). By Lemma 2.4, $\text{Id} P = \mathcal{D}(\Gamma P) \subseteq \mathcal{D}(\Gamma P) \subseteq K(\Gamma P) \subseteq \text{Sob}(\Gamma P) = \text{irr}_s(\Gamma P) = \text{Id} P$. Thus $K(\Gamma P) = \text{Id} P$.

Theorem 6.4. Let $K$ be a full subcategory of $\text{Top}_d$ containing $\text{Sob}$ which is adequate and closed with respect to homeomorphisms. Then for any poset $P$, the $K$-reflection $(\Gamma P)^K$ of $\Gamma P$ exists and it is a Scott space. More precisely, the Scott space $\Sigma \text{Id} P$ with the canonical mapping $\phi_P : \Gamma P \to \Sigma \text{Id} P$, $x \mapsto \text{cl}_{\Gamma P}(x) = \downarrow x$, is a $K$-reflection of $\Gamma P$.

Proof. By Lemma 6.3, $K(\Gamma P) = \text{Id} P$. Since $\text{Id} P$ is an algebraic domain, by Proposition 3.2, $\Sigma \text{Id} P$ is sober and hence a $K$-space. Clearly, the map $\phi_P : \Gamma P \to \Sigma K(\Gamma P) = \Sigma \text{Id} P$, $x \mapsto \text{cl}_{\Gamma P}(x) = \downarrow x$, is monotone; whence by Lemma 6.2, $\phi_P : \Gamma P \to \Sigma K(\Gamma P)$ is continuous. Therefore, Conditions (1) and (2) of Theorem 1.3 hold for the space $X = \Gamma P$. By Theorem 6.3, the Scott space $\Sigma \text{Id} P$ with the canonical mapping $\phi_P : \Gamma P \to \Sigma \text{Id} P$ is a $K$-reflection of $\Gamma P$.

Remark 6.5. We can present a direct proof of Theorem 6.4.
Proof. By Proposition 6.2 and Lemma 6.3 \( K(\Gamma P) = \text{Id} P \) and \( \Sigma \text{Id} P \) is sober and hence a K-space since \( \text{Id} P \) is an algebraic domain. Clearly, the canonical mapping \( \phi_P : \Gamma P \to \Sigma K(\Gamma P) = \Sigma \text{Id} P, x \mapsto c_{\chi P}\{x\} = \downarrow x, \) is continuous. Now we show that for each K-space \( Y \) and each continuous mapping \( f : \Gamma P \to Y \), there is a unique continuous mapping \( f^* : \Sigma \text{Id} P \to Y \) such that \( f^* \circ \phi_P = f \), that is, the following diagram commutes.

\[
\begin{array}{ccc}
\Gamma P & \xrightarrow{\phi_P} & \Sigma \text{Id} P \\
\downarrow f & & \downarrow f^* \\
Y & & Y
\end{array}
\]

We firstly prove the existence of \( f^* \). Since \( Y \) is a K-space, \( Y \) is a d-space. Therefore, \( \forall E \) exists in \( Y \) for each directed subset \( E \) of \( Y \) (with the specialization order). As \( f : \Gamma P \to Y \) is continuous, \( f : \Gamma P \to \Omega Y \) is monotone. Define a mapping \( f^* : \Sigma \text{Id} P \to Y \) by \( f^*(I) = \lor f(I) \) for each \( I \in \text{Id} P \). For every \( \{I_d : d \in D\} \in \mathcal{D}(\text{Id} P) \), we have that \( f^* \left( \bigvee_{d \in D} I_d \right) = \lor f \left( \bigvee_{d \in D} I_d \right) = \lor f(I) = \lor f(I_d) = \lor f(I) \). By Lemma 2.1 and Lemma 3.1, we have that \( f^* \circ \phi_P = f \).

Now we prove the uniqueness of \( f^* \). Suppose that \( g : \Gamma P \to Y \) is another continuous mapping satisfying \( g \circ \phi_P = f \). Then for each \( I \in \text{Id} P \), by Lemma 2.1 and Lemma 3.1 we have that \( g(I) = g \left( \bigvee_{x \in I} \downarrow x \right) = g(\bigvee_{x \in I} g(x) = \lor f(I) = \lor f(I) \), and hence \( g = f^* \).

Therefore, \( \Sigma \text{Id} P, \phi_P \) is a reflection of \( \Gamma P \) since \( \phi_P(x) = \downarrow x \) for each \( x \in P \).

From Theorem 5.3 we immediately deduce the following result.

Corollary 6.6. Let \( K \) be a full subcategory of Topd containing Sob which is adequate and closed with respect to homeomorphisms. Then for any poset \( P \), the K-completion of \( P \) exists and it is the pair \( (\text{Id} P, \phi_P) \), where \( \phi_P : P \to \text{Id} P \) is defined by \( \phi_P(x) = \downarrow x \) for each \( x \in P \).

Corollary 6.7. Let \( K \) be a full subcategory of Topd containing Sob which is adequate and closed with respect to homeomorphisms. Then K-DCPOs is reflective in Poset.

Finally, by Lemma 3.6 Corollary 6.6 and Corollary 5.7 we have the following three corollaries.

Corollary 6.8. Let \( P \) be a poset. Then the d-reflection of \( \Gamma P \), the well-filtered reflection of \( \Gamma P \) and the sobrification of \( \Gamma P \) agree. They are all the Scott space \( \Sigma \text{Id} P \) with the canonical mapping \( \phi_P : \Gamma P \to P \).

Corollary 6.9. Let \( P \) be a poset. Then the D-completion of \( P \), the WF-completion of \( P \) and the Sob-completion of \( P \) agree. They are all the pair \( (\text{Id} P, \phi_P) \), where \( \phi_P : P \to \text{Id} P \) is defined by \( \phi_P(x) = \downarrow x \) for each \( x \in P \).

Corollary 6.10. DCPOs, WF-DCPOs and Sob-DCPOs all are reflective in Poset.

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