EXTREMAL RESULTS ON $G$-FREE COLORINGS OF GRAPHS

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Abstract. Let $H = (V(H), E(H))$ be a graph. A $k$-coloring of $H$ is a mapping $\pi : V(H) \rightarrow \{1, 2, \ldots, k\}$ so that each color class induces a $K_2$-free subgraph. For a graph $G$ of order at least 2, a $G$-free $k$-coloring of $H$ is a mapping $\pi : V(H) \rightarrow \{1, 2, \ldots, k\}$ so that the subgraph of $H$ induced by each color class of $\pi$ is $G$-free, i.e. contains no copy of $G$. The $G$-free chromatic number of $H$ is the minimum number $k$ so that there is a $G$-free $k$-coloring of $H$, denoted by $\chi_G(H)$. A graph $H$ is uniquely $k$-G-free coloring if $\chi_G(H) = k$ and every $k$-G-free colouring of $H$ produces the same color classes. A graph $H$ is minimal with respect to $G$-free, or $G$-free-minimal, if for every edges of $E(H)$ we have $\chi_G(H \setminus \{e\}) = \chi_G(H) - 1$. In this paper we give some bounds and attribute about uniquely $k$-G-free colouring and $k$-G-free-minimal.

1. Introduction

All graphs $G$ considered in this paper are undirected, simple, and finite graphs. For given graphs $G$, we denote its vertex set, edge set, maximum degree, and minimum degree by $V(G)$, $E(G)$, $\Delta(G)$, and $\delta(G)$, respectively. The number of vertices of $G$ is define by $|V(G)|$. For a vertex $v \in V(G)$, we use $\deg_G(v)$ (deg $v$) and $N_G(v)$ to denote the degree and neighbors of $v$ in $G$, respectively.

The join of two graphs $G$ and $H$, define by $G \oplus H$, is a graph obtained from $G$ and $H$ by joining each vertex of $G$ to all vertices of $H$. The union of two graphs $G$ and $H$, define by $G \cup H$, is a graph obtained from $G$ and $H$, where $V(G \cup H) = V(G) \cup V(H)$ and $E(G \cup H) = E(G) \cup E(H)$. For convenience, we use $[n]$ instead of $\{1, 2, \ldots, n\}$. A $k$-vertex coloring of $H$ is a partition of $V(H)$ into $k$ color classes such that vertices in the same class are not adjacent. Moreover, $H$ is called uniquely vertex $k$-colorable if every $k$-coloring of it induces the same partition on $V(H)$. Also, $H$ is minimal with respect to $k$-vertex coloring of $H$, if for every edges of $E(H)$ we have $\chi(H \setminus \{e\}) = \chi(H) - 1$.

Uniquely vertex $k$-colorable have been studied by Chartrand and Geller [2], Aksionov [1], Harary, Hedetniemi, and Robinson [12], Bollobás [3,4], Borowiecki and Burchardt [5], Alishahi and Taherkhani [2, and Xu [15]. An $(n, k)$-coloring of a $H$ corresponds to the partition of $V(H)$ into $n$ color classes, so that the induces a subgraph with each color class whose maximum degree does not exceed $k$. A graph $H$ is uniquely $(n, k)$-colorable if it is $(n, k)$-colorable and every $(n, k)$-coloring of $H$ produces the same color classes. M. Frick and M. A. Henning show that the following results is true.

Theorem 1. [17] Suppose $H$ be a uniquely $(n, k)$-colorable graph, where $n \geq 2$ and $k \geq 1$. Hence:

$$|V(H)| \geq n(k + 1) - 1.$$  

Theorem 2. [14] For each $n \geq 2$ and $k \geq 1$, there exists a uniquely $(n, k)$-colorable graph with $n(k + 1) - 1$ member.

1.1. $G$-free coloring. The conditional chromatic number $\chi(H, P)$ of $H$, is the smallest integer $k$ for which there is a decomposition of $V(H)$ into $k$ color class say $V_1, \ldots, V_k$, so that $H[V_i]$ satisfies the property $P$, where $P$, is a graphical property and $H[V_i]$ is a the induced subgraph on $V_i$, for each $1 \leq i \leq k$. This extension of graph coloring was presented by Harary in 1985 [11]. Suppose

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that \( \mathcal{G} \) be a families of graphs, when \( P \) is the feature that a subgraph induced by each color class does not contain each copy of members of \( \mathcal{G} \), we write \( \chi^\mathcal{G}(H) \) instead of \( \chi(H, P) \). In this regard, we say a graph \( H \) has a \( \mathcal{G} \)-free \( k \)-coloring if there is a map \( \pi : V(H) \rightarrow \{1, 2, \ldots, k\} \) such that each color class \( V_i = \pi^{-1}(i) \) does not contain any members of \( \mathcal{G} \). For simplicity of notation if \( \mathcal{G} = \{G\} \), then we write \( \chi^\mathcal{G}(H) \) instead of \( \chi^\mathcal{G}(H) \).

An ordinary \( k \)-coloring of \( H \) can be viewed as \( \mathcal{G} \)-free \( k \)-coloring of a graph \( H \) by taking \( \mathcal{G} = \{K_2\} \). It was shown that for each graph \( H \), \( \chi(H) \leq \Delta(H) + 1 \). The well-known Brooks theorem states that for any connected graph \( H \), \( \chi(H) \leq \Delta(H) \) if \( H \) is a connected and is neither an odd \( C_n \) nor a \( K_n \), then \( \chi(H) \leq \Delta \) [6]. One can refer to [2, 7, 8, 13, 14] and it references for further studies.

A graph \( H \) is uniquely \( k \)-G-free colouring if \( \chi^G(H) = k \) and every \( k \)-G-free colouring of \( H \) produces the same color classes, and a graph \( H \) is minimal with respect to \( G \)-free, or \( G \)-free-minimal, if for every edges of \( E(H) \) we have \( \chi^G(H \setminus \{e\}) \leq \chi^G(H) - 1 \).

In this article we investigate some properties of uniquely \( k \)-G-free and \( G \)-free-minimal coloring of graphs as follow:

**Theorem 3.** Let \( H \) and \( G \) be two graph and \( k \geq 1 \) be a integers. Assume that \( H \) is uniquely \( k \)-G-free colourable graph, hence we have:

\[
|V(H)| \geq k|V(G)| - 1.
\]

**Theorem 4.** Let \( G \) be a graph with \( m \) vertices and \( k \geq 1 \) be an integers. Then:

I : There exists a \( k \)-G-free colourable graph, with \( km - 1 \) members where \( k - 1 \) class has \( m \) member and one class has a \( m - 1 \) member.

II : If \( m \geq 2\delta(G) - 1 \) and \( \delta(G) = 1 \), then there exists a uniquely \( k \)-G-free colourable graph with \( km - 1 \) members and if \( \delta \geq 2 \) then there exist many \( k \)-G-free colourable of a graph with \( km - 1 \) members.

III : If there exists a uniquely \( k \)-G-free colourable graph with \( km - 1 \) members, then for each \( i \geq 1 \) there exists a uniquely \( k \)-G-free colourable graph with \( km + i - 1 \) members.

**Theorem 5.** Let \( H \) and \( G \) be two graph and \( k \geq 1 \) be a integers. Assume that \( H \) is uniquely \( k \)-G-free colourable graph, \( |V(G)| = m \), \( \delta(G) = \delta \), \( |E(H)| = e_1 \) and \( |E(G)| = e_2 \), hence:

\[
e_1 \geq k(e_2 - \delta + \frac{k+1}{2}(m\delta)).
\]

**Theorem 6.** \( H \) is \( k \)-G-free-minimal iff either \( H \) is a graph \( K_{(k-1)(m-1)+1} \) minus the edges of a 1-factor, when \( k \) is even, or \( H \cong H_1 \oplus K_1 \), where \( H_1 \) is a graph \( K_{(k-1)(m-1)} \) minus the edges of a 1-factor, when \( k \) is odd, where \( G \) be a such connected graph with \( m \)-members.

2. **Uniquely \( G \)-Free Graph**

**Definition 7.** A graph \( H \) is uniquely \( k \)-G-free colouring if \( \chi^G(H) = k \) and every \( k \)-G-free colouring of \( H \) produces the same color classes.

In this section we investigate some properties of uniquely \( k \)-G-free coloring of graphs and give a lower bounds on the vertices of uniquely \( k \)-G-free graphs \( H \). In the following theorem we determine a lower bound for the size of uniquely \( k \)-G-free coloring of \( V(H) \).

**Theorem 8.** Let \( H \) and \( G \) be two graph and \( k \geq 1 \) be a integers. Assume that \( H \) is uniquely \( k \)-G-free colourable graph (\( u-k-G-f \)), then:

\[
|V(H)| \geq k|V(G)| - 1.
\]

*Proof.* Assume that \( |V(H)| = n \) and \( |V(G)| = m \) and let \( V_1, V_2, \ldots, V_k \) be a uniquely \( k \)-partition of \( V(H) \) where \( H[V_i] \) is a G-free class. Now, considering the following claim.
Claim 9. $|V_i| \geq m - 1$ for each $i \in [k]$.

Proof. By contrary, suppose that there exist at least one $i$ say $i = 1$, such that, $|V_1| \leq m - 2$. Set $v \in V_2$ and set $V'_1 = V_1 \cup \{v\}$, $V'_2 = V_2 \setminus \{v\}$ and $V'_i = V_i$ for $i \geq 3$, hence we have $H[V'_i]$ is $G$-free, a contradiction.

Claim 10. For at most one $i$, we have $|V_i| = m - 1$.

Proof. By contrary, suppose that $|V_1| = |V_2| = m - 1$. In this case set $v_i \in V_i$ for $i = 1, 2$, and set $V'_1 = (V_1 \setminus \{v_1\}) \cup \{v_2\}$, $V'_2 = (V_2 \setminus \{v_2\}) \cup \{v_1\}$ and $V'_i = V_i$ for $i \geq 3$, hence we have $H[V'_i]$ is $G$-free, a contradiction again.

Now by Claim 1, 2 we have $|V(H)| \geq k|V(G)| - 1$ and the proof is complete.

In the next theorem we establish that the bounds in Theorem 8 is best possible. Also we show that if there exists a uniquely $k$-G-free colourable graph with $km - 1$ members, then for each $i \geq 1$ there exists a u-k-G-f with $km + i - 1$ members.

Theorem 11. Let $G$ be a graph with $m$ vertices and $k \geq 1$ be an integers. Then:

I : There exists a $k$-G-free colourable graph, with $km - 1$ members where $k - 1$ class has $m$ member and one class has a $m - 1$ member.

II : If $m \geq 2\delta(G) - 1$ and $\delta(G) = 1$, then there exists a $u$-k-G-f with $km - 1$ members and if $\delta \geq 2$ then there exist many $k$-G-free colourable of a graph with $km - 1$ members.

III : If there exists a $u$-k-G-f with $km - 1$ members, then for each $i \geq 1$ there exists a $u$-k-G-f with $km + i - 1$ members.

Proof. Suppose that $H_i$ for $i = 1, 2, \ldots, k + 1$ be a graph, where $H_i \cong K_{m-1}$ for each $i = 1, 2, \ldots, k$, and $H_{k+1} \cong K_{k-1}$ where $V(H_{k+1}) = \{v_1, v_2, \ldots, v_{k-1}\}$. For each $i = 1, 2, \ldots, k$ define $H'_i = H_i \cup \{v_i\}$, so that $v_i$ is adjacent to at most $\delta(G) - 1$ vertices of $V(H_i)$, and define $H'_1 = H_k$. Set $H = \bigoplus_{i=1}^{k+1} H'_i$. Hence, we can say that $|V(H)| = km - 1$. Since $|V(H'_i)| = m$ and $|N_{H'_i}(v_i)| \leq \delta - 1$, we can check that for each $i$, $H'_i$ is $G$-free, that is $\chi_G(H) \leq k$. Now we have the following claim:

Claim 12. $\chi_G(H) = k$.

Proof. For $k = 1$ its clearly, hence assume that $k \geq 2$. By contradiction assume that $\chi_G(H) \leq k - 1$, and let $V_1, V_2, \ldots, V_{k-1}$ be a $(k - 1)$-partition of $V(H)$ where $H[V_i]$ is a $G$-free class. Since $H[i \cap \cup_{i=1}^{k} H_i] \cong K_{k(m-1)}$ we have $\omega(H) \geq k(m-1)$, hence for at least one $i$ we have $|V_i \cap \cup_{i=1}^{k} H_i| \geq m$ a contradiction. That is $\chi_G(H) \geq k$ and the proof is complete.

Therefore by Claim 12 and by definition $H$ one can say that the part (I) is true. To prove (II), by Claim 12 assume that $V_1, V_2, \ldots, V_k$ be a $k$-partition of $V(H)$, where $H[V_i]$ is a $G$-free for each $i \in [k]$ and $H$ is a graph constructed in part I. Hence, it easy to check that the following claim is true.

Claim 13. $|V_i \cap \cup_{i=1}^{k} H_i| = m - 1$ for each $i \in [k]$.

Therefore, by Claim 13 we have the claims as follow:

Claim 14. $|V_i| \leq m$ for $i = 1, 2, \ldots, k$.

Proof. By contrary, suppose that $|V_i| \geq m + 1$ for at least one i, say $i = 1$, and assume that $\{x_1, x_2, \ldots, x_{m+1}\} \subseteq V_1$. Since $H[V_1]$ is $G$-free, $\chi_G(H) = k$ and by Claim 13 we have $|V_1 \cap \cup_{i=1}^{k} H_i| = m - 1$. W.l.g let $V_1 \cap \cup_{i=1}^{k} H_i = \{x_1, x_2, \ldots, x_{m-1}\}$, that is $x_m, x_{m+1} \in V(H_{k+1})$. W.l.g suppose that $x_m = v_1$ and $x_{m+1} = v_2$. So, since $H[V_1]$ is $G$-free, $|N_H(v_1) \cap \{x_1, x_2, \ldots, x_{m-1}\}| \leq \delta - 1$, that is $\{|x_1, x_2, \ldots, x_{m-1}| \cap V(H_i)| \geq m - \delta$ for $i = 1, 2$. As $m \geq 2\delta - 1$ one can say that there exist at least one $i \in [2]$, say $i = 1$ so that $|N_H(v_1) \cap \{x_1, x_2, \ldots, x_{m-1}\}| \geq \delta$, a contradiction to
\[ |N_H(v_i) \cap \{x_1, x_2, \ldots, x_{m-1}\}| \leq \delta - 1, \text{ that is the assumption does not hold and the proof of claim is complete.} \]

Now since \( n = km - 1 \) and by Claim 12, 13 and 14 let \( |V_i| = m \) for \( i = 1, 2, \ldots, k-1 \) and \( |V_k| = m - 1 \). So, suppose that \( \delta = 1 \) and consider the following claim:

**Claim 15.** The coloring provided is unique.

**Proof.** Consider \( V_{i'} \). Since \( |V_{i'}| = m \) for each \( i' \in [k-1] \) then \( |V_{i'} \cap (\bigcup_{j=1}^{i} H_j)| = m - 1 \) by Claim 13 and \( |V_{i'} \cap V(H_j)| = 1 \) for each \( i' \in [k-1] \), let \( v_{i'} \in V_{i'} \cap V(H_j) \) for each \( i' \in [k-1] \). If there exists \( i, j \in [k] \) where \( |V_{i'} \cap V(H_j)| \neq 1 \), then one can check that \( \deg_{H[H_{i'}]}(v_{i'}) \geq 1 \), therefore \( G \subseteq H[V_{i'}] \), a contradiction. Hence, \( V_{i'} = V(H_j) \) for one \( i \in [k] \), so, w.l.g suppose that \( i' = i \) for each \( i' \in [k-1] \). Now, as \( H[V_{i'}] \) is \( G \)-free and \( \delta = 1 \), we have \( v_{i'} = v_i \), otherwise \( G \subseteq H[V_{i'}] \) a contradiction again. Hence it easy to check that \( V_{i'} = V(H_j') \) for each \( i' \in [k] \) and this coloring is unique.

Therefore by Claim by 15 if \( \delta(G) = 1 \) then the coloring provided is unique. Now, by considering \( \delta(G) \) we have the following claim:

**Claim 16.** If, \( \delta(G) \geq 2 \), then the coloring provided \( H \) is not unique, where \( H \) is a graph as defined in part I.

**Proof.** As \( \chi(H, V_i) = k \), suppose that \( V_1, V_2, \ldots, V_k \) be are coloring of \( V(H) \), where, \( V_i = V(H_i) \) for each \( i = 1, 2, \ldots, k \), therefore, \( H[V_i] \) is \( G \)-free. Consider \( V_i \), for \( i = 1, 2 \), as \( v_i \in V_i \) and \( |N(v_i) \cap V_i| = \delta - 1 \) for each \( i \in \{1, 2\} \) and \( \delta \geq 2 \), w.l.g suppose that \( v_1 \in N(v_i) \cap V_i \) for each \( i \in \{1, 2\} \). Now, considering \( V'_1, V'_2, \ldots, V'_k \) be are partition of \( V(H) \), where \( V'_i = V_i \) for each \( i = 3, 4, \ldots, k \), and \( V'_1 = V_1 \setminus \{v'_1\} \cup \{v'_2\} \) and \( V'_2 = V_2 \setminus \{v'_2\} \cup \{v'_1\} \). Therefore, by definition \( H \), one can check that \( v_1v'_2 \in E(H), v_2v'_1 \in E(H), \) and \( |N(v_i) \cap V'_i| = \delta - 1 \) for each \( i \in \{1, 2\} \). Hence, as, \( V_i = V'_i \) for each \( i \geq 3 \), and for \( i = 1, 2, |V'_i| = m \), and \( |N(v_i) \cap V'_i| = \delta - 1 \), we can say that \( H[V'_i] \) is \( G \)-free. Hence, since \( V_i \neq V'_i \) for \( i = 1, 2 \), we have the coloring provided \( H \) is not unique.

Therefore by Claim 16 if \( \delta(G) \geq 2 \) then the coloring provided is not unique and by considering any \( v \in V_i \) and \( v' \in V_j \) for each \( i, j \in [k] \) we can say there exist many \( k \)-G-free colourable of a graphs \( H \) with \( km \) \( m \)-members.

To prove (III), assume that \( H \) be a \( u \)-k-G-f with \( km - 1 \) members. Suppose that \( V_1, V_2, \ldots, V_k \) be the partition of \( V(H) \), so that \( H[V_i] \) is \( G \)-free for each \( i \in [k] \) and suppose that \( |V_k| = m - 1 \). Therefore, since \( H \) be a \( u \)-k-G-f, one can say that \( H[V_i \cup \{v\}] \) contain a copy of \( G \) say \( G_i \) consist of \( v \), for each \( v \in V_k \) and each \( i \leq k - 1 \). W.l.g assume that \( v_1 = N(u) \cap G_i \). Set \( H' = H \cup \{u\} \), where \( u \) is adjacent to all vertices of \( V'_1 \), for each \( i = 1, 2, \ldots, k - 1 \). Now, in \( H' \), for \( i = 1, \ldots, k \) set \( W_i \), so that \( W_i = V_i \) for each \( i \leq k - 1 \) and \( W_k = V_k \cup \{u\} \). As, \( W_i = V_i \) for each \( i \leq k - 1 \), \( N(u) \cup V_k = \emptyset \) and \( H \) be a \( u \)-k-G-f, one can check that, \( H' \) be a \( u \)-k-G-f graph with \( km \) members. By this way it is easy to check that for each \( i \geq 1 \) there exists a uniquely \( k \)-G-free colourable graph with \( km + i - 1 \) members.

Suppose that \( H \) is uniquely \( k \)-G-free graph where \( |E(H)| = e_1 \) and \( |E(G)| = e_2 \), in the following theorem we determine a lower bound for the size of \( E(H) \).

**Theorem 17.** Let \( H \) and \( G \) be two graph and \( k \geq 1 \) be a integers. Suppose that \( H \) is \( u \)-k-G-f, where \( |V(G)| = m, \delta(G) = \delta, |E(H)| = e_1 \) and \( |E(G)| = e_2 \), then:

\[
e_1 \geq k(e_2 - \delta + \frac{k+1}{2}(m\delta)).
\]

**Proof.** Suppose that \( |V(H)| = n, |V(G)| = m \) and let \( V_1, V_2, \ldots, V_k \) be a uniquely \( k \)-partition of \( V(H) \) where \( H[V_i] \) is a \( G \)-free class. As \( H \) is \( u \)-k-G-f, one can say that for any \( i \in \{1, 2, \ldots, k\} \)
and each $v \in V_i$, we have $G \subseteq H[V_j \cup \{v\}]$, for each $j \neq i$, that is $|E(H[V_i])| \geq e_2 - \delta$ and $|N_{V_j}(v)| \geq \delta$. Therefore it is easy to check that, $e_1 = \sum_{i=1}^{k} |E(H[V_i])| + \sum_{i=1}^{k-1} e_{ij}$ where $e_{ij} = \sum_{j=i+1}^{k} |E(H[V_i, V_j])|$ for each $i \in \{1, 2, \ldots, k\}$. It is easy to say that $e_{ij} \geq (k - i)m\delta$ for each $i$. Hence, one can check that:

$$e_1 \geq k(e_2 - \delta) + \frac{k(k+1)}{2}(m\delta) = k(e_2 - \delta + \frac{k+1}{2}(m\delta)).$$

Which means that the proof is complete. ■

In the next results we give some attribute about uniquely $k$-G-free colouring. It is easy to check that the following results is true.

**Corollary 18.** Let $H$ and $G$ be two graph and $k \geq 1$ be a integers. Assume that $H$ is $u$-$k$-$G$-$f$ where $|V(G)| = m$, $\delta(G) = \delta$, hence we have:

- For each $t \leq k - 1$, the subgraph induced on the union of any $t$ colour-classes of the unique colouring is an uniquely $t$-G-free colourable graph.
- Each vertex $v \in V(H)$ is adjacent with at least $\delta$ vertex in every colour class other than the colour class containing $v$, which means that in $H$, $\delta(H) \geq (k - 1)\delta$.

**Theorem 19.** Let $H$ and $G$ be two graph and $k \geq 1$ be a integers. Assume that $H$ is $u$-$k$-$G$-$f$ where $|V(G)| = m$, $\delta(G) = \delta$, hence; If for a vertex $v$ of $H$ we have $\deg(v) = (k - 1)\delta$ and the colour class of $v$ contains more than $m$ vertex, then $H \setminus \{v\}$ is also $u$-$k$-$G$-$f$.

**Proof.** Suppose that $H$ be a $u$-$k$-$G$-$f$, and assume that $V_1, V_2, \ldots, V_k$ be the partition of $V(H)$, so that $H[V_i]$ is $G$-free, $|V_i| = m+1$ and there exist a vertex of $V_i$ say $v$ so that $\deg(v) = (k - 1)\delta$. As $H$ be a $u$-$k$-$G$-$f$, one can say that $|N(v) \cap V_i| = \delta$ for each $i \in \{2, 3, \ldots, k\}$. Otherwise if there exist at least one $i$, so that $|N(v) \cap V_i| \leq \delta - 1$, then one can say that $H[V_i \cup \{v\}]$ is a $G$-class, hence set $V'_i = V_i \setminus \{v\}$, $V'_i = V_i \cup \{v\}$ and $V'_j = V_j$ for each $j \in [k], j \neq 1, i$. Therefore, for each $i \in [k], H[V'_i]$ is $G$-free, a contradiction. Now considering the following claim:

**Claim 20.** $H' = H \setminus \{v\}$ is $u$-$k$-$G$-$f$.

**Proof.** As $|V_i| \geq m+1$, and $H$ be a $u$-$k$-$G$-$f$, one can say that $H' = H \setminus \{v\}$ is $k$-$G$-$f$-colourable, by considering $V_1 \setminus \{v\}, V_2, \ldots, V_k$. Now, by contrary suppose that the coloring provided $H'$ is not unique. Therefore as $H$ be a $u$-$k$-$G$-$f$ and the coloring provided $H'$ is not unique, then there exist a vertex $u$ of some $V_i(i \geq 2)$ say $u \in V_i$ so that $V'_i = V_i \setminus \{v\}$, $V'_i = V_i \setminus \{u\}$ and $V'_j = V_j$ for each $j \geq 2$ and $j \neq i$ be a new $k$-$G$-$f$-color class of $V(H')$. W.l.g we may suppose that $i = 2$. As $H$ be a $u$-$k$-$G$-$f$, one can say that $uvw \in E(H)$. Now as $uw \in E(H)$ and $|N(v) \cap V_i| = \delta$ for each $i \in \{2, 3, \ldots, k\}$, one can check that $V'_i = (V_i \setminus \{u\}) \setminus \{v\}$, $V'_2 = (V_2 \setminus \{u\}) \cup \{v\}$ and $V'_j = V_j$ for each $j \geq 3$ be a new $k$-$G$-$f$-color class of $V(H)$, a contradiction. ■

Hence, by Claim 20 the proof is complete. ■

3. $G$-free minimal

**Definition 21.** A graph $H$ is vertex-minimal with respect to $G$-free, or $G$-free-vertex-minimal for short, if for every vertex of $V(H)$, $\chi_G(H \setminus \{v\}) \leq \chi_G(H) - 1$.

In this section we investigate some properties of $G$-free-vertex-minimal ($G$-$f$-$v$-$m$) coloring of graphs and give some lower bounds on the vertices of $G$-$f$-$v$-$m$ graphs $H$. The following theorem establishes a lower bound for the size of $G$-$f$-$v$-$m$ coloring of $H$. 
Theorem 22. Let $H$ and $G$ be two graph and $k \geq 1$ be a integers. Assume that $H$ is $k$-G-f-v-m graphs, then:

$$|V(H)| \geq (k-1)(|V(G)| - 1) + 1.$$  

Proof. Suppose that $|V(H)| = n$, $|V(G)| = m$ and let $V_1, V_2, \ldots, V_k$ denote the partite sets of $V(H)$, so that $H[V_i]$ for each $i \in [k]$ is G-free and $H[V_i]$ is a maximal G-free class of $H \setminus \bigcup_{j=1}^{i-1} V_j$ for each $i = i, \ldots, k$. As $\chi_G(H) = k$, we can say that $|V_k| \geq 1$. Assume that $v \in V_k$. Therefore by maximality $V_i$, one can say that $H[V_i \cup \{v\}]$ for each $i \leq k - 1$, contain a copy of $G$, that is $|V_i| \geq m - 1$ for each $1 \leq i \leq k - 1$. Hence:

$$|V(H)| \geq (k-1)(|V(G)| - 1) + 1.$$  

By Theorem 22, it is easy to say that the following results is true.

Theorem 23. Let $H$ and $G$ be two graph where $|V(H)| = (k-1)(|V(G)| - 1) + 1$ for some $k \geq 2$. If there exist a subsets of $V(H)$ say $S$, where $|S| \geq |V(G)|$, then:

$$\chi_G(H) \leq k - 1.$$  

By Theorem 23, one can say that $K_{(k-1)(|G|-1)+1}$ be a $k$-G-free graphs, for each $G$. We shall now construct k-G-f-v-m graphs with order $(k-1)(|V(G)| - 1) + 1$. Suppose that for $i = 1, 2, \ldots k - 1$, $H_i$ denote a $K_{(m-2)}$ and $H_k \cong K_k$, where $k \leq m - 1$. Set $H_1 = \bigoplus_{i=1}^{k-1} H_i \cong K_{(k-1)(m-2)}$ and suppose that $V(H_k) = \{v_1, v_2, \ldots, v_k\}$. Let for each $i \in \{1, 2, \ldots, k - 1\}$ denote a $H_i' = H_i \cup \{v_i\}$, where $v_i$ is adjacent to $m - 3$ vertices of $V(H_i)$ and $H_k' = \{v_k\}$, say $v_i w_{m-2} \not\in E(H')$, where $w_{m-2}$ be a vertex of $H_i$ for each $i \in \{1, 2, \ldots, k - 1\}$. Now, set $H^* = \bigoplus_{i=1}^{k-1} H_i'$. Not that $|V(H^*)| = (k-1)(m-1)+1$ and $\delta(H^*) = (k-1)-(m-1)-1$. Assume that $F$ denote a family of connected subgraph of $K_m$ with $m$ vertices and minus the edges of a $tK_2$ where $t = \frac{m}{2}$. Now, we have the following theorems:

Theorem 24. For each $G \in F$, $H^*$ is $k$-G-free vertex-minimal.

Proof. Since $k \leq m - 1$, we can say that $H_i$ is G-free, that is $\chi_G(H) \leq k$. As $H_1 \cong K_{(k-1)(m-2)} \subseteq H^*$, $|\omega(H^*)| \geq (k-1)(m-2)$ and by denote $H^*$, we can check that $|\omega(H^*)| = (k-1)(m-2)+1$. W.l.g, suppose that $V' = \{x_1, \ldots, x_{(k-1)(m-2)}\} \subseteq V(H^*)$, where $H^*[V'] \cong K_{(k-1)(m-2)}$. Now, we have the claims as follow:

Claim 25. For any $S$ subsets of $V(H^*)$ with $m$ member, $G \subseteq H^*[S]$.

Proof. Suppose that $S = \{s_1, \ldots, s_m\}$. Consider $|S \cap V'| \geq m - 1$, then one can say that $K_m \setminus e \subseteq H^*[S]$, therefore $G \subseteq H^*[S]$. Now, suppose that $|S \cap V'| \leq m - 2$. Assume that $|\omega(H^*[S])| = t$, and w.l.g suppose that $S' = \{s_1, \ldots, s_t\} \subseteq S$, where $H^*[S'] \cong K_t$. Consider $\overline{S'} = S \setminus S'$. As $|\omega(H^*[S])| = t$, we can say that $N_H(s) \cap S' \leq t - 1$ for each $s \in \overline{S'} = \{s_{t+1}, \ldots, s_m\}$. As, $\delta(H^*) = (k-1)(m-1) - 1$ and $H^*[S] \subseteq H^*$, we can say that $N_H(s) \cap S' = t - 1$ for each $s \in S'$ and $H^*[\overline{S'}] \cong K_{m-t}$, otherwise we can find a vertex of $S$ say $w$, so that $deg_{H^*}(w) \leq (k-1)(m-1)-2$, a contradiction. Since $|\omega(H^*[S])| = t$ and $H^*[\overline{S'}] \cong K_{m-t}$, $t \geq m - t$, that is $m \leq 2t$. Therefore we can check that $H^*[S] \supseteq (K_m \setminus \frac{m-2}{2}K_2)$. Now, as $G \in F$ and $t \geq \frac{m}{2}$, we have $G \subseteq K_m \setminus \frac{m-2}{2}K_2 \subseteq H^*[S]$.  

Therefore, as $|V(H^*)| = (k-1)(m-1)+1$ and by Claim 25, we can say that $\chi_G(H) = k$. Assume that $v$ be any vertex of $V(H^*)$, set $H'' = H^* \setminus \{v\}$. As $|V(H'')| = (k-1)(m-1)$, and $|G| = m$, we can decomposition of $V(H'')$ into $k - 1$ class, where each class have $m - 1$ member, that is $\chi_G(H'') \leq k - 1$, which means that, $H^*$ is k-G-f-v-m, and the proof of theorem is complete.

As $H^*$ is k-G-f-v-m, it is easy to say that, for each graphs $G$ with $m$ vertices and each subgraphs $H$ of $K_{(k-1)(m-1)+1}$, such that $H^* \subseteq H$, we have $H$ is k-G-f-v-m.
Definition 26. A graph $H$ is minimal with respect to $G$-free, or $G$-free-minimal, if for every edges of $E(H)$ we have $\chi_G(H \setminus \{e\}) = \chi_G(H) - 1$.

In this following results we investigate some properties of $G$-free-minimal coloring of graphs and give a lower bounds on the vertices of $G$-free-minimal graphs $H$. By argument similar to the proof of Theorem 22 in the following theorem establishes a lower bound for the size of $G$-free-minimal coloring of $H$.

Theorem 27. Let $H$ and $G$ be two graph and $k \geq 1$ be a integers. Suppose that $H$ is $k$-G-free-minimal graphs, then:

$$|V(H)| \geq (k - 1)(|V(G)| - 1) + 1.$$ 

By Theorem 23 one can say that $K_{(k-1)(m-1)+1}$ be a $k$-G-free-minimal graphs, for $G = K_m$. We shall now construct $k$-G-free-minimal graphs with order $(k - 1)(|V(G)| - 1) + 1$ for some graph $G$. Suppose that $R$ denote a subgraphs of $K_{(k-1)(m-1)+1}$ minus the edges of a $tK_2$, and $G$ denote a subgraphs of $K_m$ minus the edges of a $tK_2$, where $t \leq \frac{m}{2} - 1$. Now, we have the following theorems:

Theorem 28. $R$ is $k$-G-free-minimal.

Proof. As $|R| = (k - 1)(m - 1) + 1$, we can say that $\chi_G(R) \leq k$. As, $R$ denote a subgraphs of $K_{(k-1)(m-1)+1}$ minus the edges of $tK_2$, assume that $V(R) = \{v_1, v_2, \ldots, v_{(k-1)(m-1)+1}\}$ and w.l.g suppose that $E(tK_2) = \{e_1, e_2, \ldots, e_t\}$, where for each $i \in \{1, 2, \ldots, t\}$, $e_i = v_i$ for $i = 1, 2, \ldots, t$. Set $V_1 = \{v_1, \ldots, v_t\}$, $V_2 = \{v_{m+1}, \ldots, v_{m+t}\}$ and $V_3 = V(R) \setminus (V_1 \cup V_2)$. Now, to prove $\chi_G(R) = k$, by Claim 29 we can check that the following claims is true:

Claim 29. For any $S$ subsets of $V(R)$ with $m$ member, $G \subseteq R[S]$.

Therefore, as $|R| = (k - 1)(m - 1) + 1$ by Claim 29 we can say that $\chi_G(R) = k$. Now, as $|V(R)| = (k - 1)(m - 1) + 1$, assume that $v$ be any vertex of $V(R)$, set $R' = R \setminus \{v\}$. As $|V(R')| = (k - 1)(m - 1)$, and $|G| = m$, we can decomposition of $V(R')$ into $k - 1$ class, where each class have $m - 1$ member, that is $\chi_G(R') = k - 1$, which means that, $R$ is $k$-G-free-vertex-minimal. Hence, to prove $R$ is $k$-G-free-minimal, we most show that for any edges of $E(R)$, say $e$, $\chi_G(R \setminus e) = k - 1$. Now, to prove $\chi_G(R \setminus e) = k - 1$, we have the following claim:

Claim 30. For any edges of $E(R)$, say $e$, $\chi_G(R \setminus e) = k - 1$.

Proof. Suppose that $e = vv' \in E(R)$, now by considering $v$ and $v'$, we have three cases as follow:

Case 1: $v, v' \in V_i$ for some $i, i \in \{1, 2\}$. W.l.g we may suppose that $v, v' \in V_1$. Set $S = V_1 \cup V_2$, one can say that $|S| \geq m$. Now, as $v, v' \in S$ and $vv' \in E(R)$, we can check that $|N(x) \cap S| \leq d - 3$, for each $x \in \{v, v'\}$, therefore $R[S]$ is $G$-free. Now as, $|V(H)| = (k - 1)(m - 1) + 1$, $|V(G)| = m$ and $|S| \geq m$, one can say that $\chi_G(R \setminus e) = k - 1$.

Case 2: $v \in V_i$ and $v' \in V_j$, where $i \neq j$ and $i, j \in \{1, 2\}$. The proof is same as Case 1.

Case 1: $v, v \in V_3$. Set $S = V_1 \cup V_2 \cup \{v, v'\}$, one can say that $|S| \geq m$. Now, as $v, v' \in S$ and $vv' \in E(R)$, we can check that $R[S] \cong K_m \setminus \frac{m}{2}K_2$, therefore as $t \leq \frac{m}{2} - 1$, $R[S]$ is $G$-free. Now, as, $|V(H)| = (k - 1)(m - 1) + 1$, $|V(G)| = m$ and $|S| \geq m$, one can say that $\chi_G(R \setminus e) = k - 1$.

Therefore by Cases 1, 2, 3 we have the claim is true, which means that the proof is complete. ■

Suppose that $H$ is a connected subgraphs of $K_{(k-1)(m-1)+1}$, and $G$ denote a subgraphs of $K_m$ minus the edges of a $\frac{m}{2}K_2$, where $m$ is even and $\chi_G(H) = k$. Now, we have the following theorems:

Theorem 31. $H$ is $k$-G-free-minimal iff either $H$ is a graph $K_{(k-1)(m-1)+1}$ minus the edges of a 1-factor, when $k$ is even, or $H \cong H_1 \oplus K_1$, where $H_1$ is a graph $K_{(k-1)(m-1)}$ minus the edges of a 1-factor, when $k$ is odd.
Proof. Suppose that, either $H$ is a graph $K_{(k-1)(m-1)+1}$ minus the edges of a 1-factor, when $k$ is even, or $H \cong H_1 \oplus K_1$, where $H_1$ is a graph $K_{(k-1)(m-1)-1}$ minus the edges of a 1-factor, when $k$ is odd. As $\chi_G(H) = k$, suppose that $e$ be a arbitrary edges of $E(H)$. Therefore, by definition $H$, one can check that $\delta(H \setminus e) = (k-1)(m-1) - 1$, that is there exists at least one vertex of $V(H)$, say $v$, so that $\deg(v) = (k-1)(m-1) - 1$, and w.l.g suppose that $v', v'' \notin N(v)$. Now, suppose that $S$, be a subset of $V(H)$ with $m$, member, where $v, v', v'' \in S$. As $v, v', v'' \in S$, and $G$ denote a subgraphs of $K_m$ minus the edges of a $\frac{m}{2}$, one can check that $H[S]$ is a $G$-free. Therefore, since $|V(H)| = (k-1)(m-1) + 1, |V(G)| = m, |S| = m$, and $H[S]$ is a $G$-free, it is easy to say that $\chi_G(H \setminus e) = k - 1$, which means that $H$ is $k$-$G$-free-minimal.

Suppose that $H$ is $k$-$G$-free-minimal, and $k$ is even. As $H$ is $k$-$G$-free-minimal and $m, k$ is even, one can say that $\delta(H) \geq (k-1)(m-1) - 1$. Therefore, as $|V(H)| = (k-1)(m-1) + 1$ and $\delta(H) \geq (k-1)(m-1) - 1$, one can check that $K_{(k-1)(m-1)+1}$ minus the edges of a 1-factor be a subgraph of $H$. Now, assume that $H' \subset H$, where $H'$ is a graph $K_{(k-1)(m-1)+1}$ minus the edges of a 1-factor. So there exist at least two vertices of $V(H)$ say $v, v'$, so that $\delta(x) = (k-1)(m-1)$, for each $x \in \{v, v', v''\}$, that is $vv' \in (H)$. Consider $e = vv'$, as $H' \subset H$ and $e \notin E(H')$, we have $H' \subseteq H \setminus e$. Which means that $k = \chi_G(H') \leq \chi_G(H \setminus e) = k - 1$, a contradiction. For the case that $k$ is add, as $k$ is add and $m$ is even, we have $(k-1)(m-1) + 1$ is add. As $H$ is $k$-$G$-free-minimal we can say that $\delta(H) \geq (k-1)(m-1) - 1$. Therefore, as, $|V(H)| = (k-1)(m-1) + 1$, $\delta(H) \geq (k-1)(m-1) - 1$ and $(k-1)(m-1) + 1$ is add, one can check that $K_{(k-1)(m-1)+1}$ minus the edges of a $tK_2$ be a subgraph of $H$, where $t = \frac{(k-1)(m-1)}{2}$. Hence, there exist a vertex of $V(H)$ say $v$ so that $\deg(v) = (k-1)(m-1)$.

Now, assume that $H' \subset H$, where $H'$ is a graph $K_{(k-1)(m-1)+1}$ minus the edges of $t'K_2$ be a subgraph of $H'$, where $t' = \frac{(k-1)(m-1)}{2} - 1$. So there exist at least three vertices of $V(H)$ say $v, v', v''$, so that $\delta(x) = (k-1)(m-1)$, for each $x \in \{v, v', v''\}$, that is $vv'' \in (H)$. Consider $e = vv''$, as $H' \subset H$ and $e \notin E(H')$, we have $H' \subseteq H \setminus e$. Which means that $k = \chi_G(H') \leq \chi_G(H \setminus e) = k - 1$, a contradiction to $H$ is $k$-$G$-free-minimal.

Therefore in any case if $H$ is $k$-$G$-free-minimal then, either $H$ is a graph $K_{(k-1)(m-1)+1}$ minus the edges of a 1-factor, when $k$ is even, or $H \cong H_1 \oplus K_1$, where $H_1$ is a graph $K_{(k-1)(m-1)-1}$ minus the edges of a 1-factor, when $k$ is odd, which means that the proof is complete.

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