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ON LIMIT THEOREMS FOR FIELDS OF MARTINGALE DIFFERENCES

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Abstract. We prove a central limit theorem for stationary multiple (random) fields of martingale differences \( f \circ T_i, \ i \in \mathbb{Z}^d \), where \( T_i \) is a \( \mathbb{Z}^d \) action. In most cases the multiple (random) fields of martingale differences is given by a completely commuting filtration. A central limit theorem proving convergence to a normal law has been known for Bernoulli random fields and in [V15] this result was extended to random fields where one of generating transformations is ergodic.

In the present paper it is proved that a convergence takes place always and the limit law is a mixture of normal laws. If the \( \mathbb{Z}^d \) action is ergodic and \( d \geq 2 \), the limit law need not be normal.

For proving the result mentioned above, a generalisation of McLeish’s CLT for arrays \((X_{n,i})\) of martingale differences is used. More precisely, sufficient conditions for a CLT are found in the case when the sums \( \sum_i X_{n,i}^2 \) converge only in distribution.

The CLT is followed by a weak invariance principle. It is shown that central limit theorems and invariance principles using martingale approximation remain valid in the non-ergodic case.

1. Introduction and central limit theorems. In the study of limit theorems for dependent random variables an important role has been played by the central limit theorem for ergodic sequences of martingale differences, found independently by P. Billingsley and I.A. Ibragimov (cf. [B61], [I]). In the non-ergodic case the theorem remains true but the limit law is a mixture of non normal laws (cf. [HaHe]), the result has been proved and reproved many times; in [V89] ergodic decomposition of an invariant measure, as in this paper, is used. Here, we will study the CLT for random fields of martingale differences. By a random field we understand a field of random variables \( f \circ T_i, \ i \in \mathbb{Z}^d \), on a probability space \((\Omega, \mathcal{A}, \mu)\) where \( f \) is a measurable function on \( \Omega \) and \( T_i \) are automorphisms of \((\Omega, \mathcal{A}, \mu)\) for which \( T_i \circ T_j = T_{i+j} \). We denote \( f \circ T_i = U_i f \). Recall that the \( \mathbb{Z}^d \) action is **ergodic** if the only measurable sets \( A \) for which \( T_i A = A \) are of measure zero or one (for \( d = 1 \) we speak of an ergodic transformation \( T = T_i \)). Limit theorems for random fields of martingale differences have been studied by e.g. Basu and Dorea [BaDo], or by Nahapetian [N]. An approach using so called projection method was used in [D]. For a random field, there are several non-equivalent definitions of martingale differences.

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For martingale approximations and for weak invariance principles we will use the notion of orthomartingales as in e.g. [Go09], [WaW] (cf. also [K]):

For $\tilde{i} = (i_1, \ldots, i_d) \in \mathbb{Z}^d$, $\tilde{i} \leq \tilde{j}$ means that $i_k \leq j_k$, $k = 1, \ldots, d$; $\tilde{i} \wedge \tilde{j} = (\min\{i_1, j_1\}, \ldots, \min\{i_d, j_d\})$. $(\mathcal{F}_\tilde{i})_{\tilde{i} \in \mathbb{Z}^d}$ is a completely commuting invariant filtration if

(i) $\mathcal{F}_{\tilde{i}} = T_{-\tilde{i}} \mathcal{F}_0$ for all $\tilde{i} \in \mathbb{Z}^d$,

(ii) $\mathcal{F}_{\tilde{i}} \subset \mathcal{F}_{\tilde{j}}$ for $\tilde{i} \leq \tilde{j}$,

(iii) $\mathcal{F}_{\tilde{i}} \cap \mathcal{F}_{\tilde{j}} = \mathcal{F}_{\tilde{i} \wedge \tilde{j}}$, $\tilde{i}, \tilde{j} \in \mathbb{Z}^d$,

(iv) $E\left(E(f|\mathcal{F}_{\tilde{i}})|\mathcal{F}_{\tilde{j}}\right) = E(f|\mathcal{F}_{\tilde{i} \wedge \tilde{j}})$, for every integrable function $f$.

By $\mathcal{F}_l^{(q)}$, $1 \leq q \leq d$, $l \in \mathbb{Z}$, we denote the $\sigma$-algebra generated by the union of all $\mathcal{F}_{\tilde{i}}$ with $i_q \leq l$. For $d = 2$, $\mathcal{F}_{\infty,j} = \mathcal{F}_2^{(2)}$ denotes the $\sigma$-algebra generated by the union of all $\mathcal{F}_{i,j}$, $i \in \mathbb{Z}$, and in the same way we define $\mathcal{F}_{i,\infty}$.

By $P_l^{(q)}$, $1 \leq q \leq d$, $l \in \mathbb{Z}$, we denote the operator in $L^p$, $1 \leq p < \infty$, which sends $f \in L^p$ to $E(f|\mathcal{F}_l^{(q)}| - E(f|\mathcal{F}_{-l+1}^{(q)}|)$. Notice that for $p = 2$, $P_l^{(q)}$ is the orthogonal projection onto the Hilbert space $L^2(\mathcal{F}_l^{(q)}) \cap L^2(\mathcal{F}_{-l+1}^{(q)})$. $P_l^{(q)}$ are mutually commuting and idempotent operators. For $\tilde{i} = (i_1, \ldots, i_d)$ we define $P_\tilde{i} = \Pi_{q=1}^{d} P_{i_q}^{(q)}$. In $L^2$, $P_\tilde{i}$ is the orthogonal projection onto $\bigcap_{1 \leq q \leq d} L^2(\mathcal{F}_{i_q}^{(q)}) \cap L^2(\mathcal{F}_{i_{q-1}}^{(q)})$. The functions $P_\tilde{i}f$ are called martingale differences. Let $e_\tilde{i}$ be the vector from $\mathbb{Z}^d$ with 1 at $i$-th coordinate and 0 elsewhere. It can be noticed that if $f = P_\tilde{i}f$ then for $1 \leq q \leq d$, $(f \circ T_{\tilde{i}}^q)$ is a martingale difference sequence with respect to the filtration $(\mathcal{F}_l^{(q)})$.

For proofs and for more properties of the operators $P_\tilde{i}$ the reader can consult (e.g.) [WaW], [VWa].

By central limit theorem we understand weak convergence of the distribution of $(1/\sqrt{n_1 \ldots n_d}) \sum_{i_1=1}^{n_1} \cdots \sum_{i_d=1}^{n_d} f \circ T_{(i_1, \ldots, i_d)}$ as $\min\{n_1, \ldots, n_d\} \to \infty$.

In one dimensional case, the central limit theorem for stationary martingale differences led to many results using martingale approximations. The same holds true for higher dimension. Pioneering results were given in [WaW] where under a multiparameter version of reinforced Maxwell-Woodroofe condition a CLT and weak invariance principle were proved, and in [Go09] where the martingale-coboundary representation was studied (an interesting application of Gordin’s result can be found in [DeGo]). The results from [WaW] were improved in [VWa] where the Hannan’s condition was generalised to random fields, in [PZ] where the CLT was proved under “classical” Maxwell-Woodroofe condition and in [Gi] where the weak invariance principle (under the same assumptions) was proved. The martingale-coboundary representation has been studied (after [Go09]) in [EGi], [V16], and [Gi]. More results were published in (e.g.) [BiDu].

For orthomartingale differences, ergodicity of the $\mathbb{Z}^d$ action (with $d \geq 2$) does not guarantee a convergence to a normal law. An example can be found in the article by Wang and Woodroofe [WaW]; we use its presentation from [V15]. The idea of the example has probably appeared already in the Ph.D thesis of Hillel Furstenberg.

Example. Let the probability space $(\Omega, \mathcal{A}, \mu)$ be a product of $(\Omega_1, \mathcal{A}_1, \mu_1)$ and $(\Omega_2, \mathcal{A}_2, \mu_2)$. On $(\Omega_1, \mathcal{A}_1, \mu_i)$ there is a bimeasurable and measure preserving bijection $T_i$, $i = 1, 2$. On $(\Omega, \mathcal{A}, \mu)$ we define an action of $\mathbb{Z}^2$ by $T_{i,j}(x, y) = (T_i^1x, T_2^jy)$. 

Let us suppose that there is a random variable $e_i$ such that $e_i \circ T^j_i$, $j \in \mathbb{Z}$ are iid $\mathcal{N}(0,1)$ random variables generating the $\sigma$-algebra $\mathcal{A}_i$, $i = 1, 2$. $T^j_i$ is then an ergodic $\mathbb{Z}^d$ action (cf. [V15]). For $e = (e_1, e_2)$, $(e \circ T^j_{i,j})(i,j) \in \mathbb{Z}^2$ is then a field of martingale differences for the natural filtration (cf. [V15]). As we can easily see, for any integers $n, m \geq 1$, the sum $(1/\sqrt{nm}) \sum_{i=1}^n \sum_{j=1}^m e \circ T^j_i$ is distributed as a product $XY$ of two independent $\mathcal{N}(0,1)$ random variables.

A convergence to a normal law is guaranteed if the $\mathbb{Z}^d$ action is Bernoulli, i.e. the $\sigma$-algebra $\mathcal{A}$ is generated by iid random variables $e \circ T^j_i$, $i \in \mathbb{Z}^d$. This assumption was used e.g. in [WaW] or [BiDu].

In [V15] it has been proved that if one of the transformations $T_{e_i}$ (recall that $e_i$ is the vector from $\mathbb{Z}^d$ with 1 at $i$-th coordinate and 0 elsewhere) is ergodic, then for a random field of square integrable martingale differences the central limit theorem takes place with a normal law for limit. In [CDV], the result was extended to reversed martingales and an invariance principle was proved.

The problem we study in this paper is whether without any ergodicity assumption there still is a convergence to a limit law (as in the example above), or whether it can happen that there is no convergence at all. Theorem 1 gives a positive answer showing that there always is a limit law which is a mixture of normal laws.

In the theorem we will use a notion of a field of martingale differences which is weaker than the notion defined above.

Let $T^j_i$ be a $\mathbb{Z}^d$ action on $(\Omega, \mathcal{A}, \mu)$. We say that $(\mathcal{F}^{(q)}_i)_{i \in \mathbb{Z}, 1 \leq q \leq d}$ is a multiple filtration if for every $1 \leq q \leq d$, $(\mathcal{F}^{(q)}_i)_{i \in \mathbb{Z}}$ is a filtration with

$$
\mathcal{F}^{(q)}_i \subset T^{-e_q} \mathcal{F}^{(q)}_i = \mathcal{F}^{(q)}_{i+1}, \quad i \in \mathbb{Z}, \quad 1 \leq q \leq d,
$$

$$
T^{-e_q} \mathcal{F}^{(q)}_i = \mathcal{F}^{(q)}_i \quad \text{for all} \quad 1 \leq q' \leq d, q' \neq q, \quad i \in \mathbb{Z}.
$$

If $f \in L^1$ and there is a multiple filtration $(\mathcal{F}^{(q)}_i)_{i \in \mathbb{Z}, 1 \leq q \leq d}$ such that $f = E(f \mid \mathcal{F}^{(q)}_0) - E(f \mid \mathcal{F}^{(q)}_{-1})$ for all $1 \leq q \leq d$, we say that $(f \circ T^j_i)_i$ is a multiple field of martingale differences. If $f \in L^2$ we then have $f \in L^2(\mathcal{F}^{(q)}_0) \otimes L^2(\mathcal{F}^{(q)}_{-1}) = U_{e,q'} \left( L^2(\mathcal{F}^{(q)}_0) \otimes L^2(\mathcal{F}^{(q)}_{-1}) \right)$ for $1 \leq q' \leq d, q' \neq q$.

**Theorem 1.** Let $f \in L^2$ be such that $(f \circ T^j_i)_i$ is a multiple field of martingale differences. If $n_j \to \infty$, $j = 1, \ldots, d$ then the random variables

$$
\frac{1}{\sqrt{n_1 \cdots n_d}} \sum_{i_1=1}^{n_1} \cdots \sum_{i_d=1}^{n_d} f \circ T^{(i_1, \ldots, i_d)}
$$

converge in distribution to a law with characteristic function $E \exp(-\eta^2 t^2/2)$ for a positive random variable $\eta^2$ such that $E\eta^2 = \|f\|_2^2$. The random variables

$$
\frac{1}{n_1 \cdots n_d} \sum_{i_1=1}^{n_1} \left( \sum_{i_2=1}^{n_2} \cdots \sum_{i_d=1}^{n_d} f \circ T^{(i_1, \ldots, i_d)} \right)^2
$$

converge in distribution to $\eta^2$. 

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In order to prove Theorem 1 we use a version of McLeish’s theorem [Mc]. McLeish’s theorem generalises the Billingsley-Ibragimov’s theorem to triangular arrays \((X_{n,i})_{1 \leq i \leq k_n}\) of martingale differences which need not be stationary. Under the assumptions (i), (ii) from the Theorem 2 below on \(\max_i |X_{n,i}|\) and the assumption of convergence in probability of \(\sum_{i=1}^{k_n} X_{n,i}^2\) to a constant it gives a CLT for the sums \(\sum_{i=1}^{k_n} X_{n,i}\). The book [HaHe] brings several new versions of the theorem, in particular it is generalised to the case when the filtrations \((\mathcal{F}_{n,i})\), of the sequences \((X_{n,i})_i\) are nested and \(\sum_{i=1}^{k_n} X_{n,i}^2\) converges to a random variable \(\eta^2\) which is measurable with respect to the intersection of all \(\mathcal{F}_{n,i}\). In [GHu] the conditions (i), (ii) were replaced by \(\max_{1 \leq i \leq k_n} |X_{n,i}| \to 0\) in \(L^2\) and in [L] the \(L^2\) convergence was replaced by \(L^1\) convergence. Another generalisation/version of McLeish’s theorem was given in [PZ]. The convergence, nevertheless, is always in probability and a counterexample in [HaHe] shows that a convergence of \(\sum_{i=1}^{k_n} X_{n,i}^2\) in distribution is not sufficient without strengthening of other assumptions. Our Theorem 2 and its corollary, Proposition 3, bring a new version of McLeish’s theorem where the sums \(\sum_{i=1}^{k_n} X_{n,i}^2\) converge in distribution only.

**Theorem 2.** Let \(X_{n,j}, j = 1, \ldots, k_n\), be an array of martingale differences with respect to increasing filtrations \((\mathcal{F}^n_j)_{j \geq 0}, n = 1, 2, \ldots,\) such that

(i) \(\max_{1 \leq i \leq k_n} |X_{n,i}| \to 0\) in probability,

(ii) there is an \(L < \infty\) such that \(E \max_{1 \leq j \leq k_n} X_{n,j}^2 \leq L\) for all \(n\),

(iii) there exist \(1 \leq \ell(n) \leq k_n\) and \(\mathcal{F}^n_{\ell(n)}\)-measurable random variables \(\eta_n^2\) such that \(\sum_{j=1}^{k_n} X_{n,j}^2 - \eta_n^2 \to 0\) in probability,

(iv) there exists a random variable \(\eta^2\) such that \(\eta_n^2 \to \eta^2\) in distribution,

(v) \(E(T_n(t)|\mathcal{F}^n_{\ell(n)}) \to 1\) in \(L^1\) for every \(t \in \mathbb{R}\), where

\[T_n(t) = \prod_{j=1}^{k_n} (1 + itX_{n,j}).\]

Then the sums \(\sum_{j=1}^{k_n} X_{n,j}\) converge in distribution to a law with characteristic function \(E \exp(-\eta^2 t^2/2)\).

As a corollary we get the next proposition.

**Proposition 3.** Let \(X_{n,j}, j = 1, \ldots, k_n\), be an array of martingale differences with respect to increasing filtrations \((\mathcal{F}^n_j)_{j \geq 0}, n = 1, 2, \ldots,\) such that assumptions (i) - (iv) are satisfied and

(vi) the random variables \(\eta_n^2\) are \(\mathcal{F}^0\)-measurable

or

(vii) the sequences \((X_{n,j})_j\) are strictly stationary and \(\eta_n^2\) are measurable with respect to the \(\sigma\)-algebras of invariant sets.

Then the conclusion of Theorem 2 holds.

**2. Proofs of Theorems 1,2 and of Proposition 3.**

**Proof of Theorem 2.** As in [HaHe] (cf. the proof of Theorem 3.2) we for a \(C > 0\) define the stopping time

\[J_n = J_{C,n} = \begin{cases} j & \text{if } 1 \leq j \leq k_n, \sum_{u=1}^{j-1} X_{n,u}^2 \leq C, \sum_{u=1}^{j} X_{n,u}^2 > C, \\
        k_n & \text{if } \sum_{u=1}^{k_n} X_{n,u}^2 \leq C \end{cases}\]
and replace $X_{n,j}$ by martingale differences

$$X'_{n,j} = X_{n,j}I_{j \leq J_n}, \quad j = 1, \ldots, k_n.$$  

Notice that $j \leq J_n$ iff $\sum_{u=1}^{j-1} X_{n,u}^2 \leq C$. We denote

$$\eta^2 = \eta^2 1_{\eta^2 \leq C} + C1_{\eta^2 > C}, \quad \eta^2_{n, C} = \eta^2 1_{\eta^2 \leq C} + C1_{\eta^2 > C}, \quad n \geq 1.$$  

From (iv) it follows $\eta^2_{n, C} \to \eta_C^2$ in distribution; from (i), (iii) it follows that $\sum_{j=1}^{k_n} X_{n,j}^2 - \eta^2_{n, C} \to 0$ in probability.

We have verified (iii), (iv) for $X'_{n,j}$ and $\eta_C^2$, $\eta^2_{n, C}$. (i) and (ii) follow from $|X'_{n,j}| \leq |X_{n,j}|$.

To see (v), notice that $J_n$ is a finite stopping time; by $\mathcal{F}_{j_n}$ we denote the corresponding $\sigma$-algebra. Then $\|E(T_n(t)\mathcal{F}_{j_n}) - 1\|_1 = \|E(E(T_n(t)\mathcal{F}_{j_n}) - 1\mathcal{F}_{j_n})\|_1$ and we get (v) for $X'_{n,j}$ by the contraction property of conditional expectation.

From (iii), (iv) we deduce

$$\lim_{C \to \infty} \limsup_{n \to \infty} \mu(\exists 1 \leq j \leq k_n, X'_{n,j} \neq X_{n,j}) = 0.$$  

It is thus sufficient to prove the theorem for $X'_{n,j}$, $\eta_C^2$ and $\eta^2_{n, C}$. For simplicity of notation we denote $X'_{n,j} = X_{n,j}$ for all $n, j$ and $\eta^2 = \eta_C^2$, $\eta^2_n = \eta^2_{n, C}$. Using (ii) we then get

(1) \hspace{1cm} E|T_n(t)|^2 = E \prod_{j=1}^{k_n} (1 + i t X_{n,j})^2 = E \prod_{j=1}^{k_n} (1 + t^2 X_{n,j}^2) \leq e^{t^2 C} (1 + t^2 L).$$

As in [Mc] we use the equality

$$e^{ix} = (1 + ix) \exp(-\frac{1}{2} x^2 + r(x))$$

where $|r(x)| \leq |x|^3$ for $|x| \leq 1$ ($x \in \mathbb{R}$) and define

$$U_n(t) = \exp \left( -\frac{1}{2} t^2 \sum_{j=1}^{k_n} X_{n,j}^2 + \sum_{j=1}^{k_n} r(tX_{n,j}) \right), \quad t \in \mathbb{R}.$$  

For $I_n(t) = \exp(it S_n) = T_n(t)U_n(t)$ we then have

$$I_n(t) = T_n(t) \left( U_n(t) - \exp(-\frac{1}{2} t^2 \eta^2_n) \right) + T_n(t) \exp(-\frac{1}{2} t^2 \eta^2_n).$$

In the same way as in [Mc] we from (i), (iii) deduce that $\sum_{j=1}^{k_n} r(tX_{n,j}) \to 0$ in probability. Therefore, $U_n(t) - \exp(-\frac{1}{2} t^2 \eta^2_n) \to 0$ in probability. $I_n(t)$ are uniformly bounded, hence uniformly integrable; by (1), $T_n(t)$ are uniformly integrable.

Therefore,

$$T_n(t) \left( U_n(t) - \exp(-\frac{1}{2} t^2 \eta^2_n) \right) \to 0 \text{ in } L^1.$$
Because \( \eta_n^2 \) are \( \mathcal{F}_{\ell(n)}^n \)-measurable (cf. (iii)) we have
\[
E \exp(-\frac{1}{2} t^2 \eta_n^2) [T_n(t) - 1] = E \left[ \exp(-\frac{1}{2} t^2 \eta_n^2) \left[ E(T_n(t) | \mathcal{F}_{\ell(n)}^n) - 1 \right] \right];
\]
by (v), uniform integrability of \( T_n(t), \) and (iv) we deduce
\[
ET_n(t) \exp(-\frac{1}{2} t^2 \eta_n^2) \rightarrow E \exp(-\frac{1}{2} t^2 \eta_n^2)
\]
which finishes the proof.

\[ \square \]

Proof of Proposition 3. If the assumptions (i) - (iv) and (vi) are fulfilled, we for all \( n \) define \( \ell(n) = 0; \) in the same way as in [Mc] or [HaHe] we can see that \( E(T_n(t) | \mathcal{F}_{\ell(n)}^n) = 1 \) for all \( n, t. \)

Let the assumptions (i) - (iv) and (vii) be fulfilled. For each of the sequences \((X_n,j) \) there exists a measure preserving transformation \( T = T(n) \) such that \( X_n,j = X_n,0 \circ T_j \) for all \( j. \) By \( \mathcal{I} = \mathcal{I}_n \) we denote the \( \sigma \)-algebra of \( T \)-invariant (measurable) sets. As shown in [V87], for every \( \mathcal{F}_{\ell(n)}^n \)-measurable and integrable function \( f \) we have
\[
E(f | \mathcal{F}_{\ell(n)}^n) = E(f | \mathcal{F}_{\ell(n)}^n \vee \mathcal{I}) \text{ where } \mathcal{F}_{\ell(n)}^n \vee \mathcal{I} \text{ is the } \sigma \text{-algebra generated by } \mathcal{F}_{\ell(n)}^n \cup \mathcal{I}.
\]
We thus can replace \( \mathcal{F}_{\ell(n)}^n \) by \( \mathcal{F}_{\ell(n)}^n \vee \mathcal{I} \) and hence get (vi).

In both cases the result follows from Theorem 2.

\[ \square \]

Proof of Theorem 1.

First, we prove the theorem for \( d = 2. \) The general case can be proved by induction. Recall that we denote \( U_{i,j} f = f \circ T_{i,j}. \)

If \( f \) is not bounded, for a \( K > 0 \) we define
\[
f' = P_{0,0}(f 1_{|f| \leq K}), \quad f'' = f - f'.
\]
Because \( U_{i,j} f \) and \( U_{i,j} f' \) are martingale differences, \( U_{i,j} f'' \) are martingale differences as well. For \( K \) big we can have the \((L^2)\) norm of \( f'' \) small while \( f' \) is bounded. Without loss of generality we thus can suppose that \( f \) is bounded (and from now on, we shall do so).

For a given positive integer \( v \) and positive integers \( n, \) define
\[
F_{i,v} = \frac{1}{\sqrt{v}} \sum_{j=1}^{v} U_{i,j} f, \quad i \in \mathbb{Z}, \text{ and } X_{n,i} = X_{n,i,v} = \frac{1}{\sqrt{n}} F_{i,v}, \quad i = 1, \ldots, n.
\]
Clearly, \( X_{n,i} \) are martingale differences for the filtration \((\mathcal{F}_{i,\infty}),\) . We will prove that there exists a random variable \( \eta^2 \) (not necessarily at the same probability space) such that for any sequence of \( v(n) \nearrow \infty \)
\[
E \exp \left( it \sum_{j=1}^{n} X_{n,j,v(n)} \right) \rightarrow E \exp(-\frac{1}{2} t^2 \eta^2)
\]
for all \( t \in \mathbb{R} \) as \( n \rightarrow \infty. \)

The proof will use several lemmas. Their proofs will be given later.
Lemma 4. The sequences \((X_{n,v})_v\) satisfy assumptions (i), (ii) of Theorem 2 with \(k_n = n\) uniformly for all \(v \geq 1\).

The next statement is adapted from [V15].

Lemma 5. The processes \((F_{i,v})_i\) weakly converge to a strictly stationary process \((V_i)_i\) of martingale differences defined on a probability space with measure \(\nu_1\). The distribution of \(V_1\) is a mixture of normal laws with zero means and uniformly bounded variances and there exists a random variable \(\eta_1\) such that

\[
EV_1^2 = Ef^2, \quad E|V_1|^p < \infty, \quad \frac{1}{m} \sum_{u=1}^{m} V_u^2 \to \eta_1^2 \quad \text{a.s. (}\nu_1\text{)} \quad \text{and in L}^p(\nu_1)
\]

for all \(1 \leq p < \infty\).

Lemma 6. There exist integers \(v(m), m \geq 1\), such that for any \(t \in \mathbb{R}, m \to \infty\), and uniformly for all \(v \geq v(m)\)

\[
E\exp\left(it \sum_{j=1}^{m} F_{j,v}\right) \to E\exp(-\frac{1}{2} t^2 \eta_1^2).
\]

Similarly as we defined the functions \(F_{i,v}\) we define

\[G_{u,j} = \frac{1}{\sqrt{m}} \sum_{i=1}^{u} U_{i,j} f\]

and by the same proof as in Lemma 5 we get that there exists a probability measure \(\nu_2\) on \(\mathbb{R}^\mathbb{Z}\) such that for coordinate projections \(W_j\) from \(\mathbb{R}^\mathbb{Z}\) to \(\mathbb{R}\), the processes \((G_{u,j})_j\) converge in distribution to a process \((W_j)_j\) and

\[
\frac{1}{m} \sum_{v=1}^{m} W_v^2 \to \eta_2^2 \quad \text{a.s. (}\nu_2\text{)} \quad \text{and in L}^p(\nu_2), \quad p \geq 1
\]

where \(\eta_2^2\) has all moments finite.

By \(\mathcal{I}_1\) let us denote the \(\sigma\)-algebra of sets \(A \in \mathcal{A}\) for which \(T_{1,0}^{-1} A = A\). Suppose that \(v\) is fixed. By Birkhoff’s ergodic theorem there exists an integrable function

\[
\eta_{v,1}^2 = E(F_{i,v}^2 | \mathcal{I}_1) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} F_{i,v}^2 \quad \text{a.s. (}\mu\text{)}.
\]

Lemma 7. For \(v \to \infty\),

\[
\eta_{v,1}^2 \overset{D}{\to} \eta_2^2
\]

where \(\overset{D}{\to}\) denotes convergence in distribution.

The next statement follows from [B68, Theorem 5.3].

Let \(X, X_n, n \geq 1\), be random variables, \(p \geq 1\).

\[
\text{If } X_n \overset{D}{\to} X \text{ then } E|X|^p \leq \liminf_{n \to \infty} E|X_n|^p.
\]
Define

\[ Y_{n,v}^2 = \frac{1}{n} \sum_{i=1}^{n} F_{i,v}^2 = \sum_{i=1}^{n} X_{n,i,v}^2; \]

in the proof of Lemma 6 it is shown that there exist \( v(n), n = 1, 2, \ldots, \) such that for any sequence of \( v_n \geq v(n), \)

\begin{equation}
Y_{n,v_n}^2 = \frac{1}{n} \sum_{i=1}^{n} F_{i,v_n}^2 \xrightarrow{D} \eta_1^2
\end{equation}

for \( n \to \infty. \) By Lemma 7, for every \( n, \)

\[ E(Y_{n,v}^2 \mid I_1) = E(F_{i,v}^2 \mid I_1) = \eta_{v,1}^2 \xrightarrow{D} \eta_2^2 \text{ for } v \to \infty. \]

From now on we denote \( Y_n^2 = Y_{n,v_n}^2. \) Without loss of generality we can suppose \( v(n) \to \infty. \)

**Lemma 8.** \( Y_n^2 - E(Y_n^2 \mid I_1) \to 0 \text{ in } L^1. \)

From Lemma 8, Lemma 7, and (5) it follows:

**Lemma 9.** \( \eta_1^2 \) and \( \eta_2^2 \) are equally distributed.

Now, we finish the proof of Theorem 1 for \( d = 2. \) Recall that

\[ Y_{n,v}^2 = \frac{1}{n} \sum_{i=1}^{n} F_{i,v}^2 = \sum_{i=1}^{n} X_{n,i,v}^2, \]

and there exist \( v(n) \to \infty \) such that for any sequence of \( v_n \geq v(n), \)

\[ Y_n^2 = Y_{n,v_n}^2 = \frac{1}{n} \sum_{i=1}^{n} F_{i,v_n}^2 \xrightarrow{D} \eta_1^2. \]

Denote \( \bar{Y}_n^2 = E(Y_n^2 \mid I_1). \) We have, for \( N > n \) and \( m = \lceil N/n \rceil \) (the integer part of \( N/n) \)

\[
\frac{1}{N} \sum_{i=1}^{N} F_{i,v}^2 = \frac{1}{N} \sum_{j=0}^{m-1} \sum_{i=1}^{n} F_{j,n+i,v}^2 + \frac{1}{N} \sum_{i=mn+1}^{N} F_{i,v}^2 = \\
\frac{mn}{N} \bar{Y}_n^2 + \frac{mn}{N} \sum_{j=0}^{m-1} \left( \frac{1}{n} \sum_{i=1}^{n} F_{j,n+i,v}^2 - \bar{Y}_n^2 \right) + \frac{1}{N} \sum_{i=mn+1}^{N} F_{i,v}^2.
\]

By Lemma 8 and stationarity, \( \|(1/n) \sum_{i=1}^{n} F_{j,n+i,v}^2 - \bar{Y}_n^2\|_1 \to 0 \) uniformly in \( j, \) and for \( N/n \to \infty \) the last term goes to zero in \( L^1 \) as well. This proves that for any sequence of \( v = v_n \geq v(n) \) and any sequence of \( N_n \) with \( N_n/n \to \infty \) there exist \( I_1 \)-measurable random variables \( \eta(n)^2 = \bar{Y}_n^2 \) such that

\[ \|\eta(n)^2 - \frac{1}{N} \sum_{i=1}^{N} F_{i,v}^2\|_1 \to 0 \text{ and } \eta(n)^2 \xrightarrow{D} \eta_1^2 \]
(the second statement follows from Lemma 7 and Lemma 9).

The conditions (i)-(iv) and (vii) of Proposition 3 are thus satisfied with \( \eta^2 = \eta_1^2 \).

The random variables \( \sum_{i_1=1}^{n_1} \sum_{i_2=1}^{n_2} f \circ T_{(i_1,i_2)}, n_1, n_2 \to \infty \), thus weakly converge to a law with characteristic function \( \varphi(t) = \int \exp(-\eta_1^2 t^2/2) \, d\nu_1 \) where the measure \( \nu_1 \) was defined in Lemma 5.

The second statement of Theorem 1 follows from (5).

This finishes the proof for \( d = 2 \).

**Proofs of Lemmas 4 - 8 (d = 2)**

**Proof of Lemma 4.** The proof is well known to follow from stationarity. For reader’s convenience we recall it here.

(i) For any \( \epsilon > 0 \), uniformly for all integers \( v \geq 1 \) we have

\[
\mu(\max_{1 \leq i \leq n} |X_{n,i}| > \epsilon) \leq \sum_{i=1}^{n} \mu(|X_{n,i}| > \epsilon) = n \mu(|F_{0,v}| > \epsilon \sqrt{n}) \leq
\]

\[
\frac{1}{\epsilon^2} E \left( \frac{1}{\sqrt{v}} \sum_{j=1}^{v} U_{0,j}f \right)^2 \leq \frac{1}{\epsilon^2} \sum_{j=1}^{v} U_{0,j}f \leq 1
\]

as \( n \to \infty \). To see that the convergence is uniform for all \( v \), notice that \( U_{0,j}f, j \in \mathbb{Z} \), are martingale differences, and hence by McLeish’s CLT, \( (1/\sqrt{v}) \sum_{j=1}^{v} U_{0,j}f \) are uniformly stochastically bounded. This proves (i).

To see (ii) we note

\[
\left( \max_{1 \leq i \leq n} |X_{n,i}| \right)^2 \leq \sum_{i=1}^{n} X_{n,i}^2 = \frac{1}{n} \sum_{i=1}^{n} \left( \frac{1}{\sqrt{v}} \sum_{j=1}^{v} U_{i,j}f \right)^2
\]

which implies \( E \left( \max_{1 \leq i \leq n} |X_{n,i}| \right)^2 \leq Ef^2 \leq 1 \).

\[ \square \]

**Proof of Lemma 5.** For a finite set \( J \subset \mathbb{Z} \) and for \( a = (a_i; i \in J) \in \mathbb{R}^J \), consider the sums

\[
\sum_{i \in J} a_i \sum_{j=1}^{v} U_{i,j}f, \quad v \to \infty.
\]

Without loss of generality (cf. [V89]) we can suppose that for the transformation \( T_{0,1} \) there exists an ergodic decomposition of \( \mu \) into ergodic components \( m_\omega \); each \( m_\omega \) is a probability measure invariant and ergodic for \( T_{0,1} \). There exists a measure \( \tau \) on \( (\Omega, \mathcal{A}) \) such that for \( A \in \mathcal{A} \), \( \mu(A) = \int m_\omega(A) \, \tau(d\omega) \). The random variables \( \sum_{i \in J} a_i U_{i,j}f, j = 1, 2, \ldots, \) are strictly stationary martingale differences and by Birkhoff’s ergodic theorem,

\[
\frac{1}{v} \sum_{j=1}^{v} \left( \sum_{i \in J} a_i U_{i,j}f \right)^2 \to \eta(a)^2 \quad \text{a.s. } (\mu)
\]

for some integrable \( T_{0,1} \)-invariant function \( \eta(a)^2 \). For almost every ergodic component \( m_\omega, \eta(a)^2 \) is a.s. a constant equal to \( \int \left( \sum_{i \in J} a_i U_{i,0}f \right)^2 \, dm_\omega \). By McLeish’s CLT
(cf. Proposition 3) the random variables \((1/\sqrt{v}) \sum_{j=1}^{v} \left( \sum_{i \in J} a_{ij} U_{i,j} \right)\) converge in distribution to \(N(0, \eta(a)^2)\) (in \((\Omega, \mathcal{A}, m_{\omega})\)). By the Cramer-Wold device, for an ergodic component \(m_{\omega}\) given and \(v \to \infty\), the distributions of the random vectors \((F_{i,v}; i \in J)\) thus weakly converge to a multidimensional normal law \(\nu_{\omega} = \nu_{J,\omega}\).

For the measure \(\mu\) we deduce that \((F_{i,v}; i \in J)\) weakly converge to a mixture of multidimensional normal laws.

To show this, denote \(F_{j,v} = (F_{i,v}; i \in J)\). We have shown that there are measures \(\nu_{\omega}\) on \(\mathbb{R}^J\) such that for a bounded and continuous function \(g\) on \(\mathbb{R}^J\), \(\int_{\Omega} g \circ F_{j,v} d m_{\omega} \to \int_{\mathbb{R}^J} g d \nu_{\omega}\). We thus have \(\int_{\Omega} g \circ F_{j,v} d \mu = \int_{\Omega} \int_{\mathbb{R}^J} g \circ F_{j,v} d m_{\omega} \tau(d\omega) \to \int_{\Omega} \int_{\mathbb{R}^J} g d \nu_{\omega} \tau(d\omega)\), and hence the measure \(\nu\) on \(\mathbb{R}^J\) defined by \(\nu(A) = \int_{\Omega} \nu_{\omega}(A) \tau(d\omega)\) is the weak limit of \(\mu \circ F_{j,v}^{-1}\) for \(v \to \infty\).

This way we get a projective system of probability measures on \(\mathbb{R}^Z\) and following Kolmogorov’s theorem there exists a measure \(\nu_1\) on \(\mathbb{R}^Z\) such that for coordinate projections \(V_u\) from \(\mathbb{R}^Z\) to \(\mathbb{R}\) and for any finite \(J \subset \mathbb{Z}\), the vectors \((F_{i,v})_{i \in J}\) converge in distribution to \((V_u)_{u \in J}\) as \(v \to \infty\). By strict stationarity of the sequences \((F_{u,v})_u, v \geq 1\), the measure \(\nu_1\) is shift-invariant hence the process \((V_u)_{u}\) is strictly stationary.

In the construction above, for \(J = \{1\}\), \(\nu_{\omega} = N(0, \int f^2 d m_{\omega})\) with \(\int f^2 d m_{\omega} \leq \|f\|_{L_2}\) \(< \infty\). Therefore \(E V_1 = 0\), \(E|V_1|^2 = \int |f|^2 d m_{\omega} \tau(d\omega) = \|f\|_{L_2}^2\), and for \(1 \leq p < \infty\), \(E|V_1|^p = \int |x|^p \nu_{\omega}(dx) \tau(d\omega) < \infty\).

By Birkhoff’s ergodic theorem there exists a random variable \(\eta_1\) such that for all \(p \geq 1\), \(\eta_1^2 \in L^p(\nu_1)\) and \((1/m) \sum_{u=1}^{m} V_u^2 \to \eta_1^2\) a.s. \(\nu_1\) and in \(L^p(\nu_1)\).

Finally, we show that \(V_i\) are martingale differences. It is sufficient to prove that for any bounded measurable function \(g\) on \(\mathbb{R}^n\) we have \(\int V_0 g(V_{-n}, \ldots, V_{-1}) d \nu_1 = 0\). Let us denote by \((F_k)_k\) the natural filtration of the process \((V_j)_j\). Because \(EF_{0,v}^2 = \int V_0^2 \, d \nu_1\), \(v \geq 1\), and the vectors \((F_{-n,v}, \ldots, F_{0,v})\) converge in distribution to \((V_{-n}, \ldots, V_0)\) for \(v \to \infty\), \(F_{-n,v}, \ldots, F_{-1,v}\) are uniformly integrable and converge in law to \(V_0 g(V_{-n}, \ldots, V_{-1})\). Therefore, \(0 = E[F_{0,v} g(F_{-n,v}, \ldots, F_{-1,v})] \to \int V_0 g(V_{-n}, \ldots, V_{-1}) \, d \nu_1\) for \(v \to \infty\).

\[\square\]

**Proof of Lemma 6.**

Recall that \(V_j\) are stationary martingale differences in \(L^2\). By McLeish’s CLT (cf. Proposition 3) the partial sums \((1/\sqrt{m}) \sum_{j=1}^{m} V_j\) thus converge in distribution to a law \(\mathcal{L}\) with characteristic function \(E \exp(-\frac{1}{2} t^2 \eta_1^2)\). We can choose \(m\) large enough so that the distribution of \((1/\sqrt{m}) \sum_{j=1}^{m} V_j\) is sufficiently close to \(\mathcal{L}\) and then \(v(m)\) so big that for \(v \geq v(m)\) the law of \((F_{1,v}, \ldots, F_{m,v})\) is sufficiently close to the law of \((V_1, \ldots, V_m)\).

\[\square\]

Remark that the lemma can also be proved using Theorem 2: By Lemma 4, for \(X_{m,i,v} = F_{i,v} / \sqrt{m}\) the assumptions (i), (ii) are satisfied (we put \(k_m = m\), uniformly for all \(v\). By Lemma 5 the random vectors \((F_{i,v}, \ldots, F_{m,v})\) converge in law to \((V_1, \ldots, V_m)\), as \(v \to \infty\). For any sequence \(\ell(m) \to \infty\) with \(1 \leq \ell(m) \leq m\) there exist integers \(v(m)\) such that

\[
\frac{1}{\ell(m)} \sum_{i=1}^{\ell(m)} F_{i,v}^2 = \eta_{\ell(m),1}^2 \to \eta_1^2 \text{ in distribution, as } m \to \infty,
\]
uniformly for all \( v \geq v(m) \). We have \( E(T_m(t) \mid F_{\ell(m)}, \infty) = \prod_{j=1}^{\ell(m)} (1 + it X_{m,j}) \). By (i), (ii) we deduce that we can choose \( \ell(m) \) growing slowly enough so that for every \( t \in \mathbb{R} \), \( E(T_m(t) \mid F_{\ell(m)}, \infty) \to 1 \) in \( L^1 \) as \( m \to \infty \). For \( \eta_{v,m}^2 = \eta_{v,\ell(m),1}^2 \) the conditions (iii), (iv), (v) of Theorem 2 are thus satisfied. □

**Proof of Lemma 7.** Let \( t \in \mathbb{R} \). By McLeish’s theorem (cf. [HaHe, Theorem 3.2] or Theorem 2), for \( v \) fixed and \( n \to \infty \)

\[
E \exp \left( it \frac{1}{\sqrt{n}} \sum_{i=1}^{n} F_{i,v} \right) \to E \exp \left( -\frac{1}{2} t^2 \eta_{v,1}^2 \right).
\]

By definition we have

\[
\frac{1}{\sqrt{n}} \sum_{i=1}^{n} F_{i,v} = \frac{1}{\sqrt{v}} \sum_{j=1}^{v} G_{n,j}
\]

and by Lemma 6 applied to the functions \( G_{n,j} \), for every \( v \) there exists an \( n(v) \) such that for \( v \to \infty \)

\[
E \exp \left( it \frac{1}{\sqrt{v}} \sum_{j=1}^{v} G_{n,j} \right) \to E \exp \left( -\frac{1}{2} t^2 \eta_2^2 \right)
\]

uniformly for \( n \geq n(v) \). For a given \( \epsilon > 0 \) we thus can find \( v_0 \) big enough so that for all \( v \geq v_0 \) and all \( n \geq n(v) \),

\[
|E \exp \left( it \frac{1}{\sqrt{v}} \sum_{j=1}^{v} G_{n,j} \right) - E \exp \left( -\frac{1}{2} t^2 \eta_2^2 \right)| < \epsilon;
\]

keeping \( v \) fixed, by (3a) there is \( n \geq n(v) \) large enough so that

\[
|E \exp \left( it \frac{1}{\sqrt{n}} \sum_{j=1}^{n} F_{j,v} \right) - E \exp \left( -\frac{1}{2} t^2 \eta_{v,1}^2 \right)| < \epsilon.
\]

Therefore, \( |E \exp \left( -\frac{1}{2} t^2 \eta_2^2 \right) - E \exp \left( -\frac{1}{2} t^2 \eta_{v,1}^2 \right)| < 2\epsilon \). By properties of the Laplace transformation (cf. [F, Chapter XIII]), the convergence \( E \exp \left( -\frac{1}{2} t^2 \eta_{v,1}^2 \right) \to E \exp \left( -\frac{1}{2} t^2 \eta_2^2 \right), \ v \to \infty, \ t \in \mathbb{R} \), implies \( \eta_{v,1}^2 \overset{D}{\to} \eta_2^2 \).

□

**Proof of Lemma 8.** For \( k \geq 0 \) define

\[
\varphi_k(x) = \begin{cases} x & \text{if } |x| \leq k, \\ k \cdot \text{sign}(x) & \text{if } |x| > k. \end{cases}
\]

By Jensen’s inequality taken conditionally we have, for \( Z = \varphi_k(Y_n^2) \), \( E(Z^2 \mid I_1) \geq [E(Z \mid I_1)]^2 \) a.s. hence for every \( k, n \geq 1 \),

\[
E \left[ \varphi_k(Y_n^2) \right] \geq E \left[ E(\varphi_k(Y_n^2) \mid I_1) \right]^2.
\]
By (5), \( E\left[\varphi_k(Y_n^2)\right]^2 \xrightarrow{n \to \infty} E[\varphi_k(\eta_1^2)]^2 \) for every \( k \). We thus have

\[
\lim_{k \to \infty} \lim_{n \to \infty} E\left[\varphi_k(Y_n^2)\right]^2 = E\eta_1^4. \tag{7}
\]

By concavity of \( \varphi_k \) (on \([0, \infty)\)), \((1/n) \sum_{i=1}^n \varphi_k(F_{i,v_n}^2) \leq \varphi_k((1/n) \sum_{i=1}^n F_{i,v_n}^2) = \varphi_k(Y_n^2)\); we deduce that for every \( k \),

\[
E(\varphi_k(F_{1,v_n}^2) | I_1) = E\left(\frac{1}{n} \sum_{i=1}^n \varphi_k(F_{i,v_n}^2) | I_1\right) \leq E(\varphi_k(Y_n^2) | I_1). \tag{8}
\]

Recall that by Lemma 5 and Lemma 7, for \( n \to \infty \),

\[
F_{1,v_n} \xrightarrow{D} V_1, \quad EF_{1,v_n}^2 = Ef^2 = EV_1^2, \quad E(F_{1,v_n}^2 | I_1) \xrightarrow{D} \eta_2^2.
\]

By uniform integrability of \( F_{1,v_n}^2 \), for every \( \epsilon > 0 \) there exist a \( k \geq 1 \) and \( n(k) \) such that for all \( n \geq n(k), E|F_{1,v_n}^2 - \varphi_k(F_{1,v_n}^2)| < \epsilon \). By contractiveness of conditional expectation we get \( E[E(F_{1,v_n}^2 | I_1) - E(\varphi_k(F_{1,v_n}^2) | I_1)] < \epsilon \). For a given \( \delta > 0 \) we therefore can choose \( \epsilon > 0 \) small enough so that

\[
\mu\{|E(F_{1,v_n}^2 | I_1) - E(\varphi_k(F_{1,v_n}^2) | I_1)| > \delta\} < \delta. \tag{9}
\]

From \([E(F_{1,v_n}^2 | I_1)]^2 \xrightarrow{D} \eta_2^4\) and (4) it follows

\[
\lim_{n \to \infty} \inf E[E(F_{1,v_n}^2 | I_1)]^2 \geq E\eta_2^4.
\]

By (2), \( E\eta_2^4 < \infty \) (we can prove (2) for \( \eta_2 \) in the same way as for \( \eta_1 \)). By (9) we have \( \lim_{k \to \infty, n \geq n(k)} E(\varphi_k(F_{1,v_n}^2) | I_1)^2 \xrightarrow{D} \eta_2^4 \) hence using (4) again we deduce that for every \( \epsilon > 0 \) there are \( k(\epsilon) \) and \( n(\epsilon, k) \) such that for all \( k \geq k(\epsilon), n \geq n(\epsilon, k) \),

\[
E\left[E(\varphi_k(F_{1,v_n}^2) | I_1)^2 \geq E\eta_2^4 - \epsilon.
\]

Therefore,

\[
\lim_{k \to \infty} \lim_{n \to \infty} \inf \left[ E(\varphi_k(F_{1,v_n}^2) | I_1)^2 \geq E\eta_2^4. \tag{10}
\]

From (7), (6), (8), (10) we deduce that for every \( \epsilon > 0 \) there are \( k(\epsilon) \) and \( n(k, \epsilon) \) such that for \( k \geq k(\epsilon) \) and \( n \geq n(k, \epsilon) \)

\[
E\eta_1^4 + \epsilon \geq E[\varphi_k(Y_n^2)]^2 \geq E\left[E(\varphi_k(Y_n^2) | I_1)^2 \geq E\eta_2^4 - \epsilon.
\]

Therefore, \( E\eta_1^4 \geq E\eta_2^4 \) and by symmetry we get

\[
E\eta_1^4 = E\eta_2^4. \tag{12}
\]
Using (11) we deduce
\[
\lim_{k \to \infty} \limsup_{n \to \infty} \left\{ E\left[ \varphi_k(Y_n^2) \right]^2 - E\left[ \varphi_k(Y_n^2) \mid I_1 \right] \right\}^2 = 0.
\]
This implies
\[
\lim_{k \to \infty} \limsup_{n \to \infty} E\left[ \varphi_k(Y_n^2) - E\left( \varphi_k(Y_n^2) \mid I_1 \right) \right]^2 = 0,
\]
hence
\[
\lim_{k \to \infty} \limsup_{n \to \infty} \left\| \varphi_k(Y_n^2) - E\left( \varphi_k(Y_n^2) \mid I_1 \right) \right\|_1 = 0.
\]
Because \( EY_n^2 = Ef^2 = E\eta_1^2 \) and \( Y_n \overset{D}{\to} \eta_1^2 \), \( Y_n^2 \) are uniformly integrable, hence
\[
\lim_{k \to \infty} \limsup_{n \to \infty} \left\| Y_n^2 - \varphi_k(Y_n^2) \right\|_1 = 0.
\]
By triangular inequality and contractiveness of conditional expectation we get
\[
\lim_{k \to \infty} \limsup_{n \to \infty} \left\| Y_n^2 - E\left( \varphi_k(Y_n^2) \mid I_1 \right) \right\|_1 = 0,
\]
\[
\lim_{k \to \infty} \limsup_{n \to \infty} \left\| E\left( Y_n^2 \mid I_1 \right) - E\left( \varphi_k(Y_n^2) \mid I_1 \right) \right\|_1 = 0,
\]
therefore
\[
\lim_{n \to \infty} \left\| Y_n^2 - E\left( Y_n^2 \mid I_1 \right) \right\|_1 = 0;
\]
this finishes the proof of Lemma 8.

\[\square\]

The case of \( d > 2 \)

Now we give an idea of a proof of the theorem for \( d > 2 \). Suppose that \( d > 2 \) and for dimension \( d - 1 \), the theorem has been proved. Using the same reasoning as before we can restrict ourselves to the case of \( f \) bounded.

For each \( k \in \{1, 2, \ldots, d\} \), let us denote by \( \mathcal{I}_k \) the vectors \( \mathbf{i} = (i_1, i_2, \ldots, i_d) \in \mathbb{Z}^d \) such that \( i_k = 0 \). To \( \mathbf{i} \in \mathbb{Z}^{d-1} \) we thus can associate a \( \mathbf{i}_k \in \mathbb{Z}^d \) and having a \( \mathbb{Z}^d \) action we thus get a \( \mathbb{Z}^{d-1} \) action by associating \( \mathbf{i} \mapsto T_{\mathbf{i}_k} \). By the induction hypothesis (cf. the proof for \( d - 1 \)), by boundedness of \( f \), the sums \( (1/\sqrt{n^2}) \sum_{1 \leq j \leq n} f \circ T_{\mathbf{i}_k} \) converge in distribution to a law with characteristic function \( E\left( -\frac{1}{2} t^2 \eta_1^2 \right) \) with \( \eta \in L^p \) for all \( 1 \leq p < \infty \).

We will prove a convergence of \( (1/\sqrt{n}) \sum_{1 \leq j \leq n} f \circ T_{\mathbf{i}_k} \) to a limit law with characteristic function \( \varphi(t) = \exp(-\tilde{\eta}^2 t^2 / 2) \) where \( \tilde{\eta}^2 \in L^p \) for all \( 1 \leq p < \infty \). By \( \mathbf{i}, \mathbf{v} \) we denote an element of \( \mathbb{Z}^d \) where \( \mathbf{i} \in \mathbb{Z} \) and \( \mathbf{v} \in \mathbb{Z}^{d-1} \). We define
\[
F_{\mathbf{i}, \mathbf{v}} = \frac{1}{\sqrt{|\mathbf{v}|}} \sum_{j=1}^n U_{\mathbf{i}, \mathbf{v}} f, \quad X_{\mathbf{n}, \mathbf{i}} = X_{\mathbf{n}, \mathbf{i}, \mathbf{v}} = \frac{1}{\sqrt{|\mathbf{v}|}} F_{\mathbf{i}, \mathbf{v}}, \quad \mathbf{i} = 1, \ldots, n.
\]
For \( \mathbf{v} \) fixed, \( F_{\mathbf{i}, \mathbf{v}} \) are martingale differences for the filtration \( \mathcal{F}_i^{(1)} \). Similarly as in the preceding case we will prove that there exists a random variable \( \eta^2 \) such that for every \( t \in \mathbb{R} \) and any sequence \( \mathbf{v}(n) \to \infty \),
\[
E \exp\left( \frac{it}{2} \sum_{j=1}^n X_{\mathbf{n}, \mathbf{i}, \mathbf{v}(n)} \right) \to E\left( -\frac{1}{2} t^2 \eta^2 \right)
\]
as \( n \to \infty \).

Lemma 4 remains valid for \( X_{\mathbf{n}, \mathbf{i}, \mathbf{v}} \) with the same proof:
Lemma 4a. The sequences \((X_{n+v})\); satisfy assumptions (i), (ii) of Theorem 2 with \(k_n = n\) uniformly for all \(v \geq 1\).

For \(u \in \mathbb{Z}\) and \(j \in \mathbb{Z}^{d-1}\) we define

\[
G_{u,j} = \frac{1}{\sqrt{u}} \sum_{i=1}^{u} U_{i,j} f.
\]

Lemma 5a. For \(v \to \infty \) \((v \in \mathbb{Z}^{d-1})\) the sequences \((F_{u,v})\) converge in distribution to a sequence of martingale differences \((V_u)_{u \in \mathbb{Z}}\) where \(E V_1^2 = Ef^2\) and \(V_1 \in L^p\) for all \(1 \leq p < \infty\). We have \((1/m)\sum_{u=1}^{m} V_u^2 \to \eta_1^2\) a.s. and in \(L^p\), \(1 \leq p < \infty\).

The random fields \((G_{u,j})\) converge in distribution to a stationary random field \((W_j)\) defined on a space with a probability measure \(\nu_2\). For every \(1 \leq p < \infty\), \(W_j \in L^p(\nu_2)\), \(EW_1^2 = Ef^2 = EG_{u,j}^2\), and \((W_j)\) is a multiple field of martingale differences.

Proof. The first part of the Lemma (existence of the sequence \((V_u)\)) can be proved in the same way as Lemma 5, only instead of McLeish's theorem we use the CLT for \((d-1)\)-dimensional random fields from our assumptions. By assumptions, as mentioned above, the law of \(V_u\) has characteristic function \(E(-\frac{1}{2}t^2\eta_1^2)\) with \(\eta \in L^p\) for all \(1 \leq p < \infty\) hence \(V_u \in L^p\).

To prove the second part of the Lemma we can in the same way as before, using McLeish’s theorem, show that the random fields \((G_{u,j})\) converge in distribution to a stationary random field \((W_j)\) of coordinate projections of \(\mathbb{R}^{d-1}\) (onto \(\mathbb{R}\)) with the product \(\sigma\)-algebra of Borel sets and a probability measure \(\nu_2\) invariant with respect to the shifts; \(W_j \in L^p(\nu_2)\) for every \(1 \leq p < \infty\), \(EW_1^2 = Ef^2 = EG_{u,j}^2\) (for every \(u, j\)).

Let us show that \((W_j)\) is a multiple field of martingale differences. The \(\sigma\)-algebras \(\mathcal{F}_i^{(q)}\) generated by the random variables \(W_j\) with \(j_q \leq i\) are a multiple filtration. By definition, \(W_j\) is \(\mathcal{F}_j^{(q)}\)-measurable for all \(1 \leq q \leq d\) and using the same argument as in the proof of Lemma 5 we can show that \(E(W_j \mid \mathcal{F}_j^{(q-1)}) = 0\).

In the same way as in the proof of Lemma 5 we can show that \(EW_j^2 = Ef^2\) and \(W_j \in L^p\) for all \(1 \leq p < \infty\).

By Theorem 1 for dimension \(d - 1\), for the random field \((W_j)\) there exists a measurable function \(\eta_2^2\) such that for \(n \to \infty\), \((1/\sqrt{n}) \sum_{1 \leq j \leq n} \mathbf{1} \circ T_j\) converge in distribution to a limit law with characteristic function \(\varphi(t) = \int \exp(-\frac{1}{2}t^2/2) d\nu_2\).

In the same way as for \(d = 2\) we can prove the following lemma.

Lemma 6a. There exist \(v(m) \in \mathbb{Z}^{d-1}, m \geq 1, \) such that for \(m \to \infty\) and uniformly for all \(v \geq v(m)\)

\[
E \exp{(it \frac{1}{\sqrt{m}} \sum_{l=1}^{m} F_{l,v})} \to E \exp(-\frac{1}{2}t^2\eta_1^2), \quad t \in \mathbb{R}.
\]
There exist \(u(v) \geq 1\) and \(v \in \mathbb{Z}^{d-1}, v \geq 1\), such that for \(v \to \infty\) and uniformly for all \(u \geq u(v)\)

\[
E \exp \left( it \frac{1}{\sqrt{n}} \sum_{1 \leq j \leq v} U_j G_{u,j} \right) \to E \exp(-\frac{1}{2} t^2 \eta_2^2), \quad t \in \mathbb{R}.
\]

By \(\mathcal{I}_1\) let us denote the \(\sigma\)-algebra of sets \(A \in \mathcal{A}\) for which \(T_{v,1}^{-1} A = A\). Suppose that \(v \in \mathbb{Z}^{d-1}\) is fixed. By Birkhoff’s ergodic theorem there exists an integrable function

\[
\eta_{v,1}^2 = E(F_{1,v}^2 | \mathcal{I}_1) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} F_{i,v}^2.
\]

**Lemma 7a.** For \(v \to \infty\),

\[
\eta_{v,1}^2 \overset{D}{\to} \eta_2^2.
\]

Lemma 7a can be proved in the same way as Lemma 7.

As in the case of \(d = 2\) we define

\[
Y_{n,v}^2 = \frac{1}{n} \sum_{i=1}^{n} F_{i,v}^2 = \sum_{i=1}^{n} X_{n,v}^2;
\]

there exist \(v(n), n = 1, 2, \ldots\), such that for any sequence of \(v_n \geq v(n)\),

\[
(5a) \quad Y_{n,v_n}^2 = \frac{1}{n} \sum_{i=1}^{n} F_{i,v_n}^2 \overset{D}{\to} \eta_1^2
\]

for \(n \to \infty\). By Lemma 7, for every \(n\),

\[
E(Y_{n,v}^2 | \mathcal{I}_1) = E(F_{1,v}^2 | \mathcal{I}_1) = \eta_{v,1}^2 \overset{D}{\to} \eta_2^2 \text{ for } v \to \infty.
\]

From now on we denote \(Y_n^2 = Y_{n,v_n}^2\). Without loss of generality we can suppose \(v(n) \to \infty\). Now, Lemma 8 can be stated and proved as for \(d = 2\) and then the rest of the proof of Theorem 1 can be done as before. Notice that the assumption of \(f\) bounded leads in the CLT to a limit distribution with characteristic function

\[
E \exp(-\frac{1}{2} t^2 \eta_1^2) \text{ where } \eta_1 \in L^p \text{ for all } 1 \leq p < \infty.
\]

\(\square\)

**3. A weak invariance principle.**

By a weak invariance principle (WIP) we shall mean the WIP as defined in [VWa]. For \(n = (n_1, \ldots, n_d) \in \mathbb{N}^d\) and \(t = (t_1, \ldots, t_d) \in [0,1]^d\) we define \(n \cdot t = (n_1 t_1, \ldots, n_d t_d), [n \cdot t] = (\lfloor n_1 t_1 \rfloor, \ldots, \lfloor n_d t_d \rfloor)\); we denote

\[
S_{n,t}(f) = \sum_{0 \leq k \leq [nt]} \prod_{i=1}^{d} (k_i \wedge (n_i t_i - 1) - k_i + 1) U_k f, \quad s_{n,t}(f) = S_{n,t}(f) / \sqrt{n}.
\]

Notice that for \(d = 1\), \(S_{n,t}(f) = \sum_{k=0}^{\lfloor nt \rfloor} U_k f + (nt - \lfloor nt \rfloor) U_{\lfloor nt \rfloor + 1} f\) (here we denote \(U_0 = 1\)).
Theorem 10. Let $f \in L^2$, be such that $(f \circ T_{i})_{i} (i \in \mathbb{Z}^d)$ is a field of martingale differences for a completely commuting filtration $(\mathcal{F}_{i})_{i} (i \in \mathbb{Z}^d)$. If $n_{j} \to \infty$, $j = 1, \ldots, d$, then there exists a random variable $\eta \geq 0$ and a Brownian sheet $(B_{t})_{t \in [0,1]^d}$ independent of $\eta$ such that for $n_{1}, \ldots, n_{d} \to \infty$ $s_{n_{1}}(f)$ converge in distribution to $(\eta B_{t})_{t}$.

Proof. The proof can be done in the same way as in [CDV]. We have to prove convergence of finite dimensional distributions and tightness. For tightness, the proof from [VWa] remains valid (note that in [CDV] the same proof, adapted to weakly martingales, is used).

The convergence of finite dimensional distributions will be proved for $d = 2$; for $d > 2$ the result can be extended by induction. As in [CDV] it is necessary and sufficient to show that for partitions $0 = t_{0} < t_{1} < \cdots < t_{K} = 1$ and $0 = s_{0} < s_{1} < \cdots < s_{K} = 1$ of the interval $[0,1]$ and real constants $a_{k,l}$, $0 \leq k, l \leq K - 1$, and $n_{1}, n_{2} \to \infty$, the sums

$$
\frac{1}{\sqrt{n_{1}n_{2}}} \sum_{k=0}^{K-1} \sum_{l=0}^{K-1} a_{k,l} \sum_{i=[n_{1}t_{k}]}^{[n_{1}t_{k+1}]-1} \sum_{j=[n_{2}s_{l}]}^{[n_{2}s_{l+1}]-1} U_{i,j}f
$$

weakly converge to

$$
\eta \sum_{k=0}^{K-1} \sum_{l=0}^{K-1} a_{k,l} (B_{t_{k+1,s_{l+1}}} - B_{t_{k+1,s_{l}}} - B_{t_{k,s_{l+1}}} + B_{t_{k,s_{l}}})
$$

where $B_{t,s}$ is a Brownian sheet independent of $\eta$; $\eta \geq 0$ is defined as in the proof of Theorem 1.

Let us fix a $0 \leq k \leq K - 1$, let $J \subset \mathbb{Z}$ be a finite set and for $\bar{a} = (\bar{a}_{i}; i \in J) (\bar{a}_{i} \in \mathbb{R}$ are constants), consider, as in the proof of Lemma 5, the sums

$$
\sum_{i \in J} \bar{a}_{i} \sum_{l=0}^{K-1} \sum_{j=[t_{l}v]}^{[t_{l+1}v]-1} U_{i,j}f, \; v \to \infty.
$$

As in the proof of Lemma 5, let $m_{\omega}$ be invariant and ergodic probability measures for the transformation $T_{0,1}$ (ergodic decomposition of $\mu$). By Birkhoff’s ergodic theorem

$$
\frac{1}{v} \sum_{j=1}^{v} \left( \sum_{i \in J} \bar{a}_{i} U_{i,j}f \right)^{2} \to \eta(\bar{a})^2,
$$

$$
\frac{1}{v} \sum_{l=0}^{K-1} \sum_{j=[t_{l}v]}^{[t_{l+1}v]-1} \left( \sum_{i \in J} \bar{a}_{i} U_{i,j}f \right)^{2} \to \eta(\bar{a})^2 \sum_{l=0}^{K-1} (t_{l+1} - t_{l}) a_{k,l}^2
$$

(a.s. for $\mu$ hence also for almost all $m_{\omega}$). In the same way as in the proof of Lemma 5 we conclude that

$$
(1/\sqrt{v}) \sum_{i \in J} \sum_{l=0}^{K-1} \sum_{j=[t_{l}v]}^{[t_{l+1}v]-1} U_{i,j}f \overset{P}{\to} N(0, \eta(\bar{a})^2 \sum_{l=0}^{K-1} (t_{l+1} - t_{l}) a_{k,l}^2).
$$
In the same way as in the proof of Lemma 5 we deduce, using the Cramer-Wold device and Kolmogorov’s theorem on projective limit, that the processes

\[
\left( \frac{1}{\sqrt{n}} \sum_{l=0}^{K-1} a_{k,l} \sum_{j=[t_{l+1}]}^{[t_{l+1}v]-1} U_{i,j}f \right)_{i}
\]

converge in distribution to a process \(((\sum_{l=0}^{K-1} (t_{l+1} - t_l) a_{k,l}^2)^{1/2} V_i)_i\) where \((V_i)_i\) is the process found in the proof of Lemma 5.

In the same way as in the proof of Theorem 1 we deduce that for \(n_1, n_2 \to \infty\) the sums

\[
\frac{1}{\sqrt{n_1 n_2}} \sum_{i=1}^{n_1} \sum_{l=0}^{K-1} \sum_{j=[t_{l+1}]}^{[t_{l+1}n_2]-1} a_{k,l} \sum_{j=[t_{l}n_2]}^{[t_{l+1}n_2]-1} U_{i,j}f
\]

converge in distribution to the law with characteristic function \(E \exp(-\eta^2 c_k^2 t^2 / 2)\) where \(\eta^2\) is the same random variable as in Theorem 1 and \(c_k^2 = \sum_{l=0}^{K-1} (t_{l+1} - t_l) a_{k,l}^2\).

Recall that \(\mathcal{I}_1\) is the \(\sigma\)-algebra of \(T_{1,0}\)-invariant sets from \(\mathcal{A}\) and denote

\[
Y_{k,n_1,n_2}^2 = \frac{1}{n_1} \sum_{i=1}^{n_1} \frac{1}{n_2} \left( \sum_{l=0}^{K-1} \sum_{j=[t_{l}n_2]}^{[t_{l+1}n_2]-1} U_{i,j}f \right)^2.
\]

In the same way as in Lemma 8 we can see that for any \(\epsilon > 0\) there are \(n_1(\epsilon), n_2(\epsilon)\) such that for \(n_1 \geq n_1(\epsilon), n_2 \geq n_2(\epsilon)\) we have \(\|Y_{k,n_1,n_2}^2 - E(Y_{k,n_1,n_2}^2 | \mathcal{I}_1)\|_1 < \epsilon\).

From this we deduce that

\[
\frac{1}{n_1 n_2} \sum_{k_0}^{K-1} \sum_{i=[n_1 k_0]}^{[n_1 t_{k+1}]-1} \left( \sum_{l=0}^{K-1} \sum_{j=[n_2 s_l]}^{[n_2 s_{l+1}]-1} U_{i,j}f \right)^2 \rightarrow \eta^2 \sum_{l=0}^{K-1} \sum_{k=0}^{K-1} (t_{l+1} - t_l)(s_{k+1} - s_k) a_{k,l}^2
\]

in \(L^1\) for \(n_1, n_2 \to \infty\), in the sense that for any \(\epsilon > 0\) there are \(n_1(\epsilon), n_2(\epsilon)\) such that for \(n_1 \geq n_1(\epsilon), n_2 \geq n_2(\epsilon)\) the norm is smaller than \(\epsilon\).

Because Lemma 4 applies to the random variables

\[
X_{n_1,n_2,i} = \frac{1}{\sqrt{n_1 n_2}} \sum_{l=0}^{K-1} a_{k,l} \sum_{j=[n_2 s_i]}^{[n_2 s_{i+1}]-1} U_{i,j}f, \quad [n_1 t_k] \leq i < [n_1 t_{k+1}], 0 \leq k < K,
\]

by Proposition 3

\[
\frac{1}{\sqrt{n_1 n_2}} \sum_{k_0}^{K-1} \sum_{l=0}^{K-1} \sum_{i=[n_1 k_0]}^{[n_1 t_{k+1}]-1} \sum_{j=[n_2 s_i]}^{[n_2 s_{i+1}]-1} U_{i,j}f
\]

converge in distribution to the law with characteristic function \(E \exp(-\eta^2 c^2 t^2 / 2)\) where

\[
c^2 = \sum_{l=0}^{K-1} \sum_{k=0}^{K-1} (t_{l+1} - t_l)(s_{k+1} - s_k) a_{k,l}^2.
\]
4. Applications.

Central limit theorems

Many limit theorems have been proved using approximations by martingales. Let us denote, for $n = (n_1, \ldots, n_d)$, $|n| = n_1 \ldots n_d$ and $S_n(f) = \sum_{i_1=1}^{n_1} \cdots \sum_{i_d=1}^{n_d} f \circ T_{(i_1, \ldots, i_d)}$. Using Theorem 1 we can in the same way as in [Go69] prove that if $m \circ T_i$ are martingale differences in $L^2 (\mathbb{Z}^d)$, and for $n_1, \ldots, n_d \to \infty$

\begin{equation}
1 \sqrt{|n|} \left\| S_n(f - m) \right\|_2 \to 0
\end{equation}

then there exists a random variable $\eta^2$ (possibly in another probability space), $E\eta^2 = \|m\|^2_2$, such that $1 / \sqrt{|n|} S_n(f)$ weakly converge to the law with characteristic function $E \exp(-\frac{1}{2} t^2 \eta^2)$.

The CLT is often accompanied with a weak inviariance principle (not always, cf. e.g. [V-Samek], [DMV], [GiV]). We will present conditions leading to (12) and a weak inviariance principle at the same time.

Weak inviariance principles (WIP)

A “closer” approximation than (12) guarantees the WIP. For $d = 1$ a classical assumption has been the martingale-coboundary decomposition $f = m + g - U g$ where $m, g \in L^2$ and $(U^i g)_i$ is a sequence of martingale differences. For $d > 1$ we define the martingale-coboundary representation

\begin{equation}
f = \sum_{S \subset \{1, \ldots, d\}} \prod_{q \in S} (I - U_{eq}) g_S \quad (f \in L^p, \; 1 \leq p < \infty)
\end{equation}

where for $S \subset \{1, \ldots, d\}$, $g_S \in \bigcap_{q \in S} L^p(\mathcal{F}_0^{(q)}) \oplus L^p(\mathcal{F}_{-1}^{(q)})$; $\prod_{q \in \emptyset} (I - U_{eq})$ is defined as $I$, the identity operator. For $q' \in S$, $U_{q'} g_S \prod_{q \in S^c} (I - U_{eq}) g_S$, $i \in \mathbb{Z}$, are martingale differences while for $q' \in S^c$, $\prod_{q \in S^c} (I - U_{eq}) g_S$ are coboundaries for the transformation $T_{eq'}$. Eq. (13) with all terms in $L^2$ implies the WIP. For the first time, this decomposition was studied by Gordin in [Go09]. Sufficient and necessary and sufficient conditions for (13) can be found in [EGi], [V17], [Gi]).

If $f = \sum_{x \in \mathbb{Z}^d} P_x f$ ($P_x$ are the orthogonal projection operators defined in Section 1 - Introduction) then we say that $f$ (or the random field of $f \circ T_i$) is regular.

As we will state in Theorem 11, for regular random fields, (12) and a WIP are guaranteed e.g. by the conditions of Hannan (see [VWa])

\begin{equation}
\text{(Hannan)} \quad f = \sum_{j \in \mathbb{Z}^d} P_j f, \quad \sum_{j \in \mathbb{Z}^d} \|P_j f\|_2 < \infty
\end{equation}

and Maxwell-Woodroofe (see [Gi]; the CLT was proved in [PZ])

\begin{equation}
\text{(Maxwell-Woodroofe)} \quad f = \sum_{j \in \mathbb{Z}^d} P_j f, \quad \sum_{n \geq 1} \frac{\|E(S_n f | \mathcal{F}_j)\|_2}{|n|^{3/2}} < \infty.
\end{equation}
Theorem 11. If \((T^z_i)\) is a \(Z^d\) action, the random field \((f \circ T^z_i)\) is regular and one of the conditions (13), (14), (15) holds ((13) with all terms in \(L^2\)) then the WIP holds.

Proof. In [EGi], [V17], [VWa], [Gi] it was shown that each of the conditions implies

\[
S_{n,t}(f) = S_{n,t}(m) + R_{n,t}(f), \quad t \in [0,1]^d,
\]

where \(m \in L^2\), \(U^z_m\) are martingale differences, and \(R_{n,t}(f)\) converge in distribution to the zero function on \([0,1]^d\). The WIP thus holds for the random field of \(U^z f\) if and only if it holds for the random field of \(U^z m\). For \(U^z m\) the WIP is guaranteed by Theorem 10.

□

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REFERENCES

[BaDo] Basu, A.K. and Dorea, C.C.Y., On functional central limit theorem for stationary martingale random fields, Acta Math. Acad. Sci. Hungar. 33(3-4) (1979), 307-316.

[BiDu] Biermé, H. and Durieu, O., Invariance principles for self-similar set-indexed random fields, Transactions Amer. Math. Soc. 366 (2014), 5963-5989.

[B61] Billingsley, P., On the Lindeberg-Lévy theorem for martingales, Proc. Amer. Math. Soc. 12 (1961), 788-792.

[B68] Billingsley, P., Convergence of Probability Measures, Wiley, New York, 1968.

[CDV] Cuny, C., Dedecker, J., and Volný, D., A functional CLT for fields of commuting transformations via martingale approximation, Zapiski Nauchnyh Seminarov POMI 441 (2015), 239-261.

[CSFo] Cornfeld, I.P., Fomin, S.V., and Sinai, Ya.G., Ergodic Theory, Springer-Verlag, Berlin, 1982.

[D] Dedecker, J., A central limit theorem for stationary random fields, Probab. Theory and Rel. Fields 110 (1998), 397-426.

[DeGo] Denker, M. and Gordin, M.I., Limit theorems for von Mises statistics of a measure preserving transformation, Probab. Theory Related Fields 160 (2014), 1-45.

[EGi] El Machkouri, M. and Giraudo, D., Orthomartingale-coboundary decomposition for stationary random fields, Stochastics and Dynamics 16/5 (2016).

[F] Feller, W., An Introduction to Probability Theory and Its Applications, Wiley, New York, 1966.

[GHu] Gänsler, P. and Häusler, E., Remarks on the functional central limit theorem for martingales, Zeitschrift Wahrscheinlichkeitstheorie verw. Gebiete 50 (1979), 237-243.

[Gi] Giraudo, D., Invariance principle via orthomartingale approximation, arXiv:1702.08288 (2017).

[G069] Gordin, M.I., The central limit theorem for stationary processes, Dokl. Acad. Nau SSSR 188 (1969), 739-741.

[G09] Gordin, M.I., Martingale coboundary representation for a class of stationary random fields, 364 (Veroyatnost i Statistika 14.2) 88-108, 236, Zap. Nauchn. Sem. S.-Petersburg Otdel. Mat. Inst. Steklov. (POMI) (2009).

[HaHe] Hall, P. and Heyde, C., Martingale Limit Theory and its Application, Academic Press, New York, 1980.

[I] Ibragimov, I.A., A central limit theorem for a class of dependent random variables, Theor. Probability Appl. 8 (1963), 83-89.

[K] Khosnevisan, D., Multiparameter processes, an introduction to random fields, Springer-Verlag, New York, 2002.

[L] Lachout, P., A note on the martingale central limit theorem, Comment. Math. Univ. Carolinae 26 (1985), 637-640.
McLeish, D.L., *Dependent central limit theorems and invariance principles*, Ann. Probab. 2 (1974), 620-628.

Nahapetian, B., *Billingsley-Ibragimov theorem for martingale-difference random fields and its applications to some models of classical statistical physics*, C. R. Acad. Sci. Paris Sér. I Math. 320(12) (1995), 1539-1544.

Peligrad M. and Zhang, N., *On the normal approximation for random fields via martingale methods*, arXiv:1702.01143 (2017).

Peligrad M. and Zhang, N., *Billingsley-Ibragimov theorem for martingale-difference random fields and its applications to some models of classical statistical physics*, C. R. Acad. Sci. Paris Sér. I Math. 320(12) (1995), 1539-1544.

Vlček, M. and Zhang, N., *On the normal approximation for random fields via martingale methods*, arXiv:1702.01143 (2017).

Volný, D., *Martingale decompositions of stationary processes*, Yokohama Math. J. 35 (1987), 113-121.

Volný, D., *On non-ergodic versions of limit theorems*, Aplikace matematiky 34 (1989), 351-363.

Volný, D., *A central limit theorem for fields of martingale differences*, Compt. Rend. Acad. Sci. Paris Ser. I 353 (2015), 1159-1163.

Volný, D., *Martingale-coboundary decomposition for stationary random fields*, arXiv: arXiv:1706.07978 (2017).

Volný, D. and Samek, P., *On the invariance principle and the law of iterated logarithm for stationary processes*, Mathematical Physics and Stochastic Analysis (Essays in Honour of Ludwig Streit), World Scientific Publishing Co., Eds. S. Albeverio, Ph. Blanchard, L. Ferreira, T. Hida, Y. Kondratiev, R. Vilela Mendes (2000), 424-438.

Volný, D. and Wang, Y., *An invariance principle for stationary random fields under Hannan’s condition*, Stoch. Proc. Appl. 124 (2014), 4012-4029.

Wang, Y. and Woodroofe, M., *A new condition on invariance principles for stationary random fields*, Statist. Sinica 23(4) (2013), 1673-1696.