co-Semi-analytic functors

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Abstract

We characterize the category of co-semi-analytic functors and describe an action of semi-analytic functors on co-semi-analytic functors.

1 Introduction

Let $I$ be the (skeleton of the) category of finite sets and monomorphisms. The functor $Set^{op} \to Cat(Set^{op}, Set)$ of the left K extension along $I^{op} \to Set^{op}$ is conservative. The essential image of this functor is a category of functors that are in a sense a dual presentation to the semi-analytic functors [SZ]. This is why we call this category the category of co-semi-analytic functors. In this note, we give an abstract characterization of this category. Moreover, we show that the category of semi-analytic functors acts on this category and we show some examples of the actions along this action.

Contravariant functors on finite sets were considered in [P], [D].

Notation

Let $[n] = \{0, \ldots, n\}$, $(n) = \{1, \ldots, n\}$, $\omega$ - denote the set of natural numbers. The set $X^n$, $n$-th power of $X$, is interpreted as $X^{(n)}$, a set of functions, when convenient. The skeletal category equivalent to the category of finite sets $Set_{fin}$ will be denoted by $F$. We will be assuming that the objects of $F$ are sets $(n)$, for $n \in \omega$. The subcategories of $F$ with the same objects as $F$ but having as objects bijection, surjections and injections will be denoted by $B$, $S$, $I$, respectively. $S_n$ is the group of permutations of $(n)$. When $S_n$ acts on a set $A$ on the right and on the set $B$ on the left, then the set $A \otimes_n B$ is the usual tensor product of $S_n$-sets. Let $Epi(X, (n))$ denote the set of epimorphisms from the set $X$ to $(n)$. $S_n$ acts on $Epi(X, (n))$ on the left by compositions. If $A : I^{op} \to Set$ is a functor, $f : (n) \to (m)$, $a \in A_m (= A(m))$, then we often write $a \cdot_A f$ instead of $A(f)(a)$. $S_n$ acts of the right on $A_n$ and, according with the previous notation, we write $a \cdot_A \sigma$ instead of $A(\sigma)(a)$. Thus we can form a set

$$A_n \otimes_n Epi(X, (n))$$
whose elements are equivalence classes of pairs $\langle a, \overrightarrow{x} \rangle$ such that $a \in A_n$ and $\overrightarrow{x} : X \to (n)$ epi. We identify pairs

$$\langle a, \sigma \circ \overrightarrow{x} \rangle \sim \langle a \cdot \sigma, \overrightarrow{x} \rangle$$

for $\sigma \in S_n$.

2 The category of co-semi-analytic functors

A natural transformation $\tau : F \to G : C \to D$ is semi-cartesian iff the naturality squares for monomorphisms in $C$ are pullbacks in $D$. Let $\mathbf{cEnd}$ denote the category $\text{Nat}(\text{Set}^{op}, \text{Set})$, i.e. the category of contravariant functors on $\text{Set}$ and natural transformations.

We define a functor

$$\tilde{\cdot} : \text{Set}^{op} \to \mathbf{cEnd}$$

Let $A : I^{op} \to \text{Set}$. We put

$$\tilde{A}(X) = \sum_{n \in \omega} A_n \otimes_n \text{Epi}(X, (n))$$

For a function $f : Y \to X$ and $[a, \overrightarrow{x}] \sim A_n \otimes_n \text{Epi}(X, (n))$ we put

$$\tilde{A}(f)([a, \overrightarrow{x}] \sim) = [a \cdot A f', \overrightarrow{x}']$$

where the square

$$\begin{array}{ccc}
X & \xrightarrow{\overrightarrow{x}} & (n) \\
\downarrow f & & \downarrow f' \\
Y & \xrightarrow{\overrightarrow{x}'} & (m)
\end{array}$$

commutes and $\overrightarrow{x}'$, $f'$ is the epi-mono factorization of $\overrightarrow{x} \circ f$.

Let $\tau : A \to B$ be a natural transformation in $\text{Set}^{op}$. For $[a, \overrightarrow{x}] \sim A_n \otimes_n \text{Epi}(X, (n)) \subseteq \tilde{A}(X)$ we put

$$\tilde{\tau}_X ([a, \overrightarrow{x}] \sim) = [\tau_n(a), \overrightarrow{x}] \sim$$

We have

**Proposition 2.1.** The functor $\tilde{\cdot} : \text{Set}^{op} \to \mathbf{cEnd}$ is well defined, and it is isomorphic to the left Kan extension along the inclusion functor $I^{op} \to \text{Set}^{op}$.

**Proof.** The fact that $\tilde{\cdot}$ is well defined is easy. It is well known that the left Kan extension can be calculated with coends. It is also easy to check that, for $A \in \text{Set}^{op}$ and a set $X$, we have the second isomorphism

$$\tilde{A}(X) = \int^{[n] \in I^{op}} A_n \times \text{Set}(X, (n)) \cong \sum_{n \in \omega} A_n \otimes_n \text{Epi}(X, (n))$$

\[\text{To emphasize the domain of an epi } X \to (n), \text{ we name it } \overrightarrow{x} \text{ as it is in a sense a dual to } (n) \to X. \text{ And it is natural to name the later map } \overrightarrow{x} \text{.}\]
\[
[a, f] \sim \mapsto [a \cdot f', q]
\]
where \( f = f' \circ q \) is the epi-mono factorization of \( f \). The details are left for the reader. \( \square \)

The canonical finitary cone \( \gamma \) under a set \( X \) in \( \text{Set} \) is the cone from the vertex \( X \) to the functor
\[
X \downarrow F \xrightarrow{\pi_X} F \longrightarrow \text{Set}
\]
\( X \rightarrow (n) \rightarrow (n) \)
such that \( \gamma_f = f \), for the \( f : X \rightarrow (n) \) in \( X \downarrow F \). The canonical finitary cocone \( \kappa \) over a set \( X \) in \( \text{Set}^{\text{op}} \) is the dual of the canonical finitary cone under \( X \) i.e. the cocone from the functor
\[
(X \downarrow F)^{\text{op}} \xrightarrow{(\pi_X)^{\text{op}}} F^{\text{op}} \longrightarrow \text{Set}^{\text{op}}
\]
to the vertex \( X \), such that \( \kappa_f = f \) for \( f : X \rightarrow (n) \) in \((X \downarrow F)^{\text{op}}\).

**Theorem 2.2.** The functors \( \check{\cdot} : \text{Set}^{\text{op}} \longrightarrow \text{cEnd} \) is conservative and its essential image consists of functors sending

1. pullbacks along monos to pullbacks;
2. canonical finitary cocone under \( X \) in \( \text{Set}^{\text{op}} \) to a colimiting cocone in \( \text{Set} \), for any set \( X \);

and semi-analytic natural transformations.

Because of the above theorem, by definition, the essential image of the functor \( \check{\cdot} \) is called the category of co-semi-analytic functors, and is denoted \( \text{cSan} \).

We shall prove the above theorem through a series of Lemmas.

**Lemma 2.3.** Let \( \tau : A \rightarrow B \) be a morphism in \( \text{Set}^{\text{op}} \). Then

1. the functor \( \check{A} : \text{Set}^{\text{op}} \rightarrow \text{Set} \) sends
   (a) pullbacks along monos to pullbacks;
   (b) canonical finitary cocone over \( X \) in \( \text{Set}^{\text{op}} \) to a colimiting cocone in \( \text{Set} \), for any set \( X \);
2. the natural transformation \( \check{\tau} : \check{A} \rightarrow \check{B} \) is semi-analytic.

**Proof.** Let the square

\[
\begin{array}{ccc}
X & \xrightarrow{g} & Z \\
\downarrow f & & \downarrow f' \\
Y & \xrightarrow{g'} & T
\end{array}
\]

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be a pushout in $\mathbf{Set}$ with $f$ (and hence also $f'$) epi. We need to show that the square

$$
\begin{array}{ccc}
A(X) & \xrightarrow{\tilde{A}(g)} & A(Z) \\
\downarrow{\tilde{A}(f)} & & \downarrow{\tilde{A}(f')} \\
A(Y) & \xleftarrow{\tilde{A}(g')} & A(T)
\end{array}
$$

is a pullback in $\mathbf{Set}$. Let $[a, \overline{\nu} : Y \to (n)]_\sim \in \tilde{A}(Y)$ and $[b, \overline{\xi} : Z \to (m)]_\sim \in \tilde{A}(Z)$ such that

$$
\tilde{A}(f)([a, \overline{\nu} : Y \to (n)]_\sim) = \tilde{A}(g)([b, \overline{\xi} : Z \to (m)]_\sim)
$$

Thus we have a mono $\tilde{f} : (n) \to (m)$ such that

$$
\tilde{f} \circ \overline{\nu} \circ f = \overline{\xi} \circ g \quad \text{and} \quad b \cdot_A \tilde{f} = a
$$

Hence there is a unique morphisms $\overline{T} : T \to (m)$ such that

$$
\overline{T} \circ g' = \tilde{f} \circ \overline{\nu} \quad \text{and} \quad \overline{T} \circ f' = \overline{\xi}
$$

As $\overline{\xi}$ is epi, $\overline{T}$ is epi, as well. Then $[b, \overline{T}]_\sim \in \tilde{A}(T)$ and we have

$$
\tilde{A}(f')(\overline{b, T})_\sim = \tilde{A}(g')(\overline{b, g})_\sim = [b, \overline{\xi}]_\sim
$$

Thus the morphisms $\tilde{A}(\overline{\varphi}) : \tilde{A}(n) \to \tilde{A}(X)$ with $\overline{\varphi} : X \to (n)$ epi, jointly cover $\tilde{A}(X)$.

If we have another element $[b, q : (m) \to (k)]_\sim \in \tilde{A}(m)$ and $f : X \to (m)$ a function such that

$$
\tilde{A}(f)([b, q])_\sim = [a, \overline{\varphi}]_\sim
$$

then with $q', f'$ being the epi-mono factorization (see diagram below) of $q \circ f$ we have

$$
\tilde{A}(f)([b, q'])_\sim = [b \cdot_A f', q']_\sim = [a, \overline{\varphi}]_\sim
$$

Hence we have a $\sigma \in S_n$ making the left triangle
commute and

\[ a = (b \cdot_A f') \cdot_A \sigma \]

Thus we have a commuting square

\[
\begin{array}{ccc}
\tilde{A}(X) & \xrightarrow{\tilde{A}(f)} & \tilde{A}(m) \\
\uparrow \tilde{\varphi} & & \uparrow \tilde{q}' \\
\tilde{A}(n) & \xrightarrow{\tilde{A}(f' \circ \sigma)} & \tilde{A}(k) \\
\end{array}
\]

and \([b, 1_k], \sim \in \tilde{A}(k)\) such that

\[ \tilde{A}(q)([b, 1_k], \sim) = [b, q], \sim \]

and

\[ \tilde{A}(f' \circ \sigma)([b, 1_k], \sim) = [a, 1_n], \sim \]

i.e. if two elements go to the same element in \(\tilde{A}(X)\), they are related in the cocone. Thus \(\tilde{A} : \text{Set}^{\text{op}} \to \text{Set}\) sends canonical finitary cocone under \(X\) in \(\text{Set}^{\text{op}}\) to a colimiting cocone in \(\text{Set}\), for any set \(X\).

It remains to show that the natural transformation \(\tilde{\tau} : \tilde{A} \to \tilde{B}\) is semi-cartesian. Let \(g : X \to Y\) be an epi in \(\text{Set}\). We shall show that the square

\[
\begin{array}{ccc}
\tilde{A}(X) & \xrightarrow{\tilde{\tau}_X} & \tilde{B}(X) \\
\uparrow \tilde{A}(g) & & \uparrow \tilde{B}(g) \\
\tilde{A}(Y) & \xrightarrow{\tilde{\tau}_Y} & \tilde{B}(Y) \\
\end{array}
\]

is a pullback. Fix \([a, \tilde{\varphi} : X \to (n)], \sim \in \tilde{A}(X)\) and \([b, \tilde{\psi} : Y \to (n)], \sim \in \tilde{B}(Y)\) such that

\[ [\tau_n(a), \tilde{\varphi}], \sim = \tilde{\tau}([a, \tilde{\varphi}], \sim) = \tilde{B}(g)([b, \tilde{\psi}], \sim) = [b, \tilde{\psi} \circ g], \sim \]

Thus we have a permutation \(\sigma : (n) \to (n)\) such that

\[ \tau_n(a) \cdot_B \sigma = b \quad \text{and} \quad \tilde{\varphi} = \sigma \circ \tilde{\psi} \circ g \]

Then \([a, \sigma \circ \tilde{\psi}], \sim \in \tilde{A}(Y)\) and we have

\[
\tilde{A}(g)([a, \sigma \circ \tilde{\psi}], \sim) = [a, \sigma \circ \tilde{\psi} \circ g], \sim = [a, \tilde{\varphi}], \sim
\]
\[ \tau_Y([a, \sigma \circ \varphi]_\sim) = [\tau_n(a), \sigma \circ \varphi]_\sim = [\tau_n(a) \cdot B\sigma, \varphi]_\sim = [b, \varphi]_\sim \]

As \( \hat{A}(g) \) is mono, the above square is a pullback. Thus \( \hat{\tau} \) is a semi-cartesian natural transformation. \( \Box \)

**Lemma 2.4.** Let \( A : \text{Set}^{op} \to \text{Set} \) be a functor that sends pullbacks along monos to pullbacks and canonical finitary cocone over \( X \) in \( \text{Set}^{op} \) to a colimiting cocone in \( \text{Set} \), for any set \( X \). Then there is a functor \( \hat{A} : \Gamma^{op} \to \text{Set} \) such that \( \hat{A} \) is isomorphic to \( A \).

**Proof.** Let \( A : \text{Set}^{op} \to \text{Set} \) be a functor with the properties described in Lemma. We define the functor \( A : \Gamma^{op} \to \text{Set} \) as follows

\[ A_n = A(n) - \bigcup_h \text{im}(A(h)) \]

where the sum is over \( 0 \leq m < n \) and (proper) epis \( h : (n) \to (m) \). The set \( \text{im}(A(h)) \) is the image \( A(h) \) in \( A(n) \). Then, for a mono \( f : (n) \to (m) \) in \( \Gamma \) we put

\[ A(f) = A(f)_{\mid A_m} \]

i.e. \( A(f) \) is a restriction of \( A(f) \) to a function \( A_m \to A_n \).

First we need to verify that \( A \) is a well defined functor, i.e. that the restriction is the function with the appropriate domain and codomain. Let \( f : (n') \to (n) \) be a mono \( x \in A(n) \) but \( A(f)(x) \notin A_{n'} \). Thus there is a proper epi \( h' : (n) \to (m') \) and \( y \in A(m') \) such that \( A(h')(y) = A(f)(x) \). The square

\[
\begin{array}{ccc}
A(n) & \xrightarrow{A(h)} & A(m) \\
| & \downarrow A(f) & | \\
A(n') & \xrightarrow{A(h')} & A(m')
\end{array}
\]

is a pullback, where the square below

\[
\begin{array}{ccc}
(n) & \xrightarrow{h} & (m) \\
f & \downarrow & f' \\
(n') & \xrightarrow{h'} & (m')
\end{array}
\]

is a pullback in \( \text{Set}^{op} \), i.e. pushout of a proper epi \( h' \) along \( f \). Hence \( h \) is also proper epi. Thus, there is \( z \in A(m) \) such that

\[ A(h)(z) = x \quad \text{and} \quad A(f')(z) = y \]

But this means that \( x \notin A_n \). This shows that \( A(f) \) is well defined.
Now we define a natural isomorphism
\[ \varphi^A : \tilde{A} \rightarrow A \]
so that
\[ \varphi^A_X([a, \bar{x}]_{\sim}) = \mathcal{A}(\bar{x})(a) \]
for any \( X, n \in \omega, a \in A_n \), and epi \( \bar{x} : X \rightarrow (n] \).

The fact that \( \phi^A \) is a natural transformation is left for the reader. \( \phi^A \) is onto as \( \mathcal{A} \) sends canonical finitary cocones under any set \( X \) in \( \text{Set}^{op} \) to colimiting cocones in \( \text{Set} \).

We shall show that \( \phi^A \) is mono. Fix a set \( X \) and let \( [a, \bar{x}] : X \rightarrow (n] \), \([a', \bar{x}'] : X \rightarrow (n'] \) \( \in \tilde{A}(X) \) such that
\[ \varphi^A_X([a, \bar{x}]_{\sim}) = \varphi^A_X([a', \bar{x}']_{\sim}) \]

By assumption, the pushout of epi \( \bar{x} \) along epi \( \bar{x}' \) in \( \text{Set} \)
\[ (n) \xrightarrow{\bar{x}} X \]
\[ f \]
\[ (m) \xrightarrow{\bar{x}'} [n'] \]

is sent to the pullback in \( \text{Set} \) by \( \mathcal{A} \). Hence there is \( a'' \in \mathcal{A}(m) \) such that
\[ \mathcal{A}(f)(a'') = a \quad \text{and} \quad \mathcal{A}(f')(a'') = a' \]
Thus, by definition of \( A_n \) and \( A_{n'} \), both \( f \) and \( f' \) are bijections (as they are epi but cannot be proper epi). Hence
\[ \mathcal{A}(f^{-1} \circ f')(a) = a' \]
that is \([a, \bar{x}] : X \rightarrow (n] \) \( = [a', \bar{x}'] : X \rightarrow (n'] \) and \( \phi^A_X \) is mono, for any set \( X \).

\[ \square \]

**Lemma 2.5.** Let \( A, B : \Gamma^{op} \rightarrow \text{Set} \) be functors and \( \psi : \tilde{A} \rightarrow \tilde{B} \) a semi-analytic natural transformation. Then there is a natural transformation \( \tau : A \rightarrow B \) in \( \text{Set}^{\Gamma^{op}} \) such that \( \tilde{\tau} = \psi \).

**Proof.** Let \( \psi : \tilde{A} \rightarrow \tilde{B} \) be a semi-analytic natural transformation. We define a natural transformation \( \tau : A \rightarrow B \in \text{Set}^{\Gamma^{op}} \), as follows. Fix \( m \in \omega \) and \( a \in A_m \). Let \( \psi(m)([a, 1]_{\sim}) = [b, p : (m) \rightarrow (k)]_{\sim} \in \tilde{B}(m) \). As \( p \) is an epi and \( \psi \) is semi-analytic, the naturality square for \( p \)
\[ \begin{array}{ccc} \tilde{A}(m) & \xrightarrow{\psi(m)} & \tilde{B}(m) \\ \downarrow \mathcal{A}(p) & & \downarrow \mathcal{B}(p) \\ \tilde{A}(k) & \xrightarrow{\psi(k)} & \tilde{B}(k) \end{array} \]
is a pullback, and $\tilde{B}(p)([b, 1_k]_\sim) = [b, p]_\sim$. Hence there is $[c, q]_\sim \in \tilde{A}(k)$ such that

$$\tilde{A}(p)([c, q]_\sim) = [a, 1_{(m)}]_\sim \text{ and } \psi_{(k)}([c, q : (k) \to (l)]_\sim) = [b, 1_{(k)}]_\sim$$

In particular $q \circ p$ is a bijection, as $1_{(m)}$ is. Hence $k = m$ and $p$ is a bijection. We put

$$\tau_m(a) = b \cdot B p$$

Thus

$$\psi_{(m)}([a, 1_{(m)}]_\sim) = [\tau_m(a), 1_{(m)}]_\sim$$

for $m \in \omega$ and $a \in A_m$. From the naturality of $\psi$ on a mono $f : (m') \to (m) \in I$, for any $a \in A_m$, we have

$$[\tau_{m'}((a \cdot_A f), 1_{(m')} \sim) = \psi_{(m')}([a \cdot_A f], 1_{(m')} \sim) =$$

$$= \psi_{(m')} \circ \tilde{A}(f)([a, 1_{(m)}]_\sim) = \tilde{B}(f) \circ \psi_{(m)}([a, 1_{(m)}]_\sim) =$$

$$= \tilde{B}(f)([\tau_m(a), 1_{(m)}]_\sim) = [\tau_m(a) \cdot_B f, 1_{(m)}]_\sim$$

Thus

$$\tau_{m'}(A(f)(a)) = B(f)(\tau_m(a))$$

and hence $\tau : A \to B$ is natural.

It remains to show that $\tilde{\tau} = \psi$. Fix set $X$, $m \in \omega$ and $[a, q : X \to (m)]_\sim \in \tilde{A}(X)$. Using naturality of $\psi$, $\tilde{\tau}$ on $q$ and the above, we have

$$\psi_X([a, q]_\sim) = \psi_X \circ \tilde{A}(q)([a, 1_{(m)}]_\sim) =$$

$$= \tilde{B}(q) \circ \psi_{(m)}([a, 1_{(m)}]_\sim) = \tilde{B}(q)([\tau_m(a), 1_{(m)}]_\sim) =$$

$$= \tilde{B}(q) \circ \tau_{(m)}([a, 1_{(m)}]_\sim) = \tilde{\tau}_X \circ \tilde{A}(q)([a, 1_{(m)}]_\sim) =$$

$$= \tilde{\tau}_X([a, q]_\sim)$$

Thus $\tilde{\tau} = \psi$, as required. $\Box$

The fact that $\tilde{\sim}$ is immediate from the definition and hence Theorem 2.2 follows from Lemmas 2.3, 2.4, 2.5.
3 The action

The category \textbf{End} of endofunctors on \textit{Set} with composition as a tensor is strict monoidal. The composition

\[ \text{End} \times \text{cEnd} \to \text{cEnd} \]

is an action of a monoidal category on a category. By Theorem 2.2 and characterization of semi-analytic functors (Theorem 2.2 of [SZ]) the composition of semi-analytic functor with co-semi-analytic functor is co-semi-analytic. Thus the above action restricts to the action

\[ \text{San} \times \text{cSan} \to \text{cSan} \]

where \text{San} is the category of semi-analytic functors defined in [SZ]. This category is equivalent to \textit{Set}$^8$ and has nice abstract characterization, see Section 2 of [SZ].

\textbf{Examples.} The examples of actions of (semi-analytic) monads on \textit{Set} on contravariant functors on \textit{Set} consist of functors that build algebras of a (semi-analytic) monad out of sets in a contravariant way. We list some such examples below.

1. Let \( R \) be any algebra for a monad \( T \) on \textit{Set} and let \( R(X) \) be the algebra of functions on \( X \) with values in \( R \) with operations of the monad defined pointwise. Then \( R \) is a contravariant functor on \textit{Set} on which we have an action of the monad \( T \). The following two examples are, in a sense, special cases of this situation.

2. The contravariant power-set functor \( \mathcal{P} : \text{Set}^{\text{op}} \to \text{Set} \) is co-semi-analytic (see next example). Its value on a set \( X \) is the universe of a Boolean algebras \( \mathcal{P}(X) \) and the inverse image functor preserves the Boolean algebra structure. Thus the monad \( T_{ba} \) on \textit{Set} for Boolean algebras acts on \( \mathcal{P} \). However, the monad \( T_{ba} \) is not semi-analytic.

3. For any \( n \in \omega \) the functor

\[ \mathcal{E}^n : \text{Set}^{\text{op}} \to \text{Set} \]

\[ X \mapsto (n)^X \]

is co-semi-analytic. The coefficient functor

\[ E^n : I^{\text{op}} \to \text{Set} \]

for \( \mathcal{E}^n \) is representable by \( (n) \). The action of the monad \( T_{ba} \) on the functor \( \mathcal{P} \) described above is a special case of the following. Let \( T \) be a monad on \textit{Set} and \( ((n), \alpha : T(n) \to (n)) \) be a \( T \)-algebra. Then on \( \mathcal{E}^n(X) \) there is a natural structure of a \( T \) algebra defined pointwise or using strength of \( T \).
Recall that if $X$ and $Y$ are sets and $x \in X$, then we have a function $\bar{x}: Y \to X \times Y$ such that $\bar{x}(y) = (x, y)$. This allows to define strength on $T$:

$$st_{X,Y}: X \times T(Y) \to T(X \times Y)$$

so that $st_{X,Y}(x, t) = \bar{x}(t)$. It is an easy exercise to show that the strength on any semi-analytic monad is semi-cartesian.

The $T$-algebra structure on $E^n(X)$ is the exponential adjoint of

$$
\begin{array}{ccc}
X \times T((n)^X) & \xrightarrow{st_{X,(n)^X}} & T(X \times (n)^X) & \xrightarrow{T(ev)} & T(n) & \xrightarrow{\alpha} & (n)
\end{array}
$$

where $ev$ is the usual evaluation map.

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