THE BLOWUP ALONG THE DIAGONAL OF THE SPECTRAL FUNCTION OF THE
LAPLACIAN

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ABSTRACT. We formulate a precise conjecture about the universal behavior near the diagonal of the spectral function of the Laplacian of a smooth compact Riemann manifold. We prove this conjecture when the manifold and the metric are real analytic, and we also present an alternate proof when the manifold is the round sphere.

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1. INTRODUCTION

Suppose that \((M, g)\) is a compact, connected \(m\)-dimensional Riemannian manifold, and \((\Psi_n)_{n \geq 0}\) is a (complete) orthonormal basis of \(L^2(M, g)\) consisting of eigenfunctions of \(\Delta_g\)

\[
\Delta_g \Psi_n = \lambda_n \Psi_n, \quad 0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots
\]

For every \(L > 0\) we define the spectral function \(E_L : M \times M \to \mathbb{R}\) by

\[
E_L(p, q) = \sum_{\lambda_n \leq L} \Psi_n(p) \Psi_n(q).
\]

Equivalently, \(E_L\) is the Schwartz kernel of the orthogonal projection onto

\[
H_L := \bigoplus_{\lambda \leq L} \ker(\lambda - \Delta).
\]

This shows that as \(L \to \infty\) the spectral function \(E_L\) converges in the sense of distributions to the Dirac-type distribution supported by the diagonal

\[
D_M = \{(p, q) \in M \times M; \quad p = q\}.
\]

The goal of this paper is to describe a universal law governing the behavior of \(E_L\) as \(L \to \infty\) in an infinitesimal neighborhood of the diagonal. Here are the specifics.

Date: Started on February 2, 2011. Completed on March 2, 2011. Latest revision January 14, 2013.
2000 Mathematics Subject Classification. 58J50, 35J15, 33C45, 32C05.
Key words and phrases. Riemann manifolds, Laplacian, eigenfunctions, spectral function, real analytic manifolds, harmonic polynomials.
This work was partially supported by the NSF grant, DMS-1005745.
We denote by $N$ the normal bundle of the diagonal embedding. For any $p \in M$ we denote by $N_p$ the fiber over $(p, p) \in \mathcal{D}_M$. Also we let $r : N \to \mathbb{R}$ denote the radial distance function along the fibers of $N$, and we set
\[
N^R := r^{-1}([0, R]), \quad N^R_p := N^R \cap N_p, \quad \forall R > 0, \quad p \in M.
\]
In other words, $N^R \subset N$ is the associated bundle of normal disks of radius $R$. If $h$ is sufficiently small, then the exponential map induces a diffeomorphism from $N^h$ onto an open neighborhood $U^h$ of the diagonal. Fix once and for all such a $h$. We denote by $\tilde{E}_L^h$ the pullback of $E_L|U^h$ to $N^h$.

For every positive real number $\lambda$ we denote by $M_\lambda : N \to \mathbb{N}$ the rescaling map described on $N_p$ by
\[
N_p \ni v \mapsto \frac{1}{\lambda} v \in N_p.
\]
We define
\[
\tilde{E}_L : N^{L^{1/2}h} \to \mathbb{R}, \quad \tilde{E}_L = L^{-\frac{\nu}{2}} M^* \cdot E_L^h.
\]
For any $p \in M$ we denote by $\tilde{E}_{L,p}$ the restriction of $\tilde{E}_L$ to the fiber $N_p$.

**The Universality Conjecture.** There exists $\rho > 0$ such that for any $p \in M$ the functions
\[
\tilde{E}_{L,p} : N^{L^{1/2}h}_p \to \mathbb{R}
\]
converge as $L \to \infty$ in the topology of $C^\infty(N^p)$ to the smooth function
\[
E_\infty : N_p \to \mathbb{R}, \quad E_\infty(u) = \frac{1}{(2\pi|u|)^{\frac{\nu}{2}}} J_m\left(\frac{|u|}{2}\right),
\]
where $J_\nu$ denotes the Bessel function of the first kind and order $\nu$. \hfill \Box

**Remark 1.1.** (a) The limit function $E_\infty(u)$ has a more suggestive description, namely
\[
E_\infty(u) = \frac{1}{(2\pi|u|)^{\frac{\nu}{2}}} J_m\left(\frac{|u|}{2}\right) = \int_{B_{r_m}^m} e^{\xi \cdot u} |d\xi|, \quad B_{r_m}^m := \{ \xi \in \mathbb{R}^m; \ |\xi|^2 \leq r \}.
\]
We denote by $E_L$ the spectral function of the Laplacian on $\mathbb{R}^m$ corresponding to (generalized) eigenvalues $\leq L$. We then have (see [6, Eq. (2.1)] or [11, Eq.(1.3)])
\[
E_L(x, y) = \frac{1}{(2\pi)^m} \int_{B_{r_m}^m} e^{\xi \cdot (x-y)} |d\xi|.
\]
This shows that
\[
E_\infty(u) = E_1(u, 0).
\]

(b) We ought to elaborate on the crux of the Universality conjecture. First of all, let us point out that known local Weyl trace formulæ ([2], [4, §17.5], [13, Thm. 1.8.5], [15, §7.8]) imply that the family $(\tilde{E}_L)_{L \geq 1}$ is precompact in the $C^\infty$-topology and as detailed in see Section 2, any limit point $\tilde{E}_\infty$ is asymptotically equivalent to $E_\infty$ at $u = 0$, i.e., for any $N > 0$,
\[
\tilde{E}_\infty (u) = E_\infty (u) + O(|u|^{-N}) \quad \text{as} \quad u \searrow 0.
\]

The Universality Conjecture makes the stronger claim that the family $(\tilde{E}_L)_{L \geq 1}$ has a unique limit point in $C^\infty$-topology and moreover, that limit point is not just asymptotically equivalent, but equal to the function $E_\infty$.

(c) Recently, Lapointe, Polterovich and Safarov [7] have described a global type of relationship between $E_L$ and $E_L$ as $L \to \infty$. \hfill \Box
The main result of this paper states that the Universality Conjecture is true in the case when $M$ and $g$ are real analytic. We achieve this in Section 2 by relying on some sharp a priori estimates that we prove in Appendix A, Theorem A.1. A weaker version of these estimates were used by Donnelly and Fefferman [3, §7] in their investigation of nodal sets of eigenfunctions on real analytic Riemannian manifolds. We believe that Theorem A.1 will find many other uses. In Section 3 we give an alternate proof of the conjecture in the special case when $(M, g)$ is the round sphere.

Acknowledgments. I want to thank Steve Zelditch for a most illuminating discussion on spectral geometry.

2. THE UNIVERSALITY CONJECTURE IN THE REAL ANALYTIC CASE

Theorem 2.1. The Universality Conjecture is true when $M$ and $g$ are real analytic.

Proof. Since both $M$ and $g$ are real analytic we deduce that the spectral function $E_L$ is real analytic; see [8, 9]. The Cauchy-Kowaleskaya theorem, [5], implies that the exponential map is also real analytic so that $E^h_L$ is real analytic.

Fix a point $p_0$ in $M$ and normal coordinates $x = (x^1, \ldots, x^m)$ at $p_0$ defined on an open neighborhood $O$ of $p_0$. With these choices we can regard the restriction of $E_L$ to $O \times O$ as a real analytic function $E_L(x, y)$ defined on a neighborhood of $(0, 0)$ in $\mathbb{R}^m \times \mathbb{R}^m$. Via the exponential map

$$\exp(p_0, p_0): \mathcal{T}(p_0, p_0)M \times M \rightarrow M \times M$$

we can identify the space $\{(x, y) \in \mathbb{R}^m \times \mathbb{R}^m; \quad x + y = 0\}$ with the fiber $N_{p_0}$.

The results of [2] show that as $L \rightarrow \infty$ we have

$$\frac{\partial^{\alpha+\beta}}{\partial x^\alpha \partial y^\beta} E_L(x, y)|_{x=y=0} = L^{m+|\alpha|+|\beta|} \left( C_{\alpha, \beta} + O(L^{-\frac{1}{2}}) \right), \quad (2.1)$$

where $C_{\alpha, \beta} = 0$ if $\alpha - \beta \not\in 2\mathbb{Z}^m$, while if $\alpha - \beta \in 2\mathbb{Z}^m$ we have

$$C_{\alpha, \beta} = (-1)^{|\alpha|-|\beta|} \frac{1}{(4\pi)^{\frac{m}{2}} \Gamma(1 + m + \frac{|\alpha|+|\beta|}{2})} \prod_{i=1}^m (\alpha_i + \beta_i - 1)!!.$$ 

In fact we can say a bit more. More precisely, according to [4, Thm. 17.5.3], for any $\alpha, \beta$ there exists a constant $K_{\alpha, \beta} > 0$ that depends on the geometry of $(M, g)$ but it is independent of $p_0$ such that

$$\left| \frac{\partial^{\alpha+\beta}}{\partial x^\alpha \partial y^\beta} E_L(x, y) \right| \leq K_{\alpha, \beta} L^{m+|\alpha|+|\beta|} \frac{1}{2}, \quad \forall x, y. \quad (2.2)$$

A priori, the constants $K_{\alpha, \beta}$ can grow really fast as $|\alpha| + |\beta| \rightarrow \infty$. Our next result provides a key upper bound on this growth. To keep the flow of arguments uninterrupted we deferred its rather sneaky proof to Appendix A.

Lemma 2.2. There exist constant $C, T > 0$ such that for any $L > 1$ and any multi-indices $\alpha, \beta$ we have

$$\sup_{(p, q) \in M \times M} |\partial^{\alpha+\beta} E_L(p, q)| \leq CT^{\|\alpha\|+\|\beta\|}(\|\alpha\| + \|\beta\|)!L^{m+|\alpha|+|\beta|} \frac{1}{2}, \quad (2.3)$$

where $(x, y)$ denote normal coordinates at $(p, q)$. In other words, in (2.2) we can choose constants $K_{\alpha, \beta}$ satisfying

$$K_{\alpha, \beta} \leq KT^{\|\alpha\|+\|\beta\|}(\|\alpha\| + \|\beta\|)!, \quad (2.4)$$

where $K$ and $T$ are independent of $\alpha, \beta$ and $L$. \hfill \Box
We deduce that for
\[ L > K, T > 0. \]
From (2.4) we deduce that there exist constants
\[ C, L > 0. \]
We set
\[ \sigma = \sum (\eta^i)^2, \]
and
\[ \mathcal{E}_{L,p_0} = \mathcal{E}_L(2^{-\frac{1}{2}} \eta, -2^{-\frac{1}{2}} \eta), \quad \mathcal{E}_{L,p_0}(\eta) = L^{-\frac{m}{2}} \mathcal{E}_L \left( \frac{1}{(2L)^{\frac{1}{2}}}, -\frac{1}{(2L)^{\frac{1}{2}}} \eta \right). \]
We set
\[ C_\alpha(L) := \frac{\partial^{\alpha}_\eta \mathcal{E}_{L,p_0}(\eta) |_{\eta = 0}}{\partial^{\alpha}_\eta}. \]
From (2.4) we deduce that there exist constants \( K, T > 0 \) such that for any multi-index \( \alpha \) and any \( L > 1 \) we have
\[ |C_\alpha(L)| \leq K|\alpha|!T^{2|\alpha|}. \] (2.5)
Using (2.1) we deduce that
\[ \partial^{\ell}_\eta \mathcal{E}_L(x, y)_{x = y = 0} = C_\ell + O \left( L^{-\frac{\ell}{2}} \right), \]
where \( C_\ell = 0 \) if \( \ell \) is odd, while if \( \ell \) is even we have
\[ C_\ell = 2^{-\frac{\ell}{2}} \sum_{k=0}^{\ell} (-1)^k \binom{\ell}{k} \partial^{\ell-k}_x \partial^{k}_y \mathcal{E}_L(x, y)_{x = y = 0} = (-1)^{\frac{\ell}{2}} \frac{1}{(4\pi)^{\frac{m}{2}} \Gamma \left( 1 + \frac{m}{2} + \frac{\ell}{2} \right)} (\ell - 1)! \Gamma \left( 1 + \frac{m}{2} + \frac{\ell}{2} \right). \]
More generally,
\[ C_\alpha(L) := O \left( L^{-\frac{1}{2}} \right) \text{ if } \alpha \not\in 2\mathbb{Z}, \] (2.6)
and
\[ C_{2\alpha}(L) = (-1)^{|\alpha|} \frac{1}{(4\pi)^{\frac{m}{2}} \Gamma \left( 1 + \frac{m}{2} + |\alpha| \right)} \prod_{j=1}^{m} (2\alpha_j - 1)!! + O \left( L^{-\frac{\ell}{2}} \right). \] (2.7)
The estimates (2.5), (2.6) and (2.7) show that there exists \( \rho > 0 \) such the family \( \{ \mathcal{E}_L \}_{L \geq 1} \) of real analytic functions converges in the topology of \( C^\infty \left( \{|\eta| < \rho\} \right) \) to a function \( \mathcal{E}_\infty \) that is real analytic on the \( \{ |\eta| < \rho \} \). Moreover,
\[ \partial^{2\alpha}_\eta \mathcal{E}_\infty(0) = (-1)^{|\alpha|} \frac{1}{(4\pi)^{\frac{m}{2}} \Gamma \left( 1 + \frac{m}{2} + |\alpha| \right)} \prod_{j=1}^{m} (2\alpha_j - 1)!! \] (2.8)
We deduce that for \( |\eta| < \rho \) we have
\[ \mathcal{E}_\infty(\eta) = \sum_{\alpha} \frac{1}{\alpha!} \partial^{\alpha}_\eta \mathcal{E}_\infty(0) \eta^\alpha = \sum_{n \geq 0} \frac{(-1)^n}{(4\pi)^{\frac{m}{2}} \Gamma \left( 1 + \frac{m}{2} + n \right)} \sum_{|\alpha| = n} \prod_{j} \frac{2\alpha_j! \eta^{2\alpha}}{(2\alpha_j)!} \eta^{2\alpha} \]
\[ = \sum_{n \geq 0} \frac{(-1)^n}{(4\pi)^{\frac{m}{2}} \Gamma \left( 1 + \frac{m}{2} + n \right)} \sum_{|\alpha| = n} \frac{1}{\alpha!} \eta^{2\alpha} = \sum_{n \geq 0} \frac{(-1)^n}{(4\pi)^{\frac{m}{2}} 2^n \Gamma \left( 1 + \frac{m}{2} + n \right)} \sum_{|\alpha| = n} \frac{1}{\alpha!} \eta^{2\alpha}. \]
We set to give an alternate proof of the conjecture in the special case when

\[ (\Psi, \cdots, \Psi) \]

As is well known, the space \( H_t \) and \( p \) where

\[ \text{The addition formula for spherical harmonics [10] implies that} \]

To appreciate the complexity involved in the Universality Conjecture we believe that it is instructive to give an alternate proof of the conjecture in the special case when \( M \) is the round sphere \( S^{d-1} \subset \mathbb{R}^d \).

The spectrum of the Laplacian on \( S^{d-1} \) is (see [10])

\[ \lambda_n = n(n + d - 2), \quad n = 0, 1, 2, \ldots \]

and

\[ \dim \ker(\lambda_n - \Delta) = \mu_n = \frac{2n + d - 2}{n + d - 2} \binom{n + d - 2}{d - 2}. \]

We set

\[ \mathcal{H}_m := \ker(\lambda_m - \Delta), \quad U_n = \bigoplus_{k=0}^{n} \mathcal{H}_m. \]

As is well known, the space \( \mathcal{H}_k \) coincides with the space of restrictions to \( S^{d-1} \) of harmonic polynomials of degree \( k \) in \( d \) variables. Fix an is an orthonormal basis \( \{ \Psi_{k,m} \}_{1 \leq \alpha \leq \mu_m} \) of \( \mathcal{H}_m \) and set

\[ \mathcal{E}_n(p, q) := \mathcal{E}_{\lambda_n}(p, q) = \sum_{m=0}^{n} \sum_{k=1}^{\mu_m} \Psi_{k,m}(p) \Psi_{k,m}(q). \]

The addition formula for spherical harmonics [10] implies that

\[ \sum_{k=1}^{\mu_m} \Psi_{k,m}(p) \Psi_{k,m}(q) = \frac{\mu_m}{\sigma_{d-1}} P_{m,d}(p \cdot q), \quad \forall p, q \in S^{d-1}, \]

where \( p \cdot q \) is the canonical inner product of \( p, q \in \mathbb{R}^d \), \( \sigma_{d-1} \) is the area of \( S^{d-1} \),

\[ \sigma_{d-1} = \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}, \]

and \( P_{n,d} \) is the Legendre polynomial of degree \( n \) and order \( d \),

\[ P_{n,d}(t) := (-1)^n \frac{1}{2^n [n + d - 3]} (1 - t^2)^{-\frac{d-3}{2}} \frac{d^n}{dt^n} (1 - t^2)^{n + \frac{d-3}{2}}, \]

\[ [x]_k := x(x - 1) \cdots (x - (k - 1)) = \frac{\Gamma(x + 1)}{\Gamma(x + 1 - k)}. \]

The collection \( \{ P_{n,d} \}_{n \geq 0} \) is an orthogonal family of polynomials with respect to the measure \( w(t) dt = (1 - t^2)^{-\frac{d-3}{2}} dt \) on \([-1, 1]\). More precisely we have, [10, §2],

\[ \int_{-1}^{1} P_{n,d}(t) P_{m,d}(t) w(t) dt = h_n \delta_{n,m}, \quad h_n = \frac{\sigma_{d-1}}{\sigma_{d-2} \mu_n}. \]
Observe that we can rephrase (3.1) as
\[
\sum_{k=1}^{\mu_m} \Psi_{k,m}(p)\Psi_{k,m}(q) = \frac{P_{m,d}(p \cdot q)}{\sigma_{d-2}h_m}.
\] (3.3)

We denote by \( k_n \) the leading coefficient of \( P_{n,d} \). Its precise value is known, [10, Eq. (7.6)], but all we need in the sequel is the equality [10, Eq. (7.7)]
\[
\frac{k_{n-1}}{k_n} = \frac{n + d - 3}{2n + d - 4}.
\]

We recall the Christoffel-Darboux formula, [1, §5.2],
\[
\sum_{m=0}^{n} \frac{P_{m,d}(s)P_{m,d}(t)}{h_m} = \frac{k_n}{k_{n+1}} \frac{P_{n+1,d}(t)P_{n,d}(s) - P_{n+1,d}(s)P_{n,d}(t)}{(t - s)h_n}.
\] (3.4)

In (3.4) we let \( s = 1 \). Using the equality \( P_{m,d}(1) = 1, \forall m \), we deduce
\[
\sum_{m=0}^{n} \frac{P_{m,d}(t)}{h_m} = \frac{k_n}{k_{n+1}} \frac{P_{n+1,d}(t) - P_{n,d}(t)}{h_n(t - 1)}.
\] (3.5)

Summing (3.3) for \( m = 0, \ldots, n \) we deduce from (3.5) that
\[
\mathcal{E}_n(p, q) := \frac{k_n}{k_{n+1}} \frac{P_{n+1,d}(t) - P_{n,d}(t)}{\sigma_{d-2}h_n(t - 1)} = \frac{\mu_n}{\sigma_{d-1}} \frac{k_n}{k_{n+1}} \frac{P_{n+1,d}(t) - P_{n,d}(t)}{(t - 1)}.
\]

Hence
\[
\mathcal{E}_n(p, q) = \mathcal{E}_n(\varphi) := \frac{\mu_n}{\sigma_{d-1}} \frac{k_n}{k_{n+1}} \frac{P_{n+1,d}(\cos \varphi) - P_{n,d}(\cos \varphi)}{(\cos \varphi - 1)}, \quad \cos \varphi = p \cdot q.
\] (3.6)

Observe that \( \varphi \) is the geodesic distance between \( p \) and \( q \). Now set \( r_n := \sqrt{n} \) and define
\[
\tilde{\mathcal{E}}_n(\varphi) := r_n^{(d-1)}\mathcal{E}_n\left(\varphi, \frac{\varphi}{r_n}\right) = \frac{\mu_n}{\sigma_{d-1}r_n^{d-2}} \frac{k_n}{k_{n+1}} \frac{P_{n+1,d}(\cos \frac{\varphi}{r_n}) - P_{n,d}(\cos \frac{\varphi}{r_n})}{r_n(\cos \frac{\varphi}{r_n} - 1)}.
\]

Taking into account Remark 1.1, we see that the Universality Conjecture will follow from the equality
\[
\tilde{\mathcal{E}}_n(\varphi) = \frac{1}{(2\pi)^{d-2}} J_{d-1}(\varphi) + O(n^{-1}), \quad \text{uniformly for } |\varphi| \leq \frac{\pi}{4}.
\] (3.7)

Observe first that
\[
P_{n,d}(t) = \frac{1}{(n+\alpha)_n} P_{n}(^{(\alpha,\alpha)}(t), \quad \alpha = \frac{d - 3}{2}, \quad \left( \frac{n + \alpha}{n} \right) = \frac{\Gamma(n + \alpha + 1)}{\Gamma(\alpha + 1)\Gamma(n + 1)},
\] (3.8)

where \( P_n^{(\alpha,\beta)} \) are the Jacobi polynomials defined in [14, §4.1]. To prove (3.7) we will use the Hilbert type asymptotic estimate for Jacobi polynomials, [14, Eq. (8.2.17)] or [12, (29)]. Here are the details. Set
\[
\theta := \frac{\varphi}{r_n}
\]
Observe that
\[
\frac{1}{(n+\alpha)_n} = O(n^{-\alpha}).
\] (3.9)
The Hilb type estimate [14, Eq. (8.2.17)] coupled with (3.8) and (3.9) yields
\[
P_{n,d}(\cos \theta) = \Gamma(\alpha + 1) \left( \frac{\theta}{\sin \theta} \right) \frac{1}{2} \frac{2^\alpha}{\alpha_n \sin \alpha \theta} J_\alpha(a_n \theta) + \frac{\theta^{\alpha+2}}{\sin \alpha \theta} O(1)
\]
where \(\alpha\) is as in (3.8) and \(a_n := n + \frac{d-2}{2}\). We set
\[
F_\alpha(x) := \frac{1}{x^\alpha} J_\alpha(x)
\]
and we deduce that
\[
P_{n,d}(\cos \theta) = \Gamma(\alpha + 1) \left( \frac{\theta}{\sin \theta} \right) \frac{1}{2} \frac{(2\theta)^\alpha}{\sin \alpha \theta} F_\alpha(a_n \theta) + \frac{\theta^{\alpha+2}}{\sin \alpha \theta} O(1).
\]
Hence
\[
\frac{P_{n+1,d}(\cos \theta) - P_{n,d}(\cos \theta)}{r_n(\cos \theta - 1)} = -\Gamma(\alpha + 1) \frac{P_{n+1,d}(\cos \theta) - P_{n,d}(\cos \theta)}{2r_n \sin^2(\frac{\theta}{2})}
\]
\[
= -\Gamma(\alpha + 1) \left( \frac{\theta}{\sin \theta} \right) \frac{1}{2} \frac{(2\theta)^\alpha}{\sin \alpha \theta} \left( \frac{F_\alpha(a_{n+1} \theta) - F_\alpha(a_n \theta)}{2r_n \sin^2(\frac{\theta}{2})} \right) + \frac{\theta^{\alpha+2}}{r_n \sin^{\alpha+2} \theta} O(1).
\]
Observe that \(a_{n+1} \theta = a_n \theta + \theta\) and we deduce
\[
F_\alpha(a_{n+1} \theta) - F_\alpha(a_n \theta) = F_\alpha'(a_n \theta) \theta + O(\theta^2).
\]
The classical identity involving Bessel functions, [1, Eq. (4.6.2)],
\[
\frac{d}{dx} \left( x^{-\nu} J_\nu(x) \right) = -x^{-\nu} J_{\nu+1}(x),
\]
implies that \(F_\alpha'(z) = -x^{-\alpha} J_{\alpha+1}(x)\). Now observe that
\[
2r_n \sin^2(\frac{\theta}{2}) = \frac{r_n \theta^2}{2} \left( 1 - O(\theta^2) \right).
\]
Since
\[
O(\theta) = O(r_n^{-1}) = O(n^{-1})
\]
we deduce
\[
\frac{P_{n+1,d}(\cos \theta) - P_{n,d}(\cos \theta)}{r_n(\cos \theta - 1)} = -\left( \frac{\theta}{\sin \theta} \right) \frac{1}{2} \frac{2^{\alpha+1} \theta^\alpha}{\sin \alpha \theta} \frac{F_\alpha'(a_n \theta)}{r_n \theta} + O(n^{-1})
\]
\[
= -2^{\alpha+1} \frac{\phi}{\varphi} F_\alpha'(\varphi) + O(n^{-1}) = \left( \frac{r}{2} \right)^{-(\alpha+1)} J_{\alpha+1}(r) + O(n^{-1}).
\]
Hence
\[
\tilde{c}_n(\varphi) = \Gamma(\alpha + 1) - \frac{\mu_n}{\sigma_{d-1}^2 r_n^{d-2}} k_n \left( \frac{r}{2} \right)^{-(\alpha+1)} J_{\alpha+1}(r) + O(n^{-1}).
\]
The equality (3.7) now follows from the estimates
\[
\frac{\mu_n}{r_n^{d-2}} = \frac{2}{(d-2)!} + O(n^{-1}), \quad \frac{k_n}{k_{n+1}} = \frac{1}{2} + O(n^{-1}),
\]
the equality (3.2), and the doubling formula
\[
\sqrt{\pi} \Gamma(2x) = 2^{2x-1} \Gamma(x) \Gamma(x + \frac{1}{2}). \quad \square
\]
Remark 3.1. In the special case $M = S^2$ we can visualize the blowup behavior of the spectral function. In this case the eigenvalues of the Laplacian are $\lambda_n = n(n + 1)$ and

$$\bar{E}_n(\varphi) = \frac{1}{\lambda_n} E_n \left( \frac{\varphi}{\sqrt{\lambda_n}} \right).$$

The function $E_n(\varphi)$ has a peak at $\varphi = 0$,

$$E_n(0) = \frac{(n + 1)^2}{4\pi} \sim \frac{1}{4\pi \lambda_n} \text{ as } n \rightarrow \infty.$$ 

The oscillations in the graph depicted in Figure 1 are pushed at $\infty$ by the rescaling in $\varphi$, and the resulting behavior becomes rather tame, Figure 2.
APPENDIX A. SHARP ELLIPTIC ESTIMATES

The main goal of this appendix is to provide a proof of the estimates (2.3). We will follow the same strategy employed in the proof of [4, Thm. 17.5.3]. We will reach a conclusion stronger than [4, Thm. 17.5.3] because we will rely on sharp a priori elliptic estimates. We obtain these sharp estimates from the precise a priori estimates for elliptic operators with real analytic coefficients in [8, Chap. 8]. For the reader’s convenience we first synchronize our notations with those in [8].

For any nonnegative integers \( s \) we set \( M_s := s! \). Observe that [8, Chap. 8, Eq. (1.10)] becomes

\[
M_{t+s} \leq d^{t+s} M_t M_s, \quad \forall s, t \in \mathbb{Z}_{\geq 0}, \quad d := 2. \tag{A.1}
\]

Denote by \( \Delta \) the Laplacian operator defined by a real analytic metric \( g \) on the ball \( B_4 \), where

\[
B_R = \{ x \in \mathbb{R}^m; \ |x| < R \}.
\]

For any \( R > 0 \) we denote by \( A(B_R) \) the space of functions real analytic on the ball \( B_R \). We set

\[
\| u \|_{k,R} = \sum_{|\alpha| = k} \left( \int_{B_R} |\partial_x^\alpha u(x)|^2 \, dx \right)^{1/2}.
\tag{A.2}
\]

**Theorem A.1.** Then there exist constants \( K, T > 0, \rho_0 \in (0, 1) \) that depend only on the real analytic metric \( g \) and satisfying the following properties. If \( u \in A(B_2) \) and \( Z_0, L_0 > 1 \) are such that

\[
\| \Delta^k u \|_{0,2} \leq Z_0 L_0^2 M_{2k}, \quad \forall k \geq 0,
\]

then

\[
\| u \|_{j,\rho_0} \leq Z_0 (T \sqrt{L_0})^j M_j, \quad \forall j \geq 0.
\]

\[
|\partial_x^\alpha u(0)| \leq K Z_0 L_0^\frac{m}{4} (2T \sqrt{L_0})^j M_j, \quad \forall j \geq 0, \quad \forall |\alpha| = j.
\]

We want to emphasize that \( K \) and \( T \) are independent of \( u, Z_0, L_0 \).

**Proof.** We follow closely the approach in [8, Chap. 8, Sec. 2]. From Theorem 2.1 op. cit. we deduce the following.

**Lemma A.2.** There exist \( \rho_1, C_1 > 0, \rho_1 < 1 \), such that if \( 0 < \rho < \rho + \delta < \rho_1 \) and any \( u \in A(B_2) \) we have

\[
\| u \|_{2,\rho} \leq C_1 \left( \| \varphi_{\rho,\delta} \Delta u \|_{0, \rho+\delta} + \sum_{\ell=0}^{2m} \frac{1}{\delta^{2m-\ell}} \| u \|_{\ell, \rho+\delta} \right).
\]

\[\square\]

Arguing as in the proof of [8, Chap. 8, Lemma 2.5] we obtain the following result.

**Lemma A.3.** Suppose that \( a \in A(B_2) \) satisfies

\[
\sup_{x \in B_3} \sum_{|\alpha| = r} |\partial^\alpha a(x)| \leq L^r M_r, \quad \forall r \geq 0.
\]

Then for every \( r, s \in \mathbb{Z}_{\geq 0}, 0 < \rho < \rho + \delta < 1 \) and \( u \in A(B_2) \) we have

\[
\sum_{|\beta| = s} \sum_{|\alpha| = r} \| \partial^\beta \varphi_{\rho,\delta} (a \partial^\alpha u - \partial^\alpha (au)) \|_{0, \rho+\delta}
\]

\[
\leq C_s^r \sum_{\ell=0}^s \frac{1}{\delta^\ell} \sum_{t=0}^{s-\ell} L^{r+t} M_{r+t} \sum_{i=0}^{r-1} \frac{1}{L^i} \| u \|_{s-\ell-t+i, \rho+\delta},
\]

where \( C_s^r \) depends only on \( s \). \[\square\]
Using Lemma A.2 and A.3 we deduce as in [8, Chap. 8, Lemma 2.6] the following result.

**Lemma A.4.** Suppose that $\rho_1, C_1$ are as in Lemma A.2. For any $\nu > 0$ there exists $\gamma(\nu) > 0$ that depends only on the metric and $\nu$ such that for any $u \in \mathcal{A}(B_2)$, $k > 0$ and $\rho < \rho_1 < \rho_1$ we have

$$
\|u\|_{2k+2, \rho} \leq C_1 \left\{ \|\Delta u\|_{2k, \rho + \delta} + \nu \|u\|_{2k+2, \rho + \delta} \right\}^{\gamma(\nu)}
$$

(A.8)

where the constant $C_1$ depends only on the metric $g$. \hfill \Box

For every $\lambda, R > 0$ such that $R < \rho_1$ we set

$$
\sigma_k(u, \lambda, R) := \frac{1}{M_2k^k+1} \sup_{R/2 \leq \rho < R} (R - \rho)^{2k+2}\|u\|_{2k+2, \rho}.
$$

Using Lemma A.4 as in the proof of [8, Chap. 8, Lemma 2.7] we obtain the following result.

**Lemma A.5.** There exists $\lambda_1 > 0$ such that for any $R < \frac{1}{4}\rho_1$, $\lambda \geq \lambda_1$, $k \geq 0$ and every $u \in \mathcal{A}(B_2)$ we have

$$
\sigma_k(u, \lambda, R) \leq \frac{M_2k-2}{4M_2k} \sigma_k^{k-1}(\Delta u, \lambda R) + \frac{1}{4} \sum_{s=1}^{k-1} \sigma_s(u, \lambda, R).
$$

(A.9)

\hfill \Box

An argument identical to the one used in the proof of [8, Chap. 8, Thm. 2.2] and based on Lemma A.5 shows that if $u \in \mathcal{A}(B_2)$ satisfies (A.3), then for $\lambda, R$ as in Lemma A.5 we have

$$
\sigma_k(u, \lambda, R) \leq \frac{M_2k+2}{M_2k} Z_0(4L_0 + 2)^{k+1}.
$$

(A.10)

If we let $\lambda = \lambda_1$ and $R_0 = \frac{1}{4}\rho_1$ in the above inequality we deduce

$$
\frac{1}{\lambda_1^{k+1}} \sup_{R_0/2 \leq \rho < R_0} (R_0 - \rho)^{2k+2}\|u\|_{2k+2, \rho} \leq M_2k+2 Z_0(4L_0 + 2)^{k+1}
$$

In particular, if $\rho_0 = 3R_0/4$ and $\mu := \sqrt{\lambda_1}$ we deduce

$$
\|u\|_{2k+2, \rho_0} \leq c_0M_2k+2 \left( \frac{4\mu}{R_0} \right)^{2k+2} (4L_0 + 2)^{k+1} = Z_0M_2k+2 \zeta^{2k+2}, \quad \zeta := \frac{4\mu}{R_0} \sqrt{4L_0 + 2}.
$$

We can assume that $\zeta > 1$.

Using the interpolation inequality [8, Chap. 8, Eq. (2.7)] we deduce that

$$
\|u\|_{2k+1, \rho_0} \leq \zeta^{-1}\|u\|_{2k+2, \rho_0} + c_m\zeta \|u\|_{2k, \rho_0},
$$

where $c_m > 0$ is a constant that depends only on the dimension $m$. Hence

$$
\|u\|_{2k+1, \rho_0} \leq Z_0M_2k+2 \zeta^{2k+1} + c_m Z_0 M_2k \zeta^{2k+1}
$$

$$
= Z_0M_2k+1 \zeta^{2k+1} \left( (2k + 2) + \frac{c_m}{(2k + 1)} \right) \leq Z_0M_2k+1 \left( (2 + c_m) \zeta \right)^{2k+1}.
$$

We deduce that for any $k \geq 0$ we have

$$
\|u\|_{k, \rho_0} \leq Z_0 M_k Z^k, \quad Z = (2 + c_m) \zeta.
$$

(A.11)
This proves (A.4) with
\[ T = \frac{Z}{\sqrt{L_0 + 1}}. \]
Using the Sobolev lemma [9, Thm. 3.5.1] we deduce\(^1\) that there exists a constant \( K_m > 0 \) that depends only on \( m \) such that for any \( v \in C^\infty(B_{r/2}) \) we have
\[ |v(0)| \leq \frac{K_m}{r^2} \left( \sum_{j=0}^{p-1} \frac{r_j}{j!} \|v\|_{j,r} + \frac{r^p}{(p-1)!} \|v\|_{p,\rho_0} \right), \quad p = p_m := \left\lfloor \frac{m}{2} \right\rfloor + 1. \]
We deduce that for any multi-index \( \alpha \) such that \(|\alpha| = k\) and any \( r \) such that \( r/2 < \rho_0 \) we have
\[
|\partial^\alpha u(0)| \leq \frac{K_m}{r^2} Z \left( \sum_{j=0}^{p-1} \frac{r^j (k+j)!}{j!} Z^{k+j} + \frac{r^p (k+p)!}{(p-1)!} Z^{k+p} \right)
\leq \frac{K_m Z_0! Z^k}{r^2} \left( \sum_{j=0}^{p-1} \binom{k+j}{j} (rZ)^j + p \binom{k+p}{p} (rZ)^p \right).
\]
Recall that \( Z = T \sqrt{L_0 + 1} \). Choose \( r \) of the form
\[ r = \frac{1}{B \sqrt{L_0 + 1}} < \frac{1}{B \sqrt{2}}, \]
where \( B \) is sufficiently large so that
\[ \frac{r}{2} = \frac{1}{2B \sqrt{L_0 + 1}} < \frac{1}{2B \sqrt{2}} < \rho_0, \]
and
\[ (rZ) = \frac{T}{B} \leq \frac{1}{2}. \]
\[
|\partial^\alpha u(0)| \leq K_m B^m Z_0(L_0 + 1)^{\frac{m}{2}} Z^k k! \left( \sum_{j=0}^{p-1} \binom{k+j}{j} 2^{-j} + p \binom{k+p}{p} 2^{-p} \right)
\leq p_m K_m B^m Z_0(L_0 + 1)^{\frac{m}{2}} Z^k k! \sum_{j=0}^{\infty} \binom{k+j}{j} 2^{-j}
\leq 2p_m K_m B^m Z_0(L_0 + 1)^{\frac{m}{2}} (2T \sqrt{L_0 + 1})^k k!
\]
This proves (A.5) with \( K = 2p_m K_m B^m \). □

Remark A.6. The above arguments extend with no changes to the case when the metric belongs to a Gevrey space, or more general to one of the spaces \( D_{M_k}(B_4) \) of [8], where the weights \( M_k \) are subject to the constraints (1.6)-(1.11) in [8, Chap.8]. □

\(^1\)We have to warn the reader that the Sobolev norms \( \|\nabla^j v\|_{2,R} \) defined in [9] do not coincide with the ones defined in (A.2) but they are equivalent. The constants implied by this equivalence of norms depend only on \( j \) and \( m \).
Theorem A.1 has the following immediate consequence.

**Corollary A.7.** Suppose $N$ is a compact real analytic manifold of dimension $n$ and $g$ is a real analytic metric on $N$ with injectivity radius $r(N)$. Denote by $\Delta_g$ the Laplace operator of the metric $g$ and by $\mathcal{A}(N)$ the space of real analytic functions on $N$. Then there exist constants $K, T > 0$ and $0 < \rho_0 < r(N)$ depending only on $g$ with the following property: for any $Z_0, L_0 > 1$ and any $u \in \mathcal{A}(N)$ such that, if

$$\| \Delta_g^k u \|_{L^2(N, g)} \leq Z_0 L_0^k k!, \quad \forall k \geq 0,$$

then

$$|\partial_x^\alpha f_L(p)| \leq K Z_0 L_0^m (2T \sqrt{L_0})^j j! \| f \|, \quad \forall |\alpha| = j, \quad \forall p \in M,$$

where $x = (x^1, \ldots, x^m)$ are normal coordinates at $p$. $\square$

**Proof of (2.3).** We follow the same strategy used in the proof of [4, Thm. 17.5.3]. Fix $L > 1$ and denote by $P_L$ the orthogonal projection onto the space

$$H_L := \bigoplus_{\lambda \leq L} \ker(\lambda - \Delta).$$

Fix $j, \ell \geq 0$ points $p_0, q_0$ in $M$, and normal coordinates $x$ at $p_0$ and $y$ at $q_0$. Let $f \in L^2(M)$ and set $f_L = P_L f$. Then

$$\| \Delta^k f_L \|_{L^2(M)} \leq L^k \| f \|, \quad \forall k \geq 0.$$

Using Corollary A.7 we deduce that

$$|\partial_x^\alpha f_L(p_0)| \leq K L^m \frac{n}{2} + \frac{1}{2} (2T j!)^j |\| f \|, \quad \forall p \in M, \quad |\alpha| = j,$$

where $K, T$ are independent of $f$, $k$, $\alpha$ and $L$. Now observe that

$$\partial_x^\alpha f_L(p_0) = (f, g_0)_{L^2(M)}, \quad g_0(q) = \partial_x^\alpha \mathcal{E}_L(p_0, q) = \sum_{\lambda \leq L} \left( \partial_x^\alpha \Psi_{n}(p_0) \right) \Psi_{n}(q).$$

The above discussion shows that for every $f \in L^2(M)$ we have

$$| (f, g_0)_{L^2(M)} | \leq K L^m \frac{n}{2} + \frac{1}{2} (2T j!)^j |\| f \|,$$

so that

$$\| g_0 \|_{L^2(M)} \leq K L^m \frac{n}{2} + \frac{1}{2} (2T j!)^j!.$$

Observe that $g_0 \in H_L$ and thus for any $k \geq 0$ we have

$$\| \Delta^k g_0 \|_{L^2(M)} \leq L^k \| g_0 \|_{L^2(M)}$$

and Corollary A.7 implies that for any $|\beta| = \ell$ we have

$$|\partial_x^\alpha \partial_y^\beta \mathcal{E}_L(p_0, q_0)| = |\partial_y^\beta g_0(q_0)| \leq K L^m \frac{n}{2} + \frac{1}{2} (4T)^\ell |\| g \|_0 \leq K^2 L^{2m} \frac{n}{2} + \frac{1}{2} (4T)^\ell T^{j+\ell} j!.$$  

The inequality (2.3) follows by observing that

$$j! \| \leq \binom{j+\ell}{j} (j+\ell)! \| \leq 2^{j+\ell} (j+\ell)!. \square$$
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