Pseudo-Riemannian almost quaternionic homogeneous spaces with irreducible isotropy

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Abstract

We show that pseudo-Riemannian almost quaternionic homogeneous spaces with index 4 and an \( H \)-irreducible isotropy group are locally isometric to a pseudo-Riemannian quaternionic Kähler symmetric space if the dimension is at least 16. In dimension 12 we give a non-symmetric example.

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1 Introduction

In [AZ] Ahmed and Zeghib studied pseudo-Riemannian almost complex homogeneous spaces of index 2 with a \( \mathbb{C} \)-irreducible isotropy group. They showed that these spaces are already pseudo-Kähler if the dimension is at least 8. If furthermore the Lie algebra of the isotropy group is \( \mathbb{C} \)-irreducible then the space is locally isometric to one of five symmetric spaces.

There are two different quaternionic analogues of Kähler manifolds, namely hyper-Kähler and quaternionic Kähler manifolds. In the first case, the complex structure is replaced by three complex structures assembling into a hyper-complex structure \((I, J, K)\), in the second by the more general notion of a quaternionic structure \( Q \subset \text{End}TM \) on the underlying manifold \( M \). Riemannian as well as pseudo-Riemannian quaternionic Kähler manifolds are Einstein and therefore of particular interest in pseudo-Riemannian geometry.

In [CM] the authors investigated the hyper-complex analogue of the topic studied by Ahmed and Zeghib, namely pseudo-Riemannian almost hyper-complex homogeneous spaces.
of index 4 with an $\mathbb{H}$-irreducible isotropy group. It turned out that these spaces of dimension greater or equal than 8 are already locally isometric to the flat space $\mathbb{H}^{1,n}$ except in dimension 12, where non-symmetric examples exist.

In this article we study the quaternionic analogue, that is we consider pseudo-Riemannian almost quaternionic homogeneous spaces of index 4 with an $\mathbb{H}$-irreducible isotropy group. The main result of our analysis is the following theorem.

**Theorem 1.1.** Let $(M, g, Q)$ be a connected almost quaternionic pseudo-Hermitian manifold of index 4 and $\dim M = 4n + 4 \geq 16$, such that there exists a connected Lie subgroup $G \subset \text{Iso}(M, g, Q)$ acting transitively on $M$. If the isotropy group $H := G_p, p \in M$, acts $\mathbb{H}$-irreducibly, then $(M, g, Q)$ is locally isometric to a quaternionic Kähler symmetric space.

Here $\text{Iso}(M, g, Q)$ denotes the subgroup of the isometry group $\text{Iso}(M, g)$ which preserves the almost quaternionic structure $Q$ of $M$. A consequence of the theorem is that the homogeneous space $M$ itself is quaternionic Kähler and locally symmetric. Notice that pseudo-Riemannian quaternionic Kähler symmetric spaces have been classified in [AC]. In Section 3.2 we show, by construction of a non-symmetric example in dimension 12, that the hypothesis $\dim M \geq 16$ in Theorem 1.1 cannot be omitted. Moreover, we classify in Proposition 3.1 all examples with the same isotropy algebra $\mathfrak{h} = \mathfrak{so}(1, 2) \oplus \mathfrak{so}(3) \subset \mathfrak{so}(1, 2) \oplus \mathfrak{so}(4) \subset \mathfrak{gl}(\mathbb{R}^{1,2} \otimes \mathbb{R}^4) \cong \mathfrak{gl}(12, \mathbb{R})$ in terms of the solutions of a system of four quadratic equations for six real variables.

The strategy of the proof of Theorem 1.1 is as follows. We consider the $\mathbb{H}$-irreducible isotropy group $H$ as a subgroup of $\text{Sp}(1, n)\text{Sp}(1)$ and classify the possible Lie algebras. Then we consider the covering $G/H^0$ of $M = G/H$ and show by taking into account the possible Lie algebras that it is a reductive homogeneous space. Finally, we show that the universal covering $\tilde{M}$ is a symmetric space. The invariance of the fundamental 4-form under $G$ then implies that the symmetric space is quaternionic Kähler.

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2 About subgroups of $\text{Sp}(1, n)\text{Sp}(1)$

**Lemma 2.1** (Goursat’s theorem). Let $\mathfrak{g}_1, \mathfrak{g}_2$ be Lie algebras. There is a one-to-one correspondence between Lie subalgebras $\mathfrak{h} \subset \mathfrak{g}_1 \oplus \mathfrak{g}_2$ and quintuples $Q(\mathfrak{h}) = (A, A_0, B, B_0, \theta)$, with $A \subset \mathfrak{g}_1, B \subset \mathfrak{g}_2$ Lie subalgebras, $A_0 \subset A, B_0 \subset B$ ideals and $\theta : A/A_0 \to B/B_0$ is a Lie algebra isomorphism.

**Proof:** Let $\mathfrak{h} \subset \mathfrak{g}_1 \oplus \mathfrak{g}_2$ be a Lie subalgebra and denote by $\pi_i : \mathfrak{g}_1 \oplus \mathfrak{g}_2 \to \mathfrak{g}_i, i = 1, 2$, the natural projections. Set $A := \pi_1(\mathfrak{h}) \subset \mathfrak{g}_1, B := \pi_2(\mathfrak{h}) \subset \mathfrak{g}_2, A_0 := \ker(\pi_2|_\mathfrak{h})$ and $B_0 := \ker(\pi_1|_\mathfrak{h})$. It is not hard to see that $A_0$ and $B_0$ can be identified with ideals in $A$.
Conversely, a quintuple $Q = (A, A_0, B, B_0, \theta)$ as above defines a Lie subalgebra $\mathfrak{h} = \mathcal{G}(Q) \subset \mathfrak{g}_1 \oplus \mathfrak{g}_2$ by setting

$$\mathfrak{h} := \{ X + Y \in A \oplus B \mid \theta(X + A_0) = Y + B_0 \}.$$  

It is not hard to see that the maps $\mathcal{G}$ and $Q$ are inverse to each other. \hfill \Box

We will use the following two classification results for $\mathbb{H}$-irreducible subgroups of $\text{Sp}(1, n)$.

**Theorem 2.1** ([CM, Corollary 2.1]). Let $H \subset \text{Sp}(1, n)$ be a connected and $\mathbb{H}$-irreducible Lie subgroup. Then $H$ is conjugate to one of the following groups:

(i) $\text{SO}^0(1, n)$, $\text{SO}^0(1, n) \cdot \text{U}(1)$, $\text{SO}^0(1, n) \cdot \text{Sp}(1)$ if $n \geq 2$,

(ii) $\text{SU}(1, n)$, $\text{U}(1, n)$,

(iii) $\text{Sp}(1, n)$,

(iv) $U^0 = \{ A \in \text{Sp}(1, 1) \mid A\Phi = \Phi A \} \cong \text{Spin}^0(1, 3)$ with $\Phi = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ if $n = 1$.

**Proposition 2.1** ([CM, Proposition 2.4]). Let $H \subset \text{Sp}(1, n)$ be an $\mathbb{H}$-irreducible subgroup. Then one of the following is true.

(i) $H$ is discrete.

(ii) $H^0 = \text{U}(1) \cdot \mathbb{1}_{n+1}$ or $H^0 = \text{Sp}(1) \cdot \mathbb{1}_{n+1}$.

(iii) $H^0$ is $\mathbb{H}$-irreducible.

(iv) $n = 1$ and $H^0$ is one of the groups $\text{SO}^0(1, 1)$, $\text{SO}^0(1, 1) \cdot \text{U}(1)$, $\text{SO}^0(1, 1) \cdot \text{Sp}(1)$ or

$$S = \left\{ e^{itd} \begin{pmatrix} \cosh(at) & \sinh(at) \\ \sinh(at) & \cosh(at) \end{pmatrix} \right\} \quad t \in \mathbb{R},$$

for some non-zero real numbers $a, b$.

We denote by $\pi_1 : \text{sp}(1, n) \oplus \text{sp}(1) \rightarrow \text{sp}(1, n)$ and $\pi_2 : \text{sp}(1, n) \oplus \text{sp}(1) \rightarrow \text{sp}(1)$ the canonical projections.

**Proposition 2.2.** Let $n \geq 2$ and $H \subset \text{Sp}(1, n)\text{Sp}(1)$ be an $\mathbb{H}$-irreducible closed subgroup. Then the Lie algebra $\mathfrak{h}$ is one of the following:

(i) $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{c}$ with $\mathfrak{h}_0 \in \{ \{0\}, \text{so}(1, n) \}$, $\mathfrak{c} \subset \text{sp}(1) \cdot \mathbb{1}_{n+1} \oplus \text{sp}(1)$ and $\pi_1(\mathfrak{c}) = \text{sp}(1) \cdot \mathbb{1}_{n+1}$, $\pi_2(\mathfrak{c}) = \text{sp}(1)$, $\mathfrak{c} \cap \text{sp}(1, n) = \{0\}$, $\mathfrak{c} \cap \text{sp}(1) = \{0\}$,

(ii) $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{c}$ with $\mathfrak{h}_0 \in \{ \{0\}, \text{so}(1, n), \text{su}(1, n) \}$, $\mathfrak{c} \subset \text{u}(1) \cdot \mathbb{1}_{n+1} \oplus \text{u}(1)$ and $\pi_1(\mathfrak{c}) = \text{u}(1) \cdot \mathbb{1}_{n+1}$, $\pi_2(\mathfrak{c}) = \text{u}(1)$, $\mathfrak{c} \cap \text{sp}(1, n) = \{0\}$, $\mathfrak{c} \cap \text{sp}(1) = \{0\}$,
(iii) \( h = h_0 \oplus c \) where \( h_0 \subset \mathfrak{sp}(1, n) \) is one of the following Lie algebras

\[
\mathfrak{sp}(1, n), \ \mathfrak{u}(1, n), \ \mathfrak{su}(1, n), \ \mathfrak{so}(1, n) \oplus \mathfrak{sp}(1) \cdot \mathbb{1}_{n+1}, \ \mathfrak{so}(1, n) \oplus \mathfrak{u}(1) \cdot \mathbb{1}_{n+1},
\]

\[
\mathfrak{so}(1, n), \ \mathfrak{sp}(1) \cdot \mathbb{1}_{n+1}, \ \mathfrak{u}(1) \cdot \mathbb{1}_{n+1}, \ \{0\},
\]

and \( c \subset \mathfrak{sp}(1) \) is \( \{0\}, \mathfrak{u}(1) \) or \( \mathfrak{sp}(1) \).

**Proof:** The idea is to apply Goursat’s theorem (Lemma 2.1) to \( h \subset \mathfrak{sp}(1, n) \oplus \mathfrak{sp}(1) \). The Lie subalgebras \( A, A_0, B \) and \( B_0 \) are given by \( \pi_1(h), h \cap \mathfrak{sp}(1), \pi_2(h) \) and \( h \cap \mathfrak{sp}(1) \). Let \( p : \mathfrak{sp}(1, n) \times \mathfrak{sp}(1) \to \mathfrak{sp}(1, n) \) be the natural projection. Notice that \( H \subset \mathfrak{sp}(1, n) \mathfrak{sp}(1) \) is \( \mathbb{H} \)-irreducible if and only if \( p(\hat{H}) \subset \mathfrak{sp}(1, n) \) is \( \mathbb{H} \)-irreducible, where \( \hat{H} \) is the preimage of \( H \) under the two-fold covering \( \mathfrak{sp}(1, n) \times \mathfrak{sp}(1) \to \mathfrak{sp}(1, n) \mathfrak{sp}(1) \). By Proposition 2.1 and Theorem 2.1 we know that \( p(\hat{H}) \) is either discrete or \( (p(\hat{H}))^0 \) is one of the following subgroups of \( \mathfrak{sp}(1, n) \):

\[
\mathfrak{sp}(1, n), \ \mathfrak{U}(1, n), \ \mathfrak{SU}(1, n), \ \mathfrak{SO}^0(1, n) (\mathfrak{Sp}(1) \cdot \mathbb{1}_{n+1}), \ \mathfrak{SO}^0(1, n) (\mathfrak{U}(1) \cdot \mathbb{1}_{n+1}),
\]

\[
\mathfrak{SO}^0(1, n), \ \mathfrak{Sp}(1) \cdot \mathbb{1}_{n+1}, \ \mathfrak{U}(1) \cdot \mathbb{1}_{n+1}.
\]

Since \( dp = \pi_1 \) we immediately obtain all possibilities for \( \pi_1(h) \). Furthermore \( h \cap \mathfrak{sp}(1, n) \) is an ideal of the Lie algebra \( \pi_1(h) \). We can read off from the above list a decomposition of \( \pi_1(h) \) into ideals, which gives us all possibilities for \( h \cap \mathfrak{sp}(1, n) \). The resulting list of pairs \( (A, A_0) \) is displayed in a table below.

On the other side there are only three Lie subalgebras of \( \mathfrak{sp}(1) \), namely \( \mathfrak{sp}(1) \) itself, \( \mathfrak{u}(1) \) and \( \{0\} \). It follows that \( \pi_2(h) \) is one of these three. Again, \( h \cap \mathfrak{sp}(1) \) is an ideal of \( \pi_2(h) \).

It follows that the only possibilities for \( h \cap \mathfrak{sp}(1) \) are the same as for \( \pi_2(h) \).

By Goursat’s theorem we have a Lie algebra isomorphism \( \theta : A/A_0 \to B/B_0 \). Since we know all possibilities for \( B \) and \( B_0 \), it follows that \( A/A_0 \) is isomorphic to \( \mathfrak{sp}(1), \mathfrak{u}(1) \) or \( \{0\} \). Therefore we need to consider all possibilities for \( A \) and \( A_0 \), as listed in the following table, and keep only those for which \( A/A_0 \) is isomorphic to \( \mathfrak{sp}(1), \mathfrak{u}(1) \) or \( \{0\} \).

| \( A \)       | \( A_0 \)       |
|--------------|-----------------|
| \( \mathfrak{sp}(1, n) \) | \( \{0\} \)      |
| \( \mathfrak{su}(1, n) \oplus \mathfrak{u}(1) \) | \( \mathfrak{su}(1, n) \oplus \mathfrak{u}(1) \) |
| \( \mathfrak{su}(1, n) \) | \( \{0\} \)      |
| \( \mathfrak{so}(1, n) \oplus \mathfrak{sp}(1) \) | \( \{0\} \)      |
| \( \mathfrak{so}(1, n) \oplus \mathfrak{u}(1) \) | \( \{0\} \)      |
If $B/B_0 \cong \mathfrak{sp}(1)$ then $B = \mathfrak{sp}(1)$ and $B_0 = \{0\}$. The possibilities for $(A, A_0)$ are

\[(\mathfrak{so}(1, n) \oplus \mathfrak{sp}(1) \cdot 1_{n+1}, \mathfrak{so}(1, n)) \text{ and } (\mathfrak{sp}(1) \cdot 1_{n+1}, \{0\})\].

This gives us case $(i)$. Analogously we get the remaining Lie algebras in $(ii)$ and $(iii)$. □

3 Main results

3.1 Proof of the main theorem

Lemma 3.1 ([CM, Lemma 3.1]). Let $n \geq 3$ and $\alpha \in \otimes^3 V^*$, where $V = \mathbb{H}^{1,n}$ is considered as real vector space. If $\alpha$ is $\mathrm{SO}^0(1, n)$-invariant, then $\alpha = 0$.

Remark 3.1. The $\mathrm{SO}^0(1, n)$-invariant elements of $\otimes^3 V^*$ are in one-to-one correspondence to the $\mathrm{SO}^0(1, n)$-equivariant bilinear maps from $V \times V$ to $V$. It follows from Lemma 3.1 that the corresponding bilinear maps also vanish.

Proof of Theorem 1.1 Let $\rho : H \to \mathrm{GL}(T_p M)$ be the isotropy representation. We identify $H$ with its image $\rho(H)$. Since $H$ preserves the metric $g$ and the almost quaternionic structure $Q$, we can consider $H$ as a subgroup of $\mathrm{Sp}(1)\mathrm{Sp}(1)$.

In our first step we consider the covering $G/H_0$ of $M = G/H$ and show that it is a reductive homogeneous space, i.e. there exists an $H^0$-invariant subspace $m \subset g$ such that $g = h \oplus m$.

We apply Proposition 2.2 to $H_0$. The existence of a subspace $m$ is clear if $h$ is one of the semi-simple Lie algebras in the list. If $h$ is one of the abelian Lie algebras contained in $u(1) \cdot 1_{n+1} \oplus u(1)$, then the closure of $\text{Ad}(H^0) \subset \text{GL}(g)$ is compact and hence there exists an $\text{Ad}(H^0)$-invariant subspace $m$. The remaining Lie algebras in the list have the form $h = s \oplus z$ where $s$ is semi-simple containing $\mathfrak{so}(1, n)$ and $z$ is the non-trivial centre.

Then $g$ decomposes into $g = s \oplus z \oplus m$ with respect to the action of $s$. If we consider the action of $s$ on $m \cong \mathbb{H}^{1,n}$ as a complex representation, then $m$ is either $\mathbb{C}$-irreducible or decomposes into two $\mathbb{C}$-irreducible subrepresentations. Since the elements of $z$ commute with $s$, they preserve the sum of all non-trivial $s$-submodules, which is precisely $m$. Thus we have shown that $G/H^0$ is a reductive homogeneous space.

Next we show that $g = h \oplus m$ is a symmetric Lie algebra. It is sufficient to show that $[m, m] \subset h$. We restrict the Lie bracket $[,]$ to $m \times m$ and denote its projection to $m$ by $\beta$. It is an antisymmetric bilinear map which is $\text{Ad}(H)$-equivariant. Since $m \cong \mathbb{H}^{1,n}$, we...
can consider $\beta$ as an element of $\otimes^3(\mathbb{H}^{1,n})^*$. It is also $H_{\text{Zar}}$-invariant, where $H_{\text{Zar}}$ denotes the Zariski closure. Since $H_{\text{Zar}}$ is an algebraic group, it has only finitely many connected components, see [Mi]. Now we show that $(H_{\text{Zar}})^0$ is non-compact.

Assume that $(H_{\text{Zar}})^0$ is compact. Since $H_{\text{Zar}}$ has only finitely many connected components it follows that $H_{\text{Zar}}$ is compact and therefore contained in a maximal compact subgroup of $\text{Sp}(1,n)\text{Sp}(1)$. Hence, $H_{\text{Zar}}$ is conjugate to a subgroup of $(\text{Sp}(1)\times\text{Sp}(n))\text{Sp}(1)$ but this contradicts the $\mathbb{H}$-irreducibility of $H_{\text{Zar}}$. So we have shown that $(H_{\text{Zar}})^0$ is non-compact.

Now we apply Proposition 2.2 to $H_{\text{Zar}}$. Since $H_{\text{Zar}}$ is non-compact we see from the list there that $(H_{\text{Zar}})^0$ contains $\text{SO}^0(1,n)$. Hence, $\beta$ is $\text{SO}^0(1,n)$-equivariant. Since $n \geq 3$ it follows from Remark 3.1 that $\beta$ vanishes. This shows that $g = \mathfrak{h} \oplus \mathfrak{m}$ is a symmetric Lie algebra and that the universal covering $\tilde{M} = \tilde{G}/\tilde{G}_p$ of $M$ is a symmetric space. The fundamental 4-form $\Omega$ of $\tilde{M}$ is $\tilde{G}$-invariant and since $\tilde{M}$ is a symmetric space $\Omega$ is parallel. In particular $\Omega$ is closed. It is known that for dimension $\geq 12$ an almost quaternionic Hermitian manifold is quaternionic Kähler if $d\Omega = 0$, see [S]. This shows that $\tilde{M}$ is furthermore a quaternionic Kähler manifold. Summarizing, we have shown that $M$ is locally isometric to a quaternionic Kähler symmetric space. □

### 3.2 A class of non-symmetric examples in dimension 12

In Theorem 1.1 we did not consider the dimension 12. This is because the arguments used in the proof to show that $M$ is a reductive homogeneous space do not apply in this dimension, although still $\text{SO}^0(1,n) \subset H_{\text{Zar}}$ holds. In fact, the proof relies on Lemma 3.1 which holds for dimension $4n + 4 \geq 16$. If $\dim M = 12$ then $n = 2$ and then there exist non-trivial anti-symmetric bilinear forms $\mathbb{H}^{1,2} \times \mathbb{H}^{1,2} \to \mathbb{H}^{1,2}$ which are invariant under $\text{SO}^0(1,2)$. Therefore in dimension 12 we cannot be sure if the manifolds are symmetric.

In the following we will give a non-symmetric example by specifying a Lie algebra $g = \mathfrak{h} \oplus \mathfrak{m}$ where $\mathfrak{h}$ is a Lie algebra of the list in Proposition 2.2. The pair $(g, \mathfrak{h})$ defines a simply connected homogeneous space $M = G/H$ where $G$ is a connected and simply connected Lie group with Lie algebra $g$ and $H$ is the closed connected Lie subgroup of $G$ with Lie algebra $\mathfrak{h}$.

Let $\mathfrak{h} = \mathfrak{so}(1,2) \oplus \mathfrak{c}$ with $\mathfrak{c} = \{(X \cdot 1_3, X) \in \mathfrak{sp}(1) \cdot 1_3 \oplus \mathfrak{sp}(1) \mid X \in \mathfrak{sp}(1)\}$, see Proposition 2.2 (i). Then we consider the vector space direct sum $\mathfrak{g} := \mathfrak{h} \oplus \mathfrak{m}$ with $\mathfrak{m} = \mathbb{H}^{1,2}$ and define a Lie bracket on $\mathfrak{g}$ in the following way. For elements $A, B \in \mathfrak{h}$ we take the standard Lie bracket of $\mathfrak{h}$, i.e. $[A, B] = AB - BA$. Then we define $[A, x] = -[x, A] = Ax$ for $A \in \mathfrak{h}$ and $x \in \mathfrak{m}$. Note that, as an $\mathfrak{h}$-module, we can decompose $\mathfrak{m} = \mathbb{H}^{1,2} = \mathbb{R}^{1,2} \otimes \mathbb{H} = \mathbb{R}^{1,2} \otimes \mathbb{R}^4$, where the action of $\mathfrak{so}(1,2)$ is by the defining representation on the first factor and trivial on the second and the action of $\mathfrak{c} \cong \mathfrak{so}(3) \subset \mathfrak{so}(4)$ is trivial on the first factor and by the standard four-dimensional representation $\mathbb{H} = \mathbb{R} \oplus \text{Im} \mathbb{H} = \mathbb{R} \oplus \mathbb{R}^3$ on the second. Finally we have to define the Lie bracket for elements in $\mathfrak{m} = \mathbb{R}^{1,2} \otimes \mathbb{R}^4$.

Let $K : \mathbb{R}^{1,2} \to \mathfrak{so}(1,2)$ be an isomorphism of Lie algebras where $\mathbb{R}^{1,2}$ is endowed with
the Lorentzian cross product, \( \iota : \text{sp}(1) \to \mathfrak{c}, X \to X \cdot 1_3 + X \), and let \( \eta \) be the standard Lorentz metric on \( \mathbb{R}^{1,2} \). Furthermore denote \( \langle \cdot, \cdot \rangle \) the standard inner product on \( \mathbb{R}^4 \). Let \( x = u \otimes p, y = v \otimes q \in \mathbb{R}^{1,2} \otimes \mathbb{R}^4 \) and write \( p = p_0 + \bar{p}, q = q_0 + \bar{q} \), where \( p_0, q_0 \in \mathbb{R} \) and \( \bar{p}, \bar{q} \in \text{Im} \mathbb{H} = \mathbb{R}^3 \). We set
\[
[x, y] = (\bar{p}, \bar{q}) \cdot K(u \times v) - \frac{1}{2} \eta(u, v) \iota(\bar{p} \times \bar{q}) + u \times v (p_0 q_0 - \langle \bar{p}, \bar{q} \rangle),
\]
where \( \bar{p} \times \bar{q} \) is the Euclidian cross product in \( \text{Im} \mathbb{H} = \text{sp}(1) \) and \( u \times v \) the Lorentzian cross product in \( \mathbb{R}^{1,2} \). This extends the partially defined bracket to an anti-symmetric bilinear map \( [\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} \), which satisfies the Jacobi-identity. Hence \( \mathfrak{g} \) becomes a Lie algebra. We claim that \( (\mathfrak{g}, \mathfrak{h}) \) is not a symmetric pair. In fact, every \( \mathfrak{h} \)-invariant complement \( \mathfrak{m'} \) of \( \mathfrak{h} \) in \( \mathfrak{g} \) contains \( \mathbb{R}^{1,2} \otimes \mathbb{R}^3 \) (there is no other equivalent \( \mathfrak{h} \)-submodule in \( \mathfrak{g} \)) and thus we see from the formula for the bracket that \([\mathfrak{m'}, \mathfrak{m'}] \not
subset \mathfrak{h} \).

For a general classification of the homogeneous spaces with \( \mathfrak{h} = \mathfrak{so}(1,2) \oplus \mathfrak{c} \) we need to classify all the Lie algebra structures on the vector \( \mathfrak{g} = \mathfrak{h} \oplus \mathbb{R}^{1,2} \otimes \mathbb{R}^3 \) such that the Lie bracket restricts to the Lie bracket of \( \mathfrak{h} \) and to the given representation of \( \mathfrak{h} \) on \( \mathbb{R}^{1,2} \otimes \mathbb{R}^4 \). For this one has to describe all the \( \mathfrak{h} \)-invariant tensors of \( \Lambda^2 \mathfrak{m}^* \otimes \mathfrak{g} \cong \Lambda^2 \mathfrak{m}^* \otimes \mathfrak{h} \oplus \Lambda^2 \mathfrak{m}^* \otimes \mathfrak{m} \) which satisfy the Jacobi-identity. With the above notation, these bilinear maps have the following form
\[
[x, y] = (a \cdot p_0 q_0 + b \langle \bar{p}, \bar{q} \rangle) \cdot K(u \times v) + \eta(u, v) \langle c \cdot \iota(\bar{p} \times \bar{q}) + d (p_0 q_0 - \langle \bar{p}, \bar{q} \rangle) \rangle + u \times v \left( a_3 p_0 q_0 + a_2 \cdot \langle \bar{p}, \bar{q} \rangle - \frac{a_3}{2} (p_0 \bar{q} + q_0 \bar{p}) \right),
\]
where \( a, b, c, d, a_1, a_2, a_3 \in \mathbb{R} \). The bracket satisfies the Jacobi-identity if and only if the following equations hold
\[
\begin{align*}
0 &= d, \\
0 &= a + \frac{a_1 a_3}{2} - \frac{a_3^2}{4}, \quad (1) \\
0 &= b + 2c + \frac{a_2 a_3}{2}, \quad (2) \\
0 &= b + a_1 a_2 - \frac{a_2 a_3}{2}, \quad (3) \\
0 &= -\frac{b a_3}{2} + a a_2. \quad (4)
\end{align*}
\]
Summarizing we obtain the following proposition.

**Proposition 3.1.** Every solution \((a, b, c, a_1, a_2, a_3)\) of the quadratic system \((1)-(4)\) defines a connected and simply connected homogeneous almost quaternionic pseudo-Hermitian manifold \(G/H\) with isotropy algebra \(\mathfrak{h} = \mathfrak{so}(1,2) \oplus \mathfrak{so}(3) \subset \mathfrak{so}(1,2) \oplus \mathfrak{so}(4) \subset \mathfrak{gl}(\mathbb{R}^{1,2} \otimes \mathbb{R}^4) \cong \mathfrak{gl}(12, \mathbb{R})\). Conversely, every such homogeneous space arises by this construction.

The above example corresponds to \(a = 0, b = 1, c = -\frac{1}{2}, d = 0, a_1 = 1, a_2 = -1\) and \(a_3 = 0\).
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