A SECOND-ORDER ENSEMBLE METHOD BASED ON A
BLENDED BACKWARD DIFFERENTIATION FORMULA
TIMESTEPPING SCHEME FOR TIME-DEPENDENT
NAVIER-STOKES EQUATIONS

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ABSTRACT. We present a second-order ensemble method based on a blended
three-step backward differentiation formula (BDF) timestepping scheme to
calculate an ensemble of Navier-Stokes equations. Compared with the only ex-
isting second-order ensemble method that combines the two-step BDF timestepp-
ing scheme and a special explicit second-order Adams-Bashforth treatment
of the advection term, this method is more accurate with nominal increase
in computational cost. We give comprehensive stability and error analysis for
the method. Numerical examples are also provided to verify theoretical results
and demonstrate the improved accuracy of the method.

1. INTRODUCTION

Uncertainty quantification in geophysical systems as well as many engineering
processes often involves computing an ensemble of nonlinear partial differential
equations (PDE), see for instance [2], [14], [15], [16], [22]. Solving these nonlinear
PDEs numerically is usually very demanding in both computer resources and com-
puting time, as even one single realization may require millions or even billions
of degrees of freedom to obtain useful approximations. If the nonlinear effect is dom-
inant, accurate approximations are hard to obtain, especially if the computational
domain is large, e.g., global forecasting systems for numerical weather predictions.
Computing ensembles inevitably results in a huge increase in the computational
cost and poses a great challenge in performing accurate ensemble calculation. In
the past few decades, most efforts have been devoted to developing data assimilation
methods to reduce the number of realizations required, [22], [23]. Only recently, an
ensemble algorithm has been developed by Jiang and Layton [12], [13] to compute
an ensemble of time dependent Navier-Stokes equations efficiently. Instead of treat-
ing the simulation of each realization as separate tasks, this novel algorithm solves
all realizations at one pass for each time level. It takes advantage of the efficiency
of computing a linear system with multiple right hands for which highly efficient
algorithms have been established and well studied, i.e., Block CG [4], Block QMR
[5], Block GMRES [6]. As a result, this new ensemble algorithm can reduce the
computational cost significantly compared to the usual routine of computing the realizations separately.

Stability and accuracy are critical aspects in the development of such algorithms. In [12], an ensemble time stepping scheme based on a combination of backward Euler and forward Euler is studied. Using the finite element method for spatial discretization, the method is proven to be long time stable and first order convergent under a CFL-like time step condition. This condition depends on Reynolds number and degrades quickly as Reynolds number grows. To relax it, two ensemble eddy viscosity numerical regularization methods are proposed in [13]. They stabilize the system by adding extra numerical dissipation parameterized by mixing length and kinetic energy in fluctuations. A time relaxation regularization is also studied in [25]. It is also reported in [25] that grad-div stabilization can significantly weaken the time step restriction. As higher order methods are more efficient and thus more desirable in real engineering problems, developing accurate higher order ensemble methods is of great scientific and engineering interest. Nevertheless, extending the usual higher order time-stepping schemes to the ensemble algorithm is not trivial, as the ensemble methods require different time discretizations for different terms to ensure its efficiency as well as stability. The only existing higher order ensemble method [10], which we will refer to as (En-BDF2), is based on a two-step Backward Differentiation Formula (BDF2) and a special explicit second order in time Adams-Bashforth (AB2) treatment of the advection term. In this paper, we study a new second order ensemble method that is more accurate than (En-BDF2).

Classical BDF time schemes are among the most popular methods in the field of computational fluid dynamics (CFD) due to their strong stability properties, [18], [20], [19]. The highest order strongly A-stable BDF method is well-known to be the two-step BDF method. Classical BDF2 has been extensively used for large scale scientific computations as it can be used with large time steps without encountering numerical instability. Higher order multi-step BDF schemes are more accurate and efficient but less stable (not A stable). Hence one obvious approach is to blend the classical BDF2 and a classical higher order BDF method to obtain a multi-step method that is more accurate than classical BDF2 but still preserves good stability properties, such as A-stability. A family of such methods is proposed in [26], which blends the classical BDF2 and BDF3 method with a tuning parameter $\gamma$.

The schemes are given by

$$D_\gamma(u^{n+1}_t) = \gamma \left[ \frac{3u^{n+1} - 4u^n + u^{n-1}}{2\Delta t} \right] + (1 - \gamma) \left[ \frac{11u^{n+1} - 3u^n + \frac{3}{2}u^{n-1} - \frac{1}{3}u^{n-2}}{\Delta t} \right],$$

where $\gamma \in [\frac{1}{2}, 1]$. These are three-step methods with smaller coefficient on the leading term of the truncation error than classical BDF2, [24]. When $\gamma = \frac{1}{2}$, the scheme has the smallest truncation error constant, which is exactly half of the classical BDF2 scheme, [21]. This time marching scheme has been extensively tested in modern CFD codes in the area of aerodynamics, such as FUN3D developed and maintained at NASA Langley. In this paper, we propose a new second order ensemble method to efficiently compute an ensemble of Navier-Stokes equations based on this blended BDF scheme.

We consider an ensemble of $J$ Navier-Stokes equations with different initial conditions and/or different body forces, $j = 1, \ldots, J$:

$$u_{j,t} + u_j \cdot \nabla u_j - \nu \nabla^2 u_j + \nabla p_j = f_j(x, t), \text{ in } \Omega,$$

where \(\nu\) is the kinematic viscosity and \(\Omega\) is the computational domain.
\[ \nabla \cdot u_j = 0, \text{ in } \Omega, \]
\[ u_j = 0, \text{ on } \partial \Omega, \]
\[ u_j(x, 0) = u_j^0(x), \text{ in } \Omega, \]

where \( \Omega \) is an open, regular domain in \( \mathbb{R}^d \) (\( d = 2 \) or 3).

To construct a stable efficient ensemble algorithm, we need to use different time discretizations for different terms. The essential idea of the efficient ensemble algorithm is that all ensemble members share the same coefficient matrix and the main difficulty arises from the nonlinear term. Thus we split the nonlinear term into two terms with one containing the mean velocity that is independent of the index of ensemble members and the other one containing the fluctuation that characterizes each realization. The nonlinear term with the fluctuation needs to be lagged to previous time levels so it will go to the right hand side of the linear systems to be solved, so that the coefficient matrix is independent of the index of ensemble members. One consequence of lagging this term is a CFL-like condition to ensure the stability of the ensemble method. Now the key is to define an ensemble mean and the corresponding fluctuation by

\[
\langle u \rangle^n := \frac{1}{J} \sum_{j=1}^{J} (3u^n_j - 3u^{n-1}_j + u^{n-2}_j),
\]
\[
u^n_j := 3u^n_j - 3u^{n-1}_j + u^{n-2}_j - \langle u \rangle^n .
\]

\( 3u^n - 3u^{n-1} + u^{n-2} \) is a third order extrapolation of \( u^{n+1} \). Taking \( \gamma = \frac{1}{2} \), we consider the following blended BDF for discretization of the time derivative of velocity \( u \).

\[
D_{\frac{1}{2}}(u^{n+1}_j) = \frac{10u^{n+1}_j - 15u^n_j + 6u^{n-1}_j - u^{n-2}_j}{6\Delta t}.
\]

Suppressing the spacial discretization, the second order ensemble method we study reads: for \( j = 1, \ldots, J \), given \( u^0_j, u^1_j \) and \( u^2_j \), find \( u^{n+1}_j \) satisfying

\[
\frac{10u^{n+1}_j - 15u^n_j + 6u^{n-1}_j - u^{n-2}_j}{6\Delta t} + \langle u \rangle^n \cdot \nabla u^{n+1}_j
\]
\[+ u^n_j \cdot \nabla \left( 3u^n_j - 3u^{n-1}_j + u^{n-2}_j \right) + \nabla p^{n+1}_j - \nu \Delta u^{n+1}_j = f^{n+1}_j,
\]
\[\nabla \cdot u^{n+1}_j = 0. \]

This is a four-level method. \( u^0_j \) comes from given initial conditions of the problem. We need to obtain \( u^1_j \) through some one-step method, such as Crank-Nicolson method. To get \( u^2_j \), one can either use a one-step method or two-step method. The errors in these first few steps will affect the overall convergence rate of the method and thus they all need to be second order methods. We emphasize here that the timestepping schemes studied in [26] are applied to single Navier-Stokes equations, while the ensemble timestepping method we study in this paper deals with multiple Navier-Stokes equations, for which the fluctuation-induced instability has to be taken into account.

The rest of the paper is organized as follows. In Section 2, we present the notation that will be used throughout the work, and the finite element formulation.
of the proposed method. In the third section, the long time stability of the method is proved under a CFL-like condition. In Section 4, we first provide upper bounds for the consistency error and then give a comprehensive error analysis for the fully discretized method. Numerical experiments and results are presented in Section 5 to confirm theoretical analysis. Finally, in Section 6, we state some concluding remarks.

2. Notation and preliminaries

Throughout this paper the $L^2(\Omega)$ norm of scalars, vectors, and tensors will be denoted by $\| \cdot \|$ with the usual $L^2$ inner product denoted by $(\cdot, \cdot)$. $H^k(\Omega)$ is the Sobolev space $W^k_2(\Omega)$, with norm $\| \cdot \|_k$. Let $X, Q$ denote the velocity, pressure, and divergence free velocity spaces:

$$X := H^1_0(\Omega)^d = \{v \in L^2(\Omega)^d : \nabla v \in L^2(\Omega)^{d \times d} \text{ and } v = 0 \text{ on } \partial \Omega\},$$

$$Q := L^2_0(\Omega) = \{q \in L^2(\Omega) : \int_\Omega q \, dx = 0\},$$

$$V := \{v \in X : (\nabla \cdot v, q) = 0, \forall q \in Q\}.$$

For $\forall u, v, w \in X$, we define the usual skew symmetric trilinear form

$$b^*(u, v, w) := \frac{1}{2}(u \cdot \nabla v, w) - \frac{1}{2}(u \cdot \nabla w, v),$$

which satisfies

$$|b^*(u, v, w)| \leq C(\Omega)\|\nabla u\|\|\nabla v\|\|\nabla w\|,$$

(2.1)

$$|b^*(u, v, w)| \leq C(\Omega)\|u\|^{1/2}\|\nabla u\|^{1/2}\|\nabla v\|\|\nabla w\|,$$

(2.2)

$$|b^*(u, v, w)| \leq C(\Omega)\|\nabla u\|\|\nabla v\|\|\nabla w\|^{1/2}\|w\|^{1/2}.$$

(2.3)

For $\forall u_h, v_h, w_h \in X_h$, we have \cite{12}

$$b^*(u_h, v_h, w_h) = \int_\Omega u_h \cdot \nabla v_h \cdot w_h \, dx + \frac{1}{2} \int_\Omega (\nabla \cdot u_h)(v_h \cdot w_h) \, dx.$$

(2.4)

**Lemma 1.** For $\forall u_h, v_h, w_h \in X_h$,

$$b^*(u_h, v_h, w_h) \leq \|u_h\|_{L^4}\|\nabla v_h\|\|w_h\|_{L^4} + C\|\nabla \cdot u_h\|_{L^4}\|\nabla v_h\|\|w_h\|_{L^4}.$$

(2.5)

**Proof.** By Hölder’s inequality,

$$\int_\Omega u_h \cdot \nabla v_h \cdot w_h \, dx$$

$$\leq \left( \int_\Omega |u_h \cdot \nabla v_h|^{4/3} \, dx \right)^{3/4} \cdot \left( \int_\Omega |w_h|^4 \, dx \right)^{1/4}$$

$$\leq \left( \left( \int_\Omega |v_h|^{4/3} \, dx \right)^{1/3} \cdot \left( \int_\Omega (\nabla v_h)^{4/3} \, dx \right)^{2/3} \cdot \left( \int_\Omega |w_h|^4 \, dx \right) \right)^{3/4}$$

$$\leq \|u_h\|_{L^4}\|\nabla v_h\|\|w_h\|_{L^4}.$$
Similarly, we have
\[\int_{\Omega} (\nabla \cdot u_h)(v_h \cdot w_h) dx \leq \left( \int_{\Omega} |\nabla \cdot u_h|^4 dx \right)^{1/4} \cdot \left( \int_{\Omega} |v_h \cdot w_h|^{4/3} dx \right)^{3/4} \]
\[\leq \left( \int_{\Omega} |\nabla \cdot u_h|^4 dx \right)^{1/4} \cdot \left( \int_{\Omega} (|w_h|^{4/3})^3 dx \right)^{1/3} \cdot \left( \int_{\Omega} (|v_h|^{4/3})^{3/2} dx \right)^{2/3} \]
\[\leq \|\nabla \cdot u_h\|_{L^4}\|v_h\|_{L^4}\|w_h\|_{L^3} \leq C\|\nabla \cdot u_h\|_{L^4}\|v_h\|_{L^4}\|w_h\|_{L^3} \]

In the two-dimensional space \((d = 2)\), Ladyzhenskaya’s inequality is
\[(2.6) \quad \|u\|_{L^4} \leq C\|u|^{1/2}\|\nabla u\|^{1/2}.\]

The norm on the dual space of \(X\) is defined by
\[\|f\|_{-1} = \sup_{0 \neq v \in X} \frac{(f, v)}{\|\nabla v\|}.
\]

We denote conforming velocity, pressure finite element spaces based on an edge to edge triangulation \((d = 2)\) or tetrahedralization \((d = 3)\) of \(\Omega\) with maximum element diameter \(h\) by
\[X_h \subset X, \quad Q_h \subset Q.\]

We also assume the finite element spaces \((X_h, Q_h)\) satisfy the usual discrete inf-sup /LBB condition for stability of the discrete pressure, see [7] for more on this condition. Taylor-Hood elements, e.g., [1, 7], are one such choice used in the tests in Section 6. The discretely divergence free subspace of \(X_h\) is
\[V_h : = \{ v_h \in X_h : (\nabla \cdot v_h, q_h) = 0, \forall q_h \in Q_h) \}.\]

We assume further that the finite element spaces satisfy the inverse inequality (typical for quasi-uniform meshes, e.g., [1]), for all \(v_h \in X_h,
\[(2.7) \quad h\|\nabla v_h\| \leq C\|v_h\| .\]

The fully discrete method is: given \(u^{n-2}_{j,h}, u^{n-1}_{j,h}, u^n_{j,h}, \) find \(u^{n+1}_{j,h} \in X_h, \) \(p^{n+1}_{j,h} \in Q_h\) satisfying
\[(2.8) \quad \begin{align*}
\frac{(10u^{n+1}_{j,h} - 15u^n_{j,h} + 6u^{n-1}_{j,h} - u^{n-2}_{j,h})}{6\Delta t} &+ b^* \left( u^n_{j,h}, u^{n+1}_{j,h}, v_h \right) \\
&+ b^* \left( u^n_{j,h}, 3u^n_{j,h} - 3u^{n-1}_{j,h} + u^{n-2}_{j,h}, v_h \right) - \left( p^{n+1}_{j,h}, \nabla \cdot v_h \right) \\
&+ \nu \left( \nabla u^{n+1}_{j,h}, \nabla v_h \right) = (f^{n+1}_{j,h}, v_h), \quad \forall v_h \in X_h, \\
\left( \nabla \cdot u^{n+1}_{j,h}, q_h \right) &\equiv 0, \quad \forall q_h \in Q_h.
\end{align*}\]
3. Stability of the method

In this section, we prove (En-BlendedBDF) is long time, nonlinearly stable under a CFL-like time step condition.

**Theorem 1** (Stability of (En-BlendedBDF)). Consider the method (2.8) with a standard spacial discretization with mesh size $h$. Suppose the following time step conditions hold:

\[(3.1) \quad C \frac{\Delta t}{\nu h} \| \nabla u_{j,h}^n \|^2 \leq 1, \quad j = 1, \ldots, J.\]

Then, for any $N > 2$

\[(3.2) \quad \frac{1}{12} \| u_{j,h}^N \|^2 + \frac{1}{12} \| 3u_{j,h}^N - u_{j,h}^{N-1} \|^2 + \frac{1}{12} \| 3u_{j,h}^N - 3u_{j,h}^{N-1} + u_{j,h}^{N-2} \|^2
\]

\[+ \frac{1}{24} \sum_{n=2}^{N-1} \| u_{j,h}^{n+1} - 3u_{j,h}^n + 3u_{j,h}^{n-1} - u_{j,h}^{n-2} \|^2 + \frac{\Delta t}{4} \sum_{n=2}^{N-1} \nu \| \nabla u_{j,h}^{n+1} \|^2 \]

\[\leq \sum_{n=2}^{N-1} \frac{\Delta t}{\nu} \| f_{j}^n \|^2 - 1 + \frac{1}{12} \| u_{j,h}^1 \|^2 + \frac{1}{12} \| 3u_{j,h}^1 - u_{j,h}^0 \|^2 + \frac{1}{12} \| 3u_{j,h}^1 - 3u_{j,h}^0 + u_{j,h}^0 \|^2.\]

**Proof.** Set $v_h = u_{j,h}^{n+1}$ in (2.8), multiply through by $\Delta t$ and apply Young’s inequality to the right hand side. This gives

\[(3.3) \quad \frac{1}{12} \left( \| u_{j,h}^{n+1} \|^2 - \| u_{j,h}^n \|^2 \right) + \frac{1}{12} \left( \| 3u_{j,h}^{n+1} - u_{j,h}^n \|^2 - \| 3u_{j,h}^n - u_{j,h}^{n-1} \|^2 \right)
\]

\[+ \frac{1}{12} \| u_{j,h}^{n+1} - 3u_{j,h}^n + 3u_{j,h}^{n-1} - u_{j,h}^{n-2} \|^2 + \| \nabla u_{j,h}^{n+1} \|^2 \]

\[\leq \frac{\nu \Delta t}{4} \| \nabla u_{j,h}^{n+1} \|^2 + \frac{\Delta t}{4} \| f_{j}^{n+1} \|^2.\]

Next, we bound the remaining trilinear term using (2.3), (2.7) and Young’s inequality.

\[(3.4) \quad \Delta t b^* \left( u_{j,h}^n, 3u_{j,h}^n - 3u_{j,h}^{n-1} + u_{j,h}^{n-2} \right)
\]

\[= \Delta t b^* \left( u_{j,h}^n, u_{j,h}^{n+1}, u_{j,h}^{n+1} - 3u_{j,h}^n + 3u_{j,h}^{n-1} - u_{j,h}^{n-2} \right)
\]

\[\leq C \Delta t \| \nabla u_{j,h}^n \| \| \nabla u_{j,h}^{n+1} \| \| \nabla (u_{j,h}^{n+1} - 3u_{j,h}^n + 3u_{j,h}^{n-1} - u_{j,h}^{n-2}) \| \frac{1}{2} \| u_{j,h}^{n+1} - 3u_{j,h}^n + 3u_{j,h}^{n-1} - u_{j,h}^{n-2} \|^2
\]

\[\leq C \Delta t h^{-\frac{1}{2}} \| \nabla u_{j,h}^n \| \| \nabla u_{j,h}^{n+1} \| \| u_{j,h}^{n+1} - 3u_{j,h}^n + 3u_{j,h}^{n-1} - u_{j,h}^{n-2} \|
\]

\[\leq C \frac{\Delta t^2}{h} \| \nabla u_{j,h}^n \|^2 \| \nabla u_{j,h}^{n+1} \|^2 + \frac{1}{24} \| u_{j,h}^{n+1} - 3u_{j,h}^n + 3u_{j,h}^{n-1} - u_{j,h}^{n-2} \|^2.\]

With this bound, combining like terms, (3.3) becomes
Equation (3.5) reduces to
\[ \gamma \text{dition can be significantly weakened by adding grad-div stabilization, i.e., Reynolds number flows. However, it is shown in our numerical tests that this con-} \]

Remark 1. This time step condition seems very restrictive especially for high Reynolds number flows. However, it is shown in our numerical tests that this condition can be significantly weakened by adding grad-div stabilization, i.e., \([\nabla \cdot \mathbf{v}_h] \). The grad-div stabilization is well known to help improve mass conservation and relax the effect of the pressure error on the velocity error. \([28], [29]\).

3.1. An improved timestep condition for two-dimensional domains. For two dimensional domains, there are better embedding estimates which can lead to improvements on the timestep restriction. In this section we give one such example by making use of the 2d version of Ladyzhenskaya’s inequality \([2.6]\). We prove \((\text{En-BlendedBDF})\) is long time, nonlinearly stable under a much less restrictive timestep condition \((3.7)\). If pointwise divergence free elements (e.g., Scott-Vogelius elements, \([30]\)) are used, this 2d timestep restriction can be further relaxed.

Theorem 2. Consider the method \((2.8)\) with a standard spacial discretization with mesh size \(h\). Suppose the computational domain is in the two-dimensional space \((d = 2)\) and the following timestep conditions hold:

\[ C \frac{\Delta t}{\nu h} (\| \mathbf{u}_j^n \| + \| \nabla \cdot \mathbf{u}_j^n \|) \leq 1, \quad j = 1, ..., J. \]
Thus, (3.3) reduces to

\[ (3.8) \quad \frac{1}{12} \| u_{j,h}^N \|^2 + \frac{1}{12} \| 3u_{j,h}^N - u_{j,h}^{N-1} \|^2 + \frac{1}{12} \| 3u_{j,h}^N - 3u_{j,h}^{N-1} + u_{j,h}^{N-2} \|^2 \]

\[ \quad + \frac{1}{24} \sum_{n=2}^{N-1} \| u_{j,h}^{n+1} - 3u_{j,h}^n + 3u_{j,h}^{n-1} - u_{j,h}^{n-2} \|^2 + \frac{\Delta t}{4} \sum_{n=2}^{N-1} \nu \| \nabla u_{j,h}^{n+1} \|^2 \]

\[ \leq \sum_{n=2}^{N-1} \frac{\Delta t}{\nu} \| f_{j}^{n+1} \|^2 + \frac{1}{12} \| u_{j,h}^2 \|^2 + \frac{1}{12} \| 3u_{j,h}^2 - u_{j,h}^1 \|^2 + \frac{1}{12} \| 3u_{j,h}^2 - 3u_{j,h}^1 + u_{j,h}^0 \|^2 . \]

Proof. By lemma 1 and Ladyzhenskaya’s inequality (2.6), in the two-dimensional space we have the following bound on the nonlinear term.

\[ (3.9) \]

\[ \Delta t \| u_{j,h}^{n+1} - 3u_{j,h}^n + 3u_{j,h}^{n-1} - u_{j,h}^{n-2} \|^2 \]

\[ \leq C \Delta t \| \nabla \cdot u_{j,h}^n \| \| \nabla u_{j,h}^{n+1} \| \| \nabla u_{j,h}^{n+1} \| \| u_{j,h}^{n+1} - 3u_{j,h}^n + 3u_{j,h}^{n-1} - u_{j,h}^{n-2} \|^2 \]

\[ \leq C \Delta t \| u_{j,h}^n \|_L^2 + \| \nabla \cdot u_{j,h}^n \|_L^2 \| \nabla u_{j,h}^n \|^2 \]

Thus, (3.3) reduces to

\[ (3.10) \]

\[ \frac{1}{12} \left( \| u_{j,h}^{n+1} \|^2 - \| u_{j,h}^n \|^2 \right) + \frac{1}{12} \left( \| 3u_{j,h}^{n+1} - u_{j,h}^n \|^2 - \| 3u_{j,h}^n - u_{j,h}^{n-1} \|^2 \right) \]

\[ \quad + \frac{1}{24} \left( \| 3u_{j,h}^{n+1} - 3u_{j,h}^n + 3u_{j,h}^{n-1} - u_{j,h}^{n-2} \|^2 \right) \]

\[ \quad + \frac{\nu \Delta t}{2} \left( 1 - C \Delta t \| u_{j,h}^n \|_L^2 + \| \nabla \cdot u_{j,h}^n \|_L^2 \right)^2 \| \nabla u_{j,h}^n \|^2 \]

\[ \quad + \frac{1}{24} \| u_{j,h}^{n+1} - 3u_{j,h}^n + 3u_{j,h}^{n-1} - u_{j,h}^{n-2} \|^2 \leq \frac{\Delta t}{\nu} \| f_{j}^{n+1} \| -1 . \]

Now if the timestep condition (3.7) holds, (3.8) follows by taking sum from \( n = 2 \) to \( n = N - 1 \).
Lemma 2. For any \( u \in H^3(0, T; H^1(\Omega)) \), the following inequalities hold.

\[
\|10u^{n+1} - 15u^n + 6u^{n-1} - u^{n-2}\| \leq \frac{7}{3} \Delta t^3 \int_{t_{n-2}}^{t_n} \|\nabla u_{ttt}\|^2 dt ,
\]

(4.1)

\[
\|\nabla (u^{n+1} - 3u^n + 3u^{n-1} - u^{n-2})\| \leq 9\Delta t^5 \int_{t_{n-2}}^{t_n} \|\nabla u_{ttt}\|^2 dt .
\]

(4.2)

Proof. The technical proof is given in Appendix A.

For functions \( v(x, t) \) defined on \( \Omega \times (0, T) \), define \((1 \leq m < \infty)\)

\[
\|v\|_{\infty,k} := \text{EssSup}_{0 \leq n \leq T} \|v(\cdot, t)\|_k \quad \text{and} \quad \|v\|_{m,k} := \left( \int_0^T \|v(\cdot, t)\|_k^m dt \right)^{1/m} .
\]

We also introduce the following discrete norms:

\[
\|\|v\|\|_{\infty,k} := \max_{0 \leq n \leq N_T} \|v^n\|_k \quad \text{and} \quad \|\|v\|\|_{m,k} := \left( \sum_{n=0}^{N_T} \|v^n\|_k^m \Delta t \right)^{1/m} .
\]

To analyze the rate of convergence of the approximation we assume that the following regularity assumptions on the NSE

\[
u_j \in L^\infty (0, T; H^1(\Omega)) \cap H^3 (0, T; H^{k+1}(\Omega)) \cap H^3 (0, T; H^1(\Omega)) ,
\]

\[
p_j \in L^2 (0, T; H^{s+1}(\Omega)) , \quad \text{and} \quad f_j \in L^2 (0, T; L^2(\Omega)) .
\]

Assume \( X_h \) and \( Q_h \) satisfy the usual \((LBB)^h\) condition, then the method is equivalent to: for \( n = 1, ..., N_T - 1 \), find \( u_j^{n+1} \in V_h \) such that

\[
\left( \frac{10u_j^{n+1} - 15u_j^n + 6u_j^{n-1} - u_j^{n-2}}{6\Delta t} , v_h \right) + b^*(u_h^n , u_j^{n+1} , v_h) + b^*(u_j^m , 3u_j^n - 3u_j^{n-1} + u_j^{n-2} , v_h) + \nu \left( \nabla u_j^{n+1} , \nabla v_h \right) = (f_j^{n+1} , v_h) , \forall v_h \in V_h .
\]

(4.3)

Let \( e_j^n = u_j^n - u_j^n \) be the error between the true solution and the approximate solution, then we have the following error estimates.

Theorem 3 (Convergence of (En-BlendedBDF)). Consider the method (En-BlendedBDF). If the following conditions hold

\[
C_e \frac{\Delta t}{\nu h} \|\nabla u_{j,h}^n\|^2 \leq 1 , \quad j = 1, ..., J ,
\]

(4.4)

where \( C_e \) is a constant that depends only on the domain and the minimum angle of the mesh and is independent of the timestep, then there is a positive constant \( C \)
independent of the mesh width and timestep such that

\begin{equation}
\frac{1}{2} \| e_j^N \|^2 + \frac{1}{2} \| 3e_j^N - e_j^{N-1} \|^2 + \frac{1}{2} \| 3e_j^N - 3e_j^{N-1} + e_j^{N-2} \|^2 + \frac{3 \nu \Delta t}{16} \| \nabla e_j^N \|^2
\end{equation}

+ \frac{1}{4} \sum_{n=2}^{N-1} \| e_j^{n+1} - 3e_j^n + 3e_j^{n-1} - e_j^{n-2} \|^2 + \frac{3 \nu \Delta t}{16} \| \nabla e_j^{N-1} \|^2

+ \nu \sum_{n=2}^{N-1} \Delta t \| \nabla e_j^{n+1} \|^2 + \frac{3 \nu \Delta t}{8} \| \nabla e_j^{N-1} \|^2 + \frac{3 \nu \Delta t}{16} \| \nabla e_j^{N-2} \|^2

\leq \exp \left( \frac{CT}{\nu^2} \right) \left\{ \frac{1}{2} \| e_j^2 \|^2 + \frac{1}{2} \| 3e_j^2 - e_j^1 \|^2 + \frac{1}{2} \| 3e_j^2 - 3e_j^1 + e_j^0 \|^2 + \frac{3 \nu \Delta t}{16} \| \nabla e_j^0 \|^2

+ \frac{3 \nu \Delta t}{8} \| \nabla e_j^2 \|^2 + \frac{3 \nu \Delta t}{16} \| \nabla e_j^1 \|^2 \right\}.

\textbf{Corollary 1.} Under the assumptions of Theorem 3 with \((X_h, Q_h)\) given by the P2-P1 Taylor-Hood approximation elements \((k = 2, s = 1)\), i.e., \(C^0\) piecewise quadratic velocity space \(X_h\) and \(C^0\) piecewise linear pressure space \(Q_h\), we have the following error estimate

\begin{equation}
\frac{1}{2} \| e_j^N \|^2 + \frac{1}{2} \| 3e_j^N - e_j^{N-1} \|^2 + \frac{1}{2} \| 3e_j^N - 3e_j^{N-1} + e_j^{N-2} \|^2 + \frac{9 \nu \Delta t}{16} \| \nabla e_j^N \|^2
\end{equation}

+ \frac{1}{4} \sum_{n=2}^{N-1} \| e_j^{n+1} - 3e_j^n + 3e_j^{n-1} - e_j^{n-2} \|^2 + \frac{3 \nu \Delta t}{8} \| \nabla e_j^{N-1} \|^2 + \frac{3 \nu \Delta t}{16} \| \nabla e_j^{N-2} \|^2

\leq C \left( \frac{h^4 + \Delta t^4}{\nu^2} \| e_j^2 \|^2 + \frac{1}{2} \| 3e_j^2 - e_j^1 \|^2 + \frac{1}{2} \| 3e_j^2 - 3e_j^1 + e_j^0 \|^2 + \frac{9 \nu \Delta t}{16} \| \nabla e_j^0 \|^2 + \frac{3 \nu \Delta t}{8} \| \nabla e_j^1 \|^2 + \frac{3 \nu \Delta t}{16} \| \nabla e_j^2 \|^2 \right).

\textbf{Proof.} The true solution \((u_j, p_j)\) of the NSE satisfies

\begin{equation}
\left( \frac{10n_j^{n+1} - 15n_j^n + 6u_j^{n-1} - u_j^{n-2}}{6\Delta t} \right) + b^* (u_j^{n+1}, u_j^{n+1}, v_h) + \nu (\nabla u_j^{n+1}, \nabla v_h) - (p_j^{n+1}, \nabla \cdot v_h) = (f_j^{n+1}, v_h) + \text{Intp} (u_j^{n+1}; v_h), \text{ for all } v_h \in V_h,
\end{equation}

where \(\text{Intp} (u_j^{n+1}; v_h)\) is defined as

\[ \text{Intp} (u_j^{n+1}; v_h) = \left( \frac{10n_j^{n+1} - 15n_j^n + 6u_j^{n-1} - u_j^{n-2}}{6\Delta t} - u_j,t(f_j^{n+1}, v_h) \right). \]

Let \(e_j^n = u_j^n - u_j^n_h = (u_j^n - I_h u_j^n) + (I_h u_j^n - u_j^n_h) = \eta_j^n + \xi_j^n\), where \(I_h u_j^n \in V_h\) is an interpolant of \(u_j^n\) in \(V_h\). Subtracting (4.3) from (4.7) gives
\[
\frac{10\xi_{j,h}^{n+1} - 15\xi_{j,h}^n + 6\xi_{j,h}^{n-1} - \xi_{j,h}^{n-2}}{6\Delta t} + b^* (u_{j}^{n+1}, u_{j}^{n+1}, v_{h}) \\
+ \nu \left( \nabla \xi_{j,h}^{n+1} \cdot \nabla v_{h} \right) - b^* \left( 3u_{j,h}^n - 3u_{j,h}^{n-1} + v_{j,h}^{n-2} - v_{j,h}^n, v_{j,h}^{n+1} \right) \\
- b^* \left( u_{j,h}^n, 3u_{j,h}^n - 3u_{j,h}^{n-1} + u_{j,h}^{n-2}, v_{h} \right) - \left( p_{j}^{n+1}, \nabla \cdot v_{h} \right) \\
= - \left( \frac{10\eta_{j}^{n+1} - 15\eta_{j}^n + 6\eta_{j}^{n-1} - \eta_{j}^{n-2}}{6\Delta t} \right) - \nu \left( \nabla \eta_{j}^{n+1} \cdot \nabla v_{h} \right) + \text{Intp} \left( u_{j}^{n+1}; v_{h} \right). 
\]

Set \( v_{h} = \xi_{j,h}^{n+1} \in V_{h} \), and rearrange the nonlinear terms, then we have

\[
\frac{1}{12\Delta t} \left( \| \xi_{j,h}^{n+1} \|^2 - \| \xi_{j,h}^{n} \|^2 \right) + b^* \left( u_{j}^{n+1}, u_{j}^{n+1}, \xi_{j,h}^{n+1} \right) + b^* \left( 3u_{j,h}^n - 3u_{j,h}^{n-1} + u_{j,h}^{n-2} - u_{j,h}^n, \xi_{j,h}^{n+1} \right) \\
+ b^* \left( u_{j,h}^n, 3u_{j,h}^n - 3u_{j,h}^{n-1} + u_{j,h}^{n-2} - u_{j,h}^n, \xi_{j,h}^{n+1} \right) + \left( p_{j}^{n+1}, \nabla \cdot \xi_{j,h}^{n+1} \right) \\
- \left( \frac{10\eta_{j}^{n+1} - 15\eta_{j}^n + 6\eta_{j}^{n-1} - \eta_{j}^{n-2}}{6\Delta t} \right) - \nu \left( \nabla \eta_{j}^{n+1} \cdot \nabla \xi_{j,h}^{n+1} \right) + \text{Intp} \left( u_{j}^{n+1}, \xi_{j,h}^{n+1} \right). 
\]

We first bound the nonlinear terms on the right hand side of equation (4.9). Adding and subtracting \( b^* (u_{j}^{n+1}, u_{j}^{n+1}, \xi_{j,h}^{n+1}) \), \( b^* (3u_{j}^n - 3u_{j}^{n-1} + u_{j}^{n-2}, u_{j}^{n+1}, \xi_{j,h}^{n+1}) \) and \( b^* (u_{j}^n, 3u_{j}^n - 3u_{j}^{n-1} + u_{j}^{n-2} - u_{j}^{n+1}, \xi_{j,h}^{n+1}) \) respectively, we rewrite the nonlinear terms as

\[
\frac{1}{12\Delta t} \left( \| \xi_{j,h}^{n+1} \|^2 - \| \xi_{j,h}^{n} \|^2 \right) + b^* \left( u_{j}^{n+1}, u_{j}^{n+1}, \xi_{j,h}^{n+1} \right) + b^* \left( 3u_{j,h}^n - 3u_{j,h}^{n-1} + u_{j,h}^{n-2} - u_{j,h}^n, \xi_{j,h}^{n+1} \right) \\
+ b^* \left( u_{j,h}^n, 3u_{j,h}^n - 3u_{j,h}^{n-1} + u_{j,h}^{n-2} - u_{j,h}^n, \xi_{j,h}^{n+1} \right) - \left( \frac{10\eta_{j}^{n+1} - 15\eta_{j}^n + 6\eta_{j}^{n-1} - \eta_{j}^{n-2}}{6\Delta t} \right) - \nu \left( \nabla \eta_{j}^{n+1} \cdot \nabla \xi_{j,h}^{n+1} \right) + \text{Intp} \left( u_{j}^{n+1}, \xi_{j,h}^{n+1} \right). 
\]
We estimate the nonlinear terms using (2.1), (2.2), Lemma 2 and Young’s inequality as follows.

\begin{align}
-b^* \left( u_{j,h}^n, 3\xi_{j,h}^n - 3\xi_{j,h}^{n-1} + \xi_{j,h}^{n-2} - \xi_{j,h}^{n+1} \right) &- b^* \left( u_{j,h}^n, 3\eta_{j,h}^n - 3\eta_{j,h}^{n-1} + \eta_{j,h}^{n+1} \right) \\
&+ b^* \left( u_{j,h}^n, 3u_{j,h}^{n-1} + u_{j,h}^{n-2} - u_{j,h}^{n+1} \right).
\end{align}

\begin{align}
(4.11) & \quad b^* \left( u_{j,h}^{n+1}, \xi_{j,h}^{n+1} \right) \leq C \left\| \nabla u_{j,h}^{n+1} \right\| \left\| \nabla \xi_{j,h}^{n+1} \right\| \left\| \nabla \xi_{j,h}^{n+1} \right\| \\
& \quad \leq \frac{\nu}{64} \left\| \nabla \xi_{j,h}^{n+1} \right\|^2 + C\nu^{-1} \left\| \nabla u_{j,h}^{n+1} \right\| \left\| \nabla \xi_{j,h}^{n+1} \right\| ^2.
\end{align}

\begin{align}
(4.12) & \quad b^* \left( u_{j,h}^{n+1} - (3u_{j,h}^n - 3u_{j,h}^{n-1} + u_{j,h}^{n-2}), u_{j,h}^n, \xi_{j,h}^{n+1} \right) \\
& \quad \leq C \left\| \nabla \left( u_{j,h}^{n+1} - (3u_{j,h}^n - 3u_{j,h}^{n-1} + u_{j,h}^{n-2}) \right) \right\| \left\| \nabla u_{j,h}^{n+1} \right\| \left\| \nabla \xi_{j,h}^{n+1} \right\| \\
& \quad \leq \frac{\nu}{64} \left\| \nabla \xi_{j,h}^{n+1} \right\|^2 + C\nu^{-1} \left\| \nabla \left( u_{j,h}^{n+1} - (3u_{j,h}^n - 3u_{j,h}^{n-1} + u_{j,h}^{n-2}) \right) \right\| ^2 \left\| \nabla u_{j,h}^{n+1} \right\|^2 \\
& \quad \leq \frac{\nu}{64} \left\| \nabla \xi_{j,h}^{n+1} \right\|^2 + C\nu^{-1} \left\| \nabla \left( u_{j,h}^{n+1} - (3u_{j,h}^n - 3u_{j,h}^{n-1} + u_{j,h}^{n-2}) \right) \right\| ^2 \left\| \nabla u_{j,h}^{n+1} \right\|^2.
\end{align}

\begin{align}
(4.13) & \quad b^* \left( 3\eta_{j,h}^n - 3\eta_{j,h}^{n-1} + \eta_{j,h}^{n-2}, u_{j,h}^n, \xi_{j,h}^{n+1} \right) \\
& \quad \leq C \left\| \nabla \left( 3\eta_{j,h}^n - 3\eta_{j,h}^{n-1} + \eta_{j,h}^{n-2} \right) \right\| \left\| \nabla u_{j,h}^{n+1} \right\| \left\| \nabla \xi_{j,h}^{n+1} \right\| \\
& \quad \leq \frac{\nu}{64} \left\| \nabla \xi_{j,h}^{n+1} \right\|^2 + C\nu^{-1} \left\| \nabla \left( 3\eta_{j,h}^n - 3\eta_{j,h}^{n-1} + \eta_{j,h}^{n-2} \right) \right\| ^2 \left\| \nabla u_{j,h}^{n+1} \right\|^2.
\end{align}

\begin{align}
(4.14) & \quad 3b^* \left( \xi_{j,h}^n, u_{j,h}^{n+1}, \xi_{j,h}^{n+1} \right) \leq C \left\| \nabla \xi_{j,h}^n \right\| \left\| \nabla u_{j,h}^{n+1} \right\| \left\| \nabla \xi_{j,h}^{n+1} \right\| \\
& \quad \leq C \left\| \nabla \xi_{j,h}^n \right\| \left\| \nabla \xi_{j,h}^{n+1} \right\| \left\| \nabla \xi_{j,h}^{n+1} \right\| \left\| \nabla \xi_{j,h}^{n+1} \right\| \\
& \quad \leq C \left\| \nabla \xi_{j,h}^n \right\| \left\| \nabla \xi_{j,h}^{n+1} \right\| \left\| \nabla \xi_{j,h}^{n+1} \right\| \left\| \nabla \xi_{j,h}^{n+1} \right\| \left\| \nabla \xi_{j,h}^{n+1} \right\| \left\| \nabla \xi_{j,h}^{n+1} \right\| \\
& \quad \leq C \left( \epsilon \left\| \nabla \xi_{j,h}^{n+1} \right\|^2 + \frac{1}{\epsilon} \left\| \nabla \xi_{j,h}^{n+1} \right\|^2 \right) \\
& \quad \leq \left( \frac{\nu}{64} \left\| \nabla \xi_{j,h}^{n+1} \right\|^2 + \frac{\nu}{32} \left\| \nabla \xi_{j,h}^{n+1} \right\|^2 \right) + C\nu^{-3} \left\| \xi_{j,h}^n \right\|^2.
\end{align}

Similarly,\n
\begin{align}
(4.15) & \quad 3b^* \left( \xi_{j,h}^n, u_{j,h}^{n+1}, \xi_{j,h}^{n+1} \right) \leq C \left\| \nabla \xi_{j,h}^n \right\| \left\| \nabla u_{j,h}^{n+1} \right\| \left\| \nabla \xi_{j,h}^{n+1} \right\| \left\| \nabla \xi_{j,h}^{n+1} \right\| \\
& \quad \leq C \left\| \nabla \xi_{j,h}^n \right\| \left\| \nabla \xi_{j,h}^{n+1} \right\| \left\| \nabla \xi_{j,h}^{n+1} \right\| \left\| \nabla \xi_{j,h}^{n+1} \right\| \left\| \nabla \xi_{j,h}^{n+1} \right\| \\
& \quad \leq C \left( \epsilon \left\| \nabla \xi_{j,h}^{n+1} \right\|^2 + \frac{1}{\epsilon} \left\| \nabla \xi_{j,h}^{n+1} \right\|^2 \right) \\
& \quad \leq \left( \frac{\nu}{64} \left\| \nabla \xi_{j,h}^{n+1} \right\|^2 + \frac{\nu}{32} \left\| \nabla \xi_{j,h}^{n+1} \right\|^2 \right) + C\nu^{-3} \left\| \xi_{j,h}^n \right\|^2.
\end{align}

\begin{align}
(4.16) & \quad 3b^* \left( \xi_{j,h}^n, u_{j,h}^{n+1}, \xi_{j,h}^{n+1} \right) \leq C \left\| \nabla \xi_{j,h}^n \right\| \left\| \nabla u_{j,h}^{n+1} \right\| \left\| \nabla \xi_{j,h}^{n+1} \right\| \left\| \nabla \xi_{j,h}^{n+1} \right\| \\
& \quad \leq C \left\| \nabla \xi_{j,h}^n \right\| \left\| \nabla \xi_{j,h}^{n+1} \right\| \left\| \nabla \xi_{j,h}^{n+1} \right\| \left\| \nabla \xi_{j,h}^{n+1} \right\| \\
& \quad \leq C \left( \epsilon \left\| \nabla \xi_{j,h}^{n+1} \right\|^2 + \frac{1}{\epsilon} \left\| \nabla \xi_{j,h}^{n+1} \right\|^2 \right) \\
& \quad \leq \left( \frac{\nu}{64} \left\| \nabla \xi_{j,h}^{n+1} \right\|^2 + \frac{\nu}{32} \left\| \nabla \xi_{j,h}^{n+1} \right\|^2 \right) + C\nu^{-3} \left\| \xi_{j,h}^n \right\|^2.
\end{align}
Using (2.3) and inverse inequality (2.7) gives

\[
\leq C \left( \epsilon \|\nabla \xi_{j,h}^{n+1}\|^2 + \frac{1}{\epsilon} \left( \delta \|\nabla \xi_{j,h}^{n-2}\|^2 + \frac{1}{\delta} \|\xi_{j,h}^{n-2}\|^2 \right) \right)
\]

\[
\leq \left( \frac{\nu}{64} \|\nabla \xi_{j,h}^{n+1}\|^2 + \frac{\nu}{32} \|\nabla \xi_{j,h}^{n-2}\|^2 \right) + C \nu^{-3} \|\xi_{j,h}^{n-2}\|^2.
\]

By skew symmetry

\[
b^* \left( u_{j,h}^n, 3\xi_{j,h}^n - 3\xi_{j,h}^{n-1} + \xi_{j,h}^{n+1} \right)
\]

\[
= -b^* \left( u_{j,h}^n, \xi_{j,h}^{n+1} - 3\xi_{j,h}^n + 3\xi_{j,h}^{n-1} - \xi_{j,h}^{n+1} \right)
\]

\[
= b^* \left( u_{j,h}^n, \xi_{j,h}^{n+1} - 3\xi_{j,h}^n + 3\xi_{j,h}^{n-1} - \xi_{j,h}^{n+1} \right).
\]

Using (4.3) and inverse inequality (2.7) gives

\[
\leq C \|\nabla u_{j,h}^n\| \|\nabla \xi_{j,h}^{n+1}\| \|\nabla (\xi_{j,h}^{n+1} - 3\xi_{j,h}^n + 3\xi_{j,h}^{n-1} - \xi_{j,h}^{n+1})\| \|\xi_{j,h}^{n+1} - 3\xi_{j,h}^n + 3\xi_{j,h}^{n-1} - \xi_{j,h}^{n+1}\| \frac{1}{\epsilon}
\]

\[
\leq C \|\nabla u_{j,h}^n\| \|\nabla \xi_{j,h}^{n+1}\| \left( \frac{h^{-2}}{\epsilon} \right) \|\xi_{j,h}^{n+1} - 3\xi_{j,h}^n + 3\xi_{j,h}^{n-1} - \xi_{j,h}^{n+1}\|
\]

\[
\leq \frac{1}{24\Delta t} \|\xi_{j,h}^{n+1} - 3\xi_{j,h}^n + 3\xi_{j,h}^{n-1} - \xi_{j,h}^{n+1}\|^2 + C \frac{\Delta t}{2h} \|\nabla u_{j,h}^n\|^2 \|\nabla \xi_{j,h}^{n+1}\|^2.
\]

(4.17)

\[
b^* \left( u_{j,h}^n, \eta_{j,h}^{n+1} - 3\eta_{j,h}^n + 3\eta_{j,h}^{n-1} - \eta_{j,h}^{n+1} \right)
\]

\[
\leq C \|\nabla u_{j,h}^n\| \|\nabla (\eta_{j,h}^{n+1} - 3\eta_{j,h}^n + 3\eta_{j,h}^{n-1} - \eta_{j,h}^{n+1})\| \|\nabla \xi_{j,h}^{n+1}\|
\]

\[
\leq \frac{\nu}{64} \|\nabla \xi_{j,h}^{n+1}\|^2 + C \nu^{-1} \|\nabla u_{j,h}^n\|^2 \|\nabla (\eta_{j,h}^{n+1} - 3\eta_{j,h}^n + 3\eta_{j,h}^{n-1} - \eta_{j,h}^{n+1})\|^2
\]

\[
\leq \frac{\nu}{64} \|\nabla \xi_{j,h}^{n+1}\|^2 + \frac{C \Delta t^5}{\nu} \|\nabla u_{j,h}^n\|^2 \left( \int_{t_{n-2}}^{t_{n+1}} \|\nabla \eta_{j,h}^n\|^2 \, dt \right).
\]

(4.18)

\[
b^* \left( u_{j,h}^n, u_{j,h}^{n+1} - 3u_{j,h}^n + 3u_{j,h}^{n-1} - u_{j,h}^{n+1} \right)
\]

\[
\leq C \|\nabla u_{j,h}^n\| \|\nabla (u_{j,h}^{n+1} - 3u_{j,h}^n + 3u_{j,h}^{n-1} - u_{j,h}^{n+1})\| \|\nabla \xi_{j,h}^{n+1}\|
\]

\[
\leq \frac{\nu}{64} \|\nabla \xi_{j,h}^{n+1}\|^2 + C \nu^{-1} \|\nabla u_{j,h}^n\|^2 \|\nabla (u_{j,h}^{n+1} - 3u_{j,h}^n + 3u_{j,h}^{n-1} - u_{j,h}^{n+1})\|^2
\]

\[
\leq \frac{\nu}{64} \|\nabla \xi_{j,h}^{n+1}\|^2 + \frac{C \nu^{-1} \Delta t^5}{\nu} \|\nabla u_{j,h}^n\|^2 \left( \int_{t_{n-2}}^{t_{n+1}} \|\nabla \eta_{j,h}^n\|^2 \, dt \right).
\]

(4.19)

As \(\xi_{j,h}^{n+1} \in V_h\) we have the following estimate for the pressure term

\[
\left( p_{j,h}^{n+1}, \nabla \cdot \xi_{j,h}^{n+1} \right) = \left( p_{j,h}^{n+1} - q_{j,h}^{n+1}, \nabla \cdot \xi_{j,h}^{n+1} \right) \leq \| p_{j,h}^{n+1} - q_{j,h}^{n+1} \| \| \nabla \cdot \xi_{j,h}^{n+1} \|
\]

\[
\leq \frac{\nu}{64} \|\nabla \xi_{j,h}^{n+1}\|^2 + C \nu^{-1} \| p_{j,h}^{n+1} - q_{j,h}^{n+1} \|^2, \quad \forall q_{j,h}^{n+1} \in Q_h.
\]

For the rest of the terms on the right hand side of (4.9) we have

\[
\left( 10\eta_{j,h}^{n+1} - 15\eta_{j,h}^n + 6\eta_{j,h}^{n-1} - \eta_{j,h}^{n-2} \right) \frac{1}{6\Delta t} \left( \xi_{j,h}^{n+1}, \xi_{j,h}^{n+1} \right)
\]
Combining the above inequalities with (4.9) yields

\[ 14 \text{ NAN JIANG} \leq \frac{10\eta_{j}^{n+1} - 15\eta_{j}^{n} + 6\eta_{j}^{n-1} - \eta_{j}^{n-2}}{6\Delta t} \| \nabla \xi_{j,h}^{n+1} \| \]

\[ \leq C\nu^{-1} \| 10\xi_{j}^{n+1} - 15\xi_{j}^{n} + 6\xi_{j}^{n-1} - \xi_{j}^{n-2} \|^{2} + \frac{\nu}{64} \| \nabla \xi_{j,h}^{n+1} \|^{2} \]

\[ \leq C\nu^{-1} \| \frac{1}{\Delta t} \int_{t_{n-2}}^{t_{n+1}} \eta_{j,t} \, dt \|^{2} + \frac{\nu}{64} \| \nabla \xi_{j,h}^{n+1} \|^{2} \leq \frac{C}{\nu \Delta t} \int_{t_{n-2}}^{t_{n+1}} \| \eta_{j,t} \|^{2} \, dt + \frac{\nu}{64} \| \nabla \xi_{j,h}^{n+1} \|^{2} , \]

for (4.23) \( 1 \leq \nabla \eta_{j}^{n+1}, \nabla \xi_{j,h}^{n+1} \leq \nu \| \nabla \eta_{j}^{n+1} \| \| \nabla \xi_{j,h}^{n+1} \| \leq C\nu \| \nabla \eta_{j}^{n+1} \|^{2} + \frac{\nu}{64} \| \nabla \xi_{j,h}^{n+1} \|^{2} , \)

and

\[ \text{Intp} \left( u_{j}^{n+1}, \xi_{j,h}^{n+1} \right) = \left( \frac{10u_{j}^{n+1} - 15u_{j}^{n} + 6u_{j}^{n-1} - u_{j}^{n-2}}{6\Delta t} - u_{j,t}(t_{n}^{n+1}), \xi_{j,h}^{n+1} \right) \]

(4.22) \[ \leq C \| \frac{10u_{j}^{n+1} - 15u_{j}^{n} + 6u_{j}^{n-1} - u_{j}^{n-2}}{6\Delta t} - u_{j,t}(t_{n}^{n+1}) \| \| \nabla \xi_{j,h}^{n+1} \| \]

\[ \leq \nu \frac{C}{64} \| \nabla \xi_{j,h}^{n+1} \|^{2} + \frac{C}{\nu} \| \frac{10u_{j}^{n+1} - 15u_{j}^{n} + 6u_{j}^{n-1} - u_{j}^{n-2}}{6\Delta t} - u_{j,t}(t_{n}^{n+1}) \|^{2} \]

\[ \leq \frac{\nu}{64} \| \nabla \xi_{j,h}^{n+1} \|^{2} + \frac{C\nu^{-1}}{\nu} \int_{t_{n-2}}^{t_{n+1}} \| u_{j,t} \|^{2} \, dt . \]

Combining the above inequalities with (4.9) yields

(4.23) \[ \frac{1}{12\Delta t} \left( \| \xi_{j,h}^{n+1} \|^{2} - \| \xi_{j,h}^{n} \|^{2} \right) + \frac{1}{12\Delta t} \left( \| 3\xi_{j,h}^{n} - \xi_{j,h}^{n} \|^{2} - 3\xi_{j,h}^{n-1} - \xi_{j,h}^{n-1} \|^{2} \right) \]

\[ + \frac{1}{24\Delta t} \left( \| 3\xi_{j,h}^{n} - 3\xi_{j,h}^{n} + 3\xi_{j,h}^{n-1} - 3\xi_{j,h}^{n-1} - \xi_{j,h}^{n-2} \|^{2} \right) + \frac{\nu}{32} \left( \| \nabla \xi_{j,h}^{n+1} \|^{2} - \| \nabla \xi_{j,h}^{n+1} \|^{2} \right) \]

\[ + \nu \frac{\nu}{16} \left( \left( \| \nabla \xi_{j,h}^{n+1} \|^{2} + \| \nabla \xi_{j,h}^{n+1} \|^{2} \right) - \left( \| \nabla \xi_{j,h}^{n} \|^{2} + \| \nabla \xi_{j,h}^{n} \|^{2} \right) \right) \]

\[ + \frac{\nu}{32} \left( \| \nabla \xi_{j,h}^{n-1} \|^{2} - \| \nabla \xi_{j,h}^{n-1} \|^{2} \right) + \left( \frac{\nu}{32} - \frac{C_{e} \Delta t}{32 \Delta t} \| \nabla u_{j,h}^{n} \|^{2} \right) \| \nabla \xi_{j,h}^{n+1} \|^{2} \]

\[ \leq C\nu^{-3} \left( \| \xi_{j,h}^{n} \|^{2} + \| \xi_{j,h}^{n-1} \|^{2} + \| \xi_{j,h}^{n-2} \|^{2} \right) + C\nu^{-1} \| \nabla u_{j,h}^{n+1} \|^{2} \| \nabla \eta_{j,h}^{n+1} \|^{2} \]

\[ + \frac{C\Delta t}{\nu} \left( \int_{t_{n-2}}^{t_{n+1}} \| u_{j,t} \|^{2} \, dt \right) \| \nabla u_{j,h}^{n+1} \|^{2} + \frac{C\Delta t}{\nu} \| \nabla u_{j,h}^{n+1} \|^{2} \left( \int_{t_{n-2}}^{t_{n+1}} \| \nabla \eta_{j,t} \|^{2} \, dt \right) \]

\[ + C\nu^{-1} \left( \| \nabla \eta_{j}^{n} \|^{2} + \| \nabla \eta_{j}^{n-1} \|^{2} + \| \nabla \eta_{j}^{n-2} \|^{2} \right) \| \nabla u_{j,h}^{n+1} \|^{2} \]

\[ + \frac{C\Delta t}{\nu} \| \nabla u_{j,h}^{n} \|^{2} \left( \int_{t_{n-2}}^{t_{n+1}} \| u_{j,t} \|^{2} \, dt \right) + C\nu^{-1} \| \eta_{j}^{n+1} \|^{2} + \frac{C\Delta t^{3}}{\nu} \int_{t_{n-2}}^{t_{n+1}} \| u_{j,t} \|^{2} \, dt . \]

\((\frac{\nu}{32} - \frac{C_{e} \Delta t}{32 \Delta t} \| \nabla u_{j,h}^{n} \|^{2})\) is nonnegative and thus can be eliminated from the left hand side of (4.23) if the timestep conditions in (4.27) hold. Taking the sum of (4.23) from \( n = 2 \) to \( n = N - 1 \) and multiplying through by \( 6\Delta t \), we obtain
Further applying the discrete Gronwall inequality (Girault and Raviart [8], p. 176)

\[
\begin{align*}
\frac{1}{2} \| \xi_j^{N+1} \|^2 &+ \frac{1}{2} \| 3 \xi_j^N - \xi_j^{N-1} \|^2 + \frac{1}{2} \| 3 \xi_j^N - 3 \xi_j^{N-1} + \xi_j^{N-2} \|^2 \\
+ \frac{1}{4} \sum_{n=2}^{N-1} \| \xi_j^{n+1} - 3 \xi_j^n + 3 \xi_j^{n-1} - \xi_j^{n-2} \|^2 &+ \frac{3\nu \Delta t}{16} \| \nabla \xi_j^N \|^2 \\
+ \nu \sum_{n=2}^{N-1} \Delta t \| \nabla \xi_j^{n+1} \|^2 &+ \frac{3\nu \Delta t}{8} \left( \| \nabla \xi_j^N \|^2 + \| \nabla \xi_j^{N-1} \|^2 + \frac{3\nu \Delta t}{16} \| \nabla \xi_j^{N-2} \|^2 \\
\leq & \frac{1}{2} \| \xi_j^2 \|^2 + \frac{1}{2} \| 3 \xi_j^2 - \xi_j^1 \|^2 + \frac{1}{2} \| 3 \xi_j^2 - 3 \xi_j^1 + \xi_j^0 \|^2 + \frac{3\nu \Delta t}{16} \| \nabla \xi_j^2 \|^2 \\
+ \frac{3\nu \Delta t}{8} & \left( \| \nabla \xi_j^2 \|^2 + \| \nabla \xi_j^1 \|^2 \right) + \frac{3\nu \Delta t}{16} \| \nabla \xi_j^0 \|^2 + \Delta t \sum_{n=0}^{N-1} C \nu^{-3} \| \xi_j^n \|^2 \\
+ \Delta t & \sum_{n=0}^{N-1} C \nu^{-1} \| \eta_j^n \|^2 + \frac{\nu^3}{C} \int_{t_{n-2}}^{t^{n+1}} \| \nabla u_j, t, t \|^2 \, dt \\
&+ C \Delta t^4 h \left( \int_{t_{n-2}}^{t^{n+1}} \| \nabla \eta_j, t, t \|^2 \, dt \right) + C \nu^{-1} \| \eta_j^n \|^2 \\
&+ \frac{C \Delta t^5}{ \nu} \int_{t_{n-2}}^{t^{n+1}} \| \eta_j, t \|^2 \, dt + C \nu \| \nabla \eta_j^n \|^2 + \frac{C \Delta t^3}{ \nu} \int_{t_{n-2}}^{t^{n+1}} \| u_j, t, t \|^2 \, dt \right) \\
\end{align*}
\]

Applying interpolation inequalities to the above inequality gives

\[
\begin{align*}
\frac{1}{2} \| \xi_j^{N+1} \|^2 &+ \frac{1}{2} \| 3 \xi_j^N - \xi_j^{N-1} \|^2 + \frac{1}{2} \| 3 \xi_j^N - 3 \xi_j^{N-1} + \xi_j^{N-2} \|^2 \\
+ \frac{1}{4} \sum_{n=2}^{N-1} \| \xi_j^{n+1} - 3 \xi_j^n + 3 \xi_j^{n-1} - \xi_j^{n-2} \|^2 &+ \frac{3\nu \Delta t}{16} \| \nabla \xi_j^N \|^2 \\
+ \nu \sum_{n=2}^{N-1} \Delta t \| \nabla \xi_j^{n+1} \|^2 &+ \frac{3\nu \Delta t}{8} \left( \| \nabla \xi_j^N \|^2 + \| \nabla \xi_j^{N-1} \|^2 + \frac{3\nu \Delta t}{16} \| \nabla \xi_j^{N-2} \|^2 \\
\leq & \frac{1}{2} \| \xi_j^2 \|^2 + \frac{1}{2} \| 3 \xi_j^2 - \xi_j^1 \|^2 + \frac{1}{2} \| 3 \xi_j^2 - 3 \xi_j^1 + \xi_j^0 \|^2 + \frac{3\nu \Delta t}{16} \| \nabla \xi_j^2 \|^2 \\
+ \frac{3\nu \Delta t}{8} & \left( \| \nabla \xi_j^2 \|^2 + \| \nabla \xi_j^1 \|^2 \right) + \frac{3\nu \Delta t}{16} \| \nabla \xi_j^0 \|^2 + \Delta t \sum_{n=0}^{N-1} C \nu^{-3} \| \xi_j^n \|^2 \\
+ C \Delta t^4 h^{2k+1} & \| \nabla u_j, t, t \| \| u_j \|_{L_2, k+1} + C \frac{\Delta t^6}{ \nu} \| \nabla u_j, t, t \|_{L_2} + C \frac{\Delta t^6}{ \nu} \| \nabla u_j \|_{L_2, k+1} \\
+ C \Delta t^4 h^{2k+1} & \| \nabla u_j, t, t \|_{L_2} + C \Delta t^5 h \| \nabla u_j, t, t \|_{L_2} + C \frac{ \nu^2}{ \nu} \| p_j \|_{L_2, k+1} \\
+ C h^{2k+2} & \| u_j, t \|_{L_2, k+1} + C \nu h^{2k} \| \nabla u_j \|_{L_2, k} + C \frac{ \nu^2}{ \nu} \| u_j, t, t \|_{L_2}.
\end{align*}
\]

Further applying the discrete Gronwall inequality (Girault and Raviart [8], p. 176) yields
our algorithm, which shows the algorithm’s ensemble mean does converge to the interest. Herein we give an error estimate of the ensemble mean computed from usually the main prediction of the future state and thus its behavior is of special interest. If the following conditions hold

\[ (4.28) \]

\[
\sum_{n=2}^{N-1} \| \xi_{j,h}^{n+1} - 3 \xi_{j,h}^n + 3 \xi_{j,h}^{n-1} - \xi_{j,h}^{n-2} \|^2 + \frac{3 \nu \Delta t}{16} \| \nabla \xi_{j,h}^N \|^2 
\]

\[
+ \nu \sum_{n=2}^{N-1} \Delta t \| \nabla \xi_{j,h}^{n+1} \|^2 + 3 \nu \Delta t \left( \| \nabla \xi_{j,h}^N \|^2 + \| \nabla \xi_{j,h}^{N-1} \|^2 \right) + \frac{3 \nu \Delta t}{16} \| \nabla \xi_{j,h}^N \|^2 
\]

\[
\leq \exp \left( \frac{CN \Delta t}{\nu^2} \right) \left\{ \frac{1}{2} \| \xi_{j,h}^n \|^2 + \frac{1}{2} \| \xi_{j,h}^1 \|^2 + \frac{1}{2} \| \xi_{j,h}^0 \|^2 + \frac{3 \nu \Delta t}{8} \left( \| \nabla \xi_{j,h}^1 \|^2 + \| \nabla \xi_{j,h}^0 \|^2 \right) + \frac{3 \nu \Delta t}{16} \| \nabla \xi_{j,h}^0 \|^2 
\]

\[
+ C \frac{\nu^{2k}}{\nu} \| \nabla u_j \|_{\nu,0} \| u_j \|_{2,k+1} \|^2 + C \frac{\Delta \nu^6}{\nu} \| \| u_j \|_{2} \|^2 + C \frac{\nu^{2k}}{\nu} \| \| u_j \|_{2,k+1} \|^2 
\]

\[
+ C \Delta t^2 \frac{\nu^{2k+1}}{\nu} \| \nabla u_j \|_{2,k+1} \|^2 + C \Delta t^2 \frac{\nu^{2k+1}}{\nu} \| \| u_j \|_{2} \|^2 
\]

\[
+ C \frac{\nu^{2k+1}}{\nu} \| \| u_j \|_{2} \|^2 + C \frac{\nu^{2k+1}}{\nu} \| \| u_j \|_{2} \|^2 
\}\cdot 
\]

Applying triangle inequality on the error and absorbing constants gives (4.5).

In many applications, e.g., numerical weather prediction, the ensemble mean is usually the main prediction of the future state and thus its behavior is of special interest. Herein we give an error estimate of the ensemble mean computed from our algorithm, which shows the algorithm’s ensemble mean does converge to the true ensemble mean with optimal convergence rate.

Let \( (e)^n = (u)^n - \langle u_h \rangle^n \) be the error between the true ensemble mean and the ensemble mean computed from (En-BlendedBDF). Then we have the following error estimate.

**Theorem 4 (Convergence of ensemble mean). Consider the method (En-BlendedBDF). If the following conditions hold

\[ (4.27) \]

\[
C_e \frac{\Delta t}{\nu h} \| \nabla u_j^n \|^2 \leq 1, \quad j = 1, ..., J, 
\]

where \( C_e \) is a constant that depends only on the domain and the minimum angle of the mesh and is independent of the timestep, then there is a positive constant \( C \) independent of the mesh width and timestep such that

\[
\frac{1}{2} \| (e)^N \|^2 \leq \exp \left( \frac{CT}{\nu^2} \right) \sum_{j=1}^{J} \left\{ \frac{1}{2} \| e_j^2 \|^2 + \frac{1}{2} \| 3e_j^2 - e_j^1 \|^2 + \frac{1}{2} \| 3e_j^2 - 3e_j^1 + e_j^0 \|^2 
\]

\[
+ \frac{3 \nu \Delta t}{16} \| \nabla e_j^2 \|^2 + \frac{3 \nu \Delta t}{8} \left( \| \nabla e_j^2 \|^2 + \| \nabla e_j^1 \|^2 \right) + \frac{3 \nu \Delta t}{16} \| \nabla e_j^0 \|^2 
\]

\[
+ C \frac{\nu^{2k}}{\nu} \| \nabla u_j \|_{\nu,0} \| u_j \|_{2,k+1} \|^2 + C \frac{\Delta \nu^6}{\nu} \| \| u_j \|_{2} \|^2 + C \frac{\nu^{2k}}{\nu} \| \| u_j \|_{2,k+1} \|^2 
\]

\[
+ C \Delta t^2 \frac{\nu^{2k+1}}{\nu} \| \nabla u_j \|_{2,k+1} \|^2 + C \Delta t^2 \frac{\nu^{2k+1}}{\nu} \| \| u_j \|_{2} \|^2 
\]

\[
+ C \frac{\nu^{2k+1}}{\nu} \| \| u_j \|_{2} \|^2 + C \frac{\nu^{2k+1}}{\nu} \| \| u_j \|_{2} \|^2 
\}\cdot 
\]
The code was implemented using the software package FreeFem++, \[27\].

Relax the time step condition. In all tests, we use Taylor-Hood P2-P1 elements.

Next, we test the ability of the method to simulate high Reynolds number, complex flows. The method is tested on the well-known 3D Ethier-Steinman flow problem.

The method is given by

\[
\frac{3\nu u_{j}^{n+1} - 4\nu u_{j}^{n} + \nu u_{j}^{n-1}}{2\Delta t} + \frac{C\Delta t^4}{\nu} \|u_{j,t}\|_{2,0}^2
\]

and

\[
\nu \sum_{n=2}^{N-1} \Delta t \|\nabla (e)^{n+1}\|^2 \leq \exp \left( \frac{C T}{\nu^2} \right) \frac{1}{j} \sum_{j=1}^{J} \left( \frac{1}{2} \|e_j^n\|^2 + \frac{1}{2} \|3e_j^n - e_j^{n-1}\|^2 + \frac{3\nu \Delta t}{16} \|\nabla e_j^n\|^2 \right).
\]

(4.29)

Similarly, we have

\[
\nu \sum_{n=2}^{N-1} \Delta t \|\nabla (e)^{n+1}\|^2 \leq \exp \left( \frac{C T}{\nu^2} \right) \frac{1}{j} \sum_{j=1}^{J} \left( \frac{1}{2} \|e_j^n\|^2 + \frac{1}{2} \|3e_j^n - e_j^{n-1}\|^2 + \frac{3\nu \Delta t}{16} \|\nabla e_j^n\|^2 \right).
\]

Proof. With

\[
\|\langle e \rangle^n\|^2 = \|\frac{1}{J} \sum_{j=1}^{J} e_j^n\|^2 \leq \frac{1}{J} \sum_{j=1}^{J} \|e_j^n\|^2,
\]

(4.28) follows directly from Theorem 3.

Similarly, we have

\[
\sum_{n=2}^{N-1} \sum_{j=1}^{J} \|\nabla (e)^{n}\|^2 \leq \sum_{n=2}^{N-1} \sum_{j=1}^{J} \|\nabla e_j^n\|^2,
\]

and thus (4.29) also follows directly from Theorem 3.

5. Numerical Experiments

We perform numerical experiments for the proposed method on two test problems. First, we verify predicted convergence rates on a 2d test problem with known analytical solution. We also compare accuracy of (En-BlendedBDF) with that of the previously studied (En-BDF2AB2) method (see \[10\]). The (En-BDF2AB2) method is given by

\[
(\text{En-BDF2AB2}) \quad \frac{3u_{j}^{n+1} - 4u_{j}^{n} + \nu u_{j}^{n-1}}{2\Delta t} + \frac{C\Delta t^4}{\nu} \|u_{j,t}\|_{2,0}^2
\]

Next, we test the ability of the method to simulate high Reynolds number, complex flows. The method is tested on the well-known 3D Ethier-Steinman flow problem with high Reynolds number and grad-div stabilization is added to the method to relax the time step condition. In all tests, we use Taylor-Hood P2-P1 elements. The code was implemented using the software package FreeFem++, \[27\].
\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|}
\hline
$\Delta t$ & $\|u_1 - u_{1,h}\|_{\infty,0}$ & rate & $\|\nabla u_1 - \nabla u_{1,h}\|_{2,0}$ & rate \\
\hline
0.05 & $2.11868 \cdot 10^{-4}$ & - & $3.33272 \cdot 10^{-4}$ & - \\
0.025 & $5.86519 \cdot 10^{-5}$ & 1.8529 & $6.46582 \cdot 10^{-4}$ & 2.3658 \\
0.0125 & $1.55198 \cdot 10^{-5}$ & 1.9181 & $1.50220 \cdot 10^{-4}$ & 2.1058 \\
0.00625 & $3.99025 \cdot 10^{-6}$ & 1.9596 & $3.72779 \cdot 10^{-5}$ & 2.0107 \\
0.003125 & $1.01142 \cdot 10^{-6}$ & 1.9800 & $9.36355 \cdot 10^{-6}$ & 1.9932 \\
\hline
\end{tabular}
\caption{(En-BlendedBDF): Errors and convergence rates for the first ensemble member}
\end{table}

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|}
\hline
$\Delta t$ & $\|u_2 - u_{2,h}\|_{\infty,0}$ & rate & $\|\nabla u_2 - \nabla u_{2,h}\|_{2,0}$ & rate \\
\hline
0.05 & $2.11487 \cdot 10^{-4}$ & - & $3.32141 \cdot 10^{-4}$ & - \\
0.025 & $5.85514 \cdot 10^{-5}$ & 1.8528 & $6.44810 \cdot 10^{-4}$ & 2.3648 \\
0.0125 & $1.54929 \cdot 10^{-5}$ & 1.9181 & $1.49864 \cdot 10^{-4}$ & 2.1052 \\
0.00625 & $3.98337 \cdot 10^{-6}$ & 1.9596 & $3.71937 \cdot 10^{-5}$ & 2.0105 \\
0.003125 & $1.00968 \cdot 10^{-6}$ & 1.9801 & $9.34265 \cdot 10^{-6}$ & 1.9932 \\
\hline
\end{tabular}
\caption{(En-BlendedBDF): Errors and convergence rates for the second ensemble member}
\end{table}

5.1. **Convergence.** Our first experiment tests the predicted convergence rates for the method. We take the analytical solution of Navier-Stokes equations from [9], prescribed in the unit square $\Omega = [0,1]^2$

\[
u_{\text{true}} = (-g(t) \cos x \sin y, +g(t) \sin x \cos y)^T,
\]

\[
u_{\text{true}} = -\frac{1}{4} [\cos(2x) + \cos(2y)] g^2(t), \quad \text{where } g(t) = \sin(2t),
\]

with source term $f(x,y,t) = [g'(t) + 2v g(t)](- \cos x \sin y, \sin x \cos y)^T$. We take the viscosity $\nu = 0.01$ and simulation time $T = 1$. Inhomogeneous Dirichlet boundary condition $u = \nu_{\text{true}}$ on $\partial\Omega$ is enforced nodally on the boundary.

We consider a set of two realizations of Navier-Stokes equations $u_{1,2} = (1 \pm \epsilon)\nu_{\text{true}}$, $\epsilon = 10^{-3}$, which correspond to two different initial conditions $u_{1,2}^0 = (1 \pm \epsilon)\nu_{\text{true}}^0$ respectively. In the simulation, the source term and boundary condition for each realization need to be adjusted accordingly. As the method is a three-step method, we need $u_1^0, u_2^0, u_1^1, u_2^1$ as well to get the algorithm start to run. For this test problem, we know the exact solution so we just take the exact solution at each corresponding instant and interpolate it in the finite element space. We then calculate errors and convergence rates by computing approximations with both (En-BlendedBDF) and (En-BDF2) on 5 successive mesh refinements with $h = 2\Delta t$.

From Table 1 and Table 2, the convergence rate is close to 2, which is optimal according to our theoretical results. In Tables 3, 4, 5 and 6, we compare the error computed with (En-BlendedBDF) and (En-BDF2). As we can see from the tables, the error computed with (En-BlendedBDF) is noticeably smaller as a consequence of smaller temporal errors.

5.2. **3D Ethier-Steinman Flow.** We test our method on the 3D Ethier-Steinman flow problem for which the analytical solutions are known. [9]. The flow has complex
structures due to its nontrivial helicity [17], and thus is often used to test numerical methods for Navier-Stokes equations. The 3D analytical solutions on a \([0,1]^3\) box

| \(\Delta t\) | En-BlendedBDF | En-BDF2 |
|---|---|---|
| 0.05 | 2.11868 \(\cdot\) 10\(^{-4}\) | 4.85642 \(\cdot\) 10\(^{-4}\) |
| 0.025 | 5.86519 \(\cdot\) 10\(^{-5}\) | 1.26128 \(\cdot\) 10\(^{-4}\) |
| 0.0125 | 1.55198 \(\cdot\) 10\(^{-5}\) | 3.21716 \(\cdot\) 10\(^{-5}\) |
| 0.00625 | 3.90025 \(\cdot\) 10\(^{-6}\) | 8.12342 \(\cdot\) 10\(^{-6}\) |
| 0.003125 | 1.01142 \(\cdot\) 10\(^{-6}\) | 2.04078 \(\cdot\) 10\(^{-6}\) |

Table 3. \(\|u_1 - u_{1,h}\|_{\infty,0}\): Comparison of (En-BlendedBDF) and (En-BDF2)

| \(\Delta t\) | En-BlendedBDF | En-BDF2 |
|---|---|---|
| 0.05 | 2.11487 \(\cdot\) 10\(^{-4}\) | 4.84794 \(\cdot\) 10\(^{-4}\) |
| 0.025 | 5.85514 \(\cdot\) 10\(^{-5}\) | 1.25913 \(\cdot\) 10\(^{-4}\) |
| 0.0125 | 1.54929 \(\cdot\) 10\(^{-5}\) | 3.21161 \(\cdot\) 10\(^{-5}\) |
| 0.00625 | 3.98337 \(\cdot\) 10\(^{-6}\) | 8.10943 \(\cdot\) 10\(^{-6}\) |
| 0.003125 | 1.00968 \(\cdot\) 10\(^{-6}\) | 2.03726 \(\cdot\) 10\(^{-6}\) |

Table 4. \(\|u_2 - u_{2,h}\|_{\infty,0}\): Comparison of (En-BlendedBDF) and (En-BDF2)

| \(\Delta t\) | En-BlendedBDF | En-BDF2 |
|---|---|---|
| 0.05 | 3.32272 \(\cdot\) 10\(^{-3}\) | 5.11092 \(\cdot\) 10\(^{-3}\) |
| 0.025 | 6.46582 \(\cdot\) 10\(^{-4}\) | 1.18810 \(\cdot\) 10\(^{-3}\) |
| 0.0125 | 1.50220 \(\cdot\) 10\(^{-4}\) | 2.92502 \(\cdot\) 10\(^{-4}\) |
| 0.00625 | 3.72779 \(\cdot\) 10\(^{-5}\) | 7.31031 \(\cdot\) 10\(^{-5}\) |
| 0.003125 | 9.36355 \(\cdot\) 10\(^{-6}\) | 1.83094 \(\cdot\) 10\(^{-5}\) |

Table 5. \(\|\nabla u_1 - \nabla u_{1,h}\|_{2,0}\): Comparison of (En-BlendedBDF) and (En-BDF2)

| \(\Delta t\) | En-BlendedBDF | En-BDF2 |
|---|---|---|
| 0.05 | 3.32141 \(\cdot\) 10\(^{-3}\) | 5.09708 \(\cdot\) 10\(^{-3}\) |
| 0.025 | 6.44810 \(\cdot\) 10\(^{-4}\) | 1.18528 \(\cdot\) 10\(^{-3}\) |
| 0.0125 | 1.49864 \(\cdot\) 10\(^{-4}\) | 2.91837 \(\cdot\) 10\(^{-4}\) |
| 0.00625 | 3.71937 \(\cdot\) 10\(^{-5}\) | 7.29391 \(\cdot\) 10\(^{-5}\) |
| 0.003125 | 9.34265 \(\cdot\) 10\(^{-6}\) | 1.82684 \(\cdot\) 10\(^{-5}\) |

Table 6. \(\|\nabla u_2 - \nabla u_{2,h}\|_{2,0}\): Comparison of (En-BlendedBDF) and (En-BDF2)
are given by

\begin{equation}
\begin{aligned}
\mathbf{u}_1 &= -a(e^{ax} \sin(ay + dz) + e^{az} \cos(ax + dy))e^{-\nu d^2t}, \\
\mathbf{u}_2 &= -a(e^{ay} \sin(az + dx) + e^{ax} \cos(ay + dz))e^{-\nu d^2t}, \\
\mathbf{u}_3 &= -a(e^{az} \sin(ax + dy) + e^{ay} \cos(az + dx))e^{-\nu d^2t}, \\
p &= -\frac{a^2}{2}(e^{2ax} + e^{2ay} + e^{2az} + 2 \sin(ax + dy) \cos(az + dx)e^{a(y+z)} \\
&+ 2 \sin(ay + dz) \cos(ax + dy)e^{a(z+x)} + 2 \sin(az + dx) \cos(ay + dz)e^{a(x+y)})e^{-2\nu d^2t}.
\end{aligned}
\end{equation}

Figure 1 shows the flow structure of the test problem with streamribbons in the box, velocity streamlines and speed contours on the sides.

We simulate two realizations this test with perturbed initial conditions generated in the way as in the Section 5.1. The purpose of this test is to show that for high Reynolds number, the time step condition of our method can be relaxed by adding grad-div stabilization \( \gamma(\nabla \cdot u_{n+1}^h, \nabla \cdot v_h) \) and the stabilized method can still give reasonable approximations. As we do not test accuracy here, all tests are run on a relatively coarse mesh and moderately large time steps to save computational time. We take \( a = 1.25, d = 2.25 \) and the kinematic viscosity \( \nu = 0.001 \) in (5.1) and consider two realizations with perturbation parameters \( \epsilon_1 = 10^{-3} \) and \( \epsilon_2 = -10^{-3} \). The test is run on a coarse mesh with mesh size \( h = 0.1 \). We take time step \( \Delta t = 0.02 \) and run the simulation from \( t = 0 \) to \( t = 1 \). (En-BlendedBDF) encounters numerical instability and the kinetic energy quickly blows up. On the other hand, adding the grad-div stabilization term stabilized the method and gave acceptable approximations. We plot kinetic energy of averaged velocity computed with different stabilization parameter \( \gamma \) in Figure 2. For \( \gamma = 0 \), which means there is no stabilization, we can see the method is unstable while adding grad-div stabilization makes the method stable and the computed averaged velocity tracks the exact solution pretty well considering the coarse mesh and relatively large time.
step used. It is worth noting that adding grad-div stabilization introduces numerical errors as one can see from Figure 2 that the method with $\gamma = 0.1$ gives better approximation than the method with $\gamma = 1$ which introduces more numerical errors. Nevertheless, if $\gamma$ is too small, it may not be able to stabilize the method, as shown in Figure 2 the stabilization with $\gamma = 0.01$ managed to stabilize the simulation for a short time but the method becomes unstable eventually. The calibration of the stabilization parameter is an essential issue in practice.

![Energy vs Time Graph](image)

**Figure 2.** Kinetic Energy for $\nu = 0.001$, $\Delta t = 0.02$

6. Conclusion

The recently developed ensemble simulation methods to efficiently compute an ensemble of fluid flow equations open a new path to quantifying uncertainty and predicting flow behaviors. In this paper, we presented a second order ensemble method based on a blended BDF time stepping scheme with the optimal error constant. This method computes all ensemble members at each timestep in one pass, taking advantage of the fact that all members have the same coefficient matrix. Compared with the only existing second order method studied in [10], this method has noticeably improved accuracy, as is shown in numerical tests. Further research will include applying the method to the computation of the probability distributions of statistics of interest, which are outputs of certain partial differential equations, and investigating regularization methods for flows at high Reynolds number.

References

[1] S. Brenner and R. Scott, *The Mathematical Theory of Finite Element Methods*, Springer, 3rd edition, 2008.
[2] M. Carney, P. Cunningham, J. Dowling and C. Lee, *Predicting Probability Distributions for Surf Height Using an Ensemble of Mixture Density Networks*, International Conference on Machine Learning, (2005).
[3] C. Ethier and D. Steinman, *Exact fully 3D Navier-Stokes solutions for benchmarking*, Int. J. Numer. Methods Fluids, 19 (5) (1994), 369-375.
[4] Y. T. Feng, D. R. J. Owen and D. Peric, *A block conjugate gradient method applied to linear systems with multiple right hand sides*, Comp. Meth. Appl. Mech. & Enng., 127 (1995), 203-215.
[5] R. W. Freund and M. Malhotra, *A block QMR algorithm for non-Hermitian linear systems with multiple right-hand sides*, Linear Algebra and its Applications, 254 (1997), 119-157.
[6] E. Gallopoulos and V. Simoncini, Convergence of BLOCK GMRES and matrix polynomials, Lin. Alg. Appl., 247 (1996), 97-119.
[7] M. D. Gunzburger, Finite Element Methods for Viscous Incompressible Flows - A Guide to Theory, Practices, and Algorithms, Academic Press, (1989).
[8] V. Girault and P. Raviart, Finite element approximation of the Navier-Stokes equations, Lecture Notes in Mathematics, Vol. 749, Springer, Berlin, 1979.
[9] J. L. Guermond and L. Quartapelle, On stability and convergence of projection methods based on pressure Poisson equation, IJNMF, 26 (1998), 1039-1053.
[10] N. Jiang, A higher order ensemble simulation algorithm for fluid flows, Journal of Scientific Computing, 64 (2015), 264-288.
[11] N. Jiang, S. Kaya and W. Layton, Analysis of model variance for ensemble based turbulence modeling, Computational Methods in Applied Mathematics, 15 (2015), 173-188.
[12] N. Jiang and W. Layton, An algorithm for fast calculation of flow ensembles, IJUQ, 4 (2014), 273-301.
[13] N. Jiang and W. Layton, Numerical analysis of two ensemble eddy viscosity numerical regularizations of fluid motion, Numerical Methods for Partial Differential Equations, 31 (2015), 630-651.
[14] J. M. Lewis, Roots of ensemble forecasting, Monthly Weather Rev., 133 (2005), 1865-1885.
[15] M. Leutbecher and T. N. Palmer, Ensemble forecasting, J. Comp. Phys., 227 (2008), 3515-3539.
[16] W. J. Martin and M. Xue, Initial condition sensitivity analysis of a mesoscale forecast using very-large ensembles, Mon. Wea. Rev., 134 (2006), 192-207.
[17] M. A. Olshanski and J. G. Rehholz, Velocity-vorticity-helicity for formulation and a solver for the Navier-Stokes equations, J. Comp. Phys., 229 (2010), 4291-4303.
[18] G. A. Baker, V. A. Dougalis and O. A. Karakashian, On a higher order accurate fully discrete Galerkin approximation to the Navier-Stokes equations, Mathematics of Computation, 39 (1982), 339-375.
[19] W. Hundsdorfer, Partially implicit BDF2 blends for convection dominated flows, SIAM Journal of Numerical Analysis, 38 (2001), 1763-1783.
[20] E. Emmrich, Error of the two-step BDF for the incompressible Navier-Stokes problem, Mathematical Modeling and Numerical Analysis, 38 (2004), 757-764.
[21] V. Vatsa, M. Carpenter and D. Lockard, Re-evaluation of an Optimized Second Order Backward Difference (BDF2OPT) scheme for unsteady flow applications, AIAA Paper 2010-0122, January 2010.
[22] Z. Toth and E. Kalney, Ensemble forecasting at NMC: The generation of perturbations, Bull. Amer. Meteor. Soc., 74 (1993), 2317-2330.
[23] R. Buizza and T. N. Palmer, The singular-vector structure of the atmospheric global circulation, Journal of the Atmospheric Sciences, 52 (1995), 1434-1454.
[24] E. J. Nielsen and W. T. Jones, Integrated design of an active flow control system using a time-dependent adjoint method, Math. Model. Nat. Phenom., 5 (2011), 141-165.
[25] A. Takhir, M. Neda and J. Waters, Time relaxation algorithm for flow ensembles, Numerical Methods for Partial Differential Equations, to appear, 2015, DOI: 10.1002/num.22024.
[26] M. Nyukhtikov, N. Smelova, B. E. Mitchell and D. G. Holmes, Optimized dual-time stepping technique for time-accurate Navier-Stokes calculation, Proceedings of the 10th Int. Sym. on Unst. Aero., Aeroac., and Aeroelas. of Turbomach. (2003).
[27] F. Hecht, New development in freefem++, J. Numer. Math., 20 (2012), no. 3-4, 251-265.
[28] T. Gelhard, G. Lube, M. A. Olshanski and J. -H. Starcke, Stabilized finite element schemes with LBB-stable elements for incompressible flows, J. Comput. Appl. Math., 177 (2005), 243-267.
[29] M. A. Olshanski, A low order Galerkin finite element method for the Navier-Stokes equations of steady incompressible flow: A stabilization issue and iterative methods, Comput. Methods Appl. Mech. Engrg., 191 (2002), 5515-5536.
[30] M. Case, V. Ervin, A. Linke and L. Rehholz, A connection between Scott-Vogelius elements and grad-div stabilization, SIAM Journal on Numerical Analysis, 49 (2011), 1461-1481.
Appendix A. Proof of Lemma 2

Proof. To prove (4.1), we first rewrite

\[
10(u^{n+1} - u^n) - 5(u^n - u^{n-1}) + (u^{n-1} - u^{n-2}) - 6\Delta t u_t^{n+1} = 10 \int_{t^n}^{t^{n+1}} u_t dt - 5 \int_{t^n}^{t^n} u_t dt + \int_{t^{n-1}}^{t^n} u_t dt - 6\Delta t u_t^{n+1} = 10 \int_{t^n}^{t^{n+1}} \frac{d}{dt}(t^n u_t) dt - 5 \int_{t^n}^{t^n} \frac{d}{dt}(t^n u_t) dt
\]

\[
+ \int_{t^{n-2}}^{t^n} \frac{d}{dt}(t^{n-2} u_t) dt - 6\Delta t u_t^{n+1} = 10 \left[ (t^n u_t)_{t^n}^{n+1} - \int_{t^n}^{t^n} (t^n u_t) dt \right] - 5 \left[ (t^{n-1} u_t)_{t^{n-1}}^{n} - \int_{t^{n-1}}^{t^n} (t^{n-1} u_t) dt \right] + \left[ (t^{n-2} u_t)_{t^{n-2}}^{n-1} - \int_{t^{n-2}}^{t^n} (t^{n-2} u_t) dt \right] - 6\Delta t u_t^{n+1}
\]

\[
= \left[ 4\Delta t u_t^{n+1} - 5\Delta t u_t^n + \Delta t u_t^{n-1} \right] - 10 \int_{t^n}^{t^{n+1}} \frac{d}{dt} \left( \frac{1}{2} (t^n)^2 \right) u_t dt + 5 \int_{t^{n-1}}^{t^n} \frac{d}{dt} \left( \frac{1}{2} (t^{n-1})^2 \right) u_t dt = \left[ 4\Delta t \int_{t^n}^{t^{n+1}} u_t dt - \Delta t \int_{t^{n-1}}^{t^n} u_t dt \right] - 10 \left[ \frac{1}{2} (t^n)^2 u_t \right]_{t^n}^{n+1} - \int_{t^n}^{t^{n+1}} \frac{1}{2} (t^n)^2 u_t dt
\]

\[
+ 5 \left[ \frac{1}{2} (t^{n-1})^2 u_t \right]_{t^{n-1}}^{n} - \int_{t^{n-1}}^{t^n} \frac{1}{2} (t^{n-1})^2 u_t dt - \left[ \frac{1}{2} (t^{n-2})^2 u_t \right]_{t^{n-2}}^{n-1} - \int_{t^{n-2}}^{t^{n-1}} \frac{1}{2} (t^{n-2})^2 u_t dt
\]

\[
= 4\Delta t \left[ (t^n u_t)_{t^n}^{n+1} - \int_{t^n}^{t^n} (t^n) u_t dt \right] - \Delta t \left[ (t^{n-1} u_t)_{t^{n-1}}^{n} - \int_{t^{n-1}}^{t^n} (t^{n-1}) u_t dt \right] - 10 \left( \frac{1}{2} \Delta t^2 u_t^{n+1} \right) + 5 \left( \frac{1}{2} \Delta t^2 u_t^n \right) - \left( \frac{1}{2} \Delta t^2 u_t^{n-1} \right) + 10 \int_{t^n}^{t^{n+1}} \frac{1}{2} (t^n)^2 u_t dt - 5 \int_{t^{n-1}}^{t^n} \frac{1}{2} (t^{n-1})^2 u_t dt + \int_{t^{n-2}}^{t^n} \frac{1}{2} (t^{n-2})^2 u_t dt
\]
Now we prove (4.2). To start, we rewrite

\[ L = t_n \int_{t_n}^{t_{n+1}} \left( -5 \frac{1}{2} (t-t^n)^2 u_{ttt} dt + 10 \int_{t_n}^{t_{n+1}} \frac{1}{2} (t-t^n)^2 u_{ttt} dt \right) \]

+ 10 \int_{t_n}^{t_{n+1}} \frac{1}{2} (t-t^n)^2 u_{ttt} dt - 5 \int_{t_n-1}^{t_n} \frac{1}{2} (t-t^{n-1})^2 u_{ttt} dt + \int_{t_n-2}^{t_{n-1}} \frac{1}{2} (t-t^{n-2})^2 u_{ttt} dt \]

Then the \( L^2 \) norm of the term of interest can be estimated as follows

(A.1) \[
\left\| \frac{1}{6t} \left[ 10u^{n+1} - 15u^n + 6u^{n-1} - u^{n-2} - u_t^{n+1} \right] \right\|^2 dt
\]

\[
= \frac{1}{36t^2} \int_{t_n}^{t_{n+1}} \left( \frac{1}{2} \Delta t^2 \left[ 2 \int_{t_n}^{t_{n+1}} u_{ttt} dt - \int_{t_{n-1}}^{t_n} u_{ttt} dt \right] \right.
\]

\[
-4 \Delta t \int_{t_n}^{t_{n+1}} (t-t^n) u_{ttt} dt + \Delta t \int_{t_n}^{t_{n+1}} (t-t^{n-1}) u_{ttt} dt + 10 \int_{t_n}^{t_{n+1}} \frac{1}{2} (t-t^n)^2 u_{ttt} dt
\]

\[
-5 \int_{t_n-1}^{t_n} \frac{1}{2} (t-t^{n-1})^2 u_{ttt} dt + \int_{t_n-2}^{t_{n-1}} \frac{1}{2} (t-t^{n-2})^2 u_{ttt} dt \right) dx
\]

\[
\leq \frac{1}{18t^2} \left( \frac{1}{4} \Delta t^4 \left[ \int_{t_n}^{t_{n+1}} u_{ttt} dt \right]^2 + \frac{1}{4} \Delta t^4 \left[ \int_{t_{n-1}}^{t_n} u_{ttt} dt \right]^2 \right.
\]

\[
+16 \Delta t^2 \left[ \int_{t_n}^{t_{n+1}} (t-t^n) u_{ttt} dt \right]^2 + \Delta t^2 \left[ \int_{t_n}^{t_{n+1}} (t-t^{n-1}) u_{ttt} dt \right]^2
\]

\[
+25 \left[ \int_{t_n}^{t_{n+1}} (t-t^n)^2 u_{ttt} dt \right]^2 + 25 \left[ \int_{t_n}^{t_{n+1}} (t-t^{n-1})^2 u_{ttt} dt \right]^2
\]

\[
+ \frac{1}{4} \left[ \int_{t_{n-2}}^{t_{n-1}} (t-t^{n-2})^2 u_{ttt} dt \right]^2 \right) dx
\]

\[
\leq \frac{1}{18t^2} \left( \frac{1}{4} \Delta t^5 \int_{t_n}^{t_{n+1}} |u_{ttt}|^2 dt + \frac{1}{4} \Delta t^5 \int_{t_{n-1}}^{t_n} |u_{ttt}|^2 dt \right.
\]

\[
+16 \Delta t^3 \int_{t_n}^{t_{n+1}} |t-t^n|^2 |u_{ttt}|^2 dt + \Delta t^3 \int_{t_{n-1}}^{t_n} |t-t^{n-1}|^2 |u_{ttt}|^2 dt \right)^2
\]

\[
+25 \Delta t \int_{t_n}^{t_{n+1}} (t-t^n)^2 |u_{ttt}|^2 dt + \frac{25}{4} \Delta t \int_{t_{n-1}}^{t_n} (t-t^{n-1})^2 |u_{ttt}|^2 dt
\]

\[
+ \frac{1}{4} \Delta t \int_{t_{n-2}}^{t_{n-1}} (t-t^{n-2})^2 |u_{ttt}|^2 dt \right) dx
\]

\[
\leq \frac{7}{3} \Delta t^3 \left( \int_{t_n}^{t_{n+1}} |u_{ttt}|^2 dt \right) dx \leq \frac{7}{3} \Delta t^3 \int_{t_{n-2}}^{t_{n-1}} \|u_{ttt}\|^2 dt.
\]

Now we prove (4.2). To start, we rewrite

(A.2) \[
(u^{n+1} - 3u^n + 3u^{n-1} - u^{n-2})
\]
\[ (u^{n+1} - u^n) - (u^n - u^{n-1}) = [(u^n - u^{n-1}) - (u^{n-1} - u^{n-2})]. \]

Using integration by parts, the terms in the first brackets in the above equation can be written as

\[ \Delta t \int_{t_{n-1}}^{t_n} d(t - t^n)u_{tt}dt - \int_{t_{n-1}}^{t_n} d(t - t^n)u_{tt}dt. \]

Similarly, we have

\[ (u^n - u^{n-1}) - (u^{n-1} - u^{n-2}) \]

\[ \Delta t \int_{t_{n-2}}^{t_{n-1}} d(t - t^{n-1})u_{ttt}dt - \int_{t_{n-2}}^{t_{n-1}} d(t - t^{n-1})u_{ttt}dt. \]
Subtracting (A.4) from (A.3) gives

\[
\begin{align*}
(A.5) \quad u^{n+1} - 3u^n + 3u^{n-1} - u^{n-2} &= \Delta t \left[ \Delta t \left( u_{tt}^{n+1} - u_{tt}^n + u_{tt}^{n-1} - u_{tt}^{n-2} \right) - \int_{t_{n-1}}^{t_n} (t-t^n) u_{ttt} dt + \int_{t_{n-2}}^{t_{n-1}} (t-t^{n-1}) u_{ttt} dt \right] \\
&\quad - \left[ \frac{1}{2} \Delta t^2 \right] u_{tt}^{n+1} - \left( \frac{1}{2} \Delta t^2 \right) u_{tt}^n + \left( \frac{1}{2} \Delta t^2 \right) u_{tt}^{n-1} - \left( \frac{1}{2} \Delta t^2 \right) u_{tt}^{n-2} \\
&\quad + \left[ \int_{t_{n-1}}^{t_n} \left( \frac{1}{2} (t-t^n)^2 \right) u_{ttt} dt - \int_{t_{n-1}}^{t_n} \left( \frac{1}{2} (t-t^n)^2 \right) u_{ttt} dt \right] \\
&\quad - \left[ \int_{t_{n-2}}^{t_{n-1}} \left( \frac{1}{2} (t-t^{n-1})^2 \right) u_{ttt} dt - \int_{t_{n-2}}^{t_{n-1}} \left( \frac{1}{2} (t-t^{n-1})^2 \right) u_{ttt} dt \right] \\
&\quad = \Delta t \left[ \frac{1}{2} \Delta t \left( u_{tt}^{n+1} - u_{tt}^n + u_{tt}^{n-1} - u_{tt}^{n-2} \right) - \int_{t_{n-1}}^{t_n} (t-t^n) u_{ttt} dt + \int_{t_{n-2}}^{t_{n-1}} (t-t^{n-1}) u_{ttt} dt \right] \\
&\quad + \left[ \int_{t_{n-1}}^{t_n} \left( \frac{1}{2} (t-t^n)^2 \right) u_{ttt} dt - \int_{t_{n-1}}^{t_n} \left( \frac{1}{2} (t-t^n)^2 \right) u_{ttt} dt \right] \\
&\quad - \left[ \int_{t_{n-2}}^{t_{n-1}} \left( \frac{1}{2} (t-t^{n-1})^2 \right) u_{ttt} dt - \int_{t_{n-2}}^{t_{n-1}} \left( \frac{1}{2} (t-t^{n-1})^2 \right) u_{ttt} dt \right] \\
&\quad = \Delta t \left[ \frac{1}{2} \Delta t \left( u_{tt}^{n+1} + u_{tt}^n + u_{tt}^{n-1} + u_{tt}^{n-2} \right) - \int_{t_{n-1}}^{t_n} (t-t^n) u_{ttt} dt + \int_{t_{n-2}}^{t_{n-1}} (t-t^{n-1}) u_{ttt} dt \right] \\
&\quad + \left[ \int_{t_{n-1}}^{t_n} \left( \frac{1}{2} (t-t^n)^2 \right) u_{ttt} dt - \int_{t_{n-1}}^{t_n} \left( \frac{1}{2} (t-t^n)^2 \right) u_{ttt} dt \right] \\
&\quad - \left[ \int_{t_{n-2}}^{t_{n-1}} \left( \frac{1}{2} (t-t^{n-1})^2 \right) u_{ttt} dt - \int_{t_{n-2}}^{t_{n-1}} \left( \frac{1}{2} (t-t^{n-1})^2 \right) u_{ttt} dt \right].
\end{align*}
\]

Then by the Cauchy-Schwarz inequality we have

\[
(A.6) \quad \| \nabla \left( u^{n+1} - 3u^n + 3u^{n-1} - u^{n-2} \right) \|^2 \\
= \int_\Omega \frac{1}{2} \Delta t^2 \left( \int_{t_{n-1}}^{t_n} \nabla u_{ttt} dt + \int_{t_{n-2}}^{t_{n-1}} \nabla u_{ttt} dt \right) \\
- \Delta t \left[ \int_{t_{n-1}}^{t_n} (t-t^n) \nabla u_{tt} dt - \int_{t_{n-2}}^{t_{n-1}} (t-t^{n-1}) \nabla u_{tt} dt \right] \\
+ \int_{t_{n-1}}^{t_n} \left( \frac{1}{2} (t-t^n)^2 \right) \nabla u_{ttt} dt - \int_{t_{n-1}}^{t_n} \left( \frac{1}{2} (t-t^n)^2 \right) \nabla u_{ttt} dt \\
- \int_{t_{n-2}}^{t_{n-1}} \left( \frac{1}{2} (t-t^{n-1})^2 \right) \nabla u_{ttt} dt - \int_{t_{n-2}}^{t_{n-1}} \left( \frac{1}{2} (t-t^{n-1})^2 \right) \nabla u_{ttt} dt \right] \, dx \\
\leq 2 \int_\Omega \frac{1}{4} \Delta t^4 \left( \int_{t_n}^{t_{n+1}} \nabla u_{tt} dt \right)^2 + \frac{1}{4} \Delta t^4 \left( \int_{t_{n-2}}^{t_{n-1}} \nabla u_{tt} dt \right)^2.
\]
\[
+ \Delta t^2 \int_{t_{n-1}}^{t_{n+1}} (t - t^n) \nabla u_{ttt} dt \bigg| \bigg| + \Delta t^2 \int_{t_{n-2}}^{t_{n}} (t - t^{n-1}) \nabla u_{ttt} dt \bigg| \bigg|^2 \\
+ \int_{t_{n}}^{t_{n+1}} \left( \frac{1}{2} (t - t^n)^2 \right) \nabla u_{ttt} dt \bigg|^2 + \int_{t_{n-1}}^{t_{n}} \left( \frac{1}{2} (t - t^{n-1})^2 \right) \nabla u_{ttt} dt \bigg|^2 \\
+ \int_{t_{n-1}}^{t_{n}} \left( \frac{1}{2} (t - t^{n-1})^2 \right) \nabla u_{ttt} dt \bigg|^2 + \int_{t_{n-2}}^{t_{n-1}} \left( \frac{1}{2} (t - t^{n-2})^2 \right) \nabla u_{ttt} dt \bigg|^2 dx \\
\leq 2 \int_{\Omega} \left\{ \frac{1}{4} \Delta t^5 \int_{t_{n}}^{t_{n+1}} |\nabla u_{ttt}|^2 dt + \frac{1}{4} \Delta t^5 \int_{t_{n-2}}^{t_{n-1}} |\nabla u_{ttt}|^2 dt \\
+ 2 \Delta t^5 \int_{t_{n-1}}^{t_{n+1}} |\nabla u_{ttt}|^2 dt + 2 \Delta t^5 \int_{t_{n-2}}^{t_{n}} |\nabla u_{ttt}|^2 dt \\
+ \frac{1}{4} \Delta t^5 \int_{t_{n-1}}^{t_{n}} |\nabla u_{ttt}|^2 dt + \frac{1}{4} \Delta t^5 \int_{t_{n-1}}^{t_{n}} |\nabla u_{ttt}|^2 dt \\
+ \frac{1}{4} \Delta t^5 \int_{t_{n-2}}^{t_{n}} |\nabla u_{ttt}|^2 dt \right\} dx \\
\leq 2 \int_{\Omega} \left\{ \frac{9}{2} \Delta t^5 \int_{t_{n-2}}^{t_{n}} |\nabla u_{ttt}|^2 dt \right\} dx \leq 9 \Delta t^5 \int_{t_{n-2}}^{t_{n+1}} \|\nabla u_{ttt}\|^2 dt.
\]

This completes the proof. \(\blacksquare\)

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