QUADRATIC BSDES WITH MEAN REFLECTION

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Abstract. The present paper is devoted to the study of the well-posedness of BSDEs with mean reflection whenever the generator has quadratic growth in the z argument. This work is the sequel of [6] in which a notion of BSDEs with mean reflection is developed to tackle the super-hedging problem under running risk management constraints. By the contraction mapping argument, we first prove that the quadratic BSDE with mean reflection admits a unique deterministic flat local solution on a small time interval whenever the terminal value is bounded. Moreover, we build the global solution on the whole time interval by stitching local solutions when the generator is uniformly bounded with respect to the y argument.

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1. **Introduction.** The nonlinear backward stochastic differential equation (BSDE) of the following form was first introduced by Pardoux and Peng [31]:

\[ Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s, \quad \forall t \in [0, T], \tag{1} \]

whose solution consists of an adapted pair of processes \((Y, Z)\). Pardoux and Peng have obtained the existence and uniqueness theorem for the BSDE (1) when the generator \(f\) is uniformly Lipschitz and the terminal value \(\xi\) is square integrable. Since then, researchers made great progresses in this field. It was seen that BSDEs have provided powerful tools for the study of mathematical finance, stochastic control and partial differential equations. In particular, El Karoui, Peng and Quenez [19] have applied the BSDE theory to pricing of European contingent claims, roughly speaking, the component \(Y\) and the component \(Z\) of the solution can be interpreted as the value process of the claim and its hedging strategy, respectively. Furthermore, El Karoui, Pardoux and Quenez have investigated the pricing of American claims in [18]. In this paper, the price of an American option can be formulated as the “minimal” solution to the following type of BSDE with constraints:

\[ Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s + K_T - K_t, \quad \forall t \in [0, T], \tag{2} \]

where \(\xi\) is the terminal payoff, the component \(Y\) is forced to stay above a given running payoff \(L\) and the component \(K\) is adapted and non-decreasing, which describes the cumulative consumption under the aforementioned constraint. This constrained BSDE (2) was called reflected BSDE and has been considered by El Karoui, Kapoudjian, Pardoux, Peng and Quenez in [17], in which the minimality of solution is explicitly characterized by the Skorokhod type condition,

\[ \int_0^T (Y_t - L_t)dK_t = 0, \]

i.e., \(K\) increases only when \(Y\) stays on the reflecting boundary \(L\).

Afterwards, the theory of constrained BSDEs has been generalized in many cases in order to tackle various pricing problems in incomplete markets, see, e.g., Buckdahn and Hu [10, 11], Cvitanic, Karatzas and Soner [16], Peng and Xu [32]. It is also observed that the constrained BSDEs have strong connections with the Dynkin game (cf. [15]) and optimal switching problems (cf. [12, 21, 22, 25]).

Generally, the formulation of constrained BSDEs in the aforementioned papers involves only pointwise constraints for solutions. In contrast, Bouchard, Elie and Réveillac [3] have introduced the so-called weak terminal condition to the BSDE framework, which says the terminal value only satisfies a mean constraint of the form

\[ \mathbb{E}[\ell(Y_T - \xi)] \geq m, \]

where \(m\) is a given threshold, \(\ell\) is a non-decreasing map and can be viewed as a loss function in quantile hedging problems or in stochastic target problems under controlled loss.

Recently, motivated by super-hedging of claims under running risk management constraints, Briand, Elie and Hu [6] have formulated a new type of BSDE with constraints, which is called the BSDE with mean reflection. In their framework, the solution \(Y\) is required to satisfy the following type of mean reflection constraint:

\[ \mathbb{E}[\ell(t, Y_t)] \geq 0, \quad \forall t \in [0, T], \]
where the running loss function \( \ell(t, \cdot) \) is a collection of (possibly random) non-decreasing real-valued map. This type of reflected equation is also closely related to interacting particles systems, see, e.g., Briand, Chaudru de Raynal, Guillin and Labart [5].

In order to establish the well-posedness of BSDEs with mean reflection, in [6] the authors have introduced the notion of deterministic flat solution, i.e., the component \( K \) is a deterministic non-decreasing process and satisfies the following type of Skorokhod condition,

\[
\int_0^T \mathbb{E}[\ell(t, Y_t)] dK_t = 0.
\]

Thanks to the restriction of non-randomness, the solution \( K \) can be constructed explicitly when the generator \( f \) is independent of \( y \) and \( z \). In this case, such a simple BSDE with mean reflection can be solved easily by applying a martingale representation type argument. Then with the help of the fixed-point theory, they have generalized the result to the Lipschitz case with square integrable terminal value when the running loss function \( \ell \) is bi-Lipschitz for the mean reflection. Moreover, they have indicated the minimality of the deterministic flat solution among all the deterministic solutions of (2) under an additional structural condition on the generator.

The main purpose of this paper is to study quadratic BSDEs with mean reflection, in which the generator has quadratic growth in \( z \) and the terminal condition is bounded. Indeed, quadratic BSDEs in the classical sense have already attracted numerous studies, they are powerful tools for many finance applications, such as utility maximization problems and risk sensitive control problems, see, e.g., Hu, Imkeller and Müller [24].

The solvability of scalar-valued quadratic BSDEs was first established by Kobylianski [28] via a PDE-based method under the boundedness assumption of the terminal value. Subsequently, Briand and Hu [8, 9] have extended the existence result to the case of unbounded terminal values with exponential moments and have studied the uniqueness whenever the generator is convex (or concave). It is worth mentioning that the comparison theorem for BSDE solutions plays a key role in these works, in which the solutions are constructed by the monotone convergence. From a different point of view, Tevzadze [33] has applied the fixed-point argument to obtain the existence and uniqueness simultaneously for quadratic BSDEs with small terminal values and has stitched “small” solutions to solve a BSDE with a general bounded terminal value. In his paper, the application of the BMO martingale theory is crucial, which was first applied in [24] for considering quadratic BSDEs. Apart from this, Briand and Elie [7] have recently used the Malliavin calculus to provide a probabilistic approach for studying the quadratic BSDEs in the spirit of [1, 5]. We also refer the reader to Morlais [29], Barrieu and El Karoui [2] for more general results beyond the Brownian framework.

Contrary to the scalar-valued case, general multi-dimensional quadratic BSDEs may not have a solution, see Frei and dos Reis [20]. However, the result of Tevzadze [33] for small terminal values holds even for multi-dimensional cases. Besides, Cheridito and Nam [14] have studied a class of multidimensional quadratic BSDEs with special structure. Hu and Tang [26] have discussed the local and global solutions for multi-dimensional BSDEs with a “diagonally” quadratic generator. More recently, for multidimensional quadratic BSDEs, Xing and Zitkovic [34] have established more general existence and uniqueness results, but in a Markovian
framework, while Harter and Richou [23] have obtained positive results in some general setting.

To consider the quadratic BSDE with mean reflection, the main difficulty is the lack of a pointwise comparison theorem for solutions (see Example 3.3). In other words, it is difficult to proceed the monotone convergence argument to construct the solution as in [28, 8, 9]. Therefore, we study the solvability of quadratic BSDE with mean reflection by the fixed-point argument. The key points of our method is based on the following observation:

- Suppose that \((Y, Z)\) and \((Y, Z, K)\) are the solution to the BSDE (1) and the deterministic flat solution to the quadratic BSDE (2) with mean reflection, respectively. Then the uniqueness of the solution to standard BSDE (1) implies that

\[
Y_t = Y_t - K_T + K_t, \quad Z_t = Z_t, \quad \forall t \in [0, T],
\]

whenever the generator \(f\) is independent of \(y\).

Therefore, for such a simple case, we can obtain the solution in two steps: (a) solving the corresponding standard quadratic BSDE to define the component \(Z\); (b) solving the BSDE with mean reflection with the generator \(f(Z)\) to find the components \(Y\) and \(K\), where \(Z\) is exactly the one obtained in the previous step.

Thanks to this preliminary result, we can define a contractive map to find the component \(Y\) for solving the equation with a general quadratic generator. Comparing with [33], our contractive map is different such that the restriction on the size of the terminal value could be removed, however, as a first step, the constructed solution lives only locally on a small time interval. We observe that the maximal length of the time interval on which the mapping is contractive depends only on the bound of the component \(Y\). Once the component \(Y\) has a uniform estimate under additional assumptions, a global solution on the whole time interval can be established by stitching the local ones.

The remainder of the paper is organized as follows. In Section 2, we recall the framework of BSDEs with mean reflection and state our main results. Section 3 is devoted to the study the case when the generator \(f\) has separable deterministic linear dependence in \(y\). The general case is investigated in Section 4, in which we start by constructing the deterministic flat local solutions and then stitch them to build a global solution.

**Notation.** We introduce the notations, which will be used throughout this paper. For each Euclidian space, we denote by \(\langle \cdot, \cdot \rangle\) and \(|\cdot|\) its scalar product and the associated norm, respectively. Then consider a finite time horizon \([0, T]\) and a complete probability space \((\Omega, \mathcal{F}, \mathbb{P})\), on which \(B = (B_t)_{0 \leq t \leq T}\) is a standard \(d\)-dimensional Brownian motion. Let \((\mathcal{F}_t)_{0 \leq t \leq T}\) be the natural filtration generated by \(B\) augmented with the family \(\mathcal{N}^{\mathbb{P}}\) of \(\mathbb{P}\)-null sets of \(\mathcal{F}\). Finally, we consider the following Banach spaces:

- \(\mathcal{L}^2\) is the space of real valued \(\mathcal{F}_T\)-measurable random variables \(Y\) satisfying

\[
\|Y\|_{\mathcal{L}^2} = \mathbb{E}[|Y|^2]^{1/2} < \infty;
\]

- \(\mathcal{L}^\infty\) is the space of real valued \(\mathcal{F}_T\)-measurable random variables \(Y\) satisfying

\[
\|Y\|_{\mathcal{L}^\infty} = \text{ess sup}_{\omega} |Y(\omega)| < \infty;
\]
• $S^\infty$ is the space of real valued progressively measurable continuous processes $Y$ satisfying

$$\|Y\|_{S^\infty} = \text{ess sup}_{(t,\omega)} |Y(t,\omega)| < \infty;$$

• $A^D$ is the closed subset of $S^\infty$ consisting of deterministic non-decreasing processes $K = (K_t)_{0 \leq t \leq T}$ starting from the origin;

• $BMO$ is the space of all progressively measurable processes $Z$ taking values in $\mathbb{R}^d$ such that

$$\|Z\|_{BMO} = \sup_{\tau \in \mathcal{T}} \left\| \mathbb{E}_\tau \left[ \int_\tau^T |Z_s|^2 \, ds \right] \right\|_{L^\infty}^{1/2} < \infty,$$

where $\mathcal{T}$ denotes the set of all $[0,T]$-valued stopping times $\tau$ and $\mathbb{E}_\tau$ is the conditional expectation with respect to $\mathcal{F}_\tau$.

We denote by $S^\infty_{[a,b]}$, $A^D_{[a,b]}$ and $BMO_{[a,b]}$ the corresponding spaces for the stochastic processes have time indexes on $[a,b]$. For each $Z \in BMO_{[a,b]}$, we set

$$\mathcal{E}^a_b(Z \cdot B)_t = \exp \left( \int_a^t Z_s dB_s - \frac{1}{2} \int_a^t |Z_s|^2 ds \right),$$

which is a martingale by [27]. Thus it follows from Girsanov’s theorem that

$$(B_t - \int_a^t Z_s dB_s 1_{(a \leq t \leq b)})_{0 \leq t \leq T}$$

is a Brownian motion under the equivalent probability measure $\mathcal{E}^a_b(\cdot \cdot B)_b d\mathbb{P}$.

2. Quadratic BSDEs with mean reflection. In this paper, we consider the following type of constrained BSDE:

$$\begin{align*}
Y_t &= \xi + \int_t^T f(s, Y_s, Z_s) \, ds - \int_t^T Z_s \, dB_s + K_T - K_t; \\
\mathbb{E}[\ell(t, Y_t)] &\geq 0,
\end{align*}$$

where the second equation is a running constraint in expectation on the component $Y$ of the solution. The above equation is called BSDE with mean reflection, which was first introduced in [6]. The parameters of the BSDE with mean reflection are the terminal condition $\xi$, the generator (or driver) $f$ as well as the running loss function $\ell$. In [6], the authors have discussed such equation under the standard Lipschitz condition on the generator and the square integrability assumption on the terminal condition.

In the sequel, we study the existence and uniqueness theorem of equation (3) with quadratic generator and bounded terminal condition. These parameters are supposed to satisfy the following standard running assumptions:

(H$_\xi$) The terminal condition $\xi$ is an $\mathcal{F}_T$-measurable random variable bounded by $L > 0$ such that

$$\mathbb{E}[\ell(T, \xi)] \geq 0.$$ 

(H$_f$) The driver $f : [0,T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ is a $\mathcal{P} \times \mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R}^d)$-measurable map such that

1. For each $t \in [0,T]$, $f(t,0,0)$ is bounded by some constant $L$, $\mathbb{P}$-a.s.
2. There exists some constant $\lambda > 0$ such that, $\mathbb{P}$-a.s., for all $t \in [0,T]$, for all $y, p, q \in \mathbb{R}$ and for all $z, q \in \mathbb{R}^d$,

$$|f(t, y, z) - f(t, p, q)| \leq \lambda(|y - p| + (1 + |z| + |q|)|z - q|),$$
Theorem 2.3. Assume that solution for the quadratic BSDE with mean reflection, which reads as follows:

\[ Y_t = H_T - 
\]

only when needed, i.e., when we have flat A triple of processes (Y, Z, K) ∈ \( S^\infty \times BMO \times A^D \) are the Borel algebras on \( R \) and \( R^d \), respectively.

\( (H) \) The running loss function \( \ell : \Omega \times [0, T] \times R \to R \) is an \( F_T \times B(\mathbb{R}) \times B(\mathbb{R}) \)-measurable map and there exists some constant \( C > 0 \) such that, \( P \)-a.s.,

1. \( (t, y) \to \ell(t, y) \) is continuous,
2. \( \forall t \in [0, T], y \to \ell(t, y) \) is strictly increasing,
3. \( \forall t \in [0, T], E[\ell(t, \infty)] > 0 \),
4. \( \forall t \in [0, T], \forall y \in \mathbb{R}, |\ell(t, y)| \leq C(1 + |y|) \).

In order to introduce another assumption for the main result of this paper, we define the operator \( L_t : \mathcal{L}^2 \to [0, \infty), t \in [0, T] \) by

\[ L_t : X \to \inf\{x \geq 0 : E[\ell(t, x + X)] \geq 0\}, \]

which is well-defined due to Assumption \((H_t)\), see also [6]. The operator \( L_t \) is crucial to build a solution to BSDEs with mean reflection.

Example 2.1. Suppose that \( \ell(t, x) = x - u_t, t \in [0, T] \), for some given deterministic continuous process \( u \). It is easy to check that

\[ L_t(X) = (E[X] - u_t)^- := \min\{E[X] - u_t, 0\}, \forall t \in [0, T] \text{ and } X \in \mathcal{L}^2. \]

In addition to the aforementioned assumptions, for the construction of the solution for the quadratic BSDE with mean reflection in Section 4, the following assumptions will be needed.

\( (H_1') \) For each \( (t, y) \in [0, T] \times \mathbb{R}, f(t, y, 0) \) is bounded by a constant \( L \), \( P \)-a.s.

\( (H_L) \) There exists a constant \( C > 0 \) such that for each \( t \in [0, T] \),

\[ |L_t(X) - L_t(Y)| \leq CE[X - Y|, \forall X, Y \in \mathcal{L}^2. \]

Remark 1. Assume that \((H_L)\) holds true. Suppose that \( \ell \) is a bi-Lipschitz function in \( x \), i.e., there exist some constants \( 0 < c_\ell \leq C_\ell \) such that, \( P \)-a.s., for all \( t \in [0, T] \) and for all \( x, y \in \mathbb{R} \),

\[ c_\ell|x - y| \leq |\ell(t, x) - \ell(t, y)| \leq C_\ell|x - y|. \]

Then \((H_L)\) holds true with \( C = \frac{C_\ell}{c_\ell} \) (see also [6]).

As in [6], we study deterministic flat solutions of quadratic BSDEs with mean reflection.

Definition 2.2. A triple of processes \((Y, Z, K) \in \mathcal{S}^\infty \times BMO \times A^D \) is said to be a deterministic solution to the BSDE (3) with mean reflection if it ensures that the equation (3) holds true. A solution is said to be "flat" if moreover that \( K \) increases only when needed, i.e., when we have

\[ \int_0^T E[\ell(t, Y_t)]dK_t = 0. \]

The first main result of this paper is on the existence and uniqueness of the local solution for the quadratic BSDE with mean reflection, which reads as follows:

Theorem 2.3. Assume that \((H_\ell) - (H_f) - (H_t) - (H_L)\) hold. Then, there exists a sufficiently large constant \( A > 0 \) and a constant \( 0 < \delta^A \leq T \) depending only on \( A, C, L \) and \( \lambda \), such that for any \( h \in (0, \delta^A) \), the quadratic BSDE (3) with mean reflection admits a unique deterministic flat solution \((Y, Z, K) \in \mathcal{S}^\infty_{[T-h, T]} \times BMO_{[T-h, T]} \times A^D_{[T-h, T]} \) such that

\[ \|Y\|_{\mathcal{S}^\infty_{[T-h, T]}} \leq A. \]
Moreover, we stitch local solutions and obtain the solvability of the quadratic BSDE (3) on the whole time interval under an additional condition on the generator $f$.

**Theorem 2.4.** Assume that $\left(H_\xi\right) - \left(H_f\right) - \left(H_f'\right) - \left(H_\ell\right) - \left(H_L\right)$ hold. Then the quadratic BSDE (3) with mean reflection has a unique deterministic flat solution $(Y, Z, K) \in \mathcal{S}_\infty \times \text{BMO} \times \mathcal{A}^D$ on $[0, T]$. Moreover, there exists a uniform bound $L$ depending only on $C, L, \lambda$ and $T$ such that

$$\|Y\|_{\mathcal{S}_\infty} \leq L.$$

3. **A simple case study.** In this section, we consider a simple case where the generator has the following particular structure:

$$Y_t = \xi + \int_t^T [a_s Y_s + f(s, Z_s)] ds + \int_t^T Z_s dB_s + K_T - K_t; \quad \forall t \in [0, T],$$

(5)

and $a$ is a deterministic and bounded measurable function. For convenience, we rewrite Assumption $(H_f)$ as follows:

$(H_f')$ The driver $f : [0, T] \times \Omega \times \mathbb{R}^d \to \mathbb{R}$ is a $\mathcal{P} \times \mathcal{B}(\mathbb{R}^d)$-measurable map such that

1. For each $t \in [0, T]$, $f(t, 0)$ is bounded by some constant $L$, $\mathbb{P}$-a.s.
2. There exists some constant $\lambda \geq 0$ such that, $\mathbb{P}$-a.s., for all $t \in [0, T]$ and for all $z, q \in \mathbb{R}^d$

$$|f(t, z) - f(t, q)| \leq \lambda(1 + |z| + |q|)|z - q|.$$

**Theorem 3.1.** Assume that $(H_\xi) - (H_f^A) - (H_\ell) - (H_L)$ hold. Then the quadratic BSDE (5) with mean reflection has a unique deterministic flat solution $(Y, Z, K) \in \mathcal{S}_\infty \times \text{BMO} \times \mathcal{A}^D$.

**Proof.** It suffices to prove the case where $a_s = 0$ for each $t \in [0, T]$. Indeed, denote $A_t = \int_0^t a_s ds$ for each $t \in [0, T]$. Then it is easy to check that $(Y, Z, K)$ is a deterministic flat solution to the BSDE (5) with mean reflection if and only if

$$(Y_t^A, Z_t^A, K_t^A) = \left( e^{A_t} Y_t, e^{A_t} Z_t, \int_0^t e^{A_s} dK_s \right)$$

is a deterministic flat solution to the BSDE (5) associated with the parameters

$$\xi^A = e^{A_T} \xi, \quad f^A(t, z) = e^{A_t} f(t, e^{-A_t} z), \quad \ell^A(t, y) = \ell(t, e^{-A_t} y).$$

We shall prove the existence and the uniqueness separately.

**Step 1.** Existence. Consider the following standard quadratic BSDE on the time interval $[0, T]$:  

$$\overline{Y}_t = \xi + \int_t^T f(s, Z_s) ds - \int_t^T Z_s dB_s. \quad (6)$$

By [7] or [28], the equation (6) has a unique solution $(\overline{Y}, \overline{Z}) \in \mathcal{S}_\infty \times \text{BMO}$, which implies that

$$\mathbb{E} \left[ \left| \int_0^T |Z_s|^2 ds \right|^p \right] < \infty, \quad \forall p \geq 1.$$
Then we recall Assumption \((H_f^*)\) and obtain that
\[
\mathbb{E} \left[ \int_0^T |f(s, Z_s)| ds \right]^2 \leq \mathbb{E} \left[ \int_0^T (|f(s, 0)| + \lambda |Z_s| + \lambda |Z_s|^2) ds \right] \\
\leq \mathbb{E} \left[ \int_0^T \left( L + \frac{1}{2} \lambda + \frac{3}{2} \lambda |Z_s|^2 \right) ds \right] < \infty.
\]

Thus from Proposition 8 in \([6]\), the following simple BSDE with mean reflection
\[
\begin{cases}
\tilde{Y}_t = \xi + \int_t^T f(s, Z_s) ds - \int_t^T \tilde{Z}_s dB_s + \tilde{K}_T - \tilde{K}_t; \\
\mathbb{E}[\ell(t, \tilde{Y}_t)] \geq 0,
\end{cases}
\forall t \in [0, T],
\]
admits a unique deterministic flat solution \((\tilde{Y}, \tilde{Z}, \tilde{K})\) such that
\[
\mathbb{E} \left[ \sup_{s \in [0, T]} |\tilde{Y}_s|^2 + \int_0^T |\tilde{Z}_s|^2 ds \right] < \infty.
\]

Moreover, for each \(t \in [0, T]\) we have
\[
\tilde{K}_t = \sup_{0 \leq s \leq T} L_s(\tilde{Y}_s) - \sup_{t \leq s \leq T} L_s(\tilde{Y}_s) \quad \text{and} \quad \tilde{Y}_t = \mathbb{E}_t \left[ \xi + \int_t^T f(s, Z_s) ds \right].
\]

Consequently, \((\tilde{Y}_t - (\tilde{K}_T - \tilde{K}_t), \tilde{Z}_t)_{0 \leq t \leq T}\) and \((\tilde{Y}_t, \tilde{Z}_t)_{0 \leq t \leq T}\) are both solutions to the following standard BSDE on the time interval \([0, T]\),
\[
\tilde{Y}_t = \xi + \int_t^T f(s, Z_s) ds - \int_t^T \tilde{Z}_s dB_s.
\]

By the uniqueness of solutions to BSDE (see \([31]\)), we deduce that
\[
(\tilde{Y}_t, \tilde{Z}_t) = (\tilde{Y}_t - (\tilde{K}_T - \tilde{K}_t), \tilde{Z}_t), \forall t \in [0, T].
\]

Since \(\tilde{K}\) is a deterministic continuous process, we have \(\tilde{Y} \in \mathcal{S}^\infty\). Thus \((\tilde{Y}, \tilde{Z}, \tilde{K})\) is a deterministic flat solution to the BSDE \((5)\) with mean reflection.

**Step 2. Uniqueness.** Assume that \((Y^*, Z^*, K^*)\) is also a deterministic flat solution to the BSDE \((5)\) with mean reflection. Then we obtain that \((Y_t - (K_T - K_t), Z_t)_{0 \leq t \leq T}\) and \((Y_t^* - (K_T^* - K_t^*), Z_t)_{0 \leq t \leq T}\) are both solutions to the standard quadratic BSDE \((6)\). It follows from the uniqueness of solution for the quadratic BSDE that \(Z = Z^*\) on \([0, T]\).

Consequently, \((Y, Z, K)\) and \((Y^*, Z^*, K^*)\) are both deterministic flat solutions to the following simple BSDE with mean reflection:
\[
\begin{cases}
\tilde{Y}_t = \xi + \int_t^T f(s, Z_s) ds - \int_t^T \tilde{Z}_s dB_s + \tilde{K}_T - \tilde{K}_t; \\
\mathbb{E}[\ell(t, \tilde{Y}_t)] \geq 0,
\end{cases}
\forall t \in [0, T].
\]

Thus recalling Proposition 8 in \([6]\) again, we derive that \((Y, Z, K) = (Y^*, Z^*, K^*)\), which ends the proof.

We also have the minimality of the deterministic flat solution and the mean comparison theorem.

**Proposition 1.** Suppose that \(\ell\) is strictly increasing, then a deterministic flat solution \((Y, Z, K)\) is minimal among all the deterministic solutions of the BSDE \((5)\) with mean reflection.
Proof. By a similar argument for proving Theorem 12 in [6], we have the desired result.

**Theorem 3.2.** Suppose that \((Y^i, Z^i, K^i), i = 1, 2,\) is the deterministic flat solution to the BSDE (5) with parameters \((\xi^i, f^i, t)\) satisfying Assumptions \((H_\xi) - (H_L),\) where \(C \leq 1.\) If one of the following conditions holds:

1. \(\mathbb{E}[\xi^1] \geq \mathbb{E}[\xi^2]\) and \(\mathbb{E}[f^1] \geq \mathbb{E}[f^2],\) where \(\xi^i\) satisfies Assumption \((H_\xi)\) except that the boundedness is replaced by the square integrability and \(f^i\) is independent of the component \(Z, i = 1, 2,\) and satisfies \((H_f^i)(1)\);

2. \(\xi^i = \xi + c^i\) for some real numbers \(c^1 \geq c^2\) and \(f^1 = f^2,\) where \(\xi\) satisfies \((H_\xi)\) and \(f^i\) satisfies \((H_f^i), i = 1, 2,\)

then \(\mathbb{E}[Y^1_t] \geq \mathbb{E}[Y^2_t],\) for each \(t \in [0, T].\)

**Proof.** We only prove the second case, since the first can be shown in a similar fashion. We remark that in the first case, the solvability of BSDEs with mean reflection is given by [6]. Without loss of generality, assume that \(a_\xi = 0\) for each \(s \in [0, T].\) Then \((Y^i_t - c^i - K^i_t + K^i_t, Z^i_t)_{0 \leq t \leq T}, i = 1, 2,\) are both solutions to the standard quadratic BSDE (6), which implies that \(Z^1 = Z^2.\) Since for each \(t \in [0, T],\)

\[
K^1_T - K^1_t = \sup_{t \leq s \leq T} L_s(X^1_s) \quad \text{with} \quad X^1_t := \mathbb{E}_t \left[ \xi + c^1 + \int_t^T f(s, Z^1_s)ds \right],
\]

we deduce that

\[
K^2_T - K^1_T - (K^1_T - K^1_t) \leq \sup_{t \leq s \leq T} (L_s(X^2_s) - L_s(X^1_s)) \leq C(c^1 - c^2),
\]

from which we obtain that for each \(t \in [0, T],\)

\[
\mathbb{E}[Y^1_t - Y^2_t] = c^1 - c^2 + K^1_T - K^1_t - (K^2_T - K^1_T) \geq (1 - C)(c^1 - c^2) \geq 0.
\]

The proof is complete.

Remark that in general we cannot have the pointwise comparison theorem for BSDEs with mean reflection. Indeed, let us look at the following counter-example.

**Example 3.3.** Consider the following BSDE with mean reflection:

\[
\begin{cases}
Y_t = \xi - \int_t^T Z_s dB_s + K_T - K_t, & \forall t \in [0, T], \\
\mathbb{E}[Y_t] \geq 2T - t,
\end{cases}
\]

where \(B\) is a 1-dimensional Brownian motion. Suppose that \((Y^{(1)}, Z^{(1)}, K^{(1)})\) and \((Y^{(2)}, Z^{(2)}, K^{(2)})\) are the deterministic flat solutions to equation (7) corresponding to the terminal conditions \(\xi^{(1)} = |B_T|^2\) and \(\xi^{(2)} = \frac{3}{2}|B_T|^2,\) respectively. Then, for \(t \in [0, T],\) the solutions of these two equations can be defined by

\[
Y^{(1)}_t = |B_t|^2 + 2T - 2t, \quad Z^{(1)}_t = 2B_t, \quad K^{(1)}_t = t;
\]

\[
Y^{(2)}_t = \frac{3}{2}|B_t|^2 + \max\left(2T - \frac{5}{2}t, \frac{3}{2}T - \frac{3}{2}t\right), \quad Z^{(2)}_t = 3B_t, \quad K^{(2)}_t = \min\left(t, \frac{T}{2}\right).
\]

Note that \(\xi^{(1)} < \xi^{(2)}\) and \(\mathbb{P}(Y^{(1)}_t > Y^{(2)}_t) = \mathbb{P}(|B_t|^2 < t) > 0, \forall t \in \left(0, \frac{T}{2}\right).\) However, we have

\[
\mathbb{P}(Y^{(1)}_t > Y^{(2)}_t) = \mathbb{P}(|B_t|^2 < t) > 0, \forall t \in \left(0, \frac{T}{2}\right).
\]
4. General case. In this section, we study the general quadratic BSDE (3) with mean reflection. Namely, we consider this type of equation under Assumption \( (H_f) \) instead of \( (H_f') \). As the first step, we prove the existence and uniqueness of the solution on a small time interval, which is called local solution. Then we stitch local solutions to build the global solution.

In order to construct a contractive map to find the local solution on a small time interval \([T-h,T]\), we assume in addition \( (H_L) \) in this section. Here \( h \in (0,1) \) will be determined later. Since \( L_t(0) \) is continuous in \( t \), without loss of generality we assume that \( |L_t(0)| \leq L \) for each \( t \in [0,T] \), see \([6]\).

For each \( U \in S_{[T-h,T]}^\infty \), it follows from Theorem 3.1 that the following quadratic BSDE with mean reflection
\[
\begin{align*}
Y^U_t &= \xi + \int_t^T f(s, U_s, Z^U_s) ds - \int_t^T Z^U_s dB_s + K^U_T - K^U_t; \\
\mathbb{E}[\ell(t, Y^U_t)] &\geq 0,
\end{align*}
\]
has a unique deterministic flat solution \((Y^U, Z^U, K^U) \in S_{[T-h,T]}^\infty \times BMO_{[T-h,T]} \times A_{[T-h,T]}^D\). Then we define the purely quadratic solution map \( \Gamma : U \to \Gamma(U) \) by
\[
\Gamma(U) := Y^U, \quad \forall U \in S_{[T-h,T]}^\infty.
\]
In order to show that \( \Gamma \) is contractive, for each real number \( A \geq A_0 \) we consider the following set:
\[
\mathcal{B}_A := \{ U \in S_{[T-h,T]}^\infty : \|U\|_{S_{[T-h,T]}^\infty} \leq A \},
\]

where
\[
A_0 = 2(C + 2)L + \frac{1}{2}C(\lambda + 4L + 4L\lambda)e^{2\delta L}.
\]

**Lemma 4.1.** Assume that \( (H_\xi) - (H_f) - (H_f) - (H_L) \) hold and \( A \geq A_0 \), where \( A_0 \) is defined by (10). Then there is a constant \( \delta > 0 \) depending only on \( L, \lambda, C, A \) such that for any \( h \in (0, \delta A] \), \( \Gamma(\mathcal{B}_A) \subset \mathcal{B}_A \).

**Proof.** In view of the proof of Theorem 3.1, we conclude that for each \( t \in [T-h,T] \),
\[
(Y^U_t, Z^U_t) = (\bar{Y}^U_t, (K^U_t - K^U_t), Z^U_t),
\]
where \((\bar{Y}^U_t, \bar{Z}^U_t) \in S_{[T-h,T]}^\infty \times BMO_{[T-h,T]} \) is the solution to the following standard BSDE on the time interval \([T-h,T]\)
\[
\bar{Y}^U_t = \xi + \int_t^T f(s, U_s, \bar{Z}^U_s) ds - \int_t^T \bar{Z}^U_s dB_s,
\]
and for each \( t \in [T-h,T] \),
\[
K^U_T - K^U_t = \sup_{t \leq s \leq T} L_s(\bar{Y}^U_s) \quad \text{with} \quad \bar{Y}^U_t = \mathbb{E}_t \left[ \xi + \int_t^T f(s, U_s, Z^U_s) ds \right].
\]
Consequently, we obtain that
\[
\|Y^U\|_{S_{[T-h,T]}^\infty} \leq \|\bar{Y}^U\|_{S_{[T-h,T]}^\infty} + \sup_{T-h \leq s \leq T} L_s(\bar{Y}^U_s).
\]
The remainder of the proof will be in two steps.

**Step 1.** The estimate of \( \bar{Y}^U \). Since \( \bar{Z}^U \in BMO_{[T-h,T]} \), we can find a vector process \( \beta \in BMO_{[T-h,T]} \) such that
\[
f(s, U_s, \bar{Z}^U_s) - f(s, U_s, 0) = \bar{Z}^U_s \beta_s, \quad \forall s \in [T-h,T].
\]
Then \( \widehat{B}_t := B_t - \int_{T-h}^T \beta_s ds \mathbb{1}_{\{t \geq T-h\}} \), defines a Brownian motion under the equivalent probability measure \( \widetilde{P} \) given by
\[
d\widetilde{P} := \delta(\beta \cdot B)_{T-h}^T d\mathbb{P}.
\]
Thus by the equation (11), we have
\[\overline{Y}_t^U = \mathbb{E}_t^{\widetilde{P}} \left[ \xi + \int_t^T f(s, U_s, 0) ds \right], \forall t \in [T-h, T],\]
which implies that
\[\|\overline{Y}_t^U\|_{S_{(T-h, T)}^\infty} \leq L + (L + \lambda A)h, \tag{13}\]
where we have used the fact that \( f(s, U_s, 0) \) is bounded by \( L + \lambda A \). Thus recalling Proposition 2.1 in [7], we have
\[\|Z_t^U\|_{BMO_{(T-h, T)}}^2 \leq 1 \left( 1 + \frac{4L}{\lambda} + 2\|\overline{Y}_t^U\|_{S_{(T-h, T)}^\infty} \right) e^{3\lambda\|\overline{Y}_t^U\|_{S_{(T-h, T)}^\infty}}. \tag{14}\]

**Step 2.** The estimate of \( Y^U \). Thanks to Assumption \((H_L)\), for each \( s \in [T-h, T] \) we have
\[|L_s(\overline{Y}_s^U) - L_s(0)| \leq CE\|\overline{Y}_s^U\|\|\overline{Y}_s^U\|_{S_{(T-h, T)}^\infty} \|
\]
Therefore from the definition \( X^U \) and Assumption \((H_f)\) we deduce that
\[
\sup_{T-h \leq s \leq T} L_s(\overline{Y}_s^U) \leq L + C \sup_{T-h \leq s \leq T} \mathbb{E} \left[ |\xi| + \int_s^T \left( |f(t, U_t, 0)| + 1 + 3\lambda|Z_t^U|^2 \right) dt \right] \\
\leq (C+1)L + Ch \left( L + \lambda A + \frac{1}{2}\lambda \right) + \frac{3}{2}C\lambda\|Z_t^U\|_{BMO_{(T-h, T)}}^2. \tag{15}\]
Then we define
\[\delta^A := \min \left( \frac{L}{L + \lambda A + \frac{1}{2}\lambda}, T \right). \tag{16}\]
Recalling equations (12), (13), (14) and (15), we derive that for each \( h \in (0, \delta^A) \),
\[\|Y^U\|_{S_{(T-h, T)}^\infty} \leq 2(C+2)L + \frac{1}{2}C\lambda + 3L + 3\|Z_t^U\|_{BMO_{(T-h, T)}}^2, \forall U^1, U^2 \in \mathcal{R}_A,\]
which is the desired result. \( \square \)

Now we show the contractive property of the purely quadratic solution map \( \Gamma \).

**Lemma 4.2.** Assume that \((H_\ell) - (H_f) - (H_L)\) hold and \( A \geq A_0 \), where \( A_0 \) is defined by (10). Then there exists a constant \( \delta^A \) such that \( 0 < \delta^A \leq \delta^A \) and for any \( h \in (0, \delta^A) \), we have
\[\|\Gamma(U^1) - \Gamma(U^2)\|_{S_{(T-h, T)}^\infty} \leq \frac{1}{2}\|U^1 - U^2\|_{S_{(T-h, T)}^\infty}, \forall U^1, U^2 \in \mathcal{R}_A.
\]

**Proof.** For each \( i = 1, 2 \), set
\[Y_t^{U^i} = \Gamma(U^i),\]
where \((Y_t^{U^1}, Z_t^{U^1}, K_t^{U^1})\) is the solution to the BSDE (8) with mean reflection associated with the data \( U^1 \). Applying Theorem 3.1 again, we conclude that for each \( t \in [T-h, T] \),
\[\left( Y_t^{U^i}, Z_t^{U^i} \right) = \left( \check{Y}_t^{U^i} + (K_t^{U^i} - K_t^{U^i}), \check{Z}_t^{U^i} \right), \tag{17}\]
where \((Y_{t}^{U^1}, Z_{t}^{U^1}) \in S_{[T-h,T]}^{\infty} \times BMO_{[T-h,T]}\) is the solution to the BSDE (11) associated with the data \(U^1\). Since for each \(t \in [T-h,T]\),

\[ K_{T}^{U^1} - K_{t}^{U^1} = \sup_{t \leq s \leq T} L_{s}(Y_{s}^{U^1}) \quad \text{and} \quad \hat{Y}_{t} = E_{t}\left[ \xi + \int_{t}^{T} f(s, U_{s}^{1}, Z_{s}^{U^1}) ds \right], \]

we have

\[ \sup_{T-h \leq t \leq T} \left( |K_{T}^{U^1} - K_{t}^{U^1}| - (K_{T}^{U^2} - K_{t}^{U^2}) \right) \leq \sup_{T-h \leq s \leq T} |L_{s}(Y_{s}^{U^1}) - L_{s}(Y_{s}^{U^2})| \]

\begin{equation}
\leq C \lambda E \left[ \int_{T-h}^{T} \left( |U_{r}^{1} - U_{r}^{2}| + (1 + |Z_{r}^{U^1}| + |Z_{r}^{U^2}|)|Z_{r}^{U^1} - Z_{r}^{U^2}| \right) dr \right].
\end{equation}

Applying Hölder’s inequality, we obtain

\[ E \left[ \int_{T-h}^{T} (1 + |Z_{r}^{U^1}| + |Z_{r}^{U^2}|)|Z_{r}^{U^1} - Z_{r}^{U^2}| dr \right] \]

\[ \leq \sqrt{3} E \left[ \int_{T-h}^{T} (1 + |Z_{r}^{U^1}|^2 + |Z_{r}^{U^2}|^2) dr \right]^{\frac{1}{2}} \|Z_{r}^{U^1} - Z_{r}^{U^2}\|_{BMO_{[T-h,T]}} \]

\begin{equation}
\leq \sqrt{3 + \frac{4A}{C \lambda}} \|Z_{r}^{U^1} - Z_{r}^{U^2}\|_{BMO_{[T-h,T]}},
\end{equation}

where the last inequality is deduced from the fact that \(\|Z_{r}^{U^1}\|_{BMO_{[T-h,T]}} \leq \frac{2A}{3C \lambda}\) (see (14)).

We recall the representation (17) and conclude that

\[ \|Y_{r}^{U^1} - Y_{r}^{U^2}\|_{s_{[T-h,T]}^{\infty}} \]

\[ \leq \|\hat{Y}_{r} - \hat{Y}_{r}^{U^2}\|_{s_{[T-h,T]}^{\infty}} + \sup_{T-h \leq t \leq T} |(K_{T}^{U^1} - K_{t}^{U^1}) - (K_{T}^{U^2} - K_{t}^{U^2})| \]

\begin{equation}
\leq \|\hat{Y}_{r} - \hat{Y}_{r}^{U^2}\|_{s_{[T-h,T]}^{\infty}} + C \lambda \left( h(U_{r}^{1} - U_{r}^{2})\|s_{[T-h,T]}^{\infty} + \sqrt{3 + \frac{4A}{C \lambda}} \|Z_{r}^{U^1} - Z_{r}^{U^2}\|_{BMO_{[T-h,T]}} \right).
\end{equation}

The remainder of the proof will be in two steps.

**Step 1.** The estimate of \(\|\hat{Y}_{r} - \hat{Y}_{r}^{U^2}\|_{s_{[T-h,T]}^{\infty}}\). By the linearization argument, we can find a vector process \(\hat{\beta} \in BMO_{[T-h,T]}\) such that

\[ f(s, U_{s}^{1}, Z_{s}^{U^1}) - f(s, U_{s}^{1}, Z_{s}^{U^2}) = (Z_{s}^{U^1} - Z_{s}^{U^2}) \hat{\beta}_{s}. \]

Then \(\hat{B}_{t} := B_{t} - \int_{T-h}^{t} \hat{\beta}_{s} ds 1_{\{t \geq T-h\}}\) defines a Brownian motion under the equivalent probability measure \(\hat{P}\) given by

\[ d\hat{P} := e'(\hat{\beta} \cdot B)_{T-h}^{T} dP. \]

Thus by equation (11), we have

\[ \hat{Y}_{t}^{U^1} - \hat{Y}_{t}^{U^2} = \hat{E}_{t} \left[ \int_{t}^{T} (f(s, U_{s}^{1}, Z_{s}^{U^1}) - f(s, U_{s}^{1}, Z_{s}^{U^2})) ds \right], \forall t \in [T-h,T], \]
which implies that
\[
\| Y_t^{U_0} - Y_t^{U_2} \|_{S^\infty_{[T-h, T]}} \leq \lambda h \| U^1 - U^2 \|_{S^\infty_{[T-h, T]}},
\]  
(21)

**Step 2. The estimate of** \( \| U^{1} - U^{2} \|_{BMO_{[T-h, T]}} \). Note that for each \( t \in [T - h, T] \),
\[
Y_t^{U_0} - Y_t^{U_2} = \int_t^T (f(s, U_1^s, Z_{s}^{U_1}) - f(s, U_2^s, Z_{s}^{U_2})) ds - \int_t^T (Z_{s}^{U_1} - Z_{s}^{U_2}) dB_s.
\]

Then applying Itô’s formula to \( Y_t^{U_0} - Y_t^{U_2} \), we have
\[
\| Z_t^{U_0} - Z_t^{U_2} \|_{BMO_{[T-h, T]}}^2 \leq 2 \sup_{\tau \in [T-h, T]} \mathbb{E}_\tau \left[ \int_\tau^T \| Y_s^{U_0} - Y_s^{U_2} \|_{S^\infty_{[T-h, T]}}^2 ds \right].
\]

By Assumption \((H_f)\), the inequalities (19) and (21), we deduce that for any \( h \in (0, \delta A) \),
\[
sup_{\tau \in [T-h, T]} \mathbb{E}_\tau \left[ \int_\tau^T \| Y_s^{U_0} - Y_s^{U_2} \|_{S^\infty_{[T-h, T]}}^2 ds \right] \leq \lambda^2 h^2 \| U^1 - U^2 \|_{BMO_{[T-h, T]}^2} + \lambda^2 h \| U^1 - U^2 \|_{S^\infty_{[T-h, T]}} \sup_{\tau \in [T-h, T]} \mathbb{E}_\tau \left[ \int_\tau^T (1 + |Z_s^{U_1}| + |Z_s^{U_2}|) |Z_s^{U_1} - Z_s^{U_2}| ds \right] \leq \lambda^2 h^2 (1 + 3\lambda^2 + 4A\alpha/C) \| U^1 - U^2 \|_{S^\infty_{[T-h, T]}}^2 + \frac{1}{4} \| Z_t^{U_0} - Z_t^{U_2} \|_{BMO_{[T-h, T]}}^2,
\]
which together with the previous inequality implies that
\[
\| Z_t^{U_0} - Z_t^{U_2} \|_{BMO_{[T-h, T]}} \leq 2 \lambda h \sqrt{1 + 3\lambda^2 + 4A\alpha/C} \| U^1 - U^2 \|_{S^\infty_{[T-h, T]}},
\]
(22)

We put the estimates (21) and (22) into (20) and obtain
\[
\| Y_t^{U_0} - Y_t^{U_2} \|_{S^\infty_{[T-h, T]}} \leq 8 \lambda h (\lambda^2 + 1)(A + C + 1) \| U^1 - U^2 \|_{S^\infty_{[T-h, T]}},
\]
where we note
\[
2\sqrt{1 + 3\lambda^2 + 4A\alpha/C} \sqrt{3\lambda^2 + 4A\alpha/C} \leq 1 + 6\lambda^2 + 8A\alpha/C.
\]

Now we define
\[
\tilde{\delta}^A := \min \left( \frac{1}{16\lambda(\lambda^2 + 1)(A + C + 1)}, \delta^A \right),
\]
(23)
and it is straightforward to check that for any \( h \in (0, \tilde{\delta}^A) \),
\[
\| Y_t^{U_0} - Y_t^{U_2} \|_{S^\infty_{[T-h, T]}} \leq \frac{1}{2} \| U^1 - U^2 \|_{S^\infty_{[T-h, T]}},
\]
which completes the proof.

Then we give the proof of Theorem 2.3.
Proof of Theorem 2.2. We take $A \geq A_0$ and choose $\hat{A}$ as (23). For $h \in (0, \hat{A})$, define $Y^0 = 0$ and by (11), define $(Y^1, Z^1, K^1) := (YY^0, ZY^0, KY^0)$. By recurrence, for each $i \in \mathbb{N}$, set

$$(Y_{i+1}, Z_{i+1}, K_{i+1}) := (YY^i, ZY^i, KY^i).$$

(24)

It follows from Lemma 4.2 that there exists $Y \in \mathcal{A}_A$ such that

$$\|Y^n - Y\|_{S_{[T-h,T]}^T} \to 0.$$

By (22) and (18), there exist $Z \in BMO_{[T-h,T]}$ and $K \in \mathcal{A}_{[T-h,T]}^D$ such that

$$\|Z^n - Z\|_{BMO_{[T-h,T]}} \to 0 \quad \text{and} \quad \|K^n - K\|_{S_{[T-h,T]}^T} \to 0. \quad (25)$$

By a standard argument, we have for each $t \in [T-h, T]$, in $\mathbb{L}^2$,

$$\int_t^T f(s, Y^n_s, Z^n_s) ds \to \int_t^T f(s, Y_s, Z_s) ds.$$

Thus, the triple $(Y, Z, K)$ is a solution to the BSDE (3) with mean reflection and we only need to prove that the solution $(Y, Z, K)$ is “flat”. Indeed, it is easy to check that

$$K_T - K_t = \lim_{n \to \infty} K^n_T - K^n_t = \lim_{n \to \infty} \sup_{t \leq s \leq T} L_s(\bar{Y}^n_s),$$

where $\bar{Y}^n_t := \mathbb{E}_t \left[ \xi + \int_t^T f(s, Y^n_s, Z^n_s) ds \right]$. By (24) and (25), we deduce

$$\|\bar{Y}^n - \bar{Y}\|_{S_{[T-h,T]}^T} \to 0,$$

where $\bar{Y}_t := \mathbb{E}_t \left[ \xi + \int_t^T f(s, Y_s, Z_s) ds \right]$. Therefore, $K_T - K_t = \sup_{t \leq s \leq T} L_s(\bar{Y}_s)$, which implies the “flatness”. Similarly to the Step 2 of the proof for Theorem 3.1, we deduce the uniqueness by recalling Theorem 9 in [6]. The proof is complete. \(\square\)

In what follows, we construct a global solution to the quadratic BSDE (3) with mean reflection on the whole interval $[0, T]$ by backward recursion in time, namely, we prove Theorem 2.4. To this end, we first observe that any local solution $(Y, Z, K)$ on $[T-h, T]$ of the BSDE (3) with mean reflection has a uniform estimate if we assume additionally ($H'_4$).

Lemma 4.3. Assume that ($H_4$), ($H_4'$), ($H_4$), ($H_4'$) hold and the BSDE (3) with mean reflection has a local solution $(Y, Z, K) \in S_{[T-h,T]}^T \times BMO_{[T-h,T]} \times \mathcal{A}_{[T-h,T]}^D$ on $[T-h, T]$ for some $0 < h \leq T$. Then there exists a constant $\mathcal{L}$ depending only on $C$, $L$, $\lambda$ and $T$ such that

$$\|Y\|_{S_{[T-h,T]}^T} \leq \mathcal{L}.$$

Proof. Note that

$$(Y_t, Z_t) = (\bar{Y}_t + (K_T - K_t), Z_t), \quad \forall t \in [T-h, T].$$

Then the couple $(\bar{Y}, Z) \in S_{[T-h,T]}^T \times BMO_{[T-h,T]}$ is the solution to the following standard quadratic BSDE on $[T-h, T]$: 

$$\bar{Y}_t = \xi + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z_s dB_s.$$
Similarly to (12), we obtain that for each \( t \in [T - h, T] \),
\[
\|Y_t\|_{L^\infty} \leq \|\overline{Y}_t\|_{L^\infty} + \sup_{t \leq s \leq T} L_s(X_s),
\]
where
\[
\overline{Y}_t := E_t \left[ \xi + \int_t^T f(s, Y_s, Z_s) ds \right].
\]
By Assumption \((H'_f)\), we have
\[
\|Y_t\|_{L^\infty} \leq \|\overline{Y}_t\|_{L^\infty} \leq (C + 1)L + CE \left[ \int_t^T \left( |f(r, Y_r, 0)| + \frac{1}{2} \lambda + \frac{3}{2} |Z_r|^2 \right) dr \right]
\leq \|\overline{Y}_t\|_{L^\infty} \leq (C + 1) + C \left( L + \frac{1}{2} \lambda \right) T + \frac{3}{2} C \|Z\|_{BMO_{(T - h, T)}}^2. \tag{26}
\]
We recall Proposition 2.2 in [7] to have
\[
|\overline{Y}_t| \leq L(T + 1)e^{\lambda T} := L^1, \quad \forall t \in [T - h, T]. \tag{27}
\]
Recalling again Proposition 2.1 in [7], we have
\[
\|Z\|_{BMO_{(T - h, T)}}^2 \leq \frac{L^1}{3} \left( 1 + \frac{4L}{\lambda} + 2 L^1 \right) e^{3\lambda T} := L^2. \tag{28}
\]
We derive from the inequalities (26), (27) and (28) that for each \( t \in [T - h, T] \),
\[
\|Y_t\|_{L^\infty} \leq L^1 + (C + 1)L + C \left( L + \frac{1}{2} \lambda \right) T + \frac{3}{2} C L^2 := L,
\]
which is the desired result.

\[\square\]

**Proof of Theorem 2.4.** We treat with the existence and the uniqueness separately.

**Step 1.** Existence. Define \( \overline{A}_0 = 2(C + 2) L + \frac{1}{2} C(\lambda + 4L + 4L\lambda)e^{3\lambda T} \). Then by Theorem 2.3, there exists some constant \( \overline{h} > 0 \) depending only on \( L, \lambda \) and \( C \) together with \( \overline{A}_0 \) such that the quadratic BSDE (3) with mean reflection admits a unique deterministic flat solution \((Y^1, Z^1, K^1) \in S_{\overline{h}, T}^\infty \times BMO_{\overline{h}, T} \times A_{\overline{T}, \overline{h}, T}^D\) on the time interval \([T - \overline{h}, T] \). Furthermore, it follows from Lemma 4.3 that
\[
||Y^1||_{S_{\overline{h}, T}^\infty} \leq L.
\]

Next we take \( T - \overline{h} \) as the terminal time and apply Theorem 2.3 again to find the unique deterministic flat solution of the BSDE (3) with mean reflection \((Y^2, Z^2, K^2) \in S_{\overline{h}, T}^\infty \times BMO_{\overline{h}, T} \times A_{\overline{T}, \overline{h}, T}^D\) on the time interval \([T - 2\overline{h}, T - \overline{h}] \). Let us set
\[
Y_t = \sum_{i=1}^{2} Y^i_t|_{[T - \overline{h}, T - (i-1)\overline{h}]} + Y^i_T(T), \quad Z_t = \sum_{i=1}^{2} Z^i_t|_{[T - \overline{h}, T - (i-1)\overline{h}]} + Z^i_T(T)
\]
on \([T - 2\overline{h}, T]\) and \( K_t = K^i_t|_{[T - 2\overline{h}, T - \overline{h}]} \), \( K_t = K^2_{T - \overline{h}} + K^1_t \) on \([T - \overline{h}, T]\). One can easily check that \((Y, Z, K) \in S_{\overline{h}, T}^\infty \times BMO_{\overline{h}, T} \times A_{\overline{T}, \overline{h}, T}^D\) is a deterministic flat solution to BSDE (3) with mean reflection. By Lemma 4.3 again, it yields that
\[
||Y||_{S_{\overline{h}, T}^\infty} \leq L.
\]
Furthermore, we repeat this procedure so that we can build a deterministic flat solution \((Y, Z, K) \in S^\infty \times BMO \times A^D\) to the quadratic BSDE (3) with mean reflection on \([0, T]\). Moreover, it follows from Lemma 4.3 that \(\|Y\|_{L^\infty} \leq T\).

**Step 2. Uniqueness.** The uniqueness of the global solution on \([0, T]\) is inherited from the uniqueness of local solution on each time interval. Indeed, for each global solution \((Y, Z, K)\) to the quadratic BSDE (3) with mean reflection, it is easy to check that \((Y 1_{[T-\hat{h}, T-\hat{h}]}, Z 1_{[T-\hat{h}, T-\hat{h}]}, (K, -K_{T-\hat{h}}) 1_{[T-\hat{h}, T-\hat{h}]})\) defines a deterministic flat solution to the BSDE (3) with mean reflection associated with the terminal value \(Y_{T-\hat{h}}\) on the time interval \([T - \hat{h}, T - \hat{h}]\), where \(0 \leq \hat{h} \leq \hat{h} \leq T\). The proof is complete.

**Remark 2.** Since the component \(K\) is a deterministic process, by a truncation argument and the Malliavin calculus technique, we can find a uniform bound for \(Z\) when the corresponding Malliavin derivatives are bounded, similar to Cheridito and Nam [13]. We state the result in the Appendix and leave the proof to interested readers. We remark that under the boundedness assumption of the corresponding Malliavin derivatives, the boundedness of the terminal condition is not necessary.

**Appendix.** Let us recall usual notations about Malliavin calculus, which can be found in [30] and [13]. We denote by \(D_t \xi, 0 \leq t \leq T\) the Malliavin derivative of a Malliavin differentiable random variable \(\xi\), by \(\mathcal{D}^{1,2}\) the completion of the class of \(\mathbb{R}\)-valued smooth random variables \(\xi\) with respect to the norm

\[
\|\xi\|_{\mathcal{D}^{1,2}} = \left(\mathbb{E} \left[ \xi^2 + \int_0^T |D_t \xi|^2 dt \right] \right)^{1/2},
\]

by \(\mathcal{H}^p, p \geq 1\) the space of all progressively measurable processes \(Y\) taking values in \(\mathbb{R}\) such that

\[
\|Y\|_{\mathcal{H}^p} = \mathbb{E} \left[ \left( \int_0^T |Y_s|^2 ds \right)^{p/2} \right] < \infty,
\]

and by \(\mathcal{L}^{1,2}_a\) the space of all processes \(Y \in \mathcal{H}^2\) such that \(Y_t \in \mathcal{D}^{1,2}\) for each \(t \in [0, T]\), the process \(DY_t\) admits a square integrable progressively measurable version and

\[
\|Y\|_{\mathcal{L}^{1,2}_a} = \|Y\|_{\mathcal{H}^2} + \mathbb{E} \left[ \int_0^T \int_0^T |D_r Y_t|^2 dr dt \right] < \infty.
\]

Then we consider the following assumptions on the parameters:

\((\tilde{H}_\xi)\) The terminal condition \(\xi \in \mathcal{D}^{1,2}\) satisfies \(|D_t \xi| \leq L, dt \otimes d\mathbb{P}\text{-a.e.}\) for all \(t \in [0, T]\) and \(\mathbb{E}|\ell(T, \xi)| \geq 0\),

\((\tilde{H}_f)\) For each pair \((y, z) \in \mathbb{R} \times \mathbb{R}^d\) with

\[
|z| \leq Q := \sqrt{d} \left( Le^{\lambda T} + \int_0^T q_t e^{\lambda t} dt \right),
\]

it holds that

1. \(f(\cdot, y, z) \in \mathcal{L}^{1,2}_a\) and \(|D_r f(t, y, z)| \leq q_t, dr \otimes d\mathbb{P}\text{-a.e.}\) for all \(t \in [0, T]\) and some Borel-measurable function \(q : [0, T] \to [0, \infty)\) satisfying \(\int_0^T |q_t|^2 dt < \infty\),
2. for every \( r \in [0, T] \), and for all \( y, p, q \in \mathbb{R} \),
\[
|D_r f(t, y, z) - D_r f(t, p, q)| \leq K'_r(|y - p| + |z - q|), \quad \forall |z|, |q| \leq Q,
\]
for all \( t \in [0, T] \) and some non-negative process \( K' \) in \( \mathcal{H}^4 \).

**Theorem 4.4.** Assume \((\tilde{H}_E) - (H_f) - (\tilde{H}_f) - (H_L) \) hold. Then quadratic BSDE (3) with mean reflection has a unique deterministic flat solution \((Y, Z, K)\) such that \( Y \) is a continuous adapted process satisfying
\[
\mathbb{E}\left[ \sup_{s \in [0, T]} |Y_s|^2 \right] < \infty, \quad Z \text{ is a bounded progressively measurable process and } K \in \mathcal{A}^D.
\]
Moreover, it holds that
\[
|Z_t| \leq \sqrt{d} \left( L e^{\lambda(T-t)} + \int_t^T q_s e^{-\lambda(t-s)} ds \right), \quad dt \otimes d\mathbb{P} \text{-a.e.}
\]

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