A quantum statistical system with energy dissipation is studied. Its statistics is governed by random complex-valued non-Hermitean Hamiltonians belonging to complex Ginibre ensemble. The eigenenergies are shown to form stable structure. Analogy of Wigner and Dyson with system of electrical charges is drawn.

1 Summary

A complex quantum system with energy dissipation is considered. The quantum Hamiltonians belong the complex Ginibre ensemble. The complex-valued eigenenergies are random variables. The second differences are complex-valued random variables. The second differences have their real and imaginary parts and also radii (moduli) and main arguments (angles). For $N=3$ dimensional Ginibre ensemble the distributions of above random variables are provided whereas for generic $N$-dimensional Ginibre ensemble second difference’s, radius’s and angle’s distributions are analytically calculated. The law of homogenization of eigenenergies is formulated. The analogy of Wigner and Dyson of Coulomb gas of electric charges is studied. The stabilisation of system of electric charges is dealt with.

2 Introduction

We study generic quantum statistical systems with energy dissipation. The quantum Hamiltonian operator $H$ is in given basis of Hilbert’s space a matrix with random elements $H_{ij}$. The Hamiltonian $H$ is not Hermitean operator, thus its eigenenergies are complex-valued random variables. We assume that distribution of $H_{ij}$ is governed by Ginibre ensemble, $H$ belongs to general linear Lie group $GL(N, \mathbb{C})$, where $N$ is dimension and $\mathbb{C}$ is
complex numbers field. Since $H$ is not Hermitean, therefore quantum system is dissipative system. Ginibre ensemble of random matrices is one of many Gaussian Random Matrix ensembles GRME. The above approach is an example of Random Matrix theory RMT\textsuperscript{1,2,3}. The other RMT ensembles are for example Gaussian orthogonal ensemble GOE, unitary GUE, symplectic GSE, as well as circular ensembles: orthogonal COE, unitary CUE, and symplectic CSE. The distributions of the eigenenergies $Z_1, ..., Z_N$ for $N \times N$ Hamiltonian matrices is given by Jean Ginibre’s formula\textsuperscript{1,2,4,5}:

\[
P(z_1, ..., z_N) = \prod_{j=1}^{N} \frac{1}{\pi \cdot j!} \prod_{i<j}^{N} |z_i - z_j|^2 \cdot \exp(- \sum_{j=1}^{N} |z_j|^2),
\]

where $z_i$ are complex-valued sample points ($z_i \in \mathbb{C}$). For Ginibre ensemble we define complex-valued spacings $\Delta^1 Z_i$ and second differences $\Delta^2 Z_i$:

\[
\Delta^1 Z_i = Z_{i+1} - Z_i, \quad i = 1, ..., (N - 1),
\]

\[
\Delta^2 Z_i = Z_{i+2} - 2Z_{i+1} + Z_i, \quad i = 1, ..., (N - 2).
\]

The $\Delta^2 Z_i$ are extensions of real-valued second differences

\[
\Delta^2 E_i = E_{i+2} - 2E_{i+1} + E_i, \quad i = 1, ..., (N - 2),
\]

of adjacent ordered increasingly real-valued energies $E_i$ defined for GOE, GUE, GSE, and Poisson ensemble PE (where Poisson ensemble is composed of uncorrelated randomly distributed eigenenergies\textsuperscript{6,7,8,9,10,11}.

There is an analogy of Coulomb gas of unit electric charges pointed out by Eugene Wigner and Freeman Dyson. A Coulomb gas of $N$ unit charges moving on complex plane (Gauss’s plane) $\mathbb{C}$ is considered. The vectors of positions of charges are $z_i$ and potential energy of the system is:

\[
U(z_1, ..., z_N) = - \sum_{i<j} \ln |z_i - z_j| + \frac{1}{2} \sum_i |z_i^2|.
\]

If gas is in thermodynamical equilibrium at temperature $T = \frac{1}{k_B}$ ($\beta = \frac{1}{k_B T} = 2$, $k_B$ is Boltzmann’s constant), then probability density function of vectors of positions is $P(z_1, ..., z_N)$ Eq. (1). Complex eigenenergies $Z_i$ of quantum system are analogous to vectors of positions of charges of Coulomb gas. Moreover, complex-valued spacings $\Delta^1 Z_i$ are analogous to vectors of relative positions of electric charges. Finally, complex-valued second differences $\Delta^2 Z_i$ are analogous to vectors of relative positions of vectors of relative positions of electric charges.

The $\Delta^2 Z_i$ have their real parts $\text{Re}\Delta^2 Z_i$, and imaginary parts $\text{Im}\Delta^2 Z_i$, as well as radii (moduli) $|\Delta^2 Z_i|$, and main arguments (angles) $\text{Arg}\Delta^2 Z_i$. 

2
3 Second Difference Distributions

We define following random variables for $N=3$ dimensional Ginibre ensemble:

$$Y_1 = \Delta^2 Z_1, A_1 = \text{Re}Y_1, B_1 = \text{Im}Y_1,$$

$$R_1 = \left|Y_1\right|, \Phi_1 = \text{Arg}Y_1,$$

and for the generic $N$-dimensional Ginibre ensemble:

$$W_1 = \Delta^2 Z_1, P_1 = \left|W_1\right|, \Psi_1 = \text{Arg}W_1.$$  

Their distributions for are given by following formulae, respectively:

$$f_{Y_1}(y_1) = f_{(A_1,B_1)}(a_1,b_1) =$$

$$= \frac{1}{576\pi^{\frac{5}{2}}} (a_1^2 + b_1^2 + 24) \exp(-\frac{1}{6}(a_1^2 + b_1^2)),$$

is second difference distribution for 3-dimensional Ginibre ensemble.

$$f_{A_1}(a_1) = \frac{\sqrt{6}}{576\sqrt{\pi}} (a_1^4 + 6a_1^2 + 51) \cdot \exp(-\frac{1}{6}a_1^2),$$

$$f_{B_1}(b_1) = \frac{\sqrt{6}}{576\sqrt{\pi}} (b_1^4 + 6b_1^2 + 51) \cdot \exp(-\frac{1}{6}b_1^2),$$

$$f_{R_1}(r_1) = \Theta(r_1) \frac{1}{288} r_1 (r_1^4 + 24) \cdot \exp(-\frac{1}{6}r_1^2),$$

$$f_{\Phi_1}(\phi_1) = \frac{1}{2\pi}, \Phi_1 \in [0,2\pi],$$

are real part’s, imaginary part’s, modulus’s and angle’s probability distribu-
tions for 3-dimensional Ginibre ensemble, where $\Theta(r_1)$ is Heaviside (step) function. Notice, that the angle $\Phi_1$ has uniform distribution. Next, second differ-
ence’s distribution for $N$-dimensional Ginibre ensemble reads $P_3(w_1)$:

$$P_3(w_1) =$$

$$= \pi^{-3} \sum_{j_1=0}^{N-1} \sum_{j_2=0}^{N-1} \sum_{j_3=0}^{N-1} \frac{1}{j_1!j_2!j_3!} I_{j_1,j_2,j_3}(w_1),$$

$$I_{j_1,j_2,j_3}(w_1) =$$

$$= 2^{-2j_2} \frac{\partial^{j_1+j_2+j_3}}{\partial^{j_1} \lambda_1 \partial^{j_2} \lambda_2 \partial^{j_3} \lambda_3} F(w_1,\lambda_1,\lambda_2,\lambda_3)|_{\lambda_i=0},$$

3
\[ F(w_1, \lambda_1, \lambda_2, \lambda_3) = A(\lambda_1, \lambda_2, \lambda_3) \exp[-B(\lambda_1, \lambda_2, \lambda_3)|w_1|^2], \]

\[ A(\lambda_1, \lambda_2, \lambda_3) = \frac{(2\pi)^2}{(\lambda_1 + \lambda_2 - \frac{4}{\beta} \cdot (\lambda_1 + \lambda_3 - \frac{4}{\beta}) - (\lambda_1 - 1)^2}, \]

\[ B(\lambda_1, \lambda_2, \lambda_3) = (\lambda_1 - 1) \cdot \frac{2\lambda_1 - \lambda_2 - \lambda_3 + \frac{1}{\beta}}{2\lambda_1 + \lambda_2 + \lambda_3 - \frac{1}{\beta}}. \]

Finally,

\[ f_{\rho_1}(\varrho_1) = 2\pi P_3(\varrho_1) = 2\pi \sum_{j_1=0}^{N-1} \sum_{j_2=0}^{N-1} \sum_{j_3=0}^{N-1} \frac{1}{j_1!j_2!j_3!} I_{j_1,j_2,j_3}(\varrho_1), \]

\[ f_{\psi_1}(\psi_1) = \frac{1}{2\pi}, \psi_1 \in [0, 2\pi], \]

are modulus’s and angle’s probability distributions for \(N\)-dimensional Ginibre ensemble. Notice, that again the angle \(\Psi_1\) has uniform distribution.

4 Conclusions

We compare second difference distributions for different ensembles by defining following dimensionless second differences:

\[ C_\beta = \frac{\Delta^2 E_1}{<S_\beta>}, \]

\[ X_1 = \frac{A_1}{<R_1>}, \]

where \(<S_\beta>\) are the mean values of spacings for GOE(3) (\(\beta = 1\)), for GUE(3) (\(\beta = 2\)), for GSE(3) (\(\beta = 4\)), for PE (\(\beta = 0\)) \(^6,7,8,9,10,11\), and \(<R_1>\) is mean value of radius \(R_1\) for \(N=3\) dimensional Ginibre ensemble\(^12\).

On the basis of comparison of results for Gaussian ensembles, Poisson ensemble, and Ginibre ensemble we formulate\(^5,7,8,9,10,11,12\):

**Homogenization Law:** Random eigenenergies of statistical systems governed by Hamiltonians belonging to Gaussian orthogonal
ensemble, to Gaussian unitary ensemble, to Gaussian symplectic ensemble, to Poisson ensemble, and finally to Ginibre ensemble tend to be homogeneously distributed.

It can be restated mathematically as follows:

If $H \in$ GOE, GUE, GSE, PE, Ginibre ensemble, then $\text{Prob}(D) = \max$, where $D$ is random event corresponding to vanishing of second difference’s probability distributions at the origin.

Both of above formulation follow from the fact that the second differences’ distributions assume global maxima at origin for above ensembles $6, 7, 8, 9, 10, 11, 12$.

For Coulomb gas’s analogy the vectors of relative positions of vectors of relative positions of charges statistically most probably vanish. It means that the vectors of relative positions tend to be equal to each other. Thus, the relative distances of electric charges are most probably equal. We call such situation stabilisation of structure of system of electric charges on complex plane.

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