Improving Human Decision-Making by Discovering Efficient Strategies for Hierarchical Planning

Saksham Consul¹ · Lovis Heindrich¹ · Jugoslav Stojcheski¹ · Falk Lieder¹, ²

Received: date / Accepted: date

Abstract To make good decisions in the real world people need efficient planning strategies because their computational resources are limited. Knowing which planning strategies would work best for people in different situations would be very useful for understanding and improving human decision-making. But our ability to compute those strategies used to be limited to very small and very simple planning tasks. To overcome this computational bottleneck, we introduce a cognitively-inspired reinforcement learning method that can overcome this limitation by exploiting the hierarchical structure of human behavior. The basic idea is to decompose sequential decision problems into two sub-problems: setting a goal and planning how to achieve it. This hierarchical decomposition enables us to discover optimal strategies for human planning in larger and more complex tasks than was previously possible. The discovered strategies outperform existing planning algorithms and achieve a super-human level of computational efficiency. We demonstrate that teaching people to use those strategies significantly improves their performance in sequential decision-making tasks that require planning up to eight steps ahead. By contrast, none of the previous approaches was able to improve human performance on these problems. These findings suggest that our cognitively-informed approach makes it possible to leverage reinforcement learning to improve human decision-making in complex sequential decision-problems. Future work can leverage our method to develop decision support systems that improve human decision making in the real world.

Keywords: decision-making; planning; automatic strategy discovery; reinforcement learning; resource rationality; boosting

Acknowledgements: This project was funded by grant number CyVy-RF-2019-02 from the Cyber Valley Research Fund. The authors would like to thank Yash Rah Jain, Frederic Becker, Aashay Mehta, and Julian Skirzynski for helpful discussions.

¹ Max Planck Institute for Intelligent Systems, Tübingen, Germany, 72076
² E-mail: falk.lieder@tuebingen.mpg.de, ORCID: 0000-0003-2746-6110
1 Introduction

To make good decisions people often plan many steps ahead. This requires efficient planning strategies because the number of possible action sequences grows exponentially with the number of steps and people’s cognitive resources are limited. Recent work has shown that teaching people clever decision strategies is a promising way to improve human decision-making (Hertwig and Grüne-Yanoff, 2017; Hafenbrädl et al., 2016); this approach is known as boosting. One of the bottlenecks of boosting is that discovering clever decision strategies that work well in the real world is very challenging and time consuming. For boosting to be effective the taught strategies have to be well-adapted to the decisions and environments in which people will use them (Simon, 1956; Gigerenzer and Selten, 2002; Todd and Gigerenzer, 2012). The cognitive modeling paradigm of resource-rational analysis (Lieder and Griffiths, 2020) can be used to mathematically define planning strategies that are optimally adapted to the problems people have to solve and the cognitive resources people can use to solve those problems (Callaway et al., 2018b, 2020). Knowing those strategies can be very useful for understanding and improving human decision-making (Callaway et al., 2018b, 2020; Lieder et al., 2019, 2020). Recent work has developed algorithms for computing such optimal strategies from a model of the problem to be solved, the cognitive operations people have available to solve that problem, and how costly those operations are (Callaway et al., 2018b, 2020; Lieder et al., 2017, Griffiths, 2020). We refer to this approach as automatic strategy discovery. This approach frames planning strategies as policies for selecting planning operations. Its methods use algorithms from dynamic programming and reinforcement learning (Sutton and Barto, 2018) to compute the policy that maximizes the expected reward of executing the resulting plan minus the cost of the computations that the policy would perform to arrive at that plan (Callaway et al., 2018a, Lieder and Griffiths, 2020). Recent work used dynamic programming to discover optimal planning strategies for different three-step planning problems, and found that it is possible to improve human planning on those problems by teaching people the automatically discovered strategies (Lieder et al., 2019, 2020). Subsequent work applied reinforcement learning to approximate strategies for planning up to six steps ahead in a task where each step entailed choosing between two options and there were only two possible rewards (Callaway et al., 2018a). But none of the existing strategy discovery methods (Callaway et al., 2018a; Kentur et al., 2020) is scalable enough to discover good planning strategies for more complex environments. This is because the run time of these methods grows exponentially with the size of the planning problem. This confined automatic strategy discovery methods to very small and very simple planning tasks. Discovering planning strategies that achieve – let alone exceed – the computational efficiency of human planning is still out of reach for virtually all practically relevant sequential decision problems.

To overcome this computational bottleneck, we developed a scalable method for discovering planning strategies that achieve a (super-)human level of computational efficiency on some of the planning problems that are too large for existing strategy discovery methods. Our approach draws inspiration from the hierarchical structure of human behavior (Botvinick, 2008; Miller et al., 1960; Carver and Scheier, 2001; Tomov et al., 2020). Research in cognitive science and neuroscience suggests that the brain decomposes long-term planning into goal-setting and planning at multiple hierarchically-nested timescales (Carver and Scheier, 2001; Botvinick,
Furthermore, Solway et al. (2014) found that human learners spontaneously discover optimal action hierarchies. Inspired by these findings, we extend the near-optimal strategy discovery method proposed in Callaway et al. (2018a) by incorporating hierarchical structure into the space of possible planning strategies. Concretely, the planning task is decomposed into first selecting one of the possible final destinations as a goal solely based on its own value and then planning the path to this selected goal.

We find that imposing hierarchical structure makes automatic strategy discovery methods significantly less computationally expensive without compromising the performance of the discovered strategies. Our hierarchical decomposition leads to a substantial reduction in the computational complexity of the strategy discovery problem, which makes it possible to scale up automatic strategy discovery to many planning problems that were prohibitively large for previous strategy discovery methods. This allowed our method to discover planning strategies that achieve a super-human level of computational efficiency on non-trivial planning problems. We demonstrate that this advance makes it possible to improve human decision-making in larger and more complex planning tasks than was previously possible.

The plan for this article is as follows: We start by introducing the frameworks and methods that our approach builds on. We then present our new reinforcement learning method for discovering hierarchical planning strategies. Next, we evaluate our method’s performance and scalability against the state of the art. We then test whether the resulting advances are sufficient to improve human decision making in complex planning problems and close by discussing the implications of our findings for the development of more intelligent agents, understanding human planning (Lieder and Griffiths, 2020), and improving human decision-making (Lieder et al., 2019).

2 Background and related work

Before we introduce, evaluate, and apply our new method for discovering hierarchical planning strategies we now briefly introduce the concepts and methods that it builds on. We start by introducing the theoretical framework we use to define what constitutes a good planning strategy.

2.1 Resource rationality

Previous work has shown that people’s planning strategies are jointly shaped by the structure of the environment and the cost of planning (Callaway et al., 2018b, 2020). This idea has been formalized within the framework of resource-rational analysis (Lieder and Griffiths, 2020). Resource-rational analysis is cognitive modeling paradigm that derives process models of people’s cognitive strategies from the assumption that the brain makes optimal use of its finite computational resources. These computational resources are model as a set of elementary information processing operations. Each of these operations has a cost that reflects how much computational resources it requires. Those operations are assumed to be the building blocks of people’s cognitive strategies. To be resource-rational a planning strategy has to achieve the optimal tradeoff between the expected return of the
resulting decision and the expected cost of the planning operation it will perform to reach that decision. Both depend on the structure of the environment. The degree to which a planning strategy \((h)\) is resource-rational in a given environment \((e)\) can be quantified by the the sum of expected rewards achieved by executing the plan it generates \((R_{\text{total}})\) minus the expected computational cost it incurs to make those choices, that is

\[
RR(h,e) = E[R_{\text{total}} | h, e] - \lambda \cdot E[N | h, e],
\]

where \(\lambda\) is the cost of performing one planning operation and \(N\) is the number of planning operations that the strategy performs. Throughout this article, we use this measure as our primary criterion for the performance of planning algorithms, automatically discovered strategies, and people.

2.2 Discovering resource-rational planning strategies by solving metalevel MDPs

Callaway et al. (2018b) developed a method to automatically derive resource-rational planning strategies by modeling the optimal planning strategy as a solution to a metalevel Markov Decision Process (metalevel MDP). In general, a metalevel MDP \(M = (B, C, T, r)\) is defined as an undiscounted MDP where \(b \in B\) represents the belief state, \(T(b, c, b')\) is the probability of transitioning from belief state \(b\) to belief state \(b'\) by performing computation \(c \in C\), and \(r(h, c)\) is a reward function that describes the costs and benefits of computation (Hay et al., 2014). It is important to note that the actions in a metalevel MDP are computations which are different from object-level actions – the former are planning operations and the latter are physical actions that move the agent through the environment. Previous methods for discovering near-optimal decision strategies (Lieder et al., 2017; Callaway et al., 2018b) have been developed for and evaluated in a planning task known as the Mouselab-MDP paradigm (Callaway et al., 2017, 2020).

2.3 The Mouselab-MDP paradigm

The Mouselab-MDP paradigm was developed to make people’s elementary planning operations observable. This is achieved by externalizing the process of planning as information seeking. Concretely, the Mouselab-MDP paradigm illustrated in Figure 1 shows the participant a map of an environment where each location harbors an occluded positive or negative reward. To find out which path to take the participant has to click on the locations they consider visiting to uncover their rewards. Each of these clicks is recorded and interpreted as the reflection of one elementary planning planning operation. The cost of planning is externalized by the fee that people have to pay for each click. People can stop planning and start navigating through the environment at any time. But once they have started to move through the environment they cannot resume planning. The participant has to follow one of the paths along the arrows to one of the outermost nodes.

To evaluate the resource-rational performance metric specified in Equation 1 in the Mouselab-MDP paradigm, we measure \(R_{\text{total}}\) by the sum of rewards along the chosen path, set \(\lambda\) to the cost of clicking, measure \(N\) by the number of clicks that a strategy made on a given trial.
Discovering Efficient Strategies for Hierarchical Planning

Fig. 1: Illustration of the Mouselab-MDP paradigm. Rewards are revealed by clicking with the mouse, prior to selecting a path using the keyboard. This figure shows one concrete task that can be created using this paradigm. Many other tasks can be created by varying the size and layout of the environment, the distributions that the rewards are drawn from, and the cost of clicking.

The structure of a Mouselab-MDP environment can be modelled as a directed acyclic graph (DAG), where each node is associated with a reward that is sampled from a probability distribution, and each edge represents a transition from one node to another. In this article, we refer to the agent’s initial position as the root node, the most distant nodes as goal nodes and all other nodes as intermediate nodes.

Figure 2 shows an instance of a Mouselab-MDP environment that we use extensively in this article. There, the variance of each node’s reward distribution increases with the node’s depth. This models that the values of distant states are more variable than the values of proximal states. Therefore, the goal nodes have a higher variance than the intermediate nodes.

Fig. 2: Mouselab-MDP environment with 2 goals. Nodes associated with each goal are denoted in green (circles) and red (triangles), respectively. The goal nodes have darker shades of green and red (diamonds), and the root node’s color is blue (square).

1 A node’s depth is defined as the length of the longest path connecting this node to the root node.
2.4 Resource-rational planning in the Mouselab-MDP paradigm

To discover optimal planning strategies, we can draw on previous work that formalized resource-rational planning in the Mouselab-MDP paradigm as the solution to a metalevel MDP (Callaway et al., 2018b, 2020). In the corresponding metalevel MDP each belief state $b \in B$ encodes probability distributions over the rewards that the nodes might harbor. The possible computations are $C = \{\xi_1, \ldots, \xi_M, c_{1,1}, \ldots, c_{M,N}, \bot\}$, where $c_{g,n}$ reveals the reward at intermediate node $n$ on the path to goal $g$, and $\xi_g$ reveals the value of the goal node $g$. For simplicity, we set the cost of each computation to 1. When the value of a node is revealed, the new belief about the value of the inspected node assigns a probability of one to the observed value. The metalevel operation $\bot$ terminates planning and triggers the execution of the plan. The agent selects one of the paths to a goal state that has the highest expected sum of rewards according to the current belief state.

2.5 Methods for solving meta-level MDPs

In their seminal paper, Russell and Wefald (1991) introduced the theory of rational metareasoning. In Russell and Wefald (1992), they define the value of computation $VOC(c, b)$ to be the expected improvement in decision quality achieved by performing computation $c$ in belief state $b$ and continuing optimally, minus the cost of computation $c$. Using this formalization, the optimal planning strategy $\pi^*_\text{meta}$ is a selection of computations which maximizes the value of computation (VOC), that is

$$\pi^*_\text{meta} = \arg \max_c VOC(c, b). \quad (2)$$

When the VOC is non-positive for all available computations, the policy terminates ($c = \bot$) and executes the best object-level action according to the current belief state. Hence, $VOC(\bot, b) = 0$. In general, the VOC is computationally intractable but it can be approximated (Callaway et al., 2018a, Lin et al., 2015) estimated VOC by the myopic value of computation ($VOI_1$), which is the expected improvement in decision quality that would be attained by terminating deliberation immediately after performing the computation. Hay et al. (2014) approximated rational metareasoning by solving multiple smaller metalevel MDPs that each define the problem of deciding between one object-level action and its best alternative.

2.5.1 Bayesian Metalevel Policy Search

Inspired by research on how people learn how to plan Krueger et al. (2017), Callaway et al. (2018a) developed a reinforcement learning method for learning when to select which computation. This method uses Bayesian optimization to find a policy that maximizes the expected return of a metalevel MDP. The policy space is parameterized by weights that determine to which extent computations are selected based on the myopic VOC versus less short-sighted approximations of the value of computation. It thereby improves upon approximating the value of computation by the myopic VOC by considering the possibility that the optimal metalevel policy might perform additional computations afterwards. Concretely,
BMPS approximates the value of computation by interpolating between the myopic VOI and the value of perfect information, that is

\[
\hat{VOC}(c, b; w) = w_1 \cdot VOI_1(c, b) + w_2 \cdot VPI(b) + w_3 \cdot VPI_{sub}(c, b) - w_4 \cdot \text{cost}(c),
\]

where \( VPI(b) \) denotes the value of perfect information. \( VPI_1(b) \) assumes that all computations possible at a given belief state would take place. Furthermore, \( VPI_{sub}(c, b) \) measures the benefit of having full information about the subset of parameters that the computation reasons about (e.g., the values of all paths that pass through the node evaluated by the computation), \( \text{cost}(c) \) is the cost of the computation \( c \), and \( w = (w_1, w_2, w_3, w_4) \) is a vector of weights. Since the VOC and \( VPI_{sub} \) are bounded by the VOI from below and by the VPI from above, the approximation of VOC (i.e. \( \hat{VOC} \)) is a convex combination of these features, and the weights associated with these features are constrained to a probability simplex set. Finally, the weight associated with the cost function \( w_4 \in [1, h] \), where \( h \) is the maximum number of available computations to be performed. The value of these weights are computed using Bayesian Optimization (Mockus, 2012). Discovery of the optimized weights, is analogous to discovering the optimal policy in the environment.

2.5.2 Alternative approaches

Alternative methods to solve metalevel MDPs include works by (Sezener and Dayan, 2020) and (Svegliato and Zilberstein, 2018). Sezener and Dayan (2020) solves a multi-arm bandit problem using a Monte Carlo Tree Search based on static and dynamic value of computations. In a bandit problem, unlike most models of planning, transitions depend purely on the chosen action and not on the current state. Svegliato and Zilberstein (2018) devised an approximate metareasoning algorithm using temporal difference (TD) learning to decide when to terminate the planning process.

2.6 Intelligent cognitive tutors

Utilising the optimal planning strategies discovered by solving metalevel MDPs, Lieder et al. (2019, 2020) have developed intelligent tutors that teach people the optimal planning strategies for a given environment. Most of the tutors let people practice planning in the Mouselab-MDP paradigm and provide them immediate feedback on each chosen planning operation. The feedback is given in two ways: (1) information about what the optimal planning strategy would have done; and (2) an affective element given as a positive feedback (e.g., “Good job!”) or negative feedback. The negative feedback included a slightly frustrating time-out penalty during which participants were forced to wait idly for a duration that was proportional to how sub-optimal their planning operation had been.

Lieder et al. (2020) found that participants were able to learn to use the automatically discovered strategies, remember them, and use them in novel environments with a similar structure. These findings suggest that automatic strategy discovery can be used to improve human decision-making if the discovered strategies are well-adapted to the situations where people might use them. Additionally, Lieder
S. Consul, L. Heindrich, J. Stojcheski, F. Lieder (2020) also found that video demonstrations of click sequences performed by the optimal strategy is an equally effective teaching method as providing immediate feedback. Here, we build on these findings to develop cognitive tutors that teach automatically discovered strategies by demonstrating them to people.

3 Discovering hierarchical planning strategies

All previous strategy discovery methods evaluate and compare the utilities of all possible computations in each step. As such, these algorithms have to explore the entire metalevel MDP’s state space which grows exponentially with the number of nodes. As a consequence, these methods do not scale well to problems with large state spaces and long planning horizons. This is especially true of the Bayesian Metalevel Policy Search algorithm (BMPS; Callaway et al., 2018a) whose run time is exponential in the number of nodes of the planning problem. In contrast to the exhaustive enumeration of all possible planning operations performed by those methods, people would not even consider making detailed low-level motor plans for navigating to a specific distant location (e.g., Terminal C of San Juan Airport) until they arrive at a high-level plan that leads them to or through that location (Tomov et al., 2020). Here, we build on insights about human planning to develop a more scalable method for discovering efficient planning strategies.

3.1 Hierarchical Problem Decomposition

To efficiently plan over long horizons, people (Botvinick, 2008; Carver and Scheier, 2001; Tomov et al., 2020) and hierarchical planning algorithms (Kaelbling and Lozano-Pérez, 2010; Sacerdoti, 1974; Marthi et al., 2007; Wolfe et al., 2010) decompose the problem into first setting goals and then planning how to achieve them. This two-stage process breaks large planning problems down to smaller problems that are easier to solve. To discover hierarchical planning strategies automatically, our proposed strategy discovery algorithm decomposes the problem of discovering planning strategies into the sub-problems of discovering a strategy for selecting a goal and discovering a strategy for planning the path to the chosen goal. A pictorial representation is given in Figure 3.

Formally, this is accomplished by decomposing the metalevel MDP defining the strategy discovery problem into two metalevel MDPs with smaller state and action spaces. Constructing metalevel MDPs for goal-setting and path planning is easy when there is a small set of candidate goals. Such candidate goals can often be identified based on prior knowledge or the structure of the domain (Schapiro et al., 2013; Solway et al., 2014). A low level controller solves the goal-achievement MDP whereas the high level controller solves the goal-setting MDP. When the controller is in control, a computation is selected from the corresponding metalevel MDP and performed. The meta controller looks at the expected reward of the current goal with the expected reward of the next best goal and decides when control from the high level controller should be switched to the low level controller. Hence, when the low level controller discovers that the current goal is not as valuable as expected, the meta controller allows for goal switching.

The results presented in the paper have up to 5^{90} possible belief states.
The metalevel MDP model of the sub-problem of goal selection (Section 3.1.1) only includes computations for estimating the values of a small set of candidate goal states \( V(g_1), \ldots, V(g_M) \). This means that goals are chosen without considering how costly it would be to achieve them. This makes sense when all goals are known to be achievable and the differences between the values of alternative goal states are substantially larger than the differences between the costs of reaching them. This is arguably true for many challenges people face in real life. For instance, when a high school student plans one’s career, the difference between the long-term values of studying computer science versus becoming a janitor is likely much larger than the difference between the costs of achieving either goal. This is to be expected when the time it will take to achieve the goals is short relative to a person’s lifetime.

The goal-achievement MDP (Section 3.1.2) only includes computations that update the estimated costs of alternative paths to the chosen goal by determining the costs or rewards of state-action pairs \( r(b,c) \) that lie on those paths. This selection of computations within a selected goal leads to a possible issue of ignoring some computations that can be irrelevant in the goal achievement MDP but be highly valuable when considering the complete problem. One such example is when considering computations which reveal the value of nodes lying on an unavoidable path to the selected goal. This problem gets further accentuated if such a node has a possibility of having a highly positive or negative reward. To rectify this problem, a meta controller has been introduced to facilitate goal switching. A real world example of the necessity to switch goals after discovering an unlikely highly negative event could be, for example, to switch from investing in the stock market to investing in real estate after discovering a likely stock market crash.

Decomposing the strategy discovery problem into these two components reduces the number of possible computations that the metareasoning method has to choose between from \( M \cdot N \) to \( M + N \), where \( M \) is the number of possible final destinations.
(goals) and \( N \) is the number of steps to the chosen goal (see Appendix A.2). Perhaps the most-promising metareasoning method for automatic strategy discovery is the Bayesian Metalevel Policy Search algorithm (BMPS; (Callaway et al., 2018a; Kemtur et al., 2020)). To solve the two types of metalevel MDPs introduced below more effectively, we also introduce an improvement of the BMPS algorithm in Section 3.2.

3.1.1 Goal-setting metalevel MDP

The optimal strategy for setting the goal can be formalized as the solution to the metalevel MDP \( M^H = (B^H, C^H, T^H, R^H) \), where the belief state \( b^H(g) \in B^H \) denotes the expected cumulative reward that the agent can attain starting from the goal state \( g \in G \). The high level computations are \( C^H = \{ \xi_1, ..., \xi_M, \bot^H \} \), where \( \xi_g \) reveals the value \( V(g) \) of the goal node \( g \). \( \bot^H \) terminates the high-level planning leading to the agent to select the goal with the highest value according to its current belief state. The reward function is \( R^H(b^H, c^H) = -\lambda^H \) for \( c^H \in \{ \xi_1, ..., \xi_M \} \) and \( R^H(b^H, \bot^H) = \max_{k \in G} E[b^H(k)] \).

3.1.2 Goal-achievement metalevel MDP

Having set a goal to pursue, the agent has to find the optimal planning strategy to achieve the goal. This planning strategy is formalized as the solution to the metalevel MDP \( M^L = (B^L, C^L, T^L, R^L) \), where the belief state \( b \in B^L \) denotes the expected reward for each node. The agent can only perform a subset of meta-actions \( C_{g,L} = \{ c_{g,1}, ..., c_{g,N}, \bot^L \} \), where \( c_{g,n} \) reveals the reward at node \( n \) in the goal set \( h_g \in H \). A goal set \( h_g \in H \) refers to all nodes, including the goal node, which lie on all paths leading to goal \( g \in G \). Furthermore, \( \bot^L \) terminates planning and leads to the agent to select the path with the highest expected sum of rewards according to the current belief state. The reward function is \( R^L(b, c_g) = -\lambda^L \) for \( c_g \in \{ c_{g,1}, ..., c_{g,N} \} \) and \( R^L(b, \bot^L) = \max_{p \in P} \sum_{n \in p} E[b_n] \), where \( P \) is the set of all paths, and \( b_n \) is the belief of the reward for node \( n \).

3.2 Hierarchical Bayesian Metalevel Policy Search

Having introduced the hierarchical problem decomposition, we now present how this decomposition can be leveraged to make BMPS and other automatic strategy discovery methods more scalable. BMPS approximates the value of computation (VOC) according to Equation 3. We propose to utilize BMPS to solve the goal selection metalevel MDP and the goal achievement metalevel MDP separately. The metacontroller then decides when the policy discovered should run. A detailed analysis of the computational time is presented in the Appendix A.2.

**High level policy search:** The VOC for the high level policy is approximated using three features: (1) the myopic utility for performing a goal state evaluation (\( \text{VOI}_H \)), (2) the value of perfect information about all goals (\( \text{VPI}_H \)), and (3) the cost of the respective computation (\( \text{cost}_H \)).

\[
\hat{\text{VOC}}_H(c^H, b^H; w^H) = w_1^H \cdot \text{VOI}_H(c^H, b^H) + w_2^H \cdot \text{VPI}_H(b^H) - w_3^H \cdot \text{cost}_H(c^H),
\]

(4)
where $w_{H1}, w_{H2}$ are constrained to a probability simplex set, $w_{H1} \in \mathbb{R}_{[1, M]}$, and $M$ is the number of goals. Additionally, the cost $\text{cost}^H(c^H)$ is defined as

$$\text{cost}^H(c^H) = \begin{cases} 
\lambda^H, & \text{if } c^H \in \{\xi_1, \ldots, \xi_M\} \\
0, & \text{if } c^H = \bot^H.
\end{cases} \quad (5)$$

**Low level policy search:** In a similar manner as for the high-level policy, the value of computation for the low-level policy is approximated by using a mixture of VOI features and the anticipated cost of the current computation and future computations, that is:

$$\hat{\text{VOC}}_L(c, b, g; w_L) = w_{L1} \cdot \text{VOI}_L(c, b, g) + w_{L2} \cdot \text{VPI}_L(b, g) + w_{L3} \cdot \text{VPI}_{sub}(c, b, g) - w_{L4} \cdot \text{cost}_L(c, g, w_L) \quad (6)$$

where $w_{Li} (i = 1, 2, 3)$ are constrained to a probability simplex set, $w_{Li} \in \mathbb{R}_{[1, |h_g|]}$, and $|h_g|$ is the number of nodes in goal set $h_g$. The weight values for both levels are optimised in 100 iterations with Bayesian Optimization (Mockus, 2012) using the GPyOpt library (The GPyOpt authors, 2016).

The cost feature of the original BMPS algorithm introduced by Callaway et al. (2018a) only considered the cost of a single computation whereas its VOI features consider the benefits of performing a sequence of computations. As a consequence, policies learned with the original version of BMPS are biased towards inspecting nodes that many paths converge on, even when the values of those nodes are irrelevant. To rectify this problem, we redefine the cost feature so that it considers the costs of all computations assumed by the VOI features. Concretely, to compute the low-level policy, we define the cost feature of BMPS as the weighted average of the costs of generating the information assumed by the VOI features $F = \{\text{VOI}_L^1, \text{VPI}_L^1, \text{VPI}_{sub}^L\}$, that is

$$\text{cost}_L(c, g, w_L) = \sum_{f \in F} w_f \cdot \sum_{n} \mathbb{I}(c, f, n) \cdot \text{cost}(c) \quad (7)$$

where $\mathbb{I}(c, f, n)$ returns 1 if node $n$ is relevant when computing feature $f$ for computation $c$ and 0 otherwise.

In the remainder of this article, we refer to the resulting strategy discovery algorithm as **hierarchical BMPS** and refer to the original version of BMPS as **non-hierarchical BMPS**.

### 3.3 Tree contraction method for faster BMPS feature computation

To further increase the scalability of BMPS, we make an additional improvement to how it computes the features used to approximate the value of computation (Callaway et al., 2018a). Specifically, we aim to improve the computational efficiency by combining nodes in the meta MDP according to a set of predefined conditions, ultimately reducing the complexity of the necessary computations. The node combination is performed by merging two nodes into a single new node with a probability distribution that represents their combined reward value.
The algorithm consists of three different operations that combine node distributions. A list of conditions determine an operation to apply and the algorithm stops when the distributions of all nodes within the MDP are collapsed to a single root node.

- **Add**: Combines the distribution of two consecutive nodes by adding their distributions. This operation can be applied to two consecutive nodes in the tree as long as the parent node does not have other child nodes and the child node does not have other parents.

- **Maximise**: Combines two parallel nodes by taking the maximum value for each combination of values of nodes can take, combining the nodes distributions while taking into account that the optimal path will always lead through the higher node of the two. This operation can be applied to two nodes that have a single identical parent and child node.

- **Split**: Splits a child node into two separate nodes by duplicating that node. The whole tree is then duplicated as many times as the node has possible values, fixing the node’s distribution to each possibility. The duplicated trees are then individually reduced to single root nodes and the individual root nodes are combined to a single tree by pairwise application of the maximise operation. This operation can be applied to nodes that have multiple parent nodes where each of the individual nodes after splitting is only connected to one its parent nodes.

The split operation is the most computationally expensive operation and is therefore only applied when the add and maximise operations are insufficient to reduce the tree to a single node. Specifically, this happens when a node that needs to be reduced by the multiply operation has an additional parent or child node. Since the structure of the environment stays identical while the rewards and discovered states vary, we precompute the necessary operations to reduce the tree and then apply the reduction individually for each problem instance.

Our adjustment is purely algorithmic and it does not change the value of computation. Therefore, it does not impair the performance of the discovered strategies. An additional effect of the tree contraction method is that it extends the types of environments solvable by BMPS. Previously, BMPS was only able to handle environments with a branching tree structure: nodes can have multiple children but never multiple parents. Our new formulation allows us to compute the BMPS features for tree structures in which nodes have multiple parent nodes as well. This is possible through the application of the maximise operation, which allows to combine multiple parent nodes into a single node, making them solvable through the value of computation calculation. The range of solvable environments is therefore extended from trees to directed acyclic graphs. This extension is especially relevant for environments containing goal nodes since it is often the case that multiple intermediate nodes converge to the same goal node.

\[3\] Two nodes are consecutive if they are in a direct parent-child relation.
4 Evaluating the performance, scalability, and robustness of our method for discovering hierarchical planning strategies

To evaluate our method for discovering hierarchical planning strategies, we benchmarked its performance, scalability, and robustness using the two types of environments illustrated in Figure 2 and Figure 6. The first type of environments conforms to the structure that motivated our hierarchical problem decomposition (i.e., the variability of rewards increases from each step to the next) and the second type does not. In the second type of environment, we introduced a high-risk node on the path to each goal (see Figure 6). This violates the assumption that motivated the hierarchical decomposition and makes goal switching essential for good performance.

For each environment, we apply the criterion defined in Equation 1 (see Sections 2.1 and 2.3) to evaluate the degree to which the resulting strategies are resource rational against the resource rationality of human planning, existing planning algorithms, and the strategies discovered by state-of-the-art strategy methods. Furthermore, we show the our method is substantially more scalable than previous methods.

To be able to compare the performance of the automatically discovered planning strategies to the performance of people, we conduct experiments on Amazon Mechanical Turk (Litman et al., 2017). In these experiments, we measure human performance in Flight Planning tasks that are analogous to the environments we use to evaluate our method (see Figure 7). For the first type of environments, we recruited 78 participants for each of the four environments (average age 34.71 years, range: 19–70 years; 46 female). Participants were paid $2.00 plus a performance-dependent bonus (average bonus $1.52). The average duration of the experiment was 25.1 min. For the second type of environments, we recruited 48 participants (average age 36.98 years, range: 19–70 years; 25 female). Participants were paid $1.75 plus a performance-dependent bonus (average bonus $0.34). The average duration of the experiment was 14.86 min. Following instructions that informed the participants about the range of possible reward values, participants were given the opportunity to familiarize themselves with the task in 5 practice trials of the Flight Planning task. After this, participants were evaluated on 15 test trials of the Flight Planning task for the first type of environments and 5 test trials for the second type of environments. To ensure high data quality, we applied the same pre-determined exclusion criterion throughout all presented experiments. We excluded participants who did not make a single click on more than half of the test trials because not clicking is highly indicative to a participant not engaging and speeding through the task. In the first environment type we excluded 3 participants and in the second environment type we excluded 22 participants.

4.1 Evaluation in environments that conform to the assumed structure

We first evaluate the performance and scalability of our method in environments whose structure conforms to the assumptions that motivated the hierarchical problem decomposition. To do so, we compare the performance of the discovered strategies against the performance of existing planning algorithms, the strategies discovered by previous strategy discover methods, and human performance in four
increasingly challenging environments of this type with 2-5 candidate goals. The
reward of each node is sampled from a normal distribution with mean \(0\). The
variance of rewards available at non-goal nodes was \(5\) for nodes reachable within a
single step (level 1) and doubled from each level to the next. The variance of the
distribution from which the reward associated with the goal node was sampled
starts from \(100\) and increases by \(20\) for every additional goal node. The environment
was partitioned into one sub-graph per goal. Each of those sub-graphs contains 17
intermediate nodes, forming 10 possible paths that reach the goal state in maximum
5 steps (see Figure 2). The cost of planning is 1 point per click (\(\lambda = 1\)).

To estimate an upper bound on the performance of existing planning algorithms
on our benchmark problems, we selected Backward Search and Bidirectional Search
(Russell and Norvig 2002) because – unlike most planning algorithms – they
start by considering potential final destinations, which is optimal for planning in
our benchmark problems. These search algorithms terminate when they find a
path whose expected return exceeds a threshold, called its aspiration value. The
aspiration was selected using Bayesian Optimization (Mockus 2012) to get the best
possible performance from the selected planning algorithm. We also evaluated the
performance of a random-search algorithm, which chooses computations uniformly
at random from the set of metalevel operations that have not been performed yet.

In addition to those planning algorithms, our baselines also include three state-
of-the-art methods for automatic strategy discovery: the greedy myopic VOC
strategy discovery algorithm (Lin et al. 2015), which approximates the VOC
by its myopic utility (VOI\(_1\)), BMPS (Callaway et al. 2018a), and the Adaptive
Metareasoning Policy Search algorithm (AMPS) (Svegliato and Zilberstein 2018)
which uses approximate metareasoning to decide when to terminate planning. Our
implementation of the AMPS algorithm uses a deep Q-network (Mnih et al. 2013)
to estimate the difference between values to stop planning and continue planning,
respectively. It learns this estimate based on the expected termination reward of the
best path. The planning operations are selected by maximizing the myopic value of
information (VOI\(_1\)). When applying hierarchical BMPS to this environment, we
disabled the goal switching component of the metacontroller since the cumulative
variance of the intermediate nodes was less than the variance of the goal nodes,
rendering goal-switching unnecessary. To illustrate the versatility of our hierarchical
problem decomposition, we also applied it to the greedy myopic VOC strategy
discovery algorithm.

4.1.1 Performance of the automatically discovered strategies

Table 1 and Figure 5a compare the performance of the strategies discovered by
hierarchical BMPS and the hierarchical greedy myopic VOC method against the
performance of the strategies discovered by the two state-of-the-art methods, two
standard planning algorithms, and human performance on the benchmark problems
described above (Section 4.1). These results show that the strategies discovered
by our new hierarchical strategy discovery methods outperform extant planning
algorithms and the strategies discovered by the AMPS algorithm across all of our
benchmark problems (\(p < .01\) for all pairwise Wilcoxon rank-sum tests). Critically,
imposing hierarchical constraints on the strategy search of BMPS and the greedy
myopic VOC method had no negative effect on the performance of the resulting
strategies (\(p > .770\) for all pairwise Wilcoxon rank-sum tests). Additionally, when
human participants were tested on 15 environments, they performed much worse than than the performance of the strategy discovered by our hierarchical method regardless of the number of goals ($p < 0.02$ for all pairwise Wilcoxon rank-sum tests).

| Type | Name                                      | 2 Goals  | 3 Goals  | 4 Goals  | 5 Goals  |
|------|-------------------------------------------|----------|----------|----------|----------|
| S    | Hierarchical BMPS                         | 108.79   | **150.63** | 178.98   | 206.45   |
| S    | Non-hierarchical BMPS                    | **111.53** | 148.01   | **182.38** | 204.37   |
| S    | Hierarchical greedy myopic VOC            | 108.48   | 150.13   | 178.81   | **206.57** |
| S    | Non-hierarchical greedy myopic VOC       | 107.98   | 150.41   | 180.35   | 205.40   |
| S    | Adaptive Metareasoning Policy Search      | 77.08    | 109.39   | 127.01   | 141.34   |
| P    | Depth-first Search                        | 74.99    | 109.13   | 129.66   | 143.45   |
| P    | Breadth-first Search                      | 87.62    | 112.83   | 127.68   | 137.40   |
| P    | Bidirectional Search                      | 88.59    | 115.07   | 134.98   | 154.24   |
| P    | Backward Search                           | 87.85    | 114.29   | 134.43   | 156.56   |
| P    | Random Policy                             | 52.73    | 80.05    | 89.31    | 101.15   |
|      | Human Baseline                            | 45.42    | 88.06    | 39.32    | 124.89   |

Table 1: Net returns of various strategy discovery methods (S) and existing planning algorithms (P). The best algorithms and the best net returns for each environment setting (column) are formatted in **bold**. The four best methods performed significantly better than the other methods but the differences between them are not statistically significant.

As illustrated in Figure 4, the planning strategy our hierarchical BMPS algorithm discovered for this type of environment is qualitatively different from all existing planning algorithms. In general, the strategy is as follows: it first evaluates the goal nodes until it finds a goal node with a sufficiently high reward. Then, it plans backward from the chosen goal to the current state. In evaluating candidate paths from the goal to the current state, it discards each path from further exploration as soon as it encounters a high negative reward on that path. This phenomenon is known as pruning and has previously been observed in human planning (Huys, et al., 2012). The non-hierarchical version of BMPS also discovered this type of planning strategy. This suggests that goal-setting with backward planning is the resource-rational strategy for this environment rather than an artifact of our hierarchical problem decomposition. Unlike this type of planning, most extant planning algorithms plan forward and the few planning algorithms that plan backward (e.g., Bidirectional Search and Backward Search) do not preemptively terminate a path exploration.

### 4.1.2 Scalability

Table 2 and Figure 5 compare the run times of our hierarchical strategy discovery methods against their non-hierarchical counterparts and adaptive metareasoning.

---

While this strategy was discovered assuming that the cost of evaluating a potential goal node is the same as the cost of evaluating an intermediate node, we found that the discovered strategy remained the same as we increased the cost of evaluating goal nodes to 2, 5, or 10.
Fig. 4: Sequence of nodes revealed in particular environment. The numbers above the nodes indicate the sequence in which the nodes were revealed. The numbers in each revealed node indicates its reward.

Fig. 5: (a) Net returns of existing planning algorithms (striped bars) versus planning strategies discovered by various strategy discovery methods (bars without stripes). (b) Comparison of the mean time for various strategy discovery methods (in seconds).

policy search. This comparison shows that imposing hierarchical structure substantially increased the scalability of BMPS and the greedy myopic VOC method. The improved run time profile reflects a reduction in the asymptotic upper bound on the algorithms’ run times when hierarchical structure is imposed on the strategy space (see Appendix A.2). As shown in the last column of Table 2, this reduction in computational complexity increases the size of environments for which we can discovery resource-rational planning strategies by a factor of 14-15, depending
on the required quality of the planning strategy. This makes it possible to automatically discover planning strategies for sequential decision problems with up to 2520 states. Consequently, our method scales to metalevel MDPs with up to \(5^{2520}\) possible belief states whereas the original version of BMPS was limited to problems with only up to \(5^{36}\) possible belief states. This shows that our approach increased the scalability of automatic algorithm discovery by a factor of \(5^{2484}\). This is a significant step towards discovering resource-rational planning strategies for the complex planning problems people face in the real world. While the hierarchical greedy myopic VOC method is the most scalable strategy discovery method, the fastest method on our four benchmarks was our hierarchical BMPS algorithm with tree contraction. Comparing the first two rows shows that the tree contraction method described in Section 3.3 significantly contributed to this speed-up (for more details see Appendix A.5).

| Strategy Discovery Algorithm | 2 Goals | 3 Goals | 4 Goals | 5 Goals | Max. \# Nodes |
|------------------------------|---------|---------|---------|---------|---------------|
| **Hierarchical BMPS**        | 0.21    | 0.23    | 0.24    | 0.27    | 540           |
| with tree contraction        |         |         |         |         |               |
| Hierarchical BMPS            | 4.18    | 6.45    | 7.45    | 9.30    | 180           |
| Non-hierarchical BMPS        | 16.09   | 44.81   | 117.36  | 203.18  | 36            |
| **Hierarchical greedy myopic VOC** | 7.35    | 8.52    | 10.03   | 14.64   | 2520          |
| Non-hierarchical greedy myopic VOC | 10.56   | 29.02   | 121.53  | 101.97  | 180           |
| Adaptive Metareasoning       | 6.45    | 13.39   | 24.43   | 64.36   | 36            |
| Policy Search                |         |         |         |         |               |

Table 2: Average time to evaluate an environment represented in seconds. The last column denotes the size of the largest environment (\# nodes) for which each method can planning strategies within a time budget of 8 h.

4.2 Robustness to violations of the assumed structure

To determine the range of planning problems for which our hierarchical decomposition can be used to discover resource-rational planning strategies at scale, we varied the variance structure of the 2-goal environment and compared the performance of BMPS with and without hierarchical structure (see Table 3). We found that the usefulness of the hierarchical decomposition depends on the variance ratios of the goal values versus path costs. Concretely, the performance loss of the algorithm utilising the hierarchical structure is below 5% when the variance of goal values is at least as high as the variance of the path costs, but increases to 10% as the variance of goal values drops to only one third of the variance of the path costs.

To accommodate environments whose structure violates the assumption that more distant rewards are more variable than more proximal ones, the hierarchical strategies discovered by our method can alternate between goal selection and goal alternation.

---

5 The continuous normal distribution is discretized to 4 bins. So including the undiscovered state, each node has 5 possible state conditions.
Table 3: Comparison in performance of BMPS with and without hierarchical structure on 2-goal environment with various variance ratios. $\sigma_\Sigma$: cumulative standard deviation of the longest path to a goal node; $\sigma_1$: standard deviation of the first goal node; $\sigma_2$: standard deviation of the second goal node; $\Delta$: absolute difference between net returns; $\%\Delta$: relative difference between net returns.

| $\sigma_\Sigma$ | $\sigma_1$ | $\sigma_2$ | Hierarchical | Non-hierarchical | $\Delta$ | $\%\Delta$ |
|-----------------|------------|------------|--------------|----------------|---------|-----------|
| 46.1            | 100        | 120        | 108.79       | 111.53         | 2.74    | 2.46      |
| 46.1            | 75         | 75         | 89.41        | 90.62          | 1.21    | 1.34      |
| 46.1            | 50         | 60         | 76.84        | 79.52          | 2.68    | 3.37      |
| 46.1            | 25         | 30         | 60.62        | 64.95          | 4.33    | 6.67      |
| 46.1            | 12         | 15         | 53.63        | 59.63          | 6.00    | 10.06     |

planning. We now demonstrate the benefits of this goal switching functionality by comparing the performance of our method with versus without goal switching. In particular, we demonstrate that switching goals leads to a better performance if the assumption of increasing variance is violated and does not harm performance when that assumption is met. Firstly, we compare the performance of the two algorithms in an environment where switching goals should lead to an improvement in performance. This environment has a total of 60 nodes split into four different goals, each consisting of 15 nodes in the low-level MDP. The difference to previously used environments is that one of the unavoidable intermediate nodes has a 10% to harbor a large loss of -1500 (see Figure 6). The cost of computation in this environment is 10 points per click (i.e., $\lambda = 10$). The optimal strategy for this environment selects a goal, checks this high-risk node on the path leading to the selected goal and switches to a different goal if it uncovers the large loss. We compare the performance of hierarchical BMPS with goal-switching to the performance of hierarchical BMPS without goal-switching, the non-hierarchical BMPS method with tree contraction and human performance. The three strategy discovery algorithms were all trained on the same environment following the same training steps. Their performance is noted in Table 4.

Since humans were evaluated on only 5 randomly selected instances of this environments and each environment instance contains some randomness in its reward values, we evaluated the performance of the strategy discovery methods on the same 5 environments. All performance scores do not follow a normal distribution as tested with a Shapiro-Wilk test ($p < .001$ for each). The performance between the individual algorithms was compared with a Wilcoxon rank-Sum rest, adjusting the critical alpha value via Bonferroni correction. Comparing the score of goal-switching to both our method without goal-switching ($W = 14.07, p < .001$) and the original BMPS algorithm ($W = 11.38, p < .001$), shows a significant benefit of goal-switching. Comparing the performance of the original BMPS method to the no goal-switching algorithm, the original BMPS version performs significantly better ($W = 18.7, p < .001$). While the average human performance was only $-79.92$ (see Table 4), our method achieved a resource-rationality score of 41 on the same environment instances. A Shapiro-Wilk detected no significant violation of the assumption that participants’ average scores are normally distributed ($p = .33$).

$^6$ Without our tree contraction method, the original version of BMPS would not have been scalable enough to handle this environment.
We therefore compared the average human performance to our method in a one-sample t-test. We found that human participants performed significantly worse than the strategy discovered by our method ($t(25) = -13.06, p < .001$). This suggests that the strategy discovered by our method achieved a superhuman level of computational efficiency.

![Environment Diagram](image)

**Fig. 6:** Environment that demonstrates the utility of goal switching. High risk nodes (8, 23, 38 and 53) follow a categorical reward distribution of -1500 with a probability of 0.1 and 0 with a probability of 0.9. Goal nodes (15, 30, 45 and 60) have a categorical equiprobable reward distribution of 0, 25, 75 or 100. The first node in each sub-tree (1, 16, 31 and 46) as well as the root node (0) have a fixed reward of 0. All other intermediate nodes follow a categorical equiprobable distribution of -10, -5, 5, 10.

| Algorithm           | N  | Reward  | Std   |
|---------------------|----|---------|-------|
| No goal-switching   | 5000 | -80.38  | 446.47|
| Goal-switching      | 5000 | 51.33   | 32.2  |
| Non-hierarchical BMPS | 5000 | 39.29   | 40.93 |
| Human baseline      | 26  | -79.92  | 74.06 |

Table 4: Mean reward and standard deviation of executing the hierarchical planning algorithm with and without goal-switching, as well as the original BMPS algorithm, over 5000 random instances of the high-risk environment with four goal states. A human baseline is gathered in an online experiment with a lower number of samples.

To show that enabling our algorithm’s capacity for goal-switching has no negative effect on its performance even when the assumption of the hierarchical decomposition is met, we perform a second comparison on the two-goal environment.
with increasing variance as in Section 4.1.1 Since in this environment the rewards are most variable at the goal nodes, switching goals should usually be unnecessary. Therefore, due to the environment structure, we do not expect the goal-switching strategy to perform better than the purely hierarchical strategy. By comparing the performance in this environment we observe that both versions of the algorithm perform similarly well. A Wilcoxon rank-Sum rest ($W = 0.03, p = .98$) shows no significant difference between the two. This demonstrates that the addition of goal switching to the algorithm does not impair performance, even when goal switching is not beneficial.

| Algorithm          | N  | Reward | Std  |
|--------------------|----|--------|------|
| No goal-switching  | 5000 | 108.84 | 95.37 |
| Goal-switching     | 5000 | 108.78 | 95.37 |

Table 5: Mean reward and standard deviation of executing the meta controller and the hierarchical planning algorithm over 5000 random instances of the increasing variance environment with two goal states.

5 Improving human decision-making by teaching automatically discovered planning strategies

Having shown that our method discovers planning strategies that achieve a super-human level of performance, we now evaluate whether we can improve human decision-making by teaching them the automatically discovered strategies. Building on the Mouselab-MDP paradigm introduced in Section 2.3, we investigate this question in the context of the Flight Planning task illustrated in Figure 7. Participants are tasked to plan the route of an airplane across a network of airports. Each flight gains a profit or a loss. Participants can find out how much profit or loss an individual flight would generate by clicking on its destination for a fee of $1. The participant’s goal is to maximize the sum of the flights’ profits minus the cost of planning. Participants can make as few or as many clicks as they like before selecting a route using their keyboard.

To teach people the automatically discovered strategies, we developed cognitive tutors (see Section 2.6) that shows people step-by-step demonstrations of what the optimal strategies for different environments would do to reach a decision (see Figure 8). In each step the strategy selects one click based on which information has already been revealed. At some point the tutor stops clicking and moves the airplane down the best route indicated by the revealed information. Moving forward we will refer to cognitive tutors teaching the hierarchical planning strategies discovered by hierarchical BMPS as hierarchical tutors and refer to the tutors teaching the strategies discovered by non-hierarchical BMPS as non-hierarchical tutors.

To evaluate the effectiveness of these demonstration-based cognitive tutors, we conduct two experiments in which participants are taught the optimal strategies for flight planning problems equivalent to the two types of environments in which we evaluated our strategy discovery method in Section 3. To assess the potential
Fig. 7: Screenshot of the Flight Planning task used to assess people’s planning skills in Experiments 1. Participants can drag and zoom into the environment to show different portions.

benefits of the hierarchical tutors enabled by our new scalable strategy discovery method, these experiments compare the performance of people who were taught by hierarchical tutors against the performance of people who were taught by non-hierarchical tutors, the performance of people who were taught original feedback-based tutor for small environments (Lieder et al., 2020, 2019), and the performance of people who practiced the task on their own. We developed the best version of each tutor possible given the limited scalability of the underlying strategy discovery method. The increased scalability of our new method enabled the hierarchical tutor to demonstrate the optimal strategy for the task participants faced whereas the other tutors could only show demonstrations on smaller versions of the task. We found that showing people a small number of demonstrations of the optimal planning strategy significantly improved their decision-making not only when the assumption underlying our method’s hierarchical problem decomposition is met (Experiment 1) but also when it is violated (Experiment 2).

5.1 Experiment 1: Teaching people the optimal strategy for an environment with increasing variance

In the Experiment 1, participants were taught the optimal planning strategy for an environment in which 10 final destinations can be reached through 9 different paths comprising between 4 and 6 steps each (see Figure 7). The most important property of this environment is that the variance of available rewards doubles from each step to the next, starting from 5 in the first step. Therefore, in this environment, the optimal planning strategy is to first inspect the values of alternative goals, then
commit to the best goal one could find, and then plan how to achieve it without ever reconsidering other goals.

5.1.1 Methods

We recruited 168 participants on Amazon Mechanical Turk (average age 34.9 years, range: 19–70 years; 98 female) (Litman et al., 2017). Participants were paid $2.50 plus a performance-dependent bonus (average bonus $2.86). The average duration of the experiment was 46.9 min. Participants could earn a performance-dependent bonus of 1 cent for every 10 points they won in the test trials.

All participants had to first agree to a consent form stating they were above 18, a US citizen residing in USA and fluent in English. After this, instructions about the range of possible rewards (−250 to 250), cost of clicking ($1), and the movement keys were presented. Then, participants went through 5 trials to familiarize with the experiment. Following this, participants were either given additional 10 practice trials or had 10 trials with the cognitive tutor depending on their experimental condition. Finally, the participant was given 15 test trials in the flight planning task with 10 possible final destinations illustrated in Figure 7. Participants started with 50 points in the beginning of the test block.

To evaluate the efficacy of cognitive tutors, participants were assigned to 4 groups. In the experimental group participants were taught by the hierarchical tutor. The first control group was taught by the non-hierarchical tutor. The second control group was taught by the feedback-based cognitive tutor (Lieder et al., 2019, 2020) illustrated in Figure 9. The third control group practiced the Flight Planning task 10 times without feedback. The hierarchical tutor taught the strategy discovered by
hierarchical BMPS algorithm discovered for the task participants had to perform in the test block. It first demonstrated 3 trials with the goal selection strategy; it then showed three demonstrations of the goal-planning strategy; and finally presented 4 demonstrations of the complete strategy combining both parts. The non-hierarchical tutor showed 10 demonstrations of the strategy that non-hierarchical BMPS discovered for the largest version of the Flight Planning task it could handle (i.e., 2 goals instead of 10 goals). Computational bottlenecks confined the feedback-based tutor to a three-step planning task with six possible final destinations shown in Figure 9 (Lieder et al., 2020, 2019). Participants received feedback on each of their clicks and their decision when the stop clicking as illustrated in Figure 9. When the participant chose a sub-optimal planning operation they were shown a message stating which planning operation the optimal strategy would have performed instead. In addition, they received a timeout penalty whose duration was proportional to how sub-optimal their planning operation had been.

Counterbalanced assignment ensured that participants were equally distributed across four experimental conditions (i.e., 42 participants per condition). To ensure high data quality, we applied a pre-determined exclusion criterion. We excluded 7 participants who did not make a single click on more than half of the test trials because not clicking is highly indicative of speeding through the experiment without engaging with the task.
5.1.2 Results

Figure 10 shows the average performance of the four groups on the test trials. According to a Shapiro-Wilk test, participants’ scores on the test trials were not normally distributed in any of the four groups (all $p < .001$). We therefore tested our hypothesis using non-parametric tests. To test if there are any significant differences between the groups in our repeated-measures design, we performed a Wald test. We found that people’s performance differed significantly across the four experimental conditions ($F = 15.68, p = .001$). Planned pair-wise Wilcoxon rank-sum tests confirmed that teaching people strategies discovered by the hierarchical method significantly improved their performance (204.48 points/trial) compared to the control condition (177.36 points/trial, $p = .006, d = 0.304$), the feedback-based cognitive tutor (167.02 points/trial, $p < .001, d = 0.4$), and the non-hierarchical tutor ($p = .025, d = 0.258$). By contrast, neither the feedback-based cognitive tutor ($p = .45, d = 0.104$) nor the non-hierarchical cognitive tutor ($p = .67, d = 0.045$) were more effective than letting people practice the task on their own. These results show that the hierarchical method is able to discover and teach the discovered strategy in environments in which previous methods failed.

5.2 Experiment 2: Teaching people the optimal strategy for a risky environment

In Experiment 2, participants were taught the strategy our method discovered for the 8-step decision problem illustrated in Figure 6. Critically, in this environment each path contains one risky node that harbors an extreme loss with a probability of 10%. Therefore, the optimal strategy for this environment inspects the risky node while planning how to achieve the selected goal and then switches to another goals when it encounters a large negative reward on the path to the initially selected goal.

5.2.1 Method

To test whether our approach can also improve people’s performance in environments with this more complex structure, we created two demonstration-based cognitive tutors that teach the strategies discovered by hierarchical BMPS with
goal-switching and hierarchical BMPS without goal-switching, respectively, and a feedback-based tutor that teaches the optimal strategy for a 3-step version of the risky environment.

This experiment used a Flight Planning Task that is analogous to the environment described in Section 4.2 (see Figure 6). Specifically, the environment comprises 4 goal nodes and 60 intermediate nodes (i.e., 15 per goal). Although each goal can be reached through multiple paths all of those paths lead through an unavoidable node that has a 10% risk of harboring a large loss of -1500. The aim of this experiment is to verify that we are still able to improve human planning even when environment requires a more complex strategy that occasionally switches goals during planning. To test this hypothesis we showed the participants demonstrations of the strategy discovered by our method in the experimental condition, and compared their performance to the performance of three control groups. The first control group was shown demonstrations of the strategy discovered by the version of our method without goal switching; the second control group discovered their own strategy in five training trials; the third control group practiced planning on a three-step task with a feedback-based tutor (see Section 2.6) (Lieder et al., 2020, 2019). The environment used by the feedback-based tutor mimicked the high-risk environment. To achieve this we changed the reward distribution of the intermediate nodes so that there is a 10% chance of a negative reward of -96, a 30% chance of -4, a 30% chance of +4, and a 30% chance of +8. We then recomputed the optimal feedback using dynamic programming (Lieder et al., 2020, 2019).

We recruited 201 participants (average age 34.01 years, range:19–70 years; 101 female) on Amazon’s Mechanical Turk (Litman et al., 2017) over three consecutive days. All but two of them completed the assignment. Applying the same pre-determined exclusion criterion as we used in Experiment 1 (i.e., excluding participants who do not engage with the environment in more than half of the test trials) led to the exclusion of 30 participants (15%), leaving us with 169 participants. Participants were paid 1.30$ and a performance dependent bonus of up to 1$. The average bonus was 0.56$ and the average time of the experiment was 16.28 minutes. Participants were randomly assigned to one of four experimental conditions determining their training in the planning task. All groups were tested in five identical test trials. The data was analysed using the robust f1.ld.f1 function of the nparLD R package (Noguchi et al., 2012). On each trial the participant’s score was calculated as the expected reward of the path they chose minus the the cost of the clicks they had made.

5.2.2 Results

The results of the experiment are summarized in Table 6. Since the Shapiro-Wilk test shows that none of the four conditions are normally distributed (p < .001 for all), we again use the non-parametric Wald test to evaluate the data. The Wald test shows significant differences between the four groups (F = 62.6, p < .001). Pairwise robust post-hoc comparisons show that participants trained with demonstrations of the strategy discovered by hierarchical BMPS with goal-switching significantly outperform all other conditions: participants trained by purely hierarchical demonstrations without goal switching (p < .001, d = 0.72), participants who did not

7 As in the simulations, the cost per click was 10.
receive demonstrations \((p < .001, d = 0.51)\), and participants who had practiced planning with optimal feedback on a smaller analogous environment \((p < .001, d = 0.53)\). The performance of the three control groups was statistically indistinguishable (all \(p \geq 0.18\)). Participants trained by purely hierarchical demonstrations did not perform significantly better than participants that trained with optimal feedback \((p = .18, d = 0.11)\) or participants that did not receive demonstrations \((p = .56, d = 0.07)\). Additionally there was no significant difference between the optimal feedback condition and the no demonstration condition \((p = .6, d = 0.03)\).

The results of this experiment show that we can significantly improve human decision-making by showing them demonstrations of the automatically discovered hierarchical planning strategy with goal-switching. This is a unique advantage of our new method because none of the other approaches was able to improve people’s decision-making in this large and risky environment. By comparing human performance to the optimal performance of our algorithm in the same environment (see Table 4) we can see that even though we were able to improve human performance, participants still did not fully understand the strategy based on the demonstrations alone. This reveals the limitations of teaching planning strategies purely with demonstrations, especially for more complex strategies. Improving upon the pedagogy of our purely demonstration-based hierarchical tutor is an important direction for future work.

| Algorithm                | N   | Reward  | Std    |
|-------------------------|-----|---------|--------|
| No demonstration        | 42  | -103.31 | 173.69 |
| Goal-switching demonstration | 45  | -26.71  | 121.87 |
| Hierarchical demonstration | 39  | -94.41  | 51.75  |
| Feedback tutor          | 43  | -108.49 | 179.64 |

Table 6: Experimental results of teaching people automatically discovered planning strategies for the high risk environment shown in Figure 6. For each condition we report the number of participants, the mean expected reward and the standard deviation of the mean expected reward.

6 General Discussion

To make good decisions in complex situations people and machines have to use efficient planning strategies because planning is costly. Efficient planning strategies can be discovered automatically. But computational challenges confined previous strategy discovery methods to tiny problems. To overcome this problem, we devised a more scalable machine learning approach to automatic strategy discovery. To overcome this problem, we devised a more scalable machine learning approach to automatic strategy discovery. The central idea of our method is to decompose the strategy discovery problem into discovering goal-setting strategies and discovering goal achievement strategies. In addition, we made a substantial algorithmic improvement to the state-of-the-art method for automatic strategy discovery \cite{Callaway2018} by introducing the tree-contraction method. We found that this hierarchical decomposition of the planning problem, together with
Discovering Efficient Strategies for Hierarchical Planning

our tree contraction method, drastically reduces the time complexity of automatic strategy discovery without compromising on the quality of the discovered strategies in many cases. Furthermore, by introducing the tree contraction method we have extended the set of environment structures that automatic strategy discovery can be applied to from trees to directed acyclic graphs. These advances significantly extend the range of strategy discovery problems that can be solved by making the algorithm faster, more scalable, and applicable to environments with more complex structure. This is an important step towards discovering efficient planning strategies for real-world problems.

Recent findings suggests that teaching people automatically discovered efficient planning strategies is promising way to improve their decisions (Lieder et al., 2020, 2019). Due to computational limitations this approach was previously confined to sequential decision problems with at most three steps (Lieder et al., 2020, 2019). The strategy discovery methods developed in this article make it possible to scale up this approach to larger and more realistic planning tasks. As a proof-of-concept, we showed that our method makes it possible to improve people’s decisions in planning tasks with up to 7 steps and up to 10 final goals. We evaluate the effectiveness of showing people demonstrations of the strategies discovered by our method in two separate experiments where the environments were so large that previous methods were unable to discover planning strategies within a time budget of eight hours. Thus, the best one could to at training people with previous methods was to construct cognitive tutors that taught people the optimal strategy for a smaller environment with similar strategy or having people practice without feedback. Evaluating our method against these alternative approaches we found that our approach was the only one that was significantly more beneficial than having people practice the task on their own. To the best of our knowledge, this makes our algorithm the only strategy discovery method that can improve human performance on sequential decision problems of this size. This suggests that our approach makes it possible to leverage reinforcement learning to improve human decision-making in problems that were out of reach for previous intelligent tutors.

Our method’s hierarchical decomposition of the planning problem exploits that people can typically identify potential mid- or long-term goals that might be much more valuable than any of the rewards they could attain in the short run. This corresponds to the assumption that the rewards available in more distant states are more variable than the rewards available in more proximal states. When this assumption is satisfied, our method discovers planning strategies much more rapidly than previous methods and the discovered strategies are as good as or better than those discovered with the best previous methods. When this assumption is violated, the goal switching mechanism of our method can compensate for that mismatch. This allows the discovered strategies to perform almost as well as the strategies discovered by BMPS. Our method relies on this mechanism more the more strongly its assumption is violated. In doing so, it automatically trades off its computational speed-up against the quality of the resulting strategy. This shows that our method is robust to violations of its assumptions about the structure of the environment; it exploits simplifying structure only when it exists.

Some aspects of our work share similarities with recent work on goal-conditioned planning (Nasiriany et al., 2019; Pertsch et al., 2020), although the problem we solved is conceptually different. For comparison, both aforementioned methods optimize the route to a given final location, whereas our method learns a strategy...
for solving sequential decision problems where the strategy chooses the final state itself. Furthermore, while Nasiriany et al. (2019) specified a fixed strategy for selecting the sequence of goals, our method learns such a strategy itself. Critically, while policies learned by Nasiriany et al. (2019) select physical actions (e.g., move left), the metalevel policies learned by our method select planning operations (i.e., simulate the outcome of taking action $a$ in state $s$ and update the plan accordingly). Finally, our method explicitly considers the cost of planning to find algorithms that achieve the optimal trade-off between the cost of planning and the quality of the resulting decisions.

Our method’s scalability has its price. Since our approach decomposes the full sequential decision problem into two sub-problems (goal-selection and goal-planning), its accuracy can be limited by the fact that it never considers the whole problem space at once. This is unproblematic when the environment’s structure matches our method’s assumption that the rewards of potential goals are more variable than more proximal rewards. But it could be problematic when this assumption is violated too strongly. We mitigated this potential problem by allowing the strategy discovery algorithm to switch goals. Even with this adaptation, the discovered strategy is not optimal in all cases: Since the representation of the alternative goal reward is defined as its average expected reward, the algorithm will only switch goals if the current goal’s reward is below average. However, if the current goal’s expected return is above average, the discovered strategy will not explore other goals even when that would lead to a higher reward. On balance, we think that the scalability of our method to large environments outweighs this minor loss in performance.

The advances presented in this article open up many exciting avenues for future work. For instance, our approach could be extended to plans with potentially many levels of hierarchically nested subgoals. Future work might also extend our method so that any state can be selected as a goal. In its current form, our algorithm always selects only the environment’s most distant states (leaf nodes) as candidate goals. Future versions might allow the set of candidate goals to be chosen more flexibly such that some leaf nodes can be ignored and some especially important intermediate nodes in the tree can be considered as potential sub-goals. A more flexible definition and potentially a dynamic selection of goal nodes could increase the strategy discovery algorithm’s performance, and possibly allow us to solve a wider range of more complex problems. This would mitigate limitations of the increasing variance assumption by considering all potentially valuable states as (sub)goals regardless of where they are located.

The advances reported in this article have potential applications in artificial intelligence, cognitive science, and human-computer interaction. First, since the hierarchical structure exploited by our method exists in many real-world problems, it may be worthwhile to apply our approach to discovering planning algorithms for other real-world applications of artificial intelligence where information is costly. This could be a promising step toward AI systems with a (super)human level of computational efficiency. Second, our method also enables cognitive scientists to scale up the resource-rational analysis methodology for understanding the cognitive mechanisms of decision-making (Lieder and Griffiths, 2020) to increasingly more naturalistic models of the decision problems people face in real life. Third, future work will apply the methods developed in this article to train and support people in making real-world decisions they frequently face. Our approach is especially
Discovering Efficient Strategies for Hierarchical Planning

relevant when acquiring information that might improve a decision is costly or time consuming. This is the case in many real-world decisions. For instance, when a medical doctor plans how to treat a patient’s symptoms acquiring an additional piece of information might mean ordering an MRI scan that costs $1000. Similarly, a holiday planning app would have to be mindful of the user’s time when deciding which series of places and activities the user should evaluate to efficiently plan their road trip or vacation. Similar tradeoffs exist in project planning, financial planning, and time management. Furthermore, our approach can also be applied to support the information collection process of hiring decisions, purchasing decisions, and investment decisions. Our approach could be used to train people how to make such decisions with intelligent tutors (Lieder et al., 2020, 2019). Alternatively, the strategies could be conveyed by decision support systems that guide people through real-life decisions by asking a series of questions. In this case, each question the system asks would correspond to an adaptively chosen information gathering operation. In summary, the reinforcement learning method developed in this article is an important step towards intelligent systems with a (super)human-level computational efficiency, understanding how people make decisions, and leveraging artificial intelligence to improve human decision-making in the real world. At a high level, our findings support the conclusion that incorporating cognitively-informed hierarchical structure into reinforcement learning methods can make them more useful for real-world applications.

Declarations

Funding This project was funded by grant number CyVy-RF-2019-02 from the Cyber Valley Research Fund.

Conflicts of interest/Competing interests The authors declare that the have no conflicts of interest or competing interests.

Availability of data and material (data transparency) All materials of the behavioral experiments we conducted are available at https://github.com/RationalityEnhancement/SSD_Hierarchical/master/Human-Experiments. Anonymized data from the experiments is available at https://github.com/RationalityEnhancement/SSD_Hierarchical/master/Human-Experiments.

Code availability (software application or custom code) The code of the machine learning methods introduced in this article is available at https://github.com/RationalityEnhancement/SSD_Hierarchical.

Ethics approval The experiments reported in this article were approved by the IEC of the University of Tübingen under IRB protocol number 667/2018BO2 (“Online-Experimente über das Erlernen von Entscheidungsstrategien”).

Consent to participate (include appropriate statements) Informed consent was obtained from all individual participants included in the study.

Consent for publication Not applicable.
References

Botvinick MM (2008) Hierarchical models of behavior and prefrontal function. Trends in cognitive sciences 12(5):201–208
Callaway F, Lieder F, Krueger PM, Griffiths TL (2017) Mouselab-MDP: A new paradigm for tracing how people plan. In: The 3rd Multidisciplinary Conference on Reinforcement Learning and Decision Making, Ann Arbor, MI, URL https://osf.io/vmkrq/
Callaway F, Gul S, Krueger PM, Griffiths TL, Lieder F (2018a) Learning to select computations. Uncertainty in Artificial Intelligence
Callaway F, Lieder F, Das P, Gul S, Krueger P, Griffiths T (2018b) A resource-rational analysis of human planning. In: Kalish C, Rau M, Zhu J, Rogers T (eds) CogSci 2018
Callaway F, van Opheusden B, Gul S, Das P, Krueger P, Lieder F, Griffiths T (2020) Human planning as optimal information seeking. Manuscript under review
Carver CS, Scheier MF (2001) On the self-regulation of behavior. Cambridge University Press
Gigerenzer G, Selten R (2002) Bounded rationality: The adaptive toolbox. MIT press
Griffiths TL (2020) Understanding human intelligence through human limitations. Trends in Cognitive Sciences
Griffiths TL, Callaway F, Chang MB, Grant E, Krueger PM, Lieder F (2019) Doing more with less: meta-reasoning and meta-learning in humans and machines. Current Opinion in Behavioral Sciences 29:24–30
Hafenbrädl S, Waeger D, Marewski JN, Gigerenzer G (2016) Applied decision making with fast-and-frugal heuristics. Journal of Applied Research in Memory and Cognition 5(2):215–231
Hay N, Russell S, Tolpin D, Shimony SE (2014) Selecting computations: Theory and applications. arXiv preprint arXiv:14082048
Hertwig R, Grüne-Yanoff T (2017) Nudging and boosting: Steering or empowering good decisions. Perspectives on Psychological Science 12(6):973–986
Huys QJ, Eshel N, O’Nions E, Sheridan L, Dayan P, Roiser JP (2012) Bonsai trees in your head: how the Pavlovian system sculpts goal-directed choices by pruning decision trees. PLoS computational biology 8(3)
Kaelbling LP, Lozano-Pérez T (2010) Hierarchical planning in the now. In: Workshops at the Twenty-Fourth AAAI Conference on Artificial Intelligence
Kemtur A, Jain Y, Mehta A, Callaway F, Consul S, Stojcheski J, Lieder F (2020) Leveraging machine learning to automatically derive robust planning strategies from biased models of the environment. In: CogSci 2020, CogSci
Krueger PM, Lieder F, Griffiths TL (2017) Enhancing metacognitive reinforcement learning using reward structures and feedback. In: Proceedings of the 39th Annual Conference of the Cognitive Science Society, Cognitive Science Society
Lieder F, Griffiths TL (2020) Resource-rational analysis: understanding human cognition as the optimal use of limited computational resources. Behavioral and Brain Sciences 43
Lieder F, Krueger PM, Griffiths T (2017) An automatic method for discovering rational heuristics for risky choice. In: CogSci
Lieder F, Callaway F, Jain Y, Krueger P, Das P, Gul S, Griffiths T (2019) A cognitive tutor for helping people overcome present bias. In: RLDM 2019
Discovering Efficient Strategies for Hierarchical Planning

Lieder F, Callaway F, Jain YR, Das P, Iwama G, Gul S, Krueger P, Griffiths TL (2020) Leveraging artificial intelligence to improve people’s planning strategies. Manuscript in revision

Lin CH, Kolobov A, Kamar E, Horvitz E (2015) Metareasoning for planning under uncertainty. In: Twenty-Fourth International Joint Conference on Artificial Intelligence

Litman L, Robinson J, Abberbock T (2017) Turkprime.com: A versatile crowdsourcing data acquisition platform for the behavioral sciences. Behavior research methods 49(2):433–442

Marthi B, Russell SJ, Wolfe JA (2007) Angelic semantics for high-level actions. In: Seventeenth International Conference on Automated Planning and Scheduling, pp 232–239

Miller GA, Galanter E, Pribram KH (1960) Plans and the structure of behavior.

Mnih V, Kavukcuoglu K, Silver D, Graves A, Antonoglou I, Wierstra D, Riedmiller M (2013) Playing atari with deep reinforcement learning. arXiv preprint arXiv:1312.5602

Mockus J (2012) Bayesian approach to global optimization: theory and applications, vol 37. Springer Science & Business Media

Nasiriany S, Pong V, Lin S, Levine S (2019) Planning with goal-conditioned policies. In: Advances in Neural Information Processing Systems, pp 14843–14854

Noguchi K, Gel YR, Brunner E, Konietzchke F (2012) nparLD: An R software package for the nonparametric analysis of longitudinal data in factorial experiments. Journal of Statistical Software 50(12):1–23, URL http://www.jstatsoft.org/v50/i12/

Pertsch K, Rybkin O, Ebert F, Finn C, Jayaraman D, Levine S (2020) Long-horizon visual planning with goal-conditioned hierarchical predictors. arXiv preprint arXiv:2006.13205

Russell S, Norvig P (2002) Artificial intelligence: a modern approach

Russell S, Wefald E (1992) ‘principles of metareasoning. Artificial intelligence 49(1-3):361–395

Russell SJ, Wefald E (1991) Do the right thing: studies in limited rationality. MIT press

Sacerdoti ED (1974) Planning in a hierarchy of abstraction spaces. Artificial intelligence 5(2):115–135

Schapiro AC, Rogers TT, Cordova NI, Turk-Browne NB, Botvinick MM (2013) Neural representations of events arise from temporal community structure. Nature neuroscience 16(4):486

Sezener E, Dayan P (2020) Static and dynamic values of computation in mcts. In: Conference on Uncertainty in Artificial Intelligence, PMLR, pp 31–40

Simon HA (1956) Rational choice and the structure of the environment. Psychological review 63(2):129

Solway A, Diuk C, Córdova N, Yee D, Barto AG, Niv Y, Botvinick MM (2014) Optimal behavioral hierarchy. PLoS computational biology 10(8)

Sutton RS, Barto AG (2018) Reinforcement learning: An introduction. MIT press

Svegliato J, Zilberstein S (2018) Adaptive metareasoning for bounded rational agents. In: CAI-ECAI Workshop on Architectures and Evaluation for Generality, Autonomy and Progress in AI (AEGAP), Stockholm, Sweden

The GPyOpt authors (2016) GPyOpt: A Bayesian optimization framework in Python. http://github.com/SheffieldML/GPyOpt
Todd PM, Gigerenzer GE (2012) Ecological rationality: Intelligence in the world. Oxford University Press
Tomov MS, Yagati S, Kumar A, Yang W, Gershman SJ (2020) Discovery of hierarchical representations for efficient planning. PLoS computational biology 16(4):e1007594
Wolfe J, Marthi B, Russell S (2010) Combined task and motion planning for mobile manipulation. In: Twentieth International Conference on Automated Planning and Scheduling
Appendix

A.1 VOC features

The features used to approximate the value of information can be explained using a simplified Mouselab-MDP environment. The ground truth values of each node is depicted in Figure A11(a). The rewards are sampled from a Categorical equiprobable distribution of \{-2, -1, 1, 2\}, \{-8, -4, 4, 8\} and \{-48, -24, 24, 48\} for nodes of depth 1, 2 and 3 respectively. In this example, the value of information for two metalevel action are considered, marked in Figure A11(b).

To compute the VOC, the possible values of a subset of nodes are considered and the cumulative reward accumulated from the maximal path is computed to find the expected return if the node values were known. Subtracting this from the cumulative reward from the maximal path given the current belief state gives the value of information of knowing performing the metalevel action. The greater the difference in the two quantities imply greater information is possibly gained for performing the computation.

In case of myopic VOC computation, the subset of nodes considered is just the node whose reward would be revealed by the metalevel action. VPI considers the entire subset of nodes in the environment. VPI$_{sub}$ considers all nodes lying on paths which pass through the node revealed by the computation. For example, when considering the VPI$_{sub}$ for the computation which reveals the node marked in blue in Figure A11(b), nodes 1, 2 and 4 would be considered. Whereas, nodes 1, 3 and 7 would be considered for the computation corresponding to revealing the value of the green node.

![Diagram](https://via.placeholder.com/150)

Fig. A11: (a) Simplified example environment with the rewards beneath each node marked in the figure. In the beginning, the rewards are hidden. Metalevel actions reveal the corresponding reward of a node. (b) State of the environment. Two metalevel actions are being considered which reveal the corresponding nodes as denoted by the blue (4) and green (7) color.

A.2 Time complexity upper bound analysis

In this section we analyse the computational time upper bound of the methods that we used. For simplicity, in the hierarchical case, we assume that goal switching is not possible. That means that the high level controller would run once, followed by the low level controller. To ensure readability, we explicitly define the notations used in this section and throughout the paper:

- $N$: number of intermediate nodes to goal
- $M$: number of goal states
- $B$: number of bins to discretize a continuous probability distribution
- $RUN$: number of unrevealed nodes relevant to compute feature
| Strategy Discovery Algorithm | Feature   | \(O(RUN)\) | \(O\) |
|------------------------------|-----------|-------------|-------|
| Hierarchical BMPS           | \(\text{VOI}_H\) | 1           | \(B\) |
|                             | \(\text{VPI}_H\) | \(M\)       | \(B^M\) |
|                             | \(\text{VOI}_T\) | 1           | \(B\) |
|                             | \(\text{VPI}_T\) | \(N\)       | \(B^N\) |
|                             | \(\text{VPI}_\text{sub}\) | \(N\)   | \(B^N\) |
| Hierarchical greedy myopic VOC | \(\text{VOI}_H\) | 1           | \(B\) |
|                             | \(\text{VOI}_T\) | 1           | \(B\) |
| Non-hierarchical BMPS       | \(\text{VOI}_1\) | 1           | \(B\) |
|                             | \(\text{VPI}_{\text{sub}}\) | \(N\)   | \(B^N\) |
|                             | \(\text{VPI}\) | \(M \cdot N\) | \(B^{M \cdot N}\) |
| Non-hierarchical greedy myopic VOC | \(\text{VOI}_1\) | 1           | \(B\) |

Table 7: Asymptotic time to compute a feature for hierarchical and non-hierarchical strategy discovery algorithms

The hierarchical decomposition reduced the time complexity of the myopic strategy discovery problem from \(O((M \cdot N)^2 \cdot B)\) to \(O((M^2 + N^2) \cdot B)\). For the BMPS algorithm, the hierarchical structure reduces the computational time upper bound from \(O((M \cdot N)^2 \cdot B^N)\) to \(O(M^2 \cdot B + B^M + N^2 \cdot B^N)\). The reduction in the upper bound implies that algorithms that use hierarchical structure are scalable to more complex environments.

As discussed in Section 2.5, metalevel actions are selected based on maximising the approximate VOC. At each step, a metalevel action that maximizes the approximate VOC is chosen. Selection of a metalevel action converts the probability distribution of the chosen node to a Dirac delta function concentrated at the revealed node value. The calculation of VOI features for continuous probability distributions requires computations of multiple cumulative distribution functions (CDF). In general, this procedure is computationally expensive and an approximation of it inherently requires discretization. Hence, the probability density function (PDF) of a continuous probability distribution associated with a node in the environment is discretized into \(B\) bins. As the number of bins \(B\) increases, the discrepancy between the approximation and the true PDF/CDF decreases and it shrinks to 0 as \(B \to \infty\) at the cost of higher computation cost.

The number of relevant unrevealed nodes (RUN) is the count of unrevealed nodes relevant to compute a VOI feature. The RUN varies for different algorithm features and directly affects the time required to compute their values for a given state-action pair. For calculating the approximate VOC, the number of possible values to be compared is \(B^{\text{RUN}}\). Each possible value set requires \(\alpha \in \mathbb{R}_{\geq 0}\) time to compute the highest cumulative return from all the possible outcomes of the set. Hence, the time required to calculate the approximate VOC scales with \(\alpha \cdot B^{\text{RUN}}\). For myopic VOC estimation, the number of relevant unrevealed nodes is 1. For BMPS, the number of relevant nodes for each of the VOI features \(\mathcal{F} = \{\text{VOI}_1, \text{VPI}, \text{VPI}_{\text{sub}}\}\) is different. The \(\text{VOI}_1\) feature requires the least time for computation since its value depends on only one node in the environment. On the other hand, the most time-consuming calculation is for the VPI feature since its value depends on the all nodes in the environment. The \(\text{VPI}_{\text{sub}}\) feature considers values of all paths that pass through a node evaluated by the metalevel action. Hence, the most time consuming calculation of \(\text{VPI}_{\text{sub}}\) is for metalevel actions that correspond to the goal node. Speaking about algorithmic complexity in terms of the big-O notation, it takes \(O(B)\) time to calculate the myopic VOC value for a given state-action pair. On the contrary, it takes \(O(B^{\text{RUN}})\) time to calculate the \(\text{VPI}_{\text{sub}}\) value for a given state-action pair.

The maximum amount of computational time to calculate the approximate VOC directly depends on the selection of all possible metalevel actions, for which we prove upper bounds for both the non-hierarchical (in Section A.3) and hierarchical strategy discovery problem (in Section A.4).
A.3 Time complexity of the non-hierarchical strategy discovery problem

In the setting of the non-hierarchical strategy discovery problem, the metalevel policy has to initially select the best metalevel action from \( M \cdot (N + 1) \) possible actions. This selection requires \( M \cdot (N + 1) \) VOI feature computations. Since computation of VPI is required only once for a given state, the next computationally most-expensive feature is \( \text{VPI}_{\text{sub}} \), which is computed for each possible action. From all possible actions, the ones that demand most of the computational time to compute \( \text{VPI}_{\text{sub}} \) are the actions that correspond to the goal nodes. In this case, the computation of \( \text{VPI}_{\text{sub}} \) takes \( O(B^N) \) time.

In general, if there are \( M \) goals in the environment and each goal consists of \( N + 1 \) nodes (i.e., \( N \) intermediate nodes + 1 goal node), the maximum number of metalevel actions performed including the termination action is \( M \cdot (N + 1) + 1 \).

In the non-hierarchical BMPS strategy discovery problem, the computational time upper bound to perform all metalevel actions and terminate is

\[
\sum_{i=0}^{M(N+1)+1} [M \cdot (N+1) - i] \cdot O(B^{(N+1)}) = \sum_{i=0}^{M(N+1)} [M \cdot (N+1) - i] \cdot O(B^N) \quad (8)
\]

\[
= O((M \cdot N)^2 \cdot B^N) \quad (9)
\]

For the greedy myopic strategy discovery algorithm, the number of relevant nodes (RUN = 1) reduces the second term in the equation to \( O(B) \). In this case, the computational time upper bound to perform all metalevel actions and terminate is

\[
\sum_{i=0}^{M(N+1)+1} [M \cdot (N+1) + 1 - i] \cdot O(B) = \sum_{i=0}^{M(N+1)} [M \cdot (N+1) + 1 - i] \cdot O(B) \quad (10)
\]

\[
= \frac{(M \cdot (N+1) + 1) \cdot (M \cdot (N+1) + 2)}{2} \cdot O(B) \quad (11)
\]

\[
= O((M \cdot N)^2 \cdot B) \quad (12)
\]

A.4 Time complexity of the hierarchical strategy discovery problem

In the setting of the hierarchical strategy discovery problem, the action space shrinks severely for each metalevel action selection. During the goal-setting phase of the procedure, the number of high-level actions including the high-level policy termination is \( M + 1 \). The selection of a high-level action, requires the 1 computation of \( \text{VPI}_H \) and at most \( M \) computations of \( \text{VOI}_H \).

Therefore, the computational time upper bound for this phase is

\[
\sum_{i=0}^{M} [(M - i) \cdot O(B) + O(B^{M-i})] = O(M^2 \cdot B) + \sum_{i=0}^{M} O(B^{M-i}) = O(M^2 \cdot B + B^M) \quad (13)
\]

Similarly, during the goal-achievement phase of the algorithmic procedure, the number of low-level metalevel actions for each goal is \( N + 1 \). The most time consuming feature calculated in the goal-achievement phase is \( \text{VPI}_{\text{sub}} \). Therefore, the computational-time upper bound for this phase per goal is

\[
\sum_{i=0}^{N+1} (N + 1 - i) \cdot O(B^{N+1-i}) = O(N^2 \cdot B^N) \quad (14)
\]

To calculate the upper bound for the hierarchical strategy discovery algorithm, the combined computational time for the high-level and low-level policy is the sum of the computational time consumed on both levels independently. Additionally, since the value of a goal node is not required in the goal-achievement procedure, the computational-time upper bound of metalevel actions at the low level is bounded by the number of intermediate nodes \( N \) for each goal separately.
The computational time upper bound of the hierarchical BMPS is

\[
\sum_{i=0}^{M} \left( (M - i) \cdot O(B) + O(B^{(M-i)}) \right) + \sum_{j=0}^{N} (N - j) \cdot O(B^{(N-j)}) = O(M^2 \cdot B + B^M + N^2 \cdot B^N)
\]  

(15)

In the case of myopic approximation, the number of relevant nodes at each level is 1. Therefore, the computational time upper bound to perform all metalevel actions and terminate is

\[
\left( \sum_{i=0}^{M} (M - i) + \sum_{j=0}^{N} (N - j) \right) \cdot O(B) = \left( \frac{M \cdot (M + 1)}{2} + \frac{N \cdot (N + 1)}{2} \right) \cdot O(B)
\]

(16)

\[
= O((M^2 + N^2) \cdot B)
\]

(17)

A.5 Analysis of the speed-up achieved by the tree-contraction method

Table 8 shows an example computation of a single VPI feature (Callaway et al., 2018a) computation. The computational speedup allows us to solve larger environments in the same amount of time and contributes to scaling the algorithm to more realistic problems.

| Environment size | With tree contraction | Without tree contraction |
|------------------|-----------------------|-------------------------|
| Branching (3,3,3), 27 nodes | 0.004s | 0.018s |
| Branching (5,5,5), 125 nodes | 0.007s | 12.27s |
| Branching (6,6,6), 216 nodes | 0.013s | 328.22s |

Table 8: Comparison of computation time for a single VPI feature calculation on different environment sizes. The environments follow a three-step branching structure where all nodes in the environment have 3, 5 or 6 child nodes depending on the environment size.