A Realistic Deterministic Quantum Theory Using Borelian-Normal Numbers

T.N. Palmer
ECMWF, Shinfield Park, Reading
RG2 9AX, UK
tim.palmer@ecmwf.int

October 28, 2018
Abstract

Motivated by studies of the predictability of turbulent fluids, the elements of a deterministic quantum theory are developed, which reformulates and extends standard quantum theory. The proposed theory is ‘realistic’ in the sense that in it, a general $M$-level quantum state is represented by a single real number $0 \leq r \leq 1$, rather than by an element of a Hilbert space over the complex numbers. Surprising as it may seem, this real number contains the same probabilistic information as the element of the Hilbert space, plus additional information from which measurement outcome is determined. A crucial concept in achieving this is that of Borelian (number-theoretic) normality. The essential role of complex numbers in standard quantum theory is subsumed by the action of a set of self-similar permutation operators on the digits and places of the base-$M$ expansion of a base-$M$ Borelian-normal $r$; these permutation operators are shown to have complex structure and leave invariant the normality of the underlying real number.

The set of real numbers generated by these permutations defines not only the Hilbert space, but also, in addition, the sample space from which quantum measurement outcomes can be objectively determined. Dynamical real-number state reduction is precisely described by deterministic number-theoretic operators that reduce the degree of normality of $r$; from the degree of normality one can infer the standard quantum-theoretic trace rule for measurement probability. The (probabilistic) equivalence with standard quantum theory is demonstrated explicitly and in detail for the 2- and 3-level system.

It is suggested that many of the foundational difficulties of standard quantum theory arise directly from the treatment of the complex Hilbert space as axiomatic. The concepts of superposition, wave-particle duality, state reduction and measurement outcome, null, weak and counterfactual measurement, the exponential speed-up of quantum computers, the many-worlds interpretation, the classical limit of quantum theory, stochastic quantum theory, entanglement, non-locality and the Bell inequalities, are all discussed with new insights. These insights arise from mathematical properties of the proposed theory, rather than from particular metaphysical assumptions. It is shown that the real-number states $r$ of the proposed theory are precisely, using Bell’s terminology, its beables.
1 Introduction

In standard quantum theory, the quantum state is defined as an element of a
Hilbert space over the complex numbers. In this paper, it is suggested that
this theory’s well-known foundational difficulties, notably the measurement
problem, arise directly from the axiomatic status of the complex Hilbert
space. An alternative theory is proposed which has no such axiom.

More specifically, the elements of a reformulation and extension of quan-
tum theory are developed, each of whose states, whether associated with
a single qubit, or the whole universe, is described by a single real number
$0 \leq r \leq 1$. At first sight, such a proposal sounds preposterous. However, $r$
has structure not usually required of a state in conventional physical theory,
and it is this structure (and deterministic but non-arithmetic operators de-
lected from it) that distinguishes the proposed theory from others based, for
example, on conventional differentiable dynamics. As an illustration, a qubit
or 2-level system in the Hilbert-space state $|0\rangle + e^{i\lambda}|1\rangle$ of standard quantum
theory is described, in the proposed theory, by a single base-2 Boreian-
normal real number. By contrast, the pure Hilbert-space eigenstates $|0\rangle$ and
$|1\rangle$ are described, in the proposed theory, by the non-normal real numbers
whose binary expansions are $.000\ldots$ and $.111\ldots$ respectively.

Borelian normality is a commonplace concept in number theory, eg Hardy
and Wright(1979), though not, as mentioned, in physical theory. Roughly
speaking, a number $0 \leq r \leq 1$ is said to be base-2 normal if the digits 0
and 1 in the binary expansion of $r$ appears with equal frequency, and if,
additionally, the proportions of all possible runs of binary digits of a given
length are also equal. See the footnote in section 2 for more details. More
generally, for an $M$-level system, a maximally-superposed Hilbert-space state
in standard quantum theory is described by a single base-$M$ normal real-
number in the proposed theory, whilst the pure Hilbert-space eigenstates
$|0\rangle, |1\rangle, \ldots |M\rangle$, where $M' = M - 1$, are described by the $M$ non-normal
real-numbers whose base-$M$ expansions are $.000\ldots, .111\ldots, .222\ldots$, up to
$.M'M'M'\ldots$.

As suggested, one of the key motivations for developing the proposed
theory was to provide a fresh approach to the measurement problem. As
discussed by Kent (2002), this is essentially the problem of finding a precise
mathematical characterisation of the sample space in which quantum mea-
asurement probabilities are defined. For an $M$-level real-number state, this
sample space is defined in the proposed theory as the countable, everywhere
discontinuous set of real numbers generated by a family of self-similar permutation operators acting on the digits and places in the base-$M$ expansion of some base-$M$ normal $r$. It is shown that these operators have a well-defined complex structure (equivalent to $M$th roots of unity), and it is this complex structure which subsumes the essential role of complex numbers in defining the Hilbert-space state in standard quantum theory. These permutation operators, discussed in detail in sections 2 and 3 for $M = 2$ and $M = 3$ respectively, define the heart of the paper.

In addition to these self-similar permutation operations (which leave invariant the Borelian normality of the associated real number), a class of number-theoretic operators are defined which reduce the degree of normality of $r$ (see sections 3 and 4). For example, if $r = .02110211\ldots$ is the base-3 expansion of a base-3 normal number, then $r_{\perp 2} = .011011\ldots$ (i.e. deleting all occurrences of the digit ‘2’), defines the binary expansion of a base-2 normal number. These reduction operators are equivalent to the Hilbert-space projection operators of standard quantum theory. However, these number-theoretic reduction operators allow quantum-state reduction to be defined as a process governed by a precise deterministic mathematical procedure (‘$R$’). In this way, the proposed theory extends standard quantum theory. For example, with $|0\rangle + e^{i\lambda}|1\rangle$ described by the base-2 normal real $r$, then, under $R$, $r \mapsto r_{\perp 1} = .000\ldots$ if $r < 1/2$, and $r \mapsto r_{\perp 0} = .111\ldots$ if $r \geq 1/2$. The numbers $.000\ldots$ and $.111\ldots$ are fixed points of $R$. Under $R$, the state space of a real-number qubit state can be described as comprising two arbitrarily-intertwined basins of attraction of point attractors $A_0$ and $A_1$, where $r = .000\ldots$ and $r = .111\ldots$ respectively.

In sections 4 and 5, the probabilistic equivalence between the Hilbert-space description of standard quantum theory and the real-number state of the proposed theory, is discussed in detail for 2- and 3-level systems. For the qubit, the real number equivalent of the Bloch sphere is constructed explicitly. It is shown that quantum measurement probabilities in the proposed theory satisfy the trace rule. The equivalent of some explicit unitary transformations in standard theory are given in section 5. The (intrinsically irreversible) properties of such number-theoretic reduction operators are consistent with properties of quantum gravity, as speculated by Penrose (1989, 1994, 1998). The notions of quantum-state superposition, polarisation and interference are discussed in section 5, from the perspective of the proposed theory. As discussed in section 7, the proposed theory provides a simple and novel relationship between the notions of reduction and measurement.
outcome, without recourse to some arbitrary classical/quantum split; in the
proposed theory, an observer (or indeed the whole cosmos) is merely an \( M \)-
level system where \( M \gg 1 \). A brief description of the (non-singular) classical
limit of the proposed theory is given.

The formalism developed in this paper was motivated by studies of the
predictability of the Navier-Stokes equations for turbulent fluids in mete-
orology (Palmer, 2000). Exploiting this, the concept of a ‘Navier-Stokes’
computer is introduced in section 8 to show how the proposed theory pro-
vides an alternative single-world view to the many-worlds explanation of the
exponential speed-up of certain quantum computational problems over their
digital counterparts.

One of the key features of the sample space comprising the reals con-
structed by the family of self-similar permutations, is that it is everywhere
discontinuous and, for a qubit, is only definable on a countable set of meridi-
ans of the Bloch-sphere equivalent. As discussed in section 9, this has impor-
tant physical consequences for the proposed theory, most importantly that
counterfactual measurements, whose outcomes are not elements of reality by
definition, cannot be associated with well-defined real-number states. In this
sense, the proposed theory is fundamentally different from any classical de-
terministic theory. With such counterfactual indefiniteness, it is shown that
the theory can violate the Bell inequalities, and yet be local, at least in the
EPR (Einstein et al, 1935) sense of the word.

In summary, for any mathematically well-defined real-number state of the
proposed theory, its transform under reduction (and therefore measurement)
is determined precisely. Conversely, measurements which by definition can-
ot be elements of reality, are not associated with real-number states. Hence
real-number states and elements of reality are in one-to-one correspondence
in the proposed theory. It is for this reason that the real-number states \( r \) of
the proposed theory that include both microscopic particles and macroscopic
observers, are precisely its beables.

---

\(^1\) To quote Bell(1993): “The beables of the theory are those elements which might
correspond to elements of reality, to things which exist. Their existence does not depend
on ‘observation’. Indeed observation and observers must be made out of beables.”
2 Complex structure from self-similar permutations

Let
\[ \mathcal{S}_r = \{ a_1, a_2, a_3, \ldots \}, \]  
(1)
\[ a_i \in \{0, 1\}, \]  
denote a bit string whose elements define the binary expansion
\[ r = .a_1a_2a_3 \ldots \]  
(2)
of some real number \(0 \leq r \leq 1\). Let \(\phi(0) = 1, \phi(1) = 0\), and
\[ -\mathcal{S}_r = \{ \phi(a_1), \phi(a_2), \phi(a_3), \ldots \}, \]  
(3)
so that \(-(-\mathcal{S}_r) = \mathcal{S}_r\). Defining
\[ i(\mathcal{S}_r) = \{ \phi(a_2), a_1, \phi(a_4), a_3, \phi(a_6), a_5, \phi(a_8), a_7, \ldots \} \]  
\[ i^{1/2}(\mathcal{S}_r) = \{ \phi(a_4), a_3, a_1, a_2, \phi(a_8), a_7, a_5, a_6, \ldots \} \]  
(4)
then it is easily shown that \(i * i(\mathcal{S}_r) = -\mathcal{S}_r\), and \(i^{1/2} * i^{1/2}(\mathcal{S}_r) = i(\mathcal{S}_r)\). The operators \(i\) and \(i^{1/2}\) induce the real-number transformations
\[ r \mapsto \tilde{i}(r) = .\phi(a_2)a_1\phi(a_4)a_3\phi(a_6)a_5\phi(a_8)a_7 \ldots \]  
\[ r \mapsto \tilde{i}^{1/2}(r) = .\phi(a_4)a_3a_1a_2\phi(a_8)a_7a_5a_6 \ldots \]  
(5)

This construction is easily generalised to a family of permutation operators \(i^{1/2^n}\) (integer \(n \geq 0\)) such that
\[ i^{1/2^n} * i^{1/2^n}(\mathcal{S}_r) = i^{1/2^{n-1}}(\mathcal{S}_r) \]  
(6)
To do this, it will be convenient to write
\[ \mathcal{S}_r = \{ \beta_1^{(n)}, \beta_2^{(n)}, \beta_3^{(n)}, \ldots \} \]  
(7)
where \(\beta_j^{(n)}\) is the \(j\)th contiguous \(2^n\)-tuplet of elements of \(\mathcal{S}_r\), ie
\[ \beta_j^{(n)} = \{ a_{(j-1)2^n+1}, a_{(j-1)2^n+2}, \ldots, a_{j2^n} \}. \]  
(8)
Consider an operator \(\chi^{(n)}\) which permutes the digits and the places of the elements of each \(\beta_j^{(n)}\) as follows: operate on the last element of \(\beta_j^{(n)}\) with
\(\phi\), then swap the last two contiguous elements of \(\beta_j^{(n)}\), then swap the last two contiguous pairs of elements of \(\beta_j^{(n)}\), then swap the last two contiguous quadruplets of elements and so on, finally swapping the two contiguous \(2^{n-1}\)-tuplets of elements of \(\beta_j^{(n)}\).

\(\chi^{(n)}\) can be expressed more succinctly as the \(2^n \times 2^n\) matrix defined by the iterative self-similar block form

\[
\chi^{(n)} = \begin{pmatrix} 0 & I \\ \chi^{(n-1)} & 0 \end{pmatrix}
\]

(9)

where \(I\) is the \(2^{n-1} \times 2^{n-1}\) identity matrix and \(\chi^{(0)} = \phi\). The matrix acts on each \(\beta_j^{(n)}\), considered as a row vector. For example, from equation 9, the effect of \(\chi^{(3)}\) on \(\beta_1^{(3)}\) is represented by

\[
\begin{pmatrix}
\phi & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\phi & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

(10)

Based on these matrix operations, define

\[
i_{1/2^{n-1}}(S_r) = \{\beta_1^{(n)}, \beta_2^{(n)}, \beta_3^{(n)}, \ldots\}
\]

(11)

where the permuted \(2^n\)-tuple \(\beta_j^{(n)}\) is given by the elements of the row vector \(\beta_j^{(n)}\chi^{(n)}\). Equation 9 immediately follows from the identity

\[
\begin{pmatrix} 0 & I \\ \chi^{(n-1)} & 0 \end{pmatrix} \times \begin{pmatrix} 0 & I \\ \chi^{(n-1)} & 0 \end{pmatrix} = \begin{pmatrix} \chi^{(n-1)} & 0 \\ 0 & \chi^{(n-1)} \end{pmatrix}
\]

(12)

For any dyadic rational \(q = m/2^n\), integer \(m\) \(n\), one can therefore define

\[
i^q(S_r) = i_{1/2^n} \ast i_{1/2^n} \ast \ldots \ast i_{1/2^n}(S_r).
\]

(13)

which in turn induces the real-number transformation \(r \mapsto \tilde{i}^q(r)\).
The numbers $r$ on which $\tilde{id}$ operate are now considered. We first note that $\tilde{id}$ preserves Borelian normality. It is well known that almost all real numbers are normal (eg Hardy and Wright, 1979)\footnote{The concept of normal numbers was introduced by Emile Borel. Suppose the digit $b \in \{0, 1, 2 \ldots M - 1\}$ occurs $n_b$ times in the first $n$ places of the the base-$M$ expansion of some real $r$. If, for all $b$, $n_b/n \to 1/M$ as $n \to \infty$, then $r$ is simply normal in base $M$. If $r$ is simply normal in all of base $M, M^2, M^3 \ldots$, then $r$ is said to be normal in base $M$. Normality is a generic property of the reals, yet it is still unknown whether numbers such as $\pi$ and $e$ are normal in any base. In the discussion below this notion of normality is generalised slightly: $r$ will be said to be normal in base $M$, if it satisfies the conditions above, irrespective of the actual symbols used to represent the $M$ digits $b$. Hence, for example, .000 \ldots, .111 \ldots, .222 \ldots$ and so on, will all be described as base-1 normal. } If $r$ is base-2 normal in $\{0, 1\}$, then there is an equal probability of finding a ‘0’ or a ‘1’ at a given place in the binary expansion of $r$. Hence, the application of $\phi$ does not change the probability of finding a ‘0’ or ‘1’ at the given place. Similarly none of the swap permutations in the definition of $\chi^{(n)}$ alter the fraction of ‘0’s and ‘1’s in each $\beta_j^{(n)}$. Hence, if $r$ is a base-2 normal real, so is $\tilde{id}(r)$.

Let $r_0$ denote some arbitrary base-2 normal real; a possible choice is the binary version $r_C = 0.011011100101 \ldots$ of Champernowne’s famous normal number defined by concatenating the integers (eg Hardy and Wright, 1979). With angular coordinate $\lambda = 2\pi q$, and $q$ a dyadic rational, we define

$$r(\lambda) = \tilde{id}(r_0)$$  \hspace{1cm} (14)

so that $r(0) = r_0$ and each $r(\lambda)$ is base-2 normal. We now define the permutation operator $e^{2i\pi q}$ acting on arbitrary bit strings $S_r$ by

$$e^{2i\pi q}(S_r) = i^{4q}(S_r)$$  \hspace{1cm} (15)

Hence equation (14) can be written in the real-number ‘phase’ form

$$r(\lambda) = e^{i\lambda}(r_0)$$  \hspace{1cm} (16)

where $e^{i\lambda}$ is the real-number operator induced by $e^i$. Note that, where used in this paper, the expression $e^{ix}$ continues to denote the complex exponential of conventional mathematics.

Note that, by normality, every conceivable bit string $B_N$ of length $N$ can be expected to occur once in the first $2^N$ places in the binary expansion of any $r(2\pi q)$. As discussed below, this is the basis of the equivalence between the notion of superposition in the Hilbert-space formulation of standard quantum
theory, and the real-number formulation of the proposed theory. Similarly, for any base-2 normal $r_0$, there exists a $\lambda$ which is a dyadic rational multiple of $\pi$, such that the binary expansion of $r(\lambda)$ begins with $B_N$. Hence, the transformation $r_0 \mapsto r_0'$ between base-2 normals, can be effected by some $\lambda \mapsto \lambda'$ with fixed $r_0$. The independence of measurement statistics to the particular choice of $r_0$ is equivalent in standard quantum theory to the independence of measurement statistics to the choice of global phase factor in the standard qubit representation (cf equation 23).

For the discussion on the origin of exponential speed up in quantum computing in section 8, on counterfactual indefiniteness and quantum nonlocality in section 9, and on the concept of weak measurement and its relation to stochastic quantum theory in section 10, it is essential to note that $r(\lambda)$ does not vary continuously or monotonically with $\lambda$; indeed $r(\lambda)$ is not defined when $\lambda$ is not a dyadic rational multiple of $\pi$. To see this, let $\Delta q = 1/2^n$ and note that, by equation 14, $S_{r(2\pi\Delta q)} = i^{4\Delta q}(S_{r(0)})$. By definition, the first element of $S_{r(2\pi\Delta q)}$ is equal to the $\phi$ permutation of the $2^n-1$th element of $S_{r(0)}$. Hence the smaller is $\Delta q$, the further back in the string $S_{r(0)}$ is drawn the first element of $S_{r(2\pi\Delta q)}$. As a result, for arbitrarily small $\Delta \lambda$, the sequence of numbers $\{r_1, r_2, \ldots\}$ where $r_j = r(\lambda + j\Delta \lambda)$ has a lag-1 correlation of zero, that is, the sequence appears as if it were stochastic white noise. As a result, it is claimed in section 10 that the proposed theory can emulate some of the features of stochastic quantum theory, without actually being stochastic itself, nor being chaotic in the conventional dynamical-system sense.

All normal numbers are irrational. However, no aspect of the theory presented below requires us to consider strictly irrational numbers. Indeed, as discussed below, there may be physical constraints which limit the bit string length of the binary expansion of $r_0$. Let $\hat{r}_0$ denote a rational approximation to $r_0$ in the sense that the digits in the first $2^N$ places in the binary expansions of $\hat{r}_0$ and $r_0$ agree. Then, with $\hat{r}(0) = \hat{r}_0$, dyadic rational numbers $\hat{r}(2\pi j/2^N)$, where $j = 1, 2, 3 \ldots 2^N$ can be defined at a finite set of $2^N$ points on the circle, with a fundamental longitudinal spacing $2\pi/2^N$. Each $\hat{r}(2\pi j/2^N)$ is a rational approximation of the corresponding normal number $r(2\pi j/2^N)$ in the sense that the bits in the first $2^N$ places of their binary expansions agree. In the analysis that follows, it can be assumed that we are dealing with such rational approximations, rather than formally irrational numbers. As such the proposed theory is essentially finite, and in particular does not utilise any of the paradoxical properties of non-measurable infinite sets (eg Pitowsky, 1983).
3 A Number-Theoretic Reduction Procedure

In this section, the second essential non-arithmetic operation for constructing real-number states is proposed: number-theoretic reduction. As a simple example of this concept, the ‘reduced’ number \( r_{\perp 0}(\lambda) = .111\ldots \) arises by deleting all places where the digit ‘0’ occurs, in the binary expansion of \( r(\lambda) \). Similarly \( r_{\perp 1}(\lambda) = .000\ldots \) arises by deleting all places where the digit ‘1’ occurs. More generally, a number-theoretic procedure is defined below which generates a one-parameter family of numbers starting with the base-2 normal \( r(\lambda) \) and finishing with either of the base-1 normals .000\ldots or .111\ldots. As discussed below, since the degree of normality of these numbers decreases monotonically with this parameter, the procedure is referred to as ‘number-theoretic reduction’.

To define this procedure explicitly, and to relate with the Bloch sphere of standard quantum theory, let \( E \) denote the equator, and \( p_N \) and \( p_S \) the corresponding poles, of a two-sphere \( S^2 \). Let \( \theta \) denote co-latitude, and \( \Lambda \) denote the set of meridians whose longitude \( \lambda \) is a dyadic rational multiple of \( 2\pi \). Start with the bit string \( S = \{a_1a_2a_3\ldots\} \) defined from the binary expansion of \( r(\lambda) \), and then delete some of the digits \( a_j \) based on the following rules. If \( a_j = 1 \), then delete \( a_j \) from \( S \) iff

\[
.a_ja_{j+1}a_{j+2}\ldots < \cos^2 \frac{\theta}{2}
\]

Conversely, if \( a_j = 0 \) then delete \( a_j \) iff

\[
.a_ja_{j+1}a_{j+2}\ldots \geq \cos^2 \frac{\theta}{2}
\]

The real-number function \( r(\theta, \lambda) \) is defined from its binary expansion

\[
r(\theta, \lambda) = .a'_1a'_2a'_3\ldots
\]

where the \( a'_j \)'s correspond to the digits which have not been deleted from \( S \).

For example, at \( \theta = 0 \) all occurrences of \( a_j = 1 \) and no occurrences of \( a_j = 0 \) are deleted. At \( \theta = \pi/2 \), none of the \( a_j \) are deleted. At \( \theta = \pi \), all occurrences of \( a_j = 0 \) are deleted. Hence

\[
\begin{align*}
r(0, \lambda) &= .000\ldots \\
r(\pi/2, \lambda) &= r(\lambda) \\
r(\pi, \lambda) &= .111\ldots
\end{align*}
\]
More generally, based on an ensemble of numbers $r(\theta, \lambda)$ obtained by sampling $\lambda$ (from $\Lambda$), $P[r(\theta, \lambda) < 1/2] = \cos^2 \theta/2$, where $P$ is a probability measure.

To see this, note that if $r(\theta, \lambda) < 1/2$, then the first digit in the binary expansion of $r(\theta, \lambda)$ must be a zero. According to the reduction procedure above, $a'_1 = 0$ if $r(\lambda) < \cos^2 \theta/2$. Hence

$$P[r(\theta, \lambda) < 1/2] = P[r(\lambda) < \cos^2 \theta/2] \quad (21)$$

The required result now follows from the fact that for any base-2 normal number $0 < r < 1$ and any $\theta$, then $P[r < \cos^2 \theta/2] = \cos^2 \theta/2$. (To show this, let the binary expansion of $\cos^2 \theta/2$ be $.c_1c_2c_3\ldots$. Suppose $c_j = 0$ except where $j = j_1, j_2, j_3\ldots$. Hence, if $r < \cos^2 \theta/2$, then $a_1, a_2, \ldots a_{j_1-1}$ must all be equal to 0. If, in addition, $a_{j_1}$ is 0, then certainly $r < \cos^2 \theta/2$. Since $r$ is base-2 normal, the probability of ‘0’s in all of the first $j_1$th positions in the binary expansion of $r$ is equal to $1/2^{j_1}$ or, in binary notation, $.000\ldots1$ where the ‘1’ occurs in the $j_1$th place. By definition, this probability is equal to $.c_1c_2\ldots c_{j_1}$. Now in addition, $r < \cos^2 \theta/2$ if $a_{j_1} = 1$, but all other $a_i$ up to and including the $j_2$th element are equal to 0. The probability of such additional occurrences is (in binary) $.000\ldots1$ where now the ‘1’ occurs in the $j_2$th place. Continuing this argument, the total probability that $r < \cos^2 \theta/2$ is equal to

$$\underbrace{.000\ldots1}_{j_1} + \underbrace{.000\ldots1}_{j_2} + \underbrace{.000\ldots1}_{j_3} + \ldots \quad (22)$$

where the underbrace value gives the length of the binary expansion. By definition this sums to $.c_1c_2c_3\ldots = \cos^2 \theta/2 \text{ QED.}$

The quantity $\cos^2 \theta/2$ which on the one hand defines the probability that $r(\theta, \lambda) < 1/2$, can also be viewed as defining the ‘degree of normality’ of $r(\theta, \lambda)$, (the relative fraction of ‘0’s and ‘1’s in the binary expansion of $r$), with maximal base-2 normality at $\theta = \pi/2$, minimal base-2 normality at $\theta = 0, \pi$. As discussed below, this links in a fundamental way in the proposed theory, the probability of measurement outcome with the degree of normality of the underlying real-number quantum state.

Finally, note that if the value of $r$ at any point $p$ is $r(p)$, then the value of $r$ at the point $p'$ antipodal to $p$ is $r(p') = 1 - r(p)$. See Figure 1.
Figure 1: In the proposed theory, a state of a qubit is a real number $0 \leq r \leq 1$ defined on a countable subset $\Lambda$ of meridians on the two-sphere $S^2$. On the equator, $r$ is base-2 normal, defined from the family of self-similar permutation operators $i^q$ (with $q$ a dyadic rational) acting on some bit string $S_r$. The further away from the equator, the less $r$ is base-2 normal. The north and south poles, where $r = .000 \ldots$ and $r = .111 \ldots$ respectively, correspond to minimally-normal fixed points of a deterministic reduction operator $R$. Under $R$, an arbitrary real-number state $r$ is attracted to one or the other of these fixed points, depending on whether $r < 1/2$ or $r \geq 1/2$. In turn this determines, for example, whether a measurement outcome will be ‘spin up’ or ‘spin down’. The orientation of the poles is determined by the qubit’s coupling to some $M$-level system, $M \gg 1$, see section 7.
4 Relation to 2-level Quantum Theory

The central thesis of this paper is that \( r \), as defined in the previous two sections, is to be viewed as a real-number state in a deterministic theory which subsumes standard quantum theory. The symbol ‘\(<\)’ is used to represent this thesis inside mathematical equations. Hence, for the 2-level quantum state, the subsumption

\[
|\psi\rangle = e^{i\gamma}(\cos \frac{\theta}{2}|0\rangle + e^{i\lambda} \sin \frac{\theta}{2}|1\rangle) \triangleleft r(\theta, \lambda)
\]

is proposed between the standard quantum-theoretic Bloch-sphere representation of the Hilbert-space qubit state, and the real-number state \( 0 \leq r(\theta, \lambda) \leq 1 \).

However, if one is only concerned with the statistics of measurement outcome, as is the case in practice in quantum experimentation, then it is claimed that the proposed theory is equivalent to standard quantum theory. In this sense, the symbol ‘\(<\)’ also denotes an equivalence between the Hilbert-space state and the real-number state at the level of probabilities. Since \( |0\rangle \) and \( |1\rangle \) denote Hilbert-space eigenstates of some given Hermitian operator \( \mathcal{O} \) (eg associated with spin in some prescribed direction), this probabilistic equivalence is specific to the choice of operator (or ‘observable’). The number \( r \) is referred to as the ‘real-number \( \mathcal{O} \)-state’, or, hereafter, ‘real-number state’ of the two level system.

The issue of what determines the orientation of the polar axis in a theory which purports to construct observables from beables (the basis problem) is addressed in section 7. For the purposes of this section, the orientation of the axis are assumed given.

4.1 Global Phase Factor

As is well known, in standard quantum theory, the statistics of measurement outcome do not depend on the choice of global phase factor \( \gamma \) in equation 23. In the proposed theory, the choice of global phase factor \( \gamma \) is equivalent to the choice of base-2 normal \( r_0 \) associated with the zero of longitude. For example, if we write

\[
|0\rangle + |1\rangle \triangleleft r(\pi/2, 0) = r_0
\]

then

\[
e^{i\gamma}(|0\rangle + |1\rangle) \triangleleft r'(\pi/2, 0) = i^{2\gamma/\pi}(r_0)
\]
As discussed above, for any \( r_0 \), there exists a \( \gamma \) commensurate with \( \pi \) such that \( \bar{r}^{2\gamma /\pi} \) starts with any given finite bit string. Hence, as discussed below, the independence of the statistics of measurement outcome to the choice of \( r_0 \) is equivalent to the fact that the statistics of \( r(\theta, \lambda) \) are independent of \( \lambda \).

### 4.2 Number-Theoretic Reduction and Hilbert-Space Projection

Reduced numbers are an important component of the proposed theory. For example, if

\[
\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \equiv r(\pi/2, 0)
\]

(26)

is a base-2 normal real, then

\[
|0\rangle \equiv r_{\bot 1}(\pi/2, 0) = .000 \ldots
\]

\[
|1\rangle \equiv r_{\bot 0}(\pi/2, 0) = .111 \ldots
\]

(27)

are base-1 normal reals, and more generally,

\[
\cos \frac{\theta}{2} |0\rangle + \sin \frac{\theta}{2} |1\rangle \equiv r(\theta, \lambda)
\]

(28)

is a partially-reduced real number, (ie neither base-2-normal nor base-1-normal). As discussed in section 3, the degree of normality of \( r \) is defined to be equal to \( \cos^2 \theta/2 \). (The generic importance of reduced numbers will become more apparent when we consider 3-level and \( M \)-level systems below.) It can be noted that the concept of number-theoretic reduction in the proposed theory is equivalent to the concept of Hilbert-space state projection in conventional quantum theory (eg \( r \mapsto .000 \ldots \) is equivalent to \( |0\rangle + |1\rangle \mapsto |0\rangle \)). However, unlike conventional quantum theory, reduction is a deterministic process, internal to the dynamics of the proposed theory. This dynamical process is described in the next subsection.

### 4.3 Dynamical Real-Number State Reduction

One of the principal features of the proposed theory is that quantum state reduction is an objective process defined by a deterministic mathematical procedure. Moreover the theory provides a precise formulation of the sample space on which quantum reduction probabilities are defined. In this section
we describe this reduction criterion on a qubit; its generalisation to a 3-level system is described in section 4. In section 5 we show how real-number state reduction can be simply and deterministically linked to the notion of measurement outcome, without recourse to any artificial ‘classical/quantum’ split. Since the state is always a definite real number, the problem is never faced of ‘when’, during quantum state reduction, the notion of a complex superposition of outcomes is no longer relevant: in the proposed theory, it is never relevant. As a result, as discussed in section 7, the proposed model has a simple non-singular classical limit.

The process of real-number state reduction is based on the number-theoretic notion of reduction discussed in section 4 above. For the qubit, we first consider the reduction operation $R_j : [0,1] \rightarrow [0,1], j \in \{0,1\}$. Specifically, if
\[ r = .a_1a_2a_3\ldots \] (29)
is the binary expansion of some qubit real-number state $r$, then
\[ R_j : r \mapsto .jjj\ldots \text{ if } a_1 = j \]
\[ R_j = \text{Identity if } a_1 \neq j \] (30)

Hence under $R_1$, the real-number state evolves to $p_S$ if $r \geq 1/2$ and remains unchanged if $r < 1/2$. Under $R_0$, the real-number state evolves to $p_N$ if $r < 1/2$ and remains unchanged if $r \geq 1/2$. The real-number states .000\ldots and .111\ldots are fixed points of $R_0$ and $R_1$. We can form the compound reduction operator $R = R_0R_1$. Under $R$ the real-number state evolves to $p_S$ if $r \geq 1/2$, and to $p_N$ if $r < 1/2$. For example, $R_1$ could be used to ask the question: ‘was a particle detected’, whilst $R$ could be used to ask the question: ‘was the particle spin up or spin down’? A null measurement (‘particle not detected’) leaves the real-number state unchanged, but clearly does not imply that the real-number state is equal to zero.

It is easy to suggest a differential equation to describe the time evolution of the real-number state under $R$. For example, let
\[ \dot{\theta} = \alpha \left( r - \frac{1}{2} \right) \sin \theta \]
\[ \dot{\lambda} = 0 \] (31)
where $r = r(\theta, \lambda)$, and $\alpha$ is a parameter discussed in section 6 which sets the fundamental timescale for reduction. During the reduction process (as
described by equation (31), the degree of normality of the real-number state \( r \) reduces continuously and irreversibly. In this way, \( p_N \) and \( p_S \) can be considered attractors \( A_0 \) and \( A_1 \) respectively, of an irreversible deterministic dynamical system. Effectively, \( R \) defines a colouring of \( \Lambda \), eg \( p \) could be coloured red iff \( r(p) \mapsto \ldots .000 \ldots \) under \( R \), blue iff \( r(p) \mapsto \ldots .111 \ldots \) under \( R \). Given the properties of \( r \) then if \( p \) is coloured red, the antipodal point \( p' \) is coloured blue (and vice versa). If \( p \notin \Lambda \) then \( p \) is not coloured. Hence, red points lie in the basin of attraction of \( A_0 \); blue points lie in the basin of attraction of \( A_1 \). Because of the discontinuous and non-monotonic nature of \( r \), then there are blue points in any \( \lambda \) interval of any red point (and vice versa). Effectively equation (31) describes a deterministic intertwined-basin dynamical system (Ott et al, 1993; Palmer, 1995; Duane, 2001; Nicolis et al, 2001). However, because of the non-differentiable nature of \( r \), (31) is qualitatively unlike any intertwined-basin system based on conventional chaotic dynamics. This qualitative difference with conventional deterministic systems will be particularly relevant when the reason why the proposed theory can violate Bell’s theorem is discussed (section 9).

As described in section 7 below, the general real-number state of an \( M \)-level system is described by a real \( 0 \leq r \leq 1 \) which is at most base-\( M \) Borelian normal. In general, it is postulated that the effect of the reduction operator on the real-number state of an isolated system, is to reduce monotonically the state’s degree of Borelian normality. Hence if \( M \) is large, and the system at time \( t \) is well approximated by a base \( M(t) \)-normal real, then, in the proposed theory, then \( M(t_1) < M(t_0) \leq M \), where \( t_1 > t_0 \). This number-theoretic reduction process cannot derive from existing theories of physics, none of which attributes any significance to the degree of normality of the state, and none of which is irreversible in the sense described. We must therefore seek some fundamental physical process which may be consistent with such a number-theoretic description. Quantum gravitation (whose dynamical formulation is still open to debate) appears the only candidate. The arguments for gravity being crucial to quantum-state reduction have been given eloquently by Penrose (1989, 1994, 1998) and will not be repeated here. However, note in particular that Penrose provides support for the notion that, unlike the other forces of physics, the gravitational force is time asymmetric (see also Prigogine and Elskens, 1987; ’t Hooft, 1999).
5 Wave-Particle Duality

5.1 Schrödinger Dynamics

Consider a source emitting a single qubit every time $\Delta t$. Let the real-number state at time $t_n$ be given by $r(t_n)$ with corresponding bit string $S(t_n)$. Suppose

$$S(t_n) = e^{i\nu\Delta t}S(t_0)$$  \hspace{1cm} (32)

That is, the real-number state at time $t_n$ is obtained from the real-number state at time $t_{n-1}$ by a (phase) rotation through $2\pi\nu\Delta t$ about the polar axis. Using the permutation operator $e^{i\lambda}$ defined in equations 15 and 16, then with $\omega = 2\pi\nu$, we can write

$$r(t_n) = e^{i\omega\Delta t}r(t_0)$$  \hspace{1cm} (33)

Compare this with wavefunction evolution in the conventional form

$$|\psi(t_n)\rangle = e^{i\omega\Delta t}|\psi(t_0)\rangle$$  \hspace{1cm} (34)

Hence, in the proposed theory, $S(t_n)$ describes a coherent monochromatic wave source, with frequency $\omega$ and intensity $1/\Delta t$. With low enough intensity, individual qubits (‘particles’) can be measured. With the basic quantum premise that $E = \hbar\omega$, then 33 is equivalent to a first integral of the Schrödinger equation for a free-particle with energy $E$.

5.2 Polarisation and Uncertainty

Imagine an ensemble of real-number states associated with points in some small neighbourhood of $(\theta, \lambda)$, so that the real-number state of the $j$th qubit in this prepared ensemble is

$$r(\theta_j, \lambda_j) = r(\theta + \delta\theta_j, \lambda + \delta\lambda_j)$$  \hspace{1cm} (35)

Now by section 4, the numerical values of the real-number states in equation 35 are not sensitive to small variations $\delta\theta_j$, but by section 2, they are sensitive to small variations $\delta\lambda_j$. As discussed in section 3, the probability that one of the qubits in this ensemble reduces to $p_N$ is equal to the probability that this real-number state is $< 1/2$, which equals $\cos^2\theta/2$, consistent with quantum polarisation statistics for prepared states. When $\theta = \pi/2$, then, from this probabilistic perspective, there is complete uncertainty as to
whether a randomly-chosen real-number state will reduce to $p_N$ or $p_S$, in no matter how small a neighbourhood of real-number state space the ensemble of states is initially prepared.

### 5.3 Unitary Transforms and Superposition

For the two level system, it is well known in standard quantum theory that unitary transformations $U : |\psi\rangle \mapsto |\psi'\rangle$ are associated with the action of arbitrary isometries of the Bloch sphere. In the proposed theory, such isometries define the equivalent unitary-like transformations $\tilde{U} : r(\theta, \lambda) \mapsto r'(\theta, \lambda)$.

As an example, the unitary transform

$$
|0\rangle \mapsto \frac{|1\rangle + |0\rangle}{\sqrt{2}} \\
|1\rangle \mapsto \frac{|1\rangle - |0\rangle}{\sqrt{2}}
$$

in standard quantum theory, corresponds, in the proposed theory, to the $\tilde{U}$ transform

$$
.000\ldots \mapsto r_0 \\
.111\ldots \mapsto \overline{r_0} = 1 - r_0
$$

Note that in the proposed theory, each of the states $r_0$ and $1 - r_0$ is a definite real number, not some complex-number weighted coexistence of alternative eigenstates. The equivalence with the notion of superposition arises from the fact that in equation 37, the binary expansions of the base-2 normal reals $r_0$ and $1 - r_0$, each contain equal numbers of ‘0’s and ‘1’s.

### 5.4 Interference

Imagine a beam of photons impinging on a half-silvered mirror. Standard quantum theory views as fundamentally non-local, the Hilbert-space state $|0\rangle + i|1\rangle$ of the photon after passing through the half-silvered mirror, where $|0\rangle$ denotes the reflected state, and $|1\rangle$ the transmitted state. In the proposed theory, the same wholistic description of the state is given by the base-2 normal $r(\pi/2, \lambda)$. If $r(\pi/2, \lambda) \geq 1/2$, then under $R$, a photon will be detected if a measurement is made in the transmitted beam, and if $r(\pi/2, \lambda) < 1/2$
then a photon will be detected if a measurement is made in the reflected beam. (The relation between reduction and measurement is described in section 7 below.) However, in addition, the real-number state \( r(\pi/2, \lambda) \) can also describe each of the component parts separately. In this more local description, the state in the transmitted beam is given by \( r(\pi/2, \lambda) \), the state in the reflected beam is given by \( r(\pi/2, \lambda + \pi) = 1 - r(\pi/2, \lambda) \). Now apply the operator \( R_1 \) to detect the presence of a photon (in either the transmitted or reflected beam). In this local view, if \( R_1 \) leads to a reduction to \( \ldots 111 \) (particle detected) in one of the beams, then the state in the other beam is not in any way required to reduce non-locally to \( \ldots 000 \), but, rather, remains in its original unreduced state. Of course, if \( r(\pi, \lambda + \pi) \geq 1/2 \), then necessarily, \( r(\pi/2, \lambda) < 1/2 \). Hence, if a particle is detected in the reflected beam, it can be deduced that the leading digit in the binary expansion of \( r(\pi/2, \lambda) \) is a ‘0’, and hence a photon would not be detected were a measurement to be made on the transmitted beam. Sampling over an ensemble of photons defined by varying \( \lambda \), then 50% of the photons will be detected in the transmitted beam, and 50% in the reflected beam. In summary, the proposed theory is capable of giving a more local description to quantum phenomena than standard quantum theory. The implications of this for Bell’s theorem is discussed in section 9.

Consider now the action of the second half-silvered mirror of a Mach-Zehnder interferometer. By time symmetry, if the global incoming state is \( r(\pi/2, \lambda) \), then the outgoing state must be \( \ldots 111 \) corresponding to particle detection in one channel only. Hence the process of constructive and destructive interference in the two output channels, respectively, is well described.

By contrast, suppose one of the two incoming beams in the second half-silvered mirror has been blocked off, so that the incoming state is \( r(\pi/2, \lambda) \) in just one beam. In the proposed theory, the corresponding outgoing global state is equal to \( r \) if \( r \geq 1/2 \), equal to \( 1 - r \) if \( r < 1/2 \). This result makes use of the property of self similarity. Specifically, \( r(\pi/2, \lambda) \) can be approximated arbitrarily closely by some reduced number \( .mmm \ldots \) in base-\( 2^N \) providing \( N \) is sufficiently big. However, in the proposed theory, we can treat ‘\( m \)’ as just a label (see section 6). Hence, if \( \ldots 111 \) transforms to \( r(\pi, \lambda) \) under the action of the half-silvered mirror, then the state \( .mmm \ldots \) will transform to the number \( r' \) whose base-\( 2^N \) expansion has the same form as the binary expansion of \( r(\pi/2, \lambda) \), but where the digits \( m \) and \( \bar{m} \) replace 1 and 0 respectively, with \( m + \bar{m} = 2^N - 1 \). In this way, the digits in the leading \( N \) places in the binary expansion of \( r' \) will be the same as in the leading \( N \)
places in the binary expansion of \( r \), if the binary expansion of \( r \) begins with a ‘1’ \( (r \geq 1/2) \). Conversely, the digits in the leading \( N \) places in the binary expansion of \( r' \) will be the \( \phi \) transform of the digits in the leading \( N \) places in the binary expansion of \( r \), if \( r < 1/2 \). Either way, sampling over an ensemble of photons defined by varying \( \lambda \), then 50% of photons would be detected in the transmitted beam, and 50% in the reflected beam.

6 The Three-Level Quantum State

In this section, using the concepts and constructions discussed above, the real-number representation of the 3-level quantum state is defined. From this, extension to a general \( M \)-level system is, in principle, straightforward.

In standard quantum theory, the 3-level state vector

\[
|\psi\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle + \alpha_2|2\rangle
\]

is an element of a complex 3-dimensional Hilbert space. Here we describe the probabilistic equivalence

\[
|\psi\rangle \sim r(\theta_1, \theta_2, \lambda_1, \lambda_2)
\]

where, as before, \( 0 \leq r \leq 1 \) and the equivalence is with respect to some particular observable. The six degrees of freedom in the three complex numbers \( \alpha_j \), less normalisation and the choice of global phase factor \( \exp i\gamma \), leads to the dependence of \( r \) on four arguments. As before the choice of \( \gamma \) is equivalent to the specific choice of base-3 normal number \( r_0 \) such that

\[
\frac{1}{\sqrt{3}}(|0\rangle + |1\rangle + |2\rangle) \sim r_0
\]

One possible choice for \( r_0 \) is the (base-3) Champernowne number \( r_C = .012101112 \ldots \).

6.1 Reduced Numbers and Dynamical State Reduction

As with the 2-level state, the reduced numbers \( 0 \leq r_\perp j \leq 1, \ j \in \{0,1,2\} \) can be defined by deleting all occurrences of the digit ‘\( j \)’ in the base-3 expansion of \( r \). For example, using the base-3 Champernowne number,
Note that if $r$ is base-3 normal, then $r \perp j$ is base-2 normal. These reduced numbers define quantum sub-systems. For example,

$$\frac{1}{\sqrt{2}}(|0⟩ + |1⟩) \perp r_0 \perp 2$$
$$\frac{1}{\sqrt{2}}(|1⟩ + |2⟩) \perp r_0 \perp 0$$
$$\frac{1}{\sqrt{2}}(|2⟩ + |0⟩) \perp r_0 \perp 1$$  \hspace{1cm} (41)

Extending this notion by deleting multiple digits, we have three base-1 normals

$$|0⟩ \perp r_0 \perp \{1,2\} = (r_0 \perp 1) \perp 2 = .000\ldots$$
$$|1⟩ \perp r_0 \perp \{2,0\} = .111\ldots$$
$$|2⟩ \perp r_0 \perp \{0,1\} = .222\ldots$$  \hspace{1cm} (42)

Associated with these reduced numbers, we can define three irreversible dynamical reduction operators $R_j$ for the 3-level system (similar to the qubit). Specifically, with base-3 expansions,

$$R_j : .a_1a_2a_3\ldots \mapsto .jjj\ldots \text{ if } a_1 = j$$
$$R_j = \text{ Identity if } a_1 \neq j$$  \hspace{1cm} (43)

the latter being equivalent to a so-called ‘null’ reduction in standard quantum theory. The compound reduction operator $R = R_0R_1R_2$ allows us to ask: under compound measurement, would the real-number state reduce to level 1, 2 or 3? Again, the reader is referred to section 16 for a discussion on the relation between reduction and measurement.

### 6.2 The Phase Function $r(\lambda_1, \lambda_2)$

Consider now a two-dimensional real-valued function $r(\lambda_1, \lambda_2)$ associated with the equivalence

$$\frac{1}{\sqrt{3}}(|0⟩ + e^{i\lambda_1}(|1⟩ + e^{i\lambda_2}|2⟩) \perp r(\lambda_1, \lambda_2)$$  \hspace{1cm} (44)

which in the proposed theory is a base-3 normal number, and generalises the corresponding base-2 normal $r(\lambda)$ for the qubit. As discussed below, the
set of reals $r(\lambda_1, \lambda_2)$ where $\lambda_1$ is a triadic rational multiple of $\pi$ and $\lambda_2$ is a
dyadic rational multiple of $\pi$, defines the sample space on which quantum
measurement probabilities are defined.

As before, $r(\lambda_1, \lambda_2)$ is constructed from a family of permutation operators
which have complex structure. Let

$$S_r = \{a_1, a_2, a_3 \ldots \}$$

with $a_i \in \{0, 1, 2\}$, define the base-3 expansion $r = a_1a_2a_3 \ldots$ of some base-3
normal $0 \leq r \leq 1$. For the 3-level system, put $\phi(0) = 1$, $\phi(1) = 2$, $\phi(2) = 0$
and define

$$\omega^{1/3}(S_r) = \{\phi(a_3), a_1, a_2, \phi(a_6), a_4, a_5 \ldots \}$$

whence

$$\omega(3)(S_r) = \{\phi(a_1), \phi(a_2), \phi(a_3) \ldots \}$$
$$\omega^2(3)(S_r) = \{\phi^2(a_1), \phi(a_2), \phi^2(a_3) \ldots \}$$
$$\omega^3(3)(S_r) = \{\phi^3(a_1), \phi(a_2), \phi^3(a_3) \ldots \} = S_r$$

so that $\omega(3)$, $\omega^2(3)$ and $\omega^3(3)$ are permutation-operator representations of third
roots of unity. For consistency, the $i$ operator from the previous section can be relabelled as

$$i = \omega^{1/2}$$

As with $\omega^{1/2}$, higher-order fractional powers of $\omega^{1/3}$ are constructed by assuming self-similarity. Hence, for example

$$\omega^{1/9}(3)(S_r) = \{\phi(a_9), a_7, a_8, a_1, a_2, a_3, a_4, a_5, a_6 \ldots \}$$

whence it is straightforward to show that

$$\omega^{1/9}(3) * \omega^{1/9}(3) * \omega^{1/9}(3)(S_r) = \omega^{1/3}(3)(S_r)$$

and so on. In this way $\omega^q(3)$ can be defined for any triadic rational $q$. (And,
more generally, $\omega^q(p)$ for any $p$-adic rational $q$.) Based on this, the real number
transformations induced by $\omega(3)$ are, for example,

$$\tilde{\omega}^{1/9}(3)(r) = \phi(a_9)a_7a_8a_1a_2a_3a_4a_5a_6 \ldots$$

Putting these construction together, then

$$\frac{1}{\sqrt{3}}(\langle 0 | + e^{i\lambda_1}(\langle 1 | + e^{i\lambda_2}|2 \rangle)) \langle r(\lambda_1, \lambda_2) = \tilde{\omega}^{3\lambda_1/2\pi}(3)(\tilde{\omega}^{2\lambda_2/2\pi}(2)(r_{0,0}) \oplus .000 \ldots$$
where $\lambda_1/2\pi$ is a triadic rational, and $\lambda_2/2\pi$ is a dyadic rational. The symbol ‘⊕’ means that ‘0’ digits are added to the base-3 expansion of $\tilde{\omega}_{(2)}^{2\lambda_2/2\pi}(r_{0\perp 0})$ in exactly the same places that the same ‘0’ digits were removed from the base-3 expansion of $r_0$ by the $\perp 0$ operator. In more expanded language, the right hand side of equation 52 can be explained as follows: take the base-3 expansion of $r_0$, and permute among the places where the digits ‘1’ and ‘2’ occur with $\omega_{(2)}^{2\lambda_2/2\pi} = i^{4\lambda_2/2\pi}$, keeping the places where the digit ‘0’ occurs unchanged. Then permute the three digits in all the resulting places according to the permutation operator $\omega_{(3)}^{3\lambda_1/2\pi}$. This function is defined on a countable set of points (where $\lambda_1$ is a triadic rational multiple of $\pi$ and $\lambda_2$ is a dyadic rational of $\pi$).

A crucial property of this construction is that all $r(\lambda_1, \lambda_2)$ are base-3 normal (providing $r_0$ is). Hence we can define a rather trivial (ie constant) probability on the sample space of real-number states spanned by $(\lambda_1, \lambda_2)$. In particular, the probability that $r(\lambda_1, \lambda_2)$ lies between 0 and 1/3, or between 1/3 and 2/3, or between 2/3 and 1, is equal to 1/3. Hence, from the reduction equation 53, the probability that $R_j$ takes $r(\lambda_1, \lambda_2)$ to the reduced real-number state $\ldots jjj$ is also equal to 1/3.

6.3 Application of The Reduction Procedure

Consider some fixed values of $(\lambda_1, \lambda_2)$, which without loss of generality, may as well be $(0, 0)$. The (partial) reduction procedure defined in section 3 is now used to define the equivalence

$$|\psi\rangle = \cos \frac{\theta_1}{2} |0\rangle + \sin \frac{\theta_1}{2} (\cos \frac{\theta_2}{2} |1\rangle + \sin \frac{\theta_2}{2} |2\rangle) \triangleleft r(\theta_1, \theta_2, 0, 0).$$

(53)

For example, with $\theta_1 = \pi$

$$|\psi\rangle = \cos \frac{\theta_2}{2} |1\rangle + \sin \frac{\theta_2}{2} |2\rangle \triangleleft r(\pi, \theta_2, 0, 0)$$

(54)

can be defined using the reduction procedure given

$$\frac{1}{\sqrt{2}}(|1\rangle + |2\rangle) \triangleleft r(\pi, \frac{\pi}{2}, 0, 0) = .a_1a_2a_3\ldots \text{ base 2 normal}$$

(55)
Note that the expansions of all the numbers in equations 55 contain the digits ‘1’ and ‘2’, rather than ‘0’s and ‘1’s. To apply the reduction procedure as in section 3, first relabel the digits, so that \( a_j \mapsto a_j - 1 \). Then, as before, starting with the (relabelled) sequence \( S = \{a_1, a_2, a_3 \ldots \} \), if \( a_j = 1 \), delete \( a_j \) from \( S \) iff
\[
.a_ja_{j+1}a_{j+2} \ldots < \cos^2 \frac{\theta}{2}
\] (56)
and if \( a_j = 0 \), delete \( a_j \) iff
\[
.a_ja_{j+1}a_{j+2} \ldots \geq \cos^2 \frac{\theta}{2}
\] (57)
Now return to the original digits ‘1’ and ‘2’ with \( a_j \mapsto a_j + 1 \) and relabel the surviving sequence of digits as \( \{a'_1, a'_2, a'_3 \ldots \} \). Finally,
\[
r(\pi, \theta_2, 0, 0) = .a'_1a'_2a'_3\ldots
\] (58)
In this way, all the real-number states in equations 41 and 42 are defined.

Now consider
\[
|\psi\rangle = \sqrt{\frac{1}{3}}|0\rangle + \sqrt{\frac{2}{3}}(\cos \frac{\theta_2}{2}|1\rangle + \sin \frac{\theta_2}{2}|2\rangle) \otimes r(\theta^*, \theta_2, 0, 0)
\] (59)
where \( \cos \theta^*/2 = 1/\sqrt{3} \). Using equation 58, this can be defined by
\[
r(\theta^*, \theta_2, 0, 0) = .a'_1a'_2a'_3\ldots \oplus .000\ldots
\] (60)
where, as before, the \( \oplus \) operator means add the digits ‘0’ into the expansion \( .a'_1a'_2a'_3\ldots \) in exactly the places they were removed from the expansion of \( r_0 \) by the \( \bot 0 \) operator.

Finally, apply the reduction procedure to define \( r(\theta_1, \theta_2, 0, 0) \) given \( r(0, \theta_2, 0, 0) = .000\ldots \) which contains only the digits ‘0’, \( r(\theta^*, \theta_2, 0, 0) = .b_1b_2b_3\ldots \) which contains all three digits, and \( r(\pi, \theta_2, 0, 0) = .a'_1a'_2a'_3\ldots \) which contains only the digits ‘1’ and ‘2’. As before, start with the sequence \( S = \{b_1, b_2, b_3 \ldots \} \). Let \( c_j = 1 \) if \( b_j = 1 \) or \( b_j = 2 \), and let \( c_j = 0 \) if \( b_j = 0 \). Then if \( c_j = 1 \), delete \( b_j \) from \( S \) iff
\[
.cjc_{j+1}c_{j+2} \ldots \leq \cos^2 \frac{\theta}{2}
\] (61)
and if \( c_j = 0 \), delete \( b_j \) from \( S \) iff
\[
.c_jc_{j+1}c_{j+2} \ldots > \cos^2 \frac{\theta}{2}.
\]
(62)
The digits that survive these deletions are relabelled as \( \{b'_1, b'_2, b'_3 \ldots \} \) and
\[
r(\theta_1, \theta_2, 0, 0) = .b'_1b'_2b'_3 \ldots.
\]
(63)
This completes the description of the equivalence
\[
|\psi\rangle = \cos \frac{\theta_1}{2} |0\rangle + \sin \frac{\theta_1}{2} e^{i\lambda_1} (\cos \frac{\theta_2}{2} |1\rangle + e^{i\lambda_2} \sin \frac{\theta_2}{2} |2\rangle) \not\propto r(\theta_1, \theta_2, \lambda_1, \lambda_2).
\]
(64)

### 6.4 Equivalence with the Trace Rule

As previously discussed, sampling over \( \lambda_1 \) and \( \lambda_2 \) defines, in the proposed theory, the probability measure \( P \) for quantum measurement outcome. For example, we can ask, what is the probability that the base-3 expansion of the real-number state \( r(\theta_1, \theta_2, \lambda_1, \lambda_2) \), for randomly-chosen \( (\lambda_1, \lambda_2) \) begins with a ‘0’, ‘1’ or a ‘2’. This can be straightforwardly calculated using the method discussed in section \( \text{[3]} \). Hence, for example, entirely equivalent to the result derived in section \( \text{[3]} \), the probability that the base-3 expansion of the partially-reduced number \( r(\pi, \theta_2, \lambda_1, \lambda_2) \) begins with a ‘0’ is equal to 0, that it begins with a ‘1’ is equal to \( \cos^2 \theta_2/2 \), and that it begins with a ‘2’ is equal to \( \sin^2 \theta_2/2 \).

More generally, defining the three functions \( \rho_j(\theta_1, \theta_2) \) which give the probability that \( r(\theta_1, \theta_2, \lambda_1, \lambda_2) \) is attracted to \( A_j \) under \( R_j \), then from the construction in section \( \text{[3]} \)
\[
\rho_0 = \cos^2 \theta_1/2 \\
\rho_1 = \sin^2 \theta_1/2 \cos^2 \theta_2/2 \\
\rho_2 = \sin^2 \theta_1/2 \sin^2 \theta_2/2
\]
(65)
consistent with the trace rule for measurement outcome in standard quantum theory.

From a number-theoretic perspective, the \( \rho_j(\theta_1, \theta_2) \) define the degree of normality of the number \( r(\theta_1, \theta_2, \lambda_1, \lambda_2) \) - specifically, \( \rho_j \) is a measure of the fraction of places in the base-3 expansion of \( r \) where the digit \( j \) occurs, compared with the fraction of places where any of the other permitted digits occur. In this way, the probability density matrix of quantum theory is subsumed, in the proposed theory, by the number-theoretic degree of normality of the corresponding real-number state.
7 Sub-Systems, Measurement Outcome and the Classical Limit

Consider an $M$-level system $\text{Sys}_M$ where $M \gg 1$. Imagine, for example, $\text{Sys}_M$ represents the entire universe. In subsection 4.3, it was speculated that degree of Borelian normality of the real-number state $r$ of an isolated system, will be monotonically decreasing due to the effect of number-theoretic reduction operators, representing the ubiquitous effects of quantum self-gravitation. Hence, in the proposed theory, the real-number $O$-state $r$ of the initial big bang could be imagined to be an (optimally-normal) base-$M$ normal real, such as the base-$M$ Champernowne number $012\ldots M'\ldots$ where $M' = M - 1$. The cosmic state would then evolve deterministically though a sequence of base-$N$ normal reals, $N < M$, to some gravitationally-clumped minimally-normal real-number state $r = j jj\ldots$, where $j \in \{0, 1, 2 \ldots, M'\}$. This time-asymmetric evolution is entirely consistent with the Weyl curvature hypothesis, as discussed by Penrose (1989, 1994).

We define an $m$-level sub-system $\text{Sys}_m \subset \text{Sys}_M$, $m < M$, by making use of the concept of reduced numbers defined in section 6. In particular, choosing $m$ digits from the original $M$, then the state of the $m$-level sub-system is merely given by the base-$M$ expansion of $r$, retaining only the chosen digits. For example, if

$$|0\rangle + |1\rangle + |2\rangle + |3\rangle + |4\rangle + |5\rangle + |6\rangle + |7\rangle < .0532017621435\ldots$$

represents the state of an 8-level system, then

$$|2\rangle + |5\rangle < r_{\perp\{0,1,3,4,6,7\}} = .5225\ldots$$

denotes the state of a qubit subset of this system. As a description solely of a qubit, without consideration of the larger system to which it belongs, the digits ‘2’ and ‘5’ are merely arbitrary (but ordered) symbols representing the two different levels of the qubit. Hence we could replace $2 \mapsto 0$ and $5 \mapsto 1$, so that $.5225\ldots \mapsto .1001\ldots$ and use the results from sections 2-4 to describe the dynamics of this qubit sub-system.

However, the reduced real-number state $.222\ldots$ and $.555\ldots$ also characterise specific levels of the larger 8-level system. Hence, conversely, in the definition of the Bloch-sphere equivalent of the real-number state space of the qubit, the real-numbers $r = .000\ldots$ and $r = .111\ldots$ should be thought of as
associated with some reduced real-number states \( \ldots mmm \) and \( \ldots m'm'm' \ldots \) of \( \text{Sys}_M \), of which the qubit is a sub-system. It is the relation of these reduced states with respect to the \( M \) possible reduced states of \( \text{Sys}_m \), that determines the absolute orientation of the polar axes of the qubit’s Bloch-sphere equivalent.

In this way, the process of measurement can be discussed without any recourse to an arbitrary classical/quantum split as is required in standard quantum theory (Bell, 1990, 1993). In the proposed theory, a measuring system is some \( M \)-level system with attractors \( A_j \) where \( r = \ldots j j j \ldots \) and \( j \in \{0, 1, 2 \ldots M - 1\} \). As discussed below, it will be important that \( M \gg 1 \). However, in addition, a crucial aspect of any measurement process is the notion of detector gain - a small signal arising from the process of state reduction is amplified to give a large output signal (eg in a photomultiplier).

In the proposed model, the notion of detector gain can be described by imagining the design of the measuring system to be such that for \( J \leq j < K \), a state initially at \( A_j \) will necessarily cascade through the sequence \( A_{j+1} \rightarrow A_{j+2} \rightarrow A_{j+3} \ldots \rightarrow A_K \). We presume that the cascade process is sufficiently extensive that the real-number states \( r = \ldots J J J \ldots \) and \( r = \ldots K K K \ldots \) are perceptibly different from one another to a human observer. For \( j < J \), imagine the real-number state \( r = \ldots j j j \ldots \) as stationary.

With \( j' = j - 1 \), we presume that \( M \) is sufficiently large that the real-number state \( r = \ldots j'j'j' \ldots \) and any real-number state whose base-\( M \) expansion comprises only the digits \( j \) and \( j' \), are not perceptibly different from one another. Imagine such a measuring system initially in the (reduced) stable real-number state \( r = \ldots J'J'J' \ldots \), where \( J' = J - 1 \). The system is then perturbed by a qubit (say) in the unreduced real-number state \( a_1a_2a_3 \ldots \) where \( a_j \in \{0, 1\} \). In the case of optimal detector efficiency, the evolution of the measuring system will be tightly coupled with that of the qubit - in this situation we can call the two real-number states entangled. Hence, by the arguments above, the real-number state \( r = \ldots b_1b_2b_3 \ldots \) where \( b_j = J' \) if \( a_j = 0 \), \( b_j = J \) if \( a_j = 1 \) will represent both the real-number state of the qubit (under the symbol replacement \( J' \mapsto 0 \), \( J \mapsto 1 \)), and the real-number state of the measuring system to which the qubit is tightly coupled. By definition, the real-number state \( r = \ldots b_1b_2b_3 \ldots \) of the perturbed measuring system is not perceptibly different from its initial real-number state \( r = \ldots J'J'J' \ldots \). If \( a_1 = 1 \), then, under reduction, the real-number state of the measuring system evolves to \( r = \ldots K K K \ldots \) which then, by the detector gain process discussed above, cascades to \( r = \ldots J J J \ldots \) and then, by the detector gain process discussed above,
state of the measuring system is unchanged.

This notion of entanglement between the single qubit and the larger measuring system, implies that the time for the qubit to attract to one of the fixed points \(0.000\ldots\) or \(0.111\ldots\) would correspond to the time it takes the perturbed state \(r = b_1b_2b_3\ldots\) of the measuring system to be attracted to either \(J'J'J'\ldots\) or \(JJJJ\ldots\). It would seem plausible that this timescale would be proportional to the density of reduced real-number states \(jjj\ldots\) in state space. Following ideas developed in Diósi (1989) and Penrose (1994, 1998), the density of reduced states might be related in some way to the difference in gravitational self-energy associated with such pairs of reduced states. Motivated by the geometric ideas in the theory of general relativity, suppose the curvature of the Bloch-sphere equivalent was determined by the density of reduced states of the system \(\text{Sys}_M\) to which the qubit was entangled, and that \(\alpha\) in equation 31 depended on the radius of curvature of the Bloch-sphere equivalent. Then the greater the density of reduced states associated with \(\text{Sys}_M\), the shorter the timescale for attraction of the qubit state to one of the fixed points. If the appropriate measure of interaction between the qubit and \(\text{Sys}_M\) is gravitational, then, in the proposed theory, the ‘basis’ problem for the qubit is solved in much the same way that Mach’s principle solves the ‘inertial frame’ problem in general relativity. Clearly such ideas require further development.

In the more general case where the evolution of the qubit and the measuring system \(\text{Sys}_M\) is not so tightly coupled, then the digits \(b_i\) and \(a_i\) would not be in simple one-to-one correspondence. In this case, the fraction of digits where \(b_j = J'\) would be larger than the fraction of digits where \(b_j = J\), by some factor depending on the degree of entanglement. As a result, there will be a greater likelihood that the corresponding measurement will be null. We discuss further this notion of weak measurement in section 10 below.

In this discussion it has not been necessary to make any arbitrary classical/quantum split: the process of measurement is, in principle, contained within the theory. However, it is possible to refer to a ‘classical limit’ for the proposed theory: one where evolutionary laws are derived solely on the basis of transformations between the sub-set of reduced real-number state \(iii\ldots\). This limit clearly makes sense when the set of such reduced real-number state is comparatively dense in state space, ie when \(M\) is large. Suppose \(r(t)\) represents the exact evolution of \(r\). Then, in the proposed theory, the classical limit corresponds to finding the best approximation \(r_c(t) \sim jjj\ldots\) where \(j = j(t)\). For example, the cascade \(A_{j+1} \rightarrow A_{j+2} \rightarrow A_{j+3}\ldots \rightarrow A_K\) as de-
scribed above is a classical process (as it would be in conventional physics). By labelling the reduced state \( jjj \ldots \) by the integer \( j \), then, in the proposed theory, the laws of classical physics would correspond to evolutionary equations defined by relationships between the integers (or, equivalently, the ordinary rationals of computational physics, ie without concern to normality).

8 Composite Systems: The Quantum Computer and The Navier-Stokes Computer

In sections 6 and 7 we discussed the use of reduced numbers to determine the real-number states of sub-systems. The rule for defining the real number state of a composite system is the simple converse. For example, if we have the real-number binary expansion equivalent

\[
\frac{(|0 > + e^{i\lambda_n}|1>)}{\sqrt{2}} \leq r(\pi/2, \lambda_n) = .a_1^{(n)}a_2^{(n)}a_3^{(n)} \ldots
\]  

(68)

for any one qubit, then, in the proposed theory, the real-number equivalent \( r^{(N)} \) of the \( 2^N \)-level \( N \)-qubit composite

\[
|\psi_N\rangle = \frac{(|0 > + e^{i\lambda_1}|1>)(|0 > + e^{i\lambda_2}|1>)(|0 > + e^{i\lambda_3}|1>) \ldots (|0 > + e^{i\lambda_N}|1>)}{2^{N/2}}
\]  

(69)

will be the real number with the base-\( 2^N \) expansion

\[
r^{(N)} = .b_1b_2b_3 \ldots
\]  

(70)

where \( b_j \in \{0, 1, 2 \ldots 2^N - 1 \} \) and the binary expansion of \( b_j \) is

\[
b_j = a_j^{(1)} a_j^{(2)} a_j^{(3)} \ldots a_j^{(N)}.
\]  

(71)

\( N \) qubit Hilbert-space states, such as in equation (69) are used, for example, in the quantum Fourier transform (eg Nielsen and Chuang, 2000), an essential part of many quantum computational algorithms. The best classical algorithm for computing discrete Fourier transforms on \( 2^N \) elements uses \( O(N2^N) \) gates, whilst the quantum Fourier transform requires \( O(N^2) \) gates. One of the gates in this count is

\[
U_N = \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i/2^N} \end{pmatrix}
\]  

(72)
acting on one of the qubit channels. In the proposed theory, the equivalent \( U_N \) gate corresponds to an application of the operator \( i^{4/2^N} \) acting on the bit string associated with the binary expansion of the real-number state of this qubit. On a digital computer, the estimation of \( U_N \), as discussed in section 2, would require individual operations on the each of the first \( 2^N-1 \) places in the binary expansion of \( r \). It is this ability to perform by a single phase gate, a process that requires exponentially many permutations on a classical digital computer, that describes the power of the quantum computer in the proposed theory.

Is it conceivable, in a single-universe world view, that a deterministic real-number system can execute, what would correspond to \( O(2^N) \) primitive operations on a digital computer, in fixed finite time (independent of \( N \))? In fact, there is a well-known classical system which does precisely this: a system based on the Navier-Stokes equations for a three dimensional turbulent fluid in the limit of large Reynolds number. (Indeed studies of the properties of these equations led the author to formulate the proposed theory!)

In the inviscid limit, scaling arguments for homogeneous isotropic turbulence (in the Kolmogorov inertial range) suggest that information contained in some arbitrarily small-scale ‘eddy’ can propagate up-scale to affect the evolution of some large-scale ‘eddy’ of interest in finite time (essentially the large-scale eddy turn-over time \( T_1 \); Lorenz, 1969; Palmer, 2000). Specific non-differentiable solutions of the Navier-Stokes equations with this property have been found (Shnirelman, 1997), though rigorous generic theorems are still lacking. More quantitatively, imagine decomposing a three-dimensional fluid (eg streamfunction) field onto some real (Galerkin) basis, so the field is represented by a string \( \{x_1, x_2, x_3 \ldots x_N\} \) of real numbers. The number \( x_j \) is the projection coefficient of the field onto the \( j \)th basis function, which describes the \( j \)th ‘eddy’. By Kolmogorov scaling, the time it takes for the \( N \)th eddy to affect the first (largest) eddy, scales as \( T_1(1 - \exp(-N)) \). On a digital computer, however, the time taken to perform a computation of the evolution of \( x_1 \) over this same timescale, will grow as \( N^4 \) (allowing for the three dimensionality of the eddies, and the need to reduce the computational time step as the minimum eddy size decreases). Hence, a Navier-Stokes computer (ie a turbulent fluid!), performs this calculation exponentially more efficiently than does a digital computer.

Of course, like a quantum computer, such a Navier-Stokes computer can only perform certain computations (based on solutions of the Navier-Stokes equations!). However, in this respect, it is worth noting that, through the
nonlinear Hopf-Cole transformation (e.g., Whitham, 1974), the Burgers’ form of the Euler equation can be cast in the form of a Schrödinger equation, though with a real, rather than complex, state vector. It is also interesting to note that the inviscid Navier-Stokes equations (the Euler equations) have an interesting geometric interpretation: they describe geodesics on the group of diffeomorphisms of the fluid 3-space.

The author’s permutation construction in section 2 captures this sense of the upscale transfer of fluid-dynamical information, and, as shown, is certainly non-differentiable. Hence, in terms of the proposed theory, the exponential speed up of the quantum computer is conceptually no different from the exponential speed up of the Navier-Stokes computer, over a conventional digital computer. On this basis, it is suggested that the power of the quantum computer can be explained without recourse to multiple universes, or indeed superposed Hilbert-space eigenstates (cf. Deutsch, 1997).

Just as viscosity limits the maximum size $N$ of a Galerkin expansion of the fluid state, and hence of the maximum processing power of a Navier-Stokes computer, so one can question whether there are fundamental physical constraints which might limit the maximum possible length $2^N$ of the binary expansion of $r$, and hence the processing power of a quantum computer. Possibly self-gravitation of the composite $N$ qubit may provide such a constraint, e.g., leading to a minimum angular resolution of the points on the Bloch-sphere equivalent over which the real-number states are defined (see section 2).

9Bell’s Theorem, Beables and Counterfactual Indefiniteness

Consider a source of pairs of entangled qubits measured by devices with relative orientation $\Delta \theta \neq 0$. Define the coordinate system $(\theta, \lambda)$ and poles $(p_N, p_S)$ associated with the real-number state space of the left-hand qubit, and coordinate system $(\theta', \lambda')$ with poles $(p_{N'}, p_{S'})$ for the right-hand qubit. The co-latitude of $p_{N'}$ with respect to $(\theta, \lambda)$ is $\Delta \theta$. As demonstrated in subsection 3.1, the proposed theory can readily account for the observed correlation statistics associated with entangled EPR states. In subsection 9.2, the reason why the proposed model violates Bell’s inequalities is discussed.
9.1 EPR-state Correlations

Imagine an ensemble \( i = 1, 2, 3 \ldots N \) of pairs of qubits such that the real-number state of the \( i \)th left-hand qubit is \( r^{(i)} \), and the real-number state of the \( i \)th right-hand qubit is \( r^{′(i)} \). In the proposed theory, the fact that these qubits are entangled implies a deterministic relationship between \( r^{(i)} \) and \( r^{′(i)} \).

To define this relationship, first write the binary expansion of \( \cos^2(\Delta \theta/2) \)

\[
\cos^2(\Delta \theta/2) = .d_1d_2d_3 \ldots
\]  

(73)

and the binary expansions of \( r^{(i)} \) and \( r^{′(i)} \) as

\[
r^{(i)} = .a_1^{(i)}a_2^{(i)}a_3^{(i)} \ldots
\]

\[
r^{′(i)} = .a_1^{′(i)}a_2^{′(i)}a_3^{′(i)} \ldots
\]  

(74)

Second, decompose the set \( I = \{1, 2, 3 \ldots N\} \) of natural numbers into the disjoint subsets

\[
I = \bigcup I_j
\]  

(75)

where, for \( j \leq N \),

\[
I_j = \{i : i = 2^{j-1} + (k - 1)2^j; k \leq N, i \leq N\}
\]  

(76)

For example, for \( N=12 \),

\[
I = \{1, 3, 5, 7, 9, 11\} \cup \{2, 6, 10\} \cup \{4, 12\}
\]  

(77)

For large \( N \), the subset \( I_j \) contains a fraction \( 1/2^j \) of the \( N \) integers.

The required relationship between \( r^{(i)} \) and \( r^{′(i)} \) is defined as follows. For \( i \in I_j \), let

\[
a_i^{′(i)} = (1 - d_j)a_i^{(i)} + d_j\phi(a_i^{(i)})
\]  

(78)

for \( l = 1, 2, 3 \ldots \). That is to say, for \( i \in I_j \), and hence for a fraction \( 1/2^j \) of the \( N \) ensemble pairs, \( a_i^{′(i)} = \phi(a_i^{(i)}) \) if \( d_j = 1 \), and \( a_i^{′(i)} = a_i^{(i)} \) if \( d_j = 0 \). Taken over all the disjoint subsets \( I_j \), the total fraction of occasions when the \( a_i^{′(i)} = \phi(a_i^{(i)}) \) is

\[
d_1/2 + d_2/2^2 + d_3/2^3 \ldots
\]  

(79)

which, by definition, is equal to \( \cos^2(\Delta \theta/2) \). The corresponding fraction of occasions where the \( a_i^{′(i)} = a_i^{(i)} \) is therefore equal to \( \sin^2(\Delta \theta/2) \). Now if
\( a_1^{(i)} = \phi(a_1^{(i)}) \), then if \( r^{(i)} \) reduces to \( p_N \), \( r'^{(i)} \) reduces to \( p_{S'} \), and vice versa. If we write \( x_i = 1 \) when \( a_1^{(i)} = a_1^{(i)} \), \( x_i = -1 \) otherwise, then it is immediate that

\[
\frac{1}{N} \sum_{i=1}^{N} x_i = \frac{1}{2} (\sin^2 \frac{\Delta \theta}{2} - \cos^2 \frac{\Delta \theta}{2}) = - \cos \Delta \theta
\]

as required by experiment and standard quantum theory.

9.2 The Bell Inequalities

Since the proposed theory is able to account for quantum correlations between entangled EPR states, statistics from the proposed theory violate the Bell inequalities. Manifestly, this implies that the proposed theory cannot be formulated as a local non-contextual theory. What is the key property of the proposed model that allows it to violate Bell’s theorem? Let us use Bell’s (1964) notation to describe a local non-contextual hidden variable form

\[ A(\mathbf{n}, \lambda) = \pm 1 \]

and

\[ B(\mathbf{n}, \lambda) = \pm 1, \]

for predicting, respectively, the outcomes of measurement of (for example) spin along direction \( \mathbf{n} \) for the left and right hand (spin-1/2) qubits of some entangled pair. Bell’s theorem requires that

\[ B(\mathbf{n}, \lambda) = -A(\mathbf{n}, \lambda) \]  

(81)

In words, equation (81) requires that if the left-hand qubit was measured as spin up (with respect to \( \mathbf{n} \equiv p_N p_S \)), then the right-hand qubit would have been measured as spin down, had it been measured with respect to \( p_N p_S \).

In the proposed theory, equation (81) would certainly hold if \( p_N p_S = p_{N'} p_{S'} \), by the property of real-number states corresponding to antipodal points. However, this hypothetical measurement on the right-hand qubit is counterfactual, since, by assumption, the right-hand qubit was actually measured with respect to \( p_{N'} p_{S'} \neq p_N p_S \).

Consider a pair of entangled qubits, as described in subsection 9.1. Let the real-number state of the left-hand qubit be given by \( r(p) \), the real-number state of the the right hand qubit be given by \( r'(p') \). Then the counterfactual measurement on the right-hand qubit, as defined in the previous paragraph, has definite outcome only if \( r(p') \) is well defined. With probability one it is not. To see this, note that according to the relationship established in subsection 9.1, \( p \) and \( p' \) are not (when \( p_{N'} p_{S'} \neq p_N p_S \)) antipodal points. Now since \( r'(p') \) is well defined, \( p' \) must (from section 9) lie on a meridian (call it \( \Lambda_j' \)), belonging to the set \( \Lambda' \) of meridians whose longitudes are dyadic
rational multiples of π. From section 3, \( r'(p') \) is defined on the continuum of all possible points on \( \Lambda'_j \). However, by the discussion at the end of section 2, \( \Lambda'_j \) is intersected only a countable number of times by the set of meridians \( \Lambda \), emanating from \( p_N \), on which the real-number state \( r(p) \) is well defined. Hence the probability that \( p' \) lies on a meridian in \( \Lambda \), is equal to zero. Hence with probability one, the counterfactual real-number state \( r(p') \) is undefined - it is therefore not a beable. (Equivalently, \( r'(p) \) is also undefined with probability one). The fact that equation 81 is neither true nor false in the proposed theory is reminiscent of non-Boolean logic in certain topos theories (Isham, 1997).

Counterfactual indefiniteness is not a property of conventional classical deterministic (eg chaotic) dynamics. For example, using a numerical weather prediction model (Palmer, 2000), it is straightforward to estimate (in the model) what the weather in London would have been like today if the temperature in Chicago had been two degrees colder a week earlier. In the proposed theory, this would not be possible; if a Turing machine was programmed to estimate \( r'(p) \) recursively, given \( r(p) \), \( r'(p') \), and the constructions of the proposed theory, then, with probability one, it would never halt. In this sense, one could view \( r'(p) \) as an uncomputable element of the theory. This notion that quantum physics may have uncomputable elements has been discussed by Penrose (1989, 1994) - the real-number states \( r'(p) \) in the proposed theory are uncomputable, because, a fortiori, they cannot be defined from the state space of the two entangled qubits.

These arguments apply to microscopic qubits, to macroscopic \( M \)-level measuring apparati and indeed to experimenters themselves. Hence, it is similarly meaningless to ask what would have been the real-number state of the measuring apparatus had the experimenter chosen a different measurement orientation to the one actually chosen (again, the proposed theory would declare the real-number state to be undefined). Indeed, the same is true of the state of the experimenter herself; effectively, the proposed theory compromises the intuitive notion of experimenter ‘free-will’ (cf Bell, 1985), at least at some fundamental level, though this does not imply that the proposed theory conflicts with the experimenter’s perception that her choices of measurement orientation are freely chosen.

The lack of well-defined counterfactual-measurement outcomes is one of the primary reasons why the real-number states in the proposed theory are beables. On the one hand, for any mathematically well-defined real-number state, we can estimate how it will transform under reduction (and therefore
measurement). On the other hand, counterfactual measurements which by
definition cannot be elements of reality, are not associated with well-defined
real-number states. Hence real-number states and elements of reality are in
one-to-one correspondence in the proposed theory.

As mentioned, the proposed theory cannot be both local and non-contextual.
We have argued that counterfactual indefiniteness implies that non-contextuality
is violated. Indeed, the proposed theory is local, at least in the original EPR
sense. We have argued in section 9 when discussing the individual measure-
ments on the two channels of an interferometer, that the process of perform-
ing a measurement which reduces the real-number state in one channel, is
not required to induce a change in the real-number state of the other chan-
nel. The same argument applies when discussing measurements on entangled
EPR real-number states - a measurement on the left-hand qubit need not im-
ply a change in the real-number state of the right-hand qubit. The proposed
theory is therefore consistent with the notion of relativistic invariance.

10 Weak Reduction and Stochastic Quantum Theory

Consider the reduction of some qubit to either \( p_N \) or \( p_S \) to be broken up into
a large number of ‘weak reductions’

\[
p \rightarrow p_1 \rightarrow p_2 \rightarrow p_3 \cdots,
\]

where each step, \( p_j \rightarrow p_{j+1} \), corresponds to a partial reduction towards either
some \( p_{N_j} \) or the antipodal \( p_{S_j} \). Each weak reduction will be defined by a
partial integration of equation (31). Specifically, let \( \theta_j, \lambda_j \) denote the co-
latitude and longitude of \( p_j \) with respect to a coordinate system with poles
\( p_{N_j}, p_{S_j} \). Let \( r(p_j) \) denote the qubit real-number state with respect to \( (\theta_j, \lambda_j) \).
Then \( p_j \rightarrow p_{j+1} \) can be expressed by the finite difference equations (31)

\[
\begin{align*}
\theta_{j+1} &= \theta_j + \alpha (r - \frac{1}{2}) \sin \theta \Delta t \\
\lambda_{j+1} &= \lambda_j
\end{align*}
\]

Before the next step is implemented, there must be a transformation to the
coordinate system based on \( p_{N_{j+1}}, p_{S_{j+1}} \), giving

\[
\begin{align*}
\lambda_{j+1} &\rightarrow \lambda_{j+1} + \delta \lambda_{j+1} \\
\theta_{j+1} &\rightarrow \theta_{j+1} + \delta \theta_{j+1}
\end{align*}
\]
As discussed in section 2, the evolution $r(\lambda) \mapsto r(\lambda + \delta\lambda)$, for small $\delta\lambda$ would be interpreted in conventional computational analysis as white-noise stochastic. The sequence of weak reductions associated with the $\theta$ evolution in equation 83 therefore appears as if generated by a random (gambler’s ruin) walk (Pearle, 1993) towards the attractors $p_N$, $p_S$. In view of the relationship between reduction and measurement outcome, one could think of this sequence of reductions as being associated with measurement by a detector whose orientation was not precisely fixed.

Hence, this deterministic model of weak reduction appears to be consistent, for all practical purposes, with stochastic quantum theory (Percival, 1998). However, in stochastic quantum theory, an additional source of stochastic noise is prescribed, and the Schrödinger equation is generalised to take the form of a stochastic differential equation in Hilbert space. However, in the proposed theory, there are no stochastic sources. Hence at a fundamental level, it is claimed that the proposed deterministic theory is intrinsically simpler than stochastic quantum theory.

11 Discussion

A theory has been developed, where the quantum state is not, axiomatically, an element of a Hilbert space over the complex numbers (and hence not in a complex linear superposition of eigenstates), but is a single real number $0 \leq r \leq 1$. A key number-theoretic notion that provides substance to this (at first sight preposterous) notion is that of number-theoretic Borelian normality; in the proposed theory, the real-number state corresponding to maximally-superposed $M$-level Hilbert-space state is (a good rational approximation to) a base-$M$ normal real, whilst eigenstates of the Hilbert-space (for a particular observable) correspond to the non-normal (trivially, base-1 normal) reals $.000\ldots$, $.111\ldots$, $.222\ldots$ and so on to $.M'M'M'\ldots$, where $M' = M - 1$. One of the key features of the proposed theory, which extends quantum theory, is that the sample space over which quantum measurement probabilities can be calculated, is precisely defined. Specifically, this space is defined by the set of base-$M$ normal numbers generated by a family of self-similar permutation operators acting on the digits and places in the base-$M$ expansion of $r$. These permutation operators have complex structure (being a representation of the $M$th roots of unity) and subsume the essential role of complex numbers in standard quantum theory. In the proposed theory,
real-number state reduction is a precisely deterministic process based on the application of number-theoretic operators which reduce the degree of normality of $r$. It is speculated that these number-theoretic reduction operators describe the (irreversible) process of gravitation at the quantum level. Through the use of these reduction operators, the relation between real-number state reduction and measurement outcome can be defined, without recourse to an arbitrary classical/quantum split. From the degree of number-theoretic normality of $r$, one can directly infer the trace rule for measurement outcome in standard quantum theory. Both observers and the observed are defined from the elements of the theory.

As mentioned, the proposed deterministic realistic theory is more than a reformulation of standard quantum theory; it is an extension of quantum theory, specifically in relation to the measurement problem. It has been shown that many of the foundational difficulties of standard quantum theory are much less problematic in the proposed theory. However, this doesn’t mean that the proposed theory is more like classical theory than standard quantum theory. One of the most profound differences between the proposed theory and classical deterministic theory concerns the notion of counterfactual indefiniteness. This (emergent) property is shown to have profound implications for the interpretation of Bell’s theorem, and allows the proposed theory to be local in the sense of EPR. As such, unlike standard quantum theory, the model is entirely consistent with relativistic invariance.

Since measurement outcomes certainly correspond to elements of reality, all real-number states correspond (through the reduction operators) to elements of reality. Conversely all counterfactual measurement outcomes, which by definition cannot correspond to elements of reality, have undefined real-number states. Moreover, within the proposed theory, observers, measuring apparati (and indeed the whole cosmos) also have real-number states albeit with very large degrees of normality (eg corresponding to base-$M$ normal reals, where $M \gg 1$. Putting these facts together, it can be seen that Bell’s (1993) notion of ‘beable’ describes this real-number state precisely.

It can be noted that it is straightforward to show that the proposed theory satisfies Hardy’s 5 ‘reasonable’ probability axioms for a quantum theory—consistent with the claim that at the level of probabilities the proposed realistic theory is equivalent to standard quantum theory. Hardy’s 5th axiom, which distinguishes quantum from classical theory, and requires probability densities to vary continuously in state space, is satisfied by virtue of the fact that the degree of normality varies continuously in real-number state space.
This work was motivated by the author’s studies of predictability in meteorology, especially of three-dimensional high Reynolds-number turbulence in the inertial sub-range (Palmer, 2000). The existence of a finite-time predictability horizon in such turbulent systems is fundamentally different from the infinite-predictability horizon associated with conventional chaotic systems. This paradigm has been used in section 8 to define the concept of a ‘Navier-Stokes’ computer, to illustrate that the exponential speed up of certain quantum computations need not imply a many-worlds interpretation.

Acknowledgement

My thanks to Lucien Hardy for a helpful discussion on the formulation of the proposed theory with respect to the 3-level system, and to Michael McIntyre for encouragement and many helpful comments on how to improve the paper’s lucidity.

References

Bell, J. S., 1964: On the Einstein-Podolsky-Rosen paradox. Physics 1, 195-200.

Bell, J.S., 1985: Free variables and local causality. Dialectica, 39, 85-96.

Bell, J.S., 1990: Against ‘Measurement’. Physics World, 3, 33-40.

Bell, J.S., 1993: Speakable and unspeakable in quantum mechanics. Cambridge University Press. 212pp.

Deutsch, D., 1997: The Fabric of Reality. Penguin Books. 390pp.

Diósi, L., 1989: Models for universal reduction of macroscopic quantum fluctuations. Phys.Rev. A, 40, 1165-1174.

Duane, G.S., 2001: Violations of Bell’s inequality in synchronized hyper-chaos. Foundations of Physics Letters, 14, 341-353.

Einstein, A., Podolsky, P. and N.Rosen, 1935: Can quantum-mechanical description of physical reality be considered complete? Phys.Rev., 47, 777-780.
Hardy, G.H. and Wright, E.M., 1979: The Theory of Numbers. Oxford University Press.

Hardy, L., 2001: Quantum theory from five reasonable axioms. quant-ph/0101012.

Isham C., 1997: Topos Theory and Consistent Histories: The Internal Logic of the Set of all Consistent Sets. Int.J.Theor.Phys., 36, 785-814

Kent, A. 2002: Locality and reality revisited. quant-ph/0202064.

Lorenz, E.N., 1969: The predictability of a flow which possesses many scales of motion. Tellus, 21, 289-307.

Nicolis, J.S., Nicolis, G. and Nicolis, C., 2001: Nonlinear dynamics and the two-slit delayed experiment. Chaos, Solitons and Fractals, 12, 407-416.

Nielsen, M.A. and I.L.Chuang, 2000: Quantum Computing and Quantum Information. Cambridge University Press. 676pp

Ott, E., Sommerer, J.C., Alexander, J.C., Kan, I., and J.A.Yorke, 1993: Scaling behavior of chaotic systems with riddled basins. Phys. Rev. Lett., 71, 4134-4137.

Palmer, T.N., 1995: A local deterministic model of quantum spin measurement. Proc.R.Soc.Lond. A, 451, 585-608.

Palmer, T.N., 2000: Predicting uncertainty in forecasts of weather and climate. Reports on Progress in Physics., 63, 71-116.

Pearle, P., 1993: Ways to describe dynamical state vector reduction. Phys. Rev., A48, 913-923.

Penrose, R., 1989: The Emperor’s New Mind. Oxford University Press. Oxford. 466pp

Penrose, R., 1994: Shadows of the mind. Oxford University Press. Oxford. 457p

Penrose, R., 1998: Quantum computation, entanglement and state reduction. Phil. Trans. Roy. Soc., A451, 1927-1939.
Pitowsky, I., 1983: Deterministic model of spin and statistics. Phys.Rev., D27, 2316-2326.

Prigogine, I. and Y. Elskens, 1987: Irreversibility, stochasticity and non-locality in classical dynamics. In ‘Quantum Implications: Essays in Honour of David Bohm’. Routledge. London. 455pp

Shnirelman, A., 1997: On the nonuniqueness of weak solutions to the Euler equation. Comm. Pure & Appl. Math., 50, 1260-1286.

‘t Hooft, G., 1999: Quantum gravity as a dissipative deterministic system. [quant-qc/9903084]

Whitham, G.B., 1974: Linear and Non-linear waves. John Wiley. New York. 636pp.