Limit behavior of the magnetic coupling coefficient for mid-range, near-field applications

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Abstract. We have derived compact algebraic bounds for the limit behavior of the magnetic coupling coefficient, \( \kappa \), for axially aligned multilayer coils. These bounds are validated experimentally. We have also found an expression based on the coil geometry that captures its long-range magnetic behavior. In particular, the limit behavior of the magnetic coupling coefficient is the same for any pair of axially aligned multilayer coils when the separation distance is normalized by this expression.

1. Introduction
The magnetic coupling coefficient \( \kappa \) is a key parameter that characterizes the performance of inductive systems. Since the efficiency of inductive power transfer is proportional to the magnetic coupling coefficient squared for large enough separation distances [1, 2], these expressions bound the performance of these systems in this range. This also suggests optimal geometric coil designs that maximize the limit behavior of the magnetic coupling coefficient.

Since the mutual inductance between two filament loops is a transcendental expression that requires elliptic integrals [3], there are no general closed-form expressions for the self and mutual inductance of coils. Furthermore, even though several approximations for both self and mutual inductances of various geometries have been found [4, 5, 8], the magnetic coupling coefficient is still calculated empirically [2], derived [6], or approximated within a specific context [7, 9]. We present a simple expression that serves as a widely applicable upper bound on the magnetic coupling coefficient for a broad class of inductive systems. Furthermore, they provide a limit test for similar design equations in other configurations, such as unaligned coils, or coils with other cross sectional geometries.

2. Derivation
Figure 1 shows the specific case under consideration, consisting of axially aligned, multi-layer coils \( i = 1, 2 \) with separation \( s \) and characterized by their inductance \( L_i \), turns density \( \rho N_i \), outer radius \( r_i \), length \( \ell_i \), and fill factor \( f_i \), where \( 0 < f_i < 1 \). In the limit as the separation between sender and receiver increases, the magnetic coupling coefficient starts to decrease inversely proportional to the distance cubed, which determines the performance of mid-range inductive systems [1, 2, 10]. At this long range, while in the near field, the magnetic coupling coefficient
can be calculated by combining the mutual inductance, given by

\[ L_M = \frac{\mu_0 \rho_1 \rho_2 \pi}{2} \int_0^{\ell_2} \int_{r_1}^{r_2} \frac{r_1^2}{r_1^2 + (s + \ell_1 f_1 + \ell_2 f_2)^2} 2 \pi r_1 \sqrt{r_1^2 + (s + \ell_1 f_1 + \ell_2 f_2)^2} dr_1 \sqrt{r_1^2 + (s + \ell_1 f_1 + \ell_2 f_2)^2} dr_2, \]

with Wheeler’s approximation for the self inductance [5],

\[ L_i = \frac{\mu_0 (\rho_1 \ell_i r_i f_i)^2 \pi r_i^2 (2 - f_i)}{3 r_i + 3.51 r_i f_i + 4.51 \ell_i}, \quad i = 1, 2. \]

The mutual inductance calculation is the integral for the cross section of each coil of the field along the center axis from a single loop multiplied by the area of a single receiving loop. Since this expression relies on Wheeler’s approximation for the self inductance of a multi-layer coil, it shares the same limitations, namely, the aspect ratio (a.r.) of the coils, \( \ell_i/(r_i f_i) \) must be close to 1, or \( 1/2 < \ell_i/(r_i f_i) < 2 \). However, there are widely known approximations for other aspect ratios that would result in similar expressions [5].

The resulting expression for the magnetic coupling coefficient \( \kappa \) is

\[ \kappa = \frac{L_M}{\sqrt{L_1 L_2}} = \frac{\sqrt{3r_1 + 3.51 r_1 f_1 + 4.51 \ell_1 f_1}}{2 - f_1} \frac{\sqrt{3r_2 + 3.51 r_2 f_2 + 4.51 \ell_2 f_2}}{2 - f_2} \frac{r_1 r_2}{4 f_1} \left( \frac{f_2^2}{3} - f_2 + 1 \right) \times \ldots \]

\[ \sum_{i=1,4} (-1)^i \left[ \frac{1}{1 + \left( \frac{s + \ell_i}{r_1} \right)^2} - \left( 1 - f_i \right) \frac{1}{1 + \left( \frac{s + \ell_i}{r_1} \right)^2} + \ldots \right] \]

\[ \frac{\left( \frac{s + \ell_i}{r_1} \right)^2 \arcsin \frac{r_1}{s + \ell_i} - \frac{\left( \frac{s + \ell_i}{r_1} \right)^2 \arcsin \frac{r_1 (1 - f_i)}{s + \ell_i}}{s + \ell_i} \right], \quad \ell_i = \{0, \ell_1, \ell_1 + \ell_2, \ell_2\}. \]

The first term of the Taylor series expansion of this expression provides an upper bound, \( \kappa_u \), for the magnetic coupling coefficient as the separation \( s \) tends to infinity, where

\[ \kappa_u = \frac{1}{2} r_1 \sqrt{3r_1 + 3.51 r_1 f_1 + 4.51 \ell_1} \left( \frac{f_2^2/3 - f_2 + 1}{2 - f_1} \right) \times \ldots \]

\[ r_2 \sqrt{3r_2 + 3.51 r_2 f_2 + 4.51 \ell_2} \left( \frac{f_2^2/3 - f_2 + 1}{2 - f_2} \right) \frac{1}{s^3}, \]

as shown in Figure 2. Adding the second term of the Taylor series expansion results in a lower bound for the magnetic coupling coefficient, \( \kappa_l \), where

\[ \kappa_l = \kappa_u \left( 1 - \frac{3 \ell_1 + \ell_2}{2 s} \right), \]

also shown in Figure 2. Even though the value of the magnetic coupling coefficient in this range is on the order of \( 10^{-2} \) to \( 10^{-4} \), large enough quality factors in resonant systems may allow the transfer of power at high enough magnitude and efficiency for some applications [11].

The upper and lower bounds for the coupling coefficient also let us determine when the upper bound approximation for \( \kappa \) is an appropriate substitute. The normalized difference between the bounds is

\[ \frac{\kappa_u - \kappa_l}{\kappa_u} = \frac{3 \ell_1 + \ell_2}{2 s}. \]

This expression can be solved for \( s \) to indicate the range of separations where the magnetic coupling coefficient is arbitrarily close to its limit \( \kappa_u \).
3. Normalization
The upper bound for the magnetic coupling coefficient, $\kappa_u$, suggests a normalized separation distance $\delta$, where

$$\delta = s / \sqrt{\left( V_{\text{eff},1}^{1/3} \cdot V_{\text{eff},2}^{1/3} \right)}$$

and

$$V_{\text{eff},i} = \left[ 3 + 3.51 f_i + 4.51 \ell_i / r_i \right] \left[ \frac{f_i^2 / 3 - f_i + 1}{2 - f_i} \right] r_i^3, \quad i = 1, 2.$$

Since the quantity $V_{\text{eff}}$ has units of volume, we call it the effective magnetic volume. As shown in Figure 3, this normalization simplifies the limit behavior of any pair of coils to

$$\kappa_u = 1 / 2 \delta^3.$$
4. Validation

The expressions derived for the magnetic coupling coefficient and its bounds were validated experimentally by measuring the voltage induced by a sender coil on an axially aligned receiver coil as shown in Figure 4. The experiment was repeated using three different coil pairs. The circuit model of the experimental system ignores the load presented by the measuring probes and assumes no current in the receiver coil. The sender coil was driven with a sinusoidal voltage $v_S$ of 1.64 Vrms. The frequency was adjusted until the phase difference between the voltages across the sender $v_S$ and receiver $v_2$ was 45 degrees, such that

$$\frac{|v_2|}{v_S} = \frac{\kappa \sqrt{2}}{\sqrt{L_2/L_1}},$$

where $v_2$ is the voltage across the receiver coil, $L_2$ is the inductance of the receiver coil, $L_1$ is the inductance of the sender coil, and $v_S$ is the voltage across the sender coil.

![Figure 4](image-url)

**Figure 4.** Experimental system for measuring the magnetic coupling coefficient between to axially aligned coils and circuit model. A sender coil of inductance $L_1$ is driven using a voltage source $v_S$. The induced voltage $v_2$ is measured across an axially aligned receiver coil of inductance $L_2$. The source coil is driven at the frequency $R_{S1}/(2\pi L_1)$ such that the measurement is independent of $R_{S1}$.

In two experiments, shown in red and blue in Figure 5, the sender coil had an outer radius $r_1 = 5$ cm, inner radius of 4.5 cm (for a resulting fill factor $f_1 = 0.1$), length $\ell_1 = 2.5$ cm and a measured inductance of $L_1 = 0.74$ H. The first experiment, shown in red in Figure 5, had a receiving coil with dimensions and inductance identical to the transmitting coil. The second experiment, shown in blue in Figure 5, had a receiver coil with an outer radius $r_2 = 7$ cm, inner radius of 6.5 cm (for a resulting fill factor $f_2 = 0.07$), length $\ell_2 = 2.5$ cm and a measured inductance of $L_2 = 2.76$ H.

In the third experiment, shown in purple in Figure 5, the sender coil was the receiver coil from the second experiment with a measured inductance of $L_1 = 2.76$ H. The receiver coil had an outer radius $r_2 = 6.5$ cm, inner radius of 4.5 cm (for a resulting fill factor $f_2 = 0.31$), length $\ell_2 = 8.9$ cm and a measured inductance of $L_2 = 2.5$ H.

5. Conclusion

The universal behavior resulting from the normalized separation $\delta$ can be used to maintain the same limit behavior of the magnetic coupling coefficient while changing the dimensions of either coil. In particular, the limit behavior will be the same as long as the geometric mean of effective magnetic volumes of the coils remains constant. Alternatively, the effective magnetic volume can be used to maximize the limit behavior of the magnetic coupling coefficient by optimizing the geometry of coils given any set of constraints.
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