A General Non-Vanishing Theorem and an Analytic Proof of
the Finite Generation of the Canonical Ring

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Abstract. On August 5, 2005 in the American Mathematical Society Summer Institute on Algebraic Geometry in Seattle and later in several conferences I gave lectures on my analytic proof of the finite generation of the canonical ring for the case of general type. After my lectures many people asked me for a copy of the slides which I used for my lectures. Since my slides were quite sketchy because of the time limitation for the lectures, I promised to post later on a preprint server my detailed notes from which my slides were extracted. Here are my detailed notes giving the techniques and the proof.

Table of Contents

§1. Technique of Skoda on Ideal Generation
§2. Reduction of Algebraic Geometric Problems to $L^2$ Estimates for Stein Domains Spread Over $\mathbb{C}^n$
§3. Stable Vanishing Orders and Their Achievement by Finite Sums.
§4. Reduction of Achievement of Vanishing Order to Non-Vanishing Theorem on Hypersurface by Fujita Conjecture Type Techniques
§5. Diophantine Approximation of Kronecker
§6. A General Non-Vanishing Theorem
§7. Holomorphic Family of Artinian Subschemes and Achievement of Stable Vanishing Orders for the Case of Higher Codimension
§8. Remark on the Approach of Extension Using Techniques of the Invariance of Plurigenera
§9. Remark on Positive Lower Bound of Curvature Current

On August 5, 2005 in the American Mathematical Society Summer Institute on Algebraic Geometry in Seattle I gave a lecture in which I first presented my analytic method of proving the finite generation of the canonical ring for the case of general type. Later in several conferences (the Birthday Conference for Skoda in Paris on September 12, 2005; the Memorial

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Conference for Vitushkin in Moscow on September 26, 2005; the Birthday Conference for Bogomolov in Miami on December 18, 2005; the Birthday Conference for Toledo in Salt Lake City on March 24, 2006; the Retirement Conference for Kiselman in Uppsala on May 15, 2006; the Birthday Conference of Lu Qikeng in Beijing on June 6, 2006; the Trento Conference on CR Geometry and PDE on September 6, 2006; the Second Chinese-German Conference on Complex Analysis in Shanghai on September 12, 2006) I lectured on the same analytic proof of the finite generation of the canonical ring for the case of general type. After my lectures many people asked me for a copy of the slides which I used for my lectures. Since my slides were quite sketchy because of the time limitation for the lectures, I promised to post later on a preprint server my detailed notes from which my slides were extracted. Here are my detailed notes giving the techniques and the proof.

These notes are selected and organized from the notes of roughly one year old which I wrote for myself over a period of several years as memoranda while I worked on the problem of the finite generation of the canonical ring. Unlike a formal preprint which follows the traditional order of history, definitions, lemmas, propositions, and theorems, these notes start out with key ideas and techniques as the backbone and then flesh out with more and more details and explanations on how to deal with the difficulties which arise in the course of the implementation of the key ideas and techniques until the complete solution is reached. I made some selections when there are several ways of handling a difficulty and I unified the notations and terminology and put in the numbering for sections, paragraphs, definitions, lemmas, propositions, theorems, remarks, equations, et cetera, but the presentation retains essentially the order and the substance of the material in my original notes. This style of presentation actually makes the geometric ideas and the techniques for the proof more transparent. I hope that the people interested in the details of the techniques and the proof of the finite generation of the canonical ring presented in my lectures in the several conferences since August 2005 will find these notes easier to read and understand than a formal preprint.

The result on the finite generation of the canonical ring for the case of general type which these notes give an analytic proof for is the following.

**Main Theorem.** Let $X$ be a compact complex algebraic manifold of complex dimension $n$ which is of general type in the sense that there exist a positive
integer $m_0$ and a positive number $c$ such that $\dim_c \Gamma (X, mK_X) \geq cm^n$ for $m \geq m_0$, where $K_X$ is the canonical line bundle of $X$. Then the canonical ring $\bigoplus_{m=1}^{\infty} \Gamma (X, mK_X)$ is finitely generated.

An important component in the analytic proof of the finite generation of the canonical ring for the case of general type is the general non-vanishing theorem in the title of these notes which is stated in (6.2) with its proof given in §6.

Since the problem of the finite generation of the canonical ring is a well-known problem in algebraic geometry, the history of the problem is not repeated in these notes.

It was brought to my attention that on October 5, 2006 Caucher Birkar, Paolo Cascini, Christopher D. Hacon, James McKernan posted a preprint on the “existence of minimal models for varieties of log general type” on the “arXiv.org” server.

I explain here the organization of these notes. The key ingredient in the analytic proof of the finite generation of the canonical ring is the theorem of Skoda on ideal generation. It is presented in §1. With $L^2$ estimates problems in algebraic geometry involving holomorphic sections of holomorphic line bundles over compact complex algebraic manifolds can be reduced to problems for Stein domains spread over $\mathbb{C}^n$. This is presented in §2, with explanations on what such a reduction means in the problem of the finite generation of the canonical ring. In §3 an infinite sum $\Phi$ of the absolute-value squares of fractional powers of pluricanonical sections is introduced and the problem of the finite generation of the canonical ring is shown to be equivalent to the precise achievement of stable vanishing orders in the sense that the vanishing orders of the infinite sum $\Phi$ can be achieved by some of its finite partial sums. The proof is done by descending induction on the dimension of the subvariety $V$ where the stable vanishing order is not yet known to be precisely achieved. In §4 by techniques for Fujita conjecture type problems, the initial induction step where $V$ is a hypersurface is reduced to a general non-vanishing theorem.

In §5 we group together results derived from Kronecker’s theorem on diophantine approximation which will be needed later for our general non-vanishing theorem. In §6 the general non-vanishing theorem is presented which gives the existence of nonzero holomorphic sections belonging locally
to an appropriate multiplier ideal sheaf, under some positive lower bound condition for the curvature current of the line bundles involved. This general non-vanishing theorem rules out the possibility of an infinite number of components for the embedded stable base point set, which is the major obstacle in getting the finite generation of the canonical ring. Essential for the proof of the general non-vanishing theorem is Shokurov’s technique of comparing two applications of the theorem of Riemann-Roch, one to a line bundle and another to its twisting by a flat line bundle.

In §7 the method of continuous variation of an Artinian subscheme without jump is introduced in order to prove the precise achievement of the stable vanishing order at a generic point of a subvariety of higher codimension. The vanishing order of $\Phi$ across a hypersurface is a single number, but in the case of a subvariety $V$ of higher codimension this rôle is played by Artinian subschemes transversal to the subvariety $V$. Each finite partial sum of $\Phi$ provides one Artinian subscheme which varies continuous along the subvariety $V$ without jump except at a subvariety $E$ of codimension $\geq 1$ in $V$. At points of $V$ outside the countable union of this kind of subvarieties $E$ the stable vanishing order is precisely achieved, making it possible to go to the next step in the induction process. The proof of the finite generation of the canonical ring is completed in §7. Also in §7 it is explained why the proof of precisely achieving the stable vanishing order at a generic point of a subvariety of higher codimension cannot simply be reduced to the hypersurface case by blow-up and how the continuous variation of an Artinian subscheme without jump handles the problem.

The development of extension techniques for the problem of the deformational invariance of plurigenera was originally intended for application to the problem of the finite generation of the canonical ring. In §8 the approach by such extension techniques to the problem of the finite generation of the canonical ring is compared to the analytic proof presented in these notes. Difficulties with the approach by extension techniques are analyzed. Finally §9 gives some remarks concerning the condition on positive lower bounds for curvature currents, including a remark about the problem of the finite generation of the canonical ring without the general type condition and the difficulty of artificially adding an ample twisting first with the taking of root-limits at the end to get rid of its contribution.

Notations. $\mathbb{N}$ is the set of all positive integers. $\mathbb{Z}$ is the set of all integers.
\( \mathbb{Q} \) is the set of all rational numbers. \( \mathbb{R} \) is the set of all real numbers. \( \mathbb{C} \) is the set of all complex numbers. For a subvariety \( V \) we use \( I_V \) to denote the coherent ideal sheaf of all germs of holomorphic functions vanishing on \( V \). The structure sheaf of a complex space \( X \) is denoted by \( \mathcal{O}_X \). The maximum ideal of \( X \) at a point \( P \) is denoted by \( \mathfrak{m}_{X,P} \). A multi-valued holomorphic section \( s \) of a \( \mathbb{Q} \)-line-bundle \( E \) means that \( s^N \) is a holomorphic section of the holomorphic line bundle \( NE \) for some positive integer \( N \). For a divisor \( Y \) we denote by \( s_Y \) the canonical section of the line bundle defined by \( Y \). When \( Y \) is a \( \mathbb{Q} \)-divisor, the canonical section \( s_Y \) is a multi-valued holomorphic section of the \( \mathbb{Q} \)-line-bundle defined by \( Y \). The space of all sections of a holomorphic bundle or a sheaf \( E \) over \( X \) is denoted by \( \Gamma (X, E) \). The term “generic” is also used in the sense of avoiding some countable union of subvarieties of codimension \( \geq 1 \) (or even some countable union of locally defined subvarieties of codimension \( \geq 1 \)). The round-down of a real number \( u \) is denoted by \( \lfloor u \rfloor \) which is the largest integer \( \leq u \). The round-up of a real number \( u \) is denoted by \( \lceil u \rceil \) which is the smallest integer \( \geq u \).

§1. Technique of Skoda on Ideal Generation

The key ingredient in the analytic proof of the finite generation of the canonical ring is the following result of Skoda on ideal generation [Skoda 1972]. (Skoda’s original statement is for a Stein domain in \( \mathbb{C}^n \), but for its application to algebraic geometry we need the version of a Stein domain spread over \( \mathbb{C}^n \).)

(1.1) Theorem (Skoda on Ideal Generation). Let \( \Omega \) be a domain spread over \( \mathbb{C}^n \) which is Stein. Let \( \psi \) be a plurisubharmonic function on \( \Omega \), \( g_1, \ldots, g_p \) be holomorphic functions on \( \Omega \), \( \alpha > 1 \), \( q = \min (n,p - 1) \), and \( f \) be a holomorphic function on \( \Omega \). Assume that

\[
\int_{\Omega} \frac{|f|^2 e^{-\psi}}{\left( \sum_{j=1}^{p} |g_j|^2 \right)^{\alpha q + 1}} < \infty.
\]

Then there exist holomorphic functions \( h_1, \ldots, h_p \) on \( \Omega \) with \( f = \sum_{j=1}^{p} h_j g_j \) on \( \Omega \) such that

\[
\int_{\Omega} \frac{|h_k|^2 e^{-\psi}}{\left( \sum_{j=1}^{p} |g_j|^2 \right)^{\alpha q}} \leq \frac{\alpha}{\alpha - 1} \int_{\Omega} \frac{|f|^2 e^{-\psi}}{\left( \sum_{j=1}^{p} |g_j|^2 \right)^{\alpha q + 1}}
\]

for \( 1 \leq k \leq p \).
Remark on the Condition of Skoda’s Theorem on Ideal Generation. The condition of Skoda’s theorem on ideal generation comes from transplanting the obvious necessary supremum condition to an $L^2$ condition. It is clear that a necessary condition to express $f$ in terms of $g_1, \cdots, g_p$ as $f = \sum_{j=1}^{p} h_j g_j$ is that $|f| \leq C \sum_{j=1}^{p} |g_j|$ on any compact subset of $\Omega$ with $C$ depending on the compact subset. However, we have to use Hilbert space arguments instead of Banach space arguments with the supremum norm. Thus we need formulation in $L^2$ bounds such as

$$\int \frac{|f|^2}{\sum_{j=1}^{p} |g_j|^2} < \infty.$$ 

The translation of the condition from supremum norm to $L^2$ norm calls for modification in the formulation, because in the case of $f = 1$ and $\{g_1, \cdots, g_p\} = \{z_1, \cdots, z_n\}$, the integral is finite by polar coordinate argument for $n \geq 2$ and yet $f$ cannot be so expressed. Skoda’s formulation modifies the integral bound to

$$\int \frac{|f|^2}{\left(\sum_{j=1}^{p} |g_j|^2\right)^{\alpha q + 1}} < \infty$$

with $\alpha > 1$ and $q = \min (n, p - 1)$ for polar coordinate reasons.

We have to pay a price for translating the formulation of the assumption to the Hilbert space context in that in the denominator there is a gap between the exponent for the sufficiency and the exponent for the necessity, illustrated in the two extreme cases of (i) $p = n$ and $g_1 = z_1, \cdots, g_n = z_n$ and (ii) $p = 1$ and $g_1 = z_1$.

§2. Reduction of Algebraic Geometric Problems to $L^2$ Estimates for Stein Domains Spread Over $\mathbb{C}^n$.

Since $L^2$ holomorphic functions can be extended across a complex hypersurface, by representing an Zariski open subset of a compact complex algebraic manifold as a Stein domain spread over $\mathbb{C}^n$ and using a meromorphic section of a holomorphic line bundle, we can reduce a problem in algebraic geometry involving line bundles to an analytic problem for $L^2$ estimates on a Stein domain spread over $\mathbb{C}^n$ as follows.
(2.1) **Description of Reduction.** Let $X$ be an $n$-dimensional complex manifold inside $\mathbb{P}_N$. We regard $\mathbb{P}_{N-n-1} \subset \mathbb{P}_N - X$ as a source of light. Take any $\mathbb{P}_n \subset \mathbb{P}_N$. Fix $x \in X$, define $\pi(x)$ as the only point in $\text{span}(x, \mathbb{P}_{N-n-1}) \cap \mathbb{P}_n$ to make $\pi : X \to \mathbb{P}_n$ a branched cover, where $\text{span}(x, \mathbb{P}_{N-n-1})$ means the projective linear subspace of dimension $N - n$ in $\mathbb{P}_N$ which contains both $x$ and $\mathbb{P}_{N-n-1}$.

Let $L$ be a holomorphic line bundle over $X$ and $s$ be a global meromorphic section with pole-set $A$ and zero-set $B$. Take some hypersurface $Z$ inside $\mathbb{P}_n$ containing $\mathbb{P}_n - \mathbb{C}^n$ and $\pi(A \cup B)$ such that $\pi : X - \pi^{-1}(Z) \to \mathbb{P}_n - Z$ is a local biholomorphism. Then $X - \pi^{-1}(Z)$ is a Stein domain spread over $\mathbb{C}^n$.

Take a metric $e^{-\varphi}$ of $L$ with $\varphi$ locally bounded from above. For any open subset $\Omega$ of $X$ and any holomorphic function $f$ on $\Omega - \pi^{-1}(Z)$ with

$$\int_{\Omega - \pi^{-1}(Z)} |f|^2 e^{2\log|s| - \varphi} < \infty,$$

the section $fs$ of $L$ can be extended to a holomorphic section of $L$ over $\Omega$. So instead of dealing with $X$ and $L$ to find elements of $\Gamma(X, L)$, we can deal with holomorphic functions on a Stein domain spread over $\mathbb{C}^n$ which satisfy certain $L^2$ estimates. This method of reduction translates Skoda’s theorem on ideal generation for Stein domains spread over $\mathbb{C}^n$ to results concerning holomorphic sections of line bundles over compact complex algebraic manifolds. We are going to use it to reduce the problem of finite generation of the canonical ring to a problem for the precise achievement of stable vanishing orders which will be explained in details later.

(2.2) **Reformation of the Ideal Generation of Skoda for Compact Algebraic Manifolds and Holomorphic Line Bundles.** We now apply, to the reformulation of the theorem of Skoda on ideal generation, the above procedure of reducing algebraic geometric problems to $L^2$ estimates for Stein domains spread over $\mathbb{C}^n$. First we introduce the following definition and notation for multiplier ideal sheaves.

(2.3) **Definition of Multiplier Ideal Sheaves.** For a local plurisubharmonic function $\varphi$ on an open subset of $\mathbb{C}^n$, the multiplier ideal sheaf $\mathcal{I}_\varphi$ is the sheaf of germs of holomorphic functions $f$ such that $|f|^2 e^{-\varphi}$ is locally integrable.

Using the notation of multiplier ideal sheaves, we can formulate the theorem of Skoda on ideal generation for compact algebraic manifolds and holomorphic line bundles as follows.
(2.4) **Theorem.** Let $X$ be a compact complex algebraic manifold of complex dimension $n$, $L$ be a holomorphic line bundle over $X$, and $E$ be a holomorphic line bundle on $X$ with metric $e^{-\psi}$ such that $\psi$ is plurisubharmonic.

Let $k \geq 1$ be an integer, $G_1, \ldots, G_p \in \Gamma(X, L)$, and $|G|^2 = \sum_{j=1}^{p} |G_j|^2$. Let $\mathcal{I} = \mathcal{I}_{(n+k+1)\log |G|^2+\psi}$ and $\mathcal{J} = \mathcal{I}_{(n+k)\log |G|^2+\psi}$. Then

$$\Gamma(X, \mathcal{I} \otimes ((n+k+1)L + E + K_X)) = \sum_{j=1}^{p} G_j \Gamma(X, \mathcal{J} \otimes ((n+k)L + E + K_X)).$$

**Proof.** Take $F \in \Gamma(X, \mathcal{I} \otimes ((n+k+1)L + E + K_X))$. Let $S$ be a meromorphic section of $E$. Take a branched cover map $\pi : X \to \mathbb{P}_n$. Let $Z_0$ be a hypersurface in $\mathbb{P}_n$ which contains the infinity hyperplane of $\mathbb{P}_n$ and the branching locus of $\pi$ in $\mathbb{P}_n$ such that $Z := \pi^{-1}(Z_0)$ contains the divisor of $G_1$ and both the pole-set and zero-set of $S$. Let $\Omega = X - Z$. Let $g_j = \frac{G_j}{G_1}$ $(1 \leq j \leq p)$ and $|g|^2 = \sum_{j=1}^{p} |g_j|^2$. Define $f$ by

$$\frac{F}{G_1^{n+k+1}S} = fdz_1 \wedge \cdots \wedge dz_n,$$

where $z_1, \cdots, z_n$ are the affine coordinates of $\mathbb{C}^n$. Use $\alpha = \frac{n+k}{n}$. Let $\psi = \varphi - \log |S|^2$. It follows from $F \in \mathcal{I}_{(n+k+1)\log |G|^2+\varphi}$ locally that

$$\int_X \frac{|F|^2}{|G_1|^{2(n+k+1)}} e^{-\varphi} < \infty,$$

which implies that

$$\int_{\Omega} \frac{|f|^2}{|g|^{2(n+k+1)}} e^{-\psi} = \int_{\Omega} \left| \frac{F}{G_1^{n+k+1}S} \right|^2 e^{-\psi} = \int_{\Omega} \frac{|F|^2}{|G|^{2(n+k+1)}} e^{-\varphi} < \infty.$$

By Skoda’s theorem on ideal generation (1.1) with $q = n$ (which we assume by adding some $F_{p+1} \equiv \cdots \equiv F_{n+1} \equiv 0$ if $p < n + 1$) so that $2\alpha q + 2 = 2 \cdot \frac{n+k}{n} \cdot n + 2 = 2(n+k+1)$, we obtain holomorphic functions $h_1, \ldots, h_p$ on $\Omega$ such that $f = \sum_{j=1}^{p} h_j g_j$ and

$$\sum_{j=1}^{p} \int_{\Omega} \frac{|h_j|^2}{|g|^{2(n+k)}} e^{-\psi} < \infty.$$
Define
\[ H_j = G_1^{n+k} h_j S d z_1 \wedge \cdots \wedge d z_n. \]

Then \( F = \sum_{j=1}^{p} H_j G_j \) and
\[ \int_{\Omega} |H_j| G_j |G_j|^{2(n+k)} e^{-\varphi} = \int_{\Omega} |h_j| g_j |g_j|^{2(n+k)} e^{-\varphi} < \infty \]
so that \( H_j \) can be extended to an element of \( \Gamma(X, (n+k)L+E+K_X) \). Q.E.D.

As an illustration, we give the following trivial immediate consequence of Theorem (2.4). It is on the effective finite generation for the simple case of a numerically effective canonical line bundle, which is interesting in its own right, but not useful for our purpose.

(2.5) Corollary. Let \( F \) be a holomorphic line bundle over a compact complex algebraic manifold \( X \) of complex dimension \( n \). Let \( a > 1 \) and \( b \geq 0 \) be integers such that \( aF \) and \( bF - K_X \) are globally free over \( X \). Then the ring \( \bigoplus_{m=1}^{\infty} \Gamma(X, mF) \) is generated by \( \bigoplus_{m=1}^{(n+2)a+b-1} \Gamma(X, mF) \).

Proof. For \( 0 \leq \ell < a \) let \( E_\ell = (b + \ell) F - K_X \) and \( L = aF \). Let \( G_1, \cdots, G_p \) be a basis of \( \Gamma(X, L) = \Gamma(X, aF) \). Let \( H_1, \cdots, H_q \) be a basis of \( \Gamma(X, bF - K_X) \).

We give \( E_\ell \) the metric
\[ \frac{1}{(\sum_{j=1}^{p} |G_j|^2)^{\frac{1}{2}} (\sum_{j=1}^{q} |H_j|^2)^{\frac{1}{2}}}. \]

Since both \( \mathcal{I} \) and \( \mathcal{J} \) from (2.4) are unit ideal sheaves, it follows that
\[ \Gamma(X, (n+k+1)L + E_\ell + K_X) = \sum_{j=1}^{p} G_j \Gamma(X, (n+k)L + E + K_X) \]
for \( k \geq 1 \) and \( 0 \leq \ell < a \), which means that
\[ \Gamma(X, ((n+k+1)a + \ell + b) F) = \sum_{j=1}^{p} G_j \Gamma(X, ((n+k)a + \ell + b) F) \]
for \( k \geq 1 \) and \( 0 \leq \ell < a \). Thus \( \bigoplus_{m=1}^{(n+2)a+b-1} \Gamma(X, mF) \) generates the ring \( \bigoplus_{m=1}^{\infty} \Gamma(X, mF) \). Q.E.D.
3. Stable Vanishing Orders and Their Achievement by Finite Sums.

Let $X$ be a compact complex algebraic manifold of complex dimension $n$ which is of general type. Let

$$\Phi = \sum_{m=1}^{\infty} \varepsilon_m \sum_{j=1}^{q_m} |s_j^{(m)}|^{\frac{2}{m}},$$

where

$$s_1^{(m)}, \ldots, s_{q_m}^{(m)} \in \Gamma(X, mK_X)$$

form a basis over $\mathbb{C}$ and $\varepsilon_m > 0$ approach 0 so fast as $m \to \infty$ that locally the infinite series which defines $\Phi$ converges uniformly. Note that the expression $\frac{1}{\Phi}$ defines a metric for $K_X$. In these notes we will reserve the symbol $\Phi$ only for this kind of infinite sums.

Likewise, more generally, for a big holomorphic line bundle over $X$, we can introduce

$$\Psi = \sum_{m=1}^{\infty} \delta_m \sum_{j=1}^{r_m} |g_j^{(m)}|^{\frac{2}{m}},$$

where

$$g_1^{(m)}, \ldots, g_{r_m}^{(m)} \in \Gamma(X, mL)$$

form a basis over $\mathbb{C}$ and $\delta_m > 0$ approach 0 so fast as $m \to \infty$ that locally the infinite series which defines $\Psi$ converges uniformly. In this case $\frac{1}{\Psi}$ defines a metric for $L$.

What plays the most important rôle in the finite generation of the canonical ring is the vanishing orders of the local function $\Phi$. Since $\Phi$ is an infinite sum of the absolute-value squares of fractional powers of holomorphic functions, the easiest way to define the vanishing orders of $\Phi$ rigorously is to use Lelong numbers.

(3.1) Definition of Lelong Numbers. A $(1, 1)$-current $\Theta$ on an open subset $G$ of $\mathbb{C}^n$ is said to be positive if for any smooth $(1, 0)$-forms $\sigma_1, \ldots, \sigma_{n-1}$ on $G$ with compact support the $(n, n)$-current

$$\Theta \wedge (\sqrt{-1} \sigma_1 \wedge \overline{\sigma_1}) \wedge \cdots \wedge (\sqrt{-1} \sigma_{n-1} \wedge \overline{\sigma_{n-1}})$$

on $G$ is a nonnegative measure. Note that though by convention the adjective “positive” is used in this definition, it actually means just “nonnegative” and not “strictly positive.”
For a complex hypersurface \( Y \) in \( G \), \( \theta \mapsto \int_{\text{Reg} Y} \theta \) for any smooth \((n-1, n-1)\)-form \( \theta \) on \( G \) with compact support defines a closed positive \((1, 1)\)-current which we denote by \([Y]\), where \( \text{Reg} Y \) is the regular part of \( Y \) consisting of all nonsingular points of \( Y \).

Every closed positive \((1, 1)\)-current \( \Theta \) can be locally written as \( \sqrt{-1} \partial \bar{\partial} \varphi \) for some plurisubharmonic function \( \varphi \) (sometimes known as a plurisubharmonic potential) and, conversely, \( \sqrt{-1} \partial \bar{\partial} \varphi \) is a closed positive \((1, 1)\)-current for any plurisubharmonic function \( \varphi \).

For a closed positive \((1, 1)\)-current \( \Theta \) on some open subset \( G \) of \( \mathbb{C}^n \), the Lelong number of \( \Theta \) at a point \( P_0 \) is the limit of

\[
\frac{\int_{B_m(P_0, r)} \text{trace} \Theta}{\text{Vol} (B_{n-1}(0, r))}
\]
as \( r \to 0 \), where \( B_m(Q, r) \) is the open ball in \( \mathbb{C}^m \) of radius \( r \) centered at \( Q \) and \( \text{Vol} (B_m(Q, r)) \) is its volume and trace \( \Theta \) is

\[
\Theta \wedge \frac{1}{(n-1)!} \left( \sum_{j=1}^{n} \frac{-1}{2} dz_j \wedge d\bar{z}_j \right)^{n-1}
\]
with \( z_1, \ldots, z_n \) being the coordinates of \( \mathbb{C}^n \).

For a complex hypersurface \( Y \) in \( G \) the Lelong number of \([Y]\) at a point \( P_0 \) of \( G \) is the multiplicity of \( Y \) at \( P_0 \).

For any \( c > 0 \) and any closed positive \((1, 1)\)-current \( \Theta \), we denote by \( E_c(\Theta) \) the set of points where the Lelong number of \( \Theta \) is no less than \( c \). The set \( E_c(\Theta) \) is always a subvariety. An irreducible branch \( E \) of the subvariety \( E_c(\Theta) \) is called an irreducible Lelong set of \( \Theta \). By the generic Lelong number of an irreducible Lelong set \( E \) for \( \Theta \) is meant the Lelong number of \( \Theta \) at any generic point \( P \) of \( E \) and is independent of the choice of \( P \). For any subvariety \( V \) (not necessarily an irreducible Lelong set), the generic Lelong number of \( \Theta \) at \( V \) means the Lelong number of \( \Theta \) at a generic point of \( V \).

For a metric \( e^{-\varphi} \) we use the notation

\[
\Theta_\varphi = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \varphi
\]
to denote the curvature current of $e^{-\varphi}$. In the case of the metric $\frac{1}{\Phi}$ for the canonical line bundle $K_X$ of $X$ its curvature current is given by

$$\Theta_{\log \Phi} := \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log \Phi.$$ (3.1.1)

**Additional Factor of Two in Vanishing Order of $\Phi$ Compared to Lelong Number.** We use the Lelong number of $2\Theta_{\log \Phi}$ at a point $P$ of $X$ to define the vanishing order of $\Phi$ at $P$. In this definition of the vanishing order of $\Phi$, a factor of 2 is introduced in $2\Theta_{\log \Phi}$, because, in the definition of $\Phi$ as an infinite series, absolute-value squares are taken of the fractional powers of holomorphic functions in the individual terms instead of just the absolute value. Note that when the complex dimension $n$ of $X$ is $> 1$, the Lelong number of $2\sqrt{\frac{-1}{2\pi}} \partial \bar{\partial} \log \Phi$ at a point $P$ only measures the vanishing order of the restriction of $\Phi$ to a generic local complex curve in $X$ passing through $P$.

Because of the confusion which this additional factor of 2 may cause, when we assign a number to the vanishing order, we always use the one computed as the Lelong number of the closed positive $(1,1)$-current $\sqrt{-1} \partial \bar{\partial} \log \Phi$. When we refer to the vanishing order of $\Phi$, it will only be for comparison with the vanishing order of its partial sum

$$\Phi_{m_0} := \sum_{m=1}^{m_0} \varepsilon_m \sum_{j=1}^{q_m} |s_j^{(m)}| \frac{2}{m}$$

so that the additional factor of 2 would not make any difference.

(3.2) **Failure of the Noetherian Argument for Irreducible Lelong Sets Defined by $\Phi$.** Though $\Phi$ is defined by an infinite sum of the absolute-value squares of fractional powers of holomorphic functions, the use of fractional powers make the Noetherian argument inapplicable and we cannot conclude from the Noetherian argument that the number of irreducible Lelong sets of $\Theta_{\log \Phi}$ is finite.

To understand how this can happen, we illustrate the situation with the following simple example. On $\mathbb{C}^2$ with coordinates $(z,w)$, the vanishing order of $z$ is 1 at $z = 0$ (measured by the vanishing order of its restriction to a generic complex line through 0). The vanishing order of $|z| + |w - \frac{1}{z^2}|^{\frac{1}{2}}$ at $(z,w) = (0,\frac{1}{z})$ is $\frac{1}{2}$ (as defined by using the Lelong number of the closed
positive \((1,1)\)-current defined by it). The collection of all the irreducible Lelong sets of the closed positive \((1,1)\)-current defined by

\[
\sum_{m=1}^{\infty} \varepsilon_m |z|^m \prod_{\ell=1}^{m} \left( |z| + \left| w - \frac{1}{\ell} \right|^\frac{1}{\ell} \right)
\]

(for positive numbers \(\varepsilon_m\) decreasing fast enough) consists of \(\{(0, \frac{1}{m})\}\) for \(m \in \mathbb{N}\) with the Lelong number \(\frac{1}{m}\) at the point \(\{(0, \frac{1}{m})\}\). When we use the finite partial sum

\[
\sum_{m=1}^{N} \varepsilon_m |z|^m \prod_{\ell=1}^{m} \left( |z| + \left| w - \frac{1}{\ell} \right|^\frac{1}{\ell} \right)
\]

instead of the infinite sum (3.2.1), the collection of all the irreducible Lelong sets of the closed positive \((1,1)\)-current defined by (3.2.2) consists of \(\{(0, \frac{1}{m})\}\) for \(1 \leq m \leq N - 1\) plus the set \(\{z = 0\}\), where the Lelong number at the point \((0, \frac{1}{m})\) is \(\frac{1}{m}\) and the Lelong number at a generic point of \(\{z = 0\}\) is \(\frac{1}{N}\).

The Lelong set \(\{(0, \frac{1}{m})\}\) with Lelong number \(\frac{1}{m}\) for \(m \geq N\) does not occur for the closed positive \((1,1)\)-current defined by (3.2.2), because it is hidden inside and incorporated into the irreducible Lelong set \(\{z = 0\}\) where the generic Lelong number is \(\frac{1}{N}\). The Noetherian argument can be applied to the case of the finite sum, ruling out an infinite number of irreducible Lelong sets (because we can raise each individual term to some finite common power to get rid of the fractional powers of all the individual terms), but cannot be applied to the case of the infinite sum. The situation is like the following picture.

For an infinite number of pebbles arranged at different heights on a slope immersed in water, when the water recedes slowly, gradually more and more (but only a finite number of) pebbles are visible at any given time until the water completely drains out to make all the infinite number of pebbles visible.

An important tool for our analytic proof of the finite generation of the canonical ring for the case of general type is the following decomposition theorem for closed positive \((1,1)\)-currents. (See [Siu 1974] and [Kiselman 1979].)

(3.3) \textbf{Theorem (Decomposition of Closed Positive \((1,1)\)-Currents).} Let \(\Theta\) be a closed positive \((1,1)\)-current on a complex manifold \(X\). Then \(\Theta\) admits a
unique decomposition of the following form

$$\Theta = \sum_{j=1}^{J} \gamma_j [V_j] + R,$$

where $\gamma_j > 0$, $J \in \mathbb{N} \cup \{0, \infty\}$, $V_j$ is a complex hypersurface and the Lelong number of the remainder $R$ is zero outside a countable union of subvarieties of codimension $\geq 2$ in $X$.

(3.4) Pullbacks of Closed Positive $(1,1)$-Currents. Though in general currents can only be pushed forward and cannot be pulled back, yet for the case of a closed positive $(1,1)$-current $\Theta$ on a complex manifold $G$ it is possible to pull back by a surjective holomorphic map $\pi : D \to G$ from $G$ to a manifold $D$, because we can locally write $\Theta = \sqrt{-1} \partial \bar{\partial} \varphi$ for some local plurisubharmonic function on $G$ and pull back $\varphi$ to $\pi^* \varphi$ and then form $\sqrt{-1} \partial \bar{\partial} \pi^* \varphi$ as the pullback of $\Theta$ to $D$.

The assumption of the surjectivity of $\pi$ is just to make sure that the pullback $\pi^* \varphi$ is not identically $-\infty$ on $D$. Pullbacks can also be defined for the case when the holomorphic map $\pi : D \to G$ is not surjective as long as locally the image is not contained in the $(-\infty)$-set of the local plurisubharmonic function $\varphi$. If a submanifold $V$ of $G$ is not entirely contained in the $(-\infty)$-set of the plurisubharmonic potential $\varphi$, we can define the restriction of the closed positive $(1,1)$-current $\Theta = \sqrt{-1} \partial \bar{\partial} \varphi$ to a submanifold $V$ of $G$ (or even a subvariety $V$ of $G$) as $\sqrt{-1} \partial \bar{\partial} (\varphi|_V)$.

(3.5) Closed Positive $(1,1)$-Currents on Complex Spaces and Their Decomposition. We can also define a closed positive $(1,1)$-current $\Theta$ on a reduced complex space $Y$ with singularities by defining it as $\sqrt{-1} \partial \bar{\partial} \varphi$ for some local plurisubharmonic function $\varphi$ on $Y$ (in the sense that $\varphi$ can be extended to some plurisubharmonic function on some complex manifold in which locally $Y$ is a subvariety). We can also consider the decomposition of the closed positive $(1,1)$-current $\Theta$ on $Y$ by pulling it back to a complex manifold $\tilde{Y}$ which is a desingularization of $Y$ and then push forward to $Y$ the decomposition on $\tilde{Y}$ of the pullback of $\Theta$.

(3.6) Finite Generation from Precise Achievement of Stable Vanishing Order. Recall that a vanishing order of $\Phi$ at a point we mean the Lelong number of

$$2\Theta_{\log \Phi} := \frac{1}{2\pi} \partial \bar{\partial} \log \Phi.$$

14
By the generic vanishing order of $\Phi$ at a subvariety $V$ we mean the Lelong number of $2\Theta_{\log \Phi}$ at a generic point of $V$.

We say that at a point $P$ of $X$ the vanishing order of $\Phi$ is precisely achieved if for some $m_0 \in \mathbb{N}$ the function

$$\Phi_{m_0} := \sum_{m=1}^{m_0} \varepsilon_m \sum_{j=1}^{q_m} |s_j^{(m)}|^2/m$$

is comparable to $\Phi$ on some open neighborhood of $P$ in $X$ in the sense that there exist some open neighborhood $U$ of $P$ in $X$ and some positive number $C$ such that

$$\frac{1}{C} \Phi_{m_0} \leq \Phi \leq C \Phi_{m_0}$$
on $U$. To indicate the value $m_0$, we also say that at the point $P$ of $X$ the vanishing order of $\Phi$ is precisely achieved by the $m_0$-th partial sum of $\Phi$ (or for the finite number $m_0$). Sometimes we drop the adverb “precisely” and simply say that the vanishing order of $\Phi$ is achieved. Sometimes we also alternatively say that the precise vanishing order of $\Phi$ is achieved.

When we say that at a point the stable vanishing order is precisely achieved, we mean that the vanishing order of $\Phi$ is precisely achieved at that point. We use the adjective “stable” when $\Phi$ is not explicitly used, because $\Phi$ involves an infinite sum and involves all $\Gamma(X, mK_X)$ as $m \to \infty$ so that we refer to the vanishing order of $\Phi$ as the stable vanishing order.

Note that if the vanishing order of $\Phi$ is precisely achieved at $P$ by the $m_0$-th partial sum of $\Phi$ and if $m_1 = (m_0)!$, then

$$\frac{1}{C'} \Phi \leq \sum_{j=1}^{q_{m_1}} |s_j^{(m_1)}|^2/m_{m_1} \leq C' \Phi$$
on some open neighborhood of $P$ for some positive number $C'$.

(3.7) **Theorem (Finite Generation as Consequence of Precise Achievement of Stable Vanishing Order).** Suppose the stable vanishing orders are precisely achieved at every point of $X$ for some $m_0 \in \mathbb{N}$. Denote $(m_0)!$ by $m_1$. Then the canonical ring

$$\bigoplus_{m=1}^{\infty} \Gamma(X, mK_X)$$

is...
is generated by

\[ \bigoplus_{m=1}^{(n+2)m_1} \Gamma (X, mK_X) \]

and hence is finitely generated by the finite set of elements

\[ \left\{ s_j^{(m)} \right\}_{1 \leq m \leq m_1, 1 \leq j \leq q_m} \]

**Proof.** Let \( e^{-\varphi} = \frac{1}{\Phi} \). For \( m > (n + 2)m_1 \) and any \( s \in \Gamma (X, mK_X) \) we have

\[ \int_X |s|^2 e^{-(m-(n+2)m_1-1)\varphi} \left( \sum_{j=1}^{q_{m_1}} \left| s_j^{(m_1)} \right|^2 \right)^{n+2} < \infty, \]

because \( |s|^2 \leq \tilde{C} \Phi^m \) on \( X \) for some \( \tilde{C} \). By Skoda’s theorem on ideal generation ((1.1) and (2.4)) there exist

\( h_1, \ldots, h_{q_{m_1}} \in \Gamma (X, (m - m_1)K_X) \)

such that \( s = \sum_{j=1}^{q_{m_1}} h_j s_j^{(m_1)} \). If \( m-(n+2)m_1 \) is still greater than \( (n + 2)m_1 \), we can apply the argument to each \( h_j \) instead of \( s \) until we get

\( h_1^{(j_1, \ldots, j_{\nu})}, \ldots, h_{q_{m_1}}^{(j_1, \ldots, j_{\nu})} \in \Gamma (X, (m - m_1(\nu + 1))K_X) \)

for \( 1 \leq j_1, \ldots, j_{\nu} \leq q_{m_1} \) with \( 0 \leq \nu < N \), where \( N = \left\lfloor \frac{m}{m_1} \right\rfloor \), such that

\[ s = \sum_{1 \leq j_1, \ldots, j_N \leq q_{m_1}} h_1^{(j_1, \ldots, j_{N-1})} \prod_{\lambda=1}^{N} s_{j_\lambda}^{(m_1)}. \]

Q.E.D.

Once we have the precise achievement of stable vanishing orders, Skoda’s theorem on ideal generation ((1.1) and (2.4)) can also be applied with coefficients in a line bundle with a metric of positive curvature current. The following theorem is Theorem (3.7) with such a twisting added.
Theorem (Twisted Finite Generation as Consequence of Precise Achievement of Stable Vanishing Order). Let $E$ be a line bundle on $X$ with a metric $e^{-\chi}$ of positive curvature current. Suppose the vanishing orders of $\Phi$ are precisely achieved at every point of $X$ for some $m_0 \in \mathbb{N}$. Denote $(m_0)!$ by $m_1$. Then $\Gamma (X, \mathcal{I}_{m_0} (mK_X + E))$ is equal to

$$(\Gamma (X, m_1 K_X))^{p_m} \Gamma (X, \mathcal{I}_{(m - m_1 p_m)} ((m - m_1 p_m) K_X + E))$$

for $m \geq (n + 2) m_1$, where $p_m = \left\lfloor \frac{m}{m_1} \right\rfloor - (n + 2)$ and $\varphi = \log \Phi$.

This theorem on twisted finite generation is for later use in the discussion in (8.1)(vi) about the alternative approach to the problem of the finite generation of the canonical ring by using the extension techniques from the proof of the deformational invariance of plurigenera.

\section*{§4. Reduction of Achievement of Vanishing Order to Non-Vanishing Theorem on Hypersurface by Fujita Conjecture Type Techniques}

(4.1) Strategy to Prove Finite Generation. Our strategy to prove the finite generation of the canonical ring is to show that the vanishing orders of $\Phi$ are achieved for some finite $m_0$. We implement this strategy by descending induction on the dimension of the subvariety $V$ of $X$ such that at points of $X - V$ the vanishing orders of $\Phi$ are achieved in the sense (as given in (3.6)) that at every point $P \in X - V$ there exist some open neighborhood $U$ of $P$ in $X - V$ and some positive number $C$ and some $m_0$ (which may depend on $P$) such that

$$\frac{1}{C} \Phi_{m_0} \leq \Phi \leq C \Phi_{m_0}$$

on $U$, where

$$\Phi_{m_0} := \sum_{m=1}^{m_0} \sum_{j=1}^{q_m} \left| \mathcal{S}_{j}^{(m)} \right|^{2}.$$ 

When we have $V$, we will proceed to prove that for each branch $V_0$ of $V$ there is a subvariety $Z$ of codimension $\geq 1$ in $V_0$ such that at every point of $V_0 - Z$ the vanishing orders of $\Phi$ are achieved for some finite $m'$. At the end of the induction process we then invoke Theorem (3.7) to finish off the proof of the finite generation of the canonical ring.
Difference Between the Hypersurface Case and the Case of a Subvariety of Higher Codimension. We will first deal with the case when \( V \) is a hypersurface. Though we can always blow up a lower-dimensional \( V \) to a hypersurface \( \tilde{V} \), yet there is the difficulty that the proper subvariety \( \tilde{Z} \) of \( \tilde{V} \) (where stable vanishing orders are not yet precisely achieved) may be projected onto \( V \). So the higher-dimensional case calls for an approach other than blow-up. There is more in-depth discussion about this in (7.8).

Across a hypersurface the vanishing order of each finite sum \( \Phi_m \) can be described by a single number, but across a local submanifold \( W \) of higher codimension in \( X \) the vanishing orders of each \( \Phi_m \) at a point \( P \) of \( W \) have to be described by an Artinian subscheme in the normal directions of \( W \) at \( P \). The Artinian subschemes used in these notes are unreduced complex subspaces of \( \mathbb{C}^n \) supported at a singular point.

We will use the moduli space of Artinian subschemes in the normal directions of \( V \) and consider the points of \( V \) where these Artinian subschemes vary without jump in order to locate, in an \( a \) priori manner, a countable union \( E \) of subvarieties of codimension \( \geq 1 \) in \( \tilde{V} \) so that \( \tilde{Z} \) is always projected to inside \( E \). The notion of the continuous variation of an Artinian subscheme without jump is introduced in (7.5.1) below.

(4.3) Sketch of the Idea of Using Fujita Conjecture Type Techniques. We now consider the case of precisely achieving vanishing orders of \( \Phi \) by some finite sum at a generic point of an irreducible hypersurface \( Y \) of \( X \). Let \( \gamma \) be the generic Lelong number of \( \Phi \) at \( Y \). To make the argument more transparent, we break it up into two parts. First, we assume that \( \gamma \) is rational and show that stable vanishing orders of \( \Phi \) are precisely achieved by some finite partial sum at a generic point of the irreducible hypersurface \( Y \). This is done by Fujita conjecture type techniques. Kawamata already used this kind of techniques for the special case when \( K_X \) is numerically effective so that \( \gamma = 0 \) in his paper [Kawamata 1985] before the introduction of Fujita conjecture type techniques. Then we assume that \( \gamma \) is irrational and show how to modify the argument for the case of a rational \( \gamma \) to handle the case of an irrational \( \gamma \).

We first sketch the main ideas here and then give the details of the argument. Let us assume \( \gamma \) rational (which may be zero). When the stable vanishing order \( \gamma \) at a generic point of \( Y \) is not precisely achieved by some finite sum \( \Phi_{m_0} \), the vanishing order \( \gamma_{m_0} \) of the finite sum \( \Phi_{m_0} \) is always \( > \gamma \).
but approaches $\gamma$ as $m_0 \to \infty$. We are going to modify the metric $\frac{1}{\Phi}$ for $K_X$. The Fujita conjecture type techniques involve constructing another metric $\frac{1}{\Phi^m}$ of $K_X$ so that for some $m$ the multiplier ideal sheaf of the metric $\frac{1}{\Phi^m - 1}$ of $(m - 1)K_X$ at a generic point of $Y$ is the ideal sheaf of $(m\gamma + 1)Y$. This actually means that instead of getting the vanishing order $\gamma_{m_0}$ to be equal to $\gamma$ for some finite $m_0$ which we cannot do, we settle for some extra order across $Y$, namely for holomorphic sections of $(m - 1)K_X$ over $X$ instead of the expected vanishing order $(m - 1)\gamma$ we allow the higher vanishing $m\gamma + 1$. The will be done in (4.4.1). This extra vanishing order enables us to use the theorem of Kawamata-Viehweg-Nadel ([Kawamata 1982], [Viehweg 1982], [Nadel 1990]) with the use of an additional $K_X$ to conclude that

$$H^1(X, \mathcal{J}(mK_X - m\gamma Y - Y)) = 0,$$

where $\mathcal{J}$ is a coherent ideal sheaf whose zero-set does not contain $Y$. This means that if we are able to get a holomorphic section of $m(K_X - \gamma Y)$ over $Y$ which is nonzero at some point of $Y$ (which locally belongs to the ideal sheaf $\mathcal{J}$), then we can extend it to a holomorphic section of $m(K_X - \gamma Y)$ over $X$, which implies that the precise vanishing order of $\gamma$ at a generic point of $Y$ is achieved by the finite sum $\Phi_m$.

In the case of an irrational $\gamma$ we will use Kronecker’s theorem on diophantine approximation to make $m\gamma - [m\gamma]$ small so that the discrepancy can be handled by using the bigness of $K_X$ and the Kodaira’s decomposition of a big line bundle into an effective $\mathbb{Q}$-line-bundle and an ample $\mathbb{Q}$-line-bundle. This will be done in (4.4.6).

(4.4) Details of the Use of Fujita Conjecture Type Techniques. The induction process to prove the finite generation of the canonical ring is a descending induction on the dimension of the subvariety in $X$ outside of which the vanishing order of $\Phi$ is achieved by some finite partial sum $\Phi_m$. The zero-set of $\Phi$ is a subvariety of $X$. Clearly at points where $\Phi$ is nonzero, the vanishing order of $\Phi$ is achieved by some finite partial sum $\Phi_m$. For the initial induction step we take an irreducible hypersurface $Y$ in the zero-set of $\Phi$ if there is any. By resolving the singularities of $Y$, we can assume without loss of generality that $Y$ is a nonsingular complex hypersurface of $X$. Let $\gamma \geq 0$ be the generic Lelong number of $\Phi$ at $Y$. We would like to show that at a generic point of $Y$ we can find some $m$-canonical section over $X$ whose vanishing order at a generic point of $Y$ is precisely $m\gamma$. Suppose the contrary and we are
going to derive a contradiction. We assume first that $\gamma$ is rational. Choose a positive integer $m_1$ such that $m_1\gamma$ is an integer. Let $L = m_1 (K_X - \gamma Y)$. Let $s_Y$ be the canonical section of the line bundle $Y$ on $X$. We introduce the metric $e^{-\tilde{\phi}} = \left( \frac{|s_Y|^2_Y}{\Phi} \right)^{m_1}$ of the line bundle $L = m_1 (K_X - \gamma Y)$. The curvature current $\Theta_{\tilde{\phi}}$ is a closed positive $(1,1)$-current on $X$.

To reduce to a non-vanishing theorem the problem of achieving precisely the stable vanishing order at a generic point of a hypersurface $Y$, we have to introduce two metrics $e^{-\chi}$ and $e^{-\xi}$ of $p_0 L - K_X$ on $X$, which we will do respectively in (4.4.1) and (4.4.2). The first metric $e^{-\chi}$ will have enough singularity across $Y$ so that its multiplier ideal sheaf $\mathcal{I}_\chi$ is contained in the ideal sheaf of $Y$ and at some generic point of $Y$ equals the ideal sheaf of $Y$. The second metric $e^{-\xi}$ will essentially have minimal singularity at a generic point of $Y$ so that its multiplier ideal sheaf $\mathcal{I}_\xi$ has no zero at a generic point of $Y$. Moreover, the curvature currents $\Theta_\chi$ and $\Theta_\xi$ of both metrics dominate some smooth positive $(1,1)$-form on $X$. We apply the vanishing theorem of Kawamata-Viehweg-Nadel to the line bundle $(p + p_0) L - K_X$ with the metric $e^{-p \tilde{\phi} - \chi}$ to get the surjectivity of

$$\Gamma (X, \mathcal{I}_{p\tilde{\phi} + \xi} ((p + p_0) L)) \to \Gamma (X, (\mathcal{I}_{p\tilde{\phi} + \xi} / \mathcal{I}_{p\tilde{\phi} + \chi}) ((p + p_0) L)).$$

If we have an element of

$$\Gamma (X, (\mathcal{I}_{p\tilde{\phi} + \xi} / \mathcal{I}_{p\tilde{\phi} + \chi}) ((p + p_0) L))$$

which as a local holomorphic function on $Y$ is nonzero at some generic point $P_0$ of $Y$, then the stable vanishing order is precisely achieved at $P_0$. This would finish the reduction to a non-vanishing theorem the problem of precisely achieving the stable vanishing order at a generic point of $Y$. However, there is a small technical problem.

The problem is that a nonzero element of

$$\Gamma (X, (\mathcal{I}_{p\tilde{\phi} + \xi} / \mathcal{I}_{p\tilde{\phi} + \chi}) ((p + p_0) L))$$

may not be nonzero at some generic point $P_0$ of $Y$ as a local holomorphic function on $Y$ when the sheaf $\mathcal{I}_{p\tilde{\phi} + \xi} / \mathcal{I}_{p\tilde{\phi} + \chi}$ cannot be regarded as an ideal sheaf on $Y$ but can only be regarded as an ideal sheaf over some unreduced structure of $Y$. (An unreduced structure means that there are nonzero nilpotent elements in the structure sheaf.) In that case we have to replace the
metric $e^{-\chi}$ with an appropriate interpolation between $e^{-\chi}$ and $e^{-\xi}$ (and possibly also apply some slight modification) to make the quotient $I_{p\tilde{\phi}+\xi}/I_{p\tilde{\phi}+\chi}$ the analog of a minimal center of log canonical singularities. The procedure of interpolation and slight modification of metrics will be explained in (4.4.4).

Concerning the analog of minimal center of log canonical singularities, we will discuss in (4.4.5) the roles of unreduced subspace and minimal centers of log canonical singularities in Fujita conjecture type problems.

After we solve the technical problem of unreduced structures using the interpolation of metrics and the analog of minimal centers of log canonical singularities, we can regard the sheaf $I_{p\tilde{\phi}+\xi}/I_{p\tilde{\phi}+\chi}$ as an ideal sheaf on $Y$ or on some complex subspace of $Y$ with reduced structure, then we need only consider the problem of the existence of nonzero elements in $\Gamma(X,(I_{p\tilde{\phi}+\xi}/I_{p\tilde{\phi}+\chi})((p+p_0)L))$. This will be handled by our general non-vanishing theorem (6.2).

(4.4.1) Lemma. For a sufficiently positive integer $p_0$ there exists some metric $e^{-\chi}$ of the line bundle $p_0L - K_X$ on $X$ such that the curvature current $\Theta_\chi = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \chi$ of $e^{-\chi}$ dominates some smooth strictly positive $(1,1)$-form on $X$ and the multiplier ideal sheaf $I_\chi$ of the metric $e^{-\chi}$ is contained in the ideal sheaf of $Y$ and at some generic point of $Y$ equals the ideal sheaf of $Y$.

Proof. Since $X$ is of general type, the canonical line bundle $K_X$ is big and we can write $K_X$ in the form $K_X = E + A$, where $E$ is an effective $\mathbb{Q}$-divisor and $A$ is an ample $\mathbb{Q}$-line-bundle. Let $s_E$ be the canonical divisor of the line bundle defined by $E$. Let $\eta$ be the coefficient of $Y$ in $E$. The number $\eta$ must be greater than $\gamma$, otherwise for some $m$ sufficiently large $\Gamma(X,mA)$ generates $A$ and the subset $(s_E)^m \Gamma(X,A)$ of $\Gamma(X,mK_X)$ contains an element whose vanishing order at a generic point of $Y$ is $m\eta$ across $Y$, which contradicts the assumption that at a generic point of $Y$ there does not exist any $m$-canonical section over $X$ with vanishing order $\leq m\gamma$. Let $h_A$ be a smooth metric of $A$ with strictly positive curvature form $\Theta_A$. Write $E = \eta Y + F$, where $F$ is an effective $\mathbb{Q}$-divisor. Let $h_A$ be a smooth metric of $A$ with strictly positive curvature form $\Theta_A$.

Let $p_0$ be a positive integer on which we will later impose more conditions. For any rational number $0 < \beta \leq p_0m_1 - 1$, the line bundle $p_0L - K_Y$ can be rewritten as

\begin{equation}
(4.4.1.1) \quad p_0L - K_Y = p_0m_1(K_X - \gamma Y) - (K_X + Y)
\end{equation}
\[ (p_0 m_1 - 1) (K_X - \gamma Y) - (\gamma + 1) Y \]
\[ = (p_0 m_1 - 1 - \beta) (K_X - \gamma Y) + \beta A + \beta F - (\gamma + 1 - \beta (\eta - \gamma)) Y. \]

Now choose \( p_0 \) and \( 0 < \beta \leq p_0 m_1 - 1 \) such that \( \gamma + 1 - \beta (\eta - \gamma) = 0 \). Since \( \eta > \gamma \), this choice of \( p_0 \) and \( \beta \) is possible when \( p_0 \) is sufficiently large. Finally we define the metric
\[ e^{-\chi} = \frac{e^{-\frac{p_0 m_1 - 1 - \beta}{m_1} \tilde{\varphi} (h_A)^{\beta}}}{|s_Y|^2 |s_F|^{2\beta}} \]
which satisfies our requirement. Q.E.D.

(4.4.2) Lemma. For a sufficiently large positive integer \( p_0 \) there exists some metric \( e^{-\xi} \) of the line bundle \( p_0 L - K_X \) on \( X \) such that the curvature current \( \Theta_\xi = \frac{1}{2\pi} \partial \bar{\partial} \xi \) of \( e^{-\xi} \) dominates some smooth strictly positive \((1,1)\)-form on \( X \) and the multiplier ideal sheaf \( I_\xi \) is equal to the structure sheaf \( \mathcal{O}_X \) of \( X \) at a generic point of \( Y \).

Proof. To construct the metric \( e^{-\xi} \) of \( p_0 L - K_X \), we modify the construction of \( e^{-\chi} \) in the proof of (4.4.1) as follows. We add \( Y \) to both sides of (4.4.1.1) and choose \( 0 \leq \beta_1 \leq p_0 m_1 - 1 \) with \( \gamma - \beta_1 (\eta - \gamma) = 0 \) so that we can write
\[ p_0 L - K_X = p_0 m_1 (K_X - \gamma Y) - K_X \]
\[ = (p_0 m_1 - 1) (K_X - \gamma Y) - \gamma Y \]
\[ = (p_0 m_1 - 1 - \beta_1) (K_X - \gamma Y) + \beta_1 A + \beta_1 F - (\gamma - \beta_1 (\eta - \gamma)) Y \]
\[ = (p_0 m_1 - 1 - \beta_1) (K_X - \gamma Y) + \beta_1 A + \beta_1 F. \]

We now define the metric
\[ e^{-\xi} = \frac{e^{-\frac{p_0 m_1 - 1 - \beta_1}{m_1} \tilde{\varphi} (h_A)^{\beta_1}}}{|s_F|^{2\beta_1}} \]
which satisfies our requirement. Q.E.D.

(4.4.3) Interpolation of Metrics, Slight Modification, and Minimal Center of Log Canonical Singularities. Suppose \( M \) is a compact complex algebraic manifold and \( E \) is a holomorphic line bundle with two metrics \( e^{-\kappa_1} \) and \( e^{-\kappa_2} \) such that each \( \kappa_j \) is locally plurisubharmonic (for \( j = 1, 2 \)). By an interpolation of the two metrics \( e^{-\kappa_1} \) and \( e^{-\kappa_2} \) of \( E \) we mean a metric \( e^{-\kappa_0} \) of \( E \) of the form \( e^{-\eta \kappa_1 + (1-\eta)\kappa_2} \) with \( 0 < \eta < 1 \). Suppose \( A \) is an ample
Q-line-bundle over $M$ with metric $h_A$ and positive curvature form $\omega_A$ and $s_A$ is a multi-valued holomorphic section of $A$ over $M$. By a slight modification of the interpolated metric $e^{-\kappa_\eta}$ of $E$ we mean a metric $e^{-\kappa_{\eta,s}}$ of $E$ either of the form

$$\frac{e^{-\kappa_\eta}}{h_A|s|^2}$$

if the curvature current $\Theta_{\kappa_\eta}$ of $e^{-\kappa_\eta}$ dominates $(1 + \varepsilon)\omega_A$ for some $\varepsilon > 0$ or of the form $e^{-\kappa_\eta}h_A|s|^2$ with $\kappa_\eta - \log |s|^2$ locally plurisubharmonic on $M$ so that in both cases the curvature current of the new metric $e^{-\kappa_{\eta,s}}$ of $E$ still dominates some positive multiple of $\omega_A$.

An interpolation of metrics is used in the following context. Suppose the multiplier ideal sheaf $I_{\kappa_{1}}$ is properly contained in the multiplier ideal sheaf $I_{\kappa_{2}}$. We choose $0 < \eta_0 < 1$ as the smallest so that $I_{\kappa_{\eta_0}}$ is different from $I_{\kappa_{2}}$. In other words, we choose $0 < \eta_0 < 1$ as the smallest so that the support $Z_{\eta_0}$ of $I_{\kappa_{2}}/I_{\kappa_{\eta_0}}$ is still nonempty.

A slight modification of an interpolated metric is used in the following context. Suppose either $\Theta_{\kappa_{1}}$ or $\Theta_{\kappa_{2}}$ dominates some smooth positive $(1,1)$-form on $M$. We choose the multi-valued holomorphic section $s$ of the ample Q-line-bundle $A$ so that the support $Z_{\eta,s}$ of $I_{\kappa_{2}}/I_{\kappa_{\eta,s}}$ satisfies some additional properties. One of such additional properties which is commonly used is that there is some $\pi : \tilde{M} \to M$ obtained by a finite number of successive monoidal transformations with nonsingular center such that $Z_{\kappa_{\eta,s}}$ is the image under $\pi$ of some nonsingular exceptional divisor $H$ in $\tilde{M}$. The slight modification helps us to go from a subvariety $Z_{\eta,s}$ to the case of a complex manifold $H$.

The use of interpolation and slight modification of metrics corresponds to the process of getting the minimal center of log canonical singularities in the techniques for Fujita conjecture type problems and Shokurov’s non-vanishing theorem (see e.g., [Shokurov 1985] and [Angehrn-Siu 1996] and [Kawamata 1997]).

One important use of the interpolation and slight modification process is to go from the case of a sheaf over an unreduced space to the case of a sheaf over a reduced space which is a subspace of the unreduced space (equivalently the structure sheaf of the reduced subspace is a quotient of the structure sheaf of the unreduced space). When it is a matter of finding a non-vanishing section, we can start with a holomorphic section on a smaller reduced space.
and extend it back. For example, when $\Theta_{\kappa_1}$ dominates some smooth positive $(1,1)$-form on $M$, we have the vanishing of $H^1(M, \mathcal{I}_{\kappa_1} (E + K_M))$ and $H^1(M, \mathcal{I}_{\kappa_{m_0,s}} (E + K_M))$ and therefore the surjectivity of both restriction maps

$$
\Gamma (M, \mathcal{I}_{\kappa_1} (E + K_M)) \rightarrow \Gamma (M, (\mathcal{I}_{\kappa_1} / \mathcal{I}_{\kappa_2}) (E + K_M)),
\Gamma (M, \mathcal{I}_{\kappa_1} (E + K_M)) \rightarrow \Gamma (M, (\mathcal{I}_{\kappa_1} / \mathcal{I}_{\kappa_{m_0,s}}) (E + K_M)).
$$

When we are interested in getting some non identically zero element of $\Gamma (M, \mathcal{I}_{\kappa_1} (E + K_M))$, we can either start with a non identically zero element of $\Gamma (M, (\mathcal{I}_{\kappa_1} / \mathcal{I}_{\kappa_2}) (E + K_M))$ or a non identically zero element of $\Gamma (M, (\mathcal{I}_{\kappa_1} / \mathcal{I}_{\kappa_{m_0,s}}) (E + K_M))$. By replacing the former by the latter, we can work with a reduced space (or even a manifold without multiplicity after using blow-up). This way of handling the problem will also be used in (6.10).

(4.4.4) Remark on the Rôles of Unreduced Subspace and Minimal Centers of Log Canonical Singularities in Fujita Conjecture Type Techniques. We would like to remark further that the idea of unreduced structures and the interpolations of metrics and slight modifications described in (4.4.3) is actually the key point in the techniques for Fujita conjecture type problems. Suppose $F$ is a holomorphic line bundle with metric $e^{-\psi}$ over a compact complex algebraic manifold $M$ so that the curvature current of $e^{-\psi}$ dominates some smooth positive $(1,1)$-form on $M$. The Fujita conjecture type problem for this context is to study the question of the generation of the multiplier ideal sheaf $\mathcal{I}_{m\psi}$ by $\Gamma (M, \mathcal{I}_{m\psi} (mF + K_M))$ for $m$ sufficiently large.

While it is easy to get the vanishing of cohomology

$$H^\nu (M, \mathcal{I}_{m\psi} (mF + K_M)) = 0 \text{ for } \nu \geq 1$$

by using the theorem of Kawamata-Viehweg-Nadel which requires only some small positive lower bound of the curvature current of the metric $e^{-\psi}$. It is difficult to construct elements of $\Gamma (M, \mathcal{I}_{m\psi} (mF + K_M))$, for which a sufficiently large positive lower bound of the curvature current of the metric $e^{-\psi}$ is usually needed (see (9.1.1)).

The way to produce sections is to find another metric $e^{-\psi_1}$ of $F$ (again with curvature current dominating some smooth positive $(1,1)$-form on $M$) such that the multiplier ideal sheaf $\mathcal{I}_{m\psi_1}$ is contains in the multiplier ideal
sheaf $\mathcal{I}_m\psi$. Then the vanishing of positive-degree cohomology for $mF + K_M$ with coefficients in both multiplier ideal sheaves $\mathcal{I}_m\psi$ and $\mathcal{I}_{m\psi_1}$ gives

$$H^1(M, (\mathcal{I}_m\psi / \mathcal{I}_{m\psi_1}) (mF + K_M)) = 0$$

and the surjectivity of

$$\Gamma (M, \mathcal{I}_m\psi (mF + K_M)) \to \Gamma (M, (\mathcal{I}_m\psi / \mathcal{I}_{m\psi_1}) (mF + K_M)).$$

Still it is difficult to produce elements of $\Gamma (M, (\mathcal{I}_m\psi / \mathcal{I}_{m\psi_1}) (mF + K_M))$ unless the support of $\mathcal{I}_m\psi / \mathcal{I}_{m\psi_1}$ is isolated. If for some point $P_0 \in M$ the stalk of $\mathcal{I}_{m\psi_1}$ at $P_0$ belongs to a sufficiently high power of the maximum ideal $m_{M,P_0}$ of $M$ at $P_0$, then when we change the metric $e^{-\psi_1}$ by the interpolation of metrics and slight modification to make the support of $\mathcal{I}_m\psi / \mathcal{I}_{m\psi_1}$ minimal, we can end up with the support of $\mathcal{I}_m\psi / \mathcal{I}_{m\psi_1}$ isolated at $P_0$.

The usual method to approach Fujita conjecture type problems is to use the theorem of Riemann-Roch to produce multi-valued holomorphic sections of $F$ vanishing to high order and then use them to modify the metric $e^{-\psi}$ to construct $e^{-\psi_1}$.

For our notes, as we will see in the proof of our general non-vanishing theorem (6.2), we are following the same strategy with $mF$ replaced by $mF - K_M$ to help create some positive lower bound for the curvature current and, much more importantly, to incorporate Shokurov’s technique [Shokurov 1985] of comparing the application of the theorem of Riemann-Roch to a line bundle and its twisting by a flat line bundle when we fail to produce a multi-valued holomorphic section of $mF - K_X$ vanishing to high order. We will inductively reduce the dimension of the support of $\mathcal{I}_m\psi / \mathcal{I}_{m\psi_1}$ until we are forced to use Shokurov’s technique. During the induction process, we do not have control over where the support of $\mathcal{I}_m\psi / \mathcal{I}_{m\psi_1}$ is when we make it minimal, except that it must contain the point where the multi-valued holomorphic section of $mF - K_X$ vanishes to high order.

(4.4.6) Modification for an Irrational Stable Vanishing Order. We now explain what modification is needed to get rid of the additional assumption in the first paragraph of (4.4) that $\gamma$ is rational. In that case we use Kronecker’s diophantine approximation and replace $m(K_X - \gamma Y)$ by $mK_X - \lfloor m\gamma \rfloor Y$. For any $m \in \mathbb{N}$ let $\varepsilon_m$ denote the positive number $m\gamma - \lfloor m\gamma \rfloor$. First of all, by Kronecker’s diophantine approximation (see (5.2.1.1)), for any given $\varepsilon > 0$ there exists some $m$ such that $\varepsilon_m < \varepsilon$. 

25
Let $h_Y$ be any smooth metric for the line bundle $Y$ over $X$. For the modification we let $m = (p + p_0) m_1$ and replace the metric

$$e^{-p \tilde{\varphi} - \chi} = \frac{e^{-\left(p + \frac{p_0 m_1 - 1 - \beta_1}{m_1}\right) \tilde{\varphi}} (h_A)^{\beta}}{|s_Y|^2 |s_F|^{2\beta}}$$

of $(p + p_0) L - K_X - Y$ given above in (4.1.1) for the case of a rational $\gamma$ by the new metric

$$e^{-p \tilde{\varphi} - \chi} = \frac{e^{-\left(p + \frac{p_0 m_1 - 1 - \beta_1}{m_1}\right) \tilde{\varphi}} (h_A)^{\beta} (h_Y)^{\varepsilon_m}}{|s_Y|^2 |s_F|^{2\beta}}$$

of $(mK_X - \lfloor m\gamma \rfloor Y) - K_X - Y$. Likewise, we replace the metric

$$e^{-p \tilde{\varphi} - \xi} = \frac{e^{-\left(p + \frac{p_0 m_1 - 1 - \beta_1}{m_1}\right) \tilde{\varphi}} (h_A)^{\beta_1}}{|s_Y|^2 |s_F|^{2\beta_1}}$$

of $(p + p_0) L - K_X - Y$ given above in (4.1.2) for the case of a rational $\gamma$ by the new metric

$$e^{-p \tilde{\varphi} - \xi} = \frac{e^{-\left(p + \frac{p_0 m_1 - 1 - \beta_1}{m_1}\right) \tilde{\varphi}} (h_A)^{\beta_1} (h_Y)^{\varepsilon_m}}{|s_Y|^2 |s_F|^{2\beta_1}}$$

of $(mK_X - \lfloor m\gamma \rfloor Y) - K_X - Y$. All we have to do is to choose $m$ so that the consequence (5.2.1.1) of Kronecker’s theorem on diophantine approximation can be applied to make $\varepsilon_m$ small enough for the curvature form of the metric $(h_A)^{\beta} (h_Y)^{\varepsilon_m}$ to be strictly positive on $X$.

The main point of the modification is to use the small positivity squeezed out from the bigness of $K_Y$ to absorb the contribution from the small left-over non-integral part $\varepsilon_m Y$ of $m\gamma Y$ when $\gamma$ is irrational.

After the use of the non-vanishing theorem to be presented below (with the modification discussed in (4.4.3) and (4.4.4)), the final result is that there exists some element of $\Gamma (X, mK_X)$ whose vanishing order across $Y$ at a generic point of $Y$ is $\lfloor m\gamma \rfloor$, which would imply that the case of an irrational $\gamma$ cannot occur.

§5. Diophantine Approximation of Kronecker
Before we present our general non-vanishing theorem (6.2), we first derive some results on diophantine approximation by using the diophantine approximation theorem of Kronecker which will be needed in the presentation of our general non-vanishing theorem in §6. The key result due to Kronecker which we will use is the following. A reference is Theorem 444 on p. 382 of [Hardy-Wright 1960].

(5.1) Theorem (Kronecker). Let \(a_1, \ldots, a_N\) be \(\mathbb{Q}\)-linearly independent real numbers. Let \(b_1, \ldots, b_N \in \mathbb{R}\). Let \(\varepsilon, T\) be positive numbers. Then we can find \(t > T\) and integers \(x_1, \ldots, x_N\) such that \(|ta_j - b_j - x_j| \leq \varepsilon\) for \(1 \leq j \leq N\).

(5.2) Remark. Note that the result implies that for the case of a single irrational real number \(\gamma\), the image of the set \(\gamma \mathbb{Z}\) in \(\mathbb{R}/\mathbb{Z}\) is dense, because we can choose

\[N = 2, \quad a_1 = 1, \quad a_2 = \gamma, \quad b_1 = b_2 = 0, \quad T = 1\]

and conclude that given any \(0 < \varepsilon < 1\) there exists some \(t > T\) such that \(|ta_j - p_j| < \varepsilon_{1+|\gamma|} (j = 1, 2)\) for some \(p_1, p_2 \in \mathbb{Z}\) and, as a consequence,

\[|p_1\gamma - p_2| = |t\gamma - p_2 - (t - p_1)\gamma| \leq \frac{\varepsilon}{1 + |\gamma|} + |\gamma| \frac{\varepsilon}{1 + |\gamma|} = \varepsilon.\]

Of course, geometrically it is simply the well-known statement that the straight line in \(\mathbb{R}^2\) whose slope is an irrational real number \(\gamma\) has dense image in \((\mathbb{R}/\mathbb{Z})^2\) (as one can see by looking at the straight lines passing through the points \((x, 0)\) with the image of \(x \in \mathbb{R}\) in \(\mathbb{R}/\mathbb{Z}\) belonging to the image of \(\gamma \mathbb{Z}\) in \(\mathbb{R}/\mathbb{Z}\)).

(5.2.1) When we choose \(b_2 = \frac{2\varepsilon}{3}\) instead of \(b_2 = 0\), we can find \(t > T\) such that \(|ta_j - p_j| < \frac{\varepsilon}{3(1 + |\gamma|)} (j = 1, 2)\) for some \(p_1, p_2 \in \mathbb{Z}\) and, as a consequence,

\[|p_1\gamma - p_2 - b_2| = |t\gamma - p_2 - (t - p_1)\gamma - b_2| \leq \frac{\varepsilon}{3(1 + |\gamma|)} + |\gamma| \frac{\varepsilon}{3(1 + |\gamma|)} = \frac{\varepsilon}{3}.\]

Thus

\[p_1\gamma \geq p_2 + b_2 - \frac{\varepsilon}{3} = p_2 + 2\frac{\varepsilon}{3} - \frac{\varepsilon}{3} = p_2 + \frac{\varepsilon}{3}\]

and \(|p_1\gamma| \geq p_2\) and

\[p_1\gamma - |p_1\gamma| \leq p_1\gamma - p_2 < b_2 + \frac{\varepsilon}{3} = \frac{2\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.\]

We conclude that
for any $\varepsilon > 0$, any irrational real number $\gamma$, and any $T > 0$, there exists some integer $m > T$ such that $m\gamma - \lfloor m\gamma \rfloor < \varepsilon$.

(5.3) Corollary. Let $1 = a_0, a_1, \ldots, a_N$ be $\mathbb{Q}$-linearly independent real numbers. Let $0 < \varepsilon < \frac{1}{2}$. Then we can find a positive integer $m$ such that $ma_j - \lfloor ma_j \rfloor > \frac{1}{2} - \varepsilon$ for $1 \leq j \leq N$.

Proof. Let $\eta = \max_{0 \leq j \leq N} |a_j|$. Choose $b_0 = 0$ and $b_j = \frac{1}{q}$ for $1 \leq j \leq N$ and $\frac{1}{q} < \frac{\varepsilon}{2\eta}$ and $T = 1$. By Kronecker’s theorem applied to $1 = a_0, a_1, \ldots, a_N$ we can find a real number $t > T$ and integers $x_0, x_1, \ldots, x_N$ such that $|ta_j - b_j - x_j| \leq \frac{1}{q}$ for $0 \leq j \leq N$. We now set $m = x_0$. Then $|t - m| < \frac{1}{q} < \frac{\varepsilon}{2\eta}$ so that

$$\left| ma_j - \frac{1}{2} - x_j \right| \leq |t - m| |a_j| + \left| ta_j - \frac{1}{2} - x_j \right|$$

$$\leq \frac{\varepsilon}{2\eta} |a_j| + \frac{1}{q} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$ 

Since $x_j$ is an integer for $1 \leq j \leq N$, we have $ma_j - \lfloor ma_j \rfloor > \frac{1}{2} - \varepsilon$ for $1 \leq j \leq N$. Since $m$ is an integer and $|t - m| < \frac{1}{q}$ and $t > 1$, it follows that $m$ is positive. Q.E.D.

(5.4) Lemma. Let $\gamma$ be an irrational positive number and let $a_1, \ldots, a_N$ be positive rational numbers. Then there exists some positive integer $m$ such that $m\gamma a_j - \lfloor m\gamma a_j \rfloor \geq \frac{1}{2}$ for $1 \leq j \leq N$.

Proof. Let $p$ be the least common multiple of the denominators of $a_1, \ldots, a_N$. By replacing each $a_j$ by $pa_j$, we can assume without loss of generality that each $a_j$ is a positive integer. Choose $\varepsilon > 0$ so that $\varepsilon |a_j| < \frac{1}{2}$ for $1 \leq j \leq N$. Since $\gamma$ is irrational, the set $\gamma \mathbb{Z}$ has dense image in $\mathbb{R}/\mathbb{Z}$. It follows that there exists some $m, n \in \mathbb{Z}$ such that $m\gamma - n \in (1 - \varepsilon, 1)$. Since $\gamma$ is positive, the integer $m$ must be positive. From $1 - \varepsilon + n < m\gamma < 1 + n$ it follows that $(n + 1)a_j - \varepsilon a_j < m\gamma a_j < (n + 1)a_j$. Then $\lfloor m\gamma a_j \rfloor = (n + 1)a_j - 1$ and

$$m\gamma a_j - \lfloor m\gamma a_j \rfloor > 1 - \varepsilon a_j \geq 1 - \varepsilon |a_j| > 1 - \frac{1}{2} = \frac{1}{2}.$$ 

Q.E.D.
(5.5) Lemma. Let $\gamma_j$ (1 $\leq$ $j$ $< \infty$) be a sequence of positive numbers and $\Lambda$ be a positive integer such that $1, \gamma_1, \cdots, \gamma_\Lambda$ are $\mathbb{Q}$-linearly independent and

\[(5.5.1) \quad \gamma_j = \sum_{\lambda=1}^{\Lambda} c_{j,\lambda}\gamma_\lambda\]

for $\Lambda < j < \infty$, where $c_{j,k} \in \mathbb{Q}$. For any positive integer $N$ there exists some positive integer $m$ such that

\[m\gamma_j - \lfloor m\gamma_j \rfloor \geq \frac{1}{4} \text{ for } 1 \leq j \leq N.\]

Proof. Choose a positive numbers $\epsilon$ which we will specify more precisely later. Choose a rational number $r_\lambda$ so close to $\frac{1}{\gamma_\lambda}$ such that $|r_\lambda \gamma_\lambda - 1| < \epsilon$ and $|\frac{1}{r_\lambda} - \gamma_\lambda| < \epsilon$ for $1 \leq \lambda \leq \Lambda$. Let $\gamma'_\lambda = r_\lambda \gamma_\lambda$ for $1 \leq \lambda \leq \Lambda$ and let $c'_{j,\lambda} = \frac{c_{j,\lambda}}{r_\lambda}$ so that by (5.5.1) we have

\[\gamma_j = \sum_{\lambda=1}^{\Lambda} c'_{j,\lambda}\gamma'_\lambda.\]

We have

\[
\sum_{\lambda=1}^{\Lambda} c'_{j,\lambda} = \sum_{\lambda=1}^{\Lambda} c'_{j,\lambda}\gamma'_\lambda + \sum_{\lambda=1}^{\Lambda} c'_{j,\lambda}(1 - \gamma'_\lambda) \\
= \gamma_j + \sum_{\lambda=1}^{\Lambda} c'_{j,\lambda}(1 - \gamma'_\lambda) \geq \gamma_j - \epsilon \sum_{\lambda=1}^{\Lambda} |c'_{j,\lambda}| \\
= \gamma_j - \epsilon \sum_{\lambda=1}^{\Lambda} \left| c_{j,\lambda} \frac{1}{r_\lambda} \right| \geq \gamma_j - \epsilon \sum_{\lambda=1}^{\Lambda} |c_{j,\lambda}| (\epsilon + \gamma_\lambda). \]

We can choose $\epsilon$ sufficiently small so that

\[\epsilon \sum_{\lambda=1}^{\Lambda} |c_{j,\lambda}| (\epsilon + \gamma_\lambda) < \frac{\gamma_j}{2} \text{ for } 1 \leq j \leq N.\]

Thus

\[\sum_{\lambda=1}^{\Lambda} c'_{j,\lambda} \geq \frac{\gamma_j}{2} > 0 \text{ for } 1 \leq j \leq N.\]
Let \( q_N \) be the least common multiple of the denominator of \( c'_{j,\lambda} \) for \( 1 \leq \lambda \leq \Lambda \) and \( 1 \leq j \leq N \). Then

\[
(5.5.2) \quad q_N \gamma_j = \sum_{\lambda=1}^{\Lambda} (q_N c'_{j,\lambda}) \gamma'_\lambda,
\]

where \( q_N c'_{j,\lambda} \) is an integer.

We choose any irrational number \( 0 < \eta < 1 \). By Lemma (5.4) there exists a positive integer \( m_1 \) such that

\[
(5.5.3) \quad m_1 \eta \sum_{\lambda=1}^{\Lambda} (q_N c'_{j,\lambda}) - \left| m_1 \eta \sum_{\lambda=1}^{\Lambda} (q_N c'_{j,\lambda}) \right| \geq \frac{1}{2}
\]

for \( 1 \leq j \leq N \). Let \( 0 < \delta < \frac{1}{4} \) such that \( \delta \) is less than

\[
\left| m_1 \eta \sum_{\lambda=1}^{\Lambda} (q_N c'_{j,\lambda}) - m_1 \eta \sum_{\lambda=1}^{\Lambda} (q_N c'_{j,\lambda}) \right|
\]

for \( 1 \leq j \leq N \). Let \( C \) be the maximum of \( m_1 \sum_{\lambda=1}^{\Lambda} |q_N c'_{j,\lambda}| \) for \( 1 \leq j \leq N \).

We apply Kronecker’s theorem in (5.1) to the \( \mathbb{Q} \)-linearly independent irrational numbers \( \gamma'_1, \cdots, \gamma'_\Lambda \) so that there exist some \( m_2 \in \mathbb{Z} \) and integers \( x_\lambda \) for \( 1 \leq \lambda \leq \Lambda \) such that

\[
|m_2 \gamma'_\lambda - (x_\lambda + \eta)| < \frac{\delta}{C} \quad \text{for} \quad 1 \leq \lambda \leq \Lambda,
\]

from which we obtain

\[
\left| \sum_{\lambda=1}^{\Lambda} q_N c'_{j,\lambda} (m_2 \gamma'_\lambda - (x_\lambda + \eta)) \right| < \frac{\delta}{C} \sum_{\lambda=1}^{\Lambda} q_N |c'_{j,\lambda}|.
\]

Using (5.5.2), we get

\[
\left| m_2 q_N \gamma_j - \sum_{\lambda=1}^{\Lambda} (q_N c'_{j,\lambda} x_\lambda) - \eta \sum_{\lambda=1}^{\Lambda} (q_N c'_{j,\lambda}) \right| < \frac{\delta}{C} \sum_{\lambda=1}^{\Lambda} |q_N c'_{j,\lambda}| \leq \frac{\delta}{m_1}
\]

for \( 1 \leq j \leq N \) and

\[
\left| m_1 m_2 q_N \gamma_j - m_1 \sum_{\lambda=1}^{\Lambda} (q_N c'_{j,\lambda} x_\lambda) - m_1 \eta \sum_{\lambda=1}^{\Lambda} (q_N c'_{j,\lambda}) \right| \leq \delta
\]
for $1 \leq j \leq N$. From (5.5.3) and the choice of $\delta$ it follows that

$$m_1m_2qN\gamma_j - \lfloor m_1m_2qN\gamma_j \rfloor \geq \frac{1}{4} \quad \text{for} \quad 1 \leq j \leq N.$$ 

The required $m$ can be set to be $m_1m_2qN$. Q.E.D.

§6 A General Non-Vanishing Theorem.

Before we introduce our general non-vanishing theorem, we first introduce some terminology. It is to avoid the situation of a change of the multiplier ideal sheaf by some small perturbation of the metric, for example in the case of a metric like $\frac{1}{|z|}$. The terminology is that of a stable metric for which a sufficiently small perturbation would not change its multiplier ideal sheaf.

(6.1) Definition. A metric $e^{-\varphi}$ of a line bundle $L$ on a compact Kähler manifold $M$ is said to be stable if there exists some $\varepsilon > 0$ with the following property. If $U$ is an open neighborhood of a point $P \in M$, and $\kappa$ and $\psi$ are plurisubharmonic functions on $U$ such that the total mass, with respect to the Kähler form of $M$, of the difference of the two closed positive $(1,1)$-currents $\Theta_\varphi$ and $\Theta_\psi$ is less than $\varepsilon$, then there exists an open neighborhood $U'$ of $P$ in $M$ such that the multiplier ideal sheaf $\mathcal{I}_{\kappa+\psi}$ of the metric $e^{-\kappa-\psi}$ on $U'$ is equal to the multiplier ideal sheaf $\mathcal{I}_{\kappa+\varphi}$ of the metric $e^{-\varphi}$ on $U'$, where $\Theta_\varphi := \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \varphi$ and $\Theta_\psi := \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \psi$ are respectively the curvature currents of the metrics $e^{-\varphi}$ and $e^{-\psi}$.

(6.2) Theorem (Non-Vanishing). Let $M$ be a compact complex projective algebraic manifold. Let $L$ be a holomorphic line bundle on $M$ with a (possibly singular) metric $e^{-\varphi}$ along its fibers whose curvature current $\Theta_\varphi$ is a closed positive $(1,1)$-current. Assume that for some positive integer $p_0$ there is a (possibly singular) metric $e^{-\chi}$ along the fibers of $p_0L - K_M$ which is stable and whose curvature current $\Theta_\chi$ is a closed positive $(1,1)$-current which dominates some strictly positive smooth $(1,1)$-form on $M$. Assume that for any nonnegative integer $p$ the multiplier ideal sheaf $\mathcal{I}_{p\varphi+\chi}$ of the metric $e^{-p\varphi-\chi}$ of the line bundle $(p+p_0)L - K_M$ contains the multiplier ideal sheaf $\mathcal{I}_{(p+p_0)\varphi}$ of the metric $e^{-(p+p_0)\varphi}$ of the line bundle $(p+p_0)L$. Let $\varepsilon > 0$. Then for some sufficiently divisible $m \in \mathbb{N}$ the line bundle $(m+p_0)L$ admits a non identically zero holomorphic section on $M$ which belongs to the multiplier ideal of
Moreover, $m$ can be chosen to satisfy the stronger condition that global holomorphic sections of $\mathcal{I}_{m\varphi+\chi}(mL)$ over $M$ generate $\mathcal{I}_{m\varphi+\chi}$ outside some subvariety of codimension at least 2 in $M$.

\textbf{(6.3) Remark on the Assumption of the Multiplier Ideal Sheaf of $e^{-p\varphi-\chi}$ to Contain That of $e^{-(p+p_0)\varphi}$.} The reason for this assumption is to tailor our general non-vanishing theorem to its application to the proof of the finite generation of the canonical ring for the case of general type. To illustrate the reason for this assumption, let us consider the case of $M=X$ and $L=K_X$ and $e^{-\varphi} = \frac{1}{\Phi}$, where $\Phi$ is as defined in the beginning part of §3. In this case we choose $p_0 > 1$ and $p_0L - K_M$ is simply $(p_0 - 1) K_X$ and we can choose a sufficiently small $\varepsilon > 0$ and set

$$e^{-\chi} = \frac{e^{-(p_0-1)\varphi} (h_A)^\varepsilon}{|s_D|^{2\varepsilon}},$$

where $K_X = A + D$ is the decomposition into an effective $\mathbb{Q}$-divisor $D$ and an ample $\mathbb{Q}$-divisor $A$. When $\varepsilon$ is sufficiently small, for any nonnegative integer $p$ the multiplier ideal sheaf $\mathcal{I}_{p\varphi+\chi}$ of the metric

$$e^{-p\varphi-\chi} = \frac{e^{-(p+p_0-1)\varphi} (h_A)^\varepsilon}{|s_D|^{2\varepsilon}}$$

contains the multiplier ideal sheaf $\mathcal{I}_{(p+p_0)\varphi}$ of the metric $e^{-(p+p_0)\varphi}$ of the line bundle $(p+p_0) L$. The reason of this relation between the two multiplier ideal sheaves is, of course, due to the fact that the metric $e^{-\varphi}$ of $K_X$ is the least singular of all the metrics of $K_X$ with positive curvature current and, in particular, the metric $e^{-\varphi}$ of $K_X$ is no more singular than the metric $\frac{h_A}{|s_D|^{\varepsilon}}$ of $K_X$.

This condition is the minimum singularity condition for the metric $e^{-\varphi}$. In the induction steps to prove the general non-vanishing theorem (.62) and also in the induction steps in our analytic proof of the finite generation of the canonical ring for the case of general type, we have to use induction. In each induction step we use the Fujita conjecture type techniques and the analog of minimal centers of log canonical singularities (or after removing the necessary minimum vanishing orders) to get to a subspace. Since the choice of the subspace is to guarantee the extendability of sections from the subspace to the original ambient space, the induced metric also inherits the minimum singularity condition.
Remark on Requiring Constructed Sections to Belong to Multiplier Ideal Sheaves. The conclusion of the general non-vanishing theorem in (6.2) requires more than the mere existence of a non identically zero holomorphic section of the line bundle in question. It requires the constructed holomorphic sections to belong to the appropriate multiplier ideal sheaves for two reasons.

The first reason is to enable us to carry out the induction argument because to extend a holomorphic section from a lower-dimensional subspace to a higher dimensional space, we need to make sure that it belongs to the appropriate multiplier ideal sheaf in order to use the theorem of Kawamata-Viehweg-Nadel for the extension.

The second reason is for the application of the general non-vanishing theorem and the extension of sections to conclude that generic stable vanishing orders are achieved by finite sums of the absolute-value squares of fractional holomorphic pluricanonical sections. Again we need to make sure that the constructed sections belong to the appropriate multiplier ideal sheaf in order to use the theorem of Kawamata-Viehweg-Nadel for the extension to the entire manifold.

Beginning of the Proof and the Dichotomy According to the Structure of the Curvature Current. We now start with the proof of the general non-vanishing theorem stated in (6.2). We will differentiate among several cases according to the structure of the curvature current $\Theta_\varphi$ of the metric $e^{-\varphi}$ of $L$. According to Theorem (3.3), we have the following decomposition of the curvature current $\Theta_\varphi$ of the metric $e^{-\varphi}$ of $L$.

$$\Theta_\varphi = \sum_{j=1}^{J} \tau_j [V_j] + R,$$

where $J \in \mathbb{N} \cup \{0, \infty\}$ and the Lelong number of $R$ is zero outside a countable union $Z$ of subvarieties of codimension at least two in $M$ and $V_j$ is an irreducible hypersurface in $M$ and $\tau_j > 0$. The assumption concerning the metric $e^{-\chi}$ of $p_0L - K_M$ whose curvature current has a positive lower bound is introduced in order to be able to apply the vanishing theorem of Kawamata-Viehweg-Nadel.

First we make some remarks about the proof. The main idea of the proof is to use
(i) the techniques for Fujita’s conjecture (see e.g., [Angehrn-Siu 1995]) together with

(ii) the technique of Shokurov of using the theorem of Riemann-Roch to compare the arithmetic genus of the line bundle and that of its twisting by a flat line bundle [Shokurov 1985].

For the arguments of the Fujita conjecture, given an ample line bundle $E$ over a compact complex algebraic manifold $Y$ we use the theorem of Riemann-Roch to get the lower bound of $\dim_\mathbb{C} \Gamma (Y, mE)$ as a function of $m$ for $m$ sufficiently large to produce a multi-valued holomorphic section of a multiple of $E$ vanishing to high order. Now in our case our line bundle $L$ is not ample but with a weaker condition of having a metric $e^{-\varphi}$ of nonnegative curvature current. We will, however, still follow the path of getting a good lower bound of the dimension of the section module.

The new ingredient is the dichotomy into the case of $R = 0$ and the case of $R \neq 0$ and, in the case of $R = 0$, the further dichotomy into the case of $J = \infty$ and the case of $J \neq \infty$. For the case of $R \neq 0$ or $J = \infty$ we still get a good lower bound for the dimension of the section module. For the case of $R = 0$ and $J \neq \infty$ we explicitly construct by hand a holomorphic section by following the technique of Shokurov of using the theorem of Riemann-Roch to compare the arithmetic genus of the line bundle and that of its twisting by a flat line bundle. For our purpose we are only interested in producing a single nontrivial holomorphic section (which belongs to the appropriate multiplier ideal sheaf).

(6.6) Slicing by an Ample Divisor. Let $A$ be a very ample line bundle over $M$ such that $A - K_M$ is ample. Let $h_A$ be a smooth metric of $A$ whose curvature form $\omega_A$ is positive on $M$. We assume that $A$ is chosen to be sufficiently ample so that for each point $P \in M$ the proper transform of $A$ in the manifold obtained from $M$ by blowing up $P$ is still very ample. This technical assumption will enable us to choose a generic element of $\Gamma (M, A)$ vanishing at $P_0$ which is not a zero-divisor of a prescribed coherent ideal sheaf.

Let $p$ and $k$ be positive integers and we will impose more conditions on $p$ and $k$ later. Take $P_0 \in M$ and we will also impose more conditions on $P_0$ later. Let $s_1$ be a generic element of $\Gamma (M, A)$ vanishing at $P_0$ so that the
short exact sequence

\[ 0 \to \mathcal{I}_{p\varphi} (pL + kA) \xrightarrow{\theta s_1} \mathcal{I}_{p\varphi} (pL + (k + 1) A) \]
\[ \to (\mathcal{I}_{p\varphi} / s_1 \mathcal{I}_{p\varphi}) (pL + (k + 1) A) \to 0 \]

is exact, where \( \theta s_1 \) is defined by multiplication by \( s_1 \). For this step we have to make sure that the maximum ideal \( m_{M, P_0} \) of \( M \) at \( P_0 \) is not an associated prime ideal in the primary decomposition of the stalk of the ideal sheaf \( \mathcal{I}_{p\varphi} \) at \( P_0 \). This means that for each \( p \) we have to impose the condition that \( P_0 \) does not belong to some finite subset \( Z_0 \) of \( M \). Let \( M_1 \) be the zero-set of \( s_1 \) and

\[ \mathcal{O}_{M_1} = (\mathcal{O}_M / s_1 \mathcal{O}_M) |_{M_1}, \]

which we can assume to be regular with ideal sheaf equal to \( s_1 \mathcal{O}_M \) because \( s_1 \) is generic element of \( \Gamma (M, A) \) vanishing at \( P_0 \). By choosing \( s_1 \) generically we can also assume that \( \mathcal{I}_{(p\varphi|_{M_1})} = \mathcal{I}_{p\varphi} / s_1 \mathcal{I}_{p\varphi} \). We use \( \chi (\cdot, \cdot) \) to denote the arithmetic genus which means

\[ \chi (\cdot, \cdot) = \sum_{\nu=0}^{\infty} (-1)^\nu \dim \mathbb{C} H^\nu (\cdot, \cdot). \]

From the long cohomology exact sequence of the above short exact sequence we obtain

\[ \chi (M, \mathcal{I}_{p\varphi} (pL + (k + 1) A)) = \]
\[ \chi (M, \mathcal{I}_{p\varphi} (pL + kA)) + \chi (M_1, \mathcal{I}_{(p\varphi|_{M_1})} (pL + (k + 1) A) |_{M_1}). \]

Since \( A - K_M \) is ample and \( 2A - K_{M_1} = A - K_M \) is also ample, when we assume \( k \geq 1 \), by the theorem of Kawamata-Viehweg-Nadel

\[ H^\nu (M, \mathcal{I}_{p\varphi} (pL + kA)) = 0 \quad \text{for } \nu \geq 1, \]
\[ H^\nu (M_1, \mathcal{I}_{(p\varphi|_{M_1})} ((pL + (k + 1) A) |_{M_1}) = 0 \quad \text{for } \nu \geq 1. \]

so that

\[ \Gamma (M, \mathcal{I}_{p\varphi} (pL + (k + 1) A)) = \]
\[ \Gamma (M, \mathcal{I}_{p\varphi} (pL + kA)) + \Gamma (M_1, \mathcal{I}_{(p\varphi|_{M_1})} ((pL + (k + 1) A) |_{M_1})). \]
(6.7) Slicing by Ample Divisors Down to a Curve. Instead of one single element \( s \in \Gamma (M, A) \), we can choose generically
\[
s_1, \ldots, s_{n-1} \in \Gamma (M, A)
\]
all vanishing at \( P_0 \) so that inductively for \( 1 \leq \nu \leq n-1 \) the common zero-set \( M_\nu \) of \( s_1, \ldots, s_\nu \) with the structure sheaf
\[
\mathcal{O}_{M_\nu} := \left( \mathcal{O}_M / \sum_{j=1}^{\nu} s_j \mathcal{O}_M \right)|_{M_\nu}
\]
is regular and we end up with the inequality
\[
\dim_{\mathbb{C}} \Gamma (M, \mathcal{I}_{p\nu} (pL + (k + n - 1) A)) \\
\geq \dim_{\mathbb{C}} \Gamma (M_{n-1}, \mathcal{I}_{(p_{\nu} \mid M_{n-1})} ((pL + (k + n - 1) A) |_{M_{n-1}})).
\]

For this step we have to exclude \( P_0 \) from a subvariety \( Z_{n-2} \) of dimension \( \leq n - 2 \) in \( M \), because we have to exclude a finite set in each \( M_1 \) which would come together as the hypersurface \( M_1 \) varies to form a subvariety \( Z_1 \) of dimension \( \leq 1 \) in \( M \) (as one can argue with the quotients of coherent ideal sheaves by non zero-divisors and with the primary decompositions for coherent ideal sheaves). Likewise we have a subvariety \( Z_k \) of dimension \( \leq k \) in \( M \) so that \( Z_k \) intersects \( M_k \) in a finite number of points and finally we have end up with a subvariety \( Z_{n-2} \) of dimension \( \leq n - 2 \) in \( M \) which intersects \( M_{n-2} \) in a finite number of points and we impose the condition that \( P_0 \) does not belong to \( Z_{n-2} \).

Since \( M_{n-1} \) is a curve, all coherent ideal sheaves on it are principal and are locally free and they come from holomorphic line bundles. We can choose \( s_1, \ldots, s_{n-1} \) so generically that \( M_{n-1} \) is disjoint from \( Z \). For this step we need to make sure that \( P_0 \) does not belong to \( Z \).

We remark that actually we can also accommodate the case of \( P_0 \in Z \) as long as the number of times \( M_{n-1} \) intersects \( Z \) is far less than \( \int_{M_{n-1}} R \). For the proof of this non-vanishing theorem we are not interested in the problem of accommodating the case \( P_0 \in Z \). We simply agree to choose \( P_0 \) outside of \( Z \). We add this remark to indicate how our general non-vanishing theorem can be strengthened.
We would like to remark also that this particular step of slicing by \( n-1 \) ample divisors to get down to a curve roughly corresponds to the step in Shokurov’s proof of his non-vanishing theorem [Shokurov 1985] where he takes the product of his numerically effective divisor in his \( n \)-dimensional manifold with the \((n-1)\)-th power of a numerically effective big line bundle.

\((6.8)\) Application of the Theorem of Riemann-Roch to a Curve, Comparing Contributions from the Curvature Current and the Multiplier Ideal Sheaves, and the Unbounded Lower Bound Condition. Let \( c \) be the nonnegative number

\[
\int_{M_{n-1}} R = \int_M R \wedge (\omega_A)^{n-1}.
\]

Then

\[
(6.8.1) \quad \dim_C \Gamma (M, \mathcal{I}_{p\varphi} (pL + (k + n - 1) A)) \\
\geq \dim_C \Gamma (M_{n-1}, \mathcal{I}_{(p\varphi|_{M_{n-1}})} ((pL + (k + n - 1) A)|_{M_{n-1}})) \\
\geq 1 - \text{genus} (M_{n-1}) + (k + n - 1) A^{n-1} \text{M}_{n-1} \\
+ \sum_{j=1}^{J} (p\tau_j - \lfloor p\tau_j \rfloor) V_j \cdot A^{n-1} + p \int_{M_{n-1}} R,
\]

where the last identity is from the theorem of Riemann-Roch applied to the regular curve \( M_{n-1} \) and the locally free sheaf

\[
\mathcal{I}_{(p\varphi|_{M_{n-1}})} ((pL + (k + n - 1) A)|_{M_{n-1}})
\]

on \( M_{n-1} \). Let \( \Xi_p \) be the right-hand side of \((6.8.1)\). We introduce the following unbounded-lower-bound condition \((6.8.2)\).

\[
(6.8.2) \quad \lim_{\nu \rightarrow \infty} \Xi_{p_{\nu}} = \infty \quad \text{for some subsequence} \quad \{p_{\nu}\}_{\nu=1}^{\infty}.
\]

For example, \((6.8.2)\) holds when \( c > 0 \) (and we can choose \( \{p_{\nu}\}_{\nu} \) to be the same as \( \{p\}_{p\in\mathbb{N}} \)). Later we will show by using diophantine approximation that \((6.8.2)\) also holds when \( J = \infty \). So we are left with the case when \( R = 0 \) and \( J < \infty \), which we will handle later with Shokurov’s technique of using the theorem of Riemann-Roch to compare the arithmetic genus of a line bundle and that of its twisting by a flat line bundle. At this point let us assume that \((6.8.2)\) holds and continue with the proof under such an assumption.

37
Construction of Sections with Extra Vanishing Order from Dimension Counting and Construction of Metrics by Canceling Contributions from Ample Divisors by Using the General Type Property. For any $\ell \in \mathbb{N}$ the number of terms in a polynomial of degree $\ell$ in $d$ variables is $\binom{d+\ell}{d}$. Take any positive integer $q$. Take a positive integer $N$ and we will impose more condition on $N$ later. By Condition (6.8.2), there exists $p \in \mathbb{Z}$ such that

$$\dim_{\mathbb{C}} \Gamma (M, \mathcal{I}_{p\varphi}(pL + (k + n - 1)A)) \geq 1 + \binom{n + N(k + n - 1)q}{n}$$

and we can find some non identically zero element $s$ of

$$\Gamma (M, \mathcal{I}_{p\varphi}(pL + (k + n - 1)A))$$

which vanishes to order at least $N(k + n - 1)q$ at $P_0$ so that $s^{\frac{1}{N(k+n-1)}}$ is a multi-valued holomorphic section of the $\mathbb{Q}$-line-bundle $\frac{p}{N(k+n-1)}L + \frac{1}{N}A$ over $M$ which vanishes to order at least $q$ at $P_0$. We assume that $N$ is chosen so large that the curvature current $\Theta_\chi$ dominates $\frac{2}{N^2} \omega_A$. Let $\hat{p}$ be the round-up of $\frac{p}{N(k+n-1)}$ and $\delta_p = \hat{p} - \frac{p}{N(k+n-1)}$. We introduce the metric

$$e^{-\tilde{\chi}} := \frac{e^{-\chi - \delta_p \varphi}}{(h_A)^{\frac{1}{N}} |s|^\frac{2}{Nk}}$$

of $(p + p_0)L - K_M$ so that the multiplier ideal at $P_0$ is contained in $(m_{M,P_0})_{\hat{q}}^{\frac{1}{N}}$.

Since the metric $e^{-\chi}$ is assumed to be stable, for any given $\hat{q}$ we can conclude by choosing $N$ and $q$ sufficiently large that the multiplier ideal sheaf of $e^{-\tilde{\chi}}$ is contained in $(m_{M,P_0})^\hat{q} \mathcal{I}_{\hat{p}\varphi + \chi}$.

Interpolation of Metrics to Get the Analog of Minimal Center of Log Canonical Singularities. We now use the techniques introduced for the proof of Fujita conjecture type results. We interpolate the two metrics $e^{-\hat{p}\varphi - \chi}$ and $e^{-\tilde{\chi}}$ for the line bundle $(\hat{p} + p_0)L - K_M$ on $M$ and use a slight modification as described in (4.4.3) to get the analog of a minimal center of log canonical singularities. We end up with a metric $e^{-\kappa}$ of $(\hat{p} + p_0)L - K_M$ with the following properties.

(i) The curvature current of the metric $e^{-\kappa}$ of $(\hat{p} + p_0)L - K_M$ dominates some smooth positive (1,1)-form on $M$. 

38
(ii) The multiplier ideal sheaf $\mathcal{I}_\kappa$ of the metric $e^{-\kappa}$ of $(\hat{p} + p_0) L - K_M$ is contained in the multiplier ideal sheaf $\mathcal{I}_{\hat{p}\varphi + \chi}$ of the metric $e^{-\hat{p}\varphi}$ of $(\hat{p} + p_0) L - K_M$.

(iii) The support of $\mathcal{I}_{\hat{p}\varphi + \chi} / \mathcal{I}_\kappa$ is an irreducible subvariety $\hat{M}$ of codimension at least one in $M$ such that after replacing $M$ by the result of applying to $M$ some finite number of successive monoidal transformations with nonsingular centers we can assume without loss of generality that $\hat{M}$ is a nonsingular hypersurface in $M$.

(iv) There is a metric $e^{-\hat{\chi}}$ of $(\hat{p} + p_0) L - K_{\hat{M}}$ on $\hat{M}$ whose curvature current dominates some smooth positive $(1,1)$-form on $\hat{M}$ such that the restriction of $\mathcal{I}_{\hat{p}\varphi + \chi} / \mathcal{I}_\kappa$ to $\hat{M}$ agrees with $\mathcal{I}_{\hat{\chi}}$ on $\hat{M}$.

(v) If $\hat{M}$ is not some $V_{j_0}$, the metric $e^{-\hat{\chi}}$ of $(\hat{p} + p_0) L - K_{\hat{M}}$ on $\hat{M}$ and the metric $e^{-\varphi}$ of $L|_{\hat{M}}$ on $\hat{M}$ satisfy the conditions of Theorem (6.1) when the metric $e^{-\hat{\chi}}$ of $(\hat{p} + p_0) L - K_{\hat{M}}$ on $\hat{M}$ replaces the metric $e^{-\chi}$ of $p_0 L - K_{\hat{M}}$ on $M$ and $L|_{\hat{M}}$ on $\hat{M}$ replaces $L$ on $M$ with the restriction of the metric $e^{-\varphi}$. If $M$ is some $V_{j_0}$, in the above description we have to replace $L|_{\hat{M}}$ by $(L - \tau_{j_0} M)|_{\hat{M}}$ with the corresponding natural modification of the metric.

(6.11) Use of Induction Hypothesis for One Dimension Less. Let us assume that $\hat{M}$ is not some $V_{j_0}$ and proceed, otherwise we simply have to make a natural modification as indicated above. Since the dimension of $\hat{M}$ is less than $n$, by induction hypothesis, outside some subvariety of codimension at least two in $M$, for $p$ sufficiently large the sheaf $\mathcal{I}_{p\varphi + \chi} ((p + \hat{p} + p_0) L)$ is generated by its global sections on $\hat{M}$. We are going to extend such sections to all of $M$ in the standard way by using a short exact sequence and the vanishing theorem of Kawamata-Viehweg-Nadel as follows. From the short exact sequence

$$0 \to \mathcal{I}_{p\varphi + \kappa} ((p + \hat{p}) L) \to \mathcal{I}_{(p + \hat{p})\varphi + \chi} ((p + \hat{p}) L) \to \left( \mathcal{I}_{(p + \hat{p})\varphi + \chi} / \mathcal{I}_{p\varphi + \kappa} \right) ((p + \hat{p}) L) \to 0$$

and the vanishing of

$$H^1 (M, \mathcal{I}_{p\varphi + \kappa} ((p + \hat{p}) L))$$

it follows that the map

$$\Gamma \left( M, \mathcal{I}_{(p + \hat{p})\varphi + \chi} ((p + \hat{p}) L) \right) \to \Gamma \left( M, \left( \mathcal{I}_{(p + \hat{p})\varphi + \chi} / \mathcal{I}_{p\varphi + \kappa} \right) ((p + \hat{p}) L) \right)$$
\[
\Gamma(\tilde{M}, \mathcal{I}_{p\varphi+\tilde{\chi}} ((p + \tilde{p}) L))
\]
is surjective.

We now go back to Condition (6.8.2). A main tool is the results given in §5 on diophantine approximation which are derived from Kronecker’s theorem on diophantine approximation.

(6.12) Different Cases for the Unbounded-Lower-Bound Condition. For the unbounded-lower-bound condition (6.8.2) we earlier introduced a dichotomy into two cases and again for one case a further dichotomy. It is the same as to differentiate among three cases. The first and the easiest case is \( R \neq 0 \) and we have seen that it guarantees the unbounded-lower-bound condition (6.8.2). When \( R = 0 \), the other two cases are (i) \( J = \infty \) and (ii) \( J < \infty \).

(6.12.1) Case of Infinite Number of Irreducible Hypersurface Lelong Sets. We now look at the case \( J = \infty \) and we are going to verify the unbounded-lower-bound condition (6.8.2) for this case. For the verification we differentiate among three possibilities. Recall the following decomposition of the curvature current \( \Theta_\varphi \) of the metric \( e^{-\varphi} \) of the line bundle \( L \) as a closed positive \((1,1)\)-current.

\[
\Theta_\varphi = \sum_{j=1}^{\infty} \tau_j [V_j] + R.
\]

We have the following three cases.

(i) for any positive number \( N \) one can find \( N \) irrational elements \( \{\tau_{j\nu}\}_{\nu=1}^{N} \) in \( \{\tau_j\}_{j=1}^{\infty} \) which are \( \mathbb{Q} \)-linearly independent.

(ii) there is some positive integer \( N_0 \) such that any subsets of \( N_0 + 1 \) elements of \( \{\tau_j\}_{j=1}^{\infty} \) are \( \mathbb{Q} \)-linearly dependent, but at least one element of \( \{\tau_j\}_{j=1}^{\infty} \) is irrational.

(iii) every element of \( \{\tau_j\}_{j=1}^{\infty} \) is rational.

For Case (i) we can simply apply Corollary (5.3) above to get the unbounded-lower-bound condition (6.8.2).

For Case (ii) we can simply apply Lemma (5.5) above to get the unbounded-lower-bound condition (6.8.2).

Case (iii) is more complicated. We have to handle it by modifying somewhat the above main argument of (6.8).
Modification of Main Argument for the Case of All Lelong Numbers of Hypersurface Lelong Sets Being Rational. For any prescribed positive integer \( N \), let \( p \) be the least common multiple for the denominators of the rational numbers \( \gamma_1, \cdots, \gamma_N \) so that each \( p\gamma_j \) is a positive integer for \( 1 \leq j \leq N \).

We choose \( \eta_0 > 0 \) sufficiently small so that \( \frac{1}{2} A + \sum_{j=1}^{N} \eta p V_j \) is an ample \( \mathbb{Q} \)-line-bundle over \( X \) with positively curved smooth metric \( h_\eta \) for any positive rational number \( \eta \leq \eta_0 \). Now choose a positive integer \( \tau \) so large that \( \frac{1}{\tau} < \eta_0 \) and \( \tau > 2p\tau_j \) for every \( 1 \leq j \leq N \). Then from \( \tau \left( \frac{1}{\tau} - 1 \right) < \eta_0 \) and \( \tau > 2p\tau_j \) for every \( 1 \leq j \leq N \), it follows that

\[
(6.12.2.1) \quad \sum_{j=1}^{N} \left( \frac{\tau - 1}{\tau} p\tau_j - \left( \frac{\tau - 1}{\tau} p\tau_j \right) \right) \geq \sum_{j=1}^{N} \frac{1}{2} \geq \frac{N}{2}.
\]

We have a metric \( h_{\frac{1}{2}} \) for the \( \mathbb{Q} \)-line-bundle \( \frac{1}{2} A + \sum_{j=1}^{N} \frac{p}{\tau} V_j \) (which is the metric \( h_\eta \) for the \( \mathbb{Q} \)-line-bundle \( \frac{1}{2} A + \sum_{j=1}^{N} \eta p V_j \) when \( \eta = \frac{1}{\tau} \)). We now define the metric

\[
e^{-\tilde{\varphi}} = e^{-\frac{(r-1)p}{\tau} \varphi} h_{\frac{1}{2}} (h_A)^{\frac{1}{2}}
\]

of \( pL + A \).

At this point we introduce our modification of the above main argument of (6.8). Instead of using the metric \( e^{-p\varphi} h_A \) for \( pL + A \), we use the metric \( e^{-\tilde{\varphi}} \) of \( pL + A \) and we have the following two conclusions, the second of which comes from (6.13.2.1).

(a) The curvature current of the metric \( e^{-\tilde{\varphi}} \) of \( pL + A \) dominates that of the metric \( (h_A)^{\frac{1}{2}} \) of \( \frac{1}{2} A \).

(b) For a generic curve \( C \) in \( M \) the Chern class of the line bundle associated to \( \mathcal{I}_{\tilde{\varphi},|C} (pL + A) \) is at least \( \left\lfloor \frac{N}{2} \right\rfloor \).

We repeat the main argument of (6.8) with the metric \( e^{-\tilde{\varphi}} \) of \( pL + A \) instead of the metric \( e^{-p\varphi} h_A \) of \( pL + A \). Let \( s_1 \) be a generic element of \( \Gamma (M, A) \) vanishing at \( P_0 \) so that the short exact sequence

\[
0 \rightarrow \mathcal{I}_{\tilde{\varphi}} (pL + kA) \xrightarrow{\theta_1} \mathcal{I}_{\tilde{\varphi}} (pL + (k + 1)A) \rightarrow \mathcal{I}_{\tilde{\varphi}} (pL + (k + 1)A) / s_1 \mathcal{I}_{\tilde{\varphi}} \\ \rightarrow 0
\]
is exact, where $\theta_s$ is defined by multiplication by $s_1$. Let $M_1$ be the zero-set of $s_1$ and we have

$$\chi (M, I_\tilde{\varphi} (pL + (k + 1) A)) = \chi (M, I_\tilde{\varphi} (pL + kA)) + \chi (M_1, I_{(\tilde{\varphi}|_Y)} (pL + (k + 1) A) |_{M_1}).$$

Again, instead of one single element $s \in \Gamma (M, A)$, we can use generically $s_1, \ldots, s_{n-1} \in \Gamma (M, A)$ all vanishing at $P_0$ so that inductively for $1 \leq \nu \leq n - 1$ we have $M_\nu$ equal to the common zero-set of $s_1, \ldots, s_\nu$ and we end up with the inequality

$$\dim \Gamma (M, I_\tilde{\varphi} (pL + (k + n - 1) A)) \geq \dim \Gamma (M_{n-1}, I_{(\tilde{\varphi}|_{M_{n-1}})} ((pL + (k + n - 1) A) |_{M_{n-1}})).$$

Since $M_{n-1}$ is a curve, all coherent ideal sheaves on it are principal and are locally free and they come from holomorphic line bundles. We can choose $s_1, \ldots, s_{n-1}$ so generically that $M_{n-1}$ is disjoint from $Z$. Then

$$\dim \Gamma (M, I_\tilde{\varphi} (pL + (k + n - 1) A)) \geq \dim \Gamma (M_{n-1}, I_{(\tilde{\varphi}|_{M_{n-1}})} ((pL + (k + n - 1) A) |_{M_{n-1}})) \geq 1 - \text{genus} (M_{n-1}) + (k + n - 3) A^{n-1} M_{n-1} + \frac{N}{2},$$

because of Property (b) of the metric $e^{-\tilde{\varphi}_p}$ of $pL + A$. Thus

$$\dim \Gamma (M, I_\tilde{\varphi} (pL + (k + n - 1) A)) \geq \frac{N}{3}$$

for $N$ sufficiently large (relative to the genus of the curve $M_{n-1}$). From this point on we just follow the rest of the main argument.

(6.13) Comparing the Use of the Theorem of Riemann-Roch for a Line Bundle and Its Twisting by a Flat Line Bundle. We now discuss the remaining case of $R \neq 0$ and $J < \infty$. For this case we cannot continue the next inductive step of reducing the dimension of the new subspace defined by the multiplier ideal sheaf. However, for this case there is no need to continue with the next step, because the purpose of inductive step is to produce a holomorphic section at the end when the dimension of the subspace defined by the multiplier ideal
sheaf becomes isolated. For this case we can construct a holomorphic section
directly from the curvature current without continuing with the inductive
process. The construction comes from the technique of modification by a
flat bundle and comparing the use of the theorem of Riemann-Roch for the
original line bundle and its twisting by the flat bundle.

For this case our curvature current is a finite \( \mathbb{R} \)-linear combination of
integration over (the regular part of) irreducible hypersurfaces. Our decom-
position of the curvature current \( \Theta_\phi \) of the metric \( e^{-\phi} \) of the line bundle \( L \)
as a closed positive \((1,1)\)-current becomes

\[
\Theta_\phi = \sum_{j=1}^J \tau_j [V_j]
\]

with \( J < \infty \) and \( \tau_j \in \mathbb{R} \) and \( \tau_j > 0 \). If \( \tau_1, \ldots, \tau_J \) are all rational, we
can simply introduce the \( \mathbb{Q} \)-line-bundle \( \sum_{j=1}^J \tau_j [V_j] \) over \( X \) whose curvature
current is the same as that of \( L \), implying that \( \mathbb{Q} \)-line-bundle \( \sum_{j=1}^J \tau_j [V_j] \)
is equal to the tensor product of some flat \( \mathbb{Q} \)-line-bundle with \( L \). This flat
\( \mathbb{Q} \)-line-bundle would then be used in our twisting of \( L \) and in comparing the
application of the theorem of Riemann-Roch to a high multiple of \( L \) and its
twisting by this flat \( \mathbb{Q} \)-line-bundle (the high multiple being used to make
sure that such a high multiple of the \( \mathbb{Q} \)-line-bundle would make it a usual
holomorphic line bundle). The strict positive lower bound for the curvature
current of the metric \( e^{-\chi} \) of \( p_0 L - K_X \) makes the application of the theorem
of Kawamata-Viehweg-Nadel possible so that the dimension of the section
module is the same as the full arithmetic genus. Now we have to deal with
the situation when some of the real numbers \( \tau_1, \ldots, \tau_J \) are not rational.

Our strategy is to use the fact that \( L \) is a usual holomorphic line bundle. If
the classes in \( H^{1,1}(M) \cap H^2(M, \mathbb{R}) \) defined by \( V_1, \ldots, V_J \) are all \( \mathbb{R} \)-linearly
independent, then the real numbers \( \tau_1, \ldots, \tau_J \) must be all rational. Hence there
is some \( \mathbb{R} \)-linearly dependence among the classes in \( H^{1,1}(M) \cap H^2(M, \mathbb{R}) \) de-
defined by \( V_1, \ldots, V_J \). We use such \( \mathbb{R} \)-linearly dependency relations to change
each \( \tau_1, \ldots, \tau_J \) slightly without changing the class in \( H^{1,1}(M) \cap H^2(M, \mathbb{R}) \)
deefined by

\[
\Theta_\phi = \sum_{j=1}^J \tau_j [V_j].
\]

This changes are so slight that the new numbers which replace \( \tau_1, \ldots, \tau_J \) are
still positive. Here is the detailed implementation of this strategy.
Implementation of Strategy of Twisting by a Flat Bundle. We denote by $\mathfrak{V}$ the $\mathbb{Q}$-linear subspace $\mathfrak{V}$ in $H^{1,1}(M) \cap H^2(M, \mathbb{R})$ spanned by $V_1, \ldots, V_J$. Let $\text{Class}(V_j)$ be the class in $H^{1,1}(M) \cap H^2(M, \mathbb{R})$ defined by $V_j$ for $1 \leq j \leq J$ and let $\text{Class}(\Theta_v)$ be the class in $H^{1,1}(M) \cap H^2(M, \mathbb{R})$ defined by $\Theta_j$. After relabeling we can assume without loss of generality that $\text{Class}(V_1), \ldots, \text{Class}(V_J)$ form a $\mathbb{R}$-basis of $\mathfrak{V}$ for some $1 \leq J_0 \leq J$. If $J_0 = J$, since the class defined in $H^{1,1}(M) \cap H^2(M, \mathbb{R})$ lies in $H^{1,1}(M) \cap H^2(M, \mathbb{Q})$, it follows that $\tau_1, \ldots, \tau_J$ must be all rational. In this case we set $\rho_j$ for $1 \leq j \leq J_0$.

Now assume that $J_0 < J$. Inside $\mathfrak{V}$ we can write

$$\text{Class}(V_j) = \sum_{k=1}^{J_0} a_{j,k} \text{Class}(V_k)$$

for some $a_{j,k} \in \mathbb{Q}$ ($J_0 + 1 \leq j \leq J$, $1 \leq k \leq J_0$) which may be positive or zero or negative. Choose $p \in \mathbb{N}$ such that

(i) the denominator of each $a_{j,k}$ is a factor of $p$ for $J_0 + 1 \leq j \leq J$ and $1 \leq k \leq J_0$,

(ii) the denominator of every rational member of $\tau_1, \ldots, \tau_J$ is a factor of $p$,

(iii) if $a_1, \ldots, a_{J_0} \in \mathbb{Q}$ and $\sum_{j=1}^{J_0} a_j \text{Class}(V_j)$ belongs to $H^{1,1}(M) \cap H^2(M, \mathbb{Z})$, then $pa_j \in \mathbb{Z}$ for $1 \leq j \leq J_0$.

Choose $C > 1$ such that $C > p\tau_j$ for $1 \leq j \leq J_0$ and $C > |a_{j,k}|$ for $J_0 + 1 \leq j \leq J$ and $1 \leq k \leq J_0$. Let $s_{V_j}$ be the canonical section of the line bundle $V_j$ over $M$ for $1 \leq j \leq J$. Choose $\delta > 0$ such that the local function

$$\frac{1}{|s_{V_j}|^{2\delta}}$$

is locally integrable on $M$ for $1 \leq j \leq J$. We choose a number

$$0 < \eta < \frac{\delta}{(J - J_0) C^2 p}$$

which is less than the minimum of $\tau_1, \ldots, \tau_J$ such that the total mass of the closed positive $(1,1)$-current $(J - J_0) C^2 p \eta \sum_{j=1}^{J} [V_j]$ is less than the positive number in the definition (6.1) for the stability of the metric $e^{-\chi}$. Choose $0 <$
\( \eta < \frac{1}{2} \) and we will impose more conditions on \( \eta \) later. If none of \( \tau_1, \ldots, \tau_{J_1} \) is rational, we apply Kronecker’s theorem (5.1) to the \( \mathbb{Q} \)-linearly independent set \( 1, \tau_1, \ldots, \tau_{J_1} \) to find \( t > 1 \) such that \( |t - 1 - x_0| < \eta \) and \( |t \tau_j - x_j| < \eta \) for \( 1 \leq j \leq J_0 \) for some integers \( x_0, \ldots, x_{J_0} \). If one of \( \tau_1, \ldots, \tau_{J_1} \) is rational and we can assume without loss of generality that \( \tau_1 \) is rational, we apply Kronecker’s theorem (5.1) to the \( \mathbb{Q} \)-linearly independent set \( p \tau_1, \ldots, p \tau_{J_1} \) to find \( t > 1 \) such that \( |t - p \tau_j - x_j| < \eta \) for \( 1 \leq j \leq J_0 \) for some integers \( x_1, \ldots, x_{J_0} \). Let \( q \) be the integer closest to \( t \). Then \( |t - q| < \eta \). Moreover, by our choice of \( t \) we have \( |q p \tau_j - \rho_j| < C \eta \) for every \( 1 \leq j \leq J_0 \) and for some integers \( \rho_j \). Let

\[
\rho_k = q p \tau_k + \sum_{j=J_0+1}^{J} (q p \tau_j - \rho_j) a_{j,k}
\]

for \( 1 \leq j \leq J_0 \). Then

\[
|q p \tau_k - \rho_k| < (J - J_0) C^2 \eta \quad \text{for} \quad 1 \leq j \leq J_0.
\]

Inside \( \mathfrak{M} \) we have

(6.13.1.2) \quad \text{qp Class}(\Theta_{\varphi}) - \sum_{j=J_0+1}^{J} \rho_j \text{Class}(V_j) = \sum_{k=1}^{J_0} \rho_k \text{Class}(V_k).

Since

\[
\text{qp Class}(\Theta_{\varphi}) - \sum_{j=J_0+1}^{J} \rho_j \text{Class}(V_j)
\]

belongs to \( H^{1,1}(M) \cap H^2(M, \mathbb{Z}) \), it follows from the choice of \( p \) and (6.13.1.2) that \( q \rho_k \) is rational for \( 1 \leq k \leq J_0 \). We now have

\[
qp^2 \text{Class}(\Theta_{\varphi}) = \sum_{j=1}^{J} p \rho_j \text{Class}(V_j)
\]

with each \( p \rho_j \in \mathbb{Z} \) and

(6.13.1.3) \quad |p \rho_j - q p^2 \tau_j| < (J - J_0) C^2 p \eta \quad \text{for} \quad 1 \leq j \leq J.

Let \( F \) be the flat holomorphic line bundle \( L - \sum_{j=1}^{J} p \rho_j V_j \) on \( M \). Consider the metric

\[
e^{-\tilde{\psi}} = \frac{1}{|\prod_{j=1}^{J} (s_{V_j})^{p \rho_j}|^{\frac{1}{pq}}}.
\]

45
of \( L \). By the theorem of Kawamata-Viehweg-Nadel we have

\[
H^\nu \left( M, \mathcal{I}_{(qp^2-p_0)}(\tilde{\psi} + \chi (qp^2 L)) \right) = 0
\]

and

\[
H^\nu \left( M, \mathcal{I}_{(qp^2-p_0)}(\tilde{\psi} + \chi (qp^2 (L + F))) \right) = 0
\]

for \( \nu \geq 1 \). By using the theorem of Riemann-Roch and the fact that \( F \) is flat, we conclude that

\[
\dim_{\mathbb{C}} \Gamma \left( M, \mathcal{I}_{(qp^2-p_0)}(\tilde{\psi} + \chi (qp^2 L)) \right) = \dim_{\mathbb{C}} \Gamma \left( M, \mathcal{I}_{(qp^2-p_0)}(\tilde{\psi} + \chi (qp^2 (L + F))) \right).
\]

Let \( s = \prod_{j=1}^J (s_{V_j})^{p_{\rho_j}} \). Then \( s \) is a non identically zero holomorphic section of \( \sum_{j=1}^J p_{\rho_j} V_j = qp^2 L + F \) over \( M \). From (6.13.1.1) and (6.13.1.3) it follows that \( s \) locally belongs to the multiplier ideal sheaf \( \mathcal{I}_{(qp^2-p_0)}(\tilde{\psi} + \chi) \) and as a consequence

\[
0 \neq s \in \Gamma \left( M, \mathcal{I}_{(qp^2-p_0)}(\tilde{\psi} + \chi (qp^2 (L + F))) \right)
\]

and

\[
\dim_{\mathbb{C}} \Gamma \left( M, \mathcal{I}_{(qp^2-p_0)}(\tilde{\psi} + \chi (qp^2 (L + F))) \right) \geq 1
\]

and

\[
\dim_{\mathbb{C}} \Gamma \left( M, \mathcal{I}_{(qp^2-p_0)}(\tilde{\psi} + \chi (qp^2 L)) \right) \geq 1.
\]

This concludes the proof of our general non-vanishing theorem (6.2) by induction.

The next step for the proof of the finite generation of the canonical ring for the case of the general type is to verify the achievement of stable vanishing order at a generic point of a subvariety of codimension at least two by a finite sum of the absolute-value squares of multi-valued holomorphic pluricanonical sections. This we are going to do in the next section using holomorphic families of Artinian subschemes.

§7. Holomorphic Family of Artinian Subschemes and Achievement of Stable Vanishing Orders for the Case of Higher Codimension.

(7.1) Graded Coherent Ideal Sheaves. To study the problem of achieving the stable vanishing orders across a higher codimensional subvariety at a generic point, we have to find the analog for the following two procedures for the codimension one case.
(i) When $Y$ is an irreducible codimension-one Lelong set in $X$ for the metric $e^{-\varphi} = \frac{1}{\Phi}$ of $K_X$ with generic Lelong number $\gamma$, we have to modify the metric $e^{-\varphi}$ of $K_X$ to get the metric $\frac{e^{-\varphi}}{|s_Y|^2}$ of $K_X - \gamma Y$ before restricting it to $Y$, where $s_Y$ is the canonical section for the line bundle $Y$ on $X$. This is the procedure to remove the vanishing along $Y$ by dividing by $(s_Y)^\gamma$. The problem is to find the corresponding procedure in the case of the higher-codimensional subvariety $V$ to remove the vanishing along the subvariety $V$. Of course, the simplest way is to blow up to replace the higher-codimensional subvariety $V$ by a hypersurface, but the difficulty of using blow-ups is the inability to know definitely that we can stop in a finite number of steps. Without blowing up we can use descending induction on the dimension of the higher-codimensional subvariety to finish in a finite number of stops. The procedure we are going to remove the vanishing along a higher-codimensional subvariety $V$ involves graded coherent ideal sheaves, which we will explain in details shortly.

(ii) The proof of the finite generation of the canonical ring uses descending induction on the dimension of the subvariety where the stable vanishing order is yet achieved by a finite partial sum of $\Phi$. For each subvariety $V$ in the induction process we have to identify a subvariety $Z$ of codimension at least 1 in $V$ so that at points of $V - Z$ we can prove that the stable vanishing order is achieved by a finite partial sum of $\Phi$. In the case of a hypersurface $Y$ with generic Lelong number $\gamma$ of $\Theta_{\varphi}$ (with $e^{-\varphi} = \frac{1}{\Phi}$) instead of a higher-codimensional subvariety $V$, we use the decomposition of $\Theta_{\varphi} - \gamma [Y] = \sum_{j=1}^J \tau_j V_j + R$ and show that the stable vanishing order is achieved at some point outside of any $V_j$ where the Lelong number of $\Theta_{\varphi} - \gamma [Y]$ (and hence of the remainder $R$ also) is zero. In the case of a higher-codimensional $V$ instead of a hypersurface $Y$, the vanishing orders across $V$ in the normal directions of $V$ are no longer given by a single number. Instead the analog is an Artinian subscheme in the normal directions of a generic point of $V$ (or more precisely an infinite sequence of Artinian subschemes because, unlike the situation with nonnegative numbers in the codimension one case, it makes no sense for us to take roots of Artinian subschemes and then go to the limit). To locate a priori a subvariety $Z$ of codimension at least 1 in $V$ for proving the achievement of stable vanishing order by a finite partial sum of $\Phi$ at points of $V - Z$, we will introduce the procedure of detecting a jump in the structure of an Artinian subscheme in a holomorphic family. We will explain later this procedure.
Definition of Graded Coherent Ideal Sheaves, Their Finite Generation, Conductors, and Order Functions.

By a sequence of graded coherent ideal sheaves we mean a sequence of coherent ideal sheaves \( J(\nu) \) indexed by \( \nu \in \mathbb{N} \) such that \( J(\lambda) J(\nu) \subset J(\lambda+\nu) \) for \( \lambda, \nu \in \mathbb{N} \).

For a sequence of graded coherent ideal sheaves \( J(\nu) \) indexed by \( \nu \in \mathbb{N} \) we define its order function as

\[
\sum_{\nu=1}^{\infty} \varepsilon_{\nu} \sum_{j=1}^{p_{\nu}} \left| g^{(\nu)}_{j} \right|^2,
\]

where \( g^{(\nu)}_{1}, \ldots, g^{(\nu)}_{p_{\nu}} \) are local generators of the coherent ideal sheaf \( J(\nu) \) and \( \{\varepsilon_{\nu}\}_{\nu \in \mathbb{N}} \) is a sequence of positive numbers decreasing to 0 so fast that the infinite sequence locally converges. An order function is only locally defined and is not unique and we are interested only in its vanishing order. For our purpose we can use any function which is comparable to an order function in the sense that locally one function is some positive constant times the other.

A sequence of graded coherent ideal sheaves \( \{J(\nu)\}_{\nu \in \mathbb{N}} \) is said to be finitely generated on an open subset \( U \) if there exists some \( \nu_0 \) such that every \( J(\nu) \) is generated on \( U \) by elements \( \Gamma(U, J(\lambda)) \) for \( 1 \leq \lambda \leq \nu_0 \) in the sense of a ring.

Let \( \{J(\nu)\}_{\nu \in \mathbb{N}} \) and \( \{K(\nu)\}_{\nu \in \mathbb{N}} \) be two sequences of graded coherent ideal sheaves. By the conductor from \( \{J(\nu)\}_{\nu \in \mathbb{N}} \) into \( \{K(\nu)\}_{\nu \in \mathbb{N}} \) we mean the sequence of graded coherent ideal sheaves \( \{L(\nu)\}_{\nu \in \mathbb{N}} \) defined as follows. A holomorphic function germ \( f \) at a point \( P_0 \) belongs to the stalk \( (L(\nu))_{P_0} \) of \( L(\nu) \) at \( P_0 \) if and only if \( f(J(\lambda))_{P_0} \subset (K(\lambda+\nu))_{P_0} \) for any \( \lambda \in \mathbb{N} \).

(7.2) Examples of Graded Coherent Ideal Sheaves and the Motivations for Them. The most important example which motivates the introduction of a sequence of graded coherent ideal sheaves comes from our compact complex algebraic manifold \( X \) of general type. Let \( J^{(\nu)}_{K_X} \) be the coherent ideal sheaf on \( X \) generated locally by elements of \( \Gamma(X, \nu K_X) \) for \( \nu \in \mathbb{N} \). The collection \( \{J^{(\nu)}_{K_X}\}_{\nu \in \mathbb{N}} \) so defined is a sequence of graded coherent ideal sheaves on \( X \). The function \( \Phi \) is an order function for this sequence of graded coherent ideal sheaves on \( X \). The main purpose of introducing sequences of graded coherent ideal sheaves is to study the achievement of the vanishing orders of an order function by its finite partial sum.
Suppose $Y$ is a Lelong hypersurface with vanishing order $\gamma$ for $\Phi$. We introduce the sequence of graded coherent ideal sheaves $\{\mathcal{J}_{Y,\gamma}^{(\nu)}\}_{\nu \in \mathbb{N}}$ so that $\mathcal{J}_{Y,\gamma}^{(\nu)}$ is the coherent ideal sheaf on $X$ locally generated by $(s_Y)^{\lceil \nu \gamma \rceil}$. Let $\{\mathcal{J}_{K_X,Y,\gamma}^{(\nu)}\}_{\nu \in \mathbb{N}}$ be the conductor from $\{\mathcal{J}_{Y,\gamma}^{(\nu)}\}_{\nu \in \mathbb{N}}$ into $\{\mathcal{J}_{K_X}^{(\nu)}\}_{\nu \in \mathbb{N}}$. This conductor $\{\mathcal{J}_{K_X,Y,\gamma}^{(\nu)}\}_{\nu \in \mathbb{N}}$ is what is left from removing the vanishing order along $Y$ from $\mathcal{J}_{K_X}^{(\nu)}$. The function $\frac{\Phi}{|s_Y|^{\nu \gamma}}$ is comparable to an order function of the conductor $\{\mathcal{J}_{K_X,Y,\gamma}^{(\nu)}\}_{\nu \in \mathbb{N}}$.

We now come to the higher codimensional case which is actually the real reason to introduce the notion of a sequence of graded coherent ideal sheaves and its order function. Suppose the precise achievement of vanishing order of $\Phi$ by one of its finite partial sum is known at points outside a countable union of subvarieties of codimension $\ell$ in $X$ (and hence outside a single subvariety of codimension $\ell$ in $X$, because the Noetherian argument applies when the precise achievement of stable vanishing order is known for subvarieties of lower codimension). For example, in the case of $\ell = 2$ we know the stable vanishing order $\gamma_j$ for a hypersurface $Y_j$ is known to be achieved and the set of all such $Y_j$ is indexed by $1 \leq j \leq N$. Just as above for the case of $N = 1$, we can introduce the sequence of graded coherent ideal sheaves $\{\mathcal{J}_{Y_1,\gamma_1,\ldots,Y_N,\gamma_N}^{(\nu)}\}_{\nu \in \mathbb{N}}$ so that $\mathcal{J}_{Y_1,\gamma_1,\ldots,Y_N,\gamma_N}^{(\nu)}$ is the coherent ideal sheaf on $X$ locally generated by $\prod_{j=1}^{N} (s_{Y_j})^{\lceil \nu \gamma_j \rceil}$.

Under the assumption of the precise achievement of stable vanishing order at points outside a subvariety of codimension $\ell$ in $X$, in the case of $\ell \geq 3$, the analog of $\{\mathcal{J}_{Y_1,\gamma_1,\ldots,Y_N,\gamma_N}^{(\nu)}\}_{\nu \in \mathbb{N}}$ will be a finitely generated sequence of graded coherent ideal sheaves $\{\mathcal{J}_{(\text{codim} \geq \ell - 1)}^{(\nu)}\}_{\nu \in \mathbb{N}}$ which will generate the precise vanishing orders of $\{\mathcal{J}_{K_X}^{(\nu)}\}_{\nu \in \mathbb{N}}$ across subvarieties of codimension $\geq \ell - 1$ in $X$, both isolated and embedded. Let $\{\mathcal{J}_{(K_X,\text{codim} \geq \ell - 1)}^{(\nu)}\}_{\nu \in \mathbb{N}}$ be the conductor from $\{\mathcal{J}_{(\text{codim} \geq \ell - 1)}^{(\nu)}\}_{\nu \in \mathbb{N}}$ into $\{\mathcal{J}_{K_X}^{(\nu)}\}_{\nu \in \mathbb{N}}$. This conductor $\{\mathcal{J}_{(K_X,\text{codim} \geq \ell - 1)}^{(\nu)}\}_{\nu \in \mathbb{N}}$ is what is left from removing the vanishing order along all subvarieties of codimension $\geq \ell - 1$ from $\mathcal{J}_{K_X}^{(\nu)}$. An order function $\Phi_{K_X,\ell}$
of the conductor \( \bigl\{ J^{(\nu)}_{(K_X, \text{codim} \geq \ell - 1)} \bigr\}_{\nu \in \mathbb{N}} \) is for us to study the achievement of the vanishing orders of \( \Phi \) by one of its finite partial sums for the next step in the descending induction process for codimension at least \( \ell \).

(7.4) Artinian Subschemes Defined by Sequences of Graded Coherent Ideal Sheaves. We continue with the notations introduced in (7.4.1). Suppose there is an irreducible subvariety \( V \) of codimension \( \ell \) which is a branch of the zero-set of an order function \( \Phi_{K_X, \ell} \) of the sequence \( \bigl\{ J^{(\nu)}_{(K_X, \text{codim} \geq \ell - 1)} \bigr\}_{\nu \in \mathbb{N}} \) of graded coherent ideal sheaves on \( X \). We are going to introduce Artinian subschemes transversal to a generic point of \( V \). For the subvariety \( V \) of codimension \( \ell \) in \( X \) these Artinian subschemes will play the rôle which the Lelong number plays for the case of a hypersurface \( Y \). The Artinian subschemes used at this point are unreduced complex subspaces of \( \mathbb{C}^\ell \) supported at the origin.

(7.4.1) Construction of Artinian Subschemes. The sequence of Artinian subschemes are introduced as follows. What we consider now is only locally in \( X \) at a generic point \( P_0 \) of \( V \). (Later when we have to go back to the global situation, we will at that point indicate our return to global consideration.) Since the environment now is local, we can consider the situation in which \( X \) is replaced by some relatively compact connected open neighborhood \( U \) of the origin in \( \mathbb{C}^n \) and \( V \) is replaced by \( \{ z_1 = \cdots = z_\ell = 0 \} \cap U \) with \( z_j (P_0) = 0 \) for \( 1 \leq j \leq n \). Let \( g_1^{(\nu)}, \cdots, g_p^{(\nu)} \) be holomorphic functions on \( U \) which generate the coherent ideal sheaf \( J^{(\nu)}_{(K_X, \text{codim} \geq \ell - 1)} \) on \( U \).

We will later let \( P_0 \) vary in \( V \). So instead of fixing \( P_0 \) in \( V \) we consider a variable point \( P \) in a neighborhood of \( P_0 \) in \( V \). Fix \( P \in V \). Let \( W_P \) be the \( \ell \)-dimensional submanifold in \( \mathbb{C}^n \) defined by

\[
z_{\ell + 1} = z_1 (P), \cdots, z_n = z_n (P)
\]

which we identify with \( \mathbb{C}^\ell \). Let \( N_0 \in \mathbb{N} \) be chosen so that the common zero-set of

\[
\bigl\{ g_j^{(\nu)} \bigr\}_{1 \leq j \leq p, 1 \leq \nu \leq N_0}
\]

in \( U \) is \( V \). (Here in some steps we may have to replace \( U \) by a slightly smaller neighborhood of the origin in \( U \) and for notational simplicity when there is no risk of confusion we will just use the same letter \( U \) to denote this slightly smaller neighborhood of the origin in \( U \).)
We consider for every $N \geq N_0$ and $P \in V$ the Artinian subscheme $A_{V,N,P}$ in $\mathbb{C}^\ell$ with coordinates $z_1, \cdots, z_\ell$ defined by the ideal $\mathcal{K}_{V,N,P}$ generated by
\[
\left\{ \left( g_j^{(\nu)} \right)^{\frac{N!}{\nu}} \bigg|_{W_P \cap U} \right\}_{1 \leq j \leq p, 1 \leq \nu \leq N}.
\]
We raise the holomorphic function $g_j^{(\nu)}$ to the $\left( \frac{N!}{\nu} \right)$-th power in the definition of $\mathcal{K}_{V,N,P}$ and $A_{V,N,P}$ solely for the technical reason of changing fractional powers of holomorphic functions to just holomorphic functions. Of course, later when we consider an order function for $\mathcal{J}_{(K_X, \text{codim} \geq \ell - 1)}^{(\nu)}_{\nu \in \mathbb{N}}$ we will have to return to the original situation by taking the $(N!)$-th root.

(7.5) Purpose for Introducing Artinian Subschemes. Our purpose of introducing the sequence of Artinian subschemes $A_{V,N,P}$ is to locate the countable union of proper subvarieties $Z_N$ in $V$ so that the structure of the Artinian subscheme $A_{V,N,P}$ varies continuously without jump when $P \in V - Z_N$. Let us first give here the precise definition of a family of Artinian scheme “varying continuously without jump.”

(7.5.1) Definition of the Continuous Variation of an Artinian Subscheme Without Jump. Let $\mathfrak{M}$ be the moduli space of all Artinian subschemes on $W_{P_0}$ supported at the single point $P_0$. We can decompose $\mathfrak{M}$ as the disjoint union of local submanifolds $\mathfrak{M}_\nu$ (indexed by $\nu \in \mathcal{J}$) inside $\mathfrak{M}$. Let $A_P$ be an Artinian subscheme on $W_P$ (supported at the single point $P$) parametrized holomorphically by $P \in V$. We identify naturally $W_P$ with $W_{P_0}$ so that we regard $A_P$ naturally as an Artinian subscheme on $W_{P_0}$ supported at the single point $P_0$. Let $\check{Z}$ be a subvariety of $V$. We say that the Artinian subscheme $A_P$ varies continuously without jump for $P \in V - \check{Z}$ if for some $\nu_0 \in \mathcal{J}$ each $A_P$ is an element in $\mathfrak{M}_{\nu_0}$ for $P \in V - \check{Z}$. Equivalently we also refer to it as the variation of the Artinian subscheme $A_P$ being continuous without jump for $P \in V - \check{Z}$.

The reason for considering the proper subvarieties $Z_N$ in $V$ is as follows. In the case of a hypersurface $Y$ in $X$ we consider the vanishing order of $\Phi$ across $Y$ and after we remove the generic vanishing order $\gamma$ we end up with the current $\Theta_{\log \Phi} - \gamma Y$ whose restriction to $Y$ has the decomposition
\[
\Theta_{\log \Phi} - \gamma Y = \sum_{j=1}^{J} \tau_j [V_j] + R,
\]
where the Lelong number of \( R \) is zero except outside a countable union of subvarieties of codimension at least two in \( Y \).

For the hypersurface case, each Artinian subscheme \( \mathcal{A}_{Y,N,P} \) (or \( \mathcal{K}_{Y,N,P} \)) is just represented by a single positive integer \( \gamma_N \) and the limit of the sequence of Artinian subschemes \( \{ \mathcal{A}_{Y,N,P} \}_{N \geq N_0} \) (each one as a single positive integer \( \gamma_N \)) after normalization by the factor \( \frac{1}{N!} \) is the number \( \gamma \) at a generic point \( P \) of \( Y \) so that \( \lim_{N \to \infty} \frac{\gamma_N}{N!} = \gamma \). We are not as interested in the number \( \gamma \) as in the union \( Z \) of \( \bigcup_{j=1}^{\infty} V_j \) and the Lelong sets of \( R \), which is a countable union of proper subvarieties of \( Y \). The significance of this countable union \( Z \) of proper subvarieties of \( Y \) is that at a point \( P \) of \( Y \) outside this countable union \( Z \) we can get a holomorphic section of \( m(K_X - \gamma Y) \) for some sufficiently large \( m \) which is nonzero at \( P \). For the hypersurface case this means that at a point of \( Y - Z \) the infinite sum \( \Phi \) is comparable to one of its finite truncations in some open neighborhood of that point.

In the case of higher codimension what goes into replacing the set \( Z \) is the union \( \bigcup_{N=N_0}^{\infty} Z_N \). Since excluding a countable union (instead of a finite union) of proper subvarieties is good enough for the purpose of showing the existence of some points where the infinite sum \( \Phi \) is comparable to one of its finite truncations, there is no need for us to consider the analog of the limit \( \gamma \) and we can just take the entire sequence of Artinian subschemes \( \{ \mathcal{A}_{V,N,P} \}_{N \geq N_0} \) and work with the union \( \bigcup_{N=N_0}^{\infty} Z_N \) without taking any limit. For the case of the hypersurface \( Y \), the key point is to be able to find beforehand a countable union \( Z \) of proper subvarieties in \( Y \) which contain the Lelong sets of \( \Theta_{\log \Phi} - \gamma Y \) without any knowledge of \( \gamma \), because in the case of higher codimension there is no single number \( \gamma \) and it is difficult to make any good sense of the limit, as \( N \to \infty \), of a sequence of Artinian subschemes \( \{ \mathcal{A}_{V,N,P} \}_{N \geq N_0} \), even after normalization by the factor \( \frac{1}{N!} \).

(7.5.2) Continuous Variation of Relative Positions of Artinian Subschemes Without Jump. In the case of handling the precise achievement of vanishing orders of \( \Phi \) at a generic point of a subvariety \( V \) of higher codimension \( \ell > 1 \), there is also another contribution which goes into the \( a \ priori \) “bad” set in \( V \) other than the union \( \bigcup_{N=N_0}^{\infty} Z_N \). The reason is as follows. When we restrict the curvature current \( \Theta_{\varphi} \) of the metric \( e^{-\varphi} = \frac{1}{\Phi} \) of \( K_X \) to an irreducible subvariety \( S \) of codimension \( \ell - 1 \) in \( X \) which contains \( V \) and when, after subtracting an appropriate multiple of \( V \) from \( \Theta_{\varphi} \big| _S \), we restrict it to \( V \), the resulting closed positive (1,1)-current \( (\Theta_{\varphi} \big| _S - \gamma_S [V]) \big| _V \) on \( V \) has Lelong
sets $Z_S$ in $V$ which may move inside $V$ according to the subvariety $S$ of $X$. These Lelong sets $Z_S$ are also “bad” sets which we would like to locate in an a priori manner to make sure that they are contained in some fixed countable union of subvarieties of codimension $\geq 1$ in $V$ as $S$ goes through a family of such $(\ell - 1)$-codimensional subvarieties $S$ in $X$ parametrized by the space $\mathbb{P}_{\ell-1}$ of normal directions of $V$. In order to accomplish this, we have to consider a generalization of the continuous variation of Artinian sub schemes without jump. The generalization is the continuous variation of relative positions of Artinian subschemes whose precise meaning is as follows.

Instead of one single Artinian subscheme $A_P$ on $W_P$ supported at a single point parametrized holomorphically by $P \in V$, we consider several Artinian subscheme $A_P^{(\lambda)}$ on $W_P$ supported at a single point parametrized holomorphically by $P \in V$ for $0 \leq \lambda \leq \rho$. We also assume that we have the following inclusion relation

$$(7.5.2.1)_P \quad A_P^{(\rho)} \subset A_P^{(\rho-1)} \subset A_P^{(2)} \subset \cdots \subset A_P^{(1)} \subset A_P^{(0)}$$

Again we naturally identify $W_P$ with $W_P^0$ so that each $A_P^{(\lambda)}$ can be naturally regarded as an Artinian subscheme on $W_0$ supported at the single point $P_0$.

Let $\mathcal{M}^{(\rho)}$ be the moduli space of all nested sequences $(7.5.2.1)_P$ of $\rho+1$ Artinian subschemes on $W_{P_0}$, all supported at the single point $P_0$. We can decompose $\mathcal{M}^{(\rho)}$ as the disjoint union of local submanifolds $\mathcal{M}^{(\rho)}_\nu$ (indexed by $\nu \in \mathcal{J}^{(\rho)}$) inside $\mathcal{M}^{(\rho)}$. Let $\tilde{Z}^{(\rho)}$ be a subvariety of $V$. We say that the nested sequence $(7.5.2.1)_P$ of $\rho+1$ Artinian subschemes varies continuously without jump for $P \in V - \tilde{Z}^{(\rho)}$ if for some $\nu_0 \in \mathcal{J}^{(\rho)}$ each nested sequence $(7.5.2.1)_P$ is an element in $\mathcal{M}^{(\rho)}_{\nu_0}$ for $P \in V - \tilde{Z}^{(\rho)}$. Equivalently we also refer to it as the variation of the nested sequence $(7.5.2.1)_P$ of $\rho+1$ Artinian subschemes being continuous without jump for $P \in V - \tilde{Z}^{(\rho)}$.

The notion of continuous nested sequences of Artinian subschemes without jump will be used in the following context. Recall that we are still in the local situation where $X$ is replaced by an open subset $U$ of $\mathbb{C}^n$. We assume that $h_1, \ldots, h_\rho$ are holomorphic functions on $U$ so that their common zero-set contains $V$ and is of codimension $\rho$ in $U$. Let $h_{j,P}$ be the restriction of $h_j$ to $W_P \cap U$. Let $A_P$ be an Artinian subscheme on $W_P$ (supported at the single point $P$) parametrized holomorphically by $P \in V$. For $0 \leq \lambda \leq \rho$ we
introduce the Artinian subscheme

\[(7.5.2.2) \quad \mathcal{A}_p^{(\lambda)} = \mathcal{A}_p / \sum_{j=1}^{\lambda} h_{j,p} \mathcal{A}_p\]

so that \(\mathcal{A}_p^{(0)} = \mathcal{A}_p\). The notion of continuous nested sequences of Artinian subschemes without jump will be applied to (7.5.2.2) and, in particular, to the case where \(\mathcal{A}_p = \mathcal{A}_{V,N,P}\) for each \(N \in \mathbb{N}\).

When \(\rho = \ell\) and \(\mathcal{A}_p = \mathcal{A}_{V,N,P}\) for each \(N \in \mathbb{N}\), let \(Z_{N,h_1,\ldots,h_\ell}\) be a subvariety of codimension \(\geq 1\) in \(V\) such that the nested sequence (7.5.2.1) of \(\rho + 1\) Artinian subschemes \(\text{varies continuously without jump}\) for \(P \in V - \hat{Z}_{N,h_1,\ldots,h_\ell}\). Let \(Z^*\) be the union of \(Z_N\) and \(Z_{N,h_1,\ldots,h_\ell}\) for all \(N \geq N_0\).

(7.6) Family of Subvarieties of One Dimension Higher Containing Embedded Component of Stable Base-Set. After using the local situation to discuss sequences of graded coherent ideal sheaves, their order functions, and the continuous variation of a nested sequence of Artinian subschemes without jump, we now come back to the global situation. Assume that vanishing orders of \(\Phi\) have been achieved outside a subvariety of codimension \(\geq \ell\) in \(X\). That is, there exists some subvariety \(\hat{V}\) of codimension \(\geq \ell\) such that for every point \(P \in X - \hat{V}\) there exist some open neighborhood \(U_P\) of \(P\) in \(X - \hat{V}\) and some positive number \(C_P\) and some \(\hat{m}_\ell \in \mathbb{N}\) such that

\[\frac{1}{C_P} \Phi_{\hat{m}_\ell} \leq \Phi \leq C_P \Phi_{\hat{m}_\ell} \quad \text{on} \quad U_P,
\]

where \(\Phi_{\hat{m}_\ell}\) is the \(\hat{m}_\ell\)-th partial sum of \(\Phi\) as explained in (3.6). Denote \(((n+2)\hat{m}_\ell)!\) by \(m_\ell\). (The factor \(n + 2\) comes from the theorem of Skoda on ideal generation (1.1) and the factorial is to get a uniform grading for the pluricanonical sections used in ideal generation.)

Let \(\mathcal{J}_\ell\) be the ideal sheaf on \(X\) generated by \(\Gamma (X, m_\ell K_X)\). We blow up \(X\) by using monoidal transformations with nonsingular centers inside the zero-set of \(\mathcal{J}_\ell\) to get \(\pi: \tilde{X} \to X\) so that

(i) \(\pi^{-1}(\mathcal{J}_\ell) = \prod_{j=1}^{N} (\mathcal{I}_{Y_j})^{b_j}\) and

(ii) \(K_{\tilde{X}} = \pi^* K_X + \sum_{j=1}^{N} b_j Y_j,\)
where $b_j, b'_j$ are nonnegative integers and \{\(Y_j\)\}_{1 \leq j \leq N} is a collection of nonsingular hypersurfaces in \(\tilde{X}\) in normal crossing and \(I_{Y_j}\) is the ideal sheaf of \(Y_j\).

Let \(\tilde{V}\) be the common zero-set of
\[
\Gamma \left( \tilde{X}, k \left( m_\ell K_{\tilde{X}} - \sum_{j=1}^{N} (b_j + m_\ell b'_j) Y_j \right) \right)
\]
for all \(k \in \mathbb{N}\). Without loss of generality we can assume that \(\tilde{V} = \pi \left( \tilde{V} \right)\), otherwise we just simply replace \(\tilde{V}\) by \(\pi \left( \tilde{V} \right)\) and replace \(\ell\) by the codimension of \(\pi \left( \tilde{V} \right)\) in \(X\). For some \(k_\ell \in \mathbb{N}\) the common zero-set of
\[
\Gamma \left( \tilde{X}, k_\ell \left( m_\ell K_{\tilde{X}} - \sum_{j=1}^{N} (b_j + m_\ell b'_j) Y_j \right) \right)
\]
is \(\tilde{V}\). We take \(\ell\) generic elements \(\tilde{\sigma}_1, \ldots, \tilde{\sigma}_\ell\) of
\[
\Gamma \left( \tilde{X}, k_\ell \left( m_\ell K_{\tilde{X}} - \sum_{j=1}^{N} (b_j + m_\ell b'_j) Y_j \right) \right)
\]
and let \(\sigma_1, \ldots, \sigma_\ell\) be the elements of \(\Gamma \left( X, k_\ell m_\ell K_X \right)\) corresponding to \(\tilde{\sigma}_1, \ldots, \tilde{\sigma}_\ell\).

Note that by blowing up \(\tilde{X}\) further we can also assume without loss of generality that \(\tilde{V}\) is a hypersurface and that all the branches of \(\tilde{V}\) and \(Y_1, \ldots, Y_N\) together are in normal crossing.

Let \(V\) be a branch of \(\tilde{V}\) of codimension \(\ell\) in \(X\). In the notations of (7.5.2) we let \(\rho = \ell\) and \(h_j = \sigma_j\) for \(1 \leq j \leq \ell\) and we get the countable union \(Z^*\) of subvarieties of codimension \(\geq 1\) in \(V\). Let \(c^{(k)} = (c_1^{(k)}, \ldots, c_\ell^{(k)})\) \((1 \leq k \leq \ell - 1)\) be \(\mathbb{C}\)-independent \(\ell\)-tuples of complex numbers and we denote \((c^{(1)}, \ldots, c^{(\ell - 1)})\) by \(\mathbf{c}\). Let \(\tilde{S}_c \subset \tilde{X}\) be the common zero-set of \(\sum_{j=1}^{\ell} c_j^{(k)} \sigma_j\) in \(X\) for \(1 \leq k \leq \ell - 1\). Let \(\tilde{S}_c\) be the topological closure of \(\tilde{S}_c - \tilde{V}\) in \(\tilde{X}\). Let \(S_c \subset X\) be the \(\pi\)-image of \(\tilde{S}_c\). We can consider the element of the Grassmannian of all \((\ell - 2)\)-dimensional linear subspaces \(\mathbb{P}_{\ell-2}\) in \(\mathbb{P}_{\ell-1}\) defined by \(\mathbf{c}\) and naturally regard \(\mathbf{c}\) as an element of \(\mathbb{P}_{\ell-1}\) so that the family of \(S_c\) is parametrized by \(\mathbf{c} \in \mathbb{P}_{\ell-1}\). Each \(S_c\) is a subvariety of pure codimension \(\ell - 1\)
in $X$ for $c \in \mathbb{P}_{\ell-1}$, possibly after excluding a proper subvariety of $\mathbb{P}_{\ell-1}$, in which case we use a finite number of $\ell$-tuples of elements of

$$\Gamma \left( \tilde{X}, k_{\ell} \left( m_{\ell} K_{\tilde{X}} - \sum_{j=1}^{N} (b_j + m_{\ell} b'_j) Y_j \right) \right)$$

instead of a single $\ell$-tuple $(\tilde{\sigma}_1, \cdots, \tilde{\sigma}_\ell)$ so that the intersection of the excluded proper subvarieties of $\mathbb{P}_{\ell-1}$ is empty. In order not to distracted from the important points of the argument, let us suppress the mention of such a proper subvariety of $\mathbb{P}_{\ell-1}$.

Let

$$\Theta_{K_X,\ell} := \Theta \left( -\log \Phi_{K_X,\ell} \right) = \frac{-1}{2\pi} \partial \overline{\partial} \log \Phi_{K_X,\ell}$$

be the curvature current from the metric $1/\Phi_{K_X,\ell}$ defined by the local order function $\Phi_{K_X,\ell}$ of the conductor $\left\{ \mathcal{J}^{(v)}_{(K_X, \text{codim} \geq \ell-1)} \right\}_{v \in \mathbb{N}}$ introduced in (7.3). Note that the order function $\Phi_{K_X,\ell}$ is only locally defined, for example, on some open neighborhood $U_P$ in $X$ of some point $P \in V$. The curvature current $\Theta_{K_X,\ell}$ is only defined on $U_P$. Let $\gamma_\ell$ be the nonnegative number such that

$$(7.6.0.1) \quad (\left( \Theta_{K_X,\ell}|_{S_\ell} \right) - \gamma_\ell [V])_V$$

is a closed positive $(1, 1)$-current on $V \cap U_P$. This we can obtain by using the decomposition of $\Theta_{K_X,\ell}|_{S_\ell}$ according to (3.3), (3.4) and (3.5). Though the order function $\Phi_{K_X,\ell}$ and $\Theta_{K_X,\ell}$ are only defined on $U_P$, yet the nonnegative number $\gamma_\ell$ is globally defined and the irreducible Lelong sets of the closed positive $(1, 1)$-current (7.6.0.1) $V \cap U_P$ are globally defined for $V$. We have the following.

$$(7.6.1) \quad \text{Lemma.} \quad \text{The irreducible Lelong sets of the closed positive (1, 1)-current (7.6.0.1) on } V \cap U_P \text{ are contained in the countable union } Z^* \cap U_P \text{ of subvarieties of codimension } \geq 1 \text{ in } U_P.$$
what genericity is required for this conclusion. The genericity condition is that the point in question has to be in \( V - Z^* \) and the important point is that the genericity condition is independent of \( c \). (Again we may have to confine \( c \) to \( \mathbb{P}_{\ell-1} \) minus a proper subvariety of \( \mathbb{P}_{\ell-1} \) first and then use different proper subvarieties with empty intersection.)

We will apply the general non-vanishing theorem (6.2). The idea is to use, for an appropriate nonnegative number \( \gamma_{\ell} \) and some sufficiently large \( m \), a line bundle \( mK_X - \lfloor m\gamma_{\ell} \rfloor V \) on \( S_\ell \) and get a holomorphic section of it over \( V \) with coefficients belonging to the appropriate multiplier ideal sheaf and then extend it first to \( S_\ell \) and then to an element of \( \Gamma (X, mK_X) \). The number \( \gamma_{\ell} \) may be greater than \( \gamma_{\ell} \) because of the contribution from the ideal sheaf \( \mathcal{J}_{\ell} \) on \( X \) generated by \( \Gamma (X, m\ell K_X) \) introduced in (7.6) from the induction assumption that the stable vanishing orders of \( \Phi \) can be precisely achieved outside of a subvariety of codimension \( \geq \ell \). Note that in order to go to a global closed positive (1,1)-current on \( V \) from the local closed positive (1,1)-current (7.6.0.1) only defined on \( V \cap U_P \), we must add the contribution from the ideal sheaf \( \mathcal{J}_{\ell} \) on \( X \) generated by \( \Gamma (X, m\ell K_X) \).

We have to worry about branches of \( S_\ell \) which are in the zero-set of \( \mathcal{J}_{\ell} \). The easiest way to handle this is to go up to \( \tilde{X} \) and do the extensions in \( \tilde{X} \) instead of in \( X \), because the pullback \( \pi^{-1} (\mathcal{J}_{\ell}) \) to \( \tilde{X} \) of the ideal sheaf \( \mathcal{J}_{\ell} \) on \( X \) is of the form \( \prod_{j=1}^{N} (\mathcal{I}_{Y_j})^{b_j} \) and can be removed from \( \tilde{X} \) as a line bundle. Let \( m = km_\ell \). Earlier, before we worry about branches of \( S_\ell \) which are in the zero-set of \( \mathcal{J}_{\ell} \),

(i) we apply the general non-vanishing theorem (6.2) to get a holomorphic section of \( mK_X - \lfloor m\gamma_{\ell} \rfloor V \) over \( V \) with coefficients in the appropriate multiplier ideal sheaf and

(ii) then extend it first to \( S_\ell \) and

(iii) finally to an element of \( \Gamma (X, mK_X) \).

After we worry about branches of \( S_\ell \) which are in the zero-set of \( \mathcal{J}_{\ell} \), we carry out the equivalent extensions in \( \tilde{X} \) in the following way.

(i) We apply the general non-vanishing theorem (6.2) to get a holomorphic section of

\[
k \left( m\ell K_{\tilde{X}} - \sum_{j=1}^{N} (b_j + m_\ell b'_j) Y_j \right) - \lfloor km\ell \gamma_{\ell} \rfloor V_\ell
\]
over $V_{c}$ with coefficients in the appropriate multiplier ideal sheaf and

(ii) then extend it first to $S_{c}$ and

(iii) then to an element of

$$\Gamma \left( X, k \left( m_{\ell} K_{X} - \sum_{j=1}^{N} (b_{j} + m_{\ell}b_{j}') Y_{j} \right) \right)$$

(iv) and finally multiply it by $\prod_{j=1}^{N} \left( s_{Y_{j}} \right)^{b_{j}}$ to get an element of $\Gamma \left( X, k \left( m_{\ell} K_{X} \right) \right)$, where $s_{Y_{j}}$ is the canonical section of $Y_{j}$.

Thus for each $c \in \mathbb{P}_{\ell-1}$ we succeed in constructing a pluricanonical section of $X$ which will achieve the stable vanishing order of $\Phi$ at a generic point of $V$ in the direction of $S_{c}$ and, moreover, the generic point of $V$ can be chosen to be independent of $c$. This finishes the descending induction on the dimension of the subvariety where the vanishing order of $\Phi$ is not achieved. It finishes also the proof of the finite generation of the canonical ring for the case of general type.

Since in order to rigorously apply the general non-vanishing theorem (6.2) we have to go to the blow-up manifold $\tilde{X}$ where the inverse image $\tilde{V}$ of $V$ under the blow-up map is a nonsingular hypersurface, the question naturally arises whether we are simply applying the hypersurface case argument to $\tilde{V}$ in $\tilde{X}$ to handle the problem of achieving the precise vanishing order of $\Phi$ at a generic point of $V$. What is precisely the rôle played by the Artinian subschemes? Is it necessary to introduce the Artinian subschemes and to use the theory of the continuous variation of an Artinian subscheme without jump? We are going to answer these questions in the next paragraph and highlight the rôle played by the Artinian subschemes and their continuous variation without jump.

(7.8) Difference Between Using Artinian Subschemes and Simply Blowing Up to Reduce to the Hypersurface Case. If we just apply the hypersurface argument to $\tilde{V}$ in $\tilde{X}$, we would consider the closed positive $(1,1)$-current

$$\left( \Theta_{\log \hat{\Phi}} - \tilde{\gamma}_{\tilde{V}} \right)_{\tilde{V}}$$

where the metric $\frac{1}{\Phi}$ of $K_{X}$ is to $X$ as the metric $\frac{1}{\Phi}$ of $K_{X}$ is to $X$. The nonnegative number $\tilde{\gamma}_{\tilde{V}}$ is the Lelong number of the closed positive $(1,1)$-current $\Theta_{\left( -\log \hat{\Phi} \right)}$ on $\tilde{X}$ at a generic point of $\tilde{V}$. We can get the precise
achievement of the vanishing order of $\Phi$ at points of $\tilde{V} - \tilde{Z}$ for some subvariety $\tilde{Z}$ of codimension $\geq 1$ in $\tilde{V}$. However, in general the $\pi$-image $\pi(\tilde{Z})$ in $X$ contains $V$ and we cannot draw any conclusion about the precise achievement of vanishing order of $\Phi$ at a generic point of $V$.

When we use $\tilde{S}_c$ and consider the line bundle

$$k \left( m_\ell K_{\tilde{X}} - \sum_{j=1}^{N} (b_j + m_\ell b'_j) Y_j \right) - [km_\ell \gamma_c] V_c$$

over $V_c$ and the closed positive $(1, 1)$-current

$$\left( k \left( m_\ell \Theta_{(-\log \phi)} - \sum_{j=1}^{N} (b_j + m_\ell b'_j) [Y_j] \right) - [km_\ell \gamma_c] [V_c] \right) |_{V_c}$$

on $V_c$, the nonnegative number $\gamma_c$ depends on $c$ and is in general different from $\tilde{\gamma}_\tilde{V}$ which is independent of $c$, even after we take into account the contributions from $\sum_{j=1}^{N} (b_j + m_\ell b'_j) Y_j$. Let us use the picture in $X$ instead of in $\tilde{X}$ and discount the contribution from the stable vanishing order at a generic point of a subvariety of codimension $\leq \ell - 1$. In this picture what is going on is that $\gamma_c$ is from the computation of the vanishing order of $\Phi$ at a generic point of $V$ along a generic curve inside $S_c$, whereas $\tilde{\gamma}_\tilde{V}$ comes from the computation of the vanishing order of $\Phi$ at a generic point of $V$ along any generic curve in $X$ (without the additional condition that the curve be on $S_c$).

The use of Artinian subschemes and their continuous variation without jump allows us to take away different generic vanishing orders $\gamma_c$ along $V_c$ depending on $c$. Each $V_c$ inside $\tilde{V}$ is mapped onto $V$ by $\pi$, but the subvariety of codimension $\geq 1$ in $V_c$ where the vanishing order along $V_c$ is higher than at a generic point of $V_c$ is mapped under $\pi$ to the countable union $Z^*$ of subvarieties of codimension $\geq 1$ in $V$ which is independent of $c$. The picture is that in the directions normal to $V$ the vanishing order of $\Phi$ depends very much on the direction. The use of Artinian subschemes and their continuous variation without jump allows us to identify precisely the generic vanishing order at a point of $V$ along a generic curve inside $S_c$ in order to achieve that particular vanishing order and, moreover, to do it at points of $V - Z^*$ with the countable union $Z^*$ of subvarieties of codimension $\geq 1$ in $V$ independent of $c$. 

59
§8. Extension Techniques from the Proof of the Invariance of Plurigenera and the Finite Generation of the Canonical Ring. The development of extension techniques for the problem of the deformational invariance of plurigenera was intended for application to the problem of the finite generation of the canonical ring.

(8.1) Heuristic Discussion. The most obvious way of applying the extension techniques from the proof of the deformational invariance of plurigenera to the finite generation of the canonical ring is to try to implement in a rigorous and precise manner the following strategy.

(i) Choose a divisor $Y$ of some pluricanonical section $s_0 \in \Gamma (X, m_0 K_X)$ with $K_Y = (m_0 + 1) K_X|_Y$ and show that

$$\rho : \Gamma (X, m (m_0 + 1) K_X) \to \Gamma (Y, m (m_0 + 1) K_X) = \Gamma (Y, m K_Y)$$

is surjective.

(ii) Since the dimension of $Y$ is one lower than that of $X$, use the induction assumption to conclude that the ring $\bigoplus_{m=1}^\infty \Gamma (Y, m K_Y)$ is finitely generated so that for some $p_1 \in \mathbb{N}$ the ring $\bigoplus_{m=1}^{p_1} \Gamma (Y, m K_Y)$ is generated by its finitely truncated part $\bigoplus_{m=1}^{p_1} \Gamma (Y, m K_Y)$.

(iii) Lift by $\rho$ the generators

$$\sigma^{(m_\nu)}_\nu \in \Gamma (Y, m_\nu K_Y) \quad 1 \leq \nu \leq N$$

(from $\bigoplus_{m=1}^{p_1} \Gamma (Y, m K_Y)$) to the following holomorphic pluricanonical sections on $X$.

$$\tilde{\sigma}^{(m_\nu)}_\nu \in \Gamma (X, m_\nu (m_0 + 1) K_X) \quad 1 \leq \nu \leq N.$$ 

(iv) Take $q \in \mathbb{N}$ and any $s \in \Gamma (X, q (m_0 + 1) K_X)$. In terms of the finite number of generators of $\bigoplus_{m=1}^\infty \Gamma (Y, m K_Y)$ we can write

$$\rho (s) = \sum_{j_1 m_1 + \cdots + j_N m_N = q} c_{j_1, \cdots, j_N} \prod_{\lambda=1}^N \left( \sigma^{(m_\nu)}_\nu \right)^{j_\nu}$$

on $Y$ for some $c_{j_1, \cdots, j_N} \in \mathbb{C}$. 

60
(v) We lift the right-hand side of the above equation via $\rho$ to $X$ and divide by $s_0$ to get

$$s - \sum_{j_1 m_1 + \cdots + j_N m_N = q} c_{j_1, \ldots, j_N} \prod_{\lambda=1}^{N} (\hat{\sigma}_{\nu}^{(m_{\nu})})^{j_{\nu}} = s_0 s'$$

for some $s' \in \Gamma(X, ((q-1)(m_0 +1) +1) K_X)$.

(vi) The natural next step is to apply the preceding argument to $s'$ instead of $s$. However, there is the difficulty of $s'$ being in $\Gamma(X, ((q-1)(m_0 +1) +1) K_X)$ instead of in $\Gamma(X, (q-1)(m_0 +1) K_X)$. To overcome this difficulty, we use the theorem (3.8) on twisted finite generation (after stable vanishing orders are known to be achieved) with $E = K_X$ endowed with the metric $\frac{1}{\hat{F}} = e^{-\psi}$ to conclude that if the stable vanishing orders for $Y$ is achieved by $\tilde{m}_Y$-pluricanonical sections on $Y$, then $\Gamma(X, (m (m_0 +1) m_Y +1) K_X)|_Y$ is contained in

$$\Gamma(Y, m_Y K_Y))^{p_m} \Gamma(Y, ((m - p_m) K_Y + K_X))$$

for $m \geq (n+1)(m_0+1)m_Y$, where $m_Y = \tilde{m}_Y!$ and $p_m = \left\lfloor \frac{m}{(m_0+1)m_Y} \right\rfloor -(n+1)$.

We now needs to extend pluricanonical sections on $Y$ with twisting by $pK_X$ together with supremum bound condition with respect to $\frac{1}{\hat{F}}$ for $p = 1$ and later also for $1 \leq p \leq m_0$. For the deformational invariance of plurigenera this twisting extension result was proved in [Siu 2002]. (Recently Paun [Paun 2005] generalized it to the case of $L^2$ bound.) Thus we can apply the modified argument to $s'$. After a finite number of application of this modified argument we will get the finite generation of the canonical ring of $X$.

The problem arises concerning the problem of extending pluricanonical sections of a hypersurface to pluricanonical sections of the ambient space. We are going to discuss the approach to this problem.

(8.2) **Extension of Pluricanonical Sections From a Hypersurface.** Let $Y$ be a regular hypersurface in $X$ so that the ideal sheaf $\mathcal{I}_Y$ on $X$ is equal to the multiplier ideal sheaf $\mathcal{I}_{\varphi_Y}$ of some metric $e^{-\varphi_Y}$ whose curvature current dominates some smooth positive $(1,1)$-form on $X$. Let $L$ be a line bundle over $X$ with a metric $e^{-\varphi_L}$ whose curvature current is nonnegative. Then the vanishing theorem of Kawamata-Viehweg-Nadel gives

$$H^1(X, \mathcal{I}_{\varphi_L + \varphi_Y} (L + Y + K_X)) = 0.$$
Take the short exact sequence
\[0 \to I_{\phi_L + \phi_Y} \hookrightarrow I_{\phi_L} \to I_{\phi_L} / I_{\phi_L + \phi_Y} \to 0\]
and tensor it with \(L + Y + K_X\) to get the short exact sequence
\[0 \to I_{\phi_L + \phi_Y} (L + Y + K_X) \to I_{\phi_L} (L + Y + K_X)\]
\[\quad \to (I_{\phi_L} / I_{\phi_L + \phi_Y})(L + Y + K_X) \to 0.\]

From its long cohomology exact sequence and the vanishing of first cohomology group in (8.2.1) it follows that
\[\Gamma(X, I_{\phi_L} (L + Y + K_X)) \to \Gamma(X, (I_{\phi_L} / I_{\phi_L + \phi_Y})(L + Y + K_X))\]
is surjective. This is an analog of the extension theorem of Ohsawa-Takegoshi type. The reason for dubbing it as an analog is that we can write \(Y + K_X\) as \(K_Y\) so that we have the surjectivity of
\[\Gamma(X, I_{\phi_L} (L + K_Y)) \to \Gamma(X, (I_{\phi_L} / I_{\phi_L + \phi_Y})(L + K_Y))\]
and we can interpret
\[\Gamma(X, (I_{\phi_L} / I_{\phi_L + \phi_Y})(L + K_Y))\]
as the analog of all top-degree forms on \(Y\) with coefficient in \(L\) which is \(L^2\) with respect to the metric \(e^{-\phi_L}\) of \(L\).

At this point the natural strategy is to apply the “two-tower” extension technique from the proof of the invariance of plurigenera (at least for the case of general type) to our situation at hand to get the surjectivity of
\[\Gamma(X, I_{\phi_L} (L + mK_Y)) \to \Gamma(X, (I_{\phi_L} / I_{\phi_L + \phi_Y})(L + mK_Y))\]
for \(m \geq 1\) (see [Siu 1998], [Siu 2002], [Siu 2003], [Siu 2005], [Paun 2005]). The “two-tower” argument involves

(i) raising the given section to be extended first to a high power \(N\),

(ii) adding the twisting by a sufficiently ample line bundle \(A\) for global generation of multiplier ideal sheaves,
(iii) adding the canonical line bundle one copy at one type from the extension theorem of Ohsawa-Takegoshi type, and

(iv) finally using the $N$-th root of the absolute value of the extension of the twisted sections (or its limit for the case of non general type) to apply the extension theorem of Ohsawa-Takegoshi type to construct the extension of the given section.

The main difficulty in implementing this precisely and rigorously is the following. The coherent sheaf $\mathcal{I}_{\varphi L}/\mathcal{I}_{\varphi L+\varphi Y}$ supported on $Y$ is in general over some unreduced structure sheaf of $Y$ (i.e., a structure sheaf with nonzero nilpotent elements) and is not the multiplier ideal sheaf for some $e^{-\kappa}$ with $\kappa$ locally plurisubharmonic on $Y$. One encounters formidable obstacles when one tries to implement the step of taking roots of absolute value of a section over an unreduced structure. Earlier we have seen this kind of problem with unreduced structures in (4.4) and (6.10), where the problem is handled by the analog of minimal centers of log canonical singularities, but this kind of handling cannot be used here.

To implement precisely and rigorously this approach to the finite generation by extension techniques from the proof of the deformational invariance of plurigenera would involve a tremendous amount of tedious geometrically-uninspiring messy bookkeeping and the task of verifying the correctness of such an implementation after it is done would also be daunting.

For the finite generation of the canonical ring we put aside the approach of extension techniques from the plurigenera problem and use instead the approach presented in these notes for the following reason. The approach presented in these notes is geometrically more enlightening. It gives us a clear geometric picture of how and why a general non-vanishing theorem can be proved from techniques for Fujita conjecture type problems with input from the diophantine-approximation contribution of infinite number of irreducible Lelong sets and Shokurov’s technique of comparing the theorem of Riemann-Roch for a general line bundle and its twisting by a flat line bundle. It also provides a clear geometric picture of how and why we can use Artinian subschemes to give a useful analog of Lelong numbers for the hypersurface and locate an $a$ priori bad set in an embedded stable base-point set of higher codimension by using the continuous deformation of Artinian subschemes without jump so that such $a$ priori bad sets can be used to finish the induction process in a finite number of steps.
§9. Remark on Positive Lower Bound of Curvature Current

The proof of the finite generation of the canonical ring for the case of general type hinges on the small positivity of the curvature currents for certain line bundles. This small positivity is needed both for the general non-vanishing theorem and for the vanishing theorem of Kawamata-Viehweg-Nadel. The question is whether it is possible to prove the finite generation of the canonical ring without the assumption of general type by first artificially introducing some positive line bundle and then later using some limit process to get rid of the artificially-introduced positivity of the curvature current.

The proof of the deformational invariance of the plurigenera for the general case without the assumption of general type follows this particular strategy of artificially introducing some positivity and then getting rid of it by taking limit. The challenge is to handle well the limiting process. This challenge manifests itself already in the proof of the deformational invariance of the plurigenera for the general case without the assumption of general type where the convergence of the metric in taking the limit is the key difficulty which has to be handled. For the generation of the canonical ring without the general type assumption the situation of such a limiting process to get a non-vanishing theorem to precisely achieve stable vanishing orders would be far more involved than the situation for the deformational invariance of plurigenera.

In this section we will make some remarks concerning the positive lower bound of the curvature current in two situations related to the proof of the finite generation of the canonical ring.

(9.1) General Non-Vanishing Theorem and Kawamata-Viehweg-Nadel Vanishing as a Pair. There are two fundamental theorems in the Oka-Cartan theory of several complex variables, called Theorem A and Theorem B. Theorem B states that $H^\nu(S, \mathcal{F}) = 0$ for $\nu \geq 1$ when $S$ is a Stein manifold (or space in general) and $\mathcal{F}$ is a coherent analytic sheaf on $S$. Theorem A states that any coherent analytic sheaf $\mathcal{F}$ on a Stein space $S$ is generated at every point $P$ over the local ring $\mathcal{O}_{S,P}$ by elements of $\Gamma(S, \mathcal{F})$. Theorem A and Theorem B come together as a pair of fundamental results in the theory of Stein spaces.

In the context of compact complex algebraic manifolds and multiplier ideal sheaves there are two results in a pair which are analogous to Theorem
A and Theorem B in the theory of Stein spaces. The first result in the pair is the vanishing theorem of Kawamata-Viehweg-Nadel. It applies to a holomorphic line bundle $L$ over a compact complex algebraic manifold $Y$ with metric $e^{-\varphi}$ whose curvature current admits a strict positive lower bound (in the sense that it dominates some positive smooth $(1,1)$-form). The other result in the pair is the theorem on the global generation of multiplier ideal sheaves. It was first introduced to prove the deformational invariance of plurigenera [Siu 1998]. Its statement is as follows.

(9.1.1) Theorem on Global Generation of Multiplier Ideal Sheaves. Let $Y$ be a compact complex projective algebraic manifold of complex dimension $m$, and $L$ be a holomorphic line bundle on $Y$ with metric $e^{-\varphi}$ where $\varphi$ is plurisubharmonic. Let $E$ be a holomorphic line bundle over $Y$ sufficiently ample in the sense that for every $P \in Y$ there exist a finite number of elements of $\Gamma(Y,E)$ which vanish to order $\geq m+1$ at $P$ and do not vanish simultaneously outside $P$. Then $\Gamma(Y, \mathcal{I}_\varphi \otimes (L+E+K_Y))$ generates $\mathcal{I}_\varphi \otimes (L+E+K_Y)$.

For the global generation of multiplier ideal sheaves, a sufficiently ample line bundle $E$ is required. If we get rid of $E$ by replacing $L$ by $L-E$, the global generation of multiplier ideal sheaves can also be formulated with a trivial line bundle $E$ but the new formulation would instead require the curvature current $e^{-\varphi}$ of $L$ to dominate the curvature form of some sufficiently ample line bundle (see the proof of (9.2.1)).

Though the vanishing theorem of Kawamata-Viehweg-Nadel and the global generation of multiplier ideal sheaves form a pair, yet their assumptions on the positive lower bound for the curvature current are not comparable. For the vanishing theorem of Kawamata-Viehweg-Nadel any small positive lower bound suffices. However, for the global generation of multiplier ideal sheaves the lower bound for the curvature current has to be sufficiently positive.

For the proof of the deformational invariance of plurigenera, the difficulty from the undesirable assumption of sufficient positivity of the curvature current for the global generation of multiplier ideal sheaves is alleviated by taking roots of sufficiently high power. This technique of alleviating the difficulty by taking roots suffices for the proof of the deformational invariance of plurigenera, because the problem of the deformational invariance of plurigenera is less delicate than the problem of the finite generation of the canonical ring.
The general non-vanishing theorem (6.2) which we introduce for the purpose of proving the finite generation of the canonical ring for the case of general type fits better with the theorem of Kawamata-Viehweg-Nadel as a pair. Like the theorem of Kawamata-Viehweg-Nadel the general non-vanishing theorem (6.2) only requires some positivity, no matter how small, for the curvature current of $e^{-\chi}$. Of course, in the proof of the general non-vanishing theorem (6.2) we have to take some high-order root of some holomorphic section vanishing to high order. However, such a root-taking step is done in the proof instead of in the application of the theorem.

(9.2) No Strict Positive Lower Bound for Curvature Current of $\frac{1}{\Phi}$ with Hypersurface Lelong Set. The metric $e^{-\phi} = \frac{1}{\Phi}$ of $K_X$ used in the proof of the finite generation of the canonical ring is the metric with the least singularity. When $X$ is of general type, it is natural to suspect that its curvature current $\Theta_{\phi}$ dominates some positive smooth $(1,1)$-form on $X$. However, this is not necessarily the case. Here is a simple statement which explains why the general type condition does not in general imply strict positivity of the curvature current $\Theta_{\phi}$.

(9.2.1) Proposition. If the curvature current $\Theta_{\phi}$ has a hypersurface Lelong set, then it cannot dominate any positive smooth $(1,1)$-form $\omega_0$ on $X$.

Proof. Suppose the contrary and there is such a positive smooth $(1,1)$-form $\omega_0$ on $X$ such that $\Theta_{\phi} \geq \omega_0$ as $(1,1)$-currents. By assumption we have the following decomposition

$$\Theta_{\phi} = \sum_{j=1}^{J} \tau_j [V_j] + R,$$

where $J \in \mathbb{N} \cup \{\infty\}$, each $\tau_j$ is a positive number, each $V_j$ is an irreducible hypersurface in $X$, and the Lelong number of $R$ is zero outside a countable union of subvarieties of codimension at least two in $X$.

Let $A$ be a holomorphic line bundle on $X$ which is sufficiently ample in the sense that for every $P \in X$ there exist a finite number of elements of $\Gamma(X, A)$ which vanish to order $\geq n + 1$ at $P$ and do not vanish simultaneously outside $P$. Let $h_A$ be a smooth metric of $A$ whose curvature form $\Theta_A$ is positive. Choose $m_0 \in \mathbb{N}$ such that $m_0 \omega_0 - \Theta_A$ is positive on $X$. For $m \geq m_0$ the curvature current of the metric $e^{-\phi} h_A$ of the line bundle $mK_X - A$ is positive.
Since the multiplier ideal sheaf of the metric $e^{-\varphi}$ is $\mathcal{I}_{m\varphi}$, it follows from the theorem on the global generation of the multiplier ideal sheaf that the multiplier ideal sheaf $\mathcal{I}_{m\varphi}$ is generated by

$$\Gamma(X, \mathcal{I}_{m\varphi}((m+1)K_X)) = \Gamma(X, \mathcal{I}_{m\varphi}((mK_X - A) + A + K_X)).$$

Pick a regular point $P_0$ of $V_1$ which is not in any $V_j$ for $j > 1$ such that the Lelong number of $R$ is zero at $P_0$. Then $\mathcal{I}_{m\varphi}$ is equal to the ideal sheaf of $[m\tau_1]V_1$ at $P_0$. Since the multiplier ideal sheaf $\mathcal{I}_{m\varphi}$ is generated by $\Gamma(X, \mathcal{I}_{m\varphi}((m+1)K_X))$, it follows that at $P_0$ the vanishing order along $V_1$ of some element $s$ of $\Gamma(X, \mathcal{I}_{m\varphi}((m+1)K_X))$ is equal to $[m\tau_j]$ which is less than $(m+1)\tau_1$, contradicting the decomposition (9.2.1.1). Q.E.D.

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