ON NORMAL STABILITY FOR NONLINEAR PARABOLIC EQUATIONS

JAN PRÜSS
Institut für Mathematik, Martin-Luther-Universität Halle-Wittenberg
D-06120 Halle, Germany

GERI SIMONETT
Department of Mathematics, Vanderbilt University
Nashville, TN 37240, USA

RICO ZACHER
Institut für Mathematik, Martin-Luther-Universität Halle-Wittenberg
D-06120 Halle, Germany

Abstract. We show convergence of solutions to equilibria for quasilinear and fully nonlinear parabolic evolution equations in situations where the set of equilibria is non-discrete, but forms a finite-dimensional \( C^1 \)-manifold which is normally stable.

1. Introduction

In this short note we consider quasilinear as well as fully nonlinear parabolic equations and we study convergence of solutions towards equilibria in situations where the set of equilibria forms a \( C^1 \)-manifold.

Our main result can be summarized as follows: suppose that for a nonlinear evolution equation we have a \( C^1 \)-manifold of equilibria \( \mathcal{E} \) such that at a point \( u_* \in \mathcal{E} \), the kernel \( N(A) \) of the linearization \( A \) is isomorphic to the tangent space of \( \mathcal{E} \) at \( u_* \), the eigenvalue 0 of \( A \) is semi-simple, and the remaining spectral part of the linearization \( A \) is stable. Then solutions starting nearby \( u_* \) exist globally and converge to some point on \( \mathcal{E} \). This situation occurs frequently in applications. We call it the generalized principle of linearized stability, and the equilibrium \( u_* \) is then termed normally stable.

A typical example for this situation to occur is the case where the equations under consideration involve symmetries, i.e. are invariant under the action of a Lie-group.

The situation where the set of equilibria forms a \( C^1 \)-manifold occurs for instance in phase transitions [13, 25], geometric evolution equations [12, 14], free boundary problems in fluid dynamics [15, 16], stability of traveling waves [26], and models of tumor growth, to mention just a few.

A standard method to handle situations as described above is to refer to center manifold theory. In fact, in that situation the center manifold of the problem

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in question will be unique, and it coincides with $E$ near $u_*$. Thus the so-called *shadowing lemma* in center manifold theory implies the result. Center manifolds are well-studied objects in the theory of nonlinear evolution equations. For the parabolic case we refer to the monographs [17, 20], and to the publications [5, 6, 10, 19, 21, 27, 28].

However, the theory of center manifolds is a technically difficult matter. Therefore it seems desirable to have a simpler, direct approach to the generalized principle of linearized stability which avoids the technicalities of center manifold theory.

Such an approach has been introduced in [26] in the framework of $L^p$-maximal regularity. It turns out that within this approach the effort to prove convergence towards equilibria in the normally stable case is only slightly larger than that for the proof of the standard linearized stability result - which is simple.

The purpose of this paper is to extend the approach given in [26] to cover a broader setting and a broader class of nonlinear parabolic equations, including fully nonlinear equations. This approach is flexible and general enough to reproduce the results contained in [7, 12, 13, 14, 15, 16, 25, 26], and it will have applications to many other problems.

Our approach makes use of the concept of maximal regularity in an essential way. As general references for this theory we refer to the monographs [1, 11, 20].

2. Abstract nonlinear problems in a general setting

Let $X_0$ and $X_1$ be Banach spaces, and suppose that $X_1$ is densely embedded in $X_0$. Suppose that $F: U_1 \subset X_1 \to X_0$ satisfies

$$F \in C^k(U_1, X_0), \quad k \in \mathbb{N}, \quad k \geq 1,$$

where $U_1$ is an open subset of $X_1$. Then we consider the autonomous (fully) nonlinear problem

$$\dot{u}(t) + F(u(t)) = 0, \quad t > 0, \quad u(0) = u_0,$$  \hspace{1cm} (2)

for $u_0 \in U_1$. In the sequel we use the notation $| \cdot |_j$ to denote the norm in the respective spaces $X_j$ for $j = 0, 1$. Moreover, for any normed space $X$, $B_X(u, r)$ denotes the open ball in $X$ with radius $r > 0$ around $u \in X$.

Let $E \subset U_1$ denote the set of equilibrium solutions of (2), which means that

$$u_* \in E \quad \text{if and only if} \quad F(u_*) = 0.$$

Given an element $u_* \in E$, we assume that $u_*$ is contained in an $m$-dimensional manifold of equilibria. This means that there is an open subset $U \subset \mathbb{R}^m$, $0 \in U$, and a $C^1$-function $\Psi: U \to X_1$ such that

- $\Psi(U) \subset E$ and $\Psi(0) = u_*$,
- the rank of $\Psi'(0)$ equals $m$, and
- $F(\Psi(\zeta)) = 0$, $\zeta \in U$.  \hspace{1cm} (3)

We assume further that near $u_*$ there are no other equilibria than those given by $\Psi(U)$, i.e. $E \cap B_{X_1}(u_*, r_1) = \Psi(U)$, for some $r_1 > 0$.

Let $u_* \in E$ be given and set $A := F'(u_*)$. Then we assume that $A \in \mathcal{H}(X_1, X_0)$, by which we mean that $-A$, considered as a linear operator in $X_0$ with domain $X_1$, generates a strongly continuous analytic semigroup $\{e^{-At}; t \geq 0\}$ on $X_0$. In particular we may take the graph norm of $A$ as the norm in $X_1$. 
For the deviation \( v := u - u_* \) from \( u_* \), equation (2) can be restated as
\[
\dot{v}(t) + Av(t) = G(v(t)), \quad t > 0, \quad v(0) = v_0,
\]
where \( v_0 = u_0 - u_* \), and \( G(z) := Az - F(z + u_*) \), \( z \in V_1 := U_1 - u_* \). It follows from (1) that \( G \in C^k(V_1, X_0) \). Moreover, we have \( G(0) = 0 \) and \( G'(0) = 0 \). Setting \( \psi(\zeta) = \Psi(\zeta) - u_* \) results in the following equilibrium equation for problem (4)
\[
Av(\zeta) = G(\psi(\zeta)), \quad \text{for all } \zeta \in U.
\]
Taking the derivative with respect to \( \zeta \) and using the fact that \( G'(0) = 0 \) we conclude that \( Av'(0) = 0 \) and this implies that the tangent space of \( E \) at \( u_* \) is contained in \( \text{ker}(A) \), the kernel of \( A \).
For \( J = [0, a), a \in (0, \infty] \), we consider a pair of Banach spaces \( (E_0(J), E_1(J)) \) such that \( E_0(J) \hookrightarrow L_{1,\text{loc}}(J; X_0) \) and \( E_1(J) \hookrightarrow H_{1,\text{loc}}(J; X_0) \cap L_{1,\text{loc}}(J; X_1) \), respectively. Denoting by \( X_{\gamma} = \gamma E_1 \) the trace space of \( E_1(J) \) we assume that
\[
(A1) \quad \gamma E_1 \text{ is independent of } J, \quad \text{and the embedding } E_1(J) \hookrightarrow BUC(J; X_{\gamma}) \text{ holds.}
\]
In addition, we assume that there is a constant \( c_0 > 0 \) independent of \( J = [0, a), a \in (0, \infty] \), such that
\[
\sup_{t \in J} |w(t)|_\gamma \leq c_0 |w|_{E_1(J)}, \quad \text{for all } w \in E_1(J), \quad w(0) = 0.
\]
We refer to [3, Section III.1.4] for further information on trace spaces. Moreover, we assume that
\[
(A2) \quad \hat{w} \in E_1(J) \text{ and } |w(t)|_0 \leq |\hat{w}(t)|, \ t \in J, \text{ imply } |w|_{E_0(J)} \leq ||\hat{w}|_{E_1(J)}; \text{ for } \omega > 0 \text{ fixed, there exists a constant } c_1 > 0 \text{ not depending on } J \text{ and such that}
\]
\[
\int_J e^{-\omega s}|w(s)|_1 \, ds \leq c_1 |w|_{E_1(J)}, \quad \text{for all } w \in E_1(J),
\]
\[
\int_J e^{-\omega s}|w(s)|_1 \, ds \leq c_1 e^{-\omega t} |w|_{E_1(\mathbb{R}_+)} , \quad \text{for all } w \in E_1(\mathbb{R}_+) \text{ and } t \geq 0.
\]
Our key assumption is that \( (E_0(J), E_1(J)) \) is a pair of maximal regularity for \( A \). To be more precise we assume that
\[
(A3) \quad \text{the linear Cauchy problem } \dot{w} + Aw = g, \ w(0) = w_0 \text{ has for each } (g, w_0) \in E_0(I) \times \gamma E_1(I) \text{ a unique solution } w \in E_1(I), \text{ where } I = [0, T] \text{ is a finite interval.}
\]
We impose the following assumption for the sake of convenience. For all examples that we have in mind the condition can be derived from (A3).
Suppose that \( \sigma(A) \), the spectrum of \( A \), admits a decomposition \( \sigma(A) = \sigma_s \cup \sigma' \), where \( \sigma_s \subset \{ z \in \mathbb{C} : \text{Re} \ z > \omega \} \) for some \( \omega > 0 \) and \( \sigma' \subset \{ z \in \mathbb{C} : \text{Re} \ z \leq 0 \} \). Let \( P_s \) denote the spectral projection corresponding to the spectral set \( \sigma_s \). Then we assume that
\[
(A4) \quad \text{there exists a constant } M_0 > 0 \text{ such that for any } J = [0, a), a \in (0, \infty], \text{ any } \sigma \in [0, \omega], \text{ and any function } g \text{ with } e^{\sigma t} P_s g \in E_0(J) \text{ there is a unique solution } w \text{ of } \dot{w} + Aw = P_s g, \ t \in J, \ w(0) = 0, \text{ satisfying}
\]
\[
|e^{\sigma t} w|_{E_1(J)} \leq M_0 |e^{\sigma t} P_s g|_{E_0(J)};
\]
there exists a constant $M_1 > 0$ such that for any $J = [0, a)$, $a \in (0, \infty]$, and for any $z \in X_\gamma$ there holds
\[
\|e^{\sigma t}e^{-A_s t}P_s z\|_{E_1(J)} + \sup_{t \in J}|e^{\sigma t}e^{-A_s t}P_s z|_{\gamma} \leq M_1|P_s z|_{\gamma}, \quad \sigma \in [0, \omega].
\]
We again refer to [1] Chapter III] for more background information on the notion of maximal regularity. In order to cover the case $X_\gamma \neq X_1$ we assume the following structure condition on the nonlinearity $G$:

(A5) there exists a uniform constant $C_1$ such that for any $\eta > 0$ there is $r > 0$ such that
\[
|G(z_1) - G(z_2)|_0 \leq C_1(\eta + |z_2|_1)|z_1 - z_2|_1, \quad z_1, z_2 \in X_1 \cap B_{X_\gamma}(0, r).
\]
Observe that condition (A5) trivially holds in the case $X_\gamma = X_1$, since $G'(0) = 0$. A short computation shows that condition (A5) is also satisfied if $F$ has a quasilinear structure, i.e. if
\[
F(u) = B(u)u + f(u) \quad \text{for } u \in U_\gamma, \quad (B, f) \in C^1(U_\gamma, B(X_1, X_0) \times X_0), \quad (8)
\]
where $U_\gamma \subset X_\gamma$ is an open set.
Lastly, concerning solvability of the nonlinear problem [1] we will assume that

(A6) given $b > 0$ there exists $r_2 > 0$ such that for any $v_0 \in B_{X_\gamma}(0, r_2)$ problem [1] admits a unique solution $v \in E_1([0, b])$.

Note that since $v = 0$ is an equilibrium of [1], condition (A6) is satisfied whenever one has existence and uniqueness of local solutions in the described class as well as continuous dependence of the maximal time of existence on the initial data.

We conclude this section by describing three important examples of admissible pairs $(E_0(J), E_1(J))$.

**Example 1:** $(L_p$-maximal regularity.) In our first example, the spaces $(E_0(J), E_1(J))$ are given by
\[
E_0(J) := L_p(J; X_0), \quad E_1(J) := H^1_p(J; X_0) \cap L_p(J; X_1). \quad (9)
\]
The trace space is a real interpolation space given by $\gamma E_1 = X_\gamma = (X_0, X_1)_{1-1/p, p}$ and we have $E_1(J) \hookrightarrow BUC(J; X_\gamma)$, see for instance [1] Theorem III.4.10.2. For a proof of [1] we refer to [23] Proposition 6.2. This yields Assumption (A1). For Assumption (A2) we note that
\[
\int_J e^{-\omega s}|w(s)|^1_1 \, ds \leq c_1(\int_J |w(s)|^p_p \, ds)^{1/p} \leq c_1\|w\|_{E_1(J)}
\]
for all $w \in E_1(J)$ by Hölder’s inequality. Moreover,
\[
\int_t^\infty e^{-\omega s}|w(s)|^1_1 \, ds \leq (\int_t^\infty e^{-\omega s} \, ds)^{1/p'} (\int_t^\infty |w(s)|^p_p \, ds)^{1/p'} \leq c_1 e^{-\omega t}\|w\|_{E_1(R_+)}
\]
for $t \geq 0$ and $w \in E_1(R_+)$. We refer to [11] [18] [24], II Section III.4.10 and the references therein for conditions guaranteeing that the crucial Assumption (A3) on maximal regularity is satisfied. It is clear that the property of maximal regularity is passed on from $A$ to $A_s$ in the spaces $E_0^s(J) := L_p(J; X_0^s)$, $E_1^s(J) := H^1_p(J; X_0^s) \cap L_p(J; X_1^s)$, and this implies Assumption (A4), see for instance [1] Remark III.4.10.9(a)]. Assumption (A5) is satisfied in case that the nonlinear mapping $F$ has a quasilinear structure, see [26]. Assumption (A6) follows in case that $F$ has a quasilinear structure from (A3) and [22] Theorem 3.1, see also [2]
Theorem 2.1, Corollary 3.3]. We remark that the case of $L_{\mu}$-maximal regularity has been considered in detail in [20].

**Example 2:** (Continuous maximal regularity).
Let $J = [0, a)$ with $0 < a \leq \infty$ and set $\hat{J} := (0, a)$. For $\mu \in (0, 1)$ and $X$ a Banach space we set
\[
BUC_{1-\mu}(J; X) := \{ u \in C(\hat{J}; X) : [t \mapsto t^{1-\mu}u] \in BUC(J; X), \quad \lim_{t \to 0^+} t^{1-\mu}|u(t)|_X = 0 \},
\]
\[
BUC_0(J; X) := BUC(J; X).
\]

$BUC_{1-\mu}(J; X)$ is turned into a Banach space by the norm
\[
\|u\|_{C_{1-\mu}(J; X)} := \sup_{t \in J} t^{1-\mu}|u(t)|_X, \quad \mu \in (0, 1).
\]
Finally, we set $BUC_{1-\mu}^1(J; X) := \{ u \in C^1(\hat{J}; X) : u, \dot{u} \in BUC_{1-\mu}(J; X) \}$. With these preparations we define
\[
\mathcal{E}_0(J) := BUC_{1-\mu}(J; X_0),
\]
\[
\mathcal{E}_1(J) := BUC_{1-\mu}^1(J; X_0) \cap BUC_{1-\mu}(J; X_1)
\]
endowed with the canonical norms.

Supposing that $\mathcal{H}(X_1, X_0) \neq \emptyset$ the trace space $\gamma\mathcal{E}_1$ is the continuous interpolation space $\gamma\mathcal{E}_1 = (X_0, X_1)_{\mu, \infty} := D_A(\mu)$, and we have the embedding $\mathcal{E}_1(J) \hookrightarrow BUC(J; \gamma\mathcal{E}_1)$, see [1] Theorem III.2.3.3. A proof for estimate (6) can be found in [8, Lemma 2.2(c)], and this shows that Assumption (A1) is satisfied. Assumption (A2) holds as
\[
\int_J e^{-\omega s}|w(s)|_1 \, ds = \int_J \frac{e^{-\omega s}}{s^{1-\mu}} s^{1-\mu}|w(s)|_1 \, ds \leq c_1\|w\|_{C_{1-\mu}(J; X_1)} \leq c_1\|w\|_{\mathcal{E}_1(J)}
\]
for all $w \in \mathcal{E}_1(J)$, and
\[
\int_t^\infty e^{-\omega s}|w(s)|_1 \, ds = \int_t^\infty \frac{e^{-\omega s}}{s^{1-\mu}} s^{1-\mu}|w(s)|_1 \, ds \leq c_1 e^{-\omega t}\|w\|_{\mathcal{E}_1(\mathbb{R}_+)}
\]
for $t \geq 0$ and $w \in \mathcal{E}_1(\mathbb{R}_+)$. It turns out that maximal regularity cannot hold in the class (10) if $X_1 \neq X_0$ and $X_0$ is reflexive. On the other side, there is an interesting class of spaces $(X_0, X_1)$ where Assumption (A3) is indeed satisfied for the pair $(\mathcal{E}_0(J), \mathcal{E}_1(J))$ given in (10), see [3, 8, 9, 20] and [1, Theorem III.3.4.1]. Ass inherits the property of maximal regularity from $A$, and this implies Assumption (A4), see [1, Remark III.3.4.2(b)]. Assumption (A5) holds in the case $\mu = 1$ for any function $G \in C^1(U_1, X_0)$ with $G(0) = G'(0) = 0$. It also holds for $\mu \in (0, 1)$ if the nonlinear function $F$ given in (2) satisfies (5).

If $\mu = 1$ and $k \geq 1$ then it follows from (A3) and [3, Theorem 2.7, Corollary 2.9], see also [20, Section 8.4], that Assumption (A6) is satisfied.

If $\mu \in (0, 1), k \geq 1$ and $F$ has a quasilinear structure, see [3], then Assumption (A6) follows from (A3) and [8, Theorem 5.1], see also [3, Theorem 6.1].
Example 3: (Hölder maximal regularity.)
Suppose $\rho \in (0, 1)$, $I \subset \mathbb{R}_+$, $J \subset \mathbb{R}_+$ are intervals with $0 \in J$. Then we set
\[
\|u\|_{C^{\rho}(J;X)} := \sup_{t\neq s} \frac{|u(t) - u(s)|}{|t-s|^\rho},
\]
\[
[u]_{C^{\rho}_{\epsilon}(J;X)} := \sup_{2\epsilon \in J} \epsilon^{\rho}[u]_{C^{\rho}(\epsilon;2\epsilon;X)},
\]
and
\[
[u]_{C^{\rho}_{\epsilon}(J;X)} := \|u\|_{BC(t;X)} + [u]_{C^{\rho}_{\epsilon}(J;X)},
\]
\[
BC^{\rho}_{\epsilon}(J;X) := \{ u \in C^{\rho}(J;X) : \|u\|_{C^{\rho}_{\epsilon}(J;X)} < \infty \}.
\]
Moreover, we set
\[
BC^{\rho}(J;X) := \{ u \in BC(J;X) \cap BC^{\rho}_{\epsilon}(J;X) : \lim_{\epsilon \to 0^+} \epsilon^{\rho}[u]_{C^{\rho}(\epsilon;2\epsilon;X)} = 0 \}
\]
and equip it with the norm $\| \cdot \|_{C^{\rho}_{\epsilon}(J;X)}$. For the pair $(E_0(J), E_1(J))$ we take
\[
E_0(J) := BC^{\rho}(J;X_0),
\]
\[
E_1(J) := BC_{1+\rho}^{\rho}(J;X_0) \cap BC^{\rho}(J;X_1),
\]
where $BC^{1+\rho}_{\rho}(J;X) := \{ u \in BC^{\rho}_{\rho}(J;X_0) : \dot{u} \in BC^{\rho}_{\rho}(J;X_0) \}$. The spaces in (11) are given their canonical norms, turning them into Banach spaces.

We have $\gamma E_1(J) = X_1$ and it is clear from the definition of (the norm of) $E_1(J)$ that $E_1(J) \hookrightarrow BC(J;X_1)$, and that (11) is satisfied for any $w \in E_1(J)$. This shows that Assumption (A1) holds. By similar arguments as above we see that Assumption (A2) is satisfied as well.

For the crucial Assumption (A3) we refer to [1] Theorem III.2.5.6 with $\mu = 1$; see also [20] Corollary 4.3.6(ii)]. It is worthwhile to mention that this maximal regularity result is true for any $A \in \mathcal{H}(X_1, X_0)$ and any pair $(X_0, X_1)$. Assumption (A4) follows then as above, see [1] Theorem III.2.5.5]. Assumption (A5) holds for any function $G \in (U_1, X_0)$ with $G(0) = G'(0) = 0$.

Finally, it follows from Theorem 8.1.1 and Theorem 8.2.3 in [20] that Assumption (A6) holds for the fully nonlinear problem (2) in case that $k \geq 2$. (In fact, it suffices to require that the derivative $F'$ of $F$ be locally Lipschitz continuous.)

3. The main result

In this section we state and prove our main theorem about convergence of solutions for the nonlinear equation (2) towards equilibria.

Theorem 3.1. Let $u_\ast \in X_1$ be an equilibrium of (2), and assume that the above conditions (A1)-(A6) are satisfied. Suppose that $u_\ast$ is normally stable, i.e. assume that

(i) near $u_\ast$ the set of equilibria $\mathcal{E}$ is a $C^1$-manifold in $X_1$ of dimension $m \in \mathbb{N}$,
(ii) the tangent space for $\mathcal{E}$ at $u_\ast$ is given by $N(A)$,
(iii) $0$ is a semi-simple eigenvalue of $A$, i.e. $N(A) \oplus R(A) = X_0$,
(iv) $\sigma(A) \cap \{0\} \subset \mathbb{C}_+ = \{ z \in \mathbb{C} : \text{Re} z > \omega \}$ for some $\omega > 0$.

Then $u_\ast$ is stable in $X_\gamma$, and there exists $\delta > 0$ such that the unique solution $u(t)$ of (2) with initial value $u_0 \in X_\gamma$ satisfying $|u_0 - u_\ast|_\gamma < \delta$ exists on $\mathbb{R}_+$ and converges at an exponential rate to some $u_\infty \in \mathcal{E}$ in $X_\gamma$ as $t \to \infty$. 

Proof. The proof to Theorem 2.1 will be carried out in several steps, as follows.

(a) We denote by \( P_l, l \in \{c, s\} \), the spectral projections corresponding to the spectral sets \( \sigma_s \) and \( \sigma_c:=\{0\} \), respectively, and let \( A_l = P_l A P_l \) be the part of \( A \) in \( X^l_0 = P_l X_0 \) for \( l \in \{c, s\} \). Note that \( A_c = 0 \). We set \( X^l_j := P_l(X_j) \) for \( l \in \{c, s\} \) and \( j \in \{0, \gamma, 1\} \). It follows from our assumptions that \( X^c_0 = X^s_0 \). In the following we set \( X^c := X^c_\circ \) and equip \( X^c \) with the norm of \( X_0 \). Moreover, we take as a norm on \( X_j \)

\[
|v|_j := |P_c v|_0 + |P_s v|_j \quad \text{for} \quad j = 0, \gamma, 1. \tag{12}
\]

(b) Next we show that the manifold \( \mathcal{E} \) can be represented as the (translated) graph of a function \( \phi: B_{X^c}(0, \rho_0) \to X^c_1 \) in a neighborhood of \( u_\circ \). In order to see this we consider the mapping

\[
g : U \subset \mathbb{R}^m \to X^c, \quad g(\zeta) := P_c \psi(\zeta), \quad \zeta \in U.
\]

It follows from our assumptions that \( g'(0) = P_c \psi'(0) : \mathbb{R}^m \to X^c \) is an isomorphism. By the inverse function theorem, \( g \) is a \( C^1 \)-diffeomorphism of a neighborhood of \( 0 \) in \( \mathbb{R}^m \) onto a neighborhood, say \( B_{X^c}(0, \rho_0) \), of \( 0 \) in \( X^c \). Let \( g^{-1} : B_{X^c}(0, \rho_0) \to U \) be the inverse mapping. Then \( g^{-1} : B_{X^c}(0, \rho_0) \to U \) is \( C^1 \) and \( g^{-1}(0) = 0 \). Next we set \( \Phi(x) := \psi(g^{-1}(x)) \) for \( x \in B_{X^c}(0, \rho_0) \) and we note that

\[
\Phi \in C^1(B_{X^c}(0, \rho_0), X^c_1), \quad \Phi(0) = 0, \quad \{ u_\circ + \Phi(x) : x \in B_{X^c}(0, \rho_0) \} = \mathcal{E} \cap W,
\]

where \( W \) is an appropriate neighborhood of \( u_\circ \) in \( X_1 \). Clearly,

\[
P_c \Phi(x) = ((P_c \circ \psi) \circ g^{-1})(x) = (g \circ g^{-1})(x) = x, \quad x \in B_{X^c}(0, \rho_0),
\]

and this yields \( \Phi(x) = P_c \Phi(x) + P_s \Phi(x) = x + P_c \Phi(x) \) for \( x \in B_{X^c}(0, \rho_0) \). Setting \( \phi(x) := P_c \Phi(x) \) we conclude that

\[
\phi \in C^1(B_{X^c}(0, \rho_0), X^c) \quad \phi(0) = \phi'(0) = 0, \tag{13}
\]

and that \( \{ u_\circ + \phi(x) : x \in B_{X^c}(0, \rho_0) \} = \mathcal{E} \cap W \), where \( W \) is a neighborhood of \( u_\circ \) in \( X_1 \). This shows that the manifold \( \mathcal{E} \) can be represented as the (translated) graph of the function \( \phi \) in a neighborhood of \( u_\circ \). Moreover, the tangent space of \( \mathcal{E} \) at \( u_\circ \) coincides with \( N(A) = X^c \). By applying the projections \( P_l, l \in \{c, s\} \), to equation \( (5) \) and using that \( x + \phi(x) = \psi(g^{-1}(x)) \) for \( x \in B_{X^c}(0, \rho_0) \), and that \( A_c \equiv 0 \), we obtain the following equivalent system of equations for the equilibria of \( (1) \)

\[
P_c G(x + \phi(x)) = 0, \quad P_s G(x + \phi(x)) = A_s \phi(x), \quad x \in B_{X^c}(0, \rho_0). \tag{14}
\]

Finally, let us also agree that \( \rho_0 \) has already been chosen small enough so that

\[
|\phi'(x)|_{B(X^c, X^c_1)} \leq 1, \quad |\phi(x)|_{1} \leq |x|, \quad x \in B_{X^c}(0, \rho_0). \tag{15}
\]

This can always be achieved, thanks to \( (13) \).

(c) Introducing the new variables

\[
x = P_c v = P_c(u - u_\circ),
\]

\[
y = P_s v - \phi(P_c v) = P_s(u - u_\circ) - \phi(P_c(u - u_\circ))
\]

we then obtain the following system of evolution equations in \( X^c \times X^c_0 \)

\[
\begin{cases}
x = T(x, y), & x(0) = x_0, \\
y + A_s y = R(x, y), & y(0) = y_0,
\end{cases}
\tag{16}
\]

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with \( x_0 = P_x v_0 \) and \( y_0 = P_y v_0 - \phi(P_x v_0) \), where the functions \( T \) and \( R \) are given by
\[
T(x, y) = P_x G(x + \phi(x) + y), \\
R(x, y) = P_y G(x + \phi(x) + y) - A_x \phi(x) - \phi'(x) T(x, y).
\]
Using the equilibrium equations (14), the expressions for \( R \) as given by (17) immediately yields
\[
T(x, y) = P_x \{ G(x + \phi(x) + y) - G(x + \phi(x)) \}, \\
R(x, y) = P_y \{ G(x + \phi(x) + y) - G(x + \phi(x)) \} - \phi'(x) T(x, y).
\]
Equation (17) immediately yields
\[
T(x, 0) = R(x, 0) = 0 \quad \text{for all } x \in B_{X^c}(0, \rho_0),
\]
showing that the equilibrium set \( \mathcal{E} \) of (2) near \( u_* \) has been reduced to the set \( B_{X^c}(0, \rho_0) \times \{0\} \subset X^c \times X^*_1 \).

Observe also that there is a unique correspondence between the solutions of (2) close to \( u_* \) in \( X_2 \) and those of (16) close to 0. We call system (16) the normal form of (2) near its normally stable equilibrium \( u_* \).

(d) Taking \( z_1 = x + \phi(x) + y \) and \( z_2 = x + \phi(x) \) it follows from (A5), (15) and (17) that
\[
|T(x, y)|, |R(x, y)|_0 \leq C_1 (\eta + |x + \phi(x)|_1) |y|_1 \leq \beta |y|_1,
\]
with \( \beta := C_2 (\eta + r) \), where the constants \( C_1 \) and \( C_2 \) are independent of \( \eta, r \) and \( x, y \), provided that \( x \in \bar{B}_{X^c}(0, \rho), y \in B_{X^c}(0, \rho) \cap X_1 \) and \( \rho \in (0, r/3) \) with \( r < 3 \rho_0 \).

Suppose that \( \eta \) and, accordingly, \( r \) were already chosen small enough so that
\[
M_0 \beta = M_0 C_2 (\eta + r) \leq 1/2.
\]
(e) Suppose now that \( v_0 \in B_{X^c}(0, \delta) \), where \( \delta < r_2 \) will be chosen later. By (A6), problem (4) has a unique solution on some maximal interval of existence \([0, t_*) \). Let \( \eta \) and \( r \) be fixed so that (19) holds and set \( \rho = r/3 \). Let then \( t_1 \) be the exit time for the ball \( \bar{B}_{X^c}(0, \rho) \), that is
\[
t_1 := \sup \{ t \in (0, t_*): |v(t)|_\gamma \leq \rho, \, \tau \in [0, t] \}.
\]
Suppose \( t_1 < t_* \) and set \( J_1 = [0, t_1] \). The definition of \( t_1 \) implies that \( |x(t)| \leq \rho \) for all \( t \in J_1 \). Assuming wlog that the embedding constant of \( X_1 \hookrightarrow X_\gamma \) is less or equal to one, we obtain from (12)
\[
\rho \geq |v(t)|_\gamma = |x(t) + \phi(x(t)) + y(t)|_\gamma = |x(t)| + |\phi(x(t)) + y(t)|_\gamma \\
\geq |x(t)| + |y(t)|_\gamma - |\phi(x(t))|_\gamma \geq |y(t)|_\gamma
\]
for \( t \in J_1 \), since \( \phi(x) \) is non-expansive for \( |x| \leq \rho_0 \). In conclusion we have shown that \( |x(t)|, |y(t)| \leq \rho \) for all \( t \in J_1 \), so that the estimate (18) holds for \((x(t), y(t))\), \( t \in J_1 \). Then, by (A4) and (18), we have for \( \sigma \in [0, \omega] \)
\[
\|e^{\sigma t} y\|_{E_1(J_1)} \leq \|e^{\sigma t} e^{-A_x t} y_0\|_{E_1(J_1)} + M_0 \|e^{\sigma t} R(x, y)\|_{E_0(J_1)} \\
\leq M_1 |y_0|_\gamma + M_0 \beta \|e^{\sigma t} y\|_{E_1(J_1)},
\]
which implies
\[
\|e^{\sigma t} y\|_{E_1(J_1)} \leq 2M_1 |y_0|_\gamma, \quad \sigma \in [0, \omega],
\]
(20)
thanks to (19). Using (A1), (A4) and (20) we then have for \( t \in J_1 \),
\[
|e^{wt} y(t)| \leq |e^{wt} y(t) - e^{wt} e^{-x \cdot t} y_0| + |e^{wt} e^{-x \cdot t} y_0| \\
\leq c_0 |e^{wt} y - e^{wt} e^{-x \cdot t} y_0| \| \mathbb{I}_i, (J_1) \| + M_1 |y_0| \\
\leq (3c_0 M_1 + M_1) |y_0|, 
\]
which yields with \( M_2 = 3c_0 M_1 + M_1, \) \( |y(t)| \leq M_2 e^{-\omega t} |y_0| \), \( t \in J_1 \).

Using (7) we deduce further from the equation for \( x \) and the estimate for \( T \) in (18), and from (19)–(20) that
\[
|x(t)| \leq |x_0| + \int_0^t |T(x(s), y(s))| ds \leq |x_0| + \beta \int_0^t |y(s)|_1 ds \\
\leq |x_0| + c_1 e^{\omega t} \| e^{-x \cdot t} y_0 \| \leq |x_0| + M_3 |y_0|, \quad t \in J_1, 
\]
where \( M_3 = M_1 c_1 / M_0 \). Since \( v(t) = x(t) + \phi(x(t)) + y(t) \), the previous estimates and (15) imply that for some constant \( M_4 \geq 1, \)
\[
|v(t)| \leq |v_0|, \quad t \in J_1. 
\]
Choosing \( \delta = \min\{ \rho, r_2 \}/(2M_4) \), we have \( |v(t)| \leq \min\{ \rho, r_2 \}/2, \) a contradiction to the definition of \( t_1 \), and hence \( t_1 = t_* \). The above argument then yields uniform bounds \( |v|_{E_i(J)} \leq C \) and \( \sup_{t \in J} |v(t)| \leq r/2 \) for all \( J = [0, a) \) with \( a < t_* \). In view of (A6), it follows that \( t_* = \infty \).

(f) Repeating the above estimates on the interval \([0, \infty)\) we obtain
\[
|x(t)| \leq |x_0| + M_3 |y_0|, \quad |y(t)| \leq M_2 e^{-\omega t} |y_0|, \quad t \in [0, \infty), 
\]
for \( v_0 \in B_{X_\gamma}(0, \delta) \). Moreover, \( \lim_{t \to \infty} x(t) = x_0 + \int_0^\infty T(x(s), y(s)) ds =: x_\infty \) exists since the integral is absolutely convergent. This yields existence of
\[
v_\infty := \lim_{t \to \infty} v(t) = \lim_{t \to \infty} x(t) + \phi(x(t)) + y(t) = x_\infty + \phi(x_\infty). 
\]
Clearly, \( v_\infty \) is an equilibrium for equation (1), and \( u_\infty := u_* + v_\infty \in E \) is an equilibrium for (2). It follows from (A2), the estimate for \( T \) in (18), and from (20) that
\[
|x(t) - x_\infty| = \int_t^\infty T(x(s), y(s)) ds \leq \beta \int_t^\infty |y(s)|_1 ds \\
\leq \beta c_1 e^{-\omega t} \| e^{-x \cdot t} y_0 \| E_\infty(\mathbb{R}_+) \leq M_4 e^{-\omega t} |y_0|, 
\]
This shows that \( x(t) \) converges to \( x_\infty \) at an exponential rate. Due to (15), (21) and the exponential estimate for \( |x(t) - x_\infty| \) we now get for the solution \( u(t) = u_* + v(t) \) of (2)
\[
|u(t) - u_\infty| \gamma = |x(t) + \phi(x(t)) + y(t) - v_\infty|_1 \\
\leq |x(t) - x_\infty|_1 + |\phi(x(t)) - \phi(x_\infty)|_1 + |y(t)|_1 \\
\leq (2M_4 + M_2) e^{-\omega t} |y_0|, \\
\leq M e^{-\omega t} |P_v v_0 - \phi(P_v v_0)|_1, 
\]
thereby completing the proof of the second part of Theorem 3.1. Concerning stability, note that given \( r > 0 \) small enough we may choose \( 0 < \delta \leq r \) such that the solution starting in \( B_{X_\gamma}(u_*, \delta) \) exists on \( \mathbb{R}_+ \) and stays within \( B_{X_\gamma}(u_*, r) \).
Remarks: (a) Theorem 3.1 has been proved in [26] in the setting of $L^p$-maximal regularity, and applications to quasilinear parabolic problems with nonlinear boundary conditions, to the Mullins-Sekerka problem, and to the stability of travelling waves for a quasilinear parabolic equation have been given.

(b) It has been shown in [26] by means of examples that conditions (i)–(iii) in Theorem 3.1 are also necessary in order to get convergence of solutions towards equilibria $u_\infty \in \mathcal{E}$.

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E-mail address: jan.pruess@mathematik.uni-halle.de
E-mail address: gieri.simonett@vanderbilt.edu
E-mail address: rico.zacher@mathematik.uni-halle.de