Fast convergent method for the $m$-point problem in Banach space

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Abstract

The $m$-point nonlocal problem for the first order differential equation with an operator coefficient in a Banach space $X$ is considered. An exponentially convergent algorithm is proposed and justified provided that the operator coefficient $A$ is strongly positive and some existence and uniqueness conditions are fulfilled. This algorithm is based on representations of operator functions by a Dunford-Cauchy integral along a hyperbola enveloping the spectrum of $A$ and on the proper quadratures involving short sums of resolvents. The efficiency of the proposed algorithms is demonstrated by numerical examples.

Keywords nonlocal problem, differential equation with an operator coefficient in Banach space, operator exponential, exponentially convergent algorithms

AMS Subject Classification 65J10, 65M70, 35K90, 35L90

1 Introduction

The $m$-point initial (nonlocal) problem for a differential equation with the nonlocal condition $u(t_0) + g(t_1;\ldots;t_p;u) = u_0$ and a given function $g$ on a given point set $P = \{0 = t_0 < t_1 < \cdots < t_p\}$, is one of the important topics in the study of differential equations. Interest in such problems originates mainly from some physical problems with a control of the solution at $P$. For

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example, when the function $g(t_1; \ldots; t_p; u)$ is linear, we will have the periodic problem $u(t_0) = u(t_1)$. Problems with nonlocal conditions arise in the theory of physics of plasma [1], nuclear physics [2], mathematical chemistry [3], waveguides [4] etc.

Differential equations with operator coefficients in some Hilbert or Banach space can be considered as meta-models for systems of partial or ordinary differential equations and are suitable for investigations using tools of the functional analysis (see e.g. [5, 6]). Nonlocal problems can also be considered within this framework [7, 8].

In this paper we consider the following nonlocal $m$-point problem:

$$u'(t) + Au = f(t), \quad t \in [0, T]$$

$$u(0) + \sum_{k=1}^{m} \alpha_k u(t_k) = u_0, \quad 0 < t_1 < t_2 < \ldots < t_m \leq T,$$

where $\alpha_k \in \mathbb{R}$, $k = 1, m$, $f(t)$ is a given vector-valued function with values in a Banach space $X$, $u_0 \in X$. The operator $A$ with the domain $D(A)$ in a Banach space $X$ is assumed to be a densely defined strongly positive (sectorial) operator, i.e. its spectrum $\Sigma(A)$ lies in a sector of the right half-plane with the vertex at the origin and the resolvent decays inversely proportional to $|z|$ at the infinity (see estimate (3) below).

Discretization methods for differential equations in Banach and Hilbert spaces were intensively studied in the last decade (see e.g. [9, 10, 11, 12, 13, 14, 15, 16, 17, 18] and the references therein). Methods from [10, 11, 12, 13, 14, 16, 17, 18] possess an exponential convergence rate, i.e. the error estimate in an appropriate norm is of the type $O(e^{-N^\alpha})$, $\alpha > 0$ with respect to a discretization parameter $N \to \infty$. For a given tolerance $\varepsilon$ such discretizations provide optimal or nearly optimal computational complexity [10]. One of the possible ways to obtain exponentially convergent approximations to abstract differential equations is based on a representation of the solution through the Dunford-Cauchy integral along a parametrized path enveloping the spectrum of the operator coefficient. Choosing a proper quadrature for this integral we obtain a short sum of resolvents. Since the treatment of such resolvents is usually the most time consuming part of any approximation this leads to a low-cost naturally parallelizable algorithms. Parameters of the algorithms from [12, 14, 16] were optimized in [19, 20] to improve the convergence rate.

The aim of this paper is to construct an exponentially convergent approximation to the problem for a differential equation with $m$-point nonlocal condition in abstract setting [1]. The paper is organized as follows. In Section 2 we discuss the existence and uniqueness of the solution as well as its representation through input data. An algorithm for the homogeneous
problem (1) is proposed in section 3. The main result of this section is theorem 2 about the exponential convergence rate of the proposed discretization. The next section 4 is devoted to the exponentially convergent discretization of the inhomogeneous problem. In section 5 we represent some numerical examples which confirm theoretical results from the previous sections.

2 Existence and representation of the solution

Let the operator $A$ in (1) be a densely defined strongly positive (sectorial) operator in a Banach space $X$ with the domain $D(A)$, i.e. its spectrum $\Sigma(A)$ lies in the sector (spectral angle)

$$\Sigma = \{ z = \rho_0 + re^{i\theta} : \quad r \in [0, \infty), \quad |\theta| < \varphi < \frac{\pi}{2} \}.$$ \hspace{1cm} (2)

Additionally outside the sector and on its boundary $\Gamma_{\Sigma}$ the following estimate for the resolvent holds true

$$\| (zI - A)^{-1} \| \leq \frac{M}{1 + |z|}.$$ \hspace{1cm} (3)

The numbers $\rho_0, \varphi$ are called the spectral characteristics of $A$.

The hyperbola

$$\Gamma_0 = \{ z(\xi) = \rho_0 \cosh \xi - ib_0 \sinh \xi : \quad \xi \in (-\infty, \infty), \quad b_0 = \rho_0 \tan \varphi \}$$ \hspace{1cm} (4)

in turn is referred as a spectral hyperbola. It has a vertex at $(\rho_0, 0)$ and asymptotes which are parallel to the rays of the spectral angle $\Sigma$.

A convenient representation of operator functions is the one through the Dunford-Cauchy integral (see e.g. [5, 6]) where the integration path plays an important role. We choose the following hyperbola

$$\Gamma_I = \{ z(\xi) = a_I \cosh \xi - ib_I \sinh \xi : \quad \xi \in (-\infty, \infty) \},$$ \hspace{1cm} (5)

as the integration contour which envelopes the spectrum of $A$.

For an arbitrary vector $u_0 \in D(A^{m+1})$ the next equality holds

$$\sum_{k=1}^{m+1} \frac{A^{k-1}u_0}{z^k} + \frac{1}{z^{m+1}}(zI - A)^{-1}A^{m+1}u_0 = (zI - A)^{-1}u_0.$$ \hspace{1cm} (6)

This formula, together with

$$A^{-(m+1)}v = \frac{1}{2\pi i} \int_{\Gamma_I} z^{-(m+1)}(zI - A)^{-1}vdz,$$ \hspace{1cm} (7)
by setting \( v = A^{m+1}u_0 \), yields the following representation

\[
u_0 = A^{-(m+1)}A^{m+1}u_0 = \frac{1}{2\pi i} \int_{\Gamma_I} z^{-(m+1)}(zI - A)^{-1}A^{m+1}u_0 \, dz
\]

\[
= \int_{\Gamma_I} \left[ (zI - A)^{-1} - \sum_{k=1}^{m+1} \frac{A^{k-1}}{z^k} \right] u_0 \, dz.
\]

Various useful properties of this representation as well as the next result were discussed in [21].

**Theorem 1** Let \( u_0 \in D(A^{m+\alpha}) \) for some \( m \in \mathbb{N} \) and \( \alpha \in [0, 1] \), then the following estimate holds true:

\[
\left\| \left( zI - A \right)^{-1} - \sum_{k=1}^{m+1} \frac{A^{k-1}}{z^k} \right\| u_0 \leq \frac{1}{|z|^{m+1}} \frac{(1 + M)K}{(1 + |z|)\alpha} \| A^{m+\alpha}u_0 \|,
\]

\[
\forall \alpha \in [0, 1], \, u_0 \in D(A^{m+\alpha}),
\]

where the constant \( K \) depends on \( \alpha \) and \( M \) only.

In [7, 8] it was proven that the solution of the problem (1) exists and is unique provided that one of the following two conditions is fulfilled:

\[
\sum_{i=1}^{m} |\alpha_i| < 1,
\]

or

\[
\sum_{i=1}^{m} |\alpha_i| e^{-\rho_0 t_i} < 1,
\]

with \( \rho_0 \) from (2).

According to the Hille-Yosida-Phillips theorem [22] the strongly positive operator \( A \) generates a one parameter semigroup \( T(t) = e^{-tA} \) and the solution of (11) can be represented by

\[
u(t) = e^{-At}u(0) + \int_0^t e^{-A(t-\tau)} f(\tau) \, d\tau.
\]

(12)

Combining the nonlocal condition from (11) and (12) we obtain

\[
u(t_i) = e^{-At_i} u_0 - \sum_{k=1}^{n} \alpha_k u(t_k) + \int_0^{t_i} e^{-A(t_i-\tau)} f(\tau) \, d\tau;
\]

\[
i = 1, m.
\]

(13)
Multiplying these equations by $\alpha_i$ and summing up over $i = 1, m$ one gets

$$
\sum_{i=1}^{m} \alpha_i u(t_i) = \sum_{i=1}^{m} \alpha_i e^{-At_i}u_0 - \sum_{i=1}^{m} \alpha_i e^{-At_i} \sum_{k=1}^{m} \alpha_k u(t_k)
+ \sum_{i=1}^{m} \alpha_i \int_0^{t_i} e^{-A(t_i-\tau)} f(\tau) d\tau.
$$

(14)

Let us denote

$$
W = \sum_{i=1}^{m} \alpha_i u(t_i),
$$

then (14) implies

$$
W = - \sum_{i=1}^{m} \alpha_i e^{-At_i} W + \sum_{i=1}^{m} \alpha_i e^{-At_i} u_0 + \sum_{i=1}^{m} \alpha_i \int_0^{t_i} e^{-A(t_i-\tau)} f(\tau) d\tau,
$$

Setting $B = I + \sum_{i=1}^{m} \alpha_i e^{-At_i}$, we obtain

$$
BW = Bu_0 - u_0 + \sum_{i=1}^{m} \alpha_i \int_0^{t_i} e^{-A(t_i-\tau)} f(\tau) d\tau,
$$

where $I$ is an identity operator. Condition (10) or (11) guarantee the existence and boundedness of $B^{-1}(A)$, therefore we have

$$
W = u_0 - B^{-1} u_0 + B^{-1} \sum_{i=1}^{m} \alpha_i \int_0^{t_i} e^{-A(t_i-\tau)} f(\tau) d\tau.
$$

Now, the equations (13) yield the representation for the solution of (1) in the form $u(t) = u_h(t) + u_{ih}(t)$, where $u_h(t) = e^{-At}B^{-1}u_0$ is the solution of the homogeneous problem with the initial condition $u_0$ and

$$
u_{ih}(t) = -e^{-At} B^{-1} \sum_{i=1}^{m} \alpha_i \int_0^{t_i} e^{-A(t_i-\tau)} f(\tau) d\tau + \int_0^{t} e^{-A(t-\tau)} f(\tau) d\tau
$$

(15)

is the solution of the inhomogeneous problem with zero initial condition.

3 Homogeneous nonlocal problem

First of all we consider the solution $u_h(t) = e^{-At}B^{-1}(A)u_0$ of the homogeneous problem (11). Our aim in this section is to construct an exponentially convergent method for its approximation.
Using the Dunford-Cauchy representation of \( u_h(t) \) analogously to \[21\] we obtain

\[
    u_h(t) = \frac{1}{2\pi i} \int_{\Gamma_f} e^{-zt} B^{-1}(z)(zI - A)^{-1} u_0 dz \\
    = \frac{1}{2\pi i} \int_{\Gamma_f} \frac{e^{-zt}}{1 + \sum_{i=1}^{n} \alpha_i e^{-zt_i}} (zI - A)^{-1} u_0 dz.
\]

(16)

Representation (16) makes sense only when the function \( e^{-zt} B^{-1}(z) \) is analytic in the region enveloped by \( \Gamma_f \). Let us show, that conditions like (10) or (11) guaranty this analyticity \[6\].

Actually, the analyticity of \( e^{-zt} B^{-1}(z) \) might only be violated when \( \sum_{i=1}^{n} \alpha_i e^{-zt_i} = -1 \), since in this case \( B^{-1}(z) \) becomes unbounded. It is easy to see that for an arbitrary \( z \) we have

\[
    \left| 1 + \sum_{k=1}^{m} \alpha_k e^{-zt_k} \right| \geq \left| 1 - \sum_{k=1}^{m} |\alpha_k| \cdot |e^{t_k a_I \cosh \xi}| \cdot |e^{-it_k b_I \sinh \xi}| \right| \\
    \geq \left| 1 - \sum_{k=1}^{m} |\alpha_k| e^{-\rho_0 t_k} \right| > 0,
\]

provided that (11) holds true.

Using (8) with \( m = 0 \) we can modify the representation of \( u_h(t) \) as follows:

\[
    u_h(t) = \frac{1}{2\pi i} \int_{\Gamma_f} e^{-zt} B^{-1}(z) \left[ (zI - A)^{-1} - \frac{1}{z} I \right] u_0 dz \\
    = \frac{1}{2\pi i} \int_{\Gamma_f} \frac{e^{-zt}}{1 + \sum_{i=1}^{n} \alpha_i e^{-zt_i}} \left[ (zI - A)^{-1} - \frac{1}{z} I \right] u_0 dz.
\]

(17)

After discretization of the integral such modified resolvent provides better convergence speed than (16) in a neighbourhood of \( t = 0 \) (see \[21\] for details).

Parameterizing the integral (17) by (5) we get

\[
    u_h(t) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \mathcal{F}(t, \xi) d\xi,
\]

(18)

with

\[
    \mathcal{F}(t, \xi) = F_A(t, \xi) u_0,
\]
\[ F_A(t, \xi) = \frac{e^{-z(\xi)t} z'(\xi)}{1 + \sum_{i=1}^{n} \alpha_i e^{-z(\xi)t_i} \left[ (z(\xi)I - A)^{-1} - \frac{1}{z(\xi)}I \right]}, \]

\[ z'(\xi) = a_I \sinh \xi - ib_I \cosh \xi. \]

Supposing \( u_0 \in D(A^\alpha), \ 0 < \alpha < 1 \) it was shown in [21] that

\[ \|e^{-z(\xi)t} z'(\xi) \left[ (z(\xi)I - A)^{-1} - \frac{1}{z(\xi)}I \right] u_0\| \leq (1 + M)K \frac{b_I}{a_I} \left( \frac{2}{a_I} \right)^{\alpha} e^{-a_I t} e^{-\rho_1 |\xi|}\|A^\alpha u_0\|, \xi \in \mathbb{R}, t \geq 0. \]

The part responsible for the nonlocal condition in (18), can be estimated in the following way

\[ \left| 1 + \sum_{k=1}^{m} \alpha_k e^{-z(\xi)t_k} \right|^{-1} \leq \left( 1 - \sum_{k=1}^{m} |\alpha_k| e^{-a_I t} e^{-\rho_1 t_k}\right)^{-1} \]

\[ \leq \left( 1 - \sum_{k=1}^{m} |\alpha_k| e^{-\rho_1 t_k}\right)^{-1} \equiv Q^{-1}, \]

where \( a_I \geq \rho_1. \)

Thus, we obtain the following estimate for \( \mathcal{F}(t, \xi): \)

\[ \|\mathcal{F}(t, \xi)\| \leq Q(1 + M)K \frac{b_I}{a_I} \left( \frac{2}{a_I} \right)^{\alpha} e^{-a_I t} e^{-\rho_1 |\xi|}\|A^\alpha u_0\|, \xi \in \mathbb{R}, t \geq 0. \] (19)

The next step toward a numerical algorithm is an approximation of (18) by the efficient quadrature formula. For this purpose we need to estimate the width of a strip around the real axis where the function \( \mathcal{F}(t, \xi) \) permit analytical extension (with respect to \( \xi \)). After changing \( \xi \) to \( \xi + i\nu \) the integration hyperbola \( \Gamma_I \) will be translated into the parametric hyperbola set

\[ \Gamma(\nu) = \{ z(w) = a_I \cosh (\xi + i\nu) - ib_I \sinh (\xi + i\nu) : \xi \in (-\infty, \infty) \} = \{ z(w) = a(\nu) \cosh \xi - ib(\nu) \sinh \xi : \xi \in (-\infty, \infty) \}, \]

with

\[ a(\nu) = a_I \cos \nu + b_I \sin \nu = \sqrt{a_I^2 + b_I^2} \sin (\nu + \phi/2), \]

\[ b(\nu) = b_I \cos \nu - a_I \sin \nu = \sqrt{a_I^2 + b_I^2} \cos (\nu + \phi/2), \]

\[ \cos \frac{\phi}{2} = \frac{b_I}{\sqrt{a_I^2 + b_I^2}}, \sin \frac{\phi}{2} = \frac{a_I}{\sqrt{a_I^2 + b_I^2}}. \]
The analyticity of the function $\mathcal{F}(t, \xi + i\nu)$, in the strip

$$D_{d_1} = \{(\xi, \nu) : \xi \in (-\infty, \infty), |\nu| < d_1/2\},$$

with some $d_1$ could be violated if the resolvent or the part related to the nonlocal condition become unbounded. To avoid this we have to choose $d_1$ in a way that for $\nu \in (-d_1/2, d_1/2)$ the hyperbola set $\Gamma(\nu)$ remains in the right half-plane of the complex plane. For $\nu = -d_1/2$ the corresponding hyperbola is going through the point $(\rho_1, 0)$, for some $0 \leq \rho_1 < \rho_0$. For $\nu = d_1/2$ it coincides with the spectral hyperbola and therefore for all $\nu \in (-d_1/2, d_1/2)$ the set $\Gamma(\nu)$ does not intersect the spectral sector. This fact justifies the choice the hyperbola $\Gamma(0) = \Gamma_I$ as the integration path.

The requirements above imply the following system of equations

\[
\begin{align*}
  a_I \cos (d_1/2) + b_I \sin (d_1/2) &= \rho_0, \\
  b_I \cos (d_1/2) - a_I \sin (d_1/2) &= b_0 = \rho_0 \tan \varphi, \\
  a_I \cos (-d_1/2) + b_I \sin (-d_1/2) &= \rho_1,
\end{align*}
\]

it lead us to the next system

\[
\begin{align*}
  a_I &= \rho_0 \cos (d_1/2) - b_0 \sin (d_1/2), \\
  b_I &= \rho_0 \sin (d_1/2) + b_0 \cos (d_1/2), \\
  2a_I \cos (d_1/2) &= \rho_0 + \rho_1.
\end{align*}
\]

Eliminating $a_I$ from the first and the third equations we get

\[
\rho_0 \cos d_1 - b_0 \sin d_1 = \rho_1,
\]

\[
\cos(d_1 + \varphi) = \frac{\rho_1}{\sqrt{\rho_0^2 + b_0^2}},
\]

i.e.

\[
d_1 = \arccos \left( \frac{\rho_1}{\sqrt{\rho_0^2 + b_0^2}} \right) - \varphi, \tag{20}
\]
with $\cos \varphi = \frac{\rho_0}{\sqrt{\rho_0^2 + b_0^2}}$, $\sin \varphi = \frac{b_0}{\sqrt{\rho_0^2 + b_0^2}}$. Thus, for $a_I$, $b_I$ we receive

$$a_I = \sqrt{\rho_0^2 + b_0^2} \cos \left( \frac{d_1}{2} + \varphi \right)$$

$$= \rho_0 \frac{\cos \left( \frac{d_1}{2} + \varphi \right)}{\cos \varphi} = \rho_0 \frac{\cos \left( \arccos \left( \frac{\rho_1}{\sqrt{\rho_0^2 + b_0^2}} \right) \right) / 2 + \varphi / 2}{\cos \varphi}, \quad (21)$$

$$b_I = \sqrt{\rho_0^2 + b_0^2} \sin \left( \frac{d_1}{2} + \varphi \right)$$

$$= \rho_0 \frac{\cos \left( \frac{d_1}{2} + \varphi \right)}{\cos \varphi} = \rho_0 \frac{\cos \left( \arccos \left( \frac{\rho_1}{\sqrt{\rho_0^2 + b_0^2}} \right) \right) / 2 + \varphi / 2}{\cos \varphi}.$$

For $a_I$ and $b_I$ defined as above the vector valued function $F(t, w)$ is analytic in the strip $D_{d_1}$ with respect to $w = \xi + i\nu$ for any $t \geq 0$. Note, that for $\rho_1 = 0$ we have $d_1 = \pi/2 - \varphi$ as in [21].

Taking into account (21) we similarly can write equations for $a(\nu), b(\nu)$ on the whole interval $-\frac{d_1}{2} \leq \nu \leq \frac{d_1}{2}$.

$$a(\nu) = a_I \cos \nu + b_I \sin \nu = \sqrt{\rho_0^2 + b_0^2} \cos \left( \frac{d_1}{2} + \varphi \right) \cos(\nu)$$

$$+ \sqrt{\rho_0^2 + b_0^2} \sin \left( \frac{d_1}{2} + \varphi \right) \sin(\nu) = \sqrt{\rho_0^2 + b_0^2} \cos \left( \frac{d_1}{2} + \varphi - \nu \right),$$

$$b(\nu) = b_I \cos \nu - a_I \sin \nu = \sqrt{\rho_0^2 + b_0^2} \sin \left( \frac{d_1}{2} + \varphi \right) \cos(\nu)$$

$$- \sqrt{\rho_0^2 + b_0^2} \cos \left( \frac{d_1}{2} + \varphi \right) \sin(\nu) = \sqrt{\rho_0^2 + b_0^2} \sin \left( \frac{d_1}{2} + \varphi - \nu \right),$$

$$\rho_1 \leq a(\nu) \leq \rho_0, \quad b_0 \leq b(\nu) \leq \sqrt{b_0^2 + \rho_0^2 - \rho_1^2},$$

with $d_1$, defined by (20).

For the part responsible for the nonlocal condition we have

$$\left| \left( 1 + \sum_{i=1}^{n} \alpha_i \text{e}^{-z(\xi,\nu) t_i} \right)^{-1} \right| \leq \left( 1 - \sum_{i=1}^{n} |\alpha_i| \text{e}^{-z(\xi,\nu) t_i} \right)^{-1} \leq \left( 1 - \sum_{i=1}^{n} |\alpha_i| \text{e}^{-a(\nu) \cosh(\xi) t_i} \right)^{-1},$$

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\[
\left(1 - \sum_{i=1}^{n} |\alpha_i| e^{-a(\nu)t_i}\right)^{-1} < \left(1 - \sum_{i=1}^{n} |\alpha_i| e^{-\rho t_i}\right)^{-1} < q^{-1} = Q,
\]

where \( z(\xi, \nu) = a(\nu) \cosh(\xi) - ib(\nu) \sinh(\xi) \).

Now, for \( w \in D_d \) we get the estimate

\[
\|F(t, w)\| \leq e^{-a(\nu)t} \frac{(1 + M)QK}{a(\nu)} \frac{1}{e^{a(\nu)t} \cosh \xi} \frac{\sqrt{a^2(\nu) \sinh^2 \xi + b^2(\nu) \cosh^2 \xi}}{a^2(\nu) \cosh^2 \xi + b^2(\nu) \sinh^2 \xi (1 + \alpha)/2} \|A^\alpha u_0\|
\]

\[
\leq (1 + M)QK \frac{b(\nu)}{a(\nu)} \frac{e^{-a(\nu)t} \cosh \xi}{a(\nu)} \|A^\alpha u_0\|
\]

\[
\leq (1 + M)QK \tan \left(\frac{d_1}{2} + \varphi - \nu\right) \frac{2 \cos \varphi}{\rho_0 \cos \left(\frac{d_1}{2} + \varphi - \nu\right)} e^{-\alpha\xi} \|A^\alpha u_0\|\]

\[
\forall w \in D_d.
\]

Similarly to [23], we introduce the space \( \mathbb{H}^p(D_d) \), \( 1 \leq p \leq \infty \) of all vector-valued functions \( F \) analytic in the strip

\[ D_d = \{ z \in \mathbb{C} : -\infty < \Re z < \infty, |\Im z| < d \}, \]

equipped by the norm

\[
\|F\|_{\mathbb{H}^p(D_d)} = \begin{cases} 
\lim_{\epsilon \to 0} (\int_{\partial D_d(\epsilon)} \|F(z)\|^p |dz|)^{1/p} & \text{if } 1 \leq p < \infty, \\
\lim_{\epsilon \to 0} \sup_{z \in \partial D_d(\epsilon)} \|F(z)\| & \text{if } p = \infty,
\end{cases}
\]

where

\[ D_d(\epsilon) = \{ z \in \mathbb{C} : |\Re(z)| < 1/\epsilon, |\Im(z)| < d(1 - \epsilon) \} \]

and \( \partial D_d(\epsilon) \) is the boundary of \( D_d(\epsilon) \).

Taking into account that the integrals over the vertical sides of the rectangle \( D_d(\epsilon) \) vanish as \( \epsilon \to 0 \), the above estimate for \( \|F(t, w)\| \) implies

\[
\|F(t, \cdot)\|_{\mathbb{H}^1(D_d)} \leq \|A^\alpha u_0\| [C_-(\varphi, \alpha) + C_+(\varphi, \alpha)] \int_{-\infty}^{\infty} e^{-\alpha\xi} d\xi = C(\varphi, \alpha) \|A^\alpha u_0\| \tag{22}
\]

with

\[
C(\varphi, \alpha) = \frac{2}{\alpha}[C_+(\varphi, \alpha) + C_-(\varphi, \alpha)],
\]

\[
C_{\pm}(\varphi, \alpha) = (1 + M)QK \tan \left(\frac{d_1}{2} + \varphi \pm \frac{d_1}{2}\right) \frac{2 \cos \varphi}{\rho_0 \cos \left(\frac{d_1}{2} + \varphi \pm \frac{d_1}{2}\right)} \alpha.
\]

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Note that the influence of both the smoothness parameter of $u_0$ given by $\alpha$ and of the spectral characteristics of the operator $A$ given by $\varphi$ and $\rho_0$ is accounted by that fact, that the constant $C(\varphi, \alpha)$ from (21) tends to $\infty$ if $\alpha \to 0$, $\varphi \to \pi/2$ or $\rho_1 \to 0$ (in this case due to [20] $d_1 \to \pi/2 - \varphi$).

We approximate integral (18) by the following Sinc-quadrature [23, 21]:

$$u_{h,N}(t) = \frac{h}{2\pi i} \sum_{k=-N}^{N} \mathcal{F}(t, z(kh)), \quad (23)$$

with an error

$$\| \eta_N(\mathcal{F}, h) \| = \| u_h(t) - u_{h,N}(t) \| \leq \frac{h}{2\pi i} \sum_{k=-\infty}^{\infty} \| \mathcal{F}(t, z(kh)) \| + \frac{h}{2\pi i} \sum_{|k|>N} \| \mathcal{F}(t, z(kh)) \| + \frac{1}{2\pi 2 \sinh(\pi d_1/h)} \| \mathcal{F} \|_{H^1(D_{d_1})}$$

$$+ \frac{C(\varphi, \alpha) h \| A^\alpha u_0 \|}{2\pi} \sum_{k=N+1}^{\infty} \exp[-a t \cosh(kh) - \alpha kh]$$

$$\leq c \frac{\| A^\alpha u_0 \|}{\alpha} \left\{ \frac{e^{-\pi d_1/h}}{\sinh(\pi d_1/h)} + \exp[-a t \cosh((N+1)h) - \alpha(N+1)h] \right\},$$

where the constant $c$ does not depend on $h, N, t$. Equalizing the both exponentials for $t = 0$ implies

$$\frac{2\pi d_1}{h} = \alpha(N+1) h,$$

or after the transformation

$$h = \frac{\sqrt{2\pi d_1}}{\alpha(N+1)} \quad (24)$$

With this step-size the following error estimate holds true

$$\| \eta_N(\mathcal{F}, h) \| \leq c \frac{\alpha \exp\left(-\sqrt{\frac{\pi d_1}{2\alpha}}(N+1)\right)}{\| A^\alpha u_0 \|}, \quad (25)$$

with a constant $c$ independent of $t, N$. In the case $t > 0$ the first summand in the argument of $\exp[-a t \cosh((N+1)h) - \alpha(N+1)h]$ from the estimate for $\| \eta_N(\mathcal{F}, h) \|$ contributes mainly to the error order. Setting in this case
h = c₁ ln N/N with some positive constant c₁ we remain, asymptotically for a fixed t, with an error

\[ \| \eta_N(F, h) \| \leq c \left[ e^{-\pi d₁ N/(c₁ \ln N)} + e^{-c₁ a₁ t N/2 - c₁ a₁ \ln N} \right] \| A^α u₀ \|, \quad (26) \]

where c is a positive constant. Thus, we have proved the following result.

**Theorem 2** Let A be a densely defined strongly positive operator and u₀ ∈ D(A^α), α ∈ (0, 1), then the Sinc-quadrature (23) represents an approximate solution of the homogeneous nonlocal value problem (1) (i.e. the case when f(t) ≡ 0) and possesses an exponential convergence rate which is uniform with respect to t ≥ 0 and is of the order \( O(e^{-c \sqrt{N}}) \) uniformly in t ≥ 0 provided that h = O(1/√N) (estimate (25)) and of the order \( O \left( \max \{ e^{-\pi d N/(c₁ \ln N)}, e^{-c₁ a₁ t N/2 - c₁ a₁ \ln N} \} \right) \) for each fixed t > 0 provided that h = c₁ ln N/N (estimate (26)).

### 4 Inhomogeneous nonlocal problem

In this section we consider the particular solution (15) of inhomogeneous problem (1), i.e. with f(t) ≠ 0.

Let us rewrite formula (15) in the form

\[ u_{ih}(t) = u_{1,ih}(t) + u_{2,ih}(t), \quad (27) \]

with

\[ u_{1,ih}(t) = \int_0^t e^{-A(t-\tau)} f(\tau) d\tau, \quad u_{2,ih}(t) = -\sum_{j=1}^m α_j u_{2,ih,j}(t), \quad (28) \]

where

\[ u_{2,ih,j}(t) = \int_0^{t_j} B^{-1} e^{-A(t+t_j-\tau)} f(\tau) d\tau. \quad (29) \]

We approximate the term \( u_{1,ih}(t) \) by the algorithm proposed in [21]:

\[ u_{1,ih}(t) \approx u_{1,N}(t) = \frac{h}{2π i} \sum_{k=-N}^N z'(kh)[(z(kh)I - A)^{-1} - \frac{1}{z(kh)}I] \]

\[ \times h \sum_{p=-N}^N \mu_{k,p}(t) f(ω_p(t)), \quad (30) \]

where

\[ \mu_{k,p}(t) = \frac{t}{2} \exp \left\{ -\frac{t}{2} z(kh)[1 - \tanh(ph)] \right\} / \cosh²(ph), \]

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\[ \omega_p(t) = \frac{t}{2} [1 + \tanh (ph)], \ h = \mathcal{O}\left(\frac{1}{\sqrt{N}}\right), \]

\[ z(\xi) = a_t \cosh \xi - ib_t \sinh \xi, \ z'(\xi) = a_t \sinh \xi - ib_t \cosh \xi. \]

The next theorem (see [21]) characterizes the error of this algorithm.

**Theorem 3** Let \( A \) be a densely defined strongly positive operator with spectral characteristics \( \rho_0, \varphi \) and a right hand side \( f(t) \in D(A^\alpha), \alpha > 0 \) for \( t \in [0, \infty) \) can be analytically extended into the sector \( \Sigma_f = \{ \rho e^{i\theta_1} : \rho \in [0, \infty], |\theta_1| < \varphi \} \) where the estimate

\[ \| A^\alpha f(w) \| \leq c_\alpha e^{-\delta_\alpha |\Re w|}, \ w \in \Sigma_f \]

with \( \delta_\alpha \in (0, \sqrt{2}\rho_0] \) holds, then algorithm (30) converges to the solution of (1) with the error estimate

\[ \| E_N(t) \| = \| u_{1,ih}(t) - u_{1,N}(t) \| \leq c e^{-c_1 \sqrt{N}} \]

uniformly in \( t \) with positive constants \( c, c_1 \) depending on \( \alpha, \varphi, \rho_0 \) and independent of \( N \).

Let us construct an exponentially convergent approximation to the term \( u_{2,ih} \). Using the representation of the operator functions by means of the Dunford-Cauchy integral for the \( j \)-th summand of \( u_{2,ih} \) we get

\[ u_{2,ih,j}(t) = \int_0^{t_j} \int_{\Gamma_I} \frac{1}{2\pi i} e^{-z(t+s)-z(s)} B^{-1}(z) [zI - A]^{-1} - \frac{1}{z} I f(s) dz ds \]

\[ = \frac{1}{2\pi i} \int_{\Gamma_I} e^{-z(\xi)t} B^{-1}(z) \left[ (z(\xi)I - A)^{-1} - \frac{1}{z(\xi)} I \right] \]

\[ \times \int_0^{t_j} e^{-z(\xi)(t+s)} f(s) ds z'(\xi) d\xi, \]

\[ z(\xi) = a_t \cosh \xi - ib_t \sinh \xi. \]

Replacing here the first integral by the Sinc-quadrature with \( h = \mathcal{O}\left( N^{-1/2} \right) \) we obtain

\[ u_{2,ih,j}(t) \approx u_{2,N}(t) = \frac{h}{2\pi i} \sum_{k=-N}^{N} e^{-z(kh)t} z'(kh) B^{-1}(z(kh)) \]

\[ \times \left[ (z(kh)I - A)^{-1} - \frac{1}{z(kh)} I \right] f_{k,j}, \]

where

\[ f_{k,j} = \int_0^{t_j} e^{-z(kh)(t+s)} f(s) ds, \ k = -N, ..., N, \ j = 1, n. \]
In order to construct an exponentially convergent quadrature for these integrals we change the variables by

\[ s = \frac{t_j}{2}(1 + \tanh \xi), \]  

(36)

and obtain the improper integrals

\[ f_{k,j} = \int_{-\infty}^{\infty} \mathcal{F}_k(t_j, \xi) d\xi, \]  

(37)

with

\[ \mathcal{F}_k(t_j, \xi) = \frac{t_j}{2 \cosh^2 \xi} \exp[-z(kh)t_j(1 - \tanh \xi)/2] f(t_j(1 + \tanh \xi)/2). \]  

(38)

Note that equation (36) represents the conformal mapping \( w = \psi(z) = t_j[1 + \tanh z]/2, z = \phi(w) = \frac{1}{2} \ln \frac{t_j-w}{w} \) of the strip \( D_\nu \) onto the eye-shaped domain \( A_\nu \) \cite{23}. On the real axes the integrand can be estimated by

\[ \|\mathcal{F}_k(t_j, \xi)\| \leq \frac{t_j}{2 \cosh^2 \xi} \exp[-a_I \cosh (kh)t_j(1 - \tanh \xi)/2] \times \|f(t_j(1 + \tanh \xi)/2)\| \]  

\[ \leq 2t_j e^{-2|\xi|} \|f(t_j(1 + \tanh \xi)/2)\|. \]  

(39)

The following result from \cite{21} is needed to justify a Sinc-quadrature for (37):

**Lemma 4** Let the right-hand side \( f(t) \) in (4) for \( t \in [0, \infty) \) can be analytically extended into the sector \( \Sigma_f = \{\rho e^{i\theta} : \rho \in [0, \infty], |\theta| < \varphi\} \) and for all complex \( w \in \Sigma_f \) we have

\[ \|f(w)\| \leq c e^{-\delta|\text{Re } w|}, \]  

(40)

with \( \delta \in (0, \sqrt{2}\rho_0] \), then the integrand \( \mathcal{F}_k(t_j, \xi) \) can be analytically extended into a strip \( D_{d_1} \), \( 0 < d_1 < \varphi/2 \) and belongs to the class \( H^1(D_{d_1}) \) with respect to \( \xi \), where \( \rho_0, \varphi \) are the spectral characteristics of \( A \).

Under the assumptions of Lemma 4 we can construct the following exponentially convergent Sinc-approximation to \( f_{k,j} \):

\[ f_{k,j} \approx f_{k,j,N} = h \sum_{p=-N}^{N} \mu_{k,p,j} f(\omega_{p,j}), \]  

(41)
where
\[ \mu_{k,p,j} = \frac{t_j \exp\left(-\frac{t_j}{2}z(kh)[1 - \tanh(ph)]\right)}{\cosh^2(ph)}, \]
\[ \omega_{p,j} = \frac{t_j}{2}[1 + \tanh(ph)], \quad h = \mathcal{O}(1/\sqrt{N}), \]
\[ z(\xi) = a_I \cosh \xi - ib_I \sinh \xi. \]

Substituting (41) into (34) we get the following algorithm to compute an approximation \( u_{2,j,N} \) to \( u_{2,j,N} \):
\[
\begin{align*}
    u_{2,j,N}(t) &\approx u_{2,j,N}(t) = \frac{h}{2\pi i} \sum_{k=-N}^{N} \frac{e^{-z(kh)t}z'(kh)B^{-1}(z(kh))}{1 - \tanh(ph)} \\
    &\times \left[ (z(kh)I - A)^{-1} - \frac{1}{z(kh)}I \right] h \sum_{p=-N}^{N} \mu_{k,p,j} f(\omega_{p,j}).
\end{align*}
\]

We represent the error in the form
\[ \mathcal{E}_N(t) = u_{2,j,N}(t) - u_{2,j,N}(t) = r_{1,N}(t) + r_{2,N}(t), \]
where
\[ r_{1,N}(t) = u_{2,j,N}(t) - u_{2,j,N}(t), \]
\[ r_{2,N}(t) = u_{2,j,N}(t) - u_{2,j,N}(t). \]

Using estimate (25) (see also Theorem 2) we obtain the estimate for \( r_{1,N}(t) \)
\[ \|r_{1,N}(t)\| \leq c \alpha \exp\left(-\sqrt{\pi d\alpha (N + 1)}\right) \int_{t_j}^{t} \|A^\alpha f(s)\| ds, \]
where \( F_A(t, \xi) \) is the operator defined in section 3. Due to (9) for \( m = 0 \) we have for the error \( r_{2,N} \)
\[ \|r_{2,N}(t)\| \leq \frac{h(1 + M)QK}{2\pi} \sum_{k=-N}^{N} \left| e^{-z(kh)t}z'(kh)B(z(kh)) \right| \left| A^\alpha R_k,j \right|, \]

15
\[ R_{k,j} = f_{k,j} - f_{k,j,N}. \]

The estimate (39) yields
\[ \| A^\alpha F_k(t_j, \xi) \| \leq 2t_je^{-2|\xi|}\| A^\alpha f(\frac{t_j}{2}(1 + \tanh \xi))\|. \] (47)

Due to Lemma 4 the assumption \( \| A^\alpha f(w)\| \leq c_\alpha e^{-\delta_\alpha |\text{Re} w|} \forall w \in \Sigma_f \) guarantees that \( A^\alpha f(w) \in H^1(D_{d_1}) \) and \( A^\alpha F_k(t_j, w) \in H^1(D_{d_1}) \). These two conditions turn us to the situation as in proposition of Theorem 3.2.1, p.144 from [23] with \( A^\alpha f(w) \) instead of \( f \) which implies
\[ \| A^\alpha R_{k,j} \| = \| A^\alpha (f_{k,j} - f_{k,j,N})\| \]
\[ = \left\| \int_{-\infty}^{\infty} A^\alpha F_k(t_j, \xi) d\xi - h \sum_{k=-\infty}^{\infty} A^\alpha F_k(t_j, kh) \right\| \]
\[ \leq \frac{e^{-\pi d_1}/h}{2\sinh(\pi d_1)/h} \| F_k(t_j, w) \|_{H^1(D_{d_1})} \]
\[ + h \sum_{|k|>N} 2t_je^{-2|kh|}\| A^\alpha f(\frac{t_j}{2}(1 + \tanh kh))\| \]
\[ \leq c_\alpha e^{-2\pi d_1}/h \| A^\alpha f(t_j, w) \|_{H^1(D_{d_1})} \]
\[ + h \sum_{|k|>N} 2t_je^{-2|kh|}c_\alpha \exp \left\{ -\delta_\alpha \frac{t_j}{2}(1 + \tanh kh) \right\}, \]
i.e., we have
\[ \| A^\alpha R_{k,j} \| \leq c_\alpha e^{-c_1 \sqrt{N}}, \] (48)
where positive constants \( c_\alpha, \delta_\alpha, c, c_1 \) do not depend on \( t, N, k \). Now, (46) takes the form
\[ \| r_{2,N}(t) \| = \frac{h}{2\pi i} \sum_{k=-N}^{N} h e^{-z(kh)t}z'(kh)B(z(kh)) \]
\[ \times \left[ (z(kh)I - A)^{-1} - \frac{1}{z(kh)I} \right] R_{k,j} \]
\[ \leq c_\alpha e^{-c_1 \sqrt{N}} S_N(t), \]
with \( S_N(t) = \sum_{k=-N}^{N} h \frac{|e^{-z(kh)t}z'(kh)|}{|z(kh)|} \). Using the estimate (4.8) from [21] and
\[ |z(kh)| = \sqrt{a_I^2 \cosh^2 (kh) + b_I^2 \sinh^2 (kh)} \]
\[ \geq a_I \cosh (kh) \geq a_I e^{kh}/2, \] (50)
the last sum can be estimated by
\[ |S_N(t)| \leq \frac{c}{\sqrt{N}} \sum_{k=-N}^{N} e^{-|k|/\sqrt{N}} \leq c \int_{-\sqrt{N}}^{\sqrt{N}} e^{-\alpha t} dt \leq c/\alpha \quad \forall \ t \in [0, \infty). \quad (51) \]
Taking into account (48), (51) we deduce from (49)
\[ \| r_{2,N}(t) \| \leq c e^{-c_1 \sqrt{N}}. \quad (52) \]
The next assertion now follows from (43), (45) and (52).

**Theorem 5** Let the assumptions of theorem 3 hold. Then algorithm (42) converges uniformly with respect to \( t \) and moreover the following error estimate holds true:
\[ \| E_N(t) \| = \| u_{2,ih,j}(t) - u_{2,j,N}(t) \| \leq c e^{-c_1 \sqrt{N}}, \quad (53) \]
with positive constants \( c, c_1 \) depend on \( \alpha, \varphi, \rho_0 \) and independent of \( N \).

Now we can use approximation (30) for the every summand and get:
\[ u_{2,ih}(t) = \sum_{i=1}^{n} \alpha_i \int_{0}^{\frac{t}{n}} B^{-1} e^{-A(t+i-\tau)} f(\tau) d\tau \approx \sum_{j=1}^{m} u_{2,j,N}(t) = u_{2,N}(t). \quad (54) \]
Theorem 5 guarantees that the error this approach will be bounded by:
\[ \| u_{2,ih}(t) - u_{2,N}(t) \| \leq \sum_{j=1}^{m} c e^{-c_1 \sqrt{N}} \leq c_2 e^{-c_1 \sqrt{N}}. \quad (55) \]
Thus, the approximations (23) together with (30) and (54) represent an exponentially convergent algorithm for the problem (1).

**5 Numerical examples**

**Example 6** We consider the homogeneous problem (1) with the operator \( A \) defined by
\[ D(A) = \{ u(x) \in H^2(0,1) : u(0) = u(1) = 0 \}, \quad \]
\[ Au = -u''(x) \quad \forall u \in D(A). \quad (56) \]
The initial nonlocal condition reads as follows:
\[ u(x,0) + 0.5u(x,0.2) + 0.3u(x,0.4) = (1 + 0.5e^{-\pi^2 0.2} + 0.3e^{-\pi^2 0.4}) \sin(\pi x), \]
with \( u_0 = (1 + 0.5e^{-\pi^20.2} + 0.3e^{-\pi^20.4}) \sin(\pi x) \in D(A) \). The exact solution of the problem is \( u(x,t) = e^{-\pi^2 t} \sin(\pi x) \). It is easy to find that

\[
(zI - A)^{-1}u_0 = \left(z + \frac{d^2}{dx^2}\right)^{-1} \sin(\pi x) = \frac{\sin(\pi x)}{z - \pi^2}.
\]

Calculations with the algorithm (42) above has been performed in Maple with \( h = N^{-1/2} \). The error at \( x = 0.5, \ t = 0.3 \) is presented in Table 1 and clearly exhibits an exponential decay according to the theoretical estimates.

| \( N \) | \( \varepsilon_N \) |
|-------|-----------------|
| 4     | .2985798347712589e-1 |
| 8     | .41823888073604986e-2 |
| 16    | .1125859468208641e-2 |
| 32    | .10042178166563831e-3 |
| 64    | .2800715839828452e-5 |
| 128   | .2098826601399176e-7 |
| 256   | .1858929920173152e-10 |
| 512   | .856837124351510e-15 |

Table 1: The error for \( x = 0.5, \ t = 0.3 \).

Due to Theorem 2 the error should not be greater then \( \varepsilon_N = O\left(e^{-c\sqrt{N}}\right) \).

The constant \( c \) in the exponent can be estimated using the following a-posteriori relation:

\[
c = \ln \left(\frac{\varepsilon_N}{\varepsilon_{2N}}\right) (\sqrt{2} - 1)^{-1} N^{-1/2} = \ln (\mu_N) (\sqrt{2} - 1)^{-1} N^{-1/2}.
\]

The numerical results are presented in the Table 2 for this estimation show that the constant can be estimated as \( c \approx 1.5 \) when \( N \to \infty \).

The next example deals again with a homogeneous problem but in this more realistic case the resolvent of \( A \) on the element \( u_0 \) cannot be calculated analytically.

Example 7 We consider the homogeneous problem (1) with the operator \( A \) defined as in (56) and with the following initial nonlocal condition:

\[
u(x,0) + u(x,0.5) = x \ln(x),
\]
Table 2: The estimate of $c$

| $N$ | $c$               |
|-----|------------------|
| 4   | 2.37265251538874558858746 |
| 8   | 1.120148732795449515627946 |
| 16  | 1.458741976765153165445005 |
| 32  | 1.52764924601130131250452  |
| 64  | 1.476794596387591759032900 |
| 128 | 1.499935011373075736075927 |
| 256 | 1.50659733981609844717370  |

where $u_0 = x\ln(x) \in A^\alpha$, $\alpha < 1/2$. In this case the resolvent can be represented using the Green function

$$(zI - A)^{-1}u_0 = \left(z + \frac{d^2}{dx^2}\right)^{-1}x\ln(x) = \int_0^1 G(x,s)s\ln(s)ds,$$

$$G(x,s) = -\frac{1}{\sqrt{z}\sin(\sqrt{z})} \begin{cases} 
\sin(x\sqrt{z})\sin((1-s)\sqrt{z}) & x \leq s, \\
\sin(s\sqrt{z})\sin((1-x)\sqrt{z}) & x \geq s
\end{cases},$$

where the integrals were computed by exponentially convergent Sinc-quadrature (see e.g. [23]) using Maple. The results for $x = 0.5$, $t = 0.3$ are presented in Table 3.

Table 3: Values of the solution $u(x,t)$ for $x = 0.5$, $t = 0.3$.

| $N$ | $u(x,t)$                |
|-----|-------------------------|
| 4   | -2.41535790017043e-1    |
| 8   | -2.28401191029108e-1    |
| 16  | -1.94273285627507e-1    |
| 32  | -1.92905848633180e-1    |
| 64  | -1.92911920318628e-1    |
| 128 | -1.92907849909929e-1    |
| 256 | -1.92907820740651e-1    |

It can be easily seen that the number of stabilized digits increases according to the theoretical prediction by Theorem 4.

Example 8 Let us consider the inhomogeneous problem (1) with the same $A$ defined by (56), and the nonlocal condition

$$u(x,0) + 0.5u(x,0.2) = (1 + 0.5e^{0.2})\sin(\pi x),$$
For $f(t,x)$ at the right-hand side of the equation (1) we set

$$f(x,t) = (1 + \pi^2)e^t\sin(\pi x).$$

The exact solution of the problem is $u(x,t) = e^t\sin(\pi x)$. We have used the algorithm defined by (23), (30), (54) and implemented in Maple. For $x = 0.5$, $t = 0.3$ the results presented in Table 4 and are again in good agreement with the theoretical predictions.

| $N$  | $\varepsilon_N$ |
|------|----------------|
| 4    | 0.202211483120243 |
| 8    | 0.726677678737409e-1 |
| 16   | 0.138993889900620e-1 |
| 32   | 0.143037059411419e-2 |
| 64   | 0.554542099757830e-4 |
| 128  | 0.532640823981411e-6 |
| 256  | 0.730569324317506e-9 |
| 512  | 0.648376079810788e-13 |

Table 4: The error for $x = 0.5$, $t = 0.3$.

Acknowledgment. The authors would like to acknowledge the support provided by the Deutsche Forschungsgemeinschaft (DFG).

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