TAUTOLOGICAL INTEGRALS ON SYMMETRIC PRODUCTS OF CURVES

ZHILAN WANG

Abstract. We propose a conjecture on the generating series of Chern numbers of tautological bundles on symmetric products of curves and establish the rank 1 and rank -1 case of this conjecture. Thus we compute explicitly the generating series of integrals of Segre classes of tautological bundles of line bundles on curves, which has a similar structure as Lehn’s conjecture for surfaces.

1. Introduction

Let $X$ be a smooth quasi-projective connected complex variety of dimension $d$, and denote by $X^{[n]}$ the Hilbert scheme of $n$ points on $X$. Let $Z_n \subset X \times X^{[n]}$ be the universal family with natural projections $p_1 : Z_n \to X$ and $\pi : Z_n \to X^{[n]}$ onto the $X$ and $X^{[n]}$ respectively. For any locally free sheaf $F$ on $X$, let $F^{[n]} = \pi_*(\mathcal{O}_{Z_n} \otimes p_1^*F)$, which is called the tautological sheaf of $F$.

When $d = 2$, many invariants of the Hilbert schemes of points on a projective surface can be determined explicitly by the corresponding invariants of the surface, including the Betti numbers [10], Hodge numbers [11], cobordism classes [8], and elliptic genus [5], etc.. G. Ellingsrud, L. Gottsche and M. Lehn showed in [8] that for a polynomial in Chern classes of tautological sheaves and the tangent bundle of $X^{[n]}$, there exists a universal polynomials in Chern classes of the corresponding sheaves and the tangent bundle on $X$ such that the integrals of these two polynomials over $X^{[n]}$ and $X$ are equal. A direct consequence is that the generating series of certain tautological integrals can be written in universal forms of infinite products; though it is not easy to find explicit expressions. For example, various authors have considered the computation of the integrals of top Segre classes of tautological sheaves of a line bundle on a surface [9, 14, 16, 22, 23, 24]. M. Lehn made a conjecture on the generating series as follows:

Conjecture 1. (M. Lehn [16]) For a smooth projective surface $S$ and a line bundle $L$ on it, define

$$N_n = \int_{S^{[n]}} s_{2n}(L^{[n]}),$$

then

$$\sum_{n \geq 0} N_n z^n = \frac{(1 - k)^a(1 - 2k)^b}{(1 - 6k + 6k^2)^c}. \quad (1)$$

Here $a = HK - 2K^2$, $b = (H - K)^2 + 3\chi(\mathcal{O}_S)$ and $c = \frac{1}{2}H(H - K) + \chi(\mathcal{O}_S)$, where $H$ is the corresponding divisor of $L$ and $K$ is the canonical divisor.
And
\[ k = z - 9z^2 + 94z^3 - \cdots \in \mathbb{Q}[[z]] \]
is the inverse of the function
\[ z = \frac{k(1 - k)(1 - 2k)^4}{(1 - 6k + 6k^2)^3}. \]

This conjecture is still open up to now. Recently M. Marian, D. Oprea and R. Pandharipande showed that this conjecture holds for K3 surfaces \[13\] by considering integrals over Quot schemes and the recursive localization relations.

J. V. Rennemo \[20\] gave a generalization of the theorem of G. Ellingsrud, L. Gottsche and M. Lehn: when \( d = 1 \) and \( d = 2 \) the universal property of polynomials in Chern classes of tautological sheaves and tangent bundles holds; when \( d > 2 \), one should consider the universal property of integrals of polynomials only in Chern classes of tautological sheaves over geometric subsets of \( X^{[n]} \).

In this article, we focus on the case of \( d = 1 \). For \( C \) a smooth projective curve, the Hilbert scheme of \( n \) points on \( C \) is isomorphic to the \( n \)-th symmetric product, so it is a smooth projective variety of dimension \( n \). We have the following conjecture:

**Conjecture 2.** For \( C \) a smooth projective curve and \( E_r \) a vector bundle of rank \( r \) on \( C \), one has

\[ \sum_{n=0}^{\infty} z^n \int_{C^{[n]}} c(E_r^{[n]}) = \exp \left( \sum_{n=1}^{\infty} \frac{z^n}{n} (A_r^n d_r + B_r^n e) \right), \]

\[ \sum_{n=0}^{\infty} \frac{z^n}{n} \int_{C^{[n]}} c(-E_r^{[n]}) = \exp \left( \sum_{n=1}^{\infty} \frac{z^n}{n} (C_r^n (-d_r) + D_r^n e) \right), \]

Here \( c(E_r^{[n]}) \) is the total Chern class, \( d_r = \int_C c(E_r) \) and \( e \) is the Euler number of \( C \), \( A_r^n, B_r^n, C_r^n \) and \( D_r^n \) are integers depending only on \( r \) and \( n \), which satisfy
\[
A_r^n = (-1)^n+1 \left( \frac{rn - 1}{n - 1} \right), \quad C_r^n = (-1)^n \left( -\frac{rn - 1}{n - 1} \right) = (-1)^{n-1} A_r^{n+1}, \quad D_r^n = (-1)^n B_r^{n+1}.
\]

Conjecture 2 shares some similarities as in the surface case, which has been established in an unpublished work by Jian Zhou and the author and will appear in a subsequent work.

We will see the existence of the universal coefficients \( A_r^n, B_r^n, C_r^n \) and \( D_r^n \) is a direct consequence of Theorem 2.3 in Section 4 and it will be explained in Section 4.2. The mysterious part of Conjecture 2 to the author is that \( A_r^n, B_r^n, C_r^n \) and \( D_r^n \) are all integers and there are relationships between them. More precisely, if two vector bundles \( E_r \) and \( E_{r+1} \) of ranks \( r \) and \( r + 1 \) respectively satisfy \( \int_C c(E_r) = \int_C c(E_{1+r}) \), then it is implied by Conjecture 2 that
\[
\sum_{n=0}^{\infty} z^n \int_{C^{[n]}} c(-E_r^{[n]}) = \sum_{n=0}^{\infty} (-z)^n \int_{C^{[n]}} c(E_{1+r}^{[n]})
\]

For some special \( r \), we can determine \( B_r^n \) explicitly and prove the conjecture. When \( r = 1 \), \( B_1^1 = 0 \). We have the following theorem:
Theorem 1.1. For $C$ a smooth projective curve and $L$ a line bundle on $C$, one has

\[ \sum_{n=0}^{\infty} z^n \int_{C[n]} c(L[n]) = \exp \left( \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} z^n \int_C c(L) \right). \tag{4} \]

For the rank -1 case, it is the generating series of the integrals of top Segre classes of tautological sheaves of a line bundle on $C$. We can also prove the conjecture in this case. Analogous to Conjecture 1, we have the following theorem:

Theorem 1.2. For $C$ a smooth projective curve and $L$ a line bundle on $C$, one has

\[ \sum_{n=0}^{\infty} z^n \int_{C[n]} s(L[n]) = \exp \left( \sum_{n=1}^{\infty} \frac{z^n}{n} \left( -\left(\frac{2n-1}{n-1}\right)d + \left(4^{n-1} - \left(\frac{2n-1}{n-1}\right)e\right) \right) \right) \]

\[ = \frac{(1 - k)^{e+d}}{(1 - 2k)^2}. \tag{5} \]

Here $s(L[n])$ is the total Segre class, $d$ is the degree of the line bundle $L$, $e$ is the Euler number of $S$ and $z = k(1 - k)$.

Theorem 1.2 is related to an enumerative problem. A. S. Tikhomirov in [22] has interpreted $N_n$ in Conjecture 1 as the number of $(n - 2)$-dimensional $n$-secant planes in the image to the surface in $\mathbb{P}^{3n-1}$. In a similar fashion, $(-1)^n \int_{C[n]} s(L[n])$ counts the number of $n$-secant $(n - 2)$-planes to $C$ in $\mathbb{P}^{2n-2}$. For example, it is easy to compute from Theorem 1.2 that $\int_{C[2]} s(L[2]) = \frac{1}{2}(d^2 - 3d + 2 - 2g)$, which coincides the classical formula of the number of nodes of a curve in $\mathbb{P}^2$.

In 2007 Le Barz [15] and E. Cotterill [7] have already independently derived the generating formula of such numbers. Le Barz’s approach is via the multisecant loci, and E. Cotterill uses a formula by Macdonald (cf. [1] Chapter VIII, Prop. 4.2) and the graph theory. However, our method is different from Le Barz’s and E. Cotterill’s.

We use the similar strategy used in [25] to prove the above theorems. Firstly we establish a universal formula theorem for curves as Theorem 4.2 in [8] for surfaces, and hence we only need to prove the cases of certain line bundles on $\mathbb{P}^1$. Using the natural torus action on $\mathbb{P}^1$ and the induced action on the Hilbert scheme, we can consider the equivariant case of $\mathbb{P}^1$, or we can reduce it further to the equivariant case of $C$, which becomes a combinatoric problem.

Remark 1.1. After the first version of this article, Professor Oprea tells the author in an email that Conjecture 2 has been solved by M. Marian and himself, and the universal coefficients are also explicitly determined by them in [17]. To be more precise, the relationships between the universal coefficients conjectured in Conjecture 2 do hold; if we write

\[ C(z) = \sum_{n=1}^{\infty} \frac{z^n}{n} c_n, \quad D(z) = \sum_{n=1}^{\infty} \frac{z^n}{n} D_n, \]
then
\[ C(-t(1-t)^r) = -\log(1 + t) \]
and
\[ D(-t(1 + t)^r) = \frac{r + 1}{2} \log(1 + t) - \frac{1}{2} \log(1 + t(r + 1)) \]
.

A direct consequence of the above is that (3) can be written in a form which is similar to Theorem 1.2:

\[ \sum_{n=0}^{\infty} z^n \int_{C^{[n]}} c(-E_r^{[n]}) = (1 - k)^{r+1} e^{d(1 - (r + 1)k)} \]

where \( z = k(1 - k)^r \).

2. Universal properties of tautological integrals over Hilbert schemes of points on curves

Let \( C \) be a smooth projective connected curve. It is well-known that \( C^{[n]} \) is smooth of dimension \( n \) and in particular isomorphic to the \( n \)-th symmetric product. In this section, we will see that the theorems on the cobordism rings of Hilbert schemes of points of surfaces established by Ellingsrud, Göttsche and Lehn in \( [8] \) can be generalized to curves.

We will follow what has been done in \( [8] \). Let \( \Omega = \Omega^U \otimes \mathbb{Q} \) be the complex cobordism ring with rational coefficients. For a smooth projective curve \( C \) we denote its cobordism class by \([C]\), and define an invertible element in the formal power series ring \( \Omega[[z]]\): 

\[ H(C) := \sum_{n=0}^{\infty} [C^{[n]}] z^n. \]

We have the following theorem:

**Theorem 2.1.** \( H(S) \) depends only on the cobordism class \([C] \in \Omega\).

Two stably complex manifolds have the same cobordism class if and only if their collection of Chern numbers are identical. Theorem 2.1 is proved in \( [20] \) and there is a generalized version of this theorem:

For a smooth projective variety \( X \), let \( K(X) \) be the Grothendieck group generated by locally free sheaves. Let \( E_1, \ldots, E_m \in K(C) \) and \( r_1, \ldots, r_m \) are the ranks respectively.

**Theorem 2.2.** (J. V. Rennemo \( [20] \)) Let \( P \) be a polynomial in the Chern classes of \( C^{[n]} \) and the Chern classes of \( E_1^{[n]}, \ldots, E_m^{[n]} \). Then there is a universal polynomial \( \tilde{P} \), depending only on \( P \), in the Chern classes of the tangent bundles of \( C^{[n]} \), the ranks \( r_1, \ldots, r_m \) and the Chern classes of \( E_1, \ldots, E_m \), such that

\[ \int_{C^{[n]}} P = \int_S \tilde{P}. \]
These theorems can be used in the computations of generating series of tautological integrals. Let $\Psi : K(X) \to H^\times$ be a group homomorphism from the additive group $K(X)$ to the multiplicative group $H^\times$ of units of $H(X; \mathbb{Q})$. We require $\Psi$ is functorial with respect to pull-backs and is a polynomial in Chern classes of its argument. Also let $\phi(x) \in \mathbb{Q}[[x]]$ be a formal power series and put $\Phi(X) := \phi(x_1) \cdots \phi(x_n) \in H^*(X; \mathbb{Q})$ with $x_1, \ldots, x_n$ the Chern roots of $T_X$. For $x \in K(X)$, define a power series in $\mathbb{Q}[[z]]$ as follows:

$$H_{\Psi, \Phi}(X, x) := \sum_{n=0}^\infty \int_{X[n]} \Psi(x[n]) \Phi(x[n]) z^n.$$  

**Theorem 2.3.** For each integer $r$ there are universal power series $A_i \in \mathbb{Q}[[z]]$, $i = 1, 2$, depending only on $\Psi$, $\Phi$ and $r$, such that for each $x \in K(C)$ of rank $r$ we have

$$H_{\Psi, \Phi}(C, x) = \exp(\int_C (c_1(x)A_1 + c_1(C)A_2)).$$

The proof of the above theorem is similar as the proof of Theorem 4.2 in [8] so we omit the details here. The main idea is that $H_{\Psi, \Phi}$ factors through $\mathbb{Q}^2$ to $\mathbb{Q}[[z]]$ and we can choose $(\mathbb{P}^1, r\mathcal{O})$ and $(\mathbb{P}^1, (r-1)\mathcal{O} \oplus \mathcal{O}(-1))$ as the “basis”.

3. Proof of theorems

3.1. **Localizations on Hilbert schemes of points.** The linear coordinates on $\mathbb{C}[n]$ are given by $p_i(z_1, \ldots, z_n) = z_1^i + \cdots + z_n^i$. The induced torus action on $\mathbb{C}[n]$ is given by

$$q \cdot p_i = q^i p_i, \quad q = \exp(t) \in \mathbb{C}^*.$$

This action has only one fixed point at $p_1 = \cdots = p_n = 0$, and the tangent bundle and the tautological bundle $\mathcal{O}_{\mathbb{C}[n]}[n]$ have the following weight decompositions at this point:

$$T_{\mathbb{C}[n]} = q^{-1} + \cdots + q^{-n},$$

$$\mathcal{O}_{\mathbb{C}[n]} = 1 + q + \cdots + q^{n-1}.$$

For $A = (a_1, \ldots, a_r)$, where $a_1, \ldots, a_r \in \mathbb{Z}$, denote by $\mathcal{E}^A = \mathcal{O}_{\mathbb{C}[n]}^a \oplus \cdots \oplus \mathcal{O}_{\mathbb{C}[n]}^a$ the rank $r$ $T$-equivariant vector bundle of weight $(a_1, \ldots, a_r)$. The tautological bundle $(\mathcal{E}^A)[n]$ has the following weight decomposition at the fixed point:

$$(\mathcal{E}^A)[n] = \sum_{i=1}^r a_i (1 + q + \cdots + q^{n-1}).$$

For a smooth (quasi-)projective curve $C$ which admits a torus action with isolated fixed points $P_1, \ldots, P_l$ and $u_i = q^{c_i}$ the weights of $T_{P_i} C$, this torus action induces a $T$-action on $S[n]$. The fixed points on $S[n]$ are parameterized by nonnegative integers $(n_1, \ldots, n_l)$ such that

$$n_1 + \cdots + n_l = n.$$
The weight decomposition of the tangent space at the fixed point is given by:

\[ \sum_{i=1}^{l} (u_i^{-1} + \cdots + u_i^{-n_i}). \]

Suppose \( E \) is a rank \( r \) equivariant vector bundle on \( C \) such that

\[ E|_{P_i} = q^{a_1} + \cdots + q^{a_r}. \]

Then the weight of \( E^{[n]} \) at the fixed point \( (n_1, \cdots, n_l) \) is given by

\[ \sum_{i=1}^{l} \sum_{j=1}^{r} (t^{n_j} (1 + u_i + \cdots + u_i^{-1})). \]

Let \( \psi(x) \in \mathbb{Q}[[x]] \) be a formal power series. For \( x \in K(C) \) of rank \( r \), Let \( \Psi(x^[n]) = \psi(e_1(x^[n])) \cdots \psi(e_{rn}(x^[n])) \), where \( e_1(x^[n]), \ldots, e_{rn}(x^[n]) \) are Chern roots of \( x^[n] \). It is obvious to see that such \( \Psi \) satisfies the conditions in Theorem 2.3.

Let \( v_i = (v_1 = a_i t, \cdots, v_r = a_r t) \) and \( w_i = c_i t \). Using the localization formula, the equivariant version of \( H_{\Psi, \Phi}(C, E^{A}_r) \) which we denote by \( H_{\Psi, \Phi}(C, E^{A}_r)(t) \) is as follows:

\[
H_{\Psi, \Phi}(C, E^{A}_r)(t) = \sum_{n=0}^{\infty} z^n \frac{\phi(-st)}{-st} \prod_{j=1}^{r} \psi(v_j + (s - 1)t).
\]

Assume that

\[
H_{\Psi, \Phi}(C, E^{A}_r)(t) = \exp \left( \sum_{n=1}^{\infty} z^n \int_{\mathbb{C}}^{t} \left( \sum_{j=0}^{r} \left( A_{0,j}^{n} c_j'(C) c_j^{(A)} + A_{1,j}^{n} c_1'(C) c_j^{(A)} \right) \right) \right)
\]

\[
= \exp \left( \sum_{n=1}^{\infty} z^n \int_{\mathbb{C}}^{t} \left( \sum_{j=0}^{r} \left( A_{0,j}^{n} \sigma_j(v_1, \cdots, v_r) / t + A_{1,j}^{n} \sigma_j(v_1, \cdots, v_r) \right) \right) \right),
\]

where we denote by \( \int_{\mathbb{C}}^{t} \) the equivariant integral, \( c_1' \) is the equivariant Chern classes and \( \sigma_j \) is the \( j \)-th elementary symmetric polynomial.
Denote by $H_{\Psi,\Phi}(C, E)(t)$ the equivariant version of $H_{\Psi,\Phi}(C, E)$. It can be computed by localization as follows:

$$H_{\Psi,\Phi}(C, E)(t) = \sum_{n=0}^{\infty} z^n \prod_{i=1}^{l} \prod_{s=1}^{n_i} \frac{\phi(-sw_i)}{-sw_i} \prod_{j=1}^{r} \psi(v_j + (s - 1)w_i)$$

$$= \prod_{i=1}^{l} \sum_{n_i=0}^{\infty} z^{n_i} \prod_{s=1}^{n_i} \frac{\phi(-sw_i)}{-sw_i} \prod_{j=1}^{r} \psi(v_j + (s - 1)w_i)$$

$$= \prod_{i=1}^{l} H_{\Psi,\Phi}(\mathbb{C}, \mathcal{E}^A)(w_i)$$

$$= \prod_{i=1}^{l} \exp(\sum_{n=1}^{\infty} z^n \int_0^t (\sum_{j=0}^{r} (A_{0,j}^n \frac{\sigma_j(v_1, \cdots, v_r)}{w_i} + A_{1,j}^n \frac{\sigma_j(v_1, \cdots, v_r)}{w_i}))$$

$$= \exp(\sum_{n=1}^{\infty} z^n \sum_{j=0}^{r} (A_{0,j}^n \sum_{i=1}^{l} \frac{\sigma_j(v_1, \cdots, v_r)}{w_i} + A_{1,j}^n \sum_{i=1}^{l} \sigma_j(v_1, \cdots, v_r)))$$

$$= \exp(\sum_{n=1}^{\infty} z^n \int_0^t (\sum_{j=0}^{r} (A_{0,j}^n c_0^j(C)c_j(E^A) + A_{1,j}^n c_1^j(C)c_j(E))).$$

If $C$ is projective, by taking nonequivariant limit one has

$$H_{\Psi,\Phi}(C, E) = \exp(\sum_{n=1}^{\infty} z^n \int_0^t (\sum_{j=0}^{r} (A_{0,j}^n c_0^j(C)c_j(E^A) + A_{1,j}^n c_1^j(C)c_j(E))).$$

### 3.2. Proof of Theorem 1.1

Let us see the general case of Conjecture 2 first. Taking $\Psi : K(X) \to H^\ast$ to be the total Chern class and $\Phi = 1$, we see that such $H_{\Psi,\Phi}$ satisfies the conditions in Theorem 2.3 and hence can be written in the desired form. So $\frac{1}{n} A_n^r, \frac{1}{n} B_n^r, \frac{1}{n} C_n^r$ and $\frac{1}{n} D_n^r$ exist as the n-th coefficients of the corresponding universal power series. Now clearly $A_n^r, B_n^r, C_n^r$ and $D_n^r$ are rational numbers depending only on $r$ and $n$, and we hope to determine them explicitly as integers. As we have discussed in Section 3.1 in order to prove Conjecture 2 it suffices to prove the equivariant version of $(C, \mathcal{E}^A)$. Using localization, we have checked Conjecture 2 for $r = 2, 3, 4, 5$ and $n < 10$. We also conjecture that

$$B_n^2 = (-1)^n \left( 4^n - \binom{2n-1}{n-1} \right),$$

and

$$B_n^3 = (-1)^n \left( \sum_{i=0}^{n-1} \frac{2n-2-i}{n} (n-i)(3n-3i-1) \binom{3n}{i} - \binom{3n-1}{n-1} \right)$$

and have checked them up to $n = 10$. However effort fails to find the explicit expressions for higher $r$. 
Before proving Theorem 1.1, recall that in Section 2.5 of [25] the following identity is established:

\[\sum_{n=0}^{\infty} z^n \chi(C^{[n]}, \Lambda - y(E_1^{[n]}))(q) = \sum_{n=0}^{\infty} z^n \prod_{i=1}^{n} \frac{1 - y^{a_i}q^i - 1}{1 - q^i} = \exp\left(\sum_{n=1}^{\infty} \frac{z^n}{n} \left(1 - q^n\right)\right) = \exp\left(\sum_{n=1}^{\infty} \frac{z^n}{n} \chi(C, \Lambda - y \cdot E_1^{[n]})(q^n)\right).\]

(6)

Here \(\Lambda_u E = \sum_{i=0}^{n} u^i \Lambda^i E\) and \(\chi(C^{[n]}, \Lambda - y(\mathcal{O}_C)^{[n]})(q)\) the equivariant Euler characteristic of \(\Lambda - y(\mathcal{O}_C)^{[n]}\) on \(C^{[n]}\).

Moreover, the discussion in the last subsection implies that (6) can be generalized as the following:

**Proposition 3.1.** For a smooth projective curve \(C\) and a line bundle \(L\) on \(C\),

\[\sum_{n=0}^{\infty} z^n \chi(C^{[n]}, \Lambda - y L^{[n]}) = \sum_{n=0}^{\infty} z^n \chi(C, \Lambda - y^n L).\]

To prove Theorem [14], we only need to prove the following lemma by using (6):

**Lemma 3.2.**

\[\sum_{n=0}^{\infty} z^n \int_{C^{[n]}} e^t((E_1^{[n]}))^n) = \exp\left(\sum_{n=1}^{\infty} (-1)^{n+1} z^n \int_{C} e^t(A_1^{[n]})\right),\]

where \(A = (a), a \in \mathbb{Z}\) is the equivariant weight of \(\mathcal{O}_C\).

**Proof.** Similarly as in [12], let \(q = \exp(\beta t), y = \exp(\beta)\), and take \(\beta \to 0\) in (6), one has

\[\sum_{n=0}^{\infty} z^n \prod_{i=1}^{n} \frac{1 + at + (i - 1)t}{it} = \exp\left(\sum_{n=1}^{\infty} \frac{z^n n + an t}{nt}\right) = \exp\left(\sum_{n=1}^{\infty} \frac{z^n 1 + at}{t}\right).\]
Hence
\[ \sum_{n=0}^{\infty} z^n \int_{\mathbb{C}[n]} e^t ((E^n_1)[n]) \]
\[ = \sum_{n=0}^{\infty} \prod_{i=1}^{n} \frac{1 + at + (i-1)t}{i} \]
\[ = \sum_{n=0}^{\infty} (-z)^n \prod_{i=1}^{n} \frac{1 + at + (i-1)t}{it} \]
\[ = \exp(\sum_{n=1}^{\infty} (-z)^n \frac{1 + at}{n} \frac{1}{t}) \]
\[ = \exp(\sum_{n=1}^{\infty} \frac{(-1)^{n+1} z^n 1 + at}{n} \frac{1}{-t}) \]
\[ = \exp(\sum_{n=1}^{\infty} \frac{(-1)^{n+1} z^n}{n} \int_{\mathbb{C}} e^t (E^n_1)). \]

\[ \square \]

3.3. Proof of Theorem 1.2. We also have an equivariant version of (6) of \( \mathbb{C}. \) However, it is also difficult to compute. We have to find some other way to give a proof.

Denote \( O(d) \) over \( \mathbb{P}^1 \) by \( L_d \) and \( \int_{(\mathbb{P}^1)^{[n]}} s(L_d^{[n]}) \) by \( N_n^d \). As it has been discussed, Theorem 1.2 is true if the following lemma holds:

Lemma 3.3.
\[ \sum_{n=0}^{\infty} z^n N_n^0 = \frac{(1-k)^2}{1-2k}, \]
\[ (7) \]
and
\[ \sum_{n=0}^{\infty} z^n N_n^{-1} = \frac{1-k}{1-2k}, \]
\[ (8) \]
\[ \text{Here } z = k(1-k). \]

We will use the localization formula to prove this lemma. Recall that the homogeneous coordinates on \( \mathbb{P}^1 \) are given by \([\zeta_1 : \zeta_2] \) and there is a torus-action on \( \mathbb{P}^1 \):
\[ q \cdot [\zeta_1 : \zeta_2] = [\zeta_1 : q \cdot \zeta_2] \cdot \]

There are two fixed points \( P_1 = [1 : 0] \) and \( P_2 = [0 : 1] \) on \( \mathbb{P}^1 \). We choose the canonical lifting to the tangent bundle of \( \mathbb{P}^1 \) and the weight decompositions of the cotangent space at \( P_1 \) and \( P_2 \) are given by \( q^{-1} \) and \( q \) respectively. We also choose a lifting to \( L_d \) such that the weight decomposition of \( L_d \) is given by \( L_d|_{P_1} = 1 \) and \( L_d|_{P_1} = q^{-d} \). Denote the equivariant integral \( \int_{(\mathbb{P}^1)^{[n]}} s_x^t(L_d^{[n]}) \) by \( N_n^d(t) \), where
\[ s_x^t(L_d^{[n]}) = \frac{1}{c_x^t(L_d^{[n]})} = \frac{1}{1 + xc_1^t(L_d^{[n]}) + \cdots + x^n c_n^t(L_d^{[n]})} \]
is the equivariant total Segre class. $N^d_n$ is the coefficient of $x^n$ in $N^d_n(t)$. By localization formula one has

$$N^d_n(t) = \sum_{k=0}^{n} \prod_{i=1}^{k} \frac{1}{1 - x(i - 1)t(it)} \prod_{i=1}^{n-k} \frac{1}{1 + x((i - 1)t - dt)(-it)}.$$ 

Here we write $\prod_{i=0}^{s-1} (-1)^i = 1$ for convenient notation.

We have the following lemma:

**Lemma 3.4.** One has

$$N^d_n(t) = \frac{(2n - 2 - d)n^n}{\prod_{i=0}^{n-1} (1 + (d - i)xt)(1 - ixt)},$$

and hence it is easy to see by comparing the coefficient that $N^d_n = \binom{2n - 2 - d}{n}.$

**Proof.**

$$N^d_n(t) = \sum_{k=0}^{n} \prod_{i=1}^{k} \frac{1}{1 - x(i - 1)t(it)} \prod_{i=1}^{n-k} \frac{1}{1 + x((i - 1)t - dt)(-it)}$$

$$= \sum_{k=0}^{n} \frac{y^n}{\prod_{i=1}^{k} (y - (i - 1))(iy)} \prod_{i=1}^{n-k} \frac{1}{1 + (i - 1 - d)(-it)}$$

$$= \frac{y^n}{\prod_{i=0}^{n} (y + (d - i))(y - i)t^n} \sum_{k=0}^{n} \frac{1}{k!(n-k)!} \prod_{i=k}^{n} (-y + i) \prod_{i=n-k}^{n-1} (y + i - d)$$

$$= \frac{y^n}{\prod_{i=0}^{n} (y + (d - i))(y - i)t^n} \sum_{k=0}^{n} \binom{-y + n - 1}{n - k} \binom{y + n - 1 - d}{k}$$

$$= \frac{y^n}{\prod_{i=0}^{n} (y + (d - i))(y - i)t^n} \binom{2n - 2 - d}{n}.$$ 

The last identity is a special case of the Chu-Vandermonde identity (cf. [13], P. 45 Exercise 3.2 (a)).

Take $x = \frac{1}{ty}$ and one gets

$$N^d_n(t) = \frac{(2n - 2 - d)n^n}{\prod_{i=0}^{n-1} (1 + (d - i)xt)(1 - ixt)}.$$
Now we are going to prove (7).

\((N^0_n)\) is the integer sequence A001791 in the on-line encyclopedia of integer sequences \[21\] and the generating series is given by

\[
\sum_{n=0}^{\infty} N^0_n z^n = \frac{1 - 2z + \sqrt{1-4z}}{2\sqrt{1-4z}}.
\]

Take \(z = k(1-k)\) and one can easily get (7).

Applying similar arguments we can prove (8), so we omit the proof here.

REFERENCES

[1] Arbarello E, Cornalba M, Griffiths P. Geometry of Algebraic Curves: Volume II with a contribution by Joseph Daniel Harris, volume 268. Springer Science & Business Media, 2011
[2] Atiyah M F, Singer I M. The index of elliptic operators: III. Annals of mathematics, 546–604 (1968)
[3] Beltrametti M, Sommese A J. Zero cycles and k-th order embeddings of smooth projective surfaces. Proceedings of Problems in the theory of surfaces and their classification, Symposia Math, volume 32, 1991. 33–48
[4] Beltrametti M, Sommese A J. On k-spannedness for projective surfaces. Springer (1990)
[5] Borisov L, Libgober A. McKay correspondence for elliptic genera. Annals of mathematics, 1521–1569 (2005)
[6] Catanese F, Gottsche L. d-very-ample line bundles and embeddings of Hilbert schemes of 0-cycles. manuscripta mathematica, 68(1):337–341 (1990)
[7] Cotterill E. Geometry of curves with exceptional secant planes: linear series along the general curve. Mathematische Zeitschrift, 267(3-4):549–582 (2011)
[8] Ellingsrud G, Gottsche L, Lehn M. On the cobordism class of the Hilbert scheme of a surface. Journal of Algebraic Geometry, 10:81–100 (2001)
[9] Ellingsrud G, Stremme S. Botts formula and enumerative geometry. Journal of the American Mathematical Society, 9(1):175–193 (1996)
[10] Gottsche L. The Betti numbers of the Hilbert scheme of points on a smooth projective surface. Mathematische Annalen, 286(1):193–207 (1990)
[11] Gottsche L, Soergel W. Perverse sheaves and the cohomology of Hilbert schemes of smooth algebraic surfaces. Mathematische Annalen, 296(1):235–245 (1993)
[12] Iqbal A, Nazir S, Raza Z, et al. Generalizations of Nekrasov-Okounkov Identity. Annals of Combinatorics, 16(4):745–753 (2012)
[13] Koepf W. Hypergeometric summation. Braunschweig/Wiesbaden: Vieweg, 2014
[14] Le Barz P. Formules pour les multiteseantes des surfaces algebriques. L’Ens. Math, 33:1–66 (1987)
[15] Le Barz P. Sur une formule de Castelnuovo pour les espaces multiseantes. Bollettino dell’unione matematica italiana. Sezione B: articoli di ricerca matematica, 10(2):381–388 (2007)
[16] Lehn M. Chern classes of tautological sheaves on Hilbert schemes of points on surfaces. Inventiones mathematicae, 136(1):157–207 (1999)
[17] Marian A, Oprea D. Tautological integrals over symmetric powers of curves. http://math.ucsd.edu/~doprea/segre-curves.pdf (2015)
[18] Marian A, Oprea D, Pandharipande R. Segre classes and Hilbert schemes of points. arXiv preprint arXiv:1507.00688 (2015)
[19] Nakajima H. Lectures on Hilbert schemes of points on surfaces. Number 18. American Mathematical Soc., 1999
[20] Rennemo J V. Universal polynomials for tautological integrals on Hilbert schemes. arXiv preprint arXiv:1205.1851 (2012)
[21] Sloane N J. The on-line encyclopedia of integer sequences. Published electronically at https://oeis.org
[22] Tikhomirov A. Standard bundles on a Hilbert scheme of points on a surface. Proceedings of Algebraic Geometry and its Applications. Vieweg+ Teubner Verlag, 1994: 183-203
[23] Tikhomirov A, Troshina T. Top Segre Class of a Standard Vector Bundle $\varepsilon^2_4$ on the Hilbert Scheme $\text{Hilb}^4S$ of a Surface $S$. Proceedings of Algebraic Geometry and its Applications. Vieweg+ Teubner Verlag, 1994: 205–226
[24] Troshina T. The degree of the top Segre class of the standard vector bundle on the Hilbert scheme $\text{Hilb}^4S$ of an algebraic surface $S$. Izvestiya Rossiiskoi Akademii Nauk. Seriya Matematicheskaya, 1993, 57(6): 106-129
[25] Wang Z, Zhou J. Tautological sheaves on Hilbert schemes of points. Journal of Algebraic Geometry, 23(4):669–692 (2014)

Institute of Mathematics, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, P. R. China
E-mail address: zlwang@amss.ac.cn