Brane-wave Duality

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Abstract
The quantum mechanical transition between a free particle Lagrangian and the Klein Gordon field description of a free particle (particle wave duality) is conjectured to extend to an analogous construction of relativistically invariant wave equations associated with strings and branes, which we propose to call brane-wave duality. Electromagnetic interactions in the two systems are discussed. It is emphasised that all integrable free field theories, including those of Dirac-Born-Infeld type, are associated with Lagrangians equivalent to divergences on the space of solutions of the equations of motion.

1 Introduction

In standard textbooks on Quantum Mechanics, the description of free particle motion by the classical point particle Lagrangian

\[ \mathcal{L}_1 = \sqrt{\sum \left( \frac{\partial X^\mu}{\partial \tau} \right)^2} \] (1)

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goes over in terms of a field theory, to that given by the Lagrangian of a Klein Gordon field;

\[ L_2 = \frac{1}{2} \sum \left( \frac{\partial \phi}{\partial x_\mu} \right)^2 \]  

(2)

(for a massless field). This may be cited as an example of particle-wave duality. To distinguish between Lagrangians which involve square roots and those which do not we use a cursive notation for the former and a capital for the latter. Is there a similar alternative description of strings and branes?

One goal of this article is to suggest that there is a natural extension. The idea is that corresponding to each direction in the world-volume there should be associated a field. For example the strings described by the Nambu-Goto String Lagrangian

\[ L_3 = \sqrt{\sum \left[ \left( \frac{\partial X^\mu}{\partial \sigma} \frac{\partial X^{\nu}}{\partial \tau} \right)^2 - \left( \frac{\partial X^\mu}{\partial \sigma} \frac{\partial X^{\nu}}{\partial \tau} \right)^2 \right] } \]  

(3)

should also admit a description in terms of two fields with Lagrangian which is some power of the following

\[ L_4 = \sum \left[ \left( \frac{\partial \phi}{\partial x_\mu} \frac{\partial \psi}{\partial x_\mu} \right)^2 - \left( \frac{\partial \phi}{\partial x_\mu} \frac{\partial \psi}{\partial x_\nu} \right)^2 \right] \]

\[ = -\frac{1}{2} \sum_{\mu, \nu} \left( \frac{\partial \phi}{\partial x_\mu} \frac{\partial \psi}{\partial x_\nu} - \frac{\partial \phi}{\partial x_\nu} \frac{\partial \psi}{\partial x_\mu} \right)^2 \]  

(4)

Here \( \phi(x_\mu), \psi(x_\mu) \) are two fields and \( \mu, \nu \) range over the dimensionality of space time. Likewise, a simple brane Lagrangian, \( \sqrt{\det \frac{\partial X^\mu}{\partial \sigma_i} \frac{\partial X^{\nu}}{\partial \sigma_j} } \) may be conjectured to be equivalent to a field theory with as many components as there are world-volume co-ordinates with Lagrangian a power of

\[ L = \det \left| \frac{\partial \phi^i}{\partial x_\mu} \frac{\partial \phi^j}{\partial x_\mu} \right| = \left( \frac{n!(d-n)!}{d!} \right) \sum \left( \frac{\partial \{ \phi^1, \phi^2, \ldots, \phi^n \} \ldots \partial \{ x_{\mu_1}, x_{\mu_2} \ldots x_{\mu_n} \} }{d!} \right)^2 . \]  

(5)

Here the sum is over all permutations of the squares of Jacobians of the \( n \) fields with respect to selections of \( n \) (the dimension of the world volume)
out of the $d$ co-ordinates $x_\mu$ of space-time. This higher dimensional analogue of particle-wave duality we might facetiously describe as brane-wave duality. Similar ideas have been advanced before by Hosotani [1][2] and Morris [3][4]. What they do is to take the Lagrangian for a string in $d$-dimensions and relate it to a Lagrangian for $d - 2$ fields in $d$ dimensions. This amounts to a change of variables in their case. They show that the classical equations of motion are equivalent under interchange of dependent and independent variables. However, we should like to advocate that since the quantum mechanical equation which describes a free particle is the Klein Gordon equation, for only one field, independent of the dimension of the embedding space, it would seem that in the case of $n$ branes the field theories describing them quantum mechanically should depend upon only $n$, rather than $d - n$ fields. We shall argue however, that it seems more attractive to take a square root as the correct power, as this, as we shall demonstrate, guarantees covariance of the equations of motion; the counterpart of the reparametrisation invariance of the original Dirac-Born-Infeld brane Lagrangians.

The two properties found for the Nambu-Goto Lagrangian ($\mathcal{L}_3$), namely that it transforms under reparametrisation of the independent variables with a factor which is the Jacobian of the transformation and that $\mathcal{L}_3$ vanishes or is constant on the space of solutions of the equations of motion of the Schild Lagrangian ($\mathcal{L}_3^2$)[5][6], have recently been shown to persist for Lagrangians of the Dirac-Born-Infeld type [7][8][9],

$$\mathcal{L} = \sqrt{\det |g_{ij} + F_{ij}|} = \sqrt{\det \left( \frac{\partial X^\mu}{\partial x_i} \frac{\partial X^\mu}{\partial x_j} + \left( \frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j} \right) \right)}.$$

(6)

...
In fact we may note that a generic property of free fields is that their Lagrangians vanish, or are divergences on the space of solutions of the equations of motion which vanish, with vanishing first derivatives on the boundary. This is trivial for $L_2$, the Klein Gordon Lagrangian, which may be rewritten after partial integration as

$$L_2 = -\frac{1}{2} \sum \phi \frac{\partial^2 \phi}{\partial x^2}.$$  \hspace{1cm} (7)

Likewise the Dirac Lagrangian, $i \bar{\psi} \gamma_\mu \partial_\mu \psi - m \bar{\psi} \psi$ which is already in null form, that for the Maxwell theory and pure gravity all vanish on the space of solutions of the equations of motion. It is not true of non abelian gauge theory, except for the self dual sector, where, by definition

$$F_{\mu\nu} F^{\mu\nu} = \frac{1}{2} \epsilon_{\mu\rho\sigma} F^{\mu\nu} F^{\rho\sigma},$$  \hspace{1cm} (8)

and the right hand side is a divergence. This principle extends to the wave Lagrangians of Born-Infeld type in a straightforward manner. Here, under field redefinitions $\phi \rightarrow \Phi(\phi, \psi)$, $\psi \rightarrow \Psi(\phi, \psi)$ the Lagrangian $L_4 = \sqrt{L_4}$ scales with a conformal factor which is the Jacobian of the transformation.

Also the quantity

$$\phi \frac{\partial}{\partial x_\mu} \left( \frac{\partial L_4}{\partial \phi} \right),$$  \hspace{1cm} (9)

considered as a Lagrangian, reproduces the same equation of motion as does $L_4$ and is zero on the space of solutions. It thus differs from $L_4$ by a divergence. In a similar fashion

$$\psi \frac{\partial}{\partial x_\mu} \left( \frac{\partial L_4}{\partial \phi} \right),$$  \hspace{1cm} (10)

is itself a divergence. Similar remarks, mutatis mutandis, can be made about the same constructions with $\phi \leftrightarrow \psi$. These statements are easy to prove and simply depend upon the determinantal nature of $L_4$. The key result, which obtains also in general dimension for $L_4$ is that

$$\sum \frac{\partial \phi^i}{\partial x_\mu} \frac{\partial L_4}{\partial \phi^j} \frac{\partial \phi^j}{\partial x_\mu} = 2 \delta^i_j L.$$  \hspace{1cm} (11)
Consider
\[ \phi \frac{\partial}{\partial x_\mu} \left( \frac{\partial L_4}{\partial \phi} \right) = \frac{\partial}{\partial x_\mu} \left( \phi \frac{\partial L_4}{\partial \phi} \right) - \frac{\partial \phi}{\partial x_\mu} \left( \frac{\partial L_4}{\partial \phi} \right). \]  
(12)
The first term on the right hand side is a divergence, the second is simply \(-2L_4\), as a consequence of the determinantal properties of \(L_4\). Thus the left hand side of (12) serves as an equivalent Lagrangian, giving the same equations of motion as the original one. In the case of (10) the corresponding term left over after extracting the divergence is zero, again as a consequence of determinantal properties. Obviously this argument extends to dual brane Lagrangians of any dimension. This property suggests that these Lagrangians have a pseudo-topological aspect. If they were equivalent to divergences without any further constraint, they would be fully topological; but here the equivalence works only for solutions of the equations of motion.

3 Equations of Motion of Born-Infeld Type

For the classical point particle Lagrangian (1) then the equations of motion can be written as
\[ \frac{\partial^2 X^\mu}{\partial \tau^2} \frac{\partial X^\nu}{\partial \tau} - \frac{\partial^2 X^\nu}{\partial \tau^2} \frac{\partial X^\mu}{\partial \tau} = 0. \]  
(13)
For \(d\) dimensions then it is easy to verify that there are \(d-1\) independent equations of motion. Now consider the string case in \(d=3\) dimensions. The Nambu-Goto Lagrangian \(L_3\) gives the single equation of motion
\[ \left( \begin{array}{ccc} \dot{J}_1 & \dot{J}_2 & \dot{J}_3 \end{array} \right) \left( \begin{array}{ccc} X^1_{\sigma\sigma} & X^1_{\sigma\tau} & X^1_{\tau\tau} \\ X^2_{\sigma\sigma} & X^2_{\sigma\tau} & X^2_{\tau\tau} \\ X^3_{\sigma\sigma} & X^3_{\sigma\tau} & X^3_{\tau\tau} \end{array} \right) \left( \begin{array}{c} (X^1_\tau)^2 + (X^2_\tau)^2 + (X^3_\tau)^2 \\ -2(X^1_\sigma X^1_\tau + X^2_\sigma X^2_\tau + X^3_\sigma X^3_\tau) \\ (X^1_\sigma)^2 + (X^2_\sigma)^2 + (X^3_\sigma)^2 \end{array} \right) = 0 \]  
(14)
where
\[ X^\mu_{ij} = \frac{\partial^2 X^\mu}{\partial \sigma^i \partial \sigma^j}, \quad X^\mu_i = \frac{\partial X^\mu}{\partial \sigma^i}, \quad \text{and} \quad \sigma^i = (\sigma, \tau), \]  
(15)
\[ \dot{J}_\rho = \epsilon_{\rho\mu\nu} X^\mu_{\sigma} X^\nu_{\tau} = \frac{1}{2} \epsilon_{\rho\mu\nu} \left| \begin{array}{cc} X^\mu_{\sigma} & X^\nu_{\tau} \\ X^\mu_{\rho} & X^\nu_{\tau} \end{array} \right| \]  
(16)
In general, a typical equation of motion, of which only \(d-2\) are independent can be written in the following form:
\[ \dot{J}_\nu X^\mu_{ij} (L^{-1})_{ij} = 0, \]  
(17)
where L is the matrix with components \([L]_{ij} = \frac{\partial X^\mu}{\partial \sigma^i} \frac{\partial X^\mu}{\partial \sigma^j}\) and \(\nu\) is chosen from three of the values \(\nu_1, \nu_2, \nu_3\) of the index \(\mu\) which runs over \(1 \ldots d\). \(\hat{J}_{\nu_1}\) denotes the Jacobian \(\frac{\partial (X^{\nu_2}, X^{\nu_3})}{\partial (\sigma_1, \sigma_2)}\), omitting \(X^{\nu_1}\) etc. This can be extended to strings in \(d\) dimensions and to branes. The only essential difference is that in the typical equation of motion, \(\nu\) is now an arbitrary choice of \(n + 1\) values and \(\hat{J}_\nu\) is now a Jacobian of a subset of \(n\) of those variables \(x^\nu\), with respect to the \(n\) world sheet co-ordinates \(\sigma_j\).

Computer calculations show that for a the Nambu-Goto Lagrangian there are \(d - 2\) independent equations of motion. In general, an object (particle/string/brane) which sweeps out an \(n\)-dimensional world volume in \(d\)-dimensional space-time has only \(d - n\) independent equations of motion. The basic reason for this is that in the case \(d = n\) the Lagrangian is a divergence, so all the equations of motion vanish.

### 4 Equations of Motion of Inverse Type

For the Lagrangians of inverse type, i.e. \(\mathcal{L}_2 = \sqrt{\mathcal{L}_2}, \mathcal{L}_4 = \sqrt{\mathcal{L}_4}\) etc. there is just one equation for \(\mathcal{L}_2\) and two for \(\mathcal{L}_4\) and so on, irrespective of the dimension of the total space; however these equations fall into sums of equations appropriate to the minimal dimension for a non trivial embedding. What this means is that for \(\mathcal{L}_2\) in 2 dimensions, the minimal case the equation is the well known Bateman equation \([12]\)

\[
\left( \frac{\partial \phi}{\partial x_1} \right)^2 \frac{\partial^2 \phi}{\partial x_2^2} + \left( \frac{\partial \phi}{\partial x_2} \right)^2 \frac{\partial^2 \phi}{\partial x_1^2} - 2 \left( \frac{\partial \phi}{\partial x_1} \right) \left( \frac{\partial \phi}{\partial x_2} \right) \frac{\partial^2 \phi}{\partial x_1 \partial x_2} = 0, \tag{18}
\]

to be discussed at greater length in section 5, while in 3 dimensions the equation is the sum of three Bateman equations, corresponding to the three ways of selecting two co-ordinates out of three. A particular class of solutions to these equations can be found by simultaneously setting these three Bateman equations to zero. It is also noteworthy that these three equations also result from the transformation of the three equations of the form \([13]\) by exchanging the rôles of dependent and independent variables. Much the same happens for \(\mathcal{L}_4\). Here the minimal dimension is 3. The two equations of motion in \(d\) dimensions fall into the sum of \(\binom{d}{3}\) copies of the minimal equations, whose solution is discussed in section 5.
5 Covariance

This chapter deals with special properties of Lagrangian densities of the previous form, but where the square root has been taken, as in the standard Born-Infeld. Here the feature of general covariance plays an important rôle. This works for arbitrary dimension; under the transformation of the Lagrangian \( L \) of square root type under the field redefinition

\[ \phi^i \rightarrow \Phi^i(\phi^1, \phi^2, \ldots, \phi^n), \]  

\( L \) acquires a factor which is the Jacobian of the transformation and the equations of motion are unaffected on account of (11) since they are given by

\[ \frac{\partial J}{\partial \Phi^i} L - J \frac{\partial L}{\partial \phi^j} \left( J \frac{\partial L}{\partial \phi^j} \right) \]

\[ = \frac{\partial J}{\partial \phi^i} L - \frac{\partial J}{\partial \phi^j} \frac{\partial L}{\partial \phi^j} \frac{\partial \phi^j}{\partial x_\mu} - J \frac{\partial}{\partial x_\mu} \left( \frac{\partial L}{\partial \phi^j} \right) \]

\[ = -J \frac{\partial}{\partial x_\mu} \left( \frac{\partial L}{\partial \phi^j} \right) = 0, \]

as the first two terms cancel on account of (11). Thus the square root of the determinantal form is a generally covariant Lagrangian, which means that any function of a solution remains a solution of the equations of motion. In fact the equations of motion arising from these Lagrangians, in the case where the number of co-ordinates \( x_\mu \) exceeds that of the number \( n \) of wave fields by one are a generalisation of the Bateman equation\(^1\) and are expected to be completely integrable, since this is the case for \( n = 1 \) and \( n = 2 \), as we shall demonstrate in the next section.

5.1 Integrability in special cases

Consider the Lagrangian

\[ L_{\text{two}} = \sqrt{\sum \left( \frac{\partial \phi}{\partial x_\mu} \right)^2} \]

\( ^1 \)for a generalisation in a somewhat different direction see [12]
in the case of 2 dimensions ($\mu = 1, 2$). This action is fully integrable as the 
equation of motion is just the Bateman equation \(^{18}\).

This equation has the general solution:

$$F(\phi)x_1 + G(\phi)x_2 = c = \text{constant} \quad (22)$$

where $F$, $G$ are two arbitrary functions. It is clearly covariant; if $\phi$ is a 
solution so is any function of $\phi$. It is equivalent to a Monge nonlinear wave 
equation

$$\frac{\partial u}{\partial x_1} = u \frac{\partial u}{\partial x_2} \quad (23)$$

where

$$u = \frac{\partial \phi}{\partial x_1} \frac{\partial \phi}{\partial x_2} \quad (24)$$

What is the situation with the next Lagrangian with 2 fields in 3 dimensions 
($\mu, \nu = 1, 2, 3$),

$$L_{\text{three}} = \sqrt{\sum \left[ \left( \frac{\partial \phi}{\partial x_\mu} \frac{\partial \psi}{\partial x_\mu} \right)^2 \left( \frac{\partial \phi}{\partial x_\nu} \right)^2 \left( \frac{\partial \psi}{\partial x_\nu} \right)^2 \right]} \quad (25)$$

We know that the equations of motion are covariant; therefore we expect 
that they are expressible in first order form in terms of two ratios of the 
Jacobians

$$u = \frac{\phi_{x_1} \psi_{x_2} - \phi_{x_2} \psi_{x_1}}{\phi_{x_3} \psi_{x_2} - \phi_{x_2} \psi_{x_3}} \quad (26)$$

$$v = \frac{\phi_{x_2} \psi_{x_1} - \phi_{x_1} \psi_{x_2}}{\phi_{x_3} \psi_{x_2} - \phi_{x_2} \psi_{x_3}}$$

where $\phi_{x_\mu}$ denotes the partial derivative $\frac{\partial \phi}{\partial x_\mu}$. The Lagrangian in the first 
case takes the form

$$L_{\text{two}} = \phi_{x_2} \sqrt{1 + u^2} \quad (27)$$

Working out the equation of motion gives

$$\frac{\partial}{\partial x_2} \frac{1}{\sqrt{1 + u^2}} + \frac{\partial}{\partial x_1} \frac{u}{\sqrt{1 + u^2}} = 0. \quad (28)$$
This equation is equivalent to (23). The remarkable feature of this Lagrangian is that any differentiable function \( f(u) \) instead of \( \sqrt{1 + u^2} \) will give the same equations of motion [12]! This generalises; In the second case

\[
\mathcal{L}_{\text{three}} = (\phi_{x_2} \psi_{x_3} - \phi_{x_3} \psi_{x_2}) \sqrt{(1 + u^2 + v^2)} \tag{29}
\]

and the two independent equations of motion are

\[
\frac{\partial}{\partial x_2} \frac{1}{\sqrt{1 + u^2 + v^2}} - \frac{\partial}{\partial x_1} \frac{v}{\sqrt{1 + u^2 + v^2}} = 0.
\]

\[
\frac{\partial}{\partial x_3} \frac{1}{\sqrt{1 + u^2 + v^2}} - \frac{\partial}{\partial x_1} \frac{u}{\sqrt{1 + u^2 + v^2}} = 0. \tag{30}
\]

\( \sqrt{(1 + u^2 + v^2)} \) may be replaced by any arbitrary differentiable function of two variables, \( f(u, v) \) and an equivalent pair of equations of motion is as follows;

\[
\frac{\partial u}{\partial x_1} = u \frac{\partial u}{\partial x_3} + v \frac{\partial u}{\partial x_2},
\]

\[
\frac{\partial v}{\partial x_1} = u \frac{\partial v}{\partial x_3} + v \frac{\partial v}{\partial x_2}. \tag{31}
\]

These equations admit an implicit solution for \((u, v)\) given by solving the equations

\[
u = F(x_3 + ux_1, x_2 + vx_1), \quad v = G(x_3 + ux_1, x_2 + vx_1) \tag{32}
\]

where \( F, G \) are two arbitrary functions of two variables. The general solution to the equations of motion is given by setting \( u = U(\phi, \psi) \) and \( v = V(\phi, \psi) \) where \( U, V \) are two further arbitrary functions, and solving (31) for \( \phi, \psi \). This demonstrates the integrability of the equations of motion. The generalisation to \( n \) fields in \( n + 1 \) dimensions will be straightforward. It is anticipated that there will be an integrable generalisation to \( n \) fields in \( d \) dimensions along the lines of the Universal Field Equation [12][13], but that would take us too far afield and away from the spirit of the present investigation.
6 Electromagnetic interactions

The subject of electromagnetic interactions in the study of these new Lagrangians is at this stage somewhat speculative. In the original theory, electromagnetic interactions are implemented by the rather ad-hoc procedure of adding an antisymmetric piece to the induced metric; \( g_{ij} \rightarrow g_{ij} + F_{ij} \). This is consistent with the picture of having gauge fields living on the brane, and presumably confined to it by some Meissner type effect. However the natural way to couple electromagnetic fields to the dual theory is through a coupling to the conserved currents in the theory. This will ensure gauge invariance.

It is easy to construct such conserved quantities, e.g.

\[
J_{ij}^{\mu} = \frac{\partial L}{\partial (\partial_{x_{\mu}} \phi^j)} - \frac{1}{2} \frac{\partial L}{\partial \phi^j} \partial_{x_{\mu}} \phi^i, \quad i \neq j
\]  

(33)

and in the case \( i = j \), the currents \( J^{ii} - J^{jj} \) are also conserved. There is however an embarrassment here as these currents carry two indices. Thus the natural coupling is to a two index gauge field \( A_{ij}^{\mu} \) transforming under \( SO(n) \), i.e. the contribution

\[
L_{\text{current}} = \sum_{i,j} A_{ij}^{\mu} J_{ij}^{\mu}
\]  

(34)

represents the part of the Lagrangian coupling a non-Abelian gauge field to the fields \( \phi^i \). This suggestion, though appealing, is rather unorthodox. It is however gauge invariant up to a divergence, as in the gauge transformation

\[
A_{ij}^{\mu} \rightarrow O^{ik} A_{kl}^{\mu} O^{lj} + O^{ik} \frac{\partial}{\partial x_{\mu}} O^{kj}
\]  

(35)

the rotation matrices may be removed by a linear transformation of the fields \( \phi^i \) and the inhomogeneous term converted to an innocuous divergence. If one wants instead to mimic the Dirac-Born-Infeld incorporation of electromagnetism, the simplest assumption would be to suppose that \( A_{ij}^{\mu} \) depends upon the co-ordinates \( x_{\nu} \) only through their dependence upon \( \phi^j \) with direction in a linear combination of the gradients of \( \phi^j \). Then we may set

\[
A_{ij}^{\mu} = \sum_{j} \frac{\partial \phi^j}{\partial x_{\mu}} A_{i}(\phi^j).
\]  

(36)
Then a new Lagrangian which adds a term to the matrix components anti-symmetric in \((i, j)\), the antisymmetry being related to that of \(F_{\mu\nu}\) takes the form with \(L^{ij} = \frac{\partial \phi^i}{\partial x_\mu} \frac{\partial \phi^j}{\partial x_\mu}\)

\[
\mathcal{L}' = \sqrt{\det \frac{\partial \phi^i}{\partial x_\mu} \frac{\partial \phi^j}{\partial x_\mu} + \frac{\partial}{\partial \frac{\partial \phi^i}{\partial x_\mu}} \log(\mathcal{L}) \frac{\partial}{\partial \frac{\partial \phi^j}{\partial x_\mu}} \log(\mathcal{L}) \left( \frac{\partial A_\mu}{\partial x_\nu} - \frac{\partial A_\nu}{\partial x_\mu} \right)}
\]

\[
= \sqrt{\det \left( L^{ij} + (L^{-1})^{ip}(L^{-1})^{jq} \frac{\partial \phi^p}{\partial x_\mu} \frac{\partial \phi^q}{\partial x_\nu} \left( \frac{\partial A_\mu}{\partial x_\nu} - \frac{\partial A_\nu}{\partial x_\mu} \right) \right)}
\]

This manifestly gauge invariant suggestion is in the same spirit as the original Born-Infeld Lagrangian especially if the square root form is taken.

An alternative suggestion to compare the Kalb-Ramond string interaction

\[
B_{\mu\nu} \frac{\partial X^\mu}{\partial \sigma} \frac{\partial X^\nu}{\partial \tau}
\]

seems less satisfactory, because on the dual brane side this would give

\[
B_{\mu\nu} \frac{\partial \phi}{\partial x_\mu} \frac{\partial \psi}{\partial x_\nu}
\]

and this interaction, though gauge invariant is more like a Pauli term than one coming from minimal coupling.

7 Conclusion

While our answer might require some modification, we believe that we have posed an interesting and important question; to find Lagrangians for fields which bear a similar relation to String and Brane Lagrangians as does the Klein Gordon to that of the classical point particle. A common characteristic of these Lagrangians, as indeed of all free theories, is that they are pseudo-topological, i.e. the Lagrangian is equivalent to a divergence on the space of solutions of the equations of motion. If the square root form is taken
then the new Lagrangians are covariant, and the equations of motion are integrable and coincide in the case that the dimensions of space-time exceeds the that of the fields by one, with the Universal Field Equations proposed earlier [12] [13]. Some ideas for the introduction of gauge fields have been discussed.

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References

[1] Y. Hosotani, *Phys.Rev.Letters* **47** (1981) 399.

[2] Y. Hosotani and R. Nakayama, ‘The Hamilton-Jacobi Equations for Strings and Membranes,’ [hep-th/9903193](https://arxiv.org/abs/hep-th/9903193) (1999)

[3] T.R. Morris, From First to Second Quantized String Theory, *Phys. Lett.* **202B** (1988) 222.

[4] D.L. Gee and T.R. Morris, From First to Second Quantized String Theory. 2. The Dilaton and other Fields, *Nucl. Phys* **B331** (1990) 675. D.L. Gee and T.R. Morris, From First to Second Quantized String Theory. 3. Gauge Fixing and Quantization, *Nucl. Phys* **B331** (1990) 694.

[5] A. Schild, *Phys. Rev.* **D16** (1977) 1722.

[6] T. Eguchi, *Phys. Rev. Lett.* **44** (1980) 126.

[7] M. Born and L. Infeld Proc. Roy. Soc. **A144** (1934) 425.

[8] P.A.M. Dirac, An extensible model of the electron Proc. Roy. Soc. **A268** (1962) 57-67.

[9] A.A. Tseytlin, Born-Infeld action, supersymmetry and string theory, [hep-th/9908105](https://arxiv.org/abs/hep-th/9908105) (1999) to appear in the Yuri Golfand memorial volume.

[10] D.B. Fairlie, Dirac-Born-Infeld Equations, *Phys. Lett.* **B456**, (1999) 141-146.

[11] I.L. Buchbinder and S.M. Kuzenko, *Ideas and Methods of Supersymmetry and Supergravity* I.O.P. Publishing, Ltd. (1995) (ISBN 0 75030 258 5).

[12] D.B. Fairlie, J. Govaerts and A. Morozov, Universal Field Equations with Covariant Solutions, *Nuclear Physics B* **373** (1992) 214-232.

[13] D.B. Fairlie and J. Govaerts, Euler Hierarchies and Universal Equations, *Journal of Mathematical Physics* **33** (1992) 3543-3566.
[14] D.B. Fairlie Integrable Systems in Higher Dimensions *Quantum Field Theory, Integrable Models and Beyond* Editors. T. Inami and R. Sasaki *Progress of Theoretical Physics Supplement* **118** (1995) 309-327.