Maximally Modulated Singular Integral Operators and their Applications to Pseudodifferential Operators on Banach Function Spaces

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Abstract. We prove that if the Hardy-Littlewood maximal operator is bounded on a separable Banach function space $X(\mathbb{R}^n)$ and on its associate space $X'(\mathbb{R}^n)$ and a maximally modulated Calderón-Zygmund singular integral operator $T_\Phi$ is of weak type $(r, r)$ for all $r \in (1, \infty)$, then $T_\Phi$ extends to a bounded operator on $X(\mathbb{R}^n)$. This theorem implies the boundedness of the maximally modulated Hilbert transform on variable Lebesgue spaces $L^p(\cdot) (\mathbb{R})$ under natural assumptions on the variable exponent $p : \mathbb{R} \to (1, \infty)$. Applications of the above result to the boundedness and compactness of pseudodifferential operators with $L^\infty(\mathbb{R}, V(\mathbb{R}))$-symbols on variable Lebesgue spaces $L^{p(\cdot)} (\mathbb{R})$ are considered. Here the Banach algebra $L^\infty(\mathbb{R}, V(\mathbb{R}))$ consists of all bounded measurable $V(\mathbb{R})$-valued functions on $\mathbb{R}$ where $V(\mathbb{R})$ is the Banach algebra of all functions of bounded total variation.

1. Introduction

In this paper we will be concerned with the boundedness of maximally modulated Calderón-Zygmund singular integral operators and its applications to the boundedness of pseudodifferential operators with non-regular symbols on separable Banach function spaces.

Let us define the main operators we are dealing with. Let $L^\infty_0(\mathbb{R}^n)$ and $C^\infty_0(\mathbb{R}^n)$ denote the sets of all bounded functions with compact support and all infinitely differentiable functions with compact support, respectively. A Calderón-Zygmund operator is a linear operator $T$ which is bounded on $L^2(\mathbb{R}^n)$ such that for every $f \in L^\infty_0(\mathbb{R}^n)$,

$$(Tf)(x) := \int_{\mathbb{R}^n} K(x, y)f(y) \, dy \quad \text{for a.e.} \quad x \in \mathbb{R}^n \setminus \text{supp} \, f,$$

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where supp $f$ denotes the support of $f$. The kernel

$$K : \mathbb{R}^n \times \mathbb{R}^n \setminus \{(x,x) : x \in \mathbb{R}^n\} \rightarrow \mathbb{C}$$

is assumed to satisfy the following standard conditions:

$$|K(x,y)| \leq \frac{c_0}{|x-y|^n} \quad \text{for } x \neq y$$

and

$$|K(x,y) - K(x,y')| + |K(y,x) - K(y',x)| \leq \frac{c_0|y-y'|^\tau}{|x-y'|^{n+\tau}}$$

for $|x-y| > 2|y-y'|$, where $c_0$ and $\tau$ are some positive constants independent of $x,y,y' \in \mathbb{R}^n$ (see, e.g., [G09 Section 8.1.1]). The most prominent example of Calderón-Zygmund operators is the Hilbert transform defined for $f \in L_0^\infty(\mathbb{R})$ by

$$(Hf)(x) := \frac{1}{\pi} \text{v.p.} \int_{\mathbb{R}} \frac{f(y)}{x-y} \, dy = \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{\mathbb{R}\setminus I(x,\varepsilon)} \frac{f(y)}{x-y} \, dy, \quad x \in \mathbb{R},$$

where $I(x,\varepsilon) := (x-\varepsilon, x+\varepsilon)$.

Suppose $\Phi = \{\phi_\alpha\}_{\alpha \in \mathcal{A}}$ is a family of measurable real-valued functions indexed by an arbitrary set $\mathcal{A}$. Then for every $\phi_\alpha \in \mathcal{A}$, the modulation operator is defined by

$$(\mathcal{M}^{\phi_\alpha} f)(x) := e^{-i\phi_\alpha(x)} f(x), \quad x \in \mathbb{R}^n.$$ 

Following [GMS05] (see also [DPL13]), the maximally modulated singular integral operator $T^\Phi$ of the Calderón-Zygmund operator $T$ with respect to the family $\Phi$ is defined for $f \in L_0^\infty(\mathbb{R}^n)$ by

$$(T^\Phi f)(x) := \sup_{\alpha \in \mathcal{A}} |T(\mathcal{M}^{\phi_\alpha} f)(x)|, \quad x \in \mathbb{R}^n.$$ 

This definition is motivated by the fact that the maximally modulated Hilbert transform

$$(C f)(x) := (H^\Psi f)(x) \quad \text{with } \Psi := \{\psi_\alpha(x) = \alpha x : \alpha, x \in \mathbb{R}\}$$

is closely related to the continuous version of the celebrated Carleson-Hunt theorem on the a.e. convergence of Fourier series (see, e.g., [D91 Chap. 2, Section 2.2], [G09 Chap. 11], and [MS13 Chap. 7]). In [GMS05, DPL13] the operator $C$ is called the Carleson operator, however in [G09 Section 11.1] and in [MS13 Section 7.1] this term is used for two different from $C$ and each other operators.

For $f \in C_0^\infty(\mathbb{R})$, consider the maximal singular integral operator given by

$$(1.1) \quad (S_* f)(x) := \sup_{-\infty < a < b < \infty} |(S_{(a,b)} f)(x)|, \quad x \in \mathbb{R},$$

where $S_{(a,b)} f$ is the integral analogue of the partial sum of the Fourier series given by

$$S_{(a,b)} f (x) := \frac{1}{2\pi} \int_a^b \hat{f}(\lambda) e^{ix\lambda} \, d\lambda, \quad x \in \mathbb{R},$$

and

$$\hat{f}(\lambda) := (\mathcal{F} f)(\lambda) := \int_{\mathbb{R}} f(x) e^{-ix\lambda} \, dx, \quad \lambda \in \mathbb{R},$$

is the Fourier transform of $f$. It is not difficult to see that if $f \in C_0^\infty(\mathbb{R})$, then

$$(1.2) \quad (S_* f)(x) \leq (C f)(x) \quad \text{for a.e. } x \in \mathbb{R}.$$
For a suitable function \( a \) on \( \mathbb{R} \times \mathbb{R} \), a pseudodifferential operator \( a(x, D) \) is defined for a function \( f \in C_0^\infty(\mathbb{R}) \) by the iterated integral
\[
(a(x, D)f)(x) := \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} a(x, \lambda)e^{i(x-y)\lambda} f(y) \, dy, \quad x \in \mathbb{R}.
\]
(1.3)

The function \( a \) is called the symbol of the pseudodifferential operator \( a(x, D) \).

Our results on the above mentioned operators will be formulated in terms of the Hardy-Littlewood maximal function, which we define next. Let \( 1 \leq r < \infty \). Given \( f \in L^r_{\text{loc}}(\mathbb{R}^n) \), the \( r \)-th maximal operator is defined by
\[
(M_r f)(x) := \sup_{Q \ni x} \left( \frac{1}{|Q|} \int_Q |f(y)|^r \, dy \right)^{1/r}, \quad x \in \mathbb{R}^n,
\]
where the supremum is taken over all cubes \( Q \) containing \( x \). Here, and throughout, all cubes will be assumed to have their sides parallel to the coordinate axes and \( |Q| \) will denote the volume of \( Q \). For \( r = 1 \) this is the usual Hardy-Littlewood maximal operator, which will be denoted by \( M \).

Let \( f \in L^1_{\text{loc}}(\mathbb{R}^n) \). For a cube \( Q \subset \mathbb{R}^n \), put
\[
f_Q := \frac{1}{|Q|} \int_Q f(x) \, dx.
\]
The Fefferman-Stein sharp maximal operator \( f \mapsto M\# f \) is defined by
\[
(M\# f)(x) := \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(x) - f_Q| \, dx, \quad x \in \mathbb{R}^n,
\]
where the supremum is taken over all cubes \( Q \) containing \( x \).

Banach function spaces \( X(\mathbb{R}^n) \) will be defined in Section 2. This is a wide class of spaces including rearrangement-invariant (r.i.) Lebesgue, Orlicz, and Lorentz spaces, as well as non-r.i. variable Lebesgue spaces \( L^{p(.)}(\mathbb{R}^n) \). The main feature of these spaces is the so-called lattice property: if \( |f(x)| \leq |g(x)| \) for a.e. \( x \in \mathbb{R}^n \), then \( \|f\|_{X(\mathbb{R}^n)} \leq \|g\|_{X(\mathbb{R}^n)} \). In Section 2 we collect preliminaries prepare the proof of main results given in Section 3. Let us briefly describe them.

The boundedness of maximally modulated Calderón-Zygmund operators \( T^\Phi \) on weighted Lebesgue spaces was studied by Grafakos, Martell, and Soria [GMS05]. A quantitative version of their results was obtained recently by Di Plinio and Lerner [DPL13]. We show that a pointwise inequality for the sharp maximal function of \( T^\Phi \) obtained in [GMS05] Proposition 4.1, combined with the Fefferman-Stein inequality for Banach function spaces due to Lerner [L10] Corollary 4.2, and with the self-improving property of the Hardy-Littlewood maximal function on Banach function spaces obtained by Lerner and Pérez [LP07] Corollary 1.3, imply the boundedness of \( T^\Phi \) on a separable Banach function space \( X(\mathbb{R}^n) \) under the natural assumptions that \( M \) is bounded on \( X(\mathbb{R}^n) \), \( M \) is bounded on its associate space \( X'(\mathbb{R}^n) \), and \( T^\Phi \) is of weak type \((r, r)\) for all \( r \in (1, \infty) \) (see Theorem 3.1). Notice that the latter hypothesis is satisfied for the maximally modulated Hilbert transform \( \mathcal{H} \). This gives the boundedness of \( \mathcal{H} \) on \( X(\mathbb{R}) \) (see Corollary 3.3). From here and the pointwise estimate \([1.2]\) we also get the boundedness of the operator \( S_\mathcal{H} \) on separable Banach functions spaces such that \( M \) is bounded on \( X(\mathbb{R}) \) and on \( X'(\mathbb{R}) \) (see Lemma 4.2).

Section 4 is devoted to applications of the above results to the boundedness of pseudodifferential operators with non-regular symbols on Banach function...
spaces. Note that the boundedness of $a(x, D)$ with smooth (regular) symbols in Hörmander’s and Miyachi’s classes on separable Banach function spaces was studied in [K-A14] (see also [KS13]). On the other hand, Yu. Karlovich [K-Yu07] introduced the class $L^\infty(\mathbb{R}, V(\mathbb{R}))$ of bounded measurable $V(\mathbb{R})$-valued functions on $\mathbb{R}$ where $V(\mathbb{R})$ is the Banach algebra of all functions of bounded total variation. Symbols in $L^\infty(\mathbb{R}, V(\mathbb{R}))$ may have jump discontinuities in both variables. By using the boundedness of the operator $S_*$ on $L^p(\mathbb{R})$ for $1 < p < \infty$, Yu. Karlovich [K-Yu07] Theorem 3.1 proved the boundedness of $a(x, D)$ with $a \in L^\infty(\mathbb{R}, V(\mathbb{R}))$ on $L^p(\mathbb{R})$ for $1 < p < \infty$. Later on he extended this result to weighted Lebesgue spaces $L^p(\mathbb{R}, w)$ with Muckenhoupt weights $w$ (see [K-Yu12] Theorem 4.1). One of the important ingredients of those proofs is the pointwise inequality

$$|(a(x, D)f)(x)| \leq 2(S_* f)(x)\|a(x, \cdot)\|_V, \quad x \in \mathbb{R},$$

for $f \in C_0^0(\mathbb{R})$. From this inequality and the boundedness of $S_*$ we obtain the boundedness of $a(x, D)$ on separable Banach function spaces $X(\mathbb{R})$ under the assumption that $M$ is bounded on $X(\mathbb{R})$ and $X'(\mathbb{R})$ (see Theorem 1.3).

In Section 5 we specify the above results to the case of variable Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$. If the variable exponent $p : \mathbb{R}^n \to [1, \infty]$ is bounded away from one and infinity, then in view of Diening’s theorem [D05] Theorem 8.1, the boundedness of $M$ is equivalent to the boundedness of $M$ on its associate space. Hence all above results have simpler formulations in the case of variable Lebesgue spaces (see Section 5).

We conclude the paper with a sufficient condition for the compactness of pseudodifferential operators $a(x, D)$ with $L^\infty(\mathbb{R}, V(\mathbb{R}))$-symbols on variable Lebesgue spaces $L^{p(\cdot)}(\mathbb{R})$ (see Corollary 6.3). This result is obtained from Yu. Karlovich’s result [K-Yu07] Theorem 4.1 for standard Lebesgue spaces by transferring the compactness property from standard to variable Lebesgue spaces with the aid of the Krasnosel’skii-type interpolation theorem (see Section 5.2).

2. Preliminaries

2.1. Banach function spaces. The set of all Lebesgue measurable complex-valued functions on $\mathbb{R}^n$ is denoted by $\mathcal{M}$. Let $\mathcal{M}^+$ be the subset of functions in $\mathcal{M}$ whose values lie in $[0, \infty]$. The characteristic function of a measurable set $E \subset \mathbb{R}^n$ is denoted by $\chi_E$ and the Lebesgue measure of $E$ is denoted by $|E|$.

**Definition 2.1** ([BS88] Chap. 1, Definition 1.1). A mapping $\rho : \mathcal{M}^+ \to [0, \infty]$ is called a *Banach function norm* if, for all functions $f, g, f_n (n \in \mathbb{N})$ in $\mathcal{M}^+$, for all constants $a \geq 0$, and for all measurable subsets $E$ of $\mathbb{R}^n$, the following properties hold:

(A1) \( \rho(f) = 0 \iff f = 0 \text{ a.e.} \), \( \rho(af) = a\rho(f) \), \( \rho(f + g) \leq \rho(f) + \rho(g) \),

(A2) \( 0 \leq g \leq f \text{ a.e.} \implies \rho(g) \leq \rho(f) \) (the lattice property),

(A3) \( 0 \leq f_n \uparrow f \text{ a.e.} \implies \rho(f_n) \uparrow \rho(f) \) (the Fatou property),

(A4) \( |E| < \infty \implies \rho(\chi_E) < \infty \),

(A5) \( |E| < \infty \implies \int_E f(x) \, dx \leq C_E \rho(f) \)

with $C_E \in (0, \infty)$ which may depend on $E$ and $\rho$ but is independent of $f$. 


When functions differing only on a set of measure zero are identified, the set \( X(\mathbb{R}^n) \) of all functions \( f \in \mathcal{M} \) for which \( \rho(|f|) < \infty \) is called a Banach function space. For each \( f \in X(\mathbb{R}^n) \), the norm of \( f \) is defined by
\[
\|f\|_{X(\mathbb{R}^n)} := \rho(|f|).
\]

The set \( X(\mathbb{R}^n) \) under the natural linear space operations and under this norm becomes a Banach space (see [BS88, Chap. 1, Theorems 1.4 and 1.6]).

The norm of a bounded sublinear operator \( A \) on a Banach function space \( X(\mathbb{R}^n) \) will be denoted by \( \|A\|_{B(X(\mathbb{R}^n))} \).

If \( \rho \) is a Banach function norm, its associate norm \( \rho' \) is defined on \( \mathcal{M}^+ \) by
\[
\rho'(g) := \sup \left\{ \int_{\mathbb{R}^n} f(x)g(x) \, dx : f \in \mathcal{M}^+, \rho(f) \leq 1 \right\}, \quad g \in \mathcal{M}^+.
\]

It is a Banach function norm itself [BS88, Chap. 1, Theorem 2.2]. The Banach function space \( X'(\mathbb{R}^n) \) determined by the Banach function norm \( \rho' \) is called the associate space (Köthe dual) of \( X(\mathbb{R}^n) \). The Lebesgue space \( L^p(\mathbb{R}^n) \), \( 1 \leq p \leq \infty \), are the archetypical example of Banach function spaces. Other classical examples of Banach function spaces are Orlicz spaces, rearrangement-invariant spaces, and variable Lebesgue spaces \( L^{p(\cdot)}(\mathbb{R}^n) \).

### 2.2. Density of bounded and smooth compactly supported functions in separable Banach function spaces.

The proof of the following fact is standard. For details, see [KS14, Lemma 2.10(b)], where it was proved for \( n = 1 \). The proof for arbitrary \( n \) is a minor modification of that one.

**Lemma 2.2.** The sets \( L_0^\infty(\mathbb{R}^n) \) and \( C_0^\infty(\mathbb{R}^n) \) are dense in a separable Banach function space \( X(\mathbb{R}^n) \).

### 2.3. Nonnegative sublinear operators on Banach function spaces.

The operators \( T_k^\delta, C, \text{ and } S_{a,b} \) although nonlinear, are examples of sublinear operators that assume only nonnegative values. Let us give a precise definition of this class of operators. Let \( D \) be a linear subspace of \( \mathcal{M} \). An operator \( T : D \to \mathcal{M} \) is said to be nonnegative sublinear (cf. [BS88, p. 230]) if
\[
0 \leq T(f + g) \leq Tf + Tg, \quad T(\lambda f) = |\lambda|Tf \quad \text{a.e. on } \mathbb{R}^n
\]
for all \( f, g \in D \) and all constants \( \lambda \in \mathbb{C} \). The following result is well known for linear operators. With the property
\[
|Tf - Tg| \leq |T(f - g)| = T(f - g), \quad f, g \in D,
\]
which is an immediate consequence of (2.1), essentially the same proof establishes the result also for nonnegative sublinear operators.

**Lemma 2.3.** Let \( D \) be a dense linear subspace of a Banach function space \( X(\mathbb{R}^n) \) and \( T : D \to \mathcal{M} \) be a nonnegative sublinear operator. If there exists a positive constant \( C \) such that
\[
\|Tf\|_{X(\mathbb{R}^n)} \leq C\|f\|_{X(\mathbb{R}^n)} \quad \text{for all } f \in D,
\]
then \( T \) has a unique extension to a nonnegative sublinear operator \( \tilde{T} : X(\mathbb{R}^n) \to \mathcal{M} \) such that
\[
\|\tilde{T}f\|_{X(\mathbb{R}^n)} \leq C\|f\|_{X(\mathbb{R}^n)} \quad \text{for all } f \in X(\mathbb{R}^n).
\]

In what follows we will use the same notation for an operator defined on a dense subspace and for its bounded extension to the whole space.
2.4. Self-improving property of maximal operators on Banach function spaces. If $1 < q < \infty$, then from the Hölder inequality one can immediately get that
\[
(Mf)(x) \leq (M_r f)(x) \quad \text{for a.e. } x \in \mathbb{R}^n.
\]
Thus, the boundedness of any $M_r$, $1 < r < \infty$, on a Banach function space $X(\mathbb{R}^n)$ immediately implies the boundedness of $M$. A partial converse of this fact, called a self-improving property of the Hardy-Littlewood maximal operator, is also true. It was proved by Lerner and Pérez [LP07] (see also [LO10] for another proof) in a more general setting of quasi-Banach function spaces.

**Theorem 2.4** ([LP07] Corollary 1.3]). Let $X(\mathbb{R}^n)$ be a Banach function space. Then $M$ is bounded on $X(\mathbb{R}^n)$ if and only if $M_r$ is bounded on $X(\mathbb{R}^n)$ for some $r \in (1, \infty)$.

2.5. The Fefferman-Stein inequality for Banach function spaces. It is obvious that $M^# f$ is pointwise dominated by $Mf$. Hence, by Axiom (A2),
\[
\|M^# f\|_{X(\mathbb{R}^n)} \leq \text{const} \|f\|_{X(\mathbb{R}^n)} \quad \text{for } f \in X(\mathbb{R}^n)
\]
whenever $M$ is bounded on $X(\mathbb{R}^n)$. The converse inequality for Lebesgue spaces $L^p(\mathbb{R}^n)$, $1 < p < \infty$, was proved by Fefferman and Stein (see [FS72] Theorem 5) and also [S93] Chap. IV, Section 2.2). The following extension of the Fefferman-Stein inequality to Banach function spaces was proved by Lerner [L10].

Let $S_0(\mathbb{R}^n)$ be the space of all measurable functions $f$ on $\mathbb{R}^n$ such that
\[
\{|x \in \mathbb{R}^n : |f(x)| > \lambda| < \infty \quad \text{for all} \quad \lambda > 0.
\]

**Theorem 2.5** ([L10] Corollary 4.2]). Let $M$ be bounded on a Banach function space $X(\mathbb{R}^n)$. Then $M$ is bounded on its associate space $X'(\mathbb{R}^n)$ if and only if there exists a constant $C_\# > 0$ such that, for all $f \in S_0(\mathbb{R}^n)$,
\[
\|f\|_{X(\mathbb{R}^n)} \leq C_\# \|M^# f\|_{X(\mathbb{R}^n)}.
\]

2.6. Pointwise inequality for the sharp maximal function of $T^\Phi$. Let $1 \leq r < \infty$. Recall that a sublinear operator $A : \mathbb{L}^r(\mathbb{R}^n) \to \mathcal{M}$ is said to be of weak type $(r, r)$ if
\[
\{|x \in \mathbb{R}^n : |(Af)(x)| > \lambda\| \leq \frac{C_r}{\lambda^r} \int_{\mathbb{R}^n} |f(y)|^r dy
\]
for all $f \in \mathbb{L}^r(\mathbb{R}^n)$ and $\lambda > 0$, where $C$ is a positive constant independent of $f$ and $\lambda$. It is well known that if $A$ is bounded on the standard Lebesgue space $\mathbb{L}^r(\mathbb{R}^n)$, then it is of weak type $(r, r)$.

Grafakos, Martell, and Soria [GMS05] developed two alternative approaches to weighted $L^r$ estimates for maximally modulated Calderón-Zygmund singular integral operators $T^{\Phi}$. One is based on good-$\lambda$ inequalities, another rests on the following pointwise estimate for the sharp maximal function of $T^{\Phi}$.

**Lemma 2.6** ([GMS05 Proposition 4.1]). Suppose $T$ is a Calderón-Zygmund operator and $\Phi = \{\phi_a\}_{a \in A}$ is a family of measurable real-valued functions indexed by an arbitrary set $A$. If $T^{\Phi}$ is of weak type $(r, r)$ for some $r \in (1, \infty)$, then there is a positive constant $C_r$ such that for every $f \in \mathbb{L}^\infty(\mathbb{R}^n)$,
\[
M^#(T^\Phi f)(x) \leq C_r M_r f(x) \quad \text{for a.e. } x \in \mathbb{R}^n.
\]
3. Maximally modulated singular integrals on Banach function spaces

3.1. Boundedness of maximally modulated Calderón-Zygmund singular integral operators on Banach function spaces. We are in a position to prove the main result of the paper.

**Theorem 3.1.** Let \( X(\mathbb{R}^n) \) be a separable Banach function space. Suppose the Hardy-Littlewood maximal operator \( M \) is bounded on \( X(\mathbb{R}^n) \) and on its associate space \( X'(\mathbb{R}^n) \). Suppose \( T \) is a Calderón-Zygmund operator and \( \Phi = \{ \phi_\alpha \}_{\alpha \in A} \) is a family of measurable real-valued functions indexed by an arbitrary set \( A \). If \( T^\Phi \) is of weak type \( (r,r) \) for all \( r \in (1,\infty) \), then \( T^\Phi \) extends to a bounded operator on the space \( X(\mathbb{R}^n) \).

**Proof.** We argue as in the proof of [K-A14 Theorem 1.2]. Since \( M \) is bounded on \( X(\mathbb{R}^n) \), by Theorem 2.3, there is an \( r \in (1,\infty) \) such that the maximal function \( M_r \) is bounded on \( X(\mathbb{R}^n) \), that is, there is a positive constant \( C \) such that

\[
\| (M_r \varphi) \|_{X(\mathbb{R}^n)} \leq C \| \varphi \|_{X(\mathbb{R}^n)} \quad \text{for all} \quad \varphi \in X(\mathbb{R}^n).
\]

Assume that \( f \in C_0^\infty(\mathbb{R}^n) \). By the hypothesis, \( T^\Phi \) is of weak type \( (r,r) \) and \( M \) is bounded on \( X'(\mathbb{R}^n) \). Therefore, \( T^\Phi f \in S_0(\mathbb{R}^n) \). Moreover, by Theorem 2.6 there exists a positive constant \( C_\# \) such that

\[
\| T^\Phi f \|_{X(\mathbb{R}^n)} \leq C_\# \| M^\#(T^\Phi f) \|_{X(\mathbb{R}^n)} \quad \text{for all} \quad f \in C_0^\infty(\mathbb{R}^n).
\]

From Lemma 2.6 and Axioms (A1)–(A2) we conclude that there exists a positive constant \( C_r \) such that

\[
\| M^\#(T^\Phi f) \|_{X(\mathbb{R}^n)} \leq C_r \| M_r f \|_{X(\mathbb{R}^n)} \quad \text{for all} \quad f \in C_0^\infty(\mathbb{R}^n).
\]

Combining inequalities (3.1)–(3.3), we arrive at

\[
\| T^\Phi f \|_{X(\mathbb{R}^n)} \leq C C_\# C_r \| f \|_{X(\mathbb{R}^n)} \quad \text{for all} \quad f \in C_0^\infty(\mathbb{R}^n).
\]

To conclude the proof, it remains to recall that \( C_0^\infty(\mathbb{R}^n) \) is dense in the separable Banach function space \( X(\mathbb{R}^n) \) in view of Lemma 2.2 and apply Lemma 2.3. \( \square \)

3.2. Boundedness of the maximally modulated Hilbert transform on standard Lebesgue spaces. Fix \( f \in L^1_{loc}(\mathbb{R}) \). Let \( H_* \) be the maximal Hilbert transform given by

\[
(H_* f)(x) := \sup_{\varepsilon > 0} \left| \frac{1}{\pi} \int_{\mathbb{R} \setminus I(x,\varepsilon)} \frac{f(y)}{x-y} \, dy \right|,
\]

where \( I(x,\varepsilon) = (x-\varepsilon,x+\varepsilon) \). Further, let \( C_* \) be the maximally modulated maximal Carleson operator defined by

\[
(C_* f)(x) := \sup_{a \in \mathbb{R}} (H_*(M^\psi_a f))(x) \quad \text{with} \quad \psi_a(x) = ax, \quad a, x \in \mathbb{R}.
\]

It is easy to see that

\[
(C f)(x) \leq (C_* f)(x), \quad x \in \mathbb{R}.
\]

The boundedness of the operator \( C_* \) on the standard Lebesgue spaces \( L^r(\mathbb{R}) \) is proved, e.g., in [G09 Theorem 11.3.3] (see also [K-Yu12 Theorem 2.7]). From this observation and [3.1] we get the following result (see also [D91 Theorem 2.1] and [K-Yu12 Theorem 2.8]).
Lemma 3.2. The maximally modulated Hilbert transform $\mathcal{H}$ is bounded on every standard Lebesgue space $L^r(\mathbb{R})$ for $1 < r < \infty$.

3.3. Boundedness of the maximally modulated Hilbert transform on separable Banach function spaces. From Theorem 3.1 and Lemma 3.2 we immediately get the following.

Corollary 3.3. Let $X(\mathbb{R})$ be a separable Banach function space. Suppose the Hardy-Littlewood maximal operator $M$ is bounded on $X(\mathbb{R})$ and on its associate space $X'(\mathbb{R})$. Then the maximally modulated Hilbert transform $\mathcal{H}$ extends to a bounded operator on $X(\mathbb{R})$.

4. Boundedness of pseudodifferential operators with non-regular symbols on Banach function spaces

4.1. Functions of bounded total variation. Let $a$ be a complex-valued function of bounded total variation $V(a)$ on $\mathbb{R}$ where

$$V(a) := \sup \left\{ \sum_{k=1}^{n} |a(x_k) - a(x_{k-1})| : -\infty < x_0 < x_1 < \cdots < x_n < +\infty, n \in \mathbb{N} \right\}$$

Hence at every point $x \in \mathbb{R} := \mathbb{R} \cup \{\infty\}$ the one-sided limits $a(x \pm 0) = \lim_{t \to x \pm} a(t)$ exist, where $a(\pm \infty) = a(\infty \mp 0)$, and the set of discontinuities of $a$ is at most countable (see, e.g., [N55] Chap. VIII, Sections 3 and 9). Without loss of generality we will assume that functions of bounded total variation are continuous from the left at every discontinuity point $x \in \mathbb{R}$. The set $V(\mathbb{R})$ of all continuous from the left functions of bounded total variation on $\mathbb{R}$ is a unital non-separable Banach algebra with the norm

$$\|a\|_V := \|a\|_{L^\infty(\mathbb{R})} + V(a).$$

By analogy with $V(a) = V_{-\infty}^+(a)$, one can define the total variations $V_{c}^d(a)$, $V_{-\infty}^c(a)$, and $V_{d}^{+\infty}(a)$ of a function $a : \mathbb{R} \to \mathbb{C}$ on $[c, d]$, $(-\infty, c]$, and $[d, +\infty)$, taking, respectively, the partitions

$$c = x_0 < x_1 < \cdots < x_n = d, \quad -\infty < x_0 < x_1 < \cdots < x_n = c,$$

and $d = x_0 < x_1 < \cdots < x_n < +\infty$.

4.2. Non-regular symbols of pseudodifferential operators. Following [K-Yu07, K-Yu12], we denote by $L^\infty(\mathbb{R}, V(\mathbb{R}))$ the set of functions $a : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ such that $\tilde{a} : x \mapsto a(x, \cdot)$ is a bounded measurable $V(\mathbb{R})$-valued function on $\mathbb{R}$. Note that in view of non-separability of the Banach space $V(\mathbb{R})$, the measurability of $\tilde{a}$ means that the map $\tilde{a} : \mathbb{R} \to V(\mathbb{R})$ possesses the Luzin property: for any compact set $K \subset \mathbb{R}$ and any $\delta$ there is a compact set $K_\delta \subset K$ such that $|K \setminus K_\delta| < \delta$ and $\tilde{a}$ is continuous on $K_\delta$ (see, e.g., [S67] Chap. IV, Section 4, p. 487). This implies that the function $x \mapsto a(x, \lambda \pm 0)$ for all $\lambda \in \mathbb{R}$ and the function $x \mapsto \|a(x, \cdot)\|_V$ are measurable on $\mathbb{R}$ as well. Note that for almost all $x \in \mathbb{R}$ the limits $a(x, \lambda \pm 0)$ the limits exist for all $\lambda \in \mathbb{R}$, $a(x, \lambda) = a(x, \lambda - 0)$ for all $\lambda \in \mathbb{R}$ and we put

$$a(x, \pm \infty) := \lim_{\lambda \to \pm \infty} a(x, \lambda).$$
Therefore, the functions \( a(\cdot, \lambda \pm 0) \) for every \( \lambda \in \mathbb{R} \) and the function \( x \mapsto \| a(x, \cdot) \|_V \), where
\[
\| a(x, \cdot) \|_V := \| a(x, \cdot) \|_{L^\infty(\mathbb{R})} + V(a(x, \cdot)),
\]
belong to \( L^\infty(\mathbb{R}) \). Clearly, \( L^\infty(\mathbb{R}, V(\mathbb{R})) \) is a unital Banach algebra with the norm
\[
\| a \|_{L^\infty(\mathbb{R}, V(\mathbb{R}))} = \text{ess sup}_{x \in \mathbb{R}} \| a(x, \cdot) \|_V.
\]

4.3. Pointwise inequality for pseudodifferential operators. The following pointwise estimate was obtained by Yuri Karlovich in the proof of [K-Yu07, Theorem 3.1] and [K-Yu12, Theorem 4.1].

**Lemma 4.1.** If \( a \in L^\infty(\mathbb{R}, V(\mathbb{R})) \) and \( f \in C^\infty_0(\mathbb{R}) \), then
\[
| (a(x, D)f)(x) | \leq 2 | S_a f(x) | a(x, \cdot) \|_V \text{ for a.e. } x \in \mathbb{R}.
\]

4.4. Boundedness of the maximal singular integral operator \( S_* \) on separable Banach function spaces. We continue with the following result on the boundedness of the maximal singular integral operator \( S_* \) initially defined for \( f \in C^\infty_0(\mathbb{R}) \) by (1.1).

**Lemma 4.2.** Let \( X(\mathbb{R}) \) be a separable Banach function space. Suppose the Hardy-Littlewood maximal operator \( M \) is bounded on \( X(\mathbb{R}) \) and on its associate space \( X'(\mathbb{R}) \). Then the operator \( S_* \), defined for the functions \( f \in C^\infty_0(\mathbb{R}) \) by (1.3), extends to a bounded operator on the space \( X(\mathbb{R}) \).

**Proof.** Fix \( f \in C^\infty_0(\mathbb{R}) \). It is not difficult to check (see, e.g., [D91, Chap. 2, Section 2.2] and also [G09, p. 475]) that
\[
(S_{a,b} f)(x) = \frac{1}{2} \left\{ M^{-\psi_a}(H(M^{\psi_a} f))(x) - M^{-\psi_b}(H(M^{\psi_b} f))(x) \right\}, \quad x \in \mathbb{R},
\]
where \( \psi_a(x) = ax, \psi_b(x) = bx \) and \( -\infty < a < b < +\infty \). Therefore,
\[
(S_* f)(x) = \sup_{-\infty < a < b < +\infty} |(S_{a,b} f)(x)|
\]
\[
\leq \frac{1}{2} \sup_{a \in \mathbb{R}} |(H(M^{\psi_a} f))(x)| + \frac{1}{2} \sup_{b \in \mathbb{R}} |(H(M^{\psi_b} f))(x)|
\]
\[
= (C f)(x), \quad x \in \mathbb{R}.
\]
From this inequality, Axioms (A1)-(A2), and Corollary 4.3, we get
\[
\| S_* f \|_{X(\mathbb{R})} \leq \| C f \|_{X(\mathbb{R})} \leq \| C \|_{B(X(\mathbb{R}))} \| f \|_{X(\mathbb{R})} \text{ for } f \in C^\infty_0(\mathbb{R}).
\]
It remains to apply Lemma 2.2. \( \square \)

4.5. Boundedness of pseudodifferential operators with \( L^\infty(\mathbb{R}, V(\mathbb{R})) \) symbols on Banach function spaces. We are ready to prove the boundedness result for pseudodifferential operators with non-regular symbols on separable Banach function spaces.

**Theorem 4.3.** Let \( X(\mathbb{R}) \) be a separable Banach function space. Suppose the Hardy-Littlewood maximal operator \( M \) is bounded on \( X(\mathbb{R}) \) and on its associate space \( X'(\mathbb{R}) \). If \( a \in L^\infty(\mathbb{R}, V(\mathbb{R})) \), then the pseudodifferential operator \( a(x, D) \), defined for the functions \( f \in C^\infty_0(\mathbb{R}) \) by the iterated integral (1.3), extends to a bounded linear operator on the space \( X(\mathbb{R}) \) and
\[
\| a(x, D) \|_{B(X(\mathbb{R}))} \leq 2 \| S_* \|_{B(X(\mathbb{R}))} \| a \|_{L^\infty(\mathbb{R}, V(\mathbb{R}))}.
\]
Proof. From Lemma \[2.1\] axioms (A1)-(A2), and Lemma \[1.2\] we obtain for $f \in C_0^\infty(\mathbb{R})$,

$$
\|a(x,D)f\|_{X(\mathbb{R})} \leq 2\|S_a f\|_{X(\mathbb{R})} \|a(x,\cdot)\|_V \\
\leq 2\|S_a\|_{\mathcal{B}(X(\mathbb{R}))}\|a\|_{L^\infty(\mathbb{R},V(\mathbb{R}))}\|f\|_{X(\mathbb{R})}.
$$

Since $C_0^\infty(\mathbb{R})$ is dense in the space $X(\mathbb{R})$ in view of Lemma \[2.2\] from the above estimate we arrive immediately at the desired conclusion. \qed

5. Boundedness of maximally modulated Calderón-Zygmund operators and pseudodifferential operators with non-regular symbols on variable Lebesgue spaces

5.1. Variable Lebesgue spaces. Let $p : \mathbb{R}^n \to [1, \infty]$ be a measurable a.e. finite function. By $L^{p(\cdot)}(\mathbb{R}^n)$ we denote the set of all complex-valued functions $f$ on $\mathbb{R}$ such that

$$
I_{p(\cdot)}(f/\lambda) := \int_{\mathbb{R}^n} |f(x)/\lambda|^{p(x)} dx < \infty
$$

for some $\lambda > 0$. This set becomes a Banach function space when equipped with the norm

$$
\|f\|_{p(\cdot)} := \inf \{ \lambda > 0 : I_{p(\cdot)}(f/\lambda) \leq 1 \}.
$$

It is easy to see that if $p$ is constant, then $L^{p(\cdot)}(\mathbb{R}^n)$ is nothing but the standard Lebesgue space $L^p(\mathbb{R}^n)$. The space $L^{p(\cdot)}(\mathbb{R}^n)$ is referred to as a variable Lebesgue space.

We will always suppose that

$$
1 < p_- := \essinf_{x \in \mathbb{R}^n} p(x), \quad \esssup_{x \in \mathbb{R}^n} p(x) =: p_+ < \infty.
$$

Under these conditions, the space $L^{p(\cdot)}(\mathbb{R}^n)$ is separable and reflexive, and its associate space is isomorphic to $L^{p'\cdot}(\mathbb{R}^n)$, where

$$
1/p(x) + 1/p'(x) = 1 \quad \text{for a.e.} \quad x \in \mathbb{R}^n
$$

(see e.g. \[CF13\] Chap. 2 or \[DHHR11\] Chap. 3).

5.2. The Hardy-Littlewood maximal function on variable Lebesgue spaces. By $\mathcal{B}_M(\mathbb{R}^n)$ denote the set of all measurable functions $p : \mathbb{R}^n \to [1, \infty]$ such that \[5.1\] holds and the Hardy-Littlewood maximal operator is bounded on the variable Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$.

To provide a simple sufficient conditions guaranteeing that $p \in \mathcal{B}_M(\mathbb{R}^n)$, we need the following definition. Given a function $r : \mathbb{R}^n \to \mathbb{R}$, one says that $r$ is locally log-Hölder continuous if there exists a constant $C_0 > 0$ such that

$$
|r(x) - r(y)| \leq \frac{C_0}{-\log |x - y|}
$$

for all $x, y \in \mathbb{R}^n$ such that $|x - y| < 1/2$. One says that $r : \mathbb{R}^n \to \mathbb{R}$ is log-Hölder continuous at infinity if there exist constants $C_{\infty}$ and $r_{\infty}$ such that for all $x \in \mathbb{R}^n$,

$$
|r(x) - r_{\infty}| \leq \frac{C_{\infty}}{\log(e + |x|)}.
$$

The class of functions $r : \mathbb{R}^n \to \mathbb{R}$ that are simultaneously locally log-Hölder continuous and log-Hölder continuous at infinity is denoted by $LH(\mathbb{R}^n)$. From \[CF13\] Proposition 2.3 and Theorem 3.16] we extract the following.
Theorem 5.1. Let $p \in LH(\mathbb{R}^n)$ satisfy (5.1). Then $p \in B_M(\mathbb{R}^n)$. 

Although the latter result provides a nice sufficient condition for the boundedness of the Hardy-Littlewood maximal operator on the variable Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$, it is not necessary. Notice that all functions in $L^1(\mathbb{R}^n)$ are continuous and have limits at infinity. Lerner [L05] (see also [CF13, Example 4.68]) proved that if $p_0 > 1$ and $\mu \in \mathbb{R}$ is sufficiently close to zero, then the following variable exponent 

$$p(x) = p_0 + \mu \sin(\log \log(1 + \max\{|x|, 1/|x|\})), \quad x \neq 0,$$

belongs to $B_M(\mathbb{R})$. It is clear that the function $p$ does not have limits at zero or infinity. We refer to the recent monographs [CF13, DHHR11] for further discussions concerning the fascinating and still mysterious class $B_M(\mathbb{R}^n)$.

We will need the following remarkable result proved by Diening [D05, Theorem 5.7.2] (see also [CF13, Corollary 4.64]).

Theorem 5.2. We have $p \in B_M(\mathbb{R}^n)$ if and only if $p' \in B_M(\mathbb{R}^n)$.

5.3. Boundedness of maximally modulated Calderón-Zygmund singular integral operators on variable Lebesgue spaces. From Theorems 3.1 and 5.2 we immediately get the following.

Corollary 5.3. Let $p \in B_M(\mathbb{R}^n)$. Suppose $T$ is a Calderón-Zygmund operator and $\Phi = \{\phi_\alpha\}_{\alpha \in \mathcal{A}}$ is a family of measurable real-valued functions indexed by an arbitrary set $\mathcal{A}$. If $T^\Phi$ is of weak type $(r,r)$ for all $r \in (1,\infty)$, then $T^\Phi$ extends to a bounded operator on the variable Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$.

In turn, Corollary 5.3 and Lemma 3.2 yield the following.

Corollary 5.4. If $p \in B_M(\mathbb{R})$, then the maximally modulated Hilbert transform $C$ extends to a bounded operator on the variable Lebesgue space $L^{p(\cdot)}(\mathbb{R})$.

5.4. Boundedness of pseudodifferential operators with $L^{\infty}(\mathbb{R}, V(\mathbb{R}))$ symbols on variable Lebesgue spaces. Combining Lemma 4.2 with Theorem 5.2 we arrive at the following.

Corollary 5.5. Suppose $p \in B_M(\mathbb{R})$. Then the operator $S_\ast$, defined for the functions $f \in C^\infty_0(\mathbb{R})$ by (1.1), extends to a bounded operator on the variable Lebesgue space $L^{p(\cdot)}(\mathbb{R})$.

From Theorems 4.3 and 5.2 taking into account Corollary 5.5 we get the following.

Corollary 5.6. If $p \in B_M(\mathbb{R})$ and $a \in L^{\infty}(\mathbb{R}, V(\mathbb{R}))$, then the pseudodifferential operator $a(x, D)$, defined for the functions $f \in C^\infty_0(\mathbb{R})$ by the iterated integral (1.3), extends to a bounded linear operator on the space $L^{p(\cdot)}(\mathbb{R})$ and 

$$\|a(x, D)\|_{B(L^{p(\cdot)}(\mathbb{R}))} \leq 2\|S_\ast\|_{B(L^{p(\cdot)}(\mathbb{R}))}\|a\|_{L^{\infty}(\mathbb{R}, V(\mathbb{R}))}.$$ 

6. Compactness of pseudodifferential operators with non-regular symbols on variable Lebesgue spaces

6.1. Compactness of pseudodifferential operators with $L^{\infty}(\mathbb{R}, V(\mathbb{R}))$ symbols on standard Lebesgue spaces. We start with the case of constant exponents.
THEOREM 6.1 ([K-Yu07 Theorem 4.1]). Let $1 < r < \infty$. If $a \in L^\infty(\mathbb{R}, V(\mathbb{R}))$ and

(a) $a(x, \pm \infty) = 0$ for almost all $x \in \mathbb{R}$;
(b) $\lim_{|x| \to \infty} V(a(x, \cdot)) = 0$;
(c) for every $N > 0$,

$$\lim_{L \to +\infty} \text{ess sup}_{|x| \leq N} \left( V_{-L}^-(a(x, \cdot)) + V_{+L}^+(a(x, \cdot)) \right) = 0;$$

then the pseudodifferential operator $a(x, D)$ is compact on the standard Lebesgue space $L^r(\mathbb{R})$.

6.2. Transferring the compactness property from standard Lebesgue spaces to variable Lebesgue spaces. For a Banach space $E$, let $\mathcal{L}(E)$ and $\mathcal{K}(E)$ denote the Banach algebra of all bounded linear operators and its ideal of all compact operators on $E$, respectively.

THEOREM 6.2. Let $p_j : \mathbb{R}^n \to [1, \infty]$, $j = 0, 1$, be a.e. finite measurable functions, and let $p_0 : \mathbb{R}^n \to [1, \infty]$ be defined for $\theta \in [0, 1]$ by

$$\frac{1}{p_0(x)} = \frac{\theta}{p_0(x)} + \frac{1 - \theta}{p_1(x)}, \quad x \in \mathbb{R}^n.$$

Suppose $A$ is a linear operator defined on $L^{p_0(\cdot)}(\mathbb{R}^n) + L^{p_1(\cdot)}(\mathbb{R}^n)$.

(a) If $A \in \mathcal{L}(L^{p_0(\cdot)}(\mathbb{R}^n))$ for $j = 0, 1$, then $A \in \mathcal{L}(L^{p_0(\cdot)}(\mathbb{R}^n))$ for all $\theta \in [0, 1]$ and

$$\|A\|_{\mathcal{L}(L^{p_0(\cdot)}(\mathbb{R}^n))} \leq 4\|A\|_{\mathcal{L}(L^{p_0(\cdot)}(\mathbb{R}^n))}^{\theta} \|A\|_{\mathcal{L}(L^{p_1(\cdot)}(\mathbb{R}^n))}^{1 - \theta};$$

(b) If $A \in \mathcal{K}(L^{p_0(\cdot)}(\mathbb{R}^n))$ and $A \in \mathcal{L}(L^{p_1(\cdot)}(\mathbb{R}^n))$, then $A \in \mathcal{K}(L^{p_0(\cdot)}(\mathbb{R}^n))$ for all $\theta \in (0, 1)$.

Part (a) is proved in [DHHR11 Corollary 7.1.4] under the more general assumption that $p_j$ may take infinite values on sets of positive measure (and in the setting of arbitrary measure spaces). Part (b) follows from a general interpolation theorem by Cobos, Kühn, and Schonbeck [CKS92 Theorem 3.2] for the complex interpolation method for Banach lattices satisfying the Fatou property. Indeed, the complex interpolation space $[L^{p_0(\cdot)}(\mathbb{R}^n), L^{p_1(\cdot)}(\mathbb{R}^n)]_{1-\theta}$ is isomorphic to the variable Lebesgue space $L^{p_0(\cdot)}(\mathbb{R}^n)$ (see [DHHR11 Theorem 7.1.2]), and $L^{p_j(\cdot)}(\mathbb{R}^n)$ have the Fatou property (see [DHHR11 p. 77]).

The following characterization of the class $\mathcal{B}_M(\mathbb{R}^n)$ was communicated to the authors of [KS13] by Diening.

THEOREM 6.3 ([KS13 Theorem 4.1]). If $p \in \mathcal{B}_M(\mathbb{R}^n)$, then there exist constants $p_0 \in (1, \infty)$, $\theta \in (0, 1)$, and a variable exponent $p_1 \in \mathcal{B}_M(\mathbb{R}^n)$ such that

$$\frac{1}{p(x)} = \frac{\theta}{p_0(x)} + \frac{1 - \theta}{p_1(x)}, \quad x \in \mathbb{R}^n.$$

From the above two theorems we obtain the following result, which allows us to transfer the compactness property from standard Lebesgue spaces to variable Lebesgue spaces.

LEMMA 6.4. Let $A \in \mathcal{L}(L^{p_j(\cdot)}(\mathbb{R}^n))$ for all $p \in \mathcal{B}_M(\mathbb{R}^n)$. If $A \in \mathcal{K}(L^r(\mathbb{R}^n))$ for some $r \in (1, \infty)$, then $A \in \mathcal{K}(L^{p_j(\cdot)}(\mathbb{R}^n))$ for all $p \in \mathcal{B}_M(\mathbb{R}^n)$. 
Proof. By the hypothesis, the operator $A$ is bounded on all standard Lebesgue spaces $L^r(\mathbb{R}^n)$ with $1 < r < \infty$. From the classical Krasnosel’skii interpolation theorem (Theorem 6.2(b) with constant exponents) it follows that $A \in \mathcal{K}(L^r(\mathbb{R}^n))$ for all $1 < r < \infty$. If $p \in \mathcal{B}_M(\mathbb{R}^n)$, then in view of Theorem 5.2 there exist $p_0 \in (1, \infty)$, $\theta \in (0, 1)$, and a variable exponent $p_1 \in \mathcal{B}_M(\mathbb{R}^n)$ such that (6.1) holds. Since $A \in \mathcal{L}(L^{p_0}(\mathbb{R}^n))$ and $A \in \mathcal{K}(L^{p_1}(\mathbb{R}^n))$, from Theorem 6.2(b) we obtain $A \in \mathcal{K}(L^{p}(\mathbb{R}^n))$. \hfill \Box

6.3. Compactness of pseudodifferential operators with $L^\infty(\mathbb{R}, V(\mathbb{R}))$ symbols on variable Lebesgue spaces. Combining Corollary 6.5 and Theorem 6.1 with Lemma 6.4 we arrive at our last result.

Corollary 6.5. Suppose $p \in \mathcal{B}_M(\mathbb{R})$. If $a \in L^\infty(\mathbb{R}, V(\mathbb{R}))$ satisfies the hypotheses (a)–(c) of Theorem 6.1, then the pseudodifferential operator $a(x, D)$ is compact on the variable Lebesgue space $L^{p}(\mathbb{R})$.

References

[BBS88] C. Bennett and R. Sharpley, Interpolation of Operators. Academic Press, New York, 1988.
[CKS92] F. Cobos, T. Kühn, and T. Schonbek, One-sided compactness results for Aronszajn-Gagliardo functors. J. Funct. Analysis 106 (1992), 274–313.
[CF13] F. Cobos, T. Kühn, and T. Schonbek, One-sided compactness results for Aronszajn-Gagliardo functors. J. Funct. Analysis 106 (1992), 274–313.
[CGS05] L. Diening, P. Harjulehto, P. Hästö, and M. Růžička, Lebesgue and Sobolev Spaces with Variable Exponents. Lecture Notes in Mathematics 2017. Springer, Berlin, 2011.
[DYN91] E. M. Dyn’kin, Methods of the theory of singular integrals (the Hilbert transform and Calderon-Zygmund theory). In “Commutative Harmonic Analysis I”, Itogi Nauki i Tekhniki. Ser. Sovrem. Probl. Mat. Fund. Napr., 15, VINITI, Moscow (1987), 197-292 (in Russian). English translation: Commutative harmonic analysis I. General survey. Classical aspects, Encycl. Math. Sci. 15 (1991), 167–259.
[FSY72] Ch. Fefferman and E. M. Stein, $H^p$ spaces of several variables. Acta Math. 129 (1972), 137–193.
[G09] L. Grafakos, Modern Fourier Analysis. 2nd ed. Graduate Texts in Mathematics 250. New York, Springer, 2009.
[GMS05] L. Grafakos, J. M. Martell, and F. Soria, Weighted norm inequalities for maximally modulated singular integral operators. Math. Ann. 331 (2005), 359–394.
[K-A14] A. Yu. Karlovich, Boundedness of pseudodifferential operators on Banach function spaces. In: “Operator Theory, Operator Algebras and Applications”. Operator Theory: Advances and Applications 242 (2014), 185–197.
[KS13] A. Yu. Karlovich and I. M. Spitkovsky, Pseudodifferential operators on variable Lebesgue spaces. In: “Operator Theory, Pseudo-Differential Equations, and Mathematical Physics. The Vladimir Rabinovich Anniversary Volume”. Operator Theory: Advances and Applications 228 (2013), 173–183.
[KS14] A. Yu. Karlovich and I. M. Spitkovsky, The Cauchy singular integral operator on weighted variable Lebesgue spaces. In: “Concrete Operators, Spectral Theory, Operators in Harmonic Analysis and Approximation”. Operator Theory: Advances and Applications 236 (2014), 275–291.
[K-Yu07] Yu. I. Karlovich, Algebras of pseudo-differential operators with discontinuous symbols. In: “Modern Trends in Pseudo-Differential Operators”. Operator Theory: Advances and Applications 172 (2007), 207–233.
[K-Yu12] Yu. I. Karlovich, Boundedness and compactness of pseudodifferential operators with non-regular symbols on weighted Lebesgue spaces. Integr. Equ. Oper. Theor. 73 (2012), 217–254.
A. K. Lerner, Some remarks on the Hardy-Littlewood maximal function on variable $L^p$ spaces. Math. Z. 251 (2005), 509-521.

A. K. Lerner, Some remarks on the Fefferman-Stein inequality. J. Anal. Math. 112 (2010), 329-349.

A. K. Lerner and S. Ombrosi, A boundedness criterion for general maximal operators. Publ. Mat. 54 (2010), 53-71.

A. K. Lerner and C. Pérez, A new characterization of the Muckenhoupt $A_p$ weights through an extension of the Lorentz-Shimogaki theorem. Indiana Univ. Math. J. 56 (2007), 2697-2722.

C. Muscalu and W. Schlag, Classical and Multilinear Harmonic Analysis. Vol. II. Cambridge Studies in Advanced Mathematics 138. Cambridge University Press, Cambridge, 2013.

I. P. Natanson, Theory of Functions of a Real Variable. Frederick Ungar Publishing Co., New York, 1955.

E. Stein, Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals. Princeton University Press, Princeton, NJ, 1993.

L. Schwartz, Analyse Mathématique. Cours I. Hermann, Paris, 1967.

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