Symmetric Ghost Lagrange Densities for the Coupling of Gravity to Gauge Theories

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Abstract

We derive and present symmetric ghost Lagrange densities for the coupling of General Relativity to Yang–Mills theories. The graviton-ghost is constructed with respect to the linearized de Donder gauge fixing and the gauge ghost is constructed with respect to the covariant Lorenz gauge fixing. Both ghost Lagrange densities together with their accompanying gauge fixing Lagrange densities are obtained from the action of the diffeomorphism and gauge BRST and anti-BRST operators on suitable gauge fixing bosons. In addition, we introduce a total gauge fixing boson and show that the complete ghost and gauge fixing Lagrange density can be generated thereof using the total BRST operator and the total anti-BRST operator, introduced by the author in a previous article (2022). This generalizes results from Baulieu and Thierry-Mieg (1982) to General Relativity and covariant Yang–Mills theories.

1 Introduction

Ghost Lagrange densities are typically calculated via the Faddeev–Popov method [1]: Given a gauge field $\phi$ with coupling constant $\alpha$, an infinitesimal gauge transformation $\delta_Z \phi$ with respect to a Lie algebra valued vector field $Z$ and a chosen gauge fixing functional $\text{GF} (\phi)$. Let furthermore $\theta$ and $\overline{\theta}$ denote the corresponding ghost and antighost fields and $\kappa$ the gauge fixing parameter. Then the Faddeev–Popov gauge fixing and ghost Lagrange density reads as follows (‘$\cdot$’ denotes a scalar product on the corresponding Lie algebra and $dV_g$ denotes the Riemannian volume form, see below for the definition):

$$L_{\text{FP-GF-Ghost}} := \left( -\frac{1}{2\alpha^2} \text{GF} (\phi)^2 + \overline{\theta} \cdot \text{GF} (\delta_\theta \phi) \right) dV_g$$

This construction has the advantages of being rather simple to calculate. In addition, it provides an immediate interpretation for the equations of motion for the ghost and antighost fields: While the ghost field is constructed to satisfy residual gauge transformations as equations of motion, the antighost field is acting as a Lagrange multiplier therefore. Unfortunately, the apparent asymmetry between the ghost and antighost results in an intransparent relationship between these two fields. This becomes in particular relevant when analyzing the longitudinal and transversal contributions of the corresponding Feynman integrals. In particular, the cancellation identities for the Faddeev–Popov ghosts are quite involved: Cancellation identities are diagrammatic cancellations resulting from longitudinal projections, cf. [2, 3, 4, 5, 6, 7, 8]. Thus, this article is devoted to a proper derivation of symmetric ghost Lagrange densities using appropriate gauge fixing bosons, BRST and anti-BRST operators. More precisely, the ghost Lagrange densities in this article are Hermitian with respect to the ghost conjugation, i.e.

$$L^\dagger_{\text{QGT}} = L_{\text{QGT}}$$

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where the ghost conjugation is defined via
\[
\theta^\dagger := \bar{\theta}
\] (2b)
and
\[
\bar{\theta}^\dagger := \theta.
\] (2c)

Thus, in this construction the antighost is actually the antiparticle of ghost. This was first constructed by Baulieu and Thierry-Mieg \[9\] for Yang–Mills theories on a flat spacetime. In this article, we generalize this approach to General Relativity and covariant Yang–Mills theories, that is Yang–Mills theories coupled to General Relativity. More precisely, this approach starts with a so-called gauge fixing boson \(B\), which is a functional in ghost degree 0. From this, we construct its associated gauge fixing fermion \(\chi\) via the action of the anti-BRST operator \(\mathcal{D}\), i.e.
\[
\chi := \mathcal{D}B.
\]
Then, the corresponding gauge fixing and ghost Lagrange density \(L_{GF-Ghost}\) is given via the action of the BRST operator \(D\), i.e.
\[
L_{GF-Ghost} := D\chi.
\]
Thus, finally we obtain:
\[
L_{GF-Ghost} := \mathcal{D}B,
\] (3)
where we have introduced the super-BRST operator \(\mathcal{D}\), defined via
\[
\mathcal{D} := D \circ \mathcal{D} \equiv -\mathcal{D} \circ D \equiv \frac{1}{2} \left( D \circ \mathcal{D} - \mathcal{D} \circ D \right).
\] (4)
The equivalent expressions are due to the property \([D, \mathcal{D}] \equiv 0\). We remark that the so-constructed Lagrange density still contains the corresponding Lautrup–Nakanishi auxiliary field \[10, 11\]. This field is neither Hermitian nor anti-Hermitian with respect to the ghost conjugation. However, it can be shifted to become anti-Hermitian. Once it is eliminated via its equations of motion after this shift, we obtain the following symmetric setup:
\[
L_{Sym-GF-Ghost} := \left( -\frac{1}{2\alpha^2} \text{GF} (\varphi)^2 + \frac{1}{2} \left( \bar{\theta} \cdot \text{GF} (\delta_\theta \varphi) - \text{GF} (\delta_\varphi \theta) \right) \right) dV_g
+ \frac{\alpha^2}{16} \left( [\bar{\theta}, \bar{\theta}] \cdot [\theta, \theta] \right) dV_g
\] (5)
We remark that the gauge fixing functional \(\text{GF} (\varphi)\) for a given gauge theory and gauge fixing boson is now determined to be an optimal gauge fixing, cf. \[8\] for the definition and further properties: In particular, we show that for General Relativity it is given via the de Donder gauge fixing functional and for Yang–Mills theory it is given via the covariant Lorenz gauge fixing functional. Additionally, we highlight the new four-ghost-interaction in addition to the symmetrized Faddeev–Popov construction.

More specifically, we consider General Relativity, given via the Lagrange density:
\[
L_{GR} := -\frac{1}{2\kappa^2} R dV_g
\] (6)
In particular, we consider its linearization with respect to the metric decomposition \(g_{\mu\nu} \equiv \eta_{\mu\nu} + \kappa h_{\mu\nu}\), where \(h_{\mu\nu}\) is the graviton field and \(\kappa := \sqrt{\kappa}\) the graviton coupling constant (with \(\kappa := 8\pi G\) the Einstein gravitational constant). In addition, \(R := g^{\rho\sigma} R_{\mu\rho\nu\sigma}\) is the Ricci scalar (with \(R_{\rho\sigma} := \partial_\rho \Gamma^\mu_{\nu\sigma} - \partial_\sigma \Gamma^\mu_{\nu\rho} + \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\mu\sigma}\) the Riemann tensor). Furthermore, \(dV_g := \sqrt{-\text{Det} (g)} \, dt \wedge dx \wedge dy \wedge dz\) denotes the Riemannian volume form and \(dV_g := dt \wedge dx \wedge dy \wedge dz\).
Setting the corresponding gauge fixing boson to
\[ F := -\frac{\xi}{4} \left( \frac{1}{\kappa} \eta^{\mu\nu} h_{\mu\nu} - \overline{C}_\rho C^\rho \right) \d V_\eta \]  
implies the following symmetric gauge fixing and ghost Lagrange density, where \( \mathcal{P} := P \circ \overline{P} \) is the diffeomorphism super-BRST operator:
\[
\mathcal{L}_{\text{GR-GF-Ghost}} := \mathcal{P} F
= \left( -\frac{1}{4\kappa^2\zeta} \eta^{\mu\nu} d\mathcal{D}_\mu^{(1)} d\mathcal{D}_\nu^{(1)} + \frac{1}{2\kappa} \eta^{\mu\nu} \left( \partial_{(\mu} \overline{C}^{\nu)} - \partial_{(\mu} C^{\nu)} \right) \right) \d V_\eta
+ \frac{1}{2} \eta^{\mu\nu} \left( \frac{1}{2} \partial_{\mu} (\Gamma_{\nu\sigma} C_{\sigma}) - \partial_{\mu} (\Gamma_{\nu\sigma} \overline{C}_\sigma) \right) \d V_\eta
- \frac{1}{2} \eta^{\mu\nu} \left( \frac{1}{2} \partial_{\mu} (\Gamma_{\nu\sigma} \overline{C}_\sigma) - \partial_{\mu} (\Gamma_{\nu\sigma} C_{\sigma}) \right) \overline{C}_\rho \d V_\eta
+ \frac{\kappa^2 \zeta}{32} \eta^{\mu\nu} \left( \overline{C}_\sigma \left( \partial_{(\mu} \overline{C}^{\nu)} \right) \right) \left( C^\sigma \left( \partial_{\nu} C^\rho \right) \right) \d V_\eta
\]  
Here, \( d\mathcal{D}_\mu^{(1)} := \eta^{\sigma\tau} \Gamma_{\mu\sigma\tau} \equiv 0 \) is the linearized de Donder gauge fixing functional and \( \zeta \) denotes the gauge fixing parameter. Moreover, \( C_\mu \) and \( \overline{C}_\mu \) are the graviton-ghost and graviton-antighost, respectively. Finally, the Lagrange density for (effective) Quantum General Relativity is then given as the sum of the two:
\[
\mathcal{L}_{\text{QGR}} := \mathcal{L}_{\text{GR}} + \mathcal{L}_{\text{GR-GF-Ghost}}
\]  
In addition to the symmetric ghost Lagrange density, we also construct a homotopy that continuously interpolates between the Faddeev–Popov construction, the symmetric setup and the opposed Faddeev–Popov construction, all with respect to the linearized de Donder gauge fixing, cf. Theorem 3.2.

Additionally, we consider Yang–Mills theory, given via the Lagrange density:
\[
\mathcal{L}_{\text{YM}} := -\frac{1}{4g^2} \delta_{ab} g^{\mu\nu} g^{\sigma\tau} F_{\mu\rho} F^{b}_{\nu\sigma} \d V_g
\]  
Here, \( F_{\mu\nu} := g (\partial_{\mu} A_{\nu}^a - \partial_{\nu} A_{\mu}^a) - g^2 f_{bc} A_{\mu}^b A_{\nu}^c \) is the local curvature form of the gauge boson \( A_{\mu}^a \). Furthermore, \( \d V_g := \sqrt{\text{Det}(g)} \, dt \wedge dx \wedge dy \wedge dz \) denotes again the Riemannian volume form. Then we obtain the following result in Proposition 3.1. Setting the corresponding gauge fixing boson to
\[
G := -\frac{\xi}{2} \left( \delta_{ab} g^{\mu\nu} A_{\mu}^a A_{\nu}^b - \tau_a c^a \right) \d V_g
\]  
implies the following symmetric gauge fixing and ghost Lagrange density, where \( \mathcal{Q} := Q \circ \overline{Q} \) is the gauge super-BRST operator:
\[
\mathcal{L}_{\text{YM-GF-Ghost}} := \mathcal{Q} G
= \frac{1}{\xi} \left( -\frac{1}{2g^2} \delta_{ab} L^a L^b + g^{\mu\nu} \left( \partial_{(\mu} \overline{c}_a \right) \left( \partial_{\nu} c^a \right) \right) \d V_g
+ \frac{g}{2} g^{\mu\nu} f_{bc} \left( \left( \partial_{(\mu} \overline{c}_a \right) c^b \right) \left( \partial_{\nu} c^a \right) \d V_g
+ \frac{g^2}{16} f_{bc} f_{ade} \overline{c}^e c^f \d V_g
\]  
Here, \( L^a := g g^{\mu\nu} \left( \nabla^T_\mu M^a_{\nu} \right) \equiv 0 \) is the covariant Lorenz gauge fixing functional and \( \xi \) denotes the gauge fixing parameter. Moreover, \( c^a \) and \( \overline{c}_a \) are the gauge ghost and gauge antighost,
respectively. Finally, the Lagrange density for Quantum Yang–Mills theory is then given as the sum of the two:

$$\mathcal{L}_{\text{QYM}} := \mathcal{L}_{\text{YM}} + \mathcal{L}_{\text{YM-GF-Ghost}}$$

In addition to the symmetric ghost Lagrange density, we also construct a homotopy that continuously interpolates between the Faddeev–Popov construction, the symmetric setup and the opposed Faddeev–Popov construction, all with respect to the covariant Lorenz gauge fixing, cf. Theorem 4.2.

Finally, for the coupling of (effective) Quantum General Relativity to Quantum Yang–Mills theory, we obtain the following results: We can generate the complete gauge fixing and ghost Lagrange densities via a total gauge fixing boson $B := F + G$ as above using the total BRST operator $D := P + Q$ and the total anti-BRST operator $\overline{D} := \overline{P} + \overline{Q}$, cf. Theorem 5.1. In addition, we also obtain a double-homotopy if we combine the two homotopies, cf. Corollary 5.2.

Additionally, we refer to [12, 13] for more detailed introductions to (effective) Quantum General Relativity coupled to Quantum Yang–Mills theories. In addition, we refer to [14] for the introduction to the diffeomorphism-gauge BRST double complex and its corresponding anti complex. We will use the ghost Lagrange densities from this article in [8] to study the cancellation identities for (effective) Quantum General Relativity coupled to the Standard Model. This provides an important ingredient to study the renormalization of Quantum Gauge Theories, cf. [15, 16, 17, 18]. Moreover, we refer the interested reader to the introductory texts on BRST cohomology and the BV formalism [19, 20, 21], the historical overview [22] and earlier works in a similar direction [9, 23, 24, 25, 26, 27, 28, 29].

## 2 General setup

We start this article with a brief summary of the diffeomorphism-gauge BRST double complex, which was introduced in [14]. This includes the definitions of the diffeomorphism BRST and anti-BRST operators, as well as the gauge BRST and anti-BRST operators. Then we introduce the ghost conjugation and discuss its action on the diffeomorphism and gauge Lautrup–Nakanishi auxiliary fields. In particular, we shift them such that they become anti-Hermitian with respect to the associated ghost conjugation.

**Definition 2.1** (Set of particle fields). We consider (effective) Quantum General Relativity coupled to Quantum Yang–Mills theory, which consists of the following particle fields:

- Graviton field $h_{\mu\nu}$
- Graviton-ghost field $C_\rho$
- Graviton-antighost field $\overline{C}^\rho$
- Graviton-Lautrup–Nakanishi auxiliary field $B^\rho$
- Gauge field $A^a_\mu$
- Gauge ghost field $\epsilon^a$
- Gauge antighost field $\overline{\epsilon}_a$
- Gauge Lautrup–Nakanishi auxiliary field $b_a$
- Any other particle field $\varphi$

More generally, we denote the set of all particle fields for a given QFT $Q$ via $\mathcal{F}_Q$. We remark that the Lorentz indices of the graviton-related fields are raised and lowered via the Minkowski
background metric $\eta_{\mu\nu}$ and its inverse $\eta^{\mu\nu}$. Contrary to the Lorentz index of the gauge field is raised and lowered using the metric $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$. Finally, the color indices of the gauge related fields are raised and lowered via the color metric $\delta_{ab}$ and its inverse $\delta^{ab}$.

**Definition 2.2 (Diffeomorphism (anti-)BRST operator).** We define the diffeomorphism BRST operator $P$ as the following odd vector field on the spacetime-matter bundle with graviton-ghost degree 1:

$$P := \frac{1}{\zeta} \left( \nabla^T_M C_\nu + \nabla^T_M C_\mu \right) \frac{\partial}{\partial h_{\mu\nu}} + \kappa C^\rho \left( \partial_\rho C_\sigma \right) \frac{\partial}{\partial C_\sigma} + \frac{1}{\zeta} B^\rho \frac{\partial}{\partial C^\rho}$$

(14)

Equivalently, its action on fundamental particle fields is given as follows:

$$P h_{\mu\nu} := \frac{1}{\zeta} \left( \nabla^T_M C_\nu + \nabla^T_M C_\mu \right) \equiv \frac{1}{\zeta} \left( C^\rho \left( \partial_\rho g_{\mu\nu} \right) + \left( \partial_\mu C_\sigma \right) g_{\rho\nu} + \left( \partial_\nu C_\rho \right) g_{\mu\rho} \right)$$

(15a)

$$P g_{\mu\nu} := \kappa \left( \nabla^T_M C_\nu + \nabla^T_M C_\mu \right) \equiv \kappa \left( C^\rho \left( \partial_\rho g_{\mu\nu} \right) + \left( \partial_\mu C_\sigma \right) g_{\rho\nu} + \left( \partial_\nu C_\rho \right) g_{\mu\rho} \right)$$

(15b)

$$P C^\rho := \kappa C^\sigma \left( \partial_\sigma C^\rho \right)$$

(15c)

$$P B^\rho := 0$$

(15d)

$$P \varphi := 0$$

(15e)

Here $\nabla \cdot$ denotes the Lie derivative with respect to diffeomorphisms, $\varphi$ any other particle field and $\mathcal{F}_Q$ the set of all such fields. Additionally, we define the diffeomorphism anti-BRST operator $\overline{P}$ as the following odd vector field on the spacetime-matter bundle with graviton-ghost degree -1:

$$\overline{P} := P \Bigg|_{C \rightarrow \overline{C}}$$

(16a)

together with the following additional changes

$$\overline{P} C^\rho := -\frac{1}{\zeta} B^\rho + \kappa \left( C^\sigma \left( \partial_\sigma C^\rho \right) - \left( \partial_\rho C^\sigma \right) C^\sigma \right)$$

(16b)

$$\overline{P} B^\rho := \kappa \left( C^\sigma \left( \partial_\sigma B^\rho \right) - \left( \partial_\rho C^\sigma \right) B^\sigma \right)$$

(16c)

We remark the characteristic identities $[P, P] = [\overline{P}, P] = [P, \overline{P}] = 0$.

**Definition 2.3 (Gauge (anti-)BRST operator).** We define the gauge BRST operator $Q$ as the following odd vector field on the spacetime-matter bundle with gauge ghost degree 1:

$$Q := \left( \frac{1}{\xi} \partial_\mu e^a + g f^a_{\ bc} c^b A^c_\mu \right) \frac{\partial}{\partial A^a_\mu} + \frac{g}{2} f^a_{\ bc} c^b_\sigma A^c_\mu \frac{\partial}{\partial c^a_\sigma} + \frac{1}{\xi} b^a \frac{\partial}{\partial b^a}$$

$$+ \kappa \sum_{\varphi \in \mathcal{F}_Q} \left( \ell_{C^C} \varphi \right) \frac{\partial}{\partial \varphi}$$

(17)
Equivalently, its action on fundamental particle fields is given as follows:

\begin{align}
QA^a_\mu &= \frac{1}{\xi} \partial_\mu c^a + g f_{bc}^a c^b A^c_\mu \\
Qc^a &= \frac{g}{2} f_{bc}^a c^b \\
Q\bar{c}^a &= \frac{1}{\xi} b^a \\
Qb^a &= 0 \\
Q\delta_{ab} &= 0 \\
Q\varphi &= g (\ell_c \varphi)
\end{align}

(18a) \hspace{1cm} (18b) \hspace{1cm} (18c) \hspace{1cm} (18d) \hspace{1cm} (18e) \hspace{1cm} (18f)

Here \( \ell \) denotes the Lie derivative with respect to gauge transformations, \( \varphi \) any other particle field and \( F_Q \) the set of all such fields. Additionally, we define the gauge anti-BRST operator \( \overline{Q} \) as the following odd vector field on the spacetime-matter bundle with gauge ghost degree -1:

\[ \overline{Q} := Q \bigg|_{c \to \bar{c}} \]

(19a)

together with the following additional changes

\begin{align}
\overline{Q}c^a &= -\frac{1}{\xi} b^a + g f_{bc}^a \bar{c}^b \bar{c}^c \\
\overline{Q}\bar{c}^a &= \frac{g}{2} f_{bc}^a \bar{c}^b \bar{c}^c \\
\overline{Q}b^a &= g f_{bc}^a \bar{c}^b \bar{c}^c
\end{align}

(19b) \hspace{1cm} (19c) \hspace{1cm} (19d)

We remark the characteristic identities \([Q, Q] = [Q, \overline{Q}] = [\overline{Q}, \overline{Q}] = 0\).

**Definition 2.4** (Total (anti-)BRST operator). Let \( P \) and \( Q \) be the diffeomorphism and gauge BRST operators from Definitions 2.2 and 2.3, we call their sum

\[ D := P + Q \]

(20)

the total BRST operator. In addition, let \( \overline{P} \) and \( \overline{Q} \) be the diffeomorphism and gauge anti-BRST operators from Definitions 2.2 and 2.3 we call their sum

\[ \overline{D} := \overline{P} + \overline{Q} \]

(21)

the total anti-BRST operator. We remark that both are indeed BRST operators due to \[14\] Theorem 5.1] and satisfy the characteristic identities \([D, D] = [D, \overline{D}] = [\overline{D}, \overline{D}] = 0\).

**Definition 2.5** (Super-BRST operator). Given the BRST operators \( D, P, Q \) and their corresponding anti-BRST operators \( \overline{D}, \overline{P}, \overline{Q} \), then we define the respective super-BRST operators as follows:

\[ \mathcal{D} := D \circ \overline{D}, \]

(22)

\[ \mathcal{P} := P \circ \overline{P} \]

(23)

and

\[ \mathcal{Q} := Q \circ \overline{Q} \]

(24)

In particular, they are even vector fields on the spacetime-matter bundle with ghost degrees 0. We remark that they are also nilpotent, i.e. satisfy \( \mathcal{D}^2 = \mathcal{P}^2 = \mathcal{Q}^2 = 0 \).
Definition 2.6 (Ghost conjugation, anti-Hermitian auxiliary field). We introduce the graviton-ghost conjugation $\dagger_{G}$ and the gauge ghost conjugation $\dagger_{c}$ as follows ($\varphi$ denotes again any other particle field):

\[
\begin{align*}
(C^\rho)_{\dagger G} & := \overline{C}^\rho \\
(\overline{C}^\rho)_{\dagger G} & := C^\rho \\
(B^\rho)_{\dagger G} & := -B^\rho - \zeta(\overline{C}^\rho (\partial_\sigma C^\sigma) - (\partial_\sigma \overline{C}^\sigma)C^\sigma) \\
(C^\rho)_{\dagger c} & := C^\rho \\
(\overline{C}^\rho)_{\dagger c} & := \overline{C}^\rho \\
(B^\rho)_{\dagger c} & := B^\rho \\
(C^\rho)_{\dagger c} & := C^\rho \\
(B^\rho)_{\dagger c} & := B^\rho \\
(c^a)_{\dagger c} & := c^a \\
(\overline{c}^a)_{\dagger c} & := \overline{c}^a \\
(b^a)_{\dagger c} & := b^a \\
(b^a)_{\dagger c} & := b^a \\
(\partial_\mu)_{\dagger c} & := -\partial_\mu \\
(\Gamma^\rho_{\mu\nu})_{\dagger c} & := -\Gamma^\rho_{\mu\nu} \\
(f^a_{bc})_{\dagger c} & := f^a_{bc} \\
(\varphi)_{\dagger c} & := \varphi
\end{align*}
\]

Here, $B^\rho$ and $b^a$ are the shifted anti-Hermitian Lautrup–Nakanishi auxiliary fields, given as follows:

\[
B^\rho := B^\rho - \frac{\zeta g^2}{2} (\overline{C}^\rho (\partial_\sigma C^\sigma) - (\partial_\sigma \overline{C}^\sigma)C^\sigma)
\]

and

\[
b^a := b^a - \frac{g \xi}{2} f^a_{bc} \overline{c}^b c^c
\]

Finally, we introduce the total ghost conjugation $\dagger$ as follows:

\[
\begin{align*}
(C^\rho)^\dagger & := \overline{C}^\rho \\
(\overline{C}^\rho)^\dagger & := C^\rho \\
(B^\rho)^\dagger & := -B^\rho - \zeta(\overline{C}^\rho (\partial_\sigma C^\sigma) - (\partial_\sigma \overline{C}^\sigma)C^\sigma) \\
(C^\rho)^\dagger & := C^\rho \\
(B^\rho)^\dagger & := B^\rho \\
(c^a)^\dagger & := c^a \\
(\overline{c}^a)^\dagger & := \overline{c}^a \\
(b^a)^\dagger & := -b^a \\
(b^a)^\dagger & := -b^a \\
(\partial_\mu)^\dagger & := -\partial_\mu \\
(\Gamma^\rho_{\mu\nu})^\dagger & := -\Gamma^\rho_{\mu\nu} \\
(f^a_{bc})^\dagger & := -f^a_{bc} \\
(\varphi)^\dagger & := \varphi
\end{align*}
\]

In particular, the total ghost conjugation reverses simultaneously graviton-ghosts and gauge ghosts.
Remark 2.7. The super-BRST operators are anti-Hermitian with respect to their associated ghost conjugation:

\[(P)^\dagger = -\mathcal{P} \quad (Q)^\dagger = -Q \quad (D)\dagger = -D\]  

(28) \quad (29) \quad (30)

3 The case of (effective) Quantum General Relativity

We calculate the symmetric gauge fixing and ghost Lagrange density for (effective) Quantum General Relativity in Proposition 3.1. Then we discuss in Theorem 3.2 how the symmetric setup relates to the Faddeev–Popov and opposed Faddeev–Popov constructions, all with respect to the linearized de Donder gauge fixing.

Proposition 3.1. The symmetric gauge fixing and ghost Lagrange density for (effective) Quantum General Relativity reads

\[
\mathcal{L}_{\mathrm{GR-GF-Ghost}} = \left( -\frac{1}{4\zeta^2} \eta^{\mu\nu} dD^{(1)}_{\nu} dD^{(1)}_{\mu} + \frac{1}{2\zeta} \eta^{\mu\nu} (\partial_\mu C_\rho)(\partial_\nu C_\rho) \right) dV_\eta
\]

\[
+ \frac{1}{2} \eta^{\mu\nu} \eta^{\rho\sigma} \left( \frac{1}{2} \partial_\rho (\Gamma^\sigma_{\mu\nu} C_\sigma) - \partial_\mu (\Gamma^\sigma_{\rho\nu} C_\sigma) \right) dV_\eta
\]

\[
- \frac{1}{2} \eta^{\mu\nu} \left( \frac{1}{2} \partial_\rho (\Gamma^\sigma_{\mu\nu} C_\sigma) - \partial_\mu (\Gamma^\sigma_{\rho\nu} C_\sigma) \right) C^\rho dV_\eta
\]

\[
+ \frac{\kappa^2}{32} \eta^{\mu\nu} \left( C^\rho (\partial_\rho C^\nu) + C^\nu (\partial_\nu C^\rho) \right) dV_\eta
\]

with the linearized de Donder gauge fixing functional \(dD^{(1)}_{\mu} := \eta^{\rho\sigma} \Gamma^\rho_{\sigma\mu}\). It can be obtained from the gauge fixing boson

\[
F := -\frac{\zeta}{4} \left( \frac{1}{\zeta} \eta^{\mu\nu} h_{\mu\nu} - C^\rho C_\rho \right) dV_\eta
\]

via \(\mathcal{P} F\), where \(\mathcal{P}\) is the diffeomorphism super-BRST operator.

Proof. The claimed statement results directly from the following calculations:

\[
\mathcal{P} F = -\frac{1}{4\zeta} \eta^{\mu\nu} \left( C^\rho (\partial_\rho g_{\mu\nu}) + (\partial_\mu C^\rho) g_{\mu\nu} + (\partial_\nu C^\rho) g_{\mu\nu} \right) dV_\eta
\]

\[
\quad + \left( \frac{\kappa}{4} \frac{C^\rho C_\rho}{dV_\eta} \right) dV_\eta
\]

and

\[
(P \circ \mathcal{P}) F = \left( \frac{1}{4\zeta} B^\rho B_\rho + \frac{1}{2\zeta} B^\rho dD_\rho - \frac{1}{4\zeta} \eta^{\mu\nu} \left( \partial_\rho C^\rho \right) \left( \nabla^T_{\mu} C_\nu + \nabla^T_{\nu} C_\mu \right) \right) dV_\eta
\]

\[
\quad + \frac{1}{2} \eta^{\mu\nu} \left( \left( \partial_\rho C^\rho \right) \left( \nabla^T_{\mu} C_\nu + \nabla^T_{\nu} C_\mu \right) + \left( \partial_\nu C^\rho \right) \left( \nabla^T_{\mu} C_\rho + \nabla^T_{\rho} C_\mu \right) \right) dV_\eta
\]

\[
\quad + \left( -\frac{\kappa}{4} \frac{B^\sigma \left( \partial_\sigma C^\rho \right) - C^\sigma \left( \partial_\sigma B^\rho \right) C_\rho + \frac{\kappa^2}{16} C^\sigma \left( \partial_\sigma C^\rho \right) C_\rho }{dV_\eta} \right) dV_\eta
\]

\[
\quad + \left( \frac{\kappa^2}{32} \frac{C^\rho (\partial_\rho C^\nu) + C^\nu (\partial_\nu C^\rho)}{dV_\eta} \right) dV_\eta
\]

In addition, all conjugated BRST operators act to the left.
together with the replacement of the Lautrup–Nakanishi auxiliary field with its anti-Hermitian shift
\[ B^\rho \equiv B'^\rho + \frac{\zeta}{2} \left( \partial_\sigma C^\rho - \left( \partial_\sigma \overline{C}^{\mu} \right) C^\sigma \right) \] (33c)
and then finally eliminating the shifted auxiliary field \( B'_\rho \) by inserting its equation of motion
\[ \text{EoM} \left( B'_\rho \right) = \frac{1}{\zeta} dD_\rho , \] (33d)
which are obtained as usual via an Euler–Lagrange variation of Equation (33b).

Theorem 3.2. We obtain the following homotopy in \( \lambda \in [0,1] \) between the Faddeev–Popov construction \( \lambda = 0 \), the symmetric setup \( \lambda = 1/2 \) and the opposed Faddeev–Popov construction \( \lambda = 1 \):

\[ L_{GR-GF-Ghost} (\lambda) = \left( -\frac{1}{4\zeta} \eta^{\mu \nu} (D^{(1)}_\mu \delta^{(1)}_\nu) \right) dV_\eta + \frac{1}{2\zeta} \eta^{\mu \nu} \left( \partial_\rho (\Gamma^\sigma_{\mu \nu} C^\rho) \partial_\eta C^\rho \right) dV_\eta + (1 - \lambda) \eta^{\mu \nu} \left( \frac{1}{2} \partial_\rho (\Gamma^\sigma_{\mu \nu} C^\rho) - \partial_\eta (\Gamma^\sigma_{\rho \nu} C^\sigma) \right) C^\rho dV_\eta \]
\[ - \lambda \eta^{\mu \nu} \left( \frac{1}{2} \partial_\rho (\Gamma^\sigma_{\mu \nu} \overline{C}^\sigma) - \partial_\eta (\Gamma^\sigma_{\rho \nu} \overline{C}^\sigma) \right) C^\rho dV_\eta \frac{\zeta^2}{8} \lambda (1 - \lambda) \eta^{\mu \nu} \left( \partial_\rho (\overline{C}^{\mu}) \right) \left( C^\rho (\partial_\eta C^\mu) \right) dV_\eta \] (34)

Proof. This follows directly from the application of the diffeomorphism BRST operator \( P \) on the following homotopy gauge fixing fermion:

\[ \zeta (\lambda) := \left( -\frac{\zeta}{4\zeta} \eta^{\mu \nu} \overline{F} (h_{\mu \nu}) + \frac{\zeta}{8} \lambda \overline{P} (\overline{C}^{\mu} C_\rho) + \left( \frac{\lambda}{4} - \frac{1}{2} \right) \overline{C}^{\rho} B_\rho \right) dV_\eta \] (35)

4 The case of covariant Quantum Yang–Mills theory

We calculate the symmetric gauge fixing and ghost Lagrange density for covariant Quantum Yang–Mills theory in Proposition 4.1. Then we discuss in Theorem 4.2 how the symmetric setup relates to the Faddeev–Popov and opposed Faddeev–Popov constructions, all with respect to the covariant Lorenz gauge fixing.

Proposition 4.1. The symmetric gauge fixing and ghost Lagrange density for Quantum Yang–Mills theory reads

\[ L_{YM-GF-Ghost} = \frac{1}{\xi} \left( -\frac{1}{2g^2} \delta^a_{bc} L^a L^b + g^{\mu \nu} \left( \partial_\mu \overline{A}_a^b \right) \left( \partial_\nu c^a \right) \right) dV_g + \frac{g}{2} g^{\mu \nu} f^a_{bc} \left( \partial_\mu \overline{A}_a^b \right) \left( \partial_\nu c^a \right) \left( \overline{A}_c^b \right) dV_g \]
\[ + \frac{g^2 \zeta}{10} f^a_{bc} f^c_{de} \overline{c}^a c^b \overline{c}^d c^e dV_g \] (36)

with the covariant Lorenz gauge fixing functional \( L^a := gg^{\mu \nu} \left( \nabla^{TM}_\mu A^b_\nu \right) \equiv 0 \). It can be obtained from the gauge fixing boson

\[ G := -\frac{\xi}{2} \left( \delta_{ab} g^{\mu \nu} A^a_\mu A^b_\nu - \overline{c}^a c^a \right) dV_g \] (37)
via \( QG \), where \( Q \) is the gauge super-BRST operator.
Proof. The claimed statement results directly from the following calculations:

$$\mathcal{Q}G = \left(-\delta_{ab}g^{\mu\nu}A^\mu_{\mu} \left(\partial_\mu c^b + \xi g f_{\mu\nu}^{\mu
u} A^d_\mu \right) - \frac{g^2 \xi}{4} f_{bc}^{\mu\nu} c^c_{\mu} + \frac{1}{2} \tau a b^a \right) dV_g \quad (38a)$$

and

$$(Q \circ Q)G = \left(\frac{1}{2} \xi b^a b^a + \frac{1}{g \xi} b_a L^a + \frac{1}{\xi} g^{\mu\nu} \left(\partial_\mu c^a \right) \left(\partial_\nu c^a \right) \right) dV_g$$

$$+ \left( g f_{bc}^{a} \left(\partial_\mu c^a \right) c^b_{\mu} A^c_\nu - \frac{g}{2} f_{bc}^{a} b_a c^c_{\nu} + \frac{g^2 \xi}{8} f_{bc}^{a} f_{ade} c^c d e^c \right) dV_g \quad (38b)$$

together with the replacement of the Lautrup–Nakanishi auxiliary field with its anti-Hermitian shift

$$b^a \equiv b^a - \frac{g \xi}{2} f_{bc}^{a} c^b_{c} c^c$$

and then finally eliminating the shifted auxiliary field $b^a$ by inserting its equation of motion

$$\text{EoM} \left( b^a \right) = \frac{1}{g} L^a \quad (38d)$$

which are obtained as usual via an Euler–Lagrange variation of Equation (38b).

Theorem 4.2. We obtain the following homotopy in $\tau \in [0, 1]$ between the Faddeev–Popov construction $\tau = 0$, the symmetric setup $\tau = 1/2$ and the opposed Faddeev–Popov construction $\tau = 1$:

$$\mathcal{L}_{YM-GF-Ghost} (\tau) = \frac{1}{\xi} \left(-\frac{1}{2g^2} \delta_{ab} L^a L^b + g^{\mu\nu} \left(\partial_\mu c^a \right) \left(\partial_\nu c^a \right) \right) dV_g$$

$$+ g g^{\mu\nu} f_{bc}^{a} \left(1 - \tau \right) \left(\partial^b L^b \right) A^a_\nu - \tau L^a A^a_\nu$$

$$+ \frac{g^2 \xi}{4} \tau (1 - \tau) f_{bc}^{a} f_{ade} c^c d e^c dV_g \quad (39)$$

Proof. This follows directly from the application of the gauge BRST operator $Q$ on the following homotopy gauge fixing fermion:

$$F(\tau) := \left(-\frac{\xi}{2} \delta_{ab} g^{\mu\nu} \mathcal{Q} A^a_{\mu} A^b_{\nu} + \frac{\xi \tau}{4} \mathcal{Q} c^a + \left(\frac{\tau}{2} - 1 \right) \tau b^a \right) dV_g \quad (40)$$

5 The total construction

Combining the results from Sections 3 and 4, we show in Theorem 5.1 that the complete symmetric gauge fixing and ghost Lagrange density for the coupling of (effective) Quantum General Relativity to Quantum Yang–Mills theory (QGR-QYM) can be generated via a total gauge fixing boson, using the total BRST operator and the total anti-BRST operator. Next, we observe in Corollary 5.2 that both homotopies in the ghost construction can be added to obtain a double homotopy for the complete gauge fixing and ghost Lagrange density of QGR-QYM.
**Theorem 5.1.** We obtain the complete gauge fixing and ghost Lagrange density for (effective) Quantum General Relativity coupled to Quantum Yang–Mills theory via

\[ \mathcal{L}_{\text{GR-GF-Ghost}} + \mathcal{L}_{\text{YM-GF-Ghost}} \equiv \mathcal{D}B, \]

where \( \mathcal{D} \) is the total super-BRST operator and \( B := F + G \) is the total gauge fixing boson.

**Proof.** This follows immediately from Propositions 3.1, 4.1 and the following equalities:

\[ PG \cong_{\text{TD}} 0, \]

\[ \overline{PG} \cong_{\text{TD}} 0, \]

\[ QF = 0 \]

and

\[ \overline{QF} = 0, \]

where \( \cong_{\text{TD}} \) means equality modulo total derivatives, which holds due to [14, Lemma 3.3]. ■

**Corollary 5.2.** Given the situation of Theorem 3.2 combined with Theorem 4.2, we obtain the following double-homotopy in \((\lambda, \tau) \in [0, 1]^2\) between the corresponding Faddeev–Popov constructions, the symmetric setups and the opposed Faddeev–Popov constructions: We apply the total BRST operator \( D := P + Q \) to the total homotopy gauge fixing fermion \( \chi(\lambda, \tau) := \zeta(\lambda) + f(\tau) \):

\[ \mathcal{L}_{\text{GR-YM-GF-Ghost}}(\lambda, \tau) := D\chi(\lambda, \tau) \]

**Proof.** This follows directly from Theoremata 3.2 and 4.2 together with the argument from the proof of Theorem 5.1. ■

6 Conclusion

We have studied symmetric gauge fixing and ghost Lagrange densities for (effective) Quantum General Relativity coupled to Quantum Yang–Mills theory. To this end, we recalled in Section 2 important notions of the diffeomorphism-gauge BRST double complex, introduced in [14], together with an extensive discussion on the ghost conjugation and the shifted anti-Hermitian Lautrup–Nakanishi auxiliary fields. Thereafter, we studied the cases of (effective) Quantum General Relativity in Section 3 and Quantum Yang–Mills theory in Section 4. Our results are Propositions 3.1 and 4.1 which provide the corresponding symmetric gauge fixing and ghost Lagrange densities, and Theoremata 3.2 and 4.2 which provide respective homotopies between the Faddeev–Popov construction, the symmetric setup and the opposed Faddeev–Popov construction. Finally, in Section 5 we consider the coupling of (effective) Quantum General Relativity to Quantum Yang–Mills theory: Our results are Theorem 5.1 which states that the complete symmetric gauge fixing and ghost Lagrange density can be generated from a total gauge fixing boson via the total BRST operator and the total anti-BRST operator. In addition, we show in Corollary 5.2 that we obtain a double homotopy if we add the corresponding homotopies of Theoremata 3.2 and 4.2. We want to use the symmetric ghost Lagrange densities in [5] to verify the diffeomorphism-gauge cancellation identities for (effective) Quantum General Relativity coupled to the Standard Model. This would be a major step towards the definition of a consistent renormalization operation for perturbative Quantum General Relativity in the sense of [12, 13] via the methods of [15, 16, 17, 18].
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References

[1] L.D. Faddeev and V.N. Popov: *Feynman Diagrams for the Yang-Mills Field*. Phys. Lett. B, 25:29–30, 1967.

[2] G. ’t Hooft and M. Veltman: *Diagrammar*, pages 177–322. Springer US, Boston, MA, 1974, ISBN 978-1-4684-2826-1.

[3] P. Cvitanović: *Field Theory*. Nordita Lecture Notes, 1983. Available at [http://chaosbook.org/FieldTheory/](http://chaosbook.org/FieldTheory/).

[4] M. Sars: *Parametric Representation of Feynman Amplitudes in Gauge Theories*. PhD thesis, Humboldt University of Berlin, 2015. Available at [https://edoc.hu-berlin.de/handle/18452/17954](https://edoc.hu-berlin.de/handle/18452/17954).

[5] H. Kißler and D. Kreimer: *Diagrammatic Cancellations and the Gauge Dependence of QED*. Phys. Lett. B, 764:318–321, 2017. arXiv:1607.05729v4 [hep-th].

[6] J. A. Gracey, H. Kißler, D. Kreimer: *On the self-consistency of off-shell Slavnov-Taylor identities in QCD*. Phys. Rev. D, 100(8):085001, 2019. arXiv:1906.07996v2 [hep-th].

[7] H. Kißler: *Off-shell diagrammatics for quantum gravity*. Phys. Lett. B, 816:136219, 2021. arXiv:2007.08894v2 [hep-th].

[8] D. Prinz: *Transversality in the Coupling of Gravity to Gauge Theories*. In preparation.

[9] L. Baulieu and J. Thierry-Mieg: *The Principle of BRS Symmetry: An Alternative Approach to Yang–Mills Theories*. Nucl. Phys. B, 197:477–508, 1982.

[10] N. Nakanishi: *Covariant Quantization of the Electromagnetic Field in the Landau Gauge*. Prog. Theor. Phys., 35:1111–1116, 1966.

[11] B. Lautrup: *Canonical Quantum Electrodynamics in Covariant Gauges*, 1967.

[12] D. Prinz: *Algebraic Structures in the Coupling of Gravity to Gauge Theories*. Annals Phys., 426:168395, 2021. arXiv:1812.09919v3 [hep-th].

[13] D. Prinz: *Gravity-Matter Feynman Rules for any Valence*. Class. Quantum Grav., 38(21):215003, 2021. arXiv:2004.09543v4 [hep-th].

[14] D. Prinz: *The BRST Double Complex for the Coupling of Gravity to Gauge Theories*. arXiv:2206.00780v1 [hep-th].

[15] D. Kreimer: *A remark on quantum gravity*. Annals Phys., 323:49–60, 2008. arXiv:0705.3897v1 [hep-th].

[16] W. D. van Suijlekom: *The structure of renormalization Hopf algebras for gauge theories I: Representing Feynman graphs on BV-algebras*. Commun. Math. Phys., 290:291–319, 2009. arXiv:0807.0999v2 [math-ph].

[17] D. Prinz: *Gauge Symmetries and Renormalization*. To appear in Math. Phys. Anal. Geom., 2022. arXiv:2001.00104v3 [math-ph].

[18] D. Prinz: *Cancellation Identities and Renormalization*. In preparation.
[19] G. Barnich, F. Brandt, and M. Henneaux: *Local BRST cohomology in gauge theories*, 2000. arXiv:hep-th/0002245v3.

[20] P. Mnev: *Quantum Field Theory: Batalin–Vilkovisky Formalism and Its Applications*. AMS University Lecture Series, 72, 2019. arXiv:1707.08096v1 [math-ph].

[21] K. Wernli: *Notes on Chern–Simons perturbation theory*. Rev. Math. Phys., 34(03):2230003, 2022. arXiv:1911.09744v1 [math-ph].

[22] C.M. Becchi: *BRS ‘Symmetry’, prehistory and history*. Pramana, 78:837–851, 2012. arXiv:1107.1070v2 [hep-th].

[23] N. Nakanishi and I. Ojima: *Covariant operator formalism of gauge theories and quantum gravity*, volume 27. World Sci. Lect. Notes Phys., 1990.

[24] M. Faizal: *BRST and Anti-BRST Symmetries in Perturbative Quantum Gravity*. Found. Phys., 41:270–277, 2011. arXiv:1010.1143v2 [gr-qc].

[25] L. Baulieu and M.P. Bellon: *A Simple Algebraic Construction of the Symmetries of Supergravity*. Phys. Lett. B, 161:96–102, 1985.

[26] L. Baulieu and M.P. Bellon: *p Forms and Supergravity: Gauge Symmetries in Curved Space*. Nucl. Phys. B, 266:75–124, 1986.

[27] T.P. Shestakova: *The role of BRST charge as a generator of gauge transformations in quantization of gauge theories and Gravity*. PoS, FFP14:175, 2016. arXiv:1410.4434v1 [gr-qc].

[28] G. Barnich, F. Brandt, and M. Henneaux: *Local BRST cohomology in Einstein–Yang–Mills theory*. Nucl. Phys. B, 455:357–408, 1995. arXiv:hep-th/9505173v2.

[29] S. Upadhyay: *Perturbative quantum gravity in Batalin-Vilkovisky formalism*. Physics Letters B 723 (2013) 470–474, 2013. arXiv:1305.4709v3 [hep-th].