Boundary stress tensor and asymptotically AdS$_3$ non-Einstein spaces at the chiral point

Gaston Giribet,†‡ Andrés Goya,†‡ and Mauricio Leston‡

†Universidad de Buenos Aires FCEN-UBA and IFIBA-CONICET,
Ciudad Universitaria, Pabellón I, 1428, Buenos Aires.
‡Instituto de Astronomía y Física del Espacio IAFE-CONICET,
Ciudad Universitaria, Pabellón IAFE, 1428 C.C. 67 Suc. 28, Buenos Aires.

Chiral gravity admits asymptotically AdS$_3$ solutions that are not locally equivalent to AdS$_3$; meaning that solutions do exist which, while obeying the strong boundary conditions usually imposed in General Relativity, happen not to be Einstein spaces. In Topologically Massive Gravity (TMG), the existence of non-Einstein solutions is particularly connected to the question about the role played by complex saddle points in the Euclidean path integral. Consequently, studying (the existence of) non-locally AdS$_3$ solutions to chiral gravity is relevant to understand the quantum theory. Here, we discuss a special family of non-locally AdS$_3$ solutions to chiral gravity. In particular, we show that such solutions persist when one deforms the theory by adding the higher-curvature terms of the so-called New Massive Gravity (NMG). Moreover, the addition of higher-curvature terms to the gravity action introduces new non-locally AdS$_3$ solutions that have no analogues in TMG. Both stationary and time-dependent, axially symmetric solutions that asymptote AdS$_3$ space without being locally equivalent to it appear. Defining the boundary stress-tensor for the full theory, we show that these non-Einstein geometries have associated vanishing conserved charges.

PACS numbers: 11.25.Tq, 11.10.Kk

I. INTRODUCTION

In the last three years there has been an interesting discussion of whether formulating three-dimensional Topologically Massive Gravity [1] about AdS$_3$ and Warped-AdS$_3$ spaces yields consistent models of quantum gravity or not. The discussion started with the proposals in [2] and [3], and rapidly turned into a discussion on the consistency (the closure) of imposing strong asymptotic boundary conditions that, by extirpating undesired ghostly graviton excitations, would finally render the background stable and the theory sensible at semiclassical level. In the case of chiral gravity [2], the specific discussion regarded the question as to whether Brown-Henneaux boundary conditions [4], usually considered in Einstein gravity, are or not consistent in Topologically Massive Gravity (TMG) at the chiral point ($\mu l = 1$) as well. It was argued in [3] that TMG at $\mu l = 1$ with the Brown-Henneaux boundary conditions is actually a consistent model.

Apart from being essential to conclude the consistency (the stability) of the theory (about a given background), the discussion on the boundary conditions is also crucial to establish which are the geometries that ultimately contribute to the partition function of the quantum theory. A precise characterization of the set of geometries one has to consider has not yet been accomplished. For example, questions like whether or not one has to take non-smooth spaces into account are still unclear. Nevertheless, with the aim of coming up with a physically sensible proposal for the partition function, several assumptions and conjectures have been made about over which geometrical configurations one has to sum. For instance, in the case of chiral gravity, it has been conjectured that, after imposing strong boundary conditions, all the saddle point contributions to the partition function would come from Einstein spaces, and, consequently, would be characterizeable in terms of quotients of AdS$_3$ [6], which would result in a remarkable simplification. This conjecture was particularly expressed in [2], where it was mentioned that the fact that at linearized level any solution of chiral gravity is locally equivalent to AdS$_3$ might lead one to suspect that something similar could happen at the full non-linear level. The suspicion was partially supported by the observation that all stationary, axially symmetric solutions of chiral gravity are indeed the solutions of General Relativity (GR). However, it has been subsequently observed that less symmetric non-Einstein spaces obeying the strong asymptotic conditions also exist in Topologically Massive Gravity at $\mu l = 1$. Then, solutions that are not locally equivalent to AdS$_3$ (hereafter called non-locally AdS$_3$ solutions) have probably to be taken into account as well.

In chiral gravity, the question about the existence of non-locally AdS$_3$ solutions is particularly connected to the question about the role played by complex saddle points in the Euclidean path integral. It was noted in [2] that non-Einstein Lorentzian solutions of chiral gravity would yield complex saddle points in the Euclidean theory. The argument goes as follows: When rotating to Euclidean signature, the Cotton tensor $C_{\mu\nu}$ picks up an imaginary factor $i$, so that the equations of motion of
chiral gravity take the form
\[ G_{\mu\nu} - \frac{1}{2} g_{\mu\nu} + iC_{\mu\nu} = 0, \] (1)
with \( G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \) being the Einstein tensor. Then, since \( C_{\mu\nu} \) vanishes for all Einstein spaces, all the solutions to (1) that are not solutions to GR would correspond to complex saddle points in the Euclidean theory.

Then, the question arises as to how to deal with the contribution coming from non-Einstein spaces in the theory. Do the complex saddle points actually contribute to the Euclidean functional? A radical way to address the problem (instead of trying to account for non-Einstein contributions) would be trying to deform the theory in a way that, while still keeping the desired properties of chiral gravity, non-locally \( \text{AdS}_3 \) spaces ultimately result excluded. The first proposal to do this would be adding to the Chern-Simons gravitational term of the TMG Lagrangian the special combination of square-curvature terms proposed in the New Massive Gravity (NMG) model of [7], which seems to be the natural candidate to yield a consistent generalization of chiral gravity. However, as we will see, this is not sufficient to exclude non-Einstein spaces from the asymptotically \( \text{AdS}_3 \) sector; such spaces actually persist. Moreover, the higher-order corrections introduce additional solutions of such kind. All these solutions, however, seem to exhibit some sort of pathologies; they present closed timelike curves or timelike geodesic incompleteness in naked regions. Nevertheless, since it is not clear whether summing only over smooth geometries in the Euclidean path integral is the appropriate way of defining the quantum theory, the existence of these non-locally \( \text{AdS}_3 \) spaces deserves attention. Then, it seems reasonable to try to live in harmony with the non-Einstein contributions, and explore the implications of taking these geometries into account. Here, we will study a particular family of non-Einstein spaces that asymptote \( \text{AdS}_3 \). In particular, we will show that the solutions found in [8] can be extended to the model consisting of coupling both TMG and NMG at the chiral point (interpolating in such a way with solutions discussed in [10]). We will further generalize these solutions by finding asymptotically \( \text{AdS}_3 \) axially symmetric deformations of the extremal Bañados-Teitelboim-Zanelli black hole (BTZ). Defining the boundary stress-tensor and resorting to standard holographic renormalization techniques, we will compute the conserved charges of these asymptotically \( \text{AdS}_3 \) solutions and show they have associated vanishing mass and vanishing angular momentum.

II. MASSIVE GRAVITY IN \( \text{AdS}_3 \)

A. The action

Let us begin by reviewing three-dimensional massive gravity about asymptotically \( \text{AdS}_3 \) spaces. The action of the theory can be written as the sum of four distinct contributions, namely
\[ S = S_{\text{EH}} + S_{\text{CS}} + S_{\text{NMG}} + S_B, \] (2)
where the first term corresponds to the Einstein-Hilbert action of GR
\[ S_{\text{EH}} = \frac{1}{16\pi G} \int d^3x \sqrt{-g} \left(R - 2\lambda\right), \] (3)
the second term is the gravitational Chern-Simons term of TMG [1]
\[ S_{\text{CS}} = \frac{1}{32\pi G\mu} \int d^3x \varepsilon^{\alpha\beta\gamma} \Gamma^\sigma_{\alpha\beta\gamma} (\partial_{\sigma} \Gamma^\sigma_{\gamma\rho} + \frac{2}{3} \Gamma^\sigma_{\beta\rho} \Gamma^\sigma_{\gamma\lambda}) , \] (4)
and the third term contains the square-curvature contributions
\[ S_{\text{NMG}} = \frac{1}{16\pi G m^2} \int d^3x \sqrt{-g} (R_{\mu\nu} R^{\mu\nu} - \frac{3}{8} R^2), \] (5)
proposed in [7]. The action also includes boundary terms \( S_B \), which will be specified below.

The field equations derived from (2) read
\[ G_{\mu\nu} + \lambda g_{\mu\nu} + \frac{1}{\mu} C_{\mu\nu} + \frac{1}{2m^2} K_{\mu\nu} = 0. \] (6)
which, apart from the Einstein tensor \( G_{\mu\nu} \) and the cosmological constant term, include the Cotton tensor
\[ C_{\mu\nu} = \frac{1}{2} \varepsilon_{\alpha\beta} \nabla_\alpha R_{\beta\nu} + \frac{1}{2} \varepsilon^{\alpha\beta} \nabla_\alpha R_{\mu\beta}, \] (7)
and the tensor \( K_{\mu\nu} \)
\[ K_{\mu\nu} = 2\Box R_{\mu\nu} - \frac{1}{2} \nabla_\mu \nabla_\nu R - \frac{1}{2} \Box g_{\mu\nu} + 4 R_{\mu\alpha\beta\gamma} R^{\alpha\beta\gamma} - \frac{3}{2} R R_{\mu\nu} - R_{\alpha\beta} R^{\alpha\beta} g_{\mu\nu} + \frac{3}{8} R^2 g_{\mu\nu}. \] (8)
The latter satisfies the remarkable property
\[ g^{\mu\nu} K_{\mu\nu} = R_{\mu\nu} R^{\mu\nu} - \frac{3}{8} R^2, \] (9)
which is one of the reasons why the theory proposed in [7] is a very special one. The theory has three mass scales, namely \( \mu, m, |\lambda|^{1/2} \). While the case \( 1/\mu = 0 \) yields NMG, the case \( 1/m^2 = 0 \) corresponds to TMG. Here, we will be concerned with the full theory (2).

Three-dimensional massive gravity (2) admits \( \text{AdS}_3 \) space as a solution. In fact, it is not hard to verify that all Einstein spaces satisfy (6). The two admissible values for the typical radius of the \( \text{AdS}_3 \) solution are given by
\[ l^2 = \frac{1}{2\lambda} \left(1 \pm \sqrt{1 + \lambda m^2}\right), \] (10)
as it can be easily seen from the trace of the equations of motion (6). If \( \lambda < 0 \) and \( 1 + \lambda m^2 > 0 \), \( \text{AdS}_3 \) space exists as solution.
B. Boundary conditions

Since we are originally motivated by AdS$_3$/CFT$_2$, we will be concerned with spaces that asymptote AdS$_3$ near its boundary. Then, the first question that appears is which definition of 'asymptotically AdS$_3$ spaces' we have to take into account in the theory [2]. The asymptotic-AdS$_3$ conditions are defined by requiring the conserved charges computed in the boundary of the space to be finite, but in a way that is still compatible with a sufficiently interesting subset of the space of solutions. The question about the asymptotic is important because, as it has been observed in different scenarios [9, 13], the adequate definition of asymptotic-AdS$_3$ conditions is a theory-dependent notion. Besides, the consistency of a given set of boundary conditions not only depends on the specific Lagrangian, but it also depends on the particular point of the parameter space in which one is interested. For instance, both TMG and NMG exhibit special points in the parameter space at which AdS$_3$ asymptotic boundary conditions can be defined with a falling-off behavior that results relaxed with respect to that in Einstein gravity. It was shown in [11] that for generic values of $m^2$ the appropriate asymptotic conditions to be considered in NMG are the Brown-Henneaux boundary conditions of [2], the same as in GR. In the system of coordinates in which the metric of (the universal covering of) AdS$_3$ space takes the form

$$ds^2 = -\left(\frac{1}{l^2} + 1\right)dt^2 + \left(\frac{r^2}{l^2} + 1\right)^{-1}dr^2 + r^2d\varphi^2,$$  \(11\)

with $\varphi \in [0, 2\pi]$, $t \in \mathbb{R}$, $r \in \mathbb{R}_{\geq 0}$, the Brown-Henneaux boundary conditions read

$$g_{tt} = -\frac{r^2}{l^2} + O(1), \quad g_{r^2} = O(r^{-4}),$$  \(12\)
$$g_{rr} = \frac{r^2}{l^2} + O(r^{-4}), \quad g_{r\varphi} = O(1), \quad g_{\varphi\varphi} = r^2 + O(1),$$  \(13\)
$$g_{\varphi r} = O(r^{-4}),$$  \(14\)

where $O(r^{-n})$ stands for terms whose $r$-dependence damps off as $1/r^n$ or faster at large $r$, with their dependence on the coordinates $t$ and $\varphi$ being arbitrary. Boundary conditions [12, 13] are consistent for the full theory [2], in the sense it yields finite charges.

The group of asymptotic Killing vectors preserving [12, 13] is generated by two copies of the deWitt algebra [4], which certainly includes the isometry algebra of AdS$_3$ as a proper subalgebra. More interesting is the fact that, as it happens in three-dimensional Einstein gravity [4], the algebra satisfied by the conserved charges associated to those asymptotic symmetries turns out to coincide with two copies of the Virasoro algebra with left and right central charges

$$c_L = \frac{3l}{2G}\left(1 - \frac{1}{\mu l} + \frac{1}{2\pi^2 l^2}\right), \quad (15)$$
$$c_R = \frac{3l}{2G}\left(1 + \frac{1}{\mu l} + \frac{1}{2\pi^2 l^2}\right), \quad (16)$$

respectively. From the AdS$_3$/CFT$_2$ perspective, these central charges acquire the interpretation as being the central charges of the dual conformal field theory (CFT).

We will focus our attention on chiral gravity, namely on the theory defined on the line $c_L = 0$ of the parameter space and by imposing Brown-Henneaux boundary conditions on the space of solutions.

In contrast with the chiral gravity of [2], where $\mu l$ gets fixed to 1 by the requirement $c_L = 0$, the theory defined by action [2] at $c_L = 0$ has the coupling of the higher-curvature terms as a free parameter to play with. This will introduces more diversity in the kind of asymptotically AdS$_3$ solutions we are interested in.

III. NON-LOCALLY ADS$_3$ SOLUTIONS

A. Persistent solutions

Three-dimensional massive gravity [2] admits all GR solutions as exact solutions. This is evident in the case of TMG, as the Cotton tensor vanishes if and only if a space is conformally flat. If the NMG contribution is also included in the action this is still true. However, the reciprocal is not true; that is, there also exist a rich set of solutions to [2] that are not solutions to GR. Finding such a non-Einstein solution is not necessarily a hard problem; what is a hard problem is to try to answer the question as to whether or not non-Einstein solutions persist after strong boundary conditions like [12, 13] are imposed. It was shown recently that such solutions actually exist both in the case of TMG [8] and in the case of NMG [10], and here we will show that these solutions also exist in the general theory [2] when $c_L = 0$. To see this, let us start by considering the ansatz

$$ds^2 = \frac{l^2}{r^2}dr^2 + \frac{l^2}{r^2}(dt^2 + l^2d\varphi^2) + \sum_{\pm} h_{\pm}(dt \pm l d\varphi)^2,$$

and consider the expansion

$$h_{\pm}(t, \varphi, r) = h_{\pm}^{(0)} + r^{-2}h_{\pm}^{(2)} + r^{-4}h_{\pm}^{(4)} + r^{-6}h_{\pm}^{(6)} \ldots$$  \(17\)

with $h_{\pm}^{(2k)}$ being functions of coordinates $t$ and $\varphi$. The solution found in [8] corresponds to $h_{-}(t, \varphi, r) = 0$, with $h_{-}^{(0)} \sim t$ and $h_{-}^{(4)} \sim \text{const.}$ As in [8, 10], one may considers the ansatz

$$h_{-}(t, \varphi, r) = 0, \quad h_{++}(t, \varphi, r) = \frac{\beta^2 l^2}{96\mu} r^{-4}$$  \(18\)
and look for exact solutions. It turns out that, without major modifications, equations (6) are solved for (15) if and only if
\[
\mu = \frac{2m^2l}{2m^2l^2 + 1}, \quad \lambda = \frac{1 - 4m^2l^2}{4m^2l^4}, \quad n = \frac{2m^2l^2 - 11}{2m^2l^2 - 15},
\]
or, equivalently, if
\[
2m^2l^2 = \frac{\mu l}{1 - \mu l}, \quad \lambda = \frac{1 - 3\mu l}{2\mu l}, \quad n = \frac{12\mu l - 11}{16\mu l - 15}.
\]

\(\beta\) is an arbitrary constant. It is easy to check that these values of the parameters precisely correspond to the following values of the central charges
\[
c_L = 0, \quad c_R = \frac{3l}{G}, \quad (19)
\]
which is a generalization of the chiral point studied in [2]. That is, the generalized chiral gravity, defined as considering \(2m^2l^2 = \mu l/(1 - \mu l)\) in (2) and imposing the asymptotic behavior \([12]-[14]\), exhibits solutions that are not solutions of general relativity. We will find more general solutions of this type below.

Space \([13]\) is a time-dependent geometry that, in spite of it, still admits \(\partial_t\) as asymptotic timelike Killing vector, in the sense that it satisfies boundary conditions \([12]-[14]\). This provides us with a notion of gravitational mass and angular momentum for this space. Being a genuine asymptotically \(AdS_3\) configuration, \([13]\) deserves to be studied and it is worthwhile asking ourselves about its implications for \(AdS_3/CFT_2\). Geometrical aspects of space \([13]\) have been analyzed in [8], where it was observed that it exhibits closed timelike curves at the time dependent radius \(r_{ctc}\) that solves the equation
\[
r_{ctc} + l^2h_{+}(t, r_{ctc}) = 0.
\]
Curvature invariants of this space are
\[
R = \frac{6}{l^2}, \quad R_{\mu\nu}R^{\mu\nu} = \frac{12}{l^4}, \quad R_{\mu\alpha\nu\beta}R^{\mu\alpha}R^{\nu\beta} = -\frac{24}{l^6}.
\]
The fact that these curvature invariants coincide with those of \(AdS_3\) space may suggest that \([13]\) correspond to a locally \(AdS_3\) space; however, this is not the case. In fact, the Cotton tensor \(C_{\mu\nu}\) associated to \([13]\) does not vanish unless \(\beta = 0\), which implies that the space is not conformally flat, and thus it is not locally equivalent to \(AdS_3\). Namely,
\[
C_{\mu\nu} = \frac{\beta}{r^4} \begin{pmatrix}
\frac{4}{3}\beta(4 - 5/n)/l & \frac{4}{3}\beta(4 - 5/n)/r/l & \frac{4}{3}\beta(4 - 5/n)/r \\
\frac{4}{3}\beta(4 - 5/n)/r/l & \frac{4}{3}\beta(4 - 5/n)/r & 0
\end{pmatrix},
\]
with the labels \(x^0 = t, x^1 = \varphi, x^2 = r\).

One can in principle solve the equations of motion iteratively considering expansion \([17]\). Solution \([13]\) corresponds to the case \(h_{-\cdot}(t, \varphi, r) = 0\) and \(h_{+\cdot}(t, \varphi, r) = \beta(t - t_0)/l^2 - \beta^2l^2/(96n^4)\). It still remains a solution if a quadratic piece \(\sim r^2\) is added to this function; however, in that case conditions \([12]-[14]\) are not obeyed.

\section*{B. New solutions}

One might ask whether one can generalize \([13]\) in a way that Brown-Henneaux asymptotic is preserved. To see that this is actually possible one has to consider a more general solution whose expansion \([17]\) is not necessarily finite. Such a solution is given by
\[
h_{++}(t, r) = h_{++}^{(0)}(t, r) - h_{+}^{(L)} \log(r) + h_{+}^{(m)} r^{3/2 - 2m^2l^2}, \quad (20)
\]
where \(h_{++}^{(L)}\) and \(h_{+}^{(m)}\) are two arbitrary constant coefficients. This generalizes \([13]\) and obeys the Brown-Henneaux conditions if \(h_{++}^{(L)} = 0\) and \(m^2l^2 \geq 3/2\). The solution with \(h_{++}^{(L)} \neq 0\), on the other hand, reduces to the Log-gravity solution studied in [12] when \(\beta = h_{+}^{(m)} = 0\), and even if it does not obey Brown-Henneaux boundary conditions \([12]-[14]\), it happens to be asymptotically \(AdS_3\) in the sense of \([13]\).

Solution \([20]\) is not locally \(AdS_3\) for generic values of the coefficients. In fact, for \([20]\) to be conformally flat it is necessary (not sufficient) to have \(\beta = 0\). Besides, even if it the t-dependent term is not present, the Cotton tensor takes the form
\[
C_{\mu\nu} = \left( h_{+}^{(L)} + \frac{P(m^2l^2)}{16r^{m^2l^2-3/2}L^{(m)}} \right) \left( \begin{array}{ccc}
1/l^3 & 1/l^2 & 0 \\
0 & 1/l^2 & 1/l \\
0 & 0 & 0
\end{array} \right),
\]
with \(P(x) = 3 - 2x - 12x^2 + 8x^3\). This gives three roots for the Cotton tensor to vanish when \(h_{++}^{(L)} = 0\); namely, it only vanishes for \(m^2l^2 = \pm 1/2\) and \(m^2l^2 = 3/2\) (and, of course, for \(m^2l^2 = \infty\)). The points \(m^2l^2 = \pm 1/2\) always exhibit special features in NMG. Another special point is \(m^2l^2 = 15/2\), where \(n\) diverges. There, only a damping \(\sim 1/r^6\) is present, namely one finds \(h_{+}^{(m)} \neq 0\) with \(h_{+}^{(L)} = 0\) in \([17]\). Notice also that the term \(\sim 1/r^7\) is present in \([20]\) coincides with the \(\sim 1/r^4\) dependence of \(h_{+}^{(m)}(t, r)\) for \(m^2l^2 = 11/2\), where \(n\) vanishes. There, one can set \(\beta^2 \propto n\) so that the time-dependent term disappears by keeping the term \(\sim 1/r^4\) in the metric. This represents a stationary, axially symmetric solution that asymptotizes \(AdS_3\). More generally, the existence of solution \([20]\) already shows that Birkhoff-like theorems that were proven for TMG do not hold when the higher terms of NMG are added to the action. In fact, for \(\beta = 0\) solution \([20]\) is an asymptotically \(AdS_3\) space which is stationary, and even when it has the same scalar curvature that \(AdS_3\) space, it is not locally equivalent to it.

\section*{C. Perturbing the extremal black hole}

It is relatively easy to generalize \([20]\) further by showing that a similar perturbation of the extremal BTZ is admitted as exact solution to \([0]\) when \(c_L = 0\). Consider
the extremal BTZ metric

\[ ds^2 = -N(r)dt^2 + \frac{dr^2}{N(r)} + r^2(d\phi + N^\phi(r)dt)^2, \]

with

\[ N(r) = \frac{r^2}{l^2} - 4M + \frac{4M^2l^2}{r^2}, \quad N^\phi(r) = \frac{2Ml}{r^2}. \]

This geometry represents an extremal black hole \[14\], which locally equivalent to AdS3. The event horizon of the black hole is located at \( r_+ = \sqrt{2M} \). It turns out that, by adding to metric (21)-(22) a term \( h_+ (dt + l d\phi)^2 \) with

\[ h_+ (r) = \frac{1}{2} h_+^{(L)} \log (r^2 - r_+^2) + h_+^{(m)} (r^2 - r_+^2)^{\frac{2-2m^2}{4}}, \]

gives a new solution to (6) when \( c_L = 0 \) (again, \( h_+^{(L)} \) and \( h_+^{(m)} \) are two arbitrary constants.) This new solution generalizes the logarithmic deformation of the BTZ solution found in \[12\]. This obeys the weakened asymptotic

\[ g_{tt} = \frac{r^2}{l^2} + O(\log(r)), \quad g_{tt} = O(r^{-4}), \]

\[ g_{rr} = \frac{l^2}{r^2} + O(r^{-4}), \quad g_{r\phi} = O(\log(r)), \]

\[ g_{\phi\phi} = r^2 + O(\log(r)), \quad g_{\phi\phi} = O(r^{-4}), \]

which are consistent with the boundary conditions in the definition of Log-gravity \[3,13\]. On the other hand, when \( h_+^{(L)} = 0 \) the logarithmic term in (23) is not present, and the solution happens to obey Brown-Henneaux boundary conditions \[12,14\] with \( h_+^{(m)} \neq 0 \) if \( m^2 l^2 \geq 3/2 \). Metric function (23) diverges at the 'would-be-horizon' radius \( r_+ = \sqrt{2M} \), and this divergence makes the geodesics of falling particles to wind infinitely rapid before reaching this radius. It is also important to mention that for \( r^2 < 2Ml \) the metric of the space can be extended by reversing the sign in the argument of the logarithm in (23).

The space described by the perturbation (23) is not conformally flat. In general, for a perturbation of the extremal BTZ metric of the form

\[ h_+ (r) = -h_+^{(0)} \log (r) + h_+^{(2)} r^{-2} + h_+^{(4)} r^{-4} + h_+^{(6)} r^{-6} \ldots \]

(24)

with the coefficients \( h_+^{(2k)} \) now being constant, the non-vanishing components of the Cotton tensor are proportional to

\[ \frac{r^2 - r_+^2}{r^6} \sum_{k=0} Q_{2k}(r_+^2, r^2) r^{-2k} h_+^{(2k)}, \]

(25)

with \( Q_{2k}(r_+^2, r^2) \) being polynomials of the form

\[ Q_0(x, y) = (1y^2 - 5xy + 4x^2), \]
\[ Q_2(x, y) = 4(3y^2 - 9xy + 6x^2), \]
\[ Q_4(x, y) = 12(5y^2 - 13xy + 8x^2), \]
\[ Q_6(x, y) = 24(7y^2 - 17xy + 10x^2), \]

More precisely, one finds \( Q_{2k}(x, y) = C_{2k}((2k + 1)y^2 - (4k + 5)xy + (2k + 4)x^2) \), where \( C_0 = 1, C_2 = 4, \) and \( C_{2k} = C_{2k-2} + 4k \) for \( k > 1 \). In the large \( r \) limit only the logarithmic term gives a non-conformally flat contribution.

D. The Poincaré patch

Solution \(13\) can be generalized in a way that the metric acquires linear dependence both in \( t \) and \( \phi \). For the metric not to be multivalued, one has to take the universal covering of coordinate \( \phi \); this is done by defining \( x = l \phi \in \mathbb{R} \). Defining light-cone coordinates \( x^\pm = t \pm x \) and the new coordinate \( z = l^2/r \), we have the more general solution

\[ ds^2 = l^2 (dz^2 - dx^+ dx^-) + \kappa h_+ (dx^+)^2 \]

(26)

with

\[ h_+ (x^+, x^-, z) = x^+ + \sigma x^- - \frac{\kappa \sigma^2}{24m^2 l^2} z^4 + c_m z^m l^2 \frac{z-\frac{1}{2}}{z}, \]

(27)

where we defined \( \kappa = \beta/(2l^2) \), we rescaled \( h_+^{(m)} \) to define \( c_m \), and introduced a new parameter \( \sigma \) which can be taken to be \( -1 \leq \sigma \leq 1 \). The case \( \sigma = 1 \) is that of (20).

Space (20)-(27) is a perturbation of AdS3 space written in Poincaré coordinates; the latter corresponds to \( \kappa = 0 \). The spaces represented by metrics with different values of \( \sigma \) are locally isometric.

To analyze the large \( z \) limit, namely the region \( r \approx 0 \) of the space, one may define a new variable \( \zeta = z^{-2} \) and rescaling \( x^+ \) by a constant factor, for \( c_m = 0 \) we find that the leading piece of the metric takes the form

\[ ds^2 \approx \left( \frac{l^2}{4\zeta} (d\zeta^2 - (dx^+)^2) + O(\zeta) dx^+ dx^- + O(1) (dx^+)^2 \right). \]

This is the limit \( (r \sim \zeta^{1/2} \sim 0) \) the timelike coordinate \( t \) and the spacelike coordinate \( r \) tends to a lightlike coordinate. This limit was considered in \[8\] to discuss the nature of geodesic incompleteness at \( r \approx 0 \).

IV. BOUNDARY STRESS TENSOR

A. The Brown-York tensor

Now, let us discuss the definition of a boundary stress tensor and holographic renormalization \[15,16\] for these spaces. The general idea is that Brown-York tensor \[17\] applied to asymptotically AdS spaces, and upon an appropriate regularization of the charges it yields when evaluating at the boundary \( r \to \infty \), gives the stress-tensor of the dual conformal field theory (CFT).
Schematically, one can identify such renormalized version of the Brown-York tensor \( T_{ij}^{(\text{ren})} \) with the quantity

\[
\lim_{r \to \infty} T_{ij}^{(\text{ren})} \equiv (T_{ij})_{\text{CFT}} = \frac{2}{\sqrt{-\gamma}} \frac{\delta S_{\text{eff}}}{\delta \gamma_{ij}}|_{\gamma_{ij}=\delta_{ij}} \tag{28}
\]

where \( S_{\text{eff}} \) refers to the effective action of the gravity theory in AdS3.

Adding boundary terms to the gravity action is necessary for the Brown-York tensor to be finite when evaluated at the boundary \( r = \infty \). Such boundary terms are of two kinds: First, those terms \( S_B \) needed for the variational principle to be defined in an appropriate way; secondly, those terms \( S_C \) that, being constituted of intrinsic quantities of the boundary, may be added to the action without changing the classical dynamics and yielding finite conserved charges at the boundary. From the dual theory point of view, the boundary terms \( S_C \) in the action can be thought of as local counterterms needed by renormalization.

To define the boundary stress-tensor it is convenient to write the metric in its Arnowitt-Deser-Misner (ADM) decomposition

\[
ds^2 = N^2 dr^2 + \gamma_{ij}(dx^i + N^i dr)(dx^j + N^j dr), \tag{29}
\]

where \( N^2 \) is the radial lapse function, and \( \gamma_{ij} \) is the two-dimensional metric on the constant-\( r \) surfaces. The Latin indices refer to the coordinates on the constant-\( r \) surfaces \( i,j = 0,1 \), while the Greek indices are \( \mu, \nu = 0,1,2 \), recall \( x^2 = r \).

### B. Boundary action

Being a theory that yields fourth-order equations of motion, NMG makes the discussion on boundary action a little bit more subtle; one has to give a prescription for how the variational principle is to be defined, for how the boundary data is to be specified. Here, we will follow the proposal in \cite{18}, which amounts to first rewriting the proposal in \cite{18}, which amounts to first rewriting the variational principle to be defined in an specific way; secondly, those terms \( S_C \) that, being constituted of intrinsic quantities of the boundary, may be added to the action without changing the classical dynamics and yielding finite conserved charges at the boundary. From the dual theory point of view, the boundary terms \( S_C \) in the action can be thought of as local counterterms needed by renormalization.

To define the boundary stress-tensor it is convenient to write the metric in its Arnowitt-Deser-Misner (ADM) decomposition

\[
ds^2 = N^2 dr^2 + \gamma_{ij}(dx^i + N^i dr)(dx^j + N^j dr), \tag{29}
\]

where \( N^2 \) is the radial lapse function, and \( \gamma_{ij} \) is the two-dimensional metric on the constant-\( r \) surfaces. The Latin indices refer to the coordinates on the constant-\( r \) surfaces \( i,j = 0,1 \), while the Greek indices are \( \mu, \nu = 0,1,2 \), recall \( x^2 = r \).

The action \( S_B \) needed for the variational principle to be defined in an appropriate way by introducing an auxiliary field \( f_{\mu\nu} \) and defining the alternative action

\[
S_B = \frac{1}{16\pi G} \int_{\Sigma} d^3x \sqrt{-g} \left( R - 2\lambda + f_{\mu\nu}(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R) \right) - \frac{1}{4} m^2 (f_{\mu\nu} f^{\mu\nu} - f_{ij} f^{ij}) + S_{\text{CS}} \tag{30}
\]

with \( f = f_{\mu\nu} g^{\mu\nu} \). In fact, on-shell, \( f_{\mu\nu} \) results proportional to the Schouten tensor, namely

\[
f_{\mu\nu} = \frac{2}{m^2} (R_{\mu\nu} - \frac{1}{4} g_{\mu\nu} R), \tag{31}
\]

and plug it back in \text{[30]} one recovers the bulk action in its original form.

Then, boundary terms may be introduced for the variational principle to be defined in such a way that both the metric \( g_{\mu\nu} \) and the auxiliary field \( f_{\mu\nu} \) are fixed on the boundary. With this prescription, the boundary action reads

\[
S_B = \frac{1}{16\pi G} \int_{\partial \Sigma} d^2x \sqrt{-\gamma} \left( -2K + \tilde{f}^{ij} K_{ij} + \tilde{f} K \right), \tag{32}
\]

where \( \gamma_{ij} \) is the metric induced on the boundary, \( K_{ij} \) is the extrinsic curvature, and \( K \) is its trace \( K = \gamma^{ij} K_{ij} \). In ADM variables, the extrinsic curvature takes the form

\[
K_{ij} = - \frac{1}{2\lambda} \left( \partial_r \gamma_{ij} - \nabla_i N_j - \nabla_j N_i \right). \tag{33}
\]

The first term in \text{[32]} is of course the Gibbons-Hawking term, corresponding to the Einstein-Hilbert bulk action. The other two terms in \text{[32]} are proper of NMG. No additional boundary terms are needed because the Chern-Simons gravitational term of TMG is included in the action. Then, the (unrenormalized) boundary stress tensor of the full theory is given by

\[
T_{ij} = \frac{2}{\sqrt{-\gamma} \partial \gamma_{ij}} (S_A + S_B), \tag{34}
\]

which, again, can be written identifying three distinct contributions, namely \( T^{ij} = T_{\text{EH}}^{ij} + T_{\text{CS}}^{ij} + T_{\text{NMG}}^{ij} \). The first contribution would be the standard contribution coming from GR,

\[
T_{\text{EH}}^{ij} = \frac{1}{8\pi G} (K_{ij} - K^{ij}), \tag{35}
\]

the second one is the contribution corresponding to the Chern-Simons gravitational term

\[
T_{\text{CS}}^{ij} = \frac{1}{16\pi G \mu} \left( \epsilon^{ik(\gamma)j} \partial_k K_{kl} + 2 \epsilon^{ik(\gamma)j} K_{kl} \right), \tag{36}
\]

where the gauge \( N^i = 0 \), \( N = 1 \) was chosen, such that \( K_{ij} = -\frac{1}{2} \partial_i \gamma_{ij} \); and in the third place we have the contribution coming from NMG, namely

\[
T_{\text{NMG}}^{ij} = - \frac{1}{8\pi G} \left( \frac{1}{2} f K^{ij} + \nabla_i h^j - \frac{1}{2} D_i f^{ij} + K^{(i} f^{j)k} \right)
- \frac{1}{8} \hat{f} K^{ij} - \frac{1}{2} \hat{f} K^{ij} - \frac{1}{2} \hat{f} K - \frac{1}{2} \hat{f}^2 + \frac{1}{2} \hat{f} f_{ij}, \tag{37}
\]

where the covariant \( r \)-derivatives \( D_r \), defined in \cite{18}, in the gauge \( N^i = 0 \), \( N = 1 \) acts simply as an ordinary derivative, namely \( D_r f_{ij} = \partial_r f_{ij} \); \( D_r f = \partial_r f \); see \cite{18} for details.
C. Counterterms

The next step to define the boundary stress-tensor is proposing the counterterms in the action; namely, the additional boundary terms that ultimately yield a regularized quantities in the boundary. This additional boundary action $S_C$ has to be made of intrinsic boundary scalars like

$$S_C = \int d^2x \sqrt{-\gamma} \left( a_0 + a_1 \, \hat{f} + a_2 \, \hat{f}^2 + \beta_2 \, \hat{f}^{ij} \hat{f}^{ij} + \ldots \right),$$

where the ellipses stand for other local contributions.

As said before, from the boundary point of view, these terms are thought of as counterterms in the dual CFT$_2$; meaning that the renormalized boundary stress tensor is

$$T^{(\text{ren})}_{ij} = T_{ij} + \frac{2}{\sqrt{-\gamma}} \frac{\delta}{\delta \gamma^{ij}} S_C.$$  (39)

It is possible to verify that, for asymptotically AdS$_3$ spaces satisfying Brown-Henneaux boundary conditions, it is sufficient to consider the Graham expansion (41) and uses the leading behavior of the gravitational Chern-Simons term has been already discussed in the literature [16, 21], so that let us focus on the case

As a non-trivial check of the boundary stress-tensor proposed, let us perform a simple calculation, let us compute the central charge $Q[\xi]$ and the angular momentum $Q[\partial_t]$. For instance, considering (17), in the large $r$ limit each of the pieces (35)-(37) contributes with a term proportional to $\beta l$. When bringing all the pieces together, one finds

$$T^{(\text{ren})}_{ij} = \ell^2 T^{(\text{reg})}_{ij} = \frac{c_L}{12\pi} h^{(4)}_{ij} = 0,$$

and the tensor vanishes, as it results proportional to $c_L$. Besides, although each individual contribution to $T^{(\text{ren})}_{ij}$ is generically non-zero, the contractions $\xi^{i} w^{j} T^{(\text{ren})}_{ij}$ of each of the three pieces vanishes independently. This yields vanishing charges

$$L_0 + T_0 \sim Q[\partial_t] = 0, \quad L_0 - T_0 \sim Q[\partial_\phi] = 0.$$  (43)

This result agrees with the result for the central charge obtained by different methods: In [11] the central charge was obtained as the central extension of the algebra of asymptotic charges; see [22, 23] for the calculation in a quite general three-dimensional theory. In [13], on the other hand, the central charge $c$ was obtained by looking at the Schwarzian derivative term in the anomalous transformation of $T^{(\text{reg})}_{ij}$ under $r$-dependent diffeomorphisms. The fact these computations match supports the interpretation of the tensor (39) as the stress-tensor of a CFT$_2$ in the boundary of the space.

D. Conserved charges

The expression for the boundary stress-tensor (39) can now be used to compute conserved charges associated to asymptotic isometries. One is mainly concerned with the conserved charges associated to asymptotic Killing vectors $\partial_t$ and $\partial_\phi$, which correspond to the mass and the angular momentum, respectively. To define the charges it is convenient to make use of the ADM formalism adapted to the boundary. Then, the charges are defined by

$$Q[\xi] = \int ds \, \xi^i \, w^j T^{(\text{ren})}_{ij},$$

where $ds$ is the volume element (i.e. the line element) of the constant-$t$ surfaces at the boundary, $u$ is a unit vector orthogonal to the constant-$t$ surfaces, and $\xi$ is the asymptotic Killing vector.

Here we concentrate in (20). Applied to such spaces, calculation (43) yields vanishing conserved charges both for the mass $Q[\partial_t]$ and the angular momentum $Q[\partial_\phi]$. For instance, considering (17), in the large $r$ limit each of the pieces (35)-(37) contributes with a term proportional to $\beta l$. When bringing all the pieces together, one finds

$$T^{(\text{ren})}_{ij} = \ell^2 T^{(\text{reg})}_{ij} = \frac{c_L}{12\pi} h^{(4)}_{ij} = 0,$$

and the tensor vanishes, as it results proportional to $c_L$. Besides, although each individual contribution to $T^{(\text{ren})}_{ij}$ is generically non-zero, the contractions $\xi^{i} w^{j} T^{(\text{ren})}_{ij}$ of each of the three pieces vanishes independently. This yields vanishing charges

$$L_0 + T_0 \sim Q[\partial_t] = 0, \quad L_0 - T_0 \sim Q[\partial_\phi] = 0.$$  (43)

This generalized the observation made in [8] about the fact that $h_{++}(t, r, \phi)$ does not appear in the boundary stress-tensor (35)-(37), cf. [10].

V. CONCLUSIONS

The question remains as to whether well-behaved non-locally AdS$_3$ solutions to chiral gravity with finite (non-vanishing) conserved charges exist. The non-Einstein solutions at the chiral point that obey Brown-Henneaux
boundary conditions found so far happen not to contribute in a substantial way to the partition function, as the conserved charges associated to them are zero. One can imagine a plausible scenario in which non-Einstein spaces do not actually count: For example, even in the case one finally finds such a non-locally AdS$_3$ solution with non-vanishing charges, one may still ask whether solutions of such type but free of closed timelike curves and free of naked singularities in the global space do exist. If non-Einstein spaces free of pathologies do not exist, and if the Euclidean path integral formulation summing only over smooth geometries results to be the adequate way of defining the quantum theory, then the partition function would still be calculable by following the approaches considered in the literature.

Still far from being able to answer these questions completely, in this paper we have made two remarks on non-locally AdS$_3$ solutions that asymptote AdS$_3$ in massive gravity. These remarks should be taken as a cautionary note when thinking of extending the chiral gravity theory of [2] by adding the NMG terms to it. First, we have shown that non-Einstein solutions found in [8] persist when the higher-curvature terms are added to the gravity action. Such solutions exhibit constant curvature-invariants, so that it is not totally surprising that they result resilient under higher-curvature deformation of the action. Furthermore, we have shown that the addition of the higher-curvature terms also entails the emergence of new non-Einstein spaces that have no counterpart in TMG, and which asymptote AdS$_3$ for $m^2l^2$ sufficiently large. In particular, the solutions we studied include time-independent, axially symmetric deformations of the extremal BTZ geometry that are not locally equivalent to it. This implies that the Birkhoff-like theorem proven for TMG is circumvented in the full massive gravity theory.

This work was supported by UBA, CONICET and ANPCyT. The authors thank Alan Garbarz and Guillem Pérez-Nadal for discussions and collaboration.

[1] S. Deser, R. Jackiw and S. Templeton, Phys. Rev. Lett. 48 (1982) 975.
[2] W. Li, W. Song and A. Strominger, JHEP 0804 (2008) 082.
[3] D. Anninos, W. Li, M. Padi, W. Song and A. Strominger, JHEP 0903 (2009) 130.
[4] J. Brown and M. Henneaux, Comm. Math. Phys. 104 (1986) 207.
[5] A. Maloney, W. Song and A. Strominger, Phys. Rev. D81 (2010) 064007.
[6] A. Maloney and E. Witten, JHEP 1002 (2010) 029.
[7] E. Bergshoeff, O. Hohm and P. Townsend, Phys. Rev. Lett. 102 (2009) 201301; Phys. Rev. D79 (2009) 124042.
[8] S. de Buyl, G. Compère and S. Detournay, JHEP 1010 (2010) 042.
[9] M. Henneaux, C. Martínez, R. Troncoso and J. Zanelli, Annals Phys. 322 (2007) 824.
[10] G. Giribet and M. Leston, JHEP 1009 (2010) 070.
[11] Y. Liu and Y-W. Sun, Phys. Rev. D79 (2009) 126001; JHEP 0904 (2009) 106; JHEP 0905 (2009) 039.
[12] A. Garbarz, G. Giribet and Y. Vásquez, Phys. Rev. D79 (2009) 044036. G. Clément, Class. Quant. Grav. 26 (2009) 165002. E. Ayón-Beato, G. Giribet and M. Hassaïne, JHEP 0905 (2009) 029.
[13] D. Grumiller and N. Johansson, JHEP 0807 (2008) 134; Int. J. Mod. Phys. D17 (2009) 2367. M. Henneaux, C. Martínez and R. Troncoso, Phys. Rev. D79 (2009) 081502R; Phys. Rev. D82 (2010) 064038.
[14] M. Bañados, C. Teitelboim and J. Zanelli, Phys. Rev. Lett. 69 (1992) 1849. M. Bañados, M. Henneaux, C. Teitelboim and J. Zanelli, Phys. Rev. D48 (1993) 1506.
[15] M. Henningson and K. Skenderis, JHEP 9807 (1998) 023.
[16] V. Balasubramanian and P. Kraus, Commun. Math. Phys. 208 (1999) 413.
[17] J. Brown and J. York, Phys. Rev. D47 (1993) 1407.
[18] O. Hohm and E. Tonni, JHEP 1004 (2010) 093.
[19] Y. Kwon, S. Nam, J-D. Park and S-H. Yi, arXiv:1106.4609.
[20] A. Pérez, D. Tempo and R. Troncoso, JHEP 1107 (2011) 093.
[21] P. Kraus and F. Larsen, JHEP 0601 (2006) 022.
[22] P. Kraus, Lect. Notes Phys. 755 (2008) 193.
[23] H. Saida and J. Soda, Phys. Lett. B471 (2000) 358.