SINGULAR EQUIVALENCES OF FUNCTOR CATEGORIES VIA
AUSLANDER-BUCHWEITZ APPROXIMATIONS

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Abstract. The aim of this paper is to construct singular equivalences between functor categories. As a special case, we show that there exists a singular equivalence arising from a cotilting module $T$, namely, the singularity category of $(\perp T)/[T]$ and that of $(\mod A)/[T]$ are triangle equivalent. In particular, the canonical module $\omega$ over a commutative Noetherian ring induces a singular equivalence between $(\CM R)[\omega]$ and $(\mod R)/[\omega]$, which generalizes Matsui-Takahashi’s theorem. Our result is based on a sufficient condition for an additive category $A$ and its subcategory $\mathcal{X}$ so that the canonical inclusion $\mathcal{X} \hookrightarrow A$ induces a singular equivalence $D_{sg}(A) \simeq D_{sg}(\mathcal{X})$, which is a functor category version of Xiao-Wu Chen’s theorem.

1. Introduction

Let $A$ be an additive category with weak-kernels. Then the functor category $\mod A$, the category of finitely presented contravariant functors from $A$ to the category of abelian groups, is abelian. The notion of singularity category of $A$ is defined to be the Verdier quotient $D_{sg}(A) := \frac{D^b(\mod A)}{K^b(\proj A)}$, where we denote by $D^b(\mod A)$ the bounded derived category, and by $K^b(\proj A)$ the homotopy category of bounded complexes whose terms are projective. This concept was introduced as a homological invariant of rings by Buchweitz [Buc86]. Recently it was applied by Orlov to study Landau-Ginzburg models [Orl04]. A lot of studies on singularity categories has been done in various approaches (e.g. [Iya18, KV, Orl09, Ric, Zim]).

For additive categories $A$ and $A'$ with weak-kernels, we say that $A$ is singularly equivalent to $A'$ if there exists a triangle equivalence $D_{sg}(A) \simeq D_{sg}(A')$ [ZZ]. If $A$ is an Iwanaga-Gorenstein ring, then the singularity category of $A$ is triangle equivalent to the stable category of Cohen-Macaulay $A$-modules. Thus the singular equivalence is a generalization of the stable equivalence for Iwanaga-Gorenstein rings.

It is a basic problem to compare homological properties of a ring $A$ with its subalgebra $eAe$ given by an idempotent $e \in A$ (e.g. [APT, CPS, DR]). In this context, Xiao-Wu Chen investigated a sufficient condition for a ring $A$ and its idempotent subalgebra $eAe$ so that they induce a triangle equivalence $D_{sg}(A) \xrightarrow{\sim} D_{sg}(eAe)$ [Che, Thm. 1.3]. The first aim of this article is to provide its functor category version by using the following observations on Serre and Verdier quotients: Let $\mathcal{X}$ be a contravariantly finite subcategory of an additive category $A$ with weak-kernels. Then $\mathcal{X}$ also admits weak-kernels, hence the canonical functor $Q : \mod A \to \mod \mathcal{X}$

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induces an equivalence

\[ \frac{\text{mod } A}{\text{mod}(A/\mathcal{X})} \cong \text{mod } \mathcal{X}, \]

where the fraction denotes the Serre quotient (e.g. \cite[Prop. 3.9]{Buc97}). Moreover, the equivalence \ref{1.0.1} induces a triangle equivalence

\[ \frac{D^b(\text{mod } A)}{D^b_{A/\mathcal{X}}(\text{mod } A)} \cong D^b(\text{mod } \mathcal{X}), \]

where \(D^b_{A/\mathcal{X}}(\text{mod } A)\) is a thick subcategory consisting of objects whose cohomologies belong to \(\text{mod}(A/\mathcal{X})\) (see \cite[Thm. 3.2]{Miy} and \cite[Thm. 2.3]{CPS}). The equivalence \ref{1.0.2} gives the following first result of this paper.

**Theorem A** (Lemma 2.1, Theorem 2.2). Let \(A\) be an additive category with weak-kernels and \(\mathcal{X}\) its contravariantly finite full subcategory. Suppose that \(\text{pd}_A(\mathcal{X}|M) < \infty\) for any \(M \in A\) and \(\text{pd}_A(F) < \infty\) for any \(F \in \text{mod}(A/\mathcal{X})\). Then the canonical inclusion \(\mathcal{X} \hookrightarrow A\) induces a triangle equivalence \(\bar{Q} : D_{sg}(A) \rightarrow D_{sg}(\mathcal{X})\).

Our second result is an application of Theorem A, which provides examples of singularly equivalent categories. We denote by \(\hat{\mathcal{X}}\) the full subcategory of \(C\) consisting of objects \(M\) which admit an exact sequence

\[ 0 \rightarrow X_n \rightarrow X_{n-1} \rightarrow \cdots \rightarrow X_0 \rightarrow M \rightarrow 0 \]

with \(X_n, \ldots, X_0 \in \mathcal{X}\) for some \(n \in \mathbb{Z}_{\geq 0}\). Our result will be stated under the following condition which is a generalization of the setting appearing in Auslander-Buchweitz theory (see Condition \ref{1.1} for details). A map \(f : N \rightarrow M\) in \(C\) is called an \(\mathcal{X}\)-epimorphism if the induced map \(C(-, N)|_{\mathcal{X}} \xrightarrow{f_{-\cdot}} C(-, M)|_{\mathcal{X}}\) is surjective.

**Condition 1.1.** Let \(C\) be an abelian category with enough projectives and let \(A \supseteq \mathcal{X} \supseteq \omega\) be a sequence of full subcategories in \(C\) such that \(\mathcal{X}\) and \(\omega\) are contravariantly finite in \(A\). We consider the following conditions:

(A1) If a morphism \(f : N \rightarrow M\) in \(A\) is an \(\omega\)-epimorphism, then the kernel of \(f\) belongs to \(A\).

(A2) \(\text{Ext}^i_C(X, I) = 0\) for any \(X \in \mathcal{X}, I \in \omega\) and \(i > 0\).

(A3) For any \(M \in A\), there exists an exact sequence \(0 \rightarrow Y_M \rightarrow X_M \xrightarrow{f} M\) in \(A\) such that \(f\) is a right \(\mathcal{X}\)-approximation of \(M\) and \(Y_M \in \bar{\omega}\).

For example, the classical Auslander-Buchweitz theory (Condition \ref{1.3}) provides us with the triple \((C = A, \mathcal{X}, \omega)\) satisfies the Condition \ref{1.1}. Note that, in contrary to Condition \ref{1.3}, they are not required that: \(\omega\) is a cogenerator of \(\mathcal{X}\); each morphism \(f\) appearing in \(0 \rightarrow Y_M \rightarrow X_M \xrightarrow{f} M\) of (AB3) is surjective.

Since \(\mathcal{X}/[\omega]\) can be regarded as an analog of the costable category, we denote by \(\overline{\mathcal{X}} := A/[\omega]\) and \(\overline{\mathcal{X}} := \mathcal{X}/[\omega]\).

Our main result is the following:
**Theorem B** (Theorem 3.1). Under Condition 1.1, the canonical inclusion $\mathcal{X} \hookrightarrow \mathcal{A}$ induces a triangle equivalence $D_{sg}(\mathcal{A}) \sim D_{sg}(\mathcal{X})$.

Typical examples satisfying Condition 1.1 come from cotilting theory. Let us recall the notion of cotilting subcategories of $\mathcal{C}$. For a subcategory $\mathcal{X}$ of $\mathcal{C}$, we denote by $\bot \mathcal{X}$ the full subcategory of objects $M$ with $\text{Ext}_C^i(M, X) = 0$ for any $i > 0$ and $X \in \mathcal{X}$.

**Definition 1.2.** Let $\mathcal{C}$ be an abelian category with enough projectives. A full subcategory $\mathcal{T}$ of $\mathcal{C}$ is called a cotilting subcategory of $\mathcal{C}$, if it satisfies the following conditions:

- There exists an integer $n \in \mathbb{Z}_{\geq 0}$ such that $\text{id} I \leq n$ for any $I \in \mathcal{T}$;
- $\text{Ext}_C^i(I, J) = 0$ for any $I, J \in \mathcal{T}$ and $i > 0$;
- For each $M \in \bot \mathcal{T}$, there exists an exact sequence
  $$0 \to M \to I \to M' \to 0$$
  with $I \in \mathcal{T}$ and $M' \in \bot \mathcal{T}$.

We call an object $T \in \mathcal{C}$ a cotilting object if $\text{add} T$ is a cotilting subcategory of $\mathcal{C}$.

The following result is immediate from Theorem B.

**Corollary C** (Corollary 3.9). Let $\mathcal{A}$ be an abelian category with enough projectives and $\mathcal{T}$ its contravariantly finite cotilting subcategory. Then the canonical inclusion $\bot \mathcal{T} \hookrightarrow \mathcal{A}$ induces a triangle equivalence $D_{sg}(\mathcal{A}) \sim D_{sg}(\bot \mathcal{T})$.

As examples of Corollary C we have the followings:

**Example 1.3.**

(a) Let $\Lambda$ be a finite dimensional $k$-algebra over a field $k$ and $T$ a cotilting $\Lambda$-module. Then the canonical inclusion $\bot T \hookrightarrow \text{mod} \Lambda$ induces a triangle equivalence $D_{sg}(\text{mod} \Lambda) \sim D_{sg}(\bot T)$.

(b) Let $R$ be a commutative Cohen-Macaulay ring with a canonical $R$-module $\omega$ and $\text{CMR}$ the full subcategory of maximal Cohen-Macaulay $R$-modules. Then the canonical inclusion $\text{CMR} \hookrightarrow \text{mod} R$ induces a triangle equivalence $D_{sg}(\text{mod} R) \sim D_{sg}(\text{CMR})$.

Theorem B also provides an alternative proof for Matsui-Takahashi’s theorem [MT, Thm. 5.4(3)] (Corollary 3.11): For an Iwanaga-Gorenstein ring $\Lambda$, the canonical inclusion $\text{CMA} \hookrightarrow \text{mod} \Lambda$ induces a triangle equivalence $D_{sg}(\text{mod} \Lambda) \sim D_{sg}(\text{CMA})$.

**Notation and convention.** Throughout the paper all categories and functors are assumed to be additive. The set of morphisms $M \to N$ in a category $\mathcal{A}$ is denoted by $\mathcal{A}(M, N)$. Morphisms are composed from right-to-left. Let $\mathcal{X}$ be a subcategory of $\mathcal{A}$. We denote by $\mathcal{A}/[\mathcal{X}]$ the ideal quotient category of $\mathcal{A}$ modulo the ideal $[\mathcal{X}]$ of $\mathcal{A}$ consisting of all morphisms which factor through an object in $\mathcal{X}$. For each $M \in \mathcal{A}$, we denote by $\text{add} M$ the full subcategory consisting of direct summands of a finite direct sum of $M$ and we abbreviate $\mathcal{A}/[M]$ to indicate $\mathcal{A}/[\text{add} M]$.

The word ring and algebra always mean ring with a unit and finite dimensional algebra over a field $k$, respectively. Let $\Lambda$ be a ring. The symbol $\text{mod} \Lambda$ denotes the category of finitely presented right $\Lambda$-modules. We denote by $\text{Hom}_\Lambda(M, N)$ the morphism-set from $M$ to $N$ instead of $(\text{mod} \Lambda)(M, N)$. The full subcategory of projective (resp. injective) modules in $\text{mod} \Lambda$ will
be denoted by \( \text{proj } A \) (resp. \( \text{inj } A \)). The projective (resp. injective) dimension of right \( A \)-module \( M \) will be denoted by \( \text{pd}_A(M) \) (resp. \( \text{id}_A(M) \)).

2. A Functor Category Version of Chen’s Theorem

The aim of this section to provide a sufficient condition for an additive category \( A \) and its subcategory \( \mathcal{X} \) so that the canonical inclusion \( \mathcal{X} \hookrightarrow A \) induces a triangle equivalence \( D_{\text{sg}}(A) \cong D_{\text{sg}}(\mathcal{X}) \), which generalizes Xiao-Wu Chen’s theorem.

The category \( \text{mod } A \) is not necessarily abelian, however, if every morphism in \( A \) has weak-kernels, then \( \text{mod } A \) is abelian ([Pre, Thm. 1.4]). Since we are interested in the case that \( \text{mod } A \) is abelian, throughout this section, let \( A \) be an additive category with weak-kernels and \( \mathcal{X} \) its contravariantly finite full subcategory. Then, the canonical functor \( Q : \text{mod } A \to \text{mod } \mathcal{X} \) induces an equivalence

\[
\text{mod } A \cong \text{mod } \mathcal{X}.
\]

Moreover, by [Miy, Thm. 3.2], it induces a triangle equivalence

\[
\text{D}^b(\text{mod } A) \cong \text{D}^b(\text{mod } \mathcal{X}).
\]

Then we have the following commutative diagram

\[
\begin{array}{ccc}
\text{mod } A & \xrightarrow{Q} & \text{mod } \mathcal{X} \\
\text{D}^b(\text{mod } A) & \xrightarrow{Q'} & \text{D}^b(\text{mod } \mathcal{X})
\end{array}
\]

where the arrows of the shape \( \hookrightarrow \) denote canonical inclusions, and \( Q' \) is the functor induced from \( Q \). Note that \( \text{D}^b_{\mathcal{X}}(\text{mod } A) \) is the thick subcategory of \( \text{D}^b(\text{mod } A) \) containing \( \text{mod } (A/\mathcal{X}) \). The following lemma gives a natural sufficient condition so that the canonical functor \( \text{D}^b(\text{mod } A) \to \text{D}^b(\text{mod } \mathcal{X}) \) induces a triangle functor \( D_{\text{sg}}(A) \to D_{\text{sg}}(\mathcal{X}) \).

Lemma 2.1. The following conditions are equivalent:

(i) \( \text{pd}_\mathcal{X}(A(-, M)|\mathcal{X}) < \infty \) for any \( M \in A \);

(ii) The canonical functor \( Q' : \text{D}^b(\text{mod } A) \to \text{D}^b(\text{mod } \mathcal{X}) \) restricts to \( Q' : \text{K}^b(\text{proj } A) \to \text{K}^b(\text{proj } \mathcal{X}) \).

If this is the case, we have an induced triangle functor \( Q : D_{\text{sg}}(A) \to D_{\text{sg}}(\mathcal{X}) \).

Proof. (i) \( \Leftrightarrow \) (ii): Since the functor \( Q' : \text{D}^b(\text{mod } A) \to \text{D}^b(\text{mod } \mathcal{X}) \) restricts to \( Q'|_{\text{mod } A} = Q : \text{mod } A \to \text{mod } \mathcal{X} \), the condition (i) holds if and only if \( Q'(\text{proj } A) \subseteq \text{K}^b(\text{proj } \mathcal{X}) \) if and only if the condition (ii) holds.

The latter statement follows from the universality of the Verdier quotient. \( \square \)

Since our aim is to compare the singularity categories \( D_{\text{sg}}(A) \) and \( D_{\text{sg}}(\mathcal{X}) \), it is natural to assume that the equivalent conditions in Lemma 2.1 are satisfied. Our main result gives a
necessary and sufficient condition so that the canonical inclusion \( X \hookrightarrow A \) induces a triangle equivalence \( D_{sg}(A) \xrightarrow{\sim} D_{sg}(X) \).

**Theorem 2.2.** We assume that \( pd_X(A(\cdot, M)|X) < \infty \) for any \( M \in A \). Then the following conditions are equivalent:

(i) \( pd_A(F) < \infty \) for any \( F \in \text{mod}(A/[X]) \);

(ii) The induced functor \( \bar{Q} : D_{sg}(A) \to D_{sg}(X) \) is a triangle equivalence.

To prove Theorem 2.2, we firstly show Proposition 2.3 in a more general framework: Let \( \mathcal{T} \) be a triangulated category with a translation \([1]\). For a class \( S \) of objects in \( \mathcal{T} \), we denote by \( \text{tri} S \) the smallest triangulated full subcategory of \( \mathcal{T} \) containing \( S \). For two classes \( U \) and \( V \) of objects in \( \mathcal{T} \), we denote by \( U \ast V \) the class of objects \( X \) occurring in a triangle \( U \to X \to V \to U[1] \) with \( U \in U \) and \( V \in V \). Note that the operation \( \ast \) is associative by the octahedral axiom.

**Proposition 2.3.** Let \( U \) and \( V \) be triangulated full subcategories of \( \mathcal{T} \) and consider the Verdier quotients with respect to them:

\[
U \to \mathcal{T} \xrightarrow{Q_1} \mathcal{T}/U \quad \text{and} \quad V \to \mathcal{T} \xrightarrow{Q_2} \mathcal{T}/V.
\]

Then, there exist natural triangle equivalences

\[
\frac{\mathcal{T}/U}{\text{tri}(Q_1 V)} \xrightarrow{\sim} \frac{\mathcal{T}}{\text{tri}(U, V)} \xrightarrow{\sim} \frac{\mathcal{T}/V}{\text{tri}(Q_2 U)},
\]

where \( Q_1 V \) is the full subcategory of \( \mathcal{T}/U \) consisting of objects isomorphic to \( Q_1 V \) for some \( V \in V \), and the symbol \( Q_2 U \) is used in a similar meaning.

**Proof.** We shall show an equality \( \text{tri}(Q_1 V) = \text{tri}(U, V)/U \), where \( \text{tri}(U, V) \) denotes the smallest triangulated full subcategory of \( \mathcal{T} \) containing \( U \) and \( V \). We set \( S := U \cup V \). Obviously we have \( Q_1 S = Q_1 V \). Since \( \text{tri}(U, V) = \bigcup_{n \geq 0} S^{*n} \), we have the following equalities:

\[
\text{tri}(U, V)/U = Q_1 \left( \bigcup_{n \geq 0} S^{*n} \right) = \bigcup_{n \geq 0} (Q_1 S)^{*n} = \bigcup_{n \geq 0} (Q_1 V)^{*n} = \text{tri}(Q_1 V).
\]

Hence we have a desired triangle equivalence \( \frac{\mathcal{T}/U}{\text{tri}(Q_1 V)} = \frac{\mathcal{T}/U}{\text{tri}(U, V)/U} \sim \frac{\mathcal{T}}{\text{tri}(U, V)} \).

Now we are ready to prove Theorem 2.2

**Proof of Theorem 2.2** Apply Proposition 2.3 for \( \mathcal{T} = D^b(\text{mod} A), U = D^b_{\text{mod}(A/[X])}(\text{mod} A) \) and \( V = K^b(\text{proj} A) \). Then \( \mathcal{T}/U = D^b(\text{mod} X) \) and \( \mathcal{T}/V = D_{sg}(A) \). The assumption gives \( Q_1 V = K^b(\text{proj} X) \). Hence \( \frac{\mathcal{T}/U}{Q_1 V} = D_{sg}(X) \). Thus we have a triangle equivalence \( D_{sg}(X) \sim \frac{D_{sg}(A)}{\text{tri}(Q_2 U)} \).

This shows the condition (i) is equivalent to \( Q_2 U = 0 \), namely \( U \subset V \), which is nothing but the condition (ii). \( \Box \)

We end this section with recovering the following Chen’s theorem as a special case of Theorem 2.2 and Lemma 2.1.
Example 2.4. [Che] Thm. 1.3] (see also [PSS, Thm. 5.2], [KY, Prop. 3.3]) Let \( \Lambda \) be a Noetherian ring and \( e \) its idempotent. Assume that \( \text{pd}_{\Lambda e}(\Lambda e) < \infty \). Then the canonical inclusion \( e \Lambda e \hookrightarrow \Lambda \) induces a triangle functor \( \bar{Q} : D_{\text{sg}}(\Lambda) \to D_{\text{sg}}(e\Lambda e) \), and the following are equivalent:

(i) \( \text{pd}_{\Lambda}(M) < \infty \) for any \( M \in \text{mod}(\Lambda/\Lambda e) \);

(ii) The induced functor \( \bar{Q} : D_{\text{sg}}(\Lambda) \xrightarrow{\sim} D_{\text{sg}}(e\Lambda e) \) is a triangle equivalence.

3. Sufficient conditions for singular equivalence

The aim of this section is to construct a singular equivalence from our generalized Auslander-Buchweitz condition (Condition 1.1). First we introduce some terminology. Let \( C \) denote an abelian category and let \( C \supseteq A \supseteq B \) be a sequence of full subcategories of \( C \). We call the kernel of a \( B \)-epimorphism the \( B \)-epikernel, for short. We assume that \( A \) is closed under \( B \)-epikernels and \( B \) is contravariantly finite in \( A \). Then the ideal-quotient category \( A/\lbrack B \rbrack \) admits weak-kernels. In fact, for a morphism \( \alpha : M \to L \) of \( A \), we obtain its weak-kernel as follows: We take a right \( B \)-approximation \( \beta : B \to L \) of \( L \), and consider an induced exact sequence

\[
0 \to N \xrightarrow{(\gamma, \delta)} M \oplus B_L \xrightarrow{(\alpha, \beta)} L
\]

in \( C \). Since \( A \) is closed under \( B \)-epikernels and the morphism \( (\alpha, \beta) \) is an \( B \)-epimorphism, we have \( N \in A \). It is basic that the morphism \( \gamma \) is a weak-kernel of \( \alpha \) in \( A/\lbrack B \rbrack \).

3.1. Singular equivalences from Auslander-Buchweitz approximation. In this subsection, we give a proof of the following main theorem.

**Theorem 3.1.** Under Condition 1.1, the canonical inclusion \( \overline{X} \hookrightarrow \overline{A} \) induces a triangle equivalence \( D_{\text{sg}}(\overline{A}) \xrightarrow{\sim} D_{\text{sg}}(\overline{X}) \).

Let \( C \) be an abelian category with enough projectives and consider a sequence \( A \supseteq X \supseteq \omega \) of full subcategories in \( C \) such that \( X \) and \( \omega \) are contravariantly finite in \( A \). We always assume (AB1) in Condition 1.1.

**Proposition 3.2.** The ideal-quotient \( \overline{A} \) admits weak-kernels and \( \overline{X} \) is its contravariantly finite full subcategory. Moreover, the canonical inclusion \( \overline{X} \hookrightarrow \overline{A} \) induces the following equivalence

\[
\frac{\text{mod} \overline{A}}{\text{mod}(\overline{A}/\lbrack \overline{X} \rbrack)} \xrightarrow{\sim} \text{mod} \overline{X}.
\]

**Proof.** Since \( A \) is closed under \( \omega \)-epikernels, \( \overline{A} \) admits weak-kernels. Since \( X \) is contravariantly finite in \( A \), so is \( \overline{X} \) in \( \overline{A} \). Note that there exists an equivalence \( A/\lbrack X \rbrack \xrightarrow{\sim} \overline{A}/\lbrack \overline{X} \rbrack \). By (2.0.1), we have a desired equivalence. \( \square \)

To prove that the inclusion \( \overline{X} \hookrightarrow \overline{A} \) induces a triangle functor \( D_{\text{sg}}(\overline{A}) \to D_{\text{sg}}(\overline{X}) \), we shall check a sufficient condition given in Lemma 2.1.

**Lemma 3.3.** Assume (AB2) and (AB3). Let \( X \in \mathcal{X} \) be given. Then,

(a) One has \( \text{Ext}^i_C(X, I) = 0 \) for any \( I \in \tilde{\omega} \) and \( i > 0 \).
(b) Every morphism \( f : X \to I \) with \( I \in \hat{\omega} \) factors through an object in \( \omega \).

Proof. We only show the assertion (b). Since \( I \in \hat{\omega} \), there exists an exact sequence \( 0 \to I' \to W \to I \to 0 \) with \( W \in \omega \) and \( I' \in \hat{\omega} \). Applying \( C(X, -) \), by (a), we conclude that \( f \) factors through \( W \). \( \square \)

Proposition 3.4. Assume (AB2) and (AB3). Then the canonical inclusion \( \text{inc} : \mathcal{X} \hookrightarrow \mathcal{A} \) admits a right adjoint \( R \). Moreover, we have \( \text{pd}_A(\mathcal{X}(-, M)|_{\mathcal{X}}) = 0 \) for any \( M \in \mathcal{A} \).

Proof. The proof is similar to one given in [BR, Ch. V Prop. 1.2], but our situation is slightly different from that in loc. cit. So we include a detailed proof. By (AB3), for each \( M \in \mathcal{A} \), there exists an exact sequence \( 0 \to Y_M \to X_M \xrightarrow{\alpha} M \) with \( \alpha \) a right \( \mathcal{X} \)-approximation of \( M \) and \( Y_M \in \hat{\omega} \). We shall show that the morphism \( X \to \mathcal{X}(X, X_M) \xrightarrow{\alpha} \mathcal{X}(X, M) \) is a functorial isomorphism in \( X \in \mathcal{X} \). Its surjectivity is clear, since \( \alpha \) is a right \( \mathcal{X} \)-approximation. To show its injectivity, take a morphism \( h \in \mathcal{X}(X, X_M) \) such that \( \alpha \circ h \) factors through an object \( I \) of \( \omega \). Thus we have the following commutative diagram:

\[
\begin{array}{ccc}
X & \xrightarrow{h'} & I \\
\downarrow h & & \downarrow h'' \\
0 & \to & Y_M & \to & X_M & \xrightarrow{\alpha} & M
\end{array}
\]

Since \( \alpha \) is a right \( \mathcal{X} \)-approximation, there exists a morphism \( \alpha' : I \to X_M \) such that \( \alpha \alpha' = h'' \). The morphism \( h - \alpha' h' \) factors through \( Y_M \in \hat{\omega} \). By Lemma 3.3(ii), this implies that \( h - \alpha' h' \) factors through \( \omega \). Hence \( h \) factors through \( \omega \). By the Yoneda lemma, the assignment \( M \mapsto X_M \) gives rise to a functor \( R : \mathcal{A} \to \mathcal{X} \). The bifunctorial isomorphism \( \mathcal{X}(X, R(M)) \xrightarrow{\alpha} \mathcal{X}(X, M) \) says the pair of functors \( (\text{inc}, R) \) forms an adjoint pair.

The latter statement is obvious. \( \square \)

Proposition 3.5. Let \( \mathcal{B} \) be a contravariantly finite full subcategory of \( \mathcal{A} \) and assume that \( \mathcal{A} \) is closed under \( \mathcal{B} \)-epikernels. Let \( F \in \text{mod}(\mathcal{A}/[\mathcal{B}]) \) be given. Then there exists an exact sequence

\[
0 \to N \xrightarrow{g} M \xrightarrow{f} L
\]

in \( \mathcal{A} \) which satisfies the following conditions:

(a) The morphism \( f \) is a \( \mathcal{B} \)-epimorphism;

(b) The induces sequence

\[
0 \to \mathcal{A}(-, N) \xrightarrow{g} \mathcal{A}(-, M) \xrightarrow{f} \mathcal{A}(-, L) \to F \to 0
\]

is exact.

In particular, \( \text{pd}_A(F) \leq 2 \).

Proof. First \( F \) is a finitely presented \( \mathcal{A} \)-module. Indeed, a right \( \mathcal{B} \)-approximation \( B_Y \to Y \) of any \( Y \in \mathcal{A} \) induces a projective presentation

\[
\mathcal{A}(-, B_Y) \to \mathcal{A}(-, Y) \to \mathcal{A}/[\mathcal{B}](-, Y) \to 0
\]
of the \( \mathcal{A} \)-module \( \mathcal{A}/\mathcal{B}(-, Y) \). This shows that \( \mathcal{A}/\mathcal{B}(-, Y) \) belongs to \( \text{mod} \, \mathcal{A} \), hence so does \( F \).

Thus we have a projective presentation \( \mathcal{A}(-, M) \xrightarrow{f} \mathcal{A}(-, L) \to F \to 0 \) of the \( \mathcal{A} \)-module \( F \). Since \( F \) vanishes on \( \mathcal{B} \), the induced morphism \( f \) is a \( \mathcal{B} \)-epimorphism. Thus we have an exact sequence \( 0 \to N \xrightarrow{g} M \xrightarrow{f} L \) in \( \mathcal{A} \). Applying the Yoneda embedding, we have a projective resolution \( 0 \to \mathcal{A}(-, N) \to \mathcal{A}(-, M) \to \mathcal{A}(-, L) \to F \to 0 \) of the \( \mathcal{A} \)-module \( F \).

\[ \square \]

Let \( M \in \mathcal{A} \) and \( f : B_M \to M \) be a right \( \mathcal{B} \)-approximation of \( M \). Then we write \( \Omega_{\mathcal{B}}(M) := \ker f \). We define \( \Omega_{\mathcal{B}}(M) \) inductively for \( n \geq 1 \). We prove the following key-proposition which generalizes the well-known result given in [AR74] Prop. 4.1, 4.2 and [AR96] Prop. 1.2. The proof is similar but a bit different from the original ones.

**Proposition 3.6.** For \( F \in \text{mod}(\mathcal{A}/\mathcal{B}) \), the exact sequence (3.5.1) in Proposition 3.5 induces a projective resolution

\[
\cdots \to \mathcal{A}/\mathcal{B}(-, \Omega_{\mathcal{B}}^2(N)) \xrightarrow{\Omega_{\mathcal{B}}^g} \mathcal{A}/\mathcal{B}(-, \Omega_{\mathcal{B}}^2(M)) \xrightarrow{\Omega_{\mathcal{B}}^f} \mathcal{A}/\mathcal{B}(-, \Omega_{\mathcal{B}}^2(L)) \to \mathcal{A}/\mathcal{B}(-, \Omega_{\mathcal{B}}(N)) \xrightarrow{\Omega_{\mathcal{B}}g} \mathcal{A}/\mathcal{B}(-, \Omega_{\mathcal{B}}(M)) \xrightarrow{\Omega_{\mathcal{B}}f} \mathcal{A}/\mathcal{B}(-, \Omega_{\mathcal{B}}(L)) \to \mathcal{A}/\mathcal{B}(-, N) \xrightarrow{g} \mathcal{A}/\mathcal{B}(-, M) \xrightarrow{f} \mathcal{A}/\mathcal{B}(-, L) \to F \to 0
\]

of the \( \mathcal{A}/\mathcal{B} \)-module \( F \).

**Proof.** For the sequence (3.5.1), we take right \( \mathcal{B} \)-approximations \( \alpha_L : B_L \to L \) and \( \alpha_N : B_N \to N \). Since the morphism \( f \) is \( \mathcal{B} \)-epimorphism, we have a morphism \( \beta : B_L \to M \) such that \( \alpha_L = f \circ \beta \). The induced morphism \( \alpha_M := \begin{pmatrix} \beta \\ g \circ \alpha_N \end{pmatrix} : B_M := B_L \oplus B_N \to M \) is a right \( \mathcal{B} \)-approximation of \( M \). Since \( \mathcal{A} \) is closed under \( \mathcal{B} \)-epikernels, we have the following commutative diagram in \( \mathcal{A} \):

\[
\begin{array}{ccccccc}
0 & \to & \Omega_{\mathcal{B}}(N) & \xrightarrow{\Omega_{\mathcal{B}}g} & \Omega_{\mathcal{B}}(M) & \xrightarrow{\Omega_{\mathcal{B}}f} & \Omega_{\mathcal{B}}(L) \\
0 & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & \to & B_N & \to & B_M & \to & B_L \\
0 & \downarrow & \text{g} & \downarrow \alpha_N & \text{f} & \downarrow \alpha_M & \downarrow \alpha_L \\
0 & \to & N & \to & M & \to & L \\
\end{array}
\]
where all columns and rows are exact, and the middle row splits. Applying the Yoneda embedding and the Snake Lemma, we have the following commutative diagram in $\text{mod} \mathcal{A}$.

\[
\begin{array}{ccc}
0 & \to & \mathcal{A}(-, \Omega_B(N)) \\
\downarrow & & \downarrow \\
0 & \to & \mathcal{A}(-, \Omega_B(M)) \\
\downarrow & & \downarrow \\
0 & \to & \mathcal{A}(-, \Omega_B(L)) \\
\end{array}
\]

\[
\begin{array}{ccc}
0 & \to & \mathcal{A}(-, B_N) \\
\downarrow \alpha_N & & \downarrow \alpha_B \\
0 & \to & \mathcal{A}(-, B_M) \\
\downarrow \alpha_M & & \downarrow \alpha_L \\
0 & \to & \mathcal{A}(-, N) \\
\downarrow g & & \downarrow f \\
\mathcal{A}/B(-, N) & \to & \mathcal{A}/B(-, M) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}
\]

In particular, we have an exact sequence

\[
0 \to \mathcal{A}(-, \Omega_B(N)) \to \mathcal{A}(-, \Omega_B(M)) \to \mathcal{A}(-, \Omega_B(L)) \to 0
\]

in $\text{mod} \mathcal{A}$. We have an exact sequence $0 \to \Omega_B(N) \to \Omega_B(M) \xrightarrow{\Omega_Bf} \Omega_B(L)$ such that $\Omega_Bf$ is a $B$-epimorphism. Inductively, we have a desired projective resolution of the $\mathcal{A}/[B]$-module $F$. $\square$

**Lemma 3.7.** Under Condition (i)

(a) For any $L \in \mathcal{A}$, there exists $n \geq 0$ such that $\Omega^n_{\mathcal{X}}(L) \in \mathcal{X}$.

(b) For each $F \in \text{mod}(\mathcal{A}/[\mathcal{X}])$, we have $\text{pd}_{\mathcal{A}/[\mathcal{X}]}(F) < \infty$.

**Proof.** (a) For an object $L \in \mathcal{A}$, due to (AB3), we get an exact sequence

\[
0 \to Y \to X_0 \xrightarrow{f_0} L
\]

such that $f_0$ is a right $\mathcal{X}$-approximation of $L$ and $Y \in \widehat{\omega}$. Since $Y \in \widehat{\omega}$, we get an exact sequence

\[
0 \to I_n \to I_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} I_1 \xrightarrow{f_0} X_0 \xrightarrow{f_0} L
\]

with $I_i \in \omega$ for $1 \leq i \leq n$. By Lemma 3.3, each morphism $f_i : I_i \to \text{Im} f_i$ is a right $\mathcal{X}$-approximation of $\text{Im} f_i$ for each $1 \leq i \leq n$ hence $I_n = \Omega^n_{\mathcal{X}}(L) \in \mathcal{X}$.

(b) We consider the projective resolution (3.6.1) of the $\mathcal{A}/[\mathcal{X}]$-module $F$ given in Proposition 3.6 by setting $B := \mathcal{X}$. Then the assertion follows from (a), since $\mathcal{A}/[\mathcal{X}](-, \Omega^n_{\mathcal{X}}(L)) = 0$. $\square$

**Proposition 3.8.** Under Condition (ii), for each $F \in \text{mod}(\mathcal{A}/[\mathcal{X}])$, one has $\text{pd}_{\mathcal{A}}(F) < \infty$.

**Proof.** Since $\text{pd}_{\mathcal{A}/[\mathcal{X}]}(F) < \infty$ by Lemma 3.4 and the canonical inclusion $\iota : \text{mod}(\mathcal{A}/[\mathcal{X}]) \hookrightarrow \text{mod}(\mathcal{A})$ is exact, it is enough to check the case of $F = \mathcal{A}/[\mathcal{X}](-, M)$ for some $M \in \mathcal{A}$. By (AB3), there exists an exact sequence $0 \to Y_M \xrightarrow{g} X_M \xrightarrow{f} M$ in $\mathcal{A}$ with $f$ a right $\mathcal{X}$-approximation of $M$ and $Y_M \in \widehat{\omega}$. Applying the Yoneda embedding yields a projective resolution

\[
0 \to \mathcal{A}(-, Y_M) \xrightarrow{g} \mathcal{A}(-, X_M) \xrightarrow{f} \mathcal{A}(-, M) \to \mathcal{A}/[\mathcal{X}](-, M) \to 0
\]
of the \( A \)-module \( A/[\mathcal{X}](-, M) \). Applying Proposition 3.6 to \( B := \omega \), we have a projective resolution of the \( A \)-module \( A/[\mathcal{X}](-, M) \):

\[
\cdots \longrightarrow A(-, \Omega^{\omega-}(Y_M)) \stackrel{\Omega^{\omega-}g^\sim}{\longrightarrow} A(-, \Omega^{\omega-}(X_M)) \stackrel{\Omega^{\omega-}f^\sim}{\longrightarrow} A(-, M) \longrightarrow A/[\mathcal{X}](-, M) \longrightarrow 0.
\]

Since \( Y_M \in \hat{\omega} \), one has \( \Omega^n\omega(Y_M) \in \omega \) for some \( n \geq 0 \). Thus \( A(-, \Omega^n\omega(Y_M)) = 0 \) and hence \( \text{pd}_A(A/[\mathcal{X}](-, M)) < \infty \). □

We are ready to prove Theorem 3.1.

**Proof of Theorem 3.1.** By Lemma 2.1 and Proposition 3.4, the canonical inclusion \( X \hookrightarrow A \) induces a triangle functor \( \bar{Q} : D_{sg}(A) \rightarrow D_{sg}(X) \). By Theorem 2.2 and Proposition 3.8, the triangle functor \( \bar{Q} \) is an equivalence. □

### 3.2. Singular equivalences from cotilting objects.

In this subsection we construct a singular equivalence from a given cotilting subcategory, using Theorem 3.1. We denote by \( P(C) \) (resp. \( GP(C) \)) the full subcategory of \( C \) consisting of projective (resp. Gorenstein projective) objects. We abbreviate \( \Omega^n := \Omega^nP(C)M \) for each \( M \in C \) and denote by \( \Omega^nA \) the full subcategory of objects isomorphic to \( \Omega^nM \) for some \( M \in A \). Moreover we define \( \Omega^{-n}M \) to be the kernel of a left \( P(C) \)-approximation of \( M \). Inductively we define \( \Omega^{-n}M \) for any \( n \geq 1 \).

**Corollary 3.9.** Let \( A \) be an abelian category with enough projectives and \( T \) its contravariantly finite cotilting subcategory. Then the canonical inclusion \( \perp T \hookrightarrow A \) induces a triangle equivalence \( D_{sg}(A) \sim D_{sg}(\perp T) \).

**Proof.** Setting \( \mathcal{X} := \perp T \) and \( \omega := T \), we shall show that the sequence \( A \supseteq \mathcal{X} \supseteq \omega \) satisfies conditions (AB1)-(AB3). The condition (AB1) is obvious, because \( A = C \). The condition (AB2) holds by definition.

(AB3): By [ABu] Thm. 1.1, for any \( M \in A \), there exists an exact sequence

\[ 0 \rightarrow Y_M \rightarrow X_M \rightarrow M \rightarrow 0 \]

with \( Y_M \in \hat{\omega} \) and \( X_M \in \mathcal{X} \). It remains to show \( \hat{\mathcal{X}} = A \). Since there exists an integer \( n \geq 0 \) such that \( \text{id}I \leq n \) for all \( I \in \omega \), it follows that \( \Omega^nM \in \mathcal{X} \) holds for all \( M \in A \). This shows \( \hat{\mathcal{X}} = A \). Thanks to Theorem 3.1 we have a desired triangle equivalence. □

### 3.3. Matsui-Takahashi’s Singular equivalence.

We provide an alternative proof for Matsui-Takahashi’s singular equivalence.

**Definition 3.10.** Let \( C \) be an abelian category with enough projectives. A full subcategory \( A \) of \( C \) is called quasi-resolving if it is closed under kernels of epimorphisms and contains all projectives. A quasi-resolving subcategory is called resolving if it is closed under extensions and direct summands.
Corollary 3.11. [MT] Thm. 5.4(3)] Let \( \mathcal{A} \) be a quasi-resolving subcategory of an abelian category \( \mathcal{C} \) with enough projectives. Assume that \( \mathcal{A} \) together with an integer \( n \in \mathbb{Z}_{\geq 0} \) satisfies the condition

\[
(\ast) \quad \mathcal{A} \text{ is contained in } \mathcal{GP}(\mathcal{C}) \text{ and closed under cosyzygies}
\]

and set \( \mathcal{X} := \mathcal{A}^n \). Then the canonical inclusion \( \mathcal{X} \hookrightarrow \mathcal{A} \) induces a triangle equivalence \( D_{\text{sg}}(\mathcal{A}) \sim \rightarrow D_{\text{sg}}(\mathcal{X}) \).

Proof. Setting \( \mathcal{X} := \mathcal{A}^n \) and \( \mathcal{w} := \mathcal{P}(\mathcal{C}) \), we shall show that the sequence \( \mathcal{A} \supseteq \mathcal{X} \supseteq \mathcal{w} \) of subcategories in \( \mathcal{C} \) satisfies the conditions (AB1)-(AB3). (AB1): Since \( \mathcal{P}(\mathcal{C}) \)-epikernels are epimorphisms, the condition (AB1) follows from the definition of quasi-resolving subcategories.

(AB2): Since \( \mathcal{X} \subseteq \mathcal{GP}(\mathcal{C}) \), we have \( \mathcal{X} \subseteq ^{\mathcal{w}} \mathcal{w} \).

(AB3): Let \( M \in \mathcal{A} \). By the condition (\( \ast \)), we have an exact sequence

\[
0 \rightarrow G \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_0 \rightarrow M \rightarrow 0
\]

with \( G \in \mathcal{X} \) and \( P_{n-1}, \cdots, P_0 \in \mathcal{P}(\mathcal{C}) \). Since \( G \in \mathcal{GP}(\mathcal{C}) \), we have an exact sequence

\[
0 \rightarrow G \xrightarrow{g_n} Q_{n-1} \xrightarrow{g_{n-1}} \cdots \rightarrow Q_0 \xrightarrow{g_0} \Omega^{-n}(G) \rightarrow 0
\]

with the canonical morphisms \( \text{Im } g_i \rightarrow Q_i \) being left \( \mathcal{P}(\mathcal{C}) \)-approximations for each \( 1 \leq i \leq n \).

Thus we have the following chain map, where \( \Omega^{-n}(G) \in \mathcal{A}^n = \mathcal{X} \) by the condition (\( \ast \)).

\[
\begin{array}{c}
0 \rightarrow G \rightarrow Q_{n-1} \rightarrow \cdots \rightarrow Q_0 \rightarrow \Omega^{-n}(G) \rightarrow 0 \\
\end{array}
\]

By taking the mapping cone of the above chain map, we have an exact sequence

\[
0 \rightarrow G \rightarrow Q_{n-1} \oplus G \rightarrow Q_{n-2} \oplus P_{n-1} \rightarrow \cdots \rightarrow Q_0 \oplus P_1 \rightarrow \Omega^{-n}(G) \oplus P_0 \rightarrow M \rightarrow 0.
\]

Since the left-most morphism \( G \rightarrow Q_{n-1} \oplus G \) is a split-monomorphism, we have the following exact sequence

\[
(3.11.1) \quad 0 \rightarrow Q_{n-1} \rightarrow Q_{n-2} \oplus P_{n-1} \rightarrow \cdots \rightarrow Q_0 \oplus P_1 \rightarrow \Omega^{-n}(G) \oplus P_0 \xrightarrow{f} M \rightarrow 0.
\]

Obviously \( \text{Ker } f \in \mathcal{w} \) holds. The exact sequence \( 0 \rightarrow \text{Ker } f \rightarrow \Omega^{-n}(G) \oplus P_0 \xrightarrow{f} M \rightarrow 0 \) is a desired one. Indeed, \( f \) is a right \( \mathcal{X} \)-approximation by Lemma [3.3] \( \square \).

Recall that an additive category \( \mathcal{A} \) with weak-kernels is said to be Iwanaga-Gorenstein if \( \text{id}_{\mathcal{A}}(\mathcal{A}(-, M)), \text{id}_{\mathcal{A}^{op}}(\mathcal{A}(M, -)) < \infty \) for any \( M \in \mathcal{A} \). Typical examples of Iwanaga-Gorenstein rings are finite dimensional selfinjective algebras over a field \( k \) and commutative Gorenstein rings of finite Krull dimension. As an obvious consequence of Corollary [3.9] or [3.11] we have:

Example 3.12. Let \( \Lambda \) be an Iwanaga-Gorenstein ring with \( \text{id}_{\Lambda}(\Lambda) = n \) and \( \text{CM} \Lambda := \frac{1}{n} \Lambda \). Then the canonical inclusion \( \text{CM} \Lambda \hookrightarrow \text{mod} \Lambda \) induces a triangle equivalence \( D_{\text{sg}}(\text{mod} \Lambda) \sim \rightarrow D_{\text{sg}}(\text{CM} \Lambda) \).
4. More Results and Examples

In this section, we provide further investigations on Condition $[\mathbb{L}]$. First we give sufficient conditions so that $\mathcal{X}/[u]$ is Iwanaga-Gorenstein and of finite global dimension, respectively.

**Theorem 4.1.** Let $\Lambda$ be a finite dimensional algebra and $T \in \mod \Lambda$ a cotilting module. We set $\perp T := \perp T/\left[\perp \Lambda\right]$ and $\perp T := \perp T/\left[\perp \Lambda\right]$. Then the followings hold:

(a) If $\Lambda$ is Iwanaga-Gorenstein, then so is $\perp T$. Moreover, one has $\text{id}(\perp T)F \leq 3\text{max}\{\text{pd}_\Lambda T, \text{id}_\Lambda \Lambda\}$ for any projective $(\perp T)$-module $F$.

(b) If $\text{gl.dim}\Lambda = n$, then we have $\text{gl.dim}(\perp T) \leq 3n - 1$.

The assertion (b) can be found in [Kim, Thm. 6.1]. Let us recall from [INP] Thm. 3.4 (see also [Eno, Jia]), there exist Auslander-Reiten translations on $\perp T$, that is, mutually equivalences

$$\tau : \perp T \sim \perp T$$

Moreover, they induce functorial isomorphisms

$$\text{D} \text{Ext}^1_\Lambda(M, N) \cong \perp T(\tau^{-}N, M) \cong \perp T(N, \tau M)$$

in $M, N \in \perp T$ which are known as Auslander-Reiten dualities, where $D := \text{Hom}_k(-, k)$.

**Proof of Theorem 4.1.** (a) Since there exists an equivalence $\perp T \sim \perp T$, we shall show that $\perp T$ is Iwanaga-Gorenstein. Thanks to Auslander-Reiten duality, every injective $(\perp T)$-module is of the form $\text{Ext}^1_\Lambda(-, M)$ for some $M \in \perp T$. Since $T$ is a cotilting module, we get an exact sequence $0 \to M \to \tau T \to N \to 0$ with $T' \in \text{add} T$ and $N \in \perp T$. The induced sequence

$$0 \to \text{Hom}_\Lambda(-, M) \to \text{Hom}_\Lambda(-, T') \to \text{Hom}_\Lambda(-, N) \to \text{Ext}^1_\Lambda(-, M) \to 0$$

gives a projective resolution of $(\perp T)$-module $\text{Ext}^1_\Lambda(-, M)$. By Proposition 3.6 we have a projective resolution

$$\cdots \longrightarrow \perp T(-, \Omega_\Lambda(M)) \longrightarrow \perp T(-, \Omega_\Lambda(T')) \longrightarrow \perp T(-, \Omega_\Lambda(N)) \longrightarrow \text{Ext}^1_\Lambda(-, M) \longrightarrow 0$$

of the $(\perp T)$-module $\text{Ext}^1_\Lambda(-, M)$. Since $\Lambda$ is Iwanaga-Gorenstein, $T$ is a tilting module, in particular $\text{pd}_\Lambda(T) < \infty$. Thus there exists an integer $n \geq 0$ such that $\Omega^n_\Lambda(T') \in \text{proj} \Lambda$. Hence every injective $(\perp T)$-module $\text{Ext}^1_\Lambda(-, M)$ is of finite projective dimension. Next we shall show that every projective $(\perp T)$-module $\perp T(-, M)$ is of finite injective dimension. Considering the first syzygy of $M$, namely an exact sequence $0 \to \Omega_\Lambda M \to P \to M \to 0$ with $P \in \text{proj} \Lambda$, we get an injective resolution

$$\cdots \longrightarrow \perp T(-, M) \longrightarrow \text{Ext}^1_\Lambda(-, \Omega_\Lambda M) \longrightarrow \text{Ext}^1_\Lambda(-, P) \longrightarrow \text{Ext}^1_\Lambda(-, M) \longrightarrow \cdots$$

of the $(\perp T)$-module $\perp T(-, M)$. Since $\Lambda$ is Iwanaga-Gorenstein, we have $\text{id}_\Lambda P < \infty$. We have thus concluded that $\perp T$ is Iwanaga-Gorenstein. The latter formula follows from the sequence (4.1.1) and (4.1.2).

(b) We shall show that $\text{gl.dim}(\perp T) \leq 3n - 1$. Let $F \in \mod(\perp T)$ with a projective presentation $\perp T(-, M) \to \perp T(-, L) \to F \to 0$. Since $F$ vanishes on $\text{proj} \Lambda$, the corresponding morphism
f : M → L is an epimorphism in \textup{mod}\Lambda. Since \perp T is closed under epimorphisms, we have an exact sequence 0 → N → M → L → 0 in \perp T which induces a projective resolution

\[ \cdots \xrightarrow{\perp T(-, \Omega\Lambda(N))} \xrightarrow{\perp T(-, \Omega\Lambda(M))} \xrightarrow{\perp T(-, \Omega\Lambda(L))} \]  

of the \((\perp T)\)-module \( F \). The assumption \( \textup{gl.dim}\Lambda = n \) implies \( \Omega^n\Lambda(L) \in \text{proj}\Lambda \). Hence \( \textup{pd}(\perp T)F \leq 3n - 1. \)

Theorem 4.1 contains the following well-known result.

**Example 4.2.** [AR74, Prop. 10.2] Let \( \Lambda \) be a finite dimensional algebra with \( \textup{gl.dim}\Lambda = n \). Then we have \( \textup{gl.dim}(\textup{mod}\Lambda) \leq 3n - 1. \)

Next we explain that (AB1)-(AB3) in Condition 1.1 are satisfied in the classical Auslander-Buchweitz theory: Let \( \mathcal{C} \) be an abelian category with enough projectives and \( \mathcal{X} \supseteq \omega \) a sequence of full subcategories in \( \mathcal{C} \). We say that \( \omega \) is a cogenerator of \( \mathcal{X} \) if, for each \( X \in \mathcal{X} \), there exists an exact sequence 0 → \( X \) → \( I \) → \( X' \) → 0 with \( I \in \omega, X' \in \mathcal{X} \).

**Condition 4.3.** [ABu, p. 9, 17] For a sequence \( \mathcal{X} \supseteq \omega \) of full subcategories in \( \mathcal{C} \), we consider the following conditions:

- \( \widehat{\mathcal{X}} = \mathcal{C} \);
- \( \mathcal{X} \) is closed under direct summands and extension;
- \( \textup{Ext}^i_{\mathcal{C}}(X, I) = 0 \) for any \( X \in \mathcal{X}, I \in \omega \) and \( i > 0 \);
- \( \omega \) is a cogenerator of \( \mathcal{X} \) which is closed under direct summands.

Under these conditions, it is known that, for each \( M \in \mathcal{C} \), there exists an exact sequence

\[ 0 \to Y_M \to X_M \to M \to 0 \]

(4.3.1) with \( X_M \in \mathcal{X}, Y_M \in \omega \) [ABu Thm. 1.1]. The sequence (4.3.1) is called the Auslander-Buchweitz approximation of \( M \). As a benefit of our generalized Auslander-Buchweitz approximation in (AB3), we shall show Proposition 4.4. Notice that, in the proposition, the subcategory \( \omega \) is not necessarily a cogenerator of \( \mathcal{X} \), and right \( \mathcal{X} \)-approximations of objects of \( \mathcal{A} \) appearing in (AB3) are not necessarily surjective.

**Proposition 4.4.** Let \( \mathcal{A} \) be an abelian category with enough projectives and \( \mathcal{X} \supseteq \omega \) a sequence of full subcategories of \( \mathcal{A} \). Suppose that \( \mathcal{X} \) is a torsion class of \( \mathcal{A} \) and \( \omega \) is contravariantly finite in \( \mathcal{A} \) and satisfies \( \textup{Ext}^i_{\mathcal{A}}(X, I) = 0 \) for any \( X \in \mathcal{X}, I \in \omega \) and \( i > 0 \). Then the sequence \( \mathcal{A} \supseteq \mathcal{X} \supseteq \omega \) satisfies (AB1)-(AB3).

**Proof.** The conditions (AB1) and (AB2) are obvious. Since \( \mathcal{X} \) is a torsion class, for any \( M \in \mathcal{A} \) there exists an exact sequence 0 → \( X \to M \) with \( X \in \mathcal{X} \), hence (AB3) holds. \( \square \)

We end this section by giving examples of singularly equivalent categories using Corollary.
**Example 4.5.** Fix an integer $n \in \mathbb{Z}_{>0}$. Let $\Lambda$ be the algebra defined by the following quiver with relations.

![Quiver diagram](image)

We describe the Auslander-Reiten quiver of $\Lambda$. Since $\Lambda$ is a Nakayama algebra, an indecomposable module is determined by the pair $(m, l)$ of the socle $l$ and the Loewy length $l$. We shall denote the module by $[m]_l$.

![Mod A diagram](image)

We can easily check that the module $T := [1]_1 \oplus [1]_{2n+2}$ is a cotilting module of $\text{id}_\Lambda(T) = 1$. Due to Corollary 3.9, we conclude that $\text{mod} \Lambda := (\text{mod} \Lambda) / [T]$ is singularly equivalent to $\overline{T} := (\perp T) / [T]$. Their Auslander-Reiten quivers are described as follows:

![Mod A and T diagram](image)

where the dotted lines stand for natural mesh relations.

**Claim.** If $n = 1$, both $\overline{\text{mod}} \Lambda$ and $\overline{\perp T}$ are of finite global dimension, otherwise they are non Iwanaga-Gorenstein.

**Proof.** We only check the case of $n \geq 2$. By calculations, the injective $\perp T$-module $D^\perp T([1]_3, -)$ has the following projective resolution:

$$\cdots \rightarrow P_5 \rightarrow P_3 \rightarrow P_{2n+1} \rightarrow P_{2n-1} \rightarrow P_{2n+1} \rightarrow P_3 \rightarrow P_4 \rightarrow P_{2n+1} \rightarrow I_3 \rightarrow 0,$$

where we set $I_3 := D^\perp T([1]_3, -)$ and $P_l := \perp T(-, [1]_l)$ for each $1 \leq l \leq 2n + 1$. We notice that $\Omega^2 I_3 \cong \Omega^8 I_3$. Hence $\perp T$ is non Iwanaga-Gorenstein. It remains to check the assertion for $\overline{\text{mod}} \Lambda$. 
We denote by $Q : \text{mod}(\text{mod} \Lambda) \to \text{mod}(\text{mod} T)$ the canonical functor. There exists an injective object $J \in \text{inj}(\text{mod} \Lambda)$ such that $QJ \cong I_3$. If $\text{mod} \Lambda$ is Iwanaga-Gorenstein, then $J$ is of finite projective dimension. Moreover, since $Q$ is exact and preserves projectives, it turns out that $I_3$ is of finite projective dimension. This is a contradiction. □

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