QCD Amplitudes: new perspectives on Feynman integral calculus

Pierpaolo Mastrolia

Dipartimento di Fisica e Astronomia, Università di Padova, and INFN Sezione di Padova, via Marzolo 8, 35131 Padova, Italy.
Max-Planck Institut für Physik, Föhringer Ring 6, 80805 München, Germany.

I analyze the algebraic patterns underlying the structure of scattering amplitudes in quantum field theory. I focus on the decomposition of amplitudes in terms of independent functions and the systems of differential equations the latter obey. In particular, I discuss the key role played by unitarity for the decomposition in terms of master integrals, by means of generalized cuts and integrand reduction, as well as for solving the corresponding differential equations, by means of Magnus exponential series.

1 Introduction

High energy particle collisions are the ideal framework for accessing new informations on matter constituents and forces of nature. The higher the energy of the colliding particles, the richer the landscape of the produced ones. The discovery of new physics interactions cannot be disentangled from the discovery of massive, heavy particles, emerging from collisions of ever increasing energy. On the other side, by increasing energy, also the probability of producing many light particles is enhanced. Therefore, advances in High Energy Particle Physics necessarily depend on our ability to describe the scattering processes involving many light and heavy particles at very high accuracy, hence they depend on our capability of evaluating Feynman diagrams. Beyond leading order (LO), Feynman diagrams represent challenging multidimensional/multivariate integrals, whose direct evaluation is often prohibitive, therefore the computation of scattering amplitudes beyond the LO is addressed in two stages: i) the decomposition in terms of a basis of functions, and ii) their evaluation of the elements of such a basis, called master integrals (MIs). In this contribution, I elaborate on the algebraic properties of Feynman integrals, which can be exploited for decomposing them in terms of MIs and for computing the latter. The techniques I discuss can be applied to generic amplitudes, and have a impact on high-accuracy prediction for collider physics, as well as for exploring the more formal aspect of quantum field theory.

Let us observe that amplitudes can be decomposed in terms of independent functions, exactly like a vector can be decomposed along basic directions. One needs a basis and a projection...
technique. The latter is necessary to extract the coefficients of the linear combination. Factorization is the basic idea we are going to elaborate on. Factorization is ubiquitous in the discovery of new mathematical and physical concepts. Complex numbers emerged from factorizing the simplest number we may think of, i.e. $1 = (-i)$; quantum mechanics relies on the factorization of the identity matrix, $\mathbb{I} = \sum_n |n\rangle\langle n|$: Dirac equation emerged from factorizing the d’Alambertian operator, i.e. $\Box = (-i\gamma^\mu \partial_\mu)(i\gamma^\nu \partial_\nu)$. What does happen when amplitudes factorize?

Cutting a virtual particle and bringing it on the mass shell ($p^2 = m^2$), turned out to be a suitable projection technique yielding amplitudes decomposition. Why multiple-cuts are important? First, because multiple-cuts yield functions identification. Since any diagram is characterized by its internal lines, a given master diagram is univocally identified by a cut-diagram where all internal particles are on-shell. Moreover, when applied to amplitudes, multiple-cuts behave like high-pass filters, which isolate only the diagrams that have those internal lines to be cut, while the others are automatically discarded. Therefore, by considering all possible cuts of an amplitude, in a top-down procedure, from the maximum number of cuts to the lowest one, it is possible to build a (triangular) system of equations from which all coefficients can be determined.

2 Integrand decomposition

Tree-level scattering amplitudes obey a quadratic recurrence relation $1$ (BCFW), depicted in Fig.1, whose derivation relies on Cauchy’s residue theorem. Since tree amplitudes are rational functions of kinematic variables, the BCFW recurrence can be understood as due simply to partial fractioning $2$, because residue theorem applied to rational functions amounts to partial fractions. Is that just accidental, and holding for tree-level amplitudes, or partial fractioning can be exploited also at higher orders?

The integrand reduction algorithm $3$ had a dramatic impact on our ability of computing one-loop amplitudes. The basic idea lies in the existence of a relation between numerators and denominators of scattering amplitudes which can be used to decompose the integrands of one-loop amplitudes in terms of integrands of MIs. The amplitude decomposition in terms of MIs is then achieved after integrating the integrand decomposition. The coefficients of the MIs are a subset of the coefficients appearing in the decomposition of the integrands. Therefore, within the integrand reduction algorithm, coefficients can be determined simply by algebraic manipulation, with the great advantage of bypassing any integration.

The idea behind the GoSam framework $4,5$ is to combine automated diagram generation and algebraic manipulation with the integrand-level reduction, implemented in SAMURAI$6,7$ and NINJA$8,9$ and the tensor decomposition of GOLEM95$10$. The code is very flexible and it has been employed in several applications at NLO QCD accuracy, studies of BSM scenarios, electroweak calculations, and recently also within NNLO calculations. It is interfaced to several MonteCarlo event generators, like SHERPA, HERWIG, AMC@NLO. GoSam was used to evaluate the NLO QCD correction to $pp \to Hjjj, Hjjjj$ (in the infinite top-mass limit) $11,12$, which required an extension of the integrand decomposition methods $8$. The evaluation of the virtual amplitudes for $pp \to Hjjj$ has been further optimized, enhancing the numerical accuracy and reducing
the computing time\textsuperscript{13}, ending up into a new phenomenological analysis\textsuperscript{14,15}, obtained with the tandem of GoSam and Sherpa, see Fig. 3.

The extension of the integrand decomposition beyond one-loop has been proposed in\textsuperscript{16}, and refined in\textsuperscript{17,18,19}, where the unitarity-based decomposition of multi-loop integrands has been addressed as a polynomial decomposition problem, and systematized within the multivariate polynomial division algorithm. Accordingly, any generic multi-loop integral with $n$ denominators, $I_{12...n} = \int \, d^d q_1 \cdots d^d q_n \, I_{12...n}$, with $I_{12...n} = N_{12...n}/(D_1 \cdots D_n)$, can undergo an integrand decomposition by means of successive polynomial divisions (modulo Gröbner basis) between the numerator and the denominators, see Fig.2. The result of the decomposition reads as,

$$I_{12...n} = \frac{\Delta_{12...n}}{D_1 \cdots D_n} + \frac{\Delta_{2...n}}{D_2 \cdots D_n} + \ldots + \frac{\Delta_{12...n-1}}{D_1 \cdots D_{n-1}} + \ldots + \frac{\Delta_n}{D_n} + \ldots + \frac{\Delta_1}{D_1},$$

(1)

where $\Delta_{i...j}$ are the remainders of the iterated divisions (w.r.t. the Gröbner basis of the ideal $(D_1, \ldots, D_n)$). Each residue $\Delta_{i...j}$ is a polynomial in the components of the loop momenta not constrained by the cut $D_1 = \ldots = D_j = 0$. Therefore, by integrating both sides, one obtains the decomposition of the original integral $I_{12...n}$ in terms of independent integrals. The integrand decomposition (1) implies that, exactly as it happens for the tree-level amplitudes, also the integrands of multi-loop amplitudes can be decomposed in terms of independent building blocks simply by partial fractioning!

While in the one-loop case the independent integrals are analytically known, in the multi-loop case, their classification and evaluation is an open problem.

3 Differential Equations and Feynman Integrals

The method of differential equations (DEs)\textsuperscript{20,21,22}, reviewed in\textsuperscript{23,24,25}, is one of the most effective techniques for computing dimensionally regulated multi-loop integrals.

In fact, any $\ell$-loop integral $\mathcal{I}$ is a homogeneous function of external momenta $p_i$ and masses $m_i$, whose degree $\gamma = \gamma(d, \ell)$ depends on the space-time dimensions $d = 4-2\epsilon$, on the number of loops $\ell$, and on the powers of denominators. Therefore, one can write the Euler scaling equation,

$$\left( \sum_i p_i \cdot \partial_{p_i} + \sum_j m_j^2 \partial_{m_j^2} \right) \mathcal{I} = \gamma(d, \ell) \mathcal{I},$$

(2)

where $\partial_{x} \equiv \partial/\partial x$. Euler relation can be engineered to show that MIs obey linear systems of first-order differential equations (DEs) in the kinematic invariants, which can be used for the determination of their actual expression. By establishing an analogy between Schrödinger Equation in the interaction picture (in presence of an Hamiltonian with a linear perturbation) and systems of DEs for Feynman integrals (whose associated matrix is linear in $\epsilon$)\textsuperscript{26}, we have recently proposed an algorithm to find the transformation matrix yielding to a canonical system\textsuperscript{27}, where the dependence on the dimensional parameter $\epsilon$ is factorized from the kinematic. In particular, we found that the canonical transformation can be obtained by means of Magnus exponential matrix\textsuperscript{28}. The integration of canonical systems is simple, and the analytic properties of its solution are manifestly inherited from the associated matrix, that becomes the kernel of the representation of the solutions in terms of repeated integrations. The latter in fact are the coefficients of a Magnus (or alternatively Dyson) series expansion in $\epsilon$. Magnus exponential is not unitary, as it happens in the quantum mechanical case, but the proposed method can be considered also inspired by unitarity.
3.1 Applications

We made use of Magnus theorem for the determination of non-trivial integrals, like the two-loop vertex diagrams for the electron form factors in QED and the two-loop box integrals for the $2 \rightarrow 2$ massless scattering \cite{26}, the two-loop corrections to the $pp \rightarrow Hj$, as well as for evaluating the three-loop ladder diagrams for $pp \rightarrow Hj$ (in the infinite top-mass approximation) \cite{20}, see Fig.4. The latter is a formidable calculation involving the solution of a system of 85 MIs. In this case, after identifying a set of MIs obeying a linear system of differential equations in $x = -s/m_H^2$ and $y = -t/m_H^2$, by means of a Magnus transform, the system can be brought in canonical form, reading as,

$$d\vec{I}(x, y) = \epsilon \ A(x, y) \ \vec{I}(x, y),$$

where $\vec{I}$ is the vector of MIs, and $df = \partial_x f dx + \partial_y f dy$. The matrix $A$ is purely logarithmic, $A(x, y) = a_1 \ln(x) + a_2 \ln(1-x) + a_3 \ln(y) + a_4 \ln(1-y) + a_5 \ln(x+y) + a_6 \ln(1-(x+y))$, where the $a_i$ ($i = 1, \ldots, 6$) are $85 \times 85$ matrices whose entries are just rational numbers. The logarithmic form of $A$ trivializes the solution, which can be written as a Dyson series in $\epsilon$, where the coefficient of the series are combinations of Multiple Polylogarithms with uniform weight (where the weight increases as the order in $\epsilon$ does).

Boundary conditions are determined by imposing the regularity of the solutions in special kinematic configurations. Surprisingly, to fix the boundary values of all 85 MIs, only 2 simple integrals have to be independently provided.

Also, we have been considering the mixed EW-QCD corrections to Drell-Yan production at NNLO, whose representative diagram is depicted in Fig.5. Also in this case, Magnus exponential can employed to reach a canonical system for the 48 MIs drawn in Fig.6, in the variables $x = -s/m_V^2$ and $y = -t/m_V^2$, with $V = W, Z$ \cite{30}.

4 Conclusions

In this contribution, I have analyzed the algebraic patterns underlying the structure of scattering amplitudes in gauge theory. Unitarity plays a central role in the context of evaluating scattering amplitudes. It not only inspired methods to perform the amplitudes decomposition, by means of unitarity-cuts, but it also suggested a technique for the evaluation of master integrals, by means of matrix exponentials, similar to the unitary time-evolution in quantum mechanics.

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