The Barut Second-Order Equation, Dynamical Invariants and Interactions

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Abstract

The second-order equation in the \((1/2, 0) \oplus (0, 1/2)\) representation of the Lorentz group has been proposed by A. Barut in the beginning of the 70s, ref. [1]. It permits to explain the mass splitting of leptons (\(e, \mu, \tau\)). Recently, the interest has grown to this model (see, for instance, the papers by S. Kruglov [2] and J. P. Vigier et al. [3]). We continue the research deriving the equation from the first principles, finding the dynamical invariants for this model, investigating the influence of the potential interactions.

The Barut main equation is

\[
i\gamma_\mu \partial_\mu - \alpha_2 \frac{\partial_\mu \partial_\mu}{m} + \kappa \Psi = 0.
\]

- It represents a theory with the conserved current that is linear in 15 generators of the 4-dimensional representation of the \(O(4,2)\) group, \(N_{ab} = \gamma_a \gamma_b, \gamma_a = \{\gamma_\mu, \gamma_5, i\}\).
- Instead of 4 solutions it has 8 solutions with the correct relativistic relation \(E = \pm \sqrt{\mathbf{p}^2 + m^2}\). In fact, it describes states of different masses (the second one is \(m_\mu = m_e(1 + \frac{3\alpha}{2})\), \(\alpha\) is the fine structure constant), provided that a certain physical condition is imposed on the \(\alpha_2\) parameter (the anomalous magnetic moment should be equal to \(4\alpha/3\)).
- One can also generalize the formalism to include the third state, the \(\tau\)-lepton [1b].

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• Barut has indicated at the possibility of including $\gamma_5$ terms (e.g., $\sim \gamma_5\kappa'$).

If we present the 4-spinor as $\Psi(p) = column(\phi_R(p)) \phi_L(p))$ then Ryder states [5] that $\phi_R(0) = \phi_L(0)$. Similar argument has been given by Faustov [6]: “the matrix $B$ exists such that $Bu_\lambda(0) = u_\lambda(0)$, $B^2 = I$ for any $(2J + 1)$-component function within the Lorentz invariant theories”[7]. The most general form of the relation in the $(1/2, 0) \oplus (0, 1/2)$ representation has been given by Dvoeglazov [7,4a]:

$$\phi_L^h(0) = a(-1)^{\frac{1}{2} - h} e^{i(\theta_1 + \theta_2)} \Theta_{1/2} [\phi_R^{-h}(0)]^* + be^{2i\theta_1} \Xi_{1/2}^{-1} [\phi_L^h(0)]^*, \quad (2)$$

with

$$\Theta_{1/2} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -i\sigma_2, \quad \Xi_{1/2} = \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix}, \quad (3)$$

$\Theta_{1/2}$ is the Wigner operator for spin $J = 1/2$, $\varphi$ is the azimuthal angle $p \to 0$ of the spherical coordinate system.

Next, we use the Lorentz transformations:

$$\Lambda_{R,L} = \exp(\pm \sigma \cdot \phi / 2), \quad \cosh \phi = E_p/m, \quad \sinh \phi = |p|/m, \quad \hat{\phi} = p/|p|. \quad (4)$$

Applying the boosts and the relations between spinors in the rest frame, one can obtain:

$$\phi_L^h(p) = a\left[ \frac{p_0}{m} - \frac{\sigma \cdot p}{m} \right] \phi_R^h(p) + b(-1)^{\frac{1}{2} + h} \Theta_{1/2} \Xi_{1/2} \phi_R^{-h}(p), \quad (5)$$

$$\phi_R^h(p) = a\left[ \frac{p_0}{m} + \frac{\sigma \cdot p}{m} \right] \phi_L^h(p) + b(-1)^{\frac{1}{2} + h} \Theta_{1/2} \Xi_{1/2} \phi_R^{-h}(p). \quad (6)$$

($\theta_1 = \theta_2 = 0$, $p_0 = E_p = \sqrt{p^2 + m^2}$). In the Dirac form we have:

$$[a\hat{p}^\mu - 1] u_h(p) + ib(-1)^{\frac{1}{2} - h} \gamma^\mu C u_{-h}(p) = 0, \quad (7)$$

where $C = \begin{pmatrix} 0 & i\Theta_{1/2} \\ -i\Theta_{1/2} & 0 \end{pmatrix}$. In the QFT form we must introduce the creation/annihilation operators. Let $b_\downarrow = -i\alpha_1$, $b_\uparrow = +i\alpha_1$, then

$$[a\alpha_\downarrow \gamma^\mu \partial_\mu - bCK - 1] \Psi(x^\nu) = 0. \quad (8)$$

If one applies the unitary transformation to the Majorana representation [8]

$$U = \frac{1}{2} \begin{pmatrix} 1 - i\Theta_{1/2} & 1 + i\Theta_{1/2} \\ -1 - i\Theta_{1/2} & 1 - i\Theta_{1/2} \end{pmatrix}, \quad UCKU^{-1} = -K, \quad (9)$$

The latter statement is more general than the Ryder one, because it admits $B = \begin{pmatrix} 0 & e^{\pm i\alpha} \\ e^{-\pm i\alpha} & 0 \end{pmatrix}$, so that $\phi_R(0) = e^{i\alpha} \phi_L(0)$. 

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then \( \gamma \)-matrices become to be pure imaginary, and the equations are pure real.

\[
\begin{bmatrix}
\frac{i \hat{\partial}}{m} - b - 1 \\
\frac{i \hat{\partial}}{m} + b - 1
\end{bmatrix} \Psi_1 = 0, \\
\begin{bmatrix}
\frac{i \hat{\partial}}{m} - b - 1 \\
\frac{i \hat{\partial}}{m} + b - 1
\end{bmatrix} \Psi_2 = 0,
\]

where \( \Psi = \Psi_1 + i \Psi_2 \). It appears as if the real and imaginary parts have different masses. Finally, for superpositions \( \phi = \Psi_1 + \Psi_2 \), \( \chi = \Psi_1 - \Psi_2 \), multiplying by \( b \neq 0 \) we have:

\[
[2a \frac{i \gamma^\mu \partial_\mu}{m} + a^2 \frac{\partial^\mu \partial_\mu}{m^2} + b^2 - 1] \frac{\phi(x^\nu)}{\chi(x^\nu)} = 0,
\]

If we put \( a/2m \rightarrow \alpha, \frac{b^2}{2a} \rightarrow \kappa \) we recover the Barut equation.

How can we get the third lepton state? See the refs. [1b,4b]:

\[
M_\tau = M_\mu + \frac{3}{2} \alpha^{-1} n^4 M_e = M_e + \frac{3}{2} \alpha^{-1} \frac{1}{4} n^4 M_e + \frac{3}{2} \alpha^{-1} \frac{1}{2} n^4 M_e = 1786.08 \text{ MeV}.
\]

The physical origin was claimed by Barut to be in the magnetic self-interaction of the electron (the factor \( n^4 \) appears due to the Bohr-Sommerfeld rule for the charge moving in circular orbits in the field of a fixed magnetic dipole \( \mu \)). One can start from (7), but as opposed to the above-mentioned, one can write the coordinate-space equation in the form:

\[
[\frac{i \gamma^\mu \partial_\mu}{m} + b_1 \sigma - 1] \Psi(x^\nu) + b_2 \gamma^5 \tilde{\Psi}(x^\nu) = 0,
\]

with \( \Psi^{MR} = \Psi_1 + i \Psi_2, \tilde{\Psi}^{MR} = \Psi_3 + i \Psi_4 \). As a result,

\[
(a \frac{i \gamma^\mu \partial_\mu}{m} - 1) \phi - b_1 \chi + ib_2 \gamma^5 \phi = 0,
\]

\[
(a \frac{i \gamma^\mu \partial_\mu}{m} - 1) \chi - b_1 \phi - ib_2 \gamma^5 \chi = 0.
\]

The operator \( \tilde{\Psi} \) may be linear-dependent on the states included in the \( \Psi \). let us apply the most simple form \( \Psi_1 = -i \gamma^5 \Psi_4, \Psi_2 = +i \gamma^5 \Psi_3 \). Then, one can recover the 3rd order Barut-like equation [4b]:

\[
[i \gamma^\mu \partial_\mu - m \frac{1 \pm b_1 \pm b_2}{a}] [i \gamma^\nu \partial_\nu + \frac{a}{2m} \partial^\nu \partial_\nu + m \frac{b_1^2 - 1}{2a}] \Psi_{1,2} = 0.
\]

Thus, we have three mass states.

Let us reveal the connections with other models. For instance, in refs. [3, 9] the following equation has been studied:

\[
[(i \hat{\partial} - e\hat{A})(i \hat{\partial} - e\hat{A}) - m^2] \Psi = [(i \partial_\mu - eA_\mu)(i \partial^\mu - eA^\mu) - \frac{1}{2} e \sigma^\mu\nu F_{\mu\nu} - m^2] \Psi = 0
\]
for the 4-component spinor $\Psi$. This is the Feynman-Gell-Mann equation. In the free case we have the Lagrangian (see Eq. (9) of ref. [3c]):

$$\mathcal{L}_0 = (i \bar{\Psi} \gamma^\mu(i \partial_\mu \Psi)) - m^2 \bar{\Psi} \Psi .$$  \hspace{1cm} (19)

We can note:

- The Barut equation is a sum of the Dirac equation and the Feynman-Gell-Mann equation.
- Recently, it was suggested to associate an analogue of Eq. (19) with the dark matter [10], provided that $\Psi$ is composed of the self/anti-self charge conjugate spinors, and it has the dimension $[\text{energy}]^1$ in $c = \hbar = 1$. The interaction Lagrangian is $\mathcal{L}^I \sim g \bar{\Psi} \Psi \phi^2$.
- The term $\sim \bar{\Psi} \sigma^{\mu\nu} \Psi F_{\mu\nu}$ will affect the photon propagation, and non-local terms will appear in higher orders.
- However, it was shown in [3b,c] that a) the Mott cross-section formula (which represents the Coulomb scattering up to the order $\sim e^2$) is still valid; b) the hydrogen spectrum is not much disturbed; if the electromagnetic field is weak the corrections are small.
- The solutions are the eigenstates of $\gamma^5$ operator.
- In general, $J_0$ is not the positive-defined quantity, since the general solution $\Psi = a \Psi_+ + b \Psi_-$, where $[i \gamma^\mu \partial_\mu \pm m] \Psi_\pm = 0$, see also [11].

The most general conserved current of the Barut-like theories is

$$J_\mu = \alpha_1 \gamma_\mu + \alpha_2 \partial_\mu + \alpha_3 \sigma_{\mu\nu} \eta^{\nu} .$$  \hspace{1cm} (20)

Let us try the Lagrangian:

$$\mathcal{L} = \mathcal{L}_{\text{Dirac}} + \mathcal{L}_{\text{add}} .$$  \hspace{1cm} (21)

$$\mathcal{L}_{\text{Dirac}} = \alpha_1 [\bar{\Psi} \gamma^\mu (\partial_\mu \Psi) - (\partial_\mu \bar{\Psi}) \gamma^\mu \Psi] - \alpha_4 \bar{\Psi} \Psi ,$$  \hspace{1cm} (22)

$$\mathcal{L}_{\text{add}} = \alpha_2 (\partial_\mu \bar{\Psi}) (\partial^\mu \Psi) + \alpha_3 \partial_\mu \bar{\Psi} \sigma^{\mu\nu} \partial_\nu \Psi .$$  \hspace{1cm} (23)

Then, the equation follows:

$$[2 \alpha_1 \gamma^\mu \partial_\mu - \alpha_2 \partial_\mu \partial^\mu - \alpha_4] \Psi = 0 ,$$  \hspace{1cm} (24)

and its Dirac-conjugate:

$$\bar{\Psi} [2 \alpha_1 \gamma^\mu \partial_\mu + \alpha_2 \partial_\mu \partial^\mu + \alpha_4] = 0 .$$  \hspace{1cm} (25)
The derivatives act to the left in the second equation. Thus, we have the Dirac equation when $\alpha_1 = \frac{i}{2}$, $\alpha_2 = 0$, and the Barut equation when $\alpha_2 = \frac{1}{m} \frac{2\alpha_3}{1+4\alpha_3}$. In the Euclidean metrics the dynamical invariants are

$$J_\mu = -i \sum_i [\frac{\partial L}{\partial (\partial_\mu \Psi_i)} \Psi_i - \Psi_i \frac{\partial L}{\partial (\partial_\mu \bar{\Psi}_i)}],$$

(26)

$$T_{\mu\nu} = -i \sum_i [\frac{\partial L}{\partial (\partial_\mu \Psi_i)} \partial_\nu \Psi_i + \partial_\nu \bar{\Psi}_i \frac{\partial L}{\partial (\partial_\mu \bar{\Psi}_i)}] + L \delta_{\mu\nu},$$

(27)

$$S_{\mu\nu,\lambda} = -i \sum_{ij} [\frac{\partial L}{\partial (\partial_\lambda \Psi_i)} N_{\mu\nu,ij} \Psi_j + \bar{\Psi}_j N_{\mu\nu,ij} \frac{\partial L}{\partial (\partial_\lambda \bar{\Psi}_j)}].$$

(28)

$N_{\mu\nu}^{\Psi,\bar{\Psi}}$ are the Lorentz group generators.

Then, the energy-momentum tensor is

$$T_{\mu\nu} = -\alpha_1 [\bar{\Psi} \gamma_\mu \partial_\nu \Psi - \partial_\nu \bar{\Psi} \gamma_\mu \Psi] - \alpha_2 [\bar{\Psi} \gamma_\mu \partial_\nu \bar{\Psi} + \partial_\nu \bar{\Psi} \gamma_\mu \bar{\Psi}] - \alpha_3 [\bar{\Psi} \gamma_\mu \sigma_{\alpha\mu} \partial_\nu \Psi + \partial_\nu \bar{\Psi} \gamma_\mu \sigma_{\alpha\mu} \Psi] + \alpha_4 \bar{\Psi} \gamma_\mu \Psi \delta_{\mu\nu},$$

(29)

Hence, the Hamiltonian $\hat{H} = -i P_4 = -\int T_{44} d^3 x$ is

$$\hat{H} = \int \{\alpha_1 [\bar{\Psi} \gamma_4 \Psi - \Psi \gamma_4 \bar{\Psi}] + \alpha_2 [\bar{\Psi} \gamma_4 \partial_4 \Psi - \partial_4 \bar{\Psi} \Psi] - \alpha_3 [\bar{\Psi} \gamma_4 \sigma_{ij} \partial_4 \Psi] - \alpha_4 \bar{\Psi} \Psi\} d^3 x.$$

(30)

The current is

$$J_\mu = -i \{2\alpha_1 [\bar{\Psi} \gamma_\mu \Psi + \alpha_2 \gamma_\mu \Psi - \Psi \gamma_\mu \bar{\Psi}] + \alpha_3 \gamma_\mu \sigma_{\alpha\mu} \Psi - \Psi \sigma_{\alpha\mu} \partial_\alpha \Psi\}.\quad (31)$$

Hence, the charge operator $\hat{Q} = -i \int J_4 d^3 x$ is

$$\hat{Q} = -\int \{2\alpha_1 \bar{\Psi} \gamma_4 \Psi + \alpha_2 \gamma_4 \Psi - \Psi \gamma_4 \bar{\Psi}] + \alpha_3 \gamma_4 \sigma_{ij} \Psi - \Psi \sigma_{ij} \partial_4 \Psi\} d^3 x.\quad (32)$$

Finally, the spin tensor is

$$S_{\mu\nu,\lambda} = -\frac{i}{2} \{\alpha_1 [\bar{\Psi} \gamma_\mu \sigma_{\mu\lambda} \Psi + \Psi \sigma_{\mu\lambda} \gamma_\mu \Psi] + \alpha_2 [\bar{\Psi} \gamma_\mu \sigma_{\mu\lambda} \Psi - \Psi \sigma_{\mu\lambda} \gamma_\mu \bar{\Psi}]} + \alpha_3 [\bar{\Psi} \gamma_\mu \sigma_{\mu\lambda} \Psi - \Psi \sigma_{\mu\lambda} \gamma_\mu \partial_\lambda \Psi].\quad (33)$$

In the quantum case the corresponding field operators are written:

$$\Psi(x^m u) = \sum_h \int \frac{d^3 p}{(2\pi)^3} [u_h(p) a_h(p) e^{ip \cdot x} + v_h(p) b_h^\dagger(p) e^{-ip \cdot x}],$$

(34)

$$\bar{\Psi}(x^m u) = \sum_h \int \frac{d^3 p}{(2\pi)^3} [\bar{u}_h(p) a_h^\dagger(p) e^{-ip \cdot x} + \bar{v}_h(p) b_h(p) e^{ip \cdot x}].\quad (35)$$
The 4-spinor normalization is
\[ \bar{u}_h u_{h'} = \delta_{hh'}, \quad \bar{v}_h v_{h'} = -\delta_{hh}. \] (36)

The commutation relations are
\[ \left[ a_h(p), a_{h'}^\dagger(k) \right]_+ = (2\pi)^3 \frac{m}{p_4} \delta^{(3)}(p-k)\delta_{hh'}, \] (37)
\[ \left[ b_h(p), b_{h'}^\dagger(k) \right]_+ = (2\pi)^3 \frac{m}{p_4} \delta^{(3)}(p-k)\delta_{hh'}, \] (38)

with all other to be equal to zero. The dimensions of the $\Psi, \bar{\Psi}$ are as usual, $[\text{energy}]^{3/2}$. Hence, the second-quantized Hamiltonian is written
\[ \hat{H} = -\sum_h \int \frac{d^3p}{(2\pi)^3} \frac{2E_p^2}{m} [\alpha_1 + m\alpha_2] : [a_h^\dagger a_{h'} - b_h b_{h'}^\dagger] : . \] (39)

(Remember that $\alpha_1 \sim \frac{i}{2}$, the commutation relations may give another $i$, so the contribution of the first term to eigenvalues will be real. But if $\alpha_2$ is real, the contribution of the second term may be imaginary). The charge is
\[ \hat{Q} = -\sum_{hh'} \int \frac{d^3p}{(2\pi)^3} \frac{2E_p^2}{m} ([\alpha_1 + m\alpha_2] \delta_{hh'} - i\alpha_3 \bar{u}_h \sigma_{\mu\nu} p_{\nu} u_{h'}] : [a_h^\dagger a_{h'} + b_h b_{h'}^\dagger] : . \] (40)

However, due to $[\Lambda_{R,L}, \sigma \cdot p]_+ = 0$ the last term with $\alpha_3$ does not contribute.

The conclusions are:

- We obtained the Barut-like equations of the 2nd order and 3rd order in derivatives. The Majorana representation has been used.
- We obtained dynamical invariants for the free Barut field on the classical and quantum level.
- We found relations with other models (such as the Feynman-Gell-Mann equation).
- As a result of analysis of dynamical invariants, we can state that at the free level the term $\sim \alpha_3 \partial_\mu \bar{\Psi} \sigma_{\mu\nu} \partial_\nu \Psi$ in the Lagrangian does not contribute.
- However, the interaction terms $\sim \alpha_3 \bar{\Psi} \sigma_{\mu\nu} \partial_\nu \Psi A_\mu$ will contribute when we construct the Feynman diagrams and the $S$-matrix. In the curved space (the 4-momentum Lobachevsky space) the influence of such terms has been investigated in the Skachkov works [12]. Briefly, the contribution will be such as if the 4-potential were interact with some “renormalized” spin. Perhaps, this explains, why did Barut use the classical anomalous magnetic moment $g \sim 4\alpha/3$ instead of $\frac{1}{2\pi}$.

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