On the large $N$ limit of $SU(N)$ lattice gauge theories in five dimensions

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Abstract

We develop the necessary tools for computing fluctuations around a mean-field background in the context of $SU(N)$ lattice gauge theories in five dimensions. In particular, expressions for the scalar observable and the Wilson Loop are given. As an application, using these observables we compute a certain quantity $k_5$ that can be viewed as Coulomb’s constant in five dimensions. We show that this quantity becomes independent of $N$ in the large $N$ limit. Furthermore, the numerical value of $k_5$ we find for $SU(\infty)$ deviates by 17% from its value predicted by holography.
1 Introduction

In four dimensions and in infinite volume $SU(N)$ gauge interactions are renormalizable and always confining. Five-dimensional (5d) $SU(N)$ gauge theories are non-renormalizable and trivial at the perturbative limit. Their weak coupling phase is Coulomb and they have a first order phase transition at a critical value of the coupling. Beyond the phase transition there is a confined phase. The appropriate analytical method to describe these theories near the phase transition from the side of the Coulomb phase is the Mean-Field (MF) expansion [1]. The motivation to consider these theories, beyond the possibility of the existence of a physical fifth dimension, is that the MF method becomes more accurate as the dimension of space-time increases. This gives us the opportunity to develop a trustable analytical probe away from the perturbative point. Then one can look at regimes where the system is dimensionally reduced and build effectively four-dimensional (4d) models. This would not be possible directly in four dimensions as the MF background vanishes identically in the confined phase. As we argue here however, there are interesting theoretical issues that can be discussed already in five dimensions, in particular related to the large $N$ limit of these theories and their possible holographic description.

In [2] the five-dimensional $SU(2)$ MF formalism was developed, formulated on a periodic and anisotropic lattice. We will be using this work’s notation and results extensively. In a first application [3] it was shown that there is a regime on the phase diagram where the system reduces dimensionally by localizing the gauge interactions on four-dimensional hyperplanes. Interactions on the hyperplanes are confining and the string tension as well as some of its corrections, such as the Luscher term, were computed. The localization of the gauge interactions in this regime was consequently supported also by Monte Carlo simulations in [4]. In the meantime there has been a steadily increasing activity in lattice simulations of five dimensional gauge theories [5]. In [6] the $SU(2)$ MF model with orbifold boundary conditions is constructed, the goal being a non-perturbative description of Gauge-Higgs Unification. These (semi)analytical computations complement earlier lattice orbifold Monte Carlo simulations [7].

Another application of the MF formalism, not relying on dimensional reduction and one which will be of our main interest here, was presented in [8] where a certain quantity, an analogue of Coulomb’s constant in five dimensions was computed. It is derived from the five-dimensional Coulomb static potential $V_5(r) = \text{const.} - c_2/r^2$, where $c_2$ is the static charge, as follows:

$$k_5 = \frac{c_2}{a^2} \beta N^2,$$

(1)

where $c_2 = c_2/a$, with $a$ the lattice spacing and $\beta$ the lattice coupling. The result, obtained from $SU(2)$ gauge theory on periodic and isotropic lattices, was compared to a holographic computation of the same quantity [9]. There it was argued that this quantity, at least in the large $N$ limit, is in fact $N$-independent and this could justify a comparison of the $N = 2$ lattice result with the large $N$ gravity result. The two calculations of $k_5$ showed a numerical agreement of $2\%$ [8], suggesting that indeed, even if $k_5$ is $N$-dependent, its dependence on $N$ is probably rather weak.
In this work we extend the $SU(2)$ MF formalism of [2, 3] to $SU(N)$. In [1] one can find expressions for the $SU(N)$ MF propagator on the isotropic lattice in the axial gauge. Here we first extend the results of [1] by computing the propagator in a covariant gauge on the anisotropic lattice. Moreover, we compute the Wilson Loop and the scalar observables, that have not been computed before for $SU(N)$. As a first application, these quantities allow us to compute $k_5$ for large $N$ and verify its weak $N$-dependence suggested by holography.

2 Mean-field formalism

The general formalism of the MF expansion was presented in detail in [2, 6] and we will not repeat it here. For details one can consult these references together with [1]. One is interested in physical observables, schematically written as

$$\langle O \rangle = \frac{1}{Z} \int DU \ O[U] e^{-S_W[U]}.$$  \hspace{1cm} (2)

Here $S_W[U]$ is the usual Wilson plaquette action with anisotropic couplings $\beta_4 = \beta / \gamma$ along four-dimensional hyperplanes and $\beta_5 = \beta \gamma$ along the fifth dimension. In this notation $\beta$ is referred to as the lattice coupling and $\gamma$ as the anisotropy parameter. The isotropic case is obtained for $\gamma = 1$. To first order in the MF expansion the expectation value of an observable takes the form

$$\langle O \rangle = O[V] + \frac{1}{2} \text{tr} \left\{ \frac{\delta^2 O}{\delta V^2} \right\} V^{-1},$$  \hspace{1cm} (3)

with

$$K = -K^{(hh)} + K^{(tv)} + K^{(gf)}.$$ \hspace{1cm} (4)

the lattice propagator. $K^{(hh)}$ is the second derivative of the MF effective action with respect to auxiliary degrees of freedom $H$ and $K^{(tv)}$ is the second derivative of the Wilson plaquette action with respect to the link variables $V$. $K^{(gf)}$ is the gauge fixing term. $O$ is a gauge invariant operator and its second derivative with respect to the $H$-variables contracted against the propagator and evaluated in the MF background defines an expectation value. The connected version of the latter is defined as $O^c(t) = O(t_0 + t)O(t_0)$ and from it the correlator

$$C(t) = \langle O^c(t) \rangle = \langle O(t_0 + t) \rangle \langle O(t_0) \rangle$$ \hspace{1cm} (5)

is formed. The ground state mass associated with the operator $O$, to first order in the fluctuations, is then extracted from

$$m = \lim_{t \to \infty} \ln \frac{C(t)}{C(t - 1)}.$$ \hspace{1cm} (6)

2.1 Some $SU(N)$ integrals

Here we review some facts related to $SU(N)$ integrals [1]. Let $f(U)$ be a function of the $SU(N)$ link variables $U$ such that $f(U) = f(VUV^{-1}), \ V \in SU(N)$. Also let $\chi_r(U)$ be the character
associated with the irreducible representation $r$ of $SU(N)$ that the group element $U$ is expressed in. Then,

$$f(U) = \sum_r f_r \chi_r(U), \quad f_r = \int DU f(U) \chi_r^*(U)$$  \hspace{1cm} (7)

A special case is the function

$$e^{\frac{\hbar}{N} \text{Re tr}(U)} = \sum_r c_r \chi_r(U).$$  \hspace{1cm} (8)

We express the irrep $r$ in terms of a set of positive integers as [10]

$$r \rightarrow \{\lambda^{(r)}\} = \{\lambda_1^{(r)} \geq \lambda_2^{(r)} \geq \lambda_3^{(r)} \geq \cdots \geq \lambda_N^{(r)} = 0\}.$$  \hspace{1cm} (9)

We will need the $\lambda^{(r)}$ only for the symmetric and anti-symmetric representations. They are:

$$\lambda^{(2)} = (2, 0, \cdots, 0, 0)$$
$$\lambda^{(1,1)} = (1, 1, 0, \cdots, 0, 0)$$  \hspace{1cm} (10)

respectively. Then,

$$e^{\frac{\hbar}{N} \text{Re tr}(U)} = \sum_{\{\lambda^{(r)}\}} \chi_{\{\lambda^{(r)}\}}(U) \sum_{n=-\infty}^{+\infty} \det \left\{ I_{\lambda^{(r)}}, -j+i+n \left( \frac{\hbar}{N} \right) \right\},$$  \hspace{1cm} (11)

with $i, j = 1, \cdots, N$ and $I_q$ the Bessel function of order $q$. Using that

$$\int DU \chi_{\{\lambda^{(r)}\}}(U) = \delta_{\{\lambda^{(r)}\},\{0\}}$$  \hspace{1cm} (12)

we have

$$\int DU e^{\frac{\hbar}{N} \text{Re tr}(U)} = \sum_{\{\lambda^{(r)}\}} \delta_{\{\lambda^{(r)}\},0} \sum_{n=-\infty}^{+\infty} \det \left\{ I_{\lambda^{(r)}}, -j+i+n \left( \frac{\hbar}{N} \right) \right\}$$
$$= \sum_{n=-\infty}^{+\infty} \det \left\{ I_{-j+i+n} \left( \frac{\hbar}{N} \right) \right\}.$$  \hspace{1cm} (13)

An example is $N = 2$ where

$$\int DU e^{\frac{\hbar}{2} \text{Re tr}(U)} = \sum_n \det \begin{pmatrix} I_n & I_{n-1} \\ I_{n+1} & I_n \end{pmatrix}.$$  \hspace{1cm} (14)

In a matrix representation we can write a link in the irrep $\{\lambda^{(r)}\}$ as the matrix

$$U(n', M') \rightarrow D^{\{\lambda^{(r')}\}}_{\alpha'\beta'}(n', M').$$  \hspace{1cm} (15)

The character of $\{\lambda^{(r)}\}$ can then be written as

$$\chi_{\{\lambda^{(r)}\}}(U) = D^{\{\lambda^{(r)}\}}_{\alpha\alpha}.$$  \hspace{1cm} (16)

We note the useful orthogonality relation:

$$\int DU D^{\{\lambda^{(r')}\}}_{\alpha'\beta'} D^{\{\lambda^{(r)}\}}_{\alpha\alpha} = \delta_{\{\lambda^{(r')}\},\{\lambda^{(r)}\}} \delta_{\alpha'\beta'} \frac{1}{d_{\{\lambda^{(r)}\}}}. $$  \hspace{1cm} (17)
2.2 $K^{(hh)}$

Let $U \in SU(N)$ the link (with explicit indices $U_{\alpha \beta}(n, M)$) and $H$ an $N \times N$ complex matrix. The indices $\alpha, \beta = 1, \ldots, N$ are the gauge indices, $n$ is the location of the link in the lattice and $M$ is its direction. Define

$$\zeta(H) = \int DU e^{\frac{1}{N} \text{Re} \text{tr}(UH)}$$

$$u(H) = -\log \zeta(H).$$

We are after derivatives of $u(H)$ with respect to $H_{\alpha'\beta'}(n', M')$, evaluated in the mean-field background

$$V = \overline{V}1, \quad H = \overline{H}1$$

with $1$ the $N \times N$ unit matrix. We define the basic integral

$$\zeta_0 = \int DU e^{\frac{h_0}{N} \text{Re \text{tr}}(U)} = \sum_{n=-\infty}^{+\infty} \det \left\{ I_{-j+i+n} \left( \frac{h_0}{N} \right) \right\},$$

where in a given parametrization $V = v_0$ and $H = h_0$. We start from the first derivative of $u(H)$

$$u'_{\alpha'\beta'} = \frac{\partial u(H)}{\partial H_{\beta'\alpha'}(n', M')}|_{H=\overline{H}1} = -\frac{1}{\zeta} \frac{\partial \zeta(H)}{\partial H_{\beta'\alpha'}(n', M')}|_{H=\overline{H}1}$$

$$= -\frac{1}{\zeta_0} \int DU \frac{\partial}{\partial H_{\beta'\alpha'}(n', M')} \left[ \frac{1}{N} \text{Re} \text{tr}(UH) \right] e^{\frac{1}{N} \text{Re} \text{tr}(UH)}|_{H=\overline{H}1}$$

$$= -\frac{1}{\zeta_0} \zeta_1(\alpha', \beta', n', M') = -\frac{1}{\zeta_0} \frac{1}{N} \zeta_1 \delta_{\alpha'\beta'},$$

where

$$\zeta_1(\alpha', \beta', n', M') = \int DU U_{\alpha'\beta'}(n', M') e^{\frac{h_0}{N} \text{Re \text{tr}}(U)} \equiv \delta_{\alpha'\beta'} \frac{\zeta_1}{N}$$

$$\zeta_1 = \int_{SU(N)} dU (\text{Tr} U) e^{\frac{h_0}{N} \text{Re \text{Tr}U}}.$$ (22)

In the above expression we have dropped the argument $(n', M')$ from $\zeta_{1,\alpha'\beta'}$ because the background is uniform. In the anisotropic background the effect of the anisotropy is encoded in the vev $\overline{H}$ (it will be $h_0$ along the $\mu$-directions and $h_{05}$ along the fifth dimension). The second derivative of $u(H)$ is

$$u''_{\alpha'',\beta'',\alpha',\beta'} = \frac{\partial}{\partial H_{\beta''\alpha''}(n'', M'')} \frac{\partial}{\partial H_{\beta'\alpha'}(n', M')} u(H)|_{H=\overline{H}1}$$

$$= \frac{1}{N^2 \zeta_0^2} \left[ \zeta_1(\alpha', \beta', n', M') \zeta_1(\alpha'', \beta'', n'', M'') - \zeta_0 \zeta_2(\alpha', \beta', n', M'; \alpha'', \beta'', n'', M'') \right]$$

(23)

with

$$\zeta_2(\alpha', \beta', n', M'; \alpha'', \beta'', n'', M'') =$$

$$\int DU \left[ \frac{\partial}{\partial H_{\beta''\alpha''}(n'', M'')} \text{Re \text{tr}}(UH) \right] \left[ \frac{\partial}{\partial H_{\beta'\alpha'}(n', M')} \text{Re \text{tr}}(UH) \right] e^{\frac{1}{N} \text{Re \text{tr}(UH)}}|_{H=\overline{H}1}$$

(24)
that is,
\[\zeta_2(\alpha', \beta', n', M'; \alpha'', \beta'', n'', M'') = \int DU U_{\alpha' \beta'}(n', M') U_{\alpha'' \beta''}(n'', M'') e^{\frac{h_0}{N} \Re \text{Tr}(U)}.\]  

Next, we compute the integrals in Eq. (22) and Eq. (25) respectively. We will make repeated use of the notations:
\[D_{n,N}(z) \equiv \det [I_{n+j-i}(z)]_{1 \leq i,j \leq N}, \quad D_n \equiv D_{n,N}(z)_{z = \frac{h_0}{N}}.\]
\[D'_n = \frac{dD_{n,N}(z)}{dz} \bigg|_{z = \frac{h_0}{N}}, \quad D''_n = \frac{d^2D_{n,N}(z)}{dz^2} \bigg|_{z = \frac{h_0}{N}}.\]

A kind of generating function for SU(N) has been presented in [11]. It reads:
\[G_{SU(N)}|_{(c,d)} \equiv \int_{SU(N)} dU e^{c \text{Tr}(U) + d \text{Tr}(U)} = \sum_{n=-\infty}^{+\infty} \left( \frac{d}{c} \right)^{\frac{N+n}{2}} \det \left[ I_{n+j-i} \left( 2\sqrt{cd} \right) \right]_{1 \leq i,j \leq N}.\]

We note the following consequences:
\[\frac{\partial G_{SU(N)}}{\partial c} \bigg|_{\frac{h_0}{2N}, \frac{h_0}{2N}} = \int_{SU(N)} dU (\text{Tr}U) e^{\frac{h_0}{N} \Re \text{Tr}(U)} = \sum_{n=-\infty}^{+\infty} \left[ D'_n - \frac{nN^2}{h_0} D_n \right] \]
\[\frac{\partial^2 G_{SU(N)}}{\partial c^2} \bigg|_{\frac{h_0}{2N}, \frac{h_0}{2N}} = \int_{SU(N)} dU (\text{Tr}U^2) e^{\frac{h_0}{N} \Re \text{Tr}(U)} \]
\[= \sum_{n=-\infty}^{+\infty} \left[ D''_n - \frac{N^3}{h_0^2} (1 + 2nN) D'_n + \frac{2nN^3}{h_0^2} \left( \frac{nN}{2} + 1 \right) D_n \right] \]
\[\frac{\partial^2 G_{SU(N)}}{\partial d^2} \bigg|_{\frac{h_0}{2N}, \frac{h_0}{2N}} = \int_{SU(N)} dU (\text{Tr}U^2) e^{\frac{h_0}{N} \Re \text{Tr}(U)} \]
\[= \sum_{n=-\infty}^{+\infty} \left[ D''_n - \frac{N^3}{h_0^2} (1 - 2nN) D'_n + \frac{2nN^3}{h_0^2} \left( \frac{nN}{2} - 1 \right) D_n \right] \]
\[\frac{\partial^2 G_{SU(N)}}{\partial c \partial d} \bigg|_{\frac{h_0}{2N}, \frac{h_0}{2N}} = \int_{SU(N)} dU (\text{Tr}U) (\text{Tr}U^*) e^{\frac{h_0}{N} \Re \text{Tr}(U)} = \sum_{n=-\infty}^{+\infty} \left[ D''_n + \frac{N}{h_0} D'_n - \frac{n^2N^4}{h_0^2} D_n \right].\]

Since \(D_n\) is even under \(n \rightarrow -n\), all terms that are odd in \(n\) vanish in the sums. For the integral \(\zeta_1(\alpha', \beta', n', M')\) we have
\[\zeta_1(\alpha', \beta', n', M') = \frac{1}{N} \int_{SU(N)} dU [\text{Tr}U(n', M')] e^{\frac{h_0}{N} \Re \text{Tr}(U)} = \frac{\delta_{\alpha' \beta'}}{N} \zeta_1, \]
\[\zeta_1 = \sum_{n=-\infty}^{+\infty} \left[ D'_n - \frac{nN^2}{h_0} D_n \right].\]
Thus, in this notation,
\[ \zeta_0 = \sum_{n=-\infty}^{\infty} D_n, \quad \zeta_1 = \sum_{n=-\infty}^{\infty} D'_n. \] (31)

For the integral \( \zeta_2 \) we have to distinguish three different cases. The first is when we take two derivatives with respect to \( H' \) and \( H'' \), the second is when we take one derivative with respect to \( H' \) and one with respect to \( H'' \) and the third is when we take derivatives with respect to \( H' \) and \( H'' \). We call the corresponding integrals as \( \zeta_2^{00} \), \( \zeta_2^{0+} \) (\( \zeta_2^{0+} = (\xi_2^{0+})' \)) and \( \zeta_2^{++} (= \xi_2^{00}) \) respectively. We take all \( H' \)’s in the fundamental representation. Again, we can drop all space-time and directional arguments since all the information about the background is contained in the exponent of the integrand in Eq. (25). We have

\[ \zeta_2(\alpha', \beta', n', M'; \alpha'', \beta'', n'', M'') = \int DU \ U_{\alpha'\beta'}(n', M') \ U_{\alpha''\beta''}(n'', M'') \ e^{\frac{h_0}{2} \text{Re tr}(U)} \] (32)

\[ = c_1^{00} \delta_{\alpha'\beta'} \delta_{\alpha''\beta''} + c_2^{00} \delta_{\alpha'\beta''} + c_2^{00} \delta_{\alpha''\beta'} . \] (33)

We have set the integral equal to the two possible tensor structures, with coefficients to be determined. Contracting with \( \delta_{\alpha'\beta'} \delta_{\alpha''\beta''} \) both sides we obtain

\[ \int DU \ [\chi(U)^2] e^{\frac{h_0}{2} \text{Re tr}(U)} = c_1^{00} N^2 + c_2^{00} N = I_{\alpha\alpha;\beta\beta}, \] (34)

while contracting with \( \delta_{\alpha'\beta''} \delta_{\alpha''\beta'} \) we obtain

\[ \int DU \ [\chi(U^2)] e^{\frac{h_0}{2} \text{Re tr}(U)} = c_1^{00} N + c_2^{00} N^2 = I_{\alpha\beta;\beta\alpha} . \] (35)

Thus the coefficients can be expressed in terms of the integrals:

\[ c_1^{00} = \frac{NI_{\alpha\alpha;\beta\beta} - I_{\alpha\beta;\beta\alpha}}{N(N^2 - 1)}, \quad c_2^{00} = \frac{NI_{\alpha\beta;\beta\alpha} - I_{\alpha\alpha;\beta\beta}}{N(N^2 - 1)} . \] (36)

Following [11] we find that

\[ I_{\alpha\alpha;\beta\beta} = \int DU \ [\chi_2(U) + \chi_{11}(U)] e^{\frac{h_0}{2} \text{Re tr}(U)} \]

\[ = \sum_{n=-\infty}^{+\infty} \text{det} \ I_{j-i+n+\lambda_i^{(2)}} \left( \frac{h_0}{N} \right) + \sum_{n=-\infty}^{+\infty} \text{det} \ I_{j-i+n+\lambda_i^{(11)}} \left( \frac{h_0}{N} \right), \] (37)

\[ I_{\alpha\beta;\beta\alpha} = \int DU \ [\chi_2(U) - \chi_{11}(U)] e^{\frac{h_0}{2} \text{Re tr}(U)} \]

\[ = \sum_{n=-\infty}^{+\infty} \text{det} \ I_{j-i+n+\lambda_i^{(2)}} \left( \frac{h_0}{N} \right) - \sum_{n=-\infty}^{+\infty} \text{det} \ I_{j-i+n+\lambda_i^{(11)}} \left( \frac{h_0}{N} \right). \] (38)

These integrals will be computed numerically. The ++ case does not require extra work, we have that \( c_1^{++} = c_1^{00} \) and \( c_2^{++} = c_2^{00} \). The 0+ case needs though separate computation. The integral to be computed is

\[ \zeta_2^{(0+)}(\alpha'; \beta'; \alpha''; \beta'') \equiv J_{\alpha'\beta';\alpha'';\beta''} \equiv \int DU \ U_{\alpha'\beta'}(n', M') U_{\alpha''\beta''}(n'', M'') e^{\frac{h_0}{2} \text{Re tr}(U)} \] (39)
where the link is parametrized as $U$ in the mean-field background and leaving the gauge index structure aside for the moment, we will rewrite this expression in a more useful basis. To begin, evaluating second derivatives

The Euclidean indices is trivial (see [2]). We introduce the index $K$ which yields:

Following the same steps as before we find:

We can finally write down $K^{(hh)}$ directly in momentum space, since the Fourier transformation is trivial (see [2]). We introduce the index $q', q'' = 0, +$. Then,

The Euclidean indices $M'M''$ are shown explicitly in (diagonal) matrix form. Before we continue, we will rewrite this expression in a more useful basis. To begin, evaluating second derivatives in the mean-field background and leaving the gauge index structure aside for the moment, corresponds to

where the link is parametrized as $U = u_0 + iu_A$. Therefore the Hermitian (H) and anti-Hermitian (AH) channels are

On the other hand, in order to handle the gauge structure, one introduces the projectors on the singlet and adjoint representations

where

(40)

(41)

(42)

(43)

(44)

(45)

(46)

(47)
and the new coefficients
\[ c^H_a = c^{(0)}_a + c^{(0+)}_a, \]
\[ c^{AH}_a = c^{(0)}_a - c^{(0+)}_a \]  
with \( a = 1, 2 \) and
\[ A^H = - \frac{1}{N^2 \zeta_0} (c^2 + N c^1 - \frac{2 \xi_1}{N \zeta_0}), \quad B^H = \frac{1}{N^2 \zeta_0} c^2 \]
\[ A^{AH} = - \frac{1}{N^2 \zeta_0} (c^2 + N c^{AH}_1), \quad B^{AH} = \frac{1}{N^2 \zeta_0} c^{AH} \]  
(49)

Now we can write the non-vanishing components of \( K^{(hh)} \)
\( (p', \alpha', \beta'; p'', \alpha'', \beta'') \) simply as
\[ K^{(hh)}_{\mu\mu'}(p', \alpha', \beta'; p'', \alpha'', \beta'') = \delta_{\mu\mu'} \left( A^2 P^{(S)}_{\alpha'\beta'\alpha''\beta''} + B^2 P^{(A)}_{\alpha'\beta'\alpha''\beta''} \right) \tau_{0, \alpha} \delta_{\mu'\mu''} \]
\[ K^{(hh)}_{\delta\delta}(p', \alpha', \beta'; p'', \alpha'', \beta'') = \delta_{\mu'\mu''} \left( A^2 P^{(S)}_{\alpha'\beta'\alpha''\beta''} + B^2 P^{(A)}_{\alpha'\beta'\alpha''\beta''} \right) \tau_{05, \alpha} \]  
(50)

where \( z = H, AH \). One can check that for any \( N \)
\[ B^{AH} = \frac{2 u'}{N h_0} = - \frac{2 \pi_0}{N h_0} = - \frac{2}{x N^3 \zeta_0(x)}. \]  
(51)

We have defined \( x = \overline{h}_0 / N \) and used the background solution given later in Eq. (67).

For large \( N \), the numerical computation of the determinants involved in the coefficients is plagued by instabilities due to large cancellations. It is then useful to consider the large \( H \) expansion of \( u(H) \) [1]
\[ u(H) = \ln \left[ \prod_{k=1}^{N-1} \frac{1}{k!} \right] + \frac{N^2 - 2}{2} \ln N + \hat{H} - \frac{N^2 - 1}{2} \ln \hat{H} - \frac{N^2 - 1}{8 \hat{H}} + \cdots \]  
(52)

with \( \hat{H} = \frac{u(H)}{N} \), which is equal to \( \overline{h}_0 \) in the mean-field background. Noticing in addition that as \( N \) increases \( \beta_c \) and \( \overline{h}_0 \) also increase we can conclude that already for relatively low values of \( N \) (\( N = 5, 6, \cdots \)) the asymptotic expansion of \( u(H) \) is a good approximation.

In this limit we can use the approximate expressions
\[ A^H \simeq - \frac{N}{\bar{h}_0}, \quad B^H \simeq 0 \]
\[ A^{AH} \simeq 0, \quad B^{AH} = - \frac{2 \pi_0}{N \bar{h}_0} \]  
(53)

and for later reference
\[ \pi_0 = \frac{\zeta_1}{N \zeta_0} \simeq 1 - \frac{N^2 - 1}{2 \overline{h}_0}. \]  
(54)

For \( N = 2 \), \( B^H \) and \( A^{AH} \) are identically zero and for large \( N \), they become negligible [1]. Notice also that the expression for \( B^{AH} \) is exact for any \( N \), see Eq. (51). For instance, on the isotropic torus we have
\[ B^{AH} = - \frac{1}{4} \frac{1}{N \beta \overline{\pi}_0^2}. \]  
(55)

When \( \overline{h}_0 \to \infty \) we have \( \pi_0 \simeq 1 \) and we obtain the asymptotic expression \( B^{AH} = - \frac{2}{N \overline{h}_0} \).
2.3 $K^{(vv)}$

The space-time structure of $K^{(vv)}$ for $SU(N)$ is similar to the $SU(2)$ case. In $q'-q''$ space it is a two by two matrix, just like $K^{(hh)}$. The hermitian (anti-hermitian) channel of the $SU(N)$ model corresponds to the hermitian (anti-hermitian) channel of the $SU(2)$ model. The gauge index structures of $K^{(vv)}$ and of the gauge fixing term can be seen from the second derivative of the Wilson plaquette action (suppressing all but the group indices)

$$\frac{\partial^2 \text{tr} [U_{a\beta}U_{\gamma\delta}U_{a\beta}U_{\gamma\delta}]}{\partial U_{a'\beta'}\partial U_{a''\beta''}} \sim \delta_{a'\beta'}\delta_{a''\beta''} N$$

and the gauge fixing term

$$\frac{\partial^2 \text{tr} [U_{a\beta}U_{a\alpha}]}{\partial U_{a'\beta'}\partial U_{a''\beta''}} \sim \delta_{a'\beta'}\delta_{a''\beta''} N$$

respectively.

We define the bond shifting operators

$$\Delta_{AH} = -(N^\beta_\gamma) \cdot \left( \begin{array}{cccccc} \sum' c_{M'} - \frac{1}{\xi} s_0' & y s_0' s_1 / 2 & y s_0' s_2 / 2 & y s_0' s_3 / 2 & y s_0' s_5' s_5 / 2 & \gamma^2 \frac{\gamma_{\alpha\beta}}{v_0} \\
 y s_1' s_0' / 2 & \sum' c_{M'} - \frac{1}{\xi} s_1' & y s_1' s_2 / 2 & y s_1' s_3 / 2 & y s_1' s_5' s_5 / 2 & \gamma^2 \frac{\gamma_{\alpha\beta}}{v_0} \\
 y s_2' s_0' / 2 & y s_2' s_1 / 2 & \sum' c_{M'} - \frac{1}{\xi} s_2' & y s_2' s_3 / 2 & y s_2' s_5' s_5 / 2 & \gamma^2 \frac{\gamma_{\alpha\beta}}{v_0} \\
 y s_3' s_0' / 2 & y s_3' s_1 / 2 & y s_3' s_2 / 2 & \sum' c_{M'} - \frac{1}{\xi} s_3' & y s_3' s_5' s_5 / 2 & \gamma^2 \frac{\gamma_{\alpha\beta}}{v_0} \\
 y s_5' s_0' s_5 / 2 & y s_5' s_1 s_1' / 2 & y s_5' s_2 s_2' / 2 & y s_5' s_3 s_3' / 2 & \gamma^2 \frac{\gamma_{\alpha\beta}}{v_0} & \gamma^2 \left( \sum' c_{M'} - \frac{1}{\xi} s_5' \right) \end{array} \right)$$

where

$$y = 2 - 1 / \xi, \quad y_5 = 2 \gamma^2 \frac{\gamma_{\alpha\beta}}{v_0} - \frac{\gamma}{\xi}$$

and

$$\Delta_{H} = -(N^\beta_\gamma) \cdot \left( \begin{array}{cccccc} \sum' c_{M'} & 2 c_0 / 2 c_{1 / 2} & 2 c_0 / 2 c_{2 / 2} & 2 c_0 / 2 c_{3 / 2} & 2 c_0 / 2 c_{5 / 2} & \gamma^2 \frac{\gamma_{\alpha\beta}}{v_0} \\
 2 c_{1 / 2} / c_0 & \sum' c_{M'} & 2 c_{1 / 2} / c_{2 / 2} & 2 c_{1 / 2} / c_{3 / 2} & 2 c_{1 / 2} / c_{5 / 2} & \gamma^2 \frac{\gamma_{\alpha\beta}}{v_0} \\
 2 c_{2 / 2} / c_0 & 2 c_{2 / 2} / c_{1 / 2} & \sum' c_{M'} & 2 c_{2 / 2} / c_{3 / 2} & 2 c_{2 / 2} / c_{5 / 2} & \gamma^2 \frac{\gamma_{\alpha\beta}}{v_0} \\
 2 c_{3 / 2} / c_0 & 2 c_{3 / 2} / c_{1 / 2} & 2 c_{3 / 2} / c_{2 / 2} & \sum' c_{M'} & 2 c_{3 / 2} / c_{5 / 2} & \gamma^2 \frac{\gamma_{\alpha\beta}}{v_0} \\
 2 c_{5 / 2} / c_0 & 2 c_{5 / 2} / c_{1 / 2} & 2 c_{5 / 2} / c_{2 / 2} & 2 c_{5 / 2} / c_{3 / 2} & \gamma^2 \frac{\gamma_{\alpha\beta}}{v_0} & \gamma^2 \sum' c_{M'} \end{array} \right).$$

We use the notation $s_{0/2} = \sin p_0 / 2$ etc. and $c_5 = \gamma^2 \frac{\gamma_{\alpha\beta}}{v_0} \cos (p_5')$. These expressions appear in [1] in the axial gauge and for isotropic lattices. Here we have generalized them to anisotropic lattices and we have fixed a covariant gauge parametrized by $\xi$ as in [2]. We have checked
numerically that our results do not depend on the value of $\xi$. Using $\delta_{\alpha'\beta'}\delta_{\alpha''\beta''} = P^{(S)} + P^{(A)}$ we can express $K^{(vv)}$ as

$$K^{(vv)}_{M'M''}(p', \alpha', \beta'; p'', \alpha'', \beta''; H) = \delta_{\alpha'\beta'}\delta_{\alpha''\beta''} \{ P^{(S)}_{\alpha'\beta';\alpha''\beta''} + P^{(A)}_{\alpha'\beta';\alpha''\beta''} \} \Delta_{H,M'M''}$$

$$K^{(vv)}_{M'M''}(p', \alpha', \beta'; p'', \alpha'', \beta''; AH) = \delta_{\alpha'\beta'}\delta_{\alpha''\beta''} \{ P^{(S)}_{\alpha'\beta';\alpha''\beta''} + P^{(A)}_{\alpha'\beta';\alpha''\beta''} \} \Delta_{AH,M'M''}.$$  \hspace{1cm} (61)

The $SU(N)$ propagator is then

$$K_{M'M''} = -K^{(hh)}_{M'M''} + K^{(vv)}_{M'M''}. \hspace{1cm} (62)$$

The notation here and from now on is that $K^{-1}_{M'M''}$ represents the $M'M''$th element of $K^{-1}$. The inverse of $K$ has four sectors, labeled by $z = H, AH$ and $w = S, A$. Schematically we represent these sectors as $K^{-1}(z, w)$ and express them in this basis as

$$K^{-1}(z, w) = K^{-1}(H, S)P^{(S)} + K^{-1}(H, A)P^{(A)} + K^{-1}(AH, S)P^{(S)} + K^{-1}(AH, A)P^{(A)}. \hspace{1cm} (63)$$

### 2.4 The phase diagram

In the anisotropic theory, there are two mean values for the links, $\bar{v}_0$ and $\bar{v}_{05}$ determined by the extremization of $(d = 5)$

$$S_{\text{eff}}[\overline{V}, \overline{H}] = -\beta_4 \frac{(d-1)(d-2)}{2} [\overline{v}_0^3 - \beta_5 (d-1)\overline{v}_0^2 \overline{v}_{05} + (d-1)u(\overline{h}_0) + u(\overline{h}_{05}) + (d-1)\overline{h}_0 \overline{v}_0 + \overline{h}_{05} \overline{v}_{05}]. \hspace{1cm} (64)$$

Here $N = L^3N_5$ is the number of spatial lattice points with $N_5$ the number of points in the fifth dimension. Eq. (64) yields conditions identical in form to the $SU(2)$ case [2]

$$\overline{v}_0 = -u(\overline{h}_0)', \hspace{0.5cm} \overline{h}_0 = \frac{6\beta}{\gamma} \overline{v}_0^3 + 2\beta\gamma \overline{v}_0^2 \overline{v}_{05}$$

$$\overline{v}_{05} = -u(\overline{h}_{05})', \hspace{0.5cm} \overline{h}_{05} = 8\beta\gamma \overline{v}_0^2 \overline{v}_{05}, \hspace{1cm} (65)$$

where $u'$, according to Eq. (21) is (here we differentiate form $N = 2$)

$$u' = \text{tr}\{u_{\alpha'\beta'}\} = -\frac{1}{\zeta_0 N^2} N \zeta_1,$$  \hspace{1cm} (66)

that is

$$\overline{v}_0 = \frac{\zeta_1(x)}{N \zeta_0(x)}. \hspace{1cm} (67)$$

For $N = 2$ this reduces to the known result $\overline{v}_0 = I_2(2x)/I_1(2x)$.

One can immediately see that when all momenta vanish, in the $(AH, A)$ sector we have (for simplicity we take $\gamma = 1$), using Eq. (51):

$$K(AH, A) = -\left( \frac{N\overline{h}_0}{2u'} + 4N\beta\overline{v}_0^2 \right) \cdot 1 \hspace{1cm} (68)$$

which vanishes by Eq. (65). These are the five expected torons, which persist even when $\gamma \neq 1$. The various phases on the phase diagram are defined as follows:
• Confined phase: \( \tau_0 = 0, \tau_{05} = 0 \)
• Coulomb phase: \( \tau_0 \neq 0, \tau_{05} \neq 0 \)
• Layered phase: \( \tau_0 \neq 0, \tau_{05} = 0 \)

We will perform our computations in the Coulomb phase where the background is non-vanishing everywhere. In addition we will stay near the phase transition where we expect that cut-off effects are suppressed. The critical value of the coupling \( \beta_c \) that signals the end of the Coulomb phase moves towards to larger values as \( N \) increases. Apart from that, the phase diagram is qualitatively similar to the \( SU(2) \) phase diagram as described in [2].

3 Observables

To first order in the fluctuations we will be computing

\[
\langle O \rangle = O[\mathcal{V}] + \frac{1}{2} \text{tr} \left\{ \sum_{w=\Sigma,\Lambda} \sum_{z=H,\Lambda H} \left( \frac{\delta^2 O}{\delta V^2} \right)_{V=\tau_0} (z,w)(K^{-1})(z,w) \right\}.
\]

We will often call the above second derivative of the observable simply as \( O_2 \) which also can be expressed as

\[
O_2(z) = O_2(z, S)P^{(S)} + O_2(z, A)P^{(A)}.
\]

Then we can expand as

\[
\langle O \rangle = O[\mathcal{V}] + \frac{1}{2} \sum_{z,w} \text{tr} \{O_2(z,w)K^{-1}(z,w)\}
= O[\mathcal{V}]
+ \frac{1}{2} \text{tr} \left[ O_2(H, S)K^{-1}(H, S) + O_2(AH, S)K^{-1}(AH, S) \right] \text{tr}\{1\}_S
+ \frac{1}{2} \text{tr} \left[ O_2(H, A)K^{-1}(H, A) + O_2(AH, A)K^{-1}(AH, A) \right] \text{tr}\{1\}_A
= O[\mathcal{V}]
+ \frac{1}{2} \text{tr} \left[ O_2(H, S)K^{-1}(H, S) + O_2(AH, S)K^{-1}(AH, S) \right] N
+ \frac{1}{2} \text{tr} \left[ O_2(H, A)K^{-1}(H, A) + O_2(AH, A)K^{-1}(AH, A) \right] N(N^2 - 1).
\]

3.1 The scalar mass

The Euclidean structure is identical to that of the \( SU(2) \) model. We recall the \( SU(2) \) result (for \( SU(2) \) the gauge index takes the values \( \alpha' = 0, 1, 2, 3 \) where \( \alpha' = 0 \) represents the Hermitian-Singlet channel and \( \alpha' = 1, 2, 3 \) the anti-Hermitian-Adjoint channel) where only the Hermitian channel, free of zero modes, is present [2]

\[
C_S(t) = \frac{2}{N}(P_0)^2 \sum_{p_0} \cos(p_0 t) \sum_{p_5} |\Delta(N)(p_5)|^2 K_{55}^{-1} \left( (p_0, \bar{0}, p_5), \alpha' = 0; (p_0, \bar{0}, p_5), \alpha' = 0 \right)\).
\]
Recall that for $SU(2)$ for any observable there are at most the two sectors, $(H,S)$ and $(AH,A)$ contributing.

For $SU(N)$ the contribution of the adjoint channel vanishes identically for basically the same reason as for $SU(2)$, that is because taking only one derivative of the observable with respect to a fluctuation along the Lie algebra and evaluating it in the background gives $O_2(H,A) = O_2(AH,A) = 0$ in Eq. (71). Also, since the double derivative on the Polyakov loop that represents the scalar has the gauge index structure

$$\delta_{\alpha'}^{\alpha''} \delta_{\beta'}^{\beta''} = P^{(S)} + P^{(A)}$$

and the schematic structure

$$C_S(t) \sim NO_2(H,S)K^{-1}(H,S) + NO_2(AH,S)K^{-1}(AH,S),$$

the final expression for the correlator becomes

$$C'_S(t) = \frac{NN^2}{N} \frac{(P_0)^2}{v_0^2} \sum_{p_0'} \cos(p_0't) \delta_{p_0,0} \delta_{p_0',0} \left[ K_{55}^{-1}(p'; p; H, S) + K_{55}^{-1}(p'; p; AH, S) \right]$$

where $k = 1, 2, 3$. Notice that for $SU(2)$ the coefficient $A^{AH}$ vanishes identically, so only the $(H,S)$ sector contributes, consistently with our comment above. We have defined

$$\Delta^{(N_5)}(p) = \frac{1}{v_0} \sum_{r=0}^{N_5-1} e^{ip(r+1/2)} = N_5 \frac{\delta_{p_0,0}}{v_0},$$

where $r = 0$ labels the first link along the fifth dimension, $r = 1$ the second, etc. $P_0$ is the mean-field length of the Polyakov loop representing the scalar observable. It is easy to see that the toron $p_0 = 0$ adds a zero to the scalar correlator since it contributes a time independent constant. Given that the Hermitian channel of the propagator does not contain any zero modes, there is no toron. The plateau in the time decay of this observable yields the mass of the mass of the scalar observable $a_4 m_S$ in lattice units.

The vector mass is a purely finite volume effect (in particular it does not depend on $N$) so we can use directly the $SU(2)$ relation [2, 3]

$$a_4 m_V = \frac{4\pi}{L}.$$  

3.2 The Wilson Loop

Here we compute the Wilson Loop (WL) along four-dimensional hyperplanes. Again, the space-time structure is the same as for $SU(2)$, see [2].

For an observable represented by a single loop it makes a difference whether the two links that are removed (by the two derivatives) are pointing in the same or opposite directions. In the
former case there is a relative minus sign with respect to the latter. A case where this remark applies is the Wilson Loop. The schematic structure in this case is

\begin{align*}
\text{exchange} : & \quad (00) + (0+) \\
\text{tadpole} : & \quad (00) - (0+) \quad (78)
\end{align*}

corresponding to the two diagrams that contribute to leading non-trivial order, an exchange between the two spatial legs and the tadpole on a given spatial leg. For general \(N\), in principle, all four sectors contribute. The final result is:

\begin{equation}
V_5(r) = -\frac{1}{7^6} \frac{N}{2N} \sum_{k=1}^{3} \sum_{p'} \delta_{p_0,0} \cdot 
\left\{ \left[ \left( \frac{1}{3} \cos(p'_k r) + 1 \right) K_{00}^{-1}(p'; p'; H, S) + (N^2 - 1)(\frac{1}{3} \cos(p'_k r) - 1) K_{00}^{-1}(p'; p'; AH, A) \right] \right.
\left. \left[ \left( \frac{1}{3} \cos(p'_k r) - 1 \right) K_{00}^{-1}(p'; p'; AH, S) + (N^2 - 1)(\frac{1}{3} \cos(p'_k r) + 1) K_{00}^{-1}(p'; p'; H, A) \right] \right\} \quad (79)
\end{equation}

where we have dropped the irrelevant additive constant originating from the zeroth order contribution.

For \(SU(2)\) only the first two sectors contribute. In particular, the two terms in the last line vanish because \(A^{AH} = B^H = 0\). Finally the toron in the propagator in the \((AH, A)\) sector is cancelled by the zero in the observable in the same sector since the factor \(\cos(p'_k r) - 1\) vanishes for \(p = 0\). Notice that for \(SU(2)\) the coefficient \(A^{AH}\) is not being used by neither the scalar nor by the WL.

4 A simple application: Coulomb’s constant in five dimensions

We present a simple application of the formalism we developed, on the isotropic lattice. As mentioned in the Introduction, the dimensionless quantity \(k_5\) defined in Eq. (1) is computable both in lattice gauge theory and in gravity. An agreement between the two computations would constitute evidence for the validity of the holographic conjecture [12]. We remind that the holographic computation of \(k_5\) defined in Eq. (1) yields [9, 8] (expressed in lattice parameters) the number

\begin{equation}
k_5 = \left[ \frac{B(2/3, 1/2)}{3\pi^{2/3}} \right]^3 = 0.0649 \quad (80)
\end{equation}

with \(B(x, y)\) the Euler Beta function, for any \(N\) for which the computation is valid. This is expected to be the case in the large \(N\) limit, therefore any agreement with the lattice computation of the same quantity is expected to occur at least in this limit.

From the point of view of the lattice, we define trajectories on the phase diagram that approach the bulk first order phase transition, with

\begin{equation}
q = \frac{a_4 m_V}{a_4 m_S} \quad (81)
\end{equation}

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kept constant [3]. The quantity $k_5 = \bar{c}_2 \beta / N^2$ on the isotropic lattice depends on $\beta$ and $L$. Using these trajectories, it is however straightforward to obtain its infinite $L$ value via extrapolation. Here we choose $q = 2$ even though the infinite $L$ extrapolations of $k_5$ are expected to be essentially $q$-independent.

In figure 1 we compare for $SU(5)$ the results obtained from the asymptotic expansions Eq. (53) for the coefficients in $K^{(hh)}$ together with the asymptotic form for the background Eq. (54) against the ones obtained by the sum over determinants Eq. (49). We expect that the former (latter) would be more reliable for large (small) values of $N$ and indeed for $N = 5$ the discrepancy between the two is already quite small. According to this result, for $N \leq 5$ we will be computing the coefficients using the sum over determinants method and for $N > 5$ using their asymptotic expressions.

In figure 2 we show the values obtained for $k_5$ for increasingly large values of $N$. For $N = 2, 3, 4, 5$ we use the sum over determinants expressions and for $N > 5$ we use their asymptotic forms. We also show quadratic fits to the data. We observe that the intercept increases slightly with $N$. The largest value depicted is $N = 100$. Data for larger $N$ than that become indistinguishable from the latter with naked eye. It is clear that the curves saturate for large $N$. This demonstrates that $k_5$ becomes $N$-independent as $N \to \infty$.

In figure 3 we depict the $N$ dependence of the intercept for $N \geq 10$ along with a quadratic fit. It cuts the vertical axis at $k_5 = 0.0757$, presumably the result for $SU(\infty)$. This should be compared against the holographic result 0.0649, resulting into a 17% disagreement. Evidently the $N = \infty$ value of $k_5$ is farther away compared to its $N = 2$ value which is just 2% away from the holographic result. This ”reverse” trend could be related to the order of the computations on both the gauge theory and gravity sides. The essential fact to keep in mind is that the mean-field prediction for $k_5$ in the large $N$ limit stabilizes to a value not too far away from the holographic prediction.

5 Discussion

In order that the comparison between the Mean-Field and Holographic computations makes sense, both should be reasonably good descriptions of the 5d $SU(N)$ gauge theory at intermediate to strong coupling. Assuming that from the holographic point of view this is a conjecture and provided that the Mean-Field is a good analytical tool at strong coupling, any agreement with the gravity side gives evidence for the validity of the holographic conjecture. Reversely, if we assume that the holographic duality holds, then any observable derived from the classical gravitational system should be in principle verifiable from the gauge theory side, provided that it can be computed reliably at the non-perturbative level. No matter what point of view is taken, a necessary condition is that the truncated Mean-Field expansion we have employed here is a good approximation to the non-perturbative system. We have mentioned in the Introduction some general arguments to this effect, such as the validity of the expansion improving as the
Figure 1: Comparison of asymptotic expressions (lower curve) versus sum over determinants (upper curve) for SU(5).

dimensionality of the system increases. The only thing that can decide about this issue is a comparison with Monte Carlo simulations. In fact, it is known for some time that the Mean-Field predicts the order of the bulk phase transition and even the numerical value of the associated critical coupling correctly [1]. Note that this is a prediction already at the level of the MF background, i.e. without taking into account fluctuations. In [4] a more sophisticated comparison between the MF and Monte Carlo methods was performed on the anisotropic $N = 2$ lattice. The MF predicts, via the Wilson Loop (which can be obtained in the MF only by taking into account fluctuations), dimensional reduction via localization for $\gamma \simeq 0.55$ [2, 3]. This was verified by MC simulations that showed [4] that in the same regime of the anisotropy, Polyakov Loops fluctuate independently along four-dimensional hyperplanes. In [6] the mass of observables with 4d scalar and vector quantum numbers are computed with orbifold boundary conditions along the fifth dimension (these however should not interfere with the issue of the convergence of the MF expansion). Also these observables can be computed with the MF method only in the presence of fluctuations. The agreement was seen to be again good. All these results give us an increasing confidence that the MF is indeed a good description of the 5d system at the non-perturbative level.

More specifically now, the computation of $k_5$ from the gravity side suggested that the dual gauge theory, at leading order in a large $N$ expansion a) must be not weakly coupled by the nature of the duality b) is in its 5d Coulomb phase since the static potential is of a $1/r^2$ form c) sits in a regime of its phase diagram where cut-off effects are suppressed since in the result there is no sign of the presence of a gauge theory regulator. The first observation forced us to
move away from the 5d perturbative point where the coupling is expected to go to zero when the cut-off is removed. Moving into the interior of the phase diagram leads us eventually on the bulk phase transition because everywhere else physics is heavily cut-off dominated. The second observation on the other hand prohibits us from crossing the phase transition. The only choice we are left with is to be as near as possible to the phase transition, which is precisely where we have computed $k_5$ from the gauge theory side. What is left to be addressed is why do we expect cut-off effects to be suppressed near the phase transition, or in lattice language, why do we generally expect the lattice spacing to decrease as $\beta \to \beta_c$. A milder version of this question, related to our quantity $k_5$ is what is its lattice spacing dependence near the phase transition. It is not easy to answer definitively the former version of the question. What one can observe though from the MF computation [2] is that the scalar mass in units of lattice spacing $a m_S$ decreases as the phase transition is approached. It is however a first order phase transition therefore most likely the lattice spacing remains finite and in fact one can see that $a m_S$ can be pushed down to approximately 0.1 but not much further. This means that finite lattice spacing effects will be present even at the phase transition, even though not very large. Regarding $k_5$, it has been

Figure 2: $k_5$ versus $1/L$ for $SU(2)$ (lowest curve), $SU(3)$, $SU(4)$, $SU(5)$, $SU(10)$, $SU(15)$, $SU(30)$ and $SU(100)$. 
Figure 3: Intercept at $L = \infty$ versus $1/N$ along with a quadratic fit, predicting the value $k_5 = 0.0757$ for $SU(\infty)$.

defined [8] by analogy to the non-abelian Coulomb’s constant in four dimensions, where one can see that by construction it is a cut-off independent quantity. Here, our working assumption was that the lattice spacing dependence in $k_5$ (via the product $\tau_2 \beta$) cancels at least near the phase transition, just like in 4d and according to what the gravitational result suggests. The fact that as we move closer to the phase transition the value of $k_5$ changes by very little, especially for large $N$, strongly supports this assumption. Notice that when $L$ is increased while keeping $q$ in Eq. (81) constant, takes us closer to the phase transition and thus changes the lattice spacing.

We now discuss other possible sources for the observed 17% discrepancy. From the gravity side, in infinite four-dimensional volume, there may be $\alpha'$ and finite $N$ corrections to the static potential. It would be interesting to see if it is possible to compute such corrections for the $D4$-brane background, which is the basis for the holographic computation of sect. 4. From the gauge theory side we also expect to have corrections. Already at leading order in the MF expansion, the computation of the coefficients $A^z, B^z$ involves truncations of series and of the ranks of the matrices the determinants of which determine the coefficients. Changing the order at which these are truncated would be a way to introduce error bars in the MF results for $N < 6$. 

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Another would be to compute the coefficient $\tilde{c}_2$ via local fits as in [3] as opposed to the global fits used here. This would require computing the Wilson Loops for increasing values of $L$ (starting from approximately $L = 200$) and reading off each time the plateau value for $\tilde{c}_2$. The variations in the plateau values would introduce additional error bars on the MF data. We did not do such an analysis here due to the lack of the necessary computing power. Finally, clearly there will be new effects if we change the order of the truncation of the MF expansion itself. Higher order effects will most certainly result in further corrections. It is hard to guess without further computations which of all possible corrections is mainly responsible for the discrepancy.

6 Conclusion

We developed the necessary formalism to perform the mean-field expansion to first non-trivial order, for five-dimensional, anisotropic $SU(N)$ lattice gauge theories in a covariant gauge. We computed the mean-field background and then the propagator, the scalar observable and the Wilson Loop. The mass of the vector observable, being a geometric quantity, is the same as for $SU(2)$. We then presented an application on the isotropic lattice. We computed Coulomb’s constant in five dimensions and demonstrated that it becomes independent of $N$ in the large $N$ limit. At $N = \infty$ it converges to the value $k_5 = 0.0757$, which amounts to approximately a 17% deviation from the value predicted by a holographic approach.

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