Hierarchy of rational order families of chaotic maps with an invariant measure

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Abstract

We introduce an interesting hierarchy of rational order chaotic maps that posses an invariant measure. In contrast to the previously introduced hierarchy of chaotic maps [1, 2, 3, 4, 5], with merely entropy production, the rational order chaotic maps can simultaneously produce and consume entropy. We compute the Kolmogorov-Sinai entropy of these maps analytically and also their Lyapunov exponent numerically, where that obtained numerical results support the analytical calculations.

Keywords: entropy production and entropy consumption, chaotic maps, chaos, Lyapunov exponent, Kolmogorov-Sinai entropy.

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1 Introduction

There has been some attempts [1, 2, 3, 4, 5, 6] at introducing the hierarchy of chaotic maps with an invariant measure in the recent years. The objective of these papers is to describe the dynamic behavior of chaotic maps using Kolmogorov-Sinai entropy. These hierarchies of chaotic maps are of interest as models for describing of behavior of dynamical systems. As an example, the random chaotic maps have attracted the attention of physicists as models of convection by temporarily irregular fluid flows [7]. Once a map is determined, the long term statistical behavior is described by a probability density function, which can be obtained either by solving the Frobenius-Perron equation [8] or can be estimated by measurement of the system. Therefore, the complexity, non-linearity and non-stationarity of physical, chemical, biological, physiological and financial systems [9, 10, 11, 12, 13, 14] have been of main interest in introducing the new hierarchy of the chaotic maps. On the other hand, the sensitively to the initial condition, control parameter and ergodicity which have tight relationships with the requirement of pseudo-random coding and cryptography [15, 16] are examples of interesting features of chaotic systems and it is natural idea to use chaos as a new source to construct new encryption systems [17].

In present paper, we introduce the rational order families of chaotic maps as a new hierarchy of chaotic map with an invariant measure. The Kolmogorov-Sinai entropy of these chaotic maps can be calculated analytically by using their invariant measure. An interesting property of these chaotic maps is their ability in simultaneous production and consumption of entropy. Additionally, being a measurable dynamically systems, so it can be studied analytically.

The paper is organized as follows: In Section 2, we introduce the rational order families of chaotic maps. In Section 3, the invariant measure of these maps are given and in Section 4, we review the Kolmogorov-Sinai entropy and compute it for the rational order chaotic maps. Finally in Section 5 we calculate the Lyapunov exponent numerically and compare the results of
simulation with analytically calculated Kolmogorov-Sinai entropy. The last sections contains our conclusion and two appendices. In these appendices we have calculated the invariant measure of the rational order families of chaotic maps via two different methods.

2 Hierarchy of rational order families of chaotic maps with an invariant measure

We first review hierarchy of one-parameter chaotic maps which can be used in the construction of families of rational order chaotic maps with an invariant measure. The one-parameter chaotic maps [1] are defined as the ratio of polynomials of degree $N$:

$$
\phi^1_N(x, a) = \frac{a^2(1 + (-1)^N \, 2F_1(-N, N, \frac{1}{2}, x))}{(a^2 + 1) + (a^2 - 1)(-1)^N \, 2F_1(-N, N, \frac{1}{2}, x)} = \frac{a^2(T_N(\sqrt{x}))^2}{1 + (a^2 - 1)(T_N(1-x))^2}
$$

$$
\phi^2_N(x, a) = \frac{a^2(1 - (-1)^N \, 2F_1(-N, N, \frac{1}{2}, (1-x)))}{(a^2 + 1) - (a^2 - 1)(-1)^N \, 2F_1(-N, N, \frac{1}{2}, (1-x))} = \frac{a^2(U_N(\sqrt{1-x}))^2}{1 + (a^2 - 1)(U_N(1-x))^2}
$$

where $N$ is an integer greater than one. Also,

$$
2F_1(-N, N, \frac{1}{2}, x) = (-1)^N \cos(2N \arccos \sqrt{x}) = (-1)^N T_{2N}(\sqrt{x})
$$

is the hypergeometric polynomials of degree $N$ and $T_N(U_n(x))$ are Chebyshev polynomials of type I (type II), respectively. Here in this paper we are concerned with their conjugate maps which are defined as:

$$
\begin{align*}
\tilde{\phi}^{(1)}_N(x, a) &= h \circ \phi^{(1)}_N(x, a) \circ h^{-1} = \frac{1}{a} \tan^2(N \arctan \sqrt{x}), \\
\tilde{\phi}^{(2)}_N(x, a) &= h \circ \phi^{(2)}_N(x, a) \circ h^{-1} = \frac{1}{a} \cot^2(N \arctan \frac{1}{\sqrt{x}}).
\end{align*}
$$

Conjugacy means that invertible map $h(x) = \frac{1-x}{x}$ maps $I = [0, 1]$ into $[0, \infty)$.

Now, in order to generalize the above hierarchy of integer order chaotic maps to the hierarchy
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of rational order chaotic maps with an interesting property of simultaneous production and consumption of entropy, we need to replace \( x_{n+1} \) with a non-linear function of \( x_{n+1} \), particularly a non-linear function of chaotic maps of above type. But in order to have a single-valued map, we will take one of inverse branches of above nonlinear functions for \( x_{n+1} \) in each step with a probability equal to the probabilities of occurrence of the branches in iteration of the maps. As we will show in section 3, the probability of occurrence of each branch is equal to the integral of invariant measure of the map over the corresponding domain of the same branch. Therefore we can define these maps as:

\[
x_{n+1,k} = g_{2,k}^{-1} \circ g_1(x_n), \quad i, j \in \{1, 2\}\]

with probability \( P_k \), (2-2)

where functions \( g_1 \) and \( g_2 \) can be chosen as one the functions given in (2-1) and \( P_k \) are probabilities of occurrence of inverse branches \( g_{2,k}^{-1} \) of the map \( g_2 \) in its iteration. As we will see at the end of this section, the existence of an invariant measure will impose a relation between their parameters.

Of course the functions given in (2-1) is not the only choice for the functions \( g_1 \) and \( g_2 \) that leads to the hierarchy of rational order chaotic maps with an invariant measure. Obviously the following choices of the functions \( g_1 \) and \( g_2 \)

a) \( \frac{1}{a^2} \tan^2(Narccot\sqrt{x}) \),

b) \( \frac{1}{a^2} \cot^2(Narctan\sqrt{x}) \),

c) \( \frac{1}{a^2} \cot^2(Narccot\sqrt{x}) \),

d) \( \frac{1}{a} |\tan(Narctan|x||) | \),

e) \( \frac{1}{a} |\tan(Narccot|x||) | \),

f) \( \frac{1}{a} |\cot(Narctan|x||) | \),

g) \( \frac{1}{a} |\cot(Narccot|x||) | \),

h) \( \frac{1}{a} \tan(Narctanx) \),

I) \( \frac{1}{a} \tan(Narccotx) \),

J) \( \frac{1}{a} \cot(Narctanx) \),

k) \( \frac{1}{a} \cot(Narccotx) \)

lead to the hierarchy of rational order chaotic maps of trigonometric types (with an invariant measure), where some of them are equivalent to each others up to conjugacy. Also with the
choices of \( g_1 \) and \( g_2 \) as \([4]\)

\[
a) \quad \frac{1}{a_1} \text{se}^2(N \text{sc}^{-1}(\sqrt{x})) \quad \text{b)} \quad \frac{1}{a_2} \text{cs}^2(N \text{cs}^{-1}(\sqrt{x})) ,
\]

we get the Hierarchy of elliptic rational order chaotic maps of \text{cs} and \text{sc} types, where their invariant measure can be obtained for small enough values of module \( k \) of elliptic functions. Also it is possible to choose the function \( g_1 \) and \( g_2 \) as one of the combined chaotic maps of Ref. \[3\].

Here in this paper we will consider the hierarchy of rational order maps with

\[
g_1(a_1, N_1, x_n) = \frac{1}{a_1} \tan(N_1 \arctan x_n) \quad \text{and} \quad g_2(a_2, N_2, x_{n+1}) = \frac{1}{a_2} \tan(N_2 \arctan x_{n+1}),
\]

i.e., we have

\[
x_{n+1,k_2} = \tan \left( \frac{\arctan\left(\frac{a_1}{a_2} \tan(N_1 \arctan x_n)\right)}{N_2} + \frac{k_2 \pi}{N_2} \right) \quad \text{with probability} \quad P_{k_2}, \quad k_2 = 1, 2, ..., N_2,
\]

where \( N_1 \) and \( N_2 \) are integer greater than one and \( a_1 \) and \( a_2 \) are control parameters. As we are going to see in section 3, the maps (2-6) posses an invariant measure provided that we choose the parameters \( a_1 \) and \( a_2 \) in the form given in Equations (3-12) and (3-13), respectively. As an example we consider the following map for \( N_1 = 3 \) and \( N_2 = 2 \):

\[
x_{n+1,\pm} = \frac{a_1}{a_2} \times \frac{1 - 3x_n^2}{3x_n - x_n^3} \pm \sqrt{1 + \left(\frac{a_1}{a_2} \times \frac{1 - 3x_n^2}{3x_n - x_n^3}\right)^2} \quad \text{with probabilities} \quad P_\pm = \frac{1}{2}.
\]

### 3 Invariant measure

A dynamical system even time-discrete one-variable system has a number of possible types of behavior. The system can be in a fixed point and nothing changes, the trajectory of the system may also be on a cycle with a certain period. Fixed point and periodic orbits may be stable or unstable. We are usually interested in an invariant measure \( \mu \), i.e. a probability measure that
does not change under the dynamics. The probability measure \( \mu \) on \([0, 1]\) is an Sinai-Ruelle-Bowen (SRB) measure as an invariant measure which describes statistically stationary states of system and absolutely continues with respect to Lebesgue measure.

Now in order to determine the invariant measure of the analytical system described by the maps given in (2-6), we can write it as combination of the maps \( g_1 \) and \( g_2^{-1} \) (as the \( k_2 \)-the inverse branch of \( g_2 \)) in the following form:

\[
x_{n+1,k_2} = g_2^{-1,k_2} \circ g_1(x_n) \quad \text{with probability } P_{k_2} \quad k_2 = 1, \ldots, N_2 \tag{3-1}
\]

with \( g_1 \) and \( g_2 \) given in (2-5).

Obviously the function \( g_2(., a_2, N_2) \) maps, its \( N_2 \) inverse branches \( x_{n+1,k_2} \ ; k_2 = 1, 2, \ldots, N_2 \) with corresponding different domains \( \Delta x_{n+1,k_2} (\Delta x_{n+1,i} \cap \Delta x_{n+1,j} = \emptyset \text{ for } i \neq j = 1, 2, \ldots, N_2) \) into the same region. Therefore, if denote its value by \( y \) for different values of its argument then the map (3-1) can be written as:

\[
g_2(x_{n+1}, a_2, N_2) = y = g_1(x_n, a_1, N_1), \tag{3-2}
\]

irrespective of to which branch or domain, the output \( x_{n+1} \) belongs (see Fig. 1). But in order to have a single output or single valued dynamical map, we have to consider only one of possible \( x_{n+1} \) in each step with some probabilities or weights. Certainly the most natural weight of a given branch is the corresponding probability of its occurrence in infinite iteration of map \( y = g_2(., a_2, N_2) \), where it can written in terms of its invariant measure \( \mu_{g_2} \) as,

\[
P(\text{occurrence of } k_2 \text{ - the branch}) = \int_{\Delta x_{n+1,k_2}} \mu_{g_2}(x) dx. \tag{3-3}
\]

Therefore the invariant measure of this map should satisfy the following Frobenius-Perron integral equations:

\[
\mu(y) = \int_0^1 \delta(y - g_1(x_n, a_1, N_1)) \mu(x_n) dx_n \tag{3-4}
\]

and

\[
\mu(y) = \int_0^1 \delta(y - g_2(x_{n+1}, a_2)) \mu(x_{n+1}) dx_{n+1}, \tag{3-5}
\]
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which are equivalent to:

\[ \mu(y) = \sum_{x_{n,k_1} \in g_1^{-1}(y)} \mu(x_{n,k_1}) \left| \frac{dx_{n,k_1}}{dy} \right| \]  

(3-6)

and

\[ \mu(y) = \sum_{x_{n+1,k_2} \in g_2^{-1}(y)} \mu(x_{n+1,k_2}) \left| \frac{dx_{n+1,k_2}}{dy} \right|, \]  

(3-7)

where

\[ x_{n,k_1} = \tan \left( \frac{1}{N_1} \arctan(a_1 y) + \frac{k_1 \pi}{N_1} \right), \quad k_2 = 1, \ldots, N_1. \]

and

\[ x_{n+1,k_2} = \tan \left( \frac{1}{N_2} \arctan(a_2 y) + \frac{k_2 \pi}{N_2} \right), \quad k_2 = 1, \ldots, N_2. \]

The invariant measure \( \mu(y) \) for \( g_i(x) \) can be written as:

\[ \mu_{g_i}(y) = \sum_{k_i=1}^{N_i} \frac{a_i}{N_i} \left( \frac{1 + x_{n,k_i}^2}{1 + (a_i y)^2} \right) \mu_{g_i}(x), \quad i = 1, 2. \]  

(3-8)

Assuming that \( \mu(x) \) has the following form:

\[ \mu(x) = \frac{\sqrt{\beta}}{\pi(1 + \beta x^2)} \]  

(3-9)

where for \( \beta = 1 \), it reduces to the invariant measure which has already applied to pushout measure [18], expression (3-6) reduces to

\[ \frac{1 + (a_i y)^2}{1 + \beta y^2} = \sum_{k_i=1}^{N_i} \frac{a_i}{N_i} \left( \frac{1 + x_{n,k_i}^2}{1 + \beta x_{n,k_i}^2} \right), \quad i = 1, 2. \]  

(3-10)

By comparing of both sides of Equation (3-8), we can determine \( a_i \), \( i = 1, 2 \) as

\[ a_i = \frac{\sum_{k=0}^{\left[ \frac{N_i}{2} \right]} C_{2k}^{N_i} \beta^k}{\sum_{k=0}^{\left[ \frac{N_i}{2} \right]} C_{2k+1}^{N_i} \beta^k}, \]  

(3-11)

for even values of \( N_i \), and

\[ a_i = \frac{\sum_{k=0}^{\left[ \frac{N_i-1}{2} \right]} C_{2k+1}^{N_i} \beta^k}{\sum_{k=0}^{\left[ \frac{N_i}{2} \right]} C_{2k}^{N_i} \beta^k}, \]  

(3-12)

for odd values of \( N_i \).(for proof see Appendix A).

Therefore \( a_1 \) and \( a_2 \) depend on the parameter \( \beta \) and integers \( N_1 \) and \( N_2 \), respectively. Also to make the paper more readable, we have derived the invariant measure of the map

\[ y = \frac{1}{4} \tan(4 \arctan x) \]  

by using Shure’s invariant polynomials in Appendix B.
4 Kolmogorov-Sinai entropy

In this section we review first, the Shannon entropy and then talk about Kolmogorov-Sinai entropy (for more details see [19]). Consider dynamical system characterized by a certain iterative map. Let $B = (B_i, B_j, \ldots, B_n)$ be a decomposition of the unit interval along $x_n$. Now we subdivide each interval $B_i$ into say $\Lambda$ points, and perform $\zeta$ iterations on each one of them so we make sure that transients have died out. Then $\Lambda$ points by then will spread to other subintervals. A percentage of them will be perhaps located within the limits of $B_j$. After transients die out the common area of $F^\xi(B_i)$ and $B_j$, e.g. $F^\xi(B_i) \cap B_j$ will be express in a non-normalized way the number of elements of $B_i$ reaching $B_j$ after $\xi$ iterations. So in normalized form:

$$W^\xi(B_j/B_i) = \frac{\mu(F^\xi(B_i) \cap B_j)}{\mu(B_j)}, \quad (4-1)$$

here $\mu(.) = \int_c \mu(x)\,dx$, where $c$ is the pertinent interval. The entropy of the chosen partition or, the average amount of information needed to locate the system in state space is given by the Shannonian entropy:

$$S = -\sum_i^{\Lambda} \mu(B_i) \log_2 \mu(B_i) \text{bits.} \quad (4-2)$$

The $\Lambda$ values $\mu(B_i)$ may be calculated from the $W_{ij}$ elements from the $(\Lambda - 1)$ equations of the linear system:

$$\mu(B_i) = \sum_{j=1}^{\Lambda} \mu(B_j) W_{ij} \quad (4-3)$$

and the normalization condition:

$$\sum_{j=1}^{\Lambda} \mu(B_j) = 1,$$

where the transition probability matrix $W_{ij}$ describes the probability of jumping in one step (iteration) from the element $B_i$ of the partition to the element $B_j$. The average amount of information created by the linguistic system by per transition per unit time is given by the
Kolmogorov-Sinai entropy for the chosen partition; namely;

\[ S_k = \sum_{i=1}^{\Lambda} \sum_{j=1}^{\Lambda} \mu(B_i)W_{ij} \log_2 W_{ij} \text{bits.} \quad (4-4) \]

The macroparameter however, characterizing the degree of grammatical coherence of the created Markovian chain is the mutual information or transinformation.

\[
I(\xi) = \sum_{i=1}^{\Lambda} \sum_{j=1}^{\Lambda} \mu(F^\xi(B_i) \cap B_j) \log_2 \frac{\mu(F^\xi(B_i) \cap B_j)}{\mu(F^\xi(B_i) \mu(B_j))} \text{bits.} \quad (4-5)
\]

It stands for the information stored in a symbol along the sequence about what is going to emerge \( \xi \) iterations (or \( \xi \) time units) later, \( I(\xi) \) gives the information transferred between two symbol \( \xi \) steps apart. As the number of the decomposition of the unit interval goes to infinity, in such a way that, the size of each intervals \( (B_i) \) goes to zero. The mutual entropy given in (4-5) reduces to the well known Kolmogorov-Sinai (KS) entropy which is given by:

\[
h(\mu, g(x, a, N)) = \int \mu(x)dx \ln \left| \frac{dx_{n+1}}{dx_n} \right| = \int_{-\infty}^{+\infty} \mu(x)dx \ln \left| \frac{d}{dx}g(x, a, N) \right| \quad (4-6)
\]

with \( g(x, a, N) = \frac{1}{a}(\tan(N \arctan x)) \) \( h(\mu, g(x, a, N)) \) can be written as:

\[
h(\mu, g(x, a), N) = \int_{-\infty}^{+\infty} \frac{\sqrt{\beta}}{\pi(1 + \beta x^2)} dx \ln \left| \frac{N}{a} \times \frac{1 + a^2 y^2}{1 + x^2} \right|. \quad (4-7)
\]

Following the calculating of Ref.[1], one can show that after a change of variable \( \sqrt{\beta}x = \tan \theta \), and using the integral of type;

\[
\frac{1}{\pi} \int_0^{\pi} \ln |a + b \cos \theta| = \begin{cases} 
\ln |\frac{a + \sqrt{a^2 - b^2}}{2}| & |a| > |b|, \\
\ln |\frac{b}{2}| & |a| \leq |b|,
\end{cases} \quad (4-8)
\]

we get the following expression the KS-entropy:

\[
h(\mu, g(x, a, N)) = \frac{1}{\sqrt{\beta}} \ln \left[ \frac{N}{a^3} \left( \frac{\sqrt{\beta}(\sum_{k=0}^{N_i} \mathcal{C}_{2k}^N x^k)a + a(\sum_{k=0}^{N_i-1} \mathcal{C}_{2k+1}^{N_i} x^k)}{(\sqrt{\beta} + 1)(\sum_{k=0}^{N_i} \mathcal{C}_{2k}^N x^k)} \right)^2 \right]. \quad (4-9)
\]

Now, we come to calculate the KS-entropy of fractional order maps. Before getting to involved with the details of calculation, we first talk about simultaneous production and consumption
of entropy in these maps. Figure 1, gives us an insight to see how this is possible. In this figure, \( N_1 \) (\( N_2 \)) corresponds to the number of branches \( x_{n+1}(x_n) \) of the map at time \( n+1(n) \).

The left hand half of figure 1 shows the contraction of \( N_1 \) identical branches of \( x_{n+1} \) into a single branch, while the right hand half of the figure shows branch out to \( N_2 \) branches \( x_n \), where they correspond to increment and decrease of entropy, respectively. It should be reminded that in contraction of branches, entropy increase due to loss of information, while in branch out it decreases due to reception of information.

Now in order to calculate KS-entropy of fractional order maps(2-6), we should notice that the right and left halves of Figure 1, correspond to the right-hand and left-hand sides of Equation (3-3)( equivalent of (2-6)). In other words, there are \( N_1 \) convergent branch \( x_{n+1} \) and \( N_2 \) divergent branch \( x_n \) in the left and right halves of Figure 1. Therefore, according to Figure 1 there are \( N_2 \) possible final states \( x_{n+1,k_2} \), \( k_2 = 1, 2, ..., N - 2 \) with corresponding weights given in (3-3), where we should take average over their corresponding KS-entropy given in (4-6). Hence, the KS-entropy of rational map (2-6) can be written as

\[
h(\mu, \text{rational order map}) = \sum_{k_2=1}^{N_2} \mu(x_{n+1}, a_2) \int_{\Delta x_{n+1,k_2}}^{+\infty} \mu(x_n, a_1) dx_n \ln \left| \frac{dx_{n+1}}{dx_n} \right|
\]

\[
= \sum_{k_2=1}^{N_2} \mu(x_{n+1}, a_2) \int_{\Delta x_{n+1,k_2}}^{+\infty} \mu(x_n, a_1) dx_n \ln \left| \frac{dx_{n+1}}{dy} \right| \frac{dy}{dx_n}
\]

\[
= \sum_{k_2=1}^{N_2} \mu(x_{n+1}, a_2) \int_{\Delta x_{n+1,k_2}}^{+\infty} \mu(x_n, a_1) dx_n \ln \left| \frac{dx_{n+1}}{dy} \right| + \ln \left| \frac{dy}{dx_n} \right|
\]

\[
= \int_{-\infty}^{+\infty} \mu(x, a_1) dx \ln \left| \frac{dy}{dx_n} \right| - \int_{-\infty}^{+\infty} \mu(x_{n+1}, a_2) dx_{n+1} \ln \left| \frac{dy}{dx_{n+1}} \right|
\]

where in the last line above we have used the following normalization relation

\[
\int_{-\infty}^{+\infty} \mu(x, a_i) dx = 1 \quad i = 1, 2.
\]

Now, comparing the last line of (4-10) with (4-6), we get the following expression for KS-entropy of fractional order map (2-6):

\[
h(\mu, \text{rational order map}) = h(\mu, \frac{1}{a_1} \tan(N_1 \arctan x_n)) - h(\mu, \frac{1}{a_2} \tan(N_2 \arctan x_n))
\]
with \( h(\mu, \frac{1}{\Delta_x} \tan(N_i \arctan x_n)) \) \( i = 1, 2 \) given in (4-9).

Obviously formula (4-11) implies the simultaneous production and consumption of the entropy, where the term with positive sign corresponds to the production of the entropy while the term with minus sign corresponds to the consumption of the entropy, respectively. Also, it is interesting to note that maximum value of the entropy is equal to \( \ln_2 \frac{N_2}{N_1} \) that corresponds to, \( a_1 = a_2 = 1 \) (actually this is the main reason for naming these maps as rational order maps).

5 Lyapunov exponent and simulation:

A useful numerical way to characterize chaotic phenomena in dynamic systems is by means of the Lyapunov exponents that describe the separation rate of systems whose initial conditions differ by a small perturbation. Suppose that there is a small change \( \delta x(0) \) in the initial state \( x(0) \). At step or time \( n \) this has changed to \( \delta x(n) \) given by:

\[
\delta x(n) \approx \delta x(0) \left| \frac{dx_n}{dx_0} \right| = \delta x(0) \left| \frac{dx_n}{dx_{n-1}} \frac{dx_{n-1}}{dx_{n-2}} \cdots \frac{dx_1}{dx_0} \right|,
\]

where we have used the chain rule to expand the derivative of \( \frac{dx_n}{dx_0} \). In the limit of infinitesimal perturbations \( \delta x(0) \) and infinite time we get an average exponential amplification, the Lyapunov exponent \( \lambda \),

\[
\lambda = \lim_{n \to \infty} \frac{1}{n} \ln \left| \frac{\delta x(n)}{\delta x(0)} \right| = \lim_{n \to \infty} \frac{1}{n} \ln \left| \frac{dx_n}{dx_0} \right| = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \ln \left| \frac{dx_k}{dx_{k-1}} \right|.
\]

Similarly the Lyapunov exponent of rational maps (2-6) can be obtained from formula (5-2) provided that we replace \( x_k \), at random with \( x_{k, \pm} \) at each step \( k \) with probabilities \( P_\pm = \frac{1}{2} \). Here in this work we have simulated Lyapunov exponent of rational order map with \( N_1 = 2 \) and \( N_2 = 3 \) for different values of \( \beta \), where the result that obtained supports the analytic calculation of KS-entropy given in (4-11) for particular case of \( N_1 = 2 \) and \( N_2 = 3 \) (see Figure 2).
6 Conclusion

We have given a new hierarchy of rational order families of chaotic maps which presents an interesting description of simultaneous production and consumption of entropy. Using the SRB measure, the Kolmogorov-Sinai entropy of chaotic maps have been calculated. It would be interesting to introduce these kinds of maps in higher dimensions which is under investigation.

7 Appendix A

Here in this appendix following the prescription of References [1, 2, 3, 4, 5], we prove that the invariant measure given in (3-9) satisfies the corresponding PF equations of the maps

\[ y_i = \frac{1}{a_i} \tan(N_i \arctan x_i), \quad i = 1, 2, \]

provided that the parameters \(a_1\) and \(a_2\) can be expressed in term of \(\beta\) as in formulas (3-11) and (3-12). To do so we can write the right hand side of Equation (3-8) as,

\[
\frac{a}{N} \sum_{k=1}^{N} \frac{1 + x_k^2}{1 + \beta x_k^2} = \frac{a}{\beta} + \frac{a(\beta - 1)}{N \beta^2} \frac{\partial}{\partial \beta^{-1}} \ln \left( \prod_{k=1}^{N} (\beta^{-1} + x_k^2) \right),
\]

(7-1)

where we have omitted the indices \(i\) and \(n\). Hence, Equation (3-8) can be written as:

\[
\frac{1 + a^2 y^2}{1 + \beta y^2} = \frac{a}{\beta} + \frac{a(\beta - 1)}{N \beta^2} \frac{\partial}{\partial \beta^{-1}} \ln \left( \prod_{k=1}^{N} (\beta^{-1} + x_k^2) \right).
\]

(7-2)

To evaluate the second term in the right hand side of above formulas we can write the equation

\[ y = \frac{1}{a} \tan(N \arctan x) \]

in the following form:

\[
0 = ay \cos(N \arctan x) - \sin(N \arctan x)
\]

\[
= \frac{1}{(1 + x^2)^N} \left( ay \sum_{k=0}^{[N/2]} C_{2k}^N (-1)^k x^{2k} - x \sum_{k=0}^{[N/2]} C_{2k+1}^N (-1)^k x^{2k} \right),
\]

\[
= \text{constant} \frac{1}{(1 + x^2)^{N/2}} \prod_{k=1}^{N} (x - x_k),
\]
where \( x_k = \tan\left(\frac{1}{N} \arctan(ay) + \frac{k\pi}{N}\right) \) \( k = 1, \ldots, N \) are its roots. Therefore, we have:

\[
\frac{\partial}{\partial \beta} \ln \left( \prod_{k=1}^{N} (\beta^{-1} + x_k) \right) = \frac{\partial}{\partial \beta} \ln \left( \prod_{k=1}^{N} (i\sqrt{\beta^{-1}} + x_k) \right) + \ln \left( \prod_{k=1}^{N} (-i\sqrt{\beta^{-1}} + x_k) \right)
\]

\[
= \frac{\partial}{\partial \beta} \ln \left[ (1 - \beta^{-1})^N (ay \cos(N \arctan(-i\sqrt{\beta^{-1}})) - \sin(N \arctan(-i\sqrt{\beta^{-1}}))) \right]
\]

\[
+ \frac{\partial}{\partial \beta} \ln \left[ (1 - \beta^{-1})^N (ay \cos(N \arctan(i\sqrt{\beta^{-1}})) - \sin(N \arctan(i\sqrt{\beta^{-1}}))) \right]
\]

\[
= \frac{\partial}{\partial \beta} \ln \left[ a^2 y^2 \left( \sum_{k=0}^{\left\lfloor \frac{N}{2} \right\rfloor} C_{2k}^N \beta^{-k} \right)^2 + \beta^{-1} \left( \sum_{k=0}^{\left\lfloor \frac{N-1}{2} \right\rfloor} C_{2k+1}^N \beta^{-k} \right)^2 \right]
\]

\[
= \frac{\partial}{\partial \beta} \ln \left[ (1 - \beta^{-1})^N \left( a^2 y^2 \cos^2(N \arctan(i\sqrt{\beta^{-1}})) - \sin^2(N \arctan(i\sqrt{\beta^{-1}})) \right) \right]
\]

\[
= -\frac{N \beta}{\beta - 1} + \frac{\beta N (1 + a^2 y^2) A\left(\frac{1}{\beta}\right) B\left(\frac{1}{\beta}\right)}{(1 - \beta^{-1}) \left( (A\left(\frac{1}{\beta}\right))^2 \beta y^2 + (B\left(\frac{1}{\beta}\right))^2 \right)}, \quad (7-3)
\]

with polynomials \( A(x) \) and \( B(x) \) defined as:

\[
A(x) = \sum_{k=0}^{\left\lfloor \frac{N}{2} \right\rfloor} C_{2k}^N x^k,
\]

\[
B(x) = \sum_{k=0}^{\left\lfloor \frac{N-1}{2} \right\rfloor} C_{2k+1}^N x^k. \quad (7-4)
\]

In deriving the above formula we have used the following identities:

\[
\cos(N \arctan i\sqrt{x}) = \frac{A(x)}{(1 - x)^{\frac{N}{2}}},
\]

\[
\sin(N \arctan i\sqrt{x}) = i\sqrt{x} \frac{B(x)}{(1 - x)^{\frac{N}{2}}}, \quad (7-5)
\]

inserting the results \((7-3)\) in \((7-2)\), we get:

\[
\frac{1 + a^2 y^2}{1 + \beta y^2} = \frac{1 + a^2 y^2}{\left( \frac{B\left(\frac{1}{\beta}\right)}{A\left(\frac{1}{\beta}\right)} + \beta \left( \frac{A\left(\frac{1}{\beta}\right) B\left(\frac{1}{\beta}\right) y^2}{B\left(\frac{1}{\beta}\right)} \right) \right)}.
\]

Hence to get the final result we have to choose the parameter \( a \) as:

\[
a = \frac{B\left(\frac{1}{\beta}\right)}{A\left(\frac{1}{\beta}\right)}.\]
8 Appendix B

Here we derive the invariant measure of chaotic maps by using Shur’s invariant polynomials in a way which is different from that of Appendix A. In order to make the paper more readable we consider only $N = 4$ case, i.e, the map,

$$y = \frac{1}{a} \tan(4 \arctan x)$$  \hfill (8-1)

which can be written as:

$$ay = \frac{4x(1 - x^2)}{1 - 6x^2 + x^4}$$  \hfill (8-2)

or

$$x^4 + \frac{4x^3}{ay} - 6x^2 - \frac{4x}{ay} + 1 = 0.$$  \hfill (8-3)

The needed Shur’s invariant polynomials of variables $x_1, x_2, ..., x_4$ are defined as:

$$S_1 = x_1 + x_2 + x_3 + x_4,$$

$$S_{11} = x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4,$$

$$S_{111} = x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4,$$

$$S_{1111} = x_1x_2x_3x_4.$$  \hfill (8-4)

and

$$S_2 = x_1^2 + x_2^2 + x_3^2 + x_4^2,$$

$$S_{22} = x_1^2x_2^2 + x_1^2x_3^2 + x_1^2x_4^2 + x_2^2x_3^2 + x_2^2x_4^2 + x_3^2x_4^2,$$

$$S_{222} = x_1^2x_2x_3^2 + x_1^2x_2x_4^2 + x_1^2x_3x_4^2 + x_2^2x_3x_4^2,$$

$$S_{1111} = x_1^2x_2^2x_3x_4.$$  \hfill (8-5)

The second set of Shur’s invariant polynomials can be expressed in terms of the first one as:

$$S_2 = S_1^2 - 2S_{11}.$$
\[ S_{22} = S_{11}^2 - 2S_1S_{111} + 2S_{1111}, \]
\[ S_{222} = S_{111}^2 - 2S_{11}S_{1111}, \]
\[ S_{2222} = S_{1111}^2. \]  
(8-6)

Considering the variables \( x_1, x_2, x_3 \) and \( x_4 \) as roots of Equation (8-3), one can obtain the first set of Shur’s invariant polynomials given in (8-4) as:

\[ S_1 = -\frac{4}{ay}, \quad S_{11} = -6, \quad S_{111} = \frac{4}{ay}, \quad S_{1111} = 1 \]  
(8-7)

and using Equation (8-6), we have:

\[ S_2 = \frac{16}{(ay)^2} + 12, \quad S_{22} = \frac{32}{(ay)^2} + 38, \quad S_{222} = \frac{16}{(ay)^2} + 12, \quad S_{2222} = 1. \]  
(8-8)

Again, writing the PF equation of map (8-1) and assuming that its invariant measure is of the form (8-3), we have:

\[ \mu(y) = \frac{a}{4} \times \left( \frac{1}{1 + \alpha y^2} \right)^4 \sum_{k=1}^{4} \frac{1 + x_k^2}{1 + \beta x_k^2}. \]  
(8-9)

The summation on the right-hand side can be written as:

\[ \sum_{k=1}^{4} \frac{1 + x_k^2}{1 + \beta x_k^2} = \frac{(1 + x_1^2)(1 + \beta x_2^2)(1 + \beta x_3^2)(1 + \beta x_4^2)}{\prod_{k=1}^{4}(1 + \beta x_k^2)}, \]  
(8-10)

where using Equation (8-8), the numerator of above fraction becomes:

\[ 4 + (3\beta + 1)S_2 + (2\beta^2 + 2\beta)S_{22} + (\beta^3 + 3\beta^2)S_{222} + 4\beta^3S_{2222} = 16(1 + \beta)(1 + 6\beta + \beta^2). \]  
(8-11)

Also using again (8-8), for its denominator

\[ 1 + \beta S_2 + \beta^2 S_{22} + \beta^3 S_{222} + \beta^4 S_{2222}, \]  
(8-12)

we get:

\[ \frac{16}{a^2y^2}(1 + \beta^2)^2. \]  
(8-13)

Using the results that obtained above, the invariant measure takes the following form:

\[ \mu(y) = \frac{4\alpha(1 + \beta)(1 + 6\beta + \beta^2)}{a^2y^2(\beta^2 + 6\beta + 1)^2 + 16\beta(2\beta + 1)^2} \]  
(8-14)
which should be equal to \( \frac{1}{1+\beta y^2} \), where this is possible only for the following choice of \( a \):

\[
a = \frac{4\beta + 4\beta^2}{1 + 6\beta + \beta^2}
\]  

(8-15)

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Figures captions

Fig. 1. The schematic diagram of forward and backward branching of rational order chaotic map for describing simultaneous production and consumption of entropy.

Fig. 2. Shows Lyapunov exponent (solid curve) and KS entropy (○) versus the control parameter $\beta$, as this figure shows, there is a maximum at $\beta$ which corresponds to $a_1 = a_2 = 1$. 
$y$ · · · · · · · · · · · ·

$X_{n+1,1}$  $X_{n,1}$

$X_{n+1,2}$  $X_{n,2}$

$X_{n+1,N_2}$  $X_{n,N_1}$

$N_2$ Branches  $N_1$ Branches
