The abstraction ability of students in understanding the concept of geometry

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Abstract. The purposes of this study are to determine the ability of learner’s abstraction in understanding concepts and principles of geometry. This study used a qualitative method. The research subjects were selected from undergraduate students of mathematics education. The selection technique is based on the ability to understand geometric concepts. We were collected data through task-based interviews. The analysis of data with the genetic decomposition techniques. Furthermore, the theoretical processes is carried out to determine the characteristics of each level of abstraction. The result was the six levels of students’ ability to understand geometrical concepts. The six levels were: Level 0 (concrete objects), Level 1 (model of semi-concrete), Level 2 (theoretical models), Level 3 (language in domain example), Level 4: (language of the math), and Level 5 (model of inference).

1. Introduction
Various efforts to improve students’ ability to understand geometrical concepts have been carried out by various parties, but until now geometry learning in schools has not been able to achieve the expected competencies [1]. One of the causes is abstract geometrical objects. Students experience difficulty in understanding it, because it takes a relatively difficult process, such as abstraction, generalization and idealization [2].

According to Widada, students still make errors [3,4]. The students’ errors can be non-systematic or random errors. Also, systematic errors, consistent or patterned errors. Another form of error is one that is memorized, such as being wrong because of forgetting, or wrong because he/she don't know [3].

The student errors can occur due to student failure in the abstraction process. In fact, abstraction is a very significant process in learning geometry [5]. Therefore, the learning of geometry must be adjusted to the students' level of intellectual development [6]. According to Djasuli, et.al. [7] and Mitchelmore and White [8] abstraction is a process carried out by students through the reorganization of objects previously owned vertically in the new geometry structure. An abstraction process was influenced by the tasks performed by students, the media used by students, personal relationships between students, between students and teachers, and the social system in the class. In cognitive psychology, the main characteristic of abstraction was as the extraction of the same things from a set of concrete examples, and correspondence from the same categories [9, 10]. Therefore, abstraction was the process of transition from concrete to abstract, for a set of the same things, and the name of the set will then be the name of the concept.
According to Bruner there are three levels of the abstraction process, namely the active level, the iconic level, and the last level is the symbolic level [11]. These three levels are a process of representation which is a foundation for children to carry out an abstraction process. If you pay attention to this level, then what should be a serious concern is the movement from the iconic level to the symbolic. Therefore, the graduality of these two levels cannot necessarily be directly symbolic of iconic. Means there must be a new level in the transition between iconic and symbolic [9, 10]. Because according to the achievement of a concept, and the use of geometry symbols must be gradual, starting from the simple ones that can be cognitively understood by new students and then slowly increasing to the more complex ones [10].

The level of abstraction (LoA) is a collection that can be measured but not empty [12]. There is no command given to the observable, which is expected to be a building block in a theory characterized by their definition [9, 12]. LoA are called discrete (each analog) if and only if all that can be observed is discrete (each analog); otherwise it is called a hybrid.

Furthermore, the concept of the Gradient of Abstractions (GoA) is stated by [12]. The GoA is a formalism that is defined to facilitate discussion of discrete systems of various LoAs. While LoA formalizes the scope or details of a single model, a GoA provides a way to vary the LoA to make observations at various levels of abstraction. Therefore, there are four levels of functionally integrated abstraction [10]. He states that the first level of abstraction which is referred to as the level of Physical Objects, includes the experience of students directly or manipulating objects and phenomena, discovering the behavior and physical properties themselves. At this level these objects and phenomena have not been manipulated symbolically, but are made according to their own knowledge. The second level is the Theoretic Model, which includes the representation of physical objects or phenomena that are adapted to the properties or attributes of the object. The third level of Math Language, includes statements related to geometrical objects. Also, the fourth level is Inference theories which include metalinguages, and find statements about statements [5].

Opinion of Kaminski et al, if the purpose of teaching mathematics is to produce knowledge that students can apply to various situations, then presenting mathematical concepts through generic instantiation, such as traditional symbolic notation, may be more effective than a series of "good examples" [13]. not to say that educational design should not include contextualized examples. What we suggest is that underlying mathematics in depth in a concrete context can potentially limit its application. Students may be better able to generalize mathematical concepts to various situations if the concept has been introduced with the use of generic instantiations.

When linked to the principles and characteristics of a realistic mathematical approach [14], the abstraction process is a mathematical activity about vertical reorganization (vertical mathematizing) of the previous geometry object schema on a new structure. Organizing on new structures of geometrical objects includes creating new hypotheses (conjecture), discovering or reinvention of more complex geometrical objects, and new strategies for problem solving. Vertical mathematization is an activity of placing geometrical objects together, structured, organized and developed on other objects that are more abstract or more formal than origin. Idealization occurs when we are dealing with objects that are not perfect (unperfect), and are considered perfect. Like a line, we are not too straight, the drawn plane is not too flat, so we think the line image is straight and the field image is flat. While generalization is the process of discovering geometrical objects (concepts / principles) that are general, not just conception that cannot be accepted in the geometrical structure. Based on the previous description, this paper discusses the influence of realistic mathematics learning based on ethnomathematics on the ability of mathematical representation.

2. Methods

The study used a qualitative method. The subject of research was the students' math undergraduate that was selected by purposive sampling technique. The main instrument is the researcher himself, guided by several guidelines. It was an interview guide sheet about understanding geometry. The guide sheet is the student assignment sheet, and the interview guide sheet. This interview process was recorded using
an audiovisual recorder. Researchers collect data through task-based interviews. Each subject was interviewed based on the work he did. The results of the interviews were directly analyzed so that the characteristics of each subject could be determined. The analysis of data with the genetic decomposition techniques [15, 16, 17]. Furthermore, the theoretical processes is carried out to determine the characteristics of each level of abstraction [10]. Genetic decomposition analysis was applied to analyze research data [3, 6]. The data analyzed forms a structured collection of mental activities carried out by a person to describe how geometrical concepts can be developed in his mind. The collection is a mental activity and physical activity of the subject which is related to the subject abstraction process about geometrical concepts and principles. To determine the characteristics of the subject used theoretical process aims to determine the characteristics of each level of abstraction. The process includes four stages, namely, first, comparison of applicable events to each category, both integration and region, third theory limitation, and fourth theory writing.

### 3. Results and discussion

Based on the analysis of genetic decomposition and theoretical processes in this study, six different characteristics were obtained which were hierarchic and functional. The six hierarchic and functional characters are then arranged in The Level of Abstraction (LLA). Six levels of LLA produced from this study are as follows: Level 0: Concrete Objects; Level 1: Semi-Concrete Models; Level 2: Theoretical Models; Level 3: Languages in the Example Domain; Level 4: Math Language; and Level 5: Inference Model.

Based on data analysis of this study, physical and mental activities carried out by several subjects in abstraction of an Euclidean Geometry principle obtained the characteristics of the subject which can be categorized as follows.

**Figure 1.** Percentage of subjects for each level in LLA.

Information:

1. Level 0: Concrete Objects
2. Level 1: Semi-concrete models
3. Level 2: Theoretical Models
4. Level 3: Language in the Example Domain
5. Level 4: Math Language
6. Level 5: Inference Model

Based on Figure 1 Percentage of Subjects for Each Level in the LLA, it can be seen that the most subjects are at Level 0, which is 33.33%, this indicates that the majority of subjects in carrying out abstraction activities only use real situations or with real objects. The data also shows that there are only
6.67% of subjects who are at the highest level, namely subjects with character: in performing the abstraction process using rules in a geometry system, this character is in accordance with the Inference Model Level.

One subject at Level 0 is AR. When AR is given the following problem: "Why will two different lines intersect at most at one point?" AR demonstrated with two rulers crossing and intersecting, he said that the two rulers intersected just at one point. Next AR demonstrates that two rulers that cross and do not intersect, AR states that the two lines do not have an intersection. The interview snippet is as follows: (P = Researcher).

\[ Q \quad : \quad \text{In a geometry, why do two different lines k and l intersect at most at one point?} \]
\[ AR \quad : \quad \ldots \quad \text{Sorry sir ... I borrowed another ruler from my friend first.} \]
\[ P \quad : \quad \text{Please!} \]
\[ AR \quad : \quad \ldots \quad \text{like this, Sir, if my ruler and my friend's ruler are crossed and stuck together, it means that these two rulers intersect, and this intersection, sir. [AR points to the intersection of two rulers] ... this.} \]
\[ AR \quad : \quad \text{Then if the crossing of these two rulers I lift means there is no intersection Pak ... means there is no intersection. ..} \]
\[ Q \quad : \quad \text{Can you prove the statement: "two different lines k and l will intersect at most at one point"?} \]
\[ AR \quad : \quad \ldots \quad \text{eee, the proof is with the two rulers, sir ...} \]

In accordance with theoretical process, the results of this study can be summarized in the decryption table and examples of each level of the Abstraction Levels (LLA). These descriptions and examples are listed in table 1 below.

| Level | Description | Example |
|-------|-------------|---------|
| Level 0 | Concrete objects | Perform an abstraction process using only real situations | Use two rulers to demonstrate the intersection of two lines. |
| Level 1 | Semi-concrete models | Perform an abstraction process using manipulative objects | Using two straight wires with a length of 1 m each to demonstrate the intersection of two lines |
| Level 2 | Theoretical Models | Perform an abstraction process using images from real situations | Using a picture of two straight wires with a length of 1 m each to demonstrate the intersection of two lines. |
| Level 3 | Language in the Example Domain | Perform an abstraction process using a pattern representation of an image that is not the same as the original image or by using everyday language | Trying several lines of two lines based on the image of the wire intersecting and using an analogy from the railroad tracks, namely two different lines never intersect. |
| Level 4 | Math Language | Abstraction process up to geometry symbols as a representation of the previous pattern | Two lines k and l as intersect in P |
| Level 5 | Inference Model | Perform an abstraction process using rules in a geometry system | Suppose that two lines are given a and b. Suppose that a and b intersect more than one point, for example two points, point P and Q. This means: \( P \in a \text{ and } P \in b \), \( Q \in a \text{ and } Q \in b \). According to the Axiom (1) the line a must equal the line b. This is contrary to what is known, namely a and b are different. Means that the above assumption is wrong, it must be a and b intersect at one point or not intersect at all. |
When compared to the Structure of Geometry, Geometry is constructed using three primitive concepts, namely points, lines, and fields. All points in geometry are collected in a set of S. Lines are subsets of S, so fields are other subsets of S [18].

In Neutral Geometry, there are five basic axioms that build it, namely [18]. (1) Through two different points there is exactly one line connecting the two points. (2) If there are three different points that are not in line, then there is exactly one plane of pliers containing the three points. (3) If there are two points located on a line, the line containing both points will lie in the field. (4) If two U and V planes intersect, the U and V slices form a line. (5) Each line contains at least two points, and the field contains at least three points that are not in line.

Starting from the set S and the five axioms above, it can be derived some logical consequences as follows. "Two different lines will intersect at most at one point." Subjects who are at Level 5, can prove correctly. The following is proof of one of the subjects at Level 5, namely TD. The evidences made by TD are as follows.

For example, the two lines a and b.

Suppose that a and b intersect more than one point, for example two points, point P and Q. This means:

\[ P \in a \text{ and } P \in b \]
\[ Q \in a \text{ and } Q \in b. \]

According to the Axiom theory the line a must equal the line b. This is contrary to what is known, namely a and b are different. Means that the above assumption is wrong, it must be a and b intersect at one point or not intersect at all.

Subjects who are at Level 5: Inference Model, are able to prove the geometry principle correctly. In this study there were only 6.67% of all research subjects who were at this highest level. This shows that only a small percentage of students have high-level thinking skills and are able to use them in critical thinking and creative thinking, as well as mathematically communicating well as shown in one of the evidences taken by TD Subjects.

The results were supporting the previous studies. According to Bruner language is important for the increased ability to deal with abstract concepts [11]. Bruner argues that language can code stimuli and is free from individual constraints of dealing only with appearances, to provide a more complex yet flexible cognition. Also, undergraduate students may benefit more from learning mathematics through a single abstract, symbolic representation than from learning multiple concrete examples [13]. Therefore, according to that to emphasize the special meaning of abstraction in mathematics, we would say that mathematical objects are abstract [8]. Their meaning is defined in the world of mathematics, and they are quite separate from external references [8].

According to students' reflective abstraction stage in solving sequence number problems will begin with the phase of in-depth observing pre-determined pattern of sequence numbers [7]. It is then followed by action planning stage towards the pre-existing pattern to authenticate a new pattern. In the next stage, the students design strategies or create new patterns by constructing the characteristics of the known pattern. Also, Widada, et.al. stated that the six levels of abstraction are models of students' thinking in understanding mathematical concepts [19].

4. Conclusion
Based on the description of the research results above, the conclusions of this study are as follows: there are six levels of abstraction carried out by students in understanding geometrical concepts. These six levels are: Level 0 (Concrete Objects), Level 1 (Semi-Concrete Models), Level 2 (Theoretical Models), Level 3 (Language in Example Domains), Level 4: Math Languages, and Level 5 (Inference Model).

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