Kähler-Ricci flow on homogeneous toric bundles

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Abstract

Assume that $X$ is a homogeneous toric bundle of the form $G^C \times_{P, \tau} F$ and is Fano, where $G$ is a compact semisimple Lie group with complexification $G^C$, $P$ a parabolic subgroup of $G^C$, $\tau : P \to (T^m)^C$ is a surjective homomorphism from $P$ to the algebraic torus $(T^m)^C$, and $F$ is a compact toric manifold of complex dimension $m$. In this note we show that the normalized Kähler-Ricci flow on $X$ with a $G \times T^m$-invariant initial Kähler form in $c_1(X)$ converges, modulo the algebraic torus action, to a Kähler-Ricci soliton. This extends a previous work of X. H. Zhu. As a consequence we recover a result of Podestà-Spiro.

Key words: Kähler-Ricci flow; homogeneous toric bundles; parabolic Monge-Ampère equation

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1 Introduction

In a seminal work [26] Wang and Zhu proved the existence of a Kähler-Ricci soliton on any toric Fano manifold. This result was recovered and generalized in some later works. Here we only mention two such works of particular relevance to this note: Podestà and Spiro [16] generalized Wang-Zhu’s result to the case of homogeneous toric bundles which are Fano, and Zhu [30] recovered his result with Wang using Kähler-Ricci flow. In this note we extend [30] to the case of Fano homogeneous toric bundles and recover Podestà and Spiro’s result in [16] via Kähler-Ricci flow. More precisely we show

Theorem 1.1. Assume that $X$ is a homogeneous toric bundle of the form $G^C \times_{P, \tau} F$ and is Fano, where $G$ is a compact semisimple Lie group with complexification $G^C$, $P$ a parabolic subgroup of $G^C$, $\tau : P \to (T^m)^C$ is a surjective homomorphism from $P$ to the algebraic torus $(T^m)^C$, and $F$ is a compact toric manifold of complex dimension $m$. The normalized Kähler-Ricci flow on $X$ with a $G \times T^m$-invariant initial Kähler form in $c_1(X)$ converges, modulo the algebraic torus action, to a Kähler-Ricci soliton.

We emphasize that in Theorem 1.1 we do not assume the existence of a Kähler-Ricci soliton on the Fano homogeneous toric bundle $X$; the existence of Kähler-Ricci soliton on $X$ is part of the conclusion (which, however, was previously known
by [16], as mentioned above). On the other hand, using a result in [21] Dervan and Székelyhidi [8] prove that if a Fano manifold admits a Kähler-Ricci soliton $\omega_{KRS}$, then the normalized Kähler-Ricci flow with any initial Kähler form in the first Chern class, converges to $\omega_{KRS}$ modulo the action of automorphisms. So the following is true: Assume that $X$ is a homogeneous toric bundle of the form $G^C \times_{F, \tau} F$ and is Fano, then the normalized Kähler-Ricci flow on $X$ with any initial Kähler form in $c_1(X)$ converges, modulo the action of automorphisms, to a Kähler-Ricci soliton. I do not know how to prove this fact directly. However, note that this fact does not imply the whole Theorem 1.1: the conclusion of Theorem 1.1 is slightly stronger; of course, the condition of Theorem 1.1 is also slightly stronger.

There are many beautiful works on the Kähler-Ricci flow, see for examples [3], [20], [25] and [27] for some recent surveys.

The proof of Theorem 1.1 follows closely the lines of [30], and we also borrow some key results from [16] (so the proof in this note is not completely independent of that in [16]). In [26], to obtain some of the key estimates Wang and Zhu converted the Kähler-Ricci soliton equation - a complex Monge-Ampère equation - on a toric Fano manifold to a real Monge-Ampère equation on the Euclidean space. Similarly, one of the key steps in [16] (see Proposition 5.2 in [16]) is to convert the Kähler-Ricci soliton equation on a homogeneous toric bundle which is Fano to an equation on the open dense orbit of the $(T^m)^C$-action on the fiber, which is actually a real Monge-Ampère equation on $\mathbb{R}^m$.

In the flow case, first Zhu [30] converted the Kähler-Ricci flow - a parabolic complex Monge-Ampère equation (PCMAE)- on a toric Fano manifold to a parabolic real Monge-Ampère equation on the Euclidean space. Then he adapted some of the key estimates on the real Monge-Ampère equation in [26] to the parabolic case with the help of a deep estimate of Perelman (see [18]).

Here (see Section 2.3) using some results in [16] we also convert the Kähler-Ricci flow on a homogeneous toric bundle which is Fano to a parabolic real Monge-Ampère equation on the Euclidean space, which is very similar to the one in [30]; the only difference is that in our case there is an extra term in the equation, which turns out to be harmless due to a result in [16]. Note that in Section 3.4 of [9] Donaldson made some modifications to the original approach in [26]. In the first step of the estimates for the parabolic real Monge-Ampère equation we follow Donaldson’s modifications (see Section 3). We also observe that under the normalized Kähler-Ricci flow on a Fano homogeneous toric bundle with a $G \times T^m$-invariant initial Kähler form in $c_1(X)$, the volume of any fiber is a constant.

In Section 2.1, we briefly recall some basic concepts about toric Fano manifolds, and give alternative proofs of some known facts related to the Kähler potentials and moment maps of toric Fano manifolds (see the proofs of Facts 4 and 5 there). In Section 2.2, we give a brief introduction to homogeneous toric bundles following [16]. In Section 2.3 we reduce the normalized Kähler-Ricci flow on a homogeneous toric bundle which is Fano to a parabolic real Monge-Ampère equation on the Euclidean space. In Section 3 we prove our Theorem 1.1 following [30] in most steps, except in the first step where we mainly follow (and expand) part of Section
First we fix some conventions and notations. For a compact Kähler manifold $X$ of complex dimension $n$ with Kähler metric $g = \sum_{i,j=1}^{n} g_{ij} dz^i \otimes d\bar{z}^j$, the corresponding Kähler form $\omega = \frac{\sqrt{-1}}{2\pi} \sum_{i,j=1}^{n} g_{ij} dz^i \wedge d\bar{z}^j$. Let $\text{Ric}(\omega) = \frac{\sqrt{-1}}{2\pi} \sum_{i,j=1}^{n} R_{ij} dz^i \wedge d\bar{z}^j$ be the Ricci form of $\omega$, where

$$R_{ij} = -\frac{\partial^2}{\partial z^i \partial \bar{z}^j} \log \det(g_{kl}).$$

$\text{Ric}(\omega)$ represents the first Chern class $c_1(X)$.

Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$. Suppose $G$ acts on a manifold $X$. For any $Y \in \mathfrak{g}$, we will denote the induced vector field on $X$ by $\check{Y}$ (see line 3 on p. 122 of [13]). We also denote the set of $G$-principal points in $X$ by $X_{\text{reg}}$.

2.1 Toric Fano manifolds

We give a brief review of toric manifolds following [2] and [14]. One can also consult [6], [10] and [15] for more details. Let $M$ be a free abelian group of rank $m$, $N = \text{Hom}(M, \mathbb{Z})$, $M_{\mathbb{R}} := M \otimes \mathbb{R}$, $N_{\mathbb{R}} := N \otimes \mathbb{R}$. Denote by $\langle \cdot, \cdot \rangle : M_{\mathbb{R}} \times N_{\mathbb{R}} \to \mathbb{R}$ the natural pairing. Let $F$ be a smooth projective toric $m$-fold defined by a complete fan $\Sigma$ of regular cones $\sigma \subset N_{\mathbb{R}}$. We have a maximal (algebraic) torus $(T^m)^C = \text{Hom}(M, \mathbb{C}^*) \subset \text{Aut}(F)$, $(T^m)^C \cong (\mathbb{C}^*)^m$. $(T^m)^C$ has an open dense orbit $U \subset F$, $U \cong (T^m)^C$. Using the above notation $U = F_{\text{reg}}$. We choose a basis for $M$ and the dual basis for $N$, so we can identify $M_{\mathbb{R}}$ ($N_{\mathbb{R}}$) with $\mathbb{R}^m$ and $(T^m)^C$ with $(\mathbb{C}^*)^m$. We also fix a point in $U$ so we can also identify $U$ with $(T^m)^C$ (and also with $(\mathbb{C}^*)^m$). For $z = (z_1, \ldots, z_m) \in (\mathbb{C}^*)^m$ let $t_i = \log |z_i|^2$. To $a = (a_1, \ldots, a_m) \in M$ we associate the algebraic character $\chi^a : (T^m)^C \to \mathbb{C}^*$:

$$\chi^a(z) := z_1^{a_1} \cdots z_m^{a_m}.$$  

Then

$$|\chi^a(z)|^2 = e^{\langle a, \check{z} \rangle},$$

where $\check{z} = (t_1, \ldots, t_m) \in N_{\mathbb{R}}$ and $t_i = \log |z_i|^2$ as above.

Now we assume that the toric manifold $F$ is Fano. Then by Demazure (see Theorem 2.1 in [14]) we get an $m$-dimensional compact convex polytope $\Delta \subset M_{\mathbb{R}}$ defined by the inequalities $\langle \cdot, f \rangle \leq 1$, where $f$ runs over primitive integral generators of all 1-dimensional cones in $\Sigma$. ($\Sigma$ is called the normal fan of $\Delta$.)

Let $\{p^{(0)}, \ldots, p^{(s)}\} = M \cap \Delta$. By Demazure (see Section 2.3 in [15]) we have the anticanonical embedding $F \hookrightarrow \mathbb{CP}^s$ defined by $\chi^{p^{(0)}}, \ldots, \chi^{p^{(s)}}$. Define a function
\[ v^0 : U \to \mathbb{R} \text{ by} \]
\[ v^0(z) = \log(\sum_{k=0}^{s} |\chi^{p(k)}(z)|^2). \]

Following the convention on p. 711 in [14], we still denote by \( v^0 \) the function \( N_\mathbb{R} \to \mathbb{R} \) giving by
\[ t \mapsto \log\left( \sum_{k=0}^{s} e^{p(k)x} \right). \]

So we have \( v^0(z) = v^0(\mathbf{t}) \), where \( z = (z_1, \ldots, z_m) \in U, \mathbf{t} = (t_1, \ldots, t_m) \in N_\mathbb{R}, \) and \( t_i = \log |z_i|^2 \) as above, and we may view \( v^0 \) either as a function defined on \( U \) or as a function on \( N_\mathbb{R} \). Note that the \( \mathbb{K} \)ähler form \( \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}v^0 \) on \( U \) extends to a \( \mathbb{K} \)ähler form on \( F \), denoted by \( \hat{\omega}_0 \), which is the pull-back of the \( \mathbb{K} \)ähler form \( \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\log \sum_{k=0}^{s} |z_k|^2 \) (corresponding to the Fubini-Study metric) on \( \mathbb{CP}^s \) via the anticanonical embedding \( F \hookrightarrow \mathbb{CP}^s \) defined by \( \chi^{p(0)}, \ldots, \chi^{p(s)} \).

Let \( z_i = e^{w_i} \), where \( w_i = \frac{1}{2i}t_i + \sqrt{-1}\theta_i \) (in particular, \( t_i = \log |z_i|^2 \)). Then we have \( \partial_{w_i} = z_i \frac{\partial}{\partial z_i}, \frac{\partial}{\partial w_i} = \bar{z}_i \frac{\partial}{\partial \bar{z}_i} \), \( dw_i = \frac{1}{z_i}d\bar{z}_i \), and \( d\bar{w}_i = \frac{1}{z_i}d\bar{z}_i \); moreover, \( \partial_{\bar{w}_i} = \frac{1}{2}\left( \frac{\partial}{\partial t_i} - \sqrt{-1}\frac{\partial}{\partial \theta_i} \right), \frac{\partial}{\partial \bar{w}_i} = \frac{1}{2}\left( \frac{\partial}{\partial t_i} + \sqrt{-1}\frac{\partial}{\partial \theta_i} \right), dw_i = \frac{1}{2}dt_i + \sqrt{-1}d\theta_i, \) and \( d\bar{w}_i = \frac{1}{2}dt_i - \sqrt{-1}d\theta_i \).

As \( v^0 : U \to \mathbb{R} \) is \( T^m \)-invariant, we have
\[ \partial \bar{\partial}v^0 = \frac{\partial^2 v^0}{\partial w_i \partial \bar{w}_j} dw_i \wedge d\bar{w}_j = \frac{\partial^2 v^0}{\partial t_i \partial t_j} dt_i \wedge dt_j = -\sqrt{-1} \frac{\partial^2 v^0}{\partial t_i \partial \theta_j} dt_i \wedge d\theta_j. \]

Sometimes we’ll denote \( \frac{\partial^2 v^0}{\partial t_i \partial \theta_j} \) by \( v^0_{ij} \).

Given a toric Fano manifold \( F \) as above, for any \( T^m \)-invariant \( \mathbb{K} \)ähler form \( \rho \in c_1(F) \), by Calabi-Yau theorem [29] (one can find a much simpler proof in our toric case in [9]) there is a unique \( \mathbb{K} \)ähler form \( \omega_\rho \) with \( \text{Ric}(\omega_\rho) = \rho \). Consider the unique moment map \( \mu_\rho : F \to \mathfrak{t}^* \) relative to \( \rho \) (that is, \( d\langle \mu_\rho(\cdot), \xi \rangle = -\iota_\xi \rho \) for any \( \xi \in \mathfrak{t} \)) with \( \int_F \mu_\rho \omega_\rho^n = 0 \), which is called metrically normalized (following [16]; compare Remark 9.4.2 in [14]). By Atiyah and Guillemin-Sternberg \( \mu_\rho(F) \) is a compact convex polytope. It is pointed out in Section 2.3 in [16] that \( \mu_\rho(F) \) does not depend on the choice of \( \rho \), which also follows from Facts 2 and 5 below. Denote \( \mu_\rho(F) \) by \( \Delta_\rho \).

Note that the set of all vertices of \( \Delta \) is contained in \( M \cap \Delta \). By Demazure (see Theorem 2.1 in [14]), the set of all vertices of \( \Delta \), denoted by \( \{p^{(1)}, \ldots, p^{(l)}\} \), corresponds one-to-one to the set of all \( m \)-dimensional cones in the fan \( \Sigma \), denoted by \( \{\sigma_1, \ldots, \sigma_l\} \), with \( \langle p^{(k)}, f \rangle = 1 \) for all fundamental generators \( f \) of \( \sigma_k \), \( k = 1, 2, \ldots, l \). Note that \( N_\mathbb{R} = \bigcup_{k=1}^{l} \sigma_k \). Let \( \tilde{v}(\mathbf{t}) := \max_{1 \leq k \leq l} \langle p^{(k)}, \mathbf{t} \rangle \). It is easy to see that \( \tilde{v} \) is a piecewise linear function with \( \tilde{v}(\mathbf{t}) = \langle p^{(k)}, \mathbf{t} \rangle \) for \( \mathbf{t} \in \sigma_k \).
The following five facts are known and will be used later.

**Fact 1** (cf. [2]). We have \(0 \leq v^0 - \bar{v} \leq \log(s + 1)\).

**Proof.** Compare [2]. Clearly \(\bar{v}(t) = \max_{0 \leq k \leq s}(p^{(k)}, t)\). Now

\[
e^{\bar{v}(t)} \leq \sum_{k=0}^{s} e^{p^{(k)}(t)} \leq (s + 1)e^{\bar{v}(t)},
\]

and the desired result follows. \(\square\)

Let \(\omega \in c_1(F)\) be a \(T^m\)-invariant Kähler form. By Section 3 of [14] (compare Theorem 4.3 in [11]) there exists a \(T^m\)-invariant function \(u \in C^\infty(F_{\text{reg}})\) such that

\[
(\sqrt{-1})^me^{-u} \prod_{i=1}^{m}(dw_i \wedge d\bar{w}_i) \text{ extends to a volume form on } F \text{ and } \omega|_{F_{\text{reg}}} = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}u.
\]

We also view \(u\) as a function on \(\mathbb{R}^m\) as we do with \(v^0\).

**Fact 2** (Atiyah and Guillemin-Sternberg, cf. Theorem 4.2 in [14]). Let \(u\) be as above. The map \(\phi(z) := Du(t)\) is the restriction to \(U\) of a moment map relative to \(2\pi\omega\) for the \(T^m\)-action on \(F\), where \(z = (z_1, \cdots, z_m) \in U\), \(t = (t_1, \cdots, t_m) \in \mathbb{R}^m\) with \(t_i = \log|z_i|^2\).

**Proof.** In fact Theorem 4.2 in [14] is slightly stronger; compare Remark 4.3 there. For more details see the proof of Theorem 8.2 in [14] (compare also for example, Exercise 12.2.8 in [6] for the case \(u = v^0\)). Let’s check the first statement in Fact 2. Suppose \(\frac{\partial}{\partial \theta_i}\) is induced by \(Y_i \in \mathfrak{t} = \text{Lie}(T^m)\). Then \(\phi(z) = \sum_{i=1}^{m} \phi_i(z)Y^*_i\), where \(\phi_i(z) = \frac{\partial u}{\partial t_i}\). We have

\[
\iota_{\frac{\partial}{\partial \theta_i}} \sqrt{-1}\partial\bar{\partial}u = \iota_{\frac{\partial}{\partial \theta_i}} \frac{\partial^2 u}{\partial t_j \partial t_k} dt_j \wedge d\theta_k
= - \frac{\partial^2 u}{\partial t_i \partial t_j} dt_j
= - d\left(\frac{\partial u}{\partial t_i}\right)
= - d\phi_i,
\]

and the first statement in Fact 2 follows.

For the second statement in Fact 2 we only briefly recall some of the key steps in the proof of Theorem 8.2 in [14] and give an alternative argument under an extra condition for one of them. On p. 720-721 of [14] it is shown that \(\phi\) naturally extends to \(F\), which will be denoted by \(\tilde{\phi}\). On p. 722 of [14] it is shown that all
vertices of $\Delta$ are in the image of $\tilde{\phi}$; below we’ll give a simple proof of this result under the extra condition that $u = v^0 + \varphi$, where $\varphi$ is a $T^m$-invariant smooth function on $F$.

Let $u = v^0 + \varphi$ as above, and $u_\lambda(t) = \lambda^{-1}U(\lambda t)$ for $\lambda > 0$. By Fact 1, $|u_\lambda - \bar{v}| \leq C\lambda^{-1}$, where the constant $C$ depends on the $C^0$-bound of $\varphi$ on $F$. Let $\lambda = 1, 2, \ldots$, and consider the sequence $u_1, u_2, \ldots$. Then we have $\lim_{i \to \infty} u_i = \bar{v}$. (Compare p. 39 in [9].) By Theorem 25.7 in [17],

$$\lim_{i \to \infty} Du(it) = \lim_{i \to \infty} Du_i(t) = D\bar{v}(t) = D\bar{v}(\mathbb{R}) = p(k), \quad \forall t \in \text{int}(\sigma_k), \quad k = 1, 2, \ldots, l,$$

where $\text{int}(\sigma_k)$ is the interior of $\sigma_k$. It follows that $p(k) \in \text{Im}(Du(\text{int}(\sigma_k))) \subset \text{Im}(Du(N\mathbb{R}))$.

On p. 723 of [14] it is shown that $\tilde{\phi}(F) \subset \Delta$; compare the proof of Theorem 3.4 in [2], where it is also shown that $D(v^0 + \varphi)(N\mathbb{R}) \subset \Delta$ by using Fact 1.

**Fact 3** (cf. [11], [1], p. 38-39 in [9] and p. 327 in [30]) Let $u$ be as above. Then the function $u - v^0$ can be extended to a $T^m$-invariant smooth function on $F$, and in particular, $|u - v^0|$ is bounded.

**Proof.** The result is implicitly contained in [11] and [1]. Since both $\omega$ and $\hat{\omega}_0$ are in $c_1(F)$, there is a $T^m$-invariant smooth function, denoted by $\varphi$, on $F$ such that

$$\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} u - \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} v^0 = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \varphi$$

on $F_{\text{reg}}$.

By Fact 2 we have $\text{Im}(Du) = \Delta \setminus \partial \Delta = \text{Im}(Dv^0)$. Then by Fact 1 and convexity we have that $\bar{v} - u$ is bounded below (compare the proof of Lemma 3.4 in [26]). Combining this with Fact 1 we get that $v^0 - u$ is bounded below. Then we have that $v^0 - u + \varphi$ is a harmonic function on $\mathbb{R}^m$ which is bounded below, so it must be a constant. \qed

It follows that $u$ as above is uniquely determined by $\omega$ up to an additive constant.

Let $\omega \in c_1(F)$ be a $T^m$-invariant Kähler form as above. Then there exists a $T^m$-invariant smooth function $\varphi$ on $F$ such that $\omega = \hat{\omega}_0 + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \varphi$. Note that $\varphi$ is uniquely determined up to an additive constant.

**Fact 4** (cf. [26] and Lemma 4.3 in [19] for the case $\varphi = 0$). We have

$$\sup_{N\mathbb{R}} |\log \det(\frac{\partial^2(v^0 + \varphi)}{\partial t_i \partial t_j}) + v^0 + \varphi| < \infty.$$

**Proof.** As recalled above, the Fano condition implies that we can suppose that the polytope $\Delta$ is defined by the inequalities $l_r(\cdot) = \langle \cdot, \lambda_r \rangle + 1 \geq 0$, where $\lambda_r \in N$, $r = 1, \ldots, d$. (We may view the $\lambda_r$ as inward-pointing normals to codimension 1
faces of $\Delta$. Let $G_0(x)$ be the Legendre transform of $(v^0 + \varphi)(t)$. By Guillemin [11] we have

$$(v^0 + \varphi)(t) = \sum_{r=1}^{d} (\log l_r(x) + l_r(x)) + h_1(x),$$

and

$$G_0(x) = \sum_{r=1}^{d} l_r(x) \log l_r(x) + h_2(x),$$

where $x = D(v^0 + \varphi)(t) \in \Delta \setminus \partial \Delta$ (by Fact 2), and $h_1$ and $h_2$ are smooth functions on (an open neighborhood of) $\Delta$.

Now we have that

$$\det(\partial^2 (v^0 + \varphi) / \partial t_i \partial t_j) = (\det \text{Hess } G_0(x))^{-1} = \delta_\Delta(x) \prod_{r=1}^{d} l_r(x),$$

where $\delta_\Delta$ is smooth and positive on $\Delta$. One way to get the last equality is as follows. Since $\Delta$ satisfies the Delzant condition [7], after an affine transformation whose linear part is given by an element in $\text{GL}(m, \mathbb{Z})$ we may assume that the origin is a vertex of $\Delta$ and near the origin the polytope $\Delta$ is given by the inequalities $x_1 \geq 0, \ldots, x_m \geq 0$. Then near the origin we have $G_0(x) = \sum_{i=1}^{m} x_i \log x_i + v(x)$, where $v$ is smooth. Similarly we can treat a boundary point which lies on a codimension $i$ face of $\Delta$; see p. 39 of [9]. Then we can compute with the help of such local expressions of $G_0(x)$ to derive the last equality. Compare Section 2.4 and Appendix of [1]. Now Fact 4 follows.

**Fact 5** (cf. [16]). The map $\mu_\omega(z) := \frac{1}{2\pi} D(v^0 + \varphi)(t)$ is the restriction to $U$ of the metrically normalized moment map relative to the Kähler form $\omega$ for the $T^m$-action on $F$, where $z = (z_1, \ldots, z_m) \in U$, $t = (t_1, \ldots, t_m) \in N_R$ with $t_i = \log |z_i|^2$.

**Proof.** By the first statement in Fact 2 it remains to verify that $\mu_\omega$ is metrically normalized. One can show this fact by inspecting the proof of Lemma 5.1 in [16]. We give a direct proof below. Choose $\psi \in c_1(F)$ such that $\text{Ric}(\psi) = \omega$. Let $h_\omega$ be a Ricci potential of $\omega$, that is, $\text{Ric}(\omega) - \omega = \sqrt{-1} 2\pi \partial \bar{\partial} h_\omega$. Then we have

$$-\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log \det((v^0 + \varphi)_{ij}) - \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} (v^0 + \varphi) = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} h_\omega$$

on $U$. Using Fact 4 and the maximum principle we see that there is a constant $C$ such that

$$e^{h_\omega} = e^C e^{-(v^0 + \varphi)} \det((v^0 + \varphi)_{ij})^{-1}.$$

On the other hand it is easy to see that there is a constant $C'$ such that

$$\psi^m = C' e^{h_\omega} \omega^m.$$

(See p. 339-340 of [29].)

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Note that on $U$ we have $(2\pi \omega)^m = \det((v^0 + \varphi)_{ij})dt_1 \wedge \cdots \wedge dt_m \wedge d\theta_1 \wedge \cdots \wedge d\theta_m$. Moreover, by Fact 1, $v^0 + \varphi \to +\infty$ as $t \to \infty$. Write $\mu_i(z) = \frac{1}{2\pi} \frac{\partial(v^0 + \varphi)}{\partial t_i}$. Now
\[
\int_U \mu_i \psi^m = C' \int_U \mu_i e^{h \omega} \omega^m = C'' \int_{N_\mathbb{R}} \frac{\partial(v^0 + \varphi)}{\partial t_i} e^{-(v^0 + \varphi)} dt_1 \cdots dt_m = -C'' \int_{N_\mathbb{R}} \frac{\partial e^{-(v^0 + \varphi)}}{\partial t_i} dt_1 \cdots dt_m = 0,
\]
and we are done.\(\square\)

From Facts 2 and 5 we have $\mu_\omega(F) = \frac{1}{2\pi} \Delta$, the $\frac{1}{2\pi}$-dilation of $\Delta$ w.r.t. the origin. So $\Delta_F = \frac{1}{2\pi} \Delta$.

**2.2 Homogeneous toric bundles**

Now we will follow [16] to give a brief introduction to homogeneous toric bundles, and refer to [16] for more details; one can also consult [13] for basics of Lie groups, Lie algebras and symmetric spaces.

Assume $G$ is a compact semisimple Lie group. Let $\mathfrak{B}$ be the Cartan-Killing form on $\mathfrak{g}$, we may view $-\mathfrak{B}$ as an inner product on $\mathfrak{g}$. Given a root system $R$ w.r.t. a fixed maximal torus, choose the root vectors $E_\alpha \in \mathfrak{g}^C$ corresponding to roots $\alpha \in R$ via Chevalley’s normalization; in particular, $[E_\alpha, E_{-\alpha}] = H_\alpha$, where $H_\alpha$ is the $\mathfrak{B}$-dual of $\alpha$.

Now let $G$ be a compact semisimple Lie group, and $(F^m, J_F)$ be a compact toric manifold with a complex structure $J_F$ and a holomorphic action of $T^m(\cong (S^1)^m)$. As in [16] we consider a homogeneous toric bundle $X$ over a generalized flag manifold $V = G^C/P = G/K$ with fiber $F$,
\[
X := G^C \times_{P, \tau} F = G \times_{K, \tau} F,
\]
where $P$ is a parabolic subgroup of $G^C$, $\tau : P \to (T^m)^\mathbb{C}$ is a surjective homomorphism from $P$ to the algebraic torus $(T^m)^\mathbb{C}(\cong (\mathbb{C}^*)^m)$, and $K = P \cap G$. Note that $V$ is a complex homogeneous space with a natural $G^C$-invariant complex structure (which will be denoted by $J_V$). Let $\pi : X \to V$ be the bundle projection. As in [16] we identify $F$ with the fiber $F_{eK} := \pi^{-1}(eK)$. There is a unique $G^C$-invariant complex structure (denoted by $J$) on $X$ such that the map $\pi$ is holomorphic and the restriction of $J$ to the fiber $F_{eK}$ is $J_F$.

Note that we have a holomorphic action of $G^C \times (T^m)^\mathbb{C}$ on $(X, J)$,
\[
(g', h) \cdot [g, x] := [g'g, h \cdot x], \quad \forall g', g \in G^C, \ h \in (T^m)^\mathbb{C} \text{ and } x \in F,
\]
compare [28].
Recall that \( g \) has an \( \text{Ad}(K) \)-invariant decomposition \( g = \mathfrak{t} \oplus \mathfrak{m} \). For any fixed Cartan subalgebra (CSA) \( \mathfrak{h} \subset \mathfrak{t}^\mathbb{C} \) of \( g^\mathbb{C} \), the associated root system \( R \) has a corresponding decomposition \( R = R_o + R_m \) with \( E_\alpha \in \mathfrak{t}^\mathbb{C} \) for \( \alpha \in R_o \) and \( E_\alpha \in \mathfrak{m}^\mathbb{C} \) for \( \alpha \in R_m \). The natural \( G^\mathbb{C} \)-invariant complex structure \( J_V \) of \( V = G^\mathbb{C}/P \) induces a splitting \( R_m = R_m^+ \cup R_m^- \), such that \( \sum_{\alpha \in R_m^+} \mathbb{C}E_\alpha \) (resp. \( \sum_{\alpha \in R_m^-} \mathbb{C}E_\alpha \)) is the \( J_V \)-holomorphic (resp. \( J_V \)-antiholomorphic) subspace \( \mathfrak{m}^{(1,0)} \) (resp. \( \mathfrak{m}^{(0,1)} \)) of \( \mathfrak{m}^\mathbb{C} \). Let

\[
Z_V = -\frac{1}{2\pi} \sum_{\alpha \in R_m^+} \sqrt{-1}H_\alpha.
\]

Let \( Z(K) \) be the center of \( K \) and \( \mathfrak{z}(\mathfrak{t}) \) the Lie algebra of \( Z(K) \). Let \( \mathfrak{t}_G := \langle \ker d\tau_o \rangle \cap \mathfrak{z}(\mathfrak{t}) \). (We can identify \( \mathfrak{t}_G \) with \( \mathfrak{t} \) via \( d\tau_o \).) Choose a \(-\mathfrak{g}\)-orthonormal basis \( \{Z_1, \ldots, Z_m\} \) of \( \mathfrak{t}_G \) so that \( \exp(\mathbb{R} \cdot Z_j) \) is closed for each \( j = 1, \ldots, m \).

In \[16\] Podestà and Spiro give a criterion for a homogeneous toric bundle \( X \) to be Fano, see Theorem A in \[16\], which says that a homogeneous toric bundle \( X = G^\mathbb{C} \times F \) is Fano if and only if \( F \) is Fano and the condition (1.1) in \[16\] holds.

Now let \( X = G^\mathbb{C} \times F \) be Fano. As in \[16\], we denote the set of \( G \times T^m \)-invariant (resp. \( T^m \)-invariant) 2-forms in \( c_1(X) \) (resp. \( c_1(F) \)) by \( c_1(X)^{G \times T^m} \) (resp. \( c_1(F)^{T^m} \)). By Lemma 5.1 in \[16\], the map \( R : c_1(X)^{G \times T^m} \to c_1(F)^{T^m} \) given by \( R(\tilde{\omega}) = \tilde{\omega}|_{TF} \) (restriction) is invertible. We’ll denote \( R(\tilde{\omega}) \) (not to be confused with the scalar curvature) by \( \omega \). Let \( E : c_1(F)^{T^m} \to c_1(X)^{G \times T^m} \) be the inverse of the map \( R \). For \( \omega \in c_1(F)^{T^m} \), \( E(\omega) \) is an extension of \( \omega \); we’ll denote \( E(\omega) \) by \( \tilde{\omega} \). By Lemma 5.1 in \[16\] \( E(\omega) \) is Kähler if and only if \( \omega \) is Kähler.

Fix a Kähler form \( \omega_0 \in c_1(F)^{T^m} \). As in Section 2.1 there is \( u_0 \in C^\infty(F^{reg})^{T^m} \) such that \( (\sqrt{-1})^m e^{-u_0} \prod_{i=1}^m (dw_i \wedge dw_i) \) extends to a volume form on \( F \) and \( \omega_0|_{F^{reg}} = \frac{1}{4\pi} \, dd^c u_0 \). For any other Kähler form \( \omega \in c_1(F)^{T^m} \), we may write \( \omega = \omega_0 + \frac{1}{4\pi} \, dd^c \varphi \), where \( \varphi \in C^\infty(F)^{T^m} \). From Facts 2 and 5 in Section 2.1 we see that the map \( \left( \frac{1}{2\pi} \frac{\partial (u_0 + \varphi)}{\partial t_1}, \ldots, \frac{1}{2\pi} \frac{\partial (u_0 + \varphi)}{\partial t_m} \right) \) is the restriction to \( F^{reg} \) of the metrically normalized moment map relative to \( \omega \) (under the basis \( \{d\tau(Z_i)^t \} \) of \( t^* \)), and its image \( (\Delta_F \setminus \partial \Delta_F) \) does not depend on the choice of \( \omega \). (Note that our coordinate system is slightly different from the one used in \[16\].) As in \[16\] we denote by \( A^{-1} \) the product

\[
\prod_{\alpha \in R_m^+} \left( a^{\alpha}_0 \frac{\partial (u_0 + \varphi)}{\partial t_i} + b_\alpha \right),
\]

where \( a^{\alpha}_0 := \alpha(\sqrt{-1} Z_i) \) and \( b_\alpha := \alpha(\sqrt{-1} Z_V) \). Then one sees that

\[
A^{-1} = \prod_{\alpha \in R_m^+} \sqrt{-1} \alpha \left( \sum_{i=1}^m f_i Z_i + Z_V \right),
\]

where \( f_i = \frac{1}{2\pi} \frac{\partial (u_0 + \varphi)}{\partial t_i} \). By the Fano condition on \( X \), the property of the image of \( (f_1, \ldots, f_m) \) noticed above, and Theorem A in \[16\], it follows that \( A \) is bounded above and below by two positive constants not depending on the choice of \( \omega \),

\[
K_1 \leq A \leq K_2,
\]

(2.1)
compare the proof of Lemma 5.3 in [16].

2.3 Kähler-Ricci flow and its reduction to PMAE

Suppose $X$ is a Fano manifold of complex dimension $n$, and let $g_0$ be a Kähler metric on $X$ such that the corresponding Kähler form $\omega_0$ represents the first Chern class $c_1(X)$. Let $h_0$ be a Ricci potential of $\omega_0$, so that

$$\text{Ric}(\omega_0) - \omega_0 = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} h_0 = \frac{1}{4\pi} dd^c h_0.$$  

$h_0$ is uniquely determined up to an additive constant.

Consider the normalized Kähler-Ricci flow (NKRF) on $X$,

$$\frac{\partial \omega}{\partial t} = - \text{Ric}(\omega) + \omega, \quad \omega(0) = \omega_0.$$  

(2.2)

The following result is well-known, cf. for example [24].

Proposition 2.1. Let $X$ be Fano. Fix a Kähler form $\omega_0 \in c_1(X)$. A smooth family of Kähler forms $\omega(t)$ ($t \in [0, T]$) on $X$ solves the NKRF (2.2) if and only if there is a smooth family of smooth functions $\varphi(t)$ ($t \in [0, T]$) on $X$ with $\omega(t) = \omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi(t)$ such that

$$\frac{\partial \varphi}{\partial t} = \log \frac{\det(g^0_{ij} + \varphi_{ij})}{\det(g^0_{ij})} + \varphi - h_0, \quad \varphi(0) = 0, \quad \omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi(t) > 0,$$  

(2.3)

where $(g^0_{ij})$ is the Kähler metric corresponding to $\omega_0$, and $h_0$ is a Ricci potential of $\omega_0$.

Proof. It is straightforward to see that if $\varphi(t)$ is a solution of (2.3), then $\omega(t) := \omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi(t)$ solves (2.2).

Conversely, suppose $\omega(t)$ solves (2.2). Write $\omega(t) = \omega_0 + \sqrt{-1} \partial \bar{\partial} \tilde{\varphi}(t)$ using $\partial \bar{\partial}$-lemma, where $\tilde{\varphi}(t)$ is a smooth family of smooth functions on $X$ with $\tilde{\varphi}(0) = 0$. Then

$$\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \frac{\partial \tilde{\varphi}}{\partial t} = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log \frac{\det(g^0_{ij} + \tilde{\varphi}_{ij})}{\det(g^0_{ij})} + \tilde{\varphi} - h_0.$$  

By the maximum principle there exist constants $C(t)$ (smoothly depending on $t$) such that

$$\frac{\partial \tilde{\varphi}(t)}{\partial t} = \log \frac{\det(g^0_{ij} + \tilde{\varphi}_{ij}(t))}{\det(g^0_{ij})} + \tilde{\varphi}(t) - h_0 + C(t).$$

Choose $b(t)$ such that

$$\frac{d}{dt} b(t) = b(t) - C(t), \quad b(0) = 0.$$
Let \( \varphi(t, \cdot) = \tilde{\varphi}(t, \cdot) + b(t) \). Then \( \varphi(t) \) solves (2.3). \( \square \)

For \((X, \omega_0)\) as above, Cao [4] showed that the solution of (2.2) (or equivalently (2.3)) exists for \( t \in [0, \infty) \).

Let \( \varphi_t \) be a solution of (2.3), and \( \omega_t = \omega(t) = \omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi(t) \). For each \( t \) we choose a constant \( c_t \) such that

\[
\int_X e^{-\frac{\partial \omega}{\partial t} + c_t} \omega^n = \int_X \omega_0^n. \tag{2.4}
\]

By a deep estimate of Perelman (see [18]), there is a constant \( C \) depending only on \( \omega_0 \) such that for all \( t \in [0, \infty) \),

\[
| - \frac{\partial \varphi}{\partial t} + c_t | \leq C. \tag{2.5}
\]

Now we assume that the Fano manifold \( X \) is a homogeneous toric bundle of the form \( C^C \times_{\mathbb{T}} F \) as above. Consider the normalized Kähler-Ricci flow on \( X \),

\[
\frac{\partial \tilde{\omega}}{\partial t} = -\text{Ric}(\tilde{\omega}) + \omega, \quad \tilde{\omega}(0) = \tilde{\omega}_0. \tag{2.6}
\]

**Proposition 2.2.** Let \( X = C^C \times_{\mathbb{T}} F \) be Fano. Fix a Kähler form \( \tilde{\omega}_0 \in c_1(X)^{G \times T_m} \) and write \( R(\tilde{\omega}_0) = \omega_0 \in c_1(F)^{T_m} \). A smooth family of Kähler forms \( \tilde{\omega}(t) \ (t \in [0, T)) \) on \( X \) solves the NKRF (2.7) if and only if there is a smooth family of functions \( \varphi(t) \in C^\infty(F)^{T_m} \) \((t \in [0, T)) \) with \( \varphi(0) = 0 \) and \( \omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi(t) > 0 \) such that \( \tilde{\omega}(t) = E(\omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi(t)) \), and at all points of \( F_{\text{reg}} \),

\[
\frac{\partial u_0 + \varphi}{\partial t} = \log[\det(\frac{\partial^2 (u_0 + \varphi)}{\partial t_i \partial t_j}) \prod_{\alpha \in R_m^+} (\frac{a_i}{2\pi} \frac{\partial (u_0 + \varphi)}{\partial t_i} + b_\alpha)] + u_0 + \varphi, \tag{2.7}
\]

where \( u_0 \in C^\infty(F_{\text{reg}})^{T_m} \) such that \((\sqrt{-1})^m e^{-u_0} \prod_{i=1}^m (dw_i \wedge d\bar{w}_i)\) extends to a volume form on \( F \) and \( \omega_0|_{F_{\text{reg}}} = \sqrt{-1} \partial \bar{\partial} u_0 \).

**Proof.** Let \( \varphi(t) \in C^\infty(F)^{T_m} \) \((t \in [0, T)) \) be a smooth family of functions with \( \varphi(0) = 0 \) and \( \omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi(t) > 0 \) such that at all points of \( F_{\text{reg}} \) equation (2.7) is satisfied. Write \( \omega(t) = \omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi(t) \). By Lemma 5.1 in [16], \( \tilde{\omega}_0 := E(\omega_0) \in c_1(X)^{G \times T_m} \), and \( \tilde{\omega}(t) := E(\omega(t)) \in c_1(X)^{G \times T_m} \). Observe that the cohomology class

\[
[\frac{\partial \tilde{\omega}(t)}{\partial t}] = [\frac{\partial [\tilde{\omega}(t)]}{\partial t}] = 0.
\]

Denote the RHS of (2.7) by \(-\Psi\). Take \( \frac{1}{4\pi} d\Psi \) of both sides of (2.7); both sides of the new equation thus obtained can be naturally and uniquely extended to be smooth.
2-forms on $F$. By Lemma 2.2 and Proposition 2.3 of [16], \( \frac{1}{4\pi}dd^c \Psi \) coincides with the restriction to $F_{\text{reg}}$ of $\text{Ric}(\tilde{\omega}) - \tilde{\omega}$. Note that

\[
\frac{\partial \tilde{\omega}(t)}{\partial t}\big|_{TF} = \frac{\partial \omega(t)}{\partial t}.
\]

Then we get that

\[
\frac{\partial \tilde{\omega}(t)}{\partial t}\big|_{TF} = (\tilde{\omega}(t) - \text{Ric}(\tilde{\omega}(t)))|_{TF},
\]

or

\[
R\left(\frac{\partial \tilde{\omega}(t)}{\partial t} + \text{Ric}(\tilde{\omega}(t))\right) = R(\tilde{\omega}(t)).
\]

Using Lemma 5.1 in [16] again, we see that

\[
\frac{\partial \tilde{\omega}(t)}{\partial t} + \text{Ric}(\tilde{\omega}(t)) = \tilde{\omega}(t),
\]

thus $\tilde{\omega}(t)$ solves equation (2.6).

Conversely assume that $\tilde{\omega}(t)$ solves equation (2.6). By Lemma 5.1 in [16] we have $\omega_0 := R(\tilde{\omega}_0) \in c_1(F)^{T_m}$ and $\omega(t) := R(\tilde{\omega}(t)) \in c_1(F)^{T_m}$. We can write $\omega_t = \omega(t) = \omega_0 + \sqrt{-1} \frac{2\pi}{2\pi} \partial \partial \psi(t)$, where $\psi(t) = \psi$ is a smooth family of functions in $C^\infty(F)^{T_m}(t \in [0,T])$. Let

\[
\Psi_t := -\log[\det\left(\frac{\partial^2(u_0 + \psi_t)}{\partial t_i \partial t_j}\right) \prod_{\alpha \in R_+^m} (a_i^\alpha \frac{\partial(u_0 + \psi_t)}{\partial t_i} + b_\alpha)] - (u_0 + \psi_t).
\]

Restricting equation (2.6) to $F_{\text{reg}}$ and using Lemma 2.2 and Proposition 2.3 in [16] again, we see that

\[
\frac{1}{4\pi}dd^c \frac{\partial \psi}{\partial t} = \frac{1}{4\pi}dd^c (-\Psi_t).
\]

It follows that $\frac{\partial \psi}{\partial t} = -\Psi_t + C(t)$. Choose $b(t)$ as in the proof of Proposition 2.1 and let $\varphi(t, \cdot) = \psi(t, \cdot) + b(t)$, then $\varphi(t)$ solves equation (2.7).

Let $X = G^C \times_{P,T} F$ be Fano. Given an initial Kähler form $\tilde{\omega}_0 \in c_1(X)^{G \times T_m}$ on $X$, the function $h_0$ in (2.3) and the function $u_0$ in (2.7) are unique up to additive constants. By Fact 3 in Section 2.1 the function $u_0 - v_0$ can be extended to a $T^m$-invariant smooth function on $F$, so from Fact 4 in Section 2.1 we have that

\[
\sup_{F_{\text{reg}}} |\log \det(\frac{\partial^2 u_0}{\partial t_i \partial t_j}) + u_0| < \infty.
\]

Compare p. 327 in [30]. On the other hand, using Lemma 2.2 and Proposition 2.3 in [16], we have

\[
\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \left\{ \log[\det(\frac{\partial^2 u_0}{\partial t_i \partial t_j}) \prod_{\alpha \in R_+^m} (a_i^\alpha \frac{\partial u_0}{\partial t_i} + b_\alpha)] + u_0 + h_0 \right\}
\]

\[
= - (\text{Ric}(\tilde{\omega}_0) - \tilde{\omega}_0)|_{F_{\text{reg}}} + \frac{\sqrt{-1}}{2\pi} (\partial \bar{\partial} h_0)|_{F_{\text{reg}}}
\]

\[
= 0.
\]
Recall also (2.1). It follows that
\[ \log[\det(\frac{\partial^2 u_0}{\partial t_i \partial t_j}) \prod_{\alpha \in \mathbb{R}_+^m} (\frac{a_i^\alpha \partial u_0}{2\pi \partial t_i} + b_\alpha)] + u_0 + h_0 \]
is a constant. So we can and will choose \( h_0 \) and \( u_0 \) such that
\[ \det(\frac{\partial^2 u_0}{\partial t_i \partial t_j}) \prod_{\alpha \in \mathbb{R}_+^m} (\frac{a_i^\alpha \partial u_0}{2\pi \partial t_i} + b_\alpha) = e^{-h_0 - u_0}. \] (2.8)

From Propositions 2.1 and 2.2 we have a correspondence between the solutions to the equation (2.3) and the solutions to the equation (2.7), where \( h_0 \) and \( u_0 \) satisfy the constraint (2.8). More precisely, for a homogeneous toric bundle \( X \) which is Fano with an initial Kähler form in \( c_1(X)^{G \times T^m} \), given a solution \( \varphi \) to equation (2.3), \( u_0 + \varphi \mid_{\text{F_{reg}}} \) solves equation (2.7), where we assume that \( u_0 \) satisfies the additional constraint (2.8); conversely, given a smooth family of functions \( \varphi(t) \in C^\infty(F)^{T^m} \) with \( \varphi(0) = 0 \) and \( \omega_0 + \sqrt{-1}{2\pi} \partial \bar{\partial} \varphi(t) > 0 \) such that at all points of \( F_{\text{reg}} \), \( u_0 + \varphi \) solves equation (2.7), the \( G \times T^m \)-invariant extension of \( \varphi \) to the whole \( X \) (which will also be denoted by \( \varphi \)) solves equation (2.3), where \( h_0 \) satisfies the additional constraint (2.8).

The equation (2.7) with \( \varphi(0) = 0 \) is actually a parabolic real Monge-Ampère equation,
\[ \frac{\partial(u_0 + \varphi)}{\partial t} = \log[\det(\frac{\partial^2(u_0 + \varphi)}{\partial t_i \partial t_j})] + u_0 + \varphi - \log A, \quad \text{on} \quad \mathbb{R}^m, \] (2.9)
with \( \varphi(0) = 0 \) (and the matrices \( (\frac{\partial^2(u_0 + \varphi)}{\partial t_i \partial t_j}) > 0 \), where \( A \) is defined in Section 2.2.

## 3 Proof of Theorem 1.1

We will write \( \mathbf{t} = (t_1, \ldots, t_m) \in \mathbb{R}^m \), and \( d\mathbf{t} = dt_1 \cdots dt_m \).

**Lemma 3.1.** (\cite{20}, see also Proposition 2 on p. 54 in \cite{9}) Assume that \( v \) is a smooth convex function on \( \mathbb{R}^m \) which attains minimal value \( 0 \), such that \( \det(D^2 v) \geq \lambda \) when \( v \leq 1 \). Let \( \Gamma \) be the set where \( v \leq 1 \). Then \( \text{Vol}(\Gamma) \leq C\lambda^{-1/2} \), where \( C \) is a constant depending only on the dimension \( m \).

**Proof.** We follow the outline given on p. 54 of \cite{9}. \( \Gamma \) is a bounded convex set in \( \mathbb{R}^m \) with nonempty interior. Let \( E \) be the ellipsoid of minimum volume containing \( \Gamma \) centered at the barycenter of \( \Gamma \). By a variant of John’s Lemma (see for example Theorem 1.8.2 in \cite{12}), \( \alpha_m E \subset \Gamma \subset E \), where \( \alpha_m = m^{-3/2} \) and \( \beta Y \) denotes the \( \beta \)-dilation of a bounded set \( Y \subset \mathbb{R}^m \) w.r.t. its barycenter, that is \( \beta Y := \{t_0 + \beta(t - t_0) \mid t \in Y\} \), where \( t_0 \) is the barycenter of \( Y \). Choose a unimodular affine transformation \( T \) such that \( B(0, \alpha_m r) \subset T(\Gamma) \subset B(0, r) \). We
write $T^k_t = A_t^k + b$ for any $t \in \mathbb{R}^m$, where $A$ is a $m \times m$ matrix with $\det A = 1$ and $b \in \mathbb{R}^m$. Let
\[
\tilde{v}(\underline{t}) = v(A^{-1}(rt - b)) - 1.
\]
Let $\Gamma' = \frac{1}{r}T(\Gamma)$. Then the barycenter of $\Gamma'$ is the origin $0$, $B(0, \alpha_m) \subset \Gamma' \subset B(0, 1)$, $\min_{\Gamma'} \tilde{v} = -1$ and $\tilde{v} = 0$ on $\partial \Gamma'$. We also have $\det D^2\tilde{v} \geq r^{2m}\lambda$ on $\Gamma'$. From the proof of Proposition 3.2.4 in [12] we have
\[
\int_{\frac{1}{2}\Gamma'} \det D^2\tilde{v} dt \leq C \min_{\frac{1}{2}\Gamma'} |\tilde{v}|^m \leq C,
\]
where $C$ depends only on the dimension $m$. It follows that
\[
r^{2m}\lambda \text{Vol}(\frac{1}{2}\Gamma') \leq C,
\]
and
\[
\frac{r^m\lambda}{2^m} \text{Vol}(\Gamma) = \frac{r^m\lambda}{2^m} \text{Vol}(T(\Gamma)) \leq C.
\]
We also have
\[
\text{Vol}(\Gamma) = \text{Vol}(T(\Gamma)) \leq \omega_m r^m,
\]
where $\omega_m$ is the volume of the unit $m$-ball. It follows that
\[
\frac{\lambda}{2^m} \frac{\text{Vol}(\Gamma)}{\omega_m} \text{Vol}(\Gamma) \leq C,
\]
and we are done. \qed

For a homogeneous toric bundle $X$ which is Fano with an initial Kähler form $\omega_0 \in c_1(X)^G \times T^m$, given a solution $\varphi$ to equation (2.3), from above we see that $u = u_t = u(t, \cdot) = u_0 + \varphi$ is a solution of equation (2.9) with $\varphi(0) = 0$, where $u_0$ is as in Section 2.3.

The following result is well known.

**Lemma 3.2.** Let $u$ be as above. Then
\[
\int_{\mathbb{R}^m} \det D^2u_t dt = \text{Vol}(\Delta).
\]

**Proof.** This follows immediately from Fact 2 and the change of variable formula for multiple integrals. \qed

**Lemma 3.3.** Let $u$ be as above and $\bar{u} = \bar{u}_t = u_t - c_t$, where $c_t$ is the constant determined by equation (2.4) with $\omega_t$ and $\omega_0$ there replaced by $\bar{\omega}_t$ and $\bar{\omega}_0$ respectively. Let $m_t = \inf_{\mathbb{R}^m} \bar{u}_t(t)$. Then there is a constant $C$ such that for any $t \in (0, \infty)$,
\[
|m_t| \leq C.
\]
Proof. We follow [30] to argue that \( m_t \) is bounded below. As noted above, \( \frac{1}{2\pi} Du = (\frac{1}{2\pi} \frac{\partial (u_0 + \varphi)}{\partial t_1}, \ldots, \frac{1}{2\pi} \frac{\partial (u_0 + \varphi)}{\partial t_m}) : \mathbb{R}^m \to \Delta_F \setminus \partial \Delta_F. \) So

\[
|Du_t| = |Du_t| \leq 2\pi \text{diam} (\Delta_F).
\]

Also note that since \( 0 \in \Delta_F \setminus \partial \Delta_F \) and \( \bar{u}_t \) is convex, \( \bar{u}_t \) attains \( m_t \).

From (2.9) we have

\[
\int_{\mathbb{R}^m} e^{-\bar{u}^t} dt = \int_{\mathbb{R}^m} A^{-1} e^{-\frac{\partial u}{\partial t} + c t} \det D^2 u dt.
\]

combining this with (2.1), (2.5) and Lemma 3.2 one sees that there are positive constants \( c_1 \) and \( c_2 \) independent of \( t \), such that

\[
c_1 \leq \int_{\mathbb{R}^m} e^{-\bar{u}^t} dt \leq c_2.
\]

Then one easily sees that \( m_t \) is uniformly bounded below.

To show \( m_t \) is bounded above we follow the argument in Section 3.4 of [9], which is a slight variant of that in [26]. Let \( v_t = v_t = \bar{u}_t - m_t \). From (2.1), (2.5) and (2.9) we get

\[
\det D^2 v_t = \det D^2 u = \exp \{ \frac{\partial u}{\partial t} - c_t - \bar{u} + \log A \} \geq c e^{-\bar{u}},
\]

where \( c \) is a positive constant independent of \( t \). It follows

\[
\det D^2 v_t \geq \frac{c}{e} e^{-m_t} \quad \text{when} \quad v_t \leq 1.
\]

Let \( \Gamma \) be the set where \( v_t \leq 1 \). By Lemma 3.1,

\[
\text{Vol}(\Gamma) \leq C \left( \frac{c}{e} e^{-m_t} \right)^{-1/2} = C' e^{m_t/2}.
\]

For each \( s > 0 \) let \( \Gamma_s \) be the set \( \{ v_t \leq s \} \) and \( V(s) = \text{Vol}(\Gamma_s) \). By convexity \( \Gamma_s \) is contained in the \( s \)-dilation of \( \Gamma \) w.r.t. the minimum point of \( v_t \). Then

\[
V(s) \leq s^m \text{Vol}(\Gamma) \leq C' s^m e^{m_t/2}. \tag{3.1}
\]

Note that from (2.1), (2.5) and (2.9) we have

\[
\det D^2 u_t \leq C e^{-m_t} e^{-v_t}. \tag{3.2}
\]

So by Lemma 3.2 and (3.2) we have

\[
C_1 = \int_{\mathbb{R}^m} \det D^2 u dt \leq C e^{-m_t} \int_{\mathbb{R}^m} e^{-v_t} dt.
\]

Using the co-area formula

\[
\int_{\mathbb{R}^m} e^{-v} dt = \int_0^\infty e^{-s} V(s) ds
\]

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and (3.1), we get from (3.3) that
\[ C_1 \leq Ce^{-mt} \int_0^\infty C'e^{-s}se^{mt/2}ds = C''e^{-mt/2}, \]
and we are done. \(\square\)

Let \(x_t\) be the minimal point of \(\bar{u}_t\), and \(\bar{u} = \bar{u}_t(\cdot) = \bar{u}_t(\cdot + x_t) - m_t\). Let \(\bar{\varphi}(\cdot) = \bar{u}(\cdot) - u_0(\cdot)\). We extend \(\bar{\varphi}\) to a \(G \times T^m\)-invariant function on \(X\), still denoted by \(\bar{\varphi}\). Using Facts 1, 2 and 3 in Section 2.1, convexity of \(\bar{u}\), (2.1), Perelman’s estimate (2.5), and Lemma 3.3, one can show that
\[ |\sup \bar{\varphi}| \leq C. \quad (3.4) \]

Compare the proof of Lemma 3.4 in [26] and Proposition 3.2 in [30].

Using Proposition 2.1 in [23], Lemma 4.4 in [16], the monotonicity (see (4.5) in [24]) of the generalized K-energy \(\tilde{\mu}\) (introduced in [23]) along certain modified Kähler-Ricci flow, Lemma 5.1 in [23], (3.4) above, Lemma 3.1 in [5], Proposition 3.1 in [24], and Lemma 3.3, one can show that
\[ ||\tilde{\varphi}||_{C^0(X)} \leq C. \quad (3.5) \]

Compare the proof of Proposition 4.1 in [30]. Using Lemma 5.1 in [23], Lemma 3.3 and (3.5) one can show that
\[ \tilde{\mu}(\varphi) \geq -C, \quad (3.6) \]
where \(\varphi\) is a solution of (2.3) as above; compare the proof of Corollary 4.2 in [30]. Then by using the monotonicity of the generalized K-energy along certain modified Kähler-Ricci flow and (3.6), one can show that
\[ |\frac{\partial \varphi}{\partial t}| \leq C, \]
after suitably normalizing the Ricci potential \(h_0\) of the initial metric. Compare Proposition 4.3 in [30].

As in Section 5 of [30] we can find \(x'_t \in \mathbb{R}^m\) such that
\[ |x_t - x'_t| \leq C \quad \text{and} \quad \frac{dx'_t}{dt} \leq C. \]

Let \(\tilde{u}(\cdot) = \tilde{u}_t(\cdot) = u_t(\cdot + x'_t)\) and \(\tilde{\varphi}_t(\cdot) = \tilde{u}_t(\cdot) - u_0(\cdot)\). The \(G \times T^m\)-invariant extension of \(\tilde{\varphi}_t\) gives a Kähler potential on \(X\) relative to \(\omega_0\), still denoted by \(\tilde{\varphi}_t\). Moreover, \(\frac{dx'_t}{dt}\) are corresponding to a family of holomorphic vector fields on \(X\), denoted by \(\tilde{Y}_t\). (The real part of) \(\tilde{Y}_t\) generate a family of elements, denoted by \(\rho_t\), in the algebraic torus subgroup \((T^m)^C\) of \(\text{Aut}(X, J)\). Then \(\rho_t^*\omega_{\varphi_t} = \omega_{\tilde{\varphi}_t}\), where \(\omega_{\varphi_t} = \omega_0 + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \varphi_t\) and \(\omega_{\tilde{\varphi}_t} = \omega_0 + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \tilde{\varphi}_t\). As in [30], using the estimates above, we can show that
\[ ||\tilde{\varphi}_t||_{C^0(X)} \leq C \quad \text{and} \quad |\frac{\partial \tilde{\varphi}_t}{\partial t}| \leq C. \]
With the above preparation we can proceed as in Section 5 of [30], using arguments as in Section 6 of [24] and the uniqueness theorem in [22, 23], and get that the Kähler metrics $\omega_{\tilde{\phi}}$ converge to a Kähler-Ricci soliton as $t \to \infty$.

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