How small an environment can be to still generate decoherence?

Nicolás Mirkin\textsuperscript{1,} and Diego Wisniacki\textsuperscript{1}

\textit{Departamento de Física “J. J. Giambiagi” and IFIBA, FCEyN, Universidad de Buenos Aires, 1428 Buenos Aires, Argentina}

(Dated: June 26, 2020)

The environment of an open quantum system is usually modelled as a large and complex many-body quantum system. However, the question of how large and complex this many-body quantum system must be in order to play the role of a genuine environment and thus generate decoherence is an often elusive key-point in the literature. In this work, by studying the reduced dynamics of a spin connected to a 1D Ising spin chain, we show that the physical mechanism responsible for generating decoherence on the reduced spin relies mostly on the chaotic behaviour of the chain. More interestingly, even in the case of an extremely short spin chain composed of two spins, by analyzing the decoherence dynamics of the reduced system we are able to reproduce the whole integrable to chaos transition. Finally, we discuss implications on quantum control experiments and show that quantum chaos reigns over the best degree of control achieved, even in small chains.

\textbf{Introduction.} Nowadays there is no discussion that in order to design universal quantum technologies to process quantum information effectively we must know how to deal with decoherence. This detrimental but unavoidably effect arises from the fact that in realistic scenarios quantum systems are not isolated but in touch with their surroundings. As a consequence, tons of effort and research have been put into understanding how to model this system-environment interaction and decipher what does ‘in touch’ and ‘surrounding’ exactly mean \cite{1–4}.

One of the most common hypotheses in this context is to consider the environment as a much larger quantum system than the system of interest. However, the real world might not be that simple in the sense that time scales introduced by the coupling to the outside world can be much slower than the time scales dictating internal equilibration, depending on how well isolated the open quantum system is \cite{5}. For instance, many of the experiments that are done today consist of working on a few well isolated qubits and executing controlled operations on some of them \cite{6–10}. In this scenario, one might wonder whether the set of qubits that are not being controlled, by interacting with the qubits that are, may affect controllability in the same sense that a large environment usually does. Within certain regime, these few qubits could be acting as an effective environment over the reduced system that is being addressed. Therefore, the question of how small and simple this environment could be to act as such and generate decoherence is absolutely relevant, not only from a fundamental point of view but also from the experimental side.

In the particular case of an isolated many-body quantum system, where the dimension of the Hilbert space is sufficiently high, this issue may be partially approached from the perspective of the Eigenstate Thermalization Hypothesis (ETH) \cite{5, 11–15}. In addition, the role of both Random Matrix Theory (RMT) and quantum chaos in the process of thermalization has been proven as crucial within the literature \cite{16–19}. Nevertheless, if we consider the opposite limit, where the many-body quantum system is not sufficiently large, both quantum chaos and decoherence seem certainly elusive to define. The study of this particular limit is the main purpose of our work.

In this Letter, by analyzing the reduced dynamics of a single spin connected to a 1D Ising spin chain, we focus on the requirements that this chain must fulfill in order to act as an effective environment and generate decoherence. Under this framework, our findings show almost an exact correspondence between the degree of decoherence suffered by the reduced system and how much chaos is present within the dynamics of the chain, i.e. the more chaos the more decoherence. More notably, just by studying the purity degradation of the reduced spin system, we are able to reconstruct the whole integrable to chaos transition even in the case of extremely short spin chains composed of two spins. We believe that the implications of our findings are two fold. First, since our method does not require to consider a large Hilbert space nor to determine a whole set of symmetrized energy eigenstates \cite{20–22}, it constitutes a novel and easy way of sensing the chaotic behaviour in complicated many-body quantum systems, which may be of experimental interest due to its simplicity \cite{23–27}. Secondly and more important, we also argue that this result has relevant implications in quantum control experiments. As we show at the end of our Letter, the optimal fidelities achieved for a simple control task over the reduced system strongly depend on the chaotic behaviour of the chain. In other words, the degree of control is subordinated to the degree of chaos present within the effective environment, even if the latter is small.

The physical system under consideration has no well-defined semiclassical limit and consists on a 1D Ising spin chain with nearest neighbor (NN) interaction and open boundary conditions, described by the Hamiltonian

\begin{equation}
H = - \sum_{k=1}^{L} \left( J_k \hat{\sigma}_z^k \hat{\sigma}_z^{k+1} + h_x \hat{\sigma}_x^k \right) - \sum_{k=1}^{L-1} J_k \hat{\sigma}_z^k \hat{\sigma}_z^{k+1},
\end{equation}
where $L$ refers to the total number of spin-1/2 sites of the chain, $\sigma^j_k$ to the Pauli operator at site $k = \{1, 2, ..., L\}$ with direction $j = \{x, y, z\}$, $h_s$ and $h_z$ to the strength of the magnetic field in the transverse and parallel direction, respectively, and finally $J_k$ represents the interaction strength within the site $k$ and $k + 1$. In general, we will consider equal couplings, i.e., $J_k = 1 \forall k = \{1, 2, ..., L-1\}$, situation in which the system presents a symmetry with respect to the parity operator $\Pi$. This implies that the spanned space is divided into odd and even subspaces with dimension $D = D^{\text{odd}} + D^{\text{even}} \approx D/2$. However, since in a realistic scenario couplings may be different due to some experimental error, we will also analyze the case with different values for $J_k$ and show the robustness of our result. In regards to the initial conditions, we will consider an initial pure random state as $|\psi(0)\rangle = |\psi_1\rangle |\psi_2\rangle \ldots |\psi_L\rangle$, where each spin at site $k$ initially points in a random direction on its Bloch sphere

$$|\psi_k\rangle = \cos \left( \frac{\theta_k}{2} \right) |\uparrow\rangle + e^{i\phi_k} \sin \left( \frac{\theta_k}{2} \right) |\downarrow\rangle,$$

with $\theta_k \in [0, \pi)$ and $\phi_k \in [0, 2\pi)$. Note that this ensemble of initial states maximizes the thermodynamic entropy and consequently equals to a situation of infinite temperature \cite{28}. From now on, we will take as the reduced system the first spin of the chain and consider the rest as an \textit{effective environment}. For example, a case with $L = 3$ will represent a situation of a single spin acting as an open system and coupled to an effective environment of only two spins. This may sound too simple but we remark that a recent experiment was able to capture chaotic behaviour on a 4-site Ising spin chain by measuring Out-of-Time Ordered Correlators (OTOC’s) of local operators on a nuclear magnetic resonance quantum simulator \cite{26}.

In order to fully characterize the integrable to chaos transition, beyond considering either the short or long time behavior of the OTOC \cite{20,29}, the standard procedure requires the limit of a high dimensional Hilbert space and the separation of the energy levels according to their symmetries \cite{30,32}. This may demand huge numerical effort or even be quite laborious to implement experimentally. Within all the standard chaos indicators used in the literature, in this work we will restrict ourselves to the so-called distribution of $\min(r_n, 1/r_n)$, where $r_n$ refers to the ratio between the two nearest neighbour spacings of a given level. More explicitly, by taking $c_n$ as an ordered set of energy levels, we can define the nearest neighbour spacings as $s_n = c_{n+1} - c_n$. With this nomenclature, it was proposed the following measure to gauge the presence of chaotic behavior \cite{33,35}

$$\tilde{r}_n = \frac{\min(s_n - s_{n-1})}{\max(s_n - s_{n-1})} = \min(r_n, 1/r_n),$$

where $r_n = s_n/s_{n-1}$. As the mean value of $\tilde{r}_n (\min(r_n, 1/r_n))$ attains a maximum when the statistics is Wigner-Dyson ($I_{WD}$) and a minimum when the statistics is Poissonian ($I_P$), we can normalize this quantity as

$$\eta = \frac{\min(r_n, 1/r_n) - I_P}{I_{WD} - I_P}.$$  \hspace{1cm} (4)

The parameter $\eta$ quantifies the chaotic behaviour of the system in the sense that $\eta \rightarrow 0$ refers to an integrable dynamics while $\eta \rightarrow 1$ to a chaotic one. While $\eta$ is based on the spectral properties of the entire system and requires to analyze separately even and odd subspaces, we will show that by studying the decoherence dynamics of a single spin we will be able to reconstruct the whole structure of the regular to chaos transition, even in the case of an extremely short spin chain and without resorting to any classification according to the energy level symmetries.

As a first approximation to the problem posed, let us focus first of all on how the purity of a single spin behaves in a typically integrable and chaotic regimes. With this purpose, we will solve the Schrödinger equation numerically for the whole system $\rho(t)$ and then trace over the environmental degrees of freedom, calculating the purity within the reduced density matrix $\rho(t)$ for the first spin of the chain. This is precisely what is shown in Fig. 1 wherein we plot the purity $\mathcal{P}(t) = \text{Tr} \{ \rho^2(t) \}$ as a function of time, considering different sizes for the environment and both chaotic and integrable regimes.

![Figure 1](image)

\textit{Left panel:} Purity of the reduced system as a function of time (in units of $J^{-1}$) for different sizes of the environment and within different regimes. While the two solid curves show the case of $L = 8$, the dotted ones the case of $L = 11$. Color orange refers to an integrable regime ($h_z = 2.5$) and violet to a chaotic one ($h_z = 0.5$). For all curves, the initial state is a pure random state for each individual spin. The rest of the parameters are taken as $h_x = 1$ and $J_k = 1 \forall k = \{1, 2, ..., L-1\}$. 

\textit{Right panel:} Time evolution of the reduced spin represented in the Bloch sphere, considering the case of $L = 11$ and both integrable (orange squares) and chaotic (violet circles) regimes.

From Fig. 1 we can see that while in the chaotic regime decoherence washes out the purity of the system, leading to a state of almost maximum uncertainty, this is certainly
not the case for the integrable regime, where the purity oscillates periodically around a mean value much greater than 1/2. Let us remark that the same systematic was observed setting different interaction parameters, initial states, environmental lengths and considering different spins of the chain as the reduced system.

Having this qualitative picture in mind, we now intend to measure the effect of decoherence in a more quantitative way. To do so, we will focus again on the purity degradation suffered by our reduced spin system $\rho(t)$ after a sufficiently large time interval, but now by considering an average purity defined as

$$\overline{P} = \frac{1}{N} \sum_{i=1}^{N} \left( \frac{1}{T} \int_0^T \text{Tr}\{\hat{\rho}_i(t)^2\} dt \right). \quad (5)$$

In the expression above, we first make a temporal average over the purity of a particular $\rho_i(t)$, defined by a given random initial state, and then we repeat this procedure for $N$ different initial random states, to finally perform a global average over all realizations. At the same time, in order to compare the averaged quantity $\overline{P}$ with the chaos measure introduced in Eq. (4), we define a normalized average purity as

$$\overline{P}_\text{Norm} = \frac{\overline{P} - \min(\overline{P})}{\max(\overline{P}) - \min(\overline{P})} \quad (0 \leq \overline{P}_\text{Norm} \leq 1). \quad (6)$$

With this definition, we have now all the necessary ingredients to raise up the following question: how does the purity degradation of the reduced system behaves as a function of the degree of chaos present in the rest of the chain? To address this issue, in Fig. 2 we plot the chaos indicator $\eta$ for a large chain composed of $L = 14$ spins ($D = 16384$) together with the average purity $\overline{P}_\text{Norm}$ of the reduced system for different sizes of the total spin chain, both as a function of the magnetic field $h_z$.

Interestingly, the behavior of the average purity of the reduced spin is quite similar regardless of the length of the environment. In fact, there is a well distinguished area within all the curves where the purity degradation is maximal. By comparing with the curve given by $1 - \eta$, we can clearly see that this region coincides almost perfectly with the region where chaos reigns, i.e. $(1 - \eta) \rightarrow 0$.

Quite remarkably, this is true even in the case when the environment is extremely short ($D = 8$), where we can observe a precise correspondence with the exception of a small deviation near $h_z \sim 0.5$. This little deviation can be smoothed by either taking more realizations over different sets of initial states or slightly increasing the size of the environment by one spin (not shown).

Various implications emerge from the analysis of Fig. 2. In first place, by using one spin as a probe and studying its purity dynamics, we were able not only to sense the chaotic behaviour present in the rest of the chain, but also to reconstruct the full integrable to chaos transition with a great degree of correspondence in comparison to other standard indicators of chaos. However, while the usual methods require a full diagonalization and classification of eigenenergies according to their symmetries within huge dimensional subspaces (see Supplemental Material), we have obtained the same results without requiring the above and even in much smaller subspaces. Moreover, the average over different realizations of the purity proved to be robust not only to the size of the environment, but also to whether we consider equal couplings or even a random set of $J_k$ modelling some hypothetical experimental error (see violet crossed curve in Fig. 2). A final interesting point to remark from the simulation is that when a small fraction from a large chaotic system is selected, some memory of the universal nature of the latter survives.

Keeping in mind the results presented so far, let us now examine the following hypothetical situation: consider an experimental scenario where it is possible to well-isolate from the external environment a 1D spin chain but also where some particular spin of this chain can be externally controlled. For instance, consider a time-dependent Hamiltonian

$$H = -\sum_{k=1}^{L} (h_z \hat{\sigma}_k^z + h_z \hat{\sigma}_k^z) - \sum_{k=1}^{L-1} J_k \hat{\sigma}_k^z \hat{\sigma}_{k+1}^z + \lambda(t) \hat{\sigma}_1^z, \quad (7)$$

where $\lambda(t)$ is a control field that can be experimentally tuned. Thus, it is possible you may want to implement...
some particular protocol over the spin you are able to control. For example, consider a population transfer protocol, where the first spin of the chain has to be addressed from the pure initial state $|\psi(0)\rangle = |0\rangle$ to the final target state $|\psi_{\text{targ}}\rangle = |1\rangle$. To do so, the time-dependent control field $\lambda(t)$ must be optimized to maximize the fidelity $F = |\langle \psi(T)|\psi_{\text{targ}} \rangle|^2$ at a given final evolution time $T$. In light of the results we presented before, you may be wondering the following question: is the maximum degree of control achievable subordinated to the degree of chaos present within the non-controlled environmental spins?

In order to answer such a question, we take the control function $\lambda(t)$ as a vector of control variables $\lambda(t) \rightarrow \{\lambda_i\} \equiv \vec{\lambda}$, i.e. a field with constant amplitude $\lambda_i$ for each time step. By dividing the evolution time $T$ into 16 equidistant time steps ($l = 1, 2, ..., 16$), the optimization was performed exploring several random initial seeds and resorting to standard optimization tools [36]. In Fig. 3 we plot the optimal fidelities achieved for our population transfer protocol, as a function of the magnetic field $h_z$ and for two different lengths for the total spin chain.

![Figure 3](image)

**Figure 3.** **Main plot:** Optimal fidelities achieved for a population transfer protocol over the first spin of the chain, as a function of the magnetic field $h_z$. The dashed curve represents a case of $L = 6$ spins and the solid one a case with $L = 9$ spins. Interaction parameters are set as $T = 20$, $h_z = 1$ and $J_k \in [0, 1, 1.5] \forall k = \{1, 2, ..., L - 1\}$. The initial state is $|0\rangle$ for the first spin of the chain and random for the rest of the system (see Eq. (2)). Only one realization for a single initial state and a random set of $J_k$ was considered. **Inset plot:** The same optimal fidelities of the main plot but as a function of the chaos parameter $\eta$.

Interestingly, we can conclude from Fig. 3 that the optimal fidelities achieved for a simple control problem, such as a population transfer protocol for the first spin of the chain, end up being very sensitive to the degree of chaos that is present within the rest of the spin chain. In fact, from the main plot we can see that the optimal fidelities behave quite similarly to the chaos parameter $1 - \eta$, as a function of the magnetic field $h_z$ (see Fig. 2). Accordingly, in the inset of Fig. 3 we plot the optimal fidelities obtained in the main plot but now as a function of the degree of chaos associated to the specific strength of the magnetic field $h_z$ (see again Fig. 2). By doing this, it is clear that the more chaos, the worse control. This last statement clearly relates to what we have been discussing before, in the sense that a greater degree of chaos is also associated with a stronger decoherence. Therefore, this means that the non-controlled system is acting as an effective environment for the spin that is being actively controlled and we argue that even in the case where this effective environment is small, its dynamics should be carefully tuned in order to minimize decoherence and thus improve the degree of control over the reduced system that is being addressed.

**Concluding remarks.** The goal of this Letter was to deepen into the interplay between decoherence, quantum chaos and optimal control in the limit of a not sufficiently high dimensional Hilbert space. Our main motivation was to analyze whether in realistic experiments, where a given many-body quantum system can be well-isolated from the external environment, decoherence can still arise due to the internal degrees of freedom of the system. In this context, by studying the purity dynamics of a spin connected to a 1D Ising spin chain, we found that its purity degradation can be used as a probe to sense the chaotic behaviour present within the rest of the chain. Quite remarkably, we show that a greater degree of decoherence is always associated with a more chaotic region, which allowed us to reconstruct the whole integrable to chaos transition even in the case where the environmental chain is merely composed of two spins. Our findings open an interesting question regarding the definition of quantum chaos, considering that when a small fraction from a large chaotic system is selected, some memory of the universal nature of the latter seems to survive. We argue that this last statement has also practical implications in quantum control experiments, as we have discussed in the last part of our work. For instance, by considering a simple protocol over a spin subject to a control field that can be experimentally tuned, we show that the best control achievable for this spin is a function of the degree of chaos present within the effective environment to which it is coupled. Consequently, in realistic experiments where a control task is sought over a reduced part of a system that is not necessarily large but that nevertheless presents signatures of quantum chaos, the interaction parameters must be carefully adjusted to avoid the chaotic regime and thus achieve a better performance of the control.

We acknowledge R. A. Jalabert for his fruitful insights about the manuscript. The work was partially supported by CONICET (PIP 112201 50100493CO), UBACyT (20020130100406BA), ANPCyT (PICT-2016-1056), and National Science Foundation (Grant No. PHY-1630114).
[1] H.-P. Breuer, F. Petruccione, et al., The theory of open quantum systems (Oxford University Press on Demand, 2002).
[2] W. H. Zurek, Reviews of modern physics 75, 715 (2003).
[3] M. A. Schlosshauer, Decoherence: and the quantum-to-classical transition (Springer Science & Business Media, 2007).
[4] A. Rivas and S. F. Huelga, Open quantum systems, vol. 13 (Springer, 2012).
[5] C. Gogolin and J. Eisert, Reports on Progress in Physics 79, 056001 (2016).
[6] L. DiCarlo, J. M. Chow, J. M. Gambetta, L. S. Bishop, B. R. Johnson, D. Schuster, J. Majer, A. Blais, L. Frunzio, S. Girvin, et al., Nature 460, 240 (2009).
[7] R. Blatt and C. F. Roos, Nature Physics 8, 277 (2012).
[8] C. Gross and I. Bloch, Science 357, 995 (2017).
[9] J. Yoneda, K. Takeda, T. Otsuka, T. Nakajima, M. R. Delbecq, G. Allison, T. Honda, T. Kodera, S. Oda, Y. Hoshi, et al., Nature nanotechnology 13, 102 (2018).
[10] C. Neill, P. Roushan, M. Kolodrubetz, Z. Chen, A. Megrant, B. Chiaro, A. Dunsworth, K. Arya, et al., Science 360, 195 (2018).
[11] M. Rigol, V. Dunjko, and M. Olshanii, Nature 452, 854 (2008).
[12] M. Rigol, Physical review letters 103, 100403 (2009).
[13] C. Neill, P. Roushan, M. Fang, Y. Chen, M. Kolodrubetz, Z. Chen, A. Megrant, R. Barends, B. Campbell, B. Chiaro, et al., Nature Physics 12, 1037 (2016).
[14] A. M. Kaufman, M. E. Tai, A. Lukin, M. Rispoli, R. Schittko, P. M. Preiss, and M. Greiner, Science 353, 794 (2016).
[15] A. Dymarsky, N. Lashkari, and H. Liu, Physical Review E 97, 012140 (2018).
[16] J. M. Deutsch, Physical Review A 43, 2046 (1991).
[17] M. Srednicki, Physical Review E 50, 888 (1994).
[18] M. Srednicki, Journal of Physics A: Mathematical and General 32, 1163 (1999).
[19] F. Borgonovi, F. M. Izrailev, L. F. Santos, and V. G. Zelevinsky, Physics Reports 626, 1 (2016).
[20] E. M. Fortes, I. García-Mata, R. A. Jalabert, and D. A. Wisniacki, Physical Review E 100, 042201 (2019).
[21] E. M. Fortes, I. García-Mata, R. A. Jalabert, and D. A. Wisniacki, arXiv preprint arXiv:2004.14440 (2020).
[22] J. de la Cruz, S. Lerma-Hernandez, and J. G. Hirsch, arXiv preprint arXiv:2005.06589 (2020).
[23] R. Barends, J. Kelly, A. Megrant, V. Iten, D. Sank, E. Jeffrey, T. C. White, J. Mutus, A. G. Fowler, B. Campbell, et al., Nature 508, 500 (2014).
[24] J. Kelly, R. Barends, A. G. Fowler, A. Megrant, E. Jeffrey, T. C. White, D. Sank, J. Y. Mutus, B. Campbell, Y. Chen, et al., Nature 519, 66 (2015).
[25] S. Debnath, N. M. Linke, C. Figgatt, K. A. Landsman, K. Wright, and C. Monroe, Nature 536, 63 (2016).
[26] J. Li, R. Fan, H. Wang, B. Ye, B. Zeng, H. Zhai, X. Peng, and J. Du, Physical Review X 7, 031011 (2017).
[27] M. K. Joshi, A. Elben, B. Vermeersch, T. Brydges, C. Maier, P. Zoller, R. Blatt, and C. F. Roos, arXiv preprint arXiv:2001.02176 (2020).
[28] H. Kim and D. A. Huse, Physical review letters 111, 127205 (2013).
[29] I. García-Mata, M. Saraceno, R. A. Jalabert, A. J. Roncaglia, and D. A. Wisniacki, Phys. Rev. Lett. 121, 210601 (2018), URL https://link.aps.org/doi/10.1103/PhysRevLett.121.210601.
[30] I. Percival, Journal of Physics B: Atomic and Molecular Physics 6, L229 (1973).
[31] M. V. Berry and M. Tabor, Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences 356, 375 (1977).
[32] O. Bohigas, M.-J. Giannoni, and C. Schmit, Physical Review Letters 52, 1 (1984).
[33] V. Oganesyan and D. A. Huse, Physical review b 75, 155111 (2007).
[34] Y. Atas, E. Bogomolny, O. Giraud, and G. Roux, Physical review letters 110, 084101 (2013).
[35] K. Kudo and T. Deguchi, Physical Review B 97, 220201 (2018).
[36] E. Jones, T. Oliphant, and P. Peterson, Web http://www.scipy.org (2001).
Supplemental Material

SENSITIVITY OF STANDARD INDICATORS TO THE DIMENSION OF HILBERT SPACE

It is well-known that the usual chaos indicators are quite sensitive to the dimension of the Hilbert space considered. Indeed, laborious diagonalizations over large subspaces are required to have reasonable statistics of the energy levels. For instance, to calculate $1 - \eta$ (see Fig. 2) we have resorted to a diagonalization within a dimension of $D = 16384$ eigenstates and then classified them according to their symmetries. Despite the huge dimension used for the calculation, this measure has the disadvantage of still being a bit noisy. As it is shown in the left panel of Fig. S1, this noise is quite sensitive to the reduction of dimensionality. If we slightly decrease the size of the system on which we perform the statistics, the fluctuations become larger and larger. However, as we have already discussed in the main text, this is not the case for the quantity $\tilde{P}_{Norm}$, which is robust to the size of the system under consideration. As can be seen from Fig. S1, as we increase the size of the system for the calculation of $1 - \eta$, the curve slowly tends to the one obtained by studying the purity dynamics of a system of much smaller dimensionality.

Figure S1. Left panel: The indicator $1 - \eta$ as a function of the magnetic field $h_z$ for different sizes of the total system. Only the odd subspace was taken into account for the calculation. Right panel: Same indicator $1 - \eta$ for a case of $L = 14$ spins ($D = 16384$) together with $\tilde{P}_{Norm}$ for a case of $L = 6$ spins ($D = 64$), both as a function of the magnetic field $h_z$. In both panels, interaction parameters are set as $h_x = 1, J_k = 1 \forall k, T = 50$. For the computation of $\tilde{P}_{Norm}$, 50 realizations considering different random initial pure states were performed.

AVERAGE OVER DIFFERENT REALIZATIONS VS A SINGLE REALIZATION

Since the standard chaos measures rely on the spectral and statistical properties of the entire system, being independent of the initial state, we have defined the quantity $\tilde{P}_{Norm}$ as an average over several random initial pure states such as to avoid privileging any particular initial condition. Nevertheless, it is also important to analyze in general what is the behaviour of this quantity for a single realization and verify that the variance of the averaged quantity is also small. This is precisely what is shown in Fig. S2 for different sizes of the total spin chain. We can see that even a single realization is sensitive to the integrable to chaos transition and the variance of the averaged quantity
is in general quite small, being a little higher as the strength of the magnetic field $h_z$ increases. We remark that this is consistent with our result illustrating the implications in quantum control experiments (see Fig. 3), where only a single realization considering a fixed random pure initial state was considered.

Figure S2. **Left panel:** Normalized average purity $\overline{P}_{Norm}$ for a single realization considering different sizes of the environment together with the chaos parameter $1 - \eta$, both as a function of the magnetic field $h_z$. For computing $1 - \eta$, a chain composed of $L = 14$ spins ($D = 16384$) was selected and only the odd subspace was taken into account ($D^{even} \approx 8192$). **Right panel:** Average purity $\overline{P}$ without normalizing, as a function of the magnetic field $h_z$. The linewidth of each curve represents the variance considering 50 different realizations of random initial pure states. In both panels, interaction parameters are set as $h_x = 1$, $J_k = 1 \forall k = \{1,2, ..., L - 1\}$ and $T = 50$. 