Matrix Models at Finite $N$

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Abstract

We summarize some aspects of matrix models from the approaches directly based on their properties at finite $N$.

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I. Introduction

Current interest in string theory in less than or equal to one dimension has arisen from the discovery of the existence of the continuum limit called double scaling limit \cite{1} which sums string perturbation theory. At the same time, it has become clear that the universal equations which govern the correlation functions near the critical points are given by hierarchical integrable differential equations \cite{1,2,3}.

By some time last year, however, more than several people had come to realize that many of the results obtained in the limit are already visible when N- the size of the matrix- is kept finite. This provides an opportunity to study the system in its original definition and as a solvable model. The progress from that direction is what we will discuss below. We will summarize some aspects of zero dimensional matrix models from the point of view based directly on results obtained at finite N.

Such analyses appear to be imperative in view of the recent status of nonperturbative 2d gravity: no limiting procedure has been found, up to now, which maintains both the reality of the partition function and the original combinatorial correspondence of a matrix model with a triangulated two dimensional surface \cite{1,3}. No satisfactory definition of continuum 2d gravity (nonperturbative) has, therefore, been given.

In the one matrix model, we will cover the following items: Virasoro constraints at finite N fully characterize the space of correlators: the equivalence of the two approaches, i.e. Dyson-Schwinger approach and the one based on orthogonal polynomials; how to take the double scaling limit of the Virasoro constraints at finite N near the (2,2k − 1) critical points \cite{3} to obtain the Virasoro constraints of a twisted boson \cite{4,5}; correspondence between the one matrix model and the classical Toda lattice equation \cite{4}: the derivation of Kazakov’s loop equation \cite{10} directly from the Virasoro constraints (See, for instance, \cite{11}). See also \cite{12} for these items.

In the case of the two matrix model, the situation has been clarified since the original developments and proposals \cite{2,13}. By now, we agree that the two matrix models capture the essential feature of the 2d gravity coupled to a general (p,p') conformal matter \cite{14,15}: it contains all the critical points indexed by a set of coprime integers (p,p'). We will discuss the new constraints of w∞ type derived in \cite{16}.
A. One matrix model

Recall the partition function of the one matrix model:

\[ Z_N^{(1)} \{ \{ g_\ell \} \} = \int d^{N^2} M e^{-\text{tr} V(M; g_\ell)} \]

\[ = \int \prod_i^N d\lambda_i \Delta (\lambda_1 \cdots \lambda_N) \Delta (\lambda_1 \cdots \lambda_N) e^{-\sum_{i=1}^N V(\lambda_i; g_\ell)}, \]  

(1)

\[ V(\lambda) = \sum_{\ell=1}^{\ell_{\text{max}}} g_\ell \lambda_\ell. \]

The Dyson-Schwinger equation is a set of consistency conditions on the space of correlators which is derived under general variations. It can be regarded as an equation for the space of string theories of a particular class (in this case, the ones described by the one matrix model). An efficient way which leads to the Virasoro constraints is the following: insert \(-\sum_{i=1}^N \lambda_i^{n+1} d d\lambda_i \) or \(\sum_{i=1}^N \frac{1}{\zeta - \lambda_i} d\lambda_i\) and express the results in terms of the couplings and the derivatives of the couplings in two different ways, using partial integrations. We obtain, after some calculation on the Vandermonde determinant,

\[ \hat{e}_n Z_N^{(1)} = 0, \quad n \geq -1, \]

\[ \hat{e}_n = \sum_{\ell=0}^{\infty} \ell g_\ell \frac{\partial}{\partial g_{\ell+n}} + \sum_{\ell=0}^{n} \frac{\partial^2}{\partial g_\ell \partial g_{n-\ell}}. \]  

(2)

These are the Virasoro constraints at finite \(N\). In this derivation, it is evident that they arise from the reparametrization of the eigenvalue coordinates.

As a natural extension of the above derivation and also as a warm-up to derive constraints of the two matrix model, consider the higher order differential operators

\[ \sum_i \lambda_i^n \left( \frac{d}{d\lambda_i} \right)^m \quad \text{or} \quad \sum_i \frac{1}{\zeta - \lambda_i} \left( \frac{d}{d\lambda_i} \right)^m. \]  

(3)

Let us introduce the following notations

\[ j \equiv -\sum_{\ell=1}^{\infty} \ell g_\ell \zeta^{\ell-1}, \]
\[
\frac{d}{dj} \equiv -\sum_{\ell=0}^{\infty} \zeta^{-\ell-1} \frac{\partial}{\partial g_\ell} .
\]

(4)

In ref. [16], we succeeded in writing the constraints in a closed form:

\[
w^{(s)}(\zeta) Z_N^{(1)} = 0 ,
\]

\[
s w^{(s)}(\zeta) = : Q_s \left[ j(\zeta) + \frac{d}{dj}(\zeta) \right] :(-) + (\cdot)^s Q_s^\dagger \left[ \frac{d}{dj}(\zeta) \right] :(-) ,
\]

(5)

where

\[
Q_\ell [f] \equiv \left( \frac{\partial}{\partial \zeta} + f(\zeta) \right) ^\ell .
\]

(6)

The existence of such constraints is peculiar in view of the fact that the Virasoro constraints in the double scaling limit fully characterize a space of correlators of topological gravity [7, 8]. We have checked [16] in the lowest few cases that these higher constraints are in fact reducible to the Virasoro constraints:

\[
w^{(2)}(\zeta) \quad \text{Virasoro}
\]

\[
w^{(3)}(\zeta) = (jw^{(2)}(\zeta))_{(-)} + \partial_\zeta w^{(2)}(\zeta) ,
\]

\[
\vdots
\]

In the approach based on the Dyson-Schwinger equation, we study a relationship of all possible singlet operators and their correlation functions. An alternative approach is the one based on orthogonal polynomials, where one studies all the matrix (both diagonal and off diagonal) elements of the eigenvalue operator \( \hat{\lambda} \) as well as its conjugate \( \frac{d}{d\lambda} \).

Let us enumerate basic properties of the orthogonal polynomials \( P_n(\lambda) = \lambda^n + c_{n-1}\lambda^{n-1} + \cdots \):

\[
\lambda P_n(\lambda) = P_{n+1} + e^{\delta_n} P_n(\lambda) + R_n P_{n-1}(\lambda) ,
\]

\[
\delta_{n,m} = \int d\lambda \ e^{-V(\lambda;g_\ell)} P_n(\lambda) P_m(\lambda) \frac{1}{\sqrt{h_n}} \frac{1}{\sqrt{h_m}} \equiv < n \mid m > ,
\]

\[
h_{n+1} = R_{n+1} h_n ,
\]

\[
Z_N = N! \prod_i h_i .
\]

(8)
The basic equations in this approach are the matrix elements of the Heisenberg algebra and a response of the system to a change of parameters, namely, a parametric derivative of the normalization constants $h_j$:

$$\delta_{i,j} = < i | \left[ \frac{d}{d\lambda}, \hat{\lambda} \right] | j > ,$$

(9)

$$\frac{d}{dt_k} \ln h_j = -\sum_\ell \left( \frac{\partial}{\partial t_\ell} g_\ell \right) < j | \hat{\lambda}_\ell | j > ,$$

(10)

where $t_k = t_k(g_i)$ are a set of parameters which are functions of the original (bare) couplings $g_i$. The first equation (9) is regarded as a discrete version of the string equation. The second one (10) may be called a discrete flow equation. The basic equations derived in the continuum theory are already visible when $N$ is kept finite.

In order to demonstrate the equivalence of the two approaches, let us rederive the Virasoro constraints from the equations (9),(10) above. Begin with

$$(n + 1) \hat{\lambda}^n = \left[ \frac{d}{d\lambda}, \hat{\lambda}^{n+1} \right] .$$

(11)

Take a trace of this relation and multiply by $Z_N$. Namely, $\sum_{j=0}^{N-1} < j | \cdots | j > Z_N$. Using eq. (10) ($t_\ell = g_\ell$), we find

$$(n + 1) \frac{\partial}{\partial g_n} Z_N = \sum_\ell \ell g_\ell \frac{\partial}{\partial g_{\ell+n}} Z_N - 2 \sum_{j=0}^{N-1} < j | \hat{\lambda}^{n+1} \frac{d}{d\lambda} | j > Z_N .$$

(12)

For the second term of the right hand side, use a formula which holds for any one-body operator

$$\sum_{j=0}^{N-1} < j | \mathcal{O} \left( \hat{\lambda}, \frac{d}{d\lambda} \right) | j >= <\sum_{i=1}^{N} \mathcal{O} \left( \lambda_i, \frac{d}{d\lambda_i} \right) >_{\text{ave}} ,$$

(13)

where $<\cdots>_\text{ave}$ denotes an averaging with respect to the partition function (eq. (1)). (The derivative is in between the two determinants and does not act on the potential.) Eq. (13) readily follows from

$$\Delta(\lambda_1, \cdots \lambda_N) = \text{det} \left( P_{i-1}(\lambda_j) \right) .$$

(14)

The calculation has reduced to the one mentioned in eq. (2). We obtain the Virasoro constraints at finite $N$ again.
The properties of the orthogonal polynomials have a curious connection to the classical Toda lattice equation. Let

\[ \phi_n = \ln h_n \quad (15) \]

From eqs. (9),(10), we find a hierarchy of classical differential equations whose first member is the classical Toda lattice equation:

\[ \left( \frac{\partial}{\partial g_1} \right)^2 \phi_n = e^{\phi_{n+1}-\phi_n} - e^{\phi_n} - e^{\phi_{n+1}-\phi_n} \quad (16) \]

We leave the details to the references [9]. It is our hope that explicit results for the physical quantities come out from this correspondence at finite \( N \).

Note that we are still dealing with quantum theory. The feature that quantum correlators obey classical integrable differential equations is shared by other exactly solvable models of quantum field theory [17].

It is instructive that the Kazakov’s original loop equation [10] can be directly derived from the Virasoro constraints without touching the properties of the measure factor i.e. the Vandermonde determinant. This fact has provided much inspiration to a recent work [11] on the construction of the supersymmetric loop equation by Alvarez-Gaumé, Mañes and the author. Let a loop with length \( \ell \) be represented by

\[ w(\ell) = \int \cdots \left( \sum_{i=1}^{N} e^{\lambda_i} \right) \frac{-N}{\sum_{j=1}^{V(\lambda_i;g)}} e^{\sum_{j=1}^{V(\lambda_i;g)}} , \]

where \( \cdots \) denotes the measure part which we do not touch. The operator which Kazakov introduced [10] is

\[ \hat{K} = \sum_{m=1}^{\infty} m g_m \left( \frac{\partial}{\partial \ell} \right)^{m-1} . \quad (18) \]

The action of \( \hat{K} \) on the loop turns out to be

\[ \hat{K} w(\ell) = -\frac{1}{Z_N} \sum_{n=0}^{\infty} \frac{\ell^n}{n!} \sum_{p=0}^{\infty} p g_p \frac{\partial}{\partial g_{p+n-1}} Z_N . \]

Use the Virasoro constraints eq. (2) and undo the procedure to carry out the second derivatives of the couplings acting on the partition function. We find

\[ \hat{K} w(\ell) = << \int_{0}^{\ell} d\ell' tre^{\ell'M} tre^{(\ell'-\ell)M} >> , \quad (19) \]
which becomes, after using the factorization property of singlet operators in the planar limit,
\[
\hat{K}w(\ell) = \int_0^\ell w(\ell')w(\ell - \ell') .
\] (20)
This is Kazakov’s loop equation \cite{Kazakov}. For more complete discussion as well as its extension to the supersymmetric loop equation and the determination of its critical points, see the recent preprint \cite{Kazakov-preprint}.

Let us finally discuss how to take a double scaling limit of the Virasoro constraints at finite \(N\) \cite{Zamolodchikov} to obtain the Virasoro constraints indexed by half integers (a single twisted boson) \cite{Zamolodchikov}. First, the potential must be fine-tuned in order to reach the \(k\)-th multicritical point of Kazakov \cite{Kazakov} describing 2d gravity coupled to \((2, 2k - 1)\) nonunitary minimal conformal matter. Let \(a\) be the lattice spacing in the level space of the orthogonal polynomials. As the leading contribution to the matrix elements of \(\hat{\lambda}\) is 2, we rescale as
\[
2 - \hat{\lambda} = a^{2/k}\hat{\lambda}_{sc} ,
\] (21)
with \(N \rightarrow \infty\) and \(a \rightarrow 0\), keeping
\[
g_{st} = 1/ \left( a^{2+1/k}N \right) = \text{finite} .
\] (22)
In order to drive the system slightly away from the critical point, we add to the original potential a source which has a nontrivial continuum limit:
\[
\sum_i \sum_\ell \tilde{j}_\ell (2 - \lambda_i) \ell^{+1/2} = \sum_i \sum_\ell \tilde{j}_\ell \lambda_i^\ell .
\] (23)
In the right hand side, we reexpanded the source in terms of the polynomial bases around the origin. This is added to the original couplings \(g_\ell\). We then undo the procedure to find a dressed (renormalized) source expressed in terms of the original bases \(\{(2 - \lambda_i)\ell^{+1/2}\}\). Schematically,
\[
g_\ell \rightarrow \tilde{j}_\ell + g_\ell \rightarrow j_\ell^{(sc)} = j_\ell + \cdots .
\] (24)
We find
\[
\hat{\ell}_{n}^{(tw)} Z_N^{(1)} [[\{g_\ell\}] + [\{j_\ell\}] = 0 , \quad n \geq 1 ,
\] (25)
\[
\hat{\ell}_{n}^{(tw)} = \sum_{\ell=0}^\infty (\ell + 1/2) j_\ell^{(sc)} \frac{\partial}{\partial j_{\ell+n}^{(sc)}} + \frac{1}{2} \sum_{\ell, n=1}^n \frac{\partial^2}{\partial j_{\ell-1}^{(sc)} \partial j_{n-\ell}^{(sc)}} , \quad n \geq 1 ,
\]
\[
\hat{\ell}_{a}^{(tw)} = \sum_{\ell=0}^\infty (\ell + 1/2) j_\ell^{(sc)} \frac{\partial}{\partial j_{\ell}^{(sc)}} + \frac{1}{16} ,
\]
This is the Virasoro constraints of a twisted boson derived in [8] in the double scaling limit. It is worthwhile to emphasize that the limiting process can be postponed until the very end [6].

**B. Two Matrix Model**

Let us now turn to the two matrix model. One starts with Mehta’s formula [18]:

\[
Z_N^{(2)} \left[ \{g^{(1)}_\ell\}, \{g^{(2)}_\ell\} \right] = \int dM_1 dM_2 e^{-\text{tr} V^{(1)}(M_1) - \text{tr} V^{(2)}(M_2) + c \text{tr} M_1 M_2},
\]

\[
= \int \prod_i d\mu_i e^{-\sum_{i=1}^N V^{(1)}(\mu_i)} \Delta (\mu_1, \cdots, \mu_N)
\]

\[
\int \prod_i d\lambda_i e^{-\sum_{i=1}^N V^{(2)}(\lambda_i) + c \sum_i \mu_i \lambda_i} \Delta (\lambda_1, \cdots, \lambda_N)
\]

This time, we first discuss the approach based on orthogonal polynomials. Let us summarize some of the properties:

\[
\delta_{ij} = \int d\lambda d\mu e^{-V^{(1)}(\mu) - V^{(2)}(\lambda) + c \mu \lambda} \tilde{P}_j(\mu) \frac{P_i(\lambda)}{\sqrt{h_j}} \frac{1}{\sqrt{h_i}} \equiv \tilde{\delta}_{ij},
\]

\[
Z_N^{(2)} = N! \prod_{i=0}^{N-1} h_i,
\]

\[
\lambda P_n(\lambda) = P_{n+1}(\lambda) + R_n P_{n-1}(\lambda) + S_n P_{n-3}(\lambda) + \cdots.
\]

Note that, unlike the case of the one matrix model, the number of terms appearing in the recursion relation (eq. (29)) depends upon the degree of the potential. This is a crucial observation of [14, 13]. (See [13, 19] for earlier results.)

It was Tada and Yamaguchi [14] (and Tada [20] later) who first examined carefully the matrix elements of the Heisenberg algebra of the two matrix model:

\[
[\hat{\lambda}, \hat{\mu}] = \frac{1}{c} 1.
\]
They were able to determine the potentials which can produce the critical points indexed by \((p, p') = (4, 5), (3, 8), (3, 5),\) and \((5, 6),\) precluding explicitly the earlier expectation that two matrix model contains the critical points of the kind \((3, *)\) alone. The same conclusion was reached by [15].

Tada and Yamaguchi [14] explicitly took the double scaling limit of the Heisenberg algebra:

\[
\hat{\mu} \to X ,
\hat{\lambda} \to Y = X^{q/p} ,
\]

\[
[Y, X] = 1 ,
\]

which is distinct from the original more heuristic analysis of Douglas [2]. A prescription to obtain \((p, p')\) critical points from an asymmetric potential of degree \(2(p' - 1)\) has been given [20]. A more general and detail analysis in the planar limit has been given recently in [21].

Let us now discuss the approach based on the Dyson-Schwinger equation. It is a useful point of view to regard the second set of integrations in eq. (26) as a Laplace transform \(L\) of \(\Delta e^{-V(2)}\):

\[
Z_{N}^{(2)} = \int \prod_{i=1}^{N} e^{-\sum_{j} V(2)_{(\mu_{j})}} \Delta (\mu_{1}, \cdots \mu_{N}) \left[ \Delta e^{-V^{(2)}} \right] (\mu_{1}, \cdots \mu_{N}) .
\]

We again insert differential operators of degree \(m\)

\[
D_{m} (p) = \sum_{i} \frac{1}{\zeta - \mu_{i}} \left( \frac{d}{d\mu_{i}} \right)^{m}.
\]

Computing the action in two different ways, and using the formulas (eqs. (8),(9)) developed in the one matrix model, we find

\[
w (n, m) Z_{N}^{(2)} = 0 ,
\]

\[
w (n, m) = - \frac{(-)^{m}}{m + 1} \text{Res}_{\zeta=0} \left( \zeta^{n} : Q_{m+1}^{\dagger} \left[ j_{1} + \frac{d}{d j_{1}} \right] : \right)_{(-)}
\]

\[
+ \frac{(-)^{n}}{n + 1} \text{Res}_{\zeta=0} \left( \zeta^{n} : Q_{n+1}^{\dagger} \left[ j_{2} + \frac{d}{d j_{2}} \right] : \right)_{(-)} .
\]
Let me suggest how to take a double scaling limit of the constraint eq. (34), (35). To my knowledge, this procedure has not been carried out explicitly yet. Some progress has been made recently [22].

First of all, a set of couplings \( g_\ell^{(1)} \) and \( g_\ell^{(2)} \) must be found which produces the \((p, p')\) critical point and we fine-tune the potential to this point. Insert a source term to the potential:

\[
- \sum_{i, \ell \neq 0 (\mod p)} (c - \mu_i)^{\ell/p} t_\ell - \sum_{i, \ell \neq 0 (\mod p)} (c - \lambda_i)^{\ell/p} t_\ell .
\]  

Express the constraint eq. (34) in terms of the “renormalized” source in which the original couplings are absorbed. By taking the limit, we try to see the reduction of our constraints to the \(W_{p+1}\) constraints of [8] proven by [23].

Other results from loop equations include [24].

C. Discussion

Here, we will discuss a few scattered items which deserve further investigations. In principle, one can apply the above method of deriving constraints to the \(c = 1\) model. In practice, this is not such an easy task: we have to introduce a set of time dependent couplings to the potential of the one dimensional matrix model. This would lead to the functional differential constraints on the partition function even if we could write them down in a closed form. If we restrict our attention to time-independent (static or zero momentum) quantities alone, it is possible to derive useful constraints [25].

The basic quantity of interest is the density of states

\[
\rho (\mu) = \frac{1}{N} \text{tr} \delta \left( \mu - \hat{h} \right) \\
= -\frac{1}{\pi N} \text{Im} \sum_{n=0}^{\infty} < n \mid x > \left( \frac{1}{\mu - \epsilon_n + i0} \right) < x \mid n > .
\]  

rather than the partition function. Or consider a once integrated quantity

\[
P (\mu) = -\frac{1}{\pi N} \text{Im} \text{tr} \ln \left( \mu - \hat{h} + i0 \right) .
\]

The coordinate \(x\) of the one particle hamiltonian \(\hat{h}\) originates from the eigenvalues of the matrix variable. Virasoro constraints are obtained from the reparametrization of eigenvalues but the procedure is not identical to that of the one matrix model [25].
Our basic formulas (eqs. (5), (6), (34), (35)) can be regarded as a kind of bosonization of nonrelativistic fermions. A ramification to W geometry has been pursued recently [26].

What we have been witnessing is the transmutation of the eigenvalue coordinates into the target space degrees of freedom. This phenomenon is also present in one dimensional matrix model [27], but conceptual understanding seems to be still lacking. Constraint equations in general provide nontrivial information on target space physics such as physics of black holes [28]. Clearly much more work must be dedicated on these issues together with the issue of the definition of nonperturbative 2d gravity.

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