On the Insertion Time of Cuckoo Hashing

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Abstract

Cuckoo hashing is an efficient technique for creating large hash tables with high space utilization and guaranteed constant access times. There, each item can be placed in a location given by any one out of $k$ different hash functions. In this paper we investigate further the random walk heuristic for inserting in an online fashion new items into the hash table. Provided that $k \geq 3$ and that the number of items in the table is below (but arbitrarily close) to the theoretically achievable load threshold, we show a polylogarithmic bound for the maximum insertion time that holds with high probability.

1 Introduction

Hash tables are widely used in applications that need efficient data structures for large sets of data. A key issue in the use of hash-tables is the handling of collisions. A technique that attracted quite a bit of attention in recent years is the so-called cuckoo hashing, cf. e.g. [2, 13, 19, 23] and the references therein, that is based upon the paradigm of the power of many choices [1, 20].

The term cuckoo hashing was coined by Pagh and Rodler in [23]. In the present work we will consider a slight variation of it, as defined in [13]. We are given a table $T$ with $n$ locations, and we assume that each location can hold only one item. Further generalizations where two or more items can be stored have also been studied, see e.g. [9, 8, 12], but we will not treat those cases. Moreover, we assume that we have $k \geq 2$ hash functions $h_1, \ldots, h_k$ that each maps an element $x$ of a universe $U$ of items to a position in the table $T$. More precisely, we assume that $h_1, \ldots, h_k$ are independent truly random functions $h_i : U \rightarrow T$.

This assumption is somehow idealized, as exponentially many bits would be needed to store such truly random functions. However, there is theoretical evidence that even “simple” hash

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functions can be sufficient in practice, provided that the underlying data stream fulfills certain natural conditions; we refer the reader to the papers [21] and [8], and the references therein.

In his recent survey [19] Mitzenmacher outlined several open problems that remain to be solved in order to better understand the power of cuckoo hashing from a theoretical point of view. Among these were the issue of space utilization and the search for good upper bounds for the time needed to insert new elements. The first point was recently solved independently by the first two authors [14], as well as Frieze and Melsted [15] (for \( k \geq 4 \)) and Dietzfelbinger et al. [7]. The aim of this paper is to address the second point. Before we can state our results formally, we need to outline the results from [7, 14, 15] in more detail.

A natural question in cuckoo hashing is the following. Let us denote by \( I \subset U \) the set of available items. As \( |I| \) increases, it becomes more and more unlikely that all of them can be inserted into the table so that each item is assigned to one of its \( k \) desired locations. In other words, if \( |I| \) is “small” compared to \( |T| \), then with high probability, there is such an assignment to the locations in the table that respects the \( k \) choices of the items. On the other hand, if \( |I| \) becomes “large”, then such an assignment does not exist with high probability (trivially, this happens at the latest when \( n + 1 \) items are available). The important question is whether there is a critical size for \( I \) where the probability for the existence of a valid assignment drops instantaneously in the limiting case from 1 to 0, i.e., whether there is a load threshold for cuckoo hashing. More precisely, we say that a value \( c^*_k \) is the load threshold for cuckoo hashing with \( k \) choices per element if

\[
P \left( \text{there is an assignment of } \lfloor cn \rfloor \text{ items to a table with } n \text{ locations that respects the choices of all items} \right) \xrightarrow{n \to \infty} \begin{cases} 1, & \text{if } c < c^*_k, \\ 0, & \text{if } c > c^*_k. \end{cases}
\]

For the case \( k = 2 \), where each item has two preferred random locations, there is a natural connection with random graphs: we think of the \( n \) locations of \( T \) as the vertices of the graph, and of the items as edges, which encode the two choices for each item. If \( |S| = m \), what we obtain is the Erdős-Rényi random (multi-)graph \( G^*_{n,m} \). The properties of this random graph are essentially those of the random graph \( G_{n,m} \) on \( n \) vertices and \( m \) distinct random edges. Moreover, it easy to see by applying Hall’s Theorem that \( G_{n,m} \) has no subgraph with more edges than vertices if and only if the corresponding items can be assigned to the corresponding locations such that the choices of all items are respected. It is well-known that the property “\( G_{n,m} \) has a subgraph with more edges than vertices” coincides with the appearance of a giant connected component that contains a linear fraction of the vertices; see e.g.

[17]. As the latter is known to happen around \( m = n/2 \), we readily obtain that the load threshold for cuckoo hashing and \( k = 2 \) is at \( c^*_2 = 1/2 \). In other words, at most half of the table can be filled in a way that respects the choices of all items.

For \( k \geq 3 \) recently results were obtained independently by the first two authors [14], as well as Frieze and Melsted [15] (for \( k \geq 4 \)) and Dietzfelbinger et al. [7].

**Theorem 1.1.** For any integer \( k \geq 3 \) let \( \xi^* \) be the unique solution of the equation

\[
k = \frac{\xi^*(1 - e^{-\xi^*})}{1 - e^{-\xi^*} - \xi^* e^{-\xi^*}}.
\]

Then \( c^*_k = \frac{\xi^*}{k(1 - e^{-\xi^*})^{k-1}} \) is the load threshold for cuckoo hashing with \( k \) choices per element. In particular, if there are \( \lfloor cn \rfloor \) items, then the following holds with probability \( 1 - o(1) \).
1. If \( c < c^*_k \), then there is an assignment of the items to a table with \( n \) locations that respects the choices of all items.

2. If \( c > c^*_k \), then such an assignment does not exist.

Numerically we obtain for example that \( c^*_1 = 0.917 \), \( c^*_2 = 0.976 \) and \( c^*_5 = 0.992 \), where “\( \approx \)” indicates that the values are truncated to the last digit shown. Moreover, a simple calculation reveals that \( c^*_k = 1 - e^{-k} + o(e^{-k}) \) for \( k \to \infty \); see [14].

Note that Theorem 1.1 is non-algorithmic: it just states that whenever the load of the hash-table is below the threshold then there exists with probability \( 1 - o(1) \) an assignment of the items that respects the choices of all items. The theorem does not, however, address the question whether we can actually find such an assignment efficiently. This is the problem that we address in this paper.

More precisely, our main aim in this paper is to bound the time needed to insert an additional item into the table, assuming that some given number of items have already been inserted properly. A natural insertion strategy is to use a randomized approach: insert the new item randomly at one of its locations, if the location is currently occupied, kick the item out and reinser it randomly at one of its other \( k - 1 \) locations; recurse if necessary (see below for a more formal statement of this algorithm). In [16] Frieze et al. studied this algorithm and showed that the running time is polylogarithmic with high probability provided \( k \geq 8 \) and the load of the hash-table is not too close to the threshold \( c^*_k \).

The main theorem of our paper states that the random insertion algorithm actually succeeds in polylogarithmic time with high probability for any number of inserted items arbitrarily close to the load threshold and for all \( k \geq 3 \).

**Theorem 1.2.** For \( k \geq 3 \) let \( c := \frac{\log((k-1)e^k)}{(k-1)\log(k-1)} \). For any set of \( m = (1-\varepsilon)c^*_k n \) items of the universe \( U \), where \( \varepsilon \in (0,1) \), if the hash functions \( h_1, \ldots, h_k \) are random, then for any \( \zeta > 0 \) sufficiently small with probability \( 1 - o(1) \) each of the items will be inserted into a table with \( n \) positions in time \( O(\log^{2+c+\zeta} n) \).

In fact, we show the following slightly stronger statements: (i) if \( m = (1-\varepsilon)c^*_k n \) elements are inserted into a hash table with \( n \) positions, for some \( \varepsilon > 0 \), then with probability \( 1 - o(1) \) the positions determined by the hash functions satisfy certain ‘nice’ structural properties, and (ii) if \( m \) elements satisfy these ‘nice’ properties then an additional element can be inserted in time \( O(\log^{2+c+\zeta} n) \) with probability \( 1 - O(n^{-1-\zeta/2}) \). Observe that \( c \approx 2.66 \) for \( k = 3 \), \( c \approx 1.54 \) for \( k = 4 \) and \( c \approx 1.15 \) for \( k = 5 \). Moreover, \( c = \frac{1}{\log k} + O(\frac{1}{k}) \) as \( k \) grows. Our exponent in the bound of the running time thus compares very favorably with that from [16].

### 2 The Insertion Algorithm and its Analysis

#### 2.1 Random Walk Insertion

Roughly speaking, the insertion procedure that we study works as follows. Assume that \( m \) items have been inserted and we are about to insert the \( (m+1) \)st item. This item is assigned \( k \) random positions from the hash table and sits on one of them. However, this position might already be occupied by a previously inserted item. If this is the case, then this item is kicked out and sits on one of the other \( k - 1 \) selected positions. In turn, this position might be occupied by another item, which is kicked out and goes to one of its remaining \( k - 1 \) positions.
positions. This process may be repeated indefinitely or until a free position is found. If each selection takes place uniformly at random among the \(k-1\) available positions, this is a random walk on the positions of the table (or, as we shall shortly see, on the vertex set of a suitably defined hypergraph). Thus, it is called the \textit{random walk insertion}.

Let us now be more formal. We assume that there are \(k \geq 3\) hash functions \(h_1, \ldots, h_k\) which map a universe \(U\) to a hash table \(T\) with \(n\) positions. We denote by \(T(i)\) the contents of the \(i\)th position of \(T\), and we write \(T(i) = \emptyset\) if the \(i\)th position is empty. Also, if \(e\) is an item that has been inserted into the table, we denote by \(H(e)\) the index of the hash function that \(e\) currently uses. With these definitions available, we are able to describe formally the insertion algorithm.

| Table 1: The random-walk insertion algorithm |
|---------------------------------------------|
| 1  SUC \(\leftarrow\) FALSE;               |
| 2  \(e \leftarrow e_{m+1};\)               |
| 3  \(j \leftarrow \emptyset;\)            |
| 4  repeat                                 |
| 5  Choose uniformly at random \(i \in \{1, \ldots, k\} \setminus j;\) |
| 6  \(H(e) \leftarrow i;\)                 |
| 7  if \(T(h_i(e)) \neq \emptyset\) then   |
| 8  \(e' \leftarrow T(h_i(e));\)           |
| 9  \(j \leftarrow H(e');\)                |
|10  \(T(h_i(e)) \leftarrow e;\)           |
|11  \(e \leftarrow e';\)                   |
|12  else                                   |
|13  \(T(h_i(e)) \leftarrow e;\)           |
|14  SUC \(\leftarrow\) TRUE;              |
|15  endif                                  |
|16  until SUC = TRUE                       |

The analysis of this algorithm reduces the allocation of elements to a hypergraph setting. The hash table of size \(n\) corresponds to a set of \(n\) vertices, and the locations of each item correspond to a hyperedge of size at most \(k\). As we assume that the hash functions are truly random, a set of \(m\) elements gives rise to a \(k\)-uniform random hypergraph with \(n\) vertices and \(m\) edges each chosen uniformly at random (and with replacement) among all \(k\)-multisubsets of the vertex set. Thus, the random walk on the positions of the hash table, which is induced by the insertion algorithm, naturally gives rise to a random walk on the vertex set of this hypergraph. The techniques that we use to prove Theorem \ref{main theorem} are a combination of the approach of Frieze et al. \cite{Frieze} together with strong structural properties of such a random hypergraph.

### 2.2 Random Hash Functions and Random Hypergraphs

Assume that we have already inserted \(m\) elements which have been allocated among the \(n\) positions of the hash table, using the random hash functions \(h_1, \ldots, h_k\). This translated into a random hypergraph setting means that a random (multi)hypergraph \(H_{n,m,k}^*\) on the vertex set \(V_n := \{1, \ldots, n\}\) has been created, where each of the \(m\) edges is an ordered \(k\)-tuple of elements of \(V_n\) chosen with probability \(1/n^k\) with replacement. Each edge corresponds to
the $k$ choices of an item. Note that in this definition of $H^*_{n,m,k}$ we actually interpret the word “multi” in two ways: (i) an edge has size $k$, but may contain a particular vertex several times, and (ii) the edge set of $H^*_{n,m,k}$ may be a multiset, i.e., a particular edge can occur multiple times.

With slight abuse of terminology, we say that a multi-hypergraph $H = H(V,E)$ is a $k$-graph if it is $k$-bounded, that is, every hyperedge is a subset of $V$ with at most $k$ vertices. Observe, that the random hypergraph $H^*_{n,m,k}$ corresponds in a natural way to a $k$-graph, by projecting each ordered $k$-tuple of vertices that forms an edge in $H^*_{n,m,k}$ to the set of vertices contained in this $k$-tuple. In what follows, we will be using the symbol $H^*_{n,m,k}$ to denote both objects; each time the interpretation should be clear from the context.

### 2.3 Orientations and the $h$-neighborhood of a Vertex

For a $k$-graph $H = H(V,E)$ with vertex set $V$ and edge set $E$ where $|E| \leq |V|$ an injective mapping $h : E \rightarrow V$ of the set of edges into the set of vertices of $H$ such that each edge is mapped to one of its vertices is called an orientation of the edges. In the setting of cuckoo hashing, an orientation corresponds to an assignment of the items to the hash table.

Assume that $H = H(V,E)$ is a $k$-graph with $|E| \leq |V|$ and let $h$ be an orientation of $E$. For $v \in V$ we define its first $h$-neighborhood to be the set of vertices apart from $v$ itself that belong to the edge which is oriented to $v$. More generally, having defined the $(t − 1)$st $h$-neighborhood, for $t > 1$, the $t$th $h$-neighborhood of $v$ is the set of vertices which belong to the edges that are oriented to the vertices of the $(t − 1)$st $h$-neighborhood of $v$ and belong neither to it nor to any one of the previous neighborhoods. Also, we define the 0th $h$-neighborhood of $v$ to be $v$ itself. Note that for any $t \geq 1$, the $t$th neighborhood may contain at most $(k − 1)^t$ vertices. Also, we denote by $N_{h,t}(v)$ the number of vertices that are within $h$-distance $t$ from $v$ and $N_{h,t}(v)$ the set of these vertices. Thus,

$$N_{h,t}(v) \leq (k − 1)^{t+1}. \quad (2.1)$$

If $v$ is a vertex of $H$ and $S$ a subset of $V(H)$, then the $h$-distance of $S$ from $v$ is $d_h(v,S) := \min\{t : N_{h,t}(v) \cap S \neq \emptyset\}$. Finally, if $h$ is an orientation, we denote by $F_h$ the set of free vertices, that is, vertices to which no edge has been oriented. We call the remaining vertices occupied.

The insertion algorithm can be viewed as a random walk on the vertex set of the corresponding $k$-uniform hypergraph. Note that every (proper) assignment of the elements to positions in the hash table corresponds to an orientation in the associated hypergraph and vice versa. If we want to stress that the random walk starts with a particular assignment, that corresponds to an orientation $h$, we also speak of an $h$-random walk on the vertex set of the hypergraph $H = H(V,E)$.

### 2.4 Proof of Theorem 1.2

Analysis of the Insertion Algorithm

Let us fix an $\varepsilon > 0$ and assume that $m = (1 − \varepsilon)c^*_k$ items have been allocated. The corresponding hypergraph is distributed as $H^*_{n,m,k}$. Let us fix the realization of $H^*_{n,m,k}$ and also let $h$ be the orientation on the edges of $H^*_{n,m,k}$ induced by the allocation of the inserted items. We will bound the running time of the $h$-random walk performed on this particular realization of $H^*_{n,m,k}$ under the assumption that the latter satisfies some high probability properties which will be stated explicitly below.
Now, suppose that the new element $e_{m+1}$ is initially inserted into vertex $v \in V_n$ and assume that $v$ is occupied for otherwise we are done. Following Frieze et al. [16], we consider a decomposition of the vertex set $V_n$ into two sets according to the $h$-distance of $F_h$ from each vertex. In particular, for some $C > 0$, we let $S \subseteq V_n$ be the set of vertices $v \in V_n$ such that $d_h(v, F_h) \leq C$ and let $B = V_n \setminus S$. In the work of Frieze et al. [16], the parameter $C$ was of order $\log \log n$. In the present work, we let $S$ be the set of vertices that have $h$-distance at most $C$ from $F_h$, for some suitable constant $C$. We show that there is a constant $C = C(\varepsilon)$ such that $S$ covers almost all of the hypergraph.

Note that, if $v \in S$ then the definition of $S$ implies that there is at least one free vertex within $h$-distance $C$ from $v$, and the $h$-random walk will thus hit a free vertex with probability at least $1/(k-1)^C$ within the next $C$ steps.

In order to treat the case $v \not\in S$ we will first show that certain expansion properties of $H^*_{n,m,k}$ (that hold with probability $1 - o(1)$) guarantee that the $h$-neighborhood up to $h$-distance roughly $\log_{k-1} n$ from $v$ grows almost like a $(k-1)$-regular hypertree. This, in turn, will then allow us to show that for vertices $v \not\in S$ the $h$-random walk will hit $S$ with reasonably high probability after a logarithmic number of steps.

Our plan for the proof of Theorem 1.2 is thus as follows. In the next two subsections we define two properties of $k$-graphs, a density and an expansion property, and show that a random hypergraph $H^*_{n,m,k}$ has these properties with probability $1 - o(1)$. We also show that the density property implies that the set $S$ is large, and that the expansion property implies that a random walk starting in a vertex not in $S$ will hit $S$ within a logarithmic number of steps with high probability. In Section 2.4.3 we then formally show how these two properties conclude the proof of Theorem 1.2.

### 2.4.1 Density Properties

Assume that $H = H(V,E)$ is a $k$-graph and for every subset $V' \subseteq V$ we denote by $E(V')$ the set of edges of $H$ induced on $V'$. Additionally, we set $e(V') := |E(V')|$. We define the following property.

**Property $D_\delta$:** There exists a $\delta > 0$ such that for all non-empty $V' \subseteq V$ we have

$$e(V') < (1 - \delta)|V'|.$$

The next proposition states that if $H$ has Property $D_\delta$, then most of the vertices are such that $F_h$ is within bounded $h$-distance from them.

**Proposition 2.1.** If a $k$-graph $H = H(V,E)$ has Property $D_\delta$ and $h$ is an orientation of $E(H)$, then for all $\alpha > 0$ there exists $C = C(\alpha, \delta) > 0$ and a set $S \subseteq V$ of size at least $(1 - \alpha)|V|$ with the property that for every $v \in S$ we have $d_h(v, F_h) \leq C$.

We defer the proof to Section 3. The next theorem states that $H^*_{n,m,k}$ has Property $D_\delta$ with high probability for some suitable $\delta$. More precisely, the following holds.

**Theorem 2.2.** Let $\varepsilon > 0$ and suppose that $m = (1 - \varepsilon) c_k^\alpha$. There exists a $\delta = \delta(\varepsilon, k) > 0$ such that $H^*_{n,m,k}$ has property $D_\delta$ with probability $1 - O(1/n)$.

We prove this theorem in Section 4. Together with Proposition 2.1 this gives us a statement about the typical structure of $H^*_{n,m,k}$.
Corollary 2.3. Let $\varepsilon > 0$ and set $m = (1 - \varepsilon)c^*_k n$. Then $H^*_{n,m,k}$ has the following property with probability $1 - O(1/n)$. For every $\alpha > 0$ there is a $C = C(\alpha, \varepsilon) > 0$ such that for every orientation $h$ there exists a set $S \subseteq V_n$ such that $|S| \geq (1 - \alpha)n$ and every $v \in S$ satisfies $d_h(v, F_h) \leq C$.

2.4.2 Expansion Properties

For a set of edges $E'$ of a hypergraph $H = H(V, E)$, we denote by $V(E')$ the set of vertices contained in edges of $E'$. We say that a $k$-graph $H = H(V, E)$ has expansion property $\mathcal{E}$, if it satisfies the following two conditions:

Property $\mathcal{E}$:

1. For all $E' \subseteq E$ with $\log \log |V| < |E'| < |V|/k$ we have
   $$|V(E')| \geq (k - 1 - x_{\ell} |E'|)|E'|,$$
   where $x_{\ell} = \frac{\log((k-1)n^k)}{\log(|V|/s)-1}$.

2. For all $E' \subseteq E$ with $|E'| \leq \log \log |V|$ we have
   $$|V(E')| \geq (k - 1)|E'|.$$

Note that the choice of the parameters in this definition is somewhat arbitrary and not best possible. Nevertheless, they suffice for our arguments. In Section 5 we show that $H^*_{n,m,k}$ has Property $\mathcal{E}$ with high probability.

Proposition 2.4. If $m \leq c^*_k n$, then $H^*_{n,m,k}$ has Property $\mathcal{E}$ with probability $1 - o(1)$.

Given an orientation $h$ of a hypergraph and a vertex $v$, recall that $N_{h,t}(v)$ is the number of vertices within $h$-distance at most $t$ from $v$. In Section 5 we show the following.

Lemma 2.5. For $k \geq 3$, let $H = H(V, E)$ be a $k$-graph on $n$ vertices which has Property $\mathcal{E}$, and let $h$ be any orientation of its edges. Then, for any $\alpha > 0$ small enough and for any $\zeta > 0$ with $T = \log_{k-1} n + (c + \zeta) \log_{k-1} \log_{k-1} n$ and $c$ as in Theorem 1.2 the following holds for $n$ sufficiently large. If $v \in V$ is such that there is no free vertex within $h$-distance $T$ from $v$, then $N_{h,T}(v) > \alpha n$.

Note that Lemma 2.5 only handles the case where there are no free vertices within $h$-distance $T$ from a given vertex $v$. Intuitively, it seems obvious that this case should dominate the running time of the insertion algorithm. To make this formal, we need an additional definition. Given a hypergraph $H = H(V, E)$ and an orientation $h$ of its edges, we define an auxiliary hypergraph $\tilde{H} = \tilde{H}(V', E')$ by replacing every free vertex of $H$ by a $(k - 1)$-regular hypertree of depth $T$ (on a new set of vertices) rooted at this vertex. We extend the orientation of the edges of $H$ to the edges of $\tilde{H}$ by orienting each new edge towards the root of its tree. Thus the leaves of each such tree are the free vertices of $\tilde{H}$. As we will see in Section 5 Proposition 2.4 together with Lemma 2.5 imply also a good neighborhood growth for $H^*_{n,m,k}$.

Corollary 2.6. For every $k \geq 3$, let $H = H(V, E)$ be a $k$-graph on $n$ vertices which has Property $\mathcal{E}$, let $h$ be any orientation of its edges, and let $\tilde{H} = \tilde{H}(V', E')$ and $h'$ be as defined above. Then, for any $\alpha > 0$ small enough and for any $\zeta > 0$ with $T = \log_{k-1} n + (c + \zeta) \log_{k-1} \log_{k-1} n$ and $c$ as in Theorem 1.2 the following holds for $n$ sufficiently large. For all $v \in V$ we have $N_{h,T}(v) > \alpha n$. 

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2.4.3 Proof of Theorem 1.2

We can now put together Corollary 2.3 and Corollary 2.6 and derive a high probability bound on the running time of the random walk insertion algorithm. Assume that $m = (1−\varepsilon)e_k^n$. From Theorem 2.2 we know that there exists a $\delta = \delta(\varepsilon,k) > 0$ such that $H^*_{n,m,k}$ satisfies Property $D_\delta$ with probability $1−o(1)$. From Proposition 2.4 we also know that $H^*_{n,m,k}$ satisfies Property $\mathcal{E}$ with probability $1−o(1)$. We thus may assume that we begin with a certain realization of $H^*_{n,m,k}$, which has both properties. For convenience, we call this hypergraph $\hat{H}$. Fix an orientation $h$ of its edges. Thereafter, choose $\alpha$ small enough so that Corollary 2.6 can be applied and set $C = C(\alpha,\varepsilon)$ as in Corollary 2.3. Note that this also specifies the set $S$.

Lemma 2.7. Assume that $H$, $h$, and $C$ are as specified above and, for $\frac{\varepsilon}{C} > 0$, let $T = T(\zeta)$ and $c$ be defined as in Corollary 2.6. Then the probability that an $h$-random walk starting at a vertex $v_0$ hits a free vertex within $T + C$ steps is at least $\frac{\alpha}{(k-1)^c}\frac{1}{\log_{k-1} n}$.

Proof. Let us consider the following stopping rule. Starting from a vertex $v_0$, we walk either for $T$ steps or until we hit a free vertex, whatever occurs earlier. If the latter is not the case, then we walk for $C$ additional steps or until we hit a free vertex. We consider the same $h$-random walk on $\hat{H}$, that is, the $h$-random walk there starts at $v_0$ and makes the same random choices as the one in $H$ (and new ones, if the random walk on $H$ stopped at a free vertex). Let $u$ be the vertex that has been reached after $T$ steps. The growth property guaranteed by Corollary 2.6 implies that there are at least $2\alpha n − \alpha n = \alpha n$ vertices within $h$-distance $T$ from $v_0$ in $\hat{H}$ that either belong to $S$ or to one of the trees we added to $H$. The probability of hitting such a vertex after $T$ steps is at least $\frac{\alpha n}{(k-1)^c}$. If $u \notin S$, then $u$ belongs to one of the trees that we added to $H$, and we conclude that the $h$-random walk on $H$ has stopped before reaching $T$ steps. If $u \in S$, then we stop the $h$-random walk in $\hat{H}$ but we continue it in $H$ for another $C$ steps or until a free vertex is found. The probability that this part of the stopping rule ends up at a free vertex is at least $1/(k−1)^C$, since the assumption that $u$ belongs to $S$ implies that there is at least one free vertex within $h$-distance $C$ from $u$. Thus the probability of success is at least $\frac{\alpha n}{(k-1)^{C+1}}$, thus concluding the proof.

To conclude the proof of Theorem 1.2 we split the $h$-random walk into phases, where a phase is either a window of duration $T+C$ or until a free vertex was hit. We repeatedly use the above lemma to bound the number of unsuccessful phases. More precisely, if the first phase is unsuccessful, then the above analysis applies with the starting vertex being the vertex in which the previous phase ended, which we call $v_1$, and with a new orientation $h_1$.

The above arguments imply that for any $\zeta > 0$ the probability that at least $\log_{k−1} n$ phases are unsuccessful given that $e_{m+1}$ is inserted into $v_0$ is $O(1/n^{1+\zeta})$. As each phase lasts for $T+C = O(\log n)$ steps, we deduce that with probability $1−O(1/n^{1+\zeta})$, the random walk inserts $e_{m+1}$ within $O(\log^{2+c+2\zeta}n)$ steps. As the total number of inserted elements is $O(n)$ this concludes the proof of Theorem 1.2.

3 Proof of Proposition 2.1: $h$-neighborhoods and the Density of a Hypergraph

The main idea of the proof is as follows. As $H$ has Property $D_\delta$ we know that $|F_h| = |V| − |E| > \delta|V|$. Consider an edge $e$ that contains a vertex from $F_h$. The orientation $h$
assigns this edge to some vertex $v_e \in e$. Suppose that we remove all free vertices and all edges that contain a free vertex. As the removal of any edge $e$ that contains a free vertex generates a new free vertex $(v_e)$, this process generates a subhypergraph of $H$, which is induced by $V \setminus F_h$, with a new set of free vertices. For this subhypergraph we can again use Property $D_\delta$ to deduce that the number of free vertices is at least a $\delta$-fraction of the number of vertices. We repeat this stripping until we are left with less than $\alpha |V|$ vertices – and show that a constant number of rounds suffice.

Let us now be more formal. Let $F_0 := F_h$ and $L_0 := V$, and define inductively $L_{i+1} = L_i \setminus F_i$ and let $F_{i+1}$ be the set of free vertices in the hypergraph induced by $L_{i+1}$. Since $H$ has Property $D_\delta$, it follows that $|F_0| \geq \delta |V| = \delta |L_0|$. We claim that for all $i \geq 0$ we have $|F_{i+1}| \geq \delta |L_{i+1}|$ as well. The two crucial observations are:

(i) $|L_{i+1}| = e(L_i)$, as $L_{i+1}$ contains exactly the vertices that are the images (under $h$) of the edges in the hypergraph induced by $L_i$.

(ii) $|F_{i+1}| = e(L_i) - e(L_{i+1})$, as every edge that belongs to the hypergraph induced by $L_i$ but not to the one induced by $L_{i+1}$ generates exactly one free vertex in $F_{i+1}$.

As $H$ has Property $D_\delta$ we also know that $e(L_{i+1}) \leq (1 - \delta)|L_{i+1}|$. Hence,

$$|F_{i+1}| = e(L_i) - e(L_{i+1}) \geq e(L_i) - (1 - \delta)|L_{i+1}| = \delta |L_{i+1}|.$$

Thus

$$|L_{i+1}| = |L_i| - |F_i| < (1 - \delta)|L_i| \quad \text{for all } i \geq 0.$$

To conclude the proof observe that this implies that for all $t \geq 1$

$$|L_t| < (1 - \delta)^t |L_0| = (1 - \delta)^t |V|.$$

Thereby, choosing $t = \lceil \log_{1-\delta} \alpha \rceil$ we deduce that $|L_t| < \alpha |V|$. Thus, we may take $S := V \setminus L_t$ and $C(\alpha, \delta) = \lceil \log_{1-\delta} \alpha \rceil$.

4 Proof of Theorem 2.2: the Subgraph Density of $H^*_{n,m,k}$

The core of a hypergraph is its maximum subhypergraph of minimum degree at least 2. Of course the core of a hypergraph might be empty (that is, the null hypergraph). We will be denoting the core of a hypergraph $H$ by $C(H)$. A standard algorithm that reveals the core of a given hypergraph is the so-called stripping process. During this process, we repeatedly choose a vertex of degree 1 and we make it isolated by deleting the edge that this is incident to. This process stops when there are no more available vertices of degree 1 and what remains is either the empty hypergraph, if the core is empty, or otherwise the core itself together with a (possibly empty) collection of isolated vertices.

To prove Theorem 2.2 we will first show a lemma stating that any given hypergraph with good expansion properties (as in Proposition 2.4) and, in addition, whose core has Property $D_\delta$, also has Property $D_{\theta'}$ for some $\theta' = \theta'(\delta, k)$.

Lemma 4.1. Let $H = H(V, E)$ be a $k$-graph where every edge contains at least two vertices such that
1. C(H) has Property $D_\delta$ for some $0 < \delta < 1/4$;

2. H has Property $E$.

Then there exists $\zeta \in (0, 1)$ such that H itself has Property $D_{\zeta \delta}$.

Proof. For any subset $S$ of $V$, we denote by $e_S$ and $v_S$ the number of edges and vertices, respectively, in $S$. We will show that for any $S \subseteq V$ we have $e_S < (1 - \zeta \delta)v_S$. Towards this goal we will make a case distinction depending on the number of edges that are induced by $S$.

The first part of Property $E$ implies that for all $\beta \in (0, 1)$ there exists a $0 < \gamma < \frac{1}{\beta}$ such that for all $E' \subseteq E$ with $\log \log |V| < |E'| \leq \gamma |V|$ we have

$$|V(E')| \geq (k - 1 - \beta)|E'|.$$  

Setting $\beta = 1/2$, let $\gamma = \gamma(1/2)$ be as above. Let us start with the case $e_S \leq \gamma |V|$. The bound on the density of $S$ can be deduced right away from Property $E$. Indeed, by using the first part of Property $E$, if $e_S > \log \log |V|$, then $v_S \geq (k - 1 - 1/2)e_S$, yielding

$$\frac{e_S}{v_S} \leq \frac{1}{k - 1 - 1/2} \leq \frac{2}{3}. \quad (4.1)$$

Now, using the second part of Property $E$, if $e_S \leq \log \log |V|$, then $v_S \geq (k - 1)e_S$. A rearrangement yields

$$\frac{e_S}{v_S} \leq \frac{1}{k - 1} \leq \frac{1}{2}. \quad (4.2)$$

Suppose now that $e_S > \gamma |V|$. We will make a further case distinction depending on the number of edges that belong to the core of $H[S]$. First, let us assume that there are at least $\gamma |V|/2$ edges of $H[S]$ that do not belong to the core of $S$. To avoid unnecessary complications, let us assume that $H[S]$ is connected; clearly it is enough to argue about connected sets. Consider the stripping process on $H[S]$. This induces an ordering on the set of vertices of $H[S]$ not belonging to the core. Namely, it is the ordering according to which these vertices are stripped off. Moreover, it might happen that the deletion of a vertex and the edge it is contained to makes some other vertices isolated. We put these vertices immediately after the deleted vertex in an arbitrary ordering.

Let $v_1, \ldots, v_t$ be this ordering for an appropriate integer $t \geq 1$. Note that whenever we delete a vertex of degree one during the stripping process, this is accompanied by an edge that is deleted too, that is, the edge that this vertex is contained to. Each one of the remaining $k - 1$ vertices of this edge either belongs to the core of $H[S]$ or it is one of the vertices that comes after the deleted vertex in the above ordering. Now, consider the ordering in reverse and let $i$ be the minimum index such that the number of edges that contain $v_1, \ldots, v_i$ is $[\gamma |V|/2]$; there are $s := t - i + 1$ vertices there. Assuming that among them there are $x$ vertices which became isolated during the stripping process, we have $s = [\gamma |V|/2] + x$. The first part of Property $E$ implies that the number of vertices that are contained in these edges is at least $(k - 1 - \beta)[\gamma |V|/2]$. Among these vertices at least $(k - 1 - \beta)[\gamma |V|/2] - s = (k - 2 - \beta)[\gamma |V|/2] - x$ vertices must belong to the core of $H[S]$. Let $S_0$ denote these vertices and let $S_i$ denote the remaining vertices of the core of $H[S]$. In other words, $|S_0| \geq (k - 2 - \beta)[\gamma |V|/2] - x$. As the core of $H[S]$ is a subgraph of
Theorem 4.2 is stated for the models of random hypergraphs. Let

\[ \sum_{i=1}^{k} x_{i} = 1 \]

We start by arguing that it suffices to consider \( H \). Then the following is true. Together with (4.1)–(4.3) the above inequality determines the value of \( \gamma \). So, \( \gamma |V|/2 \geq \gamma |V|/4 \) implies that

\[ \frac{e_S}{v_S} \leq 1 - \frac{x}{|V|} - \delta \frac{|S_0|}{|V|} \leq 1 - \frac{x}{|V|} - \delta \frac{(k-2-\beta)|V|}{4|V|} \leq 1 - \delta \frac{(k-2-\beta)|V|}{4|V|} \leq 1 - \delta \frac{(k-2-\beta)|V|}{4|V|}. \] (4.3)

Finally, assume that less than \( \gamma |V|/2 \) edges of \( H[S] \) do not belong to the core of \( H[S] \). With \( e_{C(S)} \) and \( v_{C(S)} \) denoting the number of edges and vertices of the core of \( H[S] \), respectively, Property \( D_\delta \) of the core of \( H \) implies that

\[ (1-\delta)v_{C(S)} > e_{C(S)} \geq \gamma |V|/2. \] (4.4)

Thus

\[ \frac{e_S}{v_S} \leq \frac{t + e_{C(S)}}{t + v_{C(S)}} \leq \frac{t + (1-\delta)v_{C(S)}}{t + v_{C(S)}} = 1 - \delta \frac{v_{C(S)}}{t + v_{C(S)}} \leq 1 - \delta \frac{\gamma |V|}{2(1-\delta)|V|} = 1 - \delta \frac{\gamma}{2(1-\delta)}. \]

Together with (4.1)–(4.3) the above inequality determines the value of \( \zeta \). Taking

\[ \zeta := \min \left\{ \frac{\gamma}{2(1-\delta)}, \frac{(k-2-1/2)\gamma}{4} \right\} \]

suffices.

The main ingredient in our proof is statement about the subgraphs of the core itself.

**Theorem 4.2.** Let \( \varepsilon > 0 \) and suppose that \( m = (1-\varepsilon)c_k n \), where \( c_k \) is given in Theorem 1.1. Then, for sufficiently small \( \varepsilon \), the core of \( H_{n,m,k}^* \) has Property \( D_{\varepsilon^3/2} \) with probability \( 1-o(1) \).

The above theorem together with Proposition 2.4 and Lemma 4.1 yield Theorem 2.2. In the remainder of this section we prove Theorem 4.2.

### 4.1 Models of Random Hypergraphs

Theorem 4.2 is stated for the \( H_{n,m,k}^* \) model, where multiple edges are allowed, and also each edge can contain a vertex more than once. We start by arguing that it suffices to consider a slightly different random graph model. Let \( H_{n,m,k} \) denote a random hypergraph that is created by selecting \( m \) edges with \( k \) different vertices in each edge without replacement. Then the following is true.

**Proposition 4.3.** Let \( k \geq 3 \) and \( \varepsilon > 0 \) be sufficiently small. Assume that \( m = |cn| \), for some \( c > 0 \). Then, if \( \mathbb{P}(H_{n,m,k} \text{ has property } D_{\varepsilon^3}) = 1-o(1) \), then \( \mathbb{P}(H_{n,m,k}^* \text{ has property } D_{\varepsilon^3/2}) = 1-o(1) \) as well.
Proof. First of all, recall that Proposition 2.4 implies that $H^*_{n,m,k}$ has Property $E$ with probability $1 - o(1)$. Therefore, sets of size at most $\gamma n$, for some sufficiently small $\gamma > 0$, do not violate Property $D_{\varepsilon^3/2}$. So it is sufficient to argue only about sets with at least $\gamma n$ vertices.

Let us call an edge in $H^*_{n,m,k}$ bad if it either has repeated vertices or if there is another edge that contains exactly the same vertices. For each of these edges we resample new edges until the resulting hypergraph contains $m$ different edges with $k$ distinct vertices in each. Note that this process yields $H_{n,m,k}$. A trivial calculation reveals that with probability $1 - o(1)$ the random hypergraph $H^*_{n,m,k}$ has at most $2 \log n$ bad edges.

In the above process, sets with at least $\gamma n$ vertices may change their number of edges by at most $2 \log n$. Thus, for $n$ large enough, if $H_{n,m,k}$ has the property $D_{\varepsilon^3}$ and there are at most $2 \log n$ bad edges, then $H^*_{n,m,k}$ must have property $D_{\varepsilon^3/2}$. The statement of the proposition now follows.

Thus, proving Theorem 4.2 for $H_{n,m,k}$ (where we use $\varepsilon^3$ instead of $\varepsilon^3/2$) is sufficient. Our remaining proof strategy is inspired by the ideas in [14]. We will use the following auxiliary statements about binomial coefficients.

Proposition 4.4. Let $H(x) = -x \log x - (1-x) \log(1-x)$ denote the entropy function. Then, for any $0 < \alpha, \delta < 1$,

$$\binom{n}{\alpha n} = \frac{1 + o(1)}{\sqrt{2\pi \alpha(1-\alpha)n}} e^{n H(\alpha)} \quad \text{and} \quad \binom{n}{\alpha n \pm \delta n} \leq \binom{n}{\alpha n} e^{n \delta \log(\max\{1/\alpha, 1/(1-\alpha)\})}.$$  

Proof. The first statement is well-known and follows immediately from Stirling’s approximation of the factorial function; we omit the details. To see the second statement, let us first consider the case with the “+”. We can assume that $\alpha + \delta \leq 1$, as otherwise the statement holds trivially. Then

$$\frac{n}{\binom{n}{\alpha n + \delta n}} = \prod_{i=1}^{\delta n} \frac{n - \alpha n - i + 1}{\alpha n + i} \leq \left(\frac{(1-\alpha)n}{\alpha n}\right)^{\delta n} \leq e^{n \delta \log(1/\alpha)}.$$  

Similarly, for the case with the “−” we obtain

$$\frac{n}{\binom{n}{\alpha n - \delta n}} = \prod_{i=0}^{\delta n - 1} \frac{\alpha n - i}{n - \alpha n + i + 1} \leq \left(\frac{\alpha n}{1-\alpha)n}\right)^{\delta n} \leq e^{n \delta \log(1/(1-\alpha))}.$$  

For the sake of convenience we will carry out our calculations in the $H_{n,p,k}$ model of random $k$-graphs. This is the “higher-dimensional” analogue of the well-studied $G_{n,p}$ model, where, given $n \geq k$ vertices, we include each $k$-tuple of vertices with probability $p$, independently of every other $k$-tuple. Standard arguments show that if we adjust $p$ suitably, then the $H_{n,p,k}$ is essentially equivalent to $H_{n,cn,k}$. The following proposition makes this more precise.

Proposition 4.5. Let $\mathcal{P}$ be any property of hypergraphs, and let $p = ck/(\binom{n-1}{k-1})$, where $c > 0$. Then

$$\mathbb{P}(H_{n,cn,k} \notin \mathcal{P}) \leq O(\sqrt{n}) \cdot \mathbb{P}(H_{n,p,k} \notin \mathcal{P}).$$
Proof. Let $N = \binom{n}{k}$, and note that $pN = cn$. Hence,

$$
P(H_{n,p,k} \text{ has } cn \text{ edges}) = \binom{N}{cn} p^{cn} (1 - p)^{N-cn} \overset{\text{Prop. 4.4}}{=} \Theta(n^{-1/2}).$$

The claim then follows from

$$
P(H_{n,cn,k} \not\in \mathcal{P}) = P(H_{n,p,k} \not\in \mathcal{P} | H_{n,p,k} \text{ has } cn \text{ edges}) \leq \frac{P(H_{n,p,k} \not\in \mathcal{P})}{P(H_{n,p,k} \text{ has } cn \text{ edges})}.
$$

In order to prove Theorem 4.2 it is therefore sufficient to show that the core of $H_{n,p,k}$ has Property $D_\varepsilon$ with probability $1 - o(n^{-1/2})$. This is accomplished in the next sections.

4.2 Working on the Core of $H_{n,p,k}$: the Cloning Model

Recall that the core of a hypergraph is its maximum subgraph that has minimum degree (at least) 2. At this point we introduce the main tool for our analysis. The cloning model with parameters $(N, D, k)$, where $N \geq 1$ and $D \geq 0$ are random variables taking integral values, is defined as follows. We generate a graph in three stages.

1. We expose the value of $N$.
2. We expose the degrees $d = (d_1, \ldots, d_N)$, where the $d_i$’s are independent identically distributed as $D$.
3. For each $1 \leq v \leq N$ we generate $d_v$ copies, which we call $v$-clones or simply clones.
   Then we choose uniformly at random a matching from all perfect $k$-matchings on the set of all clones. Note that such a matching may not exist – in this case we choose a random matching that leaves less than $k$ clones unmatched. Finally, we construct the graph $H_{d,k}$ by contracting the clones to vertices, i.e., by projecting the clones of $v$ onto $v$ itself for every $1 \leq v \leq N$.

Note that the last stage in the above procedure is equivalent to the configuration model $H_{d,k}$ for random hypergraphs with degree sequence $d = (d_1, \ldots, d_n)$. In other words, $H_{d,k}$ is a random multigraph where the $i$th vertex has degree $d_i$.

One special instantiation of the cloning model is the so-called Poisson cloning model $\tilde{H}_{n,p,k}$ for $k$-graphs with $n$ vertices and parameter $p \in [0,1]$, which was introduced by Kim [18]. There, we choose $N = n$ with probability 1, and the distribution $D$ is the Poisson distribution with parameter $\lambda := p^{(n-1)}$. Note that here $D$ is essentially the vertex degree distribution in the binomial random graph $H_{n,p,k}$, so we would expect that the two models behave similarly. The following statement confirms this, and is implied by Theorem 1.1 in [18].

**Theorem 4.6.** Let $k \geq 2$ and suppose that $p = \Theta(n^{-k+1})$. Then there is a $C > 0$ such that for any property $\mathcal{P}$ of $k$-graphs

$$
P(H_{n,p,k} \not\in \mathcal{P}) \leq C \mathbb{P}(\tilde{H}_{n,p,k} \not\in \mathcal{P})^{1/k} + e^{-n}.
$$
One big advantage of the Poisson cloning model is that it provides a very precise description of the core. In particular, Theorem 6.2 in [18] implies the following statement, where we write “$x \pm y$” for the interval of numbers $(x - y, x + y)$.

**Theorem 4.7.** Let $\lambda_k := \min_{x > 0} \frac{x}{(1 - e^{-x})^k}$, $0 < \delta < 1$, and $c$ be such that $ck = \binom{n-1}{k-1} > \lambda_k$. Moreover, let $\tilde{x}$ be the largest solution of the equation $x = (1 - e^{-xck})^{k-1}$, and set $\xi := \tilde{x}ck$. Then the following is true with probability $1 - n^{-\omega(1)}$. If $\tilde{N}_2$ denotes the number of vertices in the core of $\tilde{H}_{n,p,k}$, then

$$\tilde{N}_2 = (1 - e^{-\xi} - \xi e^{-\xi})n \pm \delta n.$$

Furthermore, the core itself is distributed like the cloning model $(\tilde{N}_2, \text{Po}_{\geq 2}(\Lambda_{c,k}), k)$, where $\text{Po}_{\geq 2}(\Lambda_{c,k})$ denotes a Poisson random variable conditioned on being at least 2 and parameter $\Lambda_{c,k} = \xi + \beta$, for some $|\beta| \leq \delta$.

We shall say that a random variable is a 2-truncated Poisson variable, if it is distributed like a Poisson variable, conditioned on being at least 2. The next statement is taken from [14, Corollary 3.4].

**Corollary 4.8.** Let $\delta > 0$. Let $\tilde{N}_2$ and $\tilde{M}_2$ denote the number of vertices and edges in the core of $\tilde{H}_{n,p,k}$, where $p = ck / \binom{n-1}{k-1}$ and $ck > \lambda_k$, where $\lambda_k$ is defined in Theorem 4.7. Then, with probability $1 - n^{-\omega(1)}$,

$$\tilde{N}_2 = (1 - e^{-\xi} - \xi e^{-\xi})n \pm \delta n \quad \text{and} \quad \tilde{M}_2 = \frac{\xi(1 - e^{-\xi})}{k(1 - e^{-\xi} - \xi e^{-\xi})} \tilde{N}_2 \pm \delta n,$$

where $\xi = \tilde{x}ck$ and $\tilde{x}$ is the largest solution of the equation $x = (1 - e^{-xck})^{k-1}$.

In the following we will collect a few basic properties of the relation of the number of vertices and edges in the core of $\tilde{H}_{n,p,k}$. First of all, define the functions

$$f(x) = \frac{x(1 - e^{-x})}{k(1 - e^{-x} - xe^{-x})} \quad \text{and} \quad g(x) = \frac{x}{k(1 - e^{-x})^{k-1}}$$

and recall that $c_k^*$ and $\xi^*$ in Theorem 4.4 and also Theorem 4.2 are given by the solution of the system

$$1 = f(\xi^*) \quad \text{and} \quad c_k^* = g(\xi^*). \quad (4.5)$$

An easy calculation shows that $f(x)$ is an increasing function of $x$ and infinitely differentiable over $\mathbb{R}^+$, and that $g(x)$ has a unique minimum, which we shall denote by $x_g$. Moreover, $g(x_g) = \lambda_k / k$, where $\lambda_k$ is defined in Theorem 4.7. We shall need the following technical claim.

**Claim 4.9.** $x_g < \xi^*$.

**Proof.** A simple calculation reveals that

$$g'(x) = \frac{1 - e^{-x} - (k - 1)xe^{-x}}{k(1 - e^{-x})^{k-1}}.$$

The numerator of $g'(x_0)$ is

$$1 - \frac{1}{(k - 1)^2} - \frac{2 \log(k - 1)}{k - 1},$$

The theorem is then proved.
which is easily checked to be greater than zero for all $k \geq 3$. Hence $g'(x_0) > 0$ and thus $x_g \leq x_0$.

In the remainder we argue that $\xi^* \geq \frac{k}{2}$, which settles the claim with $x_0 = 2 \log(k-1) < \frac{k}{2}$. Note that the monotonicity of $f$ guarantees that it is enough to show that $f(k) \leq 1$. Using the estimate $e^x \geq 1 + x + x^2/2$, which is valid for all $x \geq 0$ we obtain

$$f \left( \frac{k}{2} \right) = \frac{1}{2} \cdot \frac{e^{k/2} - 1}{e^{k/2} - 1 - k/2} = \frac{1}{2} \left( 1 + \frac{k}{2(e^{k/2} - 1 - k/2)} \right) \leq \frac{1}{2} \left( 1 + \frac{4}{k} \right).$$

Note that for $k \geq 4$ this expression is $\leq 1$, thus concluding the proof in these cases. Finally, if $k = 3$, then numerical calculations imply that $\xi^* > 2.14 > 2 \log 2$. \hfill $\mathcal{Q}$

Let us assume that $p = c k / (n^{-1})$, where $c = (1 - \varepsilon)c_k^* > \lambda_k/k$, and set $\xi = \bar{x}ck$, where $\bar{x}$ is the largest solution of the equation $x = (1 - e^{-\bar{x}ck}) - 1$. So, $\xi$ is the largest solution of $c = g(\xi)$, implying with the above claim that $\xi < \xi^*$. Therefore, we have $f(\xi) < 1$, and Corollary 4.8 guarantees that with probability $1 - n^{-\omega(1)}$ the density of the core of $H_{n,p,k}$ is $< 1$. This argument can be extended to obtain the following finer statement.

**Corollary 4.10.** Let $\delta > 0$ be sufficiently small and choose $c > \lambda_k/k$ such that the largest solution $\xi$ of the equation $c = g(\xi)$ satisfies $\xi = \xi^* - \delta$, where $\xi^*$ is as in Theorem 1.1. Then there is an $\varepsilon > 0$ such that $\varepsilon = \Theta(\delta)$ and $c = (1 - \varepsilon)c_k^*$. Moreover, there is a constant $e_k > 0$ such with probability $1 - n^{-\omega(1)}$

$$\tilde{M}_2 = \tilde{N}_2(1 - e_k \delta + \Theta(\delta^2)).$$

**Proof.** We first show that there is an $\varepsilon > 0$ with the claimed properties. Note that $\xi$ is defined through the equation $c = g(\xi)$ and $\xi^*$ through $c_k^* = g(\xi^*)$. Let $x_g$ be the minimizer of $g$, i.e., $g(x_g) = \lambda_k/k$, and note that whenever $x > x_g$ we have $g'(x) > 0$. By applying Taylor’s Theorem and using (4.5) we infer that there is a $\mu \in [\xi, \xi^*]$ such that

$$c = g(\xi) = g(\xi^*) + g'(\mu)(\xi - \xi^*) = c_k^* - g'(\mu)\delta.$$

So $c = (1 - \varepsilon)c_k^*$, where $\varepsilon = \frac{g'(\mu)}{c_k^*} \delta$, and note that $g'(\mu)$ remains bounded for $\mu \in [\xi, \xi^*]$, whenever $\delta$ is sufficiently small.

To see the claim for $\tilde{M}_2$, note that Corollary 4.8 (where we use $\delta^2$ for $\delta$) guarantees that with probability $1 - n^{-\omega(1)}$ we may assume that $\tilde{M}_2 = (f(\xi) \pm \delta^2)\tilde{N}_2$. Moreover, Taylor’s Theorem, this time applied to $f$, implies that

$$f(\xi) = f(\xi^*) + f'(\xi^*)(\xi - \xi^*) + O((\xi - \xi^*)^2) = 1 - f'(\xi^*)\delta + O(\delta^2),$$

thus concluding the proof with $e_k = f'(\xi^*)$ and the fact that $f$ is increasing. \hfill $\mathcal{Q}$

We immediately obtain the following corollary.

**Corollary 4.11.** Let $k \geq 3$. Let $\varepsilon > 0$ be sufficiently small and suppose that $p = (1 - \varepsilon)c_k^* k / (n^{-1})$. Then, with probability $1 - n^{-\omega(1)}$

$$\tilde{M}_2 \leq (1 - \varepsilon^2)\tilde{N}_2.$$
4.3 Subgraphs of the 2-Core

In order to obtain tight bounds for the probability that there are \((1 - \varepsilon^3)\)-dense subsets in the core of \(\tilde{H}_{n,p,k}\), we will exploit the following statement.

**Proposition 4.12.** Let \(H\) be a \(k\)-uniform hypergraph such that \(e_H < \lfloor (1 - \gamma)v_H \rfloor\). Moreover, let \(U\) be an inclusion maximal subset of \(V_H\) such that \(e_U \geq (1 - \gamma)v_U\). Then \(e_U = \lfloor (1 - \gamma)v_U \rfloor\) and all edges \(e \in E_H\) satisfy \(|e \cap U| \neq k - 1\).

**Proof.** If \(e_U > \lfloor (1 - \gamma)v_U \rfloor\) then \(e_U \geq (1 - \gamma)v_U + 1\), and let \(U' = U \cup \{v\}\), where \(v\) is any vertex in \(V_H \setminus U\). Note that such a vertex always exists, as \(U \neq V_H\). Moreover, denote by \(d\) the degree of \(v\) in \(U\), i.e., the number of edges in \(H\) that contain \(v\) and all other vertices only from \(U\). Then
\[
e_U = e_U + d \geq e_U \geq (1 - \gamma)v_U + 1.
\]
Note that \(v_U = v_U + 1\). Hence, the above inequality implies that
\[
e_U \geq (1 - \gamma)(v_U + 1) - (1 - \gamma) + 1 \geq (1 - \gamma)v_U,
\]
which contradicts the maximality of \(U\). Similarly, if there was an edge \(e\) such that \(|e \cap U| = k - 1\), then we could construct a larger subset of \(V_H\) that also satisfies the density requirement by adding the vertex in \(e \setminus U\) to \(U\). \(\square\)

Let \(\gamma > 0\). The following lemma bounds the probability that a given set of the core is maximal and \((1 - \gamma)\)-dense, assuming that the degree sequence has been exposed. That is, the randomness is that of the 3rd stage of the exposure process in the Poisson cloning model. A similar statement was shown in [14] for the special case \(\gamma = 0\).

**Lemma 4.13.** Let \(k \geq 2\), \(d = (d_1, \ldots, d_N)\) be a degree sequence and \(U \subseteq \{1, \ldots, N\}\) such that \(|U| = \lfloor \beta N \rfloor\), where \(1/2 < \beta \leq 1\). Moreover, set \(M = k^{-1} \sum_{i=1}^{N} d_i\) and \(q = (kM)^{-1} \sum_{i \in U} d_i\). Let \(0 < \gamma < 1/4\) and assume that \(3\beta N/4 < M < (1 - \gamma)N\). If \(\mathcal{B}(\beta, q; \gamma)\) denotes the event that \(U\) is an inclusion maximal set of \(H_{d,k}\) such that \(e_U \geq (1 - \gamma)|U|\), then
\[
\mathbb{P}_{d,k}(\mathcal{B}(\beta, q; \gamma)) \leq \max\left\{1, \left(\frac{M}{\beta N}\right)^M \cdot (2^k - k - 1)^{M - \beta N} \cdot e^{-kM-H(q)} \cdot e^{O(\gamma \log(1/\gamma)N)}\right\},
\]
where \(H(x) = -x \ln x - (1 - x) \ln(1 - x)\) denotes the entropy function, and \(\mathbb{P}_{d,k}\) denotes the probability measure on the space of Stage 3, given the outcomes of the first two stages.

**Proof.** The graph \(H_{d,k}\) is obtained by creating \(d_i\) clones for each \(1 \leq i \leq N\) and by choosing uniformly at random a perfect \(k\)-matching on this set of clones. Note that this is the same as throwing \(kM\) balls into \(M\) bins, such that every bin contains \(k\) balls. We use this analogy to prove the claim as follows. Assume that we color the \(kqM\) clones of the vertices in \(U\) with red, and the remaining \(k(1 - q)M\) clones with blue. So, by applying Proposition 4.12 we are interested in the probability for the event that there are exactly \(\lfloor (1 - \gamma)|U| \rfloor\) bins with \(k\) red balls and no bin that contains exactly one blue ball.

We estimate the probability for this event as follows. We start by putting into each bin \(k\) balls, labeled with the numbers \(1, \ldots, k\). Let \(K = \{1, \ldots, k\}\), and let \(X_1, \ldots, X_M\) be independent random sets such that for \(1 \leq i \leq M\)
\[
\forall K' \subseteq K : \mathbb{P}(X_i = K') = q^{|K'|}(1 - q)^{k - |K'|}.
\]
Note that \(|X_i|\) is distributed like Bin\((k, q)\). We then recolor the balls in the \(i\)th bin that are in \(X_i\) with red, and all others with blue. We infer that the total number of red balls is 
\[ X = \sum_{i=1}^{M} |X_i|. \]
Set 
\[ Z = \mathbb{P}(X = kqM). \]
Note that \(\mathbb{E}(X) = kqM\), and that \(X\) is distributed like Bin\((kM, q)\). By applying Proposition 4.4 we infer that 
\[ Z = \mathbb{P}(X = \mathbb{E}(X)) = (1 + o(1))(2\pi q(1 - q)kM)^{-1/2}. \]
Let \(R_j\) be the number of \(X_i\)'s that contain \(j\) elements, and set 
\[ Z = \mathbb{P}(X = kqM \wedge R_k = \lceil(1 - \gamma)|U|\rangle \wedge R_{k-1} = 0). \]
Let \(e_U = \lceil(1 - \gamma)|U|\rangle\). By using the above notation we may estimate 
\[ \mathbb{P}(B(\beta, q; \gamma)) = \frac{P}{Z} \leq \sqrt{2M} \cdot \mathbb{P}(X = kqM \wedge R_k = e_U \wedge R_{k-1} = 0). \tag{4.6} \]
Let \(p_j = \mathbb{P}(|X_i| = j) = \binom{k}{j}q^j(1 - q)^{k-j}\). Moreover, define the set of integer sequences 
\[ \mathcal{A} = \{ (b_0, \ldots, b_{k-2}) \in \mathbb{N}^{k-1} : \sum_{j=0}^{k-2} b_j = M - e_U \text{ and } \sum_{j=0}^{k-2} jb_j = kqM - ke_U \}. \]
Then 
\[ P = \sum_{(b_0, \ldots, b_{k-2}) \in \mathcal{A}} \binom{M}{b_0, \ldots, b_{k-2}, 0, e_U} \cdot \left( \prod_{j=0}^{k-2} \binom{k}{j} p_j^b_j \right) \cdot p_k^{e_U}. \]
Observe that the summand can be rewritten as 
\[ \binom{M}{e_U} q^{kqM}(1 - q)^{k(1-q)M} \cdot \binom{M - e_U}{b_0, \ldots, b_{k-2}} \prod_{j=0}^{k-2} \binom{k}{j}^{b_j}. \]
By applying the multinomial theorem we obtain the bound 
\[ \sum_{(b_0, \ldots, b_{k-2}) \in \mathcal{A}} \binom{M - e_U}{b_0, \ldots, b_{k-2}} \prod_{j=0}^{k-2} \binom{k}{j}^{b_j} \leq (2^k - 1 - k)^{M - e_U}. \]
Thus, from (4.6) we infer that for large \(M\) 
\[ \mathbb{P}(B(\beta, q; \gamma)) \leq 2\sqrt{M} \binom{M}{e_U} q^{kqM}(1 - q)^{k(1-q)M}(2^k - k - 1)^{M - e_U}. \]
The proof is completed by estimating \(\binom{M}{e_U}\). More specifically, assume first that \(|U| \geq (1 - \gamma)M\). Then \(\gamma < 1/4\) guarantees that 
\[ \binom{M}{2\gamma M} \leq \left( \frac{M}{2\gamma M} \right)^{2\gamma M} = e^{O(\gamma \log(1/\gamma) \cdot N)}. \]
Moreover, let us write \(|U| = |\beta N| = \eta M\), for some appropriate \(\eta \leq 1 - \gamma\). Note that \(\beta > 1/2\) and \(M < (1 - \gamma)N\) guarantee that \(\eta > 1/2\). By applying Proposition 4.4 we obtain
\[
\left( \frac{M}{|U|} \right) = \left( \frac{M}{(1 - \gamma)\eta M} \right) e^{M\gamma\eta \log\max\{1/\eta, 1/(1 - \eta)\}} = \left( \frac{M}{|U|} \right) e^{O(\gamma \log(1/\gamma)N)}.
\]

With the above lemma at hand we are ready to estimate the number of \((1 - \varepsilon^3)\)-dense sets in the core of \(\tilde{H}_{n,p,k}\). Let us make some auxiliary preparations first. Suppose that the degree sequence of the core \(C\) is given by \(d = (d_1, \ldots, d_{N_2})\). Thus, the number of edges in \(C\) is \(M_2 = k^{-1} \sum_{i=1}^{N_2} d_i\). For \(q, \beta \in [0, 1]\) let \(X_{q,\beta} = X_{q,\beta}(C) = X_{q,\beta}(d)\) denote the number of subsets of \(C\) with \(|\beta N_2|\) vertices and total degree \(|q \cdot kM_2|\). (We will omit writing “\(\lfloor \cdot \rfloor\)” from now on.) Note that \(X_{q,\beta}\) is a random variable that depends only on the outcomes of the first two stages of the exposure of the core. Let also \(Y_{q,\beta}\) denote the number of these sets that are maximal and \((1 - \varepsilon^3)\)-dense.

Let \(\delta > 0\). Moreover, let \(p = ck/(n - 1)\) be such that the largest solution \(\xi\) of the equation \(g(\xi) = c\) satisfies \(\xi = \xi^* - \delta\), where \(g(\xi^*) = c_k^*\). By applying Corollary 4.10 we infer that there is a \(\varepsilon = \Theta(\delta)\) such that \(c = (1 - \varepsilon)c_k^*\). Moreover, Theorem 4.7 (where we use \(\delta^3\) for \(\delta\)) guarantees that with probability \(1 - n^{-\omega(1)}\)
\[
N_2 = n(1 - e^{-\xi} - \xi e^{-\xi}) \pm \delta^3 n \quad \text{and} \quad \Lambda_{\varepsilon, k} = \xi \pm \delta^3,
\]
where \(\xi = \xi^* - \delta\).

Set \(n_2 = (1 - e^{-\xi} - \xi e^{-\xi}) n\) and \(m_2 = \frac{\xi(1 - e^{-\xi})}{k(1 - e^{-\xi} - \xi e^{-\xi})} n_2\) and let \(A\) be the event
\[
A : \quad N_2 = n_2 \pm \delta^3 n \quad \text{and} \quad M_2 = m_2 \pm \delta^3 n.
\]

(4.7)

Corollary 4.8 implies that \(\mathbb{P}(A) = 1 - n^{-\omega(1)}\). Moreover, Corollary 4.10 guarantees the existence of a \(\varepsilon_k > 0\) such that
\[
m_2 = (1 - \varepsilon_k \delta + \Theta(\delta^2)) n_2.
\]

(4.8)

We shall assume all the above facts in the remainder. We are ready to prove the main result of this section, which addresses sets with more than \(0.7N_2\) vertices.

**Lemma 4.14.** Let \(\beta \in [0.7, 1 - \varepsilon_k \delta/2]\) and let \(\beta \leq q \leq 1 - 2(1 - \beta)/k\). Then, for sufficiently small \(\delta > 0\)
\[
\mathbb{P}(Y_{q,\beta} > 0) = n^{-\omega(1)}.
\]

Moreover, when \(q < \beta\) or \(q > 1 - 2(1 - \beta)/k\), the above probability is 0.

**Proof.** The proof follows the arguments in [14], see Corollary 4.5 – Lemma 4.11 there, so we only outline the relevant steps. First of all, suppose that we have exposed the degree sequence \(d\) of the core. Markov’s inequality implies that
\[
\mathbb{P}(Y_{q,\beta} > 0 | d) \leq X_{q,\beta}(d) \mathbb{P}(d)\mathcal{B}(\beta, q; \varepsilon^3),
\]
where $\mathcal{B}(\beta, q; \varepsilon^3)$ is as in Lemma 4.13. Note that for sufficiently small $\delta > 0$, the upper bound on $\beta$, and Proposition 4.4 imply

$$\left(\frac{m_2 + \delta^3 n}{\beta(n_2 + \delta^3 n)} \right) \leq \left(\frac{n_2}{\beta(n_2 + \delta^3 n)}\right) \leq \left(\frac{n_2}{\beta n_2}\right) \cdot e^{O(\delta^3 \log(1/\delta) n_2)}.$$ 

By conditioning on $\mathcal{A}$, taking expectations on both sides, and applying Lemma 4.13 we infer that (see also Lemma 4.6 in [14]) for $\delta > 0$ sufficiently small

$$\mathbb{P}(Y_{q, \delta} > 0) \leq \mathbb{E}(X_{q, \delta} | \mathcal{A}) \cdot \left(\frac{n_2}{\beta n_2}\right) \cdot (2^k - k - 1)^{m_2 - \beta n_2} \cdot e^{-km_2 H(q) + O(\delta^2 n_2) + \mathbb{P}(\mathcal{A})}. \quad (4.9)$$

The expectation of $X_{q, \beta}$, given the event $\mathcal{A}$, is determined by calculating the probability that a specific set with $\beta N_2$ vertices has total degree $qkM$. This task was performed in [14], see Lemma 4.10 there. We omit the details. It follows that

$$\mathbb{E}(X_{q, \beta} | \mathcal{A}) = \exp \left(n_2 H(\beta) - n_2(1 - \beta) I \left(\frac{k(1 - q)}{1 - \beta}\right) + O(n_2 \delta^2)\right),$$

where

$$I(z) = \begin{cases} z \ln T_z - \ln \xi - \ln (e^T_z - T_z - 1) + \ln (e^\xi - \xi - 1), & \text{if } z > 2, \\ 2 - 2 \ln \xi + \ln(e^\xi - \xi - 1), & \text{if } z = 2, \end{cases}$$

and $T_z$ is the unique solution of $z = \frac{T_z(1-e^{-T_z})}{1-e^{-1/T_z}e^{-1/T_z}}$. Let

$$f(\beta, q) := 2^k H(\beta) + (1 - \beta) \ln(2^k - k - 1) - kH(q) - (1 - \beta) I \left(\frac{k(1 - q)}{1 - \beta}\right).$$

By using (4.9) we infer that

$$\mathbb{P}(Y_{q, \delta} > 0) \leq \exp \left(n_2 \left(f(\beta, q) + e_k \delta (kH(q) - \ln(2^k - k - 1)) + O(\delta^2)\right)\right) + n^{-\omega(1)}.$$ 

In [14] the following was shown.

**Claim 4.15.** There exists a $C > 0$ such that for any small enough $\nu > 0$ the following is true. Let $0.7 \leq \beta \leq 1 - \nu$ and $\beta \leq q \leq 1 - 2(1 - \beta)/k$. Then

$$f(\beta, q) \leq -C \nu + O(\delta^2).$$

We distinguish between the following cases. First, note that if $0.7 \leq \beta \leq 1 - \sqrt{\delta}$, then the above claim yields for sufficiently small $\delta > 0$

$$\mathbb{P}(Y_{q, \delta} > 0) \leq e^{n_2(-C\sqrt{\delta} + O(\delta))} + n^{-\omega(1)} = n^{-\omega(1)}.$$

Finally, if $1 - \sqrt{\delta} \leq \beta \leq 1 - e_k \delta/2$, then the above claim implies that there is a $C' > 0$ such that $f(\beta, q) < C' \delta^2$. Moreover, by the monotonicity of the entropy function and $q \geq \beta$ we have for sufficiently small $\delta > 0$

$$kH(q) - \ln(2^k - k - 1) \leq kH(0.99) - \ln(2^k - k - 1).$$

A simple calculation and the fact $H(0.99) < 0.06$ show that the above expression is negative for all $k \geq 3$.  \(\square\)
This completes the proof of Theorem 4.2 for the case $0.7 \leq \beta \leq 1 - e_k \delta / 2$. Now if $\beta \geq 1 - e_k \delta / 2$, then (4.7) together with (4.8) imply that for small $\delta$ all larger subsets have density smaller than $1 - \varepsilon^5$.

In order to cover also the remaining cases we use straightforward first moment arguments. We begin with the case $k \geq 5$.

**Lemma 4.16.** Let $k \geq 5, c < 1$ and $0 < \gamma < 0.001$. Then $H_{n,cn,k}$ contains no $(1 - \gamma)$-dense subset with less than $0.7n$ vertices with probability at least $1 - n^{-(1-\gamma)k^2 + 2k + 1}$.

**Proof.** The probability that an edge of $H_{n,cn,k}$ is contained completely in a subset $U$ of the vertex set is $\binom{|U|}{k} / \binom{n}{k} \leq \binom{cnU}{k} k^{(1-\gamma)u} \leq e^{n(2H(u) + (1-\gamma)ku \ln u)}$,

where $H(x) = -x \ln x - (1-x) \ln (1-x)$ denotes the entropy function. It can easily be seen that the exponent has a unique minimum with respect to $u$ in $[0, 0.7]$, implying that it is maximized either at $u = k/n$ or at $u = 0.7$. Note that

$$2H(0.7) + (1 - \gamma) k 0.7 \ln(0.7) \leq 2H(0.7) + (1 - \gamma) 5 \cdot 0.7 \ln(0.7) \leq -0.01$$

and that

$$2H\left(\frac{k}{n}\right) + (1 - \gamma) k^2 n \ln \left(\frac{k}{n}\right) = -\frac{(1 - \gamma)k^2 - 2k}{n} \ln \frac{1}{n} + O\left(\frac{1}{n}\right).$$

So, the maximum is obtained at $u = k/n$, and we conclude for large $n$ the proof with

$$\mathbb{P}(\exists (1 - \gamma)$-dense subset with \leq 0.7n vertices) \leq \sum_{k/n \leq u \leq 0.7} n^{-(1-\gamma)k^2 + 2k} \leq n^{-(1-\gamma)k^2 + 2k + 1}.$$

$\Box$

The cases $k \in \{3, 4\}$ are slightly more involved. There we will exploit Proposition 4.12.

**Lemma 4.17.** Let $0 < \gamma < 0.001$. Let $H$ be a $k$-graph, where $k \in \{3, 4\}$ and call a set $U \subset V_H$ bad if $e_U = [(1-\gamma)|U|]$ and $\forall e \in E_H : |e \cap U| \neq k - 1$.

Then, for any $c \leq 0.95$ and sufficiently large $n$

$$\mathbb{P}(H_{n,cn,3} \text{ contains a bad subset } U \text{ with } \leq n/2 \text{ vertices}) \leq n^{-2},$$

and for any $c \leq 0.98$ and sufficiently large $n$

$$\mathbb{P}(H_{n,cn,4} \text{ contains a bad subset } U \text{ with } \leq 3n/4 \text{ vertices}) \leq n^{-2}.$$

The proof is essentially the same as the proof of Lemma 4.2 in [13], and thus omitted.
Proof of Theorem 4.2: Recall that it is sufficient to show that the core $C$ of $H_{n,p,k}$ contains with probability $1 - o(n^{-1/2})$ no maximal $(1 - \varepsilon^3)$-dense subsets (see also the discussion in Section 4.1), where $p = ck/(n^2)$.

First of all, recall that the above discussion implies that it is sufficient to consider $\beta \geq 0.7$. Let $k \geq 5$. By applying Lemma 4.16 we obtain that $H_{n,cn,k}$ does not contain any $(1 - \varepsilon^3)$-dense set with less than $0.7n$ vertices, and the same is true for $\tilde{H}_{n,p,k}$, by Proposition 4.15 and Theorem 4.6. In particular, $C$ does not contain such a subset, and the proof is completed.

The case $k = 3$ requires slightly more work. Lemma 4.17 and the fact $c_3^* < 0.95$ guarantee that $C$ has no subset $S$ with $\leq n/2$ vertices such that $e_S = \lfloor (1 - \varepsilon^3)|S| \rfloor$, and there is no edge that contains precisely two vertices in that set. In other words, by using Proposition 4.12, $C$ does not contain a maximal $(1 - \varepsilon^3)$-dense set with $n/2$ vertices. However, we know that with probability $1 - o(1)$

$$N_2 = (1 - e^{-\xi^*} - \xi^* e^{-\xi^*} + O(\delta))n,$$

where $3 = \xi^* (e^{\xi^*} - 1) / e^{\xi^*} - 1 - \xi^*$. Numerical calculations imply that $\tilde{N}_2 \geq 0.63n$ for any $\delta$ that is small enough. So, $C$ does not contain any maximal $(1 - \varepsilon^3)$-dense subset with less than $n/2 \leq \tilde{N}_2 / (2 \cdot 0.63) \leq 0.77\tilde{N}_2$ vertices. This completes the proof for $k = 3$; the case $k = 4$ follows similarly by using the second part of the conclusion of Lemma 4.17 and the fact that $c_4^* < 0.98$. \qed

5 Spanning properties of $H^*_{n,m,k}$

5.1 Proof of Proposition 2.4

Observe that any set $S$ of edges of a $k$-uniform hypergraph that spans a connected hypergraph can contain at most $(k - 1)|S| + 1$ vertices. Proposition 2.4 states that with high probability every set of edges of $H^*_{n,m,k}$ that is not too large contains a number of vertices that almost matches this upper bound. We prove this using a first moment argument, providing an upper bound on the expected number of subsets of edges of size $s$ that span at most $t$ vertices. The proof is similar to that of Lemma 5 in [16], but suitably adjusted to our parameters.

For ease of notation we write $t = (k - 1)s - \delta$. (Later we will set $\delta = x_s s$, and $\delta = 1$, respectively.) The expected number of sets of edges of size $s$ that span at most $t$ vertices is at most

$$\binom{m}{s} \binom{n}{t} \left(\frac{tk}{n^2}\right)^s \left(\frac{ne}{t}\right)^t \left(\frac{t}{n}\right)^{ks} = n^{-\delta} e^{ks - \delta (c^*_k s^{-1})^s t s + \delta}

\leq n^{-\delta} e^{ks (c^*_k s^{-1})^s ((k - 1)s)^s (ks)^s} = \left(\frac{ks}{n}\right)^{\delta} \left(\frac{c^*_k (k - 1) e^k}{n}\right)^s.

Let $\xi > 0$ be such that $(1 + \xi) e^k = 1$. To deduce the first claim we observe that for $\delta = x_s s$ the definition of $x_s$ implies that

$$\left(\frac{ks}{n}\right)^{\delta} \left(\frac{c^*_k (k - 1) e^k}{n}\right)^s = \left[\left(\frac{ks}{n}\right)^{x_s} \left(\frac{c^*_k (k - 1) e^k}{n}\right)^s\right] = (1 + \xi)^{-s}.

Taking the sum over all $s$ in the required range, we deduce that the probability that there exists a set of edges of size $s$, where $\log \log n \leq s \leq n/k$, spanning at most $(k - 1 - x_s s)$ vertices is $O\left((1 + \xi)^{-\log \log n}\right) = o(1)$. 21
The proof of the second part follows the same lines, except that we use slightly cruder upper bounds. In particular, we bound $c_k^* \leq 1$. Setting $\delta = 1$, we deduce from the formula above that the expected number of sets of edges of size $s \leq \log \log n$ which span at most $t = (k - 1)s - 1$ vertices is at most

$$\frac{k \log \log n}{n} \cdot \left(ke^k\right)^{\log \log n} = o(1).$$

5.2 Proof of Lemma 2.5

We will use the following auxiliary fact.

Proposition 5.1. For any constants $a, b > 0$ we have that whenever $D = D(a, b)$ is sufficiently large then

$$\prod_{i=1}^{j} \left(1 - \frac{a}{ib + D}\right) \geq j^{-a/b} \cdot (bD)^{-a/b} \cdot e^{-\frac{a^2}{2bD}} \quad \text{for all } j \geq 2/b.
$$

Proof. Assume that $D \geq 2$ is large enough so that $\frac{a}{b+D} \leq 0.5$. As $1 - x \geq e^{-x-x^2}$ for $x \leq 0.5$ we thus obtain that

$$\prod_{i=1}^{j} \left(1 - \frac{a}{ib + D}\right) \geq \exp\left(- \sum_{i=1}^{j} \frac{a}{ib + D} - \sum_{i=1}^{j} \left(\frac{a}{ib + D}\right)^2\right).$$

Further, we have

$$\sum_{i=1}^{j} \frac{a}{ib + D} \leq a \int_{0}^{j} \frac{1}{bx + D} \, dx = \frac{a}{b} \cdot (\log(bj + D) - \log(D)) \leq \frac{a}{b} \cdot \log(bj + D).$$

Similarly,

$$\sum_{i=1}^{j} \left(\frac{a}{ib + D}\right)^2 \leq a^2 \int_{0}^{j} \frac{1}{(bx + D)^2} \, dx = \frac{a^2}{b} \cdot \left(\frac{1}{D} - \frac{1}{b(j + D)}\right) \leq \frac{a^2}{bD}.$$

Now observe that for $j \geq 2/b$ we have $bj + D \leq bjD$ and thus $\log(bj + D) \leq \log(bj) + \log(D)$. The substitution of these two bounds into (5.1) thus yields

$$\prod_{i=1}^{j} \left(1 - \frac{a}{ib + D}\right) \geq j^{-a/b} \cdot (bD)^{-a/b} \cdot e^{-\frac{a^2}{2bD}}.$$

\[ \square \]

Let $H = H(V, E)$ be a $k$-graph on $n$ vertices having Property $E$. We also fix a vertex $v \in V$ and an orientation $h$ of the edges, and for all $i \geq 0$ we let $s_i$ be the number of vertices that are within $h$-distance at most $i$ from $v$. Note that $s_i = N_{h,i}(v)$, but we shall be using this symbol throughout this section to avoid an unnecessary notational burden. If all vertices within $h$-distance at most $i$ from $v$ are occupied, then Property $E$ implies that

$$s_{i+1} \begin{cases} (k - 1 - x_{s_i})s_i, & \text{if } s_i > \log \log n \\ (k - 1)s_i, & \text{if } s_i \leq \log \log n. \end{cases}$$

(5.2)
Claim 5.2. Let $i_0 := \min \{i : s_i > \log \log n\}$. Then $i_0 \leq \log_{k-1} \log \log n + 1$.

Proof. Observe that for all $2 \leq i < i_0$, we have $s_i \geq (k-1)s_{i-1} \geq \ldots \geq (k-1)^i s_1 = (k-1)^i$, and the claim follows.

Claim 5.3. Let $t_\varepsilon := \lceil (1 - \varepsilon) \log_{k-1} n \rceil$. Then there exists a $d_k > 0$ such that whenever $\varepsilon > 0$ is sufficiently small and $n$ is sufficiently large we have

$$s_{t_\varepsilon} \geq n^{1-\varepsilon} \cdot e^{-d_k/\varepsilon}.$$ 

Proof. Observe that by (2.1) we have $s_{t_\varepsilon} \leq (k-1)^{t_\varepsilon+1}=(k-1)n^{1-\varepsilon}$. Let $i_0$ be defined as in the previous claim. Hence, we have for all $i_0 \leq i \leq t_\varepsilon$ that $x_{s_i} \leq \frac{\log_k((k-1)e^k)}{\varepsilon \log_k n - \log_k (k-1)-1} \leq \frac{2 \log_k((k-1)e^k)}{\varepsilon \log_k n}$, for any $n$ sufficiently large. Thus, for all such $i$ the first part of (5.2) yields

$$s_{i+1} \geq (k-1-x_{s_i})s_i \geq (k-1)\left(1 - \frac{2 \log_k((k-1)e^k)}{(k-1)\varepsilon \log_k n}\right)s_i.$$ 

Set $\phi = \phi(\varepsilon, k, n) = 1 - \frac{2 \log_k((k-1)e^k)}{(k-1)\varepsilon \log_k n}$. By applying the above estimate repeatedly and using Claim 5.2 we obtain

$$s_{t_\varepsilon} \geq (k-1)^{t_\varepsilon-i_0} \phi^{t_\varepsilon-i_0} s_{i_0} \geq (k-1)^{t_\varepsilon-\log_{k-1} \log \log n \phi \log_{k-1} n} \cdot \log \log n \geq n^{1-\varepsilon} \phi^{\log_{k-1} n}.$$ 

Also, since $\frac{\log_k}{(k-1) \log (k-1)} < 1$ for all $k \geq 3$, we obtain that for $n$ large enough and for all $k \geq 3$ we have

$$(1 - \frac{2 \log_k((k-1)e^k)}{(k-1)\varepsilon \log_k n})^{\log_{k-1} n} \geq e^{-2\varepsilon^{-1} \log_k((k-1)e^k)}.$$ 

Thus, if $\varepsilon > 0$ is small enough,

$$s_{t_\varepsilon} \geq n^{1-\varepsilon} e^{-2\varepsilon^{-1} \log_k((k-1)e^k)} \geq n^{1-\varepsilon} e^{-4\varepsilon^{-1} \log_k((k-1)e^k)}.$$ 

Claim 5.4. Let $t \geq i_0$. For every $\varepsilon$ sufficiently small, if $s_t \leq \varepsilon n$, then for all $0 \leq i \leq t - i_0$ (where $i_0$ is as defined in Claim 5.2), we have

$$x_{s_{t-i}} \leq \frac{\log_k((k-1)e^k)}{i \log_k (k-1 - \gamma) + \log_k (1/\varepsilon) - 1} \leq \gamma,$$

where $\gamma = \frac{\log_k((k-1)e^k)}{\log_k (1/\varepsilon) - 1}$.

Proof. We will show the statement by induction on $i$. For $i = 0$ this is obtained directly from the definition of $x_{s_i}$:

$$x_{s_t} = \frac{\log_k((k-1)e^k)}{\log_k(n/s_t) - 1} \leq \frac{\log_k((k-1)e^k)}{\log_k (1/\varepsilon) - 1} = \gamma.$$ 

In general, we have

$$s_{t-(i+1)} \leq \frac{x_{s_{t-(i+1)}}}{k - 1 - x_{s_{t-(i+1)}}} \leq \frac{s_{t-i}}{k - 1 - \gamma} \leq \frac{s_t}{(k-1-\gamma)^{i+1}} \leq \frac{\varepsilon n}{(k-1-\gamma)^{i+1}}.$$ 

Thus the definition of $x_{s_{t-i-1}}$ yields

$$x_{s_{t-i-1}} = \frac{\log_k((k-1)e^k)}{\log_k(n/s_{t-i-1}) - 1} \leq \frac{\log_k((k-1)e^k)}{(i+1) \log_k (k-1 - \gamma) + \log_k (1/\varepsilon) - 1} < \gamma.$$ 

\qed
Claim 5.5. For every $k \geq 3$ and for every $\zeta > 0$, $\varepsilon = \varepsilon(\zeta, k) > 0$ sufficiently small and $n$ sufficiently large, if all vertices within $h$-distance $T := \log_{k-1} n + \left( \frac{\log_k((k-1)e^k)}{(k-1)\log_k(k-1)} + \zeta \right) \log_{k-1} \log_{k-1} n$ from $v$ are occupied, then $s_T > \varepsilon n$.

Proof. We prove the claim by contradiction. Assume that $s_T \leq \varepsilon n$. Claim 5.4 implies that
\[ x_{s_{T-j}} \leq \frac{\log_k((k-1)e^k)}{i \log_k(k-1 - \gamma) + \log_k(1/\varepsilon) - 1} \]
for all $0 \leq i \leq T - i_0$ and the first part of (5.2) thus implies that
\[ s_{T-j} \leq \frac{\varepsilon n}{\prod_{i=1}^{j} (k-1 - x_{s_{T-i}})} \leq \frac{\varepsilon n}{(k-1)^j} \frac{1}{\prod_{i=1}^{j} \left(1 - \frac{\log_k((k-1)e^k)/(k-1)}{i \log_k(k-1 - \gamma) + \log_k(1/\varepsilon) - 1}\right)}. \]
for all $0 \leq j \leq T - i_0$. We now apply Proposition 5.1 for $a = \log_k((k-1)e^k)/(k-1)$ and $b = \log_k(k-1 - \gamma)$. Then Proposition 5.1 implies such that whenever $\varepsilon$ is sufficiently small (such that $\log_k(1/\varepsilon) - 1 \geq D$, where $D = D(a, b)$ is as defined in Proposition 5.1), then
\[ \prod_{i=1}^{j} \left(1 - \frac{\log_k((k-1)e^k)/(k-1)}{i \log_k(k-1 - \gamma) + \log_k(1/\varepsilon) - 1}\right) \geq j^{-\frac{\log_k((k-1)e^k)}{(k-1)\log_k(k-1 - \gamma)}} \cdot \frac{1}{C_{\varepsilon, k}}, \]
where $C_{\varepsilon, k}$ is an appropriately defined constant depending only on $\varepsilon$ and $k$. Then substituting the lower bound from (5.4) into (5.3) we obtain that for all $j \leq \log_{k-1} n$ we have
\[ s_{T-j} \leq \frac{\varepsilon n}{(k-1)^j} \left( \log_{k-1} n \right)^{\frac{\log_k((k-1)e^k)}{(k-1)\log_k(k-1 - \gamma)}} C_{\varepsilon, k}. \]

If we now set $R_{\varepsilon, k} := C_{\varepsilon, k} \cdot \varepsilon_k^{d_k/\varepsilon}$, where $d_k$ is the constant from Claim 5.3 we deduce that for $j := \varepsilon \log_{k-1} n + \frac{\log_k((k-1)e^k)}{(k-1)\log_k(k-1 - \gamma)} \log_{k-1} \log_{k-1} n + \log_{k-1} R_{\varepsilon, k}$ we have
\[ s_{T-j} \leq \varepsilon e^{-d_k/\varepsilon \cdot n^{1-\varepsilon}}. \]

If $\varepsilon$ is small enough, then in turn $\gamma$ is small enough so that for $n$ sufficiently large $T - j \geq [(1 - \varepsilon) \log_{k-1} n]$; this, however, contradicts the lower bound from Claim 5.3. \qed

Note that Claim 5.5 completes the proof of Lemma 2.5.

5.3 Proof of Corollary 2.6
By Proposition 2.4, $\hat{H}$ still has Property $\mathcal{E}$ with $n$ instead of $N = |V(\hat{H})|$. Using this, the proof of Lemma 2.6 follows exactly along the lines of the proof of Lemma 2.5.

6 Conclusion
The main result of this paper asserts that for all $k \geq 3$ the random insertion algorithm succeeds in polylogarithmic time with high probability, for any number of inserted items arbitrarily close to the theoretically achievable load threshold for cuckoo hashing. In particular, we
showed that the maximum number of steps performed by the algorithm is at most \( \log^{c(k)} + \zeta n \) with high probability, where \( c(k) = 2 + \frac{1}{\log k} + O(\frac{1}{k}) \) and \( \zeta > 0 \) is arbitrary.

One of the main ingredients of our proof is a precise description of the expansion properties of an associated random hypergraph, from which we were able to derive conclusions regarding the structure of any orientation of its edges. In particular, our analysis implies that there might be vertices that are far away from any free vertex, that is, a vertex such that no edge is oriented to it. In turn, this implies that \( \Omega(\log n) \) steps are required to reach a free vertex, if the initial insertion takes place in such a vertex. This immediately also implies a logarithmic lower bound on the maximum insertion time.

Clearly, if we need \( \Omega(\log n) \) steps in the best case, then a random algorithm is likely to miss this best case. In our analysis, we thus divide the insertion process into phases: each phase lasts \( O(\log n) \) steps, and it either ends successfully at a free vertex, or it fails. In the latter case, we assume that the worst-case has occurred, namely that we start again at the furthest possible vertex. This clearly need not be the case (and it is probably not), but currently our methods do not seem strong enough to provide a way to analyze this.

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