Classification of the entangled states of $2 \times L \times M \times N$

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Abstract We present a practical entanglement classification scheme for pure state in form of $2 \times L \times M \times N$ under the stochastic local operation and classical communication (SLOCC), where every inequivalent class of the entangled quantum states may be sorted out according to its standard form and the corresponding transformation matrix. This provides a practical method for determining the interconverting matrix between two SLOCC equivalent entangled states, and classification examples for some $2 \times 4 \times M \times N$ systems are also presented.

Keywords Quantum entanglement · Entanglement classification · Matrix decomposition

1 Introduction

Quantum theory stands as a unique pillar of physics. One of the essential aspects providing quantum technologies an advantage over classical methods is quantum entanglement. Quantum entanglement has practical applications in such quantum information processing as quantum teleportation [1], quantum cryptography [2], and dense coding [3,4]. Based on the various functions in carrying out quan-
quantum information tasks, entanglement is classified. If two quantum states are interconverted via stochastic local operation and classical communication (SLOCC), they belong to the same class and are able to carry out the same quantum information task [5]. Mathematically, this is expressed such that the two quantum states in one SLOCC class are connected by invertible local operators. The operator formalism of the entanglement equivalence problem is, therefore, the foundation of the qualitative and quantitative characterizations of quantum entanglement.

Although the entanglement classification is a well-defined physical problem, generally it is mathematically difficult, especially with the partite and dimensions of the Hilbert space growing. Unlike the entanglement classification under local unitary operators [6], the full classification under SLOCC for general multipartite states has solely been obtained for up to four qubits [5,7]. For the symmetric $N$-qubit state, an operational classification scheme is presented in [8]. While in the high-dimensional and less partite cases, matrix decomposition turns out to be an effective tool for the entanglement classification under the SLOCC [9], e.g., the classification of the $2 \times M \times N$ system was completed in [10–12] and the entanglement classes of the $L \times N \times N$ system have found to be tractable [13]. Although an inductive method was introduced in [14,15] to process entangled states with more particles, its complexity substantially grows with the increasing number of particles. By using the rank coefficient matrices (RCM) technique [16], the arbitrary dimensional multipartite entangled states have been partitioned into discrete entanglement families [17,18]. As the multipartite entanglement classes generally contain continuous parameters which grow exponentially as the partite increases [5], such discrete families represent a coarse grained discrimination over the multipartite entanglement classes. Two SLOCC inequivalent quantum states were indistinguishable when falling into the same discrete family. Therefore, a general scheme that is able to completely identify the different entanglement classes and determine the transformation matrices connecting two equivalent states under SLOCC for arbitrary dimensional four-partite states remains a significant unachieved challenge of quantum information theory.

In this work, we present a general classification scheme for the four-partite $2 \times L \times M \times N$ pure system, where the entangled states are sorted into different entanglement classes under SLOCC by utilizing the tripartite entanglement classification [10–12] and the matrix realignment technique [19,20]. The structure of the paper goes as follows. In Sect. 2, the quantum states are first expressed in the matrix-pair forms. Then the entanglement classification method is accomplished by the construction of the standard forms from the matrix-pairs and the determination of the transformation matrices via the matrix realignment technique. In Sect. 3, operational considerations and some representative examples of $2 \times 4 \times M \times N$ entanglement classes are presented, where the comparison with existing results is also discussed. Summary and conclusions are given in Sect. 4.
2 The classification of $2 \times L \times M \times N$

2.1 The representation of the quantum state

A quantum state of $2 \times L \times M \times N$ takes the following form

$$|\psi\rangle = \sum_{i,l,m,n=1}^{2LMN} \gamma_{ilmn} |i, l, m, n\rangle,$$

where $\gamma_{ilmn} \in \mathbb{C}$ are complex numbers. In this form, the quantum state may be represented by a high-dimensional complex tensor $\psi$ whose elements are $\gamma_{ilmn}$. Two such quantum states $\psi'$ and $\psi$ are said to be SLOCC equivalent if [5]

$$\psi' = A^{(1)} \otimes A^{(2)} \otimes A^{(3)} \otimes A^{(4)} \psi.$$

Here, $A^{(1)} \in \mathbb{C}^{2 \times 2}$, $A^{(2)} \in \mathbb{C}^{L \times L}$, $A^{(3)} \in \mathbb{C}^{M \times M}$, $A^{(4)} \in \mathbb{C}^{N \times N}$ are invertible matrices of $2 \times 2$, $L \times L$, $M \times M$, $N \times N$ separately, which act on the corresponding particles. The transformation of the tensor elements reads

$$\gamma'_{i'l'm'n'} = \sum_{i,l,m,n} A^{(1)}_{i'i} A^{(2)}_{l'l} A^{(3)}_{m'm} A^{(4)}_{n'n} \gamma_{ilmn},$$

where $\gamma'_{i'l'm'n'}$ are the tensor elements of $\psi'$, and $A^{(k)}_{ij}$ are the matrix elements of the invertible operators $A^{(k)}$, $k \in \{1, 2, 3, 4\}$.

As a tensor, the quantum state $\psi$ may also be represented in the form of a matrix-pair representation, that is $\psi \doteq \begin{pmatrix} \Gamma_1 \\ \Gamma_2 \end{pmatrix}$. To be specific, for the $2 \times L \times M \times N$ system, we have the following

$$\psi \doteq \begin{pmatrix} \Gamma_1 \\ \Gamma_2 \end{pmatrix} = \begin{pmatrix} \gamma_{1111} & \gamma_{1112} & \cdots & \gamma_{11MN} \\ \gamma_{1211} & \gamma_{1212} & \cdots & \gamma_{12MN} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{1LM1} & \gamma_{1LM2} & \cdots & \gamma_{1LMN} \\ \gamma_{2111} & \gamma_{2112} & \cdots & \gamma_{21MN} \\ \gamma_{2211} & \gamma_{2212} & \cdots & \gamma_{22MN} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{2LM1} & \gamma_{2LM2} & \cdots & \gamma_{2LMN} \end{pmatrix}.$$

Here, $\Gamma_i \in \mathbb{C}^{L \times MN}$, i.e., complex matrices of $L$ columns and $M \cdot N$ rows. For the sake of convenience, here we assume $L < MN$ for $\Gamma_i \in \mathbb{C}^{L \times MN}$, while for $L \geq MN$ case, a $2 \times (M \times N) \times L$ system state is represented in the matrix-pair form of $\Gamma_i \in \mathbb{C}^{MN \times L}$. This ensures that the matrix columns being always more than or equal to the rows.
In this matrix-pair representation, the SLOCC equivalence of two states $\psi'$ and $\psi$ in Eq. (2) transforms into the following form

$$\begin{pmatrix} \Gamma_1' \\ \Gamma_2' \end{pmatrix} = A^{(1)} \begin{pmatrix} P\Gamma_1Q \\ P\Gamma_2Q \end{pmatrix},$$

(5)

where $P = A^{(2)}$, $Q^T = A^{(3)} \otimes A^{(4)}$, $T$ stands for matrix transposition, $A^{(1)}$ acts on the two matrices $\Gamma_{1,2}$, and $P$ and $Q$ act on the rows and columns of the $\Gamma_{1,2}$ matrices. The SLOCC equivalence of two $2 \times L \times M \times N$ quantum states in Eq. (5) has a similar form as that of the tripartite $2 \times L \times MN$ pure state [11]. The sole difference lies in that here $Q$ is not only an invertible operator but also a direct product of two invertible matrices, $A^{(3)}$ and $A^{(4)}$.

2.2 Standard forms for the $2 \times L \times M \times N$ system

The entanglement classification of the tripartite state $2 \times L \times MN$ under SLOCC has already been completed in [10,11]. Two tripartite entangled states are SLOCC equivalent if and only if their standard forms coincide. We define such standard forms of $2 \times L \times MN$ to be the standard forms of the matrix-pair of a $2 \times L \times M \times N$ system, i.e.,

$$T \otimes P \otimes Q^T \psi = T \begin{pmatrix} P\Gamma_1Q \\ P\Gamma_2Q \end{pmatrix} = \begin{pmatrix} E \\ J \end{pmatrix}.$$  

(6)

Here $T \in \mathbb{C}^{2 \times 2}$, $P \in \mathbb{C}^{L \times L}$, $Q \in \mathbb{C}^{MN \times MN}$ are all invertible matrices, and $E$ is the unit matrix, $J$ is in Jordan canonical form (we refer to [11] for the general case of the standard form). The Jordan canonical form $J$ has a typical expression of

$$J = \bigoplus_i J_{n_i}(\lambda_i),$$

(7)

wherein $\lambda_i \in \mathbb{C}$, $J_{n_i}(\lambda_i)$ are $n_i \times n_i$ Jordan blocks

$$J_{n_i}(\lambda_i) = \begin{pmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & \cdots & 0 \\ 0 & 0 & \lambda_i & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_i \end{pmatrix}.$$  

(8)

For the $2 \times L \times M \times N$ quantum state $\psi$ in the form of Eq. (4), the following proposition exists:

**Proposition 1** If two quantum states of $2 \times L \times M \times N$ are SLOCC equivalent then their corresponding matrix-pairs have the same standard forms under the invertible operators $T \in \mathbb{C}^{2 \times 2}$, $P \in \mathbb{C}^{L \times L}$, $Q \in \mathbb{C}^{MN \times MN}$.
Proof Suppose that two quantum states of $2 \times L \times M \times N$, $\psi$ and $\psi'$ are represented in the matrix-pairs

$$
\psi = \begin{pmatrix} \Gamma_1 \\ \Gamma_2 \end{pmatrix}, \psi' = \begin{pmatrix} \Gamma'_1 \\ \Gamma'_2 \end{pmatrix}.
$$
(9)

The standard form of $\psi$ under the invertible operators of $T_0 \in \mathbb{C}^{2 \times 2}$, $P_0 \in \mathbb{C}^{L \times L}$, $Q_0 \in \mathbb{C}^{MN \times MN}$ is constructed as that of a $2 \times L \times MN$ system, which is

$$
T_0 \otimes P_0 \otimes Q_0^T \psi = T_0 \begin{pmatrix} P_0 \Gamma_1 Q_0 \\ P_0 \Gamma_2 Q_0 \end{pmatrix} = \begin{pmatrix} E \\ J \end{pmatrix}.
$$
(10)

If $\psi'$ is SLOCC equivalent to $\psi$, then there exists the invertible matrices $A^{(i)}$, such that

$$
A^{(1)} \otimes A^{(2)} \otimes A^{(3)} \otimes A^{(4)} \psi' = \psi.
$$
(11)

The matrix-pair form of $\psi'$ could also be transformed into $\begin{pmatrix} E \\ J \end{pmatrix}$ via invertible matrices, because

$$
T_0 A^{(1)} \otimes P_0 A^{(2)} \otimes Q_0^T (A^{(3)} \otimes A^{(4)}) \psi'
= T_0 \otimes P_0 \otimes Q_0^T \psi'
= \begin{pmatrix} E \\ J \end{pmatrix}.
$$
(12)

This proposition serves as a necessary condition for the SLOCC equivalence of the entangled states of the $2 \times L \times M \times N$ system. That is, if their matrix-pair representation do not have the same standard form, the two $2 \times L \times M \times N$ entangled states are SLOCC inequivalent. The converse of Proposition 1 is not true, which means that different entanglement classes of $2 \times L \times M \times N$ system may have the same standard form under the SLOCC.

2.3 The transformation matrices to standard form

The standard forms of the tripartite $2 \times L \times MN$ system have been regarded as the standard forms of the corresponding $2 \times L \times M \times N$ system, or more accurately, the entanglement families of the $2 \times L \times M \times N$ system, each of which may be transformed from entangled states of different entanglement classes under SLOCC. In addition, the transforming matrices $T$, $P$, $Q$ for the standard form in Eq. (6) were also obtained.

Generally, the transformation matrices for the standard form are not unique. For example, if $T_0$, $P_0$, $Q_0$ in Eq. (10) are the matrices that transform $\psi$ into its standard...
form, then the following matrices will do likewise

\[ T_0 \otimes SP_0 \otimes (Q_0S^{-1})^T \psi = \begin{pmatrix} E \\ J \end{pmatrix}, \]  

(13)

where \( SJ S^{-1} = J \), i.e., \([S, J] = 0\). The commutative relation implies that if all the \( \lambda_i \) in the Jordan form \( J \) of Eq. (7) have geometric multiplicity 1, then the invertible matrix \( S \) may be expressed as \( S = \bigoplus S_{n_i} \), where \( S_{n_i} \) are the \( n_i \times n_i \) upper triangular Toeplitz matrices conformal to the blocks of Eq. (7)

\[ S_{n_i} = \begin{pmatrix} s_{i1} & s_{i2} & s_{i3} & \cdots & s_{in_i} \\ 0 & s_{i1} & s_{i2} & \cdots & s_{i(n_i-1)} \\ 0 & 0 & s_{i1} & \cdots & s_{i(n_i-2)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & s_{i1} \end{pmatrix}. \]  

(14)

For the general case of the geometric multiplicity of \( \lambda_i \), we refer to [13] and the references therein. There may also be an invertible operation \( S_1 \in \mathbb{C}^{2 \times 2} \) which acts on the first particle and leave the ranks of the pair of matrices invariant. This operation could be compensated by the operations on the second and third particles which leave the standard form invariant

\[ S_1 \begin{pmatrix} S_2ES_3 \\ S_2JS_3 \end{pmatrix} = \begin{pmatrix} E \\ J \end{pmatrix}. \]  

(15)

Here, the parameters in matrices \( S_2 \in \mathbb{C}^{L \times L}, S_3 \in \mathbb{C}^{MN \times MN} \) solely depend on that of \( S_1 \), as shown in the proof of the two theorems in [10].

Combining Eqs. (13) and (15), the matrices that keep the tripartite standard forms invariant are

\[ S_1 \begin{pmatrix} SS_2ES_3S^{-1} \\ SS_2JS_3S^{-1} \end{pmatrix} = \begin{pmatrix} E \\ J \end{pmatrix}. \]  

(16)

Hence, the transformation matrices which connect the two quantum states \( \psi \) and \( \psi' \), which have the same standard form of matrix-pair, could generally be written as

\[ T_0 \otimes P_0 \otimes Q_0^T \psi = \begin{pmatrix} E \\ J \end{pmatrix} = T_0' \otimes P_0' \otimes Q_0^T \psi' \Rightarrow \psi' = T \otimes P \otimes Q^T \psi, \]  

(17)

where \( T = T_0'^{-1}S_1T_0 \in \mathbb{C}^{2 \times 2}, P = P_0'^{-1}SS_2P_0 \in \mathbb{C}^{L \times L}, Q^T = Q_0S_3S^{-1}Q_0'^{-1} \in \mathbb{C}^{MN \times MN} \), see Fig. 1. These matrices may be obtained when the standard forms are constructed and their nonuniqueness comes from the symmetries of standard forms. A detailed example for the construction of these matrices is presented in Sect. 3.3.
Classification of the entangled states of $2 \times L \times M \times N$

Two quantum states $\psi, \psi'$ of $2 \times L \times M \times N$ have the same standard form $(E, J)$ under the operations $(T_0, P_0, Q_0)$, and $(E', J)$ is invariant under $(S_1, S_2, S_3^{-1})$, where all the triples of the transformation matrices have the dimensions of $(2 \times 2, L \times L, MN \times MN)$. If there exists a route (bold line) where $Q_0 S_3^{-1} Q_0^{-1}$ may be written as the Kronecker product of two invertible matrices of $C^{M \times M}$ and $C^{N \times N}$, then $\psi'$ and $\psi$ are the SLOCC equivalent $2 \times L \times M \times N$ entangled states.

2.4 The matrix realignment method

To complete the entanglement classification, we introduce the matrix realignment technique. With each matrix $A \in C^{m \times n}$, the matrix vectorization is defined to be [21]

$$\text{vec}(A) \equiv (a_{11}, \ldots, a_{m1}, a_{12}, \ldots, a_{m2}, a_{1n}, \ldots, a_{mn})^T.$$  (18)

If the dimensions of $A$ have $m = m_1 m_2$, $n = n_1 n_2$, then it may be expressed in the following block-form

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n_1} \\ A_{21} & A_{22} & \cdots & A_{2n_1} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m_11} & A_{m_12} & \cdots & A_{m_1 n_1} \end{pmatrix}. \quad (19)$$

Here, $A_{ij}$ are $m_2 \times n_2$ submatrices. The realignment of the matrix $A \in C^{m_1 m_2 \times n_1 n_2}$ according to the blocks $A_{ij} \in C^{m_2 \times n_2}$ is defined to be

$$\mathcal{R}(A) \equiv (\text{vec}(A_{11}), \ldots, \text{vec}(A_{m_11}), \text{vec}(A_{12}), \ldots, \text{vec}(A_{m_12}), \ldots, \text{vec}(A_{m_1 n_1}))^T,$$

where $\mathcal{R}(A) \in C^{m_1 n_1 \times m_2 n_2}$. It has been proved that there exists a Kronecker Product singular value decomposition (KPSVD) for the matrix $A \in C^{m \times n}$ with the integer factorizations $m = m_1 m_2$ and $n = n_1 n_2$, which tells [19]:

Lemma 2 For a matrix $A \in C^{m_1 m_2 \times n_1 n_2}$, if $\mathcal{R}(A) \in C^{m_1 n_1 \times m_2 n_2}$ has the singular value decomposition (SVD) $\mathcal{R}(A) = U \Sigma V^\dagger$, where $\Sigma = \text{diag}[\sigma_1, \ldots, \sigma_r]$, $\sigma_i > 0$ are the singular values and $r$ is the rank of $\mathcal{R}(A)$, then $A = \sum_{k=1}^{r} \sigma_k U_k \otimes V_k$, where
$U_k \in \mathbb{C}^{m_1 \times n_1}$, $V_k \in \mathbb{C}^{m_2 \times n_2}$, \( \text{vec}(U_k) = \sqrt{\sigma_k/\alpha_k} u_k \), \( \text{vec}(V_k) = \sqrt{\alpha_k \sigma_k} v_k^* \), the scaling parameters $\alpha_k \neq 0$ are arbitrary and $u_k$, $v_k$ are the left and right singular vectors of $\mathcal{R}(A)$.

This technique has been applied for recognizing bipartite entanglement [22] and determining the local unitary equivalence of two quantum states [20, 23]. From Lemma 2, we have the following corollary:

**Lemma 3** An $MN \times MN$ invertible matrix $A$ may be expressed as the Kronecker product of an $M \times M$ invertible matrix and an $N \times N$ invertible matrix iff the rank of $\mathcal{R}(A)$ is 1.

### 2.5 The complete classification of the $2 \times L \times M \times N$ system

Following the preparation of Sect. 2.4, the following theorem for the entanglement classification of $2 \times L \times M \times N$ pure states under SLOCC is presented.

**Theorem 4** Two $2 \times L \times M \times N$ quantum states $\psi$ and $\psi'$ are SLOCC equivalent iff their corresponding matrix-pair representations have the same standard forms and the realignment of the transformation matrices $Q$ in Eq. (17) could have rank one.

**Proof** If two $2 \times L \times M \times N$ quantum states $\psi$ and $\psi'$ are SLOCC equivalent with the connecting matrices between $\psi$ and $\psi'$ are $A^{(i)}$, $i \in \{1, 2, 3, 4\}$

\[
\psi' = A^{(1)} \otimes A^{(2)} \otimes A^{(3)} \otimes A^{(4)} \psi, \tag{20}
\]

then they have the same standard form in the matrix-pair form according to Proposition 1. Through this standard form, there is another connecting route between $\psi$ and $\psi'$ in addition to Eq. (20), i.e.,

\[
\psi' = T \otimes P \otimes Q^T \psi. \tag{21}
\]

Combining Eqs. (20) and (21) yields

\[
T^{-1} A^{(1)} \otimes P^{-1} A^{(2)} \otimes ((Q^T)^{-1} A^{(3)} \otimes A^{(4)}) \psi = \psi. \tag{22}
\]

As the unit matrices $E \otimes E \otimes E$ must be one of the operators which stabilizes the quantum state $\psi$ in the matrix-pair form, $Q^T$ has the solution of $Q^T = A^{(3)} \otimes A^{(4)}$. Thus, $\mathcal{R}(Q)$ could have rank one according to Lemma 3.

If the two quantum states have the same standard forms, then we will have Eq. (17). If the matrix realignment $\mathcal{R}(Q)$ according to the factorization $MN = M \times N$ has rank one, then $Q$ may be decomposed as $Q = Q_1 \otimes Q_2$ where $Q_1 \in \mathbb{C}^{M \times M}$, $Q_2 \in \mathbb{C}^{N \times N}$. As matrix $Q$ is invertible if and only if both $Q_1$ and $Q_2$ are invertible, thus

\[
\psi' = T \otimes P \otimes (Q_1 \otimes Q_2)^T \psi. \tag{23}
\]

Therefore, $\psi'$ and $\psi$ are SLOCC equivalent entangled states of a $2 \times L \times M \times N$ system. \(\square\)
To summarize, the entanglement classification scheme for the $2 \times L \times M \times N$ consists of two steps. First, the standard forms of the matrix-pair form $2 \times L \times M \times N$ quantum state $\psi$ are constructed. By utilizing the standard forms, the entangled families of $2 \times L \times M \times N$ and the interconverting matrices between two quantum states in the same family, $T$, $P$, $Q$, are obtained. Second, by determining whether or not the connecting matrix $Q$ may be decomposed as the Kronecker product of two invertible matrices via the matrix realignment technique the SLOCC equivalence of the two quantum states is asserted. Thus, the standard form together with the route (for the connecting matrices, see Fig. 1) between the quantum states form a complete classification of the $2 \times L \times M \times N$ quantum states.

3 Examples for the entanglement classification of $2 \times L \times M \times N$

3.1 Physical considerations and the genuine entangled families

In the field of entanglement classification, it is of great interest if we may establish the so-called operational classifications of entanglement [8], i.e., the different entanglement classes are related to some experimental configuration in real physical systems. Among the possible implementations, a two-level atom with the multimode radiation fields may be generally considered as a system of $2 \times L \times M \times N$ [24–26]. This is of particular importance as the theoretical model describing the interaction, the Jaynes–Cummings model [27], is exactly solvable and now has been extended to various situations [28, 29].

Here, we consider the genuine entanglement in the $2 \times L \times M \times N$ pure system. A necessary condition for the genuine entanglement of a $2 \times L \times M \times N$ system is that all dimensions of the four particles shall be involved in the entanglement. This requires

$$L \leq 2MN,$$

where without loss of generality we assume the largest value of the dimensions to be $L$. For example, a particle with dimension 25 in a $2 \times 3 \times 4 \times 25$ system would always have one effective dimension unentangled and it would have at most the genuine entanglement of $2 \times 3 \times 4 \times 24$. For $L = 4$, i.e., where the largest value of the dimensions is four, the entangled systems which satisfy Eq. (24) include

$$2 \times 2 \times 2 \times 4, \quad 2 \times 4 \times 3 \times 2, \quad 2 \times 4 \times 4 \times 2,$$

$$2 \times 4 \times 3 \times 3, \quad 2 \times 4 \times 4 \times 3, \quad 2 \times 4 \times 4 \times 4.$$

In the construction of the standard forms (entanglement families) of $2 \times L \times M \times N$, only the operator $Q \in \mathbb{C}^{MN \times MN}$ acts on the bipartite Hilbert spaces. As the standard forms give the genuine entanglement of the $2 \times L \times MN$ system [11], genuine entanglement families of the $2 \times L \times M \times N$ system are obtained if all the dimensions of $M$ and $N$ appear in the standard forms. Therefore, the total number of such families is calculated to be
\[
N_f = \sum_{i=d}^{D} \Omega_{L,i}
\]  

(26)

where \( i \in \mathbb{N}, d = \max\{M, N, \lceil L/2 \rceil \}, D = \min\{2L, MN\}, \Omega_{L,i} \) are the numbers of genuine entanglement classes of a \( 2 \times L \times i \) system (\( \Omega_{L,i} \)s are calculated from Eq. (29) in [12], with the class containing parameters is counted as being one family). From Eq. (26), the numbers of entanglement families \( N_f \) for the systems in Eq. (25) are obtained as

\[
\begin{align*}
N_f(2224) &= 22, \\
N_f(2432) &= 39, \\
N_f(2442) &= 37, \\
N_f(2433) &= 42, \\
N_f(2443) &= 37, \\
N_f(2444) &= 37.
\end{align*}
\]  

(27)

Here, \( N_f(2LMN) \) stands for the number genuine entanglement families of a \( 2 \times L \times M \times N \) system obtained from our method.

For the sake of comparison, we first list all of the entanglement families for a \( 2 \times 2 \times (2 \times 2) \) system resulting from our method. The \( N_f(2222) = 5 \) families includes: Two families from \( 2 \times 2 \times 2 \) (GHZ and W)

\[
|\psi\rangle = |11(11)\rangle + |12(22)\rangle + |21(11)\rangle, \\
|\psi\rangle = |11(11)\rangle + |12(22)\rangle + |21(22)\rangle,
\]

two families from \( 2 \times 2 \times 3 \)

\[
|\psi\rangle = |11(11)\rangle + |12(12)\rangle + |22(21)\rangle, \\
|\psi\rangle = |11(11)\rangle + |12(12)\rangle + |21(12)\rangle + |22(21)\rangle,
\]

and one family from \( 2 \times 2 \times 4 \)

\[
|\psi\rangle = |11(11)\rangle + |12(12)\rangle + |21(21)\rangle + |22(22)\rangle,
\]

where the bracket in the ket packages the particles 3 and 4.

The number of entangled families here differs from that of [7] where an accidental symmetry of \( \text{SU}(2) \otimes \text{SU}(2) \simeq \text{SO}(4) \) specific to 4-qubit states is explored, which could not be applied in more general cases of \( 2 \times L \times M \times N \). Within our scheme, any genuine entangled states of \( 2 \times 2 \times 2 \times 2 \) system may be transformed into one of the above standard forms (entangled families). However, according to Theorem 4, further analysis of their transformation matrices is needed in determining the SLOCC equivalence for the two quantum states which are assorted into the same entanglement family in our scheme. In the following, we give examples of how our method is applied in the \( 2 \times L \times M \times N \) system.

3.2 Examples of \( 2 \times 2 \times 2 \times 4 \)

We may package the 2 and 3 particles in the representation of the quantum states. The genuine entangled families of \( 2 \times (2 \times 2) \times 4 \) quantum states are listed as follows.
One family comes from the tripartite $2 \times 2 \times 4$ system

$$|\psi\rangle = |1(11)1\rangle + |1(22)2\rangle + |2(11)3\rangle + |2(22)4\rangle. \quad (28)$$

Five families come from $2 \times 3 \times 4$ system

$$|\psi\rangle = |1(11)1\rangle + |1(12)2\rangle + |1(21)3\rangle + |2(21)4\rangle,$$
$$|\psi\rangle = |1(11)1\rangle + |1(12)2\rangle + |1(21)3\rangle + |2(11)2\rangle + |2(21)4\rangle,$$
$$|\psi\rangle = |1(11)1\rangle + |1(12)2\rangle + |1(21)3\rangle + |2(11)1\rangle + |2(21)4\rangle,$$
$$|\psi\rangle = |1(11)1\rangle + |1(12)2\rangle + |1(21)3\rangle + |2(12)3\rangle + |2(21)4\rangle,$$
$$|\psi\rangle = |1(11)1\rangle + |1(21)2\rangle + |1(21)3\rangle + |2(11)2\rangle + |2(12)3\rangle + |2(21)4\rangle.$$

The other 16 families come from the standard forms of a $2 \times 4 \times 4$ system.

Therefore, there are totally 22 inequivalent families for the genuine $2 \times 2 \times 2 \times 4$ entangled classes according to the present method, while 15 distinct genuine entanglement families have been identified in [17]. There are two merits within our method. First, the 22 nonequivalent entanglement families correspond to a finer grained entanglement classification under SLOCC than that of the 15 families. Second, after obtaining the entanglement families, our method also provides a general procedure to find out the connecting matrices for two entangled states assorted into the same family, from which the SLOCC equivalence of the two states may be determined. While no further assessment of equivalence of two entangled states could be made if they fall into the same entanglement family in the coefficient matrices method.

Based on the method presented here, there also exist the continuous entanglement families. That is, different entanglement families arise from the different values of the characterization parameters. Here, we present an example of this kind. Among the 16 standard forms of $2 \times 4 \times 4$, there is the following

$$|\psi\rangle = |111\rangle + |122\rangle + |133\rangle + |144\rangle + \lambda_1|211\rangle + \lambda_2|222\rangle + \lambda_3|233\rangle, \quad (29)$$

where $\forall i \neq j, \lambda_j \neq \lambda_j$ and $\lambda_{i,j} \neq 0, 1$. This corresponds to the following entanglement family of $2 \times (2 \times 2) \times 4$ system

$$|\psi\rangle = |1(11)1\rangle + |1(12)2\rangle + |1(21)3\rangle + |1(22)4\rangle + \lambda_1|2(11)1\rangle + \lambda_2|2(12)2\rangle + \lambda_3|2(21)3\rangle, \quad (30)$$

According to the RCM method [17], this state would be regarded as one single family \(\mathcal{F}_{\sigma_0, \sigma_1, \sigma_2}\) regardless of the values of \(\lambda_i\) (still satisfying the condition of Eq. (29)), \(i \in \{1, 2, 3\}\), and no further assessment of the SLOCC equivalence for the states \(|\psi\rangle\) with the parameters of different values may be made. Here, we show that the state of Eq. (30) corresponds to a continuous entanglement family of $2 \times 2 \times 2 \times 4$ system according to our scheme.
First, as a $2 \times 4 \times 4$ state, the matrix-pair form of the state $|\psi\rangle$ is

$$
\psi = \begin{pmatrix} \Gamma_1 \\ \Gamma_2 \end{pmatrix} = \begin{pmatrix} E \\ J \end{pmatrix}.
$$

(31)

Here

$$
E = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad J = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
$$

(32)

It has already been the standard form of a $2 \times 4 \times 4$ system. From [12], we have the following two facts concerning this standard form. First, the operations of

$$
T = \begin{pmatrix} \lambda_2 \\ -\frac{\lambda_2}{\lambda_1-\lambda_2} \\ 0 \\ \frac{1}{\lambda_1} \end{pmatrix}, \quad P = \text{diag} \left\{ 1, \frac{\lambda_1}{\lambda_2}, \frac{\lambda_1}{\lambda_3}, \frac{\lambda_1-\lambda_2}{\lambda_2} \right\}, \quad Q = E
$$

(33)

will transform the state into

$$
\psi(\lambda) = \begin{pmatrix} \Gamma_1 \\ \Gamma_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
$$

(34)

The continuous parameter $\lambda = [\lambda_2(\lambda_1-\lambda_3)]/[\lambda_3(\lambda_1-\lambda_2)]$, the cross ratio for the quadruple $(0, \lambda_1, \lambda_2, \lambda_3)$, is invariant under $T \in \mathbb{C}^{2\times 2}$, $P \in \mathbb{C}^{4\times 4}$ and $Q \in \mathbb{C}^{4\times 4}$ which are the invertible operators maintaining the invariance of the standard form. Second, there is a residual symmetry for $\lambda$ whose generators are $F(\lambda) = 1/\lambda$, $G(\lambda) = 1 - \lambda$. Thus, $\psi(\lambda)$ with $\lambda \in S_\lambda = \{ \lambda, 1/\lambda, 1-\lambda, \lambda/(\lambda-1), 1/(1-\lambda), 1-1/\lambda \}$ are all SLOCC equivalent. The transformation matrices for $F$ and $G$ are

$$
G = T \otimes P \otimes Q = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},
$$

(35)

$$
F = T \otimes P \otimes Q = \begin{pmatrix} 1/\lambda & 0 \\ 0 & 1/\lambda \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
$$

(36)
\(\psi(\lambda)\) is the continuous entanglement class for \(2 \times 4 \times 4\) system, that is different values of \(\lambda\) correspond to different entanglement classes. However, when the values of \(\lambda\) are in the set \(S_\lambda\), \(\psi(\lambda)\)s will belong to the same entanglement class, e.g., \(\psi(2), \psi(1/2),\) and \(\psi(-1)\) belong to the same \(2 \times 4 \times 4\) entanglement class.

Now according to the scheme of Theorem 4, the standard forms (the entanglement classes) of \(2 \times 4 \times 4\) system would turn to the entanglement families of \(2 \times 2 \times 2 \times 4\) system. Therefore, \(\psi(\lambda)\) becomes the continuous entanglement family of \(2 \times 2 \times 2 \times 4\) system, where different values of \(\lambda\) give rise to different entanglement families. A subtle question arises: whether \(\psi(\lambda)\) with \(\lambda \in S_\lambda\) correspond to different entanglement families of \(2 \times 2 \times 2 \times 4\) system or not? To this end, we shall apply the matrix realignment method to the transformation matrices of \(P_s\) which connect the different states \(\psi(\lambda)\) with distinct values of \(\lambda\) where \(\lambda \in S_\lambda\). As the \(P_s\) act on the bipartite Hilbert space of \(2 \times 2\), their matrix realignment according to the factorization \(4 = 2 \times 2\) is

\[
R(P_G) = \begin{pmatrix}
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & -1
\end{pmatrix}, \quad R(P_F) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & \lambda
\end{pmatrix},
\]

where \(P_G, F\) are just the \(P\) operators that bring about the symmetry operations \(G, F\) in Eqs. (35) and (36). It is clear that none of them in Eq. (37) can have rank one; therefore, the transformation operations relating the \(\lambda\)s in the set \(S_\lambda\) cannot be decomposed into direct products of two submatrices. We conclude that the standard forms \(\psi(\lambda)\) with different values of \(\lambda\) correspond to different entanglement families, e.g., although \(\psi(2), \psi(1/2),\) and \(\psi(-1)\) belong to the same entanglement class of \(2 \times 4 \times 4\) system, but correspond to different entanglement families of \(2 \times 2 \times 2 \times 4\) system.

3.3 Examples of a \(2 \times 4 \times 3 \times 2\) state

In order to show the generalities of the method, we generate a random quantum state for \(2 \times 4 \times 3 \times 2\) system (using built-in function \RandomInteger[1,{4,6}] of Mathematica), which is \(\psi = \left(\begin{array}{c}
\Gamma_1 \\
\Gamma_2
\end{array}\right)\), where

\[
\Gamma_1 = \begin{pmatrix}
1 & 1 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1
\end{pmatrix}, \quad \Gamma_2 = \begin{pmatrix}
0 & 0 & 1 & 1 & 1 & 0 \\
1 & 0 & 1 & 1 & 0 & 1 \\
1 & 0 & 1 & 1 & 0 & 1 \\
1 & 0 & 0 & 1 & 0 & 0
\end{pmatrix}.
\]
In the quantum state notation, it is

$$|\psi\rangle = |1111\rangle + |1112\rangle + |1122\rangle + |1131\rangle + |1212\rangle + |1312\rangle + |1332\rangle + |1422\rangle + |1432\rangle + |2122\rangle + |2131\rangle + |2211\rangle + |2221\rangle + |2222\rangle + |2232\rangle + |2311\rangle + |2321\rangle + |2322\rangle + |2332\rangle + |2411\rangle + |2422\rangle.$$ \hspace{1cm} (38)

The rank of $\Gamma_1$ is 4, and the following operations

$$T_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad P_0 = \begin{pmatrix} 0 & 1 & -1 & 0 \\ 1 & 2 & -3 & 2 \\ 0 & -1 & 2 & -1 \\ 1 & 1 & -2 & 1 \end{pmatrix}, \quad Q_0 = \begin{pmatrix} 0 & -1 & 0 & 2 & 0 & -1 \\ 1 & 1 & 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ -1 & -1 & 0 & 1 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$ \hspace{1cm} (39)

will make

$$\Lambda = P_0 \Gamma_1 Q_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad B = P_0 \Gamma_2 Q_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$ \hspace{1cm} (40)

The matrix-pair $\begin{pmatrix} \Lambda \\ B \end{pmatrix}$ is the standard form for the randomly generated state $\psi$. It is invariant under the following operations

$$S_1 = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \alpha \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$S_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 2 \alpha \\ 0 & 0 & 1 & 0 & -\alpha & 0 \\ 0 & 0 & 0 & 1 & 0 & -\alpha \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$ \hspace{1cm} (41)
The operations stated in Eq. (13) are

\[
S = \begin{pmatrix}
\frac{1}{a_{11}} & 0 & 0 & 0 \\
-a_{21} & \frac{1}{a_{22}} & 0 & 0 \\
a_{21} & a_{22} - a_{21} & \frac{1}{a_{11}a_{22}} & a_{31} \\
a_{11}a_{22} - a_{21} & \frac{1}{a_{22}} & a_{32} & \frac{1}{a_{33}} - \frac{a_{22}}{a_{22}a_{33}} \\
0 & 0 & 0 & \frac{1}{a_{22}}
\end{pmatrix},
\]

\[
S' = \begin{pmatrix}
a_{11} & 0 & 0 & 0 & 0 \\
a_{21} & a_{22} & 0 & 0 & 0 \\
a_{31} & a_{32} & a_{33} & a_{34} & 0 \\
0 & 0 & 0 & a_{22} & 0 \\
0 & 0 & 0 & a_{32} & a_{33} & a_{34} \\
0 & 0 & 0 & 0 & a_{22}
\end{pmatrix},
\]

(42)

where \( a_{ij} \in \mathbb{C} \) are arbitrary parameters which keep \( S, S' \) invertible. The transformation matrices \((T_0, P_0, Q_0), (S_1, SS_2, S_3S')\) are thus readily obtained from the construction of the standard form. We refer to [11] for the details of the construction of the standard form of a tripartite state with one qubit.

Suppose another quantum state \( \psi' \) of \( 2 \times 4 \times 3 \times 2 \) has the same standard form as that of \( \psi \) in Eq. (40). We would obtain the corresponding transformation matrices \( T'_0, P'_0, Q'_0 \) while constructing the standard form from \( \psi' \). Thus by Theorem 4, \( \psi \) and \( \psi' \) are SLOCC equivalent if and only if the matrix realignment \( R(Q_0^{-1}S_3S'Q_0) \) could have rank one according to the dimensional factorization 6 = 2 × 3. The example suggests that the scheme works better for higher dimensions, especially for the case of \( L = MN \).

4 Conclusion

In conclusion, we propose a practical scheme for the entanglement classification of a \( 2 \times L \times M \times N \) pure system under SLOCC. The method functions by distinguishing the entanglement classes via their standard forms together with their transformation routes to the standard forms. Not only all the different entanglement classes but also the transformation matrices are obtained with the method. This gives the complete classification of the entangled states of \( 2 \times L \times M \times N \) under SLOCC, which has not yet been addressed in recent literature. The method also reveals that the combination of the standard form and the routes to the standard form may greatly reduce the complexity of the entanglement classifications. As the entanglement generally has been considered to be the key physical resource in quantum information science, our method may also shed new light on the operational classifications of multipartite entanglement with real physical systems.

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