Bag Query Containment and Information Theory

Mahmoud Abo Khamis  Phokion G. Kolaitis  Hung Q. Ngo
relationalAI  UC Santa Cruz  relationalAI
Dan Suciu
University of Washington

Abstract

The query containment problem is a fundamental algorithmic problem in data management. While this problem is well understood under set semantics, it is by far less understood under bag semantics. In particular, it is a long-standing open question whether or not the conjunctive query containment problem under bag semantics is decidable. We unveil tight connections between information theory and the conjunctive query containment under bag semantics. These connections are established using information inequalities, which are considered to be the laws of information theory. Our first main result asserts that deciding the validity of a generalization of information inequalities is many-one equivalent to the restricted case of conjunctive query containment in which the containing query is acyclic; thus, either both these problems are decidable or both are undecidable. Our second main result identifies a new decidable case of the conjunctive query containment problem under bag semantics. Specifically, we give an exponential-time algorithm for conjunctive query containment under bag semantics, provided the containing query is chordal and admits a simple junction tree.

1 Introduction

Since the early days of relational databases, the query containment problem has been recognized as a fundamental algorithmic problem in data management. This problem asks: given two queries $Q_1$ and $Q_2$, is it true that $Q_1(D) \subseteq Q_2(D)$, for every database $D$? Here, $Q_i(D)$ is the result of evaluating the query $Q_i$ on the database $D$. Thus, the query containment problem has several different variants, depending on whether the evaluation uses set semantics or bag semantics, and whether $D$ is a set database or a bag database. Query containment under set semantics on set databases is the most extensively studied and well understood such variant. In particular, Chandra and Merlin [7] showed that, for this variant, the containment problem for conjunctive queries under bag semantics. There are two variants of this problem: in the bag-bag variant, the evaluation uses bag semantics and the input database is a bag, while in the bag-set variant, the evaluation uses bag semantics and the input database is a set. It is known that for conjunctive queries, the bag-bag variant and the bag-set variant are polynomial-time reducible to each other (see, e.g., [16]); in particular, either both variants are decidable or both are undecidable. Which of the two is the case, however, remains an outstanding open question to date.

Chaudhuri and Vardi [8] were the first to raise the importance of studying the query containment problem under bag semantics. In particular, they raised the question of the decidability of the containment problem for conjunctive queries under bag semantics. There are two variants of this problem: in the bag-bag variant, the evaluation uses bag semantics and the input database is a bag, while in the bag-set variant, the evaluation uses bag semantics and the input database is a set. It is known that for conjunctive queries, the bag-bag variant and the bag-set variant are polynomial-time reducible to each other (see, e.g., [16]); in particular, either both variants are decidable or both are undecidable. Which of the two is the case, however, remains an outstanding open question to date.

During the past twenty five years, the research on the query containment problem under bag semantics has produced a number of results about extensions of conjunctive queries and also about restricted classes of conjunctive queries. Specifically, using different reductions from Hilbert’s 10th Problem, it has been shown that the containment problem under bag semantics is undecidable for both the class of unions of conjunctive queries [15] and the class of conjunctive queries with inequalities [16]. It should be noted that, under set semantics, the containment problem for these two classes of queries is decidable; in fact, it is NP-complete for unions of conjunctive queries [26], and it is $\Pi^P_2$-complete for conjunctive queries with inequalities [19, 27].
As regards to restricted classes of conjunctive queries, several decidable cases of the bag-bag variant were identified in [1], including the case where both \( Q_1 \) and \( Q_2 \) are projection-free conjunctive queries, i.e., no variable is existentially quantified. Quite recently, this decidability result was extended to the case where \( Q_1 \) is a projection-free conjunctive query and \( Q_2 \) is an arbitrary conjunctive query \([20]\); the proof is via a reduction to a decidable class of Diophantine inequalities. In a different direction, information-theoretic methods were used in \([21]\) to study the homomorphism domination exponent problem, which generalizes the conjunctive query containment problem under bag semantics on graphs. In particular, it was shown in \([21]\) that the conjunctive query containment problem under bag semantics is decidable when \( Q_1 \) is a series-parallel graph and \( Q_2 \) is a chordal graph. This was the first time that notions and techniques from information theory were applied to the study of the containment problem under bag semantics.

Notions and techniques from information theory have found a number of uses in other areas of database theory. For example, entropy and mutual information have been used to characterize database dependencies \([22, 23]\) and normal forms in relational and XML databases \([2]\). More recently, information inequalities were used with much success to obtain tight bounds on the size of the output of a query on a given database \([3, 13, 14, 17, 18]\), and to devise query plans for worst-case optimal join algorithms \([17, 18]\).

This paper unveils deeper connections between information theory and the query containment problem under bag semantics. These connections are established through the systematic use of information inequalities, which have been called the “laws of information theory” \([25]\) as they express constraints on the entropy and thus “govern the impossibilities in information theory” \([30]\).

An information inequality is an inequality of the form

\[
0 \leq \sum_{X \subseteq V} c_X h(X),
\]

where \( V \) is a set of \( n \) random variables over finite domains, each coefficient \( c_X \) is a real number, i.e. \( c = (c_X)_{X \subseteq V} \) is a \( 2^n \)-dimensional real vector, \( h \) is the entropy function of a joint distribution over \( V \) (\( V \)-distribution henceforth). In particular, \( h(X) \) denotes the marginal entropy of the variables in the set \( X \subseteq V \).

An information inequality may hold for the entropy function of some \( V \)-distribution, but may not hold for all \( V \)-distributions. Following \([4]\), we say that an information inequality is valid if it holds for the entropy function of every \( V \)-distribution. This notion gives rise to the following natural decision problem, which we denote as IIP: given integer coefficients \( c_X \in \mathbb{Z} \) for all \( X \subseteq V \), is the information inequality \([1]\) valid?\footnote{Equivalently, one can allow the input coefficients to be rational numbers.}

In this paper, we will also study a generalization of this problem that involves taking maxima of linear combinations of entropies. A max-information inequality is an expression of the form

\[
0 \leq \max_{\ell \in [k]} \sum_{X \subseteq V} c_{\ell, X} h(X),
\]

where \( V, X, h(X) \) are as before, and for each \( \ell \in [k], c_{\ell} := (c_{\ell, X})_{X \subseteq V} \) is a \( 2^n \)-dimensional real vector. We say that a max information inequality is valid if it holds for the entropy function of every \( V \)-distribution. We write \( \text{Max-IIP} \) to denote the following decision problem: given \( k \) integer vectors \( c_{\ell} \) of dimension \( 2^n \), is the max information inequality \([2]\) valid? Clearly, IIP is the special case of \( \text{Max-IIP} \) in which \( k = 1 \).

Our first main result asserts that \( \text{Max-IIP} \) is many-one equivalent to the restricted case of the conjunctive query containment problem under bag semantics in which \( Q_1 \) is an arbitrary conjunctive query and \( Q_2 \) is an acyclic conjunctive query. In fact, we show that these two problems are reducible to each other via exponential-time many-one reductions. This result establishes a new and tight connection between information theory and database theory, showing that \( \text{Max-IIP} \) and the conjunctive query containment problem under bag semantics with acyclic \( Q_2 \) are equally hard.

To the best of our knowledge, it is not known whether \( \text{Max-IIP} \) is decidable. In fact, even IIP is not known to be decidable; in other words, it is not known if there is an algorithm for telling whether a given information inequality with integer coefficients is valid. Even though the decidability question about IIP and
about Max-IIP does not seem to have been raised explicitly by researchers in information theory, we note that there is a growing body of research aiming to “characterize” all valid information inequalities; moreover, finding such a “characterization” is regarded as a central problem in modern information theory (see, e.g., the survey [1]). It is reasonable to expect that a “good characterization” of valid information inequalities will also give an algorithmic criterion for the validity of information inequalities. Thus, showing that IIP is undecidable would imply that no “good characterization” of valid information inequalities exists.

Our second main result identifies a new decidable case of the conjunctive query containment problem under bag semantics. Specifically, we show that there is an exponential-time algorithm for testing whether \( Q_1 \) is contained in \( Q_2 \) under bag semantics, where \( Q_1 \) is an arbitrary conjunctive query and \( Q_2 \) is a conjunctive query that is chordal and admits a junction tree that is simple. Here, a query is chordal if its Gaifman graph \( G \) is chordal, i.e., \( G \) admits a tree decomposition whose bags induce (maximal) cliques of \( G \); such a tree decomposition is called a junction tree. A tree decomposition is simple if every pair of adjacent bags in the tree decomposition share at most one common variable. The result follows from a new class of decidable Max-IIP problems. Note that this result is incomparable to the aforementioned decidability result about series-parallel and chordal graphs in [21], in two ways. First, the result in [21] applies only to graphs (i.e., databases with a single binary relation symbol), while our result applies to arbitrary relational schemas. Second, our result imposes more restrictions on \( Q_2 \), but no restrictions on \( Q_1 \).

The work reported here reveals that the conjunctive query containment problem under bag semantics is tightly intertwined with the validity problem for information inequalities. Thus, our work sheds new light on both these problems and, in particular, implies that any progress made in one of these problems will translate to similar progress in the other.

2 Definitions

We describe here the two problems whose connection forms the main result of this paper.

2.1 Query Containment Under Bag Semantics

Homomorphisms between relational structures We fix a relational vocabulary, which is a tuple \( \mathcal{R} = (R_1, \ldots, R_m) \), where each symbol \( R_i \) has an associated arity \( a_i \). A relational structure is \( \mathcal{A} = (A, R_1^A, \ldots, R_m^A) \), where \( A \) is a finite set (called domain) and each \( R_i^A \) is a relation of arity \( a_i \) over the domain \( A \). Given two relational structures \( \mathcal{A} \) and \( \mathcal{B} \) with domains \( A \) and \( B \) respectively, a homomorphism from \( \mathcal{B} \) to \( \mathcal{A} \) is a function \( f : B \to A \) such that for all \( i \), we have \( f(R_i^B) \subseteq R_i^A \). We write \( \text{hom}(\mathcal{B}, \mathcal{A}) \) for the set of all homomorphisms from \( \mathcal{B} \) to \( \mathcal{A} \), and denote by \(|\text{hom}(\mathcal{B}, \mathcal{A})|\) its cardinality.

Bag-Set Semantics A conjunctive query \( Q \) with variables \( \text{vars}(Q) \) and atom set \( \text{atoms}(Q) = \{A_1, \ldots, A_k\} \) is a conjunction:

\[
Q(\mathbf{x}) = A_1 \land A_2 \land \cdots \land A_k.
\]

For each \( j \in [k] \), the atom \( A_j \) is of the form \( R_{ij}(\mathbf{x}_j) \), where \( \text{rel}(A_j) \triangleq R_{ij} \) is a relation name, and \( \text{vars}(A_j) \triangleq x_j \) is a function,

\[
\text{vars}(A_j) : [\text{arity}(\text{rel}(A_j))] \to \text{vars}(Q)
\]

associating a variable to each attribute position of \( \text{rel}(A_j) \). We allow repeated variables in an atom. The variables \( \mathbf{x} \) are called head variables, and must occur in the body.

A database instance is a structure \( \mathcal{D} \) with domain \( D \). The answer of a query (3) with head variables \( \mathbf{x} \) is a set of \( \mathbf{x} \)-tuples with multiplicities. Formally, for each \( d \in D^\mathbf{x} \), denote \( Q(\mathcal{D})[d] \triangleq \{f \in \text{hom}(Q, \mathcal{D}) : f(d) = \mathbf{x}\} \).

\(^2\)An \( \mathbf{x} \)-tuple is a tuple that assigns each variable in \( \mathbf{x} \) a value in \( D \).
2.2 Information Inequality Problems

The answer to $Q$ on $\mathcal{D}$ under the bag-set semantics is the mapping $d \mapsto |Q(\mathcal{D})[d]|$. The bag-set semantics corresponds to a `count(*)-groupby` query in SQL.

Given two queries $Q_1, Q_2$ with the same number of head variables, we say that $Q_1$ is contained in $Q_2$ under bag-set semantics, and denote with $Q_1 \preceq Q_2$, if for every $\mathcal{D}$, we have $Q_1(\mathcal{D}) \leq Q_2(\mathcal{D})$, where $\leq$ compares functions point-wise, $\forall d, |Q_1(\mathcal{D})[d]| \leq |Q_2(\mathcal{D})[d]|$.

**Problem 2.1** (Query containment problem under bag-set semantics). Given $Q_1, Q_2$, check whether $Q_1 \preceq Q_2$.

A query $Q$ is called a Boolean query if it has no head variables, $|x| = 0$. It is known that the query containment problem under bag semantics can be reduced to that of Boolean queries under bag semantics. For completeness, we provide the proof in Appendix A, and only mention here that the reduction preserves all special properties discussed later in this paper: acyclicity, chordality, simplicity. For that reason, in this paper we only consider Boolean queries, and denote Problem 2.1 by BagCQC.

**Bag-bag Semantics** In our setting the input database $\mathcal{D}$ is a set, only the query’s output is a bag. This semantics is known under the term bag-set semantics. Query containment has also been studied under the bag-bag semantics, where the database may also have duplicates. This problem is known to be reducible to the containment problem under bag-set semantics [10], by adding a new attribute to each relation, and for that reason we do not consider it further in this paper. One aspect of the bag-bag semantics is that repeated atoms change the meaning of the query, while repeated atoms can be eliminated under bag-set semantics. For example $R(x) \land R(x) \land S(x,y)$ and $R(x) \land S(x,y)$ are different queries under bag-bag semantics, but represent the same query under bag-set semantics. Since we restrict to bag-set semantics we assume no repeated atoms in the query.

**The Domination Problem** We briefly review two related problems that are equivalent to BagCQC. Given two relational structures $\mathcal{A}$ and $\mathcal{B}$, we say that $\mathcal{B}$ dominates $\mathcal{A}$, and write $\mathcal{A} \preceq \mathcal{B}$, if $\forall \mathcal{D}$, $|\hom(\mathcal{A}, \mathcal{D})| \leq |\hom(\mathcal{B}, \mathcal{D})|$. 

**Problem 2.2** (The domination problem, DOM). Given a vocabulary $\mathcal{R}$, and two structures $\mathcal{A}, \mathcal{B}$, check if $\mathcal{B}$ dominates $\mathcal{A}$: $\mathcal{A} \preceq \mathcal{B}$.

DOM and BagCQC are essentially the same problem. Kopparty and Rossman [21] considered the following generalization:

**Problem 2.3** (The exponent-domination problem). Given a rational number $c \geq 0$ and two structures $\mathcal{A}$ and $\mathcal{B}$, check whether $|\hom(\mathcal{A}, \mathcal{D})|^c \leq |\hom(\mathcal{B}, \mathcal{D})|$ for all structures $\mathcal{D}$.

This problem is equivalent to DOM, because it can be reduced to DOM by observing that $|\hom(n \cdot \mathcal{A}, \mathcal{D})| = |\hom(\mathcal{A}, \mathcal{D})|^n$, where $n \cdot \mathcal{A}$ represents $n$ disjoint copies of $\mathcal{A}$ [21] Lemma 2.2. Conversely, DOM is the special case $c = 1$.

### 2.2 Information Inequality Problems

In this paper all logarithms are in base 2. For a random variable $X$ with values that are in a finite domain $\mathcal{D}$, its (binary) entropy is defined by

$$H(X) := - \sum_{x \in \mathcal{D}} \Pr[X = x] \cdot \log \Pr[X = x] \quad (5)$$

Note that in the above definition, $X$ can be a tuple of random variables, in which case $H(X)$ is their joint entropy. The entropy $H(X)$ is a non-negative real number.

Let $V = \{X_1, \ldots, X_n\}$ be a set of $n$ random variables jointly distributed over finite domains. For each $\alpha \subseteq [n]$, the joint distribution induces a marginal distribution for the tuple of variables $X_\alpha = (X_i : i \in \alpha)$. One can also equivalently think of $X_\alpha$ as a vector-valued random variable. Either way, the marginal entropy
on \( X_\alpha \) is defined by \({}\) too, where we replace \( X \) by \( X_\alpha \). Define the function \( h : 2^{[n]} \to \mathbb{R}_+ \) as \( h(\alpha) \defeq H(X_\alpha) \), for all \( \alpha \subseteq [n] \). We call \( h \) an entropic function (associated with the joint distribution on \( V \)) and identify it with a vector \( h \in \mathbb{R}^{2^n}_+ \).

The set of all entropic functions is denoted\(^3\) by \( \Gamma^*_n \subseteq \mathbb{R}^{2^n}_+ \). With some abuse, we blur the distinction between the set \([n]\) and the set of variables \( V = \{X_1, \ldots, X_n\} \), and write \( h(X_\alpha) \) instead of \( h(\alpha) \).

An information inequality, or \( \text{II} \), defined by a vector \( c = (c_X)_{X \subseteq V} \in \mathbb{R}^{2^n} \), is an inequality of the form

\[
0 \leq \sum_{X \subseteq V} c_X h(X) \tag{6}
\]

The information inequality is valid if it holds for all \( h \in \Gamma^*_n \).\(^4\)

**Problem 2.4** (\( \text{II} \)-Problem). Given a set \( V \) and a collection of integers \( c_X \), for \( X \subseteq V \), check whether the information inequality \( \text{(6)} \) is valid.

A max-information inequality, or \( \text{Max-II} \), is defined by \( k \) vectors \( c_\ell := (c_{\ell,X})_{X \subseteq V} \in \mathbb{R}^{2^n} \), \( \ell \in [k] \), and is written as:

\[
0 \leq \max_{\ell \in [k]} \sum_{X \subseteq V} c_{\ell,X} h(X) \tag{7}
\]

The Max-II is valid if it holds for all entropic functions \( h \in \Gamma^*_n \).

**Problem 2.5** (Max-II Problem). Given a set \( V \) and integers \( c_{\ell,X} \), for \( \ell \in [k] \) and \( X \subseteq V \), check whether the Max-II \( \text{(7)} \) is valid.

We denote the \( \text{II} \)- and Max-II problems by \( \text{IIP} \) and Max-IIP respectively. Both are co-recursively enumerable (Appendix \( \text{B} \)) and it is open if any of them is decidable.

### 3 Main Results

#### 3.1 Connecting BagCQC to Information Theory

We state our first main result, and defer its proofs to Sec. \( \text{I} \) and \( \text{E} \). Recall that a many-one reduction of a decision problem \( A \) to another decision problem \( B \), denoted by \( A \leq_m B \), is a computable function \( f \) such that for every input \( X \), the yes/no answer to problem \( A \) on \( X \) is the same as the yes/no answer to the problem \( B \) on \( f(X) \). This is a special case of a Turing reduction, \( A \leq_T B \), which means an algorithm that solves \( B \) and can access an oracle that solves \( B \). Two problems are many-one equivalent, denoted by \( A \equiv_m B \), if \( A \leq_m B \) and \( B \leq_m A \).

Our main result is that the Max-IIP is many-one equivalent to the query containment problem under bag semantics, when the containing query is restricted to be acyclic. We briefly review acyclic queries here (we only consider \( \alpha \)-acyclicity in this paper \( \text{I} \)).

**Definition 3.1.** A tree decomposition of a query \( Q \) is a pair \((T, \chi)\) where \( T \) is an undirected forest\(^5\) and \( \chi : \text{nodes}(T) \to 2^{\text{vars}(Q)} \) satisfies (a) the running intersection property: \( \forall x \in \text{vars}(Q), \{t \in \text{nodes}(T) \mid x \in \chi(t)\} \) is connected in \( T \), and (b) the coverage property: for every \( A \in \text{atoms}(Q) \), there exists \( t \in \text{nodes}(T) \) s.t. \( \text{vars}(A) \subseteq \chi(t) \). The sets \( \chi(t) \) are called the bags\(^5\) of the tree decomposition. A query \( Q \) is acyclic if there exists a tree decomposition \((T, \chi)\) such that, for all \( t \in \text{nodes}(T) \), \( \chi(t) = \text{vars}(A) \) for some \( A \in \text{atoms}(Q) \).

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\(^3\) Most texts drop the component \( h(\emptyset) \), which is always 0, and define \( \Gamma^*_n \subseteq \mathbb{R}^{2^n}_+ \). We prefer to keep the \( \emptyset \)-coordinate to simplify notations.

\(^4\) We allow \( Q \) to be disconnected, in which case \( T \) can be a forest, but we continue to call it a tree decomposition.

\(^5\) Not to be confused with the bag semantics.
Theorem 3.2. Let BagCQC-A denote the BagCQC problem \( Q_1 \leq Q_2 \), where \( Q_2 \) is restricted to acyclic queries. Then \( \text{Max-IIP} \equiv_m \text{BagCQC-A} \).

The proof of the theorem consists of three steps. First, we describe in Sec. [4,1] a Max-IIP inequality that is sufficient for containment, which is quite similar to, and inspired by an inequality by Kopparty and Rossman [21]. Second, we prove in Sec. [4,2] that, when \( Q_2 \) is acyclic, then this inequality is also necessary, thus solving the conjecture in [21, Sec.3]; our proof is based on Chan-Yeung’s group-characterizable entropic functions [5,6]. In particular, BagCQC-A ≤_m Max-IIP. We do not know if this can be strengthened to BagCQC and/or IIP respectively. Finally, we give the many-one reduction Max-IIP ≤_m BagCQC-A in Sec. [6].

3.2 Novel Decidable Class of BagCQC

Our next two results consist of a novel decidable class of query containment under bag semantics, and, correspondingly, a novel decidable class of max-information inequalities. We state here the results, and defer their proofs to Section 6.2.

We show that containment is decidable when \( Q_2 \) is chordal and admits a simple junction tree (decomposition); to formally state the result, we define chordality, simplicity, and junction tree next.

A query \( Q \) is said to be chordal if its Gaifman graph \( G \) is chordal, i.e., there is a tree decomposition of \( G \) in which every bag induces a clique of \( G \). A tree decomposition of \( G \) (and thus of \( Q \)) where all bags induce maximal cliques of \( G \) is called a junction tree in the graphical models literature (see Def. 2.1 in [25]).

Fix a tree decomposition of a query \( Q \), and let \( t \in \text{nodes}(T) \). A tree decomposition is called simple if \( \forall (t_1, t_2) \in \text{edges}(T), |\chi(t_1) \cap \chi(t_2)| \leq 1 \), and is called totally disconnected if \( \forall (t_1, t_2) \in \text{edges}(T), \chi(t_1) \cap \chi(t_2) = \emptyset \). As an example of a totally disconnected tree decomposition, consider the query \( Q() \leftarrow R(a), S(b) \) and a tree decomposition of \( Q \) with only two nodes \( t_1 \) and \( t_2 \) where \( \chi(t_1) = \{a\} \) and \( \chi(t_2) = \{b\} \).

Note that every acyclic query is chordal, but not necessarily simple; for example, the query \( Q() \leftarrow R(a, b, c), S(b, c, e) \) is a non-simple acyclic query. Conversely a chordal query is not necessarily acyclic; for example, any \( k \)-clique query with \( k \geq 3 \) is chordal.

Theorem 3.3. Checking \( Q_1 \leq Q_2 \) is decidable in exponential time when \( Q_2 \) is chordal and admits a simple junction tree.

Next, we complement Theorem 3.3 by showing that, if \( Q_1 \not\leq Q_2 \) then there exists a “witness” with a simple structure. This result is similar in spirit to other results where a decision problem can be restricted to special databases: for example, query containment under set semantics holds iff it holds on the canonical database of \( Q_1 \) [7], and implication between functional dependencies holds iff it holds on all relations with two tuples.

Let \( Q_1 \) be a query and \( V = \text{vars}(Q_1) \). A relation \( P \subseteq D^V \) is called a \( V \)-relation. A \( V \)-relation \( P \) and \( Q_1 \) induce a database instance \( \Pi_{Q_1}(P) \overset{\text{def}}{=} (D, R_1^P, \ldots, R_m^P) \) where,

\[
\forall \ell \in [m] : \quad R_\ell^P \overset{\text{def}}{=} \bigcup_{A \in \text{atoms}(Q_1) \cap \text{rel}(A) = R_\ell} \Pi_{\text{vars}(A)}(P)
\]  

In other words, we project \( P \) on each atom, and define \( R_\ell^P \) as the union of projections on atoms with relation name \( R_\ell \).

The notation \( \Pi_{\text{vars}(A)}(P) \) requires some explanation, because the atom \( A \) may have repeated variables, thus \( \text{vars}(A) \) is a function (described in [4]). Given a set of integer indices \( Y \) and a function \( \varphi : Y \to V \), the generalized projection is \( \Pi_{\varphi}(P) \overset{\text{def}}{=} \{ f \circ \varphi \mid f \in D^V \} \). A tuple \( f \in D^V \) is a function \( V \to D \), hence \( f \circ \varphi \) just denotes function composition. For example, if \( Q_1 = R(x, x, y) \) and \( P = \{(a, b)\} \), then \( R^D = \Pi_{(x, x, y)}(P) = \{(a, a, b)\} \). Obviously \( P \subseteq \text{hom}(Q_1, \Pi_{Q_1}(P)) \), which means \( |P| \leq |\text{hom}(Q_1, \Pi_{Q_1}(P))| \), and this implies:

Fact 3.4 (Witness). If there exists a \( \text{vars}(Q_1) \)-relation \( P \) such that \( |P| > |\text{hom}(Q_2, \Pi_{Q_1}(P))| \), then \( Q_1 \not\leq Q_2 \), in which case \( P \) is said to be a witness (for the fact that \( Q_1 \not\leq Q_2 \)).

\( ^6 \)Equivalently, \( \text{edges}(T) = \emptyset \), because any edge s.t. \( \chi(t_1) \cap \chi(t_2) = \emptyset \) can be removed.
We next define two special types of relations (and witnesses) that have interesting analogues in information theory and thus arise naturally when doing reductions between the database world and the information theory world. Let $W$ be a set of integer indices. Fix $\psi : W \rightarrow 2^V$ and a tuple $f \in D^V$. For any index $y \in W$, we view $f(\psi(y))$ as an atomic value in the domain $D^\psi(y)$. Define the $W$-tuple $\psi \cdot f = (f(\psi(y)))_{y \in W}$: its components may belong to different domains.

**Definition 3.5** (Product and normal relations). A $V$-relation $P$ is a *product relation* if $P = \prod_{x \in V} S_x$, where each $S_x$ is a unary relation. A $W$-relation is called a *normal relation* if it is of the form $\{ \psi \cdot f \ | \ f \in P \}$ where $P$ is some product $V$-relation and $\psi : W \rightarrow 2^V$ is some function.

One can verify that every product relation is a normal relation. For a simple illustration, consider the case when $V = \{X_1, X_2\}$. A product relation on $V$ is $\{(u, v) \mid u, v \in [N]\} = [N] \times [N]$. A normal relation with four attributes is $\{(uv, u, v, v) \mid u, v \in [N]\}$, where $uv$ denotes the concatenation of $u$ and $v$. This normal relation corresponds to the map $\psi : [4] \rightarrow 2^V$ where $\psi(1) = \{X_1, X_2\}$, $\psi(2) = \{X_1\}$, and $\psi(3) = \psi(4) = \{X_2\}$. In a product relation all attributes are independent, while a normal relation may have dependencies: in our example the first attribute $uv$ is a key, and the last two attributes are equal.

**Theorem 3.6.** Let $Q_2$ be chordal,

(i) If $Q_2$ admits a totally disconnected junction tree, then $Q_1 \not\leq Q_2$ if and only if there is a product witness.

(ii) If $Q_2$ admits a simple junction tree, then $Q_1 \not\leq Q_2$ if and only if there exists a normal witness.

We prove both theorems in Section 3.3, using the novel results on information-theoretic inequalities described next, in Section 3.3.

**Example 3.7.** We illustrate with the following queries:

\[
Q_1 = A(x_1, x_2) \land B(x_1, x_2) \land C(x_1, x_2) \land A(x'_1, x'_2) \land B(x'_1, x'_2) \land C(x'_1, x'_2)
\]

\[
Q_2 = A(y_1, y_2) \land B(y_1, y_3) \land C(y_4, y_2)
\]

$Q_2$ is acyclic with a simple junction tree: $\{y_1, y_3\} - \{y_1, y_2\} - \{y_2, y_4\}$. We prove that $Q_1 \not\leq Q_2$ has a normal witness:

\[
P = \{(u, u, v, v) \mid u \in [n], v \in [n]\} \subseteq D^{(x_1, x_2, x'_1, x'_2)}
\]

$P$ induces the database $\Pi_{Q_1}(P) = ([n], A^D, B^D, C^D)$, where $A^D = B^D = C^D = \{(u, u) \mid u \in [n]\}$, and $|P| = n^2 > |\hom(Q_2, \Pi_{Q_1}(P))| = n$ when $n > 1$, proving $Q_1 \not\leq Q_2$.

On the other hand, there is no product relation $P$ that can witness $Q_1 \not\leq Q_2$. Indeed, if $P = S_1 \times S_2 \times S_3 \times S_4$ where $S_1, \ldots, S_4$ are unary relations, then the associated database $\Pi_{Q_1}(P)$ has relations $A^D = B^D = C^D = (S_1 \times S_2) \cup (S_3 \times S_4)$, and therefore $|\hom(Q_2, \Pi_{Q_1}(P))| \geq \max(|S_1 \times S_2|^2, |S_3 \times S_4|^2) \geq |S_1 \times S_2 \times S_3 \times S_4| = |P|$.

### 3.3 Novel Class of Shannon-Inequalities

Our decidability results are based on a new result on information-theoretic inequalities, proving that certain max-linear inequalities are essentially Shannon inequalities. To present it, we need to review some known facts about entropic functions. We refer to Appendix B and to 29 for additional information. Recall that the set of entropic functions over $n$ variables is denoted $\Gamma^*_n \subseteq \mathbb{R}^2$, and that we blur the distinction between a set $V$ of $n$ variables and $[n]$.

We begin by discussing closure properties of entropic functions and then introduce certain special classes of entropic functions. For the benefit of the readers familiar with database theory, we give in Table 1 the mapping between some of the database concepts used in this paper and their information-theoretic counterparts. For our discussion, it is useful to define the notion of the *entropy of a relation*. Given a
| Database Theory | Information Theory |
|-----------------|-------------------|
| $P \subseteq D^V$ | $h \in \Gamma^*_n$ |
| A relation $P$ over a set of $n$ variables $V$, each of which has domain $D$ | An entropic function $h : 2^V \to \mathbb{R}_+$ over a set of $n$ variables $V$. $h$ is defined by a uniform probability distribution $p$ over $P$. |
| $P = S_1 \times \cdots \times S_n \subseteq D^V$ | $h(X) = \sum_{i \in X} h(i)$, for all $X \subseteq V$ |
| A product relation $P$ (Definition 3.5) | A modular function $h \in M_n$ |
| $P = P_1 \otimes P_2$, where $P_1 \subseteq D_1^V$, $P_2 \subseteq D_2^V$, $P \subseteq (D_1 \times D_2)^V$ | $h = h_1 + h_2$, where $h,h_1,h_2 \in \Gamma^*_n$ |
| A domain product $P$ of two relations $P_1,P_2$, all of which are over the same variable set $V$ (Definition 6.9) | A sum $h$ of two entropic functions $h_1,h_2$, all of which are over $n$ variables |
| $P_W \triangleq \{ f_1, f_2 \} \subseteq D^V$, for some $W \subseteq V$, where $f_1 \triangleq (1,1,\ldots,1)$, $f_2 \triangleq (2,\ldots,2,1,\ldots,1)$, $V-W \atop \uparrow \downarrow \uparrow \downarrow W$ | $h_W(X) \triangleq \begin{cases} 0 & \text{if } X \subseteq W \\ 1 & \text{otherwise} \end{cases}$ |
| Given $W \subseteq V$, the relation $P_W$ has two tuples $f_1,f_2$ differing only in positions $V-W$. (See Section 3.3) | Given $W \subseteq V$, a step function $h_W$. |
| $P = P_{W_1} \otimes P_{W_2} \otimes \cdots \otimes P_{W_m}$ | $h = \sum_{W \subseteq V} c_W h_W$, where $c_W \geq 0$ |
| A normal relation $P$ over variable set $V$ is a domain product of $m$ (not necessarily distinct) relations $P_{W_i}$, for $W_i \subseteq V$ (Another way to phrase Definition 3.5) | A normal entropy $h \in \mathcal{N}_n$ is a non-negative weighted sum of step functions $h_W$ |
| The set of normal relations | The set of normal functions $\mathcal{N}_n = \text{the cone closure of step functions}$ |
| $P_W$, when $|V-W| = 1$, becomes a product relation | $h_W$, when $|V-W| = 1$, becomes a modular function |
| Product relations are a proper subclass of normal relations | Modular functions are a proper subclass of normal functions $M_n \subseteq \mathcal{N}_n$ |
| A group-characterizable relation $[5]$ | An entropic function $h \in \Gamma^*_n$ |
| $P \triangleq \{(aG_1,\ldots,aG_n) \mid a \in G\}$, where $G$ is a group and $G_1,\ldots,G_n$ are subgroups | $\Gamma^*_n$ |
| The set of group-characterizable relations | $\Gamma^*_n - \Gamma^*_n$ |
| Polymatroids that are not entropic have no analog in databases |

Table 1: Translation between the database world and the information theory world.
Every step function \( h \) is entropic, if \( h \) is a non-negative linear combination of step functions, i.e.,
\[ \sum_{i=1}^{n} c_i h(X_i) \]
where \( c_i \geq 0 \). It is not hard to verify that \( h \) is indeed entropic.

As mentioned above, if \( c > 0 \) is a positive real number and \( h \) is a step function, then the function \( h' = ch \) is also entropic; the proof of this fact is given in Appendix B. A normal entropic function is a non-negative linear combination of step functions, i.e.,
\[ \sum_{W \subseteq V} c_W h_W \]
for \( c_W \geq 0 \). We write \( \mathcal{N}_n \) to denote the set of all normal functions. Since, as mentioned earlier, the sum of two entropic functions is entropic, it follows that every normal function is entropic; thus, we have that \( \mathcal{N}_n \subseteq \Gamma_n^* \). In Appendix B we show that the normal functions are precisely the entropic functions with a non-negative I-measure (defined by Yeung [29]). The term “normal” was introduced in [17]. One can check that the entropy of every normal relation (Def. 3.5) is a normal function.

**Example 3.8.** The parity function is the entropy of the following relation with 3 variables: \( P = \{(X, Y, Z) \mid X, Y, Z \in \{0, 1\}, X \oplus Y \oplus Z = 0\} \) where \( \oplus \) is the exclusive OR. More precisely, the entropy is \( h(X) = h(Y) = h(Z) = 1, h(XY) = h(XZ) = h(YZ) = h(XYZ) = 2 \). We show in Sec. 6.1 that \( h \) is not normal.

A function \( h : 2^V \to \mathbb{R}_+ \) is called modular if it satisfies \( h(X \cup Y) + h(X \cap Y) = h(X) + h(Y) \) for all \( X, Y \subseteq V \), and \( h(\emptyset) = 0 \). It is easy to show that \( h \) is modular iff \( h(X_\alpha) = \sum_{i \in \alpha} h(X_i) \) for all \( \alpha \subseteq V \). It is immediate to check that the entropy of any product relation (Def. 3.5) is modular. We write \( \mathcal{M}_n \) to denote the set of all modular functions. Every modular function is normal, hence it is also entropic. To see this, given a modular function \( h \), for each \( i \leq n \), define \( W_i = V \setminus \{X_i\} \) and let \( h_{W_i} \) be the associated step function at \( W_i \). It is now easy to verify that \( h = \sum_{i=1}^{n} h(X_i) \cdot h_{W_i} \), thus \( h \) is a normal function. In summary, we have \( \mathcal{M}_n \subseteq \mathcal{N}_n \subseteq \Gamma_n^* \).

All entropic functions satisfy Shannon’s basic inequalities, called monotonicity and submodularity,
\[ h(X) \leq h(X \cup Y) \quad \quad h(X \cup Y) + h(X \cap Y) \leq h(X) + h(Y) \quad (9) \]
for all \( X, Y \subseteq V \). (Since \( h(\emptyset) = 0 \), monotonicity implies non-negativity too.) A function \( h : 2^V \to \mathbb{R}_+ \), \( h(\emptyset) = 0 \), that satisfies Eq. (9) is called a polymatroid, and the set of all polymatroids is denoted by \( \Gamma_n \).

Thus, \( \Gamma_n^* \subseteq \Gamma_n \). Zhang and Yeung [31] showed that \( \Gamma_n^* \) is properly contained in \( \Gamma_n \), for every \( n \geq 4 \). Any inequality derived by taking a non-negative linear combination of inequalities (9) is called a Shannon inequality. In a follow-up paper [32], Zhang and Yeung gave the first example of a 4-variable valid information inequality which is non-Shannon.

In summary, we have considered the chain of the following four sets: \( \mathcal{M}_n \subseteq \mathcal{N}_n \subseteq \Gamma_n^* \subseteq \Gamma_n \). Except for \( \Gamma_n^* \), each of these sets is a polyhedral cone. Using basic linear programming, one can show that it is decidable whether a max-linear inequality holds on a polyhedral set. In contrast, (even) the topological closure of \( \Gamma_n^* \) is not polyhedral [24]; in fact, it is conjectured to not even be semi-algebraic [12], and it is an open question whether linear inequalities or max-linear inequalities on \( \Gamma_n^* \) are decidable.
For a given vector \((c_X)_{X \subseteq V} \subseteq \mathbb{R}^{|V|}\) where \(c_\emptyset = 0\), we associate a linear expression \(E\) which is the linear function \(E(h) \overset{\text{def}}{=} \sum_{X \subseteq V} c_X h(X)\). As stated earlier, a linear inequality \(E(h) \geq 0\) that is valid for all \(h \in \Gamma_n^*\) is called an information inequality; furthermore, a max information inequality is one of the form \(\max \ell E_\ell(h) \geq 0\), where \(\forall \ell, E_\ell\) is a linear expression.

In this paper, for any variable sets \(X, Y \subseteq V\), we write \(h(XY)\) as a shorthand for \(h(X \cup Y)\), and define the conditional entropy to be \(h(Y|X) \overset{\text{def}}{=} h(XY) - h(X)\). Despite its name, the mapping \(Y \mapsto h(Y|X)\) is not always an entropic function (Appendix B), but it is always a limit of entropic functions. The submodularity law \([9]\) can be written using conditional entropies as

\[
h(XY|X) \leq h(Y|X \cap Y)
\]  

\[
\text{Definition 3.9 (Simple and unconditioned linear expressions). We call the term } h(Y|X) \text{ simple if } |X| \leq 1. \text{ A simple term } h(Y|X) \text{ is unconditioned if } X = \emptyset. \text{ A conditional linear expression is a linear expression } E \text{ of the form } E(h) = \sum_{X \subseteq Y \subseteq V} d_{Y \setminus X} \cdot h(Y|X), \text{ where } d_{Y \setminus X} \text{ are non-negative coefficients. A conditional linear expression is said to be simple (respectively, unconditioned) if } d_{Y \setminus X} > 0 \text{ implies } h(Y|X) \text{ is simple (respectively, unconditioned).}
\]

\[
\text{Definition 3.10 (Decidable classes of inequalities). A class } \mathcal{I} \text{ of inequalities over variables } h : 2^{[n]} \to \mathbb{R}_+ \text{ is decidable if the problem of determining whether a given inequality } I \in \mathcal{I} \text{ holds for all } h \in \Gamma_n^* \text{ is decidable.}
\]

\[
\text{Definition 3.11 (Essentially Shannon inequalities). Let } \mathcal{I} \text{ be a class of max-linear inequalities. We say that } \mathcal{I} \text{ is essentially Shannon if, for every inequality } I \text{ in } \mathcal{I}, \text{ } I \text{ holds for every } h \in \Gamma_n^* \text{ if and only if } I \text{ holds for every } h \in \Gamma_n. \text{ Any essentially Shannon class is decidable, because } \Gamma_n \text{ is polyhedral.}
\]

\[
\text{Theorem 3.12. Consider a max-linear inequality of the following form, where } q > 0, \text{ and } E_\ell \text{ are conditional linear expressions:}
\]

\[
q \cdot h(V) \leq \max_{\ell \in [k]} E_\ell(h)
\]

\[
(i) \text{ Suppose that } E_\ell \text{ is unconditioned, } \forall \ell \in [k]; \text{ then inequality (11) holds } \forall h \in \mathcal{M}_n \text{ if and only if it holds } \forall h \in \Gamma_n.
\]

\[
(ii) \text{ Suppose that } E_\ell \text{ is simple, } \forall \ell \in [k]; \text{ then, inequality (11) holds } \forall h \in \mathcal{N}_n \text{ if and only if it holds } \forall h \in \Gamma_n.
\]

In particular, the class of inequalities (11), where each \(E_\ell\) is simple, is essentially Shannon and decidable. (Recall Definitions 3.9, 3.10 and 3.11)

\[
\text{The proof of the theorem follows from a technical lemma, which is of independent interest:}
\]

\[
\text{Lemma 3.13. Let } h : 2^{[n]} \to \mathbb{R}_+ \text{ be any polymatroid. Then there exists a normal polymatroid } h' \in \mathcal{N}_n \text{ with the following properties:}
\]

\[
1. h'(X) \leq h(X), \text{ forall } X \subseteq [n],
\]

\[
2. h'([n]) = h([n]),
\]

\[
3. h'([i]) = h([i]), \text{ forall } i \in [n].
\]

In addition, there exists a modular function \(h'' \in \mathcal{M}_n\) that satisfies conditions \([1]\) and \([2]\).

This lemma says that every polymatroid \(h\) can be decreased to become a normal polymatroid \(h'\), while preserving the values at \([n]\) (all variables) and at all singletons \([i]\). If we drop the last condition, then the existence of a modular function \(h''\) follows from the modularization lemma \([18]\), which is based on Lovasz’s monotonization of submodular functions:

\[
h''(X) \overset{\text{def}}{=} \sum_{i \in X} h([i]) [i - 1]
\]

The proof that one can also satisfy condition \([3]\), by relaxing from a modular function to a normal one, is non-trivial and given in Sec. 6.4.
Proof of Theorem 3.12. We prove the second item. Let $E(h) \overset{\text{def}}{=} \max_{\ell} E_{\ell}(h) - q \cdot h(V)$, where each $E_{\ell}$ has the form $\sum_i h(Y_i|X_i)$ with $|X_i| \leq 1$. Let $h \in \Gamma_n$, and let $h' \in N_3$ be the normal polymatroid in Lemma 3.13. For every $\ell$, we have $E_{\ell}(h') = \sum_i h'(X'_i|Y_i) - \sum_i h'(X'_i) \leq \sum_i h(X_i) - \sum_i h(X'_i) = E_{\ell}(h)$, because $|X'_i| \leq 1$ and therefore $h'(X'_i) = h(X_i)$. Since $E(h') \geq 0$, we obtain $q \cdot h(V) = q \cdot h'(V) \leq \max_{\ell} E_{\ell}(h') \leq \max_{\ell} E_{\ell}(h)$ completing the proof. The first item of the theorem is proven similarly, and omitted. □

Example 3.14. We illustrate Theorem 3.12 here with an inequality needed later in Ex. 3.8. Consider $h(X_1X_2X_3) \leq \max(E_1, E_2, E_3)$, where:

\[
\begin{align*}
E_1 &= h(X_1X_2) + h(X_2|X_1) \\
E_2 &= h(X_2X_3) + h(X_3|X_2) \\
E_3 &= h(X_1X_3) + h(X_1|X_3)
\end{align*}
\]

Notice that all three expressions are simple, hence part (ii) of the theorem applies. In particular according to Theorem 3.12 in order to check whether the above inequality holds for all entropic $h \in \Gamma_3^* \supseteq N_3$, it is sufficient to check that the inequality holds for all polymatroids $h \in \Gamma_3$. (This latter check is much easier than the former because $\Gamma_3$ is polyhedral while $\Gamma_3^*$ is not. The non-trivial direction of the theorem is proving that if the inequality fails on some $h \in \Gamma_3$, then it must fail on some $h' \in N_3 \subseteq \Gamma_3^*$.) In this example, it turns out the above inequality does indeed hold for all $h \in \Gamma_3$. In particular, using Shannon’s submodularity law (10), we infer $E_1 = h(X_1X_2) + h(X_2|X_1) \geq h(X_1X_2) + h(X_2|X_1X_3)$ and, similarly for $E_2$ and $E_3$; therefore,

\[
\max(E_1, E_2, E_3) \geq \frac{1}{3}(E_1 + E_2 + E_3) \geq \frac{1}{3}[h(X_1X_2) + h(X_2|X_1X_3) + h(X_2X_3) + h(X_3|X_1X_2) + h(X_1X_3) + h(X_1|X_2X_3)] = h(X_1X_2X_3).
\]

4 Reducing BagCQC-A to Max-IIP

In this section, we prove that BagCQC-A $\leq_m$ Max-IIP, showing half of the equivalence claimed in Theorem 3.12.

We start by associating to each query containment problem a max-information inequality. We then prove, two results: the inequality is always a sufficient condition for containment, and it is also necessary when the containing query is acyclic. From now on, we will use only upper case to denote variables, both random variables and query variables.

Before we begin, we need to introduce some notations. Fix a relation $P \subseteq D^V$ and a probability distribution with mass function $p : P \rightarrow [0, 1]$. If $X \subseteq V$ is a set of variables, and $\varphi : Y \rightarrow V$ is a function, then recall that $\Pi_X(P)$ and $\Pi_\varphi(P)$ denote the standard, and the generalized projections respectively. We write $\Pi_X(p)$ for the standard $X$-marginal of $p$, and write $\Pi_\varphi(p)$ for the $\varphi$-pullback\(^\dagger\). The latter is a probability distribution on $\Pi_\varphi(P)$ defined as follows. Start from the standard marginal $\Pi_{\varphi(Y)}(p)$ on $\Pi_{\varphi(Y)}(P)$, then apply the isomorphism $\Pi_{\varphi(Y)} \rightarrow \Pi_{\varphi(V)}$ defined as $\Pi_{\varphi(f)} \mapsto \Pi_{\varphi(V)}(f)$, $\forall f \in P$. Finally, if $E = \sum_i c_i h(Y_i)$ is a linear expression of entropic terms, where each $Y_i \subseteq Y$, then we denote by $E \circ \varphi \overset{\text{def}}{=} \sum_i c_i h(\varphi(Y_i))$ the result of applying the substitution $\varphi$ to each term in $E$.

Example 4.1. Let $V = \{X_1, X_2, X_3\}$, $P \subseteq D^V$, $\varphi(Y_1) = X_1, \varphi(Y_2) = X_2, \varphi(Y_3) = X_2$. The generalized projection is $\Pi_{\varphi}(P) = \{(a, b, b) \mid (a, b, c) \in P\} \subseteq D^{Y_1Y_2Y_3}$. Its tuples are in 1-1 correspondence with the standard projection $\Pi_{\varphi(Y)}(P) = \Pi_{X_1X_2}(P) = \{(a, b) \mid (a, b, c) \in P\}$. If $p$ is a distribution on $P$, then the $\varphi$-pullback is $\Pi_{\varphi}(p)(Y_1Y_2Y_3 = abb) \overset{\text{def}}{=} p(X_1X_2 = ab) = \sum_c p(X_1X_2X_3 = abc)$. Notice that we do not need to define the pullback for $(a, b, c)$ where $b \neq c$, because $(a, b, c) \notin \Pi_\varphi(P)$. Consider now the linear expression $E = 3h(Y_1) + 4h(Y_2Y_3) - 6h(Y_3)$. Then $E \circ \varphi = 3h(X_1) + 4h(X_2) - 6h(X_2) = 3h(X_1) - 2h(X_2)$.

\(^\dagger\)This is a generalization of the pullback in [21] Sec.4.
We will introduce now a fundamental expression, $E_T$, that connects query containment to information inequalities; we discuss its history in Sec. 7. Fix a tree decomposition $(T, \chi)$ of some query $Q$, and recall that $T$ may be a forest. Choose a root node in each connected component, thus giving an orientation of $T$’s edges, where each node $t$ has a unique parent$(t)$. We associate to $T$ the following linear expression of entropic terms:

$$E_{(T, \chi)}(h) \overset{\text{def}}{=} \sum_{t \in \text{nodes}(T)} h(\chi(t) \mid \chi(\text{parent}(t)))$$

(12)

where $\chi(\text{parent}(t)) = \emptyset$ when $t$ is a root node. We abbreviate $E_{(T, \chi)}$ with $E_T$ when $\chi$ is clear from the context. Expression (12) is independent of the choice of the root nodes, because one can check that $E_T = \sum_{t \in \text{nodes}(T)} h(\chi(t)) - \sum_{(t, t_2) \in \text{edges}(T)} h(\chi(t_1) \cap \chi(t_2))$.

4.1 A Sufficient Condition

Henceforth, let $\text{TD}(Q)$ denote the set of all tree decompositions of a given query $Q$.

Theorem 4.2. Let $Q_1, Q_2$ be two conjunctive queries, $n = |\text{vars}(Q_1)|$. If the following Max-II inequality holds $\forall h \in \Gamma_n$:

$$h(\text{vars}(Q_1)) \leq \max_{(T, \chi) \in \text{TD}(Q_2)} \max_{\varphi \in \text{hom}(Q_2, Q_1)} (E_T \circ \varphi)(h)$$

(13)

then $Q_1 \preceq Q_2$.

The theorem is inspired by, and is similar to Theorem 3.1 by Kopparty and Rossman [21], with three differences. First, the result in [21] applies only to graphs (i.e., databases with a single binary relation symbol), while our result applies to arbitrary relational schemas. Second, we do not restrict $Q_2$ to be chordal. Finally, [21] restrict $h$ to entropies satisfying the independence constraints defined by $Q_1$; while this restriction is not needed to prove Theorem 4.2, it was needed in [21] to prove necessity in a special case (Theorem 3.3 in [21]). We will prove necessity in Theorem 4.7 in the next section without needing this restriction. Our proof of Theorem 4.2 in this section is an extension of the proof in [21]. The proofs of both Theorem 4.2 and 4.7 use the following notation. Give a node $t \in \text{nodes}(T)$ of tree decomposition of $Q$, we denote by $Q_t$ the “subquery at $t$”, consisting of all atoms $A \in \text{atoms}(Q)$ s.t. vars$(A) \subseteq \chi(t)$. We can assume w.l.o.g. (Appendix A) that vars$(Q_t) = \chi(t)$. Before we present our proof of Theorem 4.2 we give an example, also from [21], that illustrates the main idea of the proof.

Example 4.3. This example is attributed to Eric Vee in [21]:

$$Q_1 = R(X_1, X_2) \land R(X_2, X_3) \land R(X_3, X_1),$$

$$Q_2 = R(Y_1, Y_2) \land R(Y_1, Y_3).$$

We show that $Q_1 \preceq Q_2$. Query $Q_2$ is acyclic, and its tree decomposition $T$ is $\{Y_1, Y_2\} - \{Y_1, Y_3\}$, therefore:

$$E_T = h(Y_1Y_2) + h(Y_3|Y_1) = h(Y_1Y_2) + h(Y_1Y_3) - h(Y_1)$$

There are three homomorphisms $\varphi : Q_2 \rightarrow Q_1$, hence inequality (13) becomes:

$$h(X_1X_2X_3) \leq \max(E_1, E_2, E_3)$$

(14)

where $E_1, E_2, E_3$ are the linear expressions in Example 3.1, where we have shown that the inequality holds for all entropic $h$. Theorem 4.2 implies $Q_1 \preceq Q_2$. Here we prove the theorem on this particular example. Consider any database $\mathcal{D}$, let $P_1 = \text{hom}(Q_1, \mathcal{D})$, $p_1$ the uniform probability space on $P_1$, and $h_1$ its entropy. Since $h_1$ satisfies inequality (13), one of the three terms on the right is larger than the left, assume w.l.o.g. that this term corresponds to the homomorphism $\varphi(Y_1) = X_1$, $\varphi(Y_2) = \varphi(X_3) = X_2$. 

12
Thus, $h_1(X_1X_2X_3) \leq h_1(X_1X_2) + h_1(X_2 \mid X_1)$. Let $P_2 = \text{hom}(Q_2 \cup X)$. This is a relation with attributes $Y_1, Y_2, Y_3$. We define a probability distribution $p_2$ on $P_2$ as follows: the marginal $p_2(Y_1, Y_2)$ is the same as $p_1(X_1, X_2)$, and the conditional $p_2(Y_3 \mid Y_1)$ is the same as $p_1(X_2 \mid X_1)$. In particular, its entropy $h_2$ satisfies $\log |P_2| \geq h_2(Y_1Y_2Y_3) = h_2(Y_1Y_2) + h_2(Y_3 \mid Y_1) = h_1(X_1X_2) + h_1(X_2 \mid X_1) \geq h_1(X_1, X_2, X_3) = \log |P_1|$ proving $Q_1 \subseteq Q_2$.

Finally, we give our general proof of Theorem 4.2. To prove the theorem we need three lemmas. The first lemma is folklore, and represents the main property of tree decomposition used for query evaluation. If $f \in D_X$, $g \in D_Y$ agree on $X \cap Y$, then $f \bowtie g$ is the unique tuple $D_X \cup Y$ that extends both $f$ and $g$. If $P_1 \subseteq D_X, P_2 \subseteq D_Y$, then $P_1 \bowtie P_2 = \{f \bowtie g \mid f \in P_1, g \in P_2\}$.

**Lemma 4.4.** Let $(T, \chi)$ be a tree decomposition for $Q$ and recall that $Q \equiv \bigwedge_{t \in \text{nodes}(T)} Q_t$ where $Q_t$ is a conjunction of atoms $A$ s.t. $\text{vars}(A) \subseteq \chi(t)$. Then, for every $D$, $\text{hom}(Q, D) = \bigwedge_{t \in \text{nodes}(T)} \text{hom}(Q_t, D)$.

**Lemma 4.5.** Fix a homomorphism $\varphi : Q_2 \to Q_1$, let $(T, \chi)$ be a tree decomposition of $Q_2$, $D$ be a database instance, and $P = \text{hom}(Q_1, D)$. Then, for every node $t \in \text{nodes}(T)$, denoting $P'_t \overset{\text{def}}{=} \Pi_{\varphi|\chi(t)}(P)$ we have:

$$P'_t \subseteq \text{hom}(Q_t, D)$$

**Proof.** Every tuple in $\Pi_{\varphi|\chi(t)}(P)$ is the composition $f \circ \varphi|\chi(t)$ for some $f \in P$. The lemma follows from the fact that both $\varphi|\chi(t) : Q_t \to Q_1$ and $f : Q_1 \to D$ are homomorphisms. \hfill $\Box$

**Lemma 4.6.** Let $p : P(\subseteq D^V) \to [0, 1]$ be a probability distribution, and $h : 2^V \to \mathbb{R}^+$ be its entropy.

(1) If $\varphi : Y \to V$ and $Z \subseteq Y$, then the $\varphi[Z]$-pullback of $p$, $\Pi_{\varphi[Z]}(p)$, is equal to the $Z$-marginal of $\Pi_{\varphi}(p)$. In particular, if $h' : 2^Z \to \mathbb{R}^+$ is the entropy of $\Pi_{\varphi}(p)$, then, $\forall Z \subseteq Y$, $h'(Z) = h(\varphi(Z))$.

(2) If $\varphi : V' \to V$ and $Y_1, Y_2 \subseteq V'$, then the pull-back distributions $\Pi_{\varphi|Y_1}(p)$ and $\Pi_{\varphi|Y_2}(p)$ agree on the common variables $Y_1 \cap Y_2$.

**Proof.** (1) The $\varphi$-pullback $\Pi_{\varphi}(p)$ is defined to be the same as the $\varphi(Y)$-marginal of $p$. Therefore its $Z$-marginal is the $\varphi(Z)$-marginal of $p$. By definition, $\Pi_{\varphi[Z]}(p)$ is also the $\varphi(Z)$-marginal of $p$, hence they are equal. Formally, given $f \in P$:

$$\Pi_{\varphi}(p)(Z = \Pi_Z(\Pi_{\varphi}(f))) = \sum_{f' : \Pi_Z(\Pi_{\varphi}(f')) = \Pi_Z(\Pi_{\varphi}(f))} p(f') = \sum_{f' : \Pi_Z(f) = \Pi_Z(f)} p(f') = \Pi_{\varphi[Z]}(p)(Z = \Pi_{\varphi[Z]}(f))$$

because $\Pi_Z \circ \Pi_{\varphi} = \Pi_{\varphi[Z]}$. This discussion immediately implies that $h'(Z) = h(\varphi(Z))$, for all $Z$.

(2) Let $Z = Y_1 \cap Y_2$. By claim (1), the $Z$-marginal of $\Pi_{\varphi|Y_1}(p)$ is $\Pi_{\varphi[Z]}(p)$ and similarly for the $Z$-marginal of $\Pi_{\varphi|Y_2}(p)$, hence they are equal. \hfill $\Box$

**Proof of Theorem 4.2.** Let $D$ be any database with domain $D$, and let $P = \text{hom}(Q_1, D)$. Consider the uniform probability distribution $p : P \to [0, 1]$, defined as $p(f) = 1/|P|$ for all tuples $f \in P$, and let $h$ be its entropy. We have $h = \log |P|$ because $p$ is uniform. By assumption of the theorem, there exists a homomorphism $\varphi : Q_2 \to Q_1$ and a tree decomposition $(T, \chi)$ of $Q_2$ such that:

$$\log |P| = h(\text{vars}(Q_1)) \leq (E_T \circ \varphi)(h)$$

(16)

For each $t \in \text{nodes}(T)$, consider the projections of $P$ and $p$ on $\chi(t)$:

$$P'_t \overset{\text{def}}{=} \Pi_{\varphi|\chi(t)}(P),$$
We will construct a probability distribution that the following hold:

\[ p \overset{\text{def}}{=} \prod_{i \in \chi(t)} (p). \]

Lemma 4.4 and Lemma 4.5 imply:

\[
P' \overset{\text{def}}{=} \bigwedge_{t \in \text{nodes}(T)} P'_t
\subseteq \bigwedge_{t \in \text{nodes}(T)} \text{hom}(Q_t, D)
= \text{hom}(Q, D) \quad (17)
\]

We will construct a probability distribution \( p' : P' \to [0, 1] \), with entropy function \( h' : 2^{\text{vars}(Q_2)} \to \mathbb{R}_+ \), such that the following hold:

\[
h'(\text{vars}(Q_2)) = E_T(h')
E_T(h') = (E_T \circ \varphi)(h) \quad (18)
\]

Assuming the existence of a distribution \( p' \) whose entropy function \( h' \) satisfies (18) and (19), the proof of the theorem follows from:

\[
\log |\text{hom}(Q, D)| = \log |P|
= h(\text{vars}(Q_1)) \leq (E_T \circ \varphi)(h) \quad \text{(by Eq. (16))}
= E_T(h') \quad \text{(by Eq. (19))}
= h'(\text{vars}(Q_2)) \quad \text{(by Eq. (18))}
\leq \log |P'| \quad \text{(Since } P' \text{ is the support of } h')
\leq \log |\text{hom}(Q, D)| \quad \text{(By Eq. (17))}
\]

It remains to show how to construct this distribution \( p' \) that satisfies (18) and (19). We will construct \( p' \) by stitching together the pull-back distributions \( p'_t \), for \( t \in \text{nodes}(T) \); this is possible because, by Lemma 4.6 (2), any two induced probabilities \( p'_{i_1}, p'_{i_2} \) agree on the common variables \( \chi(t_1) \cap \chi(t_2) \).

Formally, we start by listing \( \text{nodes}(T) \) in some order, \( t_1, t_2, \ldots, t_m \), such that each child is listed after its parent. Let \( P'_t \overset{\text{def}}{=} \bigwedge_{j=1}^{i} P'_j \), let \( T_i \) be the subtree induced by the nodes \( \{t_1, \ldots, t_i\} \), and \( \text{vars}(T_i) = \bigcup_{j=1}^{i} \chi(t_j) \) its variables. We construct by induction on \( i \) a probability distribution \( p'_i : P'_i \to [0, 1] \) such it agrees with \( p'_i, \ldots, p'_i \) on \( \chi(t_1), \ldots, \chi(t_i) \) respectively, and its entropy function \( h'_i : 2^{\text{vars}(T_i)} \to \mathbb{R}_+ \) satisfies:

\[
h'_i(\text{vars}(T_i)) = E_{T_i}(h'_i)
E_{T_i}(h'_i) = (E_{T_i} \circ \varphi)(h) \quad (20)
\]

To define \( p'_{i-1} \), we need to extend \( p'_{i-1} \) to the variables \( \text{vars}(T_{i-1}) = \text{vars}(T_i) \setminus \chi(\text{parent}(t_i)) \). We define \( p'_{i-1} \) to satisfy the following: (1) \( p'_{i-1} \) agrees with \( p'_{i-1} \) on \( \chi(t_i) \), (2) \( p'_{i-1} \) agrees with \( p'_{i-1} \) on the \( \text{vars}(T_{i-1}) \), and (3) \( \chi(t_i) \) is independent of \( \text{vars}(T_{i-1}) \) given \( \chi(t_i) \cap \chi(\text{parent}(t_i)) \). Notice that (1) and (2) are not conflicting because \( p'_{i-1} \) agrees with any other \( p'_j \) on their common variables. Formally, we define \( p'_{i-1} \) through a sequence of three equations:

\[
p'_{i}(\chi(t_i)|\chi(t_i) \cap \chi(\text{parent}(t_i))) \overset{\text{def}}{=} p'_{i-1}(\chi(t_i)|\chi(t_i) \cap \chi(\text{parent}(t_i))) \quad (22)
p'_{i}(\chi(t_i)|\text{vars}(T_{i-1})) \overset{\text{def}}{=} p'_{i}(\chi(t_i)|\chi(t_i) \cap \chi(\text{parent}(t_i))) \quad (23)
p'_{i}(\text{vars}(T_i)) \overset{\text{def}}{=} p'_{i}(\chi(t_i)|\text{vars}(T_{i-1}))p'_{i-1}(\text{vars}(T_{i-1})) \quad (24)
\]

We check Eq. (20):

\[
h'_i(\text{vars}(T_i)) = h'_i(\chi(t_i)|\text{vars}(T_{i-1}))+h'_{i-1}(\text{vars}(T_{i-1}))
= h'_i(\chi(t_i)|\text{vars}(T_{i-1}))+E_{T_{i-1}}(h'_{i-1}) \quad \text{(by Eq. (24))}
\]

(Induction)
We check Eq. (21).

\[
E_T(h') = h_i'(\chi(t_i)|\chi(t_i) \cap \chi(\text{parent}(t_i))) + E_{T_{i-1}}(h'_i) \quad \text{(Definition of } E_T) \\
= h_i'(\chi(t_i)|\chi(t_i) \cap \chi(\text{parent}(t_i))) + (E_{T_{i-1}} \circ \varphi)(h) \quad \text{(Induction)} \\
= h_i'(\chi(t_i)|\chi(t_i) \cap \chi(\text{parent}(t_i))) + (E_{T_{i-1}} \circ \varphi)(h) \quad \text{(by Eq. (22))} \\
= h(\varphi(\chi(t_i))|\varphi(\chi(t_i) \cap \chi(\text{parent}(t_i)))) + (E_{T_{i-1}} \circ \varphi)(h) \quad \text{(Lemma 1.6(1))} \\
= (E_T \circ \varphi)(h) \quad \text{(Definition of } E_T)
\]

This completes the inductive proof.

By setting \(i = m\) (the number of nodes in \(T\)) in Eq. (20) and (21), we derive Eq. (18) and (19).

\[\Box\]

### 4.2 A Necessary Condition

Next we prove that inequality (13) is also a necessary condition for containment \(Q_1 \preceq Q_2\), when \(Q_2\) is acyclic. Our result answers positively the conjecture by Kopparty and Rossman [21 Sect. 3, Discussion 1], in the case when \(Q_2\) is acyclic. To prove the theorem, we consider some entropy definitions and lemmas, where we fix a relation \(\text{hom}(Q_2, D)\), and show that \(P\) can be chosen to be “totally uniform”. This allows us to relate \(|\text{hom}(Q_2, D)|\) to the right-hand-side of Eq. (13). More precisely, we prove the following.

**Theorem 4.7.** Let \(Q_2\) be acyclic. If there exists an entropic function \(h\) such that (13) does not hold, namely,

\[
h(\text{vars}(Q_1)) > \max_{(T, \chi) \in \text{TD}(Q_2)} \max_{\varphi \in \text{hom}(Q_2, Q_3)} (E_T \circ \varphi)(h)
\]

then there exists a database \(D\) such that \(|\text{hom}(Q_1, D)| > |\text{hom}(Q_2, D)|\).

Together, Theorems 4.2 and 4.7 prove that \(\text{BagCQC-A} \leq_m \text{Max-IIP}\). To prove Theorem 4.7, we need some definitions and lemmas, where we fix a relation \(P \subseteq D^V\), for some set of variables \(V\), let \(p : P \rightarrow [0, 1]\) be its uniform distribution \((p(f) \overset{\text{def}}{=} 1/|P|\), for all \(f \in P\), and \(h : 2^V \rightarrow \mathbb{R}_+\) its entropy.

**Definition 4.8.** We call \(P\) totally uniform if every marginal of \(p\) is also uniform.

For any two sets \(X, Y \subseteq V\), and any tuple \(f_0 \in \Pi_Z(P)\), define the \(Y\)-degree of \(f_0\) as

\[
\deg_P(Y|X = f_0) \overset{\text{def}}{=} |\{f \in P \mid \Pi_Y(f) = f_0\}|.
\]

**Lemma 4.9.** Let \(P\) be totally uniform. Then, for any two sets \(X, Y \subseteq V\), the following hold:

1. \(\deg_P(Y|X = f_0)\) is independent of the choice of \(f_0\), and we denote it by \(\deg_P(Y|X)\).
2. \(\deg_P(Y|X) = |\Pi_{XY}(P)|/|\Pi_X(P)|\) and \(h(Y|X) = \log(\deg_P(Y|X))\).

**Proof.** Item 1 follows from the fact that the \(X\)-marginal of \(p\) is uniform and, therefore, \(p(X = f_0) = \deg(Y|X = f_0)/|\Pi_{XY}(P)|\) is independent of \(f_0\). For item 2

\[
|\Pi_{XY}(P)| = \sum_{f_0 \in \Pi_X(P)} \deg_P(Y|X = f_0) = |\Pi_X(P)| \cdot \deg_P(Y|X),
\]

15
and
\[ h(Y|X) = h(XY) - h(X) = \log |\Pi_{XY}(P)| - \log |\Pi_X(P)| = \log(\deg_P(Y|X)). \]

**Lemma 4.10.** If \( P_1 \subseteq D^X, P_2 \subseteq D^Y \) and \( P_2 \) is totally uniform, then \(|P_1 \Join P_2| \leq |P_1| \cdot \deg_{P_2}(Y|X \cap Y)\).

**Proof.** By Chan-Yeung’s density result, we can assume that if there exists a group \( G \) such that \( \deg_{P_2}(Y|X \cap Y) \). Let \( \Gamma \) be any entropic function witnessing the failure: \( \max_i E_i(h) = \max_{i=1}^n e_i(h) = E_i(h_0) \geq 0 \) fails, then it fails with a gap \( \Delta > 0 \) on a group-characterizable entropy. Let \( h_0 \) be any entropic function witnessing the failure: \( \max_{i=1}^n E_i(h_0) < 0 \). Choose any \( \delta > 0 \) s.t. \( \max_{i=1}^n E_i(h_0) + \delta < 0 \), and define \( k \), the limit of group-characterizable entropic functions is dense in \( \Gamma^n \): in other words, every \( h \in \Gamma^n \) is the group-characterizable. In particular, if a max-linear inequality is valid for all group-characterizable entropic functions, then it is also valid for all entropic functions.

We show that, if \( \max_{i=1}^n E_i(h) \geq 0 \), then it fails with a gap \( \Delta > 0 \) on a group-characterizable entropy. Let \( h_0 \) be any entropic function witnessing the failure: \( \max_{i=1}^n E_i(h_0) < 0 \). Choose any \( \delta > 0 \) s.t. \( \max_{i=1}^n E_i(h_0) + \delta < 0 \), and define \( k \) s.t. \( \max_{i=1}^n E_i(h_0) + \delta < 0 \), and therefore \( \max_{i=1}^n E_i(h) + \Delta < 0 \). By Chan-Yeung’s density result, we can assume that \( h \) is group-characterizable.

Finally, we prove that the set \( P \) defining a group-characterizable entropy is totally uniform. This follows immediately from the fact that, under the uniform distribution, every tuple \((aG_1, \ldots, aG_n) \in P \) has probability \([G_1 \cap \cdots \cap G_n]/|G|\), and the marginal probability of any tuple \((aG_1, \ldots, aG_i) \in \Pi_{i \in \vars} P \) has probability \([G_1 \cap \cdots \cap G_i]/|G|\). (See Theorem 1 from [6].)

**Proof of Theorem 4.11.** Let \((T, \chi)\) be a junction tree (decomposition) of \( Q_2 \), which exists because acyclic queries are chordal. Then,
\[
\begin{align*}
\log |P| &= \log(h(\vars(Q_1))) + \max_{(T',\chi) \in \TD(Q_2)} \max_{\phi \in \hom(\vars(Q_2), T')} (E_{T'} \circ \phi)(h) \\
&\geq \max_{\phi \in \hom(\vars(Q_2), T')} (E_{T'} \circ \phi)(h).
\end{align*}
\]

Fix \( \Delta \) such that \( \Delta > \log |\hom(Q_2, Q_1)| \), and let \( P \subseteq D^{\vars(Q_1)} \) be the totally uniform relation given by Lemma 4.11 whose entropy \( h \) satisfies:
\[
\log |P| = h(\vars(Q_1)) > \Delta + \max_{\phi \in \hom(\vars(Q_2, Q_1))} (E_{T'} \circ \phi)(h)
\]

\( P \)'s columns are in 1-1 correspondence with \( \vars(Q_1) = \{X_1, \ldots, X_n\} \). We annotate each value with the column name, thus a tuple \( f = (c_1, c_2, \ldots, c_n) \in P \) becomes
\[
f = ("X_1", c_1), ("X_2", c_2), \ldots, ("X_n", c_n)).
\]
The annotated $P$ is isomorphic with the original $P$, hence still totally uniform. Let $D = \Pi_{Q_1}(P)$ be the database obtained by projecting the annotated $P$ on the atoms of $Q_1$ (Eq. \ref{eq:project}). We have seen that $|\text{hom}(Q_1, \Pi_{Q_1}(P))| \geq |P|$. We will show that $|P| > |\text{hom}(Q_2, D)|$, thus $P$ is a witness for $Q_1 \not\preceq Q_2$. To do this we need to upper bound $|\text{hom}(Q_2, D)|$.

Let $e : D \rightarrow Q_1$ be the homomorphism mapping every value ("$X$", $e$) to the variable $X$: this is a homomorphism because, by the definition of $D$, Eq. \ref{eq:project}, each tuple $f_0 = R_e(("X_1", c_1), ("X_2", c_2), \ldots)$ in $D$ is the projection of some $f \in P$ on the variables $\text{vars}(A)$ of some $A \in \text{atoms}(Q_1)$; then $e$ maps $f_0$ to $A$. If we view a tuple $f \in P$ as a function $\text{vars}(Q_1) \rightarrow D$, where $D$ is the domain, then $e \circ f$ is the identity function on $\text{vars}(Q_1)$. Fix $\varphi \in \text{hom}(Q_2, Q_1)$ and denote:

$$\text{hom}_\varphi(Q_2, D) \overset{\text{def}}{=} \{ g \in \text{hom}(Q_2, D) \mid e \circ g = \varphi \}$$

We have

$$\text{hom}(Q_2, D) = \bigcup_{\varphi \in \text{hom}(Q_2, Q_1)} \text{hom}_\varphi(Q_2, D)$$

$$|\text{hom}(Q_2, D)| = \sum_{\varphi \in \text{hom}(Q_2, Q_1)} |\text{hom}_\varphi(Q_2, D)|$$

We will compute an upper bound for $|\text{hom}_\varphi(Q_2, D)|$, for each homomorphism $\varphi$. We claim:

$$\text{hom}_\varphi(Q_2, D) \subseteq \forall t \in \text{nodes}(T) \Pi_{\varphi|\chi(t)}(P)$$

where $\varphi|\chi(t)$ is the restriction of $\varphi$ to $\chi(t)$, and $\Pi_{\varphi|\chi(t)}(P)$ is the generalized projection (Sec. \ref{sec:join}), i.e. it is a relation with attributes $\chi(t)$. The reason for partitioning $\text{hom}(Q_2, D)$ into subsets $\text{hom}_\varphi(Q_2, D)$ is so we can apply inequality \ref{eq:upper-bound} to each set: notice that the right-hand-side depends on $\varphi$. To prove the claim \ref{eq:upper-bound}, we first observe:

$$\text{hom}_\varphi(Q_2, D) \subseteq \forall t \in \text{nodes}(T) \text{hom}_{\varphi|\chi(t)}(Q_t, D)$$

This is a standard property of any join decomposition (not necessarily acyclic): every tuple $g \in \text{hom}(Q_2, D)$ is the join of its fragments $\Pi_{\varphi|\chi(t)}(g) \in \text{hom}(Q_t, D)$, as long as the fragments cover all attributes of $g$. Next we prove the following locality property:

$$\text{hom}_{\varphi|\chi(t)}(Q_t, D) \subseteq \Pi_{\varphi|\chi(t)}(P)$$

It says that every answer of $Q_t$ on $D$ can be found in a single row of $P$. Here we use the fact that $Q_2$ is acyclic therefore there exists some $B \in \text{atoms}(Q_2)$ s.t. $\text{vars}(B) = \chi(t)$. Then, any homomorphism $g_0 \in \text{hom}_{\varphi|\chi(t)}(Q_t, D)$ maps $B$ to some tuple $f_0 \in D$. By construction of $D$, there exists some $A \in \text{atoms}(Q_1)$ such that $f_0 \in \Pi_{\text{vars}(A)}(P)$; in particular, $f_0 = \Pi_{\text{vars}(A)}(f)$ for some $f \in P$. Thus $g_0$, when viewed as a tuple over variables $\chi(t)$, can be found in a single row $f \in P$, more precisely $g_0 = \Pi_{\psi}(f)$, from some function $\psi : \chi(t) \rightarrow \text{vars}(Q_1)$. Noticed that we have used in an essential way the fact that $\chi(t)$ is covered by a single atom $B$: we will need to remove this restriction later when we prove Theorem \ref{thm:main} (Lemma \ref{lem:main}) in Section \ref{sec:main}. From here it is immediate to show that $\psi = \varphi|\chi(t)$, by composing with $e$: $\varphi|\chi(t) = e \circ g_0 = e \circ f \circ \psi = \psi$ because $e \circ f$ is the identity on $\text{vars}(Q_1)$. This completes the proof of Eq. \ref{eq:locality}, which, together with Eq. \ref{eq:upper-bound}, proves the claim Eq. \ref{eq:upper-bound}.

Finally, we will upper bound the size of the join in \ref{eq:upper-bound}, by applying repeatedly Lemma \ref{lem:upper-bound}. This is possible because each projection $\Pi_{\varphi|\chi(t)}(P)$ is totally uniform. Formally, fix an order of $\text{nodes}(T)$,
Consider a general Max-IIP $\mathcal{Q}$, acyclic. Then we construct $\mathcal{Q}$ using junction trees. Let $\mathcal{Q}$ be a junction tree. Then we can restrict the expression to $\mathcal{Q}$, which gives $\mathcal{Q}$. Then we can restrict the expression to $\mathcal{Q}$, which gives $\mathcal{Q}$.

Let $p'$ be the pullback of $p$. Its entropy satisfies $h'(Z) = h(\varphi(Z)) = (h \circ \varphi)(Z)$ for all $Z \subseteq \chi(t_i)$. We then have

$$\log |\text{hom}_\varphi(Q_2, D)| \leq \sum_{i=1}^m \log \deg_{\Pi_{\varphi|\chi(t_i)}(P)}(\chi(t_i)|\chi(t_i) \cap \chi(\text{parent}(t_i)))$$

$$= \sum_{i=1}^m (h \circ \varphi)((\chi(t_i)|\chi(t_i) \cap \chi(\text{parent}(t_i))) = (E_T \circ \varphi)(h)$$

$$< h(\text{vars}(Q_1)) - \Delta = \log |P| - \Delta \quad \text{(By Eq. (28))}$$

Equivalently, $|\text{hom}_\varphi(Q_2, D)| < |P|/2^\Delta$. We sum up (29):

$$|\text{hom}(Q_2, D)| < |\text{hom}(Q_2, Q_1)| \frac{|P|}{2^\Delta} < |P|$$

completing the proof.

We remark that inequality (28) is slightly stronger than necessary to prove containment. In the proof, we only need the inequality to hold for some junction tree. Conversely, Theorem 4.2 can also be stated such that we only consider non-redundant tree decompositions, of which junction trees are a special case.

## 5 Reducing Max-IIP to BagCQC-A

The results of the previous section imply $\text{BagCQC-A} \leq_m \text{Max-IIP}$. We now prove the converse, $\text{Max-IIP} \leq_m \text{BagCQC-A}$: in other words we show that Max-IIP can be reduced to the containment problem $Q_1 \leq Q_2$, with acyclic $Q_2$.

**Theorem 5.1.** Max-IIP $\leq_m$ BagCQC-A.

The proof has two parts. First, we convert the Max-IIP in Eq. (7) into a form that resembles Eq. (13), then we construct $Q_1$ and $Q_2$.

### 5.1 Max-IIP $\leq_m$ Uniform-Max-IIP

Consider a general Max-IIP (Eq. (7)), which we repeat here:

$$0 \leq \max_{t \in [k]} E_t(h)\quad (34)$$

where $E_t(h) \triangleq \sum_{X \subseteq V} c_{t,X,t} h(X)$. In order to reduce it to a query containment problem, we start by making the expressions $E_t$ uniform. More precisely, for fixed natural numbers $n, p, q$, we say that an expression $E$ is $(n, p, q)$-uniform if:

$$E(h) = n \cdot h(U) + \sum_{j=0}^p h(Y_j|X_j) - q \cdot h(V)\quad (35)$$

where $V$ is the set of all variables, $U$ is a single variable called the *distinguished variable*, and $X_j, Y_j$, for $j = 0, p$, are (not necessarily distinct) sets of variables, satisfying the following conditions:
**Chain condition** \(X_0 = \emptyset\) and \(X_j \subseteq Y_{j-1} \cap Y_j\) for \(j = 1, p\).

**Connectedness** \(U \in X_j\) for \(j = 1, p\)

A Uniform–Max–IIP is a Max–IIP, Eq. (34), such that there exist numbers \(n, p, q\) and a variable \(U\) s.t. all expressions \(E_\ell\) in Eq. (34) are \((n, p, q)\)-uniform, and have \(U\) as a distinguished variable. Notice that \(n, p, q\) and \(U\) are the same in all expressions \(E_\ell\). Clearly, a Uniform–Max–IIP is a special case of a Max–IIP. We prove:

**Lemma 5.2.** Max–IIP \(\leq_m\) Uniform–Max–IIP. Moreover, the reduction can be done in polynomial time.

**Proof.** Every \(E_\ell\) in Eq. (34) has the form \(\sum_{X \subseteq V} c_{\ell,X} h(X)\). By expanding each positive coefficient as \(c_{\ell,X} = 1 + 1 + \cdots\) and each negative coefficient as \(c_{\ell,X} = -1 -1 - \cdots\), we can write:

\[
E_\ell(h) = \sum_{i=1}^{m_\ell} h(Y_i) - \sum_{j=1}^{n_\ell} h(X_j) = \sum_{i=1}^{m_\ell} h(Y_i) + \sum_{j=0}^{n_\ell} h(V|X_j) - n_\ell \cdot h(V)
\]

Define \(X_0 \overset{\text{def}}{=} \emptyset\) and add \(h(V|X_0) - h(V) (= 0)\) to \(E_\ell\):

\[
E_\ell(h) = \sum_{i=1}^{m_\ell} h(Y_i) + \sum_{j=0}^{n_\ell} h(V|X_j) - (n_\ell + 1) \cdot h(V)
\]

(36)

The second sum is a chain, because \(X_0 = \emptyset\) and every \(X_j\) is contained in \(V\). Let \(n \overset{\text{def}}{=} \max_\ell n_\ell\). We add \(n - n_\ell\) terms \(h(V) - h(V)\) to the expression \(E_\ell\), resulting in two changes to the expression (36): the term \(- (n_\ell + 1) \cdot h(V)\) is replaced by \(- (n + 1) \cdot h(V)\), and the sum \(\sum_{i=1}^{m_\ell} h(Y_i)\) becomes \(\sum_{i=1}^{m_\ell} h(Y_i) + (n - n_\ell) \cdot h(Y_i)\) where the \(n - n_\ell\) new terms are \(Y_1 \overset{\text{def}}{=} V\). We combine the two sums \(\sum_i h(Y_i) + \sum_j h(V|X_j)\) into a single sum by writing \(h(Y_i)\) as \(h(Y_i|\emptyset)\), and thus \(E_\ell\) becomes:

\[
E_\ell(h) = \sum_{j=0}^{p_\ell} h(Y_j|X_j) - (n + 1) \cdot h(V)
\]

(37)

Notice that Eq. (37) still satisfies the chain condition: \(X_0 = \emptyset\), and \(X_j \subseteq Y_{j-1} \cap Y_j\) for \(j = 1, p_\ell\). Our next step is to enforce the connectedness condition.

Let \(U\) be a fresh variable. We will denote by \(h\) an entropic function over the variables \(V\), and by \(h'\) an entropic function over the variables \(UV\). For \(\ell \in [k]\), denote by \(E_\ell'(h')\) the following expression:

\[
E_\ell'(h') = (n + 1) \cdot h'(U) + \sum_{j=0}^{p_\ell} h'(UY_j|UX_j) - (n + 1) \cdot h'(V)
\]

(38)

We claim: \(\forall h, 0 \leq max_\ell E_\ell(h)\) iff \(\forall h', 0 \leq max_\ell E_\ell'(h')\). For the \(\Leftarrow\) direction, assume \(\forall h' : 0 \leq max_\ell E_\ell'(h')\) and let \(h\) be any entropic function over the variables \(V\). We extended it to an entropic function \(h'\) over the variables \(UV\), by defining \(U\) to be a constant random variable. In other words, \(h'(X) \overset{\text{def}}{=} h(X - \{U\})\) for all \(X \subseteq UV\): in particular \(h'(U) = 0\). Then \(E_\ell'(h') = E_\ell(h)\), for all \(\ell \in [k]\), and the claim follows from \(0 \leq max_\ell E_\ell'(h') = max_\ell E_\ell(h)\). For the \(\Rightarrow\) direction, let \(h'\) be any entropic function over the variables \(UV\), and denote \(h(-) \overset{\text{def}}{=} h'(-|U)\) the conditional entropy. The conditional entropy \(h\) is not necessarily entropic, but it is the limit of entropic functions (see Appendix [B]), hence it satisfies \(0 \leq max_\ell E_\ell(h)\). Then, \(E_\ell'(h') = \sum_{j=0}^{p_\ell} h'(UY_j|UX_j) - (n + 1) \cdot h'(V) = \sum_{j=0}^{p_\ell} h(Y_j|X_j) - (n + 1) \cdot h(V) = E_\ell(h)\), and the claim follows from \(0 \leq max_\ell E_\ell(h) = max_\ell E_\ell'(h')\).

To enforce \(X_0 = \emptyset\) in the chain condition, we write \(E_\ell'\) as:

\[
E_\ell'(h') = n \cdot h'(U) + h'(U) + \sum_{j=0}^{p_\ell} h'(UY_j|UX_j) - (n + 1) \cdot h'(V)
\]

To enforce connectedness, we can use a similar technique to enforce the connectedness condition. We define \(E_\ell'(h')\) as:

\[
E_\ell'(h') = n \cdot h'(U) + h'(U) + \sum_{j=0}^{p_\ell} h'(UY_j|UX_j) - (n + 1) \cdot h'(V)
\]
Finally, we need to ensure that all numbers $p_\ell$ are equal, and, for that, we set $p \overset{\text{def}}{=} 1 + \max_\ell p_\ell$ and add $p - p_\ell - 1$ terms $h'(U|U)$ to $E'_\ell(h')$. Comparing it with Eq. (39), the new $E'_\ell$ is an $(n, p, n + 1)$-uniform expression, proving the lemma.

\section{A Technical Lemma}

The Uniform-Max-IIP has some arbitrary $q$, while Eq. (39) has $q = 1$. We prove here a technical lemma showing that an $(n, p, q)$-uniform Max-IIP is equivalent to some Uniform-Max-IIP with $q = 1$. We do this by introducing new random variables.

Let $V$ be a set of variables. For each variable $Z \in V$, we create $q$ fresh copies $Z^{(\ell)}$, $\ell = 1 \ldots q$, called adornments of $Z$. If $X$ is a set of variables, then $X^{(\ell)}$ is the set where all variables are adorned with $\ell$. We will denote by $h$ an entropic function over the original variables $V$, and by $h'$ an entropic function over the adorned variables $V^{(1)} \ldots V^{(q)}$. If $F = \sum_i c_i h'(X^{(\ell)}_i)$ is a linear expression over adorned variables, then its erasure, $F \overset{\text{def}}{=} \sum_i c_i h(X_i)$, is defined as the expression obtained by erasing every adornment; we also say that $F$ is an adornment of $\epsilon(F)$. Conversely, if $E = \sum_i c_i h(X_i)$ is an expression over the original variables, then a constant adornment is an expression of the form $E^{(\ell)} = \sum_i c_i h'(X^{(\ell)}_i)$, i.e. all terms are adorned by the same $\ell$; clearly $\epsilon(E^{(\ell)}) = E$.

**Lemma 5.3.** Let $E_1, \ldots, E_k$ be linear expressions over variables $V$, and $F_1, \ldots, F_m$ be linear expressions over adorned variables $V^{(1)}, \ldots, V^{(q)}$ for some $q \geq 1$, such that (a) each $F_j$ is an adornment of some $E_i$, i.e. $\epsilon(F_j) = E_i$, and (b) all constant adornments are included, i.e. for every $E_i$ and every $\ell$ there exists $F_j = E_i^{(\ell)}$. Then the following two statements are equivalent:

\begin{equation}
\forall h : q \cdot h(V) \leq \max_{i \in [k]} E_i(h) \tag{39}
\end{equation}

\begin{equation}
\forall h' : h'(V^{(1)} \ldots V^{(q)}) \leq \max_{j=1,m} F_j(h') \tag{40}
\end{equation}

**Proof.** (39) $\Rightarrow$ (40) follows from:

\[
\begin{align*}
h'(V^{(1)} \ldots V^{(q)}) &\leq \sum_{\ell=1,q} h'(V^{(\ell)}) \\
&\leq q \max_{\ell=1,q} h'(V^{(\ell)}) \\
&\leq \max_{\ell=1,q} \max_{i \in [k]} E_i^{(\ell)}(h') \quad \text{(Eq. (39) applied to } V^{(\ell)}) \\
&\leq \max_{j=1,m} F_j(h') \quad \text{(Assumption (b))}
\end{align*}
\]

(40) $\Rightarrow$ (39) Let $h$ be an entropic function over variables $V$. That means that there exists a joint distribution over random variables $V$ whose entropy is given by $h$. For each random variable $Z$, create $q$ i.i.d. copies $Z^{(\ell)}$, for $\ell = 1, q$, and denote by $h'$ the entropy function of the new random variables $V^{(1)}, \ldots, V^{(q)}$. Thus, for any adorned set $X^{(\ell)}$, $h'(X^{(\ell)}) = h(X)$, and, if $E_i = \epsilon(F_j)$, then $E_i(h) = F_j(h')$. The claim follows from:

\[
\begin{align*}
q \cdot h(V) &= h'(V^{(1)}) + \cdots + h'(V^{(q)}) \quad \text{(By } h(V) = h'(V^{(\ell)}), \text{ for all } \ell) \\
&= h'(V^{(1)} \ldots V^{(q)}) \quad \text{(Independence)} \\
&\leq \max_{j=1,m} F_j(h') \quad \text{(Eq. (40))} \\
&\leq \max_{i \in [k]} E_i(h) \quad \text{(Assumption (a))}
\end{align*}
\]
5.3 Uniform-Max-IIP $\leq_m$ BagCQC-A

Given an $(n, p, q)$-uniform Max-IIP problem \((39)\), \(q \cdot h(V) \leq \max_i E_i\), where

$$E_i = n \cdot h(U) + \sum_{j=0,p} h(Y_{ij}|X_{ij}), \quad (41)$$

we will construct two queries \(Q_1, Q_2\) such that \(Q_1 \preceq Q_2\) iff condition \((40)\) holds, which we have proven is equivalent to \((39)\). Recall that the distinguished variable \(U\) occurs everywhere, except in the sets \(X_{i0}\) which, by definition, are \(\emptyset\). We first substitute everywhere the single variable \(U\) with two variables, \(U = U_1 U_2\). This does not affect the Max-IIP, since we can simply treat \(U_1 U_2\) as a joint variable.

The query \(Q_2\) will have one atom for each term of the expression \(E_i\) in \((41)\), which is possible because, by uniformity, all expressions \(E_i\) have the same number of terms. In particular, there will be an atom \(R_j\) corresponding to the term \(h(Y_{ij}|X_{ij})\), however, the number of variables \(Y_{ij}\) depends on \(i\). For that reason, we consider their disjoint union, as follows. For each variable \(V \in V\) and each \(i, j\), let \(V^{ij}\) be a fresh copy of \(V\); if \(W = \{V_1, V_2, \ldots\}\) is a set, then we denote by \(W^{ij} = \{V^{ij}_1, V^{ij}_2, \ldots\}\). We define \(\hat{Y}_j \defeq \bigcup_{i \in [k]} Y_{ij}\), for \(j = 0, p\), and \(\hat{X}_j \defeq \bigcup_{i \in [k]} X_{ij}^{(j-1)}\), for it \(j = 1, p\), and \(\hat{X}_0 \defeq \emptyset\). We notice that \(|\hat{Y}_j| = \sum_i |Y_{ij}|\), the sets \(\hat{Y}_0, \ldots, \hat{Y}_p\) are disjoint, and, since the chain condition \(X_{ij} \subseteq Y_{(j-1)}\) holds in \((41)\), we also have \(\hat{X}_j \subseteq \hat{Y}_{j-1}\); of course, \(\hat{X}_j\) is disjoint from \(\hat{Y}_j\). We define \(Q_2\) as:

$$Q_2 = S_1(\hat{U}_1) \land \cdots \land S_n(\hat{U}_n) \land R_0(\hat{X}_0 \hat{Y}_0 \hat{Z}) \land \cdots \land R_p(\hat{X}_p \hat{Y}_p \hat{Z})$$

All relation symbols are distinct. The relations \(S_1, \ldots, S_n\) are binary, and \(\hat{U}_1, \ldots, \hat{U}_n\) are disjoint sets of two fresh variables each, and \(\hat{Z}\) is a fresh set of \(k\) variables. Thus, each relation \(R_j\) has arity \((\sum_i (|X_{ij}|+|Y_{ij}|))+k\). All variables occurring in \(R_j\) are distinct (since \(\hat{X}_j \subseteq \hat{Y}_{j-1}\) which is disjoint from \(\hat{Y}_j\)) and they occur in the order that corresponds to the order \(X_{1j} \ldots X_{kj} Y_{1j} \ldots Y_{kj}\) of the original variables, followed by the \(k\) variables \(\hat{Z}\). Any two consecutive atoms \(R_{j-1}, R_j\) share the variables \(\hat{X}_j\) and \(\hat{Z}\), and therefore the tree decomposition of \(Q_2\) consists of \(n\) isolated components plus a chain:

$$T:\quad \{\hat{U}_1\} \cdots \{\hat{U}_n\} \quad \{\hat{X}_0, \hat{Y}_0, \hat{Z}\} \cdots \{\hat{X}_p, \hat{Y}_p, \hat{Z}\}$$

(42)

The query \(Q_1\) consists of \(q\) isomorphic sub-queries:

$$Q_1 = Q_1^{(1)} \land \cdots \land Q_1^{(q)}$$

which have disjoint sets of variables. We describe now the subquery \(Q_1^{(i)}\). Its variables consist of adorned copies \(V^{(i)}\) of the variables \(V\), and the query is in turn a conjunction of \(k\) sub-queries (which are no longer disjoint):

$$Q_1^{(i)} = Q_1^{(i,1)} \land \cdots \land Q_1^{(i,k)}$$

To define its atoms, we need some notations. Recall that the distinguished variables \(U_1 U_2\) occur everywhere (except \(X_{i0}\) which is empty). Then, for each \(i\), we define the the following sequences of variables:

$$\hat{X}_{ij}^{(i)} = U_1^{(i)} \ldots U_1^{(i)} \cdot U_1^{(i)} \ldots U_1^{(i)} \cdot \ldots \cdot U_1^{(i)} \ldots U_1^{(i)}$$

$$\hat{Y}_{ij}^{(i)} = U_1^{(i)} \ldots U_1^{(i)} \cdot U_1^{(i)} \ldots U_1^{(i)} \cdot \ldots \cdot U_1^{(i)} \ldots U_1^{(i)}$$
That is, the length of $\hat{X}_{ij}^{(t)}$ is the same as that of the concatenation $X_{1j}X_{2j} \ldots X_{kj}$, and has the distinguished variables $U_1^{(t)}$ on all positions except the positions of $X_{ij}$, where it has the adornment $X_{ij}^{(t)}$. (As a special case, $\hat{X}_{i0}^{(t)} = \emptyset$.) Note that the length of $\hat{X}_{ij}^{(t)}$ is independent of $i$, and $|\hat{X}_{ij}^{(t)}| = |\hat{X}_{ij}|$ (the variables from $Q_2$). Similarly for $\hat{Y}_{ij}^{(t)}$. The sequence $Z_i$ has length $k$ and contains $U_1^{(t)}$ everywhere except for position $i$ where it has $U_2^{(t)}$. Then, query $Q_{1,i}^{(t)}$ is:

$$Q_{1,i}^{(t)} = S_1(U_1^{(t)}) \wedge \ldots \wedge S_n(U_n^{(t)}) \wedge R_0(\hat{X}_{i0}^{(t)} \hat{Y}_{i0}^{(t)} Z_i^{(t)}) \wedge R_1(\hat{X}_{i1}^{(t)} \hat{Y}_{i1}^{(t)} Z_i^{(t)}) \wedge \ldots \wedge R_p(\hat{X}_{ip}^{(t)} \hat{Y}_{ip}^{(t)} Z_i^{(t)})$$

Notice that the variables of the atom $R_j$ are just $Y_{ij}^{(t)}$ (which contains $U_1^{(t)}$, $U_2^{(t)}$, and $X_{ij}^{(t)}$), and some variables are repeated several times.

We start by noticing that every homomorphism $\varphi : Q_2 \rightarrow Q_1$ must map all atoms in the chain $R_0 \cdots R_p$ to the same sub-query $Q_{1,i}^{(t)}$: this is because the chain is connected and, if one atom is mapped to an atom whose variables are adorned with $\ell$, then all atoms must be mapped to atoms adorned similarly with $\ell$. We claim something stronger, that $\varphi$ maps the entire chain to the same sub-query $Q_{1,i}^{(t)}$. This is enforced by the variables $\hat{Z}$ of $Q_2$: if one atoms is mapped to the sub-query $Q_{1,i}^{(t)}$, then $\varphi(\hat{Z}_i) = U_2^{(t)}$ and $\varphi(\hat{Z}_{i'}) = U_1^{(t)}$ for all $i' \neq i$, implying that all other atoms are mapped to the same sub-query.

By Theorems 4.2 and 4.7 we have:

$$Q_1 \preceq Q_2 \quad \text{iff} \quad \forall h', h'(\text{vars}(Q_1)) \leq \max_{\varphi \in \text{hom}(Q_2, Q_1)} (E_T \circ \varphi)(h') \quad (43)$$

We claim that the following are equivalent:

$$\forall h', h'(\text{vars}(Q_1)) \leq \max_{\varphi \in \text{hom}(Q_2, Q_1)} (E_T \circ \varphi)(h') \quad \text{iff} \quad \forall h, \forall q : h(V) \leq \max_i E_i(h), \quad (44)$$

where $E_i$ is given by (11). The claim implies the theorem: $Q_1 \preceq Q_2$ iff $\forall h, h(V) \leq \max_i E_i(h)$. To prove the claim, we will use Lemma 5.3 and, for that, we need to verify the conditions of the lemma. We start by applying the definition of $E_T$ (Eq.(12)), where $T$ is the tree decomposition of $Q_2$, Eq.(42), and obtain (recall that $\hat{X}_0 = \emptyset$):

$$E_T = h(\hat{U}_1) + \ldots + h(\hat{U}_n) + h(\hat{Y}_0 \hat{Z}) + \sum_{k=1,p} h(\hat{X}_i \hat{Y}_i \hat{Z} | \hat{X}_j \hat{Z})$$

Consider a homomorphism $\varphi \in \text{hom}(Q_2, Q_1)$. By the previous discussion, it maps all atoms in the chain to the same subquery $Q_{1,i}^{(t)}$ for some $\ell$ and $i$. We illustrate it by showing $Q_2$ and $\varphi(Q_2)$ next to each other:

$$Q_2 = S_1(\hat{U}_1) \wedge \ldots \wedge S_n(\hat{U}_n) \wedge R_0(\hat{X}_0 \hat{Y}_0 \hat{Z}) \wedge \ldots \wedge R_p(\hat{X}_p \hat{Y}_p \hat{Z})$$

$$\varphi(Q_2) = S_1(U_1^{(t)}) \wedge \ldots \wedge S_n(U_n^{(t)}) \wedge R_0(\hat{X}_{i0}^{(t)} \hat{Y}_{i0}^{(t)} Z_i^{(t)}) \wedge \ldots \wedge R_p(\hat{X}_{ip}^{(t)} \hat{Y}_{ip}^{(t)} Z_i^{(t)})$$

Next, we apply the substitution $\varphi$ to $E_T$ to obtain $E_T \circ \varphi$. Since each of the original expressions $E_i$ in Eq.(11) was $(n, p, q)$-uniform, $U$ occurs in every set $Y_{ij}$ and $X_{ij}$ (except for $X_{i0}$). By construction, $\hat{Z}_i^{(t)}$ is a sequence consisting only of the variables $U_1^{(t)}$ and $U_2^{(t)}$, thus the following set inclusions hold (except for $Z_i^{(t)} \subseteq \hat{X}_{i0}^{(t)}$): $\hat{Z}_i^{(t)} \subseteq \hat{X}_{ij}^{(t)} \subseteq \hat{Y}_{ij}^{(t)}$, and we obtain:

$$E_T \circ \varphi = h(U_1^{(t)}) + \ldots + h(U_n^{(t)}) + h(\hat{Y}_0 \hat{Z}) + \sum_{k=1,p} h(\hat{X}_i \hat{Y}_i \hat{Z} | \hat{X}_j \hat{Z})$$

22
In this section, we aim to prove the decidability of our novel class of BagCQC. Proving Decidability of a Novel Class of BagCQC

while reducing from (45) to the BagCQC-A both theorems in Section 6.2.

From the right-hand-side we derive two queries \( Q \).

We apply Eq. (13) to \( Q \), and

\[
0 \leq h(X_1) + 2h(X_2) + h(X_3) - h(X_1X_2) - h(X_2X_3)
\]

(45)

We start by rewriting the inequality as:

\[
3h(X_1X_2X_3) \leq h(X_1) + h(X_2) + h(X_3) + h(X_1X_2X_3)
\]

From the right-hand-side we derive two queries \( Q_1, Q_2 \). Query \( Q_1 \) has 9 variables, \( X_i^{(\ell)} \), \( i = 1, 3 \), \( \ell = 1, 3 \), while \( Q_2 \) has 13 variables:

\[
\begin{align*}
Q_1 &= Q_1^{(1)} \land Q_1^{(2)} \land Q_1^{(3)} \\
\ell = 1, 3: & \quad Q_1^{(\ell)} = S_1(X_1^{(\ell)}) \land S_2(X_2^{(\ell)}) \land S_3(X_3^{(\ell)}) \\
& \quad \land R_1(X_1^{(\ell)}, X_2^{(\ell)}, X_3^{(\ell)}) \land R_2(X_1^{(\ell)}, X_2^{(\ell)}, X_3^{(\ell)}, X_3^{(\ell)}) \\
& \quad \land R_3(X_1^{(\ell)}, X_2^{(\ell)}, X_3^{(\ell)}, X_3^{(\ell)}) \\
Q_2 &= S_1(U_1) \land S_2(U_2) \land S_3(U_3) \land S_4(U_4) \\
& \quad \land R_1(Y_0^0, Y_0^1, Y_0^2) \land R_2(Y_0^0, Y_0^1, Y_1^0, Y_1^1, Y_1^2) \land R_3(Y_1^0, Y_1^1, Y_1^2, Y_2^0, Y_3^0, Y_3^1, Y_3^2)
\end{align*}
\]

We apply Eq. (13) to \( Q_1 \) and \( Q_2 \). TD(Q2) has a single tree because \( Q_2 \) is acyclic. \( Q_1 \) has 3 connected components, and \( Q_2 \) has 5, therefore there are 3\(^3\) homomorphisms \( Q_2 \rightarrow Q_1 \). Eq. (13) becomes:

\[
h(X_1^{(1)}X_2^{(1)}X_3^{(1)}X_1^{(2)}X_2^{(2)}X_3^{(2)}X_1^{(3)}X_2^{(3)}X_3^{(3)}) \leq \max_{\ell_1, \ldots, \ell_5 = 1,3} (h(X_1^{(\ell_1)}) + h(X_2^{(\ell_2)}) + h(X_3^{(\ell_3)}) + h(X_3^{(\ell_4)}) + h(X_1^{(\ell_5)}X_2^{(\ell_5)}X_3^{(\ell_5)})) + h(X_1^{(\ell_5)}|X_1^{(\ell_5)}X_2^{(\ell_5)}X_3^{(\ell_5)})
\]

(47)

By Theorems 4.2 and 4.7 and because \( Q_2 \) is acyclic, the Max-IIP (47) holds for all entropic \( h \) if and only if \( Q_1 \leq Q_2 \). Moreover Lemma 5.3 proves that this Max-IIP is equivalent to the IIP in Eq. (10), completing the reduction from (10) to the BagCQC-A instance \( Q_1 \leq Q_2 \). Our example only illustrated the reduction from IIP; Lemma 6.2 addresses the challenges introduced by Max-IIP.

6 Proving Decidability of a Novel Class of BagCQC

In this section, we aim to prove the decidability of our novel class of BagCQC that was presented earlier in Section 6.2. In particular, we prove Theorems 5.3 and 5.6. The proofs of both theorems rely on Theorem 3.12 which in turn relies on Lemma 3.13. Therefore, we first prove that lemma in Section 6.1 and then we prove both theorems in Section 6.2.

\(^{10}\)This IIP holds, but our goal is not to check it, but to reduce it to BagCQC-A.
6.1 Proof of Lemma 3.13

Lemma 6.1 (Re-statement of Lemma 3.13). Let \( h : 2^{[n]} \to \mathbb{R}_+ \) be any polymatroid. Then there exists a normal polymatroid \( h' \in \mathcal{N}_n \) with the following properties:

1. \( h'(X) \leq h(X) \), for all \( X \subseteq [n] \),
2. \( h'([n]) = h([n]) \),
3. \( h'({i}) = h({i}) \), for all \( i \in [n] \).

In addition, there exists a modular function \( h'' \in \mathcal{M}_n \) that satisfies conditions (1) and (2).

Before we prove the lemma, we need some preliminaries. Recall that we blurred the distinction between a set of \( n \) variables \( V \) and the set \([n] \). In this section we will use only \([n] \). Let \( L \overset{\text{def}}{=} 2^{[n]} \) be the lattice of subsets of \([n] \). Given a function \( h : L \to \mathbb{R}_+ \), we define its dual \( g : L \to \mathbb{R}_+ \) as its Möbius inverse [17]:

\[
\forall X : \quad h(X) = \sum_{Y : Y \supseteq X} g(Y), \quad g(X) = \sum_{Y : Y \supseteq X} (-1)^{|Y| - |X|} h(Y) \quad (48)
\]

For any set \( S \subseteq L \) we define:

\[
g(S) \overset{\text{def}}{=} \sum_{X \in S} g(X) \quad (49)
\]

Notice that \( g(L) = h(\emptyset) \).

Fact 6.2. Let \( h : L \to \mathbb{R}_+ \) be any function. Then \( h \) is a normal polymatroid (i.e. \( h \in \mathcal{N}_n \)) iff its Möbius inverse \( g \) satisfies: \( g(L) = 0 \), \( g([n]) \geq 0 \) and \( g(X) \leq 0 \) for all \( X \neq [n] \).

Proof. First we check that the Möbius inverse of a step function \( h_W \) satisfies the required properties, for \( W \subseteq V \):

\[
h_W(X) = \begin{cases} 0 & \text{if } X \subseteq W \\ 1 & \text{otherwise} \end{cases} \quad g_W(X) = \begin{cases} 1 & \text{if } X = V \\ -1 & \text{if } X = W \\ 0 & \text{otherwise} \end{cases}
\]

The converse follows by observing that every \( g \) with the required properties is a non-negative linear combination of the \( g_W \)’s: \( g = \sum_{W \subseteq [n]} (-g(W)) \cdot g_W \), therefore \( h = \sum_{W \subseteq [n]} (-g(W)) \cdot h_W \).

Fact 6.2 can be used, for example, to show that the parity function \( h \) (Example 3.8) is not normal. Indeed, it’s Möbius inverse given by Eq. (48) at \( \emptyset \) is \( g(\emptyset) = 1 \), which implies that \( h \) is not normal. Fact 6.2 will be our key ingredient to prove Lemma 3.13 in order to construct the required normal polymatroid \( h' \), we will instead construct its dual \( g' \) and check that it satisfies the conditions in Fact 6.2. We also need a technical lemma:

Lemma 6.3. Let \( a_1, \ldots, a_n \geq 0 \) be \( n \) non-negative numbers. Define:

\[
h(X) = \max \{ a_i \mid i \in X \} \quad (50)
\]

Then \( h \) is a normal polymatroid.

Proof. Assume w.l.o.g. \( a_1 \leq a_2 \leq \cdots \leq a_n \) and define \( \delta_i = a_{i+1} - a_i \) for \( i = 0, 1, \ldots, n-1 \), where \( a_0 = 0 \). Define \( g : 2^{[n]} \to \mathbb{R} \):

\[
g(X) \overset{\text{def}}{=} \begin{cases} a_n & \text{if } X = [n] \\ -\delta_i & \text{if } X = [i] \ (= \{1,2,\ldots,i\}), \text{ for some } i < n \\ 0 & \text{otherwise} \end{cases}
\]

We check that \(g\) is the dual of \(h\) by verifying:
\[
  h(X) = a_{\max(X)} = -\delta_{\max(X)} - \delta_{\max(X)+1} - \cdots - \delta_{n-1} + a_n = \sum_{Y : X \subseteq Y} g(Y)
\]

We assumed above that \(\max(\emptyset) = 0\).

Finally, we need to recall the definitions of the conditional entropy and the conditional mutual information:
\[
  h(i|X) = h(\{i\} \cup X) - h(X)
  
  I(i;j|X) = h(\{i\} \cup X) + h(\{j\} \cup Y) - h(X) - h(\{i\} \cup X) - h(\{i,j\} \cup X)
\]

and observe that, denoting \([X,Y] \overset{\text{def}}{=} \{Z \mid X \subseteq Z \subseteq Y\}\), we have:
\[
  h(X) = g([X,[n]]) \quad (52)
  h(i|X) = -g([X,[n] \setminus \{i\}]) \quad (53)
  I(i;j|X) = -g([X,[n] \setminus \{i,j\}]) \quad (54)
\]

We are now ready to prove Lemma 3.13.

**Proof of Lemma 3.13** We will proceed by induction on \(L\). Split the lattice \(L = 2^{[n]}\) into two disjoint sets \(L = L_1 \cup L_2\) where:
\[
  L_1 = [\emptyset,[n-1]] \\
  L_2 = [\{n\},[n]]
\]

In other words, \(L_1\) contains all subsets without \(n\), while \(L_2\) contains all subsets that include \(n\). Then:

- \(g(L_2) = h(\{n\})\). It follows \(g(L_1) = -h(\{n\})\).
- Subtract \(h(\{n\})\) from \(g([n])\) and add it to \(g([n-1])\), and call \(g_1, g_2\) the new functions on \(L_1, L_2\) respectively. Formally:
  \[
  g_1(X) = \begin{cases} 
    g([n-1]) + h(\{n\}) & \text{if } X = [n-1] \\
    g(X) & \text{if } X \subset [n-1]
  \end{cases}
  \\
  g_2(X \cup \{n\}) = \begin{cases} 
    g([n]) - h(\{n\}) & \text{if } X = [n-1] \\
    g(X \cup \{n\}) & \text{if } X \subset [n-1]
  \end{cases}
\]

Notice that \(g_1(L_1) = 0\) and \(g_2(L_2) = 0\).

- One can check that the dual\(^{11}\) of \(g_2\) is the conditional polymatroid\(^{12}\) defined as \(h_2 : L_2 \to \mathbb{R}\):
  \[
  \forall X \in L_2 : h_2(X) \overset{\text{def}}{=} h(X|\{n\})
  \]

- We apply induction to \(h_2\) and obtain a normal polymatroid \(h_2' : L_2 \to \mathbb{R}\) satisfying properties \(\overset{\text{11}}{\text{11}}, \overset{\text{12}}{\text{12}}\), and \(\overset{\text{3}}{\text{3}}\) that are stated in Lemma 3.13:
  \[
  h_2'(X) \leq h_2(X) = h(X|\{n\})
  
  h_2'([n]) = h_2([n]) = h([n]|\{n\})
  
  h_2'(\{i,n\}) = h_2(\{i\}) = h(\{i\}|\{n\})
  \]

Notice that \(h_2'(\{n\}) = 0\), since \(\{n\}\) is the bottom of \(L_2\). Let \(g_2'\) be the dual of \(h_2'\), thus \(g_2'(X) \leq 0\) for all \(X \neq [n]\) (because \(h_2'\) is normal).

---

\(^{11}\) Strictly speaking we cannot talk about the dual of \(g_2\) because we defined the dual only for functions \(g : 2^{[n]} \to \mathbb{R}\). However, with some abuse, we identify the lattice \(L_2\) with \(2^{[n-1]}\), and in that sense the dual of \(g_2 : L_2 \to \mathbb{R}\) is a function \(h_2 : L_2 \to \mathbb{R}\).

\(^{12}\) Proof: \(h_2(X) = \sum_{Y : X \subseteq Y \subseteq [n]} g_2(Y) = \sum_{Y : X \subseteq Y \subseteq [n]} g(Y) - h(\{n\}) = h(X) - h(\{n\}) = h(X|\{n\})\).
• One can check that the dual of $g_1$ is the function\footnote{Proof:
\[
 h_1(X) = \sum_{Y : X \subseteq Y \subseteq [n]} g_1(Y) = h(\{n\}) + \sum_{Y : X \subseteq Y \subseteq [n-1]} g(Y)
 = h(\{n\}) + \sum_{Y : X \subseteq Y \subseteq [n-1]} g(Y) - \sum_{Y : X \subseteq Y \subseteq [n-1]} g(Y \cup \{n\})
 = h(\{n\}) + h(X) - h(X \cup \{n\}) = I(X; \{n\})
\]}

\[
h_1(X) \overset{\text{def}}{=} I(X; \{n\})
\]

This is no longer a polymatroid. Instead, here we use Lemma 6.3 and define the normal polymatroid $h'_1 : L_1 \to \mathbb{R}$:

\[
h'_1(X) \overset{\text{def}}{=} \max_{i \in X} h_1(\{i\}) = \max_{i \in X} I(\{i\}; \{n\})
\]

Let $g'_1 : L_1 \to \mathbb{R}$ be its dual. Thus, $g'_1(X) \leq 0$ for all $X \neq [n-1]$, and $g'_1([n-1]) = \max_{i \in [n-1]} I(\{i\}; \{n\})$.

• We combine $g'_1, g'_2$ into a single function $g' : L = L_1 \cup L_2 \to \mathbb{R}$ as follows. $g'$ agrees with $g'_1$ on $L_1$ and with $g'_2$ on $L_2$ except that we subtract a mass of $h(\{n\})$ from $g'_1([n-1])$ and add it to $g'_2([n])$.

Formally:

\[
g'(X) = \begin{cases} 
 g'_2([n]) + h(\{n\}) & \text{if } X = [n] \\
 g'_1([n-1]) - h(\{n\}) & \text{if } X = [n-1] \\
 g'_1(X) & \text{if } X \in L_1, X \neq [n-1] \\
 g'_2(X) & \text{if } X \in L_2, X \neq [n]
\end{cases}
\]

We claim that for every $X \neq [n]$, $g'(X) \leq 0$. This is obvious for all cases above (since $g'_1, g'_2$ are normal), except when $X = [n-1]$. Here we check: $g'([n-1]) = g'_1([n-1]) - h(\{n\}) = \max_{i \in [n-1]} I(\{i\}; \{n\}) - h(\{n\}) \leq 0$ because $I(\{i\}; \{n\}) \leq h(\{n\})$.

• Denote $h' : L = L_1 \cup L_2 \to \mathbb{R}$ the dual of $g'$; we have established that $h'$ is a normal polymatroid. The following hold:

\[
\forall x \in L_1 : h'(X) = \sum_{Y : X \subseteq Y \subseteq [n]} g'(Y)
= \sum_{Y : X \subseteq Y \subseteq [n-1]} g'(Y) + \sum_{Y : X \subseteq Y \subseteq [n-1]} g'(Y \cup \{n\})
= \sum_{Y : X \subseteq Y \subseteq [n-1]} g'_1(Y) + \sum_{Y : X \subseteq Y \subseteq [n-1]} g'_2(Y \cup \{n\})
= h'_1(X) + h'_2(X \cup \{n\})
\tag{55}
\]

\[
\forall X \in L_2 : h'(X) = \sum_{Y : X \subseteq Y \subseteq [n]} g'(Y)
= h(\{n\}) + \sum_{Y : X \subseteq Y \subseteq [n]} g'_2(Y) = h(\{n\}) + h'_2(X)
\tag{56}
\]

• We check that $h'$ satisfies properties (1), (2), and (8) that are stated in Lemma 6.13:

\[
\forall X \in L_1 : h'(X) = h'_1(X) + h'_2(X \cup \{n\}) \quad \text{by Eq. (55)}
\]

\[
 h_1(X) = \sum_{Y : X \subseteq Y \subseteq [n-1]} g_1(Y) = h(\{n\}) + \sum_{Y : X \subseteq Y \subseteq [n-1]} g(Y)
= h(\{n\}) + \sum_{Y : X \subseteq Y \subseteq [n-1]} g(Y) - \sum_{Y : X \subseteq Y \subseteq [n-1]} g(Y \cup \{n\})
= h(\{n\}) + h(X) - h(X \cup \{n\}) = I(X; \{n\})
\]
\[ \leq h_1(X) + h_2(X \cup \{n\}) \]
\[ = I(X; \{n\}) + h(X|\{n\}) = h(X) \]
\[ \forall X \in L_2 : h'(X) = h(\{n\}) + h'_2(X) \] by Eq. (55)
\[ \leq h(\{n\}) + h'_2(X) \]
\[ = h(\{n\}) + h(X|\{n\}) = h(X) \]
\[ h'(\{n\}) = h(\{n\}) + h'_2(\{n\}) \] by Eq. (55)
\[ = h(\{n\}) + h_2(\{n\}) \]
\[ = h(\{n\}) + h(\{n\}|\{n\}) = h(\{n\}) \]
\[ \forall i \in [n-1] : h'(\{i\}) = h'_1(\{i\}) + h'_2(\{i, n\}) \] by Eq. (55)
\[ = h_1(\{i\}) + h_2(\{i, n\}) \]
\[ = I(\{i\}; \{n\}) + h(\{i\}|\{n\}) = h(\{i\}) \]
\[ h'(\{n\}) = h(\{n\}) + h'_2(\{n\}) = h(\{n\}) + 0 \] by Eq. (55)

This completes the proof.

We illustrate the main idea of the above proof using the following example, which is based on the parity function, also shown in Fig. 1.

**Example 6.4.** Recall the parity function, and it’s Möbius inverse:
\[
\begin{align*}
    h(\emptyset) &= 0, & h(1) &= h(2) &= h(3) &= 1, \\
    h(12) &= h(13) &= h(23) &= h(123) &= 2, \\
    g(123) &= 2, & g(12) &= g(13) &= g(23) &= 0, \\
    g(1) &= g(2) &= g(3) &= -1, & g(\emptyset) &= +1.
\end{align*}
\]

The parity function is not normal, because \( g(\emptyset) > 0 \). The lattice \( L = 2^{[3]} \) is shown on the top left of Fig. 1.

We partition \( L = L_1 \cup L_2 \), and move a mass of +1 from \( g(123) \) to \( g(12) \) (so that both lattices are balanced, i.e. \( g_1(L_1) = 0, g_2(L_2) = 0 \)); this is show in the top right. We compute \( h_1, h_2 \) from \( g_1, g_2 \). Notice that \( h_1 \) is not a polymatroid.

We define \( h'_1 \) using the max-construction (Lemma 6.3) and define \( h'_2 = h_2 \) (since it is already normal). Notice that \( h'_1 = 0 \). From \( h'_1, h'_2 \) we compute \( g'_1, g'_2 \). Lower right of Figure 1.

Finally we combine the two functions \( g'_1, g'_2 \) and obtain the functions \( h', g' \) shown in the lower left. \( h' \) is normal, is dominated by \( h \), and agrees with \( h \) on the atoms and the maximum element of the lattice.

### 6.2 Proof of Theorem 3.3 and 3.6

**Theorem 6.5** (Re-statement of Theorem 3.3). Checking \( Q_1 \preceq Q_2 \) is decidable in exponential time when \( Q_2 \) is chordal and admits a simple junction tree.

**Theorem 6.6** (Re-statement of Theorem 3.6). Let \( Q_2 \) be chordal,

(i) If \( Q_2 \) admits a totally disconnected junction tree, then \( Q_1 \not\preceq Q_2 \) if and only if there is a product witness.

(ii) If \( Q_2 \) admits a simple junction tree, then \( Q_1 \not\preceq Q_2 \) if and only if there exists a normal witness.

In order to prove the above theorems, we need a technical lemma. In Theorem 4.7 we proved that, when \( Q_2 \) is acyclic and Eq. (15) fails, then \( Q_1 \not\preceq Q_2 \). Our next lemma is a variation of that result: when \( Q_2 \) is chordal and Eq. (13) fails on a normal entropic function, then \( Q_1 \not\preceq Q_2 \). Recall that a junction tree is a special tree decomposition.
Lemma 6.7. Let $Q_2$ be chordal and admit a simple junction tree $T$, and let $E_T$ be its linear expression, Eq.(12). If there exists a normal entropic function $h$ (i.e. with a non-negative I-measure) such that:

$$h(\text{vars}(Q_1)) > \max_{\varphi \in \text{hom}(Q_2, Q_1)} (E_T \circ \varphi)(h)$$

(57)

then there exists a database instance $D$ such that $|\text{hom}(Q_1, D)| > |\text{hom}(Q_2, D)|$.

We first show how to use the lemma and the essentially-Shannon inequalities in Theorem 3.12 to prove Theorems 3.3 and 3.6. Assume $Q_2$ is chordal and has a simple junction tree $T$. We prove: $Q_1 \preceq Q_2$ iff Eq. (13) holds. It suffices to prove that Eq. (13) is necessary, because sufficiency follows from Theorem 4.2. Suppose Eq. (13) fails. Then there exists an entropic function $h$ such that (57) holds where $T$ in (57) is a simple junction tree of $Q_2$. Since $T$ is simple, the conditional linear expressions on the right-hand-side of (57) are also simple. By Theorem 3.12 there exists a normal entropic function $h$ such that (57) holds. Then, by Lemma 6.7, $Q_1 \npreceq Q_2$. This proves that Eq. (13) is necessary and sufficient for containment.
Furthermore, Eq. (13) is decidable, since it is an essentially-Shannon inequality, and this completes the proof of Theorems 3.3. The proof of Theorem 3.6 follows immediately from the fact that the set of normal entropic functions \( \mathcal{N}_n \) is the cone generated by the entropies of normal relations, and the set of modular functions \( \mathcal{M}_n \) is the cone generated by the entropies of product relations.

It remains to prove Lemma 6.7: the lemma generalizes Theorem 3.2 of [21] to arbitrary vocabularies (beyond graphs). To prove the theorem, we will update the proof of Theorem 4.7 where we used acyclicity of \( Q_2 \): more precisely we need to re-prove the locality property, Eq. (32). We repeat it here:

\[
\text{hom}_{\varphi_{|x(t)}}(Q_t, D) \subseteq \Pi_{\varphi_{|x(t)}}(P)
\]

We start by observing that this property fails in general.

**Example 6.8.** Let \( Q_1 = R(X_1, X_2), S(X_2, X_3), T(X_3, X_1) \) and \( Q_2 = R(Y_1, Y_2), S(Y_2, Y_3), T(Y_3, Y_1) \) (they are identical). Consider the parity function in Example 3.8; more precisely, this is the entropy of the relation \( P = \{(X_1, X_2, X_3) \mid X_1, X_2, X_3 \in \{0, 1\}, X_1 \oplus X_2 \oplus X_3 = 0\} \), which we show here for clarity:

\[
P = 
\begin{bmatrix}
0 & 0 & 0 \\
0 & 1 & 1 \\
1 & 0 & 1 \\
1 & 1 & 0
\end{bmatrix}
\]

Recall that the entropy of \( P \) is not a normal entropic function (Sec. 6.1). This relation is perfectly uniform (in fact it is a group characterization). Computing \( D = \Pi_{Q_1}(P) \) we obtain \( R^D = S^D = T^D = \{(0, 0), (0, 1), (1, 0), (1, 1)\} \). \( Q_2 \) is a clique, with a bag \( Q_t = Q_2 \), and \( \text{hom}(Q_t, D) \) contains one extra triangle, \((1, 1, 1)\), which is in no single row of \( P \).

The example shows that we need to use in a critical way the fact that the counterexample \( h \) is a normal entropic function, \( h \in \mathcal{N}_n \). To use this fact, we will describe a class of relations whose entropic functions generate precisely the cone \( \mathcal{N}_n \), and prove that these are precisely the normal relations (Def. 3.5).

Before we start, we review a basic concept, which we call “domain-product”, first introduced by Fagin [9] to prove the existence of an Armstrong relation for constraints defined by Horn clauses, and later used by Geiger and Pearl [11] to prove that Conditional Independence constraints on probability distributions also admit an Armstrong relation. The same construction appears under the name “fibered product” in [21].

**Definition 6.9.** Fix two domains \( D_1, D_2 \). For any two tuples \( f \in D_1^V, g \in D_2^V \) we define \( f \odot g \in (D_1 \times D_2)^V \) as the function \((f \odot g)(x) \overset{\text{def}}{=} (f(x), g(x))\) for all \( x \in V \). The domain product of two relations \( P_1 \subseteq D_1^V, P_2 \subseteq D_2^V \) is \( P_1 \odot P_2 \overset{\text{def}}{=} \{ f \odot g \mid f \in P_1, g \in P_2 \} \). If \( p_1, p_2 \) are probability distributions on \( P_1, P_2 \) respectively, then their product \( p_1 \cdot p_2 \) is the probability distribution \((p_1 \cdot p_2)(f, g) \overset{\text{def}}{=} p_1(f) \cdot p_2(g)\) on \( P_1 \odot P_2 \).

The following basic fact relates to the above definition: if \( h_1 \) and \( h_2 \) are two entropic functions, then \( h_1 + h_2 \) is also entropic. In particular, if \( h_i \) is the entropy of \( p_i : P_i \to [0, 1] \), then \( h_1 + h_2 \) is the entropy of \( p_1 \cdot p_2 : P_1 \odot P_2 \to [0, 1] \), where \( P_1 \odot P_2 \) is the domain product.

Now we are ready to prove Lemma 6.7. Consider the normal entropic function \( h \) given by Lemma 6.7. We can assume w.l.o.g. that \( h \) is a sum of step functions \( 14 \), \( h = \sum_{i} h_{W_i} \), where each \( h_{W_i} \) is a step function (not necessarily distinct). Recall from Section 3.3 that \( P_{W_i} \) is the 2-tuple relation whose entropy is \( h_{W_i} \); to reduce clutter we denote here \( P_{W_i} \) by \( P_i \). Then \( h \) is the entropy of their domain-product (Def 6.9), \( P = P_1 \odot P_2 \odot \cdots \odot P_m \). One can check that \( P \) is totally uniform (it is even a group realization). We now prove the locality property, Eq. (32), using the fact that \( P \) is a domain product, which allows us to rewrite Eq. (32) as:

\[
\text{hom}_{\varphi_{|x(t)}}(Q_t, D_1 \odot \cdots \odot D_m) \subseteq \Pi_{\varphi_{|x(t)}}(P_1 \odot \cdots \odot P_m)
\]

\(14\) Suppose the contrary, that the inequality holds for all functions \( h \) that are sums of step functions. Then it holds for all linear combinations \( \sum_{W} c_W h_{W} \) where \( c_W \geq 0 \) are integer coefficients. If an inequality holds for \( h \), then it also holds for \( \lambda \cdot h \) for any constant \( \lambda > 0 \); it follows that the inequality holds for all linear combinations \( \sum_{W} c_W h_{W} \) where \( c_W \geq 0 \) are rationals. The topological closure of these expressions is \( \mathcal{N}_n \), contradicting the fact that the inequality fails on some \( h \in \mathcal{N}_n \).
It suffices prove that $\text{hom}_{\phi(x)}(Q_i, D_i) \subseteq \Pi_{\phi(x)}(P_i)$ for each $i$. Recall that $P_i$ has two tuples, $P_i = \{ f_1, f_2 \}$, where $f_1 = (1, 1, \ldots, 1)$ and $f_2$ has values 1 on positions $\in W$ and values 2 on positions $\notin W$, for some set of attributes $W$. Fix a tuple $g \in \text{hom}_{\phi(x)}(Q_i, D_i)$; we must prove that either $g \in \Pi_{\phi(x)}(f_1)$ or $g \in \Pi_{\phi(x)}(f_2)$. If $g$ maps every variable in $\text{vars}(Q_i)$ to 1, then the first condition holds, so assume that $g$ maps some variable $Y \in \text{vars}(Q_i)$ to 2; in particular, $\varphi(Y) \notin W$. We must prove that, for every variable $Y'$, if $\varphi(Y') \notin W$ then $g(Y') = 2$. Here we use the fact that $Q_2$ is chordal, hence $Q_2$ is a clique, thanks to Fact A.3. Therefore, there exists $B \in \text{atoms}(Q_2)$ that contains both $Y$ and $Y'$. Since $g$ is a homomorphism, it maps $B$ to some tuple in $\Pi_{\phi(x)}(\text{vars}(B))(P)$; since both $\varphi(Y), \varphi(Y') \notin W$, this tuple must have the value 2 on both positions (they can be identical: $\varphi(Y) = \varphi(Y')$). It follows that all variables $Y'$ s.t. $\varphi(Y') \notin W$ are mapped to 2, proving that $g \in \Pi_{\phi(x)}(f_2)$. This proves the local property, Eq.(32). The rest of the proof of Theorem 4.7 remains unchanged, and this completes the proof of Lemma 6.7.

7 Conclusion and Discussion

In this paper we established a fundamental connection between information inequalities and query containment under bag semantics. In particular, we proved that the max-information-inequality problem is many-one equivalent to the query containment where the containing query is acyclic. It is open whether these problems are decidable. Our results help in the sense that, progress on one of these open questions will immediately carry over to the other. We end with a discussion of our results and a list of open problems.

**Beyond Chordal** Our results showed that the query containment problem $Q_1 \preceq Q_2$ is equivalent to a Max-IIP when $Q_2$ is either acyclic, or when it is chordal and has a simple junction tree. In all other cases, condition (13) is only sufficient, and we do not know if it is also necessary.

**Repeated Variables, Unbounded Arities** Our reduction form Max-IIP to query containment constructs two queries $Q_1, Q_2$ where the atoms have repeated variables, and the arities of some of the relation names depend on the size of the Max-IIP. We leave open the question whether the reduction can be strengthened to atoms without repeated variables, and/or queries over vocabularies of bounded arity.

**Max-Linear Information Inequalities** Linear information inequalities have been studied extensively in the literature, while Max-linear ones much less. Our result proves the equivalence of BagCQC-A and Max-IIP, and this raises the question of whether IIP and Max-IIP are different. The following theorem (Appendix C) suggests that they might be computationally equivalent.

**Theorem 7.1.** Let $E_\ell$, $\ell = 1, m$ be linear expressions of entropic terms. Then the following conditions are equivalent:

- **This max-linear inequality holds:** $\forall h \in \Gamma_n^*, 0 \leq \max_\ell E_\ell(h)$.

- **There exists $\lambda_\ell \geq 0, s.t. \sum_\ell \lambda_\ell = 1$ and, denoting $E \text{ def } \sum_\ell \lambda_\ell E_\ell$, this linear inequality holds:** $\forall h \in \Gamma_n^*, 0 \leq E(h)$.

The second item implies the first, because $\max_\ell E_\ell \geq \sum_\ell \lambda_\ell E_\ell$; the proof of the other direction is in the appendix. Suppose we could strengthen the theorem and prove that the $\lambda$’s can be chosen to be rationals. Then there exists a simple Turing-reduction from the Max-IIP to IIP: given a Max-IIP, search in parallel for a counter example by iterating over all finite probability spaces, and for rational $\lambda$’s such that $\sum_\ell \lambda_\ell E_\ell(h) \geq 0$ (which can be checked using the IIP oracle). However, we do not know if the $\lambda$’s can always be chosen to be rational.

**The remarkable formula $E_T$ (Eq.(12))** The first to introduce the expression $E_T$ was Tony Lee [22]. This early paper established several fundamental connections between the entropy $h$ of the uniform distribution of a relation $P$, and constraints on $P$: it showed that an FD $X \rightarrow Y$ holds iff $h(Y|X) = 0$, that an MVD $X \rightarrow Y$ holds iff $I(Y; V - (X \cup Y)|X) = 0$, and, finally, that $P$ admits an acyclic join decomposition given by a tree $T$ iff $E_T(h) = h(V)$. It also proved that $E_T$ is equivalent to an inclusion-exclusion expression, which, in our notation becomes:

$$E_\ell = \sum_{S \subseteq \text{nodes}(T)} (-1)^{|S|+1} CC(T \cap S) \cdot h(\chi(S))$$

(58)
where \( \chi(S) \) \( \stackrel{\text{def}}{=} \bigcap_{t \in S} \chi(t) \), and \( CC(T \cap S) \) denotes the number of connected components of the subgraph of \( T \) consisting of the nodes \( \{ t \mid t \in \text{nodes}(T), \chi(t) \cap \bigcup_{t' \in S} \chi(t') \neq \emptyset \} \).

**Discussion of Kopparty and Rossman** [21] We now re-state the results in [21] using the notions introduced in this paper in order to describe their connection. Theorem 3.1 in [21] essentially states that Eq. (13) is sufficient for containment, thus it is a special case of our Theorem 4.2 for graph queries; they use an inclusion-exclusion formula for \( E_T \), similar to (58), but given for chordal queries only. Theorem 3.2 in [21] essentially states that, if Eq. (13) fails on a normal polymatroid, then there exists a database \( \mathcal{D} \) witnessing \( Q_1 \npreceq Q_2 \), thus it is a special case of our Lemma 6.7 for the case when the queries are graphs; they use a different expression for \( E_T \), based on the Möbius inversion of \( h \). This inversion is precisely the I-measure of \( h \), as we explain in Appendix B. Finally, Theorem 3.3 in [21] proves essentially that Eq. (13) is necessary and sufficient when \( Q_1 \) is series-parallel and \( Q_2 \) is chordal. This differs from our Theorem 3.3 in that it imposes more restrictions on \( Q_1 \) and fewer on \( Q_2 \). The proof of our Theorem 3.3 relies on the fact that any counterexample of Eq. (13) is a normal entropic function, but this does not hold in the setting of Theorem 3.3 [21]; however, the only exception is given by the parity function (Appendix B), a case that [21] handles directly.

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A Background on CQ’s

Lemma A.1. The containment problem under bag-set semantics $Q_1 \preceq Q_2$ is reducible in polynomial time to the containment problem under bag-set semantics for Boolean queries, $Q_1' \preceq Q_2'$. Moreover, this reduction preserves any property of queries discussed in this paper: acyclicity, chordality, simplicity.

Proof. Assume w.l.o.g. that $Q_1, Q_2$ have the same head variables $x$ (rename them otherwise). Define two Boolean queries $Q_1', Q_2'$ by adding new unary atoms $U_i(x_i)$ to $Q_1, Q_2$, one atom for each $x_i \in x$. We prove: $Q_1 \preceq Q_2 \iff Q_1' \preceq Q_2'$. For the $\Rightarrow$ direction, fix a database instance $D'$, denote the product of the unary relations by $U \defeq \prod U_i$, and let $D$ be obtained from $D'$ by removing the unary relations $U_i$. It follows that $\bigcup_{d \in U} Q_1[d](D) = \text{hom}(Q_1, D')$, for $\ell = 1, 2$. Since $Q_1 \preceq Q_2$, and the sets $Q[d](D), d \in U$ are disjoint, for $\ell = 1, 2$, we conclude $|\text{hom}(Q_1, D')|= \sum_{d \in U} |Q_1[d](D)| \leq \sum_{d \in U} |Q_2[d](D)| = |\text{hom}(Q_2, D')|$. For the $\Leftarrow$ direction, let $D$ be a database instance, and let $d \in D^x$. Define $D'$ to be the database obtained by adding to $D$ unary relations with one element, $U_i \defeq \{d_i\}$ for each $x_i \in x$. Then, $Q[d](D) = \text{hom}(Q, D')$ for $\ell = 1, 2$. By assumption $Q_1' \preceq Q_2'$, which implies $|Q_1[d](D)| = |\text{hom}(Q_1, D')| \leq |\text{hom}(Q_2, D')| = |Q_2[d](D)|$. \qed

Example A.2. We illustrate with this example from [8]:

\begin{align*}
Q_1(x, z) &= P(x) \land S(u, x) \land S(v, z) \land R(z) \\
Q_2(x, z) &= P(x) \land S(u, y) \land S(v, y) \land R(z)
\end{align*}

We associate them to the following two Boolean queries:

\begin{align*}
Q_1'() &= P(x) \land S(u, x) \land S(v, z) \land R(z) \land U_1(x) \land U_2(z) \\
Q_2'() &= P(x) \land S(u, y) \land S(v, y) \land R(z) \land U_1(x) \land U_2(z)
\end{align*}

Then $Q_1 \preceq Q_2$ iff $Q_1' \preceq Q_2'$; the latter can be shown using Theorems 4.2 and 4.7.

We prove now a claim that we made in Sec 4.4, namely that, for any node $t$ of a tree decomposition, we can assume $\text{vars}(Q_t) = \chi(t)$, where $Q_t$ is the query obtained by taking the conjunction of all atoms with $\text{vars}(A) \subseteq \chi(t)$.

Fact A.3. (Informal) Let $(T, \chi)$ be a tree decomposition of some query $Q$, and for all $t \in \text{nodes}(T)$, let $Q_t$ denote the conjunction of $A \in \text{atoms}(Q)$ s.t. $\text{vars}(A) \subseteq \chi(t)$. Then, for the purpose of query containment, we can assume that $\text{vars}(Q_t) = \chi(t)$, for every $t \in \text{nodes}(T)$. More specifically, we can assume that for every $t \in \text{nodes}(T)$ and every $A \in \text{atoms}(Q)$ such that $\text{vars}(A) \cap \chi(t) \neq \emptyset$, there exists $A' \in \text{atoms}(Q)$ such that $\text{vars}(A') = \text{vars}(A) \cap \chi(t)$, hence $A' \in \text{atoms}(Q_t)$.

Proof. To see an example where this property fails, consider $Q = R(x, y, z) \land S(y, z) \land R(x, z, v)$. Let $T$ be the tree decomposition $\{x, y, u\} - \{x, y, z\} - \{x, z, v\}$, and let $t$ be the middle node, $\chi(t) = \{x, y, z\}$. Then $Q_t = S(y, z)$ and its variables do not cover $\chi(t)$.

We prove that the property can be satisfied w.l.o.g. We first modify the vocabulary, by adding for each relation name $R$ of arity $a$ and for each $S \subset [a]$, a new relation name $R_S$ of arity $|S|$. Similarly, we modify a query $Q$ by adding, for each atom $R(X_1, \ldots, X_a)$ and for each $S \subset [a]$, a new atom $R_S(x_S)$, where $x_S \defeq (X_i)_{i \in S}$. Denote by $\tilde{Q}$ the modified query. Obviously $\tilde{Q}$ satisfies the desired property. We claim that this change does not affect query containment, more precisely $Q_1 \preceq Q_2 \iff \tilde{Q}_1 \preceq \tilde{Q}_2$. The $\Rightarrow$ direction follows by expanding an input database $D$ for $Q_1, Q_2$ with extra predicates $R_S^D \defeq \Pi_S(R^D)$ for every relation symbol $R$ and every $S \subset [a]$ where $a$ is the arity of $R$. The $\Leftarrow$ direction follows from modifying an input database $D$ for $Q_1, Q_2$ by replacing every $(a, a)$ relation $R^D$ by $R^D \Join (\mathcal{M}_{S \subset [a]} R_S^D)$. \qed

B Background on Information Theory

In this section, we review some additional background in information theory used in this paper, continuing the brief introduction in Sec 3.3.
Fact B.1. If \( n = 1 \) (i.e. there is a single random variable) and \( h \) is entropic, then \( c \cdot h \) is also entropic for every \( c > 0 \).

Proof. Start with a distribution \( p \) whose entropy is \([c] \cdot h\). Let \( n \) be the number of outcomes, and \( p_1, \ldots, p_n \) their probabilities. For each \( \lambda \in [0,1] \) define \( p^{(\lambda)} \) to be the distribution \( p^{(\lambda)}_i = p_i + (1-p_i)(1-\lambda) \), \( p^{(\lambda)}_1 = p_1 \cdot \lambda \) for \( i > 1 \), and \( h^{(\lambda)} \) its entropy. Then \( h^{(0)} = 0 \), \( h^{(1)} = [c] \cdot h \), and, by continuity, there exists \( \lambda \) s.t. \( h^{(\lambda)} = c \cdot h \). \( \square \)

Corollary B.2. For every \( W \subseteq V \) and every \( c > 0 \), the function \( c \cdot h_W \) is entropic, where \( h_W \) is the step function. It follows that every normal function is entropic (because it is a sum \( \sum_W c \cdot h(W) \) and \( c \cdot h(W) \) is entropic).

Proof. By the previous fact, there exists a random variable \( Z \) whose entropy is \( h_0(Z) = c \). Let \( h \) be the entropy of the following \( n \) random variables: for all \( U \in V - W \), define \( U \overset{\text{def}}{=} Z \) (hence, forall \( X \subseteq V - W \), \( h(X) = h_0(Z) = c \)), and for every \( U \in W \), define \( U \) to be a constant (hence for every \( X \subseteq W \), \( h(X) = 0 \)). Therefore, \( h = c \cdot h_W \).

However, when \( n \geq 3 \), then Zhang and Yeung \cite{31} proved that \( c \cdot h \) is not necessarily entropic. Their proof is based on the parity function, introduced in Example 3.3.

Fact B.3. \( \Gamma_n^* \) is not convex.

Proof. Zhang and Yeung \cite{31} prove this fact as follows. Let \( h \) be the entropy of the parity function in Example 3.3. For every \( c > 0 \), consider the function \( h' = c \cdot h \). They prove that \( h' \) is entropic iff \( c = \log M \), for some integer \( M \), which implies that \( \Gamma_n^* \) is not convex. We include here their proof for completeness. Assuming \( h' \) is entropic let \( p' \) be its probability distribution, then the following independence constraints hold: \( X \perp Y \), because \( h'(XY) = h'(X) + h'(Y) \), and similarly \( X \perp Z \) and \( Y \perp Z \). The following functional dependencies also hold: \( XY \rightarrow Z \) (because \( h'(XY) = h'(XYZ) \)) and similarly \( XZ \rightarrow Y \), \( YZ \rightarrow X \). Let \( x,y,z \) be any three values s.t. \( p'(x,y,z) > 0 \). Then \( p'(x,y,z) = p'(x,y) = p'(x)p'(y) \). Similarly \( p'(x,y,z) = p'(y)p'(z) \), which implies \( p'(x) = p'(z) \). Therefore, for any other value \( x' \), \( p'(x') = p'(z) \). This means that the variable \( X \) is uniformly distributed, because \( p'(x) = p'(x') \) forall \( x, x' \), hence \( p'(x) = 1/M \) where \( M \) is the size of the domain of \( X \). It follows that \( h'(X) = \log M \), proving the claim. \( \square \)

Yeung \cite{29} proves that the topological closure \( \bar{\Gamma}_n^* \) is a convex set, for every \( n \). Thus, \( \Gamma_n^* \subseteq \bar{\Gamma}_n^* \) and the inclusion is strict for \( n \geq 3 \). The elements of \( \bar{\Gamma}_n^* \) are called almost entropic functions. We note that if a linear information inequality, or a max-linear information inequality is valid for all entropic functions \( h \in \Gamma_n^* \), then, by continuity, it is also valid for all almost entropic functions \( h \in \bar{\Gamma}_n^* \).

Let \( h \) be an entropic function, and \( X, Y \subseteq V \) two sets of variables. For every outcome \( X = x \), we denote by \( h(Y|X = x) \) the entropy of \( Y \) conditioned on \( X = x \). The function \( Y \mapsto h(Y|X = x) \) is an entropic function (by definition). Recall that we have defined \( h(Y|X) \overset{\text{def}}{=} h(XY) - h(X) \). It can be shown by direct calculation that \( h(Y|X) = \sum_x h(Y|X = x) \cdot p(X = x) \), in other words it is a convex combination of entropic functions. Thus, \( h(Y|X) \) is the expectation, over the outcomes \( x \), of \( h(Y|X = x) \), justifying the name “conditional entropy”.

Fact B.4. In general, the mapping \( Y \mapsto h(Y|X) \) is not entropic.

Proof. To see an example, consider two probability spaces on \( X, Y, Z \), with probabilities \( p, p' \) and entropies \( h, h' \) such that \( h \) is the entropy of the parity (Example 3.3) and \( h' = 2h \). Consider a 4'th variable \( U \), whose outcomes are \( U = 0 \) or \( U = 1 \) with probabilities \( 1/2 \), and consider the mixture model: if \( U = 0 \) then sample \( X, Y, Z \) using \( p \), if \( U = 1 \) then sample \( X, Y, Z \) using \( p' \). Let \( h'' \) be the entropy over the variables \( X, Y, Z, U \). Then the conditional entropy \( h''(W|U) = 3/2h(W) \), for all \( W \subseteq \{X, Y, Z\} \), and thus it is not entropic. \( \square \)

Yeung \cite{29} defines the I-measure as follows. Fix a set of variables \( V \), which we identify with \([n]\). Let \( \Omega = 2^{[n]} - \{\emptyset\} \). An I-measure is any function \( \mu : 2^\Omega \rightarrow \mathbb{R} \) such that \( \mu(X \cup Y) = \mu(X) + \mu(Y) \) whenever \( X \cap Y = \emptyset \).
Notice that $\mu$ is not necessarily positive. For each variable $V_i \in V$ we denote by $\hat{\mathcal{V}}_i \eqdef \{ \omega \in \Omega \mid i \in \omega \} \subseteq \Omega$, and extend this notation to sets $X \subseteq V$ by setting $\hat{X} \eqdef \bigcup_{V_i \in X} \hat{\mathcal{V}}_i$. For each variable $V_i$ denote $\hat{V}_i^{-1} \eqdef \hat{\mathcal{V}}_i$ and $\hat{V}_i^{0} \eqdef \text{the complement of } \hat{V}_i$. An atomic cell is an intersection $C \eqdef \bigcap_{j=1}^{n} \hat{V}_j^{\epsilon_j}$, where $\epsilon_j \in \{0,1\}$ for all $j$, where at least one $\epsilon_j = 1$. Obviously, $\mu$ is uniquely defined by its values on the atomic cells.

Given $h \in \mathbb{R}^2$ (not necessarily entropic), the I-measure associated to $h$ is the unique $\mu$ satisfying the following, for all $X \subseteq V$:

$$h(X) = \sum_{C : C \subseteq \hat{X}} \mu(C) \tag{59}$$

The normal entropic functions $\mathcal{N}_n$ are precisely those with a non-negative I-measure. This can be seen immediately by observing that, for any step function $h_W$, it’s I-measure $\mu_W$ assigns the value 1 to the cell $(\bigcap_{V \in W} V^1) \cap (\bigcap_{V \in W} V^0)$, and 0 to everything else. In fact, there is a tight connection between the I-measure $\mu$ and the Möbius inverse function $g$ (Eq. (48) in Sec. 6.1), which we explain next. First, we notice that Equation (48) implies:

$$h(X) = -\sum_{Y : Y \not\subseteq X} g(Y) \tag{60}$$

The connection between $\mu$ and $g$ follows by a careful inspection of Eq. (59) and Eq (60). Each atomic cell $C$ in Eq. (59) is uniquely defined by the set of its negatively occurring variables, denote this by $\text{neg}(C)$. Then, $C \subseteq \hat{X}$ iff $X \not\subseteq \text{neg}(C)$. Define the function $g : 2^V \to \mathbb{R}$ as $g(\text{neg}(C)) \eqdef -\mu(C)$ and $g(V) = h(V)$ (recall that $\text{neg}(C) \neq V$). Then Eq.(59) becomes $h(X) = \sum_{C : X \not\subseteq \text{neg}(C)} \mu(C) = -\sum_{Y : Y \not\subseteq X} g(Y)$ which is precisely Eq. (60).

We end our background with a proof that the Max-IIP problem is co-recursively enumerable. Recall that a set $A \subseteq \mathbb{Z}^k$ is called recursively enumerable, or r.e., if there exists a Turning computable function $f$ whose image is $A$. Equivalently, there exists a computable function that, given $x \in \mathbb{Z}^k$ returns “true” if $x \in A$ and does not terminate if $x \not\in A$. The set $A$ is called co-recursively enumerable, or co-r.e., if its complement is r.e.

**Lemma B.5.** Max-IIP is co-r.e.

**Proof.** (Sketch) Enumerate all finite probability distributions where the probabilities are given by rational numbers, and check Eq. (6) on each of them. This is possible because each entropy value $h(X)$ is the log of a number of the form $\prod_i \left( \frac{1}{p_i^{(X)}} \right)^{p_i^{(X)}}$, where $i$ ranges over all possible assignments of the variable set $X$, and $p_i^{(X)}$ is the probability that $X$ takes the $i$-th assignment. Therefore the inequality becomes

$$\exists \ell \in [k] \text{ s.t. } \prod_{X \subseteq \ell} \prod_{i} \left( \frac{1}{p_i^{(X)}} \right)^{c_{\ell,X} p_i^{(X)}} \geq 1.$$

In the above, $c_{\ell,X}$ are integers while $p_i^{(X)}$ are rational numbers. We can raise both sides of the above inequality to a power of $d$, which is the common denominator among all $p_i^{(X)}$. If the inequality fails, then return “false”, otherwise continue with the next finite probability distribution.

**C  Proof of Theorem 7.1**

We note that we can replace $\Gamma_n^{*}$ in the statement of Theorem 7.1 by $\hat{\Gamma}_n^{*}$, because any max-information inequality holds on $\Gamma_n^{*}$ iff it holds on $\hat{\Gamma}_n^{*}$. We will then prove that the theorem holds more generally, for any closed, convex cone $K$; the claim follows from the fact that $K = \hat{\Gamma}_n^{*}$ is a closed, convex cone.

Recall that a cone is a subset $K \subseteq \mathbb{R}^N$ such that $x \in K$ implies $c \cdot x \in K$ for all $c \geq 0$. All four sets $\mathcal{M}_n, \hat{\mathcal{N}}_n, \hat{\Gamma}_n, \Gamma_n$ defined in Sec. 3.3 are closed, convex cones. We prove:
Theorem C.1. Let $K \subseteq \mathbb{R}^N$ be a closed, convex cone, and let $y_1, \ldots, y_m \in \mathbb{R}^N$. Then the following two conditions are equivalent:

1. $\forall x \in K : \max_i \langle x, y_i \rangle \geq 0$.
2. There exist $\lambda_1, \ldots, \lambda_m \geq 0$ such that $\sum_i \lambda_i = 1$ and $\forall x \in K : \sum_i \lambda_i \langle x, y_i \rangle \geq 0$. Equivalently, $\sum_i \lambda_i y_i \in K^*$ (the dual of $K$).

Notice that the coefficients $\lambda_i$ need not necessarily be rational numbers. The theorem says that every Max-IIP can be reduced to an IIP with, possibly irrational coefficients.

Proof. Obviously (2) implies (1) because $\max_i \langle x, y_i \rangle \geq \sum_i \lambda_i \langle x, y_i \rangle \geq 0$. We will prove that (1) implies (2).

First, we prove that (1) implies (2) when $K$ is a finitely generated cone: $K = \{ x \mid Ax \geq 0 \}$ for some $P \times N$ matrix $A$. Condition (1) implies that the following optimization problem has a value $\geq 0$:

$$\min \max_i \langle x, y_i \rangle$$

where: $Ax \geq 0$

$x \in \mathbb{R}^N$

This optimization problem is equivalent to the following, where $x_0$ is a fresh variable, and $B$ is the $m \times N$ matrix whose rows are the vectors $y_1, \ldots, y_m$:

$$\min x_0$$

where: $Ax \geq 0$

$$\begin{bmatrix} x_0 \\ \vdots \\ x_0 \end{bmatrix} - Bx \geq 0$$

This is a linear optimization problem whose solution is equal to that of the dual, which is a linear program over variables $\mu_1, \ldots, \mu_P, \lambda_1, \ldots, \lambda_m$:

$$\max 0$$

where: $\lambda_1 + \ldots + \lambda_m = 1$

$$\mu^i A - \lambda^i B = 0$$

$x_0$

$x_1, \ldots, x_N$

$\lambda \geq 0, \mu \geq 0$

Since the optimal solution of the primal is $\geq 0$, the dual must have a feasible solution $\lambda, \mu$. To prove Condition (1), assume $x \in K$. Then $Ax \geq 0$, therefore $\mu^i A x \geq 0$, thus $\lambda^i B x = \langle \sum_i \lambda_i y_i, x \rangle \geq 0$ proving the theorem for the case when $K$ is a finitely generated cone.

We prove now the general case. Let $K' = K \cap \mathbb{Q}$ be the vectors in $K$ with rational coordinates, and let $K' = \{ x_1, x_2, \ldots, x_n, \ldots \}$ be an enumeration of $K'$. For each $n \geq 0$, let $K_n \subseteq K$ be the closed, convex cone generated by $\{ x_1, \ldots, x_n \}$. Let $\Lambda_n \subseteq \mathbb{R}^m$ be the set of all vectors $\lambda$ satisfying Condition (1) for the cone $K_n$. Since $K_n$ is finitely generated, we have $\Lambda_n \neq \emptyset$. Furthermore it is easy to check that $\Lambda_n$ is topologically closed. Since $\Lambda_n$ is bounded, it follows that $\Lambda_n$ is a compact subset of $\mathbb{R}^N$. Since $K_1 \subseteq K_2 \subseteq \ldots \subseteq K_n \subseteq \ldots$ it follows that $\Lambda_1 \supseteq \ldots \supseteq \Lambda_n \supseteq \ldots$ This implies that any finite family has a nonempty intersection: $\Lambda_{n_1} \cap \ldots \cap \Lambda_{n_s} = \Lambda_{\max(n_1, \ldots, n_s)} \neq \emptyset$. It follows that the entire family has a non-empty intersection, i.e. there exists $\lambda \in \bigcap_{n \geq 0} \Lambda_n$. We prove that $\lambda$ satisfies Condition (1). Indeed, let $x \in K$, and consider any sequence $(x_n)_{n \geq 0}$ such that $x_n \in K_n$ and $\lim_{n} x_n = x$. For all $n \geq 0$, $\lambda \in \Lambda_n$, which implies $\sum_i \lambda_i \langle x_n, y_i \rangle \geq 0$, therefore $\sum_i \lambda_i < x, y_i >= \lim_{n} \sum_i \lambda_i < x_n, y_i >\geq 0$ proving the claim. \qed