Complex structure and characterization of multi-photon split states in integrated circuits

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Multi-photon split states, where each photon is in a different spatial mode, represent an essential resource for various quantum applications, yet their efficient characterization remains an open problem. Here, we formulate the general structure of their reduced spatial density matrices and identify the number of real and complex-valued independent coefficients, which in particular completely determine the distinguishability of all photons. Then, we show that this density matrix can be fully characterized by measuring correlations after photon interference in a static integrated circuit, where the required outputs scale sub-quadratically versus the number of photons. We present optimized circuit designs composed of segmented coupled waveguides, representing a linear optical neural network, which minimize the reconstruction error and facilitate robustness to fabrication deviations.

I. INTRODUCTION

The multi-photon split states (M-PSS), where each photon is in a different spatial mode of free-space beams, fibers, or waveguides, represent an essential resource for fundamental tests of quantum mechanics and various applications in quantum simulations and computations \cite{1}. For example, many photon interference and distinguishability experiments are based on M-PSSs where each photon is injected from a different spatial port \cite{2–8}. Furthermore, photon boson sampling experiments that can demonstrating quantum advantage commonly employ PSS sources that provide multiple indistinguishable photons with each of the photons injected to a different port of a linear optical network \cite{9–14}. Remarkably, PSSs with more than two photons were recently shown to possess a multi-photon collective phase beyond the real-valued pair-wise photon distinguishability measure, opening new degrees of freedom for quantum information \cite{4, 15, 16}. Therefore, the characterisation of PSSs is of great importance from fundamental and practical perspectives.

Importantly, the indistinguishability of all photons in M-PSS cannot be inferred from the distinguishability of constituent photon pairs, which demands the characterisation based on multi-photon interference. Efficient protocols for witnessing multiphoton indistinguishability were developed and realized with reconfigurable multiport interferometers \cite{7} under the assumption of a specific density matrix form, whereas this remains an open problem for general states (see a discussion in the Supplementary of Ref. \cite{6}).

Beyond the quantification of indistinguishability, the measurement of the full density matrix can provide comprehensive information about the state, including the collective photon phase. This is typically done by a process called quantum state tomography, where the density matrix of the input quantum state is reconstructed after a series of projection measurements \cite{17–19}. To fully reconstruct the density matrix, the number of distinct measurements which are in the form of multi-photon correlations should exceed the number of free parameters in the density matrix that increases exponentially with the number of photons \cite{20}. To satisfy this requirement, conventional tomography approaches are based on free-space setups or integrated circuits that are reconfigured multiple times \cite{7, 21, 22}, yet the reconfiguration can be a source of experimental inaccuracies and also make the characterization time-consuming for larger numbers of photons. On the other hand, static tomography approaches have been suggested \cite{23–27} and realized experimentally \cite{28, 29}, where the measurements at the outputs of a fixed photonic circuit enable the full state reconstruction. However, such methods have been developed for general states without taking into account the specific structure of PSSs. It remained an outstanding question of how to perform optimal characterization of PSSs with the minimum number of measurements, high robustness to fabrication inaccuracy, and measurement noise while using the most compact and practical photonic circuit design.

In this work, we formulate the general properties of the spatial density matrix structure for the PSSs without any assumptions. Then, we present a scalable approach for single-shot complete state measurement with a static integrated photonic circuit, without a need for reconfigurability. Specifically, we first theoretically derive the number of free parameters and the structure of the reduced spatial density matrix as a function of the number of photons. Furthermore, we obtain the number of free real and imaginary parts, in which the imaginary values of the density matrix are associated with the multi-photon collective phases. To realize the state tomog-
experiments can realize multi-photon interference that does not explicitly depend on the internal structure of photons \[4, 28, 29\], and such transformations are mathematically described by a unitary operator that mixes different internal structure, such as frequency spectra, the conventional detectors only register the arrival time of the photons, the proposed scheme can realize the tomography in a single shot without reconfigurability. When compared with previous static approaches developed for general states, the performance is better and the complexity of the photonic circuit is reduced. This makes the proposed scheme scalable to larger photon numbers.

The paper is organized as follows. In Sec. II, we formulate the general structure of the reduced spatial density matrix for PSSs after tracing out the internal spectral degree of freedom and determine the numbers of independent real- and complex-valued coefficients as a function of the number of photons. In the following Sec. III, we introduce a circuit design based on coupled waveguides, representing a linear artificial neural network, and describe its application for split-state tomography. Then, in Sec. IV, we present the circuit designs for two-, three- and four-photon split states, optimized for accurate state reconstruction in presence of measurement noise or fabrication imperfections. Finally, we present conclusions and outlook in Sec. V.

### II. MULTI-PHOTON SPLIT STATES AND THE SPATIAL DENSITY MATRIX

We define a multi-photon split state formed by \( N \) photons where each photon is located on a different spatial path with orthogonal states labeled by \( |0\rangle, |1\rangle, \cdots, |N-1\rangle \), as shown in Fig. 1(a). The frequency dependent wavefunction of such a PSS can be expressed as

\[
|\Psi\rangle = \int d\omega_0 d\omega_1 \cdots d\omega_{N-1} \psi(\omega_0, \omega_1, \cdots, \omega_{N-1}) a_0^\dagger(\omega_0) a_1^\dagger(\omega_1) \cdots a_{N-1}^\dagger(\omega_{N-1}) |0\rangle, \tag{1}
\]

where \( \psi(\omega_0, \omega_1, \cdots, \omega_{N-1}) \) is the joint spatial and spectral distribution of \( N \) photons. Note that the \( |0\rangle \) in Eq. (1) stands for the photon vacuum state instead of the spatial mode of the first path mentioned earlier.

We consider the setups using commonly available single-photon click detectors. The coincidences of signals from several detectors then provide a measure of the photon correlations. While the photons might have different internal structure, such as frequency spectra, the conventional detectors only register the arrival time within specific time bins. Such detection and correlation measurements in photonic circuits do not distinguish the photons by their spectral properties \[22, 30\]. Also, experiments can realize multi-photon interference that does not explicitly depend on the internal structure of photons \[4, 16, 28, 29\], and such transformations are mathematically described by a unitary operator that mixes different input ports but does not depend on frequency spectra of photons \[15, 31\]. In addition, the single-photon click detectors cannot resolve the number of photons that arrived on the detector.

For the case where the experimental detectors do not distinguish the photons by their spectrum, a PSS can be characterized by a reduced density matrix, where the internal spectrum degree of freedom of the photons is traced out via integration. In the reduced spatial density matrix, which has a dimension of \( N^N \times N^N \), each element is determined by \[28\]:

\[
\rho_{s'_0, s'_1 \cdots s'_{N-1}; s_0, s_1 \cdots s_{N-1}} = \text{Tr}\left(\hat{\rho} \hat{O}_{s'_0, s'_1 \cdots s'_{N-1}; s_0, s_1 \cdots s_{N-1}}\right), \tag{2}
\]

where \( \hat{\rho} = |\Psi\rangle \langle \Psi| \) is the full density matrix, and the \( N \)-photon density matrix projection operator is defined as

\[
\hat{O}_{s'_0, s'_1 \cdots s'_{N-1}; s_0, s_1 \cdots s_{N-1}} = \frac{1}{N!} \int d\omega_0 d\omega_1 \cdots d\omega_{N-1} \hat{a}_{s'_0}^\dagger(\omega_0) \hat{a}_{s'_1}^\dagger(\omega_1) \cdots \hat{a}_{s'_{N-1}}^\dagger(\omega_{N-1}) |0\rangle \\
\langle 0| \hat{a}_{s_0}(\omega_0) \hat{a}_{s_1}(\omega_1) \cdots \hat{a}_{s_{N-1}}(\omega_{N-1}). \tag{3}
\]

For a split state, the nonzero density matrix elements can only be associated with indices \((s'_0, s'_1, \cdots, s'_{N-1})\) and \((s_0, s_1, \cdots, s_{N-1})\) that are permutations in the set \(\{0, 1, \cdots, N-1\}\) without repetitions. We also note that
the reduced density matrix is invariant under the simultaneous exchange of indices $s_i' \leftrightarrow s_j'$ and $s_i \leftrightarrow s_j$ for arbitrary $i$ and $j$, since the photons are indistinguishable after the internal spectrum degree of freedom is traced out. Using this property, we can map all the elements to just the first row of the density matrix with elements $ho_{0,1}, \cdots, N-1, s_0, s_1, \cdots, s_{N-1}$, where $(s_0', s_1', \cdots, s_{N-1}') = (0,1, \cdots, N-1)$. Therefore, the number of nonzero and independent elements of the spatial density matrix is the number of permutations of $(s_0, s_1, \cdots, s_{N-1})$ in the set $(0,1, \cdots, N-1)$ without repetition, which is $N!$.

Within this formulation, we can associate the appearance of collective multi-photon phase [15] with the presence of complex-valued density matrix elements. Since the ordering of $|0\rangle\langle s_0|, |1\rangle\langle s_1|, \cdots, |N-1\rangle\langle s_{N-1}|$ doesn’t affect the values of density matrix elements, we have

$$
\rho_{0,1, \cdots, N-1, s_0, s_1, \cdots, s_{N-1}} = \rho_{q_0, q_1, \cdots, q_{N-1}; 0,1, \cdots, N-1} = \rho_{0,1, \cdots, N-1, q_0, q_1, \cdots, q_{N-1}},
$$

(4)

where $(s_0, s_1, \cdots, s_{N-1})$ is reordered into $(0,1, \cdots, N-1)$ and $(q_0, q_1, \cdots, q_{N-1})$ is the new order from $(0,1, \cdots, N-1)$ after the same permutation operation. When $(q_0, q_1, \cdots, q_{N-1}) \equiv (s_0, s_1, \cdots, s_{N-1})$, then it follows from Eq. (4) that $\rho_{0,1, \cdots, N-1, s_0, s_1, \cdots, s_{N-1}} = \rho_{0,1, \cdots, N-1; s_0, s_1, \cdots, s_{N-1}}$, i.e. this element is real-valued.

Let us denote the number of such cases by $A_N$. The remaining $N! - A_N$ elements will have complex values and include $(N! - A_N)/2$ complex-conjugate pairs. Correspondingly, the number of independent real and imaginary parts of the density matrix are $(N! + A_N)/2$ and
approach \[23–29\]. We consider an states can be performed by adopting a static tomography linearly up to five-photon states, \( M \) the required minimum number of waveguides are above the noise level \[35\], which can be satisfied experimentally for at least \( N = 5 \) photons \[40\]. Essentially, the configuration in Fig. 2(a) represents a linear artificial neural-network \[41\] with \( S − 1 \) hidden layers, where each hidden layer has \( M \) neurons. The waveguide couplings in each section function as the weights and the local phase shifts play similar roles to the bias.

The overall linear system transformation of the multi-photon state, provided that the losses are negligible, can be determined by a classical or one-photon unitary transfer matrix

\[
U = W_S \mathbf{B}_S \mathbf{B}_S^{-1} \cdots \mathbf{B}_2 \mathbf{W}_2 \mathbf{B}_1 \mathbf{W}_1.
\]  

Here \( \mathbf{W}_j = \exp(i\mathbf{C} \mathbf{L}_j) \), calculated through the matrix exponential, is the weight matrix of the layer \( j \), where the coupling matrix elements are \( \mathbf{C}_{n,m} = \kappa \delta_{n,m±1} \). The bias matrix acting on the \( j \)-th hidden layer is \( \mathbf{B}_j = \exp(i\Phi_j) \), where the exponent is applied element-wise to the phase shift matrix \( \Phi_j = \text{diag}(\varphi_{1,j}, \varphi_{2,j}, \ldots, \varphi_{M,j}) \). The \( M \times M \) unitary matrix \( \mathbf{U} \) can be flexibly tuned by varying the lengths of sections and local phase shifts.

For an \( N \)-photon split state, where the photons are coupled to specific \( N \) ports at the input, its transformation is governed by \( N\text{-in}-M\text{-out} \) matrix \( \mathbf{U}_r \), which contains \( N \) columns of \( \mathbf{U} \) corresponding to the selected inputs. Then, we use a standard procedure \[28, 29\] to calculate the output \( N \)-photon correlations, which values can be represented by a vector \( \mathbf{\bar{r}} \) of the length equal to different combinations \( M!/[N!(M − N)!] \). The correlations can be expressed through the independent elements of the input density matrix arranged in a vector \( \bar{\rho}_{\text{free}} \) of length \( N! \),

\[
\mathbf{\bar{r}} = \mathbf{T} \bar{\rho}_{\text{free}},
\]
where the matrix $T$ is determined by $U_r$ and the structure of the split state density matrix.

Finally, we can reconstruct the input density matrix based on the measured $N$-photon correlations as

$$\tilde{\rho}_{\text{free}} = T^+ \Gamma,$$

where $T^+$ is the pseudoinverse of $T$.

It is essential to optimize the integrated circuit to reduce the reconstruction sensitivity to noise and measurement errors. Mathematically, this is achieved by minimizing the condition number of the matrix $T$, defined as the ratio of the transformation’s maximum and minimum singular values $\sigma_{\text{max}}(T)/\sigma_{\text{min}}(T)$ [42]. For our structure design, we perform numerical optimization of the waveguide section lengths and the local phase shifts based on the Nelder-Mead simplex direct search algorithm realized using the fminsearch function in Matlab.

**IV. RESULTS AND DISCUSSIONS**

We performed extensive simulations of coupled-waveguide neural networks and found that tomography of split states with the photon number at least up to four can be efficiently performed in structures where all sections have the same length, all waveguides have the same propagation constants and thus zero detunings, and all the near-neighbour waveguide couplings are equal to each other. These conditions make the photonic circuit design and fabrication simpler, where all the waveguides have the same widths and the spacings between them are identical. We show in the following that optimization of the local phase shifts and the total waveguide length ($L$) allows us to reach low condition numbers, corresponding to low sensitivity to noise during the reconstruction. To simplify the notations, we consider the scaling of the waveguide length in the units of $\kappa^{-1}$, such that the coupling coefficient is normalized to one.

We first analyze the tomography of two-PSS. We choose the minimum required number of $M = 3$ waveguides according to Eq. (5) and Fig. 2(b), select the first and third waveguides as the input ports, and consider a circuit structure with one hidden layer ($S = 2$) as sketched in Fig. 3(a). We perform the optimization for different waveguide lengths and show the best condition number values in Fig. 3(b). One can see that the condition number reaches a minimum value of $\simeq 2.3$ when the waveguide length is longer than 0.84. This minimized condition number is smaller than the previously reported values for tomography of general two-photon states [26, 28]. The corresponding optimized phase shifts at the hidden layer are shown in Fig. 3(c), where we assign zero to one of the phases since the global phase does not affect the output correlations. Interestingly, all the three phase shifts are zero for the waveguide length shorter than 0.84, which effectively corresponds to the absence of hidden layer. For longer waveguides, the minimum value of condition number is achieved for circuits with an optimal hidden layer. For comparison, Fig. 3(d) shows the condition number for a structure without a hidden layer. We see that the circuit can allow for optimal performance over a broad range of structure lengths, offering more flexibility in integrating with other photonic components.

Next, we investigate the three-PSS tomography. Then, we use Eq. (5) to determine the required number of waveguides as $M = 5$ and choose the first, third, and fifth waveguides as the input ports, see an illustration in Fig. 4(a). We check that without hidden layers, the condition number is very high, which would prevent a state reconstruction. The condition number dependencies on the structure length with the optimized one or two hidden layers are presented in Fig. 4(b). Overall, for a certain length, more hidden layers can provide lower condition numbers due to more tuning parameters. The smallest condition numbers are $\simeq 4.1$ at $L = 2.5$ and $\simeq 3.9$. 

![Diagram of M-port coupled waveguide neural network](image-url)
Variation of phase (\(\phi\)) at \(L = 3\) for one and two hidden layers, respectively. These values are much smaller than the ones for general three-photon states \([43]\). The corresponding optimized phase shifts are \(\phi_{1,1} = (0, 1.083, 1.167, 0.973, 5.509)\) for one hidden layer and \(\phi_{j,1} = (0, 4.248, 3.808, 1.442, 5.098)\), \(\phi_{j,2} = (0, 1.844, 1.948, 2.988, 4.155)\) for two hidden layers.

We confirm the practicality of the designs by quantifying the tolerance of the optimal structures for three-PSS tomography to variations of the phase shifts due to potential fabrication errors. Figures 4(c) and 4(d) show the normalized probability density of the condition number values for random deviations of the phase shifts from the optimal values in different variation ranges. The white-colored numbers are the average condition numbers for different variation magnitudes. At small deviations, the structure with two hidden layers has better performance with the smaller condition number. In case of phase shift variations of 0.04 \(\pi\) or larger, the structure with one hidden layer is better. This is because there are fewer phase shifts, and the performance is more robust to their variations. Overall, the condition numbers are smaller than 7, even when the phase shifts vary from the optimized values by a magnitude up to 0.1\(\pi\). This confirms the high fabrication tolerance of the circuits.

Next, we demonstrate numerically the density matrix reconstruction of three-PSS. As an example, we consider the three-PSS composed of photons with uncorrelated frequency spectra, defined by the wavefunction \(|\Psi\rangle = \int d\omega_1 d\omega_2 d\omega_3 \rho_0(\omega_0)\phi_1(\omega_1)\phi_2(\omega_2)\hat{a}_0^\dagger(\omega_0)\hat{a}_1^\dagger(\omega_1)\hat{a}_2^\dagger(\omega_2)|0\rangle\), where \(\phi_j(\omega)\) represents the spectral wavefunction of one photon in the \(j\)-th spatial path. In simulations, we assume the pairwise spectral overlaps as \(\langle \phi_0 | \phi_1 \rangle = 0.7e^{-i\pi/3}\), \(\langle \phi_1 | \phi_2 \rangle = 0.65\), and \(\langle \phi_2 | \phi_0 \rangle = 0.6\), which define all the spatial density matrix elements and the collective photon phase of \(-\pi/3\) as formulated in Appendix B. The real and imaginary parts of the density matrix of this state are presented in Figs. 5(a) and 5(b), respectively. Figures 5(c) and 5(e) show the predicted three-photon correlation probabilities at the output of the optimized circuits with one and two hidden layers, respectively. We see that the correlations are different for each structure. Based on the output correlations, one can reconstruct the input density matrix. In order to verify the low sensitivity to the measurement noise, we apply a Gaussian noise to the correlations and use them to reconstruct the input density matrix. We quantify the quality of the tomography procedure by the fidelity between the reconstructed (\(\rho_{rec}\)) and the input (\(\rho_{th}\)) density matrices, defined as \(\text{Tr}(\sqrt{\rho_{th}\rho_{rec}\sqrt{\rho_{th}}}\)\). Figures 5(d) and 5(f) show the corresponding statistical distributions of the reconstruction fidelity for 5000 simulations when a Gaussian noise with a standard deviation of 5% is added to the predicted correlation probabilities. We find that the fidelity stays above 0.95 for both one- and two- hidden layer structures, with the average values of \(\approx 0.99\). We also confirm similarly high fidelity for different three-PSSs, including those with a zero collective phase. These results indicate the high accuracy of the tomographic reconstruction of split states under the presence of measurement noise.
V. CONCLUSION

To conclude, we have formulated the general structure of spatial density matrix for multi-photon split states, which are an important resource for various quantum applications and whose resource-efficient characterization is a sought-after capability. We then proposed a coupled waveguide array forming a photonic neural network for the quantum tomography of such states with low sensitivity to noise and high tolerance to fabrication errors. The state measurement can be performed using a static photonic circuit and this approach is scalable to high photon numbers.

We anticipate that the proposed platform, enabling simple and robust characterization of such commonly used quantum states, will stimulate further developments and applications of quantum optical circuits. In particular, since our scheme does not require reconfigurability, it is especially suitable for integration with on-chip superconducting nanowire single-photon detectors operating at cryogenic temperatures to facilitate plug-and-play split-state measurements. Furthermore, the theoretical methodology can be extended to the split states where photons are separated not only in spatial ports but also in other degrees of freedom, such as polarization and orbital angular momentum.

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Appendix A: Number of real and imaginary free parameters in density matrices of split states

As derived above, the number of nonzero elements in the reduced density matrix of N-photon split state equals to the distinct projection operators in the form of Eq. (3), and it is $N!$. The numbers of real and imaginary parts are determined by the numbers of nonzero distinct $(\hat{O} + \hat{O}_{H.C.})/2$ and $(\hat{O} - \hat{O}_{H.C.})/2$, respectively, where $H.C.$ stands for Hermitian conjugate [28]. To perform the counting, we define by $A_N$ the number of cases when $\hat{O} = \hat{O}_{H.C.}$, such that the related imaginary parts are zero. Next, we derive the recurrence relation for $A_N$ as a function of $N$. Let us consider the value of $s_{N-1}$. When $s_{N-1} = N - 1$, the number of cases where $\hat{O} = \hat{O}_{H.C.}$ is $A_{N-1}$. When $s_{N-1} = \tilde{n}$ for $\tilde{n} = (0, 1, \ldots, N-2)$, the condition $\hat{O} = \hat{O}_{H.C.}$ can only be satisfied when $s_{\tilde{n}} = N - 1$. In this case, the number of cases where $\hat{O} = \hat{O}_{H.C.}$ becomes $A_{N-2}$. Since $\tilde{n}$ can take $N - 1$ values, the total number is $(N - 1)A_{N-2}$. Therefore, we obtain the relation

$$A_N = A_{N-1} + (N - 1)A_{N-2} \quad \text{for } N \geq 3, \quad (A1)$$
and the values for one- and two-photon states are $A_1 = 1$ and $A_2 = 2$.

Appendix B: The spatial split-state density matrix for photons with uncorrelated frequency spectra

Whereas our approach is applicable to arbitrary multi-photon split states, here we discuss an example of states composed of photons with uncorrelated frequency spectra. Specifically, we consider a pure $N$-photon state

$$|\Psi\rangle = \int d\omega_0 \, d\omega_1 \cdots d\omega_{N-1} \psi(\omega_0, \omega_1, \ldots, \omega_{N-1}) \hat{a}^\dagger_0(\omega_0) \hat{a}^\dagger_1(\omega_1) \cdots \hat{a}^\dagger_{N-1}(\omega_{N-1}) |0\rangle$$

with the frequency-dependent wavefunction featuring no correlations between the individual spectra of photons,

$$\psi(\omega_0, \omega_1, \ldots, \omega_{N-1}) = \phi_0(\omega_0) \phi_1(\omega_1) \cdots \phi_{N-1}(\omega_{N-1}).$$

Here $\phi_j(\omega_j)$ is an individual spectral wavefunction of the photon coupled to a spatial mode number $j$.

We calculate the $N!$ nonzero elements of the first row of the reduced density matrix for the $N$-photon split state as

$$\rho_{0,1,\ldots,N-1:s_0,s_1,\ldots,s_{N-1}} = \text{Tr} (\hat{\rho} \hat{O}_{0,1,\ldots,N-1:s_0,s_1,\ldots,s_{N-1}})$$

$$= \frac{1}{N!} \int d\omega_0 d\omega_1 \cdots d\omega_{N-1} \phi^*_0(\omega_0) \phi^*_1(\omega_1) \cdots \phi^*_{N-1}(\omega_{N-1}) \phi_{s_0}(\omega_0) \phi_{s_1}(\omega_1) \cdots \phi_{s_{N-1}}(\omega_{N-1})$$

(B3)

where $(s_0, s_1, \ldots, s_{N-1})$ are permutations in the set $(0, 1, \ldots, N - 1)$ without repetition and we define the spectral overlaps between different photon pairs as

$$I_{i,j} = \langle \phi_i | \phi_j \rangle = \int d\omega \phi^*_i(\omega) \phi_j(\omega),$$

(B4)

with the normalization $I_{i,j} = 1$.

For an $N = 3$ photon case, we have

$$\rho_{0,1,2:0,1,2} = \frac{1}{6}, \quad \rho_{0,1,2:0,2,1} = \frac{1}{6} |I_{1,2}|^2, \quad \rho_{0,1,2:1,0,2} = \frac{1}{6} |I_{0,1}|^2, \quad \rho_{0,1,2:2,1,0} = \frac{1}{6} |I_{2,0}|^2,$$

$$\rho_{0,1,2:1,2,0} = \frac{1}{6} I_{0,1} I_{1,2} I_{2,0}, \quad \rho_{0,1,2:2,0,1} = \frac{1}{6} I^*_{0,1} I^*_{1,2} I^*_{2,0}.$$

(B5)

We see that the elements $\rho_{0,1,2:201} = \rho^*_{0,1,2:120}$ are related to the collective phase of three photons. Correspondingly, the six free parameters in Fig. 1(d) are

$$\rho_1 = \rho_{0,1,2:0,1,2}, \quad \rho_2 = \rho_{0,1,2:0,2,1}, \quad \rho_3 = \rho_{0,1,2:1,2,0},$$

$$\rho_4 = \text{Re}(\rho_{0,1,2:1,2,0}), \quad \rho_5 = \text{Im}(\rho_{0,1,2:1,2,0}), \quad \rho_6 = \rho_{0,1,2:2,1,0}.$$

(B6)

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