Tyranny-of-the-minority regression adjustment in randomized experiments

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Abstract

Regression adjustment is widely used for the analysis of randomized experiments to improve the estimation efficiency of the treatment effect. This paper reexamines a weighted regression adjustment method termed as tyranny-of-the-minority (ToM), wherein units in the minority group are given greater weights. We demonstrate that the ToM regression adjustment is more robust than Lin (2013)’s regression adjustment with treatment-covariate interactions, even though these two regression adjustment methods are asymptotically equivalent in completely randomized experiments. Moreover, we extend ToM regression adjustment to stratified randomized experiments, completely randomized survey experiments, and cluster randomized experiments. We obtain design-based properties of the ToM regression-adjusted average treatment effect estimator under such designs. In particular, we show that ToM regression-adjusted estimator improves the asymptotic estimation efficiency compared to the unadjusted estimator even when the regression model is misspecified, and is optimal in the class of linearly adjusted estimators. We also study the asymptotic properties of various heteroscedasticity-robust standard error estimators and provide recommendations for practitioners. Simulation studies and real data analysis demonstrate ToM regression adjustment’s superiority over existing methods.

Key words: cluster randomized experiments, covariate adjustment, design-based theory, randomization-based inference, randomized block experiments, survey experiments

1. Introduction

Since the seminal work of Fisher (1935), randomized experiments have been the gold standard for drawing causal inference. Complete randomization balances confounding factors on average such that the treatment effects can be identified without untestable assumptions as in observational studies. Different experimental designs have been proposed to improve the efficiency or address practical concerns regarding completely randomized experiments. For example, stratified randomized experiments further balance important discrete covariates and improve the efficiency of treatment effect estimation (Fisher, 1926; Imai et al., 2008; Imbens and Rubin, 2015). Cluster randomized experiments are conducted when the individual-level treatment assignment is logistically unrealistic or when there are concerns regarding interference within clusters (Hayes and Moulton, 2017). Completely randomized survey experiments address the lack of generalizability of completely randomized experiments (Imai et al., 2008; Yang et al., 2021).

Regression adjustment is widely used during the analysis stage to utilize covariate information to improve efficiency. Fisher (1935) used covariates by adding them directly in the linear regression of outcome on treatment indicator and estimated the average treatment effect using the ordinary least squares (OLS). However, Freedman (2008) criticized this practice by demonstrating that this

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may degrade efficiency compared to the simple difference-in-means estimator under an unbalanced
design or in the presence of heterogeneity between treatment and control groups. Echoing the
critique of Freedman (2008) and to fix the efficiency loss issue, Lin (2013) recommended the addition
of both covariates and treatment-covariate interactions in the regression adjustment. Since then,
Lin’s with-interaction regression adjustment has witnessed significant advances in the field of causal
inference (Bloniarz et al., 2016; Liu and Yang, 2020; Li and Ding, 2020; Ma et al., 2022; Zhao and
Ding, 2021a; Su and Ding, 2021; Liu et al., 2021; Lei and Ding, 2021; Zhao and Ding, 2021b; Liu
et al., 2022; Lu et al., 2022; Zhao and Ding, 2022a,b).

However, practitioners may be wary of using the with-interaction regression adjustment because
it doubles the degrees of freedom used for the coefficients of covariates (Schochet et al., 2021; Negi
and Wooldridge, 2021). Although this regression adjustment method can be extended to other
experimental designs, it may degrade the efficiency compared to the unadjusted estimator (Ma
et al., 2022; Liu and Yang, 2020; Liu et al., 2021, 2022). One strategy to remedy this issue is to ap-
proach covariate adjustment from the perspective of projection or conditional inference and plug-in
unknown projection coefficients using several regressions (Yang et al., 2021; Liu et al., 2021; Wang
et al., 2021); however, this is more complicated and less robust than the weighted regression adjust-
ment introduced later (see our simulation results). Additionally, heteroskedasticity-robust variance
estimators from Lin’s with-interaction regression adjustment can be anti-conservative, under the
superpopulation framework (Negi and Wooldridge, 2021; Zhao and Ding, 2021a), completely ran-
domized survey experiments (Yang et al., 2021), or when the dimension of covariates is relatively
large compared to the sample size (Lei and Ding, 2021).

Lin (2013) discussed a weighted regression adjustment method named tyranny-of-the-minority
(ToM), which embodies the principle of giving more weights to the units in the minority group.
This method saves half of the degrees of freedom and is asymptotically equivalent to the with-
interaction regression adjustment in completely randomized experiments (Lin, 2013). However, Lin
(2013) and other follow-up research have not assessed the robustness of the method in completely
randomized experiments and potential application in other experimental designs.

To address the gap and drawbacks of the with-interaction regression adjustment, we re-examine
the ToM regression adjustment method in completely randomized experiments. We demonstrate
the robustness of the ToM regression-adjusted average treatment effect estimator using theoretical
justifications and simulation studies. Simulation results reveal that ToM regression adjustment
dramatically enhances the estimation efficiency and inference reliability when the design is away
from balance or the number of covariates is relatively large compared to the sample size.

ToM regression adjustment can be applied under other experimental designs to enhance the
efficiency. We illustrate its use and design-based properties in stratified randomized experiments,
completely randomized survey experiments, and cluster randomized experiments. Under mild mo-
moment conditions, we show that the ToM regression-adjusted average treatment effect estimator
is asymptotically normal and optimal in the class of linearly adjusted estimators for each of the
aforementioned experimental designs. Moreover, we study the asymptotic properties of various
heteroscedasticity-robust standard error estimators. Our analysis is design-based, that is, the anal-
ysis is conducted by conditioning on the potential outcomes and covariates, along with treatment
assignment as the only source of randomness. Our theoretical results allow the linear regression
model to be arbitrarily misspecified. Finally, we conduct simulation to evaluate the finite-sample
performance of the ToM regression-adjusted estimator. Simulation results demonstrate the supe-
riority of the ToM regression-adjusted estimator compared to existing estimators. Based on the
theoretical and finite-sample results, we provide practical suggestions for choosing point and vari-
ance estimators to analyze the experimental results. These suggested estimators can be conveniently
obtained using off-the-shelf statistical software packages.
The remaining paper is structured as follows. In Section 2, we introduce ToM regression adjustment in the context of completely randomized experiments and compare it with Lin’s with-interaction regression adjustment to assess its robustness. In Section 3, we extend the application of ToM regression adjustment under stratified randomized experiments, demonstrating its optimality for this design. In Section 4, we extend ToM regression adjustment and demonstrate its optimality for completely randomized survey experiments. In Section 5, we conduct simulation to compare the finite-sample performance of ToM regression adjustment with that of the existing methods. In Section 6, we use ToM regression adjustment to analyze two real datasets. We discuss the combination of ToM regression adjustment and rerandomization in Section 7 and conclude the paper in Section 8. The application of ToM regression adjustment under cluster randomized experiments and proofs are provided in the Supplementary Material.

2. ToM regression adjustment in completely randomized experiments

2.1. Notation and framework

Consider a completely randomized experiment with \( n \) units. We randomly assign \( n_1 \) units to the treatment group and \( n_0 \) to the control group, with \( n_0 + n_1 = n \). Let \( Z_i \) be the treatment indicator of the \( i \)th unit with \( Z_i = 0 \) when it is assigned to the control group and \( Z_i = 1 \) when it is assigned to the treatment group. By design, \( \sum_{i=1}^{n} Z_i = n_1 \). Let \( S_1 \) and \( S_0 \) be the set of units in the treatment and control groups, respectively. We use \( Y_i(z) \) to denote the potential outcome of unit \( i \) under treatment \( z \), for \( z = 0, 1 \), with \( Y_i = Z_iY_i(1) + (1 - Z_i)Y_i(0) \) as the observed outcome. Let \( \bar{x}_i = (x_{i1}, \ldots, x_{ik})^\top \) be the covariates of unit \( i \) of length \( k \). In a realized experiment, we observe \( \{(Y_i, x_i, Z_i)\}_{i=1}^{n} \). We consider a design-based or randomization-based inference framework, under which \( \{(Y_i(1), Y_i(0), x_i)\}_{i=1}^{n} \) are all fixed finite-population quantities and treatment assignment, \( Z = (Z_1, \ldots, Z_n) \), is the only source of randomness. Throughout the study, we assume the validity of the stable unit treatment value assumption (SUTVA) (Rubin, 1980).

Let \( \tau_i = Y_i(1) - Y_i(0) \) be the unit-level treatment effect. We are interested in the population average treatment effect \( \tau = \sum_{i=1}^{n} \tau_i/n \). An unbiased estimator of \( \tau \) is the difference in the observed means of the potential outcomes in the treatment and control groups (Imbens and Rubin, 2015), which is referred to as the “difference-in-means” estimator:

\[
\hat{\tau} = \frac{1}{n} \sum_{i=1}^{n} Z_i Y_i/n_1 - \frac{1}{n} \sum_{i=1}^{n} (1 - Z_i) Y_i/n_0.
\]

We use the following notation. Let \( \bar{Y}(z) = n^{-1} \sum_{i=1}^{n} Y_i(z) \) (\( z = 0, 1 \)) and \( \bar{x} = n^{-1} \sum_{i=1}^{n} x_i \) be the population means of potential outcomes and covariates, respectively. The population variances and covariances are defined as

\[
S_x^2 = (n - 1)^{-1} \sum_{i=1}^{n} (x_i - \bar{x})(x_i - \bar{x})^\top, \quad S_{zx} = S_{xz}^\top = (n - 1)^{-1} \sum_{i=1}^{n} (x_i - \bar{x})\{Y_i(z) - \bar{Y}(z)\},
\]

\[
S_z^2 = (n - 1)^{-1} \sum_{i=1}^{n} \{Y_i(z) - \bar{Y}(z)\}^2, \quad S_{\tau}^2 = (n - 1)^{-1} \sum_{i=1}^{n} (\tau_i - \bar{\tau})^2, \quad z = 0, 1.
\]

Let \( \| \cdot \|_\infty \) be the infinity norm of a vector. Let \( Y_i \sim 1 + x_i \) denote the ordinary least squares (OLS) regression of \( Y_i \) on \( x_i \) with an intercept. Let \( Y_i^{\text{WLS}} \sim 1 + x_i \) denote the weighted least squares (WLS) regression of \( Y_i \) on \( x_i \) with an intercept and weight \( w_i \).
2.2. Regression without and with treatment-covariate interactions

The covariates may be predictive of the potential outcomes. The difference-in-means estimator does not use the covariate information, which negatively affects the efficiency. Regression adjustment is widely used at the analysis stage to improve the efficiency by adjusting for the covariate imbalance between the treatment and control groups.

The difference-in-means estimator can be derived as the OLS estimator of the coefficient of \( Z_i \) in the regression \( Y_i \sim 1 + Z_i \). Thus, the easiest way of using covariates, which dates back to Fisher (1935), is to directly add \( x_i \) in the regression formula, \( Y_i \sim 1 + Z_i + x_i \). The resulting regression-adjusted average treatment effect estimator is the OLS estimator of the coefficient of \( Z_i \). We refer to this regression method as “Fisher’s regression.” Fisher’s regression has been constantly used in observational studies (Sloczynski, 2018), completely randomized experiments (Negi and Wooldridge, 2021), cluster randomized experiments (Schochet et al., 2021), and so on.

Freedman (2008) criticized Fisher’s regression for its lack of guarantee regarding the improvement in efficiency compared to the difference-in-means estimator under unbalanced design or in the presence of heterogeneity between treatment and control groups. Echoing the critique of Freedman (2008), Lin (2013) discussed the possibility of remedying this problem by adding the treatment-covariate interactions in the regression,

\[
Y_i \sim 1 + Z_i + (x_i - \bar{x}) + Z_i(x_i - \bar{x}). \tag{1}
\]

The OLS estimator of the coefficient of \( Z_i \), denoted by \( \hat{\tau}_{\text{lin}} \), is used as the average treatment effect estimator. Note that the covariates must be centered in the interaction term.

Schochet et al. (2021) pointed out that, to include the interaction term, we risk the loss of the degrees of freedom that could seriously reduce power. Researchers may feel uncomfortable in the absence of sufficient degrees of freedom in a with-interaction model that analyzes experiments with 20–100 units, such as clinics and schools, which is very common in development economics (Negi and Wooldridge, 2021). In the same paper, Lin (2013) commented on the ToM regression and demonstrated its asymptotic equivalence to the with-interaction regression for point estimation. However, most of the follow-up work focused on Fisher’s regression and Lin’s with-interaction regression. ToM regression was barely studied. Consequently, it is essential to re-examine ToM regression because it saves half of the degrees of freedom with respective to covariates.

2.3. ToM regression

ToM regression accounts for the drawback in Fisher’s regression by giving larger weights to the units in the minority group. This regression-adjusted estimator \( \hat{\tau}_{\text{tom}} \) is derived as the WLS estimator of the coefficient of \( Z_i \) in the regression of \( Y_i \) on \((1, Z_i, x_i)\) with weights \( w_i = Z_i/p_1^2 + (1 - Z_i)/p_0^2 \), where \( p_1 = n_1/n \) and \( p_0 = n_0/n \) are the proportions of units assigned to the treatment and control groups, respectively. We denote the WLS regression as

\[
Y_i \overset{\text{w}}{\sim} 1 + Z_i + x_i. \tag{2}
\]

Remark 1. Lin (2013) used the following weights: \( Z_ip_0/p_1 + (1 - Z_i)p_1/p_0 \). These are equivalent to the weights \( w_i \)'s. We use \( w_i \)'s because they can be conveniently extended to other experimental designs.

Lin (2013) observed that \( \hat{\tau}_{\text{lin}} \) and \( \hat{\tau}_{\text{tom}} \) have the same asymptotic distribution. In the remaining of this section, we demonstrate the optimality of \( \hat{\tau}_{\text{tom}} \), derive the asymptotic property of its
heteroskedasticity-robust standard error, and show that $\hat{\tau}^{\text{tom}}$ is more robust than $\hat{\tau}^{\text{lin}}$ through the perspectives of calibrated estimator and leverage score.

In fact, all regression-adjusted estimators are linearly adjusted estimators, with the following form in completely randomized experiments: $\hat{\tau}(\beta) = \hat{\tau} - \beta^T \bar{z}_x$, where $\bar{\tau}_x = \sum_{i=1}^{n} z_i x_i / n - \sum_{i=1}^{n} (1 - Z_i) x_i / n_0$ and $\beta$ is some adjusted vector. Let $\beta^{\text{opt}}_\tau$ correspond to linearly adjusted estimator with minimum sampling variance, that is, $\text{var}\{\hat{\tau}(\beta^{\text{opt}}_\tau)\} = \min_{\beta \in \mathbb{R}^p} \text{var}\{\hat{\tau}(\beta)\}$. As shown by Li and Ding (2017), the covariance of $\sqrt{n}(\hat{\tau} - \tau, \hat{\tau}_x^\top)$ is

$$\begin{pmatrix} V_{\tau\tau} & V_{\tau x} \\ V_{x\tau} & V_{xx} \end{pmatrix} = \begin{pmatrix} p_1^{-1} S^2_1 + p_0^{-1} S^2_0 - S^2_\tau & p_1^{-1} S_{1x} + p_0^{-1} S_{0x} \\ p_1^{-1} S_{x1} + p_0^{-1} S_{x0} & (p_0 p_1)^{-1} S_x^2 \end{pmatrix},$$

Simple calculation gives $\beta^{\text{opt}}_{\tau} = V_{xx}^{-1} V_{\tau x}$ and $\text{var}\{\hat{\tau}(\beta^{\text{opt}}_{\tau})\} = V_{\tau\tau} - V_{\tau x} V_{xx}^{-1} V_{x\tau}$. It has been shown that $\hat{\tau}^{\text{lin}}$ has the same asymptotic distribution as the optimal linearly adjusted estimator $\hat{\tau}(\beta^{\text{opt}}_{\tau})$ (Lin, 2013; Li et al., 2018; Li and Ding, 2020). Under mild conditions, Lin (2013) showed that $\hat{\tau}^{\text{lin}}$ and $\hat{\tau}^{\text{tom}}$ have the same asymptotic distribution, and therefore are both optimal. Proposition 1 presented below indicates this property.

**Assumption 1.** As $n \to \infty$, for $z = 0, 1$, (i) $p_z$ has a positive limit; (ii) $S^2_1, S^2_2, S_{x1}, S^2_x$ have finite limits, the limit of $\text{var}\{\hat{\tau}(\beta^{\text{opt}}_{\tau})\}$ is positive and the limit of $S_x^2$ is nonsingular; and (iii) $\max_{1 \leq i \leq n} | Y_i(z) - \bar{Y}(z) |^2 = o(n)$, $\max_{1 \leq i \leq n} \| x_i - \bar{x} \|^2 = o(n)$.

**Proposition 1.** Under Assumption 1, both $n^{1/2}(\hat{\tau}^{\text{tom}} - \tau)$ and $n^{1/2}(\hat{\tau}^{\text{lin}} - \tau)$ are asymptotically normal with zero mean and variance $V_{\tau\tau} - V_{\tau x} V_{xx}^{-1} V_{x\tau}$.

The heteroskedasticity-robust standard errors (Huber, 1967; White, 1980) are frequently used to approximate the true asymptotic standard errors and can be conveniently obtained by standard statistical software packages. The classical linear regression literature suggests different ways of correcting the degrees of freedom loss, which leads to HC$_j$ $(j = 0, 1, 2, 3)$. HC$_0$ corresponds to the heteroskedasticity-robust standard error without correction. We have included the explicit formulas of HC$_j$ $(j = 0, 1, 2, 3)$ in the Supplementary Material.

Let $(\hat{V}_{\text{HC} j}^{\text{tom}})^{1/2}$ be the heteroskedasticity-robust standard error of $\hat{\tau}^{\text{tom}}$ of regression (2) corresponding to HC$_j$. Theorem 1 below depicts the conservativeness of the heteroskedasticity-robust standard error.

**Theorem 1.** Under Assumption 1, for $j = 0, 1, 2, 3$,

$$\hat{V}_{\text{HC} j}^{\text{tom}} = n^{-1} \min_{\beta} \left[ p_1^{-1} S^2_1(\beta) + p_0^{-1} S^2_0(\beta) \right] + o_P(n^{-1}),$$

where $S^2_0(\beta) = (n - 1)^{-1} \sum_{i=1}^{n} \{ Y_i(z) - \bar{Y}(z) - (x_i - \bar{x})^T \beta \}^2$, $z = 0, 1$.

Let $q_{\alpha}$ be the $\alpha$th quantile of a standard normal distribution. We can construct Wald-type $1 - \alpha$ $(0 < \alpha < 1)$ confidence intervals of $\tau$:

$$\left[ \hat{\tau}^{\text{tom}} + (\hat{V}_{\text{HC} j}^{\text{tom}})^{1/2} q_{\alpha/2}, \hat{\tau}^{\text{tom}} + (\hat{V}_{\text{HC} j}^{\text{tom}})^{1/2} q_{1 - \alpha/2} \right], \quad j = 0, 1, 2, 3,$$

whose asymptotic coverage rates are greater than or equal to $1 - \alpha$.

**Remark 2.** Let $(\hat{V}_{\text{HC} j}^{\text{lin}})^{1/2}$ be the heteroskedasticity-robust standard error of $\hat{\tau}^{\text{lin}}$ in the with-interaction regression $Y_i \sim 1 + Z_i + (x_i - \bar{x}) + Z_i(x_i - \bar{x})$. Li and Ding (2020) and Lei and Ding (2021) showed that, under Assumption 1,

$$\hat{V}_{\text{HC} j}^{\text{lin}} = n^{-1} \left\{ p_1^{-1} \min_{\beta_1} S^2_1(\beta_1) + p_0^{-1} \min_{\beta_0} S^2_0(\beta_0) \right\} + o_P(n^{-1}).$$
Theorem 2. \( \hat{F} \) level of efficiency improvement. In other words, \( \hat{\tau} \) indicates that \( \hat{\tau} \) has high efficiency.

\[ \frac{1}{\beta} \min \beta_1 S^2(\beta_1) + \frac{1}{\beta_0} \min \beta_0 S^2(\beta_0) \leq \min \beta \left[ p^{-1}_1 S^2(\beta) + p^{-1}_0 S^2(\beta) \right], \]

\( \hat{\tau}^{\text{tom}} \) produces a more conservative inference than \( \left( \hat{\tau}^{\text{lin}} \right)^{1/2} \). However, \( \left( \hat{\tau}^{\text{lin}} \right)^{1/2} \) may produce a finite-sample confidence interval with coverage probability lower than the nominal level when the design is not balanced or the number of covariates is relatively large compared to the sample size; see Lei and Ding (2021) and Section 5. Meanwhile, the classic Neyman-type variance estimator for the difference-in-means estimator is asymptotically equal to \( p^{-1}_1 S^2 + p^{-1}_0 S^2 \). Since

\[ p^{-1}_1 S^2 + p^{-1}_0 S^2 \geq \min \beta \left[ p^{-1}_1 S^2(\beta) + p^{-1}_0 S^2(\beta) \right], \]

\( \hat{\tau}^{\text{tom}} \) still improves the inference efficiency compared to the classic Neyman-type variance estimator.

ToM regression is more robust than the with-interaction regression because of the following two reasons. First, both \( \hat{\tau}^{\text{tom}} \) and \( \hat{\tau}^{\text{lin}} \) are special cases of calibrated estimators of the form \( \hat{\tau}^{\text{cal}} = \sum_{i \in S_1} c_i Y_i - \sum_{i \in S_0} c_i Y_i \), where \( c_i \)'s are the calibrated weights (Deville and Särndal, 1992; Deville et al., 1993). Let \( \tilde{x}(z) \) be the sample mean of \( x_i \) under treatment \( z \). As presented in the proof of Theorem 2, the calibrated weights for \( \hat{\tau}^{\text{tom}} \) are

\[ c_i^{\text{tom}} = \begin{cases} n_i^{-1} - \bar{x}_z \left\{ p_0^{-2}(n_0 - 1) s^2_{x0} + p_i^{-2}(n_1 - 1) s^2_{x1} \right\}^{-1} \{ x_i - \tilde{x}(1) \} p_0^{-2}, & i \in S_1, \\ n_i^{-1} + \bar{x}_z \left\{ p_0^{-2}(n_0 - 1) s^2_{x0} + p_i^{-2}(n_1 - 1) s^2_{x1} \right\}^{-1} \{ x_i - \tilde{x}(0) \} p_0^{-2}, & i \in S_0. \end{cases} \]

In contrast, the calibrated weights for \( \hat{\tau}^{\text{lin}} \) are

\[ c_i^{\text{lin}} = \begin{cases} n_i^{-1} - \bar{x}_z \left( n_1 - 1 \right) s^2_{x1} \left\{ x_i - \tilde{x}(1) \right\}^{-1}, & i \in S_1, \\ n_i^{-1} + \bar{x}_z \left( n_0 - 1 \right) s^2_{x0} \left\{ x_i - \tilde{x}(0) \right\}^{-1}, & i \in S_0. \end{cases} \]

We use \( \epsilon^{\text{tom}} \) and \( \epsilon^{\text{lin}} \) to denote the vector of \( c_i^{\text{tom}} \)'s and \( c_i^{\text{lin}} \)'s, respectively.

The non-calibrated weights used for the difference-in-means estimator are \( n^{-1}_z (z = 0, 1) \) for units in the treatment arm \( z \). Although both \( \hat{\tau}^{\text{tom}} \) and \( \hat{\tau}^{\text{lin}} \) are asymptotically optimal, the calibrated weights of \( \hat{\tau}^{\text{lin}} \) are not satisfactory. For example, negative or large weights may occur, which affect the robustness of the regression-adjusted treatment effect estimator. Deville and Särndal (1992) proposed a distance between the calibrated and non-calibrated weights to measure the calibrated weights’ robustness. For complete randomization, the distance measure is derived as

\[ F(c) = \sum_{i \in S_1} G(c_i n_1) + \sum_{i \in S_0} G(c_i n_0), \quad \text{where} \quad G(x) = (x - 1)^2 / 2. \]

Here \( G(c_i n_z) \) is the distance between the ratio of the calibrated and non-calibrated weights and 1. Large value of \( F(c) \) suggests the existence of extreme calibrated weights. Theorem 2 below indicates that \( \hat{\tau}^{\text{tom}} \) is better than \( \hat{\tau}^{\text{lin}} \) in the sense of embodying non-extreme calibrated weights. In other words, \( \hat{\tau}^{\text{tom}} \) makes fewer changes to the calibrated weights than \( \hat{\tau}^{\text{lin}} \) to achieve the same level of efficiency improvement.

**Theorem 2.** \( F(\epsilon^{\text{tom}}) \leq F(\epsilon^{\text{lin}}) \).
Second, for model-based inference, Huber (2004) observed that the inverse of leverage score measures the number of units required to determine the fitted value of $Y$. The gross error is not reflected in the residuals of high leverage score points. Leverage score also plays an important role for design-based inference. High leverage score negatively affects the asymptotic theory and corresponding inferences (Dorfman, 1991; Lei and Ding, 2021). Theorem 3 below indicates that $\hat{\tau}^{tom}$ is better than $\hat{\tau}^{lin}$ in terms of having smaller leverage score.

**Theorem 3.** Leverage scores in the with-interaction regression are

$$h_i^{lin} = \left\{ \begin{array}{ll}
{n_1}^{-1} + \{x_i - \hat{x}(1)\} \left\{ (n_1 - 1) s_{z(1)}^2 \right\}^{-1} \{x_i - \hat{x}(1)\}, & i \in S_1,
{n_0}^{-1} + \{x_i - \hat{x}(0)\} \left\{ (n_0 - 1) s_{z(0)}^2 \right\}^{-1} \{x_i - \hat{x}(0)\}, & i \in S_0.
\end{array} \right.$$  

In contrast, leverage scores in ToM regression are

$$h_i^{tom} = \left\{ \begin{array}{ll}
{n_1}^{-1} + \{x_i - \hat{x}(1)\} \left\{ (n_1 - 1) s_{z(1)}^2 + (p_1/p_0)^2 (n_0 - 1) s_{z(0)}^2 \right\}^{-1} \{x_i - \hat{x}(1)\}, & i \in S_1,
{n_0}^{-1} + \{x_i - \hat{x}(0)\} \left\{ (n_0 - 1) s_{z(0)}^2 + (p_0/p_1)^2 (n_1 - 1) s_{z(1)}^2 \right\}^{-1} \{x_i - \hat{x}(0)\}, & i \in S_0.
\end{array} \right.$$  

Moreover, for $i = 1, \ldots, n$, $h_i^{tom} \leq h_i^{lin}$.

Because cluster randomized experiments can be viewed as complete randomized experiments at the cluster level, we obtained results that correspond to cluster randomized experiments; see the Supplementary Material for more details. In the following two sections, we extend ToM regression to stratified randomized experiments and completely randomized survey experiments, respectively.

### 3. ToM regression adjustment in stratified randomized experiments

Stratified randomized experiments are a combination of several completely randomized experiments conducted independently in each stratum. It is natural to extend ToM regression adjustment to this experimental design. For simplicity, we use the same $k$ to denote covariate dimension. Consider a stratified randomized experiment with $H$ strata. We use index $h$ to denote quantities with respect to population in stratum $h$ ($h = 1, \ldots, H$), which leads to the stratum-specific analogs of $n$, $n_z$, $p_z$, $\bar{Y}(z)$, $\bar{x}$, $S_{xz}$, $S_{xzz}$, $S_{xzz}^2$, $z = 0, 1$, denoted by $n_h$, $n_{hz}$, $p_{hz}$, $\bar{Y}_h(z)$, $\bar{x}_h$, $S_{hxz}$, $S_{hxx}$, $S_{hxz}^2$, $z = 0, 1$. Throughout this section, we assume that $2 \leq n_{hz} \leq n_h - 2$ for all $h = 1, \ldots, H$. We use double index $hi$ ($h = 1, \ldots, H$, $i = 1, \ldots, n_h$) to denote unit $i$ in stratum $h$. Let $Y_{hi}(z)$ ($z = 0, 1$), $Y_{hi}$, $\tau_{hi}$, $x_{hi}$ and $Z_{hi}$ be the potential outcomes, observed outcome, unit-level treatment effect, covariates, and treatment indicator of unit $hi$, respectively. Denote the total population size by $n_{str} = \sum_{h=1}^{H} n_h$ and the proportion of population size of stratum $h$ by $\pi_h = n_h/n_{str}$. The average treatment effect is

$$\tau_{str} = \sum_{h=1}^{H} \sum_{i=1}^{n_h} (Y_{hi}(1) - Y_{hi}(0)) = \sum_{h=1}^{H} \sum_{i=1}^{n_h} \tau_{hi} = \sum_{h=1}^{H} \pi_h \tau_h,$$

where $\tau_h = \sum_{i=1}^{n_h} \tau_{hi}/n_h$ is the average treatment effect in stratum $h$.

Replacing $\tau_h$ in equation (3) by its unbiased estimator

$$\hat{\tau}_h = \sum_{i=1}^{n_h} Z_{hi} Y_{hi}/n_{h1} - \sum_{i=1}^{n_h} (1 - Z_{hi}) Y_{hi}/n_{h0},$$
we obtain an unbiased estimator of $\tau_{\text{str}}$, $\hat{\tau}_{\text{str}} = \sum_{h=1}^{H} \pi_{h} \hat{\tau}_{h}$. As demonstrated by Liu and Yang (2020), $\hat{\tau}_{\text{str}}$ is the OLS estimator of the coefficient of $Z_{hi}$ in the following regression:

$$Y_{hi} \sim 1 + Z_{hi} + \sum_{q=2}^{H} (\delta_{hq} - \pi_{q}) + Z_{hi} \sum_{q=2}^{H} (\delta_{hq} - \pi_{q}),$$

where $\delta_{hq}$ is the stratum indicator, $\delta_{hq} = 1$ if $q = h$ and $\delta_{hq} = 0$ otherwise.

The straightforward extension of Lin’s with-interaction regression to stratified randomized experiments is as follows:

$$Y_{hi} \sim 1 + Z_{hi} + \sum_{q=2}^{H} (\delta_{hq} - \pi_{q}) + Z_{hi} \sum_{q=2}^{H} (\delta_{hq} - \pi_{q}) + (x_{hi} - \bar{x}_{h}) + Z_{hi}(x_{hi} - \bar{x}_{h}).$$

However, it can be showed that this regression-adjusted estimator, that is, the OLS estimator of the coefficient of $Z_{hi}$, guarantees the improvement of efficiency if the following Assumption 2 is true. Otherwise, it may degrade the efficiency.

**Assumption 2.** (i) Propensity scores are the same across strata, that is, $p_{hi} = p_{12}$ for all $h = 1, \ldots, H$, (ii) $n_{h} = n_{1}$ or $n_{h} \to \infty$ for all $h = 1, \ldots, H$.

Equal propensity scores can be ensured across strata through the design; however, Assumption 2(ii) may be unrealistic in many stratified randomized experiments. To remedy this condition, Liu and Yang (2020) proposed the following weighted regression:

$$Y_{hi} \overset{w_{hi}}{\sim} 1 + Z_{hi} + \sum_{q=2}^{H} (\delta_{hq} - \pi_{q}) + Z_{hi} \sum_{q=2}^{H} (\delta_{hq} - \pi_{q}) + (x_{hi} - \bar{x}_{h}) + Z_{hi}(x_{hi} - \bar{x}_{h}),$$

where $w_{hi} = Z_{hi}n_{h}/(n_{h1} - 1) + (1 - Z_{hi})n_{h}/(n_{h0} - 1)$. They demonstrated that the resulting regression-adjusted estimator can guarantee the improvement of efficiency without Assumption 2(ii); however, Assumption 2(i) must still hold true. In this section, we apply ToM regression adjustment to stratified randomized experiments and demonstrate that this regression-adjusted estimator, denoted by $\hat{\tau}_{\text{str}}^{\text{Tom}}$, improves the efficiency without Assumption 2.

We define $\hat{\tau}_{\text{str}}^{\text{Tom}}$ as the WLS estimator of the coefficient of $Z_{hi}$ in the following weighted regression:

$$Y_{hi} \overset{w_{hi}}{\sim} 1 + Z_{hi} + \sum_{q=2}^{H} (\delta_{hq} - \pi_{q}) + Z_{hi} \sum_{q=2}^{H} (\delta_{hq} - \pi_{q}) + x_{hi},$$

with weights

$$w_{hi} = Z_{hi}p_{h1}^{-2} \frac{n_{h1}}{n_{h1} - 1} + (1 - Z_{hi})p_{h0}^{-2} \frac{n_{h0}}{n_{h0} - 1}.$$

**Remark 3.** Although the weights $w_{hi} = Z_{hi}/p_{h1} + (1 - Z_{hi})/p_{h0}^{2}$ seem like a straightforward extension of $w_{i}$ to stratified randomized experiments, only $w_{hi}$ can guarantee the improvement of efficiency and optimality of $\hat{\tau}_{\text{str}}^{\text{Tom}}$. Moreover, when $\min \{n_{h1}, n_{h0}\} \to \infty$ for $h = 1, \ldots, H$, $w_{hi}$’s are asymptotically equivalent to $w_{i}$’s.

Let $\hat{\tau}_{\text{str},x} = \sum_{h=1}^{H} \pi_{h} \hat{\tau}_{hx}$ with $\hat{\tau}_{hx} = \sum_{i=1}^{n_{h}} Z_{hi}x_{hi}/n_{h1} - \sum_{i=1}^{n_{h}} (1 - Z_{hi})x_{hi}/n_{h0}$. We define linearly adjusted estimator as $\hat{\tau}_{\text{str}}(\beta) = \hat{\tau}_{\text{str}}^{\top} \hat{\tau}_{\text{str},x}$ for some adjusted vector $\beta$. By Wang et al. (2021, Proposition 2), $n_{\text{str}}^{1/2} (\hat{\tau}_{\text{str}} - \tau_{\text{str}}, \hat{\tau}_{\text{str}}^{\top})$ has mean zero and covariance matrix

$$
\begin{pmatrix}
V_{\text{str},rr} & V_{\text{str},rx} \\
V_{\text{str},xr} & V_{\text{str},xx}
\end{pmatrix} = 
\begin{pmatrix}
\sum_{h=1}^{H} \pi_{h}p_{h1}^{-2} \sigma_{h1}^{2} + \pi_{h}p_{h0}^{-1}S_{h0}^{2} - \pi_{h}S_{h1}^{2} & \sum_{h=1}^{H} \pi_{h}p_{h1}^{-1}S_{h1x} + \pi_{h}p_{h0}^{-1}S_{h0x} \\
\sum_{h=1}^{H} \pi_{h}p_{h1}^{-1}S_{hx1} + \pi_{h}p_{h0}^{-1}S_{hx0} & \sum_{h=1}^{H} \pi_{h}(p_{h0}p_{h1})^{-1}S_{hxx}^{2}
\end{pmatrix}.
$$
Let \( \beta_{\text{str}}^{\text{opt}} \) be the optimal linear projection coefficient defined as \( \beta_{\text{str}}^{\text{opt}} = \arg \min_{\beta} \text{var}\{ \hat{\tau}_{\text{str}}(\beta) \} \). Through simple calculation, we obtain \( \beta_{\text{str}}^{\text{opt}} = V_{\text{str},xx}^{-1} V_{\text{str},x\tau} \), with \( \text{var}\{ \hat{\tau}_{\text{str}}(\beta_{\text{str}}^{\text{opt}}) \} = V_{\text{str},\tau\tau} - V_{\text{str},x\tau} V_{\text{str},xx}^{-1} V_{\text{str},x\tau} \).

To investigate the asymptotic normality and optimality of \( \hat{\tau}_{\text{str}}^{\text{tom}} \), we require Assumption 3 below.

**Assumption 3.** As \( n_{\text{str}} \to \infty \), for \( z = 0, 1 \),

(i) \( c \leq \min_{1 \leq h \leq H} p_{h1} \leq \max_{1 \leq h \leq H} p_{h1} \leq 1 - c \) for some constant \( c \in (0, 0.5] \) independent of \( n_{\text{str}} \);

(ii) \( \sum_{h=1}^{H} \pi_h p_{h1}^{-1} S_{h1}^2 + \sum_{h=1}^{H} \pi_h (p_{h1} - 1) S_{h1}^2 + \sum_{h=1}^{H} \pi_h p_{h1}^{-1} S_{hxx} + \sum_{h=1}^{H} \pi_h S_{h\tau}^2 \) have limiting values, the limit of \( \text{var}\{ \hat{\tau}_{\text{str}}(\beta_{\text{str}}^{\text{opt}}) \} \) is positive and the limit of \( \sum_{h=1}^{H} \pi_h (p_{h1} p_{h0})^{-1} S_{h\tau}^2 \) is nonsingular;

(iii) \( \max_{1 \leq h \leq H} \max_{1 \leq i \leq n_h} \| Y_{hi}(z) - \hat{Y}_h(z) \|^2 = o(n_{\text{str}}) \), \( \max_{1 \leq h \leq H} \max_{1 \leq i \leq n_h} \| x_{hi} - \hat{x}_h \|_\infty = o(n_{\text{str}}) \).

Assumption 3 is quite general, with few requirements related to the number of strata, stratum sizes, and propensity scores across strata.

**Theorem 4.** Under Assumption 3, \( n_{\text{str}}^{1/2} (\hat{\tau}_{\text{str}}^{\text{tom}} - \tau_{\text{str}}) \) is asymptotically normal with zero mean and variance \( V_{\text{str},\tau\tau} - V_{\text{str},x\tau} V_{\text{str},xx}^{-1} V_{\text{str},x\tau} \). Moreover, \( \hat{\tau}_{\text{str}}^{\text{tom}} \) is optimal with minimum asymptotic variance in the class of linearly adjusted estimators \( \{ \hat{\tau}_{\text{str}}(\beta) : \beta \in \mathbb{R}^k \} \).

Let \( \hat{V}_{\text{HC},j,\text{str}} \) denote the variance estimator of \( \hat{\tau}_{\text{str}}^{\text{tom}} \) from the regression formula (4) corresponding to HC \( j \). Theorem 5 below presents the asymptotic property of \( \hat{V}_{\text{HC},2,\text{str}} \).

**Theorem 5.** Under Assumption 3,

\[
\hat{V}_{\text{HC},2,\text{str}} = \min_{\beta} n_{\text{str}}^{-1} \sum_{h=1}^{H} \left\{ \pi_h p_{h1}^{-1} S_{h1}^2(\beta) + \pi_h p_{h0}^{-1} S_{h0}^2(\beta) \right\} + o_P(n_{\text{str}}^{-1}),
\]

where

\[
S_{h1}^2(\beta) = (n_h - 1)^{-1} \sum_{i=1}^{n_h} \{ Y_{hi}(z) - \hat{Y}_h(z) - (x_{hi} - \hat{x}_h) \}^2.
\]

The variance of \( \hat{\tau}_{\text{str}}(\beta_{\text{str}}^{\text{opt}}) \) can be derived by replacing \( Y_{hi}(z) \) by \( Y_{hi}(z) - x_i \beta_{\text{str}}^{\text{opt}} \) in the formula of \( \text{var}(\hat{\tau}_{\text{str}}) \). The optimality of \( \beta_{\text{str}}^{\text{opt}} \) implies that

\[
\text{var}\{ \hat{\tau}_{\text{str}}(\beta_{\text{str}}^{\text{opt}}) \} = \min_{\beta} n_{\text{str}}^{-1} \sum_{h=1}^{H} \left\{ \pi_h p_{h1}^{-1} S_{h1}^2(\beta) + \pi_h p_{h0}^{-1} S_{h0}^2(\beta) - S_{h\tau}^2 \right\}.
\]

Equations (5) and (6) indicate that \( \hat{V}_{\text{HC},2,\text{str}} \) is an asymptotic conservative estimator of \( \text{var}\{ \hat{\tau}_{\text{str}}(\beta_{\text{str}}^{\text{opt}}) \} \). Since \( \hat{V}_{\text{HC},2,\text{str}} \leq \hat{V}_{\text{HC},3,\text{str}} \), \( \hat{V}_{\text{HC},3,\text{str}} \) is also a conservative estimator. Therefore, the Wald-type confidence intervals

\[
\left[ \hat{\tau}_{j,\text{str}}^{\text{tom}} + (\hat{V}_{\text{HC},j,\text{str}})^{1/2} q_{\alpha/2}, \hat{\tau}_{j,\text{str}}^{\text{tom}} + (\hat{V}_{\text{HC},j,\text{str}})^{1/2} q_{1-\alpha/2} \right], \quad j = 2, 3,
\]

have asymptotic coverage rates greater than or equal to \( 1 - \alpha \).

**Remark 4.** \( \hat{V}_{\text{HC},j,\text{str}} \) (\( j = 0, 1 \)) can be anti-conservative and produce invalid confidence intervals. See the Supplementary Material for more details.
4. ToM regression adjustment in completely randomized survey experiments

Survey experiments usually comprise two stages: random sampling of units from a target population and random assignment of treatments to the sampled units. These experiments are widely used for estimating treatment effect of a target population (Imai et al., 2008). The standard survey experiments, completely randomized survey experiments, conduct simple random sampling without replacement to obtain a subset of units before assignment of sampled units through complete randomization into different treatment arms; see, for example Imbens and Rubin (2015, chap. 6) and Yang et al. (2021).

In a completely randomized survey experiment, suppose \( n \) units in the experiment are a simple random sample without replacement from a target population of size \( N \), with sampling fraction \( f = n/N \). When \( f = 1 \), it reduces to the completely randomized experiment. Let \( R_i \) and \( Z_i \) be the sampling indicator and treatment assignment indicator with \( R_i = 1 \) if unit \( i \) is sampled, and \( 0 \) otherwise, and \( Z_i = 1 \) if unit \( i \) is assigned to the treatment group, and \( 0 \) otherwise. Denote the set of the sampled units by \( S = \{ i \in \{1, \ldots, N \} : R_i = 1 \} \). By design, \( Z_i \) is not defined if \( i \not\in S \). Let \( Y_i = Z_iY_i(1) + (1 - Z_i)Y_i(0) \) be the observed potential outcome for the sampled unit \( i \). The average treatment effect of interest in completely randomized survey experiments is \( \tau_{\text{crs}} = \frac{\sum_{i \in S} Y_i(1) - Y_i(0)}{n} \). The difference-in-means estimator \( \hat{\tau}_{\text{crs}} = \sum_{i \in S} Y_i/n - \sum_{i \in S} (1 - Z_i)Y_i/n_0 \) is an unbiased estimator of \( \tau \) (Imbens and Rubin, 2015; Yang et al., 2021). Here \( n_1 = \sum_{i \in S} Z_i \) and \( n_0 = \sum_{i \in S} (1 - Z_i) \) are the (fixed) numbers of treated and control units, respectively. Let \( p_z = n_z/n \) \((z = 0, 1)\).

We can observe two kinds of covariates: \( v_i \in \mathbb{R}^{k_1} \) \((1 \leq i \leq N)\) which is available at the sampling stage and usually collected from baseline survey conducted by some investigators or previous studies on the same target population, and \( x_i \in \mathbb{R}^{k_2} \) \((i \in S)\) which is available at the treatment assignment stage and usually collected after the experiment units are sampled. Here, \( v_i \) can be a subset of \( x_i \).

By a slight abuse of the notation, we define the following finite-population quantities of the \( N \) units. We use \( S_x^2, S_z^2, S_{xv}^2, S_{xz}^2 \) to denote corresponding finite-population variances and \( S_{xx} = S_{xz}^T, S_{uv} = S_{xz}^T, S_{vx} = S_{xz}^T, S_{uv} = S_{xz}^T \) to denote the corresponding finite-population covariances. Let \( \bar{Y}(z), \bar{x}, \bar{v} \) be the finite-population means.

To motivate the form of weighted regression adjustment, we consider a general form of linearly adjusted estimator proposed by Yang et al. (2021):

\[
\hat{\tau}_{\text{crs}}(\beta, \gamma) = \hat{\tau}_{\text{crs}} - \beta^T \hat{\tau}_{\text{crs}, x} - \gamma^T \hat{\delta}_v,
\]

where

\[
\hat{\tau}_{\text{crs}, x} = \sum_{i \in S} Z_i x_i/n_1 - \sum_{i \in S} (1 - Z_i) x_i/n_0,
\]

\[
\hat{\delta}_v = \hat{v} - \bar{v}, \quad \hat{v} = \sum_{i \in S} v_i/n.
\]

The linearly adjusted estimator adjusts two kinds of covariate imbalances: the difference between the sample mean and population mean of the covariates measured by \( \hat{\delta}_v \), and the difference between the covariate means in the treatment and control groups measured by \( \hat{\tau}_{\text{crs}, x} \). Note that \( \hat{\tau}_{\text{crs}}(\beta, \gamma) \) is equal to the difference-in-means estimator applied to the observed adjusted potential outcomes,

\[
Y_i(z; \beta, \gamma) = Y_i(z) - ((z - p_0)(v_i - \bar{v}))^T \gamma - x_i^T \beta.
\]

Equation (7) catalyzes the use of covariates \((z - p_0)(v_i - \bar{v})\) and \( x_i \) in the regression adjustment. Therefore, we propose a WLS regression adjustment of the following form

\[
Y_i \overset{\text{wls}}{\sim} 1 + Z_i + x_i + (Z_i - p_0)(v_i - \bar{v})
\]
with weights \( w_i = Z_i/p_i^2 + (1 - Z_i)/p_0^2 \). Define \( \hat{\tau}_{en} \) as the estimated coefficient of \( Z_i \) through the WLS.

**Remark 5.** Note that the regression formula only needs to center \( v_i \) at its finite-population mean \( \bar{v} \). In practice, it is very difficult to collect \( v_i \) and \( x_i \) for the units that are not in the sample, that is, \( i \notin S \). Fortunately, \( \bar{v} \) is still available from some baseline surveys. Thus, ToM regression adjustment is still applicable.

By Yang et al. (2021, Lemma B1), \( n^{1/2}(\hat{\tau}_{en} - \tau_{en}, \tau_{en}^\top, \delta_v^\top) \) has mean zero and covariance

\[
\begin{pmatrix}
V_{en,\tau \tau} & V_{en,\tau v} & V_{en, v v} \\
V_{en, v \tau} & V_{en, v x} & V_{en, v v} \\
V_{en, v v} & V_{en, v v} & V_{en, v v}
\end{pmatrix}
= \begin{pmatrix}
p_1^{-1}S^2 + p_0^{-1}S^2 - fS^2_v & p_1^{-1}S_{1x} + p_0^{-1}S_{0x} & (1 - f)S_{v v}
p_1^{-1}S_{1x} + p_0^{-1}S_{0x} & (p_1p_0)^{-1}S^2 & 0 \\
(1 - f)S_{v v} & 0 & (1 - f)S^2_v
\end{pmatrix}.
\]

The optimal projection coefficients \( \beta_{en} = V_{en, xx}^{-1}V_{en, tert} \) and \( \gamma_{en} = V_{en, v v}^{-1}V_{en, tert} \) produce the minimum variance,

\[
\text{var}\{\hat{\tau}_{en}(\beta_{en} ; \gamma_{en})\} = V_{en,\tau \tau} - V_{en,\tau x}V_{en,xx}^{-1}V_{en,\tau x} - V_{en, v v}V_{en, v v}^{-1}V_{en, v v} - V_{en, tert}V_{en, tert}.
\]

Under Assumption 4 below, we demonstrate the asymptotic normality and optimality of \( \hat{\tau}_{en} \) in Theorem 6.

**Assumption 4.** As \( n \to \infty \), for \( z = 0, 1 \),

(i) \( f \) has a limit in \([0, 1]\) and \( p_1 \) has a limit in \((0, 1)\);

(ii) \( S_{x z}^2, S_{v z}, S_{x x}^2, S_{v v}^2, S_{x z}^2 \) have finite limits, and the limit of \( \text{var}\{\hat{\tau}_{en}(\beta_{en} ; \gamma_{en})\} \) is positive while the limits of \( S_{v v}^2 \) and \( S_{x z}^2 \) are nonsingular;

(iii) \( \max_{i=1}^n |Y_i(z) - \bar{Y}(z)|^2 = o(n) \), \( \max_{i=1}^n |x_i - \bar{x}|^2 = o(n) \), \( \max_{i=1}^n \|v_i - \bar{v}\|_\infty^2 = o(n) \).

**Theorem 6.** Under Assumption 4, \( n^{1/2}(\hat{\tau}_{en} - \tau_{en}) \) is asymptotic normal with zero mean and variance \( V_{en, \tau \tau} - V_{en, \tau x}V_{en, xx}^{-1}V_{en, \tau x} - V_{en, v v}V_{en, v v}^{-1}V_{en, v v} - V_{en, tert}V_{en, tert} \). Moreover, \( \hat{\tau}_{en} \) is optimal with minimum asymptotic variance in the class of linearly adjusted estimators \( \{\hat{\tau}_{en}(\beta, \gamma) : \beta \in \mathbb{R}^{k_2}, \gamma \in \mathbb{R}^{k_1}\} \).

We can estimate the variance of \( \hat{\tau}_{en} \) by the heteroscedasticity-robust standard error. Let \( \hat{V}_{HC(j, en)} \) for \( j = 0, 1, 2, 3 \) denote the variance estimator of \( \hat{\tau}_{en} \) from the regression formula (8) corresponding to HC(j).

**Theorem 7.** Under Assumption 4, for \( j = 0, 1, 2, 3 \),

\[
\hat{V}_{HC(j, en)} = n^{-1} \min_{\beta, \gamma} \left\{ p_1^{-1}S^2_{1}(\beta, \gamma) + p_0^{-1}S^2(\beta, \gamma) \right\} + o_P(n^{-1}),
\]

where \( S^2_{z}(\beta, \gamma) = (N - 1)^{-1} \sum_{i=1}^N \{Y_i(z) - \bar{Y}(z) - (x_i - \bar{x})^\top \beta - (z - p_0)(v_i - \bar{v})^\top \gamma\}^2 \).

It is easy to show that \( \text{var}\{\hat{\tau}_{en}(\beta_{en} ; \gamma_{en})\} \) can be derived by replacing \( Y_i(z) \) by the adjusted potential outcome \( Y_i(z; \beta_{en} ; \gamma_{en}) \) in the formula of \( V_{str, \tau \tau} \). The optimality of \( (\beta_{en} ; \gamma_{en}) \) implies that

\[
\text{var}\{\hat{\tau}_{en}(\beta_{en} ; \gamma_{en})\} = n^{-1} \min_{\beta, \gamma} \left\{ p_1^{-1}S^2_{1}(\beta, \gamma) + p_0^{-1}S^2(\beta, \gamma) - fS^2_v(\gamma) \right\},
\]

(10)
where
\[ S_\tau^2(\gamma) = (N - 1)^{-1} \sum_{i=1}^{N} \left\{ \tau_i - \tau - (v_i - \bar{v})^\top \gamma \right\}^2. \]

Equations (9) and (10) indicate that \( \hat{V}_{HC_{j,crs}} \) is an asymptotic conservative estimator of \( \text{var}\{\hat{\tau}_{crs}(\beta_{opt_{crs}}, \gamma_{opt_{crs}})\} \), and thus an asymptotic conservative estimator of \( \text{var}(\hat{\tau}_{tom_{crs}}) \). Therefore, the Wald-type confidence intervals
\[
\left[ \hat{\tau}_{tom_{crs}} + (\hat{V}_{HC_{j,crs}})^{1/2} q_{\alpha/2}, \hat{\tau}_{tom_{crs}} + (\hat{V}_{HC_{j,crs}})^{1/2} q_{1-\alpha/2} \right], \quad j = 0, 1, 2, 3,
\]
have asymptotic coverage rates greater than or equal to \( 1 - \alpha \).

With the assumption that the units are a random sample from an infinite superpopulation, Negi and Wooldridge (2021) demonstrated that the variance estimator constructed by the with-interaction regression is anti-conservative if the covariates \( x_i \) are not centered at their finite-population mean but at their sample mean which introduces an extra variability. This conclusion holds for completely randomized survey experiments with \( 0 < f < 1 \). In practice, \( \bar{x} \) is often not available; consequently, the with-interaction regression adjustment is not applicable for \( 0 < f < 1 \). In contrast, ToM regression adjustment does not require the centering of covariates \( x_i \) at \( \bar{x} \). The resulting point estimator is consistent and asymptotically normal and the variance estimator is asymptotically conservative regardless of \( f \).

5. Numerical studies

In this section, we compare the finite-sample performance of the point estimator and variance estimator derived by ToM regression adjustment with existing competitors in the literature in completely randomized experiments, stratified randomized experiments, and completely randomized survey experiments.

5.1. Complete randomized experiments

In his seminal paper, Lin (2013) demonstrated the equivalence of with-interaction regression adjustment and ToM regression adjustment in a low-dimensional and large-sample setting that the asymptotic theory works perfectly. In this section, we consider a relatively large dimension of covariates compared to the sample size. We further investigate how “imbalance in information” between treatment and control groups can influence the performance of the estimators, which is reflected by \( p_z \) and the signal-to-noise ratio defined later.

We set \( n = 100 \) and generate data using the following model:
\[ Y_i(z) = f_z(x_i) + e_i(z), \quad \text{with} \quad f_z(x_i) = \alpha_z + x_i^\top \beta_z, \quad z = 0, 1, \quad i = 1, \ldots, n, \quad (11) \]
where \((\alpha_z, \beta_z)\) has independent and identically distributed (i.i.d.) entries generated from \( t_3, t\)-distribution with three degrees of freedom, for \( z = 0, 1 \). Thus, there is heterogeneity between treatment and control groups. The covariates \( x_i \)'s are realizations of independent random vectors drawn from \( \mathcal{N}(0, \Sigma) \) with \( \Sigma_{ij} = 0.6 \delta_{ij} + 0.4 (1 \leq i, j \leq k) \), where \( \delta_{ij} = 1 \) if \( i = j \), and 0 otherwise. The errors \( e_i(z) \)'s are realizations of i.i.d. normal random variables with zero mean and variance fulfilling a given signal-to-noise ratio \( \text{SNR}_z \), that is, ratio of the finite-population variance of \( f_z(x_i) \) to that of \( e_i(z) \). After generation, \( \{(Y_i(1), Y_i(0), x_i)\}_{i=1}^{N} \) are fixed. The treatment assignment stage assigns \( n_1 = p_1 n \) units to the treatment group and a completely randomized experiment is simulated 1000 times.
We focus on the root mean squared errors (RMSE) of point estimators and empirical coverage probabilities of 95% confidence intervals. We vary the SNR \( z \) (\( z = 0, 1 \)), \( k \), and \( p_1 \) in each scenario. Table 1 presents the values of the factors considered in the simulation. Each scenario is repeated under 100 different random seeds.

Table 1: Parameters in simulation

|                         | \( p_1 \) | \( k \) | SNR0 | SNR1  |
|-------------------------|----------|--------|------|-------|
| completely randomized experiments | 0.3, 0.4, 0.5 | \{1, 5, 9, 13, 17, 21, 25, 29\} | \{0.25, 0.5, 1, 2\} | \{0.25, 0.5, 1, 2\} |
| stratified randomized experiments | 0.3, 0.4, 0.5 | \{1, 5, 9, 13, 17, 21, 25, 29\} | \{0.25, 0.5, 1, 2\} | \{0.25, 0.5, 1, 2\} |
| completely randomized survey experiments | 0.3, 0.4, 0.5 | \{2, 5, 8, 11, 14, 17\} | \{0.25, 0.5, 1, 2\} | \{0.25, 0.5, 1, 2\} |

Figure 1 depicts the percentage reduction in RMSE of \( \hat{\tau}_{\text{tom}} \) versus \( \hat{\tau}_{\text{lin}} \), that is, \( \frac{\text{RMSE}(\hat{\tau}_{\text{lin}})}{\text{RMSE}(\hat{\tau}_{\text{tom}})} - 1 \) when \( p_1 = 0.3 \). The results for \( p_1 = 0.4, 0.5 \) are provided in the Supplementary Material. It can be observed that the RMSE of \( \hat{\tau}_{\text{tom}} \) is overall smaller than that of \( \hat{\tau}_{\text{lin}} \) and the percentage reduction in RMSE increases as \( k \) becomes larger. ToM regression-adjusted estimator \( \hat{\tau}_{\text{tom}} \) is clearly advantageous when the majority group (control group) has a larger SNR and the minority group (treatment group) has a smaller SNR. This is because \( \hat{\tau}_{\text{tom}} \) uses the data from both groups in a pooled fashion, with larger weights bestowed to the minority group and \( \hat{\tau}_{\text{lin}} \) in a separate fashion with equal weights. Therefore, the performance of \( \hat{\tau}_{\text{lin}} \) heavily depends on how well it estimates the adjusted coefficient in the minority group. When the minority group has a small SNR, the adjusted coefficient may be poorly estimated by \( \hat{\tau}_{\text{lin}} \).

Figure 2 depicts the coverage probabilities of 95% confidence intervals constructed by \((\hat{\tau}_{\text{tom}}, \hat{V}_{\text{tom}}^{\text{HC0}})\), \((\hat{\tau}_{\text{tom}}, \hat{V}_{\text{lin}}^{\text{HC0}})\), and \((\hat{\tau}_{\text{lin}}, \hat{V}_{\text{lin}}^{\text{HC0}})\) when \( p = 0.3 \). It can be observed that these three methods tend to have worse coverage probabilities when \( k \) becomes larger. Combination of \((\hat{\tau}_{\text{tom}}, \hat{V}_{\text{tom}}^{\text{HC0}})\) is the most robust under all scenarios. Similar results were observed by Lei and Ding (2021): \( \hat{V}_{\text{lin}}^{\text{HC0}} \) tends to underestimate the variance for large \( k \). In contrast, \( \hat{V}_{\text{tom}}^{\text{HC0}} \) provides a better variance estimation for large \( k \). Combination of \((\hat{\tau}_{\text{tom}}, \hat{V}_{\text{tom}}^{\text{HC0}})\) has larger coverage probabilities on average than \((\hat{\tau}_{\text{lin}}, \hat{V}_{\text{lin}}^{\text{HC0}})\) when \( k \) is large. Therefore, its use is recommended. If one prefers a less conservative inference when \( k \) is small, combination of \((\hat{\tau}_{\text{tom}}, \hat{V}_{\text{lin}}^{\text{HC0}})\) is recommended. Moreover, all combinations have better coverage probabilities if the minority group has a larger SNR and majority group has a smaller SNR. In contrast, all combinations have worse coverage probabilities if the minority group has a smaller SNR and majority group has a larger SNR.
5.2. Stratified randomized experiments

We consider three kinds of strata size distributions: (1) many small strata (MS); (2) a few large strata (FL); and (3) many small strata compounded with a few large strata (MS+FL). For each scenario, strata sizes \( \{n_h\}_{h=1}^{H} \) are generated as independent samples with (1) \( H = 20 \) from uniform distribution on \( \{10, 11, \ldots, 20\} \); (2) \( H = 2 \) from uniform distribution on \( \{140, 141, \ldots, 160\} \); and (3) \( H = 12 \) with 10 strata sizes from uniform distribution on \( \{10, 11, \ldots, 20\} \) and 2 strata sizes from uniform distribution on \( \{140, 141, \ldots, 160\} \).

The potential outcomes are generated from the following random effect model:

\[
Y_{hi}(z) = f_{hz}(x_{hi}) + e_{hi}(z), \quad \text{with} \quad f_{hz}(x_{hi}) = \alpha_{hz} + x_{hi}^\top \beta_{hz},
\]

\( z = 0, 1, \quad h = 1, \ldots, H, \quad i = 1, \ldots, n_h, \)

where the intercepts and slopes are generated by \( \beta_{hz} = \beta_z + \zeta_{hz} \) and \( \alpha_{hz} = \alpha_z + \eta_{hz} \) with \( (\alpha_z, \beta_z) \) and \( (\eta_{hz}, \zeta_{hz}) \) \( z = 0, 1 \) embodying i.i.d. entries generated from \( t_3 \) and standard normal distribution, respectively. The covariates \( x_i \)'s are realizations of independent random vectors of length \( k \) from \( N(0, \Sigma) \) with \( \Sigma_{ij} = 0.6\delta_{ij} + 0.4 \). \( e_{hi}(z) \)'s are realizations of i.i.d. normal random variables with zero mean and variance fulfilling a given signal-to-noise ratio SNR\( z \), that is, the ratio of the finite-population variance of \( f_{hz}(x_{hi}) \) to that of \( e_{hi}(z) \).

We ensure at least two units in each treatment arm for each stratum. The number of units assigned to treatment \( n_{h1} \)'s are generated by \( n_{h1} = \lfloor c_h n_h \rfloor \), truncated at 2 and \( n_h - 2 \), where \( c_h \)'s are i.i.d. samples from Beta distribution Beta(4, 5). We vary the strata size distribution, SNR\( z \), and \( k \) in each scenario. Values of these factors are presented in Table 1. Each scenario is repeated under 100 random seeds. For each seed and each scenario, we simulate the stratified...
randomized experiments 1000 times and compute the empirical RMSE of point estimators and empirical coverage probabilities of 95% confidence intervals.

So far, Lin’s with-interaction regression adjustment has not been extended to stratified randomized experiments. Therefore, we consider constructing point and variance estimators from the conditional inference or projection perspective and using a plug-in principle (Yang et al., 2021; Wang et al., 2021; Liu et al., 2021). Recall that the optimal linearly adjusted coefficient is \[ \beta_{\text{opt}}^{\text{str}} = V_{\text{str},xx}^{-1} V_{\text{str},x}. \] Let \( s_{h1}, s_{h0}, s_{h1}^2, \) and \( s_{h0}^2 \) be the sample analogs of \( S_{h1}, S_{h0}, S_{h1}^2, \) and \( S_{h0}^2. \) We estimate \( \beta_{\text{str}}^{\text{opt}} \) by \( \hat{\beta}_{\text{str}}^{\text{plg}} = V_{\text{str},xx}^{-1} \hat{V}_{\text{str},x}. \) The plug-in principle is also used to estimate the normal component’s variance in the asymptotic distribution of \( \hat{\tau}_{\text{str}} \) under stratified rerandomization (Wang et al., 2021). This is equal to the variance of the optimal linearly adjusted estimator \( \hat{\tau}_{\text{str}}(\beta_{\text{str}}^{\text{opt}}). \) We follow their procedure to derive a conservative variance estimator of \( \hat{\tau}_{\text{str}}^{\text{plg}} \),

\[
\hat{V}_{\text{str},x} = \sum_{h=1}^{H} \pi_h p_{h1}^{-1} s_{h1} x + \pi_h p_{h0}^{-1} s_{h0} x.
\]

Therefore, \( \hat{\tau}_{\text{str}}(\beta_{\text{str}}^{\text{opt}}) \) can be estimated by \( \hat{\tau}_{\text{str}}^{\text{plg}} = \hat{\tau}_{\text{str}} - (\hat{\beta}_{\text{str}}^{\text{plg}}) \hat{\beta}_{\text{str}}^{\text{opt}}. \) The plug-in principle is also used to estimate the normal component’s variance in the asymptotic distribution of \( \hat{\tau}_{\text{str}} \) under stratified rerandomization (Wang et al., 2021). This is equal to the variance of the optimal linearly adjusted estimator \( \hat{\tau}_{\text{str}}(\beta_{\text{str}}^{\text{opt}}). \) We follow their procedure to derive a conservative variance estimator of \( \hat{\tau}_{\text{str}}^{\text{plg}}, \)

\[
\hat{V}_{\text{str},x} = n_{\text{str}}^{-1}(\hat{V}_{\text{str},xx} - \hat{V}_{\text{str},x} V_{\text{str},xx}^{-1} \hat{V}_{\text{str},x}), \quad \hat{V}_{\text{str},x} = \sum_{h=1}^{H} \pi_h \{ s_{h1}^2 p_{h1}^{-1} + s_{h0}^2 p_{h0}^{-1} \}.
\]

Figure 2 depicts the percentage reduction in RMSE of \( \hat{\tau}_{\text{str}}^{\text{tom}} \) versus \( \hat{\tau}_{\text{str}}^{\text{plg}}. \) Figure 4 presents the empirical coverage probabilities of 95% confidence intervals constructed by \( (\hat{\tau}_{\text{str}}^{\text{tom}}, \hat{V}_{\text{HC2, str}}) \) and \( (\hat{\tau}_{\text{str}}^{\text{plg}}, \hat{V}_{\text{str}}^{\text{plg}}). \) Both figures are results of many small strata scenario. The results of other scenarios
are similar so we degrade them to the Supplementary Material. It can be observed that, \( \hat{\tau}_{\text{tom}}^{\text{str}} \) dominates the plug-in estimator under all scenarios, especially when the dimension of covariates grows. The increasing outliers in the boxplot as SNR\( z \) and dimension of covariates grow imply that \( \hat{\tau}_{\text{tom}}^{\text{str}} \) is more robust than the plug-in estimator under these scenarios. Moreover, Figure 4 shows that the plug-in variance estimator tends to underestimate the true sampling variance and produce confidence intervals with coverage probabilities lower than the nominal level when the dimension of covariates is large. Therefore, we recommend \( (\hat{\tau}_{\text{tom}}^{\text{str}}, \hat{V}_{\text{HC2, str}}) \) for stratified randomized experiments.

### 5.3. Completely randomized survey experiments

We set the population size \( N = 10000 \) and sampling fraction \( f = 0.01 \) to generate data using the same model as (11). Let \( v_i = (x_{i1}, x_{i2}) \) be the covariates available at the sampling stage. We use \( (x_i, (Z_i - p_0)v_i) \) in ToM regression adjustment, with \( k + 2 \) dimensions. We set \( n = Nf \) for the sampling stage and \( p = 0.3 \) for the treatment assignment stage. We simulate the completely randomized survey experiments 1000 times to compute the empirical RMSE of point estimators and empirical coverage probabilities of 95% confidence intervals. We vary the SNR\( z \) and \( k \) in each scenario. Table 1 presents the values of these factors considered in the simulation. Each scenario is repeated under 100 different random seeds.

Let \( s^2_{x(z)} \) and \( s^2_{v(z)} \) be the sample covariances of covariates under treatment arm \( z \). Let \( s^2_{z} \) be the sample variance of \( Y_i(z) \) and \( s_{uz}, s_{xz} \) (\( z = 0, 1 \)) be the sample covariances between covariates and outcomes. Yang et al. (2021) used the plug-in principle to derive linearly adjusted point and variance estimators. The point estimator is derived by replacing the optimal projection coefficients \( (\rho_{\text{opt}}^{\text{crs}}, \gamma_{\text{opt}}^{\text{crs}}) \) in the optimal linearly adjusted estimator with their consistent estimators \( (\hat{\rho}_{\text{plg}}^{\text{crs}}, \hat{\gamma}_{\text{plg}}^{\text{crs}}) \),
Figure 4: Coverage probabilities in stratified randomized experiments with many small strata.

\[
\hat{\beta}_{\text{crs}}^{\text{plg}} = p_0\{s_{x(1)}^2\}^{-1}s_{x1} + p_1\{s_{x(0)}^2\}^{-1}s_{x0}, \quad \hat{\gamma}_{\text{crs}}^{\text{plg}} = \{s_{v(1)}^2\}^{-1}s_{v1} - \{s_{v(0)}^2\}^{-1}s_{v0}.
\]

The variance estimator \(\hat{V}_{\text{HC3,crs}}\) is derived using the estimated adjusted potential outcomes \(Y_i(z; \hat{\beta}_{\text{crs}}^{\text{plg}}, \hat{\gamma}_{\text{crs}}^{\text{plg}})\) to replace \(Y_i(z)\) in \(n^{-1}(s_{p1}^2p_1^{-1} + s_{p0}^2p_0^{-1})\). Both \(\hat{V}_{\text{HC0,crs}}\) and Yang et al.’s variance estimator tend to underestimate the true sampling variance for large \(k\) in finite samples. To remedy this issue, we use the HC3 type estimator \(\hat{V}_{\text{HC3,crs}}\) suggested by Lei and Ding (2021).

Figure 5 depicts the percentage reduction in RMSE of \(\hat{\tau}_{\text{crs}}^{\text{tom}}\) versus \(\hat{\tau}_{\text{crs}}^{\text{plg}}\). Similar to the completely randomized experiments, \(\hat{\tau}_{\text{crs}}^{\text{tom}}\) outperforms \(\hat{\tau}_{\text{crs}}^{\text{plg}}\) when the dimension of covariates grows. The trend becomes more evident when the majority group has a larger SNR and the minority group has a smaller SNR. Figure 6 depicts the coverage probabilities of 95% confidence intervals constructed by \((\hat{\tau}_{\text{crs}}^{\text{tom}}, \hat{V}_{\text{HC3,crs}})\) and \((\hat{\tau}_{\text{crs}}^{\text{plg}}, \hat{V}_{\text{crs}}^{\text{plg}})\). It can be observed that the combination of \((\hat{\tau}_{\text{crs}}^{\text{tom}}, \hat{V}_{\text{HC3,crs}})\) maintains an average of 95% coverage probabilities, while the combination of \((\hat{\tau}_{\text{crs}}^{\text{plg}}, \hat{V}_{\text{crs}}^{\text{plg}})\) tends to have low coverage probabilities for large \(k\) and performs worse when the majority group has a larger SNR and the minority group has a smaller SNR. Therefore, we recommend \((\hat{\tau}_{\text{crs}}^{\text{tom}}, \hat{V}_{\text{HC3,crs}})\) for analyzing completely randomized survey experiments.

6. Applications

6.1. The “opportunity knocks” experiment

The “opportunity knocks” (OK) experiment (Angrist et al., 2014) was a stratified randomized experiment launched to evaluate the effect of financial incentive on college students’ academic
performance. The experiment included first- and second-year students who applied for financial aid at a large Canadian commuter university. Based on sex and discretized high school grades, the students were grouped into 8 strata with strata sizes ranging from 46 to 95. In each stratum, approximately 25 students received the treatment. Therefore, the $p_{h1}$'s varied across strata. The grade point average (GPA) at the end of the fall semester was the outcome of interest. We consider 6 covariates in ToM regression adjustment: high school grade, previous year GPA, age, whether the student’s mother tongue is English, whether the student lives at home, and whether the student has high concern about the funds.

Table 2 presents $\hat{\tau}_{str,x}$, the adjusted coefficient $\hat{\beta}_{str}$, and their hadamard product. We can see that $\hat{\tau}_{str}^{tom}$ adjusts $\hat{\tau}_{str}$ because the treatment group’s previous year GPA is lower on average, and more students live at home and have high concerns about the funds.

Figure 7 depicts the average treatment effect estimators, standard errors, and 95% confidence intervals. Both ToM regression-adjusted and unadjusted estimators show that the average treatment effect is insignificant. That is, we do not have sufficient evidence to support the following: financial incentive affects students’ academic performance. However, it is interesting to see that

Figure 5: Percentage reduction in RMSE of $\hat{\tau}_{crs}^{tom}$ versus $\hat{\tau}_{crs}^{plg}$ for completely randomized survey experiments.

Table 2: $\hat{\tau}_{str,x}$, adjusted coefficient, and their hadamard product

| High school grade | Previous year GPA | Age | Whether the student’s mother tongue is English | Whether the student lives at home | Whether the student has high concern about the funds |
|-------------------|-------------------|-----|-----------------------------------------------|----------------------------------|--------------------------------------------------|
| $\hat{\tau}_{str,x}$ | 0.003             | -0.010 | 0.028                                          | 0.000                           | 0.011                                            | 0.023                                           |
| $\hat{\beta}_{str}$ | 0.186             | 7.543   | -0.089                                         | -0.201                          | -2.447                                          | -1.914                                          |
| $\hat{\tau}_{str,x} \circ \hat{\beta}_{str}$ | 0.001             | -0.075  | -0.002                                         | -0.000                          | -0.027                                          | -0.044                                          |
ToM regression adjustment provides a larger average treatment effect estimator and decreases the estimated standard error by 22.7%.

6.2. Social Trust in Polarized Times

We re-analyze the experimental dataset from Lee (2022) to evaluate the impact of perceived polarization on social trust levels. In this experiment, 1006 Americans over 18 years old were recruited from an online survey panel. We treat the experimental units as a simple random sample from the target population, that is, the entire American population over 18 years old. The experimental units are randomly assigned to read one of the three news articles designed to either promote perceived polarization (more-polarization), reduce perceived polarization (less-polarization), or serve as a control article. We evaluate the treatment effects of more-polarization and less-polarization versus the control. The outcome is an index ranging from 0 to 1, with higher values indicating higher generalized social trust. The following types of covariates are used:

- $x_i$: whether the individual is white and non-Hispanic (race1), whether the individual is black or African American (race2), whether the individual is Hispanic (race3), whether the individual is female (sex), education type (education), household income type (income), marital status (marital), whether the individual does not go to college (nocollege), and age.

- $v_i$: race1, race2, race3, age, and sex. We obtain $\bar{v}$ of the target population from the website of United States Census Bureau.

First, we add the main effect of $x_i$ and $v_i$, quadratic terms of the continuous covariates of $x_i$, and two-way interactions of $x_i$ in the full regression model, which produced a design matrix with
50 columns. Then we use forward-backward stepwise regression to obtain a reduced model with 4 and 9 covariates entering ToM regression adjustment for the treatment effects of more-polarization and less-polarization versus the control, respectively. For both regression adjustments, none of the \( v_i \) enters the model.

For the treatment effect of less-polarization versus control, Table 3 and Figure 7 show that ToM regression adjusts upwards \( \hat{\tau}_{crs,x} \) mainly because the treatment group is 0.2 years younger than the control group. Both ToM regression-adjusted and unadjusted estimators indicate that the average treatment effect is significant, that is, less-polarization articles significantly affect people’s social trust. In contrast, the treatment effect of more-polarization versus control is insignificant as presented by Figure 7. ToM regression slightly adjusts \( \hat{\tau}_{crs,x} \); see Table 4. Compared to the unadjusted estimator, ToM regression adjustment decreases the estimated standard error by 7.4\% and 4.6\%, respectively, for the less-polarization versus control and more-polarization versus control.

### 7. Extension to rerandomization

Regression adjustment is used at the analysis stage to adjust for covariate imbalance. Rerandomization is an alternative approach achieving covariate balance in the design stage (see, e.g., Morgan and Rubin, 2012, 2015; Li et al., 2018; Li and Ding, 2020; Li et al., 2020; Wang et al., 2021; Zhao and Ding, 2021b; Lu et al., 2022). Recent work by Li and Ding (2020), Wang et al. (2021), and Zhao and Ding (2021b) showed that the combination of rerandomization and Lin’s with-interaction regression adjustment can further improve the efficiency if the analysis stage utilizes more covariate information than the design stage. The same conclusion holds true for the combination of rerandomization and ToM regression adjustment.

In randomized experiments, it is common that the covariates available at the design stage are a subset or linear combinations of the covariates available at the analysis stage. In this case, the asymptotic normality and optimality of the ToM regression-adjusted estimator and the asympt-
8. Discussion

We re-examine ToM regression adjustment and justify its robustness compared to the with-interaction regression adjustment from three perspectives: first, ToM regression adjustment produces less extreme calibrated-weights; second, ToM regression adjustment produces smaller leverage scores; third, when the dimension of covariates is large or there is an imbalance in information between treatment and control groups, ToM regression adjustment produces estimator with smaller mean squared errors and better coverage probabilities. We proved the applicability of ToM regression adjustment to stratified randomized experiments, completely randomized survey experiments and cluster randomized experiments. Under each design, we showed that the ToM regression-adjusted average treatment effect estimator is asymptotically normal and optimal in the class of linearly adjusted estimators. We also studied the asymptotic properties of several heteroscedasticity-robust variance estimators derived from the ToM regression adjustment and found that some of these variance estimators may be anti-conservative. Our results are design-based and allow model misspecification. Lastly, the inferential procedure can be easily implemented by standard statistical software packages.

The asymptotic theory may not be applicable when the number of experimental units is small. In such cases, we suggest using Fisher-randomization tests with studentized test statistics obtained from ToM regression adjustment (Zhao and Ding, 2021a). The Fisher-randomization tests yield finite-sample exact p-values under the sharp null hypothesis and are asymptotically valid under the
weak null hypothesis, with the average treatment effect as zero.

Our asymptotic analysis assumes that the number of covariates is fixed. However, in many randomized experiments, such as A/B tests, the number of covariates can be very large, even larger than the sample size (Bloniarz et al., 2016; Lei and Ding, 2021). ToM regression adjustment can be easily extended to high-dimensional settings by adding an appropriate penalty on the adjusted coefficient. It would be interesting to study the design-based properties of this extension.

Finally, our theory focuses on experimental designs with binary treatment and perfect compliance. In practice, researchers may be interested in the effects of multiple-valued treatments in the presence of noncompliance. It is interesting to extend the applicability of ToM regression adjustment to analyze randomized experiments with multiple-valued treatments (Fisher, 1935; Liu et al., 2021; Ye et al., 2022) and/or noncompliance (Imbens and Angrist, 1994; Angrist and Imbens, 1995; Angrist et al., 1996; Ding and Lu, 2017).

Acknowledgement

This research is supported by the National Natural Science Foundation of China (12071242) and the Guo Qiang Institute of Tsinghua University.

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Supplementary Material

Section A provides parallel results for cluster randomized experiments.
Section B provides additional simulation results.
Section C provides formulas of the heteroskedasticity-robust standard errors HC$_j$ ($j = 0, 1, 2, 3$).
Section D provides proofs for the results under completely randomized experiments.
Section E provides proofs for the results under stratified randomized experiments.
Section F provides proofs for the results under completely randomized survey experiments.

A. ToM regression adjustment in cluster randomized experiments

Cluster randomized experiments randomly assign the treatment at the cluster level with units in the same cluster receiving the same treatment status (Hayes and Moulton, 2017). Cluster randomized experiments have been widely used in empirical research when individual-level treatment assignment is infeasible or inconvenient.

Consider $n_{cl}$ units nested in $m$ clusters of sizes $n_i$ ($i = 1, \ldots, m$, $\sum_{i=1}^{m} n_i = n_{cl}$). By design, $m_1$ clusters are randomly assigned to the treatment group and $m_0 = m - m_1$ clusters are assigned to the control group. Let $Z_i$ be the treatment assignment indicator for cluster $i$. With a slight abuse of notation, let $p_z = m_z/m$. We use $ij$ to index unit $j$ in cluster $i$ ($i = 1, \ldots, m$, $j = 1, \ldots, n_i$). Let $x_{ij}$ and $Y_{ij}(z)$ ($z = 0, 1$) be the covariates and potential outcomes for units $ij$. Let $c_i$ be the cluster-level covariates. The average treatment effect is

$$\tau_{cl} = n_{cl}^{-1} \sum_{i=1}^{m} \sum_{j=1}^{n_i} \{ Y_{ij}(1) - Y_{ij}(0) \}.$$ 

Let $\bar{n} = n_{cl}/m$ be the average cluster size. Let $\bar{Y}_{i}(z) = \bar{n}^{-1} \sum_{j=1}^{n_i} Y_{ij}(z)$ ($z = 0, 1$) be the potential outcome total of cluster $i$ scaled by $\bar{n}^{-1}$ and $\bar{Y}_{i} = Z_i \bar{Y}_{i}(1) + (1 - Z_i) \bar{Y}_{i}(0)$ be the observed scaled potential outcome total. Then, the average treatment effect can be rewritten as

$$\tau_{cl} = m^{-1} \sum_{i=1}^{m} \{ \bar{Y}_{i}(1) - \bar{Y}_{i}(0) \}.$$ 

Similarly, we define scaled covariate total $\bar{x}_{i}$. We can view cluster randomized experiments as complete randomized experiments on the cluster level with cluster-level data $\{(\bar{Y}_{i}, c_{i}, \bar{x}_{i}, Z_{i})\}_{i=1}^{m}$ (Li and Ding, 2017; Middleton and Aronow, 2015). Su and Ding (2021) showed that regression adjustment using scaled covariate total together with cluster size $n_i$ leads to larger variance reduction compared with individual-level regression adjustment. Given assumption similar to Assumption 1 on $\{(\bar{Y}_{i}(1), \bar{Y}_{i}(0), c_{i}, \bar{x}_{i}, n_{i})\}_{i=1}^{m}$, we have results in parallel with those in Section 2 in the main text.

Let $(\hat{\tau}_{cl}^{tom}, \hat{V}_{HC,j,cl})$ be the estimated coefficient and heteroscedasticity-robust variance estimator of $Z_i$ in the following weighted regression:

$$\bar{Y}_{i} \overset{w_i}{\sim} 1 + Z_i + c_i + \bar{x}_{i} + n_{i},$$

where $w_i = Z_i/p_z^2 + (1 - Z_i)/p_0^2$. Corollary 1 below is a direct result of Proposition 1 and Theorem 1.

**Corollary 1.** Under Assumption 1 with $n = m$, $Y_i(z) = \bar{Y}_{i}(z)$ ($z = 0, 1$), $x_i = (c_i, \bar{x}_{i}, n_i)$, (i) $\hat{\tau}_{cl}^{tom}$ is consistent for $\tau_{cl}$, asymptotically normal, and optimal in the class of linearly adjusted estimators,
(ii) the probability limit of $m\hat{V}_{HC,j,cl} (j = 0, 1, 2, 3)$ is larger than or equal to the true asymptotic variance of $\sqrt{m}\hat{\tau}_{tom}$, and (iii) the Wald-type $1 - \alpha$ confidence intervals

$$\left[\hat{\tau}_{cl, j} + \hat{V}_{HC,j,cl}^{1/2} q_{\alpha/2}, \hat{\tau}_{cl, j} + \hat{V}_{HC,j,cl}^{1/2} q_{1 - \alpha/2}\right], \quad j = 0, 1, 2, 3,$$

have asymptotic coverage rates greater than or equal to $1 - \alpha$.

### B. Additional simulation results

Figures 8–11 show the simulation results for completely randomized experiments when $p_1 = 0.4$. Under these two more balanced scenarios, the advantages of $\hat{\tau}_{tom}$ over $\hat{\tau}_{lin}$ are not as significant as that when $p = 0.3$. In particular, when both SNR1 and SNR0 are large, $\hat{\tau}_{tom}$ performs worse than $\hat{\tau}_{lin}$. This may be because the adjusted coefficients in both the treatment and control groups are well estimated by $\hat{\tau}_{lin}$. Therefore, for a nearly balanced design, we still recommend the use of $(\hat{\tau}_{lin}, \hat{V}_{lin})$.

Figures 12–15 show the simulation results for stratified randomized experiments with a few large strata and many small strata compounded with a few large strata. The conclusions are similar to those in the main text.

We also conduct simulation for cluster randomized experiments. The potential outcomes are generated by the following random effect model:

$$Y_{ij}(z) = f_{zi}(x_{ij}) + e_{ij}(z), \quad \text{with} \quad f_{zi}(x_{ij}) = \alpha_{zi} + x_{ij}^T \beta_{zi}, \quad z = 0, 1, \quad i = 1, \ldots, m, \quad j = 1, \ldots, n_i.$$

We set the number of clusters $m = 50$. The cluster sizes $\{n_i\}_{i=1}^m$ are generated uniformly from the set $\{n \in \mathbb{N} \mid 4 \leq n \leq 10\}$. The intercepts and slopes are generated by $\beta_{zi} = \beta_z + \zeta_{zi}$ and

![Figure 8: Percentage reduction in RMSE of $\hat{\tau}_{tom}$ versus $\hat{\tau}_{lin}$ when $p_1 = 0.4$.](image-url)
SNR0: 0.25
SNR0: 0.5
SNR0: 1
SNR0: 2

SNR1: 0.25
SNR1: 0.5
SNR1: 1
SNR1: 2

1 4 7 10 13 16 19
1 4 7 10 13 16 19
1 4 7 10 13 16 19

\[
\alpha_{zi} = \alpha_z + \eta_{zi}, \quad \text{where } (\alpha_z, \beta_z) \text{ and } (\eta_{zi}, \zeta_{zi}) \text{ have i.i.d. entries generated from } t_3 \text{ and standard normal distribution, respectively.}
\]

The covariates \( x_{ij} \)'s are realizations of independent random vectors of length \( k \) from \( N(0, \Sigma) \) with \( \Sigma_{ij} = 0.6 \delta_{ij} + 0.4 \), and \( e_{ij}(z) \)'s are realizations of i.i.d. normal random variables with zero mean and variance fulfilling a given signal-to-noise ratio \( \text{SNR}_z \), i.e., the ratio of the finite-population variance of \( f_{zi}(x_{ij}) \) to that of \( e_{ij}(z) \).

We set the proportion of clusters assigned to the treatment group \( p_1 = 0.3 \). After we have generated the data, we use the scaled cluster totals in the analysis stage. We use \( k + 1 \) covariates \( (\tilde{x}_i, n_i) \) in the regression adjustment as suggested by Su and Ding (2021). Again 1000 cluster randomized experiments are simulated and empirical RMSE and coverage probabilities are computed. We consider scenarios with all parameter values presented in Table 5.

Figures 16 and 17 show the results. The conclusions are similar to those in completely randomized experiments. Despite a few outliers, the trend is more obvious when the data is generated with clustering feature.

---

**Table 5: Parameters in simulation under cluster randomized experiments**

| random seed | 1 : 100          |
|-------------|------------------|
| \( k \)     | \{1, 3, 5, 7, 9\}|
| \( \text{SNR0} \) | \{0.25, 0.5, 1, 2\} |
| \( \text{SNR1} \) | \{0.25, 0.5, 1, 2\} |
Figure 10: Percentage reduction in RMSE of $\hat{\tau}^{\text{tom}}$ versus $\hat{\tau}^{\text{lin}}$ when $p_1 = 0.5$.

C. Heteroskedasticity-robust standard error and notation

Let $Y \in \mathbb{R}^n$ be the outcome vector, $X \in \mathbb{R}^{n \times k}$ be the covariate matrix and $W$ be a diagonal matrix. Consider a weighted regression with working model

$$Y = X\beta + e, \quad e \sim N(0, W^{-1}).$$

The leverage score of the $i$th unit, denoted by $h_i$, is the $i$th diagonal entry of the following matrix:

$$X(X^TWX)^{-1}X^TW.$$

Denote the estimated regression coefficient as $\hat{\beta}$, with

$$\hat{\beta} = (X^TWX)^{-1}X^TWY.$$ 

Let $\hat{e}_i$ be the regression residual of unit $i$. Suppose that the target estimand is $d^T\beta$, where $d$ is a known vector. Then the point estimator is $d^T\hat{\beta}$ and the heteroskedasticity-robust variance estimator is

$$d^T(X^TWX)^{-1}X^TW\Delta WX(X^TWX)^{-1}d,$$

where $\Delta$ is a diagonal matrix consisting of squared scaled residuals $\hat{e}_i^2 = (\eta_i\hat{e}_i)^2$, with $\eta_i$ varying for different estimating methods. In particular, $\eta_i = 1$ for HC$_0$, $\eta_i = \{n/(n - k)\}^{-1/2}$ for HC$_1$, $\eta_i = (1 - h_i)^{-1/2}$ for HC$_2$, and $\eta_i = (1 - h_i)^{-1}$ for HC$_3$.

We use lower case letter "s" to denote sample variance and covariance. For example, $s_{x0}$ is the sample covariance of $x_i$ and $Y_i(0)$, and $s_1^2$ is the sample variance of $Y_i(1)$. We use "($z$)" ($z = 0, 1$) to denote sample mean, variance or covariance computed using samples from treatment arm $z$. For
Figure 11: Coverage probabilities for $p_1 = 0.5$ in completely randomized experiments.

D. Proofs for the results under completely randomized experiments

D.1. Preliminary results

Proposition 2. \( \hat{\tau}^\text{tom} = \hat{\tau} - \hat{\beta}^\top \hat{r}_x \), where

\[
\hat{\beta}^\text{cr} = \left\{ p_1^{-1}(1 - n_1^{-1})s^2_{x(1)} + p_0^{-1}(1 - n_0^{-1})s^2_{x(0)} \right\}^{-1}\left\{ p_1^{-1}(1 - n_1^{-1})s_{x1} + p_0^{-1}(1 - n_0^{-1})s_{x0} \right\}.
\]

Proof. Note that regression with weights $w_i$ is equivalent to OLS regression with data multiplied by $w_i^{1/2}$. By Frisch–Waugh–Lovell (FWL) theorem (Ding, 2021), the estimated coefficient of $x_i$ in
Figure 12: Percentage reduction in RMSE of $\hat{\tau}^{\text{tom}}_{\text{str}}$ versus $\hat{\tau}^{\text{plg}}_{\text{str}}$ in stratified randomized experiments when there are a few large strata.

The weighted regression can be derived by the OLS regression $w_i^{1/2} \tilde{Y}_i \sim w_i^{1/2} \tilde{x}_i$, where

$$\tilde{Y}_i = Y_i - Z_i \tilde{Y}(1) - (1 - Z_i) \tilde{Y}(0), \quad \tilde{x}_i = x_i - Z_i \hat{x}(1) - (1 - Z_i) \hat{x}(0).$$

Then

$$\hat{\beta}_{\text{cr}} = \left( \sum_{i=1}^{n} w_i \tilde{x}_i \tilde{x}_i^\top \right)^{-1} \left( \sum_{i=1}^{n} w_i \tilde{x}_i \tilde{Y}_i \right).$$

Simple algebra yields that

$$\sum_{i=1}^{n} w_i \tilde{x}_i \tilde{x}_i^\top = p_1^{-2}(n_1 - 1)s_{x(1)}^2 + p_0^{-2}(n_0 - 1)s_{x(0)}^2,$$

$$\sum_{i=1}^{n} w_i \tilde{x}_i \tilde{Y}_i = p_1^{-2}(n_1 - 1)s_{x1} + p_0^{-2}(n_0 - 1)s_{x0}.$$

It follows that

$$\hat{\beta}_{\text{cr}} = \{p_1^{-1}(1 - n_1^{-1})s_{x(1)}^2 + p_0^{-1}(1 - n_0^{-1})s_{x(0)}^2\}^{-1} \{p_1^{-1}(1 - n_1^{-1})s_{x1} + p_0^{-1}(1 - n_0^{-1})s_{x0}\}.$$

By the property of OLS regression, $\hat{\tau}^{\text{tom}}$ is the estimated coefficient of $Z_i$ in the WLS regression of $Y_i - x_i^\top \hat{\beta}_{\text{cr}} w_i \approx 1 + Z_i$.

Therefore, $\hat{\tau}^{\text{tom}} = \hat{\tau} - \hat{\beta}_{\text{cr}}^\top \hat{x}$. 

\[\Box\]
Lemma 1 below is from Li et al. (2018, Lemma A16).

**Lemma 1.** Under Assumption 1, 
\[ s^2 - S^2 = o_P(1), \quad s^2_{x(z)} - S^2_x = o_P(1), \quad s_{xz} - S_{xz} = o_P(1), \quad z = 0, 1. \]

**Lemma 2.** Under Assumption 1, \( \hat{\beta}_{cr} - \beta_{cr}^{opt} = o_P(1) \).

**Proof.** By Proposition 2, 
\[ \hat{\beta}_{cr} = \{p_1^{-1}(1 - n_1^{-1})s^2_{x(1)} + p_0^{-1}(1 - n_0^{-1})s^2_{x(0)}\}^{-1}\{p_1^{-1}(1 - n_1^{-1})s_{x1} + p_0^{-1}(1 - n_0^{-1})s_{x0}\}. \]

Under Assumption 1 and by Lemma 1, we have 
\[ (p_1p_0)^{-1}S^2_x - \{p_1^{-1}(1 - n_1^{-1})s^2_{x(1)} + p_0^{-1}(1 - n_0^{-1})s^2_{x(0)}\} = o_P(1), \]
\[ p_1^{-1}S_{x1} + p_0^{-1}S_{x0} - \{p_1^{-1}(1 - n_1^{-1})s_{x1} + p_0^{-1}(1 - n_0^{-1})s_{x0}\} = o_P(1). \]

Therefore, 
\[ \hat{\beta}_{cr} - \{ (p_1p_0)^{-1}S^2_x \} (p_1^{-1}S_{x1} + p_0^{-1}S_{x0}) = o_P(1). \]

By definition, the second term in the left-hand side of the above equation is equal to \( \beta_{cr}^{opt} \). Therefore, 
\[ \hat{\beta}_{cr} - \beta_{cr}^{opt} = o_P(1). \]

---

**Figure 13:** Coverage probabilities in stratified randomized experiments when there are a few large strata.
Figure 14: Percentage reduction in RMSE of \( \hat{\tau}_{\text{str}}^{\text{tom}} \) versus \( \hat{\tau}_{\text{str}}^{\text{plg}} \) in stratified randomized experiments when there are many small strata compounded with a few large strata.

**Lemma 3.** Under Assumption 1,  
\[
\beta_{\text{opt}}^{\text{cr}} = \arg\min_{\beta} \left\{ p_1^{-1} S_{11}(\beta) + p_0^{-1} S_{00}(\beta) \right\}.
\]

**Proof.** Note that  
\[
p_1^{-1} S_{11}(\beta) + p_0^{-1} S_{00}(\beta) = p_1^{-1} \left( S_{11}^2 - 2\beta^T S_x1 + \beta^T S_x^2 \beta \right) + p_0^{-1} \left( S_{00}^2 - 2\beta^T S_x0 + \beta^T S_x^2 \beta \right)
= (p_1^{-1} S_{11}^2 + p_0^{-1} S_{00}^2) - 2\beta^T \left( p_1^{-1} S_x1 + p_0^{-1} S_x0 \right) + (p_1 p_0)^{-1} \beta^T S_x^2 \beta
= V_{\tau\tau} + S_x^2 - 2\beta^T V_{\tau x} + \beta^T V_{xx} \beta,
\]
where the last equality is due to the definition of \( V_{\tau\tau}, V_{\tau x}, \) and \( V_{xx} \).

Taking derivative with respect to \( \beta \), we have  
\[
\arg\min_{\beta} \left\{ p_1^{-1} S_{11}(\beta) + p_0^{-1} S_{00}(\beta) \right\} = V_{xx}^{-1} V_{\tau x} = \beta_{\text{opt}}^{\text{cr}}.
\]

Let \( \hat{\epsilon}_i \) denote the residual of unit \( i \) derived from the ToM regression adjustment. Let \( s_{\epsilon(z)}^2 \) denote the sample variance of the residuals corresponding to treatment arm \( z \), i.e.,  
\[
s_{\epsilon(1)}^2 = (n_1 - 1)^{-1} \sum_{i=1}^{n} Z_i \hat{\epsilon}_i^2, \quad s_{\epsilon(0)}^2 = (n_0 - 1)^{-1} \sum_{i=1}^{n} (1 - Z_i) \hat{\epsilon}_i^2.
\]
Figure 15: Coverage probabilities in stratified randomized experiments when there are many small strata compounded with a few large strata.

**Lemma 4.** Under Assumption 1,
\[
s_{e(1)}^2 - S_{1}^2(\beta_{\text{opt}}) = o_p(1), \quad s_{e(0)}^2 - S_{0}^2(\beta_{\text{opt}}) = o_p(1).
\]

**Proof.** Note that \( \hat{e}_i = \bar{Y}_i - \bar{x}_i^\top \hat{\beta}_{\text{cr}}. \) Therefore,
\[
(n_1 - 1)^{-1} \sum_{i \in S_1} e_i^2 = s_{e(1)}^2 - 2\hat{\beta}_{\text{cr}}^\top s_{x1} + \hat{\beta}_{\text{cr}}^\top S_{x1}^2(\beta_{\text{opt}})^\top S_{x1} + (\beta_{\text{opt}})^\top S_{x1}^2(\beta_{\text{opt}}) + o_p(1)
\]
\[
= S_{1}^2(\beta_{\text{opt}}) + o_p(1).
\]
The second equality is obtained by Lemmas 1 and 2. Similarly, we can prove the second half of Lemma 4.

**Proposition 3 (Li and Ding (2017)).** Under Assumption 1,
\[
n^{1/2} \begin{pmatrix} \hat{\tau} - \tau \\ \hat{\tau}_x \end{pmatrix} \sim N \left( 0, \begin{pmatrix} V_{\tau\tau} & V_{\tau x} \\ V_{x\tau} & V_{xx} \end{pmatrix} \right).
\]

**D.2. Proof of Proposition 1**

**Proof.** Note that
\[
n^{1/2}(\hat{\tau}_{\text{tom}} - \tau) = n^{1/2}(\hat{\tau} - \tau - (\beta_{\text{opt}})^\top \hat{\tau}_x) + n^{1/2}(\beta_{\text{opt}} - \hat{\beta}_{\text{cr}})^\top \hat{\tau}_x \\
= n^{1/2}(\hat{\tau} - \tau - (\beta_{\text{opt}})^\top \hat{\tau}_x) + n^{1/2}o_p(1)O_p(n^{-1/2}) \\
= n^{1/2}(\hat{\tau} - \tau - (\beta_{\text{opt}})^\top \hat{\tau}_x) + o_p(1),
\]
Figure 16: Percentage reduction in RMSE for cluster randomized experiments.

where the first equality is due to Proposition 2 and the second equality is due to Lemma 2 and Proposition 3.

By Proposition 3 and the definition of $\beta_{opt}^{cr}$, we have

$$n^{1/2}\{\hat{\tau} - (\beta_{opt}^{cr})^\top \hat{\tau}_x\} \sim N(0, V_{\tau\tau} - V_{\tau x}V_{xx}^{-1}V_{x\tau}).$$

Compounded with Slusky’s theorem, the conclusion follows. □

D.3. Proof of Theorem 1

Proof. Because completely randomized experiment is a special case of stratified randomized experiment with $H = 1$. The conclusion for $j = 2$ is a direct result of Theorem 5, so we omit its proof. The conclusions for $j = 0, 1, 3$ can be proved with slight modifications of the proof of Theorem 5, so we omit them. □
Figure 17: Coverage probabilities for cluster randomized experiments.

D.4. Proof of Theorem 2

Proof. By Proposition 2,

$$\hat{\tau}_{\text{tom}} = \hat{\tau} - \hat{\tau}_x^T \hat{\beta}_{\text{cr}},$$

$$\hat{\beta}_{\text{cr}} = \left\{ \sum_{i \in S_1} p_1^{-2} \bar{x}_i \bar{x}_i^T + \sum_{i \in S_0} p_0^{-2} \bar{x}_i \bar{x}_i^T \right\}^{-1} \left\{ \sum_{i \in S_1} p_1^{-2} \bar{x}_i Y_i + \sum_{i \in S_0} p_0^{-2} \bar{x}_i Y_i \right\}.$$

Rewritten $\tau_{\text{tom}}$ as $\tau_{\text{tom}} = \sum_{i \in S_1} c_i^{\text{tom}} Y_i - \sum_{i \in S_0} c_i^{\text{tom}} Y_i$, where

$$c_i^{\text{tom}} = n_i^{-1} - \bar{x}_x^T \left\{ p_1^{-2}(n_1 - 1) s_x^2(1) + p_0^{-2}(n_0 - 1) s_x^2(0) \right\}^{-1} p_1^{-2} \bar{x}_i, \quad i \in S_1,$$

$$c_i^{\text{tom}} = n_0^{-1} + \bar{x}_x^T \left\{ p_1^{-2}(n_1 - 1) s_x^2(1) + p_0^{-2}(n_0 - 1) s_x^2(0) \right\}^{-1} p_0^{-2} \bar{x}_i, \quad i \in S_0.$$
Note that
\[ \hat{\tau}^{\text{lin}} = \hat{\tau} - \hat{\tau}_x^\top (p_0 \hat{\beta}_1 + p_1 \hat{\beta}_0), \]
\[ \hat{\beta}_1 = \left\{ \sum_{i \in S_1} \hat{x}_i \hat{x}_j \right\}^{-1} \left\{ \sum_{i \in S_1} \hat{x}_i Y_i \right\} = \left\{ (n_1 - 1) s_{x(1)}^2 \right\}^{-1} \sum_{i \in S_1} \hat{x}_i Y_i, \]
\[ \hat{\beta}_0 = \left\{ \sum_{i \in S_0} \hat{x}_i \hat{x}_j \right\}^{-1} \left\{ \sum_{i \in S_0} \hat{x}_i Y_i \right\} = \left\{ (n_0 - 1) s_{x(0)}^2 \right\}^{-1} \sum_{i \in S_0} \hat{x}_i Y_i. \]

Rewritten \( \hat{\tau}^{\text{lin}} \) as \( \hat{\tau}^{\text{lin}} = \sum_{i \in S_1} c_i^{\text{lin}} Y_i - \sum_{i \in S_0} c_i^{\text{lin}} Y_i \), where
\[ c_i^{\text{lin}} = n_1^{-1} - p_0 \hat{\tau}_x^\top \left\{ (n_1 - 1) s_{x(1)}^2 \right\}^{-1} \hat{x}_i, \quad i \in S_1, \]
\[ c_i^{\text{lin}} = n_0^{-1} + p_1 \hat{\tau}_x^\top \left\{ (n_0 - 1) s_{x(0)}^2 \right\}^{-1} \hat{x}_i, \quad i \in S_0. \]

Next, we prove that \( c_{\text{tom}} \) minimizes the total distance
\[ F(c) = \sum_{i \in S_1} G(c_i n_1) + \sum_{i \in S_0} G(c_i n_0), \quad \text{where} \quad G(x) = (x - 1)^2 / 2, \]
under the constraints (12) and (13) below.
\[ \sum_{i \in S_1} c_i = 1, \quad \sum_{i \in S_0} c_i = 1, \tag{12} \]
\[ \sum_{i \in S_1} c_i x_i - \sum_{i \in S_0} c_i x_i = 0. \tag{13} \]

In contrast, \( c^{\text{lin}} \) minimizes the total distance under the constraints (12) and (14) below.
\[ \sum_{i \in S_1} c_i x_i = \bar{x}, \quad \sum_{i \in S_0} c_i x_i = \bar{x}, \tag{14} \]

Because (14) implies (13), \( F(c_{\text{tom}}) \leq F(c^{\text{lin}}) \).

Denote \( c \) the vector of \( c_i \)'s. Consider the following Lagrangian function:
\[ L^{\text{tom}}(c, \lambda_1, \lambda_0, \lambda_x) = \sum_{i \in S_1} 2^{-1} (c_i n_1 - 1)^2 + \sum_{i \in S_0} 2^{-1} (c_i n_0 - 1)^2 - \]
\[ \lambda_1 \left( \sum_{i \in S_1} c_i - 1 \right) - \lambda_0 \left( \sum_{i \in S_0} c_i - 1 \right) - \lambda_x^\top \left( \sum_{i \in S_1} c_i x_i - \sum_{i \in S_0} c_i x_i \right). \]

Setting the gradient of \( L^{\text{tom}}(c, \lambda_1, \lambda_0, \lambda_x) \) to 0, we have
\[ n_1 (c_i n_1 - 1) - \lambda_1 - \lambda_x^\top x_i = 0, \quad i \in S_1, \tag{15} \]
\[ n_0 (c_i n_0 - 1) - \lambda_0 + \lambda_x^\top x_i = 0, \quad i \in S_0. \tag{16} \]

36
Summarizing equation (15) for \( i \in S_1 \) and by the constraint (12), we have
\[
\lambda_1 = -\lambda_x^\top \hat{x}(1).
\] (17)

Summarizing equation (16) for \( i \in S_0 \) and by the constraint (12), we have
\[
\lambda_0 = \lambda_x^\top \hat{x}(0).
\] (18)

Plugging (17) into (15) and (18) into (16),
\[
\begin{align*}
n_1(c_i n_1 - 1) - \lambda_x^\top \hat{x}_i &= 0, \quad i \in S_1, \quad (19) \\
n_0(c_i n_0 - 1) + \lambda_x^\top \hat{x}_i &= 0, \quad i \in S_0. \quad (20)
\end{align*}
\]

Therefore,
\[
\begin{align*}
c_i &= n_1^{-1} + n_1^{-2} \lambda_x^\top \hat{x}_i, \quad i \in S_1 \quad (21) \\
c_i &= n_0^{-1} - n_0^{-2} \lambda_x^\top \hat{x}_i, \quad i \in S_0 \quad (22)
\end{align*}
\]

Plugging (21) and (22) into (13),
\[
\hat{\tau}_x + \left( \sum_{i \in S_1} n_1^{-2} \hat{x}_i \hat{x}_i^\top + \sum_{i \in S_0} n_0^{-2} \hat{x}_i \hat{x}_i^\top \right) \lambda_x = 0.
\]

Therefore,
\[
\lambda_x = -\left( \sum_{i \in S_1} n_1^{-2} \hat{x}_i \hat{x}_i^\top + \sum_{i \in S_0} n_0^{-2} \hat{x}_i \hat{x}_i^\top \right)^{-1} \hat{\tau}_x = -\left( n_1^{-2}(n_1 - 1)s_{x(1)}^2 + n_0^{-2}(n_0 - 1)s_{x(0)}^2 \right)^{-1} \hat{\tau}_x. \quad (23)
\]

Plugging (23) into (21) and (22), the minimizer of \( F(c) \) under constraints (12) and (13) is
\[
\begin{align*}
c_i &= n_1^{-1} - \hat{\tau}_x^\top \left\{ n_0^{-2}(n_0 - 1)s_{x(0)}^2 + n_1^{-2}(n_1 - 1)s_{x(1)}^2 \right\}^{-1} n_1^{-2} \hat{x}_i \\
&= n_1^{-1} - \hat{\tau}_x^\top \left\{ p_0^{-2}(n_0 - 1)s_{x(0)}^2 + p_1^{-2}(n_1 - 1)s_{x(1)}^2 \right\}^{-1} p_1^{-2} \hat{x}_i, \quad i \in S_1, \\
c_i &= n_0^{-1} + \hat{\tau}_x^\top \left\{ n_0^{-2}(n_0 - 1)s_{x(0)}^2 + n_1^{-2}(n_1 - 1)s_{x(1)}^2 \right\}^{-1} n_0^{-2} \hat{x}_i \\
&= n_0^{-1} + \hat{\tau}_x^\top \left\{ p_0^{-2}(n_0 - 1)s_{x(0)}^2 + p_1^{-2}(n_1 - 1)s_{x(1)}^2 \right\}^{-1} p_0^{-2} \hat{x}_i, \quad i \in S_0.
\end{align*}
\]

Similarly, consider the following lagrangian function:
\[
\mathcal{L}^{\text{lin}}(c, \lambda_1, \lambda_0, \lambda_{x1}, \lambda_{x0}) = \sum_{i \in S_1} 2^{-1}(c_i n_1 - 1)^2 + \sum_{i \in S_0} 2^{-1}(c_i n_0 - 1)^2 - \\
\lambda_1 \left( \sum_{i \in S_1} c_i - 1 \right) - \lambda_0 \left( \sum_{i \in S_0} c_i - 1 \right) - \lambda_{x1}^\top \left( \sum_{i \in S_1} c_i x_i - \bar{x} \right) - \lambda_{x0}^\top \left( \sum_{i \in S_0} c_i x_i - \bar{x} \right).
\]

Setting the gradient of \( \mathcal{L}^{\text{lin}}(c, \lambda_1, \lambda_0, \lambda_{x1}, \lambda_{x0}) \) to 0, we have
\[
\begin{align*}
n_1(c_i n_1 - 1) - \lambda_1 - \lambda_{x1}^\top x_i &= 0, \quad i \in S_1, \quad (24) \\
n_0(c_i n_0 - 1) - \lambda_0 - \lambda_{x0}^\top x_i &= 0, \quad i \in S_0. \quad (25)
\end{align*}
\]
Summarizing equation (24) for \(i \in S_1\) and by the constraint (12), we have
\[
\lambda_1 = -\lambda_{x1}^\top \hat{x}(1).
\]
(26)

Summarizing equation (25) for \(i \in S_0\) and by the constraint (12), we have
\[
\lambda_0 = -\lambda_{x0}^\top \hat{x}(0).
\]
(27)

Plugging (26) into (24) and (27) into (25),
\[
n_1(c_in_1 - 1) - \lambda_{x1}^\top \hat{x}_i = 0, \quad i \in S_1, \\
n_0(c_in_0 - 1) - \lambda_{x0}^\top \hat{x}_i = 0, \quad i \in S_0.
\]
(28)
(29)

Therefore,
\[
c_i = n_1^{-1} + n_1^{-2} \lambda_{x1}^\top \hat{x}_i, \quad i \in S_1, \\
c_i = n_0^{-1} + n_0^{-2} \lambda_{x0}^\top \hat{x}_i, \quad i \in S_0.
\]
(30)
(31)

Plugging (30) and (31) into (14),
\[
\{\hat{x}(1) - \bar{x}\} + \left\{\sum_{i \in S_1} n_1^{-2} \hat{x}_i \hat{x}_i^\top\right\} \lambda_{x1} = 0,
\]
\[
\{\hat{x}(0) - \bar{x}\} + \left\{\sum_{i \in S_0} n_0^{-2} \hat{x}_i \hat{x}_i^\top\right\} \lambda_{x0} = 0.
\]

Therefore,
\[
\lambda_{x1} = -\left\{\sum_{i \in S_1} n_1^{-2} \hat{x}_i \hat{x}_i^\top\right\}^{-1} \{\hat{x}(1) - \bar{x}\} = -\left\{n_1^{-2}(n_1 - 1)s_{x(1)}^2\right\}^{-1} p_0 \hat{\tau}_x,
\]
(32)
\[
\lambda_{x0} = -\left\{\sum_{i \in S_0} n_0^{-2} \hat{x}_i \hat{x}_i^\top\right\}^{-1} \{\hat{x}(0) - \bar{x}\} = \left\{n_0^{-2}(n_0 - 1)s_{x(0)}^2\right\}^{-1} p_1 \hat{\tau}_x.
\]
(33)

Plugging (32) into (30) and (33) into (31), the minimizer of \(F(c)\) under constraints (12) and (14) is
\[
c_i = n_1^{-1} - p_0 \hat{\tau}_x^\top \left\{(n_1 - 1)s_{x(1)}^2\right\}^{-1} \hat{x}_i, \quad i \in S_1,
\]
\[
c_i = n_0^{-1} + p_1 \hat{\tau}_x^\top \left\{(n_0 - 1)s_{x(0)}^2\right\}^{-1} \hat{x}_i, \quad i \in S_0.
\]

D.5. Proof of Theorem 3

Proof. By definition, the leverage score \(h_{tom}^{i}\) is the \(i\)th diagonal element of
\[
X_{tom} \left\{(X_{tom})^\top W X_{tom}\right\}^{-1} (X_{tom})^\top W,
\]

38
where \(X_{\text{tom}}\) is an \(n \times (2 + k)\) matrix with the \(i\)th row of \(X_{\text{tom}}\) being \((1, Z_i, x_i^T)\).

The leverage score \(h_{i}^{\text{lin}}\) is the \(i\)th diagonal element of
\[
X_{\text{lin}} \left\{(X_{\text{lin}}^\top X_{\text{lin}})^{-1} (X_{\text{lin}}^\top),
\right.
\]
where \(X_{\text{lin}} \in \mathbb{R}^{n \times (2+2k)}\) with the \(i\)th row of \(X_{\text{lin}}\) being \((1, Z_i, (x_i - \hat{x})^T, Z_i(x_i - \hat{x})^T)\).

Let \(\tilde{X}_{\text{tom}} \in \mathbb{R}^{n \times (2+k)}\) with the \(i\)th row being \((1 - Z_i, Z_i, \tilde{x}_i^T)\). Let \(X_{\text{lin}} \in \mathbb{R}^{n \times (2+2k)}\) with the \(i\)th row being \((1 - Z_i, Z_i, (1 - Z_i)\tilde{x}_i^T, Z_i\tilde{x}_i^T)\). Since
\[
\tilde{X}_{\text{tom}} = X_{\text{tom}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & I_k & 0 \\ 0 & 0 & 0 & I_k \end{pmatrix} \begin{pmatrix} 1 & 0 & -\hat{x}(0)^T \\ -1 & 1 & 0 & -\hat{x}(1)^T \\ 0 & 0 & I_k & 0 \\ 0 & 0 & 0 & I_k \end{pmatrix},
\]
\[
\tilde{X}_{\text{lin}} = X_{\text{lin}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & I_k & 0 \\ 0 & 0 & 0 & I_k \end{pmatrix} \begin{pmatrix} 1 & 0 & \tilde{x}^T - \hat{x}(0)^T & 0 \\ -1 & 1 & 0 & \tilde{x}^T - \hat{x}(1)^T \\ 0 & 0 & I_k & 0 \\ 0 & 0 & 0 & I_k \end{pmatrix},
\]
then
\[
X_{\text{lin}} \left\{(X_{\text{lin}}^\top X_{\text{lin}})^{-1} (X_{\text{lin}}^\top),
\right.
\]
\[
\tilde{X}_{\text{tom}} = X_{\text{tom}} \left\{(X_{\text{tom}}^\top W X_{\text{tom}})^{-1} (X_{\text{tom}}^\top W,
\right.
\]
\[
\tilde{X}_{\text{lin}} = X_{\text{lin}} \left\{(X_{\text{lin}}^\top W X_{\text{lin}})^{-1} (X_{\text{lin}}^\top W,
\right.
\]

Note that
\[
(\tilde{X}_{\text{lin}}^\top \tilde{X}_{\text{lin}}) = \begin{pmatrix} n_0 & 0 & 0 & 0 \\ 0 & n_1 & 0 & 0 \\ 0 & 0 & (n_1 - 1)s_{x(1)}^2 & 0 \\ 0 & 0 & 0 & (n_0 + 1)s_{x(0)}^2 \end{pmatrix},
\]
\[
(X_{\text{tom}}^\top W X_{\text{tom}})^{-1} = \begin{pmatrix} n_0p_0^2 & 0 & 0 & 0 \\ 0 & n_1p_1^{-2} & 0 & 0 \\ 0 & 0 & p_1^{-2}(n_1 - 1)s_{x(1)}^2 + p_0^{-2}(n_0 - 1)s_{x(0)}^2 \end{pmatrix}.
\]

Therefore,
\[
h_{i}^{\text{lin}} = \begin{cases} n_1^{-1} + \tilde{x}_i^\top \left\{(n_1 - 1)s_{x(1)}^2 \right\}^{-1} \tilde{x}_i, & \text{for } i \in S_1, \\
0 & \text{for } i \in S_0, \end{cases}
\]
\[
h_{i}^{\text{tom}} = \begin{cases} n_1^{-1} + \tilde{x}_i^\top \left\{(n_1 - 1)s_{x(1)}^2 + (p_1/p_0)^2(n_0 - 1)s_{x(0)}^2 \right\}^{-1} \tilde{x}_i, & \text{for } i \in S_1, \\
0 & \text{for } i \in S_0, \end{cases}
\]

Since \(s_{x(1)}^2 \geq 0\) and \(s_{x(0)}^2 \geq 0\), then
\[
\left\{(n_1 - 1)s_{x(1)}^2 \right\}^{-1} \geq \left\{(n_1 - 1)s_{x(1)}^2 + (p_1/p_0)^2(n_0 - 1)s_{x(0)}^2 \right\}^{-1},
\]
\[
\left\{(n_0 - 1)s_{x(0)}^2 \right\}^{-1} \geq \left\{(n_0 - 1)s_{x(0)}^2 + (p_0/p_1)^2(n_1 - 1)s_{x(1)}^2 \right\}^{-1}.
\]

Therefore, \(h_{i}^{\text{tom}} \leq h_{i}^{\text{lin}}\).
E. Proofs for the results under stratified randomized experiments

E.1. Preliminary results

Let \( \bar{x}_{hi} = x_{hi} - Z_i \hat{x}_h(1) - (1 - Z_i) \hat{x}_h(0) \) and \( \bar{Y}_{hi} = Y_{hi} - Z_i \hat{Y}_h(1) - (1 - Z_i) \hat{Y}_h(0) \). Let \( S_{hz} = \{i = 1, \ldots, n_h : Z_{hi} = z\} \) for \( z = 0, 1, h = 1, \ldots, H \).

Proposition 4. \( \hat{\tau}_{str}^{tom} = \hat{\tau}_{str} + \hat{\beta}_{str}^\top \hat{\tau}_{str,x} \), where

\[
\hat{\beta}_{str} = \left( \sum_{h=1}^{H} \sum_{i=1}^{n_h} w_{hi} \bar{x}_{hi} \bar{x}_{hi}^\top \right)^{-1} \left( \sum_{h=1}^{H} \sum_{i=1}^{n_h} w_{hi} \bar{x}_{hi} \bar{Y}_{hi} \right).
\]

Simple algebra gives that

\[
\sum_{h=1}^{H} \sum_{i=1}^{n_h} w_{hi} \bar{x}_{hi} \bar{x}_{hi}^\top = \sum_{h=1}^{H} \left\{ p_h^{-1} n_{h1} s_{hx(1)}^2 + p_h^{-2} n_{h0} s_{hx(0)}^2 \right\},
\]

\[
\sum_{h=1}^{H} \sum_{i=1}^{n_h} w_{hi} \bar{x}_{hi} \bar{Y}_{hi}^\top = \sum_{h=1}^{H} \left\{ p_h^{-2} n_{h1} s_{hx1} + p_h^{-2} n_{h0} s_{hx0} \right\}.
\]

Therefore,

\[
\hat{\beta}_{str} = \left[ \sum_{h=1}^{H} \left\{ \pi_h p_{h1}^{-1} s_{hx(1)}^2 + \pi_h p_{h0}^{-2} s_{hx(0)}^2 \right\} \right]^{-1} \left[ \sum_{h=1}^{H} \left\{ \pi_h p_{h1}^{-1} s_{hx1} + \pi_h p_{h0}^{-1} s_{hx0} \right\} \right].
\]

By the property of OLS, \( \hat{\tau}_{str}^{tom} \) is the estimated coefficient of \( Z_{hi} \) in the WLS regression:

\[
Y_{hi} - \bar{x}_{hi}^\top \hat{\beta}_{str} w_{hi} 1 + Z_{hi} + \sum_{q=2}^{H} (\delta_{hq} - \pi_q) + Z_{hi} \sum_{q=2}^{H} (\delta_{hq} - \pi_q).
\]

It follows that \( \hat{\tau}_{str}^{tom} = \hat{\tau}_{str} + \hat{\beta}_{str}^\top \hat{\tau}_{str,x} \).

Lemma 5. Under Assumption 3, for \( z = 0, 1 \), we have

\[
\sum_{h=1}^{H} \pi_h p_{hz}^{-1} s_{hz}^2 - \sum_{h=1}^{H} \pi_h p_{hz}^{-1} s_{hx(z)}^2 = o_p(1),
\]

\[
\sum_{h=1}^{H} \pi_h p_{hz}^{-1} s_{hz}^2 - \sum_{h=1}^{H} \pi_h p_{hz}^{-2} s_{hx(z)}^2 = o_p(1),
\]

\[
\sum_{h=1}^{H} \pi_h p_{hz}^{-1} s_{hxz} - \sum_{h=1}^{H} \pi_h p_{hz}^{-1} S_{hxz} = o_p(1).
\]
Proof. These are direct results of Lemma 7 in Wang et al. (2021), although Assumption 3 is slightly weaker than that used by Wang et al. (2021).

Lemma 6. Under Assumption 3,
\[ \hat{\beta}_{str} - \beta_{str}^{opt} = o_P(1). \]

Proof. By Lemma 5,
\[ \sum_{h=1}^{H} \pi_h p_{h1}^{-1} s_{hx(1)}^2 - \sum_{h=1}^{H} \pi_h p_{h0}^{-1} s_{hx(0)}^2 = o_P(1), \]
\[ \sum_{h=1}^{H} \pi_h h_{h1}^{-1} s_{hz}^2 - \sum_{h=1}^{H} \pi_h h_{h0}^{-1} s_{hz}^2 = o_P(1). \]

Therefore,
\[ \left\{ \sum_{h=1}^{H} \pi_h p_{h1}^{-1} s_{hx(1)}^2 + \pi_h p_{h0}^{-1} s_{hx(0)}^2 \right\}^{-1} \left( \sum_{h=1}^{H} \pi_h p_{h1}^{-1} s_{hx1} + \pi_h p_{h0}^{-1} s_{hx0} \right) - \]
\[ \left\{ \sum_{h=1}^{H} \pi_h (p_{h1} p_{h0})^{-1} s_{hx}^2 \right\}^{-1} \left( \sum_{h=1}^{H} \pi_h p_{h1}^{-1} s_{hx1} + \pi_h p_{h0}^{-1} s_{hx0} \right) = o_P(1). \]

The first term is \( \hat{\beta}_{str} \) and the second term is \( \beta_{str}^{opt} \). Therefore, the conclusion follows.

Let \( \hat{e}_{hi} \) be the residuals from the weighted regression (4). One of the variance estimator can be derived as
\[ \hat{V}_{str} = n_{str}^{-1} \sum_{h=1}^{H} \left\{ \pi_h p_{h1}^{-1} s_{he(1)}^2 + \pi_h p_{h0}^{-1} s_{he(0)}^2 \right\}, \tag{34} \]
where
\[ s_{he(1)}^2 = (n_{h1} - 1)^{-1} \sum_{i=1}^{n_{h1}} Z_{hi} \hat{e}_{hi}^2, \quad s_{he(0)}^2 = (n_{h0} - 1)^{-1} \sum_{i=1}^{n_{h0}} (1 - Z_{hi}) \hat{e}_{hi}^2. \]

Lemma 7 below shows that (34) is a conservative estimator of the variance of \( \tilde{z}_{str}^{tom} \).

Lemma 7. Under Assumption 3,
\[ n_{str} \hat{V}_{str} = \min_{\beta} \sum_{h=1}^{H} \left\{ \pi_h p_{h1}^{-1} S_{hz1}^2(\beta) + \pi_h p_{h0}^{-1} S_{hz0}^2(\beta) \right\} + o_P(1), \]
where
\[ S_{hz}^2(\beta) = (n_h - 1)^{-1} \sum_{i=1}^{n_h} \{ Y_{hi}(z) - \hat{Y}_{hi}(z) - (x_{hi} - \bar{x}_h)^\top \beta \}^2. \]
Proof. Note that
\[ n_{str} \hat{V}_{str} = \sum_{h=1}^{H} \left\{ \pi_{h} p_{h1}^{-1} s_{he(1)} + \pi_{h} p_{h0}^{-1} s_{he(0)} \right\} \]
\[ = \sum_{h=1}^{H} \pi_{h} p_{h1}^{-1} \left\{ s_{h1} - 2\hat{\beta}_{str}^{\top} s_{hx1} + \hat{\beta}_{str}^{\top} s_{hx(1)} \hat{\beta}_{str} \right\} + \pi_{h} p_{h0}^{-1} \left\{ s_{h0} - 2\hat{\beta}_{str}^{\top} s_{hx0} + \hat{\beta}_{str}^{\top} s_{hx(0)} \hat{\beta}_{str} \right\} \]
\[ = \sum_{h=1}^{H} \pi_{h} p_{h1}^{-1} \left\{ S_{h1}^{2} - 2(\beta_{str}^{opt})^{\top} S_{hx1} + (\beta_{str}^{opt})^{\top} S_{hx} \beta_{str}^{opt} \right\} \]
\[ + \pi_{h} p_{h0}^{-1} \left\{ S_{h0}^{2} - 2(\beta_{str}^{opt})^{\top} S_{hx0} + (\beta_{str}^{opt})^{\top} S_{hx} \beta_{str}^{opt} \right\} + o_{P}(1) \]
\[ = \sum_{h=1}^{H} \pi_{h} \left\{ p_{h1}^{-1} S_{h1}^{2}(\beta_{str}^{opt}) + p_{h0}^{-1} S_{h0}^{2}(\beta_{str}^{opt}) \right\} + o_{P}(1). \quad (35) \]

The second equality is derived by Lemmas 5 and 6. By the optimality of \( \beta_{str}^{opt} \),
\[ \beta_{str}^{opt} = \arg \min_{\beta} \sum_{h=1}^{H} \pi_{h} \left\{ p_{h1}^{-1} S_{h1}^{2}(\beta) + p_{h0}^{-1} S_{h0}^{2}(\beta) \right\}. \quad (36) \]

The conclusion follows from (35) and (36).

The following proposition is from Wang et al. (2021). Assumption 3 is slightly weaker than that used by Wang et al. (2021), but it does not affect the conclusion.

**Proposition 5 (Wang et al. (2021)).** Under Assumption 3,
\[ n_{str}^{1/2} \left( \hat{r}_{str} - \tau_{str} \right) \sim N \left( 0, \begin{bmatrix} V_{str,\tau \tau} & V_{str,\tau x} \\ V_{str,\tau x} & V_{str,x x} \end{bmatrix} \right). \]

### E.2. An equivalent form of regression formula

In this section, we prove that two regression formulas (37) and (38) below are equivalent in terms of point and variance estimators for the average treatment effect under stratified randomized experiments. It is useful for proving Theorem 5.

Recall the regression formula we use in the main text
\[ Y_{hi} \sim 1 + Z_{hi} + \sum_{q=2}^{H} (\delta_{hq} - \pi_{q}) + Z_{hi} \sum_{q=2}^{H} (\delta_{hq} - \pi_{q}) + x_{hi}, \quad (37) \]
where \( \delta_{hq} = 1 \) if \( q = h \) and \( \delta_{hq} = 0 \) otherwise, and
\[ w_{hi} = Z_{hi} p_{h1}^{-2} \frac{n_{h1}}{n_{h1} - 1} + (1 - Z_{hi}) p_{h0}^{-2} \frac{n_{h0}}{n_{h0} - 1}. \]
It is equivalent to the following weighed regression
\[ Y_{hi} \sim w_{hi} \sum_{q=1}^{H} Z_{hi} \delta_{hq} + \sum_{q=1}^{H} (1 - Z_{hi}) \delta_{hq} + (x_{hi} - \bar{x}_{h}). \quad (38) \]
Let $X_{\text{str}} = (X_{\text{str},1}^\top, \ldots, X_{\text{str},H}^\top)^\top \in \mathbb{R}^{n_{\text{str}} \times (2H+k)}$ be the design matrix of regression (37) with the $i$th row of $X_{\text{str},h}$ being
\[
(1, Z_{hi}, \delta_{h2} - \pi_2, \ldots, \delta_{hH} - \pi_H, Z_{hi}(\delta_{h2} - \pi_2), \ldots, Z_{hi}(\delta_{hH} - \pi_H), x_{hi}^\top).
\]
Let $E = (E_1^\top, \ldots, E_H^\top)^\top \in \mathbb{R}^{n_{\text{str}} \times (2H+k)}$ be the design matrix of regression (38) with the $i$th row of $E_h$ being
\[
(Z_{hi}\delta_{h1}, \ldots, Z_{hi}\delta_{hH}, (1 - Z_{hi})\delta_{h1}, \ldots, (1 - Z_{hi})\delta_{hH}, (x_{hi} - \bar{x}_h)^\top).
\]
Let $W$ be the digonal matrix of $w_{hi}$'s and $Y$ be the vector of $Y_{hi}$'s ($h = 1, \ldots, H, i = 1, \ldots, n_h$). Let $\hat{\beta}_1$ and $\hat{\beta}_2$ be the estimated coefficients of regression (37) and (38), respectively. Then
\[
\hat{\beta}_1 = (X_{\text{str}}^\top WX_{\text{str}})^{-1}X_{\text{str}}^\top WY, \quad \hat{\beta}_2 = (E^\top WE)^{-1}E^\top WY.
\]
Next, we prove some lemmas to build the equivalence between these two regressions. Let $\ell = (0_2^\top H, 1_1^\top)^\top$. Let $d = (\pi_1, \ldots, \pi_H, -\pi_1, \ldots, -\pi_H, 0_1^\top)^\top$ be a vector of length $2H + k$. Lemma 8 below shows that they have the same estimated coefficient for the covariates.

**Lemma 8.**
\[
\ell^\top \hat{\beta}_1 = \ell^\top \hat{\beta}_2 = \hat{\beta}_{\text{str}}.
\]

**Proof.** In the proof of Proposition 4, we have shown that
\[
\ell^\top \hat{\beta}_1 = \hat{\beta}_{\text{str}}.
\]
Similar to the proof of Proposition 4 with FWL theorem, we have
\[
\ell^\top \hat{\beta}_2 = \hat{\beta}_{\text{str}}.
\]
\]

Lemma 9 below shows that we can derive the same average treatment effect estimator. Recall that $\xi_2 \in \mathbb{R}^{2H+k}$ is a vector with 1 at the second dimension and 0 at other dimensions.

**Lemma 9.**
\[
\xi_2^\top \hat{\beta}_1 = d^\top \hat{\beta}_2 = \hat{\tau}_{\text{tom}}^\text{str}.
\]

**Proof.** In the proof of Proposition 4, we have shown that
\[
\xi_2^\top \hat{\beta}_1 = \hat{\tau}_{\text{tom}}^\text{str}.
\]
It suffices to show that
\[
d^\top \hat{\beta}_2 = \hat{\tau}_{\text{tom}}^\text{str}.
\]
By the property of OLS and Lemma 8, the estimated coefficient of $(1 - Z_{hi})\delta_{hq}$ and $Z_{hi}\delta_{hq}$ ($q = 1, \ldots, H$) can be derived in the WLS regression of
\[
Y_{hi} - (x_{hi} - \bar{x}_h)^\top \hat{\beta}_{\text{str}} \sim \sum_{q=1}^H Z_{hi}\delta_{hq} + \sum_{q=1}^H (1 - Z_{hi})\delta_{hq}.
\]
It follows that the estimated coefficients of $Z_{hi}\delta_{hq}$ and $(1 - Z_{hi})\delta_{hq}$ are, respectively,

$$\hat{Y}_q(1) - \{\hat{x}_q(1) - \bar{x}_q\}^\top \hat{\beta}_{str}, \quad \hat{Y}_q(0) - \{\hat{x}_q(0) - \bar{x}_q\}^\top \hat{\beta}_{str},$$

for $q = 1, \ldots, H$. Therefore,

$$d^\top \hat{\beta}_2 = \sum_{h=1}^{H} \pi_h \left[ \hat{Y}_h(1) - \{\hat{x}_h(1) - \bar{x}_h\}^\top \hat{\beta}_{str} - \hat{Y}_h(0) + \{\hat{x}_h(0) - \bar{x}_h\}^\top \hat{\beta}_{str} \right]$$

$$= \sum_{h=1}^{H} \pi_h \left[ \hat{Y}_h(1) - \hat{Y}_h(0) - \{\hat{x}_h(1) - \hat{x}_h(0)\}^\top \hat{\beta}_{str} \right] = \tilde{z}_{str}^\top.$$

Lemma 10. The residuals from regressions (37) and (38) are the same, which are equal to $\hat{Y}_{hi} - \bar{x}_{hi}^\top \hat{\beta}_{str}$ for unit $hi$.

Proof. By the property of OLS, the residuals of regression (38) are equal to those of the following regression:

$$Y_{hi} - (x_{hi} - \bar{x}_h)^\top \hat{\beta}_{str} \approx \sum_{q=1}^{H} Z_{hi}\delta_{hq} + \sum_{q=1}^{H} (1 - Z_{hi})\delta_{hq}.$$

The residuals of regression (37) are equal to those of the following regression:

$$Y_{hi} - x_{hi}^\top \hat{\beta}_{str} \approx 1 + Z_{hi} + \sum_{q=2}^{H} (\delta_{hq} - \pi_q) + Z_{hi} \sum_{q=2}^{H} (\delta_{hq} - \pi_q).$$

Note that the fitted values of the above two regressions are the same for units in the same stratum under the same treatment arm. Therefore, the fitted value of unit $hi$ is the mean value over the units in the same stratum under the same treatment arm with $hi$. Thus, the residuals of unit $hi$ of regressions (37) and (38) are both equal to $\hat{Y}_{hi} - \bar{x}_{hi}^\top \hat{\beta}_{str}$.

The leverage scores of these two regressions are the diagonal elements of the following matrices

$$X_{str} \left( X_{str}^\top W X_{str} \right)^{-1} X_{str}^\top W, \quad E \left( E^\top W E \right)^{-1} E^\top W.$$

As shown in the proof of Lemma 11, $E = X_{str} Q$ (The explicit formula of $Q$ can be found in the proof of Lemma 11). The fact that $Q$ is an invertible matrix indicates that

$$X_{str} \left( X_{str}^\top W X_{str} \right)^{-1} X_{str}^\top W = E \left( E^\top W E \right)^{-1} E^\top W.$$

Therefore, the leverage scores of these two regression formulas are the same. We denote by $h_{hi,str}$ the leverage score corresponding to unit $hi$. We will derive the formula of $h_{hi,str}$ in Section E.3.

Let $\hat{c}_{hi} = \hat{Y}_{hi} - \bar{x}_{hi}^\top \hat{\beta}_{str}$ be the regression residual of unit $hi$. Let $\hat{c}_{hi} = \eta_{hi} \hat{c}_{hi}$ be the scaled residual where $\eta_{hi} = 1$ for HC0, $\eta_{hi} = (n_{str}/(n_{str} - 2H - k))^{1/2}$ for HC1, $\eta_{hi} = (1 - h_{hi,str})^{-1/2}$ for HC2, $\eta_{hi} = (1 - h_{hi,str})^{-1}$ for HC3. Let $\Delta$ be the diagonal matrix of $\hat{\epsilon}_{hi}^2 (h = 1, \ldots, H, i = 1, \ldots, n_h)$. By Lemma 9, regressions (37) and (38) lead to two variance estimators for $\hat{\tau}_{str}^\top$, which are derived as

$$\xi_2 \left( X_{str}^\top W X_{str} \right)^{-1} X_{str}^\top W \Delta W X_{str} \left( X_{str}^\top W X_{str} \right)^{-1} \xi_2,$$

$$d^\top \left( E^\top W E \right)^{-1} E^\top W \Delta W E \left( E^\top W E \right)^{-1} d.$$

Lemma 11 below shows the equivalence of these two variance estimators.
Lemma 11.
\[
\xi_2^\top (X_{\text{str}}^\top W X_{\text{str}})^{-1} X_{\text{str}}^\top W \Delta W X_{\text{str}}^\top (X_{\text{str}}^\top W X_{\text{str}})^{-1} \xi_2 \\
= d^\top (E^\top W E)^{-1} E^\top W \Delta W E (E^\top W E)^{-1} d.
\]

Proof. First, we give the explicit formula of \( Q \) subject to \( E = X_{\text{str}} Q \). Let \( P_{i,j} \in \mathbb{R}^{(2H+k)\times(2H+k)} \) denote the matrix with the \((i, j)\)th element being 1 and the other elements being 0. Let \( I \) denote identify matrix of size \( 2H + k \). We can verify that
\[
Q = \prod_{q=2}^{H} (I + \pi_q P_{1,q+1}) \prod_{q=2}^{H} (I + \pi_q P_{2,q+H}) \prod_{q=2}^{H} (I - P_{H+q,q+1}) \prod_{q=2}^{H} (I - P_{H+q,2}) \prod_{t=2}^{2H} (I - P_{t,1}) Q_1 Q_2,
\]
where
\[
Q_1 = (\xi_2, \xi_{H+2}, \ldots, \xi_2 H, \xi_1, \xi_3, \ldots, \xi_{H+1}), \quad Q_2 = \begin{pmatrix} I_{2H} & A \\ 0 & I_k \end{pmatrix},
\]
\[
A = (\bar{x}_1, \ldots, \bar{x}_H, \bar{x}_1, \ldots, \bar{x}_H)^\top.
\]
Here \((I + \pi_q P_{1,q+1})(q = 2, \ldots, H)\) corresponds to the operation of changing \( \delta_{hq} - \pi_q \) to \( \delta_{hq} \); \((I + \pi_q P_{2,q+H})(q = 2, \ldots, H)\) corresponds to the operation of changing \( Z_{hi}(\delta_{hq} - \pi_q) \) to \( Z_{hi} \delta_{hq} \); \((I - P_{H+q,q+1})(q = 2, \ldots, H)\) corresponds to the operation of changing \( \delta_{hq} \) to \((1 - Z_{hi}) \delta_{hq}; \prod_{q=2}^{H} (I - P_{H+q,2})\) corresponds the operation of changing \( Z_{hi} \) to \( Z_{hi} \delta_{h1}; \prod_{t=2}^{2H} (I - P_{t,1})\) corresponds to the operation of changing 1 to \( (1 - Z_{hi}) \delta_{h1}; Q_1\) corresponds to the operation of reordering the positions of the regressors; and \( Q_2\) corresponds to the operation of centering \( x_{hi} \) at \( \bar{x}_h \).

After some calculation, we can verify that
\[
\xi_2^\top Q = d^\top.
\]
Therefore,
\[
\xi_2^\top (X_{\text{str}}^\top W X_{\text{str}})^{-1} X_{\text{str}}^\top W \Delta W X_{\text{str}}^\top (X_{\text{str}}^\top W X_{\text{str}})^{-1} \xi_2 \\
= \xi_2^\top Q (E^\top W E)^{-1} E^\top W \Delta W E (E^\top W E)^{-1} Q^\top \xi_2 \\
= d^\top (E^\top W E)^{-1} E^\top W \Delta W E (E^\top W E)^{-1} d.
\]

\[\square\]

E.3. Leverage scores of ToM regression in stratified randomized experiments

Define \( \hat{v}_{\text{str},xx} = \sum_{h=1}^{H} \pi_h \left\{ \bar{p}_{h1}^{-1} s_{hx(1)}^2 + \bar{p}_{h0}^{-1} s_{hx(0)}^2 \right\} \). Define \( w_h(z) \) the regression weights for units in stratum \( h \) under treatment arm \( z \) with
\[
w_{hi} = Z_{hi} w_h(1) + (1 - Z_{hi}) w_h(0).
\]

Proposition 6 below provides the formula of leverage scores of ToM regression in stratified randomized experiments.

Proposition 6.
\[
h_{hi,\text{str}} = \begin{cases} 
  n_{h1}^{-1} + \bar{x}_{hi} \hat{v}_{\text{str},xx} \bar{v}_{hi} w_h(1)n_{\text{str}}^{-1}, & i \in S_{h1}, \\
  n_{h0}^{-1} + \bar{x}_{hi} \hat{v}_{\text{str},xx} \bar{v}_{hi} w_h(0)n_{\text{str}}^{-1}, & i \in S_{h0}.
\end{cases}
\]
Proof. Let \( \dot{X}_{\text{str}} = (\dot{X}_{\text{str},1}, \ldots, \dot{X}_{\text{str},H})^\top \in \mathbb{R}^{n_{\text{str}} \times (2H + k)} \) with the \( i \)th row of \( \dot{X}_{\text{str},h} \) being
\[
(Z_{hi}\delta_{h1}, \ldots, Z_{hi}\delta_{hH}, (1 - Z_{hi})\delta_{h1}, \ldots, (1 - Z_{hi})\delta_{hH}, (\dot{x}_{hi})^\top).
\]
There exists a squared and invertible matrix \( Q \) such that \( \dot{X}_{\text{str}} = X_{\text{str}}Q \). Therefore,
\[
\dot{X}_{\text{str}} (\dot{X}_{\text{str}}^\top W \dot{X}_{\text{str}})^{-1} \dot{X}_{\text{str}}^\top W = X_{\text{str}} (X_{\text{str}}^\top W X_{\text{str}})^{-1} X_{\text{str}}^\top W.
\]
Note that
\[
\dot{X}_{\text{str}}^\top W \dot{X}_{\text{str}} =
\begin{pmatrix}
n_{11}w_1(1) \\
\vdots \\
n_{H1}w_H(1) \\
n_{10}w_1(0) \\
\vdots \\
n_{H0}w_H(0)
\end{pmatrix}
\sum_{h=1}^H \left\{ \sum_{i \in S_1} w_h(1) \dot{x}_{hi} \dot{x}_{hi}^\top + \sum_{i \in S_0} w_h(0) \dot{x}_{hi} \dot{x}_{hi}^\top \right\}.
\]
Moreover,
\[
\frac{1}{n_{\text{str}}} \sum_{h=1}^H \left\{ \sum_{i \in S_1} w_h(1) \dot{x}_{hi} \dot{x}_{hi}^\top + \sum_{i \in S_0} w_h(0) \dot{x}_{hi} \dot{x}_{hi}^\top \right\} = \sum_{h=1}^H \pi_h \left\{ p_{h1}^{-1} s_{h1}^2(1) + p_{h0}^{-1} s_{h0}^2(0) \right\} = \hat{V}_{\text{str},xx}.
\]
Therefore,
\[
\dot{h}_{hi,\text{str}} =
\begin{cases}
n_{h1}^{-1} + \dot{x}_{hi}^\top \hat{V}_{\text{str},xx}\dot{x}_{hi} w_h(1) n_{\text{str}}^{-1}, & i \in S_{h1}, \\
n_{h0}^{-1} + \dot{x}_{hi}^\top \hat{V}_{\text{str},xx}\dot{x}_{hi} w_h(0) n_{\text{str}}^{-1}, & i \in S_{h0}.
\end{cases}
\]

**Lemma 12.** Under Assumption 3,
\[
\|\hat{V}_{\text{str},xx}\|_{\text{op}} = O_P(1), \quad \|\hat{V}_{\text{str},xx}\|_{\infty} = O_P(1).
\]

Proof. Let
\[
\hat{V}_1 = \sum_{h=1}^H \pi_h p_{h1}^{-1} s_{h1}^2(1), \quad V_1 = \sum_{h=1}^H \pi_h p_{h1}^{-1} s_{h1}^2,
\]
\[
\hat{V}_0 = \sum_{h=1}^H \pi_h p_{h0}^{-1} s_{h0}^2(0), \quad V_0 = \sum_{h=1}^H \pi_h p_{h0}^{-1} s_{h0}^2.
\]
By Lemma 5, for \( j = 1, \ldots, k, \ j' = 1, \ldots, k, \)
\[
||\hat{V}_1 - V_1||_{(j,j')} = o_P(1), \quad ||\hat{V}_0 - V_0||_{(j,j')} = o_P(1).
\]
Therefore,

\[ \|\hat{V}_1 - V_1\|_\infty = \max_{j,j'} |[\hat{V}_1 - V_1]_{(j,j')}| \leq \sum_{j,j'} |[\hat{V}_1 - V_1]_{(j,j')}| = o_P(1). \]

Thus,

\[ \|\hat{V}_1 - V_1\|_{op} \leq \left[ \text{tr} \left\{ (\hat{V}_1 - V_1)^2 \right\} \right]^{1/2} \leq k\|\hat{V}_1 - V_1\|_\infty = o_P(1). \]

Similarly,

\[ \|\hat{V}_0 - V_0\|_{op} = o_P(1). \]

Thus,

\[ \|\hat{V}_{str,xx} - V_{str,xx}\|_{op} \leq \|\hat{V}_1 - V_1\|_{op} + \|\hat{V}_0 - V_0\|_{op} = o_P(1). \]

By Assumption 3, the limit of \( V_{str,xx} \) is an invertible matrix. Let \( \lambda_{\text{min}}(V_{str,xx}) > 0 \) be the smallest eigenvalue of \( V_{str,xx} \) and there exists a constant \( c \) such that \( \lambda_{\text{min}}(V_{str,xx}) > c \) for sufficiently large \( n_{str} \). By Weyl’s inequality, with probability tending to one,

\[ \|\hat{V}_{str,xx} - V_{str,xx}\|_{op} < c/2 \iff \lambda_{\text{min}}(V_{str,xx}) - \lambda_{\text{min}}(\hat{V}_{str,xx}) < c/2 \]

\[ \iff \lambda_{\text{min}}(\hat{V}_{str,xx}) > \lambda_{\text{min}}(V_{str,xx}) - \frac{c}{2} > \frac{c}{2}. \]

Therefore, with probability tending to one,

\[ \|\hat{V}_{str,xx}^{-1}\|_{op} = \lambda_{\text{min}}(\hat{V}_{str,xx})^{-1} \leq \frac{2}{c}. \]

Thus, \( \|\hat{V}_{str,xx}^{-1}\|_{op} = O_P(1) \). Since \( \|\hat{V}_{str,xx}^{-1}\|_\infty \leq \|\hat{V}_{str,xx}^{-1}\|_{op} \), then \( \|\hat{V}_{str,xx}^{-1}\|_\infty = O_P(1) \).

Define

\[ g_{hi} = \begin{cases} n_{h1}^{-1}, & i \in S_{h1}, \\ n_{h0}^{-1}, & i \in S_{h0}. \end{cases} \]

**Lemma 13.** Under Assumption 3,

\[ \max_{h,i} |h_{hi,xx} - g_{hi}| = o_P(1). \]

**Proof.** By Lemma 12,

\[ \max_{h,i} |h_{hi,xx} - g_{hi}| \leq \|\hat{V}_{str,xx}^{-1}\|_{op} \max_{h,z} |w_h(z)| \max_{h,i} \tilde{x}_h^\top \tilde{x}_hi n_{str}^{-1} \]

\[ \leq \|\hat{V}_{str,xx}^{-1}\|_{op} \max_{h,z} |w_h(z)| \max_{h,i} k \|x_h - \tilde{x}_h\|_\infty n_{str}^{-1} \]

\[ = O_P(1)O(1)o(n_{str})n_{str}^{-1} = o_P(1). \]
E.4. Proof of Theorem 4

Proof. Note that
\[ n^{1/2} (\tau_{\text{str}} - \tau) = n^{1/2} (\hat{\tau}_{\text{str}} - \tau_{\text{str}} - (\beta_{\text{opt}}^{\text{str}})^\top \hat{\tau}_{\text{str},x}) + n^{1/2} (\beta_{\text{str}}^{\text{opt}} - \hat{\beta}_{\text{str}}) \hat{\tau}_{\text{str},x} \]
\[ = n^{1/2} (\hat{\tau}_{\text{str}} - \tau_{\text{str}}) + n^{1/2} (\beta_{\text{opt}}^{\text{str}}) \hat{\tau}_{\text{str},x} \]
\[ = n^{1/2} (\hat{\tau}_{\text{str}} - \tau_{\text{str}}) + o_p(1), \]
where the first equality is due to Proposition 4 and the second equality is due to Lemma 6 and Proposition 5.

By Proposition 5 and the definition of $\beta_{\text{opt}}^{\text{str}}$, we have
\[ n^{1/2} (\hat{\tau}_{\text{str}} - \tau_{\text{str}}) \sim N(0, V_{\text{str}}^{\tau} - V_{\text{str},xx} V_{\text{str},x}^{-1} V_{\text{str},x}). \]

Compounded with Slusky’s theorem, the conclusion follows. \hfill \square

E.5. Proof of Theorem 5

Proof. We use the following formula of the variance estimator
\[ d^\top (E^\top WE)^{-1} E^\top WDE(E^\top WE)^{-1} d. \]

Let $u_{hi} = x_{hi} - \bar{x}_h$. Define $H$ by
\[ H = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} = E^\top WDE/n_{\text{str}}, \]
where
\[ H_{11} = n_{\text{str}}^{-1} \text{diag} \left( w_{11}^2(1) \sum_{i \in S_{11}} \hat{e}_{i1}^2, \ldots, w_{12}^2(1) \sum_{i \in S_{12}} \hat{e}_{i1}^2, w_{12}^2(0) \sum_{i \in S_{10}} \hat{e}_{i1}^2, \ldots, w_{12}^2(0) \sum_{i \in S_{10}} \hat{e}_{i1}^2 \right), \]
\[ H_{21} = H_{12}^\top = n_{\text{str}}^{-1} \left( w_{11}^2(1) \sum_{i \in S_{11}} \hat{e}_{i1}^2 u_{i1} + \sum_{i \in S_{12}} \hat{e}_{i1}^2 u_{i1}, \ldots, w_{12}^2(0) \sum_{i \in S_{10}} \hat{e}_{i1}^2 u_{i1} + \sum_{i \in S_{10}} \hat{e}_{i1}^2 u_{i1} \right), \]
\[ H_{22} = n_{\text{str}}^{-1} \sum_{h=1}^H \left( w_{h1}^2(1) \sum_{i \in S_{h1}} u_{hi} \hat{e}_{hi}^2 + w_{h2}^2(0) \sum_{i \in S_{h0}} u_{hi} \hat{e}_{hi}^2 \right). \]

Define $G$ by
\[ G = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} = E^\top WE/n_{\text{str}}. \]
where

\[ G_{11} = n_{str}^{-1} \text{diag} (n_{11}w_1, \ldots, n_{H1}w_H, n_{10}w_1, \ldots, n_{H0}w_H), \]

\[ G_{21} = G_{12}^T = n_{str}^{-1} \left( \sum_{i \in S_{11}} u_{1i}, \ldots, w_{H1} \left( \sum_{i \in S_{H1}} u_{Hi}, w_1 \left( \sum_{i \in S_{H0}} u_{1i}, \ldots, w_H \left( \sum_{i \in S_{H0}} u_{Hi} \right) \right) \right) \right), \]

\[ G_{22} = n_{str}^{-1} \sum_{h=1}^{H} \left\{ w_h(1) \sum_{i \in S_{h1}} u_{hi} u_{hi}^T + w_h(0) \sum_{i \in S_{h0}} u_{hi} u_{hi}^T \right\}. \]

Define \( \Lambda \) by

\[ \Lambda = G^{-1} = \left( \begin{array}{cc} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{array} \right). \]

By the formula of inverse of a 2 \( \times \) 2 block matrix, we have

\[ \Lambda_{11} = G_{11}^{-1} + G_{11}^{-1} G_{12} (G_{22} - G_{21} G_{11}^{-1} G_{12})^{-1} G_{21} G_{11}^{-1}, \]

\[ \Lambda_{21} = -G_{11}^{-1} G_{12} (G_{22} - G_{21} G_{11}^{-1} G_{12})^{-1}. \]

Let \( d_1 = (\pi_1, \ldots, \pi_H, -\pi_1, \ldots, -\pi_H) \), it is easy to see that

\[ n_{str} \hat{V}_{HC, \text{str}} = d_1^T \left( \begin{array}{cc} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{array} \right) \hat{H} \left( \begin{array}{c} \Lambda_{11} \\ \Lambda_{21} \end{array} \right) d_1. \quad (39) \]

To derive the formula of \( \hat{V}_{HC, \text{str}} \), we calculate the following two quantities:

(i) \( G_{11}^{-1} G_{12}, \)  \( G_{22} - G_{21} G_{11}^{-1} G_{12}. \)

For (i), we have

\[ G_{11}^{-1} G_{12} = (\hat{u}_1(1), \ldots, \hat{u}_H(1), \hat{u}_1(0), \ldots, \hat{u}_H(0))^T. \]

Denote \( G_{11}^{-1} G_{12} \) by \( \hat{U} \).

For (ii), we have

\[ G_{22} - G_{21} G_{11}^{-1} G_{12} = \frac{1}{n_{str}} \sum_{h=1}^{H} \left\{ w_h(1) \sum_{i \in S_{h1}} u_{hi} u_{hi}^T + w_h(0) \sum_{i \in S_{h0}} u_{hi} u_{hi}^T \right\} - \hat{U}^T G_{12} \]

\[ = \frac{1}{n_{str}} \sum_{h=1}^{H} \left\{ w_h(1) \sum_{i \in S_{h1}} u_{hi} u_{hi}^T + w_h(0) \sum_{i \in S_{h0}} u_{hi} u_{hi}^T \right\} - \]

\[ = \frac{1}{n_{str}} \sum_{h=1}^{H} \left\{ w_h(1) n_{h1} \hat{u}_h(1) \hat{u}_h(1)^T + w_h(0) n_{h0} \hat{u}_h(0) \hat{u}_h(0)^T \right\} \]

\[ = \frac{1}{n_{str}} \sum_{h=1}^{H} \left\{ w_h(1) \sum_{i \in S_{h1}} \hat{x}_{hi} \hat{x}_{hi}^T + w_h(0) \sum_{i \in S_{h0}} \hat{x}_{hi} \hat{x}_{hi}^T \right\} \]

\[ = \sum_{h=1}^{H} \pi_h \left\{ p_{h1}^{-1} s_{h1}(1) + p_{h0}^{-1} s_{h0}(0) \right\} = \hat{V}_{\text{str,xx}}. \]
Next, we derive the formula related to $T$. Expanding (39), we have

$$
\begin{align*}
\hat{d}_1^\top \left( \begin{array}{cc}
\Lambda_{11} & \Lambda_{12} \\
\Lambda_{21} & \Lambda_{22}
\end{array} \right) H \left( \begin{array}{cc}
\Lambda_{11} & 0 \\
0 & \Lambda_{22}
\end{array} \right) \hat{d}_1 \\
= \hat{d}_1^\top \left( G_{11}^{-1} + \hat{U} \hat{V}_{str,xx}^{-1} \hat{U}^\top - \hat{U} \hat{V}_{str,xx}^{-1} \hat{U}^\top \right) H \left( G_{11}^{-1} + \hat{U} \hat{V}_{str,xx}^{-1} \hat{U}^\top - \hat{U} \hat{V}_{str,xx}^{-1} \hat{U}^\top \right) \hat{d}_1 \\
= \hat{d}_1^\top \left( G_{11}^{-1} + \hat{U} \hat{V}_{str,xx}^{-1} \hat{U}^\top \right) H_1 \left( G_{11}^{-1} + \hat{U} \hat{V}_{str,xx}^{-1} \hat{U}^\top \right) d_1 + \hat{d}_1^\top \hat{U} \hat{V}_{str,xx}^{-1} H_2 \hat{V}_{str,xx}^{-1} \hat{U}^\top d_1 \\
- 2 \hat{d}_1^\top \left( G_{11}^{-1} + \hat{U} \hat{V}_{str,xx}^{-1} \hat{U}^\top \right) H_2 \hat{V}_{str,xx}^{-1} \hat{U}^\top d_1.
\end{align*}
$$

Let

$$
\begin{align*}
T_1 &= d_1^\top G_{11}^{-1} H_1 G_{11}^{-1} d_1, \\
T_2 &= d_1^\top G_{11}^{-1} H_1 \hat{U} \hat{V}_{str,xx}^{-1} \hat{U}^\top d_1, \\
T_3 &= d_1^\top \hat{U} \hat{V}_{str,xx}^{-1} \hat{U}^\top H_1 \hat{U} \hat{V}_{str,xx}^{-1} \hat{U}^\top d_1, \\
T_4 &= d_1^\top G_{11}^{-1} H_2 \hat{V}_{str,xx}^{-1} \hat{U}^\top d_1, \\
T_5 &= d_1^\top \hat{U} \hat{V}_{str,xx}^{-1} \hat{U}^\top H_2 \hat{V}_{str,xx}^{-1} \hat{U}^\top d_1, \\
T_6 &= d_1^\top \hat{U} \hat{V}_{str,xx}^{-1} H_2 \hat{V}_{str,xx}^{-1} \hat{U}^\top d_1.
\end{align*}
$$

Next, we derive the formula related to $T_i$ ($i = 1, \ldots, 6$).

$$
\begin{align*}
T_1 &= d_1^\top G_{11}^{-1} H_1 G_{11}^{-1} d_1 \\
&= \sum_{h=1}^H \pi_h \sum_{i \in S_{h1}} \frac{1}{n_{str}} \tilde{e}_{hi}^2 n_{h1} \left\{ \frac{n_{h1}}{n_{str}} w_h(1) \right\} -2 + \sum_{h=1}^H \pi_h \sum_{i \in S_{h0}} \frac{1}{n_{str}} \tilde{e}_{hi}^2 n_{h0} \left\{ \frac{n_{h0}}{n_{str}} w_h(0) \right\} -2 \\
&= \sum_{h=1}^H \pi_h \left\{ \left( \sum_{i \in S_{h1}} \frac{\tilde{e}_{hi}^2}{n_{h1}} \right) p_{h1} - \left( \sum_{i \in S_{h0}} \frac{\tilde{e}_{hi}^2}{n_{h0}} \right) p_{h0} \right\},
\end{align*}
$$

$$
\begin{align*}
d_1^\top G_{11}^{-1} H_1 \hat{U} &= \sum_{h=1}^H \pi_h \left\{ \frac{n_{h1}}{n_{str}} w_h(1) \right\} -1 \sum_{i \in S_{h1}} w_h^2(1) \tilde{e}_{hi}^2 \frac{1}{n_{str}} \hat{u}_h(1)^\top \\
&- \sum_{h=1}^H \pi_h \left\{ \frac{n_{h0}}{n_{str}} w_h(0) \right\} -1 \sum_{i \in S_{h0}} w_h^2(0) \tilde{e}_{hi}^2 \frac{1}{n_{str}} \hat{u}_h(0)^\top \\
&= \sum_{h=1}^H \pi_h n_{h1} \sum_{i \in S_{h1}} w_h(1) \tilde{e}_{hi}^2 \hat{u}_h(1)^\top - \sum_{h=1}^H \pi_h n_{h0} \sum_{i \in S_{h0}} w_h(0) \tilde{e}_{hi}^2 \hat{u}_h(0)^\top,
\end{align*}
$$

$$
\begin{align*}
\hat{U}^\top H_1 \hat{U} &= \sum_{h=1}^H \left\{ \hat{u}_h(1) \hat{u}_h(1)^\top \right\} \left\{ \sum_{i \in S_{h1}} \tilde{e}_{hi}^2 w_h^2(1) \frac{1}{n_{str}} \right\} + \\
&\sum_{h=1}^H \left\{ \hat{u}_h(0) \hat{u}_h(0)^\top \right\} \left\{ \sum_{i \in S_{h0}} \tilde{e}_{hi}^2 w_h^2(0) \frac{1}{n_{str}} \right\},
\end{align*}
$$

50
\[ d_1^T G_{11}^{-1} H_{12} = \sum_{h=1}^{H} \pi_h \left\{ \frac{n_{h1}}{n_{str}} w_h(1) \right\}^{-1} \sum_{i \in S_{h1}} \frac{e_{hi}^2 u_{hi}^\top}{n_{str}} \]

\[ - \sum_{h=1}^{H} \pi_h \left\{ \frac{n_{h0}}{n_{str}} w_h(0) \right\}^{-1} \sum_{i \in S_{h0}} \frac{e_{hi}^2 u_{hi}^\top}{n_{str}} \]

\[ = \sum_{h=1}^{H} \pi_h n_{h1}^{-1} w_h(1) \sum_{i \in S_{h1}} \frac{e_{hi}^2 u_{hi}^\top}{n_{str}} - \sum_{h=1}^{H} \pi_h n_{h0}^{-1} w_h(0) \sum_{i \in S_{h0}} \frac{e_{hi}^2 u_{hi}^\top}{n_{str}}, \]

\[ \hat{U}^\top H_{12} = \sum_{h=1}^{H} \hat{u}_h(1) w_h(1) \sum_{i \in S_{h1}} \frac{e_{hi}^2 u_{hi}^\top}{n_{str}} + \sum_{h=1}^{H} \hat{u}_h(0) w_h(0) \sum_{i \in S_{h0}} \frac{e_{hi}^2 u_{hi}^\top}{n_{str}}. \]

Next, we prove that \( T_1 = n_{str} \hat{V}_{str} \{1 + o_P(1)\} \) and \( T_i = o_P(1) \) \( (i = 2, \ldots, 6) \). Note that for HC2, \( \eta_{hi}^2 = (1 - h_{hi,str})^{-1} \) and by Lemma 13,

\[ \max_{h,i} |\hat{e}_{hi}^2 / \{e_{hi}^2 (1 - g_{hi})^{-1}\}| = \max_{h,i} |\eta_{hi}^2 (1 - g_{hi})| = 1 + o_P(1). \]

Therefore,

\[ T_1 = \sum_{h=1}^{H} \pi_h \left\{ \sum_{i \in S_{h1}} \frac{e_{hi}^2 (1 - g_{hi})^{-1}}{n_{h1}p_{h1}^{-1}} + \sum_{i \in S_{h0}} \frac{e_{hi}^2 (1 - g_{hi})^{-1}}{n_{h0}p_{h0}^{-1}} \right\} \{1 + o_P(1)\} \]

\[ = \sum_{h=1}^{H} \pi_h \left\{ (n_{h1} - 1)^{-1} \sum_{i \in S_{h1}} e_{hi}^2 p_{h1}^{-1} + (n_{h0} - 1)^{-1} \sum_{i \in S_{h0}} e_{hi}^2 p_{h0}^{-1} \right\} \{1 + o_P(1)\} \]

\[ = n_{str} \hat{V}_{str} \{1 + o_P(1)\}. \]

Note that \((1 - g_{hi})^{-1} \leq 2, \max_{h,i} \eta_{hi}^2 = O_P(1), n_{str} \hat{V}_{str} = O_P(1), \max_{h,z} |p_{h}^{-1}| = O(1), \max_{h,i} \|u_{hi}\|_\infty = o(n_{str}), \max_{h} \|w_{h}(z)\| \leq 2 \max_{h} \|p_{h}^{-1}\| = O(1). \) Therefore, we derive the following stochastic order
for terms related to $T_i$ ($i = 2, \ldots, 6$),

\[
\|d_i^G H_1^1 \hat{U}\|_\infty \leq \max_{h,i} \|u_{hi}\|_\infty \sum_{h=1}^{H} \pi_h \left\{ (n_{hi} - 1)^{-1} \sum_{i \in S_{hi}} \tilde{e}_{hi}^2 p_{hi}^{-1} + (n_{hi} - 1)^{-1} \sum_{i \in S_{hi}} \tilde{e}_{hi}^2 p_{hi}^{-1} \right\} \\
\leq \max_{h,i} \|u_{hi}\|_\infty \sum_{h=1}^{H} \left| \sum_{i \in S_{hi}} \pi_h p_{hi}^{-1} s_{he(1)} + \sum_{h=1}^{H} \pi_h p_{h0}^{-1} s_{he(0)} \right| \max_{h,i} \eta_{hi}^2 \\
= o(n_{str}^{1/2}) O_F(1) O_F(1) = o_F(n_{str}^{1/2}),
\]

\[
\|\hat{U}^T H_1^1 \hat{U}\|_\infty \leq k \max_{h,i} \|u_{hi}\|_\infty \left\{ \sum_{h=1}^{H} \left\{ \sum_{i \in S_{hi}} \tilde{e}_{hi}^2 w_{hi}^2(1) \frac{1}{n_{str}} \right\} + \sum_{h=1}^{H} \left\{ \sum_{i \in S_{hi}} \tilde{e}_{hi}^2 w_{hi}^2(0) \frac{1}{n_{str}} \right\} \right\} \\
\leq k \max_{h,i} \|u_{hi}\|_\infty \max_{h,z} |w_h(z)| \left\{ \sum_{h=1}^{H} \pi_h p_{hi}^{-1} s_{he(1)}^2 + \sum_{h=1}^{H} \pi_h p_{h0}^{-1} s_{he(0)}^2 \right\} \max_{h,i} \eta_{hi}^2 \\
= o(n_{str}) O_F(1) O_F(1) = o_F(n_{str}),
\]

\[
\|d_i^G H_1^2\|_\infty \leq \max_{h,i} \|u_{hi}\|_\infty \max_{h,z} |p_{hi}^{-1}| \left\{ \sum_{h=1}^{H} \pi_h p_{hi}^{-1} s_{he(1)}^2 + \sum_{h=1}^{H} \pi_h p_{h0}^{-1} s_{he(0)}^2 \right\} \max_{h,i} \eta_{hi}^2 \\
= o(n_{str}^{1/2}) O_F(1) O_F(1) = o_F(n_{str}^{1/2}),
\]

\[
\|\hat{U}^T H_2\|_\infty \leq \max_{h,i} \|u_{hi}\|_\infty \max_{h,z} |w_h(z)| \left\{ \sum_{h=1}^{H} \pi_h p_{hi}^{-1} s_{he(1)}^2 + \sum_{h=1}^{H} \pi_h p_{h0}^{-1} s_{he(0)}^2 \right\} \max_{h,i} \eta_{hi}^2 \\
= o(n_{str}) O_F(1) O_F(1) = o_F(n_{str}),
\]

\[
\|H_2\|_\infty \leq \max_{h,i} \|u_{hi}\|_\infty \max_{h,z} |w_h(z)| n_{str}^{-1} \sum_{h=1}^{H} \left\{ w_h(1) \sum_{i \in S_{hi}} \tilde{e}_{hi}^2 + w_h(0) \sum_{i \in S_{hi}} \tilde{e}_{hi}^2 \right\} \\
\leq \max_{h,i} \|u_{hi}\|_\infty \max_{h,z} |w_h(z)| \left\{ \sum_{h=1}^{H} \pi_h p_{hi}^{-1} s_{he(1)}^2 + \sum_{h=1}^{H} \pi_h p_{h0}^{-1} s_{he(0)}^2 \right\} \max_{h,i} \eta_{hi}^2 \\
= o(n_{str}) O_F(1) O_F(1) = o_F(n_{str}).
\]
Note that $d_1^\top \hat{U} = \hat{\tau}_{str,x}$. Therefore,

$$|T_2| = |d_1^\top G_{11}^{-1} H_{11} \hat{U} \hat{V}_{str,x}^{-1} \hat{U}^\top d_1| = |d_1^\top G_{11}^{-1} H_{11} \hat{U} \hat{V}_{str,x}^{-1} \hat{\tau}_{str,x}|$$

$$\leq k^2 \|\hat{\tau}_{str,x}\|_\infty \|d_1^\top G_{11}^{-1} H_{11} \hat{U}\|_\infty \|\hat{V}_{str,x}\|_\infty = O_P(n_{str}^{-1/2})O_P(n_{str}^{-1/2})O_P(1) = o_P(1),$$

$$|T_3| = |d_1^\top \hat{U} \hat{V}_{str,x}^{-1} \hat{U}^\top H_{11} \hat{U} \hat{V}_{str,x}^{-1} \hat{U}^\top d_1| = |\hat{\tau}_{str,x} \hat{V}_{str,x}^{-1} \hat{U}^\top H_{11} \hat{U} \hat{V}_{str,x}^{-1} \hat{\tau}_{str,x}|$$

$$\leq k^2 \|\hat{\tau}_{str,x}\|_\infty \|\hat{V}_{str,x}^{-1} \hat{U}^\top H_{11} \hat{U} \hat{V}_{str,x}^{-1} \hat{\tau}_{str,x}\|_\infty = O_P(n_{str}^{-1/2})O_P(n_{str})O_P(n_{str}^{-1/2}) = o_P(1),$$

$$|T_4| = |d_1^\top G_{11}^{-1} H_{12} \hat{V}_{str,x}^{-1} \hat{U}^\top d_1| = |d_1^\top G_{11}^{-1} H_{12} \hat{V}_{str,x}^{-1} \hat{\tau}_{str,x}| \leq k^2 \|d_1^\top G_{11}^{-1} H_{12}\|_\infty \|\hat{V}_{str,x}^{-1}\|_\infty \|\hat{\tau}_{str,x}\|_\infty$$

$$= o_P(n_{str}^{-1/2})O_P(1)O_P(n_{str}^{-1/2}) = o_P(1),$$

$$|T_5| = |d_1^\top \hat{U} \hat{V}_{str,x}^{-1} \hat{U}^\top H_{12} \hat{V}_{str,x}^{-1} \hat{U}^\top d_1| = |\hat{\tau}_{str,x} \hat{V}_{str,x}^{-1} \hat{U}^\top H_{12} \hat{V}_{str,x}^{-1} \hat{\tau}_{str,x}|$$

$$\leq k^4 \|\hat{\tau}_{str,x}\|_\infty \|\hat{V}_{str,x}^{-1}\|_\infty \|\hat{U}^\top H_{12}\|_\infty \|\hat{V}_{str,x}^{-1}\|_\infty \|\hat{\tau}_{str,x}\|_\infty$$

$$= O_P(n_{str}^{-1/2})O_P(1)O_P(n_{str})O_P(n_{str}^{-1/2}) = o_P(1),$$

$$|T_6| = |d_1^\top \hat{U} \hat{V}_{str,x}^{-1} \hat{U}^\top H_{22} \hat{V}_{str,x}^{-1} \hat{U}^\top d_1| = |\hat{\tau}_{str,x} \hat{V}_{str,x}^{-1} H_{22} \hat{V}_{str,x}^{-1} \hat{\tau}_{str,x}|$$

$$\leq k^4 \|\hat{\tau}_{str,x}\|_\infty \|\hat{V}_{str,x}^{-1}\|_\infty \|H_{22}\|_\infty \|\hat{V}_{str,x}^{-1}\|_\infty \|\hat{\tau}_{str,x}\|_\infty$$

$$= O_P(n_{str}^{-1/2})O_P(1)O_P(n_{str})O_P(1)O_P(n_{str}^{-1/2}) = o_P(1).$$

Thus,

$$n_{str} \hat{V}_{HC2, str} = n_{str} \hat{V}_{str}\{1 + o_P(1)\}.$$

Combining with Lemma 7, we complete the proof.

\[ \square \]

### E.6. Proof for Remark 4

We give an example to show that $\hat{V}_{HC, str}$ for $j = 0, 1$ are anti-conservative. Similar to the proof of Theorem 5, we have, for $j = 0, 1$,

$$\hat{V}_{HC,j, str} = \sum_{h=1}^{H} \pi_h \left\{ \left( \sum_{i \in S_{h1}} \hat{e}_{hi}^2 / n_{h1} \right) p_{h1}^{-1} + \left( \sum_{i \in S_{h0}} \hat{e}_{hi}^2 / n_{h0} \right) p_{h0}^{-1} \right\} + o_P(1).$$

Therefore

$$\hat{V}_{HC0, str} = \sum_{h=1}^{H} \pi_h \left\{ \left( \sum_{i \in S_{h1}} \hat{e}_{hi}^2 / n_{h1} \right) p_{h1}^{-1} + \left( \sum_{i \in S_{h0}} \hat{e}_{hi}^2 / n_{h0} \right) p_{h0}^{-1} \right\} + o_P(1),$$

$$\hat{V}_{HC1, str} = \frac{n_{str}}{n_{str} - 2H - k} \sum_{h=1}^{H} \pi_h \left\{ \left( \sum_{i \in S_{h1}} \hat{e}_{hi}^2 / n_{h1} \right) p_{h1}^{-1} + \left( \sum_{i \in S_{h0}} \hat{e}_{hi}^2 / n_{h0} \right) p_{h0}^{-1} \right\} + o_P(1).$$

53
Let \( n_{h1} = 3 \) and \( n_{h0} = 2 \) for \( h = 1, \ldots, H \). By Lemmas 5 and 6, we have

\[
\hat{V}_{HC0, str} = \sum_{h=1}^{H} \pi_h \left\{ \left( \sum_{i \in S_{h1}} \varepsilon_{hi}^2 / 3 \right) p_{h1}^{-1} + \left( \sum_{i \in S_{h0}} \varepsilon_{hi}^2 / 2 \right) p_{h0}^{-1} \right\} + o_p(1)
\]

\[
= 2/3 \sum_{h=1}^{H} \left\{ \pi_h p_{h1}^{-1} \varepsilon_{he(1)}^2 \right\} + 1/2 \sum_{h=1}^{H} \left\{ \pi_h p_{h0}^{-1} \varepsilon_{he(0)}^2 \right\} + o_p(1)
\]

\[
= 2/3 \sum_{h=1}^{H} \left\{ \pi_h p_{h1}^{-1} S_{h1}(\beta_{str}^{opt}) \right\} + 1/2 \sum_{h=1}^{H} \left\{ \pi_h p_{h0}^{-1} S_{h0}(\beta_{str}^{opt}) \right\} + o_p(1),
\]

\[
\hat{V}_{HC1, str} = \frac{5H}{3H - k} \sum_{h=1}^{H} \pi_h \left\{ \left( \sum_{i \in S_{h1}} \varepsilon_{hi}^2 / 3 \right) p_{h1}^{-1} + \left( \sum_{i \in S_{h0}} \varepsilon_{hi}^2 / 2 \right) p_{h0}^{-1} \right\} + o_p(1)
\]

\[
= 10/9 \sum_{h=1}^{H} \left\{ \pi_h p_{h1}^{-1} S_{h1}(\beta_{str}^{opt}) \right\} + 5/6 \sum_{h=1}^{H} \left\{ \pi_h p_{h0}^{-1} S_{h0}(\beta_{str}^{opt}) \right\} + o_p(1)
\]

\[
= 10/9 \sum_{h=1}^{H} \left\{ \pi_h p_{h1}^{-1} S_{h1}(\beta_{str}^{opt}) \right\} + 5/6 \sum_{h=1}^{H} \left\{ \pi_h p_{h0}^{-1} S_{h0}(\beta_{str}^{opt}) \right\} + o_p(1),
\]

Therefore, \( \hat{V}_{HC0, str} \) is anti-conservative when

\[
\frac{1}{3} \sum_{h=1}^{H} \left\{ \pi_h p_{h1}^{-1} S_{h1}(\beta_{str}^{opt}) \right\} + \frac{1}{2} \sum_{h=1}^{H} \left\{ \pi_h p_{h0}^{-1} S_{h0}(\beta_{str}^{opt}) \right\} - \sum_{h=1}^{H} \pi_h S_{h\tau}^2 > 0;
\]

\( \hat{V}_{HC1, str} \) is anti-conservative when

\[
-\frac{1}{9} \sum_{h=1}^{H} \left\{ \pi_h p_{h1}^{-1} S_{h1}(\beta_{str}^{opt}) \right\} + \frac{1}{6} \sum_{h=1}^{H} \left\{ \pi_h p_{h0}^{-1} S_{h0}(\beta_{str}^{opt}) \right\} - \sum_{h=1}^{H} \pi_h S_{h\tau}^2 > 0.
\]

F. Proofs for the results under completely randomized survey experiments

F.1. Preliminary results

Proposition 7. \( \hat{\tau}_{\text{tom}}^{\text{crs}} = \hat{\tau}^{\text{crs}} + \hat{\tau}_{\text{crs}, x}^{\text{crs}} \hat{\beta}_{\text{crs}} - \hat{\delta}_{\text{crs}}^{\top} \hat{\gamma}_{\text{crs}} \), where

\[
\begin{pmatrix}
\hat{\beta}_{\text{crs}}
\hat{\gamma}_{\text{crs}}
\end{pmatrix} = \begin{pmatrix}
p_1^{-1}(1 - n_{1}^{-1})s_{x(1)}^2 + p_0^{-1}(1 - n_{0}^{-1})s_{x(0)}^2 & (1 - n_{1}^{-1})s_{x(1)} - (1 - n_{0}^{-1})s_{x(0)} \\
(1 - n_{1}^{-1})s_{x(1)} - (1 - n_{0}^{-1})s_{x(0)} & (p_1 - n_{1}^{-1})s_{v(1)}^2 + (p_0 - n_{0}^{-1})s_{v(0)}^2
\end{pmatrix}^{-1}
\begin{pmatrix}
p_1^{-1}(1 - n_{1}^{-1})s_{x(1)} + p_0^{-1}(1 - n_{0}^{-1})s_{x(0)} \\
(1 - n_{1}^{-1})s_{v(1)} - (1 - n_{0}^{-1})s_{v(0)}
\end{pmatrix}.
\]

Proof. Recall the regression

\[Y_i \sim 1 + Z_i + x_i + (Z_i - p_0)(v_i - \bar{v}), \quad \text{where } w_i = p_1^{-2}Z_i + p_0^{-2}(1 - Z_i).\]
Let \( \hat{v}_i = v_i - Z_i\hat{\nu}(1) - (1 - Z_i)\hat{\nu}(0) \). Recall that \( S \) is the set of sampled units. By FWL theorem,
\[
\begin{pmatrix}
\hat{\beta}_{crs} \\
\hat{\gamma}_{crs}
\end{pmatrix} = \left( \sum_{i \in S} w_i \hat{x}_i \hat{x}_i^\top \right)^{-1} \left( \sum_{i \in S} w_i (Z_i - p_0) \hat{x}_i \hat{v}_i^\top \right) \left( \sum_{i \in S} w_i (Z_i - p_0)^2 \hat{v}_i \hat{v}_i^\top \right)^{-1} \left( \sum_{i \in S} w_i (Z_i - p_0) \hat{v}_i \hat{Y}_i \right).
\]
Simple algebra gives that
\[
\sum_{i \in S} w_i \hat{x}_i \hat{x}_i^\top = p_1^{-2}(n_1 - 1) s^2_{x(1)} + p_0^{-2}(n_0 - 1) s^2_{x(0)},
\]
\[
\sum_{i \in S} w_i \hat{x}_i \hat{Y}_i = p_1^{-2}(n_1 - 1) s_{x1} + p_0^{-2}(n_0 - 1) s_{x0},
\]
\[
\sum_{i \in S} w_i (Z_i - p_0)^2 \hat{v}_i \hat{v}_i^\top = (n_1 - 1) s^2_{v(1)} + (n_0 - 1) s^2_{v(0)},
\]
\[
\sum_{i \in S} w_i (Z_i - p_0) \hat{v}_i \hat{Y}_i = p_1^{-1}(n_1 - 1) s_{v1} - p_0^{-1}(n_0 - 1) s_{v0},
\]
\[
\sum_{i \in S} w_i (Z_i - p_0) \hat{x}_i \hat{v}_i^\top = p_1^{-1}(n_1 - 1) s_{xv(1)} - p_0^{-1}(n_0 - 1) s_{xv(0)}.
\]
Therefore,
\[
\begin{pmatrix}
\hat{\beta}_{crs} \\
\hat{\gamma}_{crs}
\end{pmatrix} = \left( \frac{p_1^{-1}(1 - n_1^{-1}) s^2_{x(1)} + p_0^{-1}(1 - n_0^{-1}) s^2_{x(0)} - (p_1 p_0)^{-1} S^2_x}{(1 - n_1^{-1}) s_{xv(1)} - (1 - n_0^{-1}) s_{xv(0)}} \right) \left( \frac{(p_1 - n_1^{-1}) s_{v(1)} - (p_0 - n_0^{-1}) s_{v(0)}}{(1 - n_1^{-1}) s_{v1} - (1 - n_0^{-1}) s_{v0}} \right).
\]

The following lemma is from Lemma B16 in Yang et al. (2021).

**Lemma 14.** Under Assumption 4, for \( z = 0, 1 \),
\[
s^2_x - S^2_x = o_p(1), \quad s^2_{x(z)} - S^2_x = o_p(1), \quad s_{xz} - S_{xz} = o_p(1),
\]
\[
s^2_{v(z)} - S^2_v = o_p(1), \quad s_{vz} - S_{vz} = o_p(1), \quad s_{uv} - S_{uv} = o_p(1).
\]

**Lemma 15.** Under Assumption 4,
\[
\hat{\beta}_{crs} = \beta^\text{opt}_{crs} + o_p(1), \quad \hat{\gamma}_{crs} = \gamma^\text{opt}_{crs} + o_p(1).
\]

**Proof.** By Lemma 14, we have
\[
(1 - n_1^{-1}) s_{xv(1)} - (1 - n_0^{-1}) s_{xv(0)} = o_p(1),
\]
\[
\left\{ p_1^{-1}(1 - n_1^{-1}) s^2_{x(1)} + p_0^{-1}(1 - n_0^{-1}) s^2_{x(0)} \right\} - (p_1 p_0)^{-1} S^2_x = o_p(1)
\]
\[
\left\{ (p_1 - n_1^{-1}) s^2_{v(1)} + (p_0 - n_0^{-1}) s^2_{v(0)} \right\} - S^2_v = o_p(1),
\]
\[
p_1^{-1}(1 - n_1^{-1}) s_{x1} + p_0^{-1}(1 - n_0^{-1}) s_{x0} - (p_1 p_0)^{-1} S^2_{x1} - (p_1 p_0)^{-1} S^2_{x0} = o_p(1),
\]
\[
(1 - n_1^{-1}) s_{v1} - (1 - n_0^{-1}) s_{v0} - (S_{v1} - S_{v0}) = o_p(1).
\]
By Proposition 7,
\[
\begin{pmatrix}
\hat{\beta}_{crs} \\
\hat{\gamma}_{crs}
\end{pmatrix} = \left( \begin{pmatrix} p_1 p_0 & 0 \\ 0 & S^2_v \end{pmatrix} \right)^{-1} \begin{pmatrix} p_1^{-1} S^2_{x1} + p_0^{-1} S^2_{x0} \\ S_{v1} - S_{v0} \end{pmatrix} = o_p(1).
\]
Recall the definition of \( \beta^\text{opt}_{crs} \) and \( \gamma^\text{opt}_{crs} \) and \( S_{v1} - S_{v0} = S_{v(1)} \). The conclusion follows.
\[\square\]
Proposition 8 below is from Yang et al. (2021).

Proposition 8. Under Assumption 4, $\sqrt{n}(\hat{\tau}_{crs} - \tau_{crs}, \hat{\gamma}_{crs}^\top, \hat{\delta}_{v})^\top$ is asymptotically normal with zero mean and covariance

$$
\begin{pmatrix}
V_{crs,\tau\tau} & V_{crs,\tau x} & V_{crs,\tau v} \\
V_{crs,\tau x} & V_{crs,xx} & V_{crs,xv} \\
V_{crs,\tau v} & V_{crs,xv} & V_{crs,vv}
\end{pmatrix} = 
\begin{pmatrix}
(p_1^{-1}S_1^2 + p_0^{-1}S_0^2 - fS_r^2) & p_1^{-1}S_{1x} + p_0^{-1}S_{x0} & (1-f)S_{vt} \\
p_1^{-1}S_{x1} + p_0^{-1}S_{x0} & (p_1p_0)^{-1}S_x^2 & 0 \\
(1-f)S_{vt} & 0 & (1-f)S_v^2
\end{pmatrix}.
$$

F.2. Proof of Theorem 6

Proof. Note that

$$
n^{1/2}(\hat{\tau}_{crs} - \tau_{crs}) = n^{1/2}(\hat{\tau}_{crs} - (\beta_{crs}^{opt})^\top \tau_{crs} - (\gamma_{crs}^{opt})^\top \hat{\delta}_{v}) + n^{1/2}(\beta_{crs}^{opt} - \hat{\beta}_{crs})^\top \tau_{crs,x} + n^{1/2}(\gamma_{crs}^{opt} - \hat{\gamma}_{crs})^\top \hat{\delta}_{v}$$

where the first equality is due to Proposition 7 and the second equality is due to Propositions 8 and 15. By Proposition 8 and the definition of $\beta_{crs}^{opt}$ and $\gamma_{crs}^{opt}$, we have

$$
n^{1/2}(\hat{\tau}_{crs} - (\beta_{crs}^{opt})^\top \tau_{crs} - (\gamma_{crs}^{opt})^\top \hat{\delta}_{v}) \sim N(0, V_{crs,\tau\tau} - V_{crs,\tau x}V^{-1}_{crs,xx}V_{crs,\tau x} - V_{crs,\tau v}V^{-1}_{crs,vv}V_{crs,\tau v})
$$

Compounded with Slusky’s theorem, the conclusion follows.

F.3. A plug-in variance estimator

With a slight abuse of notation, let $\hat{e}_i$ be the residual of unit $i$ from the WLS regression (8). One of the variance estimators of $\hat{\tau}_{crs}^\text{tom}$ can be derived by

$$
V_{crs} = n^{-1} \left\{p_1^{-1}s_{e(1)}^2 + p_0^{-1}s_{e(0)}^2\right\},
$$

where

$$
s_{e(1)}^2 = (n_1 - 1)^{-1} \sum_{i \in \mathcal{S}_1} \hat{e}_i^2, \quad s_{e(0)}^2 = (n_0 - 1)^{-1} \sum_{i \in \mathcal{S}_0} \hat{e}_i^2.
$$

Proposition 9 below demonstrates the asymptotic conservativeness of $V_{crs}$.

Proposition 9. Under Assumption 4,

$$
V_{crs} = n^{-1} \min_{\beta, \gamma} \left\{p_1^{-1}S_1^2(\beta, \gamma) + p_0^{-1}S_0^2(\beta, \gamma)\right\} + o_P(n^{-1}),
$$

where

$$
S_z^2(\beta, \gamma) = (N - 1)^{-1} \sum_{i=1}^N \{Y_i(z) - \bar{Y}(z) - (x_i - \bar{x})^\top \beta - (z - p_0)(v_i - \bar{v})^\top \gamma\}^2, \quad z = 0, 1.
$$
Proof. By Lemmas 14 and 15, and similar to the proof of Proposition 7, we have

$$n\hat{v}_{crs} = p_1^{-1} S_1^2(\beta_{crs}^{opt}, \gamma_{crs}^{opt}) + p_0^{-1} S_0^2(\beta_{crs}^{opt}, \gamma_{crs}^{opt}) + o_2(1).$$

Next, we show that

$$(\beta_{crs}^{opt}, \gamma_{crs}^{opt}) = \arg \min_{(\beta, \gamma)} \{ p_1^{-1} S_1^2(\beta, \gamma) + p_0^{-1} S_0^2(\beta, \gamma) \}. \quad (43)$$

Note that \( \text{var}(\hat{v}_{crs}(\beta_{crs}^{opt}, \gamma_{crs}^{opt})) \) can be derived by replacing \( Y_i(z) \) by the adjusted potential outcome \( Y_i(z; \beta_{crs}^{opt}, \gamma_{crs}^{opt}) \) in the formula of \( V_{str,\tau} \). The optimality of \((\beta_{crs}^{opt}, \gamma_{crs}^{opt})\) implies that

$$(\beta_{crs}^{opt}, \gamma_{crs}^{opt}) = \arg \min_{\beta, \gamma} \{ p_1^{-1} S_1^2(\beta, \gamma) + p_0^{-1} S_0^2(\beta, \gamma) - f S_f^2(\gamma) \}, \quad (44)$$

Since \((\beta_{crs}^{opt}, \gamma_{crs}^{opt})\) does not depend on \( f \), then (44) holds for any \( f \). Let \( f = 0 \), we have

$$(\beta_{crs}^{opt}, \gamma_{crs}^{opt}) = \arg \min_{\beta, \gamma} \{ p_1^{-1} S_1^2(\beta, \gamma) + p_0^{-1} S_0^2(\beta, \gamma) \}.$$}

The conclusion follows from (42) and (43).

\[ \square \]

F.4. Leverage scores of ToM regression in completely randomized survey experiments

Recall that \( \hat{x}_i = x_i - Z_i \hat{x}(1) - (1 - Z_i) \hat{x}(0) \) and we similarly define \( \hat{v}_i = v_i - Z_i \hat{v}(1) - (1 - Z_i) \hat{v}(0). \)

Define \( \hat{V}_{crs,(x,v)} \) by

$$\hat{V}_{crs,(x,v)} = \left( p_1^{-1}(1 - n^{-1})S_{x1}(1) + p_0^{-1}(1 - n^{-1})S_{x0}(1) - (1 - n^{-1})S_{xv}(1) - (1 - n^{-1})S_{xv}(0) \right),$$

$$= n^{-1} \left( \sum_{i \in S} w_i \hat{x}_i \hat{x}_i^\top - \sum_{i \in S} w_i (Z_i - p_0) \hat{v}_i \hat{v}_i^\top \right).$$

We define the weights for treatment arm \( z \) as \( w(z) = p_z^{-2} \).

Proposition 10. The leverage score of ToM regression for unit \( i \) under completely randomized survey experiments is

$$h_{i, crs} = \left\{ \begin{array}{ll} n_1^{-1} + (\hat{x}_i^\top, p_1 \hat{v}_i^\top)^\top V_{crs,(x,v)}^{-1} (\hat{x}_i^\top, p_1 \hat{v}_i^\top) w(1)n^{-1}, & i \in S_1, \\ n_0^{-1} + (\hat{x}_i^\top, -p_0 \hat{v}_i^\top)^\top V_{crs,(x,v)}^{-1} (\hat{x}_i^\top, -p_0 \hat{v}_i^\top) w(0)n^{-1}, & i \in S_0. \end{array} \right.$$}

Proof. Let \( \tilde{X}_{crs} \in \mathbb{R}^{n \times (k_1 + k_2 + 2)} \) with the \( i \)th row of \( \tilde{X}_{crs} \) being

$$(Z_i, 1 - Z_i, \hat{x}_i^\top, (Z_i - p_0) \hat{v}_i^\top).$$

There exists an invertible matrix \( Q \) such that \( \tilde{X}_{crs} = X_{crs} Q \). Therefore,

$$X_{crs} (X_{crs}^\top W X_{crs})^{-1} X_{crs}^\top W = \tilde{X}_{crs} (\tilde{X}_{crs}^\top W \tilde{X}_{crs})^{-1} \tilde{X}_{crs}^\top W.$$
Therefore,
\[
h_{i,\text{crs}} = \begin{cases} 
  n_1^{-1} + (\hat{x}_i^T, p_1 \hat{v}_i^T)\hat{V}^{-1}_{\text{crs}}(\hat{x}_i^T, p_1 \hat{v}_i^T)w(1)n^{-1}, & i \in S_1, \\
  n_0^{-1} + (\hat{x}_i^T, -p_0 \hat{v}_i^T)\hat{V}^{-1}_{\text{crs}}(\hat{x}_i^T, -p_0 \hat{v}_i^T)w(0)n^{-1}, & i \in S_0.
\end{cases}
\]

\hfill \square

**Lemma 16.** Under Assumption 4,
\[
\|\hat{V}^{-1}_{\text{crs}(x,v)}\|_\infty = O_P(1), \quad \|\hat{V}^{-1}_{\text{crs}(x,v)}\|_{\text{op}} = O_P(1).
\]

The proof of Lemma 16 is similar to that of Lemma 12, so we omit it.

**Lemma 17.** Under Assumption 4,
\[
\max_i h_{i,\text{crs}} = o_P(1).
\]

The proof of Lemma 17 is similar to that of 13, so we omit it.

**F.5. Proof of Theorem 7**

Let \( \hat{e}_i \) be the residual of unit \( i \). Let \( \hat{\bar{e}}_i = \eta_i \hat{e}_i \), where \( \eta_i = 1 \) for HC0, \( \eta_i = \{n/(n-k_1-k_2-2)\}^{1/2} \) for HC1, \( \eta_i = (1-h_{i,\text{crs}})^{-1/2} \) for HC2, and \( \eta_i = (1-h_{i,\text{crs}})^{-1} \) for HC3. The variance estimator HC\(_j\) \((j = 0,1,2,3)\) derives as
\[
\beta_2^T(X_{\text{crs}}^TWX_{\text{crs}})^{-1}X_{\text{crs}}^TW\Delta X_{\text{crs}}(X_{\text{crs}}^TWX_{\text{crs}})^{-1}\beta_2,
\]
where \( X_{\text{crs}} \in \mathbb{R}^{n \times (2+k_1+k_2)} \) with the \( i \)th row being \((1, Z_i, x_i^T, (Z_i - p_0)(v_i - \bar{v})^T)\), \( W \) is the diagonal matrix of \( w_i \), and \( \Delta \) is the diagonal matrix of scaled residual squares \( \xi_i^2 \).

Motivated by the following equivalent regression
\[
Y_i \sim Z_i + (1-Z_i) + (x_i-\bar{x}) + (Z_i-p_0)(v_i-\bar{v})
\]
An equivalent variance estimator derives as
\[
d^T(E^TW)E^{-1}E^TW\Delta WE(E^TWE)^{-1}d,
\]
where \( d = (1,-1,0_{k_1+k_2}^T)^T, \ E \in \mathbb{R}^{n \times (2+k_1+k_2)} \) with the \( i \)th row being \((Z_i,1-Z_i, (x_i-\bar{x})^T, (Z_i-p_0)(v_i-\bar{v})^T)\). Note that \( \bar{x} \) is unknown, and therefore the regression is infeasible, but it is useful for proving Theorem 7.

The proof of equivalence is similar to that in Section E.2, so we omit it. We will base our proof of Theorem 7 on this equivalent variance estimator.

**Proof.** Let \( u_i = x_i - \bar{x} \) and \( r_i = v_i - \bar{v} \). Define \( H \) by
\[
H = \begin{pmatrix}
  H_{11} & H_{12} \\
  H_{21} & H_{22}
\end{pmatrix} = E^TW\Delta WE/n,
\]

58
where

\[ H_{11} = n^{-1} \text{diag} \left( w^2(1) \sum_{i \in S_1} e_i^2, w^2(0) \sum_{i \in S_0} e_i^2 \right), \]

\[ H_{21} = H_{12}^T = n^{-1} \left( \begin{array}{cc}
 w^2(1) \sum_{i \in S_1} e_i^2 u_i & w^2(0) \sum_{i \in S_0} e_i^2 u_i \\
 w^2(0) \sum_{i \in S_1} e_i^2 r_i & -w^2(0) \sum_{i \in S_0} e_i^2 r_i
\end{array} \right), \]

\[ H_{22} = n^{-1} \left( \begin{array}{cc}
 w^2(1) \sum_{i \in S_1} e_i^2 u_i + w^2(1) p_1 \sum_{i \in S_1} e_i^2 r_i & w^2(1) p_1 \sum_{i \in S_1} e_i^2 r_i^T - w^2(0) p_0 \sum_{i \in S_0} e_i^2 u_i \\
 w^2(0) p_1 \sum_{i \in S_1} e_i^2 r_i^T - w^2(0) p_0 \sum_{i \in S_0} e_i^2 u_i & w^2(0) p_0 \sum_{i \in S_0} e_i^2 r_i
\end{array} \right). \]

Define \( G \) by

\[ G = \left( \begin{array}{cc}
 G_{11} & G_{12} \\
 G_{21} & G_{22}
\end{array} \right) = E^T W E / n, \]

where

\[ G_{11} = n^{-1} \text{diag} \ (w(1) n_1, w(0) n_0), \quad G_{21} = G_{12}^T = n^{-1} \left( \begin{array}{cc}
 w(1) \sum_{i \in S_1} u_i & w(0) \sum_{i \in S_0} u_i \\
 w(1) p_1 \sum_{i \in S_1} r_i & -w(0) p_0 \sum_{i \in S_0} r_i
\end{array} \right), \]

\[ G_{22} = n^{-1} \left( \begin{array}{cc}
 w(1) \sum_{i \in S_1} u_i u_i^T + w(0) \sum_{i \in S_0} u_i u_i^T & w(1) p_1 \sum_{i \in S_1} u_i r_i^T - w(0) p_0 \sum_{i \in S_0} u_i r_i^T \\
 w(1) p_1 \sum_{i \in S_1} u_i r_i^T - w(0) p_0 \sum_{i \in S_0} u_i r_i^T & w(1) p_1^2 \sum_{i \in S_1} r_i r_i^T + w(0) p_0^2 \sum_{i \in S_0} r_i r_i^T
\end{array} \right). \]

Define \( \Lambda \) by

\[ \Lambda = G^{-1} = \left( \begin{array}{cc}
 \Lambda_{11} & \Lambda_{12} \\
 \Lambda_{21} & \Lambda_{22}
\end{array} \right). \]

By the formula of inverse of 2 \( \times \) 2 block matrix, we have

\[ \Lambda_{11} = G_{11}^{-1} + G_{11}^{-1} G_{12} (G_{22} - G_{21} G_{11}^{-1} G_{12})^{-1} G_{21} G_{11}^{-1}, \]

\[ \Lambda_{21} = -G_{11}^{-1} G_{12} (G_{22} - G_{21} G_{11}^{-1} G_{12})^{-1}. \]

Let \( d_1 = (1, -1)^T \), it is easy to see that

\[ n \hat{V}_{HC,j,crs} = d_1^T \left( \begin{array}{cc}
 \Lambda_{11} & \Lambda_{12} \\
 \Lambda_{21} & \Lambda_{22}
\end{array} \right) H \left( \begin{array}{cc}
 \Lambda_{11} & \Lambda_{12} \\
 \Lambda_{21} & \Lambda_{22}
\end{array} \right) d_1. \]  

(45)

Recall that

\[ \hat{V}_{crs,(x,v)} = \left( \begin{array}{cc}
 p_1^{-1} (1 - n_1^{-1}) s_{x(1)}^2 + p_0^{-1} (1 - n_0^{-1}) s_{x(0)}^2 & (1 - n_1^{-1}) s_{xv(1)} - (1 - n_0^{-1}) s_{xv(0)} \\
 (1 - n_1^{-1}) s_{vx(1)} - (1 - n_0^{-1}) s_{vx(0)} & (p_1 - n_1^{-1}) s_{v(1)}^2 + (p_0 - n_0^{-1}) s_{v(0)}^2
\end{array} \right). \]

After some calculation, we have

(i) \( G_{11}^{-1} G_{12} = \hat{U} \), \quad (ii) \( G_{22} - G_{21} G_{11}^{-1} G_{12} = \hat{V}_{crs,(x,v)}, \)

where

\[ \hat{U} = \left( \begin{array}{cc}
 \hat{u}(1)^T & \hat{u}(1)^T p_1 \\
 \hat{u}(0)^T & -\hat{u}(0)^T p_0
\end{array} \right). \]
We expand equation (45) as follows:

\[ d_1^T ( \begin{array}{c} \Lambda_{11} \\ \Lambda_{12} \end{array} ) H \left( \begin{array}{c} \Lambda_{11} \\ \Lambda_{21} \end{array} \right) d_1 \]

\[ = d_1^T \left( G_{11}^{-1} + \hat{U} \hat{V}_{\text{crs},(x,v)}^{-1} \hat{U}^T - \hat{U} \hat{V}_{\text{crs},(x,v)}^{-1} \hat{U}^T \right) H \left( G_{11}^{-1} + \hat{U} \hat{V}_{\text{crs},(x,v)}^{-1} \hat{U}^T \right)^{-1} d_1 \]

\[ = d_1^T \left( G_{11}^{-1} + \hat{U} \hat{V}_{\text{crs},(x,v)}^{-1} \hat{U}^T \right) H_{11} \left( G_{11}^{-1} + \hat{U} \hat{V}_{\text{crs},(x,v)}^{-1} \hat{U}^T \right)^{-1} d_1 + d_1^T \hat{U} \hat{V}_{\text{crs},(x,v)}^{-1} \hat{U}^T d_1 \]

\[ - 2d_1^T \left( G_{11}^{-1} + \hat{U} \hat{V}_{\text{crs},(x,v)}^{-1} \hat{U}^T \right) H_{12} \hat{V}_{\text{crs},(x,v)}^{-1} \hat{U}^T d_1. \]

Let

\[ T_1 = d_1^T G_{11}^{-1} H_{11} G_{11}^{-1} d_1, \quad T_2 = d_1^T G_{11}^{-1} H_{11} \hat{U} \hat{V}_{\text{crs},(x,v)}^{-1} \hat{U}^T d_1, \]

\[ T_3 = d_1^T \hat{U} \hat{V}_{\text{crs},(x,v)}^{-1} \hat{U}^T H_{11} \hat{U} \hat{V}_{\text{crs},(x,v)}^{-1} \hat{U}^T d_1, \quad T_4 = d_1^T G_{11}^{-1} H_{12} \hat{V}_{\text{crs},(x,v)}^{-1} \hat{U}^T d_1, \]

\[ T_5 = d_1^T \hat{U} \hat{V}_{\text{crs},(x,v)}^{-1} \hat{U}^T H_{12} \hat{V}_{\text{crs},(x,v)}^{-1} \hat{U}^T d_1, \quad T_6 = d_1^T \hat{U} \hat{V}_{\text{crs},(x,v)}^{-1} H_{22} \hat{V}_{\text{crs},(x,v)}^{-1} \hat{U}^T d_1. \]

Then,

\[ T_1 = d_1^T G_{11}^{-1} H_{11} G_{11}^{-1} d_1 \]

\[ = \sum_{i \in S_1} n^{-1} e_i^2 w^2(1) \left\{ \frac{n_1}{n} w(1) \right\}^{-2} + \sum_{i \in S_0} n^{-1} e_i^2 w^2(0) \left\{ \frac{n_0}{n} w(0) \right\}^{-2} \]

\[ = \sum_{i \in S_1} n^{-1} \hat{e}_i^2 n_1^{-2} + \sum_{i \in S_0} n^{-1} \hat{e}_i^2 n_0^{-2} \]

\[ = \left( \sum_{i \in S_1} \frac{\hat{e}_i^2 / n_1}{n} \right) p_1^{-1} + \left( \sum_{i \in S_0} \frac{\hat{e}_i^2 / n_0}{n} \right) p_0^{-1}, \]

\[ d_1^T G_{11}^{-1} H_{11} \hat{U} = \left\{ \frac{n_1}{n} w(1) \right\}^{-1} \sum_{i \in S_1} w^2(1)e_i^2 \left( \frac{\hat{u}(1)}{p_1 \hat{r}(1)} \right)^T - \]

\[ \left\{ \frac{n_0}{n} w(0) \right\}^{-1} \sum_{i \in S_0} w^2(0)e_i^2 \left( \frac{\hat{u}(0)}{-p_0 \hat{r}(0)} \right)^T \]

\[ = n_1^{-1} \sum_{i \in S_1} w(1)e_i^2 \left( \frac{\hat{u}(1)}{p_1 \hat{r}(1)} \right)^T - n_0^{-1} \sum_{i \in S_0} w(0)e_i^2 \left( \frac{\hat{u}(0)}{-p_0 \hat{r}(0)} \right)^T \]

\[ = n_1^{-1} \sum_{i \in S_1} p_1^{-2} e_i^2 \left( \frac{\hat{u}(1)}{p_1 \hat{r}(1)} \right)^T - n_0^{-1} \sum_{i \in S_0} p_0^{-2} e_i^2 \left( \frac{\hat{u}(0)}{-p_0 \hat{r}(0)} \right)^T, \]

\[ \hat{U}^T H_{11} \hat{U} = \left( \frac{\hat{u}(1)}{p_1 \hat{r}(1)} \right)^T \left\{ n_1^{-1} \sum_{i \in S_1} e_i^2 w^2(1) \right\} + \left( \frac{\hat{u}(0)}{-p_0 \hat{r}(0)} \right)^T \left\{ n_0^{-1} \sum_{i \in S_0} e_i^2 w^2(0) \right\}, \]

where \( S_1 \) and \( S_0 \) are subsets of the sample data.
\[ d_1^\top G_{11}^{-1} H_{12} = \left\{ \frac{n_1}{n} w(1) \right\}^{-1} w^2(1) \sum_{i \in S_1} n^{-1} e_i^2 \left( \frac{u_i}{p_1 r_i} \right)^\top - \left\{ \frac{n_0}{n} w(0) \right\}^{-1} w^2(0) \sum_{i \in S_0} n^{-1} e_i^2 \left( \frac{u_i}{-p_0 r_i} \right)^\top. \]

\[ = n_1^{-1} w(1) \sum_{i \in S_1} e_i^2 \left( \frac{u_i}{p_1 r_i} \right)^\top - n_0^{-1} w(0) \sum_{i \in S_0} e_i^2 \left( \frac{u_i}{-p_0 r_i} \right)^\top. \]

\[ = p_1^{-2} n_1^{-1} \sum_{i \in S_1} e_i^2 \left( \frac{u_i}{p_1 r_i} \right)^\top - p_0^{-2} n_0^{-1} \sum_{i \in S_0} e_i^2 \left( \frac{u_i}{-p_0 r_i} \right)^\top. \]

\[ \hat{U}^\top H_{12} = \left( \frac{\hat{u}(1)}{p_1 \hat{r}(1)} \right) w^2(1) \sum_{i \in S_1} e_i^2 \left( \frac{u_i}{p_1 r_i} \right)^\top n^{-1} + \left( \frac{\hat{u}(0)}{-p_0 \hat{r}(0)} \right) w^2(0) \sum_{i \in S_0} e_i^2 \left( \frac{u_i}{-p_0 r_i} \right)^\top n^{-1}. \]

Note that \( \{n/(n - 2 - k_1 - k_2)\}^{1/2} = 1 + o_F(1) \), by Lemma 17,

\[ \max_i (1 - h_{i,crs})^{-1/2} = 1 + o_F(1), \quad \max_i (1 - h_{i,crs})^{-1} = 1 + o_F(1). \]

Therefore, \( \max_i \eta_i^2 = 1 + o_F(1) \) for HC\(_j \) (\( j = 0, 1, 2, 3 \)). Moreover, \( \max_i n_z/(n_z - 1) = 1 + O(1) \) and by Proposition 9, \( n \hat{V}_{crs} = p_1^{-1} s_{e(1)}^2 + p_0^{-1} s_{e(0)}^2 = O_F(1) \). Therefore,

\[ T_1 = \left\{ \sum_{i \in S_1} e_i^2 / n_1 p_1^{-1} + \sum_{i \in S_0} e_i^2 / n_0 p_0^{-1} \right\} \{1 + o_F(1)\} \]

\[ = \left\{ (n_1 - 1)^{-1} \sum_{i \in S_1} e_i^2 p_1^{-1} + (n_0 - 1)^{-1} \sum_{i \in S_0} e_i^2 p_0^{-1} \right\} \{1 + o_F(1)\} \]

\[ = p_1^{-1} s_{e(1)}^2 + p_0^{-1} s_{e(0)}^2 + o_F(1) = n \hat{V}_{crs} + o_F(1). \]

Note that \( \max_x |p_z^{-1}| = O(1), \max_i ||u_i||_{\infty} = o(n), \max_i ||r_i||_{\infty}^2 = o(n), \max_x |w(z)| \leq 2 \max_x |p_z^{-2}| = O(1) \). Therefore,
\[
\|d_1^T G_{11}^{-1} H_{11} \dot{U}\|_\infty \leq \max \left\{ \|\dot{\hat{u}}(1)\|_\infty, \|p_1 \dot{\hat{r}}(1)\|_\infty, \|\dot{\hat{u}}(0)\|_\infty, \|p_0 \dot{\hat{r}}(0)\|_\infty \right\} \left( n_1^{-1} \sum_{i \in S_1} p_1^{-2} e_i^2 + n_0^{-1} \sum_{i \in S_0} p_0^{-2} e_i^2 \right)
\]
\[
\leq \max \left\{ \max_i \|u_i\|_\infty, \max_i \|r_i\|_\infty \right\} \max_z p_z^{-1} n \hat{V}_{\text{crs}} \max_i \eta_i^2
\]
\[
= o(n^{1/2}) O(1) O_P(1) = o_P(n^{1/2}).
\]
\[
\|\dot{U}^T H_{11} \dot{U}\|_\infty \leq \max \left\{ \|\dot{\hat{u}}(1)\|_\infty, \|p_1 \dot{\hat{r}}(1)\|_\infty, \|\dot{\hat{u}}(0)\|_\infty, \|p_0 \dot{\hat{r}}(0)\|_\infty \right\}^2 n^{-1} \left( \sum_{i \in S_1} e_i^2 w_1^2(1) + \sum_{i \in S_0} e_i^2 w_1^2(0) \right)
\]
\[
\leq \max \left\{ \max_i \|u_i\|_2^2, \max_i \|r_i\|_\infty^2 \right\} \max_z |w(z)| n \hat{V}_{\text{crs}} \max_i \eta_i^2
\]
\[
= o(n) O(1) O_P(1) = o_P(n),
\]
\[
\|d_1^T G_{11}^{-1} H_{12}\|_\infty \leq \max \left\{ \max_i \|u_i\|_\infty, \max_i \|r_i\|_\infty \right\} \max_z |w(z)| n \hat{V}_{\text{crs}} \max_i \eta_i^2
\]
\[
= o(n^{1/2}) O(1) O_P(1) = o_P(n^{1/2}).
\]
\[
\|\dot{U}^T H_{12}\|_\infty \leq \max \left\{ \max_i \|u_i\|_2^2, \max_i \|r_i\|_\infty^2 \right\} \max_z |w(z)| n \hat{V}_{\text{crs}} \max_i \eta_i^2
\]
\[
= o(n) O(1) O_P(1) = o_P(n),
\]
\[
\|H_{22}\|_\infty \leq \max \left\{ \max_i \|u_i\|_\infty^2, \max_i \|r_i\|_\infty^2 \right\} n \hat{V}_{\text{crs}} \max_i \eta_i^2
\]
\[
= o(n) O_P(1) = o_P(n).
\]
Hence, combining with Lemma 9, we complete the proof.

\[ |T_2| = |d_1^\top G_{11}^{-1} H_{11} \hat{\mathcal{U}} \hat{\mathcal{V}}_{crs,(x,v)} \hat{\mathcal{U}}^\top d_1| \leq (k_1 + k_2)^2 \max \{\|\hat{\tau}_{crs,x}\|_{\infty}, \|\hat{\delta}_v\|_{\infty}\} |d_1^\top G_{11}^{-1} H_{11} \hat{\mathcal{U}} \hat{\mathcal{V}}_{crs,(x,v)} \hat{\mathcal{U}}^\top d_1|
\]
\[ = O_{\mathbb{P}}(n^{-1/2}) O_{\mathbb{P}}(n^{1/2}) O_{\mathbb{P}}(1) = o_{\mathbb{P}}(1), \]

\[ |T_3| = |d_1^\top \hat{\mathcal{U}} \hat{\mathcal{V}}_{crs,(x,v)} \hat{\mathcal{U}}^\top H_{11} \hat{\mathcal{U}} \hat{\mathcal{V}}_{crs,(x,v)} \hat{\mathcal{U}}^\top d_1|
\]
\[ \leq (k_1 + k_2)^2 \max \{\|\hat{\tau}_{crs,x}\|_{\infty}, \|\hat{\delta}_v\|_{\infty}\} |\hat{\mathcal{V}}_{crs,(x,v)} \hat{\mathcal{U}}^\top H_{11} \hat{\mathcal{U}} \hat{\mathcal{V}}_{crs,(x,v)} \hat{\mathcal{U}}^\top d_1|
\]
\[ = O_{\mathbb{P}}(n^{-1/2}) O_{\mathbb{P}}(n^{-1/2}) = o_{\mathbb{P}}(1), \]

\[ |T_4| = |d_1^\top G_{11}^{-1} H_{12} \hat{\mathcal{V}}_{crs,(x,v)} \hat{\mathcal{U}}^\top d_1| \leq (k_1 + k_2)^2 |d_1^\top G_{11}^{-1} H_{12} | |\hat{\mathcal{V}}_{crs,(x,v)} \hat{\mathcal{U}}^\top d_1|
\]
\[ = o_{\mathbb{P}}(n^{1/2}) O_{\mathbb{P}}(1) O_{\mathbb{P}}(n^{-1/2}) = o_{\mathbb{P}}(1), \]

\[ |T_5| = |d_1^\top \hat{\mathcal{U}} \hat{\mathcal{V}}_{crs,(x,v)} \hat{\mathcal{U}}^\top H_{12} \hat{\mathcal{V}}_{crs,(x,v)} \hat{\mathcal{U}}^\top d_1|
\]
\[ \leq (k_1 + k_2)^4 \max \{\|\hat{\tau}_{crs,x}\|_{\infty}, \|\hat{\delta}_v\|_{\infty}\} |\hat{\mathcal{U}}^\top H_{12} | |\hat{\mathcal{V}}_{crs,(x,v)} \hat{\mathcal{U}}^\top d_1|
\]
\[ = O_{\mathbb{P}}(n^{-1/2}) O_{\mathbb{P}}(1) O_{\mathbb{P}}(n^{-1/2}) = o_{\mathbb{P}}(1), \]

\[ |T_6| = |d_1^\top \hat{\mathcal{U}} \hat{\mathcal{V}}_{crs,(x,v)} H_{22} \hat{\mathcal{V}}_{crs,(x,v)} \hat{\mathcal{U}}^\top d_1|
\]
\[ \leq (k_1 + k_2)^4 \max \{\|\hat{\tau}_{crs,x}\|_{\infty}, \|\hat{\delta}_v\|_{\infty}\} |\hat{\mathcal{U}}^\top H_{22} | |\hat{\mathcal{V}}_{crs,(x,v)} \hat{\mathcal{U}}^\top d_1|
\]
\[ = O_{\mathbb{P}}(n^{-1/2}) O_{\mathbb{P}}(1) O_{\mathbb{P}}(n^{-1/2}) = o_{\mathbb{P}}(1). \]

Therefore,

\[ n \hat{\mathcal{V}}_{H_{C_j},crs} = n \hat{\mathcal{V}}_{crs} + o_{\mathbb{P}}(1), \quad j = 0, 1, 2, 3. \]

Hence, combining with Lemma 9, we complete the proof. 

\[ \square \]