Static Potential in $\mathcal{N} = 4$ Supersymmetric Yang-Mills Theory

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Abstract

We compute the leading order perturbative correction to the static potential in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. We show that the perturbative expansion contains infrared logarithms which, when resummed, become logarithms of the coupling constant. The resulting correction goes in the right direction to match the strong coupling behavior obtained from the AdS/CFT correspondence. We find that the strong coupling extrapolation of the sum of ladder diagrams goes as $\sqrt{g^2 N}$, as in the supergravity approach.
Yang-Mills theory with $\mathcal{N} = 4$ supersymmetry in four dimensions is an extremely interesting quantum field theory. It is a conformal field theory for any value of its coupling constant and is conjectured to have a Montonen-Olive strong-weak coupling duality. It is now believed to have an additional, exact duality to type IIB superstring theory on an $AdS_5 \times S^5$ background. This duality, which is known as the AdS/CFT correspondence [1, 2, 3, 4], is an explicit realization of the long anticipated but elusive mapping between gauge theory and string theory [3]. In this duality, the weak coupling, classical limit of string theory corresponds to the large $N$ limit of gauge theory. Classical string theory in its low energy, weak curvature limit is accurately described by classical IIB supergravity. This corresponds to the large $N$, large ’t Hooft coupling (denoted $\lambda \equiv g^2 N$) limit of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. This gives a new, computationally tractable scheme for extracting the large $\lambda$ limit of certain correlators in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory.

It is difficult to check the results of these computations since the only other quantitative approach to Yang-Mills theory is its perturbative expansion in small $g^2$. An important exception is correlation functions which are protected from gaining quantum corrections by non-renormalization theorems. The agreement between supergravity calculations of these correlators and weak coupling results is a powerful consistency check of the AdS/CFT correspondence [4]. In fact, some surprising new non-renormalization conjectures were found in this way [6, 7].

However, the AdS/CFT correspondence predicts a generically non-trivial dependence of various quantities on the coupling constant. The sum over planar diagrams which emerges in the large $N$ limit of Yang-Mills theory has a finite radius of convergence [8]. This suggests that the dependence of various quantities on the coupling constant is smooth near $\lambda = 0$. Furthermore, since evidence from matrix models suggests that the singularity in the sum of planar graphs usually occurs for negative $\lambda$ [4], one might extrapolate this smooth behavior to the strong coupling limit, so that some quantities have a smooth interpolation between $\lambda = 0$ and $\lambda = \infty$. An example is the free energy of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory at finite temperature which, because of conformal invariance, has the form

$$F/V = -f(g^2, N)\pi^2 N^2 T^4/6.$$  

In the large $N$ limit $f$ is a function of the ’t Hooft coupling $\lambda$, which, to agree with perturbative computation, must go to 1 at small $\lambda$ and, to agree with the AdS/CFT correspondence, must go to 3/4 as $\lambda$ goes to infinity [10]. Smoothness of this function implies that its perturbative expansion has alternating signs and that corrections to
the supergravity result are positive. Explicit calculations confirmed that this is indeed
the case: for large $\lambda$ \[1\]
\[ f = \frac{3}{4} + \frac{45}{32} \zeta(3) \lambda^{-3/2} + \ldots \]
and, for small $\lambda$ \[2\],
\[ f = 1 - \frac{3\lambda}{2\pi^2} + (3 + \sqrt{2})\lambda^{3/2} + \ldots . \]
The latter expression contains the slight surprise that the final term is non-analytic
in small $\lambda$. This non-analyticity comes from infrared divergences that occur at finite
temperature.

It is interesting to ask whether infrared divergences lead to non-analyticity in
other quantities. If we take the duality to the IIB superstring seriously, this would
imply something about the strong curvature limit of tree level string theory. Local
operators and extensive variables such as the free energy in Yang-Mills theory are
related to the low-energy supergravity degrees of freedom in string theory. An object
which couples more directly to stringy degrees of freedom is the Wilson loop. The
loop operator which is a source for classical strings is \[13\]
\[ W(C) = \frac{1}{N} \text{Tr} \text{Pexp} \int_C d\tau \left( iA_\mu \dot{x}_\mu + \Phi_i \theta_i \sqrt{\dot{x}^2} \right). \] (1)
It contains a coupling to the six scalar fields $\Phi_i$ of the $\mathcal{N} = 4$ supermultiplet (with
$\theta_i$, $i = 1, \ldots, 6$ a point on the 5-sphere, $\theta_i^2 = 1$) as well as the familiar dependence
on the Yang-Mills field. It is related to the holonomy of the wavefunction of a heavy
particle with the quantum numbers of a W-boson.

The simplest quantity which can be extracted from the Wilson loop is the inter-
action potential between static W-bosons. For this, we consider a rectangular loop
of length $T$ and width $L$ and the limit
\[ V(L) = -\lim_{T \to \infty} \frac{1}{T} \ln \langle W(C) \rangle . \] (2)
Conformal symmetry of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory implies that
\[ V(L) = -\frac{\alpha(g^2, N)}{L} , \] (3)
and, in the large $N$ limit, the effective Coulomb charge $\alpha$ is a function of the ’t Hooft
coupling $\lambda$.

The exchange of scalar bosons mediates an attractive force which is equal to that
due to gauge fields, so that, at weak coupling, the potential is twice as large as
that in pure Yang-Mills theory. At large $\lambda$, the AdS/CFT correspondence predicts a square-root dependence of the Coulomb charge \cite{13}, so that

$$\alpha(\lambda) = \begin{cases} 
\frac{\lambda}{4\pi} + \cdots, & \text{for } \lambda \to 0; \\
\frac{4\pi^2 \sqrt{2\lambda}}{\Gamma^4(1/4)} + O(1), & \text{for } \lambda \to \infty.
\end{cases}$$  \hspace{1cm} (4)

If $\alpha(\lambda)$ is a smooth function, we expect that perturbative corrections should decrease the weak coupling value of the Coulomb charge. In this Letter, we shall check this assertion by computing the correction to $\alpha(\lambda)$ to the next order in $\lambda$. Similar computations for non-supersymmetric Yang-Mills theory have been done in \cite{14, 15}.

Feynman rules for $N = 4$ supersymmetric Yang-Mills theory follow from the action

$$S = \frac{1}{g^2} \int d^4x \left\{ \frac{1}{4} (F_{\mu\nu}^a)^2 + \frac{1}{2} (D_\mu \Phi_i^a)^2 + \frac{i}{2} \bar{\Psi}^a \Gamma_\mu D_\mu \Psi^a \\
+ \frac{1}{2} f^{abc} \bar{\Psi}^a \Gamma_i \Phi_j^b \Psi^c + \frac{1}{2} f^{abc} f^{ade} \sum_{i<j} \Phi_i^e \Phi_j^f \Phi_i^d \Phi_j^e \right\},$$

where $i, j$ range from 1 to 6, $D_\mu (\cdot)^a \equiv \partial_\mu (\cdot)^a + f^{abc} A_\mu^b (\cdot)^c$, and $f^{abc}$ are the structure constants of the $SU(N)$ Lie algebra (we use the normalization of generators $\text{Tr} T^a T^b = \delta^{ab}/2$). The $\Psi^a$ are four-component Majorana fermion fields transforming as a four dimensional representation of the $SO(6)$ R-symmetry group. We use Feynman gauge where the gluon propagator is $D_{\mu\nu}(x) = \delta_{\mu\nu} g^2 / 4\pi^2 x^2$, and consider the contribution of all planar diagrams to the Wilson loop to order $g^4$. We have found that individual diagrams contain ultraviolet divergences, which cancel when all diagrams to order $g^4 N^2$ are summed. Nevertheless, the sum of all one-loop contributions to the Wilson loop expectation value does not yield a well-defined correction to the potential. The planar ladder diagram contains an extra logarithm of $T/L$:

$$- \ln \left\{ 1 + \underbrace{\cdots} + \underbrace{\cdots} + \cdots \right\} = \frac{\lambda}{4\pi} \frac{T}{L} - \frac{\lambda^2}{(2\pi)^2} \frac{T}{L} \ln \frac{T}{L} + \cdots.$$  \hspace{1cm} (5)

This logarithmic term threatens to ruin perturbation theory as well as the definition of static potential \cite{2}. A similar behavior (beginning at order $g^6$) has been observed for Wilson loops in ordinary Yang-Mills theory. According to \cite{15} the logarithm is due to an infrared divergence coming from soft gluons traveling along the Wilson loop for a long period of (Euclidean) time $t \propto T$. Diagrams with more soft gluons will have a higher degree of divergence so it is necessary to take all orders into account. It was argued in \cite{13} that the effect of the soft gluon resummation is to cut off the divergent integral over $t$ at $t \sim 1/\lambda$. Prototypically, an integral like $\int dt/t$ is replaced
by $\int (dt/t) \exp(-\lambda t)$. The perturbative expansion of the latter produces successively worse infrared divergences, but resummation of perturbation series makes the result finite and replaces $\ln(T/L)$ by $\ln(1/\lambda)$.

The consistent resummation prescription which removes the infrared divergences in the static potential is to sum up all (planar) ladder diagrams. The sum of ladder diagrams can be obtained from a Bethe-Salpeter equation, which in diagrammatic form is

\[
\sum_{\text{ladders}} \begin{array}{c}
S \\
\end{array} = 1 + \sum_{\text{ladders}} \begin{array}{c}
S \\
\end{array}
\]

and analytically

\[
\Gamma(S,T) = 1 + \int_0^S ds \int_0^T dt \frac{\lambda}{4\pi^2[(s-t)^2 + L^2]} \Gamma(s,t).
\]  
(6)

The static potential is extracted from $\Gamma(S,T)$ as $V(L) = -\lim_{T \to \infty} \frac{1}{T} \ln \Gamma(T,T)$. We note here that, generally, the sum of ladder diagrams will not produce a gauge invariant result. However, the few leading terms (the two terms in (9)) are independent of the choice of gauge.

The kernel in (8) obeys the differential equation

\[
\frac{\partial^2 \Gamma(S,T)}{\partial S \partial T} = \frac{\lambda}{4\pi^2 [(S-T)^2 + L^2]} \Gamma(S,T)
\]

and the boundary conditions $\Gamma(0,T) = 1 = \Gamma(S,0)$. This equation is separable in the variables $x = (S-T)/L$ and $y = (S+T)/L$. The Laplace transform of $\Gamma$ can be expressed in terms of eigenfunctions of the Schrödinger equation

\[
\left[ -\frac{d^2}{dx^2} - \frac{\lambda}{4\pi^2(x^2 + 1)} \right] \psi_n(x) = -\frac{\Omega^2}{4} \psi_n(x).
\]  
(7)

Explicitly,

\[
\int_{|x|}^\infty dy e^{-py} \Gamma(x,y) = 2\sum_n \frac{\psi_n(0)}{p^2 - \Omega^2/4} \psi_n(x).
\]  
(8)

The asymptotic behavior of $\Gamma(x,y)$ at large $y$ is determined by the rightmost singularity of its Laplace transform in the complex $p$ plane. The right hand side of (8) has poles at the eigenvalues of the Schrödinger equation. The pole with the largest
real component is associated with the ground state energy. Consequently, the sum of ladder diagrams grows exponentially: $\Gamma(0, y) \sim \exp(\Omega_0 y/2) = \exp(\Omega_0 T/L)$. Thus, the Coulomb charge is given by the ground state eigenvalue $\alpha(\lambda) = \Omega_0(\lambda)$.

It is straightforward to find $\alpha(\lambda)$ in both the large and small $\lambda$ limits. If $\lambda$ is small, it is necessary to rescale $x$ by $\lambda$, so that $\chi(z) = \psi(z/\lambda)$. In that case, the limit of (7) is

$$-\chi''(z) - \frac{1}{4\pi} \delta(z) \chi(z) = -\frac{\alpha^2}{4\lambda^2} \chi(z).$$

Using the lowest eigenvalue of the delta-function potential, we reproduce the leading behavior, $\alpha(\lambda) = \lambda/4\pi + \cdots$. A straightforward perturbative computation produces the next order,

$$\alpha(\lambda) = \frac{\lambda}{4\pi} - \frac{\lambda^2}{8\pi^3} \ln \frac{1}{\lambda} + \cdots,$$

including the logarithm. There is also a (smaller) term of order $\lambda^2$ which, to this order, has a gauge dependent coefficient. In any case, infrared divergences make it impossible to compute the term of order $\lambda^2$ in perturbation theory.

Even though the result is not independent of the choice of gauge, it is interesting to study the large $\lambda$ behavior of the potential arising in the infinite sum of ladder diagrams. If $\lambda$ is large, the potential in (7) can be expanded in $x$ about $x = 0$. The wavefunction in the quadratic approximation is a Gaussian

$$\psi(x) \propto e^{-\sqrt{\lambda} x^2/4\pi},$$

whose ground state energy is $\Omega^2 = \lambda/\pi^2 + \cdots$. This approximation is consistent; the expectation value of corrections to the quadratic approximation for the ground state energy is of higher order in $1/\sqrt{\lambda}$. This yields the strong coupling behavior

$$\alpha(\lambda) = \frac{\sqrt{\lambda}}{\pi} - 1 + O(1/\sqrt{\lambda}),$$

which reproduces the same power of the coupling constant as in the the AdS/CFT computation in (4), but has the wrong coefficient. The source of the discrepancy in the coefficient is clear: The ladder diagrams do not contain all of the important contributions at strong coupling.

The main result of this Letter is the one-loop correction to the static potential (9). It is satisfying to see that this correction goes in the right direction to match the prediction of the AdS/CFT correspondence. (The sub-leading corrections on the supergravity side have been studied in [16].) We also found that the $\sqrt{\lambda}$ dependence
of the effective Coulomb charge predicted by the AdS/CFT correspondence arises in the resummed perturbation theory and that $1/\sqrt{\lambda}$ is a natural expansion parameter at strong coupling, just as in the supergravity approach. These results are in good agreement with the AdS/CFT correspondence.

The infrared divergences due to soft gauge and scalar bosons in the static potential appear to be stronger in $\mathcal{N}=4$ supersymmetric Yang-Mills theory than in QCD. These divergences make the Coulomb potential non-analytic at weak coupling and lead to the breakdown of perturbation theory beyond second order. This behavior is similar to what happens at finite temperature, but the infrared divergences are milder in the case of the static potential and have a different physical origin.

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