Observability and Controllability of Nonlinear Networks: The Role of Symmetry

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Observability and controllability are essential concepts to the design of predictive observer models and feedback controllers of networked systems. For example, noncontrollable mathematical models of real systems have subspaces that influence model behavior, but cannot be controlled by an input. Such subspaces can be difficult to determine in complex nonlinear networks. Since almost all of the present theory was developed for linear networks without symmetries, here we present a numerical and group representational framework, to quantify the observability and controllability of nonlinear networks with explicit symmetries that shows the connection between symmetries and nonlinear measures of observability and controllability. We numerically observe and theoretically predict that not all symmetries have the same effect on network observation and control. Our analysis shows that the presence of symmetry in a network may decrease observability and controllability, although networks containing only rotational symmetries remain controllable and observable. These results alter our view of the nature of observability and controllability in complex networks, change our understanding of structural controllability, and affect the design of mathematical models to observe and control such networks.

I. INTRODUCTION

An observer model of a natural system has many useful applications in science and engineering, including understanding and predicting weather or controlling dynamics from robotics to neuronal systems [1]. A fundamental question that arises when utilizing filters to estimate the future states of a system is how to choose a model and measurement function that faithfully captures the system dynamics and can predict future states [2, 3]. An observer is a model of a system or process that assimilates data from the natural system being modeled [4], and reconstructs unmeasured or inaccessible variables. In linear systems, the key concept to employ a well designed observer is observability, which quantifies whether there is sufficient information contained in the measurement to adequately reconstruct the full system dynamics [5, 6].

An important problem when studying networks is how best to observe and control the entire network when only limited observation and control input nodes are available. In classic work, Lin [7] described the topologies of graph directed linear networks that were structurally controllable. Incorporating Lin’s framework, Liu et al [8] described an efficient strategy to count the number of control points required for a complex network, which have an interesting dependence on time constant [9]. Structural observability is dual to structural controllability [10]. In [11], the requirements of structural observability incorporated explicit use of transitive components of directed graphs - fully connected subgraphs where paths lead from any node to any other node - to identify the minimal number of sites required to observe from a network.

All of these prior works depend critically on the dynamics being linear and generic, in the sense that network connections are essentially random. Joly [12] showed that transitive generic networks with nonlinear nodal dynamics are observable from any node. Nevertheless, symmetries are present in natural networks, as evident from their known structures [13] as well as the presence of synchrony. Recently, Golubitsky et al [14] proved the rigid phase conjecture - that the presence of synchrony in networks implies the presence of symmetries and vice versa. In particular, synchrony is an intrinsic component of brain dynamics in normal and pathological brain dynamics [14].

Our present work is motivated by the question: what role do the symmetries and network coupling strengths play when reconstructing or controlling network dynamics? The intuition here is straightforward: consider 3 linear systems with identical dynamics (diagonal terms of the system matrix $A$ in $\dot{x}(t) = Ax(t)$), if the coupling terms are identical (off-diagonal terms of $A$), it is easy to
show that the resulting observability of individual states becomes degenerate as the rows and columns of the system matrix become linearly dependent under elementary matrix operations. For example, consider the trivial case of a 3x3 system matrix of ones:

\[ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}. \]  

The system is degenerate in the sense that there is only one dynamic, as the rows and columns of A are not independent. This lack of independent rows and columns of the system matrix has direct implications for the controllability and observability of the system. For example, in this trivial system the difference between any two of the states is constrained to a constant \( x_1 - x_2 = c \), thus there is no input coupled to the third state \( x_3 \) that could control both \( x_1 \) and \( x_2 \) independently from each other.

In fact, for the more general case of linear time-varying networks, group representation theory [16] has been utilized to show that linear time-varying networks can be non-controllable or non-observable due to the presence of symmetry in the network [17]. Brought into context, in networks with symmetry Rubin & Meadows [17] defines a coordinate transform which decomposes the network into decoupled observable (controllable) and unobservable ( uncontrollable) subspaces, which can be determined by inspection like our previous trivial example. Recently, Pecora et al. [18] utilized this same method to show how separate subsets of complex networks could synchronize and desynchronize according to these same symmetry-defined subspaces. Interestingly, while [17] has been a rather obscure work, it is based on Wigner’s work in the 1930’s applying group representation theory to the mechanics of atomic spectra [19]. Thus, just as the structural symmetry of the Hamiltonian can be used to simplify the solution to the Schrödinger equation [20], the topology of the coupling in a network can have a profound impact on its observation and control.

In this article, we extend the exploration of observability and controllability to network motifs with explicit nonlinearities and symmetries. We further explore the effect of coupling strength within such networks, as well as spatial and temporal effects on observability and controllability. Lastly, we demonstrate the utility of the linear analysis of group representation theory as a tool with which to gain insights into the effects of symmetry in nonlinear networks.

II. BACKGROUND

From the theories of differential embeddings [21] and nonlinear reconstruction [22, 23] we can create a nonlinear measure of observability comprised of a measurement function and its higher Lie derivatives employing the differential embedding map [24]. The differential embedding map of an observer provides the information contained in a given measurement function and model, which can be quantified by an index [25, 26]. Computed from the Jacobian of the differential embedding map, the observability index is a matrix condition number which quantifies the perturbation sensitivity (closeness to singularity) of the mapping created by the measurement function used to observe the system. There is a dual theory for controllability, where the differential embedding map is constructed from the control input function and its higher Lie brackets with respect to the nonlinear model function [27, 28]. Singularities in the map cause information about the system to be lost and observability to decrease. Additionally, the presence of symmetries in the system’s differential equations makes observation difficult from variables around which the invariance of the symmetry is manifested [29, 30]. We extend this analysis to networks of ordinary differential equations and investigate the effects of symmetries on observability and controllability of such networks as a function of connection topology, measurement function, and connection strength.

A. Linear Observability and Controllability

In the early 1960s, Rudolph Kalman introduced the notions of state space decomposition, controllability and observability into the theory of linear systems [5]. From this work comes the classic concept of observability for a linear time-invariant (LTI) dynamic system, which defines a ‘yes’ or ‘no’ answer whether a state can be reconstructed from a measurement using a rank condition check.

A dynamic model for a linear (time-invariant) system can be represented by

\[ \begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t),
\end{align*} \]  

where \( x \in \mathbb{R}^n \) represents the state variable, \( u \in \mathbb{R}^m \) is the external input to the system and \( y \in \mathbb{R}^p \) is the output (measurement) function of the state variable. Typically there are less measurements than states, so \( p < n \). The intuition for observability comes from asking whether an initial condition can be determined from a finite period of measuring the system dynamics from one or more sensors. That is, given the system in (2), with \( x(t) = e^{At}x_0 \) and \( Bu = 0 \), determine the initial condition \( x_0 \) from measurement \( y(t), 0 \leq t \leq T \). To evaluate this locally,
we take the higher derivatives of \( y(t) \):

\[
y(t) = Cx(t) \\
\dot{y} = Cx(t) = CAx(t) \\
\ddot{y} = CAx(t) = CA^2x(t) \\
\vdots \\
y^{(\alpha-1)}(t) = CA^{\alpha-1}x(t).
\]

Factoring the \( x \) terms and putting \( y \) and its higher derivatives in matrix form, we have a mapping from outputs to states

\[
\begin{bmatrix}
y \\
\dot{y} \\
\ddot{y} \\
\vdots \\
y^{(\alpha-1)}
\end{bmatrix} =
\begin{bmatrix}
C \\
CA \\
CA^2 \\
\vdots \\
CA^{\alpha-1}
\end{bmatrix}
\begin{bmatrix}x\end{bmatrix},
\]

where the linear observability matrix \([31]\) is defined as

\[
O \equiv \begin{bmatrix}
C \\
CA \\
CA^2 \\
\vdots \\
CA^{\alpha-1}
\end{bmatrix}
\]

(5)

The finite limit of taking derivatives in \([3]\) comes from the Cayley-Hamilton theorem, which specifies that any square matrix \( A \) satisfies its own characteristic equation, which is the polynomial \( p(\lambda) = 0 \) where \( p(\lambda) = (\lambda I_n - A) \).

In other words, \( A^n \) is spanned by the lower powers of \( A \), from \( A^0 \) to \( A^{\alpha-1} \),

\[
y(t) = Ce^{At}x_0, \quad e^{At} \equiv \sum_{k=0}^{n-1} \alpha_k(t)A^k
\]

(6)

Thus, if the observability matrix spans \( n \) space (\( \text{rank}(O) = n \)), the initial condition \( x_0 \) can be determined, as the mapping \( x_0 = O^{-1}y(t) \) from output to states exists and is unique. More formally, the system \([2]\) is locally observable ( distinguishable at a point \( x_0 \)) if there exists a neighborhood of \( x_0 \) such that \( x_0 \neq x_1 \implies y(x_0) \neq y(x_1) \).

In a similar fashion, the linear controllability matrix is derived from asking whether an input \( u(t) \) can be found to take any initial condition \( x(0) = x_0 \) to arbitrary position \( x(T) = x_f \) in a finite period of time \( T \). For the sake of simplicity, we assume a single input \( u(t) \) and take the higher derivatives of \( x(T) = Ax(t) + Bu(t) \) up to the \((n-1)^{th}\) derivative of \( u(t) \) (again using the Cayley-Hamilton theorem):

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) \\
\ddot{x}(t) &= A^2x(t) + ABu(t) + B\dot{u}(t) \\
\vdots \\
x^{(n)}(t) &= A^nx(t) + A^{n-1}Bu(t) + A^{n-2}B\dot{u}(t) + \cdots + Bu^{(n-1)}(t)
\end{align*}
\]

(7)

which gives us a mapping from input to states

\[
\begin{bmatrix}
\dot{x}(t) \\
\ddot{x}(t) \\
\vdots \\
x^{(n-1)}(t) \\
x^{(n)}(t)
\end{bmatrix} =
\begin{bmatrix}
A \\
A^2 \\
\vdots \\
A^{n-1} \\
A^n
\end{bmatrix}
\begin{bmatrix}x(t)\end{bmatrix}
\]

(8)

where the linear controllability matrix is defined \([31]\) as

\[
Q \equiv \begin{bmatrix}B, AB, A^2B, \ldots, A^{n-1}B\end{bmatrix}.
\]

B. Differential Embeddings and Nonlinear Observability

From early work on the nonlinear extensions of observability in the 1970s \([27, 28]\), it was shown that the observability matrix for nonlinear systems could be expressed using the measurement function and its higher order Lie derivatives with respect to the nonlinear system equations. The core idea is to evaluate a mapping \( \phi \) from the measurements to the states \( \phi : \mathbb{R}^p \rightarrow \mathbb{R}^n \). In particular, Hermann and Krener \([28]\) showed that the space of the measurement function is embedded in \( \mathbb{R}^n \) when the mapping from measurement to states is everywhere differentiable and injective by the Whitney Embedding Theorem \([21, 22]\). An embedding is a map involving differential structure that does not collapse points or tangent directions \([23]\), thus a map \( \phi \) is an embedding when the determinant of the map Jacobian \( \text{Det}(\frac{\partial \phi}{\partial x} \bigg|_{x \in \mathbb{R}^n}) \) is non-vanishing and one-to-one (injective). In a recent series of papers \([24, 29, 32]\), Letellier et al. computed the nonlinear observability matrices for the well-known Lorenz and Rössler systems \([33, 34]\) and demonstrated that the order of the singularities present in the observability matrix (and thus the amount of intersection between the singularities and the phase space trajectories) was related to the decrease in observability. It is worth noting that the calculation of the observability matrix and locally evaluating the conditioning of the matrix over a state trajectory is a straightforward process and much more tractable than analytically determining the singularities (and thus their order) of the observability matrix of a system of arbitrary order. The former is limited only by computational capacity and the differentiability of the
Taking the Jacobian of the map φ

where as a vector example the first Lie derivative will take the function (2) and its higher Lie derivatives (10), the differ-

simplify the display of equations. As in the linear case, we evaluate locally by taking the higher Lie derivatives

is the order of the system

\[ \phi(x(t)) = \begin{bmatrix} f_1(x) \\ \vdots \\ f_n(x) \end{bmatrix} \]

With formal definitions of the measurement (output) function \( f(x) \) and its higher Lie derivatives \( \mathcal{L}^k_f(y(x)) \), the differential embedding map \( \phi \) is defined as the Lie derivatives \( \mathcal{L}^0_f(y(x)) \ldots \mathcal{L}^{n-1}_f(y(x)) \), where the superscripts represent the order of the Lie derivative from 0 to \( n-1 \), where \( n \) is the order of the system \( A_{NL}(x(t)) \).

Taking the Jacobian of the map \( \phi \) we arrive at the observability matrix

\[ O = \frac{\partial \phi}{\partial x} = \begin{bmatrix} \frac{\partial \mathcal{L}^0_f(y(x))}{\partial x_1} & \ldots & \frac{\partial \mathcal{L}^0_f(y(x))}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial \mathcal{L}^{n-1}_f(y(x))}{\partial x_1} & \ldots & \frac{\partial \mathcal{L}^{n-1}_f(y(x))}{\partial x_n} \end{bmatrix} \]

\(^1\) If \( Bu \neq 0 \) then as long as the input is known the mapping from output to states can be solved, and the determination of observability still relies on the conditioning of the matrix \( O \).
where \( \sigma_{\text{min}} \) and \( \sigma_{\text{max}} \) are the minimum and maximum singular values of \( O^T O \) respectively and \( \delta(x) = 1 \) indicates full observability while \( \delta(x) = 0 \) indicates no observability \[36\]. Similarly the controllability index is just \[16\] with the substitution of \( Q \) for \( O \).

III. OBSERVABILITY AND CONTROLLABILITY OF 3-NODE FITZHUGH-NAGUMO NETWORK MOTIFS

A. Fitzhugh-Nagumo System Dynamics

The Fitzhugh-Nagumo (FN) equations \[37, 38\], comprise a general representation of excitable neuronal membrane. The model is a 2-dimensional analogue of the well known Hodgkin-Huxley model \[39\] of an axonal excitable membrane. The nonlinear FN model can exhibit a variety of dynamical modes which include active transients, limit cycles, relaxation oscillations with multiple time scales, and chaos \[37, 40\]. A nonlinear connection function will be used to emulate properties of neuronal synapses.

The system dynamics at a node are given by the (local 2nd order) state space

\[
\begin{align*}
\dot{v}_i &= c(v_i - \frac{v_i^3}{3} - w_i + \sum_{j=1}^{n} f_{NL}(v_j, d_{ij}) + I) \\
\dot{w}_i &= v_i - bw_i + a,
\end{align*}
\]  

where \( i = 1, 2, 3 \) for the 3-node system, \( v_i \) represents membrane voltage of node \( i \), \( w_i \) is recovery, \( d_{ij} \) the intranodal distance from node \( j \) to \( i \), \( v_j \) the voltage of neighbor nodes with \( j = 1, 2, 3 \) and \( j \neq i \), input current \( I \), and the system parameters \( a = 0.7, b = 0.8, c = 10 \). As defined above in Eqs. \ref{eq:13} and \ref{eq:15}, the observability and controllability matrices are a function of the states which means a dependence on the particular trajectory taken in phase space. In the following analysis, we are interested in directed information flow between nodes as a function of various topological connection motifs, connection strengths and input forcing functions (which provide different trajectories through phase space). Each motif is representative of a unique combination of directed connections between the 3 nodes with and without latent symmetries. The nonlinear connection function commonly used in neuronal modeling \[41\] takes the form of the sigmoidal activation function of neighboring activity (a hyperbolic tangent) and an exponential decay with inter-nodal distance. We utilize various coupling strengths to determine the effects on the observability (controllability) of the network. Our coupling function takes the form

\[
f_{NL}(v, d) = \frac{k}{2} \tanh\left(\frac{v - h}{2m}\right) + 1 \ e^{-d}.
\]  

The sigmoid parameters \( k = 1, h = 0, m = 1/4 \), are set such that \( f_{NL}(v, d) \) has an output range \([0, 1]\) for the input interval \([-2, 2]\), which is the range of the typical FN voltage variable. To introduce heterogeneity for symmetry breaking a 10% variance noise term was added to each of the \( d_{ij} \) terms (there are 6 total possible coupling terms \( d_{12}, d_{13}, \ldots \) etc.).

In this configuration, inputs from neighboring nodes act in an excitatory-only manner, while the driving input current was a square wave \( I = 0.25[\sum_{n=-\infty}^{\infty} \cap(\omega t - nT) + 1] \) (where \( \cap \) is the rectangular function, \( \omega = 2\pi/5 \) and \( T = 16\pi^2 \)) applied to all three nodes to provide a limit cycle regime to the network; for the limit cycle regime generated in the original paper by Fitzhugh \[37\], the driving current input was constant \( I = -0.45 \) (with the system parameters mentioned above) which we will also explore. Chaotic dynamics were generated with a slightly different square wave input \[40\] \( I = 0.1225[\sum_{n=-\infty}^{\infty} \cap(\omega t - nT) + 1] \) (with \( \omega = 2\pi/1.23 \) and \( T = 2.7891 \)) also applied to all three nodes. These various driving input regimes allow a wider exploration of the phase space of the system as each driving input command a different trajectory, which will in turn influence the observability and controllability matrices.

![FIG. 1. The eight different 3-node network connection motifs studied.](image)

B. Network Motifs and Simulated Data

As we are interested in the effect of connection topology on observability and controllability, we study the simplest nontrivial network: a 3-node network. Such small network motifs are highly overrepresented in neuronal networks \[12, 43\]. For each network motif shown in Figure 1, we compute the observability (controllability) indices for various measurement nodes, connection strengths, and driving inputs (dynamic regimes). Measurements of \( v_i \) for each motif were from each one of the nodes \( i = 1, 2, 3 \). Simulated network data were used to compute the observability (controllability) index for two cases: 1) where the system parameters for all 3 nodes and connections were identical, and 2) where the nodes had a heterogeneous (10% variance) symmetry-breaking set of coupling parameters. To create simulated data, the full six-dimensional FN network equations were in-
tegrated from the same initial conditions with the same driving inputs for each node via a Runge-Kutta 4th order (RK4) method with time step $\Delta t = 0.10$ for 12000 time steps. Convergence of solutions was achieved when $\Delta t$ was decreased to 0.004. Data were then imported into Mathematica and inserted into symbolic observability and controllability matrices (computed for each node), which were then numerically computed to obtain the observability (controllability) indices for each coupling strength. The indices were then averaged over the integration paths starting from random initial conditions. These calculations were summarized in Figures 2 - 4 for motif 1 and 3, plots a and b in Figures 2 and 3). For motif 1, the data show that a system with full $S_3$ symmetry (due to the connection topology and identical nodal and coupling parameters) generates zero observability (controllability) over the entire range of coupling strengths (Figure 2a and 2c). Similarly, no observability (controllability) is seen from node 2 in motif 3 which has a reflection $S_2$ symmetry across the plane through node 2 (Figure 3b). Interestingly, the cyclic symmetry of motif 7 does not cause loss of observability (controllability) as shown in Figure 4; motif 7 has rotational $C_3$ symmetry and valance 1 connectivity (1 input, 1 output). In motifs 1 and 3 the effect of the symmetry is partially broken by introducing a variation in the coupling terms, and the results show non-zero observability (controllability) indices in the plots for such heterogeneous coupling (plots a and b in Figures 2 and 3) with a dependence on the coupling strength.

Of particular interest is the substantial loss of observability (controllability) as the coupling strengths increase to critical levels for systems containing latent structural symmetries in the presence of heterogeneity (motifs 1 and 3, plots a and b in Figures 2 and 3). That is, increasing the coupling strengths when recording (stimulating) from any node in motif 1 or node 2 in motif 3, degrades observability (controllability) as coupling strength increases. A study of the 3D phase plots of the FN voltage variable in motif 1 (as a function of coupling strength for chaotic dynamics) reveals a blowout bifurcation [44] at lower values of coupling strengths (Figure 7), which is remedied by the disambiguating effect of parameter variation. Additionally, as in the motifs with symmetry, the broken local symmetries lose controllability as coupling strength further increases evident in motifs 2 and 6 in Figure 6. In the cases where the indices are zero without symmetries (motifs 5, 6, and 8 in Figures 5 and 6), the motif must contain

IV. RESULTS

A. Motifs with Symmetry

Local output symmetries occur in motifs 2, and 6 when controlling from the first and second node respectively (green and blue traces in Figure 6), which is remedied by the disambiguating effect of parameter variation. Additionally, as in the motifs with symmetry, the broken local symmetries lose controllability as coupling strength further increases evident in motifs 2 and 6 in Figure 6. In the cases where the indices are zero without symmetries (motifs 5, 6, and 8 in Figures 5 and 6), the motif must contain
one or more structurally isolated nodes and hence are not structurally controllable or observable. From the view-point of observability this means that information from the isolated node(s) cannot reach the measured node as the two are not connected in that direction [10, 12]; for controllability, this means that the isolated node(s) is not reached by the controlled node due to the two not being connected in that direction [7]. This structural nodal isolation is exemplified in motif 8 (in Figures 3 and 6), where the network is only observable from node 1, and only controllable from node 3.

Additionally, the plots in Figures 5 and 6 show counter-intuitively that as coupling strength increases the observability (controllability) indices can increase to an optimal value, and then begin to decrease as coupling strength increases past this critical coupling value.
FIG. 5. Calculation of observability indices for each of the FN network motifs with no underlying group symmetries for a pulsed input limit cycle dynamical regime, as measured from each node (green □ =1, blue × =2, red • =3). The thick lines and symbols mark the mean values while the smaller symbols and dotted lines represent the ±1 standard deviation confidence intervals. Plots in the top row are computed with heterogeneous couplings while identical coupling strengths are in the bottom row. The calculations show the effect of network coupling strength on observability; motifs 5, 6 and 8 show no observability from node 3 in motif 5, and nodes 2 and 3 in motifs 6 and 8 due to structural isolation.

FIG. 6. Calculation of controllability indices for each of the FN network motifs with no underlying group symmetries for a limit cycle dynamical regime with constant input current $I = -0.45$, all other details are the same as in Figure 5. In particular, notice that local input-output symmetries cause zero controllability when controlling motif 2 from node 1 or motif 6 from node 2.
FIG. 7. The 3-dimensional phase space for $v$ and $w$, showing trajectories in motif 1 as measured from node 1 for a range of connection strengths (weak to strong heterogeneous coupling $K$, from left to right respectively). In the first row, blue triangles mark locations in phase space where observability is higher than the mean for the trajectory, while the second row contains a phase space trajectory for $w$ and red triangles mark the higher than average controllability. The broken symmetry of the heterogeneous network has trajectories that visit locations in the phase space that vary widely in observability and controllability with a log-normal distribution.
V. SYMMETRIC NETWORK OBSERVABILITY AND CONTROLLABILITY VIA GROUP REPRESENTATION THEORY

For linear time-varying systems, Rubin & Meadows [17] used the theory of group representations [16,19,20] to show how a (circuit) network containing group symmetries would be non-controllable or non-observable due to symmetries (termed NCS or NOS respectively). The analysis involves first determining the irreducible representations of the symmetry group of the system equations, then constructing an orthogonal basis (called a symmetry basis) from the irreducible representations which transforms the system matrix \( A \) into block diagonal form (also called modal form). Inspection of the fully transformed system from [4] will reveal if the NCS or NOS property is present via zeros in a critical location of decoupled block-diagonal decomposition \( (A, B, C) \) i.e. the form

\[
\begin{align*}
\frac{d}{dt} \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} &= \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u(t) \\
y(t) &= \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix},
\end{align*}
\]

where the transformed system [19] in partitioned form above is non-controllable and non-observable (not completely controllable or observable). This can be seen by inspection, as the zeros present in the partitioned measurement and control functions \( C \) and \( B \) leave the transformed system unable to measure or control the mode associated with \( Z_2 \) as neither \( u(t) \) or \( Z_1 \) is present in the equation for \( Z_2 \) and \( Z_2 \) does not appear in the output. In the next section we summarize the minimum background components of groups and representations (without proofs) in order to further gain insight into how symmetry effects the controllability and observability of our networks.

A. Symmetric Groups and Representations

A symmetry operation on a network is a permutation (in this case nodes) that results in exactly the same configuration as before the transformation was applied. The symmetric group \( S_n \) consists of all permutations on \( n \) symbols - called the order of the group \( g \). The shorthand method of denoting a permutation operation \( R \) of nodes in a network will be written \((123)\), where node 1 is replaced by node 2 and node 2 by node 3. This is called a cycle of the permutation \((123)\), and with it we can define all of the permutations of \( S_n \). Three of the network motifs studied here contain topological symmetries (Figures 2, 3 and 4): motif 1 has \( S_3 \) symmetry, motif 3 has \( S_2 \) symmetry and motif 7 contains \( C_3 \) symmetry and each of these groups comprise the following sets of permutation operations \( R \)

\[
R : S_3 = \{ E, \sigma_1, \sigma_2, \sigma_3, C_3, C_3^3 \} = \{ E = (1)(2)(3), \sigma_1 = (23), \sigma_2 = (13), \sigma_3 = (21) \},
\]

where \( E \) is the identity operation, \( \sigma_n \) is a reflection across the \( n \)th axis in Figure 5 and \( C_3 \) and \( C_3^3 \) are two cyclic rotations where \( C_n \) denotes a rotation of the system by \( 2\pi/n \) radians where the system remains invariant after rotation \((20)\). \( S_2 \) and \( C_3 \) symmetry in motifs 3 and 7 respectively are subgroups of \( S_3 \)

\[
S_2 = \{ E, \sigma_2 \}
\]

\[
C_3 = \{ E, C_3, C_3^2 \}
\]

The permutation operations \( R \) in these symmetric groups can also be represented by monomial matrices \( D(R) \):

\[
D(R) = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

\[
E \quad \sigma_1 \quad \sigma_2 \quad \sigma_3 \quad C_3 \quad C_3^2
\]

where \( D(R) \) in \( (22) \) is a 3-dimensional representation of \( S_3 \) group symmetry (for our 3 node motifs); a representation \( D(R) \) for \( S_2 \) and \( C_3 \) group symmetry are just the matrices above in \( (22) \) corresponding to the sets of group elements given in \( (21) \).

A group of matrices \( D(\cdot) \) is said to form a representation of a group \( S_n \) if a correspondence (denoted \( \sim \)) exists between the matrices and the group elements such that products correspond to products, i.e., if \( R_1 \sim D(R_1) \) and \( R_2 \sim D(R_2) \), then the composition \((R_1R_2) \sim \)

2 See [20] for a rigorous classification of various forms of symmetry.

3 A monomial matrix has only one non-zero entry per row and column. In this case permutation operations limit those values to either +1 or -1.
$D(R_1)D(R_2) = D(R_1R_2)$ (Definition 12 in [17]); this is known as a homomorphism of the group to be represented, and if the correspondence is one-to-one the representation is isomorphic and called a “faithful” representation of the group.

Theorem 2 from [17] establishes the connection between group theory and the linear network system equations ([2]), by demonstrating that the monomial representation $D(R)$ of symmetry operations $R$ is conjugate (commutes) with the network system matrix $A$ in ([2]):

$$D^{-1}(R)A(t)D(R) = A(t), \quad \forall R \in S_n \quad (23)$$

where $D(R)$ shows how the states of the system equations transform under the symmetry operation $R$, and form a reducible representation [16, 17] of the symmetric group $S_n$. A representation is said to be reducible if it can be transformed into a block diagonal form via a similarity transformation $\alpha$, and irreducible if it is already in diagonal form; a reducible representation $D(R)$ that has been reduced to block diagonal form $\hat{D}(R)$ will have $k$ non-zero submatrices along the diagonal that define the irreducible representations $D^{(p)}(R), p = 1 \ldots k$ of the group $S_n$ [17].

$$\alpha^{\dagger} D(R)\alpha = \hat{D}(R), \quad \forall R \in S_n$$

$$\hat{D}(R) = \begin{bmatrix} D^{(1)}_{l_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & D^{(k)}_{l_k} \end{bmatrix}, \quad (24)$$

where $\dagger$ represents the complex conjugate transpose of $\alpha$, $l_p$ is the dimension of $D^{(p)}(R)$ and the number of irreducible representations $k$ equals the number of classes the group elements $R$ are partitioned into. This can be found by computing the trace of each representation in $D(R)$, $\forall R$ - called the character of the representation - and collecting those that have the same trace into separate classes $\gamma_p, p = 1 \ldots k$, which define sets of conjugate elements [20]. The character of $D(R)$ is defined as

$$\chi(R) = \text{Tr}(D(R)), \quad \forall R \in S_n. \quad (25)$$

The key to forming irreducible representations in (24) is that the transform $\alpha$ needs to reduce each representation matrix $D(R)$ to diagonal form for every group element $R$ in $S_n$.

In (24) the dimension of each irreducible representation $l_p$ can be found from the fact that the irreducible representations of the group form an orthogonal basis in the $g$-dimensional space of the group, and since there can be no more than $g$ independent vectors in the orthogonal basis it can be shown [46] that

$$\sum_{p=1}^{k} l_p^2 = g, \quad (26)$$

where the sum is over the number of irreducible representations (or classes of conjugate group elements) $k$. Some of the irreducible representations $D^{(p)}(R)$ will appear in $\hat{D}(R)$ more than once while others may not appear at all; the character of the representation completely determines this and the number of times, $a_p$, that $D^{(p)}(R)$ appears in $\hat{D}(R)$ is defined in [20] as

$$a_p = \frac{1}{g} \sum_R \chi^{(p)}(R)^*\chi(R), \quad (27)$$

where $\chi^{(p)}(R)$ is the trace of $D^{(p)}(R)$, the asterisk denotes complex conjugate and $\chi(R)$ is the trace of $D(R)$.

B. Construction of the Similarity Transform $\alpha$

We examine motif 3 in Figure 3 which has $S_2$ symmetry. Determined from (23), there are 2 classes of group elements $\gamma_1 = \{E\}$ and $\gamma_2 = \{\sigma_2\}$, and reduction of $D(R)$ yields the two, 1-dimensional ($l_1 = l_2 = 1$ computed from (26)) irreducible representations $D^{(1)}(R)$ and $D^{(2)}(R)$ of $S_2$:

\[
\begin{array}{c|cc}
R & E & \sigma_2 \\
\hline
D^{(1)}(R) & 1 & 1 \\
D^{(2)}(R) & 1 & -1 \\
\end{array}
\quad (28)
\]

where each entry in $D^{(p)}$ corresponds to the elements of $D(R)$ above in equation (22), where $R = \{E, \sigma_2\}$ as in equation (21), and from equation (27), $D^{(1)}(R)$ appears two times while $D^{(2)}(R)$ appears once in $D(R)$.

A procedure for transforming the reducible representation $D(R)$ of a symmetry group $S_n$ to block diagonal form is presented in [17, 17]. A unitary transformation $\alpha$ is constructed from the normalized linearly independent columns of the $n \times n$ generating matrix $G_i^{(p)}$

$$G_i^{(p)} = \sum_R D^{(p)}(R)_{ii}^* D(R), \quad (29)$$

where $D^{(p)}(R)_{ii}$ is the $(i,i)$th diagonal entry of a $l_p$-dimensional irreducible representation $p$ (hence $i = 1 \ldots l_p$) of the symmetry group $S_n$ and the asterisk denotes complex conjugate. Each matrix $G_i^{(p)}$ will contribute $a_p$ linearly independent columns from (27) to form the coordinate transformation matrix $\alpha$. Using equations (28) and (29) and iterating through all $l_p$ rows of each of the $k$ irreducible representations in (24), we

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4 For purposes of clarity, we simplified the presentation of the computation of $\alpha$ for our motifs where there is only one set of network nodes that can permuted amongst themselves. For the more general case where the group operations $R$ are separated into subgroups corresponding to different sets of permutable network nodes (e.g. RLC networks, or different neuron types) see [17].
construct $\alpha$ for motif 3

$$G_1^{(1)} = \sum_{R \in S_2} D^{(1)}(R)^\dagger_1 D(R)$$

$$= 1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + 1 \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix},$$  

(30)

where each linearly independent column of $G$ is a column of $\alpha$. After normalizing we have

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \xrightarrow{\text{normalize}} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \alpha_{11} & \alpha_{21} \\ \alpha_{12} & \alpha_{22} \\ \alpha_{13} & \alpha_{23} \end{bmatrix},$$  

(31)

which defines the first and second columns of $\alpha$. Continuing, we have

$$G_1^{(2)} = \sum_{R \in S_2} D^{(2)}(R)^\dagger_{11} D(R)$$

$$= 1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} - 1 \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix},$$  

(32)

which yields the final column of $\alpha$ (after normalization)

$$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \xrightarrow{\text{normalize}} \begin{bmatrix} \frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} \alpha_{31} \\ \alpha_{32} \\ \alpha_{33} \end{bmatrix}. $$  

(33)

Now the coordinate transformation matrix $\alpha$ is

$$\alpha = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix}. $$  

(34)

Motif 3 in Figure 3 has connection matrix $A_3$

$$A_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}. $$  

(35)

To control from node 1, 2 and 3 respectively, the $B$ matrix takes the form

$$B_{1,2,3} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, $$  

(36)

and to observe from node 1, 2 and 3 respectively, the $C$ matrix takes the form

$$C_{1,2,3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. $$  

(37)

The block diagonalized system $(\hat{A}_3, \hat{B}, \hat{C})$ is formed with the substitution $Z = \alpha^T x$, and $(A_3, B, C)$ in (35) to (37) becomes

$$\hat{A}_3 : \alpha^T A_3 \alpha = \begin{bmatrix} 0 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, $$  

(38)

$$\hat{B} : \alpha^T B_{1,2,3} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, $$  

$$\hat{C} : C_{1,2,3} \alpha = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{bmatrix}. $$  

(39)

By inspection of the transformed system (38) it becomes clear that motif 3 is non-controllable and non-observable from node 2 due to symmetry alone (NCS and NOS), i.e. the transformed system in modal coordinates

$$\frac{d}{dt} \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix} = \begin{bmatrix} 0 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u(t) $$  

(40)

is NCS and NOS as the mode associated with $Z_3$ cannot be reached by the input $B_2$ nor can its measurement be inferred from the output $C_3$ as in (19).

The procedure to reduce motif 1 is accomplished in similar fashion and the connection matrix $A_1$ and its reduced form $A_1$ is:

$$A_1 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \hat{A}_1 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} $$  

(41)

while the transformed $B$ and $C$ matrices in (36) and (37) are:

$$\hat{B}_{1,2,3} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} \sqrt{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \hat{C}_{123} = \hat{B}_{1,2,3}^T $$  

(42)

At first glance it appears that motif 1 is NCS and NOS for measurement and control from node 1 only, and fully controllable and observable from node 2 and 3, however there is a subtle nuance to the controllability and observability of the diagonal form used in (17) and consolidated in (19) to show non-controllability and non-observability by inspection.

It is well known that every non-singular $n \times n$ matrix has $n$ eigenvalues $\lambda_n$ and $n$ linearly independent eigenvectors, and that a matrix with repeated eigenvalues of algebraic multiplicity $m_i$ will have a degeneracy $1 \leq q_i \leq m_i$ associated with the number of linearly independent eigenvectors for repeated eigenvalue $\lambda_i$. This

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5 Full computation of $\alpha$ is detailed in the appendix.
degeneracy \( q_i \) is also called the geometric multiplicity of \( \lambda_i \), and is equal to the dimension of the null space of \( A - I \lambda_i \). When utilizing similarity transforms to reduce a matrix to diagonal (modal) form this degeneracy in the eigenvectors (brought about by repeated eigenvalues) results in a transformed matrix that is almost diagonal, called the Jordan form matrix. The Jordan form is comprised of submatrices of dimension \( m_i \) - called Jordan blocks - that have ones on the super-diagonal of each Jordan block \( J_i \) associated with the generalized eigenvectors of a repeated eigenvalue \( \lambda_i \). The diagonal form in (19) is a special case of Jordan form where the matrices on the diagonal are Jordan blocks of dimension one. This is known as the fully degenerate case with \( q_i = m_i \), and the Jordan form will have \( m_i \) separate 1 x 1 Jordan blocks associated with each eigenvalue \( \lambda_i \).

The observability and controllability of systems in Jordan form hinges on where the zeros appear in the partitioned \( C_i \) and \( B_i \) matrices, where subscript \( i \) indicates a partition associated with a particular Jordan block \( J_i \).

Given in [48, 49] the conditions for controllability and observability of a system in Jordan form are:

1. The first columns of \( C_i \) or the last rows of \( B_i \) must form a linearly independent set of vectors \( \{c_{i1} \ldots c_{i,q_i}\} \) or \( \{b_{i1} \ldots b_{i,e}\} \) (subscript \( e \) indicates the last row) corresponding to the \( q_i \) Jordan blocks \( J_{\lambda_i} \ldots J_{\lambda_i} \) for repeated eigenvalue \( \lambda_i \).
2. \( c_{ip} \neq 0 \) or \( b_{pe} \neq 0 \) when there is only one Jordan block \( J_{\lambda_i} \) associated with eigenvalue \( \lambda_i \).
3. For single output and single input systems, the partitions of \( C_i \) and \( B_i \) are scalars - which are never linearly independent - thus each repeated eigenvalue must only have one Jordan block \( J_{\lambda_i} \) associated with it for observability or controllability respectively.

From these criteria, we can now see that the transformed system for motif 1 in [40] contains three 1 x 1 Jordan blocks, two of which are associated with the repeated eigenvalue \( \lambda_2 = -1 \), which violates condition 3); thus we conclude it is NCS and NOS.

C. Motif 7 and Networks Containing Only Rotation Groups

In [17], it was shown how the \( r \)th component of \( \alpha \) vanishes according to the matrices \( D^{(p)}(R^r) \), where \( R^r \) represents a subgroup of the group operations \( (R) \) that transform the \( r \)th state variable into itself. Subsequently, two theorems were proven that make use of this fact to simplify the analysis of networks that have a single input or output coupled only to the \( r \)th state variable, which is precisely parallel to our analysis in section [15]. A paraphrasing of Theorem 6 and 12 from [17] for controllability and observability states that such a single input or output network is NCS or NOS if and only if there is an irreducible representation \( D^{(p)}(R) \) that appears in \( D(R) \) and

\[
\sum_{R} s_i^{(p)} D^{(p)}(R^r) s_i^{(p)} = 0
\]

for some value of \( i \), where \( s_i^{(p)} \) is +1 or −1 as \( R^r \) transforms state variable \( x_r \) into itself with a plus or minus sign. For this theorem to hold, the equality in (42) must be checked for all possible \( p \) for \( D^{(p)}(R) \) that appear in \( D(R) \) via (27).

Applying (42) to motif 7, the irreducible representations for \( C_3 \) symmetry are:

| \( R \) | \( E \) | \( C_3 \) | \( C_3^2 \) |
|-------|------|-------|-------|
| \( D^{(1)}(R) \) | 1 | 1 | 1 |
| \( D^{(2)}(R) \) | 1 | \( \omega \) | \( \omega^2 \) |
| \( D^{(3)}(R) \) | 1 | \( \omega^2 \) | \( \omega \) |

where \( \omega = e^{\frac{2\pi i}{3}} \). From the subset (21) of (22) we find that the only operation \( R^r \) that leaves either node 1, 2 or 3 (state variables \( x_1, x_2, \) or \( x_3 \)) invariant is just the identity operation \( E \), and it is straightforward to see that (42) \( \neq 0 \) for all choices of \( p, i \) and \( r \) since there is only one group operation that leaves the \( r \)th state variable invariant, \( R^r = E \), for \( r = 1, 2, 3 \). Thus, motif 7 cannot be NCS or NOS and must be controllable and observable from any node. Corollary 1 to Theorem 6 from [17] contains and expands this result directly to any network with only rotational symmetry (i.e. \( C_n \) groups), with the caveat that a network with a state variable that is invariant under all the group operations (motif 7 doesn’t have such a state variable) will be NCS and NOS if the input and output are coupled to that variable.

These representational group theoretic results explain our nonlinear results in section [15] and clearly demonstrate that different types of symmetry have different effects on the controllability and observability of the networks containing them. While we explicitly assume system matrices with zeros on the diagonal (for simplicity of the calculations) these results hold with generic entries on the diagonal as long as those entries are chosen to preserve the symmetry (e.g. the system matrix \( A \) for motif 1 and 7 has \( a_{11} = a_{22} = a_{33} \) and motif 3 has \( a_{11} = a_{33}, \) not shown). Linearization of the system equations in [17] would result in a system matrix \( A \) with a non-zero diagonal [9], and is typically done in the analysis of nonlinear networks [18] when utilizing such linear analysis techniques. Our computational results demonstrate the utility of this approach in providing insight into the controllability and observability of complex nonlinear networks that have not been linearized.

\[ \text{in our motifs } D(R) \text{ is a permutation representation, thus } s^{(p)}_i = \pm 1. \]
D. Application to Structurally Controllability (Observability)

It is interesting to note that the demonstration of our results above and those in [17] complement and expand Lin’s seminal theorems on structural controllability [7]. Essentially, a network with system matrix $A$ and input function $B$ (the pair $(A, B)$) are assumed to have two types of entries, non-zero generic entries, and fixed entries which are zero. The position of the zero entries leads to the notion of the structure of the system, where different systems with zeros in the same locations are considered structurally equivalent. With this definition of structure, we arrive at the definition for structurally controllability which states that a pair $(A', B')$ is structurally controllable if and only if there exists a controllable pair $(A'', B'')$ with the same structure as $(A', B')$. The major assumption of this work is that a system deemed to be structurally controllable could indeed be uncontrollable due to the specific entries in $A$ and $B$, which for a practical application are assumed to be uncertain estimates of the system parameters and thus subject to modification. While Lin’s theorems did not explicitly cover symmetry, any network pair $(A, B$ containing symmetry implies constraints on the non-zero entries in $(A, B)$, which is necessary to guarantee that symmetry is present. Thus considering only [7], a network with symmetry could be structurally controllable (observable [10]) as long as the graph of the system contains no dilations or isolated nodes, but NCS (NOS) due to the symmetry. These two theorems together paint a more complete picture of controllability (observability) than either alone as shown in section [IV] and [V] where both are used in concert to explain and understand why certain network motifs were not controllable or observable from particular nodes. Structural controllability (observability) is a more general result, as it does not depend on the explicit non-zero entries of the system pair $(A, B)$ (necessary, but not sufficient), while a network that has the NCS (NOS) property is due to specific sets of the non-zero entries in $(A, B)$ that define the symmetry contained by the system.

Additionally, [7] defined two structures called a “stem” (our motif 8 controlled from node 3) and a “bud” (our motif 7 controlled from any node) which are always structurally controllable. While both are easily shown to be structurally controllable [7], including Theorem 6 and its Corollary 1 from [17] we can take this a step further and declare that any “bud” network (of arbitrary size) containing only rotations is not only structurally controllable, but also fully controllable (or never NCS). The dual of these structures for observability is also defined in [10], and Theorem 12 and its Corollary 1 from [17] completes the statement in a similar fashion for observability. Since networks containing only rotation groups or “buds” in Lin’s terminology are always controllable, we see that in some cases, symmetries alone will not destroy the controllability of structurally controllable networks.

VI. DISCUSSION

Despite the growing importance of exploring observability and controllability in complex graph directed networks, there has been little exploration of nonlinear networks with explicit symmetries. We here report, to our knowledge, the first exploration of symmetries in nonlinear networks, and show that observability and controllability are a function of the specific type of symmetry, the spatial location of nodes sampled or controlled, the strength of the coupling, and the time evolution of the system.

In networks with structural symmetries, group representation theory provides deep insights into how the specific set of symmetry operations possessed by a network will influence its observability and controllability, and can aid in controller or observer design by obtaining a modal decomposition of the network equations into decoupled controllable and uncontrollable (observable and unobservable) subspaces. This knowledge will permit the intelligent placement of the minimum number of sensors and actuators that render a system containing symmetry fully controllable and observable. Additionally, breaking symmetry through randomly altering the coupling strengths established substantial observability or controllability that was absent in the fully symmetric case. In cases where increasing the overall level of coupling strength decreased the observability (controllability), such strong coupling eventually pushed the system towards or through a reverse Hopf bifurcation from limit cycle to a stable equilibrium point, where the lack of dynamic movement of the system then severely decreased the observability (controllability). Intuitively this results from the Lie derivatives (brackets) becoming small as the rate of change of the system trajectories goes to zero. The sensitivity of observability and controllability to the trajectories taken through phase space implies that the choice of control input to a system has to be selected carefully as a poor choice could drive the system into a region that has little to no controllability or observability, thereby thwarting further control effort and/or causing observation of the full system to be lost or limited. Furthermore, when using an observer model for observation or control the regions of local high observability could be utilized to optimize the coupling of the model to a real system by only estimating the full system state when the system transverses observable regions of phase space.

Observation (control) in motifs 2, 3, 4, 5 and 6 suggests a relationship between the degree of connections into and out of a node and its effective observability (controllability). In general, the more direct connections into an observed node, the higher the observability from that node, and the duality suggests that the more direct num-

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7 Defined in the appendix.
ber of outgoing connections from a controlled node leads to higher controllability than from other less connected nodes. The high degree ‘hub’ nodes were not the most effective driver nodes in complex networks using linear theory \cite{5}, and extending nonlinear results to more complex networks with symmetries is a challenge for future work, which may benefit from linear analysis of the connection topology utilizing group representation theory.

When observing kinematics and dynamics of rigid body mechanics obeying Newton’s laws with SE(3) group symmetry, such symmetries must be preserved in constructing an observer (controller) \cite{51}. In the observation of graph directed networks containing transitive networks, one can observe from any point equivalently within such transitive components \cite{11}. In the control of graph directed networks, the minimum number of control points were related to the maximal matching nodes \cite{51}. In \cite{51}, contraction theory was used to determine symmetric synchronous subspaces - these spaces act as detectors of observability \footnote{ACKNOWLEDGMENTS}. In fact, the proof of observability is that initial conditions and trajectories do not contract \cite{12}.

Furthermore, it is clear that the groupoid input equivalence classes (such as our motifs 6 and 7, see figure 21 in \cite{52}) are not equivalently observable or controllable - note that only 1 node can serve as an observer node in motif 6 regardless of coupling strength (our Figure 5). Indeed, whether virtual networks \cite{51} with particular groupoid equivalent symmetries serve as detectors of observability and controllability remains unresolved at this time.

Our deep knowledge of symmetries and observers in classical mechanics \cite{56} do not readily translate to graph directed networks. Further development of a theory of observability and controllability for nonlinear networks with symmetries is a vital open problem for future work.

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\begin{thebibliography}{99}
\bibitem{1} S. J. Schiff, \textit{Nerual Control Engineering} (MIT Press, Cambridge, 2012).
\bibitem{2} H. Voss, J. Timmer, and J. Kurths, International Journal of Bifurcation and Chaos \textbf{14}, 1905 (2004).
\bibitem{3} T. D. Sauer and S. J. Schiff, \textit{Phys. Rev. E} \textbf{79}, 051909 (2009).
\bibitem{4} E. Kalnay, \textit{Atmospheric Modeling, Data Assimilation and Predictability} (University Press, Cambridge, 2003).
\bibitem{5} R. Kalman, SIAM Journal on Control \textbf{1}, 152 (1963).
\bibitem{6} D. G. Luenberger, IEEE Transactions on Automatic Control \textbf{AC-16}, 596 (1971).
\bibitem{7} C.-T. Lin, \textit{Automatic Control, IEEE Transactions on} \textbf{19}, 201 (1974).
\bibitem{8} Y. Liu, J. Slotine, and A. Barabási, \textit{Nature} , 1 (2011)
\bibitem{9} N. J. Cowan, E. J. Chastain, D. a. Vilhena, J. S. Freudenberg, and C. T. Bergstrom, \textit{PloS one} \textbf{7}, 1 (2012).
\bibitem{10} C. Rech and R. Perret, \textit{International Journal of Systems Science} \textbf{21}, 1881 (1990).
\bibitem{11} Y. Liu, J. Slotine, and A. Barabási, \textit{Proceedings of the National Academy of Sciences} (2012).
\bibitem{12} R. Joly, \textit{Nonlinearity} \textbf{25}, 657 (2012).
\bibitem{13} H. Weyl, \textit{Symmetry} (Princeton University Press, New Jersey, 1952).
\bibitem{14} M. Golubitsky, D. Romano, and Y. Wang, \textit{Nonlinearity} \textbf{25}, 1045 (2012).
\bibitem{15} P. J. Uhlhaas and W. Singer, \textit{Neuron} \textbf{52}, 155 (2006).
\bibitem{16} W. Burnside, \textit{Theory of Groups of Finite Order} (Dover Publications Inc., New York, 1955).
\bibitem{17} H. Rubin and H. Meadows, \textit{Bell System Technical Journal} (1972).
\bibitem{18} L. M. Pecora, F. Sorrentino, A. M. Hagerstrom, T. E. Murphy, and R. Roy, \textit{Nature communications} \textbf{5}, 4079 (2014).
\bibitem{19} E. P. Wigner, \textit{Group Theory And Its Application To The Quantum Mechanics Of Atomic Spectra} (Academic Press Inc., New York, 1959) pp. 58–124.
\bibitem{20} M. Tinkham, \textit{Group Theory And Quantum Mechanics} (McGraw-Hill Inc., San Francisco, 1964) pp. 50–61.
\bibitem{21} H. Whitney, \textit{The Annals of Mathematics} \textbf{37}, 645 (1936).
\bibitem{22} F. Takens, \textit{Lecture Notes in Mathematics} \textbf{898}, 366 (1981).
\bibitem{23} T. Sauer, J. A. Yorke., and M. Casdagli, \textit{Journal of Statistical Physics} \textbf{65}, 579 (1991).
\bibitem{24} C. Letellier, L. Aguirre, and J. Maquet, \textit{Physical Review E} \textbf{71}, 1 (2005).
\bibitem{25} B. Friedland, \textit{Journal of Dynamic Systems, Measurement, and Control} \textbf{97}, 444 (1975).
\bibitem{26} J. Gibson, J. Doyne Farmer, M. Casdagli, and S. Eubank, \textit{Physica D: Nonlinear Phenomena} \textbf{57}, 1 (1992).
\bibitem{27} G. Haynes and H. Hermes, \textit{SIAM Journal on Control} \textbf{8}, 450 (1970).
\bibitem{28} R. Hermann and A. Krener, \textit{Automatic Control, IEEE Transactions on} \textbf{22}, 728 (1977).
\bibitem{29} C. Letellier and L. a. Aguirre, \textit{Chaos} \textbf{12}, 549 (2002).
\bibitem{30} L. Pecora and T. Carroll, \textit{Physical review letters} \textbf{64}, 821 (1990).
\bibitem{31} T. Kailath, \textit{Linear Systems} (Prentice-Hall, Upper Saddle River, 1980).
\bibitem{32} C. Letellier, J. Maquet, L. S. Sceller, G. Gouesbet, and L. Aguirre, \textit{Journal of Physics A: Mathematical and General} \textbf{31}, 7913 (1998).
\bibitem{33} E. Lorenz, \textit{Journal of the atmospheric sciences} \textbf{20}, 130 (1963).
\bibitem{34} O. Rössler, \textit{Physics Letters A} \textbf{57}, 397 (1976).
\bibitem{35} G. Strang, \textit{Linear Algebra And Its Applications 4ed} (Brooks Cole, St. Paul, 2005).
\bibitem{36} L. Aguirre, \textit{IEEE Transactions on Education} \textbf{38}, 33 (1995).
\bibitem{37} R. FitzHugh, \textit{Biophysical Journal} \textbf{1}, 445 (1961).
\bibitem{38} J. Nagumo, S. Arimoto, and S. Yoshizawa, \textit{Proceedings

NIH grant 1R01EB014641.
of the IRE 50, 2061 (1962).

[39] A. L. Hodgkin and A. F. Huxley, J. Physiol 117, 500 (1952).
[40] S. Doi and S. Sato, Mathematical Biosciences 250, 229 (2015).
[41] C. Koch and I. Segev, Methods in Neuronal Modeling: From Ions to Networks, 2nd ed. (MIT Press, Cambridge, 2003).
[42] R. Milo, S. Shen-Orr, S. Itzkovitz, N. Kashtan, D. Chklovskii, and U. Alon, Science 298, 824 (2002).
[43] S. Song, P. J. Sjöström, M. Reigl, S. Nelson, and D. B. Chklovskii, PLoS biology 3, 0507 (2005).
[44] E. Ott and J. Sommerer, Physics Letters A 188, 39 (1994).
[45] S. Schiff, P. So, T. Chang, R. Burke, and T. Sauer, Physical review. E, Statistical physics, plasmas, fluids, and related interdisciplinary topics 54, 6708 (1996).
[46] M. Hamermesh, Group Theory (Addison-Wesley Publishing Company Inc., Massachusetts, 1962) pp. 1–127.
[47] D. Kerns, Journal of Research of the National Bureau of Standards 46, 267 (1951).
[48] W. L. Brogan, Modern Control Theory (Prentice-Hall Inc., New Jersey, 1974) pp. 321–326.
[49] J. S. Bay, Fundamentals Of Linear State Space Systems (McGraw-Hill Inc., San Francisco, 1999) pp. 321–326.
[50] S. Bonnabel, P. Martin, and P. Rouche, IEEE Transactions on Automatic Control 53, 2514 (2008).
[51] G. Russo and J.-J. E. Slotine, Physical Review E 84, 041029 (2011).
[52] M. Golubitsky and I. Stewart, Bulletin of the American Mathematical Society 43, 365 (2006).
[53] J. Aitchison and J. Brown, The Lognormal Distribution (Cambridge University Press, London, 1957) pp. 94–97.

Appendix A: Supplemental Information

1. Construction of Differential Embedding Map and Lie Brackets

As an example case we begin constructing the observability matrix for motif 1 (shown in Figure 2), where the Fitzhugh-Nagumo (FN) network equations form the nonlinear vector field \( f \):

\[
\begin{align*}
  f_1 &= c(v_1 - \frac{v_1^3}{3} - w_1 + \sum_{j=2,3} f_{NL}(v_j, d_{1j})) \\
  f_2 &= v_1 - bw_1 + a \\
  f_3 &= c(v_2 - \frac{v_2^3}{3} - w_2 + \sum_{j=1,3} f_{NL}(v_j, d_{2j})) \\
  f_4 &= v_2 - bw_2 + a \\
  f_5 &= c(v_3 - \frac{v_3^3}{3} - w_3 + \sum_{j=1,2} f_{NL}(v_j, d_{3j})) \\
  f_6 &= v_3 - bw_3 + a
\end{align*}
\]

and the measurement function for node 1 in motif 1 is \( y = Cx(t) = [1, 0, 0, 0, 0]x(t) = v_1 \). We construct the differential embedding map by taking the Lie derivatives \([\partial f / \partial x]_{\phi} \) from \( L_j^0(y) \) to \( L_j^0(y) \) as:

\[
\phi_i = v_i
\]

\[
\phi_2 = \frac{\partial y}{\partial v_1} \cdot f_1 = f_1
\]

\[
\phi_3 = \frac{\partial y}{\partial v_1} f_1 + \frac{\partial y}{\partial v_2} f_2 + \frac{\partial y}{\partial v_3} f_3 + \ldots + \frac{\partial y}{\partial v_6} f_6
\]

(\[A2\])

where \( \frac{\partial y}{\partial x_j} \) is the partial derivative of the \( i \)th row of the embedding map \( \phi \), with respect to the \( j \)th state variable. We obtain the observability matrix by taking the Jacobian of \( [A2] \). In this FN network the observability matrix is dependent on the state variables and is thus a function of the location in phase space as the system evolves in time. Leteller et al. \[32\] used averages of the observability index over the state trajectories in phase space as a qualitative measure of observability. We adopt this convention when computing observability of various network motifs. The indices are computed for each time point in the trajectory, and then the average is taken over all of the trajectories.

Constructing the nonlinear controllability matrix for motif 1 from node 1 begins with the control input function \( g = Bu(t) = [1, 0, 0, 0, 0, 0]^T \) and its Lie Bracket with respect to the nonlinear vector field \( f \) in \( [A1] \). We exclude the internal driving square wave function here since it is connected to all three nodes, would provide no contribution in the Lie bracket mapping, and we are interested in the mapping from the control input \( g \) to the states in order to determine if the system can be controlled,

\[
[f, g] = \frac{\partial g}{\partial x} f - \frac{\partial f}{\partial x} g = \begin{bmatrix}
\frac{\partial f_1}{\partial v_1} g_1 & \cdots & \frac{\partial f_1}{\partial v_6} g_6 \\
\frac{\partial f_2}{\partial v_1} g_1 & \cdots & \frac{\partial f_2}{\partial v_6} g_6 \\
\frac{\partial f_3}{\partial v_1} g_1 & \cdots & \frac{\partial f_3}{\partial v_6} g_6 \\
\frac{\partial f_4}{\partial v_1} g_1 & \cdots & \frac{\partial f_4}{\partial v_6} g_6 \\
\frac{\partial f_5}{\partial v_1} g_1 & \cdots & \frac{\partial f_5}{\partial v_6} g_6 \\
\frac{\partial f_6}{\partial v_1} g_1 & \cdots & \frac{\partial f_6}{\partial v_6} g_6
\end{bmatrix}
\]

(\[A3\])

where \( \frac{\partial g}{\partial x} = 0 \) since \( g \) is the same at each node, \( \frac{\partial f}{\partial x} \) is the partial derivative of the \( i \)th row of the nonlinear vector field \( f(x) \) with respect to the \( j \)th state variable, and \( g_i \) is the \( i \)th component of the input vector \( g \). We construct the controllability matrix from the definitions in equations \([14\) and \(15\)], as the control input function \( g \) and its higher Lie Brackets from \( (ad_{f_1} g) \) to \( (ad_{f_1} g) \) with respect to the nonlinear vector field system equations,

\[
Q = [g, [f, g], [f, [f, g]], (ad_{f_1} g), (ad_{f_1} g), (ad_{f_1} g), (ad_{f_1} g)]
\]

(\[A4\])

2. Observability and Controllability Index Distribution

Log-scaled histograms (Figure 9) of the index distributions reveal that the local observability (controllability)
along the trajectories in phase space are close to a log-normal distribution. After removing zeros from the data, these log-normal distribution fits were computed and verified with the χ² test metric for all of the observability and controllability computation cases that contained an adequate number of data points to accurately compute the fit (over 90% of the data). The χ² test for goodness of fit confirmed that the data come from a log-normal distribution with 95% confidence. This type of zeros-censored log-normal distribution is known as a delta distribution with 95% confidence. This type of zeros-censored log-normal distribution fits were computed and verified from the 2-dimensional irreducible representations (computed from (26)) and their contributions to χ² were determined to account for the proportion of data points that are zero, as follows

\[
\delta = \frac{\#\{i : x_i = 0\}}{n},
\]

\[
\kappa = (1 - \delta) e^{\mu + 0.5 \sigma^2},
\]

\[
\rho^2 = (1 - \delta) e^{2\mu + 2\sigma^2}(e^{\sigma^2} - (1 - \delta)),
\]

where \(\mu\) and \(\sigma\) are the mean and variance associated with the lognormal distribution computed from the non-zero data. We use these equations to compute the statistics in the plots in the results section (Figures 2 to 7).

### 3. Group Representation Analysis of Symmetries

in Motif 1

We examine motif 1 in Figure 8 which has \(S_3\) symmetry. Determined from (25), there are 3 classes of group elements \(C_1 = \{E\}, C_2 = \{\sigma_1, \sigma_2, \sigma_3\}\) and \(C_3 = \{C_3, C_3^2\}\). Reduction of \(D(R)\) yields the two, 1-dimensional and one 2-dimensional (\(l_1 = l_2 = 1, l_3 = 2\)) irreducible representations (computed from (26)) \(D(1)(R), D(2)(R)\) and \(D(3)(R)\) of \(S_3\), which are found in Table 1 and from (27) appear 1, 0 and 2 times in \(D(R)\) respectively. Forming the generating matrix in equation (29) we construct \(\alpha\) for motif 1 as follows

\[
G^{(1)}_1 = \sum_{R \in S_3} D^{(1)}(R)_{11} D(R) I
\]

\[
= 1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + 1 \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} + \cdots
\]

\[
= \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix},
\]

where each linearly independent row of \(G\) is a column of \(\alpha\), and thus

\[
\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \xrightarrow{\text{normalize}} \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} = \begin{bmatrix} \alpha_{11} \\ \alpha_{12} \\ \alpha_{13} \end{bmatrix}
\]

defines the first column of \(\alpha\). We know from (27) that \(D^{(2)}(R)\) appears zero times in \(D(R)\) and thus yields no contribution to \(\alpha\). Continuing, we have the last two computations from the 2-dimensional irreducible representation \(D^{(3)}(R)\) (one for each row)

\[
G^{(3)}_1 = \sum_{R \in S_3} D^{(3)}(R)_{11} D(R) I
\]

\[
= 1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} + \cdots
\]

\[
= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},
\]

which after normalization yields

\[
\begin{bmatrix} 0 \\ \frac{3}{\sqrt{2}} \\ -\frac{3}{\sqrt{2}} \end{bmatrix} \xrightarrow{\text{normalize}} \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \alpha_{21} \\ \alpha_{22} \\ \alpha_{23} \end{bmatrix}
\]

and

\[
G^{(3)}_2 = \sum_{R \in S_3} D^{(3)}(R)_{22} D(R) I
\]

\[
= 1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} + \cdots
\]

\[
= \begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}
\]
yields the last column of $\alpha$ (after normalization)

$$
\begin{bmatrix}
2 \\
-1 \\
-1
\end{bmatrix}
\xrightarrow{\text{normalize}}
\begin{bmatrix}
\frac{2}{\sqrt{6}} \\
\frac{-1}{\sqrt{6}} \\
\frac{-1}{\sqrt{6}}
\end{bmatrix}
= \begin{bmatrix}
\alpha_{31} \\
\alpha_{32} \\
\alpha_{33}
\end{bmatrix}.
$$

(A11)

Finally, the coordinate transformation matrix $\alpha$ is

$$
\alpha =
\begin{bmatrix}
\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & 0 \\
\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\
\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}}
\end{bmatrix}.
$$

(A12)

and the computation is concluded in section [V B]

4. Dilations of the graph of (A,B)

In [7], the graph $G$ of the pair $(A, B)$ is defined as a graph of $n + 1$ nodes $e_1, e_2, \ldots, e_{n+1}$, where $n$ is the dimension of $A$, and $e_{n+1}$ is called the "origin" (the input). The vertex set $S = \{e_1, e_2, \ldots, e_n\}$ is defined as the set of all nodes in $G$ excluding the origin ($e_{n+1}$). A dilation is present in $G$ if and only if $|T(S)| < |S|$, where $T(S)$ is defined as the set of all nodes that have a directed edge pointing to a node in the set $S$. 
| R   | E | $\sigma_1$ | $\sigma_2$ | $\sigma_3$ | $C_3$ | $C_3^2$ |
|-----|---|------------|------------|------------|-------|---------|
| $D^{(1)}(R)$ | 1 | 1 | 1 | 1 | 1 | 1 |
| $D^{(2)}(R)$ | 1 | -1 | -1 | -1 | 1 | 1 |
| $D^{(3)}(R)$ | \[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix},
\begin{bmatrix}
-1 & 0 \\
0 & 1
\end{bmatrix},
\begin{bmatrix}
\frac{1}{2} & -\frac{\sqrt{3}}{2} \\
\frac{1}{2} & \frac{\sqrt{3}}{2}
\end{bmatrix},
\begin{bmatrix}
\frac{1}{2} & \frac{\sqrt{3}}{2} \\
\frac{1}{2} & -\frac{\sqrt{3}}{2} \n\end{bmatrix},
\begin{bmatrix}
\frac{1}{2} & -\frac{\sqrt{3}}{2} \\
\frac{1}{2} & \frac{\sqrt{3}}{2}
\end{bmatrix},
\begin{bmatrix}
\frac{1}{2} & \frac{\sqrt{3}}{2} \\
\frac{1}{2} & -\frac{\sqrt{3}}{2} \n\end{bmatrix},
\begin{bmatrix}
-\frac{1}{2} & -\frac{\sqrt{3}}{2} \\
-\frac{1}{2} & \frac{\sqrt{3}}{2}
\end{bmatrix},
\begin{bmatrix}
-\frac{1}{2} & \frac{\sqrt{3}}{2} \\
-\frac{1}{2} & -\frac{\sqrt{3}}{2} \n\end{bmatrix}
\end{bmatrix}