UPPER AND LOWER TIME DECAY BOUNDS FOR SOLUTIONS OF DISSIPATIVE NONLINEAR SCHRÖDINGER EQUATIONS

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Abstract. We study the upper and lower time decay bounds for solutions of dissipative nonlinear Schrödinger equations

\[ i\partial_t u + \frac{1}{2} \Delta u = \lambda |u|^{p-1} u, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^n, \]

\[ u(0, x) = u_0(x), \quad x \in \mathbb{R}^n, \]

in space dimensions \( n = 1, 2 \) or \( 3 \), where \( \lambda = \lambda_1 + i\lambda_2, \lambda_j \in \mathbb{R}, j = 1, 2, \lambda_2 < 0 \)

and the subcritical order of nonlinearity

\[ p = 1 + \frac{2}{n} - \mu, \]

where \( \mu > 0 \) is small enough.

1. Introduction and main results. We consider the initial value problem for the following nonlinear Schrödinger equations

\[ i\partial_t u + \frac{1}{2} \Delta u = \lambda |u|^{p-1} u, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^n, \]

\[ u(0, x) = u_0(x), \quad x \in \mathbb{R}^n \]

(1)

in space dimensions \( n = 1, 2 \) or \( 3 \), where \( \lambda = \lambda_1 + i\lambda_2, \lambda_j \in \mathbb{R}, j = 1, 2, \lambda_2 < 0 \)

and the order of nonlinearity is subcritical

\[ p = 1 + \frac{2}{n} - \mu, \]

where \( \mu > 0 \) is small enough.

There are some works concerning the physical applications of (1) (see e.g., [1] and [10]). We note that the condition \( \lambda_2 < 0 \) implies the dissipation property of \( |u(t, x)| \) by a nonlinear Ohm’s law (see e.g., [1]). In this paper we consider the initial function which yields the sharp time decay estimates of solutions of (1). In papers

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[6, 8, 9] and [10], asymptotic behavior of solutions to (1) was studied. However the sharp time decay of solutions was not obtained. It was shown in [9] that the leading term of the large time asymptotic profile is almost equal to

$$|u(t, x)| \approx \frac{1}{t^{\frac{n}{2}}} \left| \hat{u}_0 \left( \frac{x}{t} \right) \right| \left( 1 + A \frac{|\hat{u}_0 \left( \frac{x}{t} \right)|^{p-1} t^{1 - \frac{n(p-1)}{2}}} {1 + |\hat{u}_0 \left( \frac{x}{t} \right)|^{p-1} t^{1 - \frac{n(p-1)}{2}}} \right)^{\frac{1}{p-1}},$$

where $A = \frac{2|\lambda_x|^{(p-1)}}{2 - n(p-1)}$. This fact means that the upper bound of solutions is given by

$$(At)^{-\frac{1}{p-1}}$$

which does not belong to any $L^p$ space except $p = \infty$. In order to justify (2), the smallness condition on the coefficient $2 - n(p-1)$ was assumed in [9]. This condition implies that $p$ must be close to the critical exponent $1 + \frac{2}{n}$. We also find that there exists a positive constant $C$ such that

$$\frac{1}{t^{\frac{n}{2}}} \left| \hat{u}_0 \left( \frac{x}{t} \right) \right| \left( 1 + A \frac{|\hat{u}_0 \left( \frac{x}{t} \right)|^{p-1} t^{1 - \frac{n(p-1)}{2}}} {1 + |\hat{u}_0 \left( \frac{x}{t} \right)|^{p-1} t^{1 - \frac{n(p-1)}{2}}} \right)^{\frac{1}{p-1}} \geq \frac{C}{t^{\frac{n}{2}}} \inf_\xi |\hat{u}_0(\xi)|.$$

If $\inf_\xi |\hat{u}_0(\xi)| > 0$, then the lower bound for solutions follows. The condition $\inf_\xi |\hat{u}_0(\xi)| > 0$ means that $\hat{u}_0$ does not belong to any $L^p$ space except $p = \infty$. Therefore we consider the problem in such a space that the Fourier transform of the initial data does not vanish. Related work can be seen in [11] in which homogeneous weighted $L^2$ space was considered. However the initial data from [11] do not satisfy the condition $\inf_\xi |\hat{u}_0(\xi)| > 0$. On the other hand, in the case of $n = 1$, the result stated in Theorem 1.1 below was obtained in [7]. However Theorem 1.1 is not enough to get the lower bound for time decay. We also note that the final state problem for nonlinear Schrödinger system in two space dimensions with non decaying final data was studied in [5].

We introduce some function spaces and notations. Let $L^\infty$ denote the usual Lebesgue space with the norm $\|\phi\|_{L^\infty} = \text{ess.sup}_{x \in \mathbb{R}^n} |\phi(x)|$. The homogeneous Sobolev space $\dot{H}^m$ is defined by

$$\dot{H}^m = \left\{ \phi; \|\phi\|_{\dot{H}^m} = \left\| (-\Delta)^{m/2} \phi \right\|_{L^2} < \infty \right\},$$

$m \geq 0$, where $\|\phi\|_{L^2}^2 = \int_{\mathbb{R}^n} |\phi(x)|^2 \, dx$. By $\dot{B}^s_{p,q}$ we denote the homogeneous Besov space with the semi-norm

$$\|\phi\|_{\dot{B}^s_{p,q}} = \left( \int_0^\infty \lambda^{-1-sq} \sup_{|\beta| \leq \lambda} \sum_{|\alpha| = |s|} \|\partial^\alpha (\phi_\beta - \phi)\|_{L^p}^q \, d\lambda \right)^{1/q},$$

where $s = [s] + \sigma$, $0 < \sigma < 1$, $\phi_\beta(y) = \phi(x + y)$, $1 \leq p, q \leq \infty$, and $[s]$ is the largest integer less than $s$. It is known that $\dot{B}^s_{2,2} = \dot{H}^s$ (see [2]). We also denote $\langle t \rangle = \sqrt{1 + t^2}$. We define the dilation operator by

$$(D_t \phi) (x) = \frac{1}{\langle t \rangle^{\frac{n}{2}}} \phi \left( \frac{x}{t} \right)$$

and denote $M(t) = e^{\frac{|x|^2}{t^2}}$ for $t \neq 0$. Schrödinger evolution group $U(t)$ is written as $U(t) = M(t) D_t F M(t)$, where $F$ denotes the Fourier transformation. The inverse Schrödinger evolution group has the form $U(-t) = M(-t) F^{-1} D_t^{-1} M(-t)$, where
$F^{-1}$ is the inverse Fourier transformation. Different positive constants might be denoted by the same letter $C$ if it does not cause any confusion.

The standard generator of Galilei transformation is given by $J = U(t)xU(-t) = x + it\nabla$. We have the commutator relation with $J$ and $L = i\partial_t + \frac{1}{2}\Delta$ such that $[L, J] = 0$. To state our main results, we use the function space

$$X = \{ u; FU(-t - \theta) u \in C([0, \infty); Y), \| u \|_X < \infty \},$$

where $Y = L^\infty \cap \dot{H}^b, \frac{n}{2} < b < \min(2, p), \ p = 1 + \frac{2}{n} - \mu, \mu > 0$ is small enough and

$$\| u \|_X = \sup_{0 \leq t < \infty} (t + \theta)^{\frac{n}{2p} - \frac{2}{p}} \| FU(-t - \theta) u(t) \|_{L^\infty}$$

$$+ \sup_{0 \leq t < \infty} (t + \theta)^{-\gamma} \| FU(-t - \theta) u(t) \|_{\dot{H}^b}$$

with small $\gamma, \theta > 0$.

We now state our results.

**Theorem 1.1.** Let $p = 1 + \frac{2}{n} - \mu, \mu > 0$. Assume that the initial data $e^{\frac{2}{\theta}|\xi|^2} \hat{u}_0 \in \dot{H}^b$ with $\frac{n}{2} < b < \min(2, p), \theta > 0$, and the norms

$$\| e^{\frac{2}{\theta}|\xi|^2} \hat{u}_0 \|_{\dot{H}^b} \leq \rho^2, \| \hat{u}_0 \|_{L^\infty} \leq \rho.$$

Then there exist sufficiently small $\rho_0 > 0$ and $\mu_0 > 0$ such that the Cauchy problem (1) has a global in time solution $u \in X$ for all $0 < \rho \leq \rho_0$ and $0 < \mu \leq \mu_0$. Moreover the time decay estimate

$$\| u(t) \|_{L^\infty} \leq C (t + \theta)^{-\frac{1}{p-1}}$$

holds for all $t > 0$. In the case of $n = 1$, the solution $u$ is unique.

**Theorem 1.2.** Let $u$ be the global solution constructed in Theorem 1.1. Then for any $u(t)$, there exist unique $y_+, \eta_+ \in L^\infty$ such that

$$\| y_+ \|_{L^\infty} \leq C \rho, \| \eta_+ \|_{L^\infty} \leq C \rho^{p-1}$$

and the asymptotics of solutions

$$\left\| FU(-t - \theta) u(t) - \frac{y_+ e^{-iA H(t)}}{(1 + A \left((t + \theta)^{\frac{1}{2} - \frac{2}{p}(p-1) - \theta^{\frac{1}{2}}(p-1)}\right) |y_+|^{p-1} + \eta_+)} \right\|_{L^\infty} \leq C \rho^{p-1} (t + \theta)^{\frac{n}{2p} - \frac{2}{p-1} - \alpha}$$

holds for all $t > 0$, where $\alpha > 0$ and

$$H(t, \xi) = \frac{1}{(p-1)|\xi_2|} \log \left(1 + A \left((t + \theta)^{\frac{1}{2} - \frac{2}{p}(p-1) - \theta^{\frac{1}{2}}(p-1)}\right) |y_+|^{p-1} + \eta_+ \right).$$

Next we give the lower bound for the time decay of solutions to (1) and uniqueness of solutions.

**Theorem 1.3.** In addition to the assumptions of Theorem 1.1, we assume that

$$\inf_{\xi \in \mathbb{R}} |\hat{u}_0(\xi)| \geq \frac{\rho}{2}.$$

Then there exists a unique global solution $u \in X$ of the Cauchy problem (1). Furthermore for any $\hat{u}_0$ satisfying conditions, there exists a positive constant $\beta > 0$ such that global solution $u$ satisfies the lower bound

$$\inf_{x \in \mathbb{R}} |u(t, x)| \geq \beta (t + \theta)^{-\frac{1}{p-1}}$$
for all \( t > 0 \).

To explain our strategy, we look for the solution of (1) in the form
\[
 u(t, x) = (it)^{-\frac{n}{2}} e^{\frac{|x|^2}{4t}} h(t), \ h(0) = \rho. \tag{3}
\]
By a direct calculation, \( h(t) \) satisfies the ordinary differential equation
\[
 i h' = \lambda t^{-\frac{2}{2}(p-1)} |h|^{p-1} h.
\]
We change the dependent variable
\[
 h(t) = re^{iw}, \ r = |h|, \ w = \arg h, \ \text{with} \ r(0) = \rho, \ \ w(0) = 0,
\]
then we have
\[
 i r' - rw' = \lambda t^{-\frac{2}{2}(p-1)} r^{p},
\]
which gives us the ordinary differential equations
\[
 r' - \lambda_2 t^{-\frac{2}{2}(p-1)} r^{p} = 0, \ r(0) = \rho
\]
and
\[
 w' + \lambda_1 t^{-\frac{2}{2}(p-1)} r^{p-1} = 0, \ w(0) = 0.
\]
The explicit solution is as follows
\[
 r(t) = \frac{\rho}{(1 + A \rho^{p-1} t^{1-\frac{2}{2}(p-1)})^{\frac{1}{p-1}}},
\]
where
\[
 A = \frac{2 |\lambda_2| (p-1)}{2 - n (p-1)}.
\]
We note that
\[
 A = \frac{2 |\lambda_2| (2 - n \mu)}{\mu n^2} = O \left( \frac{1}{\mu} \right), \ \text{as} \ \mu \to 0.
\]
Then we have
\[
 w(t) = -\lambda_1 \int_{0}^{t} \tau^{-\frac{2}{2}(p-1)} r^{p-1} (\tau) d\tau
\]
\[
 = -\lambda_1 A^{-1} \int_{0}^{t} \rho^{-1} \tau^{-\frac{n(p-1)}{2}} \frac{d\tau}{1 + A \rho^{p-1} \tau^{1-\frac{2}{2}(p-1)}}
\]
\[
 = -\frac{2\lambda_1}{2 - n (p-1)} A^{-1} \log \left( 1 + A \rho^{p-1} t^{1-\frac{n(p-1)}{2}} \right)
\]
\[
 = -\frac{\lambda_1}{(p-1)|\lambda_2|} \log \left( 1 + A \rho^{p-1} t^{1-\frac{n(p-1)}{2}} \right).
\]
The right-hand side coincides with \( H(t, \xi) \) if we choose, \( |y_+(\xi)| = \rho, \ \theta = 0 \) and \( \eta_+(\xi) = 0 \). Thus the solution of (3) in the form \( h(t) = r(t) e^{iw(t)} \) with \( r(0) = \rho \) and \( w(0) = 0 \) can be represented as
\[
 h(t) = \frac{\rho}{(1 + A \rho^{p-1} t^{1-\frac{n(p-1)}{2}})^{\frac{1}{p-1}}} \exp \left( -\frac{\lambda_1}{(p-1)|\lambda_2|} \log \left( 1 + A \rho^{p-1} t^{1-\frac{n(p-1)}{2}} \right) \right).
\]
\[
 \tag{4}
\]
It is expected that solutions of (1) behave like (4) if the Fourier transforms of the initial data do not vanish. Our purpose is to prove the results including a particular solution (4).
2. Local existence. In the next lemma we obtain the estimates of the remainder terms

\[ R_1 = |V(t + \theta)\varphi|^{p-1}V(t + \theta)\varphi - |\varphi|^{p-1}\varphi, \]
\[ R_2 = (V^*(t + \theta) - 1)|V(t + \theta)\varphi|^{p-1}V(t + \theta)\varphi, \]

where we denote \( \varphi = F\mathcal{U}(-\theta - t)u \) and \( V(t + \theta) = F\mathcal{M}(t + \theta)F^{-1}, \) \( V^*(t + \theta) = V(-t - \theta). \)

**Lemma 2.1.** Let \( p > \max\left(1, \frac{2}{\sigma}\right) \) and \( \frac{n}{2} < b < \min(2, p) \). Then the estimate

\[ \|R_j\|_{L^\infty} \leq C(t + \theta)^{-\frac{1}{2}(b-\frac{n}{2})}\|\varphi\|_{L^\infty} + (t + \theta)^{-\frac{1}{2}(b-\frac{n}{2})}\|\varphi\|_{\dot{H}^b}, \]

is true for all \( t \geq 0, \theta \geq 0, j = 1, 2, \) provided that the right-hand side is finite.

**Proof.** Choosing \( \nu = \|\varphi\|_{L^2}^\frac{1}{2} \) \( \|\varphi\|_{\dot{H}^b}^\frac{1}{2} \) for \( \frac{n}{2} < b \leq 2, \) by the Cauchy-Schwarz inequality we get the Sobolev embedding inequality

\[ \|\varphi\|_{L^\infty} \leq \frac{\|\varphi\|_{L^2}}{2} + C \|\nu\|_{L^2} \|\varphi\|_{L^2} + C \|\nu\|_{L^2} \|\varphi\|_{L^2} \leq C \|\varphi\|_{L^2} \|\varphi\|_{\dot{H}^b} \]

Also we have

\[ \|\varphi\|_{L^\infty} \leq \frac{\|\varphi\|_{L^2}}{2} + C \|\varphi\|_{L^2} \]

for \( 0 \leq \sigma \leq 2. \) Hence choosing \( \sigma = b, \frac{n}{2} < b \leq 2, \) we find

\[ \|(V(t + \theta) - 1)\varphi\|_{L^\infty} \leq C \|(V(t + \theta) - 1)\varphi\|_{L^{\frac{2}{1+b-\frac{n}{2}}}} \|(V(t + \theta) - 1)\varphi\|_{\dot{H}^b} \leq C(t + \theta)^{-\frac{1}{2}(b-\frac{n}{2})} \|\varphi\|_{\dot{H}^b} \]

for \( t \geq 1. \) Hence

\[ \|(V(t + \theta) - 1)\varphi\|_{L^\infty} \leq C(t + \theta)^{-\frac{1}{2}(b-\frac{n}{2})} \|\varphi\|_{\dot{H}^b} \]

for all \( t \geq 0, \) where for \( \frac{n}{2} < b < 2. \) Also we write

\[ \|V(t + \theta)\varphi\|_{L^\infty} \leq \|\varphi\|_{L^\infty} + \|(V(t + \theta) - 1)\varphi\|_{L^\infty} \leq \|\varphi\|_{L^\infty} + C(t + \theta)^{-\frac{1}{2}(b-\frac{n}{2})} \|\varphi\|_{\dot{H}^b} \]

for \( \frac{n}{2} < b < 2. \) Therefore the first term \( R_1 \) is estimated as

\[ \|R_1\|_{L^\infty} \leq C \|\varphi\|_{L^\infty} + \|\varphi\|_{L^\infty} \leq C(t + \theta)^{-\frac{1}{2}(b-\frac{n}{2})} \|\varphi\|_{\dot{H}^b} \]

for \( \frac{n}{2} < b < 2. \) By Lemma 3.4 from [4], since the norm of the homogeneous Sobolev space \( \dot{H}_x^\sigma \) is equivalent to that of the homogeneous Besov space \( \dot{B}_{2,2}^\sigma \) (see [2]), we have

\[ \|\varphi\|_{L^\infty} \leq C \|\varphi\|_{\dot{B}_{2,2}^\sigma} \leq C \|\varphi\|_{L^\infty} \|\varphi\|_{\dot{B}_{2,2}^\sigma} \leq C \|\varphi\|_{L^\infty} \|\varphi\|_{\dot{H}^b} \]

(7)
where \(0 \leq \sigma < \min(2, p)\). Hence using (5) with \(\frac{n}{2} < b \leq 2\), (7) with \(\sigma = b\), and (6), we estimate the second term \(R_2\) as follows

\[
\|R_2\|_{L^\infty} \leq C \left\| (V^* (t + \theta) - 1) |V(t + \theta)\varphi|^{p-1} V(t + \theta)\varphi \right\|_{L^\infty} \\
\leq C(t + \theta)^{-\frac{1}{2}(b-\frac{2}{n})} \left\| \|V(t + \theta)\varphi|^{p-1} V(t + \theta)\varphi \right\|_{\dot{H}^b} \\
\leq C(t + \theta)^{-\frac{1}{2}(b-\frac{2}{n})} \|V(t + \theta)\varphi\|_{L^\infty} \|V(t + \theta)\varphi\|_{\dot{H}^b} \\
\leq C(t + \theta)^{-\frac{1}{2}(b-\frac{2}{n})} \left( \|\varphi\|_{L^\infty} + (t + \theta)^{-\frac{1}{2}(b-\frac{2}{n})} \|\varphi\|_{\dot{H}^b} \right)^{p-1} \|\varphi\|_{\dot{H}^b}
\]

for \(\frac{n}{2} < b < \min(2, p)\). This completes the proof of the lemma.

To prove local existence we introduce the function space \(X_T\) such that

\[
X_T = \{ v \in C([0, T]; L^\infty \cap C) : \|v\|_{X_T} < \infty \},
\]

where

\[
\|v\|_{X_T} = \sup_{0 \leq t \leq T} (\|\mathcal{F}u(-t - \theta) v(t)\|_{L^\infty} + \|\mathcal{F}u(-t - \theta) v(t)\|_{\dot{H}^b}),
\]

with \(\frac{n}{2} < b < \min(2, p)\). We are now in a position to prove the local existence theorem.

**Lemma 2.2.** Let \(\max(1, \frac{n}{2}) < p < 1 + \frac{2}{n}\). Assume that the initial data \(e^{-\frac{2}{n}t} \|\xi\|^2 \tilde{u}_0 \in L^\infty \cap \dot{H}^b\) with \(\frac{n}{2} < b < \min(2, p)\). Then for some time \(T > 0\) there exists a solution \(u \in X_T\) for the Cauchy problem (1).

**Proof.** Let us consider the Cauchy problem (1) with the regularized initial data as follows:

\[
i \partial_t u_m + \frac{1}{2} \Delta u_m = \lambda |u_m|^{p-1} u_m, \quad u_m(0, x) = u_{m,0}(x) = \mathcal{F}^{-1}\left( \left\langle \frac{\xi}{m} \right\rangle^{-n} \tilde{u}_0 \right), \tag{8}
\]

where \(m = 1, 2, ...\) The local existence of solution \(u_m \in X_T\) to the Cauchy problem (8) can be obtained as in book [3]. Let us prove that there exists a time \(T > 0\) such that the estimate

\[
\|u_m\|_{X_T} < 2\rho
\]

is true for all \(m\), where \(\rho = \|e^{-\frac{2}{n}t} \|\xi\|^2 \tilde{u}_0\|_{L^\infty \cap \dot{H}^b} > 0\). We argue by the contradiction.

In view of the continuity of the norm \(X_T\) with respect to \(T\) we can find the first time \(T > 0\) such that

\[
\|u_m\|_{X_T} \leq 2\rho.
\]

Consider the integral equation associated with (8)

\[
\varphi_m(t) \equiv \mathcal{F}u(-t - \theta) u_m(t) = \varphi_m(0) \\
- i \lambda \int_0^t \mathcal{F}u(-\tau - \theta) |U(\tau + \theta)\mathcal{F}^{-1}\varphi_m(\tau)|^{p-1} U(\tau + \theta) \mathcal{F}^{-1}\varphi_m(\tau) d\tau \\
= \varphi_m(0) - i \lambda \int_0^t (\tau + \theta)^{-\frac{2}{n}(p-1)} \\
\times V^* (\tau + \theta) |V(\tau + \theta)\varphi_m(\tau)|^{p-1} V(\tau + \theta) \varphi_m(\tau) d\tau, \tag{9}
\]

since \( U (t + \theta) F^{-1} = M (t + \theta) D_{t+\theta} V (t + \theta) \), \( F U (-t - \theta) = V^* (t + \theta) D_{t+\theta}^{-1} \overline{M} (t + \theta) \), \( V (t + \theta) = FM (t + \theta) F^{-1}, V^* (t + \theta) = V (-t - \theta) \). Using the estimate (7) we obtain

\[
\| \varphi_m (t) \|_{H^\sigma} \leq \left\| \left\langle \frac{\xi}{m} \right\rangle^{n} e^{i \theta |\xi|^2 u_0} \right\|_{H^\sigma} + C \int_0^t (\tau + \theta)^{-\frac{\sigma}{2} (p-1)} \| \varphi (\tau + \theta) \varphi_m (\tau) \|_{L^\infty} \| \varphi_m (\tau) \|_{H^\sigma} d\tau, \quad (10)
\]

where \( 0 \leq \sigma < \min (2, p) \). In view of (6), we have

\[
\| \varphi (t + \theta) \|_{L^\infty} \leq \| \varphi \|_{L^\infty} + \| \varphi (t + \theta) - 1 \|_{L^\infty} \leq \| \varphi \|_{L^\infty} + C (t + \theta)^{-\frac{1}{2} (b - \frac{p}{2})} \| \varphi \|_{H^\sigma} \leq C \rho.
\]

Hence we find

\[
\| \varphi_m (t) \|_{H^\sigma} \leq \left\| \left\langle \frac{\xi}{m} \right\rangle^{n} e^{i \theta |\xi|^2 u_0} \right\|_{H^\sigma} + C \rho \int_0^t (\tau + \theta)^{-\frac{\sigma}{2} (p-1)} d\tau
\]

Thus, there exists a time \( T = T (\rho) > 0 \) such that

\[
\| \varphi_m \|_{X_T} \leq \rho + C \rho^p T^{1 - \frac{n}{2} (p-1)} < 2 \rho
\]

for all \( m \). This is the desired contradiction. By the compact embedding of \( L^\infty \cap \dot{H}^b \) into \( L^\infty \cap \dot{H}^{b-\epsilon} \) which follows from the fact that Hölder class of order \( \alpha \) denoted by \( C^{0, \alpha} \) is included in \( L^\infty \cap \dot{H}^b \) if \( 1 > b - \frac{\epsilon}{2} > \alpha \), there exists a subsequence \( u_{m_k} \) which converges in \( \dot{H}^{b-\epsilon} \) to some function \( u \in \dot{H}^{b-\epsilon} \). Passing to the limit \( m_k \to \infty \) in the integral equation (9) we find that \( u \) is a solution of (1) (see [12]). This completes the proof of the lemma. \( \Box \)

For the one dimensional case \( n = 1 \) the nonlinearity has a sufficient regularity when the order of the nonlinearity \( p \) is close to 3. Hence we can apply the contraction mapping principle considering to the linearized version

\[
i \partial_t u + \frac{1}{2} \Delta u = \lambda |u|^{p-1} v, \quad u(0, x) = u_0 (x)
\]

with \( v \in X_T \) and \( \| v \|_{X_T} < 2 \rho \). As above we find that there exists a time \( T = T (\rho) > 0 \) such that \( \| u \|_{X_T} < 2 \rho \). Also in the same way as above, we find that there exists \( T = T (\rho) > 0 \) such that

\[
\| u_1 - u_2 \|_{X_T} \leq \frac{1}{2} \| v_1 - v_2 \|_{X_T}
\]

for the difference of two solutions

\[
i \partial_t u_j + \frac{1}{2} \Delta u_j = \lambda |v_j|^{p-1} v_j, \quad u_j (0, x) = u_0 (x), \quad j = 1, 2.
\]

Thus the transformation \( u = Sv \) is a contraction mapping from \( X_T \) into itself and there exists a unique solution \( u \in X_T \) for the Cauchy problem (1) in the case of \( n = 1 \). In the case of \( n = 2, 3 \) the nonlinearity does not possess sufficient regularity since the order \( p < 2 \). Then to prove the uniqueness we need to make some additional assumptions on the lower bound for the solutions.
Lemma 2.3. Let \( n = 2, 3 \). Assume that there exists a positive constant \( \delta > 0 \) such that the solutions \( u \in X_T \) of the Cauchy problem (1) satisfy

\[
\begin{align*}
\inf_{0 \leq t \leq \tau, \xi \in \mathbb{R}} |\varphi(t, \xi)| & \geq \frac{\delta}{2}, \|\varphi\|_{L^\infty} \leq 2\delta \\
\|\varphi\|_{\dot{H}^s} & \leq \frac{\delta}{4C\theta^{\frac{1}{2}(b-\frac{2}{p})}},
\end{align*}
\]

where \( \varphi(t) = \mathcal{F}u(-t - \theta)u(t) \) and \( C > 0 \) is the constant from the Sobolev embedding inequality (5). Then the solution \( u \) is unique.

Proof. By the contrary suppose that there exist two solutions \( \varphi_1 \) and \( \varphi_2 \) of the integral equation associated with (1)

\[
\varphi_j(t) = e^{-\frac{i}{2}\theta|\xi|^2\hat{u}_0} - i\lambda \int_0^t (\tau + \theta)^{-\frac{2}{p}(p-1)} \mathcal{V}^* (\tau + \theta) F(\mathcal{V}(\tau + \theta) \varphi_j(\tau)) d\tau,
\]

where \( F(\phi) = |\phi|^{p-1} \phi \). Then we obtain

\[
\varphi_1(t) - \varphi_2(t) = -i\lambda \int_0^t (\tau + \theta)^{-\frac{2}{p}(p-1)} \mathcal{V}^* (\tau + \theta) (F(\mathcal{V}(\tau + \theta) \varphi_1(\tau)) - F(\mathcal{V}(\tau + \theta) \varphi_2(\tau))) d\tau.
\]

Therefore by (7) we have

\[
\|\varphi_1(t) - \varphi_2(t)\|_{\dot{H}^s} \leq C \int_0^t (\tau + \theta)^{-\frac{2}{p}(p-1)} \|F(\mathcal{V}(\tau + \theta) \varphi_1(\tau)) - F(\mathcal{V}(\tau + \theta) \varphi_2(\tau))\|_{\dot{H}^s} d\tau,
\]

where \( 0 \leq \sigma < \min(2, p) \). By (5) and the assumptions on \( \varphi \) we find

\[
|\mathcal{V}(t+\theta) \varphi| \geq |\varphi| - \|\mathcal{V}(t+\theta) - 1\|_{L^\infty} \varphi \geq |\varphi| - C(\tau + \theta)^{-\frac{2}{p}(p-1)} \|\varphi\|_{\dot{H}^s} \geq \frac{\delta}{4} > 0
\]

where \( \frac{2}{p} < b \leq 2 \). By the condition on the lower bound for the solutions, \( F(\phi) \) has enough regularity to estimate

\[
\begin{align*}
\|F(\mathcal{V}(t+\theta) \varphi_1) - F(\mathcal{V}(t+\theta) \varphi_2)\|_{\dot{H}^s} & \leq C (\|\varphi_1\|_{L^\infty} + \|\varphi_2\|_{L^\infty}) \|\varphi_1 - \varphi_2\|_{\dot{H}^s} \\
& + C (\|\varphi_1\|_{\dot{H}^s} + \|\varphi_2\|_{\dot{H}^s}) \|\varphi_1 - \varphi_2\|_{L^\infty}
\end{align*}
\]

for \( n = 2, 3 \). Then we obtain

\[
\|\varphi_1(t) - \varphi_2(t)\|_{L^\infty} + \|\varphi_1(t) - \varphi_2(t)\|_{\dot{H}^s} \leq C\delta \int_0^t \left(\|\varphi_1(\tau) - \varphi_2(\tau)\|_{L^\infty} + \|\varphi_1(\tau) - \varphi_2(\tau)\|_{\dot{H}^s}\right) d\tau,
\]

where \( \frac{2}{p} < b < 2 \). Therefore by the Granwall inequality we get

\[
\|\varphi_1(t) - \varphi_2(t)\|_{L^\infty} + \|\varphi_1(t) - \varphi_2(t)\|_{\dot{H}^s} = 0.
\]

This completes the proof of the lemma. \qed
3. **Proof of Theorem 1.1.** We prove Theorem 1.1 by showing a-priori estimates of local solutions stated in Lemma 2.2.

**Lemma 3.1.** Assume that the assumptions of Theorem 1.1 hold. Also suppose that
\[\sup_{t \in [0, T]} (t + \theta)^{-\gamma} \| \varphi(t) \|_{H^n} \leq 2 \rho^2.\]
Then the estimate
\[\sup_{t \in [0, T]} (t + \theta)^{\frac{1}{p-1} - \frac{2}{p}} \| \varphi(t) \|_{L^\infty} < 2 \left( \rho + A^{-\frac{1}{p-1}} \right)\]
is true for sufficiently small \(\rho > 0\) and \(A^{-\frac{1}{p-1}}\), where \(\varphi(t) = \mathcal{F}u(t)\).

**Proof.** By the contrary we may assume that there exists a time \(T > 0\) such that
\[\sup_{t \in [0, T]} (t + \theta)^{\frac{1}{p-1} - \frac{2}{p}} \| \varphi(t) \|_{L^\infty} = 2 \left( \rho + A^{-\frac{1}{p-1}} \right).\]
We also have \(\| \varphi(0) \|_{L^\infty} \leq \rho\) from the assumptions of Theorem 1.1. We represent the solution of (1) in the form
\[\varphi(t) = \mathcal{F}u(-t - \theta)u(t) = re^{iw}, \ r = |\varphi|, \ w = \arg \varphi,\]
then we find
\[\left\{\begin{array}{l}
\partial_t r = \lambda_2 (t + \theta)^{-\frac{2}{2} (p-1)} r^p + \text{Im} \left( \lambda (t + \theta)^{-\frac{2}{2} (p-1)} e^{-iw} (R_1 + R_2) \right), \\
\partial_t w = -\lambda_1 (t + \theta)^{-\frac{2}{2} (p-1)} r^{p-1} - r^{-1} \text{Re} \left( \lambda (t + \theta)^{-\frac{2}{2} (p-1)} e^{-iw} (R_1 + R_2) \right).
\end{array}\right.\]
Therefore we obtain
\[\partial_t r - \lambda_2 (t + \theta)^{-\frac{2}{2} (p-1)} r^p = g(t), \quad (11)\]
where \(g(t) = O \left( \rho^{p+1} (t + \theta)^{\gamma - \frac{1}{2} (b - \frac{1}{2})} \right)\), since by Lemma 2.1 we have
\[\| R_j \|_{L^\infty} \leq C \left( \| \varphi \|_{L^\infty} + (t + \theta)^{-\frac{1}{2} (b - \frac{1}{2})} \| \varphi \|_{H^p} \right)^{p-1} \| \varphi \|_{H^p} \leq C \rho^{p+1} (t + \theta)^{\gamma - \frac{1}{2} (b - \frac{1}{2})}\]
for \(j = 1, 2\). Let \(r_1, r_2\) be solutions to (11) with \(r_1(0) = r_2(0) = |\varphi(0)|\), then we have
\[\partial_t (r_1 - r_2) - \lambda_2 (t + \theta)^{-\frac{2}{2} (p-1)} (r_1^p - r_2^p) = 0\]
from which it follows that
\[\frac{d}{dt} |r_1 - r_2|^2 \leq 0.\]
This implies the uniqueness of the solution for (11). Since \(0 < r(0) \leq \rho\), to find the upper bound for the solution \(r(t)\), it is enough to consider the case \(r(0) = \rho\). Define
\[f(t) = \rho \left( 1 + A \rho^{p-1} \left( (t + \theta)^{1 - \frac{2}{2} (p-1)} - \theta^{1 - \frac{2}{2} (p-1)} \right) \right)^{-\frac{1}{p-1}},\]
where
\[A = \frac{\| \lambda_2 \|_{L^\infty}}{\frac{1}{p-1} - \frac{1}{2}}, \ A = \frac{|\lambda_2| (p-1)}{1 - \frac{1}{2} (p-1)} \]
By a direct calculation we see that \(f(t)\) satisfies
\[\frac{d}{dt} f(t) - \lambda_2 (t + \theta)^{-\frac{2}{2} (p-1)} f^p = 0, \ f(0) = \rho.\]
Multiplying both sides of (11) by $f^{-p}$, we obtain
\[
\frac{d}{dt} (f^{-p}r) + \lambda_2 (t + \theta)^{-\frac{n}{2(p-1)}} (pf^{-1}r - f^{-p}r^p) = O (gf^{-p}).
\]

By the Young inequality $pf^{-1}r \leq f^{-p}r^p + (p - 1)$ and by the dissipative condition $\lambda_2 < 0$, we find
\[
\frac{d}{dt} (f^{-p}r) \leq -\lambda_2 (p - 1) (t + \theta)^{-\frac{n}{2(p-1)}} + Cgf^{-p}.
\]

Integrating in time, we get
\[
r (t) \leq f^p (t) \left( \rho_1 - p + A \left( (t + \theta)^{1-\frac{n}{2(p-1)}} - \theta^{1-\frac{n}{2(p-1)}} \right) \right)
\]
\[
+ Cf^p (t) \int_0^t f^{-p} (\tau) g (\tau) d\tau
\]
\[
= f (t) + Cf^p (t) \int_0^t f^{-p} (\tau) g (\tau) d\tau,
\]

(12)
since by the definition of $f (t)$, we see that
\[
f^{1-p} (t) = \rho^{1-p} + A \left( (t + \theta)^{1-\frac{n}{2(p-1)}} - \theta^{1-\frac{n}{2(p-1)}} \right).
\]

Let us consider the second term of the right-hand side of (12). We find
\[
\int_0^t f^{-p} (\tau) g (\tau) d\tau
\]
\[
\leq \rho \int_0^t \left( (\tau + \theta)^{\gamma-\frac{1}{2}(b-\frac{n}{2})-\frac{n}{2(p-1)}} + A\frac{\rho}{\rho+1} (\tau + \theta)^{\gamma-\frac{1}{2}(b-\frac{n}{2})-\frac{n}{2(p-1)}} \right) d\tau
\]
\[
\leq \rho \int_0^t \left( (\tau + \theta)^{-1+\frac{1}{2}\mu+\gamma-\frac{1}{2}(b-\frac{n}{2})} + A\frac{\rho}{\rho+1} (\tau + \theta)^{-1+\frac{1}{2}\mu+\gamma-\frac{1}{2}(b-\frac{n}{2})} \right) d\tau
\]
\[
\leq \rho + A\frac{\rho}{\rho+1} \rho^{p+1}
\]

for small $\mu > 0, \gamma > 0, p = 1 + \frac{2}{n} - \mu$. Hence
\[
f^p (t) \int_0^t f^{-p} (\tau) g (\tau) d\tau
\]
\[
\leq \frac{\rho^{p+1} \left( 1 + A\frac{\rho}{\rho+1} \rho^p \right)}{\left( 1 + A\rho^{p-1} (t + \theta)^{1-\frac{n}{2(p-1)}} - \theta^{1-\frac{n}{2(p-1)}} \right) \rho^{p+1} \rho^p}
\]
\[
\leq \frac{\rho^{p+1} \left( 1 + A\frac{\rho}{\rho+1} \rho^p \right)}{1 + A\frac{\rho}{\rho+1} \rho^p \left( (t + \theta)^{1-\frac{n}{2(p-1)}} - \theta^{1-\frac{n}{2(p-1)}} \right) \rho^{p+1} \rho^p}
\]
\[
\leq C\rho \left( A^{-\frac{\rho}{\rho+1}} + \rho^p \right) (t + \theta)^{\frac{n}{2}-\frac{1}{p+1}}.
\]

Therefore
\[
r (t) \leq f (t) + Cf^p (t) \int_0^t f^{-p} (\tau) g (\tau) d\tau
\]
\[
\leq \left( \rho + A^{-\frac{\rho}{\rho+1}} \right) \left( 1 + \rho \left( \rho + A^{-\frac{1}{p+1}} \right)^{p-1} \right) (t + \theta)^{\frac{n}{2}-\frac{1}{p+1}}.
\]
Hence
\[(t + \theta)^{\frac{1}{m-1} - \frac{\alpha}{2}} \| \mathcal{F}u \|_{L^\infty} < 2 \left( \rho + A^{-\frac{1}{p-1}} \right)\]
for sufficiently small \(\rho > 0\) and \(A^{-\frac{1}{p-1}}\). This is the desired contradiction and the lemma is proved.

**Lemma 3.2.** Assume that the assumptions of Theorem 1.1 hold. Also suppose that the estimate
\[\sup_{t \leq T} (t + \theta)^{-\gamma} \| \varphi(t) \|_{H^s} \leq 2 \rho^2\]
is true. Then the following estimate
\[\sup_{t \leq T} (t + \theta)^{-\gamma} \| \varphi(t) \|_{H^s} < 2 \rho^2\]
is valid for sufficiently small \(\gamma > 0\), where \(b > \frac{n}{2}\), and \(\varphi(t) = \mathcal{F}u \|_{L^\infty} \).  

**Proof.** We assume that there exists a time \(T\) such that
\[\sup_{t \leq T} (t + \theta)^{-\gamma} \| \varphi(t) \|_{H^s} = 2 \rho^2.\]

By (6) we have
\[\| \mathcal{V}(\tau + \theta) \varphi \|_{L^\infty} \leq C \| \varphi(\tau) \|_{L^\infty} + C (\tau + \theta)^{-\gamma \left( b - \frac{n}{2} \right)} \| \varphi \|_{H^s} \]
\[\leq C \left( \rho + A^{-\frac{1}{p-1}} \right) (\tau + \theta)^{-\gamma \left( b - \frac{n}{2} \right)} + 2 \rho^2 (\tau + \theta)^{\gamma - \frac{\alpha}{2} \left( b - \frac{n}{2} \right)} \]
\[\leq C \left( \rho + A^{-\frac{1}{p-1}} \right) (\tau + \theta)^{-\gamma \left( b - \frac{n}{2} \right)} \]  
(13)
if \(\frac{1}{p-1} - \frac{n}{2} \leq \frac{1}{2} \left( b - \frac{n}{2} \right) - \gamma\). We now turn to the integral equation (10) for \(\varphi\) and use (13) to get
\[\| \varphi(t) \|_{H^s} \leq \| \varphi(0) \|_{H^s} + C \int_0^t (\tau + \theta)^{-\gamma \left( p-1 \right)} \| \mathcal{V}(\tau + \theta) \varphi \|_{L^\infty} \| \varphi(\tau) \|_{H^s} d\tau \]
\[\leq \| \varphi(0) \|_{H^s} + C \rho^2 (\rho^{p-1} + A^{-1}) \int_0^t (\tau + \theta)^{\gamma - 1} d\tau \]
\[\leq \rho^2 \left( 1 + C (\rho^{p-1} + A^{-1}) (\tau + \theta)^{\gamma} \right) \leq \rho^2 (\tau + \theta)^{\gamma}\]
for sufficiently small \(\rho\) and \(A^{-1}\). This is a desired contradiction, which completes the proof of the lemma.

**Proof of Theorem 1.1:** By Lemmas 3.1 and 3.2, we have a priori estimates of solutions in the space \(X\). Therefore the global in time existence of small solutions follows. The desired time decay of solutions can be obtained by factorization technique \(u(t) = \mathcal{U}(t + \theta) \mathcal{F}^{-1} \varphi = M(t + \theta) D_{t+\theta} \mathcal{V}(t + \theta) \varphi\) in view of estimate (13).

4. **Proof of Theorem 1.2.** First we estimate the large time asymptotics of \(y(t) = e^{i\lambda f_0^{\theta}(\tau + \theta)^{-\frac{\alpha}{2} (p-1) | \varphi |^{p-1}} d\tau \varphi(t)\).

**Lemma 4.1.** Let the initial data satisfy the assumptions of Theorem 1.1 and \(u\) be the global solution constructed in Theorem 1.1. Then there exists a unique \(y_+ \in L^\infty\) such that
\[\| y(t) - y_+ \|_{L^\infty} \leq C \rho^{p+1} (t + \theta)^{-\alpha},\]
for all \( t > 0 \), where
\[
(n + 2) |\lambda_2| \mu < \alpha < \min \left\{ 1, 1 + \frac{2}{n} - \frac{n}{2} \right\}.
\]

**Proof.** Multiplying both sides of (1) by \( \mathcal{F} u (-t - \theta) \), we obtain
\[
 i \partial_t \varphi (t) = \lambda (t + \theta)^{- \frac{2}{p(p-1)} |\varphi (t)|^{p-1} \varphi (t) + \lambda (t + \theta)^{- \frac{2}{2(p-1)} (R_1 + R_2)},
\]
where
\[
R_1 = |\mathcal{V} (t + \theta) \varphi |^{p-1} \mathcal{V} (t + \theta) \varphi - |\varphi |^{p-1} \varphi,
\]
\[
R_2 = (\mathcal{V}^* (t + \theta) - 1) |\mathcal{V} (t + \theta) \varphi |^{p-1} \mathcal{V} (t + \theta) \varphi.
\]
We multiply both sides of (14) by \( e^{i \lambda_1 \int_0^t (\tau + \theta)^{- \frac{2}{p(p-1)} |\varphi |^{p-1} d\tau} \) to obtain
\[
i \partial_t y (t) = \lambda e^{i \lambda_1 \int_0^t (\tau + \theta)^{- \frac{2}{p(p-1)} |\varphi |^{p-1} d\tau} (t + \theta)^{- \frac{2}{2(p-1)} (R_1 + R_2)}
\]
from which it follows that
\[
|y (t, \xi) - y (s, \xi)| \leq |\lambda| \int_s^t e^{i |\lambda_2| \int_0^{\xi} (\tau + \theta)^{- \frac{2}{2(p-1)} (\tilde{s} + \theta)^{- \frac{2}{2(p-1)} |\varphi |^{p-1} d\tau}} (t + \theta)^{- \frac{2}{2(p-1)} (\tilde{s} + \theta)^{- \frac{2}{2(p-1)} |R_1 + R_2| d\tilde{s}}.
\]
By Theorem 1.1 we have
\[
||\varphi (t)||_{L^\infty} \leq 2 \left( \rho + A^{-\frac{1}{p-1}} \right) (t + \theta)^{\frac{\gamma}{2} - \frac{1}{p-1}}
\]
for all \( t > 0 \). Hence using Lemma 2.1, we get
\[
|y (t, \xi) - y (s, \xi)| \leq |\lambda| \int_s^t e^{C (p^{p-1} - A)} \int_0^t (\tau + \theta)^{- \frac{2}{2(p-1)} |R_1 + R_2| d\tilde{s}}
\]
\[
\leq C \int_s^t e^{C (p^{p-1} - A)} (\log (\tilde{s} + \theta))^{- \frac{2}{2(p-1)} |R_1 + R_2| d\tilde{s}}
\]
\[
\leq C \rho^{p+1} \int_s^t (\tilde{s} + \theta)^{C (p^{p-1} - A) + \gamma - \frac{1}{2} (b - \frac{2}{2}) - \frac{2}{2} (p-1)} d\tilde{s} \leq C \rho^{p+1} (t + \theta)^{- \alpha},
\]
with some \( \alpha > 0 \), if \( \rho \) and \( A^{-1} \) are sufficiently small. Hence there exists a unique limit \( y_+ \in L^\infty \) such that
\[
||y (t) - y_+||_{L^\infty} \leq C \rho^{p+1} (t + \theta)^{- \alpha}
\]
for all \( t \geq 0 \). Since \( |y (0)| \leq C \rho \) and \( ||y (0) - y_+||_{L^\infty} \leq C \rho^{p+1} \theta^{- \alpha} \), we have the estimate \( ||y_+||_{L^\infty} \leq C \rho \). This completes the proof of the lemma. \( \square \)

We next consider the asymptotic behavior of
\[
\Gamma (t) = e^{(p-1) |\lambda_2| \int_0^t (\tau + \theta)^{- \frac{2}{2(p-1)} |\varphi |^{p-1} d\tau}.
\]
Denote
\[
H (t, \xi) = \frac{1}{(p-1) |\lambda_2|} \log \left( 1 + A \left( (t + \theta)^{1 - \frac{2}{2(p-1)}} - \theta^{1 - \frac{2}{2(p-1)}} \right) |y_+|^{p-1} + \eta_+ \right).
\]

**Lemma 4.2.** Let the initial data satisfy the assumptions of Theorem 1.1 and \( \eta \) be the solution constructed in Theorem 1.1. Then there exists a unique \( \eta_+ \) such that \( ||\eta_+||_{L^\infty} \leq C \rho^{-1} \) and the following estimate
\[
\left\| \Gamma^{-\frac{1}{p-1}} (t) - e^{i |\lambda_2| H (t)} \right\|_{L^\infty} \leq C \rho^{-1} t^{- \alpha}
\]
holds for all \( t > 0 \), where \( \alpha > 0 \).
Proof. We have by a direct calculation
\[
\partial_t \Gamma(t) = (p-1) |\lambda_2| (t + \theta)^{-\frac{1}{2} (p-1)} \left( |\phi| \Gamma(t) \right)_{\frac{-1}{p-1}}^{p-1} \\
= (p-1) |\lambda_2| (t + \theta)^{-\frac{1}{2} (p-1)} |y|^{p-1} \\
= (p-1) |\lambda_2| (t + \theta)^{-\frac{1}{2} (p-1)} |y_+|^{p-1} \\
+ (p-1) |\lambda_2| (t + \theta)^{-\frac{1}{2} (p-1)} \left( |y(t)|^{p-1} - |y_+|^{p-1} \right) .
\]
Integrating in time, we obtain
\[
\Gamma(t) = 1 + A \left( (t + \theta)^{1 - \frac{1}{2} (p-1)} - \theta^{1 - \frac{1}{2} (p-1)} \right) |y_+|^{p-1} \\
+ (p-1) |\lambda_2| \int_0^t (s + \theta)^{-\frac{1}{2} (p-1)} \left( |y(s)|^{p-1} - |y_+|^{p-1} \right) ds .
\]
(15)
We denote
\[
V(t) = \Gamma(t) - \left( 1 + A \left( (t + \theta)^{1 - \frac{1}{2} (p-1)} - \theta^{1 - \frac{1}{2} (p-1)} \right) |y_+|^{p-1} \right) ,
\]
then by Lemma 4.1 and (15), we find that
\[
\| V(t) - V(s) \|_{L^\infty} \leq C \left\{ \begin{array}{ll}
\rho^{p^2-1} (t + \theta)^{-\alpha}, & \text{if } p \leq 2 \\
\rho^{2p-1} (t + \theta)^{-\alpha}, & \text{if } p > 2
\end{array} \right.
\]
for \( t > s > 0 \). Therefore there exists a unique \( \eta_+ \) such that
\[
\begin{align*}
\left\| \Gamma(t) - e^{(p-1)|\lambda_2| H(t)} \right\|_{L^\infty} &= \left\| \Gamma(t) - \left( 1 + A \left( (t + \theta)^{1 - \frac{1}{2} (p-1)} - \theta^{1 - \frac{1}{2} (p-1)} \right) |y_+|^{p-1} + \eta_+ \right) \right\|_{L^\infty} \\
&\leq C \left( \rho^{p^2-1} + \rho^{2p-1} \right) (t + \theta)^{-\alpha} .
\end{align*}
\]
(16)
We also have by (15) and (16) with \( t = 0 \)
\[
\| \Gamma(0) - (1 + \eta_+) \|_{L^\infty} = \| \eta_+ \|_{L^\infty} \leq C \left( \rho^{p^2-1} + \rho^{2p-1} \right) \theta^{-\alpha} .
\]
(17)
Then by (16) we find
\[
\begin{align*}
&\left| \Gamma \left( \frac{1}{p-1} \right) - e^{\lambda_2|H(t)|} \right| \\
\leq& C \left\{ \begin{array}{ll}
\left| \Gamma(t) - e^{(p-1)|\lambda_2| H(t)} \right| \left( \left\| \Gamma(t) \right\|_{\frac{2}{p-1}}^{\frac{2}{p-1}} + e^{(2-p)|\lambda_2| H(t)} \right), & \text{if } 1 < p \leq 2, \\
\left| \Gamma(t) - e^{(p-1)|\lambda_2| H(t)} \right|_{\frac{1}{p-1}}, & \text{if } 2 \leq p < 3 \\
\rho^{p^2-1} (t + \theta)^{-\alpha} \left( \Gamma(t) \right)_{\frac{2}{p-1}}^{\frac{2}{p-1}} + e^{(2-p)|\lambda_2| H(t)}), & \text{if } 1 < p \leq 2, \\
\rho^{2p-1} (t + \theta)^{-\alpha} \left( \Gamma(t) \right)_{\frac{2}{p-1}}^{\frac{2}{p-1}} & \text{if } 2 \leq p < 3.
\end{array} \right.
\end{align*}
\]
(18)
When \( 1 < p \leq 2 \), we have
\[
\left| e^{(2-p)|\lambda_2| H(t)} \right| = \left| 1 + A \left( (t + \theta)^{1 - \frac{1}{2} (p-1)} - \theta^{1 - \frac{1}{2} (p-1)} \right) |y_+|^{p-1} + \eta_+ \right|_{\frac{2}{p-1}}^{\frac{2}{p-1}} \\
\leq C \left( 1 + \| \eta_+ \|_{\frac{2}{p-1}} + A \left( \frac{2}{p-1} \right) (t + \theta)^{1 - \frac{1}{2} (p-1)} |y_+|^{2-\alpha} \right) \\
\leq C \left( 1 + (t + \theta) \left( \frac{2}{p-1} \right)^{2(p-1)} \right)
\]
for small $\rho > 0$ and $A^{-1}$, and by (16) we get
\[ |\Gamma (t)|^{\frac{2}{p-1}} \leq C \left( 1 + (t + \theta)^{\frac{1}{p-1} - \frac{p}{2}} \right). \]

Therefore by (18)
\[ |\Gamma^{\frac{1}{p-1}} (t) - e^{i\lambda_2|H(t)|}| \leq C \begin{cases} \rho^{p-1} (t + \theta)^{\frac{1}{p-1} - \frac{p}{2}} (2-p)^{\alpha}, & \text{if } 1 < p \leq 2, \\
\rho^{\frac{p}{p-1}} (t + \theta)^{-\frac{p}{p-1}}, & \text{if } 2 \leq p < 3. \end{cases} \]

The lemma is proved.

We prove the asymptotic behavior of $\varphi (t) = \mathcal{F} u (-t - \theta) u (t)$.

\textbf{Lemma 4.3.} Let $u$ be the global solution constructed in Theorem 1.1. Then the following estimate
\[ \left\| \mathcal{F} u (-t - \theta) u (t) - e^{-i\lambda H(t)} y_+ \right\|_{L^\infty} \leq C \rho^{p-1} (t + \theta)^{\frac{2}{p-1} - \frac{p}{p-1} - \alpha} \]
holds for all $t > 0$, where $\alpha > 0$.

\textbf{Proof.} Since
\[ \varphi (t) = \mathcal{F} u (-t - \theta) u (t) = e^{-i\lambda f_0^{(\tau + \theta)}} e^{\frac{p}{2} (p-1) \varphi |^{p-1} d\tau} y (t) \]
we have
\[ \varphi (t) - e^{-i\lambda H(t)} y_+ = e^{-i\lambda f_0^{(\tau + \theta)}} e^{\frac{p}{2} (p-1) \varphi |^{p-1} d\tau} (y (t) - y_+) \]
\[ + \left( e^{-i\lambda f_0^{(\tau + \theta)}} e^{\frac{p}{2} (p-1) \varphi |^{p-1} d\tau} - e^{-i\lambda H(t)} \right) y_+. \tag{19} \]

Let us consider the second term of the right-hand side of (19). We have
\[ \left\| \left( e^{-i\lambda f_0^{(\tau + \theta)}} e^{\frac{p}{2} (p-1) \varphi |^{p-1} d\tau} - e^{-i\lambda H(t)} \right) y_+ \right\|_{L^\infty} \]
\[ \leq \left| e^{\frac{\lambda_2}{2} H(t)} - e^{\lambda f_0^{(\tau + \theta)}} e^{\frac{p}{2} (p-1) \varphi |^{p-1} d\tau} \right|_{L^\infty} \left\| e^{-\lambda f_0^{(\tau + \theta)}} e^{\frac{p}{2} (p-1) \varphi |^{p-1} d\tau} \right\|_{L^\infty} \]
\[ + |\lambda_1| \left\| \int_0^t (\tau + \theta)^{-\frac{2}{2} (p-1)} |\varphi| |^{p-1} d\tau - H (t) \right\|_{L^\infty} \left\| e^{-\lambda f_0^{(\tau + \theta)}} e^{\frac{p}{2} (p-1) \varphi |^{p-1} d\tau} \right\|_{L^\infty}. \tag{20} \]

By the mean value theorem, we find
\[ \left| \lambda_1 \right| \left\| \int_0^t (\tau + \theta)^{-\frac{2}{2} (p-1)} |\varphi| |^{p-1} d\tau - H (t) \right\|_{L^\infty} \]
\[ \leq \left| \lambda_1 \right| \left\| e^{\lambda f_0^{(\tau + \theta)}} e^{\frac{p}{2} (p-1) \varphi |^{p-1} d\tau} \right\|_{L^\infty}. \]

By (20), Lemma 4.2 and the estimate
\[ e^{-\lambda f_0^{(\tau + \theta)}} e^{\frac{p}{2} (p-1) \varphi |^{p-1} d\tau} \]
\[ \leq C (t + \theta)^{-\frac{p}{p-1} + \frac{p}{2}}, \]
we obtain
\[ \left\| e^{-i\lambda H(t)} (t) - e^{-i\lambda H(t)} y_+ \right\|_{L^\infty} \leq C\rho^{p-1} (t + \theta)^{\frac{3}{2} - \frac{1}{p-1} - \alpha}. \] (21)

The first term of the right-hand side of (19) is estimated by Lemma 4.1
\[ \left\| e^{-i\lambda H(t)} (t) - e^{-i\lambda H(t)} y_+ \right\|_{L^\infty} \leq \| y (t) - y_+ \|_{L^\infty} \leq C\rho^{p-1} (t + \theta)^{-\alpha}. \] (22)

By (19) -(22) we have the result of lemma.

**Proof of Theorem 1.2**: By Lemma 4.3, the first estimate of the theorem follows. The estimates of \( y_+ \) and \( \eta_+ \) are given in Lemmas 4.1 and 4.2, respectively.

5. **Proof of Theorem 1.3.** We have
\[ \left\| \mathcal{F} u (-t - \theta) y_+ \right\|_{L^\infty} \leq C\rho^{p-1} (t + \theta)^{\frac{3}{2} - \frac{1}{p-1} - \alpha}. \]

Since
\[ e^{-i\lambda H(t, \xi)} y_+ (\xi) = e^{-i\lambda_1 \log \left( 1 + A \left( (t + \theta)^{1-\frac{3}{2}(p-1)} - \theta^{1-\frac{3}{2}(p-1)} \right) \left| y_+ \right|^p + \eta_+ \right)}^{\frac{1}{p-1}} \times \left( 1 + A \left( (t + \theta)^{1-\frac{3}{2}(p-1)} - \theta^{1-\frac{3}{2}(p-1)} \right) \left| y_+ \right|^p + \eta_+ \right)^{-\frac{1}{p-1}} \]
we obtain
\[ \left| \mathcal{F} u (-t - \theta) y_+ (\xi) \right| \geq \frac{\left| y_+ (\xi) \right|}{\left( 1 + A \left( (t + \theta)^{1-\frac{3}{2}(p-1)} \left| y_+ \right|^p + \eta_+ \right) \right)^{\frac{1}{p-1}} - C\rho^{p-1} (t + \theta)^{\frac{3}{2} - \frac{1}{p-1} - \alpha}. \]

By Lemmas 4.1 and 4.2, we have
\[ \| y_+ \|_{L^\infty} \leq C\rho, \| \eta_+ \|_{L^\infty} \leq C\rho^p. \]

Therefore we find that there exists a positive constant \( C \) such that
\[ \left| \mathcal{F} u (-t - \theta) y_+ (\xi) \right| \geq \frac{C \left| y_+ (\xi) \right|}{\left( 1 + (t + \theta)^{1-\frac{3}{2}(p-1)} \right)^{\frac{1}{p-1}}}. \] (23)

By Lemma 4.1 we have the estimate
\[ \left| y_+ (\xi) \right| \geq \left| y (0, \xi) \right| - \left| y (0, \xi) - y_+ (\xi) \right| \geq \left| \varphi (0) \right| - C\rho^p, \]
where \( \varphi (t) = \mathcal{F} u (-t - \theta) u. \) If \( \inf |\tilde{\omega}_0| > \frac{p}{2}, \) then \( \inf |\varphi (0, \xi)| > \frac{p}{2}. \) Therefore we obtain
\[ \inf \left| y_+ (\xi) \right| > \frac{p}{4}. \]

Hence
\[ \left| \mathcal{F} u (-t - \theta) y_+ (\xi) \right| \geq \frac{C\rho}{4 \left( 1 + (t + \theta)^{1-\frac{3}{2}(p-1)} \right)^{\frac{1}{p-1}}} \geq \frac{C\rho}{4} (t + \theta)^{\frac{3}{2} - \frac{1}{p-1}}. \]
Uniqueness of solutions come from Lemma 2.3. This completes the proof of Theorem 1.3.

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