TensorPack: a Maple-based software package for the manipulation of algebraic expressions of tensors in general relativity

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Abstract. In this paper we: (1) introduce TensorPack, a software package for the algebraic manipulation of tensors in covariant index format in Maple; (2) briefly demonstrate the use of the package with an orthonormal tensor proof of the shearfree conjecture for dust. TensorPack is based on the Riemann and Canon tensor software packages and uses their functions to express tensors in an indexed covariant format. TensorPack uses a string representation as input and provides functions for output in index form. It extends the functionality to basic algebra of tensors, substitution, covariant differentiation, contraction, raising/lowering indices, symmetry functions and other accessory functions. The output can be merged with text in the Maple environment to create a full working document with embedded dynamic functionality. The package offers potential for manipulation of indexed algebraic tensor expressions in a flexible software environment.

1. Introduction
Since the 1970s there has been a steady increase in the use of algebraic computing for manipulating tensor equations, expressions and components. A summary some common packages is shown at [1]. In general many of these systems compute components and tensors related to general relativity (GR); fewer packages provide the facility to develop fluent algebraic arguments. In this context we introduce TensorPack, an indexed string-based system that generates tensor expressions in covariant formalism consistent with the major literature and texts [2-5].

The main features of the new TensorPack package include:

- an indexed-string based format for tensorial algebra
- presentation of tensors in standard 1+3 covariant format
- functions for many common algebraic operations on tensors
- mapping algebraic expressions to tensor components
- an interface with the functionality of the Riemann package (in addition to compatibility with other Maple packages eg. physics)
• flexibility to be used as both a research and an instructional tool
• a structure that can be used in text or computational styles, or as a combination

We discuss the main features in this paper (section 2) and demonstrate its use in the proof of the theorem $\sigma = 0 \Rightarrow \Theta = 0$ for dust, based on [6] (section 3). Full downloads of the TensorPack package, and proofs of the examples of the shearfree theorem can be found at [1]. The Riemann and Canon packages are described in [7,8].

2. Summary of the package

2.1. General
The current set of functions available in TensorPack are shown using the ‘with’ command:

```maple
with(TensorPack);
[Absorbd, Absorbg, CDF, CDS, PrintArray, PrintSubArray, T, TEDS,
TELS, TPset, anglesymm, antisymm, cod, contract, dotT, fpcodiff, raise, symm]
```

2.2. The programming format of TensorPack - the T() function
The essence of the package is that tensor expressions are stored as strings using an indexed structure. The T() processes the string for visual and computational output. For example, the input (>) line below followed by T(%) results in:

```maple
> u[a, -b] = (1/3)*theta*P[a, -b]+omega[a, -b]+sigma[a, -b]: T(%);
```

```maple
u[a b] = \frac{1}{3} \theta P[a b] + \omega[a b] + \sigma[a b]
```

The above example indicates that mixed tensors are represented by the combination of positive (contravariant) and negative (covariant) indices as required. In addition, a covariant derivative is formed by the use of an uppercase index. Furthermore, in the above and following examples, it will be seen that the T() function obeys the basic linear algebra of tensors.

2.3. Index operations: raising, lowering, contraction
Changing the type of indices simply involves the Maple substitution function. The sign of the index to be raised or lowered is reversed. The following example displays raising the index b, then contracting b with a:

```maple
> expr := X[-a, -b] = Y[-a,-b] + W[-a]*Z[-b]: T(%); expr2 := subs(b = -b, expr):T(%);
```

```maple
X[a a] = Y[a a] + W[a] Z[a]
```

2.4. Substitution of a tensor equation into a tensor expression
A tensor expression can be substituted as a string into the string representing another tensor, using the TEDS() function. For example, replacing the contracted tensor with a scalar in the previous equation:

```maple
> TEDS(X[-a, a] = X, expr3); T(%);
```

```maple
X = Y[a] + W[a] Z[a]
```

2.5. Covariant and time differentiation
The format of the covariant derivative of a tensor was introduced in section 2.2. An equation can be differentiated (in this case with respect to b, according to the Liebniz rule) using the cod() function, as follows:
>expr := X[-a, -b] = Y[-a, -b] + W[-a]*Z[-b]; cod(expr, -b); T(%);

Time differentiation is of significance in GR. The time derivative of a tensor A[a,b] is expressed in TensorPack as dotA[a,b]. It is obtained by the function dotT(). For an equation (using the Liebniz rule) we have the resulting time derivative of the previous expression:

>dotT(expr); T(%);

\[ \dot{X}^{ab} = \dot{Y}^{ab} + \dot{W}^{a}Z^{b} + W^{a}\dot{Z}^{b} \]

2.6. Symmetry functions:symm(),antisymm(),anglesymm()

Symmetric and antisymmetric tensors can be fully expressed for tensors with 4 indices or less. For example, the symmetric expression is found using symm():

>symm(X[a, b, c, d]; a, c); T(%);

\[ \frac{1}{6}X^{abc} + \frac{1}{6}X^{bca} + \frac{1}{6}X^{cab} + \frac{1}{6}X^{acb} + \frac{1}{6}X^{cba} + \frac{1}{6}X^{bac} \]

2.7. Absorbing the metric and kronecker delta

Tensor terms containing the metric or kronecker delta can usually be simplified:

>expr := delta[-a, b]*X[-b]+g[-b, -a]*X[b]; T(%);

\[ \delta^{a}_{\ b}X^{b} + g_{\ b\ a}X^{b} \]

>Absorbg(Absorbd(expr)); T(%);

\[ 2X^{a} \]

2.8. General functions & interfaces

It is possible to interface TensorPack with general functions in many ways:

a. The output of a list of expressions. eg. outputting a sublist of equations, uses the PrintSubArray command:

b. The general maple file and interface functions can be applied, including save, read, printing, copying and pasting code etc...

c. Displaying components of tensor expressions, usually after setting the dimension, defining coordinates and the metric.

For a complete description of these functions, see [1].

3. Demonstration of a theorem proof using TensorPack: shearfree conjecture for dust

In this section we present a brief example of the use of the package by discussing a sample proof of the shearfree conjecture \( \sigma = 0 = \omega \Theta = 0 \) for dust as given by Senovilla et al. [6]. Our version of the proof follows the same sequential order of proofs as for the original authors. The complete set of proofs using TensorPack is available at [1] but is too long to show in complete form in this document. Instead we demonstrate the approach, and the use of the software, in the following short example.

Example. Proof of equation 11b of [6] (Vorticity vector and tensor are orthogonal)

We proceed to give a proof of the identity \( \omega^{a}_{\ b} \omega^{b}_{\ a} = 0 \)

Consider the left side of the identity:

>eq1 := omega[-a, -b]*omega[b]; T(%);

\[ \omega^{a}_{\ b} \omega^{b}_{\ a} \]

We commence by substituting:
\[ eq2 := \omega[a] = (1/2) \eta[a, b, c, d]*u[-b]*\omega[-c, -d];T(\%); \]

an identity for the vorticity vector into expression (1). We first re-arrange the indices in (2) to facilitate the substitution:

\[ eq3 := \text{subs}(b = e, c = f, d = g, a = b, eq2);T(\%); \]

Completing the substitution, using the TEDS command in TensorPack, we find the expression:

\[ eq4 := \text{expand}(\text{TEDS}(eq3, eq1));T(\%); \]

In a similar way we substitute the following expression for the vorticity tensor into expression (1):

\[ eq5 := \omega[-a, -b] = \eta[-a, -b, -c, -d]*u[d]*\omega[c];T(\%); \]

to obtain:

\[ eq6 := \text{expand}(\text{TEDS}(eq5, eq4));T(\%); \]

The skew pseudotensor is fully antisymmetric, so the expression (6) is equivalent to:

\[ eq7 := \text{expand}(\text{TEDS}(\eta[-a, -b, -c, -d] = -\eta[-b, -a, -c, -d], eq6));T(\%); \]

and hence we have, equating expressions (1) and (7), we obtain equation (8):

\[ eq8 := \text{eq1 = eq7};T(\%); \]

Now we substitute an identity for the skew pseudotensor into (8) which then yields:

\[ eq9 := \text{expand}(\text{TEDS}(\eta[b, e, f, g]*\eta[-b, -a, -c, -d] = -6*\text{antisymm}(\delta[e, -a]*\delta[f, -c]*\delta[g, -d], e, g), eq8));T(\%); \]

Next we use the Absorbd function of TensorPack to simplify terms with kronecker delta:

\[ eq10 := \text{Absorbd}(\text{Absorbd}(\text{Absorbd}(eq9)));T(\%); \]

Using \( u[d]*u[-d] = -1 \) we obtain:

\[ eq11 := \text{expand}(\text{TEDS}(u[d]*u[-d] = -1, eq10));T(\%); \]
and the repeated use of the orthogonality of the velocity and vorticity (in three sets involving the TEDS command) yields:

\begin{align*}
\text{eq12a} & := \text{expand(TEDS}(u[d] \ast \text{omega}[-a, -d] = 0, \text{eq11})); -1; \\
\text{eq12b} & := \text{expand(TEDS}(u[d] \ast \text{omega}[-c, -d] = 0, \text{eq12a})); -1; \\
\text{eq12c} & := \text{expand(TEDS}(u[d] \ast \text{omega}[-d, -a] = 0, \text{eq12b})); -1; \\
\text{eq12d} & := \text{expand(TEDS}(u[d] \ast \text{omega}[-d, -c] = 0, \text{eq12c})); -1;
\end{align*}

\begin{align*}
\text{T(12)}
\end{align*}

\begin{align*}
\omega_{a\ b} \omega^b &= -\frac{1}{2} \omega_{a\ c} \omega^c + \frac{1}{2} \omega_{c\ a} \omega^c
\end{align*}

The asymmetry of the vorticity tensor gives

\begin{align*}
\text{eq13} & := \text{expand(TEDS}(\text{omega}[-c, -a] = -\text{omega}[-a, -c], \text{eq12d})); \text{T(13)};
\end{align*}

\begin{align*}
\omega_{a\ b} \omega^b = -\omega_{a\ c} \omega^c
\end{align*}

Finally, switching dummy indices yields:

\begin{align*}
\text{eq14} & := \text{subs}(c = b, \text{eq13}); \text{T(14)};
\end{align*}

\begin{align*}
\omega_{a\ b} \omega^b = -\omega_{a\ b} \omega^b
\end{align*}

and

\begin{align*}
\text{eq15} & := \text{eq14-omega}[-a, -b] \ast \text{omega}[b]; \text{T(15)};
\end{align*}

\begin{align*}
0 = -2 \omega_{a\ b} \omega^b
\end{align*}

which completes the proof of the example (equation 11b of [6]).

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**References**

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