PROBABILISTIC DIVIDE-AND-CONQUER: DETERMINISTIC SECOND HALF

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Abstract. We present a method for exact sampling of distributions of the form

$$\mathcal{L}(X_1, \ldots, X_n | E),$$

where random variables $X_i, i = 1, \ldots, n$, can be sampled independently subject to membership in an event $E$ with mild assumptions. Our method includes cases where the event $E$ has probability 0.

1. Introduction

Probabilistic divide-and-conquer (PDC) is an exact sampling technique first explored in [9] that divides a target object into two parts, samples each part separately, and then combines them to form an exact sample from a target distribution. A large class of target distributions, as well as a particular class of divisions, is described below.

For each integer $n \geq 1$, let $X = (X_1, X_2, \ldots, X_n)$ denote an $\mathbb{R}^n$–valued stochastic process, with coordinates independent random variables with known marginal distributions $\mathcal{L}(X_1), \mathcal{L}(X_2), \ldots, \mathcal{L}(X_n)$; denote by $\mathcal{F}$ the Borel $\sigma$-algebra of measurable sets on $\mathbb{R}^n$. Given a set $E_n \in \mathcal{F}$, we define the distribution of $X'_n$ as

$$\mathcal{L}(X'_n) := \mathcal{L}\left(\left(X_1, X_2, \ldots, X_n \mid X \in E_n\right)\right).$$

Many random sampling problems of interest can be described in this manner; see for example [4, 3, 16, 13].

Goal: Generate exact samples as “efficiently” as possible from $\mathcal{L}(X'_n)$.

This goal of exact sampling intentionally excludes many popular techniques that provide approximate samples to a target distribution in finite time; see for example [26, 17]. We again emphasize that the algorithms presented are exact sampling methods in finite time for finite values of the parameters. Thus, any reference to a sampling algorithm should be interpreted as exact sampling.

Our view of “efficient” is the same as that specified in the introduction in [13]. In particular, we address the following practical problem: how much memory and time are required to sample from $\mathcal{L}(X'_n)$? We will assume that the main measure of an algorithm’s time is the number of calls to a random number generator which produces random values in the range $(0, 1)$; thus, we assume that arithmetic is largely negligible, and that precision is fixed. We also assume that all numbers occupy the same amount of space, regardless of
the magnitude. Finally, since the run times of our algorithms will themselves be random variables, most with unbounded ranges, we only consider the expected amount of time\(^3\) of an algorithm.

To simplify notation, when the values of the parameters are understood from context we adopt the conventions \(X' \equiv X'_n, E \equiv E_n, X \equiv X_n \equiv (X_1, \ldots, X_n), [n] = \{1, \ldots, n\}\). In order to describe the class of divisions, we now state the following definitions and assumptions.

**Definition 1.1.** For any subset of indices \(I \subset \{1, \ldots, n\}\),

1. let \(X_I = \pi_I(X) = (X_i)_{i \in I}\) denote the \(\mathbb{R}^{|I|}\)-valued projection;
2. let \(X^{(I)} = \pi_{[n]\setminus I}(X)\) denote the \(\mathbb{R}^{n-|I|}\)-valued projection;
3. define the \(I\)-completable-set of \(E_n\) as
   \[E^{(I)} := \pi_{[n]\setminus I}^{-1}(E_n) = \{x \in \mathbb{R}^{n-|I|} : \exists y \in \mathbb{R}^{|I|} \text{ such that } (y, x) \in E_n\};\]
4. define the \(I\)-section of \(E_n\) given \(x^{(I)} \in E^{(I)}\) as
   \[E_I \equiv E_{I[x^{(I)}]} := \pi_I^{-1}(E_n | x^{(I)}) = \{y \in \mathbb{R}^{|I|} : (y, x^{(I)}) \in E_n\}.\]

**Assumptions.** In what follows, the set \(I\) denotes some subset of indices \(I \subset \{1, \ldots, n\}\), \(T\) and \(T_I\) denote random variables, and \(k \in \text{range}(T)\) is a non–random scalar.

A1) \(\{X \in E\} = \{T = k\};\)
A2) \(\mathbb{P}(X^{(I)} \in E^{(I)}) > 0;\)
A3) \(\mathcal{L}(T_I) \equiv \mathcal{L}(T_I | X^{(I)}) = \mathcal{L}(T | X^{(I)});\)

D1) \(T\) is a discrete random variable and \(\mathbb{P}(T = k) > 0;\)
D2) \(T_I\) is discrete for each \(X^{(I)} \in E^{(I)};\)

C1) \(T\) has a density, denoted by \(f_T\), and \(0 < f_T(k) < \infty;\)
C2) \(T_I\) has a density for each \(X^{(I)} \in E^{(I)};\)

Note that when \(\mathbb{P}(X \in E) > 0\), we can let \(T = 1(X \in E)\) and \(k = 1\), which satisfies assumption (D1). Also, even though \(T_I\) depends on \(X^{(I)}\), when \(X^{(I)}\) can be inferred from context we prefer the shorter \(T_I\).

### 2. Hard rejection sampling and PDC

Suppose \(I_A\) and \(I_B\) partition \(\{1, \ldots, n\}\). We define random variables \(A, B, U, V\) with distributions given by

\[
\mathcal{L}(A) = \mathcal{L}(X_{I_A}), \quad \mathcal{L}(B) = \mathcal{L}(X_{I_B}), \quad \mathcal{L}(U) = \mathcal{L}(A | X \in E_n), \quad \mathcal{L}(V | U = a) = \mathcal{L}(B | X \in E_n, U = a).
\]

\(^3\)Since the algorithms are probabilistic, there may exist outcomes for which the run–times are arbitrarily large, but only with increasingly small probability. We shall only consider algorithms which for each fixed value of \(n\) have finite run times with probability 1.
A class of algorithms known as \textit{rejection sampling} refers to any algorithm which utilizes von Neumann’s approach to sampling from a distribution, see [29]. When the rejection probability is given by 0 or 1, depending on whether the sample lies within a certain region, we call this \textit{hard rejection sampling}. When the rejection probability is given by some $0 \leq \alpha \leq 1$, which typically depends on the particular sample generated, we call this \textit{soft rejection sampling}.

The most generic approach to sampling from the distribution (1) is hard rejection sampling, given by Algorithm 1. The PDC random sampling algorithm is given by Algorithm 2.

\textbf{Algorithm 1} Hard rejection sampling

1. Generate sample from $\mathcal{L}(A)$, call it $a$.
2. Generate sample from $\mathcal{L}(B)$, call it $b$.
3. Check if $X \in E$; if so, return $(a, b)$, otherwise restart.

\textbf{Algorithm 2} Probabilistic Divide-and-Conquer

1. Generate sample from $\mathcal{L}(U)$, call it $a$.
2. Generate sample from $\mathcal{L}(V | U = a)$ call it $b$.
3. Return $(a, b)$.

The PDC Lemma [2, Lemma 2.2] affirms that the resulting pair $(a, b)$ from Algorithm 2 is an exact sample from $\mathcal{L}(X_n')$ when $P(X \in E_n) > 0$. The lemma below is a specialization to the parameterization defined above when the random variables $T$ and $T_I$ are discrete.

\textbf{Lemma 2.1.} Assume (A1)--(A3), (D1), (D2). Let

$$
\mathcal{L}(U) = \mathcal{L}\left(X^{(I)} | X \in E \right), \quad \mathcal{L}\left(V | U = x^{(I)} \right) = \mathcal{L}\left(T | X^{(I)} = x^{(I)}, X \in E \right).
$$

Then $(U, V) =^d X_n'$.

Since $T$ is discrete, we have $P(X \in E) = P(T = k) > 0$. It was pointed out in [2] that the assumption $P(X \in E) > 0$ is not necessary, but makes for a simpler exposition, since in general the conditioning may be ill-defined. Indeed, we now enlarge this set of events to those for which $P(X^{(I)} \in E^{(I)}) > 0$, and some type of regularity condition holds on the distributions $\mathcal{L}(T)$ and $\mathcal{L}(T_I)$. The regularity conditions allow us to state an entirely new lemma applicable to continuous random variables with densities.

\textbf{Lemma 2.2.} Assume (A1)--(A3), (C1), (C2). Let

$$
\mathcal{L}(U) = \mathcal{L}\left(X^{(I)} | X \in E \right), \quad \mathcal{L}\left(V | U = x^{(I)} \right) = \mathcal{L}\left(T | X^{(I)} = x^{(I)}, X \in E \right).
$$

Then $(U, V) =^d X_n'$.
Under the assumed regularity conditions, we have
\[ S_\delta = \max_{T \in \mathbb{Z}} \left( t \in [k, k + \delta] \right), \]
whence \( \mathbb{E} h_\delta(X_I, X^{(I)}) = \mathbb{P}(G_\delta) > 0. \) We also define for each \( x^{(I)} \in E^{(I)} \) the set \( D_{\delta, x^{(I)}} = \{ T_I \in [t_I, t_I + \delta] \}, \) hence \( \mathbb{E} h_\delta(X_I, x^{(I)}) = P(D_{\delta, x^{(I)}}) > 0. \) Since we assume \( T_I \) has a density, it follows that
\[ \lim_{\delta \searrow 0} \mathbb{E} h(X_I, x^{(I)}) = f_{T_I|x^{(I)}}(t_I). \]

When \( T \) or \( T_I \) are vector–valued, we can interpret the intervals of the form \( [a, a + \delta] \) as cartesian products of intervals \( [a_1, a_1 + \delta_1] \times \ldots \times [a_d, a_d + \delta_d] \) in \( \mathbb{R}^d \) for \( d > 1 \), and let \( \delta = \max(\delta_1, \ldots, \delta_d). \)

Next, we define the random variables \( A = d X_I, B = d X_I, \) and \( \tilde{U}_{n, \delta} \equiv \tilde{U}, \tilde{V}_{n, \delta} \equiv \tilde{V}, \tilde{S}_{n, \delta} \equiv \tilde{S}, \) with distributions
\[ \mathcal{L}(\tilde{U}) = \mathcal{L}(A | h_\delta(A, B) = 1), \]
\[ \mathcal{L}(\tilde{V} | \tilde{U} = a) = \mathcal{L}(B | h_\delta(a, B) = 1), \]
\[ \mathcal{L}(\tilde{S}) = \mathcal{L}(A, B | h_\delta(A, B) = 1). \]

Under the assumed regularity conditions, we have
\[ P(U \in da) = P(A \in da | h(A, B) = 1) = \lim_{\delta \searrow 0} \mathbb{P}(A \in da; h_\delta(A, B) = 1) = \lim_{\delta \searrow 0} \mathbb{P}(\tilde{U} \in da), \]
whence \( \mathcal{L}(U) = \lim_{\delta \searrow 0} \mathcal{L}(\tilde{U}). \) Similarly, we have \( \mathcal{L}(V | U = a) = \lim_{\delta \searrow 0} \mathcal{L}(\tilde{V} | \tilde{U} = a) \) and \( \mathcal{L}(S) = \lim_{\delta \searrow 0} \mathcal{L}(\tilde{S}). \) Then, for any bounded measurable \( g : A \times B \to \mathbb{R}, \) we have
\[ \mathbb{E} g(U, V) = \mathbb{E} (\mathbb{E} g(U, V) | U) = \lim_{\delta \searrow 0} \mathbb{E} \left( \mathbb{E} g(\tilde{U}, \tilde{V}) | \tilde{U} \right) = \lim_{\delta \searrow 0} \mathbb{E} g(\tilde{S}) = \mathbb{E} g(S). \]

The third equality follows by [2, Lemma 2.2] since the conditioning event has positive probability. \( \square \)

**Remark 2.1.** Note that although \( T \) and \( T_I \) are required to satisfy regularity conditions, there is no such requirement on the distribution of \( X^{(I)}. \)

**Remark 2.2.** We can also mix and match regularity conditions coordinate–wise, as long as each coordinate of \( T \) and \( T_I \) satisfy the regularity conditions.

When \( E_n \) has positive probability, we can always use the hard rejection sampling algorithm given in Algorithm 1, although the number of times to restart the algorithm may be impractical.

In the case when \( E_n \) has probability 0, we are not aware of the existence of any such hard rejection sampling algorithm in general. We are aware of several ad hoc approaches for particular forms of the random variables and conditioning events, for example sampling from the surface of an \( n \)–sphere (see [11, 5]), or convex polytope sampling (see [10, 15]), but nothing as general as hard rejection sampling.
At this point we also remind the reader that we are not considering algorithms which generate approximate distributions in finite time, such as standard Markov chains. There is one Markov chain approach we are aware of, coupling from the past [24], which is an exact sampling method; though, as far as we are aware, in order to apply it practically requires either a complete enumeration of the objects or some kind of monotonicity, which is not required for PDC.

3. PDC deterministic second half

In addition to assumptions (A1)–(A3), we now include assumption (DSH), which makes Stage 2 in Algorithm 2 deterministic, i.e., uniquely determined by the conditioning events.

(DSH) For each \( x^{(i)} \in E^{(i)} \), \( |E_{x^{(i)}}| = |\{y_I\}| = 1 \).

The assumption (DSH) stands for deterministic second half, and is not required for PDC in general, nor for Lemma 2.1 or Lemma 2.2, yet it provides a highly desirable simplicity in the algorithms that follow without requiring detailed information about the distributions.

Remark 3.1. It is possible that \( L(T_I | X^{(I)}, X \in E) \) is trivial but \( X_I \) is not; see Section 7.5. In general, \( T_I = g(X_I) \) for some non–random function \( g \), not necessarily one-to-one, which depends on how \( T \) depends on \( X_I \) and \( X^{(I)} \). Again, we remind the reader that the regularity assumptions are placed on \( T_I \) and not on \( X_I \) directly.

We now fix the method of sampling from the conditional distribution \( L(X^{(I)} | E_n) \) in Stage 1 to be soft rejection sampling using the distribution \( L(X^{(I)}) \); ergo, as we shall see shortly in the continuous case, we must also impose the added assumption

(C2') For each \( x^{(I)} \in E^{(I)} \), the density of \( T_I \) is bounded.

To summarize, we suppose that for each \( n \geq 1 \), there exists some subset of indices \( I \) such that rejection sampling of points \( X^{(I)} \in E^{(I)} \) terminates in finite time, and that for each \( x^{(I)} \in E^{(I)} \), there exists a unique completion, say \( y_I \in E_I \), generated from a distribution satisfying sufficient regularity conditions, such that \( (y_I, x^{(I)}) \) lies in the target set \( E_n \).

In what follows, \( u \) will denote a uniform random variable between 0 and 1, independent of all other random variables.

Theorem 3.1. Algorithm 3 samples from the distribution given in Equation (1).

Proof. The algorithm is an application of soft rejection sampling. Rather than sample directly from the conditional distribution \( L(X^{(I)} | E_n) \), which we assume is not possible, we instead sample from \( L(X^{(I)}) \) and reject with the right proportion. We now demonstrate that \( \frac{P(X_I = y_I)}{\max_{x_I} P(X_I = x_I)} \) is the right proportion.

We let \( f \) denote the joint mass function of \( L(X^{(I)} | E_n) \), and \( g \) denote the joint mass function of \( L(X^{(I)}) \). Our rejection proportion is of the form: suppose we observe \( j \) under the distribution \( g \), then we reject if

\[ u > \frac{f(j)}{Cg(j)} \]
Algorithm 3 Discrete PDC DSH

procedure DISCRETE_PDC_DSH($X_1, X_2, \ldots, X_n, E, I$)

Assumptions: (A1)–(A3), (D1), (D2), (DSH)

Sample from $\mathcal{L}(X(I))$, denote the observation by $x^{(i)}$.

if $x^{(i)} \in E(I)$ and $u < \frac{P(X_I = y_I)}{\max_{\ell} P(X_I = \ell)}$ then

return $(x^{(i)}, y_I)$

else

restart

end if

end procedure

Algorithm 4 Continuous PDC DSH

procedure CONTINUOUS_PDC_DSH($X_1, X_2, \ldots, X_n, E, I$)

Assumptions: (A1)–(A3), (C1), (C2), (C2'), (DSH)

Sample from $\mathcal{L}(X(I))$, denote the observation by $x^{(i)}$.

if $x^{(i)} \in E(I)$ and $u < \frac{f_{T_I}(t_I)}{\sup_{\ell} f_{T_{I\ell}}(\ell)}$ then

return $(x^{(i)}, y_I)$

else

restart

end if

end procedure

where $C$ is any constant such that

$$f(\ell) \leq C g(\ell), \quad \text{for all } \ell.$$ 

The quantity $C$ is the expected number of iterations of the acceptance/rejection procedure before we accept a sample, see for example [10], and so in particular we would like to find the smallest $C$. Since our distributions are already specified, we obtain

$$\frac{1}{C} = \min_{\ell} \frac{g(\ell)}{f(\ell)} = \min_{\ell} \frac{\mathbb{P}(T = k)}{\mathbb{P}(T = k | X^{(I)} = \ell) \max_{\ell} \mathbb{P}(T_I = t_I(\ell) | X^{(I)} = \ell)},$$

and our rejection step reduces to

$$u > \frac{f(j)}{C g(j)} = \frac{\mathbb{P}(T_I = t_I(j))}{\max_{\ell \in E_{I(j)}} \mathbb{P}(T_I = \ell)}.$$ 

By assumption (DSH), once we accept $x^{(i)}$, the completion $y_I$ is unique. Since $T_I$ is assumed discrete, we have

$$\frac{\mathbb{P}(T_I = t_I(j))}{\max_{\ell} \mathbb{P}(T_I = \ell)} = \frac{\mathbb{P}(X_I = y_I(j))}{\max_{\ell} \mathbb{P}(X_I = \ell)}.$$ 

□
Theorem 3.2. Algorithm 4 samples from the distribution given in Equation (1).

Proof. The proof is the same application of soft rejection sampling, with probabilities replaced with probability density functions where appropriate. We have

$$\frac{1}{C} = \min_{\ell} \frac{g(\ell)}{f(\ell)} = \min_{\ell} \frac{f_T(k)}{f_{T|I}(t_I(\ell))} = \max_{\ell} \frac{f_T(k)}{f_{T|I}(t_I(\ell))},$$

and the rejection step reduces to

$$u > \frac{f(j)}{Cg(j)} = \frac{f_{T|I}(t_I(j))}{\max_{\ell} f_{T|I}(t_I(\ell))}.$$

Note that while our rejection probability is in terms of the distribution $L(T|X = x)$, we return the value $y_I$, which is the corresponding value in the range of $X_I$. □

Remark 3.2. There is a key difference between the discrete and continuous PDC deterministic second half algorithms. When $X_I$ and $T_I$ are discrete, the random variables can be rearranged before the soft rejection step from $\{T_I = t_I(x_I)\}$ to $\{X_I = y_I(x_I)\}$, whereas when $X_I$ and $T_I$ are continuous, one must first appeal to a standard change of variables formula to determine the conditional distribution $L(T|X)$, and then after a sample is accepted one can find the value of $y_I$ corresponding to $t_I$; see for example [25] and our discussion in Section 5.

Remark 3.3. The PDC deterministic second half algorithm is a soft rejection sampling algorithm.

In fact, not only is PDC deterministic second half itself a soft rejection sampling algorithm, it is easily shown to have a lower rejection rate than hard rejection sampling. The hard rejection sampling acceptance condition is $\{X \in E\}$, which for $X_I$ discrete can be written as

$$\{X(I) \in E(I) \text{ and } u < P(X_I = y_I)\}.$$

The PDC deterministic second half acceptance condition for discrete random variables is

$$(2) \quad \left\{ X(I) \in E(I) \text{ and } u < \frac{\mathbb{P}(X_I = y_I)}{\max_{\ell} \mathbb{P}(X_I = \ell)} \right\}.$$

It is easy to see that there exists a coupling of the random variables such that all events giving an acceptance in the hard rejection sampling algorithm would also be accepted in the PDC deterministic second half algorithm. The added efficiency in this case comes from the use of soft rejection sampling, which enlarges the space by the factor $\max_{\ell} \mathbb{P}(X_I = \ell)^{-1}$.

For continuous random variables, PDC deterministic second half transforms the otherwise infinite-time hard rejection sampling algorithm into an acceptance condition of the form

$$(3) \quad \left\{ X(I) \in E(I) \text{ and } u < \frac{f_{T|X(I)}(y_I)}{\sup_{\ell} f_{T|X(I)}(\ell)} \right\},$$

which is an event of positive probability as long as the density $f_{T|X(I)}$ is bounded.
4. Speedup Analysis

All of the algorithms presented require the same order of magnitude amount of memory as hard rejection sampling, hence we focus solely on run–time. We assume that arithmetic operations are negligible, whereas the cost of generating a single random uniform variate from a given interval is $O(1)$, and the cost of generating $n$ independent uniform random variables from a given interval is $O(n)$, regardless of the magnitudes of the values.

**Definition 4.1.** For a given algorithm $P$, which generates a sample from a distribution $\mathcal{L}(X)$, we denote the expected time of completion by $\text{Time}_P(X)$. When there is no subscript, we assume by default there is a direct sampling method available.

When $P$ is the hard rejection sampling algorithm, we denote the expected time of completion by $\text{Time}_{\text{rej}}(X')$. The speedup of $P$ relative to hard rejection sampling is defined by

\begin{equation}
\text{speedup} := \frac{\text{Time}_{\text{rej}}(X')}{\text{Time}_P(X')},
\end{equation}

**Proposition 4.1.** We have

\[ \text{Time}_{\text{rej}}(X') = O\left(\frac{\text{Time}(X)}{\mathbb{P}(X \in E)}\right). \]

**Remark 4.1.** For all algorithms $P$, we have speedup $\in (0, \infty)$, with speedup $> 1$ representing an improvement to hard rejection sampling.

**Theorem 4.1.** Assume $(A1)$–$(A3)$, $(D1)$, $(D2)$, $(DSH)$. Let $\text{Time}_{\text{DSH}}(X')$ denote the discrete PDC deterministic second half algorithm given in Algorithm $\text{E}$. We have

\[ \text{Time}_{\text{DSH}}(X') = O\left(\text{Time}_{\text{rej}}(X') \max_j \mathbb{P}(X_I = j)\right), \]

whence

\[ \text{speedup} = \mathcal{O}\left(\left(\max_j \mathbb{P}(X_I = j)\right)^{-1}\right). \]

Thus, the optimal choice of $I$ is one that minimizes the maximal point mass in the distribution of $X_I$.

**Proof.** Recall the optimal value of $C$ is given by

\[ C = \max_j \frac{\mathbb{P}(X_I = j)}{\mathbb{P}(T = k)}. \]

The cost of this algorithm with the optimal $C$ is then

\[ \text{Time}(X^{(t)}) C = \mathcal{O}\left(\frac{\text{Time}(X^{(t)})}{\mathbb{P}(T = k)} \max \mathbb{P}(X_I = j)\right) = \mathcal{O}\left(\text{Time}_{\text{rej}}(X') \max_j \mathbb{P}(X_I = j)\right). \]
which by Equation (4) implies the speedup is $O(\max_j \mathbb{P}(X_I = j))^{-1}$. □

**Theorem 4.2.** Assume (A1)–(A3), (C1), (C2), (C2'), (DSH). Let $\text{Time}_{C\text{DSH}}(X')$ denote the continuous PDC deterministic second half algorithm given in Algorithm 4. We have

$$\text{Time}_{C\text{DSH}}(X') = O\left(\text{Time}_{\text{rej}}(X^{(I)}|E^{(I)}) \sup_{\ell} f_T(\ell)\right).$$

**Proof.** The optimal value of $C$ is given by

$$C = \sup_{\eta, \ell} f_T(\eta).$$

The cost of this algorithm with the optimal $C$ is then

$$\text{Time}(X^{(I)}) C = O\left(\text{Time}(X^{(I)}) \sup_{\eta, \ell} f_T(\eta)\right)$$

$$= O\left(\text{Time}_{\text{rej}}(X^{(I)}|E^{(I)}) \sup_{\eta, \ell} f_T(\eta)\right).$$ □

5. An illustrative example

An example which demonstrates that care must be taken when conditioning on events of probability 0 is given by the following example of [25] (see also [7, Section 4.9.3]). The problem stated on a particular exam is as follows: If $X$ and $Y$ are independent standard normals, what is the conditional distribution of $Y$ given that $Y = X$? The three ways in which this event was interpreted were as follows

1. $((X, Y) | X - Y = 0)$;
2. $((X, Y) | \frac{X}{Y} = 1)$;
3. $((X, Y) | 1(X = Y))$.

In our notation, this is the same as $((X, Y) | T = k)$, where

1. $T = d Y - X$ and $k = 0$;
2. $T = d X/Y$ and $k = 1$;
3. $T = d 1(X = Y)$ and $k = 1$.

To apply Algorithm 4, we first sample $X$ from a standard normal distribution, and apply a rejection depending on the distribution $\mathcal{L}(T_I | X^{(I)} = a)$; we have

1. $T_I = d Y - a$, reject if $u > e^{-a^2/2}$;
2. $T_I = d a/Y$, reject if $u > |a| e^{-a^2/\sqrt{2}e}$;
3. $T_I = d 1(Y = a)$. Not applicable.

Note that the last case, $T = 1(X = Y)$, does not satisfy either of (D1),(D2) or (C1),(C2); that is, it is not an event of positive probability, nor is it a random variable with a density.

This example also serves to demonstrate that PDC is not simply rejection sampling. Rather than applying Algorithm 4, we can instead determine the conditional distribution
(Y(T = k), as the original problem demands, sample according to that distribution directly, and then appeal to Lemma 2.2.

(1) \( T_1 = Y - a, \) \( f_{Y|T=0}(y) = e^{-y^2}/\sqrt{\pi}, \ -\infty < y < \infty; \)

(2) \( T_1 = a/Y, \) \( f_{Y|T=1}(y) = |y|e^{-y^2}, \ -\infty < y < \infty; \)

(3) \( T_1 = 1(Y = a). \) \( f_{Y|T=1}(y) = e^{-y^2}/\sqrt{2\pi}, \ -\infty < y < \infty. \)

Note that, despite the fact that we can specify the conditional distribution \( \mathcal{L}(Y|T = 1) \) in item (3) above, Lemma 2.2 does not apply directly.

6. Applications to Combinatorial Classes

6.1. Table Methods. In any discussion of random sampling of combinatorial structures, invariably one is led to the recursive method of Nijenhuis and Wilf [19], which uses a table of values to calculate conditional probability distributions based on recursive properties of the random variables, and it is generally applicable to random combinatorial structures defined recursively. This method has several costs:

1. Computational cost to create the table
2. Storage cost to store the table
3. Computational cost to generate samples from the table.

If one is capable of handling items 1 and 2, then the table methods are by far the fastest known methods for random generation, since they provide fast lookups equivalent to un-ranking algorithms; a good survey of these and other similar algorithms is [27].

We do not pursue this approach in this paper, namely because the size of the table typically grows much larger than the memory requirements for the random sample, and our algorithms are designed to push the limits of what can be effectively stored on a personal computer. For integer partitions, for example, to store a random partition is asymptotically of order \( \sqrt{n} \), whereas the table would need to be at least asymptotically of order \( n^{3/2}/\ln(n) \).

6.2. Assemblies, Multisets, and Selections. Let \( w : \mathbb{N} \to \mathbb{R} \) denote a weighting function, and define the sequence \( w_i := w(i), i \geq 1, \) so that we may interpret \( a \cdot b \) as the usual dot product. For the remainder of this section, in order to keep the notation in [4], we define \( \mathbf{Z} \equiv \mathbf{X} \) and let \( T = \mathbf{Z} \cdot w. \) In the examples that follow, the process \( \mathbf{Z} \) is discrete, hence the event \( \{T = k\} \) has strictly positive probability for each \( k \in \text{range}(T). \)

There are many combinatorial stochastic processes that can be described in this fashion. In [4], a unified framework is presented for three main types of combinatorial structures. We recount some of the basic definitions, and refer the interested reader to [4] for a more inspiring exposition.

For \( a = (a_1, a_2, \ldots, a_n) \) a sequence of nonnegative integers, and weights \( w = (w_1, \ldots, w_n), \) let \( N(n, a, w) \) denote the number of combinatorial objects of weight \( n \) having \( a_i \) components of size \( w_i, i = 1, \ldots, n. \) Since each component has size \( w_i \), the total contribution to the weight of the object by component \( i \) is \( w_i a_i \), and summing over all \( i \) gives us the total weight \( n = \sum_{i=1}^{n} w_i a_i \) of the object. Suppose \( J \subset \{1, \ldots, n\}. \) The examples of interest
will have the following form:

$$N(n, a, w) = 1(w \cdot a = n) f(J, n) \prod_{i \in I} g_i(a_i),$$

for some functions $f$ and $g_i, i \in J$, with

$$p(n) = \sum_{a \in \mathbb{Z}_+^n} N(n, a, w)$$

denoting the total number of objects of weight $n$. We now suppose that our combinatorial objects are chosen uniformly at random. Then the number of components of size $i$ is a random variable, say with distribution $C_i, i = 1, \ldots, n$, and $C = (C_1, \ldots, C_n)$ is the dependent stochastic process of random component sizes that satisfies $C \cdot w = n$. The distribution of $C$ is given by

$$\mathbb{P}(C = a) = 1(w \cdot a = n) f(J, n) \prod_{i \in J} g_i(a_i).$$

For each $x > 0$, let independent random variables $Z_i, i \in J$, have distributions

$$\mathbb{P}(Z_i = k) = c_i(x) g_i(k) x^{w_i k}, \quad i \in J,$$

where $c_i, i \in J$, are the normalization constants, given by

$$c_i = \left( \sum_{k \geq 0} g_i(k) x^{w_i k} \right)^{-1}.$$ 

Now we can state the following theorem.

**Theorem 6.1.** Assume $J \subset \{1, \ldots, n\}$. Let $Z_J = (Z_i)_{i \in J}$ denote a vector of independent random variables with distributions given by Equation (6). Let $C_J = (C_i)_{i \in J}$ denote the stochastic process of random component sizes with distribution given by Equation (5). Then

$$C_J = (Z_J | T = n).$$

Furthermore,

$$\mathbb{P}(T = n) = \frac{p(n)}{f(J, n)} x^n \prod_{i \in J} c_i(x).$$

The hard rejection sampling algorithm for such combinatorial classes has cost

$$\text{Time}_{\text{rej}}(Z_J) = O\left( \text{Time}(Z_J) \frac{f(J, n)}{p(n)} x^{-n} \prod_{i \in J} c_i(x)^{-1} \right).$$

The form of the condition $\{T = n\}$ implies that the PDC deterministic second half algorithm has $|I| = 1$, i.e., $I$ consists of a single index. By Theorem 4.1 we have for any $i \in J,$

$$\text{speedup} = O\left( \max_k c_i(x) g_i(k) x^{w_i k} \right)^{-1}.$$
The optimal choice of $i$ is one that minimizes this maximal probability. In fact, any choice of $i$ will provide a speedup, which is an even more compelling reason to use PDC.

6.3. Example: Integer Partitions. A random unrestricted integer partition of (non-random) size $n$ is described by $J = \{1, \ldots, n\}$, $w(i) = i$, $f(J, n) = 1$, $g_i(a_i) = 1$, $p(n)$ is the number of partitions of $n$, usually denoted by $p(n)$. Hence, for each $0 < x < 1$, we have normalization factors $c_i = (1 - x^i)$, so Equation (6) specializes to

$$\mathbb{P}(Z_i = k) = x^k (1 - x^i), \quad 0 < x < 1, \quad i = 1, \ldots, n,$$

i.e., $Z_i$ is geometrically distributed with parameter $1 - x^i$. In this example $Z_i$ denotes the number of parts of size $i$ in a random partition (of random size).

The probability that we generate a particular partition of $n$, with multiplicities $(c_1, \ldots, c_n)$, is given by

$$\mathbb{P}(Z_1 = c_1, \ldots, Z_n = c_n) = \prod_{i=1}^{n} \mathbb{P}(Z_i = c_i) = \prod_{i=1}^{n} x^{i c_i} (1 - x^i) = x^{\sum_{i=1}^{n} i c_i} \prod_{i=1}^{n} (1 - x^i).$$

Then, since each partition of $n$ satisfies $\sum_{i=1}^{n} i c_i = n$, we have

$$\mathbb{P}(Z_1 = c_1, \ldots, Z_n = c_n) = x^{n} \prod_{i=1}^{n} (1 - x^i).$$

In other words, only the size of the partition determines its likelihood of being generated using this approach, and all partitions of the same size are equally likely to appear. Thus, either by summing over all partitions of $n$, or by Equation (7) directly, we have

$$\mathbb{P}(T = n) = p(n) x^{n} \prod_{i=1}^{n} (1 - x^i).$$

We would like to maximize this probability, and since this formula holds for all $0 < x < 1$, we can find an expression for $x$ for which this expression is at its maximum. It was shown in [16] (see also [28]) that the choice $x = e^{-c/\sqrt{n}}$, $c = \pi/\sqrt{6}$, is particularly optimal, and produces

$$\mathbb{P}(T = n) \sim \frac{1}{\sqrt{96} n^{3/4}}.$$

Thus, the hard rejection sampling algorithm has cost

$$\text{Time}_{\text{rej}}(X') = O \left( \text{Time}(X) n^{3/4} \right).$$

It was shown in [2] that for any selected index $i \in \{1, \ldots, n\}$, the PDC deterministic second half algorithm obtains a speedup to hard rejection sampling of size

$$\text{speedup} = O \left( \max_{j \geq 0} \left( x^j (1 - x^i) \right)^{-1} \right) = O \left( (1 - x^i)^{-1} \right) \sim \frac{\sqrt{n}}{i c}.$$
Thus, the optimal choice is \( I = \{1\} \), and the total time for the PDC deterministic second half algorithm is
\[
\text{Time}_{\text{DDSH}}(X') = O\left(\text{Time}(X) n^{1/4}\right).
\]

**Remark 6.1.** It was also shown in [2] that by using \( I = \{1, 3, 5, \ldots\} \), i.e., all odd parts, one could apply a self–similar recursive PDC algorithm for which the total time of the algorithm is \( O(\text{Time}(X)) \); this method is PDC but not PDC deterministic second half.

The algorithm relied on the fact that the even and odd parts of a random partition contain roughly the same amount of entropy, and that a partition of size \( 2m \) into even parts is in one-to-one correspondence with an unrestricted partition of \( m \) (simply divide each of the even parts by 2). Thus, for *unrestricted* integer partitions we prefer the self–similar algorithm, but in general such an advantageous structure may not be present.

When partitions are restricted to have distinct parts, then we have \( J = \{1, \ldots, n\} \), \( w(i) = i, f(J, n) = 1, g_i(a_i) = (a_i \leq 1) \), \( p(n) \) is the number of partitions of \( n \) into distinct parts, usually denoted by \( q(n) \). Then we have normalization constants \( c_i = (1 + x^i)^{-1} \), so
\[
P(Z_i = k) = \frac{x^{ik}}{1+x^{ik}}, \quad k \in \{0, 1\}, \quad 0 < x < 1, \quad i = 1, \ldots, n,
\]
i.e., \( Z_i \) is Bernoulli with parameter \( \frac{x^i}{1+x^i} \). For any \( i \), the speedup is thus
\[
\text{speedup} = O\left(\max_{k \in \{0, 1\}} \frac{x^{ik}}{1+x^{ik}}\right)^{-1} = O\left(\frac{x}{1+x}\right)^{-1} \leq O(1).
\]
Thus for partitions into distinct parts, PDC deterministic second half can only offer a constant factor improvement.

**6.4. Example: Selections.** Integer partitions of size \( n \) into distinct parts is an example of a selection: each element \( \{1, 2, \ldots, n\} \) is either in the partition or not in the partition. Selections in general allow \( m_i \) different types of a component with weight \( i \). For integer partitions, this would be similar to assigning \( m_i \) colors to integer \( i \), and allowing at most one component of size \( i \) of each color. Then we have for all \( 0 < x < 1 \)
\[
P(Z_i = k) = \binom{m_i}{k} \left(\frac{x^i}{1+x^i}\right)^k \left(\frac{1}{1+x^i}\right)^{m_i-k},
\]
which is binomial. By the (local) central limit theorem (see for example [20]), the maximum value is obtained at its expectation, \( EZ_i = m_i x^i/(1+x^i) \), and so the PDC deterministic second half algorithm using \( I = \{i\} \) has
\[
\text{speedup} = O\left(\max_i \mathbb{P}\left(Z_i = \frac{m_i x^i}{1+x^i}\right)^{-1}\right) = O\left(\sqrt{\frac{m_i x^i}{(1+x^i)^2}}\right) = O\left(\sqrt{\text{Var}(Z_i)}\right).
\]

Thus, the optimal PDC deterministic second half algorithm is the one that uses the index \( i \) for which the variance of \( Z_i \) is the largest; this is due to the fact that this family of random variables is asymptotically normally distributed, hence a larger variance implies an asymptotically smaller largest point probability.
6.5. **Example: Multisets.** Unrestricted integer partitions of size \(n\) is an example of a multiset: each element \(\{1, 2, \ldots, n\}\) can appear any number of times in the partition. Multisets in general allow \(m_i\) different types of a component with weight \(i\), similar to selections. We have for all \(0 < x < 1\)

\[
P(Z_i = k) = \binom{m_i + k - 1}{k} (1 - x)^{m_i} x^k, \quad k = 0, 1, \ldots,
\]

which is negative binomial. The mode of the negative binomial distribution is given by the mass at \(\lceil (m_i - 1)x/(1 - x) \rceil\). By the (local) central limit theorem, similarly as with selections, we have

\[
\text{speedup} = O \left( \max_i \Pr \left( Z_i = \frac{m_i x^i}{1 - x^i} \right)^{-1} \right) = O \left( \sqrt{\frac{m_i x^i}{(1 - x^i)^2}} \right) = O \left( \sqrt{\text{Var}(Z_i)} \right).
\]

6.6. **Assemblies.** Assemblies are described using \(Z_i\) as Poisson(\(\lambda_i\)), where \(\lambda_i = \frac{m_i x^i}{i!}\), \(i = 1, \ldots, n\), and where \(m_i\) denotes the number of components of size \(i\), and \(x > 0\). We have

\[
\Pr(Z_1 = c_1, \ldots, Z_n = c_n) = \prod_{i=1}^{n} \frac{m_i^{c_i} x^{i c_i}}{i!^{c_i} c_i!} e^{-\lambda_i} = x^n e^{-\sum_{i=1}^{n} \lambda_i} \prod_{i=1}^{n} \frac{m_i^{c_i}}{i!^{c_i} c_i!}.
\]

Here again since the random variables are Poisson, the (local) central limit theorem implies

\[
\text{speedup} = O \left( \sqrt{\text{Var}(Z_i)} \right),
\]

hence we should select the index \(i\) with the largest variance to obtain the largest speedup.

For set partitions, \(C_1\) denotes the number of blocks of size \(i\) in a random set partition of size \(n\), \(i = 1, \ldots, n\). In this case \(m_i = 1\), and we have \(\lambda_i = x^i / i!\), for \(x > 0\). In this case the probability of the event \(\{T = n\}\) is maximized by the value of \(x\) such that \(x e^x = n\), for which \(x = \log(n)\) serves as a reasonable approximation for large \(n\).\(^2\) It was shown in \([2]\) that \(I = \lceil \log(n) \rceil\) is a particularly good choice, since

\[
\lambda_I \sim \frac{\log(n) \log(n)}{(\log(n))!} \sim \frac{e^{\log(n)}}{\sqrt{2\pi \log(n)}} \sim \frac{n}{\sqrt{2\pi \log(n)}},
\]

hence,

\[
\text{speedup} = O \left( \sqrt{2\pi \lambda_I} \right) = O \left( \sqrt{n} / \sqrt[4]{\log(n)} \right).
\]

It fact, it was shown in \([22]\) that

\[
\Pr(T = n) = O \left( \sqrt{n \log(n)} \right),
\]

\(^2\)In \([3]\), it is shown that

\[
x \sim \log(n) - \log(\log(n)) + O \left( \frac{\log \log n}{\log n} \right).
\]
whence the total time for PDC deterministic second half is $O(\log^{5/4}(n))$. While we can possibly improve upon this rejection rate, perhaps by a self–similar iterative PDC algorithm, it hardly seems necessary.

6.7. **Non-uniform measures.** There are also other measures of interest, considered, for example, in \cite[Section 8]{}. In addition to the condition that \{\sum_i w_i Z_i = n\}, one can further condition, e.g., on the event \{\sum_i Z_i = k\}, which demands that the total number of components in the random structure with weight $n$ is $k$. For integer partitions, this is equivalent to sampling from partitions of size $n$ with exactly $k$ parts; or, set partitions of size $n$ with exactly $k$ blocks. In general, this condition takes the form \{\sum_i u_i Z_i = k\} for some set of coefficients $u = (u_i)_{i \geq 1}$. Then, for any $\theta > 0$, we let $Z_i$ have distribution given by

$$
\mathbb{P}_\theta(Z_i = c_i) = \frac{\theta^{u_i c_i}}{\mathbb{E} \theta^{u_i Z_i}} \mathbb{P}(Z_i = c_i), \quad i = 1, \ldots, n,
$$

where we assume $\mathbb{E} \theta^{u_i Z_i} < \infty$. Then, by choosing a value of $\theta$, we can effectively tilt the distribution.

For example, take $u_i = 1$ for all $i$ and $Z_i$ to be Poisson with parameter $\lambda_i = \frac{\theta x_i}{r}$, $i = 1, \ldots, n$, and some $\theta > 0$. Then

$$
P(Z_1 = c_1, \ldots, Z_n = c_n) = \frac{x^n e^{-\theta \sum_{i=1}^n \lambda_i}}{n!} \theta^{\theta(\theta + 1)(\theta + 2)\ldots(\theta + n - 1)} \prod_{i=1}^n \frac{\theta^{c_i}}{i^{c_i} c_i!}
$$

(9)

$$
P(Z_1 = c_1, \ldots, Z_n = c_n) = \frac{x^n e^{-\theta \sum_{i=1}^n \lambda_i}}{\theta(\theta + 1)(\theta + 2)\ldots(\theta + n - 1)} \prod_{i=1}^n \frac{\theta^{c_i}}{i^{c_i} c_i!}
$$

(10)

Letting $t_A = \sum_{i=3}^n i Z_i$ and $s_A = \sum_{i=3}^n Z_i$, there is a PDC deterministic second half algorithm for this family; namely, let $I = \{1, 2\}$, then we have

$$
E = \left\{ \sum_{i=1}^n i Z_i = n, \sum_{i=1}^n Z_i = k \right\},
$$

$$
E^{(1)} = \left\{ (x_3, \ldots, x_n) : \sum_{i=3}^n i x_i \leq n, \sum_{i=3}^n x_i \leq k \right\},
$$

$$
E_I = E_{I|X^{(1)}} = \left\{ (y, z) : y + 2z + \sum_{i=3}^n i Z_i = n, \ y + z + \sum_{i=3}^n Z_i = k \right\},
$$

$$
\mathcal{L} \left( X^{(1)} \mid E \right) = \mathcal{L} \left( Z_3, Z_4, \ldots, Z_n \right) \left| \sum_{i=3}^n i Z_i = n, \sum_{i=1}^n Z_i = m \right.,
$$

$$
\mathcal{L} \left( T_I \mid X^{(1)} = x^{(1)}, E \right) = \mathcal{L} \left( (Z_1, Z_2) \mid Z_1 + 2Z_2 = n - t_A, \ Z_1 + Z_2 = m - s_A \right).
$$
Thus, conditional on accepting a set of values for $X(I) = (Z_3, \ldots, Z_n)$, the values of $Z_1$ and $Z_2$, say $y_I$ and $z_I$, are uniquely determined, hence deterministic. The rejection condition is then given by

$$\left\{ X(I) \in E(I) \text{ and } U < \frac{\mathbb{P}(Z_1 = y_I)\mathbb{P}(Z_2 = z_I)}{\max_{j_1, j_2} \mathbb{P}(Z_1 = j_1)\mathbb{P}(Z_2 = j_2)} \right\},$$

and the speedup is

$$\text{speedup} = O\left(\left(\max_{j_1, j_2} \mathbb{P}(Z_1 = j_1)\mathbb{P}(Z_2 = j_2)\right)^{-1}\right).$$

There is also a two-parameter family which generalizes the Ewen’s sampling formula given in [21]. In general, the same kind of PDC deterministic second half algorithm can be applied to any set of restrictions, as long as they satisfy the regularity conditions, and the choice of $I$ uniquely determines a completion in the second stage of PDC.

7. Other Applications

7.1. Uniform Distribution. This section contains the most compelling application of PDC, one which is implicit in many other applications.

**Theorem 7.1.** Suppose assumptions (A1)–(A3) and (DSH) hold, and that the distribution $\mathcal{L}(T_I)$ is uniform over a bounded measurable set $J$ with positive measure. Assume also one of (D1) or (C1) holds. Then Algorithm 5 randomly generates a sample from the distribution $\mathcal{L}(X')$.

**Proof.** Assuming $X(I) \in E(I)$, the rejection step is given by: reject if

$$u > \frac{P(T_I \in dx)}{\sup P(T_I \in dy)} = 1,$$

thus we accept with probability 1 any sample that is completable.

\[ \Box \]

**Algorithm 5** Uniform PDC DSH

```
procedure UNIFORM_PDC_DSH(X_1, \ldots, X_n, E, I)
    Assumptions: (A1)–(A3), (DSH), and one of (D1) or (C1); $T_I$ is uniform over some bounded measurable set $J$ of positive measure.
    Sample from $\mathcal{L}(X^{(I)})$, denote the observation by $x^{(I)}$.
    if $x^{(I)} \in E^{(I)}$ then
        return $(y_I, x^{(I)})$
    else
        restart
    end if
end procedure
```
We thus obtain an automatic speedup over hard rejection sampling simply by the PDC observation that the rejection step is deterministic, which is equivalent to applying hard rejection sampling on a larger-sized target; in the case of continuous random variables this provides an algorithm which terminates in finite time.

7.2. **Exponential Distribution.** When the $X_i$, $i = 1, \ldots, n$, are independent exponentially distributed random variables with parameters $\lambda_i$, $i \geq 1$, the marginal density function is

$$f_{X_i}(x) = \lambda_i e^{-\lambda_i x}, \quad x > 0.$$  

When the event $E$ is of the form $E = \{\sum_{i=1}^n X_i = k\}$, then $\mathbb{P}(E) = 0$, and there is no hard rejection sampling algorithm. This problem in particular was handled using Markov chains, see [12]. We take $I = \{i\}$, then since the density $f_{T_i}$ is bounded by $\lambda_i$, we can apply Algorithm 4. The rejection condition is thus

$$\left\{ \sum_{j \neq i} X_j \leq k \text{ and } U < e^{-\lambda_i y_I} \right\},$$

where $y_I = k - \sum_{j \neq i} x_j$. By Theorem 4.2 we have

$$\text{Time}_{\text{CDSH}}(X') = O \left( \frac{\lambda_i}{\mathbb{P}(\sum_{j \neq i} X_j \leq k)} \right).$$

7.3. **Beta Distribution.** A continuous random variable $X$ is said to have Beta$(\alpha, \beta)$ distribution, $\alpha > 0, \beta > 0$, if it has density

$$f_X(x) = c_{\alpha,\beta} x^{\alpha-1} (1-x)^{\beta-1}, \quad 0 < x < 1,$$

where $c_{\alpha,\beta}$ is the normalization constant. When at least one of $\alpha, \beta$ is less than 1, the density $f_X(x)$ is not bounded. When both $\alpha$ and $\beta$ are at least 1, then we have

$$\max_x f_X(x) = \frac{\alpha - 1}{\alpha + \beta - 2}, \quad \alpha, \beta > 1.$$  

Thus, if we consider $X_1, X_2, \ldots, X_n$ independent Beta$(\alpha_j, \beta_j)$, $j = 1, \ldots, n$, with $E = \{\sum_{j=1}^n X_j = k\}$, then as long as there exists an index $i$ such that $\alpha_i, \beta_i$ are both greater than 1, we can apply PDC deterministic second half. The rejection condition is given by

$$\left\{ \sum_{j \neq i} X_i \in [k, k-1] \text{ and } U < \frac{c_{\alpha,\beta}(\alpha + \beta - 2)}{\alpha - 1} y_I^{\alpha-1} (1-y_I)^{\beta-1} \right\},$$

where $y_I = k - \sum_{j \neq i} x_i$. 
7.4. Small Ball Probabilities. Suppose $X_i, i \geq 1$ are i.i.d. with distribution $P(X_i = 1) = P(X_i = -1) = \frac{1}{2}$. Let $w_i$ denote real-valued weights with $|w_i| \geq 1, i \geq 1$. Define $T := \sum_{i=1}^{n} w_i X_i$. Then for some open set $G$, $P(T \in G)$ is known as the small ball probability, see for example [18]. To obtain sample paths, the simplest approach is to apply hard rejection sampling. However, noting that $X_i$ is actually a discrete uniform distribution over the set $\{-1, 1\}$, we can apply PDC deterministic second half and select any index $I$, and sample until $\{X(i) \in G(I)\}$, where $G(I) = \bigcup_{g \in G} (g + w_1) \cup (g - w_1)$.

When $G$ is an open set of length $2r$, then it was shown in [14] that $P(T \in G)$ is at most $2^{-n}$ times the sum of the largest $r$ binomial coefficients in $n$. Let us assume that the values $w_i$ are all integer-valued, and $G = (-1, 1)$. Let $S_r(n)$ denote the sum of the largest $r$ binomial coefficients in $n$. We have

$$P \left(T \in (-1, 1)\right) \leq 2^{-n}S_1(n) = 2^{-n} \binom{n}{n/2}.$$

If we apply PDC deterministic second half, this becomes

$$P \left(X(i) \in (-w_1 - 1, -w_1 + 1) \cup (w_1 - 1, w_1 + 1)\right) \leq 2^{-(n-1)}S_2(n-1) = 2^{-(n-1)} \binom{n}{n/2},$$

which saves at most an anticipated factor of 2.

Also, as was exploited in [11], if in addition there exist two distinct elements $w_j \neq w_\ell$, then we let $I = \{j, \ell\}$, and we have the range of $w_j X_j + w_\ell X_\ell$ is uniform over four distinct elements, say $\{v_1, v_2, v_3, v_4\}$; let $V = (v_1 - 1, v_1 + 1)$, then $V := \cup_{i=1}^{4} V_i$ is an open set of length 8, whence

$$P(T - w_j X_j - w_\ell X_\ell \in V) \leq 2^{-(n-2)}S_4(n-2).$$

One can keep going with this idea. If we let $I = \{j_1, j_2, \ldots\}$, then we must have that $\sum_{\ell} w_{j_\ell} X_{j_\ell}$ is uniform over distinct elements; i.e., each combination of $\pm 1$ in $X_{j_\ell}$ must yield a distinct element for the sum. This is true, e.g., if $w_{j_1} = 1, w_{j_2} = 2, \ldots, w_{j_\ell} = 2^\ell$.

7.5. Sampling from the surface of the $n$–sphere. Consider the distribution

$$(X_1, \ldots, X_n | X_1^2 + \ldots + X_n^2 = k),$$

where all random variables are continuous. This corresponds to some distribution on the surface of an $n$–sphere. It is known how to obtain the uniform distribution and certain other distributions over the surface of an $n$–sphere, see e.g. [11] [11]; however, if we change the form of the conditioning slightly (for example, replace just $X_1^2$ with $X_1$), or place a particular demand on any of the marginal laws of $X_1, \ldots, X_n$, then these techniques do not apply directly.

The PDC deterministic second half algorithm is robust with respect to small changes. In this case, we take $I = \{i\}$, for some $i \in \{1, \ldots, n\}$. Then we have $L(T_I | X(i) = x(I), E) = L(X_i^2 | X_i^2 = y_I(x(I)))$. The rejection has the form:

$$\left\{ y_I \in \text{range}(X_i^2) \text{ and } U \leq \frac{1}{\sup_{\ell} f_{X_i}(\ell)} \left( f_{X_i}(\sqrt{y_I}) + f_{X_i}(-\sqrt{y_I}) \right) \right\}.$$
Note that we have calculated the transformation from the distribution \( \mathcal{L}(T_I | X(I)) \) to the distribution \( \mathcal{L}(X_I) \). This is almost PDC deterministic second half because there are actually two possible values for \( X_I \), even though the distribution \( \mathcal{L}(T_I | X(I), E) \) is trivial. Once a sample is accepted, we simply choose an outcome, \( \sqrt{y_I} \) or \( -\sqrt{y_I} \), in proportion to its value in the density function \( f_{X_I} \).

7.6. Uniform Spacings. Suppose we randomly place \( m \) points uniformly distributed over the interval \([0, 1]\), call them \( u_1, u_2, \ldots, u_m \). Let \( u(1), u(2), \ldots, u(m) \) denote the ordering of the points in ascending order. Then it is well-known, see for example [10], that the marginal distributions are given by

\[
u(i) \sim \text{Beta}(i, m + 1 - i), \quad i = 1, \ldots, n.
\]

The differences between consecutive points in the interval are in fact i.i.d. with \( u(0) = 0, u(m + 1) = 1 \)

\[
u(i) - \nu(i-1) = \frac{E_i}{\sum_{i=1}^{m+1} E_i}, \quad i = 1, \ldots, m + 1,
\]

where \( E_i \) are exponential distributions with parameter 1. In other words, to obtain a sample from \( (E_1, E_2, \ldots, E_n | \sum_{i=1}^n E_i = 1) \), one can follow the steps above, which does not use PDC. Of course, as alluded to in Section 7.5, any deviation from this very specific form of distribution renders this approach effectively useless, whereas PDC can still be applied.

7.7. Convex Polytope Sampling. Suppose we wish to sample from a convex polytope \( P \subset \mathbb{R}^n \) with vertices \( \{v_1, \ldots, v_m\} \). Assuming none of the points \( v_i \) are degenerate, i.e., there do not exist any \( v_i \) such that \( v_i \in \text{ConvexHull}\{v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_m\} \), we can sample uniformly from \( P \) via Algorithm [6] (see [15]; see also [10]).

**Algorithm 6 [15] Convex polytope sampling**

```plaintext
procedure CONVEX_POLYTOPE_SAMPLER(X_1, X_2, ..., X_n, P)
Generate \((U_1, U_2, \ldots, U_{m-1})\) i.i.d in the interval \([0, 1]\).
Sort the points \( U(1), \ldots, U(m-1) \).
Compute the adjacent differences \( Y_1, \ldots, Y_m \).
Let \( X = \sum_{i=1}^m Y_i v_i \).
Return \( X \).
end procedure
```

The key aspect of this algorithm is that it has a time and memory requirement on the order of \( m \), the number of vertices, which means that it is not efficient for polytopes with a large number of vertices compared to their dimension, which we now demonstrate.

The hypersimplex \( H_{n,k} \) is defined as

\[ H_{n,k} = \left\{ (x_1, \ldots, x_n) \in [0, 1]^n : \sum_{i=1}^n x_i = k \right\}. \]
We can obtain a random point inside the hypersimplex using Section 7.1. Each coordinate is uniformly distributed over the interval $[0, 1]$, so by Theorem 7.1, the PDC deterministic second half algorithm is simply to sample $(u_2, \ldots, u_n)$ from independent uniform distributions over $[0, 1]$ until $u_1 := k - \sum_{i=2}^{n} u_i \in [k-1, k]$.

The Permutahedron $P_n$ is the convex hull of all $m = n!$ permutations of the coordinates of the point $(1, 2, \ldots, n)$. It can be described as follows:

$$P_n = \left\{ (x_1, \ldots, x_n) \in [1, n]^n : \sum_{i=1}^{n} x_i = \binom{n+1}{2} \right\},$$

where $R$ denotes Rado’s condition, which is

$$R = \{ \text{for all } j \geq 1, x_{(n)} + \ldots + x_{(j)} \leq n + (n-1) + \ldots + j \}.$$

Let

$$Q_n = \left\{ (x_1, \ldots, x_n) \in [1, n]^n : \sum_{i=1}^{n} x_i = \binom{n+1}{2} \right\}.$$

This is a scaled version of the hypersimplex, which can be sampled using Section 7.1 where each coordinate is uniformly distributed over the interval $[1, n]$. Its asymptotic volume is given in [6] as

$$\text{Vol}(Q_n) \sim (n-1)^n \sqrt{\frac{6}{\pi n}}.$$

It is well-known that $\text{Vol}(P_n) = n^{n-2}$, i.e., the number of spanning trees on $n$ vertices, see for example [23]. Hence, the probability that a point in $Q_n$ is also in $P_n$ is given by

$$\mathbb{P}(R|Q_n) \sim \frac{n^{n-2}}{(n-1)^n \sqrt{\frac{6}{\pi n}}} \sim n^{-3/2} e^{\sqrt{\frac{n}{6}}}. $$

Thus, to sample points from inside the permutahedron, one can first sample points from inside the scaled hypersimplex using PDC deterministic second half, and then apply hard rejection sampling with respect to Rado’s condition $R$.

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