Transitive Subgroups of Transvections Acting on Some Symplectic Symmetric Spaces of Ricci Type

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Abstract

Symmetric symplectic spaces of Ricci type are a class of symmetric symplectic spaces which can be entirely described by reduction of certain quadratic Hamiltonian systems in a symplectic vector space. We determine, in a large number of cases, if such a space admits a subgroup of its transvection group acting simply transitively. We observe that the simply transitive subgroups obtained are one dimensional extensions of the Heisenberg group.
1 Introduction

This paper is devoted to the study of a class of symmetric symplectic manifolds; those whose canonical connection is of Ricci type. More precisely, we address the question of determining which of those manifolds admit a simply transitive subgroup of transvections. This very particular problem is motivated by quantization; in this case, one has at one’s disposal a number of techniques to construct an invariant quantization, either formal or convergent (see, for instance, [3, 4]). When such a transitive subgroup $H$ exists, there is another isomorphic one which is the image of $H$ by the involutive automorphism $\sigma$ of the transvection group $G$ ($\sigma$ is conjugation in $G$ by the symmetry at a base point of the space). Observe that the transvection group is generated by $H$ and $\sigma(H)$; in particular, any object which is invariant under $H$ and a symmetry will be automatically invariant under the whole of $G$.

The choice of these symmetric symplectic manifolds with Ricci-type connections has two reasons. The first one is that it is one of the few classes of symmetric symplectic manifolds which is known and completely classified. The second is that these symplectic manifolds with connection are in some sense the analogue in symplectic geometry of the classical space forms of Riemannian geometry.

A simply connected symmetric symplectic manifold of dimension $2n$ ($n \geq 2$) of Ricci type is entirely determined by the conjugacy class of a non-zero element $A$ of the symplectic Lie algebra $\mathfrak{sp}(\mathbb{R}^{2(n+1)}, \Omega)$ such that $A^2 = \mu \text{Id}$. More precisely, connected symmetric symplectic spaces of Ricci-type of dimension $2n$ ($n \geq 2$) are quotients of (the universal cover of) (the connected component of) model manifolds obtained by reduction from the standard symplectic vector space $(\mathbb{R}^{2(n+1)}, \Omega)$ in the following way. Let $A$ be a non-zero element of $\mathfrak{sp}(\mathbb{R}^{2(n+1)}, \Omega)$ (with $n > 1$) such that $A^2 = \mu \text{Id}$ and such that
\[
\Sigma_A = \{ x \in \mathbb{R}^{2(n+1)} \mid \Omega(x, Ax) = 1 \} \neq \emptyset.
\]

The 1-parameter group $\{ \exp tA \}$ stabilizes $\Sigma_A$ and one considers the space $M_A$ of orbits of the group $\{ \exp tA \}$ and the canonical projection
\[
\pi : \Sigma_A \to M_A = \Sigma_A/\{ \exp tA \}.
\]
The space $M_A$ has a manifold structure such that $\pi$ is a smooth submersion, and is naturally endowed with a “reduced” symplectic structure and a “reduced” symplectic connection which is the canonical connection for a natural “reduced” symmetric space structure.

Our main result can be described in terms of this characteristic element.

**Theorem 1** Let $0 \neq A \in \mathfrak{sp}(\mathbb{R}^{2(n+1)}, \Omega)$ be such that $\Sigma_A \neq \emptyset$ and $A^2 = \mu \text{Id}$.

1. If $\mu > 0$, the space of orbits $M_A$ is diffeomorphic to $TS^n$ (hence is always connected and simply connected when $n > 1$) and never admits a simply transitive subgroup of the transvection group.
2. If \( \mu < 0 \), the symmetric bilinear form \( g(X, Y) := \Omega(X, AY) \) has \( 2p \) positive eigenvalues with \( 1 \leq p \leq n+1 \); the symmetric space \( M_A \) is connected and simply connected; it is diffeomorphic to \( \mathbb{C}^n \) if \( p = 1 \) and to a complex vector bundle of rank \( n+1-p \) over \( \mathbb{P}^{p-1}(\mathbb{C}) \) if \( p > 1 \). It admits a simply transitive subgroup of the transvection group if and only if \( p = 1 \); non-isomorphic simply transitive subgroups arise in this case.

3. If \( \mu = 0 \), let \( p (1 \leq p \leq n+1) \) be the rank of \( A \) and let \( q (1 \leq q \leq p) \) be the number of positive eigenvalues of the symmetric bilinear form \( g(X, Y) := \Omega(X, AY) \). The symmetric space \( M_A \) is diffeomorphic to \( T(S^{q-1} \times \mathbb{R}^{p-q}) \times \mathbb{R}^{2(n+1-p)} \). If \( p = 1 \), each of the two connected components of \( M_A \) is diffeomorphic to the flat symplectic vector space and hence admits a simply transitive subgroup of the transvection group. If \( q \neq 1, 2 \) or \( 4 \), \( M_A \) does not admit a simply transitive subgroup of the transvection group. If \( p = 2 \), \( M_A \) admits a simply transitive subgroup of the transvection group if and only if \( q = 1 \).

Let us mention that all simply transitive subgroups obtained here are 1-dimensional extensions of the Heisenberg group of dimension \( 2n-1 \). The case of a solvable transvection group, i.e. when \( A^2 = 0 \) and \( p = 2 \), was considered in 2006 in the mémoires de DEA of Amin Malik and Yannick Voglaire and was the framework of the quantization scheme developed by Bieliavsky in [4].

The paper is organised as follows: in Section 2, we recall definitions and known results concerning symplectic symmetric spaces of Ricci-type. In particular, we describe the manifold structure and the transvection group of each \( M_A \). Section 3 is a summary of the facts about the topology of Lie groups which we need in the later sections. We prove property 1 in Section 4, property 2 in Section 5 and property 3 in Section 6.

2 Symmetric symplectic spaces whose curvature is of Ricci type

We recall in this section known results which can be found for instance in the review [5].

**Definition 1** A smooth symplectic manifold \( (M, \omega) \) is a symmetric symplectic space if there are symmetries, that is if there exists a smooth map

\[
s : M \times M \to M \quad (x, y) \mapsto s(x, y) = s_x y
\]

such that each “symmetry” \( s_x \) squares to the identity \( (s_x)^2 = \text{Id}_M \), has \( x \) as isolated fixed point, and is a symplectic diffeomorphism \( (s_x^* \omega = \omega) \), such that \( s_x s_y s_x = s_{s_x y} \forall x, y \in M \).

The canonical symmetric connection, \( \nabla \), is defined by

\[
\omega_x(\nabla_X Y, Z) = \frac{1}{2} X_x \omega(Y + s_x Y, Z), \quad X, Y, Z \in \chi(M);
\]
it is symplectic (i.e. torsion-free and such that $\nabla \omega = 0$) and invariant by all symmetries.

A symmetric symplectic space $(M, \omega, s)$ is a homogeneous space $M = G/K$ where $G$ is the group generated by products of even number of symmetries ($G$ is called the transvection group) and $K$ is the stabilizer in $G$ of a certain point $o$ (chosen as base point).

Symmetric symplectic spaces are not classified in general, but the following cases are known:

1. those whose transvection group $G$ is semisimple [2];
2. those whose curvature of the canonical connection is of Ricci type (see definition 3 below) [6];
3. a class of symplectic symmetric spaces with nilpotent transvection group appearing as symmetric subspaces of a symplectic vector space [7];
4. symmetric symplectic spaces of dimension 2 and 4 [2].

**Definition 2** Let $(V, \nu)$ be a symplectic vector space of dimension $2n$. An algebraic symplectic curvature tensor $R$ on $(V, \nu)$ is an element of $\Lambda^2 V^* \otimes S^2 V^*$ such that

$$
\sum_{X,Y,Z} R(X, Y, Z, T) = 0
$$

where $\sum_{X,Y,Z}$ denotes the sum over cyclic permutations of $X, Y$ and $Z$.

The space $\mathcal{R}$ of algebraic symplectic curvature tensors on $(V, \nu)$ splits (if $n \geq 2$) into two subspaces which are irreducible under the natural action of the symplectic group $Sp(V, \nu)$, [9]. One writes:

$$
\mathcal{R} = \mathcal{E} + \mathcal{W}.
$$

The $\mathcal{E}$ component of the curvature tensor $R$ can be expressed in terms of the Ricci tensor $r$ associated to $R$. If $R(X, Y)$ is the endomorphism of $(V, \nu)$ defined by

$$
R(X, Y, Z, T) = \nu(R(X, Y)Z, T)
$$

the Ricci curvature associated to $R$ is the element of $S^2 V^*$

$$
r(X, Y) = \text{Tr}(Z \mapsto R(X, Z)Y).
$$

The $\mathcal{E}$ component of the curvature $R$ has the form

$$
E(X, Y, Z, T) = \frac{-1}{2(n + 1)}[2\nu(X, Y)r(Z, T) + \nu(X, Z)r(Y, T) + \nu(Y, Z)r(X, T)] - \nu(Y, Z)r(X, T) - \nu(Y, T)r(X, Z).
$$
Definition 3 Let \((M, \omega, \nabla)\) be a smooth symplectic manifold of dimension \(2n(n \geq 2)\) endowed with a symplectic connection \(\nabla\). The connection \(\nabla\) is said to be of Ricci type if, \(\forall x \in M\), the curvature is of the form \(R_x = E_x\).

The following lemma is a direct consequence of the definitions.

**Lemma 1** Let \((M, \omega, \nabla)\) be a connected smooth symplectic manifold of dimension \(2n, n \geq 2\) endowed with a smooth symplectic connection \(\nabla\) of Ricci type. Then the curvature endomorphism \(R(X,Y)\) is given by:

\[
R_x(X,Y) = \frac{1}{2(n+1)} (2\omega_x(X,Y)\rho_x + \rho_x Y \otimes X + X \otimes \rho_x Y - \rho_x X \otimes Y - Y \otimes \rho_x X) \tag{2.1}
\]

where \(\rho\) is the Ricci endomorphism, i.e.

\[
\omega(X, \rho Y) = r(X,Y)
\]

and where, for \(X \in \chi(M)\), \(\underline{X} = \omega(X, \cdot)\). Furthermore, there exists a vector field \(U\) such that

\[
\nabla_X \rho = -\frac{1}{2n+1} (X \otimes U + U \otimes X); \tag{2.2}
\]

there exists a function \(f\) such that:

\[
\nabla_X U = -\frac{2n+1}{2(n+1)} \rho^2 X + f X;
\]

and there exists a constant \(K\) such that

\[
\text{Tr} \rho^2 + \frac{4(n+1)}{2n+1} f = K. \tag{2.3}
\]

**Corollary 1** Two Ricci type connections on a real analytic connected symplectic manifold \((M, \omega)\) coincide if they have the same values \((\rho_o, U_o)\) of the Ricci endomorphism and of the vector field \(U\) at a point \(o\) of \(M\) (with \(U\) defined be 2.2) and the same constant \(K\) defined by 2.3.

**Corollary 2** Let \((M, \omega, s)\) be a connected symplectic symmetric space of dimension \(2n\) whose canonical connection is of Ricci type. Then the vector field \(U\) vanishes; the function \(f\) is a constant, \(f = \frac{2n+1}{4(n+1)^2} K\), and the Ricci endomorphism is such that

\[
\rho^2 = \frac{K}{2(n+1)} \text{Id}.
\]

Furthermore given the value \(\rho_o\) of \(\rho\) at a base point \(o\) of \(M\), the canonical connection \(\nabla\) of \((M, \omega, s)\) is uniquely determined.
Let \((M_i, \omega_i, s_i), i = 1, 2\) be connected, simply-connected, symmetric symplectic spaces of Ricci-type; let \(\alpha_i \in M_i\) and let \(\Psi : T_{\alpha_i}M_1 \to T_{\alpha_2}M_2\) be a linear isomorphism such that \(\Psi^*\omega_2 = \omega_1\) and \(\Psi \circ \rho_1 = \rho_2 \circ \Psi\), where \(\rho_i\) is the Ricci endomorphism of the canonical connection of \((M_i, \omega_i, s_i)\). Then \(\Psi\) extends to a global symplectic diffeomorphism \(\tilde{\Psi} : (M_1, \omega_1) \to (M_2, \omega_2)\) such that \(\tilde{\Psi} \circ s_1 = s_2(\tilde{\Psi}(x))\), i.e. these two symmetric spaces are isomorphic. In view of Corollary 2, we have an injective map of the set of isomorphism classes of connected, simply-connected symmetric symplectic spaces of Ricci-type and of dimension \(2n\) into the set of conjugacy classes of elements \(\rho\) of the symplectic algebra \(\mathfrak{sp}(n, \mathbb{R})\) whose square is a multiple of the identity (under the adjoint action of \(Sp(n, \mathbb{R})\)). Indeed, recall that the Ricci endomorphism \(\rho_o\) at a point \(o\) of \((M, \omega)\) is such that \(r_o(X, Y) = \omega_o(X, \rho_o Y) = r_o(Y, X) = \omega_o(Y, \rho_o X)\) and hence \(\rho_o \in \mathfrak{sp}(T_oM, \omega_o)\). The examples which follow show that this map is also surjective.

Let \((\mathbb{R}^{2(n+1)}, \Omega)\) be the standard symplectic vector space of dimension \(2(n + 1)\) and let \(o \neq A\) be an element of \(\mathfrak{sp}(n + 1, \mathbb{R})\) such that \(A^2 = \mu \text{Id}\) and such that
\[
\Sigma_A = \{x \in \mathbb{R}^{2(n+1)} \mid \Omega(x, Ax) = 1\} \neq \emptyset.
\]
Then \(\Sigma_A\) is an embedded \((n + 1)\)-dimensional submanifold of \(\mathbb{R}^{2(n+1)}\). The group \(\exp tA\) stabilizes \(\Sigma_A\) and has no fixed point in \(\Sigma_A\). The space \(M_A\) of orbits of the group \(\exp tA\) has a shape depending on the sign of \(\mu\):

i) if \(\mu = k^2, k > 0\), \(\exp tA = \text{ch} kt I + \frac{1}{k} \text{sh} kt A\);

ii) if \(\mu = -k^2, k > 0\), \(\exp tA = \cos kt I + \frac{1}{k} \sin kt A\);

iii) if \(\mu = 0\), \(\exp tA = \text{Id} + tA\).

• In the first case \((\mu = k^2)\), \(A\) admits \(\pm k\) as eigenvalues and
\[
\mathbb{R}^{2(n+1)} = V^+ \oplus V^-, \quad A|_{V^\pm} = \pm k \text{Id}
\]
with \(V^\pm\) lagrangian subspaces. In a basis \(\{e_i, e'_i \mid 1 \leq i \leq n + 1\}\) adapted to this decomposition and such that \(\Omega(e_i, e'_i) = \delta_{ij}\), we can write
\[
x = x^+ + x^-,
\]
so that \(\Omega(x, Ax) = -2k < x_+, x_>\)

where \(<,>\) is the standard scalar product on \(\mathbb{R}^{n+1}\). The orbit of \(x_0 = x_0^+ + x_0^-\) is
\[
x(t) = e^{kt}x_0^+ + e^{-kt}x_0^-.
\]
An element \( x_0 \) is in \( \Sigma_A \) if and only if
\[
<x_0^+, x_0^-> = -\frac{1}{2k}
\]
and we can choose a unique element in its orbit, \( x(t) \) such that
\[
<x(t)^+, x(t)^-> = -\frac{1}{2k}, \quad <x(t)^+, x(t)^+> = 1
\]
so that \( M_A = \{(u, v) \mid <u, u>= 1, <u, v>= -\frac{1}{2k}\} \) can be seen as an embedded submanifold. The map \( \pi : \Sigma_A \to M_A \) given by \((x_+, x_-) \mapsto (u, v) := (\frac{x_+}{<x_+, x_+>^{1/2}}, <x_+, x_+>^{1/2} x_-)\) is a surjective submersion. The manifold \( M_A \) can be identified with \( TS^n \) where \( w = v + \frac{1}{2k} u. \)

- In the second case \((\mu = -k^2), A/k = J\) is a complex structure compatible with \( \Omega\), \( \Omega(Jx, Jy) = \Omega(x, y)\), and the corresponding bilinear form \( g(x, y) = \Omega(x, Ay) \) has signature \((2p, 2q)\) where \( p + q = n + 1 \) (if \( g(x, x) \neq 0 \), then \( g(x, x) \) and \( g(Ax, Ax) \) have the same sign). Thus there exists a basis \( \{e_j, f_j \mid 1 \leq j \leq n + 1\} \) and a diagonal matrix \( \text{Id}_{pq} = \text{diag}(\epsilon_1, \ldots, \epsilon_{n+1}) \), where \( \epsilon_r = 1, r \leq p \) and \( \epsilon_r = -1, r > p \) such that
\[
\Omega(e_i, f_j) = \epsilon_i \delta_{ij}, \quad Ae_i = kf_i, \quad Af_i = -ke_i.
\]
The hypersurface \( \Sigma_A = \{u \mid \Omega(u, Au) = 1\} \) has equation
\[
x \cdot \text{Id}_{pq} x + y \cdot \text{Id}_{pq} y = 1
\]
if \( u = \sum_i x^i e_i + y^i f_i, \quad x = (x^1, \ldots, x^{n+1}), \quad y = (y^1, \ldots, y^{n+1}) \). To have \( \Sigma_A \) non-empty we must have \( p \geq 1 \).

The group \( \exp tA \) acts by
\[
x \mapsto \cos kt x - \sin kt y, \quad y \mapsto \sin kt x + \cos kt y.
\]

If \( p = 1, \Sigma_A \) is diffeomorphic to \( S^1 \times \mathbb{R}^{2n} \) and the quotient manifold is diffeomorphic to \( \mathbb{C}^n \). If \( 1 < p < n + 1, \Sigma_A \) is diffeomorphic to \( S^{2p-1} \times \mathbb{R}^{2q} \) and the quotient manifold \( M_A \) can be identified with a complex vector bundle of rank \( q \) over \( \mathbb{P}_{p-1}(\mathbb{C}) \). If \( p = n + 1, \Sigma_A \) is diffeomorphic to \( S^{2n+1} \) and the quotient manifold \( M_A \) is \( \mathbb{P}_n(\mathbb{C}) \).

- In the third case \((\mu = 0), \) let \( V = \text{Im} A; \) then Ker \( A = (\text{Im} A)^\perp \supset \text{Im} A. \) Let \( p = \dim V, 1 \leq p \leq n + 1; \) let \( W \) be an arbitrary subspace of Ker \( A, \) supplementary to \( V; \) then \( W \) is symplectic and \( W^\perp = V \oplus V^*, \) where \( V^* \) is a lagrangian subspace of \( W^\perp \) in duality with \( V. \) Choose a basis \( \{e_i, i \leq p\} \) of \( V \), \( \{e_i^*, i \leq p\} \) of \( V^* \) and \( \{f_a, a \leq 2(n + 1 - p)\} \) of \( W \) such that
\[
Ae_i^* = e_i, \quad \Omega(e_i^*, e_j) = \epsilon_i \delta_{ij}, \quad \epsilon_i = \begin{cases} 1 & \text{for } 1 \leq j \leq q \\ -1 & \text{for } q < j \leq p \end{cases}
\]
Denote by $\Omega^0$ the restriction of $\Omega$ to the symplectic vector space $W : \Omega^0_{ab} = \Omega(f_a, f_b)$.

If $u = \sum_{i=1}^{p} x^i e_i + \sum_{a=1}^{(2(n+1)-p)} X^a f_a + \sum_{j=1}^{p} x^j e^*_j \overset{\text{def}}{=} x + X + x^*$, then

$$\Omega(u, Au) = \sum_{i=1}^{p} \epsilon_i (x^i_s(t))^2.$$

Since $q$ is the number of indices $i$ such that $\epsilon_i = 1$, $\Sigma_A$ is not empty iff $1 \leq q$. The orbit of a point $(x_o, X_0, x_0^*)$ under the action of $\exp tA$ has the form:

$$\exp tA \cdot (x_o + X_o + x_o^*) = (x_o + tx_o^*, X_o, x_o^*) =: (x_o(t), X_o(t), x_o^*(t)).$$

Hence, for any point in $\Sigma_A$:

$$\sum_{i=1}^{p} \epsilon_i x^i(t)x^i_s(t) = \sum_{i=1}^{p} \epsilon_i x^i(0)x^i_s(0) + t$$

so that each orbit of $\exp tA$ in $\Sigma_A$ contains a unique point satisfying

$$\sum_{i=1}^{p} \epsilon_i x^i(t)x^i_s(t) = 0, \quad \left(\text{namely for } t = -\sum_{i=1}^{p} \epsilon_i x^i(o)x^i_s(0)\right).$$

The projection $\pi : \Sigma \to M_A : (x, X, x^*) \mapsto (x - (\sum_{i=1}^{p} \epsilon_i x^i x^i_s)x, X, x^*)$ is a surjective submersion on the space of orbits which is identified to the submanifold of $\mathbb{R}^{2n+2}$ defined by

$$\sum_{i=1}^{p} \epsilon_i (x^i_s)^2 = 1, \quad \sum_{i=1}^{p} \epsilon_i x^i x^i_s = 0.$$

Hence $M_A$ is diffeomorphic to $T(S^{p-1} \times \mathbb{R}^{p-q}) \times \mathbb{R}^{2(n+1)-p}$. If $q = 1$, the manifold has two connected components, each diffeomorphic to $\mathbb{R}^{2n}$. If $q = p$, $M_A$ is diffeomorphic to $T S^{p-1} \times \mathbb{R}^{2(n+1)-p}$.

**Lemma 2** Let $(\mathbb{R}^{2(n+1)}, \Omega)$ be the standard symplectic vector space of dimension $2(n+1)$; let $o \neq A$ be an element of $\mathfrak{sp}(n+1, \mathbb{R})$ such that

i) $A^2 = \mu \text{Id}$, $\mu \in \mathbb{R}$;

ii) $\Sigma_A = \{x \in \mathbb{R}^{2(n+1)} \mid \Omega(x, Ax) = 1\} \neq \emptyset$

then $M_A = \Sigma_A / \exp tA$ has a canonical structure of a smooth manifold of dimension $2n$ and the canonical map $\Sigma \to \Sigma / \exp tA$ is a smooth submersion. Furthermore

- if $\mu = k^2$, $M_A$ is diffeomorphic to $TS^n$, hence is connected and simply connected for $n > 1$;
• if $\mu = -k^2$, the signature of the symmetric bilinear form $g(X,Y) := \Omega(X,AY)$ has $2p$ positive eigenvalues with $1 \leq p \leq n + 1$; the symmetric space $M_A$ is connected and simply connected; if $p = 1$, it is diffeomorphic to $\mathbb{C}^n$; if $1 < p < n + 1$ it is diffeomorphic to a complex vector bundle of rank $n + 1 - p$ over $\mathbb{P}^{p-1}$; and if $p = n + 1$, it is diffeomorphic to $\mathbb{P}^n(\mathbb{C})$;

• if $\mu = 0$, let $p (1 \leq p \leq n + 1)$ be the rank of $A$ and let $q (1 \leq q \leq p)$ be the number of positive eigenvalues in the symmetric bilinear form $g(X,Y) := \Omega(X,AY)$. The space $M_A$ is diffeomorphic to $T(S^{q-1} \times \mathbb{R}^{p-q}) \times \mathbb{R}^{2(n+1-p)}$. If $q = 1$, $M_A$ has two connected components diffeomorphic to $\mathbb{R}^{2n}$. The space $M_A$ is connected if $q > 1$ and simply connected if $q > 2$.

We now show that on any such $M_A = \Sigma_A/\exp tA$, there exists a symplectic structure $\omega$ and a connection $\nabla$ which is symplectic, of Ricci type, and locally symmetric.

Let $\pi : \Sigma_A \to M_A = \Sigma_A/\exp tA$ be the natural projection; let $x \in \Sigma_A$ and $y = \pi(x)$. The tangent space to $\Sigma_A$ is

$$T_x\Sigma_A = \{Y \in T_x\mathbb{R}^{2(n+1)} | \Omega(Y,Ax) = 0\}.$$ 

Observe that $\forall x \in \Sigma_A$, the vector space $\mathbb{R}x$ is transversal to $\Sigma_A$; we have an orthogonal decomposition:

$$T_x\mathbb{R}^{2(n+1)} = T_x\Sigma_A \oplus \mathbb{R}x = \text{Span}\{x, Ax\} \oplus \text{Span}\{x, Ax\}^\perp.$$ 

The subspace $H_x = \text{Span}\{x, Ax\}^\perp$ is symplectic and $\pi_x : H_x \to T_y M_A$ is a linear isomorphism. Since $\exp tA : x \mapsto \exp tA.x$ maps $H_x$ on $H_{\exp tAx}$, one may define a 2–form on $M_A$ by

$$\omega_y(X,Y) = \Omega_x(X,\overline{Y})$$

where $X,Y \in T_y M_A, \overline{X}, \overline{Y} \in H_x$ and $\pi_x \overline{X} = X, \pi_x \overline{Y} = Y$. One checks readily that this 2–form is indeed symplectic. Let $\nabla^0$ be the standard flat symplectic connection on $(\mathbb{R}^{2(n+1)}, \Omega)$; define (as in [1])

$$\nabla_X Y(x) = \nabla^0_X \overline{Y} - \Omega(A\overline{X}, \overline{Y})x + \Omega(\overline{X}, \overline{Y})Ax.$$ 

This is a linear connection on $M_A$, which is torsion free and has the property that $\nabla \omega = 0$ and hence is symplectic.

The curvature of this connection is given by:

$$R(X,Y)\overline{Z} = -2\Omega(X,\overline{Y})AZ - \Omega(\overline{X}, \overline{Z})AY + \Omega(\overline{Y}, \overline{Z})AX$$

$$+ \Omega(AX, \overline{Z})Y - \Omega(A\overline{Y}, \overline{Z})\overline{X}.$$ 

(2.4)
Comparing 2.1 and 2.4 one gets
\[ R(X, Y) = -\frac{1}{2(n+1)}[-2\omega(X, Y)(-2(n+1)\tilde{A}) + 2(n+1)\tilde{A}Y \otimes X - 2(n+1)\tilde{A}X \otimes Y + X \otimes 2(n+1)\tilde{A}Y - Y \otimes 2(n+1)\tilde{A}X] \]

where \( \tilde{A}(\in \text{End } T_y M_A) \) is defined by
\[ (\tilde{A}X)_y = \pi^*(A\overline{x})_x. \]

In particular this shows that the canonical connection is of Ricci type and that the Ricci endomorphism is:
\[ \rho = -2(n+1)\tilde{A}. \]

Since \( \tilde{A}\nabla X Y = \nabla X \tilde{A}Y \), we have \( \nabla X \rho = 0 \) and hence the spaces \( M_A \) are locally symmetric.

**Lemma 3** Let \( 0 \neq A \) be an element of \( \mathfrak{sp}(n+1, \mathbb{R}) \) such that \( A^2 = \mu \text{Id} \) and \( \Sigma_A \neq \emptyset \). Then the orbit space \( M_A = \Sigma_A/\exp tA \) is a locally symmetric symplectic manifold of dimension \( 2n \).

**Lemma 4** The manifold \( M_A = \Sigma_A/\exp tA \) is a globally symmetric symplectic manifold and its canonical connection is \( \nabla \).

**Proof** Let \( x \in \Sigma_A \) and let \( y \in \mathbb{R}^{2(n+1)} \); if
\[ S_x y := -y + 2\Omega(y, Ax)x - 2\Omega(y, x)Ax. \]

Then \( S_x \) belongs to \( Sp(n+1, \mathbb{R}) \) and commutes with \( A \), hence stabilizes \( \Sigma_A \). If \( \pi : \Sigma_A \to M_A \) is the canonical projection, define:
\[ s_{\pi(x)} \pi(y) = \pi(S_x y) \quad y \in \Sigma_A. \]

This is well defined as the right hand side does not depend on the choice of \( x \) (resp. \( y \)) in the fibre over \( \pi(x) \) (resp. \( \pi(y) \)). The diffeomorphism \( s_{\pi(x)} \) of \( M_A \) is symplectic and one checks that it is affine (for the canonical connection \( \nabla \) on \( M_A \)). Hence the conclusion.

**Theorem 2** Let \( (N, \nu, s) \) be a symmetric symplectic space of dimension \( 2n \), \( n \geq 2 \), whose curvature is of Ricci type. Then there exists \( A \neq 0 \) in \( \mathfrak{sp}(n+1, \mathbb{R}) \) such that \( A^2 = \mu \text{Id} \) and \( \Sigma_A = \{ x \in \mathbb{R}^{2n+2} | \Omega(x, Ax) = 1 \} \neq \emptyset \) and \( (N, \nu, s) \) is locally isomorphic to the symmetric symplectic space \( (M_A = \Sigma_A/\exp tA, \omega, s) \).

If \( (N, \nu, s) \) is connected, then its universal cover is globally isomorphic to (the universal cover of) (a connected component of) \( M_A \).
A ∈ \text{vectors} be the symplectic embedding of $R$. The kernel of the homomorphism $\alpha$ of $\sigma$ and the function $\tilde{\sigma}(\beta)$ corresponding involutive automorphism of $\sigma$.

**Remark 1** Any linear symplectic endomorphism $g \in Sp(\mathbb{R}^{2(n+1)}, \Omega)$ which commutes with $A$ induces a symplectic diffeomorphism $\alpha(g)$ of $M_A$, which is an affine map for $\nabla$ through $(\alpha(g))(\pi(x)) := \pi(gx), \ x \in \Sigma_A$

The action of $G_1 := \{ g \in Sp(\mathbb{R}^{2(n+1)}, \Omega) \mid gA = Ag \}$ via $\alpha$ on $(M, \omega)$ is strongly Hamiltonian, and the function $\tilde{f}_B$ on $M_A$ corresponding to an element $B \in sp(\mathbb{R}^{2(n+1)}, \Omega)$ is defined through $\left( \pi^*(\tilde{f}_B) \right)(x) = \frac{1}{2}\Omega(x, Bx) \text{ for } x \in \Sigma_A \subset \mathbb{R}^{2(n+1)}$.

**Lemma 5** The kernel of the homomorphism $\alpha$ from $G_1 = \{ g \in Sp(n+1, \mathbb{R}) \mid gAg^{-1} = A \}$ into the group of affine symplectic diffeomorphisms of $M_A$ is given by

$\text{Ker } \alpha = \{ \exp tA \mid t \in \mathbb{R} \}$.

**Proof** Consider an element $B \in Sp(n+1, \mathbb{R})$ so that $BA = AB$ and assume that $\alpha(B) = \text{Id}_{M_A}$. This means that for any $x \in \Sigma_A$ there exists $t_x \in \mathbb{R}$ so that $Bx = \exp t_xAx$. There exists a basis $u_i, i \leq 2(n+1)$, of $\mathbb{R}^{2(n+1)}$ such that $u_i \in \Sigma_A$ for all $i$. If $B \in Sp(n+1, \mathbb{R})$ and $BA = AB$, there exist $\tau_i \in \mathbb{R}$ such that $Bu_i = e^{\tau_iA}u_i$

then $\omega(u_i, u_j) = \omega(Bu_i, Bu_j) = \omega(e^{(\tau_i-\tau_j)A}u_i, u_j)$; that is $e^{(\tau_i-\tau_j)A}u_i = u_i$ and hence $e^{\tau_iA}u_i = e^{\tau_iA}u_i \forall i$. Thus $B = e^{\tau_iA}$.

The group $\alpha(G_1)$ being transitive on $M_A$ and stable by conjugation by the symmetry at a base point $o = \pi(x_o)$ of $M_A$ contains the transvection group of $M_A$, denoted by $G(M_A)$.

If $\tilde{\sigma}$ denotes the involutive automorphism of $G(M_A)$, $\tilde{\sigma}(h) = s_o hs_o$ and $\sigma := \tilde{\sigma}_{s_1}$ the corresponding involutive automorphism of $g$ (the Lie algebra of $G(M_A)$), one knows that $g$ is generated by the subspace $p$ of $g$:

$$p = \{ x \in g \mid \sigma X = -X \}, \quad g = p \oplus [p, p] =: p \oplus \mathfrak{k}. $$
The group $G_1$ is stable by conjugation by $S_{x_o}$. Denote by $\mathfrak{g}_1$ the Lie algebra of $G_1$, by $\tilde{\sigma}_1$ the automorphism of $G_1$ defined by $\tilde{\sigma}_1 g := S_{x_o} g S_{x_o}$ and by $\sigma_1 := \tilde{\sigma}_1|_p$. Introduce $p_1 := \{Y \in \mathfrak{g}_1 | \sigma_1 Y = -Y; \}$. Consider $p_1 + [p_1, p_1] \subset \mathfrak{g}_1$. Remark that $\alpha_* P_1 = p$ and $\alpha_{*|p_1}$ is injective, since $\text{Ker} \, \alpha_* = \mathbb{R} A$ and $\alpha A = A$. Hence

**Lemma 6** With the notations defined above, the algebra $\mathfrak{g}$ of the transvection group of $M_A$ is isomorphic to $p_1 + [p_1, p_1]$ if $A \not\in [p_1, p_1]$ and to $p_1 \oplus [p_1, p_1]/\mathbb{R} A$ if $A \in [p_1, p_1]$.

We determine this case by case.

If $A^2 = k^2 \text{Id}$, $k > 0$, we have, as indicated above, a basis $\{e_i; i \leq n + 1, e'_i; i \leq n + 1\}$ such that in this basis:

$$\Omega = \begin{pmatrix} 0 & I_{n+1} \\ -I_{n+1} & 0 \end{pmatrix}, \quad A = \begin{pmatrix} kI_{n+1} & 0 \\ 0 & -kI_{n+1} \end{pmatrix}.$$  

The algebra $\mathfrak{g}_1 = \{X \in \mathfrak{sp}(n + 1, \mathbb{R}) \mid [X, A] = 0\}$ is composed of elements $X = \begin{pmatrix} X_1 & 0 \\ 0 & -\tau X_1 \end{pmatrix}, \quad X_1 \in \mathfrak{gl}(n + 1, \mathbb{R})$

Choose as base point $x_o \in \Sigma_A$, $x_o = -\frac{1}{\sqrt{2k}} e_1 + \frac{1}{\sqrt{2k}} e'_1$. The symmetry $S_{x_o}$ has matrix $S_{x_o} = \begin{pmatrix} I_{1,n} & 0 \\ 0 & I_{1,n} \end{pmatrix}$ and thus $\sigma_1 X = S_{x_o} X S_{x_o} = \begin{pmatrix} I_{1,n} & 0 \\ 0 & -\tau I_{1,n} X_1 I_{1,n} \end{pmatrix}$ and $X \in p_1$ if and only if

$$X_1 = \begin{pmatrix} 0 & \tau b \\ a & 0 \end{pmatrix}, \quad a, b \in \mathbb{R}^n \quad (2.5)$$

Consider the algebra $p_1 \oplus [p_1, p_1] =: p_1 \oplus \mathfrak{t}_1$ so that

$$\mathfrak{t}_1 = \left\{ X = \begin{pmatrix} X_1 & 0 \\ 0 & -\tau X_1 \end{pmatrix} \ \bigg| \ X_1 = \begin{pmatrix} \tau b a - \tau b' a \\ a \otimes \tau b' - a' \otimes \tau b \end{pmatrix}, \quad a, b, a', b' \in \mathbb{R}^n \right\}. \quad (2.6)$$

The matrices $X_1$ corresponding to elements of $\mathfrak{t}_1$ have zero trace. From (2.5) and (2.6), one sees that $p_1 \oplus \mathfrak{t}_1$ is isomorphic to $\mathfrak{sl}(n + 1, \mathbb{R})$ and $\mathfrak{t}_1$ to $\mathfrak{gl}(n, \mathbb{R})$. As $A \not\in \mathfrak{t}_1$, $p_1 \oplus \mathfrak{t}_1 = \mathfrak{sl}(n + 1, \mathbb{R})$ is the transvection algebra.

If $A^2 = -k^2 \text{Id}$, $k > 0$, we have a basis $\{\tilde{e}_j; j \leq 2(n + 1)\}$ such that in this basis

$$\Omega = \begin{pmatrix} 0 & I_{n+1} \\ -I_{n+1} & 0 \end{pmatrix}, \quad A = k \begin{pmatrix} 0 & -I_{p,q} \\ I_{p,q} & 0 \end{pmatrix}.$$
The algebra $\mathfrak{g}_1 = \{ X \in \mathfrak{sp}(n+1, \mathbb{R}) \mid [X,A] = 0 \}$ is composed of elements

$$X = \begin{pmatrix} X_1 & X_2 \\ X_3 & -\tau X_1 \end{pmatrix}$$

with $\tau X_2 = X_2$, $\tau X_3 = X_3$, $\tau X_{1,p,q} + I_{p,q} X_1 = 0$.

Let us introduce on $\mathbb{R}^{2(n+1)}$ the complex structure $J = \frac{1}{k} A$ and the hermitian form

$$h = g - i\Omega, \quad \text{where} \quad g(X,Y) = \Omega(X,JY).$$

Identifying $\mathbb{R}^{2(n+1)}$ with $\mathbb{C}^{n+1}$ by $z = u_1 + iI_{p,q} u_2$ with $u_1, u_2 \in \mathbb{R}^{n+1}$ one has

$$h(z, z') = \tau z I_{p,q} z'.$$

Thus $\mathfrak{g}_1$ is isomorphic to $u(p,q)$ and $Az = -ikz$. Choose as base point $x_o = \frac{1}{\sqrt{k}} \tilde{e}_1 \in \Sigma_A$.

The symmetry $S_{x_o}$ has matrix

$$S_{x_o} = \begin{pmatrix} I_{1,n} & 0 \\ 0 & I_{1,n} \end{pmatrix}$$

and thus

$$\sigma_1 X = S_{x_o} X S_{x_o} = \begin{pmatrix} I_{1,n} X_1 I_{1,n} & I_{1,n} X_2 I_{1,n} \\ I_{1,n} X_3 I_{1,n} & -I_{1,n} \tau X_1 I_{1,n} \end{pmatrix}$$

and $X \in \mathfrak{p}_1$ if and only if

$$X_1 = \begin{pmatrix} 0 & -\tau a I_{p-1,q} \\ a & 0 \end{pmatrix}, \quad a \in \mathbb{R}^n$$

$$X_2 = \begin{pmatrix} 0 & \tau c \\ c & 0 \end{pmatrix}, \quad c \in \mathbb{R}^n$$

$$X_3 = -I_{p,q} X_2 I_{p,q}.$$

One sees that $\mathfrak{k}_1 = [\mathfrak{p}_1, \mathfrak{p}_1]$ is equal to $u(p-1,q)$ and that $\mathfrak{p}_1 \oplus \mathfrak{k}_1$ is equal to $\mathfrak{su}(p,q)$. As $A \notin \mathfrak{k}_1$, the transvection algebra is $\mathfrak{g} = \mathfrak{su}(p,q)$.

If $A^2 = 0$, $A \neq 0$, we have a basis $\{e_i, i \leq p; f_a, a \leq 2(n+1-p); e_i^*, i \leq p\}$ of $\mathbb{R}^{2n+2}$ such that in this basis

$$\Omega = \begin{pmatrix} 0 & 0 & -I_{q,p-q} \\ 0 & \Omega^0 & 0 \\ I_{q,p-q} & 0 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 0 & I_p \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad 1 \leq p \leq n+1, 1 \leq q \leq p.$$
where
\[ \Omega^0_{ab} := \Omega(f_a, f_b). \]

The algebra \( \mathfrak{g}_1 = \{ X \in \mathfrak{sp}(n+1, \mathbb{R}) \mid [X, A] = 0 \} \) is composed of elements
\[
X = \begin{pmatrix}
X_1 & X_2 & X_3 \\
0 & X_5 & X_6 \\
0 & 0 & X_1
\end{pmatrix}
\]

where
\[
\tau X_1 I_{q, p-q} + I_{q, p-q} X_1 = 0, \quad \tau X_5 \Omega^0 + \Omega^0 X_5 = 0 \\
\tau X_2 = \Omega^0 X_6 I_{q, p-q}, \quad \tau X_3 I_{q, p-q} - I_{q, p-q} X_3 = 0
\]

Choose as point \( x_0 \in \Sigma_A, x_0 = e_1^*. \) The symmetry \( S_{x_0} \) has matrix
\[
S_{x_0} = \begin{pmatrix}
I_{1, p-1} & 0 & 0 \\
0 & -I_{2(n+1-p)} & 0 \\
0 & 0 & I_{1, p-1}
\end{pmatrix}
\]

and thus
\[
\sigma_1 X = S_{x_0} X S_{x_0} = \begin{pmatrix}
I_{1, p-1} X_1 I_{1, p-1} - I_{1, p-1} X_2 & I_{1, p-1} X_3 I_{1, p-1} \\
0 & X_5 & -X_6 I_{1, p-1} \\
0 & 0 & I_{1, p-1} X_1 I_{1, p-1}
\end{pmatrix}
\]

and hence \( X \in \mathfrak{p}_1 \) if and only if
\[
X_1 = \begin{pmatrix}
0 & -\tau p'' I_{q-1, p-q} \\
p'' & 0
\end{pmatrix} \quad p'' \in \mathbb{R}^{p-1} \\
X_6 = \begin{pmatrix}
P & 0
\end{pmatrix} \quad P \in \mathbb{R}^{2(n+1-p)} \\
X_3 = \begin{pmatrix}
0 & \tau p' I_{q-1, p-q} \\
p' & 0
\end{pmatrix} \quad p' \in \mathbb{R}^{p-1} \\
X_5 = 0.
\]

Notice that an element \( X \in \mathfrak{p}_1 \) is determined by a triple of matrices \( (X_1, X_6, X_3) \), as specified above. An element of \( [\mathfrak{p}_1, \mathfrak{p}_1] \) is also determined by matrices \( (X_1, X_6, X_3) \) and
\[
\left( (X_1, X_6, X_3), \left( X_1', X_6', X_3' \right) \right) = (X_1, X_6', X_3] - X_6' X_1, [X_1, X_3'] + [X_3, X_1'] + \tau (\Omega_0 X_6 I_{q, p-q}) X_6' - \tau (\Omega_0 X_6' I_{q, p-q}) X_6.
\]

The algebra \( \mathfrak{p}_1 + [\mathfrak{p}_1, \mathfrak{p}_1] = \mathfrak{p}_1 + \mathfrak{e}_1 \) is thus composed of elements
\[
X = \begin{pmatrix}
X_1 & \tau (\Omega_0 X_6 I_{q, p-q}) & X_3 \\
0 & 0 & X_6 \\
0 & 0 & X_1
\end{pmatrix}
\]
where
\[ X_1 \in \mathfrak{so}(q, p - q), \quad X_6 \in \mathfrak{gl}(\mathbb{R}^p, \mathbb{R}^{2(n+1-p)}), \quad \tau X_3 I_{q,p-q} - I_{q,p-q} X_3 = 0. \]

Observe that the subalgebra generated by the \( X_6 \)'s and \( X_3 \)'s is a nilpotent ideal and that \( A \in \mathfrak{k}_1 \). Recall that the transvection Lie algebra is \( \mathfrak{g} = (\mathfrak{p}_1 + \mathfrak{k}_1)/\mathbb{R}A \). Thus

**Lemma 7** Let \( A \in \mathfrak{sp}(n+1, \mathbb{R}), A \neq 0, A^2 = 0 \); assume \( \Sigma_A \neq \emptyset \) and let \( M_A = \Sigma_A/\exp tA \).

Let \( p (1 \leq p \leq n+1) \) be the rank of \( A \) and let \( q (1 \leq q \leq p) \) be the number of positive eigenvalues in the symmetric bilinear form \( g(X, Y) := \Omega(X, AY) \). Then

i) if \( p = 1 \) (and hence also \( q = 1 \)), \( M_A \) has two connected components diffeomorphic to \( \mathbb{R}^{2a} \) and the transvection algebra is the \( 2n \)-dimensional abelian algebra;

ii) if \( p = 2 \), the transvection algebra is solvable and admits a codimension 1 nilpotent ideal;

iii) if \( p > 2 \), the transvection algebra has a Levi factor isomorphic to \( \mathfrak{so}(q, p-q) \) and a nilpotent radical.

We shall now investigate the existence of a simply-transitive subgroup \( H \) of the transvection group \( G(M_A) \) for the symmetric spaces \( M_A = \Sigma_A/\exp tA \). We separate the discussion into three cases \( A^2 = k^2 I, A^2 = -k^2 I \) and \( A^2 = 0 \).

### 3 Topology of Lie Groups

In this section we summarize the facts about the topology of Lie groups which we need in the later sections. All the results cited can be found in the survey article [8] by Hans Samelson and its bibliography.

**Theorem 3** Let \( H \) be a Lie group diffeomorphic to \( S^a \times \mathbb{R}^b \) or \( TS^a \times \mathbb{R}^b \). Then \( a = 1 \) or \( a = 3 \) and \( H \) has maximal compact subgroup isomorphic to \( S^1 \) or \( SU(2) \).

**Proof** Note that \( TS^a \times \mathbb{R} \) is diffeomorphic to \( S^a \times \mathbb{R}^{a+1} \) so we have to consider, \( S^a \times \mathbb{R}^b \) and \( TS^a \).

By the Iwasawa–Malcev theorem, \( H \) is diffeomorphic to a product of its maximal compact subgroup \( K \) and a euclidean space. If \( H \) is diffeomorphic to \( S^a \times \mathbb{R}^b \) and \( a \geq 2 \), \( H \) must be simply connected and hence so is \( K \). The latter is then a product of simply connected compact Lie groups with simple Lie algebras. By a theorem of Hopf each factor has the same real cohomology as a bouquet of odd dimensional spheres with one copy of \( S^3 \) always occurring and with as many spheres as the rank of the Lie algebra. Since \( S^a \times \mathbb{R}^b \) has the real cohomology of a single sphere, there can only be one simple factor and it must
have rank 1. Thus $K$ is isomorphic to $SU(2)$ and $a = 3$. If $a = 1$ then $K$ is compact, connected, 1-dimensional and so $K = S^1$.

Suppose now that $H$ is diffeomorphic to $T S^a$ then $H \times \mathbb{R}$ is diffeomorphic to $S^a \times \mathbb{R}^{a+1}$ so by the previous result $a = 1$ or $a = 3$ and $H \times \mathbb{R}$ has maximal compact subgroup $S^1$ or $SU(2)$ and hence so has $H$. Since $S^1$ and $S^3$ have trivial tangent bundles, we are done.

4 Simply-transitive subgroups of the group of transvections $G(M_A)$ for $M_A = \Sigma_A/\exp tA$, when $A^2 = k^2 I$

We have shown

i) that $M_A = TS^n$ (Lemma 2);

ii) that the transvection algebra of $M_A$ is isomorphic to $\mathfrak{sl}(n+1, \mathbb{R})$;

iii) that in an appropriate basis of $\mathbb{R}^{2(n+1)}$, $\{e^+_i, i \leq n + 1; e^-_i, i \leq n + 1\}$

$$\Sigma_A = \{x = \sum x^+_i e^+_i + x^-_i e^-_i \mid \sum_{i=1}^{n+1} x^+_i x^-_i = -\frac{1}{2k}\};$$

iv) that the projection $\pi : \Sigma_A \to TS^n = \Sigma_A/\exp tA$ has the form

$$\pi(x_+, x_-) = \left(\frac{x_+}{<x_+, x_+>^{1/2}} = u, <x^+, x^+_+>^{1/2} x_- + \frac{1}{2k} <x_+, x_>^{1/2} = w\right).$$

The subgroup of $Sp(n+1, \mathbb{R})$, $G_1 = \{C \mid [C, A] = 0\}$ is $Gl(n+1, \mathbb{R})$. It acts on $TS^n$ by

$$B.(u, w) = \left(\frac{Bu}{<Bu, Bu>^{1/2}}, <Bu, Bu>^{1/2} B^{-1}(w - \frac{1}{2k} u) + \frac{1}{2k} Bu\right)$$

and the kernel of effectivity is $\{B = \lambda I, \lambda > 0\}$; the connected subgroup $GL^+(n+1, \mathbb{R})$ modulo the kernel of effectivity is $SL(n+1, \mathbb{R})$. Hence the transvection group of $TS^n$ is $SL(n+1, \mathbb{R})$ and the stabilizer of the point $(u = e^+_1, w = 0)$ is the group $GL^+(n, \mathbb{R})$.

Theorem 4 No symplectic symmetric space $M_A = \Sigma_A/\exp tA$, where $A^2 = k^2 I$, admits a subgroup of its transvection group acting simply-transitively.

Equivalently,

Proposition 1 There is no Lie subgroup of $SL(n+1, \mathbb{R})$ which acts simply-transitively on $TS^n$, $n \geq 1$.

Proof If there is a Lie subgroup of $SL(n+1, \mathbb{R})$ acting simply-transitively then $TS^n$ is diffeomorphic to a Lie group and hence by Theorem 3 $n = 1$ or $n = 3$.  


(i) $n = 1$. $TS^1$ is diffeomorphic to $S^1 \times \mathbb{R}$. If there exists a subgroup $H$ of $SL(2, \mathbb{R})$ acting simply-transitively on $TS^1$, $H$ must be non-abelian since $SL(2, \mathbb{R})$ has no 2-dimensional abelian subgroups. $H$ is diffeomorphic to $TS^1$ so must have $S^1$ as maximal compact subgroup. We consider the Lie algebra $\mathfrak{h}$ of $H$ as a representation of this circle subgroup. Real irreducibles of $S^1$ are either the trivial one dimensional representation or non-trivial two dimensional representations. Since $\mathfrak{h}$ already contains one trivial representation it must be the sum of two trivial representations and hence be abelian, so we have a contradiction showing no such $H$ exists.

(ii) $n = 3$. This is the most complicated case and the rest of this section is devoted to its examination.

Let $H$ be a subgroup of $SL(4, \mathbb{R})$ acting simply-transitively on $TS^3$. By Theorem 3, a maximal compact subgroup $K$ of $H$ is isomorphic to $SU(2)$. Up to conjugation, we may assume $K \subset SO(4, \mathbb{R})$. Let us investigate the $SU(2)$ subgroups of $SO(4, \mathbb{R})$. For this, consider $\mathbb{H}$ the space of quaternions. We have a natural map

$$(SU(2) \times SU(2)) \times \mathbb{H} \to \mathbb{H} : ((q_1, q_2), x) \mapsto q_1 x q_2^{-1}$$

which is norm preserving:

$$(q_1 x q_2^{-1})(\overline{q_1 x q_2^{-1}}) = x \overline{x}, \quad \overline{x} = \text{quaternionic conjugate of } x$$

and hence we have a homomorphism

$$SU(2) \times SU(2) \to O(4)$$

which by connectedness takes it values in $SO(4)$. The kernel of this homomorphism is $\mathbb{Z}_2 = \{(1, 1), (-1, -1)\}$. Hence

a) $SU(2) \times SU(2)/\mathbb{Z}_2$ is isomorphic to $SO(4)$;

b) $SU(2) \times \{1\}$ and $\{1\} \times SU(2)$ project isomorphically into $SO(4)$.

We shall denote these subgroups $SU(2)_L$ and $SU(2)_R$. They are normal subgroups of $SO(4)$, hence not conjugate. One checks that they are conjugate in $SL(4, \mathbb{R})$; indeed if $C$ denotes conjugation in $\mathbb{H}$:

$$C \circ R_{q_2^{-1}} \circ C = L_{q_2}.$$  

Any other compact 3-dimensional subgroup of $SO(4)$ must have a non-trivial component in each of $SU(2)_L$ and $SU(2)_R$, and since there are no 2-dimensional subgroups of $SU(2)$ we can use the projection into one factor to parametrise, and the other factor must then be related by an (inner) automorphism from which we see that such a subgroup must be
conjugate to the image in $SO(4)$ of the diagonal $SU(2) = \{(q, q) \mid q \in SU(2)\}$ and this projects to an $SO(3)$ subgroup.

Thus $SL(4, \mathbb{R})$ has two conjugacy classes of compact 3-dimensional subgroups whose members are either isomorphic to $SU(2)$ (and we may take $SU(2)_L$ as a representative) or isomorphic to $SO(3)$. Hence in looking for subgroups $H$ diffeomorphic to $TS^3$ we may assume $H$ has maximal compact subgroup $SU(2)_L$ without any loss of generality.

We now describe the adjoint action of $SU(2)_L$ on $sl(4, \mathbb{R})$. The Lie algebra $so(4)$ consists of all skew-symmetric matrices in $sl(4, \mathbb{R})$ and has an invariant 9-dimensional complement $p$ given by all traceless symmetric matrices. These form an irreducible representation of $SO(4)$ under the adjoint action (indeed if it were not one could construct a non-trivial ideal in $sl(4, \mathbb{R})$) and hence of $SU(2)_L \times SU(2)_R$. Since both factors act non-trivially (if it were not, one could construct a non trivial ideal of $sl(4, \mathbb{R})$ in $so(4)$), $p$ must be isomorphic to a tensor product of the 3-dimensional irreducible representation of each factor as $9 = 3 \times 3$ is the only non-trivial factorisation. Thus as $su(2)_L$ module, $sl(4, \mathbb{R})$ can be written as

$$sl(4, \mathbb{R}) = \mathfrak{k} \oplus p = (su(2)_L \oplus (\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R})) \oplus \mathbb{R}^3 \oplus \mathbb{R}^3 \oplus \mathbb{R}^3$$

where $\mathbb{R}^3$ denotes the irreducible 3 dimensional $su(2)_L$ module.

The algebra $\mathfrak{h}$ of $H$ contains $su(2)_L$ and is stable by the $su(2)_L$ action. Hence

$$\mathfrak{h} = su(2)_L \oplus \mathfrak{h}_1$$

where $\mathfrak{h}_1$ is a 3 dimensional representation of $su(2)_L$. In view of the above, it is either the sum of three trivial 1 dimensional representations or a 3 dimensional irreducible representation. In the first case, $\mathfrak{h}$ would be isomorphic to $so(4)$ and $H$ can not be diffeomorphic to $TS^3$. Hence $\mathfrak{h}_1$ is a 3 dimensional irreducible representation of $su(2)_L$. Thus $\mathfrak{h}_1 \subseteq \mathfrak{p}$. Now $\mathfrak{p}$ is isomorphic as $su(2)_L$ module to $su(2)_L \otimes \mathcal{W}$, where $\mathcal{W}$ is a trivial 3 dimensional $su(2)_L$ module. We can view the injection $\mathfrak{h}_1 \rightarrow \mathfrak{p}$ as an intertwining map $\phi$ from $su(2)_L$ into $su(2)_L \otimes \mathcal{W}$. If $v \in su(2)_L$ and $e_i (i \leq 3)$ is a basis of $\mathcal{W}$, we have

$$\phi(v) = \sum_{i=1}^3 \phi_i(v) \otimes e_i$$

For any $v' \in su(2)_L$:

$$[v', \phi(v)] = \sum_{i=1}^3 [v', \phi_i(v)] \otimes e_i = \phi([v', v])$$

$$= \sum_{i=1}^3 \phi_i([v', v]) \otimes e_i$$

Hence each $\phi_i$ is an intertwining operator and by Schur’s lemma, is a multiple of the identity; hence

$$\phi(v) = \sum_{i=1}^3 \lambda_i v \otimes e_i = v \otimes \sum_{i=1}^3 \lambda_i e_i =: v \otimes w$$
Thus $h_1 = \mathfrak{su}(2)_L \otimes w$.

Now the left action of $SU(2)_L$ on $\mathbb{R}^4 (\sim \mathbb{H})$ has the form

$$qxq^{-1} = (q_0, q)(x_0, x)(\overline{q_0}, -q) = (x_0, (q_0^2 - q^2)x + 2q_0q \wedge x + 2q_0x_0q) =: (x_0, R(q)x)$$

where $q_0^2 + |q|^2 = 1$. Similarly the map $x \mapsto qx$ has matrix

$$q_L = \begin{pmatrix} q_0 & -\tau \\ q & q_0I + q_\wedge \end{pmatrix}$$

and the map $x \mapsto xq^{-1}$ has matrix

$$q_R = \begin{pmatrix} q_0 & \tau \\ -q & q_0I + q_\wedge \end{pmatrix}$$

Define $\eta : \mathbb{R}^3 \otimes \mathbb{R}^3 \to P$

$$\eta(x, y) = \begin{pmatrix} x.y & \tau(y \wedge x) \\ y \wedge x & x \otimes \tau y + y \otimes \tau x - xy1 \end{pmatrix}$$

This is indeed in $p$ as $\tau \eta(x, y) = \eta(x, y)$ and $tr\eta(x, y) = 0$. Furthermore the map is clearly surjective. The following relations show that the map $\eta$ exhibits the $SU(2)_L$ and $SU(2)_R$ action on $p$

$$q_L \eta(x, y)q_L^{-1} = \eta(R(q)x, y)$$

$$q_R \eta(x, y)q_R^{-1} = \eta(x, R(q)y)$$

To conclude, we observe that the element $t^\tau w \otimes w$ of $h_1(t \in \mathbb{R})$ corresponds to the element of $p$

$$\eta(tw, w) = \begin{pmatrix} t|w|^2 & 0 \\ 0 & 2tw \otimes \tau w - t|w|^21 \end{pmatrix}.$$ 

This belongs to the Lie algebra of the subgroup $GL^+(3, \mathbb{R})$ which is the stabilizer of the point $(u = e_1^+, w = 0)$. Hence the orbit of $H$ on $TS^3$ is at most of dimension 5; a contradiction.

\[ \blacksquare \]

5 Simply-transitive subgroups of the group of transvections $G(M_A)$ for $M_A = \Sigma_A/ \exp tA$, when $A^2 = -k^2I$

We have shown that such spaces of dimension $2n$ are characterized by an integer $p$, (the symmetric bilinear form $g(X, Y) := \Omega(X, AY)$ has $2p$ positive eigenvalues) $1 \leq p \leq n + 1$. 
If $p = 1$ $M_A = \mathbb{C}^n$; $M_A$ is a complex vector bundle of rank $q = n + 1 - p$ over $\mathbb{P}^{p-1}(\mathbb{C})$ if $1 < p < n + 1$; and $M_A = \mathbb{P}^n(\mathbb{C})$ if $p = n + 1$. All these spaces are simply-connected. We have also shown that the transvection algebra is $su(p, q)$ and that the isotropy algebra is $u(p - 1, q)$.

The subgroup of $Sp(\mathbb{R}^{2(n+1)}, \Omega)$ which commutes with $A$ acts on $z \in \mathbb{C}^{n+1}$ as $U(p, q)$ and $\exp tA$ acts as multiplication by $e^{-ikt}$. Hence on the quotient we obtain the projective pseudo-unitary group $PU(p, q) = U(p, q)/e^{it}I_{n+1} = SU(p, q)/\mathbb{Z}_{n+1}$ as a symmetry group. However $PU(p, q)$ is simple and acts faithfully so it must coincide with the transvection group of $M_A$. The stabiliser of $z = e_1$ is isomorphic to $U(p - 1, q)$.

**Theorem 5** The symplectic symmetric space $M_A = \Sigma_A/\exp tA$, $A^2 = -k^2I$, admits a simply-transitive subgroup of its transvection group if and only if $p = 1$.

**Proof** To determine whether there is a simply-transitive subgroup of the transvection group, we note that all the above manifolds are simply-connected, and when $p > 1$ they have the homotopy type of $\mathbb{P}_{p-1}(\mathbb{C})$ which has first non-vanishing cohomology group in degree 2. However a simply-connected Lie group which is not contractible will have the cohomology of a product of odd dimensional spheres, with at least one copy of $S^3$ occurring. Thus the first non-zero cohomology group will be in degree 3. Hence for $p > 1$ there can be no simply-transitive subgroup of the transvection group. Thus we have shown $p = 1$ is necessary.

If $p = 1$, $M_A = SU(1, n)/U(n)$ (since the kernel of effectivity is contained in the $U(n)$ subgroup). Now $U(n)$ is the maximal compact subgroup of $SU(1, n)$ and we have an Iwasawa decomposition

$$SU(1, n) = U(n)AN$$

where $A$ is a real closed 1-parameter subgroup isomorphic to $\mathbb{R}$. The group $AN$ acts simply-transitively on $M_A$.

**Remark 2** The subgroup $H = AN$ above is isomorphic to the extension $K_{2n}$ of the $2n - 1$-dimensional Heisenberg group $H_{2n-1}$ by dilations; the Heisenberg Lie algebra $\mathfrak{h}_{2n-1}$ is defined by

$$\mathfrak{h}_{2n-1} = \{(X, a) \mid X \in \mathbb{R}^{2(n-1)}, a \in \mathbb{R}\}$$

for non degenerate skew symmetric 2-form $\Omega^0$ on $\mathbb{R}^{2n-2}$; and the Lie algebra of its extension by dilations is $K_{2n} = \mathfrak{h}_{2n-1} \oplus \mathbb{R}D$ with $D((X, a)) := (X, 2a)$.

The question is whether there are any subgroups other than $AN$ which act simply-transitively on $G/K$ in this case. That there are follows from the following Proposition:
Proposition 2 Let $G$ be a connected semisimple Lie group; let $g = \mathfrak{k} + \mathfrak{p}$ be a Cartan decomposition of its Lie algebra. Let $K$ be the analytic subgroup with algebra $\mathfrak{k}$; let $\mathfrak{a}$ be the maximal abelian subalgebra of $\mathfrak{p}$; let $A$ be the analytic subgroup with algebra $\mathfrak{a}$. Let $n$ be the subalgebra of $g$ spanned by the positive root vectors corresponding to a choice of positive roots of the pair $(\mathfrak{g}, \mathfrak{a})$. Let $N$ be the analytic subgroup with algebra $\mathfrak{n}$. Let $m$ be the centraliser of $\mathfrak{a}$ in $\mathfrak{k}$; let $\varphi : \mathfrak{a} \rightarrow m$ be a homomorphism; let $a_\varphi = \{X + \varphi(X) \mid X \in \mathfrak{a}\}$. Then $a_\varphi$ is an abelian subalgebra of $g$ of the same dimension as $\mathfrak{a}$ and that $[a_\varphi, \mathfrak{n}] \subset \mathfrak{n}$; hence $h_\varphi = a_\varphi + \mathfrak{n}$ is a solvable algebra. Let $A_\varphi$ be the analytic subgroup with algebra $a_\varphi$. Then the map $K \times A_\varphi \times N \rightarrow G : (k, a_\varphi, n) \mapsto k a_\varphi n$ is a global decomposition of $G$ as a product of three closed subgroups. Finally, $H_\varphi = A_\varphi N$ is a closed solvable subgroup of $G$ acting simply-transitively on $G/K$.

Proof It is clear that $a_\varphi$ is an abelian subalgebra of $g$ of the same dimension as $\mathfrak{a}$ and that $[a_\varphi, \mathfrak{n}] \subset \mathfrak{n}$; hence $h_\varphi = a_\varphi + \mathfrak{n}$ is a solvable algebra. Let $g \in G$ and let

$$g = k a_\varphi n$$

be its Iwasawa decomposition. Then, if $a = \exp H$,

$$g = k \exp -\varphi(H) \exp(H + \varphi(H)) n =: k a'n.$$

Hence any $g$ can be written as a product of an element of $K$, an element of $A_\varphi$ and an element of $N$. Uniqueness of the decomposition comes from the uniqueness of the Iwasawa decomposition and

$$k_1 a_1'^\varphi n_1 = k_1 \exp(H + \varphi(H)) n_1 = k_1 \exp \varphi(H) \exp H n_1 = k_1 \exp \varphi(H) a_1 n_1$$

and $k_1 \exp \varphi(H) \in K$.

Let $(a_i')$, $i \in \mathbb{N}$ be a sequence of elements of $A_\varphi$ converging in $G$; then $a'_i = \exp(H_i + \varphi(H_i))$, $H_i \in \mathfrak{a}$; then $(H_i)$ converges to $H$ as the projection on an Iwasawa factor is continuous. Hence $\lim_{i \rightarrow \infty} \exp(H_i + \varphi(H_i)) = \exp(H + \varphi(H))$ and $A_\varphi$ is closed. Thus also $H_\varphi$ is a closed solvable subgroup of $G$.

Remark 3 The adjoint action of $a_\varphi$ on $\mathfrak{n}^C$ is semi-simple. The eigenvalues have an imaginary part determined by $\varphi$. Thus in general distinct $\varphi$’s will give rise to non-isomorphic subgroups $H_\varphi$. These subgroups are all 1-dimensional extensions of $N$ which is isomorphic to the Heisenberg group.

Lemma 8 Let $H$ be a linear Lie group acting simply-transitively on $\mathbb{R}^n$; then $H$ is solvable.

Proof $H$ is connected, simply-connected and has no non-trivial compact subgroup. Hence $H$ is the semi-direct product of a simply-connected solvable group by a simply-connected semi-simple Lie group $L$. This group $L$ is linear and has no non-trivial compact
subgroup. Hence $L$ is contractible and thus isomorphic to $\widetilde{SO}(2,1)$, $\widetilde{SO}(2,2)$ or $\widetilde{SU}(1,1)$ ($\widetilde{\cdot}$ denotes universal cover) or a product of these. But such a group can not be linear as its centre is not finite. Hence $L$ is trivial and $H$ is solvable.

**Remark 4** In the situation $M = SU(1, n)/U(n)$, we have

\[
\mathfrak{su}(1, n) = \left\{ \begin{pmatrix} il & b \\ b & D \end{pmatrix} \mid l \in \mathbb{R}, b \in \mathbb{C}^n, D + \tau D = 0, il + \text{Tr} D = 0 \right\};
\]

\[
P = \left\{ \begin{pmatrix} 0 & \tau b \\ b & 0 \end{pmatrix} \mid b \in \mathbb{C}^n \right\};
\]

\[
K = \left\{ \begin{pmatrix} -\text{tr} D & 0 \\ 0 & D \end{pmatrix} \mid D + \tau D = 0 \right\}.
\]

Up to conjugation, one can choose

\[
a = \mathbb{R}\begin{pmatrix} 0 & \tau e_1 \\ e_1 & 0 \end{pmatrix} =: \mathbb{R}a
\]

where $e_1$ is the first basis vector of $\mathbb{C}^n$. Then

\[
m = \left\{ \begin{pmatrix} -\frac{1}{2} \text{tr} E & 0 & 0 \\ 0 & -\frac{1}{2} \text{tr} E & 0 \\ 0 & 0 & E \end{pmatrix} \left| E \in \mathfrak{u}(n-1) \right\} \right\}.
\]

The homomorphism $\varphi$ is determined by the element $\varphi(a)$ of $m$; up to conjugation, we may assume that $\varphi(a)$ belongs to the standard maximal torus of $\mathfrak{u}(n-1)$.

### 6 Simply-transitive subgroups of the group of transvections $G(M_A)$ for $M_A = \Sigma_A/\exp tA$, when $A^2 = 0$

We have seen that for $A \neq 0$, $M_A = \Sigma_A/\exp tA$ is diffeomorphic to $T(S^{q-1} \times \mathbb{R}^{p-q}) \times \mathbb{R}^{2(n+1-p)}$ where $p$ and $q$ are integers such that $1 \leq p \leq n+1, 1 \leq q \leq p$. If $q = 1$, the manifold has two connected components, each diffeomorphic to $\mathbb{R}^{2n}$. A reasoning completely analogous to the one used in Theorem 4 gives

**Proposition 3** Let $M_A = \Sigma_A/\exp tA$, when $A^2 = 0$, be characterized by the two integers $p, q$ where $1 \leq p \leq n+1$ and $1 \leq q \leq p$. If $p = 1$, $M_A$ is diffeomorphic to two copies of $\mathbb{R}^{2n}$; $M_A$ has the flat symplectic connection and the translation group (which is the transvection group) acts simply-transitively on each component. If $q \neq 1, 2, 4$, $M_A$ does not admit a simply-transitive subgroup.
Of the remaining cases we look at the case where \( p = 2 \) in detail.

**Theorem 6** Assume \( A^2 = 0 \) and \( p = 2 \). Let \( M_{A_0} \) be a connected component of \( M_A = \Sigma_A/\exp tA \). Then \( M_{A_0} \) admits a simply-transitive subgroup if and only if \( q = 1 \).

The proof of this theorem is split into two lemmas. We first describe all the subalgebras \( h \) of \( g = p_1 + [p_1, p_1]/\mathbb{R}A \) which are supplementary to \([p_1, p_1]/\mathbb{R}A\) in \( g \); the condition \( q = 1 \) is necessary and sufficient to have such algebras. We then show that the connected subgroup of the transvection group with algebra \( h \) acts simply-transitively on \( M_{A_0} \).

We have shown in section 2 that one can choose a basis \( \{e_1, e_2; f_a, a \leq 2(n - 1); e^*_1, e^*_2\} \) of \( \mathbb{R}^{2n+2} \) in which \( A = \begin{pmatrix} 0 & 0 & I_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \) and

\[
\Omega = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & \Omega^0 & 0 & -\epsilon \\ 1 & 0 & 0 & 0 \\ 0 & \epsilon & 0 & 0 \end{pmatrix}
\]

with \( \epsilon = \begin{cases} 1 & \text{if } q = 2 \\ -1 & \text{if } q = 1 \end{cases} \)

and a base point \( x_0 = e^*_1 \in \Sigma_A = \{(x, X, x^*_1) \mid (x^1)^2 + \epsilon(x^2)^2 = 1\} \) so that

\[
p_1 = \left\{ \begin{pmatrix} 0 & -\epsilon p \\ p & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \epsilon p' \\ p' & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -\epsilon p \\ p & 0 \\ 0 & 0 \end{pmatrix} \right\}
\]

with \( p, p' \in \mathbb{R}, P \in \mathbb{R}^{2(n-1)} \) and \( P = \iota(P)\Omega^0 \).

A subspace which is supplementary to \( [p_1, p_1]/\mathbb{R}A \) in \( g = p_1 + [p_1, p_1]/\mathbb{R}A \) is of the form

\[
h_{B,\tilde{a},\tilde{b},\tilde{c},a,c} = \{ K_{B,\tilde{a},\tilde{b},\tilde{c},a,c}(p, P, p') + \mathbb{R}A \mid p, p' \in \mathbb{R}, P \in \mathbb{R}^{2n-2} \} \]
for a matrix $B \in \mathfrak{gl}(\mathbb{R}^{2(n-1)})$, vectors $\tilde{a}, \tilde{b}, \tilde{c} \in \mathbb{R}^{2(n-1)}$ and real numbers $a, c \in \mathbb{R}$, with the matrix $K_{B,\tilde{a},\tilde{b},\tilde{c},a,c}(p, P, p')$ defined by

$$
\begin{pmatrix}
0 & -cp \\
p & 0
\end{pmatrix}
\begin{pmatrix}
-P \\
-\epsilon(p\tilde{a} + BP + p'\tilde{c})
\end{pmatrix}
\begin{pmatrix}
-p'' & cp' \\
p' & p''
\end{pmatrix}
$$

where $p'' := ap + \Omega^0(\tilde{b}, P) + cp'$. Observe that the bracket is given by

$$
[K_{B,\tilde{a},\tilde{b},\tilde{c},a,c}(p, P, p'), K_{B,\tilde{a},\tilde{b},\tilde{c},a,c}(q, Q, q')] =
\begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
-R \\
-\epsilon S
\end{pmatrix}
\begin{pmatrix}
-r_1 & \epsilon r' \\
r' & r_2
\end{pmatrix}
$$

with

$$
\begin{cases}
R := qBP - pBQ + (qp' - pq')\tilde{c}, \\
S = \epsilon(-qP + pQ), \\
r' = -2p\Omega^0(\tilde{b}, Q) + 2q\Omega^0(\tilde{b}, P) - 2\epsilon(pq' - pq') - \epsilon\Omega^0(p\tilde{a} + BP + p'\tilde{c}, Q) + \epsilon\Omega^0(q\tilde{a} + BQ + q'\tilde{c}, P), \\
r_1 = 2\epsilon(pq' - pq') + 2\Omega_0(P, Q), \\
r_2 = 2\epsilon(pq' - pq') - 2\epsilon\Omega^0(p\tilde{a} + BP + p'\tilde{c}, q\tilde{a} + BQ + q'\tilde{c}).
\end{cases}
$$

The subspace $\mathfrak{h}_{B,\tilde{a},\tilde{b},\tilde{c},a,c}$ is a Lie subalgebra of $\mathfrak{g}$ if and only if $S = BR + r'\tilde{c}$ and $\frac{1}{2}(r_2 + r_1) = \Omega^0(\tilde{b}, R) + cr'$, i.e. if and only if for any $p, p', q, q' \in \mathbb{R}$ and any $P, Q \in \mathbb{R}^{2n-2}$ one has

$$
\begin{align*}
\epsilon(-qP + pQ) &= qB^2P - pB^2Q + (qp' - pq')B\tilde{c} + \left(-2p\Omega^0(\tilde{b}, Q) + 2q\Omega^0(\tilde{b}, P)\right)\tilde{c} \\ &\quad + \left(-2\epsilon(pq' - p'q') - \epsilon\Omega^0(p\tilde{a} + BP + p'\tilde{c}, Q) + \epsilon\Omega^0(q\tilde{a} + BQ + q'\tilde{c}, P)\right)\tilde{c} \\ 2\epsilon(pq' - pq') &= \Omega^0(P, Q) - \epsilon\Omega_0(p\tilde{a} + BP + p'\tilde{c}, q\tilde{a} + BQ + q'\tilde{c}) \\ &= \Omega^0(\tilde{b}, qBP - pBQ + (qp' - pq')\tilde{c}) + c\left(-2p\Omega^0(\tilde{b}, Q) + 2q\Omega^0(\tilde{b}, P)\right) \\ &\quad + c\left(-2\epsilon(pq' - p'q') - \epsilon\Omega^0(p\tilde{a} + BP + p'\tilde{c}, Q) + \epsilon\Omega^0(q\tilde{a} + BQ + q'\tilde{c}, P)\right).
\end{align*}
$$
The terms in $q'P$ in equation (6.1), $\epsilon \Omega^0(q'\tilde{c}, P)\tilde{c}$, imply that

$$\tilde{c} = 0$$

and equation (6.1) is fulfilled provided we also have

$$B^2 = -\epsilon \text{Id}.$$ 

The terms in $pq - p'q'$ in equation (6.2) then lead to

$$\epsilon = -1 \ (\text{hence } q = 1), \quad \text{and } c^2 = 1. \quad (6.3)$$

Terms in $qP$ in equation (6.2), $q\Omega^0(BP, \tilde{a}) = \Omega^0(\tilde{b}, qBP) + c\left(2q\Omega^0(\tilde{b}, P) - \Omega^0(q\tilde{a}, P)\right)$ yield

$$\Omega^0(\tilde{b}, (B + 2c)\cdot) = \Omega^0(\tilde{a}, (c\text{Id} - B)\cdot) \quad (6.4)$$

and terms in $P, Q$ in equation (6.2) yield

$$\Omega^0((B - c\text{Id})X, (B - c\text{Id})Y) = 0 \quad \forall X, Y \in \mathbb{R}^{2(n-1)}. \quad (6.5)$$

Since $(B + 2c\text{Id})(B - 2c\text{Id}) = -3\text{Id}$ and $(c\text{Id} - B)(B - 2c\text{Id}) = 3(cB - \text{Id})$, $\tilde{b}$ is defined by equation (6.4) through

$$\Omega^0(\tilde{b}, \cdot) = \Omega^0(\tilde{a}, (c\text{Id} - B)\cdot). \quad (6.6)$$

Hence

**Lemma 9** A symmetric space of dimension $2n$, $M_A = \Sigma_A/\exp tA$, $A^2 = 0$, with non-abelian solvable transvection group (i.e. $p=2$) admitting a locally simply-transitive subgroup of the transvection group corresponds to the value $q = 1$.

**Lemma 10** A symmetric space $M_A = \Sigma_A/\exp tA$, with $A^2 = 0$ and with a non-abelian solvable transvection group and $q = 1$ admits a family of subgroups of the transvection group acting locally simply-transitively on $M_A$. The algebra of such a subgroup is determined by an endomorphism $B$ of $\mathbb{R}^{2(n-2)}$ satisfying

$$B^2 = \text{Id} \quad \text{and} \quad \Omega^0((B - c\text{Id})X, (B - c\text{Id})Y) = 0 \quad \forall X, Y \in \mathbb{R}^{2(n-1)}, \quad (6.7)$$

a vector $\tilde{a} \in \mathbb{R}^{2(n-2)}$, a real number $a$ and a sign $c$, (i.e. $c^2 = 1$); it is of the form

$$b_{B,\tilde{a},a,c} = \{ K_{B,\tilde{a},a,c}(p, p') + \mathbb{R}A \mid p, p' \in \mathbb{R}, P \in \mathbb{R}^{2n-2} \}$$
where

\[
K_{B,\tilde{a},a,c}(p, P, p') = \begin{pmatrix}
0 & p & 0 \\
p & 0 & 0 \\
0 & 0 & 0 \\
\end{pmatrix} \begin{pmatrix}
-\frac{P}{\tilde{p}\tilde{a} + BP} & 0 & 0 \\
0 & \frac{P}{\tilde{p}\tilde{a} + BP} & 0 \\
0 & 0 & P \\
\end{pmatrix} \begin{pmatrix}
-p'' & -p' \\
p' & p'' \\
0 & 0 \\
\end{pmatrix}
\]

with \( p'' = ap + \Omega^0(\tilde{a}, (\text{Id} - cB)P) + cp'. \)

Observe that

\[
\text{Ad} \begin{pmatrix}
\text{Id} & 0 & 0 \\
0 & S & 0 \\
0 & 0 & \text{Id} \\
\end{pmatrix} K_{B,\tilde{a},a,c}(p, P, p') = K_{SBS^{-1},\tilde{a},a,c}(p, SP, p')
\]

for any \( S \in Sp(\mathbb{R}^{2(n-1)}, \Omega^0) \) so that, up to isomorphism given by conjugation in the group of affine symplectomorphisms of \( M_A \), \( B \) can be defined up to conjugation in \( Sp(\mathbb{R}^{2(n-1)}, \Omega^0) \).

Similarly

\[
\text{Ad} \begin{pmatrix}
\text{Id} & \frac{-u}{0} & 0 \\
0 & \text{Id} & \frac{u}{0} \\
0 & 0 & \text{Id} \\
\end{pmatrix} K_{B,\tilde{a},a,c}(p, P, p') = K_{B,\tilde{a}+u,a-c\Omega^0(u,\tilde{a}),c}(p, P, p'')
\]

up to the addition of a multiple of \( A \), with \( p''' = p' + \Omega^0(u, \tilde{p}\tilde{a} + BP) \), so that one can assume, up to conjugation in the group of affine symplectomorphisms of \( M_A \), that \( \tilde{a} = 0 \), and

\[
\text{Ad} \begin{pmatrix}
\text{Id} & 0 & \frac{r}{0} \\
0 & \text{Id} & \frac{-r}{0} \\
0 & 0 & \text{Id} \\
\end{pmatrix} K_{B,0,a,c}(p, P, p') = K_{B,0,a+2r,c}(p, P, p' - 2rp)
\]

so that one can assume \( a = 0 \).

We now proceed to prove that the connected subgroup of the transvection group with algebra \( h_{B,\tilde{a},a,c} \) acts simply-transitively on a connected component \( M_{A_0} \) of \( M_A \).
Transitive groups acting on Symplectic Symmetric Spaces of Ricci Type

In view of the remarks above, it is enough to consider the case $\mathfrak{h}_{B,c} := \mathfrak{h}_{B,0,0,c}$ where $\bar{a} = 0$ and $a = 0$.

As above, let $\Sigma_A = \{(x^1, x^2, X^1, \ldots, X^{2(n-1)}, x^1_x, x^2_x) \mid x^1_x - x^2_x = 1\}$ and choose the connected component $\Sigma^o_A = \{(x^1, x^2, X^1, \ldots, X^{2(n-1)}, x^1_x = \text{ch} \alpha, x^2_x = \text{sh} \alpha)\}$. Observe that $\partial_\alpha = \text{sh} \alpha \partial_{x^1} + \text{ch} \alpha \partial_{x^2}$. Consider $M_{A_0} = \Sigma^o_A / \exp tA$ and the canonical projection $\pi : \Sigma^o_A \rightarrow M_{A_0} = \Sigma^o_A / \exp tA$. We endow $M_{A_0}$ with coordinates $(y^0, y^1, \ldots, y^{2(n-1)}, \gamma)$ defined by

\[
y^0(\pi(x^1, x^2, X^1, \ldots, X^{2(n-1), \text{ch} \alpha, \text{sh} \alpha})) = -x^1 \text{sh} \alpha + x^2 \text{ch} \alpha \n y^a(\pi(x^1, x^2, X^1, \ldots, X^{2(n-1), \text{ch} \alpha, \text{sh} \alpha})) = \text{ch} x^a, \quad 1 \leq a \leq 2(n-1) \n \gamma(\pi(x^1, x^2, X^1, \ldots, X^{2(n-1), \text{ch} \alpha, \text{sh} \alpha})) = \alpha.
\]

This shows in particular that $M_{A_0}$ is diffeomorphic to $\mathbb{R}^{2n}$. Furthermore,

\[
\pi_* \partial_{x^1} = - \text{sh} \alpha \partial_{y^0} \quad \pi_* \partial_{x^2} = \text{ch} \alpha \partial_{y^0} \n \pi_* \partial_{X^a} = \partial_{y^a} \quad \pi_* \partial_{\alpha} = \partial_{\gamma} + (x^2 \text{sh} \alpha - x^1 \text{ch} \alpha) \partial_{y^0}.
\]

If a bar denotes the horizontal lift on $\Sigma^o_A$ of vectors on $M_{A_0}$, then one has

\[
\overline{\partial_{y^0}} = \text{sh} \alpha \partial_{x^1} + \text{ch} \alpha \partial_{x^2} \n \overline{\partial_{y^a}} = \partial_{X^a} + \sum_b \Omega^0_{ab} X^b (\text{ch} \alpha \partial_{x^1} + \text{sh} \alpha \partial_{x^2}) \n \overline{\partial_{\gamma}} = \partial_{\alpha} + (2x^1 \text{sh} \alpha \text{ch} \alpha - x^2 (\text{ch}^2 \alpha + \text{sh}^2 \alpha)) \partial_{x^1} \n \quad + (x^1 (\text{ch}^2 \alpha + \text{sh}^2 \alpha) - 2x^2 \text{sh} \alpha \text{ch} \alpha) \partial_{x^2}
\]

Hence the symplectic form on $M_{A_0}$ has the form

\[
\omega = dy^0 \wedge d\gamma + \frac{1}{2} \sum_{1 \leq a,b \leq 2(n-1)} \Omega^0_{ab} dy^a \wedge dy^b
\]

showing in particular that these are global Darboux coordinates.

Since the projection $\pi : \Sigma_A \rightarrow \Sigma^o_A / \exp tA = M_{A_0}$ is equivariant with respect to the group of linear symplectic transformations of $\mathbb{R}^{2(n+2)}$ commuting with $A$, the fundamental vector fields on $M_{A_0}$ associated to the elements of $\mathfrak{h}_{B,c}$ are the projections of the corresponding vector fields on $\Sigma^o_A$.

The fundamental vector field on $\Sigma^o_A$ associated to

\[
K_{B,c}(p, P, p') = \begin{pmatrix}
0 & p \\
p & 0
\end{pmatrix} \begin{pmatrix}
-P \\
(BP)
\end{pmatrix} \begin{pmatrix}
-cp' & -p' \\
p' & cp'
\end{pmatrix}
\]

\[
= \begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix} \begin{pmatrix}
P & BP \\
p & 0
\end{pmatrix}
\]

\[
= \begin{pmatrix}
0 & p \\
p & 0
\end{pmatrix}
\]

is given by

\[
(K_{B,c}(p, P, p'))^{\Sigma_{A}}_{(x,X,\alpha)} = \left( -px^2 + \Omega^0(P, X) + p'(c \tanh \alpha + \tanh \alpha) \right) \partial_{x_1} \\
+ \left( -px^1 - \Omega^0(BP, X) - p'(c \tanh \alpha + c \tanh \alpha) \right) \partial_{x_2} \\
- \sum_{a=1}^{2(n-1)} (c \tanh \alpha P^a + \tanh \alpha (BP)^a) \partial_{X^a} \\
- p \partial_{\alpha}
\]

and the fundamental vector field on \(M_{A_0}\) is \((K_{B,c}(p, P, p'))^{*M_{A_0}}_{\pi(x,X,\alpha)} = \pi_{*}(K_{B,c}(p, P, p'))^{*\Sigma_{A}}_{(x,X,\alpha)}\) so that

\[
(K_{B,c}(p, P, p'))^{*M_{A_0}}_{(y^0, Y = (y^1, \ldots, y^{2(n-1)}), \gamma)} = -p\partial_{\gamma} - \sum_{a=1}^{2(n-1)} (c \tanh \alpha P^a + \tanh \alpha (BP)^a) \partial_{y^a} \\
- (\Omega^0(P, Y) \tanh \gamma + \Omega^0(BP, Y) \tanh \gamma + p'(\tanh \gamma + c \tanh \gamma)^2) \partial_{y^0}.
\]

Recall that \(B^2 = \text{Id}\) so that the matrix \((c \tanh \gamma \text{Id} + \tanh \gamma B)\) is invertible for all \(\gamma\). Hence the fundamental vector fields are linearly independent at each point of \(M_{A_0}\). Thus any orbit of the connected subgroup with algebra \(\mathfrak{h}_{B,c}\) is open. The connectedness of \(M_{A_0}\) implies that the action is transitive. Hence

**Theorem 7** Each connected subgroup \(H_{B,\tilde{a},a,c}\) of the transvection group of \(M_{A_0}\), with algebra \(\mathfrak{h}_{B,\tilde{a},a,c}\) acts globally simply-transitively on \(M_{A_0}\). In particular, the groups \(H_{B,a,\tilde{a},c}\) are symplectic groups and symmetric spaces.

To conclude this paragraph, we investigate the question of the strongly Hamiltonian character of the action of the group \(H_{B,c}\) on \(M_{A_0}\).

**Proposition 4** The action of the group \(H_{B,c}\) on \(M_{A_0}\) is strongly Hamiltonian if and only if \(B = c \text{Id}\). The group \(H_{c \text{Id},c}\) is the 1-dimensional extension of the Heisenberg group of dimension \(2n - 1\) by the dilation automorphism.

**Proof** The contraction of the 2-form \(\omega = dy^0 \wedge d\gamma + \frac{1}{2} \sum_{1 \leq a, b \leq 2(n-1)} \Omega^0_{ab} dy^a \wedge dy^b\) on \(M_{A_0}\) by the fundamental vector field \((K_{B,c}(p, P, p'))^{*M_{A_0}}\) is the differential of a function \(f_{(p, P, p')}^{B,c}(p, P, p')\)

\[
\iota \left( (K_{B,c}(p, P, p'))^{*M_{A_0}} \right) \omega = pdy^0 - p' e^{2c\gamma} d\gamma \\
- \left( \Omega^0(P, Y) \tanh \gamma + \Omega^0(BP, Y) \tanh \gamma \right) d\gamma \\
- \sum_{a,b=1}^{2(n-1)} (P^a \tanh \gamma + (BP)^a \tanh \gamma) \Omega^0_{ab} dy^b \\
= df_{B,c}^{(p, P, p')}
\]

with

\[
f_{B,c}^{(p, P, p')}((y^0, Y, \gamma)) := py^0 - \frac{1}{2c} p' e^{2c\gamma} - \Omega^0(P, Y) \tanh \gamma - \Omega^0(BP, Y) \tanh \gamma.
\]
The Lie algebra structure on $\mathfrak{h}_{B,c}$ is given by

$$[K_{B,c}(p, P, p'), K_{B,c}(q, Q, q')] = K_{B,c}(0, qBP - pBQ, -2c(pq' - qp') + \Omega^0(BP, Q) + \Omega^0(P, BQ))$$

and

$$(K_{B,c}(p, P, p'))^{\*M_{Ao}}(f^{(g, Q, q')}_{B,c}) = f^{(0, qBP - pBQ, -2c(pq' - qp') + \Omega^0(BP, Q) + \Omega^0(P, BQ))}_{B,c}$$

so the action is strongly Hamiltonian if and only if

$$\Omega^0(BP, BQ) = \Omega^0(P, Q) \quad \forall P, Q; \quad (6.8)$$

since $B^2 = \text{Id}$, equation (6.8) also implies $\Omega^0(BP, Q) = \Omega^0(P, BQ)$; the relation (6.5) $\Omega^0((B - c\text{Id})P, (B - c\text{Id})Q) = 0$, becomes $2\Omega^0(P, Q) - 2c\Omega^0(BP, Q) = 0$ and this is equivalent to

$$B = c\text{Id}.$$ 

Note that the Lie algebra structure of $\mathfrak{h}_{c\text{Id},c}$ with $c^2 = 1$ is

$$[K_{c\text{Id},c}(p, P, p'), K_{c\text{Id},c}(q, Q, q')] = K_{c\text{Id},c}(0, c(qP - pQ), 2c(qp' - q'p + \Omega^0(P, Q))).$$

Hence the derived algebra $\mathfrak{h}'_{c\text{Id},c} = [\mathfrak{h}_{c\text{Id},c}, \mathfrak{h}_{c\text{Id},c}]$ is isomorphic to the Heisenberg algebra $\mathfrak{h}_{2n-1}$ in dimension $2n - 1$

$$\mathfrak{h}'_{c\text{Id},c} \simeq \{(P, p') \mid P \in \mathbb{R}^{2(n-1)}, p' \in \mathbb{R}\} \quad \text{with} \quad [(P, p'), (Q, q')] = (0, 2c\Omega^0(P, Q)) \simeq \mathfrak{h}_{2n-1},$$

and

$$\mathfrak{h}_{c\text{Id},c} = \mathfrak{h}'_{c\text{Id},c} \oplus \mathbb{R}D \quad \text{with} \quad D((P, p')) := (-cP, -2cp') \simeq \mathcal{K}_{2n}.$$

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