Diagonal $K$-matrices and transfer matrix eigenspectra associated with the $G_2^{(1)}$ $R$-matrix

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Abstract

We find all the diagonal $K$-matrices for the $R$-matrix associated with the minimal representation of the exceptional affine algebra $G_2^{(1)}$. The corresponding transfer matrices are diagonalized with a variation of the analytic Bethe ansatz. We find many similarities with the case of the Izergin-Korepin $R$-matrix associated with the affine algebra $A_2^{(2)}$. 
1 Introduction

A systematic method, which extends the Quantum Inverse Scattering Method, has been developed by Sklyanin [1] and others [2] to construct integrable models with open boundaries. The procedure begins with an \( R \)-matrix \( R(u) \), which by definition is a solution of the Yang-Baxter equation (YBE). As is well-known, given such a solution it is possible to construct a commuting transfer matrix which defines an integrable vertex model with periodic boundaries. For open boundary versions of this model one requires solutions of a boundary version of the YBE, or ‘reflection-factorization equation’. Given \( R(u) \) and a solution \( K^{-}(u) \) of this equation, known as a \( K \)-matrix, a commuting transfer matrix \( t(u, \omega) \) can be defined – under certain fairly general technical assumptions on \( R(u) \). The parameters \( \omega = (\omega_1, \ldots, \omega_N) \) in the transfer matrix are “inhomogeneities”; setting them to zero and taking the logarithmic derivative yields an integrable open spin chain [1], whereas setting them to alternate as \( \omega_i = (-)^i u \) yields an integrable vertex model with open boundaries [3, 4].

The boundary YBE has been investigated for various \( R \)-matrices. In particular, it is known [5] that \( K^{-}(u) = 1 \) is a solution for all the \( R \)-matrices associated with the vector representations of the non-exceptional affine Lie algebras [6, 7] in the homogeneous gauge, except for the ones associated with \( D_{n+1}^{(2)} \). For the \( A_n^{(1)} \) family, all the diagonal \( K \)-matrices are known [8] and the corresponding transfer matrices have been diagonalized [9]. All diagonal \( K \)-matrices and transfer matrix eigenspectra are also known for the three-state models associated with \( A_2^{(2)} \) (the Izergin-Korepin model) [10, 4] and the spin-1 representation of \( A_1^{(1)} \) (the Zamolodchikov-Fateev model) [11]. Very recently, the transfer matrices corresponding to the \( K^{-}(u) = 1 \) solution for \( A_{2n}^{(2)} \) have been diagonalized [12].

In this paper we study the \( R \)-matrix [13] associated with the minimal representation of \( G_2^{(1)} \), being the smallest \( R \)-matrix associated with an exceptional affine algebra. We show that the \( R \)-matrix satisfies all the requirements given in [2] for the construction of a commuting Sklyanin transfer matrix. In particular, \( K^{-}(u) = 1 \) is a solution to the boundary YBE. We find, in addition, that there are only two other inequivalent diagonal solutions. For all three \( K \)-matrices we obtain the corresponding transfer matrix eigenspectra, using a modification of the analytic Bethe ansatz very similar to that employed in [12]. We find that the Bethe ansatz equations obtained are very similar to those for the Izergin-Korepin model [10, 4], which have recently [14] found applications to surface critical phenomena in self-avoiding walks.
2 The $G_2^{(1)}$ $R$-matrix and related $K$-matrices

The $G_2^{(1)}$ $R$-matrix $R(u)$ acts in the tensor product of the (seven-dimensional) representation $V_{\Lambda_2}$ of $U_q(G_2)$ with itself. In terms of the projectors $P_\Lambda$ onto the irreducible representations $V_\Lambda$ occurring in the $U_q(G_2)$-module decomposition $V_{\Lambda_2} \otimes V_{\Lambda_2} = V_0 \oplus V_{\Lambda_1} \oplus V_{\Lambda_2} \oplus V_{2\Lambda_2}$, we have the expression \[ \tilde{R}(u) = \sum_{\Lambda = 2\Lambda_2, \Lambda_1, \Lambda_2, 0} \rho_\Lambda(u) P_\Lambda \] (1) for $\tilde{R}(u) = PR(u)$, with $P$ being the permutation operator and the functions $\rho_\Lambda(u)$ given by

\[
\rho_\Lambda(u) = \begin{cases} 
[1 + u][4 + u][6 + u] & \Lambda = 2\Lambda_2 \\
[1 - u][4 + u][6 + u] & \Lambda = \Lambda_1 \\
[1 + u][4 - u][6 + u] & \Lambda = \Lambda_2 \\
[1 - u][4 + u][6 - u] & \Lambda = 0 
\end{cases}
\]

where $[x] \equiv (q^x - q^{-x})(q - q^{-1})^{-1}$. Explicit expressions for the projectors $P_\Lambda$ in terms of $q$-Wigner coefficients can also be found in [13]. The $R$-matrix defines an integrable seven-state 175-vertex model with periodic boundaries. There also exists a related integrable RSOS model [13], with an elliptic generalization.

For the transfer matrix $t_P(u, \omega) = tr_a T_a(u, \omega)$ with periodic boundary conditions (the monodromy matrix $T_a(u, \omega)$ is defined in [11]) the eigenvalue expression $\Lambda_\wp(u, \omega)$ was conjectured in [16] on the basis of Dynkin diagram considerations, before the explicit form of the $R$-matrix was known. It has since been confirmed using the analytic Bethe ansatz in a recent study [17] in which the fused models were also considered. We present it here for later comparison. Introduce the function $f(u) = \prod_{j=1}^{N} [u + \omega_j]$, through which are defined

\[
\begin{align*}
\phi_{-3}(u, \omega) &= f(1 + u) f(4 + u) f(6 + u), \\
\phi_{-2}(u, \omega) &= \phi_{-1}(u, \omega) = f(u) f(4 + u) f(6 + u), \\
\phi_0(u, \omega) &= f(u) f(3 + u) f(6 + u), \\
\phi_1(u, \omega) &= \phi_2(u, \omega) = f(u) f(2 + u) f(6 + u), \\
\phi_3(u, \omega) &= f(u) f(2 + u) f(5 + u).
\end{align*}
\] (2)

Define also the functions $d^{(i)}(u) = \prod_{j=1}^{N_i} [u - iu_j^{(i)}]$ for $i = 1, 2$. The eigenvalue expression is then given by

\[
\Lambda_\wp(u, \omega) = \sum_{j=-3}^{3} \phi_j(u, \omega) a_j(u),
\] (3)
where we have defined
\[
\begin{align*}
a_{-3}(u) &= \frac{d^{(2)}(u-1/2)}{d^{(2)}(u+1/2)}, \\
a_{-2}(u) &= \frac{d^{(1)}(u-1) d^{(2)}(u+3/2)}{d^{(1)}(u+2) d^{(2)}(u+1/2)}, \\
a_{0}(u) &= \frac{d^{(2)}(u+9/2) d^{(2)}(u+3/2)}{d^{(2)}(u+5/2) d^{(2)}(u+7/2)}, \\
a_{1}(u) &= \frac{d^{(1)}(u+1) d^{(2)}(u+9/2)}{d^{(1)}(u+4) d^{(2)}(u+5/2)}, \\
a_{2}(u) &= \frac{d^{(1)}(u+7) d^{(2)}(u+9/2)}{d^{(1)}(u+4) d^{(2)}(u+11/2)}, \\
a_{3}(u) &= \frac{d^{(2)}(u+13/2)}{d^{(2)}(u+11/2)}.
\end{align*}
\]

(4)

The following are properties [13] of the $R$-matrix:

\begin{align*}
\text{commutativity} & : [\hat{R}_{12}(u), \hat{R}_{12}(v)] = 0, \\
\text{unitarity} & : R_{12}(u) R_{12}^{t12}(-u) = \xi(u), \\
\text{regularity} & : R_{12}(0) = \xi(0)^{1/2} P_{12}, \\
\text{PT} – symmetry & : R_{21}(u) \equiv P_{12} R_{12}(u) P_{12} = R_{12}^{t12}(u), \\
\text{crossing} – symmetry & : R_{12}(u) = – V R_{12}^{t12}(-u – \rho) \frac{1}{V}.
\end{align*}

(5) (6) (7) (8) (9)

In the above relations, we have $\xi(u) = [1 + u][4 + u][6 + u][1 - u][4 - u][6 - u]$; the crossing-parameter is $\rho = 6$, and the crossing-matrix $V = F(\sigma g)^2$ satisfying $V^2 = 1$.

We also use the notation $\hat{A}$ to denote $A \otimes 1$ etc. In a basis for $V_{A_2}$ with the ordering of weight vectors given by \{v_{-3}, v_{-2}, \ldots , v_{3}\} we have the explicit expressions

\[
\sigma = \text{diag}(-i, 1, -i, 1, i, i), \\
g = \text{diag}(q^{-5/2}, q^{-2}, q^{-1/2}, 1, q^{1/2}, q^{2}, q^{5/2}),
\]

and $F$ is the matrix with 1 along the anti-diagonal and 0 elsewhere. The crossing-relation (3), which is in a “standard form” (apart from the minus sign), can be obtained from the one given in [15]

\[
R_{12}(u) = -\left((\sigma g)^{-1} \otimes (\sigma g)\right) \frac{1}{F} R_{12}^{t12}(-u - \rho) \frac{1}{F} \left((\sigma g) \otimes (\sigma g)^{-1}\right),
\]

by using the symmetry relation $[R(u), (\sigma g) \otimes (\sigma g)] = 0$, which in turn can be checked explicitly.

For an $R$-matrix satisfying properties (3) to (9) the Sklyanin transfer matrix

\[
t(u, \omega) = \text{tr}_a \ K^a (u) T_a (u, \omega) K^a (u) \tilde{T}_a (u, \omega),
\]

(10)
with monodromy matrices defined as

\[
T_a(u, \omega) = R_{a_1}(u + \omega_1) R_{a_2}(u + \omega_2) \cdots R_{a_N}(u + \omega_N),
\]

\[
\widetilde{T}_a(u, \omega) = R_{N a}(u - \omega_N) \cdots R_{2 a}(u - \omega_2) R_{1 a}(u - \omega_1),
\]

forms a commuting family \([t(u, \omega), t(v, \omega)] = 0\) if the \(K\)-matrices \(K^\pm(u)\) satisfy the boundary YBEs [1, 2]

\[
\begin{align*}
R_{12}(u - v) K^1(u) R_{21}(u + v) K^2(v) &= K^2(v) R_{12}(u + v) K^1(u) R_{21}(u - v), \\
R_{12}(-u + v) (K^+)^{t_1}(u) &M^{-1} R_{21}(-u - v - 2\rho) \tilde{M}(K^+)^{t_2}(v) = \\
(2K^+)^{t_2}(v) &M^{-1} R_{12}(-u - v - 2\rho) \tilde{M}^{-1}(K^+)^{t_1}(u) R_{21}(-u + v),
\end{align*}
\]

with \(M \equiv -V^t V = M^t\). Only the \(K^-(u)\) equation (12) needs consideration since the \(K^+(u)\) equation is related to it by the automorphism \(K^+(u) = K^-(u - \rho)^t M\).

Due to the property (3) we have immediately the “trivial” solution \(\{K^-(u) = 1, K^+(u) = M\}\) [3], which leads to a \(U_q(G_2)\)-invariant open spin chain. To obtain the most general diagonal solution we need to solve the boundary YBE (12) with the explicit form of \(R(u)\). This leads to a system of 63 coupled functional equations for the non-zero entries of \(K^-(u)\) which we solve with the help of Mathematica. The simplest two equations couple only the entry \(K^-(u)_{11}\) to \(K^-(u)_{22}\) and \(K^-(u)_{33}\) respectively. Without loss of generality we set \(K^-(u)_{11} = 1\). We are also interested only in solutions with “initial condition” \(K^-(0) = 1\). These equations can then be easily solved to obtain

\[
K^-(u)_{22} = \frac{1 + c_2 q^{2u}}{1 + c_2 q^{-2u}}, \quad K^-(u)_{33} = \frac{1 + c_3 q^{2u}}{1 + c_3 q^{-2u}},
\]

where \(c_2\) and \(c_3\) are arbitrary parameters. Consideration of the equations which couple \(K^-(u)_{22}\) and \(K^-(u)_{33}\) to \(K^-(u)_{44}\) leads to restrictions on the coefficients \(c_i\); we find that either \(c_2 = 0, c_2 = c_3 = 0\) or \(c_2 = c_3 = \pm q\). The last two choices are ruled out by consideration of the equations which couple in \(K^-(u)_{55}\). The second choice eventually leads to \(K^-(u) = 1\) while the first choice leads to only two other inequivalent solutions which can be expressed in the form

\[
K^-(u) = \Gamma_\pm(u) = \text{diag}(1, 1, \Psi_\pm, \Psi_\pm, q^{4u}, q^{4u}),
\]

with \(\Psi_\pm = (q \pm q^{2u})(q \pm q^{-2u})^{-1}\). The existence of three diagonal \(K\)-matrices for \(G_2^{(1)}\) is directly analogous to the \(A_2^{(2)}\) model [3], and in contrast to models like \(A_n^{(1)}\) for which there are free parameters in the \(K\)-matrices [8].
3 Transfer matrix diagonalization

We consider the diagonalization of the Sklyanin transfer matrix corresponding to the three cases

(i) \( K^-(u) = 1, \ K^+(u) = M, \)

(ii) \( K^-(u) = \Gamma_+(u), \ K^+(u) = \Gamma_+(-u - \rho)^t M, \)

(iii) \( K^-(u) = \Gamma_-(u), \ K^+(u) = \Gamma_-(-u - \rho)^t M, \)

i.e. boundaries of “non-mixed” type. From the properties (5) to (9) of the \( R \)-matrix can be inferred several properties of the Sklyanin transfer matrix (10). Firstly, due to (9), there is crossing-symmetry

\[
t(u, \omega) = t(-u - \rho, \omega).
\] (16)

The proof is a simple generalization of that given in [10] for the case \( K^-(u) = 1 \). We also have the fusion equation

\[
\tilde{t}(u, \omega) = \xi(2u + 2\rho)t(u, \omega)t(u + \rho, \omega) - \prod_{j=1}^{N} \xi(u + \rho + \omega_j)\xi(u + \rho - \omega_j)\Delta\{K^-(u)\}\Delta\{K^+(u)\},
\] (17)

which relates \( t(u, \omega) \) to the transfer matrix \( \tilde{t}(u, \omega) \) for the fused model [18]. In equation (17) \( \Delta\{K^\pm(u)\} \) are quantum determinants given by

\[
\Delta\{K^+(u)\} = \text{tr}_{12} \left\{ \tilde{P}_{12}^{12} V V K^+(u + \rho) M^{-1} R_{12}(-2u - 3\rho)^t M K^+(u) \right\},
\]

\[
\Delta\{K^-(u)\} = \text{tr}_{12} \left\{ \tilde{P}_{12}^{12} K^-(u) R_{21}(2u + \rho)^2 K^-(u + \rho)^t V V \right\},
\] (18)

with the projector \( \tilde{P}_{12} = \frac{1}{7} V P_{12}^{12} V \). Define the function \( g(u) = [1 - u][6 - u][4 + u] \). By explicit calculation we find

\[
\Delta\{K^-(u)\} = \beta_+(u) g(2u + \rho)
\]

\[
\Delta\{K^+(u)\} = \beta_-(u) g(-2u - 3\rho)
\] (19)

where for each case (in an obvious notation)

\[
\beta_+(u) = \begin{cases}
1 & q^{6+4u}(1 \pm q^{7+2u})(1 \pm q^{11+2u})(1 \pm q^{13+2u})(1 \pm q^{5+2u})^{-1}, \\
q^{-64-8u}(1 \pm q^{11+2u})(1 \pm q^{13+2u})(1 \pm q^{17+2u})(1 \pm q^{19+2u})^{-1}
\end{cases}
\]

\[
\beta_-(u) = \begin{cases}
q^{6+4u}(1 \pm q^{7+2u})(1 \pm q^{11+2u})(1 \pm q^{13+2u})(1 \pm q^{5+2u})^{-1}, \\
q^{-64-8u}(1 \pm q^{11+2u})(1 \pm q^{13+2u})(1 \pm q^{17+2u})(1 \pm q^{19+2u})^{-1}
\end{cases}
\]
The two relations (16) and (17) provide powerful constraints on the eigenvalues \( \Lambda(u, \omega) \) of \( t(u, \omega) \). They are key ingredients in the analytic Bethe ansatz for open boundaries formulated in [10]. In fact, together with the condition of periodicity, an analysis of asymptotic behaviour and the “dressing hypothesis”, the eigenvalue expression for the Izergin-Korepin model where \( K^{-1}(u) \) can be derived. As we will soon explain (see also [12]), this procedure is not adequate in general and has to be supplemented by an extra assumption.

Define \( F(u) = \prod_{j=1}^{N} [u + \omega_j][u - \omega_j] \) and let \( \Phi_j(u, \omega) \) be related to \( F(u) \) as \( \phi_j(u, \omega) \) is related to \( f(u) \) in (2). Define also

\[
D^{(i)}(u) = \prod_{j=1}^{N_i} [u + iu_{j}^{(i)}][u - iu_{j}^{(i)}] \tag{20}
\]

for \( i = 1, 2 \), and let \( A_j(u) \) be related to \( D^{(i)}(u) \) as \( a_j(u) \) is related to \( d^{(i)}(u) \) in (4). The functions \( \Phi_j(u, \omega) \) and \( A_j(u) \) are the “doubled” versions of \( \phi_j(u, \omega) \) and \( a_j(u) \) respectively. Our ansatz for the eigenvalue \( \Lambda(u, \omega) \) is

\[
\Lambda(u, \omega) = \sum_{j=-3}^{3} \alpha_j(u)\Phi_j(u, \omega)A_j(u), \tag{21}
\]

where \( \alpha_j(u) \) are functions independent of lattice size \( N \) and inhomogeneities \( \omega \). This is the correct form in all known cases, and can be referred to as the “doubling hypothesis” (cf. [12]).

In the analytic Bethe ansatz approach introduced in [10] \( \alpha_j(u) \) and \( A_j(u) \) in [21] are unspecified to begin with. The functions \( \alpha_j(u) \) are determined by calculating the eigenvalue on the reference state \( |\Omega\rangle \), for which \( A_j(u) = 1 \). For a model like the Izergin-Korepin model where the corresponding \( \Phi_j(u, \omega) \) are all distinct this gives \( \alpha_j(u) \) unambiguously [10]. The functions \( A_j(u) \) can then be determined by using the fusion equation, crossing-symmetry etc., as described earlier. In our case, and also the case [12] for \( A_{2n}^{(2)} \) for \( n > 1 \), the \( \Phi_j(u, \omega) \) are not all distinct. In particular, since \( \Phi_{-2}(u, \omega) \) and \( \Phi_{-1}(u, \omega) \) are identical, it is possible only to obtain the combination \( \alpha_{-2}(u) + \alpha_{-1}(u) \) unambiguously. Nevertheless, we can obtain some of the \( \alpha_j(u) \); this we do by explicitly calculating \( \langle \Omega | t(u, \omega) | \Omega \rangle \) (|\Omega\rangle being the state \( v_{-3} \otimes \cdots \otimes v_{-3} \) for small \( N \) and choosing the inhomogeneities appropriately to cancel out relevant \( \Phi_j(u, \omega) \). For instance, to obtain \( \alpha_{-3}(u) \) it is sufficient to have \( N = 1 \) and \( \omega_1 = u \) whereas for \( \alpha_0(u) \) we choose \( N = 2 \) and \( \omega_1 = 4 + u, \omega_2 = 2 + u \). In this way we find

\[
\alpha_{-3}(u) = \frac{[2 + 2u][7 + 2u][12 + 2u]}{[1 + 2u][6 + 2u][4 + 2u]} \epsilon_{-3}(u), \tag{22}
\]

\[
\alpha_0(u) = \frac{[12 + 2u][2u]}{[8 + 2u][4 + 2u]} \epsilon_0(u), \tag{23}
\]
where

\[ \epsilon_{-3}(u) = \begin{cases} 1 & \mp q^{-17-2u}(1 \pm q^{1+2u})^2(1 \pm q^{5+2u})(1 \pm q^{7+2u})^{-1}, \\ 1 & \mp q^{-23-2u}(1 \pm q^{5+2u})(1 \pm q^{7+2u}), \end{cases} \]

\[ \epsilon_0(u) = \begin{cases} 1 & \mp q^{-17-2u}(1 \pm q^{1+2u})^2(1 \pm q^{5+2u})(1 \pm q^{7+2u})^{-1}, \\ 1 & \mp q^{-23-2u}(1 \pm q^{5+2u})(1 \pm q^{7+2u}), \end{cases} \]

together with \( \alpha_3(u) = \alpha_{-3}(-u - \rho) \).

At this stage we can perform several highly non-trivial checks. They come from crossing-symmetry (16), the fusion equation (17) – more specifically, analyticity of its right-hand side at \( u = -\rho \), and from choosing alternating inhomogeneities [4], and are, respectively,

\[ \alpha_j(u) = \alpha_{-j}(-u - \rho), \quad \xi(2u + 2\rho)\alpha_3(u)\alpha_{-3}(u + \rho) = \Delta\{K^-(u)\}\Delta\{K^+(u)\}, \]

\[ \alpha_{-3}(u) = \frac{l_1(\rho)\rho}{R_{11}^{bc}(2u)}. \]

These equations (24) to (26) are indeed satisfied in all cases. It now remains to resolve \( \alpha_{-2}(u) + \alpha_{-1}(u) \) and \( \alpha_1(u) + \alpha_2(u) \) into their parts. This can be done by imposing analyticity of the eigenvalue expression (21). Thus we find that the resulting system of Bethe ansatz equations is

\[ \delta_1 \prod_{j=1}^N \frac{[iu^{(2)}_k + \frac{1}{2} + \omega_j][iu^{(2)}_k + \frac{1}{2} - \omega_j]}{[iu^{(2)}_k - \frac{1}{2} + \omega_j][iu^{(2)}_k - \frac{1}{2} - \omega_j]} = \]

\[ \times \prod_{j=1}^N \frac{[iu^{(2)}_k - iu^{(1)}_j - \frac{3}{2}][iu^{(2)}_k + iu^{(1)}_j - \frac{3}{2}]}{[iu^{(2)}_k - iu^{(1)}_j + \frac{3}{2}][iu^{(2)}_k + iu^{(1)}_j + \frac{3}{2}]} \]

\[ \times \prod_{j \neq k}^N \frac{[iu^{(2)}_k - iu^{(2)}_j + 1][iu^{(2)}_k + iu^{(2)}_j + 1]}{[iu^{(2)}_k - iu^{(2)}_j - 1][iu^{(2)}_k + iu^{(2)}_j - 1]}, \quad (27) \]

\[ \delta_2 = \prod_{j \neq k}^N \frac{[iu^{(1)}_k - iu^{(1)}_j + 3][iu^{(1)}_k + iu^{(1)}_j + 3]}{[iu^{(1)}_k - iu^{(1)}_j - 3][iu^{(1)}_k + iu^{(1)}_j - 3]} \]

\[ \times \prod_{j=1}^N \frac{[iu^{(1)}_k - iu^{(2)}_j - \frac{3}{2}][iu^{(1)}_k + iu^{(2)}_j - \frac{3}{2}]}{[iu^{(1)}_k - iu^{(2)}_j + \frac{3}{2}][iu^{(1)}_k + iu^{(2)}_j + \frac{3}{2}]} \quad (28) \]

where the functions \( \delta_i \) take different forms in terms of \( \alpha_j \), depending on the specific point at which we consider analyticity. For instance, analyticity at \( u = iu^{(2)}_k - \frac{1}{2} \) and at
have already determined. Solving the resulting functional equation we find that

\[ \delta_1 = \frac{\alpha_{-3}(i u_k^{(2)} - \frac{3}{2})[2i u_k^{(2)} - 1]}{\alpha_{-2}(i u_k^{(2)} - \frac{3}{2})[2i u_k^{(2)} + 1]} = \frac{\alpha_{-1}(i u_k^{(2)} - \frac{3}{2})[2i u_k^{(2)} - 1]}{\alpha_{0}(i u_k^{(2)} - \frac{3}{2})[2i u_k^{(2)} + 1]} \]  

(29)

whereas analyticity at \( u = i u_k^{(1)} - 2 \) and at \( u = -i u_k^{(1)} - 2 \) gives

\[ \delta_2 = \frac{\alpha_{-2}(i u_k^{(1)} - 2)[2i u_k^{(1)} - 3]}{\alpha_{-1}(i u_k^{(1)} - 2)[2i u_k^{(1)} + 3]} = \frac{\alpha_{-1}(-i u_k^{(1)} - 2)[2i u_k^{(1)} - 3]}{\alpha_{-2}(-i u_k^{(1)} - 2)[2i u_k^{(1)} + 3]} \]  

(30)

Such equalities allow \( \alpha_{\pm 1}(u) \) and \( \alpha_{\pm 2}(u) \) to be related to \( \alpha_{\pm 3}(u) \) and \( \alpha_0(u) \) which we have already determined. Solving the resulting functional equations we find that

\[ \alpha_{-2}(u) = \frac{[12 + 2u][2u][7 + 2u]}{[1 + 2u][4 + 2u][6 + 2u]} \epsilon_{-2}(u), \]  

(31)

\[ \alpha_{-1}(u) = \frac{[12 + 2u][2u]}{[4 + 2u][6 + 2u]} \epsilon_{-1}(u), \]  

(32)

with \( \epsilon_{-2}(u) = \epsilon_{-3}(u) \) and \( \epsilon_{-1}(u) = \epsilon_0(u) \), together with \( \alpha_1(u) = \alpha_{-1}(-u - \rho) \) and \( \alpha_2(u) = \alpha_{-2}(-u - \rho) \). This completes the determination of the eigenvalue expression \( \Lambda(u, \omega) \). We have shown that it satisfies all the checks mentioned in this paper. The corresponding results for the “boundary factors” \( \delta_i \) in the Bethe ansatz equations (27) and (28) are found to be \( \delta_1 = 1 \) in all cases, while \( \delta_2 = 1 \) for case (i) and

\[ \delta_2 = \left( \frac{q^{i u_k^{(1)} - 3/2} \pm q^{-i u_k^{(1)} + 3/2}}{q^{i u_k^{(1)} + 3/2} \pm q^{-i u_k^{(1)} - 3/2}} \right)^2 \]  

(33)

in the remaining two cases. We note that there is a striking resemblance to the corresponding “boundary factors” for the Izergin-Korepin model [4].

4 Discussion

We have seen how the general considerations of [1, 2] for obtaining integrable models with open boundaries can in principle be applied to \( R \)-matrices based on exceptional affine algebras. In particular we have obtained all the diagonal \( K \)-matrices for the model based on \( G_2^{(1)} \). With \( K^-(u) = 1 \) and \( K^+(u) = M \), the corresponding spin chain has Hamiltonian \( H = \sum_{k=1}^{N-1} \tilde{R}_{k,k+1}(u) \) [9] and is \( U_q(G_2) \)-invariant. In the rational limit \( q \to 1 \) the Hamiltonian is both \( G_2 \) and \( su(2) \)-invariant [20, 21] with its energy spectrum determined by the Bethe ansatz equations (27) and (28) with \( [x] \to x \), \( \delta_i = 1 \) and \( \omega_i = 0 \).

The method we have used to diagonalize the transfer matrices can be considered a variation of the analytic Bethe ansatz [11], with an extra (unproven) assumption,
namely the “doubling hypothesis”. A rigorous alternative method is probably the (nested) Bethe ansatz along the lines of [9] which we expect to be much more complicated to apply here. We have seen how the Bethe ansatz equations for the $G_2^{(1)}$ model – in particular, the “boundary factors” – resemble those for $A_2^{(2)}$. It would be interesting to see if this can be explained on Lie algebraic grounds alone, analogous to the periodic boundary situation [19].

Acknowledgements

We are grateful to A. Kuniba and J. Suzuki for helpful correspondence. This work is supported by the Australian Research Council.

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