On timelike surfaces in Lorentzian manifolds

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Abstract. We discuss the geometry of timelike surfaces (two-dimensional submanifolds) in a Lorentzian manifold and its interpretation in terms of general relativity. A classification of such surfaces is presented which distinguishes four cases of special algebraic properties of the second fundamental form from the generic case. In the physical interpretation a timelike surface \( \Sigma \) can be viewed as the worldsheet of a “track”, and timelike curves in \( \Sigma \) can be viewed as the worldlines of observers who are bound to the track, like someone sitting in a roller-coaster car. With this interpretation, our classification turns out to be closely related to (i) the visual appearance of the track, (ii) gyroscopic transport along the track, and (iii) inertial forces perpendicular to the track. We illustrate our general results with timelike surfaces in the Kerr-Newman spacetime.

1. Introduction

A standard tool for investigating the geometry of a submanifold in a semi-Riemannian manifold is the second fundamental form, or shape tensor, see e.g. O’Neill [O’N83]. In this paper we will discuss some aspects of the second fundamental form for the case that the ambient semi-Riemannian manifold has Lorentzian signature \((-,+,...,+\)) and that the submanifold is two-dimensional with Lorentzian signature \((-,+\)). We call such a submanifold a timelike surface for short. Timelike surfaces are interesting objects not only from a mathematical point of view but also in view of physics. A four-dimensional Lorentzian manifold can be interpreted as a spacetime in the sense of general relativity, and a timelike surface \( \Sigma \) in such a manifold can be interpreted as the worldsheet of an object with one spatial dimension. It is often helpful to think of \( \Sigma \) as being realized by a “track”, and of timelike curves in \( \Sigma \) as being the worldlines of observers who are bound to the track like someone sitting in a roller-coaster car, cf. Abramowicz [Abr90].

It is the main goal of this paper to give a classification of timelike surfaces in terms of their second fundamental form, and to discuss the physical relevance of this classification in view of the roller-coaster interpretation. As we want to concentrate on properties of timelike surfaces which are conformally invariant, we base our classification on the trace-free part of the second fundamental form. We call a timelike surface “generic” if this trace-free part is non-degenerate, in a sense specified below, and “special” otherwise. It turns out that a degeneracy can occur

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in four different ways, giving rise to four different types of special timelike surfaces. Using the roller-coaster interpretation, we will see how each of the four special types can be distinguished from generic timelike surfaces by three observational features: (i) the visual appearance of the track, (ii) gyroscopic transport along the track, and (iii) inertial forces perpendicular to the track. For the latter we use the definition of inertial forces given in \[\text{FHP03}\]. It can be viewed as an adaptation to general relativity of Huygens’ definition of centrifugal force, which was based on the curvature of a track. For the history of this notion see Abramowicz \[\text{Abr90}\].

Our discussion of the physics of timelike surfaces is kinematic, as opposed to dynamic, in the sense that Einstein’s field equation is not used and that we do not specify an equation of motion for our timelike surfaces. As outlined above, our main physical motivation is to give an operational approach to inertial forces and its relation with the visual appearance of a track and with gyroscopic transport. However, we would like to mention that our results might also be of interest in view of applications to strings. The worldsheet of a (classical) string is a timelike surface, and the second fundamental form of this surface gives some information on the physical properties of the string, see e.g. Carter \[\text{Car95}\].

The paper is organized as follows. In Section 2 we introduce orthonormal basis vector fields and lightlike basis vector fields on timelike surfaces and we classify such surfaces in terms of their second fundamental form. The physical interpretation of timelike surfaces, based on the roller-coaster picture, and its relation to the second fundamental form is discussed in Section 3. Among other things, in this section we consider two quite different splittings of the total inertial force perpendicular to the track into three terms and we discuss the invariance properties of these terms. Generic timelike surfaces are treated in Section 4. We show that the relevant properties of the second fundamental form are encoded in a characteristic hyperbola and that every generic timelike surface admits a distinguished reference frame (timelike vector field). The next four short sections are devoted to the four non-generic cases and to the observable features by which each of them differs from the generic case. Finally, in Section 9 we illustrate our results with timelike surfaces in the Kerr-Newman spacetime.

2. Timelike surfaces

Let \((M, g)\) be an \(n\)-dimensional Lorentzian manifold. We assume that the metric \(g\) is of class \(C^\infty\) and we choose the signature of \(g\) to be \((-+, +, \ldots, +)\). It is our goal to investigate the geometry of surfaces (i.e., two-dimensional \(C^\infty\)-submanifolds) of \(M\) that are timelike everywhere, i.e., the metric pulled back to the surface is supposed to have signature \((-+, +)\). If we fix such a surface \(\Sigma\), we may choose at each point \(p \in \Sigma\) an orthonormal basis for the tangent space \(T_p\Sigma\). We assume that this can be done globally on \(\Sigma\) with the basis depending smoothly on the foot-point. This means that we assume \(\Sigma\) to be time-orientable and space-orientable. This gives us two \(C^\infty\) vector fields \(n\) and \(\tau\) on \(\Sigma\) that satisfy

\[g(\tau, \tau) = -g(n, n) = 1, \quad g(n, \tau) = 0.\]

At each point, \(n\) and \(\tau\) are unique up to a two-dimensional Lorentz transformation \((n, \tau) \mapsto (n', \tau')\). If we restrict to transformations that preserve the time-orientation, i.e., if we require that the two timelike vectors \(n\) and \(n'\) point into
the same half of the light cone, any such Lorentz transformation is of the form
\[(2.2)\]
\[n' = \frac{n + v\tau}{\sqrt{1 - v^2}}, \quad \tau' = \frac{vn + \tau}{\sqrt{1 - v^2}},\]
where the number \(v\) gives the velocity of the \(n'\)-observers relative to the \(n\)-observers, in units of the velocity of light, \(-1 < v < 1\). Of course, \(v\) may vary from point to point.

From the orthonormal basis \((n, \tau)\) we may switch to a lightlike basis \((l^+, l^-)\) via
\[(2.3)\]
\[l_{\pm} = n \pm \tau .\]
Under a Lorentz transformation (2.2) this lightlike basis transforms according to
\[(2.4)\]
\[l_{\pm}' = \frac{1 \pm v}{\sqrt{1 - v^2}} l_{\pm} .\]
Thus, the directions of \(l^+\) and \(l^-\) are invariant with respect to Lorentz transformations. This reflects the obvious fact that, at each point of the timelike surface \(\Sigma\), there are precisely two lightlike directions tangent to \(\Sigma\). The integral curves of \(l^+\) and \(l^-\) give two families of lightlike curves each of which rules the surface \(\Sigma\). We want to call \(\Sigma\) a photon surface if both families are geodesic, and we want to call \(\Sigma\) a one-way photon surface if one of the two families is geodesic but the other is not. Clearly, this terminology refers to the fact that in general relativity a lightlike geodesic is interpreted as the worldline of a (classical) photon. More generally, one can define a photon \(k\)-surface to be a \(k\)-dimensional submanifold of a Lorentzian manifold for which every lightlike geodesic that starts tangent to \(\Sigma\) remains tangent to \(\Sigma\). This notion was introduced, for the case \(k = n - 1\), in a paper by Claudel, Virbhadra and Ellis [CVE01]; here we are interested in the case \(k = 2\) which was already treated in [FHP03] and [Per05].

Before discussing photon surfaces and one-way photon surfaces, we will demonstrate that these notions appear naturally when timelike surfaces are classified with respect to their second fundamental form. To work this out, we recall (cf. e.g. O'Neill [O’N83]) that the second fundamental form, or shape tensor field, \(\Pi\) is well-defined for any nowhere lightlike submanifold of a semi-Riemannian manifold, in particular for a timelike surface \(\Sigma\) of a Lorentzian manifold, by the equation
\[(2.5)\]
\[\Pi(u, w) = P^\perp (\nabla_u w)\]
where \(u\) and \(w\) are vector fields on \(\Sigma\). Here \(\nabla\) is the Levi-Civita connection of the metric \(g\) and \(P^\perp\) denotes the orthogonal projection onto the orthocomplement of \(\Sigma,
\[(2.6)\]
\[P^\perp(Y) = Y - g(\tau, Y) \tau + g(n, Y) n .\]
As \(\nabla_u w - \nabla_w u = [u, w]\) must be tangent to \(\Sigma\), we can read from (2.5) the well-known fact that \(\Pi\) is a symmetric tensor field along \(\Sigma\).

With respect to the lightlike basis \((l^+, l^-)\), the second fundamental form is characterized by its three components
\[(2.7)\]
\[\Pi_+ = \Pi(l^+, l^+), \quad \Pi_- = \Pi(l^-, l^-), \quad \Pi_0 = \Pi(l^+, l^-) = \Pi(l^-, l^+) .\]
Note that these three vectors lie in the orthocomplement of \(\Sigma\), so they are necessarily spacelike. In a 4-dimensional Lorentzian manifold, this orthocomplement is two-dimensional, so \(\Pi_+, \Pi_-\) and \(\Pi_0\) must be linearly dependent. In higher-dimensional Lorentzian manifolds, however, these three vectors may be linearly independent.
From (2.4) we can read the transformation behaviour of \( \Pi_+ \), \( \Pi_- \) and \( \Pi_0 \) under Lorentz transformations,

\[
\Pi'_\pm = \frac{1 \pm v}{1 \mp v} \Pi_\pm, \quad \Pi'_0 = \Pi_0.
\]

Thus, \( \Pi_+ \) and \( \Pi_- \) are invariant up to multiplication with a positive factor whereas \( \Pi_0 \) is invariant. In particular, the conditions \( \Pi_+ = 0 \) and \( \Pi_- = 0 \) have an invariant meaning. Similarly, the statement that \( \Pi_+ \) and \( \Pi_- \) are parallel (or anti-parallel, respectively) has an invariant meaning.

Note that \( \Pi_0 \) is related to the trace of the second fundamental form by

\[
(2.9) \quad \Pi_0 = -\text{trace}(\Pi).
\]

We now introduce the following terminology.

**Definition 2.1.** A timelike surface \( \Sigma \) is called generic (at \( p \)) if \( \Pi_+ \) and \( \Pi_- \) are linearly independent (at \( p \)). Otherwise it is called special (at \( p \)).

Clearly, the class of special timelike surfaces can be subdivided into four subclasses, according to the following definition.

**Definition 2.2.** A timelike surface \( \Sigma \) is called

- (a) special of the first kind (at \( p \)) if \( \Pi_+ \) and \( \Pi_- \) are both non-zero and parallel, \( \Pi_- = \alpha \Pi_+ \) with \( \alpha > 0 \) (at \( p \));
- (b) special of the second kind (at \( p \)) if \( \Pi_+ \) and \( \Pi_- \) are both non-zero and anti-parallel, \( \Pi_- = \alpha \Pi_+ \) with \( \alpha < 0 \) (at \( p \));
- (c) special of the third kind (at \( p \)) if one of the vectors \( \Pi_+ \) and \( \Pi_- \) is zero and the other is non-zero (at \( p \));
- (d) special of the fourth kind (at \( p \)) if both \( \Pi_+ \) and \( \Pi_- \) are zero (at \( p \)).

Photon surfaces are timelike surfaces that are special of the fourth kind, whereas one-way photon surfaces are timelike surfaces that are special of the third kind.

It is obvious from (2.8) that the property of being generic or special of the \( N \)th kind is independent of the chosen orthonormal frame. Moreover, it is preserved under conformal transformations. If we multiply the metric \( g \) with a conformal factor \( e^{2f} \), where \( f \) is a function on \( M \), and rescale the basis vectors accordingly,

\[
(2.10) \quad \tilde{g} = e^{2f} g, \quad \tilde{n} = e^{-f} n, \quad \tilde{\tau} = e^{-f} \tau,
\]

\( \Pi_+ \) and \( \Pi_- \) are unchanged, whereas \( \Pi_0 \) transforms inhomogeneously,

\[
(2.11) \quad \tilde{\Pi}_+ = \Pi_+, \quad \tilde{\Pi}_- = \Pi_- \quad \tilde{\Pi}_0 = \Pi_0 + 2P_\perp(U),
\]

where \( df = g(U, \cdot) \). Thus, it is always possible to make \( \Pi_0 \) equal to zero by a conformal transformation. This is the reason why we based our classification on \( \Pi_+ \) and \( \Pi_- \) alone.

We will now review the physical interpretation connected with \( \Pi_+ \) and \( \Pi_- \), and then discuss the different types of timelike surfaces one by one.

### 3. Physical interpretation

If \( M \) is 4-dimensional, \( (M, g) \) can be interpreted as a spacetime in the sense of general relativity. As indicated already in the introduction, we may interpret each timelike surface \( \Sigma \) as the worldsheet of a track and each timelike curve in \( \Sigma \) as the worldline of an observer who sits in a roller-coaster car that is bound to the track.

We want to discuss three types of “experiments” such an observer can carry out,
viz. (i) sending and receiving light rays, (ii) measuring the precession of gyroscopes, and (iii) measuring inertial accelerations. All three types of experiments turn out to be closely related to the second fundamental form.

In general relativity light rays (i.e. worldlines of classical photons) are to be identified with lightlike geodesics. If an observer at one point of the track receives a light ray from an observer at some other point of the track, the corresponding lightlike geodesic will, in general, not arrive tangentially to the track. Thus, the observer who receives the light ray will get the visual impression that the track is curved. Photon surfaces are characterized by the property that such a light ray always arrives tangentially to the track, i.e., a photon surface is the worldsheet of a track that appears straight. In the case of a one-way photon surface this is true only when looking in one direction (“forward”), but not when looking in the other direction (“backward”).

If we have chosen an orthonormal basis \((n, \tau)\) on \(\Sigma\), we can interpret the integral curves of \(n\) as observers distributed along the track described by \(\Sigma\). If we want to give an interpretation to \(\tau\), we may think of each of these observers holding a rod in the direction of the track. We want to investigate whether \(\tau\) can be realized as the axis of a gyroscope that is free to follow its inertia. According to general relativity, this is true if and only if \(\tau\) remains Fermi-Walker parallel to itself along each integral curve of \(n\) (see e.g. Misner, Thorne and Wheeler \[MTW73\], Sect. 40.7), i.e., if and only if \(\nabla_n \tau\) is a linear combination of \(n\) and \(\tau\). This is true if and only if \(\Pi(n, \tau) = 0\), which can be rewritten, in terms of the lightlike vector fields \((2.3)\), as \(\Pi(l_+ + l_-, l_+ - l_-) = 0\). Using the notation from \((2.7)\), we find that

\[
(3.1) \quad \Pi_+ = \Pi_-
\]

is the necessary and sufficient condition for \(\tau\) being Fermi-Walker parallel along \(n\). If \((3.1)\) holds along an integral curve of \(n\), a gyroscope carried by the respective observer will remain parallel to the track if it is so initially. Note that \((3.1)\) is preserved under Lorentz transformations \((2.8)\) if and only if \(\Pi_+ = \Pi_- = 0\).

Having chosen an orthonormal basis \((n, \tau)\) on \(\Sigma\), we can write any timelike curve in \(\Sigma\) as the integral curve of a vector field \(n'\) that is related to \(n\) by a Lorentz transformation according to \((2.2)\), with a relative velocity \(v\) that depends on the foot-point. We want to calculate the vector \(\Pi(n', n') = P^\perp(\nabla_{n'} n')\). Using \((2.3)\), \((2.2)\) and \((2.7)\), we find

\[
(3.2) \quad \Pi(n', n') = \frac{1}{2} \Pi_0 + \frac{1}{4} \frac{1+v}{1-v} \Pi_+ + \frac{1}{4} \frac{1-v}{1+v} \Pi_-
\]

According to general relativity, the vector \(\nabla_{n'} n'\) gives the acceleration of an observer traveling on an integral curve of \(n'\), measured relatively to a freely falling object. If we think of this observer as sitting in a roller-coaster car bound to the track modeled by \(\Sigma\), the vector \(-\Pi(n', n')\) gives the acceleration perpendicular to the track of a freely falling particle relative to the car. This relative acceleration is what an observer on a roller-coaster feels in his or her stomach, because the stomach wants to follow its inertia and move in free fall, whereas the frame of the observer’s body cannot follow this motion as it is strapped to the car. For this reason, \(-\Pi(n', n')\) is to be interpreted as the (relativistic) inertial acceleration of the \(n'\)-observers. Multiplication with the mass gives the (relativistic) inertial force onto these observer. Following \[FHP03\], we can decompose the vector \(-\Pi(n', n')\) into...
gravitational acceleration $a_{\text{grav}}$, Coriolis acceleration $a_{\text{Cor}}$ and centrifugal acceleration $a_{\text{cent}}$ by rearranging the right-hand side of (3.2) according to the following rule. $a_{\text{grav}}$ comprises all terms which are independent of $v$, $a_{\text{Cor}}$ comprises all terms of odd powers of $v$, and $a_{\text{cent}}$ comprises all terms of even powers of $v$.

\[
(3.3) \Pi(n', n') = \frac{1}{2} \Pi_0 + \frac{1}{4}(\Pi_++\Pi_-)+\frac{v}{1-v^2}\left(\frac{1}{2}(\Pi_+-\Pi_-)+\frac{v^2}{1-v^2}\frac{1}{2}(\Pi_++\Pi_-)\right).
\]

(In [FHP03] we found it convenient to work with the corresponding covectors $A_{\text{grav}} = g(a_{\text{grav}}, \cdot)$, etc.) This definition of gravitational, Coriolis and centrifugal accelerations with respect to a timelike surface has the advantage that it is unambiguous in an arbitrary Lorentzian manifold and that it corresponds, as closely as possible, with the traditional non-relativistic notions. (For alternative definitions of inertial accelerations in arbitrary general-relativistic spacetimes see, e.g. Abramowicz, Nurowski and Wex [ANW93] or Jonsson [Jon06].)

Whereas the decomposition (3.3) depends on $n$, the splitting (3.2) of the inertial acceleration into three terms is invariant under Lorentz transformations. This follows from the transformation properties (2.8). For that and for some other calculations in this paper it is convenient to substitute

\[
(3.4) \frac{1}{\sqrt{1-v^2}} = \cosh \eta \quad \text{and} \quad \frac{v}{\sqrt{1-v^2}} = \sinh \eta.
\]

Then the first equation in (2.8) reads $\Pi_+ = e^{\pm 2\eta} \Pi_\pm$. Similarly to the decomposition (3.3) one can, owing to the different dependencies on $v$, operationally separate the three terms of the sum in (3.2) by measuring the inertial acceleration for different velocities.

4. Generic timelike surfaces

In this section we consider a timelike surface which is generic at all points. We can then characterize the second fundamental form at each point by the three non-zero vectors

\[
(4.1) \quad \Pi_0, \quad I_+ = \sqrt{g(\Pi_-, \Pi_-)} \Pi_+, \quad I_- = \sqrt{g(\Pi_+, \Pi_+)} \Pi_-
\]

which, according to (2.8), are invariant with respect to Lorentz transformations. Moreover, the two linearly independent vectors $I_+$ and $I_-$ are conformally invariant, see (2.11).

We now fix an orthonormal basis $(n, \tau)$. Then we find all future-oriented vector fields $n'$ with $g(n', n') = -1$ by a Lorentz transformation (2.2). If $v$ runs from $-1$ to 1, at each point $p \in \Sigma$ the vector $\Pi(n', n')$ runs through a hyperbola, according to (3.2), see Figure 1. We call this the characteristic hyperbola of the second fundamental form at $p$. The characteristic hyperbola lies in the orthocomplement $P^\perp(T_pM)$ of the tangent space $T_p\Sigma$, which is an $(n-2)$-dimensional Euclidean vector space.

The asymptotes of the characteristic hyperbola are spanned by the linearly independent vectors $\Pi_+$ and $\Pi_-$ (or, what is the same, by the invariant vectors $I_+$ and $I_-$. The characteristic hyperbola is invariant with respect to Lorentz transformations, see (2.8), whereas a conformal transformation produces a translation of the characteristic hyperbola, see (2.11).
The points on the characteristic hyperbola are in one-to-one correspondence with future-oriented vectors normalized to $-1$ at $p$. Clearly, the vertex of the hyperbola, indicated by 1 in Figure 1, defines a distinguished observer field on every generic timelike surface. From (3.2) we find that the arrow from the origin to the vertex of the hyperbola is given by the vector

$$
\frac{1}{2} \Pi_0 + \frac{1}{4} \sqrt{\frac{g(\Pi_-, \Pi_-)}{g(\Pi_+ , \Pi_+)}} \Pi_+ + \frac{1}{4} \sqrt{\frac{g(\Pi_+, \Pi_+)}{g(\Pi_- , \Pi_-)}} \Pi_-
$$

which is Lorentz invariant, by (2.8). If we choose the distinguished observer field for our $n$, we have in the orthonormal basis $(n, \tau)$

$$
g(\Pi_+, \Pi_+) = g(\Pi_-, \Pi_-) .
$$

This property characterizes the distinguished observer field uniquely.

The distinguished observer field can be determined as the solution of an eigenvalue problem. Using the invariant vectors $I_\pm$ from (4.1), we can introduce the real-valued bilinear form

$$
\pi(u, w) = \frac{1}{2} g(I_+ + I_-, \Pi(u, w))
$$

where $u$ and $v$ are tangent to $\Sigma$. If $n$ is the distinguished observer field, the basis vectors $n$ and $\tau$ satisfy the eigenvalue equations

$$
\pi(n, \cdot) = \lambda_1 g(n, \cdot) , \quad \pi(\tau, \cdot) = \lambda_2 g(\tau, \cdot) .
$$

To prove this we observe that, for an arbitrary orthonormal frame $(n, \tau)$,

$$
\pi(n, \tau) = \frac{1}{8} \left( \sqrt{g(\Pi_-, \Pi_-)} - \sqrt{g(\Pi_+, \Pi_+)} \right) \left( \sqrt{g(\Pi_-, \Pi_-)} g(\Pi_+, \Pi_+) - g(\Pi_+, \Pi_-) \right) .
$$
Clearly, (4.5) holds if and only if \( \pi(n, \tau) = 0 \). As \( \Pi_+ \) and \( \Pi_- \) are linearly independent, the last bracket in (4.6) is different from zero. So (4.5) is, indeed, equivalent to (4.3).

A generic timelike surface \( \Sigma \) is the worldsheet of a track that appears curved to the eye of any observer, because \( \Pi_+ \) and \( \Pi_- \) are non-zero. As (5.1) cannot be satisfied, a gyroscope carried along the track cannot stay parallel to the track. With respect to any observer on the track, Coriolis and centrifugal acceleration are linearly independent. The distinguished observer field is characterized by producing a symmetry between the backward and the forward directions.

Finally, we note that the vertex is not the only point on the characteristic hyperbola that is distinguished. As an alternative, we may consider the point which is closest to the origin, denoted by 2 in Figure 1. This gives us a second distinguished observer field. Physically, it is characterized by the fact that the total inertial acceleration perpendicular to the track becomes minimal. In contrast to the (first) distinguished observer field, the second distinguished observer field is not necessarily unique; there may be one or two such observer fields, corresponding to the fact that a circle can be tangent to a hyperbola in one or two points. More importantly, the second distinguished observer field is not invariant under conformal transformations; this follows from our earlier observation that a conformal transformation corresponds to a translation of the characteristic hyperbola. As in this article we focus on conformally invariant properties, the second distinguished observer field is of less interest to us.

5. Special timelike surfaces of the first kind

If \( \Sigma \) is special of the first kind, the angle between the two asymptotes in Figure 1 is zero. Thus, the characteristic hyperbola degenerates into a straight line which is run through twice, with a turning point at the tip of the arrow (4.2). This turning point corresponds to the distinguished observer field which is still determined by (4.3). However, now it satisfies even the stronger condition \( \Pi_+ = \Pi_- \). The distinguished observer field is no longer characterized by the eigenvalue equations (4.5) because these equations are now satisfied by any orthonormal basis \((n, \tau)\). Note, however, that now (and only in this case) the distinguished observer field satisfies the “strong” eigenvalue equation \( \Pi(n, \cdot) = \Lambda \otimes g(n, \cdot) \), with an “eigenvalue” \( \Lambda \in P_{\perp}(T\Sigma) \).

If \( \Sigma \) is special of the first kind everywhere, a track modeled by \( \Sigma \) appears curved to the eye of any observer, because \( \Pi_+ \) and \( \Pi_- \) are non-zero. The distinguished observer field satisfies (5.1) which means that a gyroscope carried by a distinguished observer remains parallel to the track if it was so initially. For all other observers this is not true. If we write (5.3) for the case that \( n \) is the distinguished observer field, we read that the Coriolis acceleration is zero for all \( v \) whereas the centrifugal acceleration is non-zero for all \( v \neq 0 \).

6. Special timelike surfaces of the second kind

If \( \Sigma \) is special of the second kind, the angle between the two asymptotes in Figure 1 is 180°. Thus, the characteristic hyperbola degenerates into a straight line which extends from infinity to infinity, passing through the tip of the arrow \( \frac{1}{2} \Pi_0 \). This point corresponds to the distinguished observer field which is still determined
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by (4.5). However, now it satisfies the stronger condition $\Pi_+ = -\Pi_-$. The distinguished observer field cannot be characterized by the eigenvalue equations (4.5) because the bilinear form $\pi$ is identically zero.

If $\Sigma$ is special of the second kind everywhere, a track modeled by $\Sigma$ appears curved to the eye of any observer on the track, as $\Pi_+$ and $\Pi_-$ are non-zero. Condition (3.1) cannot be satisfied, so a gyroscope does not remain parallel to the track if it was so initially, for any observer. If we write (3.3) for the case that $n$ is the distinguished observer field, we read that the centrifugal acceleration is zero for all $v$ whereas the Coriolis acceleration is non-zero for all $v \neq 0$. The vanishing of the centrifugal force is a measurable property by which the distinguished observer field is uniquely determined.

### 7. Special timelike surfaces of the third kind

Recall that $\Sigma$ is a one-way photon surface if it is special of the third kind everywhere. A one-way photon surface is a timelike surface $\Sigma$ that is ruled by one family of lightlike geodesics, whereas the other family of lightlike curves in $\Sigma$ is non-geodesic. This implies that, if $\Sigma$ is the worldsheet of a track, the track visually appears straight in one direction but curved in the other.

For a one-way photon surface one of the two vectors that span the asymptotes in Figure 1 becomes zero. Thus, the characteristic hyperbola degenerates into a straight line which is run through once, beginning at infinity and then asymptotically approaching the tip of the arrow $\frac{1}{2}\Pi_0$. There is no distinguished observer field because one side of equation (4.5) is non-zero and the other is zero for all orthonormal bases. By the same token, (3.1) is never satisfied because the vector on one side of this equation is always zero whereas that on the other side is never zero. Hence, it is impossible to carry a gyroscope along the track modelled by $\Sigma$ in such a way that its axis stays parallel to the track.

From (3.3) we read that, on a one-way photon surface, the Coriolis acceleration $a_{\text{Cor}}$ and the centrifugal acceleration $a_{\text{cent}}$ are always parallel. Also, we read that both $a_{\text{Cor}}$ and $a_{\text{cent}}$ are necessarily non-zero for $v \neq 0$.

One-way photon surfaces can be easily constructed, on any Lorentzian manifold, in the following way. Choose at each point of a timelike curve a lightlike vector, smoothly depending on the foot-point. With each of these lightlike vectors as the initial condition, solve the geodesic equation. The resulting lightlike geodesics generate a smooth timelike surface in the neighborhood of the timelike curve. Generically, this is a one-way photon surface. (In special cases it may be a photon surface.)

### 8. Special timelike surfaces of the fourth kind

We now turn to photon surfaces, i.e. to timelike surfaces which are everywhere special of the fourth kind. The worldline of a track is a photon surface if and only if it is ruled by two families of lightlike geodesics. As already outlined, this implies that the track appears straight to the eye of any observer on the track.

For a photon surface the characteristic hyperbola degenerates into a single point, situated at the tip of the arrow $\frac{1}{2}\Pi_0$. This fact clearly indicates that all observer fields have equal rights, i.e., there is no distinguished observer field.

If $\Sigma$ is special of the fourth kind at a point $p$, of the three components $\Pi_+$, $\Pi_-$ and $\Pi_0$ only the last one is different from zero at $p$. Thus, $\Pi(u, w) = -\frac{1}{2}g(u, w)\Pi_0$...
at \( p \), i.e., the second fundamental form \( \Pi \) is a multiple of the first fundamental form \( g \). Points where this happens are called \textit{umbilic points}. A submanifold is called \textit{totally umbilic} if all of its points are umbilic. Thus, a timelike surface is a photon surface if and only if it is totally umbilic. For a more detailed discussion of totally umbilic submanifolds of semi-Riemannian manifolds see [Per05].

The defining property \( \Pi_+ = \Pi_- = 0 \) of photon surfaces implies that (3.1) is satisfied for all orthonormal bases \((n, \tau)\). Hence, a gyroscope that is initially tangent to the track modelled by \( \Sigma \) remains tangent to the track forever, independent of its motion along the track. This property characterizes photon surfaces uniquely.

From (3.3) we read that for a photon surface Coriolis acceleration \( a_{\text{Cor}} \) and centrifugal acceleration \( a_{\text{cent}} \) vanish identically. Again, this property characterizes photon surfaces uniquely.

The most obvious example for a photon surface is a timelike plane in Minkowski spacetime. A less trivial example is the timelike surface \( \vartheta = \pi/2, r = 3m \) in Schwarzschild spacetime. Inertial forces and gyroscopic transport on this circular track were discussed in several papers by Marek Abramowicz with various co-authors, see e.g. [ACL88] and [Abr90].

The existence of a photon surface requires a non-trivial integrability condition, so it is not guaranteed in arbitrary Lorentzian manifolds, see [FHP03]. In the same paper we have given several methods of how to construct photon surfaces. Also, we have determined all photon surfaces in conformally flat Lorentzian manifolds, and some examples of photon surfaces in Schwarzschild and Goedel spacetimes.

9. Example: Timelike surfaces in Kerr-Newman spacetime

As an example, let \( g \) be the Kerr-Newman metric in Boyer-Lindquist coordinates (see, e.g., Misner, Thorne and Wheeler [MTW73], p.877)

\[
(9.1) \quad g = -\frac{\Delta}{\rho^2} \left( dt - a \sin^2 \vartheta \, d\varphi \right)^2 + \frac{\sin^2 \vartheta \, \rho^2}{\rho^2} \left( (r^2 + a^2) \, d\varphi - a \, dt \right)^2 + \frac{\rho^2}{\Delta} \, dr^2 + \rho^2 \, d\vartheta^2,
\]

where \( \rho \) and \( \Delta \) are defined by

\[
(9.2) \quad \rho^2 = r^2 + a^2 \cos^2 \vartheta \quad \text{and} \quad \Delta = r^2 - 2mr + a^2 + q^2,
\]

and \( m, q \) and \( a \) are real constants. We shall assume that

\[
(9.3) \quad 0 < m, \quad 0 \leq a, \quad \sqrt{a^2 + q^2} \leq m.
\]

In this case, the Kerr-Newman metric describes the spacetime around a rotating black hole with mass \( m \), charge \( q \), and specific angular momentum \( a \). The Kerr-Newman metric (9.1) contains the Kerr metric \((q = 0)\), the Reissner-Nordström metric \((a = 0)\) and the Schwarzschild metric \((q = 0 \text{ and } a = 0)\) as special cases which are all discussed, in great detail, in Chandrasekhar [Cha83].

By (9.3), the equation \( \Delta = 0 \) has two real roots,

\[
(9.4) \quad r_{\pm} = m \pm \sqrt{m^2 - a^2 - q^2},
\]

which determine the two horizons. We shall restrict to the region

\[
(9.5) \quad M : \quad r_+ < r < \infty,
\]

which is called the \textit{domain of outer communication} of the Kerr-Newman black hole.

For \( 0 < \vartheta < \pi \) and \( r_+ < r < \infty \), let \( \Sigma_{\vartheta, r} \) denote the set of all points in \( M \) where \( \vartheta \) and \( r \) take the respective values. Clearly, \( \Sigma_{\vartheta, r} \) is a smooth two-dimensional
timelike submanifold of $M$ homeomorphic to the cylinder $\mathbb{R} \times S^1$, parametrized by the coordinates $t$ and $\varphi$. We may interpret $\Sigma_{\vartheta, r}$ as the worldsheet of a circular track around the rotation axis of the black hole. We want to investigate for which values of $\vartheta$ and $r$ the timelike surface $\Sigma_{\vartheta, r}$ is special. We choose the orthonormal basis

$$n = \frac{1}{\rho \sqrt{\Delta}} \left( (r^2 + a^2) \partial_t + a \partial_\varphi \right), \quad \tau = \frac{1}{\rho \sin \vartheta} \left( \partial_\varphi + a \sin^2 \vartheta \partial_t \right).$$

By a straight-forward calculation, we find the components $\Pi_{\pm}$ of the second fundamental form with respect to the lightlike basis $l_{\pm} = n \pm \tau$,

$$\Pi_{\pm} = - \left( 2a^2 \sin^2 \vartheta + \rho^2 \pm 2a \sqrt{\Delta} \sin \vartheta \right) \frac{\cos \vartheta}{\rho^2 \sin \vartheta} \partial_\vartheta \left( r \Delta - (r - m) \rho^2 \pm 2ar \sqrt{\Delta} \sin \vartheta \right) \frac{1}{\rho^4} \partial_r.$$

$\Pi_+$ and $\Pi_-$ are linearly dependent if and only if $\vartheta = \pi/2$. Thus, our circular track gives a special timelike surface if and only if it is in the equatorial plane. The equation $\Pi_{\pm} = 0$ is equivalent to $\vartheta = \pi/2$ and

$$2r \Delta - (r - m) r^2 \pm 2a r \sqrt{\Delta} = 0.$$

For each sign, there is precisely one real solution $r_{\pm}^{ph}$ to this equation in the domain of outer communication. Thus, among our timelike surfaces $\Sigma_{\vartheta, r}$ there are precisely two one-way photon surfaces. They correspond to the well known co-rotating and counter-rotating circular photon paths in the Kerr-Newman metric. In the Reissner-Nordström case, $a = 0$, they coincide and give a photon surface at $r = \frac{3m}{2} + \sqrt{\frac{9m^2}{4} - 2q^2}$ (cf., e.g., Chandrasekhar [Cha83], p.218).

For $\vartheta = \pi/2$ and $r > r_{\pm}^{ph}$ or $r < r_{-}^{ph}$, the timelike surface $\Sigma_{\vartheta, r}$ is special of the first kind, for $\vartheta = \pi/2$ and $r_{+}^{ph} < r < r_{+}^{ph}$ it is special of the second kind.

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