Skyrmions, Hadrons and isospin chemical potential

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Abstract

Using the Hamiltonian formulation, in terms of collective variables, we explore the evolution of different skyrmionic parameters as function of the isospin chemical potential ($\mu$), such as the energy density, the charge density, the isoscalar radius and the isoscalar magnetic radius. We found that the radii start to grow very fast for $\mu \gtrsim 140$ MeV, suggesting the occurrence of a phase transition.

PACS numbers:

Keywords:
The skyrmionic approach to baryon dynamics is an interesting attempt to discuss the occurrence of phase transitions induced by temperature and/or density effects in the hadronic sector \[1\]. The idea is to study the stability properties of such objects by an analysis of their mass behavior and the spatial extension, i.e. mean square radius, associated to different currents. Recently this idea has been extrapolated also to the analysis of Skyrmions in curved spaces \[2\].

In a previous article, we discussed the static properties of the Skyrmion solutions in the presence of a finite isospin chemical potential \(\mu\) \[1\]. We were able, among other results, to find a critical chemical potential \(\mu_c = 222.8\) MeV, where the Skyrmion mass vanishes.

In this article we will extend our analysis by using the Hamiltonian formulation introduced by Adkins et al. \[3\]. Note that the identification of baryons as Skyrmion states emerges only in the Hamiltonian formalism. The difference between the static analysis and the present one, is that now we are able to look into the energy spectrum of non strange nucleons as function of \(\mu\). In the static approach we cannot distinguish between different nucleon states.

Using collective coordinates \(a_i (i = 1..4)\), following \[3\], we establish the Lagrangian in terms of \(a_i\) and \(\dot{a}_i\). Then we infer the Hamiltonian operator which gives us the energy levels of the nucleon, the charge densities, the isoscalar mean square radius and the magnetic mean square radius. In particular, we emphasize the growing behavior of the mean squared radius, associated to different conserved currents, as function of \(\mu\), suggesting the occurrence of a phase transition.

The Skyrme lagrangian with isospin chemical potential \((\mu)\) is given by

\[
\mathcal{L} = \frac{F^2}{16} Tr \left[D_\mu U D^\mu U^\dagger\right] + \frac{1}{32e^2} Tr \left[(D_\mu U)^\dagger (D_\nu U)^\dagger \right]^2. 
\] (1)

As usual, the isospin chemical potential is introduced through the covariant derivative \[2\]

\[D_\mu = \partial_\mu - i\frac{\mu}{2} [\sigma_3, U] \delta_{\mu,0}.\] (2)

In ref \[1\], the static solution \(U_0(r)\) of \(1\) was found using a radial symmetric “Hedgehog”
ansatz \ref{eq:ansatz}, finding the minimum of the Skyrmion mass with the parametrization

\[ U_0 = \exp\{iF(r)\sigma_i\hat{n}_i\}, \tag{3} \]

where the $\sigma_i$’s are the Pauli matrices, $\hat{n}_i$ denotes the spatial unitary vector and $F(r)$ is a numerically determined function, that satisfies $F(0) = \pi$ and $F(\infty) = 0$. When the isospin chemical potential is turned on, the profile acquires also a dependence on $\mu$.

\[ F(r) \to F(r, \mu). \tag{4} \]

With this profile, the energy density acquires the following form \cite{1}

\[ M_\mu = M_\mu=0 - \frac{\mu^2}{4e^3F_\pi}I_2 - \frac{\mu^2}{32e^3F_\pi}I_4, \tag{5} \]

with

\[ M_\mu=0 = \frac{F_\pi}{4e} \left\{ 4\pi \int_0^\infty d\hat{r} \left[ \frac{\hat{r}^2}{2} \left( \frac{dF}{d\hat{r}} \right)^2 + \sin^2 F \right] \right. \]

\[ + 4\pi \int_0^\infty d\hat{r} \frac{\sin^2 F}{\hat{r}^2} \times \left[ 4\hat{r}^2 \left( \frac{dF}{d\hat{r}} \right)^2 + 2 \sin^2 F \right] \right\}, \tag{6} \]

where we introduced the dimensionless parameter

$\hat{r} = eF_\pi r$ and the integrals $I_2, I_4$

\[ I_2 = \int d^3\hat{r} \text{Tr} \left[ \sigma_0 - U\sigma_3U^\dagger\sigma_3 \right], \]

\[ I_4 = \int d^3\hat{r} \text{Tr} \left[ \varrho, L_\nu \right]^2, \tag{7} \]

with $\varrho \equiv \sigma_3 - U\sigma_3U^\dagger$, $\sigma_0$ is the $2 \times 2$ identity matrix and $L_\nu \equiv (\partial_\nu U)U^\dagger$. The variational equation for the static Skyrme Lagrangian given in \cite{1}, allows us to find a numerical solution for the profile $F(r)$ \cite{1}. As it was already mentioned, the mass of the Skyrmion vanishes for a critical chemical potential.
As usual, in order to obtain the hadronic spectra, it is convenient to introduce $SU(2)$ collective coordinates $A(t)$, such that

$$U = A(t)U_0 A^\dagger(t).$$

(8)

The $SU(2)$ matrix $A(t)$ is parameterized by the Pauli $\vec{\sigma}$ matrices and the identity $\sigma_0$, according to

$$A(t) = a_0(t)\sigma_0 + i\vec{a}(t) \cdot \vec{\sigma},$$

(9)

where the $a$’s obey the constraint

$$a_0^2(t) + \vec{a}^2(t) = 1.$$  

(10)

Introducing (8) and (9) into (1), a direct (but rather involved) computation leads us to the Lagrangian

$$L = -M_\mu + 2\lambda \left[ \left( \dot{a}_0 + \frac{\mu a_3}{2} \right)^2 + \left( \dot{a}_1 - \frac{\mu a_2}{2} \right)^2 
\right. $$

$$ + \left. \left( \dot{a}_2 + \frac{\mu a_1}{2} \right)^2 + \left( \dot{a}_3 - \frac{\mu a_0}{2} \right)^2 \right]$$

$$\equiv -M_\mu + 2\lambda \left( \dot{a}_i + \frac{\vec{A}_i}{2} \right)^2,$$

(11)

where $\lambda = (2\pi/3e^3F_\pi)\Lambda$, with

$$\Lambda = \int r^2 \sin^2 F \left[ 1 + 4 \left( F^2 + \frac{\sin^2 F}{r^2} \right) \right].$$

(12)

In equation (11) we have defined

$$\vec{A}_i = g_{ij}a_j,$$

(13)

where

$$g_{ij} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$  

(14)
Explicitly,

\[ \tilde{A}_0 = a_3, \quad \tilde{A}_1 = -a_2, \]
\[ \tilde{A}_2 = a_1, \quad \tilde{A}_3 = -a_0. \] (15)

Notice that the \( \lambda \) has the same functional dependance on \( F \) as the equivalent parameter defined in [3]. In our case \( F \) depends also on \( \mu \). \( M_\mu \) is the chemical potential dependent mass given in (5).

From the lagrangian (11), we get the Hamiltonian

\[ H = M_\mu - 2\lambda \mu^2 + \frac{\pi_i^2}{8\lambda}, \] (16)

where the canonical momentum is given through a minimal coupling

\[ \pi_i = p_i - 4\lambda \mu \tilde{A}_i. \] (17)

The Hamiltonian can be expressed in the following way

\[ H = M_\mu + \frac{p_i^2}{8\lambda} - \mu \tilde{A}_i p_i, \] (18)

and considering the canonical quantization procedure \( p_i \rightarrow \hat{p}_i = -i\delta/\delta a_i \), we get

\[ H = M_\mu + \frac{1}{8\lambda} \left( -\frac{\delta^2}{\delta a_i^2} \right) + i\mu g_{ij} a_j \frac{\delta}{\delta a_i}, \]
\[ = M_\mu - \frac{1}{8\lambda} \frac{\delta^2}{\delta a_i^2} - 2\mu \hat{I}_3, \] (19)

where \( \hat{I}_3 \) is the third component of the isospin operator [3]

\[ \hat{I}_k = \frac{i}{2} \left( a_0 \frac{\delta}{\delta a_k} - a_k \frac{\delta}{\delta a_0} - \varepsilon_{klm} a_l \frac{\delta}{\delta a_m} \right). \] (20)

Following the usual procedure, we may associate a wave function to the Skyrme Hamiltonian. In order to identify baryons in this model, these wave functions have to be odd, i.e. \( \psi(A) = -\psi(-A) \). In particular, nucleons correspond to linear terms in the \( a \)'s, whereas the quartet of \( \Delta \)'s are given by cubic terms.
FIG. 1: The nucleon static energy as function of the isospin chemical potential $\mu$. The splitting between neutron (solid line) and proton (dotted line) is due to the last term in (19).

The energy spectra of nucleons as function of $\mu$ is shown in figure 1. We can see that an energy splitting between neutrons and protons is induced. This Hamiltonian remind us the Zeeman effect, where the generation of the energy spectra is broken, by an external magnetic field. In our case the isospin chemical potential plays the same role.

Here we are interested to establish the relation between the relevant physical parameters and the isospin chemical potential, for that purpose, we will consider the Baryonic, vector and axial currents, exploring the behavior of the effective radii associated to them.

The Skyrmion model allows the existence of different conserved currents and their respective charges [8]. Using different charge densities we may define several effective radii. The evolution of those radii as function of chemical potential provides information about the critical behavior close to the phase transition, where the Skyrmion is no longer stable.

Let us first start with the topological baryonic current

$$B^{\mu} = \frac{\varepsilon^{\mu\nu\alpha\beta}}{24\pi^2} Tr [(U^\dagger \partial_\nu U)(U^\dagger \partial_\alpha U)(U^\dagger \partial_\beta U)].$$  (21)

The baryonic charge density for the Skyrmion is given by

$$\rho_B = 4\pi r^2 B^0 (r) = -\frac{2}{\pi} \sin^2 F(r) F'(r).$$  (22)

Obviously, $\int_0^{\infty} dr \rho_B = 1$, independently of the shape of the skyrmionic profile.
FIG. 2: The isoscalar mean square radius as function of \( \mu \).

The isoscalar mean square radius is defined by

\[
\langle r^2 \rangle_{I=0} = \int_0^\infty drr^2 \rho_B.
\]  

This radius seems to be quite stable up to the value of \( \mu \approx 120 \text{ MeV} \), starting then to grow dramatically. Although we do not have a formal proof that this radius diverges at a certain critical \( \mu = \mu_c \), the numerical evidence supports such claim, as it is shown in figure 2. Divergent behavior for several radii, associated to different currents, has also been observed in different hadronic effective couplings as function of temperature in the frame of thermal QCD sum rules.

The same kind of behavior is found for the mean square radius that emerges from the isoscalar magnetic density

\[
\rho_M^{I=0}(r) = \frac{r^2 F' \sin^2 F}{\int dr' r^2 F' \sin^2 F'},
\]  

which is plotted in figure 3.

This growing behavior of the mean square radius has its counterpart in the electric charge distribution. It turns out that the proton charge distribution becomes broader for higher chemical potentials.

Following the usual convention, we introduce the isoscalar and isovector magnetic moments
FIG. 3: The isoscalar magnetic mean square radius as function of $\mu$. This figure suggests the same divergent behavior as the isoscalar mean square radius with the same critical chemical potential (figure 2).

$$\vec{\mu}_{I=0} = \frac{1}{2} \int \vec{r} \times \vec{B} \, d^3x,$$

$$\vec{\mu}_{I=1} = \frac{1}{2} \int \vec{r} \times \vec{V} \, d^3x,$$

where $\vec{B}$ is the vector part of (21) and $\vec{V}$ is the Noether current associated to the vector charge. Following [3], we consider

$$\langle r^2 \rangle_{I=0} = \frac{e}{\Lambda} \frac{1}{F_{\pi}^2 4\pi},$$

$$\langle r^2 \rangle_{I=1} = \frac{2}{4\pi} \frac{\Lambda}{F_{\pi}^2 e^3}.$$  

From the definition of the Bohr magneton for nucleons

$$\vec{\mu} = \left( \frac{g}{4M} \right) \vec{\sigma},$$

and making the identification $g_{I=0} = g_p + g_n$ and $g_{I=1} = g_p - g_n$, it is possible to obtain the magnetic moments for the proton and neutron; $\mu_p = g_p/2$ and $\mu_n = g_n/2$. The behavior of both magnetic moments is presented in figure 4. Besides, figure 5 shows the ratio $|\mu_p/\mu_n|$, the figure suggests that such ratio goes to one for $\mu \rightarrow \mu_c$. 

8
FIG. 4: The magnetic moments $\mu_p$ and $\mu_n$ for the proton, (solid line) and the neutron (dotted line) respectively, as function of $\mu$.

FIG. 5: The ratio $|\mu_p/\mu_n|$ as function of $\mu$.

As a conclusion, we would like to remark the divergent behavior for the isoscalar and mean square magnetic radii as function of the chemical potential. This suggests the occurrence of a phase transition. In fact in the QCD sum rules approach, the radii are phenomenological order parameters for thermal deconfinement. In this case, however, this behavior is induced by the isospin chemical potential. Finally, it is interesting to notice that the Hamiltonian approach allows us to distinguish between the mass evolution of neutrons and protons.
ACKNOWLEDGMENTS

The authors would like to thank financial support from FONDECYT grants 1051067 and 1060653.

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