Anomalous diffusion in random dynamical systems
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1. Diffusion in dynamical systems

Let \((x_n)_{n=0}^\infty\) be a real time series with

\[ x_{n+1} = F(x_n), \, n \in \mathbb{N}_0, \]

where the map \(F\) either represents

- a **deterministic** dynamical system or
- a **stochastic** dynamical system.
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- a **stochastic** dynamical system.

We study the **diffusion** of particles under the system’s dynamics measured by the **mean square displacement (MSD)** \(\text{msd}(n)\) defined by the expectation

\[ \mathbb{E}[(x_n - x_0)^2] \quad (\text{or} \quad \langle (x_n - x_0)^2 \rangle) \]

at time \(n \in \mathbb{N}_0\).
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Diffusion is described in terms of the growth rate of the MSD over time by

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- \( 0 < \alpha < 1 \) subdiffusion
- \( 1 < \alpha < 2 \) superdiffusion
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\alpha = 2 & \quad \text{ballistic motion (like motion on constant velocity)} \\
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0 < \alpha < 1 & \quad \text{subdiffusion}
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Our numerical approximations of the MSD are ensemble averages defined by

$$\text{msd}(n) := \frac{1}{S} \sum_{i=1}^{S} (x_{n,i} - x_{0,i})^2$$

for a number $S \in \mathbb{N}$ of sample trajectories $(x_{n,i})_{n=0}^{N}$, $N \in \mathbb{N}$, $i = 1, \ldots, S$. 
A deterministic piecewise linear map

On the half-open unit interval we consider the map

\[
f(x) := \begin{cases} 
4x, & x \in [0, \frac{1}{2}) \\
4x - 3, & x \in \left[\frac{1}{2}, 1\right)
\end{cases}
\]

[Grossmann, Fujisawa, 1982]
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\end{cases} \]

and by

\[ f(x + k) := f(x) + k \]

for \( x \in [0, 1) \) and \( k \in \mathbb{Z} \) we extend the map to the entire real line.

A time series is then constructed by the iterations \( x_n = f^n(x_0), \ n \in \mathbb{N} \).

[Grossmann, Fujisawa, 1982]
A cobweb plot for the deterministic piecewise linear map $f$

$\begin{align*}
x_0 &= 0.085398, \text{ 5 iterations}
\end{align*}$

[Grossmann, Fujisawa, 1982]
A cobweb plot for the deterministic piecewise linear map $f$

$x_0 = 0.085398, 20$ iterations

[Grossmann, Fujisawa, 1982]
A cobweb plot for the deterministic piecewise linear map \( f \)

\[ x_0 = 0.085398, \text{ 250 iterations} \]

[Grossmann, Fujisawa, 1982]
A time series of the deterministic piecewise linear map $f$

$\text{x0 = 0.1532 (2000 digits)}$

[Reference: Grossmann, Fujisawa, 1982]
Diffusion by the deterministic piecewise linear map \( f \)

Fig. : An ensemble of sample trajectories of the map \( f \)

[Grossmann, Fujisawa, 1982]
Diffusion by the deterministic piecewise linear map $f$

Fig. : An ensemble of sample trajectories of the map $f$

$\Rightarrow$ The piecewise linear map $f$ exhibits normal diffusion: $\alpha = 1$.

[Grossmann, Fujisawa, 1982]
Comments on the numerics for linear maps

- Double precision provides 64 bits, 52 for the mantissa (fractional part of floating point numbers).
- Each multiplication by $4 = 2^2$ causes loss of two bits.

$\Rightarrow$ After at most 26 iterations of the map $f$ we run out of bits.

Fig. : Trajectories of the map $f$, computed with double precision
2. Deterministic Pomeau-Manneville type maps \((x + ax^z)\)

For \(z = 3\), we define the map

\[
h_1(x) := \begin{cases} 
  x + 2^z x^z, & x \in \left[0, \frac{1}{2}\right) \\
  x - 2^z (1 - x)^z, & x \in \left[\frac{1}{2}, 1\right) 
\end{cases}
\]

and extend it to the real line by

\[
h_1(x + k) := h_1(x) + k,
\]

\(x \in [0, 1), k \in \mathbb{Z}.
\)

A time series is then constructed by the iterations \(x_n = h_1^n(x_0), n \in \mathbb{N}.
\)

[Pomeau, Manneville, 1980]
Different durations of trapping

Fig. : the piecewise linear map $f$

$x_0 = 0.011835$, 3 iterations

Fig. : the Pomeau-Manneville map $h_1$

$x_0 = 0.011835$, 451 iterations

[Pomeau, Manneville, 1980]
A cobweb plot for the Pomeau-Manneville map $h_1$

$x_0 = 0.95599, 20$ iterations

[Geisel, Thomae, 1984]
A cobweb plot for the Pomeau-Manneville map $h_1$

$\begin{align*}
x_0 &= 0.95599, \text{ 20 iterations} \\
x_0 &= 0.95599, \text{ 250 iterations}
\end{align*}$

[Geisel, Thomae, 1984]
Diffusion by the Pomeau-Manneville map $h_1$ with $z = 3$

The Pomeau-Manneville map $h_1$ exhibits subdiffusion: $\alpha = \frac{1}{2}$.

[Geisel, Thomae, 1984]
2. Deterministic Pomeau-Manneville type maps \((x + ax^z)\)

For \(z = \frac{5}{3}\), consider the map

\[
h_2(x) := \begin{cases} 
  x + 2^z x^z - 1, & x \in \left[0, \frac{1}{2}\right) \\
  x - 2^z (1-x)^z + 1, & x \in \left[\frac{1}{2}, 1\right)
\end{cases}
\]

extended to the real line by

\[
h_2(x + k) := h_2(x) + k,
\]

\(x \in [0, 1), k \in \mathbb{Z}.
\)

A time series is then constructed by the iterations \(x_n = h_2^n(x_0), n \in \mathbb{N}.
\)

[Geisel, Nierwetberg, Zachary, 1985]
Diffusion by the Pomeau-Manneville map $h_2$ with $z = \frac{5}{3}$

The Pomeau-Manneville map $h_2$ exhibits **superdiffusion**: $\alpha = \frac{3}{2}$.

[Geisel, Nierwetberg, Zachary, 1985]
Intermittency in deterministic dynamical systems

\[ \alpha = 1 \]

\[ \alpha = \frac{1}{2} \]

\[ \alpha = \frac{3}{2} \]
Fig. 1: Ensemble of trajectories of the normaldiffusive pw. lin. map $f$ (red), the subdiffusive P.-M. map (green) and the superdiffusive P.-M. map (blue)
3. Random dynamical systems

For $p = \frac{2}{3}$ we define the random piecewise linear map

$$T(x) := \begin{cases} f(x), & \text{with prob. } 1 - p \\ g(x), & \text{with prob. } p \end{cases}$$

with

$$f(x) := \begin{cases} 4x, & x \in [0, \frac{1}{2}) \\ 4x - 3, & x \in \left[\frac{1}{2}, 1\right) \end{cases},$$

$$g(x) := \begin{cases} \frac{1}{2}x, & x \in [0, \frac{1}{2}) \\ \frac{1}{2}x - \frac{1}{2}, & x \in \left[\frac{1}{2}, 1\right) \end{cases}.$$

A time series is then constructed by the iterations $x_n = T^n(x_0)$, $n \in \mathbb{N}$.

[Y. Sato and R. Klages, submitting, 2016]
A cobweb plot for the trapping map $g$

$0 \leq x_0 \leq 1$ for $x_0 = 0.45, x_0 = 0.55$

[Y. Sato and R. Klages, submitting, 2016]
A cobweb plot for the random piecewise linear map $T$

$\begin{align*}
x_0 &= 0.45, x_0 = 0.55
\end{align*}$

$\begin{align*}
x_0 &= 0.45, 40 \text{ iterations}
\end{align*}$

[Y. Sato and R. Klages, submitting, 2016]
Intermittency by the random piecewise linear map $T$

Sample trajectory of random p.-w. lin. map, $x_0 = 0.19687$

[Y. Sato and R. Klages, submitting, 2016]
Intermittency and MSD of the random pw. lin. map $T$

The random piecewise linear map $T$ exhibits subdiffusion: $\alpha = \frac{1}{2}$.

[Y. Sato and R. Klages, submitting, 2016]
Comments on the choice of the probabilities

The Lyapunov exponent $\lambda$ measures the rate of the separation of close initial values over time

$$|x_{n,1} - x_{n,2}| \approx e^{\lambda n} |x_{0,1} - x_{0,2}|, \ n \in \mathbb{N}.$$ 

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$$|x_{n,1} - x_{n,2}| \approx e^{\lambda n} |x_{0,1} - x_{0,2}|, n \in \mathbb{N}.$$ 

The expanding map $f$ has Lyapunov exponent $\ln(4)$ since

$$|f^n(x_{0,1}) - f^n(x_{0,2})| = 4^n |x_{0,1} - x_{0,2}|.$$ 

Analogously, the contracting map $g$ has Lyapunov exponent $\ln\left(\frac{1}{2}\right).$

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$$|f^n(x_{0,1}) - f^n(x_{0,2})| = 4^n |x_{0,1} - x_{0,2}|.$$

Analogously, the contracting map $g$ has Lyapunov exponent $\ln\left(\frac{1}{2}\right)$.

The probability $p$ of the map $g$ is chosen, such that the averaged Lyapunov exponent vanishes by

$$0 = p \ln\left(\frac{1}{2}\right) + (1 - p) \ln(4).$$

Therefore, we choose $p = \frac{2}{3}$.

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[Y. Sato and R. Klages, submitting, 2016]
Comments on the choice of the probabilities

Fig. : Influence of the probability of the map $g$ on the MSD of the random map $T$

[Y. Sato and R. Klages, submitting, 2016]
Search for superdiffusion in random piecewise linear maps

\[ \alpha = \frac{1}{2} \]

\[ \alpha = \frac{1}{2} \]

\[ \alpha = \frac{3}{2} \]
Search for superdiffusion in random piecewise linear maps

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\[ \alpha = \frac{3}{2} ? \]
MSD of random piecewise linear candidate for superdiffusion

The map exhibits ballistic motion: $\alpha = 2$. 

Katja Polotzek, MPI PKS, Dresden, Anomalous diffusion, August 5th, 2016
Other hopeless candidates for superdiffusion

None of these models shows superdiffusion.
Relevant time scales

- All spatial configurations share the same mod 1 map (see left).
- Therefore, we only have access to the two time scales given by the maps $f$ and $g$, which yield a heavy-tailed waiting time distribution $\phi(n) \sim n^{-\frac{3}{2}}$.

(numerical observation)

- This holding time in either trapping or jumping motion prevents super-diffusion. (numerical observation)
**CTRW model for the subdiffusive random map**

![Graph](image)

Fig. : CTRW, waiting time distrib. density \( \phi(n) \) and jump length \( W(n) \)

For the subdiffusive random piecewise linear map we have \( \phi(n) \sim n^{-3/2} \) and \( W(n) \in \{-1, 0, 1\} \).

It is known that \( \phi(n) \sim n^{-(1+\alpha)} \) and for subdiffusive CTRW model fits observations for subdiffusive piecewise linear map \( (\alpha = 1/2) \).
For the subdiffusive random piecewise linear map we have

\[ \phi(n) \sim n^{-\frac{3}{2}} \quad \text{and} \quad W(n) \in \{-1, 0, 1\}. \]

It is known that

\[ \phi(n) \sim n^{-(1+\alpha)} \quad \Rightarrow \quad \text{msd}(n) \sim n^\alpha. \]

\(\Rightarrow\) CTRW model fits observations for subdiffusive piecewise linear map \((\alpha = \frac{1}{2})\).
Lévy walk model for the ballistic random maps

Lévy walk, waiting time distrib. density $\phi(n)$ and coupled jump length distrib. density $w(n)$. Particles fly on constant velocity as long as they are trapped in the jump zone.
Lévy walk model for the ballistic random maps

It is known that if \( \phi(n) \sim n^{-(1+\alpha)} \),

\[
\begin{aligned}
\text{msd}(n) &\sim \begin{cases} 
  n^2 & 0 < \alpha < 1 \\
  n^2/\ln(n) & \alpha = 1 \\
  n^{3-\alpha} & 1 < \alpha < 2 \\
  n\ln(n) & \alpha = 2 \\
  n & 2 < \alpha 
\end{cases} 
\end{aligned}
\]

Lévy walk, waiting time distrib. density \( \phi(n) \) and coupled jump length distrib. density \( w(n) \).

Particles fly on constant velocity as long as they are trapped in the jump zone.

We still have a waiting time distribution density \( \phi(n) \sim n^{-\frac{3}{2}} \) with \( \alpha = \frac{1}{2} \) and therefore we have

\[
\text{msd}(n) \sim n^2. 
\]

[J. Klafter, I. M. Sokolov, 2011]
References

1. Y. Sato, R. Klages, "Anomalous diffusion in random dynamical systems", submitting, 2016.
2. G. Zumofen, J. Klafter, "Scale-invariant motion in intermittent chaotic systems", 1993.
3. J. Klafter, I. M. Sokolov, "First steps in random walks - from tools to application", 2011.