STRONG STATIONARITY CONDITIONS FOR OPTIMAL CONTROL PROBLEMS GOVERNED BY A RATE-INDEPENDENT EVOLUTION VARIATIONAL INEQUALITY∗

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Abstract. We prove strong stationarity conditions for optimal control problems that are governed by a prototypical rate-independent evolution variational inequality, i.e., first-order necessary optimality conditions in the form of a primal-dual multiplier system that are equivalent to the purely primal notion of Bouligand stationarity. Our analysis relies on recent results on the Hadamard directional differentiability of the scalar stop operator and a new concept of temporal polyhedricity that generalizes classical ideas of Mignot. The established strong stationarity system is compared with known optimality conditions for optimal control problems governed by elliptic obstacle-type variational inequalities and stationarity systems obtained by regularization.

Key words. optimal control, rate independence, stop operator, variational inequality, sweeping process, strong stationarity, Bouligand stationarity, Kurzweil integral, polyhedricity, hysteresis

AMS subject classifications. 49J40, 47J40, 34C55, 49K21, 49K27

1. Introduction and summary of results. This paper is concerned with the derivation of first-order necessary optimality conditions for optimal control problems of the type

\[
\begin{align*}
\text{Minimize} & \quad J(y, y(T), u) \\
\text{w.r.t.} & \quad y \in CBV[0,T], \quad u \in U_{\text{ad}}, \\
\text{s.t.} & \quad \int_0^T (v - y) \, d(y - u) \geq 0 \quad \forall v \in C([0,T];Z), \\
& \quad y(t) \in Z \quad \forall t \in [0,T], \quad y(0) = y_0.
\end{align*}
\]

(P)

Here, \(y\) denotes the state; \(u\) denotes the control; \(T > 0\) is given; \(CBV[0,T]\) is the space of real-valued continuous functions of bounded variation on \([0,T]\); \(U_{\text{ad}}\) is a subset of a suitable control space \(U \subset CBV[0,T]\); \(J : L^\infty(0,T) \times \mathbb{R} \times U \to \mathbb{R}\) is a sufficiently smooth objective function; \(Z = [-r,r]\) is a given interval with \(r > 0\); \(C([0,T];Z)\) is the set of continuous functions on \([0,T]\) with values in \(Z\); \(y_0 \in Z\) is a given initial value; and the integral in the governing variational inequality is understood in the sense of Kurzweil-Stieltjes (see [43] and the appendix of this paper for details on this type of integral). For the precise assumptions on the quantities in (P), we refer to section 3. The main result of this work – Theorem 7.1 – establishes a so-called strong stationarity system for the problem (P). This is a first-order necessary optimality condition in primal-dual form that is satisfied by a control \(\bar{u} \in U_{\text{ad}}\) if and only if \(\bar{u}\) is a Bouligand stationary point of (P), i.e., if and only if the directional derivative of the reduced objective function of (P) at \(\bar{u}\) is nonnegative in all admissible directions. See also (1.5) below for the resulting stationarity system.

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1.1. Background and relation to prior work. Before we present and discuss the strong stationarity system derived in Theorem 7.1 in more detail, let us give some background. To keep the discussion concise, we focus on strong stationarity conditions for infinite-dimensional optimization problems arising in optimal control. For related results in finite dimensions, see [22, 25, 28, 37, 47] and the references therein.

In the field of infinite-dimensional nonsmooth optimization, strong stationarity conditions (although originally not referred to as such) have first been derived for optimal control problems governed by elliptic obstacle-type variational inequalities in the seminal works [40, 42] of Mignot and Puel in the nineteen-seventies and -eighties. If we use a notation analogous to that in (P), then this kind of problem can be formulated (in its most primitive form) as follows:

\[
\begin{align*}
\text{Minimize} & \quad J(y,u) \\
\text{w.r.t.} & \quad y \in H^1_0(\Omega), \quad u \in U_{\text{ad}} \subset L^2(\Omega), \\
\text{s.t.} & \quad y \in Z, \quad \int_\Omega \nabla y \cdot \nabla (v-y) \, dx \geq \int_\Omega u(v-y) \, dx \quad \forall v \in Z.
\end{align*}
\]

Here, \( \Omega \subset \mathbb{R}^d, d \in \mathbb{N} \), is a nonempty open bounded set; \( H^1_0(\Omega) \) and \( L^2(\Omega) \) are defined as usual, see [21, 23]; \( F: H^1_0(\Omega) \times L^2(\Omega) \rightarrow \mathbb{R} \) is a Fréchet differentiable objective function with partial derivatives \( \partial_1 F(y,u) \in H^{-1}(\Omega) \) and \( \partial_2 F(y,u) \in L^2(\Omega) \) (where \( H^{-1}(\Omega) \) denotes the topological dual of \( H^1_0(\Omega) \)); \( U_{\text{ad}} \subset L^2(\Omega) \) is a convex, nonempty, and closed set; \( \nabla \) is the weak gradient; and \( Z \) is a nonempty set of the type

\[
Z := \{ v \in H^1_0(\Omega) : \psi_1 \leq v \leq \psi_2 \text{ a.e. in } \Omega \}
\]

involving two given measurable functions \( \psi_1, \psi_2 : \Omega \rightarrow [\mathbb{R} \cap (0,\infty), \mathbb{R} \cap (-\infty,0)] \). The main difficulty that arises when deriving first-order necessary optimality conditions for problems like (1.1) is that the governing variational inequality causes the control-to-state operator \( S: L^2(\Omega) \rightarrow H^1_0(\Omega), u \mapsto y \), to be nondifferentiable (in the sense of Gâteaux and Fréchet). This nonsmoothness prevents classical adjoint-based approaches as found, e.g., in [50] from being applicable and makes it necessary to develop tailored strategies to establish stationarity systems for local minimizers. In [40, 42], the problem of deriving first-order optimality conditions for (1.1) was tackled by exploiting that the solution mapping \( S: L^2(\Omega) \rightarrow H^1_0(\Omega), u \mapsto y \), of the lower-level variational inequality in (1.1) is Hadamard directionally differentiable with directional derivatives \( \delta := S'(u; h) \), \( u, h \in L^2(\Omega) \), that are uniquely characterized by the auxiliary problem

(1.2) \[
\delta \in K_{\text{crit}}(y,u), \quad \int_\Omega \nabla \delta \cdot \nabla (z - \delta) \, dx \geq \int_\Omega h(z - \delta) \, dx \quad \forall z \in K_{\text{crit}}(y,u).
\]

Here, \( K_{\text{crit}}(y,u) := K_{\text{tan}}(y) \cap (u + \Delta y)^\perp \) denotes the so-called critical cone associated with \( u \) and \( y := S(u) \), i.e., the intersection of the kernel

\[
(u + \Delta y)^\perp := \left\{ z \in H^1_0(\Omega) : \int_\Omega uz - \nabla y \cdot \nabla z \, dx = 0 \right\}
\]

of the functional \( u + \Delta y \in H^{-1}(\Omega) \) and the tangent cone \( K_{\text{tan}}(y) \subset H^1_0(\Omega) \) to \( Z \) at \( y \) which is obtained by taking the closure of the radial cone \( K_{\text{rad}}(y) := \mathbb{R}_+ (Z - y) \) in \( H^1_0(\Omega) \), cf. [25, section 2] and [24, 40]. By proceeding along the lines of [40, 42], one obtains the following main result for the optimal control problem (1.1): If a control
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\( \tilde{u} \in U_{ad} \) with state \( \tilde{y} := S(\tilde{u}) \) is given such that the set \( \mathbb{R}_+(U_{ad} - \tilde{u}) \) is dense in \( L^2(\Omega) \), then \( \tilde{u} \) is a Bouligand stationary point of (1.1) in the sense that

\[
(1.3) \quad \langle \partial_1 J(\tilde{y}, \tilde{u}), S'(\tilde{u}; h) \rangle_{H^1_0} + \langle \partial_2 J(\tilde{y}, \tilde{u}), h \rangle_{L^2} \geq 0 \quad \forall h \in \mathbb{R}_+(U_{ad} - \tilde{u})
\]

holds if and only if there exist an adjoint state \( \bar{p} \in H^1_0(\Omega) \) and a multiplier \( \bar{\mu} \in H^{-1}(\Omega) \) such that \( \tilde{u}, \tilde{y}, \bar{p}, \) and \( \bar{\mu} \) satisfy the system

\[
(1.4) \quad \begin{align*}
\bar{p} + \partial_2 J(\tilde{y}, \tilde{u}) &= 0 \quad \text{in} \ L^2(\Omega), \\
-\Delta \bar{p} &= \partial_1 J(\tilde{y}, \tilde{u}) - \bar{\mu} \quad \text{in} \ H^{-1}(\Omega), \\
\bar{p} &\in K_{\text{crit}}(\tilde{y}, \tilde{u}), \quad \langle \bar{\mu}, z \rangle_{H^1_0} \geq 0 \quad \forall z \in K_{\text{crit}}(\tilde{y}, \tilde{u}).
\end{align*}
\]

Here and in what follows, the symbols \( \langle \cdot, \cdot \rangle \) and \( \langle \cdot, \cdot \rangle \) denote a dual pairing and a scalar product, respectively. For a proof of the above result, see [11, Corollary 6.1.11]. Note that, since the inequality (1.3) expresses that the directional derivatives of the reduced objective function \( L^2(\Omega) \ni u \mapsto J(S(u), u) \in \mathbb{R} \) of (1.1) are nonnegative in all admissible directions \( h \in \mathbb{R}_+(U_{ad} - \tilde{u}) \) at \( \tilde{u} \) and thus corresponds to the most natural first-order necessary optimality condition obtainable for a directionally differentiable function, and since the conditions (1.3) and (1.4) are equivalent, the system (1.4) can be considered the most precise first-order primal-dual necessary optimality condition possible for (1.1). This is the reason why systems of the type (1.4) became known as strong stationarity conditions since their initial appearance in [40, 42].

The main appeal of the system (1.4) is, of course, its equivalence to the Bouligand stationarity condition (1.3). This characteristic property distinguishes (1.4) from other first-order necessary optimality conditions and makes (1.4) an important tool, e.g., for assessing which information about \( \bar{p} \) and \( \bar{\mu} \) is lost when a stationarity system is derived by means of a regularization or discretization approach. For details on this topic, we refer to the survey article [25]. Because of these advantageous properties, strong stationarity conditions have come to play a distinct role in the field of optimal control of nonsmooth systems and have received considerable attention in the recent past. See, e.g., [4, 11, 13, 19, 26, 27, 51, 54] for contributions on strong stationarity conditions for optimal control problems governed by various elliptic variational inequalities of the first and the second kind, [3, 14, 17, 38] for extensions to optimal control problems governed by nonsmooth semi- and quasilinear PDEs, and [15] for a generalization to the multiobjective setting. Note that all of these works on the concept of strong stationarity have in common that they are only concerned with elliptic variational inequalities or PDEs involving nonsmooth terms. What has – at least to the best of our knowledge – not been accomplished so far in the literature is the derivation of a necessary optimality condition analogous to (1.4) for an optimal control problem that is governed by a true evolution variational inequality (where with “true” we mean that the inequality cannot be reformulated as a nonsmooth PDE or an elliptic problem, cf. [3]). In fact, such an extension is even mentioned as an open problem in the seminal works of Mignot and Puel; see [42, section 4] and [41] where strong stationarity conditions for parabolic obstacle problems are conjectured upon. This absence of results on strong stationarity systems for evolution variational inequalities is very unsatisfying in view of the multitude of processes that are modeled by this type of variational problem in finance, mechanics, and physics; see [39, 48].

The main reason for the lack of contributions on strong stationarity conditions for evolution variational inequalities since the nineteen-seventies is that directional differentiability results analogous to that for the elliptic obstacle problem in (1.2)
have not been available in the instationary setting for a long period of time. See, e.g., [5, p. 582] where this problem is still referred to as open. Only recently, progress in this direction has been made. In [6, 7], it could be proved by means of a semi-explicit solution formula involving the cumulated maximum that the control-to-state operator of the problem \((P)\) – the so-called scalar stop operator – is Hadamard directionally differentiable in a pointwise manner; see Theorem 4.11 below. In [12], it could further be shown by means of pointwise-a.e. convexity properties that the solution mapping of the parabolic obstacle problem is Hadamard directionally differentiable as a function into all Lebesgue spaces. This paper also establishes that the directional derivatives of the solution operator of the parabolic obstacle problem are the (not necessarily unique) solutions of a weakly formulated auxiliary variational inequality analogous to (1.2), see [12, Theorem 4.1]. Very recently, in [8], an auxiliary problem for the directional derivatives of the scalar stop operator in \((P)\) has also been obtained by means of a careful analysis of jump directions and approximation arguments. This auxiliary problem even yields a unique characterization, see Theorem 4.11 below.

1.2. Main result and contribution of the paper. The purpose of the present paper is to show that the recent developments in [6, 7, 8] make it possible to prove a strong stationarity system for the optimal control problem \((P)\). As far as we are aware, our analysis is the first to establish such a system for a true evolution variational inequality. The result in the literature that comes closest to the one derived in this paper is, at least to the best of our knowledge, [12, Theorem 5.5] which establishes a multiplier system for optimal control problems governed by parabolic obstacle-type variational inequalities that is equivalent to Bouligand stationarity if the adjoint state enjoys additional regularity properties – a deficit that is caused by a mismatch between certain notions of capacity, see the discussion in [12, section 5]. In the present work, we do not require such additional regularity assumptions and obtain a strong stationarity system for \((P)\) that is fully equivalent to the notion of Bouligand stationarity. Our main result can be summarized as follows: If \(\bar{u} \in U_{ad}\) is a control of \((P)\) with associated state \(\bar{y}\) such that the set \(\mathbb{R}_+(U_{ad} - \bar{u})\) is dense in the control space \(U\), then \(\bar{u}\) is a Bouligand stationary point of \((P)\) (in a sense analogous to that of (1.3), see Definition 5.3 below) if and only if there exist an adjoint state \(\bar{p} \in BV[0,T]\) and a multiplier \(\bar{\mu} \in G_{ad}[0,T]^*\) such that \(\bar{u}, \bar{y}, \bar{p}, \) and \(\bar{\mu}\) satisfy the system

\[
\begin{align*}
\dot{\bar{p}}(0) = \bar{p}(T) = 0, \quad \bar{p}(t) = \bar{p}(t-) \quad &\forall t \in [0,T), \\
\bar{p}(t-) \in K_{crit}^{ptw}(\bar{y}, \bar{u})(t) \quad &\forall t \in [0,T], \\
\langle \bar{\mu}, z \rangle_{G_{ad}} \geq 0 \quad &\forall z \in K_{G_{ad}}^{red,crit}(\bar{y}, \bar{u}), \\
\int_0^T h \, d\bar{p} = \langle \partial_3 J(\bar{y}, \bar{y}(T), \bar{u}), h \rangle_U \quad &\forall h \in U, \\
- \int_0^T z \, d\bar{p} = \langle \partial_1 J(\bar{y}, \bar{y}(T), \bar{u}), z \rangle_L^{+} + \partial_2 J(\bar{y}, \bar{y}(T), \bar{u}) z(T) - \langle \bar{\mu}, z \rangle_{G_{ad}} \quad &\forall z \in G_{ad}[0,T].
\end{align*}
\]

Here, \(BV[0,T]\) denotes the space of real-valued functions of bounded variation on \([0,T]\); \(G_{ad}[0,T]\) is the space of real-valued, regulated, and right-continuous functions on \([0,T]\); \(G_{ad}[0,T]^*\) is the topological dual space of \(G_{ad}[0,T]\); the partial derivatives of \(J\) are denoted by \(\partial_i J, \ i = 1, 2, 3;\) the minus in the argument of \(\bar{p}\) denotes a left limit; and \(K_{crit}^{ptw}(\bar{y}, \bar{u})(t), t \in [0,T],\) and \(K_{G_{ad}}^{red,crit}(\bar{y}, \bar{u})\) are suitably defined cones (see Definitions 4.7 and 6.1). For the precise statement of the above result, see Theorem 7.1.
Several things are noteworthy regarding the system (1.5):

First of all, it can be seen that the adjoint state \( \bar{p} \) lacks regularity in comparison with the optimal state \( \bar{y} \) \((BV[0,T] \) instead of \( CBV[0,T] \)). This reduced regularity reflects that the directional derivatives of the control-to-state mapping of \((P)\) are not continuous in time and thus significantly less regular than the states \( y \) – a behavior that is completely absent in the elliptic problem (1.1). For details on this topic, see also \([12, \text{section 3}] \) and \([8, \text{Example 4.1}] \) which demonstrate that all types of jump discontinuities of the derivatives are possible in the situation of \((P)\) and that the derivatives cannot be expected to possess, e.g., \( H^{1/2}(0,T)\)-regularity, cf. \([30]\).

Second, one observes that not the adjoint state \( \bar{p} \) but its left limits are contained in the critical cone \( K_{\text{crit}}^{\text{ptw}}(\bar{y}, \bar{u})(t) \) for all \( t \in [0,T] \) in (1.5). As we will see below, this condition on the limiting behavior – along with the left-continuity of \( \bar{p} \) in the first line of (1.5) – arises from certain properties of the jumps of the directional derivatives of the control-to-state mapping and the fact that the adjoint system evolves backwards in time (in contrast to the variational inequality for the directional derivatives of the control-to-state mapping which evolves in a forward manner). Note that these additional properties of the left limit of the adjoint state are not visible in stationarity systems derived by regularization, cf. \([2, 10, 18, 20, 29, 49, 52]\). This shows that (1.5) contains information that is not recoverable with regularization approaches.

Lastly, it should be noted that the coupling between the adjoint state \( \bar{p} \) and the partial derivative \( \partial_2 \mathcal{J}(\bar{y}, \bar{y}(T), \bar{u}) \) of the objective \( \mathcal{J} \) w.r.t. the control in (1.5) is not as direct as in (1.4) but involves an integration step. This is a consequence of the rate-independence of the variational inequality governing \((P)\) and ultimately also the reason for the nonstandard start- and endpoint conditions \( \bar{p}(0) = \bar{p}(T) = 0 \) for \( \bar{p} \) in (1.5). We remark that these conditions reflect that the partial derivative \( \partial_2 \mathcal{J}(\bar{y}, \bar{y}(T), \bar{u}) \) manifests itself – in a distributional sense – in the jump of \( \bar{p} \) at the terminal time \( T \), see the comments at the end of section 7. A similar behavior can also be observed in optimal control problems for parabolic PDEs, see \([50, \text{section 5.5.1}] \).

Regarding the derivation of the strong stationarity system in Theorem 7.1, we would like to point out that – even with the results of \([6, 7, 8]\) at hand and even though the variational inequality in \((P)\) is one of the simplest evolution variational inequalities imaginable – the proof of (1.5) is still quite involved. The main difficulty in the context of \((P)\) is that, due to the lack of weak-star continuity properties of the scalar stop operator, one has to discuss this problem in a control space \( U \) whose topology is significantly stronger than that of \( BV[0,T] \) to be able to ensure that \((P)\) is well posed; see the comments in section 5 below. Since the directional derivatives of the scalar stop are only in \( BV[0,T] \), the need for such a “small” control space \( U \) makes it necessary to employ a careful limit analysis to ensure that the control space is ample enough to be able to arrive at a strong stationarity system. Compare also with the comments on this topic in \([13, 26]\) and the results in \([51]\) in this context. In our analysis, we tackle this problem by generalizing the classical concept of polyhedricity to the time-dependent setting. This is a density property which, in the situation of the elliptic problem (1.1), ensures that the set of critical radial directions \( K_{\text{rad}}(y) \cap (u + \Delta y)^+ \) is \( H^1_0(\Omega)\)-dense in \( K_{\text{tan}}(y) \cap (u + \Delta y)^+ \) and which plays an important role in the sensitivity analysis of elliptic obstacle-type variational inequalities as well as the theory of second-order optimality conditions, see \([24, 16, 53]\). For the approximation result that we establish in this context and that we refer to as “temporal polyhedricity”, see Theorem 6.5.

We expect that Theorem 6.5, along with the insights provided by (1.5), is also helpful for the analysis of optimal control problems governed by more complicated evolution variational inequalities, cf. the problems studied in \([12, 44, 46]\).
1.3. Structure of the remainder of the paper. We conclude this section with an overview of the content and the structure of the remainder of the paper.

Sections 2 and 3 are concerned with preliminaries. Here, we introduce the notation and the standing assumptions that we use throughout this work. In section 4, we collect basic results on the properties of the control-to-state mapping of (P) – the scalar stop operator. This section also recalls the directional differentiability results of [6, 7, 8] and discusses some of their consequences. Section 5 addresses the solvability of (P) and introduces the concept of Bouligand stationarity for this problem. This section also contains an example which shows that, to be able to prove the existence of solutions for (P) by means of the direct method of the calculus of variations, one indeed has to consider a control space significantly smaller than $BV[0, T]$. In section 6, we prove the already mentioned temporal polyhedricity property for (P). The main result of this section is Theorem 6.5. Section 7 is concerned with the proof of the strong stationarity system (1.5), see Theorem 7.1. The appendix of the paper collects some results on the Kurzweil-Stieltjes integral that are needed for our analysis.

2. Notation. Throughout this work, $T > 0$ is a given and fixed number. We denote the space of real-valued continuous functions on $[0, T]$ by $C[0, T]$ and the space of real-valued regulated functions on $[0, T]$ (i.e., the space of all functions that are uniform limits of step functions, see [43, Definition 4.1.1, Theorem 4.1.5]) by $G[0, T]$. We equip both $C[0, T]$ and $G[0, T]$ with the supremum norm $\| \cdot \|_{\infty}$. Recall that this makes $C[0, T]$ and $G[0, T]$ Banach spaces and that every $v \in G[0, T]$ possesses left and right limits, see [43, chapter 4].

Given $v \in G[0, T]$, we denote these limits by $v(t–)$ and $v(t+)$, respectively, with the usual conventions at the endpoints of $[0, T]$, i.e.,

$$v(t–) := \lim_{[0, T] \ni s \to t–} v(s) \quad \forall t \in (0, T], \quad v(0–) := v(0),$$

$$v(t+) := \lim_{[0, T] \ni s \to t+} v(s) \quad \forall t \in [0, T), \quad v(T+) := v(T).$$

For the left- and the right-limit function associated with a function $v \in G[0, T]$, we use the symbols $v_-$ and $v_+$, i.e., $v_–(t) := v(t–)$ and $v_+(t) := v(t+)$ for all $t \in [0, T]$. We further define $G_r[0, T] := \{ v \in G[0, T] : v = v_+ \}$. It is easy to check that this set of right-continuous regulated functions is a closed subspace of $(G[0, T], \| \cdot \|_{\infty})$.

The space of real-valued functions of bounded variation on $[0, T]$ is denoted by $BV[0, T]$. We emphasize that we do not consider elements of $BV[0, T]$ as equivalence classes in this paper but as classical functions $v : [0, T] \to \mathbb{R}$, as in [43, chapter 2]. For a discussion of different approaches to $BV[0, T]$, see [1]. We denote the variation of a function $v : [0, T] \to \mathbb{R}$ by $\text{var}(v)$, and we define the total variation norm on $BV[0, T]$ as $\| v \|_{BV} := \| v(0) \| + \text{var}(v)$. Recall that $(BV[0, T], \| \cdot \|_{BV})$ is a Banach space that is continuously embedded into $(G[0, T], \| \cdot \|_{\infty})$; see [43, Theorem 2.2.2]. We define $CBV[0, T] := BV[0, T] \cap C[0, T]$ and $BV_r[0, T] := BV[0, T] \cap G_r[0, T]$. Note that both of these sets are closed subspaces of $(BV[0, T], \| \cdot \|_{BV})$.

Given a set-valued function $K : [0, T] \to \mathbb{R}$ and $0 \leq s < \tau \leq T$, we use the symbols $C([s, \tau]; K)$ and $G([s, \tau]; K)$ to denote the sets of continuous and regulated functions $v$ on $[s, \tau]$ which satisfy $v(t) \in K(t)$ for all $t \in [s, \tau]$, respectively. Sets $K \subseteq \mathbb{R}$ are interpreted as set-valued functions that are constant in time in this notation. We further set $C^\infty([0, T]) := \{ v \in C([0, T]) : \exists \tilde{v} \in C^\infty(\mathbb{R}) \text{ s.t. } v(t) = \tilde{v}(t) \ \forall t \in [0, T] \}$. For the classical Lebesgue and Sobolev spaces, we use the standard notation $(L^p(0, T), \| \cdot \|_{L^p})$ and $(W^{k, p}(0, T), \| \cdot \|_{W^{k, p}})$, $1 \leq p \leq \infty$, $k \in \mathbb{N}$. The weak derivative of a function $v \in W^{1, p}(0, T)$ is denoted by $v' \in L^p(0, T)$. For the topological dual of a normed space $(X, \| \cdot \|_X)$, we use the symbol $X^*$, and for a dual pairing, the brackets $\langle \cdot, \cdot \rangle$. 

In
equipped with a subscript that clarifies the space. A closure is denoted by \( \text{cl}(\cdot) \). Weak, weak-star, and strong convergence are indicated by \( \rightharpoonup \), \( \rightharpoonup^* \), and \( \to \), respectively. Given a set \( D \subset [0,T] \), we define \( \mathbf{1}_D : [0,T] \to \{0,1\} \) to be the characteristic function of \( D \), i.e., the function that equals 1 on \( D \) and 0 everywhere else.

3. Main problem and standing assumptions. As already mentioned in the introduction, the aim of this paper is to study optimal control problems of the type

\[
\begin{align*}
\text{(P)} \\
\begin{cases}
\text{Minimize} & \mathcal{J}(y, y(T), u) \\
\text{w.r.t.} & y \in CBV[0,T], \quad u \in U_{ad}, \\
\text{s.t.} & y = S(u),
\end{cases}
\end{align*}
\]

where \( S \) is the scalar stop operator, i.e., the solution map \( S : CBV[0,T] \to CBV[0,T], u \mapsto y \), of the rate-independent evolution variational inequality

\[
\begin{align*}
\text{(V)} \\
\begin{cases}
\int_0^T (v - y) \, d(y - u) \geq 0 & \forall v \in C([0,T]; Z), \\
y(t) \in Z & \forall t \in [0,T], \\
y(0) = y_0.
\end{cases}
\end{align*}
\]

General references for the properties of the function \( S \) are [9, 31, 32]; some of them will be discussed in detail in section 4.

Note that, from the application point of view, (P) can be interpreted as an optimal control problem for a one-dimensional sweeping process with characteristic set \( Z = [-r,r] \), i.e., a problem that aims to control the trajectory of a body with one degree of freedom that is placed on a slippery surface within \( Z \) and moved (swept) by moving \( Z \) back and forth, see [39, section 1.1]. (In this case, the trajectory is described by the scalar play operator \( P(u) := u - S(u) \) and the control function \( u \) models the movement of \( Z \).) This physical interpretation, however, is mainly secondary in this work. We are primarily interested in the problem (P) because it is the instationary counterpart of the optimal control problem (1.1) for the elliptic obstacle problem and captures the effects of “pure” evolution without any additional spatial dependencies (as present, e.g., in the parabolic obstacle problem, cf. [16]). We hope that the insights provided by our analysis are also helpful for the analysis of optimal control problems governed by more complicated systems arising, e.g., in the field of elasto-plasticity, which often involve the play and stop operator to incorporate hysteresis effects, cf. [39, 44, 46].

We would like to emphasize that the integral in (V) — along with all other integrals appearing in the remainder of this paper — is to be understood in the sense of Kurzweil-Stieltjes. For an in-depth introduction to the integration theory for this type of integral, we refer to [43]. A collection of basic definitions, elementary properties, and fundamental results related to the Kurzweil-Stieltjes integral can also be found in the appendix of this paper. The use of the Kurzweil-Stieltjes integral for the variational inequality approach to rate-independent evolutions goes back to [33, 34, 36] where it was employed for the study of discontinuous input functions \( u \). For this kind of \( u \), the integrand and the integrator (i.e., the function behind the “\( d \)”) in (V) usually have discontinuities at common points \( t \in [0,T] \) so that the Riemann-Stieltjes integral no longer works. Such common discontinuities also appear naturally in the variational inequality that characterizes the directional derivatives of \( S \), cf. Theorem 4.11 below. For a treatment based on the Young integral, see [35]. Alternatively, the Lebesgue-Stieltjes integral can be used as for the types of integrands and integrators appearing in this paper it is equivalent to the Kurzweil-Stieltjes integral, see [43, section 6.12].
However, for this type of integral, a careful handling of statements involving “almost everywhere” is necessary since the $\sigma$-algebra and the family of its sets of measure zero depend on the integrator. In particular, a singleton $\{t\}$ has nonzero measure if the integrator is discontinuous at $t$.

For the ease of reference, we collect our standing assumptions on the quantities in the optimal control problem (P) and the variational inequality (V) in:

**Assumption 3.1** (standing assumptions).
- $T > 0$ is given and fixed.
- $U \subset CBV[0,T]$ is a real vector space that is endowed with a norm $\| \cdot \|_U$ and that is continuously and densely embedded into $(C[0,T],\| \cdot \|_\infty)$.
- $U_{\text{ad}}$ is a nonempty and convex subset of $U$.
- $J: L^\infty(0,T) \times \mathbb{R} \times U \to \mathbb{R}$ is a Fréchet differentiable function whose partial derivative w.r.t. the first argument satisfies $\partial_1 J(y,y(T),u) \in L^1(0,T)$ for all $(y,u) \in CBV[0,T] \times U$. Here, $L^1(0,T)$ is interpreted as a subset of $L^\infty(0,T)^*$ via the canonical embedding into the bidual.
- $Z$ is an interval of the form $Z = [-r,r]$ with an arbitrary but fixed $r > 0$.
- $y_0 \in Z$ is a given and fixed starting value.

The above assumptions are always assumed to hold in the following sections, even when not explicitly mentioned. We remark that, to be able to prove the existence of solutions for (P), one requires more information about $J$, $U_{\text{ad}}$, etc. than provided by Assumption 3.1; see Corollary 5.1. For the derivation of the strong stationarity system (1.5), however, this is not relevant. An example of a control space $U$ that satisfies the conditions in Assumption 3.1 and that allows to prove the existence of minimizers for (P) is the space $H^1(0,T)$, see section 5 and the comments therein.

4. Properties of the scalar stop operator $S$. In this section, we collect properties of the solution map $S: u \mapsto y$ of the variational inequality (V) that are needed for our analysis. We begin with fundamental results on the well-definedness, monotonicity, and directional differentiability of $S$.

**Theorem 4.1** (well-definedness and Lipschitz continuity). The variational inequality (V) possesses a unique solution $S(u) := y \in CBV[0,T]$ for all $u \in CBV[0,T]$. For all $u \in W^{1,1}(0,T)$, it holds $y = S(u) \in W^{1,1}(0,T)$ and

\[ (4.1) \quad (v - y(t))(y'(t) - u'(t)) \geq 0 \quad \forall v \in Z \quad \text{for a.a. } t \in (0,T). \]

Further, $S$ satisfies the Lipschitz estimate

\[ (4.2) \quad \|S(u_1) - S(u_2)\|_\infty \leq 2\|u_1 - u_2\|_\infty \quad \forall u_1, u_2 \in CBV[0,T]. \]

**Proof.** Proofs of the unique solvability of (V) in $CBV[0,T]$ and of (4.1) can be found in [32, Theorem 4.1, Proposition 4.1]. The Lipschitz estimate (4.2) follows from [31, p. 49f.] and [9, Proposition 2.3.4].

**Lemma 4.2** (general test functions). Let $u \in CBV[0,T]$ and $0 \leq s < \tau \leq T$. Then $y := S(u)$ satisfies

\[ (4.3) \quad \int_s^\tau (v - y) \, d(y - u) \geq 0 \quad \forall v \in G([s,\tau];Z). \]

**Proof.** Since $y + 1_{[s,\tau]}(v - y) \in G([0,T];Z)$ for all $v \in G([s,\tau];Z)$ and due to Lemma A.1, it suffices to consider the case $[s,\tau] = [0,T]$. Let $v: [0,T] \to Z$ be a step function.
function of the form
\[ v = \sum_{j=1}^{N} \mathbb{I}_{(t_{j-1}, t_j)}(u_j) + \sum_{j=0}^{N} \mathbb{I}_{(t_j, t_{j+1})}(\hat{u}_j) \]

with \( \zeta_j, \hat{\zeta}_j \in \mathbb{Z} \) and \( 0 = t_0 < ... < t_N = T \). Since \( v = \lim_{n \to \infty} v_n \) pointwise for suitable \( v_n \in C([0, T]; \mathbb{Z}) \), (4.3) for \( v \) follows from the bounded convergence theorem, Theorem A.4. As step functions are dense in \( G([0, T]; \mathbb{Z}) \) by [43, Theorem 4.1.5], (4.3) holds for arbitrary \( v \in G([0, T]; \mathbb{Z}) \), again by the bounded convergence theorem.

**Lemma 4.3** (piecewise monotonicity). Let \( u \in CBV[0, T] \) and set \( y := S(u) \). Let \( J \) be an open nonempty subinterval of \([0, T]\).

i) If \( J \subset \{ t \in [0, T]: y(t) > -r \} \), then \( y-u \) is nonincreasing on \( \text{cl}(J) \).

ii) If \( J \subset \{ t \in [0, T]: y(t) < r \} \), then \( y-u \) is nondecreasing on \( \text{cl}(J) \).

**Proof.** We prove i). (The proof of ii) is analogous.) Let \( s, \tau \in J \) with \( s < \tau \). Then \( y \geq -r + \epsilon \) on \([s, \tau]\) for some \( \epsilon > 0 \). As \( v := y - \epsilon \in G([s, \tau]; \mathbb{Z}) \), we can apply Lemma 4.2 to obtain
\[ 0 \leq \int_s^\tau (v-y) \, dy = -\epsilon((y-u)(\tau) - (y-u)(s)). \]
Thus, \( y-u \) is nonincreasing on \( J \), and hence on \( \text{cl}(J) \) since \( y-u \) is continuous.

A proof of the foregoing lemma based on an explicit representation of \( y-u \) can be found in [7, section 5].

**Lemma 4.4** (comparison principle). Let \( u_1, u_2 \in CBV[0, T] \) be given such that \( u_2 - u_1 \) is nondecreasing in \([0, T]\). Then it holds \( S(u_2)(t) \geq S(u_1)(t) \) for all \( t \in [0, T] \).

**Proof.** First, let us assume that \( u_1, u_2 \in W^{1,1}(0, T) \). From (4.1), we obtain that \( y_1 := S(u_1) \) and \( y_2 := S(u_2) \) satisfy
\[ (v - y_i(t))(y'_i(t) - u'_i(t)) \geq 0 \quad \forall v \in Z \quad \text{for a.a. } t \in (0, T) \quad i = 1, 2. \]
Testing (4.4) for \( i = 1 \) with \( v = y_1(t) - \max\{0, y_1(t) - y_2(t)\} \in Z \) and for \( i = 2 \) with \( v = y_2(t) - \max\{0, y_1(t) - y_2(t)\} \in Z \) and adding the resulting inequalities gives
\[ \max\{0, y_1 - y_2\} \cdot (y'_1 - y'_2) \leq \max\{0, y_1 - y_2\} \cdot (u'_1 - u'_2) \leq 0 \quad \text{a.e. in } (0, T) \]
as \( u_2 - u_1 \) is nondecreasing. By a classical result of Stampacchia, see, for instance, [23, Lemmas 7.5 and 7.6], we have
\[ \frac{d}{dt} \frac{1}{2} (\max\{0, y_1 - y_2\})^2 = \max\{0, y_1 - y_2\} \cdot (y'_1 - y'_2) \quad \text{a.e. in } (0, T). \]
Since \( y_2(0) = y_1(0) \), we conclude that \( \max\{0, y_1 - y_2\} \leq 0 \) on \([0, T]\). Thus, \( y_2 \geq y_1 \) on \([0, T]\) as claimed. In the general case \( u_1, u_2 \in CBV[0, T] \), we choose piecewise affine interpolants \( u^n_1, u^n_2 \) of \( u_1, u_2 \) on partitions \( \Delta_n \) of \([0, T]\) whose widths go to zero for \( n \to \infty \). Since \( u^n_2 - u^n_1 \) is nondecreasing too, it follows that \( S(u^n_2) \geq S(u^n_1) \) on \([0, T]\) for all \( n \). As \( u^n_i \to u_i \) uniformly, by virtue of (4.2), we may pass to the limit, and the claim follows.

**Theorem 4.5** (pointwise directional differentiability of \( S \)). The solution operator \( S: CBV[0, T] \to CBV[0, T] \) of (V) is pointwise directionally differentiable in the sense that, for all \( u, h \in CBV[0, T] \), there is a unique \( S'(u; h) \in BV[0, T] \) satisfying
\[ \lim_{\alpha \to 0^+} \frac{S(u + \alpha h)(t) - S(u)(t)}{\alpha} = S'(u; h)(t) \quad \forall t \in [0, T]. \]

**Proof.** See [7, Corollary 5.4, Proposition 6.3] and also [8, Theorem 2.1].
Similarly to the classical result \((1.2)\) for the obstacle problem, the derivatives 
\(S'(u; h)\) in Theorem 4.5 are characterized by an auxiliary variational inequality. To be able to state this inequality, we require some additional notation from \([8]\).

**Definition 4.6 (inactive, biactive, and strictly active set).** Let \(u \in CBV[0, T]\) be a control with state \(y := S(u) \in CBV[0, T]\). We introduce:

- **the inactive set:** 
  \[ I(y) := \{ t \in [0, T] : |y(t)| < r \}, \]

- **the biactive set associated with the upper bound of \(Z\):**
  \[ B_+(y, u) := \{ t \in [0, T] : y(t) = r \text{ and } \exists \varepsilon > 0 \text{ s.t. } y - u = \text{const on } [t, t + \varepsilon) \}, \]

- **the biactive set associated with the lower bound of \(Z\):**
  \[ B_-(y, u) := \{ t \in [0, T] : y(t) = -r \text{ and } \exists \varepsilon > 0 \text{ s.t. } y - u = \text{const on } [t, t + \varepsilon) \}, \]

- **the biactive set:**
  \[ B(y, u) := B_+(y, u) \cup B_-(y, u), \]

- **the strictly active set:**
  \[ A(y, u) := \{ t \in [0, T] : |y(t)| = r \text{ and } \exists \varepsilon > 0 \text{ s.t. } y - u = \text{const on } [t, t + \varepsilon) \}. \]

Here and in what follows, we use the convention \(T \in B_{\pm}(y, u)\) in the case \(y(T) = \pm r\).

**Definition 4.7 (radial and critical cone mapping).** Given an input function \(u \in CBV[0, T]\) with state \(y := S(u) \in CBV[0, T]\), we define:

- **the set-valued pointwise radial cone mapping:**
  \[
  K_{\text{rad}}^{\text{ptw}}(y) : [0, T] \to \mathbb{R}, \\
  K_{\text{rad}}^{\text{ptw}}(y)(t) := \begin{cases} 
  \mathbb{R} & \text{if } |y(t)| < r, \\
  (-\infty, 0] & \text{if } y(t) = r, \\
  [0, \infty) & \text{if } y(t) = -r.
  \end{cases}
  \]

- **the set-valued pointwise critical cone mapping:**
  \[
  K_{\text{crit}}^{\text{ptw}}(y, u) : [0, T] \to \mathbb{R}, \\
  K_{\text{crit}}^{\text{ptw}}(y, u)(t) := \begin{cases} 
  \mathbb{R} & \text{if } t \in I(y), \\
  (-\infty, 0] & \text{if } t \in B_+(y, u), \\
  [0, \infty) & \text{if } t \in B_-(y, u), \\
  \{0\} & \text{if } t \in A(y, u).
  \end{cases}
  \]

Obviously,
\[
K_{\text{crit}}^{\text{ptw}}(y, u)(t) \subset K_{\text{rad}}^{\text{ptw}}(y)(t) \quad \forall t \in [0, T].
\]

Note that a function \(z \in C^\infty[0, T]\) satisfying \(z(t) \in K_{\text{rad}}^{\text{ptw}}(y)(t)\) for all \(t \in [0, T]\) is not necessarily an element of the “global” radial cone associated with (V), i.e., does not necessarily satisfy \(y(t) + \alpha z(t) \in Z\) for all \(t \in [0, T]\) for a number \(\alpha > 0\) independent of \(t\). A possible counterexample here is \(r = 1, y_0 = 0, T = \pi/2\), \(y(t) = u(t) = \sin(t)\), and \(z(t) = \sin(2t)\). Indeed, for these \(r, y_0, T, y, u, \) and \(z\), we clearly have \(y = S(u), z(T) = 0 \in K_{\text{rad}}^{\text{ptw}}(y)(T) = (-\infty, 0]\), and \(z(t) \in K_{\text{rad}}^{\text{ptw}}(y)(t) = \mathbb{R}\) for all \(t \in [0, T]\). Due to the identities \(y(T) = 1 = r, y'(T) = 0, \) and \(z'(T) = -2\), it further holds \(y'(T) + \alpha z(T) = (-\alpha + 2\alpha) = -2\alpha < 0\) for all \(\alpha > 0\). This implies that, for all \(\alpha > 0\), there exists \(t \in [0, T]\) satisfying \(y(t) + \alpha z(t) > r\). We thus have \(z(t) \in K_{\text{rad}}^{\text{ptw}}(y)(t)\) for all \(t \in [0, T]\) but there does not exist \(\alpha > 0\) satisfying \(y(t) + \alpha z(t) \in Z\) for all \(t \in [0, T]\).
Proposition 4.9 and Corollary 4.10 below establish a connection between the pointwise critical cone mapping \( K_{\text{crit}}^{\text{ptw}}(y, u) : [0, T] \to \mathbb{R} \) and the classical notion of criticality, that is, the property of being an element of the kernel of the multiplier that appears in the variational inequality (V), cf. the definition of \( K_{\text{crit}}(y, u) \) in (1.2). As a preparation for Proposition 4.9 and Corollary 4.10, we prove the following lemma.

**Lemma 4.8.** Let \( u \in CBV[0, T] \) be a control with state \( y := S(u) \in CBV[0, T] \) and let \( z \in G[0, T] \) be a function satisfying \( z = 0 \) on \( A(y, u) \). Then

\[
\int_0^T z \, dy - u = 0 \quad \forall \, 0 \leq s < \tau \leq T.
\]

**Proof.** Define \( D := (I(y) \cup B(y, u)) \setminus \{ T \} \). The continuity of \( y \) and the definitions of \( I(y) \) and \( B(y, u) \) imply that, for every \( t \in D \), there exists \( \varepsilon > 0 \) with \( \{ t, t + \varepsilon \} \subset D \). This entails that the set \( D \) decomposes into disjoint connected components \( \{ D_i \}_{i \in I} \) with \( I \) being finite or equal to \( \mathbb{N} \) and \( D_i \) being an interval with a nonempty interior for all \( i \in I \). Using Lemma 4.3 and again the definition of \( B(y, u) \), one easily checks that, for each \( t \in D \), there exists \( \varepsilon > 0 \) such that \( y - u \) is constant on \( \{ t, t + \varepsilon \} \). Since \( y - u \) is continuous, this implies \( y - u := c_i = \text{const} \) on each \( [a_i, b_i] := \text{cl}(D_i) \). Now \( \int_0^T 1_{[T]} z \, dy - u = 0 \) by (A.3). Using this identity, the fact that \( z = 0 \) holds on \( A(y, u) \), Lemma A.1, and (in the case \( I = \mathbb{N} \) the bounded convergence theorem (Theorem A.4), we see that

\[
\int_0^T z \, dy - u = \int_0^T \sum_{i \in I} 1_{D_i} z \, dy - u = \sum_{i \in I} \int_0^T 1_{D_i} z \, dy - u = \sum_{i \in I} \int_{a_i}^{b_i} 1_{D_i} z \, dc_i = 0.
\]

Choosing \( 1_{[s, \tau]} z \) instead of \( z \) in (4.6) yields (4.5), again due to Lemma A.1. \( \square \)

**Proposition 4.9.** (relation to the classical notion of criticality). Suppose that a control \( u \in CBV[0, T] \) with state \( y := S(u) \in CBV[0, T] \) and a function \( z \in G[0, T] \) satisfying \( z(t) \in K_{\text{rad}}^{\text{ptw}}(y(t)) \) for all \( t \in [0, T] \) are given. Then it holds

\[
\int_s^\tau z \, dy - u \geq 0 \quad \forall \, 0 \leq s < \tau \leq T.
\]

Moreover, it is true that

\[
\int_s^\tau z \, dy - u = 0 \quad \Rightarrow \quad \int_s^\tau z \, dy - u = 0 \quad \forall \, 0 \leq s < \tau \leq T,
\]

and, if \( z \) possesses the additional regularity \( z \in G_r[0, T] \), then we also have

\[
\int_0^T z \, dy - u = 0 \quad \Rightarrow \quad z(t) \in K_{\text{rad}}^{\text{ptw}}(y(t)) \forall t \in [0, T].
\]

**Proof.** In order to prove (4.7), let \( 0 \leq s < \tau \leq T \) be given. We first assume that \( y(t) \in (-\infty, r] \) holds for all \( t \in [s, \tau] \). By Lemma 4.3, \( u - y \) is nondecreasing on \( [s, \tau] \). Using the definition of \( K_{\text{rad}}^{\text{ptw}}(y) \), it is easy to check that \( \tilde{z}(t) := \max\{0, z(t)\} 1_{[s, \tau]}(t) \) satisfies the assumptions of Lemma 4.8. Therefore,

\[
\int_s^\tau z \, dy - u = \int_s^\tau \min\{0, z\} \, dy - u = \int_s^\tau \max\{0, -z\} \, d(u - y) \geq 0.
\]
This proves (4.7) in the case \( y(t) \in (-r, r) \) for all \( t \in [s, \tau] \). In the case \( y(t) \in [-r, r) \) for all \( t \in [s, \tau] \), we can use the exact same arguments as above with reversed signs to establish (4.7). To finally obtain (4.7) for arbitrary \([s, \tau]\), it suffices to consider a subdivision of \([s, \tau]\) into subintervals of the above two types and to use (A.2).

The implication (4.8) follows directly from Lemma 4.8 since \( z(t) \in K_{\text{crit}}^{\text{ptw}}(y, u)(t) \) for all \( t \in [0, T] \) implies \( z = 0 \) on \( A(y, u) \).

It remains to prove (4.9). Since \( z(t) \in K_{\text{rad}}^{\text{ptw}}(y)(t) \setminus K_{\text{crit}}^{\text{ptw}}(y, u)(t) \) for some \( t \in [0, T] \) and only if \( z(t) \neq 0 \) and \( t \in A(y, u) \), it suffices to show that the integral on the left side of (4.9) is nonzero if a time \( t \) with the latter property exists. So let \( t \in A(y, u) \) be arbitrary but fixed and suppose that \( z(t) \neq 0 \). We assume w.l.o.g. that \( y(t) = r \). (The case \( y(t) = -r \) is analogous.) From \( 0 \neq z(t) \in K_{\text{rad}}^{\text{ptw}}(y)(t) \), we obtain that \( z(t) < 0 \) holds from the right-continuity of \( z \), the definition of \( A(y, u) \), and the continuity of \( y \), that \( t \neq T \) and that there exist numbers \( c, \varepsilon > 0 \) such that \( z(s) = -c \) and \( y(s) \in (-r, r] \) holds for all \( s \in [t, t + \varepsilon] \subset [0, T] \) and such that \( y - u \) is not constant on \([t, t + \varepsilon] \). By Lemma 4.3, \( y - u \) is nonincreasing on \([t, t + \varepsilon] \). It thus follows that

\[
\int_{t}^{t+\varepsilon} z \, d(y - u) \geq c \int_{t}^{t+\varepsilon} d(u - y) = c((u - y)(t + \varepsilon) - (u - y)(t)) > 0.
\]

Using (4.7), we conclude

\[
\int_0^T z \, d(y - u) = \int_0^t z \, d(y - u) + \int_t^{t+\varepsilon} z \, d(y - u) + \int_{t+\varepsilon}^T z \, d(y - u) > 0.
\]

**Corollary 4.10.** Let \( u \in CBV[0, T] \) be a control with state \( y := S(u) \) and let \( z \in G_r[0, T] \) be a given function. Then

\[
z(t) \in K_{\text{crit}}^{\text{ptw}}(y, u)(t) \forall t \in [0, T] \quad \iff \quad \begin{cases} z(t) \in K_{\text{rad}}^{\text{ptw}}(y)(t) \forall t \in [0, T] \text{ and} \\ \int_s^T z \, d(y - u) = 0 \quad \forall 0 \leq s < \tau \leq T. \end{cases}
\]

As Corollary 4.10 shows, a function \( z \in G_r[0, T] \) is “critical in the pointwise sense” if and only if it takes values in \( K_{\text{rad}}^{\text{ptw}}(y)(t) \) for all \( t \in [0, T] \) and is contained in the kernel of the linear and continuous function \( G[0, T] \ni v \mapsto \int_s^T v \, d(y - u) \in \mathbb{R} \) for all \( 0 \leq s < \tau \leq T \). For elements of \( G_r[0, T] \), the pointwise notion of criticality introduced in Definition 4.7 is thus closely related to the notion of criticality appearing in the context of the classical obstacle problem, cf. (1.2). This relation does not exist anymore in general when the assumption of right-continuity is dropped. Indeed, as the integrator \( y - u \) of the integrals in Proposition 4.9 and Corollary 4.10 does not assign mass to singletons due to the continuity of \( u \) and \( y \) and (A.3), for every \( t \in A(y, u) \), the function \( z(s) := -\text{sgn}(y(t)) \mathbb{I}_{t \leq x} \) satisfies \( z \in G[0, T] \), \( z(s) \in K_{\text{rad}}^{\text{ptw}}(y)(t) \) for all \( s \in [0, T] \), and \( \int_s^T z \, d(y - u) = 0 \) for all \( 0 \leq s < \tau \leq T \) but does not vanish on the strictly active set \( A(y, u) \). In all situations in which \( A(y, u) \) is nonempty, the pointwise notion of criticality in Definition 4.7 thus differs from the ordinary, multiplier-based one as soon as the regularity of the considered functions is too poor.

We are now in the position to state the auxiliary problem that characterizes the pointwise directional derivatives \( S^\prime(u; h) \) of \( S \) in the situation of Theorem 4.5.

**Theorem 4.11 (variational inequality for directional derivatives).** Consider a fixed control \( u \in CBV[0, T] \) with associated state \( y := S(u) \in CBV[0, T] \). Then,
for every $h \in CBV[0,T]$, the pointwise directional derivative $\delta := S'(u; h) \in BV[0,T]$ of $S$ at $u$ in direction $h$ is the unique solution in $BV[0,T]$ of the system
\begin{align}
\int_0^s (z - \delta_+) \, d(\delta - h) & \geq 0 \quad \forall z \in G \left( [0,s]; K_{\text{crit}}^{\text{ptw}}(y,u) \right) \quad \forall s \in (0,T], \\
\delta_+(t) & \in K_{\text{crit}}^{\text{ptw}}(y,u)(t) \quad \forall t \in [0,T], \quad \delta(0) = 0.
\end{align}
Moreover, it holds $\delta(t) \in \{\delta(t+), \delta(t-)\}$ for all $t \in [0,T]$ and $\var{\delta} \leq 2 \var h$.

Proof. This follows from [8, Theorem 2.1], where the result is stated for the scalar play operator $\mathcal{P}(u) := u - S(u)$.

As $z = 0$ and $z = 2\delta_+$ are admissible test functions in (4.10), this variational inequality implies in particular that
\begin{equation}
\int_0^s \delta_+ \, d(\delta - h) = 0 \quad \forall s \in (0,T].
\end{equation}

We remark that, using the inclusion $\delta(t) \in \{\delta(t+), \delta(t-)\}$ and [43, Lemma 6.3.3], it is easy to check that the inequality in (4.10) is satisfied by $\delta$ regardless of whether the right limit $\delta_+$ in the integral is defined w.r.t. $[0,s]$ or w.r.t. $[0,T]$. To achieve that $\delta$ is uniquely characterized by (4.10), the definition w.r.t. $[0,s]$ and the corresponding convention for the endpoint $s$ have to be used, see [8, proof of Theorem 2.1].

Regarding the regularity properties of the derivatives $S'(u; h)$ in Theorem 4.11, it should be noted that $S'(u; h)$ can satisfy $S'(u; h)_+ \neq S'(u; h) \neq S'(u; h)_-$ even when $u$ and $h$ are smooth, see [8, Example 4.1]. There is, however, a logic behind the jumps of $S'(u; h)$ as the following corollary shows.

**Corollary 4.12** (direction of jumps). Consider the situation in Theorem 4.11 for some fixed $u, h \in CBV[0,T]$. Then, for all $t \in [0,T]$, it holds
\begin{align}
\delta(t+) - \delta(t-) & \geq 0 \quad \forall \zeta \in K_{\text{crit}}^{\text{ptw}}(y,u)(t), \\
\delta(t+)(\delta(t+) - \delta(t-)) - \delta(t-)(\delta(t+) - \delta(t-)) & = 0.
\end{align}

In particular, if $t \in [0,T]$ is a point of discontinuity of $\delta = S'(u; h) \in BV[0,T]$, i.e., if $\delta(t+) \neq \delta(t-)$, then it holds $\delta(t+) = 0$. Moreover, we have $\delta(0+) = \delta(0) = 0$.

Proof. For the test function $z = \mathbb{I}_{\{t\}} \zeta$ with $t \in [0,T]$ and $\zeta \in K_{\text{crit}}^{\text{ptw}}(y,u)(t)$, we obtain from (4.10), using (4.11) as well as (A.3),
\begin{equation}
0 \leq \int_0^T \mathbb{I}_{\{t\}} \zeta \, d(\delta - h) = \zeta((\delta - h)(t+) - (\delta - h)(t-)) = \zeta(\delta(t+) - \delta(t-))
\end{equation}

with the conventions $\delta(0-) = \delta(0)$ and $\delta(T+) = \delta(T)$. This proves (4.12). Using the test functions $z = \delta_+ \pm \mathbb{I}_{\{t\}} \delta_+(t)$ in (4.10), we obtain analogously
\begin{equation}
0 \leq \int_0^T \pm \mathbb{I}_{\{t\}} \delta_+(t) \, d(\delta - h) = \pm \delta(t+)(\delta(t+) - \delta(t-)).
\end{equation}

Since $\delta(t) \in \{\delta(t-), \delta(t+)\}$, both equalities in (4.13) follow. All other assertions are immediate consequences of (4.12), (4.13), and the initial condition $\delta(0) = 0$.

We would like to point out that jump conditions similar to those in Corollary 4.12 also have to be studied in order to establish the system (4.10), see [8, section 5]. We deduce Corollary 4.12 from Theorem 4.11 here to simplify the presentation and to avoid recalling major parts of the analysis in [8]. As an immediate consequence of Theorem 4.11 and Corollary 4.12, we obtain:

\[ \]
Corollary 4.13 (variational inequality for the right limits of the derivatives). Consider an arbitrary but fixed $u \in CBV[0,T]$ with state $y := S(u) \in CBV[0,T]$. Then, for every $h \in CBV[0,T]$, the right limit $\eta := S'(u;h)_+ \in BV_r[0,T]$ of the pointwise directional derivative $S'(u;h)$ of $S$ at $u$ in direction $h$ is the unique solution in $BV_r[0,T]$ of the variational inequality

\begin{equation}
\int_0^T (z - \eta) \, d(\eta - h) \geq 0 \quad \forall z \in G \left( [0,T]; K_{\text{ptw}}(y,u) \right),
\end{equation}

\begin{equation}
\eta(t) \in K_{\text{ptw}}(y,u) \quad \forall t \in [0,T],
\end{equation}

Moreover, for all $s \in (0,T)$, it is true that

\begin{equation}
\int_0^s (z - \eta) \, d(\eta - h) \geq 0 \quad \forall z \in G \left( [0,s]; K_{\text{ptw}}(y,u) \right).
\end{equation}

Proof. That $\eta$ satisfies the second line of (4.14) follows from Theorem 4.11 and Corollary 4.12. Since $S'(u;h) \in BV[0,T]$ has at most countably many discontinuity points by [43, Theorem 2.3.2], and because $(\eta - S'(u;h))(T) = 0$ by convention and $(\eta - S'(u;h))(0) = 0$ by Corollary 4.12, it follows from Lemma A.2 that

\begin{equation}
\int_0^T f \, d(\eta - S'(u;h)) = 0 \quad \forall f \in G[0,T].
\end{equation}

If we combine this identity with (4.10) for $s = T$ and the linearity of the Kurzweil-Stieltjes integral, then the variational inequality in (4.14) follows immediately. To establish (4.15), it suffices to consider functions of the form $z := 1_{[0,s]} \tilde{z} + 1_{[s,T]} \eta$, $s \in (0,T)$, $\tilde{z} \in G \left( [0,s]; K_{\text{ptw}}(y,u) \right)$, in (4.14) and to exploit (A.2) and (A.3).

Suppose now that there are two $\eta_1, \eta_2 \in BV_r[0,T]$ satisfying (4.14). In this case, we can consider functions of the form $z := 1_{[0,s]} \eta_2 + 1_{[s,T]} \eta_1$ and $z := 1_{[0,s]} \eta_1 + 1_{[s,T]} \eta_2$ in the inequalities for $\eta_1$ and $\eta_2$, respectively, and add the resulting estimates to obtain with (A.2) and (A.3) that $\int_0^s (\eta_2 - \eta_1) \, d(\eta_2 - \eta_1) \leq 0$ holds for all $s \in (0,T)$. Due to Proposition A.3 and $\eta_1(0) = \eta_2(0) = 0$, this yields $(\eta_1(s) - \eta_2(s))^2 \leq 0$ for all $s \in [0,T]$. This proves that (4.14) possesses at most one solution in $BV_r[0,T]$. 

Note that the system (4.14) has the same structure as “usual” rate-independent systems posed in $BV_r[0,T]$, cf. [45, Theorem 3.3]. Because of this, (4.14) is easier to work with than (4.10), which involves the additional varying parameter $s \in (0,T)$.

5. First consequences for the optimal control problem (P). As a direct consequence of the results for $S$ in the last section, we obtain:

Corollary 5.1 (existence of solutions). Assume, in addition to the conditions in our standing Assumption 3.1, that:

- $(U, \| \cdot \|_U)$ is a reflexive Banach space that is compactly embedded into $C[0,T]$, 
- $U_{ad}$ is a closed subset of $(U, \| \cdot \|_U)$,
- $J$ is lower semicontinuous in the sense that, for all $\{(y_n, z_n, u_n)\} \subset C[0,T] \times \mathbb{R} \times U$ satisfying $y_n \to y$ in $C[0,T]$, $z_n \to z$ in $\mathbb{R}$, and $u_n \to u$ in $U$, we have $\liminf_{n \to \infty} J(y_n, z_n, u_n) \geq J(y, z, u)$,
- $J$ is radially unbounded in the sense that there exists a function $\rho: [0, \infty) \to \mathbb{R}$ satisfying $\rho(s) \to \infty$ for $s \to \infty$ and $J(y, z, u) \geq \rho(\|u\|_U) \quad \forall (y, z, u) \in C[0,T] \times \mathbb{R} \times U$.

Then the problem (P) possesses at least one globally optimal control-state pair $(\bar{u}, \bar{y})$. 

Assumption 3.1

Corollary 5.1

If the control $P$ is not very helpful in practice. This is one of the main motivations for the derivation of the Bouligand stationarity condition (5.1). We will circumvent this problem by using the direct method of the calculus of variations to establish the solvability of (5.1) and (5.2). We will study:

\begin{align}
\langle \partial_1 J(\tilde{y}, \tilde{y}(T), \tilde{u}), S'(\tilde{u}; h) \rangle & + \langle \partial_2 J(\tilde{y}, \tilde{y}(T), \tilde{u}), S'(\tilde{u}; h) \rangle_U \geq 0 \quad \forall h \in \mathbb{R}^+ \\
\end{align}

Here, $\partial_1 J(\tilde{y}, \tilde{y}(T), \tilde{u}) \in L^1(0, T)$, $\partial_2 J(\tilde{y}, \tilde{y}(T), \tilde{u}) \in \mathbb{R}$, and $\partial_3 J(\tilde{y}, \tilde{y}(T), \tilde{u}) \in U^*$ are the partial Fréchet derivatives of the objective function $J: L^\infty(0, T) \times \mathbb{R} \times U \to \mathbb{R}$.

Proof. This follows along standard lines from the convexity of $U_{ad}$, the Fréchet differentiability of $J$, Theorem 4.5, the Lipschitz estimate (4.2), and the $L^1$-regularity of $\partial_1 J(\tilde{y}, \tilde{y}(T), \tilde{u})$. See, e.g., [11, Proposition 6.1.2] or [26, section 3].

The last result motivates:

Definition 5.3 (Bouligand stationary point). A control $\tilde{u} \in U_{ad}$ with associated state $\tilde{y} := S(\tilde{u})$ is called a Bouligand stationary point of (P) if $(\tilde{u}, \tilde{y})$ satisfies (5.1).

Due to its implicit nature, the Bouligand stationarity condition (5.1) is typically not very helpful in practice. This is one of the main motivations for the derivation of strong stationarity systems. To establish such a system for (P), we study:

6. Temporal polyhedricity properties. Throughout this section, we assume that an arbitrary but fixed $u \in CBV[0, T]$ with state $y := S(u) \in CBV[0, T]$ is given. For these $u$ and $y$, we introduce:
DEFINITION 6.1 (reduced critical cone and smooth critical radial directions). We define the reduced critical cone in $G_r[0,T]$ associated with $(y,u)$ to be the set 

$$
\mathcal{K}_{G_r}^{\text{red,crit}}(y,u) := \{ z \in G_r[0,T] : z(t) \in \mathcal{K}_{\text{crit}}^{\text{ptw}}(y,u)(t) \forall t \in [0,T],
\quad z(0) = 0,
\quad \text{and } z(t) = 0 \forall t \in [0,T] \text{ with } z(t-) \neq z(t) \}
$$

and the cone of smooth critical radial directions associated with $(y,u)$ to be the set 

$$
\mathcal{K}_{C_\infty}^{\text{rad,crit}}(y,u) := \{ z \in C_\infty[0,T] : z(t) \in \mathcal{K}_{\text{crit}}^{\text{ptw}}(y,u)(t) \forall t \in [0,T],
\quad z(0) = 0,
\quad \exists \alpha > 0 \text{ s.t. } y(t) + \alpha z(t) \in Z \forall t \in [0,T] \}.
$$

Note that $\mathcal{K}_{C_\infty}^{\text{rad,crit}}(y,u)$ is a subset of $\mathcal{K}_{G_r}^{\text{red,crit}}(y,u)$, that both $\mathcal{K}_{G_r}^{\text{red,crit}}(y,u)$ and $\mathcal{K}_{G_r}^{\text{rad,crit}}(y,u)$ are cones containing the zero function, and that $\mathcal{K}_{C_\infty}^{\text{rad,crit}}(y,u)$ is convex. The cone $\mathcal{K}_{G_r}^{\text{red,crit}}(y,u)$ is typically not convex due to the additional conditions on the points of discontinuity. From Corollaries 4.12 and 4.13, it follows that $\mathcal{S}'(u;h)_+$ is an element of $\mathcal{K}_{G_r}^{\text{red,crit}}(y,u)$ for all $h \in CBV[0,T]$. In fact, $\mathcal{K}_{G_r}^{\text{red,crit}}(y,u)$ collects all information about the pointwise properties of the right limits of the derivatives $\mathcal{S}'(u;h)$ that we have derived so far. This motivates the name “reduced critical cone”, cf. the analysis for elliptic variational inequalities in [11]. From Proposition 4.9, we obtain that 

$$
\mathcal{K}_{G_r}^{\text{red,crit}}(y,u) = \left\{ z \in G_r[0,T] : z(t) \in \mathcal{K}_{\text{rad}}^{\text{ptw}}(y,u)(t) \forall t \in [0,T],
\quad \int_0^T z d(y - u) = 0,
\quad z(0) = 0, z(t) = 0 \forall t \in [0,T] \text{ with } z(t-) \neq z(t) \right\}
$$

and 

$$
\mathcal{K}_{C_\infty}^{\text{rad,crit}}(y,u) = \left\{ z \in C_\infty[0,T] : z(0) = 0, \quad \int_0^T z d(y - u) = 0, \quad \exists \alpha > 0 \text{ s.t. } y(t) + \alpha z(t) \in Z \forall t \in [0,T] \right\}.
$$

The main result of this section – Theorem 6.5 – shows that the cone $\mathcal{K}_{C_\infty}^{\text{rad,crit}}(y,u)$ is, in a suitably defined sense, dense in $\mathcal{K}_{G_r}^{\text{red,crit}}(y,u)$. This density property extends the concept of polyhedricity to the setting considered in this paper. In the case of the elliptic problem (1.1), polyhedricity expresses that the set $\mathcal{K}_{\text{rad}}^{\text{ptw}}(y) \cap (u + \Delta y)^\perp$ is $H_0^1(\Omega)$-dense in the critical cone $\mathcal{K}_{\text{rad}}^{\text{ptw}}(y) \cap (u + \Delta y)^\perp$, see [24, 53]. For the study of the inequality (V), the set $\mathcal{K}_{C_\infty}^{\text{rad,crit}}(y,u)$ is relevant because of the following observation.

LEMMA 6.2 (directional derivative in smooth critical radial directions). Let $h$ be an arbitrary but fixed element of the set $\mathcal{K}_{C_\infty}^{\text{rad,crit}}(y,u)$. Then there exists $\alpha > 0$ such that $\mathcal{S}(u + \beta h) = \mathcal{S}(u) + \beta h$ holds for all $\beta \in (0,\alpha)$. In particular, $\mathcal{S}'(u;h) = h$.

Proof. According to the definition of the set $\mathcal{K}_{C_\infty}^{\text{rad,crit}}(y,u)$, we can find a number $\alpha > 0$ such that $y(t) + \alpha h(t) \in Z$ holds for all $t \in [0,T]$. Since $Z$ is convex, this also yields $y(t) + \beta h(t) \in Z$ for all $t \in [0,T]$ and all $\beta \in (0,\alpha)$. From Proposition 4.9 and the variational inequality (V) for $y$, we moreover obtain that 

$$
\int_0^T h d(y - u) = 0 \quad \text{and} \quad \int_0^T (v - y) d(y - u) \geq 0 \quad \forall v \in C([0,T]; Z).
$$
If we combine the above with the initial conditions \( y(0) = y_0 \) and \( h(0) = 0 \) and our previous considerations, then it follows that

\[
\int_0^T (v - (y + \beta h)) \, d(y + \beta h - (u + \beta h)) \geq 0 \quad \forall v \in C([0, T]; Z),
\]

\[
y(t) + \beta h(t) \in Z \quad \forall t \in [0, T],
\]

\[
y(0) + \beta h(0) = y_0,
\]

holds for all \( \beta \in (0, \alpha) \). Thus, \( S(u + \beta h) = y + \beta h \) for all \( \beta \in (0, \alpha) \) by Theorem 4.1 as claimed. The assertion about the directional derivative follows immediately from this identity. This completes the proof.

Note that Lemma 6.2 remains valid when the space \( C^\infty[0, T] \) in the definition of \( K_{C^\infty}^{rad, crit}(y, u) \) is replaced with the space \( CBV[0, T] \). We consider smooth critical radial directions in our analysis because this gives rise to a stronger density result in Theorem 6.5. As we will see in section 7, Lemma 6.2 makes it possible to prove the strong stationarity system (1.5) once the polyhedricity property in Theorem 6.5 is established. To obtain the latter, we require the following result.

**Lemma 6.3.** Suppose that \( z \in K_{G_r}^{rad, crit}(y, u) \) and \( \xi > 0 \) are given. Let \( t \in [0, T] \) be an arbitrary but fixed point of continuity of \( z \), i.e., a point with \( z(t) = z(t^-) \). Then there exists \( \varepsilon > 0 \) such that the step function

\[
\zeta: [0, T] \to \mathbb{R}, \quad \zeta(s) := z(t) \mathbf{1}_{J_z(t)}(s), \quad J_z(t) := [t - \varepsilon, t + \varepsilon] \cap [0, T],
\]

possesses all of the following properties:

i) It is true that

\[
\sup_{s \in [t - \varepsilon, t + \varepsilon] \cap [0, T]} |z(s) - \zeta(s)| \leq \xi.
\]

ii) It holds

\[
\zeta(s) \in K_{G_r}^{ptw}(y, u)(s) \quad \forall s \in [0, T].
\]

iii) For every \( 0 \leq \psi \in C_c^\infty(\mathbb{R}) \) with support \( \text{supp}(\psi) \subset (t - \varepsilon, t + \varepsilon) \), the function

\[
\psi \zeta \in C[0, T] \quad \text{is an element of the cone } K_{G_r}^{rad, crit}(y, u).
\]

**Proof.** Since \( z \) is continuous at \( t \), we can find \( \varepsilon > 0 \) such that i) holds. If \( z(t) = 0 \), then \( \zeta = 0 \) and ii) and iii) hold trivially for this \( \varepsilon \). Due to the definition of the set \( K_{G_r}^{rad, crit}(y, u) \) and the continuity of \( z \) at \( t \), this case covers in particular the situations \( t = 0 \) and \( t \in \text{cl}(A(y, u)) \). In what follows, we may thus assume that

\[
(6.1) \quad z(t) \neq 0 \quad \text{and} \quad J_z(t) \subset \left( I(y) \cup B(y, u) \right) \cap (0, T]
\]

and have to prove that, for a potentially smaller \( \varepsilon \), we have ii) and iii). To this end, we distinguish between three cases.

Case 1: \( t \in I(y) \). In this case, it follows from the continuity of \( y \) that, after possibly making \( \varepsilon \) smaller, we have \( J_z(t) \subset I(y) \) and \( |y| \leq r - \gamma \) on \( J_z(t) \) for some \( \gamma > 0 \). This implies in particular that \( K_{G_r}^{ptw}(y, u)(s) = \mathbb{R} \) for all \( s \in J_z(t) \).

Case 2: \( t \in B_+(y, u) \). In this case, it follows from the continuity of \( y \) that, after possibly making \( \varepsilon \) smaller, we have \( J_z(t) \subset I(y) \cup B_+(y, u) \) and \( y \geq -r + \gamma \) on \( J_z(t) \) for some \( \gamma > 0 \). Due to the definition of \( K_{G_r}^{ptw}(y, u) \), this implies in particular that \( z(t) \in K_{G_r}^{ptw}(y, u)(t) = (-\infty, 0] \subset K_{G_r}^{ptw}(y, u)(s) \) for all \( s \in J_z(t) \).

Case 3: \( t \in B_-(y, u) \). In this case, it follows from the continuity of \( y \) that, after possibly making \( \varepsilon \) smaller, we have \( J_z(t) \subset I(y) \cup B_-(y, u) \) and \( y \leq r - \gamma \) on \( J_z(t) \).
for some \( \gamma > 0 \). Due to the definition of \( \mathcal{K}_{\text{ptw}}^{\text{crit}}(y,u) \), this implies in particular that \( z(t) \in \mathcal{K}_{\text{ptw}}^{\text{crit}}(y,u)(t) = (0,\infty) \subseteq \mathcal{K}_{\text{crit}}^{\text{ptw}}(y,u)(s) \) for all \( s \in J_{\varepsilon}(t) \).

In all of the above cases, the resulting \( \varepsilon > 0 \) satisfies \( z(t) = \zeta(s) \in \mathcal{K}_{\text{crit}}^{\text{ptw}}(y,u)(s) \) and \( (y + \alpha \zeta)(s) \in Z \) for all \( s \in J_{\varepsilon}(t) \) and all \( 0 < \alpha \leq \gamma \| \zeta \|_{\infty}^{-1} \). Since \( \zeta(s) = 0 \) for \( s \notin J_{\varepsilon}(t) \), these inclusions for \( \zeta \) are also true for all \( s \in [0,T] \). This proves ii). Consider now a function \( t \leq \psi \in C^\infty_c(\mathbb{R}) \) with \( \text{supp}(\psi) \subset (t - \varepsilon, t + \varepsilon) \). Then \( \psi \zeta \in C^\infty_c(\mathbb{R}) \) and it follows from the nonnegativity of \( \psi \), the properties of \( \zeta \), the cone property of \( \mathcal{K}_{\text{crit}}^{\text{ptw}}(y,u)(s) \), and (6.1) that \( (\psi \zeta)(0) = 0 \) holds and that \( (\psi \zeta)(s) \in \mathcal{K}_{\text{crit}}^{\text{ptw}}(y,u)(s) \) and \( (y + \alpha \psi \zeta)(s) \in Z \) for all \( s \in [0,T] \) and all \( 0 < \alpha < \gamma \| \psi \|_{\infty}^{-1} \| \zeta \|_{\infty}^{-1} \). This shows \( \psi \zeta \in \mathcal{K}_{C^\infty_c}^{\text{rad, crit}}(y,u), \) establishes iii), and completes the proof.

The next lemma is a version of Lemma 6.3 for points of discontinuity.

**Lemma 6.4.** Suppose that \( z \in \mathcal{K}_{G_{\varepsilon}}^{\text{red, crit}}(y,u) \) and \( \xi > 0 \) are given. Let \( t \in [0,T] \) be an arbitrary but fixed point of discontinuity of \( z \), i.e., a point with \( z(t) \neq z(t-) \).

Then there exists \( \varepsilon > 0 \) such that the step function

\[
\zeta : [0,T] \to \mathbb{R}, \quad \zeta(s) := z(t-) \mathbb{1}_{J_{\varepsilon}^{-}(t)}(s), \quad J_{\varepsilon}^{-}(t) := [t - \varepsilon, t] \cap [0,T],
\]

possesses the following properties:

i) It is true that

\[
\sup_{s \in [t - \varepsilon, t] \cap [0,T]} |z(s) - \zeta(s)| \leq \xi.
\]

ii) It holds

\[
\zeta(s) \in \mathcal{K}_{\text{crit}}^{\text{ptw}}(y,u)(s) \quad \forall s \in [0,T].
\]

iii) For every \( 0 \leq \psi \in C^\infty_c(\mathbb{R}) \) with support \( \text{supp}(\psi) \subset (t - \varepsilon, t) \), the function \( \psi \zeta \in G[0,T] \) is an element of the cone \( \mathcal{K}_{C^\infty_c}^{\text{rad, crit}}(y,u) \).

**Proof.** Since \( z \in \mathcal{K}_{G_{\varepsilon}}^{\text{red, crit}}(y,u) \), it necessarily holds \( t > 0 \) and \( z(t) = 0 \). As \( z \) is right-continuous, this implies that there exists \( \varepsilon > 0 \) such that i) is satisfied. Moreover, \( z(t-) \neq 0 \) because \( z \) is assumed to be discontinuous at \( t \). Since \( z = 0 \) on \( A(y,u) \), it follows that, for a potentially smaller \( \varepsilon \), we have

(6.2)

\[
J_{\varepsilon}^{-}(t) \subset (I(y) \cup B(y,u)) \cap (0,T].
\]

We now again distinguish between three cases.

Case 1: After possibly making \( \varepsilon \) smaller, we have \( J_{\varepsilon}^{-}(t) \subset I(y) \). In this case, it holds \( \mathcal{K}_{\text{ptw}}^{\text{crit}}(y,u)(s) = \mathbb{R} \) for all \( s \in J_{\varepsilon}^{-}(t) \) and it follows from the continuity of \( y \) that, for every compact set \( E \subset J_{\varepsilon}^{-}(t) \), we can find a number \( \gamma > 0 \) with \( |y| \leq r - \gamma \) on \( E \).

Case 2: There exists a sequence \( \{ s_n \} \subset B_+(y,u) \) with \( s_n \to t^- \). In this case, we have \( y(s_n) = r \) and \( z(s_n) \in \mathcal{K}_{\text{crit}}^{\text{ptw}}(y,u)(s_n) = (\infty, 0) \) for all \( n \) and it follows that \( y(t) = r \) and \( z(t-) \leq 0 \). Due to the continuity of \( y \) and (6.2), this implies that, after possibly making \( \varepsilon \) smaller, we have \( J_{\varepsilon}^{-}(t) \subset I(y) \cup B_+(y,u) \). In particular, it holds \( z(t-) \in (-\infty, 0] \subset \mathcal{K}_{\text{ptw}}^{\text{crit}}(y,u)(s) \) for all \( s \in J_{\varepsilon}^{-}(t) \) and, for all compact \( E \subset J_{\varepsilon}^{-}(t) \), we can find a number \( \gamma > 0 \) with \( y \geq -r + \gamma \) on \( E \).

Case 3: There exists a sequence \( \{ s_n \} \subset B_-(y,u) \) with \( s_n \to t^- \). In this case, we have \( y(s_n) = -r \) and \( z(s_n) \in \mathcal{K}_{\text{crit}}^{\text{ptw}}(y,u)(s_n) = (0, \infty) \) for all \( n \) and it follows that \( y(t) = -r \) and \( z(t-) \geq 0 \). Due to the continuity of \( y \) and (6.2), this implies that, after possibly making \( \varepsilon \) smaller, we have \( J_{\varepsilon}^{-}(t) \subset I(y) \cup B_-(y,u) \). In particular, it holds \( z(t-) \in [0, \infty) \subset \mathcal{K}_{\text{ptw}}^{\text{crit}}(y,u)(s) \) for all \( s \in J_{\varepsilon}^{-}(t) \) and, for all compact \( E \subset J_{\varepsilon}^{-}(t) \), we can find a number \( \gamma > 0 \) with \( y \leq r - \gamma \) on \( E \).
In all of the above cases, the resulting \( \varepsilon > 0 \) satisfies \( z(t-\varepsilon) = \zeta(s) \in K_{\text{crit}}^{\text{ptw}}(y, u)(s) \) for all \( s \in J^-(t) \). Since \( \zeta(s) = 0 \) for \( s \notin J^-(t) \), this proves \( \text{ii} \). Moreover, we obtain from the above construction that, for every compact set \( E \subset J^-(t) \), there exists a number \( \gamma > 0 \) with \( (y + \alpha \zeta)(s) \in Z \) for all \( s \in E \) and all \( 0 < \alpha \leq \gamma \|\zeta\|^{-1}_z \). If a function \( \psi \in C^\infty_c(\mathbb{R}) \) with \( \psi \geq 0 \) and support \( E := \text{supp}(\psi) \subset (t-\varepsilon, t) \) is given, then this implies that \( (y + \alpha \zeta)(s) \in Z \) holds for all \( s \in [0, T] \) and all \( 0 < \alpha \leq \gamma \|\psi\|^{-1}_z \|\zeta\|^{-1}_z \). Due to the nonnegativity of \( \psi \), the properties of \( \zeta \), and the cone property of \( K_{\text{crit}}^{\text{ptw}}(y, u)(s) \), one further obtains that \( (\psi \zeta)(s) \in K_{\text{crit}}^{\text{ptw}}(y, u)(s) \) holds for all \( s \in [0, T] \), and due to (6.2) and the properties of \( \text{supp}(\psi) \), that \( (\psi \zeta)(0) = 0 \) and \( \psi \zeta \in C^\infty_c[0, T] \). Thus, \( \psi \zeta \in K_{C^\infty}^{\text{rad, crit}}(y, u) \). This establishes \( \text{iii} \) and completes the proof.

We can now prove the main result of this section.

**Theorem 6.5 (temporal polyhedricity).** Let \( z \in K_{G_r}^{\text{red, crit}}(y, u) \) be given. Then there exist functions \( z_{i,j}, z_j \in G_r[0, T], i, j \in \mathbb{N} \), such that the following is true:

\[
\begin{align*}
\|z_{i,j}\|_\infty &\leq \|z\|_\infty \quad \forall i, j, \\
\|z_j\|_\infty &\leq \|z\|_\infty \quad \forall j, \\
z_{i,j} &\to z_j \text{ pointwise in } [0, T] \text{ for } i \to \infty \text{ for all } j, \\
z_j &\to z \text{ uniformly in } [0, T] \text{ for } j \to \infty.
\end{align*}
\]

**Proof.** Consider an arbitrary but fixed \( j \in \mathbb{N} \) and define \( \xi := 1/j \). For every \( t \in [0, T] \), we choose \( \varepsilon_t > 0 \) for this \( \xi \) as in Lemmas 6.3 and 6.4. This results in a collection of open intervals \( (t - \varepsilon_t, t + \varepsilon_t) \) that covers \([0, T]\). By compactness, we can choose a finite subcover of this collection. We denote the time points of this cover with \( t_k, k = 1, \ldots, N, N \in \mathbb{N} \), and the associated \( \varepsilon_k \) with \( \varepsilon_k, k = 1, \ldots, N \). We assume w.l.o.g. that there are no \( k, l \) satisfying \((t_k - \varepsilon_k, t_k + \varepsilon_k) \subset (t_l - \varepsilon_l, t_l + \varepsilon_l) \) and \( k \neq l \). In this case, by possibly making the intervals \((t_k - \varepsilon_k, t_k + \varepsilon_k) \) smaller, we can construct intervals \((t_k - a_k, t_k + b_k) \), \( a_k, b_k > 0 \), such that

\[
(t_k - a_k, t_k + b_k) \subset (t_k - \varepsilon_k, t_k + \varepsilon_k) \quad \forall k = 1, \ldots, N, \\
[0, T] \subset \bigcup_{k=1}^{N} (t_k - a_k, t_k + b_k),
\]

and \( t_k \notin (t_l - a_l, t_l + b_l) \) \( \forall k \neq l \).

Consider now a smooth partition of unity on \([0, T]\) subordinate to the modified cover \((t_k - a_k, t_k + b_k), k = 1, \ldots, N\), i.e., a collection of functions \( \psi_k, k = 1, \ldots, N \), satisfying \( \psi_k \in C^\infty_c(\mathbb{R}), \quad 0 \leq \psi_k(t) \leq 1 \quad \forall t \in \mathbb{R}, \quad \text{supp}(\psi_k) \subset (t_k - a_k, t_k + b_k) \quad \forall k = 1, \ldots, N, \)

\[
\sum_{k=1}^{N} \psi_k(t) = 1 \quad \forall t \in [0, T],
\]

see, e.g., [21], and choose an arbitrary but fixed function \( \varphi \in C^\infty_c(\mathbb{R}) \) satisfying

\[
0 \leq \varphi(t) \leq 1 \quad \forall t \in \mathbb{R}, \quad \varphi(t) = 1 \quad \forall t \in (-\infty, -1], \quad \varphi(t) = 0 \quad \forall t \in [0, \infty).
\]

Define

\[
z_{i,j}(s) := \sum_{k: z(t_k) = z(t_k-)} z(t_k)\psi_k(s) + \sum_{k: z(t_k) \neq z(t_k-)} z(t_k-)\psi_k(s)\varphi\left(\frac{s - t_k + 1/i}{1/i}\right)
\]
for all $i \in \mathbb{N}$ and $s \in [0, T]$. We claim that $z_{i,j} \in K_{G, \infty}^{\text{rad, crit}}(y, u)$ holds for all $i \in \mathbb{N}$. To see this, we first note that we have

$$z(t_k)\psi_k(\cdot)\big|_{[0, T]} \in K_{G, \infty}^{\text{rad, crit}}(y, u) \quad \forall k: z(t_k) = z(t_k^-)$$

by Lemma 6.3iii) and the condition $\text{supp}(\psi_k) \subset (t_k - a_k, t_k + b_k) \subset (t_k - \varepsilon_k, t_k + \varepsilon_k)$ for all $k$. Analogously, we also have

$$z(t_k^-)\psi_k(\cdot)\varphi\left(\frac{\cdot - t_k + 1/i}{1/i}\right)\big|_{[0, T]} \in K_{G, \infty}^{\text{rad, crit}}(y, u) \quad \forall k: z(t_k) \neq z(t_k^-)$$

by the properties of $\psi_k$ and $\varphi$ and Lemma 6.4iii). By combining these facts with the observation that $K_{G, \infty}^{\text{rad, crit}}(y, u)$ is a convex cone, the inclusion $z_{i,j} \in K_{G, \infty}^{\text{rad, crit}}(y, u)$ follows immediately. Due to the properties of $\varphi$, we further have

$$z_{i,j}(s) \to \sum_{k: z(t_k) = z(t_k^-)} z(t_k)\psi_k(s) + \sum_{k: z(t_k) \neq z(t_k^-)} z(t_k^-)\psi_k(s)\mathbb{I}_{(-\infty, t_k)}(s)$$

for all $s \in [0, T]$ for $i \to \infty$. Let us denote the function on the right of the last limit with $z_j$. By construction, the points of discontinuity of this function $z_j$ are precisely the points $t_k$ with $z(t_k) \neq z(t_k^-)$. Further, at these points, the function $z_j$ is clearly right-continuous and, by the choice of the functions $\psi_k$ and the condition $t_k \notin (t_l - a_l, t_l + b_l)$ for all $k \neq l$, we have

$$z_j(t_k) = z(t_k^-)\psi_k(t_k)\mathbb{I}_{(-\infty, t_k)}(t_k) = 0$$

for all $k$ with $z(t_k) \neq z(t_k^-)$. In combination with the choice of the functions $\psi_k$, this yields $z_j \in G_r[0, T], z_j(t) = z_j(t^-) = 0$ for all $t \in [0, T]$ with $z_j(t) \neq z_j(t^-)$, and $z_j(0) = 0$. Due to the properties of $\psi_k$, the inclusion $(t_k - a_k, t_k + b_k) \subset (t_k - \varepsilon_k, t_k + \varepsilon_k)$ for all $k$, the second points of Lemmas 6.3 and 6.4, and the fact that $K_{G, \infty}^{\text{rad, crit}}(y, u)(s)$ is a convex cone for all $s \in [0, T]$, we also have $z_j(s) \in K_{G, \infty}^{\text{rad, crit}}(y, u)(s)$ for all $s \in [0, T]$. In summary, this allows us to conclude that $z_j \in K_{G, \infty}^{\text{rad, crit}}(y, u)$ holds as desired. It remains to establish the uniform convergence of $z_j$ to $z$ for $j \to \infty$. To this end, we note that, due to the properties of the partition of unity $\{\psi_k\}$, we have

$$\sup_{s \in [0, T]} |z(s) - z_j(s)|$$

$$= \sup_{s \in [0, T]} \left| \sum_{k: z(t_k) = z(t_k^-)} z(t_k)\psi_k(s) - \sum_{k: z(t_k) \neq z(t_k^-)} z(t_k^-)\psi_k(s)\mathbb{I}_{(-\infty, t_k)}(s) \right|$$

$$= \sup_{s \in [0, T]} \left| \sum_{k: z(t_k) = z(t_k^-)} (z(s) - z(t_k^-))\psi_k(s) \right|$$

$$\quad + \sum_{k: z(t_k) \neq z(t_k^-)} \left| (z(s) - z(t_k^-))\mathbb{I}_{(-\infty, t_k)}(s)\psi_k(s) \right|$$

$$\leq \sup_{s \in [0, T]} \left( \sum_{k: z(t_k) = z(t_k^-)} \sup_{\tau \in [t_k - \varepsilon_k, t_k + \varepsilon_k] \cap [0, T]} |z(\tau) - z(t_k^-)|\psi_k(s) \right)$$

$$\quad + \sum_{k: z(t_k) \neq z(t_k^-)} \sup_{\tau \in [t_k - \varepsilon_k, t_k + \varepsilon_k] \cap [0, T]} \left| (z(\tau) - z(t_k^-))\mathbb{I}_{(-\infty, t_k)}(\tau)\psi_k(s) \right|.$$
Due to the inequalities in Lemma 6.3i) and Lemma 6.4i), our choice \( \xi = 1/j \), and the properties of \( \psi_k \), the last estimate yields
\[
\sup_{s \in [0,T]} |z(s) - z_j(s)| \leq \sup_{s \in [0,T]} \left( \sum_{k: z(t_k) = z(t_k-)} \frac{\psi_k(s)}{j} + \sum_{k: z(t_k) \neq z(t_k-)} \frac{\psi_k(s)}{j} \right) = \frac{1}{j}.
\]
This shows that the sequence \( \{z_j\} \) indeed converges uniformly to \( z \) for \( j \to \infty \).

That we have \( \|z_{i,j}\|_\infty \leq \|z\|_\infty \) and \( \|z_j\|_\infty \leq \|z\|_\infty \) follows immediately from our construction and the properties of \( \psi_k \) and \( \varphi \). This completes the proof. \( \square \)

Note that, to be able to establish that \( K_{C_{\infty}}^{\text{rad, crit}}(y, u) \) is dense in \( K_{G_{\infty}}^{\text{rad, crit}}(y, u) \), one necessarily has to consider a type of convergence weaker than uniform convergence since otherwise it is not possible to leave the space \( C[0,T] \supset K_{C_{\infty}}^{\text{rad, crit}}(y, u) \). This is a major difference between the temporal polyhedricity result in Theorem 6.5 and the classical notion of polyhedricity for the elliptic obstacle problem in (1.1) which yields the density of the set of critical radial directions \( K_{\text{rad}}(y) \cap (u + \Delta y)^\perp \) in the critical cone \( K_{\text{tan}}(y) \cap (u + \Delta y)^\perp \) in \( (H^1_0(\Omega), \| \cdot \|_{H^1_0}) \) and thus in the topology that is natural for the underlying variational inequality. For (V), this natural choice of the topology would be that of uniform convergence as the Lipschitz estimate (4.2) shows.

Before we apply Theorem 6.5 to derive strong stationarity conditions for (P), we prove a further auxiliary result.

**Lemma 6.6.** Suppose that \( t \in [0,T] \) is given and let \( c \in \mathbb{R} \) be an element of the polar cone \( K_{\text{crit}}^{\text{ptw}}(y, u)(t) \), i.e., the set

\[
(6.3) \quad K_{\text{crit}}^{\text{ptw}}(y, u)(t) := \begin{cases} 
\{0\} & \text{if } t \in I(y), \\
[0, \infty) & \text{if } t \in B_+(y, u), \\
(-\infty, 0] & \text{if } t \in B_-(y, u), \\
\mathbb{R} & \text{if } t \in A(y, u).
\end{cases}
\]

Then there exists a sequence \( \{h_i\} \subset C^\infty[0,T] \) such that the following holds:

\[
\|h_i\|_\infty \leq |c| \text{ and } \|S'(u; h_i)\|_\infty \leq 2|c| \forall i \in \mathbb{N},
\]

\[
S'(u; h_i)_+ \to 0 \text{ pointwise in } [0,T] \text{ for } i \to \infty,
\]

\[
h_i \to c \mathbf{1}_{[t,T]} \text{ pointwise in } [0,T] \text{ for } i \to \infty.
\]

**Proof.** If \( t \in I(y) \), then we necessarily have \( c = 0 \) and we can simply choose the sequence \( h_i = 0 \) for all \( i \). If \( t = 0 \), then the sequence defined by \( h_i = c \) for all \( i \) satisfies all assertions because \( S'(u; c \mathbf{1}_{[0,T]}) = S'(u; c \mathbf{1}_{[0,T]})_+ = 0 \) by Theorem 4.11 in view of (A.1). We may thus assume that

\[
0 < t \in B(y, u) \cup A(y, u).
\]

Consider an arbitrary but fixed function \( \varphi \) with the following properties

\[
\varphi \in C^\infty(\mathbb{R}), \quad \varphi(s) = 0 \forall s \in (-\infty, -1), \quad \varphi(s) = 1 \forall s \in [0, \infty), \quad \varphi'(s) \geq 0 \forall s \in \mathbb{R}.
\]

We define \( \{h_i\} \) via

\[
h_i(s) := c\varphi\left(\frac{s - t}{1/i}\right) \quad \forall s \in [0,T] \quad \forall i \in \mathbb{N}.
\]
This sequence clearly satisfies \( \{ h_i \} \subset C^\infty [0,T], h_i(s) \to c\mathbb{1}_{[0,T)}(s) \) for all \( s \in [0,T] \) and \( i \to \infty \), and \( \| h_i \|_{\infty} = |c| \) for all \( i \). Due to the Lipschitz estimate (4.2), this also implies that \( \| S'(u; h_i) \|_{\infty} \leq 2|c| \) holds for all \( i \).

It remains to establish the pointwise convergence of \( S'(u; h_i) \) to \( S'(u; h) \). For this to hold, it suffices to prove that \( \eta_i := S'(u; h_i) \) satisfies \( \eta_i = 0 \) on \([0,t-1/i) \cup [t,T]\) for all \( i \) with \( 1/i < t \). That \( \eta_i \) vanishes on \([0,t-1/i) \) follows easily from the fact that \( h_i \) is zero on \([0,t-1/i), (A.5), \) and (4.15) with \( z = 0, z = 2\eta_i \), and \( 0 < s \leq t-1/i \). Next, we prove that \( \eta_i(t) = 0 \) by distinguishing three cases.

Case 1: \( t \in A(y,u) \). In this case, we have \( \eta_i(t) \in K^{\text{ptw}}(y,u)(t) = \{ 0 \} \).

Case 2: \( t \in B_+(y,u) \). In this case, we have \( \eta_i(t) \in K^{\text{ptw}}(y,u)(t) = (-\infty,0] \), it holds \( c \in [0,\infty) \), and \( h_i \) is nondecreasing on \([0,T] \). By Lemma 4.4, this yields \( S(u + \alpha h_i) \geq S(u) \) in \([0,T] \) for all \( \alpha > 0 \) and all \( i \in \mathbb{N} \). Hence, \( S'(u; h_i) \geq 0 \) in \([0,T] \) and, consequently, \( \eta_i = S'(u; h)_i \) \( \geq 0 \) in \([0,T] \). It follows that \( \eta_i(t) = 0 \).

Case 3: \( t \in B_-(y,u) \). In this case, it holds \( \eta_i(t) \in \mathcal{K}^\text{crit}(y,u)(t) = [0,\infty) \) and \( c \in (-\infty,0] \), and we can proceed completely analogously to Case 2 (with reversed signs) to obtain that \( \eta_i(t) = 0 \).

It remains to prove that \( \eta_i = 0 \) on \([t,T] \) if \( t < T \). Let \( \tilde{\eta}_i := \mathbb{1}_{[0,t]} \eta_i = \mathbb{1}_{[0,t]} \eta \). By the definition of \( h_i \), the function \( \tilde{\eta}_i - h_i \) has the constant value \(-c\) on \([t,T] \). Using (A.2) combined with (A.1), we obtain that, for all \( z \in G([0,T]; K^{\text{ptw}}(y,u)) \), we have

\[
\int_0^T (z - \tilde{\eta}_i) d(\tilde{\eta}_i - h_i) = \int_t^T (z - \tilde{\eta}_i) d(\tilde{\eta}_i - h_i) + \int_0^t (z - \tilde{\eta}_i) d(\tilde{\eta}_i - h_i)
\]

\[
= \int_0^t (z - \tilde{\eta}_i) d(\tilde{\eta}_i - h_i)
\]

\[
= \int_0^t (z - \eta_i) d(\eta_i - h_i) \geq 0,
\]

where the last inequality holds by Corollary 4.13. Since \( \tilde{\eta}_i(s) \in K^{\text{ptw}}(y,u)(s) \) for all \( s \in [0,T] \), we conclude that \( \tilde{\eta}_i \) solves (4.14) for \( h = h_i \). As \( \eta_i \) is the unique solution of (4.14), we must have \( \tilde{\eta}_i = \eta_i \). Thus, \( \eta_i = 0 \) on \([t,T] \) and the proof is complete. \( \square \)

**7. Strong stationarity condition.** We are now in the position to prove the strong stationarity system (1.5).

**Theorem 7.1 (strong stationarity).** Consider the situation in Assumption 3.1 and suppose that \( \tilde{u} \in U_{ad} \) is a control with state \( \tilde{y} := S(\tilde{u}) \) such that the set \( \mathbb{R}_+(U_{ad} - \tilde{u}) \) is dense in \( U \). Then \( \tilde{u} \) is a Bouligand stationary point of \( (\mathbb{F}) \), i.e., satisfies (5.1) if and only if there exist an adjoint state \( \tilde{p} \in BV[0,T] \) and a multiplier \( \tilde{\mu} \in G_{r}[0,T]^* \) such that the following system is satisfied:

\[
\tilde{p}(0) = \tilde{p}(T) = 0, \quad \tilde{p}(t) = \tilde{p}(t-) \quad \forall t \in [0,T),
\]

\[
\tilde{p}(t-) \in K^{\text{ptw}}(\tilde{y}, \tilde{u})(t) \quad \forall t \in [0,T),
\]

\[
\langle \tilde{\mu}, z \rangle_{G_{r}} \geq 0 \quad \forall z \in K_{G_{r}}^{\text{red,crit}}(\tilde{y}, \tilde{u}),
\]

\[
\int_0^T h \, d\tilde{p} = \langle \partial_{h}S(\tilde{y}, \tilde{y}(T), \tilde{u}), h \rangle_U \quad \forall h \in U,
\]

\[
- \int_0^T z \, d\tilde{p} = \langle \partial_{1}S(\tilde{y}, \tilde{y}(T), \tilde{u}), z \rangle_{L_{\infty}} + \partial_{2}S(\tilde{y}, \tilde{y}(T), \tilde{u})z(T) - \langle \tilde{\mu}, z \rangle_{G_{r}} \quad \forall z \in G_{r}[0,T].
\]
Proof. We begin with the proof of the implication “(5.1) ⇒ (7.1)”: Suppose that a control \( \tilde{u} \in U_{ad} \) with state \( \tilde{y} := S(\tilde{u}) \) is given such that the set \( \mathbb{R}_+(U_{ad} - \tilde{u}) \) is dense in \( U \) and such that (5.1) holds. Then it follows from (5.1), the fact that (4.2) implies that \( \|S'(\tilde{u}; h_1) - S'(\tilde{u}; h_2)\|_{\infty} \leq 2\|h_1 - h_2\|_{\infty} \) holds for all \( h_1, h_2 \in CBV[0, T] \), the inclusion \( U \subset CBV[0, T] \), and the continuity of the embedding \( U \hookrightarrow C[0, T] \) that

\[
(7.2) \quad \langle \partial_1 J(\tilde{y}, \tilde{y}(T), \tilde{u}), S'(\tilde{u}; h) \rangle_{L^\infty} + \langle \partial_2 J(\tilde{y}, \tilde{y}(T), \tilde{u}), S'(\tilde{u}; h) \rangle_{U} \geq 0 \quad \forall h \in U.
\]

Again due to (4.2) and since \( -h \in U \) holds for all \( h \in U \), (7.2) yields

\[
\inf \langle \partial_1 J(\tilde{y}, \tilde{y}(T), \tilde{u}), h \rangle_{U} \leq 2 \inf (\inf \langle \partial_1 J(\tilde{y}, \tilde{y}(T), \tilde{u}), h \rangle_{L^1} + \inf \langle \partial_2 J(\tilde{y}, \tilde{y}(T), \tilde{u}), h \rangle_{U}) \|h\|_{\infty}
\]

for all \( h \in U \). In combination with the Hahn-Banach theorem, this shows that the linear functional \( U \ni h \mapsto \langle \partial_1 J(\tilde{y}, \tilde{y}(T), \tilde{u}), h \rangle_{U} \in \mathbb{R} \) can be extended to an element of the dual space \( C[0, T]^* \). In view of the classical Riesz representation theorem (see, e.g., [43, section 8.1]) and Lemma A.2, this means that there exists a function \( \tilde{p} \in BV[0, T] \) satisfying \( \tilde{p}(t) = \tilde{p}(t^-) \) for all \( t \in (0, T) \), \( \tilde{p}(T) = 0 \), and

\[
\int_0^T h \, d\tilde{p} = \langle \partial_1 J(\tilde{y}, \tilde{y}(T), \tilde{u}), h \rangle_{U} \quad \forall h \in U.
\]

Since \( \partial_1 J(\tilde{y}, \tilde{y}(T), \tilde{u}) \in L^1(0, T), S'(\tilde{u}; h)\) a.e. in \( (0, T) \) by [43, Theorem 2.3.2], and \( S'(\tilde{u}; h)h(T) = S'(\tilde{u}; h)_+(T) = S'(\tilde{u}; h_-)(T) \) by definition, we may now rewrite (7.2) as follows:

\[
(7.3) \quad \int_0^T h \, d\tilde{p} + \langle \partial_1 J(\tilde{y}, \tilde{y}(T), \tilde{u}), S'(\tilde{u}; h)_+ \rangle_{L^\infty} + \langle \partial_2 J(\tilde{y}, \tilde{y}(T), \tilde{u}), S'(\tilde{u}; h)_+ \rangle_{U} \geq 0 \quad \forall h \in U.
\]

Note that, again due to the Lipschitz estimate \( \|S'(\tilde{u}; h_1) - S'(\tilde{u}; h_2)\|_{\infty} \leq 2\|h_1 - h_2\|_{\infty} \) for \( h_1, h_2 \in CBV[0, T] \) and since \( U \) is dense in \( C[0, T] \), (7.3) remains valid when the test space \( U \) is replaced by \( CBV[0, T] \). We define \( \tilde{\mu} \in G_r[0, T]^* \) via

\[
\langle \tilde{\mu}, z \rangle_{G_r} := \int_0^T z \, d\tilde{p} + \langle \partial_1 J(\tilde{y}, \tilde{y}(T), \tilde{u}), z \rangle_{L^\infty} + \langle \partial_2 J(\tilde{y}, \tilde{y}(T), \tilde{u}), z \rangle_{U} \forall z \in G_r[0, T].
\]

Then the last line in (7.1) holds, and it follows from (7.3) with test space \( CBV[0, T] \) and Lemma 6.2 that

\[
\int_0^T z \, d\tilde{p} + \langle \partial_1 J(\tilde{y}, \tilde{y}(T), \tilde{u}), z \rangle_{L^\infty} + \langle \partial_2 J(\tilde{y}, \tilde{y}(T), \tilde{u}), z \rangle_{U} \geq 0 \quad \forall z \in K_{G_r}^{rad, crit}(\tilde{y}, \tilde{u}).
\]

Due to Theorem 6.5 and the bounded convergence theorem (Theorem A.4), we can extend the last inequality to the set \( K_{G_r}^{rad, crit}(\tilde{y}, \tilde{u}) \) by approximation, i.e., we have

\[
\int_0^T z \, d\tilde{p} + \langle \partial_1 J(\tilde{y}, \tilde{y}(T), \tilde{u}), z \rangle_{L^\infty} + \langle \partial_2 J(\tilde{y}, \tilde{y}(T), \tilde{u}), z \rangle_{U} = \langle \tilde{\mu}, z \rangle_{G_r} \geq 0
\]

for all \( z \in K_{G_r}^{rad, crit}(\tilde{y}, \tilde{u}) \). This proves the third line in (7.1). It remains to establish the pointwise properties of \( \tilde{p} \) in (7.1). To this end, we again use that Corollary 4.13
and (A.1) imply that \( S'(\bar{u}; c1_{[0,T]})_+ = 0 \) holds for all \( c \in \mathbb{R} \). By (7.3) with test space \( CBV[0,T] \), this yields
\[
0 \leq c \int_0^T \mathrm{d}\bar{p} = c (\bar{p}(T) - \bar{p}(0)) \quad \forall c \in \mathbb{R}.
\]
Thus, \( \bar{p}(0) = \bar{p}(T) \). Since \( \bar{p}(T) = 0 \) and \( \bar{p}(t) = \bar{p}(t-) \) for all \( t \in (0,T) \), and since \( \bar{p}(0) = \bar{p}(0-) \) holds by definition, this establishes the first line of (7.1). Next, by invoking Lemma 6.6, by setting \( h = h_i \) in (7.3) with test space \( CBV[0,T] \), and by passing to the limit \( i \to \infty \) by means of Theorem A.4 and the dominated convergence theorem, we obtain that, for every \( t \in [0,T] \) and every \( c \in K_{\text{crit}}^{\text{piv}}(\bar{y}, \bar{u})(t)^0 \), we have
\[
0 \leq \int_0^T c1_{[t,T]} \mathrm{d}\bar{p} = c (\bar{p}(T) - \bar{p}(t-)) = -c\bar{p}(t-).
\]
Here, the last two equations follow from [43, Lemma 6.3.3] and the identity \( \bar{p}(T) = 0 \). By using the definition (6.3) of the polar cone \( K_{\text{crit}}^{\text{piv}}(\bar{y}, \bar{u})(t)^0 \) in (7.4), one readily obtains that \( \bar{p}(t-) \in K_{\text{crit}}^{\text{piv}}(\bar{y}, \bar{u})(t) \) holds for all \( t \in [0,T] \). This establishes the second line in (7.1) and proves, in combination with the previous steps, that the strong stationarity system (7.1) is indeed a necessary condition for Bouligand stationarity.

Next, we prove the implication \( "(7.1) \Rightarrow (5.1)" \). Suppose that \( \bar{u} \in U_{\text{ad}} \) is a control with state \( \bar{y} := S(\bar{u}) \) such that there exist \( \bar{p} \in BV[0,T] \) and \( \bar{p} \in G_r[0,T]^+ \) satisfying (7.1). Assume further that a direction \( h \in U \) is given and define \( \eta := S'(\bar{u}; h)_+ \). Then it follows from the properties of \( \bar{p} \), (4.14) with \( z := \eta + \bar{p} \), and the integration by parts formula for the Kurzweil-Stieltjes integral [43, Theorem 6.4.2] that
\[
0 \leq \int_0^T \bar{p} \mathrm{d}(\eta - h) = \int_0^T (h - \eta) \mathrm{d}\bar{p} + \bar{p}(T)(\eta - h)(T) - \bar{p}(0)(\eta - h)(0)
\]
\[
+ \sum_{t \in [0,T]} (\bar{p}(t) - \bar{p}(t-)) ((\eta - h)(t) - (\eta - h)(t-))
\]
\[
- \sum_{t \in [0,T]} (\bar{p}(t) - \bar{p}(t+)) ((\eta - h)(t) - (\eta - h)(t+)).
\]
Due to the identities \( \bar{p}(0) = \bar{p}(T) = 0 \) and \( \eta(0) = \eta(0-) = 0 \) and due to the left- and right-continuity properties of \( \bar{p}, h \), and \( \eta = S'(\bar{u}; h)_+ \), the last estimate simplifies to
\[
0 \leq \int_0^T (h - \eta) \mathrm{d}\bar{p} + \bar{p}(T-)(\eta(T) - \eta(T-))
\]
Note that (4.12), \( \bar{p}(T-) \in K_{\text{crit}}^{\text{piv}}(\bar{y}, \bar{u})(T) \), and the convention \( \eta(T) = \eta(T+) \) imply that \( \bar{p}(T-)(\eta(T) - \eta(T-)) = \bar{p}(T-)(\eta(T+) - \eta(T-)) \geq 0 \) holds. We thus obtain
\[
0 \leq \int_0^T (h - \eta) \mathrm{d}\bar{p} = \int_0^T h \mathrm{d}\bar{p} - \int_0^T \eta \mathrm{d}\bar{p},
\]
and, by the last three lines of (7.1) and the properties of \( \eta \),
\[
0 \leq \langle \partial_3 J(\bar{y}, \bar{y}(T), \bar{u}), h \rangle_U + \langle \partial_3 J(\bar{y}, \bar{y}(T), \bar{u}), \eta \rangle_{L^\infty} + \partial_2 J(\bar{y}, \bar{y}(T), \bar{u}) \eta(T) - \langle \bar{p}, \eta \rangle_{G_r}
\]
\[
\leq \langle \partial_3 J(\bar{y}, \bar{y}(T), \bar{u}), h \rangle_U + \langle \partial_3 J(\bar{y}, \bar{y}(T), \bar{u}), \eta \rangle_{L^\infty} + \partial_2 J(\bar{y}, \bar{y}(T), \bar{u}) \eta(T).\]
If we now exploit that \( \partial_3 J(\bar{y}, \bar{y}(T), \bar{u}) \in L^1(0,T) \), that \( S'(\bar{u}; h)(T) = S'(\bar{u}; h)(T+) \), and that \( \eta = S'(\bar{u}; h) \) a.e., then (5.1) follows. This completes the proof.
Note that, in the case $T \in I(\bar{y})$, there exists $m > 0$ such that the function 
$z(t) := c \mathbf{1}_{[T-\varepsilon,T]}(t) (T - t + \varepsilon) / \varepsilon$ is an element of $K^{\text{red, crit}}_{G_r}(\bar{y}, \bar{u})$ for all $c \in \mathbb{R}$ and all $0 < \varepsilon < m$. For such a function $z$, the third line of (7.1) becomes $\langle \bar{\mu}, z \rangle_{G_r} = 0$. Using
this in the fifth line of (7.1) and subsequently passing to the limit $\varepsilon \to 0^+$ by means
of Theorem A.4 yields, due to the $L^1$-regularity of $\partial_1 \mathcal{J}(\bar{y}, \bar{y}(T), \bar{u})$ and (A.3), that

$$-c \int_0^T \mathbf{1}_{\{T\}} \, d\bar{p} = -c (\bar{p}(T) - \bar{p}(T^-)) = \partial_2 \mathcal{J}(\bar{y}, \bar{y}(T), \bar{u})c \quad \forall c \in \mathbb{R}.$$ 

Thus, $\bar{p}(T^-) - \bar{p}(T) = \partial_2 \mathcal{J}(\bar{y}, \bar{y}(T), \bar{u})$ and we obtain that the partial derivative
$\partial_2 \mathcal{J}(\bar{y}, \bar{y}(T), \bar{u}) \in \mathbb{R}$ affects the jump of $\bar{p}$ at $T$, as mentioned in section 1. We remark
that, by redefining $\bar{p}$, this implicit jump condition on the adjoint state in (7.1) can also
be transformed into a condition on the function value at $T$. Indeed, by introducing
the modified adjoint state $\bar{q} := \bar{p} + \partial_2 \mathcal{J}(\bar{y}, \bar{y}(T), \bar{u}) \mathbf{1}_{\{T\}} \in BV[0, T]$, by using
the integration by parts formula in [43, Theorem 6.4.2] in the fourth line of (7.1), and by
employing (A.3) and [43, Lemma 6.3.2], one easily checks that the strong stationarity
system in Theorem 7.1 can also be formulated as follows:

$$\bar{q}(0) = 0, \quad \bar{q}(T) = \partial_2 \mathcal{J}(\bar{y}, \bar{y}(T), \bar{u}), \quad \bar{q}(t) = \bar{q}(t^-) \forall t \in [0, T),$$

$$\bar{q}(t) \in K^{\text{ptw}}_{\text{crit}}(\bar{y}, \bar{u}) \forall t \in [0, T],$$

$$\langle \bar{\mu}, z \rangle_{G_r} \geq 0 \quad \forall z \in K_{G_r}^{\text{red, crit}}(\bar{y}, \bar{u}),$$

$$- \int_0^T \bar{q} \, dh = \langle \partial_3 \mathcal{J}(\bar{y}, \bar{y}(T), \bar{u}), h \rangle_U \forall h \in U,$$

$$- \int_0^T \bar{z} \, d\bar{q} = \langle \partial_1 \mathcal{J}(\bar{y}, \bar{y}(T), \bar{u}), z \rangle_{L^\infty} - \langle \bar{\mu}, z \rangle_{G_r} \quad \forall z \in G_r[0, T].$$

Regarding the assumption that the set $\mathbb{R}_+(U_{\text{ad}} - \bar{u})$ is dense in $U$, we would
like to point out that this so-called “ample control” condition in Theorem 7.1 is
rather restrictive and rarely satisfied if $U_{\text{ad}} \neq U$. Using techniques from [51],
it might be possible to establish a strong stationarity system for (P) also under weaker
assumptions on the control constraints. We leave this topic for future research.

Appendix A. Results on the Kurzweil-Stieltjes integral. Let $a, b \in \mathbb{R}$ with
$a < b$ be given. For $f, g \in G[a, b]$, the Kurzweil-Stieltjes integral with integrand $f$
and integrator $g$ exists if at least one of the functions $f$ and $g$ has bounded variation, see
[43, Theorem 6.3.11]. In this case, it yields a real number which we denote by

$$\int_a^b f \, dg \quad \text{or} \quad \int_a^b f(t) \, dg(t).$$

The Kurzweil-Stieltjes integral coincides with the Riemann-Stieltjes integral whenever
the latter exists, see [43, Theorem 6.2.12]. This holds in particular if $f \in C[a, b]$ and
$g \in BV[a, b]$, see [43, Theorem 5.6.1]. If $c \in \mathbb{R}$ is interpreted as a constant function,
then it holds

$$(A.1) \quad \int_a^b c \, dg = c(g(b) - g(a)) \quad \text{and} \quad \int_a^b f \, dc = 0$$

for all $f, g \in G[a, b]$, see [43, Remark 6.3.1].
The Kurzweil-Stieltjes integral is linear w.r.t. the integrand $f$ and w.r.t. the integrator $g$, see [43, Theorem 6.2.7]. Further, for all $c \in (a, b)$, it holds

$$\int_a^b f \, dg = \int_a^c f \, dg + \int_c^b f \, dg$$

provided the first integral exists, see [43, Theorems 6.2.9, 6.2.10]. For $t \in [a, b]$ and $g \in G[a, b]$, we have (see [43, Lemma 6.3.3])

$$\int_a^b 1_{\{t\}} \, dg = g(t+) - g(t-)$$

with the conventions $g(b+) := g(b)$ and $g(a-) := g(a)$. In particular, the integral in (A.3) equals zero if $g$ is continuous at $t$.

**Lemma A.1.** Let $f \in G[a, b]$, $g \in BV_r[a, b]$, $a \leq s < \tau \leq b$, and $J := (s, \tau]$. Then

$$\int_s^\tau f \, dg = \int_a^b 1_J f \, dg.$$  

If $g \in CBV[a, b]$, then (A.4) is also true for $J = [s, \tau]$, $J = (s, \tau)$, and $J = [s, \tau)$.  

**Proof.** This is a special case of [43, Theorem 6.9.7].

**Lemma A.2.** Let $g \in BV[a, b]$ be given such that $g(a) = g(b) = 0$ holds and such that the set $\{t \in [a, b] : g(t) \neq 0\}$ is finite or countably infinite. Then

$$\int_a^b f \, dg = 0 \quad \forall f \in G[a, b].$$

**Proof.** This is a special case of [43, Lemma 6.3.15].

**Proposition A.3.** Let $g \in BV_r[a, b]$. Then

$$\int_a^b g \, dg = \frac{1}{2} (g(b)^2 - g(a)^2) + \frac{1}{2} \sum_{t \in [a, b]} (g(t) - g(t-))^2.$$ 

**Proof.** This is a special case of [33, Corollary 2.12] or of [36, Corollary 1.13].

**Theorem A.4** (bounded convergence theorem). Let $g \in BV[a, b]$, $f_n \in G[a, b]$ with $\sup_n \|f_n\|_{\infty} < \infty$ and $f_n \to f$ pointwise in $[a, b]$ be given. Then the integral $\int_a^b f \, dg$ exists and it holds

$$\lim_{n \to \infty} \int_a^b f_n \, dg = \int_a^b f \, dg.$$ 

**Proof.** This is a special case of [43, Theorem 6.8.13].

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