Multiplicative properties of a quantum Caldero-Chapoton map associated to valued quivers

Ming Ding\textsuperscript{a}, Jie Sheng\textsuperscript{b,∗}, Xueqing Chen\textsuperscript{c}

\textsuperscript{a}School of Mathematical Sciences and LPMC, Nankai University, Tianjin, P.R.China
\textsuperscript{b}Department of Applied Mathematics, China Agricultural University, Beijing, P.R.China
\textsuperscript{c}Department of Mathematics, University of Wisconsin-Whitewater
800 W.Main Street, Whitewater,WI.53190.USA

Abstract

Following Hubery’s approach [15], we prove a multiplication theorem of a quantum Caldero-Chapoton map associated to acyclic valued quivers which extends the results in [8, 7]. As an application, some integral bases of the quantum cluster algebras for affine valued quivers can be constructed.

Keywords: quantum cluster algebra, affine valued quiver, integral basis

1. Background

Quantum cluster algebras were introduced in [2] to study the canonical basis. A quantum cluster algebra is generated by a set of generators called the cluster variables inside an ambient skew-field $F$. Specializing $q$ to 1, the quantum cluster algebras are exactly cluster algebras which were introduced by Fomin and Zelevinsky (e.g. see [12, 13]).

Ever since the emergency of cluster algebras, the close relation between them and quiver representations has always been emphasized. One interesting viewpoint is to consider cluster algebra as some kind of Hall algebra of quiver representations, that was particularly enhanced by the Caldero-Chapoton map (or character) invented in [3]. Since then the multiplication formulas ([4, 5, 19]) of the Caldero-Chapoton map became important, especially in the construction of integral bases of cluster algebras (e.g. see [9, 10]).

In [17], Rupel obtained a quantum analogue of the Caldero-Chapoton map, which is crucial for the study of quantum cluster algebras. This quantum cluster character assigns to a representation $M$ of an acyclic valued quiver the element $X_M$ in some quantum torus. Rupel proved that this character provides cluster variables for finite-type and rank 2 valued quivers, and conjectured it holds for all acyclic valued quivers. Recently it was proved by Rupel [18] for all acyclic valued quivers. In the quantum cluster algebras, it is not always true that $X_NX_M = [k]^{\pm d_{N\oplus M}}X_{N\oplus M}$ for some $d_{N\oplus M} \in \mathbb{Z}$ where $M$, $N$ lie in the corresponding cluster category. A natural question to ask is whether the quantized Caldero-Chapoton formula could be extended to the cluster category. In [7], Ding reformulated

\begin{itemize}
  \item[∗]Corresponding author
  Email addresses: m-ding04@mails.tsinghua.edu.cn (Ming Ding), shengjie@amss.ac.cn (Jie Sheng),
  chenx@uw.edu (Xueqing Chen)

  Ming Ding was supported by NSF of China (No. 11301282) and Specialized Research Fund for the Doctoral Program of Higher Education (No. 20130031120004). Jie Sheng was supported by NSF of China (No. 11301533).
\end{itemize}
the quantum Caldero-Chapoton map defined in [17, 16] and then this aim was achieved for equally valued quivers.

In this paper, we define a new quantum Caldero-Chapoton map associated to cluster categories of acyclic valued quivers and then prove the multiplication formulas which can be viewed as a quantum analogue of $X_N X_M = X_{N \oplus M}$. Our multiplication formulas generalize those in [8, 7] and are similar to the multiplication in the dual Hall algebras. Moreover, we construct some integral bases of the quantum cluster algebras for affine valued quivers by using the standard monomials in [2].

2. Quantum cluster algebras, quantum Caldero-Chapoton maps and the main result

2.1. Quantum cluster algebras

We briefly recall the definition of quantum cluster algebras. Let $L$ be a lattice of rank $m$ and $\Lambda : L \times L \to \mathbb{Z}$ a skew-symmetric bilinear form. We will need a formal variable $q$ and consider the ring of integral Laurent polynomials $\mathbb{Z}[q^{\pm 1/2}]$. Define the based quantum torus associated to the quantum quantum Caldero-Chapoton map $T$ with a distinguished $\mathbb{Z}[q^{\pm 1/2}]$-basis $\{X^e : e \in L\}$ and the multiplication given by

$$X^e X^f = q^{\Lambda(e,f)/2} X^{e+f}.$$ 

It is easy to see that $T$ is associative and the basis elements satisfy the following relations:

$$X^e X^f = q^{\Lambda(e,f)} X^f X^e, \quad X^0 = 1 \quad \text{and} \quad (X^e)^{-1} = X^{-e}.$$ 

It is known that $T$ is an Ore domain, i.e., is contained in its skew-field of fractions $\mathcal{F}$. The quantum cluster algebra will be defined as a $\mathbb{Z}[q^{\pm 1/2}]$-subalgebra of $\mathcal{F}$.

A toric frame in $\mathcal{F}$ is a map $M : \mathbb{Z}^m \to \mathcal{F} \setminus \{0\}$ of the form

$$M(c) = \varphi(X^{\eta(c)})$$

where $\varphi$ is an automorphism of $\mathcal{F}$ and $\eta : \mathbb{Z}^m \to L$ is an isomorphism of lattices. By definition, the elements $M(c)$ form a $\mathbb{Z}[q^{\pm 1/2}]$-basis of the based quantum torus $T_M := \varphi(T)$ and satisfy the following relations:

$$M(c)M(d) = q^{\Lambda_M(c,d)/2} M(c + d), \quad M(c)M(d) = q^{\Lambda_M(c,d)} M(d)M(c),$$

$$M(0) = 1 \quad \text{and} \quad M(c)^{-1} = M(-c),$$

where $\Lambda_M$ is the skew-symmetric bilinear form on $\mathbb{Z}^m$ obtained from the lattice isomorphism $\eta$. Let $\Lambda_M$ also denote the skew-symmetric $m \times m$ matrix defined by $\lambda_{ij} = \Lambda_M(e_i, e_j)$ where $\{e_1, \ldots, e_m\}$ is the standard basis of $\mathbb{Z}^m$. Given a toric frame $M$, let $X_i = M(e_i)$. Then we have

$$T_M = \mathbb{Z}[q^{\pm 1/2}](X_1^{\pm 1}, \ldots, X_m^{\pm 1}, X_iX_j = q^{\lambda_{ij}} X_jX_i).$$

Let $\Lambda$ be an $m \times m$ skew-symmetric matrix and let $\tilde{B}$ be an $m \times n$ matrix with $m \geq n$, whose principal part is denoted by $B$. We call the pair $(\Lambda, \tilde{B})$ compatible if $B^T \Lambda = (D)(0)$ is an $n \times m$ matrix with $D = \text{diag}(d_1, \ldots, d_n)$ where $d_i \in \mathbb{N}$ for $1 \leq i \leq n$. The pair $(\Lambda, \tilde{B})$ is called a quantum seed if the pair $(\Lambda_M, \tilde{B})$ is compatible. Define the $m \times m$ matrix $E = (e_{ij})$ by

$$e_{ij} = \begin{cases} 
\delta_{ij} & \text{if } j \neq k; \\
-1 & \text{if } i = j = k; \\
\max(0, -b_{ik}) & \text{if } i \neq j = k.
\end{cases}$$
For \( n, k \in \mathbb{Z}, k \geq 0 \), denote \( [\frac{n}{k}]_q = \frac{(q^n - q^{-n}) \cdots (q^{n-r+1} - q^{-n+r-1})}{(q^r - q^{-r}) \cdots (q^{n-r} - q^{-n+r-1})} \). Let \( k \in [1, n] \) and \( \mathbf{c} = (c_1, \ldots, c_m) \in \mathbb{Z}^m \) with \( c_k \geq 0 \). Define the toric frame \( M' : \mathbb{Z}^m \to \mathcal{F} \setminus \{0\} \) as follows:

\[
M'(\mathbf{c}) = \sum_{p=0}^{c_k} \left[ \frac{c_k}{p} \right] q^{pk/2} M(E\mathbf{c} + pb^k) \quad \text{and} \quad M'(-\mathbf{c}) = M'(\mathbf{c})^{-1}
\]

(2.1)

where the vector \( b^k \in \mathbb{Z}^m \) is the \( k \)-th column of \( \tilde{B} \).

Define the \( m \times n \) matrix \( B' = (b'_{ij}) \) by

\[
b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k; \\ b_{ij} + \frac{|h_{ik}h_{kj} + h_{ik}h_{kj}|}{2} & \text{otherwise}. \end{cases}
\]

Then the quantum seed \( (M', \tilde{B}') \) is defined to be the mutation of \( (M, \tilde{B}) \) in direction \( k \). Two quantum seeds \( (M, \tilde{B}) \) and \( (M', \tilde{B}') \) are mutation-equivalent if they can be obtained from each other by a sequence of mutations, denoted by \( (M, \tilde{B}) \sim (M', \tilde{B}') \). Let \( \mathcal{C} := \{ M'(e_i) : (M, \tilde{B}) \sim (M', \tilde{B}'), i \in [1, n] \} \). Let \( \mathbb{Z}\mathbb{P} \) be the ring of integral Laurent polynomials in the (quasi-commuting) variables in \( \{ q^{1/2}, X_{n+1}, \ldots, X_m \} \). The quantum cluster algebra \( \mathcal{A}_q(\Lambda_M, \tilde{B}) \) is the \( \mathbb{Z}\mathbb{P} \)-subalgebra of \( \mathcal{F} \) generated by \( \mathcal{C} \).

The following proposition demonstrates the mutation of quantum cluster variables which can be viewed as a quantum analogue of cluster mutation.

**Proposition 2.1.** [2] *The toric frame \( X' \) is determined by*

\[
X' = \begin{cases} -X & \text{if } i = k; \\ X & \text{otherwise}. \end{cases}
\]

The quantum Laurent phenomenon proved by Berenstein and Zelevinsky is the important result concerning quantum cluster algebras.

**Theorem 2.2.** [2] *The quantum cluster algebra \( \mathcal{A}_q(\Lambda_M, \tilde{B}) \) is a subalgebra of \( \mathcal{T}_M \).*

The standard monomial in \( \{ X_1, \ldots, X_n, X'_1, \ldots, X'_n \} \) is an element of the form

\[
X_1^{a_1} \cdots X_n^{a_n} (X_1')^{a_1'} \cdots (X_n')^{a_n'},
\]

where all exponents are nonnegative integers and \( a_k a'_k = 0 \) for any \( 1 \leq k \leq n \). The following theorem will be used to construct \( \mathbb{Z}\mathbb{P} \)-bases of the quantum cluster algebras for affine valued quivers in the last section.

**Theorem 2.3.** [2] *The standard monomials in \( \{ X_1, \ldots, X_n, X'_1, \ldots, X'_n \} \) form a \( \mathbb{Z}\mathbb{P} \)-basis of the quantum cluster algebra \( \mathcal{A}_q(\Lambda_M, \tilde{B}) \) if and only if the principal matrix \( \tilde{B} \) is acyclic.*

### 2.2. Quantum Caldero-Chapoton maps and the main result

Let \( k \) be a finite field with cardinality \( |k| = q \) and \( n \) a positive integer. Let \( \Delta \) be a valued graph without vertex loops and with vertex set \( \{1, \ldots, n\} \). The edges of \( \Delta \) are of the form \( \frac{i}{a_{ij}}, \frac{j}{a_{ij}}, \frac{k}{a_{ij}} \), in which the positive integers \( a_{ij} \) form a symmetric matrix.

Let \( Q \) be an orientation of \( \Delta \) containing no oriented cycles: that is, we replace each valued edge by a valued arrow. Thus \( Q \) is called a valued quiver. Note that any finite dimensional basic hereditary \( k \)-algebra can be obtained by taking the tensor algebra of the \( k \)-species associated to \( Q \).
In what follows we will denote by $\mathcal{S}$ the $k$-species of type $Q$ in the sense of [15], which identified a $k$-species with its corresponding tensor algebra. We define a new quiver $\tilde{Q}$ by attaching additional vertices $n + 1, \ldots, m$ to $Q$ where $m \geq n$. The full subquiver $Q$ is called the principal part of $\tilde{Q}$. For $1 \leq i \leq m$, let $S_i$ be the $i$-th simple module for $\mathcal{S}$ which is the $k$-species of type $Q$.

Let $\bar{B}$ be the $m \times n$ matrix associated to the quiver $\tilde{Q}$ whose entry in position $(i, j)$ is given by

$$b_{ij} = \dim_{\text{End}_{\mathcal{S}}(S_i)} \text{Ext}^1_{\mathcal{S}}(S_i, S_j) - \dim_{\text{End}_{\mathcal{S}}(S_i)} \text{Ext}^1_{\mathcal{S}}(S_j, S_i)$$

for $1 \leq i \leq m, 1 \leq j \leq n$. Denote by $\tilde{I}$ the left $m \times n$ submatrix of the identity matrix of size $m \times m$.

Assume that there exists some skew-symmetric $m \times m$ integer matrix $\Lambda$ such that

$$\Lambda(-\bar{B}) = \begin{bmatrix} D_n \\ 0 \end{bmatrix},$$

where $D_n = \text{diag}(d_1, \ldots, d_n)$, $d_i \in \mathbb{N}$ for $1 \leq i \leq n$. Such $\tilde{Q}$ and $\Lambda$ exist for a given $Q$ (see [17]). Thus, the pair $(\Lambda, \bar{B})$ is compatible. Consequently the matrix $\bar{B}$ is of full rank. Let $\bar{R} = \bar{R}_{\tilde{Q}}$ be the $m \times n$ matrix with its entry in position $(i, j)$ given by

$$\bar{r}_{ij} := \dim_{\text{End}_{\mathcal{S}}(S_i)} \text{Ext}^1_{\mathcal{S}}(S_j, S_i)$$

for $1 \leq i \leq m, 1 \leq j \leq n$. And define $\bar{R} := \bar{R}_{\tilde{Q}}$. Denote the principal parts of the matrices $\bar{B}$ and $\bar{R}$ by $B$ and $R$, respectively. Note that $\bar{B} = \bar{R} - \bar{R}$ and $B = R' - R$.

Let $\mathcal{C}_{\tilde{Q}}$ be the cluster category (see [1]) of the valued quiver $\tilde{Q}$. We note that each object $M$ in $\mathcal{C}_{\tilde{Q}}$ can be uniquely decomposed in the following way:

$$M = M_0 \oplus P_M[1]$$

where $M_0$ is a $\mathcal{S}$-module and $P_M$ is a projective $\mathcal{S}$-module. Let $P_M = \bigoplus_{1 \leq i \leq m} l_i P_i$. We extend the definition of the dimension vector $\dim$ on modules in mod$\mathcal{S}$ to objects in $\mathcal{C}_{\tilde{Q}}$ by setting

$$\dim M = \dim M_0 - (l_i)_{1 \leq i \leq m}.$$ 

The Euler form on $\mathcal{S}$-modules $M$ and $N$ is given by

$$\langle M, N \rangle = \dim_k \text{Hom}_{\mathcal{S}}(M, N) - \dim_k \text{Ext}^1_{\mathcal{S}}(M, N).$$

Note that the Euler form only depends on the dimension vectors of $M$ and $N$, and for those $M$ and $N$ in mod$\mathcal{S}$, the matrix representing this form is $(I_n - R^T)D_n = D_n(I_n - R^T)$. In the following, we will denote by the corresponding underlined lower case letter $\underline{\varepsilon}$ the dimension vector of a $\mathcal{S}$-module $X$ and view $\underline{\varepsilon}$ as a column vector in $\mathbb{Z}^n$. If $X$ is a $\mathcal{S}$-module, we denote by the vector $\underline{\varepsilon} \in \mathbb{Z}^n$ with no confusions.

The quantum Caldero-Chapoton map of an acyclic equally quiver $\tilde{Q}$ has been defined in [17, 16, 7]. For valued quivers, we define

$$X_{M \oplus I[-1]} = \sum_\varepsilon [\text{Gr}_\varepsilon M]q^{-\frac{1}{2}(\underline{\varepsilon}^T \underline{\varepsilon} - 2\langle \underline{\varepsilon}, \underline{\varepsilon} \rangle)}X^{\underline{\varepsilon} - (\bar{r} - \bar{R}) \underline{\varepsilon} + \dim soc I},$$

where $M$ is a $\mathcal{S}$-module with $\dim M = \underline{m}$, $I$ is an injective $\mathcal{S}$-module with $\dim I = 1$ and $\text{Gr}_\varepsilon M$ denotes the set of all submodules $V$ of $M$ with $\dim V = \varepsilon$. Note that

$$X_{P[1]} = X_{rP} = X^{\dim P/\text{rad} P} = X^{\dim soc I} = X_{I[-1]} = X_{r^{-1} I}.$$
for any projective \( \tilde{S} \)-module \( P \) and injective \( \tilde{S} \)-module \( I \) with \( \text{soc} I = P/\text{rad} P \).

Now let us recall some notations. For any \( \tilde{S} \)-modules \( M, N \) and \( E \), denote by \( \varepsilon_{MN}^E \) the cardinality of the set \( \text{Ext}^1_{\tilde{S}}(M, N)_E \) which is the subset of \( \text{Ext}^1_{\tilde{S}}(M, N) \) consisting of those equivalence classes of short exact sequences with middle term isomorphic to \( E \) ([15, Section 4]). Denote by \([M, N] = \dim_k \text{Ext}^1_{\tilde{S}}(M, N) \) and \([M, N] = \dim_k \text{Hom}_{\tilde{S}}(M, N) \).

Let \( M, N \) and \( B \) be \( \tilde{S} \)-modules and \( I, I' \) be injective \( \tilde{S} \)-modules. Define

\[
\text{Hom}_{\tilde{S}}(M, I)_{BI'} := \{ f : M \to I | \ker f \cong B \text{ and } \coker f \cong I' \}.
\]

The main result of this paper is the following theorem:

**Theorem 2.4.** Let \( \tilde{Q} \) be an acyclic quiver, then

\[
\begin{align*}
1 & \quad q^{[M, N]} X_M X_N = q^{1 \Lambda((\tilde{I} - \tilde{R}')(\tilde{m} - \tilde{m}')} \sum E \varepsilon_{MN}^E X_E, \\
2 & \quad q^{[M, I]} X_M X_{[I]^{-1}} = q^{1 \Lambda((\tilde{I} - \tilde{R}')(\tilde{m} - \dim \text{soc} I)} \sum_{B, I'} [\text{Hom}_{\tilde{S}}(M, I)_{BI'} | X_{[B \oplus I']^{-1}}].
\end{align*}
\]

Let \( A_{\mathbb{k}}(\tilde{Q}) \) be the corresponding specialized quantum cluster algebra of \( Q \) with coefficients. Then the main theorem in [18] shows that \( A_{\mathbb{k}}(\tilde{Q}) \) is the \( \mathbb{Z}_P \)-subalgebra of \( F \) generated by

\[
\{ X_M | M \text{ is indecomposable rigid } \tilde{S} \text{ module} \} \cup \\
\{ X_{I_i^{-1}}, 1 \leq i \leq n | I_i \text{ is indecomposable injective } \tilde{S} \text{ module} \}.
\]

### 3. Proof of the main theorem

In this section, we fix a valued quiver \( Q \) with \( n \) vertices.

**Lemma 3.1.** For any dimension vector \( \underline{m}, \underline{e} \) and \( \underline{f} \) in \( \mathbb{Z}_{\geq 0}^n \), we have

\[
\begin{align*}
1 & \quad \Lambda((\tilde{I} - \tilde{R}')(\tilde{m} - \tilde{e})) = -\langle \underline{e}, \underline{m} \rangle; \\
2 & \quad \Lambda(\tilde{B} \underline{e} - \tilde{B} \underline{f}) = \langle \underline{f}, \underline{e} \rangle - \langle \underline{e}, \underline{f} \rangle.
\end{align*}
\]

**Proof.** By definition, we have

\[
\begin{align*}
\Lambda((\tilde{I} - \tilde{R}')(\tilde{m} - \tilde{e})) &= m^{tr}(\tilde{I} - \tilde{R}')^{tr} \Lambda \tilde{B} \underline{e} = -m^{tr}(\tilde{I} - \tilde{R}')^{tr} \begin{bmatrix} D_n \\ 0 \end{bmatrix} \underline{e} \\
&= -m^{tr}(I_n - (R')^{tr})D_n \underline{e} = -\underline{e}^{tr} D_n(I_n - \tilde{R'}) \underline{m} \\
&= -\langle \underline{e}, \underline{m} \rangle.
\end{align*}
\]

As for (2), the left hand side of the desired equation is equal to

\[
\begin{align*}
\underline{e}^{tr} \tilde{B}^{tr} \Lambda \tilde{B} \underline{f} &= -\underline{e}^{tr} \tilde{B}^{tr} \begin{bmatrix} D_n \\ 0 \end{bmatrix} \underline{f} = -\underline{e}^{tr} B^{tr} D_n \underline{f}.
\end{align*}
\]
The right hand side is
\[
\langle \tilde{f}, e \rangle - \langle e, \tilde{f} \rangle \\
= \int_{\mathcal{E}} \mathcal{H}n(I_n - \tilde{R})e - \mathcal{H}n(I_n - \tilde{R})f \\
= \mathcal{H}n(I_n - \tilde{R})^{tr}Dn\tilde{f} - \mathcal{H}n(I_n - \tilde{R})Dn\tilde{f} \\
= \mathcal{H}n(\tilde{R}^{tr} - (\tilde{R})\tilde{R})Dn\tilde{f} = -\mathcal{H}nB^{tr}Dn\tilde{f}.
\]
Thus we prove the lemma. \[\square\]

**Corollary 3.2.** For any dimension vector \(m, l, e\) and \(f\) in \(\mathbb{Z}_{\geq 0}^n\), we have
\[
\Lambda(-\tilde{B}e - (\tilde{I} - \tilde{R})m, \tilde{B}f - (\tilde{I} - \tilde{R})l) = \Lambda((\tilde{I} - \tilde{R})m, (\tilde{I} - \tilde{R})l) + \langle f, e \rangle - \langle e, f \rangle + \langle e, l \rangle - \langle f, m \rangle.
\]

**Proof.** It follows from Lemma 3.1 directly. \[\square\]

**Proof of Theorem 2.4:** For \(\tilde{S}\)–modules \(M, A\) and \(B\), we denote by \(F_{MB}^M\) the number of submodules \(U\) of \(M\) such that \(U\) is isomorphic to \(B\) and \(M/U\) is isomorphic to \(A\), which is the so-called Hall number.

By Green’s formula \([14]\), we have
\[
\sum_{E} E_{MN} F_{XY} = \sum_{A,B,C,D} q^{[M,N] - [A,C] - [B,D] - \langle A,D \rangle} F_{AB}^M F_{CD}^N X_{AC}^Y e_{BD}.
\]
Then
\[
\sum_{E} E_{MN} X_{E} = \sum_{E, X,Y} E_{MN} \left( -\frac{1}{2} \langle Y, X \rangle F_{XY} X - \tilde{B}u - (\tilde{I} - \tilde{R}) \right) e
\]
\[
= \sum_{A,B,C,D,X,Y} q^{[M,N] - [A,C] - [B,D] - \langle A,D \rangle - \frac{1}{2} (B + D, A + C)} F_{AB}^M F_{CD}^N X_{AC}^Y e_{BD}.
\]
By Corollary 3.2, we have
\[
X_{-\tilde{B}u - (\tilde{I} - \tilde{R}) (m+n)} = X_{-\tilde{B}u - (\tilde{I} - \tilde{R}) (m+n)} = q^{\frac{1}{2} \Lambda((\tilde{I} - \tilde{R})m, (\tilde{I} - \tilde{R})m) + \frac{1}{2} (B, D) + (B, D) + (D, M) - \langle B, N \rangle} X_{-\tilde{B}u - (\tilde{I} - \tilde{R}) (m+n)}
\]
\[
= q^{\frac{1}{2} \Lambda((\tilde{I} - \tilde{R})m, (\tilde{I} - \tilde{R})m)} F_{AB}^M F_{CD}^N X_{AC}^Y - \tilde{B}u - (\tilde{I} - \tilde{R}) (m+n).
\]
Thus
\[
\sum_{E} E_{MN} X_{E} = q^{\frac{1}{2} \Lambda((\tilde{I} - \tilde{R})m, (\tilde{I} - \tilde{R})m)} \sum_{A,B,C,D} q^{[M,N] - [A,C] - [B,D] - \langle A,D \rangle - \frac{1}{2} (B + D, A + C) + [A,C] + [B,D]}
\]
\[
= q^{\frac{1}{2} (D, A) - \frac{1}{2} (B, C)} F_{AB}^M F_{CD}^N X_{AC}^Y - \tilde{B}u - (\tilde{I} - \tilde{R}) (m+n).
\]
Here we use facts
\[ \sum_X \varepsilon_{AC}^X = q^{[A,C]} \] and \[ \sum_Y \varepsilon_{BD}^Y = q^{[B,D]} \].

An easy calculation shows that
\[
[M, N] - [A, C] - [B, D] - \langle A, D \rangle + [A, C] + [B, D] = [M, N] + [B, C].
\]

Hence
\[
\sum_E \varepsilon_{MN}^E X_E
= q^{-\frac{1}{2}} \Lambda((-\tilde{I} - \tilde{R}) m, (-\tilde{I} - \tilde{R}) m) q^{[M,N]} X_{M,N}.
\]

This finishes the proof.

**Proof of Theorem 2.4 (2):** We calculate
\[
X_{M,N} I[-1]
= \sum_{G,H} q^{-\frac{1}{2}} \Lambda(H,G) F_{GH}^M X - \tilde{B} m - \tilde{R} m \dim_{soc I}
= \sum_{G,H} q^{-\frac{1}{2}} \Lambda(H,G) F_{GH}^M X - \tilde{B} m - \tilde{R} m \dim_{soc I}
= q^{-\frac{1}{2}} \Lambda(-(-\tilde{I} - \tilde{R}) m, \dim_{soc I}) \sum_{G,H} q^{-\frac{1}{2}} \Lambda(-\tilde{B} m, \dim_{soc I}) F_{GH}^M X - \tilde{B} m - \tilde{R} m \dim_{soc I}
= q^{-\frac{1}{2}} \Lambda(\tilde{I} - \tilde{R}) m, \dim_{soc I}) \sum_{G,H} q^{-\frac{1}{2}} \Lambda(\tilde{B} m, \dim_{soc I}) F_{GH}^M X - \tilde{B} m - \tilde{R} m \dim_{soc I}.
\]

Here we use the fact that
\[ \Lambda(-\tilde{B} m, \dim_{soc I}) = -k^{tr} \Lambda(\tilde{B} m, \dim_{soc I}) = -[H, I]. \]

Let \( B \) be a \( S \)-module and \( I' \) be an injective \( \tilde{S} \)-module. Let \( f \in \text{Hom}_{\tilde{S}}(M, I) \). We obtain two short exact sequences
\[
0 \rightarrow B \rightarrow M \rightarrow A \rightarrow 0 \quad \text{and} \quad 0 \rightarrow A \rightarrow I \rightarrow I' \rightarrow 0.
\]

Thus by [15],
\[
|\text{Hom}_{\tilde{S}}(M, I)_{BI'}| = \sum_A |\text{Aut}(A)| F_{AB}^M F_{IA}^I.
\]

Consider now two short exact sequences
\[
0 \rightarrow H \rightarrow B \rightarrow X \rightarrow 0 \quad \text{and} \quad 0 \rightarrow B \rightarrow M \rightarrow A \rightarrow 0.
\]
We can uniquely complete the following diagram:

```
0 0
↓  ↓
H  H
↓  ↓
0 → B → M → A → 0
↓  ↓
0 → X → G → A → 0.
```

Thus by [15],

\[
\sum_B F_{XH}^B F_{AB}^M = \sum_G F_{AX}^G F_{GH}^M.
\]

Note that

\[
\sum_{A,P,X} |\text{Aut}(A)| F_{I'A}^P F_{AX}^P = \sum_{I',X} |\text{Hom}(G,I)_{X,I'}| = q^{(G,I)} = q^{(G,I')}.
\]

By [15, Lemma 1], we have \((\tilde{I} - \tilde{R})I = \dim \text{soc} I\). Now we can calculate the term

\[
\sum_{B,I'} |\text{Hom}(M,I)_{B'I'}| X_{B\oplus I'}[-1]
\]

\[
= \sum_{A,B,I',X,H} |\text{Aut}(A)| F_{I'A}^P F_{AX}^P q^{-\frac{1}{2}(H,X-I')} F_{XH}^B X^{-\overline{B}h - (\tilde{I} - \tilde{R})I'} + \dim \text{soc} I'
\]

\[
= \sum_{A,G,I',X,H} q^{-\frac{1}{2}(H,X-I')} |\text{Aut}(A)| F_{I'A}^P F_{AX}^P F_{GH}^M X^{-\overline{B}h - (\tilde{I} - \tilde{R})I'} + \dim \text{soc} I'.
\]

We have the following facts

\[
\overline{i} + a = \overline{i}, \ x + a = q \implies x - \overline{i} = q - \overline{i}
\]

and

\[
-\overline{B}h - (\tilde{I} - \tilde{R})I + \dim \text{soc} I'
\]

\[
= -\overline{B}h - (\tilde{I} - \tilde{R})(m - \overline{i} + \overline{i}') + \dim \text{soc} I'
\]

\[
= -\overline{B}h - (\tilde{I} - \tilde{R})m + (\tilde{I} - \tilde{R})(\overline{i} - \overline{i}') + \dim \text{soc} I'
\]

\[
= -\overline{B}h - (\tilde{I} - \tilde{R})m + (\tilde{I} - \tilde{R})I
\]

\[
= -\overline{B}h - (\tilde{I} - \tilde{R})m + \dim \text{soc} I.
\]
Hence

\[
\sum_{B, I'} |\text{Hom}_{\mathcal{B}}(M, I)_{B, I'}| X_{B \oplus I'[-1]}
= \sum_{G, H} q^{G, I} q^{-\frac{1}{2}(H, G - I)} F_{G H}^M X - \tilde{B}h - (\tilde{I} - \tilde{R}) \dim \text{soc} I
= \sum_{G, H} q^{(M, I)} q^{-\frac{1}{2}(H, I)} q^{-\frac{1}{2}(H, G)} F_{G H}^M X - \tilde{B}h - (\tilde{I} - \tilde{R}) \dim \text{soc I}
= \sum_{G, H} q^{(M, I)} q^{-\frac{1}{2}(H, I)} q^{-\frac{1}{2}(H, G)} F_{G H}^M X - \tilde{B}h - (\tilde{I} - \tilde{R}) \dim \text{soc I}.
\]

This finishes the proof.

Note that Theorem 2.4 implies the following corollary which was proved independently in [18, Theorem 5.12 and Theorem 5.13] by different methods.

**Corollary 3.3.** Let \( \tilde{Q} \) be an acyclic quiver, then

1. when \([M, N]\) = 0, we have

\[
X_M X_N = q^{\frac{1}{2} A((I - R') \dim (I - R') \dim \text{soc} I)} X_M \oplus N;
\]

If in addition \([N, M]\) = 0, we have

\[
X_M X_N = q^{A((I - R') \dim (I - R') \dim \text{soc} I)} X_N X_M;
\]

2. when \([M, I]\) = 0, we have

\[
X_M X_{I[-1]} = q^{\frac{1}{2} A((I - R') \dim (I - R') \dim \text{soc} I)} X_{M \oplus I[-1]}.
\]

**4. Bases of the quantum cluster algebras for affine valued quivers**

Let \( \tilde{Q} \) be an acyclic valued quiver with \( n \) vertices and \( i \) a sink or a source in \( \tilde{Q} \). We define the reflected quiver \( \sigma_i(\tilde{Q}) \) by reversing all the arrows ending at \( i \). An *admissible sequence of sinks* (resp. *sources*) is a sequence \((i_1, \ldots, i_l)\) such that \( i_1 \) is a sink (resp. source) in \( \tilde{Q} \) and \( i_j \) is a sink (resp. source) in \( \sigma_{i_{j-1}} \cdots \sigma_{i_1}(\tilde{Q}) \) for any \( j = 2, \ldots, l \). A quiver \( \tilde{Q}' \) is called *reflection-equivalent* to \( \tilde{Q} \) if there exists an admissible sequence of sinks or sources \((i_1, \ldots, i_l)\) such that \( \tilde{Q}' = \sigma_{i_l} \cdots \sigma_{i_1}(\tilde{Q}) \). A quiver \( \tilde{Q}' \) is called *reachable* from \( \tilde{Q} \) if \( \tilde{Q}' = \sigma_{i_l} \cdots \sigma_{i_1}(\tilde{Q}) \) where \( 1 \leq i_1, \ldots, i_l \leq n \). Note that if \( \tilde{Q}' \) is a quiver reflection-equivalent to \( \tilde{Q} \) in the direction \( i \) where \( 1 \leq i \leq n \), there is a natural canonical isomorphism between \( A_{\{i\}}(\tilde{Q}) \) and \( A_{\{i\}}(\tilde{Q}') \), denoted by

\[
\Phi_i : A_{\{i\}}(\tilde{Q}) \to A_{\{i\}}(\tilde{Q}').
\]

sending every initial cluster variable in \( A_{\{i\}}(\tilde{Q}) \) to its expansion in the initial cluster variables in \( A_{\{i\}}(\tilde{Q}') \). Denote the category of all finite-dimensional valued representations of \( \tilde{Q} \) by \( \text{rep}(\tilde{Q}) \). Let \( \Sigma_i^+ : \text{rep}(\tilde{Q}) \to \text{rep}(\tilde{Q}') \) be the standard BGP-reflection functor and \( R_i^+ : C_{\tilde{Q}} \to C_{\tilde{Q}'} \) be the extended BGP-reflection functor defined in [20]:

\[
R_i^+ : \begin{cases}
X & \mapsto \Sigma_i^+(X) \quad \text{if } X \notin S_i \text{ is a module} \\
S_i & \mapsto P_i[1] \\
P_j[1] & \mapsto P_j[1] \quad \text{if } j \neq i \\
P_i[1] & \mapsto S_i
\end{cases}
\]
By Rupel [17], the following holds:

**Theorem 4.1.** [17] Suppose $M$ is an indecomposable object in $\mathcal{C}_Q$ and $X_M^Q$ is in the corresponding quantum cluster algebra $\mathcal{A}_{|k|(Q)}$, then $\Phi_i(X_M^Q) = X_{R_i^+M}^Q$.

Now let $Q$ be an affine valued quiver with $n$ vertices and $E_1, \cdots, E_r$ be regular simple modules in a nonhomogeneous tube with rank $r$ such that $\tau E_2 = E_1, \cdots, \tau E_1 = E_r$. Given a regular simple module $E$ in a tube, let $E[i]$ be the indecomposable regular module with quasi-socle $E$ and quasi-length $i$ for any $i \in \mathbb{N}$. The minimal imaginary root of $Q$ is denoted by $\delta$. Let $Q$ be the quiver obtained from $Q$ by adding frozen vertices $\{n + 1, \cdots , 2n\}$ and arrows $n + i \rightarrow i$ for any $1 \leq i \leq n$.

**Definition 4.2.** For any objects $M$ and $Q$ obtained from $Q$ respectively, where $E$ and $\tau E$ are the quiver corresponding to two non-split triangles in $\mathcal{C}_Q$. Thus we can finish it by induction on $\text{rad} P_k$.

**Lemma 4.3.** Let $Q$ be an alternating affine valued quiver (i.e., whose vertex is a sink or a source). Then for any cluster monomial $X_M \in \mathcal{A}_{|k|(Q)}$, its Laurent expansion has the minimal term $a_M X^{(-\text{dim} M)}$ with respect to the above partial order in Definition 4.2, where $a_M$ is some nonzero monomial in $\{q^{\frac{1}{2}}, X_{n+1}^{\pm 1}, \cdots, X_{2n}^{\pm 1}\}$.

**Proof.** By Corollary 3.3, it is enough to consider the case when $M$ is an indecomposable object.

Firstly, we consider the case $M = P_k$ with $1 \leq k \leq n$ which is projective module. The following exact sequences

$$0 \rightarrow \text{rad} P_k \rightarrow P_k \rightarrow I_k \rightarrow I' \rightarrow 0$$

and

$$0 \rightarrow 0 \rightarrow P_k \rightarrow P_k \rightarrow 0 \rightarrow 0 \rightarrow 0 .$$

correspond to two non-split triangles in $\mathcal{C}_Q$

$$I_k[-1] \rightarrow E \rightarrow P_k \rightarrow I_k,$$

and

$$P_k \rightarrow 0 \rightarrow P_k[1] \rightarrow P_k[1]$$

respectively, where $E \simeq \text{rad} P_k \oplus I'[-1]$.

1) If $k$ is a sink, we have $\text{rad} P_k = 0$. Thus by [8, Theorem 3.8] and [18, Theorem 5.8], we have

$$X_{P_k} X_{I_k[-1]} = q^{\frac{1}{2}A((I'-R')P_k, (I'-R')(\pm i_2))} X_{I'[1]} + q^{\frac{1}{2}A((I'-R')P_k, (I'-R')(\pm i_2))} - \frac{1}{2}I_k, I_k].$$

It is obvious that the statement holds for $X_{P_k}$.

2) If $k$ is a source, we have $I' = 0$ and thus by [8, Theorem 3.8] and [18, Theorem 5.8], we have

$$X_{P_k} X_{I_k[-1]} = q^{\frac{1}{2}A((I'-R')P_k, (I'-R')(\pm i_2))} X_{\text{rad} P_k} + q^{\frac{1}{2}A((I'-R')P_k, (I'-R')(\pm i_2))} - \frac{1}{2}I_k, I_k].$$

Thus we can finish it by induction on $\text{rad} P_k$.
Now assume that $R$ is a cluster-tilting object and $R'$ a cluster-tilting object in $C_Q$ next to $R$ in
the tilting graph. Then there exists an exchange pair $(U, U^*)$ such that $R = U \oplus U^*$ and $R' = U^* \oplus U$.
We denote by
\[ U \rightarrow B \rightarrow U^* \rightarrow U[1] \]
and
\[ U^* \rightarrow B' \rightarrow U \rightarrow U^*[1] \]
the corresponding triangles. Thus, $B$ and $B'$ are in $add\tilde{T}$. According to [8, Theorem 3.5, 3.8] and
[18, Theorem 5.5, 5.8], we have
\[ \mathcal{X}_U \mathcal{X}_{U^*} = \tilde{a}_B \mathcal{X}_B + \tilde{a}_{B'} \mathcal{X}_{B'} \]
where $\tilde{a}_B, \tilde{a}_{B'}$ are some power of \{\( q^{\pm \frac{1}{2}} \)}. Thus the proof follows from induction and the fact that
the tilting graph is connected (see [1] and [15]).

The following result asserts that the quantum Caldero-Chapoton maps associated to some imaginary
modules in nonhomogeneous tubes lie in the corresponding quantum cluster algebras.

**Lemma 4.4.**
\[
\mathcal{X}_{E_{i-1}} \mathcal{X}_{E_i[r-1]} = q^{\lambda (\tilde{I}-\tilde{R})_{E_{i-1},(\tilde{I}-\tilde{R})_{E_i[r-1]}}} \mathcal{X}_{E_i[r]}
+ q^{\lambda (\tilde{I}-\tilde{R})_{E_{i-1},(\tilde{I}-\tilde{R})_{E_i[r-1]}}} + \frac{1}{2} (E_{i-1}, E_i[r-1]) \mathcal{X}_{E_i[r-2]} \mathcal{X}_{I[r-1]}
\]
where $I$ is an injective $\tilde{S}$–module associated to frozen vertices.

**Proof.** We have the following exact sequences
\[
0 \rightarrow E_i[r-1] \rightarrow E_i[r] \rightarrow E_{i-1} \rightarrow 0 \quad \text{and}
0 \rightarrow E_i[r-2] \rightarrow E_i[r-1] \rightarrow \tau Q E_{i-1} \rightarrow I \rightarrow 0.
\]
Hence the proof follows from [8, Theorem 3.5] and [18, Theorem 5.5].

**Remark 4.5.** By Lemma 4.3 and Lemma 4.4, it is easy to see that the Laurent expansion in $\mathcal{X}_{E_i[r]}$
has the minimal term $a_{E_i[r]} X^{\tau \dim E_i[r]}$ with respect to the above partial order, where $a_{E_i[r]}$ is
some nonzero monomial in \{\( q^{\pm \frac{1}{2}}, X_{n+1}^{\pm 1}, \ldots, X_{2n}^{\pm 1} \)}.

Define the set
\[
D(Q) = \{ \mathbf{d} \in \mathbb{N}^{Q_0} | \exists \text{ a regular rigid module } R \text{ and a regular module } E \text{ in a nonhomogeneous tube with dimension vector } \delta \text{ such that } \dim (E^\oplus \oplus R) = \mathbf{d} \}
\]
and
\[
E(Q) = \{ \mathbf{d} \in \mathbb{Z}^{Q_0} | \exists M = M_0 \oplus P_M[1] \text{ with } M_0 \text{ an } \tilde{S}\text{-module, } P_M \text{ projective } \tilde{S}\text{-module,}
M \text{ rigid object in } C_Q \text{ with } \dim M = \mathbf{d} \}
\]
Note that $\mathbb{Z}^{Q_0}$ is the disjoint union of $D(Q)$ and $E(Q)$.
If $\mathbf{d} \in E(Q)$, we set
\[
\mathcal{X}_\mathbf{d} := \mathcal{X}_M;
\]
If $\mathbf{d} \in D(Q)$ and $|Q_0| > 2$,
\[
\mathcal{X}_\mathbf{d} := (\mathcal{X}_E)^{n} \mathcal{X}_R;
\]
If $\mathbf{d} \in D(Q)$ and $|Q_0| = 2$,
\[
\mathcal{X}_\mathbf{d} := \mathcal{X}_E^n
\]
for some $E$ in a homogeneous tube of degree 1.
Theorem 4.6. Let $Q'$ be any alternating affine valued quiver and $\widetilde{Q}$ be an acyclic valued quiver which is reachable from $Q'$. Then the set

$$B(\widetilde{Q}) := \{ X_{\underline{d}} | \underline{d} \in \mathbb{Z}^n \}$$

is a $\mathbb{ZP}$-basis of $A_{|k|}(\widetilde{Q})$.

Proof. In the case of $|Q_0| = 2$, by [8, Lemma 5.5] and the following Lemma 4.7, we can deduce that $X_{R_{p(1)}}$ is in $A_{|k|}(\widetilde{Q})$. Now let $Q'$ be an alternating affine valued quiver with at least three vertices. By Lemma 4.4, $X_{E_{1|n|}}$ is in $A_{|k|}(\widetilde{Q})$. It follows that for any $\underline{m} = (m_1, \cdots, m_n) \in \mathbb{Z}^n$, $X_{\underline{m}} \in A_{|k|}(\widetilde{Q})$. By the standard monomials constructed in [8, see Theorem 4.3], we know that the set $\{ \prod_{i=1}^n X_{S_i} X_{P_i[1]} | (d_1, \cdots, d_n) \in \mathbb{Z}^n \}$ is a $\mathbb{ZP}$-basis of $A_{|k|}(\widetilde{Q})$. As $Q'$ is an alternating quiver, then by Lemma 4.3 and Remark 4.5, we have

$$X_{\underline{m}}^{Q'} = b_{\underline{m}} \prod_{i=1}^n (X_{S_i}^{Q'})^{m_i} (X_{P_i[1]}^{Q'})^{m_i} + \sum_{\underline{l} \preceq \underline{m}} b_{\underline{l}} \prod_{i=1}^n (X_{S_i}^{Q'})^{l_i} (X_{P_i[1]}^{Q'})^{l_i}$$

where $b_{\underline{m}}, b_{\underline{l}} \in \mathbb{ZP}$. Thus $B(Q')$ is a $\mathbb{QP}$-basis of $A_{|k|}(\widetilde{Q})$, where $\mathbb{QP}$ is the skew field of fraction for $\mathbb{ZP}$.

Similarly we have

$$(X_{S_i}^{Q'})^{m_i} (X_{P_i[1]}^{Q'})^{m_i} = a_{\underline{m}} X_{\underline{m}}^{Q'} + \sum_{\underline{l} \preceq \underline{m}} a_{\underline{l}} X_{\underline{l}}^{Q'}$$

where $a_{\underline{m}}, a_{\underline{l}} \in \mathbb{QP}$.

Again by Lemma 4.3 and Remark 4.5, we know that $a_{\underline{m}}$ must be some nonzero monomial in $\{ q^{L_1}, X_{n+1}, \cdots, X_{2n} \}$. Note that there exists a partial order on the remaining dimension vectors $L_1$, so we can choose the maximal elements denoted by $L_1, \ldots, L_m$ respectively. Then by $a_{\underline{m}} \in \mathbb{ZP}$ and the fact that coefficients of Laurent expansions in cluster variables belong to $\mathbb{ZP}$, we obtain that $a_{L_1}, \ldots, a_{L_m}$ must be in $\mathbb{ZP}$. Using the same method, we have all remaining $a_{\underline{m}} \in \mathbb{ZP}$.

Therefore, $B(Q')$ is a $\mathbb{ZP}$-basis of $A_{|k|}(\widetilde{Q})$. By Theorem 4.1, we obtain that $B(\widetilde{Q})$ is a $\mathbb{ZP}$-basis of $A_{|k|}(\widetilde{Q})$. □

4.1. An example

Consider the following valued quiver $Q$ of type $A_2^{(2)}:

$$
\begin{array}{c}
1 \\
\end{array} \overset{(4,1)}{\longrightarrow} \begin{array}{c}
2 \\
\end{array}$$

It is well-known [11] that indecomposable $\mathcal{G}$-modules are divided into (up to isomorphism) three families: the indecomposable regular modules with dimension vector $(nd_p, 2nd_p)$ for $p \in \mathbb{P}_k$ of degree $d_p$ and $n \in \mathbb{N}$ (in particular, denote by $R_p(n)$ the indecomposable regular module with dimension vector $(n, 2n)$ for $d_p = 1$), the preprojective modules and the preinjective modules. We consider the following quiver $Q$ with frozen vertices 3 and 4:

$$
\begin{array}{c}
1 \\
\end{array} \overset{(4,1)}{\longrightarrow} \begin{array}{c}
2 \\
\end{array} \quad 3 \quad 4
$$
Thus we have
\[
\begin{align*}
\tilde{R}' &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \\
\tilde{I} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \\
\tilde{B} &= \begin{pmatrix} 0 & 1 \\ -4 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.
\end{align*}
\]

An easy calculation shows that the following antisymmetric $4 \times 4$ integer matrix
\[
\Lambda = \begin{pmatrix} 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & -1 \\ 4 & 0 & 0 & -4 \\ 0 & 1 & 4 & 0 \end{pmatrix}
\]
satisfying
\[
\Lambda(-\tilde{B}) = \begin{pmatrix} D_2 \\ 0 \end{pmatrix},
\]
where $D_2 = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$.

Lemma 4.7. Let $R_p(1)$ be the indecomposable regular module of degree 1 as above. Then
\[
\mathbb{X}_{R_p(1)} = X_{S_2}^2 X_{S_1} - q^6 X_1 X_2^2 X_4^2 - (q^2 + q^{-2})q^3 X_2^2 X_4.
\]

Proof. By definition, we have
\[
\begin{align*}
\mathbb{X}_{S_2} &= X^{(1,-1,0,1)} + X^{(0,-1,0,0)}; \\
\mathbb{X}_{S_1} &= X^{(-1,0,1,0)} + X^{(-1,4,0,0)}; \\
\mathbb{X}_{R_p(1)} &= X^{(1,-2,1,2)} + (q^2 + q^{-2})X^{(0,-2,1,1)} + X^{(-1,-2,1,0)} + X^{(-1,2,0,0)}.
\end{align*}
\]
Hence the lemma follows from a direct calculation.

Remark 4.8. In the case of coefficient-free quantum cluster algebra of type $A_2^{(2)}$, we have already given a concrete discussion in [6].

References

References

[1] A. Buan, R. Marsh, M. Reineke, I. Reiten and G. Todorov, *Tilting theory and cluster combinatorics*, Adv. Math. 204 (2006), 572–618.

[2] A. Berenstein and A. Zelevinsky, *Quantum cluster algebras*, Adv. Math. 195 (2005), 405–455.

[3] P. Caldero and F. Chapoton, *Cluster algebras as Hall algebras of quiver representations*, Comm. Math. Helv. 81 (2006), 595–616.

[4] P. Caldero and B. Keller, *From triangulated categories to cluster algebras*, Invent. math. 172 (2008), no. 1, 169-211.
[5] P. Caldero and B. Keller, *From triangulated categories to cluster algebras II*, Ann. Sci. École Norm. Sup. (4) 39 (2006), no. 6, 983-1009.

[6] X. Chen, M. Ding and J. Sheng, *Bar-invariant bases of the quantum cluster algebra of $A_2^{(2)}$*, Czech. Math. J., 2011, 61 (4): 1077–1090.

[7] M. Ding, *On quantum cluster algebras of finite type*, Front. Math. China 2011, 6(2): 231–240.

[8] M. Ding and F. Xu, *A quantum analogue of generic bases for affine cluster algebras*, Science China Mathematics. 55 (2012), no. 10, 2045–2066.

[9] M. Ding, J. Xiao and F. Xu, *Integral bases of cluster algebras and representations of tame quivers*, Algebr Represent Theory. 16 (2013), no. 2, 491–525.

[10] G. Dupont, *Generic variables in acyclic cluster algebras and bases in affine cluster algebras*, arXiv:0811.2909.

[11] V. Dlab and C. M. Ringel, Indecomposable representations of graphs and algebras, *Mem. Amer. Math. Soc.* 173 (1976).

[12] S. Fomin and A. Zelevinsky, *Cluster algebras. I. Foundations*, J. Amer. Math. Soc. 15 (2002), no. 2, 497–529.

[13] S. Fomin and A. Zelevinsky, *Cluster algebras. II. Finite type classification*, Invent. Math. 154 (2003), no.1, 63–121.

[14] J. A. Green, *Hall algebras, hereditary algebras and quantum groups*, Inv. Math. 120 (1995), 361–377.

[15] A. Hubery, *Acyclic cluster algebras via Ringel-Hall algebras*, preprint (2005).

[16] F. Qin, *Quantum cluster variables via Serre polynomials*, J. reine angew. Math. 668 (2012), 149–190.

[17] D. Rupel, *On a quantum analogue of the Caldero–Chapoton formula*, Int Math Res Notices., (2011), no. 14, 3207–3236.

[18] D. Rupel, *Quantum cluster characters*, arXiv:1109.6694v2 [math.QA].

[19] J. Xiao and F. Xu, *Green’s formula with C*-action and Caldero–Keller’s formula for cluster algebras*, Representation Theory of Algebraic Groups and Quantum Groups, Progress in Mathematics Volume 284, 2010, 313–348.

[20] B. Zhu, *Equivalence between cluster categories*, J. Algebra 304 (2006), 832–850.