ON THE CAUCHY PROBLEM OF THE MODIFIED HUNTER-SAXTON EQUATION

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ABSTRACT. This paper is concerned with the Cauchy problem of the modified Hunter-Saxton equation, which was proposed by J. Hunter and R. Saxton [SIAM J. Appl. Math. 51(1991) 1498-1521]. Using the approximate solution method, the local well-posedness of the model equation is obtained in Sobolev spaces $H^s$ with $s > 3/2$, in the sense of Hadamard, and its data-to-solution map is continuous but not uniformly continuous. However, if a weaker $H^r$-topology is used then it is shown that the solution map becomes Hölder continuous in $H^s$.

1. Introduction. In this paper we study the periodic Cauchy problem of the modified Hunter-Saxton equation

$$\begin{align*}
\partial_t u + u^k \partial_x u &= \frac{1}{2} \partial_x^{-1} (\partial_x (u^k) \partial_x u), & x \in \mathbb{T}, t > 0, \\
u(0, x) &= u_0(x), & x \in \mathbb{T}.
\end{align*}$$

where $k \geq 1$ is a positive integer, $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ is the torus, $H^s(\mathbb{T})$ is the Sobolev space on the torus $T$ with exponent $s$. For $s > 3/2$ we prove that the Cauchy problem (1)-(2) is locally well posed in $\mathbb{T}$ and that the data-to-solution map is continuous but not uniformly continuous. Furthermore, we show that the solution map is Hölder continuous in $\mathbb{T}$ if it is equipped with an $H^r$-norm, $0 \leq r < s$. The equation (1) was proposed by J. Hunter and R. Saxton [48]. In [63], the author established local well-posedness in the Sobolev spaces in Sobolev space $H^s(\mathbb{T})$ for $s > 3/2$ and in $C^1(\mathbb{T})$ also studied the analytic regularity (both in space and time variables) of this problem. In [54], Kohlmann considered the initial boundary value problem of the modified Hunter-Saxton equation.

2010 Mathematics Subject Classification. 35B30, 35Q53.
Key words and phrases. Sobolev spaces, the modified Hunter-Saxton equation, local well-posedness.

This work was supported by NSF of China (11401050,11671055,11371384,11601053).

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For $k = 1$, Eq. (1) reduces to the Hunter-Saxton equation \cite{48}
\begin{equation}
\partial_t u + u \partial_x u = \frac{1}{2} \partial_x^{-1} \left( \partial_x (u) \right)^2, \quad x \in \mathbb{T}, t > 0,
\end{equation}
\begin{equation}
u(0, x) = u_0(x), \quad x \in \mathbb{T}.
\end{equation}
which describes the propagation of weakly nonlinear orientation waves in a massive nematic liquid crystal director field. Here, $u(t, x)$ describes the director field of a nematic liquid crystal, $x$ is a space variable in a reference frame moving with the linearized wave velocity, and $t$ is a slow time variable. More precisely, the orientation of the molecules is described by the field of unit vectors $(\cos(u(t, x)), \sin(u(t, x)))$ \cite{48, 49}.

The Hunter-Saxton equation also has a bi-Hamiltonian structure \cite{48, 59} and is completely integrable \cite{2, 49}, and it is also the Euler equation for the geodesic flow on the quotient space of the infinite-dimensional group $D(S)$ of orientation-preserving diffeomorphisms of the unit circle $S = \mathbb{R}/\mathbb{Z}$ modulo the subgroup of rotations Rot$(S)$ equipped with the $H^1$ right-invariant metric \cite{55, 56, 64}. The initial value problem for the Hunter-Saxton equation on the line (nonperiodic case) was studied by Hunter and Saxton in \cite{48}. Using the method of characteristics, they showed that smooth solutions exist locally and break down in finite time. The occurrence of blow-up can be interpreted physically as the phenomenon by which waves that propagate away from the perturbation “knock” the director field out of its unperturbed state \cite{48}. The initial value problem for the periodic Hunter-Saxton equation was studied in \cite{65}. Using the Kato theorem, the author obtained that this equation has solutions for initial data $u_0 \in H^s(S), s > \frac{3}{2}$ and showed that all the nonconstant solutions blow up in finite time \cite{65}. Recently, global dissipative and conservative weak solutions for the Hunter-Saxton equation on the line were investigated extensively, cf. \cite{4, 24, 50, 51, 66, 67, 68}.

The Hunter-Saxton equation can be regarded as a non-local perturbation of the Burgers equation. Also, it is formally obtained in the short-wave limit $(t, x) \to (\varepsilon t, \varepsilon x)$ for $\varepsilon \to 0$ of the famous Camassa-Holm equation
\begin{equation}
\begin{cases}
u_t - u_{txx} + 3u u_x = 2u_x u_{xx} + uu_{xxx}, & t > 0, x \in \mathbb{R}, \\
u(x, 0) = u_0(x), & x \in \mathbb{R},
\end{cases}
\end{equation}
modelling the unidirectional propagation of shallow water waves over a flat bottom, $u(t, x)$ stands for the fluid velocity at time $t$ in the spatial direction $x$. It is a well-known integrable equation describing the velocity dynamics of shallow water waves. This equation spontaneously exhibits emergence of singular solutions from smooth initial conditions. It has a bi-Hamilton structure \cite{30} and is completely integrable \cite{7, 13}. In particular, it possesses an infinity of conservation laws and is solvable by its corresponding inverse scattering transform. After the birth of the Camassa-Holm equation, many works have been carried out to probe its dynamic properties. Such as, Eq. (5) has travelling wave solutions of the form $ce^{-|x-c t|}$, called peakons, which describes an essential feature of the travelling waves of largest amplitude (see \cite{14, 15, 22, 17}). It is shown in \cite{21, 12, 18} that the inverse spectral or scattering approach is a powerful tool to handle the Camassa-Holm equation and analyze its dynamics. It is worthwhile to mention that Eq. (5) gives rise to geodesic flow of a certain invariant metric on the Bott-Virasoro group \cite{19, 58}, and this geometric illustration leads to a proof that the Least Action Principle holds. It is shown in \cite{16} that the blow-up occurs in the form of breaking waves, namely, the solution remains
bounded but its slope becomes unbounded in finite time. Moreover, the Camassa-Holm equation has global conservative solutions \[5, 45\] and dissipative solutions \[6, 46\]. For other methods to handle the problems relating to various dynamic properties of the Camassa-Holm equation and other shallow water equations, the reader is referred to \[3, 23, 20, 11, 32, 33, 31, 27, 25, 26\] and the references therein.

Non-uniform dependence of the Camassa-Holm type equation has been studied by several authors. In \[44\], Himonas and Misiolek provides the first nonuniform dependence result for CH on the circle for \(s \geq 2\) using explicitly constructed travelling wave solutions. This result was sharpened to \(s > \frac{3}{2}\) in \[39\] on the line and \[40\] on the circle. Both of these more recent works utilize the method of approximate solutions in conjunction with delicate commutator and multiplier estimates.

For the DP equation, the first result for nonuniform dependence can be found in Christov and Hakkaev \[10\]. In the periodic case with \(s \geq 2\), nonuniform dependence of the data-to-solution map for DP is established following the method of travelling wave solutions developed in \[44\] for the CH equation. This result has been recently sharpened in \[37\], where nonuniform dependence on the initial data for DP is proven on both the circle and line for \(s > \frac{3}{2}\) using the method of approximate solutions in tandem with a twisted \(L_2\)-norm that is conserved by the DP equation.

This method of approximate solutions has also been adapted to a homogeneous setting in Holliman \[47\] to prove nonuniform dependence of the flow map for the Hunter-Saxton equation on the circle with \(s > \frac{3}{2}\), and to the higher dimensional setting in \[43\] to prove nonuniform dependence for the Euler equations.

Motivated by the references cited above, we establish the local well-posedness for the strong solutions to the Cauchy problem (1)-(2) in \(H^{s}, s > \frac{3}{2}\) and that the data-to-solution map is Hölder continuous in \(T\) if it is equipped with an \(H^{r}\)-norm, \(0 \leq r < s\).

Our main results in this paper are stated as follows.

**Theorem 1.1.** If \(s > \frac{3}{2}\) and \(u_0 \in H^s\) then there exists \(T > 0\) and a unique solution \(u \in C([0,T];H^s)\) of the i.v.p. (1)-(2), which depends continuously on the initial data \(u_0\). Furthermore, we have the estimate

\[
\|u(t)\|_{H^s} \leq 2\|u_0\|_{H^s}, \quad \text{for} \quad 0 \leq T \leq \frac{1}{2kc_s}\|u_0\|_{H^s},
\]

where \(c_s > 0\) is a constant depending on \(s\).

The well-posedness of the solutions for the i.v.p. (1)-(2) is obtained by Kato’s semigroup approach \[63\] and by classical Friedrichs’s regularization method \[57\], respectively. However, we have not been able to find the estimates (6) in the literatures. Here, we shall give a proof of local well-posedness of the solutions for the i.v.p. (1)-(2) in the sense of Hadamard, including estimates (6) by the different which are key ingredients in our work. Our proof of Theorem 1.1 is based on a Galerkin-type approximation method, which for quasi-linear symmetric hyperbolic systems can be found in Taylor \[61\].

Using this result of Theorem 1.1 and the method of approximate solutions we prove the following nonuniform dependence result.

**Theorem 1.2.** If \(s > \frac{3}{2}\) then the data-to-solution map for both the periodic and the nonperiodic the generalized Camassa-Holm equation defined by the Cauchy problem (1)-(2) is not uniformly continuous from any bounded subset in \(H^s\) into \(C([0,T];H^s)\).
The proof of Theorem 1.2 is based on the method of approximate solutions and well-posedness estimates for the solution and its lifespan, which is motivated by the works of the [37, 38, 39, 40, 42, 28, 34, 41, 47, 43]. We will choose approximate solutions to the the generalized Camassa-Holm equation such that the size of the difference between approximate and actual solutions with identical initial data is negligible. Hence, to understand the degree of dependence, it will suffice to focus on the behavior of the approximate solutions (which will be simple in form), rather than on the behavior of the actual solutions. In order for the method to go through, we will need well-posedness estimates for the size of the actual solutions to the generalized Camassa-Holm equation, as well a lower bound for their lifespan. This will permit us to obtain an upper bound for the size of the difference of approximate and actual solutions.

Theorem 1.1 and 1.2 show that the generalized Camassa-Holm equation is well-posed in Sobolev spaces \( H^s \) on both the line and the circle for \( s > 3/2 \) and its data-to-solution map is continuous but not uniformly continuous. Here, we show that the solution map for the generalized Camassa-Holm equation is Hölder continuous in \( H^s \)-topology for all \( 0 \leq r < s \). More precisely, we prove the following result.

**Theorem 1.3.** If \( s > 3/2 \) and \( 0 \leq r < s \), then the data-to-solution map for the Cauchy problem (1)-(2), on both the line and the circle, is Hölder continuous on the space \( H^s \) equipped with the \( H^r \) norm. More precisely, for initial data \( u(0), w(0) \) in a ball \( B(0, \rho) = \{ \varphi \in H^s : \| \varphi \|_{H^s} \leq \rho \} \) of \( H^s \), the corresponding solutions \( u(t), w(t) \) satisfy the inequality

\[
\| u(t) - w(t) \|_{C([0,T];H^r)} \leq c \| u(0) - w(0) \|_{H^r},
\]

where the exponent \( \alpha \) is given by

\[
\alpha = \begin{cases} 
1, & \text{if } (s,r) \in A_1, \\
\frac{2(s-1)}{s-r}, & \text{if } (s,r) \in A_1, \\
s-r, & \text{if } (s,r) \in A_1,
\end{cases}
\]

and the regions \( A_1, A_2 \) and \( A_3 \) in the \( sr \)-plane are defined by

\[
A_1 = \{ (s,r) : s > \frac{3}{2}, 0 \leq r \leq s-1, r+s \geq 2 \},
\]

\[
A_1 = \{ (s,r) : 2 > s > \frac{3}{2}, 0 \leq r \leq 2-s \},
\]

\[
A_1 = \{ (s,r) : s > \frac{3}{2}, s-1 \leq r \leq s \}.
\]

The lifespan \( T \) and the constant \( c \) depend on \( s, r \) and \( \rho \).

The proof of Theorem 1.3 follows the work of [35, 9, 43].

The rest of this paper is organized as follows. In Section 2, using the method of approximate solutions, we show that the solution map is not uniformly continuous. Section 3 is devoted to the study of the local wellposedness result for \( s > 3/2 \) with an accompanying solution size estimate. Finally, in Section 4, we prove Theorem 1.3.

**2. Non-uniform dependence.** Before giving the proof of the non-uniform dependence for modified Hunter-Saxton equation, we introduce some definition and known results for later proof.

We define the relation

\[
x \lesssim y \iff \text{there is a constant } c > 0 \text{ s.t. } x \leq cy,
\]

(8)
and

\[ x \approx y \leftrightarrow x \lesssim y \text{ and } y \lesssim y. \] (9)

The Fourier transform \( \hat{f} \) of the function \( f \) is taken to be

\[ \hat{f}(k) = \int_{\mathbb{T}} e^{-ixk} f(x) dx. \] (10)

Therefore, the inverse relation is given by the Fourier series

\[ f(x) = \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} \hat{f}(k)e^{-ixk}. \] (11)

For any \( s \in \mathbb{R} \) we take the operator \( D^s \) to be defined by

\[ \hat{D}^s f(k) = (1 + k^2)^{s/2} \hat{f}(k). \] (12)

The Sobolev space of exponent \( s \) is defined by

\[ H^s(\mathbb{T}) = \{ f \in \mathcal{D}'(\mathbb{T}) : (1 + k^2)^{s/2} \hat{f}(k) \in \ell^2(\mathbb{Z}) \}, \]

\[ \|f\|_{H^s(\mathbb{T})} = \left( \sum_{k \in \mathbb{Z}} (1 + k^2)^s |\hat{f}(k)|^2 \right)^{1/2}. \] (13)

The homogeneous Sobolev space of exponent \( s \) to be the subspace \( \dot{H}^s \subset H^s \) defined by

\[ \dot{H}^s(\mathbb{T}) = \{ f \in H^s(\mathbb{T}) : \hat{f}(0) = 0 \}, \]

\[ \|f\|_{\dot{H}^s(\mathbb{T})} = \left( \sum_{k \in \mathbb{Z}} k^{2s} |\hat{f}(k)|^2 \right)^{1/2}. \] (14)

**Remark.** As we will be working exclusively in the periodic case, we will often use the notation \( H^s = H^s(\mathbb{T}) \), \( \dot{H}^s = \dot{H}^s(\mathbb{T}) \).

Keeping in mind that when we define the inverse of \( \partial_x \) we will need the resulting function to be periodic, we define the operator as follows

For \( f \in H^s \) we set

\[ \partial_x^{-1}f(x) = \int_0^x f(y)dy - \frac{x}{2\pi} \int_0^{2\pi} f(y)dy 
- \frac{1}{2\pi} \int_0^{2\pi} \left[ \int_0^y f(z)dz - \frac{y}{2\pi} \int_0^{2\pi} f(z)dz \right] dy. \] (15)

It defines a continuous linear map \( \partial_x^{-1} : H^s \to \dot{H}^{s+1} \) satisfying the estimate

\[ \|\partial_x^{-1}f\|_{\dot{H}^{s+1}} \leq \|f\|_{\dot{H}^s} \leq \|f\|_{H^s}. \] (16)

Also, it is both a left and right inverse of the operator \( \partial_x : H^{s+1} \to \dot{H}^s \). Moreover, it satisfies the relations

\[ \partial_x^{-1}(\partial_x f)(x) = \hat{f}(x) - f(0), \quad f \in H^s, \]
\[ \partial_x \partial_x^{-1}(f)(x) = \hat{f}(x) - \frac{1}{2\pi} f(0), \quad f \in H^s. \] (17)

Since for any \( r \geq 0 \) and \( g \in \dot{H}^r \) we have \( \|g\|_{\dot{H}^r} \leq 2^{r/2}\|g\|_{\dot{H}^{r-1}} \), we also obtain the following inequality

\[ \|\partial_x^{-1}f\|_{H^r} \leq 2^{r/2}\|f\|_{\dot{H}^r} \leq 2^{r/2}\|f\|_{\dot{H}^{r-1}}. \] (18)
Next, we will prove Theorem 1.2 for Sobolev exponents $s > 3/2$. The basis of our proof rests upon finding two sequences of solutions, $\{u_n\}, \{v_n\} \in C([0,T] : H^s(T))$, to (1)-(2) that share a common lifespan and satisfy
\[
\|u_n(t)\|_{H^s} + \|v_n(t)\|_{H^s} \lesssim 1,
\]
and
\[
\lim_{n \to \infty} \|u_n(0) - v_n(0)\|_{H^s} = 0.
\]

2.1. Approximate solutions

For any $n$, a positive integer, we define the approximate solution $u^{\omega,n} = u^{\omega,n}(x,t)$ as
\[
u^{\omega,n}(x,t) = \omega n^{-\frac{k}{s}} + n^{-s} \cos(nx - \omega t),
\] (19)
where
\[
\omega = \begin{cases} 
-1,1 & \text{if } k \text{ is odd,} \\
0,1 & \text{if } k \text{ is even.}
\end{cases}
\] (20)

Now we compute the error of the approximate solutions (19). Note that
\[
\partial_t u^{\omega,n} + (u^{\omega,n})^k \partial_x u^{\omega,n} = \omega n^{-s} \sin(nx - \omega t)
\]
\[
+ [\omega n^{-\frac{k}{s}} + n^{-s} \cos(nx - \omega t)]^k [-n^{-s+1} \sin(nx - \omega t)]
\]
\[
\triangleq E_1
\] (21)
and
\[
- \frac{1}{2} \partial_x^{-1} (\partial_x (u^k) \partial_x u)
\]
\[
= - \frac{k}{2} \partial_x^{-1} \left[ (\omega n^{-\frac{k}{s}} + n^{-s} \cos(nx - \omega t))^k n^{-2s+2} \sin^2(nx - \omega t) \right]
\]
\[
= - \frac{k}{2} \partial_x^{-1} \left( \omega^{k-1} n^{k-2s+2} \sin^2(nx - \omega t) \right)
\]
\[
- \frac{k}{4} \partial_x^{-1} \left( \sum_{j=1}^{k-1} C_j^k \omega^{k-1-j} n^{\frac{k-1-j}{s} - 2s+2} \cos^{j-1}(nx - \omega t) \right)
\]
\[
\triangleq E_2 + E_3.
\] (22)
Thus, the error $E$ of the approximate solutions (19) is
\[
E(t) = E_1(t) + E_2(t) + E_3(t).
\] (23)

We will now proceed to estimate the size of the error $E$ in $H^s$. For the remainder of this proof, we will be taking $\sigma \in \left(\frac{1}{2}, 1\right)$ with the additional condition $\sigma + 1 < s$.

Lemma 2.1. Let $\sigma \in \left(\frac{1}{2}, 1\right)$, then the $H^s$-norm of $E$ is bounded by
\[
\|E(t)\|_{H^s} \lesssim n^{1-2s+\sigma}.
\] (24)
Proof. Applying the triangle inequality, we have
\[
\|E(t)\|_{H^s} \leq \|E_1(t)\|_{H^s} + \|E_2(t)\|_{H^s} + \|E_3(t)\|_{H^s}.
\] (25)
Using the formulas
\[
\|\cos(nx - \alpha)\|_{H^s} \approx n^\sigma \quad \text{and} \quad \|\sin(nx - \alpha)\|_{H^s} \approx n^\sigma,
\] (26)
the Algebra Property, \(\omega^k = \omega\) and \(|\omega| \leq 1\), we have
\[
\|E_1\|_{H^s} = \left\| \sum_{j=1}^{k} C_j^k \omega^{k-j} n^{-\frac{1}{4}(k-j)-js+s+1} \cos^j(nx - \omega t) \sin(nx - \omega t) \right\|_{H^s}
\]
\[
= \left\| \sum_{j=1}^{k} C_j^k \omega^{k-j} n^{-\frac{1}{4}(k-j)-js-s+1} \cos^j(nx - \omega t) \sin(2nx - 2\omega t) \right\|_{H^s}
\]
\[
\leq \sum_{j=1}^{k} \sigma^j \cdot n^{\frac{1}{4}(\frac{j}{2} - s + \sigma) - s}. \tag{27}
\]
Recall that \(\sigma < s - 1 < s - \frac{1}{k}\). Thus, \(\sigma - s + \frac{1}{k} \leq s - \frac{1}{k} - s + \frac{1}{k} = 0\). Along with \(k \geq 1\), we obtain
\[
\|E_1\|_{H^s} \lesssim \sum_{j=1}^{k} n^{j(\frac{1}{2} - s + \sigma) - s} \lesssim n^{\frac{1}{2} - s + \sigma - s} \lesssim n^{1-s+\sigma-s} = n^{1-2s+\sigma}. \tag{28}
\]
Using inequality (16), we have
\[
\|E_2\|_{H^s} = \left\| \frac{1}{2} \partial_x^{-1} \left( \omega^{k-1} n^{\frac{1}{4} - 2s + 1} \sin^2(nx - \omega t) \right) \right\|_{H^s}
\]
\[
\lesssim \left\| \omega^{k-1} n^{\frac{1}{2} - 2s + 1} \sin^2(nx - \omega t) \right\|_{H^{s-1}}
\]
\[
\lesssim n^{\frac{1}{2} - 2s + 1 + \sigma - 1}
\]
\[
\lesssim n^{1-2s+\sigma}. \tag{29}
\]
\[
\|E_3\|_{H^s} = \left\| \frac{1}{4} \partial_x^{-1} \left( C_{k-1}^j \omega^{k-1-j} n^{-\frac{1}{4} - 2s + 2 - js} \cos^{j-1}(nx - \omega t) \right. \right. \times \sin(2nx - 2\omega t) \sin(nx - \omega t) \right\|_{H^s}
\]
\[
\lesssim \left\| \frac{1}{4} \partial_x^{-1} \left( C_{k-1}^j \omega^{k-1-j} n^{-\frac{1}{4} - 2s + 2 - js} \cos^{j-1}(nx - \omega t) \right. \right. \times \sin(2nx - 2\omega t) \sin(nx - \omega t) \right\|_{H^{s-1}}
\]
\[
\lesssim n^{\frac{1}{4} - 1 - 2s + 2 - js} n^{j(j-1) + 2(\sigma - 1)}
\]
\[
\lesssim n^{1-2s+\sigma}. \tag{30}
\]

2.2. Estimating the difference between approximate and actual solutions
Now that we have these initial estimates for the approximate solutions, we will
The Cauchy problem (31)-(32) has a unique solution in $C^s$. For ease of notation, we rewrite the i.v.p. as

$$u(x,t) = \omega n^{-\frac{1}{2}} + n^{-s} \cos nx.$$  

Notice that (26) implies that the initial data $u(x,0)$ for all $s \geq 0$, since

$$\|u(\cdot,0)\|_{H^s} = \omega n^{-\frac{1}{2}} + n^{-s} \|\cos nx\|_{H^s} \approx 1,$$  

for $n$ sufficiently large. Hence, by Theorem 1.1, there is a $T > 0$ such that for $n > 1$, the Cauchy problem (31)-(32) has a unique solution in $C([0,T] : H^s)$ with lifespan $T > 0$ such that $u(x,t)$ satisfies (6) for $t \in [0,T]$.

To estimate the difference between the actual solutions and the approximate solutions, we define $v = u^{\omega,n} - u_{\omega,n}$ which satisfies the following i.v.p.

$$\partial_t v = E - \frac{1}{k+1} \left[ \partial_x (u^{\omega,n})^{k+1} - \partial_x (u_{\omega,n})^{k+1} \right]$$

$$+ \frac{k}{2} \partial_x^{-1} \left[ (u^{\omega,n})^{k-1}(u_{\omega,n})^2 - (u_{\omega,n})^{k-1}(\partial_x u_{\omega,n}) \right],$$

$$v(x,0) = 0,$$  

For ease of notation, we rewrite the i.v.p. as

$$\partial_t v = E - \frac{1}{k+1} \partial_x [w_1 v] + \frac{k}{2} \partial_x^{-1} [w_2 v + w_3 v],$$

$$v(x,0) = 0,$$  

where

$$w_1 = (u^{\omega,n})^k + (u^{\omega,n})^{k-1} u_{\omega,n} + \cdots + u^{\omega,n} (u_{\omega,n})^{k-1} + (u_{\omega,n})^k,$$  

$$w_2 = \partial_x u^{\omega,n} \left[ (u^{\omega,n})^{k-2} + (u^{\omega,n})^{k-3} u_{\omega,n} + \cdots + u^{\omega,n} (u_{\omega,n})^{k-3} + (u_{\omega,n})^{k-2} \right],$$  

$$w_3 = (u^{\omega,n})^{k-1} (\partial_x u_{\omega,n} + \partial_x u_{\omega,n}),$$  

and $E$ satisfies the estimate in (24).

We will now show that the $H^s$ norm of the difference $v$ decays to zero as $n$ goes to infinity.

**Lemma 2.2.** The $H^s$ norm of the difference $v$ can be estimated by

$$\|v(t)\|_{H^s} \lesssim n^{1-2s+\sigma}.$$  

**Proof.** Apply the operator $D^\sigma$ to both sides of (34), multiply by $D^\sigma v$, and integrate over the torus to get

$$\frac{1}{2} \frac{d}{dt} \|v(t)\|^2_{H^s} = \int_T \langle D^\sigma v, D^\sigma v \rangle dx - \frac{1}{k+1} \int_T D^\sigma \partial_x (w_1 v) D^\sigma vdx$$

$$+ \frac{k}{2} \int_T D^\sigma \partial_x^{-1} (w_2 v + w_3 v) D^\sigma vdx.$$  

We now estimate the three integrals in the right-hand side of (36).
Estimate of the first integral. Applying the Cauchy-Schwarz inequality, we have

\[ \left| \int T D^\sigma E D^\sigma v dx \right| \leq \| E \|_{H^\sigma} \| v \|_{H^\sigma}. \]  

(37)

Estimate of the second integral. We begin by rewriting this term by commuting \( v \) with \( D^\sigma \partial_x \) to arrive at

\[ \int T D^\sigma \partial_x (w_1 v) D^\sigma v dx = \int T [D^\sigma \partial_x, w_1] v D^\sigma v dx + \int T w_1 D^\sigma \partial_x v D^\sigma v dx. \]  

(38)

The first integral can be handled by the following Calderon-Coifman-Meyer type commutator estimate that can be found in [40].

Lemma 2.3. If \( \sigma + 1 \geq 0 \), then

\[ \| [D^\sigma \partial_x, w] v \|_{L^2} \leq C \| w \|_{H^\sigma} \| v \|_{H^\sigma}, \]  

(39)

provided that \( \rho > \frac{3}{2} \) and \( \sigma + 1 \leq \rho \).

Applying Lemma 2.3 with \( \sigma + 1 \geq 0, \rho = s > \frac{3}{2} \) and \( \sigma + 1 \leq s \) tells us

\[ \left| \int T [D^\sigma \partial_x, w_1] v D^\sigma v dx \right| \lesssim \| w_1 \|_{H^\sigma} \| v \|_{H^\sigma}^2. \]  

(40)

Integrating by parts and using the Sobolev Theorem, we have

\[ \left| \int T w_1 D^\sigma \partial_x v D^\sigma v dx \right| = \frac{1}{2} \left| \int T (D^\sigma v)^2 \partial_x w_1 dx \right| \lesssim \| \partial_x w_1 \|_{L^\infty} \| v \|_{H^\sigma}^2 \lesssim \| w_1 \|_{H^\sigma} \| v \|_{H^\sigma}^2. \]  

(41)

We obtain

\[ \int T D^\sigma \partial_x (w_1 v) D^\sigma v dx \lesssim \| w_1 \|_{H^\sigma} \| v \|_{H^\sigma}^2. \]  

(42)

Estimate of the third integral. For the third term, we observe that after applying the Cauchy-Schwarz inequality, the quantity \( w_2 v + w_3 \partial_x v \) will be in the \( H^{\sigma-1} \) space, which precludes use of the algebra property. To overcome this obstacle, we will apply the following multiplier estimate also found in [40].

Lemma 2.4. Let \( \sigma \in \left( \frac{1}{2}, 1 \right) \) then

\[ \| fg \|_{H^{\sigma-1}} \lesssim \| f \|_{H^{\sigma-1}} \| g \|_{H^\sigma}. \]  

(43)

Applying Lemma 2.4, we obtain

\[ \int T D^\sigma \partial_x^{-1} (w_2 v + w_3 \partial_x v) D^\sigma v dx \leq \| \partial_x^{-1} (w_2 v + w_3 \partial_x v) \|_{H^\sigma} \| v \|_{H^\sigma} \]

\[ \leq \| \partial_x^{-1} (w_2 v) \|_{H^\sigma} \| v \|_{H^\sigma} + \| \partial_x^{-1} (w_3 \partial_x v) \|_{H^\sigma} \| v \|_{H^\sigma} \]

\[ \lesssim \| w_2 v \|_{H^{\sigma-1}} \| v \|_{H^\sigma} + \| w_3 \|_{H^{\sigma-1}} \| v \|_{H^\sigma} \]

\[ \lesssim \| w_2 \|_{H^{\sigma-1}} \| v \|_{H^\sigma}^2 + \| w_3 \|_{H^\sigma} \| v \|_{H^\sigma}^2. \]  

(44)

Finally, combining (37), (42) and (44), we obtain an energy estimate for \( v \)

\[ \frac{1}{2} \frac{d}{dt} \| v(t) \|_{H^\sigma} \]

\[ \lesssim \| E \|_{H^\sigma} \| v \|_{H^\sigma} + \| w_1 \|_{H^\sigma} \| v \|_{H^\sigma}^2 + \| w_2 \|_{H^{\sigma-1}} \| v \|_{H^\sigma}^2 + \| w_3 \|_{H^\sigma} \| v \|_{H^\sigma}^2. \]  

(45)
We shall show that \( \|w_1\|_{H^s} \lesssim 1, \|w_2\|_{H^{s-1}} \lesssim 1 \) and \( \|w_3\|_{H^s} \lesssim 1 \). From (26), we have
\[
\|w^{\omega,n}(x,t)\|_{H^s} = |\omega n^{-\frac{1}{2}} + n^{-s} \cos(nx - \omega t)|_{H^s} \lesssim n^{-\frac{1}{2}} + 1 \lesssim 1. \tag{46}
\]
The actual solutions \( u_{\omega,n}(x,t) \) are bounded by the solution size estimate in Theorem 1.1
\[
\|u_{\omega,n}(x,t)\|_{H^s} \lesssim \|u_{\omega,n}(x,0)\|_{H^s} \lesssim n^{-\frac{1}{2}} + 1 \lesssim 1. \tag{47}
\]
Thus, we have
\[
\|w_1\|_{H^s} \lesssim \|u_{\omega,n}\|_{H^s}^k + \|u_{\omega,n}\|_{H^s}^{k-1} \|u_{\omega,n}\|_{H^s} + \cdots + \|u_{\omega,n}\|_{H^s} + \|v\|_{H^s} \lesssim 1. \tag{48}
\]
Similarly, we have
\[
\|w_2\|_{H^{s-1}} \lesssim 1, \quad \|w_3\|_{H^s} \lesssim 1. \tag{49}
\]
We can refine (45) by using (47)-(49) to write
\[
\frac{1}{2} \frac{d}{dt} \|v(t)\|_{H^s}^2 \lesssim \|E\|_{H^s} \|v\|_{H^s} + \|v\|_{H^s}^2. \tag{50}
\]
Solving (49) and using the error estimate (35), we arrive at the desired estimate (35).

The examination of the \( H^{s+1} \) norm of the difference \( v \) is summarized in the next lemma.

**Lemma 2.5.** Let \( r > s \). The \( H^r \) norm of the difference \( v \) can be estimated by
\[
\|v(t)\|_{H^r} \lesssim n^{r-s}. \tag{51}
\]

**Proof.** From the definition of \( v \) and the triangle inequality, we have
\[
\|v(t)\|_{H^r} = \|u_{\omega,n} - u_{\omega,n}\|_{H^r} \leq \|u_{\omega,n}\|_{H^r} + \|u_{\omega,n}\|_{H^r}. \tag{52}
\]
The bound on the \( H^r \)-norm of the approximate solution is achieved by a straightforward calculation. We have
\[
\|u_{\omega,n}\|_{H^r} \leq n^{-\frac{1}{2}} + n^{-s} \|\cos(nx - \omega t)\|_{H^r} \lesssim n^{r-s}. \tag{53}
\]
From our hypothesis, we have \( r > \frac{3}{2} \). As our initial data is smooth and therefore in \( H^r \), we may appeal to estimate (6) to bound the \( H^r \) norm of the actual solution. Now using the fact that \( u_{\omega,n(0)} = u_{\omega,n} \), we have
\[
\|u_{\omega,n}\|_{H^r} \lesssim n^{r-s}. \tag{54}
\]
Combining (53) and (54) yields the desired bound.

Next, using interpolation between \( \sigma \) and \( s+1 \) we show that the \( H^\sigma \)-norm of \( v(t) \) decays. In fact, using (35) and (51) we can bound the \( H^\sigma \) norm of \( v \) as follows
\[
\|v\|_{H^\sigma} \leq \|v\|_{H^r}^{\frac{\sigma}{r}} \|v\|_{H^{s+1}}^{\frac{r-\sigma}{r}} \lesssim (n^{1-2s+\sigma})^{\frac{\sigma}{r}} \cdot n^{\frac{s+1}{s+1}} \lesssim n^{-\varepsilon}. \tag{55}
\]
where have the exponent \( \varepsilon > 0 \) given by
\[
\varepsilon = \frac{s-1}{s+1-\sigma} > 0. \tag{56}
\]

**Proof of non-uniform dependence.** Here we will prove Theorem 1.2 for Sobolev exponents \( s > \frac{3}{2} \). The basis of our proof rests upon finding two sequences of
solutions to the MHS i.v.p. (1)-(2) that share a common lifespan and satisfy three conditions. For $k$-odd, we take the sequence of solutions with $\omega = \pm 1$ and for $k$-even the sequence of solutions with $\omega = 0, 1$. The three conditions they satisfy are as follows

\begin{align}
(1) \quad \|u_{\omega, n}(t)\|_{H^s} & \lesssim 1 \quad \text{for} \quad t \in [0, T], \\
(2) \quad \|u_{1,n}(0) - u_{-1,n}(0)\|_{H^s} & \to 0 \quad \text{as} \quad n \to \infty \quad \text{for} \quad k \quad \text{-} \text{odd}, \\
\|u_{1,n}(0) - u_{0,n}(0)\|_{H^s} & \to 0 \quad \text{as} \quad n \to \infty \quad \text{for} \quad k \quad \text{-} \text{even}, \\
(3) \quad \lim_{n \to \infty} \|u_{1,n}(t) - u_{-1,n}(t)\|_{H^s} & \gtrsim \|\sin t\| \quad \text{for} \quad t \in (0, T) \quad \text{for} \quad k \quad \text{-} \text{odd}, \\
\lim_{n \to \infty} \|u_{1,n}(t) - u_{0,n}(t)\|_{H^s} & \gtrsim \left|\sin \frac{t}{2}\right| \quad \text{for} \quad t \in (0, T) \quad \text{for} \quad k \quad \text{-} \text{even}.
\end{align}

Property (1), for $k$-even or odd follows from the solution size estimate in Theorem 1.1. We have

$$
\|u_{\omega, n}(t)\|_{H^s} \lesssim \|u_{\omega, n}(0)\|_{H^s} \lesssim 1. \quad (57)
$$

Property (2), for $k$-odd, follows from the definition of our approximate solutions (19). We have

$$
\|u_{1,n}(0) - u_{-1,n}(0)\|_{H^s} = \|u^{1,n}(0) - u^{-1,n}(0)\|_{H^s} \\
= \|n^{-\frac{1}{4}} + n^{-1} \cos nx + n^{-\frac{1}{4}} - n^{-1} \cos nx\|_{H^s} \\
= 4\pi n^{-\frac{1}{2}} \to 0 \quad \text{as} \quad n \to \infty.
$$

Similarly, for $k$-even we have

$$
\|u_{1,n}(0) - u_{0,n}(0)\|_{H^s} = \|u^{1,n}(0) - u^{0,n}(0)\|_{H^s} \\
= \|n^{-\frac{1}{4}} + n^{-1} \cos nx - n^{-\frac{1}{4}} \cos nx\|_{H^s} \\
= 2\pi n^{-\frac{1}{2}} \to 0 \quad \text{as} \quad n \to \infty.
$$

For property (3), we consider $k$-odd first. Using the reverse triangle inequality we get

$$
\|u_{1,n}(t) - u_{-1,n}(t)\|_{H^s} \gtrsim \|u^{1,n}(t) - u^{-1,n}(t)\|_{H^s} - \|u^{1,n}(t) - u_{1,n}(t)\|_{H^s} \\
- \|u^{-1,n}(t) - u_{-1,n}(t)\|_{H^s} \\
\gtrsim \|u^{1,n}(t) - u^{-1,n}(t)\|_{H^s} - n^{-\varepsilon},
$$

from which we obtain that

$$
\lim_{n \to \infty} \|u_{1,n}(t) - u_{-1,n}(t)\|_{H^s} \gtrsim \lim_{n \to \infty} \|u^{1,n}(t) - u^{-1,n}(t)\|_{H^s}. \quad (58)
$$

Since, by the trigonometric identity $\cos(\alpha - \beta) - \cos(\alpha + \beta) = 2 \sin \alpha \sin \beta$, we have

$$
u^{1,n}(t) - u^{-1,n}(t) = 2n^{-\frac{1}{2}} + 2n^{-1} \sin nx \sin t,
$$

inequality (58) gives

$$
\lim_{n \to \infty} \|u_{1,n}(t) - u_{-1,n}(t)\|_{H^s} \gtrsim \lim_{n \to \infty} \|\sin t| - n^{-\frac{1}{2}}\| \gtrsim |\sin t| \quad (59)
$$

which completes the proof of property (3) in the case that $k$ is odd. \qed
We now consider $k$-even. Similarly, using the reverse triangle inequality, the definition of approximate solutions, and the fact that the difference between solutions and approximate solutions decays, we get
\[
\liminf_{n \to \infty} \|u_{1,n}(t) - u_{0,n}(t)\|_{H^s} \geq \liminf_{n \to \infty} \|u^{1,n}(t) - u^{0,n}(t)\|_{H^s},
\]
where
\[
u^{1,n}(t) - u^{0,n}(t) = n^{-\frac{1}{2}} + 2n^{-s} \sin(nx) - \frac{t}{2} \sin\frac{t}{2}.
\]
Therefore (60) gives
\[
\liminf_{n \to \infty} \|u_{1,n}(t) - u_{0,n}(t)\|_{H^s} \geq \liminf_{n \to \infty} (|\sin\frac{t}{2}| - n^{-\frac{1}{2}}) \geq |\sin\frac{t}{2}|,
\]
which completes the proof of Theorem 1.2.

3. Proof of Theorem 1.1. We observe that for $u \in H^s$ the product $u^k \partial_x u$ is in $H^{s-1}$. Thus the MHS equation (as is) can not be thought as an ODE on the space $H^s$. To deal with this problem we replace the MHS i.v.p. (1)-(2) by a mollified version
\[
\partial_t u = -J_\epsilon [(J_\epsilon u)^k \partial_x J_\epsilon u] + \frac{1}{2} \partial_x^{-1} (\partial_x (u^k \partial_x u)) = F_\epsilon,
\]
where for each $\epsilon \in (0,1]$, the operator $J_\epsilon$ is the Friedrichs mollifier. We fix a Schwartz function $j \in \mathcal{S}(\mathbb{R})$ that satisfies $0 \leq \hat{j}(\xi) \leq 1$ for every $\xi \in \mathbb{R}$ and $\hat{j}(\xi)$ for $\xi \in [-1,1]$ This allows us to define the periodic functions $j_\epsilon$, as
\[
j_\epsilon(x) = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \hat{j}(en) e^{inx}.
\]
Then $J_\epsilon$ is defined by
\[
J_\epsilon f(x) = j_\epsilon f(x).
\]
This construction of $j_\epsilon$ results in two lemmas that will prove repeatedly useful throughout the paper.

**Lemma 3.1.** [47] Let $s > 0$ and $J_\epsilon$ be defined as in (65). Then for any $f \in H^s$, we have $J_\epsilon f \to f$ in $H^s$.

**Lemma 3.2.** [47] Let $r < s$, the map $I - J_\epsilon : H^s \to H^r$ and $J_\epsilon$ be defined as in (65). Then for any $f \in H^s$, we have $J_\epsilon f \to f$ in $H^r$. satisfies the operator norm estimate
\[
\|I - J_\epsilon\|_{\mathcal{L}(H^s, H^r)} = o(\epsilon^{s-r}).
\]

Hence, for each $\epsilon \in (0,1]$, (62)-(63) has a unique solution $u_\epsilon$ with lifespan $T_\epsilon > 0$.

Our strategy is now to demonstrate that the Cauchy problem (62)-(63) satisfies the hypotheses of the Fundamental ODE Theorem. We will therefore obtain a unique solution $u_\epsilon(\cdot, t) \in H^s, |t| < T_\epsilon$, for some $T_\epsilon > 0$. This idea is summarized in the following lemma.

**Lemma 3.3.** For all $\epsilon \in (0,1]$, the mollified i.v.p. (62)-(63) has a unique solution $u_\epsilon \in C([0, T]; H^s)$ with lifespan $T_\epsilon > 0$.

**Proof.** The map $F_\epsilon : H^s \to H^s$ is well-defined. The only remaining hypothesis of the Fundamental ODE Theorem, we need to show is satisfied is that $F_\epsilon$ is a
continuously differentiable map. In this case, derivative of \( F \) can be explicitly calculated at each \( u_0 \in H^s \) as
\[
F'(u_0)u = -J_\epsilon \left[ k(J_\epsilon u_0)^{k-1}J_\epsilon u \cdot \partial_x J_\epsilon u_0 + (J_\epsilon u_0)^k \cdot \partial_x J_\epsilon u \right] \\
+ \frac{k(k-1)}{2} \partial_x^{-1} \left( u_0^{k-2}u(\partial_x u_0)^2 \right) + \frac{k}{2} \partial_x^{-1} \left( u_0^{k-1}\partial_x u_0 \partial_x u \right).
\]
Hence, for each \( \epsilon \in (0,1] \), (62)-(63) has a unique solution \( u_\epsilon \) with lifespan \( T_\epsilon > 0 \).

### 3.1. Energy estimate and lifespan of solution \( u_\epsilon \)

For each \( \epsilon \), there is a solution \( u_\epsilon \) to the mollified MHS (62)-(63). The lifespan of each of these solutions has a lower bound \( T_\epsilon \). In this subsection, we shall demonstrate that there is actually a lower bound \( T_\epsilon > 0 \) that does not depend upon \( \epsilon \). This estimate is crucial in our proofs. To show the existence of \( T_\epsilon \), we shall derive an energy estimate for the \( u_\epsilon \). Applying the operator \( D^s \) to both sides of (62), multiplying by \( D^s u_\epsilon \), and integrating over the torus yields the \( H^s \)-energy of \( u_\epsilon \)
\[
\frac{d}{dt} \| u_\epsilon(t) \|_{H^s}^2 = -\int_T D^s \left[ J_\epsilon ((J_\epsilon u_\epsilon)^k \partial_x J_\epsilon u_\epsilon) \right] D^s u_\epsilon \, dx \\
+ \frac{1}{2} \int_T D^s \partial_x^{-1} \left( \partial_x (u_\epsilon^k) \partial_x u_\epsilon \right) \cdot D^s u_\epsilon \, dx.
\]
(67)

To bound the energy, we will need the following Kato-Ponce [53] commutator estimate.

**Lemma 3.4 (Kato-Ponce).** If \( s > 0 \) then there is \( c_s > 0 \) such that
\[
\| [D^s, f] g \|_{L^2} \leq c_s (\| D^s f \|_{L^2} \| g \|_{L^\infty} + \| \partial_x g \|_{L^\infty} \| D^{s-1} f \|_{L^2}).
\]
(68)

We now rewrite the first term of (67) by first commuting the exterior \( J_\epsilon \) and then commuting the operator \( D^s \) with \((J_\epsilon u_\epsilon)^k\) arriving at
\[
\int_T D^s [(J_\epsilon u_\epsilon)^k \partial_x J_\epsilon u_\epsilon] D^s J_\epsilon u_\epsilon \, dx = \int_T [D^s, (J_\epsilon u_\epsilon)^k] \partial_x J_\epsilon u_\epsilon D^s J_\epsilon u_\epsilon \, dx \\
+ \int_T (J_\epsilon u_\epsilon)^k \partial_x D^s J_\epsilon u_\epsilon D^s J_\epsilon u_\epsilon \, dx.
\]
(69)

Setting \( v = J_\epsilon u_\epsilon \), we can bound the first term of (69) by first using the Cauchy-Schwarz inequality and then applying lemma \( 3.4 \) to arrive at
\[
\int_T [D^s, v^k] \partial_x v D^s v \, dx \lesssim \left( \| v^k \|_{H^s} \| \partial_x v \|_{L^\infty} + \| \partial_x v \|_{H^{s-1}} \| \partial_x v^k \|_{L^\infty} \right) \| v \|_{H^s}.
\]
\[
\lesssim \| v \|_{C^1} \| v \|_{H^{s+1}}^{k+1} + \| v \|_{C^1}^k \| v \|_{H^{s}}^2.
\]
(70)

For the second term of (69), we have
\[
\int_T v^k \partial_x D^s v D^s v \, dx = -\frac{1}{2} \int_T \partial_x (v^k) (D^s v)^2 \, dx \\
\lesssim \sup |\partial_x (v^k)| \int_T (D^s v)^2 \, dx \\
\lesssim \| v \|_{C^1} \| v \|_{H^{s}}^2.
\]
(71)

The second term on the right hand side of (67) is bounded by first applying the Cauchy-Schwarz inequality and then using of the operator norm of \( \partial_x^{-1} \) and the
algebra property of $H^s$. Here we have

$$\int_T D^s \partial_x^{-1} ( \partial_x (u^k_\epsilon) \partial_x u_\epsilon ) \cdot D^s u_\epsilon \, dx \leq \| \partial_x^{-1} ( \partial_x (u^k_\epsilon) \partial_x u_\epsilon ) \|_{H^s} \| u_\epsilon \|_{H^s}$$

$$\leq \| ( \partial_x (u^k_\epsilon) \partial_x u_\epsilon ) \|_{H^{s-1}} \| u_\epsilon \|_{H^s}$$

$$\lesssim \| u_\epsilon \|_{C^1} \| u_\epsilon \|_{H^s}.$$  \hspace{1cm} (72)

Using the fact that

$$\| J_\epsilon u_\epsilon \|_{H^s} \leq \| u_\epsilon \|_{H^s}.$$  \hspace{1cm} (73)

and combining these results we conclude that $u_\epsilon$ satisfies the differential inequality

$$\frac{1}{2} \frac{d}{dt} \| u_\epsilon (t) \|_{H^s}^2 \leq \| u_\epsilon (t) \|_{C^1} \| u_\epsilon (t) \|_{H^s}.$$  \hspace{1cm} (74)

This together with Sobolev’s lemma gives

$$\frac{1}{2} \frac{d}{dt} \| u_\epsilon (t) \|_{H^s}^2 \leq c_s \| u_\epsilon (t) \|_{H^s}^{k+2}. \hspace{1cm} (75)$$

Set $y = \| u_\epsilon (t) \|_{H^s}^2$, the differential inequality (75)

$$\frac{1}{2} y^{\frac{k+2}{k}} \frac{dy}{dt} \leq c_s, \hspace{1cm} y(0) = \| u_0 \|_{H^s}^2.$$  \hspace{1cm} (76)

Integrating (76) from 0 to $t$ gives

$$\frac{1}{u_\epsilon^2(0)} - \frac{1}{u_\epsilon^2(t)} \leq c_s k t.$$  \hspace{1cm} (77)

Solving for $y(t)$ and substituting back in $y = \| u_\epsilon (t) \|_{H^s}^2$, gives us the inequality

$$\| u_\epsilon (t) \|_{H^s} \leq \frac{\| u_0 \|_{H^s}}{(1 - c_s k \| u_0 \|_{H^s}^2 t)^{\frac{1}{k}}}.$$  \hspace{1cm} (78)

Setting $T = \frac{1}{2 c_s k \| u_0 \|_{H^s}^2}$, we see from (78) that the solution $u_\epsilon$ exists for $0 \leq t \leq T$ and satisfies a solution size bound

$$\| u_\epsilon (t) \|_{H^s} \leq 2 \| u_0 \|_{H^s}, \hspace{1cm} 0 \leq t \leq T.$$  \hspace{1cm} (79)

Therefore we see that $T$ is a lower bound for the lifespan of $u_\epsilon$ independent of $\epsilon$.

3.2. Existence

**Proposition 3.1.** There exists a solution $u$ to the MHS i.v.p. (1)-(2) in the space $L^\infty([0, 1]; H^s) \cap \text{Lip}([0, 1]; H^{s-1})$. Furthermore, the $H^s$ norm of $u$ satisfies (1.6)

**Proof.** Our proof revolves around refining the convergence of the family $\{ u_{\epsilon} \}$ several times by extracting subsequences $\{ u_{\epsilon} \}$. After each such extraction, it is assumed that the resulting subsequence is relabelled as $\{ u_{\epsilon} \}$.

**Weak* convergence in $L^\infty([0, T]; H^s)$.** The family $\{ u_{\epsilon} \}_{\epsilon \in (0, 1]}$ is bounded in the space $C([0, T]; H^s) \subset L^\infty([0, T]; H^s)$. By the inequality (83), we have

$$\| u_{\epsilon} \|_{L^\infty([0, 1], H^s)} = \sup_{t \in [0, T]} \| u_{\epsilon} (t) \|_{H^s} \leq 2 \| u_0 \|_{H^s}, \hspace{1cm} (80)$$

$$\Rightarrow \{ u_{\epsilon} \}_{\epsilon \in (0, 1]} \subset \overline{B}(0, 2 \| u_0 \|_{H^s}) \subset L^\infty([0, T]; H^s).$$
where we define the duality relation for \( \varphi \) by
\[
\langle u, \varphi \rangle = \int_0^T \sum_{k \in \mathbb{Z}} (1 + k^2)^s \hat{u}_k(t, k) \hat{\varphi}(t, k) dt.
\] (81)

We may thus apply Alaoglu’s theorem deduce that \( \{ u \} \) is be precompact in \( \mathcal{B} (0, 2\|u_0\|_{H^s}) \subset L^\infty([0, T]; H^s) \) with respect to the weak* topology. Therefore we may extract a subsequence \( \{ u_{\epsilon_n} \} \) that converges to an element \( u \in \mathcal{B}(0, 2\|u_0\|_{H^s}) \) weakly*.

**Strong convergence in** \( C([0, T]; H^{s-1}) \). We will prove that the family \( \{ u_{\epsilon} \}_{\epsilon \in (0, 1]} \) satisfies the hypotheses of Ascoli’s Theorem. We begin with the equicontinuity condition. For \( t_1, t_2 \in [0, T] \), we have
\[
\| u_{\epsilon}(t_1) - u_{\epsilon}(t_2) \|_{H^{s-1}} \leq \sup_{t \in [0, T]} \| \partial_t u_{\epsilon}(t) \|_{H^{s-1}} |t_1 - t_2|.
\] (82)

Using the fact that \( u_{\epsilon} \) satisfies (62)-(63), we see that the \( H^{s-1} \) norm of \( \partial_t u_{\epsilon} \) can be bounded independently of \( \epsilon \) as follows
\[
\sup_{t \in [0, T]} \| \partial_t u_{\epsilon}(t) \|_{H^{s-1}} = \sup_{t \in [0, T]} \| J_{\epsilon}[(J_{\epsilon}u)^k \partial_x J_{\epsilon}u] + \frac{1}{2} \partial_x^{-1} (\partial_x (u^k) \partial_x u) \|_{H^{s-1}}
\] \[ \lesssim \sup_{t \in [0, T]} (\| J_{\epsilon}[(J_{\epsilon}u)^k \partial_x J_{\epsilon}u] \|_{H^{s-1}} + \| \partial_x (u^k) \partial_x u \|_{H^{s-2}})
\] \[ \lesssim \| u_0 \|_{H^{k+1}}.
\] (83)

Next, we observe that for each \( t \in [0, T] \) the set \( U(t) = \{ u_{\epsilon}, \epsilon \in (0, 1] \} \) is bounded in \( H^s \). Since \( T \) is a compact manifold, the inclusion mapping \( i : H^s \to H^{s-1} \) is a compact operator, and therefore we may deduce that \( U(t) \) is a precompact set in \( H^{s-1} \). As the two hypotheses of Ascoli’s Theorem have been satisfied, we have a subsequence \( \{ u_{\epsilon_n} \} \) that converges in \( ([0, T]; H^{s-1}) \). By uniqueness of limits, this subsequence must converge to \( u \).

**Strong convergence in** \( C([0, T]; H^{s-\sigma}) \) for \( \sigma \in (0, 1] \). As in the previous case, we will prove that the family \( \{ u_{\epsilon} \} \) satisfies the hypotheses of Ascoli’s Theorem. To establish the equicontinuity condition, we see that for \( t_1, t_2 \in [0, T] \) that we have
\[
\| u_{\epsilon}(t_1) - u_{\epsilon}(t_2) \|_{H^{s-\sigma}} \leq \| u_{\epsilon} \|_{C^\sigma([0, T]; H^{s-\sigma})} |t_1 - t_2|^{\sigma}.
\] (84)

Our objective therefore is to bound \( \| u_{\epsilon} \|_{C^\sigma([0, T]; H^{s-\sigma})} \) independently of \( \epsilon \). We begin with the definition of this norm as
\[
\| u_{\epsilon} \|_{C^\sigma([0, T]; H^{s-\sigma})} \equiv \sup_{t \in [0, T]} \| u_{\epsilon}(t) \|_{H^{s-\sigma}} + \sup_{t \neq t'} \frac{\| u_{\epsilon}(t) - u_{\epsilon}(t') \|_{H^{s-\sigma}}}{|t_1 - t_2|^{\sigma}}.
\] (85)

The first term on the right hand side of (85) is bounded by application of (79), giving us
\[
\sup_{t \in [0, T]} \| u_{\epsilon}(t) \|_{H^{s-\sigma}} \leq \sup_{t \in [0, T]} \| u_{\epsilon}(t) \|_{H^{s-\sigma}} \leq 2\| u_0 \|_{H^s}.
\] (86)

For the second term, we begin with two elementary inequalities. First, as \( \sigma \in (0, 1) \) we have
\[
\frac{1}{(1 + k^2)\sigma|t - t'|^2} \leq \left( 1 + \frac{1}{(1 + k^2)|t - t'|^2} \right)^\sigma \leq 1 + \frac{1}{(1 + k^2)|t - t'|^2}.
\] (87)
Using this inequality, we may further deduce that
\[
\frac{(1 + k^2)^{s-\sigma}}{|t - t'|^{2\sigma}} \leq \frac{(1 + k^2)^s}{|t - t'|^{2\sigma}} + \frac{(1 + k^2)^{s-\sigma}}{|t - t'|^{2\sigma}}. \tag{88}
\]
We therefore can therefore bound this term by
\[
\sup_{t \neq t'} \frac{\|u_\epsilon(t) - u_\epsilon(t')\|_{H^{s-\sigma}}}{|t_1 - t_2|^{2\sigma}} = \sup_{t \neq t'} \sum_{k \in \mathbb{Z}} \frac{(1 + k^2)^{s-\sigma}}{|t - t'|^{2\sigma}} |\tilde{u}_\epsilon(k, t) - \tilde{u}_\epsilon(k, t')|^2
\leq \sup_{t \neq t'} \sum_{k \in \mathbb{Z}} 1 + k^2)^{s-\sigma} |\tilde{u}_\epsilon(k, t) - \tilde{u}_\epsilon(k, t')|^2
\leq \sup_{t \in [0, T]} \|u_\epsilon(t)\|_{H^s} + \sup_{t \in [0, T]} \|\partial_x u_\epsilon(t)\|_{H^{s-1}}
\leq \|u_0\|_{H^s} + \|u_0\|_{H^{s-1}}^{2(k+1)}. \tag{89}
\]
Combining (86) and (89), we have the \(\epsilon\) independent bound
\[
\|u_\epsilon\|_{C([0, T]; H^{s-\sigma})} \lesssim \|u_0\|_{H^s} + \|u_0\|_{H^{s-1}}^{k+1}. \tag{90}
\]
The precompactness condition is established in exactly the same fashion as the previous case as the inclusion mapping of \(H^s\) into \(H^{s-\sigma}\) is a compact operator. As the two hypotheses of Ascoli have been satisfied, we may extract a subsequence that converges to \(u\) in \(C([0, T]; H^{s-\sigma})\).

**Strong convergence in** \(C([0, T]; C^1(\mathbb{T}))\). We will now \(\sigma \in (0, 1)\) so that \(s - \sigma > \frac{3}{2}\). The Sobolev lemma then tells us that \(H^{s-\sigma}\) embeds continuously into \(C^1(\mathbb{T})\), which therefore implies that \(u_\epsilon \to u\) in \(C([0, T]; C^1(\mathbb{T}))\). We will next prove that \(\partial_t u_\epsilon \to -w^k \partial_x u + \frac{1}{2} \partial_x^{-1}(\partial_x(u^k) \partial_x u)\) in \(C([0, T]; C(\mathbb{T}))\).

**Strong convergence of** \(\partial_t u_\epsilon\) **in** \(C([0, T]; C^1(\mathbb{T}))\). From (62) we have
\[
\partial_t u_\epsilon = -J_r(\partial_x u_\epsilon^k) \partial_x J_r u_\epsilon + \frac{1}{2} \partial_x^{-1}(\partial_x(u_\epsilon^k) \partial_x u_\epsilon). \tag{91}
\]
As we have already established that \(u_\epsilon \to u\) in \(C([0, T]; C^1(\mathbb{T}))\) it follows that \(\partial_t u_\epsilon \to \partial_t u\). Using the fact that this space is an algebra, then continuity of \(\partial_x^{-1}\) implies the convergence of the nonlocal term
\[
\frac{1}{2} \partial_x^{-1}(\partial_x(u_\epsilon^k) \partial_x u_\epsilon) \to \frac{1}{2} \partial_x^{-1}(\partial_x(u^k) \partial_x u) \text{ in } C([0, T]; C(\mathbb{T})). \tag{92}
\]
Next, we observe that \(J_r u_\epsilon \to u\) in \(C([0, T]; C(\mathbb{T}))\) as
\[
\|J_r u_\epsilon - u\|_{C([0, T]; C(\mathbb{T}))} = \|J_r u_\epsilon \pm u_\epsilon - u\|_{C([0, T]; C(\mathbb{T}))}
\leq \|J_r u_\epsilon - u_\epsilon\|_{C([0, T], C(\mathbb{T}))} + \|u_\epsilon - u\|_{C([0, T], C(\mathbb{T}))}. \tag{93}
\]
For the first term of this sum, choose \(r\) with \(\frac{1}{2} < r < s\). Then applying Lemma 3.2, we see that for \(t \in [0, T]\)
\[
\|J_r u_\epsilon(t) - u(t)\|_{C(\mathbb{T})} \lesssim \|J_r u_\epsilon(t) \pm u_\epsilon(t) - u(t)\|_{H^s}
\lesssim \|I - J_r\|_{\mathcal{L}(H^s, H^s)}
= o(\epsilon^{s-r})\|u_\epsilon\|_{H^s}. \tag{94}
\]
Estimating the second term immediately follows from the fact that we have established \(u_\epsilon \to u\) in \(C([0, T]; C(\mathbb{T}))\) We then examine \(\partial_x\) as above, and similarly conclude...
that \( J_{\epsilon} \partial u_{\epsilon} \rightarrow \partial u \) in \( C([0, T]; C(\mathbb{T})) \). Finally, proceeding via additive and multiplicative properties of limits we may deduce that \( \partial u_{\epsilon} \rightarrow -u^k \partial x u + \frac{1}{2} \partial x^{-1} (\partial x (u^k) \partial x u) \) in \( C([0, T]; C(\mathbb{T})) \).

We are now ready to complete the proof of Proposition 3.1. We have established that in the space \( C([0, T]; C(\mathbb{T})) \) we have \( u_{\epsilon} \rightarrow u \) in \( C([0, T]; C(\mathbb{T})) \) and \( \partial u_{\epsilon} \rightarrow -u^k \partial x u + \frac{1}{2} \partial x^{-1} (\partial x (u^k) \partial x u) \) in \( C([0, T]; C(\mathbb{T})) \). Therefore we have that \( u \mapsto u(t) \) is differentiable map from \([0, T] \rightarrow C(\mathbb{T})\) with \( \partial u \rightarrow -u^k \partial x u + \frac{1}{2} \partial x^{-1} (\partial x (u^k) \partial x u) \) in \( C([0, T]; C(\mathbb{T})) \). Thus \( u \in L^\infty([0, T]; H^s) \) is a solution to (1)-(2). With this information, we can now establish that \( u \in L^\infty([0, 1]; H^s) \cap Lip([0, 1]; H^{s-1}) \) as we have

\[
\|u_t - u(t_2)\|^2_{H^{s-1}} \sup_t \|\partial_x u\|^2_{H^{s-1}} |t_1 - t_2| \lesssim \|u_0\|_{H^{s+1}} |t_1 - t_2|. \tag{95}
\]

Now that we have established the existence of a solution \( u \) we will improve the conclusions we have established about its regularity. \( \square \)

**Proposition 3.2.** The solution \( u \) to the MHS i.v.p. constructed in Proposition 3.1 is an element of the space \( C([0, T]; H^s) \).

**Proof.** Fix \( t \in [0, T] \) and take a sequence \( t_n \rightarrow t \). For the solution \( u \) to be continuous at \( t \), we must have \( \|u_{t_n} - u(t)\|_{H^s} \rightarrow 0 \). This property is equivalent to \( \|u_{t_n} - u(t)\|^2_{H^s} \rightarrow 0 \). Using the definition of the norm, we have

\[
\|u_{t_n} - u(t)\|^2_{H^s} = \langle u_{t_n} - u(t), u_{t_n} - u(t) \rangle_{H^s} = \|u_{t_n}\|^2_{H^s} + \|u(t)\|^2_{H^s} - \langle u_{t_n}, u(t) \rangle_{H^s} - \langle u(t), u(t) \rangle_{H^s}. \tag{96}
\]

It suffices to show that \( u \) is continuous in \( t \) with respect to the weak topology on \( H^s \), or

\[
\lim_{n \rightarrow \infty} \langle u_{t_n}, u(t) \rangle_{H^s} = \lim_{n \rightarrow \infty} \langle u(t), u_{t_n} \rangle_{H^s} = \|u(t)\|^2_{H^s}. \tag{97}
\]

and that the map \( t \mapsto \|u(t)\|^2_{H^s} \) is continuous. \( \square \)

**Verifying \( u \) is continuous given the weak topology on \( H^s \).** Let \( \varphi \in H^s \). We must demonstrate that given a sequence \( t_n \rightarrow t \) that we have

\[
\lim_{n \rightarrow \infty} \langle u(t_n) - u(t), \varphi \rangle_{H^s} = 0. \tag{98}
\]

Let \( \epsilon > 0 \). We will choose a \( \psi \in C^\infty(\mathbb{T}) \) with \( \|\varphi - \psi\|_{H^s} < \frac{\epsilon}{2\|u_0\|_{H^s}} \). We then see that

\[
|\langle u(t_n) - u(t), \varphi - \psi \rangle_{H^s}| \leq |\langle u(t_n) - u(t), \varphi - \psi \rangle_{H^s}| + |\langle u(t_n) - u(t), \psi \rangle_{H^s}|. \tag{99}
\]

For the first term we have

\[
|\langle u(t_n) - u(t), \varphi - \psi \rangle_{H^s}| \leq \|u(t_n) - u(t)\|_{H^s} \|\varphi - \psi\|_{H^s} < \frac{\epsilon}{2}. \tag{100}
\]

For the second term, using the Lipschitz property of \( u \) in \( H^{s-1} \) we have

\[
|\langle u(t_n) - u(t), \psi \rangle_{H^s}| \leq \|u(t_n) - u(t)\|_{H^{s-1}} \|\psi\|_{H^{s+1}} \lesssim \|u_0\|^k_{H^{s+1}} \|\psi\|_{H^{s+1}} |t_n - t|, \tag{101}
\]

which is bounded by \( \epsilon/2 \) for sufficiently large \( n \).

**Verifying \( t \mapsto \|u(t)\|^2_{H^s} \) is continuous.** We begin by defining the functions

\[
G(t) = \|u(t)\|^2_{H^s}, \quad G_\epsilon(t) = \|J_{\epsilon} u(t)\|^2_{H^s}. \tag{103}
\]

Lemma 3.1 implies that \( G_\epsilon \rightarrow G \) pointwise in \( t \) as \( \epsilon \rightarrow 0 \). Thus it suffices to show that each \( G_\epsilon \) is Lipschitz, and that the Lipschitz constants for this family of
functions are bounded. Applying the operator $J_t$ to the Novikov equation (1) we obtain the following $H^s$ energy inequality for $G_t$

\[
\frac{1}{2}|G'_t(t)| \leq \left| \int_T D^s J_t(u^k \partial_x u) D^s J_t u dx \right| + \frac{1}{2} \left| \int_T D^s J_t (\partial_x^{-1} (\partial_x (u^k) \partial_x u)) D^s J_t u dx \right|.
\]

(104)

To bound the first term on the right-hand side of (104) we first commute the operator $D^s$ with $u^k$ to obtain

\[
\left| \int_T D^s J_t(u^k \partial_x u) D^s J_t u dx \right| \leq \int_T [D^s, u^k] \partial_x u \cdot D^s J_t^2 u dx + \int_T J_t u^k D^s \partial_x u \cdot D^s J_t u dx.
\]

(105)

Using the Cauchy-Schwarz inequality and the lemma 3.4 (Kato-Ponce) we have

\[
\left| \int_T [D^s, u^k] \partial_x u \cdot D^s J_t^2 u dx \right| \lesssim \left\| [D^s, u^k] \partial_x u \right\|_{L^2} \left\| D^s J_t^2 u \right\|_{L^2} \lesssim (\|u^k\|_{H^s} \|\partial_x u\|_{L^\infty} + \|\partial_x u\|_{H^{s-1}} \|\partial_x u^k\|_{L^\infty}) \lesssim \|u_0\|_{H^{s+2}}^2,
\]

(106)

where in the last inequality we used the solution size estimate (6).

For the second term, we commute $J_t$ with $u^k$ and make an integration by parts to obtain

\[
\left| \int_T J_t u^k D^s \partial_x u \cdot D^s J_t u dx \right| \leq \int_T [J_t, u^k] D^s \partial_x u D^s J_t u dx + \int_T \partial_x u^k \cdot (D^s J_t u)^2 dx \lesssim \|u_0\|_{H^{s+2}}^2,
\]

(107)

where for the estimation of the first integral we used the following lemma applied with $w = u^k$ and $f = D^s u$, whose proof can be found in [62].

**Lemma 3.5.** Let $w$ be such that $\|\partial_x w\|_{L^\infty}$. Then, there is a constant $c > 0$ such that for any $f \in L^2$ we have

\[
\left\| [J_t, w] \partial_x f \right\|_{L^2} \leq c \|f\|_{L^2} \|\partial_x w\|_{L^2}.
\]

(108)

To estimate the third term of (104), we apply the Cauchy-Schwarz inequality, the operator norm of $\partial_x^{-1}$ and the well-posedness estimate (6) to obtain

\[
\frac{1}{2} \left| \int_T D^s J_t (\partial_x^{-1} (\partial_x (u^k) \partial_x u)) D^s J_t u dx \right| \lesssim \|u_0\|_{H^{s+2}}^{k+2}.
\]

(109)

Putting these results together, we have the Lipschitz constants for the family $G_t$ bounded by $c_s \|u_0\|_{H^{s+2}}^{k+2}$ for some constant $c_s$. We may therefore conclude that $G$ is Lipschitz and that the solution.

**3.3. Uniqueness.** Having established the existence of a solution $u$ in $C([0, T]; H^s)$ to the Cauchy problem of the MHS equation with given initial data $u_0$ in $H^s$, in this section we shall show that this solution is unique.
Proposition 3.2 (Uniqueness). For initial data \( u_0 \in H^s, s > \frac{3}{2} \), the Cauchy problem (1.1)-(1.2) has a unique solution in the space \( C([0,T];H^s) \).

Proof. Let \( u_0 \in H^s \) and let \( u \) and \( w \) be two solutions to the Cauchy problem (1.1)-(1.2) with \( u(x,0) = u_0(x) = w(x,0) \). Then the difference \( v = u - w \) satisfies the following Cauchy problem

\[
\begin{aligned}
\partial_t v &= -\frac{1}{k+1} \partial_x[w_1v] + \frac{k}{2} \partial_{x}^{-1}[w_2v + w_3v_x], \\
v(x,0) &= 0,
\end{aligned}
\]

(110)

where

\[
\begin{aligned}
w_1 &= (u)^k + (u)^{k-1} + \cdots + (u) + (w)^{k-1} + (w)^k, \\
w_2 &= (\partial_x u)^2 \left[ (u)^{k-2} + (u)^{k-3} + \cdots + (w)^{k-3} + (w)^{k-2} \right], \\
w_3 &= u^{k-1}(\partial_x u + \partial_x w),
\end{aligned}
\]

Let \( \frac{1}{2} < \sigma < \min\{s - 1, 1\} \). The \( H^\sigma \)-energy estimate is then given by

\[
\frac{1}{2} \frac{d}{dt} \|v(t)\|_{H^\sigma}^2 = -\frac{1}{k+1} \int D^\sigma \partial_x(w_1v)D^\sigma vdx + \frac{k}{2} \int D^\sigma \partial_x^{-1}(w_2v + w_3v_x)D^\sigma vdx.
\]

(111)

To bound (111), we commute \( D^\sigma \partial_x \) and \( v \), which results in two integrals. The commutator integral is estimated by applying the Cauchy-Schwarz inequality followed by Lemma 2.3 and the solution size estimate (6). The second integral is bounded using integration by parts, the Sobolev Embedding Theorem and the solution size bound (6). The non-local term is estimated by the Cauchy-Schwarz inequality and the continuity of \( \partial_x^{-1} \). The resulting energy estimate

\[
\frac{1}{2} \frac{d}{dt} \|v(t)\|_{H^\sigma}^2 \lesssim \|v(t)\|_{H^\sigma}^2,
\]

which we solve to find the inequality

\[
\|v(t)\|_{H^\sigma}^2 \lesssim \|v(0)\|_{H^\sigma}^2 e^{2\epsilon \sigma T}.
\]

(113)

We recall that \( v = u - w \) where \( u \) and \( w \) are both solutions to the MHS i.v.p. (1)-(2). This means we have

\[
\|v(t)\|_{H^\sigma} \leq \|v(0)\|_{H^\sigma} e^{2\epsilon \sigma T} \leq \|u_0 - u_0\|_{H^\sigma} e^{2\epsilon \sigma T} = 0.
\]

(114)

Thus, we have uniqueness. \( \square \)

3.4. Continuity of the data-to-solution map

Proposition 3.3. The data-to-solution map for the MHS i.v.p. (1)-(2) from \( H^s \) to \( C([0,T];H^s) \) given by \( u_0 \to u \) is continuous.

Proof. Fix \( u_0 \in H^s \) and let \( \{u_{0,n}\} \in H^s \) be a sequence such that

\[
\lim_{n \to \infty} u_{0,n} = u_0 \quad \text{in} \quad H^s.
\]

(115)

Then if \( u \) is the solution to the MHS i.v.p. (1)-(2) with initial data \( u_0 \) and if \( u_n \) is the solution to the MHS i.v.p. with initial data \( u_{0,n} \), we will prove that

\[
\lim_{n \to \infty} u_n = u \quad \text{in} \quad C([0,T];H^s).
\]

(116)

Our approach is to use energy estimates. To avoid some of the difficulties of estimating the term involving \( u^k \partial_x u \), we will use the \( J_k \) convolution operator to smooth.
out the initial data. Let $\epsilon \in (0, 1]$. We take $u^\epsilon$ to be the solution to the MHS i.v.p. with smoothed initial data $J_\epsilon u_0 = j_\epsilon * u_0$. Similarly, let $v_\epsilon$ be the solution with initial data $J_\epsilon u_{0,n}$. Applying the triangle inequality, we arrive at

$$\|u - u_\epsilon\|_{C([0,T];H^s)} \leq \|u - u^\epsilon\|_{C([0,T];H^s)} + \|u^\epsilon - u_\epsilon\|_{C([0,T];H^s)} + \|u_\epsilon - u\|_{C([0,T];H^s)}.$$  

(117)

We will prove that for any $\eta > 0$, there exists an $N$ such that for all $n > N$, each of these terms can be bounded by $\frac{\eta}{3}$ for suitable choices of $\epsilon$ and $N$. We note that the choice of a sufficiently small $\epsilon$ will be independent of $N$ and will only depend on $\eta$; whereas, the choice of $N$ will depend on both $\eta$ and $\epsilon$. However, after $\epsilon$ has been chosen, $N$ can be chosen so as to force each of the three terms to be small.

**Estimation of $\|u^\epsilon - u_\epsilon\|_{C([0,T];H^s)}$.** We can bound this term directly using an $H^s$-energy estimate. Let $v = u^\epsilon - u_\epsilon$. Then $v$ satisfies the following Cauchy problem

$$\partial_t v = -\frac{1}{k+1} \partial_x [\bar{w}_1 v] + \frac{k}{2} \partial_x^{-1} [\bar{w}_2 v + \bar{w}_3 v],$$

$$v(x,0) = u^\epsilon(0) - u_\epsilon(0) = J_\epsilon u_0 - J_\epsilon u_{0,n},$$  

(118)

where

$$\bar{w}_1 = (u^\epsilon)^k + (u^\epsilon)^{k-1} u_\epsilon^{\epsilon} + \cdots + u^\epsilon (u_\epsilon)^{k-1} + (u_\epsilon)^{k},$$

$$\bar{w}_2 = (\partial_x u^\epsilon)^2 \left[ (u^\epsilon)^{k-2} + (u^\epsilon)^{k-3} u_\epsilon^{\epsilon} + \cdots + u^\epsilon (u_\epsilon)^{k-3} + (u_\epsilon)^{k-2} \right],$$

$$\bar{w}_3 = (u^\epsilon)^{k-1} (\partial_x u^\epsilon + \partial_x u_\epsilon).$$

Apply the operator $D^s$ to both sides of (118), multiply by $D^s v$ and integrate over the torus to obtain the $H^s$-energy

$$\frac{1}{2} \frac{d}{dt} \|v(t)\|_{H^s}^2 = -\frac{1}{k+1} \int T D^s \partial_x (\bar{w}_1 v) D^s v dx + \frac{k}{2} \int T D^s \partial_x^{-1} (\bar{w}_2 v + \bar{w}_3 \partial_x v) D^s v dx.$$  

(119)

To estimate the first integral of (119), we commute the operator $D^s \partial_x$ with the function $\bar{w}_1$ and apply Lemma 2.3 and the Sobolev Embedding theorem to get

$$\left| -\frac{1}{k+1} \int T D^s \partial_x (\bar{w}_1 v) D^s v dx \right| \lesssim \|\bar{w}_1\|_{H^{s+1}} \|v\|_{H^s}^2.$$  

(120)

We shall consider $\|\bar{w}_1\|_{H^{s+1}}$. From our construction of $J_\epsilon$, we can see that our initial data $J_\epsilon u_0, J_\epsilon u_{0,n} \in C^\infty$. Therefore, we may apply our solution size estimate (6), in conjunction with the Algebra Property for $s + 1 > \frac{d}{2}$, to the definition of $\|\bar{w}_1\|_{H^{s+1}}$ to find

$$\|\bar{w}_1\|_{H^{s+1}} \lesssim \sum_{j=0}^k \|J_\epsilon u_0\|_{H^{s+1}} \|J_\epsilon u_{0,n}\|_{H^{s+1}}.$$  

In examining $\|J_\epsilon u_0\|_{H^{s+1}}$, we shall use that if $f \in H^{s+1}$, then

$$\|f\|_{H^{s+1}} \leq \|f\|_{H^s} + \|\partial_x f\|_{H^s}.$$  

Using this inequality and $\|\partial_x J_\epsilon f\|_{H^s} \lesssim \frac{\eta}{\epsilon} \|f\|_{H^s}$, we can write

$$\|J_\epsilon u_0\|_{H^{s+1}} \leq \|J_\epsilon u_0\|_{H^s} + \|\partial_x J_\epsilon u_0\|_{H^s} \lesssim \frac{1}{\epsilon}.$$
Similarly, we have

\[ \|J_{n} u_{0,n}\|_{H^{s+1}} \leq \frac{1}{\epsilon} \|u_{0}\|_{H^{s}} \leq \frac{1}{\epsilon}. \]

Therefore, we have

\[ -\frac{1}{k + 1} \int_{\mathbb{T}} D^{s} \partial_{x} (\tilde{w}_{1} v) D^{s} v dx \leq \frac{1}{\epsilon} \|v\|^{2}_{H^{s}}. \]  

(121)

For the second integral of (119), applying the Cauchy-Schwarz inequality and operator norm of \( \partial_{x}^{-1} \) allows us to obtain

\[ \frac{k}{2} \int_{\mathbb{T}} D^{s} \partial_{x}^{-1} (\tilde{w}_{2} v + \tilde{w}_{3} \partial_{x} v) D^{s} v dx \epsilon \leq (\|u^{\epsilon}\|_{H^{s}} + \|u_{n}^{\epsilon}\|_{H^{s}})^{k} \|v\|^{2}_{H^{s}} \approx \|v\|^{2}_{H^{s}}. \]  

(122)

Combining (120) and (122), we have the following energy estimate

\[ \frac{1}{2} \frac{d}{dt} \|v(t)\|^{2}_{H^{s}} \leq \frac{1}{\epsilon} \|v\|^{2}_{H^{s}}, \]

which implies

\[ \|v(t)\|^{2}_{H^{s}} \leq \|v(0)\|^{2}_{H^{s}} e^{2 \frac{c_{\epsilon} t}{\epsilon}} = \|u_{0} - u_{0,n}\|^{2}_{H^{s}} e^{2 \frac{c_{\epsilon} t}{\epsilon}}. \]

Recalling that the solutions are mollified, we write

\[ \|u_{0} - u_{0,n}\|_{H^{s}} e^{2 \frac{c_{\epsilon} t}{\epsilon}} \leq \|J_{n} (u_{0} - u_{0,n})\|_{H^{s}} e^{2 \frac{c_{\epsilon} t}{\epsilon}} \leq \|u_{0} - u_{0,n}\|_{H^{s}} e^{2 \frac{c_{\epsilon} t}{\epsilon}}. \]  

(123)

When we bound the first and third terms of (117), we will force \( \epsilon \) to be small. After \( \epsilon \) (independent of \( N \)) is chosen, we can bound \( \|u^{\epsilon} - u_{n}^{\epsilon}\|_{C([0,T];H^{s})} \) by taking \( N \) large enough that

\[ \|u_{0} - u_{0,n}\|_{H^{s}} < \frac{\eta}{3} e^{-\frac{c_{\epsilon} T}{\epsilon}}. \]

Then we have

\[ \|u^{\epsilon} - u_{n}^{\epsilon}\|_{C([0,T];H^{s})} \leq \frac{\eta}{3}. \]

**Estimation of** \( \|u^{\epsilon} - u\|_{C([0,T];H^{s})} \) **and** \( \|u_{n}^{\epsilon} - u_{n}\|_{C([0,T];H^{s})} \). Since the arguments will be largely the same for both terms, we will omit the subscript \( n \) until such a time as differences in their handling emerge. For convenience, let \( v = u^{\epsilon} - u \), a direct calculation verifies that \( v \) solves the Cauchy problem

\[ \partial_{t} v = \frac{1}{k + 1} \left[ \sum_{j=1}^{k+1} C_{k+1}^{j} (-1)^{j} j(u^{\epsilon})^{k+1-j} v^{j-1} \partial_{x} v \right. \]

\[ + \sum_{j=1}^{k+1} C_{k+1}^{j} (-1)^{j} (k + 1 - j)(u^{\epsilon})^{k-j} v^{j} \partial_{x} u^{\epsilon} \]

\[ + \frac{k}{2} \partial_{x}^{-1} \left( (u^{\epsilon})^{k-1} (\partial_{x} u^{\epsilon}) + \partial_{x} u^{\epsilon} \right) + v(\partial_{x} u^{\epsilon})^{2} \left( \sum_{j=0}^{k-2} (u^{\epsilon})^{j} u^{k-2-j} \right) \]

with initial condition \( v_{0} = J_{\epsilon} u_{0} - u_{0} \).

We begin by calculating the \( H^{s} \) energy of \( v \). We have

\[ \frac{1}{2} \|v(t)\|^{2}_{H^{s}} = \frac{1}{k + 1} \int_{\mathbb{T}} D^{s} \left[ \sum_{j=1}^{k+1} C_{k+1}^{j} (-1)^{j} j(u^{\epsilon})^{k+1-j} v^{j-1} \partial_{x} v \right] \]

(124)
Following Cauchy problem \( u \) and let \( u^\epsilon \) be two solutions to the Cauchy problem for MHS equation with \( w(x,0) = u_0(x) \) and \( w(x,0) = w_0 \). Then the difference \( v = u - w \) satisfies the following Cauchy problem

\[
\partial_t v = -\frac{1}{k+1} \partial_x [w_1 v] + \frac{k}{2} \partial_x^{-1} [w_2 v + w_3 v],
\]

\[
v(x,0) = u_0 - w_0,
\]

We rewrite each term as a commutator and then apply the Cauchy-Schwarz inequality before using Lemma 3.4, the Sobolev Embedding Theorem, and the Algebra Property. For the non-local terms, we employ the Cauchy-Schwarz inequality and the continuity of \( \partial_x^{-1} \). The resulting energy estimate is

\[
\frac{1}{2} \frac{d}{dt} \| v \|_{H^s}^2 \leq \sum_{j=1}^{k+1} \left( \| u^\epsilon \|_{H^{s+j}} \| v \|_{H^{s+j}} + \| u^\epsilon \|_{H^s} \| u^\epsilon \|_{H^{s+1}} \| v \|_{H^{s-1}} \| v \|_{H^{s}} + \| v \|_{L^2}^2 \right).
\]

By interpolating between 0 and \( s \), we have

\[
\| v \|_{H^{s+1}} \leq \| v \|_{H^s}^{\frac{1}{2}} \| v \|_{H^{s+1}}^{\frac{1}{2}} \leq \| v \|_{L^2}^{\frac{1}{2}}.
\]

Note that the solution size estimate (6) implies that \( \| v(t) \|_{H^s} \lesssim 1 \). By utilizing an \( L^2 \)-energy estimate, it can be shown that \( \| v \|_{L^2} = O(\epsilon^s) \). This is used to reduce the energy estimate to the differential inequality

\[
\frac{dy}{dt} \lesssim y + \delta,
\]

where \( y = y(t) = \| v(t) \|_{H^s} \) and \( \delta = \delta(\epsilon) \to 0 \) as \( \epsilon \to 0 \). Solving (127) gives

\[
y(t) \lesssim y(0) + \delta.
\]

From here we will treat the cases \( y = \| u^\epsilon - u \|_{H^s} \) and \( y = \| u^\epsilon_n - u_n \|_{H^s} \) separately.

**Case of** \( y = \| u^\epsilon - u \|_{H^s} \). Since \( y(0) = \| J_{\epsilon} u_0 - u_0 \|_{H^s} \to 0 \) as \( \epsilon \to 0 \) and \( \delta = \delta(\epsilon) \to 0 \) as \( \epsilon \to 0 \), we see that for sufficiently small \( \epsilon \) we can bound the first term of (117) by \( \frac{\delta}{3} \).

**Case of** \( y = \| u^\epsilon_n - u_n \|_{H^s} \). Since

\[
y(0) = \| J_{\epsilon} u_{0,n} - u_{0,n} \|_{H^s} \leq 2 \| u_{0,n} - u_0 \|_{H^s} + 2 \| J_{\epsilon} u_0 - u_0 \|_{H^s},
\]

we may now further refine the choice of \( \epsilon \) and \( N \) so that \( y(t) < \frac{\delta}{4} \), completing this case. Collecting our results completes the proof of continuous dependence. \( \square \)

4. **Proof of Theorem 1.3.** Proof. Lipschitz Continuity in \( A_1. \) \( u_0, w_0 \in H^s \) and let \( u \) and \( w \) be two solutions to the Cauchy problem for MHS equation with \( u(x,0) = u_0(x) \) and \( w(x,0) = w_0 \). Then the difference \( v = u - w \) satisfies the following Cauchy problem

\[
\partial_t v = -\frac{1}{k+1} \partial_x [w_1 v] + \frac{k}{2} \partial_x^{-1} [w_2 v + w_3 v],
\]

\[
v(x,0) = u_0 - w_0,
\]
where
\[ w_1 = (u)^k + (u)^{k-1}w + \cdots + u(w)^{k-1} + (w)^k, \]
\[ w_2 = (\partial_x u)^2 \left[ (u)^{k-2} + (u)^{k-3}w + \cdots + u(w)^{k-3} + (w)^{k-2} \right], \]
\[ w_3 = u^{k-1} (\partial_x u + \partial_x w), \]

Let \( 0 \leq r \leq s - 1 \) and \( r + s > 2 \). The \( H^r \)-energy estimate is then given by
\[
\frac{1}{2} \frac{d}{dt} \|v(t)\|_{H^r}^2 = - \frac{1}{k + 1} \int_T \partial_x^r \partial_x^r (w_1 v) D^r v dx + \frac{k}{2} \int_T \partial_x^r (w_2 v + w_3 \partial_x v) D^r v dx.
\]

(130)

To bound (130), we commute \( D^r \partial_x \) and \( v \), which results in two integrals. The commutator integral is estimated by applying the Cauchy-Schwarz inequality followed by Lemma 2.3 and the solution size estimate (6). The second integral is bounded using integration by parts, the Sobolev Embedding Theorem and the solution size bound (6). The non-local term is estimated by the Cauchy-Schwarz inequality and the continuity of \( \partial_x \).

Hölder Continuity in \( A_2 \). By the Lipschitz continuity in \( A_1 \) and the condition \( r < 2 - s \), we have
\[
\|u(t) - w(t)\|_{H^r} \leq \|u(t) - w(t)\|_{H^2} \leq \|u_0 - w_0\|_{H^2} e^{cT}.
\]

(134)

By interpolating between the \( H^r \) and the \( H^s \) norms, we have
\[
\|v(0)\|_{H^2} \leq \|v(0)\|_{H^r}^{2(s-1)} \|v(0)\|_{H^s}^{2-r} \leq \|v(0)\|_{H^r}^{2(s-1)},
\]

(135)

which shows the Hölder Continuity in \( A_2 \).

Hölder Continuity in \( A_3 \). Since \( s - 1 \leq r \leq s \) by interpolating between \( H^{s-1} \) and \( H^s \) norms, we get
\[
\|v(t)\|_{H^r} \leq \|v(t)\|_{H^{s-1}}^{s-r} \|v(t)\|_{H^r}^{s-1}. \]

(136)

Furthermore, from the well-posedness size estimate (6), we have
\[
\|v(t)\|_{H^r} \leq \|u_0\|_{H^r} + \|w_0\|_{H^r} \leq 1,
\]

(137)

and therefore
\[
\|v(t)\|_{H^r} \leq c\|v(t)\|_{H^{s-1}}^{s-r}.
\]

(138)

By the Lipschitz continuity in \( A_1 \) and the condition \( s - 1 \leq r \), we get
\[
\|v(t)\|_{H^r} \leq c\|v(0)\|_{H^{s-1}}^{s-r} \leq c\|v(0)\|_{H^r}^{s-r},
\]

(139)

which is the desired Hölder continuity.

\( \square \)

Acknowledgments. We would like to thank the referees very much for their valuable comments and suggestions.
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Received July 2015; revised September 2016.

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