Proof Techniques in Quasi-Monte Carlo Theory

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Abstract

In this survey paper we discuss some tools and methods which are of use in quasi-Monte Carlo (QMC) theory. We group them in chapters on Numerical Analysis, Harmonic Analysis, Algebra and Number Theory, and Probability Theory. We do not provide a comprehensive survey of all tools, but focus on a few of them, including reproducing and covariance kernels, Littlewood-Paley theory, Riesz products, Minkowski’s fundamental theorem, exponential sums, diophantine approximation, Hoeffding’s inequality, empirical processes and the Lovász local lemma, as well as other tools. We illustrate the use of these methods in QMC using examples.

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1 Introduction

Quasi-Monte Carlo (QMC) rules are quadrature rules which can be used to approximate integrals defined on the s-dimensional unit cube \([0, 1]^s\)

\[
\int_{[0,1]^s} f(x) \, dx \approx \frac{1}{N} \sum_{n=0}^{N-1} f(x_n),
\]

where \(\mathcal{P} = \{x_0, x_1, \ldots, x_{N-1}\}\) are deterministically chosen quadrature points in \([0, 1]^s\). In QMC theory one is interested in a number of questions. Of importance is the integration error

\[
\left| \int_{[0,1]^s} f(x) \, dx - \frac{1}{N} \sum_{n=0}^{N-1} f(x_n) \right|
\]

and how it behaves as \(N\) and/or \(s\) increases. Various settings can be defined to analyze this error. For instance, one can consider the worst-case error: Here one uses a Banach space \((\mathcal{H}, \| \cdot \|)\) and considers

\[
\text{wce}(\mathcal{H}, \mathcal{P}) = \sup_{f \in \mathcal{H}, \|f\| \leq 1} \left| \int_{[0,1]^s} f(x) \, dx - \frac{1}{N} \sum_{n=0}^{N-1} f(x_n) \right|.
\]

Particular nice examples of such function spaces are so-called reproducing kernel Hilbert spaces. We review essential properties of reproducing kernel Hilbert spaces in Section 2. Other settings include the average case error: In this case one defines a probability measure \(\mu\) on the function space \(\mathcal{H}\) and then studies the expectation value of the integration error

\[
\text{ace}_p(\mathcal{H}, \mathcal{P}) = \left( \mathbb{E} \left| \int_{[0,1]^s} f(x) \, dx - \frac{1}{N} \sum_{n=0}^{N-1} f(x_n) \right|^p \right)^{1/p}.
\]

Such an investigation can be carried out with the help of covariance kernels. There are a number of relations to reproducing kernels, which we also discuss in Section 2.
Covariance kernels also appear in stochastic processes, which themselves are important in applications in financial mathematics and partial differential equations (PDEs) with random coefficients, for instance. We discuss all these connections in the section on Numerical Analysis, Section 2, in which we also treat some further useful tools, like the use of bump functions to prove lower bounds and the Rader transform. Also the connection between the integration error and discrepancy of the quadrature points is shown and the Koksma-Hlawka inequality is described in this context.

The analysis of the integration error is often greatly helped by using orthogonal expansions. These can be Fourier series, Walsh series or Haar series for instance. Tools from Harmonic Analysis are important here. For instance the proof of strong lower bounds is facilitated by using the Littlewood-Paley inequality and Riesz products. We devote a section on Harmonic Analysis (Section 3) to this topic to give the reader an idea of how those methods are applied in QMC.

Another important topic in QMC is the construction of good quadrature points which can be used in computation. This area makes fundamental use of Algebra and Number Theory. Finite fields, characters and duality theory are of importance here, as well as a number of other topics including exponential sums, \(b\)-adic numbers, and diophantine approximation. These tools are reviewed and illustrated in the context of QMC in Section 4.

Although many useful explicit constructions are known based on algebraic and number theoretic methods, in some instance one can show stronger results by switching to methods which only proof the existence of some point sets, rather than explicit constructions. The simplest instance of proving an existence result can be illustrated by the principle that for a given set of real numbers \(a_1, a_2, \ldots, a_N\), at least one of those numbers is bounded above by the average \(\frac{1}{N}\sum_{n=1}^{N} a_n\). This can be rephrased in terms of random variables and expectation values and leads to the probabilistic method. There are a number of sophisticated tools available from this area which go much further than the simple averaging argument described above, for instance Hoeffding’s inequality, VC-classes and empirical processes. These methods are illustrated in Section 5 which is devoted to the use of Probability Theory in QMC. Also discussed there is the Lovász local lemma.

This article does not provide an introduction to QMC theory per se. The main goal is to illustrate the use of the tools mentioned above in QMC theory via some examples. The results in QMC theory which we use to illustrate these ideas are not always the most interesting cases since the emphasis is mainly on the tools and not the QMC results. Often we use results from QMC theory which highlight the concepts from the areas of Numerical Analysis, Harmonic Analysis, Algebra and Number Theory and Probability Theory, and not the particular results from QMC theory.

The motivation for the approach taken in this paper lies in the fact that introductions to various aspects of QMC theory have already appeared in a number of monographs and major survey articles in recent years. We mention those which are in preparation, to appear or appeared in the last ten years at the writing of this paper in chronological order. Strauch and Porubský provide a sampler of results on the distribution of sequences. This book includes many of the older results which are not included in other publications. The series of monographs by Novak and Woźniakowski is devoted to Information-Based Complexity. QMC plays some role in their since it can be used to show tractability results in high dimensional integration problems. It also provides the necessary background on various settings, from function spaces to different
error criteria, which can be used to study QMC methods. Lemieux’s work \[53\] discusses Monte Carlo methods, including pseudo random number generation, QMC and Markov chain Monte Carlo, and various aspects of their use in applications. The monograph \[17\] by Dick and Pillichshammer studies digital nets and sequences. These point sets and sequences can be used in QMC integration. Results on numerical integration and their connection to discrepancy theory are also explained in there. Triebel \[91, 93\] studies connections of discrepancy theory and numerical integration via the study of function spaces. Another introductory book on Monte Carlo methods is by Müller-Gronbach, Novak and Ritter \[61\] (in German). It discusses algorithmic aspects, simulation techniques, variance reduction, Markov chain Monte Carlo and numerical integration. The survey article \[13\] by Dick, Kuo and Sloan focuses on high dimensional numerical integration using QMC rules. Numerical integration in infinite dimensional spaces is also briefly discussed. The textbook \[54\] by Leobacher and Pillichshammer provides an introduction to QMC theory and discusses applications to various areas. A number of articles covering various aspects of discrepancy theory is provided in the monograph \[11\], edited by Chen, Srivastav and Travaglini. One of those articles relates discrepancy theory to QMC methods and shows how various parts of discrepancy theory can be used in QMC theory. Also deep results on discrepancy theory are discussed in various articles. Kritzer, Niederreiter, Pillichshammer and Winterhof \[41\] are editors of a further book consisting of survey articles focusing on number theoretic constructions of point sets and sequences, uniform distribution theory, and quasi-Monte Carlo methods. Owen \[77\] is preparing a comprehensive introduction to Monte Carlo methods covering anything from Monte Carlo, quasi-Monte Carlo to Markov chain Monte Carlo, non-uniform random number generation, variance reduction and importance sampling as well as other aspects.

Given that many aspects of QMC theory have been surveyed or covered in textbooks and research monographs, we aim to provide a survey of proof techniques and tools which are used in QMC theory. Although these tools often appear as part of proofs of theorems in QMC theory, they have usually not been the focus themselves in these other works. We do so here by introducing various methods and illustrating them via examples.

2 Numerical Analysis

Numerical integration is a classical topic in numerical analysis. The Koksma-Hlawka inequality is a basic result in QMC theory. Its establishment (1941 in dimension one by Koksma and 1961 in arbitrary dimension by Hlawka) can be considered as a starting point for the analysis of QMC methods. In the modern context, such inequalities can be considered as bounds for worst-case errors in reproducing kernel Hilbert spaces or more general function spaces. Thus reproducing kernel functions play a significant role in studying QMC methods. Reproducing kernel functions themselves have many similarities to covariance kernels. The latter are important when studying average case errors, or problems defined over random fields or stochastic processes. Stochastic processes are for instance used in financial mathematics to model the stock price, or in physical applications to model the permeability of porous media. These applications lead to stochastic differential equations and partial differential equations with random coefficients. In some of these applications, QMC is used successfully as a sampling technique to obtain estimations of the expectation value of, for instance, the payoff function of an option or a linear
functional of a solution of a PDE. In the following we survey some of the essential tools in this area.

2.1 Reproducing kernel Hilbert spaces

Reproducing kernel Hilbert spaces play a fundamental role in QMC theory nowadays. The basic reference for reproducing kernel Hilbert spaces is [3]. Since we consider QMC in this paper, we restrict the domain to the unit cube $[0,1]^s$.

**Definition 2.1** A function $K : [0,1]^s \times [0,1]^s \to \mathbb{R}$ is a reproducing kernel if

1. Symmetry
   \[ K(x,y) = K(y,x) \text{ for all } x,y \in [0,1]^s, \text{ and} \]
2. Positive semi-definite
   
   for all $a_1,a_2,...,a_N \in \mathbb{C}$ and all $x_1,x_2,...,x_N \in [0,1]^s$ we have
   \[ \sum_{n,m=1}^{N} a_n \overline{a_m} K(x_n,x_m) \geq 0. \]

A reproducing kernel $K$ uniquely defines a space $H_K$ of functions on $[0,1]^s$ and an inner product $\langle \cdot, \cdot \rangle_K$ on $H_K$. The corresponding norm is denoted by $\| \cdot \|_K$. The following properties are equivalent to the symmetry and positive semi-definiteness above.

i) $K(\cdot,y) \in H_K$ for each fixed $y \in [0,1]^s$;
ii) $\langle f, K(\cdot,y) \rangle_K = f(y)$ for all $y \in [0,1]^s$ and $f \in H_K$;
iii) if $L : [0,1]^s \times [0,1]^s \to \mathbb{R}$ satisfies i) and ii), then $L = K$.

**Examples: Reproducing kernel Hilbert spaces derived from expansions**

1. **Polynomial space**

   In the first example we consider a space $\mathcal{H}$ of polynomials $f(x) = a_0 + a_1 x + \cdots + a_r x^r$ of degree at most $r$, where $a_i \in \mathbb{R}$. The basic functions are the monomials $x^i$, $0 \leq i \leq r$, and each polynomial can be represented as a linear combination of these functions.

   We can define an inner product for polynomials $f_i = a_{i,0} + a_{i,1} x + \cdots + a_{i,r} x^r$ by
   \[ \langle f_1, f_2 \rangle_1 = \sum_{t=0}^{r} a_{1,t} \overline{a_{2,t}}. \]

   With this inner product, the monomials $x^i$ are orthonormal, that is
   \[ \langle x^i, x^j \rangle_1 = \delta_{i,j}, \]

   where $\delta_{i,j}$ is the Kronecker $\delta$-symbol.

   The task now is to find a function $K_1(x,y) : [0,1] \times [0,1] \to \mathbb{R}$ which satisfies the reproducing property $\langle f, K_1(\cdot,y) \rangle_1 = f(y)$. This function is given by $K_1(x,y) = 1 + xy + x^2 y^2 + \cdots + x^r y^r$ as can easily be verified.
An alternative way of defining an inner product on the space of polynomials of degree at most \( r \) is the following approach. For \( i \in \mathbb{N}_0 \) let \( B_i \) denote the Bernoulli polynomial of degree \( i \). Use the expansion \( g(x) = b_0 B_0 + b_1 B_1(x) + \cdots + b_r B_r(x) \), where \( b_i \in \mathbb{R} \). Again one obtains polynomials of degree at most \( r \) this way. We can define the inner product

\[
\langle g_1, g_2 \rangle_2 = \sum_{\ell=0}^{r} b_{1,\ell} b_{2,\ell},
\]

for

\[
g_i(x) = b_{i,0} + b_{i,1} B_1(x) + \cdots + b_{i,r} B_r(x).
\]  

(1)

This inner product differs from the first case. In fact, now the Bernoulli polynomials are an orthonormal basis \( \langle B_i, B_j \rangle_2 = \delta_{i,j} \). The reproducing kernel is now given by

\[
K_2(x, y) = B_0(x) B_0(y) + B_1(x) B_1(y) + \cdots + B_r(x) B_r(y).
\]

**Caution:** We provide an example where the above principles fail. Consider all polynomials of degree at most 1 of the form

\[
f_i(x) = a_{i,0} + a_{i,1} x + b_{i,1} B_1(x).
\]  

(2)

One could define the inner product \( \langle f_1, f_2 \rangle_3 = a_{1,0} a_{2,0} + a_{1,1} a_{2,1} + b_{1,1} b_{2,1} \). However, this is not well defined, since in the expansion (2) the values of \( a_{i,0}, a_{i,1}, b_{i,1} \) are not uniquely defined.

2. **Korobov space**

This space is a space of Fourier series

\[
f(x) = \sum_{k \in \mathbb{Z}} \hat{f}(k) \exp(2\pi i k x),
\]

where \( i = \sqrt{-1} \) and \( \hat{f}(k) = \int_{0}^{1} f(x) \exp(-2\pi i k x) \, dx \). For \( \alpha > 1/2 \) we define an inner product by

\[
\langle f, g \rangle_{K_\alpha} = \sum_{k \in \mathbb{Z}} \hat{f}(k) \overline{\hat{g}(k)} \max(1, |k|)^{2\alpha}.
\]

Its reproducing kernel \( K_\alpha : [0, 1] \times [0, 1] \rightarrow \mathbb{R} \) is given by

\[
K_\alpha(x, y) = \sum_{k \in \mathbb{Z}} \max(1, |k|)^{-2\alpha} \exp(2\pi i k (x - y)).
\]

3. **Unanchored Sobolev space**

The unanchored Sobolev space is the direct sum of the Korobov space and the polynomial space using the Bernoulli expansion (1).

For \( i = 1, 2 \) let \( h_i \) be a function in the Korobov space where \( \alpha = 1 \) such that \( \int_{0}^{1} h_i(x) \, dx = 0 \). Let

\[
f_i(x) = b_{i,0} B_0(x) + b_{i,1} B_1(x) + h_i(x) = b_{i,0} B_0(x) + b_{i,1} B_1(x) + \sum_{k \in \mathbb{Z}\{0\}} \hat{h}_i(k) \exp(2\pi i k x),
\]

where \( B_0(x) = 1 \) and \( B_1(x) = x - 1/2 \) are the Bernoulli polynomials. By assuming that \( \int_{0}^{1} h_i(x) \, dx = 0 \) this representation is unique, since the constant part is in \( b_{i,0} \).
and $B_1(x) = x - 1/2$ is not in the Korobov space. We can define an inner product by
\[ \langle f_1, f_2 \rangle_K = b_{1,0}b_{2,0} + b_{1,1}b_{2,1} + \frac{1}{(2\pi)^2} \sum_{k \in \mathbb{Z}\setminus\{0\}} \hat{h}_1(k)\hat{h}_2(k) |k|^2. \]

The role of the normalizing factor $(2\pi)^{-2}$ will soon become clear, but has otherwise no bearings on the principles used to define the inner product. The reproducing kernel is given by
\[ K(x, y) = B_0(x)B_0(y) + B_1(x)B_1(y) + (2\pi)^2 \sum_{k \in \mathbb{Z}\setminus\{0\}} |k|^{-2} \exp(2\pi ik(x - y)). \]

The representation above can be simplified. The inner product is then given by
\[ \langle f_1, f_2 \rangle_K = \int_0^1 f_1(x) \, dx \int_0^1 f_2(x) \, dx + \int_0^1 f'_1(x)f'_2(x) \, dx. \]

For $\ell \in \mathbb{N}$, the Bernoulli polynomial $B_\ell$ has the Fourier series expansion
\[ B_\ell(x) = -\frac{\ell!}{(2\pi i)^\ell} \sum_{k \in \mathbb{Z}\setminus\{0\}} k^{-\ell} \exp(2\pi ikx). \]

Thus we can write the reproducing kernel as
\[ K(x, y) = 1 + B_1(x)B_1(y) + \frac{1}{2}B_2(|x - y|). \]

This approach can be extended to smoothness $\alpha > 1$ with $\alpha \in \mathbb{N}$, by using the Korobov space of smoothness $\alpha$ and the space of Bernoulli polynomials of degree up to $\alpha$. (Note that the Bernoulli polynomials of degree $\ell \leq \alpha$ are not in the Korobov space of smoothness $\alpha$, thus this approach is well defined.)

4. Anchored Sobolev space

The anchored Sobolev space is based on the Taylor series expansion with integral remainder
\[ f(y) = f(0) + \int_0^1 f'(t)1_{[0,y]}(t) \, dt. \]

We define an inner product by
\[ \langle f, g \rangle_K = f(0)g(0) + \int_0^1 f'(t)g'(t) \, dt. \]

The reproducing kernel is given by
\[ K(x, y) = 1 + \int_0^1 1_{[0,x]}(t)1_{[0,y]}(t) \, dt = 1 + \min(x, y). \]

By using the same principle as above but with a Taylor series expansion with integral remainder involving derivatives up to order $r$, we obtain the anchored Sobolev space of order $r$. 

7
To define $s$-variate function spaces we can use the $s$-fold tensor product $\mathcal{H} \otimes \mathcal{H} \otimes \cdots \otimes \mathcal{H}$. The reproducing kernel is in this case given by the $s$-fold product of the one-dimensional reproducing kernels, i.e.,

$$K(x, y) = \prod_{j=1}^{s} K(x_j, y_j).$$

An important property

The following property, valid for any reproducing kernel Hilbert space, is frequently used in QMC theory. Let $T : \mathcal{H} \to \mathbb{R}$ be a continuous linear functional. Then the order of inner product and linear functional can always be interchanged, that is

$$\langle T(f), K(\cdot, x) \rangle_K = T(\langle f, K(\cdot, x) \rangle_K).$$

Reproducing kernels and the worst-case error

The worst-case integration error of a QMC rule

$$\frac{1}{N} \sum_{n=0}^{N-1} f(x_n) \quad \text{for} \quad f \in \mathcal{H}$$

based on a point set $\mathcal{P} = \{x_0, \ldots, x_{N-1}\}$ over a certain function space $\mathcal{H}$ with norm $\| \cdot \|$ is an important tool for assessing the quality of the quadrature point set. It is defined as

$$\text{wce}(\mathcal{H}, \mathcal{P}) = \sup_{f \in \mathcal{H}, \|f\| \leq 1} \left| \int_{[0,1]^s} f(x) \, dx - \frac{1}{N} \sum_{n=0}^{N-1} f(x_n) \right|.$$

If $\mathcal{H} = \mathcal{H}_K$ is a reproducing kernel Hilbert space, then the worst-case error can be stated explicitly in terms of the reproducing kernel $K$. Indeed we have for any $f \in \mathcal{H}_K$ that

$$\int_{[0,1]^s} f(x) \, dx - \frac{1}{N} \sum_{n=0}^{N-1} f(x_n) = \int_{[0,1]^s} \langle f, K(\cdot, x) \rangle_K \, dx - \frac{1}{N} \sum_{n=0}^{N-1} \langle f, K(\cdot, x_n) \rangle_K$$

$$= \left\langle f, \int_{[0,1]^s} K(\cdot, x) - \frac{1}{N} \sum_{n=0}^{N-1} K(\cdot, x_n) \right\rangle_K = \langle f, h \rangle_K, \quad (3)$$

where

$$h(y) = \int_{[0,1]^s} K(y, x) \, dx - \frac{1}{N} \sum_{n=0}^{N-1} K(y, x_n). \quad (4)$$

Thus, for any function $f \in \mathcal{H}_K$ with $f \neq 0$ we have

$$\frac{1}{\|f\|_K} \left| \int_{[0,1]^s} f(x) \, dx - \frac{1}{N} \sum_{n=0}^{N-1} f(x_n) \right| \leq \|h\|_K.$$
On the other hand, we can achieve equality by considering the integration error of the function \( h \). Thus we obtain that

\[
\text{wce}^2(\mathcal{H}_K, \mathcal{P}) = \|h\|^2_K = \langle h, h \rangle_K
\]

\[
= \int_{[0,1]^s} \int_{[0,1]^s} K(x, y) \, dx \, dy - \frac{2}{N} \sum_{n=0}^{N-1} \int_{[0,1]^s} K(x, x_n) \, dx + \frac{1}{N^2} \sum_{n,n'=0}^{N-1} K(x_n, x_{n'}). \tag{5}
\]

\[
2.2 \text{ Koksma-Hlawka inequality}
\]

The Koksma-Hlawka inequality is a classic bound on the integration error of QMC rules. We give an example of this type of inequality using reproducing kernel Hilbert spaces.

We start by introducing the reproducing kernel \( K : [0, 1] \times [0, 1] \to \mathbb{R} \) given by

\[
K(x, y) = 1 + \int_0^1 1_{[x, 1]}(t)1_{[y, 1]}(t) \, dt = 1 + \min(1 - x, 1 - y).
\]

The inner product in the corresponding reproducing kernel Hilbert space is given by

\[
\langle f, g \rangle_K = f(1)g(1) + \int_0^1 f'(t)g'(t) \, dt.
\]

For dimensions \( s > 1 \) we use the kernel

\[
K(x, y) = \prod_{j=1}^s K(x_j, y_j).
\]

Then the inner product is given by

\[
\langle f, g \rangle_K = \sum_{u \subseteq [s]} \int_{[0,1]^{|u|}} \frac{\partial^{|u|} f}{\partial x_u}(x_u; 1) \frac{\partial g}{\partial x_u}(x_u; 1) \, dx_u.
\]

where \( [s] := \{1, 2, \ldots, s\} \) and where for \( u \subseteq [s] \) and \( x = (x_1, x_2, \ldots, x_s) \) we write \( x_u = (x_j)_{j \in u} \) and \( (x_u; 1) = (z_1, z_2, \ldots, z_s) \) with

\[
z_j = \begin{cases} x_j & \text{if } j \in u, \\ 1 & \text{if } j \notin u. \end{cases}
\]

Eq. (3) provides a representation of the integration error in terms of the reproducing kernel. Of essence here is the function \( h \), which for our specific reproducing kernel \( K \) is given by

\[
h(y) = \prod_{j=1}^s \int_0^1 K(y_j, x_j) \, dx_j - \frac{1}{N} \sum_{n=0}^{N-1} \prod_{j=1}^s K(y_j, x_{n,j})
\]

\[
= \prod_{j=1}^s \frac{3 - y_j^2}{2} - \frac{1}{N} \sum_{n=0}^{N-1} \prod_{j=1}^s (1 + \min(1 - y_j, 1 - x_{n,j})).
\]
From (5) we have that the worst-case error for integration in $\mathcal{H}_K$ is given by $\|h\|_K$. We now compute this norm explicitly. To do so, we need the partial derivatives

$$\frac{\partial^u h}{\partial y_u}(y_u; 1) = (-1)^{|u|+1} \left( \frac{1}{N} \sum_{z \in \mathcal{P}} 1_{B(y_u, 1)}(z) - \text{vol}(B(y_u, 1)) \right).$$

Here $\text{vol}(B_y) = y_1 \cdots y_s$ denotes the volume of the rectangular box $B_y = [0, y_1) \times \ldots \times [0, y_s)$ for $y = (y_1, \ldots, y_s) \in [0, 1]^s$ and $1_{B_y}$ is the characteristic function of the box $B_y$.

Thus (5) implies that

$$\text{wce}(\mathcal{H}_K, \mathcal{P}) = \left( \sum_{u \subseteq [s]} \int_{[0, 1]^u} \left( \frac{1}{N} \sum_{z \in \mathcal{P}} 1_{B_y}(z) - \text{vol}(B_y) \right)^2 dy \right)^{1/2}. \quad (7)$$

For an $N$-element point set $\mathcal{P}$ in the $s$-dimensional unit cube $[0, 1]^s$ the discrepancy function $D_N$ is defined as

$$D_N(\mathcal{P}, y) := \frac{1}{N} \sum_{z \in \mathcal{P}} 1_{B_y}(z) - \text{vol}(B_y). \quad (8)$$

The sum in the discrepancy function counts the number of points of $\mathcal{P}$ contained in $B_y$ and the discrepancy function measures the deviation of this number from the fair number of points $N \text{vol}(B_y)$ which would be achieved by a perfect (but impossible) uniform distribution of the points of $\mathcal{P}$.

Since (7) is the $L_2$ norm of the discrepancy function, it is also called the $L_2$ discrepancy of the point set $\mathcal{P}$. The $L_p$ version of the discrepancy function also makes sense and can also be motivated by numerical integration. Using (3) we have

$$\int_{[0, 1]^s} f(x) \, dx - \frac{1}{N} \sum_{n=0}^{N-1} f(x_n) = (f, h)_K \quad = \sum_{u \subseteq [s]} (-1)^{|u|+1} \int_{[0, 1]^u} \frac{\partial^u f}{\partial x_u}(x_u; 1) D_N(\mathcal{P}, (x_u; 1)) \, dx.$$

Taking the absolute value and applying Hölder’s inequality for integrals and sums, we obtain that

$$\left| \int_{[0, 1]^s} f(x) \, dx - \frac{1}{N} \sum_{n=0}^{N-1} f(x_n) \right| \leq L_{p,q}(\mathcal{P}) \|f\|_{p', q'}, \quad (9)$$

where $1/p + 1/p' = 1$ and $1/q + 1/q' = 1$,

$$L_{p,q}(\mathcal{P}) = \left( \sum_{u \subseteq [s]} \left( \int_{[0, 1]^u} (D_N(\mathcal{P}, (y_u; 1)))^q dy \right)^{p/q} \right)^{1/p}$$

and

$$\|f\|_{p', q'} = \left( \sum_{u \subseteq [s]} \left( \int_{[0, 1]^u} \left| \frac{\partial^u f}{\partial x_u}(x_u; 1) \right|^{q'} dx_u \right)^{p'/q'} \right)^{1/p'},$$
with the obvious modifications if \( p, p', q \) or \( q' \) are \( \infty \). The error estimate \([9]\) is called a Koksma-Hlawka inequality. In its classical form it uses \( q = p = \infty \) and the variation of \( f \) in the sense of Hardy and Krause instead of the norm \( \| f \|_{1,1} \) (see, e.g., \([44]\)).

Let \( \mathcal{H} \) be a normed function space which contains the discrepancy function of any point set. Then we denote the norm of the discrepancy function \( D_N(\mathcal{P}, \cdot) \), as defined in \((8)\), by \( D_N(\mathcal{P}, \mathcal{H}) \). For \( 0 < p \leq \infty \), the (quasi)-norm \( D_N(\mathcal{P}, L^p) \) is called the \( L^p \)-discrepancy of the point set \( \mathcal{P} \). In particular, we abbreviate \( D_N(\mathcal{P}) = D_N(\mathcal{P}, L^\infty) = \sup_{x \in [0,1]^s} |D_N(\mathcal{P}, x)| \), which is often called the star discrepancy of \( \mathcal{P} \).

### 2.3 Mercer’s theorem

In the examples of reproducing kernel Hilbert spaces we have seen that some expansions of functions (polynomials or Fourier series for instance) yield reproducing kernel Hilbert spaces in a natural way. One may ask whether such expansions exist for any reproducing kernel (i.e., any symmetric and positive semi-definite function). An affirmative answer to this question for continuous reproducing kernels is given by Mercer’s theorem.

Let \( K : [0,1]^s \times [0,1]^s \to \mathbb{R} \) be a reproducing kernel. Assume that \( K \) is continuous. We define the linear operator \( T_K : L^2([0,1]^s) \to L^2([0,1]^s) \) by

\[
T_K(f)(x) = \int_{[0,1]^s} K(x, y) f(y) \, dy.
\]

Then \( T_K \) is a self-adjoint, positive, compact operator on \( L^2([0,1]^s) \). In the following we state a version of Mercer’s theorem \([60]\) which we adapt to our situation.

**Theorem 2.2 (Mercer)** Let the reproducing kernel \( K : [0,1]^s \times [0,1]^s \to \mathbb{C} \) be a continuous function. Then there exists a sequence of \( L^2 \) orthonormal eigenfunctions \( \psi_\ell : [0,1]^s \to \mathbb{C}, \ell \in \mathbb{N}, \) with corresponding nonnegative eigenvalues \( \lambda_\ell \in \mathbb{R}^+ \) of the operator \( T_K \)

\[
T_K(\psi_\ell)(x) = \lambda_\ell \psi_\ell(x) \quad \text{for all } \ell \in \mathbb{N}.
\]

The reproducing kernel \( K \) has the representation

\[
K(x, y) = \sum_{\ell=1}^{\infty} \lambda_\ell \psi_\ell(x) \overline{\psi_\ell(y)}.
\]

**Examples**

We now show some examples of reproducing kernels and their expansions. We have already seen an example where the eigenvalues and eigenfunctions are obvious:

1. **Korobov space**
   
   The reproducing kernel is given by \( K(x, y) = \sum_{k \in \mathbb{Z}} \max(1, |k|)^{-2\alpha} \exp(2\pi i k (x-y)) \);
   
   here the eigenvalues are \( \max(1, |k|)^{-2\alpha} \) for \( k \in \mathbb{Z} \).

We consider now the unanchored and anchored Sobolev spaces.
2. Unanchored Sobolev space
The eigenvalues and eigenfunctions of the reproducing kernel \( K(x, y) = 1 + B_1(x)B_1(y) + \frac{1}{2}B_2(|x - y|) \) have been found in [14]. The eigenvalues are 1, \( \pi^{-2} \), \( (2\pi)^{-2} \), \( (3\pi)^{-2} \), \ldots and the eigenfunctions are 1, \( \sqrt{2} \cos(\pi x) \), \( \sqrt{2} \cos(2\pi x) \), \( \sqrt{2} \cos(3\pi x) \), \ldots.

3. Anchored Sobolev space
The eigenvalues and eigenfunctions of the reproducing kernel \( K(x, y) = 1 + \min(x, y) \) have been found in [96]. The eigenvalues are \( \lambda_\ell = \alpha_\ell^{-2} \) for all \( \ell \in \mathbb{N} \), where \( \alpha_\ell \in ((\ell - 1)\pi, \ell\pi) \) is the unique solution of the equation

\[
\tan \alpha_\ell = \frac{1}{\alpha_\ell}.
\]

We consider another related example where we derive the eigenvalues and eigenfunctions via a solution to an ODE. Namely, consider the function

\[
K(x, y) = \min(x, y).
\]  

This function is symmetric and positive semi-definite and therefore a reproducing kernel. We are interested in obtaining the eigenvalues and eigenfunctions of the operator

\[
T_K(f)(x) = \int_0^1 K(x, y)f(y) \, dy = \int_0^1 \min(x, y)f(y) \, dy.
\]

Let \( \lambda_\ell \) be an eigenvalue and \( \psi_\ell \) the corresponding eigenfunction. Then

\[
\lambda_\ell \psi_\ell(x) = \int_0^1 \min(x, y)\psi_\ell(y) \, dy = \int_0^x y\psi_\ell(y) \, dy + \int_x^1 x\psi_\ell(y) \, dy.
\]

By setting \( x = 0 \) we obtain

\[
\lambda_\ell \psi_\ell(0) = 0.
\]

By differentiating with respect to \( x \) we obtain

\[
\lambda_\ell \psi_\ell'(x) = \int_x^1 \psi_\ell(y) \, dy.
\]

Setting \( x = 1 \) in the above equation yields \( \lambda_\ell \psi_\ell'(1) = 0 \). By twice differentiating with respect to \( x \) we obtain

\[
\lambda_\ell \psi_\ell''(x) = -\psi_\ell(x).
\]

The function \( \psi_\ell \) which satisfies the two boundary conditions and the last ODE is given by

\[
\psi_\ell(x) = \sqrt{2} \sin \left( \left( \ell - \frac{1}{2} \right) \pi x \right)
\]

with corresponding eigenvalue

\[
\lambda_\ell = \left( \left( \ell - \frac{1}{2} \right) \pi \right)^{-2}
\]

for \( \ell \in \mathbb{N} \). The normalizing factor \( \sqrt{2} \) is introduced, such that the functions \( \psi_\ell \) are \( L_2 \) orthonormal.
Thus the reproducing kernel \((10)\) can be written as
\[
K(x, y) = \sum_{\ell=1}^{\infty} \sqrt{2} \sin((\ell - 1/2)\pi x) \sqrt{2} \sin((\ell - 1/2)\pi y). 
\]

Functions \(f_i\) in the corresponding reproducing kernel Hilbert space \(H_K\) have an expansion of the form
\[
f_i(x) = \sum_{\ell=1}^{\infty} \hat{f}_i(\ell) \sqrt{2} \sin((\ell - 1/2)\pi x) \tag{11}
\]
and the inner product is given by
\[
\langle f_1, f_2 \rangle_K = \sum_{\ell=1}^{\infty} \hat{f}_1(\ell) \overline{\hat{f}_2(\ell)} (\ell - 1/2)^2 \pi^2.
\]

2.4 Covariance kernel

The covariance kernel has many similarities with the reproducing kernel. We restrict ourselves again to the domain \([0, 1]^s\). A covariance kernel \(C : [0, 1]^s \times [0, 1]^s \to \mathbb{R}\) is again a symmetric and positive semi-definite function (and is therefore also a reproducing kernel).

In QMC theory, the covariance kernel has two different uses. One is the study of the so-called average-case error and the other appears in the study of PDEs with random coefficients, where the covariance kernel describes the underlying random coefficients (or random field). These two cases are based on different interpretations of the covariance kernel.

In the following we use the term ‘stochastic process’ in an informal manner without giving a definition of what a stochastic process is. However, we will provide some concrete examples below. More information on stochastic processes, martingales and stochastic differential equations can for instance be found in \([40, 80, 81]\).

1. Random function

Let \(\mathcal{H}\) be a function class defined on \([0, 1]^s\) and \(\mathcal{B}(\mathcal{H})\) be a \(\sigma\) algebra on \(\mathcal{H}\). Further let \(\mu\) be a probability measure defined on \((\mathcal{H}, \mathcal{B}(\mathcal{H}))\). Then we define the covariance kernel \(C : [0, 1]^s \times [0, 1]^s \to \mathbb{R}\) by
\[
C(x, y) = \int_{\mathcal{H}} f(x)f(y) \, \mu(df).
\]

That is, the covariance kernel is the expectation value over all functions in the class \(\mathcal{H}\) evaluated at the points \(x\) and \(y\). The functions \(f \in \mathcal{H}\) themselves are not random variables, but we choose \(f \in \mathcal{H}\) randomly, i.e., once a function \(f\) is chosen it is entirely deterministic.

2. Stochastic process

Let \(Z(x)\) be a stochastic process (or random field) defined on \([0, 1]^s\). Then the covariance kernel gives the covariance of the values of the process \(Z(x)\) at the locations \(x, y \in [0, 1]^s\)
\[
C(x, y) = \text{cov}(Z(x), Z(y)).
\]
For each value \( x \) in the domain \([0, 1]^{\mathbb{R}}\), the values \( Z(x) \) are random variables with some given distribution.

In the remainder of this subsection we deal with the first case of random functions.

**Example: Continuous functions and the Wiener sheet measure**

A classic result in QMC theory is concerned with the average case error of the set of continuous functions which vanish at 0 endowed with the Wiener sheet measure [98].

We give an example of how one can define a probability measure on a function space \( \mathcal{H} \). Let \( \mathcal{H} \) be the class of functions given by

\[
\mathcal{H} = \{ f : [0, 1] \to \mathbb{R} : f(0) = 0, f \text{ is continuous} \}.
\]

The functions \( \sqrt{2} \sin((\ell - 1/2)\pi x) \) \( \ell \in \mathbb{N} \) are \( L_2 \) orthonormal. First note that the functions in \( \mathcal{H} \) permit expansions of the form

\[
f(x) = \sum_{\ell=1}^{\infty} a_{\ell} \sqrt{2} \sin((\ell - 1/2)\pi x) / ((\ell - 1/2)\pi),
\]

i.e., every continuous function \( f \) which vanishes at 0 can be described by Eq. (12). We can identify a function \( f \in \mathcal{H} \) with the sequence of coefficients \( a = (a_\ell)_{\ell \in \mathbb{N}} \) via the injective mapping \( T : \mathcal{H} \to \mathbb{R}^N \), where \( T(f) = a \). To define a probability measure on \( \mathcal{H} \), it thus suffices to define a probability measure on the set of sequences \( a \).

In one dimension, we use the Gaussian distribution with mean 0 and variance 1 and for sequences we use the infinite product measure. That is, the measure of any interval \([b, c] := \prod_{j=1}^{\infty} [b_j, c_j] \), with \( b_j \leq c_j \), is given by

\[
Q([b, c]) := \prod_{j=1}^{\infty} \frac{1}{\sqrt{2\pi}} \int_{b_j}^{c_j} \exp \left( -\frac{x^2}{2} \right) \, dx.
\]

In other words, the probability that \( a \in [b, c] \) is given by \( Q([b, c]) \). This can then be extended to any Borel set \( A \in \mathbb{R}^N \). The Borel \( \sigma \)-algebra on \( \mathbb{R}^N \) defines a \( \sigma \)-algebra \( \mathcal{F} \) on \( \mathcal{H} \) via the mapping \( T \). The probability measure on \((\mathcal{H}, \mathcal{F})\) is now given by

\[
P(F) = Q(T(f)) \text{ for any } f \in \mathcal{F}.
\]

It is known that if one chooses the coefficients \( a_\ell \) in (12) i.i.d. with Gaussian distribution with mean 0 and variance 1, then the function \( f \) is almost surely continuous. This follows since a Wiener process or Brownian motion is almost surely continuous. This means that

\[
Q(\mathbb{R}^N \setminus T(\mathcal{H})) = 0.
\]

The covariance kernel is now given by

\[
C(x, y) = \int_{\mathcal{H}} f(x)f(y) \, P(df) = \sum_{k, \ell=1}^{\infty} E(a_k a_\ell) \frac{\sqrt{2} \sin((k - 1/2)\pi x)}{(k - 1/2)\pi} \frac{\sqrt{2} \sin((\ell - 1/2)\pi y)}{(\ell - 1/2)\pi}.
\]
The expectation value for \( k \neq \ell \) is 0, whereas for \( k = \ell \) it is 1, since the mean of \( a_k \) is 0 and the variance is 1. Thus

\[
C(x, y) = \sum_{\ell=1}^{\infty} \frac{\sqrt{2} \sin((\ell - 1/2)\pi x) \sqrt{2} \sin((\ell - 1/2)\pi y)}{(\ell - 1/2)\pi} = \min(x, y).
\]

**Average-case error**

We have seen how reproducing kernels can be used to give a formula for the worst-case error. We now provide an analogue for the covariance kernel and the average-case error.

Let \( \mathcal{H} \) be a function space defined on \([0, 1]^s\) and let \((\mathcal{H}, \mathcal{F}, \mathbb{P})\) be a probability space. For \(1 \leq p \leq \infty\) we define the \(L_p\) average-case error by

\[
\text{ace}_p(\mathcal{H}, \mathbb{P}) = \left( \int_{\mathcal{H}} \left| \int_{[0,1]^s} f(x) \, dx - \frac{1}{N} \sum_{x \in \mathcal{P}} f(x) \right|^p \mathbb{P}(df) \right)^{1/p},
\]

with the obvious modifications for \( p = \infty \).

We consider now the case \( p = 2 \). Let \( C : [0, 1]^s \times [0, 1]^s \to \mathbb{R} \) be the covariance kernel, that is

\[
C(x, y) = \int_{\mathcal{H}} f(x)f(y) \mathbb{P}(df).
\]

Then we have

\[
\text{ace}_2^2(\mathcal{H}, \mathbb{P}) = \int_{[0,1]^s} \int_{[0,1]^s} C(x, y) \, dx \, dy \mathbb{P}(df)
\]

\[
- \int_{\mathcal{H}} \frac{1}{N} \sum_{x \in \mathcal{P}} \left( \int_{[0,1]^s} f(x) \, dx \right) \mathbb{P}(df) + \int_{\mathcal{H}} \frac{1}{N^2} \sum_{x, y \in \mathcal{P}} f(x)f(y) \mathbb{P}(df)
\]

\[
= \int_{[0,1]^s} \int_{[0,1]^s} C(x, y) \mathbb{P}(df) \, dx \, dy
\]

\[
- \frac{1}{N} \sum_{x \in \mathcal{P}} \left( \int_{[0,1]^s} C(x, y) \mathbb{P}(df) \, dy \right) + \frac{1}{N^2} \sum_{x, y \in \mathcal{P}} C(x, y) \mathbb{P}(df)
\]

\[
= \int_{[0,1]^s} \int_{[0,1]^s} C(x, y) \, dx \, dy - \frac{1}{N} \sum_{x \in \mathcal{P}} \int_{[0,1]^s} C(x, y) \, dy + \frac{1}{N^2} \sum_{x, y \in \mathcal{P}} C(x, y).
\]

This formula is analogous to \((6)\). Since symmetric positive definite functions can be interpreted as reproducing kernels or covariance kernels, this allows one to interpret the error either as worst-case error or as average-case error (for a different function space). We refer the reader to [73, Chapter 24] for more information on covariance kernels and average-case errors.

### 2.5 Karhunen-Loéve expansion

The Karhunen-Loéve expansion of the covariance kernel follows from Mercer’s theorem by using the fact that the covariance kernel is also a reproducing kernel.
Theorem 2.3 Let \( Z(x) \) be a zero-mean square integrable stochastic process defined over a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) and indexed over the interval \([0, 1]^s\), with continuous covariance kernel \( C : [0, 1]^s \times [0, 1]^s \to \mathbb{R} \). Then \( C \) satisfies the conditions in Mercer’s theorem and we have the expansion

\[
C(x, y) = \sum_{\ell = 1}^{\infty} \lambda_{\ell} \psi_{\ell}(x) \psi_{\ell}(y),
\]

where \( \psi_{\ell} \) are \( L_2([0, 1]^s) \) orthonormal eigenfunctions with corresponding eigenvalues \((\lambda_{\ell})\). Then the stochastic process \( Z(x) \) admits the presentation

\[
Z(x) = \sum_{\ell = 1}^{\infty} \xi_{\ell} \sqrt{\lambda_{\ell}} \psi_{\ell}(x),
\]

where the convergence is in \( L_2 \) norm, uniform in \( x \) and

\[
\xi_{\ell} = \frac{1}{\sqrt{\lambda_{\ell}}} \int_0^1 Z(x) \overline{\psi_{\ell}(x)} \, dx.
\]

The random variables \( \xi_{\ell} \) have zero-mean, are uncorrelated and have variance 1.

The Karhunen-Loève expansion yields a bi-orthogonal expansion of a stochastic process, since the random variables \( \xi_{\ell} \) are uncorrelated and hence \( \mathbb{E}(\xi_{\ell} \xi_{k}) = \delta_{\ell,k} \), the Kronecker \( \delta \) symbol, and the eigenfunctions are \( L_2 \) orthonormal.

The Wiener process or Brownian motion can be expanded in terms of its Karhunen-Loève expansion, which we describe in the following.

Example: Karhunen-Loève expansion of Wiener process or Brownian motion

The covariance kernel of the Wiener process is given by

\[
W(x, y) = \min(x, y).
\]

We have analyzed the corresponding reproducing kernel in Section 2.3. Functions in the corresponding reproducing kernel Hilbert space have the expansion given in (11).

We can now use this expansion to describe a Wiener process (or also called Brownian motion) on the interval \([0, 1]\). Compared to its deterministic counterpart (i.e. functions in the corresponding reproducing kernel Hilbert space), the coefficients in the expansion are now random variables.

Let \( \xi_{\ell} \in \mathcal{N}(0, 1) \) for \( \ell \in \mathbb{N} \) be independent Gaussian random variables with mean 0 and variance 1. Then the Wiener process \( Z(x) \) has the expansion

\[
Z(x) = \sum_{\ell = 1}^{\infty} \xi_{\ell} \sqrt{\frac{2}{\pi}} \sin((\ell - 1/2)\pi x) \quad (\ell - 1/2)\pi.
\]

It is easy to see that the expectation value of \( Z(x) \) satisfies \( \mathbb{E}(Z(x)) = 0 \), since all \( \xi_{\ell} \) have mean 0. The covariance is now given by

\[
\text{cov}(Z(x), Z(y)) = \mathbb{E} \left( \sum_{\ell = 1}^{\infty} \xi_{\ell} \sqrt{\frac{2}{\pi}} \sin((\ell - 1/2)\pi x) \right) \left( \sum_{k = 1}^{\infty} \xi_{k} \sqrt{\frac{2}{\pi}} \sin((k - 1/2)\pi y) \right)
\]

\[
= \sum_{k, \ell = 1}^{\infty} \mathbb{E}(\xi_{\ell} \xi_{k}) \sqrt{\frac{2}{\pi}} \sin((\ell - 1/2)\pi x) \sqrt{\frac{2}{\pi}} \sin((k - 1/2)\pi y) \quad (\ell - 1/2)\pi \quad (k - 1/2)\pi.
\]
Since the random variables $\xi_\ell$ are independent with mean 0 we have $\mathbb{E}(\xi_\ell \xi_k) = 0$ for $k \neq \ell$. If $k = \ell$ it follows that $\mathbb{E}(\xi_\ell \xi_\ell) = 1$, since the variance of $\xi_\ell$ is also 1. Thus we have

$$\text{cov}(Z(x), Z(y)) = \sum_{\ell=1}^{\infty} \frac{\sqrt{2} \sin((\ell - 1/2)\pi x)}{(\ell - 1/2)\pi} \frac{\sqrt{2} \sin((\ell - 1/2)\pi y)}{(k - 1/2)\pi} = \min(x, y).$$

A smooth version of the Brownian motion can be obtained via integration. The covariance kernel of the integrated Brownian motion is discussed in [24].

**Partial differential equations with random coefficients**

As an application of stochastic processes we describe partial differential equations (PDE) with random coefficients.

We consider the physical domain $[0, 1]^d$ (usually $d = 1, 2, 3$). Let

$$a(x, z) = a_0(x) + \sum_{\ell=1}^{\infty} z_\ell \lambda_\ell \psi_\ell(x),$$

where $z = (z_1, z_2, \ldots)$. The $z_\ell$ are i.i.d. random variables with mean 0 and finite variance $\sigma$. In the simplest case, the $z_\ell$ are uniformly distributed in $[-1/2, 1/2]$, but other distributions can be studied as well. Then $a - a_0$ is a stochastic process with mean 0, or, in other words, the mean of $a$ is $a_0$. The underlying covariance kernel $C$ corresponding to $a - a_0$ is given by

$$C(x, y) = \sum_{\ell=1}^{\infty} \sigma \lambda_\ell^2 \psi_\ell(x) \overline{\psi_\ell(y)}.$$

We consider now the PDE

$$-\nabla \cdot (a(x, z) \nabla u(x, z)) = f(x) \text{ in } D = [0, 1]^d, \quad u(x, z) = 0 \text{ on } \partial D.$$

Since the $z_\ell$ are random variables, the solution $u$ of the PDE also depends on the random variables $z_\ell$, and is therefore also a random variable. One is for instance interested in approximating the expectation value of $u$ (or a linear functional of $u$). To approximate the expectation value of the solution $u$, one ansatz is to use QMC points to sample $(z_1, z_2, \ldots, z_s)$ for some large enough $s$, set $z_{s+1} = z_{s+2} = \ldots = 0$ and use a PDE solver to approximate $u(x, (z_1, z_2, \ldots, z_s, 0, 0, \ldots))$. Averaging the solution $u$ over all QMC points yields an approximation of the expectation value. Such a study is carried out in [46]. See also [25] where the covariance kernel was used directly to sample from the random field.

**2.6 Lower bounds using bump functions**

A standard approach to proving lower bounds involves so-called bump functions. Let $\mathcal{H}$ be a Banach space with norm $\| \cdot \|$. To prove a lower bound on the worst-case error one possible strategy is to construct a bump function. Let $\mathcal{P} = \{x_0, x_1, \ldots, x_{N-1}\} \subseteq [0, 1]^s$ be an arbitrary but fixed point set. The idea is to construct a function $f$ with the following properties:

1. $f(x_n) = 0$ for all $0 \leq n < N$;
2. \( \|f\| = 1 \);
3. \( \int_{[0,1]^s} f(x) \, dx \) is large.

If we can construct such a function \( f \) which satisfies those three properties, with \( \int_{[0,1]^s} f(x) \, dx \geq \varepsilon(N,s) \), say, then

\[
\inf_{\mathcal{P} \subseteq [0,1]^s} \text{wce}(\mathcal{H}, \mathcal{P}) \geq \varepsilon(N,s).
\]

We illustrate the idea in a simple example.

**Theorem 2.4** Let \( \mathcal{H}_K \) be the reproducing kernel Hilbert space with reproducing kernel \( K(x,y) = \prod_{j=1}^{s}(1 + \min(x_j, y_j)) \). Let \( \mathcal{P} = \{x_0, x_1, \ldots, x_{N-1}\} \subseteq [0,1]^s \) be an arbitrary point set. Then there exists a constant \( c_s > 0 \) independent of \( N \) and \( \mathcal{P} \) such that

\[
\text{wce}(\mathcal{H}_K, \mathcal{P}) \geq c_s \left( \frac{\log N}{N} \right)^{\frac{1}{s}}.
\]

**Proof.** To construct \( f \), we start with the one-dimensional case. One choice of a basic function \( \phi : \mathbb{R} \to \mathbb{R} \) is

\[
\phi(t) = \begin{cases} 
    t(1-t) & \text{if } 0 < t < 1, \\
    0 & \text{otherwise.}
\end{cases}
\]

(If one considers function spaces of smoothness \( r \), then one could use \( t^r(1-t)^r \).) The scaled and shifted versions are

\[
\phi(2^m t - a)
\]

for integers \( m \in \mathbb{N}_0 \) and \( 0 \leq a < 2^m \). The support of this scaled and shifted function is \([a/2^m, (a+1)/2^m]\).

Choose the integer \( m \) such that \( 2^{m-2} \leq N < 2^m \). Let \( m = (m_1, m_2, \ldots, m_s) \in \mathbb{N}_0^s \) and let \( |m| = m_1 + m_2 + \cdots + m_s \). Define \( D_j = \{0,1,\ldots,2^j-1\} \) and \( D_m = D_{m_1} \times \cdots \times D_{m_s} \). We can now define a function \( g_m \) which satisfies 1. by setting

\[
g_m(x) = \sum_{a \in D_m} \prod_{j=1}^{s} \phi(2^m x_j - a_j),
\]

where \( (a/2^m, (a+1)/2^m) = \prod_{j=1}^{s}(a_j/2^{m_j}, (a_j+1)/2^{m_j}) \). The condition \( (a/2^m, (a+1)/2^m) \cap \mathcal{P} = \emptyset \) ensures that \( g_m(x_n) = 0 \) for all \( 0 \leq n < N \). We define the function

\[
g(x) = \sum_{m \in \mathbb{N}_0^s \atop |m| = m} g_m(x).
\]

Again we have \( g(x_n) = 0 \) for all \( 0 \leq n < N \).

In the next step, we estimate the norm of \( g \) and then set \( f = g/\|g\|_K \). Then \( f \) also satisfies the second condition. The squared norm in our particular function space is given by

\[
\|h\|^2_K = \sum_{u \leq |s|} \int_{[0,1]^s} \left| \frac{\partial^u h}{\partial x_u}(x_u; 0) \right|^2 \, dx_u,
\]

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where \([s] = \{1, 2, \ldots, s\}\), and for \(u \subseteq [s]\) and \(x = (x_1, x_2, \ldots, x_s)\) we write \((x_u; 0)\) for the \(s\)-dimensional vector whose \(j\)th component is \(x_j\) for \(j \in u\) and 0 otherwise.

We consider now the norm of (13). Since for \(t = 0\) we have \(\phi(2^m t - a) = \phi(-a) = 0\) for all integers \(a\), we obtain that

\[
\|g\|_K^2 = \int_{[0,1]^s} \left| \frac{\partial^s g}{\partial x} (x) \right|^2 \, dx
\]

\[
= \sum_{m \in \mathbb{N}_0^s} \sum_{m' \in \mathbb{N}_0^s} \int_{[0,1]^s} \frac{\partial^s g_m}{\partial x} (x) \frac{\partial^s g_{m'}}{\partial x} (x) \, dx
\]

\[
= \sum_{m \in \mathbb{N}_0^s} \sum_{m' \in \mathbb{N}_0^s} \sum_{a \in \mathbb{D}_m} \sum_{a' \in \mathbb{D}_m'} \prod_{j=1}^s \int_{0}^{1} \phi' (2^{m_j} x - a_j) \phi' (2^{m'_j} x - a'_j) \, dx_j.
\]

For \(m_j \geq m'_j\) we have

\[
\int_{0}^{1} \phi' (2^{m_j} x - a_j) \phi' (2^{m'_j} x - a'_j) \, dx_j = \begin{cases} \frac{1}{2} 2^{m_j - m'_j} & \text{if } \left[ \frac{a_j}{2^{m_j}}, \frac{a_j + 1}{2^{m_j}} \right] \subseteq \left[ \frac{a'_j}{2^{m'_j}}, \frac{a'_j + 1}{2^{m'_j}} \right], \\ 0 & \text{otherwise.} \end{cases}
\]

Note that \([a_j 2^{-m_j}, (a_j + 1)2^{-m_j}]\) is the support of \(\phi' (2^{m_j} x - a_j)\). The condition that the support of \(\phi' (2^{m_j} x - a_j)\) is contained in the support of \(\phi' (2^{m'_j} x - a'_j)\) is equivalent to \(2^{m_j - m'_j} a_j \leq a'_j < 2^{m_j - m'_j} (a_j + 1)\). Thus for given \(a_j, m_j, m'_j\) there are \(2^{m_j - m'_j}\) possible choices for \(a'_j\). Thus we have

\[
\|g\|_K^2 \leq \frac{1}{3^s} \sum_{m \in \mathbb{N}_0^s} \sum_{m' \in \mathbb{N}_0^s} \prod_{j=1}^s 2^{\min\{m_j, m'_j\} - \max\{m_j, m'_j\}} 2^{m'_j} 2^{\max\{m_j, m'_j\} - \min\{m_j, m'_j\}}
\]

\[
= \frac{2^{2m}}{3^s} \sum_{m \in \mathbb{N}_0^s} \sum_{m' \in \mathbb{N}_0^s} \prod_{j=1}^s 2^{-|m_j - m'_j|/2}.
\]

For any fixed \(m \in \mathbb{N}_0^s\) we have

\[
\sum_{m' \in \mathbb{N}_0^s} \prod_{j=1}^s 2^{-|m_j - m'_j|/2} \leq \sum_{k \in \mathbb{Z}^s} 2^{-|k|/2} \leq \left( \sum_{k=-\infty}^{\infty} 2^{-|k|/2} \right)^s = \left( 1 + \sqrt{2} \right)^{2s}.
\]

This implies that

\[
\|g\|_K^2 \leq 2^{2m} \left( \frac{1 + \sqrt{2}}{3} \right)^s \sum_{m \in \mathbb{N}_0^s} 1 \leq 2^{2m} \left( \frac{m + s - 1}{s - 1} \right) \left( \frac{1 + \sqrt{2}}{3} \right)^s.
\]
Further we have

\[
\int_{[0,1]^s} g(x) \, dx = \sum_{m \in \mathbb{N}_0} \sum_{a \in \mathbb{D}_m} \prod_{j=1}^s \int_0^1 \phi(2^{m_j} x_j - a_j) \, dx_j
\]

\[= \sum_{m \in \mathbb{N}_0} \sum_{a \in \mathbb{D}_m} \frac{1}{2^m 6^s} \geq \left( \frac{m + s - 1}{s - 1} \right) \frac{2^m - N}{2^m 6^s} \geq \left( \frac{m + s - 1}{s - 1} \right) \frac{1}{2 \cdot 6^s}.
\]

Let now \( f = g/\|g\|_K \). Then we have \( \|f\|_K = 1 \) and there is a constant \( c_s > 0 \) such that

\[
\int_{[0,1]^s} f(x) \, dx = \frac{1}{\|g\|} \int_{[0,1]^s} g(x) \, dx \geq \frac{1}{2} \left( \frac{1}{2(\sqrt{3} + \sqrt{6})} \right)^s \frac{1}{2^m} \sqrt{\left( \frac{m + s - 1}{s - 1} \right)} \geq c_s \frac{(\log N)^{s-1}}{N}.
\]

Since \( f \) satisfies all three conditions, we obtain

\[\text{wce}(\mathcal{H}, \mathcal{P}) \geq c_s \frac{(\log N)^{s-1}}{N}\]

for any \( N \)-element point set \( \mathcal{P} \subseteq [0,1]^s \).

\[\square\]

### 2.7 The Rader transform

The Rader transform can be used to permute certain matrices such that the resulting matrices are circulant. Circulant matrices are very useful since a fast matrix-vector multiplication using the fast Fourier transform exists in this case. The Rader transform is used in the fast component-by-component construction of lattice rules (see Section 4.1) and polynomial lattice rules (see Section 4.3). The Rader transform in the context of the component-by-component construction was introduced in \([44, 75, 76]\).

We explain a special case of the Rader transform in the context of lattice rules. Let \( N \) be a prime number and let \( \omega : \{0, 1, \ldots, N-1\} \to \mathbb{R} \) be an arbitrary mapping. Let \( C = (c_{k,\ell})_{1 \leq k,\ell \leq N} \) be the \((N-1) \times (N-1)\) matrix with

\[c_{k,\ell} = \omega(k\ell \pmod{N}).\]

In the following we show how the Rader transform can be used to obtain permutation matrices \( P \) and \( Q \) such that \( PCQ \) is a circulant matrix. A matrix \( D = (d_{k,\ell}) \) is circulant if \( d_{k,\ell} = e_{k-\ell \pmod{N-1}} \) for some numbers \( e_0, e_1, \ldots, e_{N-2} \in \mathbb{R} \).

Let \( \mathbb{F}_N = \{0, 1, \ldots, N-1\} \) be the finite field of order \( N \) (we identify the elements in \( \mathbb{Z}_N \) with the integers \( 0, 1, \ldots, N-1 \)). Then there exists a primitive element \( g \in \mathbb{F}_N \), that is, the multiplicative group \( \mathbb{F}_N^\times \) of \( \mathbb{F}_N \) is given by

\[\mathbb{F}_N^\times = \{g^0, g^1, g^2, \ldots, g^{N-2}\}.
\]

Note that we always have \( g^{N-1} = 1 \). Let \( D = (d_{k,\ell}) \) where

\[d_{k,\ell} = e_{k-\ell \pmod{N-1}} = \omega(g^{k-\ell \pmod{N}}).\]
We define now the permutation matrix \( \Pi(g) = (\pi_{k,\ell}(g))_{1 \leq k,\ell < N} \) by

\[
\pi_{k,\ell}(g) = \begin{cases} 
1 & \text{if } \ell = g^k \pmod{N}, \\
0 & \text{otherwise}.
\end{cases}
\]

Then we have

\[
D = \Pi(g)C\Pi(g^{-1})^\top
\]

and the matrix \( D \) is a circulant matrix, since

\[
d_{k,\ell} = \sum_{u,v=1}^{N-1} \pi_{k,u}(g)c_{u,v}\pi_{\ell,v}(g^{-1}) = c_{g^k,g^{-\ell}} = \omega(g^{k-\ell} \pmod{N}).
\]

3 Harmonic Analysis

Methods from harmonic analysis used in QMC range from basic applications of orthogonality like Parseval’s equality and Bessel’s inequality to sophisticated tools like Riesz products and Littlewood-Paley theory. In this section we explain some of the tools by showing some central results in simplified settings.

3.1 Orthogonal bases - error bounds for QMC

Orthogonal bases in \( L_2([0,1]^s) \) useful for the analysis of errors of QMC rules and discrepancy estimates are

- the trigonometric bases
- Walsh bases
- Haar bases.

The first two are systems of characters on \([0,1]^s\) with respect to different group structures which makes them very suitable for the analysis of point sets respecting that group structure (see Section 4.4). The Haar bases have the advantage that the orthogonal functions are local and can be used to characterize function spaces through wavelet decompositions.

The trigonometric system contains the trigonometric functions defined by \( e_k : [0,1]^s \to \mathbb{C} \) for \( k \in \mathbb{Z}^s \) by

\[
e_k(x) = \exp(2\pi ik \cdot x) \quad \text{for } x \in [0,1]^s,
\]

where \( \cdot,\cdot \) denotes the usual inner product in \( \mathbb{R}^s \). The trigonometric system is an orthonormal bases of the Hilbert space \( L_2([0,1]^s) \) whose inner product we denote with \( \langle \cdot,\cdot \rangle \). One main application of the trigonometric system in QMC is the error analysis of lattice rules. Lattices and lattice rules are discussed in more detail in Section 4.1. Here we consider for simplicity just rank-1 lattice rules, which are of the form

\[
\mathcal{P}(g,N) = \left\{ \left\{ \frac{n}{N}g \right\} : n = 0,1,\ldots,N-1 \right\}
\]

for some \( N \in \mathbb{N}, N \geq 2 \) and some generator \( g \in \mathbb{Z}^s \), where the fractional part function \( \{\cdot\} \) is applied component-wise.
Example: Error analysis of rank-1 lattice rules

Let $f: \mathbb{R}^s \to \mathbb{C}$ be a 1-periodic function (in each variable) with absolutely convergent Fourier series

$$f = \sum_{k \in \mathbb{Z}^s} \hat{f}(k)e_k$$

with the Fourier coefficients $\hat{f} = \langle f, e_k \rangle$. By periodicity, the rank-1 lattice rule with generator $g$ can be written as

$$\int_{[0,1]^s} f(x) \, dx \approx \frac{1}{N} \sum_{n=0}^{N-1} f\left(\frac{ng}{N}\right).$$

Since the integral is just $\hat{f}(0)$, we get for the error

$$\frac{1}{N} \sum_{n=0}^{N-1} f\left(\frac{ng}{N}\right) - \int_{[0,1]^s} f(x) \, dx = \frac{1}{N} \sum_{n=0}^{N-1} \sum_{k \in \mathbb{Z}^s} \hat{f}(k)e_k\left(\frac{ng}{N}\right) - \hat{f}(0)$$

$$= \sum_{k \in \mathbb{Z}^s} \hat{f}(k) \frac{1}{N} \sum_{n=0}^{N-1} e_k\left(\frac{ng}{N}\right) - \hat{f}(0)$$

$$= \sum_{k \in \mathbb{Z}^s \setminus \{0\}} \hat{f}(k) \frac{1}{N} \sum_{n=0}^{N-1} e_k\left(\frac{ng}{N}\right).$$

Now the orthogonality of the trigonometric functions implies that the inner sum is 1 if $k \cdot g \equiv 0 \pmod{N}$ and 0 otherwise (see also Lemma 1.3 in Section 4.4), hence

$$\frac{1}{N} \sum_{n=0}^{N-1} f\left(\frac{ng}{N}\right) - \int_{[0,1]^s} f(x) \, dx = \sum_k \hat{f}(k),$$

where the last sum runs only over those $k \neq 0$ with $k \cdot g \equiv 0 \pmod{N}$. This condition defines the dual lattice (cf. Section 4.4), and the error characterization can be extended accordingly to general lattices. Smoothness conditions on $f$ can be encoded in decay conditions for the Fourier coefficients. So, to get a small error for the integration of smooth functions, the lattice generator should be chosen such that the dual lattice avoids the Fourier coefficients with large $k$. For more information we refer to [68, 86].

As the trigonometric system is well adapted to study lattice rules, Walsh bases can be similarly used for digital constructions, see Section 4.2.

We now turn to the Haar system. We restrict to the base 2 case, applications of Haar bases in base $b \geq 2$ can be found in [57, 58, 59]. A dyadic interval of length $2^{-j}$, $j \in \mathbb{N}_0$, in $[0,1)$ is an interval of the form

$$I = I_{j,m} := \left[\frac{m}{2^j}, \frac{m+1}{2^j}\right) \text{ for } m = 0, 1, \ldots, 2^j - 1.$$ 

The left and right half of $I = I_{j,m}$ are the dyadic intervals $I^+ = I^+_{j,m} = I_{j+1,2m}$ and $I^- = I^-_{j,m} = I_{j+1,2m+1}$, respectively. The Haar function $h_I = h_{j,m}$ with support $I$ is the
function on $[0,1)$ which is $+1$ on the left half of $I$, $-1$ on the right half of $I$ and $0$ outside of $I$. The $L_{\infty}$-normalized Haar system consists of all Haar functions $h_{j,m}$ with $j \in \mathbb{N}_0$ and $m = 0, 1, \ldots, 2^j - 1$ together with the indicator function $h_{-1,0}$ of $[0,1)$. Normalized in $L_2([0,1))$ we obtain the orthonormal Haar basis of $L_2([0,1))$.

Let $\mathbb{N}_{-1} = \{-1, 0, 1, 2, \ldots\}$ and define $\mathbb{D}_j = \{0, 1, \ldots, 2^j-1\}$ for $j \in \mathbb{N}_0$ and $\mathbb{D}_{-1} = \{\emptyset\}$ for $j = -1$. For $j = (j_1, \ldots, j_s) \in \mathbb{N}_{-1}^s$ and $m = (m_1, \ldots, m_s) \in \mathbb{D}_j := \mathbb{D}_{j_1} \times \ldots \times \mathbb{D}_{j_s}$, the Haar function $h_{j,m}$ is given as the tensor product

$$h_{j,m}(x) = h_{j_1,m_1}(x_1) \cdots h_{j_s,m_s}(x_s) \quad \text{for } x = (x_1, \ldots, x_s) \in [0,1]^s.$$ 

The boxes

$$I_{j,m} = I_{j_1,m_1} \times \ldots \times I_{j_s,m_s}$$

are called dyadic boxes. Two boxes $I_{j_1,m_1}$ and $I_{j_2,m_2}$ have the same shape if $j_1 = j_2$. A crucial combinatorial property is that for $j = (j_1, \ldots, j_s) \in \mathbb{N}_0^s$, there are exactly $2^{j_1 + \cdots + j_s}$ boxes of that shape which are mutually disjoint. If we fix the level $\ell = j_1 + \cdots + j_s$, then there are

$$\binom{\ell + s - 2}{s - 1} \approx \ell^s$$

different shapes of boxes with level $\ell$.

The $L_{\infty}$-normalized tensor Haar system consists of all Haar functions $h_{j,m}$ with $j \in \mathbb{N}_{-1}^s$ and $m \in \mathbb{D}_j$. Normalized in $L_2([0,1)^s)$ we obtain the orthonormal Haar basis of $L_2([0,1)^s)$.

**Example: Error analysis of QMC with Hammersley point sets**

The Haar coefficients can be used directly to compute and estimate the norm of the discrepancy function. As an example, we compute the $L_2$-discrepancy (see Section 2.2) of the two-dimensional symmetrized Hammersley type point set given by

$$\mathcal{R}_n = \left\{ \left( \frac{t_1}{2} + \frac{t_{n-1}}{2^2} + \cdots + \frac{t_1}{2^n}, \frac{s_1}{2} + \frac{s_2}{2^2} + \cdots + \frac{s_n}{2^n} \right) : t_1, \ldots, t_n \in \{0,1\} \right\}$$

where $s_i = t_i$ if $i$ is even and $s_i = 1 - t_i$ if $i$ is odd. The cardinality of this set is $N = 2^n$. It was shown in [27] that these sets satisfy the $L_2$ discrepancy estimate

$$D_N(\mathcal{R}_n, L_2) \ll \frac{\sqrt{\log N}}{N},$$

which is optimal according to Theorem 3.2 in the next section. An exact formula for $D_N(\mathcal{R}_n, L_2)$ and a generalization of the result can be found in [42].

Direct, but in some cases a little tedious computations, for which we refer to [36], give the Haar coefficients $\mu_{j,m} = \langle D_N(\mathcal{R}_n, \cdot), h_{j,m} \rangle$ as follows:

**Lemma 3.1** Let $j = (j_1, j_2) \in \mathbb{N}_0^2$. Then

(i) if $j_1 + j_2 < n - 1$ and $j_1, j_2 \geq 0$ then $|\mu_{j,m}| = 2^{-(n+1)}$.

(ii) if $j_1 + j_2 \geq n - 1$ and $0 \leq j_1, j_2 \leq n$ then $|\mu_{j,m}| \leq 2^{-(n+j_1+j_2+1)}$ and $|\mu_{j,m}| = 2^{-(j_1+j_2+2)}$ for all but at most $2^n$ coefficients $\mu_{j,m}$ with $m \in \mathbb{D}_j$.

(iii) if $j_1 \geq n$ or $j_2 \geq n$ then $|\mu_{j,m}| = 2^{-(j_1+j_2+2)}$. 

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Now let $j = (-1, k)$ or $j = (k, -1)$ with $k \in \mathbb{N}_0$. Then

(iii) if $k < n$ then $|\mu_{j,m}| \leq 2^{-(n+k)}$. 

(iv) if $k \geq n$ then $|\mu_{j,m}| = 2^{-(2k+3)}$.

Finally,

(v) $|\mu_{(-1,-1),(0,0)}| = a 2^{-(n+3)} + 2^{-2(n+1)}$ with $a = 4$ if $n$ is even and $a = 3$ if $n$ is odd.

Then using these Haar coefficients in Parseval’s equality

$$D_N(R_n, L_2)^2 = \sum_{j \in \mathbb{N}^2_0} \sum_{m \in \mathbb{D}_j} \frac{\mu_{j,m}^2}{\|h_{j,m}\|_2^2}$$

gives the upper bound

$$D_N(R_n, L_2)^2 \ll \frac{n}{2^{2n}} = \log \frac{N}{N^2}.$$ 

Using the Littlewood-Paley inequality, which is explained in Section 3.3 as replacement for Parseval’s equality also provides optimality of the symmetrized Hammersley set for the $L_p$-discrepancy for $1 < p < \infty$. Similarly, optimality can be shown in Besov spaces of dominating mixed smoothness for certain parameter values, as these can be characterized by an equivalent norm via Haar coefficients, see [36, 91, 92]. For generalizations to higher dimensions, see [58, 59]. Faber bases can be used to derive error bounds in cases where the Haar functions do not work, see e.g. [93, 94].

3.2 Orthogonal functions - lower bounds

The crucial idea for proving lower bounds of norms of the discrepancy function is that the contribution of dyadic boxes containing no point can be amplified with the help of orthogonality. This idea is due to Roth [82].

**Example: Roth’s lower bound for the $L_2$-discrepancy**

**Theorem 3.2 (Roth)** The $L_2$-discrepancy of any $N$-element point set $\mathcal{P} \subseteq [0,1)^s$ satisfies the lower bound

$$D_N(\mathcal{P}, L_2) \gg_s \frac{(\log N)^{(s-1)/2}}{N}.$$ 

**Proof.** Roth used, together with orthogonality, also duality and the Cauchy-Schwarz inequality. We present here a version of the proof which just uses Bessel’s inequality and Haar functions. To this end, we need the inner products of the discrepancy function with the Haar functions. The following two lemmas separately deal with the volume part $\text{vol}(B_x)$ and the counting part $\frac{1}{N} \sum_{z \in \mathcal{P}} 1_{B_x}(z)$ of the discrepancy function. Both are easy calculations which can be reduced to the one-dimensional case using the product structure of the involved functions.

**Lemma 3.3 (Volume part)** Let $j = (j_1, \ldots, j_s) \in \mathbb{N}_0^s$ and $m \in \mathbb{D}_j$. Then

$$\langle x_1 \cdots x_s, h_{j,m}(x) \rangle = 2^{-2j_1-\cdots-2j_s-2}.$$ 

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Lemma 3.4 (Counting part) Let $j = (j_1, \ldots, j_s) \in \mathbb{N}_0^s$ and $m \in \mathbb{D}_j$. Then

$$\langle 1_{B_k}(z), h_{j,m}(x) \rangle = 0$$

whenever $z$ is not contained in the dyadic box supporting $h_{j,m}$.

Now we choose a level $\ell$ such that $2^{\ell-1} < 2N \leq 2^\ell$, so that $\ell \approx \log N$. Then, for each shape in level $\ell$, at least half of the $2^\ell$ dyadic boxes of this shape do not contain any points of $\mathcal{P}$. So, in the computation of the corresponding Haar coefficients of the discrepancy function, the counting part does not count. Let $S$ be the set of all pairs $(j, m)$ such that $I_{j,m}$ does not contain any points of $\mathcal{P}$ and is of level $\ell$. We then obtain from Bessel’s inequality and Lemma 3.3 that

$$D_N(\mathcal{P}, L_2)^2 \geq \sum_{(j,m) \in S} 2^\ell |D_N(\mathcal{P}, \cdot), h_{j,m}|^2 = 2^{-3\ell-4} \#S \approx s 2^{-2\ell} \ell^{s-1}$$

proving the theorem.

By taking more care of the number of empty boxes the best known lower bounds for the $L_2$-discrepancy are derived in [37].

3.3 Littlewood-Paley inequality

The Littlewood-Paley inequality provides a tool which can be used to replace Parseval’s equality and Bessel’s inequality for functions in $L_p(\mathbb{R})$ with $1 < p < \infty$. It involves the square function $S(f)$ of a function $f \in L_p([0,1])$ which is given as

$$S(f) = \left( \sum_{j,m} 2^{2j} \langle f, h_{j,m} \rangle^2 1_{I_{j,m}} \right)^{1/2}.$$

Theorem 3.5 (Littlewood-Paley inequality) Let $1 < p < \infty$ and let $f \in L_p([0,1])$. Then

$$\|S(f)\|_p \approx_p \|f\|_p.$$

This equivalence of norms between the function and its square function can be generalized to arbitrary dimension $s \in \mathbb{N}$ in the obvious way. This leads to a short direct proof of the lower bound of Schmidt [84] for the $L_p$-discrepancy.

Example: Schmidt’s lower bound for the $L_p$-discrepancy

Theorem 3.6 (Schmidt) Let $1 < p < \infty$. The $L_p$-discrepancy of any $N$-element point set $\mathcal{P} \subseteq [0,1)^s$ satisfies the lower bound

$$D_N(\mathcal{P}, L_p) \gg_{s,p} \frac{\log N (s-1)/2}{N}.$$
Proof. Of course, for \( p \geq 2 \) this follows immediately from Roth’s Theorem 3.2. For the general case we proceed as in the proof of that theorem but we use the Littlewood-Paley inequality instead of Bessel’s inequality and obtain

\[
D_N(\mathcal{P}, L_p)^p \gg_{s,p} \int_{[0,1]^d} \left| \sum_{(j,m) \in S} 2^{2\ell}(D_N(\mathcal{P}, \cdot), h_{j,m})^2 1_{I_{j,m}(x)} \right|^{p/2} dx.
\]

Now the Haar coefficients from Lemma 3.3 and Lemma 3.4 show that

\[
\langle D_N(\mathcal{P}, \cdot), h_{j,m} \rangle = 2^{-2\ell-2}
\]

which implies

\[
D_N(\mathcal{P}, L_p)^p \gg_{s,p} \int_{[0,1]^d} \left( \sum_{(j,m) \in S} 1_{I_{j,m}(x)} \right)^{p/2} dx.
\]

Now observe that for each fixed \( j \), the sum \( \sum_{m: (j,m) \in S} 1_{I_{j,m}(x)} \) is the indicator function of a set of measure at least \( \frac{1}{2} \). Hence

\[
\sum_{(j,m) \in S} 1_{I_{j,m}(x)} = \sum_{k=1}^M 1_{A_k}(x)
\]

where each \( A_k \) has measure at least \( \frac{1}{2} \) and \( M = \binom{\ell + s - 2}{s - 1} \approx 2^s \ell^{s-1} \) is the number of different shapes of boxes with level \( \ell \). But then \( \sum_{k=1}^M 1_{A_k}(x) \geq \frac{M}{4} \) on a set of measure at least \( \frac{1}{4} \), so that we obtain

\[
D_N(\mathcal{P}, L_p)^p \gg_{s,p} 2^{-(\ell+2)p} \frac{1}{4} \left( \frac{M}{4} \right)^{p/2} \gg_{s,p} \left( \frac{(\log N)^{(s-1)/2}}{N} \right)^{p}
\]

proving the theorem.

The Littlewood-Paley decomposition lends itself to the analysis of functions in further function spaces in harmonic analysis like BMO and \( \exp(L^\alpha) \), see [7], Hardy spaces \( H_p \) for \( 0 < p < 1 \), see [47], and spaces of dominating mixed smoothness, see [36, 57, 58, 59]. A recent survey of Roth’s method and its extensions is [6].

### 3.4 Riesz products

The Littlewood-Paley approach from the previous section is not directly applicable to the endpoints \( p = 1, \infty \). But Riesz products, another tool from harmonic analysis, can be used to prove sharp lower bounds in the case \( p = 1 \) and \( s = 2 \). This approach is due to Halász [26].

**Example: Halász’ lower bound for the \( L_1 \)-discrepancy**

**Theorem 3.7 (Halász)** The \( L_1 \)-discrepancy of any \( N \)-element point set \( \mathcal{P} \subseteq [0,1]^2 \) satisfies the lower bound

\[
D_N(\mathcal{P}, L_1) \gg \sqrt{\frac{\log N}{N}}.
\]
Proof. [Sketch] We again start as in the proof of Theorem 3.2 and choose a level \( \ell \) such that \( 2^{\ell-1} < 2N \leq 2^\ell \), so that \( \ell \approx \log N \). Observe that the shape of a rectangle in level \( \ell \) is now fixed by the parameter \( j = j_1 \) fixing the size in the first coordinate direction. Now for each such \( j = 0, 1, \ldots, \ell \) we add up the Haar functions of all dyadic rectangles \( I_{(j, \ell-j), m} \) which do not contain points of \( \mathcal{P} \) and obtain orthogonal functions \( f_0, f_1, \ldots, f_\ell \) which only take values \( \pm 1 \) and 0. Moreover, since we add up at least \( 2^{\ell-1} \) such Haar functions, we obtain from Lemma 3.3 and Lemma 3.4 that

\[
\langle D_N(\mathcal{P}, \cdot), f_j \rangle \geq 2^{\ell-1} 2^{2\ell-2} = 2^{\ell-3} \approx \frac{1}{N}.
\]

These functions are now used to build up the Riesz product

\[
F := \prod_{j=0}^{\ell} \left(1 + \frac{ic}{\sqrt{\ell + 1}} f_j \right) - 1 = \frac{ic}{\sqrt{\ell + 1}} \sum_{j=0}^{\ell} f_j + R
\]

with some small \( c > 0 \). Here the function \( R \) collects all the products of two and more Haar functions involved. It follows that

\[
||\langle D_N(\mathcal{P}, \cdot), F \rangle|| \geq c \frac{\sqrt{\ell + 1}}{2^{\ell+3}} - ||\langle D_N(\mathcal{P}, \cdot), R \rangle||.
\]

Now the property that arbitrary products of the Haar functions involved are again Haar functions on a higher level, one can show that \( ||\langle D_N(\mathcal{P}, \cdot), R \rangle|| \) is small compared with \( c \frac{\sqrt{\ell + 1}}{2^{\ell+3}} \) if \( c \) is chosen sufficiently small, but independent of \( N \). The second crucial property of \( F \) is that

\[
||F||_{\infty} \leq \left|1 + \frac{ic}{\sqrt{\ell + 1}} \right|^{\ell + 1} + 1 = \left(1 + \frac{c^2}{\ell + 1} \right)^{\ell + 1} + 1 \leq \exp \left(\frac{c^2}{2}\right) + 1,
\]

which motivates the use of complex numbers. It follows that

\[
D_N(\mathcal{P}, L_1) \geq \frac{||\langle D_N(\mathcal{P}, \cdot), F \rangle||}{||F||_{\infty}} \gg \frac{\sqrt{\ell + 1}}{2^\ell} \gg \frac{\log N}{N}.
\]

\[\square\]

The proof of Halász provides the sharp lower bound for the \( L_1 \)-discrepancy in dimension \( s = 2 \). The same bound is the best known lower bound also for higher dimensions. It is one of the main open problems in discrepancy theory to improve this lower bound.

4 Algebra and Number Theory

Algebra and Number Theory enter the stage of QMC through the various constructions of point sets with good equidistribution properties, which are required as sample nodes for QMC algorithms, and their analysis. Almost all constructions of point sets and sequences relevant for QMC are based on number theoretic or algebraic concepts.
4.1 Lattices

Lattices are an important concept in number theory, especially in the geometry of numbers which play also an important role in the construction of point sets and QMC rules.

Definition 4.1 A lattice \( L \) in \( \mathbb{R}^s \) is a discrete subset of \( \mathbb{R}^s \) which is closed under addition and subtraction.

Note that a lattice contains the origin. For every lattice \( L \) in \( \mathbb{R}^s \) there exists a lattice basis which is a set \( \{ w_1, w_2, \ldots, w_s \} \) of linearly independent vectors such that the lattice consists exactly of all integer linear combinations of \( w_1, w_2, \ldots, w_s \). The \( s \times s \) matrix \( W \) with rows \( w_1, w_2, \ldots, w_s \) is called the generator matrix of \( L \) and the determinant of \( L \) denoted by \( \det(L) \) is the absolute value of the determinant of the generator matrix \( W \). We note that the lattice bases, and therefore also \( W \), are not uniquely determined but it can be shown that \( \det(L) \) is an invariant for the lattice \( L \).

Information on lattice rules in the context of QMC can be found in [54, 68, 86]. In the following we present the two basic examples.

Example: General lattice rules

For \( x, y \in \mathbb{R}^s \) we say that the equivalence relation \( x \sim y \) holds iff there exists some \( z \in \mathbb{Z}^s \) such that \( x = y + z \). We define the equivalence classes \( x + \mathbb{Z}^s = \{ x + z \in \mathbb{R}^s : z \in \mathbb{Z}^s \} \). By \( \mathbb{R}^s/\mathbb{Z}^s \) we denote the set of all equivalence classes \( x + \mathbb{Z}^s \) of \( \mathbb{R}^s \) modulo \( \mathbb{Z}^s \), equipped with the addition \(( x + \mathbb{Z}^s ) + ( y + \mathbb{Z}^s ) := ( x + y ) + \mathbb{Z}^s \), where \( x + y \) denotes the usual addition in \( \mathbb{R}^s \). With these definitions \( \mathbb{R}^s/\mathbb{Z}^s \) becomes an abelian group.

Let \( L/\mathbb{Z}^s \) be any finite subgroup of \( \mathbb{R}^s/\mathbb{Z}^s \) and let \( x_n + \mathbb{Z}^s \) with \( x_n \in [0,1)^s \) for \( n = 0, 1, \ldots, N - 1 \) be the distinct residue classes which form the group \( L/\mathbb{Z}^s \). Then the set \( \{ x_0, x_1, \ldots, x_{N-1} \} \) is said to be the node set of the lattice rule \( L \). If we view \( L = \bigcup_{n=0}^{N-1} ( x_n + \mathbb{Z}^s ) \) as a subset of \( \mathbb{R}^s \), then \( L \) is an \( s \)-dimensional lattice.

Example: Rank-1 lattice rules

For \( N \in \mathbb{N}, N \geq 2, s \in \mathbb{N} \) and \( g \in \mathbb{Z}^s \) an \( N \)-element rank-1 lattice point set \( \mathcal{P}(g, N) = \{ x_0, x_1, \ldots, x_{N-1} \} \) is defined by

\[
x_n = \left\{ \frac{n}{N} g \right\} \quad \text{for} \quad n = 0, 1, \ldots, N - 1,
\]

where the fractional part \( \left\{ \cdot \right\} \) is applied component-wise. QMC rules that use rank-1 lattice point sets as underlying nodes are called (rank-1) lattice rules. The residue classes \( x_n + \mathbb{Z}^s = (n/N)g + \mathbb{Z}^s \) for \( n = 0, 1, \ldots, N - 1 \) corresponding to a lattice point set as defined in (14) form a finite cyclic subgroup of the additive group \( \mathbb{R}^s/\mathbb{Z}^s \) generated by \((1/N)g + \mathbb{Z}^s\). Hence (rank-1) lattice rules are a sub-class of general lattice rules.

Rank-1 lattice point sets can also be viewed as finite versions of Kronecker sequences \( S_\alpha = (x_n)_{n \geq 0} \) which are defined as

\[
x_n = \left\{ n \alpha \right\} \quad \text{for} \quad n \in \mathbb{N}_0,
\]

where \( \alpha \in \mathbb{R}^s \) and where the fractional part \( \left\{ \cdot \right\} \) is again applied component-wise. See [21, 44] or Section 4.8 for more information. The discrepancy of rank-1 lattice point sets will be discussed in Section 4.6 and the one of Kronecker sequences in Section 4.8.
4.2 Digital constructions

Digit expansions are a basic concept in Number Theory which also have applications in QMC or, in more detail, in the construction of QMC points and sequences.

Let $b \geq 2$ be an integer. Every $n \in \mathbb{N}_0$ can be expanded in its $b$-adic digit expansion $n = n_0 + n_1 b + n_2 b^2 + \cdots$ with $b$-adic digits $n_i \in \mathbb{Z}_b$, where we set $\mathbb{Z}_b := \{0, 1, \ldots, b-1\}$. A large class of constructions of QMC point sets is based on manipulations of these $b$-adic digit expansions. We remark that such constructions not only exist for $b$-adic expansions but also for more general expansions such as, e.g., Ostrowski expansions, $\beta$-adic expansions, $Q$-adic expansions, etc. However, the $b$-adic expansions are the most important ones in this context. In the following we present some examples. More information on the following examples can be found in [17, 54, 68] and the references therein.

Example: Van der Corput sequences

For an integer $b \geq 2$ the $b$-adic radical inverse function $\phi_b : \mathbb{N}_0 \to [0, 1)$ is defined by

$$\phi_b(n) = \frac{n_0}{b} + \frac{n_1}{b^2} + \frac{n_2}{b^3} + \cdots$$

whenever $n \in \mathbb{N}_0$ has $b$-adic digit expansion $n = n_0 + n_1 b + n_2 b^2 + \cdots$ (which is of course finite) with all digits $n_j \in \mathbb{Z}_b$. The $b$-adic van der Corput sequence is the one-dimensional sequence $S_b = (x_n)_{n \geq 0}$, where $x_n = \phi_b(n)$. This sequence is the prototype of many other digital constructions of point sets and sequences. It is well-known that the discrepancy of van der Corput sequences satisfies $D_N(S_b) \ll_b (\log N)/N$ (see, e.g., [17, 44, 54, 68]).

Example: Halton sequences

For $s \in \mathbb{N}$, $s \geq 2$, and for integers $b_1, \ldots, b_s \geq 2$ the Halton sequence $S_{b_1, \ldots, b_s} = (x_n)_{n \geq 0}$ in bases $b_1, \ldots, b_s$ is defined by

$$x_n = (\phi_{b_1}(n), \ldots, \phi_{b_s}(n)) \quad \text{for} \quad n = 0, 1, \ldots,$$

where $\phi_b$ is the $b$-adic radical inverse function. A Halton sequence in bases $b_1, \ldots, b_s$ is uniformly distributed in $[0, 1)^s$ if and only if $b_1, \ldots, b_s$ are mutually co-prime. In this case the discrepancy of the Halton sequence satisfies $D_N(S_{b_1, \ldots, b_s}) \ll_{s, b_1, \ldots, b_s} (\log N)^s/N$ (see, e.g., [17, 54, 68]).

Example: Hammersley point sets

For $s, N \in \mathbb{N}$, $s \geq 2$ and for pairwise coprime integers $b_1, \ldots, b_{s-1} \geq 2$ the $N$-element Hammersley point set $P_{b_1, \ldots, b_{s-1}} = \{x_0, x_1, \ldots, x_{N-1}\}$ in bases $b_1, \ldots, b_{s-1}$ is defined by

$$x_n := \left(\frac{n}{N}, \phi_{b_1}(n), \ldots, \phi_{b_{s-1}}(n)\right) \quad \text{for} \quad n = 0, 1, \ldots, N - 1.$$

If $b_1, \ldots, b_{s-1}$ are mutually co-prime, then the discrepancy of the Hammersley point set satisfies $D_N(P_{b_1, \ldots, b_{s-1}}) \ll_{s, b_1, \ldots, b_s} (\log N)^{s-1}/N$ (see, e.g., [17, 54, 68]).
Example: Digital nets

The construction of digital nets is based on finite rings \( R_b \) of order \( b \). Here we restrict our discussion to the case where \( R_b \) is the finite field \( \mathbb{F}_b \) of prime-power order \( b \). First one requires a bijection \( \varphi: \mathbb{Z}_b \to \mathbb{F}_b \) and \( m \times m \) matrices \( C_1, \ldots, C_s \) over \( \mathbb{F}_b \) (one per component). A digital net \( \{x_0, x_1, \ldots, x_{b^m-1}\} \) over \( \mathbb{F}_b \) with generating matrices \( C_1, \ldots, C_s \) is constructed in the following way: for \( n = 0, 1, \ldots, b^m-1 \) write \( n \) in its base \( b \) expansion \( n = n_0 + n_1 b + \cdots + n_{m-1} b^{m-1} \) with digits \( n_j \in \mathbb{Z}_b \). For \( j \in [s] \) compute the matrix vector product

\[
C_j \begin{pmatrix} \varphi(n_0) \\ \varphi(n_1) \\ \vdots \\ \varphi(n_{m-1}) \end{pmatrix} =: \begin{pmatrix} \overline{y}_{n,j,1} \\ \overline{y}_{n,j,2} \\ \vdots \\ \overline{y}_{n,j,m} \end{pmatrix},
\]

where all arithmetic operations are carried out in \( \mathbb{F}_b \), set

\[
x_{n,j} := \varphi^{-1}(\overline{y}_{n,j,1}) + \varphi^{-1}(\overline{y}_{n,j,2}) + \cdots + \varphi^{-1}(\overline{y}_{n,j,m})
\]

and put

\[
x_n := (x_{n,1}, \ldots, x_{n,s}).
\]

If the order \( b \) of the underlying finite field is a prime number, then one often identifies \( \mathbb{F}_b \) with the set \( \mathbb{Z}_b \) equipped with arithmetic modulo \( b \). In this case it is convenient to choose for the bijection \( \varphi \) the identity.

Depending on the choice of the generating matrices, digital nets can achieve a discrepancy of order \( (\log N)^{s-1}/N \). We refer to [17, 54, 68] for more information on the discrepancy of digital nets.

Example: Digital sequences

The construction of digital sequences over \( \mathbb{F}_b \) is analogous to the one of digital nets over \( \mathbb{F}_b \) with the difference that one requires \( C_1, \ldots, C_s \) to be \( \mathbb{N} \times \mathbb{N} \) matrices over \( \mathbb{F}_b \). For technical reasons the bijection \( \varphi \) has to map 0 to the zero element of \( \mathbb{F}_b \). For every \( m \in \mathbb{N} \) the initial \( b^m \) elements of a digital sequence form a digital net with \( b^m \) elements.

Depending on the choice of the generating matrices, digital sequences can achieve a discrepancy of order \( (\log N)^s/N \) for all \( N \geq 2 \). We refer to [17, 54, 68] for more information on the discrepancy of digital sequences.

4.3 Polynomial arithmetic and formal Laurent series

Polynomial arithmetic and formal Laurent series over a finite field play also an important role in the construction of QMC point sets and sequences.

Let \( b \) be a prime power and let \( \mathbb{F}_b \) be the finite field of order \( b \). If \( b \) is a prime number, then we identify \( \mathbb{F}_b \) with the set \( \mathbb{Z}_b = \{0, 1, \ldots, b-1\} \) equipped with arithmetic operations modulo \( b \). Let \( \mathbb{F}_b[x] \) be the set of all polynomials over \( \mathbb{F}_b \) and let \( \mathbb{F}_b((x^{-1})) \) be the field of formal Laurent series

\[
g = \sum_{k=w}^{\infty} a_k x^{-k} \quad \text{with } a_k \in \mathbb{F}_b \text{ and } w \in \mathbb{Z} \text{ with } a_w \neq 0.
\]
For $g \in \mathbb{F}_b((x^{-1}))$ and $m \in \mathbb{N} \cup \{\infty\}$ we define the “fractional part” function $\mathbb{F}_b((x^{-1})) \to [0, 1)$ by

$$\{g\}_{b,m} := \sum_{k=\max(1,w)}^{m} a_k b^{-k}.$$  

In the following we present some examples of constructions based on the concepts of polynomial arithmetic and formal Laurent series. More information can be found in [17, 68].

Example: Polynomial lattice point sets

Let $m \in \mathbb{N}$ and let $b$ be a prime number. Given a $p \in \mathbb{F}_b[x]$ with deg$(p) = m$ and $q = (q_1, \ldots, q_s) \in \mathbb{F}_b[x]^s$ a polynomial lattice point set $\mathcal{P}(q, p)$ is given by the points

$$\mathbf{x}_h = \left(\left\{ \frac{h q_1}{p} \right\}_{b,m}, \ldots, \left\{ \frac{h q_s}{p} \right\}_{b,m} \right),$$

for $h \in \mathbb{F}_b[x]$ with deg$(h) < m$. QMC rules that use polynomial lattice point sets as underlying nodes are called polynomial lattice rules.

Polynomial lattice point sets have been first introduced by Niederreiter [67] and can be viewed as polynomial analogs of lattice point sets (see Section 4.1). They are also polynomial lattice rules. The latter viewpoint also allows for constructions of “polynomial lattice point sets” in the prime-power base case.

For prime $b$ it is known that for any $p \in \mathbb{F}_b[x]$ with the property $p(x) = x^m$ or gcd$(p, x) = 1$ and deg$(p) = m$ there exists a generating vector $q \in \mathbb{F}_b[x]^s$ such that

$$D_N(\mathcal{P}(q, p)) \ll_{s,b} \frac{(\log N)^{s-1} \log \log N}{N},$$

where $w_j \leq 1$, and put $C_j = (c_{i,j}^{(j)})_{i,j=1}^{m}$ where the elements $c_{i,r}^{(j)}$ of the matrix $C_j$ are given by $c_{i,r}^{(j)} = u_{r+j-1}^{(j)} \in \mathbb{F}_b$ for $j = 1, \ldots, s$ and $i, r = 1, \ldots, m$. The latter viewpoint also allows for constructions of “polynomial lattice point sets” in the prime-power base case.

See [43, 49] for more information.

Example: Digital Kronecker sequences

Let $b$ be a prime number. For every $s$-tuple $f = (f_1, \ldots, f_s)$ of elements of $\mathbb{F}_b((x^{-1}))$ we define the sequence $\mathcal{S}(f) = (\mathbf{x}_n)_{n \geq 0}$ by

$$\mathbf{x}_n = \{n f_1\}_b, \ldots, \{n f_s\}_b \text{ for } n \in \mathbb{N}_0,$$

where we associate a nonnegative integer $n$ with $b$-adic expansion $n = n_0 + n_1 b + \cdots + n_r b^r$ with the polynomial $n(x) = n_0 + n_1 x + \cdots + n_r x^r$ in $\mathbb{F}_b[x]$ and vice versa and where $\{g\}_b := \left\{ g \right\}_{b,\infty}$.
The sequence $S(f)$ can be viewed as an analogue of the classical Kronecker-sequence and is therefore called a \textit{digital Kronecker-sequence}.

Digital Kronecker sequences are special examples of digital sequences (see Section 4.2).

Consider $f = (f_1, \ldots, f_s)$ with $f_j = \frac{f_{j,1}}{x} + \frac{f_{j,2}}{x^2} + \frac{f_{j,3}}{x^3} + \cdots \in \mathbb{F}_b((x^{-1}))$. Then the digital Kronecker sequence $S(f)$ is a digital sequence generated by the $\mathbb{N} \times \mathbb{N}$ matrices $C_1, \ldots, C_s$ over $\mathbb{F}_b$ given by

$$C_j = \begin{pmatrix} f_{j,1} & f_{j,2} & f_{j,3} & \cdots \\ f_{j,2} & f_{j,3} & f_{j,4} & \cdots \\ f_{j,3} & f_{j,4} & \vdots \end{pmatrix}.$$ 

**Example: Generalized Niederreiter sequences.**

These are special instances of digital sequences over $\mathbb{F}_b$ where the generating matrices $C_1, C_2, \ldots, C_s$ are constructed as follows: let $p_1, \ldots, p_s \in \mathbb{F}_b[x]$ be distinct monic irreducible polynomials over $\mathbb{F}_b$. For each $i \in \mathbb{N}$ and $j = 1, 2, \ldots, s$ choose a set of polynomials $\{y_{j,i,k}(x) : 0 \leq k < e_j\}$ which has to be linearly independent modulo $p_j(x)$ over $\mathbb{F}_b$. Consider the expansion

$$\frac{y_{j,i,k}(x)}{p_j(x)^i} = \sum_{r=1}^{\infty} \frac{a^{(j)}(i, k, r)}{x^r}$$

over $\mathbb{F}_b((x^{-1}))$ and define the matrix $C_j = (c_{i,r}^{(j)})_{i,r \in \mathbb{N}}$ by

$$c_{i,r}^{(j)} = a^{(j)}(Q + 1, k, r) \in \mathbb{F}_b \quad \text{for} \quad j \in [s], \; i, r \in \mathbb{N},$$

where $i - 1 = Qe_j + k$ with integers $Q = Q(j,i)$ and $k = k(j,i)$ satisfying $0 \leq k < e_j$. Generalized Niederreiter sequences comprise Sobol'-, Faure- and Niederreiter-sequences as special cases.

### 4.4 Groups, characters and duality

Let $(G, \circ)$ be a finite abelian group. A \textit{character} of $G$ is a groupomorphism $\chi : G \to \mathbb{C}^*$, that is, for all $x, y \in G$ we have $\chi(x \circ y) = \chi(x) \chi(y)$. This already implies $\chi(1_G) = 1$, where $1_G$ is the identity in $G$. Every finite abelian group of order $N$ has exactly $N$ distinct characters denoted by $\chi_0, \chi_1, \ldots, \chi_{N-1}$ where the character $\chi_0 \equiv 1$, which is 1 for all $x \in G$, is called the \textit{trivial character} or the \textit{principal character}. The set $\hat{G}$ of all characters of $G$ forms an abelian group under the multiplication $(\chi \psi)(x) = \chi(x) \psi(x)$ for all $x \in G$, for $\chi, \psi \in \hat{G}$.

Characters have the following important property which can be exploited in many applications.

**Lemma 4.2 (Character properties)** Let $\chi$ be a character of a finite abelian group $(G, \circ)$. Then we have

$$\sum_{x \in G} \chi(x) = \begin{cases} |G| & \text{if } \chi \text{ is the trivial character,} \\ 0 & \text{otherwise.} \end{cases}$$

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Let \( x \in G \). Then we have

\[
\sum_{\chi \in \hat{G}} \chi(x) = \begin{cases} 
|\hat{G}| & \text{if } x = 1_G, \\
0 & \text{otherwise.}
\end{cases}
\]

**Proof.** We just prove the first identity, the second one follows by a similar reasoning. The result is clear when \( \chi \) is the trivial character. Otherwise there exists some \( a \in G \) for which we have \( \chi(a) \neq 1 \). Then we have

\[
\chi(a) \sum_{x \in G} \chi(x) = \sum_{x \in G} \chi(a \circ x) = \sum_{x \in G} \chi(x),
\]

since as \( x \) runs through all elements of \( G \) so does \( a \circ x \). Hence we have

\[
(\chi(a) - 1) \sum_{x \in G} \chi(x) = 0
\]

and the result follows since \( \chi(a) \neq 1 \). \( \square \)

More information on characters of finite abelian groups can be found in [55, Chapter 5, Section 1]. Many constructions of QMC point sets have an inherent group structure and for these instances the above character property is an important tool for their analysis. We present the two most important examples.

**Example: General lattice rules**

Let \( L/\mathbb{Z}^s \) be any finite subgroup of \( \mathbb{R}^s/\mathbb{Z}^s \) and let \( \{x_0, x_1, \ldots, x_{N-1}\} \) be the node set of the lattice rule \( L \) (see Section 4.1). Recall the definition of the \( k \)th trigonometric functions \( e_k : [0, 1)^s \to \mathbb{C} \) from Section 3.1 given by

\[
e_k(x) = \exp(2\pi i k \cdot x) \quad \text{for } x \in [0, 1)^s. \tag{15}\n\]

Then \( \chi_k(x + \mathbb{Z}^s) = e_k(x) \) for \( x \in L \) is a well-defined character of the additive group \( L/\mathbb{Z}^s \). This character is trivial if and only if \( k \in L^\perp \), where

\[
L^\perp = \{ h \in \mathbb{Z}^s : h \cdot x \in \mathbb{Z} \text{ for all } x \in L \}.
\]

For rank-1 lattice point sets as defined in (14) it is clear that

\[
L^\perp = \{ h \in \mathbb{Z}^s : h \cdot g \equiv 0 \pmod{N} \}.
\]

The set \( L^\perp \) is again a lattice in \( \mathbb{R}^s \) which is called the dual lattice of \( L \).

Now Lemma 4.2 yields the following important result:

**Lemma 4.3** Let \( \{x_0, x_1, \ldots, x_{N-1}\} \) be the node set of an \( N \)-element lattice rule \( L \). Then for \( k \in \mathbb{Z}^s \) we have

\[
\sum_{n=0}^{N-1} e_k(x_n) = \begin{cases} 
N & \text{if } k \in L^\perp, \\
0 & \text{if } k \not\in L^\perp.
\end{cases}
\]

This basic property is exploited in the analysis of the worst-case error of lattice rules (see Section 3.1 and [68, Chapter 5]) or of discrepancy estimates of the corresponding node sets (see Section 4.6).
Example: Digital nets

Let \( b \) be a prime-power and let \( \varphi : \mathbb{Z}_b \to \mathbb{F}_b \) be a bijection with \( \varphi(0) = \overline{0} \) be fixed. For \( x, y \in [0, 1) \) let \( x = \frac{x_1}{b} + \frac{x_2}{b^2} + \cdots \) and \( y = \frac{y_1}{b} + \frac{y_2}{b^2} + \cdots \) be their \( b \)-adic expansions (with \( \xi_i \neq b - 1 \) for infinitely many \( i \) and \( \eta_j \neq b - 1 \) for infinitely many \( j \)). Then \( x \oplus y := \frac{x_1 + y_1}{b} + \frac{x_2 + y_2}{b^2} + \cdots \) with

\[
\zeta_j = \varphi^{-1}(\varphi(\xi_j) + \varphi(\eta_j)) \quad \text{for} \quad j \in \mathbb{N}.
\]

(A case which has to be excluded is, for instance (for prime \( b \), \( \mathbb{Z}_b = \mathbb{F}_b \) and \( \varphi = \text{id} \)), when \( x = (b-1)(b^{-1} + b^{-3} + b^{-5} + \cdots) \) and \( y = (b-1)(b^{-2} + b^{-4} + b^{-6} + \cdots) \). In this case \( x \oplus y = (b-1)(b^{-1} + b^{-2} + b^{-3} + \cdots) = 1 \).) For vectors \( \mathbf{x}, \mathbf{y} \in [0, 1)^s \) the \( b \)-adic addition \( \mathbf{x} \oplus \mathbf{y} \) is defined component wise. Note that this way \( \oplus \) is defined for almost all \( \mathbf{x}, \mathbf{y} \in [0, 1)^s \). Let \( \mathcal{D} = \{ \mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_{b^m-1} \} \) be a digital net over \( \mathbb{F}_b \) with \( m \times m \) generating matrices \( \mathcal{C}_1, \ldots, \mathcal{C}_s \) as defined in Section 4.2. Any vector \( \mathbf{n} = (\mathbf{n}_0, \mathbf{n}_1, \ldots, \mathbf{n}_{m-1})^\top \in \mathbb{F}_b^m \) uniquely represents an integer \( n := n_0 + n_1 b + \cdots + n_{m-1} b^{m-1} \) from \( \{0, \ldots, b^m-1\} \) via \( n_i = \varphi^{-1}(\mathbf{n}_i) \) for \( i = 0, 1, \ldots, m-1 \), and to any such integer belongs an element \( \mathbf{x}_n \) of \( \mathcal{D} \). Then the mapping

\[
\Psi : \mathbb{F}_b^m \to \mathcal{D}, \quad \mathbf{n} \mapsto \mathbf{x}_n
\]

is a group-isomorphism from the additive group of \( \mathbb{F}_b^m \) to \( \mathcal{D} \). In fact, for \( \mathbf{n}, \mathbf{l} \in \mathbb{F}_b^m \) the property \( \Psi(\mathbf{n} + \mathbf{l}) = \Psi(\mathbf{n}) \oplus \Psi(\mathbf{l}) \) easily follows from the fact that for any \( m \times m \) matrix \( \mathcal{C} \) over \( \mathbb{F}_b \) we have \( \mathcal{C}(\mathbf{n} + \mathbf{l}) = \mathcal{C} \mathbf{n} + \mathcal{C} \mathbf{l} \). Therefore we have:

**Lemma 4.4** Any digital net \( (\mathcal{D}, \oplus) \) is a finite abelian group.

For the sake of simplicity let in the following \( b \) be a prime number and identify the finite field \( \mathbb{F}_b \) with \( \mathbb{Z}_b \) and choose \( \varphi = \text{id} \).

For \( k \in \mathbb{N}_0 \) with \( b \)-adic expansion \( k = \kappa_0 + \kappa_1 b + \kappa_2 b^2 + \cdots \), where \( \kappa_i \in \mathbb{Z}_b \), the \( k \)-th \( b \)-adic Walsh function \( b \text{wal}_k : [0, 1) \to \mathbb{C} \) is defined as

\[
b \text{wal}_k(x) = \exp(2\pi i (\kappa_0 \xi_1 + \kappa_1 \xi_2 + \kappa_2 \xi_3 + \cdots )/b),
\]

for \( x \in [0, 1) \) with \( b \)-adic expansion \( x = \xi_1 b^{-1} + \xi_2 b^{-2} + \xi_3 b^{-3} + \cdots \) (unique in the sense that infinitely many of the digits \( \xi_i \) must be different from \( b - 1 \)). For vectors \( \mathbf{k} = (k_1, \ldots, k_s) \in \mathbb{N}_0^s \) and \( \mathbf{x} = (x_1, \ldots, x_s) \in [0, 1)^s \) we write

\[
b \text{wal}_\mathbf{k}(\mathbf{x}) := b \text{wal}_{k_1}(x_1, \ldots, x_s) = \prod_{j=1}^s b \text{wal}_{k_j}(x_j).
\]

The system \( \{ b \text{wal}_\mathbf{k} : \mathbf{k} \in \mathbb{N}_0^s \} \) is called the \( s \)-dimensional \( b \)-adic Walsh function system. For all \( \mathbf{x}, \mathbf{y} \in [0, 1)^s \), for which \( \mathbf{x} \oplus \mathbf{y} \) is defined we have

\[
b \text{wal}_\mathbf{k}(\mathbf{x}) b \text{wal}_\mathbf{k}(\mathbf{y}) = b \text{wal}_\mathbf{k}(\mathbf{x} \oplus \mathbf{y}) \quad \text{for all} \quad \mathbf{k} \in \mathbb{N}_0^s.
\]

In particular, \( b \text{wal}_\mathbf{k} \) is a character of the finite abelian group \( (\mathcal{D}, \oplus) \). For \( \mathbf{k} = (k_1, \ldots, k_s) \in \mathbb{N}_0^s \) we have \( b \text{wal}_\mathbf{k}(\mathbf{x}_n) = 1 \) for all \( n = 0, 1, \ldots, b^m - 1 \) if and only if

\[
\sum_{j=1}^s k_j \cdot x_{n,j} = 0 \quad \text{for all} \quad n = 0, 1, \ldots, b^m - 1,
\]

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where $k_j$ is the $m$-dimensional column vector of $b$-adic digits of $k_j$ and $x_{n,j}$ denotes the $m$-dimensional column vector of $b$-adic digits of the $j$th component of $x_n$. From the construction of the digital net we find that $x_{n,j} = C_j n$, where $n$ denotes the column vector of $b$-adic digits of $n$, and hence $b\text{wal}_k(x_n) = 1$ for all $n = 0, 1, \ldots, b^m - 1$ if and only if
\[
\sum_{j=1}^{s} k_j \cdot C_j n = 0 \quad \text{for all} \quad n = 0, 1, \ldots, b^m - 1.
\]

This is satisfied if and only if
\[
C_1^T k_1 + \cdots + C_s^T k_s = 0.
\]

Thus we have shown that $b\text{wal}_k$ is a trivial character of $D$ if and only if $k \in D^\perp$, where
\[
D^\perp = \{ k \in \{0, \ldots, b^m - 1\}^s : C_1^T k_1 + \cdots + C_s^T k_s = 0 \}.
\]

The set $D^\perp$ is called the dual net of the digital net $D$.

Now Lemma 4.2 yields the following important result:

**Lemma 4.5** Let $b$ be a prime number and let $D$ be a digital net over $\mathbb{F}_b$. Then for $k \in \{0, \ldots, b^m - 1\}^s$ we have
\[
\sum_{n=0}^{b^m-1} b\text{wal}_k(x_n) = \begin{cases} 
b^m & \text{if } k \in D^\perp, 
0 & \text{if } k \notin D^\perp.
\end{cases}
\]

This basic property is exploited in the analysis of the worst-case error of QMC rules based on digital nets or of discrepancy estimates (see, e.g., [15, 16, 17]). A generalization to the case of digital nets over $\mathbb{F}_b$ with prime-power $b$ can be found in [79, Lemma 2.5]. In this case one requires the more general concept of Walsh functions over the finite field $\mathbb{F}_b$.

### 4.5 Minkowski’s fundamental theorem

Methods from the geometry of numbers play an important role in the analysis of lattice point sets. One of the most fundamental theorems in this area is due to Minkowski from 1896.

**Theorem 4.6 (Minkowski)** Let $L$ be a lattice in $\mathbb{R}^s$. Then any convex set in $\mathbb{R}^s$ which is symmetric with respect to the origin and with volume greater than $2^s \det(L)$ contains a non-zero lattice point of $L$.

See Cassels [10] for a proof and for more information regarding this theorem. In the following we give an application of Minkowski’s result to the enhanced trigonometric degree of lattice rules. In Section 4.8 we will apply Minkowski’s theorem in the context of Diophantine Approximation.
**Example: The enhanced trigonometric degree of lattice rules**

A cubature rule is said to have *trigonometric degree* \( d \), if it integrates correctly all \( s \)-dimensional trigonometric polynomials of degree \( d \). The *enhanced trigonometric degree* is the trigonometric degree increased by one. It is known (see [56]) that the enhanced trigonometric degree of a lattice rule generated by \( g \in \mathbb{Z}^s \) and consisting of \( N \) nodes is

\[
\rho(g, N) = \min_{h \in L^\perp \setminus \{0\}} |h|,
\]

where \( |h| \) is the one-norm of the vector \( h \in \mathbb{Z}^s \) and where \( L^\perp \) is the corresponding dual lattice as defined in Section 4.4.

**Theorem 4.7** For all \( g \in \mathbb{Z}^s \) and integers \( N \geq 2 \) we have \( \rho(g, N) \leq (s!N)^{1/s} \).

**Proof.** Let \( L \) be an integration lattice generated by \( g \in \mathbb{Z}^s \) yielding an \( N \)-point lattice rule and let \( L^\perp \) be the dual lattice. According to [68, Theorem 5.30] we have \( \det(L^\perp) = N \).

Now consider the convex region

\[
C^s_\rho = \{ x \in \mathbb{R}^s : |x_1| + \cdots + |x_s| \leq \rho \},
\]

where \( \rho > 0 \). Then \( C^s_\rho \) is symmetric with respect to the origin and the volume of \( C^s_\rho \) is

\[
\text{Vol}(C^s_\rho) = \frac{2^s \rho^s}{s!}.
\]

Hence, by Minkowski’s theorem applied to \( L^\perp \), we have that if

\[
\frac{2^s \rho^s}{s!} \geq 2^s \det(L^\perp) = 2^s N,
\]

i.e., if \( \rho \geq (s!N)^{1/s} \), then \( C^s_\rho \) contains a non-zero point from \( L^\perp \). In other words, \( L^\perp \) contains a non-zero lattice point which belongs to \( C^s_{(s!N)^{1/s}} \) and therefore we have \( \rho(g, N) \leq (s!N)^{1/s} \).

4.6 **Exponential sums**

*Exponential sums* are objects of the form

\[
S(X, F) = \sum_{x \in X} \exp(2\pi i F(x))
\]

where \( X \) is an arbitrary finite set and \( F \) is a real valued function on \( X \). They have important applications in many branches of mathematics. For example, the famous Weyl criterion (see, e.g., [17, 21, 44]) states that a sequence \( S = (x_n)_{n \geq 0} \) of points in \([0, 1)^s\) is uniformly distributed modulo one if and only if for all \( h \in \mathbb{Z}^s \setminus \{0\} \) and \( F_h(x) = h \cdot x \) we have

\[
S(P_N, F_h) = o(N) \quad \text{for} \quad N \to \infty,
\]

where \( P_N \) is the point set consisting of the first \( N \) terms of \( S \). A quantitative version of this result is the inequality of Erdős-Turán-Koksma.
Theorem 4.8 (Erdős-Turán-Koksma) For the discrepancy of every $N$-element point set $P_N$ in $[0,1)^s$ we have

$$D_N(P_N) \ll_s \frac{1}{m} + \sum_{0<|h|_\infty \leq m} \frac{1}{r(h)} \frac{|S(P_N, F_h)|}{N},$$

where $m \in \mathbb{N}$ and where $r(h) = \prod_{j=1}^s \max(1, |h_j|)$ and $|h|_\infty = \max_{j=1,...,s} |h_j|$ for $h = (h_1, \ldots, h_s) \in \mathbb{Z}^s$.

A proof of this theorem can be found in [21] (and also in [44], but there only for the one-dimensional case).

We present two examples which are based on the Erdős-Turán-Koksma inequality and which illustrate the power of exponential sums for estimating discrepancy. More information on exponential sums can be found in [55, 85, 97].

Example: The star discrepancy of lattice point sets

Combining Lemma 4.3 and Theorem 4.8 with $m = N$ we find that the discrepancy of a rank-1 lattice point set $P(g, N)$ (cf. Section 4.1) satisfies

$$D_N(P(g, N)) \ll_s \frac{1}{N} + R(g, N),$$

where

$$R(g, N) := \sum_{0<|h|_\infty \leq N \atop h \in L^1} \frac{1}{r(h)}.$$ 

For simplicity let $N$ be a prime number. We average $R(g, N)$ over all $g \in G_N$, where $G_N : = \{1, \ldots, N - 1\}$, and obtain

$$\frac{1}{(N-1)^s} \sum_{g \in G_N} R(g, N) = \frac{1}{(N-1)^s} \sum_{0<|h|_\infty \leq m} \frac{1}{r(h)} \sum_{g \in L^1 \cap G_N} 1.$$ 

Now $g \in L^1 \cap G_N$ means in particular that $g_1 h_1 + \cdots + g_s h_s \equiv 0 \pmod{N}$. If at least one of the $h_i$’s is different from zero, then there are at most $(N-1)^{s-1}$ elements $(g_1, \ldots, g_s) \in G_N$ which satisfy this condition. Hence we find that

$$\frac{1}{(N-1)^s} \sum_{g \in G_N} R(g, N) \leq \frac{1}{N-1} \sum_{0<|h|_\infty \leq m} \frac{1}{r(h)} \sum_{g_1 \cdots g_s \equiv 0 \pmod{N}} 1 \leq (\log N)^s \frac{1}{N}. \quad (17)$$

Combining (16) and (17) we obtain the following result.

Theorem 4.9 For every prime number $N$ there exists a lattice point $g \in G_N$ such that

$$D_N(P(g, N)) \ll_s \frac{(\log N)^s}{N}.$$
For a more general and accurate result we refer to the book by Niederreiter [68, Chapter 5]. The currently best result for the discrepancy of rank-1 lattice point sets was proved by Larcher [48] for dimension \( s = 2 \) and by Bykovskii [9] for arbitrary dimension \( s \).

**Theorem 4.10 (Bykovskii, Larcher)** For every integer \( N \geq 3 \), there exists a lattice point \( g \in \mathbb{Z}^s \) such that

\[
D_N(\mathcal{P}(g, N)) \ll_s \frac{(\log N)^{s-1} \log \log N}{N}.
\]

**Example: Gauss sums and linear congruential pseudorandom numbers**

Discrepancy is a measure for the deviation of the distribution of a given point set from perfect uniform distribution. Hence it is also an important test criterion for pseudorandom numbers which are required for Monte Carlo integration. The following example is taken from [97].

Let \( b \) be a prime number. Let \( \chi \) be a multiplicative character and \( \psi \) be an additive character of \( \mathbb{F}_b \). Then

\[
G(\chi, \psi) = \sum_{c \in \mathbb{F}_b^\times} \chi(c) \psi(c)
\]

is called a *Gauss sum of type I*. Here and in the following \( \mathbb{F}_b^\times \) denotes the multiplicative group of \( \mathbb{F}_b \).

**Lemma 4.11** We have

\[
G(\chi, \psi) = \begin{cases} 
-1 & \text{if } \chi = \chi_0 \text{ and } \psi \neq \psi_0, \\
0 & \text{if } \chi \neq \chi_0 \text{ and } \psi = \psi_0, \\
q - 1 & \text{if } \chi = \chi_0 \text{ and } \psi = \psi_0.
\end{cases}
\]

If \( \chi \) and \( \psi \) are both nontrivial, then \( |G(\chi, \psi)| = \sqrt{b} \).

**Proof.** We only show the case where \( \chi \) and \( \psi \) are both nontrivial. Then we have

\[
|G(\chi, \psi)|^2 = G(\chi, \psi)\overline{G(\chi, \psi)} = \sum_{c,d \in \mathbb{F}_b^\times} \chi(cd^{-1})\psi(c - d) = \sum_{c \in \mathbb{F}_b^\times} \chi(e) \sum_{c \in \mathbb{F}_b^\times} \psi(c(1 - e^{-1})).
\]

From Lemma 4.12 we obtain

\[
\sum_{c \in \mathbb{F}_b^\times} \psi(c(1 - e^{-1})) = \begin{cases} 
b - 1 & \text{if } e = 1, \\
-1 & \text{if } e \neq 1,
\end{cases}
\]

and hence

\[
|G(\chi, \psi)|^2 = b - 1 - \sum_{e \in \mathbb{F}_b^\times \setminus \{1\}} \chi(e) = b,
\]

again according to Lemma 4.12. Hence \( |G(\chi, \psi)| = \sqrt{b} \). \( \square \)
We remark that Lemma 4.11 holds even if \( b \) is a prime-power. More information on Gauss sums of type I can be found in [55, Chapter 5, Section 2].

For divisors \( n \) of \( b - 1 \) a sum of the form

\[
S_n(\psi) = \sum_{c \in \mathbb{F}_b^\times} \psi(c^n)
\]

is called a Gauss sum of type II. (For arbitrary \( n \) one defines \( S_n = S_{\gcd(n,b-1)} \).)

**Lemma 4.12** If \( \psi \neq \psi_0 \) then we have \( |S_n(\psi)| \leq (n - 1)\sqrt{b} + 1 \).

**Proof.** Let \( \chi \) be a multiplicative character of \( \mathbb{F}_b \) of order \( n \), i.e., \( n \) is the least positive integer such that \( \chi^n(x) = 1 \) for all \( x \in \mathbb{F}_b^\times \). Then for \( x \in \mathbb{F}_b^\times \) we have

\[
\sum_{j=0}^{n-1} \chi^j(x) = \begin{cases} n & \text{if } \chi(x) = 1, \\ 0 & \text{otherwise.} \end{cases}
\]

Since the order of \( \chi \) is \( n \) it follows that \( \chi(x) = 1 \) if and only if \( x = c^n \) for some \( c \in \mathbb{F}_b^\times \). Note that for given \( x \in \mathbb{F}_b^\times \) the equation \( x = c^n \) has zero or exactly \( \gcd(n,b-1) = n \) solutions \( c \in \mathbb{F}_b^\times \), since \( n|(b-1) \). Then we have

\[
|S_n(\psi)| = \left| \sum_{c \in \mathbb{F}_b^\times} \psi(c^n) \right| = \left| n \sum_{x \in \mathbb{F}_b^\times} \psi(x) \right| = \left| \sum_{x \in \mathbb{F}_b^\times} \sum_{j=0}^{n-1} \chi^j(x) \psi(x) \right| \leq \sum_{j=0}^{n-1} G(\chi^j, \psi) \leq (n - 1)\sqrt{b} + 1,
\]

where we used Lemma 4.11. \( \square \)

Now we apply Gauss sums of type II to linear congruential pseudorandom numbers.

**Definition 4.13** A sequence given by the recursion

\[
x_{n+1} = ax_n + c \quad \text{for } n \in \mathbb{N}_0,
\]

where \( x_0, a, c \in \mathbb{F}_b \) with \( a \neq 0, 1 \) and all algebraic operations carried out in \( \mathbb{F}_b \) is called a linear congruential pseudorandom number generator.

Since \( a \neq 1 \), the elements \( x_n \) are given explicitly by the formula

\[
x_n = a^n x_0 + \frac{a^n - 1}{a - 1} c \quad \text{for } n \in \mathbb{N}_0.
\]

(18)

If \( c \neq (1 - a)x_0 \), then the sequence \( (x_n)_{n \geq 0} \) is \( T \)-periodic, where \( T \) is the order of \( a \pmod{b} \).

Consider now the \( T \)-element point set \( \mathcal{P}_T = \{x_0/b, x_1/b, \ldots, x_{T-1}/b\} \) in \([0,1)\) derived from a linear congruential pseudorandom number generator. Here \( \mathbb{F}_b \) is identified with the
integers \{0, 1, \ldots, b - 1\}. For simplicity we assume that \(c = 0\). Then it follows from (18) that
\[
|S(\mathcal{P}_T, F_h)| = \left| \sum_{n=0}^{T-1} \exp(2\pi i x_0 h a^n / b) \right|.
\]

Let \(w\) be a primitive root modulo \(b\) and let \(a = w^i\). Then we have
\[
T = \text{ord}_{\mathbb{F}_b^\times}(a) = \text{ord}_{\mathbb{F}_b^\times}(w^i) = \frac{b-1}{\gcd(b-1, i)}
\]
and hence \(\frac{b-1}{T} | i\). For fixed \(n \in \{0, 1, \ldots, T-1\}\) we have \(a^n = w^{in} = w^{\frac{b-1}{T} i n}\) if and only if \(k \frac{b-1}{T} \equiv in \pmod{b-1}\). Since \(\frac{b-1}{T} | i\), the last congruence has exactly \(\frac{b-1}{T}\) incongruent solutions \(k\) modulo \(b-1\). This shows that for fixed \(n\) there are exactly \(\frac{b-1}{T}\) different \(x \in \mathbb{F}_b^\times\) such that \(a^n = x^{\frac{b-1}{T}}\). Therefore
\[
|S(\mathcal{P}_T, F_h)| = \frac{T}{b-1} \left| \sum_{x \in \mathbb{F}_b^\times} \exp(2\pi i x_0 h x^{\frac{b-1}{T}} / b) \right|.
\]

The last exponential sum is a Gauss sum of type II and hence we can apply Lemma 4.12 and obtain
\[
|S(\mathcal{P}_T, F_h)| \leq \sqrt{b}
\] whenever \(h \not\equiv 0 \pmod{b}\). We remark that the bound on the Gauss sum of type II is only nontrivial if \(T > \sqrt{b}\). However, there are several nontrivial estimates known for smaller \(T\). In particular in [8] the authors proved nontrivial bounds for any \(T \geq b^\delta\) and \(\delta > 0\).

Inserting the estimate (19) into the Erdös-Turán-Koksma inequality (Theorem 4.8) finally we obtain
\[
D_T(\mathcal{P}_T) \ll b \frac{\log T}{T}.
\]

Dealing with incomplete Gauss sums Niederreiter [64, Theorem 1] showed a more general result which considers also parts of the period.

**Theorem 4.14 (Niederreiter)** For the sequence \(S = \{x_n/b : n = 0, 1, \ldots, N-1\}\) where \(x_n\) are linear congruential pseudorandom numbers, \(N < T\) and \(T\) is the order of \(a\), we have \(D_N(S) \ll \sqrt{b(\log b)^2/N}\).

### 4.7 \(b\)-adic numbers

Within this section let \(b \geq 2\) be a prime number. The set of \(b\)-adic numbers is defined as the set of formal sums
\[
\mathbb{Z}_b = \left\{ z = \sum_{r=0}^{\infty} z_r b^r : z_r \in \{0, \ldots, b-1\} \text{ for all } r \in \mathbb{N}_0 \right\}.
\]

The set \(\mathbb{N}_0\) of non-negative integers is a subset of \(\mathbb{Z}_b\). For two non-negative integers \(y, z \in \mathbb{Z}_b\), the sum \(y + z \in \mathbb{Z}_b\) is defined as the usual sum of integers. The addition can be extended to all \(b\)-adic numbers with the addition carried out in the usual manner. For instance, the inverse of 1 \(\in \mathbb{Z}_b\) is given by the formal sum
\[
(b-1) + (b-1)b + (b-1)b^2 + \cdots.
\]
Then we have

\[ 1 + [(b - 1) + (b - 1)b + (b - 1)b^2 + \cdots] = 0 + (1 + (b - 1))b + (b - 1)b^2 + \cdots \\
= 0 + 0b + (1 + (b - 1))b^2 + \cdots \\
= 0b + 0b^2 + \cdots = 0. \]

The set \( \mathbb{Z}_b \) with this addition then forms an abelian group.

The set of \( b \)-adic numbers has various applications to QMC theory. In the following we present one example in the context of lattice point sets. Other examples are to be found, for example, in \([31, 32, 78]\).

**Example: extensible lattice point sets**

One disadvantage of rank-1 lattice point sets is their dependence on the cardinality \( N \) of the resulting point set. If one constructs a generating vector of a lattice rule of cardinality \( N \) with good quality, it does not mean that the same vector can be used to generate a lattice point set of good quality which uses \( N' \neq N \) points.

Extensible lattice rules have the property that the number \( N \) of points in the node set may be increased while retaining the existing points. Their definition is based on \( b \)-adic numbers. Let \( a \in \mathbb{Z}_s^b \) and define the infinite sequence \( S_a = (x_n)_{n \geq 0} \) by \( x_n = \{a \phi_b(n)\} \), where \( \phi_b \) is the \( b \)-adic radical inverse function as defined in Section 4.2 and where the fractional part function \( \{\cdot\} \) is applied component-wise.

The so constructed sequence has the property, that any initial segment with \( N = b^m \) points is a rank-1 lattice point set. Indeed, for \( m \in \mathbb{N} \) and \( a_m := a \pmod{b^m} \) (applied component-wise) we have

\[ \{a \phi_b(n)\} : n = 0, 1, \ldots, b^m - 1 \}

Furthermore, for \( m \geq m \) we have \( \mathcal{P}(a_m, b^m) \subseteq \mathcal{P}(a_m, b^m) \).

It has been shown by Hickernell and Niederreiter \([34]\) that there exist \( a \in \mathbb{Z}_s^b \) such that for all \( \varepsilon > 0 \)

\[ D_N^*(S_a) \ll s, \varepsilon \frac{(\log N)^{s+1}(\log \log N)^{1+\varepsilon}}{N} \text{ for all } N = b, b^2, b^3, \ldots. \]

Extensible lattice point sets are also discussed in Section 5.3. More results on extensible lattice point sets can be found in \([12, 18, 33, 70]\).

**4.8 Diophantine approximation**

Diophantine approximation deals with the problem of approximating real numbers by rational numbers, or, in the multivariate case, of approximating real vectors by rational vectors. In dimension one the theory of continued fractions plays an utmost important role in this field. But also in the multivariate case there are many important theorems in this area such as Dirichlet’s approximation theorem or Minkowski’s theorem on linear forms which is a corollary to Minkowski’s fundamental theorem (Theorem 4.6); see, for example, \([4, 10, 28]\).
Theorem 4.15 (Dirichlet) Let $\alpha_1, \ldots, \alpha_s \in \mathbb{R}$. Then there exists a vector $(p_1, \ldots, p_s, q) \in \mathbb{Z}^s \times \mathbb{N}$, such that
\[
|q\alpha_j - p_j| \leq q^{1/s} \quad \text{for all } j = 1, 2, \ldots, s.
\]
Moreover, if at least one $\alpha_j$ is irrational, then there are infinitely many tuples $(p_1, \ldots, p_s, q) \in \mathbb{Z}^s \times \mathbb{N}$ with this property.

Theorem 4.16 (Minkowski) Let $A = (a_{i,j})_{i,j=1}^n$ be a real matrix and let $c_1, \ldots, c_n \in \mathbb{R}^+$. Consider the $n$ linear forms
\[
L_i(x_1, \ldots, x_n) = \sum_{j=1}^n a_{i,j} x_j \quad \text{for } i = 1, 2, \ldots, n.
\]
Then the following holds: if $c_1 \cdots c_n \geq |\det(A)|$, then there exists a vector $(h_1, \ldots, h_n) \in \mathbb{Z}^n \setminus \{0\}$ such that $|L_1(h_1, \ldots, h_n)| \leq c_1$ and $|L_i(h_1, \ldots, h_n)| < c_i$ for all $i = 2, \ldots, s$.

The applications of Diophantine approximation to QMC, in particular to discrepancy theory, are various and numerous and cannot all be cited here. We just mention some examples such as, [5, 21, 44, 62, 63, 65, 68]. Furthermore, applications of Diophantine approximation to QMC are not only restricted to the archimedean case. Many results have non-archimedean analogs, for example in the context of approximations of Laurent series over finite fields by rational functions, which can also be applied to problems in QMC. This plays a major role, e.g., in the analysis of digital nets and sequences such as polynomial lattice point sets or digital Kronecker sequences. See [50, 51, 52, 68, 69] for examples.

Here we present one classical application which is taken from [62] and which deals with the discrepancy of Kronecker sequences (see Section 4.1).

Example: Discrepancy of Kronecker sequences

One main problem in the theory of Diophantine approximation is to find bounds for $\|h \cdot \alpha\|$, where $\|\cdot\|$ denotes the distance to the nearest integer function, i.e., $\|x\| = \min(\{x\}, 1-\{x\})$ for $x \in \mathbb{R}$ and where $\alpha = (\alpha_1, \ldots, \alpha_s)$ and $h \in \mathbb{Z}^s$. This problem is directly linked to the discrepancy of Kronecker sequences $\{n\alpha\}_{n \geq 0}$. It is well known (and can easily be deduced from Weyl’s criterion) that a Kronecker sequence is uniformly distributed if and only if $1, \alpha_1, \ldots, \alpha_s$ are linearly independent over the rationals.

Definition 4.17 For a real number $\eta$, an $s$-tuple $\alpha \in (\mathbb{R} \setminus \mathbb{Q})^s$ is said to be of type $\eta$, if $\eta$ is the infimum of all numbers $\sigma$ for which there exists a positive constant $c = c(\sigma, \alpha)$ such that
\[
r(h)^\sigma \|h \cdot \alpha\| \geq c \quad \text{for all } h \in \mathbb{Z}^s \setminus \{0\},
\]
where $r(h) = \prod_{j=1}^s \max(1, |h_j|)$ for $h = (h_1, \ldots, h_s) \in \mathbb{R}^s$.

It follows easily from the above two theorems, that the type $\eta$ of an irrational vector $\alpha$ is at least one.
Proof. Assume that the type \( \eta \) of \( \alpha \in (\mathbb{R} \setminus \mathbb{Q})^s \) is less then one. According to Theorem 4.13 there exist infinitely many \((p_1, \ldots, p_s, q) \in \mathbb{Z}^s \times \mathbb{N}\) such that \(|q \alpha_i - p_i| \leq q^{-1/s}\) for all \(i = 1, 2, \ldots, s\). Now consider the linear forms \(L_i(x_1, \ldots, x_s, x) = x_i\) for \(i = 1, 2, \ldots, s\) and \(L_{s+1}(x_1, \ldots, x_s, x) = p_1 x_1 + \cdots + p_s x_s - q x\) with \(q\) as absolute value of the corresponding determinant. According to Theorem 4.16 there exists a vector \((h_1, \ldots, h_s, h) \in \mathbb{Z}^{s+1} \setminus \{0\}\) such that

\[
|h_j| \leq q^{1/s} \quad \text{for all } j = 1, 2, \ldots, s \quad \text{and} \quad |h_1 p_1 + \cdots + h_s p_s - q h| < 1.
\]

Since \(h_1 p_1 + \cdots + h_s p_s - q h \in \mathbb{Z}\) we obtain that \(q h = h_1 p_1 + \cdots + h_s p_s\).

Now for any \(\sigma \in (\eta, 1)\) we have (recall that \(q \geq 1\))

\[
|h_1 \alpha_1 + \cdots + h_s \alpha_s - h |r(\mathbf{h})|^\sigma \leq |h_1 q \alpha_1 + \cdots + h_s q \alpha_s - q h| \frac{1}{q} \prod_{j=1}^{s} \max(1, q^{1/s})^\sigma = |h_1 (q \alpha_1 - p_1) + \cdots + h_s (q \alpha_s - p_s)| \frac{1}{q^{1-\sigma}} \leq \frac{s}{q^{1-\sigma}}
\]

and hence

\[
\inf_{h \neq 0} r(\mathbf{h})^\sigma \| \mathbf{h} \cdot \alpha \| \leq \frac{s}{q^{1-\sigma}}
\]

for infinitely many \(q \in \mathbb{N}\). Thus the infimum is zero and the result follows. \(\square\)

On the other hand it has been shown by Schmidt [83] that \(\alpha = (\alpha_1, \ldots, \alpha_s)\), with real algebraic components for which \(1, \alpha_1, \ldots, \alpha_s\) are linearly independent over \(\mathbb{Q}\), is of type \(\eta = 1\). In particular, \((e^{r_1}, \ldots, e^{r_s})\) with distinct nonzero rationals \(r_1, \ldots, r_s\) or \((\sqrt{p_1}, \ldots, \sqrt{p_s})\) with distinct prime numbers \(p_1, \ldots, p_s\) are of type \(\eta = 1\).

Theorem 4.18 (Niederreiter) Let \(\alpha\) be an \(s\)-tuple of irrationals of type \(\eta = 1\). Then the discrepancy of the Kronecker sequence \(S_\alpha = \{n \alpha\}_{n \geq 0}\) satisfies for all \(\varepsilon > 0\)

\[
D_N(S_\alpha) \ll_{s, \varepsilon} \frac{1}{N^{1-\varepsilon}}.
\]

Proof. The proof is according to [62]. Using the formula for a geometric sum we obtain

\[
\left| \sum_{n=0}^{N-1} \exp(2\pi i \mathbf{h} \cdot x_n) \right| = \left| \sum_{n=0}^{N-1} \exp(2\pi i \mathbf{h} \cdot \alpha)^n \right| \leq \frac{2}{|\exp(2\pi i \mathbf{h} \cdot \alpha) - 1|} = \frac{1}{|\sin(2\pi \mathbf{h} \cdot \alpha)|} \leq \frac{1}{2 \| \mathbf{h} \cdot \alpha \|}. \tag{1.8}
\]

Inserting this into the Erdős-Turán-Koksma inequality (Theorem 1.8) we obtain for all \(m \in \mathbb{N}\)

\[
D_N(S_\alpha) \ll_s \frac{1}{m} + \frac{1}{N} \sum_{0 < |\mathbf{h}|_{\infty} \leq m} \frac{1}{r(\mathbf{h})} \frac{1}{\| \mathbf{h} \cdot \alpha \|}.
\]

Now we use the identity

\[
\sum_{0 < |\mathbf{h}|_{\infty} \leq m} \frac{1}{r(\mathbf{h})} \frac{1}{\| \mathbf{h} \cdot \alpha \|} = \sum_{n_1, \ldots, n_s = 1}^{m} f(n_1, \ldots, n_s) \sum_{h \in \mathbb{Z}^s \setminus \{0\}} \frac{1}{|h_j|} \frac{1}{\| \mathbf{h} \cdot \alpha \|},
\]

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where \( f(n_1, \ldots, n_s) = \prod_{j=1}^s g_m(n_j) \) and where \( g_m(n) = 1/(n(n+1)) \) if \( n \in \{1, \ldots, m-1\} \) and \( g_m(m) = 1/m \). This can be shown by computing the total coefficient of \( 1/\|h \cdot \alpha\| \) on the right-hand side of the equation (see [62, p. 222] for details).

In a first step we estimate the inner sum of the above double sum. Since \( \alpha \) is of type one we obtain for all \( h, h' \in \mathbb{Z}^s \setminus \{0\} \) satisfying \( |h_j|, |h'_j| \leq n_j \) for all \( j = 1, \ldots, s \), and \( h \neq \pm h' \) that

\[
\|h \cdot \alpha \pm h' \cdot \alpha\| = \|(h \pm h') \cdot \alpha\| \geq cr(h + h')^{-1-\epsilon} \geq cr(2n)^{-1-\epsilon} =: d
\]

for all \( \epsilon > 0 \), where \( c = c(\epsilon, \alpha) \) and where \( n = (n_1, \ldots, n_s) \). Since \( \|x \pm y\| \leq \|x\| - \|y\| \) we obtain

\[
|\|h \cdot \alpha\| - \|h' \cdot \alpha\|| \geq d.
\]

Hence in each of the intervals \([kd, (k+1)d)\) for \( k = 0, 1, \ldots, [1/(2d)]\), there can lie at most two numbers of the form \( \|h \cdot \alpha\| \), with no such number in the interval \([0,d)\), since we also have \( \|h \cdot \alpha\| \geq d \). Therefore

\[
\sum_{h \in \mathbb{Z}^s \setminus \{0\}, |h_j| \leq n_j} \frac{1}{\|h \cdot \alpha\|} \leq 2 \sum_{k=1}^{[1/(2d)]} \frac{1}{kd} \leq \frac{2}{d} (1 + \log[1/(2d)]) \ll_{s, \epsilon} r(n)^{1+2\epsilon}.
\]

Now we obtain

\[
\sum_{0 < |h|_\infty \leq m} \frac{1}{r(h)} \frac{1}{\|h \cdot \alpha\|} \ll_{s, \epsilon} \sum_{n_1, \ldots, n_s=1}^m f(n_1, \ldots, n_s)(n_1n_2 \cdots n_s)^{1+2\epsilon} = \left( \sum_{n=1}^m g_m(n)n^{1+2\epsilon} \right)^s \ll m^{2s\epsilon},
\]

where the last estimate easily follows from the definition of \( g_m \). Finally we obtain

\[
D_N(S_\alpha) \ll_{s} \frac{1}{m} + \frac{m^{2s\epsilon}}{N}
\]

and the result follows by choosing \( m = N \). \( \square \)

5 Probability Theory

The probabilistic method in general is used to show the existence of mathematical objects with certain properties by considering a probability measure on a class of objects and proving that the probability that a random object has the desired properties is positive or even close to 1. This concept is crucially used in many existence proofs in QMC.

5.1 Hoeffding’s inequality

Often, one wants to construct an object satisfying many constraints. Using the probabilistic method, the simplest way to achieve this is to show that the probability that one constraint is not satisfied is extremely small and then applying a union bound over all constraints. Extremely small probabilities can be obtained for the deviation from the mean for sums of independent random variables. A general and useful tool in the case of bounded random variables is Hoeffding’s inequality [39].
**Theorem 5.1 (Hoeffding)** Let $X_1, \ldots, X_N$ be independent real valued random variables such that $a_i \leq X_i - \mathbb{E}(X_i) \leq b_i$ for $i = 1, \ldots, N$ almost surely. Then for all $t > 0$

$$\text{Prob}\left(\left|\sum_{i=1}^{N} (X_i - \mathbb{E}(X_i))\right| > t\right) \leq 2 \exp\left(-\frac{2t^2}{\sum_{i=1}^{N} (b_i - a_i)^2}\right).$$

In particular, if $\mathbb{E}(X_i) = 0$ and $|X_i| \leq 1$ almost surely for $i = 1, \ldots, N$, then

$$\text{Prob}\left(\left|\sum_{i=1}^{N} X_i\right| > t\right) \leq 2 \exp\left(-\frac{t^2}{2N}\right).$$

**Example: Discrepancy of random points**

This approach was used in [30] to give an explicit bound for the star discrepancy showing polynomial tractability of the star discrepancy. For different notions of tractability and their extensive studies we refer to [71, 72, 73].

**Theorem 5.2 (Heinrich, Novak, Wasilkowski, Woźniakowski)** For $N, s \in \mathbb{N}$, there exists an $N$-element point set $\mathcal{P}$ in $[0,1)^s$ satisfying the discrepancy bound

$$D_N(\mathcal{P}) \ll \left(\frac{s}{N}\right)^{1/2} (\log s + \log N)^{1/2}.$$

**Proof.** [Sketch] Let $\mathcal{P} = \{t_1, \ldots, t_N\}$ where $t_1, \ldots, t_N$ are independent and uniformly distributed in $[0,1)^s$. We want to show that

$$\text{Prob} (D_N(\mathcal{P}) \leq 2\varepsilon) > 0$$

where $2\varepsilon$ is the right hand side in Theorem 5.1. That amounts to the task to show that the event

$$D_N(\mathcal{P}, x) > 2\varepsilon \quad \text{at least for one} \quad x \in [0,1)^s$$

has a probability smaller than 1. These are infinitely many constraints, but it can be shown that $D_N(\mathcal{P}, x) > 2\varepsilon$ implies $D_N(\mathcal{P}, y) > \varepsilon$ for one of the points in a rectangular equidistant grid of mesh size $\frac{1}{m}$ with $m = \lceil s/\varepsilon \rceil$. Actually, this holds either for the grid point directly below left or up right from $x$. Since this grid has cardinality $(m+1)^s$, a union bound shows that it is enough to prove

$$\text{Prob} (D_N(\mathcal{P}, x) > \varepsilon) < (m+1)^{-s}$$

for every $x \in [0,1)^s$. But now

$$ND_N(\mathcal{P}, x) = \sum_{i=1}^{N} (1_{B_{x}}(t_i) - \text{vol}(B_{x}))$$

is the sum of the $N$ random variables $X_i = 1_{B_{x}}(t_i) - \text{vol}(B_{x})$, which have mean 0 and obviously satisfy $|X_i| \leq 1$. So we can apply Hoeffding’s inequality and obtain

$$\text{Prob} (D_N(\mathcal{P}, x) > \varepsilon) = \text{Prob} \left(\left|\sum_{i=1}^{N} X_i\right| > N\varepsilon\right) \leq 2 \exp\left(-\frac{N\varepsilon^2}{2}\right) < (m+1)^{-s},$$

where the last inequality is satisfied for the chosen values of the parameters. \qed
5.2 Vapnik-Červonenkis classes and empirical processes

The behavior of the discrepancy function $D_N(P, \cdot)$ for a point set $P = \{t_1, \ldots, t_N\}$ with independent and uniformly distributed $t_1, \ldots, t_n$ as already considered in the previous section is intimately connected with the theory of empirical processes. In particular, this yields an essential improvement of Theorem 5.2 in [30]. Very general notions of the discrepancy function are related to empirical processes. Average discrepancies are then expectations of certain norms of such empirical processes as we explain below.

Let us first explain what an empirical process is. For a fixed integer $N$, let $X_1, \ldots, X_N$ be independent and identically distributed random variables defined on the same probability space with values in some measurable space $M$. Assume that we are given a sufficiently small class $\mathcal{F}$ of measurable real functions on $M$. The empirical process indexed by $\mathcal{F}$ is given by

$$\alpha_N(f) = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} (f(X_i) - \mathbb{E}(f(X_i)))$$

for $f \in \mathcal{F}$.

Now let $X_i = t_i$ and let $\mathcal{F}$ be the class of functions $1_{B(x)}$ with $x \in [0,1]^s$. Then

$$\alpha_N(1_{B(x)}) = \sqrt{N} D_N(P, x)$$

for $x \in [0,1]^s$, so $\sqrt{N} D_N(P, \cdot)$ is an empirical process. The expectation of the star discrepancy is related to the expectation of the supremum of this empirical process via

$$\sqrt{N} \mathbb{E}(D_N(P)) = \mathbb{E}\left( \sup_x |\alpha_N(1_{B(x)})| \right).$$

Now Donsker’s theorem [20] from empirical process theory tells us that for any fixed $s \in \mathbb{N}$ we have

$$\sqrt{N} \mathbb{E}(D_N(P)) \to \sup_{t \in [0,1]^s} |B_s(t)|$$

for $N \to \infty$. Here $B_s$ refers to the $s$-dimensional pinned Brownian sheet. It seems to be open what the value on the right hand side is for $s > 1$, so also the exact determination of $\mathbb{E}(D_N(P))$ is probably difficult.

But estimates for the supremum of empirical processes are important and available for certain classes of index sets. One example are Vapnik-Červonenkis classes which we introduce now. Let $(X, \text{Prob})$ be a probability space. A countable family $\mathcal{C}$ of measurable subsets of $X$ is called a Vapnik-Červonenkis class (for short VC-class) if there exists a nonnegative integer $s$ such that

$$\#\{A \cap C : C \in \mathcal{C}\} < 2^{s+1}$$

for any subset $A \subseteq X$ with $|A| = s + 1$. The smallest such $s$ is called VC-dimension of $\mathcal{C}$.

Also the discrepancy function can be generalized to this setting as follows. The discrepancy of an $N$-element set $P = \{t_1, \ldots, t_N\} \subseteq X$ with respect to $C \in \mathcal{C}$ is given as

$$D_N(P, C) = \frac{1}{N} \sum_{i=1}^{N} 1_C(t_i) - \text{Prob}(C).$$

Furthermore, let

$$D_N(P) = \sup_{C \in \mathcal{C}} |D_N(P, C)|.$$
If we choose for $C$ the class of boxes $B(x)$ with $x \in [0, 1]^s$, then we obtain the classical notion of the star discrepancy. Moreover, this class is a VC-class of dimension $s$, see \cite{22}. Choosing $t_i = X_i$ as independent random variables identically distributed according to $\text{Prob}$, we can again treat the $D_N(P)$ as the supremum of an empirical process indexed by the VC-class $C$.

The following theorem is a crucial large deviation inequality for empirical processes on VC-classes due to Talagrand \cite{89} and Haussler \cite{29}.

**Theorem 5.3 (Talagrand, Haussler)** There is a positive number $K$ such that for all VC-classes of dimension $s$, probabilities $\text{Prob}$, $c \geq Ks^{1/2}$ and $N \in \mathbb{N}$

$$\text{Prob} \left( D_N(P) \geq cN^{-1/2} \right) \leq \frac{1}{c} \left( \frac{Kc^2}{s} \right)^s \exp(-2s^2).$$

Using this estimate instead of Hoeffding’s inequality as in the previous section, one arrives at the following sharpening of Theorem 5.2 also proved in \cite{30}.

**Theorem 5.4 (Heinrich, Novak, Wasilkowski, Woźniakowski)** For $N, s \in \mathbb{N}$, there exists an $N$-element point set $P$ in $[0, 1]^s$ satisfying the discrepancy bound

$$D_N(P) \ll \left( \frac{s}{N} \right)^{1/2}.$$

For a version with an explicit constant in this inequality we refer to \cite{1}, for a lower bound for arbitrary sets to \cite{35}, and for a corresponding lower bound of the expectation of the star discrepancy of a random point set to \cite{19}. A standard reference for empirical processes is \cite{95}. There are also some related results on average $L_p$-discrepancies in \cite{38, 87}.

### 5.3 The Lovász local lemma

The simple principle of showing the existence of a mathematical object by proving a bound on the average and then concluding there is at least one instance which is at least as good as average, has found many important applications in QMC theory. However, this approach fails if the objects one considers are not independent. If the dependency structure is in a certain sense weak, then one way out is the Lovász local lemma.

We introduce now a simple version of the well-known Lovász local lemma \cite{23}. The following version is \cite{2, Lemma 5.11}. A more general form of the Lovász local lemma can, for instance, be found in \cite{90, Chapter 1}.

**Theorem 5.5 (Lovász local lemma)** Let $A = \{A_1, A_2, \ldots, A_m\}$ be a finite set of events in a probability space $\Omega$. For $A \in A$ let $\Gamma(A)$ denote a subset of $A$ such that $A$ is independent from the collection of events $A \setminus \{A \cup \Gamma(A)\}$. If there exist $x_1, \ldots, x_m \in (0, 1)$ such that

$$\forall \, i = 1, 2, \ldots, m : \quad \text{Prob}(A_i) \leq x_i \prod_{j=1 \atop j \neq i}^m (1 - x_j),$$

then the probability of avoiding all events in $A$ is positive, in particular

$$\text{Prob} \left( \bigcap_{i=1}^m A_i^c \right) \geq \prod_{i=1}^m (1 - x_i),$$

where $A^c$ denotes the complement of the event $A$. 
We illustrate the usefulness of the Lovász local lemma with two examples. The first one shows the existence of points on the torus whose minimal distance satisfies a certain lower bound and the second one deals with extensible lattice rules with a certain upper bound on the worst-case error in Korobov spaces.

**Example 1: Separation of points**

The main purpose of the following result is to show how to apply the Lovász local lemma (rather than obtaining a theorem of importance).

**Theorem 5.6** Let \( \Gamma \) be the Gamma function. Then there exists a set of points \( \mathcal{P}_N = \{ y_1, y_2, \ldots, y_N \} \) in the torus \([0, 1)^s\) such that

\[
\min_{1 \leq k < \ell \leq N} \| y_k - y_\ell \| \geq \frac{1}{\sqrt{\pi}} \left( \frac{2\Gamma(1 + s/2)}{N(N - 1)} \right)^{1/s},
\]

where \( \| \cdot \| \) denotes the Euclidean distance on the torus, i.e., for \( y = (y_1, y_2, \ldots, y_s) \in [0, 1)^s \) we define

\[
\| y \| = \sqrt{s \sum_{j=1}^s \min(y_j, 1 - y_j)^2}.
\]

**Proof.** Let \( \gamma > 0 \). Choose \( y_1 \in [0, 1)^s \) uniformly distributed. For \( k = 1, 2, \ldots, N - 1 \) we successively define the following events \( A_k \): Assume \( y_1, y_2, \ldots, y_k \in [0, 1)^s \) are already chosen. Then we choose \( y_{k+1} \in [0, 1)^s \) uniformly distributed and we say that the event \( A_k \) holds if

\[
y_{k+1} \in \bigcup_{\ell=1}^{k} B(y_\ell, \gamma),
\]

where

\[
B(y, \gamma) = \{ z \in [0, 1)^s : \| z - y \| < \gamma \}
\]

is the Euclidean ball with center \( y \) and radius \( \gamma \).

By inductively choosing the points \( y_k \), it is clear that the event \( A_k \) is independent of the event \( A_\ell \) for \( \ell > k \), since \( y_\ell \) is only chosen after \( y_k \). Thus the event \( A_k \) is independent of \( A_{k+1}, A_{k+2}, \ldots, A_{N-1} \). Hence we can choose \( \Gamma(A_k) = \{ A_1, A_2, \ldots, A_{k-1} \} \) in the Lovász local lemma.

We now define inductively \( x_1, \ldots, x_{N-1} \in (0, 1) \) which satisfy (20). The Lebesgue measure \( \lambda_s \) of the Euclidean ball \( B(y, \gamma) \) is given by

\[
\lambda_s(B(y, \gamma)) = \frac{\pi^{s/2}}{\Gamma(1 + s/2)} \gamma^s =: \kappa_{s, \gamma}.
\]

We have

\[
\text{Prob}(A_1) = \kappa_{s, \gamma} =: x_1.
\]

We claim that for \( k > 1 \) we can choose

\[
x_k = \frac{k \kappa_{s, \gamma}}{1 - \kappa_{s, \gamma} \sum_{i=1}^{k-1} i}.
\]
By setting $\sum_{i=1}^{0} i = 0$, the above formula also includes the case when $k = 1$.

To prove the claim, observe that
\[
\text{Prob}(A_k) = \lambda_s \left( \bigcup_{\ell=1}^{k} B(x_\ell, \gamma) \right) \leq k \kappa_{s, \gamma} = x_k \prod_{\ell=1}^{k-1} (1 - x_\ell).
\]

Thus (20) holds for our particular choice of $x$ and the conditions of Lemma 5.5 are all satisfied. The event $A_k^c$ now means that $y_{k+1}$ is chosen such that $y_{k+1} \notin \bigcup_{\ell=1}^{k} B(y_\ell, \gamma)$, which is equivalent to stating that $\min_{1 \leq \ell < k} \| y_k - y_\ell \| \geq \gamma$. Thus the event that all $A_1^c, \ldots, A_{N-1}^c$ hold simultaneously, that is, that $A_1^c \cap \ldots \cap A_{N-1}^c$ holds, is equivalent to
\[
\min_{1 \leq \ell < k \leq N} \| y_k - y_\ell \| \geq \gamma.
\]

From the Lovász local lemma we now obtain
\[
\text{Prob} \left( \min_{1 \leq \ell < k \leq N} \| y_k - y_\ell \| \geq \gamma \right) \geq \prod_{k=1}^{N-1} \left( 1 - \frac{k \kappa_{s, \gamma}}{1 - \kappa_{s, \gamma} \sum_{i=1}^{k-1} i} \right)
\]
\[
= \prod_{k=1}^{N-1} \frac{1 - \kappa_{s, \gamma} \sum_{i=1}^{k-1} i}{1 - \kappa_{s, \gamma} \sum_{i=1}^{k-1} i} = 1 - \kappa_{s, \gamma} \sum_{i=1}^{N-1} i
\]
\[
= 1 - \frac{\pi^{s/2}}{\Gamma(1 + s/2)} \gamma^s \frac{N(N-1)}{2}.
\]

The last expression is strictly positive if
\[
\gamma < \frac{1}{\sqrt{\pi}} \left( \frac{2(1 + s/2)}{N(N-1)} \right)^{1/s}
\]

For any $\gamma > 0$ which satisfies the last expression we have $\text{Prob}(\min_{1 \leq \ell < k \leq N} \| y_k - y_\ell \| \geq \gamma) > 0$. Since the torus is a compact set, there exists a point set which achieves the infimum. This shows the existence of a point set with the required property. \qed

**Example 2: Extensible lattice rules**

In this example we show that the Lovász local lemma can also be combined with Jensen’s inequality, which in its simplest form states that for a sequence of nonnegative real numbers $(a_k)$ and any $0 < \lambda \leq 1$ we have
\[
\left( \sum_k a_k \right)^\lambda \leq \sum_k a_k^\lambda.
\]

As an example, we show how the Lovász local lemma implies the existence of lattice rules which are extensible in dimension. Again, our focus is on illustrating the method. In doing so we restrict ourselves to a very simple case, namely the worst-case error of lattice rules in the unweighted Korobov space with smoothness parameter $\alpha > 1/2$ (see
Section 2). Let $N$ be a prime number. For an $N$-element lattice rule with generating vector $g \in \mathbb{Z}^s$ it can be shown that this worst-case error is given by

$$wce^2(H_{K_\alpha}, \mathcal{P}(g, N)) = \sum_{h \in L^s \setminus \{0\}} r_\alpha(h),$$

where $r_\alpha(h) = \prod_{j=1}^s r_\alpha(h_j)$ for $h = (h_1, \ldots, h_s) \in \mathbb{Z}^s$ and where for $h \in \mathbb{Z}$

$$r_\alpha(h) = \begin{cases} 1 & \text{if } h = 0, \\ |h|^{-2\alpha} & \text{if } h \neq 0. \end{cases}$$

Jensen’s inequality implies that

$$[wce^2(H_{K_\alpha}, \mathcal{P}(g, N))]^\lambda \leq wce^2(H_{K_{\alpha\lambda}}, \mathcal{P}(g, N)) \quad \text{for all } \frac{1}{2\alpha} < \lambda \leq 1,$$

(21)

where the restriction $\frac{1}{2\alpha} < \lambda$ is added to ensure that $wce^2(H_{K_{\alpha\lambda}}, \mathcal{P}(g, N))$ is well defined.

A popular way to show the existence of a generating vector $g$ with small worst-case error is to average over all lattice points from a certain finite set of lattice points. Let $G_N = \{1, \ldots, N - 1\}$. Then it can be shown that

$$\frac{1}{(N-1)^s} \sum_{g \in G_N^s} wce^2(H_{K_\alpha}, \mathcal{P}(g, N)) \leq \frac{(1 + 2\zeta(2\alpha))^s}{N-1},$$

(22)

where $\zeta(x) = \sum_{j=1}^{\infty} j^{-x}$ is the Riemann zeta function. Then Markov’s inequality guarantees the existence of a lattice point $g_* \in G_N^s$ which satisfies

$$wce^2(H_{K_\alpha}, \mathcal{P}(g, N)) \leq \frac{(1 + 2\zeta(2\alpha))^s}{N-1}.$$

We now use the Lovász local lemma to show the existence of a lattice rule which is extensible in the dimension. For $g = (g_1, \ldots, g_s) \in G_N^s$ and $d = 1, 2, \ldots, s$ we set $g_d = (g_1, \ldots, g_d)$.

**Theorem 5.7** Let $\alpha > 1/2$ be a real number, let $N$ be a prime number and let $s$ be a natural number. Let $c_1, c_2, \ldots, c_s > 0$ be such that

$$\sum_{d=1}^s \frac{1}{c_d} < 1.$$

Then there exists a lattice point $g \in G_N^s$ such that

$$wce^2(H_{K_\alpha}, \mathcal{P}(g_d, N)) < c_d^{1/\lambda} \frac{(1 + 2\zeta(\alpha))^{d/\lambda}}{(N-1)^{1/\lambda}} \quad \text{for all } d = 1, 2, \ldots, s \text{ and all } \frac{1}{2\alpha} < \lambda \leq 1.$$

**Proof.** Let $\frac{1}{2\alpha} < \lambda_1, \ldots, \lambda_s \leq 1$. For $d = 1, 2, \ldots, s$ define the events

$$A_d = \left\{ g \in G_N^s : wce^2(H_{K_{\alpha\lambda_d}}, \mathcal{P}(g, N)) \geq c_d \frac{(1 + 2\zeta(2\alpha\lambda_d))^{d}}{N-1} \right\}.$$
Then it follows easily from (22) and from Markov’s inequality, that
\[
\text{Prob}(A_d) < \frac{1}{c_d}.
\]
(23)

If we define
\[
x_d = \frac{1}{c_d \left( 1 - \sum_{k=1}^{d-1} \frac{1}{c_k} \right)},
\]
then it follows from (23) with some straightforward algebra that
\[
\text{Prob}(A_d) < x_d \prod_{j=1}^{d-1} (1 - x_j).
\]

Note that event \( A_d \) depends only on the events \( A_1, A_2, \ldots, A_{d-1} \), but not on the events \( A_{d+1}, A_{d+2}, \ldots, A_s \). Thus we can again use Lovász local lemma, where we choose \( \Gamma(A_d) = \{ A_1, \ldots, A_{d-1} \} \), and obtain
\[
\text{Prob} \left( \bigcap_{d=1}^{s} A_d^c \right) \geq \prod_{j=1}^{s} (1 - x_j) = 1 - \sum_{d=1}^{s} \frac{1}{c_d} > 0
\]
according to the choice of the \( c_d \)'s. Hence we have shown that there exists a lattice point \( g \in G_s^\lambda \) which is in the set \( \bigcap_{d=1}^{s} A_d^c \). Thus we have
\[
\text{wce}^2(H_{K_\alpha \lambda_d}, \mathcal{P}(g_d, N)) < c_d \left( \frac{1 + 2 \zeta(2\alpha \lambda_d)}{N - 1} \right)^d
\]
for all \( d = 1, 2, \ldots, s \).

From (21) we therefore obtain that there exists a lattice point which satisfies
\[
\text{wce}^2(H_{K_\alpha}, \mathcal{P}(g_d, N)) < c_d^{1/\lambda} \frac{(1 + 2 \zeta(2\alpha))^{d/\lambda}}{(N - 1)^{1/\lambda}}
\]
By choosing \( \frac{1}{2\alpha} < \lambda_d \leq 1 \) which minimizes \( c_d^{1/\lambda} \frac{(1 + 2 \zeta(2\alpha))^{d/\lambda}}{(N - 1)^{1/\lambda}} \), it follows that there is a lattice point such that
\[
\text{wce}^2(H_{K_\alpha}, \mathcal{P}(g_d, N)) < c_d^{1/\lambda} \frac{(1 + 2 \zeta(2\alpha))^{d/\lambda}}{(N - 1)^{1/\lambda}}
\]
for all \( d = 1, 2, \ldots, s \) and all \( \frac{1}{2\alpha} < \lambda \leq 1 \).

\[\square\]

References

[1] C. Aistleitner: Covering numbers, dyadic chaining and discrepancy. J. Complexity 27: 531–540, 2011.

[2] N. Alon and J.H. Spencer: The probabilistic method. Second edition. Wiley-Interscience Series in Discrete Mathematics and Optimization. Wiley-Interscience, New York, 2000.
[3] N. Aronszajn: Theory of reproducing kernels. Trans. Amer. Math. Soc. 68: 337–404, 1950.

[4] A. Baker: A Comprehensive Course in Number Theory. Cambridge University Press, Cambridge, 2012.

[5] J. Beck: Probabilistic diophantine approximation, I. Kronecker-sequences. Ann. Math. 140: 451–502, 1994.

[6] D. Bilyk: On Roth’s orthogonal function method in discrepancy theory. Unif. Distrib. Theory 6: 143–184, 2011.

[7] D. Bilyk: Exponential Squared Integrability for the Discrepancy Function in Two Dimensions. Mathematika 55: 1–27, 2009.

[8] J. Bourgain, A.A. Glibichuk, and S.V. Konyagin: Estimates for the number of sums and products and for exponential sums in fields of prime order. J. London Math. Soc. 73: 380–398, 2006.

[9] V.A. Bykovskii: The discrepancy of the Korobov lattice points. Izv. Math. 76: 446–465, 2012.

[10] J.W.S. Cassels: An introduction to the Geometry of Numbers. Springer-Verlag, Berlin, 1971.

[11] W. Chen, A. Srivastav, and G. Travaglini (eds.): A Panorama of Discrepancy Theory. Springer-Verlag, to appear 2014.

[12] R. Cools, F.Y. Kuo, and D. Nuyens: Constructing embedded lattice rules for multivariate integration. SIAM J. Sci. Comput. 28: 2162–2188, 2006.

[13] J. Dick, F.Y. Kuo, and I.H. Sloan: High-dimensional integration: the quasi–Monte Carlo way. Acta Numer. 22: 133–288, 2013.

[14] J. Dick, D. Nuyens, and F. Pillichshammer: Lattice rules for nonperiodic smooth integrands. Numer. Math. 126: 259–291, 2014.

[15] J. Dick and F. Pillichshammer: Multivariate integration in weighted Hilbert spaces based on Walsh functions and weighted Sobolev spaces. J. Complexity 21: 149–195, 2005.

[16] J. Dick and F. Pillichshammer: On the mean square weighted $L_2$ discrepancy of randomized digital $(t, m, s)$-nets over $\mathbb{Z}_2$. Acta Arith. 117: 371–403, 2005.

[17] J. Dick and F. Pillichshammer: Digital Nets and Sequences. Discrepancy Theory and Quasi-Monte Carlo Integration. Cambridge University Press, Cambridge, 2010.

[18] J. Dick, F. Pillichshammer, and B.J. Waterhouse: The construction of good extendible rank-1 lattices. Math. Comp. 77: 1345–1373, 2008.

[19] B. Doerr: A lower bound for the discrepancy of a random point set. J. Complexity 30: 16–20, 2014.
[20] M.D. Donsker: Justification and extension of Doob's heuristic approach to the Kolmogorov-Smirnov theorems. Ann. Math. Statist. 23: 277–281, 1952.

[21] M. Drmota and R.F. Tichy: Sequences, Discrepancies and Applications. Lecture Notes in Mathematics 1651, Springer-Verlag, Berlin, 1997.

[22] R.M. Dudley: A Course on Empirical Processes. Lecture Notes in Mathematics 1097, Springer-Verlag, New York, 1984.

[23] P. Erdős and L. Lovász: Problems and results on 3-chromatic hypergraphs and some related questions. In: Infinite and Finite Sets (A. Hajnal, R. Rado, and V. T. Sós, eds.), Vol. II, 609–627. Colloq. Math. Soc. János Bolyai, Vol. 10, North-Holland, Amsterdam, 1975.

[24] F. Gao, J. Hannig, and F. Torcaso: Integrated Brownian motions and exact $L_2$-small balls. Ann. Probab. 31: 1320–1337, 2003.

[25] I.G. Graham, F.Y. Kuo, D. Nuyens, R. Scheichl, and I.H. Sloan: Quasi-Monte Carlo methods for elliptic PDEs with random coefficients and applications. J. Comput. Phys. 230: 3668–3694, 2011.

[26] G. Halász: On Roth's method in the theory of irregularities of point distributions. In: Recent progress in analytic number theory, Vol. 2, 79–94. Academic Press, London-New York, 1981.

[27] J.H. Halton and S.K. Zaremba: The extreme and $L^2$ discrepancies of some plane sets. Monatsh. Math. 73: 316–328, 1969.

[28] G.H. Hardy and E.M. Wright: An Introduction to the Theory of Numbers. Oxford Science Publications, 5. Edition, Oxford, 1979.

[29] D. Haussler: Sphere packing numbers for subsets of the Boolean $n$-cube with bounded Vapnik-Červonenkis dimension. J. Combinatorial Theory A 69: 217–232, 1995.

[30] S. Heinrich, E. Novak, G. Wasilkowski, and H. Woźniakowski: The inverse of the star-discrepancy depends linearly on the dimension. Acta Arith. 96: 279–302, 2001.

[31] P. Hellekalek: A general discrepancy estimate based on $p$-adic arithmetics. Acta Arith. 139: 117–129, 2009.

[32] P. Hellekalek: A notion of diaphony based on $p$-adic arithmetic. Acta Arith. 145: 273–284, 2010.

[33] F.J. Hickernell, H.S. Hong, P. L’Ecuyer, and C. Lemieux: Extensible lattice sequences for quasi-Monte Carlo quadrature. SIAM J. Sci. Comput. 22: 1117–1138, 2000.

[34] F.J. Hickernell and H. Niederreiter: The existence of good extensible rank-1 lattices. J. Complexity 19: 286–300, 2003.

[35] A. Hinrichs: Covering numbers, Vapnik-Červonenkis classes and bounds for the star-discrepancy. J. Complexity 20: 477–483, 2004.
[36] A. Hinrichs: Discrepancy of Hammersley points in Besov spaces of dominating mixed smoothness. Math. Nachr. 283: 478–488, 2010.

[37] A. Hinrichs and L. Markhasin: On lower bounds for the $L_2$-discrepancy. J. Complexity 27: 127–132, 2011.

[38] A. Hinrichs and H. Weyhausen: Asymptotic behavior of average $L_p$-discrepancies. J. Complexity 28: 425–439, 2012.

[39] W. Hoeffding: Probability Inequalities for Sums of Bounded Random Variables. J. Amer. Statist. Assoc. 58: 13–30, 1963.

[40] I. Karatzas and S. E. Shreve: *Brownian motion and stochastic calculus*. Second edition. Graduate Texts in Mathematics, 113. Springer-Verlag, New York, 1991.

[41] P. Kritzer, H. Niederreiter, F. Pillichshammer, and A. Winterhof (eds.): *Uniform Distribution and Quasi-Monte Carlo Methods*. De Gruyter, to appear 2014.

[42] P. Kritzer and F. Pillichshammer: An exact formula for the $L_2$ discrepancy of the shifted Hammersley point set. Unif. Distrib. Theory 1: 1–13, 2006.

[43] P. Kritzer and F. Pillichshammer: Low discrepancy polynomial lattice point sets. J. Number Theory 132: 2510–2534, 2012.

[44] L. Kuipers and H. Niederreiter: *Uniform Distribution of Sequences*. John Wiley, New York, 1974.

[45] F.Y. Kuo: Component-by-component constructions achieve the optimal rate of convergence for multivariate integration in weighted Korobov and Sobolev spaces. J. Complexity 19: 301–320, 2003.

[46] F.Y. Kuo, Ch. Schwab, and I.H. Sloan: Quasi–Monte Carlo finite element methods for a class of elliptic partial differential equations with random coefficients. SIAM J. Numer. Anal. 50: 3351–3374, 2012.

[47] M. Lacey: On the discrepancy function in arbitrary dimension, close to $L^1$. Analysis Math. 34: 119–136, 2008.

[48] G. Larcher: On the distribution of sequences connected with good lattice points. Monatsh. Math. 101: 135–150, 1986.

[49] G. Larcher: Nets obtained from rational functions over finite fields. Acta Arith. 63: 1–13, 1993.

[50] G. Larcher and H. Niederreiter: Kronecker-type sequences and nonarchimedean diophantine approximation. Acta Arith. 63: 380–396, 1993.

[51] G. Larcher and F. Pillichshammer: A metrical best possible lower bound on the star discrepancy of digital sequences. Monatsh. Math., to appear.

[52] G. Larcher and F. Pillichshammer: Metrical lower bounds on the discrepancy of digital Kronecker-sequences. J. Number. Th. 135: 262–283, 2014.
[53] C. Lemieux: *Monte Carlo and quasi-Monte Carlo sampling*. Springer Series in Statistics. Springer-Verlag, New York, 2009.

[54] G. Leobacher and F. Pillichshammer: *Introduction to quasi-Monte Carlo Integration and Applications*. Birkhäuser Verlag AG, 2014.

[55] R. Lidl and H. Niederreiter: *Introduction to finite fields and their applications*. Cambridge University Press, Cambridge, 1994. (revised edition)

[56] J.N. Lyness: Notes on lattice rules. J. Complexity 19: 321–331, 2003.

[57] L. Markhasin: Discrepancy of generalized Hammersley type point sets in Besov spaces with dominating mixed smoothness. Unif. Distrib. Theory 8: 135–164, 2013.

[58] L. Markhasin: Quasi-Monte Carlo methods for integration of functions with dominating mixed smoothness in arbitrary dimension. J. Complexity 29: 370–388, 2013.

[59] L. Markhasin: Discrepancy and integration in function spaces with dominating mixed smoothness, Dissertationes Mathematicae 494: 1–81, 2013.

[60] J. Mercer: Functions of positive and negative type and their connection with the theory of integral equations. Philosophical Transactions of the Royal Society A 209: 415–446, 1909.

[61] T. Müller-Gronbach, E. Novak, and K. Ritter: *Monte-Carlo Algorithmen*. Springer-Verlag, Berlin Heidelberg, 2012.

[62] H. Niederreiter: Methods for estimating discrepancy. In: *Applications of number theory to numerical analysis* (S.K. Zaremba, ed.), pp. 203–236, Academic Press, New York, 1972.

[63] H. Niederreiter: Application of Diophantine approximations to numerical integration. Diophantine approximation and its applications (Proc. Conf., Washington, D.C., 1972), pp. 129–199. Academic Press, New York, 1973.

[64] H. Niederreiter: On the distribution of pseudo-random numbers generated by the linear congruential method II. Math. Comp. 28: 1117–1132, 1974.

[65] H. Niederreiter: Quasi-Monte Carlo methods and pseudo-random numbers. Bull. Amer. Math. Soc. 84: 957–1041, 1978.

[66] H. Niederreiter: Point sets and sequences with small discrepancy. Monatsh. Math. 104: 273–337, 1987.

[67] H. Niederreiter: Low-discrepancy point sets obtained by digital constructions over finite fields. Czechoslovak Math. J. 42: 143–166, 1992.

[68] H. Niederreiter: *Random Number Generation and Quasi-Monte Carlo Methods*. No. 63 in CBMS-NSF Series in Applied Mathematics. SIAM, Philadelphia, 1992.

[69] H. Niederreiter: Low-discrepancy sequences and non-Archimedean Diophantine approximations. Studia Sci. Math. Hungar. 30: 111–122, 1995.
[70] H. Niederreiter and F. Pillichshammer: Construction algorithms for good extensible lattice rules. Constr. Approx. 30: 361–393, 2009.

[71] E. Novak and H. Woźniakowski: *Tractability of Multivariate Problems. Volume I: Linear Information*. European Math. Soc. Publ. House, Zürich, 2008.

[72] E. Novak and H. Woźniakowski: *Tractability of Multivariate Problems. Volume II: Standard Information for Functionals*. European Math. Soc. Publ. House, Zürich, 2010.

[73] E. Novak and H. Woźniakowski: *Tractability of Multivariate Problems. Volume III: Standard Information for Operators*. European Math. Soc. Publ. House, Zürich, 2012.

[74] D. Nuyens and R. Cools: Fast algorithms for component-by-component construction of rank-1 lattice rules in shift-invariant reproducing kernel Hilbert spaces. Math. Comp. 75: 903–920, 2006.

[75] D. Nuyens and R. Cools: Fast component-by-component construction of rank-1 lattice rules with a non-prime number of points. J. Complexity 22: 4–28, 2006.

[76] D. Nuyens and R. Cools: Fast component-by-component construction, a reprise for different kernels. In: *Monte Carlo and Quasi-Monte Carlo Methods 2004* (H. Niederreiter and D. Talay, eds.), 373–387, Springer, Berlin, 2006.

[77] A. B. Owen: *Monte Carlo Theory, Methods and Examples*. In preparation, 2014.

[78] F. Pillichshammer: The \( p \)-adic diaphony of the Halton sequence. Funct. Approx. Comment. Math. 49: 91–102, 2013.

[79] G. Pirsic, J. Dick, and F. Pillichshammer: Cyclic digital nets, hyperplane nets and multivariate integration in Sobolev spaces. SIAM J. Numer. Anal. 44: 385–411, 2006.

[80] L. C. G. Rogers and D. Williams: *Diffusions, Markov processes, and martingales. Vol. 1. Foundations*. Reprint of the second (1994) edition. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 2000.

[81] L. C. G. Rogers and D. Williams: *Diffusions, Markov processes, and martingales. Vol. 2. Itô calculus*. Reprint of the second (1994) edition. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 2000.

[82] K.F. Roth: On irregularities of distribution. Mathematika 1: 73–79, 1954.

[83] W.M. Schmidt: Simultaneous approximation to algebraic numbers by rationals. Acta Math. 125: 189–201, 1970.

[84] W.M. Schmidt: Irregularities of distribution X. In: Number Theory and Algebra, Academic Press, New York, 311–329, 1977.

[85] I.E. Shparlinski: Exponential sums in coding theory, cryptology and algorithms. Coding theory and cryptology (Singapore, 2001), pages 323–383, Lect. Notes Ser. Inst. Math. Sci. Natl. Univ. Singap., 1, World Sci. Publ., River Edge, NJ, 2002.
[86] I.H. Sloan and S. Joe: *Lattice Methods for Multiple Integration*. Clarendon Press, Oxford, 1994.

[87] S. Steinerberger: The asymptotic behavior of the average $L^p$-discrepancies and a randomized discrepancy. Electron. J. Combin. 17: Research Paper 106, 18pp, 2010.

[88] O. Strauch and Š Porubský: *Distribution of Sequences: a Sampler*. Schriftenreihe der Slowakischen Akademie der Wissenschaften [Series of the Slovak Academy of Sciences], 1. Peter Lang, Frankfurt am Main, 2005.

[89] M. Talagrand: Sharper bounds for Gaussian and empirical processes. Ann. Probability 22: 28–76, 1994.

[90] T. Tao and V. Vu: *Additive Combinatorics*. Cambridge Studies in Advanced Mathematics, 105. Cambridge University Press, Cambridge, 2006.

[91] H. Triebel: *Bases in Function Spaces, Sampling, Discrepancy, Numerical Integration*. European Mathematical Society Publishing House, Zürich, 2010.

[92] H. Triebel: Numerical integration and discrepancy, a new approach. Math. Nachr. 283: 139–159, 2010.

[93] H. Triebel: *Faber Systems and Their Use in Sampling, Discrepancy, Numerical Integration*. EMS Series of Lectures in Mathematics. European Mathematical Society (EMS), Zürich, 2012.

[94] T. Ullrich: Optimal cubature in Besov spaces with dominating mixed smoothness on the unit square. J. Complexity 30: 72–94, 2014.

[95] A.W. van der Vaart and J.A. Wellner: *Weak Convergence and Empirical Processes*. Springer Series in Statistics. Springer-Verlag, New York, 1996.

[96] G. Wasilkowski and H. Woźniakowski: Weighted tensor product algorithms for linear multivariate problems. J. Complexity 15: 402–447, 1999.

[97] A. Winterhof: Topics related to character sums. Internat. Math. Nachrichten 220: 1–27, 2012.

[98] H. Woźniakowski: Average case complexity of multivariate integration. Bull. Amer. Math. Soc. (N.S.) 24: 185–194, 1991.

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