Lagrangian submanifolds in Hyperkähler manifolds, Legendre transformation

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Abstract

We develop the foundation of the complex symplectic geometry of Lagrangian subvarieties in a hyperkähler manifold. We establish a characterization, a Chern number inequality, topological and geometrical properties of Lagrangian submanifolds. We discuss a category of Lagrangian subvarieties and its relationship with the theory of Lagrangian intersection.

We also introduce and study extensively a normalized Legendre transformation of Lagrangian subvarieties under a birational transformation of projective hyperkahler manifolds. We give a Plücker type formula for Lagrangian intersections under this transformation.
1 Introduction

A Riemannian manifold $M$ of real dimension $4n$ is hyperkähler if its holonomy group is $Sp(n)$. Its has three complex structures $I, J$ and $K$ satisfying the Hamilton relation $I^2 = J^2 = K^2 = IJK = -1$. We will fix one complex structure $J$ on $M$. We denote its Kähler form as $\omega$ and its holomorphic two form as $\Omega$ which is nondegenerate and defines a (holomorphic) symplectic structure on $M$.

A submanifold $C$ in $M$ of dimension $n$ is called a Lagrangian if the restriction of $\Omega$ to it is zero. In the real symplectic geometry, Lagrangian submanifolds plays a very key role, for example in geometric quantization, Floer theory, Kontsevich’s homological mirror conjecture and Strominger, Yau and Zaslow geometric mirror conjecture.

The objective of this article is twofold: (1) We develop the foundation of the complex symplectic geometry of Lagrangian submanifolds in a hyperkähler manifold. This subject was previously studied by Donagi and Markman [DM], [DM2], Hitchin [Hi2] and others. (2) We introduce a Legendre transformation of Lagrangian subvarieties along $\mathbb{P}^n$ in $M$ and we establish a Plücker type formula.

Here we summarize our results in this paper: First it is not difficult to recognize a Lagrangian: (i) $C$ is Lagrangian if and only if its Poincaré dual $[C]$ represents a $\Omega$-primitive class in $H^{2n}(M, \mathbb{Z})$. In particular being Lagrangian is invariant under deformation. This also enable us to define singular Lagrangian subvarieties. (ii) When $C$ is a complete intersection in $M$, we have

$$\int_C c_2(M) \omega^{n-2} \geq 0,$$

moreover the equally sign holds if and only if $C$ is a Lagrangian. (iii) The rational cobordism class of the manifold $C$ is completely determined by the cohomology class $[C] \in H^{2n}(M, \mathbb{Z})$.

Second the second fundamental form of $C$ in $M$ defines a cubic vector field $\tilde{A} \in \Gamma(C, \text{Sym}^3 T_C)$ and a cubic form,

$$c_C : \text{Sym}^3 H^0(C, T_C^\ast) \to \mathbb{C},$$

$$c_C(\phi, \eta, \zeta) = \int_C \phi_i \eta_j \zeta_k \tilde{A}^{ijk} \sqrt{n!}.$$

$H^0(C, T_C^\ast)$ can be identified as the tangent space of the moduli space $\mathcal{M}$ of Lagrangian submanifolds. By varying $C$, the cubic tensor $c_C$ gives a holomorphic cubic tensor $c \in H^0(\mathcal{M}, \text{Sym}^3 T\mathcal{M})$. This cubic form defines a torsion free flat symplectic connection $\nabla$ on the tangent bundle of $\mathcal{M}$ and it satisfies $\nabla \wedge J_M = 0$. Namely it is a special Kähler structure on $\mathcal{M}$.

\footnote{Throughout this paper $M$ is a complex manifold with a fixed complex structure $J$ unless specified otherwise and all submanifolds are complex submanifolds with respect to $J$.}

\footnote{Many results presented here can be applied to any holomorphic symplectic manifold.}
Third the category $\mathcal{C}_M$ of Lagrangian subvarieties in $M$ with $Hom_{\mathcal{C}_M} (C_1, C_2) = \Sigma (-1)^q Ext^q_{O_M} (O_{C_1}, O_{C_2})$ refines the intersection theory of Lagrangians. For example

$$\dim Hom_{\mathcal{C}_M} (C_1, C_2) = (-1)^n C_1 \cdot C_2.$$ 

When $C_1$ and $C_2$ intersects cleanly, this equals the Euler characteristic of the intersection up to sign,

$$C_1 \cdot C_2 = \pm e (C_1 \cap C_2).$$

For each individual Ext group, we have (i) when $C$ is a complete intersection Lagrangian submanifold in $M$ then

$$Ext^q_{O_M} (O_C, O_C) = H^q (C, C),$$

for all $q$,

(ii) when $C_1$ and $C_2$ intersects transversely then $Ext^q_{O_M} (O_{C_1}, O_{C_2}) = 0$ for $q < n$ and equals $\mathbb{C}^n$ for $q = n$.

We also study the derived category of Lagrangian coherent sheaves on $M$, denote $D_{db}^b Lag (M)$. For example the structure sheaf of any Lagrangian subvariety in $M$ defines an object in this category.

Fourth we have good understanding of coisotropic subvarieties and their corresponding symplectic reductions in hyperkähler manifolds. Coisotropic submanifolds share some of Lagrangian properties. They can be characterized by the $\Omega$-primitivity property of their Poincaré duals. There is also a Chern number inequality in the complete intersection case. Furthermore for a generic complex structure in the twistor family of $M$ there is no isotropic or coisotropic subvarieties.

In real symplectic geometry, any well-behaved coisotropic submanifold gives rise to a reduction $\pi_D : D \to B$ with $B$ another symplectic manifold. Lagrangians in $M$ can be reduced to Lagrangians in $B$ or projected to Lagrangians of $M$ inside $D$. In the complex case we can define a reduction functor $R_D : D^b_{Lag} (M) \to D^b_{Lag} (B)$ and a projection functor $P_D : D^b_{Lag} (M) \to D^b_{Lag} (M)$ using Fourier-Mukai type functors.

Suppose the reduction process occurs inside $M$, namely there is a birational contraction $\pi : M \to Z$ such that $D$ is the exceptional locus and $B$ is the discriminant locus. In this case $\pi_D : D \to B$ is a $\mathbb{P}^k$-bundle and its relative cotangent bundle is the normal bundle of $D$ in $M ([\text{Wi}], [\text{Na}], [\text{HY}])$. Moreover one can replace $D$ by its dual $\mathbb{P}^k$-bundle and produce another holomorphic symplectic manifold $M'$, called the Mukai elementary modification $[\text{Mu1}].$

Fifth we can define Legendre transformation on any Mukai elementary modification. For example when $D \cong \mathbb{P}^n$ we associate to each Lagrangian subvariety $C$ in $M$ (not equals to $D$) another Lagrangian subvariety $C^\vee$ in $M'$.

\[3\] There is also stratified version of this construction by Markman $[\text{Ma}].$
We will explain its relationship with the classical Legendre transformation. Roughly speaking it is the hyperkähler quotient by $S^1$ of the Legendre transformation of defining functions of $C$ on the linear symplectic manifold $T^*\mathbb{C}^{n+1}$. This can also be regarded as a generalization of the dual varieties construction.

We establish the following **Plücker type formula** for the Legendre transformation,

$$C_1 \cdot C_2 + \frac{(C_1 \cdot \mathbb{P}^n) (C_2 \cdot \mathbb{P}^n)}{(-1)^{n+1}(n+1)} = C_1^{\vee} \cdot C_2^{\vee} + \frac{(C_1^{\vee} \cdot \mathbb{P}^{n*}) (C_2^{\vee} \cdot \mathbb{P}^{n*})}{(-1)^{n+1}(n+1)}.$$

When $n = 2$ this formula is essentially equivalent to a classical Plücker formula for plane curves.

Motivated from this formula we define a **normalized Legendre transformation**

$$\mathcal{L}(C) = C^{\vee} + \frac{(C \cdot \mathbb{P}^n) + (-1)^{n+1} (C^{\vee} \cdot \mathbb{P}^{n*})}{n+1} \mathbb{P}^{n*} \text{ if } C \neq \mathbb{P}^n$$

$$\mathcal{L}(\mathbb{P}^n) = (-1)^n \mathbb{P}^{n*}.$$

This transformation $\mathcal{L}$ preserves the intersection product:

$$C_1 \cdot C_2 = \mathcal{L}(C_2) \cdot \mathcal{L}(C_2).$$

When $n = 1$ we simply have $M = M'$ and $C = C^{\vee}$, however the normalized Legendre transformation is interesting, it coincides with the Dehn twist along a $(-2)$-curve in $M$.

In the general case, the Plucker type formula and the definition of the normalized Legendre transformation will involve the reduction and the projection of a Lagrangian with respect to the coisotropic exceptional submanifold $D$.

Next we are going to look at an **explicit** example of a hyperkähler manifold and its flop.

**The cotangent bundle of $\mathbb{P}^n$**

In [Ca], [Ca2] Calabi showed that the cotangent bundle of the complex projective space is a hyperkähler manifold and its metric can be described explicitly as follow: Let $z^1, ..., z^n$ be a local inhomogeneous coordinate system in $\mathbb{P}^n$ and $\zeta_1, ..., \zeta_n$’s be the corresponding coordinate system on the fibers of $T^*\mathbb{P}^n$, i.e. $\Sigma \zeta_j dz^j$ represents a point in $T^*\mathbb{P}^n$. The symplectic form on $T^*\mathbb{P}^n$ is given by $\Sigma dz^j \wedge d\zeta_j$. The hyperkähler Kähler form on $T^*\mathbb{P}^n$ is given by

$$\omega = \partial \bar{\partial} \left( \log \left( 1 + |z|^2 \right) + f(t) \right),$$

where $f(t) = \sqrt{1+4t} - \log (1 + \sqrt{1+4t})$ and $t = \left( 1 + |z|^2 \right) \left( |\zeta|^2 + |z \cdot \zeta|^2 \right)$. Calabi’s approach is to look for a $U(n+1)$-invariant hyperkähler metric and reduces the problem to solving an ODE for $f(t)$.

Another approach by Hitchin [Hi] is to construct the hyperkähler structure on $T^*\mathbb{P}^n$ by the method of hyperkähler quotient, which is analogous to the
symplectic quotient (or symplectic reduction) construction. Consider \( V = \mathbb{C}^{n+1} \) with the diagonal \( S^1 \)-action by multiplication. Its induced action on \( \mathbb{H}^{n+1} = V \times V^* = T^*V \) is given by
\[
S^1 \times T^*V \to T^*V; \quad e^{i\theta} \cdot (x, \xi) = (e^{i\theta} x, e^{-i\theta} \xi).
\]
This \( S^1 \)-action preserves both the Kähler form \( dx \wedge d\bar{x} + d\xi \wedge d\bar{\xi} \) and the natural holomorphic symplectic form \( dx \wedge d\xi \) on \( T^*V \). Namely it preserves the hyperkähler structure on \( \mathbb{H}^{n+1} = T^*V \). The real and complex moment maps are given respectively by
\[
\mu_J = i |x|^2 - i |\xi|^2 \in i\mathbb{R}, \quad \mu_c = \xi(x) \in \mathbb{C}.
\]
We can also combine them to form the hyperkähler moment map:
\[
\mu : \mathbb{H}^{n+1} \to i\mathbb{R} + \mathbb{C} = i\mathbb{R}^3.
\]
\[
\mu = (\mu_J, \mu_c) = (\mu_J, \mu_t, \mu_K).
\]
If we take \( \lambda = i (1, 0, 0) \in i\mathbb{R}^3 \) for the hyperkähler quotient construction, then \( S^1 \) acts freely on \( \mu^{-1}(\lambda) \) and the quotient is a smooth hyperkähler manifold \( M \),
\[
M = T^*V/_{\text{HK}} S^1 = \mu^{-1}(\lambda) / S^1 = \left\{(x, \xi) : \xi(x) = 0, |x|^2 - |\xi|^2 = 1\right\} / S^1.
\]
We can identify \( M \) with \( T^*\mathbb{P}^n \) explicitly as follows: It is not difficult to see that
\[
p : M \to \mathbb{P}^n
\]
\[
(x, \xi) \to y = x \left(1 + |\xi|^2\right)^{-1/2}
\]
defines a map from \( M \) to \( \mathbb{P}^n \) with a section given by \( \xi = 0 \). The fiber of \( p \) over the point \( y = [1, 0, \ldots, 0] \in \mathbb{P}^n \) consists of those \((x, \xi)\)'s of the form
\[
x = (a, 0, \cdots, 0),
\]
\[
\xi = (0, b_1, \ldots, b_n),
\]
and satisfying \( |a|^2 = 1 + \sum |b_j|^2 \), i.e. \( p^{-1}(y) \) is parametrized by \((b_1, \ldots, b_n) \in \mathbb{C}^n \). Note that \( p^{-1}(y) \) can be naturally identified with \( T^*_y\mathbb{P}^n = Hom(T_y\mathbb{P}^n, \mathbb{C}) \) via
\[
(0, c_1, \cdots, c_n) \to \Sigma b_j c_j.
\]
By using the \( U(n+1) \)-symmetry, this gives a natural identification between \( M \) and \( T^*\mathbb{P}^n \). Moreover \( \{\xi = 0\} \) in \( M \) corresponds to the zero section in \( T^*\mathbb{P}^n \), we simply denote it as \( \mathbb{P}^n \).
The only compact Lagrangian submanifold in $T^*\mathbb{P}^n$ is $\mathbb{P}^n$. However there are many non-compact Lagrangian submanifolds. For any submanifold $S$ in $\mathbb{P}^n$, its conormal bundle $N^*_S/\mathbb{P}^n$ is a Lagrangian submanifold in $T^*\mathbb{P}^n$, a well-known construction in symplectic geometry.

There is a natural isomorphism $T^*V \cong T^*(V^*)$ because of $(V^*)^* \cong V$. We can therefore carry out the above construction on $T^*(V^*)$ exactly as before and obtain another hyperkähler manifold $M'$, which is of course isomorphic to $T^*\mathbb{P}^n$. In terms of the above coordinate system on $V$, the only difference is to replace the original $S^1$-action by its inverse action. This would change the sign of the moment maps and therefore we can identify $M'$ as follow,

$$M' = T^*(V^*)/_{HK}S^1 = \{(\xi, x) \in V^* \times V : \xi(x) = 1, |\xi|^2 - |x|^2 = 1\}/S^1.$$

The two holomorphic symplectic manifolds $M$ and $M'$ are isomorphic outside their zero sections, the birational map is given explicitly as follow,

$$\Phi : M' \rightarrow M'$$

$$(x, \xi) \rightarrow \left(\begin{array}{c} x \\ \xi \\ \frac{\xi}{x} \end{array}\right).$$

In fact the zero section $\mathbb{P}^n$ inside $M = T^*\mathbb{P}^n$ can be blown down and we obtain a variety $M_0$. Both $M$ and $M'$ are two different crepant resolutions [Ca2] of the isolated singularity of $M_0$ and is usually called a flop. This is also the basic structure in the Mukai’s elementary modification [Mu1]. Explicitly we have,

$$\pi : M \rightarrow M_0,$$

$$\pi ([x, \xi]) \rightarrow x \otimes \xi,$$

where $M_0 = \{ A \in \text{End} (\mathbb{C}^{n+1}) : \text{Tr}A = 0, \text{rank} (A) \leq 1\}$. It is rather easy to check that $\pi$ is the blown down morphism of $\mathbb{P}^n$ inside $M$. The situation for $M'$ is identical.

2 Isotropic and coisotropic submanifolds

The most natural class of submanifolds in a symplectic manifold $M$ consists of those $C$ in $M$ with the property that the restriction of the symplectic form $\Omega$ to $C$ is as degenerate as possible. Such a submanifold $C$ is called isotropic, coisotropic or Lagrangian according to $\dim C \leq n$, $\dim C \geq n$ or $\dim C = n$ respectively.
2.1 Definitions and properties

We first look at the linear case, i.e. \( M \) and \( C \) are vector space and its linear subspace respectively. The complement of \( C \) in \( M \) is defined as follow,

\[
C^\perp = \{ v \in M : \Omega (v, w) = 0 \text{ for all } w \in C \}. 
\]

\( C \) is called *isotropic* (resp. *coisotropic* or *Lagrangian*) if \( C \subset C^\perp \) (resp. \( C^\perp \subset C \) or \( C = C^\perp \)). Here is the standard example: when \( M = \mathbb{C}^{2n} \) with \( \Omega = dz^1 \wedge dz^{n+1} + \cdots + dz^n \wedge dz^{2n} \), then the linear span of \( \frac{\partial}{\partial z^1}, \ldots, \frac{\partial}{\partial z^m} \) is an isotropic (resp. coisotropic or Lagrangian) subspace if \( m \leq n \) (resp. \( m \geq n \) or \( m = n \)). A useful linear algebra fact is any isotropic or coisotropic subspace of \( M \) is equivalent to the one in the standard example up to an automorphism of \( M \) which preserves \( \Omega \), namely a symplectomorphism. For general symplectic manifolds we have the following standard definition.

**Definition 1** If \((M, \Omega)\) is a symplectic manifold and \( C \) is a submanifold of \( M \), then \( C \) is called isotropic (resp. coisotropic or Lagrangian) if \( TC \subset TC^\perp \) (resp. \( TC^\perp \subset TC \) or \( TC = TC^\perp \)). Here

\[
TC^\perp = \{ v \in TM|_C : \Omega (v, w) = 0 \text{ for all } w \in TC \}. 
\]

When \( \dim C = 1 \) (resp. \( 2n-1 \)), \( C \) is automatically isotropic (resp. coisotropic). From the definition it is clear that \( C \) being isotropic is equivalent to \( \Omega|_C = 0 \). This happens if \( C \) has no nontrivial holomorphic two form, i.e. \( H^{2,0}(C) = 0 \).

In particular any submanifold \( C \) in \( M \) with dimension greater than \( n \) has \( H^{2,0}(C) \neq 0 \). More generally if \( \dim C = n + k \) then \( H^{2j,0}(C) \neq 0 \) for \( j = 0, \cdots, k \).

**Lemma 2** If \( M \) is a hyperkähler manifold and \( C \) is a submanifold in \( M \) of dimension \( n + k \) then the restriction of \( (\Omega)^k \) to \( C \) is always nowhere vanishing and moreover \( (\Omega)^{k+1} = 0 \) if and only if \( C \) is coisotropic.

Proof of lemma: In the standard example the above assertion can be verified directly. For the general case the assertion follows from the fact that every coisotropic subspace in a symplectic vector space can be conjugated by a symplectomorphism to the one in the standard example.

With the help of the above lemma we can prove the following useful result.

**Theorem 3** If \( M \) is a compact hyperkähler manifold and \( C \) is a submanifold in \( M \) then

\[
(i) \quad C \text{ isotropic} \iff L_\Omega [C] = 0,
(ii) \quad C \text{ coisotropic} \iff \Lambda_\Omega [C] = 0.
\]

Here \( [C] \in H^*(M, \mathbb{Z}) \) denotes the Poincaré dual of \( C \).
Proof: We need to use the Hard Lefschetz \( sl_2 \)-action induced by \( L_\Omega \) and \( \Lambda_\Omega \) (see appendix for details). For (i) when \( C \) is isotropic we have \( \Omega|_C = 0 \) and therefore \( L_\Omega [C] = 0 \) by Poincaré duality. For the converse we suppose that \( L_\Omega [C] = 0 \), i.e. \( [C] \cup \Omega = 0 \). By the Hard Lefschetz \( sl_2 \)-action the dimension of \( C \) must be strictly less than \( n \), say \( m \).

If \( m = 1 \) then \( C \) is already isotropic. So we assume \( m \geq 2 \) and we have

\[
0 = \int_M [C] \Omega \bar{\Omega} \omega^{m-2},
\]

where \( \omega \) is the Kähler form on \( M \). This is the same as \( \int_C \Omega \bar{\Omega} \omega^{n-2} = 0 \). Recall that \( \omega|_C \) defines a Kähler structure on \( C \). By the Riemann bilinear relation on the Kähler manifold \( C \), the function \( \Omega \bar{\Omega} \omega^{n-2} / \omega^n \) is proportional to the norm square of \( \Omega|_C \). Therefore the holomorphic form \( \Omega|_C \) must vanish, namely \( C \) is isotropic.

For (ii) we first suppose that \( \Lambda_\Omega [C] = 0 \). By the Hard Lefschetz \( sl_2 \)-action induced by \( L_\Omega \) and \( \Lambda_\Omega \), we have \( \dim C = m = n + k > n \) and

\[
[C] \cup (\Omega)^{k+1} = 0 \in H^{2n+2k+2} \left( M, \mathbb{C} \right).
\]

Therefore

\[
0 = \int_M [C] \Omega^{k+1} \bar{\Omega}^{k+1} \omega^{2n-2} = \int_C \Omega^{k+1} \bar{\Omega}^{k+1} \omega^{2n-2}.
\]

Using the Riemann bilinear relation as before, we have \( (\Omega)^{k+1}|_C = 0 \). By the above lemma, \( C \) is a coisotropic submanifold. The converse is clear. Hence the result. \( \blacksquare \)

The theorem says that a submanifold of \( M \) being isotropic (or coisotropic) can be detected cohomologically. An immediate corollary is that such property is invariant under deformation. It also enables us to define isotropic (or coisotropic) subvarieties which might be singular, or even with non-reduced scheme structure.

**Definition 4** Suppose \( M \) is a compact hyperkähler manifold. A subscheme \( C \) of \( M \) is called isotropic (resp. coisotropic) if \( L_\Omega [C] = 0 \) (resp. \( \Lambda_\Omega [C] = 0 \)).

The next proposition gives a good justification of this definition.

**Proposition 5** Suppose \( M \) is a compact hyperkähler manifold and \( C \) is an irreducible subvariety. Then \( C \) is isotropic (resp. coisotropic) if and only if \( T_x C \) is an isotropic (resp. coisotropic) subspace of \( T_x M \) for every smooth point \( x \in C \).
Proof: For the isotropic case, the if part of the assertion is obvious. For the only if part it suffices for us to check $T_xC$ being an isotropic subspace of $T_xM$ for a generic point $x \in C$ because $\Omega|_{T_xC} = 0$ is a closed condition among those $x$'s with constant dim $T_xC$. If $x$ is a smooth generic point in $C$ at which $\Omega|_{T_xC} \neq 0$, then
\[
\frac{\Omega \bar{\Omega} \omega^{k-2}}{\omega^k}|_x > 0
\]
by the Riemann bilinear relation, where $k = \dim C$. Therefore we have
\[
\int_C \Omega \bar{\Omega} \omega^{k-2} > 0
\]
violating $L_{\Omega}[C] = 0$ assumption. Hence the claim. The coisotropic case can be treated in a similar way. ■

Another corollary of the previous theorem is the following proposition which generalizes Fujiki’s observation that a generic complex structure in the twistor family of $M$ has no curves or hypersurfaces.

**Proposition 6** Suppose $M$ is a compact hyperkähler manifold. For a general complex structure in its twistor family, $M$ has no nontrivial isotropic or coisotropic submanifold.

Proof: Suppose $C \subset M$ is an isotropic submanifold of positive dimension $k$, then $[C] \cup \text{Re } \Omega = [C] \cup \text{Im } \Omega = 0$ but $[C] \cup \omega^k > 0$ in $H^*(M, \mathbb{R})$ because $C$ is a complex submanifold. Other Kähler structures in the same twistor family can be written as $a \text{Re } \Omega + b \text{Im } \Omega + c \omega$ for $(a, b, c) \in \mathbb{R}^3$ satisfying $a^2 + b^2 + c^2 = 1$. This implies that $[C]$ can not be represented by an isotropic complex submanifold in any other Kähler structures in this uncountable family, except possibly $-\omega$ for $k$ even. On the other hand $[C]$ belongs to the integral cohomology of $M$, which is a countable set. Hence we have our claim for the isotropic case. The coisotropic case can also be argued in a similar way. ■

We have the following immediate corollary of the above proof.

**Corollary 7** For any given $c \in H^*(M, \mathbb{Z})$ in a compact hyperkähler manifold $M$, there is at most two complex structures in the twistor family of $M$ such that $c$ can be represented by an isotropic or coisotropic subvariety.

Note that isotropic or coisotropic submanifolds are plentiful when the whole twistor family of complex structures on $M$ is considered. In the case of a K3 surface or an Abelian surface, they are complex curves and the numbers of such form a beautiful generating function in terms of modular forms, as conjectured by Yau and Zaslow in [YZ] and proved by Bryan and the author in [BL1], [BL2] (also see [BL3]).
Coisotropic complete intersections

When $M$ is projective, there are many ample hypersurfaces $C$ and they are all coisotropic for trivial reasons. They are varieties of general type and their Chern classes satisfy\footnote{These can be proven using the adjunction formula, $c_{2k+1} (M) = 0$ and the Chern number inequality $\int_M c_2 (M) |C|^{2n-2} \geq 0$.} (i) $c_{2k+1} (C) = c_1 (C) c_{2k} (C)$ for all $k$ and (ii) $\int_C c_2 (C) c_1 (C)^{2n-3} \leq \int_C c_1 (C)^{2n-1}$.

The next theorem says that complete intersection coisotropic subvarieties can be characterized by the vanishing of a Chern number. In particular intersecting ample hypersurfaces will not give higher codimension coisotropic subvarieties.

**Theorem 8** Suppose $M$ is a compact hyperkähler manifold and $C$ is a complete intersection subvariety of dimension $n+k$ with $k \leq n-2$. Then

$$\int_C c_2 (M) (\Omega \bar{\Omega})^k \omega^{n-k-2} \geq 0.$$  

Moreover the equality sign holds if and only if $C$ is coisotropic.

Proof: We can express the above quantity in term of the Bogomolov-Beauville quadratic form $q$ by Fujiki’s result (\cite{Fu}, see also the appendix) which says for any $D_1, \cdots, D_{2n-2} \in H^2 (M, \mathbb{C})$ we have

$$\int_M c_2 (M) D_1 \cdots D_{2n-2} = c \sum_{\{i_1, \cdots, i_{2n-2}\} = \{1, \cdots, 2n-2\}} q (D_{i_1}, D_{i_2}) \cdots q (D_{i_{2n-3}}, D_{i_{2n-2}}),$$

for some constant $c$. When $D_i$’s are ample divisors, the left hand side is strictly positive by the Chern number inequality for Kähler-Einstein manifolds. On the other hand, $q (D, D') > 0$ if $D$ is an ample divisor and $D'$ is an effective divisor. Therefore we have $c > 0$. More generally $q (D, D') \geq 0$ when $D$ and $D'$ are effective divisors and they intersect transversely, moreover the equality sign holds if and only if $(\Omega)^{n-1} |_{D \cap D'} = 0$. This is because

$$q (D, D') = c' \int_M [D] [D'] \Omega^{n-1} \bar{\Omega}^{n-1}$$

for some explicit positive constant $c'$.

We recall other basic properties of the Bogomolov-Beauville quadratic form: $q (\Omega, \bar{\Omega}) > 0$; $q (\Omega, \omega) = q (\Omega, D) = q (\bar{\Omega}, D) = 0$ for any effective divisor $D$.

Now $C$ is a complete intersection subvariety of $M$, we write

$$C = D_1 \cap \cdots \cap D_{n-k}$$

for some effective divisors $D_i$’s. We have

$$\int_C c_2 (M) (\Omega \bar{\Omega})^k \omega^{n-k-2} = \int_M c_2 (M) [D_1] \cdots [D_{n-k}] \Omega^k \bar{\Omega}^k \omega^{n-k-2}. $$
We apply Fujiki’s result and above properties of \( q \) and we get

\[
\int C c_2 (M) (\Omega \omega)^k \omega^{n-k-2} = cq (\Omega, \Omega)^k \sum q (D_{i_1}, D_{i_2}) q (\omega, D_{i_3}) \cdots q (\omega, D_{i_{n-k}}) + cq (\omega, \omega) q (\Omega, \Omega)^k \sum q (D_{i_1}, D_{i_2}) q (D_{i_3}, D_{i_4}) q (\omega, D_{i_5}) \cdots q (\omega, D_{i_{n-k}}) + ....
\]

Each term on the right hand side is non-negative. This implies

\[
\int C c_2 (M) (\Omega \omega)^k \omega^{n-k-2} \geq 0.
\]

Moreover it is zero if and only if \( q (D_i, D_j) = 0 \) for all \( i \neq j \). This is equivalent to \( \Omega^{n-1} |_{D_i \cap D_j} = 0 \) for all \( i \neq j \) because \( D_i \cap D_j \) is a complete intersection. We will prove in the next lemma that this is equivalent to \( \Omega^{k+1} |_{D_i \cap \cdots \cap D_{n-k}} = 0 \), \( k \geq 2 \), i.e. \( C = D_1 \cap \cdots \cap D_{n-k} \) is a coisotropic subvariety of \( M \). Hence the result.

The next lemma on linear algebra is needed in the proof of the above theorem and it is also of independent interest.

**Lemma 9** Let \( M \cong \mathbb{C}^n \) be a symplectic vector space with its symplectic form \( \Omega \) and \( C \) is a codimension \( m \) linear subspace in \( M \). If we write \( C = D_1 \cap \cdots \cap D_m \) for some hyperplanes \( D_i \)’s in \( M \), then \( C \) is coisotropic in \( M \) if and only if \( \Omega^{n-1} |_{D_i \cap D_j} = 0 \) for all \( i \neq j \).

Proof: We first prove the if part by induction on the dimension of \( C \). The claim is trivial for \( m = 1 \). When \( m = 2 \), it says that a codimension two subspace \( C \) in \( M \) is coisotropic if \( \Omega^{n-1} |_C = 0 \). This is true and in general we have any codimension \( m \) subspace \( C \) in \( M \) is coisotropic if and only if \( \Omega^{n-m+1} |_C = 0 \). By induction we assume \( C = D_1 \cap \cdots \cap D_{m+1} \) and the claim is true for \( m \), i.e. \( D_1 \cap \cdots \cap D_m \) is coisotropic. We can choose coordinates on \( M \) such that \( \Omega = dz^1 dz^{n+1} + \cdots + dz^n dz^{2n} \) and \( D_i = \{ z^i = 0 \} \) for \( i = 1, \ldots, m \). Suppose \( \sum_{i=1}^{2n} a_i z^i = 0 \) is the defining equation for \( D_{m+1} \). In order for \( C \) to be coisotropic, it suffices to show that \( a_i = 0 \) for \( n+1 \leq i \leq n+m \). For instance if \( a_{n+1} \neq 0 \) then \( \Omega^{n-1} |_{D_1 \cap D_{m+1}} \) would be nonzero. It is because \( dz^2 dz^{n+2} dz^{n+3} \cdots dz^n dz^{2n} |_{D_1 \cap D_{m+1}} \neq 0 \) and all other summands of \( \Omega^{n-1} \) would restrict to zero on \( D_1 \cap D_{m+1} \). If any other \( a_{n+j} \neq 0 \) the same argument applies by replacing \( D_1 \) with \( D_j \). This contradicts our assumption \( \Omega^{n-1} |_{D_1 \cap D_j} = 0 \) for all \( i \neq j \). Therefore \( a_i = 0 \) when \( n+1 \leq i \leq n+m \) and hence the claim.

For the only if part, we need to prove that \( \Omega^{n-1} |_{D_1 \cap D_2} \neq 0 \) implies \( \Omega^{n-1} |_{D_1 \cap \cdots \cap D_l} \neq 0 \) for any complete intersection \( D_1 \cap \cdots \cap D_l \). This is a simple linear algebra exercise. Hence the lemma.
2.2 Constructions and birational transformations

In this subsection we recall known results on the birational geometry of hyperkähler manifolds. The exceptional locus of any birational morphism is always a coisotropic subvariety. The reduction and the projection of a Lagrangian subvariety with respect to this coisotropic subvariety will be used to obtain the Plücker type formula. Moreover the Legendre transformation that we will discuss in section [4] fits nicely into this picture.

In real symplectic geometry, coisotropic submanifolds play a role in the symplectic reductions which produce symplectic manifolds of smaller dimensions under favorable conditions (see for example section 5.1 of [BW]). For example when there is a Hamiltonian group action one can construct a symplectic quotient by the reduction method.

In complex symplectic geometry a coisotropic subvariety arises naturally as the exceptional set of any birational contraction, and its geometric structures can even be described rather explicitly. Roughly speaking it is a general $P^k$-bundle over another symplectic manifold of dimension $2(n-k)$ when the exceptional locus is a honest $P^k$-bundle, Mukai [Mu1] showed that one can produce another symplectic manifold of the same dimension by replacing the coisotropic $P^k$-bundle by its dual $P^k$-bundle, the Mukai elementary modification. In dimension four this turns out to be the only possible way to generate birational transformations provided certain normality conditions hold ([BHL]).

All these structures will give important functors among categories of Lagrangian subvarieties that we will define in later sections.

Characteristic foliations and reductions

For any submanifold $C$ in $M$, there is an exact sequence relating their tangent bundles. Using the isomorphism between $T_M$ and $T^*_M$ induced from the symplectic form $\Omega$, we have the following exact diagram,

$$
\begin{array}{cccccccc}
0 & \rightarrow & T_C & \rightarrow & T_M|_C & \rightarrow & N_{C/M} & \rightarrow & 0 \\
\downarrow & & \downarrow & \cong & \uparrow & & \uparrow & \\
0 & \leftarrow & T^*_C & \leftarrow & T^*_M|_C & \leftarrow & N^*_C/M & \leftarrow & 0,
\end{array}
$$

where $N_{C/M}$ is the normal bundle of $C$ inside $M$. The coisotropic condition on $C$ is equivalent to the triviality of the composition homomorphism $N^*_C/M \rightarrow N_{C/M}$. When this happens we have an injective homomorphism $N^*_C/M \rightarrow T_C$ and we denote the quotient bundle as $S$,

$$
0 \rightarrow N^*_C/M \rightarrow T_C \rightarrow S \rightarrow 0.
$$

A different viewpoint is that the restriction of $\Omega$ to $T_C$ is no longer non-degenerate, and the kernel is precisely given by $N^*_C/M$. It is then clear that the bundle $S$ inherits a natural symplectic structure. In particular $c_{2k+1}(S) = 0$ for all $k$. This implies that $T_C$ cannot be a stable bundle with nonnegative degree.

When $k = 1$ the generic fiber can also be an ADE configuration of $P^1$. 

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with respect to any polarization. For instance a coisotropic submanifold in $M$ can never be a Calabi-Yau manifold or an irreducible hyperkähler manifold unless it is a Lagrangian. If $C$ is an Abelian variety, then the above exact sequence splits and both $N^*_{C/M}$ and $S$ are trivial bundles over $C$. This happens in Lagrangian fibrations on $M$.

The distribution on $C$ given by $N^*_{C/M} \subset TC$ is always integrable (see e.g. p.67 of [BW]). This is called the characteristic foliation of the coisotropic submanifold. When the leaf space $B$ is a smooth manifold, it carries a natural symplectic form, moreover this form pulls back to the symplectic structure on the bundle $S$ over $C$. This symplectic manifold $B$ if exists, is called the reduction of the coisotropic submanifold $C$ (see section 3.3.3 for further details).

Roughly speaking what we did is to contract degenerate directions in a coisotropic submanifold to construct the symplectic form on the quotient space $B$. A natural question is whether we can perform this inside $M$ rather than $C$. This would correspond to having a birational contraction of $M$.

**Birational contraction**

Suppose that

$$\pi : M \rightarrow Z$$

is a projective birational morphism, which we call a birational contraction. Let $C$ be the exceptional locus in $M$ and $B = \pi(C)$. Using the Grauert-Riemenschneider vanishing result, one can show that $Z$ has at worst rational singularities and $\pi$ is a crepant resolution with fibers uniruled varieties. In particular every fiber $F$ of $\pi_C : C \rightarrow B$ has no nontrivial holomorphic form and therefore $F$ is isotropic inside $M$. Similarly $(\iota_v \Omega)|_F = 0$ for any $v \in TB$ and this implies that $T_{vert}C \subseteq (TC)^\perp$.

Wierzba, Namikawa and Hu-Yau ([Wi], [Na], [HY]) show that generically, $T_{vert}C = (TC)^\perp$. This implies that $C$ is a coisotropic subvariety and its generic characteristic foliation is given by fibers of $\pi_C$, in particular $B$ has a natural symplectic structure outside its singular locus. The proof goes roughly as follow: It suffices to show that $\dim F \geq \dim M - \dim C$; we consider deformations of rational curves in $C$ which cannot bend and break. On the one hand any such curve cannot move away from the contractible locus $C$ and therefore the dimension of its deformation space cannot be too big compare to $\dim F$ by a result of Cho and Miyaoka. On the other hand, by a result of Ran, every rational curve in $M$ has at least $2n - 2$ deformation directions even though the expected dimension is only $2n - 3$. The fact that generic complex structures in the twistor family of $M$ has no curve is responsible for the extra dimension. Combining this we have $\dim F \geq \dim M - \dim C$ and therefore $T_{vert}C = (TC)^\perp$.
In fact each smooth irreducible component of $F$ is a projective space $\mathbb{P}^k$. The proof of this combines above arguments with Cho-Miyaoka’s characterization of $\mathbb{P}^k$ as the only smooth variety with the property that every rational curve moves in at least $2k - 2$ dimensional family.

Note that $\pi_C : C \to B$ is not necessarily a honest bundle, the dimension of fibers can jump and $B$ could be singular too. There are some structure theorems describing such behaviors and we expect them to admit good stratifications as defined by Markman [Mar].

**Mukai elementary modification**

Using previous arguments, one can show that every $\mathbb{P}^k$-bundle

$$\pi_C : C \to B$$

inside $M$ has dimension less than or equal to $2n - k$. When $\dim C = 2n - k$, $C$ is coisotropic and its characteristic foliation is precisely given by fibers of $\pi_C$. When this happens Mukai [Mu1] constructs another symplectic manifold $M'$ by replacing each fiber $\mathbb{P}^k$ with its dual projective space $(\mathbb{P}^k)^*$. This is called the **Mukai elementary modification**.

When $C = \mathbb{P}^n$ it has a neighborhood which is symplectomorphic to a neighborhood of the zero section in $T^*\mathbb{P}^n$. In this case this modification is simply the birational map $\Phi$ in the introduction. We will study a Legendre transformation along any $\mathbb{P}^n$ in $M$ in the section [1]. The general case can be regarded as a family version of this.

### 3 Lagrangian submanifolds

Lagrangian submanifolds are the smallest coisotropic submanifolds or the biggest isotropic submanifolds. They are the most important objects in symplectic geometry, ‘Everything is a Lagrangian manifold’ as Weinstein described their roles in real symplectic geometry.

In this section we will first establish some basic topological and geometrical properties of Lagrangian submanifolds in a hyperkähler manifold. We will explain the relationship between the second fundamental form and the special Kähler structure on the moduli space of Lagrangian submanifolds. Then we will study a Lagrangian category and intersection theory of Lagrangian subvarieties, a reduction functor and a projection functor. These structures will be needed to study the Legendre transformation and the Plucker type formula in the next section.

#### 3.1 Properties of Lagrangian submanifolds

When $C$ is a Lagrangian submanifold of $M$, then $N_{C/M} \cong T_C^*$ and we have the following exact sequence

$$0 \to T_C \to T_M|_C \to T_C^* \to 0.$$  

$^6$Normality is already sufficient by the work of Shepherd-Barron [14].
By the Whitney sum formula we have

$$\iota^* c(T_M) = c(T_C \oplus T_C^*) = c(T_C \otimes \mathbb{C}),$$

where $\iota : C \to M$ is the inclusion morphism.

This implies that Pontrjagin classes of $C$ are determined by Chern classes of $M$ as follows,

$$p_k (C) = (-1)^k \iota^* c_{2k} (M).$$

In particular, Pontrjagin numbers of $C$ depends only on the cohomology class $[C] \in H^{2n} (M, \mathbb{Z})$. By the celebrated theorem of Thom, this determines the rational cobordism type of $C$ and we have proven the following theorem.

**Theorem 10** Given a fixed cohomology class $c \in H^{2n} (M, \mathbb{Z})$, it determines the rational cobordism class of any possible Lagrangian submanifold in $M$ representing $c$.

In particular, the signature of the intersection product on $H^* (C, \mathbb{R})$ is also determined by $c$. Using the Hirzebruch signature formula we have

$$\text{Signature} (C) = \int_M c \cup \sqrt{L_M}.$$

Using $N_{C/M} \cong T_C^*$ we also have the following formula for the Euler characteristic of $C$,

$$\chi (C) = (-1)^n \int_M c \cup c.$$ 

Similarly if $C$ is spin, the $\hat{A}$-genus is given by

$$\hat{A} [C] = \int_M c \cup \sqrt{Td_M}.$$

The reason is when restricting to $C$, using the multiplicative property of the $\hat{A}$-genus, we have

$$Td_M = \hat{A} (T_M) = \hat{A} (T_C + T_C^*) = \hat{A} (T_C) \hat{A} (T_C^*) = \hat{A} (T_C)^2.$$

As an immediate corollary of this and the standard Bochner arguments, we can show that if $C$ is an even dimensional spin Lagrangian submanifold of $M$, then $T_C$ cannot have positive scalar curvature unless $\int_C \sqrt{Td_M} = 0$.

Recall from section 2.1 that we have following characterizations of a $n$ dimensional submanifold $C$ in $M$ being a Lagrangian. First $C$ is Lagrangian if $H^{2,0} (C) = 0$. Second if $C$ is a complete intersection then

$$\int_C c_2 (M) \omega^{n-2} \geq 0.$$
Moreover, the equality sign holds if and only if $C$ is a Lagrangian. When this happens, the tangent bundle of $C$ splits into direct sum of line bundles, this is a strong constraint upon $C$. For example when $n = 2$ the signature of $C$ would have to be zero. Third, $C$ is Lagrangian if and only if its Poincaré dual is a $\Omega$-primitive class, i.e. $\Lambda_\Omega [C] = 0$, or equivalently, $L_\Omega [C] = 0$. This implies that Lagrangian property is invariant under deformations of $C$ and it also allows us to define singular, and possibly non-reduced, Lagrangian subvarieties of $M$.

Remark: When $C$ is a Lagrangian submanifold, then $[C] \in H^{n,n} (M, \mathbb{Z}) \cap \text{Ker} \ (L_\Omega)$. In fact being a class of type $(n, n)$ follows from the $\Omega$-primitivity. More precisely, we have

$$Ker \ (L_\Omega) \cap \Omega^{2n} (M, \mathbb{R}) \subset \Omega^{n,n} (M),$$

$$Ker \ (L_\Omega) \cap H^{2n} (M, \mathbb{R}) \subset H^{n,n} (M).$$

The proof of these inclusions simply uses the hard Lefschetz decomposition of $\Omega^{\ast, \ast}$ for the $\mathfrak{sl}(2)$-action generated by $L_\Omega$ and $\Lambda_\Omega$. A similar result of Hitchin [Hi2] says that if $C$ is a real submanifold of $M$ which is a real Lagrangian submanifold with respect to both $\text{Re} \, \Omega$ and $\text{Im} \, \Omega$, then $C$ is a complex submanifold and therefore a Lagrangian submanifold of $M$ with respect to $\Omega$.

Some examples

(1) When $M$ is a K3 surface or an Abelian surface, any curve $C$ in $M$ is Lagrangian for dimensional reason. The Hilbert scheme of $n$ points in $M$, denote $S^{[n]} M$, is again a hyperkähler manifold as shown by Fujiki and Beauville. Moreover $S^{[n]} C \subset S^{[n]} M$ is a Lagrangian submanifold. Notice that $S^{[n]} C$ is simply the symmetric product of the curve $C$, i.e. $S^n C$.

(2) For any Kähler manifold $X$, its cotangent bundle $T^* X$ carries a natural symplectic structure. Moreover the conormal bundle of any submanifold of $X$ is a Lagrangian submanifold of $T^* X$. In fact they can be characterized as those closed submanifolds of $T^* X$ which are invariant under the natural $\mathbb{C}^\times$-action of scaling along fibers on $T^* X$. They are called conical submanifolds.

3.2 Second fundamental form and moduli space

In this subsection we study the differential geometry of a Lagrangian submanifold $C$ in $M$. The second fundamental form of a submanifold is a symmetric two tensor on $C$ with valued in the normal bundle and it is defined as follow, $X \otimes Y \rightarrow (\nabla_X Y)^N$ for any tangent vectors $X, Y$ on $C$. Here $\nabla$ is the Levi-Civita connection on $M$ and $(\bullet)^N$ is the orthogonal projection to normal directions.

The Lagrangian condition gives us a symmetric three tensor and it defines a cubic form on $H^0 (C, T_C^\ast)$. By varying $C$ this cubic form will determine a special Kähler structure on the moduli space of Lagrangian submanifolds. We will also give a brief physical explanations of such structure using supersymmetry.
3.2.1 Second fundamental form and a cubic form

If \( C \) is a Lagrangian submanifold of \( M \), then \( T_M|_C \) is a symplectic vector bundle with a Lagrangian subbundle \( T_C \).

**Extension class of a Lagrangian subbundle**

Suppose \( E \) is a symplectic vector bundle over a complex manifold \( C \). Given any Lagrangian subbundle \( S \), by contracting the symplectic form \( \Omega \) with elements in \( S \), we have a naturally identification between quotient bundle \( E/S \) and the dual bundle \( S^* \) of \( S \). Therefore we have the following short exact sequence,

\[
0 \to S \to E \to S^* \to 0.
\]

We consider the image of \( 1_{S^*} \) under the homomorphism \( \text{Hom}_{O_C} (S^*, S^*) \to \text{Ext}^{1}_{O_C} (S^*, S) \), coming from the corresponding long exact sequence, and call it \( \alpha \). This is the extension class \( \alpha \in \text{Ext}^{1}_{O_C} (S^*, S) \cong H^1 (C, S \otimes S) \), associated to the above short exact sequence. The next lemma says \( \alpha \) lies inside the symmetric tensor product of \( S \).

**Lemma 11** If \( S \) is a Lagrangian subbundle of a symplectic vector bundle \( E \) over \( C \), then the corresponding extension class \( \alpha \) lies inside \( H^1 (C, \text{Sym}^2 S) \).

**Proof:** For any non-holomorphic splitting \( E = S \oplus S^* \) of the above exact sequence, we write the \( \partial \)-operator of \( E \) in terms of this decomposition,

\[
\bar{\partial}_E = \begin{pmatrix} \partial_S & A \\ 0 & \partial_{S^*} \end{pmatrix}
\]

then the off-diagonal term \( A \in \Omega^{0,1} (C, S \otimes S) \) represents the extension class \( \alpha \). On \( S \oplus S^* \), there is a canonical symplectic form constructed by the natural pairing between \( S \) and \( S^* \). Using linear algebra we can choose a splitting suitably such that \( \Omega \) on \( E \) becomes the canonical symplectic form on \( S \oplus S^* \). It is easy to verify that \( \partial_E \Omega = 0 \) is then equivalent to \( A \in \Omega^{0,1} (C, \text{Sym}^2 S) \). Therefore we have \( \alpha \in H^1 (C, \text{Sym}^2 S) \). □

Given any Hermitian metric on \( E \) there is unique Hermitian connection \( D_E \) on \( E \) satisfying \( (D_E)^{0,1} = \partial_E \). Using the metric we can also decompose \( E \) into \( S \oplus S^* \) as an orthogonal decomposition. The above differential form \( A \in \Omega^{0,1} (C, S \otimes S) \) is simply the second fundamental form of the subbundle \( S \subset E \) in the context of differential geometry. If the symplectic form \( \Omega \) on \( E \) becomes the canonical symplectic form on the orthogonal decomposition \( S \oplus S^* \) then the above proof gives \( A \in \Omega^{0,1} (C, \text{Sym}^2 S) \). To achieve this we need the Hermitian metric \( h \) on \( E \) to be compatible with \( \Omega \) in the sense that if we define operators \( I \) and \( K \) by the following,

\[
\Omega (v, w) = h (Iv, w) + ih (Kv, w),
\]

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then $I, J, K$ defines a fiberwise quaternionic structure on $E$, i.e. $I^2 = J^2 = K^2 = IJK = -1$.

**Proposition 12** Suppose that $E$ is a symplectic bundle over $C$ with compatible quaternionic structure. Then the second fundamental form of any Lagrangian subbundle $S \subset E$ lies inside $\Omega^{0,1}(C, \text{Sym}^2 S)$.

Proof: Let us denote the orthogonal complement of $S$ in $E$ as $T$. One can verify using the quaternionic structure on $E$ that $T = I(S)$. This implies that $T$ is a Lagrangian subbundle of $E$. Therefore the symplectic form $\Omega$ on $E$ can be identified as the canonical symplectic form on $S \oplus S^*$ via the natural identification $T \cong E/S \cong S^*$. By the proof of the previous lemma, we have our proposition. ■

**A cubic form**

When $C$ is a Lagrangian submanifold in $M$, then its tangent bundle $S = T_C$ is a Lagrangian subbundle of the symplectic bundle $E = T_M|_C$. The hyperkähler metric on $M$ does give us a compatible quaternionic structure on $T_M|_C$. Therefore the second fundamental form $A$ satisfies,

$$A \in \Omega^{0,1}(C, \text{Sym}^2 T_C).$$

On the other hand the induced metric on $C$ determines a natural identification between $(T_C^*)^{0,1}$ and $(T_C)^{1,0} = T_C$. Hence we can identify $A$ as an element in $\Gamma(C, T_C \otimes \text{Sym}^2 T_C)$ and we denote it as $\tilde{A}$. Explicitly if $z^i$’s is a local coordinate system on $C$ and the induced metric has components $g_{ij}$’s, then

$$\tilde{A} = \sum \tilde{A}^{ijk} \frac{\partial}{\partial z^i} \otimes \frac{\partial}{\partial z^j} \otimes \frac{\partial}{\partial z^k},$$

$$\tilde{A}^{ijk} = \sum g^{il} (A^{jk})_l.$$

From basic differential geometry we know that the second fundamental form of any submanifold is a symmetric tensor with valued in the normal bundle, this reflects the fact the Levi-Civita connection on $M$ is torsion free. As a consequence $\tilde{A}^{ijk}$ is completely symmetric in $i, j$ and $k$, i.e. $\tilde{A}$ is a trilinear symmetric multi-vector field on $C$.

$$\tilde{A} \in \Gamma(C, \text{Sym}^3 T_C).$$

We use $\tilde{A}$ to define a natural cubic form on $H^0(C, T_C^*)$ as follows.

**Definition 13** If $M$ is a hyperkähler manifold and $C$ is a compact Lagrangian submanifold of $M$ then we define a cubic form on $H^0(C, T_C^*)$,

$$c_C : \text{Sym}^3 H^0(C, T_C^*) \to \mathbb{C}$$

$$c_C(\phi, \eta, \zeta) = \int_C \phi \eta_j \zeta_k A^{ijk} \omega^n / n!.$$
Remark: The above cubic form \( c_C \) coincides with the composite of natural homomorphisms

\[
H^0 \left( C, T^*_C \right) \otimes^3 \xrightarrow{\alpha} H^1 \left( C, O_C \right) \otimes H^0 \left( C, T^*_C \right) \otimes H^1 \left( C, T^*_C \right) = H^{1,1} \left( C \right) \xrightarrow{\Delta} \mathbb{C}.
\]

In particular it depends only on the extension class \( \alpha \).

### 3.2.2 Moduli space of Lagrangian submanifolds

If \( C \) is Lagrangian submanifold in \( M \) then any deformation of \( C \) inside \( M \) remains a Lagrangian. Therefore the moduli space of Lagrangian submanifolds is the same as the moduli space of submanifolds representing the same cohomology class in \( M \). We denote it by \( \mathcal{M} \). This moduli space is always smooth, namely any infinitesimal deformation of \( C \) inside \( M \) is always unobstructed. This can be proven by applying a twistor rotation on McLean’s result \([Mc]\) on unobstructedness of deformations of special Lagrangian submanifolds in Calabi-Yau manifolds. This can also be proved by algebraic geometric means using \( T^1 \)-lifting method developed by Ran.

We recall that infinitesimal deformations of \( C \) in \( M \) are parametrized by \( H^0 \left( C, T^*_C \right) \) because the Lagrangian condition gives a natural identification between \( N_{C/M} \) and \( T^*_C \). It is not difficult to see that contracting with the extension class \( \alpha \in H^1 \left( C, \text{Sym}^2 T_C \right) \) gives a homomorphism

\[
\iota_\alpha : H^0 \left( C, T^*_C \right) \to H^1 \left( C, T_C \right)
\]

which associates to an infinitesimal deformation of \( C \) in \( M \) to the corresponding infinitesimal deformation of the complex structure on \( C \). Next we will show that this cubic form \( \epsilon \), constructed from the second fundamental form, determines the special Kähler structure on \( \mathcal{M} \).

### Moduli space as a special Kähler manifold

We first give a physical reason for the existence of a natural special Kähler structure on \( \mathcal{M} \): \( \sigma \)-model studies maps from Riemann surfaces to a fix target manifold. When the target manifold \( M \) is hyperkähler, it is well-known that its \( \sigma \)-model has \( N = 4 \) supersymmetry (or SUSY). If domain Riemann surfaces have boundary components then we would require their images lie inside a fix Lagrangian submanifold \( C \subset M \). In this case only half of the SUSY can be preserved and we have a \( N = 2 \) SUSY theory. Moduli space of such theories are generally known (physically) to process special Kähler geometry. In our situation, this is simply the moduli space of Lagrangian submanifolds in \( M \).

Let us recall the definition of a special Kähler manifold (see \([Br]\) for details). A special Kähler structure on a Kähler manifold \( \mathcal{M} \) is carried by a holomorphic cubic tensor

\[
\Xi \in H^0 \left( \mathcal{M}, \text{Sym}^3 T^*_{\mathcal{M}} \right).
\]
Using the Kähler metric $g_M$ on $\mathcal{M}$ we can identify $\Xi$ with a tensor $A \in \Omega^{1,0}(\mathcal{M}, \text{End} T\mathcal{C}_M)$ as follows

$$\Xi = -\omega_M(\pi_{1,0}^{\mathcal{M}}, [A, \pi_{1,0}^{\mathcal{M}}])$$

where $\pi_{1,0}^{\mathcal{M}} \in \Omega^{1,0}(T\mathcal{C}_M)$ is constructed from the inclusion homomorphism $T_{1,0}^{\mathcal{C}} \subset T\mathcal{C}$. In terms of local coordinates we have $(A^l)_{jk} = i\Xi_{ljm}g^m_{\bar{k}}$. If we denote the Levi-Civita connection on $\mathcal{M}$ as $\nabla^{\text{LC}}$, then the special Kähler condition is $\nabla = \nabla^{\text{LC}} + A + \bar{A}$ defines a torsion free flat symplectic connection on the tangent bundle and it satisfies $\nabla \wedge J = 0$.

When $\mathcal{M}$ is the moduli space of Lagrangian submanifolds in $M$, Hitchin [Hi] shows that it has a natural special Kähler structure. We are going to show that this special Kähler structure on $\mathcal{M}$ is given by the previous cubic form.

At any given point $[C] \in \mathcal{M}$ with $C$ a Lagrangian submanifold in $M$, we have a natural identification of the tangent space $T_{\mathcal{M},[C]} = H^0(C, T^*_C)$. From previous discussions we have a cubic form on $H^0(C, T^*_C)$.

$$c_C : \text{Sym}^3 H^0(C, T^*_C) \to \mathbb{C}.$$ 

By varying the point $[C]$ in $\mathcal{M}$ this determines a cubic tensor,

$$c \in \Gamma(\mathcal{M}, \text{Sym}^3 T^*_\mathcal{M}).$$

**Theorem 14** Suppose $\mathcal{M}$ is the moduli space of Lagrangian submanifolds in a compact hyperkähler manifold $M$. Then $c \in H^0(\mathcal{M}, \text{Sym}^3 T^*_\mathcal{M})$ and it determines a special Kähler structure on $\mathcal{M}$.

Proof: We are going to prove this theorem by identifying the cubic form in the special Kähler structure on $\mathcal{M}$ with the above cubic form $c$.

The cubic form that determines the special Kähler structure on $\mathcal{M}$ was described by Donagi and Markman in [DM]. First $\mathcal{M}$ can be viewed it as the base space of the moduli space of universal compactified Jacobian $J$ for Lagrangian submanifolds in $M$. The space $J$ has a natural holomorphic symplectic structure and the natural morphism $J \to \mathcal{M}$ is a Lagrangian fibration. We consider the variation of Hodge structures, i.e. periods, for this family of Abelian varieties over $\mathcal{M}$. Its differential at a point $[C] \in \mathcal{M}$ gives a homomorphism,

$$T_{\mathcal{M},[C]} \to S^2 V,$$

where

$$V = H^1(Jac(C), O_{Jac(C)}) = H^1(C, O_C),$$ 

$$T_{\mathcal{M},[C]} = H^0(C, T^*_C).$$

Using the induced Kähler structure on $C$, we can identify $H^1(C, O_C)^*$ with $H^0(C, T^*_C)$. We thus obtain a homomorphism

$$\otimes^3 T_{\mathcal{M},[C]} \to \mathbb{C}$$

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This tensor is a symmetric tensor. The corresponding cubic form $\Gamma(M, \text{Sym}^3 T^*_M)$ is the one that determines the special Kähler structure on $M$ (see [Fr]).

The above homomorphism $T_{M,[C]} \rightarrow S^2 V$ can be identified as the composition of the natural homomorphism $H^0(C, N_{C/M}) \rightarrow H^1(C, T_C)$ and the natural homomorphism between variation of complex structures on $C$ and on its Jacobian variety,

$$H^1(C, T_C) \rightarrow H^1(Jac(C), T_{Jac(C)}) .$$

From standard Hodge theory, an Abelian variety is determined by its period, or equivalently its weight one Hodge structure, and its variation is simply given by the usual cup product homomorphism,

$$H^1(C, T_C) \otimes H^{1,0}(C) \xrightarrow{\cup} H^{0,1}(C) ,$$

or equivalently,

$$H^1(C, T_C) \otimes H^0(C, T^*_C) \xrightarrow{\cup} H^1(C, O_C) ,$$

$$(\phi^i_j \partial/\partial z^i) \otimes (\beta_k dz^k) \rightarrow \phi^i_j \beta_i d\bar{z}^j .$$

Recall under the identification $H^0(C, N_{C/M}) \cong H^0(C, T^*_C)$ we have

$$H^0(C, T^*_C) \rightarrow H^1(C, T_C) ,$$

$$\alpha_i dz^i \rightarrow \alpha_i A^i_{jk} \partial/\partial z^k \otimes d\bar{z}^j ,$$

where $A = \Sigma A^{ik}_{jk} \partial/\partial z^i \otimes \partial/\partial z^k \otimes d\bar{z}^j$ is the second fundamental form of $C$ in $M$. Therefore the corresponding cubic form $\otimes^3 H^0(C, T^*_C) \rightarrow \mathbb{C}$ is given by

$$(\alpha, \beta, \gamma) \rightarrow \int_C \alpha_i A^i_{jk} \beta^k \gamma^l g^{ij} \omega^n / n! .$$

This is precisely the cubic form $c_C$. Hence the result. ■

Remark: On a special Kähler manifold $M$ its Riemannian curvature tensor is given by

$$R_{ijkl} = -g^p_{M,ij} \Xi_{kp} \Xi_{lq} .$$

This implies that the Ricci curvature of $M$ is non-negative and the scalar curvature equals $4 |\Xi|^2 \geq 0$. Combining this with our earlier discussions on the second fundamental form, we show that the special Kähler metric on the moduli space of Lagrangian submanifolds is flat at $[C] \in M$ if the natural exact sequence $0 \rightarrow T_C \rightarrow T_{M|C} \rightarrow N_{C/M} \rightarrow 0$ splits.
3.3 Lagrangian category

On a real symplectic manifold \( (M, \omega) \), Fukaya proposed a category of Lagrangian submanifolds. The space of morphisms between two Lagrangian submanifolds \( L_1, L_2 \) is the Floer cohomology group \( HF(L_1, L_2) \). It is defined in terms of the number of holomorphic disks (i.e. instantons) bounding \( L_1 \) and \( L_2 \). The dimension of the space of such holomorphic disks can be computed using the index theorem, and expressed in terms of their Maslov indexes. Kontsevich conjectured that Fukaya category on a Calabi-Yau manifold is equivalent to the derived category of coherent sheaves of the mirror Calabi-Yau manifold, the so-called homological mirror symmetry conjecture. In this paper we study the holomorphic analog of the Fukaya category. The definition of this category is probably well-known to experts in this subject. We will establish some basic properties of it and they will play an important role in the discussions of the Plücker type formula in section 4.

3.3.1 Definition of the category

Non-existence of instanton corrections

Before we give the definition of the Lagrangian category we first demonstrate the absent of instanton corrections in the hyperkähler setting. Suppose \( M \) is a hyperkähler manifold with a preferred complex structure \( J \) among \( I, J, K \). When \( C \) is a Lagrangian submanifold in \( M \) with respect to the symplectic structure \( \Omega = \omega_I + i\omega_K \) then it is a real Lagrangian submanifold of the real symplectic manifold \( (M, \omega_\theta) \) with \( \omega_\theta = \cos \theta \omega_I + \sin \theta \omega_K \) for any \( \theta \in [0, 2\pi) \). For any fixed \( \theta \), instantons means \( J_\theta \)-holomorphic disks bounding \( C \), where \( J_\theta = \cos \theta I + \sin \theta K \). In Physics they contribute to the correlation functions, which is independent of \( \theta \) in any TQFT. In particular there is no instanton effects if there is no \( J_\theta \)-holomorphic disks for some \( \theta \).

In the hyperkähler case we have the following.

Lemma 15 If \( C \) is a Lagrangian submanifold of a compact hyperkähler manifold \( M \), then for all \( \theta \in [0, 2\pi) \), with at most one exception, there is no \( J_\theta \)-holomorphic disk in \( M \) bounding \( C \).

Proof: Suppose \( D \) is a \( J_\theta \)-holomorphic disk in \( M \) with \( \partial D \subset C \), say \( \theta = 0 \). For the complex structure \( J_0 = I \), \( \omega_K + i\omega_J \) is a holomorphic two form on \( M \) and therefore restricts to zero of on any \( I \)-holomorphic disk. On the other hand \( \omega_I \) is a Kähler form and hence it is positive on \( D \). Therefore we have

\[
\omega_J = \omega_K = 0, \omega_I > 0,
\]
on \( D \). We consider the integration of \( \omega_I \) and \( \omega_K \) on \( D \), since these two forms restrict to zero on \( C \) we have well-defined homomorphisms

\[
\int \omega_I : H_2(M, C; \mathbb{Z}) \to \mathbb{R},
\]

\[
\int \omega_K : H_2(M, C; \mathbb{Z}) \to \mathbb{R}.
\]
From earlier discussions $[D, \partial D]$ represents a class in $H_2(M, C; \mathbb{Z})$ which must lie in the kernel of $\int_\omega K$, and not in the kernel of $\int_\omega I$, in fact $\int_\omega I > 0$. Therefore, with a fix class in $H_2(M, C; \mathbb{Z})$, there is at most one $\theta$ which can support $J_\theta$-holomorphic disks representing the given class. ■

Remark on the vanishing of the Maslov index: Recall in the real symplectic geometry, the Maslov index plays a very important role, for instance in determining the dimension of the space of holomorphic disks. The origin of the Maslov index is the isomorphism $\pi_1(U(n)/O(n)) \cong \mathbb{Z}$, where the space $U(n)/O(n)$ parametrizes linear Lagrangian subspaces in $\mathbb{R}^{2n}$. For complex linear Lagrangian subspaces in $\mathbb{C}^{2n}$, their parameter space equals $Sp(n)/U(n) \subset U(2n)/O(2n)$, which has trivial fundamental group

$$\pi_1\left(\frac{Sp(n)}{U(n)}\right) = 0.$$ 

Therefore the Maslov index is always zero in our situation.

**Definition of Lagrangian category**

**Definition 16** Given any hyperkähler manifold $M$, we define a category $\mathcal{C}_M$ or simply $\mathcal{C}$, called the Lagrangian category of $M$ as follows: An object in $\mathcal{C}$ is a Lagrangian subvariety of $M$. Given two objects $C_1, C_2 \in \text{obj}$ we define the space of morphisms to be the $\mathbb{Z}$-graded Abelian group

$$\text{Hom}_{\mathcal{C}}(C_1, C_2)^{[k]} = \text{Ext}_{O^M}^k(O_{C_1}, O_{C_2}).$$

The composition of morphisms is given by the natural product structure on the $\text{Ext}$’s groups.

$$\text{Ext}_{O^M}^k(O_{C_1}, O_{C_2}) \otimes \text{Ext}_{O^M}^l(O_{C_2}, O_{C_3}) \to \text{Ext}_{O^M}^{k+l}(O_{C_1}, O_{C_3}).$$

Because of the Serre duality and $K^M = O_M$, the above category carries a natural duality property,

$$\text{Ext}_{O^M}^k(O_{C_1}, O_{C_2}) \cong \text{Ext}_{O^M}^{2n-k}(O_{C_2}, O_{C_1})^*,$$

provided that $M$ is compact.

**A symplectic 2-category**

In our situation an object is a Lagrangian $C$ in a fix hyperkähler manifold $M$. In real symplectic geometry Weinstein ([BW] and [We]) defines a symplectic category whose objects are symplectic manifolds and morphisms are immersed Lagrangian submanifolds inside $M_2 \times M_1$. Here $\tilde{M}$ denote the symplectic manifold $\tilde{M}$ with the symplectic form $-\omega$. When $M$ is a hyperkähler manifold, the complex structure of $\tilde{M}$ becomes $-J$. In fact we can combine the two approaches
together and define a symplectic 2-category: The objects are hyperkähler manifolds, 1-morphisms from \( M_1 \) to \( M_2 \) are Lagrangian subvarieties in \( M_2 \times \bar{M}_1 \), 2-morphisms between two Lagrangian subvarieties \( C_1, C_2 \subset M_2 \times \bar{M}_1 \) are given by \( \text{Ext}^*_{O_{M_2} \times \bar{O}_{M_1}}(O_{C_1}, O_{C_2}) \).

The category of Lagrangian coherent sheaves

The Lagrangian category of \( M \) is geometric in nature but it does not have very good functorial properties. Therefore we also need another category.

**Definition 17** Let \( M \) be a projective hyperkähler manifold. Let \( D^b(M) \) be the derived category of coherent sheaves on \( M \). We define the category of Lagrangian coherent sheaves \( D^b_{Lag}(M) \) to be the subcategory of \( D^b(M) \) generated by those coherent sheaves \( S \) satisfying \( \text{ch}(S) \cup \Omega \in \bigoplus_{k>2n+2} H^k(M, \mathbb{C}) \).

By the Hard Lefschetz \( sl_2 \)-action using \( L_\Omega \) and \( \Lambda_\Omega \), the condition \( \text{ch}(S) \cup \Omega \in \bigoplus_{k>2n+2} H^k(M, \mathbb{C}) \) implies that \( \text{ch}_k(S) = 0 \) for \( k < n \) and \( \text{ch}_n(S) \) is a \( \Omega \)-primitive cohomology class. In particular the \( n \) dimensional support of \( S \) is a Lagrangian in \( M \). For example if \( C \) is a Lagrangian subvariety in \( M \) then \( O_C \) is an object in \( D^b_{Lag}(M) \).

### 3.3.2 Lagrangians intersection

The Lagrangian category \( C_M \) (and similar for \( D^b_{Lag}(M) \)) is closely related to the intersection theory for Lagrangian subvarieties in \( M \).

**Theorem 18** If \( C_1 \) and \( C_2 \) are two Lagrangian subvarieties of a compact hyperkähler manifold \( M \) then

\[
\sum_k \dim (-1)^k \text{Ext}^k_{O_M}(O_{C_1}, O_{C_2}) = (-1)^n C_1 \cdot C_2.
\]

Proof: We recall the Riemann-Roch formula for the global Ext groups: For any coherent sheaves \( S_1 \) and \( S_2 \) on \( M \) we have,

\[
\dim (-1)^k \text{Ext}^k_{O_M}(S_1, S_2) = \int_M \overline{\text{ch}}(S_1) \text{ch}(S_2) Td_M
\]

where \( \overline{\text{ch}}(S_1) = \sum (-1)^k \text{ch}_k(S_1) \). For \( S_i = O_{C_i} \) the structure sheaf of a subvariety \( C_i \) of dimension \( n \), we have

\[
\text{ch}_k(O_{C_i}) = 0 \text{ for } k < n,
\]

\[
\text{ch}_n(O_{C_i}) = [C_i].
\]
Therefore
\[
\dim (-1)^k \text{Ext}^k_{O_M} (O_{C_1}, O_{C_2}) = \int_M \overline{\text{ch}} (O_{C_1}) \text{ch} (O_{C_2}) T dM = \int_M ((-1)^n [C_1] + \text{h.o.t.}) ([C_2] + \text{h.o.t.}) (1 + \text{h.o.t.}) = (-1)^n \int_M [C_1] \cup [C_2] = (-1)^n C_1 \cdot C_2.
\]

Here \text{h.o.t.} refers to higher order terms which do not contribute to the outcome of the integral. Hence the result. ■

If \( C_1 \) and \( C_2 \) intersect cleanly along \( D \) then \( C_1 \cdot C_2 \) equals the Euler characteristic of \( D \) up to sign. To prove this we use the following useful lemma whose proof is standard.

**Lemma 19** If \( C_1 \) and \( C_2 \) are two Lagrangian submanifolds of a hyperkähler manifold \( M \) which intersect cleanly along \( D = C_1 \cap C_2 \), then the symplectic form on \( M \) induces a non-degenerate pairing
\[
N_{D/C_1} \otimes N_{D/C_2} \to O_D.
\]

Proof: For any \( v \in T_{C_1} \) and \( w \in T_D \subset T_{C_2} \) we have \( \Omega (v, w) = 0 \) because \( D \subset C_1 \). This implies that the above pairing is well-defined. Suppose that \( v \in T_{C_1} \) is such that \( \Omega (v, u) = 0 \) for every \( u \in T_{C_2} \). Then \( v \in (T_{C_2})^\perp = T_D \) because \( C_2 \) is Lagrangian. That is \( v \in T_{C_1} \cap T_{C_2} = T_D \). This shows the nondegeneracy of the pairing. Hence the lemma. ■

In particular we have a natural isomorphism
\[
N_{D/C_1} \cong N_{D/C_2}^*.
\]

We also have natural exact sequences,
\[
0 \to N_{D/C_2} \to (N_{C_1/M})|_{D} \to T_D^* \to 0,
0 \to N_{D/C_1} \to (N_{C_2/M})|_{D} \to T_D^* \to 0,
0 \to N_{D/M} \to (N_{C_1/M} \oplus N_{C_2/M})|_{D} \to T_D^* \to 0,
\]
comparing relative normal bundles. One should recall that \( N_{C_i/M} \cong T_{C_i}^* \) for \( i = 1, 2 \).

On the other hand the standard intersection theory gives us that
\[
C_1 \cdot C_2 = e \left( (N_{C_1/M} \oplus N_{C_2/M})|_{D} - N_{D/M} \right)
= e (T_D^*)
= (-1)^{\dim D} e (C_1 \cap C_2).
\]
Corollary 20 If $C_1$ and $C_2$ are two Lagrangian submanifolds in a compact hyperkähler manifold $M$ which intersect cleanly, then

$$C_1 \cdot C_2 = (-1)^{\dim C_1 \cap C_2} e(C_1 \cap C_2).$$

Remark: When $M = T^*X$ and $C_i$ is the conormal bundle $N_{S_i/X}$ of a submanifold $S_i \subset X$. The intersection $C_1 \cap C_2$ is compact if and only if it lies inside the zero section $X \subset M$. This happens precisely when $T_2S_1 + T_2S_2 = T_xX$ for all $x \in S_1 \cap S_2$, i.e. $S_1$ intersects $S_2$ transversely. In this situation $C_1 \cdot C_2$ makes sense and equals to $(-1)^{\dim S_1 \cap S_2} e(S_1 \cap S_2)$.

When $C_1$ and $C_2$ intersect cleanly we expect that individual $Ext^k_{O_M}(O_{C_1}, O_{C_2})$ can be computed, not just their Euler characteristics which is given by $(-1)^n C_1 \cdot C_2$. For example we have following results.

Theorem 21 If $C$ is a Lagrangian submanifold of a compact hyperkähler manifold $M$, then there is an isomorphism of vector spaces,

$$Ext^k_{O_M}(O_C, O_C) \cong H^k(C, \mathbb{C}),$$

for all $k$ provided that the normal bundle of $C$ can be extended to the whole $M$. For instance it holds true when $C$ is a complete intersection in $M$.

Proof: We consider a Koszul resolution of $O_C$ in $M$,

$$0 \to \Lambda^n E \to \Lambda^{n-1} E \to \cdots \to E \to O_M \to O_C \to 0,$$

where $E$ is a vector bundle on $M$ whose restriction to $C$ is the conormal bundle $N_{C/M}$ which is isomorphic to the tangent bundle of $C$ by the Lagrangian condition. The groups $Ext^k_{O_M}(O_C, O_C)$ can be computed as the hypercohomology of the complex of sheaves $\mathcal{H}om_{O_M}(\Lambda^* E, O_C)$ (\cite{GH}). Note that

$$\mathcal{H}om_{O_M}(\Lambda^q E, O_C) \cong \mathcal{H}om_{O_M}(O_M, \Lambda^q E^* \otimes O_C)$$

$$\cong \mathcal{H}om_{O_C}(O_C, \Lambda^q T^*_C)$$

$$\cong \Omega^q(C).$$

From the definition of the Koszul complex, the restriction of its dual complex $\Lambda^* E^*$ to $C$ has trivial differentials. Therefore

$$Ext^k_{O_M}(O_C, O_C) \cong H\left(0 \to O_C \xrightarrow{0} \Omega^1(C) \xrightarrow{0} \cdots \xrightarrow{0} \Omega^n(C) \to 0\right)$$

$$\cong \bigoplus_{p+q=k} H^{p,q}(C)$$

$$\cong H^k(C, \mathbb{C}).$$

The last equality uses the Kählerian property of $C$. Hence the result. $\blacksquare$
Theorem 22 If $C_1$ and $C_2$ are two Lagrangian submanifolds of a compact hyperkähler manifold $M$ which intersect transversely along $C_1 \cap C_2 = \{p_1, \ldots, p_s\}$, then there is an isomorphism of vector spaces,

$$Ext^k_{O_M}(O_{C_1}, O_{C_2}) \cong \begin{cases} 0 & \text{if } k \neq n \\ \bigoplus_{p_i} \mathbb{C} \cong \mathbb{C}^s & \text{if } k = n. \end{cases}$$

Proof: Without loss of generality we can assume that $C_1 \cap C_2 = \{p\}$. We take a Koszul resolution of $O_{C_1}$ as before,

$$0 \to \Lambda^n E \to \Lambda^{n-1} E \to \cdots \to E \to O_M \to O_{C_1} \to 0.$$ 

Because $C_1$ and $C_2$ intersect transversely, the restriction of this resolution to $C_2$ gives a Koszul resolution of $O_p$ in $C_2$,

$$0 \to \Lambda^n F \to \Lambda^{n-1} F \to \cdots \to F \to O_{C_2} \to O_p \to 0,$$

where $F$ is the restriction of $E$ to $C_2$. Now

$$Hom_{O_M}(\Lambda^q E, O_{C_2}) \cong Hom_{O_M}(O_M, \Lambda^q E^* \otimes O_{C_2}) \cong Hom_{O_{C_2}}(O_{C_2}, \Lambda^q F^*) \cong Hom_{O_{C_2}}(\Lambda^q F, O_{C_2}).$$

Therefore

$$Ext^k_{O_M}(O_{C_1}, O_{C_2}) \cong H^k(Hom_{O_M}(\Lambda^q E, O_{C_2})) \cong H^k(Hom_{O_{C_2}}(\Lambda^q F, O_{C_2})) \cong Ext^k_{O_{C_2}}(O_p, O_{C_2}).$$

The theorem follows from the above equation and basic properties of the extension groups $\mathbb{H}$. ■

Intersection Euler characteristics

From previous discussions the Euler characteristic of any smooth subvariety $S$ in $\mathbb{P}^n$ equals

$$\chi (S) = (-1)^{\dim S} N^*_{S/\mathbb{P}^n} \cdot \mathbb{P}^n.$$

Even though the intersection of the two Lagrangians $N^*_{S/\mathbb{P}^n}$ and $\mathbb{P}^n$ is taken place in a non-compact manifold, namely $T^*\mathbb{P}^n$, their intersection number is well-defined in this case because their intersection occurs inside a compact region and moreover they do not intersect even at infinity. For singular variety in $\mathbb{P}^n$ we use this as a definition.

Definition 23 For any subvariety $S$ in $\mathbb{P}^n$ we define its intersection Euler characteristic to be the following intersection number inside $T^*\mathbb{P}^n$,

$$\bar{\chi} (S) = (-1)^{\dim S} N^*_{S/\mathbb{P}^n} \cdot \mathbb{P}^n.$$
It is not difficult to see that if \( S \) is a smooth simple normal crossing subvariety of \( \mathbb{P}^n \) then its intersection Euler characteristic \( \bar{\chi}(S) \) equals to the usual Euler characteristic of its normalization. For plane curves, such \( S \) would have only double point singularities. By local computations, we can prove the following formula for intersection Euler characteristics for plane curves which might even have cusp singularities: For any plane curve \( S \) of degree \( d \) with \( \delta \) double points, \( \kappa \) cusps and no other singularities, we have

\[
\bar{\chi}(S) = d^2 - 3d + 2\delta + 3\kappa.
\]

### 3.3.3 Reduction functor and projection functor

Reduction of a symplectic manifold \( M \) is induced from a coisotropic submanifold \( D \) in \( M \). Given any Lagrangian subvariety in \( M \) we can construct another one which lives inside \( D \), called the *projection* and also a Lagrangian subvariety in the reduced symplectic space, called the *reduction*. We will review these constructions and we will used them later in the normalized Legendre transformation.

Given any coisotropic submanifold \( D \) in \( M \) it has an integrable distribution \( (T_D)^\perp \subset T_D \). We denote the natural projection to the leave space \( B \) as

\[
\pi_D : D \to B.
\]

If we assume \( B \) is smooth then it has a natural holomorphic symplectic structure and \( B \) is called the *reduction* of \( D \), for simplicity we assume \( B \) is also hyperkähler. The existence of such symplectic structure on \( B \) and also our later construction of two transformations it induced are based on the following linear algebra lemma,

**Lemma 24** Suppose \( M \) is a symplectic vector space with symplectic form \( \Omega \), \( D \) is a coisotropic subspace and \( C \) is a Lagrangian subspace. Then

1. \( \Omega|_D \) induced a symplectic structure on \( D/D^\perp \);
2. \( C \cap D + D^\perp \subset D \) is a Lagrangian subspace in \( M \);
3. \( (C \cap D) / (C \cap D^\perp) \) is a Lagrangian subspace in \( D/D^\perp \).

The proof of this lemma is standard and readers can find it in chapter 5 of [BW] for instance.

**Reduction and projection of a Lagrangian**

Suppose that \( C \) is a Lagrangian subvariety of \( M \) and we denote its smooth locus as \( C^{sm} \). We construct a subvariety \( C^{red} \) in \( B \) (resp. \( C^{proj} \) in \( M \)) called the *reduction of \( C \) (resp. *projection of \( C \)) as follow,

\[
C^{red} = \pi_D(C^{sm} \cap D) \subset B.
\]
and

\[
C^{proj} \subset D \subset M \\
\downarrow \quad \downarrow \\
C^{red} \subset B.
\]

Because of the above linear algebra lemma, both \(C^{red}\) in \(B\) and \(C^{proj}\) in \(M\) are Lagrangian subvarieties. This has not yet define functors on Lagrangian categories because it is not so easy to see how to construct the functor on the morphism level. For this purpose the derived category of Lagrangian coherent sheaves \(D^b_{Lag}(\bullet)\) serves a better role.

We define a functor between derived categories called the reduction functor as follow: We consider the subvariety \(B \times B \subset B \times M\) and denote the projection morphism from \(B \times M\) to its first and second factor as \(\pi_B\) and \(\pi_M\) respectively then we define

\[
R_D : D^b_{Lag}(M) \to D^b_{Lag}(B) \\
R_D(\bullet) = R\pi_B^* \left( O_{B \times B} \otimes \pi_M^* (\bullet) \right).
\]

We can also define a projection functor as follow: We consider a subvariety \(D \times B \subset M \times M\) and denote the projection morphisms from \(M \times M\) to its first and second factors as \(\pi_1\) and \(\pi_2\) then we define

\[
P_D : D^b_{Lag}(M) \to D^b_{Lag}(M) \\
P_D(\bullet) = R\pi_1^* \left( O_{D \times B} \otimes \pi_2^* (\bullet) \right).
\]

It can be checked that the image of any Lagrangian coherent sheaf under \(P_D\) or \(R_D\) is indeed a Lagrangian coherent sheaf.\(^7\)

Remark: We expect that the functors \(R_D\) and \(P_D\) induced from a coisotropic submanifold \(D\) to enjoy many good properties. For example \(R_D \circ R_D^* = 1\), \(P_D \circ P_D = P_D\) and \(P_D = P_D^*\) for some suitably defined adjoint functors \(R_D^*\) and \(P_D^*\). We also expect that \(R_D\) is an injective functor and \(P_D\) is a surjective functor. Moreover one should be able to relax the smoothness assumption on \(D\) and \(B\).

4 Legendre transformation

In this section we will study a Legendre transformation of Lagrangian subvarieties along a coisotropic exceptional subvariety in \(M\) and a Plucker type formula. We will start by recalling the Legendre transformation in the classical mechanics. It can be reinterpreted as a transformation of conormal bundles. Then we

\(^7\)To be precise we should talk about complex of sheaves in the derived category.
descend this transformation to the hyperkähler quotient $T^*\mathbb{P}^n$. It has a natural
generalization to any Mukai elementary modification.

We establish a Plucker type formula which relates intersection numbers of
Lagrangian subvarieties under the Legendre transformation. Then we define a
normalized Legendre transformation which enjoys much better functorial prop-
erties. The definition of this normalized transformation uses the reduction and
the projection of the Lagrangian subvariety with respect to the coisotropic ex-
ceptional locus.

4.1 Classical Legendre transformation

The origin of the Legendre transformation is from the classical mechanics. It
is a transformation from the Lagrangian mechanics on $T X$ to the Hamiltonian
mechanics on $T^* X$. Given a function $L = L (q, v) : T X \to \mathbb{R}$ usually called a
Lagrangian, we can form a closed two form on $T X$ as follows,

$$\omega_L = \sum \frac{\partial^2 L}{\partial q^i \partial v_j} dq^i \wedge dq^j + \frac{\partial^2 L}{\partial v_i \partial v_j} dv_i \wedge dq^j.$$  

Suppose $L$ is non-degenerate in the sense that $\det \left( \frac{\partial^2 L}{\partial v_i \partial v_j} \right) \neq 0$ at every point
of $T X$, the Legendre transformation is defined as the following map,

$${\mathcal L} : T X \to T^* X$$

$$(q, v) \mapsto \left( q, \frac{\partial L}{\partial v_i} dq^i \right).$$

It pulls back the canonical symplectic form $\omega$ on $T^* X$ to the above $\omega_L$ on $T X$,

$$\mathcal{L}^* \omega = \omega_L.$$  

We also define a Hamiltonian,

$$H : T^* X \to \mathbb{R}$$

$$H = \sum v^i \frac{\partial L}{\partial v^i} - L,$$

and it is called the Legendre transformation of the function $L$.

**Legendre transform on vector spaces**

From above discussions, we saw that the Legendre transformation is really
a fiberwise transformation from $T_q X$ to $T^*_q X$ using a function $L (q, \cdot)$ whose
Hessian is non-degenerate at every point. We can reformulate it as follow: Suppose $V$ is a finite dimensional vector space and $f : V \to \mathbb{C}$ is a function such
that its Hessian is non-degenerate at every point,

$$\det \left( \frac{\partial^2 f}{\partial x^i \partial x^j} \right) \neq 0.$$  

We look at the graph of $df$ in $T^* V = V \times V^*$ then the Legendre transformation
induced by $f$ is the map,

$${\mathcal L}_f : V \to V^*,$$

$${\mathcal L}_f = \pi_{V^*} \circ df,$$
where \( \pi_{V^*} \) is the projection from \( V \times V^* \) to its second factor \( V^* \). That is \( L_f(x) = \xi \) if and only if \( \xi_i = \frac{\partial f}{\partial x_i} \) for all \( i \) in local coordinates.

We also define the Legendre transformation of the function \( f \) on \( V \) as a function

\[
f^\vee : V^* \to \mathbb{C},
\]

\[
f^\vee(\xi) = \Sigma x^i \xi_i - f(x),
\]

with \( \xi = L_f(x) \).

Geometrically this transformation arises from the natural isomorphism,

\[
T^*V \cong T^* (V^*),
\]

for any finite dimensional vector space \( V \). The reason is \((V^*)^* = V\) and therefore both sides are isomorphic to \( V \times V^* \).

Now the graph \( C \) of \( df \) in \( T^*V \) is a Lagrangian submanifold of \( T^*V \) with its canonical symplectic form. Using the above isomorphism we can treat

\[
C \subset T^* (V^*),
\]

as another Lagrangian submanifold. \( ^8 \)

Under the non-degeneracy assumption \( C \) is also a graph of a function on \( V^* \), this function is precisely the above \( f^\vee \).

It is obvious that the Legendre transformation is involutive, i.e.

\[
L_f \circ L_{f^\vee} = 1_V
\]

\[
L_{f^\vee} \circ L_f = 1_{V^*}.
\]

\[
(f^\vee)^\vee = f.
\]

Remark: There is another point of view for the Legendre transformation, explained in Guillemín’s book \( ^{[Gu]} \) as follow: Suppose, on a real vector space, \( f : V \to \mathbb{R} \) is a strictly convex function with a critical point, which is necessarily unique and a global minimum, such a \( f \) is called stable. Then \( L_f(x) = \xi \) if and only if \( x \) is the unique critical point for the function \( f(x) - \xi(x) \). Moreover \( f^\vee(\xi) = -\min_{x \in V} f_\xi(x) \). We can also identify the image of \( L_f \) as \( \{ \xi \in V^* : f(x) - \xi(x) \text{ is stable} \} \).

A different perspective: conormal bundles

Recall that we associate to any function \( f \) on \( V \) a Lagrangian submanifold in \( T^*V \) given by the graph of \( df \), then we use the natural isomorphism \( T^*V \cong T^* (V^*) \) to define the Legendre transformation \( f^\vee \). Besides the graph of \( df \), there is another natural Lagrangian submanifold in \( T^*V \) associated to \( f \), namely the conormal bundle of the zero set of \( f \),

\[
N^*_{S/V} \subset T^*V,
\]

\( ^8 \) Under the natural isomorphism \( T^*V \cong T^* (V^*) \), their canonical symplectic forms are identified up to a minus sign. In particular they have identical Lagrangian submanifolds.

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where \( S = \{ x \in V : f ( x ) = 0 \} \). When \( f \) is homogeneous this Lagrangian submanifold, considered inside \( T^* (V^*) \), turns out to be the conormal bundle of the zero set of \( f^\vee \). This allows us to define the Legendre transformation in much greater generality, at least in the homogeneous case which is just the right setting for the projective geometry.

**Theorem 25** Suppose \( f : V \to \mathbb{C} \) is a homogenous polynomial such that its zero set \( S = \{ x \in V : f ( x ) = 0 \} \) is smooth. Then its conormal bundle \( N^*_S / V \subset T^* V \), when viewed as a submanifold of \( T^* (V^*) \), equals \( N^*_{S^\vee / V^*} \), the conormal variety of \( S^\vee = \{ \xi \in V^* : f^\vee ( \xi ) = 0 \} \subset V^* \).

**Proof:** When \( f \) is a homogeneous polynomial of degree \( p \), we have
\[
\sum x^i \frac{\partial f}{\partial x^i} = pf ( x ) .
\]
Under the Legendre transformation \( \xi_i = \frac{\partial f}{\partial x^i} ( x ) \) this gives,
\[
f^\vee ( \xi ) = \sum x^i \xi_i - f ( x ) \\
= pf ( x ) - f ( x ) \\
= (p - 1) f ( x ) .
\]

The conormal bundle of \( S \) is given by,
\[
N^*_S / V = \left\{ (x, \eta) \in V \times V^* : f ( x ) = 0 \text{ and } \eta_i = \frac{\partial f}{\partial x^i} ( x ) \text{ for all } i, \text{ for some } c \right\}.
\]
So we need to verify that the same set can be described as
\[
\left\{ (x, \eta) \in V \times V^* : f^\vee ( \eta ) = 0 \text{ and } x^i = b \frac{\partial f^\vee}{\partial \xi_i} ( \eta ) \text{ for all } i, \text{ for some } b \right\}.
\]

Now we suppose \( (x, \eta) \in N^*_S / V \). Since \( f \) is homogenous of degree \( p \), we have
\[
\frac{\partial f}{\partial x^i} (ex) = e^{p - 1} \frac{\partial f}{\partial x^i} (x) ,
\]
for any number \( e \). Therefore \( \eta_i = c e \frac{\partial f}{\partial x^i} (x) = \frac{\partial f}{\partial x^i} (c'x) \) for some constant \( c' \). That is
\[
\eta = L_f (c'x) .
\]
Hence
\[
f^\vee ( \eta ) = f^\vee ( L_f (c'x) ) = (p - 1) f ( c'x ) = (p - 1) (c')^p f ( x ) = 0 .
\]
Similarly using the inverse Legendre transformation, we have
\[
\frac{\partial f^\vee}{\partial \xi_i} ( \eta ) = c' x^i .
\]
Hence the result.

One can also find an indirect proof of this in [GKZ]. Because of this result, we can now define the Legendre transformation for any finite set of homogenous polynomials via the conormal bundle of their common zero set. This approach works particularly well for projective spaces and it is closely related to the dual variety construction.

4.2 Legendre transform in hyperkähler manifolds

4.2.1 Legendre transform in \( T^*\mathbb{P}^n \) and dual varieties

Recall that the hyperkähler structure on \( T^*\mathbb{P}^n \) can be constructed as the hyperkähler quotient of \( T^*V \) with \( V = \mathbb{C}^{n+1} \) by the natural \( S^1 \)-action,

\[
T^*\mathbb{P}^n = \left\{ (x, \xi) \in V \times V^*: \xi(x) = 1, |x|^2 - |\xi|^2 = 1 \right\} / S^1.
\]

The Legendre transformation on a linear symplectic space \( T^*V \) comes from the natural isomorphism,

\[
T^*V \cong T^* (V^*).
\]

We are going to descend this transformation to their hyperkähler quotients \( T^*\mathbb{P}^n \) and \( T^*\mathbb{P}^{n*} \). In particular we only need to look at those functions on \( V \) which are homogenous.

The first issue is the natural isomorphism between \( T^*V \) and \( T^* (V^*) \) does not descend to their hyperkähler quotients. Instead we have a natural birational map between \( T^*\mathbb{P}^n \) and \( T^*\mathbb{P}^{n*} \) which preserves their symplectic structures. To see this we recall that

\[
T^*\mathbb{P}^{n*} = \left\{ (\xi, x) \in V^* \times V: \xi(x) = 1, |\xi|^2 - |x|^2 = 1 \right\} / S^1,
\]

and the birational map is given by

\[
\Phi: T^*\mathbb{P}^n \dashrightarrow T^*\mathbb{P}^{n*}
\]

\[
\Phi (x, \xi) = \left( \begin{array}{c|c} x & \xi \\ \hline \xi & x \end{array} \right).
\]

It can be verified directly \( \Phi \) is an isomorphism outside their zero sections, which are given by \( \xi = 0 \) in \( T^*\mathbb{P}^n \) and \( x = 0 \) in \( T^*\mathbb{P}^{n*} \). Moreover \( \Phi \) pullbacks the canonical symplectic structure on \( T^*\mathbb{P}^{n*} \) to the one on \( T^*\mathbb{P}^n \) outside their zero sections.

Because of the theorem we define the Legendre transformation in the projective setting as follow: For any homogenous function \( f: V \to \mathbb{C} \) which defines a smooth hypersurface \( S = \{ f = 0 \} \subseteq \mathbb{P}^n \), the function itself can be
recovered from its conormal bundle $N_{S/P^n}$ inside $T^*\mathbb{P}^n$. The Legendre transform $f^\vee : V^* \to \mathbb{C}$ defines the dual hypersurface $S^\vee \subset \mathbb{P}^{n*}$ under the non-degenerate assumption and we have

$$N_{S^\vee/\mathbb{P}^{n*}} = \Phi (N_{S/P^n}\setminus \mathbb{P}^n),$$

where $\mathbb{P}^n$ denote the zero section in $T^*\mathbb{P}^n$.

An arbitrary Lagrangian subvariety $C$ in $T^*\mathbb{P}^n$ can be regarded as a generalized homogenous function on $V$ unless $C$ is the zero section $\mathbb{P}^n$. Motivated from above discussions, we define the Legendre transformation $C^\vee$ as follow,

$$C^\vee = \Phi (C\setminus \mathbb{P}^n).$$

It has the following immediate properties: (i) $C^\vee$ is a Lagrangian subvariety of $T^*\mathbb{P}^{n*}$; (ii) $C \subset T^*\mathbb{P}^n$ and $C^\vee \subset T^*\mathbb{P}^{n*}$ are isomorphic outside the zero sections; (iii) the inversion property $(C^\vee)^\vee = C$.

Remark: Recall that $T^*\mathbb{P}^n$ is the hyperkähler quotient of $T^*V$ by the natural $S^1$ action. In fact the Legendre transformation on $T^*\mathbb{P}^n$ can be regarded as a $S^1$-invariant Legendre transformation on $T^*V$ using the symplectic quotient by $S^1$: If $C$ is a Lagrangian in $T^*\mathbb{P}^n$ then there is a unique $C^\vee$-invariant Lagrangian subvariety of $T^*V$, denote $D$ such that $C$ is the symplectic quotient of $D$ by $S^1$. The Legendre transformation of $D$ will be a Lagrangian subvariety $D^\vee$ in $T^*(V^*)$ which is again $C^\vee$-invariant. The symplectic quotient of $D^\vee$ by $S^1$ would be our transformation $C^\vee$ in $T^*\mathbb{P}^{n*}$.

Dual varieties in $\mathbb{P}^n$ and a Plucker formula

For any subvariety $S$ in $\mathbb{P}^n$ we can associated a dual variety $S^\vee$ in the dual projective space $\mathbb{P}^{n*}$. The dual variety $S^\vee$ is the closure of all hyperplanes in $\mathbb{P}^n$ which are tangent to some smooth point in $S$. From our previous discussions, the conormal variety of $S^\vee$ is simply the Legendre transformation of the conormal variety of $S$. In this sense our Legendre transformation is a generalization of the dual variety construction. The relationship between dual varieties and the Legendre transformation has been briefly addressed before in various places.

As an example the dual variety of the Fermat hypersurface

$$S = \{ x_0^p = x_1^p + \cdots + x_n^p \} \subset \mathbb{P}^n$$

is

$$S^\vee = \{ \xi_0^q = \xi_1^q + \cdots + \xi_n^q \} \subset \mathbb{P}^{n*}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Notice that $q$ is no longer an integer if $p > 2$, in fact $S^\vee$ is a hypersurface of degree $p(p - 1)^{n-1}$.

The study of dual varieties is a very rich classical subject in algebraic geometry (see [GKZ] for instance). We list a few known facts (1) the inversion property $(S^\vee)^\vee = S$, (2) $S$ is irreducible if and only if $S^\vee$ is irreducible, (3) If
\(x \in S\) and \(\xi \in S^\vee\) are smooth points, then \(\xi\) is tangent to \(S\) at \(x\) if and only if \(x\) is tangent to \(S^\vee\) at \(\xi\).

For plane curves, there are Plücker formulae which related various geometric quantities between \(S\) and \(S^\vee\): Suppose \(S \subset \mathbb{P}^2\) is a plane curve of degree \(d\) with \(\delta\) double points, \(\kappa\) cusps and no other singularities, we denote the corresponding quantities for \(S^\vee\) as \(d^\vee, \delta^\vee, \kappa^\vee\). Then Plücker formulae say,

\[
\begin{align*}
d^\vee &= d(d - 1) - 2\delta - 3\kappa \\
\kappa^\vee &= 3d^2 - 6d - 6\delta - 8\kappa.
\end{align*}
\]

A similar formula in the higher dimensional setting is obtained by Kleiman in [Kl].

In the next section we will discuss a similar formula for the Legendre transformation in any hyperkähler manifold. In the case of conormal varieties inside \(T^*\mathbb{P}^n\) the formula says: For any subvarieties \(S_i \subset \mathbb{P}^n\) of dimension \(s_i\) we denote their conormal varieties as \(C_i \subset T^*\mathbb{P}^n\) then we have

\[
C_1 \cdot C_2 + \frac{(C_1 \cdot \mathbb{P}^n \cdot \mathbb{P}^n)}{(-1)^{n+1}(n+1)}(C_2 \cdot \mathbb{P}^n) = C_1^\vee \cdot C_2^\vee + \frac{(C_1^\vee \cdot \mathbb{P}^n \cdot \mathbb{P}^n^\vee)(C_2^\vee \cdot \mathbb{P}^n^\vee)}{(-1)^{n+1}(n+1)},
\]

or in terms of the intersection Euler characteristics \(\bar{\chi}\) we have

\[
C_1 \cdot C_2 \pm \frac{1}{n+1} \bar{\chi}(S_1) \bar{\chi}(S_2) = C_1^\vee \cdot C_2^\vee \pm \frac{1}{n+1} \bar{\chi}(S_1^\vee) \bar{\chi}(S_2^\vee).
\]

provided that \(S_1\) and \(S_2\) intersect transversely and the same for their duals.

In the special case of plane curves, the intersection number of their conormal varieties \(C_1 \cdot C_2\) is simply the product of the degree of the curves \(d_1d_2\). In fact the proof of the formula can be reduced to the case when \(S_2\) is a point and it reads as follow: For any plane curve \(S\) we have

\[
3d^\vee = -\bar{\chi}(S) - 2\bar{\chi}(S^\vee).
\]

When \(S\) has only double point and cusp singularities, the above formula can be proven by the Plücker formulae. Conversely the Plücker formula \(d^\vee = d(d - 1) - 2\delta - 3\kappa\) also follows from it and our earlier formula for \(\bar{\chi}(S)\).

### 4.2.2 Legendre transform along \(\mathbb{P}^n\) and a Plücker type formula

Now we study the Legendre transformation on a general hyperkähler manifold \(M\) of dimension \(2n \geq 4\).

**Flop along \(\mathbb{P}^n\)**

Recall that every embedded \(\mathbb{P}^n\) in \(M\) is a Lagrangian submanifold and it has a neighborhood \(U\) which is symplectomorphic to a neighborhood of the zero section in \(T^*\mathbb{P}^n\) and we continue to denote it by \(U\). Therefore we can flop such a \(\mathbb{P}^n\) in \(M\) to obtain another holomorphic symplectic manifold \(M'\). To see how
this surgery work, we look at the birational transformation \( \Phi : T^*\mathbb{P}^n \rightarrow T^*\mathbb{P}^n \)
and let \( U' \) be the image of \( U \), i.e. \( U' = \Phi(U) \). Then \( M' = (M \setminus U) \cup U' \)
Since \( M \) and \( M' \) are isomorphic outside a codimension \( n \) subspace, \( M' \) inherits
a holomorphic two form \( \Omega' \) from \( M \) by the Hartog’s theorem. Moreover being a
section of the canonical line bundle and non-vanishing outside a codimension \( n \)
subset, \( (\Omega')^n \) must be non-vanishing everywhere. That is \( M' \) is a holomorphic
symplectic manifold with the symplectic form \( \Omega' \)

We will denote the natural birational map between \( M \) and \( M' \) as
\[ \Phi_M : M \rightarrow M'. \]

**Examples of flop**

(1) (Mukai) Suppose \( X \) is a degree two K3 surface, i.e. \( X \) is a double cover of \( \mathbb{P}^2 \) branched along a sextic curve, \( \pi : X \rightarrow \mathbb{P}^2 \). We denote \( J_0 \) the degree zero compactified Picard scheme for degree 2 curves in \( X \). There is a birational map
\[ \Phi : S^{[2]}X \rightarrow J_0 \]
which associates to any 2 generic points \((p_1, p_2)\) in \( X \) to the unique degree two curve \( C \) passing through them together with the line bundle \( \omega_C \otimes O(-p_1 - p_2) \).

Note that \( \mathbb{P}^2 \) embeds inside \( S^{[2]}X \) via the preimage of \( \pi \). In fact \( \Phi \) is the flop of \( S^{[2]}X \) along this \( \mathbb{P}^2 \).

(2) (Beauville): Let \( X \subset \mathbb{P}^3 \) be a smooth quartic surface. Any 2 points on \( X \) defines a line in \( \mathbb{P}^3 \) which intersects \( X \) at two other points. This defines a birational map
\[ \Phi : S^{[2]}X \rightarrow S^{[2]}X. \]
If \( X \) does not contain any line, this is an isomorphism. If \( X \) contains \( k \) lines \( L_1, \ldots, L_k \), then \( \Phi \) is the flop of \( S^{[2]}X \) along \( S^{[2]}L_j \cong \mathbb{P}^2 \).

**Legendre transformation and a Plücker type formula**

Suppose \( M \) and \( M' \) are hyperkähler manifolds which are related by a flop \( \Phi_M : M \rightarrow M' \) along a \( \mathbb{P}^n \subset M \). For any Lagrangian subvariety \( C \) in \( M \) which does not contain \( \mathbb{P}^n \) we define its Legendre transform to be the Lagrangian subvariety \( C^\vee \) in \( M' \) defined as,
\[ C^\vee = \Phi_M(C \setminus \mathbb{P}^n). \]

Since \((C_1 \cup C_2)^\vee = C_1^\vee \cup C_2^\vee\), we can extend the definition of the Legendre transformation to the free Abelian group generated by all Lagrangian subvarieties of \( M \) except \( \mathbb{P}^n \).

---

To be precise with the holomorphic structure on \( M' \) we should write \( M' = (M \setminus U_0) \cup U' \)
for some open set \( U_0 \supset \mathbb{P}^n \) satisfying \( \overline{U_0} \subset U \).
Clearly the Legendre transformation has the inversion property, namely \((C^\vee)^\vee = C\). However the Legendre transformation does not preserve the intersection numbers, i.e. \(C_1 \cdot C_2 \neq C_1^\vee \cdot C_2^\vee\). Instead they satisfy the following Plücker type formula \([Le]\).

**Theorem 26** Suppose \(\Phi_M : M \rightarrow M'\) is a flop along \(\mathbb{P}^n\) between projective hyperkähler manifolds then

\[
C_1 \cdot C_2 + \frac{(C_1 \cdot \mathbb{P}^n) (C_2 \cdot \mathbb{P}^n)}{(-1)^{n+1} (n+1)} = C_1^\vee \cdot C_2^\vee + \frac{(C_1^\vee \cdot \mathbb{P}^n^*) (C_2^\vee \cdot \mathbb{P}^n^*)}{(-1)^{n+1} (n+1)},
\]

for any Lagrangian subvarieties \(C_1\) and \(C_2\) not containing \(\mathbb{P}^n\).

In the above Plücker type formula, it is more natural to interpret the LHS (and similar for the RHS) as follow,

\[
C_1 \cdot C_2 + \frac{(C_1 \cdot \mathbb{P}^n) (C_2 \cdot \mathbb{P}^n)}{(-1)^{n+1} (n+1)} = \left( C_1 - \frac{(C_1 \cdot \mathbb{P}^n)}{(\mathbb{P}^n \cdot \mathbb{P}^n^*)} \mathbb{P}^n \right) \cdot \left( C_2 - \frac{(C_2 \cdot \mathbb{P}^n)}{(\mathbb{P}^n \cdot \mathbb{P}^n^*)} \mathbb{P}^n^* \right).
\]

Note that \(\left( C - \frac{(C \cdot \mathbb{P}^n)}{(\mathbb{P}^n \cdot \mathbb{P}^n^*)} \mathbb{P}^n \right) \cdot \mathbb{P}^n = 0\) for any Lagrangian subvariety \(C\) in \(M\), including \(C = \mathbb{P}^n\). Roughly speaking the LHS (resp. RHS) of the Plücker type formula is the intersection number of two Lagrangian subvarieties which do not intersect \(\mathbb{P}^n\), and therefore the Legendre transformation along such a \(\mathbb{P}^n\) should have no effect to their intersection numbers, thus giving the Plücker type formula in a heuristic way.

**Normalized Legendre transformation**

It is suggested from above discussions that we should modify the transformation so that

\[
C = \frac{(C \cdot \mathbb{P}^n)}{(\mathbb{P}^n \cdot \mathbb{P}^n^*)} \mathbb{P}^n \rightarrow C^\vee = \frac{(C^\vee \cdot \mathbb{P}^n^*)}{(\mathbb{P}^n^* \cdot \mathbb{P}^n^*)} \mathbb{P}^n^*.
\]

We also want transform \(\mathbb{P}^n\), the center of the flop. Thus we arrive to the following definition of a *normalized Legendre transformation* \(\mathcal{L}\):

\[
\mathcal{L}(C) = C^\vee + \frac{(C \cdot \mathbb{P}^n) + (-1)^{n+1} (C^\vee \cdot \mathbb{P}^n^*)}{n+1} \mathbb{P}^n^* \text{ if } C \neq \mathbb{P}^n
\]

\[
\mathcal{L}(\mathbb{P}^n) = (-1)^n \mathbb{P}^n^*.
\]

Now our Plücker type formula can be rephrased as the following simple identity,

\[
C_1 \cdot C_2 = \mathcal{L}(C_1) \cdot \mathcal{L}(C_2).
\]

Unlike our earlier Legendre transformation, this normalized Legendre transformation is rather non-trivial even when \(n = 1\). When \(M\) is a K3 surface every
embedded $\mathbb{P}^1$ is also called an $(-2)$-curve because $\mathbb{P}^1 \cdot \mathbb{P}^1 = -2$. Flopping $\mathbb{P}^1$ in $M$ is trivial, i.e. $M = M'$. This is because a point in $\mathbb{P}^1$ is also a hyperplane. Therefore the Legendre transformation $C \to C' = C$ is just the identity transformation. However the normalized Legendre transformation is given by

$$\mathcal{L}(C) = C - (C \cdot \mathbb{P}^1) \mathbb{P}^1,$$

which is well-defined for any cohomology class of the K3 surface $M$ and induces an automorphism of $H^2(M, \mathbb{Z})$, namely the reflection with respect to the class $[\mathbb{P}^1]$. We can identify $\mathcal{L}(C)$ as the Dehn twist of $C$ along the Lagrangian $\mathbb{P}^1$, or $S^2$, in $M$. If we use all the $(-2)$-curves in the whole twistor family then their corresponding Dehn twists generate $\text{Aut}(H^2(M, \mathbb{Z}))$, which gives all diffeomorphisms of $M$ up to isotopy by a result of Donaldson.

Remark: It would be interesting to compare our normalized Legendre transformation with the Dehn twist along Lagrangian $S^n$ in the real symplectic geometry as studied by P. Seidel [Se]. On the level of derived category, Thomas suggests that $\mathcal{L}$ could be the mirror object to a Dehn twisted operation defined by him and Seidel [ST].

Suppose $M$ is a projective hyperkähler manifold of dimension $2n$ and

$$\pi : M \to Z$$

is a projective contraction with normal exceptional locus $D$ in $M$. We assume that $\pi(D)$ is a single point, i.e. $Z$ has isolated singularity. Then (i) $D \cong \mathbb{P}^n$ if $n \geq 2$ and (ii) $D$ is an ADE configuration of $\mathbb{P}^1$ if $n = 1$. Conversely every such $D$ can be contracted inside $M$. Moreover the normalized Legendre transformation is defined in all these cases and very interesting.

A categorical transformation

Now we have a transformation $\mathcal{L}$ which takes Lagrangians in $M$ to Lagrangians in $M'$ and it respects their intersection numbers. It is then natural to wonder if we have a categorical transformation between the Lagrangian categories of $M$ and $M'$, namely we want to have an isomorphism on the cohomology groups,

$$\text{Ext}^q_{\mathcal{O}_M} (\mathcal{O}_{C_1}, \mathcal{O}_{C_2}) \cong \text{Ext}^q_{\mathcal{O}_{M'}} (\mathcal{O}_{\mathcal{L}(C_1)}, \mathcal{O}_{\mathcal{L}(C_2)})$$

rather than just their Euler characteristics, i.e. intersection numbers.

Suppose $\Phi : M \dashrightarrow M'$ is a flop along $P = \mathbb{P}^n$ as before. Let $\tilde{M}$ be the blow up of $M$ along $P$ and we denote its exceptional divisor as $\tilde{P}$. $\tilde{P}$ admits two $\mathbb{P}^{n-1}$-fibration over $\mathbb{P}^n$ provided that $n \geq 2$. A point in $\tilde{P}$ is a pair $(p, H)$ with $H \subset \mathbb{P}^n$ a hyperplane and $p \in H$. The two $\mathbb{P}^{n-1}$-fibrations correspond to sending the above point to $p$ and $H$ respectively. We can blow down $\tilde{P}$ along the second fibration to obtain $M'$, we write these morphisms as

$$M \xleftarrow{\cong} \tilde{M} \xrightarrow{\pi} M'.$$
Now we define a Legendre functor
\[ L : D^b_{Lag}(M) \to D^b_{Lag}(M') \]
\[ L(\bullet) = \pi_\ast' \pi^\ast(\bullet). \]

It is not difficult to show that the image of any Lagrangian coherent sheaf on \( M \) under \( L \) is a Lagrangian coherent sheaf on \( M' \). Using the machinery developed by Bondal and Orlov in \([BO]\) \( L \) defines an equivalence of categories \([Le]\). When \( S = O_C \) for a Lagrangian subvariety \( C \) in \( M \), then \( L(S) \) should be closely related to \( O_{L(C)} \). This transformation will play an important role in the proof of the Plücker type formula \([Le]\).

### 4.2.3 Legendre transform on Mukai elementary modification

Now we consider a general projective contraction on \( M \), as discussed in section 2.2, with smooth discriminant locus \( B \),
\[ D \subset M \]
\[ \downarrow \quad \downarrow \pi \]
\[ B \subset Z. \]

The exceptional locus \( D \) is a \( \mathbb{P}^k \)-bundle over \( B \), which has a natural symplectic form. Moreover, the normal bundle of \( D \) in \( M \) is the relative cotangent bundle of \( D \to B \). The Mukai elementary modification produces another symplectic manifold \( M' \) by replacing the \( \mathbb{P}^k \)-bundle \( D \to B \) with the dual \( \mathbb{P}^k \)-bundle over \( B \).

\[ \begin{array}{ccc}
D \subset M & \xrightarrow{\Phi_M} & M' \supset D' \\
\mathbb{P}^k \downarrow \quad \downarrow \pi & & \quad \downarrow \mathbb{P}^k \\
B \subset Z & = & Z \supset B
\end{array} \]

We define the Legendre transformation of a Lagrangian subvariety \( C \) in \( M \) not lying inside \( D \) as follow,
\[ C' = \Phi(C \setminus D). \]

In order to write down the Plücker type formula in this general situation, we need to recall from section 3.3.3 that \( C \) determines (i) a Lagrangian subvariety \( C^{proj} \) in \( M \) lying inside \( D \), called the projection and (ii) a Lagrangian subvariety \( C^{red} \) in \( B \), called the reduction. The restriction of \( \pi \) to \( C^{proj} \) is a \( \mathbb{P}^k \)-bundle over \( C^{red} \subset B \) and \( C^{proj} = \pi^{-1}(C^{red}) \subset M \). It is not difficult to see that we have \( (C')^{proj} = (\pi')^{-1}(C^{red}) \subset M' \) and
\[ C^{proj} \cdot C^{proj} = C'^{proj} \cdot C'^{proj}. \]

The Plücker type formula in this general case reads as follow,
\[
\left( C_1 - \frac{C_1 \cdot C_1^{\text{proj}}}{C_1^{\text{proj}}}, C_1^{\text{proj}} \right) \cdot \left( C_2 - \frac{C_2 \cdot C_2^{\text{proj}}}{C_2^{\text{proj}}}, C_2^{\text{proj}} \right) = \left( C_1^\vee - \frac{C_1^\vee \cdot C_1^{\text{proj}}^{\vee \text{proj}}}{C_1^{\text{proj}}^{\vee \text{proj}}}, C_1^{\text{proj}}^{\vee \text{proj}} \right) \cdot \left( C_2^\vee - \frac{C_2^\vee \cdot C_2^{\text{proj}}^{\vee \text{proj}}}{C_2^{\text{proj}}^{\vee \text{proj}}}, C_2^{\text{proj}}^{\vee \text{proj}} \right). 
\]

It can be proven using the same method as in the previous situation. The normalized Legendre transformation \( L \) for this general case is defined as follow,

\[
L(C) = C^\vee + (-1)^k \frac{C \cdot C^{\text{proj}} - C^\vee \cdot C^{\text{proj}}^{\vee \text{proj}}}{C^{\text{proj}} \cdot C^{\text{proj}}^{\vee \text{proj}}} \text{ when } C \notin D,
\]

\[
= (-1)^k (\pi')^{-1} (C^{\text{red}}) \text{ when } C \subset D.
\]

It preserves intersection products of Lagrangian subvarieties in \( M \) and \( M' \),

\[
C_1 \cdot C_2 = L(C_1) \cdot L(C_2)
\]

5 Appendix

In this appendix we will recall some facts about hyperkähler geometry that we used in this article. We assume \( M \) is a compact hyperkähler manifold.

**Hard Lefschetz \( sl_2 \)-action using \( \Omega \)**

We consider the homomorphism

\[
L_\Omega : \Omega^{p,q}(M, \mathbb{C}) \to \Omega^{p+2,q}(M, \mathbb{C})
\]

\[
L_\Omega(\phi) = \phi \cup \Omega,
\]

and its adjoint homomorphism

\[
\Lambda_\Omega : \Omega^{p+2,q}(M, \mathbb{C}) \to \Omega^{p,q}(M, \mathbb{C}).
\]

As studied by Fujiki in [Fu], they define a \( sl_2(\mathbb{C}) \) action on \( \Omega^{*,*}(M, \mathbb{C}) \). Moreover this action can be descended to the cohomology groups \( H^{*,*}(M, \mathbb{C}) \) because \( \Omega \) is a parallel form. We call this the Hard Lefschetz \( sl_2 \)-action on \( M \) using \( \Omega \). A cohomology class \( \phi \in H^{*,*}(M, \mathbb{C}) \) is called \( \Omega \)-primitive if \( \Lambda_\Omega \phi = 0 \). As in the standard Hodge theory for Kähler manifolds, we have an \( \Omega \)-primitive decomposition of the cohomology groups of \( M \). In particular \( L_\Omega \) in injective on \( H^{p,q}(M, \mathbb{C}) \) provided that \( p < n \). This \( sl_2 \)-action forms part of the \( so(4,1) \) action defined by Verbitsky.

**Bogomolov-Beauville quadratic form**

We normalize \( \Omega \) with \( \int_M \Omega^n \bar{\Omega}^n = 1 \). The Bogomolov-Beauville (unnormalized) quadratic form \( q \) on \( H^2(M, \mathbb{R}) \) is defined as follows (see [Be] for more details):

\[
q(\phi) = \frac{n}{2} \int \Omega^{n-1} \bar{\Omega}^{n-1} \phi^2 + (1 - n) \int \Omega^{n-1} \bar{\Omega}^{n-1} \phi \int \Omega^n \bar{\Omega}^{n-1} \phi.
\]

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It is nondegenerate and has signature \((3, b_2 - 3)\). It is not difficult to check that if \(\beta_1\) is an ample class and \(\beta_2\) is an effective divisor class then \(q(\beta_1, \beta_2) > 0\).

Using Torelli theorem for hyperkähler manifolds, one can obtain the following result of Fujiki [Fu]: If \(\alpha \in H^{4k}(M)\) is a polynomial in Chern classes of \(M\), then there exists a constant \(c_\alpha\) such that

\[
\int_M \alpha \phi^{2n-2k} = c_\alpha q(\phi)^{n-k}.
\]

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