GRASSMANN CONVEXITY AND
MULTIPLICATIVE STURM THEORY, REVISITED

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To the late Vladimir Arnold, who started all this

Abstract. In this paper we settle a special case of the Grassmann convexity conjecture formulated in [11]. We present a conjectural formula for the maximal total number of real zeros of the consecutive Wronskians of an arbitrary fundamental solution to a disconjugate linear ordinary differential equation with real time, comp. [13]. We show that this formula gives the lower bound for the required total number of real zeros for equations of an arbitrary order and, using our results on the Grassmann convexity, we prove that the aforementioned formula is correct for equations of orders 4 and 5.

1. Introduction and main results

Our subject of study is related to the PhD theses of the second and third authors defended in the early 90s (see [10, 15]). Namely, the thesis of the second author contains Conjecture 1.2, see below, but the presented argument which is supposed to prove it is false. The statement itself is still open and (if proven) would be of fundamental importance to the general qualitative theory of linear ordinary differential equations with real time. The thesis of the third author contains a number of Schubert calculus problems relevant to Conjecture 1.2. Over the years the authors made several attempts to settle it and, in particular, worked out some reformulations and special cases. This paper contains a number of new results in that direction. (In what follows, we will label conjectures, theorems and lemmas borrowed from the existing literature by letters. Results and conjectures labelled by numbers are new).

We start with the following classical definition, see e.g. [4].

Definition 1.1. A linear ordinary homogeneous differential equation

\[ y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_n(x)y = 0 \]

of order \( n \) with real-valued continuous coefficients \( p_i(x) \) defined on an interval \( I \subseteq \mathbb{R} \) is called disconjugate on \( I \) if any of its nontrivial solutions has at most \( (n-1) \) zeros on \( I \) counting multiplicities. (\( I \) can either be open or closed).

Conjecture 1.2 (Upper bound on the number of real zeros of a Wronskian). Given any equation (1.1) disconjugate on \( I \), a positive integer \( 1 \leq k \leq n-1 \), and an arbitrary \( k \)-tuple \( (y_1(x), y_2(x), \ldots, y_k(x)) \) of its linearly independent solutions,
the number of real zeros of $W(y_1(x), y_2(x), \ldots, y_k(x))$ on $I$ counting multiplicities does not exceed $k(n - k)$. Here

$$W(y_1(x), y_2(x), \ldots, y_k(x)) := \begin{pmatrix} y_1(x) & y_1'(x) & \cdots & y_1^{(k-1)}(x) \\ y_2(x) & y_2'(x) & \cdots & y_2^{(k-1)}(x) \\ \vdots & \vdots & \ddots & \vdots \\ y_k(x) & y_k'(x) & \cdots & y_k^{(k-1)}(x) \end{pmatrix}$$

is the Wronskian of the $k$-tuple $(y_1(x), y_2(x), \ldots, y_k(x))$.

Cases $k = 1$ and $k = n - 1$ are straightforward, but not very illuminating. The simplest non-trivial case $k = 2$, $n = 4$ of Conjecture 1.2 has been settled in [11]. Conjecture 1.2 has an equivalent reformulation called the Grassmann convexity conjecture first suggested in [11], Main Conjecture 1.1. To state it, we need some further definitions.

**Definition 1.3.** A smooth closed curve $\gamma : \mathbb{S}^1 \to \mathbb{R}^{P^{n-1}}$ is called locally convex if, for any hyperplane $H \subset \mathbb{R}^{P^{n-1}}$, the local multiplicity of the intersection of $\gamma$ with $H$ at any of the intersection points $p \in \gamma \cap H$ does not exceed $n - 1 = \dim \mathbb{R}^{P^{n-1}}$ and globally convex if the above condition holds for the sum of all local multiplicities, see e.g. [11].

Below we will often refer to globally convex curves as convex. The above notions directly generalize to smooth non-closed curves, i.e. $\gamma : I \to \mathbb{R}^{P^{n-1}}$.

**Remark 1.4.** Local convexity of $\gamma$ is a simple requirement equivalent to the non-degeneracy of the osculating Frenet $(n - 1)$-frame of $\gamma$, i.e. to the linear independence of $\gamma'(t), \ldots, \gamma^{n-1}(t)$ at all points $t \in \mathbb{S}^1$. Global convexity is a rather nontrivial property studied under different names since the beginning of the last century. (There exists a vast literature on convexity and the classical achievements are well summarized in [4]. For more recent developments see e.g. [1]).

Denote by $G_{k,n}$ the usual Grassmannian of real $k$-dimensional linear subspaces in $\mathbb{R}^n$ (or equivalently, of real $(k - 1)$-dimensional projective subspaces in $\mathbb{R}^{P^{n-1}}$).

**Definition 1.5.** Given an $(n - k)$-dimensional linear subspace $L \subset \mathbb{R}^n$, we define the Grassmann hyperplane $H_L \subset G_{k,n}$ associated to $L$ as the set of all $k$-dimensional linear subspaces in $\mathbb{R}^n$ non-transversal to $L$.

**Remark 1.6.** The concept of Grassmann hyperplanes is well-known in Schubert calculus, (see e.g. [3] and [11]). More exactly, $H_L \subset G_{k,n}$ coincides with the union of all Schubert cells of positive codimension constructed using any complete flag containing $L$ as a linear subspace. The complement $G_{k,n} \setminus H_L$ is the open Schubert cell isomorphic to the standard affine chart in $G_{k,n}$. By duality, $H_L \subset G_{k,n}$ is isomorphic to $H_{L'} \subset G_{n-k,n}$ where $L'$ is a $k$-dimensional linear subspace in $\mathbb{R}^n$.

**Remark 1.7.** A usual hyperplane $H \subset \mathbb{R}^{P^{n-1}}$ is a particular case of a Grassmann hyperplane if we interpret $H$ as the set of all points non-transversal (i.e. belonging) to $H$. $H$ itself can be considered as a point in $(\mathbb{R}^{P^{n-1}})^*$.

**Definition 1.8.** A smooth closed curve $\Gamma : \mathbb{S}^1 \to G_{k,n}$ is called locally Grassmann-convex if the local multiplicity of the intersection of $\Gamma$ with any Grassmann hyperplane $H_L \subset G_{k,n}$ at any of its intersection points does not exceed $k(n - k) = \dim G_{k,n}$, and globally Grassmann-convex if the above condition holds for the sum of all local multiplicities, see [11].
Below we refer to globally Grassmann-convex curves as Grassmann-convex. As above the latter notions directly generalize to smooth non-closed curves, i.e. $\Gamma : I \to G_{k,n}$.

**Definition 1.9.** Given a locally convex curve $\gamma : S^1 \to \mathbb{R}P^{n-1}$ and a positive integer $1 \leq k \leq n-1$, we define its $k$th osculating Grassmann curve $\text{osc}_k \gamma : S^1 \to G_{k,n}$ as the curve formed by the $(k-1)$-dimensional projective subspaces osculating the initial $\gamma$.

For any $k = 1, \ldots, n-1$, the curve $\text{osc}_k \gamma$ is well-defined due to the local convexity of $\gamma$.

**Conjecture 1.10** (Grassmann convexity conjecture). *For any convex curve $\gamma : S^1 \to \mathbb{R}P^{n-1}$ (resp. $\gamma : I \to \mathbb{R}P^{n-1}$) and any $1 \leq k \leq n-1$, its osculating curve $\text{osc}_k \gamma : S^1 \to G_{k,n}$ (resp. $\text{osc}_k \gamma : I \to G_{k,n}$) is Grassmann-convex.*

The equivalence of Conjectures 1.2 and 1.10 is straightforward and, in particular, is explained in [11]. Namely, we call a curve $\gamma = (\gamma_1, \ldots, \gamma_n) : I \to \mathbb{R}^n$ non-degenerate if at every point $t \in I$, its osculating frame $\{\gamma(t), \gamma'(t), \gamma''(t), \ldots, \gamma^{n-1}(t)\}$ is non-degenerate which is equivalent to the fact that its Wronskian matrix $W(t) = W(\gamma_1(t), \ldots, \gamma_n(t))$ has full rank.

Non-degenerate curves can be trivially identified with fundamental solutions of linear differential equations (1.1). In particular, we call a non-degenerate $\gamma$ disconjugate if the corresponding equation (1.1) is disconjugate. On the other hand, it is obvious that $\gamma$ is non-degenerate/disconjugate if and only if its projectivization is locally convex/convex.

Moreover, given a non-degenerate curve $\gamma = (\gamma_1, \ldots, \gamma_n) : I \to \mathbb{R}^n$ and an integer $1 \leq k < n$, the zeros of the Wronskian $W(\gamma_1, \ldots, \gamma_k)$ can be interpreted as the moments when the $k$th osculating Grassmann curve $\text{osc}_k \gamma : I \to G_{k,n}$ intersects an appropriate Grassmann hyperplane; for more details on $k = 2$, see Section 2. Observe that Conjecture 1.10 is trivially satisfied for $k = 1$ and $k = n - 1$.

The main result of the present paper which extends the case $k = 2$, $n = 4$ settled in [11] is as follows.

**Theorem 1.** *Conjecture 1.10 holds for $k = 2$ and any positive integer $n \geq 3$.*

Notice additionally that Theorem 1 admits the following natural interpretation, compare loc. cit.

**Definition 1.11.** Given a generic curve $\gamma : S^1 \to \mathbb{R}P^{n-1}$, we define its standard discriminant $D_\gamma \subset \mathbb{R}P^{n-1}$ to be the hypersurface consisting of all subspaces of codimension 2 osculating $\gamma$. (Here ‘generic’ means having a non-degenerate osculating $(n-2)$-frame at every point.)

**Definition 1.12.** By the $\mathbb{R}$-degree of a real closed projective (algebraic or non-algebraic) hypersurface $H \subset \mathbb{R}^n$ (resp. $H \subset \mathbb{R}P^{n-1}$) without boundary we mean the supremum of the cardinality of $H \cap L$ taken over all lines $L \subset \mathbb{R}^n$ (resp. $L \subset \mathbb{R}P^{n-1}$) such that $L$ intersects $H$ transversally. (Observe that the $\mathbb{R}$-degree of a hypersurface can be infinite. Discussions of this notion can be found in [7]).

**Corollary 1.13.** *For any closed convex curve $\gamma : S^1 \to \mathbb{R}P^{n-1}$, the $\mathbb{R}$-degree of its discriminant $D_\gamma$ equals $2n - 4$.*
Basic notions of the multiplicative Sturm separation theory. Following [11], let us now recall the set-up of this theory, an early version of which can be found in [10]. Denote by $\text{Fl}_n$ the space of complete real flags in $\mathbb{R}^n$. We say that two complete flags $f_1, f_2 \in \text{Fl}_n$ are transversal if for any $1 \leq i \leq n - 1$ the intersection of the $i$-dimensional subspace of $f_1$ with the $(n - i)$-dimensional subspace of $f_2$ coincides with the origin. Otherwise the flags $f_1, f_2 \in \text{Fl}_n$ are called non-transversal.

Definition 1.14. Given a locally convex curve $\gamma : \mathbb{S}^1 \rightarrow \mathbb{R}P^{n-1}$, define its osculating flag curve $\gamma F : \mathbb{S}^1 \rightarrow \text{Fl}_n$ to be the curve formed by the complete flags osculating $\gamma$, see e.g. [10]. (The curve $\gamma F$ is well-defined due to the local convexity of $\gamma$; similar notion obviously exists for non-closed locally convex curves).

For a non-degenerate curve $\Gamma : I \rightarrow \mathbb{R}^n$ (or, equivalently, for its projectivization $\gamma : I \rightarrow \mathbb{R}P^{n-1}$) and any fixed flag $f \in \text{Fl}_n$, denote by $\sharp_{\gamma,f}$ the number of moments of non-transversality between $\gamma F(t)$ and $f$ are non-transversal. Define $\sharp_{\gamma} := \sup_{f \in \text{Fl}_n} \sharp_{\gamma,f}$.

The following two lemmas provide criteria for (non-)disconjugacy of linear ordinary differential equation (or, equivalently, (non-)convexity of projective curves) on an interval $I$, compare [8].

Lemma 1.15 (see [10]). A locally convex curve $\gamma : \mathbb{S}^1 \rightarrow \mathbb{R}P^{n-1}$ (resp. $\gamma : I \rightarrow \mathbb{R}P^{n-1}$) is (globally) convex if and only if, for all $t_1 \neq t_2 \in \mathbb{S}^1$ (resp. $t_1 \neq t_2 \in I$), the flags $\gamma F(t_1)$ and $\gamma F(t_2)$ are transversal.

Lemma 1.16 (see [10]). A locally convex curve $\gamma : \mathbb{S}^1 \rightarrow \mathbb{R}P^{n-1}$ (resp. $\gamma : I \rightarrow \mathbb{R}P^{n-1}$) is not (globally) convex if and only if, for any complete flag $f \in \text{Fl}_n$, there exists $t \in \mathbb{S}^1$ (resp. $t \in I$) such that $f$ and $\gamma F(t)$ are non-transversal.

The next claim appears to be new.

Conjecture 1.17. For any convex curve $\gamma : \mathbb{S}^1 \rightarrow \mathbb{R}P^{n-1}$ (resp. $\gamma : I \rightarrow \mathbb{R}P^{n-1}$), one has

$$\sharp_{\gamma} = \frac{n^3 - n}{6}.$$ 

Conjecture 1.17 is obvious for $n = 2$ and easy for $n = 3$. Our next two results support it.

Theorem 2. For any convex curve $\gamma : \mathbb{S}^1 \rightarrow \mathbb{R}P^{n-1}$ (resp. $\gamma : I \rightarrow \mathbb{R}P^{n-1}$), one has

$$\sharp_{\gamma} \geq \frac{n^3 - n}{6}.$$ 

Combining Theorems 1 and 2 we get the following.

Corollary 1.18. Conjecture 1.17 holds for $n = 4$ and $n = 5$.

To finish the introduction, let us mention that it is well-known that, for general equations (1.1) of order exceeding 2, there is no relation between the number and location of the zeros of their different individual solutions. On the other hand, for any equation (1.1), one can split its time interval $I$ into maximal disjoint subintervals on each of which (1.1) is disconjugate. In order to get a meaningful
comparison theory, instead of looking at the individual solutions one should compare different fundamental solutions of (1.1), i.e. count the number of moments of non-transversality of the flag curve of (1.1) with different complete flags. This approach leads to the following claim which is a conceptually new generalization of the classical Sturm separation theorem to linear ordinary differential equations of arbitrary order implicitly suggested in [10].

**Conjecture 1.19** (see [11]). For \( n \geq 2 \), let \( \gamma : S^1 \to \mathbb{R}P^{n-1} \) (resp. \( \gamma : I \to \mathbb{R}P^{n-1} \)) be a locally, but not globally convex curve. Then, for any pair of complete flags \( f_1 \) and \( f_2 \),

\[
\#_{\gamma,f_1} \leq \frac{n^3 - n + 6}{6} \cdot \#_{\gamma,f_2}.
\]

Observe that (if settled) Conjecture 1.17 combined with Lemma 1.16 will imply Conjecture 1.19.

The structure of this paper is as follows. In §2 we will introduce our main technical tool which is the rank function for a certain type of cyclic words using which we prove Theorem 1. In §3 we recall several results from [6], prove some additional statements and settle Theorem 2. (Notice that besides the references we already mentioned other relevant results can be found in e.g. [9] and [2]).

2. Proof of Theorem 1

In this section we follow the notation of [5, 6] and use matrix realizations of flag curves obtained as osculating curves of convex projective curves. (Such realizations were frequently used in the earlier papers by the authors).

Observe that we can assume that for any convex curve \( \gamma : S^1 \to \mathbb{R}P^{n-1} \) (or \( \gamma : I \to \mathbb{R}P^{n-1} \)), its osculating flag curve \( \gamma_F : S^1 \to Fl_n \) (resp. \( \gamma_F : I \to Fl_n \)) lies completely in some top-dimensional Schubert cell in \( Fl_n \). To see that, depending on whether one considers the case of \( S^1 \) or \( I \), let us either fix an arbitrary point \( \tau \in S^1 \) or the left endpoint \( \tau \in I \). Take the flag \( \gamma_F(\tau) \in Fl_n \) as the reference complete flag defining the top-dimensional Schubert cell in \( Fl_n \). By Lemma 1.15, for any \( \nu \in S^1 \) (resp. \( \nu \in I \)) different from \( \tau \), the flags \( \gamma_F(\tau) \) and \( \gamma_F(\nu) \) are transversal, which means that the latter flag lies in the top-dimensional Schubert cell in \( Fl_n \) with respect to the former flag. Thus the whole flag curve \( \gamma_F \), except for one point \( \gamma_F(\tau) \), lies in this top-dimensional cell.

Top-dimensional cells in \( Fl_n \) are standardly identified with \( Lo_1^n \), where \( Lo_1^n \) is the nilpotent Lie group of real lower triangular \( n \times n \) matrices with diagonal entries equal to 1. This group can be interpreted as the tangent space to \( Fl_n \) at any fixed chosen flag \( f \). Alternatively, the usual \( LU \) and \( QR \) decompositions define diffeomorphisms \( Q : Lo_1^n \to U_1 \) and \( L : U_1 \to Lo_1^n \), where \( U_1 \subset Fl_n \) is a top dimensional cell (see [6]). The following statement can be found in e.g. [10, 5, 6].

**Lemma 2.1.** Consider an interval \( I \subseteq \mathbb{R} \) and a smooth curve \( \Gamma : I \to Lo_1^n \). Consider the diffeomorphism \( Q : Lo_1^n \to U_1 \subset Fl_n \) discussed above. Then \( \Gamma \) is an osculating flag curve of a convex projective curve \( \gamma : I \to Fl_n \) (i.e., \( \gamma_F = Q \circ \Gamma \)) if and only if, for every \( t \in I \), the logarithmic derivative \( (\Gamma(t))^{-1} \Gamma'(t) \) has strictly positive subdiagonal entries (i.e. entries in positions \( (j+1,j) \)) and zero entries elsewhere.
We sometimes abuse notation by identifying \( \text{Lo}_n^1 \) with \( \mathcal{U}_1 \) (through \( Q \)) and therefore \( \Gamma \) with \( Q \circ \Gamma \). Let us call the osculating flag curves obtained by taking the flags osculating convex projective curves flag-convex (or sometimes just convex). Notice that Lemma 1.15 implies that if \( \gamma : I \to \mathbb{R}P^{n-1} \) is such that \( \gamma_F = \Gamma, \Gamma : I \to \text{Lo}_n^1 \) (or, more precisely, \( \gamma_F = Q \circ \Gamma \)) then \( \gamma \) is (globally) convex if and only if \( \Gamma \) is flag-convex.

Given a flag-convex curve \( \Gamma : I \to \text{Lo}_n^1 \), define the function \( m_{I,k} := m_k : I \to \mathbb{R} \), given by

\[
m_k(t) := \det(\text{swminor}(\Gamma(t), k)),
\]

where \( \text{swminor}(L, k) \) is the \( k \times k \) submatrix of \( L \) formed by its last \( k \) rows and its first \( k \) columns.

Observe now that if we interpret \( \text{Lo}_n^1 \) as the top-dimensional cell in \( \mathcal{F}_n \) with respect to some fixed flag \( g \in \mathcal{F}_n \), then the moments of non-transversality of the flag curve \( \Gamma \) to the \((n-k)\)-dimensional linear subspace belonging to \( g \) are exactly the zeros of \( m_k(t) \).

Thus Conjecture 1.2 is equivalent to saying that for any \( n \), for any \( k \leq n \) and for any flag-convex curve \( \Gamma : I \to \text{Lo}_n^1 \), the number of real zeroes of the function \( m_k(t), t \in I \), is at most \( k(n-k) \).

Define the open and dense subset \( \text{Lo}_n^o \subset \text{Lo}_n^1 \) given by

\[
\text{Lo}_n^o = \{ X \in \text{Lo}_n^1 \mid \forall k \in [1, n-1], m_k(X) \neq 0 \}.
\]

In the notation of [6], we have \( \text{Lo}_n^o = \bigcup_{q \in \text{Quat}} \mathbb{Q}^{-1} [\text{Bru}_q] \). The set \( \text{Lo}_n^o \) is a disjoint union of finitely many connected components. These connected components were counted in [14] and several follow-up papers. In particular, their number equals 2, 6, 20, 52 for \( n = 2, 3, 4, 5 \) resp. and it is equal to \( 3 \times 2^{n-1} \) for all \( n \geq 6 \).

We will specially distinguish two of these connected components. Recall that a matrix \( L_0 \in \text{Lo}_n^1 \) is totally positive provided that, if a minor is nonzero for some \( L \in \text{Lo}_n^1 \), then the corresponding minor is strictly positive for \( L_0 \) (see [12]); the set \( \text{Pos} \subset \text{Lo}_n^1 \) of totally positive matrices is a contractible connected component of \( \text{Lo}_n^o \). Similarly, the set \( \text{Neg} \subset \text{Lo}_n^1 \) of totally negative matrices is another contractible connected component of \( \text{Lo}_n^o \). For \( L \in \text{Lo}_n^1 \), we have that \( L \in \text{Neg} \) if and only if \( PLP \in \text{Pos} \), where the diagonal matrix \( P \) is given by \( P = \text{diag}(1, -1, 1, \ldots, (-1)^{n-1}) \). Equivalently, for \( L \in \text{Lo}_n^1 \), \( L \in \text{Neg} \) if and only if \( L^{-1} \in \text{Pos} \). In the following Lemma we provide an alternative characterization of the subsets \( \text{Pos}, \text{Neg} \subset \text{Lo}_n^1 \); here \( \text{id} \in \text{Lo}_n^1 \) is the identity matrix.

**Lemma 2.2** (see [10], [6]). If \( \Gamma : I \to \text{Lo}_n^1 \) is flag-convex and \( \Gamma(0) = \text{id} \) then \( \Gamma(t) \in \text{Pos} \) for \( t > 0 \) and \( \Gamma(t) \in \text{Neg} \) for \( t < 0 \). Conversely, if \( L_1 \in \text{Pos} \subset \text{Lo}_n^1 \) and \( L_{-1} \in \text{Neg} \subset \text{Lo}_n^1 \) then there exists a smooth flag-convex curve \( \Gamma : \mathbb{R} \to \text{Lo}_n^1 \) such that \( \Gamma(-1) = L_{-1}, \Gamma(0) = \text{id} \) and \( \Gamma(1) = L_1 \).

Recall that, for any locally convex curve \( \gamma : I \to \mathbb{R}P^{n-1} \), we denote by \( \gamma_F(t) \) its osculating flag curve and by \( \text{osc}_{2} \gamma(t) : I \to G_{2,n} \) the osculating Grassmann curve obtained by taking the span of the first two columns of \( \gamma_F(t) \). Fix the subspace \( L = \text{span}(e_1, \ldots, e_{n-2}) \subset \mathbb{R}^n \) of codimension 2. Observe that the intersection of \( \text{osc}_{2} \gamma \) with \( H_L \) is given by the equation \( m_2(t) = 0 \). Here \( H_L \subset G_{2,n} \) is the Grassmann hyperplane associate with the latter \((n-2)\)-dimensional \( L \).

In what follows, instead of considering the curve \( \text{osc}_{2} \gamma(t) : I \to G_{2,n} \) we present a related construction.
Let $C = Lo_{1}^{2} \subset \mathbb{R}^{2 \times n}$ be the space of real $(2 \times n)$ matrices $X$ satisfying $X_{1,n-1} = X_{2,n} = 1$ and $X_{1,n} = 0$. There is a natural projection $\Pi : Lo_{1}^{n} \to C$ taking $L$ to the submatrix formed by taking the last two rows: $(\Pi(L))_{i,j} = L_{i+n-2, j}$. Alternatively, $\Pi(L) = X_{0}L$ where $X_{0} = \Pi(id) \in C$ is the matrix whose only nonzero entries are $(X_{0})_{1,n-1} = (X_{0})_{2,n} = 1$. Equivalently, let $H_{0}, H_{1} \subset Lo_{1}^{n}$ be the subgroups defined by

$$H_{0} = \{ L \in Lo_{1}^{n} \mid \forall (i,j), 1 \leq j < i \leq n, i > n - 2 \to L_{i,j} = 0 \};$$

$$H_{1} = \{ L \in Lo_{1}^{n} \mid \forall (i,j), 1 \leq j < i \leq n - 2 \to L_{i,j} = 0 \}. $$

If $L_{0} \in H_{0}, L_{1} \in H_{1}$ and $L = L_{0}L_{1}$ then $L_{i,j} = (L_{0})_{i,j}$ if $i \leq n - 2$ and $L_{i,j} = (L_{1})_{i,j}$ if $i > n - 2$. Thus, any $L \in Lo_{1}^{n}$ can be uniquely written as a product $L = L_{0}L_{1}$ with $L_{0} \in H_{0}$ and $L_{1} \in H_{1}$. The restriction $\Pi|_{H_{1}} : H_{1} \to C$ is thus a bijection.

The space $C$ is naturally identified with $H_{0}\backslash Lo_{1}^{n}$, the set of right cosets of the form $H_{0}L$, $L \in Lo_{1}^{n}$; the map $\Pi$ is now the natural quotient map $Lo_{1}^{n} \to H_{0}\backslash Lo_{1}^{n}$.

Below we will treat $X \in C$ as an $n$-tuple of real column vectors: $X = (v_{1}, \ldots, v_{n})$, $v_{i} \in \mathbb{R}^{2}$. In other words, $C \subset (\mathbb{R}^{2})^{n}$ is the set of $n$-tuples $X = (v_{1}, \ldots, v_{n})$ satisfying

$$v_{n-1} = \begin{pmatrix} 1 \\ a \end{pmatrix}, \quad v_{n} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

for some $a \in \mathbb{R}$. Clearly, $m_{2}(L) = m_{2}(\Pi(L))$ for all $L \in Lo_{1}^{n}$; the map $m_{2}$ is thus well defined as $m_{2} : C \to \mathbb{R}$. For any set $Y = \{ i < j \} \subset \{ 1, 2, \ldots, n \}$ (with $|Y| = 2$) we define a matrix

$$\gamma_{Y} = \begin{pmatrix} 1 & \ldots & 0 & \ldots & 1 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \ldots & 1 & \ldots & 0 \\ 1 & \ldots & 0 & \ldots & 1 \end{pmatrix}.$$
Notice that the complement of \( C_2 \) is a union of finitely many submanifolds of codimension at least 2. Thus \( C_2 \) is path connected and generic flag-convex curves \( \Gamma_2 : I \to \mathcal{C} \) are of the form \( \Gamma_2 : I \to C_2 \). As we shall see, \( C_3 \) has exactly \( 2n-2 \cdot (n-1)! \) connected components, all contractible. Connected components of \( C_3 \) are labeled by signed cyclic words, as we proceed to explain.

Consider cyclic words \( w \) of length \( 2n \) in the alphabet \( 1, \ldots, n, 1', \ldots, n' \) such that \( w \) contains each letter exactly once. We say that such a word is admissible (or odd) if, for all \( i \), there are exactly \( n - 1 \) other letters between the letters \( i \) and \( i' \): let \( \mathfrak{W} \) denote the set of admissible words. For \( n = 3 \) we have

\[
\mathfrak{W} = \{123'1'2'3, 123'1'23, 1'23'123, 1'23'123, 123'2'1'3, 2'13'2'13, 2'13'2'13, 2'13'2'13\}.
\]

In general, we have \( |\mathfrak{W}| = 2n-1 \cdot (n-1)! \) (fix \( n \) and \( n' \); choose one among the \( (n-1)! \) permutations of \( \{1, \ldots, n-1\} \) to fill in the gap between \( n \) and \( n' \); for \( i \) from 1 to \( n-1 \) choose the positions of \( i \) and \( i' \).

Words in \( \mathfrak{W} \) should be imagined written along a circle, always counter-clockwise. Given \( i < j \), we say that we walk from \( i \) to \( j \) counter-clockwise in \( w \) if there are fewer than \( n-1 \) letters after \( i \) and before \( j \): in this case we write \( m_{i,j}(w) > 0 \) (otherwise \( m_{i,j}(w) < 0 \)). Equivalently, \( m_{i,j}(w) > 0 \) if and only if one encounters the triple \( i, \ldots, j, \ldots, i' \) when reading \( w \) (counter-clockwise); of course, \( m_{i,j}(w) < 0 \) if and only if one encounters instead the triple \( i, \ldots, j', \ldots, i' \). Thus, for instance, if \( n = 5 \) and \( w = 13'4'25'1'3'4'^25' \) then \( m_{1,2}(w) > 0, m_{1,3}(w) < 0, m_{1,4}(w) < 0, m_{1,5}(w) < 0, m_{2,3}(w) > 0, m_{2,4}(w) > 0, m_{2,5}(w) > 0, m_{3,4}(w) > 0, m_{3,5}(w) > 0 \) and \( m_{4,5}(w) > 0 \). Let \( \mathfrak{W}^+ \subset \mathfrak{W} \) be the set of admissible words \( w \) for which \( m_{(n-1,n)}(w) > 0 \); we have \( |\mathfrak{W}^+| = 2n-2 \cdot (n-1)! \).

We now show how to assing a word \( w(X) \in \mathfrak{W}^+ \) to each \( X \in \mathcal{C}_3 \). Given \( X = (v_1, \ldots, v_n) \), set \( \nu(X) = (\hat{v}_1, \ldots, \hat{v}_n) \in S^1^n \) where \( \hat{v}_i = v_i/|v_i| \in S^1 \). Let \( \Omega_n = \nu[C_3] \); in other words, \( \Omega_n \subset (S^1)^n \) is the set of configurations of \( n \) pairwise linear independent labeled points on \( S^1 \) such that point \( n \) is \((0,1)\) and point \( n - 1 \) has coordinates \((x, y)\) with \( x > 0 \). Given \( X = (v_1, \ldots, v_n) \), label the point \( \hat{v}_i \) by \( i \) and the point \( -\hat{v}_i \) by \( i' \). Finally, read the unit circle \( S^1 \) counter-clockwise, picking up the labels as you read, to obtain the desired word \( w(X) \). Notice that \( m_Y(X) > 0 \) if and only if \( m_Y(w(X)) > 0 \); in particular, \( w(X) \in \mathfrak{W}^+ \), as desired.

**Remark 2.4.** The above discussion implies that the cyclic word \( w(M) \) corresponding to any totally positive \( 2 \times n \)-matrix \( M \) coincides with (the cyclic word given by) \((1, 2, \ldots, n, 1', 2', \ldots, n')\). We will call this cyclic word **totally positive**. The cyclic word corresponding to a totally negative matrix is obtained from the totally positive word by interchanging its every even entry with its opposite and reading it backwards. We will call this cyclic word **totally negative**.

**Example 2.5.** The cyclic word \((123'1'2'3'4)\) is totally positive while the cyclic word \((43'2'1'4'32'1)\) is totally negative.

In all figures in the remaining part of this section the cyclic words should be read counter-clockwise along the circle.

Given \( w_0 \in \mathfrak{W}^+ \), let \( \mathcal{C}[w_0] \subset \mathcal{C}_3 \) be the set of matrices \( X \in \mathcal{C}_3 \) for which \( w(X) = w_0 \). Notice that \( \mathcal{C}_3 = \bigcup_{w \in \mathfrak{W}^+} \mathcal{C}[w] \); the following lemma shows that these subsets are all well behaved.
Lemma 2.6. Given \( w_0 \in \mathbb{W}^+ \), the set \( \mathcal{C}[w_0] \) is contractible (and nonempty).

Proof. We proceed by induction on \( n \); the cases \( n \leq 3 \) are easy. In this proof, we write \( \mathbb{W}_n^+ \) in order to avoid confusion.

Given a word \( w_0 \in \mathbb{W}_n^+ \), let \( w_1 \in \mathbb{W}_{n-1}^+ \) be obtained by removing 1 and 1’ from \( w_0 \) and then subtracting 1 from each remaining label (thus, for instance, if \( w_0 = 13'452'1'34'5'2 \) then \( w_1 = 2'34'1'23'4'1 \)). Similarly, given \( X_0 = (v_1, v_2, \ldots, v_n) \in \text{Lo}_1^{3\times(n-1)} \), we obtain \( X_1 = (v_2, \ldots, v_n) \in \text{Lo}_1^{3\times(n-1)} \) by removing the first column.

By induction hypothesis, \( \mathcal{C}[w_1] \) is contractible. Given \( X_1 \in \mathcal{C}[w_1] \), the set of vectors \( v_1 \in \mathbb{R}^2 \) which can be placed at the left of \( X_1 \) to obtain \( X_0 \in \mathcal{C}[w_0] \) is a convex cone. Thus \( \mathcal{C}[w_0] \) is also contractible, as desired.

Given a flag-convex curve \( \Gamma_2 : I \rightarrow \mathcal{C}_2 \), we are now interested in following the sequence of words \( w \) corresponding to the sets \( \mathcal{C}[w] \) traversed by \( \Gamma_2 \). We first present a combinatorial description. Let us now define admissible moves on the set of all admissible cyclic words. (Below \( \ldots \) stands for an arbitrary sequence of entries in an admissible word.)

Definition 2.7. Below, we denote by \( \alpha \) either the point \( j \) or its opposite point \( j' \). If \( \alpha = j \), then \( -\alpha = j' \), if \( \alpha = j' \) then \( -\alpha = j \). For every \( k > 2 \), the following two moves are called admissible:

1) clockwise rotation of point \((k-1)\) toward point \( k\)

\[ \ldots, k', \ldots, k-1, j, \ldots, k, \ldots, (k-1)', j', \ldots \rightarrow \ldots, k', \ldots, j, \ldots, k, \ldots, (k-1)', j', \ldots \]

2) counter-clockwise rotation of point \((k-1)\) toward point \( k\)

\[ \ldots, k, \ldots, j, k-1, \ldots, k', \ldots, j', (k-1)', \ldots \rightarrow \ldots, k, \ldots, j, k-1, \ldots, k', (k-1)', j', \ldots \]

The first admissible move describes the change of a cyclic word when the point \((k-1)\) rotates clockwise toward the point \( k \) and passes the position of the point \( j \) (or \( j' \)) while the second admissible move describes the similar change when the point \((k-1)\) rotates counterclockwise.

![Figure 1](image1.png)

**Figure 1.** The first admissible move. The point \((k-1)\) rotates counterclockwise towards the point \( k \).

![Figure 2](image2.png)

**Figure 2.** The second admissible move. The point \((k-1)\) rotates clockwise towards the point \( k \).

For any flag-convex curve \( \Gamma_2 : I \rightarrow \mathcal{C}_2 \) there are finitely many \( t \in I \) for which \( \Gamma_2(t) \notin \mathcal{C}_2 \).
Lemma 2.8. Given a flag-convex curve $\Gamma_2 : I \to C_2$, consider the sequence of words $w \in \mathcal{W}^+$ for which $\Gamma_2$ traverses $C[w]$. Then such sequence of words consists of admissible moves.

Proof. By Lemma 2.1, the tangent vector to the matrix curve $\rho F(t)$ at $t_0$ belongs to the cone spanned by the vectors $\rho F(t_0) \cdot I_j$, where $I_j$ is the matrix whose only nonzero entry is located at the position $(j + 1, j)$ and equals to 1. Note that the right multiplication of an arbitrary $n \times n$-matrix by $I_j$ acts as a column operation adding the $j + 1$st column to the $j$th column. Hence, the projection of the tangent vector $\nu_*(\rho F(t_0) \cdot I_j)$ is the tangent vector to $\Omega_n$ that infinitesimally moves the point labelled $j$ of the configuration corresponding to $\nu(\rho F(t_0))$ towards the point labelled $j + 1$ along the shortest of the two arcs of $\mathbb{S}^1$ connecting them. Since any infinitesimal motion of the point configuration induced by the osculating flag curve $\rho F(t)$ is represented as a positive linear combination of such infinitesimal elementary moves we can approximate the whole time evolution of the point configuration as a sequence of consecutive elementary moves described in Definition 2.7.

Example 2.9. Set

$$\Gamma(t) = L_0 \exp(tN), \quad L_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1/6 & 0 & 1 & 0 \\ 1/8 & 1/5 & 0 & 0 \end{pmatrix}, \quad N = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}. $$

The curve $\Gamma : \mathbb{R} \to L_0^1$ is flag-convex. A simple computation verifies that the flag-convex curve $\Gamma_2 : \mathbb{R} \to C$ is of the form $\Gamma_2 : \mathbb{R} \to C_2$. Indeed, the values of $t$ for which $m_Y(\Gamma_2(t)) = 0$ for some $Y$ are: $t_1 \approx -0.63$ (for $Y = \{2, 3\}$); $t_2 = 0$ (for $Y = \{2, 4\}$); $t_3 \approx 0.26$ (for $Y = \{1, 2\}$); $t_4 \approx 0.63$ (for $Y = \{2, 3\}$); $t_5 \approx 0.77$ (for $Y = \{1, 3\}$); $t_6 \approx 1.11$ (for $Y = \{1, 2\}$). The corresponding sequence of words is: $w_0 = 143'21'4'3'2'$, $w_1 = 1423'1'4'2'3'$, $w_2 = 31243'1'2'4'$, $w_3 = 32143'2'1'4'$, $w_4 = 23142'3'1'4'$, $w_5 = 21342'1'3'4'$, $w_6 = 12341'2'3'4'$; moves are admissible, as expected.

As discussed above, the set of admissible words labels the set of connected components of $C_3$ (or of $\Omega_n$). The connected components are separated by codimension 1 walls in $C_2 \smallsetminus C_3$. Admissible moves correspond to crossing walls between connected components following flag-convex curves (i.e., curves of the form $(\rho F(t))_{[n-1,n]}$ for some convex projective curve $\rho$).

Remark 2.10. We explained above that when a (locally) convex curve osc$\rho$ intersects the divisor $H_L$ where $L = \text{span}(e_1, \ldots, e_{n-2}) \subset \mathbb{R}^n$, the respective admissible configuration of labelled points on $\mathbb{S}^1$ is acted upon by an admissible move which either interchanges the relative order of the points labelled 1 and 2 or the points 1 and $2'$.

Definition 2.11. For an admissible cyclic word $w$ and any of its two distinct entries $a$ and $b$ (belonging to $\{1, \ldots, n, 1', \ldots, n'\}$), we denote by $[a, b]$ the shortest closed arc in $\mathbb{S}^1$ starting at the point labeled $a$ and ending at the point labeled $b$. In other words, of two possible arcs connecting $a$ and $b$, we choose the one whose length does not exceed half a turn (or $(n-1)$ positions in the word $w$).

We call an arc $[a, b]$ (increasing) decreasing if $a < b$ and $b$ is obtained from $a$ by a rotation in the (counter-)clockwise direction. Given $i < j$, we say that an admissible
cyclic word $w$ contains a monotone subsequence $[i \ldots j] = [i, i + 1, i + 2, \ldots, j]$ if, for $i \leq k \leq j - 1$, all of the arcs $[k, k + 1]$ are either simultaneously increasing or simultaneously decreasing. A monotone sequence $[i \ldots j]$ can be interpreted as an immersed arc $I = [i, j] \to S^1$, taking $i \in I$ to the point of $S^1$ labeled $i$ and taking $j \in I$ to the point labeled $j$; notice that, unlike the subarcs $[k, k + 1]$, such an immersed arc can be (much) longer than a half-turn. The content of an immersed arc in $S^1$ is the number of complete half-turns contained in the arc; we denote the content of a monotone sequence $[i \ldots j]$ by $\text{Cont}([i \ldots j])$.

We call a monotone subsequence $[i \ldots j]$ maximal if neither $[i - 1 \ldots j]$ for $i > 1$ nor $[i \ldots j + 1]$ for $j < n$ is monotone. An admissible word $w$ can be interpreted as the concatenation of its maximal monotone arcs $[1 \ldots k_1], [k_1 \ldots k_2], \ldots, [k_{s-1} \ldots n]$; here $k_0 = 1, k_s = n$ and $s$ is the total number of maximal monotone subsequences in $w$. The total content $\text{Cont}(w)$ of an admissible word $w$ is $\text{Cont}(w) = \text{Cont}([1 \ldots k_1]) + \cdots + \text{Cont}([k_{s-1} \ldots n])$, the sum of the contents of all maximal monotone subsequences of $w$. We define $\text{rk}(w)$, the rank of the admissible word $w$, by

$$\text{rk}(w) = 2 \text{Cont}(w) + s - 1.$$ 

For any matrix $X \in C_3$, we define its rank $\text{rk}(X) = \text{rk}(w(X))$.

**Example 2.12.** Consider the following cyclic words:

1. for $w_1 = 123451'2'3'4'5'$, we get $s = 1, \text{Cont}(w) = 0$, and $\text{rk}(w) = 0$;
2. for $w_2 = 15'4'3'2'1'5'4'3'2'$, we get $s = 1, \text{Cont}(w) = 3$, and $\text{rk}(w) = 6$;
3. in $w_3 = 145231'4'5'2'3'$ there are $s = 3$ maximal monotone subsequences: $[1, 2, 3], [3, 4]$ and $[4, 5]$. Hence, $\text{Cont}(w) = 0$, and $\text{rk}(w) = 2$;
4. for $w_4 = 415234'1'5'2'3'$, one gets $s = 3, \text{Cont}(w) = 0$ and $\text{rk}(w) = 2$.

**Remark 2.13.** The word $w_1$ is totally positive, while $w_2$ is totally negative. The word $w_3$ is obtained from $w_3$ by one admissible move which shifts 1 closer to 2; $w_3$ can be obtained from $w_4$ by an admissible move which shifts 4 closer to 5. Observe that $\text{rk}(w_3) = \text{rk}(w_4)$.

**Lemma 2.14.** Fix $n$ so that admissible words $w \in \mathcal{W}^+$ have length $2n$. For the totally positive word $w_+ \in \mathcal{W}^+$ we have $\text{rk}(w_+) = 0$. For the totally negative word $w_- \in \mathcal{W}^+$ we have $\text{rk}(w_-) = 2(n - 2)$. For any other admissible cyclic permutation $w \in \mathcal{W}^+ \setminus \{w_+, w_-\}$ we have $0 < \text{rk}(w) < 2(n - 2)$. Furthermore, $m_{\{1,2\}}(w) > 0$ if and only if $\text{rk}(w)$ is even.

**Proof.** The proof is by induction on $n$; the cases $n \leq 3$ are easy.

Given $w_0 \in \mathcal{W}^+_n$, let $w_1 \in \mathcal{W}^+_{n-1}$ be obtained by removing 1 and 1' and by decreasing by 1 the remaining labels (as in the proof of Lemma 2.6). Let $[1 \ldots k_1]$ be the first maximal monotonic subarc for $w_0$. Notice that $k_1 = 2$ if and only if $m_{\{1,2\}}(w_0) \neq m_{\{2,3\}}(w_0)$. On the other hand, if $m_{\{1,2\}}(w_0) = m_{\{2,3\}}(w_0)$ then the first maximal monotonic arc for $w_1$ is $[1 \ldots (k_1 - 1)]$. Let $c_0 = \text{Cont}([1 \ldots k_1])$ and $c_1 = \text{Cont}([2 \ldots k_1])$ (both for $w_0$). Notice that $c_1 = \text{Cont}([1 \ldots (k_1 - 1)])$ for $w_1$. We have either $c_0 = c_1$ (if, for $w_0$, the points labeled $k_1$ and $k_1'$ are both outside the arc $[1, 2]$) or $c_0 = c_1 + 1$ (otherwise). We thus have

$$\text{rk}(w_0) = \text{rk}(w_1) + \begin{cases} 0, & m_{\{1,2\}}(w_0) = m_{\{2,3\}}(w_0), c_0 = c_1, \\ 1, & m_{\{1,2\}}(w_0) \neq m_{\{2,3\}}(w_0), \\ 2, & m_{\{1,2\}}(w_0) = m_{\{2,3\}}(w_0), c_0 = c_1 + 1; \end{cases}$$
this provides us with the desired induction step. □

The next statement is the most important technical step in our proof of Theorem 1. The proof is simple but a little long, and is done case by case; it is presented in the series of ten figures shown below.

**Proposition 2.15.** Consider $w_0, w_1 \in \mathfrak{W}_n^+$. Assume that an admissible move takes $w_0$ to $w_1$: then $\text{rk}(w_1) \leq \text{rk}(w_0)$. Furthermore, if $m_{\{1,2\}}(w_1) \neq m_{\{1,2\}}(w_0)$ then $\text{rk}(w_1) < \text{rk}(w_0)$.

Notice that the last claim follows from the first claim together with the parity remark in Lemma 2.14.

**Proof.** Below we present all possible types of elementary moves and, for each of them, we analyze what happens with the rank of $w$. Observe that during the evolution of the point configuration in $(S^1)^n$ following some curve $(\rho_{x^t})_{t=[n-1,n]}(t)$ the rank of configuration does not change until two or more configuration points collide. We consider admissible moves and list all cases when a moving point $i$ collides with one of the remaining points $j$ or $j'$.

Detailed consideration of all possible cases led us to their subdivision into the following types of collisions. (This subdivision is an artifact of our proof).

- **Type Ia**: the moving point 1 collides with the point $k$, $k > 2$;
- **Type Ib**: the moving point 1 collides with the point $k'$, $k > 2$;
- **Type IIa**: the moving point $i$, $i > 1$ collides with the point 1;
- **Type IIb**: the moving point $i$, $i > 1$ collides with the point $1'$;
- **Type IIIa**: the moving point $i$ collides with $j$ when both $i, j > 1, j \neq i - 1$;
  - If $j = i - 1$ the case needs to be subdivided into two subcases by the location of point $i - 1$ in one of the following two intervals:
    - **Type IIIb**: the moving point $i$ collides with $j = i - 1, i > 1$ and the point $i - 2$ belongs to the shortest arc between $(i - 1)'$ and $i$;
    - **Type IIIc**: the moving point $i$ collides with $j = i - 1, i > 1$ and the point $i - 2$ belongs to the shortest arc between $i - 1$ and $i''$.
- **Type IVa**: the moving point $i$ collides with $j', i, j > 1, j \neq i - 1$;
  - If $j = i - 1$ then case needs to be subdivided into two subcases also by the location of point $i - 2$ in one of the following two intervals:
    - **Type IVb**: the moving point $i$ collides with $j = i - 1, i > 1$ and the point $i - 2$ belongs to the shortest arc between $(i - 1)$ and $i$;
    - **Type IVc**: the moving point $i$ collides with $j = i - 1, i > 1$ and the point $i - 2$ belongs to the shortest arc between $(i - 1)'$ and $i'$.

The above types exhaust all possible situations of collision and we discuss below what happens with the rank under these collisions.
Figure 3. Elementary admissible move of type Ia with $k > 2$.

The move changes the relative order of points 1 and $k$ in cyclic word $w$. Rank $\text{rk}(w)$ does not change.

Figure 4. Elementary admissible move of type Ib.

The move changes the relative order of points 1 and $k'$ in word $w$. If the maximal element $k_1$ of the first maximal monotone subsequence $[1, 2, \ldots, k_1]$ is different from $k$, then $\text{rk}(w)$ does not change. If $k_1 = k$, then $\text{rk}(w)$ decreases by 1.

Figure 5. Elementary admissible move of type IIa.

If $i > 2$ and the first monotone subsequence is $\text{seq}_1 = [1, 2, \ldots, k_1]$ with $k_1 \neq i$, then $\text{rk}(w)$ does not change. If $i > 2$ and $k_1 = i$, then $\text{seq}_i$ is increasing (since otherwise, $k_1 \geq i + 1$) and $\text{Cont}(w)$ decreases by 1, hence $\text{rk}(w)$ drops by 2. Finally, if $i = 2$, then $\text{rk}(w)$ drops by 1.

Figure 6. Elementary admissible move of type IIb.
As in type III, if $i > 2$ and the first monotone subsequence is $\text{seq}_1 = [1, 2, \ldots, k_1]$ with $k_1 \neq i$, then $\text{rk}(w)$ does not change. If $i > 2$ and $k_1 = i$, then $\text{seq}_1$ is increasing (since otherwise, $k_1 \geq i + 1$) and $\text{rk}(w)$ drops by 2. Finally, if $i = 2$, then $\text{rk}(w)$ drops by 1.

**Figure 7.** Elementary admissible move of type IIIa with $i, j > 1$ and $j \notin \{i, i+1\}$.

In this case $\text{rk}(w)$ can only change if either $i < j$ and there exists an increasing maximal monotone subsequence $[i, i+1, \ldots, j]$ or if $i > j$ and there exists a decreasing maximal monotone subsequence $[j, j+1, \ldots, i]$. In both cases, $\text{Cont}(w)$ decreases by 1 and $\text{rk}(w)$ decreases by 2. The remaining situation $j = i - 1$ is considered in detail below.

We split the case $j = i - 1$ in Figure 7 into several subcases according to the relative position of $i - 2$. The point $i - 2$ can be located either in the interval $>((i-1)', i)$ or in $(i-1, i')$, as below.

**Figure 8.** Elementary admissible move of type IIIb.

The point $i - 2$ belongs to $>((i-1)', i)$. $\text{Cont}(w)$ does not change. Both the number of maximal monotone subsequences and $\text{rk}(w)$ decrease by 2.

**Figure 9.** Elementary admissible move of type IIIc.

The point $i - 2$ belongs to $(i - 1, i')$. The rank $\text{rk}(w)$ does not change. Finally,
Here $i, j > 1$, and $j$ is not in $\{i, i+1\}$. If $j \neq i-1$, then $\text{rk}(w)$ does not change unless either $i < j$ and there exists a maximal decreasing subsequence $[i, i+1, \ldots, j]$ or $i > j$ and there exists a maximal increasing subsequence $[j, j+1, \ldots, i]$. In both cases $\text{Cont}(w)$ decreases by 1 and $\text{rk}(w)$ decreases by 2. The remaining situation $j = i-1$ is considered in detail below.

As above, we split the case $j = i-1$ in the last figure into several subcases according to the relative position of $i-2$. The point $i-2$ can be located either in the interval $((i-1), i)$ or in the interval $((i-1)', i')$, as below.

The point $i-2$ belongs to $(i-1, i)$. $\text{Cont}(w)$ and the number $s$ of maximal monotone subsequences do not change. Hence, $\text{rk}(w)$ does not change either.

The point $i-2$ belongs to $((i-1)', i')$. Either $\text{rk}(w)$ does not change or it decreases by 2.

We have analyzed all the possible types of admissible elementary moves and concluded that, for each admissible move which changes the sign of $m_{\{1,2\}}$, $\text{rk}(w)$ decreases.

Recall the unipotent matrix $N$ in Example 2.9; notice that

$$(\exp(tN))_{i,j} = \begin{cases} 0, & i < j; \\ t^{i-j}/(i-j)!, & i \geq j. \end{cases}$$

**Lemma 2.16.** For any $n \times n$-matrix $G \in \text{Lo}_n^1$,
(1) there exists $t_+ > 0$ such that $G \exp(t N)$ is totally positive for any $t > t_+$;
(2) there exists $t_- < 0$ such that $G \exp(t N)$ is totally negative for any $t < t_-$.

Proof. Write $T(t) = \exp(tN)$. Note that $G_{ij} = 1$ if $i = j$ and $G_{ij} = 0$ if $i < j$. Hence, $$GT(t))_{ij} = T_{ij}(t) + \text{lower order terms of in } t.$$ Any minor $m_T$ of $T(t)$ equals $m_T(t) = at^d$ for some positive $a$ and $d$. The corresponding minor $m_G$ of $G \cdot T(t)$ equals $m_G(t) = at^d + p_{m,G}(t)$, where $p_{m,G}$ is a polynomial of degree strictly less than $d$. Hence, for $t$ such that $|t| \gg 0$ the sign of $m_G(t)$ coincides with that of $m_T(t)$. It remains to notice that $T(t)$ is totally positive for positive $t$ and totally negative for negative $t$ and Lemma follows.

Corollary 2.17. Let $\gamma : I \to \mathbb{RP}^{n-1}$ be a globally convex curve and $\gamma_F : I \to \text{Lo}_1^n$ be its osculating flag curve (considered in the appropriate open Schubert cell identified with $\text{Lo}_1^n$). Then, $\gamma$ can be extended to a globally convex $\tilde{\gamma} : [a, b] \to \mathbb{RP}^{n-1}$, $I \subseteq [a, b]$ such that $\gamma_F : [a, b] \to \text{Lo}_1^n$, $\tilde{\gamma}_F(a) \in \text{Neg}$, $\tilde{\gamma}_F(b) \in \text{Pos}$.

Proof. Take $I = [s, f]$ and set $b = f + t_+ (\gamma_F(f))$, $a = s + t_- (\gamma_F(s))$. Define
$$\tilde{\gamma}_F(t) := \begin{cases} \gamma_F(t), & \text{for } t \in I; \\ \gamma_F(f) \cdot T(t - f), & \text{for } t \in [f, b]; \\ \gamma_F(s) \cdot T(t - s), & \text{for } t \in [a, s], \end{cases}$$
where $T(t)$ is defined in Lemma 2.16. We define $\tilde{\gamma}(t)$ by taking the first column of the matrix $\tilde{\gamma}_F(t)$. Lemma 2.1 implies that the curve $\tilde{\gamma}(t)$ is globally convex.

Finally, by definition of $t_-$ and $t_+$, one has that $\tilde{\gamma}_F(a) \in \text{Neg}$, $\tilde{\gamma}_F(b) \in \text{Pos}$. □

Proof of Theorem 1. We prove the statement for a convex curve $\rho : I \to \mathbb{RP}^{n-1}$. Then, the statement for $S^1$ follows by taking the limit. By Corollary 2.17, without loss of generality, we can assume that $\rho : [a, b] \to \mathbb{RP}^{n-1}$ is a globally convex curve such that $A = \gamma_F(a) \in \text{Neg}$, $B = \gamma_F(b) \in \text{Pos}$.

Take the pair of $n \times 2$-matrices $\hat{A}$ and $\hat{B}$ formed by the two first columns of a totally negative $(n \times n)$-matrix $A \in \text{Lo}_1^n$ and a totally positive $(n \times n)$-matrix $B \in \text{Lo}_1^n$, respectively. Then osc$_{2\rho}$ has $\hat{A}$ as its starting point and $\hat{B}$ as its terminal point. Clearly, $A_{[n-1,n]} \in \text{Neg}_2$, $B_{[n-1,n]} \in \text{Pos}_2$, and the curve $(\rho_F)_{[n-1,n]}$ connects $A_{[n-1,n]}$ and $B_{[n-1,n]}$. Note that $\text{rk}(A_{[n-1,n]}) = 2n - 2$ and $\text{rk}(B_{[n-1,n]}) = 0$.

By Proposition 2.15, the rank of the point configuration does not change when the positions of all configuration points remain distinct: It does increase when two different configuration points collide, and the rank decreases at least by 1 when the points 1 and 2 (or the points 1 and 2') collide. In other words, this happens when the projections of the first and the second osculating vectors to the plane spanned by the last two basis vectors become collinear which exactly corresponds to the vanishing of the minor $m_{\{1,2\}}$. Since the total change of the rank does not exceed $2(n - 2)$, the number of times the projections of the first two osculating vectors become collinear does not exceed $2(n - 2)$ and therefore osc$_{2\rho}$ intersects $\{m_{\{1,2\}} = 0\}$ at most $2(n - 2)$ times which settles the theorem. □
3. Proof of Theorem 2

We start by introducing a special class of matrix curves. Namely, given a pair of matrices \((N_0, L_0)\), where \(N_0\) is a nilpotent lower triangular matrix with positive subdiagonal entries and zero entries elsewhere, and \(L_0 \in L_{0}^{1}\), define the curve \(\Gamma_{N_0, L_0} : \mathbb{R} \to L_{0}^{1}\) as given by

\[
\Gamma_{N_0, L_0}(t) := L_0 \exp(tN_0).
\]

One can easily see that \(\Gamma_{N_0, L_0}\) is flag-convex and, for \(i > j\), its entry \((i,j)\) is a polynomial of degree \(i - j\). We call such flag-convex curves \textit{polynomial}. (They are closely related to the fundamental solutions of the simplest differential equation \(y^{(n)} = 0\).)

For a polynomial curve \(\Gamma_{N_0, L_0}\), the function \(m_k(t)\) is indeed a real polynomial of degree \(k(n - k)\) in \(t\). So for polynomial curves, Conjecture 1.2 trivially holds.

In this section we will first prove two essential preliminary results, namely Theorems 3 and 4. Theorem 3 shows that there exist polynomial flag-convex curves which are non-transversal to the reference flag at \((n_0 + 1)^{m_0}\). Theorem 3 shows that there exist polynomial flag-convex curves \textit{real} and \textit{simple}. Furthermore, \(L_0\) can be taken so that all such roots are distinct, implying that there are totally exactly \((n^3 - n)\) distinct roots, all real and distinct.

To settle Theorem 3 we need more notation. As in [6] (see especially Sections 2 and 7), let \(S_n\) be the symmetric group with generators \(a_i = (i i + 1)\). The symmetric group is endowed with the usual Bruhat order. The top permutation (or the Coxeter element) of \(S_n\) is denoted by \(\eta\) (another common notation is \(w_0\)). For a permutation \(\sigma \in S_n\), define its \textit{multiplicity vector} with coordinates \(\text{mult}_k(\sigma) = (1^\sigma + \cdots + k^\sigma) - (1 + \cdots + k)\); thus, \(\text{mult}_k(\eta) = k(n - k)\). If \(\sigma_0 < \sigma_1 = \sigma_0 a_j = (i_0 i_1) \sigma_0\), then

\[
\text{mult}_k(\sigma_1) = \begin{cases} 
\text{mult}_k(\sigma_0) + 1, & i_0 \leq k < i_1, \\
\text{mult}_k(\sigma_0), & \text{otherwise}; 
\end{cases}
\]

this is Lemma 2.4 in [6].

For \(\rho \in S_n\), the permutation matrix \(P_\rho\) has nonzero entries in positions \((i, i')\) so that \(c_i^\top P_\rho = c_{i'}^\top\). Apply the Bruhat factorization to decompose \(L_{0}^{1}\) as a disjoint union of subsets \(\text{Bru}_\rho\), \(\rho \in S_n\). More precisely, for \(L \in L_{0}^{1}\), write \(L \in \text{Bru}_\rho\) if and only if there exist upper triangular matrices \(U_1\) and \(U_2\) such that \(L = U_1 P_\rho U_2\). In particular, \(\text{Bru}_1 = \{I\}\) and \(\text{Bru}_\eta\) is open and dense. If \(\Gamma : I \to L_{0}^{1}\) is smooth and flag-convex, \(\Gamma(0) \in \text{Bru}_\rho\) and \(\sigma = \eta \rho\) then \(t = 0\) is a root of multiplicity \(\text{mult}_k(\sigma)\) of \(m_k(t) = 0\); this is Theorem 4 in [6].

Recall that \(f_j\) is the matrix whose only nonzero entry equals 1 in position \((j + 1, j)\). Let \(\lambda_j(t) = \exp(t f_j) \in L_{0}^{1}\) so that \(\lambda_j(t)\) has an entry equal to \(t\) in position \((j + 1, j)\);
the remaining entries equal 1 (on the main diagonal) and 0 (elsewhere). If \( \rho_0 \lhd \rho_1 = \rho_0 a_j, L_0 \in \text{Bru}_{\rho_0} \) and \( t \neq 0 \), then \( L_1 = L_0 \lambda_j(t) \in \text{Bru}_{\rho_1} \) (see Section 5 in [6]).

Consider \( N_0 \) arbitrary but fixed, as in the statement of Theorem 3. Given \( L_0 \in L_{01}^1 \), construct the curve \( \Gamma_{N_0, L_0}(t) := L_0 \exp(tN_0) \) and the real polynomials \( m_k(t) \in \mathbb{R}[t] \) as above. We say that a matrix \( L_0 \) is \( \rho \)-good if and only if \( L_0 \in \text{Bru}_\rho \subset L_{01}^1 \) and, for all \( k \), all nonzero roots of \( m_k \) are real and simple. Notice that \( \text{Id} \) is (vacuously) \( \epsilon \)-good.

**Lemma 3.1.** Consider \( \rho_0, \rho_1 \in S_n \), \( \rho_0 \lhd \rho_1 = \rho_0 a_j \). Let \( L_0 \in \text{Bru}_{\rho_0} \subset L_{01}^1 \) be a \( \rho_0 \)-good matrix. Then there exists \( \epsilon > 0 \) such that, for all \( t \in \mathbb{R} \) satisfying the restriction \( 0 < |t| < \epsilon \) one has that \( L_1 = L_0 \lambda_j(t) \) is \( \rho_1 \)-good.

**Proof.** As above, we have \( L_1 \in \text{Bru}_{\rho_1} \). For \( t \) near 0, nonzero real simple roots of \( m_k \) remain nonzero, real and simple.

Let \( \sigma_0 = \eta \rho_0 \) and \( \sigma_1 = \eta \rho_1 \) so that \( \sigma_1 \lhd \sigma_0 = \sigma_1 a_j = (i_0 i_1) \sigma_1 \). As above, \( \text{mult}_k(\sigma_1) = \text{mult}_k(\sigma_0) - 1 \) for \( i_0 < k < i_1 \) and \( \text{mult}_k(\sigma_1) = \text{mult}_k(\sigma_0) \) otherwise. Originally (i.e. for \( L_0 \)) the root \( t = 0 \) has multiplicity \( \text{mult}_k(\sigma_0) \); after perturbation (i.e. for \( L_1 \)) it has multiplicity \( \text{mult}_k(\sigma_1) \). Thus, for \( k < i_0 \) or \( k > i_1 \) new root is born and we are done. For \( i_0 \leq k < i_1 \) exactly one new root is born: it must therefore be real and, for small \( |t| \), simple. \( \square \)

**Lemma 3.2.** For all \( \rho \in S_n \) there exist \( \rho \)-good matrices.

**Proof.** Consider a reduced word \( \rho = a_{i_1} \cdots a_{i_l} \) where \( l = \text{inv}(\rho) \) is the number of inversions of \( \rho \). For \( k \leq l \), define \( \rho_k = a_{i_1} \cdots a_{i_k} \); in particular, \( \rho_0 = \epsilon \) and \( \rho_1 = \rho \). As mentioned above, \( I \) is \( \rho_0 \)-good. Apply Lemma 3.1 to deduce that if there exists a \( \rho_k \)-good matrix then there exists a \( \rho_{k+1} \)-good matrix. The result follows by induction. \( \square \)

**Proof of Theorem 3.** By Lemma 3.2, there exists an \( \eta \)-good matrix \( L_0 \). The roots of every polynomial \( m_k(t) \) are real and simple. The same holds for any \( \tilde{L}_0 \in A \) where \( A \) is some sufficiently small open neighborhood of \( L_0 \). It suffices to show that for some such \( \tilde{L}_0 \) all roots are distinct.

Let \( \rho \in S_n \) be different from \( \eta \) and \( \eta a_i \), \( 1 \leq i < n \). Then \( \text{Bru}_\rho \subset L_{01}^1 \) is a submanifold of codimension at least 2. Define

\[
X_\rho := \{ L \exp(tN_0); L \in \text{Bru}_\rho, t \in \mathbb{R} \}, \quad Y = L_{01}^1 \setminus \bigcup_{\rho \in S_n \setminus \{\eta, \eta a_1, \ldots, \eta a_{n-1}\}} X_\rho;
\]

each set \( X_\rho \) has measure zero. The set \( Y \) has total measure and is therefore dense. Take \( \tilde{L}_0 \in A \cap Y \). We claim that all roots of the polynomials \( m_k \) are real, simple and distinct, as desired.

Indeed, assume by contradiction that \( m_{k_1}(t_0) = m_{k_2}(t_0) \), \( k_1 < k_2 \). Take \( \rho \in S_n \) such that \( \Gamma_{N_0, L_0}(t_0) = \tilde{L}_0 \exp(t_0 N_0) \in \text{Bru}_\rho \); set \( \sigma = \eta \rho \). We have that \( \text{mult}_{k_1}(\sigma) \geq 1 \) and \( \text{mult}_{k_2}(\sigma) \geq 1 \) whence \( \sigma \notin \{\epsilon, a_1, \ldots, a_{n-1}\} \) and therefore \( \rho \in S_n \setminus \{\eta, \eta a_1, \ldots, \eta a_{n-1}\} \). Thus \( \tilde{L}_0 = \Gamma_{N_0, L_0}(t_0) \exp(-t_0 N_0) \in X_\rho \) and therefore \( \tilde{L}_0 \notin Y \), a contradiction. \( \square \)

**Example 3.3.** For \( n = 5 \), let \( N_0 \) be the matrix with subdiagonal entries equal to 1. Write \( \eta = a_1 a_2 a_1 a_3 a_4 a_2 a_1 a_3 a_2 = abacdbabc \), an arbitrary reduced word. The matrices

\[
\lambda_1(1), \quad \lambda_1(1)\lambda_2(-1), \quad \lambda_1(1)\lambda_2(-1)\lambda_1(1)
\]
are easily seen to be $a_-$, $ab$- and $aba$-good, respectively (signs are chosen in an arbitrary manner). If we thus proceed from left to right, at each step taking a number of sufficiently small absolute value, we obtain the following example of an $(qb)$-good matrix:

$$L_0 = \lambda_1(1)\lambda_2(-1)\lambda_1(1)\lambda_3\left(\frac{1}{8}\right)\lambda_4(-\frac{1}{8})\lambda_3\left(\frac{1}{64}\right)\lambda_2\left(\frac{1}{512}\right)\lambda_1(-\frac{1}{512})\lambda_3\left(-\frac{1}{4096}\right).$$

For $\Gamma_{N_0,L_0}(t) = L_0 \exp(tN_0)$, all roots of the polynomials $m_k$ are real, simple and distinct.

Now we formulate and prove a more general result equivalent to Theorem 2.

**Theorem 4.** Consider a smooth flag-convex curve $\Gamma_0 : I \to \mathbb{L}^1$ (where $I \subset \mathbb{R}$ is a non-degenerate interval). Then there exist $I_1 \subset I$ and $L_1 \in \mathbb{L}^1$ such that, for $\Gamma_1(t) = L_1\Gamma_0(t)$ and $m_k = m_{1;k}$, the following properties hold:

1. all roots of each $m_k$ in $I_1$ are simple and belong to the interior of $I_1$;
2. roots are distinct: if $k_1 \neq k_2$ then $m_{k_1}$ and $m_{k_2}$ have no common roots;
3. for each $k$, the function $m_k$ admits precisely $k(n-k)$ roots in $I_1$. 

In Theorem 4, assume without loss of generality that 0 is an interior point of $I$ and that $\Gamma_0(0) = Id$. Take $I_1 \subset I$, $I_1$ compact, 0 in the interior of $I_1$ such that $t = 0$ is the only root of $m_{\Gamma_0,k}$ in $I_1$: recall that this root has multiplicity $k(n-k)$. Given $L_1 \in \mathbb{L}^1_\alpha$, set $\Gamma_1(t) = L_1\Gamma_0(t)$; write $m_{L_1;k} = m_{1;k}$. Thus, if $L_1 \in \text{Bru}_\rho$ and $\sigma = \eta \rho_0$ then $t = 0$ is a root of multiplicity $\text{mult}_k(\sigma)$ of $m_{L_1;k}$. A matrix $L_1$ is $\rho$-good (for $\Gamma_0$ and $I_1$ fixed) if and only if $L_1 \in \text{Bru}_\rho \subset \mathbb{L}^1_\alpha$ and, for all $k$, the function $m_{L_1;k}$ admits precisely $k(n-k) - \text{mult}_k(\sigma)$ nonzero roots in $I_1$, all in the interior of $I_1$ and all simple. Notice that $Id$ is $e$-good.

**Lemma 3.4.** Consider $\rho_0, \rho_1 \in S_n$, $\rho_0 \circ \rho_1 = \rho_0 a_j$. Let $L_0 \in \text{Bru}_{\rho_0} \subset \mathbb{L}^1_\alpha$ be a $\rho_0$-good matrix. Then there exists $\epsilon > 0$ such that, for all $\tau \in \mathbb{R}$, if $0 < |\tau| < \epsilon$, then $L_\tau = L_0\lambda_3(\tau)$ is $\rho_1$-good.

**Proof.** Let $\sigma_\tau = \eta \rho_0$ and $\sigma_1 = \eta \rho_1$ so that $\sigma_1 \circ \sigma_\tau = \sigma_1 a_j = (i_0 i_1) \sigma_1$. As above, we have $L_\tau \in \text{Bru}_{\rho_1}$ for $\tau \neq 0$. Write $\mu_k = \text{mult}_k(\sigma_0)$. As above, $\text{mult}_k(\sigma_1) = \mu_k - 1$ for $i_0 \leq k < i_1$, and $\text{mult}_k(\sigma_1) = \mu_k$ otherwise.

For $\tau$ near 0, the $k(n-k) - \text{mult}_k(\sigma_0)$ nonzero simple roots of $m_{L_{\tau,k}}$ remain nonzero, simple and in the interior of $I_1$. By compactness, for small $|\tau|$, there are no new roots away from a small neighborhood of $t = 0$.

The root $t = 0$ has multiplicity $\mu_k$ for $m_{L_{0;k}}$. Let $s_k \in \{\pm 1\}$ be the sign of $m_{L_{0;k}}^{(s_k)}(0) \neq 0$ so that $s_k m_{L_{0;k}}^{(s_k)}(t) > 0$ in a small neighborhood $I_0 \subset I_1$ of $t = 0$. For small $|\tau|$, we likewise have $s_k m_{L_{\tau;k}}^{(s_k)}(t) > 0$ in $I_0$. For $\tau \neq 0$, the root $t = 0$ has multiplicity $\text{mult}_k(\sigma_1)$. For $k < i_0$ or $k \geq i_1$, we have $\text{mult}_k(\sigma_1) = \mu_k$, and therefore $t = 0$ is the only root in $I_0$ and we are done. For $i_0 \leq k < i_1$ we have $\text{mult}_k(\sigma_1) = \mu_k - 1$. The signs of $m_k$ at the extrema of $I_0$ together with the sign of $m_k^{(s_k)}$ and the multiplicity of the zero at $t = 0$ imply that, for small $|\tau|$, there is exactly one new nonzero root of $m_{L_{\tau;k}}$ in $I_0$; this root is simple, as desired. 

**Lemma 3.5.** Consider $\Gamma_0 : I \to \mathbb{L}^1$ and $I_1 \subseteq I$ fixed, as above. For all $\rho \in S_n$ there exist $\rho$-good matrices.

**Proof.** Consider a reduced word $\rho = a_{i_1} \cdots a_{i_l}$ where $l = \text{inv}(\rho)$ is the number of inversions of $\rho$. For $k \leq l$ define $\rho_k = a_{i_1} \cdots a_{i_k}$; in particular, $\rho_0 = e$ and $\rho_1 = \rho$. As
remarked, $Id$ is $\rho_0$-good. Apply Lemma 3.4 to deduce that if there exists a $\rho_k$-good matrix, then there exists a $\rho_{k+1}$-good matrix. The result follows by induction. □

Proof of Theorem 4. For $L \in L_0^1$, set $\Gamma_L(t) = L\Gamma_0(t)$ and $m_{L:0} = m_{\Gamma_L:0}$. By Lemma 3.2, there exists an $\eta$-good matrix $L_0$. Each function $m_{L_0:0}$ has exactly $k(n - k)$ roots in $I_1$, all in the interior of $I_1$ and all simple. The same holds for the functions $m_{L_0:0}$ for any $L_0 \in A$ where $A$ is some sufficiently small open neighborhood of $L_0$. It suffices to show that, for some such $\hat{L}_0$, all roots are distinct.

Consider the quotient map $L_0^1 \to \mathbb{R}$ taking $L$ to $L_{2,1} + L_{3,2} + \cdots + L_{n,n-1}$; pre-images of points form a family of parallel hyperplanes. Let $S \subset A$, $L_0 \in S$, be a convex neighborhood of $L_0$ in its hyperplane; notice that $S$ is transversal to $\Gamma_{L_0}(0)$. The function $\Phi : S \times I_1 \to L_0^1$ defined by $\Phi(L,t) = \Gamma_L(t)$ is a tubular neighborhood of the image $\Gamma_{L_0}[I_1]$.

Let $\rho \in S_n$ be different from $\eta$ and $\eta a_i$, $1 \leq i < n$. Then $\text{Bru}_\rho \subset L_0^1$ is a submanifold of codimension at least 2, and therefore so is $\Phi^{-1}[\text{Bru}_\rho] \subset S \times I_1$. Let $X_\rho \subset S$ be its image under the projection onto $S$: the subset $X_\rho \subset S$ has measure zero. Let

$$Y = S \setminus \bigcup_{\rho \in S_n \setminus \{\eta, \eta a_1, \ldots, \eta a_{n-1}\}} X_\rho :$$

the subset $Y \subset S$ has total measure and is therefore dense. Notice that since $Y \subset S \subset A$, if $\hat{L}_0 \in Y$, then the function $m_{L_0:0}$ has precisely $k(n - k)$ roots in $I_1$, all simple and all in the interior of $I_1$. We claim that in this case all roots of the functions $m_{L_0:0}$ are also distinct, as desired.

Indeed, assume by contradiction that $m_{k_1}(t_0) = m_{k_2}(t_0)$, $k_1 < k_2$. Take $\rho \in S_n$ such that $\Gamma_{L_0}(t_0) = \Phi(L_0,t_0) \in \text{Bru}_\rho$; set $\sigma = \rho \eta$. We have $\text{mult}_{k_1}(\sigma) \geq 1$ and $\text{mult}_{k_2}(\sigma) \geq 1$ whence $\sigma \notin \{e, a_1, \ldots, a_{n-1}\}$; thus, $\rho \in S_n \setminus \{\eta, \eta a_1, \ldots, \eta a_{n-1}\}$. Thus $(L_0,t_0) \in \Phi^{-1}[\text{Bru}_\rho]$ and therefore $\hat{L}_0 \in X_\rho$ and therefore $\hat{L}_0 \not\in Y$, a contradiction. □

Remark 3.6. For the curve $\Gamma_1$ and the interval $I_1 = [t_-, t_+]$ constructed in the proof above we have $\Gamma_1(t_+) \in \text{Pos}$ and $\Gamma_1(t_-) \in \text{Neg}$ (this is easily proved by induction on $\rho$). There exist therefore no real roots of $m_k(\Gamma_1(t))$ for $t \notin I_1$ and the number of real roots of $m_k(\Gamma_1(t))$ is therefore exactly equal to $k(n - k)$.

Notice that this remark does not imply some kind of global estimate on the number of real roots of $m_k(\Gamma_1(t))$ for some other flag-convex curve $\Gamma_\ast$. No such estimate is clear.

We finish the paper with the following tantalizing question.

*Is it possible to extend the above approach from the case of $G_{2,n}$ to other Grassmannians?*

Acknowledgements. The first author wants to acknowledge support of CNPq, CAPES and Faperj (Brazil) and to express his sincere gratitude to the Department of Mathematics, Stockholm University for the hospitality in November 2018. The second author wants to acknowledge the financial support of his research provided by the Swedish Research Council grant 2016-04416. The third author is supported by the NSF grant DMS-1702115.
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