A 1-qubit gate is defined as an arbitrary element of the group $SU(2)$. The problem of finding good approximations of arbitrary 1-qubit gates is identical to that of finding a dense group generated by a universal subset of $SU(2)$ to approximate an arbitrary element of $SU(2)$ within specified accuracy $\varepsilon > 0$. In this note we study a quantitative description of this theorem in the following sense. We will work with a universal gate set $T$, a subset of $SU(2)$ such that the group generated by the elements of $T$ is dense in $SU(2)$. For $\varepsilon > 0$ small enough, we define $t_\varepsilon$ as the minimum reduced word length such that every point of $SU(2)$ lies within a ball of radius $\varepsilon$ centered at the points in the dense subgroup generated by $T$. For a measure of efficiency (covering exponent) on $T$, which we denote $K(T)$, we prove the following theorem: Fix a $\delta$ in the interval $[0, \frac{2}{3}]$. Choose an $f : (0, \infty) \rightarrow (1, \infty)$ satisfying $\lim_{\varepsilon \rightarrow 0^+} \frac{\log(f(t_\varepsilon))}{t_\varepsilon}$ exists with value 0. Assume that the inequality $\varepsilon \leq f(t_\varepsilon) \cdot 5^{t_\varepsilon/6 - 3\delta}$ holds. Our conjecture implies the following: Let $\nu(5^{t_\varepsilon})$ denote the set of integer solutions of the quadratic form: $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 5^{t_\varepsilon}$. Let $M \equiv M_3(N)$ denote the covering radius of the points $N = \nu(5^{t_\varepsilon}) \cup \nu(5^{t_\varepsilon-1})$ on the sphere $S^3$ in $\mathbb{R}^4$. Then $M \sim f(\log N)N^{\frac{1}{3t_\varepsilon}}$. Here $N \equiv N(\varepsilon) = 6 \cdot 5^{t_\varepsilon/6 - 2}$.

Keywords: Quantum gate, Quantum Algorithm, Solovay-Kitaev Theorem, Discrepancy

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1 Introduction

The problem of finding an efficient universal gate set has been studied in detail and in many forms for decades now [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13]. A quantum gate is mathematically described by an appropriate unitary matrix that acts linearly on a quantum state to produce a new state. An arbitrary 1-qubit quantum gate is an element of $SU(2)$, and acts on a state that is a linear combination of two basis states. A universal gate set is one that generates a
group dense in $SU(2)$. A symmetric gate set contains the inverse of all its elements. Much of the difficulty arises in determining the asymptotic behavior of the distribution of points generated from the universal gate set. The Solovay-Kitaev Theorem, stated below, shows that a symmetric universal gate set can approximate any arbitrary gate in $SU(2)$ with a finite sequence. However, the theorem does not determine the maximum possible efficiency of the approximation, though it does provide an approach for forming the sequence $\gamma$. 

**Theorem 1 [Solovay-Kitaev]:** Let $\varepsilon > 0$ and a symmetric, universal gate set of $SU(2)$ be specified. Then there exists a constant $c$ such that for all $X \in SU(2)$, there exists $\gamma$, a finite sequence of gates from the universal gate set of length $O\left(\log^c \left(\frac{1}{\varepsilon}\right)\right)$ approximating $X$ within $\varepsilon$ error. Typically, we take $c = 2.71$ and bringing down this value of $c$ to 1 has an interesting history, see [11] for a nice survey.

It is clear from the statement of the theorem that the length of the approximating sequence depends on the specified fault-tolerance, $\varepsilon$, as well as an undetermined constant, $c$. It turns out that $c$ depends both on the choice of universal gate set as well as the algorithm being used to determine the approximating sequence, $\gamma$, as will be discussed shortly. A more robust measure of the efficiency of a gate set is the covering exponent $K$, defined below and discussed in more detail in [3, 12, 15].

**Definition:** Let $\varepsilon > 0$ be small enough and a universal gate set $T \subset SU(2)$ be specified. Let $\Omega = \langle T \rangle$. Then $t_\varepsilon$ is the smallest integer $t$ such that $V(t)$, the set of all elements in $\Omega$ of length $t$ or less, covers $SU(2)$. Let $\mu$ be a normalized Haar measure on $SU(2)$ and $B(\varepsilon)$ be an arbitrary ball of radius $\varepsilon$. Then we define the covering exponent as

$$K(T) \equiv \limsup_{\varepsilon \to 0} \frac{\log |V(t_\varepsilon)|}{\log \left(\frac{1}{\mu(B(\varepsilon))}\right)}.$$  

The covering exponent can be understood as related to the volume of empty spherical caps on $S^3$, the sphere in $\mathbb{R}^4$, connected to the gates of $SU(2)$ by taking advantage of the diffeomorphism between $SU(2)$ and $S^3$. Specifically, the volume $V$ is such that the covering exponent is given by $V^{-\frac{1}{K(T)}}$. This provides a quantitative measure of discrepancy that is independent of the approximation algorithm.

2 Main Results

2.1 Statement of Theorem

We focus on using the symmetric, universal gate set $T$ defined below and used in [2,3]. First, consider

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad iX = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad iY = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad iZ = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$

(2)

elements of $SU(2)$. We form the elements

$$s_1 = \frac{1}{\sqrt{5}}(I + 2iX), \quad s_2 = \frac{1}{\sqrt{5}}(I + 2iY), \quad s_3 = \frac{1}{\sqrt{5}}(I + 2iZ),$$

(3)
also of $SU(2)$ and let $T \subset SU(2)$ such that
\[
T = \{ s_1, s_2, s_3, s_1^{-1}, s_2^{-1}, s_3^{-1}, iX, iY, iZ \}. \tag{4}
\]
This is similar to the set $S$ used in [12] as well as the $V$ basis discussed in [2]. From here on, $T$ refers to this specific set. In order to prove the main theorem, we assume the following conjecture.

**Conjecture 1:** Let $\varepsilon > 0$ be small enough and write $t_\varepsilon = \min \left( G \subset \bigcup_{\gamma \in V(t)} B(\gamma, \varepsilon) \right)$. We conjecture that for some $\delta \in [0, \frac{2}{3}]$,
\[
\varepsilon \leq f(t_\varepsilon) \cdot 5^{\frac{i\varepsilon}{\pi}}. \tag{5}
\]
where $f : (0, \infty) \to (1, \infty)$ satisfies $\lim_{\varepsilon \to 0^+} \frac{\log(f(t_\varepsilon))}{t_\varepsilon}$ exists with value 0.

The main theorem is stated below.

**Theorem 2:** Let $\varepsilon > 0$ be small enough and fix $\delta \in [0, 2/3]$. Then assuming Conjecture 1 for this $\delta$, $K(T) \leq 2 - \delta$ holds.

### 2.2 Remarks and Discussion

#### 2.2.1 Discrepancy Conjectures

An equivalent description of Conjecture 1 can be stated in terms of the mesh norm of a certain point set generated by $T$. Given $\varepsilon > 0$ small enough and corresponding $t_\varepsilon$, consider the quadratic forms
\[
x_1^2 + x_2^2 + x_3^2 + x_4^2 = 5^{t_\varepsilon} \tag{6}
\]
and
\[
x_1^2 + x_2^2 + x_3^2 + x_4^2 = 5^{t_\varepsilon-1} \tag{7}
\]
where $x_i \in \mathbb{Z}$. We denote the union of the sets of solutions to (6) and (7) as $\mathcal{N}$. We denote the projection of $\mathcal{N}$ onto the unit sphere $S^3$ as $\hat{\mathcal{N}} = \{ \frac{x}{\sqrt{|x|}} \mid x \in \mathcal{N} \}$. Then for some $\delta \in [0, 2/3]$, $S^3$ can be covered by balls centered at each $x \in \hat{\mathcal{N}}$ with radius $\varepsilon$: Here, $\varepsilon \leq f(t_\varepsilon) \cdot 5^{\frac{i\varepsilon}{\pi}}$ for a $f : (0, \infty) \to (1, \infty)$ with
\[
\lim_{t_\varepsilon \to \infty} \frac{\log(f(t_\varepsilon))}{t_\varepsilon}
\]
exists with value 0 (or equivalently: $f : (0, \infty) \to (1, \infty)$ satisfies $\lim_{\varepsilon \to 0^+} \frac{\log(f(t_\varepsilon))}{t_\varepsilon}$ exists with value 0). That is,
\[
S^3 \subset \bigcup_{x \in \hat{\mathcal{N}}} B_G \left( \frac{x}{\sqrt{|x|}}, \varepsilon \right).
\]
Now let $M \equiv M_{S^3}(\hat{\mathcal{N}})$ denote the mesh norm (covering radius) of the points in $\hat{\mathcal{N}}$:
\[
M_{S^3}(\hat{\mathcal{N}}) = \max_{y \in S^3} \min_{x \in \hat{\mathcal{N}}} |y - x|. \tag{8}
\]
We now have an equivalent conjecture to Conjecture 1, namely:

**Conjecture 2:** Let \( \varepsilon \) be small enough. Then we conjecture that for some \( \delta \in [0, 2/3] \),

\[
M_{S^3}(\hat{N}) \sim f(\log N) N^{\frac{1}{2\delta}}
\]

for a function \( f : (0, \infty) \rightarrow (1, \infty) \) satisfying \( \lim_{\varepsilon \rightarrow 0^+} \frac{\log(f(t\varepsilon))}{t\varepsilon} \) exists with value 0 and \( N \equiv N(\varepsilon) = 6 \cdot 5^{t\varepsilon} - 2 \). Here, for real valued functions \( f, g, f(x) \sim g(x) \) if \( \frac{|f(x)|}{|g(x)|} \) is bounded above and below by positive constants uniformly in \( x \).

It is known, see [14] that \( M_{S^3}(\hat{N}) \geq N(\frac{1}{2} + o(1)) \) and in [2] it is conjectured that \( M_{S^3}(\hat{N}) \geq N(\frac{1}{4} + o(1)) \).

The points of \( \hat{N} \) for two different values of \( t\varepsilon \) are shown mapped to \( S^2 \) in Figures 1 and 2. Coincidentally, the computational complexity of projecting these points onto the unit sphere is discussed in [7] along with the computational complexity of the strong approximation problem. This complexity largely stems from the increasing difficulty of finding all solutions as well as using previously constructed elements to navigate to new solutions.

**Remark:** While the conjectures, theorem, and proofs are stated in terms of \( T \), we believe that the efficient properties of \( T \) are universally true for golden-gates and super-golden-gates, such as those discussed in [7]. These universal gate sets can be viewed as universal sets in the quaternion group, a similarity not lost on the authors. This is suggestive of more common properties of these sets than just the navigational properties of golden-gates. Though we
cannot prove it, we believe that for any golden-gate set $A$, it is the case that $K(A) = 2 - \delta$ for one specific value of $\delta \in [2, 2/3]$.

2.2.2 Discussion of Theorem

For the asymptotic behavior of the covering exponent of balls of volume $V$ we have $V^{-\frac{1}{2+\delta}} = V^{-\frac{1}{2}}$. Recall the constant $c$ given in the Solovay-Kitaev Theorem. The relation of $K$ to $c$ is dependent upon the algorithm used to find an approximation. For the current algorithm and a common Clifford+T gate set, discussed in detail in [4], $c$ is different depending on whether the matrix to be approximated is diagonal or not. The algorithm is optimal for diagonal unitary matrices, but requires that non-diagonal matrices be diagonalized. While this is certainly possible for any unitary matrix, it causes the approximation to be three times longer than is optimal for a diagonal matrix. Thus, for diagonal matrices, $c = 1$, optimally, while for arbitrary matrices and optimal approximation, one has $c = 3$. On the other hand, as Sarnak has shown, not all arbitrary elements of $SU(2)$ can be approximated optimally. The best case scenario occurs wherein $K = \frac{4}{3}$ gives $c = 4$, while the worst case scenario would be $K = 2$ and $c = 6$. Thus, the relationship between $K$ and $c$ for the current best algorithm is $c = 3 \cdot K$. Further discussion can be found in [7].

Remark: If one chooses that $\delta = \frac{2}{3}$, the resulting theorem is that $K(T) \leq \frac{4}{3}$, which is proven in [3] as a special case of Conjecture 1. Further, let $S = T \setminus \{iX, iY, iZ\}$. Sarnak shows in [12] that $\frac{4}{3} \leq K(S) \leq 2$, whereas Damelin, Liang, and Mode show in [3] that $\frac{4}{3} \leq K(T) \leq K(S) \leq 2$. It follows from this that the $K(T) \leq 2 - \delta$ construction is applicable to $S$, though of course, it is possible that $\delta$ is different for $S$ than $T$. 

Fig. 2. The points for $t_\varepsilon = 4$
Remark: In [15], Sardari discusses the quantum approximation problem in terms of strong approximation for quadratic forms of different numbers of variables. Let the number of variables in a quadratic form be \( d \). Then the covering exponent for the set of integral solutions to a quadratic form on \( d \) variables projected onto the sphere \( S^d \) is \( \frac{d}{3} \) if \( d > 4 \). On the other hand, for the case \( d = 4 \), it is known, and discussed throughout this article, that the covering exponent, \( K \), is bounded above by 2 and below by \( \frac{4}{3} \), i.e. \( \frac{4}{3} \leq K \leq 2 \) for any quadratic form on 4 variables. His technique uses an adaptation of the Hardy-Littlewood circle method, commonly used in proofs in analytic number theory, known as the Kloosterman circle method.

The covering exponent problem for quadratic forms on \( d \) variables becomes more difficult as the \( d \) becomes smaller. Ultimately, the results derived by Sardari are essentially the same as those discussed in [12], though the techniques used are markedly different. It is interesting to compare the mesh norm of integral lattice points on \( S^3 \) to other low discrepancy points such as minimal energy points. It is known that the asymptotic covering radius for certain minimal energy point configurations on \( S^3 \) is of the order \( N^{-\frac{1}{3}} \), where \( N \) is the number of points [16]. Generating such minimal energy points from a finite set of generators would optimally solve the quantum approximation problem. It is well known that random configurations for various distributions on the sphere also produce coverings with smaller holes and it is conjectured in [2] that lattice type points as studied here are locally random for certain distributions.

3 Proof of Main Result

Proof of Theorem 2: Let \( \varepsilon > 0 \) and all \( \log \) be \( \log_5 \), for convenience only. Assume Conjecture 1 for a fixed \( \delta \). Then,

\[
K(T) = \limsup_{\varepsilon \to 0} \frac{\log[V_T(t_\varepsilon)]}{\log \left( \frac{1}{\mu(B_G(\varepsilon))} \right)}
= \limsup_{\varepsilon \to 0} \frac{\log[V_T(t_\varepsilon)]}{\log \left( \frac{1}{\varepsilon^3} \right)}
= \lim_{\varepsilon \to 0} \frac{\log(6 \cdot 5^{t_\varepsilon} - 2)}{3 \log \left( \frac{1}{\varepsilon} \right)}
\leq \lim_{\varepsilon \to 0} \frac{t_\varepsilon + \log \left( \frac{6}{5^{t_\varepsilon}} \right)}{3 \log \left( \frac{1}{f(t_\varepsilon) \cdot 5^{\varepsilon - \frac{1}{3}}} \right)}
= \lim_{\varepsilon \to 0} \frac{t_\varepsilon + \log \left( \frac{6}{5^{t_\varepsilon}} \right)}{6 - 3\delta t_\varepsilon - 3 \log(f(t_\varepsilon))}
= \lim_{\varepsilon \to 0} \left[ \frac{3}{6 - 3\delta} - \frac{1}{3 \log f(t_\varepsilon)} \cdot \frac{t_\varepsilon}{t_\varepsilon} + \frac{\log(6 - \frac{2}{5^{t_\varepsilon}})}{6 - 3\delta t_\varepsilon - 3 \log(f(t_\varepsilon))} \right].
\]
Since \( t_\varepsilon \to \infty \) as \( \varepsilon \to 0 \), we have that

\[
K(T) \leq \lim_{t_\varepsilon \to \infty} \left[ \frac{1}{6 - 3\delta} - \frac{3 \log(f(t_\varepsilon))}{t_\varepsilon} \right] + \frac{\log\left(\frac{6 - 2}{5t_\varepsilon}\right)}{6 - 3\delta} - \frac{3 \log(f(t_\varepsilon))}{t_\varepsilon}
\]

\[
= \frac{1}{3} + 0
\]

\[
= 2 - \delta.
\]

Therefore,

\[
K(T) \leq 2 - \delta. \quad (11)
\]

\[
(12)
\]

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