Hilbert Spaces for Nonrelativistic and Relativistic “Free” Plektons (Particles with Braid Group Statistics)*

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Dedicated to Gianfausto Dell’Antonio on the occasion of his 60th birthday

Abstract

Using the theory of fibre bundles, we provide several equivalent intrinsic descriptions for the Hilbert spaces of n “free” nonrelativistic and relativistic plektons in two space dimensions. These spaces carry a ray representation of the Galilei group and a unitary representation of the Poincaré group respectively. In the relativistic case we also discuss the situation where the braid group is replaced by the ribbon braid group.

1 Introduction

The last years have seen a rising interest in the theory of particles in space time dimensions two and three with strange statistics. The possibility for this was first discovered by Leinaas and Myrheim [LM], who realized that the braid group has to replace the permutation group. Models for particles with a one-dimensional representation of the braid group where first discussed by F.Wilczek [Wi], who coined the name anyons for particles with these new statistics (see

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also \([3MS]\). In the general case, where the finite dimensional irreducible representation of the braid group is not one-dimensional, one speaks of plektons \([FRS]\).

Unfortunately so far (non) relativistic fields describing “free” plektons and having suitable localization properties have not yet been constructed. However, plektonic structures have been discovered and analyzed within the context of algebraic quantum field theory and one has an understanding of the associated Haag - Ruelle scattering theory \([FM2,FGR]\).

It is the aim of this article to provide a direct intrinsic construction of the Hilbert spaces of “free” nonrelativistic and relativistic plektons in two space dimensions. These Hilbert spaces will carry a unitary ray representation of the Galilei group and a unitary representation of the universal covering group of the Poincaré group respectively. We will employ the theory of vector bundles.

In the \(n\) particle case the base space will be given as the set of all momenta of \(n\) indistinguishable particles. The universal covering space will be a principal bundle with the braid group \(B_n\) as structure group. For any finite dimensional unitary representation of \(B_n\) there is an associated bundle. Given such a representation, by definition the Hilbert space for \(n\) plektons is the space of square integrable sections in this vector bundle.

Also these bundles will be shown to be homogeneous bundles with respect to the homogeneous Galilei and Poincaré group respectively. This will be the key ingredient for constructing the unitary (ray) representations.

If one starts with an \(AFD (=approximately\ finite\ dimensional)\) representation \(\varrho_\infty\) of the infinite braid group \(B_\infty\) (see e.g. \([WF]\)), then \(\varrho_\infty\) gives rise to a finite dimensional representation \(\varrho_n\) of \(B_n\) for each \(n\) and this in turn defines Hilbert spaces \(\mathcal{H}_n\) via the above construction. The resulting Hilbert space

\[
\mathcal{H} = C \oplus \bigoplus_{n \geq 1} \mathcal{H}_n
\]

then might serve as a substitute for the bosonic or fermionic Fock spaces. It remains to construct “free” fields which transform correctly under the Galilei group and Poincaré group respectively and which in the relativistic case have suitable localization properties. This problem is still open and the difficulties associated to this program will also become apparent within our set-up. In fact, as suggested by the work of Buchholz and Fredenhagen, these relativistic fields should be localized in space-like cones \([BF]\). As is well known in the ordinary case of spin \(\frac{1}{2}\) particles, in the momentum formulation of the one particle theory one has to go from a spin basis to a spinor basis as a necessary step in obtaining anticommuting local spinor fields. At the moment it is unclear to us what the corresponding procedure should be in two space dimensions, where the spin is not quantized. It is known that in case the spin is not integer or half integer the fields necessarily have to obey braid group statistics \([FM1,F]\). However, there is the possibility that the problem may be tackled if instead of the braid
group one considers the ribbon braid group. This corresponds to the situation where each plekton carries an additional degree of freedom given by a point on the circle.

The article is organized as follows. In section 2 we present the Hilbert space construction and provide several equivalent formulations. In the case of anyons the associated bundles are known to be trivial \[D\]c, but we provide a new constructive proof. In the nonrelativistic case these line bundles are also trivial when considered as homogeneous bundles with respect to the homogeneous Galilei group. In the relativistic case, however, these bundles are not trivial when viewed as homogeneous bundles with respect to the Lorentz group, unless the representation of \(B_n\) is the trivial one. The proof will be given in appendix B.

In section 3 and 4 we construct the unitary (ray) representations of the Galilei and Poincaré groups respectively. In section 5 we give the ribbon braid construction in the relativistic case and formulate the relativistic covariance.

This work is based in part on previously unpublished remarks \[S\] and a diploma thesis \[Mu\].

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## 2 The Plektonic Hilbert Spaces

Let \(M\) be a smooth manifold of dimension \(\geq 2\) which is connected and simply connected, i.e. both \(\pi_0(M)\) and \(\pi_1(M)\) are trivial. In the \(n\) fold product \(M^{\times n}\) let \(D_n\) be the set of points \(m = (m_1, \ldots, m_n)\) with \(m_i = m_j\) for at least one pair \((m_i, m_j)\) with \(i \neq j\). Let \(S_n\) be the permutation group of \(n\) elements. \(S_n\) obviously acts as a transformation group on \(M^{\times n}\) (on the right), leaving \(D_n\) invariant:

\[(m\pi)_i = m_{\pi(i)}, \pi \in S_n, m \in M^{\times n}.
\]

We introduce the space

\[nM = (M^{\times n} \setminus D_n) / S_n.
\]

As is well known \(\pi_1(nM) \cong S_n\) such that \(M^{\times n} \setminus D_n\) is the universal covering space of \(nM\) except when \(\text{dim}M = 2\). Thus the interesting cases arise when \(M \cong \mathbb{R}^2\) or \(M \cong S^2\) with \(\pi_1(nM)\) being called the corresponding braid groups for \(n\) elements (see e.g. [1]). For the case \(M = \mathbb{R}^2\) we will write this group as \(B_n\), following standard notation.
Example (2.2) In the case $M = \mathbb{R}^2$ we will view $M$ as the momentum space of a nonrelativistic particle in two space dimensions with points being denoted by $p = (p_1, p_2)$.

Example (2.3) In the case $M = V^{+,m} = \{ p = (p^0, p^1, p^2) \in \mathbb{R}^3, \ p^0 = \left( (p^1)^2 + (p^2)^2 + m^2 \right)^{1/2} \} \cong \mathbb{R}^2$, we will view $M$ as the energy-momentum space (the forward mass shell) of a relativistic free particle of mass $m > 0$.

The spaces $^{n}R^2$ and $^{n}V^{+,m}$ will be viewed as the momentum space for $n$ identical nonrelativistic and relativistic free particles, respectively. Points in these spaces will be denoted by $p$. Let $^{n}\tilde{R}^2$ and $^{n}\tilde{V}^{+,m}$ be their universal covering spaces, with points being denoted by $\tilde{p}$. Thus we have the principal $B_n$ bundles

$$
\begin{align*}
B_n & \rightarrow \quad ^{n}\tilde{R}^2 \\
\tilde{pr} \downarrow & , \\
^{n}R^2 & \quad \quad \quad ,
\end{align*}
\begin{align*}
B_n & \rightarrow \quad ^{n}\tilde{V}^{+,m} \\
\tilde{pr} \downarrow & , \\
^{n}V^{+,m} & \quad \quad \quad ,
\end{align*}
$$

which are of course diffeomorphic.

Let $F$ be a finite dimensional Hilbert space and $\varrho : b \mapsto \varrho(b)$ ($b \in B_n$) a unitary representation of $B_n$ in $F$ (not necessarily irreducible). This define associated hermitian vector bundles $^{n}\tilde{R}^2 \times_{\varrho,B_n} F$ and $^{n}\tilde{V}^{+,m} \times_{\varrho,B_n} F$. In order not to burden the notation, we will simply write $\tilde{F}$ for these hermitean vector bundles, since $n, F$ and $\varrho$ will be fixed in what follows and since it will be clear from the context whether we are dealing with the nonrelativistic or the relativistic case. Similarly, $M$ will from now on stand for $\mathbb{R}^2$ or $V^{+,m}$.

By definition, $\tilde{F}$ is the set of orbits in $^{n}\tilde{M} \times F$ under the following right action of $B_n$ on this space: $b : (\tilde{p}, f) \mapsto (\tilde{p} \cdot b, \varrho(b^{-1})f)$ ($\tilde{p} \in ^{n}\tilde{M}, f \in F, b \in B_n$). We denote by

$$
\tilde{\chi} : ^{n}\tilde{M} \times F \rightarrow \tilde{F}
$$

the canonical projection, which by definition is the map associating to each point $(\tilde{p}, f)$ the orbit on which it lies.

The following well known lemma is the main motivation for our ansatz of a quantum mechanical description of particles with braid group statistics.

Lemma (2.4) There is a one-to-one linear correspondence between $C^\infty$ sections $\xi$ in $\tilde{F}$ and $C^\infty$ functions $\psi$ on $^{n}\tilde{M}$ with values in $F$ which obey the equivariance relation

$$
\psi(\tilde{p}) = \varrho(b) \psi(\tilde{p} \cdot b)
$$

(2.5)

for all $\tilde{p} \in ^{n}\tilde{M}$ and $b \in B_n$. 
We briefly recall this correspondence. Given \( \tilde{\xi}, \psi \), \( \psi = \psi_{\tilde{\xi}} \) is defined as follows. For \( \tilde{p} \in \tilde{n}M \) let \( p = \tilde{\text{pr}}(\tilde{p}) \in nM \) denote the corresponding base point. Then there is a unique \( f \in F \) with \( \tilde{\chi}(p) = \tilde{\chi}(\tilde{p}, f) \) and we set \( \psi_{\tilde{\xi}}(\tilde{p}) = f \). Conversely, given \( \psi \), the corresponding \( \tilde{\xi} = \tilde{\xi}_\psi \) is given by \( \tilde{\xi}_\psi(\tilde{p}) = \tilde{\chi}(\tilde{p}, \psi(\tilde{p})) \) for any \( \tilde{p} \) with \( \tilde{\text{pr}}(\tilde{p}) = p \). These correspondences are easily seen to be well defined and inverse to each other, and satisfying (2.5). By going to local trivializations, it follows that \( \xi_\psi \) is smooth if \( \psi \) is and \( \psi_{\tilde{\xi}} \) is smooth if \( \xi \) is.

Furthermore, let \( \langle \cdot, \cdot \rangle_{\tilde{p}} \) be the canonical scalar product on the fibre in \( \tilde{F} \) over \( p \), i.e.
\[
\langle \tilde{\chi}(\tilde{p}, f), \tilde{\chi}(\tilde{p}, f') \rangle_{\tilde{\text{pr}}(\tilde{p})} = \langle f, f' \rangle
\]
for all \( \tilde{p} \) with \( p = \tilde{\text{pr}}(\tilde{p}) \). Let \( d\mu(p) \) denote the canonical volume form on \( nM \) inherited from Lebesgue measure on \( \mathbb{R}^2 \) if \( M = \mathbb{R}^2 \) and from the Lorentz invariant measure \( d\mu(p) = \frac{1}{2} \left( p_1^2 + (p_2)^2 + m^2 \right)^{-\frac{1}{2}} dp_1 dp_2 \) on \( V^+m \) if \( M = V^+m \). By \( L^2(\tilde{F}) \) we denote the Hilbert space completion of the space of smooth sections of \( \tilde{F} \) having finite norm with respect to the scalar product
\[
\langle \tilde{\xi}^1(p), \tilde{\xi}^2(p) \rangle_{\tilde{p}} = \int_{nM} \langle \tilde{\xi}(p), \tilde{\xi}'(p) \rangle_{\tilde{p}} d\mu(p).
\]
Similarly, let \( L^2_{eq}(nM, F) \) denote the Hilbert space completion of the space of smooth functions \( \psi \) from \( nM \) into \( F \) satisfying (2.5) and having finite norm with respect to the scalar product
\[
\langle \psi, \psi' \rangle = \int_{nM} \langle \psi(p), \psi'(p) \rangle d\mu(p).
\]
Note that by (2.7) the integrand on the r.h.s. is a function of \( p = \tilde{\text{pr}}(\tilde{p}) \) only, such that the definition (2.9) of the scalar product makes sense.

Also by (2.7) the above map \( \tilde{\xi} \mapsto \psi_{\tilde{\xi}} \) extends to a unitary map \( V \) from \( L^2_{eq}(nM, F) \) onto \( L^2_{eq}(nM, F) \).

**Postulate (2.10)** The quantum mechanical Hilbert space for \( n \) “free” plektons and for a given unitary representation \( \varrho \) of \( B_n \) on \( F \) is given by the space \( L^2(\tilde{F}) \) or alternatively by \( L^2_{eq}(nM, F) \). In other words, we view a square integrable section \( \tilde{\xi} \in L^2(\tilde{F}) \) as the mathematical formulation of a wave function over the space of \( n \) indistinguishable momenta. Also in terms of the equivalent description by the function \( \psi_{\tilde{\xi}} = V\tilde{\xi} \), condition (2.3) replaces the familiar condition where the wave functions are required to be symmetric (Bose-Einstein statistics) or antisymmetric (Fermi-Dirac
statistics). In fact, we may recover these cases as special cases in the present context as follows.

Let $S_{b\text{os}}$ denote the trivial representation of $S_n$ and $S_{a\text{ter}}$ the alternating (one-dimensional) representation $\pi \mapsto \text{sgn}(\pi)$ of $S_n$. Also let $\tau : B_n \to S_n$ be the canonical homomorphism with kernel being the pure braid group $PB_n$. Then we have one-dimensional representations $\rho_{b\text{os}}(b) = S_{b\text{os}}(\tau(b))$ and $\rho_{a\text{ter}}(b) = S_{a\text{ter}}(\tau(b))$ of $B_n$. Functions $\psi$ on $^n M$ satisfying (2.5) descend to functions on $M^{\times n} \setminus D_n$, since $^n M$ is a principal $PB_n$ bundle over this space (see below). The resulting functions $\psi_{b\text{os}}$ and $\psi_{a\text{ter}}$ satisfy the relations

$$
\psi_{b\text{os}}(p) = S_{b\text{os}}(\pi) \psi_{b\text{os}}(p\pi) \quad \text{and} \quad \psi_{a\text{ter}}(p) = S_{a\text{ter}}(\pi) \psi_{a\text{ter}}(p\pi)
$$

for $p \in M^{\times n} \setminus D_n$, $\pi \in S_n$.

We turn to an alternative description of a Hilbert space for $n$ “free” plektons. Recall that $M$ is $\mathbb{R}^2$ or $V_{+,m}$. Consider the space $M^{\times n} \setminus D_n$ which is a regular covering space of $^n M$

$$
M^{\times n} \setminus D_n \overset{\text{pr}}{\longrightarrow} ^n M.
$$

Its universal covering space with projection $\tilde{\text{pr}}$ may be identified with $\tilde{^n M}$ such that we have a commutative diagram

$$
\begin{array}{ccc}
^n M & \xrightarrow{\tilde{\text{pr}}} & \tilde{^n M} \\
\downarrow \text{pr} & & \downarrow \tilde{\text{pr}} \\
M^{\times n} \setminus D_n & \overset{\text{pr}}{\longrightarrow} & ^n M
\end{array}
$$

More precisely, $\tilde{^n M}$ is a principal $PB_n$ bundle over $M^{\times n} \setminus D_n$, and $M^{\times n} \setminus D_n$ is an $S_n$ bundle over $^n M$.

Given the representation $\rho$ of $B_n$, we may form the hermitean vector bundle $\tilde{\mathcal{F}} = \tilde{^n M} \times_{\tilde{\rho},PB_n} F$ over $M^{\times n} \setminus D_n$, whose projection we also denote by $\tilde{\text{pr}}$. The map $\tilde{\chi} : \tilde{^n M} \times F \to \tilde{\mathcal{F}}$ is defined in analogy to $\tilde{\chi}$. The map $\text{Pr} : \tilde{\mathcal{F}} \to \tilde{\mathcal{F}}$ is defined by associating to each $PB_n$ orbit in $\tilde{^n M} \times F$ its $B_n$ orbit. Thus $\text{Pr}$ is a vector bundle map lifting $\text{pr}$:

$$
\begin{array}{ccc}
\tilde{\mathcal{F}} & \xrightarrow{\text{Pr}} & \tilde{\mathcal{F}} \\
\downarrow \tilde{\text{pr}} & & \downarrow \tilde{\text{pr}} \\
M^{\times n} \setminus D_n & \overset{\text{pr}}{\longrightarrow} & ^n M
\end{array}
$$

Here we have again denoted by $\tilde{\text{pr}}$ and $\tilde{\text{pr}}$ the canonical projections induced by the projections in the corresponding principal bundles.
Since by construction the fibres in \( \tilde{F} \) are isomorphic to those in \( \check{F} \), there exists a linear pullback \( \Pr^* \) on the space of \( C^\infty \) sections

\[
\Pr^* : \Gamma(\check{F}) \to \Gamma(\tilde{F}).
\]

By construction, \( \frac{1}{n!} \Pr^* \) is isometric with respect to the scalar product (2.3). We want to characterize its image. The following result is well known in the theory of covering spaces. Its relevance in the present context was first observed by F.Nill, extending a previous remark in [3].

We claim that \( \tilde{F} \) is a homogeneous \( S_n \) bundle, i.e. the action of \( S_n \) on \( M \times n \setminus D_n \) lifts to a (right) action on \( \tilde{F} \) which we will write as \( \approx \pi S(\pi) \approx \bar{\chi}(\tilde{p}, f) = \tilde{\chi}(\tilde{p} \cdot b(\pi), \varphi(b(\pi))^{-1} f) \). (2.14)

Since \( \approx PB_n \) is a normal subgroup of \( B_n \), this definition is easily seen to make sense and to be independent of the particular choice of the family \( \{b(\pi)\}_{\pi \in S_n} \).

Also the diagram (2.13) is commutative, since obviously \( \tilde{\varphi} S(\pi) \approx \chi \approx (\tilde{p}, f) = \approx p \pi \). This defines a unitary representation of \( S_n \) on \( \Gamma(\tilde{F}) \) via

\[
\left( U(\pi) \approx \xi \right) (p) = S(\pi)^{-1} \approx \xi (p \pi). \quad (2.15)
\]

Let \( \Gamma_{\text{inv}}(\tilde{F}) \) be the linear subspace of \( \Gamma(\tilde{F}) \) consisting of all sections \( \approx \xi \) satisfying

\[
\approx \xi = U(\pi) \approx \xi. \quad (2.16)
\]

This compares with the special situation described in [2.11]. Define the linear operator \( P \) on \( \Gamma(\tilde{F}) \) by

\[
P \approx \xi = \frac{1}{n!} \sum_{\pi \in S_n} U(\pi) \approx \xi. \quad (2.17)
\]

It is easy to see that \( P^2 = P \) and that \( P^* = P \) with respect to the scalar product in \( \Gamma(\tilde{F}) \). Furthermore by standard arguments

\[
\Gamma_{\text{inv}}(\tilde{F}) = P \Gamma(\tilde{F}). \quad (2.18)
\]
We now have the

Lemma (2.19) The following equality is valid:

\[ \Gamma_{inv}(\tilde{F}) = \Pr^* \Gamma(\tilde{F}), \]  

(2.20)
such that \( \frac{1}{n!} \Pr^* \) defines an isometry between \( \Gamma(\tilde{F}) \) and \( \Gamma_{inv}(\tilde{F}) \).

Proof Given \( \approx \xi \in \Gamma_{inv}(\tilde{F}) \), define \( \tilde{\xi} \in \Gamma(\tilde{F}) \) as follows. For \( p' \in nM \) choose \( p \in \text{pr}^{-1} \{ p' \} \subseteq M^{\times n} \setminus D_n \). Now write \( \approx \xi(p) = \tilde{\chi}(\tilde{p}, f) \) for a suitable \( f \in F \) and \( \tilde{p} \in \tilde{n}M \) with \( \tilde{\text{pr}}(\tilde{p}) = p \) and hence \( p' = \tilde{\text{pr}}(\tilde{p}) \). If we set \( \tilde{\xi}(p') = \tilde{\chi}(\tilde{p}, f) \) then by (2.16) it is easy to see that \( \tilde{\xi} \) is well defined. Going to a local coordinate system, \( \tilde{\xi} \) is also seen to be smooth. Obviously we have \( \approx \xi = \Pr^* \tilde{\xi} \). Conversely, given \( \tilde{\xi} \in \Gamma(\tilde{F}) \), \( \approx \xi = \Pr^* \tilde{\xi} \) is easily seen to satisfy (2.16), q.e.d.

Let \( L^2_{inv}(\tilde{F}) \) be the closed subspace of \( L^2(\tilde{F}) \) spanned by \( \Gamma_{inv}(\tilde{F}) \). Also \( P \) extends to an orthogonal projection on \( L^2(\tilde{F}) \) such that \( L^2_{inv}(\tilde{F}) = \text{Range} P \). Furthermore, \( \frac{1}{n!} \Pr^* \) extends to a unitary map from \( L^2(\tilde{F}) \) onto \( L^2_{inv}(\tilde{F}) \). We collect these results in the third characterization of the Hilbert space for \( n \) plektons.

Corollary (2.21) Via the map \( \frac{1}{n!} \Pr^* \) the space \( L^2(\tilde{F}) \) is unitarily equivalent to the linear subspace \( L^2_{inv}(\tilde{F}) = \text{Range} P \) of \( L^2(\tilde{F}) \) consisting of square integrable sections \( \approx \xi \) in \( \tilde{n}F \) satisfying (2.16) for almost all \( p \in M^{\times n} \setminus D_n \).

In the context of algebraic quantum field theory, the space of Haag-Ruelle scattering states describing \( n \) identical particles has been shown by Fredenhagen, Gaberdiel and Rüger [FGR] to have the same structure as \( L^2(\tilde{F}) \) for a certain class of massive theories. The anyonic case is also covered in [FM2].

In case the representation \( \varrho \) of \( B_n \) is one-dimensional, which is the anyonic situation, we have the

Theorem (2.22) In the anyonic case the line bundles \( \tilde{F} \) are trivial.

This observation was first made by J.S.Dowker [Dc], based on Arnol’d’s result that \( H^2(\tilde{n}M, \mathbb{Z}) = 0 \) [Ar] and the classification theorem of Cartan, Kostant, Souriau and Isham. It was rediscovered by M.Gaberdiel [1A] and is implicitly contained in [Wu] and in [FM2, p.564]. Here we provide an alternative, constructive

Proof : The triviality stems from the fact that the first homology group of \( \tilde{n}M \) with integer coefficients is free, that is \( \text{Tor}H_1(\tilde{n}M) = 0 \). This can be seen as follows.

\( H_1(\tilde{n}M) \) is the abelianized fundamental group \( B_n/[B_n, B_n] \) of \( \tilde{n}M \), and so the set of unitary one-dimensional representations of \( B_n \) is just the group
Hom(H_1(^nM), S^1). This group is easily seen to be isomorphic to

\[ \text{Tor}H_1(^nM) \oplus \frac{\text{Hom}(H_1(^nM), \mathbb{R})}{\text{Hom}(H_1(^nM), \mathbb{Z})} \]  

and hence, using the universal coefficient theorem, we have the isomorphism

\[ \text{Hom}(H_1(^nM), S^1) \cong \text{Tor}H_1(^nM) \oplus \frac{H^1(^nM, \mathbb{R})}{H^1(^nM, \mathbb{Z})} \]  

The braid group is generated by the elementary braids \( b_k \) \( (k = 1, \ldots, n - 1) \), where \( b_k \) is the homotopy class of a closed path in \(^nM\) through a fixed base point \( p_0 \), whose lift to \( M^{\times n} \setminus D_n \) interchanges the \( k \)th and \((k+1)\)st components (see e.g. [3]). They obey the relations

\[
\begin{align*}
'b_k b_l &= b_l b_k & \text{for } k, l \in \{1, \ldots, n-1\} \text{ and } |k-l| > 1; \\
b_k b_{k+1} b_k &= b_{k+1} b_k b_{k+1} & \text{for } k \in \{1, \ldots, n-1\}.
\end{align*}
\]

From these relations we infer that \( H_1(^nM) \) is freely generated by the equivalence class \([b_1]\) of any one of the braid group generators \( b_1, \ldots, b_{n-1} \), which are all homology-equivalent. In particular, \( \text{Tor}H_1(^nM) = 0 \), and so (2.24) implies that every one-dimensional unitary representation \( \varrho \) of the braid group can be written as

\[ \varrho([\beta]) = \exp 2\pi i \int_{\beta} \omega, \]  

where \( \beta \) is a closed path in \(^nM\) through the base point \( p_0 \), \([\beta]\) \( \in B_n \) its homotopy class and \( \omega \) is a closed 1-form, which is uniquely determined modulo a closed integer 1-form by the representation \( \varrho \).

To be more constructive, consider the 1-form \( \frac{1}{n} \sum_{k l} d\theta^{kl} \) on \( M^{\times n} \setminus D_n \), where \( \theta^{kl} \) is the angle between the \( l \)th and \( k \)th point in \( \mathbb{R}^2 \). This 1-form is the pullback of a unique 1-form \( \omega_1 \) on \(^nM\), whose cohomology class is the dual base to the base \([b_1]\) of \( H_1(^nM) \), in the sense that \( \int_{b_1} \omega_1 = 1 \). Hence the cohomology class of the 1-form \( \omega \) of formula (2.26) can be expressed as \([\omega] = r : [\omega_1]\), where \( r \in \mathbb{R} \mod \mathbb{Z} \) is uniquely determined by the representation \( \varrho \). This form was also implicitly used in [Wu] and [FM2], p.564.

We can exploit formula (2.26) to construct a nowhere vanishing section \( \xi \) of the line bundle \( \tilde{F} \). To this end we consider the universal covering space \(^nM\) as the set of (fixed end point) homotopy classes of paths in \(^nM\) starting from the base point \( p_0 \). The homotopy class of a path \( \alpha \) will be denoted by \( \text{cls}(\alpha) \). The covering projection is then given as \( \tilde{\pi} : \text{cls}(\alpha) \mapsto \alpha(1) \), and the braid group acts on \(^nM\) on the right via \([\beta] : \text{cls}(\alpha) \mapsto \text{cls}(\alpha) \cdot [\beta] = \text{cls}(\beta^{-1} \ast \alpha) \), where \( \ast \) denotes the canonical composition of paths and \( \beta^{-1} \) is the inverse path defined by \( \beta^{-1}(t) := \beta(1-t) \). Now we can define a section \( \tilde{\xi} \in \Gamma(\tilde{F}) \) by setting

\[ \tilde{\xi}(p) = \tilde{\chi} \left( \text{cls}(\alpha), e^{2\pi i \int_{\alpha} \omega} \right), \]  

(2.27)
where \( \omega \) is the 1-form of formula (2.26) corresponding to the representation \( \rho \), and \( \alpha \) is any path in \( ^nM \) starting from the base point \( p_0 \) and ending in \( p \). The r.h.s. is easily seen to be independent of the path \( \alpha \), and so the definition makes sense. The section \( \tilde{\xi} \) vanishes nowhere and hence trivializes the line bundle \( \tilde{F} \), q.e.d.

Remark: The manifold \(^n\mathbb{R}^2\) has another description which is algebraic geometric and which is given as follows. In \( \mathbb{C}^n \) with points denoted by \((z_1, \ldots, z_n)\) \((z_i \in \mathbb{C})\) consider the polynomial
\[
Q(z_1, \ldots, z_n) = \prod_{i<j}(z_i - z_j)^2. \tag{2.28}
\]
Using the elementary symmetric functions
\[
\sigma_1 = z_1 + \cdots + z_n, \\
\sigma_2 = z_1z_2 + z_1z_3 + \cdots + z_{n-1}z_n, \\
\quad \ldots \\
\sigma_n = z_1z_2 \cdots z_n,
\]
\(Q(z_1, \ldots, z_n)\) may be written as a polynomial \(\tilde{Q}(\sigma_1, \ldots, \sigma_n)\) in the \(\sigma_i\)'s (see e.g. \([v.d.W], \text{p.102}\)). \(\tilde{Q}(\sigma_1, \ldots, \sigma_n)\) is obviously weighted homogeneous in the \(\sigma_i\)'s of type \((n(n-1), \frac{1}{2}n(n-1), \ldots, n-1)\), i.e. the relation
\[
\tilde{Q}(e^{\frac{1}{n(n-1)}\sigma_1}, e^{\frac{2}{n(n-1)}\sigma_2}, \ldots, e^{\frac{n-1}{n(n-1)}\sigma_n}) = e^c \cdot \tilde{Q}(\sigma_1, \sigma_2, \ldots, \sigma_n) \tag{2.29}
\]
holds for all complex \(c\) (see e.g. \([M], \text{p.75}\)).

Now \( ^n\mathbb{R}^2 \) is diffeomorphic to the set \( \{(\sigma_1, \ldots, \sigma_n) \in \mathbb{C}^n \mid \tilde{Q}(\sigma_1, \ldots, \sigma_n) \neq 0 \} \), such that \( ^n\mathbb{R}^2 \) is diffeomorphic to the complement of a complex hypersurface in \( \mathbb{C}^n \). Moreover, this set is fibred over the circle via the map
\[
(\sigma_1, \ldots, \sigma_n) \mapsto \frac{\tilde{Q}(\sigma_1, \ldots, \sigma_n)}{|\tilde{Q}(\sigma_1, \ldots, \sigma_n)|}. \tag{2.30}
\]

As an example, we have e.g. \( ^2\mathbb{R}^2 \cong \mathbb{R}^2 \times \mathbb{R}^+ \times T \) (where \( T \) is the unit circle). In particular \( ^2\mathbb{R}^2 \) has the homotopy type of the unit circle on which all line bundles are trivial.

This observation might lead to a classification of the possible nontrivial vector bundles over this space. Note, however, that the critical points of \( \tilde{Q} \) in general are not isolated, such that the results in \([M]\) concerning the structure of the fibres are not applicable.

We also remark that some of the cohomology of \( ^n\mathbb{R}^2 \) is known (see e.g. \([A\cdot R], \text{p.29}\), whose results are cited in \([B\cdot R]\)). In particular, for \( i > 1 \) all of the cohomology groups \( H^i(^nM, \mathbb{Z}) \) are finite, and hence for every vector bundle over \( ^nM \) there is a \( k \) such that the \( k \)-fold Whitney sum is trivial.
3 Galilei Covariant Plektons

In this section we will show that in a natural way $L^2(\tilde{\mathcal{F}})$ for the case $M = \mathbb{R}^2$ carries a unitary representation of a central extension of the (universal covering of the) Galilei group. Recall that in general quantum mechanical covariance under a symmetry group requires only ray representations. Provided the symmetry group is a connected Lie group and provided certain continuity requirements are fulfilled, ray representations are equivalent to unitary representations of a suitable central extension (see e.g. [BA2] and [VA]).

The Galilei group $G_3$ in 3 space-time dimensions is a Lie group and consists of all quadruples $(t, a, v, R)$ with unit element $(0, 0, 0, 1)$ and the multiplication law

$$(t, a, v, R)(t', a', v', R') = (t + t', a + Ra' + t'v, v + Rv', RR'). \quad (3.31)$$

This group is the semidirect product $G_3 \bowtie \mathbb{R}^3$ of the homogeneous Galilei group $G_3$ consisting of elements $(0, 0, v, R)$ and the space-time translation subgroup $\sim = \mathbb{R}^3$ consisting of elements $(t, a, 0, 1)$. Its universal covering group $\tilde{G}_3$ is given by all quadruples of the form $(t, a, v, \varphi)$ $(a, v \in \mathbb{R}^2, v, \varphi \in \mathbb{R})$ with the multiplication law

$$(t, a, v, \varphi)(t', a', v', \varphi') = \left(t + t', a + R(\varphi)a' + t'v, v + R(\varphi)v', \varphi + \varphi' \right) \quad (3.32)$$

where $R : \mathbb{R} \to SO(2)$ is the standard homomorphism

$$\varphi \mapsto \begin{pmatrix} \cos \varphi & \sin \varphi \\ -\sin \varphi & \cos \varphi \end{pmatrix}. \quad (3.33)$$

Again $\tilde{G}_3$ is a semidirect product $\tilde{G}_3 \bowtie \mathbb{R}^3$ where $\tilde{G}_3$ is the subgroup consisting of all elements of the form $(0, 0, v, \varphi)$. In a natural way $\tilde{G}_3$ is the universal covering group of $G_3$. Now it turns out that the set of central extensions of $\tilde{G}_3$ forms a three dimensional manifold. The proof is given in appendix A. This result contrasts with the higher dimensional case, where the central extensions form a one parameter family. At the moment the physical relevance of these central extensions is not clear to us. Therefore we will restrict attention to the central extensions which correspond to those in higher dimensions.

The resulting central extensions $\tilde{G}_3^m$ of $\tilde{G}_3$ are parametrized by a real number $m$. For fixed $m$, $\tilde{G}_3^m$ is given as the set of quintuples $(\theta, t, a, v, \varphi)$ $(\theta, t, \varphi \in \mathbb{R}, a, v \in \mathbb{R}^2)$ and the composition law

$$(\theta, t, a, v, \varphi)(\theta', t', a', v', \varphi') = \left(\theta'' + t + t', a + R(\varphi)a' + t'v, v + R(\varphi)v', \varphi + \varphi' \right) \quad (3.34)$$
\[
\theta'' = \theta + \theta' + \frac{m^2}{2} \left( \langle a, R(\varphi)v' \rangle - \langle v, R(\varphi)a' \rangle + t' \langle v, R(\varphi)v' \rangle \right). \tag{3.35}
\]

Here \(\langle \cdot, \cdot \rangle\) denotes the canonical scalar product on \(\mathbb{R}^2\). Again \(\tilde{G}_3\) is a subgroup of \(\tilde{G}_3^m\) and the elements \((\theta, 0, 0, 0, 1)\) form the central subgroup of \(\tilde{G}_3^m\). In an irreducible unitary representation the central elements will be represented by \(\exp i\tau\theta\), and such a representation leads to a ray representation of \(\tilde{G}_3\) with multiplier \((-\tau)\) times the multiplier defined by the last summand in (3.35) (see e.g. [19], theorem 10.16). If \(m > 0\), a choice we will make, then this parameter has the physical interpretation of a mass. Given this \(m\), we let \(\tilde{G}_3\) act on \(\mathbb{R}^2\) via

\[
(v, \varphi) : p \mapsto (v, \varphi) \cdot p = R(\varphi)p - mv. \tag{3.36}
\]

This induces in a canonical way an action of \(\tilde{G}_3\) on \((\mathbb{R}^2)^\times n\) which leaves \(D_n\) invariant and which commutes with the action of \(S_n\). This implies that the action of \(\tilde{G}_3\) descends to an action on \(^n\mathbb{R}^2\) which we write symbolically as

\[
(v, \varphi) : p \mapsto (v, \varphi) \cdot p = R(\varphi)p - mv. \tag{3.37}
\]

Now this action of \(\tilde{G}_3\) on \(^n\mathbb{R}^2\) lifts to an action on the universal covering space, making the principal \(B_n\) bundle \(^n\mathbb{R}^2\) a homogeneous \(\tilde{G}_3\)-bundle (see [19], p.63). In other words, if we write the action of \(\tilde{G}_3\) on \(^n\mathbb{R}^2\) as a left action

\[
(v, \varphi) : \tilde{p} \mapsto (v, \varphi) \cdot \tilde{p}
\]

then it commutes with the right action of \(B_n\):

\[
\left( (v, \varphi) \cdot \tilde{p} \right) \cdot b = (v, \varphi) \cdot (\tilde{p} \cdot b). \tag{3.38}
\]

Furthermore, this induces an action of \(\tilde{G}_3\) on the associated bundle \(\tilde{F}\), again lifting the action on the base \(^n\mathbb{R}^2\), by setting

\[
(v, \varphi) \cdot \tilde{\chi}(\tilde{p}, f) = \tilde{\chi} \left( (v, \varphi) \cdot \tilde{p}, f \right).
\]

We need some further notation. The pairing

\[
(v, p) \mapsto \langle v, p \rangle = \langle v, \sum_{i=1}^n p_i \rangle
\]

from \(\mathbb{R}^2 \times (\mathbb{R}^2)^\times n \setminus D_n\) into \(\mathbb{R}\) is invariant under the action of \(S_n\) and hence descends to a pairing from \(\mathbb{R}^2 \times \mathbb{R}^2\) into \(\mathbb{R}\) denoted by the same symbol. Similarly the map \(p \mapsto \langle p, p \rangle = \sum_{i=1}^n \langle p_i, p_i \rangle\) from \((\mathbb{R}^2)^\times n \setminus D_n\) into \(\mathbb{R}\) descends to a map from \(^n\mathbb{R}^2\) into \(\mathbb{R}\) denoted by the same symbol. Physically, this means...
of course that for given mass the total energy and the total momentum of a plektonic configuration $\textbf{p} \in n\mathbb{R}^2$ is well defined.

With these preliminaries we now define a unitary representation of $\tilde{G}_3^n$ on $L^2(\tilde{F})$. We first consider the case where $\tilde{\varphi}$ is irreducible. For $\tilde{\varphi} \in L^2(\tilde{F})$ set

$$\left(U(g)\tilde{\varphi}\right)(\textbf{p}) = e^{ins\varphi}e^{i\left(-n\theta+(a,\textbf{p})+\frac{c_2}{2}(a,\textbf{v})+\frac{1}{2m}(\textbf{p},\textbf{p})\right)}(v,\varphi) \cdot \tilde{\varphi}\left((v,\varphi)^{-1} \cdot \textbf{p}\right),$$

(3.39)

where $g = (\theta, t, a, v, \varphi)$ and where $s$ is an arbitrary real parameter having the physical interpretation of a one-particle spin.

We may finally formulate a condition under which the representation (3.39) of $\tilde{G}_3^n$ descends to a representation of $G_3^n$. In particular, $G_3^n$ is the universal covering group of $\tilde{G}_3^n$. The representation descends iff $g = (0, 0, 0, 2\pi)$ is represented by the identity. Now we have $\tilde{\varphi}\left((0,2\pi) \cdot \tilde{\textbf{p}}\right) = \tilde{\varphi}(\tilde{\textbf{p}})$ for all $\tilde{\textbf{p}} \in n\mathbb{R}^2$. Hence the action of $(0,2\pi)$ is an element of the deck transformation group, which is, on the other hand, given by the right action of the structure group $B_n$. Consequently, there exists $b \in B_n$ such that $(0,2\pi) \cdot \tilde{\textbf{p}} = \tilde{\textbf{p}} \cdot b$ for all $\tilde{\textbf{p}}$. By looking at suitable $\tilde{\textbf{p}}$ it is easily seen that $b = c_n$ for $n \geq 2$, where $c_n = (b_1 \cdots b_{n-1})^n$ is the generator of the center of $B_n$ (see e.g. [3]). Hence the representation (3.39) descends to a representation of $G_3^n$ iff $\exp 2\pi ins = \varrho(e^{-1}_n)$.

If $\varrho$ is not irreducible, then $\varrho$ may be decomposed into irreducible components $\varrho = \varrho_1 \oplus \cdots \oplus \varrho_k$ on $\tilde{F} = F_1 \oplus \cdots \oplus F_k$. Accordingly $\tilde{F}$ decomposes into a Whitney sum $\tilde{F} = \tilde{F}_1 \oplus \cdots \oplus \tilde{F}_k$ giving the decomposition $L^2(\tilde{F}) = L^2(\tilde{F}_1) \oplus \cdots \oplus L^2(\tilde{F}_k)$. On each of these subspaces the above construction may be carried out with possibly different choices of the total spin in each of the components.

## 4 Relativistic Plektons

We turn to the relativistic case and start by recalling some well known facts in order to establish notation. Elements of the orthochronous Poincaré group $\mathcal{P}_3^\uparrow$ of the 3-dimensional Minkowski space $M_3$ may be written as $(a, \Lambda)$ with $a \in M_3$ and $\Lambda \in L_3^\uparrow$, the orthochronous Lorentz group. Group multiplication is given by $(a, \Lambda)(a', \Lambda') = (a + \alpha a', \Lambda\Lambda')$ with unit element $(0, 1)$ such that $\mathcal{P}_3^\uparrow$ is the semidirect product of $M_3$ and $L_3^\uparrow$. A twofold covering of $L_3^\uparrow$ is given as the subgroup of $SL(2, \mathbb{C})$ (conjugate to $SL(2, \mathbb{R})$) consisting of elements of the form

$$\begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}, \alpha\bar{\alpha} - \beta\bar{\beta} = 1.$$

(4.40)

The corresponding Lorentz transformation $\Lambda = \Lambda(\alpha, \beta) \in L_3^\uparrow$ is given as follows. For $a = (a^0, a^1, a^2) \in M_3$ we set

$$a = \begin{pmatrix} a^0 & a^1 - ia^2 \\ a^1 + ia^2 & a^0 \end{pmatrix}$$

(4.41)
and define $\Lambda(\alpha, \beta) a$ by

$$\Lambda a = \begin{pmatrix} \alpha & \beta \\ \frac{\alpha}{\beta} & \frac{\beta}{\bar{\alpha}} \end{pmatrix} a \begin{pmatrix} \frac{\alpha}{\beta} & \beta \\ \alpha & \frac{\beta}{\alpha} \end{pmatrix}.$$  \hspace{1cm} (4.42)

In particular for given $p = (p^0, p^1, p^2) \in V^{+, m}$ the element of the form

$$\left(2m(p^0 + m)\right)^{-\frac{1}{2}} \begin{pmatrix} p^0 + m & p^1 - ip^2 \\ p^1 + ip^2 & p^0 + m \end{pmatrix}$$  \hspace{1cm} (4.43)

gives rise to an element in $L^{+3}_3$ called a boost and is denoted by $\Lambda(p)$. One has $\Lambda(p) (m, 0, 0) = p$. The universal covering group $\tilde{L}^{+3}_3$ of $L^{+3}_3$ can be explicitly written as the set

$$\left\{ (\gamma, \omega) \mid \gamma \in \mathbb{C}, |\gamma| < 1, \omega \in \mathbb{R} \right\}$$  \hspace{1cm} (4.44)

with the group multiplication $(\gamma, \omega)(\gamma', \omega') = (\gamma'', \omega'')$ being given by

$$\gamma'' = (\gamma' + \gamma e^{-i\omega'})(1 + \gamma' \gamma e^{-i\omega'})^{-1}$$  \hspace{1cm} (4.45)

$$\omega'' = \omega + \omega' + \frac{1}{i} \log \left\{ (1 + \gamma' \gamma e^{-i\omega'})(1 + \bar{\gamma} \gamma' e^{i\omega'})^{-1} \right\}.$$  

Here the logarithm is defined in terms of its power series [3A1, p.594]. The corresponding element in the twofold covering of $L^{+3}_3$ described above is then given as

$$(1 - \gamma \bar{\gamma})^{-\frac{1}{2}} \begin{pmatrix} e^{i\omega} & \gamma e^{i\omega} \\ \bar{\gamma} e^{-i\omega} & e^{-i\omega} \end{pmatrix}.$$  \hspace{1cm} (4.46)

The resulting element in $\tilde{L}^{+3}_3$ will be denoted by $\Lambda(\gamma, \omega)$. For given $p \in V^{+, m}$, the element $h(p) = \left( \gamma = \gamma(p) = \frac{p^1 - ip^2}{p^0 + m}, \omega = 0 \right)$ in $\tilde{L}^{+3}_3$ is such that $\Lambda(h(p)) = \Lambda(p)$. The universal covering group $\tilde{P}^{+3}_3$ of $P^{+3}_3$ is now the semidirect product of $M_3$ with $\tilde{L}^{+3}_3$, the group multiplication being given by

$$\left( a, (\gamma, \omega) \right) \left( a', (\gamma', \omega') \right) = \left( a + \Lambda(\gamma, \omega)a', (\gamma, \omega)(\gamma', \omega') \right).$$  \hspace{1cm} (4.47)

In contrast to the nonrelativistic case, but in analogy with the higher dimensional case, $\tilde{P}^{+3}_3$ has no nontrivial central extensions. $\tilde{L}^{+3}_3$ acts as a transformation group on $(V^{+, m})^n$ via

$$(\gamma, \omega) : p = (p_1, \ldots, p_n) \mapsto \left( \Lambda(\gamma, \omega)p_1, \ldots, \Lambda(\gamma, \omega)p_n \right)$$

leaving $D_n$ invariant and commuting with the action of $S_n$. Hence this action of $\tilde{L}^{+3}_3$ descends to an action on $^nV^{+, m}$. Analogous to the situation in the nonrelativistic case, this action lifts to an action on $^n\tilde{V}^{+, m}$ written as
\((\gamma, \omega) : \hat{\mathbf{p}} \mapsto (\gamma, \omega) \cdot \hat{\mathbf{p}}\) such that \((\gamma, \omega) \cdot (\hat{\mathbf{p}} \cdot b)\), i.e. the principal bundle \(n\hat{V}^{+m}\) over \(nV^{+m}\) is a homogeneous \(\hat{L}^{\uparrow}_{3}\) bundle.

The pairing

\[
(a, \mathbf{p}) \mapsto \langle a, \mathbf{p} \rangle = a^{0} \sum_{j=1}^{n} p_{j}^{0} - \sum_{k=1}^{2} a^{k} \sum_{j=1}^{n} p_{j}^{k}
\]

(4.48)

from \(\mathcal{M}_{3} \times (V^{+m})^{\times n} \setminus D_{n}\) into \(\mathbb{R}\) descends to a pairing from \(\mathcal{M}_{3} \times n\hat{V}^{+m}\) into \(\mathbb{R}\), again denoted by the same symbol.

We denote by \(\hat{U}(1) = \mathbb{R}\) the abelian subgroup of \(\hat{L}^{\uparrow}_{3}\) consisting of elements of the form \((0, \omega)\). For arbitrary \(p \in V^{+m}\) and \((\gamma, \omega) \in \hat{L}^{\uparrow}_{3}\), the element

\[
t(\gamma, \omega); p) = h(p)^{-1} (\gamma, \omega) h\left(\Lambda(\gamma, \omega)^{-1} p\right)
\]

(4.49)

is in \(\hat{U}(1)\) and hence may be written in the form

\[
t(\gamma, \omega); p) = (0, \Omega((\gamma, \omega); p))
\]

(4.50)

In fact, by (4.49) and (4.45)

\[
\Omega((\gamma, \omega); p) = \omega + \frac{i}{3} \log\left\{ (1 - \gamma(p)\gamma e^{-i\omega})(1 - \bar{\gamma}(p)\gamma e^{i\omega})^{-1}\right\}
\]

\[
+ \frac{i}{3} \log\left\{ \left(1 + \frac{\gamma - \gamma(p)e^{-i\omega}}{1 - \gamma(p)\gamma e^{-i\omega}}\bar{\gamma}\left(\Lambda(\gamma, \omega)^{-1} p\right)\right)\bar{\gamma}\left(\Lambda(\gamma, \omega)^{-1} p\right)^{-1}\right\}
\]

(4.51)

with

\[
\gamma\left(\Lambda(\gamma, \omega)^{-1} p\right) = \left( -2\gamma p^{0} + \gamma^{2} e^{2i\omega}(p^{1} + ip^{2}) + e^{-i\omega}(p^{1} - ip^{2})\right).
\]

(4.52)

Note that \(\Omega((\gamma = 0, \omega); p) = \omega\) for all \(\omega\) and \(p\).

With \(p \in (V^{+m})^{\times n} \setminus D_{n}\) the element

\[
\Omega((\gamma, \omega); p) = \sum_{i=1}^{n} \Omega((\gamma, \omega); p_{i})
\]

(4.53)

defines a map \(\Omega\) from \(\hat{L}^{\uparrow}_{3} \times (V^{+m})^{\times n} \setminus D_{n}\) into \(\mathbb{R}\) which is invariant under the action of \(S_{n}\) on \((V^{+m})^{\times n} \setminus D_{n}\). Hence \(\Omega\) descends to a map from \(\hat{L}^{\uparrow}_{3} \times nV^{+m}\) into \(\mathbb{R}\) denoted by the same symbol. Again we start with the case where \(\varrho\) is
irreducible. Fix $s \in \mathbb{R}$. Then a unitary representation of $\tilde{P}^+_3$ on $L^2(\tilde{F})$ is defined by

$$
(U(a, (\gamma, \omega)) \tilde{\xi})(p) = e^{i(a, p) + i\Omega(\gamma, \omega); p} \gamma, \omega) \cdot \tilde{\xi}(\gamma, \omega)^{-1} p). \quad (4.54)
$$

For the one-particle case, this is the usual irreducible representation of $\tilde{P}^+_3$ with mass $m > 0$ and spin $s$. The extension to the general case where $\varrho$ need not be irreducible may be treated as in the nonrelativistic case.

Similarly (when $\varrho$ is irreducible) we may also formulate a necessary and sufficient condition that this representation descends to a representation of $P^+_3$.

In fact, the kernel of the covering homomorphism from $\tilde{P}^+_3$ onto $P^+_3$ is generated by the element $(\gamma = 0, \omega = 2\pi)$. But then by (4.51), $\Omega(0, 2\pi; p) = n \cdot 2\pi$ for all $p \in nV^+m$. By the same arguments as in the previous chapter, the equality

$$e^{2\pi ins} = \varrho(e_n^{-1}) \quad (4.55)$$

is necessary and sufficient to have a representation of $P^+_3$.

We want to point out that Fröhlich and Marchetti [FM2] have shown that for a certain class of massive quantum field theories the space of Haag-Ruelle scattering states in the anyonic case carries a representation of $P^+_3$, which we expect to agree with (4.54) for the anyonic case.

As indicated in the introduction, the difficulties in constructing relativistic “free” fields on the “Fock” space given by (1.1) and with suitable localization properties become visible in the construction (4.54). In fact, within the present set-up, it is not clear how to disentangle the factor $e^{is\Omega(\gamma, \omega); p}$. We recall that in four space-time dimensions this is achieved by switching from the spin basis to the spinor basis.

In the next section we shall propose an alternative set-up which at least avoids the aforementioned difficulty.

5 Relativistic Particles with Ribbon Braid Statistics

The ribbon braid group arises in the following context (we restrict ourselves to the relativistic case, see e.g. [NM]). Let $T$ be the unit circle in $\mathbb{C}$ and consider the set $(V^+m)^{\times n} \times T^{\times n}$. Again $S_n$ acts in a canonical way on this set, leaving the subset $D_n \times T^{\times n}$ invariant. We set

$$n(V^+m \times T) = \left((V^+m)^{\times n} \times T^{\times n} \setminus (D_n \times T^{\times n})\right) / S_n \quad (5.56)$$

and we will write points in this set as $(p, t)$. Its universal covering space will be denoted by $n(V^+m \times T)$ with elements $(\tilde{p}, \tilde{t})$ and canonical projection $\tilde{pr}$. 


Its structure group $RB_n$ is called the ribbon braid group and can be described as follows. On $\mathbb{Z}^n$, the structure group of $T^{\times n}$, $S_n$ and hence $B_n$ acts as an automorphism group. Then we have $RB_n = B_n \rtimes \mathbb{Z}^n$ as a semidirect product.

There are canonical maps from $n(V^{+m} \times T)$ onto $nV^{+m}$ and $T^{\times n}/S_n$ and the images of $(p, t)$ will be denoted by $p$ and $t$ respectively.

We may now proceed as before. Thus let $\rho$ be a finite dimensional unitary representation of $RB_n$ in a Hilbert space $F$ defining an associated bundle $\tilde{F}$. $L^2(\tilde{F})$ is now the Hilbert space for $n$ relativistic plektons with ribbon braid group statistics. Alternatively we may speak of framed plektons. In defining $L^2(\tilde{F})$ we of course make use of the canonical measure $d\mu(p, t)$ on $n(V^{+m} \times T)$ induced by the Lorentz invariant measure $d\mu(p)$ on $V^{+m}$ and the Haar measure $d\nu(t)$ on $T$.

In the anyonic case, i.e. when $\rho$ is a one dimensional representation, the resulting line bundle is again trivial. This may be shown by the methods used in section 2.

The main observation we need for describing relativistic invariance is that $L_3^+$ acts on $n(V^{+m} \times T)$ leaving $D_n \times T^{\times n}$ invariant and commuting with the action of $S_n$. Hence this action of $L_3^+$ descends to an action on $n(V^{+m} \times T)$ written as $(\gamma, \omega) : (p, t) \mapsto (\gamma, \omega) \cdot (p, t)$. Again, $L_3^+$ lifts to an action on the principal $RB_n$ bundle $n(V^{+m} \times T)$ written as $(\gamma, \omega) : (\tilde{p}, \tilde{t}) \mapsto (\gamma, \omega) \cdot (\tilde{p}, \tilde{t})$ such that

$$(\gamma, \omega) \cdot t = e^{-i\omega} \frac{t - \gamma e^{i\omega}}{1 - t\gamma e^{-i\omega}}.$$ (5.57)

This defines an action of $L_3^+$ on $(V^{+m} \times T)^{\times n}$ leaving $D_n \times T^{\times n}$ invariant and commuting with the action of $S_n$. Hence this action of $L_3^+$ descends to an action on $n(V^{+m} \times T)$ written as $(\gamma, \omega) : (p, t) \mapsto (\gamma, \omega) \cdot (p, t)$. Again, $L_3^+$ lifts to an action on the principal $RB_n$ bundle $n(V^{+m} \times T)$ written as $(\gamma, \omega) : (\tilde{p}, \tilde{t}) \mapsto (\gamma, \omega) \cdot (\tilde{p}, \tilde{t})$ such that

$$(\gamma, \omega) \cdot (\tilde{p}, \tilde{t}) \cdot b = (\gamma, \omega) \cdot (\tilde{p}, \tilde{t})$$

for all $b \in RB_n$.

To construct unitary representations of $\tilde{L}_3^+$ we will take recourse to a certain class of unitary representations of $L_3^+$ in $L^2(T)$, called the principal series. They were first suggested by Bargmann \cite{B} and then analyzed by Pukanszky \cite{Pu}. These representations are parametrized by the pair $(h, \sigma)$ \( -\frac{1}{2} < h \leq \frac{1}{2}, \ \sigma \in i\mathbb{R} \) (pure imaginary). Unless $h = \frac{1}{2}$ and $\sigma = 0$, these representations are irreducible. The principal series of $SL(2, \mathbb{R})$ are obtained by setting $h = 0$ or $h = \frac{1}{2}$. With a slight modification of the notation in \cite{S}, these representations have the form

$$U(\gamma, \omega) f(t) = \tau(\gamma, \omega; t) f((\gamma, \omega)^{-1} \cdot t)$$ (5.58)
with \( \tau = \tau(h, \sigma) \) given as
\[
\tau\left((\gamma, \omega); t\right) = e^{-i\omega h \left(1 + t\bar{\gamma}\right)^{h-1} \left|1 + t\bar{\gamma}\right|^{-1-2\sigma} \left(1 + |\gamma|^2\right)^{\frac{s}{2} + \sigma}}. \tag{5.59}
\]

We start by constructing a unitary representation of \( \tilde{P}_3^\dagger \) in the one particle case, thus motivating our ansatz. By definition, the Hilbert space is then given as \( L^2\left(V^+, M \times T, d\mu(p)dv(t)\right) \). For \( \psi \) in this space we set
\[
(U(a, (\gamma, \omega)) \psi)(p, t) = e^{i(p,a)} \tau\left((\gamma, \omega); t\right) \psi\left(p, (\gamma(p), 0)^{-1} t\right) \tag{5.60}
\]
giving rise to a unitary representation of \( \tilde{P}_3^\dagger \). In order to see how this representation is related to the irreducible unitary representation given in the previous section for the one-particle case, we decompose the representation (5.60) as follows. Let \( W = W(h, \sigma) \) be the unitary operator defined by
\[
(W\psi)(p, t) = \tau\left((\gamma(p), 0); t\right) \psi\left(p, (\gamma(p), 0)^{-1} t\right) \tag{5.61}
\]
and set
\[
\hat{U}\left(a, (\gamma, \omega)\right) = W^{-1} U\left(a, (\gamma, \omega)\right) W. \tag{5.62}
\]
Then a short calculation gives
\[
\left(\hat{U}\left(a, (\gamma, \omega)\right) \psi\right)(p, t) = e^{i(p,a) - i\Omega\left((\gamma, \omega)^p\right) h} \psi\left(\Lambda(\gamma, \omega)^{-1} p, t e^{i\Omega\left((\gamma, \omega)^p\right)}\right) \tag{5.63}
\]
For \( k \in \mathbb{Z} \) let \( L_k \subset L^2\left(V^+, M \times T, d\mu(p)dv(t)\right) \) be the image of \( L^2\left(V^+, d\mu(p)\right) \) under the linear isometric map
\[
\psi(p) \mapsto \psi(p, t) = \psi(p)t^k. \tag{5.64}
\]
We have the direct sum decomposition
\[
L^2\left(V^+, M \times T, d\mu(p)dv(t)\right) = \bigoplus_{k \in \mathbb{Z}} L_k. \tag{5.65}
\]

The above formula shows that \( L_k \) is invariant under \( \hat{U}\left(a, (\gamma, \omega)\right) \). More precisely,
\[
\left(\hat{U}\left(a, (\gamma, \omega)\right) \psi\right)(p, t) = e^{i(p,a) + i(k-h)\Omega\left((\gamma, \omega)^p\right)} \psi\left(\Lambda(\gamma, \omega)^{-1} p, t\right) \tag{5.66}
\]
for \( \psi \in L_k \). By comparison with \( 4.54 \) we see that we have an irreducible representation of \( \tilde{P}_3^\dagger \) on \( L_k \) of spin \( s = k - h \). In the language of physicists by analogy to the higher dimensional case, one may say that \( W \) provides the transition from the spinor basis to the spin basis.
We now generalize this construction to the $n$ particle sector as follows. For $\mathbf{t} = (t_1, \ldots, t_n) \in T^{\times n}$ define

$$
\tau((\gamma, \omega); \mathbf{t}) = \prod_{j=1}^{n} \tau((\gamma, \omega); t_j).
$$

This quantity is invariant under the action of $S_n$ on $T^{\times n}$ and hence descends to a function on $T^{\times n} / S_n$ denoted by the same symbol. A unitary representation of $\tilde{P}_3^\uparrow$ on $L^2(\tilde{F})$ is now given by

$$
(U(a, (\gamma, \omega)) \hat{\xi})(p, t) = e^{i(a, p)} \tau((\gamma, \omega); t) \cdot \hat{\xi}((\gamma, \omega)^{-1}(p, t)).
$$

### Appendix

#### A The Central Extensions of the Galilei Group in 3 Space-Time Dimensions

Here we want to prove our claim that the set of (equivalence classes of) central extensions of $\tilde{G}_3$, and hence of its Lie algebra, forms a three dimensional manifold.

As is well known, the set of central extensions of a Lie algebra $\mathfrak{a}$ is in one-to-one correspondence with its second cohomology space $H^2(\mathfrak{a})$, which is defined as follows. For $q \in \mathbb{N}$, let $\Lambda^q(\mathfrak{a}^*)$ denote the linear space of real valued antisymmetric $q$-linear forms on $\mathfrak{a} \times q$. We have the coboundary operators, defined by

$$
\delta_1 : \mathfrak{a}^* \to \Lambda^2(\mathfrak{a}^*), \quad (\delta_1 \lambda)(x, y) = \lambda([x, y])
$$

and

$$
\delta_2 : \Lambda^2(\mathfrak{a}^*) \to \Lambda^3(\mathfrak{a}^*), \quad (\delta_2 \Xi)(x, y, z) = \Xi([x, y], z) + \Xi([y, z], x) + \Xi([z, x], y)
$$

for all $x, y$ and $z \in \mathfrak{a}$. The kernel of $\delta_2$ is denoted by $Z^2(\mathfrak{a})$, and the image of $\delta_1$ by $B^2(\mathfrak{a})$. The second cohomology space of $\mathfrak{a}$ is then defined as $H^2(\mathfrak{a}) := Z^2(\mathfrak{a}) / B^2(\mathfrak{a})$.

If we denote the generators of rotations, space translations, pure Galilei transformations and time translations by $l, p_i, n_i$ and $e$ respectively ($i = 1, 2$), the multiplication law (3.32) of $\tilde{G}_3$ implies the following structure of its Lie algebra, which will be denoted by $\mathfrak{g}$ in the sequel:

$$
[l, p_1] = -p_2, \quad [l, p_2] = p_1, \quad (A.69)
$$

$$
[l, n_1] = -n_2, \quad [l, n_2] = n_1, \quad (A.70)
$$

$$
[n_i, e] = p_i \quad (i = 1, 2), \quad (A.71)
$$

and all other commutators vanish.
From these relations we conclude that an antisymmetric bilinear form \( \Xi \) in \( \Lambda^2(\mathfrak{g}^*) \) satisfies the cocycle condition \( \delta_2 \Xi = 0 \) iff all of the following equalities hold:

\[
\begin{align*}
\Xi(l,p_1) &= -\Xi(n_2,e) \quad \Xi(l,p_2) = \Xi(n_1,e) \quad (A.72) \\
\Xi(p_1,p_2) &= 0 \quad (A.73) \\
\Xi(p_1,n_2) &= \Xi(p_2,n_1) \quad \Xi(p_1,n_1) = \Xi(p_2,n_2) \quad (A.74) \\
\text{and } \Xi(p_i,e) &= 0 \; (i = 1,2) \quad (A.75)
\end{align*}
\]

The natural grading of the Lie algebra \( \mathfrak{g} \), which is given by the subalgebras

\[
\begin{align*}
\mathfrak{g}_1 &:= \text{span}\{l\}, \quad \mathfrak{g}_2 := \text{span}\{p_1,p_2\}, \quad \mathfrak{g}_3 := \text{span}\{n_1,n_2\} \quad \text{and} \quad \mathfrak{g}_4 := \text{span}\{e\},
\end{align*}
\]

exhibits \( \Lambda^2(\mathfrak{g}^*) \) as a direct sum

\[
\Lambda^2(\mathfrak{g}^*) = \bigoplus_{i=2}^3 \Lambda^2(\mathfrak{g}_i^*) \oplus \bigoplus_{i,j=1}^4 (\mathfrak{g}_i \otimes \mathfrak{g}_j)^* ,
\]

where \( (\mathfrak{g}_i \otimes \mathfrak{g}_j)^* \) denotes the space of bilinear forms on \( \mathfrak{g}_i \times \mathfrak{g}_j \). Using this decomposition, we examine the restrictions \( \Xi_{i,j} \) of an arbitrary cocycle \( \Xi \in Z^2(\mathfrak{g}) \) to the various subspaces \( \mathfrak{g}_i \times \mathfrak{g}_j \) and note that if \( \Xi_{i,j} \) is in \( B^2(\mathfrak{g}) \), we can set it equal to zero without changing the equivalence class of \( \Xi \) in \( H^2(\mathfrak{g}) \).

The restrictions of \( \Xi \) to \( \mathfrak{g}_2 \times \mathfrak{g}_2 \) and \( \mathfrak{g}_2 \times \mathfrak{g}_4 \) are zero due to equations (A.73) and (A.75), respectively.

\( \Xi_{1,2} \) is the (restriction of the) coboundary of \( \lambda \in \mathfrak{g}^* \) defined by \( \lambda(l) = 0 \), \( \lambda(p_1) = \Xi(l,p_2) \) and \( \lambda(p_2) = -\Xi(l,p_1) \).

Equation (A.72) shows that \( \Xi_{3,4} = \delta_1 \lambda \) on \( \mathfrak{g}_3 \times \mathfrak{g}_4 \) for the same \( \lambda \).

\( \Xi_{1,3} \) is seen to be a coboundary, i.e. in \( B^2(\mathfrak{g}) \), with a similar argument.

\( \Xi_{3,3} \) and \( \Xi_{1,4} \) are multiples of the cocycles \( \Xi^{(1)}(n_1,n_2) = \Xi^{(1)}(n_2,n_1) = 1 \) and \( \Xi^{(2)}(l,e) = 1 \), respectively. (A.76)

Finally, with respect to the three linearly independent equations (A.74), \( \Xi_{2,3} \) is determined by the cocycle

\[
\Xi^{(3)}(p_1,n_1) = \Xi^{(3)}(p_2,n_2) = 1 \quad , \quad \Xi^{(3)}(p_1,n_2) = 0 = \Xi^{(3)}(p_2,n_1) \quad (A.78)
\]

None of the cocycles \( \Xi^{(1)} \), \( \Xi^{(2)} \) and \( \Xi^{(3)} \) is a coboundary, since their respective arguments commute. Similarly, we see by inspection that the only linear combination which is a coboundary, is the trivial one.

Summing up, \( H^2(\mathfrak{g}) \) is three-dimensional, spanned by the equivalence classes of the bilinear forms defined in (A.76) to (A.78).
The corresponding central extensions of $\hat{G}_3$ can be found following p.127: Every element $\Xi \in H^2(g)$ determines a multiplier $\omega$ on $\hat{G}_3 \times \hat{G}_3$ via
\[
\Xi(x,y) = \frac{\partial^2}{\partial s \partial t} \left( \omega(\exp sx, \exp ty) - \omega(\exp sy, \exp tx) \right) \bigg|_{s=t=0}
\]
for all $x, y$ in $g$, which in turn defines a central extension $\hat{G}_3^\omega$ with the multiplication law
\[
(\theta, g)(\theta', g') = \left( \theta + \theta' + \omega(g, g'), gg' \right) \ (\theta, \theta' \in \mathbb{R}, g, g' \in \hat{G}_3).
\]
$\omega$ and $\hat{G}_3^\omega$ are determined by $\Xi$ uniquely modulo the relevant equivalence relations. Using the above formula, we find the following multipliers $\omega_1$, $\omega_2$ and $\omega_3$ corresponding to $\Xi^{(1)}$, $\Xi^{(2)}$ and $\Xi^{(3)}$ respectively, where $\sigma(\cdot, \cdot)$ denotes the standard symplectic form on $\mathbb{R}^2$:
\[
\omega_1(g, g') = 1/2 \sigma(v, R(\varphi)v'),
\]
\[
\omega_2(g, g') = 1/2 t'\varphi' \quad \text{and}
\]
\[
\omega_3(g, g') = 1/2 \left( \langle a, R(\varphi)v' \rangle - \langle v, R(\varphi)a' \rangle + t'\langle v, R(\varphi)v' \rangle \right)
\]
for any $g = (t, a, v, \varphi)$ and $g' = (t', a', v', \varphi')$ in $\hat{G}_3$. Obviously, the multiplier $m \cdot \omega_3$ ($m > 0$) corresponds to the central extension $\hat{G}_3^m$ considered in section 3.

The possible physical relevance of an arbitrary central extension has not been clarified yet. We note that D.R. Grigore has determined the corresponding projective unitary irreducible representations of $G_3$ in a recent paper.

B The Anyonic Line Bundles as (Non) Trivial G-Bundles

In this appendix we will show that the anyonic line bundles $\hat{F}$ in the non-relativistic case are also trivial when considered as $\hat{G}_3$-bundles (where $G_3$ is the homogeneous Galilei group). In the relativistic case, however, they are not trivial when viewed as $\hat{L}_3^\uparrow$-bundles ($\hat{L}_3^\uparrow$ denoting the Lorentz group), unless the representation $\hat{\rho}$ defining the line bundle is trivial.

We recall that a $G$-bundle $E$ over a $G$-manifold $M$ is defined to be trivial iff $E$ is diffeomorphic to $M \times E$, where $E$ is a vector space and the action of $G$ on $E$ corresponds to a product action $g : (m, e) \mapsto (g \cdot m, D(g)e)$, where $g \mapsto D(g)$ is a representation of $G$ on $E$. Also $D(g)$ is supposed to be smooth in $g$, if $G$ is a Lie group and if the action of $G$ on $E$ is smooth.

Any equivariant function $F$ on $\mathbb{R}^m$ with values in $\mathbb{C}^\times$ defines a trivialization of the associated anyonic line bundle $\hat{F}$ via the map $\chi(p, c) \mapsto (\tilde{p}(p), F(\tilde{p})c)$, and the action of $G$ on $\hat{F}$ corresponds to the following action on $\mathbb{R}^m \times \mathbb{C}$:
\[
g : (p, c) \mapsto (g \cdot p, F(\tilde{g}(p))F(\tilde{p})^{-1}c),
\]
where $\tilde{p} \in n\tilde{M}$ is any point in the fibre over $p$. $F$ trivializes $\tilde{F}$ as a $G$-bundle if and only if for all $g \in G$ the function $F(\tilde{p}) F(g \cdot \tilde{p})^{-1}$, which only depends on $p = \tilde{p}(\tilde{p})$, is actually independent of $p$. In that case,

$$D(g) = F(\tilde{p}) F(g \cdot \tilde{p})^{-1} \quad \text{(B.80)}$$

defines a representation of $G$. Identifying, as done in section 2, points in $n\tilde{M}$ with homotopy classes of paths in $nM$ starting at the base point $p_0$, the action of $G$ on $n\tilde{M}$ is given as follows. Let $\tilde{p} = \text{cls}(\alpha) \in n\tilde{M}$, and let $\gamma$ be any path in $G$ starting at the identity and ending at $g \in G$. Then $g \cdot \tilde{p}$ is the homotopy class of the path $(\gamma \cdot \alpha)(t) := \gamma(t) \cdot \alpha(t)$, or, equivalently, of the path $\alpha \ast (\gamma \cdot i_p)$, where $i_p$ is the constant path at $p = \alpha(1)$.

**B.1 The Nonrelativistic Case**

Here $G = \tilde{G}_2$, elements of which are denoted by $(v, \varphi)$, and $M = \mathbb{R}^2$, which will be identified with $\mathbb{C}$. We claim that in this case the equivariant function

$$F_r (\text{cls}(\alpha)) = \exp \left( 2ir \sum_{k<l} \int_{\tilde{\alpha}} d\theta^{kl} \right) \quad \text{(where } \text{pr} \circ \tilde{\alpha} = \alpha \text{)} \quad \text{(B.81)}$$

from equation (2.27) satisfies (B.80) with $D(v, \varphi) = e^{-irn(n-1)\varphi}$. To see this, we have to write down explicitly the action of an element $(v, \varphi) \in \tilde{G}_2$ on $\tilde{p} \in n\tilde{M}$. We set $\tilde{p} = \text{cls}(\alpha)$ with $\alpha(t) = \text{pr}(\alpha_1(t), \ldots, \alpha_n(t))$ and take $\gamma(t) = (tv, t\varphi)$ as a path in $G$ starting at the identity and ending at $(v, \varphi)$. Then the above definition of the action yields

$$(v, \varphi) \cdot \text{cls}(\alpha) = \text{cls}(\gamma \cdot \alpha) \text{ with } (\gamma \cdot \alpha)(t) = \text{pr} \left( e^{it\varphi} \alpha_1(t) - tmv, \ldots, e^{it\varphi} \alpha_n(t) - tmv \right) \quad \text{(B.82)}$$

Denoting by $\alpha_{kl}(t)$ the path $\alpha_l(t) - \alpha_k(t)$, we obtain

$$\int_{\tilde{\alpha}} d\theta^{kl} = \int_{\alpha_{kl}} d\theta = \text{Im} \int_{\alpha_{kl}} \frac{dz}{z} = \text{Im} \int_0^1 \frac{\alpha'_{kl}(t)}{\alpha_{kl}(t)} dt,$$

and hence

$$\int_{\gamma \cdot \alpha} d\theta^{kl} = \varphi + \int_{\tilde{\alpha}} d\theta^{kl}.$$

Inserting these formulas into (B.81), we see that for any $\tilde{p} \in n\tilde{M}$, the desired relation $F_r (\tilde{p}) F_r ((v, \varphi) \cdot \tilde{p})^{-1} = e^{-irn(n-1)\varphi}$ follows.

**B.2 The Relativistic Case**

Here $G = \tilde{L}_3^1$, which is the covering group of a simple group, and therefore the one dimensional representation $g \mapsto D(g)$ of equation (B.80) is necessarily
trivial. Hence the line bundle $\mathcal{F}$ is trivial as an $\mathbb{L}^+$-bundle if and only if there is a smooth nowhere vanishing function $F$ from $\mathbb{N}^n$ into the complex numbers satisfying

$$
F(\tilde{p} \cdot b) = g(b)^{-1} F(\tilde{p}) \quad \text{for all } \tilde{p} \in \mathbb{N}^n, b \in B_n, \quad \text{and}
$$

$$
F(g \cdot \tilde{p}) = F(\tilde{p}) \quad \text{for all } \tilde{p} \in \mathbb{N}^n, g \in \mathbb{L}^n.
$$

A necessary condition for such an $F$ to exist is that the representation $g$ maps any braid $b \in B_n$, for which there are $\tilde{p} \in \mathbb{N}^n$ and $g \in \mathbb{L}_2$ with $\tilde{p} \cdot b = g \cdot \tilde{p}$, to the identity. Now we claim that the generator $b_1$ of the braid group can be written as the product of two braids $b_{(1)}$ and $b_{(2)}$, each of which satisfies the above condition, i.e. there are $\tilde{p}_1, \tilde{p}_2 \in \mathbb{N}^n$ and $g_1, g_2 \in \mathbb{L}^n$ such that $\tilde{p}_j \cdot b_{(j)} = g_j \cdot \tilde{p}_j \ (j = 1, 2)$. Consequently, $g$ has to act trivially on $b_{(1)}b_{(2)} = b_1$.

Since, as a consequence of the defining relations (2.25), each $f_{(j)}$ satisfying (B.83) (for $j = 1, 2$), each $b_k$ is conjugate to $b_1$, $g$ is trivial on the generators and hence on the whole group $B_n$.

To prove our claim, we have to find two points $p_1, p_2 \in \mathbb{N}^n$, paths $\alpha_1$ and $\alpha_2$ from the base point $p_0$ to $p_1$ and $p_2$ respectively, $g_1$ and $g_2 \in \mathbb{L}^n$, and $b_{(1)}, b_{(2)} \in B_n = \pi_1(\mathbb{N}, p_0)$ such that $b_{(1)}b_{(2)} = b_1$ and

$$
\text{cls}(\beta_{(j)} \ast \alpha_j) = \text{cls}(\alpha_j \ast (\gamma_j \cdot i_{p_j})) \quad (j = 1, 2).
$$

(B.83)

Here $\beta_{(j)}$ is a loop at the base point $p_0$ whose homotopy class is $b_{(j)}$, and $\gamma_j$ is a path in $\mathbb{L}^2$ from the identity to $g_j$.

Taking $p_1 := \text{pr}(1, 0, e^{i\theta_1}, e^{i2\theta_1}, \ldots, e^{i(n-2)\theta_1})$ with $\theta_1 = \frac{2\pi}{n}$, and $\gamma_1(t) := t\theta_1$, we get a path $\gamma_1 \cdot i_{p_1}$ which is free homotopic as a closed path in $\mathbb{N}$ to the following loop $\beta_{(1)}$ at the base point $p_0$:

Fig. 1

This is equivalent to the statement that there is a path $\alpha_1$ from $p_0$ to $p_1$ satisfying (B.83) (for $j = 1$).

Let $p_2 := \text{pr}(1, e^{i\theta_2}, e^{i2\theta_2}, \ldots, e^{i(n-1)\theta_2})$ with $\theta_2 = \frac{2\pi}{m}$, and let $\gamma_2(t) := -t\theta_2$. Then the closed path $\gamma_2 \cdot i_{p_2}$ is free homotopic to the loop $\beta_{(2)}$ given by

Fig. 2

Hence there is a path $\alpha_2$ from $p_0$ to $p_2$ satisfying (B.83) for $j = 2$, and it remains to be shown that the loop $\beta_{(1)} \ast \beta_{(2)}$ represents the generator $b_1$ of the braid group. Pictorially this can be checked as follows:

Fig. 3

which corresponds to the relations

$$
b_{(1)} = [\beta_{(1)}] = b_1b_{n-1}b_{n-2}\cdots b_2b_1,
$$

$$
b_{(2)} = [\beta_{(2)}] = b_1^{-1}\cdots b_{n-1}^{-1}, \text{ and hence}
$$

$$
b_{(1)}b_{(2)} = b_1.
$$

This proves the claim.
References

[Ar] Arnol'd, V.I., O nekto ri topologitcheski invari anta algebraicheskih funkci (On Some Topological Invariants of Algebraic Functions), Trudi Moskovskogo Mathematicheskogo Obshchestva. 21, 27-46 (1970)

[BA1] Bargmann, V., Irreducible Unitary Representations of the Lorentz Group, Ann.Math. 48 568-640 (1947)

[BA2] Bargmann, V., On Unitary Ray Representations of Continuous Groups, Ann.Math. 59 1-46 (1954)

[BF] Buchholz D., Fredenhagen, K., Locality and the Structure of Particle States, Commun.Math.Phys. 84, 1-54 (1982)

[Br] Bredon, G.E., Introduction to Compact Transformation Groups, Academic Press (1972)

[Bri] Brieskorn, E., Sur les groupes de tresses, Lect. Notes Math. 317, 21-44 (1971/72)

[Bi] Birman, J.S., Braids, Links and Mapping Class Groups, Princeton University Press, Princeton 1975

[Do] Dowker, J.S., Remarks on non-Standard Statistics, J.Phys. A 18, 3521-3530 (1985)

[F] Fredenhagen, K., Structure of Superselection Sectors in Low Dimensional Quantum Field Theory, in: Chau, L.L., Nahm, W. (eds): Proceedings, Lake Tahoe City 1989

[FG] Fröhlich, J, Gabbiani, F, Braid Statistics in Local Quantum Field Theory, Rev.Math.Phys. 2, 251-353 (1990)

[FGR] Fredenhagen, K., Gaberdiel, M., Rüger, S.M., Scattering States of Plektons (Particles with Braid Group Statistics) in 2+1 Dimensional Field Theory, University of Hamburg preprint (1992)

[FM1] Fröhlich, J., Marchetti, P.A., Quantum Field Theories of Vortices and Anyons, Commun.Math.Phys. 121, 177-223 (1989)

[FM2] Fröhlich, J., Marchetti, P.A., Spin-Statistics Theorem and Scattering in Planar Quantum Field Theories with Braid Statistics, Nucl.Phys.B 356 533-573 (1991)
[FRS] Fredenhagen, K., Rehren, K.-H., Schroer, B., Superselection Sectors with Braid Group Statistics and Exchange Algebras I: General Theory, II: Geometric Aspects and Conformal Covariance, Commun.Math.Phys 125 201-226 (1989) and preprint (to appear in: Rev.Math.Phys.)

[GA] Gaberdiel, M., private communication

[GMS] Goldin, G., Menikoff, R., Sharp, D.H., Representations of a Local Current Algebra in Nonsimply Connected Space and the Aharonov-Bohm Effect, J.Math.Phys. 22, 1664-1668 (1981)
Goldin, G., Sharp, D.H., Rotation Generators in Two-Dimensional Space and Particles Obeying Unusual Statistics, Phys.Rev. D 28, 830-832 (1983)
Goldin, G., Menikoff, R., Sharp, D.H., Phys.Rev.Letts. 54, 603 (1985)

[Gr] Grigore, D.R., The Projective Unitary Irreducible Representations of the Galilei Group in 1+2 Dimensions, preprint IFA-FT-391-1993, Bucharest, Romania, and hep-th/9312048

[LM] Leinaas, J.M., Myrheim, J., On the Theory of Identical Particles, Il Nuovo Cimento 37 b 1-23 (1977)

[Mi] Mihor, J., Singular Points of Complex Hypersurfaces, Princeton University Press, Princeton 1968

[Mu] Mund., J., Quantenmechanik von nichtrelativistischen Teilchen mit Zopfgruppenstatistik, diploma thesis (Freie Universität Berlin, 1992)

[Ni] Nill, F., A Constructive Quantum Field Theoretic Approach to Chern Simons Theory, Int.J.of Mod.Phys. 6 2159-2198 (1992)

[Pu] Pukanszky, L., The Plancherel Formula for the Universal Covering Group of SL(2, R), Math.Ann. 156 96-143 (1964)

[S] Schrader, R., Zur mathematischen Formulierung von Quantensystemen mit Zopfgruppenstatistik, unpublished notes (1989)

[Sa] Sally, P.J., Jr., Uniformly Bounded Representations of the Universal Covering Group of SL(2, R), Bull.Am.Math.Soc. 72, 269-273 (1966)

[Va] Varadarajan, V.S. The Geometry of Quantum Theory, Vol.II, Van Nostrand (1970)

[v.d.W] van der Waerden, Algebra I, Springer Verlag (1971)
[We] Wenzl, H., *Representations of Braid Groups and the Quantum Young Baxter Equation*, Pacific Journal of Mathematics **145** No.1 153-180 (1990)

[Wi] Wilczek, F., *Quantum Mechanics of Fractional-Spin Particles*, Phys.Rev.Lett. **49** No.14 957-1149 (1982)

[Wu] Wu, Y.S., *Multiparticle Quantum Mechanics Obeying Fractional Statistics*, Phy.Rev.Lett. **53** No.2, 111-114 (1984)