Generalized Hitchin systems and Knizhnik-Zamolodchikov-Bernard equation on elliptic curves

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Abstract

Knizhnik-Zamolodchikov-Bernard (KZB) equation on an elliptic curve with a marked point is derived by the classical Hamiltonian reduction and further quantization. We consider classical Hamiltonian systems on cotangent bundle to the loop group $L(GL(N, \mathbb{C}))$ extended by the shift operators, to be related to the elliptic module. After the reduction we obtain the Hamiltonian system on cotangent bundle to the moduli of holomorphic principle bundles and the elliptic module. It is a particular example of generalized Hitchin systems (GHS) which are defined as hamiltonian systems on cotangent bundles to the moduli of holomorphic bundles and to the moduli of curves. They are extensions of the Hitchin systems by the inclusion the moduli of curves. In contrast with the Hitchin systems the algebra of integrals are noncommutative on GHS. We discuss the quantization procedure in our example. The quantization of the quadratic integral leads to the KZB equation. We present the explicit form of higher quantum Hitchin integrals, which upon on reducing from GHS phase space to the Hitchin phase space gives a particular example of the Belinson-Drinfeld commutative algebra of differential operators on the moduli of holomorphic bundles.

1 Introduction

The Knizhnik-Zamolodchikov system of equations [1] relates the dependence of conformal blocks in the Wess-Zumino-Witten-Novikov (WZWN) theory on the positions of the insertion points on a sphere. Later this system was generalized on higher genera curves [2]. In the generic situation we will call it Knizhnik-Zamolodchikov-Bernard (KZB) system. The further progress in the investigation of KZB was achieved in [3, 4]. Generically it describes the dependence of conformal blocks in WZWN theory on the moduli of curves.

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with marked points. Another type of independent variables in the KZB system is the moduli of holomorphic bundles on a curve \( \Sigma \). The connection between the moduli of curves and the moduli of bundles brings together by the Sugawara construction, which is a starting point in the derivation of KZB.

KZB can be derived in a completely geometric way from the Chern-Simons theory on \( \Sigma \times R \) [5, 6]. The physical space of this theory is the moduli space \( \mathcal{M}^{fl} \) of flat \( G \) bundles over \( \Sigma \), where \( G \) is a compact simple group. It is a finite-dimensional symplectic manifold. It can be endowed with a complex structure induced from \( \Sigma \) and therefore \( \mathcal{M}^{fl} \) becomes a Kähler manifold. There is a family of Kähler structures on \( \mathcal{M}^{fl} \) depending on \( \Sigma \) we picked up. Let \( \mathcal{L} = \det \bar{\partial}A \) be the determinant line bundle of family of operators on \( \Sigma \) parametrized by \( G \) connections \( A \), \( k \) is the "level", \( h^\vee \) is the dual Coxeter number of \( G \). In the geometric quantization of \( \mathcal{M}^{fl} \) for a Kähler polarization one takes as a physical Hilbert space the space of global holomorphic sections of \( \mathcal{L}^{k+h^\vee} \) over \( \mathcal{M}^{fl} \). This space formally depends on the choice of the complex structure. It leads to a bundle of quantum Hilbert spaces over the Teichmüller space \( \mathcal{T} \). According to [5, 6] there exists a projective identification of different fibers which makes the bundle projectively flat. It gives rise to the canonical (independent on the choice of complex structures) quantization. It turns out that this bundle is the same as the bundle of conformal blocks in the WZWN theory. In these terms the KZB operators are identified with a projectively flat connections in the bundle. The connection has a form of a heat type operator

\[
\nabla^{KZB} = (k + h^\vee) \frac{\partial}{\partial \tau} + B.
\]  

Here the first term is derivation along the Teichmüller space and \( B \) is the second order differential operator on \( \mathcal{M}^{fl} \). Solutions to the corresponding heat equation for \( G = U(1) \) are the theta functions on the Jacobian. For nonabelian gauge group they are called the nonabelian theta functions.

Here we adopt a different point of view on the KZB system (1). Consider the classical Hitchin system [7]. It is defined as a classical system on the cotangent bundles \( T^* \mathcal{M}^{hol} \) to the moduli of holomorphic \( G^C \) bundles to a Riemann curve. Here \( G^C \) is the complex form of \( G \). It is an integrable system since it has \( (\dim \mathcal{M}^{hol}) \) integrals in involution. As it happens in the theory of interable systems, the Hitchin system is obtained from "a big free system" by the Hamiltonian reduction. The second order Hitchin integral after quantization coincides with the operator \( B \) in (1) and therefore it gives rise the KZB on the critical level \( k = -h^\vee \). Our main goal is a continuation of this correspondence between the KZB operators and quadratic integrals in some quantum dynamical system beyond the critical level. The integrals should depend on \( (k + h^\vee) \) in a such way that for \( k = -h^\vee \) they coincide with the Hitchin Hamiltonian. In spite of the presence of the first order derivatives in the KZB operator it will be identified with a stationary Schrödinger operator and conformal blocks will be described as the ground states wave functions.

We start from a classical Hamiltonian picture based on some generalization of the Hitchin system. We extend this approach by the introducing the dynamics along the Teichmüller space \( \mathcal{T} \) as well. The cotangent bundle to the both type of variables \( T^*(\mathcal{M}^{hol} \times \mathcal{T}) \) is obtained by the Hamiltonian reduction from "the big phase space", which is an extension of the Hitchin phase space by including parameters of curves.
The gauge group acting on it is the semidirect product of the diffeomorphisms of curve and the gauge group of the bundle. There is a set of invariant Hamiltonians on this phase space. We will call the reduced system the Generalized Hitchin system (GHS). We push down the quadratic Hamiltonian from the big phase space to the phase space of GHS and quantize it. The Schrödinger operator take the parabolic form. It can be identify with the KZB operator up to a fixing some parameters. The role of later is still unclear and this is one of reasons which does not allow to identify the both kinds of constructions completely. Another reason is that, strictly speaking, the quantization procedure is incomplete. Since the symplectic form of our system is of the (2, 0) type the direct application of the geometric quantization does not work. Therefore we did not succeed to define the physical Hilbert space.

Our construction is closed to [6]. In this work the physical phase space $M^{fl}$ is constructed as the symplectic quotient of the space of $G$ connections on $\Sigma$ by the group of bundle automorphisms. The Teichmüller space is included in the game on the next stage, as it was described above. In our construction the physical phase space $T^*(M^{hol} \times T)$ is also the symplectic quotient of cotangent bundle to the set of curves and holomorphic bundles over them. The gauge group is the semidirect product of diffeomorphisms of curves and holomorphic bundles automorphisms. Since for stable bundles there exists a map from $M^{fl}$ to $M^{hol}$ (Narasimhan-Seshardi theorem) our physical phase space being projected on the $T^*M^{hol}$ component is similar to the cotangent bundle to $M^{fl}$. The main difference is the consideration of the moduli of bundles and the Teichmüller space on equal terms.

We use in our construction of the moduli of holomorphic bundles the Schottky parametrization of curves. It leads to the Čech like description of the moduli space, while the original construction [7] is based on the Dolbeault cohomologies. At the moment this difference is looked rather technical and leads eventually to the same result, but it may have some advantages in the quantization of generic systems which is postponed for future. Here we restrict ourselves by the simplest nontrivial example - an elliptic curve with a marked point. The KZB equation in this case has been written explicitly in [8, 9]. Its derivation is the direct application of the Sugawara construction for the representation of level $k$ of the loop groups. It takes the form of the heat equation with the additional elliptic Calogero potential. The appearing of well known integrable systems in the Hitchin approach is not a new phenomena [10, 11, 12]. Among them is $N$-body elliptic Calogero system. Though, due to the exploiting the Schottky parameters, our derivation is different from [11], where it has been derived, we obtain the same system with the additional degree of freedom along the module of elliptic curves as it should be.

We also consider the higher quantum integrals. Upon reducing them to the conventional Hitchin subspace they leads to a commuting algebra of differential operators on the moduli of holomorphic bundles over elliptic curves with a marked point. It is particular example of the Beilinson-Drinfeld algebra [13]. The later is defined on the moduli of holomorphic bundles over curves of genus $g > 1$ as a result of quantization of the Hitchin system.
2 Classical system

Let

$$\Sigma_\tau = \mathbb{C}^*/q^\mathbb{Z}, \ q = \exp 2\pi i \tau,$$

be a family of elliptic curves. The holomorphic principal $GL(N, \mathbb{C})$ bundle $P$ over $\Sigma_\tau$ with sections $h(z) \in \Omega^0(\Sigma_\tau, P)$ can be defined by the transition map $g(z) \in L(GL(N, \mathbb{C}))$

$$h(z) = g(z)h(qz).$$

Two bundles are equivalent if their transition maps are conjugated in the following sense

$$g_1(z) = f(z)g(z)f^{-1}(qz). \quad (2)$$

This action defines the gauge group $\{f(z)\}$, which is also $L(GL(N, \mathbb{C}))$. The last relation suggests that instead of $\{g(z)\} \sim L(GL(N, \mathbb{C}))$ we should consider the semidirect product

$$\mathcal{G} = \tilde{L}(GL(N, \mathbb{C})) = \{(\exp(2\pi i \tau z \partial, g(z))\},$$

where $\partial = \partial_z$. The group element can be represented as the product $g(z)T_q$, $q = \exp(2\pi i \tau)$, where $T_q$ is the shift operator $T_qg(z) = g(qz)$ with the evident multiplication $g_1(z)T_{q_1} \cdot g_2(z)T_{q_2} = g_1(z)g_2(q_1z)T_{q_1q_2}$. Note that the adjoint action of $L(GL(N, \mathbb{C}))$ preserves $q$. We have $\text{Lie}(\mathcal{G}) = \{2\pi i \tau z \partial + x(z), \ x(z) \in \Omega^0(S^1, \text{End}P)\}$. The dual space $\text{Lie}^*(\mathcal{G}) = \{\frac{1}{2\pi i} z^{-2}d^2 + \phi(z), \ \xi \in \mathbb{C}, \ \phi(z) \in \Omega^1(S^1, \text{End}P)\}$ is defined by the pairing $\xi \tau + \frac{1}{2\pi i} \text{tr}(S^1 \phi(z)dx).$

Consider the cotangent bundle $T^*\mathcal{G}$. The gauge transformation (2) can be lifted on $T^*\mathcal{G}$

$$x(z) \rightarrow f(z)x(z)f^{-1}(z) + 2\pi i \tau f(z)z \partial f^{-1}(z), \ \phi \rightarrow f(z)\phi(z)f^{-1}(z), \ \tau \rightarrow \tau, \ \xi \rightarrow \xi - \frac{1}{2\pi i} \text{Res}[z \text{tr}(df f^{-1}(z)\phi(z))]. \quad (3)$$

In addition we have a marked point on $\Sigma_\tau$ at $z = 1$. Classically we can put in it additional degrees of freedom in the same way as it was done in ($[11, 15, 16, 17]$). They are defined by a symplectic manifold which is a cotangent bundle to $G = GL(N, \mathbb{C})$. In other words it is the pair $p \in \text{Lie}^*(GL(N, \mathbb{C}))$, $h \in GL(N, \mathbb{C})$. The gauge transform $f(z)$ acts as the evaluation map at $z = 1 \ p \rightarrow f(1)p f^{-1}(1)$ and $h \rightarrow f(1)h$.

Therefore we have the manifold with fields $\mathcal{R}' = \{[T^*\mathcal{G} = (\xi, \tau, \phi(z), g(z))] \cup \{T^*G = (p, h)\}$. We endow it with the standard symplectic structure. By means of the Maurer-Cartan form on $\mathcal{G}$ $Y(\tau, g) = 2\pi i D\tau z \partial + g^{-1}Dg(q^{-1}z)$ we can define the canonical Liouville two form on the cotangent bundle $T^*\mathcal{G}$. Adding the same form on $T^*G$ we obtain the symplectic form which takes in account the contribution from the marked point

$$\omega = D\xi D\tau + \frac{1}{2\pi i} \text{tr} \int_{S^1} D(\phi(qz), g^{-1}Dg(z)) + D\text{tr}(ph^{-1}Dh). \quad (4)$$

It is invariant under the gauge action of $L(GL(N, \mathbb{C}))$. There is the set of $N + 1$ independent invariant Hamiltonians. They have the following form. Represent the element $g(z)$
as \( g(z) = \exp(2\pi i \tau z \partial + x(z))T_q^{-1} \). Then the following quantities are invariant under the gauge action (3)

\[
I_n = \frac{\tau^{n-1}}{2\pi in} \text{tr} \int_{S^1} \phi^n(z), \quad n = 1, \ldots, N, \tag{5}
\]

\[
L = \xi \tau + \frac{1}{2\pi i} \text{tr} \int_{S^1} (\phi(z)x(z)).
\]

Hamiltonians \( I_n \) are the same as in the Hitchin system [7]. They define motions along the moduli of the bundle. The additional Hamiltonian \( L \) includes also a motion along the elliptic module in the consistent way. While \( I_n \) are well defined on the moduli of holomorphic bundles, \( L \) is defined only on the covering of the both kind of moduli. The Hamiltonians form the closed noncommutative algebra with respect the Poisson brackets coming from (4)

\[
\{I_j, I_l\} = 0, \quad \{L, I_n\} = -I_n. \tag{6}
\]

The algebra is defined also only on the covering of the moduli space. The gauge action on \( R' \) of \( L(GL(N, C)) \) produces the moment map \( \mu : R' \to \text{Lie}^*(L(GL(N, C))) \), which we put equal to zero

\[
\mu = g(z)\phi(qz)g^{-1}(z) - \phi(z) + hph^{-1}\delta(z) = 0, \tag{7}
\]

where \( \delta(z) \) is defined as \( \frac{1}{2\pi i} \int_{S^1} F(z)\delta(z) = F(1) \). For higher genera curves \( \Sigma \) it is possible to add to the gauge transform the action of the diffeomorphisms or by what they are replaced in specific realizations of curves. They produce the additional moment constraints which lie in \( \text{Lie}^*(\text{Diff}\Sigma) \). In the present construction this action has been fixed already, since we have the forth mentioned realization of the elliptic curves as an annulus with the parameter \( \tau \).

It is instructive to write down the action corresponding to the theory defined by the symplectic form [4], constraints (7) and a Hamiltonian that can be an arbitrary linear combination of the invariant Hamiltonians (3)

\[
S = \int dt[\xi \partial_t \tau + \text{tr}(ph^{-1}\partial_\tau h)] + \frac{1}{2\pi i} \int_{S^1} (\phi(qz)A_0(z)^g(z) - \phi(z)A_0(z) - \delta(z)hph^{-1}(z)A_0(z)) - H(L, I_1, \ldots I_N)]. \tag{8}
\]

Here \( A_0(z) \) is the Lagrange multiplier, \( A_0^g = g^{-1}\partial_\tau g + g^{-1}A_0g \). The action describes nonlocal theory on the cylinder \( S^1 \times R \). The nonlocality is a result of the Schottky parametrization. The Hitchin approach based on the Dolbeault cohomologies leads to a local theory but in 2 + 1 dimension [14]. Our description seems to be more preferable in quantum calculations.

We will construct the reduced space \( R = R'/\!/L(GL(N, C)) = \mu^{-1}(0)/L(GL(N, C)) \). By the gauge transform (3) we can diagonalize \( g(z) \). Moreover it can be chosen as a constant matrix since the vector bundles over elliptic curves are reduced to a sum of linear bundles. Let represent it as \( f(z)g(z)f^{-1}(qz) = \text{diag}\exp(2\pi i \bar{u}), \quad \bar{u} = (u_1, \ldots, u_N) \in \mathcal{H} \), where \( \mathcal{H} \) is the Cartan subalgebra of \( \text{Lie}(GL(N, C)) \). The remnant gauge symmetries that preserves the diagonal \( z \)-independent form of matrix \( \text{diag}\exp(2\pi i \bar{u}) \) will be discussed at
the end of this section. The gauge invariant subvariety in $T^*G$ at the marked point is the coadjoint orbit, which is fixed by the invariants $O_p = \{ \chi = h p h^{-1} | tr p^j = c_j \}$. We still have a freedom to act by the diagonal gauge transformations $(C^*)^N$ on the points of the orbit $O_p$. This action is not free - only $(C^*)^{N-1}$ part acts effectively, where $l + 1$ is the number non equal eigenvalues of $p$. Thus the reduced space is
\[
R = T^*N, (O_p//(C^*)^N),
\]
where $T^*N = T^*G//L(GL(N, C))$ is the cotangent bundle to the moduli of elliptic curves and the moduli of holomorphic bundles. The space $O_p//(C^*)^N$ defines the parabolic structure at the marked point $z = 1$. The dimension of $R$ is equal $\dim R = 2N + 2 + [\dim O_p - 2(N - l)]$. Here the first term is responsible for the holomorphic bundles moduli, the second for the elliptic module and the last comes from the orbit in the marked point.

After the resolving the moment constraints (7)
\[
\phi_{j,j}(qz) - \phi_{j,j}(z) = \chi_{j,j} \delta(z), \ (\chi = h p h^{-1}),
\]
\[
\phi_{j,k}(qz) \exp 2\pi i (u_j - u_k) - \phi_{j,k}(z) = \chi_{j,k} \delta(z)
\]
we find
\[
\phi_{j,j}(z) = p_j, \ \chi_{j,j} = 0,
\]
\[
\phi_{j,k} = -\frac{\chi_{j,k} \theta(u_j - u_k - \zeta)\theta'(0)}{2\pi i \theta(u_j - u_k)\theta(\zeta)}, \ z = \exp 2\pi i \zeta
\]
where $p_j, j = 1, \ldots, N$ are new free parameters and $\theta(\zeta) = \sum_{n \in \mathbb{Z}} e^{\pi i (n^2 \tau + 2n \zeta)}$. The symplectic form (4) on the reduced space takes the form
\[
\omega^{red} = D\xi D\tau + D\vec{p} \cdot D\vec{u} + \text{tr} D(h^{-1}\chi Dh).
\]

Consider the quadratic Hamiltonian which is the linear combination $H = \alpha L + I_2$, $\alpha \in C$. Its form resembles the Sugawara construction. We can also consider higher Hamiltonians as well. After the reduction $H$ takes the form of the N-body elliptic Calogero Hamiltonian with the spins [13, 14, 17] and the additional terms coming from $L$
\[
H = \alpha(\xi \tau + \vec{p} \cdot \vec{u}) + \frac{\tau}{2}(\vec{p} \cdot \vec{p} + \sum_{j>k} \chi_{j,k} \chi_{k,j} U(u_j - u_k, \tau)
\]
\[
U(u, \tau) = \frac{1}{4\pi^2} [\varphi(u|\tau) + \frac{\pi^2}{3} E_2(\tau)].
\]
Here $\varphi(u|\tau)$ is the Weierstrass elliptic function and $E_2(\tau)$ is the normalized Eisenstein series which is defined as [18]
\[
E_2(\tau) = \frac{3}{\pi^2} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty'} \frac{1}{(m\tau + n)^2} = \frac{24}{2\pi i} \eta'(\tau) \frac{\eta(\tau)}{\eta(\tau)}
\]
where ' means that $n \neq 0$ if $m = 0$ and $\eta(\tau)$ is the Dedekind $\eta$-function. The spin degrees of freedom $\chi$ can be described in a more explicit form (see details in [11]). Consider the
flag variety corresponding to the orbit $O_p V^0 = C^N \supset V^1 \supset \ldots V^l$. Then $\chi_{j,k} \chi_{k,j}$ can be replaced by $\text{tr}_{V^i} S_j S_k$, where $S_j, S_k$ are matrices in $V^1$.

The symmetries of the Hamiltonian are simultaneous permutations of $u_k$ and $p_k$ extended by the shifts

$$\vec{u} \rightarrow \vec{u} + n\vec{\beta}, \quad p_i \rightarrow p_j, \quad \tau \rightarrow \tau, \quad \xi \rightarrow \xi - n(\vec{\beta}, \vec{p}),$$

(11)

$$n \in \mathbb{Z}, \quad \vec{\beta} \in \mathbb{R}^\vee$$

The symmetries of the symplectic form can be extended by the additional real shifts of $\vec{u}$:

$$\vec{u} \rightarrow \vec{u} + m\vec{\gamma}, \quad \vec{\gamma} \in \mathbb{R}^\vee.$$ The group $\hat{W}$ which is the semidirect extension of the Weyl group $W$ (the permutations at hand) by shifts $\mathbb{Z} R^\vee \tau + \mathbb{Z} R^\vee$ is a particular example of complex crystallographic Coxeter groups \[19\]. The factor $\mathcal{H}/\hat{W}$ describes almost all holomorphic $GL(N, \mathbb{C})$ bundles over $\Sigma$ (see, for example \[20\]). Thus the symplectic structure is well defined on $\mathcal{M}^\text{hol} \times T$. Let $W'$ be the extension of $W$ by the part of the shifts $\mathbb{Z} R^\vee \tau$. In other words $\hat{W} = W' \odot \mathbb{Z} R^\vee$. Consider the covering $\mathcal{M}^\text{hol} = \mathcal{H}/W'$ of $\mathcal{M}^\text{hol}$. The Hamiltonian (10) due to the second term lives on the covering $\mathcal{M}^\text{hol} \times T$. We will see in the next section that after the quantization that Hamiltonian can be push down on $\mathcal{M}^\text{hol} \times T$ as well.

3 Quantum system

We should quantize the classical Hamiltonian system with symplectic structure (8) and Hamiltonian (10) and with additional symmetries (11). There is also the algebra of higher Hamiltonians (6) which is desirable to preserve on the quantum level.

There are two type of variables - those that related to the coorbit and to the moduli spaces. The quantization of coadjoint orbits is well developed procedure \[21\]. The symplectic form on the orbit $O_p$ is the Kirillov-Kostant form. Moreover coadjoint orbits are complex manifolds such that the symplectic form has type $(1,1)$ and so it allows to use the Kähler polarization after the holomorphic prequantization. The only restriction is the integrability of the orbit $O_p$ which gives rise to the correspondence of $p$ to a dominant weight $\Lambda_p$. After the quantization we obtain the representation of the dominant weight $\Lambda_p$. Functions on the orbit become operators in the representation space. The factorization of $O_p$ under the action of the diagonal subgroup leads on the quantum level to the additional restriction - only vectors from the zero weight subspace contributes to the quantum Hilbert state.

The quantization of others degrees of freedom is a rather subtle procedure. Note first of all that the symplectic structure (8) is the holomorphic $(2,0)$ form on $T^* \mathcal{N}$. This property follows from our definition of the phase space as the cotangent bundle to the complex manifold. This form defines the polarization from the very beginning. But in contrast with the $(1,1)$ forms it does not lead to topological restrictions on the physical Hilbert space since it defines only trivial line bundles. Another unpleasant property of the system is the definition of the "coordinate type" variables $\tau$ and $\vec{u}$ which do not belong to the moduli spaces, but only their coverings. As we have mentioned above it comes from our definition of the hamiltonian $L$. We will see later that the quantization
improves partly the situation - the quantum Hamiltonian is well defined on the moduli of holomorphic bundles.

There are evidences that the first difficulty can be overcome as well at least in the original Hitchin systems in which we encounter with the same problem of the quantization of $(2,0)$ form.

A. Beilinson and V. Drinfeld [13] proposed the quantization of the original Hitchin system, defined on the cotangent bundles to the moduli space of holomorphic $G$-bundles over arbitrary curves $\Sigma$ of genus $g > 1$ without marked points. They constructed a canonical morphisms from the ring of polynomial functions generated by the Hitchin integrals of type $(6)$ to a sheaf of differential operators on the moduli of holomorphic bundles $\mathcal{M}_G(\Sigma)$. The algebra of differential operators is commutative and globally defined on the moduli space. Let $K$ be a canonical bundle on $\mathcal{M}_G(\Sigma)$. It is defined as the tensor degree of the determinant line bundle $L = \det \partial_A$, $K = L^{\otimes h^\vee}$, where $h^\vee$ is the dual Coxeter number. Then the physical Hilbert space is the space of section of the bundle $K^{1/2}$ of half-forms. They are generated by the center $\mathcal{C}_{-h^\vee}$ of the universal enveloping algebra $U_k(\text{Lie} \hat{L}G)$ on the critical level $k = -h^\vee$. It was proved in [22] that $\mathcal{C}_{-h^\vee}$ is isomorphic to the classical $W_G'$ for the dual group $G'$. This construction eventually leads to a consistent system of differential equations on $\mathcal{M}_G(\Sigma)$. The second order differential operators coincide with the KZB operators depending on the moduli of holomorphic bundles. Thus one way to derive the full KZB system as the Hamiltonian system is the reproducing the Beilinson-Drinfeld procedure for the space unifying the both kind of moduli. Note that essential part of their construction is a representation of the moduli of holomorphic bundles as a double coset space of the loop group $L(G)$. On the other hand the moduli of curves is also can be represented as a double coset space of the diffeomorphisms of a circle. Their unification does not look very natural. It seems that a more relevant construction is the unification of the moduli of holomorphic $G$-bundles with the moduli of corresponding $W_G$ structures. In contrast with the original Hitchin system the family of integrals are noncommutative on the unified space, as we have seen in our simplest example (6). Of course noncommutativity holds in a generic case.

We stop here the discussion about the rigorous quantization of the GH system. We shall not attempt to define a Hilbert space in which our observables act and restrict ourselves to the consideration of a formal algebra of operators. Since the quantization of degrees related to the orbit is straightforward we will consider the system with a simplest nontrivial orbit. The classical Hamiltonian in this case is

$$H_\alpha = \alpha(\xi \tau + \vec{p} \cdot \vec{u}) + \frac{\tau}{2}(\vec{p} \cdot \vec{p} + \nu^2 U(\vec{u}|\tau)),$$

where we have chosen the most degenerated orbit $\mathbb{C}P^{N-1}$. It is conjugate to the matrix $\nu(e^T \otimes e - \text{diag}(1, \ldots, 1))$, $e = (1, \ldots, 1)$ . Define the following quantization of the classical quantities

$$\rho(u_j) = u_j, \quad \rho(p_j) = \nabla_j = i\partial_j + \frac{i\alpha u_j}{\tau}, \quad (\partial_j = \frac{\partial}{\partial u_j})$$

$$\rho(\tau) = \tau, \quad \rho(\xi) = \nabla_\tau = i\partial_\tau - \frac{i\alpha(\vec{u} \cdot \vec{u})}{2\tau^2}.$$  (12)
These covariant derivatives can be gauge transformed from the canonical operators by the Gaussian twist $\Pi = \exp \frac{i\alpha(u, \tilde{u})}{2\tau}$. Namely,

$$\nabla_j = \Pi^{-1} \circ i\partial_j \circ \Pi, \quad \nabla_\tau = \Pi^{-1} \circ i\partial_\tau \circ \Pi.$$ 

Thus (12) is the quantum counterpart of the classical algebra of observables (9). The quantization leads to the quantum Hamiltonian

$$\hat{H}_\alpha = \alpha(\tau \nabla_\tau + \sum_{j=1}^N u_j \nabla_j) + \frac{\tau}{2} \left( \sum_{j=1}^N \nabla_j^2 + \nu^2 U(\tilde{u}|\tau) \right),$$

where $\nu^2 = \frac{m(m+1)}{2}$, $m \in \mathbb{N}$ due to the quantization of the orbit. We study the eigenvalue problem

$$\hat{H}_\alpha \Psi_E(\tilde{u}, \tau) = E \Psi_E(\tilde{u}, \tau),$$

where $\Psi_E(\tilde{u}, \tau)$ is defined on the product of $\{\tilde{u}, |M^{\text{hol}} = \mathcal{H}/W \sim \mathbb{C}^{N}/\mathbb{Z}R^\tau \}$ and the halfplane $\text{Im} \tau > 0$. After some algebra it can be transformed to

$$[i\alpha \partial_\tau + \frac{1}{2} \left( \sum_{j=1}^N -\partial_j^2 + \nu^2 U(\tilde{u}|\tau) \right)] \Psi_E(\tilde{u}, \tau) = E' \Psi_E(\tilde{u}, \tau),$$

where $E' = \frac{1}{2}(E - i\alpha N/2)$. Now the quantum Hamiltonian $\hat{H}_\alpha$ in contrast with classical one is well defined on the moduli of bundle, since it is invariant under the shift $\tilde{u} \rightarrow \tilde{u} + \mathbb{Z}R^\nu$. Therefore it lives on $\mathcal{H}/\hat{W}$, which defines the moduli of the $GL(N, \mathbb{C})$ bundles over the elliptic curve. The Hamiltonian $\hat{H}_\alpha$ for $i\alpha = k + h^\vee (h^\vee = N)$ after the gauge transform $\mathcal{D}^{-1}\hat{H}_\alpha \mathcal{D}$, ($\mathcal{D}$ is the Weyl-Kac denominator) coincides with the KZB operator (11) for the elliptic curve with a marked point [8, 9]. Gauge transformed groundstate wave functions $\mathcal{D}^{-1}\Psi_{i\alpha N/2}(\tilde{u}, \tau)$ gives the conformal blocks. They are sections of the line bundle $\mathcal{L}^{k + h^\vee}$ over $\mathcal{H}/\hat{W}$. Recently they were analized in [23, 24]. For $\nu = 0, E' = 0$ we come to the standard heat equations, which solutions are the level $i\alpha$ theta functions on $\mathbb{C}^{N}/\mathbb{Z}R^\nu + \mathbb{Z}R^\tau$.

Though the higher hamiltonians (13) are beyond the problems around the KZB system they can be quantized as well. It can be done by the using old results [25] about quantum Calogero systems. The corrections to the integrals independing on $\tilde{u}$ and depending on $\tau$ can be done in such way that the whole algebra (11) will be satisfied for commutators on $\mathcal{M}^{\text{hol}} \times \mathcal{T}$.

For the quantization it is more convinient to consider another basis in the integrals instead of $\tilde{I}_n$. Let $\hat{J}_n$ be operators which symbols are symmetric polynomials of the form $p_1 p_2 \ldots p_n +$ permutations. The reason is that $\hat{J}_n$ as well as their commutators contain only well defined terms with commuting multipliers. It is easy to show that the following recurrence relation holds

$$i\hat{J}_{n-1} = \frac{1}{N-n+1} \left[ \sum_{j=1}^N u_j, \hat{J}_n \right].$$

On the other hand the highest integral has the following form

$$\hat{J}_N = \exp\{-\frac{\nu}{2} \sum_{k,l} x^2(u_k - u_l)\partial_{\hat{p}_k} \partial_{\hat{p}_l}\} \hat{p}_1 \ldots \hat{p}_N,$$
where $\hat{p}_k = \partial u_k$ and $x(u) = \frac{\wp^{(1/2)} - \wp^{(\tau/2)}}{\sqrt{\wp(u) - \wp^{(\tau/2)}}} + c(\tau)$.

There is also possible to write down the explicit form of the lowest integral $\hat{I}_n$. Introduce the following notations $x_{k,l} = x(u_k - u_l)$, $< x_{k,l} >$ is the trace of the matrix $(x_{k,l})$. Then

$$\hat{I}_3 = \sum_{k=1}^{N} \hat{p}_k^3 + 3\nu^2 \sum_{k \neq l} x_{k,l}^2 \hat{p}_l,$$

$$\hat{I}_4 = \sum_{k=1}^{N} \hat{p}_k^4 + 2\nu^2 \sum_{k \neq l} x_{k,l}^2 (2\hat{p}_l^2 + \hat{p}_k \hat{p}_l) + \nu^4 < x_{k,l} >$$

$$+ \nu^2 \sum_{k \neq l} [2(x_{k,l})' \hat{p}_l - (x_{k,l})''],$$

$$\hat{I}_5 = \sum_{k=1}^{N} \hat{p}_k^5 + 5\nu^2 \sum_{k \neq l} x_{k,l}^2 (2\hat{p}_l^3 + \hat{p}_k^2 \hat{p}_l) + 5\nu^5 < x_{k,l} > \text{diag}(\hat{p}_1, \ldots, \hat{p}_N)$$

$$+ 5\nu^2 \sum_{k \neq l} [2(x_{k,l})' \hat{p}_l - (x_{k,l})'' \hat{p}_l].$$

The last lines in the expressions for $\hat{I}_4$ and $\hat{I}_5$ are quantum corrections to the classical integrals.

These formulae give the explicit form of the Belinson-Drinfeld commutative algebra of global differential operators on the moduli of holomorphic bundles over the elliptic curve with a marked point.

To conclude, we established the connection between the KZB equation on an elliptic curve with a marked point and the heat type extension of the elliptic Calogero quantum Hamiltonian. The later was constructed by means of the Hamiltonian reduction and lives on the moduli of holomorphic bundles and the Teichmüller space. The description of the system is suffered from the absence of correct definition of the quantum Hilbert space. It seems that the construction can be improved by including in the game from the very beginning on the classical level the central charge of the loop group. In this case the classical Hamiltonians can be defined on the cotangent bundle to the holomorphic moduli directly before the quantization. Apparently, it can help to define the quantum Hilbert space as well.

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