A Note on Superconformal $\mathcal{N} = 2$ theories and Orientifolds

Jaemo Park 1, Angel M. Uranga 2

School of Natural Sciences, Institute for Advanced Study
Princeton NJ 08540, USA.

Abstract

We construct the T duals of certain type IIA brane configurations with one compact dimension (elliptic models) which contain orientifold planes. These configurations realize four-dimensional $\mathcal{N} = 2$ finite field theories. For elliptic models with two negatively charged orientifold six-planes, the T duals are given by D3 branes at singularities in the presence of O7-planes and D7-branes. For elliptic models with two oppositely charged orientifold planes, the T duals are D3 branes at a different kind of orientifold singularities, which do not require D7 branes. We construct the adequate orientifold groups, and show that the cancellation of twisted tadpoles is equivalent to the finiteness of the corresponding field theory. One family of models contains orthogonal and symplectic gauge factors at the same time. These new orientifolds can also be used to define some six-dimensional RG fixed points which have been discussed from the type IIA brane configuration perspective.

1E-mail: jaemo@sns.ias.edu
2E-mail: uranga@sns.ias.edu
1 Introduction

There are basically two kinds of brane configurations in string theory that have been used to study four-dimensional $N = 2$ gauge theories. The first, introduced in \cite{1} in the spirit of \cite{2}, makes use of sets of type IIA D4 branes suspended between NS branes. Also, D6 branes may be added without breaking further supersymmetries. The gauge theory is realized in the non-compact part of the D4 brane world-volume. This picture is very intuitive geometrically, and has been particularly useful in finding the exact solution of a large class of $N = 2$ theories. Also, it provides a simple construction of superconformal theories. A particularly nice such family, the so-called elliptic models, can be obtained upon taking the coordinate $x^6$, along which the D4 branes are finite, to be compact.

A second type of construction is realized by type IIB D3 branes probing a certain background \cite{3, 4}. One simple example is that of a set of D3 branes sitting at an $A_k$ singularity \cite{5}. In fact, this system is related to the elliptic models described above by a T-duality along the compact direction in the type IIA brane configuration \cite{6}. Even though this picture has not proved so suitable for solving the $N = 2$ theories, it has several advantages. First, since it does not contain any NS branes, it can be analyzed perturbatively using string theory techniques. Also, it allows a simple realization of superconformal theories, by considering backgrounds in which the type IIB coupling constant (which controls the gauge theory coupling constant) does not depend on the position in spacetime. Finally, the near horizon geometry in these configurations is a quotient of $AdS_5 \times S^5$, and allows the use of the AdS/CFT correspondence \cite{7} to study their supergravity/string theory description of their large $N$ limit.

Other families of elliptic models can be obtained by the introduction of orientifold six-planes \cite{8} (configurations with O4-planes \cite{9, 10} will not be studied in the present paper). It is a natural question what their T-dual type IIB versions are. There are two main classes of models, depending on the charges of the O6-planes. If both are negatively charged, (O6$^-$-O6$^-$ configuration) finiteness of the $N = 2$ field theory requires the presence of D6 branes. Their T-duals are certain type IIB orientifolds, containing O7-planes and D7 branes (a particular case in this family reproduces the F-theory background introduced in \cite{11}, which was used in \cite{3, 12, 13} to the study of some $N = 2$ theories). These are analogous to the orientifolds introduced in \cite{5, 14, 15} to study six-dimensional theories.

The remaining case has oppositely charged O6-planes (O6$^+$-O6$^-$ configuration) \footnote{The configuration with two positively charged orientifolds cannot lead to finite theories.}.

1
Finiteness of the field theory on the D3 branes requires in this case the absence of D6 branes. The T-dual of these configurations has not been determined, and the purpose of this note is to construct such type IIB orientifolds. They will have the property that cancellation of tadpoles is achieved without D7 branes.

In all cases we will show that cancellation of twisted tadpoles is equivalent to the finiteness of the field theory. This follows from the fact that tadpoles are sources for twisted closed string modes, which propagate in two dimensions. The logarithmic solutions to the corresponding Laplace equation correspond to the one-loop evolution of the gauge coupling with the scale. Thus, finiteness of the $\mathcal{N} = 2$ theory follows from tadpole cancellation.

We would like to stress that brane configurations T-dual to those yielding the finite four dimensional theories have been used to construct consistent six-dimensional field theories in \cite{16, 17, 18}. They contain NS fivebranes, D6 and D8 branes, and possibly O6- and O8-planes. By T-dualizing along the compact dimension, the configurations with two negatively charged O8-planes are related to D5 branes at the orientifold singularities introduced in \cite{5, 14, 15}. The orientifolds we will construct in Section 3 can be used to provide the T-dual of the brane configurations with oppositely charged O8-planes. This gives an alternative construction for the six-dimensional field theories introduced in \cite{17, 18}. Comments concerning the six-dimensional version of these models will be made throughout the paper.

The structure of this paper is as follows. In Section 2 we review the brane constructions of the elliptic models with orientifold planes. In Section 3 we briefly comment how the orientifolds in \cite{15} provide the T-dual version of the models with two negatively charged orientifold six-planes. In Section 4 we describe the type IIB orientifolds which are T-dual to the O6$^+$-O6$^-$ configuration, and show explicitly that cancellation of twisted tadpole implies the finiteness of the four-dimensional $\mathcal{N} = 2$ theories. Section 5 contains our conclusions.

2 Description of the brane configurations

We will be studying brane configurations of NS branes along 012345, D4 branes along 01236, and D6 branes along 0123789 \cite{1}. The D4 branes will have finite extent along the direction $x^6$, which is taken to be compact. Orientifold six- and four-planes, if present, are parallel to the D branes of the corresponding dimension. The brane models with

\footnote{In the six-dimensional case, twisted modes have no dimensions to propagate. Thus, tadpole cancellation is a \textit{consistency} requirement, rather than a choice.}
Figure 1: The three families of brane configurations in the background of two negatively charged O6-planes. The short vertical lines represent the NS branes, the crossed circles are the orientifold planes, while uncircled crosses denote the D6 branes. The D4 branes stretch in the interval along the circle in $x^6$. In order to yield finite theories, the number of D4 branes is generically different at each interval. For the sake of clarity we have not attempted to draw them.

O6- and O4-planes have been discussed in [8] and [9, 10], respectively.

Let us center the models with O6-planes. Since the direction $x^6$ is compact, there are two orientifold planes. The configuration in the cover space is a set of $N$ NS branes located at points in the circle parametrized by $x^6$. In the $i^{th}$ interval they define, there are $v_i$ D4 branes suspended between NS branes, and $w_i$ D6 branes. The whole brane configuration must be invariant under the $Z_2$ symmetry inverting the coordinates 456. The basic rules to read the spectrum are discussed in [8, 17, 18], using previous results in [19].

When both O6-planes are negatively charged, finiteness of the field theory requires the presence of 8 D6 branes, $\sum_j w_j = 8$. Notice that this condition is equivalent to the cancellation of the RR charge in the brane picture. There are three main families within this class, depending on the positions of the NS branes.

i) The number of NS branes is odd, $N = 2P + 1$. One typical brane configuration is depicted in Figure 1a. Notice that the $Z_2$ symmetry forces one of the NS branes to intersect one O6$^-$. It also requires $v_{-i} = v_i$, $w_{-i} = w_i$. The gauge theory is

$$USp(v_0) \times SU(v_1) \times \ldots \times SU(v_P)$$

$$\oplus_{j=0}^{P-1} (\otimes_{i=1}^{P+1} w_i) + \oplus_{j=1}^{P} w_j \otimes_{i=1}^{P-1} (w_i) + \frac{1}{2} \sum_{j=1}^{P} w_j \otimes_{i=1}^{P-1} (w_i)$$

(2.1)

Our convention is that the fundamental representation of $USp(k)$ has dimension $k$. 

3
where the subindices denote the corresponding group factor. In this and the following spectra we have taken into account that the $U(1)$ factors are frozen at low energies \cite{[1]}. The condition for this theory to be finite is the vanishing of the one-loop beta functions,

$$b_j \equiv -2v_j + v_{j-1} + v_{j+1} + w_j - 4\delta_{j,0} - 2\delta_{j,P} - 2\delta_{j,P+1} = 0. \quad (2.2)$$

From the brane configuration point of view, these conditions are obtained by requiring the linking numbers of all the NS branes to be equal (so that their asymptotic bending is identical). A similar comment applies to all other models.

**ii)** The number of NS branes is even, $N = 2P$, and there are no NS branes intersecting the O6-planes. One example is shown in Figure 1b. Here we again have $v_{-i} = v_i, w_{-i} = w_i$. The gauge theory in this family of models is

$$USp(v_0) \times SU(v_1) \times \ldots \times SU(v_{P-1}) \times USp(v_P)$$

$$\bigoplus_{j=0}^{P-1} (\square_j, \square_{j+1}) + \frac{1}{2} w_0 \square_0 + \bigoplus_{j=1}^{P-1} w_j \square_j + \frac{1}{2} w_P \square_P \quad (2.3)$$

The finiteness conditions read

$$b_j \equiv -2v_j + v_{j-1} + v_{j+1} + w_j - 4\delta_{j,0} - 4\delta_{j,P} = 0. \quad (2.4)$$

**iii)** The number of NS branes is even, $N = 2P$, but two NS branes intersect the O6-planes (see Figure 1c). The $Z_2$ symmetry imposes $v_{-i+1} = v_i, w_{-i+1} = w_i$. The gauge theory is

$$SU(v_1) \times SU(v_2) \times \ldots \times SU(v_{P-1}) \times SU(v_P)$$

$$\square_1 + \bigoplus_{j=1}^{P-1} (\square_j, \square_{j+1}) + \square_P + \bigoplus_{j=1}^{P} w_j \square_j \quad (2.5)$$

In this case, finiteness is achieved when

$$b_j \equiv -2v_j + v_{j-1} + v_{j+1} + w_j - 2\delta_{j,1} - 2\delta_{j,2P} - 2\delta_{j,P} - 2\delta_{j,P+1} = 0. \quad (2.6)$$

For the O6$^+$-O6$^-$ configuration the RR charge already cancels, so finiteness does not allow the presence of D6 branes, thus $w_j = 0$. There are four families of models depending on the positions of the NS branes.

**i)** The number of NS branes is odd, $N = 2P + 1$, and the unpaired NS brane intersects the O6$^+$-plane. One such example is shown in Figure 2a. The resulting gauge theory is

$$USp(v_0) \times SU(v_1) \times \ldots \times SU(v_P)$$

$$\bigoplus_{j=0}^{P-1} (\square_j, \square_{j+1}) + \square_P \quad (2.7)$$
The finiteness conditions are

\[ b_j \equiv -2v_j + v_{j-1} + v_{j+1} - 4\delta_{j,0} + 2\delta_{j,P} + 2\delta_{j,P+1} = 0. \]  

(2.8)

Comparing them with (2.2) we clearly see the effect of the sign of the orientifold six-planes in the signs of the last contributions. A similar comment applies to the remaining models.

These conditions constrain the gauge group to be \( USp(2k) \times SU(2k+2) \times \ldots \times SU(2k+2P) \).

i) The number of NS branes is odd, \( N = 2P + 1 \), and the unpaired NS brane intersects the O6\(^{-}\)-plane, as depicted in Figure 2b. The field theory spectrum is

\[ SO(v_0) \times SU(v_1) \times \ldots \times SU(v_P) \oplus_{j=0}^{P-1} (\mathbb{Q}, \mathbb{Q}_{j+1}) + \mathbb{Q}_P \]  

(2.9)

The vanishing of the beta functions reads

\[ b_j \equiv -2v_j + v_{j-1} + v_{j+1} + 4\delta_{j,0} - 2\delta_{j,P} - 2\delta_{j,P+1} = 0. \]  

(2.10)

The most general solution has gauge group \( SO(n) \times SU(n-2) \times \ldots \times SU(n-2P) \).

ii) The number of NS branes is even, \( N = 2P \), and there are no NS branes intersecting the O6-planes (see Figure 2c). The gauge theory is

\[ SO(v_0) \times SU(v_1) \times \ldots \times SU(v_{P-1}) \times USp(v_P) \oplus_{j=0}^{P-1} (\mathbb{Q}, \mathbb{Q}_{j+1}) \]  

(2.11)
The finiteness conditions in this case are

\[ b_j \equiv -2v_j + v_{j-1} + v_{j+1} + 4\delta_{j,0} - 4\delta_{j,P} = 0. \]  

(2.12)

The finite theory has gauge group \( SO(2k) \times SU(2k-2) \times \ldots \times SU(2k-2P+2) \times USp(2k-2P) \).

iii) The number of NS branes is even, \( N = 2P \), but two NS branes intersect the O6-planes, as in Figure 2d. This construction yields the spectrum

\[ SU(v_1) \times SU(v_2) \times \ldots \times SU(v_{P-1}) \times SU(v_P) \]

\[ \Phi_1 + \bigoplus_{j=1}^{P-1} (\Box_j \oplus \Box_{j+1}) + \Box_P \]  

(2.13)

Finite field theories are obtained when

\[ b_j \equiv -2v_j + v_{j-1} + v_{j+1} - 2\delta_{j,1} - 2\delta_{j,2P} + 2\delta_{j,P} + 2\delta_{j,P+1} = 0. \]  

(2.14)

These conditions imply the group is \( SU(n) \times \ldots \times SU(n+2P-2) \).

These field theories can also be considered in six dimensions. The conditions for cancellation of six-dimensional anomalies are obtained from the four-dimensional finiteness conditions by merely replacing \( 2 \rightarrow 8 \), and \( 4 \rightarrow 16 \) in the Kronecker delta contributions of the quantities \( b_j \). Also, in the O8\(^-\)-O8\(^-\) configuration, we must require the presence of 32 D8 branes, \( \sum_j w_j = 32 \).

3 The O6\(^-\)-O6\(^-\) configuration

In this section we discuss the type IIB orientifolds T dual to the brane configurations with both O6-planes negatively charged. If we T-dualize along the \( x^6 \)-direction, the two O6-planes are mapped to one O7-plane located at \( x^4 = x^5 = 0 \), say. Also the \( N \) NS5-branes are mapped to \( Z_N \)-singularity located at \( x^6 = x^7 = x^8 = x^9 = 0 \). The D4-branes are turned into D3-branes. So the natural guess is that the T-dual of the O6\(^-\)-O6\(^-\) configuration is just the composition of the individual T-duality maps. The T-dual is thus described by the D3-branes with the orientifold projection,

\[ (1 + \theta + \theta^2 + \cdots + \theta^{N-1})(1 + \Omega'). \]  

(3.1)

where \( \theta \) is the generator of \( Z_N \) action

\[ z_1 \rightarrow e^{\frac{2\pi i}{N}} z_1 \]  

(3.2)

\[ z_2 \rightarrow e^{-\frac{2\pi i}{N}} z_2 \]  

(3.3)

\[ ^4 \text{Being more precise, one should also include the dynamics of the NS branes in this case [17, 18].} \]
with \( z_1 = x^6 + ix^7 \) and \( z_2 = x^8 + ix^9 \). Also, \( \Omega' = \Omega R_{45}(-1)^F_L \), where \( R_{45} \) is a reflection in \( x^4, x^5 \)-directions and \((-1)^F_L\) acts as \(-1\) on the Ramond sector of the left movers. For one O7-plane, we need 8 D7-branes in order to neutralize the RR-charge of the O7-plane.

If we compactify \( x^4, x^5 \)-directions and T-dualize along these directions, we have a system of D5-branes with the orientifold projection

\[
(1 + \theta + \theta^2 + \cdots + \theta^{N-1})(1 + \Omega).
\]

Blum and Intriligator have already considered this orientifold projection for the realization of six-dimensional field theories, and the three cases of \( \text{O}6^- - \text{O}6^- \) have counterparts in their construction. So our discussion will be brief and mainly refer to the construction in [13].

### 3.1 The odd order case

The spectrum (2.1) can be reproduced by the following choice of Chan-Paton matrices. We have

\[
\gamma_{\theta,3} = \text{diag} (1_{v_0}, \theta 1_{v_1}, \theta^P 1_{v_P}, \theta^{P+1} 1_{v_P}, \cdots \theta^{2P} 1_{v_1})
\]

(with \( \theta = e^{\frac{2\pi i}{N}} \)) and a similar expression for \( \gamma_{\theta,7} \). Here we choose \( \gamma_{\theta^k} = (\gamma_{\theta})^k \). Also we have

\[
\gamma_{\Omega',3} = \begin{pmatrix}
\varepsilon_{v_0} & 1_{v_1} & & \\
& 1_{v_1} & & \\
& & \ddots & \\
& & & 1_{v_P} \\
& & & -1_{v_P} \\
& & & \\
& & & -1_{v_2}
\end{pmatrix}
\]

\[
\varepsilon_{v_0} = \begin{pmatrix}
0 & 1_{v_P} \\
-1_{v_P} & 0
\end{pmatrix}.
\]

For D7 branes, \( \gamma_{\Omega',7} \) has the same block structure, but is symmetric (i.e. \( 1_{v_0} \) replaces \( \varepsilon_{v_0} \), and all unit matrices appear with positive sign). These matrices yield the spectrum shown in (2.1).

\(^5\text{Compact six-dimensional examples involving orientifold projections of this kind have been considered in [20, 21, 22]. Also, some particular examples of D3 branes at these orientifold singularities have appeared in [23].}\)
For this case the twisted tadpole is \[21, 22\]

\[
\sum_{k=1}^{N-1} \frac{1}{4 \sin^2 \frac{2 \pi k}{N}} \left( \sum_j w_j e^{\frac{4 \pi i k j}{N}} - 4 \sin^2 \frac{2 \pi k}{N} \sum_j v_j e^{\frac{4 \pi i k j}{N}} - 8 \cos^2 \frac{\pi k}{N} \right)^2 = 0. \tag{3.8}
\]

Here \(\gamma_{2k,7} = \sum_j w_j e^{\frac{4 \pi i k j}{N}}\) and \(\gamma_{2k,3} = \sum_j v_j e^{\frac{4 \pi i k j}{N}}\).

By expanding the sine and cosine functions in exponentials, it can be shown that the equations of cancellation of tadpoles are equivalent to the finiteness conditions \[2, 2\].

The difference between (3.8) and the tadpole expression in the six-dimensional context of \[15\] is that we have 8 instead of 32 in the last term of the tadpole equation. This factor-4 difference reflects the T-dual relation between our model and six-dimensional model considered in \[15\]. The crosscap states are products of the crosscap coming from the twisted action and the crosscap coming from the toroidal direction. The part of the crosscap from the twisted part remains the same and the crosscap from the toroidal direction shows the usual behavior under the T-duality. That is the origin of the factor-4 difference. Same thing happens for the corresponding orientifolds in cases ii) and iii) to be discussed below.

### 3.2 The even order cases

One apparently curious fact is that the brane configurations ii) and iii) lead to the same orientifold projection. The difference has been discussed in the six-dimensional context \[17, 18, 15\], as we presently discuss \[1\].

Gauge couplings in six-dimensional \(N = 1\) theories belong to tensor multiplets. In the brane configuration, the gauge coupling is proportional to the distance between two NS5-branes in \(x^6\)-direction. On the other hand, motion in \(x^7, x^8, x^9\)-directions of a NS-brane correspond to a hypermultiplet.

In the configuration ii), \(P\) NS5-branes are located within the interval between two O6-planes. By choosing one particular NS-brane, we can take the independent gauge couplings to be the distance of the remaining \(P - 1\) NS-branes and the particular NS-brane and the distance between this particular NS-brane and its mirror image. Hence, we have \(P\) tensor multiplets. We can also move NS5-branes in \(x^7, x^8, x^9\)-directions pairwise under the orientifolding. Excluding the overall motion of the \(P\) pairs, we have \(P-1\) hypermultiplets.

---

\[\text{The argument follows for the four-dimensional case by replacing ‘tensor multiplet’ by ‘vector multiplet’}.\]
In the configuration iii), we have $P-1$ NS-branes located within the interval between two O6-planes. In addition, we have a NS-brane stuck at each O6-plane. We can take the independent gauge couplings to be the distance of the $P-1$ NS-branes and one of the stuck branes. The distance between the two stuck branes is fixed and cannot be an independent gauge coupling. Thus we get $P-1$ tensor multiplets. On the other hand, motion of the $P-1$ pairs of NS branes along $x^7, x^8, x^9$ contribute $P-2$ hypermultiplets. The independent motions of the unpaired branes contributes two more hypermultiplets. We have $P$ hypermultiplets in total.

Thus the configuration ii) lead to $P$ tensor multiplets and $P-1$ hypermultiplets, while configuration iii) leads to $P-1$ tensor multiplets and $P$ hypermultiplets.

These two possibilities are indeed discussed in [15] in the T-dual type IIB orientifold version. The difference between both models is that for $Z_2$ twisted sector, we can keep either the hypermultiplet or the tensor multiplet under the orientifold projection. According to [15], this fact is related to the distinction between Type I configurations with (possible) vector structure and without vector structure. This distinction originates from the fact that the gauge group of Type I string theory is $Spin(32)/Z_2$. Thus if we choose the Chan-Paton matrices $\gamma$ for the $Z_N$ representation, $\gamma^N = 1$ in $Spin(32)/Z_2$. Thus we can have $\gamma^N = 1$ or $\gamma^N = w$ in $Spin(32)$ where $w$ is the generator of $Z_2$ in $Spin(32)/Z_2$. The different action on the $Z_2$ twisted sector leads to different Chan-Paton matrices corresponding to a configuration with (without) vector structure.

The number of tensor multiplets and hypermultiplets of the configuration ii) matches the spectrum with vector structure and the configuration iii) matches the spectrum without vector structure. The gauge group and matter content on the D-brane worldvolume also agree with this identification, as we will see shortly. Finally, we anticipate that the argument above also applies to O6$^+$-O6$^-$ configurations.

For the case with vector structure, the Chan-Paton matrices have the structure

$$
\gamma_{\theta,3} = \text{diag} \left( 1_{v_0}, \theta^1_{v_1}, \cdots, \theta^{P-1} v_{v_{P-1}} \theta^P 1_{v_P}, \theta^{P+1} v_{v_{P-1}}, \cdots, \theta^{2P-1} v_{v_1} \right)
$$

(3.9)
(and analogously for $\gamma_{\theta, 7}$) and

$$
\begin{align*}
\gamma_{\Omega', 3} &= \\
&= \begin{pmatrix}
\varepsilon_{v_0} & & & & & 1_{v_1} \\
& & & & 1_{v_{p-1}} & \\
& & & \varepsilon_{v_{p}} & & \\
& & -1_{v_{p-1}} & & & \\
& \vdots & & & & & \\
-1_{v_1} & & & & & \\
\end{pmatrix}.
\end{align*}
$$

(3.10)

(and a symmetric version of this for $\gamma_{\Omega', 7}$). These matrices reproduce the spectrum $\{2, 3\}$.

Notice that these matrices have the property

$$
\text{Tr} \left( \gamma_{\theta k \Omega'}^T \gamma_{\theta k \Omega'}^{-1} \right) = \mp \text{Tr} \left( \gamma_{(\theta)^{2k}} \right) = \text{Tr} \left( \gamma_{\theta k + P \Omega'}^T \gamma_{\theta k + P \Omega'}^{-1} \right)
$$

(3.11)

with the upper (lower) sign for D3 (D7) branes. The positive relative sign between the first and third terms characterizes the action of the $Z_2$ twist to correspond to models with vector structure [15]. This is also reflected in a positive relative sign between the untwisted and $Z_2$-twisted contributions to the Klein bottle.

The twisted tadpole equation is

$$
\sum_{k=1}^{N-1} \frac{1}{4 \sin^2 \frac{\pi k}{N}} \left[ \sum_j u_j e^{\frac{2\pi k j}{N}} - 4 \sin^2 \frac{\pi k}{N} \sum_j v_j e^{\frac{2\pi k j}{N}} - 8 \delta_{k, \theta \mod 2} \right]^2 = 0.
$$

(3.12)

It is easy to show that the cancellation of tadpoles is equivalent to the finiteness conditions $\{2, 3\}$.

For $N = 2P$ without vector structure, the Chan-Paton matrices are given by

$$
\gamma_{\theta, 3} = e^{-\frac{2\pi i}{N}} \text{diag} \left( \begin{array}{cccccc}
\varepsilon_{v_0} & e^{\frac{4\pi i}{N}} 1_{v_1} & e^{\frac{4\pi i}{N}} 1_{v_2} & \cdots & e^{\frac{2\pi i P}{N}} 1_{v_p} & \cdots & e^{\frac{2\pi i (P-1)}{N}} 1_{v_{P-1}} & e^{\frac{2\pi i (2P-1)}{N}} 1_{v_1}
\end{array} \right)
$$

(3.13)

(analogously for $\gamma_{\theta, 7}$) and

$$
\begin{align*}
\gamma_{\Omega', 3} &= \\
&= \begin{pmatrix}
1_{v_1} & & & & & \cdots & & 1_{v_1} \\
& & & & & & & & & \\
& & & & & & & \vdots & & \\
& & & & & & -1_{v_p} & & \\
& & & & & & & & & \\
& & & & & & & \vdots & & \\
& & & & & & -1_{v_2} & & \\
& & & & & & & & & \\
& & & & & & & & & \\
-1_{v_1} & & & & & & & & \\
\end{pmatrix}.
\end{align*}
$$

(3.14)
(and a symmetric version for $\gamma_{\Omega',\tau}$)

These matrices yield the field theory spectrum in (2.3). The matrices satisfy

$$\text{Tr} \left( \gamma^T \theta k \Omega \gamma^{-1} \right) = \mp \text{Tr} \left( \gamma (\theta)^2 k \right) = -\text{Tr} \left( \gamma^T \theta k + P \Omega \gamma^{-1} \theta k + P \Omega \right)$$

(3.15)

with the upper (lower) sign for D3 (D7) branes. The relative minus sign between the first and third contributions implies these are models without vector structure.

The tadpole equation can be taken directly from [21, 22], and read

$$N - 1 \sum_{k=1}^{N-1} \frac{1}{4 \sin^2 \frac{\pi k}{N}} \left[ \sum_j w_j e^{\frac{\pi(2j-1)k}{N}} - 4 \sin^2 \frac{\pi k}{N} \sum_j v_j e^{\frac{\pi(2j-1)k}{N}} - 8 \delta_{k,0 \mod 2} \cos \frac{\pi k}{N} \right]^2 = 0.$$  

(3.16)

We have different choice of $\gamma_\theta$ and different tadpole expression from those used in [15]. Basically we absorb the phase factor appearing in their tadpole expression into the redefinition of the $\gamma$ matrices.

Again, the tadpole cancellation conditions can be recast as the vanishing of the beta functions for the corresponding four-dimensional $\mathcal{N}=2$ field theory, eq (2.6).

Note that all the above Chan-Paton matrices satisfy the constraint imposed by Polchinski [24]. That is, if $R$ denotes the $Z_2$ twist, we have

$$\gamma_R = \mp \gamma_{\Omega'} \gamma^T_R \gamma^{-1}_{\Omega'}$$

(3.17)

with the negative (positive) sign for models that keep the hypermultiplet (tensor multiplet) in their $Z_2$ twisted sector.

Conversely if we know $\gamma_R$ and $\gamma_{\Omega}$, all Chan-Paton matrices can be determined from the tadpole equation and the Chan-Paton algebra relation. Thus the conditions (3.11), (3.15) are the corollary of (3.17).

It is interesting to note that in six dimensions, the above orientifold construction gives theories free of six-dimensional anomaly, while the T-dual construction in four dimensions gives finite $\mathcal{N}=2$ gauge theories. This nicely illustrates how tadpole conditions encode the relevant quantum effects of the field theory.

4 The $O6^+\text{-}O6^-$ configuration

The main feature of the brane configurations with oppositely charged orientifold planes is the absence of D6 branes. This fact follows from finiteness in the field theory, or from cancellation of RR charge. The natural interpretation in terms of the T-dual picture of D3 branes at orientifold singularities is that tadpole cancellation in the Type IIB orientifolds is achieved without D7 branes.
In the previous section, we have seen that all models corresponding to the $O6^-\bar{O}6^-$ background are obtained in the T-dual picture from orientifold groups with the structure $G_{\text{orient.}} = Z_N + Z_N \Omega'$. The presence of the element $\Omega'$ in $G_{\text{orient.}}$ induces the appearance of D7 branes.

The analogy between the brane constructions with the $O6^-\bar{O}6^-$ and $O6^+\bar{O}6^-$ configurations suggests there must be a natural family of IIB orientifolds which does not require D7 branes. Indeed, this is achieved by constructing a different $Z_2 \times Z_2$ extension of the orbifold group $Z_N$. If we take an element $\alpha$ generating $Z_2$, this can be constructed as

$$G_{\text{orient.}} = \{ \alpha^{2k}, \alpha^{2k+1} \Omega' \}, \quad (4.1)$$

where $k = 0, \ldots, N - 1$. That this is the correct structure can be understood as follows. In the absence of NS branes, it follows from [25] that the configuration of two oppositely charged O6-planes on a circle is T dual to a type IIB string theory compactified on the dual circle with an orientifold projection $\Omega' S$, with $S$ a half shift on the circle [22]. When the NS branes are present, before the orientifold projection the T dual is given by a $Z_N$ singularity (more precisely, a $N$-center Taub-NUT space). A shift around the whole $U(1)$ orbit in this space is associated to the generator $\theta(\equiv \alpha^2)$ of $Z_N$, and a half-shift is associated to $\alpha$. In performing the orientifold projection, the fact that the O6-planes are oppositely charged implies that $\Omega'$ must be accompanied by $\alpha$. The full group we have quotiented by is

$$G_{\text{orient.}} = Z_N + Z_N \alpha \Omega' \quad (4.2)$$

which is equal to (4.1).

In the following we show that D3 brane probes on this kind of orientifold singularities actually reproduce the $\mathcal{N} = 2$ theories arising from brane construction in the $O6^+\bar{O}6^-$ background.

### 4.1 The odd order case

When the number of NS branes is odd, $N = 2P + 1$, the orientifold group (1.1) can also be described as $G_{\text{orient.}} = Z_N + Z_N R \Omega'$, where $R$ is a $Z_2$ twist inverting the coordinates.

In this case, the absence of D7 branes allows two possible projections on the D3 branes. We will denote them as the ‘$Sp$’ and ‘$SO$’ projections, and they correspond to

---

7This orientifield projection has appeared in [21, 22] in the compact case.
the models (2.7) and (2.9) of Section 2, respectively. Let \( \theta = \alpha^2 \) generate \( Z_N \). A choice of D3 brane Chan-Paton matrices that reproduce the spectra of these theories is

\[
\gamma_\theta = \text{diag}(1_{v_0}, \theta_1 v_1, \ldots, \theta^{P} 1_{v_P}, \theta^{P+1} 1_{v_P}, \ldots, \theta^{2P} 1_{v_1})
\]

\[
\gamma_{R\bar{\Omega} Y} = \begin{pmatrix}
1_{v_0} & & & \\
& \ddots & & \\
& & 1_{v_P} & \\
& & & 1_{v_1}
\end{pmatrix},
\quad \gamma_{R\bar{\Omega} Y'} = \begin{pmatrix}
\varepsilon_{v_0} & & & \\
& \ddots & & \\
& & -1_{v_P} & \\
& & & -1_{v_1}
\end{pmatrix}
\]

where \( \theta = e^{2i\pi/N} \). The two possibilities for \( \gamma_{R\bar{\Omega} Y} \) correspond to the spectra (2.9) and (2.7), respectively.

These matrices are consistent with the group law, and verify the property

\[
\text{Tr} \left( \gamma_\theta^{-1} \gamma_{R\bar{\Omega} Y}^T \right) = \pm \text{Tr} \left( \gamma_\theta^2 \right) \tag{4.3}
\]

with the upper (lower) sign for the \( SO (Sp) \) projections, respectively.

Since \( \Omega' \) is always accompanied by some twist, the final expression for the tadpoles differs from the familiar ones. The techniques to derive them, however, are standard \[21, 22\] and we just sketch the computation.

We have the following contributions from the Klein bottle, Möbius strip and cylinder

\[
\mathcal{K} = \sum_{k=1}^{N-1} 16 \sin^2[2\pi(k/N + 1/2)] / [4 \sin^2[\pi(k/N + 1/2)]]^2
\]

\[
\mathcal{M} = \sum_{k=1}^{N-1} (-16) \cos^2[\pi(k/N + 1/2)] \text{Tr} \left( \gamma_{R\bar{\Omega} Y'}^T \gamma_{R\bar{\Omega} Y}^{-1} \right)
\]

\[
\mathcal{C} = \sum_{k=1}^{N-1} 4 \sin^2(\pi k/N) (\text{Tr} \gamma_\theta^2)^2 \tag{4.4}
\]

Notice that we have included the zero mode integration factor in the denominator of \( \mathcal{K} \), as discussed in \[21\]. Also, in the diagrams involving crosscaps we have included the effects of the \( Z_2 \) twists \( R, R_{45} \) in \( R\bar{\Omega}' \). The total contribution is equal to

\[
\sum_{k=1}^{N-1} \left[ 4 \sin^2(2\pi k/N)(\text{Tr} \gamma_\theta^2)^2 \mp 16 \sin^2(\pi k/N) \text{Tr} \gamma_\theta^2 + 4 \frac{\sin^2(\pi k/N)}{\cos^2(\pi k/N)} \right] \tag{4.5}
\]

where the upper (lower) sign is for the \( SO (Sp) \) projection. It factorizes as

\[
\sum_{k=1}^{N-1} \frac{1}{4 \sin^2(2\pi k/N)} \left[ 4 \sin^2(2\pi k/N) \text{Tr} \gamma_\theta^2 \mp 8 \sin^2(\pi k/N) \right]^2 \tag{4.6}
\]
The tadpole cancellation conditions are

\[ 4 \sin^2(2\pi k/N) \text{Tr} \gamma_{g2k} \mp 8 \sin^2(\pi k/N) = 0. \quad (4.7) \]

It is easy to check that, by expanding the sine functions in exponentials, and recasting the \( \text{Tr} \gamma_{g2k} \) in terms of the \( v_j \), the tadpole cancellation conditions are equivalent to

\[ \sum_{j=0}^{N-1} e^{2\pi i \frac{2k}{N}} [-2v_j + v_{j-1} + v_{j+1} \pm (4\delta_{j,0} - 2\delta_{j,P} - 2\delta_{j,P+1})] = 0. \quad (4.8) \]

Namely, the finiteness conditions for the \( D = 4, N = 2 \) theory, eqs. (2.10), (2.8).

The \( Z_3 \) example has appeared in [23]. Notice that the indirect construction technique employed there (considering first a system of D9 branes and then T dualizing) does not allow the construction of the whole infinite family.

The tadpoles for the six-dimensional theory are given by eq. (4.7) after replacing 8 \( \rightarrow \) 32. The corresponding orientifolds yield six-dimensional theories free of anomalies. A similar comment applies to the following models.

4.2 The even order cases

These families are very interesting from the orientifold point of view, since the spectra (2.11) and (2.13) suggest there are two opposite projections acting simultaneously on the D3 branes. We will show how the structure of the orientifold group (4.1) allows for Chan-Paton matrices yielding these spectra.

4.2.1 Case with vector structure

The brane configuration yielding the spectrum in (2.11) is clearly reminiscent of the O6\(^-\)-O6\(^-\) configuration with spectrum (2.3). This suggests the theory (2.11) is obtained through an orientifold with possible vector structure. As discussed in Section 3, this implies certain signs in the \( Z_2 \) twisted sectors that we should take into account in the tadpole computation.

The Chan-Paton matrices we use are

\[ \gamma_{\alpha^2} = \text{diag}(1_{v_0}, \theta_1_{v_1}, \ldots, \theta^{P-1}1_{v_{P-1}}, \theta^P1_{v_P}, \theta^{P+1}1_{v_{P+1}}, \ldots, \theta^{2P-1}1_{v_1}) \]
\[ \gamma_\alpha \Omega' = \begin{pmatrix} 1_{v_0} & \cdots & \alpha 1_{v_1} \\ \alpha^P 1_{v_{P-1}} & \cdots \\ \alpha^{-P-1} 1_{v_{P-1}} & \cdots \\ \alpha^{-1} 1_{v_1} \end{pmatrix} \] (4.9)

where \( \alpha = e^{2\pi i/N} \). These matrices verify the group law, and it is easy to check they give the spectrum (2.11) on the world-volume of the D3 branes. They also verify the properties

\[ \text{Tr} \left( \gamma^T \alpha^2_k \right) = \text{Tr} \left( \gamma^{\alpha^2_k} \right) = \text{Tr} \left( \gamma^{\alpha^2_{k+N-1}} \right) \] (4.10)

The positive relative sign between the first and third terms in this equation agrees with our above comment concerning the vector structure.

The contributions from the Klein bottle, Möbius strip and cylinder, are

\[ \mathcal{K} = \sum_{k=0}^{N-1} 16 \left[ \cos^2 \left( \frac{\pi \cdot 2k - 1}{2N} \right) + 1 \right] \]  
\[ \mathcal{M} = \sum_{k=0}^{N-1} (-16) \cos^2 \left( \frac{2k - 1}{2N} \right) \text{Tr} \left( \gamma^T \alpha^{2k} \right) \]  
\[ \mathcal{C} = \sum_{k=0}^{N-1} 4 \sin^2 \left( \frac{2k}{2N} \right) (\text{Tr} \gamma^{\alpha^2})^2 \] (4.11)

We have taken into account that different twists act on diagrams with and without crosscaps. Notice also the positive relative sign between the two contributions to the Klein bottle, in correlation with the positive sign we mentioned concerning (4.10).

Using the property (4.10), the total contribution can be factorized as

\[ \sum_{k=1}^{N-1} \frac{1}{4 \sin^2 (\pi k/N)} \left[ 4 \sin^2 (\pi k/N) \text{Tr} \gamma^{(\alpha^2)_k} - 8 \delta_{k,1 \mod 2} \right]^2. \] (4.12)

The cancellation of the tadpoles reads

\[ 4 \sin^2 (\pi k/N) \text{Tr} \gamma^{(\alpha^2)_k} - 8 \delta_{k,1 \mod 2} = 0. \] (4.13)

These conditions can be seen to be equivalent to the condition of finiteness for the gauge theory (2.12).
4.2.2 Case ‘without vector structure’

The brane configurations yielding the theories in the family (2.13) are analogous to those realizing the theories (2.5). Thus we expect the T-dual orientifolds to correspond to theories without vector structure.

Our choice of Chan-Paton matrices for these cases is

\[
\gamma^{\alpha^2} = \text{diag}(e^{i\pi N_{1}v_{1}}, e^{i\pi N_{1}v_{2}}, \ldots, e^{i\pi(2P+1) N_{1}v_{P}}, e^{i\pi(4P-1) N_{1}v_{2}}, \ldots, e^{i\pi(4P-1) N_{1}v_{1}})
\]

\[
\gamma^{\alpha \Omega'} = \begin{pmatrix}
\vdots \\
1_{v_{p}} \\
\ddots \\
1_{v_{2}} \\
1_{v_{1}}
\end{pmatrix}
\]

(4.14)

It is a nice exercise to check the spectrum that arises from the projection is the field theory (2.13). The matrices satisfy the group law, and have the properties

\[
\text{Tr} \left( \gamma^{T \alpha^2-1} \gamma^{-1} \right) = -\text{Tr} \left( \gamma^{(\alpha^2)^2k-1} \right) = -\text{Tr} \left( \gamma^{T \alpha^2+N\Omega\gamma^{-1} \alpha^2-N\Omega} \right) \quad (4.15)
\]

Notice the relative minus sign between the first and third terms. Consequently, there will be a relative minus sign between the two contributions to the Klein bottle.

The total contribution \( \mathcal{C} + \mathcal{M} + \mathcal{K} \) is

\[
\sum_{k=1}^{N-1} \left[ 4 \sin^{2} \left( \frac{\pi k}{2N} \right) (\text{Tr} \gamma^{\alpha k})^{2} - 16 \cos^{2} \left[ \frac{\pi k}{2N} - 1 \right] \text{Tr} \left( \gamma^{T \alpha^2-1} \gamma^{-1} \right) + 16 \left( \frac{\cos^{2} \left[ \frac{\pi k}{2N} \right]}{\sin^{2} \left[ \frac{\pi k}{2N} \right]} - 1 \right) \right]
\]

Using the properties (4.15), it factorizes as

\[
\sum_{k=1}^{N-1} \frac{1}{4 \sin^{2} \left( \frac{\pi k}{N} \right)} \left[ 4 \sin^{2} (\pi k/N) \text{Tr} \gamma^{(\alpha^2)^k} + 8 \delta_{k,1 \mod 2} \cos (\pi k/N) \right]^{2}.
\]

(4.16)

This is very similar to the tadpoles found in [21, 22]. The only difference arises from the different twist present in diagrams with and without crosscaps.

The equations for the cancellation of tadpoles

\[
4 \sin^{2} (\pi k/N) \text{Tr} \gamma^{(\alpha^2)^k} + 8 \delta_{k,1 \mod 2} \cos (\pi k/N) = 0,
\]

(4.17)

can be shown to be equivalent to the finiteness conditions (2.14).
5 Final comments

In this paper we have considered the T-duals of elliptic models with two O6-planes. They provide the construction of large families of finite four dimensional $\mathcal{N} = 2$ theories. These orientifolds can also be used to define anomaly-free six-dimensional field theories with RG fixed points at the origin of their Coulomb branches.

For the configuration with two negatively charged orientifolds, the T-duals had already been considered in [15]. Our aim has been to construct the T-duals of the configurations with oppositely charged orientifold planes. They are realized by type IIB orientifolds which do not require the presence of D7 branes. The different families nicely match the different orientifold groups that can be defined. We have shown that the tadpoles for these orientifolds are proportional to the beta functions, so cancellation of tadpoles ensures the finiteness of the field theories (cancellation of irreducible anomalies in the six-dimensional version).

Notice that the vanishing of the beta functions generically forces the different factors in the gauge group to have different rank, so that the Chan-Paton matrices are generically not traceless. From the brane picture point of view, these ranks can be determined by linking number arguments, so the non-tracelessness can be easily tracked down to the O6 and D6 charges. Thus, a nice result is how the tadpole conditions encode the linking numbers of the T-dual brane configurations.

We hope that the realization of these new superconformal field theories from D3 branes at orientifold singularities facilitates their study using the recent developments on the AdS/CFT correspondence, along the lines of [26].

It would also be very interesting to explore other theories from D3 branes at orientifold singularities. We hope that further generalizations, by considering non-abelian singularities [27], or considering singularities which only preserve $\mathcal{N} = 1$ supersymmetry [28, 29], enlarge the number of these extremely interesting field theories.

Acknowledgements

We are grateful to L. E. Ibáñez, K. Intriligator and A. Kapustin for useful discussion. A.M.U. thanks M. González for her kind encouragement. The work of J.P. is supported by the U.S. Department of Energy under Grant No. DE-FG02-90-ER40542. The work of A.M.U. is supported by the Ramón Areces Foundation (Spain).
References

[1] E. Witten, ‘Solutions of four-dimensional field theories via M theory’, Nucl.Phys. B500(1997)3, hep-th/9703166.

[2] A. Hanany, E. Witten, ‘Type IIB superstrings, BPS monopoles, and three-dimensional gauge dynamics’, Nucl.Phys. B492(1997)152, hep-th/9611230.

[3] T. Banks, M. R. Douglas, N. Seiberg, ‘Probing F theory with branes’, Phys.Lett. B387(1996)278-281, hep-th/9605199.

[4] M. R. Douglas, M. Li, ‘D-brane realization of $N = 2$ super Yang-Mills theory in four dimensions’, hep-th/9604041.

[5] M. R. Douglas, G. Moore, ‘D-branes, quivers, and ALE instantons’, hep-th/9603167.

[6] A. Karch, D. Lust, D. Smith, ‘Equivalence of geometric engineering and Hanany-Witten via fractional branes’, hep-th/9803232.

[7] J. Maldacena, ‘The Large N limit of superconformal field theories and supergravity’, hep-th/9711200. S. S. Gubser, I. R. Klebanov, A. M. Polyakov, ‘Gauge theory correlators from noncritical string theory’, Phys.Lett. B428(1998)105, hep-th/9802109. E. Witten, ‘Anti-de Sitter space and holography’, hep-th/9802150.

[8] A. M. Uranga, ‘Towards mass deformed N=4 SO(n) and Sp(k) gauge theories from brane configurations’, Nucl.Phys. B526(1998)241, hep-th/9803054.

[9] K. Landsteiner, E. Lopez, D. A. Lowe, ‘N=2 supersymmetric gauge theories, branes and orientifolds’, Nucl.Phys. B507(1997)197, hep-th/9705199.

[10] A. Brandhuber, J. Sonnenschein, S. Theisen, S. Yankielowicz, ‘M theory and Seiberg-Witten curves: Orthogonal and symplectic groups’, Nucl.Phys. B504(1997)175, hep-th/9705232.

[11] A. Sen, ‘F theory and orientifolds’, Nucl.Phys. B475(1996)562, hep-th/9605150.

[12] O. Aharony, J. Sonnenschein, S. Yankielowicz, S. Theisen, ‘Field theory questions for string theory answers’, Nucl.Phys. B493(1997)177, hep-th/9611222.

[13] M. R. Douglas, D. A. Lowe, J. H. Schwarz, ‘Probing F theory with multiple branes’, Phys.Lett. B394(1997)297, hep-th/9612062.
[14] K. Intriligator, ‘RG fixed points in six-dimensions via branes at orbifold singularities’, Nucl.Phys. B496(1997)177, hep-th/9702038.

[15] J. D. Blum, K. Intriligator, ‘Consistency conditions for branes at orbifold singularities’, Nucl.Phys. B506(1997)223, hep-th/9705030.

[16] I. Brunner, A. Karch, ‘Branes and six-dimensional fixed points’, Phys.Lett. B409(1997)109, hep-th/9705022.

[17] I. Brunner, A. Karch, ‘Branes at orbifolds versus Hanany Witten in six-dimensions’, JHEP 03(1998)003, hep-th/9712143.

[18] A. Hanany, A. Zaffaroni, ‘Branes and six-dimensional supersymmetric theories’, hep-th/9712145.

[19] K. Landsteiner, E. L´opez, ‘New curves from branes’, Nucl.Phys. B516(1998)273, hep-th/9708118.

[20] E. G. Gimon, J. Polchinski, ‘Consistency Conditions for Orientifolds and D-Manifolds’, Phys.Rev. D54(1996)1667, hep-th/9610038.

[21] E. G. Gimon, C. V. Johnson, ‘K3 orientifolds’, Nucl.Phys. B477(1996)715, hep-th/9604123.

[22] A. Dabholkar, J. Park, ‘Strings on orientifolds’, Nucl.Phys. B477(1996)701, hep-th/9604178.

[23] Z. Kakushadze, ‘Gauge theories from orientifolds and large N limit’, hep-th/9803214.

[24] J. Polchinski, ‘Tensors from K3 orientifolds’, Phys.Rev.D55(1997)6423, hep-th/9606165.

[25] E. Witten, ‘Toroidal compactification without vector structure’, JHEP 02(1998)006, hep-th/9712028.

[26] A. Fayyazuddin, M. Spalinski, ‘Large N superconformal gauge theories and supergravity orientifolds’, hep-th/9805096; O. Aharony, A. Fayyazuddin, J. Maldacena, ‘The Large N limit of N=2, N=1 field theories from three-branes in F theory’, JHEP 07(1998)013, hep-th/9806159.

[27] J. D. Blum, K. Intriligator, ‘New phases of string theory and 6-D RG fixed points via branes at orbifold singularities’, Nucl.Phys. B506(1997)199, hep-th/9705044.
[28] Z. Kakushadze, ‘On Large N Gauge Theories from Orientifolds’, hep-th/9804184.

[29] L. E. Ibáñez, R. Rabadán, A. M. Uranga, ‘Anomalous U(1)’s in Type I and Type IIB D=4, N=1 string vacua’, hep-th/9808139.