Quantum dot in a two-dimensional topological insulator: The two-channel Kondo fixed point

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A quantum dot coupled to two helical edge states of a two-dimensional topological insulator through electron tunneling is studied. We show that if the electron interactions on the edge states are repulsive, with Luttinger liquid parameter $K < 1$, the system reaches a stable two-channel Kondo fixed point at low temperatures. This is in contrast to the Luttinger liquid leads case in which $K < 1/2$ is needed. This two-channel fixed point is described by a boundary sine-Gordon Hamiltonian with a $K$ dependent boundary term. The impurity entropy, the impurity specific heat and the conductance are calculated.

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The two-channel Kondo model has been under intense theoretical study1 after the seminal work of Nozieres and Blandin,2 in which they pointed out that a non-Fermi liquid fixed point exists for the two-channel Kondo model with noninteracting electrons. Recently, the authors of Ref. 3 demonstrated experimentally that a quantum dot coupled to an infinite reservoir with noninteracting electrons and a finite reservoir with interacting electrons may display two-channel Kondo effect. The key idea is that Coulomb blockade suppresses the exchange of electrons between the finite reservoir and the infinite reservoir.4 Consequently, the two reservoirs couple to the quantum dot independently.

Indeed, using electron-electron repulsion to suppress interreservoir tunnelings was suggested earlier by Fabrizio and Gogolin, in the context of a quantum dot coupled to two Luttinger liquid leads.5–7 They show that the two-channel Kondo fixed point can be reached when the repulsive interactions of the electrons in the leads are strong enough, i.e., with Luttinger liquid parameter $K < 1/2$.

Recently, a class of materials called topological insulators were first theoretically proposed and then experimentally fabricated.8–12 Two-dimensional topological insulators have gapless helical edge states despite the presence of a bulk gap.8 The edge states are called helical because the directions of the spins and the momentum of the electrons are tied together.13 The one-channel Kondo effect of helical edge states was first studied in Ref. 14 as a truncated model of the boundary sine-Gordon Hamiltonian.16,17 However, we show that a weak repulsive system can be described by a one-channel Kondo Hamiltonian.

![Diagram](image-url)

FIG. 1. (a) A quantum dot coupled to two helical edges of a 2D topological insulator through electron tunneling. S and D denote the source and drain, respectively. (b) The schematic picture of the linear conductance $G$ from S1 to D2 versus temperature $T$. The solid line and the dashed line depict the 1/2 $K < 1$ and $K < 1/2$ cases, respectively.

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\[ H_I = \sum_{i=1,2} \left( t d_i^\dagger \psi_{R|i} + d_i^\dagger \psi_{L|i} \right) + \text{h.c.} \]

\[ H_0 \text{ and } H_I \text{ denote the kinetic energy and the interaction of the } \]

\[ \text{helical edge states, respectively. From } H_0, \text{ we see that the } \]

\[ \text{right/left moving electrons are spin up/down. } H_I \text{ is the } \]

\[ \text{Hamiltonian of the quantum dot where } \epsilon \text{ is the single particle } \]

\[ \text{energy level at the dot and } U \text{ is the on-site interaction energy. } \]

\[ H, \text{ describes the coupling between the dot and the edges and we have } \]

\[ \text{tuned to the channel symmetric point such that } t \text{ is independent of the edge index. In the presence of spin-orbit coupling, spin (1 and 1) denotes } \]

\[ \text{pseudospin of the edge modes. In the large } U \text{ limit, one obtains the Kondo } \]

\[ \text{Hamiltonian } H = H_0 + H_K \text{ through perturbation theory,} \]

\[ \text{where} \]

\[ H_K = \sum_{i=1,2} J_i \delta^i \left( \psi_{ia}^\dagger \frac{a a}{4} \psi_{ib} + \psi_{ia} \frac{a a}{4} \psi_{ib} \right) + \sum_{i \neq j} J_2 \delta^i \left( \psi_{ia}^\dagger \frac{a a}{4} \psi_{jb} + \psi_{ia} \frac{a a}{4} \psi_{jb} \right). \]

\[ (1) \]

Initially, \( J_1 = J_2 > 0 \) and without interaction, Eq. (1) reduces to a single channel Kondo problem.\(^{16,17} \) Our goal is to show that \( J_2 \) normalizes to zero at low temperatures when \( g_z > 0. \)

Because of helicity, one of the R/L and the spin \( \uparrow \downarrow \) indexes are redundant, we keep only one of them starting from Eq. (1).

It is important to note that, in general, there are extra backscattering terms of the form \( \psi_{ia}^\dagger \psi_{ib} + \psi_{ia} \psi_{ib} \) and the boson 

\[ \text{caused by the quantum dot in the Hamiltonian } H. \text{ The backscattering } \]

\[ \text{terms cut each edge into two separate parts at low temperatures if } K < 1 \text{ and render the geometry in Fig. 1(a) unstable. However, in our case, single particle back-scattering terms are } \]

\[ \text{forbidden because of time-reversal symmetry. Two-particle back scatterings preserve time-reversal symmetry but they are irrelevant so long as } K < 1/4. \]^\(13 \)

Thus, our discussion below is valid for \( K > 1/4. \)

By Abelian bosonization,\(^ {19,20} \) the electron operators can be written as: \( \psi_{ia}^\dagger \psi_{ib} = \frac{1}{\sqrt{2}} e^{i \pi \phi_{ia \phi_{ib}}} e^{i \pi \phi_{ib \phi_{ia}}} \).

We define bosonic fields \( \phi \text{ and } \theta \text{ over the quantum dot, which satisfy the commutation relations } [\phi, \theta] = 0. \)

The dual fields that satisfy the commutation relations are denoted as: \( \phi_{ia} = \frac{1}{2}(\phi_1 \pm \phi_2) \) and \( \theta_{ia} = \frac{1}{2}(\theta_1 \pm \theta_2). \)

In terms of \( \phi_{ia} \) and \( \theta_{ia} \), the Hamiltonian can be written as: \( H_0 + H_I = \frac{U}{2} \int d x \left( \delta \phi_{ia}^2 + K \delta \phi_{ia} \delta \theta_{ia} + \frac{1}{K} \delta \theta_{ia}^2 \right) \)

\[ \text{and } \frac{2}{a} \int \left( S^e e^{i \frac{2}{K} \phi_0(0)} + S^e e^{i \frac{2}{K} \phi_0(0)} \right) \cos \left( \frac{2}{K} \varphi_0(0) \right) \]

\[ \frac{2}{a} \int \left( S^e \sin \left( \frac{2}{K} \varphi_0(0) \right) \sin \left( \frac{2}{K} \varphi_0(0) \right) \right) \]

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Here, the Luttinger liquid parameter is defined as

\[ K = \sqrt{\frac{1 + g_z/2\pi\varphi_F - g_z/2\pi\varphi_F}{1 + g_z/2\pi\varphi_F + g_z/2\pi\varphi_F}} \]

\[ v_F = v_F(1 + g_z/2\pi\varphi_F)^2 - (g_z/2\pi\varphi_F)^2. \]

The bosonic form of \( H_K \) is spin anisotropic, we denote the two components of \( J_i \) as \( J_i^{\uparrow\downarrow} \), respectively. This is reasonable for our problem, because spin-orbit coupling breaks \( SU(2) \) symmetry on the edges. The spin-coupling terms acquire different scaling dimensions when \( K \neq 1 \) as we see below.

With the bosonic Hamiltonian \( H = H_0 + H_I + H_K \), we can calculate the scaling dimensions of the operators \( H_K \) at the vicinity of the fixed point where \( J_1^\uparrow = J_2^\uparrow = 0 \), the scaling dimensions of the \( J_1^\uparrow, J_1^\downarrow \) terms are 1, and \( K \), respectively. On the other hand, the scaling dimensions of the \( J_2^\uparrow, J_2^\downarrow \) terms are \( 1/(K + 1/K) > 1 \), thus, when \( K < 1 \), the \( J_2 \) terms decrease but the \( J_1 \) terms grow when temperature is lowered.

In order to study the physics at the strong coupling regime when \( J_1 < 1 \), we use the Emery-Kivelson method by applying a unitary transformation \( U = e^{i 2 \pi K \phi_0(0)/2} \) to \( H \). After the unitary transformation and the rescaling of the bosonic fields, we have

\[ \bar{H} = U^\dagger H U \]

\[ = H_0 + \lambda S^e \delta \phi_{ia}(0) + \frac{2 J_1}{a} \int d x \left( \frac{\varphi_0}{K} \sin \left( \frac{2}{K} \varphi_0(0) \right) \right) \left[ \left( S^e \cos \left( \frac{2}{K} \varphi_0(0) \right) \right) + \left( S^e \cos \left( \frac{2}{K} \varphi_0(0) \right) \right) \right] \]

\[ \left[ \left( S^e \cos \left( \frac{2}{K} \varphi_0(0) \right) \right) - \left( S^e \cos \left( \frac{2}{K} \varphi_0(0) \right) \right) \right] \]

where \( \lambda = \sqrt{2 \pi K} v_F - \sqrt{2 \pi K} / K \).

At the vicinity of the point \( \lambda = J_1^\uparrow = J_2^\uparrow = 0 \), we may calculate the scaling dimensions of the \( J_i \) terms as before. The \( J_1^\uparrow \) and \( J_2^\uparrow \) terms become more relevant because of the elimination of the \( e \pm i \delta \phi(0) \) factors by the transformation. The \( J_1^\uparrow \) term has scaling dimension \( K \).

It is relevant when \( K < 1 \). The \( J_2^\uparrow \) term has scaling dimension \( \frac{1}{K} \) and it is relevant when \( 1/2 < K < 1 \).

A quantum dot coupled to two Luttinger liquid leads is described by a Hamiltonian \( H_L \) which is very similar to \( \bar{H} \) of Eq. (2). However, there is one crucial difference between \( H_L \) and \( \bar{H} \). In \( H_L \), the two terms attached to the spin operators \( S^e + S^e \) are \( \cos \left( \frac{2}{K} \varphi_0(0) \right) \) and \( \cos \left( \frac{2}{K} \varphi_0(0) \right) \), respectively. In \( H_L \), the corresponding terms attached to \( S^e + S^e \) are \( \cos \left( \frac{2}{K} \varphi_0(0) \right) \) and \( \cos \left( \frac{2}{K} \varphi_0(0) \right) \), where \( \phi_0 \) and \( \theta_0 \) are charge and spin bosonic fields, respectively, which are independent from each other.\(^5 \)

In the present problem, \( \phi_0 \) and \( \theta_0 \) are dual fields of each other, pinning the value of the \( \phi_0 \) field increases the fluctuation of the \( \theta_0 \) field. We will see below that this is responsible for the fact that the two-channel Kondo fixed point is stable for \( K < 1 \).

When \( K < 1 \), the \( J_1^\uparrow \) term is the most relevant term. At low temperatures, one expect the effective Hamiltonian of \( \bar{H} \) is described by the fixed point Hamiltonian:
| Op.| FP | \(J_z = 0\) | \(\lambda = J^y_z = 0\) | \(\lambda = 0, J^y_z \to \infty\) |
|---|---|---|---|---|
| \(J^y_1(\delta_\lambda)\) | 1 | 1 | 1 + 1/2K |
| \(J^y_1\) | K | K/2 | N/A |
| \(J^y_2\) | \(\frac{1}{2}(K+1/K)\) | \(\frac{1}{2}(K+1/K)\) | 1/K |
| \(J^y_2\) | \(\frac{1}{2}(K+1/K)\) | 1/2K | 1/K |

\[
\tilde{H}_{\text{eff}} = H_0 + \frac{2J^y_1}{\pi a} S^y \cos(2\pi K \phi_a(0)).
\]  

To show that this fixed point is indeed stable, we need to calculate the scaling dimensions of the operators \(J^y_2 S^y \cos(\sqrt{2} \pi K \theta_a(0))\), \(\delta S_a, \delta \theta_a(0)\), and \(J^y_2 S_a, \sin(\sqrt{2} \pi K \phi_a(0))\) at the vicinity of the fixed point. \(\delta \lambda\) denotes the deviation from the Emery-Kivelson line. The results are given in the third column in Table I. We see that all these operators are irrelevant if \(K < 1\). Which confirms that Eq. (3) is the low-temperature effective Hamiltonian. Below we give some details of the calculation.

To calculate the scaling dimension of \(J^y_2 S^y \cos(\sqrt{2} \pi K \theta_a(0))\), we note that \(S^y\) commutes with \(\tilde{H}_{\text{eff}}\), so that we may set \(S^y\) to \(\pm 1/2\). As a result, one needs to calculate the scaling dimension of \(\cos(\sqrt{2} \pi K \theta_a(0))\) with the Hamiltonian

\[
\tilde{H}_\perp = H_0 \pm \frac{J^y_1}{\pi a} \cos(2\pi K \phi_a(0)).
\]

Equation (4) has the form of the Hamiltonian of a spinless Luttinger liquid wire with an impurity backscattering term at point \(x = 0\). One may interpret the \(\cos(\sqrt{2} \pi K \phi_a(0))\) as an impurity back scattering term with Luttinger liquid parameter \(K' = K/2\). At low temperatures, the “back scattering term” is relevant for \(K' < 1\). Thus, the \(\cos(\sqrt{2} \pi K \phi_a(0))\) term cuts the Luttinger liquid wire into two separate pieces at \(x = 0\). As a result, the point \(x = 0\) can be regarded as a point at the boundary when \(J^y_1 \to \infty\).

The effective Hamiltonian for half of the wire is

\[
H'_0 = \frac{1}{2} \int_0^L \left[ \frac{\hbar}{\pi a} (\partial_x \phi_a)^2 + \frac{\hbar}{\pi a} (\partial_x \phi_a) \right] dx.
\]

Define chiral fields \(\phi_{\alpha}\) and \(\psi_{\alpha}\) by the relations \(\partial_t \phi_{\alpha}(x) = \phi_{\alpha}(x) - \psi_{\alpha}(x)\) and \(\phi_{\alpha}(x) = \phi_{\alpha}(x) + \psi_{\alpha}(x)\) and apply the conditions \(\partial_t \psi_{\alpha}(x) = \partial_t \phi_a(0)\) and \(\phi_{\alpha}(x) = \phi_{\alpha}(x)\) we have

\[
H' = \frac{\hbar}{\pi a} \int_0^L \left[ \frac{\hbar}{\pi a} (\partial_x \phi_{\alpha}) \right] dx.
\]

Define free bosonic fields which are defined from \(-L\) to \(L\), \(\delta \theta_a(x) = \phi_a(x) - \psi_a(x)\) and \(\delta \phi_a(x) = \phi_a(x) + \psi_a(x)\), we have

\[
H_{\text{fixed}} = \frac{\hbar}{\pi a} \int_0^L \left[ \frac{\hbar}{\pi a} (\partial_x \phi_{\alpha}) \right] dx.
\]

As a result, the scaling dimension of \(\cos(\sqrt{2} \pi K \theta_a(0))\) at the boundary can be easily calculated. It is \(1/2\) instead of \(1/2K\) as is the case in the bulk. This well known result is crucial because the \(J^y_2 S^y \cos(\sqrt{2} \pi K \theta_a(0))\) term is now irrelevant when \(K < 1\).

The calculations of the scaling dimension of the \(S^y \delta \theta_a(0)\) term is similar to what has been done before in Ref. 23. Since \(\theta_a(0)\) is governed by a free Hamiltonian, we have

\[
\langle S^y \delta \theta_a(0) S^y \delta \theta_a(0) \rangle = \langle S^y \delta \theta_a(0) S^y \delta \theta_a(0) \rangle = \langle S^y \delta \theta_a(0) S^y \delta \theta_a(0) \rangle = \langle S^y \delta \theta_a(0) S^y \delta \theta_a(0) \rangle.
\]

In order to calculate \(\langle S^y \delta \theta_a S^y \delta \theta_a \rangle, \) we note that \(S^y\) is the sum of the raising and lowering operators of the eigenstates of \(S^y, \pm 1\). Thus, we have

\[
\langle -|e^{-\tilde{H}_{\text{eff}}^* S^y(0)} e^{\tilde{H}_{\text{eff}}^* S^y(0)} - \rangle = \langle -|e^{-\tilde{H}_{\text{eff}}^* W(0)} e^{\tilde{H}_{\text{eff}}^* W(0)} - \rangle = \langle -| W'(\tau) W(0) - \rangle \tilde{H}_L \quad (5)
\]

where \(W\) is a unitary transformation which satisfies

\[
W = e^{-\tilde{H}_{\text{eff}}^* \tau} W = e^{-\tilde{H}_{\text{eff}}^* \tau}.
\]

Actually, we need to find a unitary operator \(W\) which transforms \(\phi_a\) to \(\phi_a + \sqrt{2} \pi K \phi_a\). Indeed, \(W = e^{-i\sqrt{2} \pi K \tilde{H}_L}, \) where \(L\) is the length of the system. In order to calculate \(\langle -| W'(\tau) W(0) - \rangle \tilde{H}_L\), we come back to the spinless Luttinger liquid Hamiltonian interpretation of \(\tilde{H}_L\).

Suppose we have a spinless Luttinger liquid wire with an impurity at \(x = 0\) and the left and right ends locate at \(-L\) and \(L\), respectively. At the strong coupling fixed point, the wire is cut into two separate parts by the impurity. However, the left half and the right half are described by free Hamiltonians \(H_{0L}\) and \(H_{0R}\), respectively. As a result, we have

\[
\langle W'(\tau) W(0) \rangle_{\tilde{H}_L} = \langle e^{-i\sqrt{2} \pi K \theta_a(L\tau) - i\sqrt{2} \pi K \theta_a(-L\tau)} \rangle_{H_{0L}} \times \langle e^{i\sqrt{2} \pi K \tilde{H}_L(L\tau) - i\sqrt{2} \pi K \theta_a(-L\tau)} \rangle_{H_{0R}} \approx \frac{1}{\tau^{1/2}}. \quad (6)
\]

Since \(\delta \theta_a(0)\) has a scaling dimension of 1, we conclude that the operator \(S^y \delta \theta_a(0)\) is irrelevant with scaling dimension 1/2K + 1.

One may calculate the scaling dimension of the \(J_a S_a, \sin(\sqrt{2} \pi K \phi_a(0))\sin(\sqrt{2} \pi K \phi_a(0))\) term in a similar way. This term has scaling dimension \(1/2K + 1\). It is irrelevant when \(K < 1\).

We have shown above that the two-channel fixed point is stable. At the vicinity of this fixed point, the system is described by \(\tilde{H}_{\text{eff}}\) in Eq. (3). It is interesting to note that \(\tilde{H}_{\text{eff}}\) can be regarded as a generalization of the fixed point Hamiltonian of the two-channel and four-channel Kondo non-Fermi liquid fixed-point Hamiltonians. The two-channel and four-channel cases are described by \(\tilde{H}_{\text{eff}}\) with \(K = 1\) and \(K = 3/2\) respectively. One of the most remarkable features of the multichannel Kondo effect is the existence of fractionally degenerate ground state. For example, the two-channel and four-channel Kondo models have residual entropy of \(\frac{1}{2}\ln 2\) and \(\frac{1}{2}\ln 3\), respectively. This motivates us to study the residual entropy of \(\tilde{H}_{\text{eff}}\) for \(K < 1\).

In order to calculate the residual entropy, we first calculate the partition function \(Z = \text{Tr} \exp[-\beta \tilde{H}_{\text{eff}}]\). Since \(S^y\) commutes with \(\tilde{H}_{\text{eff}}\), the states can be labeled as \(\pm \frac{1}{2}, \phi_a\). We may write \(Z = \sum_\phi Z + \text{Tr} \exp[-\beta \tilde{H}_{\text{eff}}]\). Remember that the integral of \(H_0\) run from \(-\infty\) to \(+\infty\) if \(L \to \infty\) is taken. However, we may define bosonic fields such that the positive axis is folded to the negative axis. After folding, the effect
of the cosine term in $\tilde{H}_\pm$ is to introduce a Dirichlet boundary condition. The actual value of $\phi_\pm(0, \tau)$, whether it is pinned at $(2n+1)\pi/\sqrt{2}\pi K$ or $2n\pi/\sqrt{2}\pi K$, does not affect the partition function.28 Thus, we have $Z_\pm=Z_-$ and $Z=2Z_\pm$. The impurity entropy of $\tilde{H}_\pm$ has been calculated in Ref. 27. It is $\ln g=\ln\sqrt{K}/2$, where $g$ is the “ground-state degeneracy” discussed in Ref. 25. Together with the contribution from $S$, the total impurity entropy is

$$S=\ln\sqrt{2K}. \quad (7)$$

Substituting $K=1$ and $K=3/2$ into Eq. (7) reproduces the impurity entropy results, calculated by the Bethe ansatz and boundary conformal field theory, for the two-channel and four-channel Kondo effects with noninteracting electrons, respectively.24,25

We show above that the residual impurity entropy is determined by the boundary term. On the other hand, the thermodynamic properties are determined by the leading irrelevant operators (LIO) near the fixed point. Since the irrelevant operators have correlation functions of the form $\langle O(\tau)O(0)\rangle=\left(\frac{\sigma T}{\sin(\pi T)}\right)^{12}T$, the second-order correction of the free energy $\delta F(T)=\lambda^2 \lambda_0^2 T J_2^2 T [\sin(\pi T)]^{-12}T$ where $\lambda_0$ is the cut-off time.23

Near the strong coupling fixed point, the $\frac{\mu J_1}{\pi^2} \tan^{1/2} x_0(0)$ term with scaling dimension $1/K$ is the LIO if $1/2<K<1$. We have $\delta F\propto T^{2/K-1}+O(T^2)$, where the $T^{2/K-1}$ term is universal and independent of the cut-off time $\lambda_0$ and the $0(T^2)$ term has a $\lambda_0$ dependent coefficient. As a result, the impurity specific heat $C_{\text{imp}}\propto T^{2/K-2}$ if $2/3<K<1$. However, when $K<2/3$, the free energy is dominated by the $0(T^2)$ term and $C_{\text{imp}}\propto T$.

In the above calculations, we have assumed that the two helical edges have equal chemical potential. If a small bias is applied across the two edges, there should be a current flowing from one edge to the other. At temperatures much higher than the Kondo temperature, the linear conductance across the edges can be written as $G\propto[2(J_2^2)^2+(J_1^2)^2]^{K+1/K-2}$ by perturbation to second order. At low temperatures, the conductance can be obtained by substituting the normalized value of $J_2^2$ into the high temperature expression of $G$. As we show above, $J_2$ decreases near the fixed point with $J_2^2=J_1=J_0=0$ and $J_1$ grows. If $1/2<k<1$, however, when the $J_1$ terms are of order 1, the $J_2^2$ term becomes relevant and grows. When the temperature decreases further, the strong coupling fixed point with $J_2^2\rightarrow\infty$ is reached. Near this fixed point, $J_2^2$ and $J_2$ are irrelevant and flow to zero. As a result, the conductance acquires a nontrivial temperature dependence. On the other hand, if $K<1/2$, the $J_2$ terms are always irrelevant and decrease to zero. Consequently, the conductance decreases monotonically as the temperature lowers. The temperature dependence of the linear conductance at different fixed points are shown in the Fig. 1(b).

We show above that a quantum dot coupled to two helical edge states of 2D topological insulators is described by a Kondo Hamiltonian. Weak repulsive interaction with $K<1$ on the edges, drives the system to a two-channel fixed point at low temperatures. At the fixed point, the residual impurity entropy is $\ln\sqrt{2K}$. The impurity specific heat and the conductance across the two edges through the quantum dot are also studied.

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