Controlled Ordinary Differential Equations with Random Path-Dependent Coefficients and Stochastic Path-Dependent Hamilton-Jacobi Equations*

Jinniao Qiu†

October 25, 2021

Abstract

This paper is devoted to the stochastic optimal control problem of ordinary differential equations allowing for both path-dependence and measurable randomness. As opposed to the deterministic path-dependent cases, the value function turns out to be a random field on the path space and it is characterized by a stochastic path-dependent Hamilton-Jacobi (SPHJ) equation. A notion of viscosity solution is proposed and the value function is proved to be the unique viscosity solution to the associated SPHJ equation.

Mathematics Subject Classification (2010): 49L20, 49L25, 93E20, 35D40, 60H15

Keywords: stochastic path-dependent Hamilton-Jacobi equation, stochastic optimal control, viscosity solution, backward stochastic partial differential equation

1 Introduction

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a complete filtered probability space. The filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfies the usual conditions and is generated by an $m$-dimensional Wiener process $W = \{W(t) : t \in [0, \infty)\}$ together with all the $\mathbb{P}$-null sets in $\mathcal{F}$. The associated predictable $\sigma$-algebra on $\Omega \times [0, T]$ is denoted by $\mathcal{P}$.

Throughout this work, the number $T \in (0, \infty)$ denotes a fixed deterministic terminal time and the set $C([0, T]; \mathbb{R}^d)$ represents the space of $\mathbb{R}^d$-valued continuous functions on $[0, T]$. For each $x \in C([0, T]; \mathbb{R}^d)$, denote by $x_t$ its restriction to time interval $[0, t]$ for each $t \in [0, T]$ and by $x(t)$ its value in $\mathbb{R}^d$ at time $t \in [0, T]$. Consider the following stochastic optimal control problem

$$\min_{\theta \in U} E \left[ \int_0^T f(s, X_s, \theta(s)) \, ds + G(X_T) \right],$$

subject to

$$\begin{cases} \frac{dX(t)}{dt} = \beta(t, X_t, \theta(t)), & t \geq 0; \\
X_0 = x_0 \in \mathbb{R}^d. \end{cases}$$

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1This work was partially supported by the National Science and Engineering Research Council of Canada (NSERC) and by the start-up funds from the University of Calgary.

2Department of Mathematics & Statistics, University of Calgary, 2500 University Drive NW, Calgary, AB T2N 1N4, Canada. E-mail: jinniao.qiu@ucalgary.ca (J. Qiu).
Here, we denote by $\mathcal{U}$ the set of all the $U$-valued and $\mathcal{F}_t$-adapted processes with $U \subset \mathbb{R}^m$ ($m \in \mathbb{N}^+$) being a nonempty set. The process $(X(t))_{t \in [0,T]}$ is the state process, governed by the control $\theta \in \mathcal{U}$. We may write $X^{r,x;i,\theta}(t)$ for $0 \leq r \leq t \leq T$ to indicate the dependence of the state process on the control $\theta$, the initial time $r$ and initial path $x_r$.

In this paper, we consider the non-Markovian cases where the coefficients $\beta, f$, and $G$ may depend not only on time and control but also explicitly on $\omega \in \Omega$ and paths/history of the state process. Inspired by the (Markovian) counterparts in [14, 23], such problems may arise naturally from controlled ordinary differential equations allowing for path-dependent and random coefficients. Two interesting examples are sketched as follows.

**Example 1.1.** In terms of the optimization (1.1)-(1.2), we may study the control problem of some general stochastic equations:

$$
\min_{\theta \in \mathcal{U}} E \left[ \int_0^T \tilde{f}(s, \tilde{X}_s, \theta(s)) \, ds + \tilde{G}(\tilde{X}_T) \right]
$$

subject to

$$
\tilde{X}(t) = \tilde{x}_0 + \int_0^t \tilde{\beta}(s, \tilde{X}_s, \theta(s)) \, ds + \eta(t), \quad t \geq 0, \quad \tilde{x}_0 = x_0 \in \mathbb{R}^d,
$$

where the new term $(\eta(t))_{t \geq 0}$ may be any general $\mathcal{F}_t$-adapted stochastic process with or without rough paths, including Wiener processes, fractional Brownian motions, and associated integrals, etc. Set $X(t) = \tilde{X}(t) - \eta(t)$, for $t \in [0,T]$. The control problem (1.1)-(1.2) may be written equivalently as (1.1)-(1.2), while the associated coefficients $(f, \beta)(t, X_t, \theta(t)) = (\tilde{f}, \tilde{\beta})(s, (X + \eta)_s, \theta(s))$ and $G(X_T) = \tilde{G}((X + \eta)_T)$ may of course be random.

**Example 1.2** (Optimal consumption with habit formation).

Adopting the von Neumann-Morgenstern preferences over time interval $[0,T]$, the utility maximization with habit formation may be generally formulated as an optimal stochastic control problem like (1.1)-(1.2):

$$
\max_{c \in \tilde{A}} E \int_0^T u(t, c(t), Z(t, C_t)) \, dt
$$

subject to

$$
\begin{cases}
    dC(t) = c(t) \, dt, & t \in [0,T]; \quad C(0) = \kappa; \\
    Z(t, C_t) = \gamma(t) + \int_0^t \xi(t, s) \, dC(s), & t \in [0,T],
\end{cases}
$$

where we denote by $(c(t))_{t \in [0,T]}$ the consumption process with $C(t)$ the cumulative consumption until time $t$ and the process $Z(t, C_t)$ represents the standard of living with random coefficients $\gamma(t)$ and $\xi(t, s)$ for $0 \leq s \leq t \leq T$. The admissible control set $\tilde{A}$ may be complicated by incorporating the budget constraints for different market models; refer to [10, 16, 25] among many others.

Coming back to the control problem (1.1)-(1.2), we define the dynamic cost functional:

$$
J(t, x_t; \theta) = E_{\mathcal{F}_t} \left[ \int_t^T f(s, X_s^{t,x;\theta}, \theta(s)) \, ds + G(X_T^{t,x;\theta}) \right], \quad t \in [0,T],
$$

and the value function is given by

$$
V(t, x_t) = \text{essinf}_{\theta \in \mathcal{U}} J(t, x_t; \theta), \quad t \in [0,T].
$$

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Here and throughout this paper, we denote by $E_{\mathcal{F}_t}[\cdot]$ the conditional expectation with respect to $\mathcal{F}_t$. Due to the randomness and path-dependence of the coefficient(s), the value function $V(t, x_t)$ is generally a function of time $t$, path $x_t$, and $\omega \in \Omega$. In fact, the value function $V$ is proved to be the unique viscosity solution to the following stochastic path-dependent Hamilton-Jacobi (SPHJ) equation:

$$
\begin{cases}
-\partial_t u(t, x_t) - \mathcal{H}(t, x_t, \nabla u(t, x_t)) = 0, & (t, x) \in [0, T) \times C([0, T]; \mathbb{R}^d); \\
u(T, x) = G(x), & x \in C([0, T]; \mathbb{R}^d),
\end{cases}
$$

(1.7)

with

$$
\mathcal{H}(t, x_t, p) = \inf_{\nu \in U} \left\{ \beta'(t, x_t, v)p + f(t, x_t, v) \right\},
$$

for $p \in \mathbb{R}^d$,

where $\nabla u(t, x_t)$ denotes the the vertical derivative of $u(t, x_t)$ at the path $x_t$ (see Definition 2.1) and the unknown adapted random field $u$ is confined to the following form:

$$
u(t, x_t) = u(T, x_t, T) - \int_t^T \partial_s u(s, x_t, s - t)\,ds - \int_t^T \partial_w u(s, x_t, s - t)\,dW(s),$$

(1.8)

where $x_{t,s-t}(s) = x_t(s)1_{[0,t]}(s) + x_t(t)1_{[t,T]}(s)$ for $0 \leq t \leq s \leq r \leq T$. The Doob-Meyer decomposition theorem indicates the uniqueness of the pair $(\partial_t u, \partial_w u)$ and thus the linear operators $\partial_t$ and $\partial_w$ may be well defined in certain spaces (see Definition 2.2). The pair $(\partial_t u, \partial_w u)$ may also be defined as two differential operators; see [7 Section 5.2] and [21 Theorem 4.3] for instance. By comparing (1.7) and (1.8), we may rewrite the SPHJ equation (1.7) formally as a path-dependent backward stochastic partial differential equation (BSPDE):

$$
\begin{cases}
-\partial u(t, x_t) = \mathcal{H}(t, x_t, \nabla u(t, x_t))\,dt - \psi(t, x_t)\,dW(t), & (t, x) \in [0, T) \times C([0, T]; \mathbb{R}^d); \\
u(T, x) = G(x), & x \in C([0, T]; \mathbb{R}^d),
\end{cases}
$$

(1.9)

where the pair $(u, \psi) = (u, \partial_w u)$ is unknown.

When all the coefficients are deterministic path-dependent functions, the control problem is first studied in [23] and the viscosity solution theory of associated deterministic path-dependent Hamilton-Jacobi equations may be found in [11, 23]; for the theory of general deterministic path-dependent PDEs, see [1, 7, 8, 26, 32] among many others. When all the coefficients are just possibly random but with state-dependence, the resulting Hamilton-Jacobi equation is just a BSPDE (see [3, 29, 30]); for related research on general BSPDEs, we refer to [2, 6, 12, 17, 25] to mention but a few. When the coefficients are both random and path-dependent, some discussions may be found in [14, 15, 32] where, nevertheless, all the coefficients are required to be continuous and deterministic in $\omega \in \Omega$ and the resulting value function satisfies, instead, a deterministic path-dependent semilinear parabolic PDE. In the present work, all the involved coefficients are just measurable w.r.t. $\omega \in \Omega$ without any topology specified on $\Omega$, which allows the general random variables to appear in the coefficients, and in this setting, the control problem (1.1)-(1.2) and associated nonlinear SPHJ equation (1.7), to the best of our knowledge, have never been studied in the literature. In this paper, we are the first to use the dynamic programming method to deal with the control problem (1.1)-(1.2) allowing for both path-dependence and measurable randomness; a notion of viscosity solution is proposed and the existence and uniqueness of viscosity solution is proved under standard Lipschitz conditions.
The main feature of viscosity solution for SPHJ equation (1.7) is twofold. On the one hand, due to the path-dependence of the coefficients, the solution \( u \) is path-wisely defined on the path space \( C([0, T]; \mathbb{R}^d) \), while the lack of local compactness of the path space prompts us to define the random test functions in certain compact subspaces via optimal stopping times, which is different from the capacity and nonlinear expectation techniques for deterministic path-dependent PDEs (see [13, 14, 31] for instance). On the other hand, as the involved coefficients are just measurable w.r.t. \( \omega \) (see [13, 14, 31] for instance). On the other hand, as the involved coefficients are just measurable w.r.t. \( \omega \) on the sample space \((\Omega, \mathcal{F})\) without any equipped topology, it is not appropriate to define the viscosity solutions in a pointwise manner w.r.t. \( \omega \in (\Omega, \mathcal{F})\); rather, we use a class of random fields of form (1.8) having sufficient spacial regularity as test functions; at each point \((\tau, \xi)\) (\(\tau\) may be stopping time and \(\xi\) a \(C([0, \tau]; \mathbb{R}^d)\)-valued \(\mathcal{F}_\tau\)-measurable variable) the classes of test functions are also parameterized by \(\Omega_\tau \in \mathcal{F}_\tau\) and the type of compact subspaces. It is worth noting that due to the nonanticipativity constraint on the unknown function, the conventional variable-doubling techniques for deterministic Hamilton-Jacobi equations are not applicable in our stochastic setting. Instead, we prove a weak comparison principle with comparison conducted on compact subspaces, and then the uniqueness of viscosity solution is proved through regular approximations.

The rest of this paper is organized as follows. In Section 2, we introduce some notations, show the standing assumption on the coefficients, and define the viscosity solution. In Section 3, some auxiliary results including the dynamic programming principle are given in the first subsection and the value function is verified to be a viscosity solution in the second subsection. In Section 4, a weak comparison theorem is given and then the uniqueness is proved via approximations. Finally, we give the proof of Theorem 3.3 in the appendix.

2 Preliminaries and definition of viscosity solution

2.1 Preliminaries

For each integer \(k > 0\) and \(r \in [0, T]\), let space \(\Lambda^0_r(\mathbb{R}^k) := C([0, r]; \mathbb{R}^k)\) be the set of all \(\mathbb{R}^k\)-valued continuous functions on \([0, r]\) and \(\Lambda_r(\mathbb{R}^k) := D([0, r]; \mathbb{R}^k)\) the space of \(\mathbb{R}^k\)-valued càdlàg (right continuous with left limits) functions on \([0, r]\). Define

\[
\Lambda^0(\mathbb{R}^k) = \bigcup_{r \in [0, T]} \Lambda^0_r(\mathbb{R}^k), \quad \Lambda(\mathbb{R}^k) = \bigcup_{r \in [0, T]} \Lambda_r(\mathbb{R}^k).
\]

Throughout this paper, for each path \(X \in \Lambda_T(\mathbb{R}^k)\) and \(t \in [0, T]\), denote by \(X_t = (X(s))_{0 \leq s \leq t}\) its restriction to time interval \([0, t]\), while using \(X(t)\) to represent its value in \(\mathbb{R}^k\) at time \(t\), and moreover, when \(k = d\), we write simply \(\Lambda^0, \Lambda^0_0, \Lambda, \text{ and } \Lambda_r\).

Both the spaces \(\Lambda\) and \(\Lambda^0\) are equipped with the following quasi-norm and metric: for each \((x_r, \bar{x}_t) \in \Lambda_r \times \Lambda_t\) or \((x_r, \bar{x}_t) \in \Lambda^0_r \times \Lambda^0_t\) with \(0 \leq r \leq t \leq T\),

\[
\|x_r\|_0 = \sup_{s \in [0, r]} |x_r(s)|; \quad \|x_r - \bar{x}_t\|_0 = \sqrt{|t - r|} + \sup_{s \in [0, t]} \{|x_r(s) - \bar{x}_t(s)|1_{[0, r]}(s) + |x_r(r) - \bar{x}_t(s)|1_{[r, t]}(s)\}.
\]

Then \((\Lambda^0, \| \cdot \|_0)\) and \((\Lambda, \| \cdot \|_0)\) are Banach spaces for each \(t \in [0, T]\), while \((\Lambda^0, d_0)\) and \((\Lambda, d_0)\) are complete metric spaces. In fact, for each \(t \in [0, T]\), \((\Lambda^0_t, \| \cdot \|_0)\) and \((\Lambda_r, \| \cdot \|_0)\) can be and throughout this paper will be thought of as the complete subspaces of \((\Lambda^0_T, \| \cdot \|_0)\) and \((\Lambda_T, \| \cdot \|_0)\),

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respectively; indeed, for each \( x_t \in \Lambda_t \) \((x_t \in \Lambda^0_t \text{, respectively})\), we may define, correspondingly, \( \bar{x} \in \Lambda_T \) \((\bar{x} \in \Lambda^0_T \text{, respectively})\) with \( \bar{x}(s) = x_r(t \land s) \) for \( s \in [0, T] \) and throughout this work, we do not distinguish between \( x \) and \( \bar{x} \). In addition, we shall use \( \mathcal{B}(\Lambda^0) \), \( \mathcal{B}(\Lambda) \), \( \mathcal{B}(\Lambda^0_t) \) and \( \mathcal{B}(\Lambda_t) \) to denote the corresponding Borel \( \sigma \)-algebras. By contrast, for each \( \delta > 0 \) and \( x_r \in \Lambda \), denote by \( B_\delta(x_r) \) the set of paths \( y_t \in \Lambda \) satisfying \( d_0(x_r, y_t) \leq \delta \).

For each \( x_t \in \Lambda_t \) and any \( h \in \mathbb{R}^d \), we define its vertical perturbation \( x^h_t \in \Lambda_t \) with \( x^h_t(s) = x_t(s)1_{[0, t)}(s) + (x_t(t) + h)1_{\{t\}}(s) \) for \( s \in [0, t] \).

**Definition 2.1.** Given a functional \( \phi : \Lambda \to \mathbb{R} \) and \( x_t \in \Lambda_t \), \( \phi \) is said to be vertically differentiable at \( x_t \) if the function

\[
\phi(x^h_t) : \mathbb{R}^d \to \mathbb{R},
\]

\[
h \mapsto \phi(x^h_t)
\]

is differentiable at 0. The gradient

\[
\nabla \phi(x^h_t) := (\nabla_1 \phi(x^h_t), \ldots, \nabla_d \phi(x^h_t))^t
\]

with

\[
\nabla_i \phi(x^h_t) := \lim_{\delta \to 0} \frac{\phi(x^h_t) - \phi(x_t)}{\delta}
\]

is called the vertical derivative of \( \phi \) at \( x_t \), where \( \{e_i\}_{i=1}^{d} \) is the canonical basis in \( \mathbb{R}^d \).

Let \( \mathbb{B} \) be a Banach space equipped with norm \( \| \cdot \|_{\mathbb{B}} \). The continuity of functionals on metric spaces \( \Lambda^0 \) and \( \Lambda \) can be defined in a standard way. Given \( x_t \in \Lambda \), we say a map \( \phi : \Lambda \to \mathbb{B} \) is continuous at \( x_t \) if for any \( \varepsilon > 0 \) there exists some \( \delta > 0 \) such that for any \( \bar{x}_t \in \Lambda \) satisfying \( d_0(\bar{x}_t, x_t) < \delta \), it holds that \( \| \phi(\bar{x}_t) - \phi(x_t) \|_{\mathbb{B}} < \varepsilon \). If the \( \mathbb{B} \)-valued functional \( \phi \) is continuous and bounded at all \( x_t \in \Lambda \), \( \phi \) is said to be continuous on \( \Lambda \) and denoted by \( \phi \in C(\Lambda; \mathbb{B}) \). Similarly, we define \( C(\Lambda^0; \mathbb{B}) \), \( C([0, T] \times \Lambda; \mathbb{B}) \), and \( C([0, T] \times \Lambda^0; \mathbb{B}) \).

Throughout this work, we denote by \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\) a complete filtered probability space. The filtration \( \{\mathcal{F}_t\}_{t \geq 0} \) satisfies the usual conditions and is generated by an \( m \)-dimensional Wiener process \( W = \{W(t) : t \in [0, \infty)\} \) together with all the \( \mathbb{P} \)-null sets in \( \mathcal{F} \). The associated predictable \( \sigma \)-algebra on \( \Omega \times [0, T] \) is denoted by \( \mathcal{P} \).

For each \( t \in [0, T] \), denote by \( L^0(\Omega \times \Lambda_t, \mathcal{F} \otimes \mathcal{B}(\Lambda_t); \mathbb{B}) \) the space of \( \mathbb{B} \)-valued \( \mathcal{F} \otimes \mathcal{B}(\Lambda_t) \)-measurable random variables. For each measurable function

\[
u : \quad (\Omega \times [0, T] \times \Lambda, \mathcal{F} \otimes \mathcal{B}([0, T]) \otimes \mathcal{B}(\Lambda)) \to (\mathbb{B}, \mathcal{B}(\mathbb{B})),
\]

we say \( \nu \) is adapted if for any time \( t \in [0, T] \), \( \nu \) is \( \mathcal{F}_t \otimes \mathcal{B}(\Lambda_t) \)-measurable. For \( p \in [1, \infty] \), denote by \( \mathcal{S}^p(\Lambda; \mathbb{B}) \) the set of all the adapted functions \( \nu : \Omega \times [0, T] \times \Lambda \to \mathbb{B} \) such that for almost all \( \omega \in \Omega \), \( \nu \) is valued in \( C([0, T] \times \Lambda; \mathbb{B}) \) and

\[
\| \nu \|_{\mathcal{S}^p(\Lambda; \mathbb{B})} := \left( \sup_{(\omega, t, x) \in \Omega \times (0, T) \times \Lambda} \left\| \nu(\omega, t, x) \right\|_{\mathbb{B}}^p \right)^{1/p} < \infty.
\]

For \( p \in [1, \infty) \), denote by \( \mathcal{L}^p(\Lambda; \mathbb{B}) \) the totality of all the adapted functions \( \mathcal{X} : \Omega \times [0, T] \times \Lambda \to \mathbb{B} \) such that for almost all \((\omega, t) \in \Omega \times [0, T] \), \( \mathcal{X}(t) \) is valued in \( C(\Lambda_t; \mathbb{B}) \), and

\[
\| \mathcal{X} \|_{\mathcal{L}^p(\Lambda; \mathbb{B})} := \left( \int_0^T \sup_{x_t \in \Lambda_t} \left\| \mathcal{X}(t, x_t) \right\|_{\mathbb{B}}^p dt \right)^{1/p} < \infty.
\]
Obviously, \((\mathcal{S}^p(\Lambda; \mathbb{B}), \| \cdot \|_{\mathcal{S}^p(\Lambda; \mathbb{B})})\) and \((\mathcal{L}^p(\Lambda; \mathbb{B}), \| \cdot \|_{\mathcal{L}^p(\Lambda; \mathbb{B})})\) are Banach spaces. In a standard way, we define \(L^0(\Omega \times \Lambda^0, \mathcal{F}_t \otimes \mathcal{B}(\Lambda^0); \mathbb{B})\), \((\mathcal{S}^p(\Lambda^0; \mathbb{B}), \| \cdot \|_{\mathcal{S}^p(\Lambda^0; \mathbb{B})})\), and \((\mathcal{L}^p(\Lambda^0; \mathbb{B}), \| \cdot \|_{\mathcal{L}^p(\Lambda^0; \mathbb{B})})\).

Following is the assumption we use throughout this paper.

(A1) \(G \in L^\infty(\Omega, \mathcal{F}_T; C(\Lambda_T; \mathbb{R}))\). For the coefficients \(g = f, \beta^i\) \((1 \leq i \leq d)\),

(i) for each \(v \in U\), \(g(\cdot, \cdot, v)\) is adapted;

(ii) for almost all \((\omega, t) \in \Omega \times [0, T]\), \(g(t, \cdot, \cdot)\) is continuous on \(\Lambda_t \times U\);

(iii) there exists \(L > 0\) such that for all \(x, \bar{x} \in \Lambda_T\), \(t \in [0, T]\) and \(\gamma_l, \bar{\gamma}_l \in \Lambda_t\), there hold

\[
\text{esssup}_{\omega \in \Omega} |G(x)| + \text{esssup}_{\omega \in \Omega} \sup_{v \in U} |g(t, \gamma_l, v)| \leq L,
\]

\[
\text{esssup}_{\omega \in \Omega} |G(x) - G(\bar{x})| + \text{esssup}_{\omega \in \Omega} \sup_{v \in U} |g(t, \gamma_l, v) - g(t, \bar{\gamma}_l, v)| \leq L \left(\|x - \bar{x}\|_0 + \|\gamma_l - \bar{\gamma}_l\|_0\right).
\]

2.2 Definition of viscosity solutions

For \(\delta \geq 0\), \(x_t \in \Lambda_t\), we define the horizontal extension \(x_{t, \delta} \in \Lambda_{t+\delta}\) by setting \(x_{t, \delta}(s) = x_t(s \wedge t)\) for all \(s \in [0, t + \delta]\).

**Definition 2.2.** For \(u \in \mathcal{S}^2(\Lambda; \mathbb{R})\) with \(\nabla u \in \mathcal{L}^2(\Lambda; \mathbb{R})\), we say \(u \in \mathcal{C}_2^1\) if

(i) there exists \((\partial_t u, \partial_x u) \in \mathcal{L}^2(\Lambda; \mathbb{R}) \times \mathcal{L}^2(\Lambda; \mathbb{R}^m)\) such that for all \(0 \leq r \leq \tau \leq T\), and \(x_t \in \Lambda_r\), it holds that

\[
u(t, x_t) = u(r, x_r) + \int_r^\tau \partial_x u(s, x_{r, s-r}) \, ds + \int_r^\tau \partial_x u(s, x_{r, s-r}) \, dW(s), \text{ a.s.;} \quad (2.1)
\]

(ii) there exists a constant \(g \in (0, \infty)\) such that for almost all \((\omega, t) \in \Omega \times [0, T]\) and all \(x_t \in \Lambda_t\), there holds \(|\nabla u(t, x_t)| \leq g\);

(iii) there exist a constant \(\alpha \in (0, 1)\) and a finite partition \(0 = t_0 < t_1 < \ldots < t_n = T\), for integer \(n \geq 1\), such that \(\nabla u\) is a.s. valued in \(C((\bar{t}_j, \bar{t}_{j+1}) \times \Lambda; \mathbb{R}^d)\) for \(j = 0, \ldots, n-1\), and for any \(0 < \delta < \min_{0 \leq j \leq n-1} |t_{j+1} - \bar{t}_j|\), and \(g = \partial_t u, \nabla_i u, (\partial_x u)^j\), \(i = 1, \ldots, d, j = 1, \ldots, m\), there exists \(L^\alpha_{n} \in (0, \infty)\) satisfying a.s. for almost all \(t \in \cup_{0 \leq j \leq n-1}(\bar{t}_j, \bar{t}_{j+1} - \delta)\) and all \(x_t, y_t \in \Lambda_t\),

\[
|g(t, x_t) - g(t, y_t)| \leq L^\alpha_{n} \|x_t - y_t\|_0^\alpha. \quad (2.2)
\]

We say the number \(\alpha\) is the exponent associated to \(u \in \mathcal{C}_2^1\) and \(0 = t_0 < t_1 < \ldots < t_n = T\) the associated partition.

Each \(u \in \mathcal{C}_2^1\) may be thought of as an Itô process and thus a semi-martingale parameterized by \(x \in \Lambda\). The uniqueness of semimartingale decomposition (by Doob-Meyer decomposition theorem) ensures the uniqueness of the pair \((\partial_t u, \partial_x u)\) at points \((\omega, t, x_{s,t-s})\) for \(0 \leq s < t \leq T\). Recall that by the definition of the space \(\mathcal{L}^2(\Lambda; \mathbb{B})\) where \(\mathbb{B}\) denotes a Banach space, for almost all \((\omega, t) \in \Omega \times [0, T]\), \(g(t)\) is valued in \(C(\Lambda_t; \mathbb{B})\) for \(g \in \mathcal{L}^p(\Lambda; \mathbb{B})\). With a standard denseness argument we may define the pair \((\partial_t u, \partial_x u)\) in \(\mathcal{L}^2(\Lambda; \mathbb{R}) \times \mathcal{L}^2(\Lambda; \mathbb{R}^m)\) with

\[
(\partial_t u, \partial_x u)(t, x_t) = \lim_{s \to t^-} \left(\partial_t u, \partial_x u\right)(t, x_{s,t-s}) = (\partial_t u, \partial_x u)(t, \lim_{s \to t^-} x_{s,t-s}) \text{ a.s., } \forall x_t \in \Lambda_t,
\]
for almost all \( t \in (0, T] \), and for each \( x_0 \in \mathbb{R}^d \), set

\[
(\partial_t u, \partial_\omega u)(0, x_0) = \lim_{h \to 0^+} \frac{1}{h} \int_0^h (\partial_s u, \partial_\omega u)(s, x_0, s) ds,
\]

if the limit exists; otherwise put \((\partial_t u, \partial_\omega u)(0, x_0) = (0, 0)\). This makes sense of the two linear operators \( \partial_t \) and \( \partial_\omega \) which are consistent with the differential operators in [7 Section 5.2] and [21 Theorem 4.3]. For the reader’s interest, we may consider an example when \( m = d = 1 \):

\[
u(t, x_t) = \sin \left( \int_0^t g(s, t) \cos (x(s) + \|W_s\|_0) ds + x(t) + 2W(t) \right), \quad t \in [0, T], \quad x_t \in \Lambda(\mathbb{R}),
\]

with \( g \in C^2([0, T] \times [0, T]) \). Then we have \( u \in C^1_{\mathbb{F}}(\Lambda) \)

\[
\nabla u(t, x_t) = \cos \left( \int_0^t g(s, t) \cos (x(s) + \|W_s\|_0) ds + x(t) + 2W(t) \right),
\]

\[
\partial_t u(t, x_t) = -2u(t, x_{t^-}) + \cos \left( \int_0^t g(s, t) \cos (x(s) + \|W_s\|_0) ds + x(t^-) + 2W(t) \right)
\]

\[
\cdot \left[ \int_0^t \frac{\partial g(s, t)}{\partial t} \cos (x(s) + \|W_s\|_0) ds + g(t, t) \cos (x(t^-) + \|W_t\|_0) \right],
\]

\[
\partial_\omega u(t, x_t) = 2\cos \left( \int_0^t g(s, t) \cos (x(s) + \|W_s\|_0) ds + x(t^-) + 2W(t) \right),
\]

where \( x(0^-) = x(0), \ x(t^-) = \lim_{s \to t^-} x(s) \) for \( t \in (0, T] \) and \( x_{t^-} = x^{t(-)}_t \).

Particularly, if \( u(t, x) \) is a deterministic function on the time-state space \([0, T] \times \mathbb{R}^d\), we may have \( \partial_\omega u \equiv 0 \) and \( \partial_t u \) coincides with the classical partial derivative in time; if the random function \( u \) on \( \Omega \times [0, T] \times \mathbb{R}^d \) is regular enough (w.r.t. \( \omega \)), the term \( \partial_\omega u \) is just the Malliavin derivative. In addition, the operators \( \partial_t \) and \( \partial_\omega \) are different from the path derivatives \( (\partial_t, \partial_\omega) \) via the functional Itô formulas (see [3 and 14 Section 2.3]); if \( u(\omega, t, x) \) is smooth enough w.r.t. \( (\omega, t) \) in the path space, for each \( x \in \Lambda^0_{\mathbb{F}} \), we have the relation

\[
\partial_t u(\omega, t, x_{s,t-s}) = \left( \partial_t + \frac{1}{2} \partial_{\omega t}^2 \right) u(\omega, t, x_{s,t-s}), \quad \partial_\omega u(\omega, t, x_{s,t-s}) = \partial_\omega u(\omega, t, x_{s,t-s}),
\]

for \( 0 \leq s < t < T \), which may be seen from [14 Section 6] and [5].

**Remark 2.1.** In condition (iii) in Definition 2.2, we endow the elements of \( C^1_{\mathbb{F}}(\Lambda) \) with a certain kind of regularity in a piecewise way, which would allow us to utilize the approximations (in \( C^1_{\mathbb{F}}(\Lambda) \)) that are just piecewisely sufficiently regular in the proof for the uniqueness of viscosity solution in Section 4.2. In fact, the test function space \( C^1_{\mathbb{F}}(\Lambda) \) could be further enlarged. For instance, we may replace the Hölder continuity (2.2) (that holds locally in time \( t \)) with a local one, i.e., let (2.2) hold when \( \|x_{t} - y_{t}\|_0 \) is small enough. Nevertheless, in this work we would not seek such a generality as it is sufficient to have the present version of \( C^1_{\mathbb{F}}(\Lambda) \).

For each stopping time \( t \leq T \), let \( \mathcal{T}^t \) be the set of stopping times \( \tau \) valued in \([t, T]\) and \( \mathcal{T}_t^+ \) the subset of \( \mathcal{T}^t \) such that \( \tau > t \) for each \( \tau \in \mathcal{T}_t^+ \). For each \( \tau \in \mathcal{T}^0 \) and \( \Omega_\tau \in \mathcal{F}_\tau \), we denote by \( L^0(\Omega_\tau, \mathcal{F}_\tau; \Lambda^0_{\mathbb{F}}) \) the set of \( \Lambda^0_{\mathbb{F}} \)-valued \( \mathcal{F}_\tau \)-measurable functions and define the restricted space

\[
L^0(\Omega_\tau, \mathcal{F}_\tau; \Lambda^0_{\mathbb{F}}) = \{ \xi(\tau \wedge \cdot) : \xi \in L^0(\Omega_\tau, \mathcal{F}_\tau; \Lambda^0_{\mathbb{F}}) \}.
\]
Here, we recall that for each \( t \in [0, T] \) we think of \( \Lambda^0_t \) as a complete subspace of \( \Lambda^0_T \), and we do not distinguish \( \eta \in \Lambda^0_t \) from \( \eta(\tau \land \cdot) \in \Lambda^0_\tau \).

For each \( k \in \mathbb{N}^+, 0 \leq t \leq s \leq T \) and \( \xi \in \Lambda_t \), define

\[
\Lambda^{0,k,\xi}_{t,s} = \left\{ x \in \Lambda_s : x(\tau) = \xi(\tau \land t) + \int_{t \land \tau}^s g(r) \, dr, \quad \tau \in [0, s], \text{ for some } g \in L^\infty([0, T]) \right\},
\]

and in particular, we set \( \Lambda^{0,k}_{0,t} = \bigcup_{\xi \in \mathbb{R}^d} \Lambda^{0,k,\xi}_{0,t} \) for each \( t \in [0, T] \). Then for each \( \xi \in \Lambda^0_t \), Arzelà-Ascoli theorem indicates that each \( \Lambda^{0,k,\xi}_{t,s} \) is compact in \( \Lambda^0_s \). Moreover, it is obvious that \( \bigcup_{k \in \mathbb{N}^+} \Lambda^{0,k}_{0,s} \) is dense in \( \Lambda^0_s \). In addition, by saying \( (s, x) \to (t^+, \xi) \) for some \( (t, \xi) \in [0, T) \times \Lambda^0_t \) we mean \( (s, x) \to (t^+, \xi) \) with \( s \in (t, T] \) and \( x \in \bigcup_{k \in \mathbb{N}^+} \Lambda^{0,k,\xi}_{t,s} \).

We now introduce the notion of viscosity solutions. For each \( (u, \tau) \in S^2(\Lambda; \mathbb{R}) \times \mathcal{T}^0, \Omega_\tau \in \mathcal{F}_\tau \) with \( \mathbb{P}(\Omega_\tau) > 0 \) and \( \xi \in L^0(\Omega_\tau, \mathcal{F}_\tau; \Lambda^0_t) \), we define for each \( k \in \mathbb{N}^+ \),

\[
\mathcal{G}u(\tau, \xi; \Omega_\tau, k) := \left\{ \phi \in \mathcal{C}^1_{\text{b}L} : \text{there exists } \hat{\tau} \in \mathcal{T}^+_\tau \text{ such that} \right. \\
(\phi - u)(\tau, \xi)1_{\Omega_\tau} = 0 = \text{essinf}_{\tau \in \mathcal{T}^+} E_{\mathcal{F}_{\hat{\tau}}} \left[ \inf_{y \in \Lambda^{0,k,\xi}_{\tau,\hat{\tau}}} (\phi - u)(\hat{\tau} \land \hat{\tau}, y) \right] 1_{\Omega_\tau}, \text{ a.s.} \},
\]

\[
\mathcal{V}u(\tau, \xi; \Omega_\tau, k) := \left\{ \phi \in \mathcal{C}^1_{\text{b}L} : \text{there exists } \hat{\tau} \in \mathcal{T}^+_\tau \text{ such that} \right. \\
(\phi - u)(\tau, \xi)1_{\Omega_\tau} = 0 = \text{esssup}_{\tau \in \mathcal{T}^+} E_{\mathcal{F}_{\hat{\tau}}} \left[ \sup_{y \in \Lambda^{0,k,\xi}_{\tau,\hat{\tau}}} (\phi - u)(\hat{\tau} \land \hat{\tau}, y) \right] 1_{\Omega_\tau}, \text{ a.s.} \}.
\]

**Remark 2.2.** In the above definitions of sub/superjets \( \mathcal{G}u(\tau, \xi; \Omega_\tau, k) \) and \( \mathcal{V}u(\tau, \xi; \Omega_\tau, k) \), due to (possibly non-Markovian type) randomness that is not addressed in Lukoyanov [28], we adopt optimal stopping times after taking maximum/minimum over the compact subspaces of paths, which is, however, obviously different from taking optimal times under certain classes of nonlinear expectations in [13] [14] [31]. As for the natural connections between optimal stopping times and non-Markovian type optimal controls, we refer to [28] [29] [31] for more discussions.

The definition of viscosity solutions then comes as follows.

**Definition 2.3.** We say \( u \in S^2(\Lambda^0; \mathbb{R}) \) is a viscosity subsolution (resp. supersolution) of SPHJ equation (1.7), if \( u(T, x) \leq (\text{resp. } \geq) G(x) \) for all \( x \in \Lambda^0_T \) a.s., and for any \( K_0 \in \mathbb{N}^+ \), there exists \( k \in \mathbb{N}^+ \) with \( k \geq K_0 \) such that for any \( \tau \in \mathcal{T}^0, \Omega_\tau \in \mathcal{F}_\tau \) with \( \mathbb{P}(\Omega_\tau) > 0 \) and \( \xi \in L^0(\Omega_\tau, \mathcal{F}_\tau; \Lambda^0_t) \) and any \( \phi \in \mathcal{G}u(\tau, \xi; \Omega_\tau, k) \) (resp. \( \phi \in \mathcal{V}u(\tau, \xi; \Omega_\tau, k) \)), (resp. \( \phi \in \mathcal{G}u(\tau, \xi; \Omega_\tau, k) \)) holds,

\[
\text{ess } \liminf_{(s, x) \to (\tau^+, \xi)} \{ -d_s \phi(s, x) - H(s, x, \nabla \phi(s, x)) \} \leq 0, \text{ for almost all } \omega \in \Omega_\tau
\]

(2.3)

\[
(\text{resp. } \text{ess } \limsup_{(s, x) \to (\tau^+, \xi)} \{ -d_s \phi(s, x) - H(s, x, \nabla \phi(s, x)) \} \geq 0, \text{ for almost all } \omega \in \Omega_\tau).
\]

(2.4)

The function \( u \) is a viscosity solution of SPHJ equation (1.7) if it is both a viscosity subsolution and a viscosity supersolution of (1.7).
Remark 2.3. In the above definition, one may see that each viscosity sub-solution (resp. supersolution) of SPHJ equation (1.7) is associated to an infinite sequence of integers $1 < k_1 < k_2 < \cdots < k_n < \cdots$ (resp. $1 \leq k_1 \leq k_2 \leq \cdots \leq k_n \leq \cdots$) such that the required properties in Definition 2.3 are holding for all the test functions in $u_{\tau}(\tau, \xi; \Omega_{\tau}, k_j)$ (resp. $\overline{u}_{\tau}(\tau, \xi; \Omega_{\tau}, k_j)$) for all $i \in \mathbb{N}^+$.

Throughout this paper, we define for each $\phi \in \mathcal{C}_{\mathcal{F}}^1$, $v \in U$, $t \in [0, T]$, and $x_t \in \Lambda_t$,

$$\mathcal{L}^u \phi(t, x_t) = \partial_t \phi(t, x_t) + \beta(t, x_t, v) \nabla \phi(t, x_t).$$

Remark 2.4. In view of the assumption (A1), for each $\phi \in \mathcal{C}_{\mathcal{F}}^1$, there exists an $\mathcal{F}_t$-adapted process $\zeta^\phi \in L^2(\Omega \times [0, T])$ such that for a.e. $(\omega, t) \in \Omega \times [0, T]$, and all $x_t \in \Lambda_t$, we have

$$\left| - \partial_t \phi(t, x_t) - \mathbb{H}(t, x_t, \nabla \phi(t, x_t)) \right| \leq \sup_{v \in U} \left| \mathcal{L}^u \phi(t, x_t) + f(t, x_t, v) \right| \leq \zeta^\phi;$$

meanwhile, there exists a finite partition $0 = t_0 < t_1 < \cdots < t_n = T$, such that for any $0 < \delta < \min_{0 \leq j \leq n-1} |t_{j+1} - t_j|$, there exists $L_\alpha^\phi \in (0, \infty)$ satisfying a.s. for almost all $t \in \cup_{0 \leq j \leq n-1}(t_j, t_{j+1} - \delta]$ and all $x_t, \bar{x}_t \in \Lambda_t$,

$$\left| \left\{ - \partial_t \phi(t, x_t) - \mathbb{H}(t, x_t, \nabla \phi(t, x_t)) \right\} - \left\{ - \partial_t \phi(t, \bar{x}_t) - \mathbb{H}(t, \bar{x}_t, \nabla \phi(t, \bar{x}_t)) \right\} \right|$$

$$\leq \sup_{v \in U} \left| \left( \mathcal{L}^u \phi(t, x_t) + f(t, x_t, v) \right) - \left( \mathcal{L}^u \phi(t, \bar{x}_t) + f(t, \bar{x}_t, v) \right) \right|$$

$$\leq L_\alpha^\phi \left( \|x_t - \bar{x}_t\|_0^\alpha + \|x_t - \bar{x}_t\|_0 \right), \quad (2.5)$$

where $\alpha$ is the exponent associated to $\phi \in \mathcal{C}_{\mathcal{F}}^1$. Therefore, the essential limits in (2.3) and (2.4) are well-defined.

In Definition 2.3, the defined viscosity solution integrates the following three main aspects:

First, due to the path-dependence of the coefficients, the solution $u$ is path-wisely defined on the path space $C([0, T]; \mathbb{R}^d)$, and the lack of local compactness of the path space prompts us to define the random test functions (tangent from above or from below) in certain compact subspaces via optimal stopping times. In view of the spaces $\mathcal{G}u(\tau, \xi; \Omega_{\tau}, k)$ and $\overline{u}(\tau, \xi; \Omega_{\tau}, k)$, one may, however, see that we avoid the usage of the capacity and nonlinear expectation techniques for deterministic path-dependent PDEs (see [13, 14] for instance).

Second, as the involved coefficients are just measurable w.r.t. $\omega$ on the sample space $(\Omega, \mathcal{F})$ without any specified topology, it is not appropriate to define the viscosity solutions in a pointwise manner w.r.t. $\omega \in (\Omega, \mathcal{F})$. This together with the nonanticipativity constraint enlightens us to use relatively regular random fields in $\mathcal{C}_{\mathcal{F}}^1$ as test functions; at each point $(\tau, \xi)$, the classes of test functions are also parameterized by measurable set $\Omega_{\tau} \in \mathcal{F}_{\tau}$ and the type of compact subspaces of $C([0, T]; \mathbb{R}^d)$.

Third, the test function space $\mathcal{C}_{\mathcal{F}}^1$ is expected to include the classical solutions. However, it is typical that the classical solutions may not be differentiable in the time variable $t$ and $(\partial_t u, \partial_u u)$ may not be time-continuous but just measurable in $t$, which is also reflected in Definition 2.2, see [11, 33] for the state-dependent BSPDEs, or one may even refer to the standard theory of BSDEs that may be though of as trivial path-independent cases. This nature leads to the usage of essential limits in (2.3) and (2.4).
3 Existence of the viscosity solution

3.1 Some auxiliary results

Under assumption (A1) with the vanishing diffusion coefficients in stochastic (ordinary) differential equation (1.2), the following assertions are standard; see [18, 34] for instance.

Lemma 3.1. Let (A1) hold. Given \( \theta \in U \), for the strong solution of (stochastic) ODE (1.2), there exists \( K > 0 \) such that, for any \( 0 \leq r \leq t \leq s \leq T \), and \( \xi \in L^0(\Omega, \mathcal{F}_t; \Lambda_r) \),

(i) the two processes \( \left(X_s^r, X_s^{r, \xi, \theta}\right)_{t \leq s \leq T} \) and \( \left(X_s^t, X_s^{t, \xi, \theta}\right)_{t \leq s \leq T} \) are indistinguishable;

(ii) \( \max_{s \leq t \leq T} \left\|X_t^{r, \xi, \theta}\right\|_0 \leq K (1 + \|\xi\|_0) \text{ a.s.} \);

(iii) \( d_0(X_s^{r, \xi, \theta}, X_s^{r, \xi, \theta}) \leq K \left(|s - t| + |s - t|^{1/2}\right) \text{ a.s.} \);

(iv) given another \( \xi \in L^0(\Omega, \mathcal{F}_t; \Lambda_r) \),

\[
\max_{r \leq t \leq T} \left\|X_t^{r, \xi, \theta} - X_t^{r, \xi, \theta}\right\|_0 \leq K\|\xi - \xi\|_0 \text{ a.s.};
\]

(v) the constant \( K \) depends only on \( L \) and \( T \).

The following regular properties of the value function \( V \) hold in a similar way to [29, Proposition 3.3] and the proof is omitted.

Proposition 3.2. Let (A1) hold.

(i) For each \( t \in [0, T] \), \( \varepsilon \in (0, \infty) \), and \( \xi \in L^0(\Omega, \mathcal{F}_t; \Lambda_t) \), there exists \( \tilde{\theta} \in U \) such that

\[
E\left[J(t, \xi; \tilde{\theta}) - V(t, \xi)\right] < \varepsilon.
\]

(ii) For each \( (\theta, x_0) \in U \times \mathbb{R}^d \), \( \left\{J(t, X_t^{0, x_0; \theta}) - V(t, X_t^{0, x_0; \theta})\right\}_{t \in [0, T]} \) is a supermartingale, i.e., for any \( 0 \leq t \leq \tilde{t} \leq T \),

\[
V(t, X_t^{0, x_0; \theta}) \leq E_{\tilde{\mathcal{F}}_t} V(\tilde{t}, X_{\tilde{t}}^{0, x_0; \theta}) + E_{\tilde{\mathcal{F}}_t} \int_t^{\tilde{t}} f(s, X_s^{0, x_0; \theta}, \theta(s)) ds, \text{ a.s.} \tag{3.1}
\]

(iii) For each \( (\theta, x_0) \in U \times \mathbb{R}^d \), \( \left\{V(s, X_s^{0, x_0; \theta})\right\}_{s \in [0, \tilde{t}]} \) is a continuous process.

(iv) There exists \( L_V > 0 \) such that for each \( (\theta, t) \in U \times [0, T] \),

\[
|V(t, x_t) - V(t, y_t)| + |J(t, x_t; \theta) - J(t, y_t; \theta)| \leq L_V \|x_t - y_t\|_0, \text{ a.s., \ \forall x_t, y_t \in \Lambda_t,}
\]

with \( L_V \) depending only on \( T \) and \( L \).

(v) With probability 1, \( V(t, x) \) and \( J(t, x; \theta) \) for each \( \theta \in U \) are continuous on \([0, T] \times \Lambda\) and

\[
\sup_{(t, x) \in [0, T] \times \Lambda} \max \left\{ \left|V(t, x_t)\right|, \left|J(t, x_t; \theta)\right| \right\} \leq L(T + 1) \quad \text{a.s.}
\]

Theorem 3.3. Let assumption (A1) hold. For any stopping times \( \tau, \bar{\tau} \) with \( \tau \leq \bar{\tau} \leq T \), and any \( \xi \in L^0(\Omega, \mathcal{F}_{\tau}; \Lambda_0^0) \), we have

\[
V(\tau, \xi) = \text{essinf}_{\theta \in \mathcal{U}_{\tau}} E_{\tilde{\mathcal{F}}_{\tau}} \left[ \int_{\tau}^{\bar{\tau}} f\left(s, X_s^{\tau, \xi, \theta}, \theta(s)\right) ds + V\left(\tau, X_{\tau}^{\tau, \xi, \theta}\right) \right] \quad \text{a.s.}
\]

The proof is similar to [29, Theorem 3.4], but with some careful compactness arguments about path (sub)spaces; therefore, the proof is postponed to the appendix.
3.2 Existence of the viscosity solution

We first generalize an Itô-Kunita formula by Kunita [20] Pages 118-119 for the composition of random fields and stochastic differential equations to our path-dependent setting. Recall that for each \( \phi \in \mathcal{C}^1_{\mathcal{F}}, \ v \in U, \ t \in [0, T], \) and \( x_t \in \Lambda_t, \)

\[ \mathcal{L}^u \phi(t, x_t) = \partial_t \phi(t, x_t) + \beta^i(t, x_t, v) \nabla \phi(t, x_t). \]

**Lemma 3.4.** Let assumption (A1) hold. Suppose \( u \in \mathcal{C}^1_{\mathcal{F}} \) with the associated partition \( 0 = l_0 < l_1 < \ldots < l_n = T. \) Then, for each \( \theta \in \mathcal{U}, \) it holds almost surely that, for each \( l_j \leq \theta \leq l_{j+1}, \) \( j = 0, \ldots, n - 1, \) and \( x_0 \in \Lambda_{\theta}, \) it holds that

\[ u(\tau, X^\theta_{\tau}) - u(\theta, X_{\theta}) = \int_\theta^\tau \mathcal{L}^\theta(s) u \left( s, X^\theta_s \right) ds + \int_\theta^\tau \partial^\omega u(\tau, X^\theta_{\tau}) dW(\tau), \ a.s. \quad (3.2) \]

**Proof.** In view of the time continuity, w.l.o.g., we only prove (3.2) for \( \tau \in (0, l_i), \) \( \theta = 0 \) and \( x_0 = x \in \mathbb{R}^d. \) For each \( N \in \mathbb{N}^+ \) with \( N > 2, \) let \( t_i = \frac{i}{N} \) for \( i = 0, 1, \ldots, N. \) Then, we get a partition of \([0, \tau]\) with \( 0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = \tau. \) For each \( \theta \in \mathcal{U}, \) let

\[ N^X(t) = \sum_{i=0}^{N-1} X^0_{t_i}1_{[t_i, t_{i+1})}(t) + X^0(\tau)1_{\{\tau\}}(t), \quad \text{for } t \in [0, \tau], \]

and \( N^X_{t-s}(s) = N^X(s)1_{[0,t)}(s) + \lim_{r \to t^-} N^X(\tau)1_{\{\tau\}}(s), \) for \( 0 \leq s \leq t \leq \tau. \) In view of the path continuity, we have the following approximation:

\[ \lim_{N \to \infty} \| X^0_{t-N} - N^X_t \|_0 + \| X^0_{t-N} - N^X_t \|_0 = 0 \quad \text{for all } t \in (0, \tau], \ a.s. \]

For each \( i \in \{0, \cdots, N-1\}, \)

\[ u(t_{i+1}, N^X_{t_{i+1}}) - u(t_i, N^X_{t_i}) = u(t_{i+1}, N^X_{t_{i+1}-}) - u(t_i, N^X_{t_i}) + u(t_{i+1}, N^X_{t_{i+1}}) - u(t_{i+1}, N^X_{t_{i+1}-}) \]

\[ := I^i_1 + I^i_2, \]

where by \( u \in \mathcal{C}^1_{\mathcal{F}}, \)

\[ I^i_1 = u(t_{i+1}, N^X_{t_{i+1}-}) - u(t_i, N^X_{t_i}) = \int_{t_i}^{t_{i+1}} \partial^\omega u(s, N^X_{s-}) ds + \int_{t_i}^{t_{i+1}} \partial^\omega u(r, N^X_{r-}) dW(r), \]

and by the definition of vertical derivative and the integration by parts formula

\[ I^i_2 = u(t_{i+1}, N^X_{t_{i+1}}) - u(t_{i+1}, N^X_{t_{i+1}-}) \]

\[ = u(t_{i+1}, N^X_{t_{i+1}} - N^X_{t_i}) - u(t_{i+1}, N^X_{t_{i+1}-}) \]

\[ = \int_{t_i}^{t_{i+1}} (\nabla u)(t_{i+1}, N^X_{t_{i+1}-}) \beta(s, X^0_{s}, \theta(s)) ds. \]

Recall that

\[ u(\tau, N^X_{\tau}) - u(0, x) = \sum_{i=0}^{N-1} I^i_1 + I^i_2. \quad (3.3) \]
Then, the dominated convergence and the dominated convergence theorem for stochastic integrals ([27 Chapter IV, Theorem 32]) imply that the Lebesgue integrals converge almost surely and the stochastic integrals in probability, with the limits being the corresponding terms in (3.2). Finally, we get

$$u(\tau, X^0_{\tau}; \theta) - u(0, x) = \int_0^\tau \mathcal{L}^\theta(s) u \left( s, X^0_s; \theta \right) ds + \int_0^\tau \partial^\theta u(r, X^0_r; \theta) dW(r), \quad \text{a.s.}$$

Following is the existence of the viscosity solution.

**Theorem 3.5.** Let (A1) hold. The value function $V$ defined by (1.4) is a viscosity solution of the stochastic path-dependent Hamilton-Jacobi equation (1.7).

**Proof.** Step 1. In view of Proposition 3.2, we have $V \in \mathcal{S}^\infty(\Lambda; \mathbb{R})$. To the contrary, suppose that for any $k \in \mathbb{N}^+$ with $k \geq K_0$ for some $K_0 \in \mathbb{N}^+$, there exists $\phi \in \mathcal{G}V(\tau, \xi; \Omega, k)$ with $\tau \in T^0, \Omega \in \mathcal{F}_T, \mathbb{P}(\Omega) > 0$, and $\xi \in L^0(\Omega, \mathcal{F}_T; \Lambda_0^0)$, such that there exist $\varepsilon > 0$ and $\Omega' \subset \mathcal{F}_T$ with $\Omega' \subset \Omega$, $\mathbb{P}(\Omega') > 0$, satisfying a.e. on $\Omega'$,

$$\lim_{\delta \to 0^+} \inf_{s \in (\tau, (\tau + \delta^2) \wedge T), \xi \in \mathcal{B}_k(\xi, \Lambda_0^0) \cap \Lambda^0_{\tau + \delta^2} \cap \Lambda_0^0} \{-\partial_s \phi(s, x, \nabla \phi(s, x)) \} \geq 2\varepsilon. \quad (3.4)$$

Notice that, associated to $\phi \in \mathcal{C}^{1,2}_x$, there is a partition $0 = t_0 < t_1 < \ldots < t_n = T$. W.l.o.g., we assume that there exists $\tilde{\delta} \in (0, 1)$ with $2\tilde{\delta}^2 < \min_{0 \leq j \leq n-1}(t_{j+1} - t_j)$ such that $\Omega' = \{[\tau, \tau + 2\tilde{\delta}^2) \subset [t_j, t_{j+1}) \}$ for some $j \in \{0, \ldots, n-1\}$.

Choose the positive integer $k > L$ and let $\hat{\tau}$ be the stopping time associated to $\phi \in \mathcal{G}V(\tau, \xi; \Omega, k)$. Note that we may think of $\phi$ valued in $\Lambda_0^0$, with $\xi(t) = \xi_{t \wedge \tau}(t)$ for all $t \in [0, T]$. By assumption (ii) of (A1) and the measurable selection theorem, there exists $\bar{\theta} \in \mathcal{U}$ such that for almost all $\omega \in \Omega'$,

$$-\mathcal{L}^{\hat{\theta}}(s) \phi(s, \xi_s, \bar{\theta}(s)) - f(s, \xi_s, \bar{\theta}(s)) \geq -\partial_s \phi(s, \xi_s, \nabla \phi(s, \xi_s)) - \varepsilon,$$

for almost all $s$ satisfying $\tau \leq s < (\tau + \hat{\delta}^2) \wedge T$. This together with (3.4) implies that

$$\lim_{\delta \to 0^+} \inf_{\tau < s < (\tau + \delta^2) \wedge T} \{-\mathcal{L}^{\hat{\theta}}(s) \phi(s, \xi_s) - f(s, \xi_s, \bar{\theta}(s)) \} \geq \varepsilon, \quad \text{a.e. on } \Omega'.$$

In view of Remark 2.4, Lemma 3.1, the dynamic programming principle of Theorem 3.3 and the generalized Itô-Kunita formula of Lemma 3.4, we have for almost all $\omega \in \Omega'$,

$$0 \geq \lim_{h \to 0^+} \frac{1}{h} E_{\mathcal{G}_\tau} \left[ (\phi - V)(\tau, \xi_{\tau}) - (\phi - V) \left( (\tau + h) \wedge \hat{\tau}, X^0_{(\tau + h) \wedge \hat{\tau}} \right) \right]$$

$$\geq \lim_{h \to 0^+} \frac{1}{h} E_{\mathcal{G}_\tau} \left[ \phi(\tau, \xi_{\tau}) - \phi \left( (\tau + h) \wedge \hat{\tau}, X^0_{(\tau + h) \wedge \hat{\tau}} \right) - \int_{\tau}^{(\tau + h) \wedge \hat{\tau}} f(s, X^0_s; \nabla \phi(s, \xi_s)) ds \right]$$

$$= \lim_{h \to 0^+} \frac{1}{h} E_{\mathcal{G}_\tau} \left[ \int_{\tau}^{(\tau + h) \wedge \hat{\tau}} \left( -\mathcal{L}^{\hat{\theta}}(s) \phi(s, X^0_s; \nabla \phi(s, \xi_s)) - f(s, X^0_s; \bar{\theta}(s)) \right) ds \right]$$

$$\geq \lim_{h \to 0^+} \frac{1}{h} E_{\mathcal{G}_\tau} \left[ \int_{\tau}^{(\tau + h) \wedge \hat{\tau}} \left( -\mathcal{L}^{\hat{\theta}}(s) \phi(s, \xi_s) - f(s, \xi_s, \bar{\theta}(s)) \right) ds \right]$$

$$- \int_{\tau}^{(\tau + h) \wedge \hat{\tau}} \left( -\mathcal{L}^{\hat{\theta}}(s) \phi(s, \xi_s) - f(s, \xi_s, \bar{\theta}(s)) + \mathcal{L}^{\hat{\theta}}(s) \phi(s, X^0_s; \nabla \phi(s, \xi_s)) + f(s, X^0_s; \bar{\theta}(s)) \right) ds \right]$$
there exists \( \phi \in \mathcal{C}_T^1 \), \( L_0^\phi \) from Remark 3.1 and the constant \( K \) from Lemma 3.1. This gives rise to a contradiction. Hence, \( V \) is a viscosity subsolution.

**Step 2.** To prove that \( V \) is a viscosity supersolution of (1.7), we argue with contradiction like in **Step 1.** To the contrary, assume that for any \( k \in \mathbb{N}^+ \) with \( k \geq K_0 \) for some \( K_0 \in \mathbb{N}^+ \), there exists \( \phi \in \overline{GV}(\tau, \xi_r; \Omega_r, k) \) with \( \tau \in T^0 \), \( \Omega_r \subset \mathcal{F}_r \), \( \mathbb{P}(\Omega_r) > 0 \), and \( \xi_r \in L^0(\Omega_r, \mathcal{F}_r; \Lambda_0^\phi) \) such that there exist \( \varepsilon > 0 \) and \( \Omega' \subset \Omega_r \) with \( \mathbb{P}(\Omega') > 0 \), satisfying a.e. on \( \Omega' \),

\[
\lim_{h \to 0^+} \text{esssup}_{s \in (\tau, (\tau + h)^\wedge T], x \in B_{2\delta}(\xi_r) \cap \Lambda_{0,k}^0} \{ -\partial_s \phi(s, x) - \mathbb{H}(s, x, \nabla \phi(s, x)) \} \leq -\varepsilon.
\]

We take the integer \( k \geq 1 \) and let \( \tilde{\tau} \) be the stopping time corresponding to the fact \( \phi \in \overline{GV}(\tau, \xi_r; \Omega_r, k) \). Again, we think of \( \xi_r \) valued in \( \Lambda_0^\phi \), with \( \xi_r(t) = \xi_r(t, \omega) \) for all \( t \in [0, T] \).

Again, notice that, associated to \( \phi \in \mathcal{C}_T^1 \), there is a partition \( 0 = \tilde{t}_0 < \tilde{t}_1 < \ldots < \tilde{t}_n = T \).

W.l.o.g., we assume that there exists \( \delta > 0 \) such that \( 2\delta^2 < \min_{0 \leq j \leq n-1} (\tilde{t}_{j+1} - \tilde{t}_j) \) and \( \Omega = \{ (\tau, \tau + 2\delta^2) \subset \{ \tilde{t}_j, \tilde{t}_{j+1} \} \) for some \( j \in \{0, \ldots, n-1 \} \).

For each \( \theta \in \mathcal{U} \), define \( \tau^\theta = \inf \{ s > \tau : X_s^{\tau, \xi_r, \theta} \notin B_{\delta}(\xi_r) \} \).

Then \( \tau^\theta > \tau \) and moreover, setting \( h = \frac{\tau^\theta - \tau}{2} \) and using Chebyshev's inequality, we obtain

\[
E_{\mathcal{F}_r} \left[ 1_{\{ \tau^\theta < \tau + h \}} \right] = E_{\mathcal{F}_r} \left[ 1_{\{ \max_{0 \leq r \leq h} |X_r^{\tau, \xi_r, \theta} - \xi_r| + \sqrt{h} \delta \}} \right] \\
\leq \frac{1}{(\delta - \sqrt{h})^8} E_{\mathcal{F}_r} \left[ \max_{\tau \leq s \leq \tau + h} |X_s^{\tau, \xi_r, \theta} - \xi_r|^8 \right] \\
\leq \frac{K^8}{(\delta - \sqrt{h})^8} (h + \sqrt{h})^8 \\
\leq \frac{256 \cdot K^8}{\delta^8} (h + \sqrt{h})^8 \\
\leq \tilde{C} h^{\frac{1}{2}} \quad \text{a.s.}
\tag{3.5}
\]

Here, the constant \( K \) is from Lemma 3.1 and independent of the control \( \theta \), and thus, the constant \( \tilde{C} \) is independent of \( \theta \) as well.

In view of Remark 2.1, Theorem 3.3, Lemma 3.4 and estimate (3.5), we have for almost all \( \omega \in \Omega' \),

\[
0 = \lim_{h \to 0^+} \frac{V(\tau, \xi_r; \omega) - \phi(\tau, \xi_r)}{h} \\
= \lim_{h \to 0^+} \frac{1}{h} \text{essinf}_{\theta \in \mathcal{U}} E_{\mathcal{F}_r} \left[ \int_{\tau}^{\tau \wedge (\tau + h)} f(s, X_s^{\tau, \xi_r, \theta}, \theta(s)) \, ds + V(\tau \wedge (\tau + h), X_{\tau \wedge (\tau + h)}^{\tau, \xi_r, \theta}) - \phi(\tau, \xi_r) \right] \\
\geq \liminf_{h \to 0^+} \frac{1}{h} \text{essinf}_{\theta \in \mathcal{U}} E_{\mathcal{F}_r} \left[ \int_{\tau}^{\tau \wedge (\tau + h)} f(s, X_s^{\tau, \xi_r, \theta}, \theta(s)) \, ds + \phi(\tau \wedge (\tau + h), X_{\tau \wedge (\tau + h)}^{\tau, \xi_r, \theta}) - \phi(\tau, \xi_r) \right]
\]
\[
\begin{align*}
&\quad = \lim \inf_{h \to 0^+} \frac{1}{h} \text{essinf}_{\theta \in \U} \left[ \int_{\tau}^{\tau \wedge (\tau + h)} \left( \mathcal{L}^\theta(s) \phi \left( s, X_s^{\tau, \xi_s ; \theta} \right) + f(s, X_s^{\tau, \xi_s ; \theta}, \theta(s)) \right) \ ds \right] \\
&\geq \lim \inf_{h \to 0^+} \frac{1}{h} \text{essinf}_{\theta \in \U} \left[ \int_{\tau}^{\tau \wedge (\tau + h) \wedge \hat{\tau}} \left( \mathcal{L}^\theta(s) \phi \left( s, X_s^{\tau, \xi_s ; \theta} \right) + f(s, X_s^{\tau, \xi_s ; \theta}, \theta(s)) \right) \ ds \right] \\
&\quad - 1_{\{\hat{\tau} > \tau\} \cap \{\tau + h > \tau\}} \int_{\tau}^{\tau \wedge (\tau + h) \wedge \hat{\tau}} \left| \mathcal{L}^\theta(s) \phi \left( s, X_s^{\tau, \xi_s ; \theta} \right) + f(s, X_s^{\tau, \xi_s ; \theta}, \theta(s)) \right| \ ds \\
&\quad - 1_{\{\tau > \tau\} \cap \{\tau + h > \tau\}} \int_{\tau}^{\tau \wedge (\tau + h) \wedge \hat{\tau}} \left| \mathcal{L}^\theta(s) \phi \left( s, X_s^{\tau, \xi_s ; \theta} \right) + f(s, X_s^{\tau, \xi_s ; \theta}, \theta(s)) \right| \ ds \\
&\geq \varepsilon - \lim \sup_{h \to 0^+} \frac{1}{h} \text{esssup}_{\theta \in \U} \left( E_{\mathcal{F}_\tau} \left[ 1_{\{\tau + h > \tau\}} \right] \right)^{1/2} \\
&\quad \cdot \left( \frac{\left( \mathcal{L}^\theta(s) \phi \left( s, X_s^{\tau, \xi_s ; \theta} \right) + f(s, X_s^{\tau, \xi_s ; \theta}, \theta(s)) \right)}{\left( \mathcal{L}^\theta(s) \phi \left( s, X_s^{\tau, \xi_s ; \theta} \right) + f(s, X_s^{\tau, \xi_s ; \theta}, \theta(s)) \right)} \right)^{1/2} \\
&\quad - \lim \sup_{h \to 0^+} \frac{1}{h} \text{esssup}_{\theta \in \U} \left( E_{\mathcal{F}_\tau} \left[ 1_{\{\tau + h > \tau\}} \right] \right)^{1/2} \\
&\quad \cdot \left( \frac{\left( \mathcal{L}^\theta(s) \phi \left( s, X_s^{\tau, \xi_s ; \theta} \right) + f(s, X_s^{\tau, \xi_s ; \theta}, \theta(s)) \right)}{\left( \mathcal{L}^\theta(s) \phi \left( s, X_s^{\tau, \xi_s ; \theta} \right) + f(s, X_s^{\tau, \xi_s ; \theta}, \theta(s)) \right)} \right)^{1/2} \\
&\geq \varepsilon - 2 \lim \sup_{h \to 0^+} h^{3/2} \mathcal{C}^{1/2} \left( E_{\mathcal{F}_\tau} \left[ \int_{\tau}^{\tau \wedge (\tau + h) \wedge \hat{\tau}} \left| \phi \right|^2 \ ds \right] \right)^{1/2} \\
&\quad = \varepsilon > 0,
\end{align*}
\]

where \( h = \frac{\hat{\tau} - \tau}{\tau} \) and we have used the analysis in Remark 2.3 and the fact that \( \tau + h < \hat{\tau} \) when \( h > 0 \) is small enough. Hence, a contradiction occurs and \( V \) is a viscosity supersolution of SPIJ equation (1.7). \qed

\section{Uniqueness}

Due to the nonanticipativity constraint on the unknown function, the conventional variable-doubling techniques for deterministic Hamilton-Jacobi equations are not applicable to the viscosity solution to SPIJ equations like (1.7), basically because in our random setting, the extreme quadruples, as random functions of \( \omega \in \Omega \), attaining the extreme values of the penalized functional “doubling the number of variables” fail to be \((\mathcal{F}_s)_{s \geq 0}\)-adapted; refer to [14, 23, 31] for the applications of variable-doubling techniques to the deterministic path-dependent PDEs. In this work, we prove instead a comparison principle that is weak in the sense that the comparison relations are just holding in compact subspaces; the uniqueness is then derived on basis of this weak comparison principle.
4.1 A weak comparison principle

Proposition 4.1. Let (A1) hold and \( u \) be a viscosity subsolution (resp. supersolution) of SPHJ equation \((1.7)\). Then there is an infinite sequence of integers \( 1 \leq k_1 < k_2 < \cdots < k_n < \cdots \) (resp., \( 1 \leq k_1 < k_2 < \cdots < \bar{k}_n < \cdots \)), such that for each \( i \in \mathbb{N}^+ \), \( \phi_i(T,x) \geq (\text{resp.} \leq) G(x) \) for all \( x \in \Lambda^0_T \) a.s. and

\[
\begin{align*}
\text{ess} \liminf_{(s,x) \to (t^+,y)} \{-\partial_s \phi_i(s,x) - \mathcal{H}(s,x,\nabla \phi_i(s,x))\} &\geq 0, \text{ a.s.,} \\
\text{(resp. ess} \limsup_{(s,x) \to (t^+,y)} \{-\partial_s \phi_i(s,x) - \mathcal{H}(s,x,\nabla \phi_i(s,x))\} &\leq 0, \text{ a.s.}
\end{align*}
\]

for each \( t \in [0,T) \) with \( y \in \Lambda^0_{0,t} \) (resp., \( y \in \Lambda^0_{0,t}\bar{k}_i \)), it holds a.s. that \( u(t,x) \leq (\text{resp.,} \geq) \phi_i(t,x) \), for each \( t \in [0,T) \) with \( x \in \Lambda^0_{0,t} \).

Proof. We prove the case when \( u \) is a viscosity supersolution, then the proof for the viscosity subsolution will be following similarly. First, by Definition 2.3 and Remark 2.3, the viscosity supersolution \( u \) is associated to an infinite sequence of integers \( 1 \leq k_1 < k_2 < \cdots < k_n < \cdots \). Given \( i \in \mathbb{N}^+ \), suppose that, to the contrary, there holds \( u(t,\bar{x}_t) < \phi_i(t,\bar{x}_t) \) with a positive probability at some point \((t,\bar{x}_t)\) with \( t \in [0,T) \) and \( \bar{x}_t \in \Lambda^0_{0,t}\bar{k}_i \). Thus, we have \( \bar{x}_t \in \Lambda^0_{0,t}\bar{k}_i \xi_0 \) for some \( \xi_0 \in \mathbb{R}^d \). W.l.o.g., we assume \( \xi_0 = 0 \). Thus, there exists \( \delta > 0 \) such that \( \mathbb{P}(\overline{\Omega}_t) > 0 \) with

\[
\overline{\Omega}_t := \{\phi_i(t,\bar{x}_t) - u(t,\bar{x}_t) > \delta\}.
\]

By the compactness of \( \Lambda^0_{0,t}\bar{k}_i \) in \( \Lambda^0_T \) and the measurable selection theorem, there exists \( \bar{\xi}_t^i \in L^0(\overline{\Omega}_t, \mathcal{F}_t; \Lambda^0_{0,t}\bar{k}_i) \) such that

\[
\phi_i(t,\bar{\xi}_t^i) - u(t,\bar{\xi}_t^i) = \max_{x_t \in \Lambda^0_{0,t}\bar{k}_i} \{\phi_i(t,x_t) - u(t,x_t)\} \geq \delta \text{ for almost all } \omega \in \overline{\Omega}_t.
\]

W.l.o.g., we take \( \overline{\Omega}_t = \Omega \) in what follows.

For each \( s \in (t,T] \), choose an \( \mathcal{F}_s \)-measurable and \( \Lambda^0_{t,s}\bar{k}_i \)-valued variable \( \bar{\xi}_s^i \) such that

\[
\left(\phi_i(s,\bar{\xi}_s^i) - u(s,\bar{\xi}_s^i)\right)^+ = \max_{x_s \in \Lambda^0_{t,s}\bar{k}_i} \left(\phi_i(s,x_s) - u(s,x_s)\right)^+, \text{ a.s., (4.1)}
\]

and set \( \bar{Y}^i_i(s) = (\phi_i(s,\xi_s^i) - u(s,\xi_s^i))^+ + \frac{\delta(s-t)}{2(T-t)} \) and \( Z^i_i(s) = \text{esssup}_{\tau \in \mathcal{T}^s} E_{\mathcal{F}_s}[Y^i_i(\tau)] \). Here, recall that \( \mathcal{T}^s \) denotes the set of stopping times valued in \([s,T]\). As \( (\phi_i - u)^+ \in \mathcal{S}^2(\Lambda^0;\mathbb{R}) \), there follows obviously the time-continuity of

\[
\max_{x_s \in \Lambda^0_{t,s}\bar{k}_i} \left(\phi_i(s,x_s) - u(s,x_s)\right)^+, \text{ for } s \in [t,T],
\]

and thus that of \( \left(\phi_i(s,\xi_s^i) - u(s,\xi_s^i)\right)^+ \), although the continuity of process \((\xi_s^i)_{s \in [t,T]}\) (as path space-valued process) can not be ensured. Therefore, the process \((Y^i_i(s))_{t \leq s \leq T}\) has continuous trajectories. Define \( \tau^i_i = \inf\{s \geq t : Y^i_i(s) = Z^i_i(s)\} \). In view of the optimal stopping theory, observe that

\[
E_{\mathcal{F}_t} \left[ Y^i_i(T) \right] = \frac{\delta}{3} < \delta \leq Y^i_i(t) \leq Z^i_i(t) = E_{\mathcal{F}_t} \left[ Y^i_i(\tau^i_i) \right] = E_{\mathcal{F}_t} \left[ Z^i_i(\tau^i_i) \right],
\]
which gives that $\mathbb{P}(\tau_{\bar{k}_i} < T) > 0$. As
\[
(\phi_i(\tau_{\bar{k}_i}, \xi^{k_i}_{\tau_{\bar{k}_i}}) - u(\tau_{\bar{k}_i}, \xi^{k_i}_{\tau_{\bar{k}_i}}))^+ + \frac{\delta(\tau_{\bar{k}_i} - t)}{3(T-t)} = Z^{\tau_{\bar{k}_i}}(\tau_{\bar{k}_i}) \geq E_{\mathcal{F}_{\tau_{\bar{k}_i}}} [Y^{\bar{k}_i}(T)] = \frac{\delta}{3}, \tag{4.2}
\]
we have
\[
\mathbb{P}((\phi_i(\tau_{\bar{k}_i}, \xi^{k_i}_{\tau_{\bar{k}_i}}) - u(\tau_{\bar{k}_i}, \xi^{k_i}_{\tau_{\bar{k}_i}}))^+ > 0) > 0.
\]
Define
\[
\tau_{\bar{k}_i} = \inf\{s \geq \tau_{\bar{k}_i} : (\phi_i(s, \xi^{k_i}_s) - u(s, \xi^{k_i}_s))^+ \leq 0\}.
\]
Obviously, $\tau_{\bar{k}_i} \leq \hat{\tau}_{\bar{k}_i} \leq T$. Put $\Omega_{\tau_{\bar{k}_i}} = \{\tau_{\bar{k}_i} < \hat{\tau}_{\bar{k}_i}\}$. Then $\Omega_{\tau_{\bar{k}_i}} \in \mathcal{F}_{\tau_{\bar{k}_i}}$ and in view of relation (4.2), and the definition of $\tau_{\bar{k}_i}$, we have $\Omega_{\tau_{\bar{k}_i}} = \{\tau_{\bar{k}_i} < T\}$ and $\mathbb{P}(\Omega_{\tau_{\bar{k}_i}}) > 0$.

Set
\[
\Phi_i(s, x_s) = \phi_i(s, x_s) + \frac{\delta(s - t)}{3(T-t)} - E_{\mathcal{F}_s} \left[ Y^{\bar{k}_i}(\tau_{\bar{k}_i}) \right], \quad s \in [0, T].
\]
Notice that the process $(E_{\mathcal{F}_s} \left[ Y^{\bar{k}_i}(\tau_{\bar{k}_i}) \right])_{s \in [0, T]}$ belongs to $\mathcal{C}^1_\mathcal{F}$. Indeed, $E_{\mathcal{F}_s} \left[ Y^{\bar{k}_i}(\tau_{\bar{k}_i}) \right]$ is defined on $\Omega \times [0, T]$ independent of $x_s \in \Lambda$ and for $s \leq \tau_{\bar{k}_i}$, the martingale representation under the assumed Brownian filtration $(\mathcal{F}_r)_{r \geq 0}$ gives the well-defined $\mathcal{A}_\omega \left( E_{\mathcal{F}_s} \left[ Y^{\bar{k}_i}(\tau_{\bar{k}_i}) \right] \right)$. Thus, $\Phi_i \in \mathcal{C}^1_\mathcal{F}$ since $\phi_i \in \mathcal{C}^1_\mathcal{F}$. For each $\bar{\tau} \in T^{\tau_{\bar{k}_i}}$, we have for almost all $\omega \in \Omega_{\tau_{\bar{k}_i}},$
\[
(\Phi_i - u)(\tau_{\bar{k}_i}, \xi^{k_i}_{\tau_{\bar{k}_i}}) = 0 = Z^{\tau_{\bar{k}_i}}(\tau_{\bar{k}_i}) - Y^{\bar{k}_i}(\tau_{\bar{k}_i}) \geq E_{\mathcal{F}_{\tau_{\bar{k}_i}}} \left[ Y^{\bar{k}_i}(\bar{\tau} \wedge \tau_{\bar{k}_i}) \right] - Y^{\bar{k}_i}(\tau_{\bar{k}_i})
\]
\[
\geq E_{\mathcal{F}_{\tau_{\bar{k}_i}}} \left[ \max_{\substack{0, \bar{k}_i, \xi^{k_i}_{\tau_{\bar{k}_i}} \in \Lambda_{\tau_{\bar{k}_i}, \bar{\tau} \wedge \tau_{\bar{k}_i}}}} (\Phi_i - u)(\bar{\tau} \wedge \tau_{\bar{k}_i}, y) \right],
\]
where we have used the obvious relation $\Lambda_{\tau_{\bar{k}_i}, \bar{\tau} \wedge \tau_{\bar{k}_i}} \subset \Lambda_{0, \bar{k}_i, \xi^{k_i}_{\tau_{\bar{k}_i}} \tau_{\bar{k}_i}}$. This together with the arbitrariness of $\bar{\tau}$ implies that $\Phi_i \in \mathcal{C}^1 u(\tau_{\bar{k}_i}, \xi^{k_i}_{\tau_{\bar{k}_i}} ; \Omega_{\tau_{\bar{k}_i}}, T)$. In view of the correspondence between the viscosity supersolution $u$ and the infinite sequence $\{\bar{k}_1, \bar{k}_2, \ldots, \bar{k}_n, \ldots\}$, we have for almost all $\omega \in \Omega_{\tau_{\bar{k}_i}},$
\[
0 \leq \limsup_{(s, x) \to ((\tau_{\bar{k}_i})^+, \xi^{k_i}_{\tau_{\bar{k}_i}})} \{-\partial_s \Phi_i(s, x) - \mathbb{H}(s, x, \nabla \Phi_i(s, x))\}
\]
\[
= -\frac{\delta}{3(T-t)} + \limsup_{(s, x) \to ((\tau_{\bar{k}_i})^+, \xi^{k_i}_{\tau_{\bar{k}_i}})} \{-\partial_s \phi_i(s, x) - \mathbb{H}(s, x, \nabla \phi_i(s, x))\}
\]
\[
\leq -\frac{\delta}{3(T-t)},
\]
which is a contradiction. \qed
4.2 Uniqueness

In addition to Assumption (A1), we assume the joint time-space continuity, i.e.,
(A2) for each \( v \in U, f(\cdot, \cdot, v), \beta(\cdot, \cdot, v) \in \mathcal{S}^\infty(\Lambda; \mathbb{R}) \), for \( i = 1, \ldots, d \).

We may approximate the coefficients \( \beta, f \), and \( G \) via regular functions.

**Lemma 4.2.** Let (A1) hold. For each \( \varepsilon > 0 \), there exist partition \( 0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T \) for some \( N > 3 \) and functions

\[
(G^N, f^N, \beta^N) \in C^3([\mathbb{R}^N \times (d+m)+d) \times C(U; C^3([0, T] \times \mathbb{R}^{(m+d)\times N+d})) \times C(U; C^3([0, T] \times \mathbb{R}^{(m+d)\times N+d}))
\]

such that for each \( k \in \mathbb{N}^+ \),

\[
f_k(t) := \text{esssup}_{(x,v) \in \Lambda_0 \times U} |f^N(W(t_1 \wedge t), \ldots, W(t_N \wedge t), t, x(t_0 \wedge t), \ldots, x(t_N \wedge t), v) - f(t, x, v)|,
\]

\[
\beta_k(t) := \text{esssup}_{(x,v) \in \Lambda_0 \times U} |\beta^N(W(t_1 \wedge t), \ldots, W(t_N \wedge t), t, x(t_0 \wedge t), \ldots, x(t_N \wedge t), v) - \beta(t, x, v)|,
\]

and \( G_k := \text{esssup}_{x \in \Lambda_0} |G^N(W(t_1), \ldots, W(t_N), x(t_0), \ldots, x(t_N)) - G(x)| \) are \( \mathcal{F}_t \)-adapted with

\[
\|G_k\|_{L^2(\Omega, \mathcal{F}_T; \mathbb{R})} + \|f_k\|_{L^2(\Omega \times [0,T]; \mathbb{R})} + \|\beta_k\|_{L^2(\Omega \times [0,T]; \mathbb{R})} < \varepsilon(1 + k). \tag{4.3}
\]

Moreover, \( G^N, f^N, \) and \( \beta^N \) are uniformly Lipschitz-continuous in the space variable \( x \) with an identical Lipschitz-constant \( L_c \) independent of \( N, k, \) and \( \varepsilon \).

Lemma 4.2 may be proved with density arguments that are more or less standard; the sketch of the proof is given for the reader’s reference.

**Sketched proof of Lemma 4.2** We consider the approximations for the function \( f \). First, the joint time-space continuity in Assumption (A2) and the dominated convergence theorem allow us to approximate \( f \) via random functions of the form:

\[
j^l(\omega, t, x, v) = f(\omega, 0, x_0, v)1_{[0,t_1]}(t) + \sum_{j=1}^{l-1} f(\omega, t_j, x_{t_j}, v)1_{(t_j, t_{j+1})}(t), \quad t \in [0, T], \tag{4.4}
\]

where \( 0 = t_0 < t_1 < \cdots < t_l < T \).

Next, for each \( j \geq 0 \), the function \( f(\omega, t_j, x_{t_j}, v) \) may be approximated monotonically (see [9] Lemma 1.2, Page 16 for instance) by simple random variables of the following form:

\[
\sum_{i=1}^{l_j} 1_{A^j_i}(\omega)h^j_i(x_{t_j}, v), \quad \text{with } h^j_i \in C(\Lambda_{t_j} \times U), \quad A^j_i \in \mathcal{F}_{t_j}, \quad i = 1, \ldots, l_j,
\]

and by [21] Lemma 4.3.1., page 50], each \( 1_{A^j_i} \) may be approximated in \( L^2(\Omega, \mathcal{F}_{t_j}) \) by functions in the following set

\[
\{g(W(\tilde{t}_1), \ldots, W(\tilde{t}_{l^j_i})): g \in C_c^\infty(\mathbb{R}^{l_j^i \times m}), \tilde{t}_r \in [0, t_j], r = 1, \ldots, l^j_i\},
\]

where \( C_c^\infty(\mathbb{R}^{l_j^i \times m}) \) denotes the space of infinitely differentiable functions with compact supports in \( \mathbb{R}^{l_j^i \times m} \). Moreover, recalling that for each \( x_{t_j} \in \Lambda_{t_j}^0 \),

\[
\lim_{M \to \infty} \max_{s \in [0, t_j]} |x_{t_j}(s) - P^M(x_{t_j})(s)| = 0, \tag{4.5}
\]
where \(0 = \bar{g}_\epsilon \in C^\infty \), by random functions of the following form:

\[
G_i(t) = \text{ess inf} \left[ -\phi(s) - \mathcal{H}(s, x, \nabla \phi(s, x)) \right] \geq 0, \quad \text{a.s.}
\]

and as a continuous function lying in \(C(\mathbb{R}^{(M+1)\times d} \times U)\), each function \(\hat{h}_i^j\) may be approached by infinitely differentiable functions (denoted by itself) lying in \(C(\mathbb{R}; C^\infty(\mathbb{R}^{(M+1)\times d}))\) with a uniform identical Lipschitz constant w.r.t. the space variable \(x\). In addition, each \(1_{(t_j, t_{j+1})}\) may be increasingly approximated by compactly-supported nonnegative functions \(\varphi_j \in C^\infty((t_j-1, t_j]; \mathbb{R})\).

To sum up, we may put all the partitions together, and the function \(f\) may be approximated by random functions of the following form:

\[
f^N(W(\bar{t}_1 \wedge t), \ldots, W(\bar{t}_N \wedge t), t, x(0), x(\bar{t}_1 \wedge t), \ldots, x(\bar{t}_N \wedge t), v)
= \sum_{j=1}^{\bar{t}} \sum_{i=1}^{t_j} g_i^j(W(\bar{t}_1), \ldots, W(\bar{t}_j)) \hat{h}_i^j(x(0), x(\bar{t}_1), \ldots, x(\bar{t}_j), v) \varphi_j(t),
\tag{4.6}
\]

where \(0 = \bar{t}_0 < \bar{t}_1 < \cdots < \bar{t}_{N-1} < \bar{t}_N = T\), and \(g_i^j, \hat{h}_i^j, \varphi_j\) are smooth functions. The required approximations for \(G\) and \(\beta\) are following in a similar way. Just note that the approximation error in (13) is given as \(\varepsilon(1 + k)\) depending on \(k\) due to the approximations of the paths in Lipschitz-continuous function spaces involved in (14) and (15). \(\square\)

**Theorem 4.3.** Let Assumptions \((A1)\) and \((A2)\) hold. The viscosity solution to SPHJ equation \((1.7)\) is unique.

**Proof.** For each \(k \in \mathbb{N}^+\), define

\[
\mathcal{F}_k = \left\{ \phi \in \mathcal{C}_T^1 : \phi(T, x) \geq G(x) \quad \forall x \in \Lambda_0^k, \quad \text{a.s.}, \right\}
\]

and set

\[
\mathcal{F}_k = \left\{ \phi \in \mathcal{C}_T^1 : \phi(T, x) \leq G(x) \quad \forall x \in \Lambda_0^k, \quad \text{a.s.}, \quad \text{ess sup}_{(s, x) \to (t, y)} \left[ -\partial_s \phi(s, x) - \mathcal{H}(s, x, \nabla \phi(s, x)) \right] \leq 0, \quad \text{a.s.} \right\},
\]

and

\[
\mathcal{F}_k = \left\{ \phi \in \mathcal{C}_T^1 : \phi(T, x) \leq \psi(x) \quad \forall x \in \Lambda_0^k, \quad \text{ess inf}_{(s, x) \to (t, y)} \left[ -\partial_s \phi(s, x) - \mathcal{H}(s, x, \nabla \phi(s, x)) \right] \geq 0, \quad \text{a.s.} \right\},
\]

and

\[
\mathcal{F}_k = \left\{ \phi \in \mathcal{C}_T^1 : \phi(T, x) \leq \psi(x) \quad \forall x \in \Lambda_0^k, \quad \text{ess sup}_{(s, x) \to (t, y)} \left[ -\partial_s \phi(s, x) - \mathcal{H}(s, x, \nabla \phi(s, x)) \right] \leq 0, \quad \text{a.s.} \right\},
\]

and set

\[
\mathcal{F}_k = \mathcal{F}_k \quad \text{ess sup}_{(s, x) \to (t, y)} \left[ -\partial_s \phi(s, x) - \mathcal{H}(s, x, \nabla \phi(s, x)) \right] \leq 0, \quad \text{a.s.}
\]

It is easy to see the monotonicity of \(\mathcal{F}_k, \mathcal{F}_k, \mathcal{F}_k\), and thus that of \(\mathcal{F}_k, \mathcal{F}_k, \mathcal{F}_k\), as \(k \to \infty\), and we may define further

\[
\mathcal{F} = \lim_{k \to \infty} \mathcal{F}_k, \quad \mathcal{F} = \lim_{k \to \infty} \mathcal{F}_k.
\]

---

When \(t \in (t_j, t_{j+1})\), we write the dependence of \(f^N\) on \(W(t)\) and \(x(t)\) just for notational convenience, though the defined function \(f^N\) does not nontrivially depend on \(W(t)\) and \(x(t)\) for \(t \in (t_j, t_{j+1})\); similarly, the functions \(g_i^j\) and \(\hat{h}_i^j\) may not nontrivially depend on some particular input(s) in expression (4.6).
By the comparison principle of Proposition 4.1, each viscosity solution \( u \) satisfies \( \underline{u} \leq u \leq \overline{u} \) on \( \bigcup_{k=1}^{\infty} A_{0,k}^0 \), that is dense in \( \Lambda_0 \). Therefore, it is sufficient to verify \( \underline{u} = V = \overline{u} \) for the uniqueness of viscosity solution. The proof will be divided into two steps.

**Step 1.** We construct functions from \( \overline{V}_k \) and \( \underline{V}_k \) to dominate the value function \( V \) from above and from below respectively. Let \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}) \) be another complete filtered probability space on which a \( d \)-dimensional standard Brownian motion \( B = \{B(t) : t \geq 0\} \) is well defined. The filtration \( \{\mathcal{F}_t\}_{t \geq 0} \) is generated by \( B \) and augmented by all the \( \mathbb{P}' \)-null sets in \( \mathcal{F}' \).

Put

\[
(\tilde{\Omega}, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}_t\}_{t \geq 0}, \tilde{\mathbb{P}}) = (\Omega \times \Omega', \mathcal{F} \otimes \mathcal{F}', \mathcal{F}_t \otimes \mathcal{F}'_t \otimes \mathcal{P} \otimes \mathcal{P}',)
\]

and denote by \( \tilde{U} \) the set of all the \( U \)-valued and \( \tilde{\mathcal{F}}_t \)-adapted processes. Then we have two Brownian motions \( B \) and \( \Gamma \) that are independent on \( (\tilde{\Omega}, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}_t\}_{t \geq 0}, \tilde{\mathbb{P}}) \), and all the theory established in previous sections still hold on the enlarged probability space.

Fix an arbitrary \( \varepsilon \in (0, 1) \) and \( k \in \mathbb{N}^+ \), and choose \( (G^k, f^k, \beta^k) \) and \( (G^N, f^N, \beta^N) \) as in Lemma 4.2. By the theory of backward SDEs (see [4] for instance), let the pairs \( (Y^\varepsilon_k, Z^\varepsilon_k) \) and \( (y, z) \) be the unique adapted solutions to backward SDEs

\[
Y^\varepsilon_k(s) = G^k_k + \int_s^T (f^\varepsilon_k(t) + C_1\beta^\varepsilon_k(t)) \, dt - \int_s^T Z^\varepsilon_k(t) \, dW(t),
\]

and

\[
y(s) = \|B_T\|_0 + \int_s^T \|B_r\|_0 \, dr - \int_s^T z(r) \, dB(r),
\]

respectively, with the constant \( C_1 \geq 0 \) to be determined later. For each \( s \in [0, T) \) and \( x \in \Lambda_s \), set

\[
V^\varepsilon(s, x) = \text{essinf}_{\theta \in \mathcal{U}} E_{\mathcal{F}_s} \left[ \int_s^T f^N(W(t_1 \wedge t), \ldots, W(t_N \wedge t), t, X^{s, x; \theta, N}(t_1 \wedge t), \ldots, X^{s, x; \theta, N}(t_N \wedge t), \theta(t)) dt \right.
\]

\[
+ G^N(W(t_1), \ldots, W(t_N), X^{s, x; \theta, N}(t_1), \ldots, X^{s, x; \theta, N}(t_N)) \right],
\]

where \( X^{s, x; \theta, N}(t) \) satisfies the SDE

\[
\begin{align*}
\{ & dX(t) = \beta^N(W(t_1 \wedge t), \ldots, W(t_N \wedge t), t, X(0), X(t_1 \wedge t), \ldots, X(t_N \wedge t), \theta(t)) \, dt \\
& + \delta dB(t), \quad t \in [s, T]; \\
& X(t) = x_s(t), \quad t \in [0, s],
\end{align*}
\]

with \( \delta \in (0, 1) \) being an arbitrary constant.

For each \( s \in [t_{N-1}, T) \), let

\[
V^\varepsilon(s, x_s) = \tilde{V}^\varepsilon(W(t_1), \ldots, W(t_{N-1}), W(s), s(0), \ldots, x(t_{N-1}), x(s))
\]

with

\[
\tilde{V}^\varepsilon(W(t_1), \ldots, W(t_{N-1}), \tilde{y}, s(0), \ldots, x(t_{N-1}), \tilde{x})
\]

\[
= \text{essinf}_{\theta \in \mathcal{U}} E_{\mathcal{F}_s} \left[ \int_s^T f^N(W(t_1), \ldots, W(t_{N-1}), W(t), t, \ldots, x(t_{N-1}), X^{s, x; \theta, N}(t), \theta(t)) \, dt \\
+ G^N(W(t_1), \ldots, W(t_N), x(0), \ldots, x(t_{N-1}), X^{s, x; \theta, N}(T)) \right].
\]
Here and in what follows, we just write $x(t_j) = x_s(t_j)$ for $j = 0, \ldots, N - 1$, as they are deemed to be fixed for $s \in (t_{N-1}, T]$.

By the viscosity solution theory of fully nonlinear parabolic PDEs (see \cite[Theorems I.1 and II.1]{[19]} for instance), the function

$$
\tilde{V}^\varepsilon(W(t_1), \ldots, W(t_{N-1}), \tilde{y}, s, x(0), \ldots, x(t_{N-1}), \tilde{x})
$$

satisfies the following HJB equation:

$$
\begin{aligned}
-D_t u(\tilde{y}, t, \tilde{x}) &= \frac{1}{2} \text{tr} (D_{\tilde{y}\tilde{y}} u(\tilde{y}, t, \tilde{x})) + \frac{\delta^2}{2} \text{tr} (D_{\tilde{x}\tilde{x}} u(\tilde{y}, t, \tilde{x})) \\
&+ \inf_{v \in U} \left\{ (\beta^N)'(W(t_1), \ldots, W(t_{N-1}), \tilde{y}, t, x(0), \ldots, x(t_{N-1}), \tilde{x}, v) D_{\tilde{x}} u(\tilde{y}, t, \tilde{x}) \\
&+ f^N(W(t_1), \ldots, W(t_{N-1}), \tilde{y}, t, x(0), \ldots, x(t_{N-1}), \tilde{x}, v) \right\}; \\
\end{aligned}
$$

(4.7)

It follows similarly on intervals $[t, T]$.

Thus, the regularity theory of viscosity solutions (see \cite[Theorem 6.4.3]{[19]} for instance\footnote{As $U \subset \mathbb{R}^n$ is a nonempty separable set, it has a denumerable subset $\mathcal{K} \subset U$ that is dense in $U$, and by the continuity of the coefficients, the essential infimum may be taken over $\mathcal{K}$. Thus, we may apply \cite[Theorem 6.4.3]{[19]} straightforwardly.}) gives for each $(x(0), \ldots, x(t_{N-1})) \in \mathbb{R}^{N \times d}$,

$$
\tilde{V}^\varepsilon(W(t_1), \ldots, W(t_{N-1}), \cdot, x(0), \ldots, x(t_{N-1}), \cdot) \\
\in \bigcap_{t \in (t_{N-1}, T]} L^\infty(\Omega, \mathcal{F}_{t_{N-1}}, C^{1+\frac{\alpha}{2}, 2+\alpha}([t_{N-1}, T] \times \mathbb{R}^{m+d})),
$$

for some $\alpha \in (0, 1)$, where the time-space Hölder space $C^{1+\frac{\alpha}{2}, 2+\alpha}([t_{N-1}, T] \times \mathbb{R}^{d+m})$ is defined as usual. Similar arguments may be conducted on time interval $[t_{N-2}, t_{N-1}]$ with the previously obtained $V^\varepsilon(t_N, x)$ as the terminal value, and recursively on intervals $[t_{N-3}, t_{N-2}], \ldots, [0, t_1]$.

On $[t_{N-1}, T]$, applying the Itô-Kunita formula of \cite[Pages 118-119]{[20]} to $\tilde{V}^\varepsilon$ yields that

$$
\begin{aligned}
- dV^\varepsilon(t, x, \delta B(t)) &= \inf_{v \in U} \left\{ (\beta^N)'(W(t_1), \ldots, W(t), t, x(0), \ldots, x(t_{N-1}) - \delta B(t_{N-1}), x(t) - \delta B(t), v) \nabla V^\varepsilon(t, x, \delta B(t)) \right\} \\
&+ f^N(W(t_1), \ldots, W(t), t, x(0), \ldots, x(t_{N-1}) - \delta B(t_{N-1}), x(t) - \delta B(t), v) \, dt \\
&- D_{\tilde{y}} \tilde{V}^\varepsilon(W(t_1), \ldots, W(t), t, x(0), \ldots, x(t_{N-1}) - \delta B(t_{N-1}), x(t) - \delta B(t), v) D_{\tilde{x}} \tilde{V}^\varepsilon(W(t_1), \ldots, W(t), t, x(0), \ldots, x(t_{N-1}) - \delta B(t_{N-1}), x(t) - \delta B(t), v) \, dt \\
&+ \delta \nabla V^\varepsilon(t, x - \delta B(t), v) \, dB(t), \quad t \in [t_{N-1}, T] \text{ and } x \in \Lambda_t; \\
V^\varepsilon(T, x_T) &= G^N(W(t_1), \ldots, W(T), x(0), \cdot, x(t_{N-1}) - \delta B(t_{N-1}), x(T) - \delta B(T)), \quad x_T \in \Lambda_T.
\end{aligned}
$$

(4.8)

It follows similarly on intervals $[t_{N-2}, t_{N-1}], \ldots, [0, t_1]$, and finally we have $V^\varepsilon(\cdot, \cdot, -\delta B(\cdot)) \in \mathcal{C}_b^2$.

In view of the approximation in Lemma \cite[4.2]{[19]} and with an analogy to (iv) in Proposition \cite[3.2]{[19]} there exists $\tilde{L} > 0$ such that for all $t \in [0, T]$ with $x_t \in \Lambda_t$,

$$
|\nabla V^\varepsilon(t, x_t)| \leq \tilde{L}, \quad \text{a.s.}
$$

with $\tilde{L}$ independent of $\varepsilon$ and $N$. Set $C_1 = \tilde{L}$, and

$$
\begin{aligned}
\nabla^\varepsilon_k(s, x) &= V^\varepsilon(s, x - \delta B(s)) + Y^\varepsilon_k(s) + \delta C_{2y}(t), \\
V^\varepsilon_k(s, x) &= V^\varepsilon(s, x - \delta B(s)) - Y^\varepsilon_k(s) - \delta C_{2y}(t),
\end{aligned}
$$

for $k = 1, \ldots, m$. 

Thus, we may apply \cite[Theorem 6.4.3]{[19]} straightforwardly.

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where $C_2 = 4L_c(\bar{L} + 1)$ and $L_c$ is the Lipschitz constant in Lemma 4.2. As the uniform Lipschitz continuity gives

$$|\beta^N(t,x,\theta,B(t),\theta(t)) - \beta^N(t,x - \delta\theta(t),\theta(t))| \leq 2\delta L_c\|B_t\|_0,$$

it holds that for all $(t,x_t)$ with $t \in (t_{N-1}, T)$ and $x_t \in A_{0,t}^{0,k}$,

$$-\partial_t V_k - \mathbb{H}(\nabla V_k)$$

$$= -\partial_t V_k - \inf_{v \in U} \left\{ (\beta^N)'\nabla V_k + f^N + f^N_k + \tilde{L}_k\beta_k^2 + \delta C_2\|B_t\|_0 \right\}$$

$$+ (\beta - \beta^N)'\nabla V_k - \beta_k^2 \tilde{L} + f - f^N - f^N_k - \delta C_2\|B_t\|_0$$

$$\geq -\partial_t V_k - \inf_{v \in U} \left\{ (\beta^N)'\nabla V_k + f^N + f^N_k + \beta_k^2 \tilde{L} + C_2\|B_t\|_0 \right\}$$

$$= 0,$$

where the inputs are omitted for involved functions. It follows similarly on intervals $(t_{N-2}, t_{N-1}), \ldots, [0,t]$ that $-\partial_t V_k - \mathbb{H}(\nabla V_k) \geq 0$, which together with the obvious relation $V_k(T) = G^N_k + C^N + \delta C_2\|B_T\|_0 \geq G$ indicates that $V_k \in \mathcal{F}_k$. Analogously, $V_k^\epsilon \in \mathcal{F}_k$. 

**Step 2.** Taking $k \geq L$, we measure the distance between $V_k^\epsilon$ (resp., $V_k$) and $V$. By the estimates for solutions of backward SDEs (see [4, Proposition 3.2] for instance), we first have

$$\|Y_k^\epsilon\|_{L^2(\Omega;X([0,T];\mathbb{R}))} + \|Z_k^\epsilon\|_{L^2(\Omega;\mathbb{F}_T;\mathbb{R}^m)} \leq C_3 \left( \|G_k\|_{L^2(\Omega;\mathbb{F}_T;\mathbb{R}^m)} + \|f^\epsilon_k + \tilde{L}_k\beta_k^2\|_{L^2(\Omega;\mathbb{F}_T;\mathbb{R}^m)} \right)$$

$$\leq C_3 \epsilon \cdot (1 + \tilde{L})(1 + k),$$

with the constant $C_3$ independent of $N, k, \delta, \epsilon$.

Fix an arbitrary $s \in [0,T)$ with $x_s \in A_{0,s}^{0,k}$ As $k \geq L$, we have

$$X^{s,x;\theta,N,\theta,B} := (X^{s,x;\theta,N} - \delta B(\cdot) + \delta B(s)) \cdot 1_{[s,T]}(\cdot) + x_s(\cdot)1_{[0,s)}(\cdot) \in A_{0,T}^{0,k},$$

and for $0 \leq s \leq t \leq T$, it holds that

$$|\beta^N(t,x_t^{s,x;\theta,N},\theta(t)) - \beta(t,x_t^{s,x;\theta,N},\theta(t))|$$

$$\leq |\beta^N(t,x_t^{s,x;\theta,N},\theta(t)) - \beta^N(t,x_t^{s,x;\theta,N,\theta,B},\theta(t))| + \beta_k^2(t)$$

$$+ |\beta(t,x_t^{s,x;\theta,N,\theta,B},\theta(t)) - \beta(t,x_t^{s,x;\theta,N},\theta(t))|$$

$$+ |\beta(t,x_t^{s,x;\theta,N},\theta(t)) - \beta(t,x_t^{s,x;\theta},\theta(t))|$$

$$\leq 2(L_c + L)\delta\|B_t^{s,0}\|_0 + \beta_k^2(t) + L\|X_t^{s,x;\theta,N} - X_t^{s,x;\theta}\|_0, \text{ a.s.,}$$

where $B_t^{s,0}(r) = B(r) - B(r \wedge s)$ for $0 \leq r \leq t$. In view of the approximations in Lemma 4.2 using Itô’s formula, Burkholder-Davis-Gundy’s inequality, and Gronwall’s inequality, we have through standard computations that for any $\theta \in \mathcal{U}$,

$$E_{\mathbb{F}_s} \left[ \sup_{s \leq t \leq T} \left| X_t^{s,x;\theta,N}(t) - X_t^{s,x;\theta}(t) \right|^2 \right] \leq C_4 \left( \delta^2 + E_{\mathbb{F}_s} \int_s^T |\beta_k^2(t)|^2 \, dt \right),$$

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with $C_4$ being independent of $s$, $x_s$, $\delta$, $N$, $k$, $\varepsilon$, and $\theta$. This together with similar calculations as in (4.11) yields
\[
E |V^\varepsilon(s,x_s) - V(s,x_s)|
\leq E \sup_{\theta \in \mathcal{U}} \left[ \int_s^T \left( |f^N(t, X^s_{t,x_s}, \theta, N, \theta(t)) - f(t, X^s_{t,x_s}, \theta(t))| \right) \, dt + |G^N(X^s_{t,x_s}, \theta, N) - G(X^s_{t,x_s}, \theta)| \right]
\leq C \left( \delta + \|\beta_k\|_{L^2(\Omega \times [0,T]; \mathbb{R}^d)} + \|f^\varepsilon_k\|_{L^2(\Omega \times [0,T]; \mathbb{R}^d)} + \|G^\varepsilon_k\|_{L^2(\Omega \times \mathbb{R}^d)} \right)
\leq C_5 (1 + k + \delta),
\]
with the constant $C_5$ being independent of $N$, $\varepsilon$, $k$, $\delta$, and $(s,x_s)$. Furthermore, in view of the definitions of $V^\varepsilon_k$ and $V^\varepsilon$, there exists some constant $C_6$ independent of $\varepsilon$, $\delta$, $k$, and $N$ such that for all $s \in [0,T]$ and $x_s \in \Lambda_{0,s}$, it holds that
\[
E |V^\varepsilon(s,x_s) - V(s,x_s)| + E |V^\varepsilon_k(s,x_s) - V(s,x_s)| \leq C_6 \{\varepsilon(1 + k) + \delta\}. \quad (4.12)
\]
Because $V$ is a viscosity solution by Theorem 3.5, there exist two infinite sequence of integers $\{k_n\}_{n \in \mathbb{N}^+}$ and $\{l_n\}_{n \in \mathbb{N}^+}$ (see Remark 2.3) such that $\lim_{n \to \infty} k_n = \lim_{n \to \infty} l_n = \infty$, $V_{k_n}(s,x_s) \geq V(s,x_s)$ a.s. for all $n \in \mathbb{N}^+$, $s \in [0,T]$, and $x_s \in \Lambda_{0,s}$, and $V(t,x_t) \geq V^\varepsilon_k(t,x_t)$ for all $n \in \mathbb{N}^+$, $t \in [0,T]$, and $x_t \in \Lambda_{0,t}$. These, together with the arbitrariness of $\varepsilon$, $k$, and $\delta$ in (4.12) and the denseness of $\cup_{n=1}^\infty \Lambda_{0,n}^0$ and $\cup_{n=1}^\infty \Lambda_{0,T}^0$ in $\Lambda_0$, finally imply that $u(t,x_t) = V(t,x_t) = \pi(t,x_t)$ a.s. for all $t \in [0,T]$, and $x_t \in \Lambda_0$. \hfill \square

The above proof is inspired by but different from the conventional Perron’s method and its modifications used, for instance, in [5, 14, 29]: the key difference is that the random fields $\pi$ and $\underline{u}$ are neither extreme points of viscosity semi-solutions nor from approximate extremality of classical semi-solutions, while they are limits of approximate extreme points of certain classes of regular random functions. Besides, by enlarging the original filtered probability space with an independent Brownian motion $B$, we have actually constructed the regular approximations of $V$ with a regular perturbation induced by $\delta B$, which corresponds to approximation to the optimization (1.1)-(1.2) with stochastic controls of Markovian type. Such approximation seems interesting even for the case where all the coefficients $\beta$, $f$, and $G$ are just deterministic and path-dependent.

**Remark 4.1.** In this work, the filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions is generated by an $m$-dimensional Wiener process $W = \{W(t) : t \in [0, \infty)\}$ and augmented by all the $\mathbb{P}$-null sets in $\mathcal{F}$. Such a Brownian filtration assumption is preferred and adopted here due to technical reason. On the one hand, in the above proof of the uniqueness, the constructed approximations squeezing the viscosity solution are based on Markovian-type optimal controls and their sufficient regularity estimate is given in Krylov’s work (see [22, Theorems I.1 and II.1] for instance) where the Brownian filtration is assumed. On the other hand, with the Brownian filtration, we could relate the random test functions in $\mathcal{C}^1_\beta$ to an Itô process in Definition 2.2 with the operators $\partial_t$ and $\partial_\omega$ defined conveniently, and the generalized Itô-Kunita formula and the existence arguments follow smoothly. Even so, we think the filtration assumption could be
loosened. Nevertheless, the extension may not be straightforward. For example, working under an arbitrary filtration satisfying the usual conditions, one could replace the stochastic integral representation in (2.1) with a parameterized martingale \( M^u(t, x_t) \) that is linear w.r.t. \( u \), then one would need to contemplate how to make sense of the composition \( M^u(t, X^\tau_\theta x; \theta) \) when dealing with the existence, and in the uniqueness part, the associated differential operator needs to be specified in the Markovian approximations which is hard without any pre-specification on the underlying stochastic process.

A Proof of Theorem 3.3

Proof of Theorem 3.3. Denote the right hand side by \( V(\tau, \xi) \). By Proposition 3.2 (iv) and (v), both \( V \) and \( \nabla \) are lying in \( S^\infty(\Lambda; \mathbb{R}) \) and the continuity indicates that it is sufficient to prove Theorem 3.3 when \( \tau, \hat{\tau} \) and \( \xi \) are deterministic.

For each \( \varepsilon > 0 \), by Proposition 3.2 (iv), there exists \( \delta = \varepsilon / L_V > 0 \) such that whenever \( \| x_{\hat{\tau}} - y_{\hat{\tau}} \|_0 < \delta \) for \( x_{\hat{\tau}}, y_{\hat{\tau}} \in \Lambda_{\hat{\tau}} \), it holds that

\[
|J(\hat{\tau}, x_{\hat{\tau}}; \theta) - J(\hat{\tau}, y_{\hat{\tau}}; \theta)| + |V(\hat{\tau}, x_{\hat{\tau}}) - V(\hat{\tau}, y_{\hat{\tau}})| \leq \varepsilon \quad \text{a.s., } \forall \theta \in \mathcal{U}.
\]

Arzelà-Ascoli theorem indicates the compactness of \( \Lambda^{0, L; \xi}_{\tau, \hat{\tau}} \) in \( \Lambda_{\hat{\tau}} \). Thus, \( \Lambda^{0, L; \xi}_{\tau, \hat{\tau}} \) is separable, and there exists a sequence \( \{x^j\}_{j \in \mathbb{N}^+} \subset \Lambda^{0, L; \xi}_{\tau, \hat{\tau}} \) such that \( \bigcup_{j \in \mathbb{N}^+} \left( \Lambda^{0, L; \xi}_{\tau, \hat{\tau}} \cap B_{\delta/3}(x^j) \right) = \Lambda^{0, L; \xi}_{\tau, \hat{\tau}} \). Set \( D_1 = B_{\delta/3}(x^1) \cap \Lambda^{0, L; \xi}_{\tau, \hat{\tau}} \), and

\[
D_j = \left( B_{\delta/3}(x^j) - (\bigcup_{i=1}^{j-1} B_{\delta/3}(x^i)) \right) \cap \Lambda^{0, L; \xi}_{\tau, \hat{\tau}}, \quad \text{for } j > 1.
\]

Then \( \{D^j\}_{j \in \mathbb{N}^+} \) is a partition of \( \Lambda^{0, L; \xi}_{\tau, \hat{\tau}} \) with diameter \( \text{diam}(D^j) < \delta \), i.e., \( D^j \subset \Lambda^{0, L; \xi}_{\tau, \hat{\tau}} \), \( \bigcup_{j \in \mathbb{N}^+} D^j = \Lambda^{0, L; \xi}_{\tau, \hat{\tau}} \), \( D^i \cap D^j = \emptyset \) if \( i \neq j \), and for any \( x, y \in D^j \), \( \| x - y \|_0 < \delta \).

Then the rest of the proof is similar to that of [24 Theorem 3.4]. For each \( j \in \mathbb{N}^+ \), take \( \bar{x}^j \in D^j \), and a straightforward application of Proposition 3.2 (i) leads to some \( \theta^j \in \mathcal{U} \) satisfying

\[
0 \leq J(\hat{\tau}, \bar{x}^j; \theta^j) - V(\hat{\tau}, \bar{x}^j) := \alpha^j \quad \text{a.s., with } E[\alpha^j] < \frac{\varepsilon}{2^j}.
\]

Thus, for each \( x \in D^j \), it holds that

\[
J(\hat{\tau}, x; \theta^j) - V(\hat{\tau}, x)
\leq |J(\hat{\tau}, x; \theta^j) - J(\hat{\tau}, \bar{x}^j; \theta^j)| + |J(\hat{\tau}, \bar{x}^j; \theta^j) - V(\hat{\tau}, \bar{x}^j)| + |V(\hat{\tau}, \bar{x}^j) - V(\hat{\tau}, x)|
\leq 2 \varepsilon + \alpha^j, \quad \text{a.s.}
\]

In view of Assumption (A1) (iii), we observe that for any \( \theta \in \mathcal{U} \), \( X^{\tau, \xi, \theta}_{\hat{\tau}} \) is almost surely valued in \( \Lambda^{0, L; \xi}_{\tau, \hat{\tau}} \). For each \( \theta \in \mathcal{U} \), put

\[
\tilde{\theta}(s) = \begin{cases} \theta(s), & \text{if } s \in [0, \hat{\tau}); \\ \sum_{j \in \mathbb{N}^+} \theta^j(s) 1_{D^j}(X^{\tau, \xi, \theta}_{\hat{\tau}}), & \text{if } s \in [\hat{\tau}, T]. \end{cases}
\]

Then it follows that

\[
V(\tau, \xi) \leq J(\tau, \xi; \tilde{\theta})
\]
\[
E_{\hat{\tau}} \left[ \int_{\tau}^{\hat{\tau}} f \left( s, X_s^{\tau,\xi,\theta}, \theta(s) \right) \, ds + J \left( \hat{\tau}, X_{\hat{\tau}}^{\tau,\xi,\theta} \right) \right] \\
\leq E_{\hat{\tau}} \left[ \int_{\tau}^{\hat{\tau}} f \left( s, X_s^{\tau,\xi,\theta}, \theta(s) \right) \, ds + V \left( \hat{\tau}, X_{\hat{\tau}}^{\tau,\xi,\theta} \right) + \sum_{j \in \mathbb{N}^+} \alpha^j \right] + 2\varepsilon,
\]

where \( \{\alpha^j\} \) is independent of the choices of \( \theta \). Taking infimums and then expectations on both sides, we arrive at \( EV(\tau,\xi) \leq E\hat{V}(\tau,\xi) + 3\varepsilon \). By the arbitrariness of \( \varepsilon > 0 \), we have \( E\hat{V}(\tau,\xi) \geq EV(\tau,\xi) \), which together with the obvious relation \( \hat{V}(\tau,\xi) \leq V(\tau,\xi) \) yields that \( V(\tau,\xi) = \hat{V}(\tau,\xi) \) a.s.

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