Hidden Conformal Symmetry of Self-Dual Warped AdS$_3$ Black Holes in Topological Massive Gravity

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Abstract

We extend the recently proposal of hidden conformal symmetry to the self-dual warped AdS$_3$ black holes in topological massive gravity. It is shown that the wave equation of massive scalar field with sufficient small angular momentum can be reproduced by the SL(2, R) Casimir quadratic operator. Due to the periodic identification in the $\phi$ direction, it is found that only the left section of hidden conformal symmetry is broken to U(1), while the right section is unbroken, which only gives the left temperature of dual CFT. As a check of the dual CFT conjecture of self-warped AdS$_3$ black hole, we further compute the Bekenstein-Hawking entropy and absorption cross section and quasinormal modes of scalar field perturbation and show these are just of the forms predicted by the dual CFT.

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I. INTRODUCTION

Topological massive gravity (TMG) is described by the theory of three dimensional Einstein gravity with a gravitational Chern-Simons correction and the cosmological constant \[1, 2\]. The well-known spacelike warped AdS$_3$ black hole \[3\] (previously obtained in \[4\]), which is a vacuum solution of topological massive gravity, is conjectured to be dual to a two dimensional conformal field theory (CFT) with non-zero left and right central charges \[5\]. The spacelike warped AdS$_3$ black hole is a quotient of warped AdS$_3$ spacetime, just as BTZ black hole is a quotient of AdS$_3$ spacetime. This leads to the breaking of the SL(2, R)$\times$SL(2, R) isometry of AdS$_3$ to the SL(2, R)$\times$U(1) isometry of warped AdS$_3$ black hole. It is shown in \[6\] that, for a certain low energy limit, the wave equation of the massive scalar field in the background of spacelike warped AdS$_3$ black hole can be written as the Casimir operator of SL(2, R)$\times$SL(2, R) Lie algebra, which uncovers the hidden SL(2, R)$\times$SL(2, R) symmetry of the wave equation of scalar field.

Recently, a new class of solutions in three dimensional topological massive gravity named as self-dual warped AdS$_3$ black hole is proposed by Chen et al in \[7\]. It is conjectured that the self-dual warped AdS$_3$ black hole is dual to a chiral CFT with only nonvanishing left central charge, which is very different from the spacelike warped AdS$_3$ black hole. The self-dual warped AdS$_3$ black hole is locally equivalent to spacelike warped AdS$_3$ black hole via a coordinates transformation. The isometry group is just U(1)$_L\times$SL(2,R)$_R$, similar to the warped AdS$_3$ black hole. Under the consistent boundary condition, the U(1)$_L$ isometry is enhanced to a Virasoro algebra with nonvanishing left central charge, while the SL(2, R)$_R$ isometry becomes trivial with the vanishing right central charge, which is similar to the case of extremal Kerr/CFT correspondence \[8, 9\]. This suggests a novel example of warped AdS/CFT dual.

In this paper, motivated by the recently proposed hidden conformal symmetry of the wave equation of scalar field propagating in the background of the general rotating black hole \[10\], we consider the case of self-dual warped AdS$_3$ black holes in TMG. It is shown that the wave equation of massive scalar field propagating in the background of self-dual warped AdS$_3$ black hole can be rewritten in the form of Casimir operator of the SL(2, R)$_L\times$SL(2, R)$_R$ Lie algebra. Unlike the higher dimensional black holes where the near-horizon limit should be taken into account to match the wave equation with the Casimir operator, in the present
case, only the condition of the small angular momentum of scalar field is imposed, which suggests that the hidden conformal symmetry is valid for the scalar field with arbitrary energy. So we have uncovered the hidden $\text{SL}(2, \mathbb{R})_L \times \text{SL}(2, \mathbb{R})_R$ symmetry of the wave equation of massive scalar field in self-dual warped $\text{AdS}_3$ black hole. Then, we show that, due to the periodic identification in the $\phi$ direction, only one copy of hidden $\text{SL}(2, \mathbb{R})$ symmetry is broken to $\text{U}(1)$, while the another copy is unbroken. This only gives the left temperature $T_L$ of dual CFT, while the right temperature $T_R$ can not read from this approach. This point is also different from the higher dimensional black holes [10–22]. Despite the right temperature can not be directly read from the periodic identification in the $\phi$ direction, one can still conjecture that self-dual warped AdS$_3$ black hole is holographically dual to a two dimensional CFT with the left temperature $T_L = \frac{\alpha}{2\pi}$ and the right temperature $T_R = \frac{x_+ - x_-}{4\pi}$, which is exactly matches with the warped AdS/CFT correspondence suggested in [7].

As a check of this conjecture, we also show the entropy of the dual conformal field theory given by the Cardy formula matches exactly with the Bekenstein-Hawking entropy of self-dual warped AdS$_3$ black hole. Furthermore, the absorption cross section of scalar field perturbation calculated from the gravity side is in perfect match with that for a finite temperature 2D CFT. At last, we present an algebraic calculation of quasinormal modes for scalar field perturbation firstly proposed by Sachs et al in [23]. It is shown that the quasinormal modes coincide with the poles in the retarded Green’s function obtained in [7], which is a prediction of AdS/CFT dual.

This paper is organized as follows. In section II, we give a brief review of self-dual warped AdS$_3$ black hole in topological massive gravity. In section III, we study the hidden conformal symmetry of this black hole by analysing the wave equation of massive scalar field. In section IV, we give some interpretations of the dual conformal field description of self-warped AdS$_3$ black hole by computing the entropy, absorption cross section and quasinormal modes of scalar field and comparing the results from both gravity and CFT sides. The last section is devoted to discussion and conclusion.

II. SELF-DUAL WARPED ADS$_3$ BLACK HOLE

In this section, we will give a brief review of self-dual warped AdS$_3$ black hole in topological massive gravity. The action of topological massive gravity with a negative cosmological
constant is given by
\[
I_{TMG} = \frac{1}{16\pi G} \int \mathcal{M} d^3x \sqrt{-g} \left( R + \frac{2}{l^2} \right) + \frac{l}{96\pi G\nu} \int \mathcal{M} d^3x \sqrt{-g} \epsilon^{\lambda \mu \nu} \Gamma^a_{\lambda \sigma} \left( \partial_\mu \Gamma^\sigma_{\alpha \nu} + \frac{2}{3} \Gamma^\sigma_{\mu \tau} \Gamma^\tau_{\nu \alpha} \right). \tag{1}
\]
Varying the above action with respect to the metric yields the equation of motion, which is given by
\[
G_{\mu \nu} - \frac{1}{l^2} g_{\mu \nu} + \frac{l}{3\nu} C_{\mu \nu} = 0, \tag{2}
\]
where \( G_{\mu \nu} = R_{\mu \nu} - \frac{1}{2} R g_{\mu \nu} \) is the Einstein tensor and \( C_{\mu \nu} \) is the Cotton tensor
\[
C_{\mu \nu} = \epsilon^\alpha_\mu \nabla_\alpha \left( R_{\beta \nu} - \frac{1}{4} R g_{\beta \nu} \right). \tag{3}
\]
Recently, a new class of solutions of topological massive gravity named as the self-dual warped AdS\(_3\) black hole is investigated by Chen et al in [7]. The metric is given by
\[
\begin{align*}
\mathcal{R}^2 = & \frac{1}{\nu^2 + 3} \left( - (x - x_+) (x - x_-) d\tau^2 + \frac{1}{(x - x_+) (x - x_-)} dx^2 \\
& + \frac{4\nu^2}{\nu^2 + 3} \left( \alpha d\phi + \frac{1}{2} (2x - x_+ - x_-) d\tau \right)^2 \right), \tag{4}
\end{align*}
\]
where \(x_+\) and \(x_-\) are the location of the outer and inner horizons respectively, and we have set \(l = 1\) for simplicity. The mass \(M\) and angular momentum \(J\) of this black hole are given by
\[
M = 0, \quad J = \frac{(\alpha^2 - 1)\nu}{6G(\nu^2 + 3)}. \tag{5}
\]
The Hawking temperature \(T_H\), angular velocity of the event horizon \(\Omega_H\) and the Bekenstein-Hawking entropy \(S_{BH}\) of this solution are respectively given by
\[
\begin{align*}
T_H = & \frac{x_+ - x_-}{4\pi}, \\
\Omega_H = & \frac{x_+ - x_-}{2\alpha}, \\
S_{BH} = & \frac{2\pi\alpha\nu}{3G(\nu^2 + 3)}. \tag{6}
\end{align*}
\]
This solution is asymptotic to the spacelike warped AdS\(_3\) spacetime. It is shown in [7] that the self-dual warped AdS\(_3\) black hole is locally equivalent to spacelike warped AdS\(_3\).
spacetime. Under coordinate transformation

\[
v = \tan^{-1} \left[ \frac{2\sqrt{(x-x_+)(x-x_-)}}{2x-x_+ + x_-} \sinh \left( \frac{x_+ - x_-}{2} \tau \right) \right],
\]

\[
\sigma = \sinh^{-1} \left[ \frac{2\sqrt{(x-x_+)(x-x_-)}}{2x-x_+ + x_-} \cosh \left( \frac{x_+ - x_-}{2} \tau \right) \right],
\]

\[
u = \alpha \phi + \tan^{-1} \left[ \frac{2x-x_+ + x_-}{x_+ - x_-} \coth \left( \frac{x_+ - x_-}{2} \tau \right) \right],
\] (7)

the metric of self-dual warped AdS\(_3\) black hole solution can be transformed to the metric of spacelike warped AdS\(_3\) spacetime

\[
ds^2 = \frac{1}{\nu^2 + 3} \left( -\cosh^2 \sigma dv^2 + d\sigma^2 + \frac{4\nu^2}{\nu^2 + 3} (du + \sinh \sigma dv)^2 \right).\] (8)

It will be shown that this coordinates transformation is just the appropriate coordinates transformation to uncover the hidden conformal symmetry. The isometry group of this solution is U(1)\(_L\) \(\times\) SL(2, R)\(_R\), which is generated by the killing vectors

\[
J_2 = 2\partial_u,
\] (9)

and

\[
\tilde{J}_1 = 2 \sin \nu \tanh \sigma \partial_v - 2 \cos \nu \partial_\sigma + \frac{2 \sin \nu}{\cosh \sigma} \partial_u,
\]

\[
\tilde{J}_2 = -2 \cos \nu \tanh \sigma \partial_v - 2 \sin \nu \partial_\sigma - \frac{2 \cos \nu}{\cosh \sigma} \partial_u,
\]

\[
\tilde{J}_0 = 2\partial_v.
\] (10)

It is also shown in [7] that, under the consistent boundary condition, the U(1)\(_L\) isometry is enhanced to a Virasoro algebra with the central charge

\[
c_L = \frac{4\nu}{\nu^2 + 3},\] (11)

while the SL(2, R)\(_R\) isometry becomes trivial with the vanishing central charge \(c_R = 0\), which is similar to the case of extremal Kerr/CFT dual [8, 9]. The entropy of self-dual warped AdS\(_3\) black hole can be reproduced by the Cardy formula. So it is conjectured that the self-dual warped AdS\(_3\) black hole is holographically dual to a two dimensional chiral conformal field theory with nonvanishing left central charge.
III. HIDDEN CONFORMAL SYMMETRY

In this section, we study the hidden conformal symmetry by analyzing the massive scalar field propagating in the background of self-dual warped AdS$_3$ black hole. Firstly, it is found that the scalar field equation can be exactly solved by the hypergeometric function. Then, by introducing the SL(2, R)$_L \times$SL(2, R)$_R$ generators and using the coordinates transformation (7), we show the wave equation can be reproduced by the SL(2, R) Casimir operator.

A. Scalar field perturbation

Let us consider the scalar field $\Phi$ with mass $m$ in the background of self-dual warped AdS$_3$ black hole, where the wave equation is given by the Klein-Gordon equation

\[
\left( \frac{1}{\sqrt{-g}} \partial_{\mu} (\sqrt{-g} g^{\mu\nu} \partial_{\nu}) - m^2 \right) \Phi = 0 .
\] (12)

The scalar field wave function $\Phi(\tau, x, \phi)$ can be expanded in eigenmodes as

\[
\Phi = e^{-i\omega\tau + ik\phi} R(x) ,
\] (13)

where $\omega$ and $k$ are the quantum numbers. Then the radial wave equation can be written as

\[
\left[ \partial_x ((x - x_+)(x - x_-)\partial_x) + \frac{(\omega + \frac{x_+ - x_-}{2\alpha} k)^2}{(x - x_+)(x_+ - x_-)} - \frac{(\omega - \frac{x_+ - x_-}{2\alpha} k)^2}{(x - x_-)(x_+ - x_-)} \right] R(x)
\]
\[
= \left( -\frac{3(\nu^2 - 1) k^2}{4\nu^2} \frac{1}{\alpha^2} + \frac{1}{\nu^2 + 3} m^2 \right) R(x) .
\] (14)

This radial wave equation can be exactly solved by the hypergeometric function. In order to solve the radial equation, it is convenient to introduce the variable

\[
z = \frac{x - x_+}{x - x_-} .
\] (15)

Then, the radial equation can be rewritten in the form of hypergeometric equation

\[
z(1 - z) \frac{d^2 R(z)}{dz^2} + (1 - z) \frac{dR(z)}{dz} + \left( \frac{A}{z} + B + \frac{C}{1 - z} \right) R(z) = 0 ,
\] (16)

with the parameters

\[
A = \left( \frac{k}{2\alpha} + \frac{\omega}{x_+ - x_-} \right)^2 ,
\]
\[
B = - \left( \frac{k}{2\alpha} - \frac{\omega}{x_+ - x_-} \right)^2 ,
\]
\[
C = \frac{3(\nu^2 - 1) k^2}{4\nu^2} \frac{1}{\alpha^2} - \frac{1}{\nu^2 + 3} m^2 .
\] (17)
For later convenience, we consider the solution with the ingoing boundary condition at the horizon which is given by the hypergeometric function

\[ R(z) = z^\alpha (1 - z)^\beta F(a, b, c, z) , \]  

(18)

where

\[ \alpha = -i\sqrt{A} , \quad \beta = \frac{1}{2} - \sqrt{\frac{1}{4} - C} , \]  

(19)

and

\[ c = 2\alpha + 1 , \]
\[ a = \alpha + \beta + i\sqrt{-B} , \]
\[ b = \alpha + \beta - i\sqrt{-B} . \]

(20)

It should be noted that, generally, the wave equation cannot be analytically solved and the solution must be obtained by matching solutions in an overlap region between the near-horizon and asymptotic regions. But, in the present case, we have shown that the radial equation can be exactly solved by hypergeometric functions. As hypergeometric functions transform in the representations of SL(2,R), this suggests the existence of a hidden conformal symmetry. In the next subsection, we will try to explore this hidden conformal symmetry.

**B. SL(2, R)_L×SL(2, R)_R**

Now, we will uncover the hidden conformal symmetry by showing that the radial equation can also be obtained by using of the SL(2, R) Casimir operator. Let us define vector fields

\[ H_0 = -\frac{i}{2} \tilde{J}_2 , \]
\[ H_1 = \frac{i}{2} (\tilde{J}_0 + \tilde{J}_1) , \]
\[ H_{-1} = \frac{i}{2} (\tilde{J}_0 - \tilde{J}_1) , \]

(21)

and

\[ \tilde{H}_0 = \frac{i}{2} J_2 , \]
\[ \tilde{H}_1 = \frac{1}{2} (J_1 + J_0) , \]
\[ \tilde{H}_{-1} = \frac{1}{2} (J_1 - J_0) , \]

(22)
with
\[ J_1 = -\frac{2 \sinh u}{\cosh \sigma} \partial_v - 2 \cosh u \partial_\sigma + 2 \tanh \sinh u \partial_u, \]
\[ J_0 = \frac{2 \cosh u}{\cosh \sigma} \partial_v + 2 \sinh u \partial_\sigma - 2 \tanh \cosh u \partial_u. \]  (23)

Note that \((J_1, J_2, J_0)\) and \((\tilde{J}_1, \tilde{J}_2, \tilde{J}_0)\) are the SL(2, R)×SL(2, R) Killing vectors of AdS\(_3\) spacetime. The vector fields \((H_1, H_0, H_{-1})\) obey the SL(2, R) Lie algebra
\[ [H_0, H_{\pm 1}] = \pm i H_{\pm 1}, \quad [H_{-1}, H_1] = 2i H_0, \]  (24)
and similarly for \((\tilde{H}_1, \tilde{H}_0, \tilde{H}_{-1})\). According to the coordinates transformation (7), the SL(2, R) generators can be expressed in terms of the black hole coordinates \((\tau, x, \phi)\) as
\[ H_0 = \frac{i}{2\pi T_R} \partial_\tau, \]
\[ H_{-1} = i e^{-2\pi T_R \tau} \left[ -\sqrt{(x-x_+)(x-x_-)} \partial_x - \frac{1}{\sqrt{(x-x_+)(x-x_-)}} \frac{T_R}{T_L} \partial_\phi \right. \]
\[ + \left. \left( x - \frac{x_+ + x_-}{2} \right) \frac{1}{\sqrt{(x-x_+)(x-x_-)}} \partial_\tau \right], \]
\[ H_1 = i e^{2\pi T_R \tau} \left[ \sqrt{(x-x_+)(x-x_-)} \partial_x - \frac{1}{\sqrt{(x-x_+)(x-x_-)}} \frac{T_R}{T_L} \partial_\phi \right. \]
\[ + \left. \left( x - \frac{x_+ + x_-}{2} \right) \frac{1}{\sqrt{(x-x_+)(x-x_-)}} \partial_\tau \right], \]  (25)
and
\[ \tilde{H}_0 = \frac{i}{2\pi T_L} \partial_\phi, \]
\[ \tilde{H}_{-1} = i e^{-2\pi T_L \phi} \left[ -\sqrt{(x-x_+)(x-x_-)} \partial_x + \left( x - \frac{x_+ + x_-}{2} \right) \frac{1}{\sqrt{(x-x_+)(x-x_-)}} \partial_\phi \right. \]
\[ - \left. \frac{1}{\sqrt{(x-x_+)(x-x_-)}} \partial_\tau \right], \]
\[ \tilde{H}_1 = i e^{2\pi T_L \phi} \left[ -\sqrt{(x-x_+)(x-x_-)} \partial_x + \left( x - \frac{x_+ + x_-}{2} \right) \frac{1}{\sqrt{(x-x_+)(x-x_-)}} \partial_\phi \right. \]
\[ - \left. \frac{1}{\sqrt{(x-x_+)(x-x_-)}} \partial_\tau \right], \]  (26)
where \(T_L\) and \(T_R\) are defined by
\[ T_L = \frac{\alpha}{2\pi}, \quad T_R = \frac{x_+ - x_-}{4\pi}. \]  (27)
The SL(2, R) quadratic Casimir operator is defined by
\[
\mathcal{H}^2 = \tilde{\mathcal{H}}^2 = -H_0^2 + \frac{1}{2}(H_1 H_{-1} + H_{-1} H_1) .
\] (28)

In terms of the \((\tau, x, \phi)\) coordinates, the SL(2, R) quadratic Casimir operator becomes
\[
\mathcal{H}^2 = \partial_x ((x - x_+)(x - x_-)\partial_x) - \frac{x_+ - x_-}{x - x_+} \left[ \frac{1}{4\pi T_R} \partial_\tau - \frac{1}{4\pi T_L} \partial_\phi \right]^2

+ \frac{x_+ - x_-}{x - x_-} \left[ \frac{1}{4\pi T_R} \partial_\tau + \frac{1}{4\pi T_L} \partial_\phi \right]^2 .
\] (29)

For the case of \(\nu = 1\), the first term of right hand side of Eq. (14) is vanishing. It should be noted that, unlike the case of higher dimensional black holes [10–22], where the near-region limit for the radial wave equation is taken into account, in the present case, no extra approximation is needed to match the wave equation of scalar field with the Casimir operator. So for the case of \(\nu = 1\), the self-dual warped black holes exhibit the local SL(2, R) \(_L \times\) SL(2, R) \(_R\) symmetry just like the BTZ black hole.

We consider the nontrivial case of \(\nu^2 > 1\), when these solutions are free of naked CTCs. Additional condition must be imposed to match the wave equation with the Casimir. In order to neglect the first term of right hand side of Eq. (14), we impose the condition that the angular momentum \(k\) of scalar field is sufficient small
\[
\frac{3(\nu^2 - 1) k^2}{4\nu^2} \ll 1 .
\] (30)

Then, we find that the wave equation of massive scalar field with the sufficient small angular momentum \(k\) can be rewritten as the Casimir operator
\[
\mathcal{H}^2 \Phi = \tilde{\mathcal{H}}^2 \Phi = \frac{1}{\nu^2 + 3} m^2 \Phi .
\] (31)

and the conformal weights of dual operator of the massive scalar field \(\Phi\) is given by
\[
(h_L, h_R) = \left( \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{m^2}{\nu^2 + 3}}, \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{m^2}{\nu^2 + 3}} \right) .
\] (32)

So we have found that, similar to the case of higher dimensional black holes, the hidden SL(2, R) \(_L \times\) SL(2, R) \(_R\) symmetry of self-dual warped AdS\(_3\) black hole is uncovered by investigating the wave equation of scalar field in its background. Note that the hidden conformal symmetry is not derived from the conformal symmetry of spacetime geometry itself.
It is also interesting that the hidden conformal symmetry of self-dual warped AdS$_3$ black hole is locally the isometry of AdS$_3$ spacetime, which means that scalar fields with sufficient small angular momentum $k$ satisfying the condition (30) do not feel the warped property of spacetime. While for the case of spacelike warped AdS$_3$ black hole investigated in [6], this observation is valid for the scalar fields with sufficient low energy.

IV. CFT INTERPRETATION

A. Temperature and entropy

As pointed out in [10], for the case of higher dimensional black hole, the vector fields of SL(2, R) generators are not globally defined. Because of the periodic identification in the $\phi$ direction, the hidden SL(2, R)$_L \times$SL(2,R)$_R$ symmetry is spontaneously broken to U(1)$_L \times$U(1)$_R$ subgroup, which gives rise to the left and right temperatures of dual CFT.

The story is somewhat different in the present case. The generators of SL(2, R) presented in Eq.(26) are affected by the periodic identification in the $\phi$ direction, while these in Eq.(25) are not because they are just the Killing vectors associated to SL(2, R) isometry of this solution. This means that only one copy of hidden conformal symmetry is broken to U(1), while the another copy is unbroken, which only gives the left temperature $T_L$ of dual CFT. The right temperature $T_R$ can not read from this approach. The periodical identification makes no contribution to the right temperature. This point can also be reached from the coordinates transformation [7] which indicates that the self-dual warped AdS$_3$ black hole can not be obtained as a quotient of warped AdS$_3$ vaccum.

As discussed in [7], the left and right temperatures of dual CFT can be defined with respect to the Frolov-Thorne vacuum [24]. Consider the quantum field with eigenmodes of the asymptotic energy $\omega$ and angular momentum $k$. After tracing over the region inside the horizon, the vacuum is a diagonal density matrix in the energy-angular momentum eigenbasis with a Boltzmann weighting factor $e^{-\frac{\omega-k\Omega}{T_H}}$. The left and right charges $\omega_L$, $\omega_R$ associated to $\partial_\phi$ and $\partial_t$ are $k$ and $\omega$ respectively. In terms of these variables, the Boltzmann factor is

$$e^{-\frac{\omega-k\Omega}{T_H}} = e^{-\frac{\omega_L}{T_L}} e^{-\frac{\omega_R}{T_R}},$$

(33)

which gives the definition of left and right temperatures (27).
So one can conjecture that the self-dual warped AdS$_3$ black hole is holographically dual to a two dimensional CFT with the left temperature $T_L = \frac{\alpha}{2\pi}$ and the right temperature $T_R = \frac{x_+ - x_-}{4\pi}$. As a check of this conjecture, we now want to calculate the microscopic entropy of the dual CFT, and compare it with the Bekenstein-Hawking entropy of self-dual warped AdS$_3$ black hole. By imposing the consistent boundary condition, Chen et al [7] have calculated the central charge of the asymptotic symmetry group, where the conclusion is presented in Eq.(11). So the microscopic entropy of the dual conformal field can be calculated by using the Cardy formula

$$S_{CFT} = \frac{\pi^2}{3}(c_L T_L + c_R T_R) = \frac{2\pi\alpha \nu}{3G(\nu^2 + 3)} = S_{BH},$$

which matches with the Bekenstein-Hawking entropy of self-dual warped AdS$_3$ black hole.

B. Absorption cross section

In this subsection, we will calculate the absorption probability for the scalar filed perturbation and compare it with result from the CFT side. For the spacelike warped AdS$_3$ black hole, this aspect has been investigated in [25] and [26].

Under the condition (30), the solution to the radial equation of scalar field perturbation with the ingoing boundary condition is explicitly given by

$$R(x) = \left(\frac{x - x_+}{x - x_-}\right)^{-i\left(\frac{k}{\alpha} + \frac{\omega}{x_+ - x_-}\right)} \left(\frac{x_+ - x_-}{x - x_-}\right)^{\frac{1}{2}} \sqrt{\frac{1}{4} + \frac{m^2}{\nu^2 + 3}} \times F\left(\frac{1}{2} - \frac{1}{4} + \frac{m^2}{\nu^2 + 3} - i\frac{2\omega}{x_+ - x_-}, \frac{1}{2} - \frac{1}{4} + \frac{m^2}{\nu^2 + 3} - i\frac{k}{\alpha}, 1 - i\left(\frac{k}{\alpha} + \frac{2\omega}{x_+ - x_-}\right), \frac{x - x_+}{x - x_-}\right).$$

At asymptotic infinity, the solution behaves as

$$R(x \to \infty) \sim Ax^{-\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{m^2}{\nu^2 + 3}}},$$

with

$$A = \frac{\Gamma\left(1 - i\left(\frac{k}{\alpha} + \frac{2\omega}{x_+ - x_-}\right)\right) \Gamma\left(2\sqrt{\frac{1}{4} + \frac{m^2}{\nu^2 + 3}}\right)}{\Gamma\left(\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{m^2}{\nu^2 + 3}} - i\frac{2\omega}{x_+ - x_-}\right) \Gamma\left(\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{m^2}{\nu^2 + 3}} - i\frac{k}{\alpha}\right)}.\]
The absorption cross section is then proportional to
\[ P_{\text{abs}} \sim |A|^{-2} \]
\[ \sim \sinh \left( \frac{\pi k}{\alpha} + \frac{2\pi \omega}{x_+ - x_-} \right) \left| \Gamma \left( \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{m^2}{\nu^2 + 3} - i \frac{2\omega}{x_+ - x_-}} \right) \right|^2 \]
\[ \times \left| \Gamma \left( \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{m^2}{\nu^2 + 3} - i \frac{k}{\alpha}} \right) \right|^2 . \] (38)

To compare it with the result from the CFT side, we need to find out the related parameters. From the first law of black hole thermodynamics
\[ \delta S_{BH} = \frac{\delta M - \Omega_H \delta J}{T_H} , \] (39)
one can calculate the conjugate charges
\[ \delta S_{BH} = \frac{\delta E_L}{T_L} + \frac{\delta E_R}{T_R} . \] (40)
The solution is given by
\[ \delta E_L = \delta J , \quad \delta E_R = \delta M . \] (41)
Identifying \( \delta M = \omega \) and \( \delta J = k \), one can find the left and right conjugate charges as
\[ \omega_L \equiv \delta E_L = k , \quad \omega_R \equiv \delta E_R = \omega , \] (42)
which are coincide with the the left and right charges given in the last subsection. Finally, the absorption cross section can be expressed as
\[ P_{\text{abs}} \sim T_L^{2h_L - 1} T_R^{2h_R - 1} \sinh \left( \frac{\omega_L}{2T_L} + \frac{\omega_R}{2T_R} \right) \left| \Gamma \left( h_L + i \frac{\omega_L}{2\pi T_L} \right) \right|^2 \left| \Gamma \left( h_R + i \frac{\omega_R}{2\pi T_R} \right) \right|^2 , \] (43)
which is precisely coincide with the the absorption cross section for a finite temperature 2D CFT.

C. Quasinormal modes

In this subsection, we will compute the quasinormal modes of scalar field perturbation by using the algebraic method firstly proposed by Sachs et al in [23] and compare the results with that presented in [27]. This method strongly depends on the observation of hidden
conformal symmetry in the last section. This method has also been employed to investigate the quasinormal modes of vector and tensor perturbation by Chen et al in [28].

Here, we consider the general case without imposing the small angular momentum condition (30). The radial equation of scalar field perturbation can be written using the SL(2, R) generators as

$$\left[ \frac{1}{2} (H_1 H_{-1} + H_{-1} H_1) - H_0^2 + \frac{3(\nu^2 - 1)}{4\nu^2} \tilde{H}_0^2 \right] \Phi = \frac{m^2}{\nu^2 + 3} \Phi . \quad (44)$$

Firstly, we consider the chiral highest weight modes satisfying the condition

$$H_1 \Phi = 0 . \quad (45)$$

Under the ansatz (13) for scalar field, this condition implies the equation

$$\frac{d}{dx} \ln R(x) = -\frac{i}{(x-x_+)(x-x_-)} \left( \frac{TR_k}{T_L} + \frac{[(x-x_+)+(x-x_-)]}{4\pi TR} \omega \right) , \quad (46)$$

which gives the solution

$$R(x) = (x-x_+)^{-i\left(\frac{\omega}{2\pi TR} + \frac{k}{2\pi TL}\right)}(x-x_-)^{-i\left(\frac{\omega}{2\pi TR} + \frac{k}{2\pi TL}\right)} . \quad (47)$$

Then the operator equation (44) can be transformed into an algebra equation

$$\left( \frac{\omega}{2\pi TR} \right)^2 + i \left( \frac{\omega}{2\pi TR} \right) - C = 0 . \quad (48)$$

The solution of this equation gives the lowest quasinormal mode in the right-moving sector

$$\frac{\omega}{2\pi TR} = -i h'_R , \quad h'_R = \frac{1}{2} + \sqrt{\frac{1}{4} - C} . \quad (49)$$

It should be noted that this conformal weight is slightly different from that given by Eq. (32). If the small angular momentum limit is taken into account, it recovers the conformal weight for scalar field given by Eq. (52). Then, the descendents of the chiral highest weight mode $\Phi$

$$\Phi^{(n)} = (H_{-1} \tilde{H}_1)^n \Phi , \quad (50)$$

give the infinite tower of quasinormal modes

$$\frac{\omega}{2\pi TR} = -i (n + h'_R) . \quad (51)$$

This is just the result obtained in [27, 28], and coincides with the poles in the retarded Green’s function obtained in [7].

The left sector of quasinormal scalar modes can not be obtained analogously. The solution constructed from the chiral highest weight condition $H_1 \Phi = 0$ falls off in time as well as at infinity, while for another highest weight condition $\tilde{H}_1 \Phi = 0$ one can not obtain the solution with the same property.
V. CONCLUSION

We have investigated the hidden conformal symmetry of self-dual warped AdS$_3$ black holes in topological massive gravity. The wave equation of massive scalar field propagating in this background with sufficient small angular momentum can be rewritten in the form of SL(2, R) Casimir operator. Interestingly, unlike the higher dimensional black holes where the near-horizon limit should be taken into account to match the wave equation with the Casimir operator, in the present case, only the condition of the small angular momentum of scalar field is imposed, which suggests that the hidden conformal symmetry is valid for the scalar field with arbitrary energy.

Despite the right temperature can not be directly read from the periodic identification in the $\phi$ direction, one can still conjecture that the self-dual warped AdS$_3$ black hole is dual to a 2D CFT with nonzero left and right temperatures. As a check of this conjecture, we also show that the entropy of the dual conformal field given by the Cardy formula matches exactly with the Bekenstein-Hawking entropy of self-dual warped AdS$_3$ black hole. Furthermore, the absorption cross section of scalar field perturbation calculated from the gravity side is in perfect match with that for a finite temperature 2D CFT. At last, an algebraic calculation of quasinormal modes for scalar field perturbation is presented and the correspondence between quasinormal modes and the poles in the retarded Green’s function is found.

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