SPARSE GENERAL WIGNER-TYPE MATRICES: LOCAL LAW AND EIGENVECTOR DELOCALIZATION

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Abstract. We prove a local law and eigenvector delocalization for general Wigner-type matrices. Our methods allow us to get the best possible interval length and optimal eigenvector delocalization in the dense case, and the first results of such kind for the sparse case down to \( p = \frac{\log n}{n} \) with \( g(n) \to \infty \). We specialize our results to the case of the Stochastic Block Model, and we also obtain a local law for the case when the number of classes is unbounded.

1. Introduction

1.1. The Stochastic Block Model. The Stochastic Block Model (SBM), first introduced by mathematical sociologists Holland, Laskey and Leinhardt [21], is a widely used random graph model for networks with communities. In the last decade, there has been considerable activity [11, 24, 1, 2, 10, 9, 8] in understanding the spectral properties of matrices associated to the SBM and to other generalized graph models, in particular in connection to spectral clustering methods.

Stochastic Block Models represent a generalization of Erdős-Rényi graphs to allow for more heterogeneity. Roughly speaking, an SBM graph starts with a partitioning of the vertices into classes, followed by placing an Erdős-Rényi graph on each class (independent edges, each occurring with the same given probability depending on the class), and connecting vertices in two different blocks by independent edges, again with the same given probability which this time depends on the pair of classes. The random matrix associated to this graph is the adjacency matrix, which is a random block matrix whose entries have Bernoulli distributions, the parameters of which are dictated by the inter- and intra-block probabilities mentioned above.

Specifically, suppose for ease of numbering that \( [n] = V_1 \cup V_2 \cup \ldots \cup V_d \) for some integer \( d \), \( |V_i| = N_i \) for \( i = 1, \ldots, d \). Suppose that for any pair \((k, l) \in [d] \times [d] \) with \( k \neq l \) there is a \( p_{kl} \in [0, 1] \) such that for any \( i \in V_k, j \in V_l \),

\[
a_{ij} = \begin{cases} 
1, & \text{with probability } p_{kl}, \\
0, & \text{otherwise}.
\end{cases}
\]

Also, if \( k = l \), there is a \( p_k \) such that for any \( i, j \in V_k \),

\[
a_{ij} = \begin{cases} 
0, & \text{if } i = j, \\
1, & \text{with probability } p_k, \\
0, & \text{otherwise.}
\end{cases}
\]

Each diagonal block is an adjacency matrix of a simple Erdős-Rényi graph and off-diagonal blocks are adjacency matrix of bipartite graphs. While there is interest in studying the \( O(1) \) variance case (corresponding to all \( p_{ij} \)s and \( p_i \)s being \( O(1) \); the “dense” case), special interest is given to the sparse case (when \( p_{ij} \)s and \( p_i \)s are \( o(1) \), and more specifically, when the average vertex degrees, given by \( np_{ij} \) as well as \( np_i \), are growing very slowly with \( n \) or may even be large and constant).

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The adjacency matrices of SBM graphs are themselves a particular form of general Wigner-type matrix, which have been shown to exhibit universal properties in [5] in the dense case. We detail the connection to the broader field of random matrix universality studies in the next section.

1.2. Universality studies, general Wigner-type matrices, and related graph-based matrix models. At the same time with the increased interest in the spectra of SBM, the universality studies in random matrix theory pioneered by Tao and Vu [29] and Erdős, Schlein, and Yau [15] had been gaining ground at tremendous pace. The pioneering work on Wigner matrices started in [29] and [15] has been now extended to cover generalized Wigner matrices [18], Erdős-Rényi matrices (including sparse ones) [14, 13, 16], and general Wigner-type matrices [5]. All such studies start by proving a local law at “optimal” scale, that is, on intervals of length $\log n/n$ or $n^{1-\varepsilon}$, which is necessary for the complicated machinery of either [29] or [17] to translate the local law into universality of eigenstatistics on “optimal-length” intervals.

In this paper we prove a main theorem about (dense) generalized Wigner matrices and then apply it to cover sparse generalized Wigner matrices; finally, we show that our results translate to graph models like the SBM with bounded or unbounded number of blocks. We provide below a brief review of universality studies related to graph-based models.

After the original work on Wigner matrices, the first step in the direction of graph models, or graph-based matrices, came with [31], where the authors proved a local law for Erdős-Rényi graphs. Subsequently, [14], [13] superseeded these results for the slightly denser cases and showed bulk universality in a $p \gg n^{-1/3}$ regime. The sparsity of the model is important here, because it makes the problem more difficult.

In a more recent paper, [23] refined the results of [14, 13] and made them applicable for $p \gg n^{-1+\varepsilon}$ for any (fixed) $\varepsilon > 0$. Subsequently, in a departure from studying adjacency matrices, [22] proved bulk universality for the eigenvalue statistics of Laplacian matrices of Erdős-Rényi graphs, in the regime when $p \gg n^{-1+\varepsilon}$ for fixed $\varepsilon > 0$.

Finally, [3] examined a large class of sparse random graph-based matrices (two-atom and three-atom entry distributions), proved a local law up to intervals of length $1/n^{1-\varepsilon}$ and deduced (by the same means employed in [23]) a bulk universality theorem. This is different from our results, since our sparse matrices have entries that are not necessarily atomic, but come from the product of a Bernoulli variable and a potentially continuous one (See Section 3.1); however, the [3] case does seem to cover the sparse SBM model for $p \gg n^{-1+\varepsilon}$. As an interesting aside, in the general case there may not be an asymptotic global law (aka the limiting empirical spectral distribution); the cases we study here (SBM with bounded and unbounded number of classes) are specific enough that we can also prove the asymptotic global law. However, as it turns out, in the case of the SBM with an unbounded number of blocks, the prediction in the local law must still be made using the $n$-step approximation to the global law, not the global law itself, since convergence to the global law is not uniform.

Some of the methods used for examining the universality of these graph models rely on the work in [5, 4], where a general (dense) Wigner-type matrix model is considered and universality is proved up to intervals of length $1/n^{1-\varepsilon}$. We will also appeal to [4], since it will help establish the existence of limiting distributions and the stability of their Stieltjes transforms.

We should mention the significant body of literature that deals with global limits for the empirical spectral distributions of block matrices. Starting with the seminal work of Girko [20], the topic was treated in [19] and [28] from a free probability perspective; more recently, [6] and [12] have examined the topic again for finitely many blocks (a claim in [12] that the method extends to a growing number of blocks is incorrect). The global law for stochastic block models with a growing number of blocks was derived in [34] via graphon theory.
1.3. This paper. The main difference in the results streaming from the seminal works of [29], respectively, [15], is in the conditions imposed on the matrix entries: the former approach to universality is based on the “four moment match” condition, but imposes relatively weak conditions on the tails, while the latter works by imposing stronger conditions on the tails. In later works, these stronger conditions have included bounded moments [18, 23, 5]. While the methods of [15] have been extended to increasingly more general matrix models, the methods in [29] have been used to focus on reaching the best (smallest) possible interval lengths for the classical, Wigner case via methods whose basis was set in [32].

This paper bridges the two approaches to obtain a local law and eigenvector delocalization for dense and sparse general Wigner-type matrices and for the SBM.

Our main result is a local law in the bulk down to interval length \( CK^2 \log n \) for general Wigner-type matrices (see Section 2) whose entries are compactly supported almost surely in \([-K, K]\), employing some of the ideas from [29] and [32]. Our result is more refined than the one from [5], where the smallest interval length was \( O(1/n^{1-\varepsilon}) \) and bounded moments were assumed. With additional assumptions (either four-moment matching, as in the case of [30], or finite moments, as in [15] and subsequent works) universality down to this smaller interval length should follow.

In addition to this main result, we also obtain the first local laws for sparse general Wigner-type matrices (see Section 3.1), down to interval length \( CK^2 \log n / np \). We specialize our results to sparse SBM with finite many blocks, where a limiting law exists. Finally, we extend these results to an unbounded number of blocks for the SBM, under certain conditions (see Section 3.2).

It should be said that our local laws for sparse general Wigner-type matrices are not sharp enough to yield universality, unless \( p \) is \( \omega(1/n^{1-\varepsilon}) \) for any \( \varepsilon > 0 \). This is an artifact of the use of the methods of [29] and is also observable in [31]. It is to be expected that they can be refined (by us or by other researchers) in the near future to a point where universality can be deduced.

2. General Wigner-type Matrices

Let \( M_n := (\xi_{ij})_{1 \leq i, j \leq n} \) be a random Hermitian matrix with variance profile \( S_n = (s_{ij})_{1 \leq i, j \leq n} \) such that \( \xi_{ij}, 1 \leq i \leq j \leq n \) are independent with

\[
\mathbb{E} \xi_{ij} = 0, \quad \mathbb{E} |\xi_{ij}|^2 = s_{ij},
\]

and compactly supported almost surely, i.e. \( |\xi_{ij}| \leq K \) for some \( K = o\left(\sqrt{\frac{n}{\log n}}\right) \).

For the variance profile \( S_n \), we assume

\[
c \leq s_{ij} \leq 1
\]

for some constant \( c > 0 \). Note this is equivalent to \( c \leq s_{ij} \leq 1 \) by scaling. Define \( W_n := \frac{M_n}{\sqrt{n}} \). The Stieltjes transform of the empirical spectral distribution of \( W_n \) is given by

\[
s_n(z) := \frac{1}{n} \text{tr}(W_n - zI)^{-1}.
\]

We will show that \( s_n(z) \) can be approximated by the solution of the following quadratic vector equation studied in [4]:

\[
m_n(z) = \frac{1}{n} \sum_{k=1}^{n} g_n^{(k)}(z),
\]

\[
-g_n^{(k)}(z) = z + \frac{1}{n} \sum_{l=1}^{n} s_{kl} g_n^{(l)}(z), \quad 1 \leq k \leq n.
\]
From Theorem 2.1 in [4], equation (2.2) has a unique set of solutions $g_n^{(k)}(z) : \mathbb{H} \to \mathbb{H}, 1 \leq k \leq n$, which are analytic functions on the complex upper half plane $\mathbb{H} := \{ z \in \mathbb{C} : \text{Im}(z) > 0 \}$. The unique solution $m_n(z)$ in equation (2.1) is the Stieltjes transform of a probability measure $\rho_n$ with $\text{supp}(\rho_n) \subset [-2, 2]$ such that

$$\rho_n(x) := \lim_{\eta \downarrow 0} \frac{1}{\pi} \text{Im}(m_n(x + i\eta)).$$

We use the following definition for bulk intervals of $\rho_n$.

**Definition 2.1.** An interval $I$ of a probability density function $\rho$ on $\mathbb{R}$ is a bulk interval if there exists some fixed $\varepsilon > 0$ such that $\rho(x) \geq \varepsilon$, for any $x \in I$.

We obtain the following local law of $M_n$ in the bulk.

**Theorem 2.2.** (Local law in the bulk) Let $M_n$ be a general Wigner-type matrix and $\rho_n$ be the probability measure corresponding to equations (2.1), (2.2). For any constant $\delta, C_1$, there exists a constant $C_2 > 0$ such that with probability at least $1 - n^{-C_1}$, the following holds. For any bulk interval $I$ of length $|I| \geq C_2 K^2 \log n n^{-\delta}$, the number of eigenvalues $N_I$ of $W_n$ in $I$ obeys the concentration estimate

$$\left| N_I - n \int_I \rho_n(x) dx \right| \leq \delta n |I|. \quad (2.3)$$

As a consequence, we obtained an optimal upper bound for eigenvectors that corresponds to eigenvalues of $W_n$ in the bulk interval.

**Theorem 2.3.** (Optimal delocalization of eigenvectors in the bulk) Let $M_n$ be a general Wigner-type matrix. For any constant $C_1 > 0$ and any bulk interval $I$ such that eigenvalue $\lambda_i(W_n) \in I$, with probability at least $1 - n^{-C_1}$, there is a constant $C_2$ such that the corresponding unit eigenvector $u_i(W_n)$ satisfies

$$\|u_i(W_n)\|_{\infty} \leq \frac{C_2 K \log^{1/2} n}{\sqrt{n}}.$$

**Remark 2.4.** Theorem 2.2 and Theorem 2.3 also hold for general Wigner-type matrix whose entries $\xi_{ij}$’s are sub-gaussian with sub-gaussian norm bounded by $K$. As mentioned in Remark 4.2 in [26], the proof follows in the same way by using the inequality in Theorem 2.1 in [27] for sub-gaussian concentration instead of Lemma 1.2 in [32] for $K$-bounded entries.

We use standard methods from [29], adapted to fit the model considered here.

2.1. **Proof of Main Results.**

2.1.1. **Proof of Theorem 2.2.** For any $0 < \varepsilon < \frac{1}{4}$, define a region

$$D_{n,\varepsilon} := \{ z \in \mathbb{C} : \rho_n(\text{Re}(z)) \geq \varepsilon, \text{Im}(z) \geq \frac{C_2^2 K^2 \log n}{n \delta^6} \} \quad (2.4)$$

for some sufficiently large constant $C_3$ (depending on $C_1$).

Let $W_{n,k}$ be the matrix $W_n$ with the $k$-th row and column removed, and $a_k$ be the $k$-th row of $W_n$ with the $k$-th element removed.

Let $(W_n - zI)^{-1} := (q_{ij}^{(n)})_{1 \leq i, j \leq n}$. From Schur’s complement lemma (Theorem A.4 in [7]), we have

$$q_{kk} = \frac{1}{\frac{-\xi_{kk}}{\sqrt{n}} - z - Y_k},$$
where
\[ Y_k = a_k^*(W_{n,k} - zI)^{-1}a_k. \]

Let
\[ f_n^{(k)}(z) := \frac{1}{-\frac{z_k}{\sqrt{n}} - z - Y_k}, \]
then we can write \( s_n(z) \) as,
\[ s_n(z) = \frac{1}{n} \text{tr}(W_n - zI)^{-1} = \frac{1}{n} \sum_{k=1}^n f_n^{(k)}(z). \]

We first estimate \( Y_k \) to derive a perturbed version of (2.2). Let \( (W_{n,k} - zI)^{-1} = (q_{ij}^{(n,k)})_{1 \leq i,j \leq n-1} \), and \( S_n^{(k)} \) be a diagonal matrix whose diagonal elements are the \( k \)-th row of \( S_n \) with the \( k \)-th entry removed. We have
\[ E[Y_k|W_{n,k}] = \frac{1}{n} \text{tr}[(W_n - zI)^{-1}S_n^{(k)}] + O\left(\frac{1}{n^\eta}\right), \]
where the constant in the \( O\left(\frac{1}{n^\eta}\right) \) term is independent of \( z \).

**Lemma 2.5.** Let \( \Sigma_n^{(k)} \) be the diagonal matrix whose diagonal elements are the \( k \)-th row of \( S_n \). For any \( k, 1 \leq k \leq n \), and any fixed \( z \) with \( \text{Im}(z) \geq \frac{K C_2^2 \log n}{n^6} \),
\[ E[Y_k|W_{n,k}] = \frac{1}{n} \text{tr}[(W_n - zI)^{-1}\Sigma_n^{(k)}] + O\left(\frac{1}{n^\eta}\right), \]
where the constant in the \( O\left(\frac{1}{n^\eta}\right) \) term is independent of \( z \).

A similar estimate holds for \( Y_k \) itself.

**Lemma 2.6.** For any \( k, 1 \leq k \leq n \) and any \( z \in D_{n,\varepsilon} \), one has
\[ Y_k - \frac{1}{n} \text{tr}[(W_n - zI)^{-1}\Sigma_n^{(k)}] = o(1) \]
with probability at least \( 1 - n^{-C-10} \), where the constant in the \( o(1) \) term is independent of \( z \).

With the help of Lemmas 2.5 and 2.6, note that, since \( \frac{|s_{kl}|}{\sqrt{n}} = o(1) \),
\[ \frac{1}{n} \text{tr}[(W_n - zI)^{-1}\Sigma_n^{(k)}] = \frac{1}{n} \sum_{l=1}^n s_{kl}f_n^{(l)}, \]
and combining (2.5), (2.8), (2.9), we have
\[ f_n^{(k)}(z) + \frac{1}{n} \sum_{l=1}^n s_{kl}f_n^{(l)}(z) + z + o(1) = 0, \quad 1 \leq k \leq n \]
with probability at least \( 1 - n^{-C-9} \).
The next step involves using the stability analysis of quadratic vector equations provided in [4] to compare the solutions to (2.10) and (2.2). We have the following estimate.

**Lemma 2.7.**

\begin{equation}
\sup_{1 \leq k \leq n} |f_n^{(k)}(z) - g_n^{(k)}(z)| = o(1),
\end{equation}

for all \( z \in D_{n,\varepsilon} \) uniformly with probability at least \( 1 - n^{-C-2} \).

Therefore

\begin{equation}
|s_n(z) - m_n(z)| = \left| \frac{1}{n} \sum_{k=1}^{n} f_n^{(k)}(z) - \frac{1}{n} \sum_{k=1}^{n} g_n^{(k)}(z) \right| = o(1)
\end{equation}

uniformly for all \( z \in D_{n,\varepsilon} \) with probability at least \( 1 - n^{-C} \).

To complete the proof of Theorem 2.2, we need the following well-known connection between the Stieltjes transform and empirical spectral distribution, shown for example in Lemma 64 in [29].

**Lemma 2.8.** Let \( M_n \) be a general Wigner-type matrix. Let \( 1/n < \eta < 1/10 \) and \( L, \varepsilon, \delta > 0 \). For any constant \( C_1 > 0 \), there exists a constant \( C > 0 \) such that suppose that one has the bound

\[ |s_n(z) - m_n(z)| \leq \delta \]

with probability at least \( 1 - n^{-C} \) uniformly for all \( z \in D_{n,\varepsilon} \), then for any bulk interval \( I \), one has

\[ \left| N_I - n \int_I \rho_n(x) dx \right| \leq \delta n |I| \]

with probability at least \( 1 - n^{-C_1} \).

2.1.2. **Proof of Theorem 2.3.** The proof is based on Lemma 41 from [29] given below.

**Lemma 2.9.** Let \( W_n \) be a \( n \times n \) Hermitian matrix, and \( W_{n,k} \) be the submatrix of \( W_n \) with \( k \)-th row and column removed, and let \( u_i(W_n) \) be a unit eigenvector of \( W_n \) corresponding to \( \lambda_i(W_n) \), and \( x_k \) be the \( k \)-th coordinate of \( u_i(W_n) \). Suppose that none of the eigenvalues of \( W_{n,k} \) are equal to \( \lambda_i(W_n) \). Let \( a_k \) be the \( k \)-th row of \( W_n \) with of \( k \)-th entry removed; then

\begin{equation}
|x_k|^2 = \frac{1}{1 + \sum_{j=1}^{n-1} (\lambda_j(W_{n,k}) - \lambda_i(W_n))^{-2} |u_j(W_{n,k})^* a_k|^2},
\end{equation}

where \( u_j(W_{n,k}) \) is a unit eigenvector corresponding to \( \lambda_j(W_{n,k}) \).

Another lemma we need is a weighted projection lemma for random vectors with different variances. It is a slight generalization of Lemma 1.2 in [32]. Note that in the below

\[ \mathbb{E}|u_j^* X|^2 = \text{tr}(u_j u_j^* \Sigma), \]

and the proof follows verbatim as in [32].

**Lemma 2.10.** Let \( X = (\xi_1, \ldots, \xi_n) \) be a \( K \)-bounded random vector in \( \mathbb{C}^n \) such that \( \text{Var}(\xi_i) = \sigma_i^2 \), \( 0 \leq \sigma_i^2 \leq 1 \). Then there are constants \( C, C' > 0 \) such that the following holds. Let \( H \) be a subspace of dimension \( d \) with an orthonormal basis \( \{u_1, \ldots, u_d\} \), and \( \Sigma = \text{diag}(\sigma_1^2, \ldots, \sigma_n^2) \). Then for any \( 1 \leq r_1, \ldots, r_d \geq 0 \),

\begin{equation}
P \left( \left| \sum_{j=1}^{d} r_j |u_j^* X|^2 - \sum_{j=1}^{d} r_j \text{tr}(u_j u_j^* \Sigma) \right| \geq t \right) \leq C \exp(-C' t^2/(K^2)).
\end{equation}
In particular, by squaring, it follows that

\begin{equation}
\sum_{j=1}^{d} r_j |u_j^* X|^2 - \sum_{j=1}^{d} r_j \text{tr}(u_j u_j^* \Sigma) \geq 2t \left[ \sum_{j=1}^{d} r_j \text{tr}(u_j u_j^* \Sigma) + t^2 \right] \leq C \exp(-C' \frac{t^2}{K^2}).
\end{equation}

Below we show how delocalization follows from Lemma 2.9, Lemma 2.10 and Theorem 2.2. For any $C_1 > 0$ and any $\lambda_i(W_n)$ in the bulk, by Theorem 2.2, one can find an interval $J$ centered at $\lambda_i(W_n)$ and $|I| = \frac{K^2 C_1 \log n}{n}$ for some sufficiently large $C_2$ such that $N_I \geq \delta_1 n|I|$ for some small $\delta_1 > 0$ with probability at least $1 - n^{-C_1 - 3}$. By Cauchy interlacing law, we can find a set $J \subset \{1, \ldots, n - 1\}$ with $|J| \geq N_I/2$ such that $|\lambda_j(W_{n,k}) - \lambda_i(W_n)| \leq |I|$ for all $j \in J$. Let $X_k$ be the $k$-th column of $M_n$ with the $k$-th entry removed. Note that from Lemma 2.10, by taking $r_j = 1, j \in J$, and $t = C_3 K \sqrt{\log n}$ for $C_3 \geq \frac{C_1 + 3}{2 C_1}$ in (2.4), using assumption $s_{ij} \geq c$, we have

\begin{equation}
\sum_{j \in J} |u_j(W_{n,k})^* X_k|^2 \geq \sqrt{\sum_{j \in J} \text{tr}(u_j(W_{n,k}) u_j^*(W_{n,k}) \Sigma) - C_3 K \sqrt{\log n}}
\geq \sqrt{c|J| - C_3 K \sqrt{\log n}}
\geq (\sqrt{c - \frac{C_3}{\sqrt{2} \delta_1 / 2}}) \sqrt{|J|}
\end{equation}

with probability at least $1 - n^{-C_1 - 3}$. By choosing $C_2$ sufficiently large, (2.16) implies

$$\sum_{j \in J} |u_j(W_{n,k})^* X_k|^2 \geq C'|J|$$

for some constant $C' > 0$ with probability at least $1 - n^{-C_1 - 3}$. By (2.13),

$$|x_k|^2 = \frac{1}{1 + \sum_{j=1}^{n-1} (\lambda_j(W_{n,k}) - \lambda_i(W_n))^{-2} |u_j(W_{n,k})^* X_k|^2}
\leq \frac{1}{1 + \sum_{j \in J} (\lambda_j(W_{n,k}) - \lambda_i(W_n))^{-2} |u_j(W_{n,k})^* X_k|^2}
\leq \frac{1}{1 + n^{-1}|I|^{-2} \sum_{j \in J} |u_j(W_{n,k})^* X_k|^2}
\leq \frac{1}{1 + n^{-1}|I|^{-2} C' |J|}
\leq \frac{2|I|}{C^5 C_4^2 \log n}
\leq \frac{K^2 C_1 \log n}{n}
$$

for some constant $C_4$ with probability at least $1 - 2n^{-C_1 - 3}$. Thus by taking a union bound, $\|u_i\|_\infty \leq \frac{C_4 K^2 \log n}{n \delta^6}$ with probability at least $1 - n^{-C_1}$ for all $1 \leq i \leq n$.

2.2. Proof of Auxiliary Lemmas. We now prove the all the lemmas in the proof of Theorem 2.2.

2.2.1. Proof of Lemma 2.5. Let $\eta := \frac{C_3^2 K^2 \log n}{n \delta^6}$ and $z := x + \sqrt{-1} \cdot \eta$. By (2.7), it suffices to show for all $1 \leq k \leq n$,

\begin{equation}
|\text{tr}[(W_n - z I)^{-1} S_n^{(k)}] - \text{tr}[(W_{n,k} - z I)^{-1} S_n^{(k)}]| \leq \frac{1}{\eta}.
\end{equation}

We will use the following result known as Lemma 1.1 in Chapter 1 of [20].
Lemma 2.11. Let \( c = (c_1, \ldots, c_n) \) be a real column vector, and \( M_n = (\xi_{ij})_{n \times n} \) be a Hermitian matrix, for any \( z \) with and \( \Im z > 0 \), we have, for any \( 1 \leq k \leq n \),
\[
\hat{c}^T(M_n - zI)^{-1}\hat{c} - \hat{c}_k^T(M_{n,k} - zI)^{-1}\hat{c}_k = \frac{c_k^2 - \hat{\xi}_k^* R_k(2c_k\hat{c}_k^*) + \hat{\xi}_k^* R_k\hat{c}_k\hat{c}_k^TR_k\hat{\xi}_k}{\xi_{kk} - z - \hat{\xi}_k^* R_k\hat{\xi}_k}
\]
where \( R_k = (M_{n,k} - zI)^{-1}, \hat{c}_k \) is the vector \( \hat{c} \) with the \( k \)-th coordinate removed, and \( \hat{\xi}_k \) is the \( k \)-th column of \( M_n \) with the \( k \)-th element removed.

We introduce a real random vector \( \vec{c} = (c_1, \ldots, c_n) \) whose coordinates are mean zero, independent variables also independent of \( W_n \) with \( \Var(c_i) = s_{ki} \) for \( 1 \leq i \leq n \).

Apply Lemma 2.11 to \( W_n \) and \( \vec{c} \). We have \( R_k = (W_{n,k} - zI)^{-1}, \) and
\[
\hat{c}^T(W_n - zI)^{-1}\hat{c} - \hat{c}_k^T(W_{n,k} - zI)^{-1}\hat{c}_k = \frac{c_k^2 - a_k^*R_k(2c_k\hat{c}_k^*) + a_k^*R_k\hat{c}_k\hat{c}_k^TR_ka_k}{\xi_{kk} - z - Y_k}.
\]
By taking the conditional expectation with respect to \( c \), conditioned on \( W_n \), we have
\[
\mathbb{E}[\hat{c}^T(W_n - zI)^{-1}\hat{c} - \hat{c}_k^T(W_{n,k} - zI)^{-1}\hat{c}_k | W_n] = \frac{s_{kk} + a_k^*R_kS_n^{(k)}R_ka_k}{\xi_{kk} - z - Y_k}.
\]
Calculating the left hand side yields
\[
\text{tr}[(W_n - zI)^{-1}\Sigma_n^{(k)}] - \text{tr}[(W_{n,k} - zI)^{-1}S_n^{(k)}] = \frac{s_{kk} + a_k^*R_kS_n^{(k)}R_ka_k}{\xi_{kk} - z - Y_k}.
\]
Since we have
\[
|s_{kk} + a_k^*R_kS_n^{(k)}R_ka_k| \leq 1 + |a_k^*R_kS_n^{(k)}R_ka_k| \\
\leq 1 + a_k^*((W_{n,k} - xI)^2 + \eta^2I)^{-1}a_k,
\]
and
\[
\Im \left( \frac{\xi_{kk} - z - Y_k}{\sqrt{n}} \right) = -\eta \left( 1 + a_k^*((W_{n,k} - xI)^2 + \eta^2I)^{-1}a_k \right),
\]
(2.17) holds. This completes the proof of Lemma 2.5.

2.2.2. Proof of Lemma 2.6. We need a preliminary bound on the number of eigenvalues in a short interval. The following Lemma is similar to Proposition 66 in [29].

Lemma 2.12. For any constant \( C_1 > 0 \), there exists a constant \( C_2 > 0 \) such that for any interval \( I \subset \mathbb{R} \) with \( |I| \geq \frac{C_2K^2\log n}{n} \), one has
\[
N(I(W_n)) = O(n|I|)
\]
with probability at least \( 1 - n^{-C_1} \).

Proof. By the union bound, it suffices to show that the failure probability for (2.18) is less than \( 1 - n^{-C_1} \) for
\[
|I| = \eta := \frac{C_2K^2\log n}{n}
\]
for some sufficiently large \( C_2 \). By
\[
\Im(s_n(x + \sqrt{-1}\eta)) = \frac{1}{n} \sum_{i=1}^{n} \frac{\eta}{\eta^2 + (\lambda_i(W_n) - x)^2},
\]
(2.19)
it suffices to show that the event
\[(2.20) \quad N_I \geq Cn\eta \]
and
\[(2.21) \quad \text{Im}(s_n(x + \eta\sqrt{-1})) \geq C \]
fails with probability at least \((1 - n^{-C_1-1})\) for some large absolute constant \(C > 1\). Suppose we have (2.20), (2.21), by (2.19),
\[\frac{1}{n} \sum_{k=1}^{n} \left| \text{Im} \left( \frac{1}{z_k - \eta + Y_k} \right) \right| \geq C.\]
Using the bound \(\left| \text{Im} \left( \frac{1}{z} \right) \right| \leq \frac{1}{|\text{Im}(z)|}\), it implies
\[(2.22) \quad \frac{1}{n} \sum_{k=1}^{n} \left| \eta + \text{Im}(Y_k) \right| \geq C.\]

Note that
\[W_{n,k} = \sum_{j=1}^{n-1} \lambda_j(W_{n,k}) u_j^*(W_{n,k}) u_j(W_{n,k}),\]
where \(u_j(W_{n,k}), 1 \leq j \leq n - 1\) are orthonormal basis of \(W_{n,k}\), one has
\[Y_k = a_k^*(W_{n,k} - zI)^{-1} a_k = \sum_{j=1}^{n-1} \frac{|u_j^*(W_{n,k}) a_k|^2}{\lambda_j(W_{n,k}) - (x + \eta\sqrt{-1})}\]
and hence
\[\text{Im}(Y_k) \geq \eta \sum_{j=1}^{n-1} \frac{|u_j^*(W_{n,k}) a_k|^2}{\eta^2 + (\lambda_j(W_{n,k}) - x)^2}.\]

On the other hand, from (2.20), by Cauchy interlacing theorem, we can find an index set \(J\) with \(|J| \geq \eta n\) such that \(\lambda_j(W_{n,k}) \in I\) for all \(j \in J\), then we have
\[(2.23) \quad \text{Im}(Y_k) \geq \frac{1}{2\eta} \sum_{j \in J} |u_j^*(W_{n,k}) a_k|^2 = \frac{1}{2\eta} \|P_{H_k} a_k\|^2,\]
where \(P_{H_k}\) is the orthogonal projection onto a subspace \(H_k\) spanned by eigenvectors \(u_j(W_{n,k}), j \in J\).
From (2.22), (2.23), we have
\[(2.24) \quad \frac{1}{n} \sum_{k=1}^{n} \frac{2\eta}{2\eta^2 + \|P_{H_k} a_k\|^2} \geq C.\]
On the other hand, taking \(r_j = 1, 1 \leq j \leq d, d = |J|\) and \(t = C_3 K \sqrt{\log n}\) for some sufficiently large \(C_3\) in (2.15), using assumption \(s_{ij} \geq c\), we have that \(\|P_{H_k}(a_k)\|^2 = \Omega(\eta)\) with probability at least \(1 - O(n^{-C_3}) \geq 1 - n^{-C_1-5}\). Taking the union bound over all possible choice of \(J\), we have (2.24) holds with probability at least \(1 - n^{-C_1-1}\). The claim then follows by taking \(C\) sufficiently large.

Now we are ready to prove Lemma 2.6. From Lemma 2.5, it suffices to show
\[(2.25) \quad Y_k - \mathbb{E}[Y_k | W_{n,k}] = o(1), \quad 1 \leq k \leq n.\]
We can write
\[ Y_k = \sum_{j=1}^{n-1} \frac{|u_j^*(W_{n,k})a_k|^2}{\lambda_j(W_{n,k}) - z}, \]
where \(\{u_j(W_{n,k})\}_{j=1}^{n-1}\) are orthonormal eigenvectors of \(W_{n,k}\). Moreover,
\[
\mathbb{E}[Y_k|W_{n,k}] = \frac{1}{n} \text{tr}[(W_{n,k} - zI)^{-1}S_n^{(k)}] \\
= \frac{1}{n} \text{tr} \left[ \sum_{j=1}^{n-1} \frac{1}{\lambda_j(W_{n,k}) - z} u_j(W_{n,k})u_j^*(W_{n,k})S_n^{(k)} \right] \\
= \frac{1}{n} \sum_{j=1}^{n-1} \frac{\text{tr}[u_j(W_{n,k})u_j^*(W_{n,k})S_n^{(k)}]}{\lambda_j(W_{n,k}) - z}.
\]

Let \(X_k = \sqrt{na_k}\), and define
\[
t_j := |u_j(W_{n,k})^*X_k|^2 - \text{tr}[u_j(W_{n,k})u_j^*(W_{n,k})S_n^{(k)}].
\]

It suffices to show that
\[
|Y_k - \mathbb{E}[Y_k|W_{n,k}]| = \frac{1}{n} \sum_{j=1}^{n-1} \frac{t_j}{\lambda_j(W_{n,k}) - x - \sqrt{-1}y} = o(1).
\]

The remaining part of the proof goes through in the same way as in the proof of Lemma 5.2 in [32] with Lemma 2.10 and Lemma 2.12. Then Lemma 2.6 follows. \(\square\)

2.2.3. **Proof of Lemma 2.7.** We define \(g_n(z, x) := g_n^{(k)}(z)\) if \(x \in \left[\frac{k-1}{n}, \frac{k}{n}\right], 1 \leq k \leq n\) and
\[
S_n(x, y) := s_{ij} \quad \text{if} \quad x \in \left[\frac{i-1}{n}, \frac{i}{n}\right], y \in \left[\frac{j-1}{n}, \frac{j}{n}\right).
\]

Then (2.2) can be written as
\[
m_n(z) = \int_0^1 g_n(z, x)dx,
\]
\[
-\frac{1}{g_n(z, x)} = z + \int_0^1 S_n(x, y)g_n(z, y)dy,
\]
for all \(x \in [0, 1]\). Similarly, define \(f_n(z, x) := f_n^{(k)}(z)\) if \(x \in \left[\frac{k-1}{n}, \frac{k}{n}\right], 1 \leq k \leq n\). Then we can write (2.6) and (2.10) as
\[
s_n(z) = \int_0^1 f_n(z, x)dx \\
-\frac{1}{f_n(z, x)} = z + \int_0^1 S_n(x, y)f_n(z, y)dy + d_n(z, x),
\]
where, for any fixed \(z\) from (2.10),
\[
\|d_n(z)\|_\infty := \sup_{x \in [0,1]} |d_n(z, x)| = o(1)
\]
with probability at least \(1 - n^{-C-9}\) for any fixed \(z \in D_{n,\varepsilon}\).
The following lemma follows from Theorem 2.12 in [4] which controls the stability of equation (2.28) in the bulk. Here we use the fact that $c \leq s_{ij} \leq 1$ to guarantee the assumptions of $S_n$ in Theorem 2.12 in [4]. Define

$$
\Lambda(z) := \sup_{x \in [0,1]} |f_n(z, x) - g_n(z, x)|.
$$

**Lemma 2.13.** For any fixed $z \in D_{n, \varepsilon}$, there exist constants $\lambda, C_5 > 0$ depending on $\varepsilon$ but independent of $n$ such that for $z \in D_{n, \varepsilon}$,

$$
\Lambda(z) 1\{\Lambda(z) \leq \lambda\} \leq C_5 \|d_n(z)\|_\infty.
$$

**Proof.** Since the variance satisfies $c \leq s_{ij} \leq 1$, we have for any bulk interval $I$, from the Proof of Corollary 1.8 in [5], there exists some constant $P > 0$ independent of $n$ such that

$$
\sup_{1 \leq k \leq n} |g_n^{(k)}(z)| \leq P,
$$

for any $z$ with $\operatorname{Re}(z) \in I$, $\operatorname{Im}(z) > 0$. Then Corollary 1.8 in [5] remains true for any $z$ with $\operatorname{Re}(z) \in I$, $\operatorname{Im}(z) > 0$. \qed

From Lemma 2.13 and (2.29), we have for any fixed $z \in D_{n, \varepsilon}$,

(2.30)

$$
\Lambda(z) 1\{\Lambda(z) \leq \lambda\} = o(1)
$$

with probability at least $1 - n^{-C-9}$.

We proceed with a continuity argument as in the proof of Theorem 3.2 in the bulk (Section 3.1 in [4]) to show (2.30) holds uniformly for $z \in D_{n, \varepsilon}$ with probability at least $1 - n^{-C-2}$.

Now for any $0 < \varepsilon' < \frac{\lambda}{4}$, we consider a line segment

$$
L = x + \sqrt{-1} \left[ \frac{K^2 C_2^2 \log n}{n \delta_n}, n \right]
$$

for some fixed $x$ with $\rho_n(x) \geq \varepsilon, 0 < \varepsilon < 1/2$, and let $n$ be large enough such that $\frac{1}{n} < \varepsilon'$ and $\|d_n(z)\|_\infty \leq \varepsilon'$. Let $L_n$ consist of $n^4$ evenly spaced points on $L$. Then we have

(2.31)

$$
\Lambda(z) 1\{\Lambda(z) \leq \lambda\} \leq \varepsilon'
$$

for all $z \in L_n$ with probability at least $1 - n^{-C-5}$.

From Theorem 2.1 in [4], $g_n(z, x)$ is the Stieltjes transform of a probability measure, hence the derivative of $g_n(z, x)$ is uniformly bounded by

$$
\frac{1}{|\operatorname{Im}(z)|^2} \leq n^2
$$

for $z \in D_{n, \varepsilon}$. Similarly, for $f_n(z, x)$, from (2.5), for $1 \leq k \leq n,

$$
\left| \frac{\partial f_n^{(k)}(z)}{\partial z} \right| = \left| \frac{1 + \frac{\partial \gamma_k}{\partial z}}{\left( \frac{\xi_{nk}}{\sqrt{n}} - z - Y_k \right)^2} \right| \leq \frac{1 + a_k^* (W_{n,k} - z I)^{-2} a_k}{\left| \frac{\xi_{nk}}{\sqrt{n}} - z - Y_k \right|} \frac{1}{\left| \frac{\xi_{nk}}{\sqrt{n}} - z - Y_k \right|}.
$$

By theorem A.6. in [7], for $z = x + \sqrt{-1} \eta$,

$$
\left| \frac{1 + a_k^* (W_{n,k} - z I)^{-2} a_k}{\left( \frac{\xi_{nk}}{\sqrt{n}} - z - Y_k \right)} \right| \leq \frac{1}{\eta},
$$

and

$$
\left| \frac{\xi_{nk}}{\sqrt{n}} - z - Y_k \right| \geq \left| \operatorname{Im}(\frac{\xi_{nk}}{\sqrt{n}} - z - Y_k) \right| = \eta(1 + a_k^* ((W_{n,k} - x I)^2 + \eta^2 I)^{-1} a_k) \geq \eta,
$$

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Note that for \( z \in D_{n, \varepsilon}, \eta \geq \frac{K^2 C^2 \log n}{n b} \geq \frac{1}{n}, \) we get

\[
\left| \frac{\partial f_n^{(k)}(z)}{\partial z} \right| \leq \frac{1}{n^2} \leq n^2, \quad 1 \leq k \leq n.
\]

So both \( f_n(z, x) \) and \( g_n(z, x) \) are \( n^2 \)-Lipschitz functions in \( z \) for \( z \in D_{n, \varepsilon} \). It follows that

\[
|\Lambda(z') - \Lambda(z)| \leq 2n^2 |z' - z|,
\]

for any \( z, z' \in L \). We first claim that

\[
|\Lambda(z') - \Lambda(z)| \leq \frac{2}{n}, \quad (2.33)
\]

which implies

\[
\Lambda(z') \leq \Lambda(z) + \frac{2}{n} < 2 \varepsilon',
\]

with probability at least \( 1 - n^{-C-5} \). From \((2.31)\), \( \Lambda(z') \leq \varepsilon' \) with probability at least \( 1 - n^{-C-5} \). From \((2.33)\),

\[
\Lambda(z) \leq \Lambda(z') + \frac{2}{n} < 2 \varepsilon', \quad (2.34)
\]

with probability at least \( 1 - n^{-C-5} \), therefore \((2.32)\) holds.

In the next step we show that the indicator function in \((2.32)\) is identically equal to 1. From \((2.31)\) we have \( \Lambda(z) \notin (2 \varepsilon', \lambda/2) \) with probability at least \( 1 - n^{-C-5} \).

Let \( E \) be the event that \( \Lambda(z) 1 \{ \Lambda(z) \leq \frac{\lambda}{2} \} \leq 2 \varepsilon' \) happens. Conditioning on \( E \), since \( \Lambda(z) \) is \( 2n^2 \)-Lipschitz in \( z \), and \( L \) is simply connected, we have

\[
\Lambda(L) := \{ \Lambda(z) : z \in L \}
\]

is simply connected. Therefore \( \Lambda(L) \) is contained either in \([0, 2\varepsilon']\) or \([\frac{\lambda}{2}, \infty)\).

From \((2.5)\) we have for \( 1 \leq k \leq n, \)

\[
|f_n^{(k)}(z)| = \frac{1}{|\frac{\varepsilon}{\sqrt{n}} - z - Y_k|} \leq \frac{1}{|\text{Im}(z)|},
\]

and since \( g_n^{(k)}(z) \) is a Stieltjes transform of a probability measure, for \( 1 \leq k \leq n, \)

\[
|g_n^{(k)}(z)| \leq \frac{1}{|\text{Im}(z)|},
\]

which implies

\[
\Lambda(z) \leq \frac{2}{|\text{Im}(z)|}.
\]

Consider the point \( z_n := x + \sqrt{-1} \cdot n \in L \), we have

\[
\Lambda(z_n) \leq \frac{2}{\text{Im}(z_n)} = \frac{2}{n} \leq 2 \varepsilon',
\]

which is equal to \( 2 \varepsilon' \).
which implies $\Lambda(z_n) \in [0, 2\varepsilon']$. Hence for all $z \in L$, $\Lambda(z) \leq 2\varepsilon'$ with probability at least $1 - n^{-C-5}$, and the indicator function in (2.32) is identically equal to 1.

Now we extend the estimate to all $z \in D_{n,\varepsilon}$. Consider $n^3$ lines segments
\[
x_k + \sqrt{-1} \left[ \frac{K^2 C_4^2 \log n}{n \delta^n}, n \right], \quad \rho_n(x_k) \geq \varepsilon, 1 \leq k \leq n^3
\]
such that the $n^2$-neighborhoods of points $\{x_k, 1 \leq k \leq n^3\}$ cover any bulk interval of $\rho_n$. By the $2n^2$-Lipschitz property of $\Lambda(z)$ again, we can show $\Lambda(z) \leq 4\varepsilon'$ for all $z$ with $\rho_n(\Re(z)) > \varepsilon$, $\frac{K^2 C_4^2 \log n}{n \delta^n} \leq \Im z \leq n$, with probability at least $1 - n^{-C-2}$.

On the other hand, for all $z$ with $\Im(z) > n$,
\[
\|f_n(z) - g_n(z)\|_\infty \leq \frac{2}{\Im z} = O \left( \frac{1}{n} \right).
\]
Combining these two cases, for all $z \in D_{n,\varepsilon}$ with probability at least $1 - n^{-C-2}$,
\[
\|f_n(z) - g_n(z)\|_\infty = o(1).
\]
This completes the proof of Lemma 2.7.

3. Applications: Sparse Matrices

3.1. Sparse General Wigner-type Matrices. Let $M_n$ be a sparse general Wigner-type matrix with independent entries $M_{ij} = \delta_{ij} \xi_{ij}$ for $1 \leq i \leq j \leq n$. Here $\delta_{ij}$ are i.i.d. Bernoulli random variables which take value 1 with probability $p = \frac{g(n) \log n}{n}$, where $g(n)$ is any function for which $g(n) \to \infty$ as $n \to \infty$, and $\xi_{ij}$ are independent random variables such that
\[
\mathbb{E} \xi_{ij} = 0, \quad \mathbb{E} |\xi_{ij}|^2 = s_{ij}, \quad c \leq s_{ij} \leq 1,
\]
and in addition, $|\xi_{ij}| \leq K$ almost surely for $K = o(\sqrt{g(n)})$.

We can regard this model as the sparsification of a general Wigner-type matrix by uniform sampling. Similar models were considered in [33, 25]. Considering the empirical spectral distribution of $W_n := \frac{M_n}{\sqrt{n \rho}}$, we specify a local law for this model.

**Corollary 3.1.** Let $M_n$ be a sparse general Wigner-type matrix, let $\rho_n$ be the probability measure corresponding to equations (2.1), (2.2). For any constants $\delta, C_1 > 0$, there exists a constant $C_2 > 0$ such that with probability at least $1 - n^{-C_1}$, the following holds. For any bulk interval $I$ of length $|I| \geq \frac{C_2 K^2 \log n}{n} n_p$, the number of eigenvalues $N_I$ of $W_n := \frac{M_n}{\sqrt{n \rho}}$ in $I$ obeys the concentration estimate
\[
|N_I - n \int_I \rho_n(x) dx | \leq \delta n |I|.
\]

**Proof.** Define
\[
H_n := \frac{M_n}{\sqrt{p}} = (h_{ij})_{1 \leq i, j \leq n}.
\]
Then $\mathbb{E} h_{ij} = 0$, $\mathbb{E} |h_{ij}|^2 = s_{ij}$, and $|h_{ij}| \leq \frac{K}{\sqrt{p}} = o \left( \sqrt{\frac{n}{\log n}} \right)$. (3.1) follows as a corollary of Theorem 2.2 for $H_n$.

The infinity norm of eigenvectors in the bulk can be estimated in a similar way.
Corollary 3.2. Let $M_n$ be a sparse general Wigner-type matrix and $W_n = \frac{M_n}{\sqrt{np}}$. For any constant $C_1 > 0$ and any bulk interval $I$ such that eigenvalue $\lambda_i(W_n) \in I$, with probability at least $1 - n^{-C_1}$, there is a constant $C_2$ such that the corresponding unit eigenvector $u_i(W_n)$ satisfies

$$\|u_i(W_n)\|_\infty \leq \frac{C_2 K \log^{1/2} n}{\sqrt{np}}.$$  

3.2. Sparse Stochastic Block Models.

3.2.1. Finite Number of Classes. Our analysis of sparse random matrices applies to the adjacency matrices of sparse stochastic block models.

Consider the adjacency matrix $A_n = (a_{ij})_{1 \leq i, j \leq n}$ of an SBM graph, where $A_n$ is a random real symmetric block matrix with $d^2$ blocks. Recall that we partition all indices $\mathbb{N}$ into $d$ sets, $[n] = V_1 \cup V_2 \cup \ldots \cup V_d$ (3.2) such that $|V_i| = N_i$. We assume $a_{ii} = 0$, $1 \leq i \leq n$, and $a_{ij}$, $i \neq j$ are Bernoulli random variables such that if $a_{ij}$ is in the $(k, l)$-th block, $a_{ij} = 1$ with probability $p_{kl}$ and $a_{ij} = 0$ with probability $1 - p_{kl}$.

Let $\sigma_{kl}^2 := p_{kl}(1 - p_{kl})$. Define $p := \max_{kl} p_{kl}$ and $\sigma^2 = p(1 - p)$. Assume

$$p = \frac{g(n) \log n}{n},$$

where $\sup_n p < 1$ and $g(n) \to \infty$ as $n \to \infty$. We also assume that

$$\frac{N_i}{n} = \alpha_i + o \left( \frac{1}{g(n)} \right),$$

$$\frac{\sigma_{kl}^2}{\sigma^2} = c_{kl} + o \left( \frac{1}{g(n)} \right),$$

where $\alpha_i > 0$, $1 \leq i \leq d$ and $c_{kl} \geq c > 0$, $1 \leq k, l \leq d$ for some constant $c$. The quadratic vector equation becomes

$$m(z) = \sum_{k=1}^{d} \alpha_k g_k(z)$$

$$- \frac{1}{g_k(z)} = z + \sum_{l=1}^{d} \alpha_l c_{kl} g_l(z).$$

(3.5)

(3.6)

We state the following local law for sparse SBM.

Corollary 3.3. Let $A_n$ be the adjacency matrix of a stochastic block model with the assumptions above, let $\rho$ be the probability measure corresponding to equation (3.5). For any constant $\delta, C_1 > 0$, there exists a constant $C_2 > 0$ such that with probability at least $1 - n^{-C_1}$, the following holds. For any bulk interval $I$ of length $|I| \geq \frac{C_2 \log n}{np}$, the number of eigenvalues $N_I$ of $\frac{A_n}{\sqrt{np}}$ in $I$ obeys the concentration estimate

$$\left| N_I - n \int_I \rho(x) dx \right| \leq \delta n |I|.$$

Proof. We have the following well-known Cauchy Interlacing Lemma, appearing for example, as Lemma 36 from [29].
Lemma 3.4. Let $A, B$ be symmetric matrices with the same size and $B$ has rank 1. Then for any interval $I$, we have

$$|N_I(A + B) - N_I(B)| \leq 1. \tag{3.7}$$

where $N_I(M)$ is the number of eigenvalues of $M$ in $I$.

Let $\tilde{A}_n$ be the matrix whose off diagonal entries are equal to $A_n$ and

$$\tilde{a}_{ii} = p_{kk} \tag{3.8}$$

if $(i, i)$ is in the $k$-th block.

From Lemma 3.4, since rank $\mathbb{E}(\tilde{A}_n) = d$, we have

$$|N_I(A_n) - N_I(A_n - \mathbb{E}(\tilde{A}_n))| \leq d = o(n |I|).$$

Therefore it suffices to prove the local law for

$$W_n = \frac{A_n - \mathbb{E} \tilde{A}_n}{\sqrt{n} \sigma}.$$ 

Let $\frac{A_n - \mathbb{E} \tilde{A}_n}{\sigma} = (\xi_{ij})_{1 \leq i, j \leq n}$. By Schur’s complement, we can write the Stieltjes transform of the empirical measure $s_n(z)$ in the following way,

$$s_n(z) = \frac{1}{n} \sum_{k=1}^{n} \frac{1}{-\xi_{kk}^{1/n} - z - Y_k}.$$ 

We do the following partition of $s_n(z)$ into $d$ parts:

$$s_n(z) := \sum_{l=1}^{d} \frac{N_l}{n} f_n^{(l)}(z),$$

where

$$f_n^{(l)}(z) := \frac{1}{N_l} \sum_{k \in V_l} \frac{1}{-\xi_{kk}^{1/n} - z - Y_k}. \tag{3.9}$$

The $k$-th diagonal element in $\frac{A_n - \mathbb{E} \tilde{A}_n}{\sigma}$ is $\frac{-p_{kk}}{\sqrt{n} \sigma} = o(1)$. Similar with (2.10), we have

$$-\frac{1}{f_n^{(l)}(z)} = \sum_{m=1}^{d} \frac{N_m}{n} c_{ml} f_n^{(m)}(z) + z + o(1), \quad 1 \leq l \leq d \tag{3.10}$$

for any $z \in D_{n, \varepsilon}$ with probability at least $1 - n^{-C-9}$. Using the assumptions (3.3), (3.4) and the fact that $|f_n^{(l)}| \leq \frac{1}{n}$, we have

$$-\frac{1}{f_n^{(l)}(z)} = z + \sum_{m=1}^{d} \alpha_m c_{ml} f_n^{(m)}(z) + o(1), \quad 1 \leq l \leq d \tag{3.11}$$

for any fixed $z \in D_{n, \varepsilon}$ with probability at least $1 - n^{-C-9}$.

Since $d$ is fixed and all coefficients $c_{kl}, 1 \leq k, l \leq d$ in (3.6) are positive and bounded, from Theorem 2.10 in [4],

$$\sup_{1 \leq i \leq d} |g_i(z)| < \infty, \forall z \in \mathbb{H}.$$
Theorem 2.12 (i) in [4] implies Lemma 2.13 holds with \( \Lambda(z) := \sup_{1 \leq i \leq d} |f_n^{(i)}(z) - g_i(z)| \) for any fixed \( z \in D_{n,\varepsilon} \). Similar to the proof of Lemma 2.7, we have

\[
|s_n(z) - m(z)| = \left| \sum_{l=1}^{d} \frac{N_l}{n} f_n^{(l)}(z) - \sum_{l=1}^{d} \alpha_l g_l(z) \right|
\]

\[
\leq \left| \sum_{l=1}^{d} \frac{N_l}{n} f_n^{(l)}(z) - \sum_{l=1}^{d} \alpha_l f_n^{(l)}(z) \right| + \left| \sum_{l=1}^{d} \alpha_l f_n^{(l)}(z) - \sum_{l=1}^{d} \alpha_l g_l(z) \right|
\]

\[
\leq \sum_{l=1}^{d} \left| \left( \frac{N_l}{n} - \alpha_l \right) f_n^{(l)}(z) \right| + \sum_{l=1}^{d} \alpha_l |f_n^{(l)}(z) - g_l(z)| = o(1)
\]

uniformly for all \( z \in D_{n,\varepsilon} \) with probability at least \( 1 - n^{-C} \). Hence the local law for \( \frac{A_n}{\sqrt{n\sigma}} \) is proved.

We have the corresponding infinity norm bound for eigenvectors in the bulk.

**Corollary 3.5.** Let \( A_n \) be an adjacency matrix of a stochastic block model. For any bulk interval \( I \) such that eigenvalue \( \lambda_i \frac{A_n}{\sqrt{n\sigma}} \in I \) and any constant \( C_1 > 0 \), with probability at least \( 1 - n^{-C_1} \), the corresponding unit eigenvector \( u_i \frac{A_n}{\sqrt{n\sigma}} \) satisfies

\[
\|u_i \frac{A_n}{\sqrt{n\sigma}}\|_\infty \leq C_2 \frac{\log n}{\sqrt{n^p}}.
\]

for some constant \( C_2 > 0 \).

**Proof.** Let \( W_n := \frac{A_n}{\sqrt{n\sigma}} \). For any \( \lambda_i(W_n) \) in the bulk, by Corollary 3.3, one can find an interval \( I \) centered at \( \lambda_i(W_n) \) and \( |I| = \frac{C_1 \log n}{n^p} \) such that \( N_I \geq \delta_1 n |I| \) for some small \( \delta_1 > 0 \) with probability at least \( 1 - n^{-C_1 - 3} \). We can find a set \( J \subset \{1, \ldots, n - 1\} \) with \( |J| \geq N_I/2 \) such that \( |\lambda_j(W_{n-1}) - \lambda_i(W_n)| \leq |I| \) for all \( j \in J \). Let \( X_k \) be the \( k \)-th column of \( \frac{A_n}{\sqrt{n\sigma}} \) with the \( k \)-th entry removed, then \( X_k = \sqrt{n\sigma} k \).

Since \( X_k \) is not centered, we need to show

\[
(3.12) \quad \sum_{j \in J} |u_j(W_{n,k})^* X_k|^2 = \|\pi_H(X_k)\|^2 = \Omega(|J|)
\]

with probability at least \( 1 - n^{-C_1 - 3} \), where \( H \) is the subspace spanned by all orthonormal eigenvectors associated to eigenvalues \( \lambda_j(W_{n,k}), j \in J \) and \( \dim(H) = |J| \).

Let \( H_1 = H \cap H_2 \), where \( H_2 \) is the subspace orthogonal to the vector \( \mathbb{E}a_k \). The dimension of \( H_1 \) is at least \( |J| - 1 \). Let \( b_k = a_k - \mathbb{E}a_k \), then the entries of \( b_k \) are centered with the same variances as \( a_k \). By Lemma 2.10, we have

\[
\|\pi_{H_1}(b_k)\|^2 = \Omega \left( \frac{|J|}{n} \right)
\]

with probability at least \( 1 - n^{-C_1 - 3} \). Moreover,

\[
\|\pi_H(a_k)\| = \|\pi_H(b_k + \mathbb{E}a_k)\| \geq \|\pi_{H_1}(b_k + \mathbb{E}a_k)\| = \|\pi_{H_1}(b_k)\|,
\]

which implies (3.12) holds. The rest of the proof follows from the proof of Theorem 2.3. □
3.2.2. Unbounded Number of Classes. For the Stochastic Block Models, if we allow the number of classes \( d \to \infty \) as \( n \to \infty \), a local law can be proved under the following assumptions

\[
(3.13) \quad d = o\left(\frac{n}{g(n)}\right),
\]

\[
(3.14) \quad \sum_{i=1}^{d} \left| \frac{\sigma_{kl}}{\sigma^2} - c_{kl} \right| = o\left(\frac{1}{g(n)}\right).
\]

We will compare the Stieltjes transform of the empirical spectral distribution to the measure whose Stieltjes transform satisfies the following equations:

\[
(3.15) \quad m_n(z) = \sum_{i=1}^{N_i} n g_{n,i}(z)
\]

\[
(3.16) \quad -\frac{1}{g_{n,i}(z)} = z + \sum_{i=1}^{d} \frac{N_i}{n} c_{ij} g_{n,j}(z).
\]

We have the following local law for SBM with unbounded number of blocks.

**Corollary 3.6.** Let \( A_n \) be an adjacency matrix of SBM with assumptions (3.13),(3.14). Let \( \rho_n \) be the probability measure corresponding to equations (3.15),(3.16). For any constants \( \delta, C_1 > 0 \), there exists a constant \( C_2 \) such that with probability at least \( 1 - n^{-C_1} \) the following holds. For any bulk interval \( I \) of length \( |I| \geq \frac{C_2 \log n}{np} \), the number of eigenvalues \( N_I \) of \( \frac{A_n}{\sqrt{n} \sigma} \) in \( I \) obeys the concentration estimate

\[
(3.17) \quad \left| N_I - n \int_I \rho_n(x)dx \right| \leq \delta n |I|.
\]

**Proof.** Since \( d = o\left(\frac{n}{g(n)}\right) \), recall the definition of \( \tilde{A}_n \) from (3.8), by Cauchy interlacing law,

\[
|N_I(A_n) - N_I(A_n - E(\tilde{A}_n))| \leq C_2 n |I|.
\]

It suffices to prove the statement for the centered matrix \( W_n := \frac{A_n - E(\tilde{A}_n)}{\sqrt{n} \sigma} \). The proof then follows from Corollary 3.3 with assumption (3.14).

\[\Box\]

**Remark 3.7.** Different from Corollary 3.3, in Corollary 3.6, we are not comparing the empirical spectral distribution to a limiting spectral distribution \( \rho \) independent of \( n \). If we assume \( \frac{N_i}{n} \to \alpha_i, \alpha_1 \geq \alpha_2 \cdots \geq \cdots \), and \( \sum_{i=1}^{\infty} \alpha_i = 1 \), one can show that \( \rho_n \) converge to some \( \rho \) (see Section 7 in [34] for further details). But there is no way to have a local law comparing \( N_I \) with \( n \int_I \rho(x)dx \). In fact, let \( S_n \) be the symmetric function on \([0,1]^2\) representing the variance profile as in (2.26) and \( S \) be its point-wise limit, since there is no upper bound for rate of convergence on sup \( x,y |S_n(x,y) - S(x,y)| \).

**Remark 3.8.** With the same argument in the proof of Corollary 3.5, the infinity norm bound for eigenvectors in Corollary 3.5 still holds for the SBM with unbounded number of classes.

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