Longitudinal and transverse bending by a cylindrical shape of the sandwich plate stiffened in the end sections by rigid bodies

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Abstract. We study the problems of deformation mechanics of sandwich constructions with taking into account the interaction with the contour reinforcing rods. To derive the basic equations of equilibrium, static boundary conditions for the shell and reinforcing rods, as well as conditions of the kinematic conjugation the carrier layers with a core, the carrier layers and a core with reinforcing rods we use a generalized variational Lagrange principle. We reduce the boundary value problem on the to the integral-algebraic system of Volterra equations of the second kind. To approximate the obtained integral equations of Volterra type a collocation method with Gaussian nodes and a method for constructing the integrating matrices are proposed. For the numerical realization of the proposed methods we have developed a software package. Numerical calculations were performed. Analyze the results of numerical experiments is carried out.

1. Introduction. Statement of the problem
Sandwich plates and shells with external (carrier) layers of rigid materials and cores relating to a transversely soft class (cores of cellular or folded structure, reinforced with polystyrene etc.) are one of the most popular structural elements for various purposes. The external layers of these elements in the contour are always reinforcing by the rods to ensure the transfer of the load on the carrier layers of interaction with other construction elements (Figure 1).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Schemes of fixing the plate and reinforcing rods and loads.}
\end{figure}
Problems of deformation mechanics of sandwich constructions without taking into account the interaction with the contour reinforcing rods are studied in many of papers [1-13], from which we select articles [8-13]. In these works for sandwich plates and shells with transversal-soft core at small strains and middle displacements was built the refined geometrically nonlinear theory, which allows to describe the process of subcritical deformation and to identify all the possible buckling of carrier layers (in-phase, antiphase mixed bending and bending-shear mixed, as well as arbitrary, which include all of the above). Summarizing the results of these studies to the sandwich plates and shells with the contour reinforcing rods (Figure 2), we introduce as unknowns, as in [5-7], the contact forces of interaction the carrier layers with the a core, and the carrier layers, and a core with reinforcing rods at all points in their conjugation surfaces [14, 15].

![Figure 2. The scheme of the plate and the reinforcing rods.](image)

To derive the basic equations of equilibrium, static boundary conditions for the shell and reinforcing rods, as well as conditions of the kinematic conjugation the carrier layers with a core, the carrier layers and a core with reinforcing rods, as in [14], we use the previously proposed a generalized variational Lagrange principle [16]. With regard to the sandwich plate experiencing a cylindrical bending, derived in this way the equilibrium equations will have the form (hereinafter $k=1,2$, is the layer number, $\delta_{(1)}=1$, $\delta_{(2)}=-1$):

$$\frac{dN_{1}^{(k)}}{dx} + \frac{\delta_{(k)}E_{3}}{2h}\left(\ddot{w}^{(2)} - \ddot{w}^{(1)}\right) + X_{3}^{(k)} = 0$$  \hspace{1cm} (1)

$$\frac{dT_{11}^{(k)}}{dx} + \delta_{(k)}q_{1} + X_{1}^{(k)} = 0,$$  \hspace{1cm} (2)

$$\mu = u_{1}^{(1)} - u_{1}^{(2)} - H_{(1)}\omega^{(1)} - H_{(2)}\omega^{(2)} + \frac{2hq_{1}}{G_{13}} - \frac{2h^{2}}{3E_{3}} \frac{d^{2}q_{1}}{dx^{2}} = 0.$$  \hspace{1cm} (3)

In these equations, introduced in consideration of the tangential forces $T_{11}^{(k)}$ and generalized shear forces $N_{1}^{(k)}$ through the unknown axial displacements $u_{1}^{(k)}$, deflections $w_{1}^{(k)}$ of medial surface of the carrier layers and the tangential stresses $q_{1}$ in the core, constant over its thickness, are expressed by the formulas
Then this cross section is necessary to satisfy the symmetry conditions of solutions of equations (1) - (4) of the following types

\[ \omega^{(k)} = 0, \quad N_{11}^{(k)} = 0, \quad u_{1}^{(k)} = 0, \quad q_{1} = 0 \]  

(6)

If there is a cylindrical bending sandwich plate, the rods, reinforcing it in the face sections \( x_{1} = x_{1}^{+} \), \( x_{1} = x_{1}^{-} \) are transformed into an absolutely solid body. Therefore, the equilibrium equations of the reinforcing rod in cross-section \( x_{1} = x_{1}^{+} \) will have the following form

\[ f_{u}^{+} = -\left( Q_{11}^{(l)} + Q_{11}^{(2)} \right) + T_{z}^{+} = 0, \]  

(7)

\[ f_{w}^{+} = -\left( Q_{13}^{(l)} + Q_{13}^{(2)} + 2h\tau_{1} \right) + T_{z}^{+} = 0, \]  

(8)

\[ f_{\phi}^{+} = -\left[ L_{11}^{(l)} + L_{11}^{(2)} - Q_{11}^{(l)} H_{(l)} + Q_{11}^{(2)} H_{(2)} + B_{2}^{(\pm)} \left( Q_{13}^{(l)} + Q_{13}^{(2)} + 2h\tau_{1} \right) \right] + m_{z}^{+} = 0, \]  

(9)

where \( T_{z}^{+}, T_{z}^{-} \) are the components of the external loop load forces \( T' = T_{z}^{+} \mathbf{n} + T_{z}^{-} \mathbf{m} \), components applied to the point of line \( \mathbf{r}_{A} = \xi_{A} \mathbf{n} + \zeta_{A} \mathbf{m} \) (Fig. 2), \( m_{z}^{+} = T_{z}^{+} \xi_{A} - T_{z}^{-} \zeta_{A} \) is the loop bending moment. Emerging in the reinforcing rods the displacements \( U, W \) and rotation \( \phi \) must satisfy the kinematic conditions of the coupling the rods with carrier layers

\[ u_{1}^{(k)} - \left( U - \delta_{k} H_{(k)} \phi \right) = 0, \quad \delta Q_{11}^{(k)} \neq 0 \]  

(10)

\[ W^{(k)} - \left( W + B_{2}^{(\pm)} \phi \right) = 0, \quad \delta Q_{13}^{(k)} \neq 0 \]  

(11)

\[ w_{1}^{(k)} + \phi = 0, \quad \delta f_{41}^{(k)} \neq 0 \]  

(12)

to which it is necessary to add the kinematic conditions

\[ w_{l}^{(l)} + w_{1}^{(2)} + \frac{2h_{l}^{2}}{3E_{3}} q_{1,1} - 2\left( W + B_{2}^{(\pm)} \phi \right) = 0, \]  

(13)
taking place at $\delta \tau_1 \neq 0$. Note that in (10) - (13) the unknowns $U, W$ and $\varphi$ are the displacements of the point $O_2'$ (Figure 2) and the rotation angle of the cross section of the reinforcing body.

Under the conditions of $Q_{11}^{(k)} = Q_{13}^{(k)} = L_{14}^{(k)} = \tau_1 = 0$ (at absence in the sections $x_i = x_i^*$ the reinforcing rods, see, eg, Figure 1) to solve the problem (1) - (5) with respect to the vector functions of unknowns $X_1 = (u_1^{(1)}, u_1^{(2)}, u_1^{(3)}, q_1, Q_{11}^{(1)}, Q_{12}^{(1)}, Q_{13}^{(1)}, Q_{14}^{(1)}, L_{11}^{(1)}, L_{12}^{(1)}, \tau_1, U, W, \varphi)$ in [8] a numerical method based on the method of summation identities is proposed. In this case, however, for the boundary value problem in the form of five differential equations (1) - (3) with the boundary conditions (5), (6) and ten algebraic equations (7) - (13) it is required to determine the vector function of unknowns $X_2 = (u_1^{(1)}, u_1^{(2)}, u_1^{(3)}, q_1, Q_{11}^{(1)}, Q_{12}^{(1)}, Q_{13}^{(1)}, Q_{14}^{(1)}, L_{11}^{(1)}, L_{12}^{(1)}, \tau_1, U, W, \varphi)$.

2. Reduction of the boundary value problem to the integro-algebraic form

We reduce the formulated boundary value problem (1) - (13) on the definition of the function $X_2(x)$ to the integral-algebraic system of Volterra equations of the second kind by using additional relationships to determine the unknown integration constants. Note that, in the original differential problem includes derivatives of order $2n$ of the unknown function, while the integral equation will contain only $n$-th derivatives. This reduction is carried out by integrating the equations (1)–(3), satisfying the conditions (5), (6) and using the following equations

$$u_1^{(k)}(x) = \frac{1}{\sigma_{00}} \int_0^x \sigma_{14}^{(k)}(s) ds, \quad q_1(x) = \frac{1}{\sigma_{00}} \int_0^x \sigma_{11}^{(k)}(s) ds,$$

$$\frac{d v_1^{(k)}}{d x} = \frac{1}{\sigma_{00}} \int_0^x \sigma_{12}^{(k)}(s) ds, \quad w_1^{(k)}(x) = v_1^{(k)}(x) - \frac{1}{\sigma_{00}} \int_0^x \sigma_{22}^{(k)}(s) ds d\xi,$$

where $v_1^{(k)} = v_1 |_{x_k}$. As a result, relatively the vector function

$$X(x) = (u_1^{(1)}, u_1^{(2)}, u_1^{(3)}, q_1, w_1^{(1)}, w_1^{(2)}, Q_{11}^{(1)}, Q_{12}^{(1)}, Q_{13}^{(1)}, Q_{14}^{(1)}, L_{11}^{(1)}, L_{12}^{(1)}, \tau_1, U, W, \varphi), \quad x \in [0, x^*] = [0, a]$$

we obtain the following system of integro-algebraic equations:

$$L_{11}^{(1)} - D_{11}^{(1)} \frac{d^2 w_1^{(1)}}{d x^2} + \int_{x_0}^{x_1} \frac{d^2 w_1^{(2)}}{d s^2} ds d\xi + H^{(1)} \int_{x_0}^{x_1} \frac{d q_1}{ds} ds d\xi +$$

$$+ \frac{\delta_{00}}{2h} \int_{x_0}^{x_1} \frac{d \varphi}{dx} dx \int_{x_0}^{x_1} \frac{d^2 w_1^{(2)}}{d s^2} ds d\xi = - \frac{1}{\sigma_{00}} \int_{x_0}^{x_1} \frac{d X_3}{ds} ds d\xi, \quad \ldots, \quad (14)$$

$$Q_{11}^{(k)} - B_{11}^{(k)} \left( \frac{d u_1^{(k)}}{dx} + \frac{1}{2} \int_{x_0}^{x_1} \frac{d^2 w_1^{(k)}}{d s^2} ds d\xi \right) + \frac{\delta_{00}}{2h} \int_{x_0}^{x_1} \frac{d q_1}{ds} ds d\xi = 0, \quad \ldots, \quad (15)$$

$$\frac{2h}{3E_3} \frac{d q_1}{dx} + \frac{2h}{G_{13}} \int_{x_0}^{x_1} \frac{d q_1}{ds} ds d\xi + \frac{1}{\sigma_{00}} \int_{x_0}^{x_1} \left( \frac{d u_1^{(k)}}{dx} - \frac{d u_1^{(2)}}{dx} - H^{(1)} \frac{d^2 w_1^{(1)}}{d s^2} ds d\xi - H^{(2)} \frac{d^2 w_1^{(2)}}{d s^2} ds d\xi \right) ds d\xi = 0, \quad \ldots, \quad (16)$$

$$\int_{x_0}^{x_1} \frac{d q_1}{dx} dx - \tau_1 = 0, \quad \ldots, \quad (17)$$

$$\frac{\delta_{00}}{2h} \int_{x_0}^{x_1} \left( \frac{d u_1^{(1)}}{dx} - \frac{d u_1^{(2)}}{dx} - H^{(1)} \frac{d^2 w_1^{(1)}}{d s^2} ds d\xi - H^{(2)} \frac{d^2 w_1^{(2)}}{d s^2} ds d\xi \right) dx + Q_{11}^{(k)} + h \tau_1 = -J_3 X_3^{(k)} \quad \ldots, \quad (18)$$

to which must be added the algebraic equilibrium equations (7)-(9) for the rod.
If the point of force application \( T^+ \) has a displacement \( W_A \) in the direction of the axis \( z \) equal to zero, and the displacement in the direction of the axis \( \zeta \) equal is \( U_A \), then by virtue of (1)

\[
u_i^{(k)} \bigg|_{x=a} = \int_0^a \frac{du_i^{(k)}}{dx} \, dx, \quad W = \zeta_A \varphi, \quad U = U_A - z_A \varphi
\]

the coupling conditions (10) - (13) take the form

\[
\int_0^a \frac{du_i^{(k)}}{dx} \, dx - U_A + \left( z_A + \delta(k)H(k) \right) \varphi = 0, \quad W_a^{(k)} - \left( \zeta_A + B_2^{(k)} \right) \varphi = 0,
\]

\[
\int_0^a \frac{d^2w_i^{(k)}}{dx^2} \, dx + \varphi = 0,
\]

Thus, to determine the introduced into consideration sixteen unknowns \( X \in H^5 \times R^{11} \) the system of sixteen resolution integro-algebraic equations (14) - (21) and the rod of equilibrium equations (7), (9) are composed, where \( H = L_2(0,a) \) and \( a \) is the half-length of the plate.

3. Approximation of integral equations by the collocation method with Gaussian nodes

To approximate the obtained integral equations of Volterra type in [17] a collocation method with Gaussian nodes and a method for constructing the integrating matrices are proposed. We introduce the integral operators \( \mathcal{J}(f) = \int_0^a f(\xi)d\xi \), \( \mathcal{J}^+(f) = \int_0^a f(\xi)d\xi \), \( \mathcal{J}(f) = \int_0^a f(\xi)d\xi \) that are replaced by their finite-dimensional analogues in the form of integrating matrices \( J_1, J_2, J_3 \), respectively. Consequently, for this purpose, in the segment \([0,a]\) we introduce a grid \( \omega = \{x_i, i = 1, 2, \ldots, N\} \) according to the Gauss quadrature formula:

\[
J_3(f) = \sum_{i=1}^{N} d_i f(x_i),
\]

where \( \{d_i\} \), \( \{x_i\} \) are the weights and collocation nodes connected with the roots of the Legendre polynomial of degree \( N \). Let \( f_i \) be a value of \( f \) in the node \( x_i : f_i = f(x_i) \), we approximate \( f \) in the segment \([0,a]\) by interpolating function \( f(x) \approx \sum_{i=1}^{N} f_i l_i(x) \). As an interpolation function the linear combination of Lagrange basis functions \( l_i \) over the nodes \( \{x_i\} \) is selected. Thus, expanding the function \( l_i \) in Legendre polynomials, we construct integrating matrix \( J_1, J_2 \). Let us consider the finite-dimensional operators

\[
\mathcal{I}_1 = (I_{1w}, I_{2w}) : H_h^5 \times R^{11} \rightarrow H_h^5, \quad \mathcal{I}_2 = (I_{1w}, I_{2w} + 2a; I_{2w}, I_{2w} + 2a; I_{2w} + 2a, \mathcal{Q}^N) : H_h^5 \times R^{11} \rightarrow H_h^5,
\]

\[
\mathcal{I}_3 = (I_{3w}, I_{3w}) : H_h^5 \times R^{11} \rightarrow R^3,
\]

\[
\mathcal{Q} = (Q_{i1}^{(1)}, Q_{i1}^{(2)}, Q_{i1}^{(3)}, Q_{i1}^{(4)}, L_{i1}^{(1)}, L_{i1}^{(2)}) : H_h^5 \times R^{11} \rightarrow R^6,
\]

by the formulas

\[
I_{1w}^{(k)} X(x) = I_{1w}^{(k)} - D_{1w}^{(k)} J_{1w}^{(k)} q_{1w} + \delta_{1w} J_{2w}^{(k)} E_a \left[ w_a^{(k)} - w_a^{(k)} - J_{2w}^{(k)} \left[ w_{1w}^{(k)} - w_{1w}^{(k)} \right] \right],
\]

\[
I_{2w}^{(k)} X(x) = J_{2w}^{(k)} J_{1w}^{(k)} w_{1w}^{(k)} + \delta_{1w} J_{2w}^{(k)} J_{2w}^{(k)} w_{1w}^{(k)},
\]

\[
I_{3w}^{(k)} X(x) = Q_{i1}^{(k)} - B_{i1}^{(k)} q_{1w} + \delta_{i1} J_{2w}^{(k)} q_{1w},
\]

5
To solve the projection scheme (22), we use the following two-layer iterative process with the lowering of non-linearity in the lower layer [8, 18–24] with a preconditioner, which is a linear part of the operator of the difference scheme:

\[ A_1 \frac{U^{(n+1)} - U^{(n)}}{\tau} + (A_1 + A_2)U^{(n+1)} = F, \]

where \( X^{(0)} \) is the given initial approximation, \( \tau > 0 \) is an iteration parameter. For the numerical realization of the iterative method (23) for solving the problem (22) of the longitudinal and transverse bending of the plate, we have developed a software package. Numerical calculations were performed for the following geometric and stiffness characteristics of sandwich plate: plate length \( 2a = 20 \) cm, \( 2h_1 = 2h_2 = 0.1 \) cm, \( h = 1 \) cm, \( G_1 = 25 \) MPa, \( E_3 = 50 \) MPa, \( E^{(2)} = 133 \times 10^3 \) MPa, \( v_{12}^{(k)} = v_{21}^{(k)} = 0.3 \), \( k = 1, 2 \), the geometric characteristics of the rod: \( z_A = \zeta_A = B_2^{(n)} = h + h_1 \) (Fig. 1), a value of loop load on the reinforcing rod, which provides transmission of the load to the carrier layers \( T_\tau^+ = -100 kN / m \). The number of grid points \( N = 512 \). The calculations according to (22) were
carried out as long as the residual norm $\| F - (A_1 + A_2) X^{(n)} \|$ remained more than a given accuracy $\varepsilon = 5 \cdot 10^{-8}$. As a norm of the vector $g = (g_1, g_2, \ldots, g_m)$ the value $\| g \| = \max \{|g_1|, |g_2|, \ldots, |g_m|\}$ is chosen. Iterative parameter was selected empirically.

Figures 3-8 shows the results of numerical experiments aimed at determining the parameters of the stress-strain state of the plate for a given value of loop load $T_\gamma^* = -100kN/m$. Under the action of this load in the end cross section of the plate and the reinforcing rod are formed by the parameters of the stress-strain state, with values

$Q_{11}^{(1)} = 0.062019kN/m$, $Q_{11}^{(2)} = -100.062kN/m$, $Q_{13}^{(1)} = 0.11091kN/m$, $Q_{13}^{(2)} = -0.23413kN/m$, $L_{11}^{(1)} = 0.63506N$, $L_{11}^{(2)} = 0.066734N$, $\tau_1 = 0.006161MPa$, $U_A = -0.0069075cm$, $\varphi = -0.0032962$

**Figure 3.** Deflections of the medial surfaces of the carrier layers ($w^{(k)}$, cm).

**Figure 4.** Axial displacements of the medial surfaces of the carrier layers ($u^{(k)}_1$, cm).

**Figure 5.** Membrane forces of the carrier layers ($T_{11}^{(k)}$, kN/m).

**Figure 6.** Transverse tangential stresses in the core ($q_1$, MPa).
Figure 3 shows the dependence of the functions of deflections points of medial surfaces of the plate carrier layers on the length. It is evident that they are practically the same in view of the smallness of deformations of the transverse compression of the core in the subcritical state. Figure 4 shows the axial displacements function of points of the medial surfaces of carrier layers. Note that at the lower layer they are substantially equal to zero, and on the upper layer vary along the along the length of the plate almost linearly. From Figure 5 it follows that the plate is in the moment state due to the action on the plate eccentrically compressive load, wherein the torsional load formed in the first layer is practically zero, and the force $T_{11}^{(2)}$ along the length of the plate is not changed substantially. Figure 6 shows the variation of the tangential stresses in the core along the length of the plate. It is seen that by virtue of the acting of the eccentrically compressive force on the rod they are not equal to zero over the entire length of the plate (at the connection points of a core with the reinforcing rod, they should take the zero value in the absence of the adhesive layer) and reaches the maximum value in the neighborhood of the end cross section. Figures 7 and 8 show the functions of the generalized shear forces and shear forces in the carrier layers. By the nature of these curves we can be judged on the inclusion in the work the core and its contribution to the perception of the transverse tangential stresses.

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