Additive and multiplicative renormalization of topological charge with improved gluon/fermion actions: A test case for 3-loop vacuum calculations, using overlap or clover fermions

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Abstract

We calculate perturbative renormalization properties of the topological charge, using the standard lattice discretization given by a product of twisted plaquettes. We use the overlap and clover action for fermions, and the Symanzik improved gluon action for 4- and 6-link loops.

We compute the multiplicative renormalization of the topological charge density to one loop; this involves only the gluon part of the action. The power divergent additive renormalization of the topological susceptibility is calculated to 3 loops.

Our work serves also as a test case of the techniques and limitations of lattice perturbation theory, it being the first 3-loop computation in the literature involving overlap fermions.

Keywords: Lattice QCD, Topology, Lattice perturbation theory, Overlap action, Improved actions.

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Introduction

Topological properties of QCD are among those most widely studied on the lattice. Various methods have been used to this end, involving renormalization, cooling, fermionic zero modes, geometric definitions, etc. In recent years, the advent of fermionic actions, such as the overlap, which do not violate chirality, has brought a new thrust to the subject.

In this work we compute the renormalization constants which are necessary in order to extract topological properties, in the “field theoretic” approach, from Monte Carlo simulations using Wilson or Symanzik improved gluons, and clover or overlap fermions. In particular, we compute the multiplicative renormalization $Z_Q$ of the topological charge density, to 1 loop in perturbation theory and the power divergent additive renormalization $M(g^2)$ of the topological susceptibility, to 3 loops.

The main motivations for doing this work are:

- To enable comparison between different approaches used in studying topology, so that a coherent picture of topology in QCD may emerge.
- To enable studies, in numerical simulations, of quantities involving the density of topological charge, $q(x)$, rather than only the integrated charge; this is necessary, e.g., for studying the spin content of nucleons.
- As a feasibility study in lattice perturbation theory: Indeed, this is the first 3-loop calculation involving overlap fermions.

Computation of $Z_Q$

Our first task is to compute the the multiplicative renormalization $Z_Q$ [1] of the topological charge density $q_L(x)$ to one loop, using the background field method. We use the standard definition of $q_L$, given by a product of twisted plaquettes

$$q_L(x) = -\frac{1}{2^9 \pi^2} \sum_{\mu \nu \rho \sigma = \pm 1}^{\pm 4} \epsilon_{\mu \nu \rho \sigma} \text{Tr} [\Pi_{\mu \nu}(x)\Pi_{\rho \sigma}(x)]$$

($\epsilon_{-\mu, \nu, \rho, \sigma} \equiv -\epsilon_{\mu, \nu, \rho, \sigma}$). $\Pi_{\mu \nu}(x)$ is the parallel transport matrix along a $1 \times 1$ Wilson loop; in standard notation

$$\Pi_{\mu \nu}(x) = U_{\mu}(x)U_{\nu}(x + \mu)U^\dagger_{\mu}(x + \nu)U^\dagger_{\nu}(x)$$

The classical limit of the operator shown in Eq.(1) must be corrected by including a renormalization function $Z_Q$, which can be expressed perturbatively as

$$Z_Q = 1 + Z_1 \cdot g^2 + \cdots , \quad Z_1 = Z_{11} \cdot N_c + Z_{12} \cdot \frac{1}{N_c}$$

We perform a calculation of $Z_1$; this involves only the gluon part of the action. In the background field method, link variables are decomposed as
\[ U_\mu(x) = V_\mu(x) U_{c\mu}(x) \]  

in terms of links for a quantum field and a classical background field, respectively

\[ V_\mu(x) = e^{i g Q_\mu(x)} , \quad U_{c\mu}(x) = e^{i a B_\mu(x)} \]  

The \( N_c \times N_c \) Hermitian matrices \( Q_\mu \) and \( B_\mu \) can be expressed as

\[ Q_\mu(x) = t^a Q^a_\mu(x) , \quad B_\mu(x) = t^a B^a_\mu(x) , \quad Tr[t^a t^b] = \frac{1}{2} \delta^{ab} \]  

The perturbative nature of our calculation requires a choice of gauge. An appropriate gauge-fixing term is

\[ S_{gf} = \frac{1}{1-\xi} \sum_{\mu,\nu} \sum_x Tr[D^-_\mu Q_\mu D^-_\nu Q_\nu] \]  

This term breaks gauge invariance with respect to \( Q_\mu \), as it should, but succeeds in keeping the path integral as a gauge invariant functional of \( B_\mu \). The definition of the lattice derivative, which is covariant with respect to background gauge transformations, is

\[ D^-_\mu(U_c)Q_\nu(x) = U_{c\mu}^{-1}(x-e_\mu)Q_\nu(x-e_\mu)U_{c\mu}(x-e_\mu) - Q_\nu(x) \]  

The diagrams involved in the one-loop calculation of \( Z_Q \) are shown in Figure 1.

![Diagrams](image_url)  

Fig. 1: Diagrams contributing to \( Z_1 \). The black bullet stands for topological charge vertices.

External lines correspond to background fields.

We use the Symanzik improved gauge field action, involving Wilson loops with 4 and 6 links. In standard notation, it reads

\[ S_G = \frac{2}{g^2} \left[ c_0 \sum_{\text{plaquette}} \text{Re Tr} (1 - U_{\text{plaquette}}) + c_1 \sum_{\text{rectangle}} \text{Re Tr} (1 - U_{\text{rectangle}}) + c_2 \sum_{\text{chair}} \text{Re Tr} (1 - U_{\text{chair}}) + c_3 \sum_{\text{parallelogram}} \text{Re Tr} (1 - U_{\text{parallelogram}}) \right] \]  

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The lowest order expansion of this action, leading to the gluon propagator, is (see, e.g., Ref. [2])

$$S_G^{(0)} = \frac{1}{2} \int_{-\pi/a}^{\pi/a} \frac{d^4 k}{(2\pi)^4} \sum_{\mu \nu} A_\mu^a(k) \left[ G_{\mu\nu}(k) - \frac{\xi}{\xi - 1} \hat{k}_\mu \hat{k}_\nu \right] A_\nu^a(-k)$$

(10)

where:

$$G_{\mu\nu}(k) = \hat{k}_\mu \hat{k}_\nu + \sum_\rho \left( \hat{k}_\rho \delta_{\mu\nu} - \hat{k}_\mu \hat{k}_\rho \delta_{\mu\nu} \right) d_{\mu\rho}$$

and:

$$d_{\mu\nu} = (1 - \delta_{\mu\nu}) \left[ C_0 - C_1 a^2 \hat{k}^2 - C_2 a^2 (\hat{k}_\mu^2 + \hat{k}_\nu^2) \right]$$

The coefficients $C_i$ are related to the Symanzik coefficients $c_i$ by

$$C_0 = c_0 + 8c_1 + 16c_2 + 8c_3, \quad C_1 = c_2 + c_3, \quad C_2 = c_1 - c_2 - c_3$$

The Symanzik coefficients must satisfy: $c_0 + 8c_1 + 16c_2 + 8c_3 = 1$, in order to reach the correct classical continuum limit.

Our calculations are performed without any assumptions on the values of the external momenta $p_1, p_2$: This is safest for the topological charge operator, otherwise one may easily end up with indeterminate expressions.

For our purpose, we must express all potentially divergent integrals $I$ in terms of a continuum counterpart $I_{cont}$ (evaluated in $D = 4 - 2\epsilon$ dimensions), plus all lattice contributions $I_{Latt}$. The latter are the ones which will determine $Z_Q$. The three expressions shown below form a basis set for all the divergent integrals encountered in this calculation; we point out the need to handle also a 3-point form factor ($C_{\mu\nu}$ below) [3]

$$B(a, p) = \frac{(ka)^{2\epsilon}}{a^0} \int \frac{d^D k}{(2\pi)^D} \frac{1}{k^2 (k + ap)^2} = B(p) + I_{0Latt}$$

(11)

$$B_\mu(a, p) = \frac{(ka)^{2\epsilon}}{a^1} \int \frac{d^D k}{(2\pi)^D} \frac{\sin k_\mu k^2 (k + ap)^2}{k^2 (k + ap)^2} = B_\mu(p) + I_{1Latt}$$

(12)

$$\tilde{C}_{\mu\nu}(a, p_1, p_2) = \frac{(ka)^{2\epsilon}}{a^0} \int \frac{d^D k}{(2\pi)^D} \frac{\sin k_\mu \sin k_\nu}{k^2 (k + ap_1)^2 (k + ap_1 + ap_2)^2}$$

$$= C_{\mu\nu}(p_1, p_2) + \frac{1}{64\pi^2} \delta_{\mu\nu} \left( -\frac{1}{\epsilon} - \ln \kappa^2 a^2 - \ln 4\pi \right. + (4\pi)^2 P_2 + \gamma_E - \left. 2\pi^2 P_1 \right)$$

(13)

where $\gamma_E$ is Euler’s constant, $B, B_\mu, C_{\mu\nu}$ are the continuum counterparts of $\tilde{B}, \tilde{B}_\mu, \tilde{C}_{\mu\nu}$, respectively; $P_1, P_2$ are [4]

$$P_1 = 0.1549339023109021(1), \quad P_2 = 0.02401318111946489(1)$$

and:

$$I_{0Latt} = \frac{1}{(4\pi)^2} \left( -\frac{1}{\epsilon} - \ln 4\pi - \ln \kappa^2 a^2 + \gamma_E \right) + P_2$$

(14)

$$I_{1Latt} = p_\mu \left( \frac{P_1}{16} - \frac{1}{2} I_{0Latt} \right)$$
As we see, \(I_{\text{Latt}}\) contains poles in \(\epsilon\); indeed, diagrams (c) and (d) of Figure 1, taken separately, exhibit such poles ( \((d) \propto -1/\epsilon - \ln \kappa^2 a^2\)). These cancel, however, upon summation, as is expected by the fact that \(Q\) does not renormalize in the continuum.

The complicated algebra of perturbation theory was carried out using our package in Mathematica. The calculation of \(Z_Q\) is particularly involved in the present case, involving propagators and vertices from the improved gluonic action. In particular, the calculation of diagram (d) involves a summation of \(> 1000000\) different algebraic expressions at intermediate stages.

Our results for \(Z_Q\) are listed in Table I. In all calculations that involve the parameters \(c_i\), we choose a standard set of values as in Ref. [2]. The choice of the sets of parameters correspond to the most popular actions: The first set corresponds to the plaquette action, the second set corresponds to the tree-level Symanzik improved action [5] and the next 6 sets correspond to the tadpole improved Lüscher-Weisz (TILW) action [6,7] for 6 values of beta

\[ \beta = 8.60, 8.45, 8.30, 8.20, 8.10, 8.00 \]

The last two sets correspond to the Iwasaki [8] and DBW2 [9] actions respectively.

In the case of the plaquette action, our result agrees with the known result of Ref. [1].

It is worth noting that the value of \(Z_1\) (and of \(e_3\) in Table II) for the DBW2 action is the smallest one, leading to a renormalization factor \(Z_Q\) closer to 1 (and \(M(g^2)\) closer to 0, see below. This would single out the DBW2 action as a better candidate for studies of topology.

### Computation of \(M(g^2)\)

The second task we attend to is the calculation of the power additive renormalization of the topological charge susceptibility, which is defined as

\[ \chi_L = \sum_x \langle q_L(x) q_L(0) \rangle \]  
(15)

\(\chi_L\) develops an unphysical background term which becomes dominant in the continuum limit

\[ \chi_L(g^2) = a^4 Z_Q(g^2)^2 \chi + M(g^2) \]  
(16)

The power divergent additive renormalization of \(\chi_L\) can be written perturbatively as

\[ M(g^2) = e_3 \cdot g^6 + e_4 \cdot g^8 + \cdots \]  
(17)

We first compute the 2-loop coefficient \(e_3\). Its dependence on the number of colors has the form:

\[ e_3 = (N_c^2 - 1) N_c e_{3,0} \]  
(18)

This result is evaluated for several sets of values of the Symanzik improvement coefficients. Figure 2 shows the diagram contributing to \(e_3\).
The 3-loop term $e_4$ of the expansion of $M(g^2)$ equals

$$e_4 = e_4^g + e_4^f \quad (19)$$

where $e_4^f$ stands for the fermionic contribution to $e_4$ ($c_{SW}$ is the coefficient in the clover action)

$$e_4^f = N_f(N_c^2 - 1)N_c \cdot (e_{4,0} + e_{4,1} c_{SW} + e_{4,2} c_{SW}^2) \quad (20)$$

and $e_4^g$ is the purely gluonic contribution, which is expressed as in Ref. [10]

$$e_4^g = \frac{1}{16}(N_c^2 - 1)(1.735N_c^2 - 10.82 + 73.83/N_c^2) \times 10^{-7} \quad (21)$$

In fact, what we are interested in, is the calculation of the parameters $e_{4,0}$, $e_{4,1}$, $e_{4,2}$. This task is performed using both overlap and clover fermions. Clearly, overlap fermions involve only the parameter $e_{4,0}$). Figure 3 shows the 3-loop diagrams contributing to the evaluation of $e_4^f$.

Fig. 3: Diagrams contributing to $e_4^f$. Straight lines correspond to overlap or clover fermions and wavy lines correspond to Wilson gluons.

The propagator and vertices for overlap fermions can be obtained from the following expression of the overlap action

$$S_{Overlap} = a^4 \sum_{n,m} \bar{\Psi}(n) D_N(n,m) \Psi(m) \quad (22)$$

where $D_N$ is the massless Neuberger-Dirac operator [11]

$$D_N = \frac{M_0}{a} \left(1 + \frac{X}{\sqrt{X^+X}}\right) \quad (23)$$

$M_0$ is a real parameter corresponding to a negative mass term. $M_0$ must lie in the range $0 < M_0 < 2r$, $r$ being the Wilson parameter (in our case $r = 1$). $X$ is a Hermitian operator which can be expressed, in momentum space, in terms of the Wilson-Dirac operator $D_W$. 
\[ X(q, p) = a \left( D_W(q, p) - \frac{M_0}{a} \right) \]
\[ = aX_0(p)(2\pi)^4\delta^4(q - p) + aX_1(q, p) + aX_2(q, p) + \mathcal{O}(g^3) \]

where:
\[ X_0(p) = \frac{i}{a} \sum_\mu \gamma_\mu \sin ap_\mu + \frac{r}{a} \sum_\mu (1 - \cos ap_\mu) - \frac{M_0}{a} \]
\[ X_1(q, p) = g \int d^4k \delta(q - p - k)A_\mu(k)V_{1,\mu}(p + \frac{k}{2}) \]

with:
\[ V_{1,\mu}(q) = i\gamma_\mu \cos aq_\mu + r \sin aq_\mu \]

and:
\[ X_2(q, p) = \frac{g^2}{2} \int \frac{d^4k_1 d^4k_2}{(2\pi)^4} \delta(q - p - k_1 - k_2) \]
\[ A_\mu(k_1)A_\mu(k_2)V_{2,\mu}(p + \frac{k_1}{2} + \frac{k_2}{2}) \]

with:
\[ V_{2,\mu}(q) = -i\gamma_\mu a \sin aq_\mu + ra \cos aq_\mu \]

The clover (SW) fermionic action [12], in standard notation, reads

\[ S_L = \frac{1}{g^2} \sum_{x, \mu, \nu} \text{Tr} [1 - \Pi_{\mu\nu}(x)] + \sum_f \sum_x (4r + m)\bar{\psi}_f(x)\psi_f(x) \]
\[ - \frac{1}{2} \sum_f \sum_{x, \mu} \left[ \bar{\psi}_f(x) (r - \gamma_\mu) U_\mu(x) \psi_f(x + \mu) \right. \]
\[ + \bar{\psi}_f(x + \mu) (r + \gamma_\mu) U_\mu(x) \psi_f(x) \]
\[ \left. + \frac{i}{4} c_{SW} \sum_f \sum_{x, \mu, \nu} \bar{\psi}_f(x) \sigma_{\mu \nu} \tilde{F}_{\mu \nu}(x) \psi_f(x) \right] \]  

(28)

where:
\[ \tilde{F}_{\mu \nu} \equiv \frac{1}{8a^2} (Q_{\mu \nu} - Q_{\nu \mu}) \]

(29)

and:
\[ Q_{\mu \nu} = U_{x, x + \mu} U_{x + \mu, x + \mu + \nu} U_{x + \mu + \nu, x + \nu} U_{x + \nu, x} + U_{x, x + \nu} U_{x + \nu, x + \nu - \mu} U_{x + \nu - \mu, x - \mu} U_{x - \mu, x} + U_{x, x - \nu} U_{x - \nu, x - \nu - \mu} U_{x - \nu - \mu, x - \nu} U_{x - \nu, x} + U_{x, x - \nu} U_{x - \nu, x - \nu + \mu} U_{x - \nu + \mu, x + \mu} U_{x + \mu, x} \]

(30)

The clover coefficient \( c_{SW} \) is treated here as a free parameter; \( r \) is the Wilson parameter; \( f \) is a flavor index; \( \sigma_{\mu \nu} = (i/2)[\gamma_\mu, \gamma_\nu] \). Powers of the lattice spacing \( a \) have been omitted and may be directly reinserted by dimensional counting.

In performing this calculation, a large effort was devoted to the creation of an efficient 3-loop “integrator”, that is, a metacode for converting lengthy 3-loop integrands into efficient code for numerical integration. Some key features of the integrator are listed in the Appendix.

Table II contains our results for \( c_3 \) (cf. Eqs.(17, 18)) for different gluonic actions. Fermions do not contribute here. Our results for the case of the plaquette action agree
with older known results (see, e.g., [13,14]). Tables III and IV list our results for \( e_4 \), using the clover and overlap actions, respectively.

Figure 4 shows the coefficients \( e_{4,0}, e_{4,1}, e_{4,2} \) of the clover result for different values of the bare fermion mass \( m \). For ease of reference, Figure 5 presents the total values of \( e_4 \) for different choices of \( c_{SW} \), with \( N_f = 2, N_c = 3 \).

Figure 6 exhibits the dependence of \( e_{4,0} \), using the overlap action, on the parameter \( M_0 \). The total value of \( e_4 \) in this case is shown in Figure 7.

**Discussion and Conclusions**

We have calculated in this work the perturbative renormalization coefficients for the topological charge and susceptibility. The topological charge, defined via a usual “twisted” product of plaquettes, was renormalized multiplicatively to one loop using a variety of gluon actions presently employed in numerical simulations. The details of the fermionic action only enter this computation at 2-loop order. The 1-loop coefficient is rather pronounced for most actions but it is quite suppressed for the particular case of the DBW2 action.

For the topological susceptibility, we performed a calculation of its power-divergent, additive renormalization \( M(g^2) \) to 2 and 3 loops. Fermions do not contribute to the 2-loop result. Among the various gluonic actions, the DBW2 action leads again to a coefficient which is quite suppressed, by more than 2 orders of magnitude as compared to the standard Wilson action. Based on these results, the DBW2 action is singled out as the favourite candidate for simulations involving measurements of topology in the “field theoretic” approach.

The 3-loop calculation was performed in the presence of clover, as well as overlap fermions, and Wilson gluons. Indeed, this is the first 3-loop calculation in the literature involving overlap fermions. To facilitate a comparison of various contributions to \( M(g^2) \), we write them in the particular case of \( N_c = 3 \) with Wilson gluons:

\[
M(g^2) = g^6 1.655 \times 10^{-5} + g^8 6.50 \times 10^{-7} + 24 g^8 N_f x
\]  

The first two summands above are clearly the purely gluonic contributions, at 2 and 3 loops. The value of \( x \) can be read from Table III for clover fermions (\( x = e_{4,0} + e_{4,1} c_{SW} + e_{4,2}^2 c_{SW}^2 \)) and from Table IV for overlap fermions (\( x = e_{4,0} \)). In all cases, fermionic contributions are of the same order of magnitude as 3-loop gluon contributions, but still only a small fraction of the 2-loop result.

Our results can be used to enhance a number of related computations. In some cases, the local definition of \( q_L(x) \) is used (e.g., [15]), with renormalization estimates coming from simulations; other investigations propose non-ultralocal definitions of \( q(x) \) (e.g., [16]), which clearly are more expensive to simulate, but can circumvent renormalization; in yet other cases, operational/numerical definitions of \( q(x) \) are proposed (e.g., [17]). Even for the integrated topological charge \( Q \), where fermions with exact chiral symmetry offer us in principle unambiguous ways of extracting \( Q \) from lattice configurations (see, e.g., Refs. [18,19]), a comparison with other time-tested definitions is called for. In all cases, it would be very important to verify that a consistent picture of topology in QCD emerges from the various approaches.
Appendix

Here we briefly describe some features of our “integrator” program; this is metacode, written in Mathematica, for converting lengthy 3-loop integrands into efficient Fortran code for numerical integration on finite lattices of different sizes $L^4$. The results $r(L)$ are subsequently extrapolated to $L \to \infty$, via a fit to a large class of functions containing powers of $L$ and $\ln L$, as is a priori expected; systematic errors on $r(\infty)$ are also produced as a result of these fits.

A number of optimizations are implemented, as listed below; the resulting output code runs many orders of magnitude faster than output of automatic code conversion programs, such as, e.g., the built-in converter of Mathematica.

The input expression to the integrator is an integrand which depends on 3 four-momenta $p_1$, $p_2$, $p_3$. It has the form of a sum of terms (typically tens of thousands); each term is a product of trigonometric functions of combinations of $p_i$, possibly including also propagators for Symanzik improved gluons and overlap fermions.

The following optimizations take place:

- A number of time consuming common ingredients are precalculated numerically and stored, for all possible momentum values on a finite lattice; such ingredients are trigonometric functions, the overlap propagator, and the Symanzik propagator. While an expression in closed form exists for the latter [2], in practice it is considerably faster to invert the propagator numerically and store it; after all, in either case the final results cannot be presented as analytic functions of the parameters $c_i$, since the dependence on $c_i$ is not polynomial (unlike the case of $c_{SW}$).
- The symmetries of the integrand are exploited to reduce the volume of the integration region, with due attention paid to correct counting of points at the borders.
- All 3-loop diagrams, with the exception of those in the form of the Mercedes emblem, contain two non-overlapping loops (i.e., loops with no shared propagators); consequently, their integrands can always be written (after some trigonometry) as a sum over expressions of the form:

$$
\int dp_1^4 f_1(p_1) \left( \int dp_2^4 f_2(p_1, p_2) \right) \left( \int dp_3^4 f_3(p_1, p_3) \right)
$$

Integrations over $p_2$ and $p_3$ can then be performed sequentially, rather than in a nested fashion (actually, we perform them simultaneously, see next item), thus reducing their execution time to that of a 2-loop integral.
- The integrand is organized as an inverse tree, i.e. summands having the same functional dependence on the innermost integration variable are grouped together, and the innermost integral is performed once for each group; the procedure is then iterated for outer integrals. In practice, this procedure can save one or two orders of magnitude in execution time of the innermost integrals, which are the most expensive ones.
- Terms with different polynomial dependence on unspecified parameters such as $N_c$ or $c_{SW}$, etc., but with otherwise similar functional form are treated simultaneously, and the result is presented as a polynomial in these parameters. Also, for parameters whose values are read on input (masses, Symanzik coefficients), the code runs in parallel for different sets of values, in order to avoid computing the same quantities several times.
TABLE I. The values of $Z_{11}$ and $Z_{12}$ (Eq.(3), Figure 1), for various values of the Symanzik coefficients $c_0, c_1, c_2, c_3$ $(c_2 = 0)$.

| Action   | $c_0$   | $c_1$   | $c_3$   | $Z_{11}$           | $Z_{12}$           |
|----------|---------|---------|---------|-------------------|-------------------|
| Plaquette| 1.0     | 0.0     | 0.0     | -0.33059398205(2) | 0.2500000000(1)   |
| Symanzik | 1.6666666 | -0.083333 | 0.0     | -0.251226240(1)   | 0.18313139233(1)   |
| TILW, $\beta = 8.60$ | 2.3168064 | -0.151791 | -0.0128098 | -0.20828371039(3) | 0.147519438874(3)   |
| TILW, $\beta = 8.45$ | 2.3460240 | -0.154846 | -0.0134070 | -0.20674100461(1) | 0.146259768983(1)   |
| TILW, $\beta = 8.30$ | 2.3869776 | -0.159128 | -0.0142442 | -0.20462181183(1) | 0.144531861677(4)   |
| TILW, $\beta = 8.20$ | 2.4127840 | -0.161827 | -0.0147710 | -0.20331455801(1) | 0.143464931830(1)   |
| TILW, $\beta = 8.10$ | 2.4465400 | -0.165353 | -0.0154645 | -0.20162651307(1) | 0.142094446611(2)   |
| TILW, $\beta = 8.00$ | 2.4891712 | -0.169805 | -0.0163414 | -0.19954339172(1) | 0.14042610424(1)    |
| Iwasaki  | 3.648   | -0.331  | 0.0     | -0.15392854668(1) | 0.105132852383(2)   |
| DBW2     | 12.2688 | -1.4086 | 0.0     | -0.0617777059(4)  | 0.038277296152(6)   |

TABLE II. Evaluation of $e_{3,0}$ (cf. Eqs.(17, 18), Figure 2) with Symanzik improved gluons, for various values of the coefficients $c_0, c_1, c_2, c_3$ $(c_2 = 0)$.

| Action   | $c_0$   | $c_1$   | $c_3$   | $e_{3,0} \times 10^7$ |
|----------|---------|---------|---------|-----------------------|
| Plaquette| 1.0     | 0.0     | 0.0     | 6.89791329(1)         |
| Symanzik | 1.6666666 | -0.083333 | 0.0     | 3.1814562840(7)       |
| TILW, $\beta = 8.60$ | 2.3168064 | -0.151791 | -0.0128098 | 1.8452250005(2)       |
| TILW, $\beta = 8.45$ | 2.3460240 | -0.154846 | -0.0134070 | 1.8054229585(4)       |
| TILW, $\beta = 8.30$ | 2.3869776 | -0.159128 | -0.0142442 | 1.7516351593(8)       |
| TILW, $\beta = 8.20$ | 2.4127840 | -0.161827 | -0.0147710 | 1.7188880608(5)       |
| TILW, $\beta = 8.10$ | 2.4465400 | -0.165353 | -0.0154645 | 1.6773505020(9)       |
| TILW, $\beta = 8.00$ | 2.4891712 | -0.169805 | -0.0163414 | 1.626880218(1)        |
| Iwasaki  | 3.648   | -0.331  | 0.0     | 0.752432061(7)        |
| DBW2     | 12.2688 | -1.4086 | 0.0     | 0.038277296152(6)     |
TABLE III. Evaluation of $e_4^f$ (cf. Eqs. (17, 19, 20), Figure 3), with Wilson gluons and clover fermions, for various values of the bare fermion mass $m$.

| $m$             | $e_{4.0} \times 10^8$ | $e_{4.1} \times 10^8$ | $e_{4.2} \times 10^8$ |
|-----------------|------------------------|------------------------|------------------------|
| -1.0149250      | -4.6273(2)             | 1.28551(1)             | -1.85010(5)            |
| -0.9512196      | -4.3888(2)             | 1.17807(9)             | -1.83818(4)            |
| -0.8749999      | -4.1089(2)             | 1.05421(9)             | -1.82240(3)            |
| -0.8253968      | -3.9299(2)             | 0.97650(7)             | -1.81156(1)            |
| -0.7948719      | -3.8210(2)             | 0.92984(5)             | -1.80457(1)            |
| -0.5181059      | -2.8759(3)             | 0.54981(2)             | -1.73390(5)            |
| -0.4234620      | -2.57089(6)            | 0.43874(5)             | -1.70723(3)            |
| -0.4157708      | -2.54658(2)            | 0.43016(6)             | -1.70500(2)            |
| -0.4028777      | -2.5061(2)             | 0.41594(7)             | -1.70128(3)            |
| -0.3140433      | -2.2325(4)             | 0.32325(7)             | -1.67496(2)            |
| -0.3099631      | -2.2202(4)             | 0.31925(9)             | -1.67373(1)            |
| -0.3017750      | -2.1956(4)             | 0.31127(7)             | -1.67125(1)            |
| -0.2962964      | -2.1793(3)             | 0.30597(2)             | -1.66958(1)            |
| -0.2852897      | -2.1460(4)             | 0.2953(2)              | -1.66621(2)            |
| -0.2825278      | -2.1379(4)             | 0.2927(2)              | -1.66536(2)            |
| -0.2769916      | -2.1215(5)             | 0.2876(1)              | -1.66366(2)            |
| -0.2686568      | -2.0972(1)             | 0.2798(2)              | -1.66108(3)            |
| -0.1482168      | -1.7519(2)             | 0.17636(9)             | -1.62264(9)            |
| 0.0000          | -1.36897(4)            | 0.077477(3)            | -1.57093(3)            |
| 0.0050          | -1.35710(1)            | 0.074754(4)            | -1.56906(3)            |
| 0.0100          | -1.34534(4)            | 0.072076(3)            | -1.56720(3)            |
| 0.0140          | -1.33599(4)            | 0.069967(1)            | -1.56570(2)            |
| 0.0160          | -1.33134(4)            | 0.068929(3)            | -1.56494(2)            |
| 0.0180          | -1.32671(4)            | 0.067895(2)            | -1.56419(1)            |
| 0.0236          | -1.31382(4)            | 0.065038(5)            | -1.56207(1)            |
| 0.0270          | -1.30603(5)            | 0.063335(4)            | -1.56077(3)            |
| 0.0350          | -1.28801(6)            | 0.09418(3)             | -1.55773(6)            |
| 0.0366          | -1.28443(5)            | 0.058649(4)            | -1.55711(6)            |
| 0.0380          | -1.28131(5)            | 0.057982(4)            | -1.55658(6)            |
| 0.0427          | -1.27089(5)            | 0.055767(4)            | -1.55476(7)            |
| 0.0460          | -1.26363(5)            | 0.054238(5)            | -1.55347(7)            |
| 0.0535          | -1.24729(4)            | 0.05084(2)             | -1.55051(5)            |
| 0.0550          | -1.24405(5)            | 0.05018(2)             | -1.54991(5)            |
| 0.0720          | -1.20807(2)            | 0.04295(2)             | -1.54317(6)            |
| 0.0927          | -1.1658(2)             | 0.03486(3)             | -1.53475(5)            |
TABLE IV. Evaluation of $e_{4,0}$ (cf. Eqs.(17, 19, 20), Figure 3), with Wilson gluons and overlap fermions, for various values of $M_0$, $0 < M_0 < 2$.

| $M_0$ | $e_{4,0} \times 10^8$          |
|-------|-------------------------------|
| 0.01  | 0.59855(5)                    |
| 0.05  | 0.63347(4)                    |
| 0.10  | 0.6769(2)                     |
| 0.20  | 0.7628(2)                     |
| 0.30  | 0.8451(2)                     |
| 0.40  | 0.92220(3)                    |
| 0.50  | 0.99357(4)                    |
| 0.60  | 1.05872(3)                    |
| 0.70  | 1.11725(2)                    |
| 0.80  | 1.16893(3)                    |
| 0.90  | 1.213650(1)                   |
| 1.00  | 1.251396(7)                   |
| 1.10  | 1.282246(6)                   |
| 1.20  | 1.306376(2)                   |
| 1.30  | 1.32406(2)                    |
| 1.40  | 1.33561(4)                    |
| 1.50  | 1.34149(2)                    |
| 1.60  | 1.34201(7)                    |
| 1.70  | 1.3373(2)                     |
| 1.80  | 1.3271(4)                     |
| 1.90  | 1.308915(8)                   |
| 1.95  | 1.294780(1)                   |
| 1.99  | 1.280580(7)                   |
Fig. 4: Variation of the terms contributing to $e_4^f$ as a function of $m$

Fig. 5: Value of $e_4$ as a function of $m$
Fig. 6: Variation of $e_{4,0}$ as a function of $M_0$

![Figure 6: Variation of $e_{4,0}$ as a function of $M_0$]

Fig. 7: Value of $e_4$ as a function $M_0$

![Figure 7: Value of $e_4$ as a function $M_0$]
REFERENCES

[1] M. Campostrini, A. Di Giacomo and H. Panagopoulos, Phys. Lett. B 212 (1988) 206.
[2] R. Horsley et al., Nucl.Phys. B693 (2004) 3 [Erratum-ibid. B713 (2005) 601].
[3] H. Panagopoulos and E. Vicari, Nucl. Phys. B332 (1990) 261.
[4] M. Lüscher and P. Weisz, Nucl.Phys. B452 (1995) 234.
[5] K. Symanzik, Nucl. Phys. B 226 (1983) 187.
[6] M. Lüscher and P. Weisz, Commun. Math. Phys. 97 (1985) 59 [Erratum-ibid. 98 (1985) 433].
[7] M. G. Alford, W. Dimm, G. P. Lepage, G. Hockney and P. B. Mackenzie, Phys. Lett. B 361 (1995) 87.
[8] Y. Iwasaki, UTHEP-118 (1983).
[9] T. Takaishi, Phys. Rev. D54 (1996) 1050.
[10] B. Allés, M. Campostrini, A. Feo and H. Panagopoulos, Nucl. Phys. B413 (1994) 553.
[11] H. Neuberger, Phys. Lett. B 417 (1998) 141; Phys. Lett. B 427 (1998) 353.
[12] B. Sheikholeslami and R. Wohlert, Nucl. Phys. B259 (1985) 572.
[13] P. Di Vecchia, K. Fabricius, G. C. Rossi and G. Veneziano, Nucl. Phys. B192 (1981) 392.
[14] C. Christou, A. Di Giacomo, H. Panagopoulos and E. Vicari, Phys. Rev. D53 (1996) 2619.
[15] B. Allés, M. D’Elia and A. Di Giacomo, Phys. Rev. D71 (2005) 034503.
[16] L. Giusti, G. C. Rossi and M. Testa, Phys. Lett. B 587 (2004) 157.
[17] MILC Collaboration (C. Aubin et al.), Nucl. Phys. B (PS) 140 (2005) 626.
[18] L. Giusti, M. Lüscher, P. Weisz, H. Wittig, J. High Energy Phys. 11 (2003) 023.
[19] L. Del Debbio, L. Giusti, C. Pica, Phys. Rev. Lett. 94 (2005) 032003.