Linear programming feedrate optimization - Adaptive path sampling and feedrate override

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Abstract This paper focuses on two aspects of feedrate optimization via linear programming methods. Namely, the effect of curve sampling on time optimality of the resultant feedrate profile and a method of feedrate profile adaptation in response to a feedrate override command. A comparison of three distinct curve sampling approaches (uniform in parameter, uniform in arc length and curvature adaptive) is performed on a series of standard tool path curves. Results show that the curvature-adaptive sampling approach leads to substantial machining time reduction for tool path curves displaying high degree of curvature variation. Secondly, a method by which a new feedrate profile can be calculated in response to a feedrate override command is developed. The method formulates a new set of boundary conditions on the control point sequence of the feedrate curve in such a way that the resulting profile is guaranteed to coincide with the currently active profile up to the moment of override command, while minimizing the arc length necessary for transition to the newly commanded feedrate.

Keywords Linear programming · Feedrate optimization · Feedrate override · NURBS curve · Curve sampling

1 Introduction

Interpolator is a component of the computer numerical control (CNC) system, which governs axial servo drives by providing position commands. Vectors of axis positions are being sent to servo drives at specified frequency. In a precision and high-speed machining environment, interpolator is required to optimize axial movement, so that maximum accuracy and shortest machining time are achieved. Considerable effort has been put by researchers into developing algorithms capable of interpolating tool paths which are determined by parametric curves such as NURBS. NURBS curves (and surfaces) are a staple in computer assisted design due to their compact representation and efficient evaluation. The necessity of generating feedrate profiles on NURBS tool paths has led to the development of a variety of distinctive interpolation methods.

To create efficient feedrate profiles for NURBS curve tool paths, it is necessary to consider the relationship between the curve’s parameter and its arc length. This relationship does not have a closed form solution in general and disregarding it can lead to undesirable feedrate oscillations. This problem was typically solved using Taylor series expansion (see, e.g. [16]). This, of course, introduces errors caused by omitting higher order terms. As a possible solution to this problem, the so-called Pythagorean hodograph curves have been proposed as an alternative to NURBS curve representa-
ations (for a comprehensive study of Pythagorean hodo-
graph curves see [15]). Pythagorean hodeograph curves
enjoy the nice property of polynomial dependence of
arc length on curve parameter, making them particu-
larly useful for CNC interpolations. However, due to
the widespread use of NURBS curves in CAD systems,
Pythagorean hodeograph curves still have not found wide
application in common practice.

In order to solve the problem of parameter-to-arc
length relationship in NURBS curves, it is necessary
to devise approximation methods. In [10], the authors
develop a recursive approximation algorithm to interpo-
late quintic spline tool paths with constant distance in-
crement. The feedrate profile is constructed by varying
the interpolation period, which is then reconstructed in
the control loop period by fitting a fifth-degree poly-
nomial. Another method that approximates the relation-
ship between the spline parameter and its arc length is
presented in [11]. The main idea is to use a seventh-
degree polynomial (feed correction polynomial) to ap-
proximate this relationship. The coefficients of this poly-
nomial are estimated using least squares. This initial
approximation is then further refined using the New-
ton-Raphson iterative method. This method has been
further modified in [17], where the parameter-to-arc
length relationship is first approximated for a series of
discrete points using the adaptive Simpson rule and this
discrete representation is then used to fit a series of fee-
drate polynomials depending on a maximal allowed de-
viation. Due to the adaptive nature of this algorithm, it
can approximate the parameter-to-arc length rela-
tionship with arbitrary precision. This particular algorithm
was used for the purposes of this paper.

The most common approaches to feedrate optimiza-
tion found in current literature are: composing the fee-
drate profile by connecting S-shaped feedrate profiles
with piecewise constant jerk (constructed analytically
or by using FIR filters), brute force optimization meth-
ods and utilization of linear programming (LP). The
last mentioned approach combines computational effi-
ciency with the ability to construct nearly time-optimal
feedrate profiles. First presented in [14], this method of
feedrate optimization is based on the use of tool path
discretization followed by reformulation of the feedrate
optimization problem where the feedrate function is
represented by a B-spline curve whose control points
serve as the free variables. Finding the control points
poses a non-linear optimization problem, which requires
the introduction of the so-called pseudojerk, that serves
as an upper bound to the jerk inequality, thus allow-
ing to linearize the optimization problem. This upper
bound is precomputed using only axial velocity and ac-
celeration limits. The proximity of this solution to true
optimal feedrate profile is limited by the finite num-ber of control points for feedrate profile curve and the
pseudo-jerk relaxation. This method was further ex-
panded on in [13], where a windowing based paralleliza-
tion method was developed in order to further improve
the computational efficiency of the original algorithm.

It is of note, that the methods of linear program-
ning have also been applied to the feedrate optimiza-
tion problem in an earlier article [1], although in a sub-
stantially different way. The feedrate profile on a spline
toolpath was defined as a function of time and was ob-
tained by minimizing a square integral of jerk via man-
ipulation of time durations of selected path segments.
This solution did not, however, consider the influence of
the parameter-to-arc length relationship. For a similar
approach using time parametrisation, see [19].

Another computationally efficient method to con-
struct the feedrate profile for NURBS tool paths is
based on connecting piecewise jerk constant S-curve
type feedrate transition segments. Feedrate can be mod-
ulated in this way to comply with kinematic limits which
are evaluated at a fixed set points (see [12]). Similarly,
in [17] feedrate profiles for spline tool paths are gen-
erated by connecting constant feedrate segments by S-
shaped curve transition segments according to a heuris-
tic algorithm. Axis jerk and acceleration limits are eval-
uated at knot locations. For a given portion of the spline
trajectory, which is bounded by two knots, the maxi-
mum feedrate is calculated based on these constraints.
The algorithm performs a search for maximum feedrate
that is both achievable and respects constraints in all
portions of the spline trajectory.

In [3], an axis movement smoothing algorithm for 5-
axis milling is developed, which utilizes a heuristic algo-
rithm to conform to contour tolerance limits. In [26],
the authors describe a look-ahead algorithm, including an-
alytical expressions, for interpolating linear segments.
The article [25] presents a corner smoothing technique
for five-axis machining using micro splines inserted be-
tween consecutive linear blocks with synchronized po-

An alternative approach to generating feedrate pro-
files relies on the application of sequences of FIR filters.
The authors of [6] show that applying a sequence of moving
average filters to the feedrate impulse produces a feedrate
function that is equivalent to the analyti-
cally expressed S-shaped feedrate profile. Their work
provides a relationship between feedrate profile and its
frequency domain, which can be utilized to suppress
vibrations on given resonant frequencies by adjusting
filter parameters. In [2], analytically generated accelera-
tion limited feedrate profile is combined with the FIR
approach to generate feedrate profile for arbitrary ve-
velocity and acceleration conditions. Finally, \cite{24} extends this method by including circular and linear toolpath blending capability with confined contour error. A transition between linear and circular segments can thus be performed at non-zero feedrates by controlling the convolution overlap time of two consecutive feed impulses. This method was recently extended to 5-axis machining in \cite{24}.

Iterative method, referred to as VPOp, can be applied to interpolate NURBS toolpaths even in 5-axis milling. This method achieves short (shortest among all the mentioned approaches) machining times while respecting axis acceleration and jerk constraints. However, its high computational demands severely limit its real world application. Articles \cite{4} and \cite{5} use the VPOp algorithm to interpolate directly into the parametric space of the surface of the workpiece CAD model. This original approach removes the issue of CAM tolerance and interpolation tolerance stacking up.

To summarize the above comparison, linear programming methods produce results which are close to optimal even on complex curves. Given a sufficient number of discretization points, the method is able to respect all the given kinematic limits while maintaining computational efficiency. The biggest disadvantage is the necessity of linearization of the jerk constraint, which results in loss of optimality of the solution. Nevertheless, when compared to other methods the conciseness of the mathematical formulation, ability to simultaneously incorporate both tangential and axial limits on kinematic variables and the possibility of parallelization all make linear programming methods a favorable choice.

This paper is organized as follows: Section \ref{2} reviews the formulation of the linear programming optimization problem (Section 2.1) as well as knot vector construction and evaluation point selection given a general sampling of the toolpath curve (Section 2.2).

In Section 3 three types of curve sampling (equidistant in curve parameter, equidistant in arc length and adaptive) are presented and the effects of sampling approach on time optimality of the resultant feedrate are discussed based on a comparison for several test curves. The main original result of this section is that an application of a suitable adaptive sampling method can lead to a significant decrease of machining time.

Section 4 is dedicated to an original method of feedrate override for linear programming feedrate optimization. The method is divided into two distinct cases: override to a higher commanded feedrate (Section 4.2) and override to a lower commanded feedrate (Section 4.3). This method describes how, and under what conditions a new feedrate profile implementing the override command can be obtained given curve parameters and the currently active feedrate profile. The method is suitable for application purposes due to low memory requirements and relative ease of implementation.

The article is concluded with several closing remarks in Section 5 and two appendices: Appendix 7 providing definitions of the test path curves used in Sections 3 & 4 and Appendix 8 in which a definition of an auxiliary function is given.

2 Feedrate optimization - a linear programming problem

This chapter recalls the formulation of feedrate optimization via linear programming (Section 2.1) while briefly discussing knot vector construction and the selection of evaluation points (Section 2.2).

2.1 Problem formulation

Assume that the tool path is defined as a NURBS curve with at least \(C^2\) continuity (in practice this typically means a curve of order three of five)

\[
r(u) = [x(u), y(u), z(u)], \quad u \in [0, 1]
\]

In order to formulate the relations between velocity, acceleration and jerk along the curve, it is beneficial to introduce the arc length parametrization so that one can also express \(r\) as

\[
r(s) = [x(s), y(s), z(s)], \quad s \in [0, L]
\]

where \(L\) represents the total length of the curve. Let \(v_{\text{max}}, a_{\text{max}}, j_{\text{max}}\) denote the maximal axial limits of velocity, acceleration and jerk, respectively. Furthermore, let \(\bar{v}_{\text{max}}, \bar{a}_{\text{max}}, \bar{j}_{\text{max}}\) denote the maximal limits of tangential velocity, tangential acceleration and tangential jerk, respectively. The axial velocity, acceleration and jerk can be expressed as:

\[
v = \frac{dr}{dt} = r' \dot{s}
\]

\[
a = \frac{d^2r}{dt^2} = r'' \dot{s}^2 + r' \ddot{s}
\]

\[
j = \frac{d^3r}{dt^3} = r''' \dot{s}^2 + 3r'' \dot{s} \ddot{s} + r' \dddot{s}
\]

where \(r, r'\) and \(r''\) denote the derivatives of \(r\) with respect to \(s\), while \(\dot{s}\), \(\ddot{s}\) and \(\dddot{s}\) denote the tangential velocity (feedrate), tangential acceleration and tangential jerk, respectively. The feedrate optimization problem,
i.e. derivation of time-optimal feedrate profile, can then be formulated as:

\[
\text{maximize } \int_0^L \hat{s} \, ds
\]
such that

\[
\begin{align*}
\nu_{\text{max}} & \geq |r'\hat{s}| \\
a_{\text{max}} & \geq |r''\hat{s}^2 + r'\hat{s}| \\
\hat{j}_{\text{max}} & \geq |r'''\hat{s}^2 + 3r''\hat{s}\hat{s} + r'\hat{s}'| \\
\hat{v}_{\text{max}} & \geq |\hat{s}| \\
\hat{a}_{\text{max}} & \geq |\hat{s}|
\end{align*}
\]

In order to linearize the above problem, the authors of [14] (see also [13]) apply the following substitution:

\[
\hat{s}^2 = q(s) = \sum_{i=1}^K N_{i,2}(s) \cdot a_i = N(s) \cdot a,
\]
i.e. the square of feedrate is expressed as a cubic B-spline where \(a = [a_1, \ldots, a_K]\) is the vector of control points and \([N_{1,2}, \ldots, N_{K,2}]\) are the basis functions.

Using the above substitution the optimization problem can be reformulated as

\[
\text{maximize } \int_0^L q(s) \, ds \quad (1)
\]
such that

\[
\begin{align*}
(\nu_{\text{max}})^2 & \geq |r|^2 q \\
(a_{\text{max}})^2 & \geq |r''q + \frac{1}{2} r'q'| \\
(\hat{j}_{\text{max}})^2 & \geq |r'''q + \frac{3}{2} r''q' + r'q''| \sqrt{q} \\
(\hat{v}_{\text{max}})^2 & \geq q \\
(\hat{a}_{\text{max}})^2 & \geq \left| \frac{1}{2} q' \right| \\
(\hat{j}_{\text{max}})^2 & \geq \left| \frac{1}{2} q' \right| \sqrt{q}
\end{align*}
\]

Except for the square root of \(q\) appearing in the inequalities (2a) and (2e), the above optimization problem can be posed as a linear optimization problem in which the control points \(a_i\) represent the free variables.

To overcome the nonlinearity, the optimization problem (1) is first solved without the tangential and axis jerk constraints, leading to the LP formulation:

\[
\text{maximize } c^T \hat{a} \text{ subject to: } \hat{M} \hat{a} \leq \hat{b} \quad \hat{a} \geq 0,
\]
where the vector \(c^T\) is a uniformly distributed weighting vector. The matrix \(\hat{M}\) is a constant matrix, whose terms are obtained via evaluation of the constraint equations (2a), (2b), (2d) and (2e) at a set of evaluation points. In this way a solution \(q\) (so-called pseudojerk) is obtained which realizes a larger feedrate than any other feasible solution. In the next step, the following constraints are substituted for the original jerk constraints (2c) and (2f), respectively.

\[
\begin{align*}
\hat{j}_{\text{max}} & \geq \left| r'''q + \frac{3}{2} r''q' + r'q'' \right| \sqrt{q} \\
\hat{j}_{\text{max}} & \geq \left| \frac{1}{2} q' \right| \sqrt{q}
\end{align*}
\]

Thus, the approximate solution of the original optimization problem (1) can be formulated as:

\[
\text{maximize } c^T a \text{ subject to: } \quad Ma \leq b \quad a \geq 0.
\]

### 2.2 Linear programming - evaluation point and knot vector construction

To the authors’ best knowledge, none of the previous works describe how exactly should the knot vector and evaluation points of the feedrate curve be constructed given a sampling of the toolpath \(r\). The process is thus briefly described below for the reader’s convenience. Given a sampling in the arc-length parameter \(s\)

\[
\Gamma = \{s_1 = 0, s_2, \ldots, s_{K-1}, s_K = L\}, \quad K \in \mathbb{N}
\]
the knot vector of the squared-feedrate function \(q\) of order \(d\) is defined as

\[
U = \left\{ \frac{0, \ldots, 0, s_2, \ldots, s_{K-1}, L, \ldots, L}{x_{d}} \right\}. \quad (4)
\]

Next, a sequence of evaluation points needs to be determined such that for every basis function \(N_{i,d}\), there exists at least one point in this sequence lying in its support. A suitable choice is the sequence of Greville points (Greville abscissae) defined as

\[
G = \{ g_1, \ldots, g_{N-d} \}, \quad N \in \mathbb{N}
\]
where \(N\) is the number of control points of \(q\) and \(d\) is the order of \(q\) (defined as degree of \(q + 1\)) and

\[
g_i = \frac{1}{d-1} \left( u_{i+1} + \cdots + u_{i+d-1} \right), i \in \{ 1, \ldots, (N-d) \}.
\]

The Greville point \(g_i\) generally lies near the parameter value corresponding to the maximum of the basis function \(N_{i,d}\) [21, p. 512]. The main computational advantage of this particular selection of evaluation points
is that the matrices $\hat{M}$ and $M$ (which comprise the evaluations of the respective constraints at points of $G$) become sparse band matrices with band equal to $d$, thus increasing stability and efficiency of linear programming optimization methods.

3 Curve sampling methods

The simplest way to sample the toolpath curve is to sample the curve parameter interval uniformly, i.e.

$$I_M^u = \left\{ 0, \frac{1}{M-1}, \ldots, \frac{M-2}{M-1}, 1 \right\}, \quad M \in \mathbb{N}.$$  

This sampling technique is denoted as PAR in the following.

Somehow surprisingly, very little attention has been dedicated to the question of sampling of the underlying toolpath curve and its effect on time optimality of the resulting feedrate profile. To the author’s best knowledge, only sampling uniform in the curve parameter (PAR) has been considered in literature dealing with feedrate optimization via the LP approach (see e.g. [14], [7], [9], [22] and [27]). It is of note, however, that in [14] the authors remark that a nonuniform subdivision of the tool path that would take into consideration its local shape could result in a better performance of the algorithm.

An alternative to the uniform parameter sampling is the uniform arc length sampling (denoted as LEN)

$$I_M^s = \left\{ s_1 = 0, \ldots, s_M = 1 \right\},$$

such that

$$\text{arc length of } r_{[s_i, s_{i+1}]} = \frac{L}{M}, \quad i \in \{1, \ldots, (M-1)\}.$$  

To apply this sampling technique the information about the parameter to arc length relation is required. As this relation cannot be computed analytically in general, it is necessary to find a sufficiently close approximation. To this end, the method described in [17] was used (note that this does not pose an extra requirement, as the arc length evaluation is also used in the formulation of the LP optimization problem).

Intuitively, one would expect that a sampling technique would produce a set of sampling points with density that is proportional to the curvature of the sampled curve, while maintaining some minimal point density in areas of zero curvature so that the degree of control over the squared feedrate function $q$ correlates with the local geometric properties of the toolpath curve.

An adaptive algorithm with precisely those properties was proposed (along with a detailed description of its implementation) in [18, p. 1471-1476]. The inputs of this algorithm are: $m$, the desired number of sampling points, non-negative weights $\lambda_s$, $\lambda_c$ satisfying $\lambda_s + \lambda_c = 1$ and a threshold parameter $\varepsilon_c > 0$ denoting a minimum integrated curvature amount to be taken into account by the algorithm. The values of $\lambda_s$ and $\lambda_c$ represent contributions of arc length and curvature, respectively, to the density of the set of sampling points. Thus, the combination $\lambda_s = 1$, $\lambda_c = 0$ results in a uniform arc length sampling, while the combination $\lambda_s = 0$, $\lambda_c = 1$ results in a sampling set with zero point density in areas of zero curvature.

In the following, this adaptive algorithm is referred to as ADA.

For a comparison of the outputs of the three sampling variants see Figures 1, 2, 3 and 4.

Fig. 1 PAR sampling with $m = 65$

Fig. 2 LEN sampling with $m = 65$
3.1 Sampling techniques - comparison of results

In this section, the effects of the PAR, LEN and ADA sampling techniques on the total machining time of the LP optimization output are presented and discussed. The algorithm was programmed in Matlab2017a software in combination with C++ code for the LP optimization and NURBS curve evaluation via the MEX interface. The COIN-OR Linear programming solver has been used to solve the LP optimization task, while the C++ openNURBS® library has been used to construct the NURBS curves and evaluate their derivatives. All computations were performed on a computer with an Intel® Core™ i7-7700K processor and Windows 10 operating system.

In order to compare the optimization results several testing curves have been used. These include the Trident curve, the Butterfly curve, the Pentacle curve and the Phobos curve (see Appendix A, Figures 14-17). With the exception of the Phobos curve, all of the testing curves have been previously used in articles concerning feedrate interpolation and optimization. All the curves are third degree continuous and display different behaviors regarding maximal and minimal values of curvature and its rate of variation. The Trident and Phobos curve comprise segments of zero curvature and segments of slowly varying curvature. The curvature of the Pentacle curve varies slowly, while the Butterfly curve displays both highest absolute values of curvature and highest rates of its variation. The Phobos curve represents a spline smoothed contour curve of a blade cross section designed in a commercial CAD system.

For each curve, the LP feedrate optimization has been performed with the kinematic limits configuration presented in Table 1. This configuration corresponds to one used on a AXA VCC 1200 machining center equipped with a MEFI CNC872 iTQ-E numerical control system (developed in part by the authors). Each curve was then sampled with increasingly larger values of sampling density (defined as number of points per mm of arc length). For every such sampling density the PAR, LEN and ADA sampling methods were used, the respective machining times were recorded and the relative percentage differences (rounded to the nearest percentage point) of the PAR and LEN methods with respect to the ADA method were calculated. The parameters of the ADA method were chosen as \( \lambda_s = \lambda_\kappa = 0.5 \) and \( \varepsilon_\kappa = 1 \cdot 10^{-3} \). The results for individual curves are presented in Figures 5, 6, 7 and 8. A comparison of feedrate functions for the Butterfly and Trident curves is presented in Figures 9 and 10, respectively.

| Parameter | Value | Units |
|-----------|-------|-------|
| \( \bar{v}_{\text{max}} \) | 10 | m/min |
| \( \bar{a}_{\text{max}} \) | 1500 | mm/s² |
| \( \bar{j}_{\text{max}} \) | 16000 | mm/s³ |
| \( v_{\text{max}} \) | [10, 10, 10] | m/min |
| \( a_{\text{max}} \) | [1500, 1500, 1500] | mm/s² |
| \( j_{\text{max}} \) | [16000, 16000, 16000] | mm/s³ |

The results presented in Figures 5-8 support the following conclusions: For curves with low curvature variation such as the Trident, Pentacle and Phobos curves, the ADA sampling technique leads to machining times that are comparable (±5%) with the PAR and LEN methods, while typically being a few percent faster. On the other hand, for curves with high curvature variation, such as the Butterfly curve, the ADA sampling technique leads to machining times that are faster (up to 12%) than the LEN and ADA methods. This behavior is due to the higher density of knot points of the squared feedrate function in areas of high curvature.
Thus, the feedrate profile can reflect the local changes of toolpath geometry more closely in these regions, leading to shorter machining times (while still respecting the kinematic limits).

In conclusion, the ADA sampling technique in combination with the LP optimization approach described in Section 2.1 can lead to significantly shorter machining times when interpolating spline toolpaths with both high curvature variation and high maximal curvature. Such toolpaths are typically encountered in practice in machining of injection molds and in side milling (specifically in trimming operations).

4 Feedrate override

One of the standard commands in machining is the feedrate override command. With this command, the machine tool operator can change the value of commanded feedrate $F_{cmd}$ to a different value expressed as a percentage of the originally programmed value.

The aim of this section is to present techniques by which the squared feedrate function that was obtained using methods described in Section 2 can be appropriately modified in response to an issued feedrate override command.

This chapter is divided into three section: notation is summarized in Section 4.1, method of override accel-
eration is presented in Section 4.2 and, finally, method of override deceleration is presented in Section 4.3.

4.1 Notation

Most of the notation in the following sections follows that introduced in Section 2. Parameters associated with pseudojerk are distinguished by the use of a hat symbol. The knot vector of the squared feedrate curve is denoted by \( q \) and the evaluation point vector is denoted by \( g \). The number of control points is denoted by \( N \), while the number of evaluation points is denoted by \( K \) (note that \( K = (N - d) \), see 5). For the sake of brevity the feedrate and acceleration at the \( i \)-th evaluation point are denoted by \( F(i) \) and \( a(i) \), respectively. The currently active commanded feedrate is denoted by \( F_{\text{cur}} \), while the commanded feedrate imposed by feedrate override is denoted as \( F_{\text{ovr}} \). The currently active feedrate profile is denoted by \( F \), while the feedrate profile realizing the override command are denoted by \( F_{\text{cur}} \) and \( F_{\text{ovr}} \), respectively.

Subscripts are used to distinguish parameters pertaining to individual feedrate profiles. Specifically, parameters of the currently active feedrate profile are denoted by the subscript \( \text{cur} \) and parameters related to the feedrate profile realizing the override command are denoted by the subscript \( \text{ovr} \). Parameters related to feedrate profiles globally limited by the override command are denoted by \( F_{\text{cur}} \) and \( F_{\text{ovr}} \), respectively. The range of control points given by (6) defines the maximal range of basis functions of the feedrate profile whose supports intersect the interval \([s_n, s_{n+1}]\) and which therefore influence the values of the feedrate profile on this neighborhood of \( s \). The range of control points is further necessary to check whether the rest of the feedrate profile reaches current commanded feedrate \( F_{\text{cur}} \) and whether the override command was not issued during, or immediately before, the final deceleration phase of the current feedrate profile. To check this, consider the point \( g_{J_s} \) where

\[
J_s = \min\{k : g_k \geq s_{I_l}\}.
\]

The point \( g_{J_s} \) is the first evaluation point after the point of override for which the corresponding value of the feedrate profile is unaffected by the choice of \( \{a_{I_1}, \ldots, a_{I_d}\} \). If the feedrate at the end of the curve is fixed, then if the rest of the feedrate profile does not reach the current commanded feedrate, i.e.

\[
\max\{q_{\text{cur}}(g_j), j \in \{J_s, \ldots, K\}\} < F^2_{\text{cur}},
\]

the computation of the override profile can be skipped. Similarly, if the override command has been issued during (or immediately before) the final deceleration phase, i.e.

\[
q_{\text{cur}}(g_{J_s}) > q_{\text{cur}}(g_{I_l+1}) > \cdots > q_{\text{cur}}(g_K),
\]

the computation of the override profile is skipped. This follows from the time-optimality of the current feedrate profile (the deceleration starts as late as possible considering the acceleration and jerk limits). Obviously, the previous two arguments do not apply if the end feedrate is allowed to rise above \( F_{\text{cur}} \).

As a second step, the feedrate profile with maximal feedrate given by \( F_{\text{high}} \) is constructed. The override profile (denoted by the subscript \( \text{ovr} \)) is then constructed in two steps. Firstly, the pseudojerk \( \hat{q}_{\text{ovr}} \) of the override profile is constructed by solving the LP optimization problem:

\[
\begin{align*}
\text{maximize } & c^T \hat{a}_{\text{ovr}} \\
\text{subject to: } & \quad \hat{M} \hat{a}_{\text{ovr}} \leq \hat{b}_{\text{ovr}} \\
& \quad \hat{b}_{\text{ovr}} \leq \hat{a}_{\text{ovr}} \leq \hat{ub}_{\text{ovr}},
\end{align*}
\]

the computation of the override profile is skipped. This follows from the time-optimality of the current feedrate profile (the deceleration starts as late as possible considering the acceleration and jerk limits). Obviously, the previous two arguments do not apply if the end feedrate is allowed to rise above \( F_{\text{cur}} \).
The process of override transition to a new feedrate limit $F_{\text{low}} = a_{\text{ovr}} \cdot F_{\text{cur}}$ requires a more involved approach than the case of override transition to a higher feedrate limit. The main idea is the following: first, as in Section 4.2 the first evaluation point $g_J$, at which the override profile is allowed to deviate from the original profile is found. Then, for each following evaluation point $g_{\text{next}}$, the arc length necessary to transition from the current profile’s feedrate at $g_J$ to the feedrate value of the $F_{\text{low}}$-commanded feedrate profile at $g_{\text{next}}$ is estimated. The first evaluation point for which this arc length estimate is not higher than the actual arc length between $g_J$ and $g_{\text{next}}$ is then used in the LP formulation of the override profile to define the range of control points to be bounded from above by the respective values of the $F_{\text{low}}$-commanded profile’s control points. The rest of this section explains the algorithm in detail.

As in Section 4.2, the first order of business is to define the range of control points of the override profile which need to be fixed (6) and the index $J_s$ of the first evaluation point at which the current profile and the override profile are allowed to deviate (7).

Secondly, as in Chapter 4.2, if the override command was issued during or immediately before the final deceleration phase (5), the override profile’s construction should be skipped. This follows from the time optimality of the current feedrate profile (the deceleration is as fast as possible considering the acceleration and jerk limits).

If the override command was issued before the final deceleration phase, a feedrate profile $q_{\text{low}}$ with commanded feedrate given by $F_{\text{low}}$ is constructed. The final velocity of $q_{\text{low}}$ is defined as

$$F_{\text{end,low}} = \min \{ F_{\text{end,cur}}, F_{\text{low}} \}.$$

Next, the feedrate and tangential acceleration values of $q_{\text{cur}}$ and $q_{\text{low}}$ are evaluated at the point $g_J$. These values are denoted as $[F_{\text{cur}}(J_s), a_{\text{cur}}(J_s)]$ and $[F_{\text{low}}(J_s), a_{\text{low}}(J_s)]$, respectively. The exact form of the override algorithm then depends on whether $F_{\text{cur}}(J_s) \leq F_{\text{low}}(J_s)$ (Section 4.3.1), or $F_{\text{cur}}(J_s) > F_{\text{low}}(J_s)$ (Section 4.3.2).

### 4.3.1 Case I: $F_{\text{cur}}(J_s) \leq F_{\text{low}}(J_s)$

In this case, it is not necessary to compute any kind of estimate of the arc length necessary to transition from the current profile’s feedrate to the feedrate of $q_{\text{low}}$. Instead, it is sufficient to take the appropriate range of
control points of the $q_{low}$ profile as the upper boundary in construction of the override profile. To this end, define an index $I_t$ as

$$I_t = I_l + d.$$ 

If $I_t \geq (N - 1)$, the rest of the trajectory is not long enough to decelerate below the commanded feedrate. Otherwise, $\tilde{q}_{ovr}$ is constructed by solving the LP problem [9], where

$$\tilde{b}_i = \begin{cases} 
F_{start}^2, & i = 1, \\
0, & i \in \{2, ..., I_f - 1\}, \\
(a_{cur}), & i \in \{I_f, ..., I_l\}, \\
0, & i \in \{I_l + 1, N - 1\}, \\
F_{end_{low}}, & i = N,
\end{cases}$$

and

$$\tilde{u}_i = \begin{cases} 
F_{start}^2, & i = 1, \\
(a_{cur}), & i \in \{2, ..., I_l\}, \\
F_{cur}^2, & i \in \{I_l + 1, I_f - 1\}, \\
(a_{low}), & i \in \{I_l, ..., N - 1\}, \\
F_{end_{low}}, & i = N.
\end{cases}$$

The feedrate limitation of the right-hand side $\tilde{b}_{ovr}$ is given by the feedrate values of the current feedrate profile up to $g_j$, and by $F_{new}$ from $g_j$ onward. The override profile $q_{ovr}$ is then found by solving the LP problem:

$$\begin{align*}
\text{maximize } c^T a_{ovr} & \text{ subject to: } \\
Ma_{ovr} & \leq b_{ovr} \\
\tilde{b}_{ovr} & \leq a_{ovr} \leq \tilde{u}_{ovr}
\end{align*}$$

where the velocity-acceleration part of $b_{ovr}$ is equal to $\tilde{b}_{ovr}$, while the jerk part is constructed using the values of $\tilde{q}_{ovr}$.

For an example of case I override, see Figure 12.

4.3.2 Case II: $F_{cur}(J_s) > F_{low}(J_s)$

To formulate the upper and lower boundaries of the LP solution of the override profile, an upper estimate of the arc length necessary to realize the transition from $F_{cur}(J_s)$ to $F_{low}(k)$ (for some as yet unknown index $k$) is required.

First, define an index $J_l$ as

$$J_l = \min \{ \{ k > J_s : q_{cur}(g_k) \leq q_{low}(g_k) \} \cup \{K\} \}$$

(i.e. the index of the first evaluation point after $g_J$, for which $q_{cur}$ is bounded from above by $q_{low}$. If no such point exists, the index is set as the index of the last evaluation point).

Next, to estimate the minimal arc length necessary for a transition from $q_{cur}$ to $q_{low}$, the following algorithm is used: (for the definition of the ArcLen function, see Appendix 8, Algorithm 3 and Algorithm 4). Algorithm 1 searches for the closest evaluation point of the feedrate transition.

**Algorithm 1: Transition arc length estimate**

**Input:** \(\{F_{cur}(k)\}_{J_s}^{\infty}, \{F_{low}(k)\}_{J_s}^{\infty}, \{a_{cur}(k)\}_{J_s}^{\infty}, \{a_{low}(k)\}_{J_s}^{\infty}\)

**Output:** $J_e$ // index of first possible end point of feedrate transition

**Initialization:** $J_e = J_s$;

for ( $k = J_s + 1$; $k \leq J_l$; $k = k + 1$ )

if $F_{cur}(J_s) > F_{low}(k)$ then

\[ s = \text{ArcLen}(F_{cur}(J_s), F_{low}(k), a_{cur}(J_s), a_{low}(J_s), a_{max}, j_{max}) \]

else

\[ s = \text{ArcLen}(F_{low}(k), F_{cur}(J_s), a_{low}(k), a_{cur}(J_s), a_{max}, j_{max}) \]

if $s \leq (g_k - g_{J_s})$ then

$J_k = k$;

return $J_k$;

return $J_e$;

If $J_e = K$, the remaining arc length of the path curve is not sufficient to realize the feedrate transition and the override command should be skipped (or postponed to the start of the following path segment). Otherwise, a control point index $I_t$ is defined as

$$I_t = \min \{ \{ k : s_k \geq g(J_e) \} \} + (d - 1).$$

Index $I_t$ denotes the first control point for which the support of the corresponding basis function lies past
the evaluation point \( g(J_e) \). If \( J_e \) is undefined or \( J_e \geq K \), the remaining arc length of the path curve is not sufficient to realize the feedrate transition and the override command should be skipped (or postponed to the start of the following path segment). Otherwise, the \( q_{\text{ovr}} \) profile is constructed using the following LP-optimization problem:

\[
\begin{align*}
\text{maximize } c^T \tilde{a}_{\text{ovr}} & \text{ subject to: } \quad \tilde{M} \hat{a}_{\text{ovr}} \leq \tilde{b}_{\text{cur}} \\
& \quad \hat{b}_{\text{ovr}} \leq \bar{a}_{\text{ovr}} \leq \bar{b}_{\text{ovr}},
\end{align*}
\]

where the lower and upper boundaries of \( \hat{a}_{\text{ovr}} \) are defined as

\[
\hat{b}_i = \begin{cases} F_{\text{start}}, & i = 1, \\ (a_{\text{cur}})_i, & i \in \{2, \ldots, I_f - 1\}, \\ 0, & i \in \{I_f, \ldots, I_l\}, \\ F_{\text{end}_{\text{low}}}, & i = N \end{cases}
\]

and

\[
\bar{b}_i = \begin{cases} F_{\text{start}}, & i = 1, \\ (a_{\text{cur}})_i, & i \in \{2, \ldots, I_l\}, \\ (a_{\text{ovr}})_i, & i \in \{I_l + 1, I_l + 1\}, \\ F_{\text{end}_{\text{low}}}, & i = N \end{cases}
\]

respectively. The feedrate solution \( q_{\text{ovr}} \) is then found by solving the LP problem:

\[
\begin{align*}
\text{maximize } c^T a_{\text{ovr}} & \text{ subject to: } \quad M a_{\text{ovr}} \leq b_{\text{ovr}} \\
& \quad \hat{b}_{\text{ovr}} \leq a_{\text{ovr}} \leq \hat{b}_{\text{ovr}},
\end{align*}
\]

where \( b_{\text{ovr}} \) is constructed from \( \hat{b}_{\text{ovr}} \) and \( \bar{b}_{\text{ovr}} \) in a standard fashion.

For an example of case II override, see Figure 13.

4.3.3 Case II: Convergence failure

The LP formulations associated with the construction of \( \hat{q}_{\text{ovr}} \) or \( q_{\text{ovr}} \) described in the previous section can occasionally fail to converge to a solution. This is due to the fact that the function \( \text{ArcLen} \) assumes a so-called “bang-bang” transition feedrate profile, which exhibits maximal absolute values of either acceleration or jerk. When applied to general spline toolpaths this behavior is not always possible, due to limitations posed by local geometry. This convergence failure thus sometimes occurs when the override command was issued during or immediately before a dip in feedrate caused by a significant change in curvature. In such cases the override profile can still be obtained by setting a new point of override as the nearest local extreme of the feedrate curve and repeating the override computation. Specifically, suppose that either of the LP problems failed to converge, the nearest local extreme point of \( q_{\text{cur}} \) defined as an evaluation point \( g_{\text{ext}} \), can be found via Algorithm 2. The index \( J_{\text{ext}} \) is well defined since the override formulation is skipped whenever the override command is issued during the final deceleration phase.

Algorithm 2: Nearest local extreme point of \( q_{\text{ovr}} \)

Input: \( J_e, \{A_{\text{cur}}(k)\}_{k}^{I_f} \)

Output: \( J_{\text{ext}} \) // index of first point of local feedrate extreme after \( g_{\text{cur}} \)

if \( (A_{J_e} < 0) \) then

\[
J_{\text{ext}} = \min_{k \in \{J_e + 1, \ldots, K\}} \{ k : A_{\text{cur}}(k) \geq 0 \}
\]

else

\[
J_{\text{ext}} = \min_{k \in \{J_e + 1, \ldots, K\}} \{ k : A_{\text{cur}}(k) \leq 0 \}
\]

return \( J_{\text{ext}} \)

5 Conclusions

This paper comprises two original results related to feedrate optimization via linear programming techniques: the effects of curve sampling methods on time optimality of the resulting feedrate profile (Section 3) and a technique by which a feedrate override profile can be implemented (Section 4).

Previous works have not dedicated much attention to the effects that toolpath sampling can have on the effectiveness of feedrate planning. This paper demonstrates that this effect can be substantial (when com-
combined with linear programming feedrate planning) and suggests a suitable sampling method. This method can be integrated into the feedrate planning process with low computational overhead, thus making it an interesting choice for practical applications. Note that while the adaptive method requires the approximation of the relation between the curve’s natural parameter and its arc length as a prerequisite, this requirement is not really restrictive. Firstly, the approximation itself is required for the formulation of the feedrate LP optimization problem. Secondly, both the approximation and the sampling itself can (should) be performed in the preprocessing stage of feedrate planning and therefore do not add to the computational complexity of the online stage.

In order to successfully apply linear programming techniques to real world feedrate planning, a method capable of recalculating feedrate profile in reaction to a feedrate override command is essential. This topic, however, has not been dealt with in the past. Thus, a possible implementation of such a method is presented in this paper. While this method does require the computation of two additional feedrate profiles (one bounded by the newly issued commanded feedrate and second that realizes the desired feedrate transition), the majority of prerequisites for the computation of these profiles are shared with the original profile. Specifically: the knot vector associated with the feedrate profile, its evaluation points and the constant matrices of both the pseudojerk and feedrate curve optimization problems can be stored in memory the first time they are computed and reused in subsequent optimizations. Main part of the computational complexity is therefore due to the use of linear programming solvers as all the additional computations required are either index manipulations, or simple analytic formulas. The method is therefore suitable for application purposes due to its low memory requirements and simplicity of auxiliary calculations, as long as a suitably fast linear programming solver is available.

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6.2 Conflicts of interest

Financial interests: The authors have no conflicts of interest to declare that are relevant to the content of this article. Non-financial interests: None.

6.3 Availability of data and material

Not applicable.

6.4 Code availability

Code used for interpolation techniques referred to in this paper is available upon request from the corresponding author.

6.5 Author’s contributions

Conceptualization: [Petr Petráček, Bořivoj Vlk, Jiří Švéda]; Formal analysis: [Petr Petráček, Bořivoj Vlk]; Funding acquisition: [Jiří Švéda]; Investigation: [Petr Petráček, Bořivoj Vlk]; Methodology: [Petr Petráček]; Project administration: [Jiří Švéda]; Software: [Petr Petráček, Bořivoj Vlk]; Supervision: [Jiří Švéda]; Visualization: [Petr Petráček, Bořivoj Vlk, Jiří Švéda]; Writing-original draft: [Petr Petráček, Bořivoj Vlk]; Writing - review & editing: [Petr Petráček, Bořivoj Vlk, Jiří Švéda]

6.6 Ethics approval

Not applicable.

6.7 Consent to participate

Not applicable.

6.8 Consent for publication

Not applicable.
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7 Appendix A

7.1 Pentacle curve

![Fig. 14 Pentacle test curve](image)

Table 2 Parameters of the Pentacle test curve

| Parameters          | Values                                                                 |
|---------------------|------------------------------------------------------------------------|
| Control points      | (0,120,0); (-30,80,0); (-80,80,0); (-40,40,0); (-50,0,0); (0,30,0);    |
|                     | (50,0,0); (40,40,0); (80,80,0); (30,80,0); (0,120,0);                  |
| Knot vector         | [0, 0, 0, 0, 0.125, 0.25, 0.375, 0.5, 0.625, 0.75, 0.875, 1, 1, 1, 1]   |
| Weight vector       | [1, 1, 1, 1, 1, 1, 1, 1, 1, 1]                                         |
| Order               | 4                                                                       |

7.2 Trident curve

![Fig. 15 Trident test curve](image)

Table 3 Parameters of the Trident test curve

| Parameters          | Values                                                                 |
|---------------------|------------------------------------------------------------------------|
| Control points      | (0,0,0); (20,40,0); (4,16,0); (0,40,0); (-4,16,0); (-20,40,0); (0,0,0); |
| Knot vector         | [0,0,0,0,0.25,0.5,0.75,1,1,1]                                          |
| Weight vector       | [1, 1, 1, 1, 1, 1, 1]                                                 |
| Order               | 4                                                                       |
### 7.3 Phobos curve

![Phobos test curve](image)

**Fig. 16** Phobos test curve

#### Table 4  Parameters of the Phobos test curve

| Parameters | Values |
|------------|--------|
| Control points | (-0.45492,6.2779,0); (-0.49971,6.3975,0); (-0.68117,6.8746,0); (-1.1578,8.0587,0); (-1.9497,9.8028,0); (-28.406,30.986,0); (-36.881,32.85,0); (-46.387,34.777,0); (-54.933,36.285,0); (-61.491,37.3,0); (-65.535,37.878,0); (-68.634,38.645,0); (-70.923,39.776,0); (-72.523,41.952,0); (-71.383,45.129,0); (-68.418,47.949,0); (-62.707,51.334,0); (-51.982,54.7,0); (-39.731,55.245,0); (-24.639,52.567,0); (-7.9015,44.909,0); (3.8241,32.068,0); (11.252,17.481,0); (13.5,11.779,0); |
| Knot vector | [0,0,0,0,0,0.0019531,0.0078125,0.019531,0.03125,0.046875,0.0625,0.078125,0.097656,0.10938,0.125,0.14844,0.17188,0.20313,0.21875,0.24219,0.25633,0.29688,0.3125,0.33594,0.35938,0.36719,0.38281,0.39063,0.39844,0.4082,0.41406,0.41992,0.4257,0.43359,0.44141,0.44922,0.46094,0.46875,0.48438,0.5,0.53125,0.5625,0.59375,0.625,0.65625,0.70313,0.75,0.79688,0.84375,0.875,0.9375,0.96875,1,1,1,1,1,1] |
| Weight vector | [1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1] |
| Order | 4 |
7.4 Butterfly curve

**Fig. 17** Butterfly test curve

| Table 5 Parameters of the Butterfly test curve |
|-----------------------------------------------|
| Parameters | Values                   |
| Control points | (0.52,139.0); (1.014,52.139,0); |
| | (1.589,49.615,0); (2.287,44.971,0); |
| | (15.082,51.358,0); (23.293,58.573,0); |
| | (36.033,67.081,0); (51.48,63.801,0); |
| | (45.907,47.326,0); (40.074,39.913,0); |
| | (37.876,30.485,0); (57.894,67.514,0); |
| | (37.489,28.509,0); (34.951,20.393,0); |
| | (143.63,77.23,0); (99.384,14.49,0); |
| | (26.452,9.267,0); (27.875,15.989,0); |
| | (21.581,8.522,0); (15.69,12.55,0); |
| | (9.678,16.865,0); (5.5,22.122,0); |
| | (1.187,36.359,0); (2.432,24.995,0); |
| | (5.272,19.828,0); (0.14,94,0); |
| | (-5.273,19.828,0); (-2.433,24.994,0); |
| | (-1.188,36.359,0); (-5.501,22.122,0); |
| | (-9.679,16.865,0); (-15.691,12.551,0); |
| | (-21.582,8.521,0); (-25.341,14.535,0); |
| | (-26.453,9.267,0); (-99.387,14.49,0); |
| | (-143.63,77.235,0); (-34.954,20.391,0); |
| | (-37.396,28.512,0); (-57.912,67.5,0); |
| | (-37.891,30.496,0); (-40.294,39.803,0); |
| | (-45.825,47.408,0); (-51.493,63.794,0); |
| | (-36.028,67.084,0); (-23.296,58.572,0); |
| | (-15.082,51.358,0); (-2.289,44.971,0); |
| | (-1.589,49.614,0); (-1.015,52.139,0); |
| | (-0.001,52.139,0); |
| Knot vector | [0,0,0,0,0,0.0083,0.015,0.0361,0.0855, |
| | 0.1293,0.1509,0.1931,0.2273,0.2435, |
| | 0.2561,0.2692,0.2889,0.317,0.3316, |
| | 0.3482,0.3553,0.3649,0.3837,0.4005, |
| | 0.4269,0.451,0.466,0.4891,0.5, |
| | 0.5109,0.534,0.5489,0.5731,0.5994, |
| | 0.6163,0.6351,0.6447,0.6518,0.6683, |
| | 0.683,0.7111,0.7307,0.7439,0.7565, |
| | 0.7729,0.8069,0.8491,0.8707,0.9145, |
| | 0.9639,0.985,0.9917,1,1,1,1,1]; |
| Weight vector | [1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1, |
| | 1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1]; |
| Order | 4 |
APPENDIX B

This section contains the definition of the ArcLen function used in Section 4.3.2 Algorithm 1. The ArcLen function calculates an estimate of the the arc length necessary to decelerate from starting feedrate $F_s$ to end feedrate $F_e$ with zero acceleration and limits on tangential acceleration and jerk given by $a_{max}$ and $j_{max}$, respectively. The arc length estimate is obtained via a so-called “bang-bang” feedrate profile, i.e. a profile that maximizes the absolute value of either jerk or acceleration at every point. In case the maximization of acceleration would lead to reaching the end feedrate $F_e$ with nonzero acceleration, a new maximal deceleration value $a_{mid}$ is calculated, so that the end feedrate can be reached with exactly zero acceleration with $a_{mid}$ replacing $a_{max}$ as a new acceleration limit (see code blocks starting at lines A and B in Algorithms 3 and 4 below).
**Algorithm 3:** Feedrate transition profile: part 1

**Input:**
- $F_s$ // Feedrate at start [mm/s]
- $F_e$ // Feedrate at the end [mm/s]
- $a_s$ // Tangential acceleration at start [mm/s^2]
- $a_{max}$ // Tangential acceleration limit [mm/s^2]
- $j_{max}$ // Tangential jerk limit [mm/s^3]

**Output:**
- $s$ // Arclength necessary for transition from $F_s$ to $F_e$ [mm]

**Function** $ArcLen(F_s, F_e, a_s, a_{max}, j_{max})$:

```
if $a_s \geq 0$ then
    // Phase 0: Decelerate from $a_s$ to zero acceleration
    $T_0 = \frac{a_s}{j_{max}}$
    $F_0 = F_s + T_0 a_s - \frac{1}{2} T_0^2 j_{max}$
    $s_0 = T_0 F_s + \frac{1}{2} T_0^2 a_s - \frac{T_0^3}{6} j_{max}$
    $F_e = F_0$
    // Starting from $F_s$ with zero acceleration, decelerate until $a = -a_{max}$
    $T_1 = \frac{a_{max}}{j_{max}}$
    $F_{1e} = F_1 - \frac{1}{2} T_1^2 j_{max}$
    // Starting from $F_e$ with zero acceleration, accelerate until $a = a_{max}$
    $F_{3s} = F_e + \frac{1}{2} T_1^2 j_{max}$
    if $F_{1e} > F_{3s}$ then
        // Phase with $a = -a_{max}$, from feedrate $F_{1e}$ to $F_{3s}$
        $T_2 = \frac{F_{1e} - F_{3s}}{a_{mid}}$
        $s_2 = T_2 F_{1e} - \frac{1}{2} T_2^2 a_{mid}$
        // Add lengths of all phases
        $s_3 = T_1 F_{1e} + \frac{T_1^3}{6} j_{max}$
        $s_1 = T_1 F_s - \frac{T_1^2}{4} j_{max}$
        $s = s_0 + s_1 + s_2 + s_3$
    else
        // Calculate $a_{mid}$ such that $F_e$ is reached with zero acceleration
        $a_{mid} = -\sqrt{(F_s - F_e) j_{max}}$
        $T_1 = -\frac{a_{mid}}{j_{max}}$
        $F_{1e} = F_s - \frac{1}{2} T_1^2 j_{max}$
        $s_1 = T_1 (F_s + F_{1e}) + \frac{1}{2} T_1^2 a_{mid}$
        $s = s_0 + s_1$
    else
        // Starting from $F_s$, decelerate until $a = -a_{max}$
        $T_1 = \frac{(a_{max} + a_s)}{j_{max}}$
        $F_{1e} = F_s + T_1 a_s - \frac{1}{2} T_1^2 j_{max}$
        // Starting from $F_e$ with zero acceleration, accelerate until $a = a_{max}$
        $T_3 = \frac{a_{max}}{j_{max}}$
        $F_{3s} = F_e + \frac{1}{2} T_3^2 j_{max}$
```
Algorithm 4: Feedrate transition profile: part 2

if $F_1 e \geq F_3 s$ then
    // Phase with $a = -a_{max}$, from feedrate $F_1 e$ to $F_3 s$
    $T_2 = \frac{F_1 e - F_3 s}{a_{max}}$
    $s_2 = T_2 F_3 s - \frac{1}{2} T_2^2 a_{max}$
    $s_1 = T_1 F_s + \frac{1}{2} T_1^2 a_s + \frac{T_3}{6} j_{max}$
    $s_3 = T_3 F_3 s + \frac{T_3^2}{12} j_{max}$
    $s = s_1 + s_2 + s_3$
else
    // Calculate $a_{mid}$ such that $F_e$ is reached with zero acceleration
    $a_{mid} = \max \left\{ -a_{max}, -\sqrt{\frac{1}{2} a_s^2 + (F_s - F_e) j_{max}} \right\}$
    $T_1 = \frac{a_{s} - a_{mid}}{j_{max}}$
    $F_1 e = F_s + T_1 a_s - \frac{1}{2} T_1^2 j_{max}$
    $s_1 e = T_1 F_s + \frac{1}{2} T_1^2 a_s - \frac{T_1^3}{6} j_{max}$
    $s = s_1 e + T_1 F_1 e + \frac{1}{2} T_1^2 a_{mid} + \frac{T_3}{12} j_{max}$
return $s$