First Reduce or First Quantize?
A Lagrangian Approach and Application to Coset Spaces

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ABSTRACT

A Lagrangian treatment of the quantization of first class Hamiltonian systems with constraints and Hamiltonian linear and quadratic in the momenta respectively is performed. The “first reduce and then quantize” and the “first quantize and then reduce” (Dirac’s) methods are compared. A new source of ambiguities in this latter approach is revealed and its relevance on issues concerning self-consistency and equivalence with the “first reduce” method is emphasized. One of our main results is the relation between the propagator obtained à la Dirac and the propagator in the full space, eq. (5.25). As an application of the formalism developed, quantization on coset spaces of compact Lie groups is presented. In this case it is shown that a natural selection of a Dirac quantization allows for full self-consistency and equivalence. Finally, the specific case of the propagator on a two-dimensional sphere $S^2$ viewed as the coset space $SU(2)/U(1)$ is worked out.
1 Introduction

As is well known, the quantization of constraint systems is plagued with many ambiguities and difficulties which are added to those encountered when dealing with regular systems (ordering problems, Groenwald-van Hove obstruction [1], etc.). In particular, in the framework of Dirac quantization [2], the preservation of the first class nature of the Hamiltonian and the constraints at the operator level is a highly non-trivial issue. Another important aspect in the quantization of constrained systems is that of the equivalence between the two standard procedures: a) Dirac’s method of quantizing the “entire” system (i.e., including gauge variables) and obtaining physical states as those annihilated by the operator version of the constraints, and b) first reducing the classical variables by solving the constraints, and then quantizing as a regular system. These procedures are usually referred to as “quantize first and then reduce” and “reduce first and then quantize” respectively. There seems to be no a-priori general principle that guarantees that they agree with each other, let alone that selects one procedure over the other from the physical point of view [3]. In fact, some of these issues have recently been discussed in the context of Chern-Simons topological field theories [4],[5],[6],[7],[8], which exemplifies the level of subtlety involved in this kind of problems.

Kuchař, in a beautiful series of papers [9], has discussed these problems in the case of quadratic Hamiltonians and linear constraints. (Recently Kuchař and Hájíček have also studied the case of parametrized theories with quadratic constraints [10].) He finds that, in order to achieve equivalence, one is forced to give up, in the general case, the full Hilbert space structure of the entire system. Hence, in a strict sense, in Dirac’s method one is not “quantizing first” and then finding physical states, since the entire space does not admit a quantum interpretation. Following Kuchař’s work, McMullan et al. [11] have used the BRST approach of Batalin, Fradkin, and Vilkovisky [12] to address the same problems. They were able to implement the “quantize first” procedure, while keeping the equivalence with the
“reduce first” method, by means of the introduction of ghost variables. Concurrently, several other groups [13] have also investigated a wide variety of aspects which arise in the study of the equivalence of these two methods.

The treatment of these problems has been done mostly with canonical (operator) quantization. However, similar issues like the relation between reduced and covariant path integrals (e.g. Faddeev-Popov) also appear. We have investigated these issues in a series of publications [14] [15] [16] and have proven equivalence in this case (see also [17]). Let us now return to the operator formulation.

The work of the groups mentioned above has been mostly done within the Hamiltonian framework. By doing this, they may have overlooked a potential new source of ambiguities stemming from the fact that the Hamiltonian for a classical constrained system is not uniquely defined off the constraint surface, which is going to have an impact upon its quantization. For this reason, in order to tackle this type of problems, we have chosen to start with a Lagrangian framework (section 2). Beyond its well known advantages—explicit display of symmetries being perhaps the most commonly quoted—this formalism enables us to encode in a single function all the features of the constrained system, including, of course, the constraints. We then proceed with the Hamiltonian formulation (section 3), and immediately afterwards we present the two types of quantizations mentioned above (sections 4 and 5). This is followed (section 6) by a discussion of the subtleties and ambiguities present in the “quantize first” approach which, to our knowledge, have not been addressed before. Here we also make some remarks on the relevance of the gauge group in this kind of questions. On section 7 we apply our general setting to the case of quantization on coset spaces of compact Lie groups and then go on to the special case of $S^2 \approx SU(2)/U(1)$ (section 8). Finally, we present conclusions and outlook in section 9.
2 The Lagrangian setting

We are interested in a model with only first class constraints (in phase space, $T^*Q$, where $Q$ is the configuration space manifold), all of which should appear as primary constraints in order for them to generate independent gauge transformations. This means that also the Hamiltonian has to be first class, i.e., its Poisson brackets with all the constraints have to vanish on the constraint surface. There is an easy way to cast this information in the Lagrangian formalism. In this paper we will consider Lagrangians of the form:

$$L = \frac{1}{2} G_{AB} \dot{Q}^A \dot{Q}^B - V,$$

where $G_{AB}$ and $V$ are functions of configuration space variables $Q^A$, $A = 1, \ldots, N$. $G_{AB}$ is a singular metric tensor of rank $n < N$.

The standard construction of the first generation of velocity space ($TQ$) constraints is as follows. The equation of motion derived from (2.1) is

$$G_{AB} \ddot{Q}^B = \alpha^A,$$  \hspace{1cm} (2.2)

where

$$\alpha^A = \frac{\partial L}{\partial Q^A} - \frac{\partial L}{\partial Q^B} \dot{Q}^B \dot{Q}^A \dot{Q}^B = \frac{1}{2} G_{BC,A} \dot{Q}^B \dot{Q}^C - G_{AC,B} \dot{Q}^B \dot{Q}^C - V_A.$$  \hspace{1cm} (2.3)

If $U^A(Q)$ is a null vector of $G_{AB}$, i.e., $G_{AB} U^B = 0$ identically, then the constraint surface corresponding to the first step of the stabilization algorithm is obtained by requiring

$$U^A \alpha_A \approx 0$$  \hspace{1cm} (2.4)

for every null vector of the metric.
In order for (2.4) not to select a constraint surface, this relation has to be satisfied identically, i.e., its solution has to be the entire velocity space \([21]\). This leads to

\[ U^C \Gamma_{CAB} = 0, \quad (2.5) \]

and

\[ U^C V_C = 0, \quad (2.6) \]

where

\[ \Gamma_{CAB} = -G_{AB,C} + G_{CB,A} + G_{CA,B}. \quad (2.7) \]

Using \( G_{AB} U^B = 0 \), (2.5) becomes

\[ U^C G_{AB,C} + G_{AC} U_C^C + G_{CB} U_A^C \equiv (\mathcal{L}_U G)_{AB} = 0 \]

(2.8)

with

\[ \hat{U} = U^A(Q) \frac{\partial}{\partial Q^A}. \quad (2.9) \]

In (2.8) we have used the standard notation for the Lie derivative.

Clearly, in this case there cannot be any further constraints. Thus, the conditions which the Lagrangian (2.1) has to satisfy in order for it to guarantee a first class Hamiltonian structure are:

a) every null vector of the degenerate metric tensor \( G \), considered as a vector field in configuration space, also has to be a Killing vector for it, and

b) every such null vector has to be tangent to the equipotential surfaces of the potential \( V \).

At this point, the following comment about the null vectors of \( G \) is in order. The set of null vectors is closed under the Lie bracket. Indeed, let \( \hat{U} \) and \( \hat{V} \) be two such vectors, i.e.,

\[ i_{\hat{U}} G = i_{\hat{V}} G = 0. \quad (2.10) \]
As we saw, they also have to be Killing vectors. Hence, using (2.8) the following equalities hold:
\[ 0 = \mathcal{L}_{\hat{U}}(i_{\hat{V}}G) = i_{(\mathcal{L}_{\hat{U}}\hat{V})}G = i_{[\hat{U},\hat{V}]}G. \] (2.11)
Therefore, \([\hat{U},\hat{V}]\) is a null vector.

If we consider a basis for the space of null vectors, \(\{\hat{U}_\alpha, \alpha = 1, \ldots, k = N - n\}\) (this means that any null vector can be written as a linear combination of the vectors in this basis, with its coefficients being functions of configuration space variables), this closure property is written
\[ [\hat{U}_\alpha, \hat{U}_\beta] = C_{\alpha\beta}(Q)\hat{U}_\gamma. \] (2.12)
As we will see later this relation prefigures the first class character of the constraints that appear in the Hamiltonian formulation.

3 Hamiltonian setting

Now we will explicitly see how our previous scheme is realized in the Hamiltonian framework.

The momenta are defined as usual by the Legendre map \(P = \partial L/\partial \dot{Q} \).

\[ P_A = G_{AB}\dot{Q}^B. \] (3.1)

This immediately implies the following set of primary constraints
\[ \varphi_\alpha \equiv U_\alpha^A P_A \approx 0, \alpha = 1, \ldots, k. \] (3.2)

Since the rank of the metric \(G\) is \(n\), (3.2) is the full set of primary constraints. Equation (2.12) shows that they are first class under Poisson bracket:
\[ \{\varphi_\alpha, \varphi_\beta\} = -C_{\alpha\beta}^\gamma \varphi_\gamma. \] (3.3)

As mentioned earlier, these constraints generate gauge transformations. For any trajectory \(\eta = (Q^A(t), P_A(t))\) in \(T^*Q\),
\[ \delta \eta = \epsilon^\alpha(t) \{\eta, \varphi_\alpha\}, \] (3.4)
where $\epsilon^\alpha(t)$ are $k$ arbitrary infinitesimal functions of $t$ which describe $k$ independent gauge transformations. For fixed $t$ these turn into infinitesimal symmetry transformations in $T^*Q$. In particular, for the coordinates of configuration space

$$\delta Q^A = \epsilon^\alpha \{ Q^A, \varphi_\alpha \} = \epsilon^\alpha U^A, \quad (3.5)$$

where $\epsilon^\alpha$ are infinitesimal parameters. Hence the vector fields $\hat{U}_\alpha$ generate symmetry transformations in configuration space. This means that configuration space $\mathcal{M}$ is divided into orbits under the action of these vector fields, and that the physical configuration space is the quotient of the original one by the orbits.

Let us now set up the dynamics on $T^*Q$. The energy function in velocity space is

$$E = \frac{\partial L}{\partial \dot{Q}^A} \dot{Q}^A - L = \frac{1}{2} G_{AB} \dot{Q}^A \dot{Q}^B + V. \quad (3.6)$$

This function has to be the pullback under the Legendre mapping of some function in phase space, which, up to functions that vanish on the constraint surface – i.e., having zero pullback – has the following general form:

$$H = \frac{1}{2} M^{AB}(Q) \dot{P}_A \dot{P}_B + V. \quad (3.7)$$

This is, of course, the Hamiltonian function. Notice that it is only uniquely defined on the constraint surface. This fact is reflected in an ambiguity in $M^{AB}$. Indeed, by construction, the only requirement that it has to satisfy is that ($M$ being symmetric)

$$M^{AB} G_{AC} G_{BD} \dot{Q}^C \dot{Q}^D = G_{CD} \dot{Q}^C \dot{Q}^D, \quad (3.8)$$

i.e.,

$$M^{AB} G_{AC} G_{BD} = G_{CD}. \quad (3.9)$$
Equation (3.9) displays the ambiguity very clearly. Any transformation of the form

\[ M^{AB} \rightarrow M'^{AB} = M^{AB} + \lambda^{A\alpha} U^B_{\alpha} + \lambda^{B\alpha} U^A_{\alpha} \]  

(3.10)

with arbitrary \( \lambda^{A\alpha} \) will define another Hamiltonian which will have the same energy function in velocity space. This lack of uniqueness in \( M^{AB} \) allows for the possibility of working with a non-singular matrix which may well be used as a metric tensor in configuration space. Nevertheless, notice that this assignment is highly ambiguous. This is to say, a first class Hamiltonian system of the type we are considering does not endow configuration space with a unique non-singular metric structure. This is the point mentioned earlier which has not been taken into account in previous work. We would like to emphasize the importance of these remarks in the light of the problems one faces when quantizing these systems. Recall that we are to try to preserve the first class nature of the constraints, and to check equivalence between the “quantize and then reduce” and the “reduce and then quantize” approaches. This freedom in the choice of a metric structure in configuration space will play an important role in our quantization program.

Finally, we would like to point out the generality of our construction. It can be shown that given a Hamiltonian of the form (3.7), and a set of the first class constraints (3.2), there always exists a Lagrangian of the form (2.1) which gives rise to the Hamiltonian and the constraints. For details see appendix.

4 Reduce first and then quantize

We now come to the description of the physical (reduced) configuration space and the quantum theory on it. At this level we may proceed from either the Hamiltonian or the Lagrangian formulation. As we will show, one obtains the same results. Let us then begin with the Hamiltonian version. There is a natural way of endowing \( \mathcal{M} \) with a non-singular metric structure. Consider the following contravariant tensor field associated to the kinetic term in
the Hamiltonian

\[ M^{AB} \frac{\partial}{\partial Q^A} \otimes \frac{\partial}{\partial Q^B}. \]  (4.1)

The projection

\[ \pi : \mathcal{Q} \to \mathcal{M} \]  (4.2)

allows us to assign to (4.1) a contravariant vector field in \( \mathcal{M} \), as long as the following “projectability” condition is fulfilled:

\[ \hat{U}_\alpha \left( M^{AB} \frac{\partial f}{\partial Q^A} \frac{\partial g}{\partial Q^B} \right) = 0, \quad \alpha = 1, \ldots, k \]  (4.3)

for any functions \( f, g \) of configuration variables \( Q^A \) such that

\[ \hat{U}_\alpha f = \hat{U}_\alpha g = 0, \quad \alpha = 1, \ldots, k. \]  (4.4)

This condition can be recast as

\[ \left( \mathcal{L}_{\hat{u}_\alpha} M \right)^{AB} G_{AC} G_{BD} = 0 \]  (4.5)

which is a trivial consequence of (3.9). Hence we have to our avail this contravariant tensor field in \( \mathcal{M} \).

In order to gain further insight into the structure of this tensor field, we now introduce coordinates \( q^a, a = 1, \ldots, n \), on \( \mathcal{M} \). Then the projection \( \pi \) is described by the functions \( q^a(Q) \) that satisfy \( \hat{U}_\alpha q^a = 0; \alpha = 1, \ldots, k, a = 1, \ldots, n \).

In these coordinates the components of the new tensor field in \( \mathcal{M} \) are

\[ \tilde{g}^{ab} = M^{AB} \frac{\partial q^a}{\partial Q^A} \frac{\partial q^b}{\partial Q^B}. \]  (4.6)

The projectability condition shows that \( \tilde{g}^{ab} \) only depends on the physical coordinates \( q^a \).

Notice that \( \tilde{g}^{ab} \) so defined is invariant under changes of \( M \) of the type (3.10). Moreover, these are the only changes that leave \( \tilde{g}^{ab} \) invariant. This result has the direct physical consequence that the ambiguities present in the Hamiltonian framework play no role in the
reduce first approach to quantization. Hence the physical phase space \( T^* \mathcal{M} \) is so naturally endowed with a Hamiltonian

\[
h = \frac{1}{2} \tilde{g}^{ab} p_a p_b + V .
\]  

(4.7)

Now we will show that \( \tilde{g} \) is non-singular. In order to prove this it is convenient to work with a coordinate system in \( Q \) adapted to the orbit structure. This is achieved by adding to the physical coordinates \( q^a, a = 1, \ldots, n \), which label the orbits, a new set of functions \( q^\alpha, \alpha = 1, \ldots, k \) such that \( \det |\hat{U}_\alpha q^\beta| \neq 0 \) (these new “gauge” coordinates parametrize the orbits. The entire set will henceforth be termed “adapted coordinate system”).

In this system the Lagrangian metric \( G_{AB} \) and the null vectors are written

\[
G = \begin{pmatrix}
g_{ab} & g_{a\beta} \\
g_{ab} & g_{\alpha\beta}
\end{pmatrix}
\]  

and

\[
\hat{U}_\alpha = U^\beta_\alpha (q^a, q^\gamma) \frac{\partial}{\partial q^\beta},
\]

(4.8)

respectively.

The condition that these vectors are null vectors for this metric immediately leads to

\[
g_{ab} = g_{a\beta} = g_{\alpha\beta} = 0
\]

(4.10)

and the Killing condition for these vectors becomes:

\[
\hat{U}_\alpha (g_{ab}) = 0, \quad \alpha = 1, \ldots, k .
\]

(4.11)

This last result guarantees that \( g_{ab} \) only depends on the physical coordinates \( q^a, a = 1, \ldots, n \).

Equations (4.8) and (4.10) display the non-singularity of \( g_{ab} \) because the rank of \( G \) is \( n \).

Equation (3.9) now becomes

\[
M = \begin{pmatrix}
g^{ab} & m^{a\beta} \\
m_{a\beta} & m^{\alpha\beta}
\end{pmatrix},
\]  

(4.12)
where $g^{ab} = (g_{ab})^{-1}$, and the remaining components of $M$ are arbitrary.

Finally, the projection $\pi$ to $M$ of the contravariant tensor $\bar{L}$ leads to the identification $\tilde{g}^{ab} = g^{ab}$. This proves that the contravariant metric tensor defined in $M$ is non-singular. The Hamiltonian formulation on $T^*M$ is now completed.

Let us next proceed with the Lagrangian version. Taking advantage of the form of $G$ in the adapted coordinate system, it is immediate to see that there is a unique function $\ell$ on $TM$ such that its pullback $\pi^*\ell$ under the derivative mapping $\pi'$ induced by the projection $\pi$, i.e.,

$$\pi' : TQ \longrightarrow TM$$

is just the original Lagrangian $L$:

$$\ell = \frac{1}{2} g_{ab} \dot{q}^a \dot{q}^b - V, \quad \pi^*\ell = L.$$  \hspace{1cm} (4.14)

Summarizing: we have found the Hamiltonian and the Lagrangian for the reduced system. They are obviously related by a Legendre transformation. In the construction of these functions it now becomes quite clear that the metric $g_{ab}$ is the one to be used to define the measure on $M$. From this point on, we can proceed within the standard quantization scheme for regular systems. (See [24] and [41] for details). Notice that in this geometrical framework no explicit gauge fixing was performed.

The inner product that defines the Hilbert space $\mathcal{H}_r = \mathcal{L}^2(M, |g|^{1/2})$ is defined as

$$\langle \psi_1 | \psi_2 \rangle = \int_M d^nq \, |g|^{1/2} \bar{\psi}_1(q) \psi_2(q),$$

where $|g|$ is the determinant of $g_{ab}$. The dynamical evolution (Schrodinger equation) will be given by the Hamiltonian operator

$$\hat{h} = -\frac{1}{2} \Delta_g + V,$$  \hspace{1cm} (4.16)

where $\Delta_g$ is the Laplace-Beltrami operator.
\[ \Delta_g = |g|^{-\frac{1}{2}} \frac{\partial}{\partial q^a} |g|^{\frac{1}{2}} g^{ab} \frac{\partial}{\partial q^b}. \]  

(4.17)

In Kuchař’s words, (4.16) implements the “principle of minimal coupling”; curvature terms are not included.

The configuration variables \( q^a \) become multiplicative operators as in the usual case, whereas, in order to retain hermiticity, the canonical momenta \( p_a \) are quantized as follows:

\[ p_a \to -i \left( \frac{\partial}{\partial q^a} + \frac{1}{2} |g|^{-\frac{1}{2}} |g|^\frac{1}{2} \right). \]  

(4.18)

In the next section we reverse the order of things and proceed to “quantize first and then reduce”.

5 Quantize first and then reduce, Dirac’s Method

As we discussed in Section 3 we always have an enormous freedom to choose a non-singular metric \( M \) with which we can immediately write down an inner product.

\[ \langle \psi_1 | \psi_2 \rangle = \int_Q d^N Q |M|^\frac{1}{2} \psi_1^*(Q) \psi_2(Q). \]  

(5.1)

This inner product defines the Hilbert space \( \mathcal{H} = \mathcal{L}^2 \left(Q, |M|^\frac{1}{2}\right) \). At this point we will assume that \( M \) allows for a consistent quantization of our classical first class Hamiltonian system, in the sense defined in Section 1. That is to say that the quantum operators associated with the classical constraint and the Hamiltonian still form a first class system with respect to the commutator algebra. For a further discussion of this assumption we refer to the next section.

The operator assignment for the classical constraints is dictated by the requirement that the physical wave function only depend upon physical variables, i.e., variables describing \( M \):

\[ \varphi_\alpha \longrightarrow \hat{\varphi}_\alpha \equiv \hat{U}_\alpha = U_\alpha^A \frac{\partial}{\partial Q^A} \]  

(5.2)

\(^1\text{Here } |M| = \det M_{AB}, \quad M_{AB} = (M^{AB})^{-1}.\)
Notice that equation (2.12) guarantees that the classical constraints \( \varphi_\alpha \) are realized as a set of first class quantum operators. The Hamiltonian in \( \mathcal{H} \) is chosen as in Section 4, i.e.,

\[
\hat{H} = -\frac{1}{2} \Delta_M + V, \tag{5.3}
\]

where

\[
\Delta_M = |M|^{-\frac{1}{2}} \frac{\partial}{\partial Q^A} |M|^\frac{1}{2} M^{AB} \frac{\partial}{\partial Q^B} \tag{5.4}
\]

and it is assumed to satisfy the first class condition:

\[
[\hat{U}_\alpha, \hat{H}] = \lambda^\beta_\alpha (Q) \hat{U}_\beta. \tag{5.5}
\]

The physical Hilbert space \( \mathcal{H}_p \) in this framework is then obtained as the subspace of \( \mathcal{H} \) defined by those states which satisfy

\[
\hat{U}_\alpha |\text{Phys}\rangle = 0, \quad \alpha = 1, \ldots, k. \tag{5.6}
\]

In wave function language

\[
\hat{U}_\alpha \langle Q | \psi \rangle \equiv \hat{U}_\alpha \psi(Q) = 0. \tag{5.7}
\]

This is equivalent to

\[
\psi(Q) = \tilde{\psi}(f^a(Q)), \tag{5.8}
\]

where \( q^a = f^a(Q) \) defines explicitly the projection \( \pi \), equation (1.2). Physical position kets \( |q^a\rangle_{\text{Ph}} \) are those states in \( \mathcal{H} \) defined by

\[
_{\text{Ph}} \langle q^a | \psi \rangle = \tilde{\psi}(q^a), \tag{5.9}
\]

with \( |\psi\rangle \) satisfying (5.6). It is convenient to expand these states in terms of the position states \( |Q\rangle \) in \( \mathcal{H} \). For this purpose consider the following string of identities, valid for arbitrary \( \tilde{\psi} \):

\[
\tilde{\psi}(q^a) = _{\text{Ph}} \langle q^a | \psi \rangle = \int d^N Q |M|^\frac{1}{2} _{\text{Ph}} \langle q^a | Q \rangle \langle Q | \psi \rangle
\]

\[
= \int d^N Q |M|^\frac{1}{2} _{\text{Ph}} \langle q^a | Q \rangle \psi(Q)
\]

\[
= \int d^N Q |M|^\frac{1}{2} _{\text{Ph}} \langle q^a | Q \rangle \tilde{\psi}(f^a(Q)) \tag{5.10}
\]
which implies that

$$\rho ( q^a | Q ) = \frac{1}{\mu ( q^a )} \delta^n \left( q^a - f^a ( Q ) \right),$$

(5.11)

where

$$\mu ( q^a ) = \int d^N Q |M|^\frac{1}{2} \delta^n \left( q^a - f^a ( Q ) \right).$$

(5.12)

Notice that (5.11) requires that $$\mu ( q^a )$$ be finite at every point $$q^a$$. This is equivalent to requiring that the physical wave functions $$\psi$$ be normalizable in $$H$$, since they are constant along each orbit. Hence, in a rigorous sense, “first quantize” approach is only applicable in the cases where this condition is satisfied. Of course, one can be more practical, and carry out the Dirac construction in a more formal fashion ignoring details of normalization in $$H$$. In our paper we are going to assume that $$\mu ( q^a )$$ is finite at every $$q^a$$.

It will prove convenient to rewrite $$\mu ( q^a )$$ without the delta function. For this purpose, let us use the adapted coordinate system $$(q^a, q^\alpha)$$ [23]. Consider the following identity [26].

$$1 = \int d^k q \delta^k ( q^a - f^a ( Q ) ) ,$$

(5.13)

where the functions $$f^\alpha$$ give the explicit realization of the coordinate change. Inserting (5.13) into (5.12) we obtain

$$\mu ( q^a ) = \int d^k q \int d^N Q |M|^\frac{1}{2} \delta^n ( q^a - f^a ( Q ) ) \delta^k ( q^a - f^a ( Q ) )$$

$$= \int d^k q \int d^N Q |M|^\frac{1}{2} \delta^N ( q - f ( Q ) )$$

$$= \int d^k q \int d^N Q |M|^\frac{1}{2} \frac{1}{|\partial q / \partial Q|} \delta^N ( Q - Q ( q ) )$$

$$= \int d^k q |M|^\frac{1}{2} \frac{1}{|\partial q / \partial Q|} = \int d^k q |m|^\frac{1}{2} ,$$

(5.14)

where $$|m|$$ is the determinant of the metric in the adapted coordinate system.

In [14] we proved the following important factorization property of $$m$$: in adapted coordinates,

$$|m| = |g_{ab}| \left| m_{\alpha\beta} \right| = \left| g \right| \left| m_{\alpha\beta} \right|$$

(5.15)
where $g_{ab}(q^a)$ is the “physical” metric of section 4. This implies that eq. (5.14) can also be written as

$$
\mu(q^a) = \int d^kq \, |g|^{1/2} \, |m_{\alpha\beta}|^{1/2} = |g|^{1/2} \int d^kq \, |m_{\alpha\beta}|^{1/2}
$$

$$
= |g|^{1/2} \, \mathcal{V}(q^a).
$$

(5.16)

$\mathcal{V}(q^a)$ is naturally identified as the volume of the orbit labeled by $q^a$ (recall the $q^\alpha$‘s are “gauge” variables). Notice that since the hypothesis $\mu(q^a)$ if finite, so is $\mathcal{V}(q^a)$ (or, as this formula shows, one can demand finiteness of $\mathcal{V}(g^q)$ instead from the outset).

Using (5.11) we can easily compute the inner product between any two physical position kets:

$$
\langle \psi | q^a_1 \rangle_{\text{Ph}} = \int d^NQ \, |M|^1/2 \, \text{Ph}(q^a_1 | Q) \, \langle Q | q^a_2 \rangle_{\text{Ph}}
$$

$$
= \int d^NQ \, |M|^{1/2} \, \frac{1}{\mu(q^a_1)} \delta^n (q^a_1 - f^a(Q)) \, \frac{1}{\mu(q^a_2)} \delta^n (q^a_2 - f^a(Q))
$$

$$
= \delta^n (q^a_1 - q^a_2) \, \frac{1}{\mu(q^a_1)} \, \frac{1}{\mu(q^a_2)} \, \int d^NQ \, |M|^{1/2} \, \delta^n (q^a_1 - f^a(Q))
$$

$$
= \frac{1}{\mu(q^a_1)} \delta^n (q^a_1 - q^a_2).
$$

(5.17)

Equation (5.17) identifies $\mu(q^a)$ as the measure on $\mathcal{M}$. This can also be seen directly if we compute the inner product of two physical states $|\psi\rangle, |\chi\rangle$ in $\mathcal{H}_p$:

$$
\langle \psi | \chi \rangle = \int d^NQ \, |M|^{1/2} \psi^*(Q)\chi(Q)
$$

$$
= \int d^NQ \, |M|^{1/2} \tilde{\psi}^* (f^a(Q)) \tilde{\chi} (f^a(Q))
$$

$$
= \int d^aq \, d^kq \left[ \frac{\partial Q}{\partial q} \right] \, |M|^{1/2} \tilde{\psi}^* (q^a) \tilde{\chi} (q^a)
$$

$$
= \int d^aq \, \left[ \int d^kq |m|^1/2 \right] \tilde{\psi}^* (q^a) \tilde{\chi} (q^a)
$$

$$
= \int d^aq \, \mu(q^a) \tilde{\psi}^* (q^a) \tilde{\chi} (q^a).
$$

(5.18)
The physical position states $|q^a\rangle_{\text{Ph}}$ can be written in terms of the position states $|Q\rangle$ in $\mathcal{H}$. Indeed, using (5.11) we obtain

$$|q^a\rangle_{\text{Ph}} = \frac{1}{\mu(q^a)} \int d^N Q |M|^{\frac{1}{2}} \delta^n \left(q^a - f^a(Q)\right) |Q\rangle.$$ (5.19)

In adapted coordinates, using the factorization property, equation (5.15), we obtain

$$|q^a\rangle_{\text{Ph}} = \frac{1}{V(q^a)} \int dq^k |m_{\alpha\beta}|^{1/2} |q^a, q^\alpha\rangle.$$ (5.20)

This formula (5.20) displays the nature of $|q^a\rangle_{\text{Ph}}$ as a kind of average over the gauge degrees of freedom of $|q^a, q^\alpha\rangle$, and this connection will be useful when we compute the propagator in $\mathcal{H}_p$. The projection onto $\mathcal{H}_p$ can readily be written as

$$\mathcal{P} = \int d^a q \mu(q^a) |q^a\rangle_{\text{Ph}} \langle q^a|.$$ (5.21)

This operator can also be expressed in terms of the states $|Q\rangle$:

$$\mathcal{P} = \int d^N Q d^N Q' |M|^{\frac{1}{2}} |M'|^{\frac{1}{2}} \frac{1}{\mu(f^a(Q))} \delta^n \left(f^a(Q) - f^a(Q')\right) |Q\rangle \langle Q'|.$$ (5.22)

As expected, $\mathcal{P}$ projects states $|Q\rangle$ into $|q^a\rangle_{\text{Ph}}$:

$$\mathcal{P} |Q\rangle = |q^a\rangle_{\text{Ph}},$$ (5.23)

where $q^a = f^a(Q)$. Finally, we are now in the position of being able to relate the propagator in $\mathcal{H}$ with the propagator in $\mathcal{H}_p$. Since we have a well defined Hamiltonian $\hat{H}$ in $\mathcal{H}$ which, due to the first class nature—even at the quantum level—of our system, evolves physical states into physical states, then we can immediately write down the extension of formula (5.19) to the Heisenberg representation:

$$|q^a, t\rangle_{\text{Ph}} = \frac{1}{V(q^a)} \int dq^k |m_{\alpha\beta}|^{1/2} |(q^a, q^\alpha, t)\rangle.$$ (5.24)

$$\mathcal{P}_{\text{h}}(q^a_2, t_2 | q^a_1, t_1)_{\text{Ph}} = \frac{1}{V(q^a_2) V(q^a_1)} \int dq^k_1 dq^k_2 |m_{\alpha\beta}(q_2)|^{1/2} |m_{\alpha\beta}(q_1)|^{1/2} \langle q^a_2, q^\alpha_2, t_2 | q^a_1, q^\alpha_1, t_1 \rangle.$$ (5.25)

This equation is one of the main results of this paper.
6 Quantum First Class Condition and the Gauge Group.

In this section we want to explore the possibilities afforded by virtue of the enormous freedom in the choice of a non-singular $M^{AB}$ satisfying equation (3.9). Its selection should be constrained by at least the requirement that the first class nature of the system be preserved at the quantum level. This is essential for the implementation of the Dirac program. Once this has been achieved, we can then address the issue of coincidence of Dirac’s method with the “first reduce” quantization scheme. It is not entirely clear to us that this is something which should limit the possible choices of $M^{AB}$, since there doesn’t seem to be a physical principle which would instruct us to favor one method over the other. In view of this we will keep an open mind on this issue, but always having present that this is an important aspect of any comparison between different methods of quantization, since the existence of inequivalent quantization schemes for constraint systems is something that should not be overlooked so lightly.

**QUANTUM FIRST CLASS CONDITION**

The expression for $M^{AB}$ in the adapted coordinate system, equation (4.12)

$$M = (m^{A'B'}) = \left( \begin{array}{cc} g^{ab} & m^{a\beta} \\ m^{a\beta} & m^{a\beta} \end{array} \right)$$

allows us to write the Laplace-Beltrami operator (acting on scalars) as

$$\Delta_M = m^{A'B'} \nabla_{A'} \partial_{B'} = m^{A'B'} (\partial_{A'} \partial_{B'} - \Gamma_{A'B'}^{C'} \partial_{C'})$$ (6.1)

or, on physical wave functions:

$$\Delta_M \mid_{\text{phys}} = g^{ab} \partial_a \partial_b - m^{A'B'} \Gamma_{A'B'}^{a} \partial_a .$$ (6.2)

The first class condition at the quantum level, equation (5.4), is equivalent to preserving the physical nature of the states under Hamiltonian evolution, which in our case, in virtue of
the gauge invariance of the potential, equation (2.6), reads

\[ \hat{U}_\alpha \Delta_M |\text{Phys} \rangle = 0, \quad \alpha = 1, \cdots, k \]  

(6.3)

which leads, considering the form of (5.2), to the following sufficient and necessary condition:

\[ \hat{U}_\alpha (m^{AB'} \Gamma^a_{\alpha A'B'}) = 0, \quad \alpha = 1, \cdots, k, \quad a = 1, \ldots, n. \]  

(6.4)

In other words, \( m^{AB'} \Gamma^a_{\alpha A'B'} \) has to be a gauge invariant object.

One criterium that immediately comes to mind for the determination of \( m^{AB'} \) in order to fulfill equation (6.4) is to demand that it be a Killing metric for the generators \( \hat{U}_\alpha, \alpha = 1, \cdots, k \) of the gauge transformations [27]. This guarantees that the Hamiltonian \( \hat{H} \) of equation (5.3) commutes with these Killing vectors, which is a strong way to satisfy the first class condition, equation (6.4). For this to be achieved, these generators have to form a Lie algebra, i.e., the structure functions in (2.12), \( C^\gamma_{\alpha \beta} (Q) \), have to be constant. It is easy to show that this is nothing but the integrability condition for the existence of a Killing metric constructed with these generators. Hitherto we have not had to mention the freedom to choose a basis of the space of null vectors of the Lagrangian metric \( G_{AB} \). This corresponds to the invariance of the constraint surface defined by (5.2) under rescaling of the primary Hamiltonian constraints [28]. By selecting the generators \( \hat{U}_\alpha, \alpha = 1, \cdots, k \) to form a Lie algebra (i.e., constant structure coefficients), we would be spoiling this rescaling invariance. Moreover, different ways of selecting the generators may lead to different Lie algebra structures, and hence different Killing metrics. This will likely lead to different quantizations. These reflexions prompt us to discuss the nature of the gauge group at more length.

**THE GAUGE GROUP**

At a fixed time, instead of considering the gauge group acting on the space of trajectories, we can consider its action on \( Q \). Here the gauge group is defined as the group of transformations (diffeomorphisms) that leave the orbits invariant. In adapted local coordinates [29],

\[ q^a, \bar{q}^a \longrightarrow \bar{q}^a, \bar{q}^\alpha \]
with
\[ \bar{q}^\alpha = q^\alpha, \quad \bar{q}^\alpha = \bar{q}^\alpha(q^\alpha, q^\alpha), \]
(6.5)
The null vector fields of the Lagrangian metric, described by the basis \( \hat{U}_\alpha, \alpha = 1, \cdots, k \), are tangent to the orbits and generate uniparametric subgroups of the gauge group. There is a great deal of freedom to choose these generators. In particular it may be possible to choose \( k \) independent generators spanning a Lie algebra structure which will be associated with a \( k \)-dimensional Lie subgroup. Locally, one can always obtain these Lie subgroups of the gauge group, but global obstructions may rule out some of them. One simple example of such a local construction is furnished by the holonomic basis \( \partial/\partial q^\alpha, \alpha = 1, \cdots, k \), which provides us with an Abelian subgroup. These considerations serve to illustrate the following important point: *any of these subgroups can be used to reconstruct the orbits, and hence, the full gauge group.* So, in this geometrical picture, there is no loss of information if we “cut down” the gauge group to one of these subgroups [30].

The possibility of “changing” the structure of the gauge group (for instance “Abelianization” of the gauge group) has been dealt with by the authors of reference [32]. Nevertheless, in the light of our previous discussion, we see that this kind of language can be somewhat misleading, since one is never changing the structure of the gauge group but just selecting different subgroups (or more general objects as for instance generators that close with non-constant structure functions) to describe the orbits.

After this discussion on the gauge group, it should be quite clear what we meant above by the statement that different ways of selecting the generators may lead to different Lie subgroups, hence different Killing metrics, and possibly inequivalent quantizations. It would be interesting to find some principle, in the case of an arbitrary manifold \( Q \), that would naturally select one particular Lie subgroup of the gauge group. At this point we haven’t found such a principle, and it seems to us very unlikely that there is one. Of course, nothing precludes the existence of such a selection procedure in some specific cases. Indeed, that will be the case of \( Q = G \) with \( G \) a compact group, which will be the subject of the next section.
If we assume the Killing condition one can try to make contact with the “reduce first” method as follows. First of all, as we saw in Section 5, we have to check for consistency of the dynamical evolution in both approaches, i.e.,

\[ \Delta_M |_{P_b} = \Delta_g \]  

(6.6)

where \( \Delta_g \) is defined by equation (4.17). In the adapted coordinate system, (6.6) implies

\[ m_{A'B'}^{\alpha\beta} \Gamma_{\alpha\beta}^{a} = g^{cb} \tilde{\Gamma}_{cb}^{a} \]  

(6.7)

In equation (6.7) \( \tilde{\Gamma}_{cb}^{a} \) is computed with the reduced metric. This would be readily guaranteed if the metric \( M \) in adapted coordinates reads as follows:

\[ m_{A'B'} = \begin{pmatrix} g^{ab}(q^a) & 0 \\ 0 & g^{\alpha\beta}(q^\alpha) \end{pmatrix} \]  

(6.8)

Consistency of this form of the metric and the Killing condition demands that

\[ U_{\alpha,a}^{\beta} = 0 \]  

(6.9)

which, considering the freedom in the choice of the \( \hat{U}_\alpha \) to form a Lie algebra does not seem to be, at least locally, too restrictive.

The other issue to study in order to check for matching between the two approaches is the relation between the measures in both cases. At this point the factorization property, eq. (5.15), is very useful. Notice that in general the volume of the orbit \( V(q^a) \) is a function of \( q^a \). In the case the metric takes the form (6.8) a \( q^a \) dependence would arise if the range of integration in eq. (5.16) were to depend on the orbit which is a global problem that one should not discard a priori. Equation (5.16) shows that in the general case, when \( V(q^a) \) is not a constant, there is a good chance for the two procedures not to agree, since the measures are not equal. One could attempt to remedy this deficiency by absorbing \( V(q^a) \) in a redefinition of the wave functions. Nevertheless, this will generally spoil the equivalence.
between the dynamics which had been previously established. In the situation where $\mathcal{V}$ is a constant, one would be tempted to conclude that equivalence has been achieved. But there might be some issues concerning the domain of the operators when the self-adjointness condition is implemented, which we have taken for granted. This clearly deserves further study on a case by case basis.

Finally, we would like to make the following remark. The consistency of the dynamical evolution of the two approaches depend only on the form of the metric (6.8), regardless of it being Killing or not. Moreover, within our formulation we can always choose $m^{AB'}$ to be of this form, in virtue of the freedom afforded to us and displayed in equation (4.12). It now seems more plausible that by playing with coordinate choices for the orbits one can obtain volumes which are orbit independent in the case the metric takes the form (6.8). Indeed, if we could find gauge coordinates with constant range, then all we would do is write down (4.12) in these coordinates in the form (6.8). In this case, again up to the resolution of issues of domains of the operators in the theory, we would have achieved equivalence.

At this point the reader may have developed the feeling that, given the multiplicity of options available in the Dirac quantization approach, equivalence with reduced quantization is almost a matter of chance. Nevertheless, there is a wide class of theories where equivalence is rigorously obtained, with a natural selection of the Lie subgroup of the gauge group and the metric. This is the case of quantization on coset spaces of compact Lie groups, which is the subject of next section. Before that it is worth to mention that Kuchař [31] as shown in a simple model the non-equivalence between Dirac and reduced quantization. His model has a field theoretical application in scalar electrodynamics but, as it is pointed out by Kuchař, renormalization problems obscure in this case the issue of the equivalence of both approaches.
7 Application: Quantization on Coset Spaces

One important example of theories where our previous results can be applied is the case of quantization on coset spaces of compact Lie groups. We will first establish notation and then explain how to set up the constraint systems of interest for us in such manifolds.

Our configuration manifold will be a compact Lie group $G$ of dimension $N$. Its Lie algebra will be realized by means of left-invariant vector fields $\hat{U}_A$, $A = 1, \ldots, N$:

$$[\hat{U}_A, \hat{U}_B] = C_{AB}^C \hat{U}_C,$$  \hspace{1cm} (7.1)

where the structure constants satisfy, besides the obvious ones, the following antisymmetry property:

$$C_{AB}^C = -C_{BA}^C$$  \hspace{1cm} (7.2)

which can always be achieved for any compact Lie algebra.

On coordinates $Q^A$, $A = 1, \ldots, N$ our left-invariant vector fields can be written as $\hat{U}_A = U_A^B(Q) \partial / \partial Q^B$, and its dual left-invariant forms as $\Omega^A = \Omega_A^B(Q) dQ^B$, $\Omega_B^A = (U_A^B)^{-1}$. $G$ is endowed with a non-singular left-invariant metric $M_{AB}$ which is a Killing metric for the vector fields $\hat{U}_A$, $A = 1, \ldots, N$. It has the form

$$M_{AB} = \Omega_A^C \Omega_B^D \delta_{CD}. $$  \hspace{1cm} (7.3)

In order to introduce a constraint system in a natural fashion, we will extract from (7.3) a singular metric which will define the kinetic term in the Lagrangian for such a system. One way to achieve this is simply to restrict the range of the summation index in (7.3) to $a = 1, \ldots, n < N$. Indeed, $\hat{U}_\alpha$, where $\alpha$ runs over the complementary set of indices, form a basis of null vectors for this new metric $G_{AB} = \Omega_A^a \Omega_B^\beta \delta_{\alpha\beta}$. At this point, we start making contact with our Lagrangian setting of Section 2. In this case, condition (2.8) implies that the structure constants have to satisfy

$$C_{\alpha\beta}^a = 0,$$  \hspace{1cm} (7.4)
and
\[ C_{ba}^a = -C_{ab}^b. \] (7.5)

Equation (7.4) is the statement that the \( \hat{U}_\alpha, \alpha = 1, \ldots, k = N - n \) form a subalgebra of the original Lie algebra:
\[ [\hat{U}_\alpha, \hat{U}_\beta] = C_{\alpha\beta}^\gamma \hat{U}_\gamma. \] (7.6)

Equation (7.5) does not give new information, since this already guaranteed by (7.2).

This subalgebra generates a Lie subgroup \( K \) of \( G \). The quotient manifold \( M \) of Section 4 is then the coset space \( G/K \). To fix ideas, it is convenient to write our singular Lagrangian:
\[ L = \frac{1}{2} G_{AB} \dot{Q}^A \dot{Q}^B - V, \] (7.7)
where \( G_{AB} \) was just defined above, and \( V \) is a potential function satisfying, according to (2.6),
\[ U_A^A V_A = 0, \quad \alpha = 1, \ldots, k. \]

Now we present the Hamiltonian formalism. According to (3.2) the constraints are
\[ \varphi_\alpha \equiv U^A_\alpha(Q) P_A \approx 0, \quad \alpha = 1, \ldots, k. \] (7.8)

Due to the particular structure of our metric \( G_{AB} \) it is straightforward to write down a solution to equation (3.9), namely, \( U^A_a U_a^B \), which is obviously singular. The freedom of choice of \( M \) as shown in equation (3.10) allows us to work with a non-singular contravariant metric \( M^{AB} = U^A_C U^B_C \). This is nothing but the inverse of the covariant non-singular metric \( M_{AB} \) that we started with. This metric structure provides us with the invariant measure on \( G \), and it is such a natural structure on it, that we will henceforth work always with it. This illustrates our point that the freedom in the choice of \( M \) can be of great importance.

Therefore, (3.7) becomes
\[ H = \frac{1}{2} M^{AB} P_A P_B + V, \] (7.9)
with \( M^{AB} = U^A_C U^B_C \). Now we have all the ingredients to quantize according to the two schemes presented in Sections 4 and 5. The “first reduce and then quantize” procedure
is readily implemented since all we have to do is use the metric tensor $M$ to obtain $\tilde{g}$ for $G/K$ (equation (4.6)). This assignment defines the Hilbert space structure and the quantum dynamics according to (4.15)–(4.18). This completes the “first reduce . . . ” program.

In order to implement the “first quantize and then reduce” scheme, we first have to check the first class character of the quantum Hamiltonian, i.e., equation (5.5). The first class nature of the constraints is already assured by (7.6). In our case the Laplacian operator is

$$\Delta_M = |M|^{-\frac{1}{2}} \frac{\partial}{\partial Q^A} |M|^\frac{1}{2} M^{AB} \frac{\partial}{\partial Q^B}. \quad (7.10)$$

Now, since the $\hat{U}_A$, $A = 1, \ldots, N$ are Killing vectors for $M$, they are divergenceless:

$$|M|^{-\frac{1}{2}} \frac{\partial}{\partial Q^A} |M| U^A_C = 0, \quad C = 1, \ldots, N \quad (7.11)$$

which allows us to write $\Delta_M$ as

$$\Delta_M = \hat{U}_C \hat{U}_C. \quad (7.12)$$

Using equation (7.12) and the antisymmetry property $C^C_{AB} = -C^B_{AC}$ is a matter of straightforward algebra to verify that

$$[\Delta_M, \hat{U}_A] = 0, \quad A = 1, \ldots, N. \quad (7.13)$$

Hence, $\hat{H}$ passes the test. This guarantees a consistent quantization in $\mathcal{H}_p$, in the sense defined in Section 5. Furthermore, since the gauge orbits (generated by $K$) are compact, all the formulae derived in that section apply in a rigorous sense. We now proceed to show equivalence between the two approaches. This will be done in two steps: first, we will show that the measure on the physical space $\mathcal{H}_p$ coincides, up to an irrelevant constant factor, with the measure obtained from the “reduced first” procedure - this shows that, in fact, the two Hilbert spaces are the same. Then, we will see that the dynamics in both cases are identical.

**THE MEASURES.**
In the “first reduce” procedure, the metric in $G$ is defined as
\[
g_{ab} = M^{AB} \frac{\partial q^a}{\partial Q^A} \frac{\partial q^b}{\partial Q^B} = U_D(q^a) U_D(q^b) = U_{\hat{d}}(q^a) U_{\hat{d}}(q^b) = U_a U_b
\] (7.14)
where we used $\hat{U}_a(q^a) = 0$. The measure is then
\[
\mu_R = |g|^\frac{1}{2} = |g_{ab}|^{-\frac{1}{2}} = |U_a^b|^{-1}
\] (7.15)

Now consider the “first quantize” approach. In the adapted coordinate system, $Q^{A'} = (q^a, q^\alpha)$, we showed factorization in general, eq. (5.15). The only possible obstruction to equivalence is the dependence of $V(q^a)$ on $q^a$. Let us now compute $V(q^a)$ in this case. From (7.3) we get
\[
|m|^{1/2} = |\hat{U}_D(Q^{A'})|^{-1} = |U_b^a|^{-1} |U_a^\beta|^{-1} = \mu_R(q^a) |U_a^\beta|^{-1} = \mu_R(q^a) |\Omega_{\alpha}^\beta| \equiv \mu_R(q^a) |\Omega|
\] (7.16)

Therefore, the volume of the orbit is:
\[
V(q^a) = \int d^k q |\Omega|.
\] (7.17)

We now prove that (7.17) is indeed a constant. To see this, it is convenient to rewrite $V$ in intrinsic notation. Consider the injection
\[
i_{q^a} : O(q^a) \longrightarrow G.
\] (7.18)

Then we have
\[
V(q^a) = \int_{O(q^a)} i_{q^a}^* \sum,
\] (7.19)
where $\sum = \Omega_1 \wedge \Omega_2 \wedge \cdots \wedge \Omega_k$ and the indices $1, 2, \cdots k$ are the $\alpha$-type indices. $i_{q^a}^* \sum$ is the pullback of $\sum$ under (7.18). We should remind ourselves that $\sum$ is by construction left-invariant. Now consider an element $g \in G$, but $g \notin K$. It maps orbits into other orbits
\[
g : O(q^a) \longrightarrow O(\tilde{q}^a)
\] (7.20)
Then the following chain of equalities holds:

\[
\mathcal{V}(\tilde{\rho}^a) \equiv \int_{\mathcal{O}(\tilde{\rho}^a)} \tilde{\rho}^a \sum = \int_{\mathcal{O}(\tilde{\rho}^a)} g^* (\tilde{\rho}^a \sum) = \int_{\mathcal{O}(\tilde{\rho}^a)} \tilde{\rho}^a (g^* \sum) = \int_{\mathcal{O}(\rho^a)} \tilde{\rho}^a \sum \equiv \mathcal{V}(\rho^a). \tag{7.21}
\]

The first equality comes from a passive interpretation of the action of \( g \) as a change of variables in the same orbit. The second one is just a consequence of the fact that \( i_{\tilde{\rho}^a} \circ g = g \circ i_{\rho^a} \). To obtain the last one we used the left-invariance of \( \sum \). This proves that all orbits have the same volume. Therefore the measures are equal up to a normalization constant.

**THE DYNAMICS**

Now we proceed to show that the Laplacian operator (7.10), \( \Delta_M \), when restricted to \( \mathcal{H}_p \) coincides with the Laplacian operator (4.17) \( \Delta_g \) of the “reduce first” approach. First we observe that (7.12) allows us to write the restriction of \( \Delta_M \) on \( \mathcal{H}_p \) as

\[
\Delta_M |_{\mathcal{H}_p} = \hat{U}_a \hat{U}_a,
\tag{7.22}
\]

where we have used \( \hat{U}_a \rho^a = 0 \). Explicitly, in the adapted coordinate system,

\[
\Delta_M |_{\mathcal{H}_p} = U^b_a \frac{\partial}{\partial q^b} U^c_a \frac{\partial}{\partial q^c} + U^a_\alpha \frac{\partial}{\partial q^\alpha} U^c_a \frac{\partial}{\partial q^c} |_{\mathcal{H}_p}
= U^b_a \frac{\partial}{\partial q^b} U^c_a \frac{\partial}{\partial q^c} + U^a_\alpha U^c_a,_{\alpha} \frac{\partial}{\partial q^c}. \tag{7.23}
\]

On the other hand,

\[
\Delta_g = |g|^{-\frac{1}{2}} \frac{\partial}{\partial b} |g|^{\frac{1}{2}} g^{bc} \frac{\partial}{\partial q^c} = |g|^{-\frac{1}{2}} \frac{\partial}{\partial b} |g|^{\frac{1}{2}} U^b_a \frac{\partial}{\partial q^c} =
= U^b_a \frac{\partial}{\partial q^b} U^c_a \frac{\partial}{\partial q^c} + |g|^{-\frac{1}{2}} \left( |g|^{\frac{1}{2}} U^b_a \right)_b U^c_a \frac{\partial}{\partial q^c}. \tag{7.24}
\]

Hence

\[
\Delta_M |_{\mathcal{H}_p} - \Delta_g = \left( U^a_\alpha U^c_a,_{\alpha} - |g|^{-\frac{1}{2}} \left( |g|^{\frac{1}{2}} U^b_a \right)_b U^c_a \right) \frac{\partial}{\partial q^c}
\equiv A^c \frac{\partial}{\partial q^c}. \tag{7.25}
\]
We are going to show that $A^c = 0$. Using the fact that $|g|, \alpha = 0$, we can write

$$|g|^\frac{1}{2} (|g|^\frac{1}{2} U_a^b) = |g|^\frac{1}{2} (|g|^\frac{1}{2} U_a^A')_{,A'} - U^\alpha_{a,\alpha}$$  \hspace{1cm} (7.26)

where $A'$ runs over the entire set of indices $a$ and $\alpha$. Then

$$A^c = U^\alpha_a U^c_{a,\alpha} - \left(|g|^\frac{1}{2} (|g|^\frac{1}{2} U_a^A')_{,A'} - U^\alpha_{a,\alpha}\right) U^c_a$$

$$= (U^\alpha_a U^c_a)_{,\alpha} - |g|^\frac{1}{2} (|g|^\frac{1}{2} U_a^A')_{,A'} U^c_a.$$  \hspace{1cm} (7.27)

The divergenceless of the vector field $\hat{U}_a$ can be reexpressed as

$$0 = |m|^\frac{1}{2} (|m|^\frac{1}{2} U_a^A')_{,A'} = |g|^\frac{1}{2} |\Omega|^{-1} (|g|^\frac{1}{2} |\Omega| U_a^A')_{,A'}$$

$$= |\Omega|^{-1} |\Omega|_{,A'} U_a^A' + |g|^\frac{1}{2} (|g|^\frac{1}{2} U_a^A')_{,A'}.$$  \hspace{1cm} (7.28)

which allows us to rewrite (7.27) as

$$A^c = (U^\alpha_a U^c_a)_{,\alpha} + |\Omega|^{-1} \hat{U}_a (|\Omega|) U^c_a.$$  \hspace{1cm} (7.29)

Now, recall that $\hat{U}_a = U^\alpha_a \partial_\alpha$, $\partial_a = \Omega^\beta_\alpha \hat{U}_a$. Also,

$$|\Omega|^{-1} \hat{U}_a (|\Omega|) = |\Omega|^{-1} |\Omega| \left(- \Omega^\beta_\alpha \hat{U}_a (U^\alpha_\beta)\right)$$

$$= -\Omega^\beta_\alpha \hat{U}_a (U^\alpha_\beta).$$  \hspace{1cm} (7.30)

Using these results (7.29) can be rewritten as

$$A^c = \Omega^\beta_\alpha \left(\hat{U}_{\beta}(U^\alpha_a U^c_a) - \hat{U}_a (U^\alpha_\beta U^c_a)\right)$$

$$= \Omega^\beta_\alpha \left(\left(\left(\hat{U}_{\beta}(U^\alpha_a) - \hat{U}_a (U^\alpha_\beta)\right) U^c_a + U^\alpha_a \hat{U}_\beta (U^c_a)\right).$$  \hspace{1cm} (7.31)

Recalling that $C^\gamma_{\beta a} = -C^a_{\beta \gamma} = 0$ (compactness and subgroup conditions) we have

$$[\hat{U}_{\beta}, \hat{U}_a] = C^d_{\beta a} \hat{U}_d,$$  \hspace{1cm} (7.32)
which implies (using $U^b_\beta = 0$) that

$$\hat{U}_\beta(U^\alpha_a) - \hat{U}_{\alpha}(U^\alpha_\beta) = C^d\beta_a U^\alpha_d, \quad (7.33)$$

and

$$\hat{U}_\beta(U^c_a) = C^d\beta_a U^c_d. \quad (7.34)$$

Hence

$$A^c = \Omega^\beta_\alpha C^d\beta_a (U^\alpha_d U^c_a + U^\alpha_a U^c_d), \quad (7.35)$$

which is zero in virtue of the antisymmetry of the structure constants. Therefore,

$$\Delta_M |_{H_p} = \Delta_g. \quad (7.36)$$

We would like to emphasize the importance of the compactness of $G$ in the derivation of (7.30).
8 Example: The Free Propagator on $S^2$ from The Free Propagator on $SU(2)$

As a concrete application of the formalism developed in Sections 5 and 7 we will now consider the motion of a free particle on a sphere $S^2$, viewed as the coset space $SU(2)/U(1)$ [30].

Making use of the isomorphism $SU(2) \sim S^3$ we may parametrize the group manifold $SU(2)$ using polar coordinates on $S^3$:

$$Q^A = (\theta, \phi, \psi), \quad 0 \leq \theta \leq \pi, \quad 0 \leq \phi < 2\pi, \quad 0 \leq \psi < 4\pi. \quad (8.1)$$

The left-invariant vector fields are:

$$\hat{U}_1 = \left( \sin \psi \partial_\theta - \frac{\cos \psi}{\sin \theta} \partial_\phi, \frac{\cos \psi}{\tan \theta} \partial_\psi \right) \equiv U_1^B \partial_B, \quad (8.2)$$

$$\hat{U}_2 = \left( \cos \psi \partial_\theta + \frac{\sin \psi}{\sin \theta} \partial_\phi - \frac{\sin \psi}{\tan \theta} \partial_\psi \right) \equiv U_2^B \partial_B, \quad (8.3)$$

$$\hat{U}_3 = \partial_\psi = U_3^B \partial_B, \quad (8.4)$$

where $\partial_B \equiv \frac{\partial}{\partial Q^B}$. They satisfy the $SU(2)$ algebra

$$[\hat{U}_A, \hat{U}_B] = \epsilon_{ABC} \hat{U}_C. \quad (8.5)$$

The $\hat{U}_A, A = 1, 2, 3$, are Killing vectors for the metric

$$G^{AB} = U_A^C U_B^D \delta^{CD}, \quad (8.6)$$

which in this coordinate system takes the form

$$G^{AB} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sin^2 \theta} & -\cos \theta/\sin^2 \theta \\ 0 & -\cos \theta/\sin^2 \theta & \frac{1}{\sin^2 \theta} \end{pmatrix}. \quad (8.7)$$

We will choose the uniparametric subgroup $U(1)$ generated by $\hat{U}_3$ as the gauge group $K$.

The simplest choice of the adapted system is then

$$q^a = (\theta, \phi) \quad \Rightarrow \quad q^\alpha = (\psi). \quad (8.8)$$
In this system, the measure, equation (5.14), becomes

$$\mu(\theta, \phi) = 4\pi \sin \theta,$$

and the propagator, equation (5.25), now reads

$$P_{\text{ph}} \langle \theta', \phi', t' | \theta, \phi, t \rangle_{\text{ph}} = \frac{1}{16\pi^2} \int_0^{4\pi} d\psi_2 \int_0^{4\pi} d\psi_1 \langle \theta', \phi', \psi_2, t' | \theta, \phi, \psi_1, t \rangle. \quad (8.8)$$

All we have to do now is to substitute for $$\langle \theta', \phi', \psi_2, t' | \theta, \phi, \psi_1, t \rangle$$ its explicit form, which can be readily found in the literature. Following Schulman [38], [39],

$$\langle \theta', \phi', \psi_2, t' | \theta, \phi, \psi_1, t \rangle = \frac{1}{16\pi^2} \frac{1}{\sin \Gamma} \sum_{j=0,\frac{1}{2},1,\cdots} (2j + 1)(\sin(2j + 1)\Gamma)e^{-\frac{j}{\pi} j(j+1)(t'-t)} \quad (8.9)$$

where $$\Gamma$$ is the geodesic distance between two points on $$S^3$$:

$$\cos \Gamma = \cos \left(\frac{\theta}{2}\right) \cos \left(\frac{\theta'}{2}\right) \cos(\psi_+ - \psi'_+) + \sin \left(\frac{\theta}{2}\right) \sin \left(\frac{\theta'}{2}\right) \cos(\psi_- - \psi'_-), \quad (8.10)$$

with $$\psi_\pm = (\psi \pm \phi)/2$$, and $$\psi' = \psi_2$$, $$\psi = \psi_1$$. We use here notation of reference [37].

Notice we are taking advantage of the isomorphism between $$SU(2)$$ and $$S^3$$. Performing the integrations in (8.8) we then obtain

$$P_{\text{ph}} \langle \theta', \phi', t' | \theta, \phi, t \rangle_{\text{ph}} = \frac{1}{16\pi^2} \sum_{j=0}^{\infty} (2j + 1)P_j(\cos \gamma) e^{-\frac{j}{\pi} j(j+1)(t'-t)} \quad (8.11)$$

where $$\gamma$$ is the geodesic distance between $$q' = (\theta', \phi')$$ and $$q = (\theta, \phi)$$ on $$S^2$$. This is in agreement with [10], after taking into account the normalizations for our states, given by equations (5.17) and (8.7).
9 Conclusions and Outlook

There are several lessons that can be drawn from this work, both at a conceptual as well as at a more technical level, keeping in mind that we are considering only systems with Hamiltonian and constraints quadratic and linear in the momenta respectively. On the one hand, the novelty of the Lagrangian setting for first class systems reveals quite clearly a source of ambiguities present in the framework of Dirac’s quantization which, to our knowledge, had not been pointed out before. Within our scheme the one-to-one correspondence between the original Lagrangian and the reduced Hamiltonian quantization is neatly seen. Hence the reduced quantization possesses a certain uniqueness that is lacking in the Dirac approach. This point of view is reinforced when one considers our work with path integrals [14] where again reduced quantization plays a central role. The discussion in section 6 shows that the ambiguities mentioned above make plausible the selection of a specific Dirac quantization which coincides with the reduced first method. Several features of the gauge group relevant to this problem are also mentioned there.

On a more practical level we show rigorously that in the case of coset spaces of compact Lie groups there is a natural selection of Dirac quantization that fully coincides with the reduced quantization. This fact allows us to use results of the quantum theory on the group $G$ to obtain the quantum theory on the coset space $G/K$, with $K$ a subgroup. To make contact with work of other people we compute the propagator on a two dimensional sphere $S^2 = SU(2)/U(1)$ by this means.

It would be interesting to extend our work to investigate similar problems in more sophisticated cases like Chern-Simons which we mentioned in the introduction. The case of non-compact Lie groups clearly demands an extension of our methods in this direction also.

We feel that a close comparison with our path-integral work would be illuminating. Finally, the treatment of constraints quadratic in the momenta -relevant in the study of reparameter-
ization invariant systems like gravity—within our framework, deserves our full consideration. Work on these topics is currently in progress.

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Appendix

We want to show that from the Hamiltonian (3.5),
\[ H = \frac{1}{2} M^{AB}(Q) P_A P_B + V \]  
and the primary constraints (3.2),
\[ U^A_\alpha(Q) P_A \approx 0, \]
we can recover the Lagrangian (2.1) which we started with. For this purpose we will perform the inverse Legendre transformation [41] on the Dirac Hamiltonian:
\[ H_D = \frac{1}{2} M^{AB} P_A P_B + \lambda^\alpha U^A_\alpha P_A + V. \]  
This means we want to find \((P_A, \lambda^\alpha)\) in terms of \((Q^A, \dot{Q}^A)\), as the solution of the following algebraic system of equations
\[ \dot{Q}^A = \{Q^A, H_D\} = M^{AB} P_B + \lambda^\alpha U^A_\alpha \]
\[ U^A_\alpha P_A = 0. \]
Once this is achieved we will just substitute the momenta as functions of velocity space variables in the expression for the Lagrangian
\[ L = P_A \dot{Q}^A - H. \]
Notice that equation (A.4) involves only half of Hamilton equations. This is so, because this is the one that contains the information about the Legendre map. In virtue of equation (A.4),
\[ \dot{Q}^A U^A_\beta = \lambda^\alpha \Theta_{\alpha\beta} \]
where
\[ \Theta_{\alpha\beta} = U^A_\alpha U^A_\beta \equiv U^A_\alpha M_{AB} U^B_\beta \]
is just the scalar product of the vector fields $\hat{U}_\alpha, \hat{U}_\beta$ with respect to the metric $M$. Since the vector fields $\hat{U}_\alpha, \alpha = 1, \ldots, k$ are independent, and the metric $M$ is non-singular, the matrix $\Theta_{\alpha\beta}$ is invertible. Hence,

$$\chi^\alpha = \Theta^{\alpha\beta} U_\beta A \dot{Q}^A,$$ \hspace{1cm} (A.8)

and using again equation (A.4) we obtain

$$\dot{Q}^A = M^{AB} P_B + \Theta^{\alpha\beta} U_\beta C \dot{Q}^C U_\alpha^A$$ \hspace{1cm} (A.9)

from which we get for $P_B$

$$P_B = M_{BA} P_C^A \dot{Q}^C,$$ \hspace{1cm} (A.10)

where

$$P_C^A = \delta_C^A - \Theta^{\alpha\beta} U_\beta C U_\alpha^A$$ \hspace{1cm} (A.11)

is the projector in the direction transverse to the orbits. Finally, the Lagrangian becomes

$$L = \frac{1}{2} M^\perp_{CD} \dot{Q}^C \dot{Q}^D - V,$$ \hspace{1cm} (A.12)

where

$$M^\perp_{CD} = P_C^A M_{AB} P_D^B.$$ \hspace{1cm} (A.13)

It only remains to show that $M^\perp_{CD} = G_{CD}$, where $G_{CD}$ is the metric in the original Lagrangian (2.1). To show this it is convenient to work in the adapted coordinate system, in which the obvious fact that the vectors $\hat{U}_\alpha, \alpha = 1, \ldots, k$ are null vectors of $M^\perp_{CD}$ reveals that it has the form

$$M^\perp = \begin{pmatrix} M^\perp_{ab} & 0 \\ 0 & 0 \end{pmatrix}.$$ \hspace{1cm} (A.14)

In view of equation (4.12), in order to check that $M^\perp_{ab} = g_{ab}$ we only need to verify $M^\perp_{A^a B^b} = \delta^\alpha_\sigma$. From the definition of $M^\perp$ we obtain

$$M^\perp_{A^a B^b} = M_{A^a B^b} - M_{A^c B^b} U_\alpha^c \Theta^{\alpha\beta} U_\beta^C M_{C \sigma B^b},$$ \hspace{1cm} (A.15)
where \( U^a_\alpha = 0 \), equation (4.9), was used. Finally, recalling that \( \delta^a_\rho = 0 \) we have

\[
M^{aA'}M^b_{A'b} = \delta^a_b - \delta^a_\rho U^\rho_\alpha \Theta^{\alpha\beta} U^\beta_\sigma M_{\sigma b} = \delta^a_b ,
\]

which proves our assertion.
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[18] Throughout this paper we will consider explicitly the case of a quantum mechanical system with a finite number of degrees of freedom. Nevertheless, our results will readily generalize to the case of a field theory, once DeWitt’s condensed notation is adopted. The important application to Yang-Mills fields requires a delicate analysis of the role played by the time component \( A_0 \) as a Lagrange multiplier. In this way we can get rid of the canonical momenta conjugate to \( A_0 \) and obtain a non-covariant formulation of Yang-Mills theories with only primary constraints, the Gauss law, which were secondary constraints in the covariant formulation. Once this is done, our results can be applied.

[19] We use the convention that a comma denotes partial derivative.

[20] The weak equality \( \approx \) stands for equality on the constraint surface (Dirac’s notation).

[21] This result is a direct consequence of the analysis of the relationship between Hamiltonian and Lagrangian constraints, which was performed by the authors of reference [22]. There it was shown that at every stage of the stabilization algorithm (i.e., the iterative procedure of generating the constraints of the theory, which appear by consistency requirements) the Lagrangian constraints can be obtained by means of either
of the following two constructions: a) Some will come as the pullback of the secondary, terciary, ..., constraints which arise from the Hamiltonian version of the algorithm mentioned above; b) The others will come as the Lagrange multipliers in the Dirac Hamiltonian are determined as specific functions in phase space. Having a first class Hamiltonian system without secondary constraints (and hence without further constraints) renders these two constructions meaningless. Hence there will be no room for Lagrangian constraints. For more details we advise the reader to consult the reference given above.

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[25] Notice that $\alpha$ runs from $n + 1$ to $N$. Nevertheless, to simplify notation, we will have it run from $g$ to $k = N - n$. Consider the following identity

[26] From here on, we will use the convention that $\int d^n q$ and $\int d^k q$ denote integration over variables $q^a$ and $q^\alpha$ respectively.

[27] In usual treatments which ignore the ambiguities present in the Hamiltonian kinetic term of a constraint system, this condition on the metric is taken as the starting point.

[28] This invariance under rescaling plays a central role in Kuchar's work.

[29] Since we are not specifying the manifold $Q$ at this point, we will ignore global considerations which may arise. Everything we do here is valid locally.
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[35] Equation (7.13) is already guaranteed by the Killing vector condition.

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