The Crawler: Two Equivalence Results for Object (Re)allocation Problems when Preferences Are Single-peaked

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Abstract

For object reallocation problems, if preferences are strict but otherwise unrestricted, the Top Trading Cycle rule (TTC) is the leading rule: It is the only rule satisfying efficiency, the endowment lower bound, and strategy-proofness; moreover, TTC coincides with the core. However, on the subdomain of single-peaked preferences, Bade (2019a) defines a new rule, the “crawler”, which also satisfies the first three properties. Our first theorem states that the crawler and a naturally defined “dual” rule are actually the same. Next, for object allocation problems, we define a probabilistic version of the crawler by choosing an endowment profile at random according to a uniform distribution, and applying the original definition. Our second theorem states that this rule is the same as the “random priority rule” which, as proved by Knuth (1996) and Abdulkadiroglu and Sönmez (1998), is equivalent to the “core from random endowments”.

Keywords: object reallocation problems, single-peaked preferences, the crawler, the random priority rule, the core.

JEL classification: C78, D47.

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1 Introduction

Consider a group of agents each of whom is endowed with an indivisible good, called an “object”. Each agent has preferences over the objects. The initial allocation may not be efficient (in the sense of Pareto efficiency) and the goal is reallocating the objects so as to achieve efficiency as well as possibly other socially desirable properties. An example of this type of problems is when the agents are households and the objects are housing units they own (what Shapley and Scarf (1974) call a “housing market”). A rule is a single-valued mapping that associates with each such problem an allocation, interpreted as a recommendation for the problem. If preferences are strict but otherwise unrestricted, the Top Trading Cycle rule (TTC) is the leading rule (Shapley and Scarf, 1974): It is the only rule satisfying three desirable properties, “efficiency”, the “endowment lower bound”,\(^1\) and “strategy-proofness” (Ma, 1994).\(^2\)

Interestingly, TTC is not the only rule satisfying these properties on the subdomain of “single-peaked” preferences. Returning to our example of a housing market, suppose that the housing units are of different sizes and that each household evaluates units based on their size. A single person may prefer a small unit; a family with children may prefer a large one. Each household has an ideal size; the further the size of a unit is from this ideal size, in either direction, the less desirable the unit is. Thus, households have single-peaked preferences with respect to the order. Instead of size, the order could be based on how expensive they are, or their proximity to a school or to the central business district. Many other examples can be found where agents have single-peaked preferences with respect to some reference order on the object set.\(^3\)

On the single-peaked domain, Bade (2019a) defines a new rule, which she calls the “crawler”, and shows that this rule, as TTC does, satisfies efficiency, the endowment lower bound, and strategy-proofness. The idea underlying the crawler is as follows. Objects are labeled in such a

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\(1\)Another common name for this property is “individual rationality”.

\(2\)Other proofs of this uniqueness result can be found in Svensson (1999), Anno (2015), Sethuraman (2016), and Bade (2019a).

\(3\)Accordingly, this domain has been studied from several other viewpoints (Liu, 2018; Damamme et al., 2015; Beynier et al., 2019). Moreover, for other types of resources allocation problems, a single-peakedness is a natural assumption. An example is when an infinitely divisible commodity has to be fully allocated among a group of agents (Sprumont, 1991).
way that preferences are single-peaked with respect to this order. Similarly, agents are ordered according to their ownership of the objects. They are visited from left to right and each agent in turn is asked if he most prefers his endowment or an object to the left of his endowment. If yes, he is asked which object he most prefers and he is assigned that object and leaves. Otherwise, the agent on his immediate right is asked the same question. At least one agent has to most prefer his endowment or an object to the left of his endowment. So some agent is eventually assigned an object, and he leaves with his assignment. Then the problem is updated as follows. Consider all of the agents whose endowments are between the object assigned to the agent who left and the object that agent owned. For each of those agents, ownership is shifted by one spot to the right. This sweeping process is repeated with the updated problem, and continue in this way until every agent has been assigned an object.

Obviously, one could define a “dual” rule by visiting agents from right to left (Bade, 2019a). One would expect the crawler and its dual to differ. Because each of the two crawlers visits agents in a specific order, one may be concerned that the specific order that is chosen confers a particular advantage to some agents based on the location of their endowments relative to the location of other agents’ endowments. Our first result is that they are in fact the same (Theorem 1). Thus, the order in which agents are visited confers no such advantage.

The “core” is one of the most important solutions concepts in economics. For object reallocation problems, an allocation is in the core if there are no coalition and assignments for the members of the coalition such that (i) the objects assigned to them are the objects they collectively own, and (ii) each member of the coalition is at least as well off and at least one of them is better off. TTC bears a strong relation to the core: For each preference profile, including each profile of single-peaked preferences, the core is a singleton and the allocation selected by TTC coincides with the allocation in the core (Shapley and Scarf, 1974; Roth and Postlewaite, 1977).

Although the core concept pertains to reallocation problems, it also relates to a family of rules for “object allocation problems”; there, instead of being owned individually, objects are owned collectively. Well-studied rules for object allocation problems are the “sequential priority rules”⁴: To each order on the agent set is associated such a rule: The agent who is first

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⁴Another common name for these rules is “serial dictatorships”.

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is assigned his most preferred object; the agent who is second is assigned his most preferred object among the remaining ones; and so on. Obviously, none of these rules achieve any punctual fairness notion. However, this lack of fairness can be alleviated by allowing rules to select probability distribution over allocations. These are “probabilistic rules”. Applying the idea to the sequential priority rules, we would choose an order on the agent set at random according to a uniform distribution and apply the induced sequential priority rule. The resulting rule is known as the “random priority rule” (Abdulkadiroglu and Sönmez, 1998).\(^5\) Another way of attempting to achieve some fairness is by choosing an endowment profile at random according to a uniform distribution and selecting the core of the induced object reallocation problem. The resulting probabilistic rule is known as the “core from random endowments”. It turns out that the random priority rule is the same as the core from random endowments (Knuth, 1996; Abdulkadiroglu and Sönmez, 1998).

We may ask whether the crawler bears a similar relationship to the sequential priority rules. The procedures underlying the definitions of the crawler and of the sequential priority rules are based on completely different considerations. Hence, one may be doubtful that such a relation exists. Yet, our second theorem provides a positive answer to this question. Given a preference profile, let us select an endowment profile at random according to a uniform distribution, and apply the crawler to the induced object reallocation problem. We call the probabilistic rule so defined the “crawler from random endowments”. Our second result is that the crawler from random endowments is the same probabilistic rule as the random priority rule (Theorem 2). Therefore, it is also the same as the core from random endowments. If there are at least three agents, the crawler differs from TTC (Bade, 2019a). Hence, the crawler does not in general select an allocation from the core. However, our equivalence result can be interpreted as mitigating this fact, and it should further strengthen the appeal of this rule.

Several other equivalence results in the same vein have been obtained (Pathak and Sethuraman, 2011; Lee and Sethuraman, 2011; Sönmez and Ünver, 2005; Ekici, 2017; Carroll, 2014; Bade, 2019b). Each of them states that a probabilistic version of a generalized TTC is equivalent to the random priority rule (or a variant). However, as we show in Section 3.1, our result cannot be deduced from any of these equivalences.

\(^5\)Another common name for this rule is the “random serial dictatorship”.

This paper is organized as follows. In Section 2, we define the model. We formally define the crawler. Also, we define a dual rule by visiting agents from right to left as opposed to from left to right, and state our first equivalence result; the crawler and this dual rule are the same. In Section 3, we define the crawler from random endowments, and state our second equivalence result: the crawler from random endowments is the same as the random priority rule, and hence the same as the core from random endowments. Proofs are collected in the appendix.

2 Model

There is a set $N = \{1, 2, \ldots, n\}$ of agents and a set $O$ of objects ($|O| = n$). Each agent is endowed with one object in $O$, no two agents being endowed with the same object. We denote by $\omega$ the endowment profile, i.e., $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$. Each agent $i \in N$ has a strict preference relation $P_i$ over $O$. Let $P$ be the set of all preferences. We write $R_i$ to denote the “at least as desirable” relation associated with $P_i$. That is, for each pair $o, o' \in O$, $o R_i o'$ if only if either $o P_i o'$ or $o = o'$. We represent $P_i$ by an ordered list of the objects, such as $P_i : o, \tilde{o}, o', \ldots.$

Let $P^N$ be the set of preference profiles for $N$. Our generic notation for a preference profile is $P = (P_i)_{i \in N}$.

A problem is defined by a preference profile and an endowment profile. An allocation is a list $x = (x_1, x_2, \ldots, x_n)$ such that for each $i \in N$, $x_i \in O$, and for each pair $i, j \in N$ such that $i \neq j$, $x_i \neq x_j$. Let $\mathcal{X}$ be the set of allocations. A rule is a single-valued mapping $\varphi: P^N \times \mathcal{X} \rightarrow \mathcal{X}$ that associates with each $(P, \omega) \in P^N \times \mathcal{X}$ an allocation $x \in \mathcal{X}$.

Let $\mathcal{L}$ be the set of strict orders on $O$. We consider the following restriction on preference profiles. There is an order $\prec \in \mathcal{L}$ on $O$ such that each agent has a unique most preferred object; the further with respect to $\prec$ an object is from his most preferred object, in either direction, the

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6Given $N' \subset N$, we denote $-N'$ the subset $N \setminus N'$ of agents.

7The $i^{th}$ coordinate of $\omega$ is the object endowed by agent $i$.

8Given $N' \subset N$, we denote by $\omega_{N'}$ the endowment profile of $N'$.

9Given $N' \subset N$, we denote by $P_{N'}$ the preference profile of $N'$.

10Given $N' \subset N$, we denote by $X_{N'}$ the set of assignments for $N'$. 
worse off he is. Formally, a preference profile $P$ is **single-peaked** if there is an order $\prec \in \mathcal{L}$ over the object set

$$o_1 \prec o_2 \prec \cdots \prec o_n$$

such that for each $i \in N$, one of the following three conditions holds:

- there is $t \in \{2, 3, \ldots, n-1\}$ such that

  $$P_i : o_t, o_{t-1}, o_{t-2}, \ldots, o_1 \text{ and } P_i : o_{t+1}, o_{t+2}, \ldots, o_n$$

- $$P_i : o_n, o_{n-1}, o_{n-2}, \ldots, o_1,$$

- $$P_i : o_1, o_2, o_3, \ldots, o_n.$$

We denote by $p(P_i)$ agent $i$’s most preferred object at $P_i$.

Throughout, we consider preference profiles that are **single-peaked** with respect to some reference order on the object set.

Given $\prec \in \mathcal{L}$, for each $o \in O$, we denote by $o - 1$ and $o + 1$ the objects that are adjacent to object $o$. (Obviously, the leftmost object has no object to its left, and the rightmost object has no object to its right.) Likewise, given $\prec \in \mathcal{L}$, for each $i \in N$, we denote by $i - 1$ and $i + 1$ the agents whose endowments are adjacent to that of agent $i$. (The agent whose endowment is leftmost has no one to his left, and the agent whose endowment is rightmost has no one to his right.) Moreover, for each pair $i, j \in N$, if agent $j$ is to the right of agent $i$, we write that $i \prec j$.

Let $f : \{1, 2, \cdots, n\} \to N$ be a bijection and let $f = (f(1), f(2), \cdots, f(n))$ be the resulting strict order on $N$. Let $\mathcal{F}$ be the set of all orders on $N$. For each $P_i \in P$ and each $O' \subseteq O$, let $X_i(O')$ be the most preferred object of agent $i \in N$ in $O'$ at $P_i$, i.e.,

$$X_i(O') = o \iff o \in O' \text{ and for each } o' \in O' \setminus \{o\}, o P_i o'.$$

We conclude this section by defining three basic properties. Recall that $\varphi$ is our generic notation for a rule. First, we require that for each problem, the chosen allocation be such that there is no other allocation that all agents find at least as desirable and at least one agent prefers:
Efficiency: For each \((P, \omega) \in \mathcal{P}^N \times \mathcal{X}\), there is no \(x \in \mathcal{X}\) such that for each \(i \in N\), \(x_i \ R_i \varphi_i(P, \omega)\), and there is \(j \in N\) such that \(x_j \ P_j \varphi_j(P, \omega)\).

Second, we require that for each problem, each agent find his assignment at least as desirable as his endowment:

Endowment lower bound: For each \((P, \omega) \in \mathcal{P}^N \times \mathcal{X}\) and each \(i \in N\),
\[
\varphi_i(P, \omega) \ R_i \omega_i.
\]

Third, we require that no agent ever benefit by misrepresenting his preferences:

Strategy-proofness: For each \((P, \omega) \in \mathcal{P}^N \times \mathcal{X}\), each \(i \in N\), and each \(P'_i \in \mathcal{P}\),
\[
\varphi_i((P_i, P_{-i}), \omega) \ R_i \varphi_i((P'_i, P_{-i}), \omega).
\]

2.1 The crawler

In the context of object reallocation problems with strict preferences, TTC has been the most central rule in the literature: It is well known that TTC is the only rule satisfying efficiency, the endowment lower bound, and strategy-proofness (Ma, 1994). However, on the single-peaked domain, TTC is not the only rule satisfying these properties. Bade (2019a) defines a new rule for this domain, and shows that this rule also satisfies these three properties.

The idea underlying the rule is as follows. Objects are labeled in such a way that preferences are single-peaked with respect to the order. Agents are visited according to the way the objects they own are ordered, from left to right, we say “in ascending order”. Each agent in turn is asked if he most prefers his endowment or an object to the left of his endowment. If yes, he is further asked which specific object it is. Depending on his answer, we take one of the following actions:

1. If an agent’s most preferred object is to the right of his endowment, we move to the next agent.
2. If an agent’s most preferred object is his endowment, he is assigned his endowment. He is removed with his assignment.
If an agent’s most preferred object is to the left of his endowment, he is assigned his most preferred object. He is removed with his assignment.

When an agent is removed, the problem is updated as follows. Consider all of the agents whose endowments are between the object assigned to the agent who left and the object that agent owned. Each of their ownership is shifted by one spot to the right. Hence, each of these agents owns a new object. This sweeping procedure is repeated with the updated problem.

In anticipation of a forthcoming definition, we refer to the rule just defined as the ascending crawler. We denote it by $ACR$.

The formal description is as follows.

**Ascending crawler:** Let $\prec \in L$, $P \in P^N$, where $P$ is single-peaked with respect to $\prec$, and $\omega \in X$.

Label the objects in $O$ in such a way that for each $t \in \{1, \ldots, n-1\}$, $o_t \prec o_{t+1}$. Label the agents in $N$ in such a way that for each $t \in \{1, \ldots, n-1\}$, we have $\omega_i \prec \omega_{i+1}$. Let $\hat{O}^0 = \{o_1, \ldots, o_n\}$ and $\hat{N}^0 = \{i_1, \ldots, i_n\}$.

At each step $k \geq 1$, let

$$k^* \equiv \min_{\{1, \ldots, n-k\}} \{t : o_t, o_{t+1} \in \hat{O}^{k-1}\}.$$

Let $ACR_{i_k^*}(P, \omega) = X_{i_k^*}(O^{k-1})$. Let $O^k \equiv O^{k-1} \setminus \{X_{i_k^*}(O^{k-1})\}$ and $N^k \equiv N^{k-1} \setminus \{i_k^*\}$.

Label the objects in $O^k$ in such a way that for each $t \in \{1, \ldots, n-k-1\}$, $o_t \prec o_{t+1}$. Label the agents in $N^k$ in such a way that for each $t \in \{1, \ldots, n-k-1\}$, at step $k-1$, we have $i_t \prec i_{t+1}$. Let $\hat{O}^k = \{o_1, \ldots, o_{n-k}\}$ and $\hat{N}^k = \{i_1, \ldots, i_{n-k}\}$.

**Example 1.** [Illustrating the ascending crawler.] Let $N = \{1, 2, 3, 4\}$. Let $\omega_1 \prec \omega_2 \prec \omega_3 \prec \omega_4$. Let $P \in P^N$ be defined by

$$P_1 : \omega_4, \omega_3, \omega_2, \omega_1$$
$$P_2 : \omega_2, \omega_1, \omega_3, \omega_4$$
$$P_3 : \omega_1, \omega_2, \omega_3, \omega_4$$
$$P_4 : \omega_2, \omega_1, \omega_3, \omega_4.$$

At each step, we ask the following question to the agent we visit:
Among the available objects, do you most prefer your endowment or an object to the left of your endowment? If the answer is yes, which object do you most prefer?  

**Step 1:** Agent 1 is queried first. Because his answer is no, agent 2 is queried next. Agent 2 answers yes, and he most prefers his endowment. Hence, he is assigned his endowment. He leaves with his assignment.

**Step 2:** Agent 1 is queried first. Because his answer is no, agent 3 is queried next. Agent 3 answers yes, and he most prefers $\omega_1$. Hence, he is assigned that object and leaves with his assignment. The endowment of agent 1 is shifted by one spot to the right.

**Step 3:** We repeat the sequence of queries. Agent 4 is the first agent who answers yes, and he most prefers object $\omega_3$. Hence, he is assigned that object and leaves with his assignment. The endowment of agent 1 is shifted by one spot to the right.

**Step 4:** Agent 1 is the only agent who has not been assigned an object yet and object $\omega_4$ is the only available object. Hence, agent 1 is assigned object $\omega_4$ and leaves with his assignment.

Because no agent remains, the algorithm terminates, yielding

$$ACR(P, \omega) = (\omega_4, \omega_2, \omega_1, \omega_3).$$

Figure 1 illustrates the process. At each step, the agent who is assigned an object at that step is circled. At each step, agents who have already been assigned objects are shown in boxes.

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11For the agent whose endowment is leftmost, we ask ‘among the available objects, do you most prefer your endowment?’
2.2 The first equivalence result: The equivalence between the ascending crawler and the descending crawler

Of course, as pointed out by Bade (2019a), one can define a “dual” rule by visiting agents from right to left, let us say “in descending order”. Let us call this dual rule the descending crawler. We denote it by DCR. We omit the formal definition and only show how to apply to our Example 1.

Example 1 (Continued). [Illustrating the descending crawler.] At each step, we ask the following question to the agent we visit;

‘Among the available objects, do you most prefer your endowment or an object to the right of your endowment? If the answer is yes, which object do you most prefer?’\footnote{For the agent whose endowment is rightmost, we ask ‘among the available objects, do you most prefer your endowment?’}

Step 1: Agent 4 is queried first. Because his answer is no, agent 3 is queried next. Because his answer is no, agent 2 is queried next. Agent 2 answers yes, and he most
prefers his endowment. Hence, he is assigned that object and leaves with his assignment.

**Step 2:** Agent 4 is queried first. Because his answer is no, agent 3 is queried next. Because his answer is no, agent 1 is queried next. Agent 1 answers yes, and he most prefers object $\omega_4$. Hence, he is assigned that object and leaves with his assignment. The endowments of agents 3 and 4 are shifted by one spot to the left.

**Step 3:** We repeat the sequence of queries. Agent 3 is the first agent who answers yes, and he most prefers object $\omega_1$. Hence, he is assigned that object and leaves with his assignment.

**Step 4:** Agent 4 is the only agent who has not been assigned an object yet and object $\omega_3$ is the only available object. Hence, agent 4 is assigned object $\omega_3$ and leaves with his assignment.

Because no agent remains, the algorithm terminates, yielding

$$DCR(P, \omega) = (\omega_4, \omega_2, \omega_1, \omega_3).$$
In the example, the assignments obtained by applying the two rules are the same. In general, one could expect the two rules to select different allocations. However, it turns out that they are always the same:

**Theorem 1.** The ascending crawler is the same rule as the descending crawler. That is, \( ACR = DCR \).

Because the ascending crawler visits agents from left to right, one may be concerned that this specific order confers a particular advantage to some agents based on the location of their endowments relative to the location of other agents’ endowments. For example, because agents whose endowments are on the left side of the order are visited first, one may suspect that these agents enjoy some advantage at the expense of the agents whose endowments are on the right side of the order. However, Theorem 1 implies that the order in which agents are visited gives no benefit to any particular agent.

Hereafter, for simplicity, we refer to these rules as the crawler, using the notation \( CR \). Without loss of generality, for each endowment profile, when we operate the crawler, we apply the ascending procedure.

3 **The second equivalence result: The equivalence between the crawler from random endowments and the random priority rule**

One of the most important solution concepts in economics is the “core”. An allocation is blocked by a coalition if the members of the coalition can exchange their endowments among themselves in such a way that each of them finds his assignment at least as desirable as his component of the allocation, and at least one agent in the coalition prefers his assignment to his component of the allocation. An allocation is in the core if there is no such coalition.\(^\text{13}\) For each preference profile, including each profile of single-peaked preferences, the core is a singleton preference profile, including each profile of single-peaked preferences, the core is a singleton

\(^\text{13}\) Formally, allocation \( x \in \mathcal{X} \) is blocked by coalition \( N' \subseteq N \) at \((P, \omega) \in \mathcal{P}^N \times \mathcal{X}\) if there is \((y_i)_{i \in N'} \in \mathcal{X}_{N'}\) such that \( \bigcup_{N'} y_i = \bigcup_{N'} \omega_i \), for each \( i \in N' \), \( y_i R_i x_i \), and there is \( j \in N' \) such that \( y_j P_j x_j \). An allocation is in the core of \((P, \omega)\) if there is no such a coalition.
and the allocation selected by TTC coincides with the allocation in the core (Shapley and Scarf, 1974; Roth and Postlewaite, 1977).

Instead of each agent being endowed with one object, agents may collectively own a set of objects and a rule then has to assign each agent one object. We refer to this type of problems as “object allocation problems”. Although the core concept is defined for reallocation problems, it can help provide solutions to object allocation problems. To explain how, we first need to define the concept of a “sequential priority rule”: (formal definition is given below) To each order on the agent set is associated such a rule. The agent who is first is assigned his most preferred object, the agent who is second is assigned his most preferred object among the remaining ones, and so on. None of these rules treats agents fairly. However, their unfairness can be mitigated by considering lotteries. Formally, a lottery is a probability distribution over allocations, \( p = (p_1, \ldots, p_n) \), such that for each \( k \), \( p_k \geq 0 \), and \( \sum_k p_k = 1 \). We denote the degenerate lottery that assigns probability 1 to allocation \( x \) by \( p^x \). Let \( \Delta(X) \) be the set of all lotteries. A probabilistic rule is a single-valued mapping which associates a lottery with each preference profile. A probabilistic version of the sequential priority rules is obtained by choosing an order on the agent set at random according to a uniform distribution and applying the induced sequential priority rule. This rule is called the “random priority rule” (Abdulkadiroglu and Sönmez, 1998).

The random priority rule is a leading rule in the context of probabilistic assignments. Interestingly, the probability distribution over the allocations derived by applying the random priority rule is the same as that derived by applying the core from random endowments: For each problem, choose an endowment profile at random according to a uniform distribution, and select the core of the induced object reallocation problem (Knuth, 1996; Abdulkadiroglu and Sönmez, 1998).

Motivated by this equivalence, we study the relationship between the random priority rule and the crawler from random endowments: For each problem, choose an endowment profile at random according to a uniform distribution, and apply the crawler to the induced object reallocation problem.

First, let us note that if there are at least three agents, the crawler and TTC disagree (Bade, 2019a). Hence, the crawler does not always select an allocation in the core. Also, the procedures underlying the crawler and the sequential priority rules differ. Thus, one should be
doubtful that there is any relationship between the random priority rule and the crawler from random endowments. However, as we show, the probability distributions over allocations obtained by applying these two rules are the same.

Let us now formally define the rules under discussion. Here, a problem is simply defined as a pair of a preference profile and an object set. However, TTC and the crawler select an allocation on the basis of both a preference profile and an endowment profile, so that the family of rules that we are defining are parametrized by an endowment profile. Reflecting this parametrization, we denote each of these rules by $TTC^\omega$ and $CR^\omega$, respectively.

Given $f \in \mathcal{F}$, the **sequential priority rule induced by $f$, $SP^f : \mathcal{P}^N \rightarrow \mathcal{X}$**, is defined by setting, for each $P \in \mathcal{P}^N$,

$$SP^f_{f(1)}(P) = X_{f(1)}(O),$$

$$SP^f_{f(2)}(P) = X_{f(2)}\left(O \setminus \{SP^f_{f(1)}(P)\}\right),$$

$$\vdots$$

$$SP^f_{f(i)}(P) = X_{f(i)}\left(O \setminus \bigcup_{j=1}^{i-1}\{SP^f_{f(j)}(P)\}\right),$$

$$\vdots$$

$$SP^f_{f(n)}(P) = X_{f(n)}\left(O \setminus \bigcup_{j=1}^{n-1}\{SP^f_{f(j)}(P)\}\right).$$

The **random priority rule, $RP : \mathcal{P}^N \rightarrow \Delta(\mathcal{X})$**, is defined by setting, for each $P \in \mathcal{P}^N$,

$$RP(P) = \sum_{f \in \mathcal{F}} \frac{1}{n!}p^{SP^f(P)}.$$

The **core from random endowments $RTTC : \mathcal{P}^N \rightarrow \Delta(\mathcal{X})$**, is defined by setting, for each $P \in \mathcal{P}^N$,

$$RTTC(P) = \sum_{\omega \in \mathcal{X}} \frac{1}{n!}p^{TTC^\omega(P)}.$$

**Theorem** (Knuth, 1996; Abdulkadiroglu and Sönmez, 1998). The core from random endowments is the same probabilistic rule as the random priority rule. That is, $RTTC = RP$.

The **crawler from random endowments $RCR : \mathcal{P}^N \rightarrow \Delta(\mathcal{X})$**, is defined by setting, for each $P \in \mathcal{P}^N$,

$$RCR(P) = \sum_{\omega \in \mathcal{X}} \frac{1}{n!}p^{CR^\omega(P)}.$$
Here is our second theorem.

**Theorem 2.** The crawler from random endowments is the same probabilistic rule as the random priority rule. That is, $RCR = RP$.

The proof involves constructing a mapping $g: X \rightarrow F$, and showing that (i) the crawler induced by each endowment profile and the sequential priority rule induced by the order on the agent set given by the mapping select the same allocation, and (ii) the mapping is one-to-one and onto. Knuth (1996), Abdulkadiroglu and Sönmez (1998), and we all identify a mapping that associates an order over the agent set with each feasible allocation, and show that this mapping is one-to-one and onto. However, how we construct our mapping differs from theirs. Their mapping is based on the cycles that come out of the application of TTC. By contrast, ours is defined by recursively finding pairs of agents one of them “envies” the other, and giving higher priority to the second agent than to the first agent. Combining Theorem 2 and the equivalence result in Knuth (1996) and Abdulkadiroglu and Sönmez (1998), we deduce another equivalence result:

**Corollary 1.** The crawler from random endowments is the same probabilistic rule as the core from random endowments. That is, $RCR = RTTC$.

Theorem 2 and Corollary 1 contribute to our understanding of how the crawler relates to the core. Although the crawler does not always select an allocation in the core, the allocations selected by the crawler and the allocations selected by the core are the same in a probabilistic sense.

### 3.1 Discussion

Following the equivalence result between the core from random endowments and the random priority rule (Knuth, 1996; Abdulkadiroglu and Sönmez, 1998), several other equivalence results have been proved (Pathak and Sethuraman, 2011; Lee and Sethuraman, 2011; Sönmez and Ünver, 2005; Ekici, 2017; Carroll, 2014; Bade, 2019b). Each of these papers generalizes TTC, and shows that the probabilistic rule associated with the generalized TTC is the same as the random priority rule (or a variant of it). None of these rules are equivalent to the crawler. Hence, our Theorem 2 cannot be deduced from any of these results.
In particular, a large family of rules, which are efficient, strategy-proof, and non-bossy, is defined by Pycia and Ünver (2017). They show that on the domain of strict preferences, any rule satisfying the above properties is a member of the family. Rules in their family, so-called the “trading cycles” rules, are parametrized by a “control-rights structure” (formal definition is given in the appendix). The procedure underlying their definition is similar to TTC, but control-rights structures add flexibility to TTC. Now, given a control-rights structure, the associated probabilistic rule is obtained by permuting the agent set at random according to a uniform distribution, and applying the induced rule.

For each rule in the trading cycles family, the probabilistic rule associated with it is the same rule as the random priority rule (Bade, 2019b). This result holds for each preference profile, and in particular of course, it holds on the subdomain of problems with profiles of single-peaked preferences. Thus, on this domain her equivalence result remains true; yet, on this subdomain, there may be rules that are not the trading cycles rules, but are still efficient, strategy-proof, and non-bossy. In fact, the crawler is an example (Appendix C). Therefore, our Theorem 2 cannot be deduced from Bade (2019b)’s previous result.

4 Conclusion

We have shown two equivalence results pertaining a new rule defined by Bade (2019a) for object reallocation problems when preferences are single-peaked. The first equivalence result states that this rule, which she calls the crawler, and which we call the ascending crawler, is the same as the dual rule that she proposes, and which we call the descending crawler. This result shows that the order in which agents are visited does not confer a particular advantage to agents based on the location of their endowments relative to the location of other agents’ endowments.

The second equivalence result states that for object allocation problems, the probability distribution over allocations selected by the crawler from random endowments is the same as the probability distribution selected by the random priority rule. Hence, it is the same as the probability distribution over allocations derived from the core from random endowments.

\[ \varphi \] is non-bossy if for each \((P, \omega) \in \mathcal{P}^N \times X\), each \(i \in N\), and each \(P'_i \in \mathcal{P}\), if \(\varphi_i ((P'_i, P_{-i}), \omega) = \varphi_i ((P_i, P_{-i}), \omega)\), then \(\varphi ((P'_i, P_{-i}), \omega) = \varphi ((P_i, P_{-i}), \omega)\).
Although the crawler does not always select an allocation from the core, our second result mitigates this fact, and should strengthen the appeal of this rule.

Appendices

Given $\prec \in \mathcal{L}$, let $|o - o'|$ be the number of objects between objects $o$ and $o'$, including these objects. So for example, the distance between objects $o$ and $o + 1$ is 2.

A Proof of Theorem 1

Let $(P, \omega) \in \mathcal{P}^N \times X$. Because both the ascending and the descending crawlers meet the endowment lower bound, for each $i \in N$ such that $\omega_i \prec ACR_i(P, \omega)$, we have $\omega_i \prec p(P_i)$. This implies that $\omega_i \preceq DCR_i(P, \omega)$.

Claim 1. For each $i \in N$ such that $\omega_i \prec ACR_i(P, \omega)$, we have $ACR_i(P, \omega) = DCR_i(P, \omega)$.

Proof of Claim 1. Let $i_1 \in N$ be such that $\omega_{i_1} \prec ACR_{i_1}(P, \omega)$. Suppose, by way of contradiction, that $DCR_{i_1}(P, \omega) \neq ACR_{i_1}(P, \omega)$. This implies that there is $N' = \{i_1, i_2, \ldots, i_{k-1}, i_k\} \subseteq N$, where $i_{k-1} \equiv i$ and $i_k \equiv j$, such that

- for each $l \in \{2, \ldots, k\}$, $DCR_{i_l}(P, \omega) = ACR_{i_{l-1}}(P, \omega)$; and
- either
  
  (1) for each $l \in \{1, \ldots, k-1\}$, $ACR_{i_l}(P, \omega) \not< DCR_{i_l}(P, \omega)$; or
  
  (2) for each $l \in \{1, \ldots, k-1\}$, $DCR_{i_l}(P, \omega) \not< ACR_{i_l}(P, \omega)$.

Suppose that Case (1) holds. Because both crawlers are efficient, $DCR_{i_j}(P, \omega) \not< ACR_{i_j}(P, \omega)$. By the definition of the crawlers, either (1-i) $\omega_i \not< ACR_i(P, \omega)$ or (1-ii) $\omega_j \not< ACR_i(P, \omega)$.

Suppose that Case (2) holds. Because both crawlers are efficient, $ACR_{i_j}(P, \omega) \not< DCR_{i_j}(P, \omega)$. By the definition of the crawlers, either (2-i) $\omega_i \not< ACR_i(P, \omega)$ or (2-ii) $\omega_j \not< ACR_i(P, \omega)$.

Arguments for Cases (1-i), (1-ii), (2-i), and (2-ii) are analogous. So without loss of generality,
we only handle Case (1-i). Let
\[ J \equiv \{ j \in N : \omega_i < \omega_j \text{ and } \omega_j < ACR_j(P, \omega) \}. \]
The proof is by contradiction. We show that \( DCR_i(P, \omega) = ACR_i(P, \omega) \) by applying induction on \(|J|\). First, let
\[ N' \equiv \{ k \in N : \omega_i < \omega_k < \omega_j \}, \]
\[ S \equiv \{ k \in N : \omega_i < \omega_k < \omega_j \text{ and } ACR_k(P, \omega) \nleq \omega_i \}, \]
\[ T \equiv \{ k \in N : \omega_i < \omega_k < \omega_j, \ ACR_k(P, \omega) \nleq \omega_k, \text{ and } \omega_i < ACR_k(P, \omega) < ACR_i(P, \omega) \}, \]
\[ S' \equiv \{ k \in N : \omega_i < \omega_k < \omega_j \text{ and } \omega_j \nleq DCR_k(P, \omega) \}, \] and
\[ T' \equiv \{ k \in N : \omega_i < \omega_k < \omega_j, \omega_k \nleq DCR_k(P, \omega), \text{ and } ACR_i(P, \omega) < DCR_k(P, \omega) < \omega_j \}. \]
Note that for each pair \( X, Y \in \{ S, T, S', T' \} \), \( X \cap Y = \emptyset \) and \( S \cup T \cup S' \cup T' \subseteq N' \).

**Base Step:** Suppose that \(|J| = 0\). The fact that agent \( j \) is assigned object \( ACR_i(P, \omega) \) by the descending crawler implies that \( S' \neq \emptyset \). However, because \(|J| = 0\), for each \( k \in N' \), we have \( ACR_k(P, \omega) \nleq \omega_k \). This implies that \( p(P_k) \nleq \omega_k \). The fact that \( S' \neq \emptyset \) implies that there is \( j' \in N \) such that \( p(P_j') \nleq \omega_j < DCR_j(P, \omega) \). This means that the descending crawler does not meet the endowment lower bound, a contradiction. Therefore, \( ACR_i(P, \omega) = DCR_i(P, \omega) \).

**Induction hypothesis:** Let \( k \in \{ 1, \ldots, n-1 \} \). For each \(|J| \leq k-1\), we have \( ACR_i(P, \omega) = DCR_i(P, \omega) \).

**Induction step:** Let \(|J| = k\). By the induction hypothesis, for each \( j' \in N' \) such that \( \omega_j < ACR_k(P, \omega) \), we have \( ACR_j(P, \omega) = DCR_j(P, \omega) \). Hence, \( ACR_i(P, \omega) \nleq \omega_j \).

The fact that agent \( i \) is assigned object \( ACR_i(P, \omega) \) by the ascending crawler implies that
\[ S \neq \emptyset \text{ and } |S \cup T| \geq |ACR_i(P, \omega) - \omega_i| - 1. \]
On the other hand, the fact that agent \( j \) is assigned object \( ACR_i(P, \omega) \) by the descending crawler implies that
\[ S' \neq \emptyset \text{ and } |\omega_j - ACR_i(P, \omega)| - 1 \leq |S' \cup T'|. \]
However, \(^{15}\)
\[ |ACR_i(P, \omega) - \omega_i| + |\omega_j - ACR_i(P, \omega)| - 3 = |N'|. \]

\(^{15}|ACR_i(P, \omega) - \omega_i| + |\omega_j - ACR_i(P, \omega)|\) includes objects \( \omega_i \) and \( \omega_j \), and counts object \( ACR_i(P, \omega) \) twice.
This implies that

\[ |N'| < |ACR_i(P, \omega) - \omega_i| + |\omega_j - ACR_i(P, \omega)| - 2 \leq |S \cup T \cup S' \cup T'|. \]

This contradicts the fact that \( S \cup T \cup S' \cup T' \subseteq N' \). Therefore, \( ACR_i(P, \omega) = DCR_i(P, \omega) \).

Consequently, \( DCR_i(P, \omega) = ACR_i(P, \omega) \).

\[ \square \]

Claim 2. For each \( i \in N \) such that \( DCR_i(P, \omega) < \omega_i \), we have \( DCR_i(P, \omega) = ACR_i(P, \omega) \).

The proof of Claim 2 is analogous to that of Claim 1. Hence, we omit it.

Claim 3. For each \( i \in N \),

\[ ACR_i(P, \omega) = \omega_i \iff DCR_i(P, \omega) = \omega_i. \]

Proof of Claim 3. Suppose, by way of contradiction that, there is \( i \in N \) such that

\[ ACR_i(P, \omega) = \omega_i \neq DCR_i(P, \omega). \]

Suppose that \( \omega_i < DCR_i(P, \omega) \). Because the descending crawler meets the endowment lower bound, the fact that \( \omega_i < DCR_i(P, \omega) \) implies that \( \omega_i < p(P_i) \). Let \( i^* \in N \) be such that \( p(P_i) < \omega_i, \omega_i, P_i \omega_i \), and for each \( j \in N \) such that \( \omega_i < \omega_j \), we have \( \omega_j P_i \omega_j \). Let

\[ J \equiv \{ o \in O : \omega_i < o \not\prec \omega_i \}. \]

Let \( k^* \in N \) be such that \( \omega_i < \omega_{k^*}, \omega_i < ACR_{k^*}(P, \omega) \not\prec \omega_{k^*}, \) and there is no \( k \in N \) such that \( \omega_{k^*} < \omega_k \) and \( \omega_i < ACR_k(P, \omega) \not\prec \omega_{k^*} \). The fact that \( ACR_i(P, \omega) = \omega_i \) implies that each \( o \in J \) is assigned to some agent whose endowment is to the right of \( \omega_i \) before agent \( i \) is assigned object \( \omega_i \). Moreover, for each \( j \in N \) such that \( \omega_i < \omega_j \not\prec \omega_{k^*} \), we have \( \omega_{i+1} P_j \omega_i \). This implies that when applying the descending crawler, each \( o \in J \) is assigned to some agent before agent \( i \) is visited. This implies that \( DCR_i(P, \omega) \not\in J \). However, \( DCR_i(P, \omega) \neq \omega_i \). This means that \( \omega_i P_i DCR_i(P, \omega) \). Hence, the descending crawler does not meet the endowment lower bound, a contradiction. Therefore, \( DCR_i(P, \omega) = \omega_i \). On the other hand, \( DCR_i(P, \omega) < \omega_i \) contradicts Claim 2.

Now suppose that there is \( i \in N \) such that

\[ ACR_i(P, \omega) \neq \omega_i = DCR_i(P, \omega). \]

The argument for this case is analogous to above. Hence, we omit the proof. \[ \square \]
Proof of Theorem 1. By Claim 3, for each \( i \in N \),
\[
ACR_i(P, \omega) = \omega_i \iff DCR_i(P, \omega) = \omega_i.
\]
This implies that for each \( i \in N \),
\[
ACR_i(P, \omega) \neq \omega_i \iff DCR_i(P, \omega) \neq \omega_i.
\]
Because both crawlers meets the endowment lower bound, for each \( i \in N \),
\[
\omega_i \prec ACR_i(P, \omega) \iff \omega_i \prec DCR_i(P, \omega)
\]
while for each \( i \in N \),
\[
ACR_i(P, \omega) \prec \omega_i \iff DCR_i(P, \omega) \prec \omega_i.
\]
Claim 1 shows that in the former case, \( ACR_i(P, \omega) = DCR_i(P, \omega) \). On the other hand, Claim 2 shows that in the later case, \( ACR_i(P, \omega) = DCR_i(P, \omega) \). Overall, \( ACR(P, \omega) = DCR(P, \omega) \).

\[\square\]

B Proof of Theorem 2

Let \( P \in \mathcal{P}^N \). The following lemma shows that for each allocation selected by the sequential priority rule induced by a given order on the agent set, there is an endowment profile for which the crawler selects the same allocation at the endowment profile. Conversely, for each allocation selected by the crawler at a given endowment profile, there is an order on the agent set such that the sequential priority rule induced by the order selects the same allocation.

Lemma 1.
\[
\{ x \in \mathcal{X} : \text{there is } f \in \mathcal{F} \text{ such that } SP^f(P) = x \} = \{ x \in \mathcal{X} : \text{there is } \omega \in \mathcal{X} \text{ such that } CR^\omega(P) = x \}.
\]

Given a preference profile, the set of allocations selected by TTC at the various endowment profiles is the same as the set of allocations selected by the sequential priority rule induced by the various orders on the agent set (Abdulkadiroglu and Sönmez, 1998). We show that the set of allocations selected by the crawler at the various endowment profiles is the same as the set of allocations selected by TTC at the various endowment profiles. Combining these results, we derive Lemma 1.
Lemma (Abdulkadiroglu and Sönmez, 1998).

\[ \{ x \in X : \text{there is } f \in \mathcal{F} \text{ such that } SP^f(P) = x \} = \{ x \in X : \text{there is } \omega \in \mathcal{X} \text{ such that } TTC^\omega(P) = x \}. \]

(1)

Proof of Lemma 1. We show that for each \( \omega \in \mathcal{X} \), there is \( \omega' \in \mathcal{X} \) such that

\[ TTC^\omega(P) = CR^{\omega'}(P). \]

This implies that

\[ \{ x \in X : \text{there is } \omega \in \mathcal{X} \text{ such that } TTC^\omega(P) = x \} \subseteq \{ x \in X : \text{there is } \omega \in \mathcal{X} \text{ such that } CR^\omega(P) = x \}. \]

The proof for the other direction is analogous. Hence, we only show one direction.

Let \( \omega \in \mathcal{X} \). Because the crawler meets the endowment lower bound, for each \( i \in N \),

\[ CR_i^{TTC^\omega}(P) R_i TTC_i^\omega(P). \]

Because TTC is efficient, there is no \( i \in N \) such that

\[ CR_i^{TTC^\omega}(P) P_i TTC_i^\omega(P). \]

This implies that

\[ CR^{TTC^\omega}(P) = TTC^\omega(P). \]

Together with (1), we have

\[ \{ x \in X : \text{there is } f \in \mathcal{F} \text{ such that } SP^f(P) = x \} = \{ x \in X : \text{there is } \omega \in \mathcal{X} \text{ such that } CR^\omega(P) = x \}. \]

We now construct a mapping from \( \mathcal{X} \) into \( \mathcal{F} \) and show that (i) the crawler induced by an endowment profile and the sequential priority rule induced by the order on the agent set given by the mapping select the same allocation, and (ii) the mapping is one-to-one and onto.

Let \( \omega \in \mathcal{X} \). Given \( g \in \mathcal{F} \), we denote by \( g_i \) the rank of agent \( i \). The order \( g(\omega) \in \mathcal{F} \) is obtained by the recursive procedure. In the procedure, at each round, we identify two types of chains formed by agents, an “envy chain” and a “generalized chain”. An envy chain is derived
by connecting “envy” relations between agents. On the other hand, a generalized chain is
derived by combining envy chains using the order over the agent set that is determined at the
previous round.

Let \( g^0(\omega) \in \mathcal{F} \) be defined by setting, for each pair \( i, j \in N \),
\( g^0_i(\omega) < g^0_j(\omega) \) if and only if
\( \omega_i < \omega_j \). For each \( i \in N \), let \( O^0_i = O \) and \( H^0_i = \emptyset \).

At each round \( r \geq 1 \),

**Step 1:** For each \( i \in N \) such that \( O_{i-1}^r \neq \emptyset \), identify \( j \in N \) such that \( CR_i^r(\omega_i) = X_i(O_{i-1}^r) \).
If \( j \neq i \), we say “agent \( i \) envies agent \( j \)”, which we write as \( i \rightarrow j \). Also, let \( H_i^r = \{X_i(O_{i-1}^r)\} \).
If \( j = i \), let \( H_i^r = O_i^{r-1} \). If there is \( N' = \{i_1, i_2, \ldots, i_k\} \) such that for each \( k' \in \{1, \ldots, k - 1\} \)
agent \( i_{k'} \) envies agent \( i_{k'=1} \), we concatenate these relations, and refer to this resulting chain as
an **envy chain formed at round** \( r \). If there are \( N', N'' \subseteq N \) such that the agents in \( N' \) form an
envy chain at round \( r \), the agents in \( N'' \) form an envy chain at round \( r' \leq r \), and \( N' \cap N'' \neq \emptyset \),
we connect these chains as follows.

**Case 1:** Suppose that \( \{i_1, \ldots, i_k, l\} \subseteq N \) form the following envy chain at round \( r \);
\[
\begin{align*}
i_1 &\rightarrow \ldots \rightarrow i_k \rightarrow l.
\end{align*}
\]
Similarly, suppose that \( \{j_1, \ldots, j_k', l\} \subseteq N \) form the following envy chain at round \( r' \leq r \);
\[
\begin{align*}
j_1 &\rightarrow \ldots \rightarrow j_k' \rightarrow l.
\end{align*}
\]
Then we connect these two chains
\[
\begin{align*}
i_1 &\rightarrow \ldots \rightarrow i_k \rightarrow l \rightarrow j_1 \rightarrow \ldots \rightarrow j_k'.
\end{align*}
\]

**Case 2:** Suppose that \( \{l, i_1, \ldots, i_k\} \subseteq N \) form the following envy chain at round round \( r \);
\[
\begin{align*}
l &\rightarrow i_1 \rightarrow \ldots \rightarrow i_k.
\end{align*}
\]
Similarly, suppose that \( \{l, j_1, \ldots, j_k'\} \subseteq N \) form the following envy chain at round \( r' \leq r \);
\[
\begin{align*}
l &\rightarrow j_1 \rightarrow \ldots \rightarrow j_k'.
\end{align*}
\]
Then we connect these two chains

\[ l \rightarrow i_1 \rightarrow \ldots \rightarrow i_k \]
\[ j_1 \rightarrow \ldots \rightarrow j_{k'} \]

Note that because the crawler is efficient, there is no cycle of envy. Let \( K \in \{1, \ldots, n\} \). Let \( C = \{C_1, \ldots, C_K\} \) be a partition of the agent set \( N \) according to the chains. That is, for each pair \( k, k' \in \{1, \ldots, K\} \) such that \( k \neq k' \), each \( i \in C_k \) is not in the same connected chain with any \( j \in C_{k'} \). Let \( C \) be the set of subsets of agents that are identified by the partition.

**Step 2:** Let \( C_k \in C \). Let \( C = N^1 \cup N^2 \cup \ldots \cup N^t \) be such that, for each \( q \in \{1, \ldots, t\} \), the agents in \( N^q \) form an envy chain. First,

\[ \bigcup_{i \in C_k} g^{r}_i(\omega) = \bigcup_{i \in C_k} g^{r-1}_i(\omega). \]

We derive a generalized chain by combining \( t \) connected envy chains. Let \( C_k^0 = \{N^1, \ldots, N^t\} \).

**Substep \( s \) (1 \( \leq \) s \( \leq \) S):** Let \( N', N'' \subset C_k^{s-1} \).

- For each pair \( i, j \in N' \) such that at substep \( s-1 \), \( i \rightarrow j \), set \( i \rightarrow \ldots \rightarrow j \).

**Case 1:** Suppose that \( \{i_1, \ldots, i_k, l\} \subset N' \) and \( \{j_1, \ldots, j_{k'}, l\} \subset N'' \) are such that

\[ i_1 \rightarrow \ldots \rightarrow i_k \rightarrow l \]
\[ j_1 \rightarrow \ldots \rightarrow j_{k'} \]

We combine the two chains in the following way.

- \( j_{k'} \rightarrow \ldots \rightarrow i_k \) if and only if \( g^{r-1}_{i_k}(\omega) < g^{r-1}_{j_{k'}}(\omega) \).
- Suppose that \( j_{k'} \rightarrow \ldots \rightarrow i_k \). For each \( q \in \{k', \ldots, 1\} \), identify \( p \in \{k-1, \ldots, 1\} \) such that for each \( \tilde{q} \in \{k', \ldots, q\} \), \( g^{r-1}_{j_{\tilde{q}}}(\omega) < g^{r-1}_{i_p}(\omega) \). If there is a unique such
value, denote it by $p^*$. If there are multiple such values, take the largest one and denote it by $p^*$. Set

$$i_{p^*} \rightarrow \ldots \rightarrow j_q.$$ 

If there is no such a value, set

$$j_1 \rightarrow \ldots \rightarrow j_q \rightarrow i_1.$$

- Suppose that $i_k \rightarrow \ldots \rightarrow j_{k'}$. For each $q \in \{k, \ldots, 1\}$, identify $p \in \{k' - 1, \ldots, 1\}$ such that for each $\tilde{q} \in \{k, \ldots, q\}$, $g_{\tilde{q}}^{-1}(\omega) < g_j^{-1}(\omega)$. If there is a unique such value, denote it by $p^*$. If there are multiple such values, take the largest one and denote it by $p^*$. Set

$$j_{p^*} \rightarrow \ldots \rightarrow i_q.$$

If there is no such a value, set

$$i_1 \rightarrow \ldots \rightarrow i_q \rightarrow j_1.$$

**Case 2:** Suppose that $\{l, i_1, \ldots, i_k\} \subseteq N'$ and $\{l, j_1, \ldots, j_{k'}\} \subseteq N''$ are such that

$$l \rightarrow i_1 \rightarrow \ldots \rightarrow i_k$$

$$\text{We derive combine the two chains in the following sway.}$$

- $j_1 \rightarrow \ldots \rightarrow i_1$ if and only if $g_{i_1}^{-1}(\omega) < g_{j_1}^{-1}(\omega)$.

- Suppose that $j_1 \rightarrow \ldots \rightarrow i_1$. For each $q \in \{2, \ldots, k'\}$, identify $p \in \{1, \ldots, k\}$ such that for each $\tilde{q} \in \{k', \ldots, q\}$, $g_{\tilde{q}}^{-1}(\omega) < g_{i_p}^{-1}(\omega)$. If there is a unique such value, denote it by $p^*$. If there are multiple such values, take the largest one and denote it by $p^*$. Set

$$i_{p^*} \rightarrow \ldots \rightarrow j_q.$$

If there is no such a value, set

$$j_1 \rightarrow \ldots \rightarrow j_q \rightarrow i_1.$$
Suppose that $i_1 \rightarrow \ldots \rightarrow j_1$. For each $q \in \{2, \ldots, k\}$, identify $p \in \{1, \ldots, k\}$ such that for each $\tilde{q} \in \{k, \ldots, q\}$, $g^{-1}_i(\omega) < g^{-1}_j(\omega)$. If there is a unique such value, denote it by $p^*$. If there are multiple such values, take the largest one and denote it by $p^*$. Set $j_{p^*} \rightarrow \ldots \rightarrow i_q$.

If there is no such a value, set $i_1 \rightarrow \ldots \rightarrow i_q \rightarrow j_1$.

- Let $C^s_k$ be the set of resulting chains formed at substep $s$.

- When there is only one chain in $C^S_k$ such that $i_1 \rightarrow i_2 \rightarrow \ldots \rightarrow i_l$, let $g^s_n(\omega) < g^s_n(\omega) < \ldots < g^s_n(\omega)$.

**Step 3:** For each $C_k \in C$ such that $C_k = \{i\}$, let $g^s_i(\omega) = g^{s-1}_i(\omega)$.

**Step 4:** For each $i \in N$, let $O^s_i = O^{s-1}_i \setminus H^s_i$.

The algorithm terminates when for each $i \in N$, we have $O^s_i = \emptyset$.

**Example 2.** [Illustrating how the procedure above selects the order over the agent set.] Let $N = \{1, 2, 3, 4, 5, 6, 7\}$. Let $\omega_1 \prec \omega_2 \prec \ldots \prec \omega_7$. Let $P \in \mathcal{P}^N$ be such that

$P_1 : \omega_7, \omega_6, \omega_5, \omega_4, \omega_3, \omega_2, \omega_1$

$P_2 : \omega_2, \ldots$

$P_3 : \omega_7, \omega_6, \omega_5, \omega_4, \omega_3, \omega_2, \omega_1$

$P_4 : \omega_3, \ldots$

$P_5 : \omega_7, \omega_6, \omega_5, \omega_4, \omega_3, \omega_2, \omega_1$

$P_6 : \omega_3, \ldots$

$P_7 : \omega_5, \ldots$

Then $CR^\omega(P) = (\omega_1, \omega_2, \omega_4, \omega_6, \omega_7, \omega_5, \omega_3)$. First, because $\omega_1 \prec \omega_2 \prec \ldots \prec \omega_7$, we have $g^0(\omega) = (1, 2, 3, 4, 5, 6, 7)$. For each $i \in N$, $O_i^0 = O$ and $H_i^0 = \emptyset$.

**Round 1:** Because for each $i \in \{1, 3\}$, $X_i(O_i^0) = \omega_7 = CR^\omega_7(P)$, we have $i \rightarrow 5$ and $H_i^1 = \{\omega_7\}$. Because, for each $j \in \{2, 4, 5, 6, 7\}$, $X_j(O_j^0) = CR^\omega_j(P)$, we have
Claim 4. \( CR^\omega(P) = SP^{g(\omega)}(P) \).

Proof of Claim 4. Suppose, by way of contradiction, that there is a sequence of agents \( \{i_1, \ldots, i_k\} \subseteq N \) such that for each \( k' \in \{1, \ldots, k-1\} \), \( SP^{g(\omega)}(P) = CR_{k',+1}^{\omega}(P) P_{i_{k'}} CR_{i_{k'}}^{\omega}(P) \). Because the crawler is efficient, \( SP^{g(\omega)}(P) \neq CR_{i_{k}}^{\omega}(P) P_{i_{k}} SP^{g(\omega)}(P) \). This implies that \( g_{i_{k-1}}(\omega) < g_{i_{k}}(\omega) \). However, because \( CR_{i_{k}}^{\omega}(P) P_{i_{k}} CR_{i_{k-1}}^{\omega}(P) \), there is \( s \in \{1, \ldots, S\} \) such

\( H_1^1 = O_1^0 \). Because \( g_1^0(\omega) < g_3^1(\omega) \), we have \( g^1(\omega) = (5, 2, 1, 4, 3, 6, 7) \). For each \( i \in \{1, 3\} \), \( O_1^1 = \{\omega_1, \omega_2, \ldots, \omega_6\} \). For each \( j \in \{2, 4, 5, 6, 7\} \), \( O_j^1 = \emptyset \).

Round 2: Because, for each \( i \in \{1, 3\} \), \( X_i(O_1^1) = \omega_6 = CR_1^\omega(P) \), we have \( i \to 4 \) and \( H_1^2 = \{\omega_6\} \). Because \( g_1^1(\omega) < g_2^1(\omega) \) and \( g_3^1(\omega) < g_4^1(\omega) \), we have \( g^2(\omega) = (5, 2, 4, 1, 3, 6, 7) \). For each \( i \in \{1, 3\} \), \( O_1^2 = \{\omega_1, \omega_2, \ldots, \omega_6\} \).

Round 3: Because, for each \( i \in \{1, 3\} \), \( X_i(O_2^1) = \omega_5 = CR_2^\omega(P) \), we have \( i \to 7 \) and \( H_1^3 = \{\omega_5\} \). Because \( g_2^1(\omega) < g_3^2(\omega) \) and \( g_3^2(\omega) < g_3^1(\omega) \), we have \( g^3(\omega) = (5, 2, 4, 1, 3, 6, 7) \). For each \( i \in \{1, 3\} \), \( O_1^3 = \{\omega_1, \omega_2, \ldots, \omega_4\} \).

Round 4: Because \( X_1(O_3^1) = \omega_4 = CR_3^\omega(P) \), we have \( 1 \to 3 \) and \( H_1^4 = \{\omega_4\} \). On the other hand, because \( X_3(O_3^1) = \omega_4 = CR_3^\omega(P) \), we have \( H_3^4 = O_3^3 \). We have \( g^4(\omega) = (5, 2, 4, 7, 3, 6, 1) \). Also, \( O_1^4 = \{\omega_1, \omega_2, \omega_3\} \) and \( O_3^4 = \emptyset \).

Round 5: Because \( X_1(O_4^1) = \omega_3 = CR_3^\omega(P) \), we have \( 1 \to 6 \) and \( H_1^5 = \{\omega_3\} \). Because \( g_3^1(\omega) < g_4^1(\omega) \), we have \( g^5(\omega) = (5, 2, 4, 7, 3, 6, 1) \). Also, \( O_1^5 = \{\omega_1, \omega_2\} \).

Round 6: Because \( X_1(O_5^1) = \omega_2 = CR_2^\omega(P) \), we have \( 1 \to 2 \) and \( H_1^6 = \{\omega_2\} \). Because \( g_2^1(\omega) < g_3^2(\omega) < g_4^1(\omega) \), we have \( g^6(\omega) = (5, 2, 4, 7, 3, 6, 1) \). Also, \( O_1^6 = \{\omega_1\} \).

Round 7: Because \( X_1(O_6^1) = \omega_1 = CR_1^\omega(P) \), we have \( H_1^7 = O_1^6 \). Because there is no envy chain, \( g^7(\omega) = g^6(\omega) = (5, 2, 4, 7, 3, 6, 1) \). We have \( O_1^7 = \emptyset \). Hence, no agent remains. So the algorithm terminates at this step.

Therefore, \( g(\omega) = (5, 2, 4, 7, 3, 6, 1) \).

We show that the crawler induced by an endowment profile and the sequential priority rule induced by the order on the agent set given by the mapping select the same allocation,
that agent \( i_{k-1} \) envies agent \( i_k \) at round \( s \). This implies that \( g_{ik}(\omega) < g_{ik-1}(\omega) \), a contradiction.

Let \( \omega \in \mathcal{X} \). Consider round \( s \) of the procedure. Take an envy chain formed at round \( s \) and label the agents in the chain in such a way that each agent, except for the last agent, envies an agent on his right at round \( s \). Consider two endowment profiles in which each agent who is not in the chain has the same endowment. We show that if the allocations selected by the crawler at each of these endowment profiles are the same, each agent in the sequence is also endowed with the same object in the two profiles.

**Claim 5.** Let \( \omega \in \mathcal{X} \). Let \( s \in \{1, \ldots, S\} \). Let \( N' = \{i_1, \ldots, i_k\} \subseteq N \) be such that for each \( k' \in \{1, \ldots, k - 1\} \), agent \( i_{k'} \) envies agent \( i_{k'+1} \) at round \( s \). Let \( \tilde{\omega} \in \mathcal{X} \) be such that \( \tilde{\omega}_{-N'} = \omega_{-N'} \). If \( CR^{\tilde{\omega}}(P) = CR^\omega(P) \), then \( \tilde{\omega} = \omega \).

**Proof of Claim 5.** Without loss of generality, suppose that \( s = 1 \). First, note that

\[(*) \text{ for each } k' \in \{1, \ldots, k - 2\}, \text{ either } p(P_{i_{k'}}) < p(P_{i_{k'+1}}) \text{ or } p(P_{i_{k'+1}}) < p(P_{i_{k'}}), \]

and there is no \( o \in O \setminus \{p(P_{i_{k'}}), p(P_{i_{k'+1}})\} \) such that \( p(P_{i_{k'}}) < o < p(P_{i_{k'+1}}) \) or \( p(P_{i_{k'+1}}) < o < p(P_{i_{k'}}) \).

The proof is by induction on \( k \).

**Base step:** Suppose that \( k = 2 \). Let \( p(P_{i_1}) = p(P_{i_2}) = o \). There are four possible configurations of the endowments of agents 1 and 2 relative to object \( o \).

**Case 1:** \( \omega_{i_1} \prec \omega_{i_2} \preceq o \). Suppose that \( \tilde{\omega}_{i_1} = \omega_{i_2} \) and \( \tilde{\omega}_{i_2} = \omega_{i_1} \). Because \( \tilde{\omega}_{i_2} \prec \tilde{\omega}_{i_1} \preceq o \), at \( \tilde{\omega} \), the ownership of agent \( i_1 \) is shifted to object \( o \). This implies that \( CR^{\tilde{\omega}}_{i_2}(P) \neq o \), a contradiction.

**Case 2:** \( o \preceq \omega_{i_2} \prec \omega_{i_1} \). Suppose that \( \tilde{\omega}_{i_1} = \omega_{i_2} \) and \( \tilde{\omega}_{i_2} = \omega_{i_1} \). Because \( o \preceq \tilde{\omega}_{i_1} \prec \tilde{\omega}_{i_2} \), at \( \tilde{\omega} \), agent \( i_1 \) is visited earlier than \( i_2 \) is. This implies that \( CR^{\tilde{\omega}}_{i_2}(P) \neq o \), a contradiction.

**Case 3:** \( \omega_{i_2} \preceq o \prec \omega_{i_1} \). Let

\[ S = \{i \in N : \omega_{i_2} \prec o < \omega_{i_1} \text{ and } CR^\omega_i(P) < \omega_{i_2}\} . \]

Because \( CR^{\tilde{\omega}}_{i_2}(P) = o \), we have \( |S| \geq |o - \omega_{i_2}| - 1 \). Suppose that \( \tilde{\omega}_{i_1} = \omega_{i_2} \) and \( \tilde{\omega}_{i_2} = \omega_{i_1} \). Hence, \( \tilde{\omega}_{i_1} \preceq o \prec \tilde{\omega}_{i_2} \). Let

\[ \tilde{S} = \{i \in N : \tilde{\omega}_{i_1} \prec \tilde{\omega}_i \prec \tilde{\omega}_{i_2} \text{ and } CR^\omega_i(P) \prec \tilde{\omega}_{i_2}\} . \]
Because $CR_{\tilde{\omega}}(P) = CR_{\omega}(P)$, we have $\tilde{S} = S$. This implies that $|\tilde{S}| \geq |o - \omega_i| - 1$. Hence, at $\tilde{\omega}$, the ownership of agent $i_1$ is shifted to object $o$ earlier than agent $i_2$ is visited. This implies that $CR_{\tilde{\omega}}^o(P) \neq o$, a contradiction.

**Case 4:** $\omega_{i_1} \prec o \not\prec \omega_{i_2}$. Let

$$T = \{i \in N : \omega_{i_1} < \omega_i < \omega_{i_2} \text{ and } CR_i^\omega(P) < \omega_i \}.$$  

Because $CR_i^\omega(P) = o$, we have $|T| < |o - \omega_{i_1}| - 1$. Suppose that $\tilde{\omega}_{i_1} = \omega_{i_2}$ and $\tilde{\omega}_{i_2} = \omega_{i_1}$. Hence, $\tilde{\omega}_{i_2} < o \not\prec \tilde{\omega}_{i_1}$. Let

$$\tilde{T} = \{i \in N : \tilde{\omega}_{i_2} < \tilde{\omega}_i < \tilde{\omega}_{i_1} \text{ and } CR_i^\omega(P) < \tilde{\omega}_{i_2} \}.$$  

Because $CR_{\tilde{\omega}}(P) = CR_{\omega}(P)$, we have $\tilde{T} = T$. This implies that $|\tilde{T}| < |o - \tilde{\omega}_{i_2}| - 1$. Hence, at $\tilde{\omega}$, agent $i_1$ is visited earlier than the ownership of agent $i_2$ is shifted to object $o$. This implies that $CR_{\tilde{\omega}}^o(P) \neq o$, a contradiction.

**Induction hypothesis:** Let $K \in \{2, \cdots, n\}$. For each $k \leq K - 1$, we have $\tilde{\omega}_{i_k} = \omega_{i_k}$.

**Induction step:** Suppose that $k = K$. We show that $\tilde{\omega}_{i_k} = \omega_{i_k}$. Then by the induction hypothesis, we have $\tilde{\omega}_{N'} = \omega_{N'}$. Let $p(P_{i_{k-1}}) = p(P_{i_k}) \equiv o$. There are four possible configurations of the endowment profile relative to object $o$.

**Case 1:** For each $i \in N'$, $\omega_i \not\prec o$. The envy chain formed by $N'$ exists only if for each $k' \in \{1, \cdots, k - 1\}$, $\omega_{k'} \prec p(R_{k'})$ and $\omega_{i_{k'}} < \omega_{k'+1}$. This implies that $CR_{\tilde{\omega}}(P) = CR_{\omega}(P)$ holds only if $\tilde{\omega}_{i_k} = \omega_{i_k}$.

**Case 2:** For each $i \in N'$, $o \not\prec \omega_i$. The envy chain formed by $N'$ exists only if for each $k' \in \{1, \cdots, k - 1\}$, $p(R_{i_k}) < \omega_{i_k}$ and $\omega_{i_{k+1}} < \omega_{i_k}$. This implies that $CR_{\tilde{\omega}}(P) = CR_{\omega}(P)$ holds only if $\tilde{\omega}_{i_k} = \omega_{i_k}$.

Otherwise, there is a pair $i, j \in N'$ such that $\omega_i \not\prec o \not\prec \omega_j$, and there is no $k \in N'$ such that either $\omega_i < \omega_k \not\prec o$ or $o \not\prec \omega_k < \omega_j$. Note that the envy chain formed by $N'$ exists only if either $i = i_k$ or $j = i_k$ holds.

**Case 3:** $\omega_{i_k} \not\prec o \prec \omega_j$. Suppose that $\tilde{\omega}_{i_k} = \omega_j$ and $\tilde{\omega}_{i_l} = \omega_{i_k}$, where $l \in \{k-1, \ldots, 1\}$. Suppose that $l = k - 1$. At $\tilde{\omega}$, the ownership of agent $i_{k-1}$ is shifted to object $o$ earlier
than agent $i_k$ is visited. This implies that $CR_{i_k}^{\omega}(P) \neq o$, a contradiction. Suppose that $l \in \{k - 2, \ldots, 1\}$. Let

$$U = \{i \in N : \omega_i \prec \omega_k \text{ and } CR_{i}^{\omega}(P) \prec \omega_i\} \text{ and}$$

$$\tilde{U} = \{i \in N : \tilde{\omega}_i \prec \omega_i \text{ and } CR_{i}^{\tilde{\omega}}(P) \prec \tilde{\omega}_i\}.$$  

Because $CR^{\omega}(P) = CR^{\tilde{\omega}}(P)$, we have $U = \tilde{U}$. This implies that at $\tilde{\omega}$, the ownership of agent $i_l$ is shifted to object $o$ earlier than agent $i_k$ is visited. Hence, if $p(P_{i_l}) \prec o$, we have $CR_{i_l}^{\omega}(P) = p(P_{i_l})$, a contradiction. Suppose that $o \prec p(P_{i_l})$. The ownership of agent $i_l$ is shifted to at least object $o$. Note that for each $t \in \{k - 1, \ldots, l + 1\}$, object $p(P_{i_{t+1}})$ is assigned to some agent before object $p(P_{i_l})$ is assigned to some agent. Moreover, for each $t \in \{k - 1, \ldots, l + 1\}$, $p(P_{i_t}) \prec p(P_{i_l})$. Then by ($\ast$), the ownership of agent $i_l$ is shifted to object $p(P_{i_l})$ earlier than agent $i_{l+1}$ is either visited or his ownership is shifted to object $p(P_{i_l})$. Hence, $CR_{i_l}^{\tilde{\omega}}(P) = p(P_{i_l})$, a contradiction.

Consequently, $\tilde{\omega}_{i_k} = \omega_{i_k}$.

**Case 4: $\omega_i \prec o \preceq \omega_{i_k}$.** Suppose that $\tilde{\omega}_{i_k} = \omega_i$ and $\tilde{\omega}_{i_l} = \omega_{i_k}$, where $l \in \{k - 1, \ldots, 1\}$.

Suppose that $l = k - 1$. At $\tilde{\omega}$, agent $i_{k-1}$ is visited earlier than the ownership of agent $i_k$ is shifted to object $o$. This implies that $CR_{i_l}^{\tilde{\omega}}(P) \neq o$, a contradiction. Suppose that $l \in \{k - 2, \ldots, 1\}$. Let

$$V = \{j \in N : \omega_i \prec \omega_j \prec \omega_k \text{ and } CR_{i}^{\omega}(P) \prec \omega_i\} \text{ and}$$

$$\tilde{V} = \{j \in N : \tilde{\omega}_{i_k} \prec \omega_j \prec \tilde{\omega}_{i_l} \text{ and } CR_{i}^{\tilde{\omega}}(P) \prec \tilde{\omega}_i\}.$$  

Because $CR^{\omega}(P) = CR^{\tilde{\omega}}(P)$, we have $V = \tilde{V}$. Suppose that $p(P_{i_{l+1}}) \preceq \omega_{i_k}$. Because at $\tilde{\omega}$, agent $i_l$ is visited and receives his assignment first among the agents in $N'$, we have $CR_{i_l}^{\tilde{\omega}}(P) = p(P_{i_l})$, a contradiction. Suppose that $\omega_{i_k} \prec p(P_{i_l})$. Note for each $t \in \{k - 1, \ldots, l + 1\}$, object $p(P_{i_t})$ is assigned to some agent before object $p(P_{i_l})$ is assigned to some agent. Moreover, for each $t \in \{k - 1, \ldots, l + 1\}$, $p(P_{i_t}) \prec p(P_{i_l})$.

Then by ($\ast$), the ownership of agent $i_l$ is shifted to object $p(P_{i_l})$ earlier than agent $i_{l+1}$ is either visited or his ownership is shifted to object $p(P_{i_l})$. Hence, $CR_{i_l}^{\tilde{\omega}}(P) = p(P_{i_l})$, a contradiction. Consequently, $\tilde{\omega}_{i_k} = \omega_{i_k}$.

\[\square\]
Given $\omega \in \mathcal{X}$, for each $s \in \{1, \ldots, S\}$, let $\omega^s \in \mathcal{X}$ be defined by setting for each $i \in N$, $\omega_i^s < \omega_{i+1}^s$ if and only if $g_i^{s-1}(\omega) < g_{i+1}^{s-1}(\omega)$. We call $\omega^s$ the endowment profile at step $s$.

**Claim 6.** $g$ is a one-to-one mapping. That is, for each pair $\omega, \tilde{\omega} \in \mathcal{X}$, if $g(\omega) = g(\tilde{\omega})$, then $\omega = \tilde{\omega}$.

**Proof of Claim 6.** Let $\omega, \tilde{\omega} \in \mathcal{X}$ be such that $g(\omega) = g(\tilde{\omega})$. Hence, $CR^\omega(P) = CR^{\tilde{\omega}}(P)$. Let $s \in \{0, \ldots, S\}$, and suppose that $g^S(\omega) = g(\omega)$. We show that $\omega = \tilde{\omega}$ by induction on $s$.

**Base step:** Suppose that $s = S - 1$. Note that round $S$ is the last round of the procedure when the endowment profile is $\omega$. Because $CR^\omega(P) = CR^{\tilde{\omega}}(P)$, when the endowment profile is $\tilde{\omega}$, round $S$ is also the last round of the procedure. Let $C^S_k \subseteq N$ form a connected chain at endowment profile $\omega$. Because $CR^\omega(P) = CR^{\tilde{\omega}}(P)$, $C^S_k$ also form the same connected chain at round $S$ when the endowment profile is $\tilde{\omega}$. Let $i \in C^S_k$. Let $j \notin C^S_k$. Suppose that $\tilde{\omega}_i^S = \omega_j^S$. Because $C^S_k$ form a connected chain at round $S$ when the endowment profile is $\tilde{\omega}$, there is $k \in C^S_k$ such that $g^S_k(\tilde{\omega}) \neq g^S_k(\omega)$. Moreover, because round $S$ is the last round of the procedure for both $\omega$ and $\tilde{\omega}$, we have $g_k(\tilde{\omega}) \neq g_k(\omega)$, a contradiction. Let $j \in C^S_k$ be such that agents $i$ and $j$ do not belong to the same chain. Suppose that $\tilde{\omega}_i = \omega_j$. Then by construction of $g$, there is $k \in C^S_k$ such that $g^S_i(\omega) < g^S_i(\omega)$ and $g^S_i(\tilde{\omega}) > g^S_i(\tilde{\omega})$, or $g^S_i(\omega) > g^S_i(\omega)$ and $g^S_i(\tilde{\omega}) < g^S_i(\tilde{\omega})$. Because round $S$ is the last round of the procedure for both $\omega$ and $\tilde{\omega}$, we have $g_k(\tilde{\omega}) \neq g_k(\omega)$, a contradiction. Finally, by Claim 5, we have $g^S_i(\tilde{\omega}) = g^S_i(\omega)$. Therefore, $g^{S-1}(\tilde{\omega}) = g^{S-1}(\omega)$.

**Induction hypothesis:** Let $s' \in \{0, \ldots, S - 1\}$. For each $s \geq s' + 1$, we have $g^s(\tilde{\omega}) = g^s(\omega)$.

**Induction step:** Suppose that $s = s'$. Suppose that $g^s(\tilde{\omega}) \neq g^s(\omega)$. This implies that $\tilde{\omega}^{s+1} \neq \omega^{s+1}$. By the induction hypothesis, for each $t \geq s + 1$, we have $g^t(\tilde{\omega}) = g^t(\omega)$. Let $C^{s+1}_k \subseteq N$ form a connected chain at endowment profile is $\omega$. Because $CR^{\tilde{\omega}}(P) = CR^\omega(P)$, $C^{s+1}_k$ form the same relation at round $s + 1$ when the endowment profile is $\tilde{\omega}$. Let $i \in C^{s+1}_k$. Let $j \notin C^{s+1}_k$. Suppose that $\tilde{\omega}^{s+1}_i = \omega^{s+1}_j$. Because $C^{s+1}_k$ form a connected chain at round $s + 1$ when the endowment profile is $\tilde{\omega}$, there is $k \in C^{s+1}_k$ such that $g^{s+1}_k(\tilde{\omega}) \neq g^{s+1}_k(\omega)$, a contradiction. Let $j \in C^{s+1}_k$ be such that agents $i$ and $j$ do not belong to the same chain. Suppose that $\tilde{\omega}_i = \omega_j$. Then by construction of $g$, there is $k \in C^{s+1}_k$ such that $g^{s+1}_i(\omega) < g^{s+1}_i(\omega)$ and $g^{s+1}_k(\tilde{\omega}) > g^{s+1}_k(\tilde{\omega})$, or $g^{s+1}_i(\omega) > g^{s+1}_k(\omega)$ and $g^{s+1}_k(\tilde{\omega}) < g^{s+1}_k(\tilde{\omega})$. This contradicts the fact that $g^{s+1}(\tilde{\omega}) = g^{s+1}(\omega)$. Finally, by Claim 5, we have $g_i^s(\tilde{\omega}) = g_i^s(\omega)$. Therefore,
\( g^{*}(\tilde{\omega}) = g^{*}(\omega) \).

Consequently, \( \omega = \tilde{\omega} \). \( \square \)

**Claim 7.** \( g \) is an onto mapping. That is, for each \( f \in F \), there is \( \omega \in X \) such that \( g(\omega) = f \).

**Proof of Claim 7.** By Claims 4 and 6, for each \( x \in X \),

\[
|\{ \omega \in X : CR^\omega(P) = x \}| \leq |\{ f \in F : SP^f(P) = x \}|.
\]

Hence,

\[
\sum_{x \in X} |\{ \omega \in X : CR^\omega(P) = x \}| \leq \sum_{x \in X} |\{ f \in F : SP^f(P) = x \}|.
\]

However, the left-hand side of the inequality is equal to the number of possible endowment profiles and its right-hand side is equal to the number of orderings over the agents. Both of these numbers equal to \( n! \). Therefore, for each \( x \in X \),

\[
|\{ \omega \in X : CR^\omega(P) = x \}| = |\{ f \in F : SP^f(P) = x \}|.
\]

This implies that for each \( f \in F \), there is \( \omega \in X \) such that \( g(\omega) = f \). \( \square \)

**C  An example showing that the crawler is not a member of the trading cycles family**

For our purpose, it will suffice to formally define the trading cycles family for problems with three agents and three objects.\(^{16}\) For each \( N' = \{i_1, \ldots, i_{n'}\} \subseteq N \) and each \( O' \subseteq O \) such that \( |N'| = |O'| \), a “partial allocation for \( N' \)” is a list \( x_{N'} = (x_{i_1}, \ldots, x_{i_{n'}}) \) such that for each \( i \in N', x_i \in O' \), and for each pair \( i, j \in N' \) such that \( i \neq j, x_i \neq x_j \). We call each \( i \in N \setminus N' \) an “unassigned agent”, and each \( o \in O \setminus O' \) an “unassigned object”. Let \( X_{N'} \) be the set of partial allocations for \( N' \). Let \( \mathcal{Y} = \bigcup_{N' \subseteq N} X_{N'} \). Obviously, \( X \subseteq \mathcal{Y} \). Let \( \overline{Y} = \mathcal{Y} \setminus X \). For each \( y \in \mathcal{Y} \), let \( \overline{N}_y \) be the set of unassigned agents at \( y \) and \( \overline{O}_y \) be the set of unassigned objects at \( y \).

A “control-rights structure” is a collection of mappings

\[
\{(c_y, b_y) : \overline{O}_y \rightarrow \overline{N}_y \times \{\text{ownership, brokerage}\}\}_{y \in \overline{Y}}
\]

such that

\(^{16}\)Refer to Pycia and Ünver (2017) for general problems.
(R1) At the empty allocation, i.e., when the set of unassigned objects is the entire set of objects, each object is brokered.\textsuperscript{17}

(R2) If agent $i$ is the only unassigned agent at $y$, he owns all of the unassigned objects at $y$.

(R3) If agent $i$ brokers an object at $y$, he does not control any other objects at $y$.

Moreover, for each pair $y, y' \in \mathcal{Y}$ such that $y \subset y'$ and each $i \in \mathcal{N}_{y'}$ such that there is $o \in \mathcal{O}_{y'}$ that agent $i$ owns at $y$,$^18$

(R4) Agent $i$ owns object $o$ at $y'$.

(R5) If agent $j \in \mathcal{N}_{y'}$ brokers object $o' \in \mathcal{O}_{y'}$ at $y$, then either agent $j$ brokers object $o'$ at $y'$ or agent $i$ owns object $o'$ at $y'$.

(R6) If agent $j \in \mathcal{N}_{y'}$ controls object $o' \in \mathcal{O}_{y'}$ at $y$, then agent $j$ owns object $o$ at $y \cup \{(i, o')\}$.$^19$

**Trading Cycles:** (For problems with three agents and three objects.) Let $(c, b)$ be a control-rights structure. Let $P \in \mathcal{P}^N$.

Let $x^0$ be the empty allocation. In each round $r = 1, 2, \ldots$, some agents are assigned objects. We denote by $x^{r-1}$ the partial allocation at round $r - 1$. Note that $x^{r-1} \subset x^r$. If $x^{r-1} \in \mathcal{Y}$, the algorithm proceeds with the following steps of round $r$.

**Step 1: Pointing.** Each $o \in \mathcal{O}_{x^{r-1}}$ points to the agent who controls it at $x^{r-1}$. Each $i \in \mathcal{N}_{x^{r-1}}$ points to his most preferred object in $\mathcal{O}_{x^{r-1}}$.

A cycle

$$o^1 \rightarrow i^1 \rightarrow \ldots o^k \rightarrow i^k \rightarrow o^1$$

is simple if there is an agent in the cycle who owns an object.

**Step 2a: Executing “simple” cycles.** Each agent in each simple cycle is assigned the object he is pointing to.

**Step 2b: Forcing brokers to downgrade their pointing.** If there are non-simple cycles, we proceed as follows:

- If there is a cycle in which there is a broker $i$ who points to a brokered object, and there is another broker or owner who points to the same object, we force broker $i$ to point to his next most preferred object in $\mathcal{O}_{x^{r-1}}$ and we return to step 2(a).

---

$^17$This assumption is only for problems with three agents and three objects. It differs for general problems.

$^18$ $y \subset y'$ means that if an agent is assigned an object at $y$, then he is assigned the same object at $y'$.

$^19$(i, o') means that agent $i$ is assigned object o'.

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• Otherwise, we clear all cycles by assigning each agent in each cycle the object he is pointing to.

**Step 3:** Each agent who has been assigned an object leaves with his assignment. If each agent is assigned an object, the algorithm terminates.

The following example shows that there is a problem such that there is no control-rights structure such that the associated rule always select the same allocation as the crawler.

**Example 3.** Let $N = \{1, 2, 3\}$ and $O = \{o_1, o_2, o_3\}$. Let $\omega = (o_1, o_2, o_3)$. Let $P \in \mathcal{P}^N$ be such that

\[
P_1 : o_2, \ldots
\]
\[
P_2 : o_1, o_2, o_3
\]
\[
P_3 : o_1, o_2, o_3.
\]

Then

\[CR^\omega(P) = (o_2, o_1, o_3).\]

There are two control-rights structure such that $TC^{(c,b)}(P) = CR^\omega(P)$;

**Case 1:** At the empty allocation, agents 1, 2, and 3 broker objects $o_1, o_3$, and $o_2$, respectively.

**Case 2:** At the empty allocation, agents 1, 2, and 3 broker objects $o_3, o_2$, and $o_1$, respectively.

Consider Case 1. For each $i \in N$, let $P'_i \in \mathcal{P}^N$ be defined by

\[P'_i : o_1, o_2, o_3.
\]

Then

\[CR^\omega(P') = (o_1, o_2, o_3) \neq (o_2, o_1, o_3) = TC^{(c,b)}(P').\]

Consider Case 2. For each $i \in N$, let $\tilde{P} \in \mathcal{P}^N$ be defined by

\[\tilde{P}_i : o_3, o_2, o_1.
\]

Then

\[CR^\omega(\tilde{P}) = (o_1, o_2, o_3) \neq (o_2, o_1, o_3) = TC^{(c,b)}(\tilde{P}).\]
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