INDEX OF Γ-EQUIVARIANT TOEPLITZ OPERATORS

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Abstract. Let Γ be a discrete subgroup of PSL(2, ℝ) of infinite covolume with infinite conjugacy classes. Let \( H_t \) be the Hilbert space consisting of analytic functions in \( L^2(\mathbb{D}, (\text{Im } z)^{t-2}d\bar{z}dz) \) and let, for \( t > 1 \), \( \pi_t \) denote the corresponding projective unitary representation of PSL(2, ℝ) on this Hilbert space. We denote by \( A_t \) the \( II_\infty \) factor given by the commutant of \( \pi_t(\Gamma) \) in \( B(\mathcal{H}_t) \). Let \( F \) denote a fundamental domain for \( \Gamma \) in \( \mathbb{D} \) and assume that \( t > 5 \). \( \partial M = \partial \mathbb{D} \cap F \) is given the topology of disjoint union of its connected components.

Suppose that \( f \) is a continuous \( \Gamma \)-invariant function on \( \mathbb{D} \) whose restriction to \( F \) extends to a continuous function on \( F \) and such that \( f|_{\partial M} \) is an invertible element of \( C_0(\partial M)^{-} \). Let \( T^f = \mathcal{P}_t \mathcal{M}_f \mathcal{P}_t \) denote the Toeplitz operator with symbol \( f \). Then \( T^f \) is Fredholm, in the Breuer sense, with respect to the \( II_\infty \) factor \( A_t \) and, moreover, its Breuer index is equal to the total winding number of \( f \) on \( \partial M \).


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1. Introduction

In this paper we study equivariant Toeplitz operators acting on the Hilbert space \( \mathcal{H}_t \) consisting of all square summable analytic functions in
\[ L^2(\mathbb{D}, (\text{Im } z)^{t-2}d\bar{z}dz). \]

Let us first recall that the classical theory of Toeplitz operators in the unit disc yields an extension of C*-algebras
\[ 0 \to \mathcal{K} \to \mathcal{T} \to C(\partial \mathbb{D}) \to 0, \]
where $\mathcal{K}$ denotes the algebra of compact operators on $\mathcal{H}_2$ and $\mathcal{T}$ the Toeplitz C*-algebra generated by compressions $T_f$ to $\mathcal{H}_2$ of multiplication operators (by $f$’s from $C(\overline{\mathbb{D}})$). In particular, for $f|_{\partial \mathbb{D}}$ invertible, $T_f$ is Fredholm and the boundary map for the K-theory six term exact sequence of this extension is equivalent, via index theorem for Toeplitz operators, to the equality:

$$\text{Index } (T_f) = \text{winding number of } f|_{\partial \mathbb{D}}.$$ 

As it turns out, all of these facts admit suitable generalisation to the equivariant case.

Let $\Gamma$ be a fuchsian subgroup of $PSL(2, \mathbb{R})$, which has infinite conjugacy classes and is of infinite covolume. Recall that the action of $PSL(2, \mathbb{R})$ on $\mathbb{D}$ by fractional linear transformations lifts to projective unitary representations of $PSL(2, \mathbb{R})$ on these Hilbert spaces (cf. ([21], [19])) and the commutant of $\pi_t(\Gamma)$ is a $II_\infty$ factor. We will denote by $A_t$ the commutant $\pi_t(\Gamma)' \cap B(\mathcal{H}_t)$ and by $\tau$ the normal positive non-zero trace on $A_t$. If $\sigma_t$ denotes the 2-group cocycle corresponding to the projective unitary representation $\pi_t$, then $A_t$ is isomorphic ([20]) to $L(\Gamma, \sigma_t) \otimes B(K)$, where $K$ is an infinite dimensional separable Hilbert space and $L(\Gamma, \sigma_t)$ is the twisted group von Neumann algebra of $\Gamma$.

Let $F$ denote a fundamental domain for the action of $\Gamma$ on $\mathbb{D}$. We will denote by $M$ the quotient space $\mathbb{D}/\Gamma$ and by $\partial M$ its boundary:

$$\partial M = (\partial \mathbb{D} \cap \overline{F})/\Gamma$$

equipped with the topology of the disjoint union of its connected components, i. e. of a countable union of disjoint circles. In particular $\overline{M} = \overline{F}/\Gamma = M \cup \partial M$ inherits a structure of locally compact space. $C_0(\partial M)^-$ denotes the unitisation of the C*-algebra of continuous, vanishing at infinity functions on $\partial M$.

For any continuous $\Gamma$-invariant function $f$ on $\mathbb{D}$ which extends to a continuous function on $\overline{M}$ we denote by $T_f$ the Toeplitz operator on $\mathcal{H}_t$ with symbol $f$, i. e. the compression to $\mathcal{H}_t$ of the operator of multiplication by $f$ on $L^2(\mathbb{D}, (\text{Im } z)^{t-2}d\text{Im } z)$. Because of the $\Gamma$ invariance of the symbol such a $T_f$ belongs to $A_t = \{\pi_t(\Gamma)' \cap B(\mathcal{H}_t)\}$ ([24]).

Suppose that $t > 5$ and that $f|_{\partial M}$ is an invertible element of $C_0(\partial M)^-$. We prove below that $T_f$ is Fredholm (in the sense of Breuer ([1]), in $A_t$. Moreover the Breuer index is (in analogy with the classical case) equal to the winding number of $f|_{\partial M} : \partial M \to \mathbb{C} \setminus \{0\}$. This can be seen as an analogue of Atiyah’s index formula for coverings ([2]).

The organisation of the paper is as follows.
In Section 2 we gather some more or less known results about nuclearity properties of Toeplitz operators on $H_t$ and prove the main technical result:

Let $t > 5$ and $f, g \in L^\infty(\mathbb{D})$ are given. Suppose that $g \in C^\infty(\mathbb{D})$ and that

$$\inf \{|z - \xi| | z \in \text{supp } f \text{ and } \xi \in \text{supp } g\} > \epsilon$$

for some positive number $\epsilon$. Then both $T_f T_g$ and $T_g T_f$ are of trace class and $\text{Tr}([T_f, T_g]) = 0$.

(cf. Theorem 2).

In Section 3 we study the $L^1(\tau)$-properties of commutators of Toeplitz operators with $\Gamma$-invariant symbol and prove the following result.

Suppose that $t > 5$ and $f$ and $g$ are $\Gamma$-invariant functions on $\mathbb{D}$ which are smooth on the closure of a fundamental domain for $\Gamma$. Then both $[T_f, T_g]$ and $T_fg - T_f T_g$ are in $\mathcal{M} \cap L^1(\tau)$ and

$$\tau([T_f, T_g]) = \frac{1}{2\pi i} \int_F df \; dg$$

(cf. Theorem 3 and the remarks following).

Let $\mathcal{T}_\Gamma$ be the $C^*$-subalgebra of $\mathcal{A}_t$ generated by Toeplitz operators $T_f$ with $f$ $\Gamma$-invariant and smooth on a fundamental domain for $\Gamma$, $\mathcal{K}_\Gamma$ be the $C^*$-ideal generated by the $\mathcal{L}^1(\tau)$-elements in $\mathcal{T}_\Gamma$ and $M = \mathbb{D}/\Gamma$.

In Section 4 we construct the extension

$$0 \to \mathcal{K}_\Gamma \to \mathcal{T}_\Gamma \to C(\partial M) \to 0.$$

$$T_f \to f|_{\partial M}$$

Let $\partial : K_1(\partial M) \to K_0(\mathcal{K}_\Gamma)$ denote the boundary map in K-theory associated to this extension. We prove that, for $T_f \in \mathcal{T}_\Gamma$ with $f$ invertible on the boundary of $M$, the following equality holds:

$$\langle \tau , \partial[T_f] \rangle = \text{winding number of } f \text{ on } \partial M$$

(cf. Theorem 4).

**Remark 1.** For notational simplicity we work throughout the paper with the case when the number of boundary components of a fundamental domain of $\Gamma$ is finite. The only difference (except for typographical complications) in the general case consists of replacing the above extension of $C(\partial M)$ by $\mathcal{K}_\Gamma$ by the extension:

$$0 \to \mathcal{K}_\Gamma \to \mathcal{T}_0^\Gamma \to C_0(\partial M) \to 0$$

where $\mathcal{T}_0^\Gamma$ stands for the (in general nonunital) $C^*$-algebra generated by $T_f$ with $f$ continuous on $\overline{F}$ and in with non-zero values on finitely many components of $\partial \mathbb{D} \cap \overline{F}$.
The method of the proof are based on the equivariant Berezin’s quantization theory for such groups ([20], [17]). Let $F$ be, as above, a fundamental domain for the action of $\Gamma$ in $\mathbb{D}$. Then for every $\Gamma$-equivariant, bounded function $g$ on $\mathbb{D}$, having compact support in the interior of $F$, the Toeplitz operator $T_g \in \mathcal{A}_t$ is in $\mathcal{L}^1(\mathcal{A}_t)$ and has trace equal to a universal constant times the integral $\int_F g(z) \langle \text{Im } \rangle^{-2} d\overline{z}dz$

Moreover we will show that the commutator of two Toeplitz operators, having symbols that are smooth and continuous on the closure of $F$, belongs to trace ideal of the $II_\infty$ factor. In particular if the symbol is invertible in the neighborhood of the intersection of the boundary of $\mathbb{D}$ with the closure of $F$, the operator is Fredholm in $\mathcal{A}_t$ in Breuer’s sense ([6]).

To identify the Breuer index of such a Toeplitz operator we use the Carey-Pincus theory ([8]). Let us first recall the pertinent facts. Given an operator $A \in \mathcal{A}_t$ such that $\tau[A^*, A] < \infty$, the bilinear map

$$\mathbb{C}[z, \bar{z}] \ni P, Q \rightarrow \tau[P(A^*, A), Q(A^*, A)]$$

defines a cyclic one-cocycle on the algebra of polynomials (in two real variables), of the form

$$(P, Q) \rightarrow \int_Z \{P, Q\} d\mu(\overline{z}, z),$$

where $Z$ is the of spectrum of the class of $A$ in $\mathcal{A}_t/\mathcal{K}_\Gamma$. $d\mu$, called the principal function of $A$, is a finite measure on the complex plane having the property that, for any connected component $Z'$ of $\mathbb{C} \setminus Z$,

$$d\mu|_{Z'} = -c\pi d\overline{z}dz$$

with a constant $c$ equal to the value of $\tau$ on the index class of $\partial[A - \lambda] \in K_0(\mathcal{K}_\Gamma)$ for any $\lambda \in Z'$. Hence one has to determine the principal function, given by the $\tau$-values on commutators of polynomials in $T_f$ and its adjoint (like in the classical case in [6], [15]).

To deal with the computation of those we apply the spatial theory of von Neumann algebras ([9]). One of its basic constructions gives an operator-valued weight $E : \mathcal{B}(\mathcal{H}_t) \rightarrow \mathcal{A}_t$ such that, for a trace-class operator $A_0$ in the domain of $E$, $\tau \circ E(A_0) = Tr(A_0)$. This allows one to replace the computation of values of $\tau$ on $\Gamma$-invariant operators by computation of value of the classical trace on certain trace-class operators on $\mathcal{H}_t$. In fact we prove in Section 3 that commutators of $\Gamma$-equivariant Toeplitz operators are of the form $E(A_0)$ for $A_0$ given by a suitable (trace class) commutator of polynomials in Toeplitz operators and hence the computation reduce to the classical case.

But in this case the principal function for a pair of Toeplitz operators that commute modulo the trace ideal in $B(\mathcal{H}_t)$ is well understood - it
is basically given by the fact that the index of $T_z$ is equal to one \([1]\) and gives explicit formulas that lead to the results stated above.

2. Some results on Toeplitz operators

Let $\mathbb{D}$ denote the unit circle in the complex plane and set

$$d\mu_t(z) = (1 - |z|^2)^t \frac{d\bar{z}dz}{(1 - |z|^2)^2}$$

We set

(2) \quad $H_t = \{ f \in L^2(\mathbb{D}, d\mu_t) \mid f \text{ holomorphic on } \mathbb{D} \}$

As is well known, $H_t$ is a closed subspace of $L^2(\mathbb{D}, d\mu_t)$ and the orthogonal projection $P_t : L^2(\mathbb{D}, d\mu_t) \to H_t$ is called the Toeplitz projection. Given a function $f \in L^\infty(\mathbb{D}, d\mu_t)$ we denote by $M_f$ the operator of multiplication by $f$ on $L^2(\mathbb{D}, d\mu_t)$ and set

- the Toeplitz operator associated to $f$: \( T_f = P_t M_f P_t \)
- the Henkel operator associated to $f$: \( H_f = (1 - P_t) M_{\bar{f}} P_t \)

Note, for future computations, that $T_f$ is an integral operator on $L^2(\mathbb{D}, d\mu_t)$ with integral kernel

(3) \quad $K_f(z, \xi) = \frac{t - 1}{2\pi i} \frac{f(\xi)}{(1 - z\bar{\xi})^t}$

and

(4) \quad $H_f^* H_g = T_f g - T_f T_g$.

We will denote by $\delta$ the absolute value of the cosine of hyperbolic distance on $\mathbb{D}$, i.e.

(5) \quad $\delta(a, b) = \frac{(1 - ||a||^2)(1 - ||b||^2)}{|1 - ab|^2}$.

As is well known,

(6) \quad $||T_f|| = ||f||_\infty$

and

(7) \quad $||f||_{S_2}^2 = ||H_f||_2^2 + ||H_{\bar{f}}||_2^2 = \left(\frac{t - 1}{2\pi i}\right)^2 \int_{\mathbb{D} \times \mathbb{D}} |f(a) - f(b)|^2 \delta^t(a, b) d\mu_0(a, b)$.

In particular, since both $H_z$ and $H_{\bar{z}}$ are of finite rank, the function $f(a, b) = (a - b)$ is square integrable with respect to the measure...
δt(a, b)dμ0(a, b) and hence all functions which are Lipschitz with exponent one on $\mathbb{D}$ have finite $S_2$-norm.

We let $\text{PSL}(2, \mathbb{R})$ act on $\mathbb{D}$ by fractional linear transformations and denote by $\pi_t$ the induced projective unitary representation on $L^2(\mathbb{D}, d\mu_t)$ (and $\mathcal{H}_t$). Both $d\mu_0$ and $\delta_t(a, b)$ are $\text{PSL}(2, \mathbb{R})$-invariant, which gives a useful formula

$$\int_{\mathbb{D}} \delta_t(a, b)d\mu_0(a) = \int_{\mathbb{D}} \delta_t(a, 0)d\mu_0(a) = \frac{4\pi}{t - 1}.$$  

(8)

The following result is probably well known to specialists, however, since we do not have a ready reference, so we will include the proof below.

**Theorem 1.** Let $f$ and $g$ belong to $C^\infty(\overline{\mathbb{D}})$. Then $T_fT_g - T_{fg}$ is a trace class operator and, moreover,

$$\text{Tr}([T_f, T_g]) = \left(\frac{t - 1}{2\pi i}\right)^2 \int_{\mathbb{D} \times \mathbb{D}} (f(a)g(b) - f(b)g(a))\delta_t(a, b)d\mu_0(a, b)$$

$$= \frac{1}{2\pi i} \int_{\mathbb{D}} df dg = \frac{1}{2\pi i} \int_{\partial\mathbb{D}} f dg.$$

Proof. By the smoothness assumption, both $f$ and $g$ belong to the $S_2$ class and hence $T_fT_g - T_{fg}$ is a trace class operator. For the computation of the trace we can just as well assume that both $f$ and $g$ are real valued. To begin with, for a real-valued function $f$, (7) gives

$$\text{Tr}(T_f^2 - T_{f^2}) = \frac{1}{2} \left(\frac{t - 1}{2\pi i}\right)^2 \int_{\mathbb{D} \times \mathbb{D}} (f(a) - f(b))^2\delta_t(a, b)d\mu_0$$

and hence, by an application of the polarisation identity,

$$\text{Tr}(T_{f^2} - T_fT_g) = \left(\frac{t - 1}{2\pi i}\right)^2 \int_{\mathbb{D} \times \mathbb{D}} f(a)(g(a) - g(b))\delta_t(a, b)d\mu_0$$

which implies immediately the first equality.

To get the second equality recall that, for a pair of (non-commutative) polynomials $P(\bar{z}, z)$ and $Q(\bar{z}, z)$, the Carey-Pincus formula holds:

$$\text{Tr}([P(T_{\bar{z}}, T_z), Q(T_{\bar{z}}, T_z)]) = \frac{1}{2\pi i} \int_{\mathbb{D}} dPdQ = \frac{1}{2\pi i} \int_{\partial\mathbb{D}} PdQ,$$

see [7]. Since $P(T_{\bar{z}}, T_z) = T_P$ mod $\mathcal{L}^1(Tr)$, this implies the second equality for $f$ and $g$ polynomial. Approximating arbitrary pair of smooth functions uniformly with their first derivatives on $\overline{\mathbb{D}}$ completes the proof of the second equality.

The following is the main technical result of this section
Theorem 2. Let $t > 5$ and $f, g \in L^\infty(\mathbb{D})$ be given. Suppose moreover, that $g \in C^\infty(\mathbb{D})$ and that

$$\inf \{ ||z - \xi|| \mid z \in \text{supp } f \text{ and } \xi \in \text{supp } g \} > \epsilon$$

for some positive number $\epsilon$. Then both $T_f T_g$ and $T_g T_f$ are of trace class and $\text{Tr}([T_f, T_g]) = 0$.

Proof. We use $\partial$ to denote the unbounded operator $H_t \ni h \to \partial z h \in H_t$, defined on the subspace of holomorphic functions $h$ such that their first derivative is smooth up to the boundary of the disc and in $H_t$, and by $\partial^{-1}$ the unique extension to a bounded operator on $H_t$ of

$$z^n \to \frac{1}{n+1} z^{n+1}. \quad (9)$$

It is easy to see that $\partial^{-1}$ is Hilbert-Schmidt, in fact, since $||z^n||_2 \sim O(1)$ as $n \to \infty$, the characteristic values of $\partial^{-1}$ are of the order $O(n^{-1})$. Moreover $\text{Id} - \partial \partial^{-1}$ is of finite rank. Since we can write

$$T_f T_g|_{\mathcal{C}[\mathbb{D}]} = T_f T_g \partial \partial^{-1} + T_f T_g (1 - \partial \partial^{-1})$$

to prove that $T_f T_g$ is trace class it is sufficient to show that the densely defined operator $T_f T_g \partial$ has a (unique) extension to a Hilbert-Schmidt operator on $H_t$. Suppose first that $h$ and $\partial h$ both belong to $H_t$ and are smooth up to the boundary of $\mathbb{D}$. Given an $a \notin \text{supp}(g)$, we have

$$(T_g \partial h)(a) = \frac{-1}{2\pi i} \int_{\mathbb{D}} d\mu_t(b) \frac{g(b)}{(1 - ab)^t} \partial_b h(b)\frac{\partial_b}{(1 - ab)^t}$$

$$= \frac{-1}{2\pi i} \int_{\mathbb{D}} g(b) \frac{1 - |b|^2}{(1 - ab)^t} \partial_b h(b)$$

$$= \frac{-1}{2\pi i} \int_{\mathbb{D}} \partial_b \left( \frac{g(b) (1 - |b|^2)^{t-2}}{(1 - ab)^t} \right) h(b) db,$$

where we used Stokes theorem and the fact that the integrand is smooth and vanishes at the boundary of $\mathbb{D}$. But this implies that the densely defined operator $T_f T_g \partial$ is in fact given by an integral operator with kernel

$$K(z, \xi) = \text{const} \int_{\mathbb{D} \times \mathbb{D}} d\lambda(a, b) \frac{(1 - |a|^2)^{t-2}(1 - |b|^2)^{t-3} F(a, b)}{(1 - \bar{a} z)^t(1 - \bar{\xi} b)^t} \frac{F(a, b)}{(1 - ba)^t}$$

where $d\lambda$ is the Lebesque measure on $\mathbb{D} \times \mathbb{D}$ and $F$ is an $L^\infty$-function which vanishes on a neighbourhood of the diagonal in $\mathbb{D} \times \mathbb{D}$ given by $\{(a, b) \mid |a - b| > \epsilon\}$. In particular,

$$\sup_{a, b} \left| \frac{F(a, b)}{(1 - ba)^t} \right| < \infty$$
and, by Cauchy-Schwartz inequality,
\[ |K(z, \xi)|^2 \leq \text{const} (\text{Vol}(D, d\lambda))^2 \int_{D \times D} d\lambda(a, b) \frac{(1 - |a|^2)^{-2}(1 - |b|^2)^{-3}}{(1 - \bar{a}z)(1 - \bar{\xi}b)} |^2. \]

To estimate the \( L^2 \)-norm of \( K(z, \xi) \) we can first integrate over \( z \) and \( \xi \) which gives, in view of (8), the estimate
\[ \|K\|^2 \leq \text{const} \int_{D \times D} (1 - |a|^2)^{-4}(1 - |b|^2)^{-6} d\lambda(a, b) \]
which is finite for \( t > 5 \).

To finish the proof \( T_g T_f = (T_f T_g)^* \) and hence is also trace class by applying the above argument to \( T_f T_g \). As a direct consequence we get \( Tr[T_f, T_g] = 0 \).

3. \( \Gamma \)-invariant Toeplitz operators

Let \( \Gamma \) be a countable, icc and discrete subgroup of \( \text{PSL}(2, \mathbb{R}) \).

The von Neumann algebra \( (\pi_t(\Gamma)'') \) is a \( II_1 \) factor with unique normal normalized trace \( \tau' \) given by
\[ \tau'(\pi_t(\gamma)) = 0 \]
for \( \gamma \neq e \). Its commutant \( \mathcal{A}_t = (\pi_t(\Gamma))' \) is a factor of type II. We will assume from now on that \( \Gamma \) has infinite covolume in \( D \), i. e. \( M = \mathbb{D}/\Gamma \)
is an open Riemannian surface which can (and will) be thought of as an open subset of an ambient closed Riemannian surface \( N \). We assume moreover that \( M \) has finitely many boundary components (the boundary in \( N \)), hence \( \partial M = \cup_i C_i \), a finite union of disjoint smooth simple closed contractible curves in \( N \). In this case \( \mathcal{M} \) is a \( II_\infty \) factor with a unique (up to the normalisation) positive normal trace \( \tau \). By general theory (see \cite{14}, \cite{4}, \cite{12}) there exists a unique, normal, semifinite operator-valued weight
\[ E : \mathcal{B}(\mathcal{H}_t) \to \mathcal{M} \]
such that, for \( A \in \mathcal{L}^1(\text{Tr}) \) in the domain of \( E \),
\[ \tau \circ E(A) = \text{Tr}(A). \]

\( E \) is uniquely determined by the equality of normal linear functionals
\[ m \to \tau(E(A)m) = \text{Tr}(Am) \]
for \( A \in \mathcal{L}^1(\text{Tr}) \) and \( m \in \mathcal{A}_t \). Below we list some of the properties of \( E \) used later.
A vector \( \xi \in \mathcal{H}_t \) is called \( \Gamma \)-bounded if the densely defined map
\[
L^2(\Gamma) \ni \{c_\gamma\}_{\gamma \in \Gamma} \mapsto \sum_{\gamma \in \Gamma} c_\gamma \pi_t(\gamma^{-1})\xi \in \mathcal{H}_t
\]
is bounded. Let \( p_\xi \) denote the orthogonal projection onto the one dimensional subspace spanned by vector \( \xi \). Then it is easy to see that
\[
R_\xi R^*_\xi = \sum_{\gamma \in \Gamma} \pi_t(\gamma) P_\xi \pi_t(\gamma^{-1})
\]
i. e. \( \xi \) is \( \Gamma \)-bounded precisely in the case when the sum \( \sum_{\gamma \in \Gamma} \pi_t(\gamma)p_\xi \pi_t(\gamma^{-1}) \) converges in the strong operator topology to a bounded operator on \( \mathcal{H}_t \), in fact equal to \( E(p_\xi) \) and in this case \( \tau(E(p_\xi)) = Tr(p_\xi) = 1 \).

Let us introduce the following.

**Definition 1.** A bounded operator \( A \) is called \( \Gamma \)-bounded if the sums
\[
\sum_{\gamma \in \Gamma} \pi_t(\gamma)A\pi_t(\gamma^{-1})
\]
converge in the strong operator topology.

Let \( A \) be a positive trace class operator of the form
\[
Ax = \sum_i \lambda_ip_{\xi_i}
\]
where \( \{\xi_i\} \) is an orthonormal system in \( \mathcal{H}_t \). \( A \) is in the domain of \( E \) if it is \( \Gamma \)-bounded and in this case
\[
E(A) = \sum_{\gamma \in \Gamma} \pi_t(\gamma)A\pi_t(\gamma^{-1}) \quad \text{and} \quad \tau(E(A)) = TrA.
\]

**Proposition 1.** Let \( f_0 \) be an \( L^\infty \) function \( \mathbb{D} \) satisfying the conditions:
- the euclidean distance from the essential support of \( f_0 \) to \( \partial \mathbb{D} \) is strictly positive;
- the sum \( \sum_{\gamma \in \Gamma} f_0 \circ \gamma \) is locally finite.

If, moreover, \( t > 2 \), the associated \( \Gamma \)-invariant \( L^\infty \)-function \( f = \sum_{\gamma \in \Gamma} f_0 \circ \gamma \) on \( \mathbb{D} \) satisfies
\[
T_f \in \mathcal{A}_t \cap \mathcal{L}^1(\tau)
\]

Proof. Since \( f \) is \( \Gamma \)-invariant and in \( L^\infty(\mathbb{D}) \), \( T_f \in \mathcal{A}_t \). for the rest of the claim it is sufficient to look at \( f_0 \) positive. But then \( T_{f_0} \) is a positive operator with smooth kernel, hence, by Lidskii theorem, it is of trace class. By the second assumption it is \( \Gamma \)-bounded and hence, according to the remarks above, it is in the domain of \( E \) and
\[
E(T_{f_0}) = T_{\sum_{\gamma} f_0 \circ \gamma} = T_f.
\]
In particular
\[ \tau(T_f) = \tau(E(T_{f_0})) = Tr(T_{f_0}) < \infty \]
as claimed.

Let us introduce some notation connected with fundamental domains for the action of \( \Gamma \) on \( \mathbb{D} \). Suppose we choose points \( P_i \) on \( \partial M \), one on each connected component \( C_i \). We’ll call this a cut of \( M \). To each such cut we can associate a fundamental domain \( F \) such that the chosen points are in bijective correspondence with end-points of the intervals \( F \cap \partial \mathbb{D} \).

From now on \( F \) will (unless explicitly stated to the contrary) denote a generic fundamental domain for \( \Gamma \) on \( \mathbb{D} \).

Our goal is to compute the \( \tau \)-trace of commutators of the form \([T_f, T_g]\), where \( f \) and \( g \) are sufficiently general \( \Gamma \)-invariant functions on \( \mathbb{D} \). To see what is the problem, suppose first that \( f_0, g_0 \in C^\infty(F) \) satisfy \( \text{supp} f_0 \subset F^{\text{int}} \) and \( \text{supp} g_0 \subset F^{\text{int}} \). Let \( f = \sum_{\gamma \in \Gamma} f_0 \circ \gamma \) and \( g = \sum_{\gamma \in \Gamma} g_0 \circ \gamma \) be the corresponding \( \Gamma \)-invariant function on \( \mathbb{D} \). Looking at kernels, we obtain that \((20), (17)\)
\[ \tau(T_{f_0}^2 - T_{g_0}^2) = \frac{1}{2} \left( t - \frac{1}{2} \right)^2 \int_{D \times F} (f(a) - f(b))^2 \delta'(a, b)d\mu_0 \]

and hence, by an application of the polarisation identity,
\[ \tau(T_{f_0} - T_{g_0}) = \left( \frac{t - 1}{2\pi i} \right)^2 \int_{D \times F} f(a)(g(a) - g(b))\delta'(a, b)d\mu_0. \]

Note that the right hand side is bounded by
\[ \frac{1}{2} \left( \frac{t - 1}{2\pi i} \right)^2 \int_{D \times F} |a - b|^2 \delta'(a, b)d\mu_0, \]
which is convergent by \([1]\).

Hence
\[ \tau([T_f, T_g]) = \left( \frac{t - 1}{2\pi i} \right)^2 \int_{D \times F} (f(a)g(b) - f(b)g(a))\delta'(a, b)d\mu_0, \]

Consequently, since the formula in Theorem 1 extends by continuity for functions \( f, g \) that are smooth and \( \Gamma \)-invariant (by replacing \( D \times D \) by \( \mathbb{D} \times F \)), it follows that, for such \( f \) and \( g \),
\[ \left( \frac{t - 1}{2\pi i} \right)^2 \int_{D \times F} (f(a)g(b) - f(b)g(a))\delta'(a, b)d\mu_0 = \frac{1}{2\pi i} \int_{\mathbb{D}} d(f_0)d\mu_0 = \frac{1}{2\pi i} \int_{F} df dg. \]

But it is not obvious from the outset neither that \( \tau([T_f, T_g]) \) is in the domain of \( \tau \) nor that its trace is approximated by the trace of commutators of Toeplitz operators associated to functions of the form
\[ f = \sum_{\gamma \in \Gamma} f_0 \circ \gamma \quad \text{and} \quad g = \sum_{\gamma \in \Gamma} g_0 \circ \gamma \quad \text{with} \quad f_0 \quad \text{and} \quad g_0 \quad \text{supported away from the boundary!} \]

**Theorem 3.** Let \( f \) and \( g \) be two smooth functions on \( \overline{M} \) (i.e. continuous with all their derivatives up to the boundary of \( M \)). We will use the same notation to denote their representatives as \( \Gamma \)-invariant functions on \( D \). Suppose that \( t > 5 \). Then \([T_f, T_g] \) is in \( \mathcal{M} \cap L^1(\tau) \) and
\[
\tau([T_f, T_g]) = \frac{1}{2\pi i} \int_{F} df dg.
\]

**Proof.**
Let us begin with the following observations.

1. Suppose that \( h_0, \ldots h_n \) is a finite family of functions on \( D \) satisfying the conditions of the proposition and we set \( A = T_{h_0} \ldots T_{h_n} \). Since
\[
|A*|^2 \leq \left( \prod_{i \neq 0} ||h_i||_{\infty} \right) T_{|h_0|^2},
\]
the averages \( \sum_{\gamma} \pi_t(\gamma)|A*|^2\pi_t(\gamma^{-1}) \) converge in the strong operator topology to \( E(|A*|^2) \). Moreover, for any normal linear functional \( \psi \) on \( \mathcal{B}(\mathcal{H}_t) \),
\[
\sum_{\gamma} \psi(\pi_t(\gamma)|A*|^2\pi_t(\gamma^{-1})) = \psi(E(|A*|^2)).
\]

To see the equality it is sufficient to consider positive \( \psi \), but then all that is involved is an exchange of the order of summation for a double series consisting of positive terms.

2. There exists a smooth partition of unity of \( D \) of the form \( \sum \phi_i \) where \( \phi_i \) are smooth, positive functions such that, for each \( i \), the family of functions \( \{ \phi_i \circ \gamma \}_{\gamma \in \Gamma} \) is locally finite. To see this it is sufficient to notice that, for any disc \( D_\epsilon = \{ z \in D ||z| \leq \epsilon \} \) with \( \epsilon < 1 \), the number of \( \gamma \in \Gamma \) such that \( \gamma(D_\epsilon) \cap D_\epsilon \neq \emptyset \) is finite. This follows from the fact that \( \Gamma \) is discrete as a subgroup of \( \text{PSL}(2, \mathbb{R}) \).

This implies that the set of normal linear functionals \( m \rightarrow Tr(BmA*) \) with \( A \) and \( B \) as above is total in \( \mathcal{B}(\mathcal{H}_t)_* \).

The proof of the theorem will be done in two steps.

**First part.**
Suppose that we are given a fundamental domain \( F \) for \( \Gamma \) such that
\[
f = \sum_{\gamma} f_0 \circ \gamma, \quad g = \sum_{\gamma} g_0 \circ \gamma
\]
where \( f_0 \) and \( g_0 \) are both smooth on \( \mathbb{D} \) and their supports have positive Euclidean distance to the complement of \( F \) in \( \mathbb{D} \).

We will compute

\[
Tr(A^*[T_f, T_g]A)
\]

where \( A = T_{h_0} \ldots T_{h_n} \). The operator under trace has a smooth kernel and the integral of its restriction to the diagonal has (up to a constant) the form

\[
\int_{D} d\mu_t(d) \int_{D} d\mu_t(c) \int_{D} d\mu_t(b) \int_{D} d\mu_t(a) \frac{A(a, d)(f(b)g(c) - g(b)f(c))}{(1 - \bar{a}b)^t(1 - bc)^t(1 - \bar{c}d)^t(1 - da)^t}
\]

where \( A(a, d) \) is a smooth kernel with support of strictly positive euclidean distance from \( \partial \mathbb{D} \times \mathbb{D} \cup \mathbb{D} \times \partial \mathbb{D} \). Hence the function

\[
F(a, b, c, d) = (1 - |b|^2)^{t/2}(1 - |c|^2)^{t/2} \frac{A(a, d)(f(b)g(c) - g(b)f(c))}{(1 - \bar{a}b)^t(1 - bc)^t(1 - \bar{c}d)^t(1 - da)^t}
\]

is uniformly bounded on \( \mathbb{D}^4 \) (the only singularity in the denominator appears for \( b = c \) and it is controlled by the fact that \( |\delta| \leq 1 \)) and our integral can be written as the integral of \( L^\infty \)-function \( F(a, b, c, d) \) with respect to the finite measure

\[
d\omega = d\mu_t \otimes d\mu_{t/2} \otimes d\mu_{t/2} \otimes d\mu_t.
\]

Hence

\[
\int Fd\omega = \sum_{\gamma \in \Gamma} \int_{D \times \gamma(F) \times D \times D} Fd\omega,
\]

i. e.

\[
Tr(A^*[T_f, T_g]A) = \sum_{\gamma} Tr(A^*[T_{f_0}^\gamma T_g - T_{g_0}^\gamma T_f]A).
\]

Since \( T_{f_0}^\gamma T_g - T_{g_0}^\gamma T_f \) is of trace class, we can exchange the summation over \( \gamma \in \Gamma \) with the trace and get the identity

\[
Tr(A^*[T_f, T_g]A) = Tr \left( \sum_{\gamma} \pi_t(\gamma)AA^*\pi_t(\gamma)^{-1}(T_{f_0}^\gamma T_g - T_{g_0}^\gamma T_f) \right).
\]

But this shows that

\[
\tau(E(|A^*|^2)[T_f, T_g]) = \tau(E(|A^*|^2)E(T_{f_0} T_g - T_{g_0} T_f)).
\]

Since for any \( m \in \mathcal{A}_t \)

\[
\tau(E(AA^*)m) = Tr((AA^*)m)
\]

and the set of such linear functionals is separating for \( \mathcal{B}(\mathcal{H}_t) \), we get

\[
E(T_{f_0} T_g - T_{g_0} T_f) = E(T_f T_{g_0} - T_{g_0} T_f) = [T_f, T_g].
\]

But, since

\[
T_f T_{g_0} - T_{g} T_{f_0} = [T_{f_0}, T_{g_0}] + T_{f-f_0} T_{g_0} - T_{g-g_0} T_{f_0}
\]
is of trace class, \([T_f, T_g]\) is in \(\mathcal{M} \cap \mathcal{L}^1(\tau)\) and

\[
\tau([T_f, T_g]) = \frac{1}{2} \text{Tr}([T_{f_0}, T_g] - [T_{g_0}, T_f])
\]

By the theorem 3

\[
\tau([T_f, T_g]) = \frac{1}{2\pi i} \int_{\Sigma} df_0 dg = \frac{1}{2\pi i} \int_{F} df dg.
\]

Second part.

By the proposition 1 we can assume that both \(f\) and \(g\) are, as functions on \(\overline{M}\), supported on a neighbourhood of the boundary of \(\overline{M}\) diffeomorphic to \((\cup_i C_i) \times ] - \epsilon, 0]\). Now, using partition of unity, we can split both \(f\) and \(g\) into finite sums

\[
f = \sum_k f_k, \quad g = \sum_s g_s
\]

so that for any pair of indices \((k, s)\) there are open intervals \(I_{k,s}^i\) of non-zero length on each of the boundary components \(C_i\) such that both \(f_k\) and \(g_s\) vanish on \((\cup_i I_{k,s}^i \times] - \epsilon, 0]\) - possibly with a smaller, but still positive value of \(\epsilon\). But then, choosing a cut of \(M\) given by a choice of points \(P_i\) in the interior of \(I_{k,s}^i\) will provide us with a fundamental domain \(F_{k,s}\) such that the conditions of the first part of this proof hold for \((f_s, g_k, F_{k,s})\) and hence \([T_{f_k}, T_{g_s}] \in \mathcal{L}^1(\tau)\) and

\[
\tau([T_{f_k}, T_{g_s}]) = \frac{1}{2\pi i} \int_{F_{k,s}} df dg
\]

To complete the proof note that the expression \(df dg\) for \(\Gamma\)-invariant functions is \(\Gamma\)-invariant, hence the integral \(\int_{F} df dg\) is independent on the choice of the fundamental domain and the result follows.

**Corollary 1.** Suppose that \(f\) and \(g\) are smooth functions on \(\overline{M}\). Then

\[
\tau([T_f, T_g]) = \int_{\partial M} f dg.
\]

Proof. This follows immediately from the fact that under the natural diffeomorphism \(F \setminus \partial F\) the integral \(\int_{F} df dg\) becomes identified with \(\int_{M} df dg\) and the Stokes theorem.

**Remarks**

1. Virtually the same proof shows that, for \(f\) and \(g\) smooth on \(\overline{M}\), the operator \(T_f T_g - T_{fg}\) is in \(\mathcal{M} \cap \mathcal{L}^1(\tau)\) and

\[
\tau(T_f T_g - T_{fg}) = \left(\frac{t - 1}{2\pi i}\right)^2 \int_{\Sigma \times F} f(a)(g(b) - g(a))\delta^1(a, b)d\mu_0(a, b).
\]
2. All of the results above can be easily extended to the case when \( f \) and \( g \) are in \( L^\infty(M) \) and Lipschitz with exponent one in a tubular neighbourhood of \( \partial M \) in \( \overline{M} \).

4. \( \Gamma \)-Fredholm operators

Let \( \mathcal{T}_\Gamma \) denote the \( C^* \)-subalgebra of \( \mathcal{A}_t \) generated by Toeplitz operators \( T_f \) with Toeplitz symbol \( f \in \mathcal{C}(\overline{M}) \) and denote by \( \mathcal{K}_\Gamma \) the \( C^* \)-ideal generated by elements in \( \mathcal{L}^1(\Gamma) \cap \mathcal{T}_\Gamma \). An element \( A \) of \( \mathcal{A}_t \) is called \( \Gamma \)-Fredholm if it has an inverse, say \( R \), modulo \( \mathcal{K}_\Gamma \) and, in this case, the commutator \( [A, R] \) has well-defined trace

\[ \Gamma \text{-index of } A = \tau([A, R]) \]

which depends only on the class of \( A \) in \( K_1(\mathcal{T}_\Gamma/\mathcal{K}_\Gamma) \).

**Remark 2.** The number “\( \Gamma \)-index of \( A \)” is also known as *Brauer index* of \( A \).

According to the proposition 1, there exists a surjective continuous map

\[ q : C(\partial M) \to \mathcal{T}_\Gamma/\mathcal{K}_\Gamma \]

sending function \( f|_{\partial M} \) to \( T_f \) mod \( \mathcal{K}_\Gamma \) - this map is well defined since \( ||T_f|| = ||f||_\infty \).

**Theorem 4.** Let \( \Gamma \) be a countable, discrete, icc subgroup of \( PSL(2,\mathbb{R}) \) such that \( \mathbb{D}/\Gamma \) has infinite covolume and \( M = \mathbb{D}/\Gamma \) is an open Riemannian surface with finitely many boundary components. Assume that \( t > 5 \) and that the trace \( \tau \) on \( \mathcal{A}_t \) is normalized by its value on a Toeplitz operators \( T_f \) with symbols \( f \in \mathcal{C}_c^\infty(M) \) by

\[ \tau(T_f) = \frac{t-1}{2\pi i} \int_F f(z)d\mu_0(z). \]

The following holds.

1. For any function \( f \in C(\overline{M}) \) such that \( f|_{\partial M} \) is invertible, the operator \( T_f \) is \( \Gamma \)-Fredholm and its \( \Gamma \)-index is equal to the sum of the winding numbers of restriction of \( f \) to the boundary of \( M \).
2. The map \( q : C(\partial M) \to \mathcal{T}_\Gamma/\mathcal{K}_\Gamma \) is injective and yields a nontrivial extension

\[ 0 \to \mathcal{K}_\Gamma \to \mathcal{T}_\Gamma \to C(\partial M) \to 0. \]

**Proof.**

**Step 1.**

Suppose that \( f \in C^\infty(\overline{M}) \) be invertible on the boundary of \( M \). Then, for any function \( g \) smooth in the closure of \( M \), and such that
supp(1 - fg) ⊂ M the theorem 3 gives 1 - T_f T_g ∈ K_Γ and hence T_f is Γ-Fredholm.

Let now P and Q be two non-commutative polynomials in (¯z, z). Then, again by the theorem 3 and its corollary,

$$\tau([P(T_f^*, T_f), Q(T_f^*, T_f)]) = \tau([T_{P(T_f^*, T_f)}, T_{Q(T_f^*, T_f)}) = \frac{1}{2\pi i} \int_{\partial M} P(\bar{f}, f) dQ(\bar{f}, f).$$

On the other hand, by Carey-Pincus formula for traces of commutators (see [7]),

$$\tau([P(T_f^*, T_f), Q(T_f^*, T_f)]) = \int_{|z| < ||T_f||} \{P, Q\} d\nu$$

where d\nu is a finite measure supported on the convex hull of the essential spectrum of T_f mod(K_Γ) and, since T_f is Γ-Fredholm, there exists an open ball B_\epsilon around the origin such that d\nu|_{B_\epsilon} = c d\lambda, where $2\pi i c = \Gamma$-index of T_f.

If we set d\nu = d\nu|_{B_\epsilon} + d\nu_1, the two formulas above give

$$\frac{1}{2\pi i} \int_{\partial M} P(\bar{f}, f) dQ(\bar{f}, f) = \frac{1}{2\pi i} \Gamma\text{-index of } T_f \int_{|z| = \epsilon} dPdQ + \int_{\epsilon \leq |z| \leq ||T_f||} \{P, Q\} d\nu_1.$$ 

Applying Stokes theorem, we get the equality

$$\frac{1}{2\pi i} \int_{\partial M} P(\bar{f}, f) dQ(\bar{f}, f) = \frac{1}{2\pi i} (\Gamma\text{-index of } T_f) \int_{|z| = \epsilon} PdQ + \int_{\epsilon \leq |z| \leq ||T_f||} \{P, Q\} d\nu_1.$$ 

If we now set P(\bar{z}, z) = z and approximate $\frac{1}{z}$ uniformly on the annulus $\epsilon \leq |z| \leq ||T_f||$ by polynomials Q, since both sides are continuous in the uniform topology on $C(\epsilon \leq |z| \leq ||T_f||)$ we get, in the limit,

$$\frac{1}{2\pi i} \int_{\partial M} f^{-1} df = \frac{1}{2\pi i} (\Gamma\text{-index of } T_f) \int_{|z| = \epsilon} z^{-1} dz = (\Gamma\text{-index of } T_f).$$

**Step 2.**

We will now prove injectivity of q. To this end it is enough to show that, for any open interval I ⊂ ∂M, we can find a function in C(M) which is zero when restricted to ∂M \ I but for which the corresponding toeplitz operator T_f is not in K_Γ. But given such an interval, we can easily find a smooth function f such that f|_{\partial M} is supported within I and such that the winding number of 1 + f on ∂M is nonzero. But then, by step 1 above, the Γ-index of 1 + T_f is nonzero and hence T_f is not an element of K_Γ, which proves the second part of the theorem.

Now, let f be continuous and with invertible restriction to the boundary of M. By part two of the theorem, this implies that the image of T_f in T_Γ/K_Γ is f|_{\partial M} and hence invertible, i.e. T_f is Γ-Fredholm, and the formula for its Γ-index follows from the fact that it is a functional
on $K_1(\mathcal{T}_\Gamma/K_\Gamma)$ and hence homotopy invariant of the class of $f|_{\partial M}$ in $K_1(C(\partial M))$.

The normalisation statement follows immediately from the proposition 1.

References

[1] Arazy, J., Fisher, S., Peetre, J., Hankel operators on weighted Bergman spaces, Amer. J. Math. 110 (1988), no. 6, 989–1053.
[2] Atiyah, M. F. Elliptic operators, discrete groups and von Neumann algebras. Colloque "Analyse et Topologie" en l’Honneur de Henri Cartan (Orsay, 1974), pp. 43–72. Asterisque, No. 32-33, Soc. Math. France, Paris, 1976.
[3] M.F. Atiyah, W. Schmidt, A geometric construction of the discrete series for semisimple Lie groups, Invent. Math., 42, (1977), 1-62.
[4] F. A. Berezin, General concept of quantization, Comm. Math. Phys., 40 (1975), 153-174.
[5] P. Bressler, R. Nest and B. Tsygan, Riemann-Roch theorems via deformation quantization, I.Math. Res. Letters 20, (1997), page 1033.
[6] Breuer, M. Fredholm theories in von Neumann algebras. II, Math. Ann. 180 1969 313–325.
[7] Carey, R. W.; Pincus, J. Mosaics, principal functions, and mean motion in von Neumann algebras, Acta Math. 138 (1977), no. 3-4, 153–218.
[8] Carey, R. W.; Pincus, J., An invariant for certain operator algebras, Proc. Nat. Acad. Sci. U.S.A. 71 (1974), 1952–1956.
[9] A. Connes, Non-commutative Differential Geometry, Publ. Math., Inst. Hautes Etud. Sci., 62, (1986), 94-144.
[10] Connes, A., On the spatial theory of von Neumann algebras, J. Funct. Anal. 35 (1980), no. 2, 153–164.
[11] A. Connes, M. Flato, D. Sternheimer, Closed star products and Cyclic Cohomology, Letters in Math. Physics, 24, (1992), 1-12.
[12] M. Enock, R. Nest, Irreducible inclusions of factors, multiplicative unitaries, and Kac algebras, J. Funct. Anal. 137 (1996), no. 2, 466–543.
[13] F. Goodman, P. de la Harpe, V.F.R. Jones, Coxeter Graphs and Towers of Algebras, Springer Verlag, New York, Berlin, Heidelberg, 1989.
[14] U. Haagerup, Operator-valued weights in von Neumann algebras. I., J. Funct. Anal. 32 (1979), no. 2, 175–206.
[15] W. J. Helton, R. Howe, Traces of commutators of integral operators. Acta Math. 135 (1975), no. 3-4, 271–305.
[16] F. J. Murray, J. von Neumann, On ring of Operators,IV, Annals of Mathematics, 44, (1943), 716-808.
[17] R. Nest, T. Natsume, Topological approach to quantum surfaces, Comm. Math. Phys. 202 (1999), no. 1, 65–87.
[18] R. Nest, B. Tsygan, Algebraic index theorem for families, Adv. Math. 113 (1995), no. 2, 151–205.
[19] Puknkszy, L., On the Plancherel theorem of the $2 \times 2$ real unimodular group, Bull. Amer. Math. Soc. 69 1963 504–512.
[20] F. Rădulescu, The $\Gamma$-equivariant form of the Berezin quantization of the upper half plane, Mem. Amer. Math. Soc. 133 (1998), no. 630,
[21] P. Sally, Analytic continuation of the irreducible unitary representations of the universal covering group of $\text{SL}(2, \mathbb{R})$, Memoirs of the American Mathematical Society, No. 69 American Mathematical Society, Providence, R. I. 196

[22] D. Voiculescu, Circular and semicircular systems and free product factors. In Operator Algebras, Unitary Representations, Enveloping algebras and Invariant Theory. *Prog. Math. Boston, Birkhauser*, 92, (1990), 45-60.

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