On the Lagrange and Markov Dynamical Spectra for Geodesic Flows in Surfaces with Negative Curvature

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Abstract
We consider the Lagrange and the Markov dynamical spectra associated with the geodesic flow on surfaces of negative curvature. We show that for a large set of real functions on the unit tangent bundle and typical metrics with negative curvature and finite volume, both the Lagrange and the Markov dynamical spectra have non-empty interiors.

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1 Introduction

The classical Lagrange spectrum from the theory of Diophantine approximations can be defined as follows: We associate, to each irrational number $\alpha$, its best constant of approximation (Lagrange value of $\alpha$), given by

$$k(\alpha) = \sup \left\{ k > 0 : \left| \alpha - \frac{p}{q} \right| < \frac{1}{kq^2} \text{ has infinite rational solutions } \frac{p}{q} \right\} = \lim_{p,q \to \infty} \inf_{p,q \in \mathbb{Z}} \left\{ q|q\alpha-p|^{-1} \in \mathbb{R} \cup \{+\infty\} \right\}.$$

The Lagrange spectrum is the set

$$\mathbb{L} = \{ k(\alpha) : \alpha \in \mathbb{R} \setminus \mathbb{Q} \text{ and } k(\alpha) < \infty \}.$$ 

Another interesting set related to Diophantine approximations is the classical Markov spectrum defined by

$$\mathbb{M} = \left\{ \left( \inf_{(x,y) \neq (0,0)} |f(x,y)| \right)^{-1} : f(x,y) = ax^2 + bxy + cy^2 \text{ with } b^2 - 4ac = 1 \right\}. \quad (1)$$

Perron (see, for instance, [CF89]) gave a dynamical characterization of the Lagrange and Markov spectra, as follows:
Let $\Sigma = (\mathbb{N}^*)^\mathbb{Z}$ and $\sigma: \Sigma \to \Sigma$ the shift defined by $\sigma((a_n)_{n \in \mathbb{Z}}) = (a_{n+1})_{n \in \mathbb{Z}}$. If $f: \Sigma \to \mathbb{R}$ is defined by $f((a_n)_{n \in \mathbb{Z}}) = \alpha_0 + \beta_0 = [a_0, a_1, \ldots] + [0, a_{-1}, a_{-2}, \ldots]$, then

$$L = \left\{ \lim sup_{n \to \infty} f(\sigma^n(\theta)) : \theta \in \Sigma \right\}$$

and

$$M = \left\{ \sup_{n \in \mathbb{Z}} f(\sigma^n(\theta)) : \theta \in \Sigma \right\}.$$ 

As a consequence of this characterization it follows that $L$ and $M$ are closed sets of real numbers and $L \subset M$.

In 1947, M. Hall [Hal47] proved that $L$ (and thus also $M$) contains a whole half-line (for instance $[6, +\infty)$).

In 1975, G. Freiman (cf. [Fre75] and [CF89]) determined the precise beginning of Hall’s ray (the biggest half-line contained in $L$, which turns out to be also the biggest half-line contained in $M$), which is

$$\frac{2221564096 + 283748\sqrt{462}}{491993569} \approx 4,52782956616 \ldots .$$

Both Hall’s and Freiman’s results are directly related to the study of arithmetic sums of regular Cantor sets.

Perron characterization of the classical Markov and Lagrange spectra inspired a more general notion of dynamical Markov and Lagrange spectra (see for instance [MR17]):

Let $M$ be a smooth manifold, $T = \mathbb{Z}$ or $\mathbb{R}$, and $\phi = (\phi^t)_{t \in T}$ be a discrete-time ($T = \mathbb{Z}$) or continuous-time ($T = \mathbb{R}$) smooth dynamical system on $M$, that is, $\phi^t : M \to M$ is a smooth diffeomorphism, where $\phi^0 = \text{id}$, and $\phi^t \circ \phi^s = \phi^{t+s}$ for all $t, s \in T$. Given a compact invariant subset $\Lambda \subset M$ and a function $f: M \to \mathbb{R}$, we define the dynamical Markov spectrum, denoted by $M(\phi, \Lambda, f)$, Lagrange spectrum, denoted by $L(\phi, \Lambda, f)$ as

$$M(\phi, \Lambda, f) = \{ m_{\phi, f}(x) : x \in \Lambda \}, \quad L(\phi, \Lambda, f) = \{ \ell_{\phi, f}(x) : x \in \Lambda \}$$

where

$$m_{\phi, f}(x) := \sup_{t \in \mathbb{R}} f(\phi^t(x)), \quad \ell_{\phi, f}(x) := \lim sup_{t \to +\infty} f(\phi^t(x)).$$

It is not difficult to show that $L(\phi, \Lambda, f) \subset M(\phi, \Lambda, f)$ [MR17].

When $\Lambda$ is the whole manifold, we denote $M(\phi, M, f) = M(\phi, f)$ and $L(\phi, M, f) = L(\phi, f)$.

A class of dynamical spectra which is closely related to the classical ones is given by spectra associated to geodesic flows on surfaces of constant negative curvature. In 1986, A. Haas [Haa86] proved the existence of an analogous to Hall’s ray (a half-line contained in the dynamical spectra) for the geodesic flow on the quotient of $\mathbb{H}^2$ by a Fuchsian group of $SL(2, \mathbb{R})$ and A. Haas and C. Series for Hecke group [HS86], this last related with Lagrange spectra and cusp excursions in modular surface (cf. [LMMR20]). In 1997, Thomas A. Schmidt and Mark Sheigorn [SS97] proved that Riemann surfaces have Hall’s
ray in every cusp. Recently, in [AMU20] it was proved that Hall’s rays are persistent on each cusp of a Riemann surface (so in constant negative curvature).

The existence of Hall’s rays for dynamical spectra was also proved in other contexts. In 2015, P. Hubert, L. Marchese, and C. Ulcigrai [HMU15] showed the existence of Hall’s ray in the context of Teichmüller dynamics, more precisely for moduli surfaces, using renormalization. In 2016, M. Artigiani, L. Marchese, C. Ulcigrai [AMU16] showed that Veech surfaces also have a Hall’s ray. In the context of non-archimedean quadratic Lagrange spectra, some new results is this direction can be found in [PP20] and [Bug14].

For manifolds of finite volume, variable negative curvature and dimension greater than or equal to 3, in [PP10] it was proved the existence of Hall’s ray for the height function associated with an end of the manifold (see the beginning of Section 7 for more details of the height function). In the same paper [PP10, page 278], J. Parkkonen and F. Paulin expected that the existence of Hall’s ray could be false for surfaces of variable negative curvature.

This paper is inspired by the question:

**Question:** Is there Hall’s ray for surfaces of variable negative curvature and finite volume?

In this work, we show a positive result in the direction of this question: we prove that these spectra have typically non-empty interiors for surfaces of variable negative curvature and finite volume.

The continuity of the Hausdorff dimension of the intersection of the spectra with the half-line $(-\infty, t)$ was studied at [CMR21] for geodesic flows on surfaces of negative curvature, generalizing results of [Mor18] and [CMM18].

We refer to the book [CF89] for results on the classical spectra and to the book [LMMR20] for more recent results on classical and dynamical spectra.

In [Rn21], the second author studied Markov and Lagrange dynamical spectra for Anosov flows on manifolds of dimension 3. It has been proved that for suitable small perturbations of the flow the dynamical Lagrange and Markov spectra have persistently non-empty interior for typical real functions. Although surfaces of pinched negative curvature have geodesic flows of the Anosov type, the perturbations constructed in [Rn21] are not necessarily perturbations by geodesic flows. Indeed, even a small local perturbation of a Riemannian metric in a neighborhood of a point affects the corresponding geodesic flow in the whole fibers of the unit tangent bundle over this neighborhood. Thus, in this paper, we study the Lagrange and Markov spectra for the geodesic flow of surfaces of pinched negative curvature. More specifically, we pursue two goals. The first goal is to extend the result in [Rn21] in the context of geodesic flows (see Theorem 1.1 and Theorem 1.2 for details), which requires sophisticated perturbation techniques of Riemannian metrics. The second goal is to obtain a version of Theorem 1.2 for the height function, associated with an end of the manifold (see Theorem 1.3).
1.1 Statement of the Main Results

Let $M$ be a complete surface, and let $g_0$ be a smooth $(C^r, r \geq 3)$ pinched negatively curved Riemannian metric on $M$, i.e., the curvature is bounded above and below by two negative constants and finite volume. Let $\phi = (\phi^t_{g_0})_{t \in \mathbb{R}}$ be the geodesic flow on the unit tangent bundle $S^{g_0}M$ of $M$ with respect to $g_0$. We denote $\mathcal{G}^3(M)$ the space of $C^3$ Riemannian metrics on $M$. For a metric $g \in \mathcal{G}^3(M)$ we denote by $S^gM$ the unit tangent bundle of the metric $g$ and $\phi_g$ is the vector field derivative of the geodesic flow $(\phi^t_g)_{t \in \mathbb{R}}$ of the metric $g$. Moreover, the above statement holds persistently: for any $g \in \mathcal{G}^3(M)$ we denote by $S^gM$ the unit tangent bundle of the metric $g$ and $\phi_g$ is the vector field derivative of the geodesic flow $(\phi^t_g)_{t \in \mathbb{R}}$ of the metric $g$.

In the sequel, given $A \subset \mathbb{R}$, int $A$ denotes the interior of $A$.

Our first result in this context is:

**Theorem 1.1.** Arbitrarily close to $g_0$ there is an open set $\mathcal{G} \subset \mathcal{G}^3(M)$ such that for any $g \in \mathcal{G}$, there exists a dense and $C^2$-open subset $\mathcal{H}_g \subset C^2(S^gM, \mathbb{R})$ so that

$$\text{int } \mathcal{M}(\phi_g, f) \neq \emptyset \quad \text{and} \quad \text{int } \mathcal{L}(\phi_g, f) \neq \emptyset,$$

whenever $f \in \mathcal{H}_g$.

Moreover, the above statement holds persistently: for any $\tilde{g} \in \mathcal{G}$, it holds for any $(f, g)$ in a suitable neighborhood of $\mathcal{H}_{\tilde{g}} \times \{\tilde{g}\}$ in $C^2(M, \mathbb{R}) \times \mathcal{G}^3(M)$.

The next result is a version of Theorem 1.1 for the restricted case, where the set of functions is a composition of functions on the manifold $M$ with the canonical projection.

**Theorem 1.2.** Arbitrarily close to $g_0$ there is an open set $\mathcal{G} \subset \mathcal{G}^3(M)$ such that for any $g \in \mathcal{G}$, there exists a dense and $C^2$-open subset $\tilde{\mathcal{H}}_g \subset C^2(M, \mathbb{R})$ so that

$$\text{int } \mathcal{M}(\phi_g, f \circ \pi) \neq \emptyset \quad \text{and} \quad \text{int } \mathcal{L}(\phi_g, f \circ \pi) \neq \emptyset,$$

whenever $f \in \tilde{\mathcal{H}}_g$. Here $\pi: S^gM \to M$ is the canonical projection. Moreover, the above statement holds persistently: for any $\tilde{g} \in \mathcal{G}$, it also holds for any $(f, g)$ in a suitable neighborhood of $\tilde{\mathcal{H}}_{\tilde{g}} \times \{\tilde{g}\}$ in $C^2(M, \mathbb{R}) \times \mathcal{G}^3(M)$.

The final result presented in this paper states that for small perturbations of the metric, the Lagrange and Markov spectra defined in [PP10] contain intervals. More precisely, we can state this as the following theorem.

**Theorem 1.3.** Let $M$ be a complete and non-compact surface. Let $e$ be an end of $M$. Then arbitrarily close to $g$ there exists an open set $\tilde{\mathcal{G}} \subset \mathcal{G}^3(M)$ such that for any $\tilde{g} \in \tilde{\mathcal{G}}$,

$$\text{int } \mathcal{M}(\phi_{\tilde{g}}, h_{\tilde{e}} \circ \pi) \neq \emptyset \quad \text{and} \quad \text{int } \mathcal{L}(\phi_{\tilde{g}}, h_{\tilde{e}} \circ \pi) \neq \emptyset,$$

where $h_{\tilde{e}}$ is the height function associated to end $e$ with the metric $\tilde{g}$.

The problem of finding intervals in the classical Lagrange and Markov spectra is closely related to the study of the fractal geometry of regular Cantor sets related to the Gauss map. However, the results about the existence of Hall’s ray cited above do not involve fractal geometry or the study of regular Cantor sets.

In the present study of the two-dimensional spectra, recent results on fractal geometry of regular Cantor sets are (again) a key ingredient in the proof of our results about
dynamical Lagrange and Markov spectra associated with geodesic flows on surfaces of negative curvature. We use and adapt in this work techniques from [MY01], [MY10], [MR17], and [Rn21].

The previous theorems are far from being direct consequences of [MR17] and [Rn21]. In fact, the proofs of the previous results depend strongly on local perturbations of diffeomorphisms or vector fields. In the present situation, we need to perform perturbations of the Riemannian metrics, which produce necessarily global modifications of the geodesic field - we need to develop delicate combinatorial and geometrical techniques in order to emulate local perturbations of the geodesic flow near hyperbolic sets by perturbing the Riemannian metrics.

Indeed, as it is well known, perturbation results within the set of Riemannian metrics are usually hard, basically because when we change the metric in a neighborhood of a point of the manifold we affect all the geodesics intersecting this neighborhood; in other words, even if the size of our neighborhood in the manifold is small, the effect of the perturbation in the unit tangent bundle could be large. However, the main difficulty in this part of the work is to produce, just by performing small perturbations on the metrics, a sufficiently rich family of perturbations (with “enough independence”) of a horseshoe invariant by a Poincaré map associated with the geodesic flow in order to produce, via the probabilistic method of [MY10], stable intersections of the stable Cantor set of this horseshoe and the unstable Cantor set of another invariant horseshoe (see Subsections 5.1.1 and A.2 for the definition of regular Cantor sets and the stable intersection of two Regular Cantor sets). We develop new techniques of Fractal Geometry of combinatorial nature to address this problem.

The paper is organized as follows: In Section 2 we recall some classical results of hyperbolic dynamics which are relevant to our work. In Section 3 we use a construction from [Rn21, Lemma 4.10, and Corollary 5] to get a basic set for the geodesic flow with Hausdorff dimension close enough to 3. Using this basic set, we construct a finite number of (disjoint) cross sections to the geodesic flow, and we show that the Poincaré (first return) map of the union of sections has a basic set - a horseshoe - with Hausdorff dimension close enough to 2 (see [Rn21, Section 3]). The last arguments allow us to reduce Theorem 1.1 and Theorem 1.2 to a bi-dimensional version (see Theorem 3.9 and Theorem 3.10). In Section 4 we construct explicitly the set $H_g$ and $\tilde{H}_g$, which has a non-trivial construction. In Section 5 we develop techniques of perturbations of Riemannian metrics together with further combinatorial techniques in order to adapt constructions of [MY10] in the context of our work, which most important and hardest part of this paper. In Section 6 using the results of Section 3, [MR17, MY01, MY10] and some combinatorial techniques develop in [Rn21, Section 4.3.2], we prove Theorem 3.9 and Theorem 3.10. Finally, in Section 7 we prove Theorem 1.3.

2 Preliminaries

Let $N$ be a complete Riemannian manifold and $\phi^t : N \to N$ a $C^r$-flow on $N$. We say that an invariant set $\Lambda \subset N$ is hyperbolic for $\phi^t$ if: there exists a continuous splitting
\[ T_\Lambda N = E^s \oplus \phi \oplus E^u \] such that for each \( \theta \in \Lambda \)

\[
\begin{align*}
d\phi_\theta'(E^s(\theta)) &= E^s(\phi'(\theta)), \\
d\phi_\theta'(E^u(\theta)) &= E^u(\phi'(\theta)), \\
||D\phi_\theta'|_{E^s}|| &\leq C\lambda^t, \\
||D\phi_\theta^{-t}|_{E^u}|| &\leq C\lambda^t,
\end{align*}
\]

for all \( t \geq 0 \) with \( C > 0 \) and \( 0 < \lambda < 1 \), where \( \phi \) is the vector field derivative of the geodesic vector flow.

The bundles \( E^s \) and \( E^u \) are called stable and unstable bundles, respectively. They are known to be uniquely integrable (see [KH95]).

A hyperbolic set \( \Lambda \) is called a basic set if satisfies the following three properties:

(a) the set of periodic orbits contained in \( \Lambda \) are dense in \( \Lambda \),

(b) \( \phi^t \) is transitive,

(c) There is an open set \( U \supset \Lambda \) so that \( \Lambda = \bigcap_{t \in \mathbb{R}} \phi^t(U) \).

Basic sets play an important role in this paper because they have well-understood fractal geometry. For diffeomorphisms on a surface, no trivial basic sets are also called horseshoe.

When the entire manifold \( N \) is a hyperbolic set, then we said that the flow is an Anosov flow. The central examples of Anosov flows, which will be treated in this work, are provided by geodesic flows.

### 2.1 Geodesic Flow

Given a Riemannian manifold \( M \), denoted by \( TM \) the tangent bundle and \( SM = \{(x, v) \in TM : \|v\| = 1\} \) the unit tangent bundle of \( M \). For \( \theta = (p, v) \in SM \), we denoted by \( \gamma_\theta(t) \) the unique geodesic with initial conditions \( \gamma_\theta(0) = p \) and \( \gamma_\theta'(0) = v \). The family of diffeomorphisms \( \phi^t : SM \to SM \) given by \( \phi^t(\theta) = (\gamma_\theta(t), \gamma_\theta'(t)) \) and satisfying \( \phi^{t+s} = \phi^t \circ \phi^s \) for all \( t, s \in \mathbb{R} \) is called geodesic flow. We endowed \( SM \) with the Sakaki metric (see Subsection A.4.1).

A classic result due to D. Anosov (cf. [Ano69], [Kli82] and [KH95]) states that complete manifolds of curvature bounded between two negative constants (pinched negative curvature) have Anosov geodesic flow. Moreover, in this condition, if \( M \) has finite volume, then the non-wandering set of the geodesic flow is equal to \( SM \), and \( \phi^t \) is transitive. (cf. [Pat99] and [Kli82, chapter 3]).

For the purposes of this paper, from now on, we consider \( (M, g_0) \) a complete \( C^r, r \geq 3 \), a Riemannian surface with pinched negative curvature, \( SM \) the unit tangent bundle, which has dimension 3, and whose geodesic flow \( \phi^t = (\phi_{g_0}^t)_{t \in \mathbb{R}} \) is Anosov.

### 3 Hyperbolic Set for Two-dimensional Dynamic

The goal of this section is to reduce Theorem 1.1 and Theorem 1.2 to two-dimensional versions (see Theorem 3.9 and Theorem 3.10).
3.1 Dimension Reduction via Poincaré Maps

The geodesic flow of a complete surface of pinched negative curvature and finite volume carries many basic sets. In [Rn21] was proved that, in this case, the geodesic flow has a basic set with Hausdorff dimension close to 3.

**Lemma 3.1.** [Rn21, Corollary 5] There is a basic set $\Lambda$ for $\phi^t$ with Hausdorff dimension arbitrarily close to 3.

To reduce the dynamic of $\phi^t$ to two dimensional dynamical, we need the concept of Good Cross-Section introduced at [Rn21, Section 3].

**Definition 3.2.** Let $\Lambda$ be a compact subset of $M$.
We say that a compact cross-section $S$ is a Good Cross-Section (or simply GCS) for $\Lambda$ if
$$d(\Lambda \cap S, \partial S) > 0$$
where $d$ is the intrinsic distance in $S$.
If $S$ is a GCS and $x \in \Lambda \cap S$, we say that $S$ is a GCS at $x$.

The transitivity of the geodesic flow allows us to construct GCS’s for any points of a hyperbolic set (see [Rn21, Section 3] for more details). Furthermore, its specific constructions allow to prove the following lemma, which allows reducing the dynamics of $\Lambda$ to a Poincaré map (see also [Rn21, Remark 7]).

**Lemma 3.3.** [Rn21, Lemma 3.5] There are $\gamma > 0$ and smooth-GCS, $\Sigma_i$, $i = 1, \ldots, k$ such that
$$\Lambda \subset \bigcup_{i=1}^{k} \phi^{(-\gamma,\gamma)}(\text{int} \Sigma_i)$$
with $\Sigma_i \cap \Sigma_j = \emptyset$.

As was exploited in [Rn21, Section 3.2], we describe the dynamics of $\phi^t$ on $\Lambda$ in terms of Poincaré maps. More precisely, the GCS $\Sigma_i$, $1 \leq i \leq k$, of Lemma 3.3 was constructed such that the $\phi^t$-orbit of any point of $\Lambda$ intersects $\Sigma := \bigcup_{i=1}^{k} \Sigma_i$, the subset $\Delta := \Lambda \cap \Sigma$ is disjoint from the boundary $\partial \Sigma := \bigcup_{i=1}^{k} \partial \Sigma_i$. Consider the Poincaré (first return) map $\mathcal{R}: D_\mathcal{R} \to \Sigma$ from a neighborhood $D_\mathcal{R} \subset \Sigma$ of $\Delta$ to $\Sigma$ sending $y \in D_\mathcal{R}$ to the point $\mathcal{R}(y) = \phi^{t_0(y)}_0(y)$ where the forward $\phi^t$-orbit of $y$ first hits $\Sigma$, and $\mathcal{R}^{-1}(y) = \phi^{t_0(y)}_0(y)$, whenever $\mathcal{R}^{-1}(y)$ is defined. Since $\Lambda$ is a basic set for $\phi^t$, it is not difficult to prove that

**Lemma 3.4.** The set $\Delta$ is a basic set for $\mathcal{R}$.

The splitting $E^s \oplus \phi \oplus E^u$ over a neighborhood of $\Lambda$ defines a continuous splitting $E^s_{\Sigma} \oplus E^u_{\Sigma}$ of the tangent bundle $T\Sigma$ given by
$$E^s_{\Sigma}(y) = E^s_y \cap T_y \Sigma \text{ and } E^u_{\Sigma}(y) = E^u_y \cap T_y \Sigma,$$
where $E^s_y = E^s_y \oplus \text{span} \phi(y)$ and $E^u_y = E^u_y \oplus \text{span} \phi(y)$. Denote by $W^s_\mathcal{R}(x)$ the stable manifold and $W^u_\mathcal{R}(x)$ the unstable manifold of $x \in \Delta$ (see Appendix A.1).

The relation between the Hausdorff dimension of $\Delta$ and $\Lambda$ is described by the following lemma (compare with Lemma 3.8 in [Rn21]):
Lemma 3.5. [Rn21] Lemma 3.8] In the previous setting, one has $HD(\Lambda) = HD(\Delta) + 1$. In particular, $HD(\Delta) > 1$.

Let $V \subseteq \mathcal{G}(M)$ a neighborhood of $g_0$ such that for every $g \in V$, the hyperbolic set $\Lambda$ has a hyperbolic continuation $\Lambda_g$ for $\phi_g = (\phi^i_g)_{i \in \mathbb{R}}$, the geodesic flow of the metric $g$, and $HD(\Lambda_g) > 2$.

Given $g, g_1 \in V$, we define the diffeomorphism $S^g_{g_1} : S^g M \to S^{g_1} M$ by

$$S^g_{g_1}(x, v) = \left( x, \frac{v}{||v||_{g_1}} \right),$$

where $S^g M$ and $S^{g_1} M$ are the unit tangent bundle associated to metric $g$ and $g_1$, respectively. It is easy to check $(S^g_{g_1})^{-1} = S^g_{g_1}$. In order not to saturate the notation, whenever $g_1 = g_0$, we write $S^g := S^g_{g_0}$. In the following sense: Let $(x, v) \in S^g M$ a chart of $S^g M$, with $\psi_{g_0}(0) = (x, v)$ and $\psi_{g_1} : V \subset \mathbb{R}^3 \to S^{g_1} M$ a chart of $S^{g_1} M$ such that $\psi_{g_0}(0) = S^g_{g_1}(x, v)$, then $\psi_{g_1}^{-1} \circ S^g_{g_1} \circ \psi_{g_0} : U \to V$ is $C^{2}$-diffeomorphism close to the identity of $\mathbb{R}^3$.

Note also that, $J^g_{g_1}(t, \cdot) = S^g_{g_1} \circ \phi^i_g \circ S^{g_1}(\cdot)$ defines a flow on $S^{g_1} M$. Since $g$ and $g_1$ are $C^3$-close, then $J^g_{g_1}(t, \cdot) = C^2$-close to $\phi^i_g$, and therefore, the Poincaré map $\mathcal{R}_{g_1}^{g_1} : \Sigma_{g_1} \to \Sigma_{g_1}$, associated to $J^g_{g_1}$, is $C^2$-close to $R^g_{g_1}$.

If $g_0 = g_1$, then $\mathcal{R}_{g_0}^{g_1} : \Sigma \to \Sigma$ is $C^2$-close to $\mathcal{R}$ and we denoted $\tilde{\Delta}_g$ the hyperbolic continuation of $\Delta$ by $\mathcal{R}_{g_0}^{g_1}$, in fact, a basic set.

Lemma 3.6. The diffeomorphisms $\mathcal{R}_{g_0}^{g_1}$ and $\mathcal{R}^g$ are conjugate, i.e.,

$$\mathcal{T}^g \circ \mathcal{R}_{g_0}^{g_1} = \mathcal{R}^g \circ \mathcal{T}^g$$

Proof. Simply we observe that $S^g \circ \phi^i_g(x, v) = J^g_{g_0}(t, S^g(x, v))$ and therefore

$$\phi^i_g \circ \mathcal{T}^g = \mathcal{T}^g \circ J^g_{g_0}(t, \cdot).$$

The last lemma implies that the set $\Delta := \mathcal{T}^g(\tilde{\Delta}_g)$ is a basic set for $\mathcal{R}^g$. Without loss of generality, from now on, we call $\Delta_g$ the hyperbolic continuation of $\Delta$ by $\mathcal{R}^g$. If $V$ is small enough, then from the continuity of the Hausdorff dimension for basic sets (cf. [PT93]), we have that $HD(\Delta_g) > 1$. Also, given a basic set $\Theta$ for $\mathcal{R}$ we denote $\Theta_g$ the hyperbolic continuation of $\Theta$ by $\mathcal{R}^g$.

Remark 3.7. As mentioned before, if $g, g_1 \in V$, then $\mathcal{R}_{g_1}^{g_0} : \Sigma_{g_1} \to \Sigma_{g_1}$ is $C^2$-close to $\mathcal{R}^{g_1}$, then we call $\tilde{\Delta}_{g_1,g}$ the hyperbolic continuation of $\Delta_{g_1}$ by $\mathcal{R}_{g_1}^{g_0}$. Analogously, as Lemma 3.6, we have that $\mathcal{R}_{g_1}^{g_0}$ and $\mathcal{R}^g$ are conjugate and

$$S^{g_1}_{g_0} \circ \mathcal{R}_{g_1}^{g_0} = \mathcal{R}^g \circ S^{g_1}_{g_0},$$

which implies that $\Delta_{g_1,g} := S^{g_1}_{g_0}(\tilde{\Delta}_{g_1,g})$ is a basic set for $\mathcal{R}^g$ with $HD(\Delta_{g_1,g}) > 1$. 

\[9\]
From now on, we only consider the metrics on \( V \).

### 3.1.1 The dynamical Lagrange and Markov spectra of \( \Lambda \) and \( \Delta \)

The dynamical Lagrange and Markov spectra of \( \Lambda \) and \( \Delta \) are related in the following way. Given a function \( F \in C^s(SM, \mathbb{R}), s \geq 1 \), let us denote by \( f = \max F_\phi: D_R \to \mathbb{R} \) the function

\[
\max F_\phi(x) := \max_{0 \leq t \leq t_+} F(\phi^t(x)).
\]

**Remark 3.8.** We have that \( f = \max F_\phi \) might not be \( C^1 \) in general.

We see that

\[
\limsup_{n \to +\infty} f(\mathcal{R}^n(x)) = \limsup_{t \to +\infty} F(\phi^t(x)),
\]

and

\[
\sup_{n \in \mathbb{Z}} f(\mathcal{R}^n(x)) = \sup_{t \in \mathbb{R}} F(\phi^t(x))
\]

for all \( x \in \Delta \). In particular,

\[
\mathbb{L}(\phi, \Lambda, F) = \mathbb{L}(\mathcal{R}, \Delta, f) \quad \text{and} \quad \mathbb{M}(\phi, \Lambda, F) = \mathbb{M}(\mathcal{R}, \Delta, f).
\]

(4) Thus, the relation (4) reduces Theorem 1.1 and Theorem 1.2 to the following theorems:

**Theorem 3.9.** In the setting of Theorem 1.1, arbitrarily close to \( g_0 \) there is an open set \( G \subset \mathcal{G}^3(M) \) such that for any \( g \in G \) one can find a dense and open \( C^2 \)-open subset \( \mathcal{H}_{g,\Lambda} \subset C^2(S^gM, \mathbb{R}) \), so that

\[
\operatorname{int} \mathbb{M}(\mathcal{R}^g, \Delta_g, \max F_{\phi_g}) \neq \emptyset \quad \text{and} \quad \operatorname{int} \mathbb{L}(\mathcal{R}^g, \Delta_{\phi_g}, \max F_{\phi_g}) \neq \emptyset
\]

whenever \( F \in \mathcal{H}_{g,\Lambda} \).

Here \( \Delta_g \) denoted the hyperbolic continuation of \( \Delta \) by the Poincaré map \( \mathcal{R}^g \).

**Theorem 3.10.** In the setting of Theorem 1.2, arbitrarily close to \( g_0 \) there is an open set \( G \subset \mathcal{G}^3(M) \) such that for any \( g \in G \) one can find a dense and open \( C^2 \)-open subset \( \tilde{\mathcal{H}}_{g,\Lambda} \subset C^2(M, \mathbb{R}) \), so that

\[
\operatorname{int} \mathbb{M}(\mathcal{R}^g, \Delta_g, \max (f \circ \pi)_{\phi_g}) \neq \emptyset \quad \text{and} \quad \operatorname{int} \mathbb{L}(\mathcal{R}^g, \Delta_g, \max (f \circ \pi)_{\phi_g}) \neq \emptyset
\]

whenever \( f \in \tilde{\mathcal{H}}_{g,\Lambda} \). Here \( \Delta_g \) denotes the hyperbolic continuation of \( \Delta \) by the Poincaré map \( \mathcal{R}^g \).

The following sections focus on doing the proofs of the above theorems.

### 4 Construction of the Typical Functions

In this section, we construct explicitly the section of real functions that work for Theorem 3.9 and Theorem 3.10.
4.1 Construction of Typical Functions for Theorem 1.1

The construction of the set of functions \( \mathcal{H}_{g, \Lambda} \) of Theorem 1.1 will be similar to the construction given in [Rn21 Section 4.3 - Lemma 4.9] with some minor changes. The following properties will be very useful for our construction.

**Definition 4.1.** Let \( \Theta \) be a basic set for \( \mathcal{R} \) and \( \beta > 0 \) small, we say that a sub-horseshoe \( \tilde{\Theta} \) of \( \Theta \) has the \( \beta \)-stable property if 

\[
HD(K_{\tilde{\Theta}}^s) > HD(K_{\tilde{\Theta}}^u) - \beta,
\]

where \( K_{\tilde{\Theta}}^s, K_{\tilde{\Theta}}^u \) are the stable Cantor sets associated to \( \tilde{\Theta} \) and \( \Theta \) (see Section A.3 to the definition of stable and unstable Cantor sets).

The definition of the \( \beta \)-unstable property is analogous, using unstable Cantor sets instead of stable Cantor sets.

**Remark 4.2.** It is worth noting that the parameter \( \beta \) in the definition above is small, i.e., \( 0 < \beta < \max\{HD(K_{\tilde{\Theta}}^s)/2, HD(K_{\tilde{\Theta}}^u)/2\} \). In fact, all the lemmas in this section that involve the parameter \( \beta \) work in this interval. However, in Section 5.2.2, we use the basic set \( \Delta \) and an interval of the parameters \( \beta \), which is related to \( HD(K_{\tilde{\Theta}}^s) \) and \( HD(K_{\tilde{\Theta}}^u) \), more specifically, since \( HD(\Delta) = HD(K_{\tilde{\Theta}}^s) + HD(K_{\tilde{\Theta}}^u) \) (see Subsection A.3), we take \( \beta \) such that

\[
HD(\Delta) - 6\beta = HD(K_{\tilde{\Theta}}^s) + HD(K_{\tilde{\Theta}}^u) - 6\beta > 1,
\]

where \( \Delta \) is the basic set given by Lemma 3.4, which satisfies Lemma 3.5.

**Remark 4.3.** All results of this section will be valid for any basic set \( \Theta \) of \( \mathcal{R} \). In Section 3 we use all these results for the specific basic set \( \Delta \).

Given a basic set \( \Theta \) for \( \mathcal{R} \), a Markov partition \( \mathcal{R} \) of \( \Theta \), we define the set

\[
H_1(\mathcal{R}, \Theta) = \left\{ f \in C^1(\Sigma \cap \mathcal{R}, \mathbb{R}) : \# M_f(\Theta) = 1, \; z \in M_f(\Theta), \; DR_z(e_z) \neq 0, j = s, u \right\},
\]

where \( M_f(\Theta) := \{ z \in \Theta : f(z) \geq f(x) \text{ for all } x \in \Theta \} \), the set of maximum points of \( f \) in \( \Theta \) and \( e_z \in E^2_z(z) \) is a unit vector, \( j = s, u \). (cf. [MR17 Section 3]).

A notion that will be useful in the next sections is the concept of the Markov partition of a horseshoe, to make the text easier to read we put its definition in the Appendix (see Subsection A.3).

**Definition 4.4.** We say that \( F \in \mathcal{H}_{\Theta, \beta}^s \subset C^2(S^0; M, \mathbb{R}) \) if there is a sub-horseshoe \( \Theta_F^s \) of \( \Theta \) with the \( \beta \)-stable property and there is a Markov partition \( \mathcal{R}_F^s \) of \( \Theta_F^s \) such that the function \( \max(F)_{\Sigma \cap \mathcal{R}_F^s} \in H_1(\mathcal{R}, \Theta_F^s) \subset C^1(\Sigma \cap \mathcal{R}_F^s, \mathbb{R}) \). The definition for the unstable case \( \mathcal{H}_{\Theta, \beta}^u \) is analogous (see Subsection A.3 for the definition of Markov Partition).

**Lemma 4.5.** [Rn21 Lemma 4.9] For any basic of \( \Theta \) of \( \mathcal{R} \) and \( \beta > 0 \) small, we have that \( \mathcal{H}_{\Theta, \beta}^j \), is dense and \( C^2 \)-open set, \( j = s, u \). Consequently, for \( g \in \mathcal{V} \) there is \( \tilde{\beta} > 0 \) such that \( \mathcal{H}_{\Theta, \beta}^j \) is a dense and \( C^2 \)-open set, \( j = s, u \).
Remark 4.6. By the construction at [MR17, Section 3] and the continuity of the Hausdorff dimension, it is easy to see that, if $V$ is small enough, then for $g \in V$ there is $\beta$ (which depends on $g$) such that

$$\mathcal{H}_{\Theta,\beta}^j = \left\{ F \circ S_g : F \in \mathcal{H}_{\Theta, \beta}^j \right\}, \text{ for } j = s, u.$$

Remark 4.7. The proof of Lemma 4.5 also allow us conclude the following: If $\Theta$ and $\Gamma$ are two basic sets of $\mathcal{R}$, then the set $\mathcal{H}_{\Theta,\Gamma,\beta} \subset \mathcal{H}_{\Theta,\beta} \cap \mathcal{H}_{\Gamma,\beta}$, which satisfies

$$\max(F)_\phi|_{\Sigma \cap (R_{F,\Theta} \cup R_{F,\Gamma})} \in H_1(\mathcal{R}, (\Theta_1 \cup \Gamma_1)) \subset C^1(\Sigma \cap (R_{F,\Theta} \cup R_{F,\Gamma}), \mathbb{R})$$

is dense and $C^2$-open.

Here $R_{F,\Theta}$ and $R_{F,\Gamma}$ are the Markov partition of $\Theta_1$ and $\Gamma_1$ given by the definition of $\mathcal{H}_{\Theta,\beta}$ and $\mathcal{H}_{\Gamma,\beta}$, respectively.

Analogously, since Remark 4.6, it is easy to see that, if $V$ is small enough, then for any $g \in V$ there is $\beta$ (which depends on $g$) such that

$$\mathcal{H}_{\Theta,\Gamma,\beta} = \left\{ F \circ S_g : F \in \mathcal{H}_{\Theta,\Gamma,\beta} \right\},$$

and consequently $\mathcal{H}_{\Theta,\Gamma,\beta}$ is a dense and $C^2$-open set, where $\Theta_g$ and $\Gamma_g$ is the hyperbolic continuation of $\Theta$ and $\Gamma$ by $\mathcal{R}_g$, respectively (see Subsection 3.1).

4.2 Construction of Typical Functions for Theorem 1.2

The construction of $\tilde{\mathcal{H}}_{g,\Lambda}$ is a little more complicated than the construction of $\mathcal{H}_{g,\Lambda}$, although we will use some tools developed at [Rn21, Lemma 4.9] to make $\max(f \circ \pi)_\phi$ a $C^1$-function, we have an additional problem because the function $f \circ \pi$ is constant along the fiber of $S_\Theta$. For the construction of $\tilde{\mathcal{H}}_{g,\Lambda}$ we need some auxiliary sets of functions, which give some differentiability and good properties to $\max(f \circ \pi)_\phi$.

Let $\Theta$ be a basic set for $\mathcal{R}$, and $\beta > 0$ small (as Remark 4.2), we consider one-parameter families of sets of functions, $\tilde{\mathcal{H}}_{\Theta,\beta}^s, \tilde{\mathcal{H}}_{\Theta,\beta}^u \subset C^2(M, \mathbb{R})$, defined as follows

$$\tilde{\mathcal{H}}_{\Theta,\beta} = \left\{ f \in C^2(M, \mathbb{R}) : f \circ \pi \in \mathcal{H}_{\Theta,\beta}^j \right\}, \text{ for } j = s, u.$$

In other words, $f \in \tilde{\mathcal{H}}_{\Theta,\beta}^j$ if there is a sub-horseshoe $\Theta_1^j$ of $\Theta$ and a Markov partition $R_{F,\Theta}$ of $\Theta_1^j$ such that $\Theta_1^j$ has the $\beta$-stable property and $\max(f \circ \pi)_\phi|_{\Sigma \cap R_{F,\Theta}^j} \in H_1(\mathcal{R}, \Theta_1^j) \subset C^1(\Sigma \cap R_{F,\Theta}^j, \mathbb{R})$. The stable case is analogous. Similar to Remark 4.6 we have

Remark 4.8. It is easy to see that, for all metric $g$, $\pi_g = \pi \circ S^g$. So, if $V$ is small enough, then the construction at [MR17, Section 3] and the continuity of the Hausdorff dimension, provides that for $g \in V$ there is $\beta$ (which depends on $g$) such that

$$\tilde{\mathcal{H}}_{\Theta_g,\beta}^j = \tilde{\mathcal{H}}_{\Theta,\beta}^j, \text{ for } j = s, u$$

where $\Theta_g$ is the hyperbolic continuation of $\Theta$ by $\mathcal{R}_g$ (see Subsection 3.1).
Although the set $\mathcal{H}^j_{\Theta,\beta}$ is dense and $C^2$-open in $C^2(SM,\mathbb{R})$, it does not easily follow that $\tilde{\mathcal{H}}^j_{\Theta,\beta}$ is dense and $C^2$-open in $C^2(M,\mathbb{R})$, $j = s, u$. Thus, the remainder of this section is devoted to showing that

**Lemma 4.9.** For any basic $\Theta$ of $\mathcal{R}$ and $\beta > 0$ small, we have that $\tilde{\mathcal{H}}^j_{\Theta,\beta}$ is dense and $C^2$-open set, $j = s, u$. Consequently, for $g \in \mathcal{V}$, $\tilde{\mathcal{H}}^j_{\Theta,s,\beta}$ is a dense and $C^2$-open set, $j = s, u$.

The proof of this lemma is extensive and we present some auxiliary lemmas to do it.

The first important ingredient, from [Rn21] Lemma 4.6, for all constructions in this section will be the following lemma. Recall that $\Sigma = \bigcup_{i=1}^{k} \Sigma_i$, then

**Lemma 4.10.** (Lemma 2.6 - [Rn21]) Let $\alpha = \{\alpha_i : [0,1] \to \Sigma, i \in \{1,\ldots,m\}\}$ be a finite family of $C^1$-curves and $\Theta$ a basic set for $\mathcal{R}$. Then for all $\epsilon > 0$ there is sub-horseshoe $\Theta^s_\alpha$ of $\Theta$ such that $\Theta^s_\alpha \cap \alpha_i([0,1]) = \emptyset$, for any $i \in \{1,\ldots,m\}$ and $\Theta^s_\alpha$ satisfies the $\epsilon$-stable property with respect to $\Theta$, that is,

$$HD(K^s_\alpha) \geq HD(K^s_\Theta) - \epsilon,$$

where $K^s_\alpha, K^s_\Theta$ are the stable regular Cantor sets that of $\Theta^s_\alpha$, and $\Theta$, respectively. An analog result to the unstable case, using the $\epsilon$-unstable property.

This lemma will allow us to delete suitable subsets of $\Theta$ without losing too much of the Hausdorff dimension of $\Theta$.

**Definition 4.11.** Let $U$ be an open set of $SM$, we say that $f \in \mathcal{N}^s_U,\Theta,\beta \subset C^2(M,\mathbb{R})$ if there is a sub-horseshoe $\Theta^s_f$ of $\Theta$ with the $\beta$-stable property and a Markov partition $R^{\phi}_f,\Theta$ of $\Theta^s_f$ so that for each $(x,v) \in \Theta^s_f$ we have

$$\#\{t \in (0,t_+(x,v)) : \phi^t(x,v) \in U \text{ and } t \text{ is a critical point of } f \circ \pi(\phi^t(x,v))\} < \infty,$$

where $\# A$ denotes the cardinality of the set $A$.

The definition for the unstable case $\mathcal{N}^u_U,\Theta,\beta$ is analogous.

When $U = SM$, then we denote $\mathcal{N}^j_{SM,\Theta,\beta} := \mathcal{N}^j_{\Theta,\beta}, j = s, u$.

**Lemma 4.12.** For any $\beta > 0$ small, the set $\mathcal{N}^j_{\Theta,\beta}$ is dense and $C^2$-open, $j = s, u$.

To prove this lemma we look at specific open sets $U$ of $SM$ as below: Let $U$ be an open set in $SM$ such that,

(i) The closure, $\bar{U}$, of $U$ is contained in the image of a parametrization $\tilde{\varphi} : V \times I \to SM$, where $V \subset \mathbb{R}^2$ and $I$ is an interval. For instance, if $\varphi$ is a parametrization of $M$, $\varphi : V \to M$, such that the set $\{\tilde{\varphi}_x, \tilde{\varphi}_y\}$ is an orthonormal basis and $\tilde{\varphi}(x,y,z) = (\varphi(x,y), \cos z \frac{\partial \varphi}{\partial x} + \sin z \frac{\partial \varphi}{\partial y})$.

(ii) $U \cap \bigcup_{(x,v) \in \Theta} \bigcup_{t \in [0,t_+(x,v)]} \phi^t(x,v) \neq \emptyset$.
Lemma 4.13. If \( U \) satisfies (i) and (ii), then the set \( \mathcal{N}_{u,\Theta,\beta}^j \) is dense and \( C^2 \)-open, \( j = s, u \).

Proof. We prove the stable case, since the unstable case is analog. The openness is a consequence of the definition of \( \mathcal{N}_{u,\Theta,\beta}^j \). Our task is to prove the density. Let \( f \in C^\infty(M, \mathbb{R}) \) and put \( F = f \circ \pi \). Then using the local coordinates given by (i) we can write

\[
F(x, y, z) = f(\varphi(x, y, z))
\]

and the vector field \( \phi \) we write as

\[
\phi(x, y, z) = (X_1(x, y, z), X_2(x, y, z), X_3(x, y, z)).
\]

Consider now the set

\[
S = \{(x, y, z) : \langle \nabla F(x, y, z), \phi(x, y, z) \rangle = 0 \}
\]

and

\[
H = \{(x, y, z) : \langle \text{Hess} F(x, y, z)\phi(x, y, z), \phi(x, y, z) \rangle = 0 \},
\]

where \( \text{Hess} F \) is the Hessian matrix of \( F \), given by

\[
\text{Hess} F(x, y, z) = \begin{pmatrix}
\frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} & 0 \\
\frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]

We have

\[
H = \left\{(x, y, z) : \frac{\partial^2 f}{\partial x^2} X_1^2 + 2 \frac{\partial^2 f}{\partial x \partial y} X_1 X_2 + \frac{\partial^2 f}{\partial y^2} X_2^2 = 0 \right\}.
\]

Now we would like to perturb \( f \) so that the sets \( S \) and \( H \) have a transversal intersection. In fact:

Since the vector field \( \phi \) is transverse to fiber (see Appendix A.4), then \( X_1 \neq 0 \) or \( X_2 \neq 0 \). Suppose that \( X_1 \neq 0 \), then put

\[
f_\delta(x, y) = f(x, y) - \frac{\delta x^2}{2} - cx
\]

(later the constant \( c \) will be chosen accordingly). Taking \( \delta \) a small regular value of

\[
L(x, y, z) := \frac{\frac{\partial^2 f}{\partial x^2} X_1^2 + 2 \frac{\partial^2 f}{\partial x \partial y} X_1 X_2 + \frac{\partial^2 f}{\partial y^2} X_2^2}{X_1^2}.
\]

Then, the set \( H_\delta := \{L(x, y, z) = \delta\} \) is a regular surface. In addition, for any choice of the parameter \( c \), we also have that

\[
H_\delta = \left\{(x, y, z) : \frac{\partial^2 f_\delta}{\partial x^2} X_1^2 + 2 \frac{\partial^2 f_\delta}{\partial x \partial y} X_1 X_2 + \frac{\partial^2 f_\delta}{\partial y^2} X_2^2 = 0 \right\}.
\]

Consider now the function

\[
G(x, y, z) := \frac{(\frac{\partial f}{\partial x} - \delta x) X_1 + \frac{\partial f}{\partial y} X_2}{X_1}
\]
From Lemma 4.13, for each $i$ we consider the projections of the curves $J_i$ along the flow on $\Sigma$. Let $\alpha_i$ be the projections of the curves $J_i$ along the flow on $\Sigma$. Then, by Lemma 4.10 there are sub-horseshoes $\Theta_i^s$ of $\Theta$ such that $\Theta_i^s \cap \alpha_i = \emptyset$ and

$$HD(K_i^s) \geq HD(K_{\Theta_i^s}^s) - \beta,$$

where $K_i^s, K_{\Theta_i^s}^s$ are the stable Cantor sets associated to $\Theta_i^s$ and $\Theta$. Moreover, let $R_i^s$ be a Markov partition of $\Theta_i^s$ such that $R_i^s \cap \alpha_i = \emptyset$. Then, if $(x, v) \in R_i^s$, then the critical points of $(f_i \circ \pi)((f_i)_{x,v}(x,y,z))$ are non-degenerate and therefore finite. Thus, $f_i \in \mathcal{N}_{U_i, \Theta_i, \beta}$ as we wish $f_i \in \mathcal{N}_{U_i, \Theta_i, \beta}$ as we wish.

**Proof of Lemma 4.12** The set $\mathcal{N}_{\Theta_i, \beta}^s$ is clearly open. Since $\bigcup_{(x,v) \in \Theta} \bigcup_{t \in [0, t_+]} \phi^t(x,v)$ is a compact set, then there are a finite number of open set $U_1, \ldots, U_n \subset SM$, which satisfies (i) and (ii) such that

$$\bigcup_{(x,v) \in \Theta} \bigcup_{t \in [0, t_+]} \phi^t(x,v) \subset \bigcup_{i=1}^n U_i. \quad (7)$$

From Lemma 4.13 for each $i$, $\mathcal{N}_{U_i, \Theta_i, \beta}^s \subset C^2(M, \mathbb{R})$ is dense and $C^2$-open.

**Claim:** $\bigcup_{i=1}^n \mathcal{N}_{U_i, \Theta_i, \beta}^s \subset \mathcal{N}_{\Theta_i, \beta}^s$.

**Proof of Claim.** Let $f \in \bigcup_{i=1}^n \mathcal{N}_{U_i, \Theta_i, \beta}^s$, then for each $i$ there is a sub-horseshoe $\Theta_{f,i}^s$ of $\Theta$ which satisfies Definition 4.11. Therefore, taking $\Theta_f^s := \bigcup_{i=1}^n \Theta_{f,i}^s$, we see that $f \in \mathcal{N}_{\Theta, \beta}^s$.

Since $\bigcup_{i=1}^n \mathcal{N}_{U_i, \Theta_i, \beta}^s$ is $C^2$-dense, then the claim completes the proof.

The unstable case is analogous.

**Remark 4.14.** Once again we cover $\Lambda$ with a finite number of tubular neighborhoods $U_r$, $1 \leq r \leq m$ whose boundaries are the good cross-sections $\Sigma = \bigcup_{i=1}^k \Sigma_i$ mentioned in Section 3.1. For each $r$, let us fix coordinates $(x_1(w), x_2(w), x_3(w))$ on $U_r$ such that $x_3(w)$ is the flow direction and $U_r \cap \Sigma = \{x_3(w) = 0\} \cup \{x_3(w) = 1\}$.

Consider $f \in \mathcal{N}_{\Theta, \beta}^s$, and $\Theta_f^j$ the sub-horseshoe given by Definition 4.11 $j = s, u$.

Then, the proof of Lemma 4.13 implies that there is a Markov partition $R_{f,\Theta}^j$ of $\Theta_f^j$. 


such that the value \( \max(f \circ \pi)_{\phi_0}(z) \), for \( z \in R_f^j \), is described by the values of a finite collection of graphs transverse to the flow direction. In other words, for each \( (x_1, x_2, 0) \in U_r \cap R_f^j \) there is neighborhood \( V \) of \( (x_1, x_2, 0) \) and a finite collection of disjoints graphs \( \{(x, y, \psi_l(x, y)) : (x, y, 0) \in V \} \), \( 1 \leq l \leq n \), such that if \( F(x, y, t) = \max(f \circ \pi)_{\phi}(x, y, t) \) with \( (x, y, 0) \in V \), then \( t = \psi_l(x, y) \) for some \( l \).

Remark 4.15. A sub-horseshoe \( \Theta \) of \( R \) has finite many fixed points. Thus, removing the fixed points of \( \Theta \), we obtain a sub-horseshoe (that we still call \( \Theta \)) with almost the same Hausdorff dimension of \( \Theta \). Hence, from now on, we assume that \( R \) has no fixed points on \( \Theta \).

Definition 4.16. We say that \( f \in M_{\Theta, \beta}^* \subset C^2(M, \mathbb{R}) \) if there is a sub-horseshoe \( \tilde{\Theta}_f^* \) of \( \Theta \) with the \( \beta \)-stable property and a Markov partition \( R_{f, \tilde{\Theta}}^* \) of \( \tilde{\Theta}_f^* \) so that for each \( (x, v) \in \tilde{\Theta}_f^* \) we have that \( \pi(\phi^0(x, v)) \neq \pi(\phi^1(x, v)) \), whenever

\[ t_0, t_1 \in \{t \in (0, t_+(x, v)) : t \text{ is a critical point of } f \circ \pi(\phi^t(x, v)) \}. \]

The definition for the unstable case \( M_{\Theta, \beta}^u \) is analogous.

Lemma 4.17. For any \( \beta > 0 \) small, the set \( M_{\Theta, \beta}^j \) is dense and \( C^2 \)-open, \( j = s, u \).

Proof. It is clear that \( M_{\Theta, \beta}^s \) is a \( C^2 \)-open set.

Consider the set \( M \) of Morse's functions of \( M \), which is a dense and \( C^2 \)-open set, then we will prove that the set \( M \cap N^*_{\alpha, \beta/2} \subset M_{\Theta, \beta}^s \) and therefore, from Lemma 4.12, \( M_{\Theta, \beta}^s \) is a dense set. Let \( f \in M \cap N^*_{\alpha, \beta/2} \), then the set of critical points of \( f \) in \( \pi(\cup_{x \in \Sigma} \cup_{x \in [0, t_+(x)]} \phi^t(x)) \) is finite, which we denoted by \( x_1^j, \ldots, x_k^j \). Consider the finite family of \( C^1 \)-curves \( \alpha_f \) given by the projections of the fibers \( \pi^{-1}(x_1^j) \) on \( \Sigma \) along of the flow. Thus, applying Lemma 4.10 to the family of curves \( \alpha_f \) and the horseshoe \( \Theta_f^* \) (given by Lemma 4.12 with \( \beta/2 \)), we obtain two sub-horseshoes \( \Theta_f^* \) of \( \Theta_f^* \), such that \( \Theta_f^* \cap \alpha_f = \emptyset \), and

\[ HD(\gamma^{s}K_{f, \Theta}^*) \geq HD(K_{f, \Theta}^*) - \beta/2 \quad \text{and} \quad HD(\gamma^{u}K_{f, \Theta}^*) \geq HD(K_{f, \Theta}^*) - \beta/2, \]

where \( \gamma^{s} \) is the stable Cantor sets associated to \( \Theta_f^* \). Therefore, as \( \Theta_f^* \) has the \( \frac{\beta}{2} \) stable property, then

\[ HD(\gamma^{s}K_{f, \Theta}^*) \geq HD(K_{\Theta}^*) - \beta \quad \text{and} \quad HD(\gamma^{u}K_{f, \Theta}^*) \geq HD(K_{\Theta}^*) - \beta. \]

Put \( \tilde{\Theta}_f^* := \gamma^{s}K_{f, \Theta}^* \) and consider a Markov partition \( R_{f, \tilde{\Theta}}^* \) of \( \gamma^{s}K_{f, \Theta}^* \) such that \( R_{f, \tilde{\Theta}}^* \cap \alpha_f = \emptyset \). We stated that \( f \in M_{\Theta, \beta}^s \), in fact: by contradiction, assume that there is \( (x, v) \in R_{f, \tilde{\Theta}}^* \) and \( t_0, t_1 \in (0, t_+(x, v)) \) critical points of \( g(t) := f \circ \pi(\phi^t(x, v)) \) such that \( \pi(\phi^0(x, v)) = \pi(\phi^1(x, v)) \), i.e., \( g'(t_0) = g'(t_1) = 0 \). Moreover, since \( g'(t) = (\nabla f(\gamma_v(t)), \gamma_v(t)) \), where \( \gamma_v(t) = \pi(\phi^t(x, v)) \), the construction of \( \Theta_f^* \) provides that \( \nabla f(\gamma_v(t)) \neq 0 \) for all \( t \in (0, t_+(x, v)) \). Thus, as \( \gamma_v(t_0) = \gamma_v(t_1) \) must be \( \gamma_v(t_0) = -\gamma_v(t_1) \) or \( \gamma_v(t_0) = \gamma_v(t_1) \). In the first case, we have a contradiction with the uniqueness of the geodesics with respect to the initial conditions. In the second case, we have that \( \gamma_v(t) \) is a closed geodesic, and this is a contradiction, since \( t_0, t_1 \in (0, t_+(x, v)) \). Thus, we complete the proof of the lemma. The unstable case is analogous. \( \Box \)
Definition 4.18. We say that $f \in C_{\Theta}^{s} \subset C^{2}(M, \mathbb{R})$ if there is a sub-horseshoe $\Theta_{i}^{s}$ of $\Theta$ with the $\beta$-stable property and there is a Markov partition $R_{f, \Theta}^{s}$ of $\Theta_{i}^{s}$ such that the function $\max(f \circ \pi)_{\phi}|_{\Sigma \cap R_{f, \Theta}^{s}} \in C^{1}(\Sigma \cap R_{f, \Theta}^{s}, \mathbb{R})$. The definition for the unstable case $C_{\Theta}^{u}$ is analogous.

Lemma 4.19. For any $\beta > 0$ small, the set $C_{\Theta}^{j}$ is dense and $C^{2}$-open, $j = s, u$.

Proof. The set $C_{\Theta}^{j}$ is clearly a $C^{2}$-open set. We will just prove the density for the stable case. The proof for the unstable case is analogous.

By Lemma 4.17, it is sufficient to prove that $C_{\Theta}^{j}$ is dense $M_{\Theta}^{0, \beta, 2}$. Let $f \in M_{\Theta}^{0, \beta, 2}$, then using coordinates $(x_{1}(w), x_{2}(w), x_{3}(w))$ given in Remark 4.14 our task is reduced to perturb $f$ in such a way that $\max(f \circ \pi)_{\phi}(x_{1}, x_{2}, t)$ is given by the values of $\max(f \circ \pi)_{\phi}$ on a unique graph $(x, y, \psi(x, y))$, $(x, y) \in V, 1 \leq i \leq n$ (see Remark 4.14). Along this lines, we employ the argument from [Rn21, Lemma 4.7]. We begin noting that by Lemma 4.17

$$\pi\{\text{graph } \psi_{i}\} \cap \pi\{\text{graph } \psi_{j}\} = \emptyset, \ i \neq j. \quad (8)$$

Let $O_{i}, \tilde{O}_{i}$ be small neighborhoods of $\pi\{(z = \psi_{i}(x, y))\}$ such that $O_{i} \subset \tilde{O}_{i}$ and $\tilde{O}_{i} \cap \tilde{O}_{j} = \emptyset$, $i \neq j$. Let $g_{ij} = f \circ \pi(x, y, \psi_{i}(x, y)) - f \circ \pi(x, y, \psi_{j}(x, y))$ for $j \neq 1$. Consider $\gamma_{1} > 0$ a small regular value of $g_{ij}$ for all $j \neq 1$. Take $\xi_{1}$ a $C^{\infty}$-bump function, equal to $-\gamma_{1}$ in $O_{1}$ and 0 outside of $\tilde{O}_{1}$. So, the function $(f + \xi_{1})$ is $C^{2}$-close to $f$. We define the function

$$g_{1j}^{1}(x, y) = ((f + \xi_{1}) \circ \pi)(x, y, \psi_{i}(x, y)) - ((f + \xi_{1}) \circ \pi)(x, y, \psi_{j}(x, y)) = g_{ij}(x, y) - \gamma_{1}.$$

Put $f_{1} := f + \xi_{1}$ and define $g_{2j}(x, y) = (f \circ \pi)(x, y, \psi_{i}(x, y)) - (f \circ \pi)(x, y, \psi_{j}(x, y))$ for $j \neq 2$. Consider $\gamma_{2} > 0$ a small regular value of $g_{2j}$ for all $j \neq 2$. Take $\xi_{2}$ a $C^{\infty}$-bump function, equal to $-\gamma_{2}$ in $O_{2}$ and 0 outside of $\tilde{O}_{2}$. So, the function $f_{1} + \xi_{2}$ is $C^{2}$-close to $f_{1}$ and define the function

$$g_{2j}^{2}(x, y) = ((f_{1} + \xi_{2}) \circ \pi)(x, y, \psi_{i}(x, y)) - ((f_{1} + \xi_{2}) \circ \pi)(x, y, \psi_{j}(x, y)) = g_{2j}(x, y) - \gamma_{2}.$$

The equation $(8)$ implies that the functions $\xi_{1} \circ \pi$ and $\xi_{2} \circ \pi$ have disjoint support and therefore the perturbation of $f, f_{1}$, does not affect the perturbation $f_{2}$. Thus, as it is explained in [Rn21, Lemma 4.7], we can perform a small perturbation, $\tilde{f}$, of $f$ on $V$ in such a way that 0 is a simultaneous regular value of the functions $(x, y) \mapsto g_{ij}(x, y) := \tilde{f} \circ \pi(x_{1}, x_{2}, \psi_{i}(x_{1}, y)) - \tilde{f} \circ \pi(x_{1}, x_{2}, \psi_{j}(x_{1}, y))$ for all choices of $1 \leq i < j \leq n$. In this situation, $L_{n} = \bigcup_{1 \leq i < j \leq n} g_{ij}^{-1}(0)$ is a finite collection of $C^{1}$-curves such that, for each $y \in V \setminus L_{n}$, the values of $\max(\tilde{f} \circ \pi)_{\phi}$ near $y$ are described by the values of $\tilde{f} \circ \pi$ at a unique graph. Hence, for each $y \in V \setminus L_{n}$, one has that $\max(\tilde{f} \circ \pi)_{\phi}(y) = (\tilde{f} \circ \pi)(\phi(y))$ for a unique $0 \leq t(y) \leq t_{i}(y)$ that depends $C^{1}$ on $y$.

Since $f \in M_{\Theta}^{s, \beta, 2}$, then $\tilde{f} \in M_{\Theta}^{s, \beta, 2}$ and let $\Theta_{i}^{s}$ the sub-horseshoe of definition 4.16 which satisfies

$$\text{HD}(K_{f, \Theta}^{s}) > \text{HD}(K_{f, \Theta}^{s}) - \beta, /2,$$

where $K_{f, \Theta}^{s}$ is the stable Cantor sets associated to $\Theta_{i}^{s}$. Moreover, for $\beta/2$, the Lemma 4.10 provides a sub-horseshoe $\star \Theta_{i}^{s}$ of $\Theta_{i}^{s}$ a Markov partition $\star R_{f, \Theta}^{s}$ of $\star \Theta_{i}^{s}$ such that $\star R_{f, \Theta}^{s} \cap L_{n} = \emptyset$ and

$$\text{HD}(\star K^{s}_{f, \Theta}) > \text{HD}(\star K^{s}_{f, \Theta}) - \beta / 2 > \text{HD}(K_{f, \Theta}^{s}) - \beta, /2$$

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where $\mathcal{K}_{\mathcal{J}}^{\ast}$ is the stable (unstable) Cantor sets associated to $\Theta_{\mathcal{J}}^{\ast}$.

The above discussion allows us to conclude that $\tilde{f} \in C_{\Theta,\beta}^{\ast}$, as we wish. \hfill \square

**Remark 4.20.** For $j = s, u$, the construction of the sub-horseshoe $\Theta_{\mathcal{J}}^{j}$ and the Markov Partition $R_{\mathcal{J},\Theta}^{j}$ at Lemma 4.19 satisfies

(i) $\{\pi(\phi^{j}(z)) : z \in \Theta_{\mathcal{J}}^{j}, \ 0 \leq t \leq t_{+}(z)\} \cap \{\text{Critical points of } f\} = \emptyset$,

(ii) For any $z \in R_{\mathcal{J},\Theta}^{j} \text{ we have that }$

$$\#\{t \in [0,t^{+}(z)] : \max(f \circ \pi)(z) = f \circ \pi(\phi^{j}(z))\} = 1,$$

(iii) Note that, for any sub-horseshoe $\Theta$, the properties (i) and (ii) are persistently for small perturbations of $\Theta$. So, if $f_{1}, f_{2} \in C_{\Theta,\beta}^{j}$ are close enough, then the sub-horseshoe $\Theta_{\mathcal{J}}^{j}$ can choose close to $\Theta_{\mathcal{J}}^{j}$ in the Hausdorff distance for compact sets, $j = s, u$.

The rest of this section will be devoted to proving Lemma 4.9. The set of functions $C_{\Theta,\beta}^{j}$ provides the $C^{1}$ property for $\max(f \circ \pi)_{\Sigma \cap R_{\mathcal{J},\Theta}^{j}}$, $j = s, u$. To finally prove Lemma 4.9, we need to find the conditions of $[5]$.

**4.2.1 The set of geodesics with transversal self-intersection**

Given any Riemannian metric $g$ on $M$, we define the set, $\mathcal{SI}_{g}$, of self-intersection of geodesics by:

$$\mathcal{SI}_{g} = \{(x,v) \in S^{g}M : \exists \ t(x,v) \text{ such that } \gamma_{v}(t(x,v)) = x \text{ and } \{v, \gamma_{v}'(t(x,v))\} \text{ are L.I.} \} \),$$

where L.I. stands for linearly independent and $\gamma_{v}(t)$ is the geodesic in the metric $g$ with $\gamma_{v}(0) = 0$ and $\gamma_{v}'(0) = v$.

**Remark 4.21.** For any Riemannian metric $g$ of finite volume, the Liouville measure is finite and invariant by the geodesic flow, then $\mathcal{SI}_{g} \neq \emptyset$.

**Proposition 4.22.** The set $\mathcal{SI}_{g}$ is a cross-section for the geodesic flow of the metric $g$, $\phi_{g}^{t}$.

**Proof.** Let $(x_{0},v_{0}) \in \mathcal{SI}_{g}$ and $t_{0}$ such that $\gamma_{v_{0}}(t_{0}) = x_{0}$ with $\{v_{0}, \gamma_{v_{0}}'(t_{0})\}$. Consider $\mathcal{L}$ a transverse section to the flow $\phi_{g}^{t}$ and the fiber $\pi_{g}^{-1}(x_{0})$ and define the following function

$$h: \mathcal{L} \times \mathbb{R} \longrightarrow M$$

$$((x,v),t) \longmapsto \pi_{g}(\phi_{g}^{t}(x,v))$$

with $h((x_{0},v_{0}),0) = x_{0}$ and $h((x_{0},v_{0}),t_{0}) = x_{0}$. Put $h_{0} := h|_{\mathcal{L} \times I_{0}}$ and $h_{t_{0}} := h|_{\mathcal{L} \times I_{t_{0}}}$, where $I_{0}$ and $I_{t_{0}}$ are small intervals containing 0 and $t_{0}$, respectively.

Let $\varphi: T_{x_{0}}M \to U_{x_{0}}$ be normal coordinates in $x_{0}$, where $U_{x_{0}}$ is a neighborhood of $x_{0}$, i.e., let $\{e_{1}, e_{2}\}$ be orthonormal basis of $T_{x_{0}}M$ and $\varphi(x_{1}, x_{2}) = \varphi(x_{1}e_{1} + x_{2}e_{2})$.
We define \( H : \mathcal{L} \times I_{t_0} \times I_0 \rightarrow V_0 \subset T_{x_0}M \) by
\[
H ((x, v), t, s) = (\varphi^{-1} \circ h_{t_0}) ((x, v), t) - (\varphi^{-1} \circ h_0) ((x, v), s)
\]
with \( H((x_0, v_0), t_0, 0) = 0 \).

Moreover,
\[
\frac{\partial H}{\partial t}((x_0, v_0), t_0, 0) = (D\varphi^{-1})_{h_{t_0}((x_0, v_0), t_0)} \left[ \frac{\partial h_{t_0}}{\partial t}((x_0, v_0), t_0) \right]
\]
and
\[
\frac{\partial H}{\partial s}((x_0, v_0), t_0, 0) = -(D\varphi^{-1})_{h_{t_0}((x_0, v_0), 0)} \left[ \frac{\partial h_0}{\partial s}((x_0, v_0), 0) \right] = -D((\exp_{x_0}^g)^{-1})_{x_0}(\gamma'_w(0)) = -v_0.
\]

Since \( \{ v_0, \gamma'_w(t_0) \} \) are linearly independent, then \( \frac{\partial H}{\partial (t,s)} \) is an isomorphism. Therefore, by the Implicit Function Theorem, there is neighborhood \( U_{\mathcal{L}} \subset \mathcal{L} \) of \( (x_0, v_0) \) and a diffeomorphism \( \xi : U_{\mathcal{L}} \rightarrow V_{(t_0,0)} \), where \( V_{(t_0,0)} \) is an open set containing \( (t_0, 0) \in \mathbb{R} \times \mathbb{R} \) and \( H ((y, w), (\xi(y, w))) = 0 \).

Without loss of generality we can assume that \( V_{(t_0,0)} = \tilde{I}_{t_0} \times \tilde{I}_0 \) and
\[
\xi(y, w) = (\xi_1(y, w), \xi_2(y, w)),
\]
with \( \xi_1 \) close to \( t_0 \) and \( \xi_2 \) close to \( 0 \). Then, by definition of \( H \) we have that
\[
\exp_{x_0}^{-1}(\pi_g(\phi^{\xi_1(y,w)}(y, w))) = \exp_{x_0}^{-1}(\pi_g(\phi^{\xi_2(y,w)}(y, w))).
\]

Therefore \( \pi_g(\phi^{\xi_1(y,w)}(y, w)) = \pi_g(\phi^{\xi_2(y,w)}(y, w)) \). Equivalently, \( \gamma_w(\xi_1(y, w)) = \gamma_w(\xi_2(y, w)) \), where \( \pi_g(\phi^{\xi_1(y,w)}(y, w)) \) is close to \( \gamma'_w(t) \) for any \( (y, w) \in U_{\mathcal{L}} \). Consider the new cross-section
\[
\tilde{U}_{\mathcal{L}} = \{ \phi^{\xi_2(y,w)}(y, w) : (y, w) \in U_{\mathcal{L}} \}.
\]

Note that \( \xi_1(x_0, v_0) = t_0 \) and \( \xi_2(x_0, v_0) = 0 \), so \( (x_0, v_0) \in \tilde{U}_{\mathcal{L}} \).

Let \( (x, v) \in \tilde{U}_{\mathcal{L}} \), then there is a unique \( (y, w) \in U_{\mathcal{L}} \) such that \( (x, v) = \phi^{\xi_2(y,w)}(y, w) \) and
\[
x = \pi_g(x, v) = \pi_g(\phi^{\xi_2(y,w)}(y, w)) = \pi_g(\phi^{\xi_1(y,w)}(y, w)) = \pi_g(\phi^{\xi_1(y,w)-\xi_2(y,w)}(\phi^{\xi_2(y,w)}(y, w)))
\]
and
\[
\eta(y, w) = \xi_1(y, w) - \xi_2(y, w) \text{ is close to } t_0. \text{ This implies that for any } (x, v) \in \tilde{U}_{\mathcal{L}} \text{ there is } \eta(y, w) \text{ such that } \phi^{\eta(y,w)}(x, v) \in \pi_g^{-1}(x) \text{ and } \{ v, \gamma'_w(\eta(y, w)) \} \text{ are linearly independent and consequently } \tilde{U}_{\mathcal{L}} \subset SI_g.
\]

Thus, the above discussion leads us to see that \( SI_g \) is a cross-section. \( \square \)
Remark 4.23. If \( ST^n_g := \{(x,v) \in SI_g : |t(x,v)| < n\} \). Clearly, \( ST^n_g \subset ST^{n+1}_g \).

Moreover, given \((x,v) \in ST^n_g\), from Proposition 4.22 there is a neighborhood \(U\) of \((x,v)\) in \(SI_g\) such that \(U \subset ST^{n+1}_g\). Therefore, we can consider that \(ST^n_g\) as a submanifold of \(SM\) of dimension 2. Moreover, since the transversality condition is open and dense, then, from now on, we assume that \(\Sigma_g\) is transverse to the surface \(SI_g\) and \(ST^n_g\).

Remark 4.24. Under the conditions of the above remark, \(\Sigma \cap SI_{g_0}\) is a finite family of smooth curves \(\alpha\). So, by Lemma 4.10 given \(\beta > 0\) we have that there is a sub-horseshoe, \(\Theta^s_\alpha\), of \(\Theta\) such that \(\Theta^s_\alpha \cap \Theta = \emptyset\) satisfying the \(\beta\)-stable property. Analogously, there is a sub-horseshoe \(\Theta^u_\alpha\) for the unstable case. So, in order to avoid any confusion in the notation, we put \(\Theta_1 := \Theta^s_\alpha\) and \(\Theta_2 := \Theta^u_\alpha\).

Let \(\beta > 0\) be small enough, let \(f \in C^s_{\Theta_1,\beta}\), then there is a sub-horseshoe \(\Theta^s_{1,f}\) of \(\Theta_1\) which satisfies the Definition 4.18. Analogously, if \(f \in C^u_{\Theta_2,\beta}\) there is a sub-horseshoe \(\Theta^u_{2,f}\) of \(\Theta_2\) satisfying of Definition 4.18.

Definition 4.25. We say that \(f \in A^s_{\Theta_1,\beta} \subset C^s_{\Theta_1,\beta}\) if there is \(z_f \in M_{\max(f \circ \pi)}(\Theta^s_{1,f})\) (the set of maximum point of \(\max(f \circ \pi)\) on \(\Theta^s_{1,f}\)) such that

\[
\max(f \circ \pi)_\phi(z_f) > \max(f \circ \pi)_{\phi}(z), \quad z \in \pi^{-1}(\pi(z_j)) \cap \Theta^s_{1,f}.
\]

Changing \(s\) by \(u\) and \(\Theta_1\) by \(\Theta_2\), we have the definition of \(A^u_{\Theta_2,\beta}\).

Lemma 4.26. The sets \(A^s_{\Theta_1,\beta}\) and \(A^u_{\Theta_2,\beta}\) are dense and \(C^2\)-open.

Proof. We prove the lemma for \(A^s_{\Theta_1,\beta}\) since the proof for \(A^u_{\Theta_2,\beta}\) is analog. By definition, the set \(A^s_{\Theta_1,\beta}\) is \(C^2\)-open. To prove the density, we have to prove simply that \(A^s_{\Theta_1,\beta}\) is dense in \(C^s_{\Theta_1,\beta}\). In fact: Let \(f \in C^s_{\Theta_1,\beta}\), and \(z_f = (x,v)\) a maximum point of \(\max(f \circ \pi)\) in \(\Theta_1\). We can assume, without loss of generality, that \(\pi^{-1}(\pi(z_j)) \cap \Sigma\) and therefore \(\#(\pi^{-1}(\pi(z_j)) \cap \Sigma) < \infty\). If \(f\) satisfies the condition of Definition 4.25 the result is proven. Otherwise, we can suppose that there is \(z = (x,w) \in \pi^{-1}(\pi(z_f)) \cap \Theta^s_{1,f}\) such that \(\max(f \circ \pi)_\phi(z_f) = \max(f \circ \pi)_\phi(z)\), put \(\tilde{z}_f := \phi^{(\ell(z_f))}(z_f)\) and \(\tilde{z} := \phi^{(\ell(z))}(z)\) such that \((f \circ \pi)(\tilde{z}_f) = (f \circ \pi)(\phi^{(\ell(z_f))}(z_f)) = \max(f \circ \pi)_\phi(z_f) = \max(f \circ \pi)_\phi(z) = \pi (\tilde{z}) = f \circ \pi (\phi^{(\ell(z))}(z))\).

Claim: The points \(\tilde{z}_f\) and \(\tilde{z}\) satisfies

1) \(\pi(\tilde{z}_f) \neq \pi(\tilde{z})\),

2) \(\pi^{-1}(\pi(\tilde{z}_f)) \cap \{\phi^l(z) : t \in [t_-(z), t_+(z)]\} = \emptyset\).

Proof of Claim. First we prove item 1) By contradiction, assume that \(\pi(\tilde{z}_f) = \pi(\tilde{z})\), then as \(df_{\pi(\tilde{z}_f)} \neq 0\) (see Remark 4.20) and \(0 = \frac{d}{dt}(f \circ \pi((\phi^l(z))))|_{t=t(z)} = df_{\pi(\tilde{z}_f)} \gamma'_u(t(z))\) and \(0 = \frac{d}{dt}(f \circ \pi((\phi^l(z))))|_{t=t(z)} = df_{\pi(\tilde{z})} \gamma'_u(t(z))\), we have that \(\gamma'_u(t(z))\) and \(\gamma'_u(t(z))\) are parallel. Thus, due the uniqueness of geodesics, we must have \(\gamma'_u(t(z)) = -\gamma'_u(t(z))\). Note that \(\Theta\) does not contain fixed points (see Remark 4.15), then \(v\) and \(w\) are Linearly Independence and therefore \(z_f \in SI_{g_0}\), which is a contradiction by Remark 4.24 and the proof of item 1) is complete.

Let us now prove item 2). Suppose by contradiction that \(\phi^l(z) \in \pi^{-1}(\pi(\tilde{z}_f))\). Then
\(f \circ \pi(\phi^t(z)) = f \circ \pi(\hat{z}) = f \circ \pi(\hat{z}) = f \circ \pi(\phi^t(z)).\) By uniqueness of the maximum points of function \(t \to f \circ \pi(\phi^t(z))\) (see Remark 4.20), we have that \(t_1 = t(z),\) i.e., \(\hat{z} = \phi^t(z) = \phi^t(z) \in \pi^{-1}(\pi(\hat{z}))\), which is not possible by item 1) and the proof of item 2) is complete. \(\square\)

Following the proof of the lemma, note that the second property of claim provides that for \(z \in (\pi^{-1}(\pi(z))) \setminus \{z\}\) and \(\Theta_{1,f}^*\) with \(\max(f \circ \pi(\phi_z)) = \max(f \circ \pi(\phi))(z)\), there are neighborhoods \(U_z\) of \(z\) in \(\Sigma\) and \(U_{\pi(\hat{z})}\) of \(\pi(\hat{z})\) such that

\[
\left( \bigcup_{y \in V_z} \{ \phi^t_y(y) : t \in [t_-(y), t_+(y)] \} \right) \cap \pi^{-1}(U_{\pi(\hat{z})}) = \emptyset. \tag{10}
\]

Let \(R_{1,f}\) be Markov partition from \(\Theta_{1,f}^*\) of Definition 4.18. Given \(\epsilon > 0\) sufficiently small, let \(0 < \varphi < \epsilon\) a \(C^\infty\)-bump function with a unique maximum point equal to \(\epsilon\) in \(\pi(\hat{z})\) and \(0\) outside of \(U_{\pi(\hat{z})}\). Then, \(z_f\) is a maximum point of \(\max(f + \varphi)\) in \(\Theta_{1,f}^*\).

Moreover, if \(z \in (\pi^{-1}(\pi(z))) \setminus \{z\}\) and \(\Theta_{1,f}^*\), then by equation (10), we have that \(\phi^t(z) \notin \pi^{-1}(U_{\pi(\hat{z})})\), thus

\[
(f + \varphi) \circ \pi(\phi^t(z)) = f(\pi(\phi^t(z))) + \varphi(\pi(\phi^t(z))) < f(\pi(\hat{z})) + \epsilon = f(\pi(\hat{z})) + \varphi(\pi(\hat{z})) = \max(f + \varphi)(z),
\]

i.e., \(\max(f + \varphi)(z) < \max(f + \varphi)(z_f)\).

The previous property together with the fact that \(\pi^{-1}(\pi(z)) \cap \Sigma\) implies that there is a neighborhood \(U_{z_f}\) of \(z_f\) in \(\Sigma\) such that

\[
\max(f + \varphi)(\hat{z}) < \max(f + \varphi)(x)\]

Finally, note that for \(\epsilon\) small enough, \(f + \varphi \in \mathcal{C}_{\Theta_{1},\beta}^t\) and \(\Theta_{1,f}^* + \varphi\) is close to \(\Theta_{1,f}^*\) (see Remark 4.20). Therefore, the last inequality holds for \(f + \varphi\) and \(\Theta_{1,f}^* + \varphi\), and then \(f + \varphi \in \mathcal{A}_{\Theta_{1},\beta}^t\) as we wished. \(\square\)

**Corollary 4.27.** With the notation of Lemma 4.26, if \(f \in \mathcal{A}_{\Theta_{1},\beta}^t\) there is a neighborhood \(U_{z_f} \subset \Sigma\) of \(z_f\) such that

\[
\max(f \circ \pi_\phi(x) > \max(f \circ \pi_\phi(x))
\]

for all \(x \in U_{z_f}\) and \(\tilde{x} \in (\pi^{-1}(\pi(x)) \setminus \{x\}) \cap \Theta_{1,f}^*\).

**Proof.** Without loss of generality, we can assume that \(\pi^{-1}(\pi(z)) \cap \Sigma\) and there is a neighborhood \(V\) of \(z_f\) such that

\[
\pi^{-1}(\pi(z)) \cap \Sigma, \text{ for all } z \in V.
\]

By contradiction, suppose that there are \(z_n \rightarrow z_f\) and \(\hat{z}_n \in \pi^{-1}(\pi(z_n)) \cap \Theta_{1,f}^*\) such that \(\max(f \circ \pi_\phi(z_n)) \leq \max(f \circ \pi_\phi(\hat{z}_n)).\) By compactness and transversality, we assume that \(\hat{z}_n \rightarrow z \in \pi^{-1}(\pi(z_f)) \setminus \{z_f\}\). Therefore, since \(z_f\) is a maximum point of \(\max(f \circ \pi_\phi)\), then \(\max(f \circ \pi_\phi(z_f)) = \max(f \circ \pi_\phi(z)),\) so \(f \notin \mathcal{A}_{\Theta_{1},\beta}^t\) and we have a contradiction. \(\square\)
Proof of Lemma 4.9. We simply prove the lemma for \( \tilde{\mathcal{H}}_{\Theta, \beta} \) since the proof for \( \mathcal{H}_{\Theta, \beta} \) is analog. The openness \( \mathcal{H}_{\Theta, \beta}^* \) is a consequence of its definition. By density of Lemma 4.26 it suffices to prove that \( \mathcal{H}_{\Theta, \beta}^* \) is dense in \( \mathcal{A}_{\Theta_1, \beta/2}^* \).

By definition, if \( f \in \mathcal{A}_{\Theta_1, \beta/2}^* \), there is \( z_f = (x, v) \in \Theta_1^* f \) a maximum point of \( (f \circ \pi)_\phi \). Then by Corollary 4.27 there is a neighborhood \( U_{\tilde{z}_f} \) of \( z_f \) such that

\[
\max(f \circ \pi)_\phi(x) > \max(f \circ \pi)_\phi(\tilde{x})
\]

for all \( x \in U_{\tilde{z}_f} \) and \( \tilde{x} \in (\pi^{-1}(\pi(x)) \setminus \{x\}) \cap \Theta_1^* f \). It is easy to see that there is \( h \in C^2(\Sigma \cap R_1^* f, \mathbb{R}) \), \( C^2 \)-close to the null function, \( h = 0 \) outside of \( U_{\tilde{z}_f} \), and such that the function \( \max(f \circ \pi)_\phi + h \in H_1(\mathcal{R}, \Theta_1^* f) \) with the maximum point \( \tilde{z}_f \) in \( U_{\tilde{z}_f} \), i.e., \( \tilde{z}_f \in M_{\max(f \circ \pi)_\phi + h}(\Theta_1^* f) \cap U_{\tilde{z}_f} \) (the set of maximum points of \( f \) in \( \Theta_1^* f \cap U_{\tilde{z}_f} \)). Without loss of generality, we can assume that \( \pi|_{U_{\tilde{z}_f}} : U_{\tilde{z}_f} \rightarrow \pi(U_{\tilde{z}_f}) \) is a diffeomorphism. Put \( \tilde{h} := h \circ (\pi|_{U_{\tilde{z}_f}})^{-1} : \pi(U_{\tilde{z}_f}) \rightarrow \mathbb{R} \) and put \( j \) equal to \( 0 \) outside of the neighborhood of \( \pi(U_{\tilde{z}_f}) \).

**Claim:** The function \( \max(f \circ \pi)_\phi + \tilde{h} \circ \pi \in H_1(\mathcal{R}, \Theta_1^* f) \).

**Proof of Claim.** We have to prove simply that \( \tilde{z}_f \) is the only maximum point of \( \max(f \circ \pi)_\phi + \tilde{h} \circ \pi \) in \( \Theta_1^* f \). Since \( \tilde{h} \circ \pi = h \) on \( U_{\tilde{z}_f} \), then \( \tilde{z}_f \in M_{\max(f \circ \pi)_\phi + h}(\Theta_1^* f) \cap U_{\tilde{z}_f} \). Now, if \( z \in (\pi^{-1}(\pi(U_{\tilde{z}_f})) \setminus U_{\tilde{z}_f}) \cap \Theta_1^* f \), then there is \( x \in U_{\tilde{z}_f} \) such that \( z \in \pi^{-1}(\pi(x)) \). Thus \( \tilde{h} \circ \pi(z) = \tilde{h} \circ \pi(x) \). Moreover, by inequality (11) we have \( \max(f \circ \pi)_\phi(x) > \max(f \circ \pi)_\phi(z) \). This implies that

\[
\max(f \circ \pi)_\phi(\tilde{z}_f) + \tilde{h} \circ \pi(\tilde{z}_f) > \max(f \circ \pi)_\phi(x) + \tilde{h} \circ \pi(x) > \max(f \circ \pi)_\phi(z) + \tilde{h} \circ \pi(z).
\]

Again, since \( \tilde{h} \circ \pi = 0 \) on \( \Sigma \setminus \pi^{-1}(\pi(U_{\tilde{z}_f})) \), then the last inequality holds for \( z \in \Theta_1^* f \setminus \pi^{-1}(\pi(U_{\tilde{z}_f})) \), and therefore \( \tilde{z}_f \) is the unique maximum point of \( \max(f \circ \pi)_\phi + \tilde{h} \circ \pi \) on \( \Theta_1^* f \) and consequently, this proves the claim. \( \square \)

To finish the proof of lemma, note that the function

\[
f_1 = \begin{cases} 
    f + \tilde{h}, & \text{on } \pi(U_{\tilde{z}_f}), \\
    f, & \text{otherwise}
\end{cases}
\]

is \( C^2 \)-close to \( f \) with

\[
\max(f_1 \circ \pi)_\phi = \max(f \circ \pi)_\phi + \tilde{h} \circ \pi \in H_1(\mathcal{R}, \Theta_1^* f).
\]

Moreover, as \( \mathcal{A}_{\Theta_1, \beta/2}^* \subset \mathcal{C}_{\Theta_1, \beta/2}^* \) then by definition of \( \mathcal{C}_{\Theta_1, \beta/2}^* \) (see Definition 4.18) and Remark 4.24 we have that \( f_1 \in \mathcal{H}_{\Theta, \beta}^* \) with the sub-horseshoe \( \Theta_1^* f \) and the Markov partition \( R_1^* f \), which concludes the proof. \( \square \)

**Remark 4.28.** Note that the proof of Lemma 4.9 also allows us to conclude that: If \( \Theta \) and \( \Gamma \) are two basic sets of \( \mathcal{R} \), then the set \( \mathcal{H}_{\Theta, \Gamma, \beta} \subset \mathcal{H}_{\Theta, \beta} \cap \mathcal{H}_{\Gamma, \beta} \), which satisfies

\[
\max(f \circ \pi)|_{\Sigma \cap (R_1^* \Theta \cup R_1^* \Gamma)} \in H_1(\mathcal{R}, \Theta_1^* f \cup \Gamma_1^* f) \subset C^1(\Sigma \cap (R_1^* \Theta \cup R_1^* \Gamma), \mathbb{R})
\]

is disjoint from \( \mathcal{C}_{\Theta_1, \beta/2}^* \setminus \mathcal{C}_{\Theta_1, \beta/2}^* \).
is dense and $C^2$-open.

Here $R_{j,\Theta}$ and $R_{j,\Gamma}$ are the Markov partition of $\Theta^j$ and $\Gamma^j$ given by the definition of $\tilde{H}_{\Theta,\beta}$ and $\tilde{H}_{\Gamma,\beta}$, respectively.

Analogously, since Remark 4.8 (compare with Remark 4.7), it is easy to see that, if $V$ is small enough, then for any $g \in V$ there is $\tilde{\beta}$ (which depends on $g$) such that

$$\tilde{H}_{\Theta_g, \Gamma, \tilde{\beta}} = \tilde{H}_{\Theta, \Gamma, \beta},$$

and consequently $\tilde{H}_{\Theta_g, \Gamma, \tilde{\beta}}$ is a dense and $C^2$-open set, where $\Theta_g$ is the hyperbolic continuation of $\Theta$ by $R^g$, respectively (see Section 3.1).

5 The Family of Perturbations of Metric

Recall that in Section 3 it was proven that there are a finite number of smooth-GCS, $\Sigma_i$ pairwise disjoint and such that the Poincaré map $R: \Sigma \to \Sigma$ satisfies:

- $\bigcap_{n \in \mathbb{Z}} R^{-n}(\Sigma) := \Delta$ is hyperbolic set for $R$.
- $HD(\Delta) \sim 2$.

5.1 Nonzero Birkhoff Invariant and Property V

The family of perturbations of our metric will be special and involve some properties of horseshoes, as well of regular Cantor sets. Since the geodesic flow is conservative, then Poincaré map $R$ (map of first return) of $\Sigma := \bigcup_{i=1}^j \Sigma_i$

$satisfies:

- $\bigcap_{n \in \mathbb{Z}} R^{-n}(\Sigma) := \Delta$ is hyperbolic set for $R$.
- $HD(\Delta) \sim 2$.

**Lemma 5.1.** There is a small perturbation of the metric $g_0$ such that the Birkhoff invariant, for the new Poincaré map (still denoted by $R$), is non-zero for some periodic orbit.

**Proof.** We can assume that $R$ has a fixed point such that the Birkhoff invariant is zero. Then Lemma A.5 implies that the set $Q \subset J^3(0)$ of mappings having non-zero Birkhoff invariant is open, dense, and invariant. Thus, the Klingenberg and Takens theorem (cf. Appendix A.5.2 and MY10 Section 4.3) for more details).

**Remark 5.2.** As the perturbation of $g_0$ of Lemma 5.1 is small enough, we can assume such perturbation is an element of $V \subset \mathcal{G}^3(M)$. Moreover, from Lemma 5.1 we can assume that, from now on, the Poincaré map $R$ has the property that the Birkhoff invariant is non-zero for some periodic orbit.
5.1.1 Property V

In this section, we will try to expose a second and important property that we need to achieve a good perturbation of $\mathcal{R}$.

**Definition 5.3.** Let $K, K'$ be two regular Cantor sets, we say that $K$ and $K'$ have a stable intersection if there is a neighborhood $U$ of $(K, K')$ in the set of pairs in the $C^{1+}$-regular Cantor sets such that $(\tilde{K}, \tilde{K}') \in V \Rightarrow \tilde{K} \cap \tilde{K}' \neq \emptyset$.

For more details on the definition of $C^{1+}$-topology of pairs of regular Cantor sets, see [MY01, Section 2] or [PT93, Chapter 2].

Given a horseshoe $\Gamma$ associated to a $C^2$ bi-dimensional diffeomorphism $\psi$, we consider $K^s$ and $K^u$ the stable and unstable Cantor sets associated to $\Gamma$ (see Section A.3).

**Definition 5.4.** We say that a pair $(\psi, \Gamma)$, where $\Gamma$ is a horseshoe for $\psi$, has the property $V$ if $K^s$ and $K^u$ have a stable intersection.

For more details on property $V$, we recommend in [MY01, Sections 1 and 2].

When a horseshoe has Hausdorff dimension bigger than 1, then we get the property $V$, in fact:

**Theorem 5.5 (Moreira-Yoccoz).** Let $\varphi$ be a $C^\infty$ diffeomorphism on a surface $M$ with a horseshoe $\Gamma$ of $HD(\Gamma) > 1$. If $U$ is a sufficiently small neighborhood of $\varphi$ in $Diff^\infty(M)$, there is an open and dense set $U^* \subset U$ such that, for every $\psi \in U^*$ the pair $(\psi, \Gamma_\psi)$ has the property $V$, where $\Gamma_\psi$ is the hyperbolic perturbation of $\Gamma$ by $\psi$.

We recall that $\Delta$ is a horseshoe for $\mathcal{R}$ with $HD(\Delta) > 1$, so the above theorem implies that we get the property $V$ with small perturbations of $\mathcal{R}$, but note that perturbations of $\mathcal{R}$ non-necessarily are Poincaré maps of small perturbation of the metric $g_0$. In other words, we can not use directly the above theorem. So, to get the property $V$ on the pair $(R^g, \Delta_g)$, for a small perturbation of $g_0$, we need to use the construction of the proof of the above theorem. Thus, in the next sections, we will explain such a construction.

5.2 Good Perturbations

In this section, we will prove Theorems 3.9 and 3.10. The main difficulty for this goal is to produce perturbations of the Riemannian metric so that, the perturbation obtained for the Poincaré map $\mathcal{R}$ has independence, i.e., allows to obtain property $V$ (see Subsection 5.1.1). It is worth pointing out that, small perturbations of the Riemannian metric produce great perturbations on the geodesic flow. For example, if we want to perturb $\mathcal{R}$ to obtain property $V$ and still be an application of the first return of the geodesic flow for a Riemannian metric near the initial Riemannian metric, we must keep in mind that perturbing a metric in a small neighborhood $U \subset M$ always affect the Sasaki’s metric on $\pi^{-1}(U) \subset SM$, where $\pi: SM \rightarrow M$ is the canonical projection. In other words, local perturbations of the metric, in general, do not produce local perturbations of $\mathcal{R}$. Thus, the more complicated part of this section is to produce local perturbations of $(\mathcal{R}, \Delta)$ with the property $V$ perturbing the metric.
5.2.1 Independent Perturbations of the Metric

In the sequel, we present the necessary tools to obtain the independence of the perturbations of the metric. We start with the following lemma.

**Lemma 5.6.** For any fixed $\theta \in \Delta$, put $\tilde{\theta} = \phi^{\hat{t}_+(\theta)}(\theta)$, then the surface

$$\mathcal{W}_{\tilde{\theta}} := \bigcup_{t \in (-\hat{t}_+(\theta), \hat{t}_+(\theta))} \phi^t(\pi^{-1}(\pi(\tilde{\theta})))$$

intersects transversely $\Sigma$, therefore $\mathcal{W}_{\tilde{\theta}} \cap \Sigma = \{\beta_1, \ldots, \beta_l\}$ is a finite family of curves. Moreover, if $z \in \beta_i \cap \Delta \cap \Sigma$ for some $i \in \{1, \ldots, l\}$, then

$$\beta_i \cap W^s(z, \Sigma) \quad \text{and} \quad \beta_i \cap W^u(z, \Sigma).$$

**Proof.** For $(p, v) \in \pi^{-1}(\pi(\tilde{\theta}))$ and a non-zero vector $\xi \in T_{(p, v)} \pi^{-1}(\pi(\tilde{\theta}))$ (the tangent space of $\pi^{-1}(\pi(\tilde{\theta}))$ in $(p, v)$), then the tangent space to $\mathcal{W}_{\tilde{\theta}}$ in $\phi^t(\xi)$ is

$$T_{\phi^t((p, v))}\mathcal{W}_{\tilde{\theta}} = \text{span}\{d\phi^t((p, v))(\xi), d\phi^t((p, v))\} = \text{span}\{d\phi^t((p, v))(\xi), \phi^t((p, v))\}.$$ 

Therefore, since $\phi \cap \Sigma$, then $\mathcal{W}_{\tilde{\theta}} \cap \Sigma := \{\beta_1, \ldots, \beta_l\}$ a finite set of $C^1$-curves.

Let $z \in \beta_i \cap \Delta \cap \Sigma$ for some $i \in \{1, \ldots, l\}$, then there is $\omega \in \pi^{-1}(\pi(\tilde{\theta}))$ such that $z = \phi^t(\omega)$, so

$$T_{\tau_i}(\mathcal{W}_{\tilde{\theta}} \cap T_{\tau_i} \Sigma) = \text{span}\{d\phi^t(\omega), \phi^t(\omega)\} \cap T_{\tau_i} \Sigma,$$

where $\tau_i \in T_{\tau_i}(\pi^{-1}(\pi(\tilde{\theta}))$ is a non-zero vector.

Remember that $T_{\tau_i} W^s(z, \Sigma) = (E^s(z) \oplus \text{span}(\phi(z))) \cap T_{\tau_i} \Sigma$ for $i = s, u$, then since $E^s$ is an invariant sub-bundle for $d\phi^t$, $i = s, u$, it follows that $T_{\tau_i} W^u(z, \Sigma) = (d\phi^t(E^u(\omega)) \oplus \text{span}\{\phi^t(\omega)\}) \cap T_{\tau_i} \Sigma$. Therefore, as $\eta$ is a vertical vector, Lemma A.3 implies that $d\phi^t(\eta) \notin (d\phi^t(E^u(\omega)) \oplus \text{span}\{\phi^t(\omega)\})$, $i = s, u$, which completes the proof of the lemma. \hfill \Box

**Definition 5.7.** Let $\theta, \tilde{\theta}$ two words and $\mathcal{R} = \{R_1, R_2, \ldots, R_k\}$ a set of words of the same alphabet $\mathcal{B}$. We say that the word $\theta \tilde{\theta}$ is prohibited by $\mathcal{R}$ if there is $R \in \mathcal{R}$ inside (as a factor of) $\theta \tilde{\theta}$.

![Diagram](image)

**Definition 5.8.** Let $\Theta_1, \Theta_2$ be two disjoint sub-horseshoes of $\Delta$ and a Markov Partition $\mathcal{R}_i$ of $\Theta_i$, $i = 1, 2$. We say that $R_1 \in \mathcal{R}_1$ disturbs $R_2 \in \mathcal{R}_2$ if

$$T_{\tau_{\tau_{R_2}}} \cap \tau_{\tau_{R_1}^{1/2}} \neq \emptyset,$$

where $T_{\tau_{R_2}} = \{\phi^t(x) : x \in R_2, 0 \leq t \leq t_+(x)\}$, $\tau_{\tau_{R_1}^{1/2}} = \pi^{-1}(\pi(\tau_{R_1}^{1/2}))$ and $\tau_{\tau_{R_1}^{1/2}} = \phi^{\hat{t}_1}(R_1)$ with $\hat{t}_1 = \sup_{x \in R_2 \cap \Delta} t_+(x)$. We say that $\Theta_1$ has no interference on $\Theta_2$ if no element of $\mathcal{R}_1$ disturbs any element of $\mathcal{R}_2$. 

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The main goal of this section is to prove the following lemma, which will give us the region where we can perturb the metric independently.

Lemma 5.9. There are two disjoint sub-horseshoe $\Delta_2$ and $\Delta_3$ of $\Delta$ such that $\Delta_2$ has no interference on $\Delta_3$ and $HD(K^s_3) + HD(K^u_3) > 1$, where $K^s_3$ is the stable Cantor set of $\Delta_3$ and $K^u_2$ is the unstable Cantor set of $\Delta_2$.

The remainder of this section is dedicated to proving Lemma 5.9. For this sake, we introduce some auxiliary lemmas which will help reach this goal.

Lemma 5.10. Let $\Delta_1 \subset \Delta$ a sub-horseshoe with $0 < HD(\Delta_1) =: \lambda < \frac{1}{2}$, then there exists another sub-horseshoe $\Delta_2 \subset \Delta$ with the following properties:

1. $HD(K^s_2)$ is sufficiently close $HD(K^u)$, where $K^s_2$ and $K^u$ are the unstable Cantor sets associated with $\Delta_2$ and $\Delta$, respectively;
2. $\Delta_1 \cap \Delta_2 = \emptyset$;
3. $\Delta_2$ has no interference on $\Delta_1$.

Proof. Consider a Markov partition $\mathcal{R}$ of $\Delta$ into squares of size $\epsilon$. Then $\mathcal{R}$ has approximately $\epsilon^{-d}$ squares, where $d = HD(\Delta)$. Consider, analogously, a set $\mathcal{R}_{\Delta_1} \subset \mathcal{R}$ of the order of $\epsilon^{-\lambda}$ squares of size $\epsilon$ forming a Markov partition of $\Delta_1$. Observe also that, given $p \in \Delta_1$ belonging to a cross-section $\Sigma_i$, the projection by the flow $\phi^t$ of the fiber $\pi^{-1}(\pi(\phi^t(p)/2(p)))$ is a curve (or a finite union of curves), and so, as in the proof of Lemma [4.10] ([Rn21 Lemma 4.6]), each square of $\mathcal{R}_{\Delta_1}$ has interference on at most the order of $\epsilon^{-d/2}$ (which is much smaller than $\epsilon^{-1}$) squares of $\mathcal{R}$. Thus, the squares of $\mathcal{R}_{\Delta_1}$ have interference on at most $\epsilon^{-\lambda} \cdot \epsilon^{-1} \leq \epsilon^{-3/2}$ squares of $\mathcal{R}$. We call $\mathcal{X} \supset \mathcal{R}_{\Delta_1}$ the set of squares that suffer interference of some square of $\mathcal{R}_{\Delta_1}$. Therefore, we have $\tilde{N} := \#(\mathcal{R} \setminus \mathcal{X}) \geq \epsilon^{-d} - \epsilon^{-3/2}$ squares of $\mathcal{R}$ which do not suffer interference of any square of $\mathcal{R}_{\Delta_1}$.

The maximal invariant set by $\mathcal{R}$ of the union of these $\tilde{N}$ remaining squares in $\mathcal{R} \setminus \mathcal{X}$, which will be the sub-horseshoe $\Delta_2' \subset \Delta$. By the above construction, this sub-horseshoe $\Delta_2'$ satisfies conditions 2 and 3 of the lemma. In what follows, we will estimate the size of $\Delta_2'$.

Let $\{\tilde{\theta}_1, \ldots, \tilde{\theta}_{\tilde{N}}\}$ be the words associated with the remaining squares. They generate intervals of the length of the order of $\epsilon^2$ in $W^u(\Delta)$ (of the construction of the unstable regular Cantor set). Without loss of generality, we can assume, as in [Rn21 Remark 13], that the transitions $\tilde{\theta}_i \tilde{\theta}_j$ are admissible for all $i, j \in \{1, \ldots, \tilde{N}\}$.

Claim: $\#(\tilde{\theta}_i \tilde{\theta}_j$ prohibited by $\mathcal{X}) = O(\tilde{N}^2)$.

Proof of Claim. Since the product of the lengths of the intervals in $W^u(\Delta)$ generated by the words $\alpha$ and $\beta$ is of the order of $\epsilon^4/\epsilon^2 = \epsilon^2$, the number of possibilities for the word $\#(\alpha \beta)$ is of the order of $\tilde{N}$. On the other hand, the size of each word $\tilde{\theta}_i$ (which gives an upper bound for the number of positions where the word $R$ begins) is of the order of $\log \epsilon^{-d}$, which is of the order of $\log \tilde{N}$. Then each word $R$ corresponding to a square in $\mathcal{X}$ prohibits $O(\tilde{N} \log \tilde{N})$ transitions. So we have in total $O(\epsilon^{-3/2} \tilde{N} \log \tilde{N})$ prohibited transitions $\tilde{\theta}_i \tilde{\theta}_j$. Since $d > 3/2$ and $\tilde{N}$ is of the order of $\epsilon^{-d}$, we have $\epsilon^{-3/2} \tilde{N} \log \tilde{N} = o(\tilde{N}^2)$. 

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This shows that the number of prohibited transitions is much smaller than the total number of transitions. So, consider the following matrix $A$ for $i, j \in \{1, \ldots, \tilde{N}\}$

$$a_{ij} = \begin{cases} 1 & \text{if } \tilde{\theta}_i \tilde{\theta}_j \text{ is not prohibited;} \\ 0 & \text{if } \tilde{\theta}_i \tilde{\theta}_j \text{ is prohibited by } \mathcal{X}. \end{cases}$$

The above claim states that $\#\{a_{ij} : a_{ij} = 1\} \geq \frac{99}{100} \tilde{N}^2$, so from [Rn21, Remark 13], there is a sub-horseshoe $\Delta_2$ of $\Delta'_2$ that satisfies condition (1). Also, as $\Delta'_2 \subset \Delta_2$, then $\Delta'_2$ also satisfies properties (2) and (3).

Notice that the sub-horseshoe $\Delta_1 \subset \Delta$ in Lemma 5.10 with $0 < H_D(\Delta_1) < \frac{1}{2}$ can be taken such that $H_D(K^s_1) \sim \frac{1}{4}$, where $K^s_1$ is the regular stable Cantor set associated to $\Delta_1$. Now we construct a family of sufficiently independent perturbations of $\mathcal{R}$ in a neighborhood of a suitable sub-horseshoe of $\Delta_1$. Fix $n \in \mathbb{N}$ large and let $\mathcal{S} T^n_{g_0}$ be as in Subsection 4.2.1 (see Remark 4.23). Since the transversality condition is open and dense, then we can suppose that the $\Sigma \cap \mathcal{S} T^n_{g_0}$ is a finite family of smooth curves and from Lemma 4.10 applied to the family of curves $\alpha_n$ and the sub-horseshoe $\Delta_1$ (compare with Remark 4.23), we have that given $\tilde{\delta} > 0$ there is a sub-horseshoe $\Delta_0$ of $\Delta_1$ such that $\Delta_0 \cap \alpha = \emptyset$ and satisfies $\tilde{\delta}$-stable property (see Definition 4.1), that is,

$$H_D(K^s_0) \geq H_D(K^s_1) - \tilde{\delta},$$

where $K^s_0$ and $K^s_1$ are the stable Cantor sets associated to $\Delta_0$ and $\Delta_1$, respectively.

Moreover, as the set of periodic points of $\mathcal{R}$ of period smaller than $n$ in $\Delta_1$ is finite, we may also assume that $\Delta_0$ does not contain any periodic point of period smaller than $n$. In short, $t_+ (\theta) \geq n$, $\theta \in \Delta_0$.

**Remark 5.11.** Given a positive integer $n$, large enough, choose a Markov partition $\mathcal{R}_0$ of $\Delta_0$ such that for each $R_a \in \mathcal{R}_0$ there is a neighborhood $U_a$ of $R_a$ with the property $\phi^r(U_a) \cap (\tau_{U_a^{1/2}} \setminus U_a^{1/2}) = \emptyset$ for

$$\inf_{x \in U_a} t_0(x) < t \leq \sup_{x \in U_a} \sum_{i=0}^n t_i(x) \text{ or } \sup_{\theta \in U_a} \sum_{i=0}^n t_i(\theta) \leq t < 0,$$

where $t_i(\theta) = t_+ (\mathcal{R}^i(\theta))$ and $\tau_{U_a^{1/2}} = \pi^{-1}(\pi(U_a^{1/2}))$. In particular, $\mathcal{R}^r(U_a) \cap \tau_{U_a^{1/2}} = \emptyset$ for $0 < r \leq n$.

The above Remark will be essential to count the number of prohibited transitions of a suitable Markov partition of $\Delta_0$ (see proof of Lemma 5.12).

Let $\mathcal{R}_0 = \{R_1, \ldots, R_N\}$ be a Markov partition by squares of size $\epsilon^{1/2}$ of $\Delta_0$ as in Remark 5.11. Note that since $\mathcal{R}$ is conservative, then each square of this Markov partition $\mathcal{R}_0$, corresponds to an interval of the size of the order of $\epsilon$ in $W^\epsilon(\Delta_0)$ (of the
constructions of the stabler Cantor set of $\Delta_0$): there is an iterate of the square which is a strip in the unstable direction, whose basis is this interval. We call $X := \{\theta_1, \ldots, \theta_N\}$ the set of words associated with the intervals corresponding to the squares of $\mathcal{R}_0$ in $W^s(\Delta_0)$. Without loss of generality (by considering, if necessary, a suitable sub-horseshoe with almost the same dimension), we can assume, as in Remark 13 at [Rn21], that the transitions $\theta_i \theta_j$ are all admissible. Remember that $\theta_i$ disturbs $\theta_j$ if $i \neq j$ and $T_{R(\theta_i)} \cap \tau_{R(\theta_j)^{1/2}} \neq \emptyset$, where $R(\theta_i)$ is the square associated to the word $\theta_i$. Consider $P_{\theta_i} := \{\theta_j : \theta_i \text{ disturbs } \theta_j\} = \{\theta_{r_1(i)}, \ldots, \theta_{r_\delta(i)}\}$, and similar to the proof of Lemma 5.10 $|P_{\theta_i}| = O(N^{1/2})$.

**Lemma 5.12.** Given a constant $\delta \in (0, 1)$, there is a positive integer $n$ such that, if $\Delta_0$ is a sub-horseshoe of $\Delta_1$ as in Remark 5.11 then, for any $i \leq N$, 

$$\#\{\theta_i \theta_j \text{ or } \theta_j \theta_i \text{ prohibited by } P_{\theta_i}, \ j \leq N\} \leq \delta N.$$ 

**Proof.** Let us consider transitions of the type $\theta_i \theta_j$. Write $\theta_i = \alpha \beta \gamma$ such that $\alpha$ is associated to an interval of the size of the order $\epsilon^{1/2}$ and $\beta$, $\gamma$ are associated to intervals of sizes of order $\epsilon^{1/4}$ in $W^s(\Delta_0)$. Also, let $\alpha = s_1 s_2 \ldots s_r$, $\beta = s'_1 s'_2 \ldots s'_t$, $\gamma = s''_1 s''_2 \ldots s''_u$, and $P_{\theta_i} = \{\theta_{r_1(i)}, \ldots, \theta_{r_\delta(i)}\}$. If $\theta_i$ prohibits the transition $\theta_i \theta_j$ then there exists a word $\theta_{r_\delta(i)} \in P_{\theta_i}$ inside (as a factor of) $\theta_i \theta_j$.

Let us first show that the word $\theta_{r_\delta(i)}$ cannot begin too close from the beginning of $\theta_i$ itself. More precisely, if it begins with a letter $s_k$ of $\alpha$ then we should have $k > n$. Indeed, if $k \leq n$ then the square $R_i$ of the Markov partition $\mathcal{R}_0$ corresponding to $\theta_i$ is such that $\mathcal{R}^{k-1}(R_i)$ intersects $R_{r_\delta(i)}$ (notice that $k > 1$ since, by definition, $r_1(i) \neq i$). Since, for some $(x, v) \in R_{r_\delta(i)}$, there is $0 \leq t \leq t_+(x, v)$ such that $\phi^t(R_{r_\delta(i)}) \cap \tau_{R_i^{1/2}} \neq \emptyset$. If we take $(y, w) \in R_i$ and $\tilde{t} = \sum_{i=0}^{k-2} t_i(y, w)$ so that $\mathcal{R}^{k-1}(y, w) = \phi^{\tilde{t}}(y, w)$, then $\phi^{\tilde{t}+t}(U_i) \cap \tau_{R_i^{1/2}} \neq \emptyset$, which is a contradiction with the stated in Remark 5.11. Now we consider three possibilities:

- Assume that a word $\theta_{r_{1(i)}}$ begins with the letter $s_k$ of $\alpha$ (with $k > n$).

Then, if $\hat{\alpha}$ is the factor of $\theta_i$ beginning by the letter $s_k$ of $\alpha$ (and also an initial factor of $\theta_{r_{1(i)}}$) associated to an interval of the size of the order $\epsilon^{1/2}$. The square associated with the word $\theta_{r_{1(i)}}$ belongs to a strip in the unstable direction corresponding to the interval (of the size of the order $\epsilon^{1/2}$ in $W^s(\Delta_0)$ associated with the word $\hat{\alpha}$. The previous Lemma implies that there exists a constant $\hat{C} > 0$ (which depends on the transversality constants in the previous Lemma, but is independent of $\epsilon$) such that $\theta_i$ disturbs at most $\hat{C}$ squares in this strip. So, given $k$ in this situation, there are at most $\hat{C}$ possibilities for $\theta_{r_{1(i)}}$.

For each such word, the largest part of it will be a factor of $\theta_i$, and the remaining will be an initial factor $\hat{\alpha}$ of $\theta_j$. Let $m'$ be the minimum size of $\hat{\alpha}$, then $m'$ is of order $m$. There is a positive constant $\lambda < 1$ (hyperbolicity constant for $\Delta$) such that, for each $q \geq m'$, if the size of $\hat{\alpha}$ is $q$, then the number of words $\theta_j$ beginning by $\hat{\alpha}$ is at most $\lambda^q N$. Therefore, the number of prohibited transitions $\theta_i \theta_j$ in this situation is at most

$$N \cdot \sum_{q \geq m'} \hat{C} \cdot \lambda^q = \frac{\hat{C} N \lambda^{m'}}{1 - \lambda} < \frac{\delta N}{4}.$$ 

- Assume that a word $\theta_{r_{1(i)}}$ begins with the letter $s'_k$ of $\beta$.

Then, if $\hat{\beta}$ is the factor of $\theta_i$ beginning with the letter $s'_k$ of $\beta$ (and also an initial factor of $\theta_{r_{1(i)}}$).
In this case, a part of the word $O\theta$. Since the number of intervals of the construction of in $W^s(\Delta_0)$ whose sizes are of order $\epsilon^{1/2}$ contained in an interval of size $\epsilon^{1/4}$ is at most of order $N^{1/4}$. Then, by the discussion of the previous step, the number of squares in this strip that are disturbed by $\theta_i$ is at most of the order of $N^{1/4}$. So, given $k$ in this situation, there are at most $N^{1/4}$ possibilities for $\theta_{r(i)}$. For each such word, a part of it will be a final factor of $\theta_i$, and the remaining will be an initial factor $\hat{\alpha}$ of $\theta_j$, which corresponds to an interval of size at most of the order $\epsilon^{1/2}$. So, the number of words $\theta_j$ beginning by $\hat{\alpha}$ is at most of the order of $N^{1/2}$. Since the number of letters in $\beta$ is of the order $\log N$, the number of prohibited transitions $\theta_j\theta_j$ in this situation is $O(\log N \cdot N^{3/4}) = o(N)$.

- Assume that a word $\theta_{r(i)}$ begins with the letter $s^{i}_{0}$ of $\gamma$. In this case, a part of the word $\theta_{r(i)}$ will be a final factor of $\theta_i$, and the remaining will be an initial factor $\hat{\alpha}$ of $\theta_j$, which corresponds to an interval of size at most of order $\epsilon^{3/4}$. So, the number of words $\theta_j$ beginning by $\hat{\alpha}$ is at most of order $N^{1/4}$. Since $|P_{\theta_i}| = O(N^{1/2})$, the number of prohibited transitions $\theta_j\theta_j$ in this situation is $O(N^{3/4}) = o(N)$. Thus, the total number of prohibited transitions $\theta_i\theta_j$ is at most

$$\frac{\delta N}{4} + o(N) < \frac{\delta N}{2}.$$ 

In any case, the total of prohibited transitions is smaller than $\delta N/2$, which implies the result.

We will only give some details of the argument corresponding to the first step: we show that the word $\theta_{r(i)}$ cannot end too close to the end of $\theta_i$: it should end at least $n$ letters before it. Indeed, if it ends $k$ letters before the end of $\theta_i$, and $k < m$ then the square $R_{r(i)}$ of the Markov partition $\mathcal{R}_0$ corresponding to $\theta_{r(i)}$ is such that $\mathcal{R}_k(R_{r(i)})$ intersects $R_i$ (notice that $k > 0$ since, by definition, $r_i(i) \neq i$). Since, for some $(x, v) \in R_{r(i)}$, there is $0 \leq t \leq t_+(x,v)$ such that $\phi^t(R_{r(i)}) \cap \tau R_i^{1/2} \neq \emptyset$. If we take $(y, v) \in R_{r(i)}$ and $t = \sum_{i=1}^{k-1} t_i(y,v)$ (so that $\mathcal{R}_k(y,w) = \phi^t(y,w)$), then $\phi^{-t}(U_i) \cap \tau R_i^{1/2} \neq \emptyset$. Thus, we come to a contradiction with what was stated in Remark 5.11.

Now we will perform a probabilistic construction. First, fix a parameter $\alpha$ with $1/4 < \alpha < 1/2$. Then, we can state the following lemma.

**Lemma 5.13.** Let $f: \{1, \ldots, \lfloor N^{\alpha} \rfloor \} \rightarrow X = \{\theta_1, \ldots, \theta_N\}$ a random function (i.e., each value $f(i)$ is chosen randomly, with the uniform distribution, and independently from the other). Then $f$ is injective with probability $1 - O_N(1)$.

**Proof.** The total number of functions $f$ is $N^{\lfloor N^{\alpha} \rfloor}$. The number of injective functions among them is

$$\frac{N!}{(N - \lfloor N^{\alpha} \rfloor)!} = \prod_{j=0}^{\lfloor N^{\alpha} \rfloor - 1} (N - j).$$

So, the desired probability is

$$\frac{1}{N^{\lfloor N^{\alpha} \rfloor}} \prod_{j=0}^{\lfloor N^{\alpha} \rfloor - 1} (N-j) = \prod_{j=0}^{\lfloor N^{\alpha} \rfloor - 1} \left(1 - \frac{j}{N}\right) \geq 1 - \sum_{j=0}^{\lfloor N^{\alpha} \rfloor - 1} \frac{j}{N} \geq 1 - \frac{(N^{\alpha})^2}{2N} = 1 - \frac{1}{2N^{1-2\alpha}} = 1 - O_N(1).$$
Given three indices $i, j, k \leq \lfloor N^\alpha \rfloor$ with $j \neq k$, we will estimate the probability that given a random function $f : \{1, \ldots, \lfloor N^\alpha \rfloor \} \to X$, $f(j)f(k)$ is prohibited by $f(i)$.

We have two cases:

i) From Lemma 5.12 if $i \in \{j, k\}$, the above probability is at most $\delta$.

ii) If $i /\in \{j, k\}$, $i \in \{1, \ldots, \lfloor N^\alpha \rfloor \}$, assume that $f(i)$ prohibits $\theta_j\theta_k$. Then the situation is as in the following diagram, where we have two representations of the same word:

```
α  f(i)  β
  \theta_j   \theta_k
```

The number of possibilities for the pair $(\alpha, \beta)$ is $O(N \log N)$ and so, since $|P_{f(i)}| = O(N^{1/2})$, we have

$$
\#\{\theta_j\theta_k : \theta_j\theta_k \text{ is prohibited by } f(i)\} = O(N^{1/2} \cdot N \log N).
$$

Therefore, the probability $P_1$ that the transition $f(j)f(k)$ is prohibited by $f(i)$ is

$$
P_1 = \frac{O(N^{1/2}N \log N)}{N^2} = O(N^{-1/2} \log N). \quad (15)
$$

The previous estimates imply that, since $\alpha < 1/2$, the expected number of prohibited transitions is at most

$$
2\delta \lfloor N^\alpha \rfloor^2 + \lfloor N^\alpha \rfloor^3 \cdot O(N^{-1/2} \log N) = 2\delta \lfloor N^\alpha \rfloor^2 + O(N^{3\alpha-1/2}) < 3\delta \lfloor N^\alpha \rfloor^2.
$$

Given a function $f$ as Lemma 5.13, we put $\theta_i = f(i)$. It follows that the probability $P$ such that the number of prohibited transitions $\theta_j\theta_k$ with $j \neq k$ is $\geq 4\delta \lfloor N^\alpha \rfloor^2$ satisfies $P \leq 3/4$.

**Proof of Lemma 5.9**. With the same notation as the last paragraphs. We consider $A = (a_{ij})$ for $i, j \in \{1, \ldots, \lfloor N^\alpha \rfloor \}$ the matrix defined by

$$
a_{ij} = \begin{cases} 
1 & \text{if } \theta_i\theta_j \text{ is not prohibited;} \\
0 & \text{if } \theta_i\theta_j \text{ is prohibited by some } \theta_k \in \text{Im}f.
\end{cases} \quad (16)
$$

Then, with probability at least $1/4$, $\#\{(i, j) : i, j \in \{1, \ldots, \lfloor N^\alpha \rfloor \} \text{ with } a_{ij} = 0\}$ is at most $\lfloor N^\alpha \rfloor + 4\delta \lfloor N^\alpha \rfloor^2 < 5\delta \lfloor N^\alpha \rfloor^2$ (for $N$ large). We assume that $f$ is injective and that $\delta < 1/500$.

Define $K$ the regular Cantor set

$$
K := \{\theta_{i_1}\theta_{i_2} \cdots \theta_{i_n} \cdots : a_{i_ki_{k+1}} = 1, \forall k \geq 1\} \subset K_0^*.
$$
By the previous discussion we have \(#\{a_{ij} : a_{ij} = 1\} \geq \frac{99}{100}([N^\alpha]^2)\), so by [Rn21, Remark 13] we have

$$HD(\overline{K}) \sim \frac{\log [N^\alpha]}{-\log \epsilon} \sim \alpha HD(K_0^s).$$

Consider the sub-horseshoe of \(\Delta_0\) defined by

$$\Delta_3 := \bigcap_{n \in \mathbb{Z}} R^n \left( \bigcup_{i,j \leq [N^\alpha], a_{ij} = 1} (R(\theta_i) \cap R^{-1}(R(\theta_j))) \right), \quad (17)$$

where \(R(\theta_i)\) is the square associated to the word \(\theta_i\).

Since the stable regular Cantor set \(K_{s3}^\alpha\) of \(\Delta_3\) is equal to \(\overline{K}\), by the above discussion we have

$$HD(K_{s3}^\alpha) \sim \alpha HD(K_0^s). \quad (18)$$

As \(HD(\Delta) \sim 2\), then \(HD(K_u^\alpha) \sim 1\), then by Lemma 5.10 the sub-horseshoe \(\Delta_2\) satisfies that \(\Delta_2 \cap \Delta_3 = \emptyset\) since \(\Delta_3 \subset \Delta_1\), and \(HD(K_{u2}^\alpha) \sim HD(K_u^\alpha) \sim 1\). Thus, from (13), (14), and (18) we arrive at \(HD(K_{s3}^\alpha) \sim \frac{1}{4}\). Therefore, since \(\alpha\) can be taken equal to \(1 - 4\epsilon\), with small \(\epsilon > 0\), then \(HD(K_{s3}^\alpha) \sim \frac{1}{8} - \epsilon\). Thus

$$HD(K_{u2}^\alpha) + HD(K_{s3}^\alpha) > 1.$$  

Therefore, this concludes the proof of the lemma. \(\square\)

### 5.2.2 Getting the Property V

We use sub-horseshoes \(\Delta_2\) and \(\Delta_3\) given by Lemma 5.9 to perform a perturbation of the metric \(g_0\) to obtain the property \(V\) (cf. Section 5.1.1 and [MY10, pp. 19-20]).

From Remark 4.28 we describe some important properties of the sets \(\mathcal{H}_{\Delta_3,\Delta_2,\beta}\) and \(\tilde{\mathcal{H}}_{\Delta_3,\Delta_2,\beta}\).

#### 5.2.2.1 Properties of \(\mathcal{H}_{\Delta_3,\Delta_2,\beta}\) and \(\tilde{\mathcal{H}}_{\Delta_3,\Delta_2,\beta}\)

By definition \(\tilde{\mathcal{H}}_{\Delta_3,\Delta_2,\beta} = \{f \in C^2(M, \mathbb{R}) : f \circ \pi \in \mathcal{H}_{\Delta_3,\Delta_2,\beta}\}\). Given \(F \in \mathcal{H}_{\Delta_3,\Delta_2,\beta}\) or \(f \in \tilde{\mathcal{H}}_{\Delta_3,\Delta_2,\beta}\), we denote by \(G\), the function \(F\) or \(f \circ \pi\). Then

(i) There is a sub-horseshoe \((\Delta_3)_G^s\) of \(\Delta_3\) and a sub-horseshoe \((\Delta_2)_G^u\) of \(\Delta_2\) that satisfies the \(\beta\)-stable (unstable) property, respectively.

(ii) There is a Markov partition \(R_{G,\Delta_3}^s\) of \((\Delta_3)_G^s\), and a Markov partition \(R_{G,\Delta_2}^u\) of \((\Delta_2)_G^u\)

$$\max(G)_{\phi \in \Sigma(n(R_{\Delta_3}^s) \cup R_{\Delta_2}^u)} \in H_1(\mathcal{R}, (\Delta_3)_G^s \cup (\Delta_2)_G^u) \subset C^1(\Sigma \cap (R_{G,\Delta_3}^s \cup R_{G,\Delta_2}^u), \mathbb{R}).$$

Note that \(\Delta_2 \cap \Delta_3 = \emptyset\), \(\Delta_2\) has no interference of \(\Delta_3\), and \(HD(K_{s3}^\alpha) + HD(K_{u2}^\alpha) > 1\), then if \(\beta\) is small enough we have that
Using the notation of Remark 5.14.

Then we need to perturb the metric in some suitable opens sets (related to \((\Delta_3)_G\)) such that, for all \(i, j\)

\[
\pi(T_{R_i}) \cap \pi(R_{1/2}^{1/2}) = \emptyset.
\]

At this point, since the properties (iii) of Subsection 5.2.2.1 fit in the above conditions, then we need to perturb the metric in some suitable opens sets (related to \((\Delta_3)_F\) and \((\Delta_3)_S\)) to produce perturbations on \(\mathcal{R}\) with the property \(V\).

For \(F \in \mathcal{H}_{\Delta_3,\Delta_2,\beta}\) or \(f \in \mathcal{H}_{\Delta_3,\Delta_2,\beta}\) we denote \(\Omega^*_s\) the sub-horseshoe of \(\Delta_3\), \((\Delta_3)_F\) or \((\Delta_3)_S\), and \(\Omega^u\) the sub-horseshoe of \(\Delta_2\), \((\Delta_2)_F\) or \((\Delta_2)_S\), which satisfies the properties (i)-(ii) of Subsection 5.2.2.1. Then by the construction of \(\Delta_3\) (see proof of Lemma 5.9 or equation (17)) we can take a small Markov partition \(\{R_1, \ldots, R_k\}\) of \(\Omega^*_s\) such that, for all \(i, j\)

\[
\pi(T_{R_i}) \cap \pi(R_{1/2}^{1/2}) = \emptyset.
\]

Our task is to perturb metric \(g_0\) in \(\pi(R_i^{1/2})\), so that this perturbation is independent, i.e., the dynamics of \(\mathcal{R}\) in \(R_j\), for \(j \neq i\), does not change. Moreover, the dynamic of \(\mathcal{R}\) in the Markov partition of \(\Delta_2\) given Lemma 5.10 also does not change (see Definition 5.8).

Since the diameter of \(R_i\) is small enough, we can assume that \(\pi(R_i^{1/2})\) is contained in a normal coordinate system, i.e., there is a point \(p \in \pi(R_i^{1/2})\), an orthonormal basis \(\{e_1, e_2\}\) of \(T_pM\) (tangent space of \(p\)) and an open set \(\tilde{U}_i \subset T_pM\) such that \(\varphi: \tilde{U}_i \rightarrow U_i^{1/2}\) is a diffeomorphism defined by \(\varphi(x, y) = \exp_p(xe_1 + ye_2)\), and \(\pi(R_i^{1/2}) \subset U_i^{1/2}\).

Let \(g\) be a Riemannian metric \(C^2\)-close to the metric \(g_0\) and such that the support of \(g - g_0\) is contained in \(U_i^{1/2}\) and satisfies

\[
\pi_{g}^{-1}(x) \cap \pi_{g_0}^{-1}(x) = \emptyset \quad \text{for all} \quad x \in \pi(R_i^{1/2}),
\]

where \(\pi_{g}^{-1}(x) = \{v \in T_xM : \|v\|_g := \sqrt{g(v, v)} = 1\}\) and \(S_{g_0}^{-1}(x) = \pi^{-1}(x)\).

**Remark 5.14.** Using the notation of Definition 5.8, consider \(V_i\) a neighborhood of \(R_i\) and put \(t_+ = \sup_{x \in R_i} t(x)\), then we define the \(\mathcal{R}_{1/2}: V_i \rightarrow \phi^{1/2}(V_i)\) as \(\mathcal{R}_{1/2}(\theta) = \phi^{t(\theta)}(\theta)\), where \(\phi^{t(\theta)}(\theta) \in \phi^{1/2}(V_i)\) is the first hit. We recall that \(S^g: S^gM \rightarrow S^gM\)
is a $C^2$-diffeomorphism and denote by ${\mathcal T}^g := (S^g)^{-1}$. Thus, analogously, we consider $R_{1/2}^g : {\mathcal T}^g(V_i) \to \phi^g_{1/2}({\mathcal T}^g(V_i))$ the first hit, where $\phi^g_t$ is the geodesic flow of the metric $g$.

**Lemma 5.15.** Let $g$ be a Riemannian metric $C^2$-close enough to $g_0$ which satisfies the equation \([19]\). Then

$$R^g(\mathcal{T}^g(W^s_R(z))) \cap \mathcal{T}^g(R(W^s_R(z))) = \emptyset \quad \text{for all} \quad z \in R_i \cap \Omega^*_s.$$  

**Proof.** First, we show that

$$R^g_{1/2}(\mathcal{T}^g(W^s_R(z))) \cap \mathcal{T}^g(R_{1/2}(W^s_R(z))) = \emptyset \quad \text{for all} \quad z \in R_i \cap \Omega^*_s. \quad (20)$$

By contradiction, assume that there is $(x, X) \in R^g_{1/2}(\mathcal{T}^g(W^s_R(z))) \cap \mathcal{T}^g(R_{1/2}(W^s_R(z)))$ and $(x, X) \in \pi_g^{-1}(U_i^{1/2})$. Thus, there are $(y, Y), (p, P) \in W^s_R(z)$ such that $\mathcal{T}^g(R_{1/2}^g(p, P)) = R^g_{1/2}(\mathcal{T}^g(y, Y)) = (x, X)$. Since the support of $g - g_0$ is contained in $U_i^{1/2}$, then $\mathcal{T}^g(y, Y) = (y, Y)$, and

$$\|\mathcal{R}^g_{1/2}(\mathcal{T}^g(y, Y))\|_g = \|\mathcal{R}^g_{1/2}(\mathcal{T}^g(y, Y))\| = \|X\|_g = 1,$$

which implies that $\|\mathcal{T}^g(R_{1/2}^g(p, P))\|_g = 1$. On the other hand, if $R_{1/2}^g(p, P) = (w, W)$, then $\mathcal{T}^g(w, W) = (w, W/\|W\|_g)$ and $1 = \|\mathcal{T}^g(w, W)\| = \|W\|_g = 1$. So, $\mathcal{T}^g(w, W) = (w, W)$ and $\|X\|_g = \|\mathcal{T}^g(w, W)\|_g = 1$. This implies that $(x, X) \in \pi_g^{-1}(x) \cap \pi_{g_0}^{-1}(x)$ which contradicts \([19]\).

Since $\|\mathcal{R}^g_{1/2}(\mathcal{T}^g(q, Q))\|_g = 1$ for all $(q, Q) \in W^s_R(z)$, then by \([19]\) we have

$$R^g_{1/2}(\mathcal{T}^g(W^s_R(z))) \cap \mathcal{T}^g(\pi^{-1}(U_i^{1/2})) = \emptyset.$$  

Therefore, $\phi^g_t(R^g_{1/2}(\mathcal{T}^g(q, Q))) = \phi^g_t(R^g_{1/2}(\mathcal{T}^g(q, Q)))$ for all $(q, Q) \in W^s_R(z)$, where $\phi^g_t$ and $\phi^t$ are the geodesic flow of the metrics $g$ and $g_0$, respectively. So, equation \((20)\) ends the proof of the lemma. \(\square\)

The next step is to exhibit the perturbations or families of perturbations of $g$ that have the property \([19]\).

For each $i \in \{1, \ldots, k\}$, let $w_i > 0$ be a small real parameter. Consider $\alpha_{w_i}(x, y)$ a continuous family of $C^\infty$-real function (bump function) with support contained $U_i$ ($U_i$ is the domain of the parametrization $\phi$), $C^2$-close to the null function (close depend of $w_i$) with $\alpha_{w_i}(0, 0) = w_i$, and $\alpha_{0}(x, y) \equiv 0$. Moreover, if $w_i \neq 0$, then $\alpha_{w_i}(x, y) \neq 0$ for all $(x, y) \in \varphi^{-1}(\pi(U_i^{1/2}))$. Thus, in local coordinates, we exhibit three family of Riemannian metric $g^{w_i}$ given by

(a) $g^{w_i} = (1 + \alpha_{w_i}(x, y))g$,
(b) $g^{w_i} = e^{\alpha_{w_i}(x, y)}g$,
(c) $g^{w_i}_{\alpha}(x, y) = g_{\alpha}(x, y) + \alpha_{w_i}(x, y)$, $(g^{w_i})_{rs}(x, y) = g_{rs}(x, y)$ $(r, s) \neq (0, 0)$

which satisfy the property \([19]\) and therefore, it satisfy Lemma 5.15.

We denote by $R^{w_i} : \Sigma_{g^{w_i}} \to \Sigma_{g^{w_i}}$ the Poincaré map given by $g^{w_i}$ (see Subsection 3.1). Define the following application $\Phi^{w_i}_R$ on $R_i$ by

$$\Phi^{w_i}_R(x, v) := R^{-1} \circ S^{g^{w_i}} \circ R^{w_i} \circ \mathcal{T}^{g^{w_i}}(x, v) \quad \text{for} \quad (x, v) \in R_i,$$  

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which is $C^2$-close to the identity if $w_i$ is small enough.

As a consequence of Lemma 5.15 we have

**Corollary 5.16.** For $w_i$ small enough, $\Phi^{w_i}(W^s_R(z)) \cap W^u_R(z) = \emptyset$, for $z \in R_i \cap \Omega^s_e$.

Since $i \in \{1, \ldots, k\}$, then we consider a vector $w \in \mathbb{R}^k$, $w = (w_1, w_2, \cdots, w_k)$. Then, if $||w||$ is small enough we define a metric $g_w$, $C^2$-close to metric $g$ by

$$g_w = \begin{cases} 
    g^{w_i}, & \text{in a small neighborhood of } \pi(R^{1/2}) \\
    g, & \text{otherwise.} 
\end{cases}$$

Put $\Phi^w(x, v) := \Phi^{w_i}(x, v)$ if $(x, v) \in R_i$. So, it is easy to see that $\Phi^w = R^{-1} \circ S^{w_0} \circ R^{w} \circ T^{w}$, where $R^w$ is the Poincaré map $R_\Sigma: \Sigma_{g_w} \to \Sigma_{g_w}$. From Lemma 3.6 (see the notation of Subsection 3.1) $P_w := R^{s_w} = R \circ \Phi^w$, which is $C^2$-close to $R$, whenever $||w||$ is small enough.

Put $\Omega^s_e := \Omega^s_e \cup \Omega^u_e$, then as $\Omega^s_e \cap \Omega^u_e = \emptyset$ and the perturbation on the metric is done in a small neighborhood of $\bigcup_{i=1}^{k} \pi(R_i^{1/2})$, where $\{R_1, \ldots, R_k\}$ is a small Markov partition of $\Omega^s_e$, then the hyperbolic continuation of $\Omega^s_e$ by $P_w$ is $\tilde{\Omega}^s_{s,w} := \Omega^s_{s,w} \cup \Omega^u_{s,w}$, with $\tilde{\Omega}^s_{s,w}$ being the hyperbolic continuation of $\Omega^s_e$ by $P_w$ and we still have $\tilde{\Omega}^s_{s,w} \cap \Omega^u_e = \emptyset$.

Therefore, as an immediate consequence of Corollary 5.16 we have the following lemma.

**Lemma 5.17.** For a set of positive Lebesgue measure of the parameter $w$, with $||w||$ small enough, the pair $(P_w, \tilde{\Omega}^s_{s,w})$ has the property $V$.

**Proof.** Observe that, by Corollary 5.16

$$P_w(W^s_R(z)) \cap R(W^u_R(z)) = \emptyset, \text{ for } z \in \Omega^s_e.$$ 

Moreover, as $\Omega^s_e$ and $\Omega^u_e$ satisfy the properties (i), (ii), and (iii) of the beginning of the Subsection 5.2.2.1 then

- $\Omega^s_e \cap \Omega^u_e = \emptyset$,
- $\text{HD}(K^s_{\Omega^s_e}) + \text{HD}(K^u_{\Omega^u_e}) > 1$, where $K^s_{\Omega^s_e}$ is stable Cantor set of $\Omega^s_e$ and $K^u_{\Omega^u_e}$ is the unstable Cantor set of $\Omega^u_e$.

Thus, the family $P_w$ satisfies the condition to get the property $V$ (cf. Section 5.1.1 and MY10 Appendix 5.3)).

By Lemma 3.6 the map $P_w$ and $R^w$ are conjugated. Thus, as in Subsection 3.1 the set $\Omega^s_{s,w} = T^{w}(\Omega^s_{s,w}) = T^{w}(\tilde{\Omega}^s_{s,w}) \cup \Omega^u_e := \Omega^s_{s,w} \cup \Omega^u_e$ is the hyperbolic continuation of $\Omega^s_{s,w}$ by $R^w$. Consequently, from Lemma 5.17 we have

**Lemma 5.18.** For a set of positive Lebesgue measure of the parameter $w$, with $||w||$ small enough, the pair $(R^w, \Omega^s_{s,w})$ has the property $V$.
Remark 5.19. Let \( G \subset V \subset G^3(M) \) a neighborhood of \( g_w \). Let \( \Omega_{*,w,g} \) be the basic set for \( \mathcal{R}^w \) as in Remark 3.7. Moreover

\[
\Omega_{*,w,g} = \Omega_{*,w,g}^s \cup \Omega_{*,g}^u \quad \text{with} \quad \Omega_{*,w,g}^s \cap \Omega_{*,g}^u = \emptyset,
\]

where \( \Omega_{*,w,g}^s \) and \( \Omega_{*,g}^u \) as in Remark 3.7.

The last discussion, and the robustness of the property \( V \) (cf. Section 5.1.1 and [MY10, Appendix 5.3]), provides that

Lemma 5.20. If \( G \) is small enough, then for all \( g \in G \), the pair \( (\mathcal{R}^g, \Omega_{*,w,g}) \) has the property \( V \).

These latter lemmas are an important tool for proving Theorems 3.9 and 3.10, since it allows us to use the Main Theorem in [MR17].

6 The Final Proofs

Before beginning with the proofs of Theorems 3.9 and 3.10 we present the

Theorem 6.1 ([MR17 Main Theorem]). Let \( \Lambda \) be a horseshoe associated to a \( C^2 \)-diffeomorphism \( \varphi \) of a surface \( M \), such that \( \text{HD}(\Lambda) > 1 \). If \( \varphi_0 \) is \( C^2 \) sufficiently close to \( \varphi \) such that the pair \( (\varphi_0, \Lambda_{\varphi_0}) \) has the property \( V \), where \( \Lambda_{\varphi_0} \) is the hyperbolic continuation of \( \Lambda \). Then, there is \( C^2 \)-neighborhood \( W \) of \( \varphi_0 \) such that, if \( \Lambda_{\psi} \) denotes the continuation of \( \Lambda \) associated to \( \psi \in W \), there is an open and dense set \( H_1(\psi, \Lambda_{\psi}) \subset C^1(M, \mathbb{R}) \) such that for all \( f \in H_1(\psi, \Lambda_{\psi}) \), we have

\[
\text{int } L(\psi, \Lambda_{\psi}, f) \neq \emptyset \quad \text{and} \quad \text{int } M(\psi, \Lambda_{\psi}, f) \neq \emptyset,
\]

where \( \text{int } A \) denotes the interior of the set \( A \).

The set \( H_1(\psi, \Lambda_{\psi}) \) of the above theorem is described by (compare with Definition 4.4)

\[
H_1(\psi, \Lambda_{\psi}) = \{ f \in C^1(M, \mathbb{R}) : \# M_f(\Lambda_{\psi}) = 1, \ z \in M_f(\Lambda_{\psi}), \ D\psi_z(e^j_z) \neq 0, j = s, u \},
\]

where \( M_f(\Lambda_{\psi}) := \{ z \in \Delta : f(z) \geq f(x) \} \) for all \( x \in \Lambda_{\psi} \) is the set of maximum points of \( f \) in \( \Lambda_{\psi} \) and \( e^j_z \in E^j(z) \) is a unit vector, \( j = s, u \).

An immediate consequence of the above theorem, which will be useful to the proof of Theorem 1.3, is the following corollary

Corollary 6.2. The above theorem is also valid for the set of functions

\[
H_\infty(\psi, \Lambda_{\psi}) = \{ f \in C^1(M, \mathbb{R}) : \# M_f(\Lambda_{\psi}) < \infty, \ D\psi_z(e^j_z) \neq 0, z \in M_f(\Lambda_{\psi}), j = s, u \}.
\]
Proof of the Theorem 3.9. Let $F \in \mathcal{H}_{\Delta_3, \Delta_2, \beta}$ for $\beta$ small, in this case, $\Omega^* = (\Delta_3)^*$ a sub-horseshoe of $\Delta_3$ and $\Omega^* = (\Delta_2)^*$ a sub-horseshoe of $\Delta_2$. Then, by Lemma 5.18 we consider the metric $g_w$ such that $(R^w, \Omega_s, w)$ has the property $V$ (see Section 5.1.1). By Remark 4.7, we have that $F \circ S^{g_w} \in \mathcal{H}_{(\Delta_3)_w, \Delta_2, \beta}$, where $(\Delta_3)_w$ is the hyperbolic continuation by $R_w$ of $\Delta_3$. Therefore, by the definition of $\mathcal{H}_{(\Delta_3)_w, \Delta_2, \beta}$ we have that

$$\max(F \circ S^{g_w})_{\phi_w} |_{\Sigma_{g_w} \cap (R^w_{F_0, S^{g_w}_w, \Omega^*_{s, w}} \cup R^w_{F_0, S^{g_w}_w, \Omega^*_{w}})} \in H_1(R^w, (\Omega^*_{s, w} \cup \Omega^*_{w})), $$

where $R^w_{F_0, S^{g_w}_w, \Omega^*_{s, w}}$ and $R^w_{F_0, S^{g_w}_w, \Omega^*_{w}}$ are the Markov partitions given by the definition of

$$\mathcal{H}_{(\Delta_3)_w, \Delta_2, \beta}.$$ 

Thus, as $(R^w, \Omega^*_{w})$ has the property $V$, then by the Main Theorem at [MR17] we can conclude that

$$\int M(R^w, \Omega^*_{w}, \max(F \circ S^{g_w})_{\phi_w} |_{\Sigma_{g_w} \cap (R^w_{F_0, S^{g_w}_w, \Omega^*_{s, w}} \cup R^w_{F_0, S^{g_w}_w, \Omega^*_{w}})}) \neq 0,$$  \hspace{1cm} (21)

$$\int L(R^w, \Omega^*_{w}, \max(F \circ S^{g_w})_{\phi_w} |_{\Sigma_{g_w} \cap (R^w_{F_0, S^{g_w}_w, \Omega^*_{s, w}} \cup R^w_{F_0, S^{g_w}_w, \Omega^*_{w}})}) \neq 0.$$

Let $G \subset V \subset G^3(M)$ a neighborhood of $g_w$ as in Lemma 5.20, then for every $g \in G$, the pair $(R^g, \Omega_{s, w, g})$ has the property $V$. So, for every $F \in \mathcal{H}_{(\Delta_3)_g, \Delta_2, \beta}$, the function $F \circ S^g \in \mathcal{H}_{(\Delta_3)_g, \Delta_2, \beta}$ and by Main Theorem at [MR17], we have that the triplet $(R^g, \Omega_{s, w, g}, \max(F \circ S^g)_{\phi_g})$ satisfies the equations (21) and (22). Therefore, put $\mathcal{H}_{g, \Lambda} := \mathcal{H}_{(\Delta_3)_g, \Delta_2, \beta}$, then as $\mathcal{H}_{(\Delta_3)_w, \Delta_2, \beta}$ is dense and $C^2$-open, then we have completed the proof of Theorem 3.9 (and, a fortiori, Theorem 1.1).

Proof of the Theorem 3.10. The proof is completely analogous to the proof of Theorem 3.9 just change $\mathcal{H}_{(\Delta_3)_w, \Delta_2, \beta}$, $\Omega^*_{s, w}$, $\Omega^*_{w}$, and $F \circ S^{g_w}$ by $\mathcal{H}_{(\Delta_3)_g, \Delta_2, \beta}$, $(\Delta_3)^g$, $(\Delta_2)^g$, and $f \circ \pi_g$, respectively. The set of functions will be $\mathcal{H}_{g, \Lambda} := \mathcal{H}_{(\Delta_3)_g, \Delta_2, \beta}$, for some $\beta$ and $g \in G \subset V \subset G^3(M)$ a neighborhood of $g_w$ as in Lemma 5.20.

7 The Spectrum of the Height Function

Let $e$ be an end of the manifold $M$ and $\Gamma(t)$ a ray that defines the end $e$. Thus, the height function associated to $e$ is defined by

$$ht_e(x) = \lim_{t \to +\infty} d(x, \Gamma(t)) - t.$$ 

Usually $-ht_e(x)$ is called the Busemann function associated to end $e$ and denoted by $b_T(x)$ (see [Ebe80] for the precise definition of an end).

7.1 Differentiability of Busemann Function

It is known that the Busemann functions are not always differentiable, however, the points where we lose the differentiability are well known and we are going to use it to show some results where we have the differentiability of the Busemann functions, at least in a region close to the set hyperbolic $\Delta$.
Definition 7.1. Let $\gamma : [0, \infty) \to M$ be a forward ray. Then a forward ray $\sigma : [0, \infty) \to M$ emanating from $x := \sigma(0) \in M$ is called a forward coray (or a forward asymptotic ray) to $\gamma$ if there exists a divergent sequence of numbers $\{t_j\}$ and a sequence of minimal geodesics $\sigma_j$ with $\sigma_j := \sigma_j(0)$, $\sigma_j(t_j) := \gamma(t_j)$, for some $t_j > 0$, and such that $\lim_{j \to +\infty} q_j = \sigma(0)$ and $\sigma'(0) = \lim_{j \to +\infty} \sigma_j'(0)$ (see Figure 1). A forward coray of $\gamma$ is called maximal if for any $\epsilon > 0$ its extension to $[-\epsilon, +\infty)$ is no longer a coray to $\gamma$. The origin points of maximal corays of $\gamma$ are called the co-points to $\gamma$ and it is denoted by $C_\gamma$.

![Figure 1: Forward Coray](image)

Proposition 7.2 ([HinS2 Proposition 1]). If $p \in M$ is not a co-point to $\gamma$, then the Busemann function $b_\gamma$ is differentiable at $p$.

Remark 7.3. Let $ND(b_\gamma) \subset M$ be the set of non-differentiable points of the Busemann function $b_\gamma$. Then, $ND(b_\gamma) \subset C_\gamma \subset ND(b_\gamma)$.

The following Lemma gives us information on what happens at the end $e$, for an asymptotic ray to $\gamma$ ([SST03 Theorem 3.8.2])

Lemma 7.4. Let $\gamma : [0, \infty) \to M$ a ray.

1. For any $x \in M$, there exists a ray $\sigma$ asymptotic to $\gamma$ such that $\sigma(0) = x$.
2. For any ray $\sigma$ asymptotic to $\gamma$ it holds $b_\gamma(\sigma(s)) = b_\gamma(\sigma(0)) + s$.
3. If $b_\gamma$ is differentiable at $x \in M$, then $\sigma(s) = \exp_x(s \nabla b_\gamma(x))$ is a unique ray asymptotic to $\gamma$ emanating from $x$, where $\nabla b_\gamma(x)$ is the Riemannian gradient of $b_\gamma$ at $x$.

Proof. The proof of item (1) is trivial. Item (3) is a consequence of the item (2). Thus, let us prove item (2). Take $\{t_i\}_{i \in \mathbb{N}}$ and $\{\sigma_i\}_{i \in \mathbb{N}}$ as in the definition of asymptotic. It holds that

$$b_\gamma(\sigma(s)) = \lim_{t_i \to \infty} \{t_i - d(\sigma(s), \gamma(t_i))\}$$

by the definition of $b_\gamma$. We can replace $\sigma(s)$ in the right hand side with $\sigma_i(s)$ since

$$|d(\sigma(s), \gamma(t_i)) - d(\sigma_i(s), \gamma(t_i))| \leq d(\sigma(s), \sigma_i(s)) \to 0$$

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as \( i \to \infty \). Hence we have, by the choice of \( \sigma_i \),

\[
\begin{align*}
\hat{b}_i(\sigma(s)) &= \lim_{t \to \infty} \{ t_i - d(\sigma_i(s), \gamma(t_i)) \} = \lim_{t \to \infty} \{ t_i - d(\sigma_i(0), \gamma(t_i)) + s \} \\
&= \lim_{t \to \infty} \{ t_i - d(0, \gamma(t_i)) + s \} = \hat{b}_i(\sigma(0)) + s.
\end{align*}
\]

In our case, since \( M \) is a surface with pinched curvature and finite volume, then Gromov [Gro78] and Heintze [Hei76] showed that \( M \) is diffeomorphic to the interior of a compact manifold with boundary. Moreover, for every point \( p \in M \), there are only finitely many distinct minimizing geodesic rays emanating from \( p \). In particular, if \( M \) has only finitely many ends and each end is parabolic, i.e., the quotient of a horoball of \( \hat{M} \) (the universal covering of \( M \)) by a group of parabolic isometries with a compact cross-section (see Ebe80). Therefore, each end of \( M \) is quasi-isometric to a ray. Thus, suppose that \( C_0, C_1, \ldots, C_k \) are simple closed geodesic polygons on \( M \), no two of which have common point, such that each \( C_i \) bounds a tube \( U_i \) homeomorphic to a circular disk punctured at the center (or a cylinder), that is, a representation of the ends of \( M \). Then \( M \setminus \bigcup_{i=0}^{k} U_i \) is a bounded open set.

**Definition 7.5.** A tube \( U_i \) is said non-expanding or expanding according to any minimal sequence of closed curves homotopic with the curve \( C_i \) being bounded or non.

**Remark 7.6.** In this case, since \( U_i \) represents a parabolic end, then any minimal sequence of closed curves homotopic to the boundary \( C_i \) is bounded, so each \( U_i \) is non-expanding (cf. Nas57).

**Theorem 7.7** (Nas57 Theorem 5.2]). Suppose that \( S \) is a surface that is homeomorphic to a cylinder. Then for a ray \( \gamma \) the set \( C_\gamma \) is an unbounded simple arc or vacuous according as the tube \( U_0 \) which contains \( \gamma \) or a subray of \( \gamma \), is expanding or non-expanding.

As a consequence of the above theorem, Lemma 7.4, Remark 7.3, and Remark 7.6 we have (see Bal03, chap 3]) that

**Lemma 7.8.** The Busemann function \( \hat{b}_\gamma \) increases in tube \( U_0 \) which contains \( \gamma \) or a subray of \( \gamma \). Moreover, it is \( C^2 \) in \( U_0 \).

**Proof.** Item (2) of Lemma 7.4 implies the first part. Now, by Remark 7.6 the tube \( U_0 \) is not expanding, then by the above theorem, we have that \( C_\gamma \cap U_0 = \emptyset \). Thus, by Lemma 7.4 the Busemann function \( \hat{b}_\gamma \) is \( C^1 \) in \( U_0 \). Now, we will prove that it is also \( C^2 \). Let \( x \in U_0 \) be and Let \( t_n \) be a sequence such that \( t_n \to \infty \), let

\[
\hat{b}_n(x) = t_n - d(\gamma(t_n), x).
\]

Then \( \hat{b}_n \to \hat{b}_\gamma \) uniformly on compact subsets of \( M \). Put \( r_n := d(x, \gamma(t_n)) \), and let \( X \) be a smooth vector field on \( M \) and let \( \alpha_x(s) \) be an integral curve of the field \( X \) with \( \alpha_x(0) = x \), then for each \( s \) there is \( v_n(s) \in \pi^{-1}(\alpha_x(s)) \) such that \( \exp_{\alpha_x(s)} t(s) v_n(s) = \gamma(t_n) \) for some \( t(s) \in \mathbb{R}, i = 1, \ldots , m \). It is easy to see that each \( v_n(s) \) is smooth in \( s \) and moreover,

\[
\frac{d}{ds} v_n(s) \bigg|_{s=0} = -J'_n(x, 0),
\]

(23)
where \( J_n(x, \cdot) \) is the Jacobi vector field along of the geodesic \( \exp_x t v_n(0) \) with \( J_n(x, 0) = X(x) \) and \( J_n(x, r_n) = 0 \). Note that, as the curvature of \( M \) is negative, then \( M \) has no conjugate points and therefore \( J_n(x, \cdot) \) is unique with these properties.

For \( s \) small \( \alpha_x(s) \in U_0 \), thus \( b_\gamma \) is differentiable in \( \alpha_x(s) \) and by Lemma 7.8 we have that
\[
\nabla b_\gamma (\alpha_x(s)) = \lim_{n \to +\infty} v_n(s). \tag{24}
\]
Then by equations (23) and (24) we have that for every \( i \in \{1, \ldots, m\} \),
\[
D_{X(x)} \nabla b_\gamma(x) = \frac{\partial}{\partial s} \left( \nabla b_\gamma(\alpha_x(s)) \right) \bigg|_{s=0} = \lim_{n \to +\infty} \frac{d}{ds} v_n(s) \bigg|_{s=0} = -\lim_{n \to +\infty} J'_n(x, 0).
\]
Since \( J_n(x, 0) = X(x) \) and \( J_n(x, r_n) = 0 \) and \( M \) has no conjugate points \( J_n(x, \cdot) \to J_X(\cdot) \) the unique stable Jacobi vector field along \( \exp_x t \nabla b_\gamma(x) \) with \( J_X(0) = X(x) \) (cf. Section A.4), and thus \( J'_n(x, \cdot) \to J'_X(\cdot) \).

Therefore, we have
\[
D_X \nabla b_\gamma(x) = -J'_X(0), \tag{25}
\]
for \( x \in U_0 \). Thus, as \( X \) is smooth, we conclude that \( b_\gamma \) is \( C^2 \) in the tube \( U_0 \).

7.2 Horseshoe for Tubes

Fix a ray \( \gamma \) and consider \( U_0 \) as in Lemma 7.8. In this section, we construct a hyperbolic set for \( \phi^t \) whose projection on \( M \) intersects the tube \( U_0 \).

**Definition 7.9.** We say that an open set \( V_0 \subset U_0 \) is a collar of \( U_0 \), if \( U_0 \setminus V_0 \) has two connected components.

**Lemma 7.10.** Let \( V_0 \) be a collar of \( U_0 \), then there is a basic set \( \Lambda_0 \subset SM \) with \( HD(\Lambda_0) \) close enough to 3 and such that
\[
\pi(\Lambda_0) \cap V_0 = \emptyset,
\]
where \( \pi: SM \to M \) is the canonical projection.

**Proof.** Since the geodesic flow is transitive, then the construction given in Lemma 3.1 provides the result. \( \square \)

7.2.1 Poincaré Map for the Tube

By Lemma 7.8 and definition of \( \Lambda_0 \), for each \( (x, v) \in \Lambda_0 \) there is a \( \bar{t}(x, v) \) such that
\[
\max_{t \in \mathbb{R}} b_\gamma(\pi(\phi^t(x, v))) = \bar{t}(x, v) \] and \( \phi^\bar{t}(x, v) \in U_0 \), since \( \Lambda_0 \) is compact invariant set. Thus \( b_\gamma \) is \( C^2 \) in \( \phi^\bar{t}(x, v) \). Consider the set
\[
A(x, v) := \{ t \in \mathbb{R} : b_\gamma(\pi(\phi^t(x, v))) = b_\gamma(\pi(\phi^\bar{t}(x, v))) \}.
\]
Is easy to see that \( 1 \leq |A(x, v)| < \infty \), where \(|A(x, v)|\) is the cardinality of \( A(x, v) \).

**Lemma 7.11.** There is \( \eta > 0 \) such that for every \( (x, v) \in \Lambda_0 \) and any \( s, r \in A(x, v) \), we have that \(|s - r| > \eta\).
Proof. The proof of the lemma is a consequence of the following claim.
Claim: For any \((x, v) \in \Lambda_0\), put \(f_\gamma(t) = b_\gamma(\pi(\phi^t(x, v)))\), then \(f''(s) \neq 0\) for all \(s \in A(x, v)\).

Proof of Claim:
By definition of \(A(x, v)\) and Lemma 7.8 we have that \(f(t)\) is \(C^2\) in a small neighborhood of \(A(x, v)\) and \(f'(s) = 0\) for all \(s \in A(x, v)\). In this neighborhood, from proof of Lemma 7.8 we can see that
\[
f''(t) = -\langle J'_{\gamma_\nu(t)}(0), \gamma_\nu'(t) \rangle,
\]
where \(J_{\gamma_\nu(t)}(\cdot)\) is a stable Jacobi vector field along \(\sigma_{\gamma_\nu(t), \gamma}\) with \(J_{\gamma_\nu(t)}(0) = \gamma_\nu'(t)\). Here \(\sigma_{\gamma_\nu(t), \gamma}\) is the only asymptotic geodesic from \(\gamma_\nu(t)\) to \(\gamma\) (see equation (25)).

If \(f''(t) = 0\), since the curvature is negative, the real function \(g(z) = \|J_{\gamma_\nu(t)}(z)\|^2\) is strictly convex. Moreover, \(g(0) = \|J_{\gamma_\nu(t)}(0)\|^2 = \|\gamma_\nu'(t)\|^2 = 1\) and \(g'(0) = 2 \cdot f''(t) = 0\). So, \(g\) is a strictly convex function with a minimum point in \(z = 0\), then \(g\) is an unbounded function. But since \(J_{\gamma_\nu(t)}(\cdot)\) is a stable Jacobi vector field, then it holds that \(\|J_{\gamma_\nu(t)}(z)\|\) is bounded, so we have a contradiction. Thus, we concluded the proof of the claim.

Now as \(f_\gamma\) is \(C^2\) in a neighborhood of \(A(x, v)\), then the claim implies that its critical points are isolated. Thus, we conclude the proof of the Lemma.

The basic set \(\Lambda_0\) satisfies all the properties of Subsection 3.1, the and put also \(R: \Sigma \to \Sigma\) the Poincaré map, when \(\Sigma = \bigsqcup_{i=1}^k \Sigma_i\), each \(\Sigma_i\) is a Good Cross-Section, and \(\Lambda_0 \cap \Sigma\) a basic set for \(R\) with \(HD(\Lambda_0 \cap \Sigma) > 1\) (in fact close to 2). The Lemma 7.11 and the construction made in [Rn21, Section 3], allow us to choose Good Cross Sections \(\Sigma_i\) such that
\[
\Lambda_0 \subset \bigcup_{i=1}^k \phi^{(-\gamma_\nu)}(\text{int}(\Sigma_i)),
\]
where \(\gamma > 0\) (which depend of \(\eta\) at Lemma 7.11) and

1. For \((x, v) \in \Lambda_0 \cap \Sigma\), then \(\phi^s(x, v) \notin \Sigma_i\) for all \(s \in A(x, v)\) and all \(i = 1, \ldots, k\).

2. Consider the set \(A = \bigcup_{s \in A(x, v)} \bigcup_{(x, v) \in \Lambda_0} \phi^s(x, v)\).

Then by item (1), \(\Lambda_0 \cap \Sigma \cap A = \emptyset\). Therefore, \(0 \notin A(y, w)\) for all \((y, w) \in \Lambda_0 \cap \Sigma\).

Also, for all \((y, w) \in \Lambda_0 \cap \Sigma\) with \(|A(y, w)| \geq 2\) and for any two consecutive values \(s < r\) in \(A(y, w)\), there are \(\Sigma_{(y,w)}^{(s)}\), \(\Sigma_{(y,w)}^{(r)}\) \(\in \{\Sigma_1, \ldots, \Sigma_k\}\) such that
\[
R(\phi^s(y, w)) \in \Sigma_{(y,w)}^{(s)} \quad \text{and} \quad R^{-1}(\phi^r(y, w)) \in \Sigma_{(y,w)}^{(r)}.
\]

Now, let us define a class of differentiable functions that contains all the information on the Lagrange and Markov Dynamical Spectra (see [MR17]) associated with the Busemann function.

For \((x, v) \in \Lambda_0 \cap \Sigma\) consider the sets \(A^+(x, v) := \{s > 0: s \in A(x, v)\}\) and \(A^-(x, v) := \{s < 0: s \in A(x, v)\}\). If \(|A^+(x, v)| \geq 1\). Then, we denote \(a^+(x, v) = \min\{s : s \in A^+(x, v)\}\) and \(a^-(x, v) = \max\{s : s \in A^-(x, v)\}\). Thus, \(a^-(x, v) < a^+(x, v)\) are consecutive values in \(A(x, v)\), then by item (2), there are \(\Sigma_{(x,v)}^{(a^-(x,v))}\) and \(\Sigma_{(x,v)}^{(a^+(x,v))}\) such that
\[
\phi^{s-(x,v)}(x, v) = R(\phi^{a^-(x,v)}(x, v)) \in \Sigma_{(x,v)}^{(a^-(x,v))}
\]
and
and
\[ \phi^{s+}(x,v)(x,v) = \mathcal{R}^{-1}(\phi^{a+}(x,v)(y,w)) \in \Sigma^{(x,v)}_{\{a\}^+(x,v)}. \]

- If \( A^- (x,v) = \emptyset \), then put \( r^+(x,v) \) such that \( \phi^{r+}(x,v) = \mathcal{R}(\phi^{a+}(x,v)(x,v)) \) and \( \mathcal{R}^{-1}(x,v) = \phi^{r-}(x,v)(x,v) \).

- If \( A^+(x,v) = \emptyset \), then put \( r^-(x,v) \) such that \( \phi^{r-}(x,v) = \mathcal{R}^{-1}(\phi^{a-}(x,v)(x,v)) \) and \( \mathcal{R}(x,v) = \phi^{r+}(x,v)(x,v) \).

Thus, we can define the following function \( \max_{b^*_\gamma}: \Sigma \to \mathbb{R} \) as
\[
\max_{b^*_\gamma}(x,v) = \begin{cases} 
\max_{[x,-(x,v),r^-(x,v)]} b_\gamma(\pi(\phi^b_\gamma(x,v))) & \text{if } |A^-(x,v)| \geq 1, \\
\max_{[x,-(x,v),r^+(x,v)]} b_\gamma(\pi(\phi^b_\gamma(x,v))) & \text{if } |A^-(x,v)| = 0, \\
\max_{[x,-(x,v),r^+(x,v)]} b_\gamma(\pi(\phi^b_\gamma(x,v))) & \text{if } |A^+(x,v)| = 0.
\end{cases}
\]

Item (2) above shows that in each interval of the definition of \( \max_{b^*_\gamma}(x,v) \) the maximum point is unique, then function \( \max_{b^*_\gamma} \) is \( C^1 \).

### 7.2.2 The Behavior of the Maximum Points of \( \max_{b^*_\gamma} \)

The next step is to show that the function \( \max_{b^*_\gamma} \) has a finite number of maximum points (restricted to \( \Lambda_0 \cap \Sigma \)) and in every one of these maximum points, its gradient is not parallel to the stable and unstable bundles.

From Lemma 7.10, the basic set \( \Lambda_0 \) has Hausdorff dimension close to 3. Thus, set \( \Delta_0 := \Lambda_0 \cap \Sigma \) a basic set for \( \mathcal{R} \) with \( H D(\Delta_0) \) is close to 2 (see Section 3).

Moreover, the splitting \( E^s \oplus \phi \oplus E^u \) over a neighborhood of \( \Lambda_0 \) defines a continuous splitting \( E^s_\Sigma \oplus E^u_\Sigma \) of the tangent bundle \( T\Sigma \), defined by
\[
E^s_\Sigma(y) = E^s_y \cap T_y \Sigma \text{ and } E^u_\Sigma(y) = E^u_y \cap T_y \Sigma,
\]
(26)
where \( E^s_y = E^s_y \oplus \langle \phi \rangle \) and \( E^u_y = E^u_y \oplus \langle \phi \rangle \). Also, \( W^s(x, \Sigma) = W^s_y \cap \Sigma \) and \( W^u(x, \Sigma) = W^u_y \cap \Sigma \) are the stable and unstable manifolds of \( x \in \Delta_0 \) (cf. [Rn21 Appendix]).

We denote the set
\[
M_{\max_{b^*_\gamma}}(\Delta_0) = \{(y,w) \in \Delta_0 : \max_{b^*_\gamma}(y,w) \geq \max_{b^*_\gamma}(x,v) \text{ for all } (x,v) \in \Delta_0 \}.
\]

It is clear that by definition of \( \max_{b^*_\gamma} \), the set \( M_{\max_{b^*_\gamma}}(\Delta_0) \subset U_0 \). The set \( U_0 \) is defined as in Lemma 7.8

**Lemma 7.12.** For any \( (y,w) \in M_{\max_{b^*_\gamma}}(\Delta_0) \cap \Sigma \), we have that
\[
\frac{\partial}{\partial v_s} \max_{b^*_\gamma}(y,w) \neq 0 \quad \text{and} \quad \frac{\partial}{\partial v_u} \max_{b^*_\gamma}(y,w) \neq 0,
\]
where \( v^j \in E^j_\Sigma(y) \) is a unit vector, \( j = s,u \).
Proof. Let \( \alpha(r) \subset W^s((y, w), \Sigma) \) be with \( \alpha'(0) = v^s \), then there is a differentiable real function \( t(r) \) such that
\[
\max_{\delta} b_\gamma(\alpha(r)) = b_\gamma \circ \pi(\phi(t(r))(\alpha(r))).
\]
Put \( \hat{\alpha}(r) = \phi(t(r))(\alpha(r)) \) e \( \beta(r) = \pi(\hat{\alpha}(r)) \). By contradiction, suppose that

\[
0 = \frac{d}{dt} b_\gamma(\alpha(r) = \frac{d}{dt} \max_{\delta} b_\gamma(\alpha(r))|_{t=0} = \langle \nabla b_\gamma(\beta(0)), \beta'(0) \rangle.
\]

Also, as the function \( t \mapsto b_\gamma \circ \pi(\phi(t(y, w))) \) has a local maximum in \( t = t(0) \), then
\[
0 = \frac{d}{dt} b_\gamma \circ \pi(\phi(t(y, w)))|_{t=t(0)} = \langle \nabla b_\gamma(\beta(0)), D\pi_{\phi(t(0))(y, w)}(\phi(\phi(t(0))(y, w))) \rangle = \langle \nabla b_\gamma(\beta(0)), \gamma'_w(t(0)) \rangle.
\]

Thus, as \( \nabla b_\gamma(\beta(0)) \neq 0 \), then \( \beta'(0) \) and \( \gamma'_w(t(0)) \) are parallel.

Now, our next task is to show that the last statement leads to a contradiction. Observe that \( \hat{\alpha}'(r) = A\phi(\hat{\alpha}(r)) + B\Sigma(r) \) for some \( \Sigma(r) \in E^s(\hat{\alpha}(r)) \setminus \{0\} \). Then, we take \( \theta > 0 \) small and consider the surface
\[
S^s_\theta := \bigcup_r \phi^{(-\theta, \theta)}(\hat{\alpha}(r)) \subset W^s_{loc}(\hat{\alpha}(0)).
\]

Then, the tangent space of \( S^s_\theta \) in \( \phi^{\delta}(\hat{\alpha}(r)) \) for \( |\delta| < \theta \) is
\[
T_{\phi^{\delta}(\hat{\alpha}(r))}S^s_\theta = \text{span}\{\phi(\phi^{\delta}(\hat{\alpha}(r)))\} \oplus \text{span}\{D\phi^{\delta}_{\hat{\alpha}(r)}(\hat{\alpha}'(r))\}
\]
\[
= \text{span}\{\phi(\phi^{\delta}(\hat{\alpha}(r)))\} \oplus \text{span}\{D\phi^{\delta}_{\hat{\alpha}(r)}(\Sigma(r))\} = E^cs(\phi^{\delta}(\hat{\alpha}(r))).
\]

Therefore, by Lemma A.3, we have that \( S^s_\theta \) is transverse to the fiber of \( \pi \), so \( \pi|_{S^s_\theta} \) is a local diffeomorphism. We know that \( D\pi_{\hat{\alpha}(0)}(\phi(\hat{\alpha}(0))) = \gamma'_w(t(0)) \) and \( D\pi_{\hat{\alpha}(0)}(\hat{\alpha}'(0)) = \beta'(0) \) are parallel and \( T_{\hat{\alpha}(0)}S^s_\theta = \langle \phi(\hat{\alpha}(0)) \rangle \oplus \langle \hat{\alpha}'(0) \rangle \). This is a contradiction since \( \pi|_{S^s_\theta} \) is a local diffeomorphism.

The proof of the unstable case is analogous. Thus, we conclude the proof of the lemma. \( \square \)

Corollary 7.13. The set of maximum points \( M_{\max, b_\gamma}(\Delta_0) \) consists of isolated points and therefore is finite.

7.3 Proof of Theorem 1.3

In this section, we use the family of perturbations, \( g_w \), of metric \( g \) as in Subsection 5.2.1 in such a way that if \( R^w \) is the Poincaré map associated with the geodesic flow of \( g_w \) (see paragraph after Corollary 5.16), then the pair \( (R^w, \Delta^w_0) \) has the property \( V \) (see Subsection 5.1.1), where \( \Delta^w_0 \) is the hyperbolic continuation of \( \Delta_0 \). Moreover, by Remark 5.2, we can assume also that \( R^w \) has the property that the Birkhoff invariant is non-zero for some periodic orbit (see Subsection 5).

Let \( \gamma \) be a ray as in Subsections 7.1 and 7.2. Since the GCS of \( \Sigma \) can be taken transverse to the fiber of \( \pi \), then the surface \( \pi^{-1}(\gamma) \) is transverse to \( \Sigma \). So \( \pi^{-1}(\gamma) \cap \Sigma \) is a finite number of \( C^2 \)-curves in \( \Sigma \). Then by Lemma 4.10, there is a sub-horseshoe \( \Delta_1 \) of \( \Delta_0 \)
such that $\Delta_1 \cap (\pi^{-1}(\gamma) \cap \Sigma) = \emptyset$, and $HD(\Delta_1)$ is close to $HD(\Delta_0)$, and therefore, $HD(\Delta_1) \sim 2$ (see paragraphs at the beginning in Subsection 7.2.2). Thus, (using $\Delta_1$ instead of $\Delta_0$) we can assume, without loss of generality, that the support of $g_w$, $\text{supp} \ g_w := \{x \in M : g(x) \neq g_w(x)\}$ does not intersect the geodesic $\gamma$, which implies that $\gamma$ is still a geodesic ray of $g_w$.

Denote by $b^w_\gamma$ the Busemann function associated to the ray $\gamma$ with the metric $g_w$ and $\phi_w$ the derivative of the geodesic flow of the metric $g_w$. It is easy to see that all Lemmas of Subsection 7.2.2 hold for the function $\max_{\phi_w} b^w_\gamma$. In particular:

**Corollary 7.14.** The set of maximum points $M_{\max_{\phi_w} b^w_\gamma}(\Delta_0^w)$ consists of isolated points and therefore is finite.

**Remark 7.15.** By the robustness of the property $V$ (cf. Section 5.1.1 and [MY10, Appendix 5.3]) we can find a small neighborhood, $\mathcal{G} \subset \mathcal{G}^3(M)$, of $g_w$ such that

(a) The pair $(\mathcal{R}^g, \Delta^w_{0,g})$ has the property $V$, where $\Delta^w_{0,g}$ is the hyperbolic continuation of $\Delta_0^w$ by $\mathcal{R}^g$ (see Subsection 3.1).

(b) Let $\gamma_g \in \mathcal{G}$ a ray of $M$ with the metric $g$, which represents the same end of $\gamma$, then we can assume (by the same argument before Corollary 7.14) that $\Delta^w_{0,g} \cap \pi^{-1}(\gamma_g) = \emptyset$.

(c) The Busemann function $b_{\gamma_g}$ satisfies all the results of Subsection 7.1 and Subsection 7.2.

**Proof of Theorem 1.3.** Consider the set

$$H^\infty_{\mathcal{R}_0^w} = \{f \in C^1(\Sigma_0, \mathbb{R}) : \#M_f(\Delta_0^w) < \infty \text{ and } Df_z(e_j^w) \neq 0, \ z \in M_f(\Delta_0^w), \ j = s, u\},$$

where $e_j^w \in E^j_{\mathcal{R}_0^w}(z)$ is a unit vector, $j = s, u$.

By Lemma 7.12 and Corollary 7.13 we have that $\max_{\phi_w} b^w_\gamma \in H^\infty_{\mathcal{R}_0^w}$. Moreover, the pair $(\mathcal{R}^w, \Delta_0^w)$ has the property $V$, then by the Main Theorem in [MR17] (see also Corollary 6.2) we have that the sets

$$\mathbb{L}(\max_{\phi_w} b^w_\gamma, \Delta_0^w) = \left\{ \limsup_{n \to \infty} \max_{\phi_w} b^w_\gamma((\mathcal{R}_0^w)^n(x)) : x \in \Delta_0^w \right\}$$

and

$$\mathbb{M}(\max_{\phi_w} b^w_\gamma, \Delta_0^w) = \left\{ \sup_{n \in \mathbb{Z}} \max_{\phi_w} b^w_\gamma((\mathcal{R}_0^w)^n(x)) : x \in \Delta_0^w \right\}$$

have nonempty interior. Moreover, since

$$\mathbb{L}(\max_{\phi_w} b^w_\gamma, \Delta_0^w) \subset L(b^w_\gamma, \phi_w) \text{ and } \mathbb{M}(\max_{\phi_w} b^w_\gamma, \Delta_0^w) \subset \mathbb{M}(b^w_\gamma, \phi_w),$$

and height function $h^w_\gamma = -b^w_\gamma$, then

$$\text{int } \mathbb{M}(h^w_\gamma, \phi_w) \neq \emptyset \text{ and } \text{int } \mathbb{L}(h^w_\gamma, \phi_w) \neq \emptyset.$$ 

Analogously, by Remark 7.15 we have that for all $g \in \mathcal{G}$

$$\text{int } \mathbb{M}(h^w_{\gamma_g}, \phi_g) \neq \emptyset \text{ and } \text{int } \mathbb{L}(h^w_{\gamma_g}, \phi_g) \neq \emptyset,$$

which completes the proof. 

\qed
A Appendix

A.1 Stable and Unstable Manifold

The Stable and Unstable Manifold Theorem [KH95] implies that, if $\Lambda$ is a hyperbolic set for a flow $\phi^t$, then there is $\epsilon > 0$ such that for every $x \in \Lambda$ the set

$$W^s_\epsilon(x) = \{ y : d(\phi^t(x), \phi^t(y)) \leq \epsilon \text{ and } d(\phi^t(x), \phi^t(y)) \to 0 \text{ as } t \to +\infty \}$$

and

$$W^u_\epsilon(x) = \{ y : d(\phi^t(x), \phi^t(y)) \leq \epsilon \text{ and } d(\phi^t(x), \phi^t(y)) \to 0 \text{ as } t \to -\infty \}$$

are invariant $C^r$-manifolds tangent to $E^s_x$ and $E^u_x$ respectively at $x$. Then we call $W^s_\epsilon(x)$ the local strong-stable manifold and $W^u_\epsilon(x)$ the local strong-unstable manifold, sometimes denoted by $W^s_{\text{loc}}(x)$ and $W^u_{\text{loc}}(x)$, respectively. Here $d$ is the distance on $M$ induced by the Riemannian metric. Moreover, the manifolds $W^s_\epsilon(x)$ and $W^u_\epsilon(x)$ vary continuously with $x$. Also, if $x \in \Lambda$ one has that

$$W^s(x) = \bigcup_{t \geq 0} \phi^{-t}(W^s_\epsilon(\phi^t(x))) \quad \text{and} \quad W^u(x) = \bigcup_{t \leq 0} \phi^{-t}W^u_\epsilon(\phi^t(x))$$

are $C^r$ invariant manifolds immerse in $M$, called of strong-stable manifold and strong-unstable manifold of $x$, respectively. Finally, the sets

$$W^{cs}(x) = \bigcup_{t \in \mathbb{R}} W^s(\phi^t(x)) \quad \text{and} \quad W^{cu}(x) = \bigcup_{t \in \mathbb{R}} W^u(\phi^t(x))$$

are invariant $C^r$ manifolds tangent to $E^s_x \oplus \phi(x)$ and $E^u \oplus \phi(x)$, respectively.

A.2 Regular Cantor Sets

To keep the notation in the text, we rewrote the definition of regular Cantor sets found in [MY01, Section 1.1], but an alternative definition can be found in [PT93, Chapter 4]. Let $\mathbb{A}$ be a finite alphabet, $\mathbb{B}$ a subset of $\mathbb{A}^2$, and $\Sigma_\mathbb{B}$ the subshift of finite type of $\mathbb{A}^\mathbb{Z}$ with allowed transitions $\mathbb{B}$. We will always assume that $\Sigma_\mathbb{B}$ is topologically mixing and that every letter in $\mathbb{A}$ occurs in $\Sigma_\mathbb{B}$.

An expansive map of type $\Sigma_\mathbb{B}$ is a map $g$ with the following properties:

(i) the domain of $g$ is a disjoint union $\bigcup_{\mathbb{B}} I(a, b)$. Where for each $(a, b)$, $I(a, b)$ is a compact subinterval of $I(a) := [0, 1] \times \{a\}$;

(ii) for each $(a, b) \in \mathbb{B}$, the restriction of $g$ to $I(a, b)$ is a smooth diffeomorphism onto $I(b)$ satisfying $|Dg(t)| > 1$ for all $t$. 

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The regular Cantor set associated with \( g \) is the maximal invariant set
\[
K = \bigcap_{n \geq 0} g^{-n} \left( \bigcup B I(a,b) \right).
\]
Let \( \Sigma_B^+ \) be the unilateral subshift associated with \( \Sigma_B \). There exists a unique homeomorphism \( h: \Sigma_B^+ \rightarrow K \) such that
\[
h(a) \in I(a_0), \text{ for } a = (a_0,a_1,\ldots) \in \Sigma_B^+ \text{ and } h \circ \sigma = g \circ h,
\]
where \( \sigma^+: \Sigma_B^+ \rightarrow \Sigma_B^+ \), is defined as \( \sigma^+((a_n)_{n \geq 0}) = (a_{n+1})_{n \geq 0} \).

### A.3 Markov Partitions and Stable and Unstable Cantor Sets

Some good references in this section are the books [PT93] and [LMMR20, Section 3.2.3].

#### A.3.1 Markov Partition

**Definition A.1.** Let \( \Lambda \) be a horseshoe associated to a \( C^2 \)-diffeomorphism \( \varphi \) on a surface \( M \). A Markov partition \( R \) for \( \Lambda \) is a finite collection of compact sets \( \{R_1,R_2,\ldots,R_k\} \) (all diffeomorphic to a square) such that:

1. \( \Lambda \subset \bigcup_{i=1}^k R_i \),
2. \( \text{int} R_i \cap \text{int} R_j = \emptyset, \text{ if } i \neq j \),
3. If \( x \in \text{int} R_i \) and \( \varphi(x) \in \text{int} R_j \), then \( \varphi(W^u(x) \cap R_i) \subset \varphi(W^u(x) \cap R_i) \) and \( \varphi(W^s(x) \cap R_i) \subset W^s(\varphi(x)) \cap R_j \),
4. There is a positive integer \( n \) such that \( \varphi^n(R_i) \cap R_j \neq \emptyset \), for all \( 1 \leq i,j \leq k \) (mixing property).

Markov partitions play an important role in dynamical systems as they allow us to encode dynamics. Moreover, in this case, the diameter of \( R_i \)’s can be chosen arbitrarily small (cf. [PT93, Theorem 2 at Appendix]).

The dynamical on a horseshoe \( \Lambda \) can be codified using a matrix of transition associated to a Markov partition \( R \) of \( \Lambda \). In fact, the matrix of transition \( A = (a_{ij}) \) is defined by
\[
a_{ij} = \begin{cases} 
1 & \text{if } \varphi(R_i) \cap R_j \neq \emptyset; \\
0 & \text{otherwise.}
\end{cases}
\]

Consider the space of symbols \( \Xi = \{1,\ldots,k\}^Z \). We say that a finite word \((x_1,x_2,\ldots,x_n)\) in \( \Xi \) is admissible if \( a_{x_ix_{i+1}} = 1 \), for \( i = 1,2,\ldots,n-1 \). The admissible words are important since it help to localize the dynamic on the Markov partition. More generally, we consider the space of bi-infinite admissible words \( \Xi_A = \{(x_n) \in \Xi : a_{x_nx_{n+1}} = 1 \} \). It is not difficult to show that the \( \varphi|_\Lambda \) and the shift \( \sigma: \Xi_A \rightarrow \Xi_A \) are conjugated, i.e., there is a homeomorphism \( h: \Lambda \rightarrow \Xi_A \) such that
\[
h \circ \varphi = \Sigma \circ h,
\]
where \( \sigma(x_n) = (x_{n+1}) \).

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A.3.2 Stable and Unstable Cantor Sets

If $\Lambda$ is a horseshoe associated to a $C^2$-diffeomorphism $\varphi$ on a surface $M$ and consider a finite collection $(R_a)_{a \in A}$ of disjoint rectangles of $M$, which are a Markov partition of $\Lambda$ (cf. [PT93] Appendix 2 for more details). Define the sets

$$W^s(\Lambda, R) = \bigcap_{n \geq 0} \varphi^{-n}(\bigcup_{a \in A} R_a),$$
$$W^u(\Lambda, R) = \bigcap_{n \leq 0} \varphi^{-n}(\bigcup_{a \in A} R_a).$$

There is a $r > 1$ and a collection of $C^r$-submersions $(\pi_a : R_a \to I(a))_{a \in A}$, satisfying the following property:

If $z, z' \in R_{a_0} \cap \varphi^{-1}(R_{a_1})$ and $\pi_{a_0}(z) = \pi_{a_0}(z')$, then we have

$$\pi_{a_1}(\varphi(z)) = \pi_{a_1}(\varphi(z')).$$

In particular, the connected components of $W^s(\Lambda, R) \cap R_a$ are the level lines of $\pi_a$. Then we define a mapping $g^u$ of class $C^r$ (expansive of type $\Sigma_B^r$) by the formula

$$g^u(\pi_{a_0}(z)) = \pi_{a_1}(\varphi(z))$$

for $(a_0, a_1) \in \mathbb{B}$, $z \in R_{a_0} \cap \varphi^{-1}(R_{a_1})$. The regular Cantor set $K^u$ defined by $g^u$, describes the geometry transverse of the stable foliation $W^s(\Lambda, R)$. Analogously, we can describe the geometry transverse of the unstable foliation $W^u(\Lambda, R)$, using a regular Cantor set $K^s$ defined by a mapping $g^s$ of class $C^r$ (expansive of type $\Sigma_B^r$).

Also, the horseshoe $\Lambda$ is locally the product of two regular Cantor sets $K^s$ and $K^u$. So, the Hausdorff dimension of $\Lambda$, $HD(\Lambda)$ is equal to $HD(K^s \times K^u)$, but for regular Cantor sets, we have that $HD(K^s \times K^u) = HD(K^s) + HD(K^u)$. Thus $HD(\Lambda) = HD(K^s) + HD(K^u)$ (cf. [PT93] Ch. 4).

A.4 Geometry of $TM$ and $SM$

The following two subsections can be found in [Pat99]:

A.4.1 Vertical and horizontal subbundles

Let $\pi : TM \to M$ the canonical projection, i.e., if $\theta = (x, v) \in TM$, then $\pi(\theta) = x$.

The vertical subbundle is defined by $V(\theta) = \ker(d\pi_\theta)$. It is easy to see that $\phi$ is transverse to the fibers $\pi^{-1}(\cdot)$.

Denote by $TTM$ the tangent bundle of $TM$. The connection map

$$K : TTM \to TM,$$

is defined as follows. Let $\xi \in T_0 TM$ and $\rho : (-\epsilon, \epsilon) \to TM$ be a curve adapted to $\xi$, that is, with initial conditions as follows:

$$\begin{align*}
\rho(0) &= \theta; \\
\rho'(0) &= \xi.
\end{align*}$$
Such a curve gives rise to a curve $\nu: (-\varepsilon, \varepsilon) \to M$, $\nu := \pi \circ \rho$, and a vector field $\Upsilon$ along to $\nu$, equivalently, $\rho(t) = (\nu(t), \Upsilon(t))$. Define

$$K_\theta(\xi) = (\nabla_\nu \Upsilon)(0),$$

where $\nabla$ is the Levi-Civita connection, and $TTM$ is the tangent bundle of $TM$. The horizontal subbundle is the subbundle of $TTM$ whose fiber at $\theta$ is given by $H(\theta) = \ker K_\theta$. The vertical and the horizontal subbundle hold that

$$T_\theta TM = H(\theta) \oplus V(\theta),$$

and that the map $j_\theta: T_\theta TM \to T_x M \times T_x M$ given by

$$j_\theta(\xi) = (d\pi_\theta(\xi), K_\theta(\xi)),$$

is a linear isomorphism.

By writing $\xi = (\xi_h, \xi_v)$, where $\xi_h = d\pi_\theta(\xi)$ and $\xi_v = K_\theta(\xi)$, we identify $\xi$ with $j_\theta(\xi)$.

Using the decomposition $T_\theta TM = H(\theta) \oplus V(\theta)$, we can define naturally a Riemannian metric on $TM$ that makes $H(\theta)$ and $V(\theta)$ orthogonal. This metric is called the Sasaki metric and is given by

$$\langle\langle \xi, \eta \rangle\rangle_\theta = \langle d\pi_\theta(\xi), d\pi_\theta(\eta) \rangle_{\pi(\theta)} + \langle K_\theta(\xi), K_\theta(\eta) \rangle_{\pi(\theta)}.$$

The one-form $\alpha$ of $TM$ is defined by

$$\alpha_\theta(\xi) = \langle\langle \xi, \phi(\theta) \rangle\rangle = \langle d\pi_\theta(\xi), v \rangle_x,$$

such that $\alpha$ restricted to $SM$ (the unit tangent bundle) it becomes a contact form whose characteristic flow is the geodesic flow restricted to $SM$.

### A.4.2 Jacobi Fields and the Differential of the Geodesic Flow

In this section, for $\theta \in SM$, we shall describe an isomorphism between the tangent space $T_\theta TM$ and the Jacobi fields along the geodesic $\gamma_\theta$. Using the decomposition of $T_\theta TM$ in vertical and horizontal subspaces, we shall give a very simple expression for the differential of the geodesic flow in terms of Jacobi fields. Recall that a Jacobi vector field along the geodesic $\gamma_\theta$ is a vector field along $\gamma_\theta$ that is obtained as the variational vector field of a variation of $\gamma_\theta$ through geodesics. It is well known that $J$ is a Jacobi vector field along $\gamma_\theta$ if and only it satisfies the Jacobi equation

$$J'' + R(\gamma_\theta', J)\gamma_\theta = 0,$$

where $R$ is the Riemann curvature tensor of $M$ and $'$ denotes covariant derivatives along $\gamma_\theta$.

Let $\xi \in T_\theta TM$ and $\rho: (-\varepsilon, \varepsilon) \to TM$ be an adapted curve to $\xi$. Then the map $(s, t) \to \pi \circ \phi^t(\rho(s))$ gives rise to a variation of $\gamma_\theta = \pi \circ \phi^t(\theta)$. The curves $t \to \pi \circ \phi^t(\rho(s))$ are geodesics and, therefore, the corresponding variational vector field $J_\xi(t) = \frac{\partial}{\partial s}|_{s=0} \pi \circ \phi^t(\rho(s))$ is a Jacobi vector field with initial conditions given by

$$\begin{cases}
J_\xi(0) = \frac{\partial}{\partial s}|_{s=0} \pi \circ \phi^t(\rho(s))|_{t=0} = d\pi_\theta(\xi); \\
J'_\xi(0) = \frac{\partial}{\partial s}|_{s=0} \frac{\partial}{\partial s}|_{t=0} \pi \circ \phi^t(\rho(s)) = \frac{\partial}{\partial s}|_{s=0} \Upsilon(s) = K_\theta(\xi).
\end{cases}$$
Using the above representation, we can describe the differential of the geodesic flow in terms of Jacobi fields and the splitting of $T_\theta TM$ into horizontal and vertical subbundles we have

**Claim:** Given $\theta \in TM$, $\xi \in T_\theta TM$, and $t \in \mathbb{R}$, we have

$$d\phi_\theta'(\xi) = (J_\xi(t), J_v'(t)).$$

The following Lemma can be found in (cf. [Ebe73]) for the compact case. For the non-compact case, the proof still holds with some adaptations.

**Lemma A.2.** Let $X \subset SM$ be a hyperbolic set. Then for any $\theta \in X$

$$E^s(\theta) \oplus E^u(\theta) = \ker_\theta.$$

**Proof.** Let us show that $E^s(\theta) \subset \ker_\theta$. For $E^u(\theta)$ the proof is analogous. Since $\phi^t$ preserves the contact form $\alpha$, then given $\eta \in E^s(\theta)$, we have

$$\alpha_\theta(\eta) = \alpha_{\phi^t(\theta)}(d\phi_\theta(\eta)) = \langle d\pi_{\phi^t(\theta)}(d\phi_\theta(\eta)), \gamma'(t) \rangle = \langle d(\pi \circ \phi^t)_\theta(\eta), \gamma'(t) \rangle$$

where $\gamma$ is a Jacobi field on $\gamma_0$ such that $\gamma_0$ is an unstable Jacobi field.

Since $\|d\phi_\theta(\eta)\|^2 = \|J_\eta(t)\|^2 + \|J'_\eta(t)\|^2$ and $\|d\phi_\theta(\eta)\| \to 0$ when $t \to \infty$, then $\|J_\eta(t)\| \to 0$ when $t \to \infty$. Also, $|\alpha_\theta(\eta)| \leq \|J_\eta(t)\|$, so we have that $\alpha_\theta(\eta) = 0$ showing that $E^s(\theta) \subset \ker_\theta$. Since $E^s(\theta) \oplus E^u(\theta)$ and $\ker_\theta$ have the same dimensions, then we conclude that $E^s(\theta) \oplus E^u(\theta) = \ker_\theta$.

**A.4.3 Stable and Unstable Jacobi Fields**

Let $\theta = (x, v)$ and $w$ orthogonal to $v$ and let $J^T_w(t)$ be the unique Jacobi filed on $\gamma_0(t)$ such that $J^T_w(0) = w$ and $J^T_w(T) = 0$.

The limit $J^s_w(t) := \lim_{T \to -\infty} J^T_w(t)$ exists and is a Jacobi vector field on $\gamma_0(t)$ (cf. [Ebe73]). Clearly, $J^s_w(0) = w$ and $J^s_w(t) \neq 0$ for all $t > 0$. We call $J^s_w(t)$ the stable Jacobi field.

The unstable Jacobi field $J^u_w(t)$ along $\gamma_v(t)$ is obtained by considering taking the limit as $T \to -\infty$, i.e.,

$$J^u_w(t) := \lim_{T \to -\infty} J^T_w(t).$$

The subspaces (using the identification $T_\theta SM = H(\theta) \oplus V(\theta)$)

$$E^s(\phi^t(\theta)) = \{(J(t), J'(t)) \in T_{\phi^t(\theta)}SM | J \text{ is a stable Jacobi field}\},$$

$$E^u(\phi^t(\theta)) = \{(J(t), J'(t)) \in T_{\phi^t(\theta)}SM | J \text{ is an unstable Jacobi field}\}$$

are called the Green subbundles on $\gamma_\theta$, which are also the stable and unstable subbundles of the definition of hyperbolicity of the geodesic flow on $SM$ (cf. [Ebe73]).

In the pinched negative condition of the curvature $(-a^2 \leq K_M \leq -b^2 < 0)$, we have that the Jacobi fields that vanish at some point are always divergent, in other words, can not be bounded (cf. [Ebe73]).
**Lemma A.3.** If \( w \in T_oSM \setminus \{0\} \) is a vertical vector, then \( w \notin E^s(\theta) \oplus \phi(\theta) \) and \( w \notin E^u(\theta) \oplus \phi(\theta) \), where \( E^s(\theta), E^u(\theta) \) are the stable and unstable space, respectively.

**Proof.** From Lemma A.2 we have \( E^s(\theta) \oplus E^u(\theta) = ker \alpha_\theta \). By contradiction, assume that \( w = \alpha \xi^s + \beta \phi(\theta) \) with \( \xi^s \in E^s(\theta) \), then

\[
0 = \langle d\pi_\theta(w), v \rangle = \langle d\pi_\theta(\xi^s), v \rangle + \beta \langle v, v \rangle = \beta.
\]

Therefore, \( \xi^s \) is a vertical vector, but

\[
E^s(\theta) = \{(J_s(0), J'_s(0)) \in H(\theta) \oplus V(\theta) : J_s \text{ is a stable Jacobi field})\}.
\]

If \( \xi^s = (J_s(0), J'_s(0)) \), then \( J_s(0) = 0 \), and hence, the condition on the curvature implies that \( J_s \) is a divergent Jacobi field, which is a contradiction since \( J_s \) is bounded because it is a stable Jacobi field.

The proof for \( E^u(\theta) \oplus \phi(\theta) \) is analogous. \( \square \)

### A.5 Generic Properties of Geodesic Flows

#### A.5.1 Klingerberg and Takens Theorem

Let \( J^k_s(2n) \) be the set of \( k \)-jets of symplectic automorphisms of \((\mathbb{R}^{2n}, 0)\) (with \( \omega = \sum dx_i \wedge dx_j \)) let \( Q \) be an invariant subset of \( J^k_s(2n) \) (i.e. for all \( \sigma \in J^k_s(2n) \), \( \sigma Q \sigma^{-1} = Q \)). Let \( Q \subset J^k_s(2n) \) and \( \gamma \) a closed geodesic for the metric \( g \), we say that \( \phi_g \) (the vector of the geodesic flow) has the property \( Q \subset J^k_s(2n) \) if, for some \( x \in \gamma \) and some cross-section \( \Sigma \) to \( \phi_g \) in \( x \), we have that the symplectic automorphism \( DP_x \in Q \), where \( P \) is the Poincaré map associate to the section \( \Sigma \), (we denote by \( P_Q \) for specifying the property \( Q \)). It is important to note that this definition is not dependent on the cross-section \( \Sigma \) (cf. [KT72]).

**Theorem A.4** (Klingerberg-Takens). Let \( Q \subset J^k_s(2n) \) be open dense and invariant. Then the following property \( P_Q \) is \( C^{k+1} \)-generic, i.e., \( \phi_g \) has property \( P_Q \) if the Poincaré map of every closed geodesic of \( g \) belongs to \( Q \).

#### A.5.2 The Birkhoff Invariant

Let \( f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0) \) be a germ of diffeomorphism area-preserving (in dimension two is symplectic) and \( 0 \) a hyperbolic fixed point with eigenvalues \( \lambda \) and \( \lambda^{-1} \), then the Birkhoff normal form (cf. [Mos56]) says that there is an area-preserving change of coordinates \( \Phi \) such that \( \Phi^{-1} \circ f \circ \Phi = N \), where \( N(x, y) = (U(xy)x, U^{-1}(xy)y) \) and \( U(xy) \) is a power series \( \lambda + U_2xy + \cdots \) convergent in a neighborhood of \( x = y = 0 \). In other words, in this coordinates \( f \) can be written as

\[
f(x, y) = (\lambda x(1 + axy + O((x, y)^4))), \lambda^{-1}y(1 - axy + O((x, y)^4))), \quad (28)
\]

where the constant \( a \) is called the **Birkhoff Invariant** of \( f \).

**Lemma A.5.** The Birkhoff invariant for diffeomorphism area-preserving in \((\mathbb{R}^2, 0)\) only depends of 3-jets in \(0, J^3(0)\). Moreover, the set of diffeomorphism area-preserving in \((\mathbb{R}^2, 0)\) such that the Birkhoff invariant is non-zero is open, dense, and invariant in \( J^3(0) \).
Proof. The proof of [Mos56, Theorem 1 and 2], show the first part of the lemma and also the opening. For density, suppose that for some $f: (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$, the Birkhoff invariant is zero, then for $\epsilon > 0$ we consider the function $N_\epsilon(x, y) := (\lambda x (1 + O(||(x, y)||^4)), \lambda^{-1}y (1 + O(||(x, y)||^4))) + \epsilon(x^2y, -xy^2)$, then the function $f_\epsilon = \Phi \circ N_\epsilon \circ \Phi^{-1}$ is area-preserving diffeomorphism close to $f$ with the Birkhoff invariant is $\epsilon$.

Let $f, g$ be as above and suppose that the Birkhoff invariant for $f$ is non-zero, then $g^{-1} \circ f \circ g$ has the Birkhoff invariant non zero. Indeed, by the Birkhoff Normal Form [Mos56, Theorem 1], there is an area-preserving change of coordinates $\Phi$ such that $\Phi^{-1} \circ g^{-1} \circ f \circ g \circ \Phi$ has the form (28), then $(g \circ \Phi)^{-1} \circ f \circ (g \circ \Phi)$ has the form (28), in other words, there is another area-preserving change of coordinates $g \circ \Phi$ such that $f$ has the form (28), but by the uniqueness of the Birkhoff normal form (see [Mos56, page 674]), we have that the Birkhoff invariant of $g^{-1} \circ f \circ g$ is equal to the Birkhoff invariant of $f$, therefore non-zero.

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