An Introduction to Effectus Theory

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Abstract

Effectus theory is a new branch of categorical logic that aims to capture the essentials of quantum logic, with probabilistic and Boolean logic as special cases. Predicates in effectus theory are not subobjects having a Heyting algebra structure, like in topos theory, but ‘characteristic’ functions, forming effect algebras. Such effect algebras are algebraic models of quantititative logic, in which double negation holds. Effects in quantum theory and fuzzy predicates in probability theory form examples of effect algebras.

This text is an account of the basics of effectus theory. It includes the fundamental duality between states and effects, with the associated Born rule for validity of an effect (predicate) in a particular state. A basic result says that effectuses can be described equivalently in both ‘total’ and ‘partial’ form. So-called ‘commutative’ and ‘Boolean’ effectuses are distinguished, for probabilistic and classical models. It is shown how these Boolean effectuses are essentially extensive categories. A large part of the theory is devoted to the logical notions of comprehension and quotient, which are described abstractly as right adjoint to truth, and as left adjoint to falsity, respectively. It is illustrated how comprehension and quotients are closely related to measurement. The paper closes with a section on ‘non-commutative’ effectus theory, where the appropriate formalisation is not entirely clear yet.
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1 Introduction

Effectus theory is a new branch of categorical logic that started with [Jac15a]. The theory is already used in several other publications, such as [JWW15, Cho15, CJWW15, Ada14, AJ15], and is being actively developed. Therefore we think that it is appropriate to provide a systematic, accessible introduction to this new area.

The aim of effectus theory is to axiomatise the categorical characteristics of quantum computation and logic. This axiomatisation should include probabilistic and classical computation and logic as special cases. The relevant logical structure is described in terms of effect algebras. Such algebras have been introduced in theoretical physics to describe generalised (quantum) probability theories, see [FB94], and also [CK95, GG94]. Effect algebras generalise both classical logic, in the form of Boolean algebras, and quantitative logic, with the unit interval $[0,1]$ of real numbers as main example. The double negation law holds in such effect algebras. One of the basic insights of [Jac15a] is that certain elementary categorical structure — involving coproducts and pullbacks — provides predicates with effect algebra structure.

In the usual set-theoretic world one can define predicates on a set $X$ as subsets $P \subseteq X$ or equivalently as characteristic functions $p: X \to 2$, where $2 = 1+1 = \{0,1\}$ is the two-element set. The ‘subset’ or ‘spatial’ approach is the basis for much of categorical logic where predicates are described as subobjects $P \hookrightarrow X$. In topos theory, there is a bijective correspondence between such subobjects of $X$ and characteristic maps $X \to \Omega$ to a distinguished ‘classifier’ object $\Omega$. These predicates on an object $X$ in a topos form a Heyting algebra, a basic formalisation of intuitionistic logic, in which the double negation law fails. Topos theory has developed into a rich area of mathematics, combining geometry and logic, see for instance the handbooks [MM92] or [Joh02].

Effectus theory breaks with this spatial approach and uses maps of the form $X \to 1+1$ as predicates. Negation then corresponds to swapping of outcomes in $1+1$, so that the double negation law immediately holds. As mentioned, predicates in an effectus are not Heyting algebras, but effect algebras: their logic is completely different.

The current text provides an introduction to this new area. It starts with some basic results from the original source [Jac15a]. But it differs from [Jac15a] in that it combines the total and partial perspectives on effectuses right from the beginning, working towards the equivalence of these two approaches, as developed in [Cho15]. Similarly, this article provides a systematic description of quotients and comprehension in effectuses, which are neatly described as a chain of four adjunctions in a row, which quotients as left adjoint to falsity, and comprehension as right adjoint to truth. This extends [CJWW15] where quotients and comprehension are described only concretely, in specific examples.

This paper contains several definitions and results that have not been published before, such as:

- the construction of effectuses from ‘grounded’ biproduct categories in Section 6
- the definition of commutative (probabilistic) effectuses and Boolean (clas-
sical) effectuses in Section 9 giving a picture:

\[
\begin{array}{ccc}
\text{Boolean} & \subseteq & \text{commutative} \\
\text{effectuses} & \subseteq & \text{(non-commutative)} \\
\text{effectuses}
\end{array}
\]

The adjective ‘non-commutative’ means ‘arbitrary’; it is only used to distinguish the general case from the two special subcases.

- the equivalence between a Boolean effectus with comprehension and the well-known notion of extensive category; this gives many examples of (Boolean) effectuses, including categories of (compact Hausdorff) topological spaces, and of measurable spaces; also, the opposite \(\text{CRng}^{\text{op}}\) of the category of commutative rings is an example, see [CJWW15, Jac 15a];

- the first steps in non-commutative — properly quantum — effectus theory.

A recurring topic in effectus theory is the association of an ‘assert’ partial map \(\text{asrt}_p: X \to X + 1\) with a predicate \(p: X \to 1 + 1\). This assert map is the ‘action’ of the predicate and is typical for a quantum setting, where an observation, determined by a predicate, may have a side-effect and disturb the system under observation. In a commutative or Boolean effectus one does have such action maps \(\text{asrt}_p\), but they are side-effect free. This is formalised by saying that these assert maps are below the identity, in a suitable partial order on partial maps. Indeed, the identity map does nothing, so maps below the identity also do not change the state. This predicate-action correspondence is one of the main topics.

Four running examples are used throughout the text, namely:

1. the category \(\text{Sets}\) of sets and functions, which is a Boolean effectus;
2. the Kleisli category \(\mathcal{K}\ell(\mathcal{D})\) of the distribution monad \(\mathcal{D}\) on \(\text{Sets}\), which is a commutative effectus, modeling discrete probabilistic computation;
3. the opposite \(\text{OUG}^{\text{op}}\) of the category \(\text{OUG}\) of order unit groups, which is used as a relatively simple toy example to illustrate some basic notions;
4. the opposite \(\text{vNA}^{\text{op}}\) of the category \(\text{vNA}\) of von Neumann algebras, which is the prime example of a non-commutative effectus.

The last two categories occur in opposite form because they incorporate Heisenberg’s view where quantum computations are predicate transformers which operate in opposite direction, transforming predicates on the post-state to predicates on the pre-state.

The insight that these diverse examples are all instance of the notion of effectus has significant value. It allows us to grasp in these examples what, for instance, the comprehension or quotient structure of an effectus amounts to. These descriptions can be non-trivial and hard to understand without the abstract categorical viewpoint. This applies in particular to von Neumann algebras, where the ‘opposite’ interpretation can be confusing, and where notions like support projection exist already for a long time (see e.g. [Sak71, Dfn. 1.10.3]), but without their proper universal properties.
This text uses von Neumann algebras (or $W^*$-algebras) instead of $C^*$-algebras. The key aspect of von Neumann algebras that we use is that path predicates/effects form a directed complete poset. This allows us to define for an arbitrary predicate/effect $p$ the least sharp predicate $[p]$ above $p$. This construction is crucial for images, comprehension and for quotients. Additionally, (normal) states separate predicates in von Neumann algebras, which is used in the quotient construction. These two properties, directed completeness and separation characterise von Neumann algebras among $C^*$-algebras, following Kadison [Kad56, Dfn. 1 and Thm. 1] (see also [Ren14]).

Before starting, we would like to put effectus theory in broader perspective. The Oxford school in categorical quantum theory started with [AC04], see the recent books [HV15] and [CKar]. This approach takes the tensor $\otimes$ for parallel composition as the main ingredient of quantum theory — following Schrödinger’s emphasis on composite systems, and not von Neumann’s focus on measurement, see [CL13, §§2.2]. In the associated graphical calculus this tensor is represented simply via parallel wires. This gives a powerful formalism in which many protocols can be described. Coproducts (or biproducts) played a role originally, but have become less prominent, probably because they do not work well in a projective (i.e. ‘scalar-free’) setting [CDH] and make the graphical calculus more complicated. The main examples for the Oxford school are so-called dagger-compact closed monoidal categories, such as Hilbert spaces, sets and relations, or finite-dimensional $C^*$-algebras and completely positive maps.

In contrast, in effectus theory the coproduct $+$ is taken as primitive constructor (and not the tensor $\otimes$), leading to a strong emphasis on logic, validity and measurement, with the category of von Neumann algebras as leading quantum example. A connection with the Oxford school exists via the $CP^*$-categories (with biproducts) of [CHK14]. Section 6 below shows that the causal maps in a ‘grounded’ biproduct category — including $CP^*$-categories — form an effectus. Tensors can be added to an effectus, see Section 10 and lead to a much richer theory, more closely connected to the Oxford school. But tensors are not essential for the definition of an effectus theory, and for the associated logic of effect algebras.

Other connections are emerging in recent work [Thu16] on operational probabilistic theories [CDP10] and effectuses. This makes it possible to transfer methods and results in existing quantum theory and effectus theory back and forth.

One may ask: is effectus theory the new topos theory? It is definitely too early to say. But what hopefully becomes clear from this introduction is that effectus theory is a rich novel direction in categorical logic that promises to capture essential aspects of the quantum world and to provide a unifying framework that includes the probabilistic and Boolean worlds as special cases.

This text gives a survey of first results in the theory of effectuses. Some parts focus more on explaining ideas and examples, and some parts serve as reference for basic facts — with long series of results. We briefly discuss the contents of the various sections. First, a separate section is devoted to notation, where distinct notation is introduced and explained for total and partial maps. Then, Section 3 introduces the central notion of effectus, as a category with coproducts $(+ , 0)$ and a final object $1$ satisfying certain elementary axioms. This same section describes categorical consequences of these axioms, in particular about partial projections and pairing, and introduces the leading examples of
effectuses. Section 4 continues the exploration and focuses on the partial monoid structure \((\otimes, 0)\) on partial maps \(X \to Y + 1\) in an effectus. Predicates \(X \to 1 + 1\) are a special case and have additional structure: it is shown that they form effect algebras with scalar multiplication — making them effect modules. Subsequently, Section 5 shows that states \(1 \to X\) of an object \(X\) in an effectus form convex sets, and that the basic adjunction between effect modules and convex sets gives what is called a state-and-effect triangle of the form:

\[
\begin{array}{ccc}
\text{Stat} &=& \text{Hom}(1, -) \\
\text{Conv}_M &=& \text{Hom}(-, \text{M}) \\
\text{EMod}_M^{op} &=& \text{Hom}(-, M) \\
B &=& \text{Pred}(-, -1 + 1)
\end{array}
\]

where \(B\) is an effectus and \(M = \text{Pred}(1)\) is the effect monoid of scalars in \(B\), see Theorem 23 for details. In this situation we can describe the Born rule for validity \(\omega \models p\) of a predicate \(p: X \to 1 + 1\) and a state \(\omega: 1 \to X\) simply via composition, as scalar \(p \circ \omega: 1 \to 1 + 1\).

Section 6 describes a new construction — inspired by \(\text{CP}^*\)-categories [CHK14] — to obtain an effectus as the subcategory of ‘causal’ maps in a biproduct category with special ground maps \(\dagger\). Many examples of effectuses can be obtained (and understood) in this way, where, for instance, causal maps correspond to the unital ones. Subsequently, Section 7 is more auxiliary: it introduces kernels and images in an effectus and collects many basic results about them that will be used later on. A basic result in the theory of effectuses, namely the equivalence of the descriptions in terms of total and partial maps, is the topic of Section 8 following [Cho15]. It plays a central role in the theory, and once we have seen it, we freely switch between the two descriptions, by using phrases like an ‘effectus in total form’, or an ‘effectus in partial form’.

Special axioms in an effectus are identified in Section 9 that capture probabilistic models and Boolean models as special cases. Section 10 describes tensors \(\otimes\) in an effectus, for parallel composition. These tensors come equipped with projections \(X \leftarrow X \otimes Y \to Y\), for weakening/discarding. But these tensors do not have copiers/diagonals \(X \to X \otimes X\), since contraction/duplication/cloning does not exist in the quantum world. We show that if copiers exist, then the effectus becomes commutative.

The subsequent two Sections 11 and 12 introduce the important notions of comprehension \(\{X | p\}\) and quotients \(X/p\) for a predicate \(p\) on an object \(X\) in an effectus. These notions are nicely captured categorically as right adjoint to truth and as left adjoints to falsity, in a chain of comprehensions:

\[
\text{quotient} \dashv \text{falsity} \dashv \text{forget} \dashv \text{truth} \dashv \text{comprehension}
\]

Such chains exist in all our leading examples, see also [CJWW15]. The combination of comprehension and quotients is studied in Section 14.

Inbetween, Section 13 returns to Boolean effectuses and gives a new characterisation: it shows that Boolean effectuses with comprehension are essentially the same as extensive categories. The latter are well-known kinds of categories in which finite coproducts \((+, 0)\) are well-behaved.

Our final Section 15 is of a different nature because it describes some unclarities in the theory of effectuses which require further research. What is unclear

6
is the precise formalisation of the general non-commutative case that captures
the essentials of quantum theory. This involves the intriguing relations between
the predicate-action correspondence for measurement on the one hand, and quo-
tients and comprehension on the other. We show that the ‘assert’ action maps
are uniquely determined by postulates in von Neumann algebras, but we cannot,
at this stage, prove uniqueness at a general, axiomatic level. More directions
for future research are collected in Section 16.

2  Notation

In the theory of effectuses, both total and partial maps play an important role.
When reasoning in an effectus one often switches between total and partial maps.
Since this may lead to confusion, we address this topic explicitly in the current
section, before introducing the notion of effectus itself in the next section.

There are two equivalent ways of defining effectuses, see Section 8 for details.

1. The ‘total’ approach starts from a category of total maps, and introduces
partial maps as special, Kleisli maps for the lift monad \((-) + 1\) in a larger
enveloping Kleisli category. This is how effectuses were originally intro-
duced in \[Jac15a\].

2. The ‘partial’ approach starts from a category of partial maps, and de-
scribes the total maps via a smaller ‘wide’ subcategory with the same
objects where the total maps are singled out via a special property. This
approach comes from \[Cho15\].

In both cases we have a faithful functor between two categories with the same
objects:

\[
\begin{array}{ccc}
\text{(total maps)} & \xrightarrow{\text{functor}} & \text{(partial maps)} \\
\end{array}
\]

These categories each have their own form of composition. In order to disam-
biguate them we shall use different notation, for the time being, namely \(\cdot\) and
\(\circ\). In fact, we shall also use different notation for morphisms, namely \(\to\) and
\(\Rightarrow\), and also for sums of maps, namely \(+\) and \(\oplus\). These notational differences
are relevant as long as we have not established the precise relationship between
total and partial maps, see Theorem 53. Once we know the relationship, we
re-evaluate the situation in Discussion 54.

Only at that stage do we see clearly how to switch back-and-forth, and can
decide whether to take total or partial maps as first class citizens.

Total maps

Let \(\mathcal{B}\) be a category with finite coproducts \((+, 0)\) and a final object \(1 \in \mathcal{B}\). We
consider the maps in \(\mathcal{B}\) as total, and we write the coprojections \(\kappa_i\) and cotupling
\([-,-]\) as maps:

\[
X_1 \xrightarrow{\kappa_1} X_1 + X_2 \xrightarrow{\kappa_2} X_2 \quad \text{and} \quad X_1 + X_2 \xrightarrow{[f_1,f_2]} Y \quad \text{for} \quad X_1 \xrightarrow{f} Y
\]

The sum of two maps \(f: X \to A, g: Y \to B\) is written as \(f + g = [\kappa_1 \cdot f, \kappa_2 \cdot \]
\(g]: X + Y \to A + B\). The \(n\)-fold coproduct \(X + \cdots + X\) of the same object
X is usually written as \( n \cdot X \), and called a copower. We use the notation \( \nabla = [\text{id}, \text{id}] : X + X \to X \) for the codiagonal.

For an arbitrary object \( X \in B \) we write \(!_X\), or simply \(!\), both for the unique map from 0 to \( X \), and for the unique map from \( X \) to 1, as in:

\[
\begin{array}{ccc}
0 & \xrightarrow{!} & X \\
X & \xrightarrow{!} & 1
\end{array}
\]

**Partial maps**

A partial map is a map of the form \( X \to Y + 1 \). As is well-known, the assignment \( X \mapsto X + 1 \) forms a monad on the category \( B \). It is often called the lift or the maybe monad. The unit of this monad is the first coprojection \( \kappa_1 : X \to X + 1 \), and the multiplication is the cotuple \([\text{id}, \kappa_2] : (X + 1) + 1 \to X + 1\). We shall write \( \text{Par}(B) \) for the Kleisli category of this lift monad. Its objects are the objects of \( B \), and its maps from \( X \) to \( Y \), written as \( X \to Y \), are morphisms \( X \to Y + 1 \) in \( B \). Thus, (ordinary) maps in \( \text{Par}(B) \) are partial maps in \( B \).

Composition of \( f : X \to Y \) and \( g : Y \to Z \) in \( \text{Par}(B) \) is written as \( g \cdot f : X \to Z \), and is described in \( \text{Par}(B) \) and in \( B \) as:

\[
g \cdot f = \left( X \xrightarrow{f} Y \xrightarrow{g} Z \right) = \left( X \xrightarrow{f} Y + 1 \xrightarrow{[g, \kappa_2]} Z + 1 \right).
\]

Each map \( h : X \to Y \) in \( B \) gives rise to a map \( \langle h \rangle : X \to Y \) in \( \text{Par}(B) \), namely:

\[
\langle h \rangle = \left( X \xrightarrow{h} Y \xrightarrow{\kappa_1} Y + 1 \right).
\]

We call such maps in \( \text{Par}(B) \) of the form \( \langle h \rangle \) *total*, and write them with this (single) guillemet notation \( \langle \cdot \rangle \). Later on, in Lemma 7, we shall see an alternative characterisation of total maps, which will be much more useful.

The identity on \( X \) in \( \text{Par}(B) \) is then the (total) map \( \langle \text{id}_X \rangle = \kappa_1 : X \to X \). There is an identity-on-objects functor \( B \to \text{Par}(B) \), which we simply write as \( \langle \cdot \rangle \). It preserves composition, since:

\[
\langle k \rangle \cdot \langle h \rangle = [\kappa_1 \cdot k, \kappa_2] \cdot \kappa_1 \cdot h = \kappa_1 \cdot k \cdot h = \langle k \cdot h \rangle.
\]

It is also not hard to see that:

\[
g \cdot \langle h \rangle = g \cdot h \quad \text{and} \quad \langle k \rangle \cdot f = (k + \text{id}) \cdot f.
\]

The initial object \( 0 \in B \) is a *zero object* in \( \text{Par}(B) \), because there are unique maps \( 0 \to X \) and \( X \to 0 \) in \( \text{Par}(B) \), namely the maps \(! : 0 \to X + 1\) and \(! : X \to 0 + 1 \cong 1 \) in \( B \). Hence, for each pair of objects \( X, Y \) there is a special *zero map* between them in \( \text{Par}(B) \), defined as:

\[
\left( X \xrightarrow{0} Y \right) = \left( X \xrightarrow{!} 1 \xrightarrow{\kappa_2} Y + 1 \right).
\]

Notice that \( 0 \circ g = 0 = f \circ 0 \) for all (partial) maps \( f, g \). Notice that we write this zero map \( 0 \) in bold face, in order to distinguish it from the zero object 0. In general, we shall write the top and bottom element of a poset as such bold
face 1 and 0. Frequently, the zero map is indeed the bottom element in a poset structure on homsets.

The presence of this zero map 0 allows us to define kernel and cokernel maps in $\text{Par}(B)$, like in Abelian categories, namely as (co)equaliser of a map $X \to Y$ and the zero map $0: X \to Y$. This will appear later.

The coproducts $(+, 0)$ in $B$ also form coproducts in $\text{Par}(B)$. The coprojections in $\text{Par}(B)$ are:

$$X_1 \xrightarrow{\langle \kappa_1 \rangle = \kappa_1 \cdot 1} X_1 + X_2 \xleftarrow{\langle \kappa_2 \rangle = \kappa_1 \cdot \kappa_2} X_2$$

The cotuple $[f, g]$ in $\text{Par}(B)$ is the same as in $B$. For two maps $f: X \to A$ and $g: Y \to B$ there is thus a sum of maps $f + g: X + Y \to A + B$ in $\text{Par}(B)$, which is defined in $B$ as:

$$f + g = \left( X + Y \xrightarrow{[(\kappa_1 + \text{id}) \cdot f, (\kappa_2 + \text{id}) \cdot g]} (A + B) + 1 \right).$$

It is not hard to see that:

$$\langle f \rangle, \langle g \rangle = \langle [f, g] \rangle \quad \text{and} \quad \langle h \rangle + \langle k \rangle = \langle h + k \rangle.$$

The coproduct $X_1 + X_2$ in $\text{Par}(B)$ comes with ‘partial projections’ $\triangleright_i$: $X_1 + X_2 \to X_i$, described in $B$ as:

$$X_1 + 1 \xrightarrow{\triangleright_1 = \text{id} + 1} X_1 + X_2 \xleftarrow{\triangleright_2 = [\kappa_2 + \text{id}, \kappa_1]} X_2 + 1$$

Equivalently they can be described in $\text{Par}(B)$ via the zero map 0 as cotuples:

$$X_1 \xrightarrow{\triangleright_1 = [\text{id}, 0]} X_1 + X_2 \xleftarrow{\triangleright_2 = [0, \text{id}]} X_2$$

These partial projections are natural in $\text{Par}(B)$, in the sense that:

$$\triangleright_1 \cdot (f_1 + f_2) = f_1 \cdot \triangleright_1.$$  (3)

It is easy to define these projections $\triangleright_i: X_1 + \cdots + X_n \to X_i$ for $n$-ary coproducts.

The coproduct $+$ in $\text{Par}(B)$ forms what is sometimes called a ‘split product’ or ‘butterfly’ product (see e.g. [Gra12, 2.1.7]): the following triangles commute in $\text{Par}(B)$, where the two diagonals are zero maps:

$$
\begin{array}{c}
\begin{tikzpicture}
\node (X) at (0,0) {$X$};
\node (Y) at (2,0) {$Y$};
\node (X+Y) at (1,1) {$X + Y$};
\node (X1) at (0,1) {$X_1$};
\node (X2) at (2,1) {$X_2$};
\draw[->] (X) -- (X+Y);
\draw[->] (X+Y) -- (Y);
\draw[->] (X1) -- (X+Y);
\draw[->] (X+Y) -- (X2);
\end{tikzpicture}
\end{array}
$$

(4)

In particular, the coprojections $\langle \kappa_i \rangle$ are split monos, and the projections $\triangleright_i$ are split epis in $\text{Par}(B)$.

In this diagram (4) we write $\langle \kappa_i \rangle: X \to X + Y$ for the coprojection in a category of partial maps $\text{Par}(B)$. Similarly we have written $\langle \text{id} \rangle$ for the identity in $\text{Par}(B)$. From now on we shall omit these guillemets $\langle \cdot \rangle$ for coprojections and identities when the context is clear. Thus we simply write $\kappa_1: X \to X + Y$ and $\text{id}: X \to X$ for a coprojection or identity map in $\text{Par}(B)$.
Predicates and predicate transformers

In the context of effectus theory a predicate on an object $X$ is a total map of the form $X \rightarrow 1 + 1$, or equivalently, a partial map $X \rightarrow 1$. The predicates $1$ for true and $0$ for false are given by:

$$1 = \begin{array}{c} X \xrightarrow{p} 1 + 1 \end{array}$$

$$0 = \begin{array}{c} X \xrightarrow{q} 1 + 1 \end{array}$$

The zero predicate on $X$ is thus the zero map $X \rightarrow 1$. The negation, called orthosupplement in this setting, of a predicate $p: X \rightarrow 1 + 1$ is obtained by swapping the outcomes:

$$p^\perp = \begin{array}{c} X \xrightarrow{p} 1 + 1 \end{array}$$

Clearly, $p^{\perp\perp} = p$ and $1^{\perp} = 0$. One may expect that these predicates $X \rightarrow 1 + 1$ form a Boolean algebra, but this is not true in general. When $B$ is an effectus, the predicates on $X$ form an effect algebra. This is one of the fundamental results of effectus theory, see Section 4 for details. The double negation law $p^{\perp\perp} = p$ always holds in an effect algebra.

We use two kinds of re-indexing or substitution operations for a map, each transferring predicates on the codomain to predicates on the domain. For total maps $f: Y \rightarrow X$ and partial maps $g: Y \rightarrow X$ we write:

$$f^\star(p) = \begin{array}{c} Y \xrightarrow{f} X \xrightarrow{p} 1 + 1 \end{array}$$

$$g^\Box(p) = \begin{array}{c} Y \xrightarrow{g} X + 1 \xrightarrow{[p,q]} 1 + 1 \end{array}$$

The idea is that $f^\star(p)$ is true on an input to $f$ iff $p$ is true on its output. Similarly, $g^\Box(p)$ is true on an input to $g$ iff either $g$ does not terminate, or $g$ terminates and $p$ is true on its output (i.e. if $g$ terminates then $p$ is true on the output). The ‘box’ notation $g^\Box$ suggests this modal reading, see also the De Morgan dual $g^\hat{\Box}$ in point (4) below.

The next result collects some basic observations. The proofs are easy and left to the reader.

**Exercise 1.** The two substitution (or modal) operations $(-)^\star$ and $(-)^\Box$ for total and partial maps $g: Y \rightarrow X$ we write:

1. $\text{id}^\star(p) = p$ and $(f_2 \circ f_1)^\star(p) = f_1^\star(f_2^\star(p))$;
2. $\text{id}^\Box(p) = p$ and $(g_2 \circ g_1)^\Box(p) = g_1^\Box(g_2^\Box(p))$;
3. $f^\Box(p) = f^\star(p)$;
4. $g^\Box(p) = (p^\perp \circ g)^\perp = g^\Box(p^\perp)^\perp$, if we define $g^\Box(q) = q \circ f$;
5. $f^\star(p^\perp) = f^\star(p)^\perp$ and $f^\star(1) = 1$, and thus also $f^\star(0) = 0$;
6. $g^\Box(1) = 1$. □
3 Effectuses

In this section we give the definition of effectus, we provide several leading examples, and we prove some basic categorical results about effectuses. The definition below is slightly simpler than the original one in [Jac15a]. We shall show that the requirements used below are equivalent to the original ones, see Lemma 3 (1). There is no clear intuition behind the nature of the axioms below: they ensure that coproducts are well-behaved and provide the logical structure that will be described in Section 4.

**Definition 2.** An effectus is a category $\mathcal{B}$ with finite coproducts $(+, 0)$ and a final object $1$, such that both:

1. diagrams of the following form are pullbacks in $\mathcal{B}$:

   $\begin{array}{ccc}
   X + Y & \xrightarrow{\text{id} + 1} & X + 1 \\
   \downarrow^{\text{id} + 1} & & \downarrow^{\text{id} + 1} \\
   1 + Y & \xrightarrow{1 + 1} & 1 + 1
   \end{array} \ 
   \begin{array}{ccc}
   X & \xrightarrow{1} & 1 \\
   \downarrow^{\kappa_1} & & \downarrow^{\kappa_1} \\
   X + Y & \xrightarrow{\text{t} + \text{id}} & 1 + 1
   \end{array}$ (7)

2. the following two maps are jointly monic in $\mathcal{B}$:

   $\begin{array}{ccc}
   (1 + 1) + 1 & \xrightarrow{\mathcal{W} = ([\kappa_1, \kappa_2], \kappa_2)} & 1 + 1 \\
   \mathcal{V} = ([\kappa_2, \kappa_1], \kappa_2)
   \end{array}$ (8)

Joint monicity means that if $f, g$ satisfy $\mathcal{V} \cdot f = \mathcal{V} \cdot g$ and $\mathcal{W} \cdot f = \mathcal{W} \cdot g$, then $f = g$.

A map of the form $X \rightarrow 1 + 1$, or equivalently $X \rightarrow 1$, will be called a predicate on $X \in \mathcal{B}$. We write $\text{Pred}(X)$ for the collection of predicates on $X$. Predicates of the special form $1 \rightarrow 1 + 1$ are also called scalars.

A state is a total map of the form $1 \rightarrow X$, and a substate is a partial map $1 \rightarrow X$. We write $\text{Stat}(X)$ and $\text{SStat}(X)$ for the collections of such states and substates of $X$.

The diagram on the left in (7) provides a form of partial pairing for compatible partial maps, see Lemma 6 below; the diagram on the right ensures that coprojections are disjoint, see Proposition 4 below. The joint monicity requirement (8) is equivalent to joint monicity of partial projections, see Lemma 5 below. It captures a form of cancellation. The symbols $\mathcal{W}$ and $\mathcal{V}$ should suggest what these two maps do, as functions from a 3-element set to a 2-element set, read downwards.

The diagram on the right in (7) only mentions the first coprojection $\kappa_1: X \rightarrow X + Y$. A corresponding pullback exists for the second coprojection $\kappa_2: Y \rightarrow X + Y$ since it is related to $\kappa_1$ via the swap isomorphism $X + Y \cong Y + X$. Similarly, the results below only mention the first (co)projection, but hold for the second one as well.

An important property of effectuses is that predicates in an effectus have the logical structure of an effect module. This will be elaborated in the next few sections.
The next few results describe some categorical consequences of the structure in an effectus. They focus especially on pullbacks. We recall the following general result, known as the Pullback Lemma: consider a situation with two commuting squares $A$ and $B$,

\[
\begin{array}{c}
\text{A} \\
\downarrow \\
\text{B}
\end{array}
\]

Then: if $B$ is a pullback, then $A$ is a pullback if and only if the outer rectangle is a pullback.

**Lemma 3.** Let $B$ be a category with finite coproducts and a final object, in which the squares (7) in the definition of an effectus are pullbacks.

1. Rectangles of the following forms are also pullbacks in $B$.

\[
\begin{array}{cc}
X + Y & \xrightarrow{id+g} X + B \\
\downarrow f+id & \downarrow f+id \\
A + Y & \xrightarrow{id+g} A + B
\end{array}
\]

\[
\begin{array}{cc}
X & \xrightarrow{f} A \\
\downarrow \kappa_1 & \downarrow \kappa_1 \\
X + Y & \xrightarrow{f+g} A + B
\end{array}
\] (9)

2. In the category $\text{Par}(B)$ of partial maps the following squares are pullbacks, where $f: X \to A$ and $g: Y \to B$ are total maps.

\[
\begin{array}{cc}
X + Y & \xrightarrow{id+g} X + B \\
\downarrow (f)+id & \downarrow (f)+id \\
A + Y & \xrightarrow{id+g} A + B
\end{array}
\]

\[
\begin{array}{cc}
X + Y & \xrightarrow{(f)+id} A + Y \\
\downarrow \triangleright_1 & \downarrow \triangleright_1 \\
X & \xrightarrow{f+y} A
\end{array}
\] (10)

Recall that we write $\triangleright_i$ for the partial projection maps from (2).

The rectangles in (7) used to define effectuses are instances of the more general formulations (9) provided in this lemma.

**Proof** We start with the diagram on the left in (7). Suppose we have $h: C \to A + Y$ and $k: C \to X + B$ satisfying $(\text{id} + g) \cdot h = (f + \text{id}) \cdot k$. Consider the following diagram.

```
``
The outer diagram commutes since:

\[(\text{id} + !) \cdot (\text{id} + g) \cdot (\text{id} + \text{id}) \cdot h = (\text{id} + !) \cdot (\text{id} + g) \cdot (\text{id} + !) \cdot k\]

The outer rectangle, consisting of all four squares, is a pullback since it is an instance of the pullback on the left in (7). Hence there is a unique map \(\ell: C \to A + B\) with \((! + \text{id}) \cdot \ell = (\text{id} + !) \cdot h\) and \((\text{id} + !) \cdot \ell = (\text{id} + !) \cdot k\). We obtain \((f + \text{id}) \cdot \ell = (f + \text{id}) \cdot k\) by uniqueness of mediating maps, since the lower two squares form a pullback, again by (7). Similarly \((\text{id} + g) \cdot \ell = k\) holds because the two square on the right together form a pullback.

In a similar way we show that the diagram on the right in (9) is a pullback: let \(h: C \to X + Y\) and \(k: C \to A\) satisfy \((f + g) \cdot h = \kappa_1 \cdot k\). Then we have a situation:

Since the two squares together and the square on the right form a pullback, as on the right in (7). Hence we obtain a unique map \(\ell: C \to X\) with \(\kappa_1 \cdot \ell = h\). We get \(f \cdot \ell = k\) by uniqueness of mediating maps, using the pullback on the right.

We turn to the diagram (10) in the category \(\text{Par}(\mathcal{B})\). We write \(\alpha\) for the standard associativity isomorphism \(U + (V + W) \to (U + V) + W\) in \(\mathcal{B}\), given explicitly by \(\alpha = [\kappa_1 \cdot \kappa_2, \kappa_2 + \text{id}]\) and \(\alpha^{-1} = [\text{id} + \kappa_1, \kappa_2 \cdot \kappa_2]\). For total maps \(f: X \to A\) and \(g: Y \to B\) we have commuting diagrams in \(\mathcal{B}\):

As a result, the rectangle on the left below is a pullback in \(\mathcal{B}\), since it is related via the associativity isomorphisms \(\alpha\) to the rectangle on the right, which is a pullback by (9).

The fact that the above rectangle on the left is a pullback in \(\mathcal{B}\) implies that the rectangle on the left in (10) is a pullback in \(\text{Par}(\mathcal{B})\).
We use basically the same trick for the diagram on the right in (10). Let $h: C \to X$ and $k: C \to A + Y$ be (partial) maps with $(g_1 \circ h ) = \triangleright_1 \circ k$. When translated to the category $B$ the latter equation becomes $(g + \text{id}) \cdot h = [\triangleright_1, \kappa_2] \cdot k$.

The rectangle is a pullback, as instance of the square on the left in (9). Then $\ell' = \alpha \cdot \ell: \ C \to X + Y$ is the required mediating map in $\text{Par}(B)$. □

**Proposition 4.** In an effectus $\quad \kappa_i: X_i \to X_1 + X_2$ are monic and disjoint, and the initial object $0$ is strict.

An easy consequence of this result is that for an effectus $\ B \to \text{Par}(B)$ from $B$ to the Kleisli category $\text{Par}(B)$ of the lift monad $(\cdot + 1)$ is faithful. This is the case because $\langle f \rangle = \kappa_1 \cdot f$, and $\kappa_1$ is monic.

**Proof** The first coprojection $\kappa_1: A \to A + B$ is monic since, using the pullback on the right in (9), we get a diagram as on the left below. Its two rectangles are pullbacks, and so their combination too.

The above diagram on the right shows that the intersection (pullback) of $\kappa_1, \kappa_2$ is the initial object $0$: the small rectangle is a pullback since it is an instance of the pullback on the left in (9). Hence the outer rectangle is a pullback via the isomorphisms in the diagram.

Strictness of the initial object $0$ means that each map $f: X \to 0$ is an isomorphism. For such a map, we have to prove that the composite $X \to 0 \to X$
is the identity. Consider the diagram:

![Diagram](image)

The rectangle is a pullback, as on the right in (9). By initiality of 0, we have \( \kappa_1 = \kappa_2 : 0 \to 0 + 0 \), so that the outer diagram commutes. Then we get the dashed map, as indicated, which must be \( f : X \to 0 \). But now we are done:

\[
! \cdot f = [!, \text{id}] \cdot \kappa_1 \cdot f = [!, \text{id}] \cdot \kappa_2 = \text{id}.
\]

The next result shows that the joint monicity requirement in the definition of an effectus has many equivalent formulations.

**Lemma 5.** Let \( B \) be a category with finite coproducts and a final object, in which the squares in (7) are pullbacks. Then the following statements are equivalent.

1. The category \( B \) is an effectus, that is, the two maps \( \cdots \leftarrow (1 + 1) + 1 \leftarrow 1 + 1 \) in (8) are jointly monic in \( B \).

2. The two partial projection maps \( \dashv_1, \dashv_2 : 1 + 1 \to 1 \) are jointly monic in \( \text{Par}(B) \).

3. For each object \( X \), the two partial projection maps \( \dashv_1, \dashv_2 : X + X \to X \) are jointly monic in \( \text{Par}(B) \).

4. For each pair of objects \( X_1, X_2 \), the two projections \( \dashv_i : X_1 + X_2 \to X_i \) are jointly monic in \( \text{Par}(B) \).

5. For each \( n \)-tuple of objects \( X_1, \ldots, X_n \), with \( n \geq 1 \), the \( n \) projections \( \dashv_i : X_1 + \cdots + X_n \to X_i \) are jointly monic in \( \text{Par}(B) \).

**Proof** The implication \( 1 \Rightarrow 2 \) is a simple reformulation, using that \( \mathcal{W} = [\dashv_1, \kappa_2] \) and \( \mathcal{W} = [\dashv_2, \kappa_2] \).

For the implication \( 2 \Rightarrow 3 \), let \( f, g : Y \to X + X \) satisfy \( \dashv_i \circ f = \dashv_i \circ g : Y \to X \) for \( i = 1, 2 \). Consider the following diagram in \( \text{Par}(B) \).

![Diagram](image)
All rectangles are pullbacks in $\text{Par}(B)$ by Lemma 3 (2). The definitions $f' = (\downarrow \circ i + \downarrow) \circ f$ and $g' = (\downarrow \circ i + \downarrow) \circ g$ yield equal maps $Y \to 1 + 1$ by point (2), since $\triangleright_i \circ f' = \downarrow \circ \triangleright_i \circ f = \downarrow \circ \triangleright_i \circ g = \triangleright_i \circ g'$ by Lemma 3 (3). But then:

\[
\begin{cases}
(\downarrow \circ i + \downarrow) \circ f = (\downarrow \circ i + \downarrow) \circ g & \text{by the upper right pullback} \\
(\downarrow \circ i + \downarrow) \circ f = (\downarrow \circ i + \downarrow) \circ g & \text{by the lower left pullback.}
\end{cases}
\]

Hence $f = g$ by the upper left pullback.

For the implication (3) $\Rightarrow$ (4) let $f, g: Z \to X_1 + X_2$ satisfy $\triangleright_i \circ f = \triangleright_i \circ g$. The trick is to consider $(\kappa_1 + \kappa_2) \circ f, (\kappa_1 + \kappa_2) \circ g: Z \to (X_1 + X_2) + (X_1 + X_2)$ instead. They satisfy, by naturality (3) of the partial projection $\downarrow$:

\[
\downarrow \circ (\kappa_1 + \kappa_2) \circ f = \kappa_1 \circ \downarrow \circ f = \downarrow \circ (\kappa_1 + \kappa_2) \circ g.
\]

We obtain $(\kappa_1 + \kappa_2) \circ f = (\kappa_1 + \kappa_2) \circ g$ from point (3). But then we are done:

\[
f = \nabla \circ (\kappa_1 + \kappa_2) \circ f = \nabla \circ (\kappa_1 + \kappa_2) \circ g = g.
\]

The implication (4) $\Rightarrow$ (5) is obtained via induction. Finally, the implication (5) $\Rightarrow$ (1) is trivial. □

In an effectus these partial projections $\triangleright_i$ are projections for a ‘partial pairing’ operation $\langle\langle-,-\rangle\rangle$ that will be described next. In fact one can understand the pullback on the left in (7) in the definition of effectus as precisely providing such a pairing for maps which are suitably orthogonal to each other. Later on, in Section 7 we can make this requirement more precise in terms of kernels that are each other’s orthosupplement.

**Lemma 6.** In an effectus, each pair of partial maps $f: Z \to X$ and $g: Z \to Y$ with $1 \circ f = (1 \circ g)^\perp$, determines a unique total map $\langle\langle f,g \rangle\rangle: Z \to X + Y$ with:

\[
\triangleright_1 \circ \langle\langle f,g \rangle\rangle = f \quad \text{and} \quad \triangleright_2 \circ \langle\langle f,g \rangle\rangle = g.
\]

Uniqueness gives equations:

\[
\langle\langle f, g \rangle\rangle \circ h = \langle\langle f \circ h, g \circ h \rangle\rangle \quad \text{and} \quad \langle\langle \triangleright_1 \circ k, \triangleright_2 \circ k \rangle\rangle = k
\]

for each $h: W \to Z$ and $k: Z \to X + Y$. Also, for total maps $h, k$ we have:

\[
(h \circ k) \circ \langle\langle f, g \rangle\rangle = \langle\langle h \circ f, k \circ g \rangle\rangle.
\]

Note that the partial composite $1 \circ f$ can be unfolded to:

\[
1 \circ f = [1, \kappa_2] \circ f = [\kappa_1 \circ i, \kappa_2] \circ f = (\downarrow + i \circ i) \circ f.
\]

This map $1 \circ (-)$ plays an important role, see also Lemma 7 below. It is the orthosupplement of the kernel operation, see Section 7 below.
Proof The assumption $1 \circ f = (1 \circ g)^\perp$ says that the outer diagram commutes in:

The (inner) rectangle is a pullback by (7), yielding the pairing $\langle\langle f, g \rangle\rangle$ as the unique total map with:

$$1 \langle\langle f, g \rangle\rangle = (\text{id} + !) \cdot \langle\langle f, g \rangle\rangle = f$$

Clearly, $\langle f, g \rangle$ is the unique (mediating) total map $Z \rightarrow X + Y$ satisfying these equations. The equations (11) follow directly from this uniqueness property. Finally, the equation (12) is obtained from uniqueness of mediating maps in pullback like the one above:

This partial pairing $\langle\langle -,- \rangle\rangle$ will be generalised later, see Lemma 43, and the end of Discussion 54. It plays an important role in the sequel, for instance in decomposition, see Lemma 83 (7), (8).

Lemma 7. Let $f : X \rightarrow Y$ be a partial map in an effectus. Then:

$$1 \circ f = 0 \iff f = 0 \quad \text{and} \quad 1 \circ f = 1 \iff f \text{ is total.}$$

Proof The implication ($\Leftarrow$) on the left is trivial, and for the implication ($\Leftarrow$) on the right, let $f$ be total, say $f = \kappa_1 \cdot g$ for $g : X \rightarrow Y$. Then:

$$1 \circ f = [\kappa_1 \cdot !, \kappa_2] \cdot \kappa_1 \cdot g = \kappa_1 \cdot ! \cdot g = \kappa_1 \cdot ! = 1.$$
Both rectangles are pullbacks, as instances of the diagram on the right in (9)—— once for the second coprojection \( \kappa_2 \).

### 3.1 Examples of effectuses

Below we describe the running examples of effectuses that will be used throughout this text. We will not go into all details and proofs for these examples, but just sketch the essentials.

**Example 8.** The category \( \textbf{Sets} \) of sets and functions has coproducts via disjoint union: \( X + Y = \{ \langle x, 1 \rangle \mid x \in X \} \cup \{ \langle y, 2 \rangle \mid y \in Y \} \). The numbers 1 and 2 in this set are used as distinct labels, to make sure that we have a disjoint union. The coprojections \( \kappa_1 : X \to X + Y \) and \( \kappa_2 : Y \to X + Y \) are given by \( \kappa_1 x = \langle x, 1 \rangle \) and \( \kappa_2 y = \langle x, 2 \rangle \). For functions \( f : X \to Z \) and \( g : Y \to Z \) there is a cotuple map \([f, g] : X + Y \to Z\) given by \([f, g](\kappa_1 x) = f(x)\) and \([f, g](\kappa_2 y) = g(y)\).

The empty set is the initial object \( 0 \in \textbf{Sets} \), precisely because for any set \( X \), there is precisely one function \( 0 \to X \), namely the empty function. Any singleton set is final in \( \textbf{Sets} \). We typically write \( 1 = \{ * \} \) for (a choice of) the final object.

Predicates on a set \( X \) are ‘characteristic’ functions \( X \to 1 + 1 = 2 \cong \{ 0, 1 \} \), which correspond to subsets of \( X \). There is thus an isomorphism \( \text{Pred}(X) \cong \mathcal{P}(X) \), where \( \mathcal{P}(X) \) is the powerset. In particular, the set \( \text{Pred}(1) \) of scalars is the two-element set of Booleans \( \mathcal{P}(1) \cong 2 \). A state is a function \( 1 \to X \), and thus corresponds to an element of the set \( X \).

Kleisli maps \( f : X \to Y + 1 \) of the lift monad correspond to partial functions from \( X \) to \( Y \), written according to our convention as maps \( f : X \rightsquigarrow Y \). We say that \( f \) is undefined at \( x \in X \) if \( f(x) = \ast \in 1 \) — or more formally, if \( f(x) = \kappa_2 * \).

It is easy to see that Kleisli composition corresponds to the usual composition of partial functions.

Later, in Section 13, we shall see that the category \( \textbf{Sets} \) is an instance of an extensive category, and that such extensive categories are instances of effectuses, that can be characterised in a certain way. In fact, every topos is an extensive category, and thereby an effectus.

**Example 9.** In order to capture (discrete) probabilistic models we use two monads on \( \textbf{Sets} \), namely the \textit{distribution} monad \( \mathcal{D} \) and the \textit{subdistribution} monad \( \mathcal{D}_{\leq 1} \).

For an arbitrary set \( X \) we write \( \mathcal{D}(X) \) for the set of formal convex combinations of elements in \( X \). There are two equivalent ways of describing such convex combinations.

- We can use expressions \( r_1 | x_1 \rangle + \cdots + r_n | x_n \rangle \) where \( x_i \in X \) and \( r_i \in [0, 1] \) satisfy \( \sum_i r_i = 1 \). The ‘ket’ notation \( | x \rangle \) is syntactic sugar, used to distinguish an element \( x \in X \) from its occurrence in such sums. The expression \( \sum_i r_i | x_i \rangle = r_1 | x_1 \rangle + \cdots + r_n | x_n \rangle \in \mathcal{D}(X) \) may be understood as: the probability of element \( x_i \) is \( r_i \).

- We also use functions \( \omega : X \to [0, 1] \) with finite support \( \text{supp}(\omega) = \{ x \in X \mid \omega(x) \neq 0 \} \), and with property \( \sum_x \omega(x) = 1 \). We can write this \( \omega \) as formal convex sum \( \sum_x \omega(x) | x \rangle \). Conversely, each formal convex sum \( \sum_i r_i | x_i \rangle \) yields a function \( X \to [0, 1] \) with \( x_i \mapsto r_i \), and \( x \mapsto 0 \) for all other \( x \in X \).
We shall freely switch between these two ways of describing formal convex combinations. Sometimes we use the phrase ‘discrete probability distribution’ for such a combination.

The subdistribution monad $\mathcal{D}_{\leq 1}$ has ‘subconvex’ combinations $\sum_i r_i |x_i\rangle \in \mathcal{D}_{\leq 1}(X)$, where $r_i \in [0, 1]$ satisfy $\sum_i r_i \leq 1$.

We write $\mathcal{K}(\mathcal{D})$ for the Kleisli category of the monad $\mathcal{D}$. Objects of this category are sets, and morphisms $X \to Y$ are functions $X \to \mathcal{D}(Y)$. If $X, Y$ are finite sets, such maps $X \to \mathcal{D}(Y)$ are precisely the stochastic matrices.

Such functions can be understood as Markov chains, describing probabilistic computations. For functions $f: X \to \mathcal{D}(Y)$ and $g: Y \to \mathcal{D}(Z)$ we define $g \cdot f: X \to \mathcal{D}(Z)$ essentially via matrix multiplication:

$$(g \cdot f)(x) = \sum_z \left( \sum_y g(y)(z) \cdot f(x)(y) \right) |z\rangle.$$

Notice that we have mixed the above two descriptions: we have used the ket-notation for the distribution $(g \cdot f)(x) \in \mathcal{D}(Z)$, but we have used the distributions $f(x) \in \mathcal{D}(Y)$ and $g(y) \in \mathcal{D}(Z)$ as functions with finite support, namely as functions $f(x): Y \to [0, 1]$ and $g(y): Z \to [0, 1]$. This allows us to multiply (and add) the probabilities $g(y)(z) \in [0, 1]$ and $f(x)(y) \in [0, 1]$. It can be checked that the above definition yields a distribution again, and that the operation $\cdot$ is associative.

We also need an identity map $X \to X$ in $\mathcal{K}(\mathcal{D})$. It is the function $X \to \mathcal{D}(X)$ that sends $x \in X$ to the ‘Dirac’ distribution $|x\rangle \in \mathcal{D}(X)$. As function $X \to [0, 1]$ this Dirac distribution sends $x$ to $1$, and any other element $x' \neq x$ to $0$. In this way $\mathcal{K}(\mathcal{D})$ becomes a category. The empty set $\emptyset$ is initial in $\mathcal{K}(\mathcal{D})$, and the singleton set $\{1\}$ is final, because $\mathcal{D}(1) \cong 1$.

The Kleisli category $\mathcal{K}(\mathcal{D})$ inherits finite coproducts from the underlying category $\text{Sets}$. On objects it is the disjoint union $X + Y$, with coprojection $X \to \mathcal{D}(X + Y)$ given by $x \mapsto 1 |\kappa_i x\rangle$. The cotuple is as in $\text{Sets}$. We leave it to the reader to check that the requirements of an effectus hold in $\mathcal{K}(\mathcal{D})$. Later, in Example 33, we shall see that this example effectus $\mathcal{K}(\mathcal{D})$ is an instance of a more general construction.

Predicates on a set $X$ are functions $X \to \mathcal{D}(1 + 1) = \mathcal{D}(2) \cong [0, 1]$. Hence predicates in this effectus are fuzzy: $\text{Pred}(X) \cong [0, 1]^X$. The scalars are the probabilities from $[0, 1]$. A state $1 \to X$ corresponds to a distribution $\omega \in \mathcal{D}(X)$.

Interestingly, the category $\text{Par}(\mathcal{K}(\mathcal{D}))$ of partial maps in the Kleisli category of the distribution monad $\mathcal{D}$ is the Kleisli category $\mathcal{K}(\mathcal{D}_{\leq 1})$ of the subdistribution monad. The reason is that subdistribution in $\mathcal{D}_{\leq 1}(Y)$ can be identified with a distribution in $\mathcal{D}(Y + 1)$. Indeed, given a subdistribution $\sum_i r_i |y_i\rangle$, we take $r = 1 - \sum_i r_i$ and get a proper distribution $\sum_i r_i |y_i\rangle + r |\ast\rangle$, where $1 = \{\ast\}$. Thus we often identify a partial map $X \to Y$ in $\mathcal{K}(\mathcal{D})$ with a function $X \to \mathcal{D}_{\leq 1}(Y)$.

We have used the distribution monad $\mathcal{D}$ on $\text{Sets}$, for discrete probability. For continuous probability one can use the Giry monad $\mathcal{G}$ on the category of measurable spaces, see [Jac15a, Jac13] for details. The Kleisli category $\mathcal{K}(\mathcal{G})$ of this monad $\mathcal{G}$ is also an effectus.

The next example of an effectus involves order unit groups. It does not capture a form of computation, like $\text{Sets}$ or $\mathcal{K}(\mathcal{D})$. Still this example is interesting because it shares some basic structure with the quantum model given by von
Neumann algebras — see the subsequent Example 11 — but it is mathematically much more elementary. Hence we use it as an intermediate stepping stone towards von Neumann algebras.

**Example 10.** We write \( \text{Ab} \) for the category of Abelian groups, with group homomorphisms between them. An Abelian group \( G \) is called *ordered* if there is a partial order \( \leq \) on \( G \) that satisfies: \( x \leq x' \) implies \( x + y \leq x' + y \), for all \( x, x', y \in G \). One can then prove, for instance \( x \leq x' \) iff \( 0 \leq x' - x \) iff \( x + y = x' \) for some \( y \geq 0 \).

A homomorphism \( f : G \to H \) of ordered Abelian groups is a group homomorphism that is monotone: \( x \leq x' \Rightarrow f(x) \leq f(x') \). Such a group homomorphism is monotone iff it is positive. The latter means: \( x \geq 0 \Rightarrow f(x) \geq 0 \). We write \( \text{OAb} \) for the category of ordered Abelian groups with monotone/positive group homomorphisms between them. There is an obvious functor \( \text{OAb} \to \text{Ab} \) which forgets the order.

An ordered Abelian group \( G \) is called an *order unit group* if there is a positive unit \( 1 \in G \) such that for each \( x \in G \) there is an \( n \in \mathbb{N} \) with \( -n \cdot 1 \leq x \leq n \cdot 1 \). A homomorphism \( f : G \to H \) of order unit groups is a positive group homomorphisms which is ‘unital’, that is, it preserves the unit: \( f(1) = 1 \). We often write ‘\( \text{PU} \)’ for positive unital. We obtain another category \( \text{OUG} \) of order unit groups and \( \text{PU} \) group homomorphisms between them. We have a functor \( \text{OUG} \to \text{OAb} \) that forgets the unit.

It is easy to see that order unit groups are closed under finite products: the cartesian product, commonly written as direct sum \( G_1 \oplus G_2 \), is again an ordered Abelian group, with componentwise operations and order, and with the pair \( (1,1) \) as unit. The trivial singleton group \( \{0\} \) with \( 1 = 0 \) is final in the category \( \text{OUG} \). The group of integers \( \mathbb{Z} \) is initial in \( \text{OUG} \), since for each order unit group \( G \) there is precisely one map \( f : \mathbb{Z} \to G \), namely \( f(k) = k \cdot 1 \).

We now claim that the *opposite* \( \text{OUG}^{\text{op}} \) of the category of order unit groups is an effectus. The direct sums \( (\oplus, \{0\}) \) form coproducts in \( \text{OUG}^{\text{op}} \), and the integers \( \mathbb{Z} \) form the final object. The effectus requirements in Definition 2 can be verified by hand, but they also follow from a general construction in Example 6. This category is in opposite form since it incorporates Heisenberg’s view where computations are predicate transformers, acting in opposite direction, transforming ‘postcondition’ predicates on the codomain of the computation to ‘precondition’ predicates on the domain.

We elaborate on predicates and on partial maps since they can be described in alternative ways that also apply in other settings.

1. A predicate on an order unit group \( G \) is formally a map \( p : G \to 1 + 1 \) in the effectus \( \text{OUG}^{\text{op}} \). In the category \( \text{OUG} \) this amounts to a map \( p : \mathbb{Z} \oplus \mathbb{Z} \to G \). We claim that such predicates correspond to *effects*, that is, to elements in the unit interval \([0,1]_G = \{x \in G \mid 0 \leq x \leq 1\} \). Indeed, the element \( p(1,0) \) is positive, and below the unit since \( p(1,0) + p(0,1) = p(1,1) = 1 \), where \( p(0,1) \) is positive too. In the other direction, an effect \( e \in [0,1]_G \) gives rise to a map \( p_e : \mathbb{Z} \oplus \mathbb{Z} \to G \) by \( p_e(k,m) = k \cdot e + m \cdot e^\perp \), where \( e^\perp = 1 - e \).

2. A partial map \( f : G \to H \) in \( \text{OUG}^{\text{op}} \) is a map \( f : G \to H + 1 \), which corresponds to a positive unital group homomorphism \( f : H \oplus \mathbb{Z} \to G \). We next claim that these homomorphisms \( f \) correspond to *subunital* positive
group homomorphisms $g: H \to G$, where subunital means that $g(1) \leq 1$.

This correspondence works as follows: given $f$, take $f(y) = f(y, 0)$; and

given $g$, take $g(y, k) = g(y) + k \cdot g(1)^\perp$, where $g(1)^\perp = 1 - g(1)$. We use

the abbreviation ‘PsU’ for ‘positive subunital’.

Summarising, there are bijective correspondences for predicates and partial maps:

\[
\begin{align*}
G & \to 1 + 1 & \text{in } \text{OUG}^{\text{op}} \\
\mathbb{Z} \oplus \mathbb{Z} & \to G & \text{in } \text{OUG} \\
G & \to H & \text{in } \text{OUG}^{\text{op}} \\
G & \to H + 1 & \text{in } \text{OUG}^{\text{op}} \\
H \oplus \mathbb{Z} & \to G & \text{in } \text{OUG} \\
H & \text{PsU} & \to G
\end{align*}
\]

(13)

A state of an order unit group $G$ is a homomorphism $G \to \mathbb{Z}$.

Order unit groups provide a simple example of an effectus, in which part
of the structure of more complex models — like von Neumann algebras, see
below — already exists. Similar structures are order unit spaces. They are
ordered vector spaces over the reals, which are at the same time an order unit

The category OUS of order unit spaces has positive unital linear maps

as homomorphisms. It plays a prominent role in generalised probabilistic

theories, see e.g. [Will12]. One can prove that the category OUS$^{\text{op}}$

is an effectus too. The states of an order unit space $V$ are the homomorphisms $V \to \mathbb{R}$. They

carry much more interesting structure than the states of an order unit group.

For instance, this set of states $\text{Stat}(V)$ forms a convex compact Hausdorff space.

For completeness, we give the definition of von Neumann algebra (in point 3),
and the morphisms of $\text{vNA}$ (in point 7), but we refer the reader to [KR97,
Pau02, Sak71] for the intricate details.

1. The structure that invoked the study of von Neumann algebra is the set

$B(\mathcal{H})$ of bounded operators $T: \mathcal{H} \to \mathcal{H}$ on a Hilbert space $\mathcal{H}$. Recall

that an operator $T$ on $\mathcal{H}$ is bounded provided that:

$$
\|T\| = \inf\{b \in [0, \infty) \mid \forall x \in \mathcal{H}, \|T(x)\| \leq b \cdot \|x\|\} < \infty.
$$

Not only is $B(\mathcal{H})$ a complex vector space (with coordinatewise opera-
tions), but it also carries a multiplication (namely composition of operators),
an involution (viz. taking adjoint), and a unit (viz. the identity operator),
which interact appropriately, and $B(\mathcal{H})$ is therefore a unital *-algebra. The adjoint of an operator $S \in B(\mathcal{H})$ is the the unique operator

$S^* \in B(\mathcal{H})$ with $\langle S(x) \mid y \rangle = \langle x \mid S^*(y) \rangle$ for all $x, y \in \mathcal{H}$.

We assume that every *-algebra has a unit (‘is unital’). A *-subalgebra
of a *-algebra $\mathcal{A}$ is a complex linear subspace $\mathcal{I}$ of $\mathcal{A}$ which is closed
under multiplication and involution, \((-\cdot)^*,\) and contains the unit 1 of \(\mathcal{A}\).

An isomorphism between \(*\)-algebras is a linear bijection which preserves multiplication, involution and unit.

2. Let \(\mathcal{H}\) be a Hilbert space. A unital \(C^*\)-algebra of operators \(\mathcal{A}\) on \(\mathcal{H}\) is a \(*\)-subalgebra of \(B(\mathcal{H})\) which is norm closed, that is, if \((T_\alpha)\) is a net in \(\mathcal{A}\) and \(S \in \mathcal{A}\) such that \(\|T_\alpha - S\| \to 0\), then \(S \in \mathcal{A}\). A von Neumann algebra of operators \(\mathcal{A}\) on \(\mathcal{H}\) is a \(*\)-subalgebra \(B(\mathcal{H})\), which is weakly closed, that is, if \((T_\alpha)\) is a net in \(\mathcal{A}\) and \(S \in \mathcal{A}\) such that \(\langle x | (S - T_\alpha)(x) \rangle \to 0\) for all \(x \in \mathcal{H}\), then \(S \in \mathcal{A}\).

3. A unital \(C^*\)-algebra \(\mathcal{A}\) is a \(*\)-algebra which is isomorphic to a unital \(C^*\)-algebra of operators on some Hilbert space. (This is not the usual definition, but it is equivalent, see [KR97] Thm. 4.5.6.) A von Neumann algebra (also \(W^*\)-algebra) \(\mathcal{A}\) is a \(*\)-algebra which is isomorphic to a von Neumann algebra of operators on some Hilbert space (cf. text above Example 5.1.6 of [KR97]). Every von Neumann algebra is also a \(C^*\)-algebra, but the converse is false. In this text we will deal mainly with von Neumann algebras; we added the definition of \(C^*\)-algebra here only for comparison.

4. For every Hilbert space \(\mathcal{H}\), the \(*\)-algebra of operators \(B(\mathcal{H})\) is a von Neumann algebra. In particular, \(\{0\}, \mathbb{C}, M_2, M_3, \ldots\) are all von Neumann algebras, where \(M_n\) is the \(*\)-algebra of \(n \times n\) complex matrices. The cartesian product, \(\mathcal{A} \otimes \mathcal{B}\), of two von Neumann algebras is again a von Neumann algebra with coordinatewise operations. In particular, the complex plane \(\mathbb{C}^2 = \mathbb{C} \oplus \mathbb{C}\) is a von Neumann algebra. Given a von Neumann algebra \(\mathcal{A}\) the \(*\)-algebra \(M_n(\mathcal{A})\) of \(n \times n\)-matrices with entries drawn from \(\mathcal{A}\) is a von Neumann algebra.

5. An element \(a\) of a von Neumann algebra \(\mathcal{A}\) is called positive if \(a = b^* \cdot b\) for some \(b \in \mathcal{A}\), and self-adjoint if \(a^* = a\). The set of positive elements of \(\mathcal{A}\) is denoted by \(\mathcal{A}_+\), and the set of self-adjoint elements of \(\mathcal{A}\) is denoted by \(\mathcal{A}_{sa}\). All positive elements of \(\mathcal{A}\) are self-adjoint.

A von Neumann algebra \(\mathcal{A}\) is partially ordered by: \(a \leq b\) iff \(b - a\) is positive. The self-adjoint elements of a von Neumann algebra \(\mathcal{A}\) form an order unit space, \(\mathcal{A}_{sa}\). For every positive element \(a \in \mathcal{A}\) of a von Neumann algebra there is a unique positive element \(b \in \mathcal{A}\) with \(b^2 = a\), which is denoted by \(\sqrt{a}\).

In a von Neumann algebra every bounded directed net of self-adjoint elements \(D\) has a supremum \(\bigvee D\) in \(\mathcal{A}_{sa}\). Both ‘bounded’ and ‘directed’ are necessary, and the statement is false for \(C^*\)-algebras.

6. A linear map \(f: \mathcal{A} \to \mathcal{B}\) between von Neumann algebras is called

(a) unital if \(f(1) = 1\);
(b) subunital if \(f(1) \leq 1\);
(c) positive if \(f(a) \in \mathcal{B}_+\) for all \(a \in \mathcal{A}_+\);
(d) involutive if \(f(a^*) = f(a)^*\) for all \(a \in \mathcal{A}\);
(e) multiplicative if \(f(a \cdot b) = f(a) \cdot f(b)\) for all \(a, b \in \mathcal{A}\);
10. Let $C$ be a von Neumann algebra $A$.

7. We denote the category of von Neumann algebras and completely positive maps between von Neumann algebras.

In the category of effect algebras, see the correspondences (13). Also, the partial maps $\text{Pred}$ correspond to elements $a$ in $\mathcal{VNA}$.

8. A von Neumann algebra $\mathcal{A}$ is commutative if $a \cdot b = b \cdot a$ for all $a, b \in \mathcal{A}$.

We write $\text{CvNA} \hookrightarrow \mathcal{VNA}$ for the full subcategory of commutative von Neumann algebras. We recall that a positive unital map $\mathcal{A} \rightarrow \mathcal{B}$ is automatically completely positive when either $\mathcal{A}$ or $\mathcal{B}$ is commutative. These commutative von Neumann algebras capture probabilistic models, as special case of the general quantum models in $\mathcal{VNA}$. Also the category $\text{CvNA}^{\text{op}}$ is an effectus.

9. For quantum computation the most important von Neumann algebra is perhaps the $\ast$-algebra $M_2$ of $2 \times 2$ matrices, because it models a qubit; it is closely related to the Hilbert space $\mathbb{C}^2$, since $M_2 = \mathcal{B}(\mathbb{C}^2)$.

A pure state of a qubit is a vector in $\mathbb{C}^2$ of length 1, and can thus be written as $\alpha |0\rangle + \beta |1\rangle$, where $|0\rangle = (1,0)$ and $|1\rangle = (0,1)$ are base vectors in $\mathbb{C}^2$, and $\alpha, \beta \in \mathbb{C}$ are scalars with $|\alpha|^2 + |\beta|^2 = 1$. One says that any pure state of a qubit is in a superposition of $|0\rangle$ and $|1\rangle$. Two pure states $(\alpha_1, \beta_1)$ and $(\alpha_2, \beta_2)$ of a qubit are equivalent if $(z \cdot \alpha_1, z \cdot \beta_1) = (\alpha_2, \beta_2)$ for some $z \in \mathbb{C}$ (with $|z| = 1$).

A sharp predicate on a qubit is a linear subspace $C$ of $\mathbb{C}^2$ which in turn corresponds to a projection $P \in M_2$ (namely the orthogonal projection onto $C$). For instance, the predicate “the qubit is in state $|0\rangle$” is represented by the linear subspace $\{(\alpha, 0) : \alpha \in \mathbb{C}\}$ (the $x$-axis), and by the projection $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

10. Let $C$ be a linear subspace of $\mathbb{C}^2$ (i.e. a sharp predicate of a qubit) with orthogonal projection $P \in M_2$. Then $P^\perp = I - P$ is again a projection, onto the orthocomplement $C^\perp = \{y \in \mathbb{C}^2 : \forall x \in C, \langle x | y \rangle = 0\}$ of $C$.

Let $x \in \mathbb{C}^2$ be a pure state of a qubit (so $||x|| = 1$). Then:

- $x \in C$ $\iff$ predicate $C$ certainly holds in state $x$
- $x \in C^\perp$ $\iff$ predicate $C$ certainly does not hold in state $x$. 

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However, if $x \notin C \cup C^\perp$, then the matter whether $C$ holds in $x$ is undecided. The matter can, however, be pressed using measurement; it forces the qubit to the state $P(x) \cdot \|P(x)\|^{-1}$ with probability $\|P(x)\|^2$, and to the state $P^\perp(x) \cdot \|P^\perp(x)\|^{-1}$ with probability $\|P^\perp(x)\|^2$.

Thus, using measurement one finds that the predicate $C$ holds in state $x$ with probability $\|P(x)\|^2$. After measurement, the state of the qubit has become a (probabilistic) mixture of $P(x) \cdot \|P(x)\|^{-1} \in C$ and $P^\perp(x) \cdot \|P^\perp(x)\|^{-1} \in C^\perp$.

11. Any pure state $x$ of a qubit gives rise to a state $\omega_x : M_2 \to \mathbb{C}$ of $\mathbf{vNA}^{\text{op}}$ given by $\omega_x(A) = \langle Ax | x \rangle$ for all $A \in M_2$. The states of the form $\omega_x$ are precisely the extreme points of the convex set $\text{Stat}(M_2)$ of all states on $M_2$. The states which are not extreme correspond to probabilistic mixtures of pure states. For example, if one measures whether the pure state $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ is in the state $|0\rangle$, then the resulting state is $\frac{1}{2}(\omega_x |0\rangle + \omega_x |1\rangle)$, which is not extreme, and corresponds to a qubit which is either in state $|0\rangle$ or $|1\rangle$ with equal probability.

Any state of $M_2$ can be written as a convex combination of extreme states. But this is not true in general: a von Neumann algebra might have no extreme states at all.

12. Any sharp predicate $C$ of the qubit gives rise to a predicate on $M_2$ in $\mathbf{vNA}^{\text{op}}$. To see this recall that the predicates on $M_2$ in $\mathbf{vNA}^{\text{op}}$ correspond to the elements $P \in M_2$ with $0 \leq P \leq 1$. Now, the sharp predicate $C$ corresponds to the orthogonal projection $P_C$ onto $C$.

The predicates on $M_2$ of the form $P_C$ are precisely those predicates $P$ with: for every $Q \in M_2$ with $0 \leq Q \leq 1$, if $Q \leq P$ and $Q \leq P^\perp$, then $Q = 0$.

13. Let us think about the conjunction of two sharp predicates $C$ and $D$ on the qubit. Surely, it should be $C \cap D$. Is measuring whether $C \cap D$ holds simply a matter of first measuring $C$ and then measuring $D$?

Let $P$ be the orthogonal projection on $C$, and let $Q$ be the orthogonal projection onto $D$. First note that if $x \in C^\perp$ is a state of the qubit, then the probability to find that $C$ holds upon measurement, and then that $D$ holds up measurement is $\|Q(P(x))\|^2$, and the resulting state is $Q(P(x)) \cdot \|Q(P(x))\|^{-1}$. By the way, $\|Q(P(x))\|^2 = (P(Q(P(x))) | x) = \varphi_x(PQP)$, so first measuring $C$ and then $D$ corresponds in this formalism to the operator $PQP \in M_2$ (and not to $QP$ as one might have guessed).
Now, (by choosing $C := \{ (\alpha,0) : \alpha \in \mathbb{C} \}$ and $D := \{ (\alpha,\alpha) : \alpha \in \mathbb{C} \}$) one easily sees the following peculiar behaviour typical for quantum systems:

(a) Measurement disturbs the system: while $P(x) \cdot \|P(x)\|^{-1}$ is in $C$, the state $Q(P(x)) \cdot \|Q(P(x))\|^{-1}$ need not be in $C$;

(b) The order of measurement matters for the probability of the outcome: we might have $\|Q(P(x))\| \neq \|P(Q(x))\|$;

(c) The composition of two sharp measurements need not be sharp: $PQP$ need not be a projection.

Thus, measuring whether $C \cap D$ holds is not always simply a matter of measuring $C$ and then measuring $D$.

The operator $PQP$ is called the sequential product of $P$ and $Q$ and is denoted by $P & Q$. More generally, if $A$ is a von Neumann algebra, and $p,q \in [0,1]_A$, then the sequential product is given by $p \& q = \sqrt{pq} \sqrt{p}$, and it represents first measuring $p$ and then measuring $q$.

We will see in Theorem 110 that $p \& q$ can be defined in $\text{vNA}^{\text{op}}$ using only the language of category theory.

14. We have the following situation for a qubit:

$$\{\text{pure states}\} \subseteq \mathbb{C}^2 \quad \{\text{sharp predicates}\} \subseteq M_2.$$ 

This situation is typical: any Hilbert space $\mathcal{H}$, representing a purely quantum system, gives rise to the von Neumann algebra $B(\mathcal{H})$ of bounded operators on $\mathcal{H}$; the elements of norm 1 in $\mathcal{H}$ are the pure states of this quantum system, while the projections in $B(\mathcal{H})$ are the sharp predicates, corresponding to closed linear subspaces of $\mathcal{H}$. For instance, $M_3$ represents a qutrit, and $B(\ell^2)$ represents a quantum integer.

15. Let $\mathcal{A}$ be a von Neumann algebra. An important technical fact is that the states on $\mathcal{A}$ in $\text{vNA}^{\text{op}}$ (usually called normal states in the literature) are separating, that is, for all $a,b \in \mathcal{A}$ we have $a \leq b$ if and only if for every $\omega : \mathcal{A} \to \mathbb{C}$ in $\text{vNA}$ we have $\omega(a) \leq \omega(b)$.

This entails that the ultraweak topology—the least topology on $\mathcal{A}$ such that all states on $\mathcal{A}$ in $\text{vNA}^{\text{op}}$ are continuous—is Hausdorff. A net $(a_i)_{i \in D}$ of elements of $\mathcal{A}$ converges ultraweakly (i.e. with respect to the ultraweak topology) to an element $a \in \mathcal{A}$ iff $(\omega(a_i))_{i \in D}$ converges to $\omega(a)$ for every state $\omega$ on $\mathcal{A}$ in $\text{vNA}^{\text{op}}$.

It is not hard to see that a positive linear map $f : \mathcal{A} \to \mathcal{B}$ between von Neumann algebras is ultraweakly continuous if and only if $f$ is normal. In
this sense the ultraweak topology plays the same role in the theory of von Neumann algebras as the Scott topology plays in domain theory.

There is much more to be said about the ultraweak topology in particular, and von Neumann algebras in general, but since we draw most (new) results about von Neumann algebras from more specialized publications, [WW15, Ren14, Cho14], we refrain from diverging any farther into the theory of operator algebras.

4 The structure of partial maps and predicates

This section concentrates on the structure that exists on partial maps $X \rightarrow Y$ in an effectus. It turns out that they come equipped with a partial binary sum operation, written as $\oplus$, which has the zero map as unit and is commutative and associative (in an appropriately partial sense). Abstractly, the relevant observation is that the category $\text{Par}(B)$ of partial maps of an effectus $B$ is enriched over the category $\text{PCM}$ of partial commutative monoids. We first describe what such PCMs are.

The section continues with predicates, as special case. The main result of this section states that these predicates in an effectus form an effect module, in a functorial way.

**Definition 12.** A partial commutative monoid, abbreviated as PCM, is given by a set $M$ with a zero element $0 \in M$ and a partial binary operation $\oplus$ on $M$, satisfying the following requirements — where we call $x, y$ and orthogonal and write $x \perp y$ when $x \oplus y$ is defined.

1. $x \perp 0$ and $x \oplus 0 = x$, for each $x \in M$;
2. if $x \perp y$ then $y \perp x$ and $x \oplus y = y \oplus x$, for all $x, y \in M$;
3. if $x \perp y$ and $(x \oplus y) \perp z$, then $y \perp z$ and $x \perp (x \oplus z)$, and moreover $(x \oplus y) \oplus z = x \oplus (y \oplus z)$, for all $x, y, z \in M$.

A homomorphism $f: M \rightarrow N$ between two PCMs is a function that preserves $0$ and $\oplus$, in the sense that:

1. $f(0) = 0$;
2. if $x \perp y$ in $M$, then $f(x) \perp f(y)$ in $N$ and $f(x \oplus y) = f(x) \oplus f(y)$.

We write $\text{PCM}$ for the resulting category.

This notion of PCM shows up in the following way. Sums $\oplus$ are defined for parallel partial maps, via bounds. These definitions work well and satisfy appropriate properties via the categorical structure in effectuses, as identified in the previous section.

**Proposition 13** (From [Cho15]). Let $B$ be an effectus.

1. Each homset $\text{Par}(B)(X, Y)$ of partial maps $X \rightarrow Y$ is a PCM in the following way.
• $f \perp g$ iff there is a ‘bound’ $b: X \rightarrow Y + Y$ with $\triangleright_1 \circ b = f$ and $\triangleright_2 \circ b = g$. By Lemma 5 such a bound $b$, if it exists, is necessarily unique.

• if $f \perp g$ via bound $b$, then we define $f \odot g = \nabla \circ b = (\nabla + \text{id}) \circ b: X \rightarrow Y$.

2. This PCM-structure is preserved both by pre- and by post-composition in $\text{Par}(B)$.

The latter statement can be stated more abstractly as: $\text{Par}(B)$ is enriched over PCM. This works since the category PCM is symmetric monoidal, see [JM12a, Thm. 8]. Another way is to say that $\text{Par}(B)$ is a finitely partially additive category (abbreviated as ‘FinPAC’), using the terminology of [AMS0] (and [Cho15]). The original definition involves countable sums, whereas we only use finite sums, see Subsection 5.1.

More results about these partial homsets follow, in Proposition 41.

Proof For an arbitrary map $f: X \rightarrow Y$ we can take $b = \kappa_1 \ast f: X \rightarrow Y + Y$ as bound for $f \perp 0$, showing $f \odot 0 = f$. If $f \perp g$ via $b: X \rightarrow Y + Y$, then swapping the outcomes of $b$ yields a bound for $g \perp f$, proving $f \odot g = g \odot f$. The more difficult case is associativity.

So assume $f \perp g$ via bound $a$, and $(f \odot g) \perp h$ via bound $b$. We then have:

\[
\begin{align*}
\triangleright_1 \circ a &= f \\
\triangleright_2 \circ a &= g \\
\nabla \circ a &= f \odot g \\
\end{align*}
\quad \text{and} \quad
\begin{align*}
\triangleright_1 \circ b &= f \odot g \\
\triangleright_2 \circ b &= h \\
\nabla \circ b &= (f \odot g) \odot h.
\end{align*}
\]

Consider the diagram below, where the rectangle is a pullback in $\text{Par}(B)$, as instance of the diagram on the right in (10). Here we use that the codiagonal $\nabla = \langle \text{id}, \langle \text{id} \rangle \rangle = \langle \text{id}, \text{id} \rangle$ is total in $\text{Par}(B)$.

We now take $c' = (\triangleright_2 + \text{id}) \ast c: X \rightarrow Y + Y$. This $c'$ is a bound for $g$ and $h$, giving $g \perp h$ since:

\[
\begin{align*}
\triangleright_1 \ast c' &= \triangleright_1 \ast (\triangleright_2 + \text{id}) \ast c \\
&= \triangleright_2 \ast \triangleright_1 \ast c \\
&= \triangleright_2 \ast a \\
&= g
\end{align*}
\quad \text{and} \quad
\begin{align*}
\triangleright_2 \ast c' &= \triangleright_2 \ast (\nabla + \text{id}) \ast c \\
&= \triangleright_2 \ast b \\
&= h.
\end{align*}
\]
Next, the map $c'' = [id, \kappa_2] \triangleright c : X \to Y + Y$ is a bound for $f$ and $g \otimes h$ since:

$$
\triangleright_1 \triangleright c'' = \triangleright_1 \triangleright [id, \kappa_2] \triangleright c \quad \triangleright_2 \triangleright c'' = \triangleright_2 \triangleright [id, \kappa_2] \triangleright c
$$

$$
= \triangleright_1 \triangleright [0] \otimes c \\
= \triangleright_1 \triangleright [id, 0] \otimes c \\
= \triangleright_1 \triangleright [\triangleright_1 \triangleright c] \\
= f
$$

$$
= \triangleright_2 \triangleright [id] \otimes c \\
= \triangleright_2 \triangleright (\triangleright_2 \triangleright [id] \otimes c) \\
= \triangleright_2 \triangleright \triangleright c' \\
= \triangleright c

$$

We now obtain the associativity of $\otimes$:

$$
f \otimes (g \otimes h) = \triangleright \triangleright c' = \triangleright \triangleright [id, \kappa_2] \triangleright c = [\triangleright, id] \otimes c
$$

$$
= \triangleright \triangleright (\triangleright \triangleright id) \otimes c

= \triangleright \triangleright \triangleright b

= f \otimes (g \otimes h).
$$

Finally we check that $\otimes$ is preserved by pre- and post-composition. Let $f \bot g$ via bound $b : X \to Y + Y$.

- For $h : Z \to X$ it is easy to see that $b \circ h$ is a bound for $f \circ h$ and $g \circ h$, giving $(f \circ h) \otimes (g \circ h) = (f \otimes g) \circ h$.

- For $k : Y \to Z$ we claim that $(k + k) \circ b$ is a bound for $k \circ f$ and $k \circ g$. This follows from naturality (3) of the partial projections $\triangleright_i$. In this way we get $(k \circ f) \otimes (k \circ g) = k \circ (f \otimes g)$.

Here is a concrete illustration: the two partial projections $\triangleright_1, \triangleright_2 : X + X \to X$ in an effectus are orthogonal, via the identity map $X + X \to X + X$. This is obvious. Hence we have as sum $\triangleright_1 \triangleright \triangleright_2 = \triangleright$.

We briefly show what this PCM-structure is in our leading examples.

**Example 14.** In the category $\text{Par}(\text{Sets})$ of partial maps of the effectus $\text{Sets}$ two partial maps $f, g : X \to Y + 1$ are orthogonal if they have disjoint domains of definition. That is:

$$
f \bot g \iff \forall x \in X. f(x) = * \lor g(x) = *.
$$

In that case we have:

$$
(f \otimes g)(x) = \begin{cases} 
  f(x) & \text{if } g(x) = * \\
  g(x) & \text{if } f(x) = *
\end{cases}
$$

In the effectus $\mathcal{K}L(D)$ for discrete probability, orthogonality of two partial maps $f, g : X \to D_{\leq 1}(Y)$ is given by:

$$
f \bot g \iff \forall x \in X. \sum_{y \in Y} f(x)(y) + g(x)(y) \leq 1.
$$

In that case we have $(f \otimes g)(x)(y) = f(x)(y) + g(x)(y)$.

In the effectus $\text{OUG}^{\text{op}}$ of order unit groups two subunital positive maps $f, g : G \to H$ are orthogonal if $f(1) + g(1) \leq 1$. In that case we simply have $(f \otimes g)(x) = f(x) + g(x)$. The same description applies in the effectus $\text{vNA}^{\text{op}}$ of von Neumann algebras.
The PCM-structure on partial maps \( X \rightarrow Y \) from Proposition 13 exists in particular for predicates — which are maps of the form \( X \rightarrow 1 \). In the language of total maps, a bound for \( p, q : X \rightarrow 1 + 1 \) is a map \( b : X \rightarrow (1 + 1) + 1 \) with \( \mathcal{W} \cdot b = p \) and \( \mathcal{W} : (1 + 1) + 1 \rightarrow 1 + 1 \) are the jointly monic maps (8) in the definition of an effectus. The actual sum is then given by \( p \otimes q = (\mathcal{V} + \text{id}) \cdot b : X \rightarrow 1 + 1 \).

In this special case of predicates there is more than PCM-structure: predicates on \( X \) form an effect algebra and also an effect module. This will be shown next. We begin with the definition.

**Definition 15.** An effect algebra is a partial commutative monoid \((M, \otimes, 0)\) with an orthosupplement operation \((-)\perp : M \rightarrow M\) such that for each \( x \in M \):

1. \( x \perp \) is the unique element with \( x \otimes x \perp = 1 \), where \( 1 = 0 \perp \);
2. \( x \perp 1 \) implies \( x = 0 \).

A homomorphism of effect algebras \( f : M \rightarrow N \) is a homomorphism of PCMs satisfying \( f(1) = 1 \). We write \( \text{EA} \) for the resulting category.

The unit interval \([0, 1]\) is an example of an effect algebra, with \( r \perp s \) iff \( r + s \leq 1 \), and in that case \( r \otimes s = r + s \). The orthosupplement is given by \( r \perp = 1 - r \). Every Boolean algebra is also an effect algebra with \( x \perp y = x \land y = 0 \), and in that case \( x \otimes y = x \lor y \). The orthosupplement is negation \( x \perp = \neg x \). In particular, the set \( 2 = \{0, 1\} \) is an effect algebra. It is initial in the category \( \text{EA} \) of effect algebras.

In [JM12a] it is shown that the category \( \text{EA} \) of effect algebras is symmetric monoidal, where the unit for the tensor \( \otimes \) is the initial effect algebra \( 2 = \{0, 1\} \).

Each effect algebra carries an order, defined by \( x \leq y \) iff \( x \otimes z = y \) for some \( z \). We collect some basic results. Proofs can be found in the literature on effect algebras, see e.g. [DP00].

**Exercise 16.** In an effect algebra one has:

1. orthosupplement is an involution: \( x^{\perp \perp} = x \);
2. cancellation: \( x \otimes y = x \otimes y' \) implies \( y = y' \);
3. positivity: \( x \otimes y = 0 \) implies \( x = y = 0 \);
4. \( \leq \) is a partial order with \( 1 \) as top and \( 0 \) as bottom element;
5. \( x \leq y \) implies \( y \perp \leq x \perp \);
6. \( x \otimes y \) is defined iff \( x \perp y \) iff \( x \leq y \perp \) iff \( y \leq x \perp \);
7. \( x \leq y \) and \( y \perp z \) implies \( x \perp z \) and \( x \otimes z \leq y \otimes z \);
8. the downset \( \downarrow y = \{x \mid x \leq y\} \) is again an effect algebra, with \( y \) as top, orthosupplement \( x^{\perp \perp} = (y \perp \otimes x)^{\perp} \), and sum \( x \otimes y x' \) which is defined iff \( x \perp x' \) and \( x \otimes x' \leq y \), and in that case \( x \otimes_y x' = x \otimes x' \). \( \square \)
The uniqueness of the orthosupplement $x^\perp$ is an important property in the proof of this exercise. For instance it gives us $f(x^\perp) = f(x)^\perp$ for a map of effect algebras, since:

$$f(x^\perp) \otimes f(x) = f(x^\perp \otimes x) = f(1) = 1 = f(x)^\perp \otimes f(x).$$

A test in an effect algebra is a finite set of elements $x_1, \ldots, x_n$ which are pairwise orthogonal and satisfy $x_1 \perp \ldots \perp x_n = 1$. Each effect algebra thus gives rise to a functor (preshelf) $\mathbb{N} \to \text{Sets}$, sending a number $n$ to the collection of $n$-element tests. This structure is investigated in [SU15].

Proposition 17. Let $B$ be an effectus. Then:

1. $\text{Pred}(X)$ is an effect algebra, for each object $X \in B$;
2. sending $X \mapsto \text{Pred}(X)$ and $f \mapsto f^*$ gives a functor $\text{Pred}: B \to \text{EA}^{\text{op}}$.

Proof From Proposition 13 (1) we know that $\text{Pred}(X)$ is a PCM. We show that it is an effect algebra with the orthosupplement $p^\perp = [\kappa_2, \kappa_1] \cdot p$ from (5).

The three equations below show that $b = \kappa_1 \cdot p: X \to (1 + 1) + 1$ is a bound for $p$ and $p^\perp$, yielding $p \perp p^\perp = 1$.

\[
\begin{align*}
W \cdot b &= [\text{id}, \kappa_2] \cdot \kappa_1 \cdot p = p \\
W \cdot b &= [[\kappa_2, \kappa_1], \kappa_2] \cdot \kappa_1 \cdot p = [\kappa_2, \kappa_1] \cdot p = p^\perp \\
(\nabla + \text{id}) \cdot b &= (\nabla + \text{id}) \cdot \kappa_1 \cdot p = \kappa_1 \cdot \nabla \cdot p = \kappa_1 = 1.
\end{align*}
\]

Next we show that $p^\perp$ is the only predicate with $p \perp p^\perp = 1$. So assume also for $q: X \to 1 + 1$ we have $p \perp q = 1$, say via bound $b$. Then $W \cdot b = p$, $W \cdot b = q$, and $(\nabla + \text{id}) \cdot b = 1 = \kappa_1 \cdot !$. We use the pullback on the right in (9):

The fact that the bound $b$ is of the form $\kappa_1 \cdot c$ is enough to obtain $q = p^\perp$ in:

\[
\begin{align*}
p^\perp &= [\kappa_2, \kappa_1] \cdot W \cdot b = [\kappa_2, \kappa_1] \cdot [\text{id}, \kappa_2] \cdot \kappa_1 \cdot c \\
&= [[\kappa_2, \kappa_1], \kappa_2] \cdot \kappa_1 \cdot c = W' \cdot b = q.
\end{align*}
\]

Next, suppose $1 \perp p$; we must prove $p = 0: X \to 1 + 1$. We may assume a bound $b: X \to (1 + 1) + 1$ with $\triangleright_1 \ast b = 1 = \kappa_1 \cdot !$ and $\triangleright_2 \ast b = p$. We use the pullback in $B$ on the right in (9) to obtain the unique map $X \to 1$ as mediating.
As before, $\alpha$ is the associativity isomorphism. Hence $b = \alpha \cdot \kappa_1 \cdot ! = \kappa_1 \cdot \kappa_1 \cdot ! = \langle 1 \rangle$.

Then, as required:

$$p = \triangleright_2 \circ b = [0, \text{id}] \circ \langle 1 \rangle = [\kappa_2, \text{id}] \cdot \kappa_1 \cdot ! = \kappa_2 \cdot ! = 0.$$ 

For the second point of the proposition we only have to prove that substitution $f^*$ preserves $\odot$ and $1$. Preservation of $\odot$ follows from Proposition 13 (2), and preservation of $1$ holds by Exercise 1 (5). □

In Example 22 below we shall describe this effect algebra structure concretely for our running examples. But we first we show that the result can be strengthened further: predicates in an effectus are not only effect algebras, but also effect modules. The latter are effect algebras with a scalar multiplication, a bit like in vector spaces. In examples the scalars are often probabilities from $[0, 1]$ or Booleans from $\{0, 1\}$. But more abstractly they are characterised as effect monoids.

**Definition 18.** An effect monoid is an effect algebra which is at the same time a monoid, in a coherent way: it is given by an effect algebra $M$ with (total) associative multiplication operation $\cdot : M \times M \to M$ which preserves $0$, $\odot$ in each coordinate separately, like in the second part of Definition 12, and satisfies $1 \cdot x = x = x \cdot 1$.

An effect monoid is commutative if its multiplication is commutative.

More abstractly, using that the category $\mathbf{EA}$ of effect algebras is symmetric monoidal [JM12a], an effect monoid $(M, \cdot, 1)$ is a monoid in the monoidal category $\mathbf{EA}$ of effect algebras of the form:

$$\begin{array}{ccc}
2 & \longrightarrow & M \\
& \searrow & \downarrow \\
& & M \otimes M
\end{array}$$

satisfying the monoid requirements. Since the tensor unit $2 = \{0, 1\}$ is initial in the category $\mathbf{EA}$, the map on the left is uniquely determined, and the multiplication map on the right is the only structure.

The unit interval $[0, 1]$ is the prime example of an effect monoid, via its standard multiplication. It is clearly commutative. This structure can be extended pointwise to fuzzy predicates $[0, 1]^{X}$, for a set $X$, and to continuous predicates $C(Y, [0, 1])$ for a topological space $Y$.

The Booleans $\{0, 1\}$ also form an effect monoid, via conjunction (multiplication). More generally, each Boolean algebra is an effect monoid, with conjunction as multiplication.
Recall that any order unit group gives rise to an effect algebra by considering its unit interval. If the order unit group carries an associative bilinear positive multiplication for which the order unit is neutral, then its unit interval is an effect monoid. We should warn that this does not apply to C*-algebras: their self-adjoint elements form an order unit group, but their multiplication is not positive and does not restrict to self-adjoint elements — except when the C*-algebra is commutative. Indeed, we shall see in Section 9 that predicates in a ‘commutative’ effectus form commutative effect monoids, see esp. Lemma 17 [4].

Most obvious examples of an effect monoid are commutative. But here is an example of a non-commutative effect monoid. Consider R^5 with standard basis e_1, ..., e_5 ordered lexicographically such that e_1 ≫ e_2 ≫ ... ≫ e_5, where v ≫ w denotes v ≥ λ w for any λ > 0. There is a unique associative bilinear positive product * fixed by e_1 * e_j = e_j * e_1 = e_j, e_2 * e_2 = e_4, e_3 * e_2 = e_5 and in the remaining cases e_i * e_j = 0. For details, see [Wes13].

Scalars in an effectus are predicates 1 → 1 + 1 on the final object 1. They form an effect algebra by Proposition 17 (1). When we view scalars as partial maps 1 ⊸ 1, we directly see that they also carry a monoid structure, namely Kleisli/partial composition ◦. The latter preserves the sums ⊕, 0 in each coordinate by Proposition 13 [4]. Since the scalar 1 is the first coprojection κ_1: 1 ⊸ 1 + 1, it is the identity map on 1 in the category of partial maps, and thus the unit for ◦. We now summarise the situation.

Definition 19. For an arbitrary effectus B with final object 1, we write Pred(1) = Par(Ω(1,1)) for the effect monoid of scalars with partial composition ◦.

Once we know what scalars are, we can define associated modules having scalar multiplication.

Definition 20. Let M be an effect monoid.

1. An effect module over M is an effect algebra E with a scalar multiplication (action) M ⊗ E → E in EA. In such a module we have elements r · e ∈ E, for r ∈ M and e ∈ E, satisfying, apart from preservation of 0, ⊕ in each coordinate, 1 · e = e and r · (s · e) = (r · s) · e.

2. A map of effect modules is a map of effect algebras f: E → D satisfying f(r · e) = r · f(e) for each r ∈ M, e ∈ E. We write EMod_M for the category of effect modules over M, with effect module maps as morphisms between them.

We can also define effect modules more abstractly: an effect monoid M is a monoid M ∈ EA in the symmetric monoidal category of effect algebras. Hence the functor M ⊗ (−) is a monad on EA. The category EMod_M of effect modules over M is the resulting category of Eilenberg-Moore algebras of this monad, i.e. the category of actions of the monoid M. There is an obvious forgetful functor EMod_M → EA.

Effect modules over the Booleans {0, 1} are just effect algebras. Almost always in examples we encounter effect modules over the probabilities [0, 1]. That’s why we often simply write EMod for EMod_{[0,1]}. In the context of effect modules, the elements of the underlying effect monoid are often called scalars.

For each set X, the collection [0, 1]^X of ‘fuzzy predicates’ on X is an effect algebra, by Proposition 17 where p ⊥ q iff p(x) + q(x) ≤ 1 for all x ∈ X, and
in that case \( p \oplus q)(x) = p(x) + q(x) \). The orthosupplement \( p^\perp \) is defined by \( p^\perp(x) = 1 - p(x) = p(x)^\perp \). This effect algebra \([0,1]^X\) is an effect module over \([0,1]\), via scalar multiplication \( r \cdot p \) defined as \((r \cdot p)(x) = r \cdot p(x)\). This mapping \( X \mapsto [0,1]^X \) yields a functor \( \text{Sets} \to \text{EMod}^{\text{op}} \).

We can now strengthen Proposition 17 in the following way.

**Theorem 21 (From [Jac15a])**. For each effectus \( B \) the mapping \( X \mapsto \text{Pred}(X) \) yields a predicate functor:

\[
B \xrightarrow{\text{Pred}} \left( \text{EMod}_M \right)^{\text{op}} \quad \text{where} \quad M = \text{Pred}(1).
\]

This functor \( \text{Pred} \) preserves finite coproducts and the final object.

This theorem says that predicates in an effectus form effect modules over the effect monoid \( \text{Pred}(1) \) of scalars in the effectus.

**Proof** For a predicate \( p: X \to 1 \) on \( X \) and a scalar \( s: 1 \to 1 \) we define scalar multiplication simply as partial composition \( s \circ p \). The PCM-structure \( \oplus, 0 \) is preserved in each coordinate by Proposition 13 (2). We have \( r \circ (s \circ p) = (r \circ s) \circ p \) by associativity of partial composition, and \( 1 \circ p = p \) because \( 1 = \kappa_1: 1 \to 1 + 1 \) is the identity on \( 1 \) for partial composition \( \oplus \).

For a (total) map \( f: Y \to X \) in \( B \) we see that substitution \( f^* \) preserves scalar multiplication:

\[
f^*(s \circ p) = (s \circ p) \cdot f = (s \circ p) \circ f = s \circ (p \circ f) = s \circ f^*(p).
\]

We still have to prove that the functor \( \text{Pred}: B \to (\text{EMod}_M)^{\text{op}} \) preserves finite coproducts and the final object. This means that it sends coproducts in \( B \) to products in \( \text{EMod}_M \), and the final object to the initial one in \( \text{EMod}_M \).

- There is precisely one predicate \( 0 \to 1 + 1 \), so \( \text{Pred}(0) \) is a singleton, which is final in \( \text{EMod}_M \).
- The scalars \( M = \text{Pred}(1) \) are initial in \( \text{EMod}_M \).
- For objects \( X, Y \in B \), there is an isomorphism of effect modules:

\[
\text{Pred}(X + Y) \cong \text{Pred}(X) \times \text{Pred}(Y)
\]

With this final object \( M \) and these finite coproducts the category \( (\text{EMod}_M)^{\text{op}} \) is an effectus. Its scalars are precisely the elements of the effect monoid \( M \).

**Example 22.** We briefly review what this theorem means concretely for the running examples from Subsection 3.1.

1. The scalars in the effects \( \text{Sets} \) of sets and functions are the Booleans \( 2 = \{0,1\} \), see Example 8. Since effect modules over 2 are just effect algebras, the predicate functor takes the form \( \text{Pred}: \text{Sets} \to \text{EA}^{\text{op}} \). It sends a set \( X \) to the effect algebra \( \mathcal{P}(X) \) of subsets of \( X \). As is well-known, these predicates \( \mathcal{P}(X) \) form a Boolean algebra, but that is a consequence
of the fact that \textbf{Sets} is a special, ‘Boolean’ effectus, see Proposition 61.

For a function \( f: Y \to X \), the associated substitution functor \( f^*: \mathcal{P}(X) \to \mathcal{P}(Y) \) is inverse image:

\[
    f^*(U) = \{ y \in Y \mid f(y) \in U \}.
\]

2. We recall from Example 9 that scalars in the Kleisli category \( K\ell(\mathcal{D}) \) are the probabilities \([0, 1]\), and predicates on a set \( X \) are functions \( X \to [0, 1] \).

Indeed, as we saw before Theorem 21, the set of fuzzy predicates \([0, 1]^X\) is an effect module over \([0, 1]\]. For a map \( f: Y \to X \) in \( K\ell(\mathcal{D}) \), that is, for a function \( f: Y \to \mathcal{D}(X) \), the associated substitution map \( f^*: [0, 1]^X \to [0, 1]^Y \) is defined as:

\[
    f^*(p)(y) = \sum_{x \in X} f(y)(x) \cdot p(x).
\]

This map \( f^* \) is indeed a homomorphism of effect modules. Hence we have a predicate functor \( K\ell(\mathcal{D}) \to \text{EMod}^{\text{op}} \).

3. The scalars in the effectus \( \text{OUG}^{\text{op}} \) of order unit groups are the Booleans \( 2 = \{0, 1\} \), and the predicates on an order unit group \( G \) are the effects in the unit interval, \([0, 1]_G = \{ x \in G \mid 0 \leq x \leq 1 \}\), see Example 10. For a map \( f: G \to H \) in \( \text{OUG}^{\text{op}} \), that is, for a homomorphism of order unit groups \( f: H \to G \) the substitution function \( f^*: [0, 1]_H \to [0, 1]_G \) is given simply by function application:

\[
    f^*(x) = f(x).
\]

This is well-defined since \( f \) is positive, so that \( f(x) \geq 0 \), and unital so that \( f(x) \leq f(1) = 1 \). In this case we have a predicate functor \( \text{OUG}^{\text{op}} \to \text{EA}^{\text{op}} \).

For the special case of order unit spaces, the scalars in the relevant effectus \( \text{OUS}^{\text{op}} \) are the probabilities \([0, 1]\). In this case the predicate functor is of the form \( \text{Pred}: \text{OUS}^{\text{op}} \to \text{EMod}^{\text{op}} \). It is full and faithful.

4. The situation is similar for the effectus \( \text{vNA}^{\text{op}} \) of von Neumann algebras, see Example 11: the scalars are the probabilities \([0, 1]\), and the predicates on a von Neumann algebra \( \mathcal{A} \) are the effects in \([0, 1]_{\mathcal{A}} = \{ a \in \mathcal{A} \mid 0 \leq a \leq 1 \}\). This is an effect module since \( r \cdot a \in [0, 1]_{\mathcal{A}} \) for \( r \in [0, 1] \) and \( a \in [0, 1]_{\mathcal{A}} \). Substitution \( f^* \) works, like for order unit groups, via function application:

\[
    f^*(a) = f(a).
\]

We thus obtain a predicate functor \( \text{vNA}^{\text{op}} \to \text{EMod}^{\text{op}} \). It is also full and faithful, see [FJL5].

In (the proof of) Theorem 21 we have seen scalar multiplication \( s \circ p \) via partial post-composition, for a scalar \( s: 1 \to 1 \) and a predicate \( p: X \to 1 \).

The same trick can be used for substates \( \omega: 1 \to X \) via partial pre-composition. Substates thus form a PCM with scalar multiplication (which is a map of PCMs.
in each coordinate). We call these structure partial commutative modules (PC-modules, for short). They are organised in a category \( \text{PCMod} \) in the obvious manner. We have to keep in mind that writing partial composition \( \circ \) in the usual order gives ‘right’ modules, with the scalar written on the right.

**Lemma 23.** The sets \( \text{SStat}(X) \) of substates \( \omega: 1 \to X \) in an effectus are partial commutative modules over the effect monoid of scalars \( \text{Pred}(1) \), via \( \omega \circ s \).

**Proof** Obviously, this definition \( \omega \circ s \) determines a right action of the monoid \( \text{Pred}(1) \) of scalars on the set \( \text{SStat}(X) \) of substates. It preserves the PCM structure in each coordinate by Proposition 13 (2). \( \square \)

Later, in Section 10 we shall see that in the presence of tensor products \( \otimes \) all partial homsets, and not just the ones of substates, become PC-modules.

This scalar multiplication on substates makes it possible to define when a substate is pure. In quantum theory a state is called pure if it is not a mixture (convex combination) of other states. In the current setting this takes the following form.

**Definition 24.** A non-zero substate \( \omega: 1 \to X \) in an effectus is called pure if for each pair of orthogonal substates \( \omega_1, \omega_2: 1 \to X \) with \( \omega_1(y) + \omega_2(y) = (\omega_1 \otimes \omega_2)(y) = 0 \) for \( y \neq x \). Hence we take \( s = \frac{r_1}{r} \in [0,1] \). Clearly:

\[
s \cdot \omega = \frac{r_1}{r} \cdot r|x\rangle = r_1|x\rangle = \omega_1.
\]

And:

\[
s^\perp \cdot \omega = (1 - \frac{r_1}{r}) \cdot r|x\rangle = (r - r_1)|x\rangle = r_2|x\rangle = \omega_2.
\]

## 5 State and effect triangles

In quantum theory there is a basic duality between states and effects, see e.g. [HZ12]. This duality can be formalised in categorical terms as an adjunction \( \text{EMod}^{op} \rightleftarrows \text{Conv} \) between the opposite of the category of effect modules and the category of convex sets. This adjunction will be described in more detail below.

The duality between states and effects is related to the different approaches introduced by two of the founders of quantum theory, namely Erwin Schrödinger and Werner Heisenberg. Schrödinger’s approach is state-based and works in a forward direction, whereas Heisenberg’s describes how quantum operations work on effects, in a backward direction. It turns out that the difference between these two approaches is closely related to a well-known distinction in the semantics of computer programs, namely between state transformer semantics and predicate
The situation can be described in terms of a triangle:

\[
\text{Log}^{op} = \begin{array}{ccc}
\text{Heisenberg} & \text{T} & \text{Schrödinger} \\
\text{Predicate} & \text{State} & \text{Computations} \\
\text{transformers} & \text{transformers} & \text{computations} \\
\end{array}
\]

(14)

The main result of this section shows that each effectus gives rise to a such a ‘state and effect’ triangle.

We start with a closer inspection of the structure of states in an effectus. Recall that a state is a map of the form \( \omega : 1 \to X \). We will show that states are closed under convex combinations \( \sum i r_i \omega_i \), where the \( r_i \) are scalars \( 1 \to 1 + 1 \). We recall from Definition 19 that these scalars form an effect monoid. Hence we need to understand convexity with respect to such effect monoids, generalising the usual form of convexity with respect to the effect monoid \([0, 1]\) of probabilities.

**Definition 25.** Let \( M \) be an effect monoid.

1. A convex set over \( M \) is a set \( X \) with sums of convex combinations with scalars from \( M \). More precisely, for each \( n \)-tuple \( r_1, \ldots, r_n \in M \) with \( \sum i r_i = 1 \) and \( n \)-tuple \( x_1, \ldots, x_n \in X \) there is an element \( \sum i r_i x_i \in X \). These convex sums must satisfy the following two properties:

\[
1x = x \quad \text{and} \quad \sum i r_i (\sum j s_{ij} x_{ij}) = \sum i j (r_i \cdot s_{ij}) x_{ij}
\]

2. A function \( f : X \to Y \) between two convex sets is called affine if it preserves sums of convex combinations: \( f(\sum i r_i x_i) = \sum i r_i f(x_i) \). Convex sets and affine functions between them form a category \( \text{Conv}_M \).

Convex sets over \( M \) can be described more abstractly as an Eilenberg-Moore algebra of a distribution monad \( D_M \) defined in terms of formal convex combinations with scalars from \( M \), see [Jac15a]. This explains the form of the above two equations.

Convex sets over the unit interval \([0, 1]\) have sums of ‘usual’ convex combinations, with scalars from \([0, 1]\). That’s why we often simply write \( \text{Conv} \) for the category \( \text{Conv}_{[0, 1]} \) — just like \( \text{EMod} \) is \( \text{EMod}_{[0, 1]} \). Convex sets over the Booleans \([0, 1]\) are ordinary sets: in an \( n \)-tuple \( r_1, \ldots, r_n \in \{0, 1\} \) with \( \sum i r_i = 1 \) there is precisely one \( i \) with \( r_i = 1 \), and \( r_j = 0 \) for \( j \neq i \). In that case we can define \( \sum j r_j x_j = x_i \). This works for any set \( X \). Hence \( \text{Conv}_2 \cong \text{Sets} \).

The next result gives a categorical formalisation of the duality between states and effects in quantum physics. It goes back to [Jac10a].

**Proposition 26.** Let \( M \) be an effect monoid. By “homming into \( M \)“ one obtains an adjunction:

\[
\begin{array}{ccc}
\text{EMod}_M \congop & \text{Hom}(-, M) & \text{Conv}_M \\
\text{Hom}(-, M) & \text{Conv}_M & \text{EMod}_M
\end{array}
\]
Proof Given a convex set \( X \in \text{Conv}_M \), the homset \( \text{Conv}(X, M) \) of affine maps is an effect module, with \( f \perp g \) iff \( \forall x \in X. f(x) \perp g(x) \) in \( M \). In that case one defines \( (f \odot g)(x) = f(x) \odot g(x) \). It is easy to see that this is again an affine function. Similarly, the pointwise scalar product \( (r \cdot f)(x) = r \cdot f(x) \) yields an affine function. This mapping \( X \mapsto \text{Conv}(X, M) \) gives a contravariant functor since for \( h: X \to X' \) in \( \text{Conv}_M \) pre-composition with \( h \) yields a map \((-) \circ h: \text{Conv}(X', M) \to \text{Conv}(X, M)\) of effect modules.

In the other direction, for an effect module \( E \in \text{EMod}_M \), the homset \( \text{EMod}(E, M) \) of effect module maps yields a convex set: a convex sum \( f = \sum_j r_j f_j \), where \( f_j: E \to M \) in \( \text{EMod}_M \) and \( r_j \in M \), can be defined as \( f(y) = \sum_j r_j \cdot f_j(y) \). This \( f \) forms a map of effect modules. Again, functoriality is obtained via pre-composition.

The dual adjunction between \( \text{EMod}_M \) and \( \text{Conv}_M \) involves a bijective correspondence that is obtained by swapping arguments. \( \square \)

Lemma 27. Let \( B \) be an effectus, with effect monoid of scalars \( \text{Pred}(1) \).

1. For each object \( X \) the set of states \( \text{Stat}(X) = B(1, X) \) is a convex set over the effect monoid \( \text{Pred}(1) \) of scalars in \( B \).

2. Each (total) map \( f: X \to Y \) in \( B \) gives rise to an affine function \( f_* = f \cdot (\cdot): \text{Stat}(X) \to \text{Stat}(Y) \).

Thus we have a state functor:

\[
\begin{array}{ccc}
\text{Stat} & \longrightarrow & \text{Conv}_M \\
\end{array}
\]

where \( M = \text{Pred}(1) \).

Proof We first have to prove that convex sums exist in the set of states \( \text{Stat}(X) \).

So let \( r_1, \ldots, r_n \in \text{Pred}(1) \) be scalars with \( \sum_i r_i = 1 \). This means that there is a bound \( b: 1 \to n \cdot 1 = 1 + \cdots + 1 \) with \( \triangleright_i s \cdot b = r_i \) and \( \triangleright s \cdot b = 1: 1 \to 1 \), where \( \triangleright: n \cdot 1 \to 1 \) is the \( n \)-ary codiagonal. The equation \( \triangleright s \cdot b = 1 \) translates to \( (\triangleright + \text{id}) \cdot b = \kappa_1 \cdot 1: 1 \to 1 + 1 \). Hence \( b \) is a total map \( 1 \to n \cdot 1 \), of the form \( b = \kappa_1 \cdot s \) in:

\[
\begin{array}{c}
1 \\
\downarrow \kappa_1 \\
n \cdot 1 + 1 \downarrow \triangleright + \text{id} \\
\downarrow b \\
1 + 1 \\
\end{array}
\]

For an \( n \)-tuple of states \( \omega_i: 1 \to X \) we now define \( [\omega_1, \ldots, \omega_n] \cdot s \) to be the convex sum \( \sum_i r_i \omega_i \).

Post-composition with \( f: X \to Y \) is clearly affine, since:

\[
f_* \left( \sum_i r_i \omega_i \right) = f \cdot [\omega_1, \ldots, \omega_n] \cdot s = [f \cdot \omega_1, \ldots, f \cdot \omega_n] \cdot s = \sum_i r_i f_*(\omega_i). \quad \square
\]

When we combine this result with Theorem 21 and Proposition 20 we obtain the following result.
**Theorem 28.** Let $B$ be an effectus, with $M = \text{Pred}(1) = \text{Stat}(1 + 1)$ its effect monoid of scalars. There is a ‘state and effect’ triangle of the form:

$$
\begin{align*}
\text{B} & \quad \text{Hom}(\cdot, \cdot) \quad \text{Stat}(\cdot) \\
\text{Stat}(\cdot) & \quad \text{Hom}(\cdot, \cdot) \\
\text{Conv}_M & \quad \text{Hom}(\cdot, \cdot)
\end{align*}
$$

For a predicate (effect) $p: X \to 1 + 1$ and a state $\omega: 1 \to X$ we define the validity $\omega \models p$ as the scalar obtained by composition:

$$(\omega \models p) = \omega \circ p: 1 \to 1 + 1.$$  \hfill (16)

We call this abstract definition the Born rule, see Example 29 below.

This validity $\models$ satisfies the following Galois correspondence:

$$f_*(\omega) \models q = q \circ f \circ \omega = \omega \models f^*(q),$$  \hfill (17)

where $f: X \to Y$, $\omega: 1 \to X$, and $q: Y \to 1 + 1$.

This validity relation $\models$ gives rise to two natural transformations in:

$$
\begin{align*}
\text{EMod}_M^{\text{op}} & \quad \text{Conv}_M \\
\text{Hom}(\cdot, M) & \quad \text{Stat}(\cdot)
\end{align*}
$$

**Proof** Most of this is a summary of what we have seen before. We concentrate on the natural transformations in the last part of the theorem. The natural transformation $\text{Hom}(\text{Stat}(\cdot), M) \Rightarrow \text{Pred}$ consists of functions $\text{Pred}(X) \to M^{\text{Stat}(X)}$ in $\text{EMod}_M$, given by $p \mapsto (\omega \mapsto \omega \models p)$. Similarly, the natural transformation $\text{Stat} \Rightarrow \text{Hom}(\text{Pred}(\cdot), M)$ consists of affine maps $\text{Stat}(X) \to M^{\text{Pred}(X)}$ given by $\omega \mapsto (p \mapsto \omega \models p)$. Naturality of these functions is given by the equations (17). □

Notice that we do not require that the triangle (15) commutes (up to isomorphism), that is, that the natural transformations in (18) are isomorphisms. In some examples they are, in some examples they are not.

**Example 29.** We shall review the running examples in the remainder of this section.

1. For the effectus $\text{Sets}$ the triangle (15) takes the following form.

$$
\begin{align*}
\text{EA}^{\text{op}} & \quad \text{Sets} \\
\text{Hom}(\cdot, 2) & \quad \text{Stat}(\cdot)
\end{align*}
$$

Here we use that convex sets over the effect monoid $2 = \{0, 1\}$ of scalars in $\text{Sets}$ are just sets. The predicate functor $2(\cdot)$ is powerset, see Example 8.
For a state $x \in X$ and a predicate $p \in 2^X$ the Born rule \[\text{[16]}\] gives an outcome in $\{0, 1\}$ determined by membership:

$$x \models p = p(x).$$

2. For the effectus $\mathcal{K}(\mathcal{D})$ we have a triangle:

For a Kleisli map $f : X \to \mathcal{D}(Y)$ the associated state transformer function $f_* : \mathcal{D}(X) \to \mathcal{D}(Y)$ is Kleisli extension, given by:

$$f_*(\omega)(y) = \sum_x \omega(x) \cdot f(x)(y).$$

This is ‘baby’ integration. For the Giry monad $\mathcal{G}$ it is proper integration, see \[\text{[Jac13]}\]. For a state $\omega \in \mathcal{D}(X)$ and a predicate $p \in [0, 1]^X$ the Born rule gives the expected value:

$$\omega \models p = \sum_x \omega(x) \cdot p(x) \in [0, 1].$$

The (Born) validity rules for discrete and continuous probability have been introduced in \[\text{[Koz81, Koz85]}\], in the context of semantics of probabilistic programs. For continuous probability, the correspondence \[\text{[17]}\] occurs in \[\text{[Jac13]}\] for the Giry monad as an equation:

$$f_*(\omega) \models q = \int q \, df_*(\omega) = \int f^*(q) \, d\omega = \omega \models f^*(q).$$

Here, $f : X \to \mathcal{G}(Y)$ is a measurable function, $q : Y \to [0, 1]$ is a (measurable) predicate on $Y$ and a measure/state $\omega \in \mathcal{G}(X)$. The operation $f_*$ is Kleisli extension and $f^*$ is substitution.

3. The state and effect triangle for the effectus $\text{OUUG}^{\text{op}}$ of order unit groups is:

For a predicate $e \in [0, 1]_G$ and a state $\omega : G \to \mathbb{Z}$ validity is:

$$\omega \models e = \omega(e) \in \{0, 1\}.$$
4. The effectus $\mathbf{vNA}^{\text{op}}$ of von Neumann algebras yields:

$$
\begin{array}{ccc}
\text{EMod}^{\text{op}} & \xrightarrow{\text{Hom}(-,[0,1])} & \text{Conv} \\
\text{Hom}(-,[0,1]) & \xleftarrow{\text{Stat} = \text{Hom}(-,\mathbb{C})} & \mathbf{vNA}^{\text{op}}
\end{array}
$$

For a predicate $e \in [0,1]$, and a state $\omega : \mathcal{A} \to \mathbb{C}$ one interpretes validity again as expected probability:

$$
\omega \models e = \omega(e) \in [0,1].
$$

An interesting special case is $\mathcal{A} = \mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space, with associated von Neumann algebra $\mathcal{B}(\mathcal{H})$ of bounded operators $\mathcal{H} \to \mathcal{H}$. It is well-known that (normal) states $\omega : \mathcal{B}(\mathcal{H}) \to \mathbb{C}$ correspond to density matrices $\rho : \mathcal{H} \to \mathcal{H}$, via $\omega = \text{tr}(-\rho)$. The Born rule formula (16), for an effect $E \in [0,1]$, $\mathcal{B}(\mathcal{H}) = \{F : \mathcal{H} \to \mathcal{H} \mid 0 \leq F \leq \text{id}\}$ becomes the Born rule:

$$
\text{tr}(\omega) \models E = \text{tr}(E\rho).
$$

**Remark 30.** In Theorem [21] we have seen that the predicate functor $\text{Pred} : \mathcal{B} \to (\text{EMod}_M)^{\text{op}}$ of an effectus $\mathcal{B}$ preserves finite coproducts. The same preservation property does not hold in general for the states functor $\text{Stat} : \mathcal{B} \to \text{Conv}_M$ from Lemma [27]. The matter is investigated in [JWW15], for the case where the effect monoid $M$ is $[0,1]$.

It turns out that preservation of finite coproducts by the states functor is closely related to normalisation of substates. This works as follows.

We say that an effectus $\mathcal{B}$ satisfies normalisation if for each non-zero substate $\omega : 1 \to X$ there is a unique state $\rho : 1 \to X$ with:

$$
\omega = \langle \rho \rangle \star 1 \star \omega.
$$

(19)

This says that the substate $\omega$ is scalar multiplication $\langle \rho \rangle \star s$ in the sense of Lemma [23], where the scalar $s = 1 \star \omega : 1 \to 1$ is determined by $\omega$ itself.

We briefly show that the effectuses $\mathcal{K}(\mathcal{D})$ and $\mathbf{vNA}^{\text{op}}$ satisfy normalisation.

1. A substate in $\mathcal{K}(\mathcal{D})$ is a subdistribution $\omega \in \mathcal{D}_{\leq 1}(X)$. Let us assume that it is non-zero, so that the associated scalar $s = 1 \star \omega = \sum_x \omega(x) \in [0,1]$ is non-zero. We take $\rho = \frac{\omega}{s} = \sum_x \frac{\omega(x)}{s} |x\rangle$. This $\rho$ is a proper state (distribution) since:

$$
\sum_x \rho(x) = \sum_x \frac{\omega(x)}{s} = 1.
$$

Moreover, we have $s \cdot \rho = \omega$ by construction.

2. A state in the effectus $\mathbf{vNA}^{\text{op}}$ is a positive subunital map $\omega : \mathcal{A} \to \mathbb{C}$. If it is non-zero, then $s = \omega(1) \in [0,1]$ is a non-zero scalar, so we can define $\rho : \mathcal{A} \to \mathbb{C}$ as $\rho(a) = \frac{\omega(a)}{s}$. By construction $\rho$ is unital, and satisfies $\omega = s \cdot \rho$. 

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6 Effectuses from biproduct categories

This section describes a construction that turns a biproduct category with a suitable ‘ground’ map into an effectus. The construction is inspired by causal maps in the context of CP*-categories, see [CHK14].

A biproduct category is a category with finite biproducts \((0, \oplus)\). This means first of all that the object 0 is both initial and final, and thus gives rise to zero maps \(0 : X \to 0 \to Y\) between arbitrary objects \(X, Y\). Next, for each pair of objects \(X_1, X_2\), the object \(X_1 \oplus X_2\) is both a product, and a coproduct, with coprojections and projections:

\[
\begin{align*}
X_i &\xrightarrow{\kappa_i} X_1 \oplus X_2 \\
&\xrightarrow{\pi_j} X_j \quad \text{satisfying} \quad \pi_j \circ \kappa_i = \begin{cases} \text{id} & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}
\end{align*}
\]

It follows immediately that \(\kappa_1 = \langle \pi_1 \circ \kappa_1, \pi_2 \circ \kappa_1 \rangle = \langle \text{id}, 0 \rangle\) and \(\pi_1 = [\pi_1 \circ \kappa_1, \pi_1 \circ \kappa_2] = [\text{id}, 0]\), and similarly for \(\kappa_2\) and \(\pi_2\). Moreover, each homset of maps \(X \to Y\) is a commutative monoid, with sum of maps \(f, g : X \to Y\) given by:

\[
f + g = \left( \begin{array}{c}
X \xrightarrow{\Delta = (\text{id}, \text{id})} X \oplus X \\
\xrightarrow{f \oplus g} Y \oplus Y \\
\xrightarrow{\nabla = [\text{id}, \text{id}]} Y
\end{array} \right)
\]

The zero element for this addition \(+\) is the zero map \(0 : X \to Y\).

Definition 31. We call a category \(A\) a grounded biproduct category if \(A\) has finite biproducts \((0, \oplus)\) and has a special object \(I\) with for each \(X \in A\) a ‘ground’ map \(\top_X : X \to I\) satisfying the four requirements below. We omit the subscript \(X\) in \(\top_X\) when it is clear from the context.

1. the ground on \(I\) is the identity: \(\top_I = \text{id}_I : I \to I\);
2. coprojections commute with ground: \(\langle X_i \xrightarrow{\kappa_i} X_1 \oplus X_2 \xrightarrow{\text{id}} I \rangle = (X_i \xrightarrow{\text{id}} I)\).
3. ground maps are ‘zero-monic’, that is \(\top \circ f = 0\) implies \(f = 0\)
4. ‘subcausal cancellation’ holds: if \(f + g = f + h = \top : X \to I\), then \(g = h\).

A map \(f : X \to Y\) in \(A\) is called causal if \(\top_Y \circ f = \top_X\). We write \(\text{Caus}(A) \hookrightarrow A\) for the subcategory with causal maps.

As we shall see in Examples 33 and 34 below, these ground maps describe a unit elements, possibly in opposite direction. Causal maps preserve this units, and may thus be called unital — or co-unital. The idea of defining them causal maps in this way occurs in [CL13], building on the causality axiom in [CDPT1]. The above second point says that coprojections are causal. Third point, about cancellation, fails in the CP*-category obtained from the category of relations, where union \(\cup\) is used as sum \(+\).

We can now state and prove the main result, which was obtained jointly with Aleks Kissinger.

Theorem 32. The category \(\text{Caus}(A)\) of causal maps, for a grounded biproduct category \(A\), is an effectus.
**Proof** The category $Caus(A)$ has coproduct $\oplus$ since the coprojections $\kappa_i$ are causal by definition, and the cotuple $[f, g]$ is causal if $f: X \to Z$, $g: Y \to Z$ are causal by requirement (2) in Definition 31.

$$\hat{\pi}_Z \circ [f, g] = [\hat{\pi}_Z \circ f, \hat{\pi}_Z \circ g] = [\hat{\pi}_X, \hat{\pi}_Y] \oplus [\hat{\pi}_{X+Y} \circ \kappa_1, \hat{\pi}_{X+Y} \circ \kappa_2] = \hat{\pi}_{X+Y}.$$  

The zero object $0$ in $A$ is initial in $Caus(A)$ since $!_X: 0 \to X$ is causal. We have $\hat{\pi}_X \circ !_X = !_I = \hat{0}$. The object $I \in A$ is final in $Caus(A)$, since: the ground map $\hat{\pi}_X: X \to I$ is causal because $\hat{\pi}_I \circ \hat{\pi}_X = id_I \circ \hat{\pi}_X = \hat{\pi}_X$. Further, any causal map $f: X \to I$ satisfies $f = id_I \circ f = \hat{\pi}_I \circ f = \hat{\pi}_X$.

We have to prove that the two diagrams from Definition 2 are pullbacks in $Caus(A)$.

\[
\begin{array}{ccc}
X \oplus Z & \xrightarrow{id \oplus \hat{\pi}} & X \oplus I \\
\hat{\pi} \oplus id & \downarrow & \hat{\pi} \oplus id \\
I \oplus Y & \xrightarrow{id \oplus \hat{\pi}} & I \oplus I \\
\end{array}
\quad
\begin{array}{ccc}
X & \xrightarrow{\hat{\pi}} & I \\
\kappa_1 & \downarrow & \kappa_1 \\
X \oplus Y & \xrightarrow{\hat{\pi} \oplus \hat{\pi}} & I \oplus I \\
\end{array}
\]

For the diagram on the left let $A \in A$ be an object with causal maps $f: A \to X \oplus I$ and $g: A \to I \oplus Y$ satisfying $(\hat{\pi} \oplus id) \circ f = (id \oplus \hat{\pi}) \circ g$. There is an obvious, unique mediating map $\langle \pi_1 \circ f, \pi_2 \circ g \rangle: A \to X \oplus Y$. We only have to prove that it is causal. First notice that:

$$\hat{\pi}_Y \circ \pi_1 \circ f = \pi_1 \circ (\hat{\pi} \oplus id) \circ f = \pi_1 \circ (id \oplus \hat{\pi}) \circ g = \pi_1 \circ g = \hat{\pi}_I \circ \pi_1 \circ g.$$  

Now we have that:

$$\hat{\pi}_{X \oplus Y} = \hat{\pi}_{X \oplus Y} \circ [\kappa_1, \kappa_2] = \hat{\pi}_{X \oplus Y} \circ \kappa_1, \hat{\pi}_{X \oplus Y} \circ \kappa_2] = [\hat{\pi}_X, \hat{\pi}_Y] = \nabla \circ (\hat{\pi}_X \oplus \hat{\pi}_Y)$$  

Hence:

$$\hat{\pi} \circ (\pi_1 \circ f, \pi_2 \circ g) \overset{(*)}{=} \nabla \circ (\hat{\pi} \oplus \hat{\pi}) \circ (\pi_1 \circ f, \pi_2 \circ g)$$  

$$= \nabla \circ (\hat{\pi} \circ \pi_1 \circ f, \hat{\pi} \circ \pi_2 \circ g)$$  

$$= \nabla \circ (\hat{\pi} \circ \pi_1 \circ g, \hat{\pi} \circ \pi_2 \circ g) \quad \text{as shown above}$$  

$$= \nabla \circ (\hat{\pi} \circ \pi_1 \circ g, \pi_2 \circ g) \overset{(*)}{=} \hat{\pi} \circ g$$  

$$= \hat{\pi}.$$  

The $(\pi_1 \circ f, \pi_2 \circ g)$ is causal, and the diagram on the left is a pullback.

For the above diagram on the right, let $f: A \to X \oplus Y$ be a causal map satisfying $(\hat{\pi} \oplus \hat{\pi}) \circ f = \kappa_1 \circ \hat{\pi}$. The obvious mediating map is $\pi_1 \circ f: A \to X$. It is causal since $\hat{\pi} \circ \pi_1 \circ f = \pi_1 \circ (\hat{\pi} \oplus \hat{\pi}) \circ f = \pi_1 \circ \kappa_1 \circ \hat{\pi} = \hat{\pi}$. We obtain $\pi_2 \circ f = 0$ via Definition 31 (3) from:

$$\hat{\pi} \circ \pi_2 \circ f = \pi_2 \circ (\hat{\pi} \oplus \hat{\pi}) \circ f = \pi_2 \circ \kappa_1 \circ \hat{\pi} = \hat{0} \circ \hat{\pi} = 0.$$  

We now get:

$$f = (\pi_1 \circ f, \pi_2 \circ f) = (\pi_1 \circ f, 0) = (id, 0) \circ \pi_1 \circ f = \kappa_1 \circ \pi_1 \circ f.$$  

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Hence $\pi_1 \circ f : A \to X$ is the required unique mediating (causal) map.

Finally we have to prove that the following two maps $W$ and $\mathcal{W}$ are jointly monic, in:

$$(I \oplus I) \oplus I \xrightarrow{W} I \oplus I$$

where

$$W = [\text{id}, \kappa_2] = \langle \pi_1 \circ \pi_1, (\pi_2 \circ \pi_1) + \pi_2 \rangle$$

$$\mathcal{W} = [\kappa_2, \kappa_1] = \langle \pi_2 \circ \pi_1, (\pi_1 \circ \pi_1) + \pi_2 \rangle.$$

Let $f, g : X \to (I \oplus I) \oplus I$ be causal maps satisfying $W \circ f = W \circ g$ and $\mathcal{W} \circ f = \mathcal{W} \circ g$. Then $\pi_1 \circ f = \pi_1 \circ g$ since:

$$\pi_1 \circ \pi_1 \circ f = \pi_1 \circ W \circ f = \pi_1 \circ W \circ g = \pi_1 \circ \pi_1 \circ g$$

$$\pi_2 \circ \pi_1 \circ f = \pi_1 \circ \mathcal{W} \circ f = \pi_1 \circ \mathcal{W} \circ g = \pi_2 \circ \pi_1 \circ g.$$

Hence it suffices to show $\pi_2 \circ f = \pi_2 \circ g : X \to I$. For this we use subcausal cancellation from Definition 31 (4):

$$(\nabla \circ \pi_1 \circ f) + (\pi_2 \circ f) = \nabla \circ (\nabla \circ \pi_2 \circ f)$$

$$= \nabla \circ (\nabla \oplus \text{id}_I) \circ f$$

$$= \xi \circ \nabla \circ (\nabla \oplus \text{id}_I) \circ f$$

since $\xi \circ \text{id}_I$.

$$= \xi \circ f,$$

where the last equality holds since $f$ is causal, and causal maps are closed under composition, cotuple and coproduct. Similarly we have $(\nabla \circ \pi_1 \circ g) + (\pi_2 \circ g) = \xi \circ f$. By $\pi_1 \circ f = \pi_1 \circ g$ and the cancellation in Definition 31 (4) we obtain $\pi_2 \circ f = \pi_2 \circ g$. Hence $f = g$, showing that $W$ and $\mathcal{W}$ are jointly monic.

In the remainder of this section we show how each of our four running example effectuses can be understood as category of causal maps in a grounded biproduct category.

**Example 33.** Let $S$ be a semiring which is positive and cancellative, that is, it satisfies $x + y = 0 \Rightarrow x = y = 0$ and $x + y = x + z \Rightarrow y = z$. We write $M_S : \text{Sets} \to \text{Sets}$ for the multiset monad with scalars from $S$. Thus, elements of $M_S(X)$ are finite formal sums $\sum_i s_i x_i$ with $s_i \in S$ and $x_i \in X$. It is easy to see that $M_S$ is a monad, with unit $\eta(x) = 1 \{ x \}$. It is an ‘additive’ monad (see [CR]), since $M_S(0) \cong 1$ and $M_S(X + Y) \cong M_S(X) \times M_S(Y)$. The Kleisli category $\mathcal{K}(M_S)$ then has finite biproducts $(+, 0)$.

We claim that $\mathcal{K}(M_S)$ is a grounded biproduct category. We take $I = 1$ and use that $M_S(1) \cong S$. We thus take as map $\Phi : X \to 1$ in $\mathcal{K}(M_S)$ the function $X \to S$ given by $\Phi(x) = 1 \in S$ for all $x \in X$. We note that each map $f$ in $\mathcal{K}(M_S)$, of the form $f = \eta \circ g$ for $g : X \to Y$ in $\text{Sets}$, is causal, since in $\mathcal{K}(M_S)$:

$$(\Phi \circ f)(x) = (\mu \circ M_S(\Phi) \circ \eta \circ g)(x)$$

$$= (\mu \circ \eta \circ \Phi \circ g)(x) = \Phi(g(x)) = 1 = \Phi(x).$$

We now briefly check that the ground maps $\Phi$ in $\mathcal{K}(M_S)$ satisfy the four requirements from Definition 31.
1. The ground map $\uparrow: 1 \to 1$ in $\mathcal{KL}(\mathcal{M}_S)$ is $\uparrow(*) = 1$, which is the unit $\eta$ of the monad $\mathcal{M}_S$, and thus the identity in $\mathcal{KL}(\mathcal{M}_S)$.

2. The coprojection $k_1: X \to X + Y$ in $\mathcal{KL}(\mathcal{M}_S)$ is $\eta \circ k_1$, where, for the moment, we write $k_1$ for the coprojection in $\mathbf{Sets}$. Hence it is causal, as noted above.

3. If $\pi \circ f = 0$ in $\mathcal{KL}(\mathcal{M}_S)$, for a Kleisli map $f: X \to Y$, then $\sum_y f(x)(y) \cdot \pi(y) = 0$ for each $x \in X$. Since $S$ is positive, we get $f(x)(y) = 0$, for each $y \in Y$. But then $f(x) = 0 \in \mathcal{M}_S(X)$, and thus $f = 0$.

4. If $f + g = f + h = \uparrow$, for $f, g, h: 1 \to \mathcal{M}_S(X)$, then we may identify $f, g, h$ with multisets in $\mathcal{M}_S(X)$ that satisfy $f(x) + g(x) = f(x) + h(x) = 1$, for each $x \in X$. But then $g(x) = h(x)$ by cancellation in $S$, for each $x \in X$, and thus $g = h$.

We now consider two special choices for the semiring $S$.

- By taking $S = \mathbb{N}$ we obtain the category $\mathbf{Sets}$ as the effectus of causal maps in $\mathcal{KL}(\mathcal{M}_\mathbb{N})$. Indeed, maps $f: X \to \mathcal{M}_\mathbb{N}(Y)$ with $\sum_y f(x)(y) = 1$, for each $x \in X$ are determined by a unique $y \in Y$ with $f(x)(y) = 1$. Hence $f$ corresponds to a function $X \to Y$.

- Next we take $S = \mathbb{R}_{\geq 0}$, the semiring of non-negative real numbers. We now obtain the Kleisli category $\mathcal{KL}(\mathcal{D})$ as effectus of causal maps in the grounded biproduct category $\mathcal{KL}(\mathcal{M}_{\mathbb{R}_{\geq 0}})$, since maps $f: X \to \mathcal{M}_{\mathbb{R}_{\geq 0}}(Y)$ with $\sum_y f(x)(y) = 1$, for each $x$, are precisely the maps $X \to \mathcal{D}(Y)$.

**Example 34.** It is well-known that the category $\mathbf{Ab}$ of Abelian groups has finite biproducts ($\oplus, \{0\}$), given by cartesian products. These biproducts restrict to the category $\mathbf{OAb}$ of ordered Abelian groups with positive/monotone group homomorphisms (described in Example 31). Let’s use the *ad hoc* notation $\mathbf{OUG}$ for the category with order unit groups as objects, but with positive group homomorphisms as maps. Hence we have a full and faithful functor $\mathbf{OUG} \to \mathbf{OAb}$ since we do not require that units are preserved. It is not hard to see that $\mathbf{OUG}$ also has biproducts.

We claim that $\mathbf{OUG}^\text{op}$ is a grounded biproduct category. It has biproducts since they are invariant under taking the opposite. We take $I = \mathbb{Z}$. For an order unit group $G$ we have to define a ground map $G \to \mathbb{Z}$ in $\mathbf{OUG}^\text{op}$. We define this function $\uparrow: \mathbb{Z} \to G$ simply as $\uparrow(k) = k \cdot 1 \in G$. We check the four requirements from Definition 31:

1. The ground map $\uparrow: \mathbb{Z} \to \mathbb{Z}$ is given by $\uparrow(k) = k \cdot 1 = k$ and is thus the identity.

2. We have a commuting diagram in $\mathbf{OUG}$:

$$
\begin{array}{ccc}
\mathbb{Z} & \xrightarrow{\uparrow} & G_1 \oplus G_2 \\
& \downarrow{\pi_i} & \downarrow{\pi_i} \\
& G_i & \\
\end{array}
$$

since $\pi_i(\uparrow(k)) = \pi_i(k \cdot (1, 1)) = \pi_i(k \cdot 1, k \cdot 1) = k \cdot 1 = \uparrow(k)$. 

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3. If \( f : G \to H \) in \( \text{OUG} \) satisfies \( f \circ \hat{} = 0 \), then \( f(1) = 0 \). From this we obtain \( f(x) = 0 \) for an arbitrary \( x \in G \) in the following way. Since \( G \) is an order unit group there is an \( n \in \mathbb{N} \) with \( -n \cdot 1 \leq x \leq n \cdot 1 \). But then because \( f \) is monotone:

\[
0 = -n \cdot f(1) = f(-n \cdot 1) \leq f(x) \leq f(n \cdot 1) = n \cdot f(1) = 0.
\]

Hence \( f = 0 \).

4. If \( f + g = f + h = \hat{} : \mathbb{Z} \to G \), then \( f, g, h \) can be identified with elements of \( G \) satisfying \( f + g = f + h = 1 \). By subtracting \( f \) on both sides we obtain \( g = h \).

It is now easy to see that \( \text{OUG}^{\text{op}} \) is the subcategory \( \text{Caus}(\text{OUG}^{\text{op}}) \) and is thus an effectus. Indeed, a map \( f : G \to H \) in \( \text{OUG} \) with \( f \circ \hat{} = \hat{} \) is unital: \( f(1) = f(\hat{}(1)) = \hat{}(1) = 1 \). Hence \( f \) is a map in the category \( \text{OUG} \) of order unit groups.

In the same way one can prove that the opposite \( \text{OUS}^{\text{op}} \) of the category \( \text{OUS} \) of order unit spaces is an effectus. Since there is an equivalence of category \( \text{OUS} \cong \text{EMod} \), see [JM12b] for details, the opposite \( \text{EMod}^{\text{op}} \) of the category of effect modules (over \([0, 1]\)) is also an effectus.

In an analogous way one can define a category \( \text{vNA} \) of von Neumann algebras with completely positive maps and show that its opposite is a grounded biproduct category. The category of von Neumann algebras with completely positive unital maps is the associated effectus of causal maps. In particular, it forces this quantum model to be ‘non-signalling’, see [CK15, §5.3].

**Remark 35.** As mentioned in the beginning of this section, the \( \text{CP}^* \)-categories from [CHK14] form an inspiration for the construction in this section. A category \( \text{CP}^*(C) \) is obtained from a dagger-compact category \( C \), and forms a grounded biproduct if:

1. \( C \) has finite biproducts;
2. the dagger in \( C \) yields a “positive definite inner product”, that is, for all \( \psi : I \to X \), if \( \psi^\dagger \circ \psi = 0 \), then \( \psi = 0 \);
3. “uniqueness of positive resolutions” holds: for all positive maps \( p,q,q' \), if \( p + q = \text{id} = p + q' \), then \( q = q' \). We recall that a map \( p \) is called positive in a dagger-category if there exists \( g \) such that \( p = g^\dagger \circ g \).

The main point of this section is to show that inside a grounded biproduct category there is an effectus of causal maps. The grounded biproduct category can then be seen as a larger, ambient category of the effectus. One can also go in the other direction, that is, produce an ambient grounded biproduct category from an effectus via a (universal) totalisation construction. This will be described elsewhere.

## 7 Kernels and images

Kernels and images (and cokernels) are well-known constructions in (categorical) algebra. Standardly, for a map \( f : A \to B \) its kernel \( \ker(f) \) and image \( \text{im}(f) \) are
understood as maps, of the form \( \ker(f) : K \rightarrow A \) and \( \im(f) : B \rightarrow C \), satisfying certain universal properties. Here we define kernels and images as predicates, namely \( \ker(f) : A \rightarrow 1 + 1 \) and \( \im(f) : B \rightarrow 1 + 1 \). Later on, in the presence of comprehensions and quotients, our kernels and images (as predicates) will give rise to kernels and images/cokernels in the traditional sense, see in particular Lemma 80 (2) and Lemma 83 (14). When confusion might occur we speak of kernel/image predicates versus kernel/image maps. But our default meaning is: predicate.

In the context of effectuses there is an important difference between kernels and images: kernels always exist, but the presence of images is an additional predicate.

We use that partial pre-composition (\( \bullet \)) preserves the effect module structure. This is different for the partial substitution map \( g^\bigcirc \). So far we only know that it preserves truth, see Exercise 11(3). But there is a bit more that we can say now.

**Lemma 36.** For a partial map \( g : X \rightarrow Y \) in an effectus the partial substitution map \( g^\bigcirc : \text{Pred}(Y) \rightarrow \text{Pred}(X) \) preserves \( 1, \vee \) and is monotone — where \( \vee \) is the De Morgan dual of \( \odot \) given by \( p \odot q = (p^\perp \odot q^\perp)^\perp \).

**Proof** We use that partial pre-composition \( (\bullet) \ast g \) preserves \( \odot \), see Proposition 13(2). If \( p^\perp \perp q^\perp \), then:

\[
g^\bigcirc(p \odot q) = \left((p \odot q)^{\perp \ast g}\right)^{\perp} = \left((p^\perp \odot q^\perp) \ast g\right)^{\perp} = \left((p^\perp \ast g) \odot (q^\perp \ast g)\right)^{\perp} = \left(p^\perp \ast g\right)^{\perp \ast (q^\perp \ast g)^{\perp}} = g^\bigcirc(p^\perp) \odot g^\bigcirc(q^\perp).
\]

As a consequence \( g^\bigcirc \) is monotone, since \( p \leq q \) iff \( q^\perp \leq p^\perp \) iff \( q^\perp \odot r^\perp = p^\perp \) for some \( r \), that is, \( q \odot r = p \). \( \square \)

### 7.1 Kernels

In linear algebra the kernel of a (linear) map \( f : A \rightarrow B \) is the subspace \( \{ a \in A \mid f(a) = 0 \} \) of those elements that get mapped to 0. This also works for partial maps \( X \rightarrow Y + 1 \) where we intuitively understand the kernel as capturing those elements of \( X \) that get sent to 1 in the outcome \( Y + 1 \).

**Definition 37.** Let \( B \) be an effectus. The kernel of a partial map \( f : X \rightarrow Y \) is the predicate \( \ker(f) : X \rightarrow 1 + 1 \) given by:

\[
\ker(f) \overset{\text{def}}{=} f^\bigcirc(0) = [0, \kappa_1] \cdot f = [\kappa_2, \kappa_1] \cdot (! + \text{id}) \cdot f = ((! + \text{id}) \cdot f)^\perp.
\]
We call \( f \) an internal mono if \( \ker(f) = 0 \).

Sometimes it is easier to work with the orthosupplement of a kernel, and so we introduce special notation \( \ker^\perp \) for it:

\[
\ker^\perp(f) \overset{\text{def}}{=} \ker(f)^\perp = (! + \text{id}) \cdot f = (! \cdot \kappa_1 \cdot !, \kappa_2) \cdot f = 1 \cdot f.
\]

This \( \ker^\perp(f) \) is sometimes called the ‘domain predicate’ since it captures the ‘domain of \( f \)’, where \( f \) is defined, that is, ‘non-undefined’. Here we call it the kernel-supplement. We shortly see that internal monos are not related to ‘external’ monos, in the underlying category, but they are part of a factorisation system, see Proposition 84. Similarly, the internal epis that will be defined later on in this section form part of a factorisation system, see Proposition 95 (7).

We review the situation for our running examples.

**Example 38.** In the effectus \( \text{Sets} \) the kernel of a partial function \( f : X \to Y + 1 \) the partial substitution map \( f \� : \mathcal{P}(Y) \to \mathcal{P}(X) \) is given by:

\[
f \�(V) = \{ x \in X \mid \forall y \in Y. f(x) = \kappa_1 \Rightarrow y \in V \}. \tag{21}
\]

Hence we obtain as kernel predicate:

\[
\ker(f) = f \�(0) = \{ x \in X \mid \forall y \in Y. f(x) = \kappa_1 \Rightarrow y \in \emptyset \}
= \{ x \in X \mid f(x) = * \},
\]

where we write \( * \) for the sole element of the final set/object 1.

In the effectus \( \mathcal{K}(\mathcal{D}) \) for discrete probability a partial map \( f : X \to Y + 1 \) may be described either as a function \( f : X \to \mathcal{D}(Y + 1) \) or as \( f : X \to \mathcal{D}_{\leq 1}(Y) \), see Example 9. We consider both cases.

1. For \( f : X \to \mathcal{D}(Y + 1) \) the partial substitution map \( f^\square : [0, 1]^Y \to [0, 1]^X \) is given by:

\[
f^\square(q)(x) = \sum_{y \in Y} f(x)(y) \cdot q(y) + f(x)(*) \tag{22}.
\]

Hence the \( \ker(f) \in [0, 1]^X \) is the fuzzy predicate:

\[
\ker(f)(x) = f^\square(0)(x) = f(x)(*) = 1 - \sum_{y \in Y} f(x)(y).
\]

The kernel thus assigns to \( x \in X \) the probability that \( f(x) \) is undefined.

2. For a function \( f : X \to \mathcal{D}_{\leq 1}(Y) \) we have:

\[
f^\square(q)(x) = \sum_{y \in Y} f(x)(y) \cdot q(y) + (1 - \sum_y f(x)(y)). \tag{23}
\]

It gives essentially the same description of the kernel:

\[
\ker(f)(x) = 1 - \sum_y f(x)(y) \quad \text{so that} \quad \ker^\perp(f)(x) = \sum_y f(x)(y).
\]
In the effectus $\textbf{OUG}^\text{op}$ of order unit groups the kernel of a partial map $f: G \to H$ can also be described in two ways, via the correspondences from Example 10. We choose to understand $f$ as a positive subunital map $H \to G$. Then, for an effect $e \in [0,1]_H$ we have:

$$f^\odot(e) = f(e) + f(1)^\perp.$$  \hfill (24)

As a result, if identify $e \in [0,1]_H$ with the corresponding map $e: H \to \mathbb{Z} \oplus \mathbb{Z}$, then:

$$e \circ f = f^\odot(e_\perp)$$

by Exercise 11 \hfill (25)

Moreover, we simply have:

$$\ker(f) = f(1)^\perp \quad \text{and so} \quad \ker^\perp(f) = f(1).$$

The same descriptions apply in the effectus $\textbf{vNA}^\text{op}$ of von Neumann algebras.

We continue with some basic properties of kernels.

**Lemma 39.** The kernel operation $\ker(-)$ in an effectus $\textbf{B}$ satisfies:

1. $\ker(p) = p^\perp$ for a predicate $p: X \to 1$, and so $\ker^\perp(p) = p$;
2. $\ker(f) = 1 \iff \ker^\perp(f) = 0 \iff f = 0$;
3. $\ker(\triangleright_1) = [0,1]$ and $\ker(\triangleright_2) = [1,0]$, for the partial projections $\triangleright_i$ defined in (1);
4. $f$ is internally monic, that is $\ker(f) = 0$, iff $f$ is total;
5. $\ker(g \circ f) = f^\odot(\ker(g))$
6. $\ker(g \circ f) \geq \ker(f)$;
7. $\ker(h \circ f) = \ker(f)$;
8. the internally monic maps form a subcategory of $\text{Par}(\textbf{B})$;
9. if $g \circ f$ and $g$ are internally monic, then so is $f$;
10. $f^\odot(p^\perp) = f^\odot(p)^\perp \sqcup \ker(f)$.

**Proof** This involves some elementary reasoning in an effectus.

1. Clearly, $\ker(p) = [0, \kappa_1] \cdot p = [\kappa_2 \cdot !, \kappa_2] \cdot p = [\kappa_2, \kappa_1] \cdot p = p^\perp$.
2. First, $\ker(0) = \ker(\kappa_2 \cdot !) = [0, \kappa_1] \cdot \kappa_2 \cdot ! = \kappa_1 \cdot ! = 1$. Next, if $1 = \ker(f)$, then $(! + \text{id}) \cdot f = \ker^\perp(f) = 0$, so that $f = 0$ by Lemma 11.
3. We simply calculate:

\[
\ker(\bot_1) = [0, \kappa_1] \cdot (\text{id} + !) = [0, \kappa_1 \cdot !] = [0, 1]
\]
\[
\ker(\bot_2) = [0, \kappa_1] \cdot [\kappa_2 \cdot !, \kappa_1] = [\kappa_2 \cdot !, 0] = [1, 0].
\]

4. On the one hand the kernel of a total map is 0 since:

\[
\ker(\langle g \rangle) = [\kappa_2 \cdot !, \kappa_1] [\kappa_1 \cdot g = \kappa_2 \cdot ! g = \kappa_2 \cdot ! = 0.
\]

On the other hand, if \( \ker(f) = 0 \), then \( 1 \circ f = \ker^\perp(f) = 1 \) so that we \( f \)

is total by Lemma 7.

5. We have by Exercise 1 (2):

\[
\ker(\circ f) = (g \circ f)^\perp(0) = f^\perp(g^\perp(0)) = f^\perp(\ker(g)).
\]

6. From \( 0 \leq \ker(g) \) we get by Lemma 36 and point 5:

\[
\ker(f) = f^\perp(0) \leq f^\perp(\ker(g)) = \ker(g \circ f).
\]

7. By a straightforward computation:

\[
\ker(\langle h \rangle \circ f) = [0, \kappa_1] \cdot (h + \text{id}) \cdot f = [0, \kappa_1] \cdot f = f^\perp(0) = \ker(f).
\]

8. Internally monic maps are total by point 4, and these total map form a

subcategory by Lemma 7.

9. Let \( g \circ f \) and \( g \) be internally monic. Write \( g = \langle h \rangle \). Then \( f \) is internally

monic since by point 7,

\[
\ker(f) = \ker(\langle h \rangle \circ f) = \ker(g \circ f) = 0.
\]

10. Let \( f : X \to Y \) be a partial map and \( p \) a predicate on \( Y \). We first observe

that:

\[
1 = (1 \circ f) \odot (1 \circ f) = (p \odot p^\perp) \odot f \odot \ker(f)
\]
\[
= (p \circ f) \odot (p^\perp \circ f) \odot \ker(f)
\]
\[
= f^\perp(p^\perp) \odot f^\perp(p) \odot \ker(f).
\]

This last equation is based on Exercise 1 (3). By uniqueness of orthosupplements we obtain \( f^\perp(p^\perp) = f^\perp(p) \odot \ker(f). \)

We can now see that ‘internally monic’ does not imply ‘monic’. Indeed

‘internally monic’ means ‘total’, and not every total function in \( \text{Par(Sets)} \) is

also monic in \( \text{Par(Sets)} \), like the truth function \( \{0, 1\} \to 1 + 1 \).

More generally, it is not hard to see that a partial map \( f : X \to Y + 1 \) is

monic in the category \( \text{Par(B)} \) of partial maps of an effectus \( B \) if and only if its

Kleisli extension \( \langle f, \kappa_2 \rangle : X + 1 \to Y + 1 \) is monic in \( B \).
Lemma 40. Let $B$ be an effectus, with objects $X, Y \in B$. The kernel-supplement map

$$\text{Par}(B)(X,Y) \xrightarrow{\ker^+} \text{Pred}(X)$$

is a map of PCMs, that preserves and reflects both 0 and orthogonality $\perp$.

Moreover, this kernel-supplement map is ‘dinatural’ in the situation:

$$B^\text{op} \times B \xrightarrow{\ker^+} \text{Sets}$$

Indeed, for each map $f: X \to Y$ in $B$ we have a commuting diagram:

$$\begin{array}{ccc}
\text{Par}(B)(X,X) & \xrightarrow{\ker^+} & \text{Pred}_1(X,X) = \text{Pred}(X) \\
\text{Par}(B)(Y,X) & \xrightarrow{\ker^+} & \text{Pred}_1(X,Y) = \text{Pred}(X) \\
\text{Par}(B)(Y,Y) & \xrightarrow{\ker^+} & \text{Pred}_1(Y,Y) = \text{Pred}(Y)
\end{array}$$

Proof First, the map $\ker^+$ preserves and reflects 0, since $\ker^+(f) = 0$ iff $f = 0$ by Lemma 33 (2) — or equivalently, Lemma 7. It also preserves $\otimes$ by Proposition 16 (2): if parallel partial maps $f, g$ are orthogonal, then:

$$\ker^+(f \otimes g) = 1 \ast (f \otimes g) = (1 \ast f) \otimes (1 \ast g) = \ker^+(f) \otimes \ker^+(g).$$

The main challenge is to prove that $\ker^+(f) \perp \ker^+(g)$ implies $f \perp g$. So let $b: X \to 1 + 1$ be a bound for $\ker^+(f)$ and $\ker^+(g)$. Then $\triangleright_1 \ast b = \ker^+(f) = 1 \ast f = \langle \rangle \ast f$ and similarly $\triangleright_2 \ast b = \langle \rangle \ast g$. We use the projection pullbacks in $\text{Par}(B)$ from 10 in Lemma 3 (2) in two steps below, first to get a map $c$, and then $d$.  

But also: not all ‘external monos’ are ‘internal monos’. For instance the partial function $f: X \to D(X + 1)$ given by $f(x) = \frac{2}{7} \cdot (x) + \frac{1}{7} \ast x$ has $\ker(f)(x) = f(x)(*) = \frac{3}{7}$, so $f(x) \neq 0$ in $[0,1]^X$, and thus $f$ is not internally monic. Still $f$ is an external mono in $\text{Par}(K(D))$: if $\omega, \omega' \in D(X + 1)$ satisfy $f_\ast(\omega) = f_\ast(\omega')$, then for each $x \in X$ we have $\omega(x) = \frac{2}{7} \cdot f_\ast(\omega)(x) = \frac{3}{7} \cdot f_\ast(\omega')(x) = \omega'(x)$, and $\omega(*) = 4 \cdot f_\ast(\omega)(*) = 4 \cdot f_\ast(\omega')(*) = \omega'(*)$.

The following technical but important result about kernel-supplements, and the subsequent proposition, are due to [Cho15].
The outer diagram on the right commutes since:

\[ \triangleright_2 \circ c = \triangleright_2 \circ (\langle \triangleright \rangle + \text{id}) \circ c = \triangleright_2 \circ b = \langle \triangleright \rangle \circ g. \]

This \( d \) is a bound for \( f, g \), showing \( f \perp g \). By construction \( \triangleright_2 \circ d = g \), and:

\[ \triangleright_1 \circ d = \triangleright_1 \circ (\text{id} + \langle \triangleright \rangle) \circ d = \triangleright_1 \circ c = f. \]

Finally we check commutation of the dinaturality diagram (26): for a total map \( f: X \to Y \) and a partial map \( g \in \text{Par}(\mathcal{B})(Y, X) \) we have:

\[
\begin{align*}
(f^* \circ \text{ker}_{X}^\perp \circ (f + \text{id}) \cdot -)(g) &= f^* (\text{ker}_{X}^\perp((f + \text{id}) \cdot g)) \\
&= (f + \text{id}) \cdot (f + \text{id}) \cdot g \cdot f \\
&= (f + \text{id}) \cdot g \cdot f \\
&= \text{ker}_{X}^\perp(g \cdot f) \\
&= (\text{ker}_{X}^\perp \circ (- \cdot f))(g).
\end{align*}
\]

This reflection of orthogonality is a quite powerful property. First we use it to say more about the PCM-structure \((\otimes, 0)\) on homsets of partial maps from Proposition 13.

**Proposition 41.** Let \( \mathcal{B} \) be an effectus. Then:

1. the sum \( \otimes \) on partial homsets \( \text{Par}(\mathcal{B})(X, Y) \) is positive: \( f \otimes g = 0 \) implies \( f = g = 0 \);
2. it is also cancellative in the sense: \( f \otimes g = g \) implies \( f = 0 \);
3. \( \text{Par}(\mathcal{B})(X, Y) \) is a partial order via \( f \leq g \) iff \( f \otimes h = g \) for some \( h \); pre- and post-composition of partial maps is thus monotone.

As a result of the last point the category \( \text{Par}(\mathcal{B}) \) of partial maps is not only enriched over PCMs, but also over pointed posets (with a bottom element, preserved under composition).

**Proof** If \( f \otimes g = 0 \), then \( \text{ker}^\perp(f) \otimes \text{ker}^\perp(g) = \text{ker}^\perp(f \otimes g) = 0 \), so that \( \text{ker}^\perp(f) = \text{ker}^\perp(g) = 0 \) since the effect module \( \text{Pred}(X) \) is positive. But then \( f = g = 0 \) by Lemma 39 (2).

If \( f \otimes g = g \), then by similarly applying \( \text{ker}^\perp(-) \) we get \( \text{ker}^\perp(f) = 0 \) in \( \text{Pred}(X) \), and thus \( f = 0 \).

Obviously, the order \( \leq \) on \( \text{Par}(\mathcal{B})(X, Y) \) is reflexive and transitive. But it is also anti-symmetric: if \( f \leq g \) and \( g \leq f \), say via \( f \otimes h = g \) and \( g \otimes k = f \), then \( f \otimes (h \otimes k) = f \), so that \( h \otimes k = 0 \) by point (2), and thus \( h = k = 0 \) by point (1). Hence \( f = g \).

We include a second result that uses reflection of orthogonality by kernel-supplements. The fact that normalisation of substates holds in the two examples \( \mathcal{KL}(\mathcal{D}) \) and \( \mathcal{vNA}_{op} \) in Remark 30 is an instance of the following result due to Sean Tull.

**Lemma 42.** Let \( \mathcal{B} \) be an effectus whose scalars are given by the unit interval \([0, 1]\) of \( \mathbb{R} \). Then normalisation holds in \( \mathcal{B} \): non-zero substates can be normalised to proper states, that is, for each non-zero \( \omega: 1 \to X \) there is a unique \( \rho: 1 \to X \) with \( \omega = \langle \rho \rangle \circ r \) for the (non-zero) scalar \( r = 1 \circ \omega: 1 \to 1 \).
In particular, this means that in the context of \cite{JWW15}, where all effectuses have $[0,1]$ as scalars, normalisation comes for free.

**Proof** Let $\omega: 1 \rightarrow X$ be a non-zero substate, with corresponding scalar $r = 1 \ast \omega \in [0,1]$. Since $r \neq 0$ — by Lemma 7 — we can find an $n \in \mathbb{N}$ and $r' \in [0,1]$ with $r' \leq r$ and $n \cdot r + r' = 1$. More abstractly, we can find scalars $s_1, \ldots, s_m \in [0,1]$ with $\bigwedge_i s_i \cdot r = 1$. We now form the scalar multiplication $\omega \ast s_i: 1 \rightarrow X$ as in Lemma 23. The scalars $1 \ast \omega \ast s_i = r \ast s_i$ are orthogonal, so the maps $\omega \ast s_i$ are orthogonal too, since $1 \ast (\cdot) = \ker^+ \ast \text{reflects orthogonality}$, by Lemma 10. But then we have the following equalities of maps $1 \rightarrow 1$.

$$1 \ast \bigwedge_i (\omega \ast s_i) = \bigwedge_i 1 \ast \omega \ast s_i = \bigwedge_i r \ast s_i = 1.$$  

Lemma 7 tells that the partial map $\bigwedge_i (\omega \ast s_i): 1 \rightarrow X$ is total, so we can write it as $\langle \rho \rangle$, for a unique state $\rho: 1 \rightarrow X$. By construction we have $\langle \rho \rangle \ast 1 \ast \omega = \omega$ as in (19).

If also $\rho': 1 \rightarrow X$ satisfies $\langle \rho' \rangle \ast r = \omega = \langle \rho \rangle \ast r$, then we obtain $\rho' = \rho$ from faithfulness of $\langle \cdot \rangle: B \rightarrow \text{Par}(B)$:

$$\langle \rho \rangle = \langle \rho \rangle \ast \bigwedge_i r \ast s_i = \bigwedge_i \langle \rho \rangle \ast r \ast s_i = \bigwedge_i \langle \rho' \rangle \ast r \ast s_i = \langle \rho' \rangle. \quad \square$$

Tull’s result is a bit more general and generalises the crucial property used in the lemma to the requirement that the scalars satisfy: for each non-zero scalar $r$ there are scalars $s_1, \ldots, s_m$ with $\bigwedge_i s_i \cdot r = r = \bigotimes_i r \cdot s_i$. This condition does not apply to effectuses that have a cube $[0,1]^n$ as scalars. For instance $(r,0) \in [0,1]^2$ is non-zero, for $r \neq 0$, but there is no way to raise $(r,0)$ to the top element $(1,1) \in [0,1]^2$ via multiply-and-add. This means that the product of two effectuses with normalisation need not have normalisation.

By generalising the requirement to all sets of predicates one can normalise all non-zero partial maps, via the scalar multiplication from Lemma 67.

Once again using reflection of orthogonality we can give an alternative description of the partial pairing $\langle\langle f,g \rangle\rangle$ from Lemma 6. It exists for maps $f, g$ satisfying $1 \ast f = (1 \ast g)^{++}$. In the terminology of the present section we can rephrase this assumption as $\ker^+ (f) = \ker (g)$, or as $\ker^+ (f) \subseteq \ker^+ (g) = 1$.

**Lemma 43.** Let $f: Z \rightarrow X$ and $g: Z \rightarrow Y$ be to partial maps in an effectus with $\ker^+ (f) = \ker (g)$. Then we can describe the resulting total pairing map $\langle\langle f, g \rangle\rangle: Z \rightarrow X + Y$ as the unique one satisfying in the homset of partial maps $Z \rightarrow X + Y$:

$$\langle\langle f, g \rangle\rangle = (\kappa_1 \ast f) \cdot (\kappa_2 \ast g). \quad (27)$$

As special case of this equation we obtain, using the equation $\langle\langle \triangleright_1, \triangleright_2 \rangle\rangle = \text{id}$ from (11), a new relationship between projections and coprojections:

$$\text{id} = (\kappa_1 \ast \triangleright_1) \cdot (\kappa_2 \ast \triangleright_2).$$

More generally, there is the following bijective correspondence (from \cite{Cho15}):

$$Z \xrightarrow{f} X_1 + \cdots + X_n$$

with $\ker^+ (f_i)$ orthogonal

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Moreover, the map \( f \) above the lines is total if and only if the maps \( f_i \) below the lines satisfy \( \bigvee_i \ker^\perp(f_i) = 1 \).

**Proof** For the sum \( \bigoplus \) on the right in the equation (27) we take as bound \( b: \mathbb{Z} \rightarrow (X + Y) + (X + Y) \) the map \( b = \langle \kappa_1 + \kappa_2 \rangle \cdot \langle f,g \rangle = \langle \kappa_1 \rangle \cdot \langle f,g \rangle + \langle \kappa_2 \rangle \cdot \langle f,g \rangle \).

Then:
\[
\bigtriangledown b = \bigtriangledown (\langle \kappa_1 \rangle \cdot \langle f,g \rangle) = \langle \kappa_1 \rangle \cdot \bigtriangledown f = \kappa_1 \cdot f.
\]

Similarly we get \( \bigtriangledown g = \kappa_2 \cdot g \). Then:
\[
\langle \kappa_1 \cdot f \rangle \bigodot \langle \kappa_2 \cdot g \rangle = \nabla \cdot b = \langle \nabla \rangle \cdot \langle \kappa_1 + \kappa_2 \rangle \cdot \langle f,g \rangle = \langle \nabla \rangle \cdot \langle f,g \rangle = \langle \langle f,g \rangle \rangle.
\]

The equation \( \langle \kappa_1 \cdot \bigtriangledown \rangle \bigodot \langle \kappa_2 \cdot \bigtriangledown \rangle = \text{id} \) holds by (11), for \( k = \text{id} \).

In the bijective correspondence (28) we send a map \( f: \mathbb{Z} \rightarrow X_1 + \cdots + X_n \) to the \( n \)-tuple of maps \( f_i = \bigtriangledown_i \cdot f \). These maps are all orthogonal, via \( f \) as bound. Hence the maps \( \ker^\perp(f_i) = 1 \circ f_i \) are also orthogonal. If \( f \) is total, then by an \( n \)-ary version of (27),
\[
1 = 1 \cdot f = 1 \cdot \bigodot_i (\kappa_i \cdot f_i) = \bigodot_i 1 \cdot \kappa_i \cdot f_i = \bigodot_i 1 \cdot f_i = \bigodot_i \ker(f_i).
\]

In the other direction, let \( f_i: \mathbb{Z} \rightarrow X_i \) be maps for which the kernel-supplements \( \ker^\perp(f_i) = 1 \circ f_i: \mathbb{Z} \rightarrow 1 \) are orthogonal. The maps \( \kappa_i \circ f_i: \mathbb{Z} \rightarrow X_1 + \cdots + X_n \) are then orthogonal too since the maps \( 1 \circ \kappa_i \circ f_i = 1 \circ f_i \) are orthogonal and \( \ker^\perp = 1 \circ (\cdot) \) reflects orthogonality, by Lemma 40. Hence we take \( f = \bigodot_i (\kappa_i \cdot f_i) \).

This map \( f \) is total if \( \bigodot_i \ker(f_i) = 1 \) by following the previous chain of equations backwards.

We get a bijective correspondence since \( \bigodot_i (\kappa_i \cdot \bigtriangledown \cdot f) = f \) like in (27), and:
\[
\bigtriangledown_j \circ \bigodot_i (\kappa_i \cdot f_i) = \bigodot_i (\bigtriangledown_j \circ \kappa_i \cdot f_i) = \bigodot_i \left\{ \begin{array}{ll} 1 \circ f_i & \text{if } j = i \\ 0 & \text{if } j \neq i \end{array} \right\} = f_j. \quad \square
\]

The bijective correspondence (28) extends the pairing from Lemma 8 to \( n \)-ary form and to partial maps, see the end for Discussion 54 for more details.

### 7.2 Images

A kernel is a predicate on the domain of a partial map. An image is a predicate on its codomain. An effectus always has kernels. But the existence of images must be required explicitly.

There is one more notion that we need in the description of images. In an effect algebra an element \( s \) is called *sharp* if \( s \land s^+ = 0 \). One may argue that the meet \( \land \) may not exist. But the definition of sharpness only requires that the particular meet \( s \land s^+ \) exists and is equal to 0. Equivalently, without meets: \( s \) is sharp if for each element \( x \) one has: \( x \leq s \) and \( x \leq s^+ \) implies \( x = 0 \). Notice that 0 and 1 are sharp elements, and that if \( s \) is sharp then its orthosupplement \( s^+ \) is sharp too.
Definition 44. We say that an effectus has images if for each partial map \( f : X \rightarrow Y \) there is a least predicate \( q \) on \( Y \) with \( f^\phi(q) = 1 \). In that case we write \( \text{im}(f) \) for this predicate \( q \). We say that the effectus has sharp images if these image predicates \( \text{im}(f) \) are sharp.

Like kernel-supplements \( \ker^\perp \) we also uses image-complements \( \text{im}^\perp \) defined as \( \text{im}^\perp(f) = \text{im}(f)^\perp \).

We call \( f \) an internal epi if \( \text{im}(f) = 1 \).

In all our examples images are sharp. Since many basic results about images can be proven without assuming sharpness, we shall not use sharp images until we really need them.

Let’s see if we have images in our running examples.

Example 45. In the effectus \( \text{Sets} \), each partial map \( f : X \rightarrow Y \) has an image, namely the subset of \( Y \) given by:

\[
\text{im}(f) = \{ y \in Y \mid \exists x \in X. f(x) = \kappa_1 y \}.
\]

It is easy to see that \( \text{im}(f) \) is the least subset \( V \subseteq Y \) with \( f^\phi(V) = 1 \), using the definition of \( f^\phi \) from \[23\].

The effectus \( \text{Kl}(D) \) also has images: for a partial map \( f : X \rightarrow D(Y + 1) \) take the (sharp) predicate \( \text{im}(f) : Y \rightarrow [0,1] \) given by:

\[
\text{im}(f)(y) = \begin{cases} 1 & \text{if there is an } x \in X \text{ with } f(x)(y) > 0 \\ 0 & \text{otherwise.} \end{cases}
\]

We have, using the description of \( f^\phi \) from \[22\],

\[
f^\phi(\text{im}(f))(x) = \sum_y f(x)(y) \cdot \text{im}(f)(y) + f(x)(\bot) = \sum_y f(x)(y) + f(x)(\bot) = 1.
\]

Further, if \( q \in [0,1]^Y \) satisfies \( f^\phi(q) = 1 \), then \( \sum_y f(x)(y) \cdot q(y) + f(x)(\bot) = 1 \) for each \( x \in X \). But this can only happen if \( q(y) = 1 \) if \( f(x)(y) > 0 \), that is, if \( \text{im}(f) \leq q \).

In the effectus \( \text{OUG}^{\text{op}} \) of order unit groups, images need not exist. To see this, let \( A = [0,1]^2 \) be the unit square, which is clearly a convex set. The set \( \text{Aff}(A) \) of bounded affine functions \( A \rightarrow \mathbb{R} \) forms an order unit group with coordinate-wise operations and order. Let \( \varphi : \text{Aff}(A) \rightarrow \mathbb{R} \) be the map given by \( \varphi(f) = f(0,0) \) for all \( f \in \text{Aff}(A) \). We claim that \( \varphi \), seen as arrow \( R \rightarrow \text{Aff}(A) \) in \( \text{OUG}^{\text{op}} \), has no image. Towards a contradiction, let \( \varphi \) have image \( f = \text{im}(\varphi) \) in the unit interval of predicates \([0,1]_{\text{Aff}(A)}\), see Example \[10\]. Then:

\[
1 = \varphi^\phi(f) = \varphi(f) + \varphi(1)^\perp = f(0,0) + 1(0,0)^\perp = f(0,0).
\]

Consider the functions \( f_1, f_2 \in \text{Aff}(A) \) given by \( f_1(x,y) = 1 - x \) and \( f_2(x,y) = 1 - y \). Clearly, \( \varphi^\phi(f_1) = f_1(0,0) = 1 \) and \( \varphi^\phi(f_2) = f_2(0,0) = 1 \). Hence \( f \leq f_1, f_2 \) by minimality of images. Then \( f(1,y) \leq f_1(1,y) = 0 \) and \( f(x,1) \leq f_2(x,1) = 0 \). In particular \( f(1,1) = 0 \). We now consider the middle point.
\( \left( \frac{1}{2}, \frac{1}{2} \right) \in A \). It can be written in two ways as convex combination of extreme points, namely as:

\[
\frac{1}{2}(1, 0) + \frac{1}{2}(0, 1) \overset{(a)}{=} \left( \frac{1}{2}, \frac{1}{2} \right) \overset{(b)}{=} \frac{1}{2}(0, 0) + \frac{1}{2}(1, 1).
\]

By applying the affine function \( f \) on both sides we obtain a contradiction. Starting from the above equation \((a)\) we get:

\[
f\left( \frac{1}{2}, \frac{1}{2} \right) = f\left( \frac{1}{2}(1, 0) + \frac{1}{2}(0, 1) \right) = \frac{1}{2}f(1, 0) + \frac{1}{2}f(0, 1) = 0.
\]

But starting from the equation \((b)\) we obtain a different outcome:

\[
f\left( \frac{1}{2}, \frac{1}{2} \right) = f\left( \frac{1}{2}(0, 0) + \frac{1}{2}(1, 1) \right) = \frac{1}{2}f(0, 0) + \frac{1}{2}f(1, 1) = \frac{1}{2}.
\]

The conclusion is that the map \( \varphi \) in the effectus \( \text{OUG}^{\text{op}} \) has no image.

In the effectus \( \text{vNA}^{\text{op}} \) of von Neumann algebra the image of a subunital map \( f: \mathcal{A} \to \mathcal{B} \) does exist, and is given by the following (sharp) effect of \( \mathcal{A} \).

\[
im(f) = \bigwedge \{ p \in \mathcal{A} \mid p \text{ is a projection with } f(p) = f(1) \}. \tag{29}
\]

Here we use that projections are the sharp elements and form a complete lattice, see e.g. [Sak71, Prop. 1.10.2]. One can prove: \( f(a) = 0 \) implies \( \im(f) \leq a^\perp \), for an arbitrary element \( a \in [0, 1]_{\mathcal{A}} \), see also Lemma 47 (3) below.

It can be shown that in a von Neumann algebra the sharp elements are precisely the projections, that is, the effects \( e \) with \( e \cdot e = e \). Later on, in Proposition 106 (3), we prove this in abstract form.

In general, images need not exist in an effectus, but they do exist for a few special maps: identity maps, coprojections, and partial projections.

**Lemma 46.** In the category \( \text{Par}(\mathcal{B}) \) of partial maps of an effectus \( \mathcal{B} \),

1. \( \im(\id) = 1 \), for the identity map \( \id: X \to X \);

2. \( \im(0) = 0 \), for the zero map \( 0: X \to Y \);

3. \( \im(\kappa_1) = [1, 0] \) and \( \im(\kappa_2) = [0, 1] \);

4. \( \im(\triangleright_1) = 1 \) and also \( \im(\triangleright_2) = 1 \), so that the partial projections \( \triangleright_1 \) are internally epic.

**Proof** We show each time that the claimed predicate has the universal property of an image (the least sharp one such that \( \ldots \)).

1. By Exercise 12 (2) we have \( \id\triangleright(p) = p \). Hence \( \id\triangleright(p) = 1 \) iff \( p = 1 \), so that \( \im(\id) = 1 \).

2. Let \( 0\triangleright(p) = 1 \). Then \( 1 = [p, \kappa_1] \cdot \kappa_2 \cdot ! = \kappa_1 \cdot ! \). This equation imposes no restrictions on \( p \), so the least predicate for which this holds is the falsity predicate \( 0 \). Hence \( \im(0) = 0 \).

3. Formally a coprojection in \( \text{Par}(\mathcal{B}) \) is of the form \( \langle \kappa_1 \rangle = \kappa_1 \cdot \kappa_\ast \). By Exercise 13 (3) we have \( \langle \kappa_1 \rangle \circ ([1, 0]) = \kappa_\ast([1, 0]) = [1, 0] \cdot \kappa_1 = 1 \). Further, if also \( \langle \kappa_1 \rangle \circ(p) = 1 \), then \( p \cdot \kappa_1 = 1 \). Since \( p \circ \kappa_2 \geq 0 \), we get \( p \geq [1, 0] \). Similarly one shows that \( \im(\kappa_2) = [0, 1] \).

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4. Let `$\triangleright^0_1(p) = 1$`. Then `$1 = [p, \kappa_1] \cdot \triangleright_1 = [p, \kappa_1] \cdot (\text{id} + !) = [p, \kappa_1 \cdot !] = [p, 1]$`. But then `p = [p, 1] \cdot \kappa_1 = 1 \cdot \kappa_1 = 1`. The `$\triangleright_2$`-case is left to the reader. □

Having seen this result we can say that the following is an ‘internal’ short exact sequence in the category of partial maps an effectus.

$$
0 \longrightarrow X \xrightarrow{\kappa_1} X + Y \xrightarrow{\triangleright_2} Y \longrightarrow 0 \quad (30)
$$

Exactness means that the image of one map is the kernel of the next one. This works because the coprojections are total in the category of partial maps, and thus internally monic, the partial projections are internally epic, and `im(\kappa_1) = [1, 0] = \ker(\triangleright_2)`. But there is more: the sequence (30) involves two splittings, making $\kappa_1$ a split mono, and $\triangleright_2$ a split epi:

$$
0 \longrightarrow X \xrightarrow{\kappa_1} X + Y \xrightarrow{\triangleright_2} Y \xrightarrow{\kappa_2} Y \longrightarrow 0 \quad (31)
$$

This is a reformulation of the ‘butterfly’ diagram (4).

**Lemma 47.** In an effectus $\mathcal{B}$ with images one has:

1. `ker(f) = f^\Box (im^\perp(f))`;
2. `im(f) = 0` iff `ker(f) = 1`;
3. `$p \ast f = 0$ iff $p \leq im^\perp(f)$; in particular, $im^\perp(f) \ast f = 0$;
4. `im(g \ast f) \leq im(g)`;
5. `im(f) \leq g^\Box (im(g \ast f))`;
6. `im(g \ast f) = im(g)` if $f$ is an internal epi;
7. `$g \ast f = 0$ iff `im(f) \leq ker(g)`;
8. if $g \ast f$ and $f$ are internally epic, then so is $g$;
9. internal epis form a subcategory;
10. a partial map $f$ is internally epic iff for each predicate $q$ on its codomain: $f^\Box(q) = 1$ $\iff$ $q = 1$;
11. ‘external’ epis are internal epis: each epi $f$ in $\text{Par}(\mathcal{B})$ satisfies `im(f) = 1`;
12. `im([f, g]) = im(f) \lor im(g)`, for predicates $f, g$ with the same codomain.

**Proof** All of the points are obtained by elementary arguments.

1. The equation `ker(f) = f^\Box (im^\perp(f))` follows directly from Lemma 39 (10):

```
\text{f}^\Box(\text{im}^\perp(f)) = \text{f}^\Box(\text{im}(f))^\perp \otimes \text{ker}(f) = 1^\perp \otimes \text{ker}(f) = \text{ker}(f).
```

2. If `im(f) = 0`, then by the previous point:

```
\text{ker}(f) = \text{f}^\Box(\text{im}^\perp(f)) = \text{f}^\Box(0^\perp) = \text{f}^\Box(1) = 1.
```

Conversely, if $1 = \ker(f) = f^\Box(0)$, then $\text{im}(f) \leq 0$ since it is minimal.
3. First, we have:
\[
\text{im}^\bot(f) \circ f = [\text{im}^\bot(f), \kappa_2] \circ f = [\kappa_2, \kappa_1] \cdot [\text{im}(f), \kappa_1] \cdot f = f^\bot(\text{im}(f))^\bot = 1^\bot = 0.
\]
Hence if \( p \leq \text{im}^\bot(f) \), then \( p \circ f \leq \text{im}^\bot(f) \circ f = 0 \), using Lemma \[10\].
In the other direction, if \( p \circ f = 0 \), then:
\[
f^\bot(p^\bot) = (p \circ f)^{\bot} = 0^\bot = 1.
\]
Hence \( \text{im}(f) \leq p^\bot \) by minimality of images, and so \( p \leq \text{im}(f)^\bot = \text{im}^\bot(f) \).

4. In order to prove \( \text{im}(g \circ f) \leq \text{im}(g) \) it suffices to show \( (g \circ f)^{\bot}(\text{im}(g)) = 1 \).
But the latter is easy, since \( (g \circ f)^{\bot}(\text{im}(g)) = f^{\bot}(g^{\bot}(\text{im}(g))) = f^{\bot}(1) = 1 \).

5. The inequality \( \text{im}(f) \leq g^{\bot}(\text{im}(g \circ f)) \) follows by minimality of images from:
\[
f^{\bot}(g^{\bot}(\text{im}(g \circ f))) = (g \circ f)^{\bot}(\text{im}(g \circ f)) = 1.
\]
6. By point \[4\] we only have to prove \( \text{im}(g \circ f) \geq \text{im}(g) \). This follows if we can show \( g^{\bot}(\text{im}(g \circ f)) = 1 \). But this is a consequence of the previous point, since \( \text{im}(f) = 1 \) because \( f \) is by assumption an internal epi.

7. Let \( g \circ f = 0 \). Then, by Lemma \[39\] \[5\], we have \( f^{\bot}(\ker(g)) = \ker(0) = 1 \), so that \( \text{im}(f) \leq \ker(g) \). Conversely, if \( \text{im}(f) \leq \ker(g) \), then \( 1 = f^{\bot}(\text{im}(f)) \leq f^{\bot}(\ker(g)) = \ker(g \circ f) \). But then \( g \circ f = 0 \) by Lemma \[39\] \[2\].

8. Let \( g \circ f \) and \( f \) be internally epic. Then \( g \) is internally epic too, by point \[6\] since: \( \text{im}(g) = \text{im}(g \circ f) = 1 \).

9. Lemma \[40\] \[1\] says that the partial identity is internally epic. And if \( g, f \) are internal epis, then by point \[4\], \( g \circ f \) too, since \( \text{im}(g \circ f) = \text{im}(g) = 1 \).

10. Let \( f \) be internally epic. Clearly, if \( q = 1 \), then \( f^{\bot}(q) = f^{\bot}(1) = 1 \). In the other direction, if \( f^{\bot}(q) = 1 \), then, by minimality of images, \( 1 = \text{im}(f) \leq q \).

Conversely, assume \( f^{\bot}(q) = 1 \iff q = 1 \) for each predicate \( q \). Take \( q = \text{im}(f) \); since by definition \( f^{\bot}(\text{im}(f)) = 1 \), we get \( q = \text{im}(f) = 1 \), so that \( f \) is internally epic.

11. Let \( f : X \to Y \) be epic in \( \text{Par}(B) \). By point \[3\] we have \( \text{im}^\bot(f) \circ f = 0 \).
But also \( 0 \circ f = 0 \). Hence \( \text{im}^\bot(f) = 0 \) since \( f \) is epic, and thus \( \text{im}(f) = 1 \), so that \( f \) is internally epic.

12. We use point \[3\] to show that for an arbitrary predicate \( p \),
\[
\text{im}([f, g]) \leq p \iff p^\bot \leq \text{im}^\bot([f, g])
\iff p^\bot \circ [f, g] = [p^\bot \circ f, p^\bot \circ g] = 0
\iff p^\bot \circ f = 0 \text{ and } p^\bot \circ g = 0
\iff p^\bot \leq \text{im}^\bot(f) = 0 \text{ and } p^\bot \leq \text{im}^\bot(g) = 0
\iff \text{im}(f) \leq p \text{ and } \text{im}(g) \leq p.
\]
8 Relating total and partial maps

This section contains the main result of [Cho15], giving a precise relation between total and partial maps in an effectus. Briefly certain requirements \( \mathcal{R} \) are identified so that:

- for a category \( C \) satisfying \( \mathcal{R} \), the subcategory \( \text{Tot}(C) \) of ‘total’ maps in \( C \) is an effectus, and \( \text{Par}(\text{Tot}(C)) \cong C \);
- for an effectus \( B \), the category of partial maps \( \text{Par}(B) \) satisfies \( \mathcal{R} \), and \( \text{Tot}(\text{Par}(B)) \cong B \).

A category satisfying \( \mathcal{R} \) is called a \textit{FinPAC with effects} in [Cho15]. This terminology is explained below.

This correspondence is made precise in [Cho15] in the form of a 2-equivalence of 2-categories. Here however, we discuss the essentials in more concrete form, and refer the reader to \textit{loc. cit.} for further details. This close correspondence between total and partial maps can be seen as a confirmation of the appropriateness of the notion of effectus. The correspondence and its consequences — for notation and terminology — are discussed at the end of this section.

We first have to explore the notion of finitely partially additive category, or FinPAC. This is based on the notion of partially additive category, introduced in [AM80].

8.1 FinPACs

Recall that an arbitrary category \( C \) is PCM-enriched if all its homsets are PCM\((\mathbb{0}, \mathbb{0})\)-enriched categories and pre- and post-composition preserve the PCM-structure.

If such a category \( C \) has coproducts \(+\), then we can define ‘projections’ \( \triangleright_i \) via the zero map \( \mathbb{0} \), like in (2), as:

\[
\begin{array}{c}
X \\
\triangleright_1 = [\text{id}, \mathbb{0}] \\
X + Y \\
\triangleright_2 = [\mathbb{0}, \text{id}] \\
Y
\end{array}
\]  \hspace{1cm} (32)

These projections are automatically natural: \( \triangleright_1 \circ (f_1 + f_2) = f_1 \circ \triangleright_1 \).

Since \( C \) is any category — not necessarily an effectus — we shall use ordinary notation, like \( \circ \) and \( \to \), for composition and maps, and not the special notation \( , , \to , \to \) for effectuses.

**Definition 48.** A finitely partially additive category (a FinPAC, for short) is a PCM-enriched category \( C \) with finite coproducts \(+, \mathbb{0}) satisfying both the:

- **Compatible Sum Axiom:** if there is a bound \( b: X \to Y + Y \) for maps \( f, g: X \to Y \) with \( \triangleright_1 \circ b = f \) and \( \triangleright_2 \circ b = g \), then \( f \perp g \) in the PCM \( C(X, Y) \);

- **Untying Axiom:** if \( f \perp g \) then \( (\kappa_1 \circ f) \perp (\kappa_2 \circ g) \).

The names for these two axioms come from the theory of partially additive categories, see [AM80] or [Hag00]. The maps \( \triangleright_i \) are often called \textit{quasi} projections in that context.

We need some basic results about FinPACs, following [AM80] [Cho15].

**Lemma 49.** In a FinPAC \( C \),
1. The initial object \( 0 \in C \) is also final, and thus a zero object; the resulting zero map \( X \to 0 \to Y \) is the zero element \( 0 \) of the PCM-structure on the homset \( C(X,Y) \);

2. the maps \( \kappa_1 \circ \triangleright_1, \kappa_2 \circ \triangleright_2 : X + Y \to X + Y \) are orthogonal with sum \((\kappa_1 \circ \triangleright_1) \oplus (\kappa_2 \circ \triangleright_2) = \text{id}; \)

3. any map \( f : Z \to X+Y \) can be written as sum \( f = (\kappa_1 \circ \triangleright_1 \circ f) \oplus (\kappa_2 \circ \triangleright_2 \circ f) \);

4. the two projection maps \( \triangleright_1 : X_1 + X_2 \to X_1 \) are jointly monic;

5. \( f_1 \perp f_2 \) iff there is a necessarily unique bound \( b \) with \( \triangleright_i \circ b = f_i \) and \( f_1 \circ f_2 = \nabla \circ b \).

**Proof** For the sake of completeness, we include the details.

1. For each object \( X \in C \) there is a map \( 0 : X \to 0 \), namely the PCM-zero. It is the only such map, since each \( f : X \to 0 \) satisfies: \( f = \text{id} \circ f = 0 \circ f = 0 \).

   The resulting map \( X \to 0 \to Y \) is thus \( !\circ 0 = 0 \).

2. Take as bound \( b = \kappa_1 + \kappa_2 : X + Y \to (X + Y) + (X + Y) \). Then:

   \[
   \triangleright_1 \circ b = [\text{id}, 0] \circ (\kappa_1 + \kappa_2) = [\kappa_1, 0] = \kappa_1 \circ [\text{id}, 0] = \kappa_1 \circ \triangleright_1.
   \]

   Similarly one obtains \( \triangleright_2 \circ b = \kappa_2 \circ \triangleright_2 \). This gives \((\kappa_1 \circ \triangleright_1) \perp (\kappa_2 \circ \triangleright_2)\) by the Compatible Sum Axiom. We take their sum \( s = (\kappa_1 \circ \triangleright_1) \oplus (\kappa_2 \circ \triangleright_2) \) in the homset of maps \( X + Y \to X + Y \) and obtain \( s = \text{id} \) from \( s \circ \kappa_i = \kappa_i \), as in:

   \[
   s \circ \kappa_2 = ((\kappa_1 \circ \triangleright_1) \oplus (\kappa_2 \circ \triangleright_2)) \circ \kappa_2 \\
   = (\kappa_1 \circ [\text{id}, 0] \circ \kappa_2) \oplus (\kappa_2 \circ [0, \text{id}] \circ \kappa_2) \\
   = (\kappa_1 \circ 0) \circ \kappa_2 \\
   = 0 \circ \kappa_2 \\
   = \kappa_2.
   \]

3. Directly by the previous point, for \( f : Z \to X + Y \),

   \[
   f = \text{id} \circ f = ((\kappa_1 \circ \triangleright_1) \oplus (\kappa_2 \circ \triangleright_2)) \circ f = (\kappa_1 \circ \triangleright_1 \circ f) \oplus (\kappa_2 \circ \triangleright_2 \circ f).
   \]

4. Suppose \( f, g : Z \to X_1 + X_2 \) satisfy \( \triangleright_i \circ f = \triangleright_i \circ g \), for \( i = 1, 2 \). Then, by the previous point:

   \[
   f = (\kappa_1 \circ \triangleright_1 \circ f) \oplus (\kappa_2 \circ \triangleright_2 \circ f) = (\kappa_1 \circ \triangleright_1 \circ g) \oplus (\kappa_2 \circ \triangleright_2 \circ g) = g.
   \]

5. If a bound exists, then it is unique because the \( \triangleright_i \) are jointly monic.

   The Compatible Sum Axiom says that existence of a bound \( b \) for \( f_1, f_2 \) gives \( f_1 \perp f_2 \). We have to prove the converse. So let \( f_1 \perp f_2 \), and thus \((\kappa_1 \circ f_1) \perp (\kappa_2 \circ f_2)\) by the Untying Axiom. We take \( b = (\kappa_1 \circ f_1) \oplus (\kappa_2 \circ f_2) \).

   Then \( \triangleright_i \circ b = f_i \), making \( b \) a bound. Moreover:

   \[
   \nabla \circ b = (\nabla \circ \kappa_1 \circ f_1) \oplus (\nabla \circ \kappa_2 \circ f_2) = f_1 \circ f_2.
   \]

\[59\]
The next result shows the relevance of FinPACs in the current setting.

**Lemma 50.** The category Par\(\mathcal{B}\) of partial maps in an effectus \(\mathcal{B}\) is a FinPAC.

**Proof** We know that Par\(\mathcal{B}\) is enriched over PCM by Proposition 13, and inherits coproducts from \(\mathcal{B}\), like any Kleisli category. The Compatible Sum Axiom holds by definition of orthogonality, see Proposition 13 (1). For the Untying Axiom, let \(f_1, f_2: X \to Y\) satisfy \(f_1 \perp f_2\) via bound \(b: X \to Y + Y\). Then \(c = (\kappa_1 + \kappa_2) \circ b: X \to (Y + Y) + (Y + Y)\) is a bound for \(\kappa_i \circ f_i\) since:

\[
\triangleright_i \circ c = \triangleright_i \circ (\kappa_1 + \kappa_2) \circ b = \kappa_i \circ \triangleright_i \circ b = \kappa_i \circ f_i.
\]

\(\square\)

### 8.2 FinPACs with effects

We now come to the axiomatisation of the category of partial maps in an effectus. We use the name ‘FinPAC with effects’ from [Cho15]. This is a temporary name, see Discussion 54 below.

**Definition 51.** A category \(\mathcal{C}\) is called a FinPAC with effects if it is a FinPAC with a special object \(I \in \mathcal{C}\) such that:

1. the homset \(\mathcal{C}(X, I)\) is not only a PCM, but an effect algebra, for each object \(X \in \mathcal{C}\);
2. the top/truth element \(1 \in \mathcal{C}(X, I)\) satisfies: for all \(f, g: Y \to X\),

\[
(1 \circ f) \perp (1 \circ g) \implies f \perp g
\]

3. the bottom/falsity element \(0 \in \mathcal{C}(X, I)\) satisfies: for all \(f: Y \to X\),

\[
1 \circ f = 0 \implies f = 0.
\]

These last two points say that the function \(1 \circ (\cdot)\) reflects orthogonality and zero.

A map \(f: X \to Y\) in such a FinPAC with effects is called total if \(1_Y \circ f = 1_X\). We write \(\operatorname{Tot}(\mathcal{C}) \hookrightarrow \mathcal{C}\) for the ‘wide’ subcategory (with the same objects) of total maps in \(\mathcal{C}\).

These top maps \(1: X \to I\) resemble the ground maps \(\dag: X \to I\) in Definition 31. Recall that causal maps \(f\) satisfy \(\dag \circ f = \dag\). The corresponding property \(1 \circ f = 1\) describes the total maps, as defined above.

Before arriving at the main result of this section, we collect a few facts about FinPACs with effects.

**Lemma 52.** In an FinPAC with effects \((\mathcal{C}, I)\),

1. split monics are total, so in particular all coprojections and isomorphisms are total;
2. \(\operatorname{id}_I = 1: I \to I\); as a result, \(I \in \mathcal{C}\) is final in \(\operatorname{Tot}(\mathcal{C})\);
3. \([1, 1] = 1: X + Y \to I\);
4. for each total map \( f : X \to Y \) pre-composition \((-) \circ f : \text{C}(Y, I) \to \text{C}(X, I)\) is a map of effect algebras;

5. \( \text{Tot}(\text{C}) \) inherits finite coproducts \((+, 0)\) from \( \text{C} \).

**Proof** We reason in the category \( \text{C} \).

1. Let \( f \circ m = \text{id} \), making \( m \) a split monic. We have \( 1 \circ f \leq 1 \), since \( 1 \) is by definition the top element. Hence by post-composing with \( m \) we get: \( 1 = 1 \circ f \circ m \leq 1 \circ m \), so that \( 1 = 1 \circ m \). Coprojections \( \kappa_i \) are split monics in \( \text{C} \), since \( \triangleright_i \circ \kappa_i = \text{id} \).

2. In the effect algebra \( \text{C}(I, I) \) we have \( \text{id} \perp \text{id}^\perp \). Hence \( (\text{id} \circ \text{id}) \perp (\text{id} \circ \text{id}^\perp) \) since post-composition is a PCM-map. But then \( 1 \circ \text{id}^\perp = 0 \) by Definition 51 (2). This gives \( \text{id}^\perp = 0 \) by Definition 51 (3), and thus \( \text{id} = 1 \).

For each object \( X \) there is a total map \( 1_X : X \to I \), since \( 1_X = \text{id}_I \circ 1_X = 1_I \circ 1_X \). If \( f : X \to I \) is total, then \( f = \text{id}_I \circ f = 1_I \circ f = 1_X \).

3. For the equation \([1, 1] = 1 : X + Y \to I\) we use that coprojections are total, by point (1):

\[
1 = 1 \circ [\kappa_1, \kappa_2] = [1 \circ \kappa_1, 1 \circ \kappa_2] = [1, 1].
\]

4. The map \((-) \circ f \) preserves the PCM-structure by definition. And it preserves truth \( 1 \) since \( f \) is total. Hence it is a map of effect algebras, see Definition 15.

5. The object \( 0 \) is initial in \( \text{Tot}(\text{C}) \), since the unique map \( ! : 0 \to X \) is total: \( 1_X \circ !_X = !_I = 1_0 \) by initiality in \( \text{C} \). Coprojections are total by point (1).

If \( f, g \) are total, then so is \([f, g] \) since \( 1 \circ [f, g] = [1 \circ f, 1 \circ g] = [1, 1] = 1 \) by point (1). \( \square \)

We now come to the main result of this section.

**Theorem 53.** (From \[Cho15\])

1. For an effectus \( \text{B} \), the category of partial maps \( \text{Par}(\text{B}) \) with special object \( 1 \) is a FinPAC with effects, and \( \text{Tot}(\text{Par}(\text{B})) \cong \text{B} \).

2. For a FinPAC with effects \( (\text{C}, I) \), the subcategory \( \text{Tot}(\text{C}) \) of total maps is an effectus, and \( \text{Par}(\text{Tot}(\text{C})) \cong \text{C} \).

**Proof** Let \( \text{B} \) be an effectus. Lemma 50 tells that \( \text{Par}(\text{B}) \) is a FinPAC. We take \( I = 1 \), so that \( \text{Par}(\text{B})(X, 1) = \text{B}(X, 1 + 1) = \text{Pred}(X) \) is an effect algebra. Next if \( 1 \circ f = \ker^+(f) \perp \ker^+(g) = 1 \circ g \), then \( f \perp g \) since \( \ker^+ \) reflects orthogonality — and zero too — by Lemma 10. Reflection of zero proves requirement (3) in Definition 51. In order to prove \( \text{Tot}(\text{Par}(\text{B})) \cong \text{B} \) we have to prove that a map \( f : X \to Y \) is total iff \( 1 \circ f = \text{1} \). But we already know this from Lemma 7.

For the second point, let \( (\text{C}, I) \) be a FinPAC with effects. From Lemma 52 we know that the category \( \text{Tot}(\text{C}) \) of total maps has \( I \) as final object, and has coproducts \((+, 0)\) as in \( \text{C} \). We first show that the rectangles in Definition 2 (1)
are pullbacks in $\text{Tot}(\mathcal{C})$. We may thus assume that we have total maps $f, g, h$ in commuting (outer) diagrams:

\[
\begin{array}{c}
Z \\
\downarrow^k \\
X + Y \\
\downarrow^g \quad \downarrow^{1+id} \\
I + Y \\
\end{array}
\quad \quad
\begin{array}{c}
W \\
\downarrow^h \quad \downarrow^{1+id} \\
X + Z \\
\downarrow^{\kappa_1} \\
X + I \\
\end{array}
\]

We first concentrate on the situation on the left. By assumption, $(1 + \text{id}) \circ f = (1 + 1) \circ g = b$, say. Then $(\nabla_2 \circ b) \perp (\nabla_2 \circ b)$, by definition of orthogonality. But:

\[
\nabla_1 \circ b = (\nabla_1 \circ (1 + \text{id}) \circ f = 1 \circ \nabla_1 \circ f = 1 \circ \kappa_1 \circ \nabla_1 \circ f \\
\nabla_2 \circ b = (\nabla_2 \circ (1 + \text{id}) \circ g = 1 \circ \nabla_2 \circ g = 1 \circ \kappa_2 \circ \nabla_2 \circ g.
\]

Hence we have $(1 \circ \kappa_1 \circ \nabla_1 \circ f) \perp (1 \circ \kappa_2 \circ \nabla_2 \circ g)$, from which we can conclude $(\kappa_1 \circ \nabla_1 \circ f) \perp (\kappa_2 \circ \nabla_2 \circ g)$. Thus we can define:

\[
k = (\kappa_1 \circ \nabla_1 \circ f) \odot (\kappa_2 \circ \nabla_2 \circ g) : W \to X + Y.
\]

Then:

\[
(\text{id} + 1) \circ k = ((\text{id} + 1) \circ \kappa_1 \circ \nabla_1 \circ f) \odot ((\text{id} + 1) \circ \kappa_2 \circ \nabla_2 \circ g) \\
= (\kappa_1 \circ \nabla_1 \circ f) \odot (\kappa_2 \circ 1 \circ \nabla_2 \circ g) \\
= (\kappa_1 \circ \nabla_1 \circ f) \odot (\kappa_2 \circ \nabla_2 \circ (1 + \text{id}) \circ g) \\
= (\kappa_1 \circ \nabla_1 \circ f) \odot (\kappa_2 \circ \nabla_2 \circ (1 + \text{id}) \circ f) \\
= (\kappa_1 \circ \nabla_1 \circ f) \odot (\kappa_2 \circ \nabla_2 \circ f) \\
= f \quad \text{by Lemma 51 (3)}.
\]

Similarly one proves $(1 + \text{id}) \circ k = g$. For uniqueness, let $\ell : W \to X + Y$ also satisfy $(\text{id} + 1) \circ \ell = f$ and $(1 + \text{id}) \circ \ell = g$, then:

\[
k = (\kappa_1 \circ \nabla_1 \circ f) \odot (\kappa_2 \circ \nabla_2 \circ g) \\
= (\kappa_1 \circ \nabla_1 \circ (1 + \text{id}) \circ \ell) \odot (\kappa_2 \circ \nabla_2 \circ (1 + \text{id}) \circ \ell) \\
= (\kappa_1 \circ \nabla_1 \circ \ell) \odot (\kappa_2 \circ \nabla_2 \circ \ell) \\
= \ell.
\]

In the above diagram on the right we have $(1 + 1) \circ h = \kappa_1 \circ 1$. We claim $\nabla_2 \circ h = 0$. This follows by Definition 51 (3) from

\[
1 \circ \nabla_2 \circ h = \nabla_2 \circ (1 + 1) \circ h = \nabla_2 \circ \kappa_1 \circ 1 = 0 \circ 1 = 0.
\]

The map $h : W \to X + Y$ then satisfies, by Lemma 51 (3),

\[
h = (\kappa_1 \circ \nabla_1 \circ h) \odot (\kappa_2 \circ \nabla_2 \circ h) = (\kappa_1 \circ \nabla_1 \circ h) \odot (\kappa_2 \circ 0) = \kappa_1 \circ \nabla_1 \circ h.
\]
Hence \( \triangleright_1 \circ h : W \to X \) is a mediating map. It is the unique one, since if also \( k : W \to X \) satisfies \( k_1 \circ k = h \), then \( \triangleright_1 \circ h = \triangleright_1 \circ k_1 \circ k = k \).

We still have to prove that the two maps \( W, W : (I + I) + I \to I + I \) are jointly monic in \( \text{Tot}(C) \), where \( W = \text{id}, \kappa_2 \) and \( W = [\kappa_2, \kappa_1], \kappa_2 \). So let \( f, g : X \to (I + I) + I \) satisfy \( W \circ f = W \circ g \) and \( W \circ f = W \circ g \). Write:

\[
\begin{align*}
f_1 &= \triangleright_1 \circ \triangleright_1 \circ f, \\
f_2 &= \triangleright_2 \circ \triangleright_1 \circ f, \\
f_3 &= \triangleright_2 \circ f.
\end{align*}
\]

Then \( f = (\kappa_1 \circ \kappa_1 \circ f_1) \circ (\kappa_1 \circ \kappa_2 \circ f_1) \circ (\kappa_2 \circ f_3) \). We can write the map \( g \) in a similar manner. The equation \( W \circ f = W \circ g \) yields,

\[
(\kappa_1 \circ f_1) \circ (\kappa_2 \circ f_2) \circ (\kappa_2 \circ f_3) = (\kappa_2 \circ g_1) \circ (\kappa_1 \circ g_2) \circ (\kappa_2 \circ g_3).
\]

Hence by post-composing with \( \triangleright_1 \) and with \( \triangleright_2 \) we get:

\[
\begin{align*}
f_1 &= g_1, \\
f_2 \circ f_3 &= g_2 \circ g_3.
\end{align*}
\]

Similarly, the equation \( W \circ f = W \circ g \) yields:

\[
(\kappa_2 \circ f_1) \circ (\kappa_1 \circ f_2) \circ (\kappa_2 \circ f_3) = (\kappa_2 \circ g_1) \circ (\kappa_1 \circ g_2) \circ (\kappa_2 \circ g_3).
\]

Post-composing with \( \triangleright_1 \) yields \( f_2 = g_2 \). By substitution in our previous finding we get \( f_2 \circ f_3 = f_2 \circ g_3 \). Cancellation in the effect algebra \( C(X, I) \) gives \( f_3 = g_3 \). Hence \( f = g \).

Finally we show that we have an identity-on-objects, full and faithful functor \( F : \text{Par}(\text{Tot}(C)) \to C \), defined on maps by \( F(g) = \triangleright_1 \circ g \). It is easy to see that \( F \) is indeed a functor. We construct an inverse functor \( G : C \to \text{Par}(\text{Tot}(C)) \).

Let \( f : X \to Y \) be a map in \( C \). We form \( 1 \circ f : X \to I \), so that \( (1 \circ f)^{\perp} (1 \circ f)^{\perp} \). Now note that \( 1 \circ f = 1 \circ \kappa_1 \circ f \), for \( \kappa_1 : Y \to Y + 1 \). Next,

\[
(1 \circ f)^{\perp} = 1 \circ (1 \circ f)^{\perp} = 1 \circ (1 \circ f)^{\perp} = 1 \circ \kappa_2 \circ (1 \circ f)^{\perp},
\]

where \( \kappa_2 : 1 \to Y + 1 \). Hence \( (1 \circ \kappa_1 \circ f)^{\perp} \circ (1 \circ \kappa_2 \circ (1 \circ f)^{\perp}) \), which gives by Definition \( \lceil \lceil 2 \rceil \rceil \) an orthogonality \( (1 \circ \kappa_1 \circ f)^{\perp} \circ (1 \circ \kappa_2 \circ (1 \circ f)^{\perp}) \) in the homset of maps \( X \to Y + 1 \). We now define:

\[
G(f) = (\kappa_1 \circ f) \circ (\kappa_2 \circ (1 \circ f)^{\perp}) : X \to Y + I.
\]

Clearly,

\[
\begin{align*}
FG(f) &= \triangleright_1 \circ G(f) = (\triangleright_1 \circ \kappa_1 \circ f) \circ (\triangleright_1 \circ \kappa_2 \circ (1 \circ f)^{\perp}) \\
&= f \circ (1 \circ f)^{\perp} = f \circ 0 = f.
\end{align*}
\]

In order to see \( GF(g) = g \), for a total map \( g : X \to Y + I \), write \( g = (\kappa_1 \circ \triangleright_1 \circ g) \circ (\kappa_2 \circ \triangleright_2 \circ g) \) by Lemma \( \lceil 4 \lceil \rceil 3 \rceil \), and compare it with:

\[
\begin{align*}
GF(g) &= (\kappa_1 \circ \triangleright_1 \circ g) \circ (\kappa_2 \circ (1 \circ \triangleright_1 \circ g)^{\perp}).
\end{align*}
\]

Hence it suffices to show \( \triangleright_2 \circ g = (1 \circ \triangleright_1 \circ g)^{\perp} \). This is done as follows. The map \( (1 + \text{id}) \circ g : X \to I + I \) satisfies \( (\triangleright_1 \circ (1 + \text{id}) \circ g) \perp (\triangleright_2 \circ (1 + \text{id}) \circ g) \) and thus:

\[
(1 \circ \triangleright_1 \circ g) \circ (\triangleright_2 \circ g) = \triangleright \circ (1 + \text{id}) \circ g = [1, 1] \circ g = 1 \circ g = 1.
\]

But then we are done by uniqueness of orthosupplements in the effect algebra \( C(X, I) \). \( \blacksquare \)
Discussion 54. Now that we have seen the equivalence of ‘effectus’ and ‘FinPAC with effects’ we have a choice — or a dilemma, if you like: which notion to use? Let’s start by listing some pros and contras.

1. The notion of effectus has the definite advantage that its definition is simple and elegant — see Definition 2. Surprisingly many results can be obtained from this relatively weak structure, which are best summarised in the resulting state-and-effect triangle [15].

A disadvantage of using the notion of effectus is that we have to explicitly distinguish total and partial maps, for which we have even introduced separate notation. Another disadvantage is that from a computational perspective the category \( \text{Par}(\mathcal{B}) \) of partial maps in an effectus \( \mathcal{B} \) is the more interesting structure, and not \( \mathcal{B} \) itself. In support of the notion of effectus one could claim that the main examples are most naturally described as effectus, and not as FinPAC with effects: thus, for instance the categories \( \text{Sets} \) and \( \text{K}\ell(\mathcal{D}) \) with total functions and distributions are in a sense more natural descriptions, than the categories \( \text{Par}(\text{Sets}) \) and \( \text{Par}(\text{K}\ell(\mathcal{D})) \cong \text{K}\ell(\mathcal{D}_{\leq 1}) \) with partial maps and subdistributions.

2. The definition of ‘FinPAC with effects’ is much less elegant, see Definitions 51 and 18; it is not only much more verbose, but also involves ‘structure’, namely the special object \( I \), of which it is even not clear that it is determined up-to-isomorphism. On the other hand, a definite advantage is that in a FinPAC with effects the total maps are a natural subclass of all the maps (understood as the partial ones), and there is no need for separate notation for total and partial maps. Moreover, the notion of FinPAC with effects gives you in many, computational situations directly the structure that is of most interest, namely partial maps. This is especially the case when we discuss comprehension and quotients later on.

How to weigh these arguments? How to proceed from here? We can choose to work from now on (1) only with effectuses, (2) only with FinPACs with effects, or (3) switch freely between them, depending on whatever works best in which situation.

The first two options are easiest, but provide limited flexibility. Therefore we will choose the third approach. We do realise that it does not make the theory of effectuses easier, since one has to been keenly aware of which description applies. But we hope that the reader will reach such a level of enlightenment that the differences become immaterial — and Wittgenstein’s proverbial ladder can be thrown away, after one has climbed it.

More concretely, in the sequel we will start definitions and results with either “let \( \mathcal{B} \) an effectus in total form”, or with “let \( \mathcal{C} \) be an effectus in partial form”. The latter expression will replace the term ‘FinPAC with effects’; it will not be used anymore in the sequel of this document.

We take another important decision: up to now we have used separate notation for total (\( \cdot, \cdot \to, + \)) and partial (\( \ast, \to, + \)) maps in effectuses (in total form). From now on:

- we use ordinary categorical notation in an effectus in total form (replacing \( \cdot, \cdot \to, + \) by \( \circ, \to, + \)) but continue to use special notation \( \ast, \to, + \) in the category of partial maps, i.e. in the Kleisli category of the lift monad.
we also use ordinary categorical notation in an effectus in partial form.

In line with such easy switching of contexts we will freely use notation that we have introduced for partial maps in an effectus in total form for ordinary maps in effectuses in partial form — where $\circ$ simply becomes $\circ$. Thus for instance, for such a map $f : X \to Y$ in an effectus in partial form we write:

$$f^\circ(q) = (q^\perp \circ f)^\perp$$
$$\ker(f) = f^\circ(0) = (1 \circ f)^\perp$$
$$\ker^\perp(f) = 1 \circ f.$$

Theorem 53 allows us to translate back and forth between the total and partial world. Thus, the properties of, for instance, Lemma 39, which are formulated for an effectus in total form, also make sense for an effectus in partial form. Further, if $f$ is total, then the two forms of substitution $f^\square$ and $f^*$ coincide:

$$f^\square(q) = q \circ f = f^*(q).$$

This follows from uniqueness of orthosupplements:

$$\overline{(q^\perp \circ f) \circ (q \circ f)} = \overline{(q^\perp \circ q) \circ f} = 1 \circ f = 1.$$

There is another topic that we can now understand in greater generality, namely the partial pairing $\langle\langle f, g \rangle\rangle$. It was introduced in Lemma 6 for maps $f, g$ satisfying $\ker^\perp(f) \circ \ker^\perp(g) = 1$, and produced a total map $\langle\langle f, g \rangle\rangle$. The bijective correspondence (28), in upwards direction, extends this pairing in two ways, namely to $n$-ary pairing and to partial maps $f_i : Z \to X_i$, for which the kernel-supplements $\ker^\perp(f_i)$ are only orthogonal (instead of adding up to 1). The resulting pairing $\langle\langle f_1, \ldots, f_n \rangle\rangle : Z \to X_1 + \cdots + X_n$ is then only a partial map, defined as $\sqcup_i (\kappa_i \circ f_i)$. It is unique in satisfying:

$$\sqcup_i \circ \langle\langle f_1, \ldots, f_n \rangle\rangle = f_i.$$

Thus, in an effectus in partial form we have partial pairing too.

Recall that this map $\langle\langle f_1, \ldots, f_n \rangle\rangle$ is total if $\sqcup_i \ker^\perp(f_i) = 1$. In that case we are back in the situation of Lemma 6. In the sequel we use this pairing in this more general form, as essentially given by the correspondence (28).

9 Commutative and Boolean effectuses

Partial endomaps $X \to X$ play an important role in effectus theory. They give rise to predicates, by taking their kernel, but they may also be obtained from predicates, as their associated ‘side effect’. This is a topic that will return many times in the sequel. For this reason we introduce special notation, and write $\text{End}(X)$ for the set of partial maps $X \to X$, in an effectus in total form. This set $\text{End}(X)$ is a partial commutative monoid (PCM) by Proposition 13 via $\sqcup, 0$, it carries a partial order by Proposition 41 (3), and it is a (total) monoid via partial composition $s, \text{id}$. We recall from Lemma 40 that the kernel-orthosupplement $\ker^\perp$ forms a PCM-homomorphism

$$\text{End}(X) \xrightarrow{\ker^\perp} \text{Pred}(X).$$
It reflects $0$ and $\perp$. Explicitly, $\ker\perp(f) = 1 * f = (l + \text{id}) \circ f$.

We shall write $\text{End}_{\leq \text{id}}(X) \hookrightarrow \text{End}(X)$ for the subset:

$$\text{End}_{\leq \text{id}}(X) = \{ f : X \rightharpoonup X \mid f \leq \text{id}_X \},$$

where $\text{id}_X = \kappa_1$ is the partial identity $X \rightharpoonup X$. We shall understand endomaps in $\text{End}_{\leq \text{id}}(X)$ as side-effect free morphisms.

**Definition 55.** An effectus in total form is called commutative if for each object $X$ both:

1. the map $\ker\perp : \text{End}_{\leq \text{id}}(X) \to \text{Pred}(X)$ is an isomorphism; we shall write the inverse as $p \mapsto \text{asrt}_p$, and call it ‘assert’;
2. $\text{asrt}_p \circ \text{asrt}_q = \text{asrt}_q \circ \text{asrt}_p$, for each pair of predicates $p, q \in \text{Pred}(X)$.

For $p, q \in \text{Pred}(X)$ we define a new ‘product’ predicate $p \& q \in \text{Pred}(X)$ via:

$$p \& q = \ker\perp(\text{asrt}_p \circ \text{asrt}_q) = \ker\perp(\text{asrt}_q \circ \text{asrt}_p) = q \& p.$$

An effectus is called Boolean if it is commutative and satisfies: $\text{asrt}_p \ast \text{asrt}_p \perp = 0$, for each predicate $p \in \text{Pred}(X)$.

We shall read the predicate $p \& q$ as ‘$p$ and then $q$’. This $\&$ is a commutative operation in the present commutative context, but it is non-commutative in a more general setting, see Section 15.

The conditions in this definition are given in such a way that they can easily be reformulated for an effectus in partial form. Hence, in the sequel, we freely speak about a commutative/Boolean effectus in partial form.

Later on in Subsection 10.1 we will show that the presence of copiers makes an effectus commutative.

**Example 56.** We describe three examples of these subclasses of effectuses.

1. The effectus $\text{Sets}$ is Boolean. For a predicate $P \subseteq X$, the partial assert function $\text{asrt}_P : X \to X + 1$ is given by:

$$\text{asrt}_P(x) = \begin{cases} \kappa_1 x & \text{if } x \in P \\ \kappa_2 * & \text{if } x \notin P \end{cases}.$$

For convenience we often omit these coprojections $\kappa_1, \kappa_2$ in such descriptions. We have $\text{asrt}_P \leq \text{id}$ since $\text{asrt}_P \perp \text{asrt}_P = 0$, see the description of $\perp$ on partial maps in Example 14. We have:

$$\ker\perp(\text{asrt}_P) = \{ x \mid \text{asrt}_P(x) \neq * \} = \{ x \mid x \in P \} = P.$$

Next, let $f : X \to X + 1$ satisfy $f \leq \text{id}$. This means $f(x) \neq * \Rightarrow f(x) = x$. Now we take as predicate $P = \{ x \mid f(x) \neq * \}$, so that $f = \text{asrt}_P$. Hence we have an isomorphism $\text{End}_{\leq \text{id}}(X) \cong \text{Pred}(X)$.

We further have:

$$(\text{asrt}_P \circ \text{asrt}_Q)(x) = \begin{cases} x & \text{if } x \in P \cap Q \\ * & \text{otherwise} \end{cases} = (\text{asrt}_Q \circ \text{asrt}_P)(x).$$

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The product predicate $P \& Q$ is thus intersection/conjunction $P \cap Q$.
The effectus $\textbf{Sets}$ is Boolean since:
\[
\left(\text{asrt}_P \circ \text{asrt}_{P^\perp}\right)(x) = \begin{cases} x & \text{if } x \in P \cap \neg P \\ \ast & \text{otherwise} \end{cases} = \ast = 0(x).
\]

2. As may be expected, the effectus $\mathcal{K}(\mathcal{D})$ is commutative. For a predicate $p \in [0,1]^X$ we have an assert map $\text{asrt}_p : X \to \mathcal{D}(X + 1)$ given by the convex sum:
\[
\text{asrt}_p(x) = p(x)|x| + (1 - p(x))|\ast|.
\]
We have $\text{asrt}_p \leq \text{id}$ since $\text{asrt}_p \otimes \text{asrt}_{p^\ast} = \text{id}$. Then, following the description of kernels in $\mathcal{K}(\mathcal{D})$ from Example 38 we get:
\[
\ker^+(\text{asrt}_p)(x) = 1 - \ker(\text{asrt}_p)(x) = 1 - \text{asrt}_p(\ast) = 1 - (1 - p(x)) = p(x).
\]
We check that we get an isomorphism $\text{End}_{\leq \text{id}}(X) \cong \text{Pred}(X)$. Let $f : X \to \mathcal{D}(X + 1)$ satisfy $f \leq \text{id}$, where $\text{id}(x) = 1|x|$. Then $f(x)(x') \neq 0 \Rightarrow x' = x$. Taking $p(x) = f(x)(x)$ then yields $f = \text{asrt}_p$.
Next, these assert maps satisfy:
\[
\left(\text{asrt}_p \circ \text{asrt}_q\right)(x) = p(x) \cdot q(x)|x| + (1 - p(x) \cdot q(x))|\ast| = (\text{asrt}_q \circ \text{asrt}_p)(x).
\]
The product predicate $p \& q$ is thus the pointwise multiplication $(p \& q) = p(x) \cdot q(x)$.
It is instructive to see why $\mathcal{K}(\mathcal{D})$ is not a Boolean effectus: It satisfies:
\[
\left(\text{asrt}_p \circ \text{asrt}_{p^\perp}\right)(x) = p(x) \cdot (1 - p(x))|x| + (1 - p(x) \cdot (1 - p(x)))|\ast|.
\]
If $\text{asrt}_p \circ \text{asrt}_{p^\perp} = 0$, then $p(x) \cdot (1 - p(x)) = 0$ for each $x$. But this requires that $p(x)$ is a Boolean predicate, with $p(x) \in \{0,1\}$ for each $x$, so that $p$ restricts to $X \to \{0,1\}$. But of course, not every predicate in $\mathcal{K}(\mathcal{D})$ is Boolean.

3. The effectus $\textbf{CvNA}^{op}$ of commutative von Neumann algebras is commutative. For an effect $e \in [0,1]_{\mathcal{A}}$ in such a commutative algebra $\mathcal{A}$, we define $\text{asrt}_e : \mathcal{A} \to \mathcal{A}$ by $\text{asrt}_e(a) = e \cdot a$. Clearly, this is a linear subunital map. We use commutativity to show that it is positive. For $a \geq 0$, say $a = b \cdot b^*$ we obtain a positive element:
\[
\text{asrt}_e(a) = e \cdot a = \sqrt{e} \cdot \sqrt{e} \cdot b \cdot b^* = \sqrt{e} \cdot b \cdot b^* \cdot \sqrt{e} = (\sqrt{e} \cdot b) \cdot (\sqrt{e} \cdot b)^*.
\]
Here we use that each positive element $x$ has a positive square root $\sqrt{x}$ and is self-adjoint, that is, satisfies $x = x^*$. 

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Like before, we have \( \text{asrt}_e \otimes \text{asrt}_e = \text{id} \), so that \( \text{asrt}_e \leq \text{id} \). Further, following the description in Example 38,

\[
\ker^+(\text{asrt}_e) = \text{asrt}_e(1) = e \cdot 1 = e.
\]

Showing that \( \ker^+ \) is an isomorphism requires more work. Let \( f: \mathcal{A} \to \mathcal{A} \) be a subunital positive map below the identity. This means \( f(a) \leq a \) for all \( a \geq 0 \). Define \( g: \mathcal{A} \oplus \mathcal{A} \to \mathcal{A} \) by \( g(x, y) = f(x) + y - f(y) \). This map is positive, because if \( x, y \geq 0 \), then \( f(x) \geq 0 \) and \( y - f(y) \geq 0 \), since \( f(y) \leq y \), and thus \( g(x, y) \geq 0 \). Clearly, \( g \) is unital. More generally, \( g(x, x) = x \), which can be written more abstractly as \( g \circ \Delta = \text{id} \), where the diagonal \( \Delta: \mathcal{A} \to \mathcal{A} \oplus \mathcal{A} \) preserves multiplication. Hence ‘Tomiyama’ [Tom57] applies, see [Jac15a, Lemma 38], so that \( z \cdot g(x, y) = g(\Delta(z) \cdot (x, y)) = g(z \cdot x, z \cdot y) \). The element \( g(1, 0) = f(1) \) is central in \( \mathcal{A} \) since:

\[
x \cdot g(1, 0) = g(\Delta(x) \cdot (1, 0)) = g(x, 0) = g((1, 0) \cdot \Delta(x)) = g(1, 0) \cdot x.
\]

We thus obtain:

\[
f(x) = g(x, 0) = g(1, 0) \cdot x = f(1) \cdot x = \text{asrt}_f(1)(x) = \text{asrt}_{\ker^+(f)}(x).
\]

Finally, we have, for arbitrary effects \( e, d \in [0, 1]_{\mathcal{A}} \),

\[
(\text{asrt}_e \ast \text{asrt}_d)(a) = e \cdot (d \cdot a) = d \cdot (e \cdot a) = (\text{asrt}_d \ast \text{asrt}_e)(a).
\]

As a result, the product predicate \( e \& d \) is given by multiplication \( \cdot \) in the von Neumann algebra.

The assert maps, if they exist, give rise to many interesting properties.

**Lemma 57.** Let \( B \) be a commutative effectus in total form.

1. Each PCM \( \text{End}_{\leq \text{id}}(X) \) is an effect algebra; in fact, with composition \( * \), \( \text{id} \) it is a commutative effect monoid.
2. The assert map \( \text{Pred}(X) \to \text{End}_{\leq \text{id}}(X) \) is an (iso)morphism of effect algebras, and makes \( \text{Pred}(X) \) with \( \& \) also a commutative effect monoid.
3. \( p \ast f = 0 \) iff \( \text{asrt}_p \ast f = 0 \).
4. \( \text{asrt}_s = s \) for each scalar \( s: 1 \to 1 \).
5. \( \text{asrt}_{p \ast q} = \text{asrt}_p + \text{asrt}_q: X + Y \to X + Y \).
6. The instrument map \( \text{instr}_p = \langle \text{asrt}_p, \text{asrt}_p \rangle: X \to X + X \) satisfies \( \nabla \circ \text{instr}_p = \text{id} \).
7. \( p \& q \leq q \circ \text{asrt}_p \), and thus \( p \& q \leq q \) and also \( p \& q \leq p \).
8. \( (\text{asrt}_p)\ominus(q) = (p \& q) \ominus p \).
9. The following points are equivalent, for an arbitrary predicate \( p \).
(a) \( \text{asrt}_p \ast \text{asrt}_p = \text{asrt}_p \)

(b) \( p \& p = p \)

(c) \( \text{asrt}_p \ast \text{asrt}_{p\perp} = 0 \)

(d) \( p \& p^\perp = 0 \)

(e) \( p \) is sharp, that is, \( p \land p^\perp = 0 \).

The instrument map \( \text{instr}_p : X \rightarrow X + X \) from point (6) can be understood as a ‘case’ expression, sending the input to the left component in \( X + X \) if \( p \) holds, and to the right component otherwise. The property \( \nabla \circ \text{instr}_p = \text{id} \) says that this instrument map has no side-effects. We will encounter instrument maps in a non-commutative setting in Section 15 where they do have side-effects. Such side-effects are an essential aspect of the quantum world. They don’t exist in the current commutative setting.

**Proof** We handle these points one-by-one.

1. First we have to produce an orthosupplement for an arbitrary \( f \in \text{End}_{\leq}^{\leq} \text{id}(X) \). Since \( f \leq \text{id} \), we have \( f \otimes g = \text{id} \) for some \( g \in \text{End}(X) \). Clearly \( g \leq \text{id} \). It is unique with this property, since if \( f \otimes h = \text{id} = f \otimes g \), then \( \ker^\perp(f) \otimes \ker^\perp(g) = 1 = \ker^\perp(f) \otimes \ker^\perp(h) \). Hence \( \ker^\perp(g) = \ker^\perp(h) \) by cancellation in the effect algebra \( \text{Pred}(X) \). But then \( g = h \) since \( \ker^\perp \) is an isomorphism.

   We see that \( \text{id} = 0^\perp \in \text{End}_{\leq}^{\leq} \text{id}(X) \). If \( \text{id} \perp f \in \text{End}_{\leq}^{\leq} \text{id}(X) \), then \( \ker^\perp(\text{id}) = 1 \perp \ker^\perp(f) \). Hence \( \ker^\perp(f) = 0 \) in \( \text{Pred}(X) \), and thus \( f = 0 \) since \( \ker^\perp \) reflects \( \perp \).

   The partial composition operation \( \ast \) on \( \text{End}_{\leq}^{\leq} \text{id}(X) \) preserves \( \otimes, 0 \) by Proposition [15] [2]. Moreover, the top element \( \text{id} \in \text{End}_{\leq}^{\leq} \text{id}(X) \) obviously satisfies \( p \ast \text{id} = p = \text{id} \ast p \). Hence \( \text{End}_{\leq}^{\leq} \text{id}(X) \) is an effect monoid, see Definition [13]. We show that it is commutative. For arbitrary maps \( f, g : X \rightarrow X \) write \( p = \ker^\perp(f) \) and \( q = \ker^\perp(g) \), so that \( f = \text{asrt}_p \) and \( g = \text{asrt}_q \). Then we are done by Definition [55] [2]:

   \[
   f \ast g = \text{asrt}_p \ast \text{asrt}_q = \text{asrt}_q \ast \text{asrt}_p = g \ast f.
   \]

2. We have \( \text{asrt}_1 = \text{id} \), since \( 1 = \ker^\perp(\text{id}) \). Next, if \( p \perp q \in \text{Pred}(X) \), then \( \ker^\perp(\text{asrt}_p) = p \perp q = \ker^\perp(\text{asrt}_q) \), and thus \( \text{asrt}_p \perp \text{asrt}_q \) since \( \ker^\perp \) reflects \( \perp \). In that case \( \text{asrt}_{p \otimes q} = \text{asrt}_p \otimes \text{asrt}_q \) since:

   \[
   \ker^\perp(\text{asrt}_{p \otimes q}) = p \otimes q = \ker^\perp(\text{asrt}_p) \otimes \ker^\perp(\text{asrt}_q) = \ker^\perp(\text{asrt}_p \otimes \text{asrt}_q).
   \]

   By construction, \( \ker^\perp : \text{End}_{\leq}^{\leq} \text{id}(X) \rightarrow \text{Pred}(X) \) sends \( \ast \) to \( \& \), so that \( \text{Pred}(X) \) becomes a commutative effect monoid, and \( \ker^\perp \) an isomorphism of effect monoids.

3. By Lemma [17] using that \( 1 \ast \text{asrt}_p = \ker^\perp(\text{asrt}_p) = p \),

   \[
   \text{asrt}_p \ast f = 0 \iff 1 \ast \text{asrt}_p \ast f = 0 \iff p \ast f = 0.
   \]
4. We obtain \( \text{asrt}_s = s \) for a scalar \( s \) from Lemma 39 (1):
\[
\ker^\perp(\text{asrt}_s) = s = \ker^\perp(s).
\]

5. The equation \( \text{asrt}_{[p,q]} = \text{asrt}_p + \text{asrt}_q \) follows from:
\[
\ker^\perp(\text{asrt}_p + \text{asrt}_q) = 1 \ast [\kappa_1 \circ \text{asrt}_p, \kappa_2 \circ \text{asrt}_q]
= [1 \circ \kappa_1 \circ \text{asrt}_p, 1 \circ \kappa_2 \circ \text{asrt}_q]
= [1 \circ \text{asrt}_p, 1 \circ \text{asrt}_q] \quad \text{since coprojections are total}
= [\ker^\perp(\text{asrt}_p), \ker^\perp(\text{asrt}_q)]
= [p, q]
= \ker^\perp(\text{asrt}_{[p,q]}).
\]

6. For a predicate \( p \in \text{Pred}(X) \) we have by definition: \( \ker(\text{asrt}_p) = p^\perp = \ker^\perp(\text{asrt}_p) \). Hence we can use the pairing from Lemma 6 and can form the (total) instrument map \( \text{instr}_p = \langle\langle \text{asrt}_p, \text{asrt}_p^\perp \rangle\rangle : X \to X + X \). We obtain \( \nabla \circ \text{instr}_p = \text{id} \) from:
\[
\langle\nabla \circ \text{instr}_p\rangle = \nabla \circ \langle\text{instr}_p\rangle = \nabla \circ ((\kappa_1 \circ \text{asrt}_p) \otimes (\kappa_2 \circ \text{asrt}_p^\perp))
= (\nabla \circ \kappa_1 \circ \text{asrt}_p) \otimes (\nabla \circ \kappa_2 \circ \text{asrt}_p^\perp)
= \text{asrt}_p \otimes \text{asrt}_p^\perp
= \langle\text{id}\rangle.
\]

7. Simply: \( p \& q = \ker^\perp(\text{asrt}_q \ast \text{asrt}_p) = 1 \ast \text{asrt}_q \ast \text{asrt}_p = q \ast \text{asrt}_p \).
Since \( \text{asrt}_p \leq \text{id} \), we get \( p \& q = q \ast \text{asrt}_p \leq q \ast \text{id} = q \). By commutativity we obtain: \( p \& q = q \& p \leq p \).

8. We have:
\[
(\text{asrt}_p)^\Box(q) = (\text{asrt}_p)^\Box(q^{\perp\perp})
= (\text{asrt}_p)^\Box(q^{\perp}) \otimes \ker(\text{asrt}_p) \quad \text{by Lemma 39 (10)}
= (q \ast \text{asrt}_p) \otimes p^\perp
= (p \& q) \otimes p^\perp.
\]

9. The equivalences \( (9a) \Leftrightarrow (9b) \) and \( (9c) \Leftrightarrow (9d) \) are obvious, via the isomorphism \( \ker^\perp \). We prove \( (9a) \Leftrightarrow (9c) \) and \( (9e) \Leftrightarrow (9a) \).
For \( (9a) \Rightarrow (9c) \), let predicate \( p \) satisfy \( \text{asrt}_p \ast \text{asrt}_p = \text{asrt}_p \). Then:
\[
\text{asrt}_p = \text{asrt}_p \ast \text{id} = \text{asrt}_p \ast (\text{asrt}_p \otimes \text{asrt}_p^\perp)
= (\text{asrt}_p \circ \text{asrt}_p) \otimes (\text{asrt}_p \circ \text{asrt}_p^\perp)
= \text{asrt}_p \otimes (\text{asrt}_p \circ \text{asrt}_p^\perp).
\]
Hence \( \text{asrt}_p \circ \text{asrt}_p^\perp = 0 \) by Lemma 11 (2).
In the reverse direction, assuming asrt\(p \odot \text{asrt}_{p^\perp} = 0\) we obtain:

\[
asrt_p = \text{asrt}_p \odot \text{id} = \text{asrt}_p \odot (\text{asrt}_p \odot \text{asrt}_{p^\perp}) = (\text{asrt}_p \odot \text{asrt}_p) \odot \text{asrt}_{p^\perp} = \text{asrt}_p \odot \text{asrt}_p.
\]

For (9c) ⇒ (9e) we assume asrt\(p \odot \text{asrt}_p = 0\). Let \(q \leq p\) and \(q \leq p^\perp\). If we show \(q = 0\), then \(p \wedge p^\perp = 0\). First, the inequality \(q \leq p\) gives:

\[
q \odot \text{asrt}_{p^\perp} \leq p \odot \text{asrt}_{p^\perp} = 1 \odot \text{asrt}_p \odot \text{asrt}_{p^\perp} = 1 \odot 0 = 0.
\]

Hence:

\[
q \odot \text{asrt}_p = (q \odot \text{asrt}_p) \odot (q \odot \text{asrt}_{p^\perp}) = q \odot (\text{asrt}_p \odot \text{asrt}_{p^\perp}) = q \odot \text{id} = q.
\]

But now the inequality \(q \leq p^\perp\) gives the required result:

\[
q = q \odot \text{asrt}_p \leq p^\perp \odot \text{asrt}_p = 1 \odot \text{asrt}_{p^\perp} \odot \text{asrt}_p = 1 \odot 0 = 0.
\]

Finally, for (9e) ⇒ (9c) let \(p \wedge p^\perp = 0\). We have \(p \& p^\perp = 0\) and also \(p \& p^\perp \leq p^\perp\) by point (7). Hence \(0 = p \& p^\perp = \ker^\perp(\text{asrt}_p \odot \text{asrt}_{p^\perp})\). Since \(\ker^\perp\) reflects \(0\), we obtain \(\text{asrt}_p \odot \text{asrt}_{p^\perp} = 0\). □

We show how Bayes’ rule can be described abstractly in the current setting. A type-theoretic formulation of these ideas is elaborated in [AJ15].

**Example 58.** Let \((C, I)\) be a commutative effectus, in partial form, with normalisation, as described in Remark [30]. Consider a total state \(\omega: 1 \to X\) and a predicate \(p: X \to I\) on the same object \(X\). We obtain a substate \(\text{asrt}_p \circ \omega: 1 \to X\) with:

\[
1 \circ \text{asrt}_p \circ \omega = p \circ \omega \quad \text{(16)} \quad \omega \models p.
\]

We thus have by Lemma [7]

\[
\text{asrt}_p \circ \omega = 0 \iff (\omega \models p) = 0.
\]

Now let the validity \(\omega \models p\) be non-zero. Then we can normalise the substate \(\text{asrt}_p \circ \omega\). We write the resulting total ‘conditional’ state as \(\omega|_p: 1 \to X\). It satisfies by construction, see (19):

\[
\omega|_p \circ (\omega \models p) = \text{asrt}_p \circ \omega, \quad \text{(33)}
\]

where composition \(\circ\) on the left is scalar multiplication, see Lemma [23]. This new ‘conditional’ state \(\omega|_p\) should be read as the update of state \(\omega\) after learning \(p\).

We claim that we now have the following abstract version of Bayes’ rule: for an arbitrary predicate \(q\) on \(X\),

\[
(\omega|_p \models q) \cdot (\omega \models p) = (\omega \models p \& q). \quad \text{(34)}
\]

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In presence of division, this equation can be recast in more familiar form:
\[ \omega \mid_p \models q = \frac{\omega \models p \& q}{\omega \models p}. \]

The proof of Bayes’ equation (34) is easy:
\[
(\omega \mid_p \models q) \cdot (\omega \models p) = q \circ \omega\mid_p \circ (\omega \models p)
= (p \& q) \circ \omega
= \omega \models p \& q.
\]

This abstract description suggests how to do conditional probability in a non-commutative setting, in presence of assert maps, see Section 15, and see also [LS13].

At this abstract level the total probability law, also known as the rule of belief propagation [LS13] also holds: for a state \( \omega \) on \( X \), an \( n \)-test \( p_1, \ldots, p_n \) on \( X \), and an arbitrary predicate \( q \) on \( X \),
\[
(\omega \models q) = \bigotimes_i (\omega \mid_{p_i} \models q) \cdot (\omega \models p_i).
\]

The proof is left to the interested reader.

In the current setting we can take the commutative effectus \( \mathcal{K}l(\mathcal{D}) \) as example. For a total state \( \omega \in \mathcal{D}(X) \) on a set \( X \) and a (fuzzy) predicate \( p \in [0,1]^X \) with \( \omega \models p = \sum_x \omega(x) \cdot p(x) \neq 0 \) we obtain as normalised state \( \omega \mid_p \in \mathcal{D}(X) \),
\[
\omega \mid_p = \sum_x \frac{\omega(x) \cdot p(x)}{\omega \models p} \mid_x.
\]

Then indeed, for \( q \in [0,1]^X \),
\[
\omega \mid_p \models q = \sum_x \omega_p(x) \cdot q(x) = \sum_x \frac{\omega(x) \cdot p(x) \cdot q(x)}{\omega \models p} = \frac{\omega \models p \& q}{\omega \models p}.
\]

We briefly mention the continuous probabilistic case, given by the Kleisli category \( \mathcal{K}l(\mathcal{G}) \) of the Giry monad \( \mathcal{G} \) on measurable spaces. For a measurable space \( X \), with set \( \Sigma_X \) of measurable subsets, let \( \omega \in \mathcal{G}(X) \) be a probability distribution. Each measurable subset \( M \in \Sigma_X \) gives rise to predicate \( 1_M : X \to [0,1] \) with \( 1_M(x) = 1 \) if \( x \in M \) and \( 1_M(x) = 0 \) otherwise. We claim that the conditional state \( \omega\mid_{1_M} \in \mathcal{G}(X) \), as described above, is the conditional probability measure \( \omega(- \mid M) \), if \( \omega(M) \neq 0 \).

Indeed, the subprobability measure \( \text{asrt}_{1_M} \circ \omega : \Sigma_X \to [0,1] \) is given by \( A \mapsto \int 1_{M \cap A} \, d\omega = \omega(M \cap A) \). Normalisation gives the conditional probability:
\[
\omega\mid_{1_M}(A) = \frac{\omega(M \cap A)}{\omega(M)} = \omega(A \mid M).
\]

We turn to Boolean effectuses and collect some basic results.

**Lemma 59.** Let \( \mathcal{B} \) now be a Boolean effectus, that is, a commutative effectus in which \( \text{asrt}_p \star \text{asrt}_p = \mathbf{0} \) holds for each predicate \( p \).

1. All assert maps are idempotent, that is, \( \text{asrt}_p \star \text{asrt}_p = \text{asrt}_p \), and all predicates \( p \) are sharp, that is, \( p \land p^\perp = \mathbf{0} \).
2. The predicate \( p \land q \) is the meet/conjunction \( p \land q \) in \( \text{Pred}(X) \). Disjunctions then also exist via De Morgan: \( p \lor q = (p^\perp \land q^\perp)^\perp \).

3. \( p \perp q \) iff \( \text{asrt}_p \ast \text{asrt}_q = 0 \) iff \( p \land q = 0 \).

4. If \( p \perp q \), then \( p \oplus q = p \lor q \).

5. Conjunction \( \land \) distributes over disjunction \( \lor \), making each effect algebra \( \text{Pred}(X) \) a Boolean algebra.

**Proof** Most of these points are relatively easy, except the last one.

1. Directly by Lemma 57 (9).

2. We have \( p \land p = \ker^\perp(\text{asrt}_p \ast \text{asrt}_p) = \ker^\perp(\text{asrt}_p) = p \). This allows us to show that \( p \land q \) is the meet of \( p, q \). We already have \( p \land q \leq p \) and \( p \land q \leq q \) by Lemma 57 (7). Next, let \( r \) be a predicate with \( r \leq p \) and \( r \leq q \). Since product \( r \land (\neg) \) preserves \( \lor \), by Lemma 57 (1), it is monotone. Hence: \( r = r \land r \leq p \land q \).

3. The equivalence \( \text{asrt}_p \ast \text{asrt}_q = 0 \) follows from the previous point and Lemma 57 (9). So let \( p \perp q \), so that \( q \leq p^\perp \). Then \( p \land q \leq p \land p^\perp = 0 \). Conversely, if \( \text{asrt}_p \ast \text{asrt}_q = 0 \) then:

\[
\text{asrt}_p = \text{asrt}_p \ast \text{id} = \text{asrt}_p \ast (\text{asrt}_q \odot \text{asrt}_q^\perp) = (\text{asrt}_p \ast \text{asrt}_q) \odot (\text{asrt}_p \ast \text{asrt}_q^\perp) = 0 \odot (\text{asrt}_p \ast \text{asrt}_q^\perp) = \text{asrt}_p \ast \text{asrt}_q^\perp.
\]

Hence \( p = p \land q^\perp \), so that \( p \leq q^\perp \), and thus \( p \perp q \).

4. Let \( p \perp q \). We intend to prove that the sum \( p \odot q \), if it exists, is the join \( p \lor q \). In any effect algebra, \( p \odot q \) is an upperbound of both \( p \) and \( q \), so we only need to prove that it is the least upperbound. Let \( p \leq r \) and \( q \leq r \). The inequality \( p \leq r \) says \( p \perp r^\perp \) and thus \( \text{asrt}_{p^\perp} \ast \text{asrt}_p = 0 \) by the previous point. Similarly \( q \leq r \) gives \( \text{asrt}_{q^\perp} \ast \text{asrt}_q = 0 \). But then:

\[
\text{asrt}_{p^\perp} \ast \text{asrt}_{p \odot q} = \text{asrt}_{p^\perp} \ast (\text{asrt}_p \odot \text{asrt}_q) = (\text{asrt}_{p^\perp} \ast \text{asrt}_p) \odot (\text{asrt}_{p^\perp} \ast \text{asrt}_q) = 0 \odot 0 = 0.
\]

Hence \( p \odot q \perp r^\perp \), again by the previous point, and thus \( p \odot q \leq r \).

5. This result can be traced back to [BF95] Thm. 3.11]. However, we give our own proof of distributivity \( p \land (q \lor r) = (p \land q) \lor (p \land r) \), for all predicates \( p, q, r \in \text{Pred}(X) \). We repeatedly use the equivalence

\[
x \leq y^\perp \iff x \land y = 0\quad (\ast)
\]

from point (3).

The inequality \( p \land (q \lor r) \geq (p \land q) \lor (p \land r) \) always holds, so we need to prove the inequality:

\[
p \land (q \lor r) \leq (p \land q) \lor (p \land r) = [(p \land q)^\perp \land (p \land r)^\perp]^\perp.
\]

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That is, by (\(\ast\)) we have to prove:

\[
p \land (q \lor r) \land (p \land q) \perp \land (p \land r) \perp = 0.
\]

We pick an arbitrary predicate \(x\) for which:

(a) \(x \leq p\) \hspace{1cm} (b) \(x \leq p \lor r\)

We need to prove \(x = 0\). The inequality (a) is equivalent to \(x \land p = x\).

By (\(\ast\)), (c) and (d) are equivalent to \(x \land p \land q = 0\) and \(x \land p \land r = 0\). But then \(x \land q = 0\) and \(x \land r = 0\), so that \(x \leq q \perp\) and \(x \leq r \perp\) again by (\(\ast\)). We now have \(x \leq q \perp \land r \perp = (p \lor r) \perp\), which, together with (b) and point (??)

gives the required conclusion:

\[
x \leq (p \lor r) \land (p \lor r) \perp = 0.
\]

**Remark 60.** The commutation requirement \(\text{asrt}_p \ast \text{asrt}_q = \text{asrt}_q \ast \text{asrt}_p\) from Definition 55 (2) holds automatically in a Boolean effectus. More precisely, it follows from the ‘Boolean’ requirement \(\text{asrt}_p \ast \text{asrt}_{p \perp} = 0\) in Definition 55 assuming the assert isomorphism in point (1). This works as follows.

Let \(p, q\) be arbitrary predicates on the same object. The associated assert maps satisfy:

\[
0 = \text{asrt}_p \circ \text{asrt}_{p \perp} = \text{asrt}_p \circ \text{id} \ast \text{asrt}_{p \perp} = \text{asrt}_p \circ (\text{asrt}_q \land \text{asrt}_{q \perp}) \ast \text{asrt}_{p \perp} = (\text{asrt}_p \ast \text{asrt}_q \ast \text{asrt}_{p \perp}) \circ (\text{asrt}_p \ast \text{asrt}_{q \perp} \ast \text{asrt}_{p \perp}).
\]

By Proposition 41 (1) we obtain:

\[
\text{asrt}_p \ast \text{asrt}_q \ast \text{asrt}_{p \perp} \overset{(a)}{=} 0. \quad \text{Similarly,} \quad \text{asrt}_{p \perp} \ast \text{asrt}_q \ast \text{asrt}_p \overset{(b)}{=} 0.
\]

We can now derive commutation of assert maps:

\[
\text{asrt}_p \ast \text{asrt}_q = \text{asrt}_p \ast \text{asrt}_q \ast (\text{asrt}_p \land \text{asrt}_{p \perp}) = (\text{asrt}_p \ast \text{asrt}_q \ast \text{asrt}_{p \perp}) \circ (\text{asrt}_p \ast \text{asrt}_q \ast \text{asrt}_{p \perp}) \overset{(a)}{=} (\text{asrt}_p \ast \text{asrt}_q \ast \text{asrt}_{p \perp}) \circ 0 \overset{(b)}{=} (\text{asrt}_p \ast \text{asrt}_q \ast \text{asrt}_{p \perp}) \circ (\text{asrt}_{p \perp} \ast \text{asrt}_q \ast \text{asrt}_p) = (\text{asrt}_p \ast \text{asrt}_{p \perp} \ast \text{asrt}_q \ast \text{asrt}_{p \perp}) = \text{asrt}_q \ast \text{asrt}_p.
\]

It is not known if this commutation property can be derived from the assert isomorphism in Definition 55 (1).

The next result justifies the name *Boolean* effectuses. It uses the category \(\text{BA}\) of Boolean algebras, which is a subcategory \(\text{BA} \hookrightarrow \text{EA}\) of the category of effect algebras.

**Proposition 61.** *For a Boolean effectus \(B\), all predicates are sharp, and the predicate functor restricts to \(\text{Pred}: B \rightarrow \text{BA}^{op}\).*
Theorem 59. For each $X \in \mathcal{B}$ the collection $\text{Pred}(X)$ contains only sharp predicates and is a Boolean algebra, by Lemma 59 (3). We have to prove that for each map $f : Y \to X$ in $\mathcal{B}$ the total substitution functor $f^* : \text{Pred}(X) \to \text{Pred}(Y)$ is a map of Boolean algebras. For this it suffices that $f^*$ preserves disjunctions $\lor$.

First, let $p, q \in \text{Pred}(X)$ be disjoint, that is, $p \land q = 0$. By Lemma 59 (3) and (4), using that $f^*$ is a map of effect algebras, $f^*(p \lor q) = f^*(p) \lor f^*(q) = f^*(p) \lor f^*(q)$.

For arbitrary $p, q$ we can always rewrite the join $p \lor q$ in a Boolean algebra as a disjoint join:

$$p \lor q = (p \land q) \lor (p \land q) \lor (q \land p).$$

Since substitution $f^*$ preserves disjoint joins we get:

$$f^*(p \lor q) = f^*(p \land q) \lor f^*(p \land q) \lor f^*(q \land p) = f^*(p) \lor f^*(q).$$

Later, in Section 13, we return to Boolean effectuses and describe their close relationship to extensive categories. But we first need the notions of comprehension and quotient, see Section 11 and 12.

10 Monoidal effectuses

In this section we shall consider effectuses with tensors $\otimes$. Such tensors are used for parallel composition, which is an important part of quantum theory. For background information on symmetric monoidal categories we refer to [Mac71].

Definition 62. An effectus in total form $\mathcal{B}$ is called monoidal if it is a symmetric monoidal category such that:

1. the tensor unit is the final object $1$;

2. the tensor distributes over finite coproducts $(+, 0)$; this means that the following canonical maps are isomorphisms:

$$(X \otimes A) + (Y \otimes A) \longrightarrow (X + Y) \otimes A \quad 0 \longrightarrow 0 \otimes A \quad (35)$$

This says that the functor $(-) \otimes A$ preserves finite coproducts $(+, 0)$. By symmetry $A \otimes (-)$ then also preserves $(+, 0)$.

We recall the notation that is standardly used for monoidal natural isomorphisms, and add notation for the distributivity isomorphism (35).

$$1 \otimes X \stackrel{\lambda}{\longrightarrow} X \quad X \otimes Y \stackrel{\gamma}{\longrightarrow} Y \otimes Y \quad X \otimes (Y \otimes Z) \stackrel{\alpha}{\longrightarrow} (X \otimes Y) \otimes Z$$

$$X \otimes I \stackrel{\rho}{\longrightarrow} X \quad (X \otimes A) + (Y \otimes A) \stackrel{\Xi}{\longrightarrow} (X + Y) \otimes A$$
Because the final object 1 is the tensor unit there are projections $X \leftarrow X \otimes Y \rightarrow Y$ defined in:

$$
\begin{array}{c}
X & \xrightarrow{\pi_1} & X \otimes Y & \xrightarrow{\pi_2} & Y \\
\rho & \approx & \text{id} \otimes ! & \approx & \lambda
\end{array}
$$

These projections take the marginal, see Example 64. They are natural in the sense that $\pi_i \circ (f_1 \otimes f_2) = f_i \circ \pi_i$. A monoidal category in which the tensor unit is final is sometimes called semicartesian.

We have described tensors in the ‘total’ case, for effectuses. But they can equivalently be described in the ‘partial’ case. This is the topic of the next result.

**Proposition 63.** Let $\mathcal{B}$ be an effectus in total form. Then: $\mathcal{B}$ is monoidal effectus if and only if

- $\text{Par}(\mathcal{B})$ is symmetric monoidal, and the monoidal and distributivity isomorphisms (35) are total — so that $\otimes$ distributes over $(+, 0)$ in $\text{Par}(\mathcal{B})$;
- the tensor $f \otimes g$ of two total morphisms $f, g$ in $\text{Par}(\mathcal{B})$ is again total;
- the final object $1 \in \mathcal{B}$ forms a tensor unit in $\text{Par}(\mathcal{B})$.

When we talk about a ‘monoidal effectus in partial form’ we mean an effectus in partial form that satisfies the properties described above for the category of partial maps $\text{Par}(\mathcal{B})$, with 1 as tensor unit. The above second point is equivalent to: the following diagram of partial maps commutes.

$$
\begin{array}{c}
X \otimes Y & \xrightarrow{1 \otimes 1} & 1 & \approx & 1 \\
1 \otimes 1 & \approx & 1
\end{array}
$$

**Proof** Given the tensor $\otimes$ on $\mathcal{B}$ we obtain $\otimes$ on $\text{Par}(\mathcal{B})$ in the same way on objects. Given partial maps $h: X \rightarrow A$ and $k: Y \rightarrow B$ in $\text{Par}(\mathcal{B})$ we obtain $h \otimes k: X \otimes Y \rightarrow A \otimes B$ in $\text{Par}(\mathcal{B})$ as:

$$
X \otimes Y
\xrightarrow{h \otimes k}
(A + 1) \otimes (B + 1) \approx (A \otimes B) + (A \otimes 1) + (1 \otimes B) + (1 \otimes 1)
\approx [\kappa_1, \kappa_2, \kappa_3, \kappa_4]
(A \otimes B) + 1
$$

Then $\langle f \rangle \otimes \langle g \rangle = \langle f \otimes g \rangle$. Remaining details are left to the reader. □

**Example 64.** We shall look at monoidal structure in three of our four running examples. It is not clear if order unit groups have tensors.

1. The effectus $\text{Sets}$ has finite products $(\times, 1)$, where functors $A \times (-)$ preserve finite coproducts. Obviously, the final object is the unit for $\times$. 

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2. The effectus $\mathcal{K}(\mathcal{D})$ for discrete probability has tensors because the distribution monad $\mathcal{D}$ is monoidal: there are natural maps $\mathcal{D}(X) \times \mathcal{D}(Y) \to \mathcal{D}(X \times Y)$ which commute appropriately with the unit and multiplication of the monad. These maps send a pair $(\omega, \rho) \in \mathcal{D}(X) \times \mathcal{D}(Y)$ to the distribution in $\mathcal{D}(X \times Y)$ given by $(x, y) \mapsto \omega(x) \cdot \rho(y)$.

The resulting tensor on $\mathcal{K}(\mathcal{D})$ is given on objects by cartesian product $\times$. For morphisms $f: A \to \mathcal{D}(X)$ and $g: B \to \mathcal{D}(Y)$ there is map $f \otimes g$ in $\mathcal{K}(\mathcal{D})$ given by the composite:

$$A \times B \xrightarrow{f \times g} \mathcal{D}(X) \times \mathcal{D}(Y) \xrightarrow{\pi_1} \mathcal{D}(X \times Y)$$

the projection map $X \otimes Y \to X$ in $\mathcal{K}(\mathcal{D})$, is, following (36), the function

$$X \times Y \xrightarrow{\pi_1} \mathcal{D}(X) \quad \text{with} \quad \pi_1(x, y) = 1|x\rangle$$

When applied to a state $\omega \in \mathcal{D}(X \times Y) = \text{Stat}(X \times Y)$ we obtain the marginal distribution $\text{Stat}(\pi_1)(\omega) = (\pi_1)_*(\omega) \in \text{Stat}(X)$ given by:

$$(\pi_1)_*(\omega) = \sum_x (\sum_y \omega(x, y))|x\rangle.$$  

3. The effectus $\mathbf{vNA}^{\text{op}}$ of von Neumann algebras is also monoidal. Here it is essential that the morphisms of $\mathbf{vNA}$ are completely positive. The details are quite complicated, see e.g. [Tak01] and [Cho14] for distributivity results using the ‘minimal’ tensor. The tensor unit is the algebra $\mathbb{C}$ of complex numbers, which is final in $\mathbf{vNA}^{\text{op}}$. Distribution of $\otimes$ over coproducts ($\oplus$, 0) is investigated in [Cho14].

The next result is proven explicitly in [Jac15a]. Here we give a more abstract argument, using partial maps.

Corollary 65. In a monoidal effectus the effect monoid of scalars $\text{Pred}(1)$ is commutative.

Proof It is a well-known fact that the endomaps $I \to I$ on the tensor unit in a monoidal category form a commutative monoid under composition. This relies on the ‘Eckmann-Hilton’ argument see e.g. [KL80]. We apply this to the monoidal category $\text{Part}(\mathcal{B})$ of a monoidal effectus $\mathcal{B}$. The scalars are precisely the partial maps $1 \to 1$ on the tensor unit 1, and their multiplication is partial/Kleisli composition, see Definition [19].

We need to know a bit more about the monoidal structure in categories of partial maps.

Lemma 66. Let $\mathcal{B}$ be a monoidal effectus in total form. Then, in $\text{Part}(\mathcal{B})$,

1. $f \otimes 0 = 0: X \otimes Y \to A \otimes B$;

2. the following diagram commutes:

$$
\begin{array}{c}
X_1 \otimes A + X_2 \otimes A \\
\xrightarrow{\cong} (X_1 + X_2) \otimes A \\
\xrightarrow{\pi_1} X_1 \otimes A \\
\xrightarrow{f \otimes \text{Id}} X_1 \otimes A
\end{array}
$$
3. $f \otimes (g_1 \otimes g_2) = (f \otimes g_1) \otimes (f \otimes g_2)$.

4. The functor $\leftarrow : \mathcal{B} \to \text{Par}(\mathcal{B})$ preserves projections: these projections can be described in $\text{Par}(\mathcal{B})$ as:

$$\langle \pi_1 \rangle = \left( X \otimes Y \xrightarrow{\id \otimes 1} X \otimes 1 \xrightarrow{\rho} X \right)$$

$$\langle \pi_2 \rangle = \left( X \otimes Y \xrightarrow{1 \otimes \id} 1 \otimes Y \xrightarrow{\lambda} Y \right)$$

These projection maps are natural only wrt. total maps.

**Proof** We follow the relevant definitions.

1. We have $f \otimes 0 = 0$ via the following diagram, using that $0$ is the zero object in $\text{Par}(\mathcal{B})$.

```
\[ \begin{array}{c}
X \otimes A \xrightarrow{f \otimes 1} Y \otimes 0 \\
\downarrow \hspace{1cm} \downarrow \\
0 \otimes B
\end{array} \]  
```

2. We calculate in $\text{Par}(\mathcal{B})$, for $i = 1$,

$$(\triangleright_1 \otimes \id) * [\kappa_1 \otimes \id, \kappa_2 \otimes \id]$$

$$= [(\id, 0) \otimes \id] * (\kappa_1 \otimes \id), (\id, 0) \otimes \id] * (\kappa_2 \otimes \id)]$$

$$= [\id \otimes \id, 0 \otimes \id]$$

$$= [\id, 0] \text{ by point (1)}$$

$$= \triangleright_1.$$  

3. Let $b : Y \to B + B$ be a bound for $g_1, g_2 : Y \to B$. Then we take as new bound:

$$c = \left( X \otimes Y \xrightarrow{f \otimes b} A \otimes (B + B) \xrightarrow{=} A \otimes B + A \otimes B \right)$$

Then, $c$ is a bound for $f \otimes g_i$, since by point (2),

$$\triangleright_i \ast c = (\id \otimes \triangleright_i) \ast (f \otimes b) = f \otimes (\triangleright_i \ast b) = f \otimes g_i.$$ 

Hence:

$$\langle (f \otimes g_1) \otimes (f \otimes g_2) \rangle = \triangledown \ast c = (\id \otimes \triangledown) \ast (f \otimes b)$$

$$= f \otimes (\triangledown \ast b)$$

$$= f \otimes (g_1 \otimes g_2).$$

4. We use that the functor $\leftarrow : \mathcal{B} \to \text{Par}(\mathcal{B})$ preserves the tensor, and that $1 : X \to 1$ is the unique map to the final object in $\text{Par}(\mathcal{B})$. Hence the first projection in $\text{Par}(\mathcal{B})$ is:

$$\rho \ast (\id \otimes 1) = \rho \ast (\langle \id \rangle \otimes \langle ! \rangle) = (\rho \otimes \id) \circ \langle \id \rangle$$

$$= \kappa_1 \circ \rho \circ (\id \otimes !)$$

$$= \langle \pi_1 \rangle. \qed$$
In Proposition 13 we have seen that homsets of partial maps $X \rightarrow Y$ in an effectus form a partial commutative monoid (PCM). In the presence of tensors one also obtains scalar multiplication on such homsets.

**Lemma 67.** If $B$ is a monoidal effectus in total form, then each partial homset $\text{Par}(B)(X,Y)$ is a partial commutative module; partial pre- and post-composition preserves scalar multiplication, and thus the module structure.

Moreover, the homsets $B(X,Y)$ are convex sets, and the partial homsets $\text{Par}(B)(X,Y)$ are subconvex sets; again this structure is preserved by pre- and post-composition.

**Proof** For a partial map $f: X \rightarrow Y$ and a scalar $s: 1 \rightarrow 1$ we define $s \cdot f: X \rightarrow Y$ in $\text{Par}(B)$ as:

$$s \cdot f = \left( X \overset{\cong}{\longrightarrow} 1 \otimes X \overset{s \otimes f}{\longrightarrow} 1 \otimes Y \overset{\cong}{\longrightarrow} Y \right)$$

It is obviously an action, and preserves $0, \odot$ in each argument by Lemma 66.

We only show that the homset $B(X,Y)$ is convex. We essentially proceed as in (the proof of) Lemma 27: for scalars $r_1, \ldots, r_n \in \text{Pred}(1)$ with $\sum_i r_i = 1$, the bound $b: 1 \rightarrow n \cdot 1$ with $r_i \odot b = r_i$ and $\nabla \otimes b = 1: 1 \rightarrow 1$ is a total map of the form $b = s_1 \cdot s$. For an $n$-tuple of total maps $f_i: X \rightarrow X$ we now define the convex sum $\sum_i r_i f_i$ to be the composite in $B$:

$$X \overset{\cong}{\longrightarrow} 1 \otimes X \overset{s \otimes \text{id}}{\longrightarrow} (n \cdot 1) \otimes X \overset{\cong}{\longrightarrow} n \cdot (1 \otimes X) \overset{\cong}{\longrightarrow} n \cdot X \overset{[f_1, \ldots, f_n]}{\longrightarrow} Y \square$$

In Definition 24 we have described when substates $1 \rightarrow X$ are pure, via scalar multiplication for such states. With the scalar multiplication for arbitrary partial maps we can extend the definition of purity.

**Definition 68.** A partial map $f: X \rightarrow Y$ in a monoidal effectus is called pure if for each pair of orthogonal partial maps $f_1, f_2: X \rightarrow Y$ with $f = f_1 \odot f_2$ there is a scalar $s$ with:

$$f_1 = s \cdot f \quad \text{and} \quad f_2 = s^\perp \cdot f.$$

An object $X$ is called pure if the partial identity map $\text{id}: X \rightarrow X$ is pure, in the above sense.

It can be shown that a von Neumann algebra $\mathcal{A}$ is pure if and only if it is a factor, that is, its center is $\mathbb{C}$, the algebra of complex numbers. The proof will be given elsewhere.

Here is another notion that requires tensors. It is seen as the key property of quantum mechanics according to [CDP11]. It will be further elaborated elsewhere.

**Definition 69.** The purification of a substate $\omega: 1 \rightarrow X$ is a pure substate $\rho: 1 \rightarrow X \otimes Y$ with $\pi_1 \ast \rho = \omega$.  

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10.1 Commutative effectuses via copiers

This section concentrates on special tensors \( \otimes \) which come equipped with copier operations \( X \to X \otimes X \). It is shown that in presence of such copiers — satisfying suitable requirements — an effectus is commutative. This result is the logical contrapositive of the familiar fact that no-cloning holds in the quantum world: if we do have cloning, we are not in the quantum, but in the probabilistic, world.

**Definition 70.** We say that a monoidal effectus \( B \) in total form has copiers if for each object \( X \in B \) there is a copier, or diagonal, map \( \delta \colon X \to X \otimes X \) such that the following diagrams commute:

and additionally, for each partial map \( f \colon X \to X \) which is side-effect-free, that is satisfies \( f \leq \text{id} \), one has in \( \text{Par}(B) \):

\[
(f \otimes \text{id}) \circ \langle \delta \rangle = \langle \delta \rangle \circ f = (\text{id} \otimes f) \circ \langle \delta \rangle.
\]

(The two equations are equivalent.)

We now define an assert map in terms of copying. It’s easiest to do this in partial form. For a predicate \( p \colon X \to 1 \) write \( \text{asrt}_p \) for the composite:

\[
\text{asrt}_p = \left( X \xrightarrow{\delta} X \otimes X \xrightarrow{p \otimes \text{id}} 1 \otimes X \xrightarrow{\lambda} X \right)
\]

**Theorem 71.** A monoidal effectus with copiers is commutative.

**Proof** Let \( B \) be a monoidal effectus in total form, with copiers. We reason in \( \text{Par}(B) \). We first show that the map \( \text{asrt}_p \colon X \to X \) from (38) is side-effect-free, i.e. that it is below the (partial) identity \( X \to X \). This follows from:

\[
\text{asrt}_p \otimes \text{asrt}_p = (\lambda \circ (p \otimes \text{id}) \otimes \langle \delta \rangle) \circ (\lambda \circ (p \otimes \text{id}) \otimes \langle \delta \rangle)
\]

by Proposition [13] (2)

\[
= \lambda \circ ((p \otimes \text{id}) \otimes (p \otimes \text{id}) \otimes \langle \delta \rangle)
\]

by Lemma [66] (3)

\[
= \lambda \circ (1 \otimes \text{id}) \otimes \langle \delta \rangle
\]

by Lemma [66] (4)

\[
= \text{id}.
\]

Next we show that the mapping \( p \mapsto \text{asrt}_p \) makes the map \( \ker^\perp \colon \text{End}_{\leq \text{id}}(X) \to \)
\[ \text{Pred}(X) \text{ bijective, see Definition 55 (1).} \]

\[
\ker^\perp(\text{asrt}_p) = 1 \ast \lambda \ast (p \otimes \text{id}) \ast \langle \delta \rangle
\]
\[
= \lambda \ast (\text{id} \otimes 1) \ast (p \otimes \text{id}) \ast \langle \delta \rangle
\]
\[
= p \ast (p \otimes \text{id}) \ast (\text{id} \otimes 1) \ast \langle \delta \rangle \quad \text{since } \lambda = p: 1 \otimes 1 \rightarrow 1
\]
\[
= p \ast (\langle \pi_1 \rangle \ast \langle \delta \rangle)
\]
\[
= p \ast (\text{id} \otimes 1) \ast \langle \delta \rangle \quad \text{by Lemma 66 (4)}
\]
\[
= p.
\]

In the other direction, let \( f \): \( X \rightarrow X \) satisfy \( f \leq \text{id} \). The associated predicate \( \ker^\perp(f) = 1 \ast f \): \( X \rightarrow 1 \) satisfies:

\[
\text{asrt}_\ker^\perp(f) = \lambda \ast (1 \otimes \text{id}) \ast (f \otimes \text{id}) \ast \langle \delta \rangle
\]
\[
= \langle \pi_1 \rangle \ast \langle \delta \rangle \ast f \quad \text{by (37) and Lemma 66 (4)}
\]
\[
= f.
\]

Finally we have to prove that the assert maps commute: \( \text{asrt}_q \circ \text{asrt}_p = \text{asrt}_p \circ \text{asrt}_q \). Since the kernel-supplement map \( \ker^\perp \) is bijective it suffices to prove the middle equation in:

\[
\ker^\perp(\text{asrt}_q \circ \text{asrt}_p) = p \& q = q \& p = \ker^\perp(\text{asrt}_p \circ \text{asrt}_q).
\]

Hence we are done by:

\[
p \& q = q \ast \text{asrt}_p
\]
\[
= q \ast \lambda \ast (p \otimes \text{id}) \ast \langle \delta \rangle
\]
\[
= \lambda \ast (\text{id} \otimes q) \ast (p \otimes \text{id}) \ast \langle \delta \rangle
\]
\[
= \lambda \ast \gamma \ast (p \otimes q) \ast \langle \delta \rangle \quad \text{since } \lambda = \rho = \lambda \circ \gamma: 1 \otimes 1 \rightarrow 1
\]
\[
= \lambda \ast (q \otimes p) \ast \gamma \ast \langle \delta \rangle
\]
\[
= p \ast \lambda \ast (q \otimes \text{id}) \ast \langle \delta \rangle
\]
\[
= q \& p.
\]

We conclude by describing copiers in two of our examples.

**Examples 72.** In the commutative effectus \( K\ell(D) \) one has copiers \( \delta: X \rightarrow X \otimes X \) given by:

\[
\delta(x) = 1| (x, x).
\]

These copiers are not natural in \( X \). But we do not need naturality for the above theorem. We do need to check equation \( 57 \). So let \( f: X \rightarrow D_{\leq 1}(X) \) be side-effect-free map. Then \( f \) is of the form \( f(x) = p(x)|x \), for a predicate \( p \in [0, 1]^X \), see Example 56 (2). But then:

\[
((f \otimes \text{id}) \circ \delta)(x) = p(x)| (x, x)) = (\delta \circ f)(x).
\]

In the commutative effectus \( \text{CvNA}^{\text{op}} \) of commutative von Neumann algebras copiers \( \delta: \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A} \) are given by multiplication:

\[
\delta(a \otimes b) = a \cdot b.
\]
This map is positive since the multiplication of two positive elements is positive again in a commutative von Neumann algebra.

If \( f: \mathcal{A} \rightarrow \mathcal{A} \) is a subunital map below the identity, then \( f(x) = f(1) \cdot x \), see Example 56 (3). Hence the equation \( 37 \) holds:

\[
(\delta \circ (f \otimes \text{id}))(a \otimes b) = \delta(f(a) \otimes b) = (f(1) \cdot a) \cdot b
= f(1) \cdot (a \cdot b) = (f \circ \delta)(a \otimes b).
\]

11 Comprehension

Comprehension is the operation that assigns to a predicate \( p \) on a type/object \( X \) a new type/object \( \{X \mid p\} \) which intuitively contains those elements of \( X \) that satisfy \( p \). This comprehension type comes equipped with an inclusion map \( \pi_p: \{X \mid p\} \rightarrow X \) which we call a comprehension map.

In categorical logic comprehension is described nicely via an adjunction, see e.g. [Jac99], like any logical operation that comes equipped with an introduction and elimination rule. The slogan is: comprehension is right adjoint to truth. Later in Section 12 we shall see a similar situation for quotients, namely: quotients are left adjoint to falsity.

This truth functor exists as functor that maps an object \( X \) to the truth/top element \( 1(X) \in \text{Pred}(X) \) of predicates on \( X \). Our first task is to combine these set \( \text{Pred}(X) \) into a single category \( \text{Pred}(B) \) of predicates in an effectus \( B \). In the partial case there is a slightly different category of predicates.

Definition 73. Let \( B \) be an effectus in total form. We write \( \text{Pred}(B) \) for the category with:

- objects are pairs \( (X, p) \) where \( p \in \text{Pred}(X) \) is a predicate \( X \rightarrow 1 + 1 \) on \( X \); often we simply talk about objects \( p \in \text{Pred}(B) \) when the carrier \( X \) is clear from the context;
- morphisms \( f: (X, p) \rightarrow (Y, q) \) in \( \text{Pred}(B) \) are maps \( f: X \rightarrow Y \) in \( B \) with an inequality:

\[
p \leq f^*(q).
\]

We recall that \( f^*(q) = q \circ f \) is total substitution, which is a map of effect modules by Theorem 21.

We obtain a category \( \text{Pred}(B) \) by Exercise 1. It comes equipped with a forgetful functor \( \text{Pred}(B) \rightarrow B \) which is so trivial that we don’t bother to give it a name.

There are truth and falsity functors \( 1, 0: B \rightarrow \text{Pred}(B) \) given by the truth/top element \( 1(X) \in \text{Pred}(X) \) and the falsity/bottom \( 0(X) \in \text{Pred}(X) \) respectively. These functors form left and right adjoints to the forgetful functor in:

\[
\begin{array}{ccc}
\text{Pred}(B) & \rightarrow & B \\
\downarrow & & \downarrow \\
1 & \rightarrow & 0
\end{array}
\]

(39)
The forgetful functor $\text{Pred}(\mathcal{B}) \rightarrow \mathcal{B}$ is an example of a ‘fibration’, or ‘indexed category’, which is a basic structure in categorical logic and type theory [Jac99]. Here however we use this functor concretely, without going into the general theory.

The adjunction $0 \dashv \text{forget} \dashv 1$ in (39) exists because there are obvious bijective correspondences:

$$X \xrightarrow{f} Y \quad \text{in} \quad \mathcal{B}$$

$$(X, p) \xrightarrow{f} (Y, 1) = 1(Y) \quad \text{in} \quad \text{Pred}(\mathcal{B})$$

And:

$$X \xrightarrow{f} Y \quad \text{in} \quad \mathcal{B}$$

$$0(Y) = (X, 0) \xrightarrow{f} (Y, q) \quad \text{in} \quad \text{Pred}(\mathcal{B})$$

These trivial correspondences exist because one always has $p \leq f^*(1) = 1$ and $0 \leq f^*(q)$.

There is a similar, but slightly different, way to combine predicates in a category in the partial case.

**Definition 74.** Let $(C, I)$ be an effectus in partial form. We write $\text{Pred}^C(C)$ for the category with:

- objects $(X, p)$, where $p: X \rightarrow I$ is a predicate on $X$;
- morphisms $f: (X, p) \rightarrow (Y, q)$ are maps $f: X \rightarrow Y$ in $C$ satisfying:

$$p \leq f^C(q)$$

where $f^C(q) = (q^\perp \circ f^\perp)$ is partial substitution, which is monotone and preserves truth by Lemma 36. This yields a category by Exercise 1 (2).

Like in the total case we have falsity and truth functors as left and right adjoints to the forgetful functor.

$$\text{Pred}^C(C)$$

$$\xrightarrow{\text{C}}$$

$$(0 \dashv \cdot \dashv 1)$$

The adjoint correspondences in (40) work just like in the total case. They use that partial substitution $f^C$ preserves truth.

The above definition for an effectus in partial form applies in particular to the category $\text{Par}(\mathcal{B})$ of partial maps of an effectus $\mathcal{B}$. This is the topic of the next result.

**Lemma 75.** Let $\mathcal{B}$ be an effectus in total form, with its inclusion functor $\langle \cdot \rangle: \mathcal{B} \rightarrow \text{Par}(\mathcal{B})$. Then we have a diagram in which everything from left to right commutes:

$$\xrightarrow{\text{Pred}(\mathcal{B}) \xrightarrow{(0 \dashv \cdot \dashv 1)} \text{Pred}^C(\text{Par}(\mathcal{B})) \xrightarrow{\langle \cdot \rangle} \text{Par}(\mathcal{B})}$$
Proof The functor $\langle \_ \rangle : B \to Par(B)$ lifts to $Pred(B) \to Pred(Par(B))$ since a map $f: (X,p) \to (Y,q)$ in $Pred(B)$ yields a map $\langle f \rangle : (X,p) \to (Y,q)$. The reason is Exercise 113, giving: $p \leq f^*(q) = \langle f \rangle \circ (q)$.

We now come to the definition of comprehension, as right adjoint to truth.

We first give separate formulations for the total and partial case, but we prove that they are equivalent later on — see Theorem 78. Comprehension will be a functor from predicates to their underlying objects. We shall write it as $(X,p) \mapsto \{X|p\}$. When we consider it as a functor we write $\{\_ \_ \}$. 

Definition 76. We give separate formulations of essentially the same notion.

1. An effectus in total form $B$ has comprehension when its truth functor $1 : B \to Pred(B)$ has a right adjoint $\{\_ \_ \}$ such that for each predicate $p : X \to 1 + 1$ and object $Y \in B$ the canonical map

$$
\{X|p\} + Y \to \{X + Y | [p, 1]\}
$$

is an isomorphism.

2. An effectus in partial form $(C, I)$ has comprehension if its truth functor $1 : C \to Pred(I)(C)$ has a right adjoint $\{\_ \_ \}$ and each counit component $\{X|p\} \to X$ in $C$ is a total map.

Formally this counit is a map in $Pred(I)(C)$ of the form $\pi_p : (1, \{X|p\}) \to (X, p)$, but in the definition we have written it as a map in $B$, for simplicity. These comprehension maps $\pi_p$ should not be confused with the tensor projections $\pi_i$ in (36).

Example 77. We describe comprehension for our four running examples. We shall alternate between the total and partial descriptions. This comprehension structure occurs already in [CJWW15], except for order unit groups.

1. Recall that predicates on an object/set $X$ in the effectus $Sets$ correspond to subsets of $X$. The category $Pred(Sets)$ has such subsets ($P \subseteq X$) as objects; morphisms $f : (P \subseteq X) \to (Q \subseteq Y)$ are functions $f : X \to Y$ satisfying $P \subseteq f^{-1}(Q)$, that is: $x \in P \Rightarrow f(x) \in Q$.

The comprehension functor $Pred(Sets) \to Sets$ is given by $(Q \subseteq Y) \mapsto Q$. The adjoint correspondences for $1 + \{\_ \_ \}$ amount to:

$$
1(X) = (X \subseteq X) \xrightarrow{f} (Q \subseteq Y) \xrightarrow{g} Q = \{Y|Q\}
$$

Obviously, if $X \subseteq f^{-1}(Q)$, then $f(x) \in Q$ for each $x \in X$, so that $f$ restricts to a unique map $\bar{f} : X \to Q = \{Y|Q\}$ with $\pi_Q \circ \bar{f} = f$, where $\pi_Q : \{Y|Q\} \to Y$ is the inclusion.

For a predicate $P \subseteq X$ the predicate $[P, 1] \subseteq X + Y$ used in (11) is given by $[P, 1] = \{\kappa_1 x \mid x \in P\} \cup \{\kappa_2 y \mid y \in Y\}$. Hence it is immediate that we have: $\{X|P\} + Y = P + Y = [P, 1] = \{X + Y | [P, 1]\}$. 

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2. For discrete probability we use the effectus in partial form $\mathcal{K}(\mathcal{D}_{\leq 1}) \cong \text{Par}(\mathcal{K}(\mathcal{D}))$. The category $\text{Pred}_0(\mathcal{K}(\mathcal{D}_{\leq 1}))$ has fuzzy predicates $p \in [0,1]^X$ as objects. A morphism $f: (p \in [0,1]^X) \to (q \in [0,1]^Y)$ is a function $f: X \to \mathcal{D}_{\leq 1}(Y)$ satisfying, for each $x \in X$,

$$p(x) \leq f(q)(x) = \sum_y f(x)(y) \cdot q(y) + 1 - \sum_y f(x)(y).$$

Comprehension sends a predicate $q \in [0,1]^Y$ to the set $\{Y|q\} = \{y \in Y \mid q(y) = 1\}$ of elements where the predicate $q$ holds with certainty. Let’s check the correspondences:

$$\begin{array}{ccc}
(1 \in [0,1]^X) & \xrightarrow{f} & (q \in [0,1]^Y) \\
X & \xrightarrow{g} & \{y \in Y \mid q(y) = 1\} \\
\end{array} \quad \text{in} \quad \text{Pred}_0(\mathcal{K}(\mathcal{D}_{\leq 1}))$$

Let $f: X \to \mathcal{D}_{\leq 1}(Y)$ satisfy $1 \leq \sum_y f(x)(y) \cdot q(y) + 1 - \sum_y f(x)(y)$ for each $x \in X$. Hence $\sum_y f(x)(y) \leq \sum_y f(x)(y) \cdot q(y)$. This can only happen if $f(x)(y) > 0$ implies $q(y) = 1$, for each $x$. But this means that we can write each $f(x) \in \mathcal{D}_{\leq 1}(Y)$ as subconvex sum $f(x) = \sum_y f(x)(y) q(y)$ with $q(y) = 1$, and thus $y \in \{Y|q\}$. Hence we have $f(x) \in \mathcal{D}_{\leq 1}(\{Y|q\})$, so that $f$ restricts to a unique function $\overline{f}: X \to \mathcal{D}_{\leq 1}(\{Y|q\})$ with $\pi_Q \circ \overline{f} = f$, where the comprehension map $\pi_Q: \{Y|q\} \to \mathcal{D}_{\leq 1}(Y)$ is given by $\pi_Q(y) = 1$. Clearly, it is total.

3. The effectus $\text{OUS}^\text{op}$ of order unit groups also has comprehension. The description is rather confusing because we work in the opposite category. It turns out that in this model comprehension is given by a quotient! We follow the relevant description meticulously, in the total case.

The category $\text{Pred}(\text{OUS}^\text{op})$ of predicates has effects $e \in [0,1]_G$ in order unit groups $G$ as objects. A morphism $f: (e \in [0,1]_G) \to (d \in [0,1]_H)$ is a map $f: G \to H$ in $\text{OUS}^\text{op}$ with $e \leq f^*(d)$. This means that $f$ is a homomorphism of order unit groups $f: H \to G$ with $e \leq f(d)$ in $G$.

For an effect $d \in H$ we write $\langle d \rangle_H \subseteq H$ for the subset given by:

$$\langle d \rangle_H = \{x \in H \mid \exists n \in \mathbb{N}. -n \cdot d \leq x \leq n \cdot d\}. \quad (42)$$

It is not hard to see that $\langle d \rangle_H$ is a subgroup of $H$. It is an ideal, since it satisfies: $-x \leq y \leq x$ and $x \in \langle d \rangle_G$ implies $y \in \langle d \rangle_H$. In fact, $\langle d \rangle_H$ is the smallest ideal containing $d$. It inherits the order from $H$ and is an order unit group itself, with $d \in \langle d \rangle_H$ as unit.

We now define comprehension as quotient group:

$$\{H|d\} = H/(d^\perp)_H \quad \text{with} \quad \begin{cases} H & \xrightarrow{\pi_d} \{H|d\} \\
x & \xrightarrow{x + (d^\perp)_H} \end{cases}$$

First we have to check that $\{H|d\}$ is an order unit group. It is easy to see that the quotient of an ordered Abelian group with an ideal is again an ordered Abelian group, so we concentrate on checking that $\{H|d\}$ has
an order unit. We claim that \( \pi_d(1) = 1 + \langle d^\perp \rangle_H \) is the order unit in \( H/\langle d^\perp \rangle_H = \{H\} \). Indeed, for an arbitrary \( x \in H \) there is an \( n \in \mathbb{N} \) with \(-n \cdot 1 \leq x \leq n \cdot 1\). But then:

\[ -n \cdot \pi_d(1) = -n \cdot 1 + \langle d^\perp \rangle_H \leq x + \langle d^\perp \rangle_H \leq n \cdot 1 + \langle d^\perp \rangle_H = n \cdot \pi_d(1). \]

The claimed comprehension adjunction involves a bijective correspondence between:

\[
\begin{align*}
1(G) = (1 \in [0,1]_G) & \quad \xmapsto{f} \quad (H,d) \quad \text{in \text{Pred}(OUG^{op})} \\
G & \xmapsto{g} \quad \{H|d\} \quad \text{in OUG^{op}}
\end{align*}
\]

That is, between maps in OUG:

\[
\begin{array}{c}
H \xmapsto{f} G \quad \text{with} \quad f(d) = 1 \\
\{H|d\} \xmapsto{g} G
\end{array}
\]

This works as follows.

- Let \( f : H \to G \) in OUG satisfy \( f(d) = 1 \). Then for each \( x \in \langle d^\perp \rangle_H \) we have \( f(x) = 0 \). Indeed, if \(-n \cdot d^\perp \leq x \leq n \cdot d^\perp \), then, because \( f(d^\perp) = f(1 - d) = f(1) - f(d) = 1 - 1 = 0 \), we get:

\[ 0 = -n \cdot f(d^\perp) = f(-n \cdot d^\perp) \leq f(x) \leq f(n \cdot d^\perp) = n \cdot f(d^\perp) = 0. \]

Thus there is a unique group homomorphism \( \overline{f} : \{H|d\} = H/\langle d^\perp \rangle_H \to G \) by \( \overline{f}(x + \langle d^\perp \rangle_H) = f(x) \). This map \( \overline{f} \) is clearly monotone and unital.

- The other direction is easy: given \( g : \{H|d\} \to G \), take \( \overline{g} = g \circ \pi_d : H \to G \). Then:

\[ \overline{g}(d) = g(\pi_d(d)) = g(d + \langle d^\perp \rangle_H) \]
\[ = g(d + d^\perp + \langle d^\perp \rangle_H) \]
\[ = g(1 + \langle d^\perp \rangle_H) = g(1_{\{H|d\}}) = 1. \]

Clearly, \( \overline{f} = \overline{g} \circ \pi_d = f \). And \( \overline{f} = g \) holds by uniqueness, since \( \overline{f}(x) = g(\pi_d(x)) = g(x + \langle d^\perp \rangle_H) \).

Finally, for \( e \in [0,1]_G \) the canonical map \( \{G \oplus H|(e,1)\} \to \{G|e\} \oplus H \) is the function:

\[ (G \oplus H)/\langle(e,1)\rangle \xrightarrow{(x,y) \mapsto (x + \langle e^\perp \rangle_H,y)} G/\langle e^\perp \rangle \oplus H \]

The equivalence relation on \( G \oplus H \) induced by the ideal \( \langle(e,1)\rangle = \langle(e^\perp,0)\rangle \) is given by \( (x,y) \sim (x',y') \iff (x-x',y-y') \in \langle(e^\perp,0)\rangle \). The latter means that there is an \( n \in \mathbb{N} \) with \(-n \cdot (e^\perp,0) \leq (x-x',y-y') \leq n \cdot (e^\perp,0) \). This is equivalent to: \(-n \cdot e^\perp \leq x-x' \leq n \cdot e^\perp \) and \( y = y' \). The above function is thus well-defined, and clearly an isomorphism, with inverse \((x + \langle e^\perp \rangle_H,y) \mapsto (x,y) + ((e,1)\rangle). \)
4. The effectus in partial form Par($\mathbf{vNA}^{op}$) of von Neumann algebras and subunital maps also has comprehension. The relevant category of predicates $\text{Pred}_{\mathcal{V}}(\text{Par}(\mathbf{vNA}^{op}))$ has effects $e \in [0, 1]_{\mathcal{V}}$ in a von Neumann algebra $\mathcal{A}$ as objects. Morphisms $f: (e \in [0, 1]_{\mathcal{V}}) \to (d \in [0, 1]_{\mathcal{V}})$ are subunital completely positive normal maps $f: \mathcal{B} \to \mathcal{A}$ with $e \leq f(d)$ in $\mathcal{A}$.

Recall the sharp predicates on a von Neumann algebra are precisely the sharp element $s \in \mathcal{A}$, with the previous point we obtain, for instance: $\sqrt{d} \cdot |d| \cdot \sqrt{d} = \sqrt{d}$. The projection $[d]$ is known as the support projection of $d$. In other texts it is often denoted as $r(d)$, as it is as the projection onto the closed range of $d$. Before we construct the comprehension in $\mathbf{vNA}^{op}$, we list a few properties of $[d]$, which will be useful later on.

(a) The ascending sequence $d \leq d^{1/2} \leq d^{1/4} \leq \cdots$ has supremum $[d]$. This can be shown using the spectral theorem. It follows $[d] = [\sqrt{d}] = [d^2]$.

(b) We have $d = [d] \cdot d = d \cdot [d]$. In fact, $[d]$ is the least projection with this property. This is a consequence of Lemma [99]. In combination with the previous point we obtain, for instance: $\sqrt{d} \cdot |d| = \sqrt{d} \cdot \sqrt{d} = \sqrt{d} = [d] \cdot \sqrt{d}$.

Now we can define comprehension as:

$$\{ \mathcal{B}[d] = [d] \mathcal{B}[d] = \{ [d] \cdot x \cdot [d] \mid x \in \mathcal{B} \}.\$$

This subset $\{ \mathcal{B}[d] \}$ is itself a von Neumann algebra, with $[d]$ as unit. The associated comprehension map is the map $\pi_d: \mathcal{B} \to \{ \mathcal{B}[d] \}$ in $\mathbf{vNA}$ given by $\pi_d(x) = [d] \cdot x \cdot [d]$.

We must show that given a von Neumann algebra $\mathcal{A}$ and a subunital map $f: \mathcal{B} \to \mathcal{A}$ with $f(d) = f(1)$ there is a unique subunital map $g: [d] \mathcal{B}[d] \to \mathcal{A}$ with $g([d] \cdot x \cdot [d]) = f(x)$. Put $g(x) = f(x)$; the difficulty it to show that $f([d] \cdot x \cdot [d]) = f(d)$. For the details, see [WW15].

We sketch the proof here. By a variant of Cauchy-Schwarz inequality for the completely positive map $f$ (see [Pau02, Exc. 3.4]) we can reduce this problem to proving that $f([d]) = f(1)$, that is, $f([d^2]) = 0$. Since $[d^2]$ is the supremum of $d^1 \leq (d^2)^{1/2} \leq (d^2)^{1/4} \leq \cdots$ and $f$ is normal, $f([d^2])$ is the supremum of $f(d^1) \leq f((d^2)^{1/2}) \leq f((d^2)^{1/4}) \leq \cdots$, all of which turn out to be zero by Cauchy-Schwarz since $f(d) = f(1)$. Thus $f([d^2]) = 0$, and we are done.

We note that with the same argument, one shows that for any 2-positive normal subunital map $h: \mathcal{A} \to \mathcal{B}$, we have, for all effects $a \in [0, 1]_{\mathcal{A}}$,

$$h(a) = 0 \iff h([a]) = 0. \quad (43)$$

We come to the promised result that relates comprehension in the total and in the partial case.
Theorem 78. An effectus in total form \( B \) has comprehension iff its effectus in partial form \( \text{Par}(B) \) has comprehension.

In both cases the comprehension maps \( \pi_p: \{X|p\} \to X \) are monic — in \( B \) and in \( \text{Par}(B) \) respectively.

Proof Since the left adjoint truth functors \( 1: B \to \text{Pred}(B) \) and \( 1: \text{Par}(B) \to \text{Pred}_2(\text{Par}(B)) \) are faithful in both cases, the counits of the adjunctions are monic by a general categorial result — see e.g. (the dual of) Mac Lane IV.3 Thm.1. These counits are monic maps \( (\{X|p\},1) \to (X,p) \) in the categories \( \text{Pred}(B) \) and \( \text{Pred}_2(\text{Par}(B)) \) respectively. It is easy to see, using the truth functor \( 1 \), that the underlying maps are then monic in \( B \) and \( \text{Par}(B) \).

First, let \( 1: B \to \text{Pred}(B) \) have a right adjoint \( \{-,-\} \). The counit is a map \( \pi_p: (\{X|p\},1) \to (X,p) \in \text{Pred}(B) \) satisfies \( 1 \leq \pi_p^\square(p) = p \circ \pi_p \) by definition. The underlying map \( \pi_p: \{X|p\} \to X \) is monic in \( B \), as just noted.

We first investigate the canonical map (11), call it \( \sigma \), is the unique one in:

\[
\begin{array}{c}
\{X|p\} + Y \\
\stackrel{\pi_p + id}{\longrightarrow} \stackrel{\pi_p}{\longrightarrow} \{X + Y | [p,1]\}
\end{array}
\] (44)

This map \( \sigma \) exists by comprehension because:

\[
(\pi_p + id)^*( [p,1]) = [p, \kappa_1 \circ !] \circ (\pi_p + id) = [p \circ \pi_p, \kappa_1 \circ !] = [\kappa_1 \circ !, \kappa_1 \circ !] = \kappa_1 \circ ! = 1.
\]

By assumption, this \( \sigma \) is an isomorphism, for each predicate \( p \) and object \( Y \).

We now show that the truth functor \( 1: B \to \text{Pred}_2(\text{Par}(B)) \) is a right adjoint, on an object \((X,p)\) given by \( \{X|p\} \). There is total map \( \langle \pi_p \rangle = \kappa_1 \circ \pi_1: \{X|p\} \to X \) in \( \text{Par}(B) \), which satisfies by Exercise 13:

\[
\langle \pi_p \rangle^\triangle(p) = (\pi_p)^*(p) = p \circ \pi_p = 1.
\]

Let \( f: Y \to X \) be an arbitrary partial map satisfying \( 1 \leq f^\triangle(p) = [p, \kappa_1] \circ f = [p, 1]\circ f = f^\square([p,1]) \). Then there is a unique total map \( \bar{f}: Y \to \{X + 1 | [p,1]\} \) with \( \pi_{[p,1]} \circ \bar{f} = f \). We now take \( \bar{f} = \sigma^{-1} \circ \bar{f}: Y \to \{X|p\} + 1 \), using the isomorphism from (11) with \( Y = 1 \). Then:

\[
\langle \pi_p \rangle \circ \bar{f} = (\pi_p + id) \circ \sigma^{-1} \circ \bar{f} \overset{13}{=} \pi_{[p,1]} \circ \bar{f} = f.
\]

If \( g: Y \to \{X|p\} \) also satisfies \( \langle \pi_1 \rangle \circ g = f \), then \( \sigma \circ g: Y \to \{X + 1 | [p,1]\} \) satisfies \( \pi_{[p,1]} \circ \sigma \circ g = (\pi_p + id) \circ g = \langle \pi_p \rangle \circ g = f \). Hence \( \sigma \circ g = \bar{f} \), and thus \( g = \sigma^{-1} \circ \bar{f} \).

In the other direction, we show how comprehension in the partial case yields comprehension in the total case with the distributivity isomorphism (11). Therefore, assume that a right adjoint \( \{-,-\} \) to \( 1: B \to \text{Pred}_2(\text{Par}(B)) \) exists so that each counit map \( \{X|p\} \to X \) is total. We shall write this counit as \( \langle \pi_p \rangle \), for \( \pi_p: \{X|p\} \to X \) in \( B \). As noted in the very beginning of this proof, the map \( \langle \pi_p \rangle \) is monic in \( \text{Par}(B) \). Further, we have:

\[
p \circ \pi_p = (\pi_p)^*(p) = \langle \pi_p \rangle^\square(p) = 1.
\]
The last equation holds because the counit is a map \( \langle \pi_p \rangle : (\{X\}|p), 1) \to (X, p) \) in \( \text{Pred}_0(\text{Par}(B)) \).

We now show that there is also a right adjoint to the truth functor \( 1 : B \to \text{Pred}(B) \), on objects given by \( (X, p) \mapsto \{X\}|p \).

Let \( f : Y \to X \) be a total map satisfying \( f^*(p) = 1 \). Then \( \langle f \rangle = \kappa_1 \circ f : Y \to X \) in \( \text{Par}(B) \) satisfies \( \langle f \rangle \circ \langle g \rangle = f^*(p) \). Hence there is a unique partial map \( \hat{f} : Y \to \{X\}|p \) with \( \langle \pi_p \rangle \circ \hat{f} = \langle f \rangle \). The latter means that the outer diagram below commutes in the effectus \( B \).

\[
\begin{array}{c}
Y \\
\downarrow \hat{f} \\
\{X\}|p \\
\downarrow \kappa_1 \\
\{X\}|p \to + \pi_p+id \\
\end{array}
\]

\[
\begin{array}{c}
X \\
\downarrow \kappa_1 \\
\{X\}|p \to + 1 \\
\end{array}
\]

The rectangle in the middle is a pullback by (9). This mediating map \( \hat{f} \) is what we need, since it satisfies \( \pi_p \circ \hat{f} = f \) by construction. Moreover, if \( g : Y \to \{X\}|p \) in \( B \) also satisfies \( \pi_p \circ g = f \), then \( \langle g \rangle = \kappa_1 \circ g : Y \to \{X\}|p \) in \( \text{Par}(B) \) satisfies \( \langle \pi_p \rangle \circ \langle g \rangle = \langle \pi_p \circ g \rangle = \langle f \rangle \). Hence \( \langle g \rangle = \hat{f} \) by uniqueness. But then \( \kappa_1 \circ g = \langle g \rangle = \hat{f} = \kappa_1 \circ f \), and so \( g = f \) since \( \kappa_1 \) is monic. Thus we have shown that the truth functor \( 1 : B \to \text{Pred}(B) \) has a right adjoint.

We still have to prove that the canonical map \( \sigma : \{X\}|p \to \{X \to [Y \to [p, 1]]\} \) from (14) is an isomorphism. The inverse \( \sigma^{-1} \) for \( \sigma \) is obtained via the following pullback (1) from the definition of effectus:

\[
\begin{array}{c}
\{X + Y | [p, 1]\} \\
\downarrow \sigma^{-1} \\
\{X\}|p \to + (\text{id} + \text{id}) \pi_{[p, 1]} \\
\downarrow f \\
\{X\}|p \to + \text{id} + 1 \\
\end{array}
\]

\[
\begin{array}{c}
\{X + Y | [p, 1]\} \\
\downarrow (\text{id} + \text{id}) \pi_{[p, 1]} \\
\{X\}|p \to + 1 \\
\downarrow \text{id} + 1 \\
\{X\}|p \to + 1 + 1 \\
\end{array}
\]

The auxiliary map \( f \) in this diagram is the unique one with \( \langle \pi_p \rangle \circ f = (\text{id} + \text{id}) \circ \pi_{[p, 1]} : \{X + Y | [p, 1]\} \to X + 1 \). It exists since:

\[
((\text{id} + \text{id}) \circ \pi_{[p, 1]} \circ \sigma^{-1}) \circ \sigma^{-1} = \pi_{[p, 1]} \circ \sigma^{-1}
\]

We get \( \pi_{p + \text{id}} \circ \sigma^{-1} = \pi_{[p, 1]} : \{X + Y | [p, 1]\} \to X + Y \) via a similar pullback,
with \( X + Y \) in the upper left corner, since:
\[
(l + id) \circ (\pi_p + id) \circ \sigma^{-1} = (l + id) \circ \sigma^{-1} \\
= (l + id) \circ \pi_{[p,1]} \\
(id + !) \circ (\pi_p + id) \circ \sigma^{-1} = (\pi_p + id) \circ (id + !) \circ \sigma^{-1} \\
= (\pi_p + id) \circ f \\
= (id + !) \circ \pi_{[p,1]}.
\]

From this we can conclude \( \sigma \circ \sigma^{-1} = id \), using that \( \pi_{[p,1]} \) is monic in \( B \), and:
\[
\pi_{[p,1]} \circ \sigma \circ \sigma^{-1} = (\pi_p + id) \circ \sigma^{-1} = \pi_{[p,1]}.
\]
We obtain \( \sigma^{-1} \circ \sigma = id \) via uniqueness of mediating maps in the above pullback defining \( \sigma^{-1} \):
\[
(l + id) \circ \sigma^{-1} \circ \sigma = (l + id) \circ \pi_{[p,1]} \circ \sigma \\
= (l + id) \circ (\pi_p + id) \\
= (l + id) \\
(id + !) \circ \sigma^{-1} \circ \sigma = f \circ \sigma \\
= (id + !).
\]

The latter, marked equation uses that the map \( \langle \pi_p \rangle \) is monic in \( \text{Par}(B) \):
\[
\langle \pi_p \rangle \circ (f \circ \sigma) = (\pi_p + id) \circ f \circ \sigma \\
= (id + !) \circ \pi_{[p,1]} \circ \sigma \\
= (id + !) \circ (\pi_p + id) \\
= \langle \pi_p \rangle \circ (id + !).
\]

In the remainder of this section we list several properties of comprehension. Some of these properties are more naturally formulated in the total case, and some in the partial case. Therefore we split these results up in two parts.

**Lemma 79.** Let \( B \) be an effectus with comprehension, in total form.

1. A (comprehension) projection map \( \pi_p : \{X|p\} \rightarrow X \) can be characterised either as the equaliser in \( B \):

   \[
   \begin{array}{c}
   \{X|p\} \\
   \pi_p
   \end{array}
   \begin{array}{c}
   X \\
   \downarrow\downarrow 1 + 1
   \end{array}
   \]

   or as pullback:

   \[
   \begin{array}{c}
   \{X|p\} \\
   \pi_p
   \end{array}
   \begin{array}{c}
   X \\
   \downarrow\downarrow 1 + 1
   \end{array}
   \]

2. One has \( p^\perp \circ \pi_p = 0 \), and moreover the diagrams below are, respectively, an equaliser and a pullback in \( B \).
3. A projection $\pi_p : \{X | p\} \rightarrow X$ is an isomorphism if and only if $p = 1$.

4. The comprehension $\{X | 0\}$ is initial in $B$.

5. Projection maps are closed under pullback in $B$: for each predicate $p$ on $X$ and map $f : Y \rightarrow X$ we have a pullback in $B$:

$$
\begin{align*}
\{Y | f^*(p)\} & \rightarrow \{X | p\} \\
\pi_{f^*(p)} & \downarrow \\
Y & \rightarrow X
\end{align*}
$$

(45)

6. The pullback of a coprojection along an arbitrary map exists in $B$, and is given as in:

$$
\begin{align*}
\{X | p\} & \rightarrow Y \\
\pi_p & \downarrow \\
X & \rightarrow Y + Z
\end{align*}
$$

where $p = (1 + !) \circ f : X \rightarrow 1 + 1$

In particular, the coprojection $Y \rightarrow Y + Z$ is itself (isomorphic to) a comprehension map, namely to $Y \cong \{Y + Z | [1, 0]\} \rightarrow Y + Z$.

7. Projections of orthogonal predicates are disjoint: if $p \perp q$, then the diagram below is a pullback.

$$
\begin{align*}
0 & \rightarrow \{X | p\} \\
\downarrow & \downarrow \\
\{X | q\} & \rightarrow X
\end{align*}
$$

8. For predicates $p$ on $X$ and $q$ on $Y$ the sum map $\pi_p + \pi_q : \{X | p\} + \{Y | q\} \rightarrow X + Y$ is monic in $B$.

9. For predicates $p$ on $X$ and $q$ on $Y$ there is a (canonical) isomorphism as on the left below. Using point (1), this implies that the square on the right is a pullback.

$$
\begin{align*}
\{X | p\} + \{Y | q\} & \cong \{X + Y | [p, q]\} \\
\pi_{p+p_q} & \downarrow \\
X + Y & \rightarrow \{X + Y | [p, q]\}
\end{align*}
$$

$$
\begin{align*}
\{X | p\} + \{Y | q\} & \rightarrow 1 \\
\pi_{p+p_q} & \downarrow \\
X + Y & \rightarrow 1 + 1
\end{align*}
$$

Proof We use the formulation of comprehension for effectuses in Definition 76 (1).

1. This is just a reformulation of the universal property of comprehension.

2. We have $p^! \circ \pi_p = [\kappa_2, \kappa_1] \circ \rho \circ \pi_p = [\kappa_2, \kappa_1] \circ \kappa_1 \circ ! = \kappa_2 \circ ! = 0$. If $f : Y \rightarrow X$ satisfies $p \circ f = 0$, then $p^! \circ f = [\kappa_2, \kappa_1] \circ \rho \circ f = [\kappa_2, \kappa_1] \circ \kappa_2 \circ ! = \kappa_1 \circ ! = 1$. Hence $f$ factors through $\{X | p^!\}$ making the diagrams in point (2) an equaliser and a pullback.
3. We first show that $\pi_1: \{X|1\} \rightarrow X$ is an isomorphism. The identity map $id: X \rightarrow X$ trivially satisfies $1 = id^*(1)$. Hence there is a unique map $f: X \rightarrow \{X|1\}$ with $\pi_1 \circ f = id$. The equation $f \circ \pi_1 = id$ follows because the projections are monic: $\pi_1 \circ (f \circ \pi_1) = \pi_1 = \pi_1 \circ id$.

Conversely, if $\pi_p: \{X|p\} \rightarrow X$ is an isomorphism, then, using the pullback in point (1) we obtain $p = 1$ by writing:

$$p = \left( X \xrightarrow{\pi_p^{-1}} \{X|p\} \xrightarrow{1} 1 \xrightarrow{\kappa_1} 1 + 1 \right) = \left( X \xrightarrow{1} 1 + 1 \right)$$

4. The projection $\pi_0: \{X|0\} \rightarrow X$ gives rise to an equality of predicates $1 = 0: \{X|0\} \rightarrow 1 + 1$ via:

$$1 = \pi_0^*(0) = 0 \circ \pi_0 = 0.$$

Hence we have a situation:

![Diagram](image_url)

Proposition 4 says that the rectangle is a pullback, and that 0 is strict. This means that the map $\{X|0\} \rightarrow 0$ is an isomorphism.

5. The dashed arrow in diagram (45) exists since:

$$(f \circ \pi_f^*(p))^*(p) = p \circ f \circ \pi_f^*(p) = f^*(p) \circ \pi_f^*(p) = 1.$$ 

The square (45) is a pullback, since if $g: Z \rightarrow Y$ and $h: Z \rightarrow \{X|p\}$ satisfy $f \circ g = \pi_p \circ h$, then $g$ factors through $\pi_{f^*(p)}$ since:

$$g^*(f^*(p)) = p \circ f \circ g = p \circ \pi_p \circ h = 1 \circ h = 1.$$ 

6. First we have to check that the rectangle in point (6) commutes. It arises in a situation:

![Diagram](image_url)

The outer rectangle is a pullback by point (1). Hence the rectangle on the left is a pullback, by the Pullback Lemma.
7. We use that diagram [45] is a pullback, with \(f = \pi_q\). It yields that \(\{X[q] \mid \pi_q^*(p)\}\) forms a pullback. But since \(p \perp q\), and so \(p \leq q^\perp\), we have \(\pi_q^*(p) \leq \pi_q^*(q^\perp) = 0\). This gives \(\{X[q] \mid \pi_q^*(p)\} \cong 0\) by point (3). Hence the rectangle in point (7) is a pullback.

8. Let \(f, g : Y \to \{X[p] \mid \{X[q]\}\}\) satisfy \((\pi_p + \pi_q) \circ f = (\pi_p + \pi_q) \circ g\). We must show that \(f = g\). The next diagram is a pullback in \(B\), see Definition 2:

\[
\begin{array}{ccc}
\{X[p]\} + \{X[q]\} & \xrightarrow{1 + \text{id}} & 1 + \{X[q]\} \\
\text{id} + \text{id} & \downarrow & \downarrow \text{id} + \text{id} \\
\{X[p]\} + 1 & \xrightarrow{1 + \text{id}} & 1 + 1
\end{array}
\]

Hence we are done if we can prove \(f_1 = g_1\) and \(f_2 = g_2\) where:

\[
\begin{aligned}
f_1 &= (\text{id} + 1) \circ f \\
g_1 &= (\text{id} + 1) \circ g
\end{aligned}
\] \[\begin{aligned}
f_2 &= ((1 + \text{id}) \circ f \\
g_2 &= ((1 + \text{id}) \circ g)
\end{aligned}
\]

We recall from Theorem 78 that the projection maps \(\langle \pi_q \rangle = \kappa_1 \circ \pi_q\) are monic in \(\text{Par}(B)\). We then get:

\[\langle \pi_p \rangle \circ f_1 = (\pi_p + \text{id}) \circ (\text{id} + 1) \circ f = (\text{id} + 1) \circ (\pi_p + \pi_q) \circ f = (\text{id} + 1) \circ (\pi_p + \pi_q) \circ g = (\pi_p + \text{id}) \circ (\text{id} + 1) \circ g = \langle \pi_p \rangle \circ g_1.\]

Similarly one gets \(f_2 = g_2\).

9. For the isomorphism \(\{X[p]\} + \{Y[q]\} \cong \{X + Y \mid [p, q]\}\) we define total maps in both directions. First, consider the map \(\kappa_1 \circ \pi_p : \{X[p]\} \to X + Y\), satisfying:

\[\langle \kappa_1 \circ \pi_p \rangle^*([p, q]) = [p, q] \circ \kappa_1 \circ \pi_p = p \circ \pi_p = 1.\]

This yields a unique map \(\varphi_1 : \{X[p]\} \to \{X + Y \mid [p, q]\}\) with \(\pi_{[p, q]} \circ \varphi_1 = \kappa_1 \circ \pi_p\). In a similar way we get \(\varphi_2 : \{Y[q]\} \to \{X + Y \mid [p, q]\}\) with \(\pi_{[p, q]} \circ \varphi_2 = \kappa_2 \circ \pi_q\). The cotuple \(\varphi = [\varphi_1, \varphi_2] : \{X[p]\} + \{Y[q]\} \to \{X + Y \mid [p, q]\}\) gives a map in one direction. It satisfies, by construction:

\[\pi_{[p, q]} \circ \varphi = [\pi_{[p, q]} \circ \varphi_1, \pi_{[p, q]} \circ \varphi_2] = [\kappa_1 \circ \pi_p, \kappa_2 \circ \pi_q] = \pi_p + \pi_q.\]

For the other direction we consider the map \(\rhd_1 \circ \pi_{[p, q]} : \{X + Y \mid [p, q]\} \to X + 1\). It satisfies:

\[(\rhd_1 \circ \pi_{[p, q]})^\circ(p) = [p, \kappa_1] \circ (\text{id} + 1) \circ \pi_{[p, q]} = [p, 1] \circ \pi_{[p, q]} = \pi_{[p, q]}([p, 1]) = \pi_{[p, q]}([p, q]) = 1.\]
Hence by (the partial version of) comprehension there is a unique map
\[ \psi_1 : \{X + Y \mid [p, q]\} \to \{X[p]\} + 1 \text{ with } (\pi_p + \text{id}) \circ \psi_1 = \triangleright_1 \circ \pi_{[p, q]} \]. In a similar way there is a unique \( \psi_2 : \{X + Y \mid [p, q]\} \to \{Y[q]\} + 1 \text{ with } (\pi_q + \text{id}) \circ \psi_2 = \triangleright_2 \circ \pi_{[p, q]} \). We then obtain the map \( \psi = \langle \psi_1, \psi_2 \rangle : \{X + Y \mid [p, q]\} \to \{X[p]\} + \{Y[q]\} \), like in Lemma 5. This pairing exists since:

\[
\ker(\psi_1) = \left[\kappa_2, \kappa_1\right] \circ \left(\triangleright + \text{id}\right) \circ \psi_1
= \left[\triangleright_2 \circ \triangleright_1 \circ \pi_{[p, q]} \right]
= \left(\triangleright + \text{id}\right) \circ \pi_{[p, q]} \circ \psi_2
= \left(\triangleright + \text{id}\right) \circ \psi_2
= \ker^\perp(\psi_2).
\]

By (12) and the uniqueness of \( \langle\langle -, -\rangle\rangle \) we obtain:

\[
(\pi_p + \pi_q) \circ \psi = \langle \langle \pi_p \circ \psi_1, \pi_q \circ \psi_2 \rangle \rangle
= \langle \langle \triangleright_1 \circ \pi_{[p, q]}, \triangleright_2 \circ \pi_{[p, q]} \rangle \rangle = \pi_{[p, q]}.
\]

Our final aim is to show that \( \varphi \) and \( \psi \) are each others inverses. This is easy since both \( \pi_{[p, q]} \) and \( \pi_p + \pi_q \) are monic — the latter by point 5). Hence we are done with:

\[
\pi_{[p, q]} \circ \varphi \circ \psi = (\pi_p + \pi_q) \circ \psi = \pi_{[p, q]}
= (\pi_p + \pi_q) \circ \psi \circ \varphi = \pi_{[p, q]} \circ \varphi = (\pi_p + \pi_q).
\]

We turn to comprehension properties in the partial case. Of particular interest is Diagram (46) below, which shows that in presence of comprehension, kernel predicates give rise to kernel maps.

**Lemma 80.** Let \((C, I)\) be an effectus with comprehension, in partial form.

1. For each predicate \( p \) one has \( p \circ \pi_p = 1 \) and \( p^\perp \circ \pi_p = 0 \).
2. The category \( C \) has ‘total’ kernel maps: for each map \( f : X \to Y \) the following diagram is an equaliser:

\[
\begin{array}{c}
\{X \mid \ker(f)\} \\
\downarrow \pi_{\ker(f)} \\
X \\
\downarrow 0 \\
Y
\end{array}
\]

where the comprehension map \( \pi_{\ker(f)} \) is total, by definition.
3. For each map \( f \), the composite \( f \circ \pi_{\ker^{-1}(f)} \) is total.
4. Comprehension maps are closed under pullback in \( C \): for each map \( f : X \to Y \) and predicate \( p \) on \( Y \) there is a pullback:

\[
\begin{array}{c}
\{Y \mid f^\perp(p)\} \\
\downarrow \pi_{f^\perp(p)} \\
X \\
\downarrow f \\
Y
\end{array}
\]

\[ \pi_{f^\perp(p)} \]

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5. In presence of images, every map $f: X \to Y$ factors through the comprehension map of its image:

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\quad \quad \quad \searrow & \quad \quad & \swarrow \im(f) \\
\end{array}
\]

**Proof** We use the formulation of comprehension in the partial case from Definition 76 (2).

1. By definition the counit is a map $\pi_p: (\{X | p\}, 1) \to (X, p)$ in $\text{Pred}_0(C)$. This means $1 \leq \pi_p^\perp(p) = (p^+ \circ \pi_p)^\perp$. Hence $p^+ \circ \pi_p = 0$.

Since $\pi_p$ is total, pre-composition $(-) \circ \pi_p$ is a map of effect algebras, see Lemma 52 (4). Therefore:

\[
p \circ \pi_p = p^+ \circ \pi_p = (p^+ \circ \pi_p)^\perp = 0^\perp = 1.
\]

2. For the kernel predicate $\ker(f) = (1 \circ f)^\perp: X \to I$ of a map $f: X \to Y$ consider the comprehension map $\pi_{\ker(f)}: \{X | \ker(f)\} \to X$. It satisfies, by the previous point:

\[
1 \circ f \circ \pi_{\ker(f)} = \ker(f) \circ \pi_{\ker(f)} = 0.
\]

Hence $f \circ \pi_{\ker(f)} = 0$ by Definition 51 (3).

Next, let $g: Z \to X$ satisfy $f \circ g = 0 \circ g = 0$. Then:

\[
g^\perp(\ker(f)) = \ker(f \circ g) \quad \text{by Lemma 59 (3)}
\]

\[
= \ker(0) \quad \text{by Lemma 59 (2)}.
\]

Hence there is a necessarily unique map $\overline{g}: Z \to \{X | \ker(f)\}$ with $\pi_{\ker(f)} \circ \overline{g} = g$.

3. For each $f: X \to Y$ we have a total map $f \circ \pi_{\ker^+(f)}$ since by point (1):

\[
1 \circ f \circ \pi_{\ker^+(f)} = \ker^+(f) \circ \pi_{\ker^+(f)} = 1.
\]

4. The dashed arrow in diagram (47) exists because by Exercise 1 (2):

\[
(f \circ \pi_{f^+(p)})^\perp(p) = \pi_{f^+(p)}^\perp(f^+(p)) = 1.
\]

Next, assume we have maps $g: Z \to X$ and $h: Z \to \{X | p\}$ with $f \circ g = \pi_p \circ h$. The map $g$ then factors through $\pi_{f^+(p)}$ since:

\[
g^\perp(f^+(p)) = (g \circ f)^\perp(p) = (\pi_p \circ h)^\perp(p) = h^\perp(\pi_p^\perp(p)) = h^\perp(1) = 1.
\]

5. By definition $f^\perp(\im(f)) = 1$, so that $f$ factors through $\pi_{\im(f)}$. \qed
12 Quotients

Comprehension sends a predicate \( p \) on \( X \) to the object \( X|p| \) of elements of \( X \) for which \( p \) holds. Quotients form a dual operation: it sends a predicate \( p \) on \( X \) to the object \( X/p \) in which elements for which \( p \) holds are identified with 0.

We briefly describe comprehension and quotients for vector spaces and Hilbert spaces (like in [CLWW15]). Let \( LSub(Vect) \) and \( CLSub(Hilb) \) be the categories of linear (resp. closed linear) subspaces \( S \subseteq V \), where \( V \) is a vector (resp. Hilbert) space. Morphisms are maps between the underlying spaces that restrict appropriately. The obvious forgetful functors \( LSub(Vect) \to Vect \) and \( CLSub(Hilb) \to Hilb \) have two adjoints on each side:

\[
\begin{array}{ccc}
\text{LSub}(\text{Vect}) & \overset{\text{Vect}}{\rightarrow} & \text{Hilb} \\
\leftarrow V/S & \overset{\leftarrow 0} \rightarrow & \subseteq S \\
(\subseteq V) \mapsto & (\subseteq V) \mapsto & (\subseteq V) \mapsto \\
(\subseteq V) & \rightarrow S & \rightarrow S
\end{array}
\]

We see that in both cases comprehension is right adjoint to the truth functor \( 0 \), and is given by mapping a subspace \( S \subseteq V \) to \( S \) itself. This is like for \( \text{Sets} \), see Example [7].

Categorically, quotients have dual description to comprehension: not as right adjoint to truth, but as left adjoint to falsity \( 0 \). We briefly describe the associated dual correspondences, on the left below for vector spaces, and on the right for Hilbert spaces.

\[
\begin{array}{ccc}
(V/S) & \overset{\leftarrow g} \rightarrow & W \\
(\subseteq V) & \overset{f} \rightarrow & (\{0\} \subseteq W) \\
&S^\perp & \rightarrow \subseteq W \\
\end{array}
\]

The correspondence on the left says that a linear map \( f: V \to W \) with \( S \subseteq f^{-1}(\{0\}) = \ker(f) \) corresponds to a map \( g: V/S \to W \). Indeed, this \( g \) is given by \( g(v + S) = f(v) \). This is well-defined, because \( f(v) = f(v') \) if \( v \sim v' \), since the latter means \( v - v' \in S \), and thus \( f(v) - f(v') = f(v - v') = 0 \). This is the standard universal property of quotients in algebra.

The situation is more interesting for Hilbert spaces. A map \( f: V \to W \) with \( S \subseteq \ker(f) \) is completely determined by what it does on the orthosupplement \( V^\perp \), since each vector \( v \in V \) can be written as sum \( v = x + y \) with \( x \in S \) and \( y \in S^\perp \). Hence \( f(v) = f(y) \). This explains why \( f \) corresponds uniquely to a map \( g: S^\perp \to W \), namely its restriction.

Below we shall see more examples where quotients are given by complements of comprehension. But first we have to say what it means for an effectus to have quotients. Recall that comprehension can be defined both for effectuses in total and partial form in an equivalent manner, see Theorem [7]. In contrast, quotients only makes sense in the partial case.

**Definition 81.** We say that an effectus in partial form \((C, I)\) has quotients if its zero functor \( 0: C \to \text{Pred}_0(C) \) has a left adjoint.

We say that an effectus in total form \( B \) has quotients if its category \( \text{Par}(B) \) of partial maps has quotients.
For an effectus in partial form $\mathbf{C}$ with both comprehension and quotients we have a ‘quotient-comprehension chain’, like for vector and Hilbert spaces:

\[
\begin{array}{c}
\text{Quotient} \\
\downarrow \\
\downarrow \\
\text{Comprehension} \\
\end{array}
\begin{array}{c}
(X, p) \mapsto X/p \\
\downarrow \\
\downarrow \\
(X, p) \mapsto \{X|p\} \\
\end{array}
\]

We shall study such combinations in Section 14.

The unit of the quotient adjunction is a map in $\text{Pred}(\mathbf{C})$ which we write as:

\[
\begin{array}{c}
\xi_p \\
\downarrow \\
\downarrow \\
\end{array}
\begin{array}{c}
(X, p) \\
\downarrow \\
\downarrow \\
(X/p, 0) \\
\end{array}
\]

Hence by definition it is a map $\xi_p: X \to X/p$ in $\mathbf{C}$ satisfying $p \leq f^0(0) = \ker(f)$.

**Example 82.** Each of our four running examples of effectuses has quotients.

1. For the effectus $\textbf{Sets}$, the associated category $\text{Par}(\textbf{Sets})$ is the category of sets and partial functions. For a predicate $P \subseteq X$ we define the quotient set $X/P$ as comprehension of the complement:

\[
X/P \overset{\text{def}}{=} \{X|\neg P\} = \{x \in X \mid x \not\in P\}, \quad (48)
\]

analogously to the Hilbert space example described above. This gives a left adjoint to the zero functor $0: \text{Par}(\textbf{Sets}) \to \text{Pred}(\text{Par}(\textbf{Sets}))$, since there are bijective correspondences:

\[
\frac{(P \subseteq X) \xrightarrow{f} 0(Y)}{\{X|\neg P\} \xrightarrow{g} Y}
\]

Indeed, given $f: (P \subseteq X) \to 0(Y)$ in $\text{Pred}(\text{Par}(\textbf{Sets}))$, then $f: X \to Y$ is a partial function satisfying $P \subseteq f^0(0) = \{x \mid f(x) = \ast\}$, see (21).

Thus $f$ is determined by its outcome on the complement $\neg P$, so that we can simply define a corresponding partial function $\overline{f}: \{X|\neg P\} \to Y$ as $\overline{f}(x) = f(x)$. And, in the other direction, for $g: \{X|\neg P\} \to Y$ we define the extension $\overline{g}: X \to Y$ as:

\[
\overline{g}(x) = \begin{cases} 
\ast \text{ i.e. undefined} & \text{if } x \in P \\
g(x) & \text{if } x \in \neg P 
\end{cases}
\]

By construction, this $\overline{g}$ is a map $(P \subseteq X) \to (\emptyset \subseteq Y)$ in $\text{Pred}(\text{Par}(\textbf{Sets}))$ since:

\[
\ker(\overline{g}) = \overline{g}^0(0) = \{x \mid g(x) = \ast\} \supseteq P.
\]

It is easy to see $\overline{f} = f$ and $\overline{g} = g$. The unit $\xi_P: X \to X/P$ is the partial function with $\xi_P(x) = x$ if $x \in P$ and $\xi_P(x)$ undefined otherwise.
2. We recall that the category of partial maps for the effectus $\mathcal{K}(\mathcal{D})$ for discrete probability is the Kleisli category $\mathcal{K}(\mathcal{D}_{\leq 1})$ of the subdistribution monad $\mathcal{D}_{\leq 1}$, see Example $\mathbb{[}$ For a predicate $p \in [0,1]^X$ on a set $X$ we take as quotient:

$$X/p \overset{\text{def}}{=} \{ x \in X \mid p(x) < 1 \}. \quad (49)$$

The quotient adjunction involves bijective correspondences:

$$\begin{array}{ccc}
(p \in [0,1]^X) & \xrightarrow{L} & (0 \in [0,1]^Y) \\
X/p & \xrightarrow{g} & Y
\end{array}$$

This works as follows.

- Given $f: (p \in [0,1]^X) \to (0 \in [0,1]^Y)$ in $\text{Pred}_0(\mathcal{K}(\mathcal{D}_{\leq 1}))$, then $f: X \to \mathcal{D}_{\leq 1}(Y)$ satisfies $p(x) \leq f^\sharp(0)(x) = \ker(f)(x) = 1 - \sum_y f(x)(y)$ for each $x \in X$, see (23). We then define $\overline{f}: X/p \to \mathcal{D}_{\leq 1}(Y)$ as:

$$\overline{f}(x) = \sum_y \frac{f(x)(y)}{p(x)}(y) = \sum_y \frac{f(x)(y)}{1-p(x)}(y)$$

This is well-defined, since $p(x) \neq 1$ for $x \in X/p$.

- In the other direction, given a function $g: X/p \to \mathcal{D}_{\leq 1}(Y)$ we define $\overline{g}: X \to \mathcal{D}_{\leq 1}(Y)$ as:

$$\overline{g}(x) = \sum_y p^\perp(x) \cdot g(x)(y)$$

This $\overline{g}$ is a morphism $(p \in [0,1]^X) \to (0 \in [0,1]^Y)$ in $\text{Pred}_0(\mathcal{K}(\mathcal{D}_{\leq 1}))$, since we have an inequality $p \leq \ker(\overline{g})$ via:

$$\ker(\overline{g})(x) = 1 - \sum_y \overline{g}(x)(y) = 1 - \sum_y p^\perp(x) \cdot g(x)(y)$$

$$= p(x) + p^\perp(x) - p^\perp(x) \cdot \sum_y g(x)(y)$$

$$= p(x) - p^\perp(x) \cdot (1 - \sum_y g(x)(y))$$

$$= p(x) + p^\perp(x) \cdot \ker(g)(x)$$

$$\geq p(x).$$

Clearly, the mappings $f \mapsto \overline{f}$ and $g \mapsto \overline{g}$ are each other’s inverses. The unit map $\xi_p: X \to X/p$ is given by $\xi_p(x) = p^\perp(x)\|x\rangle = (1 - p(x))\|x\rangle$.

We notice that also in this case we can express the quotient object as comprehension, namely:

$$X/p = \{ x \in X \mid p(x) < 1 \} = \{ x \in X \mid \lceil p \rceil(x) = 1 \} = \{ X\lceil p \rceil \},$$

where, in general, $\lceil q \rceil$ is the least sharp predicate above $q$, given by:

$$\lceil q \rceil(x) = \begin{cases} 1 & \text{if } q(x) > 0 \\ 0 & \text{if } q(x) = 0 \end{cases} \quad (50)$$

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3. For the effectus of order unit groups we recall from Example 10 that the category $\text{Par}(\text{OUG}^{\text{op}})$ of partial maps in $\text{OUG}^{\text{op}}$ contains positive subunital group homomorphisms as maps. Because of the ‘opposite’ involved, we define quotients via subgroups, in the following way. For an effect $e \in [0,1]_G$ in an order unit group $G$ we take:

$$G/e \overset{\text{def}}{=} \langle e^\perp \rangle_G, \quad (51)$$

where $\langle e^\perp \rangle_G \subseteq G$ is the subgroup given by $\langle e \rangle_G = \{ x \in G \mid \exists n \in \mathbb{N}, -n \cdot e^\perp \leq x \leq n \cdot e^\perp \}$, see Example 77 (3). We write the inclusion $\langle e^\perp \rangle_G \subseteq G$ as a map $\xi_e : \langle e^\perp \rangle_G \rightarrow G$. This is a positive group homomorphism, which is subunital since:

$$\xi_e(1_G/e) = \xi_e(e^\perp) = e^\perp \in [0,1]_G.$$

The quotient adjunction involves a bijective correspondence between:

$$\begin{array}{ccc}
(G,e) & \xrightarrow{f} & 0(H) \\
G/e & \xrightarrow{g} & H
\end{array} \quad \text{in } \text{Pred}_0(\text{OUG}^{\text{op}}) \quad \text{in } \text{Par}(\text{OUG}^{\text{op}})$$

That is, between positive subunital (PsU) group homomorphisms:

$$\begin{array}{ccc}
H & \xrightarrow{f} & G \text{ with } e \leq f^0(0) \\
H & \xrightarrow{g} & G/e
\end{array}$$

where $f^0(0) = \ker(f) = f(1)^\perp$, see (21). This works as follows.

- Given $f : H \rightarrow G$ with $e \leq \ker(f) = f(1)^\perp$, we have $f(1) \leq e^\perp$. For each $x \in H$ there is an $n$ with $-n \cdot 1 \leq x \leq n \cdot 1$. Using that $f$ is positive/monotone we get:

$$-n \cdot e^\perp \leq -n \cdot f(1) \leq f(x) \leq f(n \cdot 1) = n \cdot f(1) \leq n \cdot e^\perp.$$

Hence $f(x) \in \langle e^\perp \rangle_G$, so that we can simply define $\overline{f} : H \rightarrow \langle e^\perp \rangle_G = G/e$ as $\overline{f}(x) = f(x)$.

- Given $g : H \rightarrow G/e = \langle e^\perp \rangle_G$, define $\overline{g} = \xi_e \circ g : H \rightarrow G$. This is a subunital map with:

$$\overline{g}^2(0) = \overline{g}(1)^\perp = \xi_e \circ (g(1))^\perp = (g(1))^\perp \geq (1_{\langle e^\perp \rangle_G})^\perp = (e^\perp)^\perp = e.$$

We have $\overline{f} = \xi_e \circ \overline{f} = f$, and $\overline{g} = g$ by injectivity of $\xi_e$. $\square$

4. In the effectus $\text{vNA}^{\text{op}}$ of von Neumann algebras the partial maps are also the subunital ones, see Example 11. The quotient of an effect $e \in [0,1]_\mathcal{A}$ in a von Neumann algebra $\mathcal{A}$ is defined as:

$$\mathcal{A}/e \overset{\text{def}}{=} [e^\perp]_\mathcal{A}[e^\perp] = \{ [e^\perp] \cdot x \cdot [e^\perp] \mid x \in \mathcal{A} \}. \quad (52)$$
It uses the ‘ceiling’ $[e^\bot] = [e]^\bot$ from Example 77. The quotient map $\xi_e : \mathcal{A} \to \mathcal{A}/e$ in $\mathbf{vNA}^{op}$ is given by the subunital function $\mathcal{A}/e \to \mathcal{A}$ with:

$$\xi_e(x) = \sqrt{e^\bot} \cdot x \cdot \sqrt{e^\bot}. \quad (53)$$

This map incorporates Lüders rule, see e.g. [BS98, Eq.(1.3)].

The proof that these constructions yield a left adjoint to the zero functor is highly non-trivial. For details we refer to [WW15]. We conclude that once again quotient objects can be expressed via comprehension:

$$\{\mathcal{A}[e^\bot]\} = \lceil \lceil e^\bot \rceil \rceil \mathcal{A} \lceil \lceil e^\bot \rceil \rceil = \lceil e^\bot \rceil \mathcal{A} \lceil e^\bot \rceil = \mathcal{A}/e.$$  

We continue with a series of small results that hold for quotients.

**Lemma 83.** Let $(C, I)$ be an effectus with quotients, in partial form.

1. The unit $\xi_p : X \to X/p$ satisfies $\ker(\xi_p) = p$ and thus $\ker^+(\xi_p) = p^\bot$.
2. The unit map $\xi_p : X \to X/p$ is epic in $C$.
3. For a predicate $q : X/p \to I$ one has $q \circ \xi_p \leq p^\bot$.
4. The functor $\text{Pred}_{\{\bot\}}(C) \xrightarrow{\xi_{(-)}} \text{Quot}(C)$

    is full and faithful — where $\text{Quot}(C)$ is the obvious category with epis $\twoheadrightarrow$ as objects and commuting squares between them as arrows.

5. The universal property of quotient amounts to: for each $f : X \to Y$ with $p \leq \ker(f)$ there is a unique $\tilde{f} : X/p \to Y$ with $\tilde{f} \circ \xi_p = f$. But we can say more: if there is an equality $p = \ker(f)$, then $\tilde{f}$ is total.

6. There are total isomorphisms $\xi_0 : X \xrightarrow{\cong} X/0$ and $0 \xrightarrow{\cong} X/1$.

7. For each predicate $p$ on $X$ there is a total ‘decomposition’ map $dc_p : X \to X/p^\bot + X/p$, namely the unique map in:

$$X/p^\bot \xrightarrow{\xi_p} X/0 \xrightarrow{\xi_p} X/p \xrightarrow{\xi_p} X/p + X/0 \xrightarrow{\xi_p} X/p.$$

7. For each predicate $p$ on $X$ there is a total ‘decomposition’ map $dc_p : X \to X/p^\bot + X/p$, namely the unique map in:

Using the pairing notation of Lemma 6 we can describe it as $dc_p = \langle \xi_p, \xi_p \rangle$, and thus also as total sum $dc_p = (\kappa_1 \circ \xi_p^+) \& (\kappa_2 \circ \xi_p)$ of two maps $X \to X/p^\bot + X/p$, see Lemma 43.

8. Each total map $f : Y \to X_1 + X_2$ can be decomposed in the subcategory $\text{Tot}(C) \hookrightarrow C$ of total maps as $f = (f_1 + f_2) \circ dc_p$ where $p = (\bot + \bot) \circ f$ and $f_1, f_2$ are total, and uniquely determined.

9. For predicates $p$ on $X$ and $q$ on $Y$ the sum map $\xi_p + \xi_q : X + Y \to X/p + Y/q$ is epic in $C$. 

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10. For predicates $p$ on $X$ and $q$ on $Y$ there is a total (canonical) isomorphism in:

\[(X/p) + (Y/q) \cong (X + Y)/[p,q]\]

11. Partial projections $\rhd_1$ are (isomorphic to) quotient maps, via total isomorphisms:

\[(X + Y)/[0,1] \cong X \quad (X + Y)/[1,0] \cong Y\]

12. For a predicate $p$ on $X$ there is an isomorphism of effect modules:

\[\text{Pred}((X/p)^\perp) \cong \downarrow p = \{ q \in \text{Pred}(X) \mid q \leq p \}\]

As a result, if $X/p \cong 0$, then $p = 1$.

13. Quotient maps are closed under composition, up to total isomorphisms. Given a predicate $p$ on $X$ and $q$ on $X/p$, there is an isomorphism as on the left below. It uses the operation $p^\perp \cdot (-) = (-) \circ \xi_p$ from point (12).

\[(X/p)/q \cong (X/(p^\perp \cdot q^\perp))^\perp \quad (X/p)/r \cong (X/(r^\perp \cdot p^\perp))^\perp \]

Equivalently, for predicates $p,r$ on $X$ with $p \leq r$, there is a total isomorphism as on the right above.

14. If our effectus $\mathbf{C}$ has images, then it has cokernel maps: for each map $f : X \to Y$ we have a coequaliser in $\mathbf{C}$ of the form:

\[X \xrightarrow{f} Y \xrightarrow{\text{coker}(f) \triangleq \xi_{\text{im}(f)}} Y/\text{im}(f)\]

Thus: $\text{coker}(f) = 0$ iff $\text{im}(f) = 1$, that is, iff $f$ is internally epic.

The decomposition property of maps from point (8) plays an important role in Section (13) when we relate extensive categories and (Boolean) effectuses.

Proof We reason in the effectus in partial form $\mathbf{C}$ and do not use different notation for partial and total maps.
1. Since the unit \( \xi_p \) is a map \((X, p) \to (X/p, 0)\) in \( \text{Pred}_0(C) \) we have an inequality \( p \leq \xi_p^0(0) = \ker(\xi_p) \) by definition. In order to get an equation \( p = \ker(\xi_p) \), notice that the predicate \( p^\perp : X \to I \) is a map \((X, p) \to (I, 0)\) in \( \text{Pred}_0(C) \), since:

\[
(p^\perp)^0(0) = (0^\perp \circ p^\perp)^\perp = (1_I \circ p^\perp)^\perp = (\text{id}_I \circ p^\perp)^\perp = p.
\]

Hence there is a unique map \( \overline{p} : X/p \to I \) with \( \overline{p} \circ \xi_p = p^\perp \). But then we are done by Lemma 39 (1) and (6):

If we have an equation \( p = \ker(p^\perp) \) using that the unit map \( \xi \) is total, by Definition 51 (3) since \( \ker(p^\perp) = \ker(\overline{p} \circ \xi_p) \geq \ker(\xi_p) \).

2. Let maps \( f, g : X/p \to Y \) satisfy \( f \circ \xi_p = g \circ \xi_p = h \), say. This \( h : X \to Y \) then satisfies \( p = \ker(\xi_p) \leq \ker(f \circ \xi_p) = \ker(h) = h^0(0) \). Hence there is a unique \( \overline{h} : X/p \to Y \) with \( \overline{h} \circ \xi_p = h \). But then \( f = \overline{h} = g \) by uniqueness.

3. For a predicate \( q \) on \( X/p \) one has:

\[
q \circ \xi_p = (q^\perp \circ \xi_p)^\perp = \xi_p^0(q^\perp)^\perp \leq \xi_p^0(0)^\perp \quad \text{since} \quad 0 \leq q^\perp \\
= \ker(\xi_p)^\perp \\
= p^\perp \quad \text{by point (1)}.
\]

4. Let’s assume we have a commuting diagram in \( C \) of the form:

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{\xi_p} & & \downarrow{\xi_q} \\
X/p & \xrightarrow{\sim} & Y/q
\end{array}
\]

We have to prove \( p \leq f^0(q) \), making \( f \) a map \((X, p) \to (Y, q)\) in \( \text{Pred}_0(C) \). This follows from:

\[
p = \ker(\xi_p) = \xi_p^0(0) \leq \xi_p^0(g^0(0)) \\
= f^0(\xi_p^0(0)) = f^0(\ker(\xi_q)) = f^0(q).
\]

5. Clearly, \( p \leq \ker(f) = f^0(q) \) means that \( f \) is a map \((X, p) \to (Y, 0)\) in \( \text{Pred}_0(C) \). Hence the unique map \( \overline{f} : X/p \to Y \) is the adjoint transpose.

If we have an equation \( p = \ker(f) \), then we can show that \( \overline{f} \) is total, by using that the unit map \( \xi_p \) is epic:

\[
1 \circ \overline{f} \circ \xi_p = 1 \circ f = \ker(f)^\perp = p^\perp = \ker(\xi_p)^\perp = 1 \circ \xi_p.
\]

6. We first show \( X/0 \cong X \). The identity map in \( X \) satisfies: \( \ker(\text{id}_X) = 0 \). Hence by point (3) there is a unique total map \( f : X/0 \to X \) satisfying \( f \circ \xi_0 = \text{id} \). The map \( \xi_0 \) is also total since \( 1 \circ \xi_0 = \ker(\xi_0) = 0^\perp = 1 \). We now get \( \xi_0 \circ f = \text{id}_X \) by using that \( \xi_0 \) is epic: \( (\xi_0 \circ f) \circ \xi_0 = \xi_0 = \text{id} \circ \xi_0 \).

Next, for the isomorphism \( X/1 \cong 0 \), we notice that the zero map \( 0 : X \to 0 \) in \( C \) satisfies \( \ker(0) = (0^\perp \circ 0)^\perp = 0^\perp = 1 \), so that there is a unique total map \( f : X/1 \to 0 \) with \( f \circ \xi_1 = 0 \). Obviously, \( f \circ 1_{X/1} = \text{id}_0 \), where \( 1_{X/1} : 0 \to X/1 \) is total. We also have \( 1_{X/1} \circ f = \text{id}_{X/1} \), since \( \xi_1 \) is epic and: \( (1_{X/1} \circ f) \circ \xi_1 = 1_{X/1} \circ 0 = 0 = \xi_1 \). This last equation holds by Definition 51 (3) since \( 1 \circ \xi_1 = \ker(\xi_1) = 1^\perp = 0 \).
7. The diagram uniquely defines $d_{cp} = \langle\langle \xi_p \perp, \xi_p \rangle\rangle$ as described in Lemma. It satisfies $\kappa_2, \kappa_1 \circ d_{cp} = d_{cp} \perp$ since:

$$\triangleright_1 \circ [\kappa_2, \kappa_1] \circ d_{cp} = [\text{id, 0}] \circ \kappa_2, [\text{id, 0}] \circ \kappa_1 \circ d_{cp} = [0, \text{id}] \circ d_{cp} = \triangleright_2 \circ d_{cp} = \xi_p.$$ 

Similarly one proves $\triangleright_2 \circ [\kappa_2, \kappa_1] \circ d_{cp} = \xi_p \perp$, so that:

$$[\kappa_2, \kappa_1] \circ d_{cp} = \langle\langle \xi_p, \xi_p \rangle\rangle = d_{cp} \perp.$$ 

8. Let $f: Y \rightarrow X_1 + X_2$ be total, and write $p = (! + !) \circ f: Y \rightarrow 1 + 1$ in the subcategory $\text{Tot}(C)$ of total maps. The partial map $\triangleright_1 \circ f: Y \rightarrow X_1$ satisfies:

$$\ker (\triangleright_1 \circ f) = ((! + \text{id}) \circ (\text{id} + !) \circ f)^\perp = p^\perp.$$ 

Hence there is a unique total map $f_1: Y/p^\perp \rightarrow X_1$ with $f_1 \circ \xi_p^\perp = \triangleright_1 \circ f$. Similarly, from $\ker (\triangleright_2 \circ f) = p$ we obtain a unique total $f_2: Y/p \rightarrow X_2$ with $f_2 \circ \xi_p = \triangleright_2 \circ f$. But then in $\text{Tot}(C)$, using Lemma:

$$(f_1 + f_2) \circ d_{cp} = (f_1 + f_2) \circ \langle\langle \xi_p^\perp, \xi_p \rangle\rangle = \langle\langle f_1 \circ \xi_p^\perp, f_2 \circ \xi_p \rangle\rangle$$

$$= \langle\langle \triangleright_1 \circ f, \triangleright_2 \circ f \rangle\rangle = f.$$ 

9. In an arbitrary category, if $f: X \rightarrow A$, $g: Y \rightarrow B$ are both epic, then so is the sum $f + g$.

10. We first have to say what the canonical map $\tau: (X + Y)/[p, q] \rightarrow (X/p) + (Y/q)$ is. Consider the sum map $\xi_p + \xi_q$ in $C$ with:

$$\ker^\perp (\xi_p + \xi_q) = 1 \circ (\xi_p + \xi_q) = [1, 1] \circ (\xi_p + \xi_q) \quad \text{by Lemma}$$

$$= [1 \circ \xi_p, 1 \circ \xi_q]$$

$$= [\ker^\perp (\xi_p), \ker^\perp (\xi_q)]$$

$$= [p^\perp, q^\perp]$$

$$= [p, q]^\perp \quad \text{by Theorem}$$

This last step uses the isomorphism of effect modules $\text{Pred}(X + Y) \cong \text{Pred}(X) \times \text{Pred}(Y)$. The resulting equation $\ker (\xi_p + \xi_q) = [p, q]$ yields by point a unique total map $\tau: (X + Y)/[p, q] \rightarrow X/p + Y/q$ with $\tau \circ \xi_{[p, q]} = \xi_p + \xi_q$.

In the other direction, consider the two composites:

$$X \xrightarrow{\kappa_1} X + Y \xrightarrow{\xi_{[p, q]}} (X + Y)/[p, q]$$

$$Y \xrightarrow{\kappa_2} X + Y \xrightarrow{\xi_{[p, q]}} (X + Y)/[p, q]$$
The first one satisfies:

\[
\ker (\xi_{[p,q]} \circ \kappa_1) = (1 \circ \xi_{[p,q]} \circ \kappa_1)^\perp = (\ker^\perp (\xi_{[p,q]} \circ \kappa_1))^\perp = ([p^\perp, q^\perp] \circ \kappa_1)^\perp = p.
\]

Similarly, \( \ker (\xi_{[p,q]} \circ \kappa_2) = q \). Hence there are unique total maps \( f : X/p \to (X + Y)/[p,q] \) and \( g : Y/q \to (X + Y)/[p,q] \) with \( f \circ \xi_p = \xi_{[p,q]} \circ \kappa_1 \) and \( g \circ \xi_q = \xi_{[p,q]} \circ \kappa_2 \). We claim that the cotuple \([f,g] : X/p + Y/q \to (X + Y)/[p,q] \) is the inverse of the canonical map \( \tau \).

We obtain \([f,g] \circ \tau = \id \) from the fact that \( \xi \)'s are epic:

\[
[f,g] \circ \tau \circ \xi_{[p,q]} = [f,g] \circ (\xi_p + \xi_q) = [f \circ \xi_p, g \circ \xi_q] = [\xi_{[p,q]} \circ \kappa_1, \xi_{[p,q]} \circ \kappa_2] = \xi_{[p,q]}.
\]

In the other direction we use that the sum map \( \xi_p + \xi_q \) in \( C \) is epic:

\[
\tau \circ [f,g] \circ (\xi_p + \xi_q) = \tau \circ [f \circ \xi_p, g \circ \xi_q] = [\tau \circ \xi_{[p,q]} \circ \kappa_1, \tau \circ \xi_{[p,q]} \circ \kappa_2] = [(\xi_p + \xi_q) \circ \kappa_1, (\xi_p + \xi_q) \circ \kappa_2] = (\xi_p + \xi_q).
\]

11. By points (10) and (13) we have isomorphisms:

\[
\begin{align*}
\begin{array}{ccc}
X + Y & \xrightarrow{\xi_{[p,1]}} & (X + Y)/[0,1] \\
& \xleftarrow{\xi_0+\xi_1} & (X/0) + (Y/1) \\
& \xrightarrow{id+1} & X + 0 \\
& \xleftarrow{1} & X \\
\end{array}
\end{align*}
\]

12. For a predicate \( q \leq p \), we have \( p^\perp \leq q^\perp = \ker(q) \). Hence there is a unique predicate, written as \( q/p : X/p^\perp \to I \), with \( (q/p) \circ \xi_{p^\perp} = q \). In the other direction, for \( r : X/p^\perp \to I \) we write \( p \cdot r = r \circ \xi_{p^\perp} : X \to I \). Then \( p \cdot r \leq p \) follows from:

\[
(p \cdot r)^\perp = \ker(p \cdot r) = \ker(r \circ \xi_{p^\perp}) \geq \ker(\xi_{p^\perp}) \quad \text{by Lemma 39 (6)}
\]

\[
= p^\perp.
\]

By construction, \( p \cdot (q/p) = q \). But also \( (p \cdot r)/p = r \), because \( \xi \)'s are epic, and: \((p \cdot r)/p \circ \xi_{p^\perp} = p \cdot r = r \circ \xi_{p^\perp} \). It requires a bit of work to verify that these mappings \((-)/p\) and \(p \cdot (-)\) preserve the effect module structure.

Finally, if \( X/p \cong 0 \), then \( p^\perp \cong \Pred(X/p) \cong \Pred(0) \cong 1 \). But then \( p^\perp = \{0\} \), so that \( p^\perp = 0 \) and thus \( p = 1 \).

13. Let \( p \) be a predicate on \( X \) and \( q \) on \( X/p \). Then by Lemma 39 (6):

\[
\ker(\xi_{q} \circ \xi_{p}) = \xi_{p}^\perp(\ker(\xi_{q})) = \xi_{p}^\perp(q) = (q^\perp \circ \xi_{p})^\perp = (p^\perp \cdot q^\perp)^\perp.
\]

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Hence there is a unique (total) map \( f: X/(p^\perp \cdot q^\perp)^\perp \to (X/p)/q \) with \( f \circ \xi_{X/(p^\perp \cdot q^\perp)^\perp} = \xi_p \circ \xi_q \). In the other direction we proceed in two steps. First, by construction, \( p^\perp \cdot q^\perp \leq p^\perp \), so \( p \leq (p^\perp \cdot q^\perp)^\perp \). This yields a map \( g: X/p \to X/(p^\perp \cdot q^\perp)^\perp \) with \( g \circ \xi_p = \xi_{(p^\perp \cdot q^\perp)^\perp} \). We wish to show that there is a total map \( h: (X/p)/q \to X/(p^\perp \cdot q^\perp)^\perp \) with \( h \circ \xi_q = g \). This works if \( \ker(g) = q: X/p \to I \). Because the \( \xi \)'s are epic in \( C \) we are done by:

\[
\ker^\perp(g) \circ \xi_p = 1 \circ g \circ \xi_p = 1 \circ \xi_{(p^\perp \cdot q^\perp)^\perp} = \ker^\perp(\xi_{(p^\perp \cdot q^\perp)^\perp}) = p^\perp \cdot q^\perp
\]
\[
= q^\perp \circ \xi_p.
\]

We now have (total) maps \( (X/p)/q \rightleftarrows X/(p^\perp \cdot q^\perp)^\perp \), which are each other's inverses because they commute with the \( \xi \)'s.

We leave it to the reader to construct similar maps \( (X/p)/(r^\perp \cdot p^\perp)^\perp \rightleftarrows X/r \) for \( p \leq r \).

14. Assuming images in \( C \) we have for each map \( f: X \to Y \) a predicate \( \text{im}(f): Y \to I \), with quotient map \( \xi_{\text{im}(f)}: Y \to Y/\text{im}(f) \). We claim that \( \xi_{\text{im}(f)} \circ f = 0: X \to X/\text{im}(f) \). This follows by Definition [51] \( 33 \) from:

\[
1 \circ \xi_{\text{im}(f)} \circ f = \ker^\perp(\xi_{\text{im}(f)}) \circ f = \text{im}^\perp(f) \circ f = f^\perp(\text{im}(f))^\perp = 1^\perp = 0.
\]

Now suppose we have a map \( g: Y \to Z \) with \( g \circ f = g \circ 0 = 0 \). Then \( \text{im}(f) \leq \ker(g) \) by Lemma [47] \( 7 \). This yields the required unique map \( \overline{g}: Y/\text{im}(f) \to Z \) with \( \overline{g} \circ \xi_{\text{im}(f)} = g \).

Finally, if \( \text{im}(f) = 1 \), then \( Y/\text{im}(f) = Y/1 \cong 0 \), by point \( 4 \), so that \( \text{coker}(f) = \xi_{\text{im}(f)}: Y \to 0 \) is the zero map. Conversely, let \( \text{coker}(f) = 0 \). By Lemma [59] \( 1 \) we have \( \ker(\text{im}(f)) = \text{im}(f): Y \to I \). Hence there is a unique map \( g: X/\text{im}(f) \to I \) with \( \text{im}(f) = g \circ \xi_{\text{im}(f)} = g \circ \text{coker}(f) = g \circ 0 = 0 \). But then \( \text{im}(f) = 1 \).

\[\square\]

Several of the points in Lemma [83] are used to prove that quotients give rise to a factorisation system. Such a system is given by two collections of map, called ‘abstract monos’ and ‘abstract epis’, satisfying certain properties, see e.g. [BWS5] for details.

**Proposition 84.** Each effectus with quotients in partial form has a factorisation system given by internal monos (i.e. total maps) and quotient maps \( \xi \).

**Proof** An arbitrary map \( f: X \to Y \) can be factored as:

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow \xi_{\text{ker}(f)} & & \downarrow f' \text{, total} \\
X/\text{ker}(f) & & \\
\end{array}
\]

The fact that the map \( f' \) is total follows from Lemma [83] \( 5 \).
By Lemma 39 (8) the total (internally monic) maps are closed under composition. The $\xi$’s are also closed under composition, by Lemma 83 (13). The intersection of these two classes consists of isomorphisms: if $\xi_p: X \to X/p$ is total, then $1 = 1 \circ \xi_p = \ker(\xi_p) = p^\perp$, so that $p = 0$, making $\xi_p = \xi_0: X \to X/0$ an isomorphism, see Lemma 83 (6).

Finally we check the diagonal-fill-in property. In a commuting diagram as on the left below, there is a unique diagonal as on the right, making both triangles commute.

We have by Lemma 59 (7) and (8), using that $f$ is total,

$$\ker(g) = \ker(f \circ g) = \ker(h \circ \xi_p) \supseteq \ker(\xi_p) = p.$$

Hence there is a unique map $k: X/p \to Y$ with $k \circ \xi_p = g$. We obtain $f \circ k = h$ by using that $\xi_p$ is epic. □

13 Boolean effectuses and extensive categories

The main result of this section says that a Boolean effectus with comprehension is the same thing as an extensive category. This is an important coincidence of two categorical notions, which gives a deeper understanding of what Boolean effectuses are. These extensive categories are well-known from the literature [CLW93] and capture well-behaved coproducts.

In this section we shall work in the ‘total’ language of effectuses, and not the ‘partial’ version. The reason is that we will work towards a correspondence between extensive categories and certain effectuses in total form— and not between extensive categories and effectuses in partial form.

We recall from Definition 55 that an effectus is Boolean if it is commutative — with assert maps $\text{asrt}_p: X \to X + 1$ for predicates $p: X \to 1 + 1$ as inverse to $\ker^\perp$ — which satisfy $\text{asrt}_p \circ \text{asrt}_p^\perp = 0$. We start with an auxiliary result about comprehension in commutative effectuses.

**Lemma 85.** In a commutative effectus $\mathbf{B}$ with comprehension the following two diagrams are equalisers in $\mathbf{B}$.

\[
\begin{array}{ccc}
\{X|p\} & \xrightarrow{\pi_p} & X \\
\downarrow \pi_{p^\perp} & \searrow \text{asrt}_p \downarrow & \searrow \kappa_1 \\
X+1 & \xrightarrow{0} & X+1
\end{array}
\quad
\begin{array}{ccc}
\{X|p\} & \xrightarrow{\pi_p} & X \\
\downarrow \kappa_1 & \swarrow \text{asrt}_p \downarrow & \swarrow \pi_p \\
X+1 & \xrightarrow{0} & X+1
\end{array}
\]

**Proof** The equaliser on the left in $\mathbf{B}$ is in essence the kernel map property in $\text{Par}(\mathbf{B})$ from Lemma 59 (2). For the one on the right we recall that $\text{asrt}_p \leq \text{id}$ in the homset of partial maps $X \to X$, by definition. Hence:

$$\text{asrt}_p \circ \pi_p = \text{asrt}_p \circ (\pi_p) \leq \text{id} \circ (\pi_p) = \kappa_1 \circ \pi_p.$$
We wish to show that this inequality is actually an equality. So let \( f : X \to X \) satisfy \((\text{asrt}_p \circ \pi_p) \otimes f = (\kappa_1 \circ \pi_p)\). Applying the map of effect algebras \( \ker^+ = (\! + \text{id}) \circ (-) \) on both sides yields in \( \text{Pred}(\{X|p\}) \),

\[
1 = \kappa_1 \circ ! = \kappa_1 \circ ! \circ \pi_p = \ker^+ (\kappa_1 \circ \pi_1) = \ker^+ (\text{asrt}_p \circ \pi_p) \otimes \ker^+(f) = ((\! + \text{id}) \circ \text{asrt}_p \circ \pi_p) \otimes \ker^+(f) = (p \circ \pi_p) \otimes \ker^+(f) = 1 \otimes \ker^+(f).
\]

Hence \( \ker^+(f) = 0 \) by cancellation in the effect algebra \( \text{Pred}(\{X|p\}) \), and thus \( f = 0 \) by Lemma 39 (2). But then \( \kappa_1 \circ \pi_p = (\text{asrt}_p \circ \pi_p) \otimes f = \text{asrt}_p \circ \pi_p \).

Next, let \( f : Y \to X \) in \( B \) satisfy \( \text{asrt}_p \circ f = \kappa_1 \circ f \). Then:

\[
f^*(p) = p \circ f = (\! + \text{id}) \circ \text{asrt}_p \circ f = (\! + \text{id}) \circ \kappa_1 \circ f = \kappa_1 \circ ! \circ f = \kappa_1 \circ ! = 1.
\]

Hence \( f \) factors in a unique way through \( \pi_p \). \( \square \)

We turn to extensive categories. There are several equivalent formulations, see [CLW93], but we use the most standard one.

**Definition 86.** A category is called extensive if it has finite coproducts \((+ , 0)\) such that pullbacks along coprojections exist, and in every diagram of the form,

\[
\begin{array}{c}
Z_1 \rightarrow \downarrow \rightarrow Z \leftarrow \downarrow Z_2 \\
\text{X} \downarrow \kappa_1 \downarrow \text{X + Y} \leftarrow \downarrow \text{Y} \\
\end{array}
\]

the two rectangles are pullbacks if and only if the top row is a coproduct, that is, the induced map \( Z_1 + Z_2 \to Z \) is an isomorphism.

There are many examples of extensive categories, with ‘well-behaved’ coproducts. For instance, every topos — including \textbf{Sets} — is an extensive category, see e.g. [CLW93]. At the end of this section we list some concrete examples.

For the record we recall the following basic observation from [CLW93]. The first point is the analogue of Proposition [4] for effectuses.

**Lemma 87.** In an extensive category,

1. coprojections are are monic and disjoint, and the initial object 0 is strict;
2. If the rectangles on the left below are pullbacks, for \( i = 1, 2 \), then the rectangle on the right is a pullback too.

\[
\begin{array}{c}
A_1 \rightarrow \downarrow X \\
\text{f}_i \downarrow \rightarrow f \\
B_i \downarrow \leftarrow \downarrow Y \\
\end{array}
\]

\[
\begin{array}{c}
A_1 + A_2 \rightarrow \downarrow X \\
\text{f}_{i+f_2} \downarrow \rightarrow f \\
B_1 + B_2 \leftarrow \downarrow Y \\
\end{array}
\]
Proof For the first point, consider the rectangles:

\[
\begin{array}{c}
0 \\
X \\
\end{array}
\begin{array}{c}
Y \\
Y \\
X + Y \\
\end{array}
\begin{array}{c}
\kappa_2 \\
\kappa_1 \\
\kappa_2 \\
\end{array}
\begin{array}{c}
Y \\
X \\
X \\
\end{array}
\begin{array}{c}
\kappa_2 \\
\kappa_1 \\
\kappa_2 \\
\end{array}
\begin{array}{c}
Y \\
Y \\
Y \\
\end{array}
\]

The top row is a coproduct diagram. Hence the two rectangles are pullbacks. The one on the left says that coprojections are disjoint, and the one on the right says that \( \kappa_2 \) is monic.

For strictness of \( 0 \), we have to prove that a map \( f: X \to 0 \) is an isomorphism. Consider the diagram:

\[
\begin{array}{c}
X \\
\end{array}
\begin{array}{c}
f \\
\kappa_1 \circ f \circ \kappa_2 \circ f \\
0 \\
\end{array}
\begin{array}{c}
Y \\
Y \\
0 + 0 \\
\end{array}
\begin{array}{c}
f \\
\kappa_1 \\
\kappa_2 \\
\end{array}
\begin{array}{c}
0 \\
\end{array}
\]

The two rectangles are pullbacks because the coprojections are monic. Hence the top row is a coproduct diagram, so that the codiagonal \( \nabla = [\text{id}, \text{id}]: X + X \to X \) is an isomorphism. Using \( \nabla \circ \kappa_1 = \text{id} = \nabla \circ \kappa_2 \) we get \( \kappa_1 = \kappa_2: X \to X + X \).

Now we can now prove that \( ! \circ f: X \to 0 \to X \) is the identity, via:

\[
! \circ f = (! \circ f, \text{id}_X) \circ \kappa_1 = (! \circ f, \text{id}_X) \circ \kappa_2 = \text{id}_X.
\]

We turn to the second point. Let \( a: Z \to X \) and \( b: Z \to B_1 + B_2 \) satisfy \( f \circ a = [h_1, h_2] \circ b \). Form the pullbacks on the left, and the mediating maps \( c_i \) on the right:

\[
\begin{array}{c}
X \\
\end{array}
\begin{array}{c}
f \\
\kappa_1 \circ f \circ \kappa_2 \circ f \\
0 \\
\end{array}
\begin{array}{c}
Y \\
Y \\
0 + 0 \\
\end{array}
\begin{array}{c}
f \\
\kappa_1 \\
\kappa_2 \\
\end{array}
\begin{array}{c}
0 \\
\end{array}
\]

The outer diagram on the right commutes since:

\[
f \circ a \circ k_i = [h_1, h_2] \circ b \circ k_i = [h_1, h_2] \circ \kappa_i \circ b_i = h_i \circ b_i.
\]

The unique mediating map is \( c = (c_1 + c_2) \circ [k_1, k_2]^{-1}: Z \to A_1 + A_2 \).

Proposition 88. Each extensive category with a final object is a Boolean effectus with comprehension.

Proof This is quite a bit of work. So let \( A \) be an extensive category with a final object \( 1 \). We start by showing that \( A \) is an effectus. We first concentrate
on the two pullbacks in Definition 2.

\[
\begin{array}{c}
X + Y \xrightarrow{id + !} X + 1 \\
1 + Y \xrightarrow{id + !} 1 + 1
\end{array}
\quad
\begin{array}{c}
X \xrightarrow{!} 1 \\
X + Y \xrightarrow{\kappa_1} X + 1
\end{array}
\quad
\begin{array}{c}
Y \xrightarrow{!} 1 \\
1 + Y \xrightarrow{! + id} 1 + 1
\end{array}
\quad
\begin{array}{c}
1 + X \xrightarrow{\kappa_2} 1 + 1
\end{array}
\]

(\ast)

For convenience, we start with the diagram on the right. We consider it as turned left-part of the following diagram:

\[
\begin{array}{c}
X \xrightarrow{\kappa_1} X + Y \xrightarrow{\kappa_2} Y \\
1 \xrightarrow{\kappa_1} 1 + 1 \xleftarrow{\kappa_2} 1
\end{array}
\]

Since the top row is a coproduct diagram, both rectangles are pullbacks in an extensive category, by definition.

For the diagram on the left in (\ast), it suffices, by Lemma 87 (2) to prove that the two diagrams below are pullbacks.

\[
\begin{array}{c}
X \xrightarrow{\kappa_1} X + 1 \\
1 \xrightarrow{\kappa_1} 1 + 1
\end{array}
\quad
\begin{array}{c}
Y \xrightarrow{!} 1 \\
1 + Y \xrightarrow{! + id} 1 + 1
\end{array}
\]

The rectangle on the left is a pullback just like the diagram on the right in (\ast). Similarly, the right-rectangle on the right is a pullback. The left-rectangle on the right is obviously a pullback, so we are done by the Pullback Lemma.

The next step is to prove that the two maps \(W = [id, \kappa_2], W' = [\kappa_2, \kappa_1, \kappa_2]\) are jointly monic. We sketch how to proceed. If \(f, g: Y \to (1 + 1) + 1\) satisfy \(W \circ f = W' \circ g\) then we decompose \(f, g\) each in three parts, via pullbacks with appropriate coprojections \(1 \to (1 + 1) + 1\), and show that these three parts are equal. This is left to the interested reader.

We now know that the extensive category \(A\) that we started from is an effectus. We turn to predicates and comprehension. For each predicate \(p: X \to 1 + 1\) in \(A\) we choose a pullback:

\[
\begin{array}{c}
\{X|p\} \xrightarrow{\pi_p} X \\
\{X|p\} \xrightarrow{\pi_p} X
\end{array}
\quad
\begin{array}{c}
p \downarrow \\
1 \xrightarrow{\kappa_1} 1 + 1
\end{array}
\]

We will need a number of facts about these maps \(\pi_p: \{X|p\} \to X\).

(a). The following diagrams are also pullbacks.

\[
\begin{array}{c}
X \xrightarrow{\pi_p^+} \{X|p\} \\
1 + 1 \xrightarrow{\kappa_2} 1
\end{array}
\quad
\begin{array}{c}
\{X|p\} + Y \xrightarrow{\pi_p + id} X + Y \\
\{X|p\} + Y \xrightarrow{\pi_p + id} X + Y
\end{array}
\quad
\begin{array}{c}
[p, 1] \xrightarrow{[\cdot, \cdot]} \{X|p\} \\
1 \xrightarrow{\kappa_1} 1 + 1
\end{array}
\]

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The combination of the left diagram and Diagram (56) tells us that the cotuple \([\pi_p, \pi_{p^⊥}]\): \(\{X|p\} + \{X|p^⊥\} \to X\) is an isomorphism. The consequence of the pullback on the right is that \(\{X|[p, 1]\} \cong \{X|p\} + Y\), so that the effectus \(A\) has comprehension, see Definition (6).

It is easy to see that the above diagram on the left is a pullback: if \(f: Y \to X\) satisfies \(p \circ f = \kappa_2 \circ !\), then:

\[
p^⊥ \circ f = [\kappa_2, \kappa_1] \circ p \circ f = [\kappa_2, \kappa_1] \circ \kappa_2 \circ ! = \kappa_1 \circ ! = 1.
\]

Hence \(f\) factors uniquely through \(\pi_{p^⊥}\) in (56).

The above rectangle on the right is a pullback diagram since one can apply Lemma 87 (2) to Diagram (56) and a trivial pullback.

(b). The pair of comprehension maps \(\pi_p, \pi_{p^⊥}\) is jointly epic.

Indeed, if \(f \circ \pi_p = g \circ \pi_p\) and \(f \circ \pi_{p^⊥} = g \circ \pi_{p^⊥}\), then:

\[
f \circ [\pi_p, \pi_{p^⊥}] = [f \circ \pi_p, f \circ \pi_{p^⊥}] = [g \circ \pi_p, g \circ \pi_{p^⊥}] = g \circ [\pi_p, \pi_{p^⊥}].
\]

Hence \(f = g\) since the cotuple \([\pi_p, \pi_{p^⊥}]\) is an isomorphism.

We now define for each predicate \(p\): \(X \to 1 + 1\) an ‘assert’ map \(X \to X + 1\) as:

\[
\text{asrt}_p \overset{\text{def}}{=} \left( X \xrightarrow{[\pi_p, \pi_{p^⊥}]^{-1}} \{X|p\} + \{X|p^⊥\} \xrightarrow{\pi_{p^⊥}^+} X + 1 \right)
\]

Again we prove a number of basic facts.

(i). \(\text{asrt}_p \circ \pi_p = \kappa_1 \circ \pi_p\) and \(\text{asrt}_p \circ \pi_{p^⊥} = 0\).

From \([\pi_p, \pi_{p^⊥}] \circ \kappa_1 = \pi_p\) we obtain \([\pi_p, \pi_{p^⊥}]^{-1} \circ \pi_p = \kappa_1\). Hence:

\[
\text{asrt}_p \circ \pi_p = (\pi_p + !) \circ [\pi_p, \pi_{p^⊥}]^{-1} \circ \pi_p = (\pi_p + !) \circ \kappa_1 = \kappa_1 \circ \pi_p.
\]

Similarly:

\[
\text{asrt}_p \circ \pi_{p^⊥} = (\pi_p + !) \circ [\pi_p, \pi_{p^⊥}]^{-1} \circ \pi_{p^⊥} = (\pi_p + !) \circ \kappa_2 = \kappa_2 \circ ! = 0.
\]

(ii). \(\text{asrt}_p \leq \text{id}\) in the homset of partial maps \(X \to X\), since \(\text{asrt}_p \circ \text{asrt}_{p^⊥} = \text{id}\), via the bound \(b = \kappa_1 \circ (\pi_p + \pi_{p^⊥}) \circ [\pi_p, \pi_{p^⊥}]^{-1}: X \to (X + X) + 1\). Indeed:

\[
\begin{align*}
\triangledown_1 \circ b &= (\text{id} + !) \circ (\pi_p + \pi_{p^⊥}) \circ [\pi_p, \pi_{p^⊥}]^{-1} \\
&= (\pi_p + !) \circ [\pi_p, \pi_{p^⊥}]^{-1} \\
&= \text{asrt}_p
\end{align*}
\]

\[
\begin{align*}
\triangledown_2 \circ b &= [\kappa_2 \circ !, \kappa_1] \circ (\pi_p + \pi_{p^⊥}) \circ [\pi_p, \pi_{p^⊥}]^{-1} \\
&= [\kappa_2 \circ !, \kappa_1 \circ \pi_{p^⊥}] \circ [\pi_p, \pi_{p^⊥}]^{-1} \\
&= (\pi_p + !) \circ [\kappa_2, \kappa_1 \circ \pi_{p^⊥}]^{-1} \\
&= (\pi_p + !) \circ [\pi_{p^⊥}, \pi_{p^⊥}]^{-1} \\
&= \text{asrt}_{p^⊥}
\end{align*}
\]

\[
\text{asrt}_p \odot \text{asrt}_{p^⊥} = \triangledown \ast b \\
= \kappa_1 \circ \triangledown \circ (\pi_p + \pi_{p^⊥}) \circ [\pi_p, \pi_{p^⊥}]^{-1} \\
= \kappa_1
\]

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(iii). This mapping \( p \mapsto \text{asrt}_p \) is the inverse of \( \ker^\perp : \text{End}_{\leq \text{id}}(X) \to \text{Pred}(X) \).

First, we have:

\[
\ker^\perp(\text{asrt}_p) = (! + \text{id}) \circ (\pi_p + !) \circ [\pi_p, \pi_p^\perp]^{-1} = (! + !) \circ [\pi_p, \pi_p^\perp]^{-1} = p.
\]

Conversely, let \( f : X \to X \) satisfy \( f \leq \text{id} \). We need to prove \( f = \text{asrt}_p \), for \( p = \ker^\perp(f) = 1 \ast f = (! + \text{id}) \circ f \). Consider the diagram:

\[
\begin{array}{ccc}
\{X[p]\} & \xrightarrow{\pi_p} & X & \xrightarrow{\pi_p^\perp} & \{X[p^\perp]\} \\
\downarrow & & \downarrow f & & \downarrow \\
X & \xrightarrow{\kappa_1} & X + 1 & \xrightarrow{\kappa_2} & Y \\
\downarrow & & \downarrow ! + \text{id} & & \\
1 & \xrightarrow{\kappa_1} & 1 + 1 & \xrightarrow{\kappa_2} & Y
\end{array}
\]

We first show \( h = \pi_p \). Since \( f \leq \text{id} \) we can write \( f \otimes g = \text{id} \) for some map \( g : X \to X \). Then:

\[
1 = 1 \ast \text{id} = 1 \ast (f \otimes g) = (1 \ast f) \otimes (1 \ast g) = p \otimes (1 \ast g).
\]

Hence \( 1 \circ g = p^\perp \). Then:

\[
1 \circ (g \circ \pi_p) = p^\perp \circ \pi_p = 0,
\]

so that \( g \circ \pi_p = 0 \) by Lemma 7. We now get \( \pi_p = h \) from:

\[
\kappa_1 \circ \pi_p = (f \otimes g) \circ \pi_p = (f \circ \pi_p) \otimes (g \circ \pi_p) = (\kappa_1 \circ h) \otimes 0 = \kappa_1 \circ h.
\]

Finally we obtain \( f = \text{asrt}_p \) from:

\[
f \circ [\pi_p, \pi_p^\perp] = [f \circ \pi_p, f \circ \pi_p^\perp] = [\kappa_1 \circ \pi_p, \kappa_2 \circ !] = \pi_p + !.
\]

(iv). \( \text{asrt}_p \ast \text{asrt}_{p^\perp} = 0 \).

We simply calculate:

\[
\text{asrt}_p \ast \text{asrt}_{p^\perp} = [\text{asrt}_p, \kappa_2] \circ (\pi_p^\perp + !) \circ [\pi_p^\perp, \pi_p]^{-1} = [\text{asrt}_p \circ \pi_p^\perp, \kappa_2 \circ !] \circ [\pi_p^\perp, \pi_p]^{-1} = [\kappa_2 \circ !, \kappa_2 \circ !] \circ [\pi_p^\perp, \pi_p]^{-1} = 0 \circ [\pi_p^\perp, \pi_p]^{-1} = 0.
\]

(v). Commutation of asserts — that is, \( \text{asrt}_p \ast \text{asrt}_q = \text{asrt}_q \ast \text{asrt}_p \) — follows from the previous point, as described in Remark 60.

We may now conclude that the extensive category \( \mathbf{A} \) is an effectus with comprehension, by point (a), which is commutative, by points (ii) – (v), and Boolean in particular, by point (iv). \qed
We now arrive at the main result of this section.

**Theorem 89.** Let $B$ be a category with finite coproducts and a final object. Then: $B$ is extensive if and only if it is a Boolean effectus with comprehension. In this situation quotients come for free.

**Proof** By Proposition 88 we only have to prove that a Boolean effectus $B$ with comprehension is extensive. From Proposition 61 we know that all predicates are sharp and form Boolean algebras.

For a predicate $p$ on $X$ we have the map $\text{asrt}_p : X \to X$ with $\ker(\text{asrt}_p) = p$. In a Boolean effectus we have:

$$\text{asrt}_p^{\perp}(p) = (p^{\perp} \circ \text{asrt}_p) = (1 \circ \text{asrt}_p) = (1 \circ 0) = 1.$$ We then have a factorisation in $\text{Par}(B)$ of the form:

\[ X \xrightarrow{\text{asrt}_p} \{X|p\} \]

The notation $\xi$ for these maps is deliberate, because they yield quotients. But first we show that:

$\xi^{\perp}_p \circ \langle \pi_p \rangle = \text{id}$ and $\xi^{\perp}_p \circ \langle \pi_p \rangle = 0$.  

The proof uses that $\langle \pi_p \rangle$ is monic in $\text{Par}(B)$:

$$\langle \pi_p \rangle \circ \xi^{\perp}_p \circ \langle \pi_p \rangle = \text{asrt}_p \circ \langle \pi_p \rangle$$
$$= \kappa_1 \circ \pi_1 \quad \text{by Lemma 85}$$
$$= \langle \pi_p \rangle$$
$$\langle \pi_p \rangle \circ \xi^{\perp}_p \circ \langle \pi_p \rangle = \text{asrt}_p \circ \langle \pi_p \rangle$$
$$= 0 \quad \text{by Lemma 85}$$
$$= \langle \pi_p \rangle \circ 0.$$  

We now show that $\xi^{\perp}_p : X \to \{X|p\}$ is a quotient map, where $\{X|p\}$ is the quotient object $X/p^{\perp}$. First we have:

$$\ker(\xi^{\perp}_p) = (1 \circ \xi^{\perp}_p)^{\perp} = (1 \circ \langle \pi_p \rangle \circ \xi^{\perp}_p)^{\perp} = (1 \circ \text{asrt}_p)^{\perp} = p^{\perp}.$$  

Next, if $f : X \to Y$ satisfies $p^{\perp} \leq \ker(f)$, then:

$$1 \circ f \circ \text{asrt}_p \leq \text{ker}(f) \circ \text{asrt}_p \leq p \circ \text{asrt}_p$$
$$= 1 \circ \text{asrt}_p \circ \text{asrt}_p = 1 \circ 0 = 0.$$  

This gives $f \circ \text{asrt}_p = 0$, by Lemma 7 and thus:

$$f = f \circ \text{id} = f \circ (\text{asrt}_p \oplus \text{asrt}_p) = (f \circ \text{asrt}_p) \oplus (f \circ \text{asrt}_p) = f \circ \text{asrt}_p.$$
We obtain \( T = f \circ \pi_p : \{ X \} \to Y \) as mediating map, since:

\[
T = \pi_p, \quad \pi_p = f \circ \pi_p = \pi_p = f, \quad \text{as rt}_p = f.
\]

This map \( T \) is unique, since if \( g : \{ X \} \to Y \) also satisfies \( g \circ \pi_p = f \), then:

\[
T = f \circ \pi_p = g \circ \pi_p = f = g \circ \text{id} = g.
\]

Now that we have quotient maps we can form the total ‘decomposition’ map \( \text{dc}_p = \langle \langle \pi_p, \pi_p \rangle \rangle : X \to \{ X \} \to \{ X \} \) from Lemma \([33, 34]\). We claim that this map \( \text{dc}_p \) is an isomorphism, with the cotuple \( [\pi_p, \pi_p] \) as inverse. We first prove \( \text{dc}_p \circ \pi_p = \kappa_1 \) and \( \text{dc}_p \circ \pi_p = \kappa_2 \) by using \([11]\):

\[
\begin{align*}
\text{dc}_p \circ \pi_p & = \langle \langle \pi_p, \pi_p \rangle \rangle \circ \pi_p = \langle \langle \pi_p \circ \pi_p, \pi_p \circ \pi_p \rangle \rangle \\
& \overset{(\ast)}{=} \langle \langle \kappa_1, 0 \rangle \rangle = \langle \langle \Delta_1 \circ \kappa_1, \Delta_2 \circ \kappa_1 \rangle \rangle = \kappa_1 \\
\text{dc}_p \circ \pi_p & = \langle \langle \pi_p \circ \pi_p, \pi_p \circ \pi_p \rangle \rangle \\
& \overset{(\ast)}{=} \langle \langle 0, \kappa_1 \rangle \rangle = \langle \langle \Delta_1 \circ \kappa_2, \Delta_2 \circ \kappa_2 \rangle \rangle = \kappa_2.
\end{align*}
\]

At this stage we can prove one part of the claim that the decomposition map \( \text{dc}_p \) is an isomorphism:

\[
\text{dc}_p \circ [\pi_p, \pi_p] = [\text{dc}_p \circ \pi_p, \text{dc}_p \circ \pi_p] = [\kappa_1, \kappa_2] = \text{id}.
\]

For the other direction, we take \( b = \kappa_1 \circ (\pi_p + \pi_p) \circ \text{dc}_p : X \to X + X \) and claim that \( b \) is a bound for the pair of maps \( \text{as rt}_p, \text{as rt}_p : X \to X \). This is the case, since:

\[
\begin{align*}
\Delta_1 \circ b & = (\text{id} + !) \circ (\pi_p + \pi_p) \circ \text{dc}_p = (\pi_p + \text{id}) \circ (\text{id} + !) \circ \text{dc}_p \\
& = (\pi_p + \text{id}) \circ \pi_p \\
& = \text{as rt}_p \\
\Delta_2 \circ b & = [\kappa_2 \circ !, \kappa_1] \circ (\pi_p + \pi_p) \circ \text{dc}_p = [\kappa_2, \kappa_1 \circ \pi_p] \circ (\text{id} + !) \circ \text{dc}_p \\
& = [\kappa_2, \kappa_1 \circ \pi_p] \circ [\kappa_2, \kappa_1] \circ \pi_p \\
& = (\pi_p + \text{id}) \circ \pi_p \\
& = \text{as rt}_p.
\end{align*}
\]

Since the assert map is a homomorphism of effect algebras we have:

\[
\begin{align*}
\kappa_1 & = \text{as rt}_p \circ \text{as rt}_p = (\nabla + \text{id}) \circ b = \kappa_1 \circ \nabla \circ (\pi_p + \pi_p) \circ \text{dc}_p \\
& = \kappa_1 \circ [\pi_p, \pi_p] \circ \text{dc}_p.
\end{align*}
\]

But then \( [\pi_p, \pi_p] \circ \text{dc}_p = \text{id} \), making \( \text{dc}_p \) an isomorphism, as required.

We are finally in a position to show that the effectus \( \mathbf{B} \) is an extensive category, see Definition \([34]\). Let \( f : Z \to X + Y \) be an arbitrary map in \( \mathbf{B} \). Write
The two lower pullbacks exist because $B$ is an effectus. The two outer ones, with the curved arrows, exist because $B$ has comprehension. The two upper squares, with the dashed arrows are then pullbacks by the Pullback Lemma. Thus, the pullbacks along $f$ of the coprojections $Y \to Y + Z \leftarrow Z$ exist. Moreover, we have just seen that the cotuple $[\pi_p, \pi_{p^+}]$ of the pulled-back maps is an isomorphism. This is one part of Definition 86.

For the other part, consider a similar diagram where the top row is already a coproduct diagram:

Then:

$$f = f \circ [\kappa_1, \kappa_2] = [f \circ \kappa_1, f \circ \kappa_2] = [\kappa_1 \circ f_1, \kappa_2 \circ f_2] = f_1 + f_2.$$  

But the above two rectangles are then pullbacks by Lemma $1$.

We thus see that a Boolean effectus with comprehension corresponds to the well-established notion of extensive category. It is an open question whether there is a similar alternative notion for a commutative effectus, capturing probabilistic computation.

Each topos is an extensive category, and thus an effectus. Notice that in a topos-as-effectus the predicates are the maps $X \to 1 + 1$, and not the more general predicates $X \to \Omega$, where $\Omega$ is the subobject classifier. There are many extensive categories that form interesting examples of effectuses, such as: the category $\textbf{Top}$ of topological spaces, and it subcategory $\textbf{CH} \hookrightarrow \textbf{Top}$ of compact Hausdorff spaces; the category $\textbf{Meas}$ of measurable spaces; the opposite $\textbf{CRng}^{op}$ of the category of commutative rings, see $[\text{CJWW15}].$

14 Combining comprehension and quotients

In previous sections we have studied comprehension and quotients separately. We now look at the combination, and derive some more results. First we need the following observation.
Lemma 90. Let $C$ be an effectus in partial form which has comprehension and also has quotients. Write for a predicate $p$ on an object $X \in C$:

$$\{X|p\} \xrightarrow{\theta_p \overset{\text{def}}{=} \xi_p \circ \pi_p} X/p$$

(57)

Then:

1. this map $\theta_p$ is total;
2. if $\theta_p$ is an isomorphism, then $p$ is sharp.

Proof The first point is easy: $\theta_p$ is total by Lemma 80 (1) and Lemma 83 (1):

$$1 \circ \theta_p = 1 \circ \xi_p \circ \pi_p = \ker(\xi_p) \circ \pi_p = p \circ \pi_p = 1.$$

For the second point let $\theta_p$ be an isomorphism, and let $q \in \text{Pred}(X)$ satisfy $q \leq p$ and $q \leq p^\perp$. In order to prove that $p$ is sharp, we must show $q = 0$. From $p^\perp \leq q^\perp = \ker(q)$ we get a predicate $q/p$ in $\text{Pred}(X/p^\perp)$ with $q/p \circ \xi_p^\perp = q$, see Lemma 83 (12). Then:

$$q/p = q/p \circ \xi_p^\perp \circ \theta^{-1}_p = q/p \circ \xi_p^\perp \circ \pi_p \circ \theta^{-1}_p = q \circ \pi_p \circ \theta^{-1}_p \leq p^\perp \circ \pi_p \circ \theta^{-1}_p = 0 \circ \theta^{-1}_p = 0.$$

Hence $q = q/p \circ \xi_p^\perp = 0 \circ \xi_p^\perp = 0$. □

We now consider the condition which enforces an equivalence in the above second point.

Definition 91. We say that an effectus in partial form has both quotients and comprehension if it has comprehension, like in Definition 76 (2), and quotients, like in Definition 81, such that for each sharp predicate $p \in \text{Pred}(X)$ the total map $\theta_p: \{X|p\} \to X/p^\perp$ in (57) is an isomorphism.

We then define, for each such sharp $p \in \text{Pred}(X)$ the map:

$$\text{asrt}_{p} \overset{\text{def}}{=} \left( X \xrightarrow{\xi_p^\perp} X/p^\perp \xrightarrow{\theta^{-1}_p \overset{\text{def}}{=} \pi_p \circ \theta^{-1}_p} \{X|p\} \xrightarrow{\pi_p} X \right).$$

(58)

Example 92. We briefly illustrate the map $\theta_p: \{X|p\} \to X/p^\perp$ from (57) in our running examples.

In the effectus $\textbf{Sets}$ we have for a predicate $P \subseteq X \{X|P\} = P$ and also $Q/P^\perp = \{X|P^\perp\} = P$. The map $\theta$ is the identity, and is thus always an isomorphism. Indeed, in this Boolean effectus $\textbf{Sets}$ every predicate is sharp.

More generally, consider an extensive category $\textbf{B}$ with final object, as a Boolean effectus with both comprehension and quotients, see Theorem 89. The quotient that is constructed in the proof is of the form $\xi_p^\perp : X \to \{X|p\}$. Hence we have $\theta_p = \text{id} : \{X|p\} \to X/p^\perp$.

In the effectus $K\ell(\mathcal{D})$ we have for a predicate $p \in [0,1]^X$ an inclusion:

$$\{X|p\} = \{x \mid p(x) = 1\} \xrightarrow{\theta_p} \{x \mid p(x) > 0\} = X/p^\perp$$
When \( p \) is sharp, we have \( p(x) \in \{0, 1\} \), so that \( \theta_p \) is the identity.

Next we consider the effectus \( \text{OUG}^{op} \) of order unit groups. For an effect \( e \in [0, 1]_G \) in an order unit group \( G \) we have a map (in the opposite direction):

\[
G/e^\perp = (e)_G \xrightarrow{\theta_e} G/(e^\perp)_G = \{G|e\}
\]

This map is unital since:

\[
\theta_e(1_{G/e^\perp}) = e + \langle e \rangle_G = e + e^\perp + \langle e^\perp \rangle_G = 1 + \langle e^\perp \rangle_G = 1_{\{G|e\}}.
\]

It can be shown that this map \( \theta_e \) is an isomorphism for sharp \( e \), if one assumes that \( G \) satisfies interpolation, see [Goo86] for details.

Finally, in our effectus \( \text{vNA}^{op} \) of von Neumann algebras we also have, for an effect \( e \in [0, 1]_A \), a map of the form:

\[
\mathcal{A}/e^\perp = \lceil e \rceil \mathcal{A} \lceil e \rceil \xrightarrow{\theta_e} \lfloor e \rfloor \mathcal{A} \lfloor e \rfloor = \{\mathcal{A}|e\}
\]

This map is well-defined since \( \lceil e \rceil \cdot \lfloor e \rfloor = \lceil e \rceil = \lfloor e \rfloor \cdot \lceil e \rceil \), using a more general fact:

\[
a \leq \lceil e \rceil \Rightarrow a \cdot \lceil e \rceil = a = \lceil e \rceil \cdot a.
\]

In case \( e \) is sharp we get \( \lceil e \rceil = e = \lfloor e \rfloor \), so that the above map \( \theta_e \) is an isomorphism.

We collect some more results about the maps \( \theta \). Since these maps are total, we switch to the total perspective.

**Lemma 93.** Let effectus in total form \( B \) have quotients and comprehension.

1. For each predicate \( p \in \text{Pred}(X) \) the two squares below are pullbacks in \( B \).

\[
\begin{array}{ccc}
X/p^\perp & \xrightarrow{\theta_p} & \{X|p\} \\
\downarrow{\kappa_1} & & \downarrow{\pi_p} \\
X/p^\perp + 1 & \xrightarrow{\xi_{p^\perp}} & X/p + 1
\end{array}
\]

2. For each predicate \( p \) on \( X \), the following triangle commutes.

\[
\begin{array}{ccc}
\{X|p\} + \{X|p^\perp\} & \xrightarrow{\theta_p + \theta_{p^\perp}} & X/p^\perp + X/p \\
\downarrow{[\pi_p, \pi_{p^\perp}]} & & \downarrow{dc_p} \\
X & & X
\end{array}
\]

Thus, for sharp \( p \) the decomposition map \( dc_p \) is a split epi, and the cotuple \( [\pi_p, \pi_{p^\perp}] \) is a split mono, in \( B \).

3. The maps \( \theta \) commute with the distributivity isomorphisms from Lemma 79 and Lemma 83, as in:

\[
\begin{array}{ccc}
\{X|p\} + \{Y|q\} & \xrightarrow{\theta_{X|p} + \theta_{Y|q}} & X/p^\perp + Y/q^\perp \\
\downarrow{||} & & \downarrow{||} \\
\{X + Y|p,q\} & \xrightarrow{\theta_{[p,q]}'} & (X + Y)/[p,q]^\perp = (X + Y)/[p^\perp,q^\perp]
\end{array}
\]
Proof Recall that we are in an effectus in total form.

1. The left square in (59) commutes because $\theta_p$ is total. It forms a pullbacks since if $f: Y \to X/p^\perp$ and $g: Y \to X$ satisfy $\kappa_1 \circ f = \xi_{p^\perp} \circ g$, then $g$ factors uniquely as $g = \pi_p \circ \varphi$ since:

\[
g^\ast(p) = p \circ g = \ker(\xi_{p^\perp}) \circ g = (! + \text{id}) \circ \xi_{p^\perp} \circ g = (! + \text{id}) \circ \kappa_1 \circ \theta_p = \kappa_1 \circ ! \circ \theta_p = \kappa_1 \circ ! = 1.
\]

Then $\theta_p \circ \varphi = f$ because the coprojection $\kappa_1$ is monic:

\[
\kappa_1 \circ \theta_p \circ \varphi = \xi_{p^\perp} \circ \pi_p \circ \varphi = \xi_{p^\perp} \circ \varphi = \kappa_1 \circ f.
\]

The rectangle on the right in (59) commutes by Lemma 7 since:

\[
(! + \text{id}) \circ \xi_p \circ \pi_p = \ker(\xi_p) \circ \pi_p = p^\perp \circ \pi_p = 0.
\]

It is a pullback: if $f: Y \to X$ satisfies $\xi_p \circ f = \kappa_2 \circ ! = 0$, then $f$ factors uniquely through $\pi_p$ since:

\[
f^\ast(p) = p \circ f = \ker(\xi_p) \circ f = [\kappa_2, \kappa_1] \circ (! + \text{id}) \circ \xi_p \circ f = [\kappa_2, \kappa_1] \circ (! + \text{id}) \circ \kappa_2 \circ ! = \kappa_1 \circ ! = 1.
\]

2. We use the commuting rectangles (59) and the equations (11) to get:

\[
dc_p \circ \pi_p = \langle \xi_{p^\perp}, \xi_p \rangle \circ \pi_p = \langle \xi_{p^\perp} \circ \pi_p, \xi_p \circ \pi_p \rangle = \langle \kappa_1 \circ \theta_p, 0 \rangle = \langle D_1 \circ \kappa_1 \circ \theta_p, D_2 \circ \kappa_1 \circ \theta_p \rangle = \kappa_1 \circ \theta_p.
\]

\[
dc_p \circ \pi_{p^\perp} = \langle \xi_p, \xi_{p^\perp} \rangle \circ \pi_{p^\perp} = \langle \xi_p \circ \pi_{p^\perp}, \xi_{p^\perp} \circ \pi_{p^\perp} \rangle = \langle 0, \kappa_1 \circ \theta_{p^\perp} \rangle = \langle D_1 \circ \kappa_2 \circ \theta_{p^\perp}, D_2 \circ \kappa_2 \circ \theta_{p^\perp} \rangle = \kappa_2 \circ \theta_{p^\perp}.
\]

Hence we see that the triangle (60) commutes:

\[
dc_p \circ [\pi_p, \pi_{p^\perp}] = [dc_p \circ \pi_p, dc_p \circ \pi_{p^\perp}] = [\kappa_1 \circ \theta_p, \kappa_2 \circ \theta_{p^\perp}] = \theta_p + \theta_{p^\perp}.
\]

3. We show that the composite:

\[
\{X + Y \mid [p, q]\} \xrightarrow{\psi} \{X|p\} + \{Y|q\} \xrightarrow{\theta_{p^\perp} - \theta_p} X/p^\perp + Y/q^\perp \xrightarrow{\psi^\ast} (X + Y)/[p, q]^\perp
\]

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satisfies the equation that defines $\theta_{[p,q]}$ as $\kappa_1 \circ \theta_{[p,q]} = \xi_{[p,q]} \perp \pi_{[p,q]}$.

$$\begin{align*}
\kappa_1 \circ \psi \circ (\theta_p + \theta_q) \circ \varphi \\
= (\psi + \id) \circ \kappa_1 \circ [\kappa_1 \circ \theta_p, \kappa_2 \circ \theta_q] \circ \varphi \\
= (\psi + \id) \circ [(\kappa_1 + \id) \circ \kappa_1 \circ \theta_p, (\kappa_2 + \id) \circ \kappa_1 \circ \theta_q] \circ \varphi \\
= (\psi + \id) \circ [(\kappa_1 + \id) \circ \xi_p \perp \pi_p, (\kappa_2 + \id) \circ \xi_q \perp \pi_q] \circ \varphi \\
= (\psi + \id) \circ [(\kappa_1 + \id) \circ \xi_p \perp, (\kappa_2 + \id) \circ \xi_q \perp] \circ (\pi_p + \pi_q) \circ \varphi \\
= (\psi \ast (\xi_p + \xi_q)) \circ \pi_{[p,q]} \\
= (\xi_{[p,q]} \circ \pi_{[p,q]}).
\end{align*}$$

We continue with properties of the assert map \((58)\) for sharp predicates \(p\). We have seen ‘assert’ maps in Section \(9\) as inverse of the kernel-supplement map \(\ker^+ : \text{End}_{\leq \id}(X) \to \text{Pred}(X)\). This isomorphism of effect algebras is a key aspect of commutative effectuses (including Boolean ones). The assert map \((57)\) in this section is at the same time more general and also more restricted: it is not necessarily side-effect free, that is, below the identity, nor a map of effect algebras; and it is defined only for sharp predicates.

**Lemma 94.** Let \((C,I)\) be an effectus with quotients and comprehension, in partial form. Let \(p\) be a sharp predicate on \(X \in C\).

1. \(\text{asrt}_p \circ \text{asrt}_p = \text{asrt}_p\) and \(\text{asrt}_p \circ \text{asrt}_p \perp = 0\).
2. \(\ker(\text{asrt}_p) = p \perp\), and so \(\ker^+ (\text{asrt}_p) = p\).
3. \(\text{asrt}_0 = 0\) and \(\text{asrt}_1 = \id\).
4. For each \(f : X \to Y\) one has:

\(p \perp \leq \ker(f) \iff f \circ \text{asrt}_p = f\).

5. The following two diagrams are equalisers in \(C\).

$$\begin{array}{c}
\{X|p\} \xrightarrow{\pi_p} X \\
\downarrow \text{asrt}_p \uparrow \text{id} \\
0 \xrightarrow{\id} X
\end{array} \quad \begin{array}{c}
\{X|p\} \xrightarrow{\pi_p} X \\
\downarrow \text{asrt}_p \uparrow \id \\
X \xrightarrow{\id} X
\end{array}$$

6. Similarly, the next diagram is a coequaliser.

$$\begin{array}{c}
X \xrightarrow{\text{asrt}_p} X \\
\downarrow \text{id} \uparrow \xi_{[p,q]} \perp \\
X/p \perp
\end{array}$$

7. Let \(f : X \to X\) be a side-effect free endomap, that is \(f \leq \id_X\) in the poset \(\text{End}_{\leq \id}(X)\) of endomaps on \(X\) below the identity. If the predicate \(p = \ker^+ (f) = 1 \circ f\) is sharp, then \(f = \text{asrt}_p\).

Moreover, in that case the decomposition map \(\text{dc}_p\) from Lemma \((83)(7)\) is a total isomorphism:

$$X \xrightarrow{\text{dc}_p \cong} X/p \perp + X/p$$
Proof Recall that \( \text{asrt}_p = \pi_p \circ \theta_p^{-1} \circ \xi_{p\perp} : X \to X \), see \([58]\).

1. We compute:

\[
\begin{align*}
\text{asrt}_p \circ \text{asrt}_p &= \pi_p \circ \theta_p^{-1} \circ \xi_{p\perp} \circ \pi_p \circ \theta_p^{-1} \circ \xi_{p\perp} \\
&= \pi_p \circ \theta_p^{-1} \circ \theta_p \circ \theta_p^{-1} \circ \xi_{p\perp} \\
&= \pi_p \circ \theta_p^{-1} \circ \xi_{p\perp} \\
&= \text{asrt}_p \\
\text{asrt}_p \circ \text{asrt}_{p\perp} &= \pi_p \circ \theta_p^{-1} \circ \xi_{p\perp} \circ \pi_{p\perp} \circ \theta_{p\perp}^{-1} \circ \xi_{p\perp} \\
&= \pi_p \circ \theta_p^{-1} \circ \theta_{p\perp} \circ \theta_{p\perp}^{-1} \circ \xi_{p} \\
&= \pi_p \circ \theta_p^{-1} \circ 0 \circ \theta_{p\perp}^{-1} \circ \xi_{p\perp} \\
&= 0.
\end{align*}
\]

2. Applying Lemma \([39\ 7]\) with the total map \( \pi_p \circ \theta_p^{-1} \) yields:

\[
\ker(\text{asrt}_p) = \ker(\pi_p \circ \theta_p^{-1} \circ \xi_{p\perp}) = \ker(\xi_{p\perp}) = p\perp.
\]

3. We have \( \text{asrt}_0 = 0 \) since \( \ker(\text{asrt}_0) = 0 \) and \( \ker(\perp) \) reflects \( 0 \), see Lemma \([40]\). Further, there are isomorphisms \( X/0 \cong X \cong \{X|1\} \) by Lemma \([33\ 9]\) and Lemma \([79\ 4]\). Hence:

\[
\text{asrt}_1 = \left(X - \frac{\xi_0}{\cong} X/0 \quad \frac{\theta_1^{-1}}{\cong} \quad \frac{\pi_1}{\cong} \quad \{X|1\} \right) = \text{id},
\]

since by definition \( \theta_1 = \xi_0 \circ \pi_1 \), and so \( \theta_1^{-1} = \pi_1^{-1} \circ \xi_0^{-1} \).

4. We need to prove the equivalence \( p\perp \leq \ker(f) \iff f \circ \text{asrt}_p = f \). This is done in two steps.

- If \( p\perp \leq \ker(f) \), then we can write \( f = \overline{f} \circ \xi_{p\perp} \), so that:

\[
\begin{align*}
f \circ \text{asrt}_p &= \overline{f} \circ \xi_{p\perp} \circ \pi_p \circ \theta_p^{-1} \circ \xi_{p\perp} \\
&= \overline{f} \circ \theta_p \circ \theta_p^{-1} \circ \xi_{p\perp} \\
&= \overline{f} \circ \xi_{p\perp} \\
&= f.
\end{align*}
\]

- And if \( f \circ \text{asrt}_p = f \), then by Lemma \([39\ 6]\),

\[
\ker(f) = \ker(f \circ \text{asrt}_p) \geq \ker(\text{asrt}_p) = p\perp.
\]

5. The first kernel map equaliser is an instance of Lemma \([80\ 2]\), since \( \ker(\text{asrt}_p) = p\perp \). For the other one, notice that:

\[
\text{asrt}_p \circ \pi_p = \pi_p \circ \theta_p^{-1} \circ \xi_{p\perp} \circ \pi_p = \pi_p \circ \theta_p^{-1} \circ \theta_p = \pi_p.
\]

And if \( f : Y \to X \) satisfies \( \text{asrt}_p \circ f = f \), then \( f \) factors uniquely through \( \pi_p \) via the composite: \( \theta_p^{-1} \circ \xi_{p\perp} \circ f : Y \to \{X|p\} \).
6. First,
\[ \xi_{p^\perp} \circ \text{asrt}_p = \xi_{p^\perp} \circ \pi_p \circ \theta_p^{-1} \circ \xi_{p^\perp} = \theta_p \circ \theta_p^{-1} \circ \xi_{p^\perp} = \xi_{p^\perp}. \]

Next, if \( f : X \to Y \) satisfies \( f \circ \text{asrt}_p = f \), then \( p^\perp \leq \ker(f) \) by point (4), so that \( f = T \circ \xi_{p^\perp} \), for a necessarily unique \( T : X/p^\perp \to Y \).

7. Let \( f : X \to X \) satisfy \( f \leq \text{id} \), say via \( f \circ g = \text{id} \). Further, the predicate \( p = \ker^\perp(f) = \text{id} \circ f = p^\perp \) is sharp, by assumption. Then \( \text{id} \circ g = p^\perp \) by uniqueness of orthosupplements:

\[ p \circ (\text{id} \circ g) = \text{id} \circ (f \circ g) = \text{id} \circ \text{id} = \text{id}. \]

As a result, \( \text{id} \circ g \circ \pi_p = p^\perp \circ \pi_p = 0 \), so that \( g \circ \pi_p = 0 \) by Definition (3).

Hence:
\[ f \circ \pi_p = (f \circ \pi_p) \circ (g \circ \pi_p) = (f \circ g) \circ \pi_p = \text{id} \circ \pi_p = \pi_p. \]

By definition, \( p^\perp = \ker(f) \), so that there is a unique total map \( T : X/p^\perp \to X \) with \( f = T \circ \xi_{p^\perp} \). But then:
\[ \pi_p = f \circ \pi_p = T \circ \xi_{p^\perp} \circ \pi_p = T \circ \theta_p. \]

Hence \( T = \pi_p \circ \theta_p^{-1} \), using that \( p \) is sharp. Now we can conclude \( f = T \circ \xi_{p^\perp} = \pi_p \circ \theta_p^{-1} \circ \xi_{p^\perp} = \text{asrt}_p \). In a similar way one obtains \( g = \text{asrt}_{p^\perp} \).

In this situation the decomposition map \( \text{dc}_p = \langle \xi_{p^\perp}, \xi_p \rangle : X \to X/p^\perp + X/p \) is an isomorphism. We already know that it has a right inverse \( [\pi_p, \pi_{p^\perp}] \circ (\theta_p^{-1} + \theta_{p^\perp}^{-1}) \) by Lemma (2). We also have:

\[
[\pi_p, \pi_{p^\perp}] \circ (\theta_p^{-1} + \theta_{p^\perp}^{-1}) \circ \text{dc}_p
= \nabla \circ (\pi_p + \pi_{p^\perp}) \circ (\theta_p^{-1} + \theta_{p^\perp}^{-1}) \circ \langle \xi_{p^\perp}, \xi_p \rangle
= \nabla \circ \langle \pi_p \circ \theta_p^{-1} \circ \xi_{p^\perp}, \pi_p \circ \theta_{p^\perp}^{-1} \circ \xi_p \rangle
= \nabla \circ \langle \text{asrt}_p, \text{asrt}_{p^\perp} \rangle
= f \circ g
= \text{id}. \]

If we add two more assumptions we can prove a lot more. At this stage we start using images that are sharp.

**Proposition 95.** Let \( C \) be an effectus with quotients and comprehension, in partial form, which additionally:

- has sharp images
- and satisfies, for all sharp predicates \( p, q \in \text{Pred}(X), \)

\[ p \leq q \iff \{X|p\} \xrightarrow{\pi_p} X \xleftarrow{\pi_q} \{X|q\} \]

(61)
Then we can prove the following series of results.

1. Take for an arbitrary predicate $p$,

$$[p] \overset{\text{def}}{=} \text{im}(\pi_p) \quad \text{and} \quad [p]^{\perp} \overset{\text{def}}{=} \pi_p(p^{\perp}).$$

Then: $[p]$ is the greatest sharp predicate below $p$, and $[p]^{\perp}$ is the least sharp predicate above $p$.

2. Let $p \in \text{Pred}(X)$. For each total map $f: Y \rightarrow X$ one has:

$$f^*(p) = 1 \iff f^*(|[p]|) = 1 \quad \text{and} \quad f^*(p) = 0 \iff f^*(|[p]|) = 0. \tag{62}$$

For an arbitrary map $g: Y \rightarrow X$ we have:

$$g^\square(p) = 1 \iff g^\square(|[p]|) = 1.$$

3. For a sharp predicate $p \in \text{Pred}(X)$, the map $\text{asrt}_p: X \rightarrow X$ from [58] satisfies $\text{im}(\text{asrt}_p) = p$. Hence by Lemma [53] [11] we have a cokernel map:

$$X \xrightarrow{\text{asrt}_p} X \xrightarrow{\xi_p} X/p$$

4. For $f: X \rightarrow Y$ and $p \in \text{Pred}(X)$ put:

$$\sum_f(p) \overset{\text{def}}{=} \text{im}(f \circ \pi_p).$$

Then there is an adjunction:

$$\begin{array}{c}
\text{ShaPred}(X) \\
\sum_f(p) \\
\downarrow[	ext{im}(\square)] \quad \downarrow \quad \downarrow[	ext{im}(\square)] \\
\text{ShaPred}(Y)
\end{array} \quad \tag{63}$$

where ShaPred describes the subposet of sharp predicates.

5. Comprehension maps $\pi$ are closed under composition, up-to-isomorphism.

6. The posets $\text{ShaPred}(X) \hookrightarrow \text{Pred}(X)$ of sharp predicates are orthomodular lattices.

7. Comprehension maps $\pi$ and internal epis form a factorisation system in $\mathbf{C}$.

**Proof** The fact that images are assumed to be sharp plays an important role.

1. By definition the image $[p] = \text{im}(\pi_p): X \rightarrow I$ is a sharp predicate. This $[p]$ is below $p$ by minimality of images, and $\pi_p(p^{\perp}) = (p^{\perp} \circ \pi_p)^{\perp}$ by $0^{\perp} = 1$. If $s \in \text{Pred}(X)$ is sharp with $s \leq p$, then $\pi_s$ factors through $\pi_{[p]}$ since:

$$\begin{array}{c}
\{X|s\} \\
\xrightarrow{\pi_s} \\
\xrightarrow{\pi_{[p]}} \\
\xrightarrow{\pi_p} \\
\xrightarrow{\pi_{[p]}} \quad \{X|[p]|\}
\end{array} \quad \sum_f(p) \overset{\text{def}}{=} \text{im}(f \circ \pi_p). \tag{64}$$

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The dashed arrow exists since by definition $\pi^\circ_p([p]) = \pi^\circ_p(\operatorname{im}(\pi_p)) = 1$. Hence we get $s \leq [p]$ by (61).

We now have $[p^\bot] \leq p^\bot$, and thus $p \leq [p^\bot] = [p]$. If $p \leq s$, where $s$ is sharp, then $s^\bot \leq p^\bot$, so that $s^\bot \leq [p^\bot]$, and thus $[p] = [p^\bot] \leq s$.

2. Let $f: Y \rightarrow X$ be a total map. If $f^*([p]) = 1$, then $f^*(p) = 1$ since $[p] \leq p$. In the other direction, if $f^*(p) = 1$, then $f$ factors through $\pi_p$. But $f$ then also factors through $\pi_{[p]}$, by the isomorphism in Diagram (64), so that $f^*([p]) = 1$.

We now get:

$$f^*(p) = 0 \iff f^*(p^\bot) = f^*(p^\bot) = 1 \iff f^*([p^\bot]) = 1 \quad \text{as just shown} \iff f^*([p]) = f^*([p^\bot]) = f^*([p^\bot]) = 0.$$  

For an arbitrary map $g: Y \rightarrow X$, if $g^\circ([p]) = 1$, then $g^\circ(p) = 1$ since $g^\circ$ is monotone by Lemma 36. In the other direction, if $g^\circ(p) = 1$, then $\operatorname{im}(g) \leq p$ by minimality of images, and thus $\operatorname{im}(g) \leq [p]$ by point (1). But then $1 = g^\circ(\operatorname{im}(g)) \leq g^\circ([p])$.

3. For a sharp predicate $p$ we have:

$$\operatorname{im}(\operatorname{asrt}_p) = \operatorname{im}(\pi_p \circ \theta_p^{-1} \circ \xi_p) = \operatorname{im}(\pi_p) \quad \text{by Lemma 17 (6), (11)}$$

$$= [p] \quad \text{since } p \text{ is sharp}.$$  

4. Let $f: X \rightarrow Y$ be an arbitrary map. We need to prove for sharp predicates $p, q \in \text{Pred}(X)$ and $q \in \text{Pred}(Y)$,

$$\sum_f(p) = \operatorname{im}(f \circ \pi_p) \leq q \iff p \leq [f^\circ(q)].$$  

This works in the following way.

- Let $\sum_f(p) = \operatorname{im}(f \circ \pi_p) \leq q$. We have:

$$1 = (f \circ \pi_p)^\circ(\operatorname{im}(f \circ \pi_p)) \leq (f \circ \pi_p)^\circ(q) = \pi^\circ_p(f^\circ(q)).$$  

This implies that we have a map $\{X|p\} \rightarrow \{X|f^\circ(q)\}$ commuting with the projections. We thus get $p \leq f^\circ(q)$ by (11), and thus $p \leq [f^\circ(q)]$.

- In the other direction, let $p \leq [f^\circ(q)] \leq f^\circ(q)$. Then:

$$1 = \pi^\circ_p(p) \leq \pi^\circ_p(f^\circ(q)) = (f \circ \pi_p)^\circ(q).$$  

But then $\sum_f(p) = \operatorname{im}(f \circ \pi_p) \leq q$ by minimality of images.
5. For two sharp predicates \( p \in \text{Pred}(X) \) and \( q \in \text{Pred}(\{X|p\}) \) consider the sharp predicate on \( X \) given by:

\[
\sum_{\pi_p}(q) = \text{im}(\pi_p \circ \pi_q).
\]

Notice that by Lemma [17][11],

\[
\sum_{\pi_p}(q) = \text{im}(\pi_p \circ \pi_q) \leq \text{im}(\pi_p) = \lfloor p \rfloor = p. \tag{\star}
\]

We are done if we can show that there are necessarily unique maps \( \varphi, \psi \) in a commuting square:

\[
\begin{array}{ccc}
\{\{X|p\}|q\} & \xrightarrow{p} & \{X|\sum_{\pi_p}(q)\} \\
\pi_p & \downarrow & \downarrow \pi_{\sum_{\pi_p}(q)} \\
\{X|p\} & \xleftarrow{\psi} & X
\end{array}
\]

The existence of the map \( \varphi \) is easy since comprehension maps are total and thus:

\[
1 = (\pi_p \circ \pi_q)\circ (\text{im}(\pi_p \circ \pi_q)) = (\pi_p \circ \pi_q)^*\left(\sum_{\pi_p}(q)\right).
\]

In order to show the existence of the map \( \psi \) we use \textit{ad hoc} notation for:

\[
p \cap q = \left(X \xrightarrow{\xi_p} X/p^\perp \xrightarrow{\theta_p^{-1}} \{X|p\} \xrightarrow{q} I\right)
\]

Then:

\[
\pi_p^*(p \cap q) = (p \cap q) \circ \pi_p = q \circ \theta_p^{-1} \circ \xi_p^* \circ \pi_p = q \circ \theta_p^{-1} \circ \theta_p = q.
\]

And thus:

\[
(\pi_p \circ \pi_q)^*(p \cap q) = \pi_q^*(\pi_p^*(p \cap q)) = \pi_q^*(q) = 1.
\]

The latter yields \( \sum_{\pi_p}(q) = \text{im}(\pi_p \circ \pi_q) \leq p \cap q \) by minimality of images. We now go back to the inequality \( \sum_{\pi_p}(q) \leq p \) from (\star). It gives a total map \( f: \{X|\sum_{\pi_p}(q)\} \to \{X|p\} \) with \( \pi_p \circ f = \pi_{\sum_{\pi_p}(q)} \). This \( f \) factors through \( \pi_q \), and thus restricts to the required map \( \psi \), since:

\[
f^*(q) = q \circ f = q \circ \theta_p^{-1} \circ \theta_p \circ f = q \circ \theta_p^{-1} \circ \xi_p^* \circ \pi_p \circ f = (p \cap q) \circ \pi_{\sum_{\pi_p}(q)} \geq \sum_{\pi_p}(q) \circ \pi_{\sum_{\pi_p}(q)} = 1.
\]
6. We first show how to obtain conjunction $\wedge$ for sharp predicates. For $p,q \in ShaPred(X)$ define:

$$p \wedge q = \sum_{\pi_p} \pi_p^*(q) = \text{im}(\pi_p \circ \pi_p^*(q)).$$

Since projections are closed under pullback — see Lemma [79] (5) — we have a total isomorphism $\varphi$ between two pullbacks in:

$$\begin{array}{c}
\{\{X|p]\}|\pi_p^*(q)\} \\
\{\{X|q]\}|\pi_q^*(p)\} \\
\{X|p\} \\
\{X|q\} \\
X
\end{array}$$

Hence we can prove that $p \wedge q$ is a lower bound of both $p$ and $q$ via Lemma [17] (3):

$$p \wedge q = \text{im}(\pi_p \circ \pi_p^*(q)) \leq \text{im}(\pi_p) = [p] = p$$

$$p \wedge q = \text{im}(\pi_p \circ \pi_p^*(q)) = \text{im}(\pi_q \circ \pi_q^*(p) \circ \varphi) \leq \text{im}(\pi_q) = [q] = q.$$

We show that $p \wedge q$ is the greatest lower bound in $ShaPred(X)$: inequalities $r \leq p$ and $r \leq q$ yield maps $f: \{X|r\} \to \{X|p\}$ and $g: \{X|r\} \to \{X|q\}$ with $\pi_p \circ f = \pi_r = \pi_q \circ g$. The above pullback gives a mediating map $h: \{X|r\} \to \{\{X|p\}|\pi_p^*(q)\}$ with $\pi_{p|q}(q) \circ h = f$. But then we are done:

$$r = [r] = \text{im}(\pi_r) = \text{im}(\pi_p \circ f) = \text{im}(\pi_p \circ \pi_{p|q}(q) \circ h) \leq \text{im}(\pi_p \circ \pi_{p|q}(q)) = p \wedge q.$$

We now also have joins $p \vee q$ in $ShaPred(X)$ via De Morgan: $p \vee q = (p^\perp \wedge q^\perp)^\perp$. We prove the orthomodularity law in the following way. Let $p \leq q$: we have to prove $p \vee (p^\perp \wedge q) = q$. This is done essentially as in [HJ10] Prop. 1.

$$p \vee (p^\perp \wedge q) = (p \wedge q) \vee (p^\perp \wedge q)$$

$$= \sum_{\pi_p} (p) \vee \sum_{\pi_p} (p^\perp)$$

$$= \sum_{\pi_p} (p \vee p^\perp)$$

$$= \sum_{\pi_p} (1)$$

$$= \sum_{\pi_p} (\pi_q^*(1))$$

$$= q \wedge 1$$

$$= q.$$
We recall the factorisation from Lemma 80 (5):

\[
X \xrightarrow{f} \xrightarrow{\text{ie}(f)} \{Y | \text{im}(f)\} \xrightarrow{\pi} \text{im}(f) \rightarrow Y
\]

The map \(\text{ie}(f)\) is the ‘internal epi’ part of \(f\). Let \(p: \{Y | \text{im}(f)\} \rightarrow I\) be a predicate with \(\text{ie}(f) \triangleup (p) = 1\). If we can show that \(p = 1\), then we know that the map \(\text{ie}(f)\) is indeed internally epic. First we have:

\[
0 = \text{ie}(f) \triangleup (p) = p \perp \circ \text{ie}(f) = p \perp \circ \theta^{-1}_{\text{im}(f)} \circ \theta_{\text{im}(f)} \circ \text{ie}(f) = p \perp \circ \theta_{\text{im}(f)} \circ \xi_{\text{im}(f)} \circ \pi_{\text{im}(f)} \circ \text{ie}(f) = p \perp \circ \theta^{-1}_{\text{im}(f)} \circ \xi_{\text{im}(f)} \circ f.
\]

Lemma 47 (3) then yields:

\[
p \perp \circ \theta^{-1}_{\text{im}(f)} \circ \xi_{\text{im}(f)} \leq \text{im}(f).
\]

But we also have, by Lemma 83 (3),

\[
p \perp \circ \theta^{-1}_{\text{im}(f)} \circ \xi_{\text{im}(f)} \leq \text{im}(f) \perp = \text{im}(f).
\]

Since images are sharp, by definition, we obtain:

\[
p \perp \circ \theta^{-1}_{\text{im}(f)} \circ \xi_{\text{im}(f)} = 0.
\]

The fact that the map \(\theta^{-1}_{\text{im}(f)} \circ \xi_{\text{im}(f)}\) is epic — see Lemma 88 (2) — yields \(p \perp = 0\), and thus \(p = 1\), as required.

There are a few more requirements of a factorisation system that we need to check: the internal epis are closed under composition, by Lemma 47 (9); the comprehension maps are closed under composition, by point (5). We show that an internally epic comprehension map is an isomorphism. Let \(\pi_p\) be internally epic, that is, \(1 = \text{im}(\pi_p) = [p]\), so that \(\pi_{[p]} = \pi_1\) is an isomorphism by Lemma 79 (3). But \(\pi_p\) is then an isomorphism too, by the isomorphism in (64).

Finally we have to check that the diagonal-fill-in property holds: in a commuting rectangle as below, we need a diagonal.

\[
\begin{array}{ccc}
Y & \xrightarrow{f} & Z \\
g & \xrightarrow{\text{internal epi}} & h \\
\{X | p\} & \xrightarrow{\pi_p} & X
\end{array}
\]

We are done if we can show that \(h\) factors through \(\pi_p\), that is, if \(h^\triangleup (p) = 1\). By Lemma 47 (10) this is equivalent to \(f^\triangleup (h^\triangleup (p)) = 1\). But:

\[
f^\triangleup (h^\triangleup (p)) = g^\triangleup (\pi^\triangleup (p)) = g^\triangleup (1) = 1.
\]

\[\square\]
As a result of points (4) and (6) in Proposition 95 the assignment \( X \mapsto \ShaPred(X) \) gives a functor \( C \to \OMLatGal \), where \( \OMLatGal \) is the dagger kernel category of orthomodular lattices, with Galois connections between them, as used in [Jac10b] (see also [HJ10]).

**Remark 96.** In an effectus as considered in Proposition 95 we have two factorisation systems, namely:

- the one with internal monos and quotient maps \( \xi \) from Proposition 84;
- the one with comprehension maps \( \pi \) and internal epis from Proposition 95 (7).

The question is how they are related. What is called the ‘first isomorphism theorem’ gives an answer. This ‘theorem’ is the familiar phenomenon, that can be expressed informally as \( X/\ker(f) \cong \text{im}(f) \), for each map \( f: X \to Y \). It holds for many algebraic structures. However, this ‘theorem’ does not hold in effectuses, in general.

Abstractly this isomorphism is obtained via a canonical map from the ‘cokernel’ to the ‘image’, which is then an isomorphism. In the setting of an effectus (in partial form), this canonical map is the dashed one described in the following diagram — which is the same as diagram (2) in [Gra92] for the ideal of zero maps.

We briefly explain this diagram, starting on the right.

- The map \( \xi_{\text{im}(f)} \) is the cokernel of \( f \), see Lemma 83 (14). The upgoing projection \( \pi_{\text{im}(f)} \) is its kernel map, because \( \ker(\xi_{\text{im}(f)}) = \text{im}(f) \).

- On the left, the projection \( \pi_{\text{ker}(f)} \) is the kernel map of \( f \), see Lemma 80 (2).
  Its cokernel is the downgoing map \( \xi_{[\ker(f)]} \) since \( \text{im}(\pi_{\text{ker}(f)}) = [\ker(f)] \).

The dashed arrow exist, because the evident maps \( X/[\ker(f)] \to Y \) and \( X \to \{Y|\text{im}(f)\} \) restrict appropriately.

This canonical dashed map is in general not an isomorphism. For instance, for a partial map \( f: X \to D(Y + 1) \) in the Kleisli category of the distribution monad \( D \) we have:

\[
X/[\ker(f)] = \{ x \in X \mid [\ker(f)] < 1 \} = \{ x \in X \mid f(x) = 0 \} = \{ x \in X \mid f(x)(*) = 0 \}
\]

\[
\{Y|\text{im}(f)\} = \{ y \in Y \mid \text{im}(f)(y) = 1 \} = \{ y \in Y \mid \exists x \in X. f(x)(y) > 0 \}.
\]

The canonical map \( X/[\ker(f)] \to \{Y|\text{im}(f)\} \) in the above diagram is given by \( x \mapsto \sum_{y,f(x)(y)>0} f(x)(y)|y \), which is not an isomorphism.
More generally, a total internally epic map need not be an isomorphism. The situation is reminiscent of Quillen model categories as used in homotopy theory (see [Qui67]), where one has two factorisation systems, and mediating maps like the dashed one above which are not necessarily isomorphisms. However, in model categories the relevant classes of maps satisfy closure properties which do not hold here.

We have already hinted a few times that an effectus, in partial form, is similar to, but more general, and less well-behaved, than an Abelian category. For such an Abelian category \( A \) we do have a quotient-comprehension chain:

\[
\begin{array}{c}
\text{Quotient} \\
(U \rightarrow X) \rightarrow X/U
\end{array}
\begin{array}{c}
\text{Sub}(A)
\end{array}
\begin{array}{c}
\Downarrow
\end{array}
\begin{array}{c}
\text{Comprehension}
\end{array}
\begin{array}{c}
(U \rightarrow X) \rightarrow U
\end{array}
\]

Quotients are obtained via cokernels: for a subobject \( m: U \rightarrow X \) one takes the codomain of the cokernel \( \text{coker}(m): X \rightarrow X/U \) as quotient unit. In such an Abelian category the first isomorphism theorem does hold.

The general theory that we are after will bear some resemblance to recent work in (non-Abelian) homological algebra, see in particular [Wei14], where adjunction chains like above are studied, but also [Jan94, Gra12]. There, part of the motivation is axiomatising the category of (non-Abelian) groups, following [Mac50]. As a result, stronger properties are used than occur in the current setting, such as the first isomorphism theorem and left adjoints to substitution, corresponding to bifibrations, which do not hold in general here.

We still have to check that the assumptions in Proposition 95 make sense. This concentrates on the equivalence in (61) saying that for sharp predicates \( p, q \) one has \( p \leq q \) iff \( \pi_p \leq \pi_q \), that is, \( \pi_p \) factors through \( \pi_q \). We check this for the effectus examples \( \text{Sets} \) — in fact, more generally, for Boolean effectuses — for \( \mathcal{K}(\mathcal{D}) \), and \( \mathsf{vNA}^{\mathsf{op}} \).

This equivalence (61) obviously holds in the effectus \( \text{Sets} \), since the comprehension of a predicate \( P \subseteq X \) is given by \( P \) itself. It is less trivial that the equivalence (61) also holds for Boolean effectuses / extensive categories, see Theorem 89. Recall that all predicates are automatically sharp in the Boolean case, see Lemma 59 (1).

**Lemma 97.** Let \( A \) be an extensive category, understood as a Boolean effectus with comprehension. Then \( p \leq q \) iff \( \pi_p \leq \pi_q \) for all predicates \( p, q \) on the same object.

**Proof** We proceed in two steps, where we first prove that \( \pi_p \leq \pi_q \) implies \( \pi_{q^\perp} \leq \pi_{p^\perp} \), and only then that it also implies \( p \leq q \).

So let \( \pi_p \leq \pi_q \), say via a (necessarily unique) map \( \varphi: \{X|p\} \rightarrow \{X|q\} \) with \( \pi_q \circ \varphi = \pi_p \).

1. We first need to produce a map \( \{X|q^\perp\} \rightarrow \{X|p^\perp\} \) commuting with the
projections. Consider the following diagram in $A$.

$$
\begin{array}{ccc}
A \xrightarrow{g_1} & \{X|q^+\} \xrightarrow{g_2} & B \\
\downarrow f_1 & & \downarrow f_2 \\
\{X|p\} \xrightarrow{\pi} & \{X|p^+\} \xrightarrow{\pi} & B
\end{array}
$$

We now have a situation:

$$
\begin{array}{ccc}
A \xrightarrow{f_1} & \{X|q^+\} \xrightarrow{g_2} & B \\
\downarrow \phi & & \downarrow \psi \\
\{X|p\} \xrightarrow{\pi} & \{X|p^+\} \xrightarrow{\pi} & Y + 1
\end{array}
$$

The outer diagram commutes since:

$$
\pi_{q^+} \circ g_1 = [\pi_{p^+}, \pi_{q^+}] \circ [\pi_{p^+}, \pi_{q^+}]^{-1} \circ \pi_{q^+} \circ g_1
= [\pi_{p^+}, \pi_{q^+}] \circ \kappa_1 \circ f_1
= \pi_{p^+} \circ f_1
= \pi_q \circ \phi \circ f_1.
$$

Thus, $A \cong 0$, since 0 is strict, by Lemma 87. Since the cotuple $[g_1, g_2]: A + B \to \{X|q^+\}$ is an isomorphism, we obtain that $g_2: B \to \{X|q^+\}$ is an isomorphism. But then $f_2 \circ g_2^{-1}: \{X|q^+\} \to \{X|p^+\}$ is the required map, since:

$$
\pi_{p^+} \circ f_2 \circ g_2^{-1} = [\pi_{p^+}, \pi_{q^+}] \circ \kappa_2 \circ f_2 \circ g_2^{-1}
= [\pi_{p^+}, \pi_{q^+}] \circ [\pi_{p^+}, \pi_{q^+}]^{-1} \circ \pi_{q^+}
= \pi_{q^+}.
$$

2. Our second aim is to prove $p \leq q$, assuming $\pi_p \leq \pi_q$ via the map $\varphi$. We define a predicate $r = \mathcal{W} \circ ((p \circ \pi_q) + !) \circ [\pi_q, \pi_{q^+}]^{-1}: X \to 1 + 1$, and claim that $p \otimes r = q$. This proves $p \leq q$.

As bound $b: X \to (1 + 1) + 1$ we take $b = ((p \circ \pi_q) + !) \circ [\pi_q, \pi_{q^+}]^{-1}$. Then $\mathcal{W} \circ b = r$ holds by construction. We have:

$$
\mathcal{W} \circ b = [\text{id}, \kappa_2] \circ ((p \circ \pi_q) + !) \circ [\pi_q, \pi_{q^+}]^{-1}
= [p \circ \pi_q, \kappa_2 \circ !] \circ [\pi_q, \pi_{q^+}]^{-1}
= p.
$$

The last equation follows from:

$$
[p \circ \pi_q, \kappa_2 \circ !] = p \circ [\pi_q, \pi_{q^+}] \quad \text{and thus from} \quad p \circ \pi_{q^+} = 0.
$$
As just shown we have \( \pi_{q^+} \leq \pi_p^+ \), say via a map \( \psi \). Then:
\[
p \circ \pi_{q^+} = p \circ \pi_{p^+} \circ \psi = 0 \circ \psi = 0.
\]

We now obtain:
\[
p \odot r = (\nabla + \text{id}) \circ b = (\nabla + \text{id}) \circ ((p \circ \pi_q) + ! \circ [\pi_q, \pi_{q^+}]^{-1} = ((! \circ \pi_q) + ! \circ [\pi_q, \pi_{q^+}]^{-1} = (! + !) \circ [\pi_q, \pi_{q^+}]^{-1} = q,
\]
where the latter equation holds since:
\[
q \circ [\pi_q, \pi_{q^+}] = [q \circ \pi_q, q \circ \pi_{q^+}] = [\kappa_1 \circ !, \kappa_2 \circ !] = ! + !.
\]

\[\text{Example 98.}\] In the effectus \( K(\mathcal{D}) \) we have \( p \leq q \) if \( \pi_p \leq \pi_q \) for sharp \( p,q \). Recall from Example 77 that \( \{X|p\} = \{x \mid p(x) = 1\} \), where \( p(x),q(x) \in \{0,1\} \) because \( p,q \) are sharp. Let \( \pi_p \leq \pi_q \), say via a function \( \varphi: \{X|p\} \rightarrow D(\{X|q\}) \) with \( \pi_q \circ \varphi = \pi_q \). We have to prove \( p(x) = 1 \Rightarrow q(x) = 1 \). So assume \( p(x) = 1 \), so that \( x \in \{X|p\} \). Write \( \varphi(x) = \sum_i r_i|x_i\rangle \) with \( x_i \in \{X|q\} \). The equation \( \pi_p = \pi_q \circ \varphi \) yields:
\[
1|x\rangle = \pi_p(x) = (\pi_q \circ \varphi)(x) = \sum_i r_i|x_i\rangle.
\]
This can only happen if \( \varphi(x) = 1|x\rangle \), so that \( x \in \{X|q\} \). Hence \( q(x) = 1 \).

Before showing the equivalence 61 for von Neumann we give a general order result.

\[\text{Lemma 99.}\] Let \( \mathcal{A} \) be a von Neumann algebra with effects \( a,p \in [0,1]_{\mathcal{A}} \). If \( p \)

is a projection (i.e. is sharp), then the following are equivalent.

1. \( a \leq p \)
2. \( ap = a \)
3. \( pa = a \)
4. \( pap = a \)

In particular, if \( a \leq p \), then \( a \) and \( p \) commute.

\[\text{Proof}\] We will prove (1) \( \Rightarrow \) (2) \( \Rightarrow \) (3) \( \Rightarrow \) (4) \( \Rightarrow \) (1). Without loss of generality, we may assume \( \mathcal{A} \) is a von Neumann algebra of operators on a Hilbert space \( \mathcal{H} \).

(1) \( \Rightarrow \) (2) Assume \( a \leq p \). By definition, \( \langle av | v \rangle \leq \langle pv | v \rangle \) for all \( v \in \mathcal{H} \). Thus \( \|\sqrt{av}\|^2 = \langle av | v \rangle \leq \langle pv | v \rangle = \|pv\|^2 \). Hence \( pv = 0 \) implies \( av = 0 \). In particular, as \( p(1-p)v = 0 \) we have \( a(1-p)v = 0 \). We compute \( av = apv + a(1-p)v = apv \). Hence \( a = ap \).

(2) \( \Rightarrow \) (3) Assume \( ap = a \). Directly \( a = a^* = (ap)^* = p^*a^* = pa \).

(3) \( \Rightarrow \) (4) Assume \( pa = a \). Clearly \( pap = ap = (pa)^* = a^* = a \).

(4) \( \Rightarrow \) (1) Assume \( pap = a \). For any \( v \in \mathcal{H} \), we have \( \langle av | v \rangle = \langle papv | v \rangle = \langle apv | pv \rangle \leq \langle pv | pv \rangle = \|pv\|^2 \). Consequently \( a \leq p \). \( \square \)
Example 100. The equivalence \((61)\), namely \(p \leq q \iff \pi p \leq \pi q\) for sharp \(p, q\), also holds in the effectus \(\text{vNA}^{\text{op}}\) of von Neumann algebra. Let \(p, q \in [0, 1]_{\mathcal{A}}\) be sharp, that is, be projections, so that \(p \cdot p = p\) and \(q \cdot q = q\). We first prove that \(\text{im}(\pi p) = p\), and similarly for \(q\). Since \(\pi p(p) = 1\) we always have \(\text{im}(\pi p) \leq p\), so we only have to prove \(\geq\). Recall from Example 77 (4) that the projection \(\pi p\) is the map \(\pi p : \mathcal{A} \to \mathcal{A}\) given by \(\pi p(x) = pxp\). Here we use that \(p\) is sharp, and thus \(\lfloor p \rfloor = p\).

According to (29) the image of \(\pi p\) is given by:
\[
\text{im}(\pi p) = \bigwedge \{s \in \mathcal{A} \mid s \text{ is a projection with } \pi p(s) = psp = p = \pi p(1)\}.
\]

We thus need to prove \(s \geq p\) for a projection \(s\) with \(psp = p\). Write \(t = s^\perp = 1 - s\), so that \(ptp = pp - psp = p - p = 0\). Hence:
\[
0 = \|ptp\| = \|pttp\| = \|(pt) \cdot (pt)^*\| = \|pt\|^2.
\]
But then \(0 = pt = p - ps\), so that \(ps = s\). Hence \(p \leq s\) by Lemma 99.

We can now reason abstractly: if \(\pi p \leq \pi q\) via a map \(\varphi : \{X|p\} \to \{X|q\}\) with \(\pi q \circ \varphi = \pi q\) in \(\text{vNA}^{\text{op}}\), then we are done by Lemma 14 (1):
\[
p = \text{im}(\pi p) = \text{im}(\pi q \circ \varphi) \leq \text{im}(\pi q) = q.
\]
This concludes the example.

15 Towards non-commutative effectus theory

In the preceding sections we have presented the first steps of the theory of effectuses. These sections provide a solid foundation. At this stage we have reached the boundaries of what we know for sure. We add one more section that is of a more preliminary nature. It contains a list of postulates for a class of effectuses that is intended to capture the essential aspects of von Neumann algebras. Although this ‘axiomatisation’ is by no means fixed, we do already have a name for this notion, namely \textit{telos}. It is an effectus satisfying the postulates 102–105 below. Thus, we are the first to admit that the notion of telos is poorly defined. But by making our current thoughts about this topic explicit, we hope to generate more research, leading eventually to a properly defined concept.

In the previous sections we have frequently associated a partial ‘assert’ map \(\text{asrt}_p : X \to X + 1\) with a predicate \(p : X \to 1 + 1\). This predicate-action correspondence occurs for instance in:

- Definition 55 where the (unique) existence of assert maps as inverse of kernel-supplements is the essence of the definitions of ‘commutative’ and ‘Boolean’ effectus;
- Definition 91 where the map \(\text{asrt}_p\) can be defined, but only for a sharp predicate \(p\).

(Proposition 94 (7) says that when these two cases overlap, the relevant assert maps coincide.)

These assert maps \(\text{asrt}_p : X \to X + 1\) incorporate the side-effect map associated with a predicate \(p\). This predicate-action correspondence is similar to
the situation in Hilbert spaces, where a projection may be described either as a closed subset (a predicate), or as a function that projects all elements into this subset (an `assert` map). These assert maps express the dynamic character of quantum computation. They incorporate the side-effect of an observation, if any. In the commutative (and Boolean) case the assert maps have no side-effect — technically, since such assert maps are below the identity. But in the proper quantum case assert maps need not be below the identity. This is already clear for the assert maps associated with sharp elements in Definition 91.

We thus have:

| Situation | Question |
|-----------|----------|
| Assert maps are fully determined in the commutative (and Boolean) case, and also for sharp elements. | Can we also define assert maps in general, in the non-commutative non-sharp case, via a certain property, or do we have to assume them as separate structure? |

This question arises in particular in the effectus $\text{vNA}^{\text{op}}$ of von Neumann algebras.

Recall from Diagram (58) in Definition 91 that for a sharp predicate $p$, the associated assert map is defined via:

$$\text{asrt}_p = \left( X \xrightarrow{\xi_p} X/p^+ \xrightarrow{\theta_p^{-1}} \{ X[p] \} \xrightarrow{\pi_p} X \right). \tag{65}$$

This definition relies on the assumption that the map $\theta_p = \xi_p \circ \pi_p : \{ X[p] \} \to X/p^+$ is an isomorphism when $p$ is sharp. That assumption does not hold in general, for non-sharp $p$.

Still, in the commutative effectus $\mathcal{K}(\mathcal{D}_{\leq 1})$, in partial form, we have, for a fuzzy predicate $p \in [0, 1]^X$, an assert map $\text{asrt}_p : X \to \mathcal{D}_{\leq 1}(X)$ given by $\text{asrt}_p(x) = p(x)|x\rangle$. Intriguingly, this assert map can also be described via quotient and comprehension, as composite in $\mathcal{K}(\mathcal{D}_{\leq 1})$ of the form:

$$X \xrightarrow{\xi_p} X/p^+ = \{ x \mid p(x) > 0 \} \xrightarrow{\pi_p} X \xrightarrow{p(x)} X.$$

Here we use that there is an equality (or isomorphism) $\{ X[|p|] \} \to X/p^+$. The map $\theta_p$ used above is a special case, since $[p] = p$ for a sharp predicate $p$.

It turns out that we can do the same for von Neumann algebras. For an effect $e \in [0, 1]_{\mathcal{A}}$ in a von Neumann algebra $\mathcal{A}$ we can define, formally in $\text{Par}(\text{vNA}^{\text{op}})$,

$$\text{asrt}_e = \left( \mathcal{A} \xrightarrow{\xi_e} \mathcal{A}/e\uparrow = [e]\mathcal{A}[e] = [\lfloor e \rfloor]\mathcal{A}[\lfloor e \rfloor] = \{ \mathcal{A}[\lfloor e \rfloor] \} \xrightarrow{\pi_{\lfloor e \rfloor}} \mathcal{A} \right).$$

The equality in the middle was already mentioned at the end of Example 82[4]. As subunital map $\text{asrt}_e : \mathcal{A} \to \mathcal{A}/e\uparrow$ this composite becomes:

$$\text{asrt}_e(a) = \xi_{e\uparrow} \left( \pi_{\lfloor e \rfloor}(a) \right) = \xi_{e\uparrow} \left( [\lfloor e \rfloor]a\lfloor e \rfloor \right) = \sqrt{e} [\lfloor e \rfloor]a\lfloor e \rfloor \sqrt{e} = \sqrt{e} a \sqrt{e}. \tag{66}$$

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In this way we obtain Lüders rule, see [BS98, Eq.(1.3)]. The problem of defining assert maps via comprehension and quotient is thus reduced to having an isomorphism \( \{ X \mid \lceil p \rceil \} \cong X/p^\perp \). But: which isomorphism?

We have not found, in general, a canonically defined isomorphism \( \{ X \mid \lceil p \rceil \} \cong X/p^\perp \) in an effectus that gives rise to the canonical description (66) in the effectus of von Neumann algebras. This remains largely an unsolved problem, although the situation is clearer in the special case of von Neumann algebras, see Subsection 15.1 below. A related question is: is this apparent lack of canonicity a ‘bug’ or a ‘feature’?

Accepting for the time being that there is no such canonical isomorphism \( \{ X \mid \lceil p \rceil \} \cong X/p^\perp \), there are two possible ways forward.

1. Simply assume some isomorphism \( \{ X \mid \lceil p \rceil \} \cong X/p^\perp \) and use it to define an assert map as in (65), but with \( \pi_{\lceil p \rceil} \) instead of \( \pi_p \). Then we can see which requirements we need for this isomorphism in order to prove reasonable properties about assert maps.

2. Simply assume maps \( \text{asrt}_p : X \to X + 1 \) satisfying some reasonable properties which induce an isomorphism \( \{ X \mid \lceil p \rceil \} \cong X/p^\perp \).

In the sequel we shall follow the second approach. We have already seen various properties of assert maps — e.g. in Lemmas 57 and 94. Hence we can check if they hold in the effectus \( \text{vNA}^{op} \), and if so, use them as a basis for our preliminary axiomatisation of what we call a telos.

(This same approach is followed in [Jac15a], where instrument maps \( \langle \langle \text{asrt}_p, \text{asrt}_p^\perp \rangle \rangle : X \to X + X \) are assumed, satisfying certain properties, instead of assert maps. But the difference between using instruments or assert maps is inessential.)

Below we describe a number of postulates that together give a preliminary description of the notion of telos. Each postulate contains a requirement, and a short discussion about its rationale and consequences.

**Postulate 101.** Each telos is a monoidal effectus with sharp images. We shall describe it in partial form (with special object \( I \), as usual).

In the sequel the monoidal structure, see Section 10, plays a modest role, but it should be included since it is important for combining operations. Similarly, images are a basic ingredient, see Subsection 7.2.

The next postulate introduces assert maps as actions that are associated with predicates. Given such maps, we define an ‘andthen’ operation \& on predicates, as before: \( p \& q = q \circ \text{asrt}_p \), for predicates \( p, q \) on the same object. In the current general situation \& is not commutative, like in Section 9.

**Postulate 102.** For each predicate \( p \) on an object \( X \) in a telos there is an assert map \( \text{asrt}_p : X \to X \) such that:

1. \( \ker(\text{asrt}_p) = p^\perp \), or equivalently, \( \ker^\perp(\text{asrt}_p) = p \);
2. \( \im(\text{asrt}_p) = \lfloor p \rfloor \), where \( \lfloor p \rfloor \) is the least sharp predicate above \( p \);
3. if \( f \leq \id_X \), then \( f = \text{asrt}_p \) for \( p = \ker^\perp(f) = 1 \circ f \);
4. \( \text{asrt}_{[p,q]} = \text{asrt}_p + \text{asrt}_q : X + Y \to X + Y \) for \( p \in \text{Pred}(X) \), \( q \in \text{Pred}(Y) \).
5. $\text{asrt}_{p \otimes q} = \text{asrt}_p \otimes \text{asrt}_q : X \otimes Y \rightarrow X \otimes Y$, where $p \otimes q : X \otimes Y \rightarrow I \otimes I \cong I$.

6. $\text{asrt}_p \circ \text{asrt}_p = \text{asrt}_{p \& p}$;

7. $\text{asrt}_p \circ f = f \circ \text{asrt}_{f \circ (p)}$ for any predicate $p$ and any map $f : Y \rightarrow X$ that preserves sharp elements: $f^{op}(q)$ is sharp if $q$ is sharp.

We check that these properties hold in our leading example of a telos: the opposite $\text{vNA}^{op}$ of the category of von Neumann algebras.

**Example 103.** The effectus $\text{vNA}^{op}$ of von Neumann algebras, with assert maps given by $\text{asrt}_p(x) = \sqrt{p} x \sqrt{p}$ as in [64], satisfies the previous postulate. Clearly our chosen $\text{asrt}_p$ map is subunital, linear, and positive. It is also completely positive [St55, Thm. 1] and normal [Sak71, Lem. 1.7.4]. See also appendix of [WW15]. We cover the different postulates one at a time.

1. Obviously, $\ker^{\perp}(\text{asrt}_p) = \text{asrt}_p(1) = \sqrt{p} \ 1 \ \sqrt{p} = p$.

2. From $\sqrt{p} [p] = \sqrt{p}$ we obtain $\sqrt{p} [p]^{\perp} \sqrt{p} = 0$ and thus $[p]^{\perp} \circ \text{asrt}_p = 0$.

   But then $\text{asrt}_p(\{p\}) = ([p]^{\perp} \circ \text{asrt}_p)^{\perp} = 1$, and thus $\dim(\text{asrt}_p) \leq [p]$ by minimality of images.

   Now we prove the reverse inequality $[p] \leq \dim(\text{asrt}_p)$. Write $b = \dim^{\perp}(\text{asrt}_p)$. We have $b \circ \text{asrt}_p = 0$ by Lemma [44][3], so that $\text{asrt}_p(b) = 0$ by [25].

   That is: $\sqrt{p} b \sqrt{p} = 0$. By the C*-identity $\|b \sqrt{p}\|^2 = \|\sqrt{p} b \sqrt{p}\| = 0$.

   Hence $b \sqrt{p} = 0$. But then also $\sqrt{p} b = (b \sqrt{p})^* = 0$. Thus $p$ and $b$ commute. By [23] we obtain $[p] b = 0 = b [p]$. Since both $[p]$ and $b$ are sharp we obtain that the sum $[p] + b$ is sharp too, and thus an effect. The latter yields $[p] + b \leq 1$, and thus $\dim(\text{asrt}_p) = b \leq [p]^{\perp}$. Consequently $[p] \leq \dim(\text{asrt}_p)$ as desired.

3. In Example [20][6] we have already shown that a subunital map $f : \mathcal{A} \rightarrow \mathcal{A}$ with $f \leq \id$ of the form $f(x) = f(1) x$, where the element $f(1) \in [0, 1]_\mathcal{A}$ is central in $\mathcal{A}$. Hence $\sqrt{f(1)}$ is central too, so that the effect $p = \ker^{\perp}(f) = f(1)$ satisfies:

   $$\text{asrt}_p(x) = \sqrt{f(1)} x \sqrt{f(1)} = f(1) x = f(x).$$

4. Simply:

   $$\text{asrt}_{[p, q]}(x, y) = \sqrt{[p, q]} (x, y) \sqrt{[p, q]}$$

   $$= (\sqrt{p} x \sqrt{p}, \sqrt{q} y \sqrt{q})$$

   $$= (\text{asrt}_p \otimes \text{asrt}_q)(x, y).$$

5. The linear span of predicates $p \otimes q$ is ultraweakly dense in $\mathcal{A} \otimes \mathcal{B}$, see e.g. [Cho14, Prop. 4.5.3]. As our chosen asrt map is ultraweakly continuous and linear, it is sufficient to show the equality for product-predicates:

   $$\text{asrt}_{p \otimes q}(x \otimes y) = \sqrt{p \otimes q} (x \otimes y) \sqrt{p \otimes q}$$

   $$= (\sqrt{p} \otimes \sqrt{q}) (x \otimes y) (\sqrt{p} \otimes \sqrt{q})$$

   $$= (\sqrt{p} x \sqrt{q}) \otimes (\sqrt{q} y \sqrt{q})$$

   $$= \text{asrt}_p(x) \otimes \text{asrt}_q(y)$$

   $$= (\text{asrt}_p \otimes \text{asrt}_q)(x \otimes y).$$

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For the last step, see e.g. [Cho14, Prop. 4.5.5].

6. Note that \( p \& p = \sqrt{p} p \sqrt{p} = p^2 \) and so:

\[
\text{asrt}_p(\text{asrt}_p(x)) = \sqrt{p} \sqrt{p} x \sqrt{p} \sqrt{p} = p x p = \sqrt{p^2} x \sqrt{p^2} = \text{asrt}_{p \& p}(x).
\]

7. A completely positive map \( f : \mathcal{A} \to \mathcal{B} \) preserves projections if and only if it is multiplicative. Hence if \( f \) is sharp, then it is a (unital) and preserves multiplication and thus square roots too. Hence:

\[
f^*(\text{asrt}_p(x)) = f(\sqrt{p} x \sqrt{p}) = \sqrt{f(p)} f(x) \sqrt{f(p)} = \text{asrt}_{f^*(p)}(f(x)).
\]

This concludes the example.

Point (1) allows us to define (total) instrument maps like in Lemma 57 (6):

\[
X \xrightarrow{\text{instr}_p} X + X
\]

The side-effect associated with the predicate \( p \) is the map \( \nabla \circ \text{instr}_p : X \to X \). We call \( p \) side-effect free if this map \( \nabla \circ \text{instr}_p \) is the identity. Also, it allows us to define sequential composition ‘andthen’ on predicates (on the same object) as:

\[
p \& q = q \circ \text{asrt}_p,
\]

see Lemma 57 (7). Moreover, we can define conditional states \( \omega \mid p \) in a quantum context as normalisation of \( \text{asrt}_p \circ \omega \), like in Example 58, if we additionally assume normalisation in a telos.

**Postulate 104.** Each assert map in a telos has a total kernel map — that is, a kernel map which is total — written as:

\[
\{X\mid p^\perp\} \xrightarrow{\pi_{p^\perp}} X \xrightarrow{\text{asrt}_p} 0 \xrightarrow{\text{total}} X
\]

The comprehension notation is deliberate, since this kernel map \( \pi_{p^\perp} \) is a comprehension map (for \( p^\perp \)), as in Definition 76 (2): let \( f : Y \to X \) satisfy \( f^\perp(p^\perp) = 1 \).

Then:

\[
0 = f^\perp(p^\perp) = p \circ f = \ker^\perp(\text{asrt}_p) \circ f = 1 \circ \text{asrt}_p \circ f.
\]

But then \( \text{asrt}_p \circ f = 0 \), by Lemma 7 so that \( f \) factors through the kernel map \( \pi_{p^\perp} \).

Notice that the assert maps are structure, but their kernel maps are determined up to isomorphism, since they form comprehension maps and thus a right adjoint to the truth functor.
Postulate 105. Postulate 102 (2) tells that im(asrt_p) = ⌈p⌉. In particular, 

\[ \text{asrt}_p(⌈p⌉) = 1 \]

so that we obtain a factorisation:

\[ \begin{array}{c}
X \\
\downarrow \pi_{[p]} \\
\{X|p\} \\
\downarrow \xi_p \\
\uparrow \text{asrt}_p \\
X
\end{array} \]

using that the kernel maps \( \pi \) are comprehension maps, see Postulate 104. Then, using Lemma 39 (7),

\[ \ker(\xi_p) = \ker(\pi_{[p]} \circ \xi_p) = \ker(\text{asrt}_p) = p^⊥. \]

We postulate that in a telos these maps \( \xi_p \) are universal, forming quotients.

In this way the equation \( X/p^⊥ = \{X|p\} \) that we discussed in the beginning of this section is built in. Moreover, if \( p \) is sharp, then \( [p] = p \), so that we have an equality \( X/p^⊥ = \{X|p\} \), as in Definition 61. Below, in Proposition 106 (6) it is shown that this means that the canonical map \( \theta_p \) from [57] is an isomorphism. Hence the properties of Lemmas 93 and 94 hold in a telos.

Since the \( \xi \)'s are quotient maps and \( \text{im}(\text{asrt}_p) = [p] \) we have a coequaliser (cokernel map) diagram by Lemma 83 (14):

\[ \begin{array}{c}
X \\
\downarrow \text{asrt}_p \\
0 \\
\downarrow \xi_p \\
X/p \\
\downarrow \uparrow \text{asrt}_p \\
\uparrow \xi_p \\
X/\{p\}
\end{array} \]

This concludes our description of the notion of telos. We continue with some basic properties that hold in a telos about the assert maps and the ‘andthen’ operator \( p \& q \), written as \( p \circ q \) in [GG02], and as \( [p?][q] \) in [Jac15a]. Below we prove the first three of the five requirements for andthen in [GG02 §3], see points (2) and (3) below. We also prove that sharpness is related to idempotency of \( \& \), like in \( C^* \)-algebras.

**Proposition 106.** Let \( C \) be a telos, that is, \( C \) is an effectus in partial form, satisfying the postulates 104 – 105. Then the following properties hold.

1. The assert maps satisfy \( \text{asrt}_1 = \text{id} : X \rightarrow X \) and \( \text{asrt}_0 = 0 : X \rightarrow X \) for the truth and falsity predicates \( 1, 0 \) on \( X \); moreover, \( \text{asrt}_s = s : I \rightarrow I \) for each scalar \( s \).

2. For each predicate \( p \) on \( X \in C \) we have a map of effect algebras:

\[ \text{Pred}(X) \xrightarrow{p \&(-)} \downarrow p \]

As a result, \( p \& q \leq p \). Moreover, by point (1):

\[ 1 \& p = p = p \& 1 \quad \text{and} \quad 0 \& p = 0 = p \& 0. \]

3. If \( p \& q = 0 \), then \( p \& q = q \& p \).

4. For a predicate \( p \in \text{Pred}(X) \) the following side-effect freeness formulations are equivalent.
(a) \(\text{asrt}_p \leq \text{id}\);
(b) \(\text{asrt}_{p^\perp} \leq \text{id}\);
(c) \(\nabla \circ \text{instr}_p = \text{id}\).

5. There are equivalences:

\[
p \text{ is sharp } \iff p \land p = p.
\]

6. For a sharp predicate \(p\), the map \(\theta_p = \xi_{p^\perp} \circ \pi_p: \{X|p\} \to X/p^\perp\) from \([57]\) is the identity. In particular, a telos has both comprehension and quotients as in Definition \([71]\).

7. For each predicate \(p\) the image of the comprehension map \(\pi_p\) is given by \(\text{im}(\pi_p) = \lceil p \rceil\). Further, for sharp predicates \(p,q\) on the same object we have \(p \leq q\) iff \(\pi_p \leq \pi_q\), like in \([61]\). Hence Proposition \([72]\) applies in a telos.

Proof We must be careful not to assume more about the assert maps than is postulated above.

1. We use Postulate \([102] (3)\) each time. First, the identity map \(\text{id} : X \to X\) evidently satisfies \(\text{id} \leq \text{id}\), so that \(\text{id} = \text{asrt}_{1_{\text{id}}} = \text{asrt}_1\). Similarly, \(0 \leq \text{id}\), so that \(0 = \text{asrt}_{1_{\text{id}}} = \text{asrt}_0\). Further, we have \(\text{id} = 1: I \to I\), see Lemma \([52] (2)\). Hence every scalar \(s : I \to I\) satisfies \(s \leq 1 = \text{id}\), and thus \(s = \text{asrt}_{1_{\text{id}}} = \text{asrt}_{1_{\text{id}}} = \text{asrt}_s\).

2. By Proposition \([13] (2)\) we have:

\[
p \land (q_1 \oplus q_2) = (q_1 \oplus q_2) \circ \text{asrt}_p
\]

\[
= (q_1 \circ \text{asrt}_p) \oplus (q_2 \circ \text{asrt}_p) = (p \land q_1) \oplus (p \land q_2).
\]

As a result, \(p \land (\neg)\) is monotone, and in particular \(p \land q \leq p\). Since \(p \land 1 = p\) we obtain that \(p \land (\neg)\) is a map of effect algebras \(\operatorname{Pred}(X) \to \downarrow p\).

3. Let \(p \land q = 0\), then \(q \circ \text{asrt}_p = p \land q = 0\), so that \(q \leq \text{im}^\perp(\text{asrt}_p) = [p]^\perp\) by Lemma \([47] (3)\). But then \([q] \leq \lceil p \rceil\), since \(\lceil p \rceil\) is sharp, and thus \(p \leq [p] \leq \lceil q \rceil = \text{im}^\perp(\text{asrt}_q)\). Hence, again by Lemma \([47] (3)\), \(q \land p = p \circ \text{asrt}_q = 0 = p \land q\).

4. For the implication \([48] \Rightarrow [49]\), let \(\text{asrt}_p \leq \text{id}\). Then there is a map \(f: X \to X\) with \(\text{asrt}_p \odot f = \text{id}\). This \(f\) then also satisfies \(f \leq \text{id}\), so that \(f = \text{asrt}_q\) for \(q = \ker^\perp(f)\) by Postulate \([102] (3)\). We have:

\[
1 = \ker^\perp(\text{id}) = \ker^\perp(\text{asrt}_p \odot f) = \ker^\perp(\text{asrt}_p) \odot \ker^\perp(f) = p \odot \ker^\perp(f).
\]

Hence \(p^\perp = \ker^\perp(f) = q\). But then \(\text{asrt}_{p^\perp} = \text{asrt}_q = f \leq \text{id}\).

For the implication \([49] \Rightarrow [40]\) we assume \(\text{asrt}_{p^\perp} \leq \text{id}\). By reasoning as before, we get \(\text{asrt}_p \odot f = \text{id}\) for \(f = \text{asrt}_p \leq \text{id}\). Hence \(\text{asrt}_p \odot \text{asrt}_{p^\perp} = \text{id}\), so that we are done as in the proof of Lemma \([57] (6)\).
Finally, for (4c) \( \Rightarrow \) (4a), assume an equality of total maps \( \nabla \circ \text{instr}_p = \text{id} : X \to X \). Then:

\[
\text{asrt}_p \odot \text{asrt}_{p^\perp} = (\nabla \circ \kappa_1 \circ \text{asrt}_p) \odot (\nabla \circ \kappa_2 \circ \text{asrt}_{p^\perp}) \nabla \circ (\kappa_1 \circ \text{asrt}_p) \odot (\kappa_2 \circ \text{asrt}_{p^\perp}) \nabla \circ \langle \text{asrt}_p, \text{asrt}_{p^\perp} \rangle = \nabla \circ \langle \text{asrt}_p, \text{asrt}_{p^\perp} \rangle = \nabla \circ \text{instr}_p = \text{id}.
\]

Hence \( \text{asrt}_p \leq \text{asrt}_p \odot \text{asrt}_{p^\perp} = \text{id} \).

5. We use equivalences:

\[
p \text{ is sharp} \iff \text{im}(\text{asrt}_p) = \lceil p \rceil \leq p
\]

\[
\iff p^\perp \leq \text{im}^\perp(\text{asrt}_p)
\]

\[
\iff p \& p^\perp = p^\perp \circ \text{asrt}_p = 0 \quad \text{by Lemma 47 (3)}
\]

\[
\iff (p \& p^\perp)^\perp = 0 \text{ in } \downarrow p
\]

\[
\iff p \& p = p.
\]

The marked equivalence uses that \( p \& (\cdot) \) is a map of effect algebras \( \text{Pred}(X) \to \downarrow p \), see point (2). Hence it preserves orthosupplements.

6. Let \( p \) be a sharp predicate. Then \( p \& p = p \) by point (5), and thus \( \text{asrt}_p \circ \text{asrt}_p = \text{asrt}_{p \& p} = \text{asrt}_p \) by Postulate 102 (6). Using that \( p = \lceil p \rceil \), this last equation yields:

\[
\pi_p \circ \xi_{p^\perp} \circ \pi_p \circ \xi_{p^\perp} = \text{asrt}_p \circ \text{asrt}_p = \text{asrt}_p = \pi_p \circ \xi_{p^\perp}.
\]

But then \( \theta_p = \xi_{p^\perp} \circ \pi_p = \text{id} \), since \( \pi_p \) is monic, and \( \xi_{p^\perp} \) is epic.

7. For an arbitrary predicate \( p \) we have \( \text{im}(\pi_{\lceil p \rceil}) = \text{im}(\pi_{\lceil p \rceil} \circ \xi_{p^\perp}) = \text{im}(\text{asrt}_p) = \lceil p \rceil \), since \( \xi_{p^\perp} \) is externally, and thus internally, epic. In particular, \( \text{im}(\pi_q) = q \) if \( q \) is sharp.

We always have \( \text{im}(\pi_p) \leq p \) by minimality of images. We show that \( \text{im}(\pi_p) \) is the greatest sharp predicate below \( p \). If \( q \) is sharp, and \( q \leq p \), then there is a (total) map \( f : \{X \mid q\} \to \{X \mid p\} \) with \( \pi_p \circ f = \pi_q \). But then we are done: \( q = \text{im}(\pi_q) \leq \text{im}(\pi_p) \), where the inequality follows from minimality of images, and:

\[
\pi_q^\perp \circ \text{im}(\pi_p) = \pi_q^* \circ \text{im}(\pi_p) = f^* \circ \pi_p^* \circ \text{im}(\pi_p) = f^* (1) = 1.
\]

Next we prove the equivalence \( p \leq q \iff \pi_p \leq \pi_q \) for sharp predicates \( p, q \) on the same object. The direction (\( \Rightarrow \)) always holds. For (\( \Leftarrow \)) we use \( p = \text{im}(\pi_p) \leq \text{im}(\pi_q) = q \).

**Remark 107.** Let us try to find out in which sense assert maps satisfying the above postulates are uniquely determined. To this end, assume we have two sets of assert maps, written as \( \text{asrt}_p \) and \( \text{asrt}_p' \).
They both have kernel maps like in Postulate 104, written as:

\[ \{ X | p \} \xrightarrow{\pi_p} X \xrightarrow{\varphi_p} \{ X | p \} \]

In Postulate 104 we have seen that the kernel maps form comprehension maps, and are thus determined up-to-isomorphism. This means that for each predicate \( p \) there is a (total) isomorphism \( \varphi_p \) in a commuting triangle:

\[ \{ X | p \} \xrightarrow{\varphi_p} \{ X | p \} \]

By factoring like in Postulate 105 we obtain two maps \( \xi_p \) and \( \xi'_p \) in:

\[ \{ X | [p] \} \]

Since both \( \xi \) and \( \xi' \) are universal quotient maps, there is a second isomorphism, written as \( \psi_p \), with \( \psi_p \circ \xi_p = \xi'_p \). The endo map \( \varphi_p^{-1} \circ \psi_p : \{ X | [p] \} \rightarrow \{ X | [p] \} \) satisfies:

\[ \pi_{[p]} \circ (\varphi_{[p]}^{-1} \circ \psi_p) \circ \xi_p = \pi'_{[p]} \circ \xi'_p = \text{asrt}'_p. \]

15.1 Uniqueness of assert maps, in von Neumann algebras

We have defined a telos to be a special type of effectus (Postulate 101) endowed with a family of assert maps (Postulate 102) that gives us comprehension (Postulate 104) and quotients (Postulate 105). We have devoted much effort to see whether this list of postules (or any extension of it) uniquely determines the assert maps. In this section, we will show that for the telos of von Neumann algebras, \( \text{vNA}^{\text{op}} \), the assert maps are uniquely determined, and are given by \( \text{asrt}_p(x) = \sqrt{\pi x} \sqrt{\pi} \), if we add Postulate 105 to the list. Whether the assert maps are uniquely determined by these postulates in general remains an open problem.

We shall call a map \( f : X \rightarrow Y \) a comprehension projection, of simply a comprehension map if there is a sharp predicate \( q \) on \( Y \) with an isomorphism:

\[ X \xrightarrow{\pi_q} \{ Y | q \} \]

Such a comprehension is automatically monic.
Postulate 108. Let \( p \) be any predicate on \( X \) and \( \pi : 1 \to X \) a state that is also a comprehension map (in a telos). Then:

\[
\text{im}(p * \pi) = [p & \text{im}(\pi)],
\]

where \( p * \pi = \text{asrt}_p \circ \pi \) and \( p & \text{im}(\pi) = \text{im}(\pi) \circ \text{asrt}_p \) as in (68).

The substate \( p * \pi \) is an unnormalised version of the conditional state \( \pi \downarrow p \) of Example 58 which makes sense even if the validity probability \( \pi \downarrow p \) is zero. If \( (\pi \downarrow p) \neq 0 \), we have

\[
\pi \downarrow p \circ (\pi \downarrow p) = p * \pi.
\]

Before we come to the main result, we show that this additional postulate holds in the telos of von Neumann algebras. In doing so we use the following two properties. For a non-zero \( r \in [0, 1] \),

\[
\text{im}(r \cdot f) = \text{im}(f) \quad \text{and} \quad [r \cdot p] = p,
\]

(69)

for a subunital map \( f \) and a projection \( p \).

Example 109. The postulate 108 is true in the effectus \( \text{vNA}^{op} \) with the standard asrt-maps \( \text{asrt}_p(x) = \sqrt{p} \cdot x \cdot \sqrt{p} \) from Example 103. To demonstrate this, we will first study states that are comprehension maps. Let \( A \) be a von Neumann algebra. We will state and prove a number of claims.

1. For a sharp predicate \( s \) on \( A \), the mapping:

\[
\downarrow s \to \text{Pred}(\{ A \mid s \}) = [0, 1]_{\{ A \mid s \}}
\]

\[
a \to \pi_s^*(a) = \pi_s(a) = sas
\]

is an order isomorphism — where the downset \( \downarrow s \) is a subset of \( \text{Pred}(\{ A \mid s \}) = [0, 1]_{\{ A \mid s \}} \). Recall that \( \{ A \mid s \} = ssA s \), see Example 77 (4).

To see this, note that \( sas \leq s \) for any \( sas \in [0, 1]_{\{ A \mid s \}} \) since \( s \) is the unit element in \( \{ A \mid s \} = ssA s \). Hence there is an inclusion-map \( j : [0, 1]_{\{ A \mid s \}} \to \downarrow s \). It is the inverse to the above map \( \pi_s^* \) since:

\[
\pi_s^*(j(sas)) = s(sas)s = sas \quad \text{since } s \text{ is a projection}
\]

\[
\pi_s^*(j(a)) = sas = a \quad \text{by Lemma 69 since } a \leq s.
\]

2. A comprehension map \( \pi_s : \mathcal{A} \to \{ A \mid s \} \) for a sharp predicate \( s \) on \( \mathcal{A} \) is a state if and only if \( s \) is a minimal projection.

First, if \( \pi_s \) is a state, then \( \{ A \mid s \} \cong \mathbb{C} \), so that by the previous point we have an order isomorphism:

\[
\downarrow s \cong \text{Pred}(\{ A \mid s \}) \cong \text{Pred}(\mathbb{C}) = [0, 1].
\]

If \( t \in \downarrow s \) is a projection, then it corresponds to sharp element in \([0, 1] \). But there are only two sharp elements in \([0, 1] \), namely 0 and 1, so that \( t = 0 \) or \( t = s \). Hence \( s \) is a minimal projection.

Conversely, if \( s \) is a minimal projection, then \( \text{Pred}(\{ A \mid s \}) \cong \downarrow s \) has only two projections. This means that the von Neumann algebra \( \{ A \mid s \} \) itself has two projections, and is thus isomorphic to the unique von Neumann algebra \( \mathbb{C} \) with two projections.
3. Let \( \mathcal{H} \) be any Hilbert space with element \( v \in \mathcal{H} \). The projection \( |v\rangle\langle v| \in \mathcal{B}(\mathcal{H}) \) is minimal, and the corresponding state \( \pi_v : \mathcal{B}(\mathcal{H}) \to \mathbb{C} \) given by \( \pi_v(t) = (tv|v) \) is a comprehension map. One calls a state of this form \( \pi_v \) a vector state. The image in \( \mathcal{B}(\mathcal{H}) \) of such a vector state \( \pi_v \) is given by the projection \( |v\rangle\langle v| \). Any state \( \mathcal{B}(\mathcal{H}) \to \mathbb{C} \) which is a comprehension map is of this form.

4. Any state that is a projection is of the form \( \pi : \mathcal{B}(\mathcal{H}) \oplus \mathcal{B} \to \mathbb{C} \), where \( \pi(a, b) = \langle av | v \rangle \) for some vector \( v \in \mathcal{H} \). The image of \( \pi \) is then the pair \( (|v\rangle\langle v|, 0) \in \mathcal{B}(\mathcal{H}) \oplus \mathcal{B} \).

To show this, assume \( \pi : \mathcal{A} \to \mathbb{C} \) is a comprehension map for a (consequently minimal) projection \( s \). Let \( c(s) \) denote the central carrier of \( s \) that is: \( c(s) \in [0, 1]_\mathcal{A} \) is the least central projection above \( s \in [0, 1]_\mathcal{A} \). We will show \( \{ \mathcal{A} | c(s) \} \cong c(s) \mathcal{A} \) is a (type I) factor, in which the projection \( c(s)s \) is minimal.

Let \( z \leq c(s) \) be any central projection in \( c(s) \mathcal{A} \). If we can show \( z = 0 \) or \( z = c(s) \), we may conclude \( c(s) \mathcal{A} \) is a factor. Note \( z \) is central in \( \mathcal{A} \) as \( za = az \). Hence \( c(s)z = az = az \) for any \( a \in \mathcal{A} \). Clearly \( z \) is a projection below \( s \). Hence by minimality of \( s \), we have \( zs = s \) or \( zs = 0 \). For the first case, assume \( zs = s \). Then \( s \leq z \) by Lemma [102]. Hence \( c(s) \leq z \leq c(s) \). Thus \( z = c(s) \), as desired. Now, we cover the other case \( zs = 0 \). That is: \( z = z^\perp \). Hence \( z \leq s^\perp \) by Lemma [102]. So \( s \leq z^\perp \). Hence \( c(s) \leq z^\perp \).

A fundamental result says that each such (type I) factor is given by bounded operators on a Hilbert space, see e.g. [Top7] Corolary 10. Thus, let \( c(s) \mathcal{A} \cong \mathcal{B}(\mathcal{H}) \) for some Hilbert space \( \mathcal{H} \). Consequently, there is an isomorphism \( \vartheta : \mathcal{A} \to \mathcal{B}(\mathcal{H}) \) such that \( \pi = \pi' \circ \pi_1 \circ \vartheta \), where \( \pi' : \mathcal{B}(\mathcal{H}) \to \mathbb{C} \) is a comprehension map for the minimal projection corresponding to \( c(s)s \) and hence a vector state.

5. We will now show that the additional postulate [103] holds in the tens \( v \mathcal{NA}^{op} \mathcal{A} \) for the canonical asr-maps from Example [103]. Let \( \pi \) be a state that is also a comprehension map. With the previous in mind, we may assume without loss of generality that \( \pi \) is of the form \( \pi : \mathcal{B}(\mathcal{H}) \oplus \mathcal{B} \to \mathbb{C} \) with \( \pi(a, b) = \langle av | v \rangle \) for some \( v \in \mathcal{H} \). Let \( e = (e_1, e_2) \) and \( d = (d_1, d_2) \) be arbitrary effects on \( \mathcal{B}(\mathcal{H}) \oplus \mathcal{B} \). Note that:

\[
(e * \pi)(d) = \pi(\text{asr}_e(d)) = \pi(\sqrt{d_1} v_1 \sqrt{d_2} v_2) = \langle d_1^{\frac{1}{2}} v_1 | v \rangle = \langle d_1^{\frac{1}{2}} v_1 | v \rangle.
\]

Suppose \( \sqrt{d_1} v = 0 \). Then \( (e * \pi)(d) = \langle d_1^{\frac{1}{2}} v_1 | v \rangle = 0 \), so that \( \text{im}(e * \pi) = 0 \). Also \( (e \& \text{im}(\pi)) = 0 \) since:

\[
e \& \text{im}(\pi) = e \& (|v\rangle\langle v|, 0) \quad \text{see point (1)}
\]

\[
= \text{asr}_e(|v\rangle|v\rangle, 0)
\]

\[
= (|\sqrt{d_1} v, \sqrt{d_2} v\rangle, 0)
\]
For the other case, assume $\sqrt{e_1}v \neq 0$. Then, using what we have seen above:

$$(e \ast \pi)(d) = \langle d, \sqrt{e_1}v \rangle \sqrt{e_1}v = \|\sqrt{e_1}v\|^2 \langle d, \frac{\sqrt{e_1}v}{\|\sqrt{e_1}v\|} \rangle \frac{\sqrt{e_1}v}{\|\sqrt{e_1}v\|}.$$ 

This means $e \ast \pi$ is a scaled vector state with:

$$\text{im}(e \ast \pi) = (\|\sqrt{e_1}v\| \langle \sqrt{e_1}v, 0 \rangle \rangle \langle \sqrt{e_1}v, 0 \rangle, 0)$$

by (69) and point 4

$$= \|\sqrt{e_1}v\| \langle \sqrt{e_1}v, 0 \rangle \rangle \langle \sqrt{e_1}v, 0 \rangle$$

by (69)

$$= [\sqrt{e_1}v \rangle \langle \sqrt{e_1}v, 0 \rangle ]$$

Hence, in both cases $\text{im}(e \ast \pi) = [\sqrt{e_1}v \rangle \langle \sqrt{e_1}v, 0 \rangle ]$, as desired.

Now we are ready to show there is only one choice of assert maps in $v\text{NA}^{\text{op}}$ that satisfies all the postulates — including 108. The result is a reformulation of a result from [WW15], which in turn is inspired by the characterization of the sequential product in Hilbert spaces by Gudder and Latrémoïlère, see [GL08]. Our Postulate 108 should be compared with their Condition 1.

**Theorem 110.** For each von Neumann algebra $\mathcal{A}$ and $p \in [0, 1]_{\mathcal{A}}$, let

$$\text{asrt}_p : \mathcal{A} \rightarrow \mathcal{A}$$

be a completely positive normal subunital map. Assume that these assert maps on $v\text{NA}^{\text{op}}$ satisfy Postulates 101, 102, 104, 105, and 108.

Then for every von Neumann algebra $\mathcal{A}$, predicate $p \in [0, 1]_{\mathcal{A}}$, and $x \in \mathcal{A}$,

$$\text{asrt}_p(x) = \sqrt{p} x \sqrt{p}. \quad (70)$$

**Proof** Let $\mathcal{H}$ be a Hilbert space. We will first show that Equation (70) holds for $\mathcal{A} = B(\mathcal{H})$. Let $p \in [0, 1]_{B(\mathcal{H})}$ be given.

By the discussion in Remark 107 we already have the following connection between the canonical assert map $a \mapsto \sqrt{p}a\sqrt{p}$ and the one, $\text{asrt}_p$, we are given: there is an automorphism $\vartheta$ on $[p]B(\mathcal{H})[p]$ such that, for all $a \in B(\mathcal{H})$,

$$\text{asrt}_p(a) = \sqrt{p} \vartheta([p]a[p]) \sqrt{p}.$$ 

Since $[p]B(\mathcal{H})[p]$ is a type I factor (i.e. isomorphic to a $B(\mathcal{H})$), it is known (see Theorem 3 of [Kap52]) that $\vartheta$ must be what is called an inner automorphism, that is, there is an unitary $u \in [p]B(\mathcal{H})[p]$ such that, $\vartheta(a) = u^*au$ for all $a \in [p]B(\mathcal{H})[p]$. Note that $[p]u = u$ since $u \in [p]B(\mathcal{H})[p]$, and thus we have, for all $a \in B(\mathcal{H})$,

$$\text{asrt}_p(a) = \sqrt{p} u^* a u \sqrt{p}.$$ 

Of course, our ultimate goal should be to show that $u = 1$, or at least that $u = \lambda \cdot 1$ for some $\lambda \in \mathbb{C}$ with $|\lambda| = 1$. Our first step is to prove $u p = p u$.

To this end, we extract some information about $u$ from Postulate 108. Let $x \in \mathcal{H}$ with $\|x\| = 1$ be given. Let $\pi : B(\mathcal{H}) \rightarrow \mathbb{C}$ be given by $\pi(a) = \text{tr}(a) x \langle x, a \rangle$, where $\text{tr}(a)$ is the trace of the matrix representation of $a$.
\( \langle x \mid ax \rangle \) for \( a \in B(\mathcal{H}) \). Then by Example 109 we know that \( \pi \) is a comprehension. Thus, by Postulate 108 we know that, in \( \mathbf{vNA}^{op} \),

\[
\text{im}(\text{asrt}_p \circ \pi) = [p \& \text{im}(\pi)].
\] (71)

Before we continue, observe that for \( y \in \mathcal{H} \) with \( \|y\| \leq 1 \) we have

\[
\|y\| \cdot \|y\| = \text{im}(\langle y \mid (-y) \rangle) = (\text{projection onto } y\mathbb{C} \equiv \{\lambda y : \lambda \in \mathbb{C}\}).
\]

Now, let us unfold Equation (71).

\[
(\text{projection onto } (\sqrt{pu^*}x)\mathbb{C}) = [\| \sqrt{pu^*}x \rangle \langle \sqrt{pu^*}x \mid ]
\]

\[
= [ \sqrt{pu^*} | x \rangle \langle x | u\sqrt{p} ]
\]

\[
= [ p \& \text{im}(\pi) ]
\]

\[
= \text{im}(\text{asrt}_p \circ \pi)
\]

\[
= \text{im}(\langle x \mid \sqrt{pu^*}(-u\sqrt{p}x) \rangle)
\]

\[
= \text{im}(\langle u\sqrt{pu}x \mid (-u\sqrt{p}x) \rangle)
\]

\[
= (\text{projection onto } (u\sqrt{p}x)\mathbb{C})
\]

Hence, for every \( x \in \mathcal{H} \) with \( \|x\| = 1 \) there is \( \alpha \in \mathbb{C} \) with \( \alpha \neq 0 \) such that

\[
\sqrt{pu^*}x = \alpha \cdot u\sqrt{p}x.
\]

By scaling it is clear that this statement is also true for all \( x \in \mathcal{H} \) (and not just for \( x \in \mathcal{H} \) with \( \|x\| = 1 \)). While a priori \( \alpha \) might depend on \( x \), we will show that there is \( \alpha \in \mathbb{C} \setminus \{0\} \) such that \( \sqrt{pu^*} = \alpha \cdot u\sqrt{p} \).

First note that \( \sqrt{pu^*}x = 0 \) iff \( u\sqrt{p}x = 0 \) for all \( x \in \mathcal{H} \). Thus we may factor \( \sqrt{pu^*} \) and \( u\sqrt{p} \) through the quotient map \( q : \mathcal{H} \to \mathcal{H}/K \), where

\[
K = \{x \in \mathcal{H} : \sqrt{pu^*}x = 0\} = \{x \in \mathcal{H} : u\sqrt{p}x = 0\}.
\]

(Here \( \mathcal{H}/K \) is just the quotient of \( \mathcal{H} \) as vector space.) Let \( t, s : \mathcal{H}/K \to \mathcal{H} \) be given by \( s \circ q = \sqrt{pu^*} \) and \( t \circ q = u\sqrt{p} \). Then \( s \) and \( t \) are injective, and writing \( V = \mathcal{H}/K \), it is not hard to see that for every \( x \in V \) there is \( \alpha \in \mathbb{C} \setminus \{0\} \) with \( s(x) = \alpha \cdot t(x) \).

We will show that there is \( \alpha \in \mathbb{C} \setminus \{0\} \) with \( s = \alpha \cdot t \). If \( V = \{0\} \) then this is clear, so assume that \( V \neq \{0\} \). Pick \( x \in V \) with \( x \neq 0 \), and let \( \alpha \in \mathbb{C} \setminus \{0\} \) be such that \( s(x) = \alpha \cdot t(x) \). Let \( y \in V \) be given; we must show that \( s(y) = \alpha \cdot t(y) \).

Now, either \( t(x) \) and \( t(y) \) are linearly independent or not.

Suppose that \( t(x) \) and \( t(y) \) are linearly independent. Let \( \beta, \gamma \in \mathbb{C} \setminus \{0\} \) be such that \( s(y) = \beta \cdot t(y) \) and \( s(x + y) = \gamma \cdot t(x + y) \). Then

\[
(\gamma - \alpha) \cdot t(x) + (\gamma - \beta) \cdot t(y) = 0.
\]

Thus, as \( t(x) \) and \( t(y) \) are linearly independent, \( \gamma - \alpha = 0 \) and \( \gamma - \beta = 0 \), and so \( \alpha = \beta \). Hence \( s(y) = \alpha \cdot t(y) \).

Suppose that \( t(x) \) and \( t(y) \) are linearly dependent. Since \( x \neq 0 \), we have \( t(x) \neq 0 \)—as \( t \) is injective—, and thus \( t(y) = \alpha t(x) \) for some \( \alpha \in \mathbb{C} \). Then \( t(y - \alpha x) = 0 \), and so \( y = \alpha x \) since \( t \) is injective. Then

\[
s(y) = \alpha s(x) = \alpha t(x) = \alpha t(y).
\]
Thus, in any case, \( s(y) = \alpha t(y) \). Hence \( s = \alpha \cdot t \). Thus \( \sqrt{p_u} = \alpha \cdot u \sqrt{p} \).

It follows that \( p = \sqrt{p_u} u \sqrt{p} = \alpha \cdot u \sqrt{p_u} \sqrt{p} = u \sqrt{p} \sqrt{p_u} = u p u \), and so \( p u = u p \). Then also \( \sqrt{p_u} = u \sqrt{p} \). Thus \( \sqrt{p_u} = \alpha u \sqrt{p} = \alpha \sqrt{p} u \).

Now, note that \( (\sqrt{p_u})^* = u \sqrt{p} \), and so \( u \sqrt{p} = \alpha^* \sqrt{p_u}^* = \alpha^* \alpha u \sqrt{p} \). Then if \( u \sqrt{p} \neq 0 \) we get \( \alpha^* \alpha = 1 \), and if \( u \sqrt{p} \) then we can put \( \alpha = 1 \) and still have both \( \sqrt{p_u}^* = \alpha u \sqrt{p} \) and \( \alpha^* \alpha = 1 \).

It follows that, for all \( b \in B(\mathcal{H}) \),

\[
\sqrt{p_u}^* b u \sqrt{p} = \sqrt{p} u b^* \sqrt{p}.
\]

By the universal property of the quotient, we conclude that \( u^*(-) u = u(-) u^* \), and thus \( u^2 b = b u^2 \) for all \( b \in B(\mathcal{H}) \). Hence \( u^2 \) is central in \( B(\mathcal{H}) \).

Since \( B(\mathcal{H}) \) is a factor, we get \( u^2 = \lambda \cdot 1 \) for some \( \lambda \in \mathbb{C} \). Then by Postulate \( 102 \) (6) and using \( \sqrt{p} u = u \sqrt{p} \), we get, for all \( b \in B(\mathcal{H}) \),

\[
p b p = \sqrt{p_u}^* \sqrt{p_u} b u \sqrt{p} \sqrt{p} = (\text{asrt}_p \circ \text{asrt}_p)(b) = \text{asrt}_{p^2 p}(b).
\]

Note that \( p \& p = \text{asrt}_{p^2}(p) = \sqrt{p_u}^* p u \sqrt{p} = p^2 \). Thus, for all \( b \in B(\mathcal{H}) \),

\[
\text{asrt}_{p^2}(b) = p b p.
\]

Since every element of \([0, 1]_{B(\mathcal{H})}\) is a square, we have proven Equation \( (70) \) when \( \mathcal{A} \equiv B(\mathcal{H}) \).

Now, let us consider the general case (so \( \mathcal{A} \) need not be of the form \( B(\mathcal{H}) \)).

Let \( \omega: \mathcal{A} \to \mathcal{C} \) be any normal state on \( \mathcal{A} \). Let \( a \in \mathcal{A} \) be given. As normal states are separating, it suffices to prove that \( \omega(\text{asrt}_p(a)) = \omega(\sqrt{p_u} \sqrt{p}) \).

Let \( p: \mathcal{A} \to B(\mathcal{H}) \) be the GNS-representation of \( \mathcal{A} \) for the state \( \omega \) with cyclic vector \( x \in \mathcal{H} \). Let \( \pi: B(\mathcal{H}) \to \mathcal{C} \) be given by \( \pi(b) = (x | b x) \) for all \( b \in \mathcal{A} \).

Then we have \( \omega = \pi \circ g \) (in \text{vNA}). Thus:

\[
\omega(\text{asrt}_p(a)) = \pi(g(\text{asrt}_p(a))) = \pi(g(\text{asrt}_p(a))) = \pi(\sqrt{\text{asrt}_p(a)} \sqrt{\text{asrt}_p(a)}) = \pi(\sqrt{\text{asrt}_p(a)} \sqrt{\text{asrt}_p(a)}) = \omega(\sqrt{\text{asrt}_p(a)} \sqrt{\text{asrt}_p(a)}) = \omega(\sqrt{p_u} \sqrt{p})
\]

Hence \( \text{asrt}_p(a) = \sqrt{p_u} \sqrt{p} \). We have proven Equation \( (70) \).

\[\square\]

16 Conclusions and future work

This text collects definitions and results about the new notion of effectus in categorical logic. Already at this early stage it is clear that the theory of effectuses includes many examples that are of interest in quantum (and probability) theory. But much remains to be done. We list a few directions for further research.

1. Which constructions exist to obtain new effectuses from old, such as products, slices, (co)algebras of a (co)monad, etc.? A related matter is the definition of an appropriate notion of morphism of effectuses: one can take a functor that preserves finite coproducts and the final object; alternatively, one can take adjoints as morphisms, like in geometric morphisms between toposes.
2. Tensors in effectuses have been discussed in Section 10 but only in a very superficial way. They deserve more attention, leading to a closer connection with the work done in the Oxford school (see the introduction). For instance, the combination of tensors \( \otimes \) and coproducts + could lead to a 3-dimensional graphical calculus that combines (parallel) composition and effect logic.

3. The approach in this text is very much logic-oriented. Connections with quantum theory are touched upon, but should be elaborated further. In particular, the formulation (and correctness!) of concrete quantum protocols in the present setting is missing.

4. An internal language for effectuses, along the lines of [Ada14, AJ15], may be useful for the verification of probabilistic and/or quantum protocols.

5. The possibility of doing homological algebra (in abstract form, see [Gra92, Gra12]) also deserves attention.

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2015.
| Notation | Meaning |
|----------|---------|
| $[f, g]$ | cotuple of maps $f, g$, 7 |
| $\mathcal{B}(\mathcal{H})$ | space of operators on a Hilbert space $\mathcal{H}$, 40 |
| $\text{End}(X)$ | homset of partial maps $X \to X$, 65 |
| $\mathcal{M}$ | 11 |
| $\text{Pred}(X)$ | collection of predicates on an object $X$, 11 |
| $\text{SStat}(X)$ | collection of substates of an object $X$, 11 |
| $\text{ShaPred}(X)$ | collection of sharp predicates on an object $X$, 121 |
| $\text{Stat}(X)$ | collection of states of an object $X$, 11 |
| $\text{End}_{\leq \text{id}}(X)$ | set of partial maps $X \to X$ below the identity, 66 |
| $\ast$ | composition, 7 |
| $\ast$, sum of total maps, 7 |
| $\text{tr}$ | trace operation, 40 |
| 0 | total map, 7 |
| $\mathbf{Ab}$ | category of Abelian groups, 20 |
| $\mathbf{BA}$ | category of Boolean algebras, 74 |
| $\text{Caus}(A)$ | category of causal maps, 41 |
| $\text{Conv}_{M}$ | category of convex sets over an effect monoid $M$, 36 |
| $\text{CvNA}$ | category of commutative von Neumann algebras, 23 |
| $\mathbf{EA}$ | category of effect algebras, 29 |
| $\mathbf{EMod}_{M}$ | category of effect modules over an effect monoid $M$, 32 |
| $\mathcal{K}$ | Kleisli category, 19 |
| $\mathbf{OAb}$ | category of ordered Abelian groups, 20 |
| $\mathbf{OUG}$ | category of order unit groups, 20 |
OUS, category of order unit spaces, 21

PCM, category of partial commutative monoids, 26

$Pred_\cup(C)$, category of predicates in an effectus in partial form $C$, 83

$Par(B)$, Kleisli category of the lift monad on $B$, 8

$Pred(B)$, category of predicates in an effectus in total form $B$, 82

Sets, category of sets, 18

$Tot(C)$, category of total maps in a FinPAC with effects $C$, 60

vNA, category of von Neumann algebras, 21

copier, 80

$D$, distribution monad, 18

$G$, Giry monad, 19

$M$, multiset monad, 43

$D_{\leq 1}$, subdistribution monad, 19

semicartesian, 76

$g^\triangledown$, modal predicate transformer for partial map $g$, 10

$g^\circ$, modal predicate transformer for partial map $g$, 10

$f^*$, substitution predicate transformer for total map $f$, 10

$f_*$, state transformer associated with total map $f$, 37
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