Long range scattering for some
Schrödinger related nonlinear systems

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Abstract

We review some results, especially recent ones, on the theory of scattering and more precisely on the local Cauchy problem at infinity in time, in long range situations, for some nonlinear systems including some form of the Schrödinger equation. We consider in particular the Wave-Schrödinger system \((WS)_3\) and the Maxwell-Schrödinger system \((MS)_3\) in space dimension \(n = 3\) and the Klein-Gordon-Schrödinger system \((KGS)_2\) in space dimension \(n = 2\). We also consider the Zakharov system \((Z)_n\) which can be studied from the same point of view in space dimension \(n = 3\) where it is short range and in space dimension \(n = 2\) where it is in a mixed situation. We concentrate on the applications of a direct method which is intrinsically restricted to the case of small Schrödinger data and to the borderline long range case, to which \((WS)_3\), \((MS)_3\) and \((KGS)_2\) belong. The main results in all cases are the existence of solutions defined for large times and with prescribed asymptotic behaviour, without any size restriction for the data of the wave, Maxwell or Klein-Gordon function. Furthermore convergence rates as negative powers of \(t\) are obtained for the solutions in suitable norms. The asymptotic forms in the long range cases are obtained by suitable modifications of the solutions of the underlying free linear system.

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Key words : Long range scattering, Wave-Schrödinger system, Maxwell-Schrödinger system, Zakharov system, Klein-Gordon-Schrödinger system.

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1 Introduction

The purpose of this paper is to review some results, especially recent ones, on the theory of scattering in long range situations for some non linear systems containing some form of the Schrödinger equation. In order to put the subject in perspective, we first list a number of equations and systems for which one can ask the same questions and to which one can apply the same methods as described below. In all this paper, \(u\) denotes a complex valued function defined in space time \(\mathbb{R}^{n+1}\), \(\Delta\) denotes the Laplacian in \(\mathbb{R}^n\) and \(\square = \partial^2_t - \Delta\) the d’Alembertian in \(\mathbb{R}^{n+1}\). When appended to the symbol of an equation or system, the subscript \(n\) means that that equation or system is considered in space dimension \(n\).

The starting point is the linear Schrödinger equation

\[
i \partial_t u = -(1/2)\Delta u + Vu \quad \text{(LS)}_n
\]

where \(V\) is a real potential defined in \(\mathbb{R}^n\), a typical form of which is

\[
V(x) = \kappa |x|^{-\gamma} \quad \text{(1.1)}
\]

with \(\kappa \in \mathbb{R}\) and \(\gamma > 0\). Next come the nonlinear Schrödinger equation, a simple form of which is

\[
i \partial_t u = -(1/2)\Delta u + \kappa |u|^{p-1}u \quad \text{(NLS)}_n
\]

with \(\kappa \in \mathbb{R}\), with \(1 < p < \infty\) for \(n = 1, 2\) and \(1 < p < (n+2)/(n-2)\) for \(n \geq 3\), and the Hartree equation

\[
i \partial_t u = -(1/2)\Delta u + (V \ast |u|^2)u \quad \text{(R3)}_n
\]

where \(V\) is a real even potential defined in \(\mathbb{R}^n\) of the same type as for (LS)\(_n\), for instance given by (1.1), and where \(\ast\) denotes the convolution in \(\mathbb{R}^n\). Next come some systems including the Schrödinger equation with a real time dependent potential \(A\) defined in \(\mathbb{R}^{n+1}\) and a second equation whereby the potential \(A\) is nonlinearly coupled to the Schrödinger function. These include the Wave-Schrödinger system

\[
\begin{align*}
i \partial_t u &= -(1/2)\Delta u + Au \\
\square A &= -|u|^2
\end{align*} \quad \text{(WS)}_n
\]

which will be of special interest for \(n = 3\), the Klein-Gordon-Schrödinger system

\[
\begin{align*}
i \partial_t u &= -(1/2)\Delta u + Au \\
(\square + 1)A &= -|u|^2
\end{align*} \quad \text{(KGS)}_n
\]
which will be of special interest for \( n = 2 \), and the Zakharov system, the simplest version of which is
\[
\begin{cases}
  i\partial_t u = -(1/2)\Delta u + Au \\
  \Box A = \Delta |u|^2
\end{cases}
\]
which will be of special interest for \( n = 2, 3 \). Finally, we shall consider the Maxwell-Schrödinger system in space dimension 3, which we write here in the Coulomb gauge
\[
\begin{cases}
  i\partial_t u = -(1/2)\Delta_A u + g(|u|^2)u \\
  \Box A = \text{P Im } \pi \nabla_A u, \quad \nabla \cdot A = 0.
\end{cases}
\]
Here \( A \) is an \( IR^3 \) valued function (the magnetic potential) defined in \( IR^{3+1} \), \( \nabla_A = \nabla - iA \) and \( \Delta_A = \nabla_A^2 \) are respectively the covariant gradient and the covariant Laplacian, \( g(|u|^2) \) is the Hartree interaction
\[
g(|u|^2) = (4\pi|x|)^{-1} * |u|^2
\]
which is of the same type as in (R3) with \( V \) given by (1.1) with \( \kappa = (4\pi)^{-1} \) and \( \gamma = 1 \), and \( P \) is the projector on divergence free vector fields
\[
P = \mathbb{1} - \nabla \Delta^{-1} \nabla.
\]
The condition \( \nabla \cdot A = 0 \) is the Coulomb gauge condition. We shall describe the \((\text{MS})_3\) system in more detail in Section 3.

All the previous equations and systems are Lagrangian, namely are the Euler-Lagrange equations associated with some suitable Lagrange function. As a consequence of that fact and of some obvious invariance properties, they possess some formally conserved quantities. In particular the \( L^2 \) norm of \( u \) is formally conserved in all cases, as well as an energy function which is closely related to and in favorable cases controls the \( H^1 \) norm of \( u \) and some norm of \( A \). This leads to the definition of an energy space which is \( H^1 \) for \( u \) for \((\text{LS})_n\), \((\text{NLS})_n\) and \((\text{R3})_n\), which is \( H^1 \oplus H^1 \oplus L^2 \) for \((u, A, \partial_t A)\) for \((\text{WS})_n\), \((\text{KGS})_n\) and \((\text{MS})_3\) and which is \( H^1 \oplus L^2 \oplus \dot{H}^{-1} \) for \((u, A, \partial_t A)\) for \((\text{Z})_n\). We shall occasionally refer to those energy spaces in the sequel, but we shall never make use of the explicit form of the energy.

We now review briefly some basic ideas of scattering theory in order to extract therefrom the mathematical problem to be addressed in this paper. In a typical scattering experiment, the experimentalist has at his/her disposal a target at rest, typically in some stationary state, and does his/her best to prepare an incoming
beam of free waves or free particles. Thus the mathematical picture is that of a
dynamical system target $\times$ beam with an asymptotic motion in the distant past
of a simple nature, typically stationary state $\times$ free motion. As the experiments
proceeds, the beam reaches the target, thereby giving rise to a complicated interacting
motion. Eventually a modified more or less free beam emerges, possibly leaving
the target in a different state, so that the asymptotic motion in a distant future is
again simple. The basic objects describing that process are the wave operators $\Omega_\pm$
for the past (−) and future (+), defined as the maps going from the asymptotic
simple motion in the past and/or future to the actual interacting motion. The ex-
perimentalist then measures the scattering operator $S = \Omega_+^{-1} \circ \Omega_-$, namely the map
from the simple asymptotic motion in the past to the simple asymptotic motion in
the future. The basic idea that we shall keep from this picture is that we have a
dynamical system with simple asymptotics in time, and that we want to classify the
possible motions of the system through their asymptotics by constructing the wave
operators.

We now give a slightly different and more formal description of the previous
picture, taking the simple case of one single Schrödinger-like equation, e.g. (LS)$_n$,
(NLS)$_n$ or (R3)$_n$. The dynamical system has the space of Cauchy data as state space,
and the evolution is described by the equation at hand. The extension to systems
of two equations is obvious. We give ourselves a set $\mathcal{U}_a$ of presumed asymptotic
motions $u_a$ for the system at hand, parametrized by some data $u_\pm$. In the present
case, a natural candidate for $\mathcal{U}_a$ would be the set of solutions of the free Schrödinger
equation, namely

$$u_a(t) = U(t)u_+$$

where $U(t)$ is the unitary group

$$U(t) = \exp(i(t/2)\Delta)$$

solving the free Schrödinger equation. The initial data $u_+$ is called the asymptotic
state. One is then faced with two natural problems.

**Problem 1.** Given $u_a \in \mathcal{U}_a$, construct a solution $u$ of the equation at hand such that
$u(t) - u_a(t)$ tends to zero as $t \to +\infty$ in a suitable sense, more precisely in suitable
norms. If that problem can be solved for “all” $u_a$, one can define the wave operator
$\Omega_+$ (for positive times) as the map $u_a \to u$ thereby obtained. The same problem can
be considered for \( t \to -\infty \), thereby giving rise to the wave operator \( \Omega_- \) for negative times. Thus Problem 1 is that of the construction of the wave operators.

Traditionally the wave operator, for instance \( \Omega_+ \) is defined as the map \( u_+ \to u(0) \) where \( u \) is the solution considered above. Thus the problem decomposes into two steps. The first step is to construct the required solution \( u \) in a neighborhood of infinity in time, namely in an interval \([T, \infty)\) for \( T \) sufficiently large and with given asymptotic behaviour \( u_a \). This is the local Cauchy problem at infinity in time. The second and rather independent step consists in extending the solution previously obtained down from \( t = T \) to \( t = 0 \), and therefore reduces to the global Cauchy problem at finite times. In this paper we shall concentrate on the first step and leave aside the second one. Actually the latter is well controlled, for instance in the energy space, for all systems in the previous list for the relevant values of \( n \), except for the \((\text{MS})_3\) system (see below in Section 3 for that system).

**Problem 2.** This is the converse to Problem 1. Given a generic solution \( u \) of the equation at hand, find an asymptotic motion \( u_a \in U_a \) such that \( u(t) - u_a(t) \) tends to zero as \( t \to +\infty \) in a suitable sense (in suitable norms). If that problem and the same one for \( t \to -\infty \) can be solved (in a suitable functional framework) for “all” \( u \), one says that asymptotic completeness holds with respect to the set \( U_a \). Asymptotic completeness is a much harder question than the existence of the wave operators. It requires in particular that all possible asymptotic behaviours of the solutions of the equation at hand have been identified and included in \( U_a \). It has been proved so far only in cases of repulsive interactions (e.g. \( \kappa > 0 \) in \((\text{NLS})_n\) or \((\text{R3})_n\)) where no solitary waves occur and where \( U_a \) can be taken as the set of solutions of the free Schrödinger equation, or for small data. We shall say nothing more about that problem in this paper.

We conclude that brief introduction to scattering theory by explaining the difference between short and long range situations. The short range case is the case where the interaction in the equation or system at hand decays sufficiently fast at infinity in space and/or time for the set of solutions of the underlying free system consisting of the free Schrödinger equation and possibly of the free wave of Klein-Gordon equation to be adequate as the set \( U_a \) of asymptotic motions. The long range case is the complementary case where that set is inadequate and has to be replaced by a set of modified asymptotic motions. The modification includes in general (but
not always, e.g. \((\text{KGS})_2\) the introduction of a phase in the asymptotic Schrödinger function, and possibly the addition of correcting terms to the Schrödinger and to the Wave or Klein-Gordon function. With respect to that classification, \((\text{LS})_n\) and \((\text{R3})_n\) with potential \((1.1)\) are short range for \(\gamma > 1\) and long range for \(\gamma \leq 1\), \((\text{NLS})_n\) is short range for \(p > 1 + 2/n\) and long range for \(p \leq 1 + 2/n\), \((\text{WS})_3\), \((\text{KGS})_2\) and \((\text{MS})_3\) are borderline long range, namely analogous to \((\text{LS})_n\) and \((\text{R3})_n\) with \(\gamma = 1\), while \((\text{WS})_n\), \((\text{KGS})_n\) are short range for larger values of \(n\). Finally \((\text{Z})_3\) is short range, while \((\text{Z})_2\) is in a mixed situation and exhibits some difficulties typical of the long range case. In this paper, we shall review some of the results available for the systems \((\text{WS})_3\), \((\text{MS})_3\), \((\text{Z})_3\), \((\text{Z})_2\) and \((\text{KGS})_2\), in that order.

The construction of the wave operators and more precisely the local Cauchy problem at infinity in the long range cases of the previous nonlinear equations and systems has been treated essentially by two methods, of which we shall concentrate on the first one. The second one will only be quoted below for completeness. The first method is intrinsically restricted to the case of small Schrödinger data and to the borderline long range case. It applies to \((\text{NLS})_n\) with \(p = 1 + 2/n\) and \(1 \leq n \leq 3\), to \((\text{R3})_n\) with \(V\) given by \((1.1)\) with \(\gamma = 1\) and \(n \geq 2\), to \((\text{WS})_3\), \((\text{MS})_3\) and \((\text{KGS})_2\). It can also be applied to \((\text{Z})_n\) for \(n = 2, 3\). That method was initiated by Ozawa in the case of \((\text{NLS})_1\) \([26]\). It was extended by Ozawa and one of the authors to \((\text{NLS})_{2,3}\) and to \((\text{R3})_n\) \([5]\), by Ozawa and Tsutsumi to \((\text{KGS})_2\) and \((\text{Z})_3\) \([28]\) \([29]\) and by Tsutsumi to \((\text{MS})_3\) \([37]\). In those early works on \((\text{KGS})_2\), \((\text{MS})_3\) and \((\text{Z})_3\) (in the last case for large data), it was assumed in addition that the Fourier transform \(\widehat{u_+}\) of the Schrödinger asymptotic state \(u_+\) satisfied a suitable support condition, in order to cope with a difficulty coming from the difference of asymptotic properties of the solutions of the Schrödinger and of the wave or Klein-Gordon equations. Furthermore a smallness assumption was made on the Klein-Gordon or Maxwell data in the \((\text{KGS})_2\) and \((\text{MS})_3\) cases respectively. Recently, the problem was revived in a series of papers by Shimomura \([30]\)-\([35]\) who considered the \((\text{KGS})_2\), \((\text{WS})_3\), \((\text{MS})_3\) and \((\text{Z})_3\) systems. The main achievement of that work is to eliminate the previous support condition on \(\widehat{u_+}\). This is made possible by using an improved form of the asymptotic \(u_a\) including an additional correction term which partly cancels the free part of \(A\) in the Schrödinger equation. Furthermore in the \((\text{KGS})_2\) case, the smallness condition of the KG data is eliminated by going to (smaller) more regular function spaces and by using a detailed asymptotic form for \((u_a, A_a)\) \([31]\) \([32]\). Finally the problem was revisited by the authors in the case of the \((\text{WS})_3\),
(MS)$_3$ and (Z)$_{2,3}$ systems [14]-[16]. In that work, the smallness condition on the wave or Maxwell field is eliminated for (WS)$_3$ and (MS)$_3$ respectively. Furthermore, a refinement of the estimates and a more systematic use of Strichartz inequalities allows for the use of (larger) less regular function spaces. It is then possible to accommodate asymptotic ($u_a, A_a$) of low accuracy, so that the problem can be treated without the Shimomura improved asymptotics and with much weaker assumptions on the asymptotic state.

For completeness and although we shall not elaborate further on this point in the present paper, we mention that the same problem, namely the local Cauchy problem at infinity, can also be treated, at least for the (R3)$_n$ equation and for the (WS)$_3$ and (MS)$_3$ systems, by a more complex method where one first applies a phase-amplitude separation to the Schrödinger function, inspired by previous work by Hayashi and Naumkin on the (R3)$_n$ equation [18] [19]. The main interest of that method is to eliminate the smallness condition on the Schrödinger function and to go beyond the $\gamma = 1$ borderline case for the (R3)$_n$ equation. It has been applied by the authors to the (R3)$_n$ equation [8] [9] with improvements by Nakanishi eliminating a loss of regularity between the asymptotic state and the solution [24] [25], to the (WS)$_3$ system [10] [11] and to the (MS)$_3$ system in a special case [12].

In this paper, as mentioned earlier, we shall review the main results available by the first method for the systems (WS)$_3$, (MS)$_3$, (Z)$_3$, (Z)$_2$ and (KGS)$_2$. Before describing the contents of this paper in more detail however, we need to introduce some notation and to give the principle of that method. The construction of solutions ($u, A$) of the system at hand with prescribed asymptotics ($u_a, A_a$) is performed in two steps.

**Step 1.** One performs a change of variables, namely one looks for ($u, A$) in the form ($u, A$) = ($u_a + v, A_a + B$) and one studies the system satisfied by ($v, B$). For instance in the (WS)$_3$ case, that auxiliary system takes the form

\[
\begin{cases}
  i\partial_t v = -(1/2)\Delta v + Av + Bu_a - R_1 \\
  \Box B = -(|v|^2 + 2 \text{ Re } \overline{u}_av) - R_2
\end{cases}
\]  

(1.6)

where the remainders $R_1, R_2$ are defined by

\[
\begin{cases}
  R_1 = i\partial_t u_a + (1/2)\Delta u_a - A_a u_a \\
  R_2 = \Box A_a + |u_a|^2
\end{cases}
\]

(1.7)
The remainders measure the failure of the asymptotic form \((u_a, A_a)\) to satisfy the original system. In particular the decay in time of the remainders measures the quality of \((u_a, A_a)\) as an asymptotic form. It plays an essential role in the problem. The first step of the method consists in solving the auxiliary system for \((v, B)\) with \((v, B)\) tending to zero at infinity in suitable norms under assumptions on \((u_a, A_a)\) of a general nature, the most important of which being decay assumptions on the remainders \(R_1\) and \(R_2\). That can be done as follows. One first linearizes partly the auxiliary system satisfied by \((v, B)\). For instance in the \((WS)_3\) case, the linearized auxiliary system takes the form

\[
\begin{cases}
i\partial_t v' = -(1/2)\Delta v' + Av' + Bu_a - R_1 \\\square B' = -(|v|^2 + 2 \text{ Re } \tau_a v) - R_2.
\end{cases}
\] (1.8)

One solves the linearized system for \((v', B')\) with \((v', B')\) tending to zero at infinity, for fixed \((v, B)\) tending to zero at infinity. This defines a map \(\phi : (v, B) \rightarrow (v', B')\). One then shows by a contraction method that the map \(\phi\) has a fixed point in some Banach space \(X(I)\) of pairs \((v, B)\) of functions defined in \(I = [T, \infty)\) for \(T\) sufficiently large. The definition of the space \(X(I)\) must include two kinds of ingredients, to be tailored after the problem at hand, namely:

(i) local space or space time regularity of the functions, in order to cope with the nonlinearity of the relevant system. We shall always assume that \(v \in C(I, H^k)\) and possibly include other norms of equivalent homogeneity. As regards \(B\), we shall include norms of the same homogeneity as \((B, \partial_t B) \in C(I, \dot{H}^\ell \oplus \dot{H}^{\ell-1})\) for some suitably high \(\ell\) and possibly for lower values. For such a choice of \(X(I)\), we shall say that \(v\) is at level \(k\), that \(B\) is at level \(\ell\) and that \(X(I)\) and the corresponding theory are at level \((k, \ell)\).

(ii) time decay of \((v, B)\) as \(t \rightarrow \infty\) in the relevant norms. Intuitively that time decay will be expressed by an exponent \(\lambda\) so that the relevant norms decay essentially as \(t^{-\lambda}\) (see Section 2 below for a more precise formulation). In order to perform Step 1, \(\lambda\) will have to be sufficiently large, namely we shall need a lower bound on \(\lambda\). The theories described in this paper will then be characterized in first rough approximation by their level \((k, \ell)\) and their decay exponent \(\lambda\).

**Step 2.** That step obviously consists in constructing asymptotic \((u_a, A_a)\) satisfying the assumptions needed for Step 1 and in particular the time decay assumptions of the remainders \(R_1\) and \(R_2\). That decay has to be sufficient to allow for the exponent
\( \lambda \) needed for Step 1. It is therefore of interest to have the lower bound in Step 1 as low as possible in order to accommodate the most general (and therefore least accurate) possible \((u_a, A_a)\).

We can now describe the contents of this paper. In Section 2, we consider the \((WS)_3\) system which is both the simplest one and the most representative. After some preliminaries of general interest, which will be used again in the subsequent sections, we implement Step 1 above at the lowest available level, namely \((k, \ell) = (0, 1/2)\) with lower bound \(\lambda > 3/8\) (Proposition 2.1). We then describe the simplest adequate choice of \((u_a, A_a)\) and the associated final result, which has level \((k, \ell) = (0, 1/2)\) and \(\lambda = 1/2\) (Proposition 2.2). We then describe the Shimomura improved asymptotic \((u_a, A_a)\) and the associated final result, which has again level \((k, \ell) = (0, 1/2)\) but now with \(\lambda = 1\) up to logarithms (Proposition 2.3). We then turn to more regular theories and in particular we implement Step 1 at the level \((k, \ell) = (2, 1)\). Remarkably enough the lower bound on \(\lambda\) remains unchanged at \(\lambda > 3/8\) (Proposition 2.4). We then describe the final results at that level, first with the simplest asymptotics and with \(\lambda = 1/2\) (Proposition 2.5) and then with the improved asymptotics and \(\lambda = 1\) up to logarithms (Proposition 2.6). In Section 3, we consider the \((MS)_3\) system in the Coulomb gauge. It turns out that the lowest level available theory for that system is very similar to although much more complicated than the level \((2, 1)\) theory for \((WS)_3\). We first implement Step 1 for \((MS)_3\) at the level \((k, \ell) = (2, 3/2)\) with the same lower bound \(\lambda > 3/8\) as for \((WS)_3\) (Proposition 3.1). We then describe the simplest adequate \((u_a, A_a)\) and the associated final result at level \((k, \ell) = (2, 3/2)\) with \(\lambda = 1\) up to logarithms (Proposition 3.2). There is no need in that case to use an improved asymptotic \((u_a, A_a)\) for the \((MS)_3\) system in the Coulomb gauge. We then comment briefly on the possible extension of the result to other gauges and in particular to the Lorentz gauge. In Section 4 we consider the \((Z)_n\) system, first for \(n = 3\). That system is short range and consequently no smallness condition is needed on any of the data. We first implement Step 1 at the lowest available level, namely \((k, \ell) = (2, 1)\), with the lower bound \(\lambda > 1/4\) (Proposition 4.1). The simplest adequate \((u_a, A_a)\) is then provided by the solutions of the free Schrödinger and wave equations, and yields immediately the corresponding final result with \(\lambda = 1/2\) (Proposition 4.2), while the improved asymptotic \((u_a, A_a)\) yields the corresponding final result with \(\lambda = 3/2\) (Proposition 4.3). We then turn to the \((Z)_2\) system. We first implement Step 1 at the level
\((k, \ell) = (2, 1)\) with \(\lambda > 1/2\) (Proposition 4.4). That result however requires again small Schrödinger data and the conditions on \(A_a\) imply that the asymptotic state for \(A\) has to be zero. We then state the final result with \(u_a\) a solution of the free Schrödinger equation and \(A_a = 0\), at the level \((k, \ell) = (2, 1)\) and with \(\lambda = 1\) (Proposition 4.5). In Section 5 we consider the (KGS)\(_2\) system. We implement Step 1 first at the lowest available level, namely \((k, \ell) = (0, 1)\) with lower bound \(\lambda > 1/2\) (Proposition 5.1) and then at the level \((k, \ell) = (2, 1)\), which is the same as that used for the (Z)\(_2\) theory, with the same lower bound \(\lambda > 1/2\). We then describe the construction of the appropriate \((u_a, A_a)\) performed in [30], and we state the final results in a qualitative way only, since additional work would be needed to optimize the combination of that construction with the preceding treatment of Step 1.

We emphasize again the fact that in the results described above, the Schrödinger data have to satisfy a smallness condition, except in the case of (Z)\(_3\), that the wave, Maxwell and Klein-Gordon data can be arbitrarily large except in the case of (Z)\(_2\) where the asymptotic state for \(A\) has to be zero, and that no support assumption is made on \(\hat{u}_+\).

We conclude this introduction by giving some notation which will be used freely in this paper. We denote by \(F\) the Fourier transform in \(\mathbb{R}^n\), with \(F f = \hat{f}\) for any function \(f\), by \(\| \cdot \|\), the norm in \(L^r = L^r(\mathbb{R}^n)\), \(1 \leq r \leq \infty\), \(n = 2, 3\), and by \(< \cdot, \cdot >\) the scalar product in \(L^2\). Beyond the standard Sobolev spaces \(H^k\) defined for any \(k \in \mathbb{R}\) by

\[
H^k = \left\{ u : \| u; H^k \| = \| < \omega >^k u \|_2 < \infty \right\}
\]

where \(\omega = (-\Delta)^{1/2}\) and \(< \cdot, \cdot >\)\(= (1 + | \cdot |^2)^{1/2}\) and their homogeneous versions

\[
\dot{H}^k = \left\{ u : \| u; \dot{H}^k \| = \| \omega^k u \|_2 < \infty \right\},
\]

we shall use the spaces \(H^{k,s}\) defined for \(k, s \in \mathbb{R}\) by

\[
H^{k,s} = \left\{ u : \| u; H^{k,s} \| = \| < x >^s < \omega >^k u \|_2 < \infty \right\}
\]

so that \(H^{k,0} = H^k\) and \(H^{0,k} = F H^k\). We shall also use the Sobolev spaces \(W^k_r\) defined for \(1 \leq r \leq \infty\) and for \(k\) a non negative integer by

\[
W^k_r = \left\{ u : \| u; W^k_r \| = \sum_{\alpha: 0 \leq |\alpha| \leq k} \| \partial^\alpha_x u \|_r < \infty \right\}
\]

so that \(W^k_2 = H^k\). For any interval \(I\), for any Banach space \(X\) and for any \(q, 1 \leq q \leq \infty\), we denote by \(L^q(I, X)\) (resp. \(L^q_{\text{loc}}(I, X)\)) the space of \(L^q\) integrable
(resp. locally $L^q$ integrable) functions from $I$ to $X$ if $q < \infty$ and the space of measurable essentially bounded (resp. locally essentially bounded) functions from $I$ to $X$ if $q = \infty$.

## 2 The Wave-Schrödinger system (WS)$_3$

In this section we review the main results available on the local Cauchy problem at infinity for the (WS)$_3$ system

$$
\begin{cases}
    i\partial_t u = -(1/2)\Delta u + Au \\
    \Box A = -|u|^2.
\end{cases}
$$

That system is known to be globally well posed in the energy space [1] [4].

The exposition is based mostly on [14] but includes also some results of [33] rephrased in the framework of [14]. We follow the sketch given in the introduction and we first consider Step 1. For a given asymptotic $(u_a, A_a)$, we look for $(u, A)$ in the form $(u, A) = (u_a + v, A_a + B)$ and we try to solve the auxiliary system (1.6) for $(v, B)$ with $(v, B)$ tending to zero at infinity. For that purpose we first solve the partly linearized system (1.8) for $(v', B')$ and we try to prove that the map $\phi: (v, B) \to (v', B')$ thereby defined is a contraction in a suitable space $X(I)$ with $I = [T, \infty)$. The crucial point of Step 1 is the choice of $X(I)$ and that choice is dictated by the available estimates. We use three types of estimates.

(i) $L^2$ or energy estimates. If $v$ satisfies the Schrödinger equation

$$
i\partial_t v = -(1/2)\Delta v + Av + f
$$

then

$$
\partial_t \| v \|_2^2 = 2 \text{ Im } < v, f >
$$

so that if $v$ tends to zero at infinity, then

$$
\| v(t) \|_2 \leq \| f; L^1([t, \infty), L^2) \|.
$$

An important feature of that estimate is that it does not involve $A$. This is the key to the elimination of smallness conditions on $A$ in the theory. More generally, if $\partial = \nabla$ or $\partial_t$, from the equation

$$
i\partial_t \partial v = -(1/2)\Delta \partial v + A\partial v + v\partial A + \partial f
$$

(2.4)
we obtain
\[ \| \partial_t v(t) \|_2 \leq \| v \partial A + \partial f; L^1([t, \infty), L^2) \| \] (2.5)
if \( \partial v \) tends to zero at infinity. The estimate (2.3) has level \( k = 0 \), while (2.5) has level \( k = 1 \) if \( \partial = \nabla \) and level \( k = 2 \) if \( \partial = \partial_t \).

We shall also use the energy estimate for the wave equation. If \( B \) satisfies
\[ \Box B = g \] (2.6)
and tends to zero at infinity, then
\[ \| \partial_t B(t) \|_2 \vee \| \nabla B(t) \|_2 \leq \| g; L^1([t, \infty), L^2) \| . \] (2.7)
That estimate has level \( \ell = 1 \).

(ii) Strichartz inequalities for the Schrödinger equation.

We recall those inequalities for completeness, in dimension \( n \geq 2 \) (see [3] [21] [38] and references). A pair of exponents \( q, r \) with \( 2 \leq q, r \leq \infty \) is called admissible if
\[ 0 \leq 2/q = n/2 - n/r \leq 1 \quad \text{for } n \geq 3 \]
\[ < 1 \quad \text{for } n = 2 . \] (2.8)

**Lemma 2.1.** Let \( (q_i, r_i), i = 1, 2, \) be two admissible pairs. Let \( v \) satisfy the equation
\[ i \partial_t v = -(1/2)\Delta v + f \]
in some interval \( I \) with \( v(t_0) = v_0 \) for some \( t_0 \in I \). Then the following estimates hold :
\[ \| v; L^{q_1}(I, L^{r_1}) \| \leq C \left( \| v_0 \|_2 + \| f; L^{q_2}(I, L^{r_2}) \| \right) \] (2.9)
where \( C \) is a constant independent of \( I \), and with \( 1/p + 1/p = 1 \).

The Strichartz inequalities do not involve any derivative and are written in (2.9) at the level \( k = 0 \). Their extension to general level \( k \) is obvious.

(iii ) Strichartz inequalities for the wave equation.

We shall use and therefore we state those inequalities only in dimension 3 and for a special case of exponents (see [7] [21] and references for the general case).
Lemma 2.2. Let $n = 3$ and let $B$ satisfy the equation (2.6) in some interval $I$ with $B(t_0) = B_0$, $\partial_t B(t_0) = B_1$ for some $t_0 \in I$. Then the following estimates hold:

\[
\| B; L^4(I, L^4) \| \leq C \left( \| \omega^{1/2} B_0 \|_2 + \| \omega^{-1/2} B_1 \|_2 + \| g; L^{4/3}(I, L^{4/3}) \| \right),
\]

(2.10)

\[
\| \nabla B; L^4(I, L^4) \| \vee \| \partial_t B; L^4(I, L^4) \| \leq C \left( \| \omega^{3/2} B_0 \|_2 + \| \omega^{1/2} B_1 \|_2 + \| \nabla g; L^{4/3}(I, L^{4/3}) \| \right)
\]

(2.11)

where $C$ is a constant independent of $I$, and $\omega = (-\Delta)^{1/2}$.

The estimates (2.10) and (2.11) have level $\ell = 1/2$ and $\ell = 3/2$ respectively.

The definition of the space $X(I)$ will involve a suitable time decay which will be characterized by a function $h \in C([1, \infty), \mathbb{R}^+)$ such that for a suitable $\lambda > 0$, the function $\overline{h}(t) = t^\lambda h(t)$ is nonincreasing and tends to zero at infinity. This means in particular that $h$ decreases slightly faster than $t^{-\lambda}$.

We can now state the result concerning Step 1 for the (WS)$_3$ system at the lowest available level, which is $(k, \ell) = (0, 1/2)$. That result is the prototype of all the results concerning Step 1 contained in this paper. The relevant function space is defined by

\[
X(I) = \left\{ (v, B) : v \in C(I, L^2), \| (v, B); X(I) \| \equiv \sup_{t \in I} h(t)^{-1} \left( \| v(t) \|_2 + \| v; L^{8/3}(J, L^4) \| + \| B; L^4(J, L^4) \| \right) < \infty \right\}
\]

(2.12)

for any interval $I \subset [1, \infty)$, with $J = [t, \infty) \cap I$. The pair of exponents $(8/3, 4)$ is Schrödinger admissible, and the norm involves the $L^2$ norm and a Strichartz norm of level $k = 0$ for $v$, and a Strichartz norm of level $\ell = 1/2$ for $B$. The result is Proposition 2.2 of [14] and can be stated as follows.

**Proposition 2.1.** Let $h$ be defined as above with $\lambda = 3/8$ and let $X(\cdot)$ be defined by (2.12). Let $(u_a, A_a)$ be sufficiently regular (for the following estimates to make sense) and satisfy the estimates

\[
\| u_a(t) \|_4 \leq c_4 \ t^{-3/4},
\]

(2.13)

\[
\| A_a(t) \|_\infty \leq a \ t^{-1},
\]

(2.14)

\[
\| R_1; L^1([t, \infty), L^2) \| \leq r_1 h(t),
\]

(2.15)
\[ \| R_2; L^{4/3}([t, \infty), L^{4/3}) \| \leq r_2 h(t), \quad (2.16) \]

for some constants \( c_4, a, r_1 \) and \( r_2 \) with \( c_4 \) sufficiently small and for all \( t \geq 1 \). Then there exists \( T, 1 \leq T < \infty \) and there exists a unique solution \((v, B)\) of the system (1.6) in the space \( X([T, \infty)) \).

**Remark 2.1.** The time decay assumed in (2.13) (2.14) for \( (u_a,A_a) \) is the optimal time decay that can be obtained for solutions of the free Schrödinger and wave equations respectively. We shall always make assumptions of this type in all subsequent results concerning Step 1.

**Remark 2.2.** There is an absolute smallness condition on \( u_a \) through \( c_4 \) but none other. In particular \( A_a \) can be arbitrarily large.

**Sketch of proof.** We follow the sketch given in the introduction. Let \( 1 \leq T \leq \infty \) and let \((v, B) \in X([T, \infty)) \), so that \((v, B)\) satisfies

\[
\begin{align*}
\| v(t) \|_2 & \leq N_0 h(t) \\
\| v; L^{8/3}([t, \infty), L^4) \| & \leq N_1 h(t) \\
\| B; L^4([t, \infty), L^4) \| & \leq N_2 h(t)
\end{align*}
\]

for some constants \( N_i \) and for all \( t \geq T \). We first solve the linearized system (1.8) for \((v', B')\) in \( X([T, \infty)) \). That can be done by first solving that system in \( X([T, t_0]) \) with initial condition \((v', B')(t_0) = 0\) for some large \( t_0 > T \), which is an easy (linear) Cauchy problem with finite initial time, and then taking the limit of the solution thereby obtained when \( t_0 \to \infty \). An essential point for taking that limit is to derive estimates of the solution in \( X(I) \) for \( I = [T, t_0] \) that are uniform in \( t_0 \). We define

\[
\begin{align*}
N'_0 & = \sup_{t \in I} h(t)^{-1} \| v'(t) \|_2 \\
N'_1 & = \sup_{t \in I} h(t)^{-1} \| v'; L^{8/3}(J, L^4) \| \\
N'_2 & = \sup_{t \in I} h(t)^{-1} \| B'; L^4(J, L^4) \|
\end{align*}
\]

where \( J = [t, \infty) \cap I \). Using an \( L^2 \) estimate of the type (2.3) and Strichartz estimates
of the type (2.9) (2.10), one can show that

\[
\begin{align*}
N_0' &\leq C_0 (c_4 N_2 + r_1) \\
N_1' &\leq C_1 (c_4 N_2 + r_1) \left(1 + a + N_2 \overline{h}(T)\right) \quad (2.19) \\
N_2' &\leq C_2 (c_4 N_0 + r_2 + N_0 N_1 \overline{h}(T)) \, ,
\end{align*}
\]

where the \(C_i, 0 \leq i \leq 2\), are absolute constants. Using the fact that those estimates are uniform in \(t_0\), one can easily take the limit \(t_0 \to \infty\) of the previous solution, thereby obtaining a solution \((v', B') \in X([T, \infty))\) also satisfying the estimates (2.19) with the \(N_i'\) defined by (2.18), now with \(I = [T, \infty)\). At this stage we have completed the construction of the map \(\phi : (v, B) \to (v', B')\) and we have proved in addition that this map is bounded in \(X(I)\). We next show that \(\phi\) is a contraction on a suitable closed subset \(\mathcal{R}\) of \(X(I)\) for \(T\) sufficiently large. We define \(\mathcal{R}\) by (2.17).

From (2.19), it follows that \(\mathcal{R}\) is stable under \(\phi\) provided

\[
\begin{align*}
C_0 (c_4 N_2 + r_1) &\leq N_0 \\
C_1 (c_4 N_2 + r_1) \left(1 + a + N_2 \overline{h}(T)\right) &\leq N_1 \quad (2.20) \\
C_2 (c_4 N_0 + r_2 + N_0 N_1 \overline{h}(T)) &\leq N_2
\end{align*}
\]

which can be ensured under the smallness condition \(C_0 C_2 c_4^2 < 1\) by choosing the \(N_i\) according to

\[
\begin{align*}
N_0 &= C_0 (c_4 N_2 + r_1) \\
N_1 &= C_1 (c_4 N_2 + r_1) (2 + a) \quad (2.21) \\
N_2 &= C_2 (c_4 N_0 + r_2 + 1)
\end{align*}
\]

and by taking \(T\) sufficiently large so that

\[
N_2 \overline{h}(T) \leq 1 \, , \, N_0 N_1 \overline{h}(T) \leq 1 \, . \quad (2.22)
\]

It remains to prove that \(\phi\) is a contraction on \(\mathcal{R}\). This is done by estimating the difference of two solutions \((v', B')\) of (1.8) corresponding to two different \((v, B)\). The estimates are minor variants of (2.19) (see the proof of Proposition 2.2 in [14] for details).
We now turn to Step 2, namely to the construction of \((u_a, A_a)\) satisfying the assumptions of Proposition 2.1. One sees immediately that taking for \((u_a, A_a)\) a pair of solutions \((u_0, A_0)\) of the free Schrödinger and wave equations is inadequate, since in that case \(R_2 = |u_0|^2\) so that at best
\[
\| R_2 \|_{4/3} \leq \| u_0 \|_2 \| u_0 \|_4 \leq C t^{-3/4} \notin L^{4/3}.
\]
However, with the weak time decay allowed by Proposition 2.1, the simplest modification available in the literature suffices. We describe that choice first. We choose
\[
A_a = A_0 + A_1 \tag{2.23}
\]
\[
A_0 = \cos \omega t A_+ + \omega^{-1} \sin \omega t \dot{A}_+ \tag{2.24}
\]
\[
A_1 = \int_t^\infty dt' \omega^{-1} \sin(\omega(t-t'))|u_a(t')|^2 \tag{2.25}
\]
where \(\omega = (-\Delta)^{1/2}\) and \((A_+, \dot{A}_+\) is the asymptotic state of \(A\). That choice ensures that \(\Box A_0 = 0, \Box A_a = \Box A_1 = -|u_a|^2\) so that \(R_2 = 0\). We next choose \(u_a\). The Schrödinger group (1.5) admits the well known factorisation
\[
U(t) \equiv \exp(i(t/2)\Delta) = M D F M \tag{2.26}
\]
where
\[
M \equiv M(t) = \exp(ix^2/2t) \tag{2.27}
\]
\[
D(t) = (it)^{-3/2} D_0(t), \quad (D_0(t)f)(x) = f(x/t) \tag{2.28}
\]
and \(F\) is the Fourier transform. We choose
\[
uu_a = M D \exp(-i\varphi)\hat{u}_+ \tag{2.29}
\]
where \(\varphi\) is a real phase, \(\hat{u}_+ = Fu_+\) and \(u_+\) is the asymptotic state of \(u\). Substituting (2.29) into (2.25) yields
\[
A_1(t) = t^{-1} D_0(t) \tilde{A}_1, \tag{2.30}
\]
where
\[
\tilde{A}_1 = - \int_1^\infty d\nu \nu^{-3} \omega^{-1} \sin(\omega(\nu - 1)) D_0(\nu)|\hat{u}_+|^2. \tag{2.31}
\]
In particular \(\tilde{A}_1\) is constant in time. Substituting (2.23) (2.29) (2.30) into the definition (1.7) of \(R_1\) and using the commutation relation
\[
(i\partial_t + (1/2)\Delta) MD = MD \left(i\partial_t + (2t^2)^{-1}\Delta\right) \tag{2.32}
\]
we obtain
\[ R_1 = \tilde{R}_1 - A_0 \ u_a \]  (2.33)

where
\[
\tilde{R}_1 = (i\partial_t + (1/2)\Delta - A_1) MD \exp(-i\varphi)\hat{u}_+ \\
= MD \left( i\partial_t + (2t^2)^{-1}\Delta - t^{-1}\tilde{A}_1 \right) \exp(-i\varphi)\hat{u}_+ .
\]  (2.34)

We finally use \( \varphi \) to cancel the long range term \( \tilde{A}_1 \) by taking
\[
\varphi = (\ell n \ t)\tilde{A}_1
\]  (2.35)

thereby obtaining
\[
\tilde{R}_1 = (2t^2)^{-1}MD\Delta \exp(-i\varphi)\hat{u}_+ .
\]  (2.36)

In order to show that \((u_a, A_a)\) satisfies the assumptions needed for Step 1, we need the well known dispersive estimate for the Schrödinger equation, namely
\[
\| U(t)u_+ \|_r \leq (2\pi|t|)^{-\delta(r)} \| u_+ \|_r
\]  (2.37)

where \(2 \leq r \leq \infty, \delta(r) = n/2 - n/r\) and \(1/r + 1/\mathfrak{p} = 1\), and some general estimates of solutions of the free wave equation, which we recall for completeness [36].

**Lemma 2.3.** Let \( A_0 \) be defined by (2.24) and let \( \ell \geq 0 \) be an integer. Let \((A_+, \dot{A}_+)\) satisfy the conditions
\[
A_+, \omega^{-1}\dot{A}_+ \in H^\ell .
\]  (2.38)

Then \( A_0 \) satisfies the estimates
\[
\begin{align*}
\| A_0(t); H^\ell \| & \leq C , \\
\| \partial_t A_0(t); H^{\ell-1} \| & \leq C \quad \text{for } \ell \geq 1.
\end{align*}
\]  (2.39)

If in addition \((A_+, \dot{A}_+)\) satisfies the conditions
\[
\nabla^2 A_+ , \nabla \dot{A}_+ \in W_1^\ell ,
\]  (2.40)

then \( A_0 \) satisfies the estimates
\[
\begin{align*}
\| A_0(t); W^\ell_r \| & \leq C \ t^{-1+2/r} , \\
\| \partial_t A_0(t); W^{\ell-1}_r \| & \leq C \ t^{-1+2/r} \quad \text{for } \ell \geq 1
\end{align*}
\]  (2.41)

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for $2 \leq r \leq \infty$ and for all $t \in \mathbb{R}$.

Using Lemma 2.3 and Sobolev inequalities, one can easily derive the estimates of $(u_a, A_a)$ needed in Proposition 2.1 (see Proposition 3.1, part (1) in [14]).

**Lemma 2.4.** Let $u_+ \in H^{0,2}$ and let $(A_+, \dot{A}_+)$ satisfy (2.38) (2.40) with $\ell = 0$. Then the following estimates hold

$$
\| u_a(t) \|_r \leq t^{-\delta(r)} \| \hat{u}_+ \|_r \quad \text{for} \ 2 \leq r \leq \infty ,
$$

where $\delta(r) = 3/2 - 3/r$,

$$
\| A_a(t) \|_\infty \leq a t^{-1} ,
$$

$$
\| \tilde{R}_1(t) \|_2 \leq r_1 t^{-2(2 + \ell n t)} ,
$$

$$
\| R_1(t) \|_2 \leq r_1 t^{-3/2} .
$$

Putting together Proposition 2.1 and Lemma 2.4, we obtain the first (and simplest) final result for the system (WS)$_3$ at the level $(k, \ell) = (0, 1/2)$ (see Proposition 1.1, part (1) in [14]).

**Proposition 2.2.** Let $h(t) = t^{-1/2}$ and let $X(\cdot)$ be defined by (2.12). Let $(u_a, A_a)$ be defined by (2.29) (2.35) (2.23) (2.24) (2.30) (2.31). Let $u_+ \in H^{0,2}$ with $c_4 = \| \hat{u}_+ \|_4$ sufficiently small. Let $A_+, \omega^{-1} \dot{A}_+ \in L^2$ and $\nabla^2 A_+, \nabla \dot{A}_+ \in L^1$. Then there exists $T$, $1 \leq T < \infty$, and there exists a unique solution $(u, A)$ of the (WS)$_3$ system (2.1) such that $(v, B) \equiv (u - u_a, A - A_a) \in X([T, \infty))$.

It is apparent from Lemma 2.4, especially from (2.44) (2.45) that the time decay $t^{-1/2}$ in Proposition 2.2 comes from the term $A_0 u_a$ in $R_1$ (see (2.33)) which is estimated only as

$$
\| A_0 u_a \|_2 \leq \| A_0 \|_2 \| u_a \|_\infty \leq C t^{-3/2}
$$

whereas $\tilde{R}_1$ would allow for the better decay $t^{-1(2 + \ell n t)}$. In [33], Shimomura has proposed an improved form of $u_a$ and has obtained that improved decay on a subspace of more regular and decaying asymptotic states. We now describe the results that can be obtained thereby. We keep the same $A_a$ as before, but we now take $u_a = u_1 + u_2$ where $u_1$ is the previous choice of $u_a$, namely

$$
u_1 = MD \exp(-i\varphi)\hat{u}_+
$$

(2.46)
and $u_2$ is chosen in such a way that $(i\partial_t + (1/2)\Delta)u_2$ in $R_1$ partly cancels the term $A_0u_1$. For that purpose, it is appropriate to look for $u_2$ in the form $u_2 = fu_1$ for a suitable real function $f$, so that now

$$u_a = (1 + f)u_1$$

with $u_1$ defined by (2.46). Substituting (2.47) into the definition of $R_1$ yields

$$R_1 = (i\partial_t + (1/2)\Delta - A_0)(1 + f)u_1$$

$$= (1 + f) (i\partial_t + (1/2)\Delta - A_1)u_1 + ((1/2)\Delta f - A_0)u_1$$

$$- fA_0u_1 + \nabla f \cdot \nabla u_1 + i(\partial_t f)u_1 .$$

(2.48)

We now choose

$$f = 2\Delta^{-1}A_0 .$$

(2.49)

Making that choice and using the operators

$$J = x + it\nabla \quad , \quad P = t\partial_t + x \cdot \nabla ,$$

(2.50)

we obtain (see (2.34))

$$R_1 = (1 + f)\tilde{R}_1 - fA_0u_1 - it^{-1}\nabla f \cdot Ju_1 + it^{-1}(Pf)u_1 .$$

(2.51)

Now $f$ and $\nabla f$ are solutions of the free wave equation. From the commutation rule

$$\Box P = (P + 2)\Box$$

(2.52)

it follows that the same is true for $Pf$. By Lemma 2.3, one can ensure that $f$, $\nabla f$ and $Pf$ are uniformly bounded in $L^2$ by making suitable assumptions on their initial data, which reduce to assumptions on $(A_+, \dot{A}_+)$. On the other hand from the commutation relation

$$JMD = iMD\nabla$$

(2.53)

it follows that

$$\| Ju_1 \|_\infty \leq C t^{-3/2}(2 + \ell n t) .$$

(2.54)

The last three terms in (2.51) are then estimated by

$$\begin{cases}
\| fA_0u_1 \|_2 \leq \| f \|_2 \| A_0 \|_\infty \| u_1 \|_\infty \leq C t^{-5/2} \\
t^{-1} \| \nabla f \cdot Ju_1 \|_2 \leq t^{-1} \| \nabla f \|_2 \| Ju_1 \|_\infty \leq C t^{-5/2}(2 + \ell n t) \\
t^{-1} \| (Pf)u_1 \|_2 \leq t^{-1} \| Pf \|_2 \| u_1 \|_\infty \leq C t^{-5/2}
\end{cases}$$

(2.55)

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and would therefore allow for a decay $h(t) = t^{-3/2}(2 + \ln t)$, so that now the dominant term in $R_1$ is $\tilde{R}_1$, thereby allowing for a decay $h(t) = t^{-1}(2 + \ln t)^2$.

With the new choice (2.46) (2.47) of $u_a$ and with the same choice of $A_a$ as before, the remainder $R_2$ is no longer zero and becomes

$$R_2 = (f^2 + 2f)|u_1|^2.$$  (2.56)

This can be estimated as

$$\| R_2 \|_{4/3} \leq (\| f \|_{\infty} + 2) \| f \|_2 \| u_1 \|_{\infty} \| u_1 \|_4 \leq C t^{-3/2-3/4}$$

so that

$$\| R_2 ; L^{4/3}([t, \infty), L^{4/3}) \| \leq C t^{-3/2},$$  (2.57)

which would allow for a decay $h(t) = t^{-3/2}$.

Making the appropriate assumptions required to implement the previous estimates, one obtains the following final result with improved decay.

**Proposition 2.3.** Let $h(t) = t^{-1}(2 + \ln t)^2$ and let $X(\cdot)$ be defined by (2.12). Let $(u_a, A_a)$ be defined by (2.46) (2.47) (2.49) and (2.35) (2.23) (2.24) (2.30) (2.31). Let $u_+ \in H^{0,2}$ with $xu_+ \in L^1$ and with $c_4 = \| \hat{u}_+ \|_4$ sufficiently small. Let $(A_+, \dot{A}_+)$ satisfy

$$A_+, \omega^{-1}\dot{A}_+ \in \dot{H}^{-2} \cap L^2, \quad \nabla^2 A_+, \nabla \dot{A}_+ \in L^1, \quad x \cdot \nabla A_+, \omega^{-1}x \cdot \nabla \dot{A}_+ \in \dot{H}^{-2}.$$  (2.58)

Then :

1. There exists $T$, $1 \leq T < \infty$, and there exists a unique solution $(u, A)$ of the (WS)$_3$ system (2.1) such that $(v, B) \equiv (u - u_a, A - A_a) \in X([T, \infty))$.

2. There exists $T$, $1 \leq T < \infty$, and there exists a unique solution $(u, A)$ of the (WS)$_3$ system (2.1) such that $(u - u_1, A - A_a) \in X([T, \infty))$. One can take the same $T$ and the solution $(u, A)$ is the same as in Part (1).

Part (2) of the Proposition follows immediately from Part (1) and from the fact that $(fu_1, 0) \in X([T, \infty))$ for any $T \geq 1$ under the assumptions made on $(u_+, A_+, \dot{A}_+)$. Thus the correcting term $u_2 = fu_1$ does not change the asymptotics. It plays however an essential role in estimating the time derivative of $v$. The same situation will appear repeatedly, in particular in Propositions 2.6 and 4.3 below.
The subspace of asymptotic states \((u_+, A_+, \dot{A}_+\)) giving rise to the improved decay of Proposition 2.3 is specified by the additional assumptions of Proposition 2.3 as compared with those of Proposition 2.2. They include the condition \(xu_+ \in L^1\), which expresses a space decay dimensionally stronger by one half power of \(x\) than \(u_+ \in H^{0.2}\), the condition that \(A_+, \omega^{-1}\dot{A}_+ \in \dot{H}^{-2}\) and a similar condition on \(x \cdot \nabla A_+, \omega^{-1}x \cdot \nabla \dot{A}_+\). The condition on \(A_+, \omega^{-1}\dot{A}_+\) expresses a space decay dimensionally stronger than the corresponding \(L^2\) condition by two powers of \(x\), but that condition cannot be ensured by space decay alone. In fact

\[
\omega^{-2}A_+ = (4\pi |x|)^{-1} * A_+
\]

and this can never be in \(L^2\) unless the space integral of \(A_+\) vanishes. The situation is even worse for \(\dot{A}_+\). As an illustration, we now give conditions on \(A_+, \dot{A}_+\) in terms of space decay and vanishing of suitable moments which suffice to ensure the \(\dot{H}^{-2}\) conditions. Similar results apply to \(x \cdot \nabla A_+, x \cdot \nabla \dot{A}_+\). The following result is a special case of Lemma 3.5 in [11].

**Lemma 2.5.** Let \(A_+, \omega^{-1}\dot{A}_+ \in L^2\). Assume in addition that

\[
<x>^{1/2+\varepsilon} A_+ \in L^1, <x>^{3/2+\varepsilon} \dot{A}_+ \in L^1 \quad \text{for some} \ \varepsilon > 0,
\]

\[
\int dx A_+ = \int dx \dot{A}_+ = \int dx x A_+ = 0.
\]

Then \(A_+, \omega^{-1}\dot{A}_+ \in \dot{H}^{-2}\).

**Remark 2.3.** In Proposition 2.3 as in Proposition 2.2, there is again a discrepancy between the time decay allowed by \(\tilde{R}_1\) and by the other terms of \(R_1\), but now in the opposite direction since now \(\tilde{R}_1\) allows for \(t^{-1}(2 + \ell \ln t)^2\) whereas the other terms allow for \(t^{-3/2}(2 + \ell \ln t)\). This can be amended in two ways. On the one hand one can weaken the assumptions on \(A_+, \dot{A}_+\) so that the last three terms of \(R_1\) have no better decay than \(\tilde{R}_1\). On the other hand one can replace \(u_1\) and in particular \(\varphi\) by better approximations in order to improve the time decay of \(\tilde{R}_1\) to \(t^{-3/2}\), possibly up to logarithms. We refer to Section 3 of [11] where similar questions are treated in a slightly different context.

We have considered the \((\text{WS})_3\) system so far at the lowest convenient level of regularity \((k, \ell) = (0, 1/2)\). However there is no difficulty in constructing theories
similar to the previous one at higher levels of regularity. It suffices to include suitable higher norms in the definition of \( X(\cdot) \) and to estimate those norms in the proofs by straightforward extensions of the estimates of the lower norms. It seems to be a general feature of the problem that, under the natural additional regularity assumption on \((u_a, A_a)\) and/or on \((u_+, A_+, A_+^\prime)\),

(i) the required time decay remains the same, and in particular the value of \( \lambda \) needed for Step 1 remains \( \lambda = 3/8 \),

(ii) no additional smallness condition appears on the higher norms of \( u_a \) and/or \( u_+ \) beyond the smallness condition of \( c_4 \) that appears at the lower level.

Two theories with higher level of regularity are of special interest:

(1) The theory at level \((k, \ell) = (1, 1)\) since that is the level of the energy for the \((WS)_3\) system. The appropriate function space is

\[
X_1(I) = \left\{ (v, B) : v \in C(I, H^1), \nabla B, \partial_t B \in C(I, L^2), \right. \\
\left. \| (v, B) \|_{X_1(I)} \equiv \sup_{t \in I} h(t)^{-1} \left( \| v(t) \|_{H^1} + \| v \cdot L^{8/3}(J, W^1_4) \| + \| B \cdot L^4(J, L^4) \| + \| \nabla B(t) \|_2 + \| \partial_t B(t) \|_2 \right) < \infty \right\}
\]

where \( J = [t, \infty) \cap I \). It differs from the previous space \( X(I) \) by the inclusion of the \( L^2 \) norm and of a Strichartz norm of \( \nabla v \), and of the energy norm of \( B \). Actually in that definition, the \( L^2 \) norm of \( \partial_t B \) is optional since it is never used to perform the contraction estimates. It can be recovered at the end as a by-product. Furthermore it turns out that the energy norm of \( B \) has a better time decay than that expressed by the definition of \( X_1(\cdot) \). The additional estimates needed for that theory as compared with the lower level one are obtained from \( L^2 \) and Strichartz estimates on the gradient of the Schrödinger equation, namely on (2.4) with \( \partial = \nabla \), and from the energy estimate (2.7) for \( B \). We refer to [14] Proposition 1.1, part (2), Proposition 2.3 and Proposition 3.1, part (2) for a description of that theory with the simplest asymptotic \((u_a, A_a)\).

(2) The theory at level \((k, \ell) = (2, 1)\) since \( k = 2 \) is the lowest level for \( v \) where we can treat the \((MS)_3\) system. That theory is therefore a simplified model for the \((MS)_3\) theory, which has a strong similarity with it. The appropriate function space is now

\[
X_2(I) = \left\{ (v, B) : v \in C(I, H^2) \cap C^1(I, L^2), \nabla B, \partial_t B \in C(I, L^2), \right. \\
\left. \| (v, B) \|_{X_2(I)} \equiv \sup_{t \in I} h(t)^{-1} \left( \| v(t) \|_{H^2} + \| \partial_t v(t) \|_2 \right) < \infty \right\}
\]
\[
+ \| v; L^{8/3}(J, W^1_4) \| + \| \partial_t v; L^{8/3}(J, L^4) \| + \| B; L^4(J, L^4) \|
+ \| \nabla B(t) \|_2 + \| \partial_t B(t) \|_2 \right) < \infty \}
\]  

(2.60)

where \( J = [t, \infty) \cap I \). It differs from the previous \( X(\cdot) \) by the inclusion of the \( L^2 \) norm and of a Strichartz norm of \( \partial_t v \) and of \( \Delta v \), and of the energy norm of \( B \). Since the Schrödinger equation directly relates \( \Delta v \) to \( \partial_t v \), the norms of \( \Delta v \), namely of level \( k = 2 \), are estimated in terms of those of \( \partial_t v \). As a result the basic estimates of \( v \) can be performed on the time derivative of the Schrödinger equation, namely on (2.4) with \( \partial = \partial_t \). It is then sufficient to apply only one (time) derivative on \( B \), so that \( B \) can still be taken at the level \( \ell = 1 \) only, as in the case of the \( H^1 \) theory. Furthermore the \( L^2 \) norm of \( \nabla B \) and the \( L^2 \) norm and Strichartz norm of \( \Delta v \) are optional in the definition of \( X_2(\cdot) \), since they are not used in the contraction proof to estimate the other norms that occur in the definition. They can be recovered at the end as a by-product. Finally, as in the case of the \( H^1 \) theory, it turns out that the energy norm of \( B \) has a better time decay than that expressed by the definition of \( X_2(\cdot) \).

In the remaining part of this section, we describe the results available at the level \((k, \ell) = (2, 1)\), as a simplified model of the \((\text{MS})_3 \) theory to be presented in the next section. We refer to [14], Proposition 1.1, part (3), Proposition 2.4 and Proposition 3.1, part (3) for a description of that theory with the simplest asymptotic \((u_a, A_a)\). The result with improved asymptotics is adapted from [33].

The basic result concerning Step 1 is the following.

**Proposition 2.4.** Let \( h \) be defined as previously with \( \lambda = 3/8 \) and let \( X_2(\cdot) \) be defined by (2.60). Let \((u_a, A_a)\) satisfy the estimates (2.13)-(2.16) of Proposition 2.1 and in addition

\[
\| u_a(t) \|_\infty \leq c \ t^{-3/2} \quad \text{and} \quad \| \partial_t u_a(t) \|_4 \leq c \ t^{-3/4},
\]  

(2.61)

\[
\| \partial_t A_a(t) \|_\infty \leq a \ \ t^{-1},
\]  

(2.62)

\[
\| \partial_t R_1; L^1([t, \infty), L^2) \| \leq r_1 \ h(t),
\]  

(2.63)

\[
\| R_1; L^{8/3}([t, \infty), L^4) \| \leq r_1 \ h(t),
\]  

(2.64)

\[
\| R_2; L^1([t, \infty), L^2) \| \leq r_2 \ t^{-1/2} \ h(t),
\]  

(2.65)

for some constants \( c_4, c, a, r_1 \) and \( r_2 \) with \( c_4 \) sufficiently small and for all \( t \geq 1 \). Then there exists \( T, \ 1 \leq T < \infty \) and there exists a unique solution \((v, B)\) of the
system (1.6) in the space $X_2([T, \infty))$. Furthermore $B$ satisfies the estimate
\[ \| \nabla B(t) \|_2 \vee \| \partial_t B(t) \|_2 \leq C \left( t^{-1/2} + t^{1/4} h(t) \right) h(t) \] (2.66)
for some constant $C$ and for all $t \geq T$.

The proof follows closely that of Proposition 2.1. One starts from $(v, B) \in X_2([T, \infty))$ for some $T$, $1 \leq T < \infty$, and therefore satisfying (2.17) and in addition
\[
\begin{cases}
\| \partial_t v(t) \|_2 \leq N_3 h(t) \\
\| \partial_t v; L^{8/3}([t, \infty), L^4) \| \leq N_4 h(t) \\
\| \nabla B(t) \|_2 \vee \| \partial_t B(t) \|_2 \leq N_5 h(t) \\
\| \Delta v(t) \|_2 \leq N_6 h(t) \\
\| \Delta v; L^{8/3}([t, \infty), L^4) \| \leq N_7 h(t)
\end{cases}
\] (2.67)
for some constants $N_i$, $0 \leq i \leq 7$ and for all $t \geq T$. For each such $(v, B)$ one constructs a solution $(v', B')$ of the system (1.8) in $X_2(I)$, first for $I = [T, t_0]$ and then for $I = [T, \infty)$, and one shows finally that the map $\phi : (v, B) \rightarrow (v', B')$ thereby defined is a contraction on the subset $\mathcal{R}_2$ of $X_2([T, \infty))$ defined by (2.17) (2.67) for suitably chosen $N_i$ and for sufficiently large $T$. The crux of the proof is to estimate the seminorms (2.18) of $(v', B')$ and the seminorms $N'_i$, $3 \leq i \leq 7$ defined in an obvious way in correspondence with (2.67). The fact that no additional smallness assumption is needed to complete the proof comes from the fact that the linear part of the system of conditions extending (2.21) and including the additional variables $N_i$, $3 \leq i \leq 7$, is essentially triangular with respect to the latter variables.

We next state the final result that can be obtained with the simple asymptotics considered above.

**Proposition 2.5.** Let $h(t) = t^{-1/2}$ and let $X_2(\cdot)$ be defined by (2.60). Let $(u_\alpha, A_\alpha)$ be defined by (2.29) (2.35) (2.23) (2.24) (2.30) (2.31). Let $u_+ \in H^{1,3} \cap H^{2,2}$ with $c_4 = \| \hat{u}_+ \|_4$ sufficiently small. Let $A_+, \omega^{-1} \hat{A}_+ \in H^1$ and $\nabla^2 A_+, \nabla \hat{A}_+ \in W^1_1$. Then there exists $T$, $1 \leq T < \infty$, and there exists a unique solution $(u, A)$ of the (WS)$_3$ system (2.1) such that $(v, B) \equiv (u - u_\alpha, A - A_\alpha) \in X_2([T, \infty))$. Furthermore $B$ satisfies the estimate
\[ \| \nabla B(t) \|_2 \vee \| \partial_t B(t) \|_2 \leq C t^{-3/4} \] (2.68)
for some constant $C$ and for all $t \geq T$.

We finally state the final result that can be obtained by using the improved asymptotics of [33].

**Proposition 2.6.** Let $h(t) = t^{-1}(2 + \ln t)^2$ and let $X_2(\cdot)$ be defined by (2.60). Let $(u_a, A_a)$ be defined by (2.46) (2.47) (2.49) and (2.35) (2.23) (2.24) (2.30) (2.31). Let $u_+ \in H^{1,3} \cap H^{2,2}$ with $xu_+ \in W^2_1$, $x^2u_+ \in W^1_1$ and with $c_4 = \| \hat{u}_+ \|_4$ sufficiently small. Let $(A_+, \dot{A}_+)$ satisfy

$$A_+, \omega^{-1} \dot{A}_+ \in \dot{H}^{-2} \cap H^1, \quad \nabla^2 A_+, \nabla \dot{A}_+ \in W^1_1,$$

$$x \cdot \nabla A_+, \omega^{-1} x \cdot \nabla \dot{A}_+ \in \dot{H}^{-2} \cap \dot{H}^{-1}. \quad (2.69)$$

Then:

1. There exists $T$, $1 \leq T < \infty$, and there exists a unique solution $(u, A)$ of the (WS)$_3$ system (2.1) such that $(u - u_1, A - A_0) \in X_2([T, \infty))$. Furthermore $B$ satisfies the estimate

$$\| \nabla B(t) \|_2 \vee \| \partial_t B(t) \|_2 \leq C t^{-3/2}(2 + \ln t)^2 \quad (2.70)$$

for some constant $C$ and for all $t \geq T$.

2. There exists $T$, $1 \leq T < \infty$, and there exists a unique solution $(u, A)$ of the (WS)$_3$ system (2.1) such that $(u - u_1, A - A_0) \in X_2([T, \infty))$. One can take the same $T$ and the solution $(u, A)$ is the same as in Part (1).

As in Proposition 2.3, Part (2) follows immediately from Part (1) and from the fact that $(fu_1, 0) \in X_2([T, \infty))$ for any $T \geq 1$ under the assumptions made on $(u_+, A_+, \dot{A}_+)$. 

### 3 The Maxwell-Schrödinger system (MS)$_3$

In this section we review the main results available on the local Cauchy problem at infinity for the (MS)$_3$ system. That system can be written as

$$\begin{cases}
i \partial_t u = -(1/2) \Delta_A u + A_e u \\
\Box A_e - \partial_t (\partial_t A_e + \nabla \cdot A) = |u|^2 \\
\Box A + \nabla (\partial_t A_e + \nabla \cdot A) = \text{Im} \nabla A u .
\end{cases} \quad (3.1)$$
Here \((A, A_e)\) is an \(\mathbb{R}^{3+1}\) valued function defined in space time \(\mathbb{R}^{3+1}\), \(\nabla_A = \nabla - iA\) and \(\Delta_A = \nabla^2_A\) are the covariant gradient and Laplacian respectively. An important property of that system is its gauge invariance, namely the invariance under the transformation

\[
(u, A, A_e) \rightarrow (u \exp(-i\varphi), A - \nabla \varphi, A_e + \partial_t \varphi),
\]

where \(\varphi\) is an arbitrary real function defined in \(\mathbb{R}^{3+1}\). As a consequence of that invariance, the system (3.1) is underdetermined as an evolution system and has to be supplemented by an additional equation, called a gauge condition. Here we shall use the Coulomb gauge condition, namely \(\nabla \cdot A = 0\). Under that condition, one can replace the system (3.1) by a formally equivalent one in the following standard way. The second equation of (3.1) can be solved for \(A_e\) by

\[
A_e = -\Delta^{-1} |u|^2 = (4\pi|x|)^{-1} * |u|^2 \equiv g(|u|^2).
\]

(3.3)

Substituting (3.3) into the first and last equation of (3.1) yields the new system

\[
\begin{cases}
  i\partial_t u = -(1/2)\Delta_A u + g(|u|^2)u \\
  \Box A = P \text{ Im } \bar{u} \nabla_A u, \quad \nabla \cdot A = 0
\end{cases}
\]

(3.4)

where \(P = \mathbb{1} - \nabla \Delta^{-1} \nabla\) is the projector on divergence free vector fields so that the gauge condition \(\nabla \cdot A = 0\) is preserved by the evolution. The system (3.4) coincides with that given in the introduction.

The Coulomb gauge has several advantages. In the expansion of the covariant Laplacian, the term \(iA \cdot \nabla\) is a transport term by the vector field \(A\). Using the Coulomb gauge consists in decomposing \(A\) into a divergence free part and a gradient and taking advantage of the gauge invariance of the system to eliminate the gradient part. It makes the transport term isometric in \(L^r\) for any \(r\) and eliminates all derivatives of \(A\) in the system. The Coulomb gauge is also strongly preferred by atomic physicists since it provides a clean separation of the electromagnetic interaction into an instantaneous electrostatic part and a propagating magnetic part.

Nevertheless other gauges can be considered for the system (3.1). The Lorentz
gauge is defined by \( \partial_t A_e + \nabla \cdot A = 0 \). In that gauge, the \((\text{MS})_3\) system becomes

\[
\begin{align*}
\begin{cases}
  i\partial_t u &= -(1/2)\Delta_A u + A_e u \\
  \Box A_e &= |u|^2 \\
  \Box A &= \text{Im} \, \overline{u} \nabla_A u \\
  \partial_t A_e + \nabla \cdot A &= 0.
\end{cases}
\end{align*}
\]

(3.5)

In particular the electric potential \( A_e \) is now coupled to \( u \) by a WS type interaction, with however a change of sign. As in the case of the Coulomb gauge, the gauge condition is preserved by the evolution.

The Lorentz gauge condition is Lorentz invariant, which is of little advantage for the MS system since the Schrödinger equation is not. That invariance becomes important for Lorentz invariant systems such as the Maxwell-Dirac or Maxwell-Klein-Gordon systems, for instance in order to ensure that Lorentz invariance is preserved term by term in formal expansions such as perturbation expansions. This property is extensively used for instance in Quantum Electrodynamics.

In this section, we shall use the Coulomb gauge and make only a brief comment at the end on the Lorentz gauge.

In contrast with the case of the other equations and systems listed in the introduction, the Cauchy problem for the \((\text{MS})_3\) system is not in a satisfactory shape. It is known that this problem is locally well posed at the level \((k, \ell)\) for sufficiently large \((k, \ell)\), namely with sufficiently high regularity \([22] [23]\), the best result available so far being \((k, \ell) = (5/3, 4/3)\) \([23]\). On the other hand, the \((\text{MS})_3\) system has global solutions at the level of the energy, namely \((k, \ell) = (1, 1)\), obtained by compactness and therefore without uniqueness \([17]\). However the \((\text{MS})_3\) system is so far not known to be globally well posed in any function space.

We now turn to the local Cauchy problem at infinity for the \((\text{MS})_3\) system in the Coulomb gauge (3.4). The exposition follows \([15]\) which is strongly inspired by \([37]\), and includes also some results from \([34]\). Note that \([34] [37]\) also consider the \((\text{MS})_3\) system in the Lorentz gauge (3.5).

The theory follows the same pattern as that for the \((\text{WS})_3\) system and strongly resembles the \(H^2\) theory for the latter. One looks for \((u, A)\) in the form \((u, A) = (u_a + v, A_a + B)\) where \((u_a, A_a)\) is the asymptotic form, which has to satisfy the gauge condition \(\nabla \cdot A_a = 0\). The auxiliary system satisfied by the new functions
\((v, B)\) is now
\[
\begin{align*}
  i\partial_t v &= -(1/2)\Delta_A v + g(|u|^2)v + G_1 - R_1 \\
  \Box B &= G_2 - R_2 , \quad \nabla \cdot B = 0 ,
\end{align*}
\] (3.6)
where \(G_1\) and \(G_2\) are defined by
\[
\begin{align*}
  G_1 &= iB \cdot \nabla_A u_a + (1/2)B^2 u_a + g (|v|^2 + 2 \text{ Re } \overline{u}_a v) u_a \\
  G_2 &= P \text{ Im } (\overline{v} \nabla_A v + 2\overline{v} \nabla_A u_a) - P B |u_a|^2
\end{align*}
\] (3.7)
and the remainders are defined by
\[
\begin{align*}
  R_1 &= i\partial_t u_a + (1/2)\Delta_A u_a - g (|u_a|^2) u_a \\
  R_2 &= \Box A_a - P \text{ Im } \overline{u}_a \nabla_A u_a .
\end{align*}
\] (3.8)
As in the case of \((\text{WS})_3\), we consider also the partly linearized system for functions \((v', B')\)
\[
\begin{align*}
  i\partial_t v' &= -(1/2)\Delta_A v' + g(|u'|^2)v' + G_1 - R_1 \\
  \Box B' &= G_2 - R_2 , \quad \nabla \cdot B' = 0 .
\end{align*}
\] (3.9)
The first step of the method consists again in solving the system (3.6) for \((v, B)\), with \((v, B)\) tending to zero at infinity in time in suitable norms, under assumptions on \((u_a, A_a)\) of a general nature, the most important of which being decay assumptions on the remainders \(R_1\) and \(R_2\).

When choosing the appropriate function space \(X(\cdot)\) to perform that step, one has to take into account the following difference with the \((\text{WS})_3\) case. If \(v\) satisfies the Schrödinger equation
\[
i\partial_t v = -(1/2)\Delta_A v + V v + f
\] (3.10)
with real \(V\), then again
\[
\partial_t \| v \|^2 = 2 \text{ Im } < v, f >
\]
and \(v\) is estimated in \(L^2\) by (2.3) independently of \(A\) and \(V\). However in order to use the Strichartz inequality (2.9), one has to expand the covariant Laplacian, so that
\[
i\partial_t v = -(1/2)\Delta v + iA \cdot \nabla v + \text{ other terms}
\] (3.11)
and the term \(A \cdot \nabla v\) has to be included in \(f\), thereby requiring the control of one more derivative than appears at the level where the Strichartz inequality is considered.
As a consequence, when using $L^2$ norms at the level $k$, one can use Strichartz norms only at the level $k - 1$. Taking that point into account, one can define the relevant space $X(\cdot)$ by

$$X(I) = \left\{(v, B) : v \in \mathcal{C}(I, H^2) \cap \mathcal{C}^1(I, L^2), \right.$$\[
\| (v, B); X(I) \| \equiv \sup_{t \in I} h(t)^{-1} \left( \| v(t); H^2 \| + \| \partial_t v(t) \|_2 + \| v; L^{8/3}(J, W^1_4) \| ight. \\
\left. + \| B; L^4(J, W^1_4) \| + \| \partial_t B; L^4(J, L^4) \| \right) < \infty \}

(3.12)

where $J = [t, \infty) \cap I$ and $h$ is defined as in Section 2. That space has level $(k, \ell) = (2, 3/2)$. More precisely it includes $L^2$ norms of $v$ at the level $k = 2$, Strichartz norms of $v$ at the level $k = 1$, in keeping with the preceding remark, and Strichartz norms of $B$ at the level $\ell = 3/2$ and $\ell = 1/2$ (compare with Lemma 2.2, especially (2.10) (2.11)). One could also include the energy norm of $B$, which is at the level $\ell = 1$, but that norm is not needed for the proof of the estimates and it comes out at the end as a by-product thereof (see Propositions 3.1 and 3.2 below). The space (3.12) is larger than that used in [34] [37] which has level $(k, \ell) = (3, 2)$.

We can now state the main result concerning Step 1, which is Proposition 2.2 of [15].

**Proposition 3.1.** Let $h$ be defined as in Section 2 with $\lambda = 3/8$ and let $X(\cdot)$ be defined by (3.12). Let $u_a$, $A_a$, $R_1$ and $R_2$ be sufficiently regular (for the following estimates to make sense) and satisfy the estimates

$$\| \partial_t^j \nabla^k u_a(t) \|_r \leq c t^{-\delta(r)} \text{ for } 2 \leq r \leq \infty$$

(3.13)

where $\delta(r) = 3/2 - 3/r$, and in particular

$$\| u_a \|_3 \leq c_3 t^{-1/2}, \quad \| \nabla u_a \|_4 \leq c_4 t^{-3/4},$$

(3.14)

$$\| \nabla^2 u_a(t) \|_4 \quad \text{and} \quad \| \partial_t \nabla u_a(t) \|_4 \leq c t^{-3/4},$$

(3.15)

$$\| \partial_t^j \nabla^k A_a(t) \|_\infty \leq c t^{-1},$$

(3.16)

$$\| \partial_t^j \nabla^k R_1; L^1([t, \infty), L^2) \| \leq r_1 h(t),$$

(3.17)

$$\| R_2; L^{4/3}([t, \infty), W^1_{4/3}) \| \leq r_2 h(t),$$

(3.18)
for all \( t \geq 1 \), then there exists \( T, 1 \leq T < \infty \) and there exists a unique solution \((v, B)\) of the system (3.6) in \( X([T, \infty))\). If in addition

\[
\| R_2; L^1([t, \infty), L^2) \| \leq r_2 t^{-1/2} h(t),
\]

for all \( t \geq T \), then \( \nabla B, \partial_t B \in \mathcal{C}([T, \infty), L^2) \) and \( B \) satisfies the estimate

\[
\| \nabla B(t) \|_2 \vee \| \partial_t B(t) \|_2 \leq C \left( t^{-1/2} + t^{1/4} h(t) \right) h(t)
\]

for some constant \( C \) and for all \( t \geq T \).

The proof follows closely those of Propositions 2.1 and 2.4. One starts from \((v, B) \in X([T, \infty))\) for some \( T, 1 \leq T < \infty \), so that \((v, B)\) satisfies

\[
\begin{aligned}
\| v(t) \|_2 &\leq N_0 h(t) \\
\| v; L^4(J, L^3) \| \vee \| v; L^{8/3}(J, L^4) \| &\leq N_1 h(t) \\
\| B; L^4(J, L^4) \| &\leq N_2 h(t) \\
\| \partial_t v(t) \|_2 &\leq N_3 h(t) \\
\| \nabla v; L^4(J, L^3) \| \vee \| \nabla v; L^{8/3}(J, L^4) \| &\leq N_4 h(t) \\
\| \Delta v(t) \|_2 &\leq N_5 h(t) \\
\| \nabla B; L^4(J, L^4) \| \vee \| \partial_t B; L^4(J, L^4) \| &\leq N_6 h(t)
\end{aligned}
\]

for some constants \( N_i, 0 \leq i \leq 6 \) and for all \( t \geq T \), with \( J = [t, \infty) \). For each such \((v, B)\) one constructs a solution \((v', B')\) of the system (3.9) in \( X([T, \infty))\) and one shows that the map \( \phi : (v, B) \to (v', B') \) thereby defined is a contraction on the subset \( \mathcal{R} \) of \( X([T, \infty))\) defined by (3.21) for suitably chosen \( N_i \) and for sufficiently large \( T \). The smallness conditions on \( c_3 \) and \( c_4 \) appear when trying to solve the extension to the present case of the system (2.1). The coefficient \( c_3 \) comes from the Hartree interaction and appears in a diagonal term in the equation for \( N_0 \). The coefficient \( c_4 \) comes from the magnetic interaction and appears in cross couplings of \( N_0 \) and \( N_2 \) as in (2.1). The rest of the system is triangular in its linear part and does not generate any additional smallness condition. Note also that the decay exponent \( \lambda = 3/8 \) is the same as for the (WS)\(_3\) system, and that the energy norm of \( B \) has a stronger decay than that occurring in the definition of \( X(\cdot) \).
We now turn to Step 2, namely to the construction of \((u_a, A_a)\) satisfying the assumptions of Proposition 3.1. Again solutions of the underlying free equations are immediately seen to be inadequate, but the weak time decay allowed by Proposition 3.1 allows for the simplest modification available in the literature. We want to ensure sufficient decay properties of the remainders. We consider first \(R_2\). Using the operator
\[
J = x + it\nabla
\] (3.22)
we rewrite \(R_2\) as
\[
R_2 = \Box A_a + P \left( t^{-1} \text{Re } \pi_a J u_a + A_a |u_a|^2 - t^{-1} x |u_a|^2 \right). \tag{3.23}
\]

We now choose \(A_a\) according to (2.23) (2.24), now however with
\[
A_1 = \int_0^\infty dt' (\omega t')^{-1} \sin(\omega(t' - t)) P x|u_a(t')|^2,
\] (3.24)
thereby ensuring that \(\Box A_0 = 0, \Box A_a = \Box A_1 = Pt^{-1}x |u_a|^2\) and therefore
\[
R_2 = P \left( t^{-1} \text{Re } \pi_a J u_a + A_a |u_a|^2 \right). \tag{3.25}
\]

With that choice, under general assumptions on \((u_a, A_a)\) of the same type as in Proposition 3.1 but not making use of their special form, one can prove that \(R_2\) satisfies the assumptions needed for that proposition with \(h(t) = t^{-1}(2 + \elln t)\).

Lemma 3.1. Let \((u_a, A_a)\) satisfy the estimates
\[
\| u_a(t); W^1_r \| \leq c t^{-\delta(r)} \quad \text{for } 2 \leq r \leq \infty,
\] (3.26)
where \(\delta(r) = 3/2 - 3/r\),
\[
\| J u_a(t) \| \leq c_1 (2 + \elln t),
\] (3.27)
\[
\| A_a(t); W^1_\infty \| \leq a t^{-1}
\] (3.28)
for all \(t \geq 1\). Then \(R_2\) satisfies the estimates
\[
\| R_2; L^{4/3}([t, \infty), W^1_{4/3}) \| \leq r_2 t^{-1}(2 + \elln t),
\] (3.29)
\[
\| R_2; L^1([t, \infty), L^2) \| \leq r_2 t^{-3/2}(2 + \elln t)
\] (3.30)
for some constant \(r_2\) and for all \(t \geq 1\).
We now turn to $R_1$. We first skim some harmless terms. Expanding the covariant Laplacian and using again $J$, we rewrite $R_1$ as

$$R_1 = R_{1,1} + R_{1,2}$$  \hspace{1cm} (3.31)

where

$$R_{1,1} = i \partial_t u_a + (1/2) \Delta u_a + t^{-1} (x \cdot A_1) u_a - g(|u_a|^2) u_a ,$$  \hspace{1cm} (3.32)

$$R_{1,2} = t^{-1} (x \cdot A_0) u_a - t^{-1} A_a \cdot J u_a - (1/2) A_a^2 u_a .$$  \hspace{1cm} (3.33)

In the same way as for $R_2$, one can show that $R_{1,2}$ satisfies the assumptions needed for Proposition 3.1 with $h(t) = t^{-1} (2 + \ell n t)$ under general assumptions on $(u_a, A_a)$ not making use of their special form.

**Lemma 3.2.** Let $u_a, A_a$ and $A_0$ satisfy the estimates

$$\| \partial_t^j \nabla^k u_a \|_2 \leq c ,$$  \hspace{1cm} (3.34)

$$\| \partial_t^j \nabla^k J u_a \|_2 \leq c(2 + \ell n t) ,$$  \hspace{1cm} (3.35)

$$\| \partial_t^j \nabla^k A_a \|_\infty \leq a t^{-1} ,$$  \hspace{1cm} (3.16) \equiv (3.36)

$$\| \partial_t^j \nabla^k (x \cdot A_0) \|_\infty \leq a_0 t^{-1} ,$$  \hspace{1cm} (3.37)

for $0 \leq j + k \leq 1$ and for all $t \geq 1$. Then $R_{1,2}$ satisfies the estimates

$$\| \partial_t^j \nabla^k R_{1,2} \|_2 \leq r_{1,2} t^{-2} (2 + \ell n t) ,$$  \hspace{1cm} (3.38)

for $0 \leq j + k \leq 1$, for some constant $r_{1,2}$ and for all $t \geq 1$.

We next choose $u_a$ according to (2.29). Substituting (2.29) into (3.24) yields again (2.30), now however with

$$\tilde{A}_1 = \int_1^\infty d\nu \nu^{-3} \omega^{-1} \sin(\omega (1 - 1)) D_0(\nu) P x |\tilde{u}_+|^2 .$$  \hspace{1cm} (3.39)

Again $\tilde{A}_1$ is constant in time. Substituting (2.29) (2.30) into the definition (3.32) of $R_{1,1}$ and using again the commutation relation (2.32), we obtain

$$R_{1,1} = MD \left( i \partial_t + (2t^2)^{-1} \Delta + t^{-1} x \cdot \tilde{A}_1 - t^{-1} g(|\tilde{u}_+|^2) \right) \exp(-i \varphi) \tilde{u}_+ .$$  \hspace{1cm} (3.40)

We finally choose $\varphi$ so as to cancel the long range terms $x \cdot \tilde{A}_1$ and $g(|\tilde{u}_+|^2)$ in (3.40). We take

$$\varphi = (\ell n t) \left( g(|\tilde{u}_+|^2) - x \cdot \tilde{A}_1 \right) ,$$  \hspace{1cm} (3.41)
thereby obtaining
\[ R_{1,1} = (2t^2)^{-1} MD \Delta \exp(-i\varphi)\hat{u}_+ \]  
which is very similar to (2.36) except for the different choice of \( \varphi \).

Using mainly Sobolev inequalities, one can derive the estimates of \((u_a, A_1)\) needed for Lemmas 3.1 and 3.2, and the remaining estimates of \(u_a\) and of \(R_{1,1}\) needed for Proposition 3.1. The following lemma is slightly stronger than needed (see Proposition 3.1 in [15]).

\textbf{Lemma 3.3.} Let \( u_a \) be defined by (2.29) with \( \varphi \) defined by (3.41) (3.3) (3.39) and let \( R_{1,1} \) be given by (3.42). Let \( u_a \in H^{1,3} \cap H^{3,1} \). Then the following estimates hold for some constants \( c, c_1, a_1 \) and \( r_{1,1} \), for \( 0 \leq j + k \leq 1 \) and for all \( t \geq 1 \)
\[ \| \partial_t^j \nabla^k u_a(t) \|_r \leq c t^{-\delta(r)} \quad \text{for} \quad 2 \leq r \leq \infty . \]  
(3.13) \( \equiv \) (3.43)

In particular
\[ \| u_a(t) \|_3 \leq \| \hat{u}_+ \|_3 \ t^{-1/2} , \]  
(3.44)
\[ \| \nabla u_a(t) \|_4 \leq ( \| x\hat{u}_+ \|_4 + O(t^{-1}\ell n \ t) ) t^{-3/4} . \]  
(3.45)
\[ \| \partial_t^j \nabla^{k+1} u_a(t) \|_r \leq c t^{-\delta(r)} \quad \text{for} \quad 2 \leq r \leq 6 , \]  
(3.46)
\[ \| \partial_t^j \nabla^k J u_a(t) \|_r \leq c_1(2 + \ell n \ t) t^{-\delta(r)} \quad \text{for} \quad 2 \leq r \leq 6 , \]  
(3.47)
\[ \| \partial_t^j \nabla^k A_1 \|_\infty \leq a_1 t^{-1} , \]  
(3.48)
\[ \| \partial_t^j \nabla^k R_{1,1}(t) \|_2 \leq r_{1,1} t^{-2}(2 + \ell n \ t)^2 . \]  
(3.49)

Finally, the contribution of \( A_0 \) to the conditions of Proposition 3.1 and of Lemmas 3.1 and 3.2, namely to (3.16) and (3.37), can be estimated by the use of Lemma 2.3 and of the remark that if \( A_0 \) is a solution of the wave equation \( \Box A_0 = 0 \) satisfying the Coulomb gauge condition \( \nabla \cdot A_0 = 0 \), then \( x \cdot A_0 \) is also a solution of the wave equation, namely \( \Box (x \cdot A_0) = 0 \).

Combining the previous estimates, especially Lemmas 3.1-3 with Proposition 3.1, we obtain the final result for the \((MS)_3\) system in the Coulomb gauge in the following form (see Proposition 1.1 in [15]).

\textbf{Proposition 3.2.} Let \( h(t) = t^{-1}(2 + \ell n \ t)^2 \) and let \( X(\cdot) \) be defined by (3.12). Let \( u_a \) be defined by (2.29) with \( \varphi \) defined by (3.41) (3.3) (3.39). Let \( A_a = A_0 + A_1 \) with
$A_0$ defined by (2.24) and $A_1$ by (2.30) (3.39). Let $u_+ \in H^{3,1} \cap H^{1,3}$ with $\| x\hat{u}_+ \|_4$ and $\| \hat{u}_+ \|_3$ sufficiently small. Let $\nabla^2 A_+, \nabla \hat{A}_+, \nabla^2 (x \cdot A_+)$ and $\nabla (x \cdot \hat{A}_+)$ belong to $W^1_1$ with $A_+, x \cdot A_+ \in L^3$ and $\hat{A}_+, x \cdot \hat{A}_+ \in L^{3/2}$ and let $\nabla \cdot A_+ = \nabla \cdot \hat{A}_+ = 0$.

Then there exists $T$, $1 \leq T < \infty$ and there exists a unique solution $(u, A)$ of the \((\text{MS})_3\) system (3.4) such that $(v, B) \equiv (u - u_a, A - A_a) \in X([T, \infty))$. Furthermore $\nabla B, \partial_t B \in C([T, \infty), L^2)$ and $B$ satisfies the estimate

$$\| \nabla B(t) \|_2 + \| \partial_t B(t) \|_2 \leq C \, t^{-3/2}(2 + \ell n t)^2$$

for some constant $C$ depending on $(u_+, A_+, \hat{A}_+)$ and for all $t \geq T$.

**Remark 3.1.** The only smallness conditions bear on $\| x\hat{u}_+ \|_4$ and on $\| \hat{u}_+ \|_3$ and are required by the magnetic interaction and the Hartree interaction (3.3) respectively. In particular there is no smallness condition on $(A_+, \hat{A}_+)$. 

**Remark 3.2.** The assumptions $A_+, x \cdot A_+ \in L^3$ and $\hat{A}_+, x \cdot \hat{A}_+ \in L^{3/2}$ serve to exclude the occurrence of constant terms in $A_+, x \cdot A_+, \hat{A}_+, x \cdot \hat{A}_+$ and of terms linear in $x$ in $A_+, x \cdot A_+$, but are otherwise implied by the $W^1_1$ assumptions on those quantities through Sobolev inequalities.

**Remark 3.3.** The assumptions on $A_+, \hat{A}_+$ imply that $\omega^{1/2} A_+, \omega^{-1/2} \hat{A}_+ \in H^1$ through Sobolev inequalities. As a consequence the free wave solution $A_0$ defined by (2.24) belongs to $L^4(I^R, W^1_1)$ by Strichartz inequalities, with $\partial_t A_0 \in L^4(I^R, L^4)$. In particular $A_0$ satisfies the local in time regularity of $B$ required in the definition of the space $X(\cdot)$. Furthermore $\nabla A_+, \hat{A}_+ \in L^2$ and therefore $\nabla A_0, \partial_t A_0 \in (C \cap L^\infty)(I^R, L^2)$, namely $A_0$ is a finite energy solution of the wave equation.

We conclude this section with some remarks on the \((\text{MS})_3\) system in the Lorentz gauge (3.5). There is every reason to expect similar results in that case. Step 1 of the method should be implementable with no smallness condition on $(A, A_e)$ and with the same decay exponent $\lambda = 3/8$ as in Proposition 3.1. This should lead to a final result with $h(t) = t^{-1/2}$ with simple asymptotics and with $h(t) = t^{-1}(2 + \ell n t)^2$ with improved asymptotics, the correcting term to $u_a$ being needed as in [34] to cancel the contribution of $\nabla \cdot A_0$ coming from $\Delta_\cdot$. The required level of regularity cannot be expected to be lower than $(k, \ell, \ell_e) = (2, 2, 1)$ for $(v, B, B_e)$, with $\ell = 2$ for $B$ required by the $\partial_t \nabla \cdot B$ term in the equation for $\partial_t v$ and $\ell_e = 1$ for $B_e$ suggested
by the \((WS)_3\) result of Proposition 2.4. Ascertaining whether that level is sufficient would require to go through the detail of the estimates and we leave that as an open question. In [34] [37] the Lorentz gauge case is treated together with the Coulomb gauge case at the level \((k, \ell, \ell_e) = (3, 2, 2)\).

4 The Zakharov system \((Z)_n\) for \(n = 2, 3\)

In this section we review the main results available on the local Cauchy problem at infinity for the \((Z)_n\) system

\[
\begin{align*}
   \begin{cases}
   i\partial_t u = -(1/2)\Delta u + Au \\
   \Box A = \Delta |u|^2
   \end{cases}
\end{align*}
\]

in space dimension \(n = 3\) and 2 (in that order). That system is known to be globally well posed in the energy space (see [2] [6] [27] and references to previous works therein quoted). The exposition is based mostly on [16] and includes some previous results from [28] [35].

We first consider the case of dimension \(n = 3\). The \((Z)_3\) system is short range, as will be clear below. We follow the sketch given in the introduction and we first consider Step 1. For a given asymptotic \((u_a, A_a)\), we look for \((u, A)\) in the form \(u = u_a + v, A = A_a + B\). The auxiliary system satisfied by the new functions \((v, B)\) is now

\[
\begin{align*}
   \begin{cases}
   i\partial_t v = -(1/2)\Delta v + Av + Bu_a - R_1 \\
   \Box B = \Delta (|v|^2 + 2 \text{Re } u_a v) - R_2
   \end{cases}
\end{align*}
\]

where the remainders are defined by

\[
\begin{align*}
   R_1 &= i\partial_t u_a + (1/2)\Delta u_a - A_a u_a \\
   R_2 &= \Box A_a - \Delta |u_a|^2 .
\end{align*}
\]

As in the case of \((WS)_3\) we consider also the partly linearized system for functions \((v', B')\)

\[
\begin{align*}
   \begin{cases}
   i\partial_t v' = -(1/2)\Delta v' + Av' + Bu_a - R_1 \\
   \Box B' = \Delta (|v'|^2 + 2 \text{Re } u_a v') - R_2
   \end{cases}
\end{align*}
\]

Step 1 of the method consists again in solving the system (4.2) for \((v, B)\) with \((v, B)\) tending to zero at infinity under assumptions on \((u_a, A_a)\) of a general nature. In
contrast with the case of the (WS)$_3$ and (MS)$_3$ systems however, the Strichartz inequalities for the wave equation do not seem to be useful for that purpose, the reason being that the Laplacian in the RHS of the equation for $B$ allows for an easy estimate of $B$ in $L^2$. The relevant space $X(\cdot)$ can be defined as follows

$$X(I) = \left\{ (v, B) : (v, B) \in C(I, H^2 \oplus H^1) \cap C^1(I, L^2 \oplus L^2), \right.$$  
$$\left. \| (v, B); X(I) \| \equiv \sup_{t \in I} h(t)^{-1} \left( \| v(t); H^2 \| + \| \partial_t v(t) \|_2 \right. \right.$$  
$$+ \| v; L^{8/3}(J, W^2_4) \| + \| \partial_t v; L^{8/3}(J, L^4) \|$$  
$$\left. + \| B(t); H^1 \| + \| \partial_t B(t) \|_2 \right) \right\} < \infty \right.$$  

(4.5)

where $J = [t, \infty) \cap I$ and $h$ is defined as in Section 2. That space has $(k, \ell) = (2, 1)$, namely 1 above the level of the energy space in both variables. The definition includes $L^2$ norms and Strichartz norms at the level $k = 2$ for $v$, and $L^2$ norms at the levels $\ell = 0$ and $\ell = 1$ for $B$. We could also have included the norm $\| \omega^{-1} \partial_t B \|_2$ which is part of the Zakharov energy for $B$, but that norm is never used to perform the estimates and it comes out at the end as a by-product thereof. We have omitted it for simplicity.

We can now state the result concerning Step 1 (see Proposition 2.1 in [16]). We recall that $\omega = (-\Delta)^{1/2}$.

**Proposition 4.1** Let $h$ be defined as in Section 2 with $\lambda = 1/4$, and let $X(\cdot)$ be defined by (4.5). Let $u_a$, $A_a$, $R_1$ and $R_2$ be sufficiently regular (for the following estimates to make sense) and satisfy the estimates

$$\| u_a(t); W^2_\infty \| \lor \| \partial_t u_a(t) \|_\infty \leq c \ t^{-3/2} ,$$  

(4.6)

$$\| \partial_t^j A_a(t) \|_\infty \leq a \ t^{-1} \quad \text{for } j = 0, 1 ,$$  

(4.7)

$$\| \partial_t^j R_1; L^1([t, \infty), L^2) \| \leq r_1 \ h(t) \quad \text{for } j = 0, 1 ,$$  

(4.8)

$$\| R_1; L^{8/3}([t, \infty), L^4) \| \leq r_1 \ t^{-\eta} \ h(t) \quad \text{for some } \eta \geq 0 ,$$  

(4.9)

$$\| \omega^{-1} R_2; L^1([t, \infty), H^1) \| \leq r_2 \ h(t) ,$$  

(4.10)

for some constants $c$, $a$, $r_1$ and $r_2$ and for all $t \geq 1$. Then there exists $T$, $1 \leq T < \infty$, and there exists a unique solution $(v, B)$ of the system (4.2) in $X([T, \infty))$. If in addition

$$\| \omega^{-1} R_2; L^1([t, \infty), L^2) \| \leq r_2 \ t^{-1/2} \ h(t)$$  

(4.11)
for all \( t \geq T \), then \( B \) satisfies the estimate

\[
\| B(t); H^1 \| \lor \| \omega^{-1} \partial_t B(t); H^1 \| \leq C \left( t^{-1/2} + t^{1/4} h(t) \right) h(t)
\]

(4.12)

for some constant \( C \) and for all \( t \geq T \).

The proof follows again those of Propositions 2.1 and 2.4. One starts from \((v, B) \in X([T, \infty))\) for some \( T, 1 \leq T < \infty \), so that \((v, B)\) satisfies

\[
\begin{aligned}
\| v(t) \|_2 &\leq N_0 h(t) \\
\| v; L^{8/3}(J, L^4) \| &\leq N_1 h(t) \\
\| B(t); H^1 \| \lor \| \partial_t B(t) \|_2 &\leq N_2 h(t) \\
\| \partial_t v(t) \|_2 &\leq N_3 h(t) \\
\| \partial_t v; L^{8/3}(J, L^4) \| &\leq N_4 h(t) \\
\| \Delta v(t) \|_2 &\leq N_5 h(t) \\
\| \Delta v; L^{8/3}(J, L^4) \| &\leq N_6 h(t)
\end{aligned}
\]

(4.13)

for some constants \( N_i, 0 \leq i \leq 6 \) and for all \( t \geq T \), with \( J = [t, \infty) \). For each such \((v, B)\) one constructs a solution \((v', B')\) of the system (4.4) in \( X([T, \infty)) \) and one shows that the map \( \phi : (v, B) \to (v', B') \) thereby defined is a contraction on the subset \( \mathcal{R} \) of \( X([T, \infty)) \) defined by (4.13) for suitably chosen \( N_i \) and for sufficiently large \( T \). The fact that the \((Z)_3\) system is short range manifests itself at this stage by the absence of smallness conditions on \( u_a \). Technically, this follows from the fact that the nontriangular linear terms in the system of equations for the \( N_i \) which ensures the contraction contain decaying powers of \( T \) and can be made small by taking \( T \) large.

We now turn to Step 2, namely to the construction of \((u_a, A_a)\) satisfying the assumptions of Proposition 4.1. The \((Z)_3\) system is short range and one can take for \((u_a, A_a)\) a pair of solutions \((u_0, A_0)\) of the underlying free linear system, namely

\[
u_0(t) = U(t)u_+
\]

(4.14)

and \( A_0 \) defined by (2.24). The remainders become

\[
\begin{aligned}
R_1 &= -A_0 u_0 \\
R_2 &= -\Delta |u_0|^2
\end{aligned}
\]

(4.15)
and the final result can be stated as follows.

**Proposition 4.2.** Let $h(t) = t^{-1/2}$ and let $X(\cdot)$ be defined by (4.5). Let $u_+ \in H^2 \cap W^1_1$, let $A_+, \omega^{-1}A_+ \in H^1$ and $\nabla^2 A_+, \nabla \dot{A}_+ \in W^1_1$. Let $(u_0, A_0)$ be defined by (4.14) (2.24). Then there exists $T, 1 \leq T < \infty$, and there exists a unique solution $(u, A)$ of the $(Z)_3$ system (4.1) such that $(v, B) \equiv (u - u_0, A - A_0) \in X([T, \infty))$. If in addition $u_+ \in H^{0,2}$, then $B$ satisfies the estimate

$$\| B(t); H^1 \| \lor \| \omega^{-1}\partial_t B(t); H^1 \| \leq C t^{-3/4}$$

(4.16)

for some constant $C$ and for all $t \geq T$.

The result follows from Proposition 4.1, from the dispersive estimate (2.37), from Lemma 2.3 and Sobolev inequalities. In particular

$$\| R_1 \|_2 \leq \| A_0 \|_2 \| u_0 \|_\infty \leq C t^{-3/2}$$

and similar estimates lead to the decay $h(t) = t^{-1/2}$, while

$$\| R_1 \|_4 \leq \| A_0 \|_4 \| u_0 \|_\infty \leq C t^{-2}$$

ensures the condition (4.9) with $\eta = 9/8$. The last statement of Proposition 4.2 requires in addition the following lemma, which we state in dimensions $n = 2, 3$, and which follows immediately from the factorisation (2.26) of $U(t)$.

**Lemma 4.1.** Let $n = 2$ or 3. Let $u_+ \in H^{0,2}(\subset L^1)$ and let $u_0 = U(t)u_+$. Then the following estimates hold:

$$\| \nabla |u_0|^2 \|_2 \leq 2(2\pi|t|)^{-n/2} t^{-1} \| u_+ \|_1 \| xu_+ \|_2 ,$$

(4.17)

$$\| \Delta |u_0|^2 \|_2 \leq 4(2\pi t)^{-n/2} t^{-2} \| u_+ \|_1 \| x^2 u_+ \|_2 .$$

(4.18)

In the same way as for the (WS)$_3$ system, by using a more accurate asymptotic form for $u_a$, one can obtain a stronger asymptotic convergence in time of the solution on a smaller subspace of asymptotic states [35]. Thus we choose $(u_a, A_a) = ((1 + f)u_0, A_0)$ with $(u_0, A_0)$ defined by (4.14) (2.24) and $f$ by (2.49), namely $f = 2\Delta^{-1}A_0$. Using the operators $J$ and $P$ defined by (2.50), we rewrite $R_1$ as

$$R_1 = (i\partial_t + (1/2)\Delta - A_0)(1 + f)u_0$$

$$= -f A_0 u_0 - it^{-1}(\nabla f) \cdot Ju_0 + it^{-1}(P f)u_0$$

(4.19)
while $R_2$ now becomes
\[
R_2 = -\Delta (1 + f)^2 |u_0|^2. \tag{4.20}
\]
The new remainders are easily estimated through the following lemmas.

**Lemma 4.2.** Let $u_+ \in W^2_1$, $xu_+ \in W^2_1$, and let $A_0$ and $f$ satisfy
\[
\| \partial_t \nabla^k A_0 \|_{\infty} \leq a t^{-1}, \tag{4.21}
\]
\[
\| \partial_t \nabla^k f; H^1 \| \vee \| \partial_t \nabla^k P f \|_2 \leq C, \tag{4.22}
\]
for $0 \leq j + k \leq 1$ and for all $t \geq 1$. Then the following estimates hold:
\[
\| \partial_t \nabla^k R_1 \|_2 \leq C t^{-5/2}, \tag{4.23}
\]
for some constant $C$, for $0 \leq j + k \leq 1$ and for all $t \geq 1$.

**Lemma 4.3.** Let $u_+ \in W^2_1 \cap H^{0,2}$ and let $f$ satisfy
\[
\| \nabla f(t) \|_2 \vee \| \Delta f(t) \|_2 \vee \| f(t) \|_{\infty} \leq C \tag{4.24}
\]
for all $t \geq 1$. Then the following estimates hold:
\[
\| \omega^{-1} R_2 \|_2 \leq C t^{-5/2}, \tag{4.25}
\]
\[
\| R_2 \|_2 \leq C t^{-3}, \tag{4.26}
\]
for some constant $C$ and for all $t \geq 1$.

Lemma 4.3 follows readily from Lemma 4.1. In practice, the bound on $\| f \|_{\infty}$ in (4.24) will follow from the Sobolev inequality
\[
\| f \|_{\infty} \leq C (\| \nabla f \|_2 \| \Delta f \|_2)^{1/2}
\]
for $f$ tending to zero at infinity in some weak sense.

We can now state the final result with improved asymptotics.

**Proposition 4.3.** Let $h(t) = t^{-3/2}$ and let $X(\cdot)$ be defined by (4.5). Let $u_+ \in H^2 \cap H^{0,2} \cap W^2_1$ with $xu_+ \in W^2_1$. Let $(A_+, \dot{A}_+)$ satisfy
\[
A_+, \omega^{-1} \dot{A}_+ \in \dot{H}^{-2} \cap H^1, \quad \nabla^2 A_+ , \nabla \dot{A}_+ \in W^1_1,
\]
\[
x \cdot \nabla A_+, \omega^{-1} x \cdot \nabla \dot{A}_+ \in \dot{H}^{-2} \cap H^{-1}. \tag{4.27}
\]
Let \((u_0, A_0)\) be defined by (4.14) (2.24) and let \(u_a = (1 + f)u_0\) with \(f = 2\Delta^{-1}A_0\). Then:

1. There exists \(T, 1 \leq T < \infty\) and there exists a unique solution \((u, A)\) of the \((Z)_3\) system (4.1) such that \((v, B) \equiv (u - u_a, A - A_0) \in X([T, \infty))\).

2. Assume in addition that \(\nabla^2\omega^{-2}A_+, \nabla\omega^{-2}\dot{A}_+ \in W^1_1\). Then there exists \(T, 1 \leq T < \infty\) and there exists a unique solution \((u, A)\) of the \((Z)_3\) system (4.1) such that \((u - u_0, A - A_0) \in X([T, \infty))\). One can take the same \(T\) and the solution \((u, A)\) is the same as in Part (1).

As in Proposition 2.3, Part (2) follows immediately from Part (1) and from the fact that \((fu_0, 0) \in X([T, \infty))\) for any \(T \geq 1\). The additional assumption on \((A_+, \dot{A}_+)^\ell\) has been made to ensure that property, which does not seem to follow immediately from the previous assumptions. It is stronger than needed. Together with the previous assumptions and through Lemma 2.3, it implies that \(f, \nabla f\) and \(\partial_t f\) also satisfy the decay estimates stated for \(A_0\) in (2.41) with \(\ell = 0, 1\). Only the special case \(r = 4\) is used for the present purpose. More economical assumptions could be made instead by using suitable Besov spaces.

Note also that the assumption (4.27) on \((A_+, \dot{A}_+)\) is the same as the assumption (2.69) of Proposition 2.6 relative to the \((WS)_3\) system, a reflection of the fact that we are treating \(A_0\) in exactly the same way in both cases.

We now turn to the case of space dimension \(n = 2\), where the situation is much less satisfactory. The free part of the asymptotic field is estimated at best as

\[
\| A_0(t) \|_{\infty} \leq C t^{-1/2}
\]

and we are unable to handle such a slow decay in Step 1, so that the final result will eventually be restricted to the special case of zero asymptotic state \((A_+, \dot{A}_+)\) for \(A\). On the other hand, in a suitable limit, the Zakharov system formally yields the cubic NLS equation, which is short range for \(n = 2\), and one might naively expect a similar situation for the \((Z)_2\) system, allowing for a treatment of that system without a smallness condition on \(u\). This turns out not to be the case, and the \((Z)_2\) system does actually require such a smallness condition at the level of Step 1. The treatment of that step is very similar to the case of \((Z)_3\). The relevant space \(X(\cdot)\) is essentially the same up to obvious changes, namely

\[
X(I) = \left\{(v, B) : (v, B) \in C(I, H^2 \oplus H^1) \cap C^1(I, L^2 \oplus L^2)\right\},
\]

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\[ \| (v, B); X(I) \| \equiv \sup_{t \in I} h(t)^{-1} \left( \| v(t); H^2 \| + \| \partial_t v(t) \|_2 \right) + \| v; L^4(J, W^2_4) \| + \| \partial_t v; L^4(J, L^4) \| + \| B(t); H^1 \| + \| \partial_t B(t) \|_2 \right) < \infty \] 

(4.29)

where \( J = [t, \infty) \cap I \), and the result can be stated as follows.

**Proposition 4.4** Let \( h \) be defined as in Section 2 with \( \lambda = 1/2 \) and let \( X(\cdot) \) be defined by (4.29). Let \( u_a, A_a, R_1 \) and \( R_2 \) be sufficiently regular and satisfy the estimates

\[ \| u_a(t); W^2_\infty \| \vee \| \partial_t u_a(t) \|_\infty \leq c \ t^{-1}, \] 

(4.30)

\[ \| \partial^j A_a(t) \| \leq a \ t^{-1-j} \theta \quad \text{for some } \theta > 0 \text{ and for } j = 0, 1, \] 

(4.31)

\[ \| \partial^j R_1; L^1 ([t, \infty), L^2) \| \leq r_1 \ h(t) \quad \text{for } j = 0, 1, \] 

(4.32)

\[ \| R_1; L^4 ([t, \infty), L^4) \| \leq r_1 \ t^{-\eta} \ h(t) \quad \text{for some } \eta \geq 0, \] 

(4.33)

\[ \| \omega^{-1} R_2; L^1 ([t, \infty), H^1) \| \leq r_2 \ h(t) \] 

(4.34)

for some constants \( c, a, r_1 \) and \( r_2 \) with \( c \) sufficiently small and for all \( t \geq 1 \). Then there exists \( T, 1 \leq T < \infty \), and there exists a unique solution \((v, B)\) of the system (4.2) in \( X([T, \infty)) \).

The proof is a minor variation of that of Proposition 4.1. The assumption (4.31) is rather arbitrary. It is too strong to accommodate a non-zero \( A_0 \) (see (4.28)) but weaker by one power of \( t \) than the condition that would be satisfied by an \( A_1 \) of the type (2.25) devised to ensure \( R_2 = 0 \). It has been chosen so as to ensure that the proof of the proposition proceeds smoothly.

The final result can be stated as follows.

**Proposition 4.5.** Let \( h(t) = t^{-1} \) and let \( X(\cdot) \) be defined by (4.29). Let \( u_+ \in H^2 \cap H^{0,2} \cap W^2_4 \) with \( \| u_+; W^2_1 \| \) sufficiently small and let \( u_0(t) = U(t)u_+ \). Then there exists \( T, 1 \leq T < \infty \), and there exists a unique solution \((u, A)\) of the \( (Z)_2 \) system (4.1) such that \((u - u_0, A) \in X([T, \infty)) \).

The result follows readily from Proposition 4.4 with \( A_a = 0 \), from the dispersive estimate (2.37) and from Lemma 4.1.
The Klein-Gordon-Schrödinger system \((\text{KGS})_2\)

In this section we review the main results available on the local Cauchy problem at infinity for the \((\text{KGS})_2\) system

\[
\begin{cases}
  i\partial_t u = -(1/2)\Delta u + Au \\
  (\Box + 1)A = -|u|^2 .
\end{cases}
\] (5.1)

That system is known to be globally well posed in the energy space \([1] [4]\). We shall first present the results concerning Step 1, which are easily obtained by minor variations from the corresponding results for the \((\text{WS})_3\) and \((\text{Z})_2\) systems, especially from Propositions 2.1, 2.4 and 4.4. We shall then outline the construction of the asymptotic form \((u_a, A_a)\) given in \([30]\). That construction is more delicate than in the \((\text{WS})_3\) case. We shall then state the final results in a partly qualitative way, but we shall refrain from a completely formal statement. The reason is that Step 1 is treated in \([30]\) by a slightly different method and in spaces smaller (more regular) than here, so that the assumptions made in \([30]\) on the asymptotic state are stronger than needed for a combination with the treatment of Step 1 given here. We leave it as an open question to perform that combination and in particular to determine the most economical and natural assumptions on the asymptotic state needed for that purpose.

We first consider Step 1. We follow again the sketch given in the introduction. For a given asymptotic \((u_a, A_a)\), we look for \((u, A)\) in the form \((u, A) = (u_a + v, A_a + B)\). The auxiliary system satisfied by \((v, B)\) is now

\[
\begin{cases}
  i\partial_t v = -(1/2)\Delta v + Av + Bu_a - R_1 \\
  (\Box + 1)B = -(|v|^2 + 2 \text{ Re } \pi_a v) - R_2
\end{cases}
\] (5.2)

where the remainders are defined by

\[
\begin{cases}
  R_1 = i\partial_t u_a + (1/2)\Delta u_a - A_u a \\
  R_2 = (\Box + 1)A_a + |u_a|^2 .
\end{cases}
\] (5.3)

We consider also the partly linearized system for functions \((v', B')\)

\[
\begin{cases}
  i\partial_t v' = -(1/2)\Delta v' + Av' + Bu_a - R_1 \\
  (\Box + 1)B' = -(|v'|^2 + 2 \text{ Re } \pi_a v) - R_2
\end{cases}
\] (5.4)
Step 1 of the method consists again in solving the system (5.2) for \((v, B)\) with \((v, B)\) tending to zero at infinity under assumptions on \((u_a, A_a)\) of a general nature. As in the case of the \((Z)_n\) system, but now for a different reason, namely because the KG energy controls the \(L^2\) norm, it is easy to estimate the \(L^2\) norm of \(B\) (through the energy), so that the Strichartz inequalities for the KG equation do not seem to be useful here. On the other hand, as in the case of the \((WS)_3\) system and in contrast with the case of \((Z)_n\), the low level of local singularity of \((KGS)_2\) allows for a treatment of Step 1 in low level spaces, and in particular with \(k = 0\). The largest suitable space \(X(\cdot)\) can be defined as follows

\[
X(I) = \left\{ (v, B) : (v, B) \in C(I, L^2 \oplus H^1), B \in C^1(I, L^2), \right. \\
\| (v, B) \| X(I) \equiv \sup_{t \in I} h(t)^{-1} \left( \| v(t) \|_2 + \| v ; L^4(J, L^4) \| + \| B(t) ; H^1 \| + \| \partial_t B(t) \|_2 \right) < \infty \right\}
\]

(5.5)

where \(J = [t, \infty) \cap I\) and \(h\) is defined as in Section 2. That space has \((k, \ell) = (0, 1)\). The definition includes the \(L^2\) norm and a Strichartz norm of level \(k = 0\) for \(v\), and the energy norm for \(B\), which has level \(\ell = 1\).

We now state the result concerning Step 1, which is a minor variation of Proposition 2.1

**Proposition 5.1** Let \(h\) be defined as in Section 2 with \(\lambda = 1/2\) and let \(X(\cdot)\) be defined by (5.5). Let \(u_a, A_a, R_1\) and \(R_2\) be sufficiently regular (for the following estimates to make sense) and satisfy the estimates

\[
\| u_a(t) \|_{\infty} \leq c_0 \ t^{-1}, \quad (5.6)
\]

\[
\| A_a(t) \|_{\infty} \leq a \ t^{-1}, \quad (5.7)
\]

\[
\| R_1 ; L^1([t, \infty), L^2) \| \leq r_1 \ h(t), \quad (5.8)
\]

\[
\| R_2 ; L^1([t, \infty), L^2) \| \leq r_2 \ h(t) \quad (5.9)
\]

for some constants \(c_0, a, r_1\) and \(r_2\) with \(c_0\) sufficiently small and for all \(t \geq 1\). Then there exists \(T, 1 \leq T < \infty\) and there exists a unique solution \((v, B)\) of the system (5.2) in \(X([T, \infty))\).
The proof follows closely that of Proposition 2.1. One starts from \((v, B) \in X([T, \infty))\) for some \(T, 1 \leq T < \infty\), so that \((v, B)\) satisfies

\[
\begin{align*}
\|v(t)\|_2 & \leq N_0 h(t) \\
\|v; L^4([t, \infty), L^4)\| & \leq N_1 h(t) \\
\|B(t); H^1\| & \vee \|\partial_t B(t)\|_2 \leq N_2 h(t)
\end{align*}
\]

for some constants \(N_i, 0 \leq i \leq 2\) and for all \(t \geq T\). For each such \((v, B)\) one constructs a solution \((v', B')\) of the system (5.4) in \(X([T, \infty))\) and one shows that the map \(\phi : (v, B) \to (v', B')\) thereby defined is a contraction on the subset \(R\) of \(X([T, \infty))\) defined by (5.10) for suitably chosen \(N_i\) and for sufficiently large \(T\). The seminorms \(N'_i\) of \((v', B')\) corresponding to (5.10) are estimated by the use of (2.3) (2.7) (2.9) as

\[
\begin{align*}
N'_0 & \leq 2c_0 N_2 + r_1 \\
N'_1 & \leq C_1 (c_0 N_2 + r_1) (1 + a) \\
N'_2 & \leq 4c_0 N_0 + r_2 + 4N_1^2 \overline{h}(T)
\end{align*}
\]

for \(T\) sufficiently large to ensure that \(C_1 N_2 \overline{h}(T) \leq 1\), for some absolute constant \(C_1\). Sufficient conditions to ensure the stability of \(R\) under \(\phi\) are obtained in analogy with (2.21) by imposing

\[
\begin{align*}
N_0 & = 2c_0 N_2 + r_1 \\
N_1 & = C_1 (c_0 N_2 + r_1) (1 + a) \\
N_2 & = 4c_0 N_0 + r_2 + 1
\end{align*}
\]

which is possible under the smallness condition \(8c_0^2 < 1\) and by taking \(T\) sufficiently large so that in addition \(4N_1^2 \overline{h}(T) \leq 1\). The rest of the proof is a minor variation of that of Proposition 2.1.

As in the case of the \((\text{WS})_3\) system, there is no difficulty to implement Step 1 at higher levels of regularity. As in the case of \((\text{WS})_3\), the time decay remains the same, namely \(\lambda = 1/2\), and no additional smallness condition appears beyond the previous one of \(c_0\). Again two theories are of special interest.
(1) The theory at the level \((k, \ell) = (1, 1)\) of the energy. The appropriate function space is now
\[
X_1(I) = \left\{ (v, B) : (v, B) \in \mathcal{C}(I, H^1 \oplus H^1), B \in \mathcal{C}^1(I, L^2), \right.
\]
\[
\| (v, B) ; X_1(I) \| \equiv \sup_{t \in I} h(t)^{-1} \left( \| v(t) ; H^1 \| + \| v ; L^4(J, W^1_4) \| \right.
\]
\[
+ \| B(t) ; H^1 \| + \| \partial_t B(t) \|_2 \left) < \infty \right\}
\] (5.13)
where \(J = [t, \infty) \cap I\). It differs from the previous \(X(I)\) by the inclusion of the \(L^2\) norm and of a Strichartz norm of \(v\) at the level \(k = 1\). The additional estimates needed in that theory are obtained from (2.5) with \( \partial = \nabla \) and from (2.9) applied to \(\nabla v\).

(2) The theory at the level \((k, \ell) = (2, 1)\) because that is the lowest convenient level appropriate for the \((Z)\) system. Actually one can make the same choice of \(X(I)\) as in the latter case, namely
\[
X_2(I) = \left\{ (v, B) : (v, B) \in \mathcal{C}(I, H^2 \oplus H^1) \cap \mathcal{C}^1(I, L^2 \oplus L^2), \right.
\]
\[
\| (v, B) ; X_2(I) \| \equiv \sup_{t \in I} h(t)^{-1} \left( \| v(t) ; H^2 \| + \| \partial_t v(t) \|_2 \right.
\]
\[
+ \| v ; L^4(J, W^2_4) \| + \| \partial_t v ; L^4(J, L^4) \|
\]
\[
+ \| B(t) ; H^1 \| + \| \partial_t B(t) \|_2 \left) < \infty \right\}
\] (5.14)
which coincides with (4.29). We state the result concerning that theory in order to allow for comparison both with the case of the \((Z)_2\) system treated in Section 4 and with the treatment of Step 1 in [30] which uses a space of level \((k, \ell) = (2, 2)\).

**Proposition 5.2.** Let \( h \) be defined as in Section 2 with \( \lambda = 1/2 \) and let \( X_2(\cdot) \) be defined by (5.14). Let \( u_a, A_a, R_1 \) and \( R_2 \) be sufficiently regular (for the following estimates to make sense) and satisfy the estimates
\[
\| \partial_t^j u_a(t) \|_\infty \leq c \ t^{-1} \quad \text{for } j = 0, 1
\] (5.15)
and in particular (5.6),
\[
\| \partial_t^j A_a(t) \|_\infty \leq a \ t^{-1} \quad \text{for } j = 0, 1 \ , \quad (5.16)
\]
\[
\| \partial_t^j R_1 ; L^1([t, \infty), L^2) \| \leq r_1 \ h(t) \quad \text{for } j = 0, 1 \ , \quad (5.17)
\]
\[
\| R_1 ; L^4([t, \infty), L^4) \| \leq r_1 \ t^{-\eta} h(t) \quad \text{for some } \eta \geq 0 \ , \quad (5.18)
\]

\[ \| R_2; L^1([t, \infty), L^2) \| \leq r_2 h(t) \]  
(5.9) \equiv (5.19)

for some constants \( c, c_0, a, r_1 \) and \( r_2 \) with \( c_0 \) sufficiently small and for all \( t \geq 1 \).

Then there exists \( T, 1 \leq T < \infty \) and there exists a unique solution \((v, B)\) of the system (5.2) in \( X_2([T, \infty)) \).

The proof is a combination of those of Propositions 5.1 and 4.4. One starts from \((v, B) \in X_2([T, \infty))\) for some \( T \), \( 1 \leq T < \infty \), so that \((v, B)\) satisfies (5.10) and in addition

\[
\begin{align*}
\| \partial_t v(t) \|_2 & \leq N_3 h(t) \\
\| \partial_t v; L^4([t, \infty), L^4) \| & \leq N_4 h(t) \\
\| \Delta v(t) \|_2 & \leq N_5 h(t) \\
\| \Delta v; L^4([t, \infty), L^4) \| & \leq N_6 h(t)
\end{align*}
\]  
(5.20)

for some constants \( N_i \), \( 0 \leq i \leq 6 \) and for all \( t \geq T \). For each such \((v, B)\) one constructs a solution \((v', B')\) of the system (5.4) in \( X([T, \infty)) \). Now the additional estimates of the norms of \( v'\) corresponding to (5.20) are identical with those that occur in the proof of Proposition 4.4 with \( \theta = 0 \), since one is estimating the same norms of \( v'\) from the same Schrödinger equation and with the same information on \( A_a, B \) and \( R_1 \). The rest of the proof proceeds as before.

We now turn to Step 2, namely the construction of \((u_a, A_a)\) satisfying the assumptions needed for Step 1, and we describe the choice made in [30]. As in the (WS)$_3$ case, one sees immediately that taking for \((u_a, A_a)\) a pair of solutions of the free equations is inadequate. Furthermore introducing a phase in \( u_a \) does not seem to be useful at this stage. One takes

\[ u_a = u_1 + u_2 \equiv (1 + f)u_1 \]  
(5.21)

where \( f \) is a complex valued function to be chosen later and

\[ u_1 = MDu_+ . \]  
(5.22)

The difference between \( u_1 \) and \( u_0 = U(t)u_+ \) can be written as

\[ u_0 - u_1 = U(t) \left( 1 - M(t) \right) u_+ \]  
(5.23)
and is easily controlled under suitable assumptions on $u_+$. One then takes

$$A_a = A_0 + A_1$$  \hspace{1cm} (5.24)

$$A_0 = \cos \omega_1 t \ A_+ + \omega_1^{-1} \sin \omega_1 t \ \dot{A}_+$$  \hspace{1cm} (5.25)

with $\omega_1 = (1 - \Delta)^{1/2}$, so that $(\Box + 1) A_0 = 0$, and one takes for $A_1$ the solution of the equation

$$(\Box + 1) A_1 = -|u_1|^2$$  \hspace{1cm} (5.26)

which vanishes at infinity. Using an integration by parts in time, one can write $A_1$ as

$$A_1(t) = \int_t^\infty dt' \ (1 - \cos (\omega_1 (t' - t))) \ \omega_1^{-2} \partial_t |u_1|^2(t') .$$  \hspace{1cm} (5.27)

This formula is justified and a good control of $A_1$ is provided by the following lemma, which is a variant of Lemma 3.2 in [30] or Lemma 2.4 in [29].

**Lemma 5.1.** Let $g \in C^1([1, \infty), L^2)$ with $\partial_t g \in L^1([1, \infty), L^2)$ so that $g(t)$ has an $L^2$ limit $g(\infty)$ as $t \to \infty$ and let $g(\infty) = 0$. Then there exists a unique solution $A$ of the equation

$$(\Box + 1) A = g$$

such that $(A, \partial_t A) \in C([1, \infty), H^1 \oplus L^2)$ and that $\| \omega_1 A \|_2 \vee \| \partial_t A \|_2 \to 0$ as $t \to \infty$. Furthermore $(A, \partial_t A) \in C([1, \infty), H^2 \oplus H^1)$ and $A$ satisfies the estimates

$$\begin{cases}
\| \omega_1 A(t) \|_2 \leq 2 \| \partial_t g; L^1([t, \infty), L^2) \| \\
\| \omega_1 \partial_t A(t) \|_2 \leq \| \partial_t g; L^1([t, \infty), L^2) \|
\end{cases}$$  \hspace{1cm} (5.28)

for all $t \geq 1$.

The proof uses an integration by parts in time and a limiting procedure. It yields in particular the representations

$$\begin{cases}
\omega_1^2 A(t) = -\int_t^\infty dt' \ (1 - \cos (\omega_1 (t' - t))) \ \partial_t g(t') \\
\omega_1 \partial_t A(t) = \int_t^\infty dt' \ \sin (\omega_1 (t' - t)) \ \partial_t g(t')
\end{cases}$$  \hspace{1cm} (5.29)

of which (5.27) is a special case and from which the estimates (5.28) follow immediately. In the case of (5.26), one has

$$g(t) = -|u_1(t)|^2 = -t^{-2} D_0(t) |\hat{u}_+|^2$$  \hspace{1cm} (5.30)

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so that by the commutation rule
\[
\partial_t D_0(t) = t^{-1} D_0(t) \left( t \partial_t - x \cdot \nabla \right)
\]
(5.31)
one obtains
\[
\partial_t g(t) = -\partial_t |u_1(t)|^2 = t^{-3} D_0(t) \left( 2 |\tilde{u}_+|^2 + x \cdot \nabla |\tilde{u}_+|^2 \right)
\]
(5.32)
and therefore by (5.28)
\[
\| \omega_1^2 A_1(t) \|_2 + \| \omega_1 \partial_t A_1(t) \|_2 \leq 3 t^{-1} \left( 2 \| \tilde{u}_+ \|_4^2 + \| x \cdot \nabla \tilde{u}_+ \|_2^2 \right).
\]
(5.33)
In particular \( A_1 \) exhibits a \( t^{-1} \) decay in norms for which \( A_0 \) has no decay.

We now turn to the remainders. Substituting (5.21) (5.24) into their definition (5.3) and using again \( J \) and \( P \) defined by (2.50), we obtain (compare with (2.48) (2.51))
\[
R_1 = (1 + f) \left( i \partial_t + (1/2) \Delta - A_1 \right) u_1 - f A_0 u_1 - i t^{-1} \nabla f \cdot J u_1
+ \left( (1/2) \Delta f + i t^{-1} P f - A_0 \right) u_1,
\]
(5.34)
\[
R_2 = \left( |f|^2 + 2 \text{ Re } f \right) |u_1|^2.
\]
(5.35)
The first three terms in the RHS of (5.34) and the RHS of (5.35) are expected and will turn out to be \( O(t^{-2}) \) in the relevant norms, thereby fulfilling the required assumptions of Propositions 5.1 and 5.2 with \( h(t) = t^{-1} \). On the other hand the last term in (5.34) has only \( t^{-1} \) decay for general \( f \) and one has to choose \( f \) in order to improve that decay to \( t^{-2} \). However this is more delicate than in the (WS)\(_3\) case, because if \( f \) is a solution of the free KG equation, in general \( P f \) is not, so that \( P f \) has to be taken into account in order to perform the cancellation. This is done by using the explicit asymptotic form of solutions of the free KG equation [20]. It is convenient to decompose \( A_0 \) into positive and negative frequency parts, namely
\[
A_0 = A_{0+} + A_{0-} = 2 \text{ Re } A_{0+}
\]
(5.36)
where
\[
A_{0\pm} = (1/2) \left( A_0 \mp i \omega_1^{-1} \partial_t A_0 \right)
\]
(5.37)
and \( A_{0-} = A_{0+} \) since \( A_0 \) is real. The equation satisfied by \( A_{0\pm} \) is
\[
(i \partial_t \pm \omega_1) A_{0\pm} = 0
\]
(5.38)
with the initial condition

\[ A_{0\pm}(0) = (1/2) \left( A_+ \mp i\omega^{-1}\hat{A}_+ \right) \equiv A_{\pm\pm}. \]  

(5.39)

The asymptotic form of \( A_{0\pm} \) is then ([20], Theorem 7.2.5)

\[ A_{0\pm} \sim \tilde{A}_{0\pm} = \pm i(2\pi t)^{-1}(t^2/\rho^2)\tilde{A}_{\pm\pm}(\mp x/\rho) \exp(\pm i\rho)\chi(|x| < t) \]  

(5.40)

where \( \rho = (t^2 - x^2)^{1/2} \) and \( \chi(|x| < t) \) denotes the characteristic function of the set \( \{(x,t) : |x| < t\} \). Corresponding to the decomposition (5.36) of \( A_0 \), we look for \( f \) in the form

\[
\begin{cases}
  f = f_+ + f_- , \\
  f_\pm = t^{-1} F_\pm(x/t) \exp(\pm i\rho).
\end{cases}
\]  

(5.41)

We compute

\[
\begin{align*}
(1/2)\Delta f_\pm + it^{-1}Pf_\pm &= t^{-1}\left\{-it^{-1}F_\pm(x/t) + (2t^2)^{-1}(\Delta F_\pm)(x/t) \\
&\pm it^{-1}(\nabla F_\pm)(x/t) \cdot \nabla \rho - \left(\pm t^{-1}P\rho + (1/2)|\nabla \rho|^2 \mp (i/2)\Delta \rho\right)F_\pm(x/t)\right\}\exp(\pm i\rho)
\end{align*}
\]  

(5.42)

where we have used the fact that \( P(F(x/t)) = 0, \)

\[
\begin{align*}
&\cdots = t^{-1}\left\{-i \left( t^{-1} \pm \rho^{-1} \pm x^2(2\rho^3)^{-1} \right) F_\pm(x/t) + (2t^2)^{-1}(\Delta F_\pm)(x/t) \\
&\mp i\rho^{-1}(x \cdot \nabla F_\pm)(x/t) - \left( \pm \rho t^{-1} + x^2(2\rho^3)^{-1} \right) F_\pm(x/t)\right\}\exp(\pm i\rho)
\end{align*}
\]  

(5.43)

where we have used the fact that \( P\rho = \rho \) and \( \Delta \rho = -(2/\rho + x^2/\rho^3) \). Now all the terms in the bracket in (5.43) are \( O(t^{-1}) \) in the sense that they become \( Ct^{-1} \) for \( x = ct, |c| < 1 \), except for the last term which is \( O(1) \) in the same sense. We now choose \( F_\pm \) in such a way that this term cancels the contribution of \( \tilde{A}_0 \) in the last term of (5.34), namely

\[
\left( \pm \rho t^{-1} + x^2(2\rho^3)^{-1} \right) F_\pm(x/t) = \pm (2\pi i)^{-1}(t^2/\rho^2)\hat{A}_{\pm\pm}(\mp x/\rho)\chi(|x| < t)
\]

or equivalently

\[
\left( \pm (1 - x^2)^{3/2} + x^2/2 \right) F_\pm(x) = \pm (2\pi i)^{-1}\hat{A}_{\pm\pm} \left( \mp x(1 - x^2)^{-1/2} \right) \chi(|x| < 1).
\]  

(5.44)

The functions \( F_\pm \) defined by (5.44) can exhibit two kinds of singularities. First, because of the occurrence of \((1 - x^2)^{1/2}\) and of \( \chi(|x| < 1) \), their derivatives could be
singular when $|x| \to 1$. However in that limit the argument of $\hat{A}_{+\pm}$ tends to infinity, and those singularities can be controlled by assuming sufficient decay of $\hat{A}_{+\pm}$ at infinity. Second, $F_-$ contains the factor

$$g(x) = \left( (1 - x^2)^{3/2} - x^2/2 \right)^{-1}$$

which is singular on the cercle $C_r = \{x : |x| = r\}$ for a suitable $r$, $0 < r < 1$. This singularity always appears in the remainders in the form of combinations of the type $\partial^\alpha f \partial^\beta u_1$ which generate combinations of the type $\partial^\alpha F_- \partial^\beta \hat{u}_+$ and it can therefore be controlled by assuming that $\hat{u}_+$ vanishes of sufficient order on $C_r$. We refer to [30] for the estimates of $u_a$ and of the remainders with the previous choice of $(u_a, A_a)$.

Combining that choice with the treatment of Step 1 provided by Propositions 5.1 and 5.2, one should obtain the following final result, which we state in a semi qualitative way.

**Pseudoproposition 5.3.** Let $h(t) = t^{-1}$ and let $X(\cdot)$ be defined by (5.5). Let $(u_a, A_a)$ be defined by (5.21) (5.22) (5.24) (5.25) (5.27) (5.41) (5.44) (5.39). Let $(u_+, A_+, \dot{A}_+)$ satisfy suitable regularity and decay properties, with $\hat{u}_+$ suitably vanishing on $C_r$ and $\|\hat{u}_+\|_\infty$ sufficiently small. Then

1. There exists $T$, $1 \leq T < \infty$, and there exists a unique solution $(u, A)$ of the $(KGS)_2$ system (5.1) such that $(v, B) \equiv (u - u_a, A - A_a) \in X([T, \infty))$.

2. There exists $T$, $1 \leq T < \infty$, and there exists a unique solution $(u, A)$ of the $(KGS)_2$ system (5.1) such that $(u - u_0, A - A_0) \in X([T, \infty))$. One can take the same $T$ and one obtains the same solution as in Part (1).

One should obtain similar results with $X(\cdot)$ replaced by $X_1(\cdot)$ or by $X_2(\cdot)$, defined by (5.13) (5.14), with suitably stronger regularity and decay assumptions on $(u_+, A_+, \dot{A}_+)$ and suitably stronger vanishing of $\hat{u}_+$ on $C_r$. The smallness condition should be the same in all cases, coming from the contribution of $u_1$ to (5.6), namely smallness of $\|\hat{u}_+\|_\infty$. In particular there should be no smallness condition on $(A_+, \dot{A}_+)$. It remains to determine appropriate assumptions on $(u_+, A_+, \dot{A}_+)$ to make the previous pseudoproposition into a genuine proposition and to remove the conditional form in the accompanying comments. We refer to [30] for a set of sufficient assumptions. On the other hand, as mentioned above, since we are treating Step 1 in larger spaces than is done in [30], those assumptions are stronger than needed here, and we
leave it as an open question to determine the most economical assumptions adapted to the present situation and to the various choices of $X(\cdot)$.

We finally mention that a stronger decay, namely $t^{-\lambda}$ with $1 < \lambda < 2$, has been obtained in [31] [32] on a smaller space of asymptotic states by the use of more precise asymptotic forms for $(u_a, A_a)$.

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