Efficient Computation of Collision Probabilities for Safe Motion Planning

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Abstract—We address the problem of safe motion planning. As mobile robots and autonomous vehicles become increasingly more prevalent in human-centered environments, the need to ensure safety in the sense of guaranteed collision free behaviour has taken renewed urgency. Achieving this when perceptual modules provide only noisy estimates of objects in the environment requires new approaches. Working within a probabilistic framework for describing the environment, we present methods for efficiently calculating a probabilistic risk of collision for a candidate path. This may be used to stratify a set of candidate trajectories by levels of a safety threshold. Given such a stratification, based on user-defined thresholds, motion synthesis techniques could optimise for secondary criteria with the assurance that a primary safety criterion is already being satisfied. A key contribution of this paper is the use of a ‘convolution trick’ to factor the calculation of integrals providing bounds on collision risk, enabling an \( O(1) \) computation even in cluttered and complex environments.

I. INTRODUCTION

Mobile robotic systems that can autonomously plan their paths in complex environments are becoming increasingly more prevalent. An example of such a rapidly emerging technology is autonomous vehicles that can navigate by themselves on urban roads. Such vehicles must not only perform complex manoeuvres among people and other vehicles, but they must do this while guaranteeing stringent constraints on the probability of adverse events occurring, such as collisions with these other agents in the environments. In one study [1], it is estimated that publicly deployable autonomous vehicles must achieve less than 1 collision in hundreds of millions of miles driven in order for their maximum failure rates to be established as acceptable.

The sensory input available to such a mobile robot are typically quite noisy. Typically, the environment is perceived through sensors such as based on stereo vision or LIDAR, requiring not only signal processing for smoothing or noise removal, but also the use of more complex ‘object finding’ algorithms to discern drivable surfaces or other agents in the environment. State of the art algorithms achieving such capability tend to be based on methods that imply a limit on the achievable accuracy and reliability of detections [2], [3]. Therefore, we can only treat such perception modules as being able to provide us with a probability distribution [4] over poses of the various objects in the scene. Achieving safe motion planning in such a setting will require the motion planning methods to turn these into probabilities of unsafe events (such as a collision of the robot with another agent), providing at least approximate assurance regarding the (non-)occurrence of these events.

A. Problem Formulation

The core problem that concerns us in this paper is that of safe motion planning in an imprecisely known environment. As input, we assume detection of objects in the environment along with probability distributions over pose, such as over their centroid position when the shape model is known. This defines the space within which we must search over paths. In our formulation, the first step is to determine the probability of a collision event occurring (i.e., the spatial extent of the robot and the spatial extent of any obstacle having an overlap) along any given path. This core computation may then be used to modify a variety of motion synthesis methods. For instance, the Rapidly-exploring Random Tree (RRT) algorithm can utilise this probability within the search process. Likewise, a variational formulation of optimal control [5] could include this within the cost terms.

B. Related Work

The issue of safety in control and motion planning has been investigated from a number of different methodological
directions. As this literature is much more extensive than we can fully survey in the present work, we mention a few salient approaches in order to situate our work within this research landscape. At a high level, our approach will be in the spirit of probabilistic robotics [6], wherein environmental uncertainty has been posed probabilistically in the form of occupancy grids. However, we will seek to relax several assumptions. Firstly, we will take into account the fact that collision probabilities are not independent from cell to cell in an occupancy grid representation. Instead, the probability of collision at one cell is directly coupled to that at a nearby cell, via the latent variable of true object pose. Secondly, we will reconsider the grid representation in itself, instead adopting an alternative computational approach that allows us to model the paths and the object shapes in a continuous fashion, deferring approximations to a much later stage in the procedure.

1) Safety in Control and Planning: In early attempts within the robotics community, treatment of safety was closely associated with the notions of robustness to modelling errors in control theory [7]. In the context of feedback control design, robust control theory provides techniques to determine the single controller that can simultaneously guarantee a desired performance level for a family of possible target system models. This basic notion can be extended to the case of hybrid nonlinear dynamical systems and optimal control specifications [8] through the concept of reachable sets. Reachable sets can be computed using techniques such as level set methods [9]. While recent extensions have included the use of Gaussian Process based learning of a robot's internal dynamics model, to enable reachability computations with physical robots [10], the focus of these methods has not been on treatment of probabilistic uncertainty in models of the environment (hence, of the cost function driving the problem).

Other current approaches consider safety in the context of human-robot interaction [11], where one is also interested in considerations such as modulating physical forces and human factors associated with the psychology of interaction. The core question of avoiding collision states in motion planning is dealt with in these works through the definition of safe zones and operating envelopes, which are not unlike the reachable sets mentioned above. In many applications of interest to us, the resulting solutions may be overly conservative - leading to ‘frozen robots’ [12].

2) Dealing with Environment/Location Uncertainty: Our work builds on a line of inquiry initiated by Laugier and collaborators [13], [14], who calculate probabilistic risk of collision of mobile robots and utilise this to define a risk guided search, e.g., in the Risk-RRT. As stated earlier, our version of a similar calculation relaxes independence assumptions and also includes a novel technique for much faster computation of the probabilistic risk.

In work drawing on robust optimization and stochastic programming, some researchers have posed the problem of motion planning as one of chance-constrained optimization [15]. Following the above mentioned approaches to robust control, these works use set-bounded models of uncertainty and pose the search for paths as a problem of convex programming. [16] is another recent example where the authors calculate rectangular bounds (shadows) associated with uncertainty in location of obstacles, to determine the safe space within which RRT style algorithms can operate. As we will discuss in later sections, our proposed approach will improve upon these computations, particularly in the case of close interactions in tight spaces - such as in urban driving.

A related but distinct body of work addresses the case where the focus is on state estimation of the robot, e.g., [17], [18]. These approaches allow for computational simplifications, such as that the collision probability at different locations are conditionally independent, which do not apply in our domain.

3) AI Safety: The problem of safety in Artificially Intelligent systems has received significant recent attention. While much of this work has been focussed on abstract mechanisms by which specifications and rewards might have unintended consequences [19], [20], some researchers have looked explicitly at the design of safe policies in robot motion domains [21]. In [22], the authors make the pragmatic observation that absolute safety is likely not achievable in general. Instead, they advocate the notion of ‘blame-free’ policy synthesis. While the examples in [22] were still mainly of a conceptual nature, and the corresponding algorithmic methods fairly limited, we build on their line of reasoning in this paper by presenting a methodology for achieving a graded notion of safe path planning - applying it to realistic situations involving non-trivial robot and path geometries.

4) Computational Geometry Tools: A key contribution of our paper is a faster procedure for calculating integrals associated with the calculation of collision risk. We achieve this through the use of convolutions over boundary shapes. The computational geometry literature includes many methods for making such computations efficient, e.g., [23], [24].

II. EFFICIENT CALCULATION OF COLLISION PROBABILITIES

We are interested in the search for safe paths. For this, we must determine the risk of collision from taking a sequence of actions (i.e., traversing along a candidate path), given a description of the objects in the environment. This risk can be posed in terms of the probability of occupation of the surface swept by the robot moving along the path during a specified time interval along that path. Our starting point is the work by Laugier and collaborators, who propose a probabilistic framework for such modelling [13]. In this section, we begin with this model for describing obstacles, and then proceed to describing improved methods for the computation of these quantities.

A. The Probabilistic Obstacle Framework: Laugier’s Model

Consider an environment consisting of a set of $K$ obstacles. These obstacles are typically detected by perception
modules whose outputs can be noisy, hence we treat them as probabilistic. The position of each obstacle centre (or suitable canonical point, with respect to which all geometric computations can be reframed) is assumed known modulo perceptual uncertainty via a density function \( p_k(x) \).

The probability of collision between the robot and the \( k \text{th} \) obstacle (which could be written as ‘obstacle \( k \)’), if that obstacle were a point obstacle, could we written as [13],

\[
P_D(k) = \int_A p_k(x)
\]

where \( A \) is the swept area of the robot, along a path \( \pi \) and over time \( t \in [t_1, t_2] \).

Most obstacles are not mere points. When an obstacle has a shape, \( B_k \subset \mathbb{R}^2 \), situated at the origin, then at position \( x \) the displaced object would be described as \( B_k(x) = B_k + x = \{x + x', x'\} \). So, the probability of collision with the obstacle may be rewritten as,

\[
P_D(k) = \int_{A_k} p_k(x)
\]

where \( A_k = A \oplus B_k \), the Minkowski sum of the robot shape and the obstacle shape, as in figure 2. This is adequate for a single obstacle, but when multiple obstacles are present simultaneously we can write the following upper bound on the total probability of at least one collision with obstacles,

\[
F_D = \sum_{k=1}^K p_D(k)
\]

which serves as the bound, \( p_D \leq F_D \).

In search based planning algorithms, the Minkowski sums for \( A_k \) must be recomputed for every new path \( A \) that is conjectured, and for every obstacle, \( B_k \). This becomes inefficient when \( K \), the number of obstacles, is large. We would therefore like to find a way to replace these repeated \( O(K) \) computations by an \( O(1) \) computation.

Even before proceeding to discuss these, we observe that - for the point obstacle case (i.e., equations 2 and 3 with \( K \) point obstacles only) - the computation of \( F_D \) could be made more efficient, i.e., of order \( O(1) \), by factoring it as:

\[
F_D \leq \int I_A(x) G(x) \text{ where } G = \sum_{k=1}^K p_k
\]

where \( I_A \) stands for an indicator function for points belonging to the set \( A \). In the following sections, we show how this simple insight could be extended and applied in more general settings.

**B. Minkowski Sum and the “Convolution Trick”**

We note that the integral in equation 2 can be rewritten as the convolution of two functions, evaluated at the origin,

\[
P_D(k) = [I_{A_k} \ast \hat{p}_k(x)](0)
\]

This motivates also noting a connection between the Minkowski sum of two sets, and the convolution operator \( \ast \) applied to the indicator functions of the two sets,

\[
Z = X \oplus Y \Rightarrow Z = \text{supp}(I_X \ast \hat{I}_Y)
\]

This gives us a method for calculating \( I_{A_k} \) when \( B_k(x) \subset A \). Next, we need the corresponding calculation for the case when \( B_k(x) \not\subset A \).

**C. Using contours to bound the integral**

The complementary component of the Minkowski sum, \( I_{A_k} \), can be bounded via convolution using the bounding

\[
I_{A_k} = \lambda_k [I_A \ast \hat{I}_{B_k}](x), \text{ when } B_k(x) \subset A
\]

where \( \lambda = \frac{1}{\text{area}(B_k)} \). The expression on the right of this equation is everywhere positive as it is a convolution of indicator functions which are positive.

This gives us a method for calculating \( I_{A_k} \) when \( B_k(x) \subset A \). Next, we need the corresponding calculation for the case when \( B_k(x) \not\subset A \).
contours of the obstacles, $\partial B_k$, and of the robot, $\partial A_k$, to obtain an upper bound on any integral of the form, $\int_{A_k} f(x)$ such as the collision probability in equation 2.

Given the set $A$, the delta function ridge around its boundary $\partial A$ is calculated as,

$$\partial A_\sigma(x) = |\nabla g_\sigma(x) \ast I_A(x)|$$  \hspace{1cm} (9)

where $g_\sigma(x)$ is a normalised isotropic 2-dim Gaussian function with a (small) diameter $\sigma$. Similarly, we define $\partial B_{k,\sigma}(x)$ as the delta function ridge around $\partial B_k$. Now, the indicator function for the Minkowski sum is bounded in the complementary condition, and in the limit that $\sigma \rightarrow 0$, by the convolution of the two delta function ridge functions as follows,

$$I_{A_k}(x) \leq \frac{1}{2} [\partial A_\sigma \ast \hat{\partial B}_{k,\sigma}](x), \text{ when } B_k(x) \not\subseteq A \hspace{1cm} (10)$$

This is illustrated in figure 4.

![Figure 4: Approximating the indicator function of the Minkowski sum of sets $A$ and $B$ via contour convolution.](image)

As with equation 8, the right hand side of the expression is everywhere positive. So, now the complementary expressions 8 and 10 can be combined into a single bound on the indicator function of the Minkowski sum,

$$I_{A_k}(x) \leq \frac{1}{2} [\partial A_\sigma \ast \hat{\partial B}_{k,\sigma}](x) + \lambda_k [I_A \ast \hat{I}_{B_k}](x)$$  \hspace{1cm} (11)

As this bounds holds everywhere, we may simply write,

$$I_{A_k} \leq \frac{1}{2} [\partial A_\sigma \ast \hat{\partial B}_{k,\sigma}] + \lambda_k [I_A \ast \hat{I}_{B_k}]$$  \hspace{1cm} (12)

Returning to the earlier expression for the collision probability, in equation 2, we have,

$$P_D(k) \leq \left[ \frac{1}{2} [\partial A_\sigma \ast \hat{\partial B}_{k,\sigma}] + \lambda_k [I_A \ast \hat{I}_{B_k}] \right] \ast \hat{p}_k$$  \hspace{1cm} (0)

(13)

Using the associativity of the convolution operator, this may be rewritten as,

$$P_D(k) \leq \frac{1}{2} [\partial A_\sigma \ast \hat{\partial B}_{k,\sigma} \ast \hat{p}_k] + \lambda_k [I_A \ast \hat{I}_{B_k} \ast \hat{p}_k]$$  \hspace{1cm} (14)

Finally,

$$P_D(k) \leq \int \partial A_\sigma \frac{1}{2} [\partial B_{k,\sigma} \ast p_k] + \int I_A \lambda_k I_{B_k} \ast p_k$$  \hspace{1cm} (15)

If we sum up over obstacles, we get the bound on number of collisions as,

$$F_D \leq \int \partial A_\sigma(x) G_\sigma(x) + \int I_A(x) G(x) \hspace{1cm} (16)$$

where $G_\sigma$ and $G$ are:

$$G_\sigma = \frac{1}{2} \sum_k \partial B_{k,\sigma} \ast p_k$$  \hspace{1cm} (17)

$$G = \sum_k \lambda_k I_{B_k} \ast p_k$$  \hspace{1cm} (18)

Note that $G$ and $G_\sigma$ are independent of $A$ and do not need to be recomputed every time $A$ changes. So, the repeated computation of the bound 16 for different $A$ would indeed be $O(1)$, i.e., independently of the number of obstacles - once the initial calculations of $G$ and $G_\sigma$ have been carried out.

1) Proof of the “Contour Convolution Trick”: We now show that the indicator function for the Minkowski sum (in the limit that $\sigma \rightarrow 0$), in inequality 10 is indeed bounded as:

$$I_{A_k}(x) \leq \frac{1}{2} [\partial A_\sigma \ast \hat{\partial B}_{k,\sigma}](x)$$  \hspace{1cm} (19)

In the case that $B_k(x) \cap A = \phi$, both sides of equation 19 are 0. Elsewhere, and given that $B_k(x) \not\subseteq A$, the contours of $A$ and $B_k$ must intersect at least twice. In that case,

$$[\partial A_\sigma \ast \hat{\partial B}_{k,\sigma}](x) = \int_{x'} \partial A_\sigma(x') \partial B_{k,\sigma}(x' - x)$$  \hspace{1cm} (20)

integrates across two or more contour intersections, and the integral at each intersection (contours crossing at an angle $\theta$) has the form,

$$\int \int g_\sigma(x) g_\sigma(x \cos \theta + y \sin \theta) dx dy = \frac{1}{\sin \theta} \geq 1$$  \hspace{1cm} (21)

In the limit when $\sigma \rightarrow 0$, since the indicator function $I_{A_k}(x)$ has value 1 whenever the contours intersect, the formula 19 does indeed hold.

An algorithmic description of the proposed method to calculate the bound on the number of collisions using the “contour convolution trick” is shown in algorithm 1.

III. EXPERIMENTS

In figure 5, we see the outcome of the calculation of the collision probabilities for a set of candidate paths between two poses of the vehicle in a typical environment with other agents present. We use a Closed Loop variant of the RRT algorithm to generate potential paths in the environment, which we then rank order according to the measure of safety described in the foregoing discussion. In the figures, the different candidate paths are coloured according to the computed value of the bound $F_D$, enabling the basis of further optimisation or search as dictated by additional motion synthesis considerations. For instance, in addition to the RRT mechanism we use here, one could also incorporate this in...
Algorithm 1: Efficiently calculate the bound on number of collisions for a given path and a set of obstacles.

Data: $A, B_{1:K}, p_{1:K}(x), \sigma = 1$

Result: $F_D$

// Compute $G_\sigma$
for $k$ in 1 to $K$ do
  $\partial B_{k, \sigma}(x) = |\nabla g_\sigma(x) \ast I_{B_k}(x)|$
end
$G_\sigma(x) = \frac{1}{2} \sum_{k=1}^{K} \partial B_{k, \sigma}(x) \ast p_k(x)$

// Compute $G$
$G(x) = \sum_{k=1}^{K} \frac{1}{\text{area}(B_k)} I_{B_k}(x) \ast p_k(x)$

// Compute $A_\sigma$
$\partial A_\sigma(x) = |\nabla g_\sigma(x) \ast I_{A(x)}|$

// Integrate over a box around $A$ with margins $4\sigma$
$F_D = \int_{R^2} \partial A_\sigma(x) G_\sigma(x) + I_A(x) G(x)$

the metric associated with variational optimisation methods for motion synthesis, e.g., [5].

We demonstrate in this section that sets of randomly-generated trajectories (where by ‘random’ we mean the mechanisms commonly adopted within the sampling based motion planning literature) for a particular environment configuration can be stratified into ranges according to safety (as measured by the bound $F_D$).

![Visualisation of the calculated bounds on number of collisions $F_D$ for a set of paths between two poses and locations of a vehicle. The coloured lines represent a ranking according to safety, as mentioned earlier. The gray objects represent obstacles whose position is known only up to probability distribution, visualised here as a shaded halo around the obstacle.](image)

**Fig. 5: Visualisation of the calculated bounds on number of collisions $F_D$ for a set of paths between two poses and locations of a vehicle.** The coloured lines represent a ranking according to safety, as mentioned earlier. The gray objects represent obstacles whose position is known only up to probability distribution, visualised here as a shaded halo around the obstacle.

### A. The Playground Environment

In our experiments, we utilise a 2-dim simulation environment in which a rectangular vehicle of size $2m \times 4m$ navigate along continuous paths (paths being defined in a configuration space involving full pose, even if the visualisations only depict the centroid). A simulation scenario is defined by a collection of obstacles within the environment, each specified with its shape (as a subset of $\mathbb{R}^2$), dimensions and true pose, as well as start and target poses for the ego vehicle.

We employ a Closed Loop RRT-based motion planner to generate trajectories between start and target poses. This motion planner takes into account the vehicle kinematic and dynamic model, discretising steering and accelerator input ranges when expanding the random tree to generate realisable trajectories, and in order to restrict abrupt steering or velocity changes. Nodes in the RRT are scored based on their proximity to the goal, similarity to its orientation and velocity. The RRT uses a fixed $\epsilon \in [0, 1]$ to randomly choose, at every step, either to expand the next node towards the target or towards a random goal in the neighbourhood of the best scored node in the tree. This setup allows collecting multiple trajectories that reach the specified target pose for any environment.

### B. Stratifying a Set of Trajectories by Safety

The two scenarios discussed in this section attempt to represent common situations in driving. First, we look at a scenario resembling a road with cars parked along the side, including some cars navigating the lanes. Second, we consider the scenario where two paths may be feasible but one seems unnatural to the observer - going in between two other cars in ways that no human driver would do. For each scenario we randomly generated 400 paths. To model the uncertainty over the obstacle’s position we used a normal distribution centred at the obstacle true position and a standard deviation $\sigma$ corresponding to 20 cm in the physical environment. For scale comparison, the obstacles in figure 5 are 2 m wide.

As can be seen from figure 6, paths with the lowest $F_D$ bounds remain further away from the obstacles throughout the trajectory while still being perfectly natural with respect to norms of regular driving. Trajectories that go closer to the obstacle in the centre have the highest $F_D$ bound.

In figure 7 we can observe that only when $F_D > 0.1$ do we have trajectories that take the route from above - through the narrower gap. In such an environment, the majority of trajectories, 126 in this experimental batch, are scored in the safest range while the more populated environment only contains 2 in that range. It is satisfying to see that aspects of naturalness in driving are compatible in this sense with our notion of safety, indicating that sometimes our choices are a matter of how we choose to set thresholds on safety. This behaviour is in line with similar observations regarding choices between distinct solution classes in optimal control problems [25].

### C. Performance Evaluation

Finally, we present experimental results to illustrate the computational benefits of the contour convolution based approach to evaluating the bound on probabilistic risk. In
Fig. 6: Segmented safety ranges based on calculated bound on number collisions $F_D$ for a set of paths in an overtaking scenario of size $24m \times 40m$.

(a) $F_D \leq 0.01$, 2 paths  
(b) $0.01 < F_D \leq 0.1$, 3 paths  
(c) $0.1 < F_D \leq 1$, 72 paths  
(d) $F_D > 1$, 323 paths

Fig. 7: Segmented safety ranges based on calculated bound on number collisions $F_D$ for a set of paths in an environment, size $40m \times 40m$, with two feasible trajectories

(a) $F_D \leq 0.01$, 126 paths  
(b) $0.01 < F_D \leq 0.1$, 16 paths  
(c) $0.1 < F_D \leq 1$, 74 paths  
(d) $F_D > 1$, 184 paths

Table I: Timing results comparing time complexity of our method to a baseline implementing the integral in equation (2) without using the ‘convolution trick’. We measure timing on scenarios with increasing numbers of obstacles. All results are presented in units of ms, for 100 paths.

| Obstacle count | Our method | Baseline [13], [14] |
|----------------|------------|---------------------|
| 1              | 1618.58    | 12103.19            |
| 4              | 2014.24    | 47474.60            |
| 7              | 2501.55    | 85374.13            |
| 11             | 3047.30    | 134678.57           |

IV. DISCUSSION

A. On the Probabilistic Characterisation

As is well understood from the history of probability [26], decision making agents must cope with two kinds of uncertainty - epistemic (i.e., beliefs about subjective aspects of uncertainty) and aleatory (inherent randomness in phenomena of interest). In domains such as automated vehicles, agents face limits along both directions - they are unsure about what other agents might do, and they face intrinsic limits to what sensors can provide. In our setup, limited as it is for the moment to environments that do not themselves exhibit complex dynamic behaviours, epistemic uncertainty about others’ policies is not yet the main issue (although it could be factored in within such a probabilistic framework). Aleatory uncertainty in the form of limits to sensing accuracy [2], [3], [27] relate directly to our discussion. While a complete absence of detection would be an issue that is beyond the scope of motion planning, diffuse probability distributions over location of objects would translate in our framework to paths that vary in shape as in the examples above - while still respecting desired bounds on safety metrics such as $F_D$.

In some related works, e.g., [16], the authors approach safe motion planning by computing intervals or shadows from the description of uncertainty in obstacle locations, so that the search for safe paths is restricted to the complement of these shadows. With many closely spaced obstacles, this can lead to an overestimate of risk and conservative policies (as already observed, for instance, in [12]).

B. On Notions of Safety in Motion Planning

In our approach, we measure the safety of entire candidate paths in configuration space. This enables a direct translation from end user requirements regarding safety [1] (such as number of acceptable collisions per million miles) to the performance of policies being synthesized. Our method is in keeping with the policy gradient approach to reinforcement learning. In the literature on safe reinforcement learning, [28], one approach to safe policy synthesis is to constrain optimisation within a ‘space of considered safe policies’. We provide one such definition of a safe set of policies.

Moreover, we seek a graded notion of safety, wherein parametrised policies can be associated with a user-defined threshold on acceptable levels of safety. This also permits, for instance, occasional deviations from a more conservative...
envelope into an aggressive regime when the higher level task specifications may demand it (e.g., when accelerating quickly is the only way to safely merge). Our problem formulation and computation approach enables this.

V. CONCLUSIONS

Safe motion planning is a problem whose importance is steadily increasing owing to the continued deployment of autonomous robots and vehicles in our daily environments. In the autonomous vehicles domain, the stringent requirements regarding potential collisions calls for both new notions of safety and efficient computational tools for ensuring the safety of plans. We present an approach that builds on a probabilistic framework for calculating collision risk, and uses a new calculation approach to rendering these computations linear time. We show that this enables us to threshold and stratify candidate paths according to a measure of safety, enabling a variety or search and optimisation based motion planning methods that respect such a graded notion of safety. This might imply, for instance, that although many paths might have been possible, none are deemed safe enough at a certain threshold value. Conversely, when a larger set of trajectories is deemed safe enough, one is free to optimise for other criteria such as rider comfort and travel time.

In related current work, we are exploring the extension of the proposed method to the case of dynamic obstacles, i.e., obstacles whose position varies over time, so that $p_k(x)$ in equation 2 would now need to be based on a spatio-temporal (stochastic) process.

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