MAGNUS SPIN HALL AND SPIN NERNST EFFECTS IN 2D RASHBA SYSTEMS WITH ZEEMAN-SPLITTING

PRIYADARSHINI KAPRI, BASHAB DEY, AND TARUN KANTI GHOSH

DEPARTMENT OF PHYSICS, INDIAN INSTITUTE OF TECHNOLOGY-KANPUR, KANPUR-208 016, INDIA

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We study the Magnus transport in a Zeeman-split 2D electron gas with Rashba spin-orbit coupling using semiclassical Boltzmann transport formalism. Apart from its signature in the charge transport coefficients, the inclusion of Magnus velocity in the spin current operator enables us to study Magnus spin transport in the system. In particular, we study the roles of Zeeman gap and Fermi surface topology on the behavior of Magnus Hall and Nernst conductivities and their spin counterparts. We find that the Magnus spin Hall conductivity vanishes in the limit of zero Zeeman gap, unlike the universal spin Hall conductivity \( \sigma_s = e/(8\pi) \). The Magnus spin currents with spin polarization perpendicular to the applied bias (electrical/thermal) are finite while with polarization along the bias vanishes. Each Magnus conductivity displays a plateau as Fermi energy sweeps through the gap and has peaks (whose magnitudes decrease with the gap) when the Fermi energy is at the gap edges.

I. INTRODUCTION

After the discovery of classical Hall effect, a wide range of Hall effects (HEs), such as quantum HE, anomalous HE, spin HE, nonlinear HE and thermal HE have been discovered in the course of time. These discoveries have played a leading role in elucidating the novel electronic states \([1,2]\) and electron dynamics \([3,4]\) and thus have drawn tremendous interest to the solid state community. The traditional (classical) Hall \([3]\) effect and quantum Hall effect \([4]\) arise only in the presence of an external magnetic field. However, anomalous \([5,17]\) and the quantum anomalous Hall effect \([5,10]\) originate from the anomalous velocity of the charge carriers, where the role of Berry curvature (BC) is involved. Further, the involvement of Berry curvature with spin index, valley index, and orbital degrees of freedom lead to the spin HE \([11,12]\), valley HE \([13,15]\) and orbital HE \([16,17]\) respectively. Furthermore, the dipole moment of BC in momentum space generates a net anomalous velocity and gives rise to the non linear Hall effect \([15]\). Moreover, the thermal analogue of these Hall effects lead to thermal Hall effects \([19,20]\), which play a pivotal role in the field of caloritronics.

As we know, the discrete symmetries of the Hamiltonian namely, inversion symmetry (IS) and time reversal symmetry (TRS) play a significant role in directing the fate of the Hall current. For example, the intrinsic anomalous Hall effect (AHE) vanishes for TRS invariant systems, because the Berry curvature of the systems follows the relation: \( \Omega(k) = -\Omega(-k) \) and hence the Hall conductivity (which is proportional to the integral of Berry curvature) vanishes. However, such a distribution of Berry curvature in a TRS invariant system raises the question whether it can give rise to any interesting phenomena in charge transport. This leads to realization of the valley Hall effect \([13]\) and the non-linear Hall effects in TRS invariant but IS systems \([13,20,22]\). Very recently, it is observed that such TRS invariant but IS broken systems with a built-in electric field and no external magnetic field manifests a new type of Hall effect namely, Magnus Hall effect (MHE) \([23]\). The Magnus Hall effect (MHE) arises from the Magnus velocity of electron which is perpendicular to the Berry curvature and the built-in electric field induced by gate voltages (see Fig. 1). Furthermore, in Refs. \([21,22]\) the appearance of Magnus Nernst effect (MNE) (transverse Magnus current produced by longitudinal thermal gradient) and Magnus thermal Hall effect (Magnus heat current in a direction transverse to a temperature gradient) are demonstrated in TRS invariant but inversion broken systems. It has been proposed that the systems having IS breaking and TRS invariant nature, such as, monolayer (ML) graphene on hBN, bilayer (BL) graphene with applied perpendicular electric field \([20,27]\), the two-dimensional (2D) transition metal dichalcogenides MX\(_2\) (\(M = Mo, W\) and \(X = S, Se, Te\)) \([13,25,29]\), hetero-structures \([30]\), surfaces of topological insulator (TI) \([31]\) and Weyl semimetals \([32,34]\) are the potential candidates for exploring the MHE and MNE. Recently, A detailed study on Thus it remains to be seen whether the Magnus Hall effect and Magnus Nernst effect exist in systems where both the IS and TRS are broken with non zero Berry curvature.

Recently, in our previous paper \([35]\), we have shown that the origin of the spin Hall current can be explained by redefining the spin current operator with inclusion of the anomalous velocity. We have also shown that the anomalous velocity can generate pure anomalous non linear spin current with in-plane polarization. Thus, our general instinct motivates us to investigate whether the inclusion of Magnus velocity in spin velocity operator can produce Magnus spin Hall current (out of plane polarization) and Magnus spin current (in-plane polarization). Also, the observation of Spin Nernst effect \([36,37]\) points towards a possibility of Magnus spin Nernst effect (MNE) on incorporation of the Magnus velocity.

Motivated by the above discussion, we study the Magnus Hall and the Magnus Nernst effect of a 2D Rashba system with Zeeman splitting that causes the TRS breaking of the system. In particular, we investigate the role of gap parameter (induced by the TRS breaking Zeeman term) and the role of Fermi surface topology in the be-
behavior of Magnus Hall and the Magnus Nernst conductivities. Furthermore, we redefine the spin current operator with including the Magnus velocity and study the spin counterparts of Magnus conductivities.

This paper is organized as follows. In Sec. II A, we present the semiclassical Boltzmann transport formalism to discuss the Magnus Hall conductivities (MHCs) in the diffusive regime as well as in the ballistic regime. Section II B presents the extension of above theory to study spin counterparts of MHCs by inclusion of Magnus velocity in spin velocity operator. Section III includes the basic information of a 2D gapped Rashba system. In Sec. IV, we present the analytical and numerical results. Finally Sec. V concludes and summarizes our main findings.

II. FORMALISM OF MAGNUS TRANSPORT

In this section, we present a theory for Magnus Hall, Magnus Nernst conductivities for a generic two-band system and finally extend the theory to calculate their spin counterparts.

A. Derivation of Magnus transport coefficients

A mesoscopic Hall bar is considered, where a slowly varying electric potential energy $U(r)$ is set along the length of the bar (See Fig. 1). The potential energy gradient is introduced by the two gate voltages, $U_L$ and $U_R$. The difference in potential energies ($\Delta U = U_L - U_R$) gives rise to an in-built electric field $\mathbf{E}_{in} = e^{-1}\nabla_x U$ ($-e$ being the electronic charge) along the bar. As mentioned earlier, to have the Magnus transport, the material of the bar should have a non-zero Berry curvature $\Omega$. Furthermore, there must be an additional applied electric field $\mathbf{E}$, or temperature gradient ($\nabla_x T$) between the source and drain for driving the charge current or spin current.

The motion of wave-packet inside the Hall bar is described by the semiclassical equations of motion [38-41]

$$\hbar \dot{r} = \nabla_k \epsilon_k + (\nabla_x U + e \mathbf{E}) \times \Omega, \quad (1)$$

$$\hbar \dot{k} = -\nabla_x U - e \mathbf{E}. \quad (2)$$

The first, second and third terms in the right hand side of Eq. (1) are associated with the semiclassical band velocity $v_b = \frac{1}{\hbar} \nabla_k \epsilon_k$, Magnus velocity $v_m = \frac{1}{\hbar} (\nabla_x U \times \Omega)$ and anomalous velocity $v_a = \frac{\hbar}{\tau} (\mathbf{E} \times \Omega)$, respectively with $\Omega$ being the Berry curvature. For a 2D system confined in $x$-$y$ plane, the Berry curvature is always in the $\hat{z}$-direction and hence both the Magnus and the anomalous velocity are along the $\hat{y}$-direction for a $\hat{x}$-directed external bias (electric field or temperature gradient).

Within the relaxation time approximation, the non-equilibrium carrier distribution function $f(r, k)$ obeys the Boltzmann transport equation (BTE) [40]

$$\frac{\partial f}{\partial t} + \dot{r} \cdot \frac{\partial f}{\partial r} + \dot{k} \frac{\partial f}{\partial k} = -\frac{f - f_0}{\tau}, \quad (3)$$

where $f_0$ and $\tau$ denote the equilibrium distribution function and the scattering time, respectively. For simplicity, $\tau$ is considered to be momentum independent. In the steady state condition ($\frac{\partial f}{\partial t} = 0$), with no external bias the equilibrium distribution function is given by the Fermi function

$$f_0(k, r) = \frac{1}{1 + e^{\beta[\epsilon(r, k)-\epsilon_F]}}, \quad (4)$$

where $\epsilon(k, r) = \epsilon_k + U(r)$, $\beta = 1/k_B T$ and $\epsilon_F$ is a constant Fermi energy. Now, in the presence of an external field ($\hat{x}$-directed), the steady state non-equilibrium distribution function up to first order in the bias fields can be obtained as

$$f = f_0 + v_{b,x} \tau (e E_x + \beta[\epsilon(k, r) - \epsilon_F] k_B \nabla_x T) \frac{\partial f_0}{\partial k}, \quad (5)$$

where $v_{b,x}$ denotes the $x$ component of band velocity $v_b$. For our setup, $v_x = v_{b,x}$, as both the anomalous and Magnus velocities are in $\hat{y}$ direction.

In the presence of external electric field $\mathbf{E}$ and temperature gradient $\nabla T$ applied between the source and drain, the charge current density $\mathbf{J}$ from linear response theory can be written as

$$J_i = \sigma_{ij} E_j + \alpha_{ij} (-\nabla_j T), \quad (6)$$

where $i$ and $j$ denote the propagation and applied electric field directions with $\sigma$ and $\alpha$ being the conductivity tensors. For our setup, $j = x$, as the external electric field or the temperature gradient is applied in $\hat{x}$ direction.
For a two-band system, the charge current in a 2D system can be written as

\[ J = -e \sum_{x} \int \frac{d^{2}k}{(2\pi)^{2}} \langle \lambda, k|\hat{v}|x, \lambda\rangle f_{n}, \]  

where \( \lambda \) denotes the band index and \( f_{n} \) denotes the \( n \)-th order distribution function. From Eq. (7), one can easily separate out the band velocity, anomalous velocity and the Magnus velocity contributions in the transport coefficients. Since the main focus of our paper is Magnus transport, we discuss the Magnus contribution only.

Thus, the Magnus Hall conductivity \( (\sigma_{m} = J_{y}/E_{x}) \), driven by electric field and the Magnus Nernst conductivity \( (\alpha_{m} = J_{y}/(-\nabla_{x}T)) \), driven by temperature gradient, of the first-order in bias fields are obtained as

\[ \sigma_{m} = -\frac{e^{2}U}{hL} \sum_{\lambda} \int \frac{d^{2}k}{(2\pi)^{2}} \Omega_{\lambda}^{2} v_{x}^{\lambda} \frac{\partial f_{0}}{\partial \epsilon_{k}}, \]  

\[ \alpha_{m} = \frac{e}{hT} \Delta U \sum_{\lambda} \int \frac{d^{2}k}{(2\pi)^{2}} \Omega_{\lambda}^{2} v_{x}^{\lambda} \frac{\partial f_{0}}{\partial \epsilon_{k}}, \]

where \( U \) is a slowly varying function of \( x \), \( \partial U/\partial x = \Delta U/L \) with \( L \) being the length of the Hall bar, \( v_{x}^{\lambda} = \langle \lambda, k|\hat{v}_{x}|x, \lambda\rangle \) and \( \epsilon_{k} = (\epsilon_{k} - \epsilon_{F}) \).

Here, it is to be noted that the Magnus conductivities of linear order (in external bias fields) arise from \( f_{1} \), whereas \( f_{0} \) is responsible for first order anomalous conductivities as \( v_{a} \propto \nabla \mathbf{E} \). However, the above Magnus conductivities can be viewed as an effective second order response as external bias and the built-in electric field both are involved in the calculation of currents.

So far the Magnus responses in the diffusive limit has been discussed. In the ballistic regime, the mean free time between two collisions is infinite \( \tau \rightarrow \infty \). Thus, no collision occurs in the transport direction along the Hall bar. Therefore, the right hand side of the Boltzmann transport equation given in Eq. (3) vanishes in the ballistic regime.

In this setup, the larger electrochemical potential of the source region causes a surplus of electrons entering the system at \( x = 0 \) interface with \( v_{x} > 0 \). These electrons propagate across the device without any scattering in the ballistic limit. Hence, only the carriers from the source with positive velocity are allowed in region \( 0 < x < L \). The non-equilibrium part of the distribution function can be written as

\[ f_{1}(k, r) = -\Delta \epsilon_{F} \frac{\partial f_{0}}{\partial \epsilon_{k}} - \frac{\epsilon_{k}}{T} \Delta \epsilon_{F} \frac{\partial f_{0}}{\partial \epsilon_{k}} \text{ for } v_{x} > 0, \]

\[ f_{1}(k, r) = 0 \text{ for } v_{x} < 0, \]

where \( \Delta \epsilon_{F} = eV_{sd} \) is the electrochemical potential difference between the source and drain with \( V_{sd} \) being the small bias voltage. These solutions become identical with the solutions in Eq. (5), if one identifies the scattering length \( v_{x} \tau = L, \Delta \epsilon_{F}/L = -eE_{x} \) and \( \Delta T/L = -\nabla_{x}T \).

Thus, the linear order (in the bias field) Magnus Hall conductivity and the Magnus Nernst conductivity in the ballistic regime can be expressed as

\[ \sigma_{m} = -\frac{e^{2}U}{h} \sum_{\lambda} \int \frac{d^{2}k}{(2\pi)^{2}} \Omega_{\lambda}^{2} v_{x}^{\lambda} \frac{\partial f_{0}}{\partial \epsilon_{k}}, \]  

\[ \alpha_{m} = \frac{e}{hT} \Delta U \sum_{\lambda} \int \frac{d^{2}k}{(2\pi)^{2}} \Omega_{\lambda}^{2} v_{x}^{\lambda} \frac{\partial f_{0}}{\partial \epsilon_{k}}. \]

As mentioned earlier, the Magnus responses are dependent on the built-in electric field.

B. Derivation of Magnus spin conductivities

Now we extend the above theory to calculate the spin counterparts of the Magnus conductivities. The general definition of spin current for a 2D system can be written as

\[ J_{ij} = \frac{\hbar}{2} \sum_{\lambda} \int \frac{d^{2}k}{(2\pi)^{2}} \langle \lambda, k|\hat{v}_{ij}|x, \lambda\rangle f_{n}, \]

where \( \hat{v}_{ij} = \hat{v}_{i,j} + \hat{v}_{m,i,j} \) with \( \hat{v}_{i,j} \), \( \hat{v}_{a,i,j} \), and \( \hat{v}_{m,i,j} \) being the band component, the anomalous component and the Magnus component of spin velocity operator, respectively. The first index \( i \) denotes the propagation direction and second index \( j \) denotes the polarization direction. By separating out the Magnus contribution, the Magnus spin current of first order (in the bias field) can be written as

\[ J_{m,i,j} = \frac{\hbar}{2} \sum_{\lambda} \int \frac{d^{2}k}{(2\pi)^{2}} \langle \lambda, k|\hat{v}_{m,i,j}|x, \lambda\rangle f_{1}. \]

From Eq. (14), one can easily find out the Magnus spin conductivity. In the diffusive regime, the electric field (\( \hat{x} \) directed) driven Magnus spin conductivity (\( \sigma_{m,yj} = J_{m,yj}/E_{x} \)) of first order with propagation and polarization in \( \hat{y} \) and \( \hat{j} \) directions, respectively, can be written as

\[ \sigma_{m,yj} = \frac{\hbar e}{2} \sum_{\lambda} \int \frac{d^{2}k}{(2\pi)^{2}} \Omega_{\lambda}^{2} v_{m,yj}^{\lambda} \frac{\partial f_{0}}{\partial \epsilon_{k}}, \]

\[ = \frac{e}{2L} \Delta U \sum_{\lambda} \int \frac{d^{2}k}{(2\pi)^{2}} \Omega_{\lambda}^{2} v_{m,yj}^{\lambda} \frac{\partial f_{0}}{\partial \epsilon_{k}}, \]

where \( v_{m,yj}^{\lambda} = \langle \lambda, k|\hat{v}_{m,yj}|x, \lambda\rangle = \frac{1}{2} \langle \lambda, k|\hat{v}_{m,y}\sigma_{j} + \sigma_{j}\hat{v}_{m,y}|x, \lambda\rangle \)+ \( \alpha_{m,yj} = J_{m,yj}/(-\nabla_{x}T) \) propagating in \( \hat{y} \) direction with polarization in \( \hat{j} \) direction can be obtained as

\[ \alpha_{m,yj} = -\frac{\Delta U}{2L} \sum_{\lambda} \int \frac{d^{2}k}{(2\pi)^{2}} \Omega_{\lambda}^{2} v_{m,yj}^{\lambda} \frac{\partial f_{0}}{\partial \epsilon_{k}}. \]
For the ballistic regime, the Eq. (15) and Eq. (16) becomes
\[ \sigma_{m,yz} = \frac{e\Delta U}{2} \sum_{\lambda} \int \frac{d^2 k}{(2\pi)^2} \lambda_2^{\lambda} \frac{\partial f_0}{\partial k}, \]
\[ \alpha_{m,yz} = -\frac{\Delta U}{2T} \sum_{\lambda} \int \frac{d^2 k}{(2\pi)^2} \lambda_2^{\lambda} \frac{\partial f_0}{\partial k}, \]
where \( v_{\lambda}^{x,y} \) is replaced by \( L \). Here, it is worth noting that \( \sigma_{m,yz} \) and \( \alpha_{m,yz} \) denote the Magnus spin Hall conductivity and Magnus spin Nernst conductivity, respectively (as \( \sigma \) of the system is given by two-dimensional electron gas (2DEG) system with the torsion radiation [42].

III. 2D GAPPED RASHBA SYSTEM

Here we provide the basic information of a gapped two-dimensional electron gas (2DEG) system with the Rashba spin-orbit interaction (RSOI). The Hamiltonian of the system is given by
\[ H = \frac{\hbar^2}{2m^*} \sigma_0 + \alpha \sigma \cdot (k \times \hat{z}) + M \sigma_z, \]
where \( m^* \) denotes the effective mass of a charge carrier with \( k = \{ \cos \phi, k \sin \phi \} \) being the wavevector of the charge carrier, \( \sigma_0 = 2 \times 2 \) identity matrix, \( \sigma_{x,y,z} \) are the Pauli’s spin matrices and \( \alpha \) is the RSOI strength which is responsible for inversion symmetry (IS) breaking of the system. The Zeeman like term \( M \sigma_z \) breaks time reversal symmetry (TRS) of the system and can be created either by application of an external magnetic field [11] or by applying a circularly polarized electromagnetic radiation [12].

The energy spectrum of the system is obtained as
\[ \epsilon_\lambda(k) = \frac{\hbar^2 k_z^2}{2m^*} + \alpha \sqrt{M^2 + \alpha^2 k^2}, \]
with \( \lambda = \pm \) denoting the band indices. The corresponding normalized wavefunctions are
\[ |+, k\rangle = \begin{bmatrix} \cos \frac{\theta}{2} e^{-i \phi} \\ -i \sin \frac{\theta}{2} \end{bmatrix}, \quad |-, k\rangle = \begin{bmatrix} \sin \frac{\theta}{2} e^{-i \phi} \\ i \cos \frac{\theta}{2} \end{bmatrix}, \]
with \( \cos \theta = \frac{M}{\sqrt{M^2 + \alpha^2 k^2}} \) and \( \sin \theta = \frac{\alpha k}{\sqrt{M^2 + \alpha^2 k^2}} \). The TRS breaking term \( M \sigma_z \) creates a finite gap \( 2M \) at \( k = 0 \). The spin orientation of an electron for this system is obtained as \( \lambda_2^{\lambda} = \lambda \{ \sigma \mid \lambda, k \rangle = \lambda \{ \sin \theta \sin \phi, -\sin \theta \cos \phi, \cos \theta \} \).

The wavevectors corresponding to regime (i) \( (\epsilon > M) \), see Fig. 2, are \( k_3 = k_0 \sqrt{(E - \lambda)^2 - M^2} \), with \( \hat{E} = \sqrt{1 + \epsilon + M^2}, \ \epsilon = \epsilon/\epsilon_\alpha, k_0 = m^* \alpha/\hbar^2 \). The wavevectors \( k_{\pm} \) represent the radii of the two concentric circular constant energy contours of convex shape. For regime (iii) \( (\epsilon < -M \) and \( M < 2\epsilon_\alpha) \), the topology of the Fermi surface has convex-concave shape on the outer and inner branches, respectively. Here, the wavevectors are presented by \( k_\nu = k_0 \sqrt{[1 + (-1)^{\nu - 1} E]^2 - M^2} \), where \( \nu = 1, 2 \) (\( \nu = 1 \rightarrow \) outer branch and \( \nu = 2 \rightarrow \) inner branch). For the regime regime (ii) \( (-M \leq \epsilon \leq M) \), only the branch \( \nu = 1 \) with \( \lambda = -1 \) exists.

For regime (i), the velocity component \( v_x \) corresponding to band \( \lambda \) is obtained as \( v^x_\lambda = \frac{\hbar k_x}{m^*} \hat{E} \left[ 1 - \frac{\hbar^2}{m^*} (\frac{\lambda}{E - \lambda})^2 \right]^{1/2} \cos \phi \). This yields the limit of \( \phi \) integration for calculating different Magnus conductivities in ballistic regime (see Eqs. (11-12) and Eqs. (17-18)). In the regime (iii), \( v_x \) can be obtained from the same equation with \( \lambda = -1 \) and \( \hat{E} \) replaced by \( (-1)^{\nu - 1} \hat{E} \). For the regime (ii), \( v_x \) has the similar form with \( \nu = 1 \).

The IS and TRS breaking terms give rise to a non zero Berry curvature in the system. The Berry curvature corresponding to band \( \lambda \) is given by
\[ \Omega_\lambda(k) = -\lambda \frac{M \alpha^2 \hat{z}}{2(M^2 + \alpha^2 k^2)^{3/2}}, \]
which is isotropic in nature. It peaks at \( k = 0 \) and decays with increasing \( k \).

IV. RESULTS AND DISCUSSION

This section presents the results of different Magnus conductivities and their spin counterparts along with corresponding discussions. The plots for different conductiv-
ties are presented as a function of scaled Fermi energy $\tilde{\epsilon}_F$ for different values of $M$. Temperature is fixed at $T = 1$ K.

A. Electric field driven Magnus conductivity and Magnus spin conductivities

We calculate the Magnus Hall conductivity using Eq. \[11\]. At zero temperature the Magnus Hall conductivity in ballistic regime has the following forms

$$\sigma_m = -\sigma_0 \frac{4\tilde{M}}{({\tilde{\epsilon}_F + M^2})^2} \text{ for regimes (i) and (iii),}$$

$$\sigma_m = \sigma_0 \frac{\tilde{M}}{E_F(E_F + 1)^2} \text{ for regime (ii),}$$

where $\tilde{E}_F = \sqrt{1 + \tilde{\epsilon}_F + M^2}$ with $\tilde{\epsilon}_F = \epsilon_F/\epsilon_{c_a}$ and $\sigma_0 = \frac{e^2}{\hbar} \frac{\Delta U}{16\pi \epsilon_{c_a}}$. For the regimes (i) ($\epsilon_F > M$) and (iii) ($\epsilon_F < -M$), the Magnus Hall conductivities have identical expressions, which is different from that of regime (ii) ($-M < \epsilon_F < M$). For regime (i), the band $\lambda = \pm$ has contribution $-\sigma_0 \frac{\tilde{M}}{E_F(E_F - 1)^2}$, and $\sigma_0 \frac{\tilde{M}}{E_F(E_F + 1)^2}$, respectively, whereas for regime (iii), the band $\nu = 1, 2$ contribute $\sigma_0 \frac{\tilde{M}}{E_F(E_F + 1)^2}$, and $-\sigma_0 \frac{\tilde{M}}{E_F(E_F - 1)^2}$, respectively. For regime (i), the opposite signs in the band contributions appear because of the opposite signs of Berry curvature of the two bands. However, for regime (iii), the opposite signs appear due to the different Fermi surface topology for the branches (here, the Berry curvatures have same signs for the two branches). Thus, summation over the band index ($\lambda$) or the branch index ($\nu$) yield the same expressions of Magnus conductivity for regime (i) and (iii). As discussed earlier, for regime (ii) only the $\nu = 1$ branch exists.

It is to be noted that $\sigma_m$ is discontinuous at $\epsilon_F = \pm M$. The reason is discussed below. The expression of $\sigma_m$ (see Eq. \[11\]) can be re written as $\sigma_m = \frac{e^2}{\hbar} \frac{\Delta U}{16\pi \epsilon_{c_a}} \sum_\lambda \int_{-\infty}^{\infty} d\epsilon_k \mathcal{H}_\lambda(\epsilon_k) \frac{\partial \epsilon_k}{\partial \epsilon} \approx -\frac{e^2}{\hbar} \frac{\Delta U}{16\pi \epsilon_{c_a}} \sum_\lambda \mathcal{H}_\lambda(\epsilon_F)$, where $\mathcal{H}_\lambda(\epsilon) = \int_{\epsilon > 0} \epsilon \partial \epsilon \Omega^\lambda$. Thus, the continuity of the Magnus Hall conductivity depends on the continuity of $\sum_\lambda \mathcal{H}_\lambda(\epsilon_F)$. From Fig. \[2\] it can be seen that '1+' band and branch '2' of '−1' band are discontinuous at $M$ and $-M$ respectively. Since $\mathcal{H}_{+}(\epsilon_F = M) \neq 0$ and $\mathcal{H}_{-2}(\epsilon_F = -M) \neq 0$, $\sigma_m$ becomes discontinuous at $\epsilon_F = \pm M$. It is worth mentioning that the linear Hall and spin Hall currents in a 2D gapped Rashba system are continuous at $\pm M$ \[15\] \[11\], although the integrands in their expressions are discontinuous at $\pm M$. This is because the linear Hall and spin Hall currents arise from $f_0$ (as opposed to $f_1 \approx \frac{\partial f_0}{\partial \epsilon} = -\delta(\epsilon - \epsilon_F)$ in Magnus Hall) due to which integration is performed from $-\infty$ to $\epsilon_F$ (at $T \to 0$) and not just the value of integrand at $\epsilon_F$ is picked up, as was the case with Magnus Hall current. For a 2D gapped system, similar to Magnus Hall current, non linear spin currents are also discontinuous at $\pm M$ because of the presence of $f_1$ in their definition.

The Magnus Hall conductivity $\sigma_m$ (in units of $\sigma_0 = \frac{e^2}{\hbar} \frac{\Delta U}{16\pi \epsilon_{c_a}}$) as a function of scaled Fermi energy $\tilde{\epsilon}_F$ for different values of $M$ is shown in Fig. \[3\]. Figure \[3\] depicts two peaks at the gap edges, i.e. at $\tilde{\epsilon}_F = \pm 2M$, where the magnitudes of the peaks decrease with the increasing strength of $M$. Furthermore, the Magnus Hall conductivity displays a plateau when Fermi energy lies between the two gap edges, i.e. $-2M < \epsilon_F < 2M$. For regime (i) and (iii), the Magnus Hall conductivity increases with $M$, which is opposite to the nature of variation of the peak values with $M$. Since, the Berry curvature decreases with momentum, Berry curvature for regime (iii) is greater than that for regime (i) (see Fig. \[2\]). Thus, the magnitude of peak at $\tilde{\epsilon}_F = -2M$ is greater than that at $\tilde{\epsilon}_F = +2M$. Furthermore, the peaks have same signs.

Now we calculate the spin counterpart of Magnus conductivity, i.e., Magnus spin conductivity for all polarization directions. The Magnus spin conductivities in ballistic regime are calculated using Eq. \[17\], where we find $\sigma_{m,xy} \neq 0$, $\sigma_{m,yz} \neq 0$ and $\sigma_{m,xz} = 0$. The Magnus spin conductivity having polarization in $\hat{x}$ direction, i.e, $\sigma_{m,xz}$ vanishes because of the $\phi$ integration.

At zero temperature the Magnus spin conductivity having polarization in $\hat{z}$ direction for ballistic regime is obtained as

$$\sigma_{m,yz} = \frac{2\tilde{M}^2(\tilde{E}_F^2 + 3)}{({\tilde{\epsilon}_F + M^2})^3} \text{ for regimes (i) and (iii)},$$

$$\sigma_{m,yz} = \frac{\tilde{M}^2}{E_F(E_F + 1)^3} \text{ for regime (ii),}$$

where $\sigma_1 = \frac{e}{4\pi \epsilon_{c_a}}$. For $\sigma_{m,yz}$, the applied electric field, the propagation and polarization directions are mutually
FIG. 4. The Magnus spin Hall conductivity $\sigma_{m,yy}$ (in units of $\sigma_1$) as a function of scaled Fermi energy $\bar{\epsilon}_F$ for three different values of $\bar{M}$. Temperature is fixed at $T = 1$ K.

FIG. 5. The Magnus spin conductivity $\sigma_{m,yz}$ (in units of $\sigma_1$) as a function of scaled Fermi energy $\bar{\epsilon}_F$ for three different values of $\bar{M}$. Temperature is fixed at $T = 1$ K.

The Magnus spin Hall conductivity $\sigma_{m,yy}$ having polarization in $y$ direction at zero temperature for ballistic regime is obtained as

$$\sigma_{m,yy} = -\frac{\bar{M}}{\bar{E}_F} \frac{(\bar{E}_F + 1)^3[(\bar{E}_F - 1)^2 - \bar{M}^2]^{1/2} + (\bar{E}_F - 1)^3[(\bar{E}_F + 1)^2 - \bar{M}^2]^{1/2}}{(\bar{E}_F + \bar{M}^2)^3} \text{ for regimes (i) and (iii)},$$

$$\sigma_{m,yy} = -\frac{\bar{M}}{\bar{E}_F} \frac{[(\bar{E}_F + 1)^2 - \bar{M}^2]^{1/2}}{(\bar{E}_F + 1)^3} \text{ for regime (ii)},$$

with $\sigma_2 = \frac{e}{16\pi^2} \frac{\Delta U}{\epsilon_n}$.

Figure 5 shows the Magnus spin conductivity $\sigma_{m,yy}$ (in units of $\sigma_2 = \frac{e}{16\pi^2} \frac{\Delta U}{\epsilon_n}$) as a function of scaled Fermi energy $\bar{\epsilon}_F$ for different values of $\bar{M}$. As expected, $\sigma_{m,yy}$ also shows the peaks at $\bar{\epsilon}_F = \pm 2\bar{M}$. Unlike the case of $\sigma_{m,yz}$, the peaks have the same signs. As earlier, the peak magnitude decreases with $\bar{M}$ and for the regime (i) and (iii), the reverse nature is obtained.

B. Thermally driven Magnus conductivity and Magnus spin conductivities

In previous section, we have studied electric field ($\hat{e}$ directed) driven Magnus conductivities, while this sec-
creases with \( \bar{\epsilon} \) in the left panel of insets), the Magnus Nernst conductivity in the peaks for Magnus Nernst conductivity have opposite trend is obtained for the regime \((\bar{\epsilon}_F \gtrsim k_B T)\) as a function of \( \bar{\epsilon}_F \) for different values of \( M \). Temperature is fixed at \( T = 1 \text{ K} \).

First we calculate the Magnus Nernst conductivity (thermal analogue of Magnus Hall conductivity) using Eq. (12). In the limit \( \epsilon_F \gg k_B T \), the Magnus Nernst conductivity for ballistic regime has the following forms

\[
\alpha_m = -\alpha_0 \frac{16M}{(\bar{\epsilon}_F + M^2)^3} \quad \text{for regimes (i) and (iii),} \tag{26}
\]

\[
\alpha_m = \alpha_0 \frac{M(1 + 3\bar{\epsilon}_F)}{E_F^2(1 + E_F)^3} \quad \text{for regime (ii)},
\]

where \( \alpha_0 = \frac{\epsilon\Delta U\pi k_B^2 T}{96\epsilon_F^4} \). Here, it is to be noted that the above expressions are valid for the Fermi energies away from the gap edges. Here also, the expressions of Magnus Nernst conductivity for regime (i) and (iii) are same, whereas for regime (ii), it is different.

The Magnus Nernst conductivity \( \alpha_m \) (in units of \( \alpha_0 = \frac{\epsilon\Delta U\pi k_B^2 T}{96\epsilon_F^4} \)) as a function of scaled Fermi energy \( \bar{\epsilon}_F \) for different values of \( M \) is shown in Fig. [6]. Unlike the previous cases, the peaks at the gap edges are accompanied by kinks, where the peak and kink values decrease with the increasing strength of \( \bar{M} \). The kinks in Fig. [6] cannot be captured by the analytical expressions in Eq. (26).

For each value of \( M \), the plot displays nearly a plateau when Fermi energy lies inside the Zeeman gap. For regime (i) (see right panel of insets) and (iii) (see left panel of insets), the Magnus Nernst conductivity increases with \( M \). Unlike the Magnus Hall conductivity, the peaks for Magnus Nernst conductivity have opposite signs because of the term \( \epsilon'_F = (\epsilon_k - \epsilon_F) \) in Eq. (12).

Now we present the results of thermally driven Magnus spin conductivities with different polarization. Similar to the case of electric field driven Magnus spin conductivities, here also we find that Magnus spin conductivities having polarization in \( \hat{z} \) and \( \hat{y} \) directions are non zero, i.e. \( \alpha_{m,zz} \neq 0 \), \( \alpha_{m,yy} \neq 0 \), whereas the Magnus spin conductivity having polarization in \( \hat{x} \)-direction \( (\alpha_{m,zx}) \) vanishes because of the \( \phi \) integration.

In the limit \( \epsilon_F \gg k_B T \), the thermally driven (for \( \hat{x} \) directed temperature gradient) Magnus spin conductivity with polarization in \( \hat{z} \) direction and propagation in \( \hat{y} \) direction for ballistic regime is obtained as

\[
\alpha_{m,zz} = \alpha_1 \frac{8M^2(E_F^2 + 5)}{(\bar{\epsilon}_F + M^2)^4} \quad \text{for regimes (i) and (iii)}, \tag{27}
\]

\[
\alpha_{m,yy} = \alpha_1 \frac{M^2(4E_F^2 + 1)}{E_F^4(\bar{\epsilon}_F + 1)^4} \quad \text{for regime (ii)},
\]

where \( \alpha_1 = \frac{\Delta U\pi k_B^2 T}{192\epsilon_F^2} \). As earlier, these expressions are valid for the Fermi energies away from gap edges. This can be viewed as the thermal analogue of Magnus spin Hall conductivity, and hence called as Magnus spin Nernst conductivity.

Figure [7] depicts the Magnus spin Nernst conductivity \( \alpha_{m,zz} \) (in units of \( \alpha_1 = \frac{\Delta U\pi k_B^2 T}{192\epsilon_F^2} \)) as a function of \( \bar{\epsilon}_F \) for different values of \( M \), where the peaks and kinks near the gap edges have same signs, unlike the cases of Magnus spin Hall conductivity \( \sigma_{m,zz} \) and Magnus Nernst conductivity \( \alpha_m \). As expected, the behavior of \( \alpha_{m,zz} \) as a function of \( M \) is similar as earlier, i.e. the peak and kink values decrease with increasing strength of \( M \) and the opposite trend is obtained for the regime (i) (see right panel of insets) and (iii) (see left panel of insets).

Now, in the limit \( \epsilon_F \gg k_B T \) and for the Fermi energies away from the gap edges, the thermally driven Magnus spin conductivity with polarization and propagation in \( \hat{y} \) direction for ballistic regime has the following forms
\begin{align}
\alpha_{m,yy} &= \alpha_2 \frac{\tilde{M}}{\tilde{E}_F} \sum_{\eta = \pm} \frac{(-3\tilde{E}_F^3 + \eta \tilde{E}_F^2 - 5\tilde{E}_F + 4\tilde{M}^2\tilde{E}_F + \eta - \eta\tilde{M}^2)}{[(\tilde{E}_F - \eta)^2 - M^2]^{1/2}(\tilde{E}_F - \eta)^4}} \\
\alpha_{m,yy} &= \alpha_2 \frac{\tilde{M}}{\tilde{E}_F} \frac{(-3\tilde{E}_F^3 - 7\tilde{E}_F^2 - 5\tilde{E}_F + 4\tilde{M}^2\tilde{E}_F - 1 + \tilde{M}^2)}{[(\tilde{E}_F + 1)^2 - M^2]^{1/2}(\tilde{E}_F + 1)^4} \\
\end{align}

for regimes (i) and (iii),

for regime (ii),

V. CONCLUSION

In this paper, we have explored the Magnus transport in a system, where both the inversion symmetry and time reversal symmetry are broken with a finite Berry curvature. The Magnus Hall effect arises because of the Magnus velocity which appears in a self-rotating quantum electronic wave-packet moving through a crystalline material having potential energy gradient. We have studied both the electric field driven and temperature gradient driven Magnus conductivities in a gapped 2D Rashba system in ballistic regime. The spin counterparts of the Magnus conductivities have also been explored by modifying the spin current operator with inclusion of the Magnus velocity. Unlike the spin Hall conductivity which has a universal value \(\sigma_s = \frac{e}{8\pi}\) as \(M \to 0\), the Magnus spin Hall conductivity vanishes in this limit. We have found that the Magnus spin currents with spin polarization perpendicular to the applied bias (electrical/thermal) are finite while with polarization along the bias vanishes because of the \(\phi\) integration. We have studied the role of gap parameter (\(M\), induced by the time reversal symmetry breaking term) and the Fermi energy (\(\epsilon_F\)) on the behavior of different conductivities. We find that all the conductivities and their spin counterparts show peaks (and kinks for thermally driven conductivities) at the gap edges, where magnitudes of peaks (and kinks) decrease with the increasing strength of gap parameter. For Magnus spin Hall conductivity (\(z\)-polarized) and Magnus Nernst conductivity, the signs of the peaks are opposite at the two gap edges. For the regime \(\epsilon_F > M\) and \(\epsilon_F < -M\), the conductivities increases with increasing strength of \(M\). All the Magnus conductivities are nearly constant when the Fermi energy varies inside the Zeeman gap.

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