Approximating local properties by tensor network states with constant bond dimension

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Abstract

Suppose we would like to approximate all local properties of a quantum many-body state to accuracy $\delta$. In one dimension, we prove that an area law for the Renyi entanglement entropy $R_\alpha$ with index $\alpha<1$ implies a matrix product state representation with bond dimension $\text{poly}(1/\delta)$. For (at most constant-fold degenerate) ground states of one-dimensional gapped Hamiltonians, it suffices that the bond dimension is almost linear in $1/\delta$. In two dimensions, an area law for $R_\alpha(\alpha<1)$ implies a projected entangled pair state representation with bond dimension $e^{O(1/\delta)}$. In the presence of logarithmic corrections to the area law, similar results are obtained in both one and two dimensions.

1 Introduction

As a variational algorithm over matrix product states (MPS) \cite{26}, the density matrix renormalization group (DMRG) \cite{34,35} has established itself as the leading numerical method for simulating one-dimensional (1D) quantum systems. Besides the practical success, significant progress has been made in explaining the working principle of DMRG. For ground states of 1D gapped Hamiltonians, notable rigorous results include

- area laws for the entanglement entropy \cite{12,1,15};
- efficient MPS approximations to the wave function \cite{12,1,15}, i.e., there exists an MPS with bond dimension less than a polynomial in the system size such that the fidelity approaches 1;
- efficient algorithms for finding such MPS approximations \cite{21,16,5,19,2}.

Note that in 1D, an area law for the von Neumann entanglement entropy does not necessarily imply efficient MPS approximations \cite{28}.

In practice, accurate results can often be obtained using MPS with quite small bond dimension. Extreme examples are the so-called infinite DMRG \cite{23} and infinite (imaginary-)time-evolving block decimation \cite{32} algorithms, which work directly in the thermodynamic limit. It was observed that a constant bond dimension is sufficient for computing expectation values of local observables. This observation cannot be explained by \cite{12,1,15}, for the upper bounds on the bond dimension proved there become infinite in the thermodynamic limit.

Here we prove that
1. In 1D, an area law for the Renyi entanglement entropy $R_\alpha$ with index $\alpha < 1$ implies an MPS representation with bond dimension poly$(1/\delta)$ such that all local properties are approximated to accuracy $\delta$.

2. For (at most constant-fold degenerate) ground states of 1D gapped Hamiltonians, it suffices that the bond dimension is almost linear in $1/\delta$.

Similar results with even stronger upper bounds on the bond dimension were previously known \cite{18, 27} for (positive semidefinite) matrix product operator (MPO) \cite{31, 37} approximations. MPS are certainly more favorable than MPO, for the latter are more difficult to work with in both theory \cite{20} and practice.

In 2D, few positive results are known. In particular, it is an open problem whether ground states of gapped Hamiltonians always obey an area law \cite{7, 24, 10, 17}, and an area law (in its strongest formulation) does not imply efficient tensor network approximations to the wave function \cite{9, 14}. Projected entangled pair states (PEPS) \cite{29} are generalizations of MPS to higher dimensions. We prove that

3. In 2D, an area law for $R_\alpha(\alpha < 1)$ implies a PEPS representation with bond dimension $e^{O(1/\delta)}$ such that all local properties are approximated to accuracy $\delta$.

Critical ground states often obey an area law with logarithmic corrections \cite{13, 33, 22, 3, 4, 36, 11}. In the presence of such corrections, similar results are obtained:

4. In 1D, there exists an MPS approximation with bond dimension poly$(1/\delta)$.

5. In 2D, there exists a PEPS approximation with bond dimension $e^{\tilde{O}(1/\delta)}$. \footnote{To simplify the notation, we use a tilde to hide a polylogarithmic factor, e.g., $\tilde{O}(x) := O(x \text{ poly log } x)$.}

2 Preliminaries

In this and the next sections, we restrict ourselves to 1D with open boundary conditions. Consider a chain of $n$ qudits or spin-$\frac{d-1}{2}$’s with $d = \Theta(1)$. Let $\mathcal{H}_i = \mathbb{C}^d$ be the Hilbert space of qudit $i$, and define

$$\mathcal{H}_{[i,j]} = \bigotimes_{k = \max\{i,1\}}^{\min\{j,n\}} \mathcal{H}_k$$

as the Hilbert space of qudits with indices in the interval $[i, j]$.

**Definition 1** (matrix product states \cite{25, 8}). Let $\{|j_i\rangle\}_{j_i=1}^{d}$ be the computational basis of $\mathcal{H}_i$ and $\{D_i\}_{i=0}^{n}$ with $D_0 = D_n = 1$ be a sequence of positive integers. An MPS $|\psi\rangle$ has the form

$$|\psi\rangle = \sum_{j_1,j_2,\ldots,j_n=1} A_{j_1}^{[1]} A_{j_2}^{[2]} \cdots A_{j_n}^{[n]} |j_1,j_2,\ldots,j_n\rangle,$$

where $A_{j_i}^{[i]}$ is a matrix of size $D_{i-1} \times D_i$. Terminologies: $D_i$ is called the bond dimension across the cut $\mathcal{H}_{[i]} \otimes \mathcal{H}_{[i+1,n]}$, and $D := \max_{0 \leq i \leq n} D_i$ is the bond dimension of the MPS $|\psi\rangle$.

Let $\lfloor \cdot \rfloor$ denote the floor function. Any state can be expressed exactly as an MPS with exponential bond dimension $D \leq d^{n/2}$. \footnotemark
Lemma 1 (canonical form [25]). Any MPS can be transformed into the so-called canonical form without increasing the bond dimension across any cut such that

$$
\sum_{j=1}^{d} A_j^{[i]} A_j^{[i] \dagger} = I, \quad \sum_{j=1}^{d} A_j^{[i]} A_j^{[i-1] \dagger} A_j^{[i]} = \Lambda^{[i]}, \quad \Lambda^{[i]} = \text{diag} \left\{ \left( \lambda_1^{[i]} \right)^2, \left( \lambda_2^{[i]} \right)^2, \ldots \right\},
$$

where $\lambda_1^{[i]} \geq \lambda_2^{[i]} \geq \cdots > 0$ are the Schmidt coefficients of the MPS across the cut $\mathcal{H}_{[1,i]} \otimes \mathcal{H}_{[i+1,n]}$ in non-ascending order.

Definition 2 (entanglement entropy). The Renyi entanglement entropy $R_\alpha$ with index $\alpha \in (0,1) \cup \{1,\infty\}$ of a bipartite pure state $\rho_{AB} = |\psi\rangle\langle\psi|$ is defined as

$$
R_\alpha(\rho_A) = \frac{1}{1-\alpha} \ln \text{tr} \rho_A^\alpha,
$$

where $\rho_A = \text{tr}_B \rho_{AB}$ is the reduced density matrix. The von Neumann entanglement entropy is

$$
S(\rho_A) = -\text{tr}(\rho_A \ln \rho_A) = \lim_{\alpha \to 1} R_\alpha(\rho_A).
$$

It is well known (and not difficult to prove) that $R_\alpha$ is monotonically non-increasing with respect to $\alpha$, i.e., $R_\alpha(\rho_A) \geq R_\beta(\rho_A)$ if $\alpha \leq \beta$.

Fix a cut, and let $\lambda_1 \geq \lambda_2 \geq \cdots > 0$ with $\sum_j \lambda_j^2 = 1$ be the Schmidt coefficients of $|\psi\rangle$ across the cut. The “truncation error” can be upper bounded as follows.

Lemma 2 ([30]). Let $R_\alpha(\alpha < 1)$ be the Renyi entanglement entropy of $|\psi\rangle$ across the cut. Then,

$$
\sum_{j \geq D+1} \lambda_j^2 \leq e^{\frac{1}{2\alpha} (R_\alpha - \ln D)}.
$$

Lemma 3 ([15]). Suppose that $|\psi\rangle$ is the (at most constant-fold degenerate) ground state of a 1D Hamiltonian $H = \sum_{i=1}^{n-1} H_i$ with a constant energy gap $\Delta = \Theta(1)$, where $H_i$ acts on $\mathcal{H}_{[i,i+1]}$ and satisfies $\|H_i\| \leq 1$. Then, $\sum_{j \geq D+1} \lambda_j^2 \leq \epsilon$ for

$$
D = e^{O \left( \frac{1}{2} + \frac{1}{2} \log^3 \frac{1}{\epsilon} \right)}.
$$

As remarked in [15], it is not necessary to assume exact ground-state degeneracy. In particular, the result remains valid in the presence of an exponentially small $e^{-\Omega(n)}$ splitting of the degeneracy (as is typically observed in physical systems).

3 Results in one dimension

Lemma 4. For any positive integer $D'$, define

$$
\epsilon_{D'} = \max_{1 \leq i \leq n-1} \sum_{j \geq D'+1} \left( \lambda_j^{[i]} \right)^2,
$$

where $\lambda_1^{[i]} \geq \lambda_2^{[i]} \geq \cdots > 0$ are the Schmidt coefficients of $|\psi\rangle$ across the cut $\mathcal{H}_{[1,i]} \otimes \mathcal{H}_{[i+1,n]}$. Then, there exists an MPS $|\phi\rangle$ with bond dimension

$$
D = O \left( D'^3 \frac{\log^2 D'}{\epsilon_{D'}} \right)
$$

3
such that
\[ \left| \langle \psi | \hat{O} | \psi \rangle - \langle \phi | \hat{O} | \phi \rangle \right| \leq O \left( \sqrt[3]{\epsilon_D} \log D' \right) \]  
(10)

for any local observable \( \hat{O} \) with \( \| \hat{O} \| \leq 1 \).

**Proof.** Schuch and Verstraete [27] proved a similar result, but only for MPO approximations. We loosely follow their approach with additional technical ingredients. We express \( |\psi\rangle \) exactly as an MPS (2) with bond dimension \( \leq d^{[n/2]} \) in the canonical form. Let \( P = \sum_{j=1}^{D'} |j\rangle \langle j| \) and

\[ |u_i\rangle = \sum_{j_1, j_2, \ldots, j_n = 1}^{d} A_{j_1}^{[1]} A_{j_2}^{[2]} \cdots A_{j_n}^{[n]} \left( \prod_{j=m+1}^{j+2} P A_{j+3}^{[j+1]} P \cdots P A_{j+m+1}^{[j+m]} P A_{j+m+2}^{[j+m+2]} \cdots A_{j_n}^{[n]} \right) |j_1 j_2 \cdots j_n\rangle \]

(11)

for \( i = 1 - m, 2 - m, \ldots, n - 1 \) be the state obtained by truncating every bond from qudit max\( \{i, 1\} \) to qudit min\( \{i + m + 1, n\} \). A minor modification of the proof of Lemma 1 in [30] leads to

**Lemma 5.**
\[ \| |\psi\rangle - |u_i\rangle \|^2 \leq 2 \sum_{k = \max\{i, 1\}}^{\min\{i+m,n-1\}} \sum_{j \geq D'+1} \left( \lambda_j^{[k]} \right)^2 \leq 2(m + 1) \epsilon D'. \]

(12)

This lemma implies that
\[ \sup_{\| \hat{O} \| \leq 1} \left| \langle \psi | \hat{O} | \psi \rangle - \langle u_i | \hat{O} | u_i \rangle \right| \leq O(\sqrt{m \epsilon D'}). \]

(13)

To simplify the notation, let \( b := \lceil \log_d D' \rceil \) with \( \lceil \cdot \rceil \) the ceiling function. Since the Schmidt ranks of \( |u_i\rangle \) across the cuts \( \mathcal{H}_{[1,i]} \otimes \mathcal{H}_{[i+1,n]} \) and \( \mathcal{H}_{[1,i+m]} \otimes \mathcal{H}_{[i+m+1,n]} \) are \( \leq D' \) by construction, there exist isometries

\[ U_i^{\text{left}} : \mathcal{H}_{[i-b+1,i]} \rightarrow \mathcal{H}_{[1,i]}, \quad U_i^{\text{right}} : \mathcal{H}_{[i+m+1,i+m+b]} \rightarrow \mathcal{H}_{[i+m+1,n]} \]

(14)

such that \( |u_i\rangle \) can be written as
\[ |u_i\rangle = U_i^{\text{left}} \otimes U_i^{\text{right}} |v_i\rangle, \]

(15)

where \( |v_i\rangle \) is some state in \( \mathcal{H}_{[i-b+1,i+m+b]} \). Let \( M := m + 2b \). For \( i = 1 - m, 2 - m, \ldots, 2b \), define

\[ |w_i\rangle = \bigotimes_{k=0}^{b-1} V_{i+kM} |v_{i+kM}\rangle, \]

(16)

where \( V_{i+kM} \) is some unitary on \( \mathcal{H}_{[i-b+1+kM,i+kM]} \). For any local observable \( \hat{O} \) on \( \mathcal{H}_{[i+1+kM,i+m+kM]} \), it is easy to see that
\[ \langle w_i | \hat{O} | w_i \rangle = \langle v_{i+kM} | \hat{O} | v_{i+kM} \rangle = \langle u_{i+kM} | \hat{O} | u_{i+kM} \rangle \quad \Rightarrow \quad \left| \langle \psi | \hat{O} | \psi \rangle - \langle w_i | \hat{O} | w_i \rangle \right| \leq O(\sqrt{m \epsilon D'}). \]

(17)

Suppose that \( i \neq j \). If every \( V_{i+kM} \) were a Haar-random unitary, then
\[ \sup_{\| \hat{O} \| \leq 1} \frac{\mathbb{E}}{\| V_{i+kM} \| \leq 1} \left| \langle w_i | \hat{O} | w_j \rangle \right|^2 \leq \sup_{\| W \| \leq 1} \frac{\mathbb{E}}{\| V_{i+kM} \| \leq 1} \left| \langle w_i | W \rangle \right|^2 \leq \sup_{\| W \| \leq 1} \langle W | \left( \mathbb{E} \left( V_{i+kM} | w_i \rangle \langle w_i | \right) \right) W \rangle \leq e^{-\Omega(nb/M)} =: t. \]

(18)
Markov’s inequality implies that
\[
\Pr \left( |\langle w_i|\hat{O}|w_j \rangle|^2 \geq \sqrt{t} \right) \leq \sqrt{t}.
\] (19)

Since the total number of independent local operators is \(O(n)\), the union bound implies the existence of a particular set of unitaries \(\{V_{i+kM}\}\) such that
\[
|\langle w_i|\hat{O}|w_j \rangle| \leq O \left( \sqrt{t} \right) \tag{20}
\]
for any local operator \(\hat{O}\). Define
\[
|\phi\rangle = \frac{1}{\sqrt{M}} \sum_{i=1-m}^{2b} |w_i\rangle
\] (21)
so that
\[
\langle \psi|\hat{O}|\psi \rangle - \langle \phi|\hat{O}|\phi \rangle \leq O(\sqrt{m\epsilon_D'}) + O(b/M) \leq O \left( \frac{3}{\sqrt{b\epsilon_D'}} \right) = O \left( \frac{3}{\sqrt{b\epsilon_D'} \log D'} \right), \tag{23}
\]
where we set \(m = \lceil \sqrt{b^2/\epsilon_D'} \rceil\). By construction, each \(|w_i\rangle\) is an MPS with bond dimension \(\leq D'\).

The bond dimension of \(|\phi\rangle\) is
\[
D \leq D'M = O \left( D' \sqrt{\frac{3b^2}{\epsilon_D'}} \right) = O \left( D' \sqrt{\frac{3\log^2 D'}{\epsilon_D'}} \right). \tag{24}
\]

**Theorem 1.** Suppose that \(|\psi\rangle\) obeys an area law in the sense that the Renyi entanglement entropy \(R_\alpha(\alpha < 1)\) across any cut is \(O(1)\). Then, there exists an MPS \(|\phi\rangle\) with bond dimension
\[
D = \tilde{O} \left( \delta^{-\frac{3\alpha}{1-\alpha}} \right) \tag{25}
\]
such that
\[
|\langle \psi|\hat{O}|\psi \rangle - \langle \phi|\hat{O}|\phi \rangle| \leq \delta \text{ for any local observable } \hat{O} \text{ with } \|\hat{O}\| \leq 1.
\]

**Proof.** This is a consequence of Lemmas 2 and 4. Lemma 2 implies that
\[
\epsilon_D' = O \left( D'^{-\frac{1-\alpha}{\alpha}} \right). \tag{26}
\]

Lemma 4 implies that
\[
\delta = O \left( \sqrt{\epsilon_D' \log D'} \right) = O \left( D'^{-\frac{1-\alpha}{\alpha}} \frac{3}{\sqrt{\log D'}} \right) \Rightarrow D' = \tilde{O} \left( \delta^{-\frac{3\alpha}{1-\alpha}} \right). \tag{27}
\]

Finally,
\[
D = O \left( D' \sqrt{\frac{\log^2 D'}{\epsilon_D'}} \right) \Rightarrow D\delta = O(D' \log D') \Rightarrow D = \tilde{O} \left( \delta^{-\frac{3\alpha}{1-\alpha}} \right). \tag{28}
\]

\[\square\]
Theorem 2. Suppose that $|\psi\rangle$ is the (at most constant-fold degenerate) ground state of a 1D Hamiltonian with a constant energy gap $\Delta$. Then, there exists an MPS $|\phi\rangle$ with bond dimension

$$D = e^{\tilde{O}(\sqrt[4]{\frac{1}{\Delta} \log^3 \frac{1}{\epsilon_{D'}}})}/\delta = O(\delta^{-1-\gamma}), \quad \forall \gamma > 0$$

such that $|\langle \psi | \hat{O} | \psi \rangle - \langle \phi | \hat{O} | \phi \rangle| \leq \delta$ for any local observable $\hat{O}$ with $\|\hat{O}\| \leq 1$.

Proof. This is a consequence of Lemmas 3 and 4. Lemma 3 implies that

$$\log D' = \tilde{O}(\frac{1}{\Delta} + \sqrt[4]{\frac{1}{\Delta} \log^3 \frac{1}{\epsilon_{D'}}})$$

Lemma 4 implies that

$$\delta = O\left(\sqrt[3]{\epsilon_{D'} \log D'}\right) = \tilde{O}(\delta^2) \implies D' = e^{\tilde{O}\left(\frac{1}{\Delta} + \sqrt[4]{\frac{1}{\Delta} \log^3 \frac{1}{\epsilon_{D'}}}\right)}.$$

Finally,

$$D = O\left(D' \sqrt{\frac{\log^2 D'}{\epsilon_{D'}}}\right) \implies D\delta = O(D' \log D') \implies D = e^{\tilde{O}\left(\frac{1}{\Delta} \log^3 \frac{1}{\epsilon_{D'}}\right)}/\delta.$$

4 Extension to two dimensions

It is straightforward to extend Theorem 1 to 2D. Consider a square lattice of $n \times n$ qudits.

Theorem 3. Suppose that $|\psi\rangle$ obeys an area law in the sense that the Renyi entanglement entropy $R_\alpha(\alpha < 1)$ between any rectangular region $X$ and its complement $X^c$ is $O(|\partial X|)$, where $\partial X$ is the boundary of $X$. Then, there exists a PEPS $|\phi\rangle$ with bond dimension

$$D = e^{O(1/\delta)}$$

such that $|\langle \psi | \hat{O} | \psi \rangle - \langle \phi | \hat{O} | \phi \rangle| \leq \delta$ for any local observable $\hat{O}$ with $\|\hat{O}\| \leq 1$.

Proof. Please refer to Fig. 1 for an illustration. Consider a region $L$ (inner dashed square) of $m \times m$ qudits in a slightly larger region $R$ (outer dashed rectangle) of $(m + b) \times m$ qudits. Let $(i, j)$ be the coordinate of the qudit in the lower left corner of $L$ (or $R$). We express $|\psi\rangle$ exactly as an MPS with bond dimension $d^\lfloor n^2/2 \rfloor$, where the qudits are ordered as follows. Starting from the qudit $(i, j)$, we follow the thick line until the first qudit outside $L$. The remaining $n^2 - m^2 - 1$ qudits outside $L$ are ordered arbitrarily. Let $|u_{i,j}\rangle$ be the state obtained by truncating the first $m^2$ bonds (marked by the thick line) to dimension $D'$. Due to Lemma 2 and the area law for the Renyi entanglement entropy $R_\alpha(\alpha < 1)$, we have

$$\sup_{\|\hat{O}\| \leq 1} \left| \langle \psi | \hat{O} | \psi \rangle - \langle u_{i,j} | \hat{O} | u_{i,j} \rangle \right| \leq \delta/2$$

for

$$D' = e^{O(m) \text{ poly}(1/\delta)}.$$
Figure 1: An illustration of the construction of the state $|u_{i,j}\rangle$ in the proof of Theorem 3, pretending that $m = 5$ and $b = 2$. The grid is the lattice. The inner dashed square and outer dashed rectangle are regions $L$ and $R$, respectively. The numbers along the thick line show the order of the qudits. The coordinate of the qudit labeled “1” is $(i,j)$.

Let $\mathcal{H}_X = \mathbb{C}^{d^{|X|}}$ be the Hilbert space of qudits in region $X$. There exists an isometry $U_{i,j} : \mathcal{H}_{R\setminus L} \to \mathcal{H}_L$ such that $|u_{i,j}\rangle = U_{i,j}|v_{i,j}\rangle$ for some state $|v_{i,j}\rangle \in \mathcal{H}_R$, provided that
\begin{equation}
    b = \left\lceil \frac{\log_d D'}{m} \right\rceil = O \left( 1 + \frac{1}{m} \log \frac{1}{\delta} \right).
\end{equation}

Since $U_{i,j}$ does not touch $L$, we have $\langle u_{i,j} | \hat{O} | u_{i,j} \rangle = \langle v_{i,j} | \hat{O} | v_{i,j} \rangle$ for any local observable $\hat{O}$ on $\mathcal{H}_L$. Let $V_{i,j}$ be some unitary on $\mathcal{H}_{R\setminus L}$. Assume without loss of generality that $n$ is a multiple of $m$ and $m + b$. Using periodic boundary conditions, we define
\begin{equation}
    |\phi\rangle = \frac{1}{\sqrt{(m+b)m}} \sum_{i=1}^{m+b} \sum_{j=1}^{m} |w_{i,j}\rangle, \quad |w_{i,j}\rangle = \sum_{k=0}^{n/m-1} \sum_{l=0}^{n/m-1} V_{i+k(m+b),j+lm} |v_{i+k(m+b),j+lm}\rangle.
\end{equation}

Similar to (23), there exists a particular set of unitaries $\{V_{i,j}\}$ such that
\begin{equation}
    |\langle \psi | \hat{O} | \psi \rangle - \langle \phi | \hat{O} | \phi \rangle| \leq \delta/2 + O(b/m) \leq \delta
\end{equation}
for any local observable $\hat{O}$, where we set $m = O(b/\delta) = O(1/\delta)$. By construction, each $|w_{i,j}\rangle$ is an MPS and thus a PEPS with bond dimension $e^{O(m)} \text{poly}(1/\delta)$. Therefore, $|\phi\rangle$ is a PEPS with bond dimension
\begin{equation}
    D = e^{O(m)} \text{poly}(1/\delta)(m + b)m = e^{O(1/\delta)}.
\end{equation}

5 Area law with logarithmic corrections

A minor modification of the proof of Theorem 3 leads to...
Corollary 1. In 2D, suppose that $|\psi\rangle$ obeys an area law with logarithmic corrections in the sense that the Renyi entanglement entropy $R_\alpha(\alpha < 1)$ between any rectangular region $X$ and its complement $\overline{X}$ is $\tilde{O}(|\partial X|)$. Then, there exists a PEPS $|\phi\rangle$ with bond dimension

$$D = e^{\tilde{O}(1/\delta)}$$

such that $|\langle \psi | \hat{O} | \psi \rangle - \langle \phi | \hat{O} | \phi \rangle| \leq \delta$ for any local observable $\hat{O}$ with $\|\hat{O}\| \leq 1$.

Corollary 2. In 1D, suppose that $|\psi\rangle$ obeys an area law with logarithmic corrections in the sense that the Renyi entanglement entropy $R_\alpha(\alpha < 1)$ between any contiguous region $X$ and its complement $\overline{X}$ is $\leq c_\alpha \ln |X| + O(1)$. Then, there exists an MPS $|\phi\rangle$ with bond dimension

$$D = \tilde{O} \left( \delta^{-1 \frac{3\alpha}{1-\alpha} - c_\alpha} \right)$$

such that $|\langle \psi | \hat{O} | \psi \rangle - \langle \phi | \hat{O} | \phi \rangle| \leq \delta$ for any local observable $\hat{O}$ with $\|\hat{O}\| \leq 1$.

Proof sketch. We reduce the proof of Theorem 3 from 2D to 1D, and set

$$m = \tilde{O}(1/\delta), \quad D' = \tilde{O} \left( \delta^{-\frac{3\alpha}{1-\alpha} - c_\alpha} \right).$$

Notes

A related paper by Dalzell and Brandão appears on arXiv simultaneously [6].

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