A lifting of an automorphism of a K3 surface over odd characteristic

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Abstract

In this paper, we prove that, over an algebraically closed field of odd characteristic, a weakly tame automorphism of a K3 surface of finite height can be lifted over the ring of Witt vectors of the base field. Also we prove that a non-symplectic tame automorphism of a supersingular K3 surface or a symplectic tame automorphism of a supersingular K3 surface of Artin-invariant at least 2 can be lifted over the ring of Witt vectors. Using these results, we prove, for a weakly tame K3 surface of finite height, there is a lifting over the ring of Witt vectors to which whole the automorphism group of the K3 surface can be lifted. Also we prove a K3 surface equipped with a purely non-symplectic automorphism of a certain order is unique up to isomorphism.

1 Introduction

For an algebraic complex K3 surface $X$, the second integral singular cohomology $H^2(X, \mathbb{Z})$ is a free abelian group of rank 22 equipped with a lattice structure induced by the cup product. As a lattice

$$H^2(X, \mathbb{Z}) = U^3 \oplus E_8,$$

here $U$ is a unimodular hyperbolic lattice of rank 2 and $E_8$ is a negative definite unimodular root lattice of rank 8. By the Lefschetz (1,1) theorem, the Neron-Severi group of $X$, $NS(X)$ is a primitive sublattice of $H^2(X, \mathbb{Z})$ and

$$NS(X) = H^{1,1}(X) \cap H^2(X, \mathbb{Z})$$

in $H^2(X, \mathbb{C})$. In particular the rank of $NS(X)$ is at most 20. We say the rank of $NS(X)$ is the Picard number of $X$ and it is denoted by $\rho(X)$. $NS(X)$ is an even integral lattice of signature $(1, \rho(X) - 1)$. We say the orthogonal complement of the embedding

$$NS(X) \hookrightarrow H^2(X, \mathbb{Z})$$

the transcendental lattice of $X$ and we denote the transcendental lattice of $X$ by $T(X)$. $T(X)$ is an integral lattice of signature $(2, 20 - \rho(X))$. By the Hodge decomposition, $H^0(X, \Omega^2_{X/\mathbb{C}})$ is a direct factor of $T(X) \otimes \mathbb{C}$ and there exists a projection

$$T(X) \otimes \mathbb{C} \to H^0(X, \Omega^2_{X/\mathbb{C}}).$$
By the Torelli theorem for complex K3 surfaces, an isometry $\psi \in O(H^2(X,\mathbb{Z}))$ is induced by an automorphism of $X$ if and only if $\psi$ preserves the line of holomorphic 2 forms $H^0(X,\Omega^2_{X/\mathbb{C}})$ in $H^2(X,\mathbb{Z}) \otimes \mathbb{C}$ and the ample cone inside $NS(X) \otimes \mathbb{R}$. Let

$$\chi_X : \text{Aut}(X) \rightarrow O(T(X))$$

and

$$\rho_X : \text{Aut}(X) \rightarrow \text{Gl}(H^0(X,\Omega^2_{X/\mathbb{C}}))$$

be the representations of the automorphism group of $X$ on the transcendental lattice and the global two forms respectively. Since $H^0(X,\Omega^2_{X/\mathbb{C}})$ is a direct factor of $T(X) \otimes \mathbb{C}$, there is a canonical projection

$$p_X : \text{Im} \chi_X \rightarrow \text{Im} \rho_X.$$ 

It is known that $p_X$ is isomorphic and $\text{Im} \chi_X$ and $\text{Im} \rho_X$ are finite cyclic groups ([21]). Assume the order of $\text{Im} \rho_X$ is $N$ and $\xi_N = \rho_X(\alpha)$ is a primitive $N$-th root of unity. Then in a natural way, $T(X)$ is a free $\mathbb{Z}[\xi_N]$-module and the rank of $T(X)$ is a multiple of $\phi(N)$ ([18]). Here $\phi$ is the Euler $\phi$-function.

An automorphism $\alpha \in \text{Aut}(X)$ is symplectic if $\rho_X(\alpha) = 1$. An automorphism $\alpha$ is purely non-symplectic if $\alpha$ is of finite order greater than 1 and the order of $\alpha$ is equal to the order of $\rho_X(\alpha)$.

Assume $k$ is an algebraically closed field of odd characteristic $p$. Let $W$ be the ring of Witt vectors of $k$ and $K$ be the fraction field of $W$. Assume $X$ is a K3 surface over $k$. The formal Brauer group of $X$, $\hat{Br}_X$ is a smooth one dimensional formal group over $k$ and the height of $\hat{Br}_X$ is an integer between 1 and 10 or $\infty$.

If the height of $X$ is $\infty$, we say $X$ is supersingular and it is known that $\rho(X) = 22$ ([8], [19], [20]). The discriminant group of $NS(X)$, $(NS(X))^*/NS(X)$ is $(\mathbb{Z}/p)^{2\sigma}$ for an integer $\sigma$ between 1 and 10. We call $\sigma$ the Artin-invariant of $X$. It is known that the lattice structure of $NS(X)$ is determined by the base characteristic $p$ and the Artin-invariant ([25]). All the supersingular K3 surfaces of Artin-invariant $\sigma$ form a family of $\sigma - 1$ dimension over $k$ and a supersingular K3 surface of Artin-invariant 1 is unique up to isomorphism ([24]).

If $X$ is of finite height $h$, the second crystalline cohomology has a slope decomposition ([9], [10], [14])

$$H^2_{\text{cris}}(X/W) = H^2_{\text{cris}}(X/W)_{[1-1/h]} \oplus H^2_{\text{cris}}(X/W)_{[1]} \oplus H^2_{\text{cris}}(X/W)_{[1+1/h]}.$$ 

Considering the slope spectral sequence, $H^2_{\text{cris}}(X/W)_{[1-1/h]}$ is $H^2(X,WO_X)$ which is isomorphic to the Dieudonné module of $\hat{Br}_X$. The Dieudonné module of a 1 dimensional smooth formal group of finite height $h$ can be express as

$$W[V,F]/(VF = p, \ F = V^{h-1}).$$
Here $F$ is a Frobenius linear operator and $V$ is a Frobenius inverse linear operator. It follows that $H^2_{\text{cris}}(X/W)_{[1-1/h]}$ is a free $W$-module of rank $h$. For the cup product pairing, $H^2_{\text{cris}}(X/W)_{[1-1/h]}$ and $H^2_{\text{cris}}(X/W)_{[1+1/h]}$ are dual to each other and $H^2_{\text{cris}}(X/W)_{[1]}$ is unimodular. Therefore the rank of $H^2_{\text{cris}}(X/W)_{[1]}$ is $22 - 2h$. Considering the cycle map

$$c : NS(X) \otimes W \hookrightarrow H^2_{\text{cris}}(X/W)_{[1]},$$

we have $\rho(X) \leq 22 - 2h$. We call the orthogonal complement of the embedding

$$c : NS(X) \otimes W \hookrightarrow H^2_{\text{cris}}(X/W)$$

the crystalline transcendental lattice of $X$ and it is denoted by $T_{\text{cris}}(X)$. Since

$$H^2_{\text{cris}}(X/W)_{[1-1/h]} \oplus H^2_{\text{cris}}(X/W)_{[1+1/h]}$$

is a direct factor of $T_{\text{cris}}(X)$ and there is an isomorphism $H^2(X, \mathcal{O}_X)/V \simeq H^2(X, \mathcal{O}_X)$, we have a canonical projection $T_{\text{cris}}(X) \rightarrow H^2(X, \mathcal{O}_X)$. We denote the representation of $\text{Aut}(X)$ on $T_{\text{cris}}$ by

$$\chi_{\text{cris},X} : \text{Aut}(X) \rightarrow O(T_{\text{cris}}(X)).$$

By the Serre duality, the representation of $\text{Aut}(X)$ on $H^2(X, \mathcal{O}_X)$ is isomorphic to $\rho_X$ and there is a compatible projection

$$p_{\text{cris},X} : \text{Im} \chi_{\text{cris},X} \rightarrow \text{Im} \rho_X.$$

For any $\alpha \in \text{Aut}(X)$, the characteristic polynomial of $\alpha^*|H^2_{\text{cris}}(X/W)$ has integer coefficients ([6], 3.7.3). Hence the characteristic polynomial of $\chi_{\text{cris},X}(\alpha)$ also has integer coefficients.

For a K3 surface $X$ of finite height over $k$, there is a Neron-Severi group preserving lifting $\mathfrak{X}/W$ ([22], [17], [11]). When $X_K = \mathfrak{X} \otimes \overline{K}$ is a geometric generic fiber of $\mathfrak{X}/W$, the reduction map $NS(X_K) \rightarrow NS(X)$ is isomorphic and the inclusion $\text{Aut}(X_K) \hookrightarrow \text{Aut}(X)$ is of finite index. Using this fact, we have that $\text{Im} \rho_X$ and $\text{Im} \chi_{\text{cris},X}$ are finite ([22]). Moreover If $n$ is the order of $\chi_{\text{cris},X}(\alpha)$, $\phi(n)$ is at most the rank of $T_{\text{cris}}(X)$.

When $X$ is a K3 surface of arbitrary height over $k$, an automorphism $\alpha \in \text{Aut}(X)$ is tame if $\alpha$ is of finite order and the order of $\alpha$ is not divisible by the base characteristic $p$. It is known that if $p$ is greater than 11, any automorphism of finite order of $X$ is tame ([6], Theorem 2.1.). If $X$ is of finite height, we say an automorphism $\alpha \in \text{Aut}(X)$ is weakly tame if the order of $\chi_{\text{cris},X}(\alpha)$ is not divisible by $p$. A tame automorphism is weakly tame. We say $X$ is weakly tame if the order of $\text{Im} \chi_{\text{cris},X}$ is not divisible by $p$. Since the rank of $T_{\text{cris}}(X)$ is less than 22, if $p \geq 23$, any K3 surface of finite height is weakly tame.

Let $X$ be a K3 surface over $k$. We say an automorphism $\alpha \in \text{Aut}(X)$ is liftable over $W$ if there is a scheme lifting $\mathfrak{X}/W$ of $X/k$ and a $W$-automorphism $\alpha : \mathfrak{X} \rightarrow \mathfrak{X}$ such
that the restriction of \( a \) on the special fiber \( a|X \) is equal to \( \alpha \). In this paper, we prove that the following theorem.

**Theorem 3.3.** Let \( X \) be a K3 surface over \( k \). If \( X \) is of finite height and \( \alpha \in \text{Aut}(X) \) is weakly tame, \( \alpha \) is liftable over \( W \). If \( X \) is supersingular and \( \alpha \in \text{Aut}(X) \) is nonsymplectic tame, \( \alpha \) is liftable over \( W \). If \( X \) is supersingular of Artin-invariant at least 2 and \( \alpha \in \text{Aut}(X) \) is symplectic tame, \( \alpha \) is liftable over \( W \).

Also, for a weakly tame K3 surface, there exists a Neron-Severi group preserving lifting which lifts all the automorphisms.

**Theorem 3.7.** Let \( X \) be a weakly tame K3 surface over \( k \). Then there exists a Neron-Severi group preserving lifting \( X/W \) of \( X \) such that the reduction map \( \text{Aut}(X \otimes K) \to \text{Aut}(X) \) is isomorphic.

In a previous work ([12]), we prove that, if \( k \) is an algebraic closure of a finite field, \( X \) is of finite height and \( N \) is the order of \( \text{Im} \rho_X \), the rank of \( T_{\text{cris}}(X) \) is a multiple of \( \phi(N) \). Moreover if \( X \) is weakly tame, \( p_{\text{cris},X} \) is isomorphic. Using Theorem 3.3, we can prove the same results holds over an arbitrary algebraically closed field.

**Corollary 3.5.** Let \( X \) be a K3 surface of finite height over \( k \). If \( \alpha \) is a weakly tame automorphism of \( X \) and \( \rho_X(\alpha) = \text{id} \), then \( \chi_X(\alpha) = \text{id} \). If \( X \) is weakly tame, the projection \( p_{\text{cris},X} : \text{Im} \chi_X \to \text{Im} \rho_X \) is an isomorphism.

**Corollary 3.6.** Let \( X \) be a K3 surface of finite height over \( k \). When \( N \) is the order of \( \text{Im} \rho_X \), the rank of \( T_{\text{cris}}(X) \) is a multiple of \( \phi(N) \).

Due to this result, for a weakly tame K3 surface \( X \), \( \text{Im} \chi_{\text{cris},X} \) is finite cyclic.

When \( \Sigma \) is a finite set of positive integers \( \{13, 17, 19, 25, 27, 32, 33, 40, 44, 50, 66\} \), it is known that, for \( N \in \Sigma \), there is a unique complex algebraic K3 surface \( X_N \) equipped with a purely non-symplectic automorphism of order \( N \), \( g_N \). A precise elliptic surface model of \( X_N \) is known and \( X_N \) is defined over \( \mathbb{Q} \). Moreover, if \( p \) does not divide \( 2N \), \( (X_N, g_N) \) has a good reduction over an algebraic closure of a prime field \( \mathbb{F}_p \). It is also known that if \( k \) is an algebraically closed field of characteristic \( p \neq 2, 3 \), there is a unique K3 surface over \( k \) equipped with an automorphism of order 66 ([15]). Using Theorem 3.3, we prove the uniqueness of a K3 surface equipped with a purely non-symplectic automorphism of order \( N \in \Sigma \) when \( p \) does not divides \( 2N \). This unique K3 surface is the reduction of \( X_N \) over a finite field.

**Theorem 4.3.** Assume \( N \in \Sigma \) and \( p \) does not divide \( 2N \). Then there exists a unique K3 surface equipped with a purely non-symplectic automorphism of order \( N \). This unique K3 surface has a model over a finite field.
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2 Deformation of a K3 surface

In this section we review some results on the deformation of K3 surfaces over odd characteristic. For the detail we refer to [24], [4], [5].

Let $k$ be an algebraically closed field of odd characteristic $p$. Let $W$ be the ring of Witt vectors of $k$ and $K$ be the fraction field of $W$. Assume $X$ is a K3 surface defined over $k$. The deformation space of $X$ over artin $W$-algebras is an affine smooth formal scheme of dimension 20 over $W$. Let $B = W[[t_1, \cdots, t_{20}]]$ and $S = \text{Spf } B$ be the deformation space of $X$. Let $\pi : X \to S$ be the universal family over $S$. For $A$, an artin $W$-algebra whose residue field is isomorphic to $k$, the set of isomorphic classes of deformation of $X$ over $A$ is $S(A) = \text{Hom}_{W, \text{cont}}(B, A)$. The second derham cohomology $H = H^2_{\text{dr}}(X/S)$ is a vector bundle of rank 22 on $S$. The vector bundle $H$ is equipped with the Hodge filtration

$$H = \mathcal{F}il^0 \supset \mathcal{F}il^1 \supset \mathcal{F}il^2 \supset 0$$

and the Gauss-Manin connection

$$\nabla : H \to H \otimes_B \Omega^1_{S/W}.$$

Here $\mathcal{F}il^1$ and $\mathcal{F}il^2$ are vector bundles on $S$ of rank 21 and of rank 1 respectively. The cup product gives a perfect paring $H \otimes H \to \mathcal{O}_S$. The graded module of the filtration $gr^i = \mathcal{F}il^i/\mathcal{F}il^{i+1}$ is a vector bundle and there is a natural isomorphism

$$gr^i \simeq R^{2-i} \pi_* \Omega^i_{X/S}.$$

With respect to the cup product paring,

$$(\mathcal{F}il^1)^\perp = \mathcal{F}il^2 \text{ and } (\mathcal{F}il^2)^\perp = \mathcal{F}il^1.$$

By the Griffith transversality, we have

$$\nabla(\mathcal{F}il^2) \subset \mathcal{F}il^1 \otimes \Omega^1_{S/W}.$$

This induces an $\mathcal{O}_S$-linear morphism

$$gr^2 \nabla : gr^2 \to gr^1 \otimes \Omega^1_{S/W}.$$

It is known that $gr^2 \nabla$ is an isomorphism ([4], Proposition 2.4.).
Any \( f \in S(W) \) is corresponding to a formal lifting of \( X \) over \( \text{Spf} \ W \), \( \mathcal{X}_f \to \text{Spf} \ W \). There is a canonical isomorphism

\[
\lambda_f : f^*H = H_{2\text{dr}}^2(\mathcal{X}_f/W) \simeq H_{\text{cris}}^2(X/W).
\]

Through \( \lambda_f \), the Hodge filtration on \( H_{2\text{dr}}^2(\mathcal{X}_f/W) \),

\[
H_{2\text{dr}}^2(\mathcal{X}_f/W) \supseteq f^*\text{Fil}^1 \supseteq f^*\text{Fil}^2
\]
gives a filtration on \( H_{\text{cris}}^2(X/W) \). Let \( M^2_f = \lambda_f(f^*\text{Fil}^i) \) be a submodule of \( H_{\text{cris}}^2(X/W) \).

A line bundle \( L \) on \( X \) extends on \( \mathcal{X}_f \) if and only if the crystalline cycle class of \( L \), \( c(L) \in H_{\text{cris}}^2(X/W) \) is contained in \( M^2_f \) ([24], Proposition 1.12). The rank 1 submodule \( M^2_f \subset H_{\text{cris}}^2(X/W) \) satisfies the following conditions.

1. \( M^2_f \otimes k = H^0(X, \Omega^2_{X/k}) \) through the isomorphism \( H_{\text{cris}}^2(X/W) \otimes k \simeq H_{\text{dr}}^2(X/k) \).

2. \( M^2_f \) is isotropic for the cup product pairing.

3. For the canonical Frobenius morphism \( \mathbf{F} : H_{\text{cris}}^2(X/W) \to H_{\text{cris}}^2(X/W) \), \( \mathbf{F}(M^2_f) \subset p^2H_{\text{cris}}^2(X/W) \) and \( \mathbf{F}(M^2_f) \not\subset p^3H_{\text{cris}}^2(X/W) \).

Let us fix a basis \( v_1, \ldots, v_{22} \) of \( H_{\text{cris}}^2(X/W) \) satisfying \( v_1 \in H^1_f \) and \( v_2, \ldots, v_{21} \in H^1_f \). Note that \( \mathbf{F}(v_1) \in p^2H_{\text{cris}}^2(X/W) - p^3H_{\text{cris}}^2(X/W) \), \( \mathbf{F}(v_i) \in p^2H_{\text{cris}}^2(X/W) - p^3H_{\text{cris}}^2(X/W) \) for \( 2 \leq i \leq 21 \) and \( \mathbf{F}(v_{22}) \not\in p^3H_{\text{cris}}^2(X/W) \). Since the cup product pairing is perfect and the orthogonal complement of \( H_{\text{cris}}^2 \) is \( H^1_f \), we may assume \( v_1, v_{22} = 1 \). Assume \( M \) is a submodule of \( H_{\text{cris}}^2(X/W) \) of rank 1 satisfying the condition 1 and the condition 2 above. There exists a unique element

\[
v_{M} = v_1 + \sum_{i=2}^{22} a_i v_i \in M, \ (a_i \in W).
\]

We can easily check that \( a_i \in pW \) for \( 2 \leq i \leq 21 \), \( a_{22} \in p^2W \) and \( a_{22} \) is uniquely determined by \( a_2, \ldots, a_{21} \). Since \( \mathbf{F}(M^2_f) \subset p^2H_{\text{cris}}^2(X/W) \), the condition 3 is automatically satisfied for \( M \). Let \( \mathcal{M} \) be the set of rank 1 submodules of \( H_{\text{cris}}^2(X/W) \) satisfying the condition 1 and the condition 2. The correspondence \( M \mapsto (v_2, \ldots, v_{21}) \) gives a bijection between \( \mathcal{M} \) and \( (pW)^{20} \), so we may regard \( \mathcal{M} = (pW)^{20} \). When \( g \) is another element in \( S(W) \) and \( \mathcal{X}_g/W \) is the corresponding lifting, \( M^2_g \) is an element of \( \mathcal{M} \).

Let \( \Phi : S(W) \to \mathcal{M} \) be the function \( g \mapsto M^2_g \).

\textbf{Proposition 2.1} (Local Torelli theorem). The function \( \Phi : S(W) \to \mathcal{M} \) is bijective.

\textit{Proof.} Let us fix a morphism \( f : B \to W \in S(W) \) such that \( f(t_i) = 0 \) for all \( i \). We choose \( x \), a generator of \( \text{Fil}^2 \). Let us denote the differential

\[
\nabla(d/dt_i) : H \to H
\]

by \( D_i \). Since \( \text{gr}^2 \nabla \) is isomorphic, we may choose a basis \( v_1, \ldots, v_{22} \) of \( H_{\text{cris}}^2(X/W) \) as above such that
\[ v_1 = \lambda_f (f^* x) \] and \[ v_i = \lambda_f (f^* D_i x) \] for \( 2 \leq i \leq 21 \).

Assume \( g \in \mathcal{S}(W) \) and \( g(t_i) = p a_i \in p W \). The \( \mathcal{O}_S \)-module \( H = H^2_{\text{dR}}(X/S) \) with the Gauss-Manin connection is an \( F \)-crystal in the sense of \[5\]. Since \( f \otimes k = g \otimes k \), there is an isomorphism

\[ \chi(g, f) : H^2_{\text{dR}}(X_g/W) = g^* H \simeq f^* H = H^2_{\text{dR}}(X_f/W). \]

Because the Gauss-Manin connection on \( H \) is the connection associated to \( R^2 \pi_{\text{cris}} \mathcal{O}_X \) (\[2\], Proposition V 3.6.4), the isomorphism \( \chi(f, g) \) makes the following diagram commute

\[ \begin{array}{ccc}
H^2_{\text{dR}}(X_g/W) & \xrightarrow{\chi(g, f)} & H^2_{\text{dR}}(X_f/W) \\
\downarrow \Lambda_g & & \downarrow \lambda_f \\
H^2_{\text{cris}}(X/W). & &
\end{array} \]

Precisely \( \chi(g, f) \) is given as follow (\[5\], Lemme 1.1.2.). When \( m = (m_1, \cdots, m_{20}) \in \mathbb{N}^{20} \) is a multi-index, we denote \( D^m = D_1^{m_1} \cdots D_{20}^{m_{20}} \). Note that since \( \nabla \) is an integrable connection, \( D_i D_j = D_j D_i \) for any \( i, j \). Let \( \gamma_i : p W \to W \) be the divided power given by \( \gamma_i(a) = a^i/i! \). Then

\[ \chi(g, f)(g^* y) = \sum_m \gamma_{m_1}(p a_1) \cdots \gamma_{m_{20}}(p a_{20}) (f^* D^m y) \]

for any \( y \in H \). The above summation is taken over all the multi index \( m \). We set

\[ \lambda_g (g^* x) = \lambda_f (\chi(g, f)(f^* x)) = \sum_i h_i v_i. \]

Here \( h_i \in W[[a_1, \cdots, a_{20}]] \) is a formal series in \( a_i \). In this case,

\[ h_1 = 1 + p^2 k_1 \]

and

\[ h_i = p a_i + p^2 k_i \ (2 \leq i \leq 21) \]

where all \( k_i(1 \leq i \leq 21) \) are formal series which begins at degree 2 terms. Since \( \lambda_g (g^* x) \) is a generator of \( M^2_g, \Phi(g) = h_1^{-1}(h_2, \cdots, h_{21}) \in (p W)^{20} \). By the Hensel lemma, the claim follows. \( \square \)

### 3 Lifting of an automorphism

Assume \( X \) is a K3 surface over \( k \) and \( \alpha \) is an automorphism of \( X \). Let \( X_f/W \) be the formal lifting of \( X \) over \( W \) associated to \( f \in \mathcal{S}(W) \).

**Lemma 3.1** (c.f. \[24\], Corollary 2.5.). An automorphism \( \alpha \in \text{Aut}(X) \) extends to \( X_f/W \) if and only if \( \alpha^* H^2_{\text{cris}}(X/W) \) preserves \( M^2_f \).
Proof. The only if part is trivial. We assume \( \alpha^*(M_f^2) = M_g^2 \). Let \( \mathfrak{X}_g/W \) be the pull back of the lifting \( \mathfrak{X}_f/W \) of \( X/k \) through the isomorphism \( \alpha \). Then there is a \( W \)-isomorphism \( \mathfrak{a} : \mathfrak{X}_g \to \mathfrak{X}_f \) and we have a Cartesian diagram

\[
\begin{array}{ccc}
X & \xleftarrow{a} & \mathfrak{X}_g \\
\uparrow \alpha & & \uparrow \mathfrak{a} \\
X & \xleftarrow{a} & \mathfrak{X}_f.
\end{array}
\]

Since the isomorphism \( \lambda_f \) and \( \lambda_g \) are functorial, the following diagram commutes.

\[
\begin{array}{ccc}
H^2_{\text{dR}}(\mathfrak{X}_f/W) & \xrightarrow{\mathfrak{a}^*} & H^2_{\text{dR}}(\mathfrak{X}_g/W) \\
\downarrow \lambda_f & & \downarrow \lambda_g \\
H^2_{\text{cris}}(X/W) & \xrightarrow{\alpha^*} & H^2_{\text{cris}}(X/W).
\end{array}
\]

Because \( \mathfrak{a}^*H^0(\mathfrak{X}_f, \Omega^1_{\mathfrak{X}_f/W}) = H^0(\mathfrak{X}_g, \Omega^2_{\mathfrak{X}_g/W}) \) and \( \alpha^*M_f^2 = M_g^2 \) by the assumption,

\[
M_g^2 = \lambda_g(\mathfrak{a}^*H^0(\mathfrak{X}_f, \Omega^1_{\mathfrak{X}_f/W})) = \alpha^*(\lambda_f(H^0(\mathfrak{X}_f, \Omega^1_{\mathfrak{X}_f/W}))) = M_f^2.
\]

By Proposition 2.1, \( f = g \) and the automorphism \( \alpha \) extends to an automorphism \( \mathfrak{a} \) of \( \mathfrak{X}_f \).

Remark 3.2. In the above lemma, if \( \mathfrak{X}_f \) is algebraizable then \( \alpha \) extends to the the algebraic model of \( \mathfrak{X}_f \).

Theorem 3.3. Let \( X \) be a K3 surface over \( k \). If \( X \) is of finite height and \( \alpha \in \text{Aut}(X) \) is weakly tame, \( \alpha \) is liftable over \( W \). If \( X \) is supersingular and \( \alpha \in \text{Aut}(X) \) is non-symplectic tame, \( \alpha \) is liftable over \( W \). If \( X \) is supersingular of Artin-invariant at least 2 and \( \alpha \in \text{Aut}(X) \) is symplectic tame, \( \alpha \) is liftable over \( W \).

Proof. By Proposition 2.2 and Lemma 3.1, in each case, it is enough to find \( M \in \mathcal{M} \) and an ample line bundle \( V \) of \( X \) such that \( \alpha^*M = M \) and \( M \) is orthogonal to \( c(V) \in H^2_{\text{cris}}(X/W) \).

Assume \( X \) is of finite height \( h \) and \( \alpha \) is weakly tame. We fix an \( F \)-crystal decomposition

\[
H^2_{\text{cris}}(X/W) = H^2_{\text{cris}}(X/W)_{[1-1/h]} \oplus H^2_{\text{cris}}(X/W)_{[1]} \oplus H^2_{\text{cris}}(X/W)_{[1+1/h]}
\]

and an identification

\[
H^2_{\text{cris}}(X/W)_{[1-1/h]} = W[F,V]/(FV = p, F = V^{h-1}).
\]

Let

\[
\pi : H^2_{\text{cris}}(X/W) \to H^2_{\text{cris}}(X/W) \otimes k \cong H^2_{\text{dR}}(X/k)
\]

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be the canonical projection. We denote the Hodge filtration on $H^2_{d_2}(X/k)$ by $F^\cdot H^2_{d_2}(X/k)$. Since
\[ F(H^2_{cris}(X/W)[1] \oplus H^2_{cris}(X/W)[1+1/h]) \subset pH^2_{cris}(X/W) \]
and
\[ H^2_{cris}(X/W)[1-1/h]/V \simeq H^2(X, \mathcal{O}_X), \]
we have
\[ \pi(VH^2_{cris}(X/W)[1-1/h] \oplus H^2_{cris}(X/W)[1] \oplus H^2_{cris}(X/W)[1+1/h]) = F^1H^2_{d_2}(X/k). \]

Let $v \in H^2_{cris}(X/W)[1+1/h]$ be the dual of $1 \in W[F, V]/(FV = p, F = V^{h-1})$ with respect to the base $1, V, \cdots, V^{h-1}$ of $H^2_{cris}(X/W)[1-1/h]$. Then
\[ \pi(v) \in (F^1H^2_{d_2}(X/k))^\perp = F^2H^2_{d_2}(X/k) \subset H^2_{d_2}(X/k) \]
and
\[ F^2H^2_{d_2}(X/k) \subset H^2_{cris}(X/W)[1+1/h] \otimes k. \]

**Lemma 3.4.** Let $L$ be a finite free $W$-module and $\psi : L \to L$ be an automorphism of $L$ of finite order coprime to $p$. Then there is a basis of $L$ consisting of eigenvectors for $\psi$. If $v \in L \otimes k$ is an eigenvector of $\psi | (L \otimes k)$, there is an eigenvector $\hat{v} \in L$ such that $\hat{v} \otimes k = v$.

**Proof.** Let $N$ be the order of $\psi$. Then the polynomial $t^N - 1 \in W[t]$ splits completely. Therefore when $L_\zeta$ is the eigenspace of $(L, \psi)$ for an eigenvalue $\zeta$, we have a decomposition
\[ L = \bigoplus_\zeta L_\zeta. \]
The claim follows easily. \(\square\)

Since $\alpha$ is weakly tame and $H^2_{cris}(X/W)[1+1/h]$ is a direct factor of $T_{cris}(X)$, the order of $\alpha^*|H^2_{cris}(X/W)[1+1/h]$ is not divisible by $p$. Because $F^2H^2_{d_2}(X/k)$ is one dimensional and is invariant for $\alpha^*$, it follows that, by the above lemma, there is a rank 1 $\alpha^*$-stable primitive submodule $M$ of $H^2_{cris}(X/W)[1+1/h]$ such that $\pi(M) = F^2H^2_{d_2}(X/k)$. Because $H^2_{cris}(X/W)[1+1/h]$ is isotropic for the cup product, $M$ is an element of $\mathcal{M}$. Let $f \in S(W)$ be the lifting of $X$ such that $M^2_f = M$. Then $M^2_f \perp H^2_{cris}(X/W)[1]$ and $c(NS(X)) \otimes W$ is a submodule of $H^2_{cris}(X/W)[1]$, so all the line bundles of $X$ extend to $\mathfrak{X}_f$. In particular, $\mathfrak{X}_f$ is algebraizable and $\alpha$ is liftable over $W$. Note that $\mathfrak{X}_f$ is a Neron-Severi group preserving lifting of $X$.

Now assume $X$ is supersingular and $\alpha$ is non-symplectic and tame. Let
\[ H^2_{cris}(X/W) = \bigoplus_\zeta L_\zeta \]
be the eigenspace decomposition for $\alpha^*|H^2_{cris}(X/W)$. We assume $F^2H^2_{d_2}(X/k) \subset \pi(L_0)$ for some eigenvalue $\zeta_0 \neq 1$. Then $\rho_X(\alpha) = \zeta_0$, where $\zeta_0$ is the reduction of $\zeta_0$ in $k$. If
\(\zeta_0 \neq -1\), \(L_{\zeta_0}\) is isotropic and there is a rank 1 primitive submodule \(M \subset L_{\zeta_0}\) satisfying 
\[\pi(M) = F^2H^2_{\text{dr}}(X/k).\] Then \(M\) is an element of \(\mathcal{M}\). If \(\zeta_0 = -1\), \(\rho_X(\alpha) = -1\) and by the Serre duality, \(\alpha^*(H^2_{\text{dr}}(X/k)/F^1) = -1\). We set \(l_{-1} = \pi(L_{-1})\). The pairing on \(l_{-1}\) is non-degenerate. The rank of \(L_{-1}\) is at least 2 and 
\[l_{-1} \not\subset F^1H^2_{\text{dr}}(X/k) = (F^2H^2_{\text{dr}}(X/k))^\perp.\]

Let us choose \(0 \neq x \in F^2H^2_{\text{dr}}(X/k)\) and \(y \in l_{-1}\) such that \(x \cdot y = 1\). Let \(u\) and \(v\) be liftings of \(x\) and \(y\) in \(L_{-1}\) satisfying \(u \cdot v = 1\). Since \(u \cdot u\) is divisible by \(p\), by the Hensel lemma, there is \(a \in W\) such that \(u + pav \in L_{-1}\) is isotropic. If \(M\) is a submodule of \(L_{-1}\) generated by \(u + pav\), \(M\) is an element of \(\mathcal{M}\). Let \(f \in S(W)\) be the formal lifting of \(X\) over \(W\) such that \(M = M_f^2\). Because \(\alpha\) is of finite order, there is an \(\alpha^*\)-stable ample line bundle of \(X\), \(V\). Then \(c(V) \in L_1\) and \(c(V) \perp L_{\zeta_0} \supset M\). Therefore the formal lifting \(X_f\) is algebraizable and \(\alpha\) is liftable over \(W\).

Assume \(X\) is supersingular of Artin-invariant at least 2 and \(\alpha\) is symplectic and tame. We set \(l_1 = \pi(L_1)\). The pairing on \(l_1\) is non-degenerate. By the assumption, \(F^2H^2_{\text{dr}}(X/k) \subset l_1\). Let \(V\) be a primitive \(\alpha^*\)-ample bundle. Then \(c(V) \in L_1\) and \(\pi(c(V)) \neq 0\). Let \(x\) be a non-zero element of \(F^2H^2_{\text{dr}}(X/k)\) and \(y = \pi(c(V))\). By [24], Proposition 2.2, \(x\) and \(y\) are linearly independent. We denote the kernel of \(l_1 \rightarrow H^2(X, O_X)\) by \(F^1l_1\). \(F^1l_1\) is of codimension 1 in \(l_1\). Note that \(x, y \in F^1l_1\) and \(F^1l_1\) is the orthogonal complement of \(x\) in \(l_1\). Let \(c(V)^\perp\) be the orthogonal complement of \(c(V)\) in \(L_1\). Suppose the self intersection of \(c(V)\) is not divisible by \(p\). Since \(x \cdot y = 0\),
\[F^2H^2_{\text{dr}}(X/k) \subset \pi(c(V)^\perp)\]
and
\[\pi(c(V)^\perp) \not\subset F^1H^2_{\text{dr}}(X/k).\]

Therefore the rank of \(c(V)^\perp\) is at least 2 and as above there a rank 1 submodule \(M\) of \(c(V)^\perp\) such that \(\pi(M) = F^2H^2_{\text{dr}}(X/k)\). The formal lifting corresponding to \(M\) is algebraizable and \(\alpha\) is liftable to the scheme lifting corresponding to \(M\). Suppose the self intersection of \(c(K)\) is divisible by \(p\). Then \(y\) is isotropic. Since \(x\) and \(y\) are linearly independent, there is \(y \neq z \in F^1l_1\) such that \(z \cdot y = 1\). Hence the dimension of \(F^1l_1\) is at least 3 and the rank of \(L_1\) is at least 4. Let \(v\) and \(u\) be arbitrary liftings of \(x\) and \(z\) in \(L_1\) respectively. Then \(v \cdot u\) is divisible by \(p\) and \(c(K) \cdot u\) is a unit. We choose \(w \in L_1\) such that \(v \cdot w\) is a unit. We can find \(a, b \in W\) satisfying
\[v + au, w + bu \in (c(K))^\perp.\]
Since \(v \cdot c(K)\) is divisible by \(p\), \(a \in pW\) and \((v + au) \cdot (w + bu)\) is a unit. Then inside \((c(K))^\perp\), we can find a rank 1 isotropic submodule \(M\) such that \(\pi(M) = F^2H^2_{\text{dr}}(X/k)\).

The formal lifting associated to \(M\) is algebraizable and \(\alpha\) is liftable to the scheme lifting associated to \(M\).

**Corollary 3.5.** Let \(X\) be a K3 surface of finite height over \(k\). If \(\alpha\) is a weakly tame automorphism of \(X\) and \(\rho_X(\alpha) = id\), then \(\chi_X(\alpha) = id\). If \(X\) is weakly tame, the projection \(p_{\text{cris}, X} : \text{Im} \chi_X \rightarrow \text{Im} \rho_X\) is an isomorphism. 

\[
\begin{array}{c}
\gamma_0 \neq -1, L_{\gamma_0} \text{ is isotropic and there is a rank 1 primitive submodule } M \subset L_{\gamma_0} \text{ satisfying } \\
\pi(M) = F^2H^2_{\text{dr}}(X/k). \text{ Then } M \text{ is an element of } \mathcal{M}. \text{ If } \gamma_0 = -1, \rho_X(\alpha) = -1 \text{ and by the Serre duality, } \alpha^*(H^2_{\text{dr}}(X/k)/F^1) = -1. \text{ We set } l_{-1} = \pi(L_{-1}). \text{ The pairing on } l_{-1} \text{ is non-degenerate. The rank of } L_{-1} \text{ is at least 2 and } \\
l_{-1} \not\subset F^1H^2_{\text{dr}}(X/k) = (F^2H^2_{\text{dr}}(X/k))^\perp. \\
\end{array}
\]
Proof. Since $\alpha$ is weakly tame, as in the proof of the above theorem, there is a Neron-Severi group preserving lifting $\mathfrak{X}/W$ of $X$ equipped with an automorphism $a : \mathfrak{X} \to \mathfrak{X}$ satisfying $a \otimes k = \alpha$. Let $X_K/K$ be the generic fiber of $\mathfrak{X}/W$. By the assumption $a^*H^0(X_K,\Omega_{X_K/K}) = id$. Since $K$ is of characteristic 0, $a^*|T(X_K) = id$. But there is a functorial isomorphism

$$H^2_{dr}(X_K/K) \simeq H^2_{cris}(X/W) \otimes K,$$

so $\alpha^*|T_{cris}(X) = id$. The later part follows easily. \hfill \square

Corollary 3.6. Let $X$ be a K3 surface of finite height over $k$. When $N$ is the order of $\text{Im} \rho_X$, the rank of $T_{cris}(X)$ is a multiple of $\phi(N)$.

Proof. Let $\alpha$ be an automorphism of $X$ such that $\rho_X(\alpha)$ generates $\text{Im} \rho_X$. We assume the order of $\chi_{cris,X}(\alpha)$ is $p^rM$ where $M$ is a positive integer which is not divisible by $p$. Then $\alpha^{p^r}$ is weakly tame and $M$ is equal to $N$ by the above corollary. Replacing $\alpha$ by $\alpha^{p^r}$, we may assume $\alpha$ is weakly tame. Then there exist a Neron-Severi group preserving lifting $\mathfrak{X}/W$ and a lifting of $\alpha$, $a : \mathfrak{X} \to \mathfrak{X}$. Since the order of $\rho_{X_K}(a)$ is $N$, the rank of $T(X_K)$ is a multiple of $\phi(N)$. The rank of $T(X_K)$ is equal to the rank of $T_{cris}(X)$ and the claim follows. \hfill \square

Theorem 3.7. Let $X$ be a weakly tame K3 surface over $k$. There exists a Neron-Severi group preserving lifting $\mathfrak{X}/W$ of $X$ such that the reduction map $\text{Aut}(\mathfrak{X} \otimes K) \to \text{Aut}(X)$ is isomorphic.

Proof. Let $h$ be the height of $X$. Let $\alpha$ be a weakly tame automorphism of $X$ such that $\rho_X(\alpha)$ generates $\text{Im} \rho_X$. Then by Corollary 3.5, $\chi_{cris,X}(\alpha)$ generates $\text{Im} \chi_{cris,X}$. As in the proof of Theorem 3.3, we can find $M \in \mathcal{M}$ inside $H^2_{cris}(X/W)_{[1+1/h]}$ which is $\alpha^*$-stable. Let $\mathfrak{X}/W$ be the lifting of $X$ corresponding to $M$. Then $\alpha$ is liftable to $\mathfrak{X}$. For any $\beta \in \text{Aut}(X)$, $\chi_{cris,X}(\beta) = \chi_{cris,X}(\alpha^i)$ for some integer $i$. Since

$$H^2_{cris}(X/W)_{[1+1/h]} \subset T_{cris}(X),$$

$M$ is stable for $\beta^*$ and $\beta$ is liftable to $\mathfrak{X}$. Therefore

$$\text{Aut}(\mathfrak{X} \otimes K) = \text{Aut}(\mathfrak{X}) \to \text{Aut}(X)$$

is surjective. \hfill \square

Remark 3.8. Assume $p$ is at least 5 and $X$ is a supersingular K3 surface of Artin-invariant 1 over $k$. Then $\text{Im} \rho_X$ is a cyclic group of order $p + 1$ ([13]). Hence if $p > 60$, $\phi(p + 1) > 21$ and there is an automorphism of $X$ which can not be lifted over characteristic 0. It is also known that for a supersingular K3 surface of Artin-invariant 1 over a field of characteristic 3, there is an automorphism which can not be lifted over characteristic 0 ([7]). We can ask whether for any supersingular K3 surface, there is an automorphism which can not lifted over characteristic 0.
4 Non-symplectic automorphisms

Let $k$ be an algebraically closed field of odd characteristic $p$ whose cardinality is equal to or less than the cardinality of the real numbers. Let $W$ be the ring of Witt-vectors of $k$ and $K$ be the fraction field of $W$. Let $\bar{K}$ be an algebraic closure of $K$. We fix an isomorphism $\bar{K} \cong \mathbb{C}$. Let $\Sigma = \{13, 17, 19, 25, 27, 32, 33, 40, 44, 50, 66\}$ be a finite set of positive integers. The following is known.

**Theorem 4.1** ([16], [18], [23], [26]). If $N \in \Sigma$, there exists a unique complex algebraic $K3$ surface $X_N$ equipped with a purely non-symplectic automorphism of order $N$, $g_N$ up to isomorphism. $X_N$ has a model over $\mathbb{Q}$ and if a prime number $p$ does not divide $2N$, $(X_N, g_N)$ has a good reduction $(X_{N,p}, g_{N,p})$ over an algebraic closure of a prime field of characteristic $p$.

In the case of $N = 66$, the following result over positive characteristic is also known.

**Theorem 4.2** ([15]). If the characteristic of $k$ is not 2 or 3, there is a unique $K3$ surface equipped with an automorphism of order 66.

Note that the above result covers a wild case of characteristic 11.

Using Theorem 3.3, we prove the uniqueness of a $K3$ surface over $k$ equipped with a purely non-symplectic tame automorphism of order $N$ for $N \in \Sigma$.

**Theorem 4.3.** Assume $N \in \Sigma$ and $p$ does not divide $2N$. Then there exists a unique $K3$ surface equipped with a purely non-symplectic automorphism of order $N$ up to isomorphism. This unique $K3$ surface has a model over a finite field.

**Proof.** The existence is guaranteed by Theorem 4.1. Now assume $X$ is a $K3$ surface over $k$ and $\alpha \in \text{Aut}(X)$ is purely non-symplectic of order $N$. Since $\alpha$ is non-symplectic tame, by Theorem 3.3, there exists a scheme lifting $X/W$ of $X$ and an automorphism $\alpha : X \to X$ such that $\alpha \otimes k = \alpha$. Then $X \otimes \mathbb{C}$ is a complex $K3$ surface equipped with a purely non-symplectic automorphism of order $N$, so $X \otimes \mathbb{C} \simeq X_N$. It follows that $X$ is isomorphic to $X_{N,p} \otimes k$. \hfill \square

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