Non-Gibrat’s Law and the size dependence of growth rate distributions

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Abstract. The authors study growth rate distributions of two types of variables by using numerical simulations. The first type data \(v\) exhibit both positive and negative values, and the second type data \(x\) only take positive values. The simulation model is defined by the Langevin equation for variables \(v\), and variables \(x\) are obtained from \(v\). Two kinds of Pareto indices, and the standard deviation \(\sigma\) of the growth rate distribution of \(x\) are acquired. Pareto indices for \(v\) and \(x\) are equal. Also, \(\sigma\) exhibits scaling behavior with a scaling exponent \(\varepsilon < 0\). The scaling exponent \(\varepsilon\) depends on the Pareto index \(\mu\).

1. Introduction
There are many studies of distributions of sizes and growth rates of firms. In those studies, several empirical laws have been described. One of these is Pareto’s Law [1]:

\[
P(v) \propto v^{-(\mu+1)} \quad \text{for} \quad v > v_{th},
\]

Here, \(v\) denotes a company’s size in terms of its assets, sales, profits, the number of employees or some similar measure (see for instance [2]), and \(P(v)\) is the probability density function (pdf)\(^1\). \(\mu\) is called the Pareto index and \(v_{th}\) is a certain threshold.

Another empirical law is Gibrat’s Law concerning growth rate distributions [3]. The growth rate \(R\) is defined as \(R = v_2/v_1\), where \(v_1\) and \(v_2\) are two successive measures of a firm’s size. Gibrat’s Law indicates that the conditional pdf \(Q(R|v_1)\) is independent of the past firm’s size \(v_1\)

\[
Q(R|v_1) = Q(R)
\]

over a large size range [4]. In Ref. [4], size is variously defined as personal income in Japan (the former of the two references), as total assets or sales in France, or as the number of employees in the UK (the later reference).

The above two laws are valid over a large size range and are related to each other\(^2\). Fujiwara et al. [4] show that Pareto’s Law is derived from Gibrat’s Law and the Law of Detailed Balance.

\(^1\)Analysis of data for Japanese listed companies reveals that the distribution of market capitalization is predominantly log-normal. This data is peculiar, in that the distribution of sales is also predominantly log-normal. The reason is that listed companies produce highly restricted sales data. Their data is, however, exhaustive with respect to market capitalization.

\(^2\)There are several studies which show that Gibrat’s Law leads to Pareto’s Law under certain conditions (for instance [5]).
Detailed Balance is the time-reversal symmetry \( (v_1 \leftrightarrow v_2) \) observed in a stable economy

\[
P_{12}(v_1, v_2) = P_{12}(v_2, v_1),
\]

where \( P_{12}(v_1, v_2) \) is the joint pdf. The study of Pareto’s Law over a large size range is important, because this encompasses a large percent of the overall measurements of assets, sales, profits, or number of employees. At the same time, the study of the distribution in the medium size range is also important, since the majority of companies are medium-sized. It is considered that the log-normal distribution agrees well with the data in this size range. The log-normal distribution is not, however, derived from Gibrat’s Law under conditions of Detailed Balance. In fact, it is reported that Gibrat’s Law is not relevant empirically (for instance [6, 7, 8, 9, 10, 11]), i.e. that \( Q(R|v_1) \) does depend on the past size \( v_1 \). In Ref. [10], it is reported that the dependence on the past size is determined by the condition that \( Q(R|v_1) \) is expressed as an exponential tent-shaped function:

\[
Q(R|v_1) = \text{Const.} \ R^{t_\pm(v_1)-1} \quad \text{for} \quad R \gtrless 1.
\]

The deduced past size dependence is as follows:

\[
t_\pm(v_1) = t_\pm(v_{th}) \pm \alpha \ln \frac{v_1}{v_{th}},
\]

where \( \alpha = 0 \) for \( v_1 > v_{th} \) and \( \alpha \neq 0 \) for \( v_1 < v_{th} \). In addition, the pdf \( P(v) \) is also deduced to be the log-normal distribution

\[
P(v) = C \ v^{-(\mu+1)} e^{-\alpha \ln^2 \frac{v}{v_{th}}} \quad \text{for} \quad v < v_{th}
\]

in the medium size range [9]. Note that Detailed Balance is a requirement in the above derivation.

The assumed \( Q(R|v_1) \) is a linear function on a log-log scale as shown in Fig. 1. Moreover, according to Eq. (5), the probability of positive growth decreases and the probability of negative growth increases symmetrically as the classification of \( v \) increases in the medium size range. Profit data for Japanese firms exhibits such behavior [10]. We call Eqs. (4) and (5) the First Non-Gibrat’s Law.

There are, however, some growth rate distributions which cannot be approximated by Eqs. (4) and (5) [6, 8, 11]. These growth rate distributions exhibit a fat tail behavior as shown in Fig. 2. Moreover, the probability of both positive and negative growth rates decreases simultaneously as the classification of \( v_1 \) increases, as shown in Fig. 2. We call this behavior the Second Non-Gibrat’s Law.

One of the authors (A.I.) proposes an assumption to account for the difference between the two Non-Gibrat’s Laws [12]. Then the economic variables may be classified into two types. One
type is defined by using a kind of subtraction, whereas the other type does not involve any subtraction in its definition. Also it is assumed that the first type of data obeys the First Non-Gibrat’s Law and the second type of data obeys the Second Non-Gibrat’s Law. Moreover, first type data are obtained as the temporal difference between two second type data. For example, profits are data of the first type, and sales and assets are data of the second type. By definition, the first type data exhibits both positive and negative values. Regarding a scatter plot of $v_1$ vs $v_2$, the data exist not only in the first quadrant, but also in all four quadrants in the $v_1$-$v_2$ plane, where $v_1$ and $v_2$ express two successive firm sizes. In this study, variables $v$ represent first type data, and $x$ represent second type data.

In Ref. [13], we verify that the Takayasu-Sato-Takayasu (TST) model [14] satisfies not only Pareto’s Law but also Detailed Balance under the First Gibrat’s Law, by using numerical simulations. In the model, multiplicative stochastic noise obeying the First Non-Gibrat’s Law has been employed. However, the subjects of the model were only positive data because both the multiplicative and the additive noise have been defined as positive. To avoid this discrepancy, an extension of the model is needed.

An extended model has been proposed in Ref. [15]. In this model, the variables $v$ exhibit both positive and negative values. Positive variables $x$ are then obtained from $v$. The growth rate distributions of $x$ are also obtained. It is verified that the First and Second Non-Gibrat’s Laws are valid for the growth rate distributions of $v$ and $x$ respectively.

In this study, the dependence of the obtained Second Non-Gibrat’s Law on the past size $x_1$ is investigated by the use of numerical simulations. This study provides a clue to a quantitative analysis because it is important to determine what the Second Non-Gibrat’s Law depends on. It can also serve as a clue to an analytic understanding.

2. Simulation model and results
We begin with a brief review of our model. First, the positive and negative stochastic variables $v$ are generated by using the Langevin equation [16]:

$$v(t + 1) = a(t) \cdot b(v(t), t) \cdot v(t) + f(t),$$

where $a = \pm 1$ with equal probability, $b$ is a non-negative multiplicative stochastic noise and $f$ is an additive stochastic noise. A normal distribution $N(0, s^2)$ is employed for the additive noise $f$. Eqs. (4) and (5) are utilized as the distribution of $b$. To avoid excessive changes of $t_\pm(v)$, $v_{th2}$ is introduced, and the parameter $\alpha$ is chosen as $\alpha = 0$ for $v > v_{th}$, $\alpha \neq 0$ for $v_{th2} < v < v_{th}$ and $\alpha = 0$ again for $v < v_{th2}$ (Fig. 3). We also introduce the threshold $v_{min}$, above which Detailed Balance is observed. Non-Gibrat’s Law is observed in the range $\max(v_{min}, v_{th2}) < v < v_{th}$. We call the combination of Gibrat’s Law in the large scale range and Non-Gibrat’s Law in the medium scale range as Extended-Gibrat’s Law [9, 15].
Next, we define positive variables $x$ by using both positive and negative variables $v$. It is proposed that $v$ is provided by the temporal change of $x$:

$$v(t) = x(t + 1) - x(t),$$

namely that the evolution of $x$ can be expressed as

$$x(t + 1) = \begin{cases} x(t) + v(t) & \text{if } x(t) + v(t) > 0, \\ x(t) & \text{if } x(t) + v(t) < 0. \end{cases}$$

Here, we restrict $x$ to be positive. This restriction is relevant for $x \sim v$, but does not contribute for $x >> v$. The restriction does not affect results because variables $x$ increase in general as the iteration proceeds. In the simulation, we do not add $v$ less than $v_{\text{min}}$ to $x$ because Detailed Balance does not hold in the range below $v_{\text{min}}$. We imposed this to avoid the effects of such data.

Let us describe the simulation results [15]. We first state a few results needed in the next section. Extended Gibrat’s Law and Detailed Balance are confirmed. It is observed that the pdfs $P(v)$ and $P(x)$ follow Pareto’s Law over a large scale range, for which the Pareto indices $\mu_v$ and $\mu_x$ are estimated.

The conditional pdfs $q(r|x_1)$ of the log growth rate $r = \log_{10} x_2/x_1$ are shown in Fig. 4. The pdf $q(r|x_1)$ is related to the pdf $Q(R|x_1)$ by $\log_{10} q(r|x_1) = \log_{10} Q(R|x_1) + r + \log_{10}(\ln 10)$. It is found that the Second Non-Gibrat’s Law for variables $x$ is valid in the medium scale range. The probability of positive and negative growth rate decrease simultaneously as the size of $x_1$ increases. The pdfs also exhibit a fat tail behavior. In the simulation, we employ the following parameters: $t_+(v_{\text{th}}) = 2.6, t_-(v_{\text{th}}) = 1.6, v_{\text{th}} = 10^6, s = 20$ and $\alpha = 0$. The total number of data is 500,000.

**Figure 4.** Conditional pdfs $q(r|x_1)$ of the log growth rate $r = \log_{10} x_2/x_1$. The data are classified into twenty bins of the past variables with equal magnitude in logarithmic scale, $x \in [10^{1+0.5(n-1)}, 10^{1+0.5n}]$ ($n = 1, 2, \ldots, 20$).
3. Analysis of simulation results

We present several relations between the distribution of $v$ and the distribution of $x$. First, let us compare $\mu_v$ and $\mu_x$. As shown in Fig. 5, $\mu_x$ are equal to $\mu_v$ as follows:

$$\mu_x = \mu_v \equiv \mu \quad \text{for } 0 < \mu < 2. \quad (10)$$

Next, let us consider the dependence of growth rate distributions on the past size $x_1$. As mentioned in section 1, the growth rate distribution of $x$ obeys the Second Non-Gibrat’s Law in the medium size range [15]. Namely, the width of the growth rate distribution of $x$ decreases as the past size $x_1$ increases. To clarify this, we examine the standard deviation $\sigma$ of the growth rate distribution of $x$.

Now we discuss the following two cases: (a) the growth rate distribution of $v$ obeys only Gibrat’s Law; (b) the growth rate distribution of $v$ obeys the Extended Gibrat’s Law. The result of case (a) is shown in Fig. 6. The dependence of $\sigma$ on the past size is obtained as

$$\log_{10} \sigma = \text{Const.} + \varepsilon n, \quad (11)$$

where $\varepsilon = -0.05\mu - 0.055$ for $1.0 < \mu < 1.5. \quad (12)$

Here, $n$ is a size parameter of $x$: $x \in [10^{1+0.5(n-1)}, 10^{1+0.5n}]$. So Eq. (11) can be expressed by

$$\sigma \propto x^{2\varepsilon}. \quad (13)$$

The standard deviation $\sigma$ decreases as the size of $x$ increases because $\varepsilon$ is negative. This result is consistent with earlier works [6, 8, 11]. Furthermore, Eq. (12) indicates that the scaling exponent $\varepsilon$ depends on $\mu$ (Fig. 7).

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{figure5.png}
\caption{Comparison of $\mu_v$ and $\mu_x$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{figure6.png}
\caption{The dependence of $\sigma$ on the past size $x_1$.}
\end{figure}

Let us consider the result of case (b). In case (b), the growth rate distribution of $v$ obeys the First Non-Gibrat’s Law, and the pdf of $v$ forms the log-normal distribution (6) in the medium scale range [9]. The relation between $\varepsilon$ and $\alpha$ is shown in Fig. 8. This dependence is obtained as

$$\varepsilon(\alpha) = 0.8\alpha + \varepsilon(\alpha = 0). \quad (14)$$

Here, we restrict ourselves to the case $\alpha < 0.1$. This restriction makes it possible to keep a certain width for the range in which the First Non-Gibrat’s Law holds with respect to $v$. For the same reason, it is required that $v_{th2}$ is less than or equal to $v_{min}$. 


4. Conclusion
Using numerical simulations, we study two types of variables and their growth rate distributions. The first type data $v$ exhibit both positive and negative values, and the second type data $x$ only take positive values. Moreover, variables $v$ are taken as the temporal change of variables $x$. Variables $x$ are obtained as the summation of variables $v$ by definition. We acquire the Pareto index, and the standard deviation $\sigma$ of the growth rate distribution. It is confirmed that the two Pareto indices $\mu_v$ and $\mu_x$ are equal. It is found that $\sigma$ exhibits scaling behavior with the scaling exponent $\varepsilon$. Because $\varepsilon$ is negative, the standard deviation $\sigma$ decreases as the size of $x$ increases. This is consistent with the Second Non-Gibrat’s Law. It is also worth noting that the scaling exponent $\varepsilon$ depends on $\mu$ and $\alpha$.

We have adopted only Eqs. (4) and (5) as the distribution of $b$. Other candidates should be considered, for example an exponential distribution. The effect of different choices should also be considered. It is also important to describe the relation between any one time step and actual time. Furthermore, analytic derivation of the statistical properties of $x$ from the statistical properties of $v$ is a significant issue. This study can serve as a clue to an analytic understanding of Non-Gibrat’s Law. It might also provide a clue to knowledge of the management of credit risk.

Acknowledgments
The authors thank APFA7 & Tokyo Tech – Hitotsubashi Interdisciplinary Conference. Discussions during the Conference were useful to the completion of this work. The work was supported in part by a Grant-in-Aid for Scientific Research (C) (No. 20510147) from the Ministry of Education, Culture, Sports, Science and Technology, Japan. We also thank to P. M. Arathoon for his careful reading of the manuscript and his suggestions.

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