TWO SOLUTIONS TO KAZDAN-WARNER’S PROBLEM ON SURFACES

LI MA

ABSTRACT. In this paper, we study the sign-changing Kazdan-Warner's problem on two dimensional closed Riemannian manifold with negative Euler number $\chi(M) < 0$. We show that once, the direct method on convex sets is used to find a minimizer of the corresponding functional, then there is another solution via a use of the variational method of mountain pass. In conclusion, we show that there are at least two solutions to the Kazdan-Warner’s problem on two dimensional Kazdan-Warner equation provided the prescribed function changes signs and with this average negative.

Mathematics Subject Classification 2010: 53C20, 35J60, 58G99. Keywords: Kazdan-Warner problem, mountain pass, direct method on convex sets, multiple solutions.

1. Introduction

The aim of this paper is to study the sign-changing Kazdan-Warner’s problem on two dimensional closed Riemannian manifold with $\chi(M) < 0$ and we show that there are at least two solutions to (1). This non-uniqueness problem is open since 1974 and the precise result is stated below.

For a given smooth function $K$ on a compact Riemannian manifold $(M, g)$, Kazdan-Warner studied the following problem

\begin{equation}
- \Delta u + \alpha = K(x)e^{2u}, \text{ in } M,
\end{equation}

where $\Delta$ is the Laplacian operator of the metric $g$ (and which is $\sum_i \partial_i^2$ on $\mathbb{R}^2$) and $\alpha$ is a given real constant. Integrating by part we have that necessarily,

$$\alpha |M| = \int_M K e^{2u} \, dv$$

where $|M| = \int_M dv$ is the volume of $(M, g)$ with $dv$ being the volume element of the metric $g$. Assume that $\alpha < 0$. Using the method of super and sub solutions, Kazdan-Warner have proved that there exists a constant $\alpha_0 < 0$ such that for each $\alpha \in (\alpha_0, 0)$ there is a solution to (1). The critical number $\alpha_0$ is defined by

$$\alpha_0 = \inf \{ \alpha; (1) \text{ is solvable for } \alpha \}.$$
Using a very beautiful argument, Kazdan and Warner [9] have showed that 
\[ \alpha_0 = \infty \text{ if and only if } K \text{ is nontrivial non-positive function on } M. \]

For the case when the given smooth function \( K \) is positive somewhere with negative average, by the result of Kazdan and Warner mentioned above we have the critical number \( \alpha_0 > -\infty \). W.Chen and C.Li [6] have proved that there is a solution \( u_0 \) to (1) at \( \alpha = \alpha_0 \) and \( u_0 \) is the \( H^1 \) limit of stable solution sequence \( \{ u_k \} \) (corresponding to \( \alpha_k \to \alpha_0^+ \)). Their argument is based on the solutions obtained by Kazdan-Warner and they have used the variational argument to get their solution. Note that in this case, there is a positive smooth function \( \psi \) such that

\[ -\Delta \psi - 2K(x)e^{2u_0}\psi = 0, \text{ in } M. \]

The remaining question is whether the solution to (1) is unique.

We show that there is no uniqueness of solutions to (1) when \( \alpha \in (\alpha_0, 0) \). We denote by \( \chi(M) \) the Euler characteristic of for the surface \( M \). We have two solutions result to 1 in this case.

**Theorem 1.** On the 2-dimensional compact Riemannian manifold \((M, g)\) with \( \chi(M) < 0 \), for \( \alpha \in (\alpha_0, 0) \), there are at least two solutions to (1) provided the given smooth function \( K \) is positive somewhere with negative average.

This result will be proved by the mountain pass argument below. There are some interesting works about prescribed sign-changing Gauss curvature on closed surfaces and related works on scalar curvature problems. We mention the interesting works such as the classical books [1] and [10], the papers of M.Berger [2], Chang-Yang [4], Kazdan-Warner [8], Chen-Li [6], [7], Borger-Luca-Struwe [3], Ma-Hong [12], etc, and one may refer to Ma [13] and Ma-Wei [16] for more related references.

The plan of this paper is below. In section 2 we consider the method of super and sub solutions (the monotone method) to obtain a solution to (1) in any dimensions. In section 3 we consider the mountain pass solution to (1) in dimension two. The key step is the verification of Palais-Smale condition to the related functional. In the last section we give a variational argument of Kazdan-Warner’s result.

2. THE FIRST SOLUTION VIA THE DIRECT METHOD ON CONVEX SETS

In this section, we use the direct method on convex sets to get a solution to (1), our construction is slightly different from the one used in (1) on a compact Riemannian manifold \((M, g)\). We have the following result and the new part in it is the local minima property, which will play a role in the mountain pass argument in next section.

**Theorem 2.** We assume that on a compact Riemannian manifold \((M, g)\) of any dimension, the smooth function \( K \) changes signs and \( \bar{K} < 0 \). Then
for any $\alpha \in (\alpha_0, 0)$, there is a solution to (I), which is a local minimizer of the functional

$$I(u) = \int_M (|\nabla u|^2 + \alpha u) dv - \int_M Ke^{2u} dv$$

on $H^1(M)$.

The proof may be outlined below. Let $u_+ = u_0$ be the solution obtained by Kazdan-Warner [9] for some $\alpha_1, \alpha_0 < \alpha < \alpha_1$. Then for any $\alpha \in (\alpha_1, 0)$,

$$-\Delta u_+ + \alpha - K(x)e^{2u_+} > -\Delta u_+ + \alpha_1 - K(x)e^{2u_+} = 0,$$

i.e., $u_+$ is the super solution to (I) for $\alpha \in (\alpha_1, 0)$.

To get a solution by the method of super and sub solutions, we need to find a sub solution $u_- < u_+$ to (I). We do this below.

Recall that $\alpha < 0$. Note that for any real number $c$ very negative and $v_c = c$, we have

$$\Delta v_c - \alpha + Ke^{v_c} = -\alpha + Ke^c > 0, \text{ in } M.$$

Then $u_- = c( < u_+)$ is a sub solution to (I). Then we have a solution $u_1$ to (I) in the interval $[c, u_+]$, which is a local minimizer of the functional

$$I(u) = \int_M (|\nabla u|^2 + \alpha u) dv - \int_M Ke^{2u} dv$$

on $H^1(M)$. We refer to [5] for related references.

3. The Second Solution: The Mountain Pass

In this section we consider the equation (I) on the closed surface $(M, g)$ with $\chi(M) = \frac{1}{2\pi} \int_M k dv$, where $k$ is the Gauss curvature of $g$. Note that Theorem 3 below implies Theorem 1.

**Theorem 3.** On the 2-dimensional compact Riemannian manifold $(M, g)$ with $\chi(M) < 0$, there is a mountain-pass solution to (I) provided the given smooth function $K$ such that $K > 0$ somewhere and $\bar{K} < 0$.

We now give the idea to prove this result. We recall the functional

$$I(u) = \int_M (|\nabla u|^2 + \alpha u) dv - \int_M Ke^{2u} dv$$

on $H^1(M)$. Note that the solution $u_1$ obtained by the method of super and sub solutions can also be described as the minimizer of the functional $I$ on $[u_-, u_+] \cap H^1(M)$. Hence, for any $\phi \in H^1(M)$, $< P'(u_1)\phi, \phi > \geq 0$, i.e.,

$$\int_M (|\nabla \phi|^2 - 2Ke^{2u_1}\phi^2) dv \geq 0.$$

Recall that $u_1$ satisfies

$$-\Delta u_1 + \alpha = Ke^{2u_1}, \text{ in } M,$$
and then
\[ \int_M Ke^{2u_1} dv = \alpha |M|. \]

We give a remark here. If \( u_1 \) is a strict minimizer, then we have some uniform constant \( c > 0 \) such that
\[ \int_M (|\nabla \phi|^2 - 2Ke^{2u_1} \phi^2) dv \geq c \int_M \phi^2 dv, \quad \forall \phi \in H^1(M). \]

Otherwise, we have a positive solution \( \phi \) to the linear equation
\[-\Delta \phi - 2Ke^{2u_1} \phi = 0, \quad \text{in } M.\]

We want to find another solution of the form \( u_1 + u \) such that \( u \neq 0 \) satisfies
\[-\Delta (u_1 + u) + \alpha = Ke^{2u_1 + 2u}, \quad \text{in } M.\]

Then we have that
\[-\Delta u = Ke^{2u_1} (e^{2u} - 1), \quad \text{in } M.\]

So we look for a non-trivial critical point of the functional
\[ J(u) = \int_M |\nabla u|^2 dv + \int_M R(x)(2u - e^{2u}) dv \]
on \( H^1(M) \), where we have set \( R(x) = Ke^{2u_1} \). Note that \( \int_M Ke^{2u_1} = \alpha |M| \).

Recall that \( u = 0 \) is a local minimizer of \( J \) on \( H^1(M) \).

For small \( \epsilon > 0 \), we let \( M_\epsilon = \{ R(x) \geq \epsilon \} \) and \( M_{-\epsilon} = \{ -R(x) \geq \epsilon \} \). As above, we denote by \( \bar{u} = \frac{1}{\text{vol}(M)} \int_M u dv \). We now choose a smooth function \( w_0 \in C^1_0(M_\epsilon \cup M_{-\epsilon}) \) which is positive in \( \{ R(x) \geq \epsilon \} \) for some \( \epsilon > 0 \) and \( \bar{w}_0 = 0 \). Note that as \( t \to \infty \),
\[ J(tw_0) = t^2 \int_M |\nabla w_0|^2 dv + \int_M R(x)(2tw_0 - e^{2tw_0}) dv \to -\infty \]
and we may choose \( t_0 > 0 \) large such that \( J(t_0w_0) < J(0) = 0 \).

Define
\[ X = H^1(M) \]
and
\[ c = \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} J(\gamma(t)), \]
where
\[ \Gamma = \{ \gamma \in C([0,1], X); \gamma(0) = 0, \gamma(1) = t_0w_0 \}. \]

Note that \( c \geq 0 \). We shall verify that \( J \) satisfies the Palais-Smale condition on \( X \). Then, there is a "mountain pass" critical point of \( J \) on \( X \) satisfies
\[-\Delta u - R(x)e^{2u} + R(x) = 0, \]
weakly in \( H^1(M) \). By the result of K.C.Chang \[5\], \( c \) is a mountain pass critical value of the function \( J \) and may be obtained by the function \( w \neq 0 \), which gives a solution, which is different from the solution \( u_1 \) obtained by the monotone method above. Thus as we have noted before, we get the proof of Theorem \[3\].
The main topic now is to check the Palais-Smale condition for the functional $J$ on $H^1(M)$. Set $M_- = \{ x \in M; R(x) < 0 \}$, which is a non-empty open set in $M$. We have the following compactness result for the functional $J$ on $H^1(M)$.

**Lemma 4.** The functional $J$ satisfies the Palais-Smale condition at the level $c \geq 0$ in the function space $X = H^1(M)$. That is, if any sequence $\{u_k\} \subset X$ satisfies $J(u_k) \to c$ and $J'(u_k) \to 0$ in the dual space $X^*$, then there is a subsequence of $\{u_k\}$ converges in $X$.

**Proof.** Assume that the sequence $\{u_k\} \subset X$ satisfies $J(u_k) \to c$ and $J'(u_k) \to 0$ in the dual space $X^*$. That is,

\begin{equation}
\int_M (|\nabla u_k|^2 + R(x)(2u_k - e^{2u_k}))dv \to c
\end{equation}

and

\begin{equation}
\int_M (\nabla u_k \cdot \nabla \phi + R(x)\phi - R(x)e^{2u_k}\phi)dv = o(||\phi||), \quad \phi \in X
\end{equation}

where $|| \cdot ||$ is the norm on $X$. Set $\phi = 1$ in (3), we have

\begin{equation}
\int_M R(x)(1 - e^{2u_k})dv \to 0, \quad \text{i.e.,} \quad \int_M R(x)e^{2u_k}dv \to \int_M R(x)dv = d < 0.
\end{equation}

By (2), we have

\begin{equation}
\int_M (|\nabla u_k|^2 + R(x)(2u_k))dv \to c + d.
\end{equation}

Let

$$
\bar{u}_k = \frac{1}{vol(M)} \int_M u_k dv := c_k.
$$

Then we have

\begin{equation}
\int_M (|\nabla u_k|^2 + 2R(x)(u_k - c_k))dv + 2c_kd \to c + d.
\end{equation}

Note that

\begin{equation}
|\int_M R(x)(u_k - c_k)dv| \leq |R|_{\infty} \int_M |u_k - c_k|dv \leq \frac{1}{2} \int_M |\nabla u_k|^2 + C_1
\end{equation}

for some $C_1$. Then

\begin{equation}
\int_M (|\nabla u_k|^2 + 2R(x)(u_k - c_k))dv + 2c_kd \geq -C_1 + 2c_kd.
\end{equation}

Let $u_k^+(x) = \sup(u_k(x), 0)$. We want to show that the sequence $\{u_k^+\}$ is locally bounded in $H^1_{loc}(M_-)$. Take any non-empty domain $D$ in $M_-$ with $\text{dist}(D, \partial M_-) := d > 0$ and $-R(x) \geq \delta > 0$ on $D$. We show that there is a constant $C(D)$ such that $||u_k^+|| \leq d(D)$. Take any $p \in M_-$ such that $B_d := B_d(p) \subset M_-$, where $B_r(p)$ is the geodesic ball centered at $p$ with radius $r > 0$. Choose the cut-off function $\eta \in C^1_0(B_{d/2})$ such that $\eta = 1$ on
$B_{d/4}$ and $|\nabla \eta|^2/\eta \leq C$ for some uniform constant $C > 0$. Set $\phi = u_k^+ \eta^2$ in (3). Then we have some uniform $C > 0$ such that

$$\int_M (\nabla u_k \cdot \nabla (u_k^+ \eta^2) + R(x)u_k^+ \eta^2 - R(x)e^{2u_k}u_k^+ \eta^2)dv \leq C(||u_k^+ \eta^2||).$$

Since

$$\int_M \nabla u_k \cdot \nabla (u_k^+ \eta^2) = \int_M |\nabla (\eta u_k^+)|^2 - \int_M (u_k^+)^2 |\nabla \eta|^2,$$

we then have

$$\int_M (|\nabla (\eta u_k^+)|^2-R(x)e^{2u_k}u_k^+ \eta^2)dv \leq C \int_M |\nabla \eta|^2 (u_k^+)^2 - \int_M R(x)u_k^+ \eta^2 + C(||u_k^+ \eta||).$$

By $e^{2t} \geq t^3$ for any real $t$ and $|\nabla \eta|^2 \leq C\eta$, we have,

$$\int_M (|\nabla (\eta u_k^+)|^2 + \delta \eta^2 (u_k^+)^4)dv \leq C \int_M \eta (u_k^+)^2 - \int_M R(x)u_k^+ \eta^2 + C(||u_k^+ \eta||).$$

Using the Holder inequality we then have

$$\int_M (|\nabla (\eta u_k^+)|^2 + \frac{1}{2} \delta \eta^2 (u_k^+)^4)dv \leq C(\delta) + C(||u_k^+ \eta||),$$

which implies that there is a uniform constant $C := C(d)$ such that

$$\int_M (|\nabla (\eta u_k^+)|^2 + \frac{1}{4} \delta \eta^2 (u_k^+)^4)dv \leq C.$$ We now show that $\int_M u_k^2$ is uniformly bounded (and is equivalent to $c_k$ being a bounded sequence). If this is true, then the Palais-Smale sequence is bounded in $X$. Then we may assume that there is a subsequence, still denoted by $u_k$, which weakly converges to $u$ in $X$. Then the subsequence converges in $X$ by (3) and the fact that for any $p > 1$,

$$e^{2u_k} \rightarrow e^{2u}, \quad \text{in } L^p(M).$$

To show $|u_k|^2 := \int_M u_k^2$ being uniformly bounded, we argue by contradiction and assume $|c_k| \rightarrow \infty$. Let

$$v_k = u_k/|u_k|_2 = (w_k + c_k)/\sqrt{|w_k|^2 + c_k^2 Vol(M)}.$$

Then by (5) we have

$$\int_M |\nabla v_k|^2 dv \rightarrow 0.$$

We may assume that $v_k$ converges to $v$ strongly in $L^2(M)$ and weakly in $X$ with $|v|_2 = 1$. Then $\int_M |\nabla v|^2 dv = 0$ and $v = \beta$ is a constant. Since $u_k^+$ is locally bounded in $H^1_{loc}(M_\pm)$, we have $v_+ = 0$ in $M_-$ and then $\beta < 0$ since $|v|^2 = 1$ and $c_k \rightarrow -\infty$. However, this is impossible by (6) and (7). We may let $u$ be the weak limit of $u_k$ in $H^1(M)$. Choosing $\phi = u_k - u$, this then shows that $(u_k)$ is a convergent PS sequence of $J$. \(\square\)

Once we have the compactness result for the functional $J$ as above, the proof of Theorem 3 is complete.
4. LOCAL MINIMIZER FOR $K \leq 0$

As we mentioned before, using a very beautiful argument, Kazdan and Warner [9] have showed that $\alpha_0 = \infty$ if and only if $K$ is nontrivial non-positive function on $M$. The question now is if one can use the variation method to get a solution in the case treated by Kazdan-Warner. Using the variational method on convex set we answer this question affirmatively. Precisely, we prove the following result by the variational method on convex set.

**Theorem 5.** On the any dimensional compact Riemannian manifold $(M, g)$, for $\alpha \in (-\infty, 0)$ and for any non-trivial smooth function $K \leq 0$ on $M$, there is a solution to (1), which is a local minimizer of the functional defined by

$$F(u) = \int_M (|\nabla u|^2 + 2\alpha u)dv - \int_M K(x)e^{2u}dv, \; u \in H^1(M).$$

This result is basically obtained in [9], where they have used the monotone method. Here we prefer to give a variational proof for completeness which also complements Berger’s program of application of variational methods to problems of prescribed non-positive Gauss curvature on closed surface, which are of the same type to (1). The variational method was used by C.Hong [11] to study the problem (1) on the two-sphere.

**Proof.** Recall that $K \leq 0$ on $M$ and $\bar{K} = \frac{1}{|M|} \int_M Kdv < 0$. It is well-known that by the direct method, we can solve the Poisson equation

$$-\Delta w = K - \bar{K}, \; \text{in } M$$

to get the smooth solution $w$ with $w > 0$ on $M$. We take $b > 0$ and $b = e^r$ and let

$$v = bw + r.$$ 

Choose $b > 0$ such that $-b\bar{K} + \alpha > 0$. Note that

$$-K(e^{bw} - 1) \geq 0.$$ 

Then we have

$$-\Delta v + \alpha - Ke^v = -b\Delta w - Ke^{bw+r} + \alpha$$

$$= -b(\bar{K} - K) - bKe^{bw} + \alpha$$

$$= -b\bar{K} - bK(e^{bw} - 1) + \alpha$$

$$> 0.$$ 

This implies that $u_+ := v$ is a super solution to (1).

Similar to [2], we define

$$F(u) = \int_M (|\nabla u|^2 + 2\alpha u)dv - \int_M K(x)e^{2u}dv$$

on the convex set $H := \{v \in H^1(M), u \leq u_+\}$, which is a convex functional on $H$ by the condition $K \leq 0$ on $M$. Here $v \leq u_+$ means that $v(x) \leq u_+(x)$. 


almost everywhere on \( M \). Set \( \mu = \inf_H F(u) \). We shall show that \( \mu > -\infty \). Assume that \( \mu = \infty \), and we shall have a sequence \( \{u_k\} \subset H \) such that \( F(u_k) \to \mu = -\infty \). Note that
\[
F(u) = \int_M |\nabla u|^2 dv - \int_M K(x)e^{2u} dv + 2\alpha |M| \bar{u}.
\]
Then we must have
\[
\bar{u}_k \to \infty,
\]
which is impossible by the constraint condition
\[
u_k \leq u_+.
\]
Then \( \mu > -\infty \) and \( \bar{u}_k \) is uniformly bounded. Hence \( \{u_k\} \subset H^1(M) \) is a bounded sequence in \( H^1(M) \) and then we have a weakly convergent subsequence, still denoted by \( u_k \) with limit \( u \). Hence \( F(u) = \mu \) and one can directly verify [14] that \( u \) is a solution to (1) on \( M \) and a local minimizer of the functional \( F \) [14]. This completes the proof of the result. 

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