The size of the nodal sets for the eigenfunctions of the smooth laplacian

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Description. A classical problem in physics and geometry is the qualitative description of the spectrum of the laplacian on functions on a domain with Dirichlet boundary conditions. The eigenfunctions determine the standing vibrational modes of a drum shaped as the given domain. It is expected that the eigenfunctions behave as polynomials of degree related to the order of the harmonic. This is manifested for instance by the size of their nodal sets, i.e. the set of zeroes. In terms of the vibrational motion of a drum it consists of points that remain stationary and can be interpreted as the locus of destructive interference of the waves with the boundary. In the late 18th century Chladni performed experiments with planar drums of that revealed the shape and size of the nodal set vary with the order of the harmonic. Similar questions can be asked for the laplacian acting on functions on an arbitrary riemannian compact, closed manifold.

Statement and known results Let \((M^n, g), n \geq 2\) be an \(n\)-dimensional compact closed manifold equipped with a smooth riemannian metric \(g\). The Laplace-Beltrami operator acting on functions \(\Delta_g\) on \(M^n\) is written in local coordinates, \(g^{ij} = (g^{-1})_{ij}, g = det(g_{ij})\):

\[
\Delta_g \phi = \frac{1}{\sqrt{g}} \sum_{i,j} \frac{\partial}{\partial x_i} \left( g^{ij} \sqrt{g} \frac{\partial \phi}{\partial x_j} \right)
\]

The spectrum of the laplacian is discrete \(\text{spec}(\Delta) = \{ \lambda_j \}_{j \in \mathbb{N}} \subset (0, \infty)\) consists of eigenvalues with eigenfunctions \(u_{\lambda}, \lambda \in \text{spec}(\Delta)\) are the solutions of the equation

\[
\Delta_g u_{\lambda} + \lambda u_{\lambda} = 0
\]

The nodal set of an eigenfunction \(u_{\lambda}\) is defined as

\[
N(u_{\lambda}) = \{ x \in M^n / u_{\lambda}(x) = 0 \}
\]

and it is proved that this set is a smooth submanifold outside a set of \((n-1)\)-Hausdorff measure zero. Brüning [1979] and Yau showed that for a smooth surface the nodal length has a lower bound as:

\[
\mathcal{H}^1(N(u_{\lambda})) \geq C \lambda^{1/2}
\]

and then Yau conjectured (Y, pr. no 74) that the Hausdorff measure of this set grows as

\[
\mathcal{H}^{n-1}(N(u_{\lambda})) \sim \lambda^{1/2}
\]

which is evidently the case for the spherical harmonics on the sphere. DFT1 established this when \((M^n, g)\) is analytic. Furthermore they showed DFT2 that for smooth surfaces one has that

\[
\mathcal{H}^1(N(u_{\lambda})) \leq C \lambda^{3/4}
\]

which was obtained by D by different methods. Note that N proved that

\[
\mathcal{H}^1(N(u_{\lambda})) \leq C' \lambda \log \lambda
\]

Hardt and Simon in their excellent work HS showed for \(C^{1,1}\) metrics that

\[
\mathcal{H}^{n-1}(N(u_{\lambda})) \leq C' \sqrt{\lambda}
\]

JM based on Donnelly-Fefferman work obtained bounds for the size of tubular neighbourhoods of nodal sets for real analytic metrics. Recently SZ, CM, M came up with new lower estimates of the nodal volume. The first authors provide an interesting formula for the size of level sets comprising an integral over the regular level sets of the eigenfunction.
**Short description of the method.** Our method is based on the construction of a cell decomposition of a riemannian manifold with the boundaries of cells consisting of pieces of geodesic spheres. This resembles the picture suggested by the Huyghens principle governing wave propagation. The cells are selected so that we are allowed to follow an inductive argument restricting the eigenfunction on the faces and get precise growth estimates for it. Recall that the area of geodesic spheres determines the absorption rate of a wave in the course of propagation in a medium. The faces are constructed through the introduction of a cluster of points: these are thought as a collection of sources of spherical waves, they define the geodesic spheres that provide the pieces forming the cells. The domains that are formed are called *geodesic pixels* while the faces are called *undulating fronts* and the geodesic sphere pieces called *elementary wave fronts*. The pixel size is arranged essentially by the mean curvature of the elementary wave front. The conjecture is then proved by an inductive argument combined in the lower bound with the isoperimetric inequality and eigenvalue estimates. The upper bound is obtained by an elaboration of the Dong formula.

**Notation** We introduce some notation. First we introduce the localized energy of a geodesic pixel:

$$E^{(1)}(R; B; \zeta) = \int_{P} \zeta^2 (|\nabla R|^2 + |B|^2)$$

and

$$E^{(0)}(R; \zeta) = \int_{P} \zeta^2 |R|^3$$

where we introduce the *Bach tensor*

$$B_{ijk} = \frac{1}{n-2} \text{curl}(R \zeta)_{ijk} - \frac{1}{2(n-1)(n-2)} (g_{ij} R_k - g_{jk} R_i)$$

The faces of the pixel are pieces of geodesic spheres called *elementary wave fronts* and are denoted by $F$ with mean curvature $h$ and localized tension $T^1(h; \vartheta)$ given for a smooth test function $\vartheta$, $\text{supp} \vartheta \subset P$:

$$T^1(h; \vartheta) = \int_{F} \vartheta^2 |\nabla h|^2$$

and

$$T^0(h; \vartheta) = \int_{F} \vartheta^2 h^2$$

Then we introduce two numbers

$$r(P) = \sup_{\zeta \in C_0^\infty(P)} \left( \frac{E^{(1)}(R; B; \zeta)}{E^{(0)}(R; \zeta)} \right)$$

and

$$t(F) = \sup_{\vartheta \in C_0^\infty(F)} \left( \frac{T^1(h; \vartheta)}{T^0(h; \vartheta)} \right)$$

The first theorem comprises Harnack estimates on such sets

**Theorem 1.** Let $(M^n, g)$ be a riemannian manifold with scalar curvature function $R$. Then in a pixel $P$, $r(P) = (\ell; i_1, \ldots, i_k) = \mu$ with generic face $F$, $t(F) = \eta$ we have that for positive constants explicitly calculated

$$\inf_{P} |h| \leq c_{11} \inf_{F} |h| + \epsilon$$

$$\inf_{F} |k|^2 \leq c_{12} \inf_{F} |k|^2 + \epsilon$$

where $c_{11}, c_{12}$ depend in an explicit way on $F$. Furthermore let $u$ be an eigenfunction with eigenvalue $\lambda$ and $\epsilon$ a regular value of $u$

$$P_\epsilon = P \cap \{|u| > \epsilon\}, \quad F_\epsilon = F \cap \{|u| > \epsilon\}$$

Then for explicitly calculated positive constants $c_{2j} = c_{2j}(P, \lambda), c_{3j} = c_{3j}(F), j = 0, 1, 2, 3, 4$ we have that

$$\sup_{P_\epsilon} |u| \leq c_{20} \epsilon$$

$$\sup_{P_\epsilon} (|\nabla u|) \leq c_{21} \inf_{P_\epsilon} |\nabla u| + c_{24}$$

$$\sup_{P_\epsilon} (|\nabla^2 u|) \leq c_{22} \inf_{P_\epsilon} |\nabla^2 u| + c_{25}$$

$$\sup_{F_\epsilon} |u| \leq c_{30} \epsilon$$

$$\sup_{F_\epsilon} (|\nabla u|) \leq c_{31} \inf_{F_\epsilon} |\nabla u| + c_{34}$$

$$\sup_{F_\epsilon} (|\nabla^2 u|) \leq c_{32} \inf_{F_\epsilon} |\nabla^2 u| + c_{35}$$
These estimates provide the basis for the inductive argument and the application of Dong’s formula:

**Theorem 2.** Let \( u \) be as above then

\[
\mathcal{H}^{n-1}(N(u_\lambda)) \sim \sqrt{\lambda}
\]

## 1 Decomposition of a manifold

### 1.1 Quantitative Huygens principle

We introduce the set of points, chosen to lie on the nodal set

\[ C_0 = \{ \{ \lambda \}_{i=1}^N \subset M^u \} \]

and lying at distance

\[
d(\mathcal{C}_0, \mathcal{C}_0') \sim d_0 \sim \frac{1}{\sqrt{\lambda}}
\]

This is possible since in every geodesic ball of radius \( O(\lambda^{-\frac{1}{2}}) \) there is always a zero. Furthermore we define \( n \)-dimensional balls of radii \( r_i \), \( B_n^r(C_0) \), bounded by geodesic spheres \( S_{n-1}^r(C_0) \). These are taken to overlap in \((n+1)\)-ples: we set as

\[
P^{0,1,\ldots,k}_0 = \bigcap_{j=1}^k B_{n}^r(C_0)^j \]

for \( k = 2, \ldots, n + 1 \). These domains are bounded by spherical regions denoted as \( F^{0,1,\ldots,k} \), and have interior curvature data

\[
Ric^{0,1,\ldots,k}
\]

and face second fundamental form

\[
h^{0,1,\ldots,k} \quad k^{0,1,\ldots,k}.
\]

We call these sets *geodesic pixels*.

We introduce the collection of new centers and arrive at the collection of pixels after generation \( j \):

\[
P^{j;1,\ldots,k}_j
\]

Its boundary consists of the elementary wave fronts \( F^{j;1,\ldots,k} \) and curvature data:

\[
Ric^{j;1,\ldots,k}, \quad h^{j;1,\ldots,k}, \quad k^{j;1,\ldots,k}.
\]

It is written in the form for

\[
\partial P^{j;1,\ldots,k} = \bigcup_{j',k',\ell'} F^{j',1,\ldots,k;\ell'}
\]

The faces \( F^{j;1,\ldots,k;\ell'} \) are called *elementary wave fronts* (EWF). Each pixel defines homothetic EWF spanned by the tubular neighbourhoods of the elementary wave fronts:

\[
K^{(j;k,\ell)} = I_{r,\varepsilon} \times F^{j;1,\ldots,k}, \quad I_{r,\varepsilon} = ((1-\varepsilon)r, (1+\varepsilon)r)
\]

We introduce the localized tension \( T(h; \vartheta) \) of an EWF given for a smooth test function \( \vartheta \), \( \text{supp} \vartheta \subset F^{j;1,\ldots,k} \):

\[
T^j(h; \vartheta) = \int_{F} \vartheta |X^j|h|^2
\]

where \( \bar{h} \) is the mean curvature of the (EWF). We introduce two numbers

\[
r(P) = \sup_{\zeta \in C^\infty_0(P)} \left( \frac{\bar{h}^{(1)}(R; B; \zeta)}{\bar{h}^{(0)}(R; \zeta)} \right)
\]

and

\[
t(F) = \sup_{\vartheta \in C^\infty_0(F)} \left( \frac{T^{1}(h; \vartheta)}{T^{0}(h; \vartheta)} \right)
\]
Let $\eta, \mu$ be positive constants. We say that a pixel $P$ with boundary consisting of EWF $\partial P = \bigcup_{\ell=1}^{n+1} F_{\ell}$ satisfies an $(\eta, \mu)$ condition if

$$t(F_{\ell}) \leq \eta, \quad r(P) \leq \mu$$

Let $(\eta_{j}, \mu_{j}, \epsilon_{j})$ be positive constants. We assume that the geodesic pixels $P(j;i_{1}, \ldots, i_{k})$ are selected so that their EWF $F(j;i_{1}, \ldots, i_{k})$ satisfy $(\eta_{j}, \epsilon_{j})$ estimate while $(\mu_{j}, \epsilon_{j})$ are the parameters in the curvature estimate. This set of pixels is a subset of compact closure in the neighbourhood of the zero section in $TM$.

In the sequel we will try to establish the equations defining this set, as consequences of the above integral inequalities.

**The structure of the metric in geodesic polar coordinates.** Every geodesic pixel has center some $C_{j}i_{obtained in the j-th generation of center selection. We consider geodesic coordinates from the neighboring pixels. Therefore let $B_{r}C_{j}i_{be a geodesic ball centered at the point $C_{j}$ and introduce polar coordinates through Gauss lemma. The metric is written then as:

$$g = dr^2 + \gamma(r)$$

where $\gamma(r)$ is a riemannian metric on the geodesic sphere $S^{n-1}$, $\partial B_{r}C_{j}$ with second fundamental form and mean curvature respectively $\hat{k}, \hat{h}$. Accordingly we have the first and second variation equations for the metric $\gamma$, if we denote the radial derivative by $\frac{d}{dr}$ while the angular ones by $; j$ while in this section the symbol $\nabla$ denotes collectively the angular derivatives:

$$\frac{d\gamma}{dr} = 2k,$$

$$\frac{d\hat{k}_{ij}}{dr} = -\hat{k}_{im}\hat{k}_{jm} - R_{00ij} - \frac{1}{2} \hat{h}^2 - k^2 = R - R_{00},$$

Then setting:

$$\psi = \frac{1}{2} \log (\det(\gamma))$$

$$\sigma = R_{00ij}k_{ij} + k_{im}k_{jm}k_{ij},$$

$$\kappa = \kappa_0 = |k|$$

we infer that

$$\frac{d\psi}{dr} = h,$$

$$\kappa \frac{ds}{dr} = \sigma$$

(2)

Based on Newton’s identities for symmetric polynomials we get the following inequality for $\sigma$:

$$(3 - C_{n}\alpha)h\kappa^3 - h^3 - |Rm|\kappa \leq \sigma \leq \frac{C_{n}\alpha + 3}{3} h\kappa^2 - \frac{1}{2} h^3 + |Rm|\kappa$$

or

$$-(C_{n}\alpha + 2n)\kappa^3 - |Rm|\kappa \leq \sigma \leq \frac{C_{n}\alpha + 3n}{3} \kappa^3 + |Rm|\kappa$$

**The structure equations** The Gauss equations that relate the curvature of $\gamma, \overline{R}$ to the ambient curvature

$$\overline{R}_{imjn} + (k_{ij}k_{mn} - k_{im}k_{jm}) = R_{imjn},$$

$$\overline{R}_{ij} + k_{ij}h - k_{im}k_{mj} = R_{ij},$$

$$\overline{R} + \hat{h}^2 - k^2 = R - R_{00},$$

(3c)

The Codazzi equations

$$\overline{\nabla}_{i}k_{jm} - \overline{\nabla}_{j}k_{im} = R_{m0ij},$$

(4)
constitute a Hodge system:

\[
\begin{align*}
\omega(k)_{ijm} &= R_{mij} \\
\delta \omega(k)_{i} &= - \nabla_i h = R_{0i}
\end{align*}
\]

where

\[
\omega(U)_{ijk} = \mathcal{X}_k U_{ij} - \mathcal{X}_j U_{ik}, \quad \delta \omega(U)_{i} = \mathcal{X}_j U^2_j
\]

1.2 The second fundamental form and the mean curvature of the fronts

1.2.1 Harnack on the slice

The Gauss-Codazzi equation for the spherical front gives that

\[
|\mathcal{X}h|^2 = \delta \omega(k)_{i} h_i - R_{0} h_i,
\]

\[
|\delta \omega(k)_{i}|^2 = \delta \omega(k)_{i} h_i + R_{0} h_i
\]

(5)

Elaborating the preceding identities with Young inequality for suitable \( p \) and obtain \( h \)-growth inequality in the spherical front domain \( \mathcal{F} = \mathcal{F}^{(U; 1, \ldots, \ell)} \) and cut-off \( \vartheta \):

\[
\int_{\mathcal{F}} |\delta \omega(k)|^2 \leq \frac{\eta \vartheta^p}{p} \int_{\mathcal{F}} \vartheta^2 |h|^{2p} + \frac{(p - 1) \text{vol}((\mathcal{F}, \ell))^{\frac{p+1}{2p}}}{p} \int_{\mathcal{F}} |\delta \omega(k)|^2
\]

or choosing \( \epsilon = \left( \frac{2(p-1)}{p} \right)^{1 - \frac{1}{p}} \text{vol}((\mathcal{F}, \ell))^{\frac{1}{2p}} \) and get that

\[
\int_{\mathcal{F}} |\delta \omega(k)|^2 \leq \eta \left( \frac{2}{p} \right)^p (p - 1)^{p-1} \text{vol}((\mathcal{F}, \ell))^{\frac{p+1}{2p}} \int_{\mathcal{F}} \vartheta^2 |h|^{2p}
\]

We recall Sobolev inequality from \([\text{SLI}]\) for the case of EWF, \( \mathcal{F} \subset \mathcal{W}, r = \frac{n+1}{n-p-1} \):

\[
\left( \int_{\mathcal{F}} |U|^r \right)^\frac{1}{r} \leq C \int_{\mathcal{F}} |\mathcal{X}U| + |h| |U|
\]

Starting from

\[
\left( \int_{\mathcal{F}} |U|^{tr} \right)^\frac{1}{r} \leq tC \int_{\mathcal{F}} |U|^{t-1} |\mathcal{X}U| + |h| |U|^t
\]

and applying Hölder’s inequality

\[
\int_{\mathcal{F}} |U|^{t-1} |\mathcal{X}U| \leq \left( \text{vol}(\mathcal{F}) \right)^{1 - \frac{1}{t}} \left( \int_{\mathcal{F}} |U|^{p(t-1)} \right)^{\frac{1}{t}} \left( \int_{\mathcal{F}} |\mathcal{X}U|^{\frac{t}{t-1}} \right)^{1 - \frac{1}{t}}
\]

(6)

where \( q < 2 \). The inequalities on the slice

\[
\int_{\mathcal{F}} |\mathcal{X}U|^2 \leq C \int_{\mathcal{F}} |\delta \omega(U)|^2 + |\omega(U)|^2 + |Rm| |U|^2
\]

read for the localization second fundamental form, \( U = \zeta |k| \):

\[
\int_{\mathcal{F}} \zeta^2 |\mathcal{X}k|^2 \leq C \int_{\mathcal{F}} |\zeta|^2 |\mathcal{R} c| + \zeta |\mathcal{X}h|^2 + (|\mathcal{X}\zeta|^2 + |Rm| \zeta^2) |k|^2
\]

The last term in the right hand side leads after the application of Young’s inequality combined with Sobolev’s inequality to

\[
\int_{\mathcal{F}} \zeta^2 |\mathcal{X}k|^2 \leq C \int_{\mathcal{F}} |\zeta|^2 |\mathcal{R} c| + \zeta |\mathcal{X}h|^2 + \int_{\mathcal{F}} (|\mathcal{X}\zeta|^2 + |Rm|) \zeta^{\frac{n+1}{n}}
\]

We consider now the slice regions \( \mathcal{F}_i \) determined by the tension energy through the sequence of constants \( \{\eta_j\} \), in which \( t(\mathcal{F}_i) = \eta_j \). The Harnack inequality is proved through Moser iteration on the domain with smooth
boundary $W \subset F$ obtained by smoothing out the boundary of $F_j$. Therefore we exhaust the domain through the harmonic approximation of the face defining function $F, \hat{F}_0$:

$$W_j(\eta) = \{ \tilde{\nu} \in F : (\theta - \theta^{j+1})\eta \leq |\hat{F}_0(\tilde{\nu})| \leq (1 - \theta + \theta^j)\eta \}$$

and then Harnack inequality takes the form:

$$\sup_{W} |h| \leq D(\eta, \eta_0) \inf_{W} |h|$$

where the quantity $D(\eta, \eta_0) > 0$ is calculated in the appendix.

### The growth of the tension integral

The preceding estimates necessitate the derivation of radial growth estimates for the tension integral $T = T^1(h; \vartheta)$.

We start differentiating and proceed with the application of the structural equations. We obtain the differential inequality

$$\frac{dT}{dr} \leq 2 \left( \int_F \vartheta^2 |h| |\nabla h|^2 + \int_F |\nabla h|^2 \left| \frac{dh}{dr} \right| + \int_F \vartheta^2 |h| \left| \frac{dh}{dr} \right| \right)$$

We obtain that

$$\left| \frac{dh}{dr} \right| \leq 2|k||\nabla k| + |\nabla R_{00}|$$

Therefore we have that:

$$\frac{dT}{dr} \leq \left( \sup_F |h| \right) T + \left( \sup_F |k| \right) \left( \int_F |\nabla h|^2 \right)^{1/2} \left( \int_F |\nabla k|^2 \right)^{1/2}$$

The last term is majorised by

$$\left( \int_F |\nabla h|^2 \right)^{1/2} \left( \int_F |\nabla k|^2 \right)^{1/2} \leq CT + \int_F (\vartheta^2 |Ric| + |Rm| |\nabla \zeta|^2)$$

Furthermore we have that:

$$\sup_F |k| \leq c_1 T^{1/2} + c_2$$

We select cutoffs satisfying for some $\eta > 0$

$$\frac{d|\nabla \vartheta|}{dr} \leq \eta |\nabla \vartheta|$$

We arrive at the inequality

$$\frac{dT}{dr} \leq c_1 T^{3/2} + c_2, \quad c_i = c_i(F, \epsilon), i = 1, 2$$

We conclude through the use of Young’s inequality:

$$\frac{dT}{dr} \leq c \left( T^2 + 1 \right)$$

and hence for $r \leq \frac{\eta}{20C}$:

$$|T(r(1 + \epsilon)) - T(r(1 - \epsilon))| \leq \tan(Cr) \leq C r$$

### 1.2.2 Radial Harnack estimates

In this section we derive the radial variation of the curvature quantities in the radial interval $I_{r, \epsilon} = ((1 - \epsilon)r, (1 + \epsilon)r)$ relative to a given value of this quantities at $r$. We write (11) in contracted form:

$$\frac{dh}{dr} = -k^2 - R_{00}$$

First we have that

$$\frac{dh}{dr} \leq -\frac{1}{n-1} h^2 - R_{00} \Leftrightarrow h^2 \leq -(n - 1) \frac{dh}{dr} - (n - 1)R_{00}$$
We derive the differential inequality for \( k \):
\[
\frac{d|k|}{dr} \leq |k|^2 + |Rm| \leq (|k|^2 + 1)(|Rm| + 1)
\]
that is written after integration and elementary trigonometry as:
\[
\frac{||k(r(1 + \varepsilon))| - |k(r(1 - \varepsilon))||}{1 + |k||(r(1 + \varepsilon))|k||r(1 - \varepsilon)|} \leq \tan(Cr)
\]
for
\[
\max_{I_{r,\varepsilon}} \left( C(\varepsilon) \int_{K} |Rm| \right), \max_{I_{r,\varepsilon}} \left( |k(r(1 + \varepsilon))| \cdot |k(r(1 - \varepsilon))| \right) \leq C
\]
For \( r \leq \frac{9}{20\varepsilon} \):
\[
||k(r(1 + \varepsilon))| - |k(r(1 - \varepsilon))|| \leq C'r
\]
The estimates for \(|\nabla k|, |\nabla^2 k|\) follow from the elementary differential inequalities:
\[
\frac{d|\nabla k|}{dr} \leq |k||\nabla k| + (|\nabla Rm| + |Rm||k|)
\]
\[
\frac{d|\nabla^2 k|}{dr} \leq C (|k||\nabla^2 k| + |\nabla k|^2 + |Rm||\nabla k| + |\nabla Rm||k| + |\nabla^2 Rm|)
\]
This lead to the estimates:
\[
|\nabla k| \leq \left[ Cr^2 \left( \int_{K} |Rm| \right) + \int_{K} |\nabla Rm| \right] e^{\frac{c_2 r^2}{\varepsilon}}
\]
Similarly
\[
|\nabla^2 k| \leq \left[ Cr^2 \left( \int_{K} |Rm| \right) + \int_{K} |\nabla Rm| \right] e^{\frac{c_2 r^2}{\varepsilon}}
\]
Let \( \chi \) be a cutoff supported in \( \text{supp}(\chi) \subset I_{r,\varepsilon} \) such that
\[
\varepsilon|\chi'| + \varepsilon^2|\chi''| \leq C
\]
as well as
\[
\mu_\varepsilon(r) = \int_{I_{r,\varepsilon}} \chi h(r', \xi) dr'
\]
**Lemma 3.** The following estimates hold for \( r \leq \frac{1}{\sqrt{C_0}} \) and are relative to a fixed value of \( h(r) \):
\[
|\mu_\varepsilon| \leq (2\varepsilon r)^{1/2} \left( |h|^{1/2} + (2\varepsilon r|C_0|)^{1/2} \right)
\]
\[
|\nabla \mu_\varepsilon| \leq 2\varepsilon r \left( |\nabla h| + 2(1 + \varepsilon) r c_1 (|Rm|, |\nabla Rm|) \right)
\]
\[
|\nabla^2 \mu_\varepsilon| \leq (2\varepsilon r) \left( |\nabla^2 h| + 2r(1 + \varepsilon) c_2 (|Rm|, |\nabla Rm|, |\nabla^2 Rm|) \right)
\]
**The estimate of \( \mu_\varepsilon \)** We start applying and Cauchy-Schwarz in the end
\[
|\mu_\varepsilon| \leq (2\varepsilon r)^{1/2} \left( \int_{I_{r,\varepsilon}} h^2 \right)^{1/2} \leq ((n-1)\varepsilon r)^{1/2} \left( |h(r(1 - \varepsilon)) - h(r(1 + \varepsilon))|^{1/2} + (2\varepsilon r|C_0|)^{1/2} \right) \leq (n-1)^{1/2}\varepsilon r (|k| + C_0)
\]
**Estimate of \( |\nabla \mu_\varepsilon|, |\nabla^2 \mu_\varepsilon| \)** We commence with the integration by parts for \( U : I_{r,\varepsilon} \rightarrow \mathbb{R} \):
\[
\left| \int_{I_{r,\varepsilon}} U \right| = \left| \int_{I_{r,\varepsilon}} \frac{dr}{dr} |U| \right| \leq \int_{I_{r,\varepsilon}} \frac{dr}{dr} |U| \right| \leq 2\varepsilon r |U| - \int_{I_{r,\varepsilon}} r \frac{d|U|}{dr}
\]
and apply this to
\[
\frac{d|\nabla h|}{dr} \leq C_1 (|k| |\nabla k| + |\nabla R_{00}|)
\]
\[
\frac{d|\nabla^2 h|}{dr} \leq C_2 (|k| |\nabla^2 k| + |\nabla k|^2 + |\nabla^2 R_{00}|)
\]
and obtain the desired estimates.
The radial-slice estimates. We start recalling the differentiation identity:

\[
\frac{d}{dr} \left( \int_U F \right) = \int_U \frac{dU}{dr} + hU
\]

We introduce a cutoff function \( \tilde{\chi} \) supported in the shell \( \text{supp} \tilde{\chi} \subset K \):

\[
K = I_{r,\varepsilon} \times F; \\
I_{r,\varepsilon} = ((1-\varepsilon)r, (1+\varepsilon)r)
\]

Therefore applying Sobolev inequality on the slice, Hölder and standard elliptic estimates we derive the differential inequalities for the quantities

\[
\frac{dU_j}{dr} \leq aU_j + b, \quad \frac{dH_j}{dr} \leq a\sqrt{U_j} H_j + c_j
\]

where

\[
U_j = \int_F |\mathbf{X}_j k|^2, \quad H_j = \int_F |\mathbf{X}_j h|^2
\]

\[
a = D(\eta) \left[ \int_F |k|^2 + h^2 \right]^{1/2}, b_0 = \int_F |Rm|,
\]

\[
b_1 = \int_F |\nabla Rm|^2 + U_0 \int_F |Rm|^2,
\]

\[
b_2 = U_1 \left[ U_1 + U_0 \int_F |Rm|^2 \right] + \left( \int_F |\nabla Rm|^2 \right) U_0 + \int_F |\nabla^2 Rm|^2
\]

The differential equations for \( U_j, H_j \) are of the form

\[
\frac{dU_j}{dr} \leq aU + b
\]

we obtain for \( |\tilde{\varrho}| \leq \varepsilon \):

\[
U(r(1 + \tilde{\varrho})) \leq e^{r \int_0^r a(1+r')dr'} \left( U(r) + r \int_0^r b(1+r')e^{-r'} \int_0^{r'} a(1+r'e')dr' \right)
\]

Lemma 4. The following hold true in \( I_{r,\varepsilon} \) for \( i, j = 1, 2 \) and constants \( c_{ij} = c_{ij}(\tau; \eta_1, \ldots, \eta_j; \varepsilon_j) > 0 \)

\[
U_{j,\pm}(r(1 \pm \varepsilon)) \leq c_{1j} U_{j,\pm}(r), \quad H_{j,\pm}(r(1 \pm \varepsilon)) \leq c_{2j} H_{j,\pm}(r)
\]

1.3 Selection of the pixels through the front estimates

1.3.1 The basic ansatz

We introduce the notation for a smooth domain \( W \) in a riemannian manifold equipped with riemannian volume \( dv \):

\[
\mathcal{D}^j(U; W) = \int_W |\nabla^j U|^2 dv
\]

We will derive the equation satisfied by \( u_\lambda \) near (EWF) i.e. in a spherical shell of the form

\[
K = I_{r,\varepsilon} \times F = ((1-\varepsilon)r, (1+\varepsilon)r) \times F
\]

for some \( F = F^{(i_1, \ldots, i_k, \ell)} \) with coordinates \( (r, \theta) \) and volume \( drd\sigma = \sqrt{\gamma} dr d\theta \), while \( h_\ell \) stands for the mean curvature of the shell.

We select the front so that its mean curvature is controlled by the eigenfunction growth near it. The eigenfunction equation is written near (EWF) in the form:

\[
\frac{d^2 u_\lambda}{dr^2} + h \frac{du_\lambda}{dr} + \Delta_{\gamma} u_\lambda = -\lambda u_\lambda
\]

We make the ansatz for the parameter \( \beta \) to be determined and the smooth functions \( \alpha, \phi \):

\[
u_\lambda(r, \theta) = A(r, \theta) \sin \left( \beta_1 \phi(r, \theta) \right),
\]

\[
\beta_1 = \frac{\lambda}{\Delta_{\gamma} A(r, \theta)}
\]

\[
A(r, \theta) = \frac{1}{\sin \left( \beta_1 \phi(r, \theta) \right)}
\]

\[
\Delta_{\gamma} A(r, \theta) = \frac{\Delta \gamma A(r, \theta)}{\sin \left( \beta_1 \phi(r, \theta) \right)}
\]

\[
\beta_1 = \frac{\lambda}{\Delta_{\gamma} A(r, \theta)}
\]

\[
u_\lambda(r, \theta) = A(r, \theta) \sin \left( \beta_1 \phi(r, \theta) \right)
\]

\[
\beta_1 = \frac{\lambda}{\Delta_{\gamma} A(r, \theta)}
\]

\[
A(r, \theta) = \frac{1}{\sin \left( \beta_1 \phi(r, \theta) \right)}
\]

\[
\Delta_{\gamma} A(r, \theta) = \frac{\Delta \gamma A(r, \theta)}{\sin \left( \beta_1 \phi(r, \theta) \right)}
\]

\[
\beta_1 = \frac{\lambda}{\Delta_{\gamma} A(r, \theta)}
\]

\[
u_\lambda(r, \theta) = A(r, \theta) \sin \left( \beta_1 \phi(r, \theta) \right)
\]
and arrange the (EWF) so that near the front $A$ is smooth and positive which implies that $\phi$ inherits the unique continuation property from $u_\lambda$. Note also that we should inspect also level sets of the form $\frac{\mu e}{\beta_1}$ for suitable $\lambda$’s. The equation then splits as:

\[ \left[ \frac{d^2 A}{dr^2} + \frac{dA}{dr} - \beta_1^2 \left( \left( \frac{d\phi}{dr} \right)^2 + |\mathbf{\nabla}\phi|^2 \right) A + (\mathbf{\nabla} A + \lambda A) \right] \sin(\beta_1 \phi) + \right. \\
+ \left. \beta_1 \left[ \left( \frac{2dA}{dr} + hA \right) \frac{d\phi}{dr} + \frac{d^2 \phi}{dr^2} \right] \right] \cos(\beta_1 \phi) = 0 \]

**(9)**

### 1.3.2 Determination of local phase and local amplitude

We require that for a function $\alpha$ and a parameter $\beta_2$ to be determined there holds in the interval $(r_0(1-r), r_0(1+r))$:

\[ \frac{dA}{dr} + \frac{h}{2} A = \frac{\beta_2 e^{\beta_2 \alpha}}{2} e^{-\frac{\mu e}{\beta_1}} \]

with the conditions

\[ A(0, \theta) = 1, \quad \alpha(0, \theta) = -\frac{2 \log \beta_2}{\beta_2} \]

We find

\[ A(\sqrt{\lambda} r, \theta) = e^\lambda, \quad \Lambda = -\frac{1}{2} \mu e + \beta_2 \alpha \]

Therefore we obtain the pair of first order equations:

\[ \left( \frac{d\phi}{dr} \right)^2 + |\mathbf{\nabla}\phi|^2 = \frac{\lambda}{\beta_1^2} - \frac{1}{\beta_1^2} s_1, \quad s_1 = \beta_2 \left( \mathbf{\nabla} \alpha + \frac{d^2 \alpha}{dr^2} \right) + \frac{\beta_2^2}{2} \left[ |\mathbf{\nabla}\alpha|^2 + \left( \frac{d\alpha}{dr} \right)^2 \right] = -\beta_2 \mathbf{\nabla} \mu \cdot \mathbf{\nabla} \alpha + \mathbf{K} \]

if

\[ \mathbf{K} = \frac{1}{4} (2|k|^2 + 2R_{00} - h^2) - \frac{1}{2} \mathbf{\nabla} \mu + \frac{1}{4} |\mathbf{\nabla}|^2 \]

The other term gives

\[ \beta_2 \left( \frac{d\phi}{dr} \frac{d\alpha}{dr} + 2 \mathbf{\nabla} \phi \cdot \mathbf{\nabla} \alpha \right) = -s_2, \quad s_2 = \frac{d^2 \phi}{dr^2} + \mathbf{\nabla} \phi - \mathbf{\nabla} \mu \cdot \mathbf{\nabla} \phi \]

We choose $\beta_1 = \sqrt{\lambda} = \frac{1}{\beta_2}$. The solution of these equations is achieved through the method of characteristics. We set

\[ \mathbf{A} = |\mathbf{\nabla}\alpha|^2 + \left( \frac{d\alpha}{dr} \right)^2 \]

\[ \mathbf{P} = |\mathbf{\nabla}\phi|^2 + \left( \frac{d\phi}{dr} \right)^2 \]

\[ \Lambda = \frac{2}{\lambda} \alpha + \mu \]

Then we will consider this system in the form

\[ \mathbf{P} = 1 + \frac{1}{\lambda \sqrt{\lambda}} \left[ \mathbf{\nabla} \alpha + \frac{d^2 \alpha}{dr^2} + \frac{1}{\lambda} \mathbf{A} - \frac{1}{\sqrt{\lambda}} \mathbf{\nabla} \mu \cdot \mathbf{\nabla} \alpha + \mathbf{K} \right] \]

\[ \frac{d\phi}{dr} \frac{d\alpha}{dr} + 2 \mathbf{\nabla} \phi \cdot \mathbf{\nabla} \alpha = -\lambda \left( \frac{d^2 \phi}{dr^2} + \mathbf{\nabla} \phi - \mathbf{\nabla} \mu \cdot \mathbf{\nabla} \phi \right) \]

We derive estimates for $\alpha, \phi$ in the form

\[ -\mathbf{\nabla} \alpha - \frac{d^2 \alpha}{dr^2} - \frac{1}{\sqrt{\lambda}} \mathbf{\nabla} \mu \cdot \mathbf{\nabla} \alpha = \lambda \sqrt{\lambda} [1 - \mathbf{P}] + \frac{1}{\lambda} \mathbf{A} + \mathbf{K}, \quad \frac{d\phi}{dr} \frac{d\alpha}{dr} = \frac{1}{\lambda} \mathbf{A} \]

\[ -\frac{d^2 \phi}{dr^2} - \mathbf{\nabla} \phi + \mathbf{\nabla} \Lambda \cdot \mathbf{\nabla} \phi = \frac{1}{\lambda} \frac{d\phi}{dr} \frac{d\alpha}{dr} \]
1.3.3 Equations for the higher derivatives of the phase and amplitude functions

Furthermore differentiating the equations (12) we obtain the necessary equations for the higher derivatives of the phase function,

\[ \Psi_{0,m} = \frac{d^m \phi}{dr^m}, \quad a_{0,m} = \frac{d^m \alpha}{dr^m}, \quad A_{0,m} = \frac{d^m A}{dr^m}, \quad A_m = \frac{d^m A}{dr^m}, \quad P_{0,m} = \frac{d^m P}{dr^m}, \quad m = 1, 2, 3 \]

and the angular derivatives

\[ \Psi_{\gamma,m} = |\nabla^m \phi|^2, \quad a_{\gamma,m} = |\nabla^m \alpha|^2, \quad A_{\gamma,m} = |\nabla^m A|^2, \quad P_{\gamma,m} = |\nabla^m P|^2, \quad m = 1, 2, 3 \]

Radial derivatives of the phase function  Specifically we obtain, applying the commutation rules, the following equations:

\[ \nabla \Psi_{0,1} + \frac{d^2 \Psi_{0,1}}{dr^2} - \nabla \Lambda \cdot \nabla \Psi_{0,1} = S_{0,1} \quad (13a) \]
\[ \nabla \Psi_{0,2} + \frac{d^2 \Psi_{0,2}}{dr^2} - \nabla \Lambda \cdot \nabla \Psi_{0,2} = S_{0,2} \quad (13b) \]
\[ \nabla \Psi_{0,3} + \frac{d^2 \Psi_{0,3}}{dr^2} - \nabla \Lambda \cdot \nabla \Psi_{0,3} = S_{0,3} \quad (13c) \]

where the "source terms" are the following:

\[ S_{0,1} = R_{1}^{0} \Psi_{1} + R_{1}^{1} \nabla \phi \cdot \nabla \phi \quad (14a) \]
\[ S_{0,2} = 2R_{1}^{1} \nabla \Psi_{1} + dR_{1}^{1} d \nabla \psi_{1} + \nabla \psi_{1} + \nabla \Lambda \cdot \nabla \phi \quad (14b) \]
\[ S_{0,3} = - (R_{1}^{1} \nabla \phi_{1} + 2R_{1}^{1} \nabla \phi_{1} \nabla \psi_{1} + 2R_{1}^{1} \nabla \phi_{1} \nabla \psi_{1} + \nabla \Lambda \cdot \nabla \phi) \quad (14c) \]

Higher angular derivatives of the phase  We obtain, applying the commutation rules, the following equations:

\[ \nabla \Psi_{\gamma,1} + \frac{d^2 \Psi_{\gamma,1}}{dr^2} - \nabla \Lambda \cdot \nabla \Psi_{\gamma,1} = S_{2,1} \quad (15a) \]
\[ \nabla \Psi_{\gamma,2} + \frac{d^2 \Psi_{\gamma,2}}{dr^2} - \nabla \Lambda \cdot \nabla \Psi_{\gamma,2} = S_{2,2} \quad (15b) \]

where the "source terms" are the following:

\[ S_{2,1} = - (2R_{1}^{0} \nabla \phi_{1} + R_{1}^{1} \nabla \phi_{1} \nabla \phi) \nabla \phi - (\tau_{j} \nabla \phi_{1} \nabla \phi_{j} + \tau_{i} \nabla \phi_{1} \nabla \phi_{j} \nabla \phi) \quad (16a) \]
\[ S_{2,2} = - R_{m} \nabla \phi_{1} \nabla \phi_{1} \nabla \phi_{1} \nabla \phi - R_{m} \nabla \phi_{1} \nabla \phi_{1} \nabla \phi_{1} \nabla \phi - R_{m} \nabla \phi_{1} \nabla \phi_{1} \nabla \phi_{1} \nabla \phi + \nabla \tau_{i} \nabla \phi_{i} \quad (16b) \]

Radial derivatives of the amplitude  Set first that:

\[ V = \frac{1}{\lambda} A + \lambda^{3/2} (1 - P) + K \]

Specifically we obtain, applying the commutation rules, the following equations:

\[ - \nabla a_{0,1} + \frac{d^2 a_{0,1}}{dr^2} + \frac{1}{\sqrt{\lambda}} \nabla \mu \cdot \nabla a_{0,1} = S_{3,1} + \frac{dV}{dr} \quad (17a) \]
\[ - \nabla a_{0,2} + \frac{d^2 a_{0,2}}{dr^2} + \frac{1}{\sqrt{\lambda}} \nabla \mu \cdot \nabla a_{0,2} = S_{3,2} + \frac{d^2 V}{dr^2} \quad (17b) \]
\[ - \nabla a_{0,3} + \frac{d^2 a_{0,3}}{dr^2} + \frac{1}{\sqrt{\lambda}} \nabla \mu \cdot \nabla a_{0,3} = S_{3,3} + \frac{d^2 V}{dr^3} \quad (17c) \]

where the "source terms" are the following:

\[ S_{3,1} = R_{0}^{0} a_{0,1} + R_{1}^{1} a_{0,1} + \nabla \mu \cdot \nabla \alpha \quad (18a) \]
\[ S_{3,2} = 2R_{1}^{1} a_{0,1} + \frac{dR_{1}^{1} d}{dr} a_{0,1} + \nabla \phi \cdot \nabla \alpha \quad (18b) \]
\[ S_{3,3} = - \left( R_{1}^{1} a_{0,2} + 2R_{1}^{1} a_{0,1} + 2R_{1}^{1} a_{0,1} \nabla \phi_{0,1} a_{0,1} + \nabla \phi \cdot \nabla \alpha \right) \quad (18c) \]
Higher angular derivatives of the amplitude  

We obtain, applying the commutation rules, the following equations:

\[
\begin{align*}
\Delta a_{\gamma,1} + \frac{d^2 a_{\gamma,1}}{dr^2} - \nabla \mu \cdot \nabla a_{\gamma,1} &= S_{4,1} \\
\Delta a_{\gamma,2} + \frac{d^2 a_{\gamma,2}}{dr^2} - \nabla \mu \cdot \nabla a_{\gamma,2} &= S_{4,2}
\end{align*}
\]

where the "source terms" are the following:

\[
\begin{align*}
S_{4,1} &= -(2R^{\alpha}_{0\beta}) \nabla_{\beta} \alpha + R^i \nabla_{\beta} \alpha \nabla_{\beta} \alpha - (\tau_i \nabla_{\beta} \alpha \nabla_{\beta} \alpha + \tau_i \nabla_{\beta} \alpha \nabla_{\beta} \alpha) \\
S_{4,2} &= -R \nabla^2 \alpha \cdot \nabla \alpha - \nabla R \nabla \alpha \cdot \nabla \alpha - R \nabla a_{\alpha,1} \cdot \nabla \alpha + \Delta \tau_i \nabla_{\beta} \alpha
\end{align*}
\]

1.3.4 Estimates  

We recall the variation identities:

\[
\begin{align*}
\int_F \frac{\partial^2 \nu}{\partial r^2} &= \frac{1}{2} \frac{d}{dr} \left( \int_F \nu^2 \right) - \frac{1}{2} \int_F h^2 \\
\int_F \frac{\partial^2 \nu}{\partial r^2} &= \frac{1}{2} \frac{d^2}{dr^2} \left( \int_F \nu^2 \right) - \int_F \left( \frac{d}{dr} \left( \frac{\partial \nu}{\partial r} \right) \right)^2 - \frac{d}{dr} \left( \int_F h^2 \right) \\
\frac{dh}{dr} &= -|k|^2 - R_{00}
\end{align*}
\]

[23] gives the pointwise estimate provided \( \alpha \geq 0 \):

\[
-\alpha \Delta \alpha - \alpha \frac{d^2 \alpha}{dr^2} - \frac{1}{\lambda} \Lambda \alpha + \frac{1}{\sqrt{\lambda}} (\nabla \mu \cdot \nabla \alpha) \alpha \leq \lambda^2 \alpha + K \alpha
\]

We employ Hardy’s inequalities for the harmonic approximation of \( \mu \) in the terms of the last parentheses:

\[
\int_F \left( \Delta \mu + \frac{1}{2} |\nabla \mu|^2 \right) \theta^2 \alpha \leq \sup_F |\mu| \int_F \left| \frac{\Delta \mu}{\mu} \right| \alpha + \frac{1}{2} \left( \sup_F |\mu| \right)^2 \int_F \left| \frac{\nabla \mu}{\mu} \right|^2 \alpha \leq \epsilon \left( c_3(H) + \frac{1}{2} c_2(H) \sup_F |\mu| \right) \int_F |\nabla \alpha|^2
\]

provided that

\[
\sup_F |\mu| \leq \epsilon \alpha
\]

We conclude that after elementary manipulations:

\[
\int_F \theta^2 |\nabla \alpha|^2 - \frac{d^2}{dr^2} \left( \int_F \theta^2 \alpha^2 \right) - \frac{d}{dr} \left( \int_F \theta^2 h \alpha \right) \leq c_\lambda \frac{3}{2} \int_F \theta^2 \alpha^2
\]

Similarly starting from

\[
- \phi \nabla \phi - \phi \frac{d^2 \phi}{dr^2} = -\nabla \mu \cdot \nabla \phi - \frac{1}{\lambda} \left( \phi \frac{d \alpha}{dr} + 2 \phi \nabla \alpha \cdot \nabla \phi \right)
\]

we get that

\[
\int_F \theta^2 |\nabla \phi|^2 - \frac{d^2}{dr^2} \left( \int_F \theta^2 \phi^2 \right) - \frac{d}{dr} \left( \int_F \theta^2 h \phi \right) \leq c_\lambda \frac{3}{2} \int_F \theta^2 \phi
\]

We introduce arbitrary test functions in the angular variables \( \theta \) supported in \( F \). Furthermore in [23], multiplying by \( \theta \) and using the harmonic approximation \( \Lambda \) of \( \alpha \) and its initial form \( \Lambda_0 \):

\[
|\Delta \Lambda| + |\nabla \Lambda|^2 \leq |\Delta \Lambda_0| + 2 |\nabla \Lambda_0|^2 + \epsilon
\]

Then for \( \eta = \sup_F |\Lambda_0| \) we apply GHI and get that

\[
\int_F \left( |\Delta \Lambda_0| + 2 |\nabla \Lambda_0|^2 + \epsilon \right) \theta^2 \leq \int_F \left( c_3(H) \eta + 2 c_2(H) \eta^2 + \epsilon \right) |\nabla \theta|^2
\]
and conclude that for suitable $\eta > 0$

$$\int_{F} \nabla^{2} |\varphi|^{2} \leq \int_{F} (|K| + \lambda) \vartheta^{2}$$

if we select $\beta_{1}^{2} = C$. Also we get that:

$$\int_{F} \vartheta^{2} \left( \frac{d\vartheta}{dr} \right)^{2} \leq C \int_{F} (|K| + \lambda) \vartheta^{2}$$

We will sketch the derivation of estimates derived from the preceding integral identities. We multiply the equation by $\vartheta^{2}$ and we obtain that

$$2 \int_{F} \vartheta^{2} |\varphi|^{2} \leq \int_{F} \vartheta^{2} \left( \frac{d\vartheta}{dr} \right)^{2} + 2 \int_{F} \vartheta^{2} \left( \frac{d\varphi}{dr} \right)^{2} \leq \int_{F} \vartheta^{2} (2\Lambda \varphi^{2} + 2\sigma |\varphi|^{2})$$

(27)

We select $\epsilon_{1} = \frac{1}{4}$, $\epsilon_{2} = \frac{1}{4}$ and obtain that:

$$2 \int_{F} \vartheta^{2} |\varphi|^{2} \leq \int_{F} \vartheta^{2} \left( \frac{d\vartheta}{dr} \right)^{2} + \frac{d}{dr} \left( \int_{F} \vartheta h^{2} \varphi \right) \leq \int_{F} \vartheta^{2} (2\Lambda \varphi^{2} + 2\sigma |\varphi|^{2})$$

(28)

and simplifies to the following inequality:

$$2 \int_{F} \vartheta^{2} |\varphi|^{2} \leq \int_{F} \vartheta^{2} \left( \frac{d\vartheta}{dr} \right)^{2} + \frac{d}{dr} \left( \int_{F} \vartheta h^{2} \varphi \right) \leq \int_{F} \vartheta^{2} (2\Lambda \varphi^{2} + 2\sigma |\varphi|^{2})$$

(29)

These are the basic equations that will be used in order to derive the necessary estimates.

**Lemma 5.** Let $\vartheta$ be a cut off along the shell $F$, then the following holds true:

$$c_{2}(r, F) \mathcal{D}^{1}(\vartheta \varphi; F) \leq \mathcal{D}^{0}(\vartheta \varphi; F) \leq C_{2}(r, F) \mathcal{D}^{1}(\vartheta \varphi; F)$$

### 1.3.5 The shell estimates

We observe that the equations (17), (19) are of the form as (17) and hence are set in the integral form:

$$\int_{F} \vartheta^{2} |\varphi|^{2} \leq \int_{F} \vartheta^{2} |\varphi|^{2} + \frac{d}{dr} \left( \int_{F} h \vartheta^{2} \varphi \right) \leq \int_{F} L \vartheta^{2} \varphi^{2}$$

(30)

where $L$ is an expression comprising $\mu, R, |\varphi| R$ etc. We introduce the quantities

$$\Pi_{0}(r) = \int_{F} \vartheta^{2} \varphi^{2} \geq 0$$

$$\Pi_{1}(r) = \int_{F} \left( \vartheta \frac{d\vartheta}{dr} \right)^{2} \geq 0$$

(31)

and we integrate (30) in the interval $I_{r, e} = ((1 - \epsilon)r, (1 + \epsilon)r)$ choosing

$$\text{supp}(\vartheta) \cap \{ r/(r, \theta) \in K \} \subset I_{r,e}$$

and obtain:

$$\int_{I_{r,e}} \left[ \frac{d\Pi_{0}(r')}{dr} \right]^{2} - \frac{d\Pi_{0}}{dr} \Pi_{1} dr' + \int_{I_{r,e}} \mathcal{D}^{1}[v; F; \vartheta] \Pi_{0}(r') dr' =$$

$$= - \int_{I_{r,e}} \Pi_{0}(r') \Pi_{2}(r') dr'$$

(32)
where

\[ D^1[v; F; \vartheta] = \int_F \vartheta^2 |\nabla v|^2, \quad (33a) \]

\[ Y_1 = \int_F h \vartheta^2 \varphi^2, \quad (33b) \]

\[ Y_2 = \int_F \bigg( L \vartheta^2 + \vartheta \frac{d^2 \vartheta}{dt^2} + \bigg( \frac{d \vartheta}{dt} \bigg)^2 + |\nabla \vartheta|^2 - \vartheta \Delta \vartheta \bigg) v^2 \quad (33c) \]

We notice first that

\[ \int_{I_r,\varepsilon} Y_1 \frac{d \Pi_0}{dr} \leq \frac{1}{2} \int_{I_r,\varepsilon} \bigg( \frac{d \Pi_0}{dr} \bigg)^2 + \frac{1}{2} \int_{I_r,\varepsilon} Y_2^2 \quad (34a) \]

\[ Y_1 \leq C(K) \left( \int_F (h \vartheta)^2 \right)^{1/2} \left[ \left( D^1[v; F; \vartheta] \right)^{1/2} + \left( \int_F |h|^2 \right)^{1/2} \right] \quad (34b) \]

In conclusion we obtain

\[ \int_{I_r,\varepsilon} \left( \frac{d \Pi_0}{dr} \right)^2 \leq C(P) \int_{I_r,\varepsilon} \Pi_0^2 \]

**Lemma 6.** Let \( \vartheta \) be a cut off along the shell \( F \), then the following holds true:

\[ c_2(r, F) D^1(\vartheta \varphi; F) \leq D^0(\vartheta \varphi; F) \leq C_2(r, F) D^1(\vartheta \varphi; F) \]

We will derive an upper bound for \( \psi^{2k} \), \( \psi = \log \Pi_0 \) then for \( r \in [\delta^{1/k}, 1] \):

\[ |\psi| \leq \frac{1}{\delta} \left( |\Pi_0|^{\delta} + \frac{1}{|\Pi_0|^\delta} \right) \]

Specifically we have by Hardy’s inequality

\[ \int_{I_r,\varepsilon} \psi^{2kp} \leq C_p (2k(1 + \epsilon))^{p} \int_{I_r,\varepsilon} |\psi|^{(2k-1)p} |\psi|^p \leq \frac{C_p}{\epsilon^{\frac{1}{k+\epsilon'}}} (2k(1 + \epsilon))^{p} \int_{I_r,\varepsilon} |\psi|^{(2k-1-\frac{1}{k+\epsilon'})p} |\Pi_0|^p \]

for \( 0 < \delta < 1 \) that we choose shortly. After Hölder inequality we obtain:

\[ \int_{I_{r,\varepsilon}} \psi^{2kp} \leq \frac{C_p C(P)}{\epsilon^{\frac{1}{k+\epsilon'}}} (2k(1 + \epsilon))^{p} \left( \int_{I_{r,\varepsilon}} |\psi|^{(2k-1-\frac{1}{k+\epsilon'})p} \right)^{\frac{2k-1+\frac{1}{k+\epsilon'}}{2k-1+\frac{1}{k+\epsilon'}}} \left( \int_{I_{r,\varepsilon}} |\Pi_0|^2 \right) \]

Then selecting \( \delta = \frac{1}{k+\epsilon' - 1}, \epsilon' > 0 \) we get:

\[ \int_{I_{r,\varepsilon}} \psi^{2kp} \leq \alpha \left( \int_{I_{r,\varepsilon}} |\Pi_0|^2 \right)^{\frac{2k-1+\frac{1}{k+\epsilon'}}{2k-1+\frac{1}{k+\epsilon'}}}, \quad \alpha_k = (2k(1 + \epsilon))^{2+\frac{1}{k+\epsilon'}} \left[ C_p C(P)(k + \epsilon' - 1)^{k+\epsilon'-1} \right]^{\frac{1}{k+\epsilon'}} \]

Then we have that

\[ \int_{I_{r,\varepsilon}} \psi^{2kp} \leq \alpha_k \left( \frac{2k-1+\frac{1}{k+\epsilon'}}{2k-1+\frac{1}{k+\epsilon'}} \left( 1 - \frac{\epsilon'}{k} \right) \left( \frac{2\epsilon'}{k} \right) \right)^{\frac{k}{k+\epsilon'}} \]

Summing up we get that:

\[ \sum_{k=0}^{\infty} \int_{I_{r,\varepsilon}} |\psi|^{2k} \leq \left( \sum_{k=0}^{\infty} \alpha_k \left( \frac{2k-1+\frac{1}{k+\epsilon'}}{2k-1+\frac{1}{k+\epsilon'}} \left( 1 - \frac{\epsilon'}{k} \right) \left( \frac{2\epsilon'}{k} \right) \right)^{\frac{k}{k+\epsilon'}} \right)^{\frac{1}{2}} \]

This is majorised after \( \Gamma \)-function duplication formula by

\[ \sum_{k=0}^{\infty} [C_p C(F)(1 + \epsilon)r]^k = c(F, r) \]
Therefore we obtain that:
\[ \sum_{j=0}^{\infty} \frac{x^{2j}}{(2j)!} \geq \frac{1}{2} e^x \]
and hence
\[ \int_{I_{r,\epsilon}} \Pi_0^2 \leq c(F, r) \]
Following the usual iteration obtained in the appendix we obtain the usual Harnack inequalities. These are culminated in the following

**Lemma 7.** The following estimates hold true:
\[ \int_{F} \phi^2 \theta^2 \leq C_{00}(K) \epsilon \]
\[ \int_{F} |\nabla \phi|^2 \theta^2 \leq C_{01}(K) \epsilon \]
\[ \int_{F} |\nabla^2 \phi|^2 \theta^2 \leq C_{02}(K) \epsilon \]
provided that
\[ \int_{F} \phi^2, \int_{F} |\nabla \phi|^2, \int_{F} |\nabla^2 \phi|^2 \geq \epsilon \]

### 1.4 The Lower Bound

The lower bound is obtained by an inductive argument based on the reduction to the boundary of a pixel. The one dimensional case indicates the method. Assume that we have a function \( \phi \) defined in the interval \([0, \mu]\), \( \phi(0) = \phi(\mu) = 0 \) satisfying
\[ c_1 \int_{0}^{\mu} \phi(x)^2 w(x) dx \leq \mu^2 \int_{0}^{\mu} \phi'(x)^2 w(x) dx \leq c_2 \int_{0}^{\mu} \phi(x)^2 w(x) dx \]
for a positive weight \( w, 0 < c_1 < c_2 \). Then we prove min-max principle allows us to assert that the roots of \( \phi \) in \([0, \mu]\) are at least \( c_1 \). Then we will derive the inequality through the min-max principle, standard eigenvalue and isoperimetric inequalities. The construction is based on the estimates of the preceding section that lead to Harnack inequalities for the restriction of the eigenfunction on the boundary of a geodesic pixel. We consider first a domain \( W_\epsilon \subset (M^n, g), n \geq 2 \) as is described in the appendix on the harmonic approximation. We drop the \( \epsilon \) subscript.

**Lemma 8.** Let \( W \) be a domain with smooth boundary \( \partial W \) equipped with a smooth metric \( \gamma \), induced from the metric \( g \). Let \( \psi : \partial W \to \mathbb{R} \) be a smooth nonnegative function satisfying the estimate for \( \gamma_j(\psi) = |\nabla^j \psi|^2 : \)
\[ c_{j0} \tau D^0(\gamma_j(\psi); \partial W) \leq D^1(\gamma_j(\psi); \partial W) \leq c_{j1} \tau D^0(\gamma_j(\psi); \partial W), \quad j = 0, 1, 2 \]

Let the zero set of \( \psi \), \( N(\psi) \) be \((n - 2)\)-rectifiable. Moreover let \( \phi : P \to \mathbb{R} \) be such that for \( \tau > 0 : \)
\[ c_{30} \tau D^0(\phi; W) \leq D^1(\phi; W) \leq c_{31} \tau D^0(\phi; W) \]

and
\[ D^0(\phi - \psi; \partial W) \leq \epsilon D^0(\psi; \partial W) \]

Then for \( C_4 = C_4(\tau, c_{10}, c_{11}, c_{20}, c_{21}, c_{30}, c_{31}) : \)
\[ m(\partial W \setminus N(\psi)) < C_4 \]

and
\[ \mathcal{H}^{n-1}(N(\phi)) \geq C_0 \tau^{-\frac{n-1}{2}} \]

\( C_0 \) is a numerical constant.
In the appendix we prove that if a smooth function satisfies estimates (35) then in every connected component of
\[ W_\varepsilon = \{ x \in W / \phi(x) > \varepsilon \} \]
the following inequalities hold:
\[
\sup_{W_\varepsilon} \phi \leq C_0(\varepsilon), \quad \sup_{W_\varepsilon} |\mathcal{X}_\varepsilon \phi| \leq C_1(\varepsilon), \quad \sup_{W_\varepsilon} |\mathcal{X}_\varepsilon^2 \phi| \leq C_2(\varepsilon)
\]
for the constants \( C_0, C_1, C_2 \) explicitly calculated. Moreover, the harmonic approximation method implies that the function \( \kappa = \phi - \hat{\phi} \) satisfies the estimate
\[
\int_W |\nabla (\kappa \phi)|^2 \leq C \tau \int_W (\kappa \phi)^2
\]

The tubular neighbourhood of a nodal set. The initial form of the harmonic function \( \hat{\phi} \) denoted by \( \hat{\phi}_0 \), is of degree \( m \leq \sqrt{\varepsilon} \). Let
\[
T_\varepsilon(N(\hat{\phi}_0)) = \{ x \in W / |\hat{\phi}_0(x)| \leq \varepsilon \}
\]
and use Hardy’s inequality [10] we obtain
\[
\int_{T_\varepsilon(\hat{\phi}_0)} \phi^2 \leq C H^{\varepsilon} \int_{T_\varepsilon(\hat{\phi}_0)} |\nabla \phi|^2 \leq c_2 \varepsilon^{\varepsilon \tau} \int_{T_\varepsilon(\hat{\phi}_0)} (\xi \phi)^2 \leq C \varepsilon^{\varepsilon \tau} \inf_{T_\varepsilon(\hat{\phi}_0)} |\phi|
\]
The usual Moser iteration gives us that
\[
\sup_{T_\varepsilon} |\kappa| \leq C \varepsilon \int_{T_\varepsilon(\hat{\phi}_0)} \left( \frac{1}{\inf_{T_\varepsilon(\hat{\phi}_0)} |\kappa|^2} \right)^{1/2}
\]
Then according to the usual Harnack estimates we have that if \( \phi, > 0 \) near \( y \)
\[
T_\varepsilon(\hat{\phi}_0) = \{ x \in W : |\hat{\phi}_0(x)| \leq \varepsilon \} \cap \{ x \in W : \phi(x) > \varepsilon \}
\]
we have that:
\[
\sup_{T_\varepsilon} |\kappa| \leq C \varepsilon \int_{T_\varepsilon(\hat{\phi}_0)} \left( \frac{1}{\inf_{T_\varepsilon(\hat{\phi}_0)} |\kappa|^2} \right)^{1/2}
\]
we conclude that \( \phi \sim \hat{\phi} \) near \( N(\hat{\phi}) \). The harmonic approximation applied on the slice allows us to construct the tubular neighbourhood of nodal sets by the Lojasiewicz inequality for the approximating function. Specifically, we have that for suitable choice of \( \varepsilon \) and the tube near the singularities of multiplicity \( m \) of the nodal set:
\[
|\phi| \leq |\kappa| + |\hat{\phi}| \leq C(\varepsilon \eta + 1)\varepsilon
\]
and selecting \( \varepsilon \eta = 1, \eta = 2 \)
\[
N(\phi) \subset T_{3\varepsilon-m}(\hat{\phi})
\]

The inductive argument. For \( n = 2 \) we reduce on a disc and then we derive estimates for the zero sets using the functions \( \hat{u} \) in order to produce test functions for the application of the min-max theorem, as it was used in the Courant nodal domain theorem. We recall here the following lemma from [HS]

**Lemma 9.** There exists \( \eta = \eta_0(n) \in (0, \frac{2}{3}) \) such that with \( \eta \in (0, \eta_0) \) if \( w_1, w_2 \in C^2(B_2(0)) \), \( |w_j|_{C^2} \leq 1 \) and if \( |w_1 - w_2|_{C^1} \leq \frac{\varepsilon_0}{2} \) then
\[
\mathcal{H}^{n-1}(B_{2-\eta}(0)) \cap \{ w_1 = 0, \nabla w_1 \geq \eta \}) \leq (1 + c\eta)\mathcal{H}^{n-1}(B_2(0)) \cap \{ w_2 = 0, \nabla w_2 \geq \frac{\eta}{2} \})
\]

**Estimates on nodal domains and eigenvalues.** We recall the definition of higher order eigenvalues as
\[
\lambda_k = \max_{s_{k-1} \subset H^1(\partial W)} \min_{v \in s_{k-1}} \left( \frac{D^1(v; \partial W)}{D^1(v; \partial W)} \right)
\]
The min-max principle for the eigenvalue, \( \lambda_k \) suggests also that
\[
\lambda_k = \min_{s_k \subset H^1(\partial W)} \max_{v \in s_k} \left( \frac{D^1(v; \partial W)}{D^1(v; \partial W)} \right)
\]
and therefore:
\[
\lambda_k \leq \max_{v \in s_k} \left( \frac{D^1(v; \partial W)}{D^1(v; \partial W)} \right)
\]
Upper bounds on eigenvalues. Let us denote that
\[ \sharp \{ \partial F \setminus N(\psi) \} = k \]
and having selected \( F \) containing a geodesic disc of radius \( \frac{1}{\sqrt{\tau}} \) then \( k > 1 \). We select as \( S_k = \{ \zeta_1, \ldots, \zeta_k \} \) where \( \zeta_j : W \to \mathbb{R}, \zeta_j > 0 \) defined as follows. Let \( \{ C_i \}_{j=1}^k \) be the nodal domains of \( \psi \). Set then the tubular neighbourhood
\[ C_{i, \epsilon} = \{ x \in C_i / d(x, N(\psi)) > \epsilon \} \]
and
\[ N_{j, \epsilon}(\psi) = \partial C_{j, \epsilon} \]
For this we approximated \( \psi \) harmonically and replaced near its nodes by \( \hat{\psi}_0 \) so that the Hardt-Simon estimate holds. We set as
\[ \zeta_j |_{C_{j, \epsilon}} = \tau_j \]
and
\[ \zeta_j |_{C_i} = 0, j \neq i, \quad \text{or} \quad \zeta_j |_{T_{\epsilon N(\hat{\psi}_0) \cap C_j} = 0} \]
and
\[ |\nabla \zeta_j| \leq \eta_j \]
for \( \tau_j, \eta_j \) to be selected. After Sard’s lemma \( N_{j, \epsilon}(\psi) \) is smooth for suitable \( \epsilon > 0 \) and hence we assume that \( \partial C_i \) is also smooth We approximate harmonically \( \psi \) and construct a smooth partition of unity, each member supported in a connected component. The Harnack inequalities in the appendix suggest that
\[ ||\hat{\psi} - \psi||_{2,F} \leq C(F) \epsilon \]
and
\[ \sup_{\partial W} |\psi| \leq \eta \]
We compute that
\[ \lambda_k \leq \frac{\sum_{j=1}^k \tilde{b}_j \eta_j^2}{\sum_{j=1}^k (\mu_j^2 \tilde{b}_j + b_j) \tau_j^2} \]
where
\[ a_j = \text{vol}(N_{j, \epsilon}(\psi)), \quad b_j = \text{vol}(C_{j, \epsilon}), \quad \tilde{b}_j = \text{vol}(C_{j, \epsilon} \setminus C_{j, \mu \epsilon}) \]
Sard’s lemma again allows us to choose \( \epsilon, \mu \) so that
\[ \tilde{b}_j \sim \mu b_j \]
Furthermore the isoperimetric inequality suggests:
\[ \tilde{b}_j^{\frac{n-2}{n-1}} \leq C \left( a_j (1 + \epsilon_j + \sup_{\partial W} |\nabla \psi|) + b_j \sup_{\partial W} |h| \right) \]
Moreover for suitable \( \epsilon > 0 \):
\[ \sum_{j=1}^k b_j = \text{vol}(W) - \text{vol}(T_{\epsilon}(\psi)) \geq (1 - \epsilon) \text{vol}(W) - (\text{vol}(N(\psi)))^{\frac{n-2}{n-1}} \]
Let \( b_1, b_k \) be such that:
\[ b_1 \geq \frac{1}{k} \text{vol}(W) \]
and similarly
\[ b_k \leq \frac{1}{k} \text{vol}(W) \]
We conclude
\[ \lambda_k \leq \frac{k \left( n^{n-2} (\text{vol}(\psi)) \right)^{\frac{n-1}{n-2}}}{\epsilon \text{vol}(W)} \]
The first eigenvalue of the laplacian in $C_{i,\epsilon}$ satisfies:

$$\lambda_1(C_{i,\epsilon}) \leq \frac{\int_{C_{i,\epsilon}} |\nabla \psi|^2}{\int_{C_{i,\epsilon}} \psi^2}$$

Therefore

$$\lambda_1(C_{i,\epsilon}) \leq c_0 \tau$$

We need now a lower bound for the first eigenvalue of $C_{i,\epsilon}$: we select $\epsilon$, so that we avoid dumbbell shape of the nodal domain. Cheeger estimate of the first eigenvalue combined with the isoperimetric inequality on the spherical piece suggest that

$$2c_0 \tau \geq \lambda_1(C_{i,\epsilon}) \geq \frac{c}{\text{vol}(C_{i,\epsilon})^{\frac{1}{n-1}}} - \inf_{C_{i,\epsilon}} |h|$$

or

$$\text{vol}(C_{i,\epsilon}) \geq \left( c_0 \tau + \inf_{C_{i,\epsilon}} |h| \right)^{-\frac{1}{n-1}}$$

Hence since at least one nodal domain should have volume at most

$$(c_0 \tau)^{-\frac{1}{n-1}} \leq \frac{1}{k} \text{vol}(\mathcal{F})$$

hence we have that

$$k \leq \text{vol}(\mathcal{F})(c_0 \tau)^{\frac{1}{n-1}}$$

**Lower bounds on eigenvalues** The max-min part suggests that

$$\lambda_k \geq \min_{S_{k-1}} \frac{\int_{\mathcal{F}} |\nabla \zeta|^2}{\int_{\mathcal{F}} \zeta^2}$$

Therefore in order to construct a test space $S_{k-1}$ we have to merge some of the nodal domains of $N(\psi)$. We start introducing two numbers for $l = 1,\ldots,k$ and for $\psi \geq 0$:

$$D_l^2 = \frac{1}{N} \int_{C_l} |\nabla \psi|^2, \quad W_l^2 = \frac{1}{N} \int_{C_l} \psi^2, \quad P_l = \frac{1}{P} \int_{C_l} \psi,$$

where

$$N = \int_{\mathcal{F}} \psi^2, \quad P = \int_{\mathcal{F}} \psi$$

We deduce after dyadic considerations $\{C_{l}\}_{l=1,\ldots,k}$'s that we can select two domains, for $l = 1, 2$ and find some $j_1 \geq 1$:

$$\frac{1}{2^{j_1}} \leq |D_1^2 - D_2^2| \leq \frac{1}{2^{j_1}} + \frac{1}{2^{j_1+1}}$$

Similarly for $j_2, j_3 \geq 1$:

$$\frac{1}{2^{j_2}} \leq |N_1^2 - N_2^2| \leq \frac{1}{2^{j_2}} + \frac{1}{2^{j_2+1}}$$

and

$$\frac{1}{2^{j_3}} \leq |P_1^2 - P_2^2| \leq \frac{1}{2^{j_3}} + \frac{1}{2^{j_3+1}}$$

As $k$ increases then we select the smallest triple $(j_1, j_2, j_3)$ with this order. We consider the space of functions of the form for $\eta$ a parameter that we select appropriately and compensates the growth of the function in the two nodal domains:

$$\Psi = a_1 \left( \frac{1}{P_1} \psi \chi_{C_1} - \frac{\eta}{P_2} \psi \chi_{C_2} \right) + \sum_{j=3,\ldots,k} a_j \psi \chi_{C_j}$$

We compute that

$$\Psi^2 = a_1^2 \left( \frac{1}{P_1} \psi \chi_{C_1} - \frac{\eta}{P_2} \psi \chi_{C_2} \right)^2 + \sum_{j=3,\ldots,k} a_j^2 \psi^2 \chi_{C_j}$$

and

$$|\nabla \Psi|^2 = a_1^2 \left( \frac{1}{P_1} \chi_{C_1} \nabla \psi - \frac{\eta}{P_2} \chi_{C_2} \nabla \psi \right)^2 + \sum_{j=3,\ldots,k} a_j^2 |\nabla \psi|^2 \chi_{C_j}$$
Furthermore we have

\[ \int_{\mathcal{F}} |\nabla \psi|^2 = a_1^2 \int_{\mathcal{F}} \left( \frac{1}{P_1} \chi C_1 \nabla \psi - \frac{\eta}{P_2} \chi C_2 \nabla \psi \right)^2 + \sum_{j=3, \ldots, k} a_j^2 D_j^2 \]

Using Lagrange’s identity we write the first term as:

\[ \int_{\mathcal{F}} \left( \frac{1}{P_1} \chi C_1 \nabla \psi - \frac{\eta}{P_2} \chi C_2 \nabla \psi \right)^2 = \left( \frac{1}{P_1 P_2} \right)^2 \left[ (D_1^2 + \eta D_2^2) (P_1^2 + P_2^2) - (P_1^2 D_1^2 - P_2^2 \eta D_2^2)^2 \right]^{\frac{1}{2}} \]

After Young’s inequality since \( D_{12}^2 = D_1^2 + D_2^2, P_{12} = P_1 + P_2 \) and setting

\[ a = \frac{P_1}{P_{12}}, \quad b = \frac{P_2}{P_{12}}, \quad v = \frac{\eta D_2}{D_{12}} \]

we deduce that the last term is minimized by the following expression:

\[ 4\lambda \left[ (4ab - a^2b^2 - 1)v^2 + 2a^2(1 - 2ab)v + \frac{1}{16} - a^4 \right]^{\frac{1}{2}} \]

After a tedious but otherwise elementary calculation we select \( a \) close to 1 and \( \eta \) sufficiently big then we arrange that the Rayleigh quotient is bounded by \( c\lambda \), for a suitable constant \( c > 0 \).

- \( \frac{3}{4} \leq D_1 \leq 1, \quad 0 \leq D_2 < \frac{1}{4} \)

which implies that

\[ \frac{1}{2} \leq D_1 - D_2 \leq 1 \]

- \( \frac{1}{2} \leq D_1 \leq \frac{3}{4}, \quad \frac{1}{4} D_2 < \frac{1}{2} \)

which implies

\[ 0 \leq D_1 - D_2 \leq \frac{1}{2} \]

Therefore we have to majorise a function of the form:

\[ Q(a_1, \ldots, a_{k-1}) = \frac{\sum_{j=1}^{k-1} D_j a_j^2}{\sum_{j=1}^{k-1} W_j a_j^2} \geq \frac{\lambda}{k} \]

We pick \( \epsilon \sim \frac{1}{\sqrt{\tau}} \) and compute that:

\[ k^2 \mathcal{H}^{n-2}(\mathcal{N}(\psi) \cap \mathcal{F}) \geq \sqrt{\tau} \text{vol(\mathcal{F})} \]

Therefore since \( k^2 \leq c \tau^{\frac{n-2}{2}} \mathcal{H}^{n-2}(\mathcal{N}(\psi) \cap \mathcal{F}), \text{vol(\mathcal{F})} \geq \tau^{\frac{n+1}{2}} \): we conclude that

\[ \mathcal{H}^{n-2}(\mathcal{N}(\psi)) \geq C \tau^{-\frac{n+1}{4}} \]

**Modification of the pixel for the singularities of \( \phi \)** Hopf’s strong maximum principle for \( \hat{\phi} \) guarantees that \( \mathcal{N}(\hat{\phi}) \) meets transversely the boundary of the pixel. The comparison of nodal sets reduces the problem to the estimation of the nodal set of \( \hat{\phi} \). Therefore using the geodesic spheres starting near \( \mathcal{N}(\psi) \) we obtain by the coarea formula, away from singularities of \( \hat{\phi} \):

\[ \mathcal{H}^{n-1}(\mathcal{N}(\phi)) = \sum_{j=0}^{m} \int_{\mathcal{F}_{j+1}} dr \int_{\mathcal{N}(\psi) \subset S_r} d\mathcal{H}^{n-2}(\mathcal{N}(\psi)) \geq c \tau^{-\frac{n-1}{4}} \]

18
The eigenfunctions. The eigenfunctions fulfill the hypothesis of the preceding theorem and therefore

Theorem 10. Let $u : (M^n, g) \to \mathbb{R}, \Delta_g u = -\lambda u, P \subset (M^n, g)$ be a pixel. Then

$$\mathcal{H}^{n-1}(N(u_\lambda) \cap P) \geq C(P) \sqrt{\lambda}$$

We are placed in the $P$ with boundary spanned by the fronts $F_\ell, \ell = 1, \ldots, m_{i_1 \cdots i_k}$:

$$\partial P = \bigcup_{k=1}^{n+1} \bigcup_{i_1, \ldots, i_k} \bigcup_{\ell=1}^{m_{i_1 \cdots i_k}} F_\ell$$

The boundary of the pixel is not smooth, but we apply the smoothing method described in the appendix. Near each front we have the representation of the eigenfunction in the form:

$$u(r, \theta) = e^{-\frac{1}{\sqrt{\lambda}} \mu(r, \theta)} + \beta_1 \alpha(\theta) \sin(\beta_1 \phi(r, \theta))$$

In the preceding paragraph we obtained the estimates that $\phi$ satisfies with the constant $c_{11} \sim \lambda$. Therefore we have that inside a pixel of diameter $\frac{1}{\sqrt{\lambda}}$:

$$\mathcal{H}^{n-1}(N(u_\lambda) \cap P) \geq C(P) \sqrt{\lambda}^{-\frac{n-1}{2}}$$

and since the manifold is split in $\lambda^n$ pixels of this size we have the required estimate.

1.5 The upper bound

[D] proved that the Hausdorff measure of the nodal set contained in a pixel $P$ with its boundary smoothed out is majorised by:

$$\mathcal{H}^{n-1}(N(u_\lambda) \cap P) \leq \frac{1}{2} \int_P |\nabla \log q| + \sqrt{\lambda n} \lambda \text{vol}(P) + \text{vol}(\partial P)$$

where

$$q(u) = |\nabla u|^2 + \frac{\lambda}{n} u^2$$

We split the set $P$ in three parts

- The part that is free of nodes:

$$P_\lambda = \{ x \in P / |u(x)| \geq \eta \} = \hat{P} \setminus T_\eta(N(u))$$

- The tubular neighbourhood of the nodal set $T_\eta(N(u) \cap P)$ is split further as:

$$T_\eta(N(u) \cap P) = R_\eta(N(u) \cap P) \cup S_\eta(N(u) \cap P)$$

The regular part of the nodal set

$$R_\eta(N(u) \cap P) = \{ x \in P / |u(x)| < \eta, |\nabla u(x)| \geq \eta \}$$

and the neighbourhood of the singular set:

$$S_\eta(N(u) \cap P) = \{ x \in P / |u(x)| < \eta, |\nabla u(x)| < \eta \}$$

The problems in (38) arise in the singular part of the nodal set. We will estimate the behaviour of $|\nabla \log q|$ near $S_\eta(P)$. We will use induction with respect to the multiplicity of the nodal set, introducing new pixels of multiplicity bounded from below and approximate harmonically the eigenfunction there:

$$T_\eta(N(u) \cap P) = \bigcup_{\ell=0}^{m(\lambda)} \bigcup_{m=1}^{c_\ell} S_{\eta,\ell,m}$$

where $S_{\eta,\ell,m}$ is a connected component of multiplicity $\ell$

$$S_{\eta,\ell,m} = \{ x \in P / |u(x)|^2 + \cdots + |\nabla^{\ell-1} u(x)| \leq \eta, |\nabla^\ell u(x)| \geq \eta \}$$
Notice that \( \text{Sing}_{0,m} = P, \quad \text{Sing}_{1,m} = \text{Reg}(N(u) \cap P) \)

The harmonic approximation of \( u_\lambda \) in \( \text{Sing}_{\ell,m} \) is denoted \( \hat{u}_{\ell,m} \). We introduce the localization functions \( \zeta_{\ell,m} \) with supports \( \text{supp}(\zeta_{\ell,m}) \subset \text{Sing}_{\ell,m} \). We split the integral as

\[
\int_{P} |\nabla \log q| \leq c \sum_{m=0}^{\infty} \int_{P} \zeta_{0,m} |\nabla \log q| + \sum_{\ell=0}^{\infty} \sum_{m=1}^{\infty} \int_{S_{\ell,m}} \zeta_{\ell,m} |\nabla \log q|
\]

Therefore we set

\[
I_{\ell,j}[\zeta_m] = \int_{S_{\ell,j}} \zeta_{\ell,j} |\nabla \log q| \leq C \int_{S_{\ell,j}} \zeta_{\ell,j} U
\]

where we set

\[
U = \frac{|\nabla|\nabla u|^2| + \lambda |\nabla u|^2}{|\nabla u|^2 + \lambda^2 u^2}, \quad IU_{\ell,j} = \int_{S_{\ell,j}} \zeta_{\ell,j} U,
\]

We will estimate the terms appearing in the right hand side of the inequality. If \( \ell = 0 \) we have that for \( D(\eta, \lambda) = \eta^{\frac{1}{n}} \lambda n^{-1} \) where \( t \) is a parameter to be selected:

\[
\eta \leq |u(x)| \leq D(\eta, \lambda) \eta
\]

\[
|\nabla u(x)| \leq D(\eta, \lambda) \sqrt{\eta}
\]

\[
|\nabla^2 u(x)| \leq C D(\eta, \lambda) \lambda \eta
\]

Therefore we have that

\[
IU_{0,j} \leq \lambda^{-\frac{s}{2} + qn + \frac{1}{2} n s}
\]

and we select as

\[
\eta = \lambda^{-m_s}, \quad m = \frac{q}{s} + \frac{1}{n s}
\]

If \( \ell = 1 \) then we split the tube around the regular \( R \) of \( N(\theta u) \) in the layers

\[
R = \bigcup_{j=1}^{\infty} R_j, \quad R_j = \{ x \in P : |u(x)|, |\nabla u| \geq \eta \}
\]

We have that in \( R_j \)

\[
|u(x)| \leq D(\eta, \lambda) \theta^j \eta
\]

\[
\eta \leq |\nabla u(x)| \leq D(\eta, \lambda) \sqrt{\eta}
\]

\[
|\nabla^2 u(x)| \leq C D(\eta, \lambda) \lambda \eta
\]

In this case we get

\[
IU_{0,j} \leq \lambda^{-\frac{s}{2} + qn + \frac{1}{2} n s}
\]

and we select \( \eta \) as before. If \( \ell > 2 \): For this we need sharper estimates and we appeal to the harmonic approximation in order to use Lojasiewicz inequalities. We exhaust \( \text{Sing}_{\ell,m} \):

\[
\text{Sing}_{\ell,m} = \bigcup_{j=0}^{\infty} Q^j_{\ell}(\theta, \eta), \quad Q^j_{\ell}(\theta, \eta) = T^j_{\ell,m} \setminus T^{j+1}_{\ell,m}
\]

for

\[
T^j_{\ell,m} = \text{Sing}_{\ell,m} \cap \{ x \in P : |u| \leq \theta^{j+1} \eta, \theta^{j+1} \eta \leq |\nabla u| \leq \theta^j \eta \}
\]

We introduce in each \( Q^j_{\ell,m}(\theta, \eta) \) then

\[
\kappa_{\ell,m; \pm} = u \pm \hat{u}_{\ell,m}
\]

and drop the indices

\[
|\nabla^2 u|^2 = |\nabla (\nabla \kappa_+ \cdot \nabla \kappa_-) + \nabla |\nabla \hat{u}|^2 = (\nabla^2 \kappa_+) \cdot \nabla \kappa_- + (\nabla^2 \kappa_-) \cdot \nabla \kappa_+ + \nabla |\nabla \hat{u}|^2
\]
The harmonic approximation method combined with Harnack inequality suggests that in $Q^j_{\ell,m}(\theta, \eta), \ell > 1$ we have Bernstein’s inequalities

$$|\nabla u| \leq D(\eta, \lambda)\sqrt{\lambda} \eta^j \quad |\nabla^2 u| \leq D(\eta, \lambda)\sqrt{\lambda} \eta^j$$

and also

$$|\nabla \kappa_-| \leq C_1 \lambda \eta^{2(j+1)} \eta$$

Set

$$\beta = D(\eta, \lambda)\eta^j$$

Furthermore since $|a + b| \geq (1 - \varepsilon)\left(a^2 - \frac{1}{4}b^2\right)$ then :

$$|u|^2 \geq (1 - \varepsilon)\left(\tilde{u}^2 - \frac{1}{\varepsilon}|\kappa_-|^2\right) \geq \left(1 - \frac{\sqrt{\lambda}}{\varepsilon}\right)\tilde{u}^2$$

and

$$|\nabla u|^2 \geq (1 - \varepsilon)\left(\frac{1 - \sqrt{\lambda}}{\varepsilon}\right) |\nabla \tilde{u}|^2 \geq (1 - \varepsilon)^2 \left(1 - \frac{\sqrt{\lambda}}{\varepsilon}\right) \tilde{u}^2$$

In $Q^j_{\ell,m}$ we have:

$$u^2 \geq \left(1 - \frac{\sqrt{\lambda}}{\varepsilon}\right) g^2(1 + j)^2 \eta^2, \quad |\nabla u|^2 \geq (1 - \varepsilon)^2 \left(1 - \frac{\sqrt{\lambda}}{\varepsilon}\right) g^2(1 + j)^2 \eta^2$$

The integrand is estimated according to the estimates derived above:

$$\zeta_{\ell,m,j} U \leq \zeta_{\ell,m,j} D(\eta, \lambda)^2 \text{vol}(Q^j_{\ell,m})\lambda \eta^{2(1 + j)} \eta^2$$

We have that due to the multiplicity bound we have that

$$\text{vol}(Q^j_{\ell,m}) \leq C \lambda^{-\frac{m+1}{2}} \sqrt{\lambda} \eta^j, \quad D(\eta, \lambda) \leq C(\eta^s \lambda)^n$$

We set $\eta = \lambda^{-m}$ and compute

$$m = \frac{4qn + 1}{2(2qn + 2\nu + 1)} + j$$

Summing up we get

$$\sum_{j=0}^{\infty} IU_{i,j} \leq C \sum_{j=0}^{\infty} S_{\ell,j} \zeta_{\ell,j} \leq \frac{C\sqrt{\lambda}}{1 - g^2} \lambda^{-j}$$

and therefore we have that

$$\sum_{j=0}^{\infty} C\sqrt{\lambda} IJ_j \leq \frac{C\sqrt{\lambda}}{1 - g^2}$$

We will use repeatedly the following method that we call **harmonic approximation method**. The domains encountered here $W$ have boundaries with normal crossings singularities: the singular set $\mathcal{H}(\partial W)$ is given by the transversal intersection of hypersurfaces: geodesic spheres with local equations $s_1, \ldots, s_\ell, \ell = n, n + 1$. The piece of the hypersurface $\mathcal{H}_\epsilon = \{x \in W(s_1, \ldots, s_\ell + \epsilon) | x = 0\}$ near $\mathcal{H}$ is for suitable $\epsilon$ a smooth hypersurface close to $\mathcal{H}(\partial W)$. We will consider the domain $\hat{W}$ obtained by replacing the singular part $\mathcal{H}(W)$ by $\mathcal{H}_\epsilon$ with repacing the defining function through cut-offs by the function given there. Let $\hat{F} : \hat{W} \to R$ be the solution of the boundary value problem:

$$\Delta \hat{F} = 0, \quad \hat{F}|_{\partial \hat{W}} = F$$

**Harmonic polynomials** We will also approximate the harmonic function defined in the pixel $\hat{F}$ by a sequence $\{F_n\}_{n \in N}$ of functions such that

$$\Delta_0 \hat{F}_0 = 0, \quad \Delta_0 \hat{F}_n = - \sum_{i,j} R_{ij} \frac{\partial^2 F_{n-1}}{\partial x_i \partial x_j} - g^{ij} \partial_i \psi \partial_j F_{n-1}$$

(39)
where \( g_{ij} = \delta_{ij} + R_{ij} \), \( \varrho = \text{diam}(W) \), \( \psi = \frac{1}{2} \log(g) \), \( g = \det(g_{ij}) \)

and for \( j = 0, 1, 2 \):

\[
|\nabla^j \varphi| \leq C \mu^j \varrho^2 - j.
\]

Integration by parts after multiplication by \( \zeta^2 F_n \) and incorporation of the preceding estimates along with Young’s inequality leads to:

\[
\int_W \zeta^2 |\nabla F_n|^2 \leq C \varrho^2 \int_W |\nabla (\zeta F_0)|^2
\]

and

\[
\text{supp}(\nabla \zeta) \subset A(W) = \{ x \in W / d(x, \partial W) < \epsilon \}
\]

and

\[
|\nabla^j \zeta| \leq C_j \epsilon^j
\]

We select \( C \mu^2 = 1 \) then

\[
\int_W \zeta^2 |\nabla F_n|^2 \leq C \int_W |\nabla (\zeta F_0)|^2
\]

Similarly we have the inequalities:

\[
\int_W \zeta^2 |\nabla^2 F_n|^2 \leq C \left( \varrho^4 \int_W \zeta^2 |\nabla^2 F_{n-1}|^2 + \varrho^2 \int_W (|\Delta \zeta| + |\nabla \zeta|^2) |\nabla F_{n-1}|^2 \right)
\]

and

\[
\int_W \zeta^2 |\nabla^3 F_n|^2 \leq C^2 \left( \varrho^4 \int_W \zeta^2 |\nabla^3 F_{n-1}|^2 + \varrho^2 \int_W \zeta^2 |\nabla^2 F_{n-1}|^2 + \varrho^4 \int_W (|\Delta \zeta| + |\nabla \zeta|^2) |\nabla^2 F_{n-1}|^2 \right)
\]

Therefore we have that after iteration:

\[
\int_W \zeta^2 |\nabla^2 F_n|^2 \leq C \int_W \zeta^2 |\nabla^2 F_0|^2
\]

and as well as

\[
\int_W \zeta^2 |\nabla^3 F_n|^2 \leq C \int_W \zeta^2 |\nabla^3 F_0|^2
\]

The Nash-Moser iteration that we describe in the sequel allows us to bound the sequence in \( C^0(W) \). Rellich lemma allows us to extract a sequence that converges in \( H^1(W) \) and therefore we use it in order to approximate the initial function by a harmonic polynomial with any accuracy we desire.

### 1.5.1 The brick localization details

Let \( P = P^{(\ell, i_1, \ldots, i_k)} \) be a brick of size determined by the parameters \( r(P), \mu(F) \). We use coordinates \( \bar{x} = \bar{x} + \bar{c} \) for \( \bar{c} \) denoting the centre of the brick. We construct cut-offs Let \( \psi \in C^\infty_0(\mathbb{R}^n) \) be the following function:

\[
\psi_\epsilon(t) = \begin{cases} 
1 & \text{if } 0 \leq t \leq 1 \\
0 & \text{if } t \geq \frac{1}{2}
\end{cases}
\]

Then the following function that localizes in the brick:

\[
\ell_0(\bar{c}) = \prod_{i=1}^n \psi \left( \frac{\sqrt{4\ell^2 + \epsilon_i^2}}{\sqrt{3}\epsilon_i} \right)
\]

Also we will use the function that localizes in the neighbourhood of the zeroes of the function \( \hat{h} : P \to \mathbb{R} \)

\[
\mathcal{N}(\hat{h}) = \{ \bar{c} \in P : \hat{h}(\bar{c}) = 0 \}
\]
then the function
\[ \ell(z) = \ell_0(z_j) \psi \left( \frac{\hat{h}(z)}{\epsilon} \right) \]
localizes in the set:
\[ N_\epsilon(\hat{h}) = \{ z \in P/|\hat{h}| \leq \epsilon \} \]
We can prove inductively that:
\[ |\nabla^j \ell_0| \leq C_j \]
and
\[ |\nabla^j \ell_\epsilon(z)| \leq C_j \sum |\nabla^{i_1} \hat{h}|^{p_1} \ldots |\nabla^{i_m} \hat{h}|^{p_m} |\nabla^{j_1} \psi|^{\alpha_1} \ldots |\nabla^{j_n} \psi|^{\alpha_n} \]
where we sum over all indices \( i_1 p_1 + \cdots + i_m p_m, j_1 q_1 + \cdots + j_n q_n = \ell \) and hence that again for indices \( i_1 p_1 + \cdots + i_m p_m = \ell, \ell = 0, \ldots, j \):
\[ |\nabla^j \ell_\epsilon(z)| \leq C_j \sum |\nabla^{i_1} \hat{h}|^{p_1} \ldots |\nabla^{i_m} \hat{h}|^{p_m} \]
These will be used successively in the sequel.

1.5.2 Lojasiewicz, Hardy for functions of the form \( \hat{h} \circ N_0^{-1} \)

Let \( \hat{h}_0 \) be a polynomial function in rectangular coordinates in \( B_0^{-1} \) then we have the following immediate result

**Nodal volume 11.** The function \( \hat{h} = \hat{h}_0 \circ N^{-1} \) is a function that satisfies the following

- The multiplicity strata \( \Sigma_{m}(\hat{h}) \) of the variety \( N(\hat{h}) \) are mapped to \( \Phi(\Sigma_{m}(\hat{h}_0)) = \Sigma(\hat{h}) \).
- The Lojasiewicz inequalities hold true
  \[ |\nabla \hat{h}| \geq c_1 |\hat{h}|^{1-\ell_1}, \ |\hat{h}(x)| \geq d(x, N(\hat{h})) \]

The first conclusion comes form the chain rule in many variables:
\[ D^\alpha \hat{h}_0 = \sum C_{\alpha, \beta, \gamma} (D^\beta \hat{h})(\Phi(x))(D^{\gamma_1} \Phi)_{\epsilon_1} \ldots (D^{\gamma_j} \Phi)_{\epsilon_j} \]
where the sum extends over all multiindices \( \alpha, \beta, \gamma \in N^n, j = 1, \ldots, \ell, \epsilon_1, \ldots, \epsilon_j \in N \) such that:
\[ |\beta| = 1, e_1 \gamma_1 + \cdots + e_\ell \gamma_\ell = |\alpha| - |\beta| \]

Exchanging the role of \( \hat{h}, \hat{h}_0, \Phi, N \) we get the defining equations of the equimultiple locus. For the second we compute:
\[ |\nabla f| = |D\Phi(z)| |\nabla \hat{h}_0(\Phi(z))| \geq C |\nabla \hat{h}_0(\Phi(z))| \geq C' |\hat{h}(\Phi(x))|^{1-\ell_1} = C' |f(z)|^{1-\ell_1} \]
where we have chosen \( R, r \) so that
\[ |D\Phi(z)| = |A + B(z)| \geq |A| - |B(z)| \geq \frac{|A|}{2} \]
where \( |B| \leq \frac{|A|}{2} \). Similarly,
\[ |\hat{h}(z)| = |\hat{h}_0(\Phi(z))| \geq c_2 d(\Phi(x), N(\hat{h}_0))^{\ell_2} \]
now due to the first inequality and the definition of the exponential map we conclude that:
\[ d(\Phi(z), N(\hat{h}_0)) \geq c' d(z, N(\hat{h})) \]
A consequence of this is that (GHI)’s hold for such functions.
1.5.3 Hardy’s inequalities

Let \( P: \mathbb{R}^n \to \mathbb{R} \) be a homogeneous polynomial of degree \( m \) and \( N(P) \) its set of zeroes

\[
N(P) = \{ \underline{z} \in \mathbb{R}^n / P(\underline{z}) = 0 \}
\]

Moreover let \( f \in C^\infty_0 (\mathbb{R}^n \setminus N(P)) \) there exist constants, \( \|f\|_P \) \( 0 < c_j(H) = \frac{1}{(n-2j)^{4} + O(\epsilon)n > 2} \)

\[
\int_{\mathbb{R}^n} |P|^{-\frac{2}{n}} f^2 \leq c_1(H) \int_{\mathbb{R}^n} |\nabla f|^2 \tag{40a}
\]

\[
\int_{\mathbb{R}^n} \left| \frac{\nabla P}{P} \right|^2 f^2 \leq c_2(H) \int_{\mathbb{R}^n} |\nabla f|^2, \tag{40b}
\]

\[
\int_{\mathbb{R}^n} \left| \frac{\Delta P}{P} \right|^2 f^2 \leq c_3(H) \int_{\mathbb{R}^n} |\nabla f|^2, \tag{40c}
\]

From the euclidean Hardy’s inequalities we obtain the riemannian versions by modifying suitably the constants by \( 1 + \epsilon \).

1.5.4 Integration formulas

Let \( T \) be a tensor field of type \((p + 1, 0)\) then we introduce:

\[
A(T)_{i_1 \ldots i_p} = T_{i_1 \ldots i_p, k} - T_{i_1 \ldots i_p, k} \tag{40d}
\]

\[
D(T)_{i_1 \ldots i_{p-1}} = g^{ij} T_{i_1 \ldots i_{p-1}, i} \tag{40e}
\]

The general integration by parts formula reads as

**Lemma 12.** Let \( T \) be a \((p, 0)\) tensor field on the riemannian manifold \((M^n, g)\) and \( \phi = |T|^{k-1} \chi \), a smooth cut-off function supported in the domain \( K \). Then we have that

\[
\int_K \phi^2 |\nabla T|^2 \leq \int_K \frac{1}{2} |A(T)|^2 + |D(T)|^2 + \int_K \phi^2 \sum_{i=1}^5 \int_N |\nabla \phi|^2 |T|^2
\]

We set \( T_{i,j} = T_{i_1 \ldots i_{p-1}, j} \) and hence we have that

\[
\phi^2 |A(T)|^2 = 2 \phi^2 (|\nabla T|^2 - T_{i_k,j} T^{i,j;k})
\]

The last term gives that

\[
\phi^2 T_{i_k,j} T^{i,j;k} = (\phi^2 T_{i_k} T^{i,j;k})_{j} - \phi^2 T_{i_k} T^{i,j;k} \]

\[
= 2 \phi T_{i_k} \phi_j T^{i,j;k} = (BT) + (I) + (II)
\]

Furthermore we have that

\[
(I) = \phi^2 T_{i_k} T^{i,j;k} = \phi^2 T_{i_k} (D(T)_{i,j,k}) + \phi^2 \text{Ric}^k_x T^{i,k} T_{i_k} + \phi^2 \sum_{i=1}^p \int_{N} R_{i_k} T^{i,j;k} T_{i_k}
\]

for \( i_{\ell} = i_1 \cdot i_{\ell-1} s i_{\ell+1} \cdot i_p \). The first term then is written as

\[
\phi^2 T_{i_k} (D(T)_{i,j,k}) = (BT) - \phi^2 |D(T)|^2 - 2 \phi T_{i_j} \phi_j (D(T))^k
\]

The term \( (II) \) is written using that \( \phi = \chi |T|^{k-1} \):

\[
(II) = 2 \phi T_{i_k} \phi_j A(T)^{i,j;k} + \frac{1}{2} \nabla \log \chi \cdot \nabla |T|^2 + (k-1) \phi^2 |\nabla \chi|^2
\]

In summary we have that:

\[
\int_K \phi^2 |\nabla T|^2 \leq \int_K \phi^2 \left( |D(T)|^2 + \frac{1}{2} |A(T)|^2 \right) + C \int_K \left( |\nabla \phi| + |\nabla \text{Ric}| + |\nabla \log |^2 \right) \phi^2 |T|^2
\]

24
1.5.5 Functions
For a function \( f : K \to \mathbb{R} \), \( \text{supp}\phi \subset K \) we have that:
\[
\int_K \phi^2 |\nabla^2 f|^2 \leq C \int_K \phi^2 (|\Delta f|^2 + |\text{Ric}| |\nabla f|^2) \tag{41a}
\]
\[
\int_K \phi^2 |\nabla^3 f|^2 \leq C \int_K \phi^2 (|\nabla(\Delta f)|^2 + |\text{Rm}||\nabla f|^2) \tag{41b}
\]

1.5.6 Curvature
The differential Bianchi identities are written as:
\[
\text{div}(\text{Rm})_{ijkl;m} = \text{Rm}_{ijkl,m}
\]
also:
\[
\text{div}(\text{Rm})_{ijkl,l} = A(\text{Ric})_{jki} + A(\text{Ric})_{ikj}
\]
We recall here the Bach tensor:
\[
B_{ijk} = A(\text{Ric})_{ijk} + \frac{1}{(n-1)(n-2)} [g_{ij} \nabla_k R - g_{ik} \nabla_j R]
\]
and the contracted identities are written as
\[
\text{div}(\text{Ric})_i = \hat{\gamma}_i R
\]
Therefore we have that
\[
\int_K \phi^2 |\nabla Rm|^2 = 4 \int_K |B|^2 + \frac{2}{(n-1)(n-2)} |\nabla R|^2 + \sum_{i=1}^5 (\text{Rm}^* \text{Rm}_i),
\]

The iterative method Let \( g, \chi : W \to \mathbb{R} \) be smooth functions and \( \hat{h} : K \to \mathbb{R} \) a polynomial weight function of degree \( m \) then set \( W_j = \{ x \in W \mid \theta(1-\theta^j) \frac{\eta}{2} \leq |\hat{h}(x)| \leq (1-\theta+\theta^j)\eta \} \) and \( \text{supp}\chi_j \subset W_j \). This for instance is given for \( \chi_j(x) = \ell \left( \frac{\hat{h}(x)}{(1-\theta+\theta^j)\eta} \right) \ell (\theta(1-\theta^j)\eta - \hat{h}(x)) \)
We suppose that the smooth function \( g \) satisfies the inequality, for positive constants \( \gamma > 1, e = 2, 4 \) and any smooth cut-off \( \chi \):
\[
\int_W \chi^2 |\nabla g|^2 \leq \gamma \int_W \chi^2 |g|^e
\]
Then Sobolev inequality suggests for \( s = \frac{np}{n-p}, 1 < p < 2, k, \ell, d > 1 \):
\[
\left( \int_{W_j} \left( \chi^d |\hat{h}|^{-\ell} |g|^k \right)^{p/s} \right)^{p/s} \leq C_0 k \int_{W_{j-1}} \left( |\hat{h}|^{-\ell} \chi^d |g|^{k-1} \right)^p |\nabla g|^p + \ell \int_{W_{j-1}} \left| \nabla \hat{h} \right|^p \left( \chi^d |\hat{h}|^{-\ell} |g|^k \right)^p + p \int_{W_{j-1}} \chi^{(d-1)p} |\nabla \chi|^p |g|^k |\hat{h}|^{-p\ell} \tag{42}
\]
The first term then gives for \( r = \frac{2p}{2p - p} \)
\[
\int_{W_j} \left( |\hat{h}|^{-\ell} \chi^d |g|^{k-1} \right)^p |\nabla g|^p \leq \left( \int_{W_{j-1}} \chi^{dp} \left( |\hat{h}|^{-\ell} |g|^{k-1} \right)^r \right)^{\frac{p}{r}} \left( \int_{W_{j-1}} \chi^{dp} |\nabla g|^p \right)^{\frac{p}{2}} \leq C_1 ((1-\theta+\theta^j)\eta)^{p\ell} \gamma^2 \frac{\text{vol}(W_j)^{\frac{(k-1)p}{2(k-1)\ell} - 1}}{(\int_{W_{j-1}} \chi^{dp} \left( |\hat{h}|^{-r} |g|^{(k-1)r} \right)^{p(\frac{(k-1)p}{2(k-1)\ell} - 1)}} \tag{44}
\]
since we have

$$\left( \int_{W_j} \chi^{2dp} |\nabla g|^2 \right)^{p/2} \leq \gamma^2 \left( (1 - \theta + \theta^j) \eta \right) \frac{\nu_{i-1}}{\nu_i} \text{vol}(W_j)^{(k-1)r} \left( \int_{W_{j-1}} \chi^{2dp} |\hat{h}^{-r} g|^{(k-1)r} \right)^{\frac{p}{2(k-1)r}}$$

The middle term after application of (40b) gives that:

$$\int_{W_j} |\nabla \chi|^p \left( \chi^d |\hat{h}|^{-r} g \right)^k \left( \int_{W_{j-1}} \chi^{dp} (h^{-r} g^{(k-1)r}) \right)^2 \left( \int_{W_{j-1}} \chi^{2dp} |\hat{h}^{-r} g|^{(k-1)r} \right)^{\frac{p}{2(k-1)r}}$$

In an analogous way the last term leads to:

$$\int_{W_j} |\nabla \chi|^p \left( \chi^d |\hat{h}|^{-r} g \right)^k \leq C_1 ((1 - \theta + \theta^j) \eta)^p \gamma^2 \text{vol}(W_j)^{(k-1)r} \left( \int_{W_{j-1}} \chi^{2dp} |\hat{h}^{-r} g|^{(k-1)r} \right)^{\frac{p}{2(k-1)r}}$$

In summary we arrive at

$$\left( \int_{W_j} \left( \chi^d |\hat{h}|^{-r} g \right)^k \right)^{\frac{1}{k}} \leq C_3 k^r \left( (1 - \theta + \theta^j) \eta \right)^{\frac{p}{2}} \gamma^2 \text{vol}(W_j)^{(k-1)r} \left( \int_{W_{j-1}} \chi^{2dp} |\hat{h}^{-r} g|^{(k-1)r} \right)^{\frac{p}{2(k-1)r}}$$

Notice that

$$r = qs, \quad q(n,p) = 1 - \frac{p}{2}, \quad 1 - \frac{2}{n} \leq q(n,p) \leq 2 - \frac{2}{n}$$

We conclude with the basic inequality that we will iterate

$$\left( \frac{1}{v_j} \int_{W_j} G^{k+1} \right)^{\frac{1}{k+1}} \leq C_j \left( \frac{1}{v_{j-1}} \int_{W_{j-1}} G^{(k+1)r} \right)^{\frac{1}{k+1}}$$

where

$$r = \frac{s}{a}, \quad a = \frac{n}{n-1}, \quad v_j = \text{vol}(W_j), \quad C_j = C_3 \left[ k \eta^{k+1} \left( (1 - \theta + \theta^j) \eta \right)^{\frac{k+1}{k}} \gamma^2 \text{vol}(W_j)^{(k-1)r} \right]^{\frac{1}{k+1}}$$

and we replace $\beta_j$ by the upper bound

$$\beta_j \leq \frac{1}{\theta^j} = \sigma$$

In order to bring this to the standard iteration form we do the following:

$$k_j = \frac{a^{j+1}}{s} + 1$$

Finally we arrive at the basic iteration inequality:

$$I_{j+1} = \left( \frac{1}{v_{j+1}} \int_{W_{j+1}} G^{k+1} \right)^{\frac{1}{k+1}} \leq C_j \left( \frac{1}{v_j} \int_{W_j} G^k \right)^{\frac{1}{k}}$$

Then we have the iteration inequality

$$I_{j+1} \leq C_j I_j \quad (45)$$
The iteration leads to the inequality
\[ \sup_{W} |G| \leq D \left( \frac{1}{\text{vol}(W_0)} \int_{W_0} |G|^p \right) ^{\frac{1}{p}}, \quad D = \lim_{j \to \infty} (C_1 \ldots C_j) \]

We select the local density parameter as \( \theta \sim \frac{1}{t}, t > 0 \). The constant is estimated through elementary inequalities of the form:
\[ \frac{1}{\gamma^t a^2 - 1} \leq -\sum_{j=0}^{\infty} \frac{1}{a^{2j}} \log \left( 1 - \frac{1}{\gamma^t a^2 - 1} \right) \leq \frac{\gamma^t}{\gamma^t - 1} \frac{1}{\gamma^t a^2 - 1} \]

We arrive finally at
\[ c = \eta^{p\gamma^{-\frac{a+1}{a(n-2)}}} \gamma^{\frac{p(n+1)}{n-1} - \frac{a+1}{a-1}} \]

We will denote the constant in the form:
\[ D(\eta, \gamma) = (\eta^s \gamma^t)^n, \quad s = \ell p + \frac{a + 1}{na(n-2)}, \quad q = \frac{p(t+1)}{2t} + \frac{t}{n(n-2)} \]

**Back to harmonic approximation** The harmonic approximation method suggests that
\[ \sup_{W} |\kappa| \leq D(\eta, \lambda) \left( \frac{1}{\text{vol}(W_0)} \int_{W_0} u^2 \right) ^{1/2} \]

Similarly we have the higher order inequalities
\[ \sup_{W} |\nabla \kappa| \leq D(\eta, \lambda^2) \left( 1 + D(\eta, \mu) \frac{1}{\text{vol}(W_0)} \int_{W_0} |Ric| \right) ^{1/2} \left( \frac{1}{\text{vol}(W_0)} \int_{W_0} u^2 \right) ^{1/2} \]

Let \( \hat{u} > 0 \) be a harmonic function. Then set for \( j = 0, 1 \)
\[ h_j = \left| \frac{\nabla^{j+1} \hat{u}}{\nabla \hat{u}} \right|^2, \quad H_0 = \left| \frac{\nabla \nabla \hat{u}}{\nabla \hat{u}} \right|^2 \]

and compute
\[ \Delta_\eta h_0 \geq -(|Ric| + 2) h_0^2 + \frac{2}{3} h_1 h_0 \geq -(|Ric| + 2) h_0^2 \]
or that
\[ \int_{W} \zeta^2 |\nabla h_0|^2 \leq \int_{W} (|Ric| + 2) \zeta^2 h_0^3 \]

We note that the integral is of the form
\[ \int_{W} \phi^2 h_0^2 \leq (1 + \epsilon^2) c_2(H) \int_{W} \phi^2 |\nabla h_0|^2 + \frac{1}{\epsilon^2} |\nabla \phi|^2 \]

majorised after application of Hardy’s inequality. We compute that
\[ |\nabla h_0|^2 \leq (1 + \epsilon)^2 (H_0 h_0^2 + \frac{1}{\epsilon^2} h_0) \]

then we find
\[ \int_{W} \phi^2 |\nabla h_0|^2 \leq (1 + \epsilon^2) \int_{W} \phi^2 (H_0 h_0^2 + \frac{1}{\epsilon^2} h_0) \leq (1 + \epsilon^2) c_2(H) \int_{W} |\nabla \phi|^2 \left( \frac{1}{\epsilon^2} + h_0^2 \right) + \phi^2 |\nabla h_0|^2 \]

If \( n > 1, \epsilon' > 0 \) then select
\[ \frac{n^2 - 2n - \epsilon'^2}{n^2 - 2n + 8} = \epsilon' < \frac{n^2 - 2n}{n^2 - 2n + 8} \]

and conclude that
\[ \int_{W} \phi^2 |\nabla h_0|^2 \leq c_n(\epsilon') \int_{W} |\nabla \phi|^2 \left( \frac{1}{\epsilon^2} + h_0^2 \right), \quad c_n(\epsilon') = \frac{8 - \epsilon'^2}{n^2 - 2n - \epsilon'^2} \]

27
and hence
\[ \int_W \phi^2 h_0^3 \leq c_0(\epsilon') \int_W \phi_1^2 (1 + \epsilon' h_0^2), \quad \phi_1 = |\nabla \phi| \]

Repeating the procedure we get
\[ \int_W \phi^2 h_0^2 \leq c_0(\epsilon') \int_W (1 + \epsilon^2 h_0) |\phi_3|^2 \]
and
\[ \int_W \phi^2 h_0 \leq c_0(\epsilon') \int_W (1 + \epsilon^2) |\phi_3|^2, \quad \phi_3 = |\nabla \phi_2| \]
We apply this formula for \( \phi = (|\text{Ric}| + 2)^{1/2} \zeta \) that
\[ \int_W \zeta^2 |\nabla h_0|^2 \leq \int_W \sum_{i=1}^{3} c_n(\epsilon')^i |\nabla^{i-j} \text{Ric}| |\nabla^j \zeta|^2 \]
and we obtain
\[ \sup W h_0 \leq D(t, \gamma^{-3}) \int_{A(W)} |\zeta|, \quad t = \max(t_1(W), t_2(W), t_3(W)) \]
where \( |\nabla \zeta|^j \leq c \gamma^{-j} \)

**Growth of a function near its nodal set**
We assume that
\[ \int_W \zeta^2 |\nabla u|^2 \leq \tau \int_W \zeta^2 u^2 \]
Let
\[ W_j = \{ x \in P/(1 - \theta^j) \theta \eta \leq |\tilde{u}(x)| \leq (1 - \theta + \theta^j) \eta \} \bigcap B_{C^{-1/2}} \]
and
\[ A(W_j) = W_j \bigcap \{ x \in P/(1 + \theta^{j+1}) \theta^2 \leq |\tilde{u}(x)| \leq \theta(1 - \theta^j) \eta \} \bigcap B_{C^{-1/2}} \]
The cut-off function \( \zeta \) satisfies the following estimate:
\[ |\nabla^\ell \zeta| \leq \frac{c_\ell}{\theta^\ell} \]
We apply Hardy’s inequality
\[ \int_{W_j} \zeta^2 u^2 = \int_{W_j} \tilde{u}^{\frac{2}{\theta^2}} (\zeta u)^2 \leq c_1(H) \left( \sup_{W_j} |\tilde{u}| \right)^{\frac{2}{\theta^2}} \int_{A(W_j)} |\nabla (\zeta u)|^2 \leq \]
\[ \leq C \left( \sup_{W_j} |\tilde{u}| \right)^{\frac{2}{\theta^2}} (1 + \epsilon) \left[ \frac{1}{\epsilon} \int_{A(W_j)} |\nabla \zeta|^2 u^2 + \tau \int_{A(W_j)} (u\zeta)^2 \right] \]
Therefore we have that close to \( T_0(\text{N}(\tilde{u})) \), for \( \eta = (2c_1(H)\tau(1 + \epsilon))^{-\frac{1}{\theta^2}} \). We have that
\[ \int_{W_j} \zeta^2 u^2 \leq \frac{1}{\tau^2 \epsilon} \int_{A(W_j)} |\nabla \zeta|^2 u^2 \leq \frac{c_1}{\tau^2 \epsilon \theta^2} \int_{A(W_j)} u^2 \]
We have that:
\[ \int_{A(W_j)} u^2 \leq 2 \int_{A(W_j)} \tilde{u}^2 + \kappa^2 \]
and the second integral is estimated again as
\[ \int_{A(W_j)} \kappa^2 \leq c_1(H)(\theta \eta)^{\frac{2}{\theta^2}} \int_{A(W_j)} u^2 \]
and hence
\[ \int_{A(W_j)} u^2 \leq \frac{4}{2 - \theta^2} \int_{A(W_j)} \tilde{u}^2 \]

28
The coarea formula suggests then after Lojasiewicz inequality that
\[ \int_{A(W_j)} \zeta^2 \hat{u}^2 = \int_{(\theta-\theta^0)(1+\xi)\eta} \frac{\hat{u}^2 d\sigma_{\mu}}{|\nabla \hat{u}|} + \int_{(1-\theta+\theta^0)\eta} \frac{\hat{u}^2 d\sigma_{\mu}}{|\nabla \hat{u}|} \leq \]
\[ \int_{(\theta-\theta^0)(1+\xi)\eta} \mu^{\nu+1} \alpha(\mu) d\mu + \int_{(1-\theta+\theta^0)\eta} \mu^{\nu+1} \alpha(\mu) d\mu \]

Inside a ball of radius \( \tau \) applying Crofton formula if the multiplicity of \( \hat{u} \) is \( m \)
\[ \alpha(\mu) = \int_{\{\hat{u}=\mu\}} d\sigma_{\mu} \leq c \tau^{-\frac{n+1}{2}} m \]

Therefore we find that for \( \sigma_j = \theta - \theta^0 \):
\[ \int_{A(W_j)} \zeta^2 \hat{u}^2 \leq \tau^{-\frac{n+1}{2}} \eta^{\nu+2} \left( \sigma_j^{\nu+1} \xi^{\nu+2} + \frac{(1 - \sigma_j)^{\nu+2}(1 - 3\sigma_j)}{1 - 2\sigma_j} \right) \]

Finally we have that
\[ \int_{W_j} \zeta^2 u^2 \leq C \tau^{-\frac{n+1}{2}} \eta^{\nu+2} \]

Moreover we recall the following identity from \([P]\):
\[ \eta^3 \frac{d}{d\eta} \eta^{-3} \int_W \zeta^2 \hat{u}^2 = -\int_W \nabla Q \cdot \nabla \hat{u} \zeta^2 \hat{u}^2 \]

for \( Q = |\nabla \hat{u}|^2 \geq c |\hat{u}|^2(1 - \nu) \). This leads to
\[ \eta^3 \frac{d}{d\eta} \eta^{-3} \int_W \zeta^2 \hat{u}^2 \leq \int_W \left| \frac{\nabla Q}{Q} \right| \zeta^2 \hat{u}^2 \leq \int_W \left| \frac{\nabla Q}{Q} \right|^2 \zeta^2 \hat{u}^{\nu+2} \]

We apply Hardy’s inequality and get
\[ \eta I'(\eta) \leq C \left( 1 + \tau^{-\frac{n+1}{2}} \eta^{\nu} \right) \tau \eta^3 I(\eta) \]

for
\[ I(\eta) = \eta^{-3} \int_W \zeta^2 \hat{u}^2 \]

Finally we get that for \( \eta > \eta_0 \):
\[ I(\eta) \leq C e^c \left( 1 + \tau^{-\frac{n+1}{2}} \eta^{\nu+1} \right) \tau \eta \eta_0^3 I(\eta_0) \]

and
\[ \int_W \zeta^2 \hat{u}^2 \leq C e^c \left( 1 + \tau^{-\frac{n+1}{2}} \eta^{\nu+1} \right) \tau \eta \left( \frac{\eta}{\eta_0} \right)^3 \int_{W_\omega} \zeta^2 \hat{u}^2 \]

Morrey estimates Let \( \epsilon < 1 \), \( 0 < \gamma < 1 \) or \( \gamma < 0 \), \( p < 2 \):
\[ u_\epsilon = \sqrt{u^2 + \epsilon^2}, \quad \psi_\epsilon = \log u_\epsilon, \quad w = u_\epsilon^\gamma \]

and for \( \zeta, \text{supp}(\zeta) \subset W \):
\[ \int_W \zeta^2 |\nabla u_\epsilon|^2 \leq \tau \int_W \zeta^2 u_\epsilon^2 \]  \hspace{1cm} (46)

Then for \( q = \frac{2}{\gamma} \):
\[ \int_W |\nabla w|^p \zeta^p \leq C_1(\tau) \int_W |\nabla \zeta|^2 w^q \] \hspace{1cm} (47a)
\[ \int_W |\nabla \psi_\epsilon|^2 \zeta^2 \leq C_2(\tau) \int_W |\nabla \zeta|^2 + \zeta^2 \] \hspace{1cm} (47b)
The inequality (47a) follows after selection for $\zeta$ as $u^{-1}$ gives since

$$\|\nabla w\| = \gamma w^{1+\frac{1}{\gamma}} \|\nabla u\|$$

that

$$\int W \zeta^2 |\nabla w|^2 \leq \frac{C_0}{\gamma^2} \int W \zeta^2 w^2$$

The inequality (47b) requires the additional assumption for $\tau > 0$:

$$\int W \zeta^2 |\nabla^2 u|^2 \leq \tau^2 \int W \zeta^2 u^2$$

We start selecting values $u_1, \ldots, u_m > 0$ and assume that

$$u = v_j + h_j$$

making the following choice:

$$|h_j| \leq \epsilon |v_j|$$

then

$$(v_j + h_j)^2 \geq (1 - \epsilon) \left( v_j^2 - \frac{h_j^2}{\epsilon} \right) \geq (1 - \epsilon) h_j^2$$

We approximate harmonically $h_j$ in suitable bricks selected so that we use the initial form of $\hat{h}_j$. Hence we have that for $\psi = \log(u_\epsilon)$, $\hat{\psi} = \log(h)$

$$\int P |\nabla \psi|^2 \zeta^2 \leq c \int P |\nabla \hat{\psi}|^2 \zeta^2$$

Therefore let $\hat{h}$ be the harmonic approximation of $h$ in $W$ and we set:

$$h = \hat{h} + \kappa$$

The standard harmonic approximation method estimates combined with partial integration leads us to

$$\int W \zeta^2 |\nabla \kappa|^2 \leq D(\eta, \tau^2) \int W \zeta^2 u^2$$

The estimate of the preceding paragraph

$$|\nabla \kappa| \leq \sup_{W_0} |\nabla \kappa| \leq c \left( \int W \zeta^2 u^2 \right)^{1/2}, \quad c = \frac{D(\eta, \tau) + D(\eta, \mu) ||Ric||_1}{\text{vol}(W_0)}$$

We compute for $\epsilon < 1$:

$$\epsilon^2 = \hat{u}^2 + 2\kappa \hat{u} + \kappa^2 + \epsilon^2 \geq (1 - \epsilon^2)\hat{u}^2 + (1 - \frac{1}{\epsilon^2})\kappa^2 + \epsilon^2$$

Then we select $\epsilon$ such that

$$\kappa^2 (1 - \frac{1}{\epsilon^2}) + \epsilon^2 > \frac{1}{2} \epsilon^2$$

and therefore we get that

$$|\kappa| \leq \frac{\epsilon^2}{\sqrt{2(1 - \epsilon^2)}}$$

Let then $m$ denote the highest multiplicity of $\hat{a}$. We apply the preceding estimates to conclude that for $\epsilon' = \frac{\epsilon^2}{\epsilon(1 - \epsilon^2)}$:

$$\int W \zeta^2 \frac{|\nabla u_\epsilon|^2}{u_\epsilon^2} \leq \frac{1}{2(1 - \epsilon^2)} \int W \frac{|\nabla \hat{u}_\epsilon|^2}{\hat{u}_\epsilon^2} \zeta^2 + c \int W \zeta^2 u_\epsilon^2 \int W \frac{\zeta^2}{\hat{u}_\epsilon^2}$$

Now we use the estimate of the preceding paragraph

$$|| \zeta u ||_{2,W} \leq C \mu \tau^{-\frac{m-1}{2}} \eta^{m+2}$$

and obtain that

$$\int W \zeta^2 |\nabla \psi| \zeta \leq C \int W |\nabla \zeta|^2 + c \tau^{-\frac{m-1}{2}} \eta^{m+2} \int W \frac{\zeta^2}{\hat{u}_\epsilon^2}$$
We set \( \epsilon_k(m) = 1 - \frac{k}{m} \) and then
\[
\int_W \frac{\zeta^2}{\tilde{u}^2} = \int_W \frac{(\tilde{u}^{-\epsilon_1} \zeta)^2}{\tilde{u}^2} \leq c_1(H) \left[ e^2 \int_W \left| \nabla \tilde{u} \right|^2 (\tilde{u}^{-\epsilon_1} \zeta)^2 + \int_W \tilde{u}^{-2\epsilon_1} |\nabla \zeta|^2 \right]
\]
The constant in Hardy’s inequality is \( c_{1,2}(H) \sim \frac{4}{(d-1)^2}, \ d \geq 3; \)
\[
\int_W |\nabla \tilde{u}|^2 \tilde{u}^{-2\epsilon_1} \zeta^2 \leq (1 + \epsilon)c_{1,2}(H) \left[ \frac{1}{\epsilon} \int_W \tilde{u}^{-2\epsilon_1} |\nabla \zeta|^2 + e^2 \int_W |\nabla \tilde{u}|^2 \tilde{u}^{-2\epsilon_1} \zeta^2 \right]
\]

We select then \( \epsilon \) so that:
\[
\epsilon \leq \frac{(d-3)m + 2}{(d + 1)m - 2}
\]
Hence we have
\[
\int_W \zeta^2 |\nabla \tilde{u}|^2 \leq \frac{(1 + \epsilon)c_1(H)}{\epsilon} \int_W \tilde{u}^{-2\epsilon_1} |\nabla \zeta|^2
\]
and we conclude that:
\[
\int_W \zeta^2 |\nabla \psi|^2 \leq \frac{(1 + \epsilon)c_1(H)}{\epsilon} \int_W \tilde{u}^{-2\epsilon_1} |\nabla \zeta|^2 \tag{48}
\]

Near the zeros of \( u, \tilde{u} \), i.e., selecting \( \eta \) suitably small then we retrieve the same inequality. Further treatment is required when we introduce in (48)
\[
\zeta = \psi^\ell_0 \check{\vartheta}
\]
and we get:
\[
\int_W \psi^2 |\nabla \tilde{u}|^2 \leq C\ell^2 \int_W \tilde{u}^{-\frac{\ell}{2}} \left( \psi^2 |\nabla \tilde{u}|^2 + \psi^2 |\nabla \vartheta|^2 \right)
\]
through the elementary inequality for \( x \in (\delta^{\frac{1}{2}}, 1) \):
\[
|\log x| \leq \frac{2}{\delta} \left( |x|^{\frac{1}{2}} + |x|^{-\frac{1}{2}} \right) \leq \frac{4}{\delta} \left( 1 + \frac{\log x}{\log \delta} \right)
\]
We obtain for \( \vartheta < 1, \delta = (\epsilon \ell - 1) \):
\[
\int_W \psi^2 |\nabla \psi|^2 \leq 2C\ell^2(1-c\ell) \int_W \psi^2 |\nabla \psi|^2 + \psi^2 |\nabla \vartheta|^2 \tag{49}
\]

The iteration for the lower bound. We follow the method of [SL2]. We select as \( \zeta \)
\[
u^{2(\ell-1)} \vartheta^{2(\alpha\ell-b)}, \ v = \psi - \sigma, \ a \geq 1 + b
\]
obtaining that:
\[
\int_W \nu^{2(\ell-b)} |\nabla \psi|^2 \leq \ell^2(1-c\ell) \int_W \nu^{2\ell-4} |\nabla \psi|^2 \vartheta^{2(\alpha\ell-b)} + \nu^{2(\ell-1)} \vartheta^{2(\alpha\ell-b-1)}
\]
We get that
\[
\int_W |\nabla |v|^{\ell} \vartheta^{2\ell-1}|^2 \leq \ell^2(\epsilon \ell + 3) \text{vol}(W) + \ell^2(\epsilon \ell + 1) \int_W \nu^{2\ell} \vartheta^{2(\alpha\ell-b)}
\]
Therefore we have that for \( a = kb, \ b = \frac{m}{m-1}, \ m = \frac{\ell}{2} \)
\[
\left( \frac{1}{\text{vol}(W)} \int_W (|v| \vartheta)^{2k} \right)^{\frac{1}{2k}} \leq \ell^{2(\epsilon \ell + 3)} + \ell^{2(\epsilon \ell + 1)} \left( \frac{1}{\text{vol}(W)} \int_W (|v| \vartheta)^{2m} \right)^{\frac{1}{2m}}
\]
Hence we have that
\[
\left( \frac{1}{\text{vol}(W)} \int_W (|v| \vartheta)^{2k} \right)^{\frac{1}{2k}} \leq \ell^{2(\epsilon \ell + 1)} + \ell^{2(\epsilon \ell + 1)} \left( \frac{1}{\text{vol}(W)} \int_W (|v| \vartheta)^{2m} \right)^{\frac{1}{2m}}
\]
Selecting a sequence \( \{\ell_j\} \):

\[
\ell_j = k^j, \quad I_j = \left( \frac{1}{v_j} \int_{W} (|v| |\eta|)^{2k^j} \right)^{1/k^j}
\]

Hence we obtain

\[
I_{j+1} \leq k^{j(e_p+3)} + k^{j(e_p+1)} I_j
\]

or that

\[
I_{j+1} \leq k^{j(e_p+3)} (1 + I_j)
\]

Examining separately the two cases: for some \( j_0 \):

\[
I_{j_0} \leq 1
\]

and the complementary case we arrive at the conclusion

\[
I_j \leq (2k^{j_0})^{j-j_0}
\]

Therefore following the reasoning in \([SL2]\) we conclude that

\[
\int_{W} e^{c_1|\psi| - \sigma} \leq C \text{vol}(W)
\]

which implies that

\[
\left( \int_{W} (u^2 + \epsilon^2)^{1/2} \right) \left( \int_{W} (u^2 + \epsilon^2)^{-1/2} \right) \leq C^2 \text{vol}(W)^{2}
\]

and for small \( p > 0 \):

\[
\inf_{W} u_{\epsilon} \geq C \left( \frac{1}{v_{\infty}} \int_{W} u_{\epsilon}^p \right)^{1/p}
\]

We follow again \([SL2]\) appealing to \([17]\) and get the bound:

\[
\inf_{W} |u| \geq C \left( \frac{1}{v_{\infty}} \int_{W} u^p \right)^{1/p}
\]

for any \( p \leq \frac{n}{n-2} \).

### 1.6 The two dimensional case

We will derive a version of Hardy’s inequality for the two dimensional situation that is not covered in the general case. Therefore we start with the radial blow-up of the plane covered in the

\[
C_1 = \mathbb{R}^2 \setminus \{|x_1| \geq \epsilon|x_2|\}, \quad C_2 = \mathbb{R}^2 \setminus \{|x_1| \geq \epsilon|x_2|\}
\]

We set in \( C_1 \)

\[
x_1 = \frac{r \xi}{\sqrt{1 + \xi^2}}, \quad x_2 = \frac{r}{\sqrt{1 + \xi^2}}
\]

and interchange the roles of \( x_1, x_2 \) in \( C_2 \). We obtain for

\[
P(x) = r^m R_j(\xi), \quad j = 1, 2
\]

the elementary identity:

\[
|\nabla P|^2 = r^{2(m-1)} \left[ \frac{m^2 R_j^2}{(1 + \xi^2)^m} + (1 + \xi^2)^2 r^2 \frac{d}{d\xi} \left( (1 + \xi^2)^{-2} R_j \right) \right]
\]

Therefore we have that:

\[
\frac{P^2(1 - \frac{d}{d\xi})}{|\nabla P|^2} = \frac{R^2}{m^2 + ((1 + \xi^2) R^2 - m \xi R)^2} \leq \frac{R^2}{m^2}
\]

Similarly:

\[
\frac{|\nabla P|^2}{P^2} = \frac{R^2}{r^2 + (1 + \xi^2) \frac{R^2}{R} - m \xi^2}
\]
We compute then that for $\delta < 1$ and $f \in C^\infty_0(\mathbb{R}^2 \setminus \{(P = 0) \cup \{\log |P| = \delta\})$ then

$$\int_{\mathbb{R}^2} \frac{1}{P^2 |\log \frac{|P|}{\delta}|} f^2 \leq \frac{C_1(P)}{|\log(\delta)|} \int_{\mathbb{R}^2} |\nabla f|^2$$

and

$$\int_{\mathbb{R}^2} \frac{2 |\nabla P|^2}{P^2 |\log \frac{|P|}{\delta}|} f^2 \leq \frac{C_2(P)}{|\log(\delta)|} \int_{\mathbb{R}^2} |\nabla f|^2$$

Then we start localizing in areas of constant sign for $R$:

$$I_{j,\epsilon} = \int_{\mathbb{R}^2} \frac{1}{r^2 |\log \frac{|P|}{\delta}|} \left| \frac{\nabla P}{P} \right|^2 \chi_{j,\epsilon} f^2 \leq C_\epsilon \int_{\mathbb{R}^2} \frac{m^2}{|\log \frac{|P|}{\delta}|} \chi_{j} f^2 + \frac{r}{|\log \frac{|P|}{\delta}|} \left( (1 + \frac{\epsilon^2}{R}) \frac{R'}{R} - m \frac{\epsilon}{2} \right) \chi_{j} f^2$$

We used the inequality:

$$(a + b)^2 \geq (1 - \epsilon)^2 \left( a^2 - \frac{1}{\epsilon^2} b^2 \right)$$

alternatively in the regions $(\log R)^2 \geq \eta^2 (\log r)^2$ and $(\log R)^2 \leq 2 \eta^2 (\log r)^2$. The inequality is majorised after integration by parts in the radial variable through the elementary inequality:

$$\int_0^\infty \frac{1}{r^2 |\log \frac{r}{\delta}|} g^2 dr \leq \frac{4}{|\log \delta|} \int_0^\infty g^2$$

This proved easily by splitting the integral after arranging $\epsilon$ according to the support of $g$

$$\int_0^\infty \frac{1}{r^2 |\log \frac{r}{\delta}|} g^2 dr = \int_0^{\delta^{1+}} \frac{1}{r^2 |\log \frac{r}{\delta}|} g^2 dr + \int_0^\infty \frac{1}{r^2 |\log \frac{r}{\delta}|} g^2 dr$$

The first integral is written after integration by parts:

$$2 \int_0^{\delta^{1+}} \frac{1}{r |\log \delta|} g d \frac{1}{r^2 (\log \frac{r}{\delta})^2} g^2 \leq \frac{1}{\epsilon} \int_0^\infty g^2 + \frac{1 + \epsilon}{|\log \delta|} \int_0^\infty \frac{1}{r^2 |\log \frac{r}{\delta}|} g^2$$

then we choose $\epsilon = \frac{|\log \delta|}{2}$ and we get:

$$\int_0^{\delta^{1+}} \frac{1}{r^2 |\log \delta|} g^2 \leq \frac{C}{|\log \delta|} \int_0^{\delta^{1+}} g^2$$

Similarly for the other integral, setting as $g^2 = r f^2$ and splitting the integral in two pieces then

$$\int_0^\infty \frac{1}{r^2 |\log \frac{r}{\delta}|} f^2 \leq \frac{2C_\epsilon}{|\log \delta|} \int_0^\infty r f^2 + \frac{2C_\epsilon}{|\log \delta|} \int_0^\infty \frac{1}{r^2 |\log \frac{r}{\delta}|} r f^2 \leq \frac{2C_\epsilon}{|\log \delta|} \int_0^\infty f^2 r dr$$

The two dimensional inequality is obtained by a direct application of the usual one dimensional inequality in the $\xi$-variable after the formula:

$$\left( \frac{R'}{R} \right)^2 \leq 2 \left( C + \sum_{j=1}^{m} \frac{m^2}{(\xi - \xi_j)^2} \right)$$

### 1.7 Curvature estimates

In the integration by parts formulas we substitute the curvature identities we find that:

$$\int_{\text{K}} \phi^2 |\nabla Rm|^2 \leq 3 \int_{\text{K}} \phi^2 (|R|^3 + |Rm|^3), \quad \int_{\text{K}} \phi^2 |\nabla \text{Ric}|^2 \leq 3 \int_{\text{K}} \phi^2 |R|^3$$

The iteration scheme suggests that

$$\sup_{\text{W}^*} |Rm| \leq D(\eta, \mu) \inf_{\text{W}^*} |R|$$

$$\sup_{\text{W}^*} |\text{Ric}| \leq D(\eta, \mu) \inf_{\text{W}^*} |R|$$
1.8 Local properties of eigenfunctions

In the case of an eigenfunction we have the following

\[ \int_K \phi^2 |\nabla^2 u_\lambda|^2 \leq \lambda (\lambda + D(\eta, \mu)||Ric||_1, \mathbf{W}) \left( \int_K \phi^2 u_\lambda^2 \right) \]  
\[ \int_K \phi^2 |\nabla^3 u_\lambda|^2 \leq \lambda^2 (\lambda + D(\eta, \mu)||Ric||_1, \mathbf{W}) \int_K \phi^2 u_\lambda^2 \]  

Performing partial integration to the term:

\[ \int_K \phi^2 |Ric||\nabla f|^2 = -2 \int_K |Ric| f \phi \nabla \cdot \nabla f - \int_K \phi^2 f \nabla |Ric| \cdot \nabla f - \int_K |Ric| \phi^2 f \Delta_g f \]

Young’s inequality along with harmonic approximation for \( \sqrt{|Ric|^2 + \epsilon} \) leads to

\[ \int_K \phi^2 |Ric||\nabla u_\lambda|^2 \leq C \lambda \int_K |Ric| \phi^2 u_\lambda^2 \]

Similarly we get for Finally we conclude that

\[ \int_K \phi^2 |\nabla^2 u_\lambda|^2 \leq \lambda^2 \int_K \phi^2 \left( 1 + \frac{|Ric|}{\lambda} \right) \]
\[ \int_K \phi^2 |\nabla^3 u_\lambda|^2 \leq C \lambda^3 \int_K \phi^2 (1 + \frac{|Rm|}{\lambda}) u_\lambda^2 + \int_K \left[ \frac{1}{\epsilon^2} (|\nabla Rm|^2 + |\nabla Ric|^2) + |Rm|^2 + |Ric|^2 \right] \phi^2 |\nabla u_\lambda|^2 \]

1.8.1 Harnack inequalities

**The eigenfunction.** We have for \( \gamma = \lambda, \tilde{u}_{\lambda, \epsilon} = \sqrt{u_\lambda^2 + \epsilon^2} \) that

\[ \sup_{\mathbf{W}} \tilde{u}_{\lambda, \epsilon} \leq D(\eta, \lambda) \inf_{\mathbf{W}} \tilde{u}_{\lambda, \epsilon} \]

**The gradient.** Now the gradient \( G_{\lambda, \epsilon} = |\nabla u_\lambda|^2 + \epsilon^2 \) requires that we use the \( \gamma_1 = \lambda + \kappa \) and we get that

\[ \sup_{\mathbf{W}} G_{\lambda, \epsilon} \leq D(\eta, \lambda) (\lambda + D(\eta, \mu)||Ric||_1, \mathbf{W}) \inf_{\mathbf{W}} G_{\lambda, \epsilon} \]

**The hessian estimate** The estimate

\[ \sup_{\mathbf{W}} H_{\lambda, \epsilon} \leq D(\eta, \lambda) \lambda^2 (\lambda + D(\eta, \mu)||Ric||_1, \mathbf{W}) \inf_{\mathbf{W}} H_{\lambda, \epsilon} \]

**The estimates for the restriction on the spherical front** The restriction of the eigenfunction

\[ u(r, \theta) = e^\lambda \sin(\beta_1 \phi) \]

of the spherical front satisfies the following inequalities for \( j = 1, 2 \):

\[ \int_F \theta^2 |\nabla^j u|^2 \leq C \int_F \theta^2 (\lambda + R^2) u^2 \]

where \( R \) is a polynomial depending on \( Rm, \nabla Rm, \ldots, \nabla^2 Rm, Ric, \nabla Ric, \nabla Ric \). This combined with Michael-Simon Sobolev inequality provides Harnack inequalities for the restriction of \( u, |Xu|, X^2 u \) on the spherical front.

1.8.2 The Bernstein inequalities

The integration by parts formulas suggest along with the Harnack inequalities the following Berstein type inequalities in geodesic pixels:

**Estimates 13.** The following estimates hold in a domain inside a geodesic pixel \( \mathbf{W} \subset \mathbf{P} \)

\[ |\nabla^2 u_\lambda| \leq \sup_{\mathbf{W}} |\nabla^2 u_\lambda| \leq C_2(\mathbf{W}) \lambda^{\frac{3}{2}} (|\nabla u_\lambda| + \epsilon) \leq C_3(\mathbf{W}) \lambda^{\frac{3}{2}+1} (|u_\lambda| + \epsilon) \]
\[ |\nabla u_\lambda| \leq \sup_{\mathbf{W}} |\nabla u_\lambda| \leq C_4(\mathbf{W}) \lambda^{\frac{3}{2}+1} (|u_\lambda| + \epsilon) \]
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