The Cauchy problem for the energy-critical inhomogeneous nonlinear Schrödinger equation with inverse-square potential

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Abstract

In this paper, we study the Cauchy problem for the energy-critical inhomogeneous nonlinear Schrödinger equation with inverse-square potential

\[ \begin{align*}
    iu_t + \Delta u - c|x|^{-2}u = \lambda|x|^{-b}|u|^\sigma u, & \quad u(0) = u_0 \in H^1, \\
    (t, x) \in \mathbb{R} \times \mathbb{R}^d,
\end{align*} \]

where \( d \geq 3, \lambda = \pm 1, 0 < b < 2, \sigma = \frac{d-2}{2} \) and \( c > -c(d) := -\left(\frac{d-2}{(d+2-2b)^2}\right)^2 \). We first prove the local well-posedness as well as small data global well-posedness and scattering in \( H^1 \) for \( c > \frac{d+2-4b}{(d+2-2b)^2} \) and \( 0 < b < \frac{d}{2} \), by using the contraction mapping principle based on the Strichartz estimates. Based on the local well-posedness result, we then establish the blowup criteria for solutions to the equation in the focusing case \( \lambda = -1 \). To this end, we derive the sharp Hardy-Sobolev inequality and virial estimates related to this equation.

Keywords: Inhomogeneous nonlinear Schrödinger equation; Inverse-square potential; Energy-critical; Well-posedness; Blowup; Hardy-Sobolev inequality; Virial estimates

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1 Introduction

In this paper, we consider the Cauchy problem for the inhomogeneous nonlinear Schrödinger equation with inverse-square potential, denoted by INLS\(_c\) equation,

\[ \begin{align*}
    \begin{cases}
    iu_t - P_c u = \lambda|x|^{-b}|u|^\sigma u, & \quad (t, x) \in \mathbb{R} \times \mathbb{R}^d, \\
    u(0, x) = u_0(x),
    \end{cases}
\end{align*} \]

where \( d \geq 3, u : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}, u_0 : \mathbb{R}^d \to \mathbb{C}, \sigma > 0, \lambda = \pm 1 \) and \( P_c = -\Delta u + c|x|^{-2} \) with \( c > -c(d) := -\left(\frac{d-2}{d+2-2b}\right)^2 \). \( \lambda = -1 \) corresponds to the focusing case and \( \lambda = 1 \) corresponds to the defocusing case. The restriction on \( c \) comes from the sharp Hardy inequality:

\[ \frac{(d-2)^2}{4} \int_{\mathbb{R}^d} |x|^{-2} |u(x)|^2 \, dx \leq \int_{\mathbb{R}^d} |\nabla u(x)| \, dx, \quad \forall u \in H^1(\mathbb{R}^d), \]

which ensures that \( P_c \) is a positive operator. The INLS\(_c\) equation appears in a variety of physical settings, for example, in nonlinear optical systems with spatially dependent interactions (see e.g. [2] and the references therein). In particular, when \( c = 0 \), it can be thought of as modeling inhomogeneities in
the medium in which the wave propagates (see e.g. [17]). When \( b = 0 \), the equation (1.1) also appears in various areas of physics, for instance in quantum field equations, or in the study of certain black hole solutions of the Einstein equations (see e.g. [2, 12]).

The case \( b = c = 0 \) is the classic nonlinear Schrödinger (NLS) equation which has been been widely studied over the last three decades (see e.g. [8, 21, 23] and the references therein). The case \( b = 0 \) and \( c \neq 0 \) is known as the NLS equation with inverse-square potential, denoted by NLS(\( \sigma \)) equation, which has also been extensively studied in recent years (see e.g. [10, 18, 20, 23, 26] and the references therein). Moreover, when \( c = 0 \) and \( b \neq 0 \), we have the inhomogeneous nonlinear Schrödinger (INLS) equation, which has also attracted a lot of interest in recent years (see e.g. [1, 2, 6, 9, 13] and the references therein).

On the other hand, the inhomogeneous nonlinear Schrödinger with potential in the following form:

\[
\begin{align*}
\left\{ 
&iu_t + \Delta u - Vu = \lambda |x|^{-b}|u|^\sigma u, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^d, \\
&u(0, x) = u_0(x).
\end{align*}
\] 

(1.3)

has also been studied by several authors in recent years. For example, Dinh [11] studied the well-posedness, scattering and blowup for (1.3) when \( d = 3 \), \( b > 0 \), \( \lambda = \pm 1 \) and \( V \) is a real-valued potential satisfying \( V \in K_0 \cap L^2 \) and \( \|V|_\infty < 4\pi \), where \( |V|_\infty = \min \{V, 0\} \) and \( K_0 \) is defined as the closure of bounded and compactly supported functions with respect to the Kato norm

\[ \|V\|_K := \sup_{x \in \mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|V(y)|}{|x - y|} dy. \]

Luo [22] also studied the stability and multiplicity of standing waves for (1.3) with \( V = |x|^2 \) (harmonic potential), \( \lambda = -1 \) and \( b < 0 \). The case \( V(x) = c|x|^{-2} \) with \( c > -c(d) \) (inverse-square potential) and \( b > 0 \) was considered by [7, 24].

In this paper, we are interested in (1.3) with \( V(x) = c|x|^{-2} \) with \( c > -c(d) \) and \( b > 0 \), i.e. we study the INLS(\( \sigma \)) equation (1.1).

Before recalling the known results for the INLS(\( \sigma \)) equation (1.1) and stating our main results, let us give some information about this equation. The INLS(\( \sigma \)) equation (1.1) is invariant under the scaling,

\[ u_\lambda(t, x) := \lambda^{\frac{2b}{d}} u\left(\lambda^2 t, \lambda x\right), \quad \lambda > 0. \]

An easy computation shows that

\[ \|u_\lambda(0)\|_{\dot{H}^\sigma} = \lambda^{\frac{2b}{d}} + \frac{2b}{d} \|u_0\|_{\dot{H}^\sigma}, \]

which implies that the critical Sobolev index is given by

\[ s_c = \frac{d}{2} - \frac{2 - b}{\sigma}. \] 

(1.4)

Note that, if \( s_c = 0 \) (alternatively \( \sigma = \sigma_* := \frac{d - 2b}{d} \)) the problem is known as the mass-critical or \( L^2 \)-critical; if \( s_c = 1 \) (alternatively \( \sigma = \sigma_* := \frac{d - 2b}{2} \)) it is called energy-critical or \( \dot{H}^1 \)-critical. The problem is known as intercritical (mass-supercritical and energy-subcritical) if \( 0 < s_c < 1 \) (alternatively \( \sigma_* < \sigma < \sigma^* \)). On the other hand, solutions to the INLS(\( \sigma \)) equation (1.1) conserve the mass and energy, defined respectively by

\[ M(u(t)) := \int_{\mathbb{R}^d} |u(t, x)|^2 dx, \]

(1.5)

\[ E_{b,c}(u(t)) := \int_{\mathbb{R}^d} \left[ \frac{1}{2} |\nabla u(t, x)|^2 + \frac{c}{2} |x|^{-2} |u(t, x)|^2 + \frac{\lambda}{\sigma + 2} |x|^{-b} |u(t, x)|^{\sigma + 2} \right] dx. \]

(1.6)
Let us recall the known results for the INLS equation \((1.1)\). Using the energy method, Suzuki \cite{24} showed that if \(d \geq 3\), \(0 < \sigma < \sigma^*\), \(c > -c(d)\) and \(0 < b < 2\), then the INLS equation \((1.1)\) is locally well-posed in \(H^1_c(\mathbb{R}^d)\) (which is equivalent to \(H^1(\mathbb{R}^d)\)). It was also proved that any local solution of \((1.1)\) with \(u_0 \in H^1_c(\mathbb{R}^d)\) extends globally in time if either \(\lambda = 1\) (defocusing case) or \(0 < \sigma < \sigma^*_\lambda\) for \(\lambda = -1\) (focusing, mass-subcritical case). Recently, Campos-Guzmán \cite{7} established the sufficient conditions for global existence and blowup in \(H^1_c(\mathbb{R}^d)\) for \(d \geq 3\), \(\lambda = -1\) and \(\sigma^*_\lambda \leq \sigma < \sigma^*\), using a Gagliardo-Nirenberg-type estimate. They also studied the local well-posedness and small data global well-posedness under some assumption on \(b\) and \(c\) in the energy-subcritical case \(\sigma < \sigma^*\) with \(d \geq 3\) by using the standard Strichartz estimates combined with the fixed point argument. Furthermore, they showed a scattering criterion and construct a wave operator in \(H^1_c(\mathbb{R}^d)\), for the intercritical case. As mentioned above, the authors in \cite{7,24} studied the local and global well-posedness as well as blowup and scattering in \(H^1_c(\mathbb{R}^d)\) with \(d \geq 3\) for the INLS equation \((1.1)\) in the energy–subcritical case \(\sigma < \sigma^*\) \(\left(= \frac{4-2b}{d-2}\right)\).

In this paper, we study the well-posedness and blowup in \(H^1(\mathbb{R}^d)\) with \(d \geq 3\) for the INLS, equation \((1.1)\) in the energy-critical case \(\sigma = \sigma^* = \frac{4-2b}{d-2}\).

First, we prove the local well-posedness as well as small data global well-posedness and scattering by using the contraction mapping principle based on Strichartz estimates.

**Theorem 1.1.** Let \(d \geq 3\), \(0 < b < \frac{4}{d}\), \(\sigma = \frac{4-2b}{d-2}\) and \(c > -\frac{(d+2-2b)^2-4}{(d+2-2b)^2}c(d)\). If \(u_0 \in H^1(\mathbb{R}^d)\), then there exists \(T = T(u_0) > 0\) such that \((1.1)\) has a unique solution
\[
\begin{align*}
u &\in L^{\gamma(r)}([-T,T), H^{1,r}(\mathbb{R}^d)), \\
\gamma(r), &\text{ is an admissible pair satisfying}
\end{align*}
\]
\[
r = \frac{2d(d + 2 - 2b)}{d^2 - 2db + 4}.
\]
Moreover, for any admissible pair \((\gamma(p), p)\), we have
\[
\begin{align*}
u &\in L^{\gamma(p)}([-T,T), H^{1,p}(\mathbb{R}^d)). \\
\gamma(p) &\text{ is an admissible pair satisfying}
\end{align*}
\]
If \(\|u_0\|_{H^1(\mathbb{R}^d)}\) is sufficiently small, then the above solution is global and scatters.

**Remark 1.2.** Theorem \((1.1)\) can be seen as the extension of the well-posedness result of NLS, equation \((1.1)\) to the INLS equation.

**Remark 1.3.** In Theorem \((1.1)\) the restriction \(b < \frac{4}{d}\) comes from the fractional Hardy inequality (Lemma \((3.1)\)). And the restriction \(c > -\frac{(d+2-2b)^2-4}{(d+2-2b)^2}c(d)\) comes from the equivalence of Sobolev spaces \(\tilde{H}^{1,r}_c \sim \tilde{H}^{1,r}\).

Based on the local well-posedness result above, we study the blowup phenomena for the focusing, energy–critical INLS equation.

Let \(0 < b < 2\), \(c > -c(d)\), and let \(C_{HS}(b, c)\) be the sharp constant in the Hardy-Sobolev inequality related to the focusing, energy–critical INLS equation \((1.1)\), namely,
\[
C_{HS}(b, c) = \inf_{f \in H^1_c(\mathbb{R}^d) \setminus \{0\}} \frac{\|f\|_{H^1_c}}{\|\lambda |x|^{-b} |f|^\sigma + 2\|\|_{L_{\gamma(p)}^{1/2}}^2}.
\]
We will see in Lemma \((3.1)\) that:

\footnote{Note that the author in \cite{24} considered \((1.1)\) with \(c = -c(d)\). The authors in \cite{7} pointed out that the proof for the case \(c > -c(d)\) is an immediate consequence of the previous one.}
1. When $-c(d) < c \leq 0$, the sharp constant $C_{HS}(b,c)$ is attained by the function
\[
W_{b,c}(x) := \frac{[\varepsilon(d-b)(d-2)]^{\frac{\beta-\frac{1}{2}}{2}}}{[\varepsilon + |x|^{2-b}\beta]^{\frac{\beta-\frac{1}{2}}{2}}} |x|^{\rho},
\]
with $\beta = 1 - \frac{2d}{d+2}$, for all $\varepsilon > 0$ (see (2.2) for the definition of $\rho$).

2. If $c > 0$, $C_{HS}(b,c) \leq C_{HS}(b,0)$.

We have the following blowup result for the focusing, energy-critical INLS$_c$ equation.

**Theorem 1.4.** Let $d \geq 3$, $0 < b < \frac{d}{2}$, $\lambda = -1$, $c > -\frac{(d+2-2b)^2}{2(d+b)}c(d)$ and $\sigma = \frac{4-b}{d+b}$. Let $u_0 \in H^1(\mathbb{R}^d)$ and $u$ be the corresponding solution to (1.1). Suppose that either $E_{b,c}(u_0) < 0$, or if $E_{b,c}(u_0) \geq 0$, we assume that $E_{b,c}(u_0) < E_{b,c}(W_{b,c})$ and $\|u_0\|_{H^1} > \|W_{b,c}\|_{H^1}$, where $\bar{c} = \min \{c, 0\}$. If $xu_0 \in L^2$ or $u_0$ is radial, then the solution $u$ blows up in finite time.

**Remark 1.5.**

1. In Theorem 1.4 the restrictions on $b$ and $c$ only come from the local well-posedness result (Theorem 1.1). If we can prove the local existence of solution for the wider range of $b$ and $c$, the result of Theorem 1.4 still holds.

2. Theorem 1.4 can be seen as the extension of the blowup result of NLS$_c$ equation (see Theorem 1.12 of [10]) to the INLS$_c$ equation.

This paper is organized as follows. In Section 2, we recall some useful facts which are used in this paper. In Section 3, we prove Theorem 1.4. In Section 4, we derive the sharp Hardy-Sobolev inequality and virial estimates related to INLS$_c$ equation to prove Theorem 1.4.

## 2 Preliminaries

Let us introduce the notation used throughout the paper. As usual, we use $\mathbb{C}$, $\mathbb{R}$ and $\mathbb{N}$ to stand for the sets of complex, real and natural numbers, respectively. $C > 0$ will denote positive universal constant, which can be different at different places. $a \lesssim b$ means $a \leq Cb$ for some constant $C > 0$. We also write $a \sim b$ if $a \lesssim b \lesssim a$. We denote by $p'$ the dual number of $p \in [1, \infty]$, i.e. $1/p + 1/p' = 1$. As in [25], for $s \in \mathbb{R}$ and $1 < p < \infty$, we denote by $H^{s,p}(\mathbb{R}^d)$ and $\dot{H}^{s,p}(\mathbb{R}^d)$ the usual nonhomogeneous and homogeneous Sobolev spaces associated to the Laplacian $-\Delta$. As usual, we abbreviate $H^{s,2}(\mathbb{R}^d)$ and $\dot{H}^{s,2}(\mathbb{R}^d)$ as $H^s(\mathbb{R}^d)$ and $\dot{H}^s(\mathbb{R}^d)$, respectively. Similarly, we define Sobolev spaces in terms of $P_c$ via
\[
\|f\|_{H^s(\mathbb{R}^d)} = \|P_c \tilde{f}\|_{L^s(\mathbb{R}^d)}, \quad \|f\|_{H^{s,p}(\mathbb{R}^d)} = \|P_c \tilde{f}\|_{L^s(\mathbb{R}^d)}.
\]

We also abbreviate $\dot{H}^s(\mathbb{R}^d) = \dot{H}^{s,2}(\mathbb{R}^d)$ and $\dot{H}^s(\mathbb{R}^d) = H^{s,2}(\mathbb{R}^d)$. Note that by sharp Hardy inequality (1.2), we see that
\[
\|f\|_{H^s(\mathbb{R}^d)} \sim \|f\|_{H^s(\mathbb{R}^d)} \quad \text{for } c > -c(d).
\]

For $I \subset \mathbb{R}$ and $\gamma \in [1, \infty]$, we will use the space-time mixed space $L^\gamma(I, X(\mathbb{R}^d))$ whose norm is defined by
\[
\|f\|_{L^\gamma(I, X(\mathbb{R}^d))} = \left( \int_I \|f(t)\|_{X(\mathbb{R}^d)}^\gamma dt \right)^{1/\gamma},
\]
with a usual modification when $\gamma = \infty$, where $X(\mathbb{R}^d)$ is a normed space on $\mathbb{R}^d$. Given normed spaces $X$ and $Y$, $X \subset Y$ means that $X$ is continuously embedded in $Y$, i.e. there exists a constant $C(>0)$
such that $\|f\|_Y \leq C \|f\|_X$ for all $f \in X$. If there is no confusion, $\mathbb{R}^d$ will be omitted in various function spaces.

Next, we recall the equivalence between the usual Sobolev space defined by $-\Delta$ and the one defined by $P_c$. For convenience, we define the following number:

$$\rho := \frac{d - 2}{2} - \sqrt{\left(\frac{d - 2}{2}\right)^2 + c}. \quad (2.2)$$

**Lemma 2.1** (Equivalence of Sobolev spaces, [19]). Let $d \geq 3$, $c > -c(d)$ and $0 < s < 2$.

1. If $1 < p < \infty$ satisfies $\frac{d + \rho}{d} < \frac{1}{p} < \min \left\{ 1, \frac{d - \rho}{d} \right\}$, then $\|f\|_{H^{s,p}} \lesssim \|f\|_{\dot{H}^{s,p}}$ for all $f \in C_0^\infty(\mathbb{R}^d \setminus \{0\})$.

2. If $1 < p < \infty$ satisfies $\max \left\{ \frac{q}{2}, \frac{d}{\rho} \right\} < \frac{1}{p} < \min \left\{ 1, \frac{d - \rho}{d} \right\}$, then $\|f\|_{H^{s,p}} \lesssim \|f\|_{H^{r,p}}$ for all $f \in C_0^\infty(\mathbb{R}^d \setminus \{0\})$.

**Remark 2.2.** Let $0 < s < 2$.

1. When $c > 0$, $\|f\|_{H^{s,p}}$ is equivalent to $\|f\|_{H^{r,p}}$, provided that $1 < p < \frac{d}{\rho}$.

2. When $-c(d) \leq c < 0$, $\|f\|_{H^{s,p}}$ is equivalent to $\|f\|_{H^{r,p}}$, provided that $\frac{d - \rho}{d - p} < p < \frac{d - \rho}{d - p}$.

We end this section by recalling the Strichartz estimates for the INLS, equation (1.1).

**Definition 2.3.** Let $d \geq 3$. We say that a pair $(\gamma(p), p)$ is admissible, if

$$2 \leq p \leq \frac{2d}{d - 2} \frac{2}{\gamma(p)} = \frac{d}{2} - \frac{d}{p}. \quad (2.3)$$

**Lemma 2.4** (Strichartz estimates, [4] [5]). Let $d \geq 3$ and $c > -c(d)$. Then for any $s \in \mathbb{R}$ and any admissible pairs $(\gamma(p), p)$, $(\gamma(r), r)$, we have

$$\left\| e^{-itP_c} f \right\|_{L^{\gamma(p)}(\mathbb{R}, H^{s,p}_c)} \lesssim \|f\|_{H^{s}_c}. \quad (2.4)$$

$$\left\| \int_0^t e^{-i(t-\tau)P_c} f(\tau) d\tau \right\|_{L^{\gamma'(r)}(\mathbb{R}, H^{s,r}_c)} \lesssim \|f\|_{L^{\gamma'(r)}(\mathbb{R}, H^{s,r}_c)}. \quad (2.5)$$

### 3 Local and global well-posedness

In this section, we prove Theorem 1.1. To establish the nonlinear estimates, we recall the following fractional Hardy inequality which is a direct consequence of Theorem 3.1 of [13].

**Lemma 3.1** (Fractional Hardy Inequality). Let $1 < p < \infty$ and $0 < s < \frac{d}{p}$. Then we have

$$\left\| |x|^{-s} f \right\|_{L^p(\mathbb{R}^d)} \lesssim \|f\|_{H^{s,p}(\mathbb{R}^d)}.$$

Using Lemma 3.1, we have the following nonlinear estimates.

**Lemma 3.2.** Let $\bar{\rho} = \frac{2d}{d - 2}$, $r = \frac{2d(d + 2 - 2b)}{d^2 - 2db + 4}$, $0 < b < \frac{1}{d}$ and $\sigma = \frac{4 - 2b}{d - 2}$. Then we have

$$\left\| |x|^{-b} |u|^\sigma u \right\|_{H^{1,r}} \lesssim \|u\|_{H^{1,r}}^{\sigma + 1}, \quad (3.1)$$

$$\left\| |x|^{-b} |u|^\sigma v \right\|_{H^{1,r}} \lesssim \|u\|_{H^{1,r}}^\sigma \|v\|_{L^r}. \quad (3.2)$$
Proof. Noticing that
\[ |\nabla (|x|^{-b}|u|^\sigma u)| \lesssim |x|^{-b-1}|u|^\sigma + |x|^{-b}|u|^\sigma |\nabla u|, \] (3.3)
we have
\[ \| |x|^{-b}|u|^\sigma u|_{H^r_\rho} = \| \nabla (|x|^{-b}|u|^\sigma u) \|_{\rho} \lesssim \| |x|^{-b-1}|u|^\sigma + |x|^{-b}|u|^\sigma |\nabla u| \|_{\rho}. \] (3.4)

First we estimate \( \| |x|^{-b-1}|u|^\sigma u \|_{\rho} \). We can see that
\[ \frac{1}{\rho} = (\sigma + 1) \left( \frac{1}{r} - \frac{1}{d} \left( 1 - \frac{b + 1}{\sigma + 1} \right) \right). \]
Putting
\[ \frac{1}{\rho} := \frac{1}{r} - \frac{1}{d} \left( 1 - \frac{b + 1}{\sigma + 1} \right), \]
we have \( \dot{H}^{1 - \frac{b + 1}{\sigma + 1}} \subset L^\rho \). Here, we use the fact \( 1 - \frac{b + 1}{\sigma + 1} > 0 \Leftrightarrow b < \frac{d}{2} \). Using Lemma 3.1, we have
\[ \| |x|^{-b-1}|u|^\sigma u \|_{\rho} = \| |x|^{-\frac{b+1}{\sigma+1}} u \|_{\rho} \lesssim \| |x|^{-\frac{b+1}{\sigma+1}} u \|_{H^{1 - \frac{b+1}{\sigma+1}}}. \] (3.5)

Next we estimate \( \| |x|^{-b}|u|^\sigma \nabla u \|_{\rho} \). We get
\[ \sigma \left( \frac{1}{r} - \frac{1}{d} \left( 1 - \frac{b}{\sigma} \right) \right) + \frac{1}{r} = \frac{1}{\rho}. \]
Putting
\[ \frac{1}{\gamma} := \frac{1}{r} - \frac{1}{d} \left( 1 - \frac{b}{\sigma} \right), \]
and noticing \( 1 - \frac{b}{\sigma} > 0 \), we have \( \dot{H}^{1 - \frac{b}{\sigma}} \subset L^\gamma \). Hence it follows from H"older inequality and Lemma 3.1 that
\[ \| |x|^{-b}|u|^\sigma \nabla u \|_{\rho} \lesssim \| |x|^{-\frac{b}{\sigma}} u \|_{\gamma} \| \nabla u \|_{\rho} \lesssim \| |x|^{-\frac{b}{\sigma}} u \|_{H^{1 - \frac{b}{\sigma}}} \| u \|_{L^\gamma} \lesssim \| u \|_{\sigma + 1}. \] (3.6)

In view of (3.4)–(3.6), we immediately have (3.1). Similarly we also have
\[ \| |x|^{-b}|u|^\sigma v \|_{L^\rho} \lesssim \| |x|^{-\frac{b}{\sigma}} u \|_{\gamma} \| v \|_{\rho} \lesssim \| |x|^{-\frac{b}{\sigma}} u \|_{H^{1 - \frac{b}{\sigma}}} \| v \|_{\rho} \lesssim \| u \|_{\sigma + 1} \| v \|_{\rho}, \]
this concludes the proof. \( \square \)

Proof of Theorem 1.1. We can easily see that \((\gamma(r), r)\) is admissible, where \( r \) is given in (1.8). Furthermore, using Remark 2.2 we can easily verify that \( \dot{H}_{c, r}^1 \) is equivalent to \( \dot{H}_{c, r}^1 \) provided that \( c > -\frac{(d + 2) - 2b + 4}{d + 2 - 2b} c(d) \). Putting \( r = \frac{2d}{d - 2} \), we can also see that \( \dot{H}_{c, r}^1 \sim \dot{H}_{1, r}^1 \). Noticing
\[ \frac{1}{\gamma(\bar{r})} = \frac{1}{\gamma(r)} + \frac{1}{\gamma(r)} \]
and using Lemma 3.2, H"older inequality, we immediately have
\[ \| |x|^{-b}|u|^\sigma u \|_{L^{\gamma(r)}(I, H^r)} \lesssim \| u \|_{L^{\gamma(r)}(I, H^r)}^{\sigma + 1}, \]
where \( I \subset \mathbb{R} \) is an interval. Using (3.7), Lemma 3.2 and H"older inequality, we also have
\[ \| |x|^{-b}|u|^\sigma u \|_{L^{\gamma(r)}(I, L^r)} \lesssim \| u \|_{L^{\gamma(r)}(I, H^r)}^{\sigma} \| u \|_{L^{\gamma(r)}(I, L^r)}. \]
(3.9)
In view of (3.8) and (3.9), we have
\[ \|x|^{-b}|u|^\sigma \|_{L^\gamma(I, H^{1'})} < \|u\|_{L^\gamma(I, \hat{H}^{1'})} \|u\|_{L^\gamma(I, H^{1'})}. \] (3.10)

On the other hand, noticing that
\[ \|x|^{-b}|u|^\sigma u - |x|^{-b}|v|^\sigma v \| \lesssim |x|^{-b}(|u|^\sigma + |v|^\sigma)|u - v|, \]
using Lemma 3.2, we have
\[ \|x|^{-b}|u|^\sigma u - |x|^{-b}|v|^\sigma v\|_{L^\gamma(I, L')} \lesssim \|x|^{-b}(|u|^\sigma + |v|^\sigma)(u - v)\|_{L^\gamma(I, L')} \leq (\|u\|_{H^1} + \|v\|_{H^1}) \|u - v\|_{L^\gamma(I, L')} . \] (3.11)

Using (3.7), (3.11) and Hölder inequality, we immediately have
\[ \|x|^{-b}|u|^\sigma u - |x|^{-b}|v|^\sigma v\|_{L^\gamma(I, L')} \lesssim (\|u\|_{L^\gamma(I, H^1)} + \|v\|_{L^\gamma(I, H^1)}) \|u - v\|_{L^\gamma(I, L')} . \] (3.12)

First, we prove the local well-posedness. Let \( T > 0 \) and \( M > 0 \) which will be chosen later. We define the following complete metric space
\[ D = \{ u \in L^\gamma(I, H^1) : \|u\|_{L^\gamma(I, H^1)} \leq M \} , \]
which is equipped with the metric
\[ d(u, v) = \|u - v\|_{L^\gamma(I, L')} , \]
where \( I = [-T, T] \). We consider the mapping
\[ T: u(t) \rightarrow e^{-itP_e}u_0 + \int_0^t e^{-i(t-\tau)P_e} |x|^{-b} |u(\tau)|^{\sigma} u(\tau) d\tau =: u_L + u_{NL} . \]

Lemma 2.4 (Strichartz estimates) yields that
\[ \|u_L\|_{L^\gamma(I, H^1)} \sim \|u_L\|_{L^\gamma(I, H^{1'})} \lesssim \|u_0\|_{H^1} \sim \|u_0\|_{H^1} , \] (3.13)
\[ \|u_{NL}\|_{L^\gamma(I, H^1)} \sim \|u_{NL}\|_{L^\gamma(I, H^{1'})} \lesssim \|x|^{-b}|u|^\sigma u\|_{L^\gamma(I, H^{1'})} , \] (3.14)
\[ \|Tu - Tv\|_{L^\gamma(I, L')} \lesssim \|x|^{-b}|u|^\sigma u - |x|^{-b}|v|^\sigma v\|_{L^\gamma(I, L')} . \] (3.15)

In view of (3.13), we can see that \( \|u_L\|_{L^\gamma(I, H^1)} \rightarrow 0 \) as \( T \rightarrow 0 \). Take \( M > 0 \) such that \( CM^\sigma \leq \frac{1}{4} \) and \( T > 0 \) such that
\[ \|u_L\|_{L^\gamma([-T, T], H^{1'})} \leq \frac{M}{2} . \] (3.16)

Using (3.10), (3.14), (3.16), and the fact \( \hat{H}^{1,\frac{\sigma}{2}} \sim \hat{H}^{1',\sigma} \), we have
\[ \|Tu\|_{L^\gamma(I, H^1)} \leq \frac{M}{2} + C \|u\|_{L^\gamma(I, H^{1'})} \leq M . \] (3.17)

In view of (3.12) and (3.17), we have
\[ \|Tu - Tv\|_{L^\gamma(I, L')} \leq 2CM^\sigma \|u - v\|_{L^\gamma(I, L')} \leq \frac{1}{2} \|u - v\|_{L^\gamma(I, L')} . \] (3.18)
and \((3.18)\) imply that \(T : (D, d) \to (D, d)\) is a contraction mapping. From Banach fixed point theorem, there exists a unique solution \(u\) of \((1.1)\) in \((D, d)\). Furthermore for any admissible pair \((\gamma (p), p)\), it follows from Lemma 2.4 (Strichartz estimates) and \((3.10)\) that
\[
\|u\|_{L^{\gamma (p)}(I, H^{1, r})} \lesssim \|u_0\|_{H^1} + \|u\|_{L^{\gamma (r)}(I, H^{1, r})}^{\sigma + 1},
\]
which implies \(u \in L^{\gamma (p)}(I, H^{1, p})\). This completes the proof of the local well-posedness.

Next we prove the global well-posedness with small initial data. We define the following complete metric space
\[
E = \left\{ u \in L^{\gamma (r)}(\mathbb{R}, H^{1, r}) : \|u\|_{L^{\gamma (r)}(\mathbb{R}, H^{1, r})} \leq m, \|u\|_{L^{\gamma (r)}(\mathbb{R}, H^{1, r})} \leq M \right\},
\]
which is equipped with the metric
\[
d(u, v) = \|u - v\|_{L^{\gamma (r)}(\mathbb{R}, L^r)}.
\]

Using Lemma 2.4 (Strichartz estimates) and \((3.18)\), it follows from the facts \(\dot{H}^{1, r} \sim \dot{H}^{1, r}\) and \(\dot{H}^{1, r} \sim \dot{H}^{1, r}\) that
\[
\|Tu\|_{L^{\gamma (r)}(\mathbb{R}, H^{1, r})} \leq C \|u_0\|_{H^1} + C \|u\|_{L^{\gamma (r)}(\mathbb{R}, H^{1, r})}^{\sigma + 1}.
\]

Similarly, using Lemma 2.4 (Strichartz estimates), \((3.10)\) and \((3.12)\), we also have
\[
\|Tu - T v\|_{L^{\gamma (r)}(\mathbb{R}, L^r)} \leq C \left( \|u\|_{L^{\gamma (r)}(\mathbb{R}, H^{1, r})} + \|v\|_{L^{\gamma (r)}(\mathbb{R}, H^{1, r})} \right) \|u - v\|_{L^{\gamma (r)}(\mathbb{R}, L^r)}.
\]

Put \(m = 2C \|u_0\|_{H^1}, M = 2C \|u_0\|_{H^1}\) and \(\delta = 2(4C)^{-\frac{\sigma + 1}{\sigma}}\). If \(\|u_0\|_{H^1} \leq \delta\), i.e. \(C m^\sigma < \frac{1}{4}\), then it follows from \((3.19)\)–\((3.21)\) that
\[
\|Tu\|_{L^{\gamma (r)}(\mathbb{R}, H^{1, r})} \leq m,
\]
\[
\|Tu - T v\|_{L^{\gamma (r)}(\mathbb{R}, L^r)} \leq \frac{1}{2} \|u - v\|_{L^{\gamma (r)}(\mathbb{R}, L^r)}.
\]

So \(T : (E, d) \to (E, d)\) is a contraction mapping and there exists a unique solution \(u\) in \(E\). The scattering result with small initial data can be proved using the standard argument and we omit the details. This concludes the proof.

4 Blowup

In this section, we prove Theorem 1.4. To arrive at this goal, we derive the sharp Hardy-Sobolev inequality as well as the standard virial identity and localized virial estimate related to the focusing, energy-critical INLS\(_c\) equation.

4.1 Sharp Hardy-Sobolev inequality

In this subsection, we consider the sharp Hardy-Sobolev inequality related to the focusing, energy-critical INLS\(_c\) equation:
\[
\left( \int_{\mathbb{R}^d} |x|^{-b} |f|^\gamma \, dx \right)^{\frac{2}{\gamma + 2}} \lesssim CHS(c) \|f\|_{H^1},
\]
where the sharp constant $C_{HS}(b, c)$ is defined by
\[
C_{HS}(b, c) = \inf_{f \in H^1 \setminus \{0\}} \frac{\|f\|_{H^1}^2}{\left(\int |x|^{-b} |f|^{\sigma^* + 2} \, dx\right)^{\frac{\sigma^* + 2}{\sigma^*}}}. \tag{4.2}
\]

**Lemma 4.1** (Sharp Hardy-Sobolev inequality). Let $d \geq 3$, $0 < b < 2$ and $c > -c(d)$.

1. If $-c(d) < c \leq 0$, then the equality in (4.1) is attained by function $W_{b,c}(x)$ given in (1.10).

2. If $c > 0$, then $C_{HS}(b, c) \leq C_{HS}(b, 0)$.

**Proof.** The proof of Item 1 can be found in [16]. Using the fact $c > 0$, we immediately have that $\|f\|_{H^1} < \|f\|_{H^1}^2$ for any $f \in H^1 \setminus \{0\}$. Hence it follows from Item 1 that
\[
\left(\int |x|^{-b} |f|^{\sigma^* + 2} \, dx\right)^{\frac{\sigma^* + 2}{\sigma^*}} \leq C_{HS}(b, 0) \|f\|_{H^1} < C_{HS}(b, 0) \|f\|_{H^1}^2,
\]
which implies that $C_{HS}(b, c) \leq C_{HS}(b, 0)$. This completes the proof. 

**Remark 4.2.** When $b = 0$ and $c > 0$, it is known that $C_{HS}(b, c) = C_{HS}(b, 0)$ and the equality in (4.1) is never attained. In fact, $C_{HS}(0, c) \geq C_{HS}(0, 0)$ can be proved by considering $f_n(x) = W_{0,c}(x - x_n)$ for any sequence $x_n \to \infty$. See [10, 18] for details. But when $b > 0$, we could not apply this argument and we don’t know whether $C_{HS}(b, c) = C_{HS}(b, 0)$.

Next, we recall some properties related to $W_{b,c}$. Lemma 2.2 of [16] also shows that $W_{b,c}$ with $-c(d) < c \leq 0$ solves the equation
\[ P_x W_{b,c} = |x|^{-b} |W_{b,c}|^{\sigma^*} W_{b,c}, \]
and satisfies
\[ \|W_{b,c}\|_{H^1}^2 = \int |x|^{-b} W_{b,c}^{\sigma^* + 2} \, dx. \tag{4.3} \]
Hence, we have for $-c(d) < c \leq 0$,
\[ \|W_{b,c}\|_{H^1}^2 = \int |x|^{-b} W_{b,c}^{\sigma^* + 2} \, dx = C_{HS}(b, c) - \frac{2(d-b)}{2(d-b)}; \quad \|W_{b,c}\|_{H^1}^{\sigma^*} = C_{HS}(b, c)^{-(\sigma^* + 2)}, \tag{4.4} \]
\[ E_{b,c}(W_{b,c}) = \frac{1}{2} \|W_{b,c}\|_{H^1}^2 - \frac{1}{\sigma^* + 2} \int |x|^{-b} W_{b,c}^{\sigma^* + 2} \, dx = \frac{2 - b}{2(d-b)} C_{HS}(b, c)^{-\frac{2(d-b)}{2(d-b)}}. \tag{4.5} \]
Moreover, for any $c > -c(d)$, we have
\[ C_{HS}(b, c) \leq C_{HS}(b, \bar{c}) = \|W_{b,\bar{c}}\|_{H^1}^{-\frac{2-b}{2(d-b)}} = \|x|^{-\frac{2-b}{2(d-b)} W_{b,\bar{c}}^{\sigma^* + 2} \|_{L^1}^{-\frac{2-b}{2(d-b)}} = \left[\frac{2(d-b)}{2-b} E_{b,\bar{c}}(W_{b,\bar{c}})\right]^{-\frac{2-b}{2(d-b)}}. \tag{4.6} \]

**4.2 Virial estimates**

In this subsection, we derive the standard virial identity and localized virial estimate related to the focusing INLS equation. Given a real valued function $a$, we define the virial potential by
\[ V_a(t) := \int a(x) |u(t, x)|^2 \, dx. \]
A simple computation shows that the following result holds.
Lemma 4.3 (III). Let \( d \geq 3 \) and \( c > -c(d) \). If \( u : I \times \mathbb{R}^d \to \mathbb{C} \) is a smooth-in-time and Schwartz-in-space solution to \( i u_t - \dot{P} u = N(u) \), with \( N(u) \) satisfying \( \text{Im}(N(u) \bar{u}) = 0 \), then we have for any \( t \in I, \)

\[
\frac{d}{dt} V_a(t) = 2 \int \nabla a(x) \cdot \text{Im}(\bar{u}(t, x) \nabla u(t, x)) dx,
\]

and

\[
\frac{d^2}{dt^2} V_a(t) = - \int \Delta^2 a(x)|u(t, x)|^2 dx + 4 \sum_{j,k=1}^d \int \partial_{j,k}^2 a(x) \text{Re}(\partial_k u(t, x) \partial_j \bar{u}(t, x)) dx
\]

\[
+ 4c \int \nabla a(x) \cdot \frac{x}{|x|^2} |u(t, x)|^2 dx + 2 \int \nabla a(x) \cdot \{N(u), u\}_p(t, x) dx,
\]

where \( \{f, g\}_p := \text{Re}(f \nabla \bar{g} - g \nabla \bar{f}) \) is the momentum bracket.

Note that if \( N(u) = -|x|^{-b}|u|^\sigma u \), then
\[
\{N(u), u\}_p = \frac{\sigma}{\sigma + 2} \nabla(|x|^{-b}|u|^{\sigma + 2}) + \frac{2}{\sigma + 2} \nabla(|x|^{-b})|u|^{\sigma + 2}.
\]

Hence, we immediately have the following result.

Corollary 4.4. If \( u \) is a smooth-in-time and Schwartz-in-space solution to the focusing INLS\(_c\) equation, then we have for any \( t \in I, \)

\[
\frac{d^2}{dt^2} V_a(t) = - \int \Delta^2 a(x)|u(t, x)|^2 dx + 4 \sum_{j,k=1}^d \int \partial_{j,k}^2 a(x) \text{Re}(\partial_k u(t, x) \partial_j \bar{u}(t, x)) dx
\]

\[
+ 4c \int \nabla a(x) \cdot \frac{x}{|x|^2} |u(t, x)|^2 dx - \frac{2\sigma}{\sigma + 2} \int \Delta a(x)|x|^{-b}|u(t, x)|^{\sigma + 2} dx
\]

\[
+ \frac{4}{\sigma + 2} \int \nabla a(x) \cdot \nabla(|x|^{-b})|u(t, x)|^{\sigma + 2} dx.
\]

We have the following standard virial identity for the focusing INLS\(_c\) equation.

Lemma 4.5 (Standard Virial Identity). Let \( d \geq 3, \) \( 0 < b < 2 \) and \( c > -c(d) \). Let \( u_0 \in H^1 \) be such that \( |x|u_0 \in L^2 \) and \( u : I \times \mathbb{R}^d \to \mathbb{C} \) be the corresponding solution to the focusing INLS\(_c\) equation. Then, \( |x|u \in C(\overline{I, L^2}) \). Moreover, for any \( t \in I, \)

\[
\frac{d^2}{dt^2} \|xu(t)\|_{L^2}^2 = 8 \|u(t)\|_{H^1}^2 - \frac{4(d\sigma + 2b)}{\sigma + 2} \int |x|^{-b}|u(t, x)|^{\sigma + 2} dx. \tag{4.7}
\]

Proof. The first claim follows from the standard approximation argument and we omit the details (see e.g. Proposition 6.5.1 of \[8\] for details). It remains to prove (4.7). Applying Corollary 4.4 with \( a(x) = |x|^2, \)

we have

\[
\frac{d^2}{dt^2} \|xu(t)\|_{L^2}^2 = \frac{d^2}{dt^2} V_{|x|^2}(t)
\]

\[
= 8 \int |\nabla u(t, x)|^2 + c|x|^{-2}|u(t, x)|^2 dx - \frac{4(d\sigma + 2b)}{\sigma + 2} \int |x|^{-b}|u(t, x)|^{\sigma + 2} dx
\]

\[
= 8 \|u(t)\|_{H^1}^2 - \frac{4(d\sigma + 2b)}{\sigma + 2} \int |x|^{-b}|u(t, x)|^{\sigma + 2} dx,
\]

this completes the proof. \(\square\)
Next we derive the localized virial estimate which is used to prove the blowup for the focusing INLS<sub>c</sub> equation with radial data. To do so, we introduce a function \( \theta : [0, \infty) \to [0, \infty) \) satisfying

\[
\theta(r) = \begin{cases} 
  r^2, & \text{if } 0 \leq r \leq 1, \\
  \text{const}, & \text{if } r \geq 2, \\
  \text{and } \theta''(r) \leq 2, & \text{for } r \geq 0.
\end{cases}
\]  

For \( R > 1 \), we define the radial function

\[
\varphi_R(x) = \varphi_R(r) := R^2 \theta(r/R), \quad r = |x|.
\]

One can easily see that

\[
2 - \varphi''_R(r) \geq 0, \quad 2 - \frac{\varphi'_R(r)}{r} \geq 0, \quad 2d - \Delta \varphi_R(x) \geq 0.
\]  

**Lemma 4.6 (Localized Virial Estimate).** Let \( d \geq 3, 0 < b < 2, c > -c(d), R > 1 \) and \( \varphi_R \) be as in \((4.9)\). Let \( u : I \times \mathbb{R}^d \to \mathbb{C} \) be a radial solution to the focusing INLS<sub>c</sub> equation. Then for any \( \varepsilon > 0 \) and any \( t \in I \),

\[
\frac{d^2}{dt^2} V_{\varphi_R}(t) \leq 8 \left\| u(t) \right\|_{H^1_i}^2 - \frac{4(d \sigma + 2b)}{\sigma + 2} \int |x|^{-b} |u(t, x)|^{\sigma+2} dx \\
+ O \left( R^{-2} + \varepsilon \frac{\sigma}{4-\sigma} R^{2(d-1)/2} + \varepsilon \left\| u(t) \right\|_{H^1_i}^2 \right).
\]  

**Proof.** We use the argument similar to that used to prove the localized virial estimates for INLS equation and NLS<sub>c</sub> equation (see [9, 10]). Applying Corollary 4.3 with \( a(x) = \varphi_R(x) \), we have

\[
\frac{d^2}{dt^2} V_{\varphi_R}(t) = -\int \Delta^2 \varphi_R(x) |u(t, x)|^2 dx + 4 \sum_{j,k=1}^d \int \partial_{jk}^2 \varphi_R(x) \text{Re}(\partial_k u(t, x) \partial_j \bar{u}(t, x)) dx \\
+ 4c \int \nabla \varphi_R(x) \cdot \frac{x}{|x|^2} |u(t, x)|^2 dx - \frac{2\sigma}{\sigma + 2} \int \Delta \varphi_R(x) |x|^{-b} |u(t, x)|^{\sigma+2} dx \\
+ \frac{4}{\sigma + 2} \int \nabla \varphi_R(x) \cdot \nabla(|x|^{-b}) |u(t, x)|^{\sigma+2} dx.
\]

Since \( \varphi_R(x) = |x|^2 \) for \( |x| < R \), it follows from Lemma 4.5 that

\[
\frac{d^2}{dt^2} V_{\varphi_R}(t) = 8 \left\| u(t) \right\|_{H^1_i}^2 - \frac{4(d \sigma + 2b)}{\sigma + 2} \int |x|^{-b} |u(t, x)|^{\sigma+2} dx - 8 \left\| u(t) \right\|_{H^1_i(|x|>R)}^2 \\
+ \frac{4(d \sigma + 2b)}{\sigma + 2} \int_{|x|>R} |x|^{-b} |u(t, x)|^{\sigma+2} dx - \int_{|x|>R} \Delta^2 \varphi_R(x) |u(t, x)|^2 dx \\
+ 4 \sum_{j,k=1}^d \int_{|x|>R} \partial_{jk}^2 \varphi_R(x) \text{Re}(\partial_k u(t, x) \partial_j \bar{u}(t, x)) dx \\
+ 4c \int_{|x|>R} \nabla \varphi_R(x) \cdot \frac{x}{|x|^4} |u(t, x)|^2 dx - \frac{2\sigma}{\sigma + 2} \int_{|x|>R} \Delta \varphi_R(x) |x|^{-b} |u(t, x)|^{\sigma+2} dx \\
+ \frac{4}{\sigma + 2} \int_{|x|>R} \nabla \varphi_R(x) \cdot \nabla(|x|^{-b}) |u(t, x)|^{\sigma+2} dx.
\]
Since $\Delta \varphi_R \lesssim 1$, $\Delta^2 \varphi_R \lesssim R^{-2}$ and $\nabla \varphi_R(x) \cdot \nabla(|x|^{-b}) \lesssim |x|^{-b}$, we have
\[
\frac{d^2}{dt^2} V_{\varphi_R}(t) = 8 \| u(t) \|_{H^1}^2 - \frac{4(ds + 2b)}{\sigma + 2} \int |x|^{-b} |u(t, x)|^{\sigma + 2} dx
\]
\[
- 8 \| u(t) \|_{H^1}^2 \| u \|_{H^1}^2 + 4 \sum_{j,k=1}^d \int_{|x| > R} \partial_{jk}^2 \varphi_R(\partial_k u \partial_j \bar{u}) dx + 4c \int_{|x| > R} \nabla \varphi_R(x) \cdot \frac{x}{|x|^2} |u(t, x)|^2 dx
\]
\[
+ 4 \left( \int_{|x| > R} R^{-2} |u(t)|^2 + |x|^{-b} |u(t)|^{\sigma + 2} dx \right).
\]
Using (4.10) and the fact
\[
\partial_{jk}^2 = \left( \frac{\delta_{jk}}{r} - \frac{x_j x_k}{r^3} \right) \partial_r + \frac{x_j x_k}{r^2} \partial_r^2,
\]
we can see that
\[
4 \sum_{j,k=1}^d \partial_{jk}^2 \varphi_R(\partial_k u \partial_j \bar{u}) dx \leq 2|\nabla u|^2, \nabla \varphi_R \cdot x \leq 2| x |^2.
\]
Hence, we have
\[
- 8 \| u(t) \|_{H^1}^2 \| u \|_{H^1}^2 + 4 \sum_{j,k=1}^d \int_{|x| > R} \partial_{jk}^2 \varphi_R(\partial_k u \partial_j \bar{u}) dx + 4c \int_{|x| > R} \nabla \varphi_R(x) \cdot \frac{x}{|x|^2} |u(t)|^2 dx
\]
\[
\leq 4c \int_{|x| > R} \left( \nabla \varphi_R(x) \cdot x - 2| x |^2 \right) \frac{|u|^2}{|x|^4} = -4c \int_{|x| > R} \left( 2 - \frac{\varphi''(r)}{r} \right) \frac{|u(t)|^2}{|x|^2} dx
\]
\[
\leq \max \{-4cS, 0\} \int_{|x| > R} R^{-2} |u(t)|^2 dx \leq \max \{-4cSM(u(t)), 0\} R^{-2},
\]
where $S = \max_{r > 1} 2 - \frac{\varphi''(r)}{r}$. The conservation of mass implies
\[
\frac{d^2}{dt^2} V_{\varphi_R}(t) \leq 8 \| u(t) \|_{H^1}^2 - \frac{4(ds + 2b)}{\sigma + 2} \int |x|^{-b} |u(t, x)|^{\sigma + 2} dx
\]
\[
+ O \left( R^{-2} + \int_{|x| > R} |x|^{-b} |u(t)|^{\sigma + 2} dx \right).
\]
Using the same argument as in the proof of Lemma 3.4 of [9], it follows from the fact $\dot{H}^1 \sim \dot{H}^1$, we have
\[
\int_{|x| > R} |x|^{-b} |u(t)|^{\sigma + 2} dx \lesssim R^{-\frac{(d-1)p+2b}{2}} \| u(t) \|_{H^1}^2,
\]
whose proof will be omitted. Next we use the Young inequality\footnote{Let $a$, $b$ be non-negative real numbers and $p$, $q$ be positive real numbers satisfying $\frac{1}{p} + \frac{1}{q} = 1$. Then for any $\varepsilon > 0$, we have $ab \leq \varepsilon a^p + \varepsilon^{-\frac{p}{p}} b^q$.} to get for any $\varepsilon > 0$,
\[
R^{-\frac{(d-1)p+2b}{2}} \| u(t) \|_{H^1}^2 \lesssim \varepsilon^{-\frac{p}{p}} R^{-\frac{2(d-1)p+2b}{2}} + \varepsilon \| u(t) \|_{H^1}^2,
\]
this completes the proof. \qed
4.3 Proof of Theorem 1.4

In this subsection, we prove Theorem 1.4. We divide the study in two cases: \( E_{b,c}(u_0) < 0 \) and \( E_{b,c}(u_0) \geq 0 \).

- **The case** \( E_{b,c}(u_0) < 0 \).

First, we consider the case \( xu_0 \in L^2 \). Applying the standard virial identity (4.10) and the conservation of energy, we have

\[
\frac{d^2}{dt^2} \| xu(t) \|_{L^2}^2 = 8 \| u(t) \|_{H^1}^2 - \frac{4(d \sigma^* + 2b)}{\sigma^* + 2} \int |x|^{-b} |u(t,x)|^{\sigma^* + 2} \, dx
\]

where we used the fact \( d \sigma^* + 2b > 0 \). By the classical argument of Glassey [12], it follows that the solution \( u \) blows up in finite time.

Next, we consider the case \( u_0 \) is radial. Using the localized virial estimate (4.11) and the conservation of energy, we have

\[
\frac{d^2}{dt^2} \| \varphi_R(t) \|_{L^2}^2 \leq 8 \| u(t) \|_{H^1}^2 - \frac{4(d \sigma^* + 2b)}{\sigma^* + 2} \int |x|^{-b} |u(t,x)|^{\sigma^* + 2} \, dx
\]

where \( \varphi_R \) is radial. Using the localized virial estimate (4.11) and the conservation of energy, we have

\[
\frac{d^2}{dt^2} \| \varphi_R(t) \|_{L^2}^2 \leq 8 \| u(t) \|_{H^1}^2 - \frac{4(d \sigma^* + 2b) E_{b,c}(u(t))}{\sigma^* + 2} - 2(d \sigma^* - 4 + 2b) \| u \|_{H^1}^2 < 0,
\]

for any \( t \) in the existence time and for any \( \varepsilon > 0 \). Since \( d \sigma^* - 4 + 2b > 0 \), we take \( \varepsilon > 0 \) small enough and \( R > 1 \) large enough depending on \( \varepsilon \) to have that

\[
\frac{d^2}{dt^2} \| \varphi_R(t) \|_{L^2}^2 \leq 2(d \sigma^* + 2b) E_{b,c}(u_0) < 0,
\]

for any \( t \) in the existence time. This shows that the solution \( u \) must blow up in finite time.

- **The case** \( E_{b,c}(u_0) \geq 0 \).

By the definition of the energy (1.6) and Lemma 4.1, we have

\[
E_{b,c}(u(t)) = \frac{1}{2} \| u(t) \|_{H^1}^2 - \frac{1}{\sigma^* + 2} \left\| |x|^{-b} |u|^{\sigma^* + 2} \right\|_{L^1}
\]

\[
\geq \frac{1}{2} \| u(t) \|_{H^1}^2 - \frac{C_{HS}(b, \bar{c}) \sigma^* + 2}{\sigma^* + 2} \| u(t) \|_{H^1}^{\sigma^* + 2} =: g \left( \| u(t) \|_{H^1} \right),
\]

where

\[
g(y) = \frac{1}{2} y^2 - \frac{C_{HS}(b, \bar{c}) \sigma^* + 2}{\sigma^* + 2} y^{\sigma^* + 2}.
\]  \hspace{1cm} (4.12)

It also follows from (4.6) that

\[
g(\| W_{b,c} \|_{H^1}) = E_{b,c}(W_{b,c}).
\]

By the conservation of energy and the assumption \( E_{b,c}(u_0) < E_{b,c}(W_{b,c}) \), we can see that

\[
g(\| u(t) \|_{H^1}) \leq E_{b,c}(u(t)) = E_{b,c}(u_0) < E_{b,c}(W_{b,c}).
\]
By the assumption \( \|u_0\|_{\dot{H}^1} > \|W_{b, \bar{c}}\|_{\dot{H}^1} \) and the continuity argument, we have

\[
\|u(t)\|_{\dot{H}^1} > \|W_{b, \bar{c}}\|_{\dot{H}^1},
\]  

for any \( t \) as long as the solution exists. (4.13) is improved as follows. Pick \( \delta > 0 \) small enough such that

\[
E_{b, \bar{c}}(u_0) \leq (1 - \delta)E_{b, \bar{c}}(W_{b, \bar{c}}),
\]  

which implies that

\[
g(\|u(t)\|_{\dot{H}^1}) \leq (1 - \delta)E_{b, \bar{c}}(W_{b, \bar{c}}).
\]  

Using (4.6), (4.12) and (4.15), we have

\[
\text{LHS}(4.17) = 4(\bar{d}\sigma^* + 2b)E_{b, \bar{c}}(u(t)) + (8 + \varepsilon - 2d\sigma^* - 4b)\|u(t)\|^2_{\dot{H}^1}
\]

\[
\leq 4(1 - \delta)(\bar{d}\sigma^* + 2b)E_{b, \bar{c}}(W_{b, \bar{c}}) + (8 + \varepsilon - 2d\sigma^* - 4b)(1 + \delta')^2\|W_{b, \bar{c}}\|^2_{\dot{H}^1}
\]

\[
= \|W_{b, \bar{c}}\|^2_{\dot{H}^1} \left[ \frac{8(2 - b)}{d - 2}(1 - \delta - (1 + \delta')^2) + \varepsilon(1 + \delta')^2 \right].
\]

Hence, by taking \( \varepsilon > 0 \) small enough, we can get (4.17).

First, we consider the case \( xu_0 \in L^2 \) satisfying \( E_{b, \bar{c}}(u_0) < E_{b, \bar{c}}(W_{b, \bar{c}}) \) and \( \|u_0\|_{\dot{H}^1} > \|W_{b, \bar{c}}\|_{\dot{H}^1} \). Using the standard virial identity (4.14) and (4.11), we have

\[
\frac{d^2}{dt^2} \|u(t)\|^2_{L^2} = 8\|u(t)\|^2_{\dot{H}^1} - \frac{4(\bar{d}\sigma^* + 2b)}{\sigma^* + 2}\| |x|^{-b}|u|^{\sigma^* + 2}\|_{L^1} \leq -c < 0,
\]

which implies that the solution blows up in finite time.

Next, we consider the case \( u_0 \) is radial, and satisfies \( E_{b, \bar{c}}(u_0) < E_{b, \bar{c}}(W_{b, \bar{c}}) \) and \( \|u_0\|_{\dot{H}^1} > \|W_{b, \bar{c}}\|_{\dot{H}^1} \). Using the localized virial estimates (4.11), we have

\[
\frac{d^2}{dt^2} V_{\varphi_R}(t) \leq 8\|u(t)\|^2_{\dot{H}^1} - \frac{4(\bar{d}\sigma^* + 2b)}{\sigma^* + 2}\| |x|^{-b}|u|^{\sigma^* + 2}\|_{L^1}
\]

\[
+ O \left( R^{-2} + \varepsilon^{-\frac{\sigma^*}{\sigma^* + 2b}} R^{-\frac{2((d-1)\sigma^* + 2b)}{\sigma^* + 2b}} + \varepsilon \|u(t)\|^2_{\dot{H}^1} \right),
\]

for any \( \varepsilon > 0 \) and any \( t \) in the existence time. Taking \( \varepsilon > 0 \) small enough and \( R > 1 \) large enough depending on \( \varepsilon \), it follows from (4.17) that

\[
\frac{d^2}{dt^2} V_{\varphi_R}(t) \leq -c/2 < 0,
\]

which implies that the solution must blow up in finite time. This completes the proof.
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