No hair for spherically symmetric neutral black holes: nonminimally coupled massive scalar fields

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It is proved that spherically symmetric asymptotically flat neutral black holes cannot support spatially regular static configurations made of massive scalar fields with non-minimal coupling to gravity. Interestingly, our compact no-hair theorem is valid for generic values of the dimensionless physical parameter $\xi$ which quantifies the strength of coupling between the scalar field and the spacetime curvature.

I. INTRODUCTION

The interplay between classical black-hole spacetimes and scalar matter fields has attracted much attention from physicists and mathematicians during the last five decades. Interestingly, early investigations of the non-linearly coupled Einstein-scalar field equations made by Bekenstein [1] and Teitelboim [2] (see also [3–8] and references therein) have revealed that spatially regular static matter configurations made of massive scalar fields with minimal coupling to gravity cannot be supported in asymptotically flat regular [9] black-hole spacetimes [10]. It is worth noting that, as opposed to the static scalar fields considered in the physically important early no-hair theorems of [1] and [2], stationary massive scalar (and, in general, bosonic) fields can be supported in asymptotically flat rotating black-hole spacetimes [11–13].

Interestingly, the rigorous derivation of an analogous no-hair theorem for the more generic case of black holes interacting with spatially regular scalar fields which are characterized by non-minimal coupling to the spacetime curvature turns out to be a mathematically challenging task. In particular, the physically influential and mathematically elegant theorems presented by Bekenstein and Mayo [7, 8] imply that spherically symmetric asymptotically flat black holes cannot support nonminimally coupled spatially regular neutral scalar fields in the restricted regimes $\xi < 0$ and $\xi \geq 1/2$ of the dimensionless field-curvature non-minimal coupling parameter $\xi$ [10, 14–20]. (It is worth emphasizing that Bekenstein and Mayo [7] have also presented an extensive study of the no-hair property for non-minimally coupled, complex, potentially charged scalar fields in the composed Einstein-Maxwell theory).

The no-hair theorem presented in [10], which is valid for generic inner boundary conditions, can be used to exclude the existence of spherically symmetric asymptotically flat static hairy black-hole configurations supporting massive scalar fields with nonminimal coupling to gravity in the dimensionless regimes $\xi < 0$ and $\xi > 1/4$.

To the best of our knowledge, no mathematically rigorous no-hair theorem has thus far been presented in the physical literature which rules out the possible existence of static hairy black-hole configurations with regular event horizons which are made of nonminimally coupled massive neutral scalar fields with a field-curvature coupling parameter in the dimensionless physical regime $0 < \xi \leq 1/4$ [8, 10].

The main goal of the present paper is to present a compact no-hair theorem which explicitly proves that spherically symmetric asymptotically flat neutral black holes with regular event horizons cannot support static matter configurations made of massive scalar fields with non-minimal coupling to gravity. Interestingly, our novel no-hair theorem, to be presented below, is valid for generic values of the physical parameter $\xi$ which characterizes the non-minimal field-curvature coupling in the composed black-hole-massive-scalar-field system (In particular, in the present paper we shall extend the important no-hair theorems of [7, 10, 16, 20] to the regime of nonminimally coupled massive neutral scalar fields with $0 < \xi \leq 1/4$).

II. DESCRIPTION OF THE SYSTEM

We consider a spherically symmetric static matter distribution described by a massive scalar field $\psi$ with nonminimal coupling to gravity which is non-linearly coupled to a neutral black hole of horizon radius $r_H$. The spherically symmetric static curved spacetime of the composed black-hole-massive-scalar-field system is characterized by the radially-dependent line element [7] (we shall use natural units in which $\hbar = c = G = 1$)

$$ds^2 = -e^{\nu}dt^2 + e^{\lambda}dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2),$$ (1)
where \( t, r, \theta, \phi \) are the Schwarzschild spacetime coordinates and \( \{ \nu = \nu(r), \lambda = \lambda(r) \} \). As proved in [7], non-extremal black-hole spacetimes with regular event horizons [9] are characterized by the simple near-horizon functional relations

\[
e^{-\lambda} = L \cdot x + O(x^2) \quad \text{where} \quad x \equiv \frac{r - r_H}{r_H} \quad ; \quad L > 0
\]  

(2)

[as explicitly shown in [7], the expansion coefficient in (2) is given by \( L \equiv 1 + 8\pi T^t_t (r_H) r^2_H > 0 \), where \(-T^t_t (r_H)\) is the energy density which characterizes the matter fields at the horizon \( r = r_H \) of the black-hole spacetime], and

\[
\lambda' r_H = -\frac{1}{x} + O(1) \quad ; \quad \nu' r_H = \frac{1}{x} + O(1) ,
\]

(3)

where a prime ' denotes a spatial derivative with respect to the radial coordinate \( r \). In addition, asymptotically flat spacetimes are characterized by the large-\( r \) functional relations [7]

\[
\nu \sim M/r \quad \text{and} \quad \lambda \sim M/r \quad \text{for} \quad r \to \infty ,
\]

(4)

where \( M \) is the total ADM mass (as measured by asymptotic observers) of the hairy black-hole spacetime.

The non-minimally coupled scalar field \( \psi \) of mass \( \mu \) [note that the mass parameter \( \mu \) of the scalar field stands for \( \mu/\hbar \) and therefore has the dimensions of (length)^{-1}] is characterized by the action [7]

\[
S = S_{EH} - \frac{1}{2} \int (\partial_\alpha \psi \partial^\alpha \psi + \mu^2 \psi^2 + \xi R \psi^2) \sqrt{-g} d^4 x ,
\]

(5)

where \( S_{EH} \) denotes the Einstein-Hilbert action of the curved spacetime and the dimensionless physical parameter \( \xi \) quantifies the strength of the nonminimal coupling between the scalar field \( \psi \) and the scalar curvature \( R(r) \) [see Eq. (23) below] of the black-hole spacetime. It is worth noting that the no-hair theorem for scalar-tensor theories presented in [18], which is based on a simple transformation of the action from the Jordan frame to the Einstein frame, is not applicable to the action (5) considered in the present paper. In particular, the scalar-curvature coupling considered in [18] is proportional to \( \psi R \), whereas the action (5) for the massive scalar field is characterized by a nonminimal scalar-curvature coupling of the functional form \( \psi^2 R \).

Note that an asymptotically flat spacetime is characterized by the simple functional relation [7]

\[
R(r \to \infty) \to 0 .
\]

(6)

In addition, as discussed in [7], physically acceptable spacetimes are characterized by finite and positive values of the effective asymptotic \((r/M \gg 1)\) gravitational constant \( G_{eff} = G[1 - 8\pi G \xi \psi^2 (r/M \gg 1)] \) [7]. This physical requirement implies that the radial eigenfunction of the nonminimally coupled massive scalar fields is bounded asymptotically by the simple relations

\[
-\infty < 8\pi \xi \psi^2 < 1 \quad \text{for} \quad r/M \gg 1 .
\]

(7)

The action (5) of the nonminimally coupled massive scalar field yields the differential equation [7]

\[
\partial_\alpha \partial^\alpha \psi - (\mu^2 + \xi R) \psi = 0 ,
\]

(8)

which, in the spherically symmetric static black-hole spacetime [11] and for a real scalar field \( \psi = \psi(r) \) whose spatial behavior depends only on the radial coordinate \( r \), can be written in the form

\[
\psi'' + \frac{1}{2}(\frac{4}{r} + \nu' - \lambda') \psi' - (\mu^2 + \xi R) e^\lambda \psi = 0 .
\]

(9)

[It is worth emphasizing that the radial differential equation (9) is valid for a real and static scalar field which inherits the symmetries of the spherically symmetric static black-hole spacetime (11)].

The characteristic action [7] of the nonminimally coupled massive scalar field also yields the following functional expressions [7]

\[
T'^t_t = e^{-\lambda} [\xi (4/r - \lambda') \psi \psi' + (2\xi - 1/2) (\psi')^2 + 2\xi \psi \psi'' - \frac{\mu^2 \psi^2}{2(1 - 8\pi \xi \psi^2)}] ,
\]

(10)

\[
T'^r_t - T'^t_r = e^{-\lambda} [2(\xi - 1) (\psi')^2 + 2\xi \psi \psi'' - \xi (\nu + \lambda') \psi \psi'] ,
\]

(11)
\[ T_t^t - T_\phi^\phi = e^{-\lambda(2/r - \nu')\psi}\psi' / (1 - 8\pi\psi^2) \]  

(12)

for the components of the energy-momentum tensor \( [It \text{ is worth noting that, except of the Einstein relation } R = -8\pi T, \text{ our no-hair theorem, to be presented below, does not rely on the explicit forms of the Einstein field equations } G_{\mu\nu} = 8\pi T_{\mu\nu}. \text{ Nevertheless, for the interested readers, we note that the corresponding functional expressions of the Einstein tensor components } G_{\mu\nu} \text{ can be found, for example, in Eqs. (2.4), (2.5) and (2.6) of [21]. As discussed in [7], causality requirements imply that, for physically acceptable systems, the components of the energy-momentum tensor are characterized by the simple inequalities} \]

\[ |T_\theta^\theta| = |T_\phi^\phi| \leq |T_t^t| \geq |T_r^r| . \]  

(13)

In particular, below we shall use the following functional relations \([7]\)

\[ \text{sgn}(T_t^t) = \text{sgn}(T_t^t - T_r^r) = \text{sgn}(T_t^t - T_\phi^\phi) , \]  

(14)

which provide necessary conditions for the validity of the energy conditions \([13]\) derived in \([7]\) for physically acceptable systems. Finally, following \([7]\) we stress the important fact that physically acceptable spacetimes are also characterized by energy-momentum tensors with finite mixed components \([7]\):

\[ \{|T_t^t|, |T_r^r|, |T_\theta^\theta|, |T_\phi^\phi|\} < \infty . \]  

(15)

**III. REVIEW OF FORMER ANALYTICAL RESULTS: NO NONMINIMALLY COUPLED MASSIVE SCALAR HAIR IN THE REGIMES } \xi < 0 \text{ AND } \xi > 1/4 \)\]

Before we proceed, it is important to mention that the no-hair theorem presented in \([16]\) rules out the existence of asymptotically flat static hairy black-hole configurations with regular event horizons supporting massive scalar fields with nonminimal coupling to gravity in the dimensionless regimes \( \xi < 0 \) and \( \xi > 1/4. \) (It is worth stressing the fact that the no-scalar-hair theorem presented in \([16]\) is valid for generic inner boundary conditions. In particular, this theorem is valid for both asymptotically flat black-hole spacetimes with absorbing horizons and for asymptotically flat stars with compact reflecting boundaries \([16]\)). For completeness of the presentation, in the present section we shall give a brief sketch of the no-hair theorem presented in \([16]\).

Taking cognizance of the characteristic asymptotic functional relations \([4]\) and \([9]\) of the static black-hole spacetime \([11]\), one finds from \([16]\) the mathematically compact radial scalar equation \([16]\)

\[ \psi'' + 2r \psi' - \mu^2 \psi = 0 \quad \text{for} \quad r/M \gg 1 , \]  

(16)

which determines the spatial behavior of the massive scalar fields in the asymptotic \( r \gg M \) region. The physically acceptable solution of the radial differential equation \([16]\) which respects the characteristic asymptotic relation \([7]\) is given by

\[ \psi(r) = A \cdot e^{-\mu r} \mu r \quad \text{for} \quad r/M \gg 1 , \]  

(17)

where \( A \) is a dimensionless normalization constant.

Taking cognizance of Eqs. \([4]\), \([10]\), \([12]\), and \([17]\), one obtains the asymptotic functional expressions \([16]\)

\[ T_t^t = (4\xi - 1)\mu^2 \psi^2 \cdot [1 + O(M/r)] \]  

(18)

and

\[ T_t^t - T_\phi^\phi = -\xi \frac{2\mu}{r} \psi^2 \cdot [1 + O(M/r)] \]  

(19)

for the energy-momentum components of the spatially regular nonminimally coupled massive scalar fields. In particular, the expressions \([13]\) and \([19]\) yield the mathematically simple and physically important asymptotic relation

\[ \text{sgn}(T_t^t) = -\text{sgn}(T_t^t - T_\phi^\phi) \quad \text{for} \quad \xi < 0 \text{ or } \xi > 1/4 . \]  

(20)
Interestingly, the relation (20) between the components of the energy-momentum tensor which characterizes the nonminimally coupled massive scalar field, violates the fundamental relation (14) which, as explicitly proved in [7], characterizes physically acceptable Einstein-matter systems. One therefore realizes that spatially regular static matter configurations made of massive scalar fields nonminimally coupled to gravity in the regimes \( \xi \leq 0 \) [note that the early no-hair theorems of [1] and [2] explicitly rule out the existence of external matter configurations (‘hair’) which are made of spatially regular minimally coupled \((\xi = 0)\) static massive scalar fields] and \( \xi > 1/4 \) cannot be supported in spherically symmetric asymptotically flat neutral black-hole spacetimes [10].

IV. THE NO-HAIR THEOREM FOR THE NONMINIMALLY COUPLED MASSIVE SCALAR FIELDS IN THE REGIME \( \xi > 0 \)

In the present section we shall complete our no-hair theorem for the composed black-hole-scalar-field system by explicitly proving that static matter configurations made of spatially regular nonminimally coupled massive scalar fields in the dimensionless physical regime \( \xi > 0 \) cannot be supported in spherically symmetric asymptotically flat neutral black-hole spacetimes.

We shall first prove that the characteristic eigenfunction \( \psi(r) \) of the nonminimally coupled self-gravitating massive scalar fields is a non-monotonic function of the radial coordinate \( r \). Taking cognizance of the characteristic asymptotic behavior [see Eq. (17)]

\[
\psi(r/M \to \infty) \to 0
\]  

(21)
of the spatially regular massive scalar fields, one deduces that if the radial scalar eigenfunction is characterized by the near-horizon behavior \( \psi(x \to 0) \to 0 \) then it must possess, in accord with the above mentioned claim, an extremum point in the exterior black-hole spacetime. We shall now prove that scalar eigenfunctions with the near-horizon behavior

\[
\psi(x \to 0) \neq 0
\]  

(22)
are also characterized by a non-monotonic radial profile.

Taking cognizance of Eqs. (13), (11), and (12), and using the Einstein relation \( R = -8\pi T \) [here the radially-dependent function \( T(r) \) denotes the trace of the energy-momentum tensor which characterizes the matter fields], one finds the functional expression

\[
R = -\frac{8\pi}{1 - 8\pi\xi}\psi^2 \left\{ e^{-\lambda} \left[ \frac{12}{r} + 3\nu' - 3\lambda' \right] \psi \psi' + 6\xi \psi \psi'' + (6\xi - 1)(\psi')^2 \right\} - 2\mu^2 \psi^2
\]  

(23)
for the Ricci scalar curvature which characterizes the spherically symmetric composed black-hole-massive-scalar-field system. Substituting (23) into (9), one obtains the (rather cumbersome) radial differential equation

\[
\psi'' \cdot \left[ 1 + 8\pi\xi(6\xi - 1)\psi^2 \right] + \psi' \cdot \left[ \frac{12}{r} + \nu' - \lambda' \right] \left[ 1 + 8\pi\xi(6\xi - 1)\psi^2 \right] + 8\pi\xi(6\xi - 1)\psi \psi' - \mu^2 e^\lambda \left( 1 + 8\pi\xi\psi^2 \right) \psi = 0
\]  

(24)
which determines the spatial behavior of the nonminimally coupled massive scalar fields in the static black-hole spacetime [11].

We note that, in the dimensionless regimes \( \xi \geq 1/6 \) and \( \xi \leq 0 \) of the coupling parameter \( \xi \), the functional expression

\[
\mathcal{F}(r; \xi) \equiv 1 + 8\pi\xi(6\xi - 1)\psi^2
\]  

(25)
that appears on the l.h.s of (24) is a positive definite function. We shall now prove that \( \mathcal{F}(r) \) is a positive definite function in the exterior black-hole spacetime also in the dimensionless physical regime \( 0 < \xi < 1/6 \). To this end, we first point out that the characteristic asymptotic behavior (21) of the massive scalar fields yields the simple relation [see Eq. (25)]

\[
\mathcal{F}(r/M \to \infty) \to 1
\]  

(26)
Let us assume that the function \( \mathcal{F}(r) \) switches signs at some radial point \( r_0 \in (r_H, \infty) \). Then, the characteristic differential equation (24) yields the simple relation

\[
8\pi\xi(6\xi - 1)(\psi')^2 = \mu^2 e^\lambda (1 + 8\pi\xi\psi^2) \quad \text{at} \quad r = r_0
\]  

(27)
for the radial eigenfunction of the nonminimally coupled massive scalar fields at the assumed root of $F(r)$. Noting that the functional expression on the l.h.s of (27) is non-positive in the physical regime $0 < \xi < 1/6$ whereas the functional expression on the r.h.s of (27) is positive definite, one deduces that, in the exterior black-hole spacetime, the function $F(r)$ cannot switch signs. In particular, from the asymptotic behavior (26), one obtains the simple relation

$$F(r) > 0 \quad \text{for} \quad r \in (r_H, \infty) \, .$$

We further note that one deduces from Eqs. (2), (3), (12), and (15) that the radial eigenfunction $\psi(x)$ of the nonminimally coupled massive scalar fields is characterized by the bounded near-horizon functional behavior [23]

$$|\psi\psi'(x \to 0)| < \infty \, .$$

Taking cognizance of Eqs. (2), (3), (28), and (29), one obtains from (24) the near-horizon ($x \ll 1$) radial equation

$$\psi'' \left[ 1 + 8\pi\xi(6\xi - 1)\psi^2 \right] + \psi' \left( \frac{1}{x r_H} \left[ 1 + 8\pi\xi(6\xi - 1)\psi^2 \right] - \frac{\mu^2}{L x} \left( 1 + 8\pi\xi\psi^2 \right) \right) = 0 \quad (30)$$

for the nonminimally coupled massive scalar fields. The physically acceptable solution of the radial scalar equation (30) which respects the relation (28) is characterized by the small-$x$ (near-horizon) functional behavior [23, 25]

$$\psi(x \to 0) = a \left[ 1 + \frac{\mu^2 x^2}{L} \frac{1 + 8\pi\xi a^2}{1 + 8\pi\xi(6\xi - 1)a^2} \right] \cdot x + O(x^2) \, ,$$

where $a$ is a constant. From Eqs. (2), (28), and (31) one deduces that, in the $\xi \geq 0$ regime, the radial scalar eigenfunction of the nonminimally coupled massive scalar fields is characterized by the near-horizon ($x \ll 1$) functional behavior [23, 25]

$$\psi\psi'(x \to 0) > 0 \, .$$

Taking cognizance of the analytically derived small-$x$ and large-$x$ spatial behaviors [21] and [32], which characterize the radial eigenfunction $\psi(x)$ of the static nonminimally coupled massive scalar field configurations, one deduces that the function $\psi(r)$ has a non-monotonic dependence on the radial coordinate $r$. In particular, the spatially regular scalar eigenfunction $\psi(r)$ must have (at least) one extremum point $r_{\text{peak}} \in (r_H, \infty)$ in the exterior black-hole spacetime which is characterized by the simple functional relations

$$\{ \psi \neq 0 \ ; \ \psi' = 0 \ ; \ \psi \cdot \psi'' < 0 \} \quad \text{for} \quad r = r_{\text{peak}} \, .$$

From Eqs. (24) and (26) one obtains the functional relation

$$F \cdot \psi'' = \mu^2 e^\lambda (1 + 8\pi\xi\psi^2)\psi^2 \quad \text{at} \quad r = r_{\text{peak}}$$

at the extremum point $r = r_{\text{peak}}$ [see Eq. (33)] which characterizes the non-monotonic radial eigenfunction of the nonminimally coupled massive scalar fields. From Eqs. (28) and (33) one deduces that the radial expression on the l.h.s of (34) is negative definite whereas the radial expression on the r.h.s of (34) is positive definite in the dimensionless physical regime $\xi \geq 0$. One therefore realizes that the functional relation (34), which characterizes the spatial behavior of the non-monotonic radial scalar eigenfunction at the extremum point (33), cannot be respected. We therefore conclude that, in the dimensionless regime $\xi \geq 0$ of the nonminimal field-curvature coupling parameter $\xi$, asymptotically flat static matter configurations made of nonminimally coupled massive scalar fields cannot be supported by spherically symmetric black holes with regular event horizons.

V. SUMMARY

The physically interesting no-hair theorems of Bekenstein and Mayo [7, 8] can be used to rule out the existence of spherically symmetric neutral black holes with regular event horizons that support static massive scalar fields nonminimally coupled to gravity in the dimensionless regimes $\xi \leq 0$ and $\xi \geq 1/2$ [10]. The no-hair theorem presented in [16] can be used to extend the validity of the no-hair property to the dimensionless regime $\xi > 1/4$. Intriguingly, however, no mathematically rigorous theorem has so far been presented in the physics literature which rules out the existence of asymptotically flat static hairy black-hole-massive-scalar-field configurations in the dimensionless complementary regime $0 < \xi \leq 1/4$. 
Studying analytically the non-linearly coupled Einstein-scalar field equations, we have explicitly proved in the present paper that spherically symmetric asymptotically flat neutral black holes with regular event horizons cannot support static matter configurations made of nonminimally coupled massive scalar fields with generic values of the dimensionless physical parameter $\xi$.

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Note that \( a \equiv \psi(x \to 0) \neq 0 \) [see Eq. (22)]. As emphasized above, if \( \psi(x \to 0) = 0 \) then the functional relation (21), which characterizes the asymptotic spatial behavior of the radial eigenfunction of the massive scalar fields, implies that \( \psi(r) \) must have an extremum point in the exterior black-hole spacetime in accord with our previous assertion.

Note that, as proved in [7], \( L > 0 \).