KDV-TYPE EQUATION LIMIT FOR ION DYNAMICS SYSTEM

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Abstract. In this paper, we consider the KdV-type limit for ion dynamics system. Under the Gardner-Morikawa type transforms, we derive the KdV-type equation by the scaling $\varepsilon^{4}(x-t) \to X$, $\varepsilon^{3}t \to T$ for ion dynamics system in one dimension. By proving the uniform estimates for the remainders system, we show that when $\varepsilon \to 0$, the solutions to the ion dynamics system converge globally in time to the solutions of the KdV-type equation.

1. Introduction. The Euler-Poisson system is a model about compressible ion-fluids and interacted electrons with their electromagnetic fields in plasma physics [8]. When ion-acoustic waves propagate in a plasma comprising cold ions and hot electrons in the presence of a uniform magnetic field, we consider the Euler-Poisson system in one-dimensional form

\begin{align}
    n_t + \partial_x (nv) &= 0, \\
    v_t + v \partial_x v &= -\partial_x \phi, \\
    \partial_x^2 \phi &= n_e - n,
\end{align}

(1.1)

where $n(x,t)$ stands for non-dimensional ion number density, $v(x,t)$ is the ion fluid velocity and $\phi(x,t)$ is the electrostatic potential at the time $t \geq 0$. In this paper, $n_e$ is the electron number density expressed as follows

$$
n_e = \exp\phi - \frac{4}{3}b\phi^2, \quad b > 0,
$$

(1.2)

where $b = (1-T_{ef}/T_{et})\pi^{-1/2}$ measures the isothermal deviation, $T_{ef}$ is the constant temperature of the free electrons and $T_{et}$ is the constant temperature of the trapped electrons [18]. Obviously, $(n,v,\phi) = (1,0,0)$ is an equilibrium solution. The global existence for the ion dynamics system around the equilibrium solution is a difficult problem. Therefore, this paper focuses on expansions around this equilibrium.

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A great deal of research work contributed to the study of the Euler-Poisson system for a plasma was obtained by mathematicians and physicists in different aspects during the past decades. KdV equation and KdV-like equations can be derived from Euler-Poisson system in appropriate limit and a variety of powerful methods were used to study the solutions, well-posedness, conservation laws and so on. Guo and Pu studied the KdV limit of the Euler-Poisson system in one dimension [9], and the higher dimensional case in [16]. Lannes and Linares established the KdV limit in one dimension and the ZK equation in three dimension from the Euler-Poisson equation system [13]. Han-Kwan gave the proof of a long wave scaling for the Vlasov-Poisson system equation and the KdV equation in one-dimension and the ZK equation in three dimensions using the modulated energy method [2]. ZK equation which as an approximate model in many physical contexts can be derived from the Euler-Poisson system [20]. Since many important nonlinear PDEs can be derived from the Euler-Poisson system, a natural question to ask is whether the KdV-type limit can be derived from the Euler-Poisson system. This is the mainly motivation of this paper.

MKdV equation comes from finding the solution of nonlinear evolution equation by perturbation parameter or series expansion methods in high order approximation and appears in many branches of nonlinear science. It has the following general form

$$u_t + \alpha u^\gamma u_x + \beta u_{xxx} = 0,$$

where $u$ is a real function, $\alpha$, $\beta$ and $\gamma$ are real numbers [3, 17]. Different values of $\alpha$, $\beta$ and $\gamma$ have different physical significance and correspond to different transformation scale. MKdV equation used to describe nonlinear optics, atmospheric fluctuation and quantum mechanics. A well-known results about the Cauchy problem studied extensively [14]. Particularly, the locally well-posed, globally well-posed, completely integrable and asymptotic behaviour of mKdV equation investigated in detail in [1, 7, 11, 12]. Fu and Chen constructed many solutions for mKdV equation in [4, 5]. Taogetusang [19] studied the exact solitary wave solutions for the GmKdV equation (1.3). However, until now there have been no rigorous mathematical justifications of KdV-type limits from the Euler-Poisson equation to the best of the authors' knowledge. In this paper our main aim is to derive the KdV-type limit from (1.1) in one dimension for the case $\alpha = b$, $\gamma = \frac{1}{2}$ and $\beta = \frac{1}{2}$, where $b$ is defined under equation (1.2), and the higher dimensional case of this problem will be further explored in future work.

Many useful methods arise in the course of the proof, the Gardner-Morikawa transformation contribute to the formal derivation of the KdV-type equations, the energy estimate is the fundamental method to the uniform estimates for the remainders. We define the triple norm (3.5) due to the classical continuity method, in the process of calculation we find the difficulty is the term $\Theta(3\varepsilon)$ out of control by common energy estimate methods, but we find the Poisson equation is crucial for controlling the term, this is the innovation of this paper, the process will be described in detail in Section 4.

The rest of this paper is organized as follows. In Section 2, we will give the formal derivation of the KdV-type equation. By means of the new transformation scale (2.1), which is determined by the linear dispersion relation and for balancing the effects of nonlinearity and dispersion [10], we can get the transformation system (2.2). Next considering the global existence of smooth solutions around the equilibrium which is expanded in a series of power $\varepsilon$, we find that its leading order
term satisfies the KdV-type equation, so making an appropriate truncation about the expansion, we can derive the KdV-type equation under the case $\alpha = b, \gamma = \frac{1}{2}$ and $\beta = \frac{1}{2}$. In Section 3, the main results is stated in Theorem 3.2. We will give the remainder system (3.3a)-(3.3c) for $(N, V, \Phi)$ in $\varepsilon$, and define the triple norm (3.5) which depends on $\varepsilon$ under the classical Gardner-Morikawa transformation.

And finally, in Section 4, the main estimates will be stated in Propositions 1, 2 and 3. We will complete the uniform estimates for the remainders (3.3a)-(3.3c) by classical continuity method. Using Gronwall inequality, we get the resulting estimates is closed in the triple norm (3.5).

2. Formal expansion. By using the classical Gardner-Morikawa transformation [6]$$X = \varepsilon^{\frac{1}{2}}(x - t), \quad T = \varepsilon^{\frac{3}{2}}t,$$where $\varepsilon$ is a small parameter of amplitude of the initial disturbance. When assume $b \gg o(\varepsilon^{1/2})$, we get the ion dynamics system (1.1) with parameter $\varepsilon$ as follows:

\[
\begin{align*}
\varepsilon^{\frac{1}{2}} n_T - n_X + \partial_X(nv) &= 0, \quad (2.2a) \\
\varepsilon^{\frac{1}{2}} v_T - v_X + v\partial_X v &= -\partial_X\phi, \quad (2.2b) \\
\varepsilon^{\frac{1}{2}} \partial_X^2\phi &= \exp \phi - \frac{4}{3} b \phi^3 - n. \quad (2.2c)
\end{align*}
\]

For system (2.2), according to perturbation theory, the equilibrium solution $(n, v, \phi) = (1, 0, 0)$ can be expressed as a series of small parameter $\varepsilon$ as follows:

\[
\begin{align*}
&n = 1 + \varepsilon n^{(1)} + \varepsilon^{\frac{3}{2}} n^{(2)} + \varepsilon^{2} n^{(3)} + \varepsilon^{\frac{5}{2}} n^{(4)} + \cdots, \\
v = \varepsilon v^{(1)} + \varepsilon^{\frac{3}{2}} v^{(2)} + \varepsilon^{2} v^{(3)} + \varepsilon^{\frac{5}{2}} v^{(4)} + \cdots, \\
&\phi = \varepsilon \phi^{(1)} + \varepsilon^{\frac{3}{2}} \phi^{(2)} + \varepsilon^{2} \phi^{(3)} + \varepsilon^{\frac{5}{2}} \phi^{(4)} + \cdots.
\end{align*}
\]

Putting the expansion system (2.3) into (2.2), we get a power series of $\varepsilon$ only depending on $(n^{(k)}, v^{(k)}, \phi^{(k)})$ for $k = 1, 2, 3, \cdots$.

Obviously, the coefficient at $O(\varepsilon^{0})$ is 0.

Let the coefficient at $O(\varepsilon^{\frac{3}{2}})$ be zero, we obtain

\[
\begin{align*}
\begin{cases} 
n^{(1)}_X = v^{(1)}_X, \\
v^{(1)}_X = \phi^{(1)}_X, \\
\phi^{(1)} = n^{(1)}. \end{cases}
\end{align*}
\]

We conclude that $v^{(1)} = \phi^{(1)} = n^{(1)}$. If we know $n^{(1)}$ we can determine $(n^{(1)}, v^{(1)}, \phi^{(1)})$.

Let the coefficient at $O(\varepsilon^{2})$ be zero, we obtain

\[
\begin{align*}
\begin{cases} 
n^{(2)}_X - n^{(1)}_X + v^{(2)}_X &= 0, \\
v^{(2)}_X - v^{(1)}_X &= -\phi^{(2)}_X, \\
\phi^{(2)}_{XX} &= \phi^{(2)} - \frac{4}{3} b (\phi^{(1)})^{\frac{3}{2}} - n^{(2)}. \end{cases}
\end{align*}
\]

Differentiating (2.5c) with respect to $x$, then adding together with (2.5a) to (2.5b), we can deduce $n^{(1)}$ satisfies the KdV-type equation

\[
n^{(1)}_T + b(n^{(1)})^{\frac{1}{2}} n^{(1)}_X + \frac{1}{2} n^{(1)}_{XXX} = 0.
\]

Using the relation (2.4) we find $n^{(2)}, v^{(2)}$ and $\phi^{(2)}$ in (2.6) vanish. That means the systems (2.5) and (2.6) are independent of $(n^{(j)}, v^{(j)}, \phi^{(j)})$ for $j \geq 2$ and are self
then using the relation (2.4) and (2.7) we can deduce the classical KdV equation due to the term \(1702\) RONG RONG AND YI PENG

inhomogeneous KdV-type equation determine \((2.4)\) with some lower order correctors \(g_0\) and \(f_0\). We only need to know \((n_2), v_2, \phi_2\) to determine \((n_2), v_2, \phi_2\).

Let the coefficient at \(O(\varepsilon^2)\) be zero, we get

\[
\begin{align*}
\frac{\partial}{\partial x}(n_2^{(1)} \frac{1}{2} n_2^{(2)}) + \frac{1}{2} n_2^{(2)} & = F^{(1)}, \\
\frac{\partial}{\partial x}(n_2^{(1)} \frac{1}{2} n_2^{(2)}) + \frac{1}{2} n_2^{(2)} & = F^{(1)},
\end{align*}
\]

where \(F^{(1)} = F^{(1)}(n_1)\) is determined by \(n_1\). Also, the system (2.8) and (2.9) about \((n_2), v_2, \phi_2\) are self contained and also independent of \((n_j), v_j, \phi_j\) for \((j \geq 3)\).

Let the coefficient at \(O(\varepsilon^k)\) be zero for \(k \geq \frac{5}{2}\), we get a system about \((n^{(k-1)}, v^{(k-1)}, \phi^{(k-1)}),\) then we can deduce

\[
\begin{align*}
\frac{\partial}{\partial x}(n^{(k)} \frac{1}{2} n^{(k)}) + \frac{1}{2} n^{(k)} & = F^{(k-1)},
\end{align*}
\]

where \(F^{(k-1)}\) is determined by \(n^{(1)}, \ldots, n^{(k-1)}\), which are all known by above equations. Again, the system (2.10) and (2.11) are self contained, and independent of \((n^{(k)}, v^{(k)}, \phi^{(k)})\) for \(j \geq k + 1\).

On account of the solvability of \((n^{(k)}, v^{(k)}, \phi^{(k)})\) for \(k \geq 1\), the initial value problem (2.4) and (2.6) has a unique solution. By the conservation laws of the mKdV equation, we can extend the solution to any time interval \([-\tau, \tau]\), see [15]. Suppose \((n^{(1)}, v^{(1)}, \phi^{(1)})\) is smooth enough for the following prove, for any given initial data
$(n^{(k)}, v^{(k)}, \phi^{(k)})$ satisfying (2.10), there exist a unique solution $(n^{(k)}, v^{(k)}, \phi^{(k)})$ for (2.11).

So, in the following paper, we assume that these solutions $(v^{(k)}, n^{(k)}, \phi^{(k)})$ for $1 \leq k \leq 6$ are as smooth enough as we want.

3. Main results. In order to justify $n^{(1)}$ converges the solution of KdV-type equation as $\varepsilon \to 0$, we let $(n, v, \phi)$ be a solution of the scaled system (2.2) in the following expansion

\[
\begin{align*}
n &= 1 + \varepsilon n^{(1)} + \varepsilon^2 n^{(2)} + \varepsilon^3 n^{(3)} + \varepsilon^4 n^{(4)} + \varepsilon^5 n^{(5)} + \varepsilon^6 n^{(6)} + \varepsilon^7 N, \\
v &= \varepsilon v^{(1)} + \varepsilon^2 v^{(2)} + \varepsilon^3 v^{(3)} + \varepsilon^4 v^{(4)} + \varepsilon^5 v^{(5)} + \varepsilon^6 v^{(6)} + \varepsilon^7 V, \\
\phi &= \varepsilon \phi^{(1)} + \varepsilon^2 \phi^{(2)} + \varepsilon^3 \phi^{(3)} + \varepsilon^4 \phi^{(4)} + \varepsilon^5 \phi^{(5)} + \varepsilon^6 \phi^{(6)} + \varepsilon^7 \Phi,
\end{align*}
\]

(3.1)

where $(n^{(1)}, v^{(1)}, \phi^{(1)})$ is the solution of equation (2.4) and (2.6), $(n^{(k)}, v^{(k)}, \phi^{(k)})$ is the solution of equation (2.10) and (2.11) for $2 \leq k \leq 6$, and $(N, V, \Phi)$ is the remainder. We need to give the energy estimates uniformly in $\varepsilon$ for the remainder $(N, V, \Phi)$.

To simplify the expression, we set

\[
\begin{align*}
\hat{n} &= n^{(1)} + \varepsilon^2 n^{(2)} + \varepsilon^3 n^{(3)} + \varepsilon^4 n^{(4)} + \varepsilon^5 n^{(5)} + \varepsilon^6 n^{(6)} , \\
\hat{v} &= v^{(1)} + \varepsilon^2 v^{(2)} + \varepsilon^3 v^{(3)} + \varepsilon^4 v^{(4)} + \varepsilon^5 v^{(5)} + \varepsilon^6 v^{(6)}, \\
\hat{\phi} &= \phi^{(1)} + \varepsilon^2 \phi^{(2)} + \varepsilon^3 \phi^{(3)} + \varepsilon^4 \phi^{(4)} + \varepsilon^5 \phi^{(5)} + \varepsilon^6 \phi^{(6)},
\end{align*}
\]

(3.2)

and then we get the remainder system for $(N, V, \Phi)$:

\[
\begin{align*}
\partial_T N - \frac{1 - v}{\varepsilon^2} \partial_X N + \frac{n}{\varepsilon^2} \partial_X V + \varepsilon^2 \partial_X \hat{n} V + \varepsilon^2 \partial_X \hat{v} N + \varepsilon^2 \partial_X \hat{\Phi} = 0, \\
\partial_T V - \frac{1 - v}{\varepsilon^2} \partial_X V + \varepsilon^2 \partial_X \hat{v} V + \frac{1}{\varepsilon^2} \partial_X \hat{\Phi} + \varepsilon^2 \partial_X \hat{\Phi} = 0, \\
\varepsilon \partial_X \hat{\Phi} = -2 \varepsilon^2 \hat{\Phi} + \varepsilon^2 \hat{\Phi} + \varepsilon \Phi = 0.
\end{align*}
\]

(3.3a, 3.3b, 3.3c)

where

\[
\begin{align*}
\mathcal{R}_1 &= \partial_T n^{(6)} + \sum_{1 \leq i \leq j \leq 5} \varepsilon^i \varepsilon^j \partial_X (n^{(i)} v^{(j)}), \\
\mathcal{R}_2 &= \partial_T v^{(6)} + \sum_{1 \leq i \leq j \leq 5} \varepsilon^i \varepsilon^j \partial_X (v^{(i)} \Phi), \\
\mathcal{R}_3 &= -\frac{1}{2} \varepsilon^2 \Phi^2 + \varepsilon^2 \Phi^2 + \varepsilon \Phi^2.
\end{align*}
\]

(3.4)

From above we know $\mathcal{R}_3$ is dependent on $\Phi$ and its derivatives. The main difficulty is to get the estimates for the remainders $(N, V, \Phi)$ uniformly in $\varepsilon$.

In order to get the uniform estimates conveniently, the triple norm is denoted as

\[
\| (V, \Phi) \|_{H^2}^2 = \| V \|_{H^2}^2 + \| \Phi \|_{H^2}^2 + \varepsilon \| \partial_X^2 V \|_{H^2}^2 + \varepsilon \| \partial_X^2 \Phi \|_{H^2}^2 + \varepsilon \| \partial_X^2 \Phi \|_{H^2}^2.
\]

(3.5)

Lemma 3.1. There exists some constant $C = C(\| \phi^{(i)} \|_{H^{\alpha}}, \varepsilon^{\frac{1}{2}} \| \Phi \|_{H^\alpha})$ for $\alpha = 0, 1, 2, \ldots$ (integers), so that

\[
\begin{align*}
\| \mathcal{R}_3 \|_{H^\alpha} &\leq C(\| \phi^{(i)} \|_{H^{\alpha}}, \varepsilon^{\frac{1}{2}} \| \Phi \|_{H^\alpha}), \\
\| \mathcal{R}_3 \|_{H^\alpha} &\leq C(\| \phi^{(i)} \|_{H^{\alpha}}, \varepsilon^{\frac{1}{2}} \| \Phi \|_{H^\alpha}), \\
\| \partial_T \mathcal{R}_3 \|_{H^\alpha} &\leq C(\| \phi^{(i)} \|_{H^{\alpha}}, \varepsilon^{\frac{1}{2}} \| \Phi \|_{H^\alpha}) \| \partial_T \Phi \|_{H^\alpha}, \\
\| \partial_T \mathcal{R}_3 \|_{H^\alpha} &\leq C(\| \phi^{(i)} \|_{H^{\alpha}}, \varepsilon^{\frac{1}{2}} \| \Phi \|_{H^\alpha}) \| \partial_T \Phi \|_{H^\alpha}.
\end{align*}
\]

(3.6)

Due to the fact that $H^1$ is a algebra, the estimate of Lemma 3.1 is straightforward, we omit the proof here.
The main result is stated in the following

**Theorem 3.2.** Let \( \hat{s}_i \geq 2 \) be sufficiently large and \((n^{(1)}, v^{(1)}, \phi^{(1)}) \in H^2 \) be a solution for the KdV-type equation with initial \((n^{(1)}, v^{(1)}, \phi^{(1)}) \in H^\hat{s}_1 \) satisfying (2.4). Let \((n^{(i)}, v^{(i)}, \phi^{(i)}) \in H^{\hat{s}_i}(i = 2, 3, 4)\) be a solution of (2.10) and (2.11) with the initial data \((n^{(i)}, v^{(i)}, \phi^{(i)}) \in H^\hat{s}_i\) satisfied (2.3). Let \((N_0, V_0) \in H^s\), \(s \geq 3\) and assume

\[
\begin{align*}
 n &= 1 + \varepsilon n^{(1)} + \varepsilon^2 n^{(2)} + \varepsilon^2 n^{(3)} + \varepsilon^2 n^{(4)} + \varepsilon^2 n^{(5)} + \varepsilon n^{(6)} + \varepsilon^2 N_0, \\
v &= \varepsilon v^{(1)} + \varepsilon^2 v^{(2)} + \varepsilon^2 v^{(3)} + \varepsilon^2 v^{(4)} + \varepsilon^2 v^{(5)} + \varepsilon^2 v^{(6)} + \varepsilon^2 V_0, \\
\phi &= \varepsilon \phi^{(1)} + \varepsilon^2 \phi^{(2)} + \varepsilon^2 \phi^{(3)} + \varepsilon^2 \phi^{(4)} + \varepsilon^2 \phi^{(5)} + \varepsilon^2 \phi^{(6)} + \varepsilon^2 \Phi_0,
\end{align*}
\]

where \( \Phi_0 \) satisfies (3.3c). Then for any \( \tau > 0 \), there exists \( \varepsilon_0 > 0 \), if \( 0 < \varepsilon < \varepsilon_0 \), the solution of the ion dynamics system (1.1) with initial data \((n_0, v_0, \phi_0)\) can be expressed as

\[
\begin{align*}
n &= 1 + \varepsilon n^{(1)} + \varepsilon^2 n^{(2)} + \varepsilon^2 n^{(3)} + \varepsilon^2 n^{(4)} + \varepsilon^2 n^{(5)} + \varepsilon n^{(6)} + \varepsilon^2 N, \\
v &= \varepsilon v^{(1)} + \varepsilon^2 v^{(2)} + \varepsilon^2 v^{(3)} + \varepsilon^2 v^{(4)} + \varepsilon^2 v^{(5)} + \varepsilon^2 v^{(6)} + \varepsilon^2 V, \\
\phi &= \varepsilon \phi^{(1)} + \varepsilon^2 \phi^{(2)} + \varepsilon^2 \phi^{(3)} + \varepsilon^2 \phi^{(4)} + \varepsilon^2 \phi^{(5)} + \varepsilon^2 \phi^{(6)} + \varepsilon^2 \Phi,
\end{align*}
\]

such that for all \( 0 < \varepsilon < \varepsilon_0 \),

\[
\begin{align*}
sup_{[0, \tau]} \| \left( (N, V, \Phi) \right) \|_{H^2}^2 + \varepsilon \| (\partial^2_X V, \partial^2_X \Phi) \|_{L^2}^2 + \varepsilon \| \partial^4_X \Phi \|_{L^2}^2 \\
\leq C_\tau (1 + \| (N_0, V_0, \Phi_0) \|_{H^2}^2 + \varepsilon \| (\partial^2_X V_0, \partial^2_X \Phi_0) \|_{L^2}^2 + \varepsilon \| \partial^4_X \Phi_0 \|_{L^2}^2).
\end{align*}
\]

Observe the equation (3.10), we see the \( H^2 \)-norm of the remainder \((V, N, \Phi)\) is uniformly in \( \varepsilon \).

4. Uniform energy estimate. In this section, we will give the estimates uniformly in \( \varepsilon \) for the remainder \((V, N, \Phi)\), the process needs some energy methods and analysis for the remainder equations in (3.3). For simplification, we assume that the system (3.3) has smooth solutions depending on \( \varepsilon \) in \([0, \tau_0]\) for any \( \tau_0 > 0 \). Let \( C_0 \) be a constant independent of \( \varepsilon \), which will be determined latter, much larger than the bound \( \| (V, \Phi) (0) \|_2^2 \) of the initial data.

Now we use the classical continuity method. When \( \tau_\varepsilon > 0 \) and belongs to \([0, \tau_\varepsilon]\), we have

\[
\begin{align*}
\| N \|_{H^2}^2, \quad \| (V, \Phi)(0) \|_2^2 \leq C_0.
\end{align*}
\]

From above we know \( n \) is bounded, there exists \( \varepsilon_1 > 0 \) when \( \varepsilon < \varepsilon_1 \) so that \( \frac{1}{2} < n < \frac{3}{2} \) and \( \| v \| < \frac{1}{2} \). We know \( R_3 \) is a smooth function about \( \Phi \), there exists some constant \( C_1 = C_1(\varepsilon C_0) \) for any \( \alpha, \beta \geq 0 \) such that

\[
\| \partial^\alpha_X \partial^\beta_X R_3 \| \leq C_1 = C_1(\varepsilon C_0),
\]

we think \( C_1(\cdot) \) is nondecreasing. There is \( \varepsilon_0 > 0 \) for any given \( \tau > 0 \) and \( 0 < \varepsilon < \varepsilon_0 \), we have the existence time \( \tau_\varepsilon > \tau \).

Our purpose is proving Proposition 1 and 2. In subsection 4.1, we will give three lemmas which are very useful in following prove. In subsection 4.2 and 4.3, we give and prove the two main Propositions, and some crucial terms will estimate in subsection 4.4 and 4.5. \( \| \cdot \| \) is the \( L^2 \) norm.
4.1. Basic estimates.

**Lemma 4.1.** Let $(N, V, \Phi)$ be a solution to (3.3) and $\alpha \geq 0$ (integer). For every $0 < \varepsilon < \varepsilon_1$, there exist some constants $0 < \varepsilon_1 < 1$ and $C_1 = C_1(\varepsilon C_0)$ so that

$$C_1^{-1} \| \partial^{\alpha}_X N \|^2 \leq \| \partial^{\alpha}_X \Phi \|^2 + \varepsilon \| \partial^{\alpha+1}_X \Phi \|^2 + \varepsilon \| \partial^{\alpha+2}_X \Phi \|^2 \leq C_1 \| \partial^{\alpha}_X N \|^2. \quad (4.2)$$

**Proof.** First we consider the case of $\alpha = 0$, taking inner product of equation (3.3c) with $\Phi$, we get

$$\| \Phi \|^2 + \varepsilon \| \partial X \Phi \|^2 = \int N \Phi + \int (2b \varepsilon \tilde{\phi}^2 \tilde{\phi} - \varepsilon \tilde{\phi})|\Phi|^2 - \int \varepsilon \mathcal{R}_3 \Phi. \quad (4.3)$$

Using Young inequality to the first term on the RHS of equation (4.3), we have

$$\left| \int N \Phi \right| \leq \frac{1}{4} \| \Phi \|^2 + \| N \|^2. \quad (4.4)$$

From Lemma 3.1, we get

$$\| \mathcal{R}_3 \|_{L^2} \leq C_1(\varepsilon C_0) \| \Phi \|. \quad (4.5)$$

There exist $\varepsilon < \varepsilon_1$ which is sufficiently small so that when $C_1(\varepsilon C_0) < C_1(1)$ we obtain

$$\left| \int \varepsilon \mathcal{R}_3 \Phi \right| \leq \frac{1}{8} \| \Phi \|^2. \quad (4.6)$$

We know $\tilde{\phi}$ are composed of $\phi^{(1)}$, $\phi^{(2)}$, $\phi^{(3)}$, $\phi^{(4)}$, $\phi^{(5)}$, $\phi^{(6)}$, which are all known and bounded in $L^\infty$, so there exists some $0 < \varepsilon_1 < 1$, when $0 < \varepsilon < \varepsilon_1$ we have

$$\left| \int (2b \varepsilon \tilde{\phi}^2 \tilde{\phi} - \varepsilon \tilde{\phi})|\Phi|^2 \right| \leq \frac{1}{8} \| \Phi \|^2. \quad (4.7)$$

From above equations, we have

$$\| \Phi \|^2 + \varepsilon \| \partial X \Phi \|^2 \leq \frac{1}{2} \| \Phi \|^2 + \| N \|^2. \quad (4.8)$$

Therefore, there exists $\varepsilon_1 > 0$ when $0 < \varepsilon < \varepsilon_1$, we get

$$\| \Phi \|^2 + \varepsilon \| \partial X \Phi \|^2 \leq 2 \| N \|^2. \quad (4.9)$$

Taking the inner product of equation (3.3c) with $\varepsilon \tilde{\phi} \partial^2 X \Phi$, then integrates by parts, we have equation similarly as follows

$$\varepsilon \| \partial^2 X \Phi \|^2 + \varepsilon \| \partial^2 X \Phi \|^2 \leq 2 \| N \|^2. \quad (4.10)$$

Besides, from the equation (3.3c), there exist $C$, we have

$$\| N \|^2 \leq \| \Phi \|^2 + \varepsilon \| \partial^2 X \Phi \|^2 + C \| \partial^2 X \Phi \|^2 \leq C(\| N \|^2 + \varepsilon \| \partial^2 X \Phi \|^2). \quad (4.11)$$

When put the equations (4.9), (4.10), (4.11) together, we deduce the inequality (4.2) for $\alpha = 0$.

The case of $\alpha > 0$, in order to get higher order inequalities, we take the same procedure as $\alpha = 0$, we differentiate the Poisson equation (3.3c) with $\partial^2 X$ and take inner product with $\partial^2 X \Phi$ and $\varepsilon \tilde{\phi} \partial^2 X \Phi$ separately. So the Lemma 4.1 is proved.

**Lemma 4.2.** Let $(N, V, \Phi)$ be a solution to (3.3). There exist some constants $C$ and $C_1 = C_1(\varepsilon C_0)$, we have

$$\| \varepsilon \tilde{\phi} \partial^2 X N \|^2 \leq C(\| \Phi \|^2_{H^1} + \| V \|^2_{H^1} + \varepsilon \| \partial^2 X \Phi \|^2 + \varepsilon \| \partial^2 X \Phi \|^2) + C \varepsilon, \quad (4.12)$$

and

$$\| \varepsilon \tilde{\phi} \partial^2 X N \|^2 \leq C_1(\| V \|^2_{H^2} + \| \Phi \|^2_{H^2} + \varepsilon \| \partial^2 X \Phi \|^2 + \varepsilon \| \partial^2 X \Phi \|^2) + C \varepsilon. \quad (4.13)$$
That is equivalent to
\[ \| \varepsilon \frac{1}{2} \partial_T N \|_{H^1}^2 \leq C_1 ||(V, \Phi)||_\varepsilon^2 + C \varepsilon. \] (4.14)

**Proof.** From equation (3.3a), we get
\[ \varepsilon \frac{1}{2} \partial_T N = (1 - v) \partial_X N - n \partial_X V - \varepsilon \partial_X \tilde{u} V - \varepsilon \partial_X \tilde{v} N - \varepsilon \tilde{R}_1, \] (4.15)
where \( \frac{1}{2} < n < \frac{3}{2} \) and \( |v| < \frac{1}{2} \), making \( L^2 \)-norm, we get
\[ \| \varepsilon \frac{1}{2} \partial_T N \|^2 \leq C(\| \partial_X N \|^2 + \| \partial_X V \|)^2 + C \varepsilon^2 (1 + \| V \|^2 + \| N \|^2). \] (4.16)
Applying Lemma 4.1, when \( \alpha = 1 \) we deduce
\[ \| \varepsilon \frac{1}{2} \partial_T N \|^2 \leq C(\| \Phi \|_{H^1}^2 + \| V \|_{H^1}^2 + \varepsilon \| \partial_X \Phi \|)^2 + C \varepsilon. \] (4.17)
Taking \( \partial_X \) of equation (3.3a), and take \( L^2 \)-norm, we get
\[ \| \varepsilon \frac{1}{2} \partial_T XN \|^2 \leq C(\| V \|_{H^2}^2 + \| N \|_{H^2}^2) + C \varepsilon^6 \int | \partial_X V |^2 | \partial_X N |^2 + C \varepsilon^2. \] (4.18)
Using Sobolev embedding \( H^2 \hookrightarrow L^\infty \), we get
\[ C \varepsilon^6 \int | \partial_X V |^2 | \partial_X N |^2 \leq C \varepsilon^6 \| V \|_{H^2}^2 \| N \|_{H^1}^2 \leq C(\varepsilon C_0) \| V \|_{H^2}^2. \] (4.19)
So using Lemma 4.1, we have
\[ \| \varepsilon \frac{1}{2} \partial_T XN \|^2 \leq C_1 (\| V \|_{H^2}^2 + \| \Phi \|_{H^2}^2 + \varepsilon \| \partial_X \Phi \|)^2 + C \varepsilon. \] (4.20)

**Lemma 4.3.** Let \( (N, V, \Phi) \) be a solution to (3.3). \( \alpha \geq 0 \) (integer), there exist some constants \( C_1 = C(\varepsilon C_0) \) and \( \varepsilon_1 > 0 \) for any \( 0 < \varepsilon < \varepsilon_1 \),
\[ \| \partial_T \partial_X^\alpha \Phi \|^2 + \varepsilon \| \partial_T \partial_X^{\alpha + 1} \Phi \|^2 \leq 2 \| \partial_T \partial_X^\alpha N \|^2 + C_1(\varepsilon C_0). \] (4.21)

**Proof.** For the case of \( \alpha = 0 \). Taking \( \partial_T \) of equation (3.3c), and then taking inner product with \( \partial_T \Phi \), we have
\[ \| \partial_T \Phi \|^2 + \varepsilon \| \partial_T X \Phi \|^2 \]
\[ \leq \int \partial_T N \partial_T \Phi - \int \varepsilon \frac{1}{2} \partial_T (\varepsilon \frac{1}{2} \partial_T \Phi - 2b \tilde{R}_3 \partial_T \Phi) - \int \varepsilon \partial_T \tilde{R}_3 \partial_T \Phi \]
\[ \leq \frac{1}{4} \| \partial_T \Phi \|^2 + \| \partial_T N \|^2 + \frac{1}{8} \| \partial_T \Phi \|^2 + C_1(\varepsilon C_0) \| \partial_T \Phi \|^2. \] (4.22)
Due to Lemma 3.1, For \( \varepsilon_1 > 0 \), when \( \varepsilon < \varepsilon_1 \), we get
\[ \| \partial_T \Phi \|^2 + \varepsilon \| \partial_T X \Phi \|^2 \leq 2 \| \partial_T N \|^2 + C_1(\varepsilon C_0). \] (4.23)
Now, consider the case \( \alpha = 1 \). We take \( \partial_T X \) of equation (3.3c), and then take inner product with \( \partial_T X \Phi \)
\[ \| \partial_T X \Phi \|^2 + \varepsilon \| \partial_T \partial_X \Phi \|^2 \leq 2 \| \partial_T X N \|^2 + C_1(\varepsilon C_0). \] (4.24)
The prove of the case \( \alpha \geq 2 \) is similar to above. \( \square \)
4.2. Zeroth, first and second order estimates.

Proposition 1. Let \((N,V,\Phi)\) be a solution to (3.3), and \(\gamma = 0, 1, 2\), then
\[
\frac{1}{2} \frac{d}{dT} \|\partial_X^\gamma V\|^2 + \frac{1}{2} \frac{d}{dT} \left[ \int \frac{1 + \varepsilon^2}{n} \partial_X^{\gamma+1} \Phi^2 \right] 
\leq C_1 (1 + \varepsilon^2 \|\Phi\|_{L^\infty}^2) (1 + \|V\|_{H^2}^2).
\] (4.25)

Proof. Taking \(\partial_X^\gamma\) of equation (3.3b), then make inner product with \(\partial_X^\gamma V\), and integrates by parts, we obtain
\[
\frac{1}{2} \partial_T \|\partial_X^\gamma V\|^2 + \varepsilon^\frac{1}{2} \int \partial_X^\gamma [\tilde{\nu} + \varepsilon^2 \nu] \partial_X^\gamma V \partial_X^\gamma V - \frac{1}{\varepsilon^\gamma} \int \partial_X^{\gamma+1} V \partial_X^\gamma V 
+ \varepsilon^\frac{1}{2} \int \partial_X^\gamma \partial_X \nu \partial_X^\gamma V + \varepsilon^\frac{1}{2} \int \partial_X^\gamma \Phi \partial_X^\gamma V = \frac{1}{\varepsilon^\gamma} \int \partial_X^\gamma \Phi \partial_X^{\gamma+1} V =: \gamma. \quad (4.26)
\]

Estimate the LHS of (4.26). For \(0 \leq \gamma \leq 2\), by integration by parts, the third term on the LHS of above system vanishes. The second term can be divided into two parts
\[
\varepsilon^\frac{1}{2} \int \partial_X^\gamma [\tilde{\nu} + \varepsilon^2 \nu] \partial_X^\gamma V \partial_X^\gamma V = \varepsilon^\frac{1}{2} \int \partial_X^\gamma \tilde{\nu} \partial_X^\gamma V \partial_X^\gamma V + \varepsilon^\frac{1}{2} \int \partial_X^\gamma [\nu \partial_X V] \partial_X^\gamma V. \quad (4.27)
\]

Estimate the first part of (4.27). For \(0 \leq \gamma \leq 2\), by integration by parts, we have
\[
\varepsilon^\frac{1}{2} \int \partial_X^\gamma \tilde{\nu} \partial_X^\gamma V \partial_X^\gamma V 
= \varepsilon^\frac{1}{2} \int \tilde{\nu} \partial_X^{\gamma+1} V \partial_X^\gamma V + \varepsilon^\frac{1}{2} \sum_{0 \leq \beta \leq \gamma-1} C_2^\beta \int \partial_X^{\gamma-\beta} \tilde{\nu} \partial_X^{\beta+1} V \partial_X^\gamma V
\]
\[
= -\frac{1}{2} \varepsilon^\frac{1}{2} \int \partial_X \tilde{\nu} \partial_X^{\gamma+1} V \partial_X^\gamma V + \varepsilon^\frac{1}{2} \sum_{0 \leq \beta \leq \gamma-1} C_2^\beta \int \partial_X^{\gamma-\beta} \tilde{\nu} \partial_X^{\beta+1} V \partial_X^\gamma V \leq C \varepsilon^\frac{1}{2} \|V\|_{H^2}^2. \quad (4.28)
\]

Estimate the second part of (4.27). For \(0 \leq \gamma \leq 2\), by integration by parts, we get
\[
\varepsilon^\frac{1}{2} \int \partial_X^\gamma [\nu \partial_X V] \partial_X^\gamma V 
= -\frac{1}{2} \varepsilon^\frac{1}{2} \int \partial_X \nu \partial_X^{\gamma+1} V \partial_X^\gamma V + \varepsilon^\frac{1}{2} \sum_{0 \leq \beta \leq \gamma-1} C_2^\beta \int \partial_X^{\gamma-\beta} \nu \partial_X^{\beta+1} V \partial_X^\gamma V
\]
\[
\leq C \varepsilon^\frac{1}{2} \|\partial_X \nu \|_{L^\infty} \|V\|_{H^\gamma}^2 \leq C (\varepsilon^\frac{1}{2} \|\nu\|_{L^\infty} \|\Phi\|_{L^\infty} \|\partial_X^\gamma V\|_{H^\gamma}^2). \quad (4.29)
\]

Estimate the last two terms on the LHS of equation (4.26). The fourth term is similar to the equation (4.27), the fifth term is integrable by (3.4), they all are bounded by \((\varepsilon^\frac{1}{2} \|V\|_{H^\gamma}^2 + C \varepsilon)\). So, the last four terms on the LHS of (4.26) are bounded by
\[
C (1 + \varepsilon^\frac{1}{2} \|\nu\|_{L^\infty} \|\Phi\|_{L^\infty} \|\partial_X^\gamma V\|_{H^\gamma}^2). \quad (4.30)
\]

Estimate the RHS term \(\gamma\) in equation (4.26). Taking \(\partial_X^\gamma\) of equation (3.3a), we get
\[
\frac{1}{\varepsilon^\frac{1}{2}} \partial_X^{\gamma+1} V = \frac{1}{n} \frac{1 - \varepsilon}{\varepsilon^\frac{1}{2}} \partial_X^{\gamma+1} N - \partial_T \partial_X^\gamma N - \frac{1}{\varepsilon^\gamma} \sum_{0 \leq \beta \leq \gamma-1} C_2^\beta \partial_X^{\gamma-\beta} \nu \partial_X^{\beta+1} N
\]
\[
- \frac{1}{\varepsilon^\gamma} \sum_{0 \leq \beta \leq \gamma-1} C_2^\beta \partial_X^{\gamma-\beta} n \partial_X^{\beta+1} V - \varepsilon^\frac{1}{2} \sum_{0 \leq \beta \leq \gamma-1} C_2^\beta \partial_X^{\gamma-\beta+1} n \partial_X^{\beta} V \quad (4.31)
\]
$-\frac{\varepsilon^2}{2} \sum_{0 \leq \beta \leq \gamma-1} C_{\gamma}^{\beta} \partial_{\gamma}^{\gamma-\beta+1} \bar{v} \partial_{\gamma}^{\beta} N - \varepsilon^2 \partial_{\gamma}^{\gamma-\beta+1} R_1 =: B_i^{(\gamma)}.$

Observed we find $A^{(\gamma)}$ can be decomposed into

$$A^{(\gamma)} = \sum_{i=1}^{7} A_i^{(\gamma)} = \sum_{i=1}^{7} \int \partial_{\gamma}^{\gamma} \Phi B_i^{(\gamma)}. \tag{4.32}$$

Now, we firstly estimate the term $A_i^{(\gamma)}$ for $3 \leq i \leq 7$ and then estimate $A_1^{(\gamma)}$ and $A_2^{(\gamma)}$ in the following Lemma 4.4 and 4.5 separately.

The estimate of $A_3^{(\gamma)}$ in (4.32).

$$A_3^{(\gamma)} = \frac{1}{\varepsilon^2} \sum_{0 \leq \beta \leq \gamma-1} C_{\gamma}^{\beta} \int \partial_{\gamma}^{\gamma} \Phi \partial_{\gamma}^{\gamma-\beta} \bar{v} \partial_{\gamma}^{\beta+1} N \tag{4.33}$$

$$= \varepsilon^2 \frac{1}{n} \sum_{0 \leq \beta \leq \gamma-1} C_{\gamma}^{\beta} \int \partial_{\gamma}^{\gamma} \Phi \partial_{\gamma}^{\gamma-\beta} \bar{v} \partial_{\gamma}^{\beta+1} N + \varepsilon^2 \frac{1}{n} \sum_{0 \leq \beta \leq \gamma-1} C_{\gamma}^{\beta} \int \partial_{\gamma}^{\gamma} \Phi \partial_{\gamma}^{\gamma-\beta} V \partial_{\gamma}^{\beta+1} N$$

$$=: A_{31}^{(\gamma)} + A_{32}^{(\gamma)}.$$

The first term $A_{31}^{(\gamma)}$ is bounded by

$$A_{31}^{(\gamma)} \leq C(\|N\|_{H^2}^2 + \varepsilon^2 \|\Phi\|_{H^2}). \tag{4.34}$$

The second term $A_{32}^{(\gamma)}$ is bounded by using H"older inequality, Lemma 4.2 and Sobolev embedding $H^1 \rightarrow L^\infty$, we have

$$A_{32}^{(\gamma)} \leq C \varepsilon^2 \|\partial_{\gamma}^{\gamma} \Phi\|_{L^\infty} \|N\|_{H^2} \|V\|_{H^2} \leq C(1 + \varepsilon^2 \|\Phi\|_{L^\infty}^2)(\|V\|_{H^2}^2 + \|N\|_{H^2}^2).$$

Hence, we get

$$A_3^{(\gamma)} \leq C(1 + \varepsilon^2 \|\Phi\|_{L^\infty}^2)(1 + \|\Phi\|_{L^\infty}^2). \tag{4.35}$$

Estimate of $A_4^{(\gamma)}$ in (4.32), the term $A_4^{(\gamma)}$ is bound similarity

$$A_4^{(\gamma)} \leq C(1 + \varepsilon^2 \|\Phi\|_{L^\infty}^2)(1 + \|\Phi\|_{L^\infty}^2). \tag{4.36}$$

Estimate of $A_5^{(\gamma)}$, $A_6^{(\gamma)}$, $A_7^{(\gamma)}$ in (4.32). Because the term $A_i^{(\gamma)} (i = 5, 6, 7)$ are bilinear and linear in the unknowns, they can be bounded by

$$A_i^{(\gamma)} \leq C(1 + \|\Phi\|_{L^\infty}^2).$$

In summary, we get

$$\sum_{i=3}^{7} A_i^{(\gamma)} \leq C(1 + \varepsilon^2 \|\Phi\|_{L^\infty}^2)(1 + \|\Phi\|_{L^\infty}^2). \tag{4.37}$$

Lemma 4.4. Let $(N, V, \Phi)$ be a solution to (3.3), we have

$$A_1^{(\gamma)} \leq C_1(1 + \varepsilon^2 \|\Phi\|_{L^\infty}^2)(1 + \|\Phi\|_{L^\infty}^2), \tag{4.38}$$

where $A_1^{(\gamma)}$ is defined in (4.32).

Proof. Taking $\partial_{\gamma}^{\gamma+1}$ of equation (3.3c), we get

$$\partial_{\gamma}^{\gamma+1} N = \partial_{\gamma}^{\gamma+1} \Phi - \varepsilon^{\frac{1}{2}} \partial_{\gamma}^{\gamma+1} \Phi + \partial_{\gamma}^{\gamma+1} (\varepsilon^{\frac{1}{2}} \Phi - 2 \varepsilon^{\frac{1}{2}} \Phi) + \varepsilon \partial_{\gamma}^{\gamma+1} R_3 =: \sum_{i=1}^{4} D_i^{(\gamma)}. \tag{4.39}$$
due to the fact that

Using Sobolev embedding and Lemma 4.2, we have

\[ A_{11}^{(\gamma)} \leq C \varepsilon^\frac{1}{2} \| \partial_X \Phi \|_2^2 + C_1 (1 + \varepsilon^2 \| |(V, \Phi)|_2 \| \| \partial_X \Phi \|_2^2). \] (4.43)

**Estimate of** \( A_{12}^{(\gamma)} \) **in** (4.40). **By integration by parts twice, we have**

\[ A_{12}^{(\gamma)} = - \int \frac{1 - v}{n} \partial_X \Phi \partial_X \gamma \phi \Phi \] (4.44)

\[ = - \frac{3}{2} \int \partial_X \left( \frac{1 - v}{n} \right) |\partial_X \gamma \phi \Phi|^2 - \int \partial_X \left( \frac{1 - v}{n} \right) \partial_X \Phi \partial_X \gamma \phi \Phi =: A_{121}^{(\gamma)} + A_{122}^{(\gamma)}, \]

due to the fact that

\[ \partial_X \frac{1 - v}{n} \leq C \varepsilon |(\partial_X v) + |\partial_X n)| + \varepsilon \left( |(\partial_X v) + |\partial_X N)|. \] (4.45)

Similar to the estimate of \( A_{11}^{(\gamma)} \), we have

\[ A_{121}^{(\gamma)} \leq C \varepsilon \| \partial_X \gamma \phi \Phi \|_2^2 + C_1 (1 + \varepsilon^2 \| |(V, \Phi)|_2 \| \| \partial_X \gamma \phi \Phi \|_2^2). \] (4.46)

**Estimate for** \( A_{122}^{(\gamma)} \) **in** (4.44). **We note that**

\[ \partial_X \frac{1 - v}{n} \leq C \varepsilon^4 \left( |(\partial_X v) + |\partial_X N)| + \varepsilon^4 \left( |(\partial_X v) + |\partial_X N)| + \varepsilon^6 \left( |(\partial_X v) + |\partial_X N)|. \] (4.47)

Using Hölder inequality, Sobolev embedding and Lemma 4.2 for \( 0 \leq \gamma \leq 2 \), we get

\[ A_{122}^{(\gamma)} \leq C \| \partial_X \Phi \|_{L^\infty} \| \partial_X \frac{1 - v}{n} \| \| \partial_X \gamma \phi \Phi \| \]

\[ \leq C_1 (1 + \varepsilon^2 \| |(V, \Phi)|_2 \|_2 \| \partial_X \Phi \|_{2, H^{\gamma+1}}^2). \] (4.48)

Due to \( A_{121}^{(\gamma)} \) and \( A_{122}^{(\gamma)} \), we get

\[ A_{12}^{(\gamma)} \leq C_1 (1 + \varepsilon^2 \| |(V, \Phi)|_2 \|_2 \| \partial_X \Phi \|_{2, H^{\gamma+1}}^2). \] (4.49)

**Estimate for** \( A_{13}^{(\gamma)} \) **in** (4.40). **The estimate for** \( A_{13}^{(\gamma)} \) **is similar to procedure for** \( A_{11}^{(\gamma)} \), **so we have**

\[ A_{13}^{(\gamma)} \leq C_1 (1 + \varepsilon^2 \| |(V, \Phi)|_2 \|_2 \| \partial_X \Phi \|_{2, H^{\gamma+1}}^2). \] (4.50)

**Estimate for** \( A_{14}^{(\gamma)} \) **in** (4.40). **By integration by parts and Young’s inequality, we obtain**

\[ A_{14}^{(\gamma)} = \varepsilon^\frac{1}{2} \int \partial_X \Phi \left( \frac{1 - v}{n} \partial_X \gamma \phi \Phi \right) \]

\[ = - \varepsilon^\frac{1}{2} \int \partial_X \left( \frac{1 - v}{n} \right) \partial_X \Phi \partial_X \gamma \phi \Phi - \varepsilon^\frac{1}{2} \int \left( \frac{1 - v}{n} \right) \partial_X \gamma \phi \Phi \partial_X \gamma \phi \Phi. \] (4.51)
\[\leq C_1\varepsilon^2(1 + \|\partial_X V\|_{L^\infty} + \|\partial_X N\|_{L^\infty})\|\Phi\|_H^2 + C_1\varepsilon^2\|\Phi\|_{H^\gamma+1}\|\Phi\|_{H^\gamma}\]
\[\leq C_1(1 + \varepsilon^2)(|\langle V, \Phi \rangle| + \epsilon\|\Phi\|_{H^\gamma+1}^2).\]

**Lemma 4.5.** Let \((N, V, \Phi)\) be a solution to (3.3) for \(0 \leq \gamma \leq 2\), the following inequality holds
\[A_2^{(\gamma)} \leq -\frac{1}{2} \frac{d}{dt} \int \frac{1 + \varepsilon}{n} |\partial_X^\gamma \Phi|^2 - \frac{1}{2} \frac{d}{dt} \int \frac{\varepsilon^2}{n} |\partial_X^{\gamma+1} \Phi|^2 + C \varepsilon^2 \|\Phi\|_{H^\gamma+1}^2 \|\Phi\|_{H^\gamma},\]
where \(A_2^{(\gamma)}\) is defined in (4.32).

**Proof.** Taking \(\partial_T \partial_X^\gamma\) of equation (3.3c), we get
\[\partial_T \partial_X^\gamma N = \partial_T \partial_X^\gamma \Phi - \varepsilon^2 \partial_T \partial_X^{\gamma+2} \Phi + \partial_T \partial_X^\gamma (\varepsilon \Phi) - 2b\varepsilon^2 \partial_X^\gamma \Phi + \varepsilon \partial_T \partial_X^\gamma \Phi\]
\[= \sum_{i=1}^4 E_i^{(\gamma)}(\Phi).\] (4.52)

Accordingly, we have the decomposition
\[A_2^\gamma = \sum_{i=1}^4 \int \frac{1}{n} \partial_X^\gamma \Phi E_i^{(\gamma)} =: A_2^\gamma_i.\] (4.53)

**Estimate of** \(A_2^{(\gamma)}\) **in** (4.53). **By integration by parts**, we have
\[A_2^{(\gamma)} = -\int \frac{1}{n} \partial_X^\gamma \Phi \partial_T \partial_X^\gamma \Phi = -\frac{1}{2} \frac{d}{dt} \int \frac{1}{n} |\partial_X^\gamma \Phi|^2 + \frac{1}{2} \int \partial_T [\frac{1}{n} |\partial_X^\gamma \Phi|^2].\] (4.54)

Thank to Young’s inequality
\[\int |\partial_T [\frac{1}{n} |\partial_X^\gamma \Phi|^2| \leq C\varepsilon \|\partial_X^\gamma \Phi\|_2^2 + \varepsilon^3 \|\partial_T N\|_2^2 \|\partial_X^\gamma \Phi\|_2^2 + \varepsilon \|\partial_X^\gamma \Phi\|_L^\infty^2 \leq C_1(1 + \varepsilon^2)(|\langle V, \Phi \rangle| + \epsilon\|\Phi\|_{H^\gamma+1}^2),\]

**Estimate of** \(A_2^{(\gamma)}\) **in** (4.53). **By integration by parts**, we have
\[A_2^{(\gamma)} = \int \partial_X^\gamma \Phi \partial_T \partial_X^{\gamma+2} \Phi\]
\[= -\int \frac{\varepsilon^2}{n} \partial_X^{\gamma+1} \Phi \partial_T \partial_X^\gamma \Phi - \partial_X (\frac{\varepsilon^2}{n} |\partial_X^\gamma \Phi \partial_T \partial_X^{\gamma+1} \Phi| =: A_2^{(\gamma)} + A_2^{(\gamma)}\]

**First, estimate of** \(A_2^{(\gamma)}\). **Using Sobolev embedding and Lemma 4.3 and 4.4**, we get
\[A_2^{(\gamma)} = \frac{1}{2} \frac{d}{dt} \int \frac{\varepsilon^2}{n} |\partial_X^\gamma \Phi|^2 + \frac{1}{2} \int \partial_T [\varepsilon^2 \|\partial_X^\gamma \Phi\|_2^2 \leq \frac{1}{2} \frac{d}{dt} \int \frac{\varepsilon^2}{n} |\partial_X^{\gamma+1} \Phi|^2 + C\varepsilon^2 (1 + \varepsilon^2 \|\partial_T N\|_{L^\infty})(\epsilon\|\Phi\|_{H^\gamma+1}^2) \leq \frac{1}{2} \frac{d}{dt} \int \frac{\varepsilon^2}{n} |\partial_X^{\gamma+1} \Phi|^2 + C\varepsilon^2 (1 + \varepsilon^2)(|\langle V, \Phi \rangle| + \epsilon\|\Phi\|_{H^\gamma+1}^2).\]

**Then estimate for** \(A_2^{(\gamma)}\). **By integration by parts**
\[A_2^{(\gamma)} = \int \partial_X (\frac{\varepsilon^2}{n} |\partial_X^\gamma \Phi \partial_T \partial_X^{\gamma+1} \Phi|\]

Corollary 1. Let \((N, V, \Phi)\) be a solution to (3.3) for \(0 \leq \gamma \leq 2\), then

\[
\frac{1}{2} \frac{d}{dt} \|\partial_N^2 V\|^2 + \frac{1}{2} \frac{d}{dt} \left[ \int \frac{1 + \varepsilon \partial_N^2}{n} |\partial_N^2 V|^2 \right] + \int \frac{\varepsilon^2}{n} |\partial_N^2 \Phi|^2 
\leq C(1 + \varepsilon^2 |||(V, \Phi)|||^2_{H^1})(1 + |||(V, \Phi)|||^2_{H^2}) + \Theta^{(2)},
\]

where

\[
\Theta^{(2)} = \int \partial_N \left[ \frac{\varepsilon^2}{n} |\partial_N^2 \Phi|^2 \right].
\]
Proof. This comes from equation (4.25) with $\gamma = 2$.

The term $\Theta^{(2)}$ is very important for us to close the proof in our proof. Especially, the term $\Theta^{(3)}$ is not controllable in terms of $\| (V, \Phi) \|_\varepsilon$. We need a combination of $\Theta^{(2)}$ and $\varepsilon \Theta^{(3)}$, this is why we estimated the third order derivatives separately. $\square$

4.3. Third order estimates.

Proposition 2. Let $(N, V, \Phi)$ be a solution to (3.3), then

$$
\frac{1}{2} \frac{d}{dT} [\varepsilon^2 \| \partial_X^3 V \|^2] + \left[ \frac{1}{2} \frac{d}{dT} \int \frac{\varepsilon}{n} (1 + \varepsilon \partial_X) \| \partial_X^3 \Phi \|^2 + \int \frac{\varepsilon}{n} \| \partial_X^4 \Phi \|^2 \right] 
\leq C (1 + \varepsilon^2 \| (V, \Phi) \|_\varepsilon) (1 + \varepsilon \| V \|_{H^2}) + \Theta^{(3 \times \varepsilon)},
$$

(4.60)

where

$$
\Theta^{(3 \times \varepsilon)} = - \int \partial_X^3 \Phi \partial_X \left[ \frac{\varepsilon}{n} \right] \partial_T \partial_X^3 \Phi.
$$

(4.61)

Proof. Taking $\partial_X^3$ of (3.3b) and making inner product with $\varepsilon^2 \partial_X^3 V$, we have

$$
\frac{1}{2} \frac{d}{dT} [\varepsilon^2 \| \partial_X^3 V \|^2] = \int \partial_X^3 \Phi \partial_X \left[ \frac{\varepsilon}{n} \right] \partial_T \partial_X^3 \Phi + \int \varepsilon \| \partial_X^4 \Phi \|^2 
+ \varepsilon \int \partial_X^3 \Phi \partial_X \left[ \frac{\varepsilon}{n} \right] \partial_T \partial_X^3 \Phi = \int \varepsilon^2 \partial_X^3 \Phi \frac{\partial_T \partial_X^3 \Phi}{\varepsilon^2} =: \Phi'(\varepsilon). \tag{4.62}
$$

Estimate of the LHS of equation (4.62). The second term of the LHS of equation (4.62) will vanish by integration by parts. Firstly, estimate the third term by integration by parts, we get

$$
\int \partial_X^3 \Phi [\varepsilon \partial_X V] \partial_X^3 V 
= \int \varepsilon \partial_X^3 \Phi [\varepsilon \partial_X \varepsilon \partial_X^3 V] 
= 3 \int \varepsilon \partial_X [\varepsilon \partial_X^3 (\varepsilon \partial_X^3 V)] \partial_X^3 V + \sum_{\beta = 0, 2, 3} C_3^\beta \int \varepsilon \partial_X^3 [\varepsilon \partial_X^3 V] \partial_X^4 \beta V \partial_X^3 V =: H_{31}^{(3 \times \varepsilon)} + H_{32}^{(3 \times \varepsilon)}.
$$

Estimate for $H_{31}^{(3 \times \varepsilon)}$.

$$
H_{31}^{(3 \times \varepsilon)} \leq C (1 + \varepsilon^2 \| \partial_X V \|_{L^\infty}) (\varepsilon \| \partial_X^3 V \|)^2 \leq C (1 + \varepsilon^2 \| (V, \Phi) \|_\varepsilon) (\varepsilon \| \partial_X^3 V \|)^2. \tag{4.64}
$$

Estimate for $H_{32}^{(3 \times \varepsilon)}$. The case of the estimate for $\beta = 0, 3$ are similar to the estimate of $H_{31}^{(3 \times \varepsilon)}$. Now we estimate the case for $\beta = 2$.

$$
H_{32}^{(3 \times \varepsilon)} = C_3^2 \int \varepsilon \partial_X^3 (\varepsilon \partial_X^3 V)^2 \partial_X^3 V \partial_X^3 V \leq C (1 + \varepsilon^2 \| \partial_X^2 V \|_{L^\infty}) \varepsilon \| \partial_X^3 V \| \leq C (1 + \varepsilon^2 \| (V, \Phi) \|_\varepsilon) (\varepsilon \| V \|_{H^2}). \tag{4.65}
$$

By Lemma 3.1, the fourth and fifth terms of the LHS of equation (4.62) are bounded by

$$
\varepsilon (1 + \| V \|_{H^2}). \tag{4.66}
$$

From above proof, the last four term of the LHS of equation (4.62) are bounded by

$$
C (1 + \varepsilon^2 \| (V, \Phi) \|_\varepsilon) (1 + \varepsilon \| V \|_{H^2}). \tag{4.67}
$$
Now, we estimate $F^{(3\times \epsilon)}$ of equation (4.62) by decomposition. Taking $\partial_X^3$ of the equation (3.3a), we obtain

$$\frac{1}{\epsilon^2} \partial_X^3 V = \frac{1}{n} \frac{1 - v}{\epsilon^2} \partial_X^3 N - \partial_T \partial_X^3 N - \frac{1}{\epsilon^2} \sum_{\beta=1}^3 C^{\beta} \partial_X^3 v \partial_X^{4-\beta} N$$

$$- \frac{1}{\epsilon^2} \sum_{\beta=0}^3 C^{\beta} \partial_X^3 \epsilon \partial_X^{3-\beta} v - \frac{1}{\epsilon^2} \sum_{\beta=1}^3 C^{\beta} \partial_X^{4-\beta} v \partial_X^3 N - \frac{1}{\epsilon^2} \sum_{\beta=0}^3 C^{\beta} \partial_X^{4-\beta} v \partial_X^3 N$$

$$- \frac{\epsilon}{\epsilon^2} \partial_X^3 [v] =: K^{(3)}_i.$$

$F^{(3\times \epsilon)}$ can be decomposed into

$$F^{(3\times \epsilon)} = \frac{7}{n} \int \frac{1}{\epsilon^2} \partial_X^3 \Phi K^{(3)}_1 =: F^{(3\times \epsilon)}_1. \quad (4.68)$$

Estimate $F^{(3\times \epsilon)}$ for $3 \leq i \leq 7$. Now, estimate for $F^{(3\times \epsilon)}_3$ in (4.68).

$$F^{(3\times \epsilon)}_3 = - \sum_{\beta=1}^3 C^{\beta} \int \frac{1}{n} \partial_X^3 \Phi \partial_X^3 (\epsilon \partial_X^3 V) \partial_X^{4-\beta} N$$

$$= - \sum_{\beta=1}^3 C^{\beta} \int \epsilon \partial_X^3 \Phi \partial_X^3 \epsilon \partial_X^{4-\beta} N - \sum_{\beta=1}^3 C^{\beta} \int \epsilon \partial_X^3 \Phi \partial_X^3 \epsilon \partial_X^{4-\beta} N. \quad (4.69)$$

The first term of equation (4.69) is bilinear in $(V, N)$ and can be bounded by

$$C\epsilon \|\partial_X^3 \Phi\|^2 + C(\|N\|_{2H^2} + \epsilon \|N\|_{2H^2}). \quad (4.70)$$

The second term on the RHS of (4.69) for $\beta = 1, 2$, it is bounded by Lemma 4.3

$$- \sum_{\beta=1,2} \int \frac{\epsilon}{n} \partial_X^3 \Phi \partial_X^3 \epsilon \partial_X^{4-\beta} N \leq C\epsilon^2 \|\partial_X^3 V\|^2 (\epsilon^2 \|\partial_X^3 \Phi\|^2_{L^\infty} + C\epsilon^2 \|\partial_X^{4-\beta} N\|^2$$

$$\leq C_1 (1 + \epsilon^2 \|(V, \Phi)\|_{2}^2 (\epsilon^2 \|N\|_{H^2})).$$

For $\beta = 3$, using Sobolev embedding and by integration by parts, we have

$$- \int \frac{\epsilon}{n} \partial_X^3 \Phi \partial_X^3 \epsilon \partial_X N$$

$$= \int \partial_X \left[ \frac{\epsilon}{n} \partial_X^3 \Phi \partial_X^3 \epsilon \partial_X N \right] + \int \epsilon \partial_X^3 \Phi \partial_X^3 \epsilon \partial_X N + \int \epsilon \partial_X^3 \Phi \partial_X^3 \epsilon \partial_X N$$

$$\leq C(\epsilon \|\partial_X^3 \Phi\|(\epsilon \|\partial_X^3 V\|)(\epsilon \|\partial_X N\|_{L^\infty})(1 + \epsilon \|\partial_X N\|_{H^2})$$

$$\leq C_1 (1 + \epsilon^2 \|(V, \Phi)\|_{2}^2 (\epsilon^2 \|N\|_{H^2} + \epsilon^2 \|\Phi\|_{H^2}^2).$$

We get the completed estimates for $F^{(3\times \epsilon)}_3$. The case for $i = 4, 5, 6, 7$ can be bounded by the same bound.

In summary, we obtain

$$\sum_{i=3}^7 F^{(3\times \epsilon)}_i \leq C_1 (1 + \epsilon^2 \|(V, \Phi)\|_{2}^2 (1 + \|(V, \Phi)\|_{2}^2). \quad (4.71)$$
4.4. Estimate for $F_1^{(3\times\varepsilon)}$.

Lemma 4.6. Let $(N, V, \Phi)$ be a solution to (3.3), then

$$F_1^{(3\times\varepsilon)} \leq C_1(1 + \varepsilon^2 \|\| (V, \Phi)\|\|_2^2)(1 + \|\| (V, \Phi)\|\|_2^5).$$

(4.72)

where $F_1^{(3\times\varepsilon)}$ is defined in (4.68).

Proof. From equation (4.62), we know

$$F_1^{(3\times\varepsilon)} = \int \partial_X^4 \Phi \left[ \frac{1 - v}{n} \partial_X^4 N \right].$$

(4.73)

Taking $\partial_X^4$ of equation (3.3c), we get

$$\partial_X^4 N = \partial_X^4 \Phi - \varepsilon \partial_X^2 \Phi + \partial_X^3 (\varepsilon \partial_X \Phi - 2 \varepsilon \partial_X \Phi) + \varepsilon \partial_X \Phi R_3 =: f^{(\gamma)}_4.$$

(4.74)

The same procedure, $F_1^{(3\times\varepsilon)}$ can be divided as follows

$$F_1^{(3\times\varepsilon)} = \int \partial_X^4 \Phi \left[ \frac{1 - v}{n} f^{(\gamma)}_4 \right] = \sum_{i=1}^{4} F_1^{(3\times\varepsilon)}.$$

(4.75)

Now, firstly estimate for $F_1^{(3\times\varepsilon)}$ in (4.75). By integration by parts, we have

$$F_1^{(3\times\varepsilon)} = \int \partial_X^4 \Phi \left[ \frac{1 - v}{n} \partial_X^4 \Phi \right] = - \frac{1}{2} \int \partial_X \left[ \frac{1 - v}{n} \right] \| \partial_X^3 \Phi \|^2$$

$$\leq C \| \partial_X^3 \Phi \|^2 + C \varepsilon^2 (\| \partial_X N \|_{L^\infty} + \| \partial_X V \|_{L^\infty}) \| \partial_X^3 \Phi \|^2$$

$$\leq C_1 (1 + \varepsilon^2 \|\| (V, \Phi)\|\|_2^2) \| \partial_X^4 \Phi \|^2).$$

Estimate for $F_{12}^{(3\times\varepsilon)}$ in (4.75).

$$F_{12}^{(3\times\varepsilon)} = - \int \varepsilon^2 \partial_X^3 \Phi \left[ \frac{1 - v}{n} \partial_X^4 \Phi \right]$$

$$- \frac{3}{2} \int \varepsilon^2 \partial_X \left[ \frac{1 - v}{n} \right] \| \partial_X^3 \Phi \|^2 - \int \varepsilon^2 \partial_X^3 \Phi \partial_X^2 \left[ \frac{1 - v}{n} \right] \| \partial_X^4 \Phi \| =: F_{121}^{(3\times\varepsilon)} + F_{122}^{(3\times\varepsilon)}.$$

Estimate for $F_{121}^{(3\times\varepsilon)}$. Using Hölder inequality and Sobolev embedding, we have

$$F_{121}^{(3\times\varepsilon)} \leq C \varepsilon^2 \| \partial_X^3 \Phi \|^2 + C \varepsilon^2 (\| \partial_X V \|_{L^\infty} + \| \partial_X N \|_{L^\infty}) \| \partial_X^3 \Phi \|^2$$

$$\leq C (1 + \varepsilon^2 \|\| (V, \Phi)\|\|_2^2) \| \partial_X^4 \Phi \|^2).$$

(4.76)

Estimate for $F_{122}^{(3\times\varepsilon)}$. Using equation (4.47), Sobolev embedding and Lemma 4.3, we have

$$\int \varepsilon^2 | \partial_X^3 \Phi | | \partial_X^4 \Phi | \leq C(\varepsilon \| \partial_X^3 \Phi \|^2 + \varepsilon^2 \| \partial_X^4 \Phi \|^2),$$

(4.77)

by Sobolev embedding, and Lemma 4.1, we get

$$\int \varepsilon^2 \partial_X^3 \Phi (| \partial_X V | + | \partial_X N |) \partial_X^4 \Phi$$

$$\leq C \varepsilon^2 (\| \partial_X V \|^2 + \| \partial_X N \|^2 \| \partial_X^3 \Phi \|_{L^\infty}^2) + \varepsilon^2 \| \partial_X^4 \Phi \|^2$$

$$\leq C (1 + \varepsilon^2 \|\| (V, \Phi)\|\|_2^2) \| \partial_X^4 \Phi \|^2,$$

(4.78)

and hence

$$\int \varepsilon^2 \partial_X^3 \Phi (| \partial_X N | + | \partial_X V | + \varepsilon^2 (| \partial_X N |^2 + | \partial_X V |^2)) \partial_X^4 \Phi$$

$$\leq C \varepsilon^2 (| \partial_X N |^2 + \| \partial_X V \|^2) \| \partial_X^4 \Phi \|^2,$$

(4.79)
Proof. Estimate for $F_{12}^{(3\times e)}$ we obtain

$$F_{12}^{(3\times e)} \leq C(1 + \varepsilon^2 \||| (V, \Phi)|||_e^2)(1 + \varepsilon \||| \Phi|||_{H^1}^2 + \varepsilon^2 \||| \Phi|||_{H^4}^2).$$  \hspace{1cm} (4.79)

Summary equations from (4.77)-(4.79), we get

$$F_{13}^{(3\times e)} \leq C(1 + \varepsilon^2 \||| (V, \Phi)|||_e^2)(1 + \varepsilon \||| \Phi|||_{H^1}^2 + \varepsilon^2 \||| \Phi|||_{H^4}^2).$$  \hspace{1cm} (4.80)

Estimate of $F_{14}^{(3\times e)}$ in (4.75). The estimate is similar to $F_{11}^{(3\times e)}$,

$$\int \partial_X^3 \Phi \frac{1 - v}{n} \partial_X^3 \Phi = - \int \varepsilon \partial_X^3 \Phi \frac{1 - v}{n} \partial_X^3 \Phi \leq C(1 + \varepsilon \||| (V, \Phi)|||_e^2)(1 + \varepsilon \||| \Phi|||_{H^1}^2 + \varepsilon^2 \||| \Phi|||_{H^4}^2)$$

Estimate of $F_{14}^{(3\times e)}$ in (4.75).

$$F_{14}^{(3\times e)} = \int \varepsilon \partial_X^3 \Phi \frac{1 - v}{n} \partial_X^3 \Phi = - \int \partial_X^3 \Phi \partial_X \frac{\varepsilon}{n} \partial_X^3 \Phi = \Theta^{(3\times e)},$$

where

$$\Theta^{(3\times e)} = - \int \partial_X^3 \Phi \partial_X \frac{\varepsilon}{n} \partial_X N.$$  \hspace{1cm} (4.82)

Proof. The same to equation (4.59).

Taking $\partial_X \partial_X^3$ of equation (3.3c) and then do decomposition for $F_{2}^{(3\times e)}$, we have

$$F_{2}^{(3\times e)} = - \int \frac{\varepsilon}{n} \partial_X^3 \Phi \partial_X \frac{\varepsilon}{n} \partial_X^3 \Phi = \sum_{i=1}^{4} F_{2i}^{(3\times e)}. \hspace{1cm} (4.83)$$

Estimate for $F_{2i}^{(3\times e)}$ in (4.83). By integration by parts and using Sobolev embedding we obtain

$$F_{21}^{(3\times e)} = - \int \frac{\varepsilon}{n} \partial_X^3 \Phi \partial_X \frac{\varepsilon}{n} \partial_X^3 \Phi = - \frac{1}{2} \frac{d}{dT} \int \frac{\varepsilon}{n} \partial_X^3 \Phi \partial_X^3 \Phi + \frac{1}{2} \int \partial_X \frac{\varepsilon}{n} \partial_X^3 \Phi \partial_X^3 \Phi$$

$$\leq - \frac{1}{2} \frac{d}{dT} \int \frac{\varepsilon}{n} \partial_X^3 \Phi \partial_X^3 \Phi + C \varepsilon \frac{1}{2} \||| \Phi \|||_e^2 + C \varepsilon \frac{1}{2} \||| \Phi \|||_{H^1}^2$$

Estimate for $F_{22}^{(3\times e)}$ in (4.83). By integration by parts, we have

$$F_{22}^{(3\times e)} = \int \partial_X^3 \Phi \frac{\varepsilon}{n} \partial_X \partial_X^3 \Phi = - \int \partial_X^3 \Phi \frac{\varepsilon}{n} \partial_X \partial_X^3 \Phi - \int \partial_X^3 \Phi \partial_X \frac{\varepsilon}{n} \partial_X \partial_X^3 \Phi = F_{221}^{(3\times e)} + \Theta^{(3\times e)}.$$  \hspace{1cm} (4.84)
Estimate for the first term $F^{(3\times e)}_{221}$ in (4.84), we have
\[
F^{(3\times e)}_{221} = - \frac{1}{2} \frac{d}{dT} \int \left[ \frac{\varepsilon}{n} \right] \partial_X^2 \Phi^2 + \frac{1}{2} \frac{d}{dT} \int \partial_T \left[ \frac{\varepsilon}{n} \right] \partial_X^2 \Phi^2
\leq - \frac{1}{2} \frac{d}{dT} \int \left[ \frac{\varepsilon}{n} \right] \partial_X^2 \Phi^2 + C_1 (1 + |||(V, \Phi)|||_\varepsilon)(\varepsilon^2 \|\partial_X^2 \Phi\|^2).
\]
The term $\Theta^{(3\times e)}$ cannot be controlled by $|||(V, \Phi)|||_\varepsilon$ so far, so we need $\Theta^{(2)}$ to estimate it.

Estimate for $F^{(3\times e)}_{23}$ in (4.83), we have
\[
F^{(3\times e)}_{23} \leq - \frac{1}{2} \frac{d}{dT} \int \left[ \frac{\varepsilon^2}{n} \right] \partial_X^3 \Phi + C\varepsilon^2 (1 + \varepsilon^2 |||(V, \Phi)|||_\varepsilon)(1 + |||(V, \Phi)|||_\varepsilon). \tag{4.85}
\]

Estimate for $F^{(3\times e)}_{24}$ in (4.83), we have
\[
F^{(3\times e)}_{24} = - \int \frac{\varepsilon^2}{n} \partial_X^3 \Phi \partial_T \partial_X^2 \Phi = \int \frac{\varepsilon^2}{n} \partial_X^3 \Phi \partial_T \partial_X^2 \Phi + \int \partial_X \left[ \frac{\varepsilon^2}{n} \right] \partial_X^3 \Phi \partial_T \partial_X^2 \Phi, \tag{4.86}
\]
by Lemma 3.1 we know
\[
\|\varepsilon \partial_T \partial_X^2 \Phi \|^2 \leq C(||\phi^{(i)}||_{H^3}, \varepsilon \|\Phi\|_{H^4})(\|\varepsilon \partial_T \Phi\|^2_{H^4}), \tag{4.87}
\]
due to Sobolev embedding and Lemma 4.2
\[
\|\partial_X \left( \frac{1}{n} \right) \|^2_{L^\infty} \leq C(1 + \varepsilon^2 |||(V, \Phi)|||_\varepsilon), \tag{4.88}
\]
therefore,
\[
F^{(3\times e)}_{24} \leq C(||\phi^{(i)}||_{H^3}, \varepsilon \|\Phi\|_{H^4})(\|\varepsilon \partial_T \Phi\|^2_{H^4}) + \varepsilon \|\Phi\|^2_{H^4}
\leq C(1 + \varepsilon^2 |||(V, \Phi)|||_\varepsilon)(1 + |||(V, \Phi)|||_\varepsilon).
\]

Now, estimate for $\Theta^{(3\times e)}$. By integration by parts for twice, we have
\[
\int \partial_X^3 \Phi \partial_X \left[ \frac{\varepsilon}{n} \right] \partial_T \partial_X^2 \Phi
\]
\[
= \int \partial_X^3 \Phi \partial_X \left[ \frac{\varepsilon}{n} \right] \partial_T \partial_X^2 \Phi + 2 \int \partial_X^4 \Phi \partial_X^2 \left[ \frac{\varepsilon}{n} \right] \partial_T \partial_X \Phi + \int \partial_X^4 \Phi \partial_X^2 \left[ \frac{\varepsilon}{n} \right] \partial_T \partial_X \Phi,
\]
by Lemma 4.4 we find $\|\partial_T \partial_X^2 \Phi\|_{L^2}$ can be controlled by $|||(V, \Phi)|||_\varepsilon$, through $\|\partial_T \Phi\|^2_{H^4}$, however the term $\int \partial_X^4 \Phi \partial_X \left[ \frac{\varepsilon}{n} \right] \partial_T \partial_X \Phi$ can not be controllable because of $\partial_X^2 \Phi$. Due to the same structure $\Theta^{(2)}$, we can put $\Theta^{(2)}$ and $\Theta^{(3\times e)}$ together,
\[
\Theta^{(2,e)} = \Theta^{(2)} + \Theta^{(3\times e)} = \int \partial_X \left[ \frac{\varepsilon^2}{n} \right] \partial_X^3 \Phi \partial_T \partial_X^2 \Phi, \tag{4.89}
\]
obsc {}, it can be controlled. \hfill \square

Proposition 3. Let $(N, V, \Phi)$ be a solution to (3.3), then
\[
\Theta^{(2,e)} \leq C(1 + \varepsilon^2 |||(V, \Phi)|||_\varepsilon)^2(1 + |||(V, \Phi)|||_\varepsilon^2), \tag{4.90}
\]
where $|||(V, \Phi)|||_\varepsilon$ is defined in (3.5).

Proof. By equation (3.3c), we have
\[
\Theta^{(2,e)} = \int \partial_X \left[ \frac{\varepsilon^2}{n} \right] \partial_X^3 \Phi \partial_T \partial_X^2 \Phi = \int \partial_X \left[ \frac{\varepsilon^2}{n} \right] \partial_X^3 \Phi \partial_T \partial_X^2 (N + (2b \frac{\varepsilon^2}{n} \Phi - \varepsilon \partial_R \Phi) - \varepsilon \partial_R \Phi) =: \Theta^{(2,e)}_i. \tag{4.91}
\]
Estimate for $\mathcal{H}_1^{(2,e)}$ in (4.91). By integration by parts, we have

$$
\mathcal{H}_1^{(2,e)} = \int \partial_X\left[\frac{\varepsilon^2}{n}\partial_X^2 \Phi \partial_X N\right] = -\int \partial_X\left[\frac{\varepsilon^2}{n}\partial_X^2 \Phi \partial_X N\right] - \int \partial_X\left[\frac{\varepsilon^2}{n}\partial_X^2 \Phi \partial_X N\right] =: \mathcal{H}_1^{(2,1,e)} + \mathcal{H}_1^{(2,2,e)}.
$$

From the expression of $n$ in (2.3), we have

$$
|\partial_X\left(\frac{\varepsilon^2}{n}\right)| \leq C\varepsilon^2 (|\partial_X N| + \varepsilon^3 |\partial_X N|), \quad (4.92)
$$

and

$$
|\partial_X^2\left(\frac{\varepsilon^2}{n}\right)| \leq C\varepsilon^2 (\varepsilon + \varepsilon^3 |\partial_X^2 N| + \varepsilon^4 |\partial_X N| + \varepsilon^6 |\partial_X N|^2). \quad (4.93)
$$

Estimate for $\mathcal{H}_1^{(2,1,e)}$.

$$
\mathcal{H}_1^{(2,1,e)} \leq C(\varepsilon^2 \varepsilon |\partial_X^2 \Phi|) + C(\varepsilon^2 |\partial_X N|) + C(\varepsilon^2 \varepsilon |\partial_X^2 \Phi|) = C(\varepsilon^2 \varepsilon |\partial_X^2 \Phi|) + C(\varepsilon^2 \varepsilon |\partial_X N|) \quad (4.94)
$$

$$
\leq C(\varepsilon^2 \varepsilon \|\partial_X^2 \Phi\| + \varepsilon^2 \|\partial_X N\|)(1 + \varepsilon \|\partial_X N\|) \leq C(1 + \varepsilon \|\partial_X N\|)(1 + \|\partial_X N\|).
$$

Estimate for $\mathcal{H}_1^{(2,2,e)}$.

Firstly,

$$
\varepsilon^2 \int |\partial_X^3 \Phi \partial_T X N| \leq C \varepsilon \|\partial_X^2 \Phi\|^2 + C \|\partial_X N\|^2 \leq C(1 + \|\partial_X N\|), \quad (4.95)
$$

Secondly,

$$
\varepsilon^2 \int |\partial_X^3 \Phi \partial_T X N| \leq C \varepsilon^2 \|\partial_X^2 \Phi\|^2 + C \varepsilon^2 \|\partial_X N\|^2 \leq C(1 + \varepsilon \|\partial_X N\|)(1 + \varepsilon \|\partial_X N\|), \quad (4.96)
$$

Finally,

$$
\varepsilon^2 \int |\partial_X^3 \Phi \partial_T X N| \leq C(1 + \varepsilon \|\partial_X N\|)(1 + \|\partial_X N\|), \quad (4.97)
$$

Summarizing equations (4.95)-(4.97), we have

$$
\mathcal{H}_1^{(2,e)} \leq C(1 + \varepsilon \|\partial_X N\|)(1 + \|\partial_X N\|). \quad (4.98)
$$

By equations (4.94) and (4.98), we have

$$
\mathcal{H}_1^{(2,e)} \leq C(1 + \varepsilon \|\partial_X N\|)(1 + \|\partial_X N\|). \quad (4.99)
$$

Estimate for $\mathcal{H}_2^{(2,e)}$ in (4.91).

$$
\mathcal{H}_2^{(2,e)} = \int \partial_X\left[\frac{\varepsilon^2}{n}\partial_X^2 \Phi \partial_X N [2\varepsilon^2 \partial_X^2 \Phi - \varepsilon \partial_X^3 \Phi]\right] \quad (4.100)
$$

$$
\leq C(1 + \varepsilon \|\partial_X N\|)(\varepsilon \|\partial_X^2 \Phi\|^2) + C \varepsilon^2 (\varepsilon \|\partial_X^2 \Phi\|^2) \leq C(1 + \varepsilon \|\partial_X N\|)(1 + \|\partial_X N\|). \quad (4.101)
$$

Estimate for $\mathcal{H}_3^{(2,e)}$ in (4.91).

$$
\mathcal{H}_3^{(2,e)} \leq C(1 + \varepsilon \|\partial_X N\|)(1 + \|\partial_X N\|). \quad (4.101)
$$
4.6. Proof for theorem 3.2.

Proof. Combining Proposition 1, 2, 3 together, we get
\[ \frac{1}{2} \frac{d}{dt} \left( |V|^2_{H^2} + \varepsilon \| \partial_X^2 V \|_{L^2}^2 \right) + \frac{1}{2} \frac{d}{dt} \left( \int \frac{1 + \varepsilon^2 \phi}{n} |\Phi|^2 + \int \frac{\varepsilon^2}{n} |\partial_X^2 \Phi|^2 \right) \quad (4.102) \]
\[ + \left( \int \frac{1 + \varepsilon^2 \phi}{n} |\partial_X \Phi|^2 + \int \frac{\varepsilon^2}{n} |\partial_X^2 \Phi|^2 \right) + \left( \int \frac{1 + \varepsilon^2 \phi}{n} |\partial_X^2 \Phi|^2 + \int \frac{\varepsilon^2}{n} |\partial_X^2 \Phi|^2 \right) \]
\[ \left( \int \frac{\varepsilon^2}{n} (1 + \varepsilon \tilde{\phi}) |\partial_X^2 \Phi|^2 + \int \frac{\varepsilon}{n} |\partial_X \Phi|^2 \right) \leq C_1 (1 + \varepsilon^2 |(V, \Phi)|^2) + (1 + |(V, \Phi)|^2). \]

Because \( \tilde{\phi} \) is consisted of \( \phi^{(1)}, \phi^{(2)}, \phi^{(3)}, \phi^{(4)}, \phi^{(5)}, \phi^{(6)} \), which are all uniformly bounded, for \( \varepsilon < \varepsilon_1, 1 + \varepsilon^2 \tilde{\phi} > \frac{1}{2} \), there exists some \( \varepsilon_1 > 0 \). Integrating the inequality (4.102) over \( (0, t) \) yields
\[ |(V, \Phi)(t)||^2 \leq C |(V, \Phi)(0)||^2 + \int_0^t C_1 (1 + \varepsilon^2 |(V, \Phi)||^2) (1 + |(V, \Phi)(t)||^2) ds \]
\[ \leq C |(V, \Phi)(0)||^2 + \int_0^t C_1 (1 + \varepsilon \tilde{C}) (1 + |(V, \Phi)||^2) ds, \quad (4.103) \]

where \( C \) is an absolute constant and \( \tilde{C} \geq \varepsilon |(V, \Phi)||^2 \).

Recall that \( C_1 \) depends on \( |(V, \Phi)||^2 \) through \( \varepsilon |(V, \Phi)||^2 \) and is nondecreasing. Let \( C_2 \geq C \sup_{\varepsilon < 1} |(V, \Phi)(0)||^2 \) and \( C_3 \) \( |(V, \Phi)||^2 \) \( C_1 (1 + \varepsilon \tilde{C}) (1 + |(V, \Phi)||^2) \). For any \( \tau > 0 \), we choose \( C_0 > 2C_1 (1 + C_3 \varepsilon \tilde{C} \tau) \).

Then there exists \( \varepsilon_0 > 0 \) such that \( \varepsilon C_0 \leq 1 \) for all \( \varepsilon < \varepsilon_0 \), we have
\[ \sup_{0 \leq t \leq \tau} |(V, \Phi)||^2 \leq C_2 (1 + C_3 \varepsilon \tilde{C} \tau) \leq \frac{C_0}{2}. \quad (4.104) \]

We know \( (V, \Phi) \) is uniform bound.
\[ \sup_{0 \leq t \leq \tau} |(V, \Phi)(t)||^2 + \varepsilon \| \partial_X^2 (V, \Phi)(t) \|_2^2 + \varepsilon \| \partial_X^2 \Phi(t) \|_2^2 \leq \frac{C_0}{2}. \quad (4.105) \]

Besides, due to Lemma 4.2, we have
\[ \sup_{0 \leq t \leq \tau} \| \phi \|_{H^2} \leq \frac{C_0}{2}. \quad (4.106) \]

Now, we complete the standard uniform estimates independent of \( \varepsilon \) by the continuity method. \( \square \)

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