A new second-order midpoint approximation formula for Riemann-Liouville derivative: algorithm and its application*

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Abstract

Compared to the classical first-order Grünwald-Letnikov formula at time \( t_{k+1} \) (or \( t_k \)), we firstly propose a second-order numerical approximate scheme for discretizing the Riemann-Liouville derivative at time \( t_{\frac{k+1}{2}} \), which is very suitable for constructing the Crank-Nicolson technique applied to the time-fractional differential equations. The established formula has the following form

\[
\mathrm{RLD}_{0,t}^\alpha u(t) \big|_{t=\frac{k+1}{2}} = \tau^{-\alpha} \sum_{\ell=0}^{k} \omega_{\ell}^{(\alpha)} u(t_k - \ell\tau) + \mathcal{O}(\tau^2), \quad k = 0, 1, \ldots, \alpha \in (0, 1),
\]

where the coefficients \( \omega_{\ell}^{(\alpha)} (\ell = 0, 1, \ldots, k) \) can be determined via the following generating function

\[
G(z) = \left( \frac{3\alpha + 1}{2\alpha} - \frac{2\alpha + 1}{\alpha} z + \frac{\alpha + 1}{2\alpha} z^2 \right)^\alpha, \quad |z| < 1.
\]

Applying this formula to the time fractional Cable equations with Riemann-Liouville derivative in one or two space dimensions. Then the high-order compact finite difference schemes are obtained. The solvability, stability and convergence with orders \( \mathcal{O}(\tau^2 + h^4) \) and \( \mathcal{O}(\tau^2 + h_x^4 + h_y^4) \) are shown, where \( \tau \) is the
temporal stepsize and $h, h_x, h_y$ are the spatial stepsizes, respectively. Finally, numerical experiments are provided to support the theoretical analysis.

**Key words:** Riemann-Liouville derivative; generating function; the energy method.

## 1 Introduction

In recent years, a great deal of attention has been focused on fractional differential equations due to well describing many physical processes and phenomena [2, 3, 18, 19, 20, 26]. Very limited analytical methods, such as the Fourier transform method, the Laplace transform method and the Green function method are used to solve the very special fractional differential equations. So to seek numerical methods is the center task for studies of fractional differential equations [1, 5, 6, 7, 11, 15, 24, 28, 30, 31, 33]. In the history of numerical methods for fractional differential equations, Liu et al. [14], and Meerschaert and Tadjeran [21] are the first ones that developed the finite difference methods for fractional partial differential equations. The Galerkin finite element methods for fractional partial differential equations is proposed by Ervin and Roop, for the stationary space fractional partial differential equations with two-sided Riemann-Liouville derivatives. They first presented a rigorous analysis of the well-posedness of the weak formulation in the framework of fractional Sobolev spaces [7].

Generally speaking, one of key issues of approximating fractional differential equations is how to numerically discretise the fractional derivatives. Although there have existed some studies on numerical approximations of fractional integrals and fractional derivatives, high-order scheme for time fractional derivatives have not been thoroughly solved. This paper aims to construct new and effective the second-order mid-point approximate formula for time Riemann-Liouville derivative. Then the established scheme is applied to time fractional Cable equations in one and two space dimensions. From bibliography available, there have existed numerical studies for the fractional Cable equations. For example, Langlans et al. [16] developed two implicit finite difference schemes with convergence orders $O(\tau + h^2)$ and $O(\tau^2 + h^2)$. Hu and Zhang proposed two implicit compact difference schemes, where the first scheme was proved to be stable and convergent with order $O(\tau + h^4)$ by the energy method [9]. In [22], Quintana-Murillo and Yuste constructed an explicit numerical scheme for fractional Cable equation which includes two temporal Riemann-Liouville derivatives, where they showed the stability and convergence conditions by using
the Von Neumann method. Zhuang et al. [33] considered the one-dimensional time fractional Cable equation by using the Galerkin finite element method, in which the proposed method was based on a semi-discrete finite difference approximation in time and Galerkin finite element method in space. The spectral method for fractional Cable equation was discussed by Lin et al. [17], where the detailed theoretical analysis was provided. As far as we know, the computational efficiency for time fractional Cable equation is not high yet. Besides, the high-dimensional time fractional Cable equations seen not to be studied. Here, we study the fractional cable equation in two space dimensions where the fractional derivative is approximated by the derived method in this paper. The unconditional stability and convergence of the established numerical algorithms are presented by the energy method.

The reminder of the paper is constructed as follows. In Section 2, we establish a new second-order approximation formula for Riemann-Liouville derivatives. Then two high-order finite difference schemes for the fractional Cable equations in one and two space dimensions are proposed in Sections 3 and 4, respectively. Numerical experiments are displayed in Section 5, where are in line with the theoretical analysis. Remarks and conclusions are included in the last section.

2 Second-order scheme for Riemann-Liouville derivative

In the section, we propose a new second-order approximation formula for computing Riemann-Liouville derivatives.

**Definition 2.1** [23, 27] Let $-\infty \leq a < t < b \leq \infty$. For given $u \in L_1([a,b])$, its Riemann-Liouville derivative of order $\alpha > 0$ with lower limit $a$ is defined as follows

$$
\text{RLD}_{a,t}^\alpha u(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{u(s)}{(t-s)^{\alpha-n+1}} ds, \ n-1 < \alpha \leq n \in \mathbb{Z}^+.
$$

**Lemma 2.1** [23, 27] Suppose $\alpha > 0$, $u(t) \in L_1(\mathbb{R})$. Then the Fourier transform of $\alpha$-th Riemann-Liouville derivative of $u(t)$ is

$$
\mathcal{F} \{ \text{RLD}_{-\infty, t}^\alpha u(t) \} (\omega) = (-i\omega)^\alpha \hat{u}(\omega),
$$

where $\hat{u}(\omega) = \int_{\mathbb{R}} e^{i\omega t} u(t) dt$ denotes the Fourier transform of $u(t)$.

Now, we start to develop the second-order numerical approximation formula.

**Theorem 2.1** Denote

$$
\mathbb{C}^{n+\alpha}(\mathbb{R}) = \left\{ u | u \in L_1(\mathbb{R}), \int_{\mathbb{R}} (1 + |\omega|)^{n+\alpha} |\hat{u}(\omega)| d\omega < \infty \right\}.
$$
Define the following difference operator
\[
\delta_{-\infty, \tau}^\alpha u(t) = \tau^{-\alpha} \sum_{\ell=0}^{\infty} \varpi_{\ell}^{(\alpha)} u(t - \ell\tau),
\]
where \(\tau\) is temporal stepsize. If \(u \in \mathcal{C}^{2+\alpha}(\mathbb{R})\), then one has
\[
\mathcal{R} \mathcal{D}_{-\infty, \tau}^\alpha u \left( t + \frac{1}{2} \right) = \delta_{-\infty, \tau}^\alpha u(t) + \mathcal{O}(\tau^2)
\]
as \(\tau \to 0\).
Here \(\varpi_{\ell}^{(\alpha)} (\ell = 0, 1, \ldots)\) are the expansion coefficients of \(G(z)\), that is,
\[
G(z) = \sum_{\ell=0}^{\infty} \varpi_{\ell}^{(\alpha)} z^\ell, \ |z| < 1.
\]

Proof. Taking the Fourier transform on both sides of equation (1) then combining with equation (3), one has
\[
\mathcal{F}\{\delta_{-\infty, t}^\alpha u(t)\}(\omega) = \tau^{-\alpha} \sum_{\ell=0}^{\infty} \varpi_{\ell}^{(\alpha)} e^{i\omega \ell \tau} \hat{u}(\omega)
\]
\[
= \tau^{-\alpha} \hat{u}(\omega) \sum_{\ell=0}^{\infty} \varpi_{\ell}^{(\alpha)} e^{i\omega \ell \tau}
\]
\[
= (-i\omega)^\alpha e^{-\frac{i\omega\tau}{2}} S(i\omega \tau) \hat{u}(\omega),
\]
where \(S(z) = (-z)^{-\alpha} e^{\frac{z}{2}} G(e^z)\).

Note that
\[
G(e^z) = \left( \frac{3\alpha + 1}{2\alpha} - \frac{2\alpha + 1}{\alpha} e^z + \frac{\alpha + 1}{2\alpha} e^{2z} \right)^\alpha
\]
\[
= (-z)^\alpha \left( 1 - \frac{1}{2\alpha} z - \frac{2\alpha + 3}{6\alpha} z^2 + \ldots \right)^\alpha
\]
\[
= (-z)^{-\alpha} \left( 1 - \frac{1}{2} z - \frac{8\alpha^2 + 9\alpha + 3}{24\alpha} z^2 + \ldots \right)
\]
and
\[
S(z) = (-z)^{-\alpha} e^{\frac{z}{2}} G(e^z) = \left( 1 + \frac{z}{2} + \frac{z^2}{8} + \ldots \right) \left( 1 - \frac{1}{2} z - \frac{8\alpha^2 + 9\alpha + 3}{24\alpha} z^2 + \ldots \right)
\]
\[
= 1 - \frac{8\alpha^2 + 12\alpha + 3}{24\alpha} z^2 + \mathcal{O}(|z|^3).
\]
So, there exists a constant $c_1 > 0$ such that $|1 - S(i\omega\tau)| \leq c_1|\omega\tau|^2$.

Then
\[
\mathcal{F}\left\{\delta_{-\infty,t}^\alpha u(t)\right\}(\omega) = (-i\omega)^\alpha e^{-i\frac{\pi}{2}} \hat{u}(\omega) + (-i\omega)^\alpha e^{-i\frac{\pi}{2}} (S(i\omega\tau) - 1) \hat{u}(\omega)
\]
\[
= \mathcal{F}\left\{\text{RLD}_{-\infty}^\alpha u\left(t + \frac{1}{2}\tau\right)\right\}(\omega) + \hat{\varphi}(\tau, \omega)
\]
by using Lemma 2.1. Hence, one has
\[
|\varphi(\tau, \omega)| = \left|\frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{-i\omega\tau} \hat{\varphi}(\tau, \omega) d\omega\right| \leq \frac{c_1 \tau^2}{2\pi} \int_{-\infty}^{\infty} (1 + |\omega|)^{2+\alpha} |\hat{u}(\omega)| d\omega = O(\tau^2).
\]
All this ends the proof. ■

Remark: If $u(t)$ is suitably smooth and has compact support for $0 < t < T$, then Riemann-Liouville derivative $\text{RLD}_{-\infty}^\alpha u(t)$ coincides with $\text{RLD}_{0,t}^\alpha u(t)$. Denote
\[
\delta_{0,t}^\alpha u(t) = \tau^{-\alpha} \sum_{\ell=0}^{[\frac{t}{\tau}]} \omega_{\ell}^{(\alpha)} u(t - \ell\tau).
\]
Then the corresponding numerical approximation formula (2) is reduced to
\[
\text{RLD}_{0,t}^\alpha u(t) \big|_{t=t_k+\frac{1}{2}} = \delta_{0,t}^\alpha u(t_k) + O(\tau^2), \ k = 0, 1, \ldots,
\]
that is,
\[
\text{RLD}_{0,t}^\alpha u(t) \big|_{t=t_k+\frac{1}{2}} = \tau^{-\alpha} \sum_{\ell=0}^{k} \omega_{\ell}^{(\alpha)} u(t_k - \ell\tau) + O(\tau^2), \ k = 0, 1, \ldots
\]
the coefficients $\omega_{\ell}^{(\alpha)} (\ell = 0, 1, \ldots)$ in equation (3) can be expressed as follows,
\[
\omega_{\ell}^{(\alpha)} = \left(\frac{3\alpha + 1}{2\alpha}\right)^\alpha \sum_{m=0}^{\ell} \left(\frac{\alpha + 1}{3\alpha + 1}\right)^m g_m^{(\alpha)} g_{\ell-m}^{(\alpha)}, \ \ell = 0, 1, \ldots
\]
where
\[
g_m^{(\alpha)} = (-1)^m \left(\frac{\alpha}{m}\right) = (-1)^m \frac{\Gamma(1 + \alpha)}{\Gamma(m + 1)\Gamma(1 + \alpha - m)}, \ m = 0, 1, \ldots, \ell.
\]
For convenience, take the place of $\alpha$ in $\omega_{\ell}^{(\alpha)}$ by $1 - \alpha$ which can not cause confusion, where $\alpha \in (0, 1)$. It is easy to get the following theorem.
Theorem 2.2 The coefficients \( \omega^{(1-\alpha)}_\ell, \ell = 0, 1, \ldots \) can be computed recursively by the formulas.

\[
\begin{align*}
\omega^{(1-\alpha)}_0 & = \left( \frac{4 - 3\alpha}{2(1-\alpha)} \right)^{1-\alpha} ; \\
\omega^{(1-\alpha)}_1 & = \frac{2(\alpha - 1)(3 - 2\alpha)}{4 - 3\alpha} \omega^{(1-\alpha)}_0 ; \\
\omega^{(1-\alpha)}_\ell & = \frac{1}{(4 - 3\alpha)\ell} \left[ 2(\ell + \alpha - 2)(3 - 2\alpha)\omega^{(1-\alpha)}_{\ell-1} + (2 - \alpha)(4 - 2\alpha - \ell)\omega^{(1-\alpha)}_{\ell-2} \right], \ell \geq 2.
\end{align*}
\]

Theorem 2.3 The coefficients \( \omega^{(1-\alpha)}_\ell \) are nonpositive if \( \ell \geq 5 \) where \( \alpha \in (0, 1) \), that is, \( \omega^{(1-\alpha)}_\ell \leq 0 \) if \( \ell \geq 5 \), where \( \alpha \in (0, 1) \).

Proof. For \( 0 < \alpha < 1 \) and \( \ell \geq 5 \), one has

\[
\begin{align*}
\omega^{(1-\alpha)}_\ell & = \left( \frac{4 - 3\alpha}{2(1-\alpha)} \right)^{1-\alpha} \sum_{m=0}^{\ell} \left( \frac{2 - \alpha}{4 - 3\alpha} \right)^m g^{(1-\alpha)}_m g^{(1-\alpha)}_{\ell-m} \\
& = \left\{ \left[ 1 + \left( \frac{2 - \alpha}{4 - 3\alpha} \right)^\ell \right] g^{(1-\alpha)}_0 g^{(1-\alpha)}_\ell + \left[ \frac{2 - \alpha}{4 - 3\alpha} + \left( \frac{2 - \alpha}{4 - 3\alpha} \right)^{\ell-1} \right] g^{(1-\alpha)}_1 g^{(1-\alpha)}_{\ell-1} \\
& \quad + \sum_{m=2}^{\ell-2} \left( \frac{2 - \alpha}{4 - 3\alpha} \right)^m g^{(1-\alpha)}_m g^{(1-\alpha)}_{\ell-m} \right\} \left( \frac{4 - 3\alpha}{2(1-\alpha)} \right)^{1-\alpha} \\
& \leq \left\{ \left[ 1 + \left( \frac{2 - \alpha}{4 - 3\alpha} \right)^\ell \right] g^{(1-\alpha)}_0 g^{(1-\alpha)}_\ell + \left[ \frac{2 - \alpha}{4 - 3\alpha} + \left( \frac{2 - \alpha}{4 - 3\alpha} \right)^{\ell-1} \right] g^{(1-\alpha)}_1 g^{(1-\alpha)}_{\ell-1} \\
& \quad + g^{(1-\alpha)}_2 g^{(1-\alpha)}_{\ell-2} \sum_{m=2}^{\infty} \left( \frac{2 - \alpha}{4 - 3\alpha} \right)^m \left( \frac{4 - 3\alpha}{2(1-\alpha)} \right)^{1-\alpha} \\
& = \left\{ \left[ 1 + \left( \frac{2 - \alpha}{4 - 3\alpha} \right)^\ell \right] \left( 1 - \frac{2 - \alpha}{\ell} \right) + \left[ \frac{2 - \alpha}{4 - 3\alpha} + \left( \frac{2 - \alpha}{4 - 3\alpha} \right)^{\ell-1} \right] (\alpha - 1) \right\} \\
& \quad - \frac{\alpha(\ell - 1)(2 - \alpha)^2}{4(4 - 3\alpha)(\alpha + \ell - 3)} \left( \frac{4 - 3\alpha}{2(1-\alpha)} \right)^{1-\alpha} g^{(1-\alpha)}_{\ell-1} \\
& \leq \left\{ \left[ 1 + \left( \frac{2 - \alpha}{4 - 3\alpha} \right)^\ell \right] \left( 1 - \frac{2 - \alpha}{\ell} \right) + \left[ \frac{2 - \alpha}{4 - 3\alpha} + \left( \frac{2 - \alpha}{4 - 3\alpha} \right)^{\ell-1} \right] (\alpha - 1) \right\} \\
& \quad - \frac{\alpha(\ell - 1)(2 - \alpha)^2}{4(4 - 3\alpha)(\alpha + \ell - 3)} \left( \frac{4 - 3\alpha}{2(1-\alpha)} \right)^{1-\alpha} g^{(1-\alpha)}_{\ell-1} \\
& \quad + \frac{3(1-\alpha)}{2} \left[ \left( \frac{2 - \alpha}{4 - 3\alpha} \right)^\ell - \left( \frac{2 - \alpha}{4 - 3\alpha} \right)^4 \right] g^{(1-\alpha)}_{\ell-1}.
\end{align*}
\]
Here
\[ P(\alpha, \ell) = \left( \frac{\ell - 2 + \alpha}{\ell} + \frac{\alpha(1 - 2 - \alpha)}{4 - 3\alpha} - \frac{\alpha(\ell - 1)(2 - \alpha)^2}{4(4 - 3\alpha)(\alpha + \ell - 3)} - \frac{3(1 - \alpha)}{2} \left( \frac{2 - \alpha}{4 - 3\alpha} \right)^4 \right), \]
and
\[ Q(\alpha, \ell) = \left( \frac{\ell - 2 + \alpha}{\ell} + \frac{3(1 - \alpha)}{2} + \frac{(\alpha - 1)(4 - 3\alpha)}{2 - \alpha} \right). \]

It is somewhat tedious but easy to check that \( P(\alpha, \ell) \) and \( Q(\alpha, \ell) \) are increasing with respect to \( \ell \). Hence,
\[ P_{\min}(\alpha, \ell) = P(\alpha, 5) = \frac{177\alpha^6 - 969\alpha^5 + 3990\alpha^4 - 12000\alpha^3 + 20000\alpha^2 - 16336\alpha + 5152}{10(2 + \alpha)(4 - 3\alpha)^4} > 0, \]
and
\[ Q_{\min}(\alpha, \ell) = Q(\alpha, 5) = \frac{-17\alpha^2 + 23\alpha + 2}{10(2 - \alpha)} > 0. \]

Noticing that \( g_{\ell - 1}^{(1 - \alpha)} \leq 0 \) for \( 0 < \alpha < 1 \) and \( \ell \geq 1 \) gives
\[ \omega_{\ell}^{(1 - \alpha)} \leq \left\{ P(\alpha, \ell) + Q(\alpha, \ell) \frac{2 - \alpha}{4 - 3\alpha} \right\} g_{\ell - 1}^{(1 - \alpha)} \]
\[ \leq \left\{ P_{\min}(\alpha, \ell) + Q_{\min}(\alpha, \ell) \frac{2 - \alpha}{4 - 3\alpha} \right\} g_{\ell - 1}^{(1 - \alpha)} \leq 0. \]

All this completes the proof. 

**Theorem 2.4** The coefficient \( \omega_{\ell}^{(1 - \alpha)} \) are increasing with respect to \( \ell \geq 7 \), where \( \alpha \in (0, 1) \), that is, \( \omega_{\ell}^{(1 - \alpha)} \geq \omega_{\ell - 1}^{(1 - \alpha)} \) if \( \ell \geq 7 \), where \( \alpha \in (0, 1) \).

**Proof.** Here, we use mathematical induction to prove this theorem. Let
\[ S_\ell = \omega_{\ell}^{(1 - \alpha)} - \omega_{\ell - 1}^{(1 - \alpha)}, \ \ell \geq 7. \]

From equation (5), one easily knows that \( S_7 \geq 0 \). Now suppose that the conclusion holds for \( \ell = k > 7 \), that is
\[ \omega_k^{(1 - \alpha)} \geq \omega_{k - 1}^{(1 - \alpha)}, \ k > 7. \]
Then for $\ell = k + 1$, according to Theorems 2.3 and 2.4, one gets
\[
S_{k+1} = \varpi_{k+1}^{(1-\alpha)} - \varpi_k^{(1-\alpha)}
\]
\[
= \frac{1}{(k + 1)(4 - 3\alpha)} \left[ 2(k + \alpha - 1)(3 - 2\alpha)\varpi_k^{(1-\alpha)} + (2 - \alpha)(3 - 2\alpha - k)\varpi_{k-1}^{(1-\alpha)} - (k + 1)(4 - 3\alpha)\varpi_k^{(1-\alpha)} \right]
\]
\[
\geq \frac{1}{(k + 1)(4 - 3\alpha)} \left[ 2(k + \alpha - 1)(3 - 2\alpha) + (2 - \alpha)(3 - 2\alpha - k) \right]
\]
\[
- (k + 1)(4 - 3\alpha) \varpi_k^{(1-\alpha)}
\]
\[
= \frac{1}{(k + 1)(4 - 3\alpha)} \left[ 2(1 - \alpha)(\alpha - 2) \right] \varpi_k^{(1-\alpha)} \geq 0.
\]
The proof is thus completed. □

3 Application to the fractional Cable equation in one space dimension

In this section, we study the following one-dimensional Cable equation
\[
\frac{\partial u(x, t)}{\partial t} = K_1 RL D_{0,t}^{1-\alpha_1} \frac{\partial^2 u(x, t)}{\partial x^2} - K_2 RL D_{0,t}^{1-\alpha_2} u(x, t) + f(x, t),
\]
\[(x, t) \in (0, L) \times (0, T], \tag{6}\]
with initial condition
\[
u(x, 0) = 0, \quad x \in [0, L], \tag{7}\]
and boundary value conditions
\[
u(0, t) = \varphi_1(t), \quad \nu(L, t) = \varphi_2(t), \quad t \in (0, T], \tag{8}\]
where $0 < \alpha_1, \alpha_2 < 1$, $K_1$ and $K_2$ are two constants, $f(x, t)$, $\varphi_1(t)$ and $\varphi_2(t)$ are suitably smooth functions.

3.1 Development of numerical algorithm

Firstly, denote $x_j = jh$, $t_k = k\tau$, $\Omega_h = \{x_j|0 \leq j \leq M\}$, $\Omega_\tau = \{t_k|0 \leq k \leq N\}$, and $\Omega_{\tau,h} = \Omega_\tau \times \Omega_h$, where $h = L/M$, $\tau = T/N$ are the uniform spatial and temporal mesh sizes respectively, and $M, N$ are two positive integers. Let
\[
S_h = \{u|u = (u_0, u_1, \ldots, u_M), \quad u_0 = u_M = 0\}
\]
be defined on $\Omega_h$.

In addition, define the following first- and second-order difference operators as,

$$\delta_x u_{j-\frac{1}{2}} = \frac{1}{h} (u_j - u_{j-1}), \quad \delta_x^2 u_j = \frac{1}{h} \left( \delta_x u_{j+\frac{1}{2}} - \delta_x u_{j-\frac{1}{2}} \right),$$

and fourth-order compact difference operator $\mathcal{L}$ as,

$$\mathcal{L} u_j^k = \begin{cases} 
( I + \frac{h^2}{12} \delta_x^2 ) u_j^k = \frac{1}{12} (u_{j+1}^k + 10u_j^k + u_{j-1}^k), & 1 \leq j \leq M - 1, \\
 u_j^k, & j = 0, M,
\end{cases}$$

where $I$ is the unit operator.

Now we turn to derive an effective finite difference scheme for solving equation (6), together with initial and boundary value conditions (7) and (8). Consider equation (6) at point $\left( x_j, t_{k+\frac{1}{2}} \right)$

$$\frac{\partial u \left( x_j, t_{k+\frac{1}{2}} \right)}{\partial t} = K_1 RL D_{0,t}^{1-\alpha_1} \left( \frac{\partial^2 u \left( x_j, t_{k+\frac{1}{2}} \right)}{\partial x^2} \right) - K_2 RL D_{0,t}^{1-\alpha_2} u \left( x_j, t_{k+\frac{1}{2}} \right) + f \left( x_j, t_{k+\frac{1}{2}} \right).$$

(9)

Applying the second-order central difference formula

$$\frac{u \left( x_j, t_{k+1} \right) - u \left( x_j, t_k \right)}{\tau} = \frac{u \left( x_j, t_{k+1} \right) - u \left( x_j, t_k \right)}{\tau} + \mathcal{O}(\tau^2),$$

and second-order approximation formula (4) to the above equation (9), one gets

$$\frac{u \left( x_j, t_{k+1} \right) - u \left( x_j, t_k \right)}{\tau} = K_1 RL D_{0,t}^{1-\alpha_1} \frac{\partial^2 u \left( x_j, t_k \right)}{\partial x^2} - K_2 RL D_{0,t}^{1-\alpha_2} u \left( x_j, t_k \right) + f \left( x_j, t_{k+\frac{1}{2}} \right) + \mathcal{O}(\tau^2).$$

(10)

Acting the operator $\mathcal{L}$ on both sides of (10) and noticing

$$\mathcal{L} \frac{\partial^2 u \left( x_j, t_k \right)}{\partial x^2} = \delta_x^2 u \left( x_j, t_k \right) + \mathcal{O}(h^4),$$

one has

$$\mathcal{L} \left( u \left( x_j, t_{k+1} \right) - u \left( x_j, t_k \right) \right) = \tau K_1 RL D_{0,t}^{1-\alpha_1} \delta_x^2 u \left( x_j, t_k \right) - \tau K_2 RL D_{0,t}^{1-\alpha_2} \mathcal{L} u \left( x_j, t_k \right) + \tau R_j^k,$$
in which there exists a positive constant \( C_1 \) such that \( |R^k_j| \leq C_1 (\tau^2 + h^4) \).

Omit the local truncation error \( R^k_j \) and denote the numerical solution of \( u(x_j, t_k) \in \Omega_{\tau,h} \) by \( u^k_j \). One can establish the following high-order compact difference scheme for equation (6), together with (7) and (8),

\[
\mathcal{L} \left( u^{k+1}_j - u^k_j \right) = \tau K_1 \delta^1 \delta_x^2 u^k_j - \tau K_2 \delta^1 \alpha^2 \mathcal{L} u^k_j + \tau \mathcal{L} f^{k+\frac{1}{2}},
\]

\[
1 \leq j \leq M - 1, \quad 1 \leq k \leq N - 1,
\]

\[
u^0_j = 0, \quad 0 \leq j \leq M,
\]

\[
u^k_0 = \varphi_1(t_k), \quad \nu^k_M = \varphi_2(t_k), \quad 1 \leq k \leq N.
\]

### 3.2 Solvability, stability and convergence analysis

For arbitrary vectors \( u, v \in \mathcal{S}_h \), we introduce the following inner products and the corresponding norms,

\[
(u, v) = h \sum_{j=1}^{M-1} u_j v_j, \quad ||u|| = \sqrt{(u, u)}, \quad (\delta_x^2 u, v) = h \sum_{j=1}^{M-1} (\delta_x^2 u_j)v_j,
\]

\[
(\delta_x u, \delta_x v) = h \sum_{j=1}^{M} (\delta_x u_j)(\delta_x v_j), \quad ||\delta_x u|| = \sqrt{(\delta_x u, \delta_x u)}.
\]

It is easy to know \( (\delta_x^2 u, v) = -(\delta_x u, \delta_x v) \).

Next several lemmas are listed which will be used later on.

**Lemma 3.1** [29] If \( a \geq 0, \ b \geq 0 \), then the following Young’s inequality holds,

\[
ab \leq \left( (\epsilon a)^p \right)^{\frac{q}{p}} + \left( \frac{b^q}{q} \right)^{\frac{1}{q}},
\]

where \( \epsilon > 0, \ p > 1, \ q > 1 \) and \( \frac{1}{p} + \frac{1}{q} = 1 \).

**Lemma 3.2** (Gronwall’s Inequality [23]) Assume that \( \{k_n\} \) and \( \{p_n\} \) are nonnegative sequences, and the sequence \( \{\phi_n\} \) satisfies

\[
\phi_0 \leq q_0, \quad \phi_n \leq q_0 + \sum_{\ell=0}^{n-1} p_\ell + \sum_{\ell=0}^{n-1} k_\ell \phi_\ell, \quad n \geq 1,
\]

where \( q_0 \geq 0 \). Then the sequence \( \{\phi_n\} \) satisfies

\[
\phi_n \leq \left( q_0 + \sum_{\ell=0}^{n-1} p_\ell \right) \exp \left( \sum_{\ell=0}^{n-1} k_\ell \right), \quad n \geq 1.
\]
Lemma 3.3 For any grid function $u \in S_h$, one has
\[
\frac{2}{3} \| u \|^2 \leq (L u, u) \leq \| u \|^2.
\]

Lemma 3.4 For any grid function $u \in S_h$, there exists a symmetric positive difference operator denoted by $L^\frac{1}{2}$, such that
\[
(L u, u) = (L^\frac{1}{2} u, L^\frac{1}{2} u).
\]

Proof. Obviously, the corresponding matrix of operator $L$ is given by
\[
D = \begin{pmatrix}
\frac{5}{6} & \frac{1}{12} & 0 & \cdots & 0 \\
\frac{1}{12} & \frac{5}{6} & \frac{1}{12} & 0 & \cdots \\
\vdots & \vdots & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & \frac{5}{6} & \frac{1}{12} \\
0 & 0 & \cdots & \frac{1}{12} & \frac{5}{6}
\end{pmatrix}
\]
Obviously, $D$ is real symmetric and positive definite. Hence, there exists a symmetric positive matrix denoted by $D^\frac{1}{2}$ such that
\[
(L u, u) = h u^T D u = h u^T \left( D^\frac{1}{2} \right)^2 u = h \left( D^\frac{1}{2} u \right)^T D^\frac{1}{2} u = (L^\frac{1}{2} u, L^\frac{1}{2} u),
\]
where $L^\frac{1}{2}$ is the associate operator of matrix $D^\frac{1}{2}$. Therefore, the proof is ended.

Lemma 3.5 For any mesh function $\{u^k | k = 0, 1, \ldots, n-1\} \in S_h$, it holds that
\[
\sum_{k=0}^{n-1} (\delta_{0,t}^{1-\alpha} u^k, u^{k+1} + u^k) \geq 0
\]
for $0 < \alpha < 1$, where $\alpha$ denotes $\alpha_1$ or $\alpha_2$.

Proof. Firstly, we have
\[
\sum_{k=0}^{n-1} (\delta_{0,t}^{1-\alpha} u^k, u^{k+1} + u^k) = h \sum_{j=1}^{M-1} \sum_{k=0}^{n-1} \left( \sum_{\ell=0}^{k} \omega_\ell^{(1-\alpha)} u_{j}^{k-\ell} \right) \left( u_{j}^{k+1} + u_{j}^{k} \right)
\]
\[
= h \sum_{j=1}^{M-1} \sum_{k=0}^{n-1} \left( \sum_{\ell=0}^{k} \omega_\ell^{(1-\alpha)} u_{j}^{k-\ell} \right) u_{j}^{k} + h \sum_{j=1}^{M-1} \sum_{k=0}^{n-1} \left( \sum_{\ell=0}^{k} \omega_{\ell-1}^{(1-\alpha)} u_{j}^{k+1-\ell} \right) u_{j}^{k+1}
\]
\[+ h \sum_{j=1}^{M-1} \sum_{k=0}^{n-1} \left( \omega_{k}^{(1-\alpha)} u_{j}^{0} - \omega_{k-1}^{(1-\alpha)} u_{j}^{k+1} \right) u_{j}^{k+1}.
\]
Letting $\mathcal{W}_{-1}^{(1-\alpha)} = 0$ and noticing $u_0^0 = 0$, one has

$$\sum_{k=0}^{n-1} (\delta_{0,t} u_k^k, u_{k+1}^k + u^k) =$$

$$= h \sum_{j=1}^{M-1} \sum_{k=0}^{n-1} \left( \sum_{\ell=0}^{k} \mathcal{W}_{\ell}^{(1-\alpha)} u_j^{k-\ell} \right) u_j^k + h \sum_{j=1}^{M-1} \sum_{k=0}^{n-1} \left( \sum_{\ell=0}^{k} \mathcal{W}_{\ell-1}^{(1-\alpha)} u_j^{k+1-\ell} \right) u_j^{k+1}$$

$$= (A_{\alpha} u, u) + (B_{\alpha} u, u) = (E_{\alpha} u, u),$$

where matrices $A_{\alpha}$ and $B_{\alpha}$ are

$$A_{\alpha} = \begin{pmatrix}
\mathcal{W}_0^{(1-\alpha)} & \frac{1}{2} \mathcal{W}_1^{(1-\alpha)} & \frac{1}{2} \mathcal{W}_2^{(1-\alpha)} & \ldots & \frac{1}{2} \mathcal{W}_n^{(1-\alpha)} \\
\frac{1}{2} \mathcal{W}_1^{(1-\alpha)} & \mathcal{W}_0^{(1-\alpha)} & \frac{1}{2} \mathcal{W}_1^{(1-\alpha)} & \frac{1}{2} \mathcal{W}_2^{(1-\alpha)} & \ldots \\
\vdots & \vdots & \ddots & \ddots & \ddots \\
\frac{1}{2} \mathcal{W}_{n-1}^{(1-\alpha)} & \frac{1}{2} \mathcal{W}_{n-2}^{(1-\alpha)} & \ldots & \mathcal{W}_0^{(1-\alpha)} & \frac{1}{2} \mathcal{W}_1^{(1-\alpha)} \\
\frac{1}{2} \mathcal{W}_n^{(1-\alpha)} & \frac{1}{2} \mathcal{W}_{n-1}^{(1-\alpha)} & \ldots & \frac{1}{2} \mathcal{W}_1^{(1-\alpha)} & \mathcal{W}_0^{(1-\alpha)}
\end{pmatrix},$$

and

$$B_{\alpha} = \begin{pmatrix}
\mathcal{W}_{-1}^{(1-\alpha)} & \frac{1}{2} \mathcal{W}_0^{(1-\alpha)} & \frac{1}{2} \mathcal{W}_1^{(1-\alpha)} & \ldots & \frac{1}{2} \mathcal{W}_n^{(1-\alpha)} \\
\frac{1}{2} \mathcal{W}_0^{(1-\alpha)} & \mathcal{W}_{-1}^{(1-\alpha)} & \frac{1}{2} \mathcal{W}_0^{(1-\alpha)} & \frac{1}{2} \mathcal{W}_1^{(1-\alpha)} & \ldots \\
\vdots & \vdots & \ddots & \ddots & \ddots \\
\frac{1}{2} \mathcal{W}_{n-2}^{(1-\alpha)} & \frac{1}{2} \mathcal{W}_{n-3}^{(1-\alpha)} & \ldots & \mathcal{W}_{-1}^{(1-\alpha)} & \frac{1}{2} \mathcal{W}_0^{(1-\alpha)} \\
\frac{1}{2} \mathcal{W}_{n-1}^{(1-\alpha)} & \frac{1}{2} \mathcal{W}_{n-2}^{(1-\alpha)} & \ldots & \frac{1}{2} \mathcal{W}_0^{(1-\alpha)} & \mathcal{W}_{-1}^{(1-\alpha)}
\end{pmatrix}.$$

By the Grenander-Szegö Theorem [10], if the generating function of matrix $E_{\alpha}$ is nonnegative, then matrix $E_{\alpha}$ is positive semi-definite. So, we only consider the
generating function of matrix $E_\alpha$ which is

$$G_{E,\alpha}(x, \alpha) = G_{A,\alpha}(x, \alpha) + G_{B,\alpha}(x, \alpha)$$

$$= \omega_0^{(1-\alpha)} + \frac{1}{2} \sum_{k=1}^{\infty} \omega_1^{(1-\alpha)} e^{ikx} + \frac{1}{2} \sum_{k=1}^{\infty} \omega_{k-1}^{(1-\alpha)} e^{-ikx}$$

$$+ \omega_{-1}^{(1-\alpha)} + \frac{1}{2} \sum_{k=1}^{\infty} \omega_{k-1}^{(1-\alpha)} e^{ikx} + \frac{1}{2} \sum_{k=1}^{\infty} \omega_{k-1}^{(1-\alpha)} e^{-ikx}$$

$$= \frac{1}{2} \sum_{k=0}^{\infty} \omega_k^{(1-\alpha)} e^{ikx} + \frac{1}{2} \sum_{k=0}^{\infty} \omega_k^{(1-\alpha)} e^{-ikx} + \frac{1}{2} \sum_{k=0}^{\infty} \omega_k^{(1-\alpha)} e^{i(k+1)x}$$

$$+ \frac{1}{2} \sum_{k=0}^{\infty} \omega_k^{(1-\alpha)} e^{-i(k+1)x}$$

$$= 2 \left( \frac{4 - 3\alpha}{2(1-\alpha)} \right)^{1-\alpha} \left( 2 \sin \frac{x}{2} \right)^{1-\alpha} \left( a^2 + b^2 \right)^{\frac{1-\alpha}{2}} \cos \left( \frac{x}{2} \right) Z(x, \alpha),$$

where

$$a = 1 - \frac{2 - \alpha}{4 - 3\alpha} \cos(x), \quad b = \frac{2 - \alpha}{4 - 3\alpha} \sin(x),$$

$$Z(x, \alpha) = \cos \left( \frac{x}{2} + (1-\alpha) \left( \theta + \frac{x - \pi}{2} \right) \right), \quad \theta = -\arctan \frac{b}{a}.$$

Because $G_{E,\alpha}(x, \alpha)$ is a real value and even function, we only consider the case of $x \in [0, \pi]$ for $G_{E,\alpha}(x, \alpha)$. From the above formula, it is easy to know that only function $Z(x, \alpha)$ needs to be studied for $0 < \alpha < 1$.

Let

$$q(x, \alpha) = \frac{x}{2} + (1-\alpha) \left( \theta + \frac{x - \pi}{2} \right), \quad x \in [0, \pi], \quad \alpha \in (0, 1).$$

One has

$$\frac{\partial q(x, \alpha)}{\partial x} = \frac{4(2 - \alpha)(1 - \alpha)^2 + 2(4 - 3\alpha)(2 - \alpha)^2 \sin^2 \left( \frac{x}{2} \right)}{[(4 - 3\alpha) - (2 - \alpha) \cos(x)]^2 + [(2 - \alpha) \sin(x)]^2} \geq 0$$

which implies that $q(x, \alpha)$ is an increasing function with respect to $x$. And

$$q_{\min}(x, \alpha) = q(0, \alpha) = -\frac{\pi}{2}(1 - \alpha), \quad q_{\max}(x, \alpha) = q(\pi, \alpha) = \frac{\pi}{2}.$$

Hence,

$$Z(x, \alpha) = \cos(q(x, \alpha)) \geq 0,$$
it immediately follows that

$$G_{E_n}(x, \alpha) \geq 0.$$ 

So the proof is completed. ■

In the following, the first step is to prove the solvability of finite difference scheme (11), together with (12) and (13).

**Theorem 3.1** The finite difference scheme (11), together with (12) and (13) is uniquely solvable.

**Proof.** Consider the homogeneous form of system (11),

$$\mathcal{L}u_j^{k+1} = 0. \quad (14)$$

Taking the inner product of (11) with $u_j^{k+1}$ gives

$$\left( \mathcal{L}u_j^{k+1}, u_j^{k+1} \right) = 0.$$

It follows from Lemma 3.3 that

$$||u_j^{k+1}||^2 = 0,$$

that is, $u_j^{k+1} = 0$. This finishes the proof. ■

Now, we give the stability result.

**Theorem 3.2** The finite difference scheme (11), together with (12) and (13) is unconditionally stable with respect to the initial value.

**Proof.** Let $u_j^k$ and $v_j^k$ be the solutions of the following two equations, respectively,

\[
\mathcal{L} \left( u_j^{k+1} - u_j^k \right) = \tau K_1 \delta_{x_0,-1}^2 u_j^k - \tau K_2 \delta_{x_0,1} \mathcal{L} u_j^k + \tau \mathcal{L} f_j^{k+\frac{1}{2}}, \quad 1 \leq j \leq M - 1, \ 1 \leq k \leq N, \\
u_j^0 = 0, \ 0 \leq j \leq M, \\
u_0^k = \varphi_1(t_k), \ u_M^k = \varphi_2(t_k), \ 1 \leq k \leq N;
\]

and

\[
\mathcal{L} \left( v_j^{k+1} - v_j^k \right) = \tau K_1 \delta_{x_0,-1}^2 v_j^k - \tau K_2 \delta_{x_0,1} \mathcal{L} v_j^k + \tau \mathcal{L} f_j^{k+\frac{1}{2}}, \quad 1 \leq j \leq M - 1, \ 1 \leq k \leq N, \\
v_j^0 = \rho_j, \ 0 \leq j \leq M, \\
v_0^k = \varphi_1(t_k), \ v_M^k = \varphi_2(t_k), \ 1 \leq k \leq N.
\]
Denote $\varepsilon^k_j = v^k_j - u^k_j$. Then one can get the following perturbation equation,
\[
\mathcal{L} (\varepsilon^{k+1}_j - \varepsilon^k_j) = \tau K_1 \delta_{0,t}^{1-\alpha_1} \delta_x^2 \varepsilon^k_j - \tau K_2 \delta_{0,t}^{1-\alpha_2} \mathcal{L} \varepsilon^k_j, 1 \leq j \leq M - 1, 1 \leq k \leq N,
\]
\[
\varepsilon^0_j = \rho_j, \quad 0 \leq j \leq M,
\]
\[
\varepsilon^k_0 = 0, \quad \varepsilon^k_M = 0, \quad 1 \leq k \leq N.
\] (15)

Taking the discrete inner product with $(\varepsilon^{k+1} + \varepsilon^k)$ on both sides of the equation (15) and summing up for $k$ from 0 to $n-1$ give
\[
\sum_{k=0}^{n-1} (\mathcal{L} (\varepsilon^{k+1} - \varepsilon^k), (\varepsilon^{k+1} + \varepsilon^k)) = \tau K_1 \sum_{k=0}^{n-1} (\delta_{0,t}^{1-\alpha_1} \delta_x^2 \varepsilon^k, (\varepsilon^{k+1} + \varepsilon^k))
\]
\[
- \tau K_2 \sum_{k=0}^{n-1} (\delta_{0,t}^{1-\alpha_2} \mathcal{L} \varepsilon^k, (\varepsilon^{k+1} + \varepsilon^k)).
\]

Furthermore, one has the following results
\[
\sum_{k=0}^{n-1} (\mathcal{L} (\varepsilon^{k+1} - \varepsilon^k), (\varepsilon^{k+1} + \varepsilon^k)) = \sum_{k=0}^{n-1} [(\mathcal{L} \varepsilon^{k+1}, \varepsilon^{k+1}) - (\mathcal{L} \varepsilon^k, \varepsilon^k)],
\]
\[
\sum_{k=0}^{n-1} (\delta_{0,t}^{1-\alpha_1} \delta_x^2 \varepsilon^k, (\varepsilon^{k+1} + \varepsilon^k)) = - \sum_{k=0}^{n-1} (\delta_{0,t}^{1-\alpha_1} \delta_x \varepsilon^k, \delta_x (\varepsilon^{k+1} + \varepsilon^k)) \leq 0,
\]
and
\[
- \sum_{k=0}^{n-1} (\delta_{0,t}^{1-\alpha_2} \mathcal{L} \varepsilon^k, (\varepsilon^{k+1} + \varepsilon^k)) = - \sum_{k=0}^{n-1} (\delta_{0,t}^{1-\alpha_2} \mathcal{L} \frac{1}{2} \varepsilon^k, \mathcal{L} \frac{1}{2} (\varepsilon^{k+1} + \varepsilon^k)) \leq 0
\]
by follows that Lemmas 3.4 and 3.5. Hence,
\[
\sum_{k=0}^{n-1} [(\mathcal{L} \varepsilon^{k+1}, \varepsilon^{k+1}) - (\mathcal{L} \varepsilon^k, \varepsilon^k)] \leq 0,
\]
i.e.,
\[
(\mathcal{L} \varepsilon^n, \varepsilon^n) \leq (\mathcal{L} \varepsilon^0, \varepsilon^0).
\]
Using Lemma 3.3 again, one has,
\[
||\varepsilon^n|| \leq \frac{\sqrt{6}}{2} ||\varepsilon^0|| = \frac{\sqrt{6}}{2} ||\rho||.
\]
This ends the proof.  

Finally, we study the convergence of the above compact difference scheme.
**Theorem 3.3** Assume that the solution $u(x, t)$ of equation (6), together with (7) and (8) is suitably smooth, and let \( \{u_k^j|0 \leq j \leq M, 0 \leq k \leq N\} \) be the solution of the finite difference scheme (11)–(13). Set $e_k^j = u(x_j, t_k) - u_j^k$, then

\[
||e^n|| \leq \frac{3}{2} C_1 \exp(T) \sqrt{2LT} (\tau^2 + h^4).
\]

**Proof.** From the above analysis, we have the following error system,

\[
\mathcal{L} (e_j^{k+1} - e_j^k) = \tau K_1 \delta_{0,t}^{1-\alpha_1} \delta_x^2 e_j^k - \tau K_2 \delta_{0,t}^{1-\alpha_2} \mathcal{L} e_j^k + \tau R_j^k,
\]

\[
1 \leq j \leq M - 1, 1 \leq k \leq N,
\]

\[
e_j^0 = 0, 0 \leq j \leq M,
\]

\[
e_k^0 = 0, e_k^M = 0, 1 \leq k \leq N.
\]

Taking the discrete inner product with \( (e_j^{k+1} + e_j^k) \) on the both sides of equation (16) and summing up for $k$ from 0 to $n - 1$, one gets

\[
\sum_{k=0}^{n-1} (\mathcal{L} (e_j^{k+1} - e_j^k), (e_j^{k+1} + e_j^k)) = \tau K_1 \sum_{k=0}^{n-1} \left( \delta_{0,t}^{1-\alpha_1} \delta_x^2 e_j^k, (e_j^{k+1} + e_j^k) \right)
\]

\[
- \tau K_2 \sum_{k=0}^{n-1} \left( \delta_{0,t}^{1-\alpha_2} \mathcal{L} e_j^k, (e_j^{k+1} + e_j^k) \right) + \tau \sum_{k=0}^{n-1} \left( R_j^k, (e_j^{k+1} + e_j^k) \right).
\]

It follows from Lemma 3.1 that

\[
\tau \sum_{k=0}^{n-1} \left( R_j^k, (e_j^{k+1} + e_j^k) \right) \leq \tau \sum_{k=0}^{n-1} ||R_j^k|| \left( ||e_j^{k+1}|| + ||e_j^k|| \right)
\]

\[
\leq \frac{3}{2} \tau \sum_{k=0}^{n-1} ||R_j^k||^2 + \frac{1}{3} \tau \sum_{k=0}^{n-1} \left( ||e_j^{k+1}||^2 + ||e_j^k||^2 \right).
\]

Hence, one further arrives at

\[
\frac{2}{3} ||e^n||^2 \leq \frac{3}{2} \tau \sum_{k=0}^{n-1} ||R_j^k||^2 + \frac{1}{3} \tau \sum_{k=0}^{n-1} \left( ||e_j^{k+1}||^2 + ||e_j^k||^2 \right)
\]

\[
\leq \frac{1}{3} ||e^n||^2 + \frac{2}{3} \tau \sum_{k=0}^{n-1} ||e_j^k||^2 + \frac{3}{2} L T C_1^2 (\tau^2 + h^4)^2,
\]

i.e.,

\[
||e^n||^2 \leq 2 \tau \sum_{k=0}^{n-1} ||e_j^k||^2 + \frac{9}{2} L T C_1^2 (\tau^2 + h^4)^2,
\]
Applying Lemma 3.2 to it yields
\[
||e^n|| \leq \frac{3}{2} C_1 \exp(T) \sqrt{2LT} (\tau^2 + h^4).
\]
The proof is thus finished. ■

4 Application to the fractional Cable equation in two space dimensions

In this section, we study the following two-dimensional Cable equation

\[
\frac{\partial u(x, y, t)}{\partial t} = K_1 RL D^{1-\alpha_1}_{0,t} \left( \frac{\partial^2 u(x, y, t)}{\partial x^2} + \frac{\partial^2 u(x, y, t)}{\partial y^2} \right) - K_2 RL D^{1-\alpha_2}_{0,t} (u(x, y, t)) + f(x, y, t),
\]

where \(0 < \alpha_1, \alpha_2 < 1\), \(K_1\) and \(K_2\) are two constants, \(f(x, y, t)\) and \(\varphi(x, y, t)\) are given suitably smooth functions.

4.1 Development of numerical algorithm

Denote \(x_i = ih_x, 0 \leq i \leq M_1, y_j = jh_y, 0 \leq j \leq M_2\), where \(h_x = L_x/M_1, h_y = L_y/M_2\) and \(M_1, M_2\) are two positive integers. Define \(\Omega_{h_x h_y} = \{(x_i, y_j) | 0 \leq i \leq M_1, 0 \leq j \leq M_2\}\), \(\Omega_{h_x h_y} = \Omega_{h_x h_y} \cap \Omega\) and \(\partial \Omega_{h_x h_y} = \Omega_{h_x h_y} \cap \partial \Omega\). For any grid function \(u \in \mathcal{V}_{h_x h_y}\), \(u|_{\mathcal{V}} = \{u_{i,j} | 0 \leq i \leq M_1, 0 \leq j \leq M_2\}\), define the difference operators as,

\[
\delta_x u_{i-\frac{1}{2},j} = \frac{1}{h_x} (u_{i,j} - u_{i-1,j}), \quad \delta_x^2 u_{i,j} = \frac{1}{h_x} \left( \delta_x u_{i+\frac{1}{2},j} - \delta_x u_{i-\frac{1}{2},j} \right),
\]

\[
\delta_y u_{i,j-\frac{1}{2}} = \frac{1}{h_y} (u_{i,j} - u_{i,j-1}), \quad \delta_y^2 u_{i,j} = \frac{1}{h_y} \left( \delta_y u_{i,j+\frac{1}{2}} - \delta_y u_{i,j-\frac{1}{2}} \right),
\]
and the spatial compact difference operators as,

\[
\mathcal{L}_x u_{i,j} = \begin{cases} 
(I + \frac{\Delta t^2}{12} \delta_x^2) u_{i,j}, & 1 \leq i \leq M_1 - 1, \ 0 \leq j \leq M_2, \\
u_{i,j}, & i = 0 \text{ or } M_1, \ 0 \leq j \leq M_2, \\
u_{i,j}, & j = 0 \text{ or } M_2, \ 0 \leq i \leq M_1.
\end{cases}
\]

\[
\mathcal{L}_y u_{i,j} = \begin{cases} 
(I + \frac{\Delta t^2}{12} \delta_y^2) u_{i,j}, & 1 \leq j \leq M_2 - 1, \ 0 \leq i \leq M_1, \\
u_{i,j}, & i = 0 \text{ or } M_1, \ 0 \leq j \leq M_2, \\
u_{i,j}, & j = 0 \text{ or } M_2, \ 0 \leq i \leq M_1.
\end{cases}
\]

Now we consider equation (17) at point \((x_i, y_j, t_{k+\frac{1}{2}})\). Operating the operator \(\mathcal{L}_x \mathcal{L}_y\) on both sides of it leads to,

\[
\mathcal{L}_x \mathcal{L}_y \frac{\partial u(x_i, y_j, t_{k+\frac{1}{2}})}{\partial t} = K_1 \mathcal{L}_x \mathcal{L}_y \frac{\partial^2 u(x_i, y_j, t_{k+\frac{1}{2}})}{\partial x^2} + L \mathcal{L}_x \mathcal{L}_y \frac{\partial^2 u(x_i, y_j, t_{k+\frac{1}{2}})}{\partial y^2} - L \mathcal{L}_x \mathcal{L}_y f(x_i, y_j, t_{k+\frac{1}{2}}) + \mathcal{L}_x \mathcal{L}_y f(x_i, y_j, t_{k+\frac{1}{2}}),
\]

\[1 \leq i \leq M_1 - 1, \ 1 \leq j \leq M_2 - 1, \ 0 \leq k \leq N - 1.
\]

Similar to the one-dimensional case, one easily has

\[
\mathcal{L}_x \mathcal{L}_y (u(x_i, y_j, t_{k+1}) - u(x_i, y_j, t_k)) = \tau K_1 \delta_{t,\alpha_1} (\mathcal{L}_x^2 + \mathcal{L}_y^2) u(x_i, y_j, t_k) - \tau K_2 \delta_{t,\alpha_2} \mathcal{L}_x \mathcal{L}_y u(x_i, y_j, t_k)
\]

\[+ \mathcal{L}_x \mathcal{L}_y f(x_i, y_j, t_{k+1}) + \mathcal{L}_x \mathcal{L}_y f(x_i, y_j, t_k),
\]

\[1 \leq i \leq M_1 - 1, \ 1 \leq j \leq M_2 - 1, \ 0 \leq k \leq N - 1,
\]

where there exists a positive constant \(C_2\) such that \(|R^k_{i,j}| \leq C_2(\tau^2 + h_x^4 + h_y^4) \).

Omitting the small term \(R^k_{i,j}\) in (20) and replacing the grid function \(u(x_i, y_j, t_k)\) with its numerical approximation \(u^k_{i,j}\), one gets a high-order compact scheme in the following form,

\[
\mathcal{L}_x \mathcal{L}_y (u^k_{i,j+1} - u^k_{i,j}) = \tau K_1 \delta_{t,\alpha_1} (\mathcal{L}_x^2 + \mathcal{L}_y^2) u^k_{i,j} - \tau K_2 \delta_{t,\alpha_2} \mathcal{L}_x \mathcal{L}_y u^k_{i,j} + \tau \mathcal{L}_x \mathcal{L}_y f^k_{i,j} + \mathcal{L}_x \mathcal{L}_y f^k_{i,j},
\]

\[1 \leq i \leq M_1 - 1, \ 1 \leq j \leq M_2 - 1, \ 0 \leq k \leq N - 1,
\]

\[u^0_{i,j} = 0, \quad 0 \leq i \leq M_1, \ 0 \leq j \leq M_2,
\]

\[u^k_{i,j} = \varphi(x_i, y_j, t_k), \quad (x_i, y_j) \in \partial \Omega_{h_x h_y}.
\]
4.2 Solvability, stability and convergence analysis

Define

\[ S_{h_x h_y} = \{ u | u = \{ u_{i,j} \}, u|_{\partial \Omega_{h_x h_y}} = 0 \} \]

Then for any \( u, v \in S_{h_x h_y} \), define the inner products as

\[
(u, v) = h_x h_y \sum_{i=1}^{M_1-1} \sum_{j=1}^{M_2-1} u_{i,j} v_{i,j},
\]

\[
(\delta_x \delta_y u, \delta_x \delta_y v) = h_x h_y \sum_{i=1}^{M_1} \sum_{j=1}^{M_2} \left( \delta_x \delta_y u_{i-\frac{1}{2},j-\frac{1}{2}} \right) \left( \delta_x \delta_y v_{i-\frac{1}{2},j-\frac{1}{2}} \right),
\]

\[
(\delta_x u, \delta_x v) = h_x h_y \sum_{i=1}^{M_1} \sum_{j=1}^{M_2-1} \left( \delta_x u_{i-\frac{1}{2},j} \right) \left( \delta_x v_{i-\frac{1}{2},j} \right),
\]

\[
(\delta_y u, \delta_y v) = h_x h_y \sum_{i=1}^{M_1-1} \sum_{j=1}^{M_2} \left( \delta_y u_{i,j-\frac{1}{2}} \right) \left( \delta_y v_{i,j-\frac{1}{2}} \right).
\]

We temporarily leave to give several definition(s) and lemmas [13] which will be utilized later on.

**Definition 4.1** If \( A = (a_{i,j}) \) is an \( m \times n \) matrix and \( B = (b_{i,j}) \) is a \( p \times q \) matrix, then the Kronecker product \( A \otimes B \) is an \( mp \times nq \) block matrix and denoted by

\[
A \otimes B = \begin{pmatrix}
  a_{11}B & a_{12}B & \cdots & a_{1n}B \\
  a_{21}B & a_{22}B & \cdots & a_{2n}B \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{m1}B & a_{m2}B & \cdots & a_{mn}B
\end{pmatrix}.
\]

**Lemma 4.1** Suppose that \( A \in \mathbb{R}^{n \times n} \) has eigenvalues \( \{ \lambda_j \}_{j=1}^n \), and \( B \in \mathbb{R}^{m \times m} \) has eigenvalues \( \{ \mu_j \}_{j=1}^m \). Then the \( mn \) eigenvalues of \( A \otimes B \) are given below,

\[
\lambda_1 \mu_1, \ldots, \lambda_1 \mu_m, \lambda_2 \mu_1, \ldots, \lambda_2 \mu_m, \ldots, \lambda_n \mu_1, \ldots, \lambda_n \mu_m.
\]

**Lemma 4.2** Let \( A \in \mathbb{R}^{m \times n}, B \in \mathbb{R}^{r \times s}, C \in \mathbb{R}^{n \times p}, D \in \mathbb{R}^{s \times t} \). Then

\[
(A \otimes B) (C \otimes D) = AC \otimes BD.
\]

Moreover, if \( A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{m \times m}, I_n \) and \( I_m \) are unit matrices of orders \( n, m \), respectively, then matrices \( I_m \otimes A \) and \( B \otimes I_n \) can commute with each other.
Lemma 4.3 For both $A$ and $B$, there holds
$$(A \otimes B)^T = A^T \otimes B^T.$$ 

Next we give an inequality for the mesh function.

Lemma 4.4 For any mesh function $u \in S_{h_xh_y}$, there holds
$$\frac{1}{3} ||u||^2 \leq (\mathcal{L}_x \mathcal{L}_y u, u) \leq ||u||^2.$$ 

Now we return to discuss the derived compact scheme.

Lemma 4.5 For any mesh functions $u, v \in S_{h_xh_y}$, there has a symmetric positive definite operator $Q_{x+y}$, such that
$$(\mathcal{L}_y \delta_x^2 + \mathcal{L}_x \delta_y^2)u, v) = -(Q_{x+y}u, Q_{x+y}v).$$

Proof. Denote
$$u = (u_{1,1}, u_{2,1}, \ldots, u_{M_1-1,1}, \ldots, u_{1,M_2-1}, u_{2,M_2-1}, \ldots, u_{M_1-1,M_2-1})^T$$
and
$$v = (v_{1,1}, v_{2,1}, \ldots, v_{M_1-1,1}, \ldots, v_{1,M_2-1}, v_{2,M_2-1}, \ldots, v_{M_1-1,M_2-1})^T.$$ 

Then $((\mathcal{L}_y \delta_x^2 + \mathcal{L}_x \delta_y^2)u, v)$ can be rewritten as
$$(\mathcal{L}_y \delta_x^2 + \mathcal{L}_x \delta_y^2)u, v) = h_1 h_2 v^T L_{x+y} u,$$
where
$$L_{x+y} = \frac{1}{h_1^2} (C_2 \otimes I_1) (I_2 \otimes D_1) + \frac{1}{h_2^2} (I_2 \otimes C_1) (D_2 \otimes I_1).$$

Here $I_p$ are the unit matrices of order $M_p - 1$ ($p = 1, 2$), $C_p$ and $D_p$ ($p = 1, 2$) are defined by
$$C_p = \begin{pmatrix}
\frac{5}{6} & \frac{1}{12} & 0 & \cdots & 0 \\
\frac{1}{12} & \frac{5}{6} & \frac{1}{12} & 0 & \cdots \\
0 & 0 & \cdots & \frac{5}{6} & \frac{1}{12} \\
0 & 0 & \cdots & \frac{1}{12} & \frac{5}{6}
\end{pmatrix}.$$ 

The size of $C_p$ is $(M_p-1) \times (M_p-1)$.
and
\[ D_p = \begin{pmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & -2 & 1 \\ 0 & 0 & \cdots & 1 & -2 \end{pmatrix}_{(M_p-1)\times(M_p-1)}. \]

By using Lemmas 4.2 and 4.3 and the fact
\[ L^T x + y + \frac{1}{h_1^2} (I_2 \otimes D_1)^T (C_2 \otimes I_1)^T + \frac{1}{h_2^2} (D_2 \otimes I_1)^T (I_2 \otimes C_1)^T \\ = \frac{1}{h_1^2} (I_2 \otimes D_1) (C_2 \otimes I_1) + \frac{1}{h_2^2} (D_2 \otimes I_1) (I_2 \otimes C_1) \\ = \frac{1}{h_1^2} (C_2 \otimes I_1) (I_2 \otimes D_1) + \frac{1}{h_2^2} (I_2 \otimes C_1) (D_2 \otimes I_1) \\ = L_{x+y}, \]

we know that matrix \( L_{x+y} \) is real symmetric. It follows from Lemma 4.1 that matrix \( L_{x+y} \) is negative definite since its eigenvalues are all negative. Hence, we can declare that matrix \( L_{x+y} \) is real symmetric and negative definite, and there exist an orthogonal matrix \( H_{x+y} \) and a diagonal matrix \( \Lambda_{x+y} \) such that
\[ L_{x+y} = -H_{x+y}^T \Lambda_{x+y} H_{x+y} = - \left( \Lambda_{x+y}^{\frac{1}{2}} H_{x+y} \right)^T \left( \Lambda_{x+y}^{\frac{1}{2}} H_{x+y} \right) = - Q_{x+y}^T Q_{x+y}. \]

Hence, we have
\[ ((\mathcal{L}_y \delta_x^2 + \mathcal{L}_x \delta_y^2) u, v) = h_1 h_2 v^T L_{x+y} u \\ = -h_1 h_2 (Q_{x+y} v)^T Q_{x+y} u = -(\mathcal{Q}_{x+y} u, \mathcal{Q}_{x+y} v), \]

where \( \mathcal{Q}_{x+y} \) is a symmetric and positive definite operator, so is \( Q_{x+y} \). Therefore, the proof is ended. ■

**Lemma 4.6** [12] For any mesh function \( u \in S_{h_x h_y} \), there exists a symmetric positive definite operator \( \mathcal{Q}_{xy} \) such that
\[ (\mathcal{L}_x \mathcal{L}_y u, v) = (\mathcal{Q}_{xy} u, \mathcal{Q}_{xy} v). \]

In the following, we firstly give the solvability of finite difference scheme (21), together with (22) and (23).
Theorem 4.1 The finite difference scheme (21), together with (22) and (23) is uniquely solvable.

Proof. As the same as the one dimensional case, we easily get the homogeneous form of difference scheme (21) is

\[ L_x L_y u_{i,j}^{k+1} = 0. \]

Taking the inner product with \( u_{i,j}^{k+1} \) and using Lemma 4.4 lead to

\[ ||u^{k+1}|| = 0, \]

i.e., \( u_{i,j}^{k+1} = 0 \), which shows that the conclusion holds. \( \blacksquare \)

Next, we study the stability result of finite difference scheme (21)–(23).

Theorem 4.2 The finite difference scheme (21), together with (22) and (23) is unconditionally stable with respect to the initial value.

Proof. Due to the perturbation equation

\[
L_x L_y (\varepsilon_{i,j}^{k+1} - \varepsilon_{i,j}^k) = \tau K_1 \delta_{0,t}^{1-\alpha_1} (L_y \delta_x^2 + L_x \delta_y^2) \varepsilon_{i,j}^k - \tau K_2 \delta_{0,t}^{1-\alpha_2} L_x L_y \varepsilon_{i,j}^k
\]

\[ 1 \leq i \leq M_1 - 1, 1 \leq j \leq M_2 - 1, 0 \leq k \leq N - 1, \]

\[ \varepsilon_{i,j}^0 = \rho_{i,j}, \quad 1 \leq i \leq M_1, 1 \leq j \leq M_2, \]

\[ \varepsilon_{i,j}^k = 0, \quad (x_i, y_j) \in \partial \Omega_h, \]

taking the inner product with \((\varepsilon^{k+1} + \varepsilon^k)\) for the first equation and summing up for \( k \) from 0 to \( n - 1 \) yield

\[
\sum_{k=0}^{n-1} (L_x L_y (\varepsilon^{k+1} - \varepsilon^k), (\varepsilon^{k+1} + \varepsilon^k))
\]

\[
= \tau K_1 \sum_{k=0}^{n-1} (\delta_{0,t}^{1-\alpha_1} (L_y \delta_x^2 + L_x \delta_y^2) \varepsilon^k, (\varepsilon^{k+1} + \varepsilon^k))
\]

\[ - \tau K_2 \sum_{k=0}^{n-1} (\delta_{0,t}^{1-\alpha_2} L_x L_y \varepsilon^k, (\varepsilon^{k+1} + \varepsilon^k)). \tag{24} \]

Obviously, the left hand side of equation (24) can be reduced to

\[
\sum_{k=0}^{n-1} (L_x L_y (\varepsilon^{k+1} - \varepsilon^k), (\varepsilon^{k+1} + \varepsilon^k)) = (L_x L_y \varepsilon^n, \varepsilon^n) - (L_x L_y \varepsilon^0, \varepsilon^0).
\]
Applying Lemmas 3.5, 4.5 and 4.6 to the right hand side of equation (24) leads to

\[ \tau K_1 \sum_{k=0}^{n-1} \left( \delta_{0,t}^{1-\alpha_1} \left( \mathcal{L}_y \delta_x^2 + \mathcal{L}_x \delta_y^2 \right) e^k + (e^{k+1} + e^k) \right) \]

\[ -\tau K_2 \sum_{k=0}^{n-1} \left( \delta_{0,t}^{1-\alpha_2} \mathcal{L}_x \mathcal{L}_y e^k + (e^{k+1} + e^k) \right) \]

\[ = -\tau K_1 \sum_{k=0}^{n-1} \left( \delta_{0,t}^{1-\alpha_1} \mathcal{Q}_x e^k, \mathcal{Q}_x \left( e^{k+1} + e^k \right) \right) \]

\[ -\tau K_2 \sum_{k=0}^{n-1} \left( \delta_{0,t}^{1-\alpha_2} \mathcal{Q}_y e^k, \mathcal{Q}_y \left( e^{k+1} + e^k \right) \right) \]

\[ \leq 0. \]

Hence, combining the above two estimations gives

\[ (\mathcal{L}_x \mathcal{L}_y e^n, e^n) \leq (\mathcal{L}_x \mathcal{L}_y e^0, e^0). \]

Using Lemma 4.4 again yields

\[ ||e^n|| \leq \sqrt{3} ||e^0|| = \sqrt{3} ||\rho||. \]

The proof is complete. ■

Finally, we list the convergence result.

**Theorem 4.3** Assume that the solution \( u(x, y, t) \) of equation (17), together with (18) and (19) is suitably smooth, and that \( \{u_{i,j}^k|0 \leq i \leq M_1, 0 \leq j \leq M_2, 0 \leq k \leq N\} \) is the solution of the finite difference scheme (21), together with (22) and (23). Let \( e_{i,j}^k = u(x_i, y_j, t_k) - u_{i,j}^k \), then

\[ ||e^n|| \leq 2C_2 \exp (3T) \sqrt{6L_x L_y} (\tau^2 + h_x^4 + h_y^4). \]

**Proof.** The error equation reads as,

\[ \mathcal{L}_x \mathcal{L}_y \left( e_{i,j}^{k+1} - e_{i,j}^k \right) = \tau K_1 \delta_{0,t}^{1-\alpha_1} \left( \mathcal{L}_y \delta_x^2 + \mathcal{L}_x \delta_y^2 \right) e_{i,j}^k - \tau K_2 \delta_{0,t}^{1-\alpha_2} \mathcal{L}_x \mathcal{L}_y e_{i,j}^k \]

\[ + \tau R_{i,j}^k, \quad 1 \leq i \leq M_1 - 1, 1 \leq j \leq M_2 - 1, 0 \leq k \leq N - 1, \]

\[ u_{i,j}^k = 0, \quad 1 \leq i \leq M_1, 1 \leq j \leq M_2, \]

\[ u_{i,j}^k = \varphi(x_i, y_j, t_k), \quad (x_i, y_j) \in \partial \Omega_{h_x,h_y}. \]

Simple calculations give

\[ (\mathcal{L}_x \mathcal{L}_y e^n, e^n) \leq (\mathcal{L}_x \mathcal{L}_y e^0, e^0) + \tau \sum_{k=0}^{n-1} (R^k, (e^{k+1} + e^k)). \]
Noticing
\[
\tau \sum_{k=0}^{n-1} (R^k, (e^k + e^k)) \leq \frac{\tau}{4} \|e^n\|^2 + \frac{1}{2} \tau \sum_{k=0}^{n-1} \|e^k\|^2 + 2\tau \sum_{k=0}^{n-1} \|R^k\|^2
\]
\[
\leq \frac{1}{4} \|e^n\|^2 + \frac{1}{2} \tau \sum_{k=0}^{n-1} \|e^k\|^2 + 2L_x L_y T C_2^2 (\tau^2 + h_x^4 + h_y^4)^2 ,
\]
on one has
\[
\|e^n\|^2 \leq 6\tau \sum_{k=0}^{n-1} \|e^k\|^2 + 24L_x L_y T C_2^2 (\tau^2 + h_x^4 + h_y^4)^2 ,
\]
i.e.,
\[
\|e^n\| \leq 2C_2 \exp(3T) \sqrt{6L_x L_y T (\tau^2 + h_x^4 + h_y^4)} .
\]
All this completes the proof. \[\blacksquare\]

5 Numerical examples

In this section, numerical experiments are carried out for the proposed numerical algorithms to illustrate our theoretical analysis.

Example 5.1 Take \(u(t) = t^{2+\alpha}, \ t \in [0,1]\). Obviously, the exact value of \(u(t)\) at \(t = 0.5\) is
\[
\text{RLD}_{0,t}^{\alpha}u(t) \big|_{t=0.5} = \frac{\Gamma(3 + \alpha)}{\Gamma(2 + 2\alpha)} \left( \frac{1}{2} \right)^{1+2\alpha} .
\]
By using formula (4) to approximate function \(u(t)\), the absolute error and convergence order are listed in Table 1. It can be seen from the table that the convergence order of formula (4) is almost 2, which is in line with our theoretical analysis.

Example 5.2 Consider the following one-dimensional fractional cable equation:

\[
\frac{\partial u(x,t)}{\partial t} = \frac{1}{\pi^8} \text{RLD}_{0,t}^{1-\alpha_1} \frac{\partial^2 u(x,t)}{\partial x^2} - \text{RLD}_{0,t}^{1-\alpha_2} u(x,t) + f(x,t), \ (x,t) \in (0,1) \times (0,1],
\]

with initial value condition
\[
u(x,0) = 0, \ x \in [0,1],
\]
and boundary value conditions
\[
u(0,t) = 0, \ u(1,t) = 0, \ t \in (0,1],
\]
where the source term is \(f(x,t) = 2 \left( t + \frac{t^{1+\gamma_1}}{\pi^4 \Gamma(2+\gamma_1)} + \frac{t^{1+\gamma_2}}{\Gamma(2+\gamma_2)} \right) \sin \pi x \). The analytical solution of the above system is \(u(x,t) = t^2 \sin \pi x \).
Table 1: The absolute error and convergence order of Example 1 by numerical formula (4).

| α   | τ  | the absolute error          | the convergence order |
|-----|----|------------------------------|-----------------------|
| 0.1 | 1/20 | 9.887230e-003 | —                     |
|     | 1/10 | 3.031228e-003 | 1.7057                |
|     | 1/50 | 8.357173e-004 | 1.8588                |
|     | 1/100 | 2.184140e-004 | 1.9359               |
|     | 1/200 | 5.576440e-005 | 1.9696               |
| 0.3 | 1/20 | 8.294679e-003 | —                     |
|     | 1/10 | 2.177047e-003 | 1.9298                |
|     | 1/50 | 5.554318e-004 | 1.9707                |
|     | 1/100 | 1.401845e-004 | 1.9863               |
|     | 1/200 | 3.520843e-005 | 1.9933               |
| 0.5 | 1/20 | 7.611138e-003 | —                     |
|     | 1/10 | 1.903986e-003 | 1.9991                |
|     | 1/50 | 4.759995e-004 | 2.0000                |
|     | 1/100 | 1.190001e-004 | 2.0000               |
|     | 1/200 | 2.975004e-005 | 2.0000               |
| 0.7 | 1/20 | 5.717212e-003 | —                     |
|     | 1/10 | 1.389258e-003 | 2.0410                |
|     | 1/50 | 3.429045e-004 | 2.0184                |
|     | 1/100 | 8.520286e-005 | 2.0088               |
|     | 1/200 | 2.123685e-005 | 2.0043               |
| 0.9 | 1/20 | 2.151549e-003 | —                     |
|     | 1/10 | 5.116059e-004 | 2.0723                |
|     | 1/50 | 1.249915e-004 | 2.0332                |
|     | 1/100 | 3.090217e-005 | 2.0160               |
|     | 1/200 | 7.683366e-006 | 2.0079               |
In this numerical test, we present the absolute errors and the corresponding temporal and spatial convergence orders in Table 2 for different $\alpha_1$ and $\alpha_2$ by using finite difference scheme (11), together with (12) and (13), which verifies that the second-order accuracy in time and fourth-order accuracy in space direction are obtained. Meanwhile, the evolutions of the absolute errors were depicted in Figs. 5.1 and 5.2 for different orders $\alpha_1, \alpha_2$ and stepsizes $\tau, h$. Obviously, all of the above numerical results are in accordance with our stability and convergence analysis of the proposed numerical algorithm (11), together with (12) and (13).

![Figure 5.1: The evolutions of the absolute errors obtained by the numerical algorithm (11), together with (12) and (13) for ($\alpha_1, \alpha_2$) = (0.3, 0.7) when $\tau = \frac{1}{20}$ and $h = \frac{1}{50}$.](image)

**Example 5.3** Consider the following two-dimensional fractional cable equation:

\[
\frac{\partial u(x, y, t)}{\partial t} = \frac{1}{\pi^8} RL D_{0,t}^{1-\alpha_1} \left( \frac{\partial^2 u(x, y, t)}{\partial x^2} + \frac{\partial^2 u(x, y, t)}{\partial y^2} \right) - RL D_{0,t}^{1-\alpha_2} u(x, y, t) + f(x, y, t), (x, y; t) \in \Omega \times [0, 1],
\]

with initial value condition

\[ u(x, y, 0) = 0, \ (x, y) \in \Omega, \]

and boundary value conditions

\[ u(x, y, t) = 0, \ (x, y) \in \partial \Omega, \ t \in [0, 1], \]
Table 2: The absolute errors (TAE), temporal convergence order (TCO) and spatial convergence order (SCO) of Example 2 by using difference scheme (11), together with (12) and (13).

| $(\alpha_1, \alpha_2)$ | $\tau, h$ | TAE       | TCO   | SCO   |
|------------------------|-----------|-----------|-------|-------|
| (0.2, 0.8)             | $\tau = \frac{1}{4}, h = \frac{1}{9}$ | 4.017482e-02 | —     | —     |
|                        | $\tau = \frac{1}{20}, h = \frac{1}{10}$ | 3.443878e-03 | 1.7721 | 3.5442 |
|                        | $\tau = \frac{1}{45}, h = \frac{1}{15}$ | 7.010198e-04 | 1.9630 | 3.9259 |
|                        | $\tau = \frac{1}{80}, h = \frac{1}{20}$ | 2.254316e-04 | 1.9718 | 3.9437 |
|                        | $\tau = \frac{1}{125}, h = \frac{1}{25}$ | 9.258968e-05 | 1.9939 | 3.9877 |
| (0.4, 0.6)             | $\tau = \frac{1}{5}, h = \frac{1}{5}$ | 3.636258e-02 | —     | —     |
|                        | $\tau = \frac{1}{20}, h = \frac{1}{10}$ | 2.709682e-03 | 1.8731 | 3.7463 |
|                        | $\tau = \frac{1}{45}, h = \frac{1}{15}$ | 5.399822e-04 | 1.9891 | 3.9783 |
|                        | $\tau = \frac{1}{80}, h = \frac{1}{20}$ | 1.725672e-04 | 1.9827 | 3.9653 |
|                        | $\tau = \frac{1}{125}, h = \frac{1}{25}$ | 7.068420e-05 | 2.0000 | 4.0000 |
| (0.5, 0.5)             | $\tau = \frac{1}{5}, h = \frac{1}{5}$ | 3.552136e-02 | —     | —     |
|                        | $\tau = \frac{1}{20}, h = \frac{1}{10}$ | 2.583580e-03 | 1.8906 | 3.7812 |
|                        | $\tau = \frac{1}{45}, h = \frac{1}{15}$ | 5.125456e-04 | 1.9947 | 3.9893 |
|                        | $\tau = \frac{1}{80}, h = \frac{1}{20}$ | 1.635583e-04 | 1.9852 | 3.9704 |
|                        | $\tau = \frac{1}{125}, h = \frac{1}{25}$ | 6.694813e-05 | 2.0015 | 4.0030 |
| (0.6, 0.4)             | $\tau = \frac{1}{5}, h = \frac{1}{5}$ | 3.463007e-02 | —     | —     |
|                        | $\tau = \frac{1}{20}, h = \frac{1}{10}$ | 2.513471e-03 | 1.8921 | 3.7843 |
|                        | $\tau = \frac{1}{45}, h = \frac{1}{15}$ | 4.969394e-04 | 1.9989 | 3.9978 |
|                        | $\tau = \frac{1}{80}, h = \frac{1}{20}$ | 1.583867e-04 | 1.9873 | 3.9746 |
|                        | $\tau = \frac{1}{125}, h = \frac{1}{25}$ | 6.479215e-05 | 2.0029 | 4.0127 |
| (0.8, 0.2)             | $\tau = \frac{1}{5}, h = \frac{1}{5}$ | 4.961263e-02 | —     | —     |
|                        | $\tau = \frac{1}{20}, h = \frac{1}{10}$ | 2.863711e-03 | 2.0574 | 4.2960 |
|                        | $\tau = \frac{1}{45}, h = \frac{1}{15}$ | 5.157444e-04 | 2.1139 | 4.0857 |
|                        | $\tau = \frac{1}{80}, h = \frac{1}{20}$ | 1.542104e-04 | 2.0983 | 4.0304 |
|                        | $\tau = \frac{1}{125}, h = \frac{1}{25}$ | 6.299963e-05 | 2.0059 | 4.0127 |
where the spatial domain $\Omega = (0, 1) \times (0, 1)$, and the source term is chosen as $f(x, y, t) = 2\left(t + \frac{2t^{1+\gamma_1}}{\pi^2\Gamma(2+\gamma_1)} + \frac{t^{1+\gamma_2}}{\Gamma(2+\gamma_2)}\right)\sin \pi x \sin \pi y$. The analytical solution is $u(x, t) = t^2 \sin \pi x \sin \pi y$.

Table 3 lists the computed errors, the temporal and spatial convergence orders respectively, which shows that the convergence order of our scheme (21)–(23) is $O(\tau^2 + h_x^4 + h_y^4)$. Finally, Figs. 5.3 and 5.4 show the evolutions of the absolute errors for Example 5.3 with different orders, temporal and spatial stepsizes, respectively. It can be seen that the numerical results are in good agreement with the theoretical results.

6 Conclusions

In this paper, two classed high-order numerical algorithms are derived to solve the one- and two-dimensional fractional Cable equations based on the derived new second-order difference operator in time direction and the compact techniques in space direction. By using the energy method, we proved that our difference schemes are unconditionally stable to the initial values. The temporal, spatial convergence orders can reach two and four respectively. Finally, numerical examples are given to show the effectiveness of the derived numerical algorithms.
Table 3: The absolute errors (TAE), temporal convergence order (TCO) and spatial convergence order (SCO) of Example 3 by using difference scheme (21), together with (22) and (23).

| $\alpha_1, \alpha_2$ | $\tau, h_x, h_y$ | TAE        | TCO   | SCO   |
|---------------------|------------------|------------|-------|-------|
| (0.2, 0.8)          | $\tau = \frac{1}{2}, h_x = h_y = \frac{1}{5}$ | 3.822113e-02 | —     | —     |
|                     | $\tau = \frac{1}{20}, h_x = h_y = \frac{1}{10}$ | 3.444335e-03 | 1.7360 | 3.4721 |
|                     | $\tau = \frac{1}{40}, h_x = h_y = \frac{1}{15}$ | 6.972544e-04 | 1.9698 | 3.9395 |
|                     | $\tau = \frac{1}{80}, h_x = h_y = \frac{1}{20}$ | 2.254541e-04 | 1.9623 | 3.9246 |
|                     | $\tau = \frac{1}{125}, h_x = h_y = \frac{1}{25}$ | 2.254541e-04 | 1.9983 | 3.9966 |
| (0.4, 0.6)          | $\tau = \frac{1}{2}, h_x = h_y = \frac{1}{5}$ | 3.460010e-02 | —     | —     |
|                     | $\tau = \frac{1}{20}, h_x = h_y = \frac{1}{10}$ | 2.710859e-03 | 1.8370 | 3.6740 |
|                     | $\tau = \frac{1}{40}, h_x = h_y = \frac{1}{15}$ | 5.372495e-04 | 1.9959 | 3.9919 |
|                     | $\tau = \frac{1}{80}, h_x = h_y = \frac{1}{20}$ | 1.726387e-04 | 1.9731 | 3.9462 |
|                     | $\tau = \frac{1}{125}, h_x = h_y = \frac{1}{25}$ | 7.057381e-05 | 2.0044 | 4.0088 |
| (0.5, 0.5)          | $\tau = \frac{1}{2}, h_x = h_y = \frac{1}{5}$ | 3.79936e-02  | —     | —     |
|                     | $\tau = \frac{1}{20}, h_x = h_y = \frac{1}{10}$ | 2.584901e-03 | 1.8544 | 3.7088 |
|                     | $\tau = \frac{1}{40}, h_x = h_y = \frac{1}{15}$ | 5.09957e-04  | 2.0015 | 4.0029 |
|                     | $\tau = \frac{1}{80}, h_x = h_y = \frac{1}{20}$ | 1.636407e-04 | 1.9757 | 3.9513 |
|                     | $\tau = \frac{1}{125}, h_x = h_y = \frac{1}{25}$ | 6.684963e-05 | 2.0059 | 4.0119 |
| (0.6, 0.4)          | $\tau = \frac{1}{2}, h_x = h_y = \frac{1}{5}$ | 3.295086e-02 | —     | —     |
|                     | $\tau = \frac{1}{20}, h_x = h_y = \frac{1}{10}$ | 2.514903e-03 | 1.8559 | 3.7117 |
|                     | $\tau = \frac{1}{40}, h_x = h_y = \frac{1}{15}$ | 4.945015e-04 | 2.0056 | 4.0113 |
|                     | $\tau = \frac{1}{80}, h_x = h_y = \frac{1}{20}$ | 1.584782e-04 | 1.9778 | 3.9555 |
|                     | $\tau = \frac{1}{125}, h_x = h_y = \frac{1}{25}$ | 6.470169e-05 | 2.0073 | 4.0146 |
| (0.8, 0.2)          | $\tau = \frac{1}{2}, h_x = h_y = \frac{1}{5}$ | 4.721574e-02 | —     | —     |
|                     | $\tau = \frac{1}{20}, h_x = h_y = \frac{1}{10}$ | 2.866291e-03 | 2.0210 | 4.0420 |
|                     | $\tau = \frac{1}{40}, h_x = h_y = \frac{1}{15}$ | 5.133508e-04 | 2.1208 | 4.2416 |
|                     | $\tau = \frac{1}{80}, h_x = h_y = \frac{1}{20}$ | 1.543393e-04 | 2.0888 | 4.1775 |
|                     | $\tau = \frac{1}{125}, h_x = h_y = \frac{1}{25}$ | 6.292841e-05 | 2.0103 | 4.0205 |
Figure 5.3: The evolutions of the absolute errors obtained by the numerical algorithm (21), together with (22) and (23) for \((\alpha_1, \alpha_2) = (0.2, 0.8)\) at \(t = 0.5\) when \(\tau = \frac{1}{20}\) and \(h = \frac{1}{40}\).

Figure 5.4: The evolutions of the absolute errors obtained by the numerical algorithm (21), together with (22) and (23) for \((\alpha_1, \alpha_2) = (0.8, 0.2)\) at \(t = 0.5\) when \(\tau = \frac{1}{40}\) and \(h = \frac{1}{20}\).
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