Zero-Hopf Polynomial Centers of Third-Order Differential Equations

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Received: 18 June 2016 / Revised: 14 September 2016 / Published online: 28 October 2016 © Springer Science+Business Media New York 2016

Abstract We study the 3-dimensional center problem at the zero-Hopf singularity in some families of polynomial vector fields arising from third-order polynomial differential equations. After proving some general properties we check that the quadratic family has no 3-dimensional centers. Later we characterize all the 3-dimensional centers in the cubic homogeneous family. Finally we give a partial classification of the 3-dimensional centers at just one singularity of the full cubic family and propose one open problem to close this classification.

Keywords Zero-Hopf singularity · Three-dimensional vector fields · Continua of periodic orbits · Poincaré map

Mathematics Subject Classification 37G15 · 37G10 · 34C07

1 Introduction and Statement of the Main Results

The Kukles systems are described by second-order ordinary differential equations \( \ddot{x} = g_3(x, \dot{x}) \) where \( g_3 \in \mathbb{R}_3[x, \dot{x}] \) is a real cubic polynomial. They were named like this since they were first studied by the Russian mathematician Kukles in [5]. The over dote denotes, as usual, derivative with respect to the time independent variable \( t \). When \( g_3(x, y) = -x + \cdots \) in [5] it is investigated the center problem at the monodromic singularity \( (x, y) = (0, 0) \) of its associated planar vector field \( \dot{x} = y, \dot{y} = g_3(x, y) \). Due to both theoretical and practical applications several papers have been published on the center problem for this kind of cubic
systems, see for example [2,10]. We recall that the characterization of the centers in analytic families of planar vector fields began with the pioneering works of Poincaré [8] and Liapunov [6]. See also the the book [9] for a modern approach based on computational algebra techniques.

In this paper we consider a more general situation, by considering higher-order Kukles systems. Indeed, we consider third-order ordinary differential equations

$$\dddot{x} = f_n(x, \dot{x}, \ddot{x})$$  \hspace{1cm} (1)

with $f_n \in \mathbb{R}_n[x, \dot{x}, \ddot{x}]$, a real polynomial in three variables of degree $n$.

We transform the differential equation (1) in the usual way as a polynomial differential system

$$\dot{x} = y, \hspace{0.2cm} \dot{y} = z, \hspace{0.2cm} \dot{z} = f_n(x, y, z).$$  \hspace{1cm} (2)

A real singularity $(x, y, z) = (x_0, 0, 0) \in \mathbb{R}^3$ of (2) is called a zero-Hopf singularity if its associated eigenvalues are $\{ \pm i, 0 \}$ with $i^2 = -1$. We say that a zero-Hopf singularity is a 3-dimensional center if there is a neighborhood of it in $\mathbb{R}^3$ completely foliated by periodic orbits of (2), including continua of equilibria as trivial periodic orbits. As far as we know the only work that completely addresses this issue is [4] although in [7] it also appears in the context of the complete analytic local integrability at the zero-Hopf singularity.

The main results of this paper are the following. First we see how a discrete symmetry acts on any zero-Hopf singularity of (2) producing a 3-dimensional center.

**Theorem 1** If $f_n(x, -y, z) = -f_n(x, y, z)$ then any zero-Hopf singularity of (2) is a 3-dimensional center.

The next result shows one reduction from the 3-dimensional center problem to the classical nondegenerate center problem of a planar vector field.

**Theorem 2** Let the origin be a zero-Hopf singularity of (2) with $\frac{\partial f_n}{\partial x} \equiv 0$. If the planar system $\dot{y} = z, \dot{z} = f_n(y, z)$ has a center at the origin in $\mathbb{R}^2$ then the origin in $\mathbb{R}^3$ is a 3-dimensional center of (2).

Next result tell us that the minimum degree $n$ needed for the appearance of 3-dimensional centers in family (2) is $n = 3$.

**Theorem 3** The quadratic $(n = 2)$ vector field (2) has no 3-dimensional centers.

Now we present the classification of all the 3-dimensional centers of family (2) when $n = 3$ and $f_3$ is a cubic homogeneous polynomial.

**Theorem 4** The cubic vector field (2) with $f_3(x, y, z)$ a cubic homogeneous polynomial satisfies the following:

(i) If $f_3(x, 0, 0) \neq 0$ then there is no zero-Hopf singularity in the whole family (2).

(ii) When $f_3(x, 0, 0) \equiv 0$ family (2) has a 3-dimensional center if and only if it has the form

$$\dot{x} = y, \hspace{0.2cm} \dot{y} = z, \hspace{0.2cm} \dot{z} = -K^2 x_2 y + Ay^3 + Bxyz + Cyz^2$$  \hspace{1cm} (3)

with $K \neq 0$. Actually it has a continuum of 3-dimensional centers at any point of the x-axis except the origin.
When \( n = 3 \) and the origin is a zero-Hopf singularity of (2) with spectrum \( \{ \pm i \omega, 0 \} \) and \( \omega \neq 0 \), the cubic polynomial \( f_3 \) is written as \( f_3(x, y, z) = -\omega^2y + \hat{f}_3(x, y, z) \) with \( \hat{f}_3 \) a polynomial of degree 3 without constant and linear terms. Performing a linear change of variables and a time rescaling \( t \mapsto t/\omega \) the linear part of system (2) is written in real Jordan canonical form and (2) becomes

\[
\dot{x} = -y + \omega^2 F(x, y, z), \quad \dot{y} = x, \quad \dot{z} = F(x, y, z)
\] (4)

where \( \omega^3 F(x, y, z) = \hat{f}_3(-x/\omega^2 + z, y/\omega, x) \).

Now we give a partial classification of the 3-dimensional centers at the origin of the full cubic \((n = 3)\) family (2).

**Theorem 5** The cubic \((n = 3)\) family (2) has a 3-dimensional center at the origin when the function \( F(x, y, z) \) of its associated family (4) is of the form:

(i) \( \omega^2 F(x, y, z) = Fx^2 + Cxy - Fy^2 + Gx^2z + Dxyz - Gy^2z \) with \( DF - CG = 0 \);

(ii) \( \omega^2 F(x, y, z) = Cxy + Hx^2y - \frac{H}{3} y^3 - \frac{4}{3} H \omega^2xyz \);

(iii) \( \omega^2 F(x, y, z) = Hx^2y - \frac{H}{3} y^3 + Gx^2z + Dxyz - Gy^2z \) with \( GH = 0 \);

(iv) \( \omega^2 F(x, y, z) = Cxy + Hx^2y - Hy^3 - H \omega^2xyz \) with \( H \neq 0 \);

(v) \( \omega^2 F(x, y, z) = Cxy + Hx^2y - \frac{1}{2} H \omega^2xyz \) with \( H \neq 0 \);

(vi) \( \omega^2 F(x, y, z) = \left( \frac{2D}{\omega^2} + H \right) y^3 + Hx^2y + Dxyz \);

(vii) \( \omega^2 F(x, y, z) = Cxy + \frac{A}{\omega^2} yz + Hx^2y + \frac{B}{\omega^2} y^3 + Dxyz + \frac{E}{\omega^2} yz^2 \).

Moreover if the origin is a 3-dimensional center then \( F(x, y, z) \) is either of the above forms or has the expressions:

(viii) \( \omega^2 F(x, y, z) = Fx^2 + Cxy - Fy^2 + Ix^3 + Hx^2y - 3Ixy^2 - \frac{H}{3} y^3 - \frac{3}{2} I \omega^2 x^2 z - \omega^2 xyz + \frac{1}{2} I \omega^2 y^2 z \) with \( 2FH - 3CI = 0 \);

(ix) \( \omega^2 F(x, y, z) = Fx^2 - \frac{DF}{IK} xy - Fy^2 + Ix^3 + Hx^2y - 3Ixy^2 + \left( \frac{2D}{\omega^2} + H \right) y^3 - I \omega^2 x^2 z + Dxyz + I \omega^2 y^2 z \) with \( I \neq 0 \) and \( 2D^2 + 3DHK^2 + H^2 K^4 - I^2 K^4 = 0 \);

(x) \( \omega^2 F(x, y, z) = Fx^2 + Cxy - Fy^2 + Hx^2y - \frac{AF}{\omega^2} xy^2 + \frac{A}{\omega^2} yz - H \omega^2 xyz + \frac{AF}{\omega^2} yz^2 \).

**Remark 6** We have proved that family (4) with the \( F(x, y, z) \) given in the statement (viii) of Theorem 5 has a 3-dimensional center at the origin in the parameter case \( I = 0 \). Otherwise if \( I \neq 0 \) then we have also showed that after a translation in the \( z \)-axis we can assume the parameters \( F = C = 0 \) in the expression of such a \( F \) without loss of generality.

On the other hand we have also checked that the origin of (4) with the \( F(x, y, z) \) stated in part (x) of Theorem 5 is a 3-dimensional center when the parameters are either \( F = 0 \) or \( A = H = 0 \).

All together Theorem 5 and Remark 6 implies that we will have the complete classification of the 3-dimensional centers at the origin in the cubic family (2) if and only if we can solve the next problem.

**Open Problem** Find out if the origin is a 3-dimensional center for family (4) where \( F(x, y, z) \) is given by one of the following tree cases of Theorem 5: (viii) with \( I \neq 0 \) and \( F = C = 0 \); (ix); (x) with either \( F \neq 0 \) or \( A^2 + H^2 \neq 0 \).

In analogy with the classical two-dimensional center problem in the qualitative theory of differential equations, the necessary 3-dimensional center conditions in the parameter space for family (4) are derived by vanishing the initial coefficients \( \delta_j \) of the series of an adequate Poincaré map, see the next section. But the sufficient 3-dimensional center conditions (which
guarantee that actually \( \delta_j \equiv 0 \) for any \( j \in \mathbb{N} \) in the forthcoming proof of Theorem 5 needs the use of symmetry-integrability arguments. The technical difficulties why the cases stated in the above open problem cannot be solved are that, although we have checked that \( \delta_j \equiv 0 \) for \( 1 \leq j \leq 11 \) so that it is very probable that the singularity is a 3-dimensional center, we have not been successful in finding the necessary symmetries or first integrals that ensure the existence of that 3-dimensional center.

2 The Background and Some Auxiliary Results

2.1 The Poincaré Map at the Zero-Hopf Singularity

In this subsection we present a brief description of the theory introduced in [4] and next developed in [3]. There, it is considered an analytic three-dimensional system

\[
\begin{align*}
\dot{x} &= -y + F_1(x, y, z) \\
\dot{y} &= x + F_2(x, y, z) \\
\dot{z} &= F_3(x, y, z),
\end{align*}
\]

(5)
defined on a neighborhood \( \mathcal{U} \subset \mathbb{R}^3 \) of the origin and having a zero-Hopf singularity at the origin. Then, the \( F_i \) are real analytic functions on \( \mathcal{U} \) without independent and linear terms.

**Theorem 7** [3,4] We consider system (5) defined on a neighborhood \( \mathcal{U} \subset \mathbb{R}^3 \) of the origin. Let \( \delta > 0 \) be sufficiently large but fixed and define \( \mathcal{C}_\delta = \{(x, y, z) \in \mathcal{U} : z^2 > \delta(x^2 + y^2)\} \), a thin solid cone with vertex at the origin surrounding the z-axis. Doing first the rescaling \( (x, y, z) \mapsto (x/\varepsilon, y/\varepsilon, z/\varepsilon) \) and later the polar blow-up \( (x, y, z) \mapsto (\theta, r, w) \) defined by

\[
\begin{align*}
x &= r \cos \theta, & y &= r \sin \theta, & z &= rw,
\end{align*}
\]

(6)

system (5) can be written in \( \mathcal{U}\setminus\mathcal{C}_\delta \), for \(|r| \) and \(|\varepsilon| \) sufficiently small, as the analytic system

\[
\begin{align*}
\frac{dr}{d\theta} &= \varepsilon R(\theta, r, w; \varepsilon), & \frac{dw}{d\theta} &= \varepsilon W(\theta, r, w; \varepsilon),
\end{align*}
\]

(7)

with invariant set \(|r = 0|\) and defined on the cylinder \(\{\theta, r, w) \in S^1 \times \mathbb{R} \times \mathcal{K}\} \) where \( S^1 = \mathbb{R}/2\pi \mathbb{Z} \) and \( \mathcal{K} = \{w \in \mathbb{R} : |w| \leq \delta^2\} \).

Denoting by \( \Psi(\theta; r_0, w_0; \varepsilon) = (r(\theta; r_0, w_0; \varepsilon), w(\theta; r_0, w_0; \varepsilon)) \) the solution of (7) satisfying the initial condition \( \Psi(0; r_0, w_0; \varepsilon) = (r_0, w_0) \in \mathbb{R} \times \mathcal{K} \) for \(|r_0| \) sufficiently small, one can define the Poincaré translation map \( \Pi(r_0, w_0; \varepsilon) = \Psi(2\pi; r_0, w_0; \varepsilon) \) and the analytic displacement map \( d(r_0, w_0; \varepsilon) = \Pi(r_0, w_0; \varepsilon) - (r_0, w_0) \).

**Remark 8** The need to restrict the values of \( w \) to the arbitrary but fixed compact set \( \mathcal{K} \) containing the origin is clarified in [3]. From the geometry associated to the polar blow-up (6), we see that \((x, y, z) \in \mathcal{U}\setminus\mathcal{C}_\delta \) when \( w \in \mathcal{K} \) and that (6) is a diffeomorphism in \( \mathcal{U}\setminus\mathcal{C}_\delta \). Although the polar blow-up (6) does not cover \( \mathcal{U} \) not intersecting the z-axis is contained in \( \mathcal{U}\setminus\mathcal{C}_\delta \) for \( \delta \) sufficiently large. In consequence, the zeros of the displacement map \( d(r_0, w_0) \), with \( w_0 \in \mathcal{K} \) and \(|r_0| \ll 1 \), pick up all these periodic orbits. Anyway, see the arguments of [3] based on the properties real analytic functions of several variables that vanish on a set of positive measure, in [3] it is proved that the origin of system (5) is a 3-dimensional center if and only if \( d(r_0, w_0; \varepsilon) \equiv 0 \).
The idea behind the rescaling \((x, y, z) \mapsto (x/\varepsilon, y/\varepsilon, z/\varepsilon)\) in Theorem 7 is that now we can compute the Taylor series about \(\varepsilon = 0\)
\[
d(r_0, w_0; \varepsilon) = \sum_{j \geq 1} \delta_j(r_0, w_0)\varepsilon^j
\]
such that condition \(d(r_0, w_0; \varepsilon) \equiv 0\) is equivalent to the vanish of all the coefficients \(\delta_j(r_0, w_0)\). Using the terminology of the analogous two-dimensional problems in the qualitative theory of differential equations (see for example [1]), we will refer to the analytic function \(\delta_j : \mathbb{R}^2 \to \mathbb{R}^2\) as the \(j\)th Melnikov function.

The computation of the Melnikov functions \(\delta_j\) is algorithmic although aid of an algebraic manipulator is highly recommended because the calculations involved are massive. First we expand in power series of \(\varepsilon\) both system (7) and its solution. Then we have
\[
\frac{dr}{d\theta} = \varepsilon R(\theta, r, w; \varepsilon) = \varepsilon \sum_{j \geq 1} R_j(\theta; r_0, w_0)\varepsilon^j,
\]
\[
\frac{dw}{d\theta} = \varepsilon W(\theta, r, w; \varepsilon) = \varepsilon \sum_{j \geq 1} W_j(\theta; r_0, w_0)\varepsilon^j,
\]
\[
(8)
\]
and
\[
\Psi(\theta; r_0, w_0; \varepsilon) = (r_0, w_0) + \left( \sum_{j \geq 1} \Psi_{1,j}(\theta; r_0, w_0)\varepsilon^j, \sum_{j \geq 1} \Psi_{2,j}(\theta; r_0, w_0)\varepsilon^j \right).
\]
Since \(\Psi\) satisfies the initial condition \(\Psi(0; r_0, w_0; \varepsilon) = (r_0, w_0)\) it follows that \(\Psi_{i,j}(0; r_0, w_0) = 0\) for all \(j \in \mathbb{N}\) and \(i \in \{1, 2\}\). In summary the displacement map is
\[
d(r_0, w_0; \varepsilon) = \Psi(2\pi; r_0, w_0; \varepsilon) - (r_0, w_0)
\]
\[
= \sum_{j \geq 1} \left( \Psi_{1,j}(2\pi; r_0, w_0), \Psi_{2,j}(2\pi; r_0, w_0) \right) \varepsilon^j,
\]
and consequently the \(j\)th Melnikov function is
\[
d_j(r_0, w_0) = \left( \Psi_{1,j}(2\pi; r_0, w_0), \Psi_{2,j}(2\pi; r_0, w_0) \right).
\]

2.2 Generalized Reversibility and Local Analytic First Integrals Near Equilibriums

In this subsection we summarize some results obtained in [7] that we will need later in the proof of Theorem 1. The framework is to detect analytic first integrals near equilibrium points (located at the origin) of analytic vector fields
\[
\dot{x} = f(x) = Ax + \cdots
\]
in some open neighborhood of the origin in \(\mathbb{C}^n\). The main argument is the well-known fact that any formal first integral of a (formal) Poincaré–Dulac normal form
\[
\dot{x} = \hat{f}(x) = Ax + \cdots
\]
of (9) is also a first integral of the linear system
\[ \dot{x} = A_s x \] (11)
where \( A_s \) is the semisimple part of the linearization \( A \) of (9). Recall that we can always decompose \( A = A_s + A_n \) with \( A_s \) semisimple (diagonalizable in \( \mathbb{C} \)) and \( A_n \) is nilpotent with commutation \([A_s, A_n] = 0\). Also we remind that the formal vector field (10) is in Poincaré–Dulac normal form if \([A_s, \hat{f}] = 0\).

In this context a very interesting case to investigate is whether all formal first integrals of \( A_s \) are conserved by a normal form because this property may ensure convergence of the normal form and also complete local analytic integrability. The following result was first proven by Zhang [11,12], see also Theorem 9 in [7].

**Theorem 9** [7,11,12] Assume that \( A_s \neq 0 \) and the linear system (11) in \( \mathbb{C}^n \) admits \( n - 1 \) independent polynomial first integrals. If some formal Poincaré–Dulac normal form of (9) admits \( n - 1 \) independent formal first integrals, then (9) admits a convergent transformation to Poincaré–Dulac normal form and also it possesses \( n - 1 \) independent analytic first integrals.

Now we adapt the general Proposition 11 of [7] to our purpose, hence we state it in the very particular case that we need. We say that (9) is **time-reversible** if there is a nonsingular \( n \times n \) matrix \( T \) such that \( T^{-1} \circ f \circ T = -f \).

**Proposition 10** [7] Consider system (9) with \( n = 3 \) and diagonal linear part \( A = A_s \) with spectrum \( \{ \pm \iota \omega, 0 \} \) where \( \omega \in \mathbb{R} \setminus \{0\} \). If (9) is time-reversible then the two independent first integrals of (11) are conserved in a Poincaré–Dulac normal form of (9).

### 3 Proof of Theorem 1

First we observe that any real singularity \((x_0, 0, 0)\) of (2) can be placed at the origin via the translation \((x, y, z) \mapsto (x - x_0, y, z)\) without affecting the form of (2). Hence without loss of generality we will assume that the origin is a zero-Hopf singularity of (2). In particular \( f_n(x, y, z) = -\omega^2 y + \hat{f}_n(x, y, z) \) with \( \omega \neq 0 \) and \( \hat{f}_n \) a polynomial of degree \( n \) without constant nor linear terms such that the eigenvalues of the linearization at the origin of (2) are \( \{ \pm \iota \omega, 0 \} \). Moreover we can perform a linear change of variables \((x, y, z)^T \mapsto P^{-1}(x, y, z)^T\) where

\[
P = \begin{pmatrix}
-\frac{1}{\omega^2} & 0 & 1 \\
0 & \frac{1}{\omega} & 0 \\
1 & 0 & 0 
\end{pmatrix}
\]

and a time rescaling \( t \mapsto t/\omega \) such that the linear part of system (2) is written in real Jordan canonical form. In short, after all these transformations (2) becomes

\[
\dot{x} = -y + \omega^2 F(x, y, z) \\
\dot{y} = x \\
\dot{z} = F(x, y, z)
\] (12)

where \( F(x, y, z) = \frac{1}{\omega^3} \hat{f}_n(-x/\omega^2 + z, y/\omega, x) \).

We also can transform system (12) via the linear complex change of variables \((x, y, z) \mapsto (X, Y, Z) = (x + iy, x - iy, z)\) diagonalizing in \( \mathbb{C} \) the linear part of (12). In this way system (12) becomes a system in \( \mathbb{C}^3 \) with the form

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\[ \dot{X} = iX + \omega^2 \mathcal{F} \left( \frac{X + Y}{2}, \frac{i(Y - X)}{2}, Z \right), \]
\[ \dot{Y} = -iY + \omega^2 \mathcal{F} \left( \frac{X + Y}{2}, \frac{i(Y - X)}{2}, Z \right), \]
\[ \dot{Z} = \mathcal{F} \left( \frac{X + Y}{2}, \frac{i(Y - X)}{2}, Z \right). \]  

\[ (13) \]

It is straightforward to check that this complex system is time-reversible with respect to the linear involution \((X, Y, Z) \mapsto (Y, X, Z)\) that exchanges \(X\) and \(Y\) if and only if \(\mathcal{F}\) is an odd function in second variable, i.e., \(\mathcal{F}(x, -y, z) = -\mathcal{F}(x, y, z)\). Notice that this symmetry is inherited by \(f_n\), hence \(f_n(x, -y, z) = -f_n(x, y, z)\). In this symmetric case both first integrals of the linear part are conserved in a Poincaré–Dulac normal form of \((13)\), see Proposition 10. Hence the complex system admits two independent analytic first integrals \(\hat{H}_1(X, Y, Z) = X^2 - Y^2 + \cdots\) and \(\hat{H}_2(X, Y, Z) = Z + \cdots\), see Theorem 9, which implies that the real system \((12)\) possesses two independent locally real analytic first integrals \(H_1(x, y, z) = x^2 + y^2 + \cdots\) and \(H_2(x, y, z) = z + \cdots\). Then the origin of \((12)\) is a 3-dimensional center and consequently the singularity \((x_0, 0, 0)\) of \((2)\) is too.

4 Proof of Theorem 2

We take \(f_n\) independent of \(x\) so that in system \((2)\) a decoupling occurs. Thus we deal with the planar subsystem
\[ \dot{y} = z, \quad \dot{z} = f_n(y, z) = -\omega^2 y + \cdots \]

\[ (14) \]

where the starting terms in \(f_n(y, z)\) are derived from the assumption of having a zero-Hopf singularity of \((2)\) at the origin, see the beginning of the proof of Theorem 1. By hypothesis \((14)\) has a nondegenerate center at the origin. Therefore the solution \(\xi(t; (y_0, z_0)) = (y(t; (y_0, z_0)), z(t; (y_0, z_0)))\) of \((14)\) passing at time \(t = 0\) through the point \((y_0, z_0) \in U \subset \mathbb{R}^2\) in a sufficiently small neighborhood \(U\) of the origin is \(T(y_0, z_0)\)-periodic for any \((y_0, z_0) \in U\) and also there is an analytic first integral \(H(y, z) = y^2 + z^2 + \cdots\) defined in \(U\) of \((14)\). Going back to the full system \((2)\) in \(\mathbb{R}^3\), it has the solution \((x(t; (x_0, y_0, z_0)), \xi(t; (y_0, z_0)))\) with initial condition \((x_0, y_0, z_0) \in \mathbb{R} \times U\) and where \(x(t; (x_0, y_0, z_0)) = x_0 + \int_0^t y(t; (y_0, z_0)) dt\). We claim that actually the function \(x(t; (x_0, y_0, z_0))\) is also a \(T(y_0, z_0)\)-periodic function implying that the origin of \((2)\) is a 3-dimensional center. To prove the claim we only need to prove that the function \(y(t; (y_0, z_0))\) has zero average in the interval \([0, T(y_0, z_0)]\). This is true because this average is

\[ \int_0^{T(y_0, z_0)} y(t; (y_0, z_0)) dt = \int_{H=h} dx \equiv 0 \text{ for all } (y_0, z_0) \in U, \]

where in the last step we integrate an exact 1-form over closed level curves \(\{H(y, z) = h\}\) where \(h = H(y_0, z_0)\).

Remark 11 We observe Theorem 2 can be restated in terms of the transformed system \((12)\) as follows. If \(\frac{\partial \mathcal{F}}{\partial x} \equiv 0\) and the planar vector field \(\dot{x} = -y + \omega^2 \mathcal{F}(x, y), \dot{y} = x\) has a center at the origin in \(\mathbb{R}^2\) then the origin in \(\mathbb{R}^3\) is a 3-dimensional center of \((12)\).
5 Proof of Theorem 3

We take \( f_2(x, y, z) = a_0(y, z) + a_1(y, z)x + a_2(y, z)x^2 \). A singularity \( p_0 = (x_0, 0, 0) \in \mathbb{R}^3 \) of (2) corresponds to a real solution of the quadratic equation \( f_2(x_0, 0, 0) = 0 \). It is easy to check that when \( a_2(0, 0) = 0 \) then either there are no singularities of (2) when \( a_1(0, 0) = 0 \) or there is just one singularity but without any zero associated eigenvalue otherwise. So we continue the proof assuming that \( a_2(0, 0) \neq 0 \). We impose the condition

\[
a_0(0, 0) = \frac{a_1(0, 0)^2}{4a_2(0, 0)} \tag{15}
\]
in order to have a null eigenvalue. Next we have the conditions

\[
\frac{\partial a_0}{\partial z}(0, 0) = \frac{a_1(0, 0)}{4a_2(0, 0)} \left[ 2a_2(0, 0) \frac{\partial a_1}{\partial z}(0, 0) - a_1(0, 0) \frac{\partial a_2}{\partial z}(0, 0) \right], \tag{16}
\]

\[
0 > 4a_2^2(0, 0) \frac{\partial a_0}{\partial y}(0, 0) - 2a_1(0, 0)a_2(0, 0) \frac{\partial a_1}{\partial y}(0, 0)
+ a_1^2(0, 0) \frac{\partial a_2}{\partial y}(0, 0) \tag{17}
\]

for having also two non-zero pure imaginary eigenvalues.

Since \( f_2 \) is a polynomial of degree 2 we have

\[
a_0(y, z) = b_0 + b_1y + b_2z + b_3y^2 + b_4yz + b_5z^2,
\]

\[
a_1(y, z) = c_0 + c_1y + c_2z,
\]

\[
a_2(y, z) = d_0.
\]

Then \( a_2(0, 0) \neq 0 \), (15) and (16) produces

\[
d_0 \neq 0, \quad b_0 = \frac{c_0^2}{4d_0}, \quad b_2 = \frac{c_0c_2}{2d_0}, \quad b_1 = \frac{2c_0c_1d_0 - K^2}{4d_0^2}
\]

where in the last condition we have introduced a new parameter \( K \neq 0 \) such that inequality (17) reads for \( 0 > -K^2 \). The coordinates of the zero-Hopf singularity are \( p_0 = (-c_0/(2d_0), 0, 0) \) whose eigenvalues are \( \{ \pm i\omega, 0 \} \) with \( \omega = K/(2d_0) \). As in the first paragraph of the proof of Theorem 1, we translate \( p_0 \) to the origin; we perform a linear transformation to get the real Jordan canonical form of the linear part of the system and we rescale the time \( t \mapsto t/\omega \). Thus we obtain that (2) is transformed into a system (12) of the form

\[
\dot{x} = -y + \omega^2 \mathcal{F}(x, y, z)
\]

\[
\dot{y} = x
\]

\[
\dot{z} = \mathcal{F}(x, y, z) \tag{18}
\]

with

\[
\mathcal{F}(x, y, z) = \frac{4Cd_0^2}{K^2} x^2 + \frac{4Bd_0^2}{K^2} xy + \frac{32b_3d_0^5}{K^5} y^2 + \frac{4Ad_0^2}{K^2} xz
+ \frac{16c_1d_0^4}{K^4} yz + \frac{8d_0^4}{K^3} z^2
\]
after the reparametrization \((c_2, b_4, b_5) \mapsto (A, B, C)\) with
\[
c_2 = \frac{16d_0^4 + AK^3}{2d_0K^2}, \quad b_4 = \frac{16c_1d_0^4 + BK^4}{4d_0^2K^2}, \quad b_5 = \frac{32d_0^6 + 4Ad_0^2K^3 + CK^5}{2d_0K^4}.
\]

Using the theory developed in [4] and summarized in Sect. 2.1 gives that family \((18)\) with \(d_0 \neq 0\) and \(K \neq 0\) has not a 3-dimensional center at the origin because the first associated Melnikov function \(\delta_1(r_0, w_0)\) is
\[
\delta_1(r_0, w_0) = \left(\frac{A \pi r_0^2w_0}{K^5}, -\frac{\pi r_0}{K^5}[−32b_3d_0^5 − 4Cd_0^2K^3 − 16d_0^4K^2w_0^2 + AK^5w_0^2]\right)
\]
and clearly \(\delta_1(r_0, w_0) \neq 0\) for all possible choice of the parameters. Hence the theorem follows.

6 Proof of Theorem 4

If \(n = 3\) and \(f_3\) is a cubic homogeneous polynomial, then is a simple exercise to check that there is no zero-Hopf singularity in family \((2)\) if \(f_3(x, 0, 0) \neq 0\). Hence statement (i) is straightforward to check and we concentrate our attention only in proving statement (ii).

Following the notation of the proof of Theorem 3 and taking into account that \(f_3(x, 0, 0) = 0\) we take \(f_3(x, y, z) = a_0(y, z) + a_1(y, z)x + a_2(y, z)x^2\) with
\[
\begin{align*}
a_0(y, z) &= b_6y^3 + b_7y^2z + b_8yz^2 + b_9z^3, \\
a_1(y, z) &= c_3y^2 + c_4yz + c_5z^2, \\
a_2(y, z) &= d_1y + d_2z.
\end{align*}
\]

Under these conditions it is easy to check that system \((2)\) has zero-Hopf singularities only when \(d_2 = 0\) and \(d_1 = -K^2\) with \(K \neq 0\). In short a continuum of singularities \((x_0, 0, 0)\) on the x-axis appear for any \(x_0 \in \mathbb{R}\) such that all of them except the origin are zero-Hopf points with eigenvalues \(\{\pm i\omega, 0\}\) where \(\omega = Kx_0\).

Pick up one of these zero-Hopf singularities \((x_0, 0, 0)\) with \(x_0 \neq 0\) and translate it to the origin by means of \((x, y, z) \mapsto (x - x_0, y, z)\). Next we follow the first paragraph of the proof of Theorem 1 and we do a linear change of variables to obtain the linear part of the system in real Jordan canonical form and also we perform the time rescaling \(t \mapsto t/\omega\). After all these transformations system \((2)\) is written as a system \((12)\) of the form
\[
\begin{align*}
\dot{x} &= -y + \omega^2F(x, y, z) \\
\dot{y} &= x \\
\dot{z} &= F(x, y, z)
\end{align*}
\]
where \(F\) is the following cubic polynomial
\[
F(x, y, z) = -\frac{c_5}{K^3x_0^2}x^2 - \frac{2 + c_4x_0^2}{K^4x_0^5}xy - \frac{c_3}{K^5x_0^4}y^2 + \frac{2}{K^2x_0^3}yz \\
+ \frac{b_9K^2x_0^2 - c_5}{K^3x_0^5}x^3 + \frac{-1 - c_4x_0^2 + b_8K^2x_0^4}{K^6x_0^8}x^2y.
\]
\[- \frac{c_3 - b_7 K^2 x_0^2}{K^7 x_0^2} y^2 + \frac{b_6}{K^6 x_0^2} y^3 + \frac{c_5}{K^3 x_0^2} x^2 z + 2 + \frac{c_4 x_0^2}{K^4 x_0^6} x y z + \frac{c_3}{K^5 x_0^2} y^2 z^2 - \frac{1}{K^2 x_0^4} y^2 z^2.\]

Using the theory developed in [4] and summarized in Sect. 2.1 gives that the first associated Melnikov functions \( \delta_1(r_0, w_0) \) at the origin for family (19) are the following.

\[\delta_1(r_0, w_0) = \left(0, -\frac{\pi r_0 (c_3 + c_5 K^2 x_0^2)}{K^5 x_0^4}\right)\]

and clearly \( \delta_1(r_0, w_0) \equiv 0 \) if and only if \( c_3 = -c_5 K^2 x_0^2 \). Next we compute

\[\delta_2(r_0, w_0) = \left(\frac{\pi r_0^3 (b_7 - 4c_5 + 3b_9 K^2 x_0^2)}{4K^3 x_0^3}, -\frac{\pi r_0^2 w_0 (b_7 - 12c_5 + 3b_9 K^2 x_0^2)}{4K^3 x_0^3}\right)\]

which vanishes identically only when \( c_5 = 0 \) and \( b_7 = -3b_9 K^2 x_0^2 \). After we get that

\[\delta_3(r_0, w_0) = \left(-\frac{3b_9 \pi r_0^4 w_0}{2K x_0^2}, -\frac{b_9 \pi r_0^3 (2 + c_4 x_0^2 + 3K^4 w_0^2 x_0^4)}{2K^5 x_0^6}\right)\]

and \( \delta_3(r_0, w_0) \equiv 0 \) if and only if \( b_9 = 0 \). Also we can check that \( \delta_4(r_0, w_0) = \delta_5(r_0, w_0) \equiv 0 \).

The resulting family (2) becomes (3) after the relabeling of parameters \( (b_6, c_4, b_8) = (A, B, C) \). The proof finishes noticing that in (3) we have that \( f_3(x, y, z) = -K^2 x^2 y + Ay^3 + Bxyz + Cyz^2 \) is an odd function in the variable \( y \) and applying Theorem 1.

### 7 Proof of Theorem 5

Let \( f_3(x, y, z) = a_0(y, z) + a_1(y, z)x + a_2(y, z)x^2 + a_3 x^3 \). The singularity \( p_0 = (x_0, 0, 0) \in \mathbb{R}^3 \) of (2) corresponds to a real solution of the cubic equation \( f_3(x_0, 0, 0) = 0 \). Since we want the origin \( (0, 0, 0) \) to be a singular point of (2) with \( n = 3 \) we must have \( a_0(0, 0) = 0 \). Then we get \( a_1(0, 0) = 0 \) to have a null eigenvalue associated to the origin and moreover

\[\frac{\partial a_0}{\partial z}(0, 0) = 0, \quad \frac{\partial a_0}{\partial y}(0, 0) < 0,\]

for having also two non-zero pure imaginary eigenvalues.

Now we take the following expression of the polynomial \( f_3 \):

\[a_0(y, z) = b_0 + b_1 y + b_2 z + b_3 y^2 + b_4 yz + b_5 z^2 + b_6 y^3 + b_7 y^2 z + b_8 yz^2 + b_9 z^3,\]

\[a_1(y, z) = c_0 + c_1 y + c_2 z + c_3 y^2 + c_4 yz + c_5 z^2,\]

\[a_2(y, z) = d_0 + d_1 y + d_2 z,\]

\[a_3 = e_0.\]

Conditions \( a_0(0, 0) = a_1(0, 0) = 0 \) and (20) yield

\[b_0 = c_0 = b_2 = 0, \quad b_1 = -K^2\]

with \( K \neq 0 \). The resulting system (2) with \( n = 3 \) has a singularity at the origin with eigenvalues \( \{\pm i \omega, 0\} \) with \( \omega = K \).
As in the first paragraph of the proof of Theorem 1, after performing a linear transformation to get the real Jordan canonical form of the linear part of the system and doing a time rescaling $t \mapsto t/\omega$ we obtain that (2) is transformed into a system (12) of the form

\[
\begin{align*}
\dot{x} &= -y + K^2 \mathcal{F}(x, y, z) \\
\dot{y} &= x \\
\dot{z} &= \mathcal{F}(x, y, z)
\end{align*}
\]

with

\[
\mathcal{F}(x, y, z) = \frac{F}{K^2} x^2 + \frac{b_3}{K^2} y^2 + \frac{A}{K^2} xz + \frac{c_1}{K^4} yz + \frac{d_0}{K^3} z^2 + \frac{I}{K^2} x^3
\]

\[
+ \frac{C}{K^2} xy + \frac{H}{K^2} x^2 y + \frac{E}{K^2} x^2 y + \frac{b_6}{K^6} y^3 + \frac{G}{K^2} x^2 z,
\]

\[
+ \frac{D}{K^2} x^2 yz + \frac{c_3}{K^5} y^2 z + \frac{B}{K^2} x^2 z^2 + \frac{d_1}{K^4} y^2 z^2 + \frac{e_0}{K^3} z^3
\]

after the reparametrization

\[
(c_2, d_2, b_4, c_4, b_7, b_5, b_8, b_9) \mapsto (A, B, C, D, E, F, G, H, I)
\]

with

\[
\begin{align*}
c_2 &= \frac{2d_0 + AK^3}{K^2}, \quad d_2 = \frac{3e_0 + B K^3}{K^2}, \quad b_4 = \frac{c_1 + CK^4}{K^2}, \\
c_4 &= \frac{2d_1 + DK^4}{K^2}, \quad b_7 = \frac{c_3 + E K^5}{K^2}, \quad b_5 = \frac{d_0 + AK^3 + FK^5}{K^4}, \\
c_5 &= \frac{3e_0 + 2B K^3 + G K^5}{K^4}, \quad b_8 = \frac{d_1 + DK^4 + HK^6}{K^4}, \\
b_9 &= \frac{e_0 + BK^3 + G K^5 + IK^7}{K^6}.
\end{align*}
\]

Now we compute Melnikov functions (as explained in Sect. 2.1) for family (21) with $K \neq 0$ in order to detect 3-dimensional centers at the origin. The first associated Melnikov function $\delta_1(r_0, w_0)$ is

\[
\delta_1(r_0, w_0) = \left( A \pi r_0^2 w_0, -\frac{\pi r_0}{K^3} \left[ -b_3 - FK^3 + K^2 \left( -2d_0 + AK^3 \right) w_0^2 \right] \right)
\]

Therefore $\delta_1(r_0, w_0) \equiv 0$ if and only if

\[
A = d_0 = 0, \quad b_3 = -FK^3.
\] (22)

Under these constrains, the second Melnikov function $\delta_2(r_0, w_0) = (\delta_{2,1}(r_0, w_0), \delta_{2,2}(r_0, w_0))$ has components

\[
\begin{align*}
\delta_{2,1}(r_0, w_0) &= -\frac{\pi}{4K^3} r_0^3 \left[ c_1 F + E K^4 + 3I K^4 + 4BK^4 w_0^2 \right], \\
\delta_{2,2}(r_0, w_0) &= -\frac{\pi}{4K^2} r_0^2 w_0 \left[ -4c_3 + 5c_1 FK - 4G K^3 + E K^5 + 3I K^5 \\
&\quad - 8e_0 K^2 w_0^2 + 4BK^5 w_0^2 \right].
\end{align*}
\]

Next it is easy to check that the second Melnikov function $\delta_2(r_0, w_0)$ vanishes identically if and only if
\[ B = e_0 = 0, \]
\[ c_3 = \frac{1}{4} (5c_1 F K - 4G K^3 + EK^5 + 3IK^5), \]
\[ E = -\frac{1}{K^4} (c_1 F + 3IK^4). \]  

(23)

Straightforward computations gives the following higher order Melnikov functions. More precisely we get
\[ \delta_3(r_0, w_0) = \left( -\Delta_1 \frac{\pi}{4K^4} r_0^4 w_0, \frac{\pi}{4K^6} r_0^3 \left[ \Delta_2 - 3K^2 \Delta_3 w_0^2 \right] \right) \]

where \( \Delta_i \) are polynomials in the parameter space of family (21) whose expressions can be found in Appendix 1. Also we obtain \( \delta_4(r_0, w_0) = (\delta_{4,1}(r_0, w_0), \delta_{4,2}(r_0, w_0)) \) with components
\[ \delta_{4,1}(r_0, w_0) = -\frac{\pi}{48K^8} r_0^3 \left[ \Delta_4 + 4\Delta_1 (3c_1 + CK^4) w_0 + 6K^2 \Delta_5 w_0^2 \right], \]
\[ \delta_{4,2}(r_0, w_0) = \frac{\pi}{48K^{10}} r_0^4 \left[ -4(3c_1 - 4CK^4) \Delta_2 + K^2 \Delta_6 w_0 + 4K^2 \Delta_7 w_0^2 + 6K^4 \Delta_8 w_0^3 \right]. \]

The fifth-order Melnikov function \( \delta_5(r_0, w_0) = (\delta_{5,1}(r_0, w_0), \delta_{5,2}(r_0, w_0)) \) has components
\[ \delta_{5,1}(r_0, w_0) = -\frac{\pi}{576K^{12}} r_0^6 \left[ 12\Delta_9 - \Delta_10 w_0 + 24K^2 \Delta_{11} w_0^2 + 18K^4 \Delta_{12} w_0^3 \right], \]
\[ \delta_{5,2}(r_0, w_0) = \frac{\pi}{48K^{14}} r_0^5 \left[ \Delta_{13} + 12K^2 \Delta_{14} w_0 - K^2 \Delta_{15} w_0^2 + 24K^4 \Delta_{16} w_0^3 + 18K^6 \Delta_{17} w_0^4 \right]. \]

The Bautin ideal \( \mathcal{B} = \langle \Delta_i : i \in \mathbb{N} \rangle \) at the origin of (21) is generated by all the Poincaré–Liapunov constants \( \Delta_i \) and is a polynomial ideal in the polynomial ring \( \mathbb{R}[\lambda] \) where
\[ \lambda = (K, C, D, F, G, H, I, b_6, c_1, d_1) \in \mathbb{R}^{10} \]
contains some of the parameters of family (21). Recall that several parameters have been fixed, see (22) and (23).

Since \( \mathcal{B} \) is Noetherian it is generated by a finite number of polynomials by the Hilbert basis theorem. But unfortunately we do not know this basis a priori. Let \( \mathcal{B}_k = \langle \Delta_1, \ldots, \Delta_k \rangle \) be the ideal generated by the first \( k \) Poincaré–Liapunov constants at the origin of (21). Before computing the center variety \( \mathbf{V}(\mathcal{B}) \) we will do some simplifications. We define \( \mathbf{B}_k = \langle \Delta_1, \ldots, \Delta_k \rangle \) be the ideal generated by some polynomials \( \Delta_i \) that we define sequentially as follows. Let \( \Delta_1 = \Delta_1 \) and \( \Delta_i \equiv \Delta_i \mod \mathbf{B}_{i-1} \) for \( i \geq 2 \). Clearly \( \mathbf{B}_k = \mathcal{B}_k \). In the former reduction we obtain that \( \Delta_7 = \Delta_{11} = \Delta_{12} = \Delta_{16} = \Delta_{17} = 0 \). Relabeling consecutively the subscripts of the \( \Delta_i \) such that we remove the above null polynomials we can work with the ideal \( \mathbf{B}_{11} = \langle \Delta_1, \ldots, \Delta_{11} \rangle \). We expect that \( \mathbf{V}(\mathcal{B}) = \mathbf{V}(\mathbf{B}_{11}) \) where the inclusion \( \mathbf{V}(\mathcal{B}) \subset \mathbf{V}(\mathbf{B}_{11}) \) is obvious. To verify the opposite inclusion we will find the minimal irreducible decomposition of the variety \( \mathbf{V}(\mathbf{B}_{11}) \). This decomposition will be performed with the help of a computer algebra system, more precisely with the procedure minAssGTZ from the library primdec.lib of the software SINGULAR using the degree-reverse lexicographic order with \( K > C > D > F > G > H > I > b_6 > c_1 > d_1 \) to find the primary decomposition of \( \sqrt{\mathbf{B}_{11}} \), the radical of \( \mathbf{B}_{11} \). The output is that \( \mathbf{V}(\mathbf{B}_{11}) = \bigcup_{i=1}^{13} \mathbf{V}(J_i) \) where the irreducible varieties \( \mathbf{V}(J_i) \) are the varieties associated to the ideals \( J_i \) listed in Appendix 2.
Finally we must verify the each point of the varieties \( V(J_i) \) corresponds with a system (21) having a 3-dimensional center at the origin. We shall see that sometimes this is not true and we need to compute higher order Melnikov functions \( \delta_j(r_0, w_0) \) with \( j \geq 6 \) at the origin of family (21) restricted to \( \lambda \in V(J_i) \).

Now we recall that in family (21) the parameter \( K \neq 0 \) and therefore we do not take into account any of the varieties \( V(J_i) \) with \( i \in \{2, 11, 12\} \) because they have the polynomial \( K \) as generator.

### 7.1 The Variety \( V(J_1) \)

Since the variety \( V(J_1) \) is defined by

\[
V(J_1) = \{ \lambda \in \mathbb{R}^{10} : d_1 = c_1 = b_6 = I = H = DF - CG = 0 \}
\]

we get the associated family (21) with

\[
K^2 F(x, y, z) = Fx^2 + Cxy - Fy^2 + Gx^2 z + Dxyz - Gy^2 z
\]

and the condition \( DF - CG = 0 \).

By Theorem 1 and Remark 11 we have that if \( F = G = 0 \) or \( D = G = 0 \) then (21) has a 3-dimensional center at the origin.

Therefore only remains the study of the case \( G \neq 0 \). In this case we can put \( C = DF / G \) and after rescaling the time by \( t = G \cdot t \) system (21) becomes

\[
\begin{align*}
\dot{x} &= -Gy + K^2 \tilde{F}(x, y, z) \\
\dot{y} &= Gx \\
\dot{z} &= \tilde{F}(x, y, z),
\end{align*}
\]

with \( K^2 \tilde{F}(x, y, z) = (Gx^2 + Dxy - Gy^2)(F + Gz) \). Performing the planar rotation in the \((x, y)\)-plane of angle \( \theta^* = -\frac{1}{2} \arccot(D/(2G)) \) system (25) is greatly simplified because

\[
2 \cos(2\theta^*) + (D \sin(2\theta^*)/G) = 0.
\]

Indeed, doing the change of variables

\[
\begin{pmatrix}
u \\ v \\ w
\end{pmatrix} =
\begin{pmatrix}
\cos \theta^* & -\sin \theta^* & 0 \\
\sin \theta^* & \cos \theta^* & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
x \\ y \\ z
\end{pmatrix}
\]

system (25) becomes

\[
\begin{align*}
\dot{u} &= -v + C_1 \tilde{G}(u, v, w), \\
\dot{v} &= u + C_2 \tilde{G}(u, v, w), \\
\dot{w} &= C_3 \tilde{G}(u, v, w),
\end{align*}
\]

where \( \tilde{G}(u, v, w) = uv(F + Gw) \) and \( C_i \) are certain constants depending on the parameters of the family. Removing common factors we see that two planar subsystems are associated to (27), namely

\[
\begin{align*}
\dot{u} &= -1 + C_1 u(F + Gw), \\
\dot{w} &= C_3 u(F + Gw),
\end{align*}
\]

and

\[
\begin{align*}
\dot{v} &= 1 + C_2 v(F + Gw), \\
\dot{w} &= C_3 v(F + Gw).
\end{align*}
\]

Since the origin is a regular point of both (28) and (29) there are around it analytic first integrals \( \tilde{H}_1(u, w) \) of (28) and \( \tilde{H}_2(v, w) \) of (29). Additionally both functions \( \tilde{H}_1(u, w) \) and \( \tilde{H}_2(v, w) \) are first integrals of the full system (27). Let \( \nabla = (\partial_u, \partial_v, \partial_w) \) be the gradient operator. Since
the gradient vectors $\nabla \hat{H}_1(u, w) = (\partial_u \hat{H}_1, 0, \partial_w \hat{H}_1)$ and $\nabla \hat{H}_2(v, w) = (0, \partial_v \hat{H}_2, \partial_w \hat{H}_2)$ are linearly independent except in the zero Lebesgue measure set $\{(u, v, w) : \partial_u \hat{H}_1 = \partial_v \hat{H}_2 = 0\}$ we conclude that $\hat{H}_1(u, w)$ and $\hat{H}_2(v, w)$ are two functionally independent analytical first integrals of (27) almost everywhere in a neighborhood of the origin, that is in a full Lebesgue measure (dense) neighborhood of the origin. In short, going back through the linear change (26) one has that the pull back $H_1(x, y, z) = \hat{H}_1(\cos \theta_x^* x - \sin \theta_y^* y, z)$ and $H_2(x, y, z) = \hat{H}_2(\sin \theta_x^* x + \cos \theta_y^* y, z)$ also are functionally independent analytical first integrals of family (21) with $\mathcal{F}$ given by (24). Taking into account that the linear part of (21) there is no restriction in assuming that $H_1(x, y, z) = x^2 + y^2 + \cdots$ and $H_2(x, y, z) = z + \cdots$ and therefore family (21) with $\mathcal{F}$ given by (24) has a center at the origin. This proves statement (i) of Theorem 5.

7.2 The Variety $V(J_3)$

Taking into account the expressions of the generators of the ideal $J_3$ we see that the variety $V(J_3)$ is given by

$$V(J_3) = \{\lambda \in \mathbb{R}^{10} : d_1 = c_1 = H = I = G = D = b_6 = 0\}$$

and therefore $V(J_3) \subset V(J_1)$.

7.3 The Variety $V(J_4)$

From the generators of the ideal $J_4$ we get that

$$V(J_4) = \left\{\lambda \in \mathbb{R}^{10} : d_1 = c_1 = F = 0, G = -2K^2 I, D = -\frac{4}{3} K^2 H, b_6 = \frac{1}{4} K^2 D\right\}.$$ 

Imposing that $\lambda \in V(J_4)$ we can compute the sixth-order Melnikov function at the origin of (21)

$$\delta_6(r_0, w_0) = I \left(H^2 + 9I^2\right) \pi \left(\frac{5}{864} r_0^7, \frac{7}{288} r_0^6 w_0\right)$$

from where we see that $\delta_6(r_0, w_0) \equiv 0$ if and only $I = 0$. Therefore its associated family (21) has

$$K^2 \mathcal{F}(x, y, z) = Cxy + Hx^2 y - \frac{H}{3} y^3 - \frac{4}{3} H K^2 xyz,$$

(30)

and by Theorem 1 family (21) has a 3-dimensional center at the origin. This proves statement (ii) of Theorem 5.

7.4 The Variety $V(J_5)$

From the generators of the ideal $J_5$ one has that

$$V(J_5) = \left\{\lambda \in \mathbb{R}^{10} : d_1 = c_1 = 0, G = -\frac{3}{2} K^2 I, D = -K^2 H, b_6 = \frac{1}{3} K^2 D, 2FH - 3CI = 0\right\}.$$
If \( \lambda / \Lambda_1 \) resultants between the \( R \) only if the following polynomials in 
and the condition \( 2FH - 3CI = 0 \). Then statement (viii) of Theorem 5 follows. 
By Theorem 1 and Remark 11 we have that if \( F = I = 0 \) or \( H = I = 0 \) then (21) has a 3-dimensional center at the origin. 
Hence only remains the study of \( I \neq 0 \) and we take \( C = 2FH/(3I) \). Doing the \( z \)-translation \( (x, y, z) \mapsto (x, y, z - \alpha) \) with \( \alpha = 2F/(3IK^2) \) we obtain the same system (21) 
with \( K^2F(x, y, z) \) given by (31) but with \( F = C = 0 \). Under these new constrains one has \( \delta_i(r_0, w_0) \equiv 0 \) for all \( i \leq 11 \). This proves Remark 6. 

7.5 The Variety \( V(J_6) \)

From the generators of the ideal \( J_6 \) we get 
\[
\mathcal{V}(J_6) = \left\{ \lambda \in \mathbb{R}^{10} : d_1 = c_1 = F = C = 0, b_6 = -\frac{1}{3}K^4H, 2GH - 3DI = 0 \right\}.
\]
If \( \lambda \in \mathcal{V}(J_6) \) we known that the first Melnikov function \( \delta_i(r_0, w_0) \equiv 0 \) for \( i = 1, \ldots, 5 \) but \( \delta_6(r_0, w_0) \neq 0 \). After some involved computations we can see that \( \delta_6(r_0, w_0) \equiv 0 \) if and only if the following polynomials in \( \mathbb{R}[\lambda] \)
\[
\Lambda_1 = 8DGH - 6D^2I + 24G^2I + 10GH^2K^2 - 6DHIK^2 + 54GI^2K^2 + 3H^2IK^4 + 27I^3K^4,
\]
\[
\Lambda_2 = 32DGH - 30D^2I + 72G^2I + 66GH^2K^2 - 78DHIK^2 + 126GI^2K^2 + 3H^2IK^4 + 27I^3K^4
\]
vanishes. If we define \( \Lambda_3 = 2GH - 3DI \), one of the generators of the ideal \( J_5 \), and compute resultants between the \( \Lambda_i \) with respect to \( G \) yields 
\[
\mathcal{R}[\Lambda_1, \Lambda_3, G] = 12I(H^2 + 9I^2)(D + HK^2)(2D + HK^2),
\]
\[
\mathcal{R}[\Lambda_2, \Lambda_3, G] = 12I(H^2 + 9I^2)(D + HK^2)(6D + HK^2).
\]
Therefore two cases arise to annul the above resultants: 
(a) Put the parameter \( I = 0 \). Then the associated family (21) has 
\[
K^2F(x, y, z) = Hx^2y - \frac{H}{3}y^3 + Gx^2z + Dxyz - Gy^2z,
\]
with the condition \( GH = 0 \). Clearly if \( G = 0 \) then by Theorem 1 family (21) has a 3-dimensional center at the origin, hence we can assume that \( H = 0 \) and \( G \neq 0 \). In this last case, family (21) has \( F(x, y, z) \) as in (24) (with the particular election of parameters \( F = C = 0 \)) and we have already proved that family (21) has a 3-dimensional center at the origin. Then statement (iii) of Theorem 5 is proved.
(b) Let \( D = -HK^2 \) and \( I \neq 0 \). Then \( \delta_i(r_0, w_0) \equiv 0 \) for \( i = 6, 7, 8 \) if and only if \( G = -3IK^2/2 \), in which case family (21) has

\[
K^2 \mathcal{F}(x, y, z) = Ix^3 + Hx^2y - 3Ixy^2 - \frac{H}{3}y^3 - \frac{3}{2}IK^2x^2z - HK^2xyz + \frac{3}{2}IK^2y^2z
\]

with \( I \neq 0 \). In this case, the associated family (21) has \( \mathcal{F}(x, y, z) \) as in (31) (with the parameters \( F = C = 0 \)) and we have already proved that in this case family (21) has a 3-dimensional center at the origin.

7.6 The Variety \( V(J_7) \)

From the generators of the ideal \( J_7 \) we obtain that

\[ V(J_7) = \{ \lambda \in \mathbb{R}^{10} : d_1 = c_1 = H = I = b_6 = C = F = D = G = 0 \} \]

giving rise to the trivial linear system \( \dot{x} = -y, \dot{y} = x, \dot{z} = 0 \).

7.7 The Variety \( V(J_8) \)

From the generators of the ideal \( J_8 \) we see that in order to express the variety \( V(J_8) \) we must impose the following parameter constraints

\[ d_1 = c_1 = 0, \quad G = -K^2I, \quad b_6 = K^4H + 2K^2D \]

but additional polynomial restrictions are needed to determine the variety \( V(J_8) \). Then we will split the problem in several subcases.

First we assume that \( I = F = H = 0 \). Then we obtain also \( D = 0 \) and \( \lambda \in V(J_8) \) but the resulting associated family (21) is a particular case of system (24).

Assume now that \( I = F = 0 \) but \( H \neq 0 \) and \( D = -HK^2 \). Then \( \lambda \in V(J_8) \) and the associated family (21) has

\[
K^2 \mathcal{F}(x, y, z) = Cxy + Hx^2y - Hy^3 - HK^2xyz,
\]

with \( H \neq 0 \). By Theorem 1 family (21) has a 3-dimensional center at the origin. This proves statement (iv) of Theorem 5.

We can suppose now that \( I = F = 0, H \neq 0 \) and \( D \neq -HK^2 \). Then we necessarily get \( D = -HK^2/2 \) to have \( \lambda \in V(J_8) \). In this case family (21) has

\[
K^2 \mathcal{F}(x, y, z) = Cxy + Hx^2y - \frac{1}{2}HK^2xyz.
\]

with \( H \neq 0 \). By Theorem 1 family (21) has a 3-dimensional center at the origin and statement (v) of Theorem 5 is proved.

Let \( I = 0 \) but \( F \neq 0 \). If \( D \neq 0 \) then \( \lambda \notin V(J_8) \), so we assume that \( D = 0 \). Then \( \lambda \in V(J_8) \) if and only if \( H = 0 \). Hence family (21) becomes a particular case of system (24).

Finally we deal with the case \( I \neq 0 \). Then we have \( C = -DF/(IK^2) \) and \( \lambda \in V(J_8) \) if and only if \( 2D^2 + 3DHK^2 + H^2K^4 - I^2K^4 = 0 \). We also check that under this constrained one has \( \delta_i(r_0, w_0) \equiv 0 \) for \( i = 1, \ldots, 8 \). The resulting family (21) has
\[ K^2 \mathcal{F}(x, y, z) = Fx^2 - \frac{DF}{IK^2} xy - Fy^2 + Ix^3 + Hx^2 y - 3Ixy^2 \]
\[ + \left( \frac{2D}{K^2} + H \right) y^3 - IK^2 x^2 z + Dxyz + IK^2 y^2 z, \] (36)

with the conditions \( I \neq 0 \) and \( 2D^2 + 3DHK^2 + H^2 K^4 - I^2 K^4 = 0 \). Then statement (ix) of Theorem 5 is proved. Notice that Theorem 1 and Remark 11 do not work in this case because \( I \neq 0 \). Also we have seen that, in the special case \( F = 0 \) then \( \delta_i(r_0, w_0) \equiv 0 \) for \( i = 1, \ldots, 11 \).

7.8 The Variety \( V(J_9) \)

From the generators of the ideal \( J_9 \) we see that
\[
V(J_9) = \{ \lambda \in \mathbb{R}^{10} : d_1 = c_1 = F = C = 0, G = -K^2 I, \]
\[
b_6 = K^4 H + 2K^2 D \}.
\]
We get that either \( I \neq 0 \) and then the resulting family (21) becomes a particular case with \( F = 0 \) of (36) or \( I = 0 \) and therefore the associated family (21) has
\[ K^2 \mathcal{F}(x, y, z) = Fx^2 + Cxy - Fy^2 + Hx^2 y - c_1 F K^4 xy^2 + c_1 K^2 yz \] (37)

Using Theorem 1 we obtain that (21) has a 3-dimensional center at the origin proving statement (vi) of Theorem 5.

7.9 The Variety \( V(J_{10}) \)

The generators of the ideal \( J_{10} \) yield
\[
V(J_{10}) = \{ \lambda \in \mathbb{R}^{10} : d_1 = b_6 = I = G = 0, D = -K^2 H \}.
\]
Hence the associated family (21) has
\[ K^2 \mathcal{F}(x, y, z) = Fx^2 + Cxy - Fy^2 + Hx^2 y - \frac{c_1 F}{K^4} xy^2 + \frac{c_1}{K^2} yz \]
\[ - HK^2 x^2 z + \frac{c_1 F}{K^2} y^2 z. \] (38)

Then statement (x) of Theorem 5 is proved changing the name \( c_1 \mapsto A \). Further computations reveal that \( \delta_i(r_0, w_0) \equiv 0 \) for \( i = 1, \ldots, 10 \). We notice that by Theorem 1 and Remark 11 we have that if \( F = 0 \) or \( c_1 = H = 0 \) then (21) has a 3-dimensional center at the origin proving thus the second part of Remark 6.

7.10 The Variety \( V(J_{13}) \)

The generators of the ideal \( J_{13} \) yield
\[
V(J_{13}) = \{ \lambda \in \mathbb{R}^{10} : I = G = F = 0 \}.
\]
Hence the associated family (21) has
\[
K^2 \mathcal{F}(x, y, z) = Cxy + \frac{c_1}{K^2} yz + H x^2 y + \frac{b_6}{K^4} y^3 + Dxyz + d_1 y^2 z^2.
\]

By Theorem 1 family (21) has a 3-dimensional center at the origin. This proves statement (vii) of Theorem 5 after changing the name of parameters \((c_1, b_6, d_1) \mapsto (A, B, E)\).

**Acknowledgements** The first author is partially supported by a MINECO Grant Number MTM2014-53703-P and an AGAUR Grant Number 2014SGR 1204. The second author is supported by Portuguese National Funds through FCT - Fundação para a Ciência e a Tecnologia within CAMGSD and the Project PTDC/MAT/117106/2010.

## 8 Appendix 1

\[
\Delta_1 = -2d_1 F + c_1 G + 3c_1 IK^2,
\]
\[
\Delta_2 = -b_6 F + DFK^2 - CGK^2 + FHK^4 - 2C1K^4,
\]
\[
\Delta_3 = 2d_1 F + c_1 G - c_1 IK^2,
\]
\[
\Delta_4 = -5b_6 c_1 F - 2c_1^2 I - 6Cd_1 FK^2 + 3c_1 DFK^2 + 16b_6 G K^2 + 2c_1 CG K^2 + 2b_6 CFK^4 + 3c_1 FHK^4 + 18b_6 IK^4 + 3c_1 CIK^4 - 2CDFK^6 + 2G^2 K^6 + 8GKH^6 - 4DIK^6 - 2CFK^8 + 4C^2IK^8 + 6HIK^8,
\]
\[
\Delta_5 = -13c_1 d_1 F - 4c_1^2 G + 9c_1^2 IK^2 + 2Cd_1 FK^4 + 2c_1 CGK^4 + 6d_1 IK^4,
\]
\[
\Delta_6 = -47b_6 c_1 F - 14c_1^2 I - 42Cd_1 FK^2 + 33c_1 DFK^2 + 4b_6 G K^2 - 52c_1 CG K^2 + 2b_6 CFK^4 + 33c_1 FHK^4 + 18b_6 IK^4 - 57c_1 CIK^4 - 2CDFK^6 + 2G^2 K^6 + 20GKH^6 - 28DIK^6 - 2CFK^8 + 4C^2IK^8 + 6HIK^8,
\]
\[
\Delta_7 = -24c_1 d_1 F - 6c_1^2 G + 18c_1^2 IK^2 - 10Cd_1 FK^4 - 7c_1 CGK^4 + 3c_1 CIK^6,
\]
\[
\Delta_8 = -33c_1 d_1 F - 24c_1^2 G - 8d_1 G K^2 + 9c_1^2 IK^2 + 2Cd_1 FK^4 + 2c_1 CGK^4 + 6d_1 IK^4,
\]
\[
\Delta_9 = 4b_6 c_1^2 F - 16b_6 d_1 FK^2 - 8c_1 Cd_1 FK^2 - 4c_1^2 DFK^2 + 8b_6 c_1 G K^2 + 8c_1^2 CGK^2 - 9b_6 c_1 CFK^4 - 4c_1^2 FHK^4 + 24b_6 c_1 IK^4 + 18c_1^2 CIK^4 - 6c_1^2 FK^6 + 4b_6 DFK^6 + 7c_1 CDFK^6 - 8d_1 F^3 K^6 + 16b_6 CGK^6 - 2c_1 C^2 GK^6 + 4c_1^2 F^2 GK^6 - 8d_1 FHK^6 + 4c_1 GKH^6 + 2b_6 C^2 FK^8 + 4D^2 FK^8 - 4CDGK^8 + 7c_1 CFHK^8 + 18b_6 CIK^8 - 5c_1 C^2 IK^8 + 12c_1 F^2 IK^8 + 12c_1 HIK^8 - 2C^2 DFK^10 + 2C^3 GIK^10 + 4DFH K^10 + 8CGHK^10 - 12CDIK^10 - 2C^2 FH K^12 + 4C^3 IK^12 + 6CHIK^12,
\]
\[
\Delta_{10} = 168c_1^2 d_1 F - 84c_1^3 G + 294b_6 c_1^2 FK^2 - 108c_1^2 IK^2 + 540b_6 d_1 FK^4 + 810c_1 Cd_1 FK^4 - 150c_1^2 DFK^4 - 246b_6 c_1 G K^4
\]
\[ +159c^2CGK^4 - 84b_6c_1CFK^6 + 48d_1DFK^6 + 24Cd_1GK^6 \\
-36c_1DGK^6 - 150c_1^2FHK^6 - 810b_6c_1IK^6 - 381c_1^2CIK^6 \\
+12C^2d_1FK^8 - 24b_6DFK^8 + 60c_1CDFK^8 + 208d_1F^3K^8 \\
-24b_6CGK^8 - 150c_1^2GK^8 + 40c_1F^2GK^8 + 84d_1FHK^8 \\
-210c_1GHK^8 - 72Cd_1IK^8 + 84c_1DIK^8 + 24D^2FK^{10} \\
-24CDGK^{10} + 60c_1CFHK^{10} - 246c_1^2IK^{10} \\
-168c_1F^2IK^{10} - 270c_1HIK^{10} + 24DFHK^{12} \\
+24CGHK^{12} - 96CDIK^{12}, \]

\[ \Delta_{11} = -78c_1^2d_1F - 12c_1^3G - 12d_1^2FK^2 + 6c_1d_1GK^2 + 66c_1^3IK^2 \\
-13c_1Cd_1FK^4 + 2c_1^2CGK^4 + 54c_1d_1IK^4 - 12d_1DFK^6 \\
-6c_1DGK^6 + 15c_1^2CIK^6 + 2c_1^2CGK^8 + 6c_1d_1IK^8 + 6c_1DIK^8, \]

\[ \Delta_{12} = -120c_1^2d_1F - 75c_1^3G - 64d_1^2FK^2 - 64c_1d_1GK^2 + 45c_1^3IK^2 \\
+20c_1Cd_1FK^4 + 20c_1^2CGK^4 + 72c_1d_1IK^4 + 8d_1DFK^6 \\
+8C_1d_1GK^6 + 8c_1DGK^6, \]

\[ \Delta_{13} = 84b_6c_1^2F - 24c_1^3I + 240b_6d_1FK^2 + 48c_1Cd_1FK^2 \\
-108c_1^2DFK^2 + 72b_6c_1GK^2 + 108c_1^2CGK^2 + 378b_2^2FK^4 \\
-63b_6c_1CFK^4 - 108c_1^2FHK^4 - 72b_6c_1IK^4 + 204c_1^2CIK^4 \\
+90C^2d_1FK^6 - 366b_6DFK^6 + 123c_1CDFK^6 + 72d_1F^3K^6 \\
+294b_6CGK^6 - 93c_1^2GK^6 - 24c_1^2F^2GK^6 + 48d_1FHK^6 \\
+72c_1GHK^6 + 48Cd_1IK^6 - 72c_1DIK^6 + 210b_6^2CFK^8 \\
+36D^2FK^8 + 200b_6F^3K^8 - 36CDGK^8 - 132b_6FHK^8 \\
+123c_1CFHK^8 + 612b_6CIK^8 - 246c_1^2IK^8 - 96c_1F^2IK^8 \\
-24c_1HIK^8 - 210c_2DFK^{10} - 200DF^3K^10 + 210c_3GK^{10} \\
+200CF^2GK^{10} - 162DFHK^{10} + 90CGHK^{10} + 48CDIK^{10} \\
-96FGIK^{10} - 210c^2FHK^{12} - 200F^3HK^{12} - 198FH^2K^{12} \\
+420c^3IK^{12} + 400CF^2IK^{12} + 348CHIK^{12} - 144FHIK^{12}, \]

\[ \Delta_{14} = 39b_6c_1^2F - 2c_1^4I - 104b_6d_1FK^2 - 70c_1Cd_1FK^2 - 41c_1^2DFK^2 \\
-16b_6c_1GK^2 + 30c_2CGK^2 - 79b_6c_1CFK^4 - 24d_1DFK^4 \\
+24Cd_1GK^4 - 41c_1^2FHK^4 + 114b_6c_1IK^4 + 121c_1^2CIK^4 \\
-54c_1^2df_1K^6 - 20b_6DFK^6 + 61c_1CDFK^6 - 64d_1F^3K^6 \\
-88c_1CGK^6 - 16c_1^2GK^6 - 88d_1FHK^6 - 8c_1GHK^6 \\
+48Cd_1IK^6 - 4c_1DIK^6 + 2b_6C^2FK^8 + 20D^2FK^8 \\
-20CDGK^8 + 61c_1CFHK^8 + 18b_6CIK^8 - 113c_1^2IK^8 \\
+48c_1F^2IK^8 + 54c_1HIK^8 - 2CF^2DFK^{10} + 2C^3GK^{10} \\
+20DFHK^{10} + 24CGHK^{10} - 76CDIK^{10} - 2CF^2FHK^{12} \\
+4C^3IK^{12} + 6CHIK^{12}, \]

\[ \Delta_{15} = 816c_1^2d_1F + 132c_1^3G + 1428b_6c_1^2FK^2 - 288d_1^2FK^2 \\
+144c_1d_1GK^2 - 36c_1^3IK^2 + 2304b_6d_1FK^4 + 3648c_1Cd_1FK^4 \]
\[-780c_1^2 DFK^4 + 1518b_6c_1 G K^4 + 2979c_1^2 C G K^4 + 432c_1 d_1 I K^4 \\
- 84b_6c_1 C F K^6 - 96d_1 D F K^6 + 888c_1 d_1 G K^6 + 36c_1 D G K^6 \\
- 780c_1^2 F H K^6 - 810b_6c_1 I K^6 + 735c_1^2 C I K^6 + 312c_1^2 d_1 F K^8 \\
- 24b_6 D F K^8 + 60c_1 C D F K^8 + 832d_1 F^3 K^8 - 24b_6 C G K^8 \\
+ 150c_1^2 C G K^8 + 664c_1 F^2 G K^8 + 192d_1 F H K^8 - 102c_1 G H K^8 \\
+ 648c_1 d_1 I K^8 + 804c_1 D I K^8 + 24D^2 F K^{10} - 24C D G K^{10} \\
+ 60c_1 C F H K^{10} - 246c_1 C I K^{10} - 168c_1 F^2 I K^{10} \\
- 270c_1 H I K^{10} + 24D F H K^{12} + 24C G H K^{12} - 96C D I K^{12},
\]

\[\Delta_{16} = -233c_1^2 d_1 F - 122c_1^2 G - 24d_1^2 F K^2 - 36c_1 d_1 G K^2 \\
+ 111c_1^2 I K^2 - 63c_1 d_1 F K^4 - 48c_1^2 C G K^4 + 90c_1 d_1 I K^4 \\
- 28d_1 D F K^6 - 16C_1 d_1 G K^6 - 22c_1 D G K^6 + 15c_1^2 C I K^6 \\
+ 2c_1^2 d_1 F K^8 + 2c_1^2 C G K^8 + 6c_1 d_1 I K^8 + 6c_1 D I K^8,
\]

\[\Delta_{17} = -260c_1^2 d_1 F - 215c_1^2 G - 144d_1^2 F K^2 - 224c_1 d_1 G K^2 \\
+ 45c_1^2 I K^2 + 20c_1 C d_1 F K^4 + 20c_1^2 C G K^4 + 72c_1 d_1 I K^4 \\
+ 8d_1 D F K^6 + 8c_1 d_1 G K^6 + 8c_1 D G K^6.
\]

9 Appendix 2

\[J_1 = \langle d_1, c_1, b_6, I, H, D F - C G \rangle,
\]

\[J_2 = \langle d_1, c_1, b_6, K \rangle,
\]

\[J_3 = \langle d_1, c_1, H^2 + 9I^2, 2G H - 3D I, \\
DH + 6GI, 3G^2 - 4Hb_6, DG + 8Ib_6, \\
3D^2 + 16Hb_6, 2K^2 I + G, 4K^2 H + 3D, \\
6K^2 D - 4b_6 \rangle,
\]

\[J_4 = \langle d_1, c_1, F, 2G H - 3D I, \\
DG + 8Ib_6, 3D^2 + 16Hb_6, 2K^2 I + G, \\
4K^2 H + 3D, K^2 D - 4b_6 \rangle,
\]

\[J_5 = \langle d_1, c_1, 2G H - 3D I, 2F H - 3C I, \\
2DG + 9Ib_6, D F - C G, D^2 + 3Hb_6, \\
3K^2 I + 2G, K^2 H + D, 2C G^2 + 9FIb_6, \\
K^2 D - 3b_6, K^2 C G - 3F b_6 \rangle,
\]

\[J_6 = \langle d_1, c_1, F, C, 2G H - 3D I, \\
K^4 H + 3b_6, K^4 D I + 2G b_6 \rangle,
\]

\[J_7 = \langle d_1, c_1, H^2 + 9I^2, 2G H - 3D I, \\
2F H - 3C I, DH + 6GI, CH + 6FI, \\
D F - C G, D^2 + 4G^2, C D + 4F G, \\
C^2 + 4F^2, K^4 H + 3b_6, 3K^4 F^2 - Hb_6, \\
2K^4 G I - D b_6, 2K^4 F I - C b_6, \\
K^4 D I + 2G b_6, K^4 C I + 2F b_6 \rangle.
\]
\[ J_8 = \langle d_1, c_1, DF - CG, K^2 I + G, \
G^2 H - 2DG I - I^2 b_6, \\
DG H - 2D^2 I + G^2 I - H b_6, \\
G^3 - Gb_6 + DI b_6, FG^2 - FH b_6 + CI b_6, \\
2DG^2 I - Gb_6^2 + DHI b_6 + GI^2 b_6, \\
2CG^2 I - FH^2 b_6 + CH I b_6 + FI^2 b_6, \\
FGH^2 - 3CGHI + 2CDI^2 - FGI^2, \\
F^2 H^2 - 3CFHI + 2C^2 I^2 - F^2 I^2, \\
K^2 GH + 2DG + 1b_6, K^2 DH + 2D^2 - G^2 + Hb_6, \\
2D^2 G^2 - D^2 Hb_6 + 3DGIb_6 - Hb_6^2 + I^2 b_6^2, \\
K^2 G^2 - K^2 Hb_6 - Db_6, \\
K^2 H^2 b_6 + 2DG^2 + DH b_6 + GI b_6, \\
4D^2 GI^2 - GH^2 b_6 + DH^2 I b_6 + GH I^2 b_6 + 2DI^3 b_6, \\
4CDGI^2 - F H^2 b_6 + CH^2 I b_6 + F H^2 b_6 + 2CI^3 b_6, \\
4D^3 GI - D^2 H^2 b_6 + 8D^2 I^2 b_6 - 3G^2 I^2 b_6 - H^3 b_6^2 + 4HI^2 b_6^2, \\
K^2 F H^2 + 3CGH - 2CDI + FGI, K^4 H + 2K^2 D - b_6, \\
8D^3 I^3 - G^4 b_6 + DHI^3 b_6 - Gb_6^2 I^2 b_6 + 8DH I^3 b_6 + 2GI^4 b_6, \\
8CD^3 I^3 - FH^4 b_6 + CH^3 I b_6 - FH^2 I^2 b_6 + 8CH I^3 b_6 + 2FI^4 b_6, \\
8D^4 I^2 - D^2 H^3 b_6 + 10D^2 H I^2 b_6 - Hb_6^4 b_6^2 + 2H^2 I^2 b_6^2 - I^4 b_6^2, \]

\[ J_9 = \langle d_1, c_1, F, C, K^2 I + G, G^2 H - 2DG I - I^2 b_6, \\
K^2 GH + 2DG + 1b_6, K^4 H + 2K^2 D - b_6, \rangle \]

\[ J_{10} = \langle d_1, b_6, I, G, K^2 H + D \rangle, \]

\[ J_{11} = \langle d_1, b_6, I, G, K \rangle, \]

\[ J_{12} = \langle c_1, F, K \rangle, \]

\[ J_{13} = \langle I, G, F \rangle. \]

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