RICHARDSON ELEMENTS FOR PARABOLIC SUBGROUPS OF CLASSICAL GROUPS IN POSITIVE CHARACTERISTIC

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Abstract. Let $G$ be a simple algebraic group of classical type over an algebraically closed field $k$. Let $P$ be a parabolic subgroup of $G$ and let $p = \text{Lie } P$ be the Lie algebra of $P$ with Levi decomposition $p = l \oplus u$, where $u$ is the Lie algebra of the unipotent radical of $P$ and $l$ is a Levi complement. Thanks to a fundamental theorem of R. W. Richardson ([16]), $P$ acts on $u$ with an open dense orbit; this orbit is called the Richardson orbit and its elements are called Richardson elements. Recently ([3]), the first author gave constructions of Richardson elements in the case $k = \mathbb{C}$ for many parabolic subgroups $P$ of $G$. In this note, we observe that these constructions remain valid for any algebraically closed field $k$ of characteristic not equal to 2 and we give constructions of Richardson elements for the remaining parabolic subgroups.

1. Introduction

Let $G$ be a reductive algebraic group over an algebraically closed field $k$ and let $P$ be a parabolic subgroup of $G$ with unipotent radical $U$. We write $g$, $p$ and $u$ for the Lie algebras of $G$, $P$ and $U$ respectively. It is well-known that $G$ has finitely many nilpotent orbits in $g$: this was first proved by R. W. Richardson ([15]), when $\text{char } k$ is zero or good for $G$ (see also [19, Thm. 5.4]); we refer the reader to [6, 5.11] for a survey of the result in bad characteristic. It follows that there is a unique nilpotent orbit $G \cdot e$ which intersects $u$ in an open dense subvariety. Richardson’s dense orbit theorem ([16]) tells us that the intersection $G \cdot e \cap u = P \cdot e$ is a single $P$-orbit (we may assume that $e \in u$). The $P$-orbit $P \cdot e$ is called the Richardson orbit and its elements are called Richardson elements; note that we also sometimes refer to $G \cdot e$ as the Richardson orbit of $P$.

The purpose of this paper is to give explicit constructions of Richardson elements $e \in u$ in case $G$ is of classical type and $\text{char } k \neq 2$; this extends work of the first author in [3]. Further, the representatives that we construct are of a “nice” form as explained in Theorem 1.1 below. We only consider orthogonal and symplectic groups in this paper, because a construction of Richardson elements for $G$ a special linear (or general linear) group was given in [5].

We note that Richardson’s dense orbit theorem also applies to the action of $P$ on its unipotent radical $U$ by conjugation, i.e. there is an open dense $P$-conjugacy class in $U$. If $\text{char } k$ is good for $G$ then, under minor restrictions on $G$, there exists a Springer isomorphism, i.e. a $G$-equivariant isomorphism from the unipotent variety of $G$ to the nilpotent variety of $g$, see [19, III, 3.12] and [1] Cor. 9.3.4. A Springer isomorphism restricts to a $P$-equivariant isomorphism of varieties $\phi : U \to u$; in fact such a $P$-equivariant isomorphism exists without any restrictions on $G$. Therefore, our constructions of Richardson elements $e \in u$ can be
used to give Richardson elements in $U$. In case $G$ is an orthogonal or symplectic group, we can take a Cayley map for a Springer isomorphism, see [19, III, 3.14].

In case $G$ is an orthogonal or symplectic group, we can take a Cayley map for a Springer isomorphism, see [19, III, 3.14]. In case $G=\text{GL}_n(k)$ or $\text{SL}_n(k)$ a construction of Richardson elements was given by T. Brüstle, L. Hille, C. M. Ringel and G. Röhrle in [5, §8] valid for any algebraically closed field $k$. This construction was a consequence of results concerning the representation theory of a certain quasi-hereditary algebra and defines the Richardson elements from certain line diagrams in the plane. It is also recalled in [3, §3] and a direct proof that the construction yields Richardson elements is given.

In [3] the first author constructed Richardson elements for many parabolic subgroups of a classical group over $k=\mathbb{C}$ using line diagrams generalizing those in [5]. The purpose of this paper is to extend these constructions to any parabolic subgroup of a classical group and further to justify that the constructions remain valid for any algebraically closed field $k$ of characteristic not equal to 2. We note that the line diagrams defined in this paper are similar to the pyramids defined in [7, §5 and 6].

In case $G$ is an exceptional group, R. Lawther has given representatives of all Richardson elements $u \in U$ in the unpublished article [12]; one can check that the tables in loc. cit. also determine Richardson elements in $x \in u$, when char $k$ is good for $G$. A. G. Elashvili has also constructed representatives; these constructions were made when Elashvili determined the Lusztig–Spaltenstein induction (see [13]) for nilpotent orbits in exceptional groups, this work is reported in the appendix to Chapter 2 of [18]. In addition, in [2] the first author gave representatives of Richardson elements for nice parabolic subalgebras.

As a consequence of our constructions in this paper along with those in [5], [3], and the representatives given in [12], we have Theorem 1.1 below, which says that one can always find Richardson elements of a “nice” form. Before stating this theorem we need to introduce some notation.

Let $G$ be a simple algebraic group over an algebraically closed field $k$. Assume that char $k$ is zero or a good prime for $G$. Let $T$ be maximal torus of $G$ and $P$ a parabolic subgroup of $G$ containing $T$. Let $\Phi$ be the root system of $G$ with respect to $T$. For $\alpha \in \Phi$, let $g_\alpha$ denote the root subspace of $g=\text{Lie} \, G$ corresponding to $\alpha$ and let $e_\alpha$ be a generator of $g_\alpha$. Let $u$ be the Lie algebra of the unipotent radical of $P$ and let $\Phi(u) \subseteq \Phi$ be defined by $u = \bigoplus_{\alpha \in \Phi(u)} g_\alpha$. Given $x \in g$, we write $C_G(x)$ for the centralizer of $x$ in $G$. For a closed subgroup $H$ of $G$, we write $N_G(H)$ and $C_G(H)$ for the normalizer and centralizer of $H$ in $G$ respectively.

We can now state Theorem 1.1. It is straightforward to verify the first assertion for classical groups from the constructions in [5], [3] and this paper; and from the tables in [12] for exceptional groups. We discuss the claim about the minimality of the size of $\Gamma$ for $G$ an orthogonal or symplectic group below; it is undemanding to verify the claim for $G$ a special linear or exceptional group from the representatives given in [5] and [12].

**Theorem 1.1.** There exists a subset $\Gamma$ of $\Phi(u)$ consisting of linearly independent roots such that

$$x = \sum_{\alpha \in \Gamma} e_\alpha \in u$$

is a Richardson element for $P$. Moreover, we can find $\Gamma$ with $|\Gamma| = \text{rank} \, G - \text{rank} \, C_G(x)$ and this is the minimal possible size of $\Gamma$. 

We call a representative of the Richardson orbit for $P$ of the form $x = \sum_{\alpha \in \Gamma} a_{\alpha} e_{\alpha} \in u$, where $a_{\alpha} \in k^\times$, and $\Gamma$ is a set of linearly independent roots with $|\Gamma| = \text{rank } G - \text{rank } C_G(x)$, a *minimal Richardson element*. We do not make any assertion about uniqueness of minimal Richardson elements here. Indeed let $L$ be the Levi subgroup of $P$ containing $T$, then for a minimal Richardson element $x$ and $g \in N_L(T)$, we have that $g \cdot x$ is a minimal Richardson element. In case $x$ is regular nilpotent in the Lie algebra of the Levi subgroup $C_G(S)$ of $G$, where $S$ is a maximal torus of $C_G(x)$, one can in fact prove that all minimal Richardson elements for $P$ are of the form $g \cdot x$ for some $g \in N_L(T)$. This need not be the case in general.

It is unclear whether Theorem 1.1 holds without the assumption that $\text{char } k$ is good for $G$. We note that for orthogonal groups the constructions given in the present paper, need not yield Richardson elements when $\text{char } k = 2$. For the symplectic groups the representatives for Richardson orbits that we give do remain valid for $\text{char } k = 2$; though this does not follow from the proofs we give in this article.

Let $S$ be as in the statement of Theorem 1.1 and further assume that $S \subseteq T$. According to the Bala–Carter classification of nilpotent orbits (see [10] §4), $x$ is distinguished nilpotent in the Levi subgroup $C_G(S)$ of $G$. In particular, this forces $\Gamma$ to have size at least the semisimple rank of $C_G(S)$, i.e. rank $T - \text{rank } S$. To verify that the Richardson elements $x$ given in this paper are minimal we determine the type of the orbit of $x$. We recall that the *type* of (the orbit of) $x$ is by definition the conjugacy class in $G$ of the derived subgroup $C_G(S)'$ of $C_G(S)$; this is usually determined by the isomorphism type of $C_G(S)'$. For the types that we give, it is rarely the case that they are not determined by just the isomorphism type of $C_G(S)'$, we make brief remarks where there are different conjugacy classes. We check that for our constructions we have that $|\Gamma|$ is equal to the rank of $C_G(S)'$; this is carried out at the end of subsections 3.2 and 4.2. The type of $x$ forms part of the Bala–Carter label of the nilpotent orbit of $x$. We note that it is straightforward to determine the Bala–Carter labels for the Richardson orbits, but this is technical so we choose not to include it in this paper.

The minimality of the size of $\Gamma$ for the representatives constructed in this paper along with those given in [5] and [12] allows one to give an alternative proof of the classification of *nice parabolic subalgebras* given by N. Wallach and the first author in [4]. Further, one can show that the classification of nice parabolic subalgebras given in loc. cit. remains valid in good positive characteristic. We explain how this can be achieved below.

Let $g = \sum_{i \in \mathbb{Z}} g_i$ be the $\mathbb{Z}$-grading of $g$ associated to $p$ (and $t$), see for example [4] §1]. So we have that $p = \sum_{i \geq 0} g_i, \quad u = \sum_{i \geq 1} g_i$. We recall that $p$ is called *nice* if there exists a Richardson element $x \in g_1$.

Let $x = \sum_{\alpha \in \Gamma} e_{\alpha} \in u$ be a minimal Richardson element for $P$. Let $\Gamma_1$ be the subset of $\Gamma$ consisting of roots $\alpha$ such that $g_{\alpha} \subseteq g_1$ and let $x_1 = \sum_{\alpha \in \Gamma_1} e_{\alpha}$. It follows from a result of Richardson ([17] Thm. E)] that $L$ acts on $g_1$ with a dense orbit, where $L$ is the Levi subgroup of $P$ containing $T$. It follows from the fact that $P \cdot x$ is dense in $u$ that $L \cdot x_1$ is dense in $g_1$. Suppose that $p$ is nice and let $y \in g_1$ be a Richardson element. Then $L \cdot y$ is dense in $g_1$ and it follows that $y$ is in the same $P$-orbit as $x_1$, thus $x_1$ is Richardson. The minimality of $|\Gamma|$ implies that $\Gamma_1 = \Gamma$. Hence, we deduce that $p$ is a nice parabolic subalgebra if and only if $g_{\alpha} \subseteq g_1$ for all $\alpha \in \Gamma$. From this equivalence and the representatives of Richardson orbits given in [5], [12] and this paper, one can determine which parabolic subalgebra are nice; and that it does not depend on the characteristic of $k$, for char $k$ zero or good for $G$. 

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In the language of \[19\], a nilpotent $G$-orbit is polarizable if it is the Richardson orbit for some parabolic subgroup of $G$. As a consequence of our constructions one can easily obtain the classification of polarizable nilpotent $G$-orbits for classical $G$ given in Theorem \[12\] below. This classification is also given by G. Kempken in \[11\] as a consequence of the determination of the Lusztig–Spaltenstein induction for nilpotent orbits in classical groups presented in \[11\]. The Lusztig–Spaltenstein induction for nilpotent orbits in classical groups is also described in \[18\] Ch. 2, §7. In the case $G = SL_n(k)$ it is well-known that every nilpotent orbit is the Richardson orbit of some parabolic subgroup. Further, we note that one can obtain the classification for exceptional groups from the tables in \[12\].

We require some notation for the statement of Theorem \[12\]. We recall that nilpotent $G$-orbits are parametrized by orthogonal partitions of $N$, i.e. partitions of $N$ for which all even parts occur with even multiplicity; the nilpotent orbits of the symplectic group $Sp_{2n}(k)$ are parametrized by symplectic partitions of $2n$, i.e. partitions of $2n$ for which all odd parts occur with even multiplicity.

Given partitions $\lambda^i = (\lambda^i_1, \ldots, \lambda^i_l)$ (for $i = 1, \ldots, m$) with $\lambda^i_j \geq \lambda^{i+1}_j$ for each $i$, we write $(\lambda^1, \ldots, \lambda^m)$ for the partition $(\lambda^1_1, \ldots, \lambda^1_l, \lambda^2_1, \ldots, \lambda^2_l, \ldots, \lambda^m_1, \ldots, \lambda^m_l)$. Let $\lambda = (\lambda_1, \ldots, \lambda_l)$ be a partition and suppose that $l$ is even. We say that $\lambda$ is: an even-pair partition if each $\lambda_i$ is even and $\lambda_{2i-1} = \lambda_{2i}$ for each $i$; an odd-pair partition if each $\lambda_i$ is odd and $\lambda_{2i-1} = \lambda_{2i}$ for each $i > 0$; an even 2-step-descending partition if each $\lambda_i$ is even and $\lambda_{2i} > \lambda_{2i+1}$ for each $i$; an odd 2-step-descending partition if each $\lambda_i$ is odd and $\lambda_{2i} > \lambda_{2i+1}$ for each $i$.

**Theorem 1.2.** Assume $\text{char } k \neq 2$.

(i) Let $G = O_N(k)$ and let $O_\lambda$ be the nilpotent orbit given by the orthogonal partition $\lambda$. Then $O_\lambda$ is the Richardson orbit for some parabolic subgroup of $G$ if and only if $\lambda$ is of the form

$$\lambda = (\lambda^0, \lambda^1, \ldots, \lambda^m),$$

where: $m \geq 0$; $\lambda^0$ is a (possibly empty) partition with all odd entries; for odd $i > 0$, $\lambda^i$ is an even-pair partition; and for even $i > 0$, $\lambda^i$ is an odd 2-step-descending partition.

(ii) Let $G = Sp_{2n}(k)$ and let $O_\lambda$ be the nilpotent orbit given by the symplectic partition $\lambda$. Then $O_\lambda$ is the Richardson orbit for some parabolic subgroup of $G$ if and only if $\lambda$ is of the form

$$\lambda = (\lambda^1, \ldots, \lambda^m, \mu),$$

where: $m \geq 0$; for odd $i > 0$, $\lambda^i$ is an odd-pair partition ($\lambda^1$ is allowed to be the empty partition); for even $i > 0$, $\lambda^i$ is an even 2-descending partition; and $\mu$ is a (possibly empty) partition with all even entries.

We now outline the structure of this paper. In Section 2 we give the required recollection about nilpotent orbits in classical groups. In Section 3 we give the construction of Richardson elements for parabolic subgroups of $O_N(k)$. Finally, in Section 4 we give the construction of Richardson elements for parabolic subgroups of $Sp_{2n}(k)$. We give complete proofs that the construction is correct for the $O_N(k)$ case; the $Sp_{2n}(k)$ case is similar (and easier) so we omit the proofs in this case.
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2. Preliminaries

Throughout this paper $k$ is an algebraically closed field of characteristic not equal to 2.

By a partition $\lambda$ we mean a sequence $\lambda = (\lambda_1, \ldots, \lambda_l)$ of non-increasing positive integers; we say $\lambda$ is a partition of $\sum_l \lambda_i$. We recall the dominance ordering on partitions is defined by $\lambda \leq \mu$ if and only if $\sum_{i=1}^{j} \lambda_i \leq \sum_{i=1}^{j} \mu_i$ for all $j$.

We recall that the nilpotent orbits of the general linear group $G = \text{GL}_n(k)$ are parametrized by the partitions of $n$. Let $x$ and $y$ lie in the nilpotent $G$-orbits parametrized by $\lambda$ and $\mu$ respectively. Then it is well-known that $G \cdot x \subseteq G \cdot y$ if and only $\lambda \leq \mu$, see [5 §3].

2.1. Orthogonal groups. Let $G = \text{O}_N(k)$ and $\mathfrak{g} = \text{Lie} G = \mathfrak{so}_N(k)$. We assume $N \geq 3$ and let $n = \lfloor \frac{N}{2} \rfloor$. Let $V = k^N$ be the natural $N$-dimensional $G$-module with standard basis $v_{-n}, \ldots, v_{-2}, v_0, v_1, \ldots, v_n$ and $G$-invariant symmetric bilinear form $(,)$ defined by $(v_0, v_i) = (v_0, v_{-i}) = 0, (v_0, v_0) = 1, (v_i, v_j) = (v_{-i}, v_{-j}) = 0$ and $(v_i, v_j) = \delta_{i,j}$ for $1 \leq i, j \leq n$ (omitting $v_0$ everywhere if $N$ is even). Let $T$ be the maximal torus of $G$, which acts diagonally on the standard basis of $V$ and let $\Phi$ be the root system of $G$ with respect to $T$. The following matrices give a Chevalley basis for $\mathfrak{g} = \mathfrak{so}_N(k)$ (omitting the last family if $N$ is even):

$$\{e_{i,j} - e_{-j,-i}\}_{1 \leq i, j \leq n} \cup \{e_{i,-j} - e_{-j,i}, e_{-j,i} - e_{i,-j}\}_{1 \leq i < j \leq n}$$

$$\cup \{e_{k,0} - e_{0,-k}, e_{0,k} - e_{-k,0}\}_{1 \leq k \leq n},$$

where $e_{i,j}$ denotes the $ij$-matrix unit for $-n \leq i, j \leq n$, i.e. $e_{i,j}$ is the linear map of $V$ sending $v_i$ to $v_j$ and $v_j$ to 0 for $i' \neq i$. We define $e_{i,j} = \pm 1$ to be the coefficient of $e_{i,j}$ in this Chevalley basis if it appears.

Let $d' = (d_1, \ldots, d_s) \in \mathbb{Z}_{\geq 0}$ satisfy $\sum_i d_i = n$. Let $t = 2s + 1$, and define $d = (d_{-s}, \ldots, d_{-s-1}, d_{s-1}, d_s) \in \mathbb{Z}_{\geq 0}^t$ by setting $d_0 = N - 2 \sum_{i=1}^s d_i$, and $d_{-i} = d_i$ for $1 \leq i \leq s$. A vector $d$ as constructed above is called an orthogonal dimension vector; we say that $d$ is proper if: $d_i \neq 0$ for all $i \neq 0$; and if $2(\sum_{i=1}^s d_i) = N$, then $d_1 \neq 1$. We write $|d| = \sum_{i=1}^s d_i$.

Let $d$ be an orthogonal dimension vector. We define $c_i = \sum_{j=0}^{i-1} d_j$ for $i = 1, \ldots, s$. For $i = 1, \ldots, s+1$, let $V_{-i}$ be the subspace of $V$ generated by $v_{-n}, \ldots, v_{-i-1}$. The stabilizer $P(d)$ of the isotropic flag $0 = V_{-(s+1)} \subseteq V_{-s} \subseteq \ldots \subseteq V_{-1}$ is a parabolic subgroup of $G$. Moreover, for any parabolic subgroup $P$ of $G$, there is a unique proper orthogonal dimension vector $d$ such that $P$ is conjugate to $P(d)$. Therefore, in what follows it suffices to just consider parabolic subgroups of the form $P(d)$. We note that if $d$ is proper, then it gives the sizes of the blocks in the Levi subgroup of $P$ containing $T$.

If $\tilde{d}$ differs from $d$ by removing zero entries, then we have $P(\tilde{d}) = P(d)$. Further, if we have $\tilde{d}_1 = \tilde{d}_{s-1} = 1$ and $\tilde{d}_0 = 0$ and $d$ is obtained by replacing the 3-tuple $(1, 0, 1)$ in the centre of $\tilde{d}$ with the single entry 2, then $P(\tilde{d}) = P(d)$. This explains why all parabolic subgroups are obtained by just considering proper $d$. We allow $d$ not to be proper as we need to consider such $d$ in the inductive construction of Richardson elements given in Definition 1.
We may identify $GL_N(k)$ with $GL(V)$. For $i > 0$, we define $V_i$ to be the subspace generated by $v_{-n}, \ldots, v_{-1}, v_0, v_1, \ldots, v_n$ (we omit $v_0$ if $N$ is even). The stabilizer $Q(d)$ of the flag $0 = V_{-(s+1)} \subseteq V_{-s+1} \subseteq \ldots \subseteq V_{-1} \subseteq V_1 \subseteq \ldots \subseteq V_{s+1}$ in $GL_N(k)$ is a parabolic subgroup of $GL_N(k)$ and moreover we have $P(d) = Q(d) \cap G$.

We recall that the nilpotent $G$-orbits are parametrized by partitions $\lambda$ of $N$ such that every even part of $\lambda$ appears with even multiplicity. We call such a partition an *orthogonal partition*. Let $x$ and $y$ lie in the nilpotent $G$-orbits parametrized by the orthogonal partitions $\lambda$ and $\mu$ respectively. Then $G \cdot x \subseteq G \cdot y$ if and only $\lambda \leq \mu$, see [§3]. This means that for nilpotent $x, y \in \mathfrak{g}$ we have $x \in G \cdot y$ if and only if $x \in GL_N(k) \cdot y$.

**Remark 2.1.** We note that if we considered the group $G = SO_N(k)$ instead of $O_N(k)$, then the description of the conjugacy classes of parabolic subgroups of $G$ and nilpotent $G$-orbits is slightly more complicated. In case $N$ is even, some conjugacy classes of parabolic subgroups of $O_N(k)$ give rise to two conjugacy classes of parabolic subgroups of $G$; and if $N$ is divisible by 4, then some nilpotent orbits for $O_N(k)$ split into two $G$-orbits.

Let $P = P(d)$ be a parabolic subgroup of $O_N(k)$. Then either $P \subseteq G$ or $P \cap G$ is a subgroup of $P$ of index 2. Let $x$ be a representative of the dense $P$-orbit in $u$, then we must have $P \cdot x = (P \cap G) \cdot x$. This is because $P \cdot x$ splits into at most two $(P \cap G)$-orbits, and if it splits into two orbits, then they must have the same dimension, namely $\dim u$, which is not possible. Therefore, the representatives of Richardson orbits given in Section §3 are valid for both $O_N(k)$ and $SO_N(k)$.

Let $P$ be a parabolic subgroup of $G$ that is not conjugate to $P(d) \cap G$ for any $d$. Let $g \in O_N(k) \setminus G$, be defined by $g = \sum_{i=2}^{n}(e_{i,i} + e_{-i,-i}) + e_{1,-1} + e_{-1,1}$. Then $gP\!g^{-1}$ is conjugate to $P(d) \cap SO_N(k)$ for some $d$, and can obtain a representative of the Richardson orbit for $P$ from one for $gP\!g^{-1}$ by conjugating by $g$.

**2.2. Symplectic groups.** Much of the notation given in this subsection is analogous to that for orthogonal groups in the previous subsection, so we are briefer.

Let $G = Sp_{2n}(k)$ and $\mathfrak{g} = \text{Lie} G = \mathfrak{sp}_{2n}(k)$. Let $V = k^{2n}$ be the natural $2n$-dimensional $G$-module with standard basis $v_{-n}, \ldots, v_{-1}, v_1, \ldots, v_n$ and $G$-invariant skew-symmetric bilinear form $(,) \text{ defined by } (v_i, v_j) = (v_{-i}, v_{-j}) = 0$ and $(v_i, v_{-j}) = \delta_{i,j}$ for $1 \leq i, j \leq n$. Let $T$ be the maximal torus of $G$, which acts diagonally on the standard basis of $V$ and let $\Phi$ be the root system of $G$ with respect to $T$. The following matrices give a Chevalley basis for $\mathfrak{g}$:

$$\{e_{i,j} - e_{-j,-i}\}_{1 \leq i, j \leq n} \cup \{e_{i,-i} + e_{-i,i}, e_{-i,j} + e_{i,-j}\}_{1 \leq i < j \leq n} \cup \{e_{k,-k}, e_{-k,k}\}_{1 \leq k \leq n},$$

where $e_{i,j}$ denotes the $ij$-matrix unit. We define $e_{i,j}$ to be the coefficient of $e_{i,j}$ in this basis if it appears.

Let $d' = (d_1, \ldots, d_s) \in \mathbb{Z}_{>0}^s$ satisfy $\sum_{i=1}^{s} d_i \leq n$. We define $d$ as in [§2.1] Such $d$ is called a *symplectic dimension vector* and is said to be *proper* if $d_i \neq 0$ for all $i \neq 0$. We define $c_i$ and $V_i$ as in [§2.1]. The stabilizer $P(d)$ of the isotropic flag $0 = V_{-s} \subseteq V_{-s+1} \subseteq \ldots \subseteq V_{-1}$ is a parabolic subgroup of $G$. Moreover, for any parabolic subgroup $P$ of $G$, there is a unique proper symplectic dimension vector $d$ such that $P$ is conjugate to $P(d)$.

In analogy with the orthogonal case, one can define a parabolic subgroup $Q(d)$ of $GL_N(k)$ such that $P(d) = Q(d) \cap G$.

We recall that the nilpotent $G$-orbits are parametrized by partitions $\lambda$ of $N$ such that every odd part of $\lambda$ appears with even multiplicity. We call such a partition a *symplectic*
partition. Let $x$ and $y$ lie in the nilpotent $G$-orbits parametrized by the symplectic partitions $\lambda$ and $\mu$ respectively. Then $G \cdot x \subseteq G \cdot y$ if and only $\lambda \leq \mu$, see [3 §3].

3. Orthogonal groups

For this section we use the notation given in [2.1] We construct Richardson elements for all parabolic subgroups $P(d)$ of $G = O_N(k)$. First we consider the case where all $d_i$ are at most 2. For the general case we explain how to decompose $d$ as $d = d^0 + d^1 + \cdots + d^m$, where the entries of each $d^i$ are all 2 or less, in such a way that we can build up a Richardson element $x$ for $P(d)$ from the Richardson elements for the $P(d^i)$. The idea is that the natural $G$-module $V$ decomposes as an $x$-stable orthogonal sum $V = V_1 \oplus \cdots \oplus V_m$, where $P(d^i)$ is a parabolic subgroup of $O(V_j)$.

3.1. Blocks of size two or less. Let $d = (d_0, \ldots, d_n)$ be an orthogonal dimension vector with $|d| = N$ and all $d_i = 0, 1, 2$, further assume that if $d_0 = 1$, then all other nonzero entries of $d$ are 1. Let $P = P(d)$ be the parabolic subgroup of $G$ corresponding to $d$. In this subsection we construct a representative $x \in \mathfrak{u}$ of the Richardson orbit of $P$.

The idea is to define $x$ from a line diagram $D(d)$ in the plane, which is defined by considering three cases. The diagram $D(d)$ consists of $N$ vertices labelled by the integers $-n, -n + 1, \ldots, n - 1, n = [\frac{N}{2}]$ (we omit 0 if $N$ is even) and arrows between certain vertices. The vertices in $D(d)$ are always labelled so that the labels increase from left to right and from bottom to top; in order to save space we write $\overline{i}$ rather than $-i$ in all the diagrams that we include below. We define $x \in \mathfrak{u}$ from $D(d)$ by

\begin{equation}
(3.1) \quad x = \sum \epsilon_{i,j} e_{i,j},
\end{equation}

where the sum is taken over all arrows from $i$ to $j$ in $D(d)$ and $\epsilon_{i,j} = \pm 1$ is as defined in [2.1].

The definition of $x$ means that $x$ sends $v_i$ to $\sum \epsilon_{i,j} v_j$, where the sum is over all $j$ such that there is an arrow from $i$ to $j$ in $D(d)$; this sum always has at most two summands. The diagram is centrally symmetric about the origin and the labelling of the vertices forces the vertex labelled $-i$ to be centrally symmetric to the vertex labelled $i$. This central symmetry means that $x \in \mathfrak{g} = \mathfrak{so}_N$; the numbering of the vertices ensures that $x \in \mathfrak{u}$.

In each of the cases considered below, we give the partition determined by the Jordan normal form of $x$; this is required for the proof of Theorem [3.3] in [3.2]. As notation for this we let $\sigma$ be the number of entries of $d$ equal to 2 and we let $\rho = |d| - \sigma$, so $\rho$ is the number of nonzero entries in $d$.

**Case 1.** $d_0 = 1$ and all nonzero entries of $d$ are 1.

We construct the diagram $D(d)$ in the plane by placing vertices at the points $(i, 0)$ for $i$ such that $d_i = 1$. For each but the leftmost vertex we draw an arrow to the vertex on its left.

For $d = (1, 1, 1, 1, 1, 1, 1, 1, 1, 1)$ the diagram $D(d)$ is illustrated below.

\[ \overline{4} \leftarrow \overline{3} \leftarrow \overline{2} \leftarrow \overline{1} \leftarrow 0 \leftarrow 1 \leftarrow 2 \leftarrow 3 \leftarrow 4 \]

For $d = (1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1)$, the diagram $D(d)$ is as below.

\[ \overline{2} \leftarrow \overline{1} \leftarrow 0 \leftarrow 1 \leftarrow 2 \leftarrow 3 \leftarrow 4 \]
We have $\rho = |d|$, $\sigma = 0$ and it is easy to see that the Jordan normal form of $x$ is given by the partition $(\rho)$.

**Case 2.** $d_0 = 2$.

To construct $D(d)$, we draw vertices at $(i, 1)$ and $(-i, -1)$ for $i \geq 0$ such that $d_i \neq 0$; and at $(i, -1)$ and $(-i, 1)$ for $i > 0$ such that $d_i = 2$. From each vertex that is not leftmost in its row we draw an arrow to the vertex on its left. If some $d_i = 1$, then we let $l > 0$ be minimal such that $d_l \neq 0$ and draw additional arrows from $(l, 1)$ to $(0, -1)$, and from $(0, 1)$ to $(-l, -1)$.

For example for $d = (2, 2, 2, 2, 2, 2, 2)$, we have $D(d)$ as below; we recall that we label the vertices so that the labels increase from left to right and bottom to top.

We illustrate the diagram $D(d)$ when some entries of $d$ are 1 in three examples. For $d = (1, 1, 1, 2, 2, 1, 1, 1)$ we have

and for $d = (2, 1, 2, 1, 2, 1, 2, 1, 2)$ we have

Finally we give an example when some entries of $d$ are 0, for $d = (2, 1, 0, 1, 0, 2, 0, 1, 0, 1, 2)$ we have $l = 2$ and $D(d)$ as below.

Next we show that the Jordan normal form of $x$ is given by the partition is $(\rho, \sigma)$. In case all nonzero entries of $d$ are 2, we have $\rho = \sigma = n$ and this is trivial. So suppose, some entries of $d$ are 1. First, we note that the kernel of $x$ is 2-dimensional: if $d_0$ is the only entry of $d$ equal to 2 then the kernel of $x$ has basis $\{e_{-n}, e_1 - e_{-1}\}$; otherwise, let $-a$ the label of the leftmost vertex on the top row of $D(d)$, then the ker $x$ has basis $\{e_{-n}, e_{-a}\}$. Therefore, $x$ has two Jordan blocks. Next we note that $x^{\rho-1}e_n = \pm 2e_{-n} \neq 0$ and $x^\rho = 0$ so that $x$ has a Jordan block of size $\rho$. It follows that the partition given by the Jordan normal form of $x$ is $(\rho, \sigma)$ as required.

**Case 3.** $d_0 = 0$. 

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To construct $D(d)$, we draw vertices at $(i, 1)$ and $(-i, -1)$ for $i > 0$ such that $d_i \neq 0$; and at $(i, -1)$ and $(-i, 1)$ for $i > 0$ such that $d_i = 2$. From each vertex that is not leftmost in its row we draw an arrow to the vertex on its left. If four or more entries of $d$ are 1, then there are additional diagonal arrows: Let $0 < l < m$ be minimal such that $d_l, d_m \neq 0$; then we draw an arrow from $(l, 1)$ to $(-m, -1)$, and an arrow from $(m, 1)$ to $(-l, -1)$.

If $d = (1, 1, 1, 0, 1, 1, 1, 1)$, then $D(d)$ is the following diagram

```
1 ← 2 ← 3 ← 4
1 ← 3 ← 2 ← 1
```

and if $d = (2, 2, 2, 2, 0, 2, 2, 2)$, then $D(d)$ is the following diagram

```
7 ← 5 ← 3 ← 1 ← 2 ← 4 ← 6 ← 8
8 ← 6 ← 4 ← 2 ← 1 ← 3 ← 5 ← 7
```

We illustrate the diagram $D(d)$ when some entries of $d$ are 1 in the following examples. For $(2, 2, 2, 1, 2, 2, 1, 2, 2, 2)$ we have

```
8 ← 6 ← 4 ← 2 ← 1 ← 3 ← 5 ← 7 ← 9
```

for $d = (1, 1, 1, 2, 2, 2, 1, 1, 1)$ we have

```
3 ← 1 ← 2 ← 4 ← 5 ← 6 ← 7
```

and for $(2, 1, 2, 1, 2, 2, 1, 2, 1, 2)$ we have

```
7 ← 6 ← 5 ← 4 ← 2 ← 1 ← 3 ← 5 ← 6 ← 8
8 ← 6 ← 5 ← 3 ← 2 ← 1 ← 4 ← 7
```

Finally we give an example where some entries of $d$ are 0. For $d = (2, 1, 0, 2, 0, 2, 0, 1, 2)$ we have $l = 1$, $m = 3$ and $D(d)$ as below.

```
4 ← 1 ← 2 ← 3 ← 5
5 ← 3 ← 2 ← 1 ← 4
```

If all nonzero entries of $d$ are 2, then $\sigma = \rho$, and the Jordan normal form of $x$ is given by the partition $(\rho, \rho)$. Next we show that, when $d$ has some entries 1, the Jordan normal form of $x$ is given by the partition is $(\rho - 1, \sigma + 1)$. If there are only two entries 1 in $d$ this
is immediate \((\rho - 1 = \sigma + 1)\), so we assume that four or more entries of \(d\) are 1. First, we note that the kernel of \(x\) is 2-dimensional: if all nonzero entries of \(d\) are 1, then the kernel of \(x\) has basis \(\{e_{-n}, e_1 - e_-\}\); otherwise, let \(-a\) be the label of the leftmost vertex on the top row of \(D(d)\), then the ker \(x\) has basis \(\{e_{-n}, e_a\}\). Next we note that \(x^{\rho - 2} e_n = \pm 2 e_{-n} \neq 0\) and \(x^{\rho - 1} = 0\) so that \(x\) has a Jordan block of size \(\rho - 1\). It follows that the partition given by the Jordan normal form of \(x\) is \((\rho - 1, \sigma + 1)\) as required.

### 3.2. General case.

In this section we show how to build up Richardson elements for arbitrary parabolic subgroups \(P(d)\) of \(G\). This construction is given in Definition 3.1 which is in two parts: first the decomposition of the (proper) orthogonal dimension vector \(d\) is given; the second part gives the diagram \(D(d)\) from which \(x\) is defined. The Richardson element \(x\) is defined from the diagram \(D(d)\) in the same way as in the previous subsection from the formula (3.1). The construction is quite technical, so the reader may wish to look at the examples given after the definition before reading it.

**Definition 3.1.** (i) First we explain how to make the decomposition \(d = d^0 + d^1 + \cdots + d^m\). We initially set \(d^0 = 0\). Suppose we have defined \(d^0, d^1, \ldots, d^{j-1}\) and consider \(c^j = d - \sum_{i=0}^{j-1} d^i\). We define \(d^j\) by considering cases:

**Case A.** \(c^0 > 0\) and all nonzero entries of \(c^j\) are 2 or greater. Then we define \(d^j\) by

\[
d^j_i = \begin{cases} 
0 & \text{if } c^j_i = 0 \\
2 & \text{if } c^j_i \geq 2 
\end{cases}
\]

**Case B.** \(c^0 > 0\) is odd and \(c^j\) has an entry equal to 1. In this case we must have \(d^0 = 0\) and we redefine it by

\[
d^0_i = \begin{cases} 
0 & \text{if } c^j_i = 0 \\
1 & \text{if } c^j_i \geq 1 
\end{cases}
\]

Then we update \(c^j\).

**Case C.** \(c^0 > 0\) is even and \(c^j\) has an entry equal to 1. We let \(a \geq 0\) be the least positive even entry of \(c^j\) and define \(d^j\) by

\[
d^j_i = \begin{cases} 
0 & \text{if } c^j_i = 0 \\
1 & \text{if } c^j_i \in \{1, 3, \ldots, a - 1\} \\
2 & \text{if } c^j_i \geq a 
\end{cases}
\]

**Case D.** \(c^0 = 0\). Then we define \(d^j\) by

\[
d^j_i = \begin{cases} 
0 & \text{if } c^j_i = 0 \\
1 & \text{if } c^j_i = 1 \\
2 & \text{if } c^j_i \geq 2 
\end{cases}
\]

We continue until \(c^j = 0\).

(ii) Now we explain how the diagram \(D(d)\) is constructed. If \(d_0\) is odd, then we draw the diagram \(D(d^0)\) (with unlabelled vertices); if \(d_0\) is even then we must have \(d^0 = 0\) and we do nothing. Suppose we have dealt with \(d^0, d^1, \ldots, d^{j-1}\). Then we insert the diagram \(D(d^j)\) (with unlabelled vertices) stretched in the vertical direction, so that a vertex at \((i, \pm j)\) is moved to \((i, \pm 1)\) and diagonal arrows are stretched accordingly.

Label \(D(d)\) so that numbers are increasing from left to right and from bottom to top. Then define \(x\) from the formula (3.1).
Examples 3.2. (1) Let \( d = (3, 4, 2, 4, 3) \). Then \( d^0 = 0 \), \( d^1 = (2, 2, 2, 2, 2) \) and \( d^2 = (1, 2, 0, 2, 1) \). The diagram \( D(d) \) is as illustrated below.

\[
\begin{array}{c}
\overline{2} & \multimap & 5 & \multimap & 8 \\
\overline{6} & \multimap & 3 & \multimap & 1 & \multimap & 4 & \multimap & 7 \\
7 & \multimap & 4 & \multimap & 1 & \multimap & 3 & \multimap & 6 \\
\overline{8} & \multimap & 5 & \multimap & 2 \\
\end{array}
\]

(2) Let \( d = (2, 5, 2, 3, 2, 5, 2) \). Then we have \( d^0 = (0, 1, 0, 1, 0, 1, 0) \), \( d^1 = (2, 2, 2, 2, 2, 2, 2) \) and \( d^2 = (0, 2, 0, 0, 0, 2, 0) \). The diagram \( D(d) \) is as illustrated below.

\[
\begin{array}{c}
\overline{4} & \multimap & 8 \\
\overline{9} & \multimap & 5 & \multimap & 2 & \multimap & 1 & \multimap & 3 & \multimap & 7 & \multimap & 10 \\
\overline{6} & \multimap & 0 & \multimap & 6 \\
\overline{10} & \multimap & 7 & \multimap & 3 & \multimap & 1 & \multimap & 2 & \multimap & 5 & \multimap & 9 \\
\overline{8} & \multimap & 4 \\
\end{array}
\]

(3) Let \( d = (2, 2, 4, 1, 4, 2, 2) \). Then we have \( d^0 = (1, 1, 1, 1, 1, 1, 1) \), \( d^1 = (1, 1, 2, 0, 2, 1, 1) \) and \( d^2 = (0, 0, 1, 0, 1, 0, 0) \). The diagram \( D(d) \) is as illustrated below.

\[
\begin{array}{c}
\overline{1} & \multimap & 3 & \multimap & 6 & \multimap & 8 \\
\overline{7} & \multimap & 5 & \multimap & 2 & \multimap & 0 & \multimap & 2 & \multimap & 5 & \multimap & 7 \\
\overline{8} & \multimap & 6 & \multimap & 3 & \multimap & 1 \\
\overline{4} \\
\end{array}
\]
(4) Let \( d = (4, 1, 3, 4, 3, 1, 4) \). Then we have \( d^0 = 0, d^1 = (2, 1, 1, 2, 1, 1, 2) \) and \( d^2 = (2, 0, 2, 2, 2, 0, 2) \). The diagram \( D(d) \) is as illustrated below.

Next we prove that our constructions do indeed give Richardson elements.

**Theorem 3.3.** The element \( x \) defined from \( D(d) \) by (5.1) is a Richardson element for \( P(d) \).

**Proof.** Let \( P = P(d) \) and let \( Q = Q(d) \) be the corresponding parabolic subgroup of \( GL_N(k) \), so we have \( Q \cap O_N(k) = P \); see (5.1). Let \( \lambda = (\lambda_1, \lambda_2, \ldots) \) be the orthogonal partition of \( N \) corresponding to the \( O_N(k) \)-orbit of \( x \) and let \( \lambda^0 = (\lambda^0_1, \lambda^0_2, \ldots) \) be the partition of \( N \) corresponding to the Richardson orbit for \( Q \). By the construction of a Richardson element for \( Q(d) \) given in [5], we know that \( \lambda^0 \) is the dual of the partition of \( N \); it is given by reordering the entries of \( d \) into non-increasing order.

We show that for any orthogonal partition \( \lambda' \) of \( N \) satisfying \( \lambda' \leq \lambda^0 \) we have \( \lambda' \leq \lambda \) (in the dominance order on partitions). The closure order on nilpotent \( GL_n(k) \)-orbits and \( G \)-orbits (see (2.1)) then tells us that \( u \subseteq G \cdot x \). Hence we have \((G \cdot x) \cap u\) is open in \( u \) so that \( x \) is Richardson as required.

Let \( a \) be the maximal such that \( \lambda^0_1, \ldots, \lambda^0_a \) are all odd. In case \( N \) is odd we necessarily have \( a \) is odd, and if \( N \) is even then so is \( a \). We let \( b = \lfloor \frac{a}{2} \rfloor \). From the construction of \( D(d) \) in Definition 3.1 and the partitions given for the \( D(d') \) in Cases 1 and 2 in (5.1) one can easily check that \( \lambda_j = \lambda^0_j \) for \( j = 1, \ldots, a \).

We define the partitions \( \mu = (\lambda_{a+1}, \lambda_{a+2}, \ldots) \) and \( \mu^0 = (\lambda^0_{a+1}, \lambda^0_{a+2}, \ldots) \). For \( j = 1, \ldots, m-b \), let \( \sigma^j \) be the number of entries 2 in \( d^{b+j} \) and let \( \rho^j = |d^{b+j}| - \sigma^j \). We define \( \delta^j \) to be the number of 2's, \( \delta^j = 0 \) if all nonzero entries of \( d^{b+j} \) are 2; and \( \delta^j = 1 \) otherwise. Then \( \mu^0 = (\rho^1, \rho^2, \rho^3, \ldots) \) and \( \mu = (\rho^1 - \delta^1, \rho^1 + \delta^1, \rho^2 - \delta^2, \rho^2 + \delta^2, \ldots) \).

We define a sequence \( \lambda^0, \lambda^1, \ldots, \lambda^{m-b} = \lambda \) of orthogonal partitions of \( N \) as follows: we define \( \lambda^j \) to have the same first \( a + 2j \) entries as \( \lambda \) and remaining entries equal to those of \( \lambda^0 \). Below we show inductively that for each \( j \), we have that:

\[
\lambda^j \leq \lambda^0 \quad \text{and for any orthogonal partition } \lambda' \text{ of } N \text{ satisfying } \lambda' \leq \lambda^0 \quad (\star)
\]

we have \( \lambda' \leq \lambda^j \).

This condition for \( j = m-b \) implies that \( x \) is Richardson for \( P \).

Assume inductively that \((\star)\) holds for \( \lambda^{j-1} \). Let \( \lambda' \) be an orthogonal partition with \( \lambda' \leq \lambda \) in the dominance ordering. By induction we have that \( \sum_{i=1}^{l} \lambda^j_i \leq \sum_{i=1}^{l} \lambda^i_i \) for all \( l \neq a + 2j - 1 \). The only way that we can have \( \sum_{i=1}^{a+2j-1} \lambda^j_i > \sum_{i=1}^{a+2j-1} \lambda^i_i \) is if \( \delta^j = 1, \lambda^j_i = \lambda^i_i \).
for \( i = 1, \ldots, a + 2j - 2 \) and \( \lambda'_{a+2j-1} = \lambda'_{a+2j-1} = \lambda''_{a+2j-1} + 1 \). However, these conditions force \( \lambda''_{a+2j-1} \) to be even and of odd multiplicity in \( \lambda' \), contrary the assumption that \( \lambda' \) is an orthogonal partition. It follows that \((*)\) holds for \( \lambda'' \).

As mentioned after the statement of Theorem 1.1 we now discuss the type of the Richardson element \( x \). Let \( \Gamma \) be the subset of \( \Phi \) such that \( x = \sum_{\alpha \in \Gamma} e_\alpha \); the elements of \( \Gamma \) correspond to pairs of arrows in \( D(d) \).

First we consider the case when \( N \) is even. For \( j = 1, \ldots, m \), we let \( \sigma^j \) be the number of entries 2 in \( d^j \) and let \( \rho^j = |d^j| - \sigma^j \). We define \( \delta^j = 0 \) if \( d_0^j = 2 \) or all nonzero entries of \( d\delta_j \) are 2, and \( \delta^j = 1 \) otherwise. We let \( I_d = \{ j \mid \rho^j - \delta^j = \sigma^j + \delta^j \} \), \( J_d = \{ j \mid \rho^j - \delta^j > \sigma^j + \delta^j \} \) and \( \eta = \frac{1}{2} \sum_{j \in I_d} (\rho^j + \sigma^j) \). Then the type of the orbit of \( x \) is

\[
D_\eta + \sum_{j \in I_d} A_{\rho^j - \delta^j - 1}.
\]

By convention we have \( D_2 = A_1 + A_1 \) (respectively \( D_3 = A_3 \)), but the corresponding Levi subgroup lies in a different conjugacy class to an \( A_1 + A_1 \) (respectively \( A_3 \)) Levi subgroup. Further, we note that if \( \eta = 0 \) and \( |I_d| = n - \sum_{j \in I_d} (\rho^j - \delta^j - 1) \), then inside \( SO_N(k) \) there are two conjugacy classes of Levi subgroups with isomorphism type \( \sum_{j \in I_d} A_{\rho^j - \delta^j} \); however, these two classes fuse in \( G = O_N(k) \). The type of \( x \) given above can be easily verified as in [14] §3 or [10] §4.8.

For \( j \in I_d \), there are \( 2(\rho^j - \delta^j - 1) \) arrows in \( D(d^j) \). Therefore, the arrows in \( D(d^j) \) contribute \( \rho^j - \delta^j - 1 \) elements to \( \Gamma \). For \( j \in I_d \), there are \( \rho^j + \sigma^j \) arrows in \( D(d^j) \). Therefore, the arrows in \( D(d^j) \) contribute \( \frac{1}{2}(\rho^j + \sigma^j) \) elements to \( \Gamma \). Hence, we have

\[
|\Gamma| = \eta + \sum_{j \in I_d} (\rho^j - \delta^j - 1).
\]

This is the minimal possible size for \( \Gamma \) as stated in Theorem 1.1 because it is the rank of the type of the orbit of \( x \).

The situation for the case where \( N \) is odd is almost exactly the same. We define \( \rho^j, \sigma^j, \delta^j, I_d \) and \( J_d \) as for the case \( N \) even. We define \( \eta = \frac{1}{2}(|d^j| - 1 + \sum_{j \in I_d} (\rho^j + \sigma^j)) \). Then the type of the orbit of \( x \) is of the form

\[
B_\eta + \sum_{j \in I_d} A_{\rho^j - \delta^j - 1},
\]

where by convention \( B_1 = A_1 \), but with a short root and therefore the corresponding Levi subgroup is in a different conjugacy class to an \( A_1 \) Levi subgroup. One can check this type is correct, and that the size of \( \Gamma \) is equal to the rank of the type of \( x \), as for the case \( N \) even.

4. Symplicial groups

In this section we use the notation from [22]. We construct Richardson elements for all parabolic subgroups \( P(d) \) of \( G = Sp_{2n}(k) \). As for the orthogonal groups, we first consider the case where all \( d_i \) are at most 2. We decompose a general symplectic dimension vector \( d \) as \( d = d^1 + \cdots + d^m \) where the entries of each \( d^j \) are all 2 or less, and then build up a Richardson element for \( P(d) \) from the Richardson elements for the \( P(d^j) \).
4.1. **Block of size two or less.** Let \( d = (d_{-s}, \ldots, d_s) \) be a symplectic dimension vector with \( |d| = 2n \) and all \( d_i = 0, 1, 2 \). Let \( P = P(d) \) be the corresponding parabolic subgroup of \( G \). In this subsection we construct a representative \( x \in u \) of the Richardson orbit of \( P \).

As for the orthogonal case, we define \( x \) from a line diagram \( D(d) \) in the plane which is given by considering four cases. The diagram consists of vertices labelled \( \pm 1, \pm 2, \ldots, \pm n \) and arrows between certain vertices. The vertices in \( D(d) \) are labelled as in the previous section: they increase from left to right and from bottom to top; again \( i \) stands for \(-i\). We define \( x \in u \) by

\[
(4.1) \quad x = \sum \epsilon_{i,j} e_{i,j},
\]

where the sum is taken over all arrows from \( i \) to \( j \) in \( D(d) \) and \( \epsilon_{i,j} = \pm 1 \) is defined as in §2.2.

The diagram \( D(d) \) is centrally symmetric about the origin and the vertex labelled \(-i\) is centrally symmetric to the vertex \(i\), which implies \( x \in \mathfrak{sp}_{2n} \). Thanks to the numbering of the vertices we have \( x \in u \).

For each of the four cases below we give the partition of the Jordan normal form of \( x \). As notation for this we let \( \sigma \) be the number of entries of \( d \) equal to 2 and we let \( \rho = |d| - \sigma \), so \( \rho \) is the number of nonzero entries in \( d \).

**Case 1.** \( d_0 = 0 \) and all nonzero entries of \( d \) are 1.

To construct \( D(d) \) we draw vertices at the points \((i, 0)\) for \( i \) with \( d_i \neq 0 \). From each vertex that has a left neighbour, we draw an arrow to the left. So for example, for \( d = (1, 1, 1, 1, 0, 1, 1, 1, 1) \), we have \( D(d) \) as below:

\[
\begin{align*}
4 & \quad \quad \quad \quad \quad \quad \quad \quad 3 \quad \quad \quad \quad \quad \quad 2 \quad \quad \quad \quad \quad \quad 1 \quad \quad \quad \quad \quad \quad 2 \quad \quad \quad \quad \quad \quad 3 \quad \quad \quad \quad \quad \quad 4
\end{align*}
\]

and for \( d = (1, 0, 1, 1, 0, 0, 0, 1, 1, 0, 1) \) we have

\[
\begin{align*}
3 & \quad \quad \quad \quad \quad \quad 2 \quad \quad \quad \quad \quad \quad 1 \quad \quad \quad \quad \quad \quad 2 \quad \quad \quad \quad \quad \quad 3
\end{align*}
\]

We have \( \rho = |d| \) and \( \sigma = 0 \), and the Jordan normal form of \( x \) is given by the partition \((\rho)\).

**Case 2.** \( d_0 = 2 \).

We draw vertices at \((i, 1)\) and \((-i, -1)\) for \( i \geq 0 \) such that \( d_i \neq 0 \) and further vertices at \((-i, 1)\) and at \((i, -1)\) for \( i > 0 \) such that \( d_i = 2 \). From each vertex that is not leftmost in its row we draw an arrow to the vertex on its left. If four or more entries of \( d \) are 1, then let \( l > 0 \) be minimal such that \( d_l > 0 \) and draw an additional arrow from \((l, 1)\) to \((-l, -1)\).

So if \( d = (2, 2, 2, 2, 2, 2) \), then we have \( D(d) \) as below:

\[
\begin{align*}
6 & \quad \quad \quad \quad \quad \quad 4 \quad \quad \quad \quad \quad \quad 2 \quad \quad \quad \quad \quad \quad 1 \quad \quad \quad \quad \quad \quad 3 \quad \quad \quad \quad \quad \quad 5 \quad \quad \quad \quad \quad \quad 7
\end{align*}
\]

\[
\begin{align*}
7 & \quad \quad \quad \quad \quad \quad 5 \quad \quad \quad \quad \quad \quad 3 \quad \quad \quad \quad \quad \quad 1 \quad \quad \quad \quad \quad \quad 2 \quad \quad \quad \quad \quad \quad 4 \quad \quad \quad \quad \quad \quad 6
\end{align*}
\]

We illustrate \( D(d) \) in cases where there are some entries 1 in \( d \) with three examples. For \((2, 1, 2, 2, 2, 1, 2)\) we have

\[
\begin{align*}
3 & \quad \quad \quad \quad \quad \quad 2 \quad \quad \quad \quad \quad \quad 1 \quad \quad \quad \quad \quad \quad 3 \quad \quad \quad \quad \quad \quad 4 \quad \quad \quad \quad \quad \quad 6
\end{align*}
\]

\[
\begin{align*}
6 & \quad \quad \quad \quad \quad \quad 4 \quad \quad \quad \quad \quad \quad 3 \quad \quad \quad \quad \quad \quad 1 \quad \quad \quad \quad \quad \quad 2 \quad \quad \quad \quad \quad \quad 5
\end{align*}
\]
for (1,1,2,2,2,1,1) we have

\[ \begin{align*}
4 & \rightarrow 2 \rightarrow 1 \rightarrow 3 \rightarrow 5 \rightarrow 6 \rightarrow 7 \\
7 & \rightarrow 6 \rightarrow 5 \rightarrow 3 \rightarrow 1 \rightarrow 2 \rightarrow 4
\end{align*} \]

and for (2,1,1,2,1,1,2) we have

\[ \begin{align*}
4 & \rightarrow 2 \rightarrow 3 \rightarrow 1 \rightarrow 5 \\
5 & \rightarrow 3 \rightarrow 2 \rightarrow 4
\end{align*} \]

Finally, we give an example when some of the entries of \( d \) are 0. For (1, 0, 2, 1, 0, 2, 0, 1, 2, 0, 1) we have \( l = 2 \) and \( D(d) \) as below

\[ \begin{align*}
3 & \rightarrow 1 \rightarrow 2 \rightarrow 4 \rightarrow 5 \\
5 & \rightarrow 4 \rightarrow 2 \rightarrow 3
\end{align*} \]

If all entries of \( d \) are 2 then \( \rho = \sigma \) and the Jordan normal form of \( x \) is given by the partition \((\rho, \rho)\); otherwise the Jordan normal form is given by \((\rho - 1, \sigma + 1)\). This can be verified using argument similar to those for orthogonal cases in §3.1.

**Case 3.** \( d_0 = 0 \) and all nonzero entries of \( d \) are 2.

To construct \( D(d) \) we draw vertices at the points \((i, 1)\) and \((i, -1)\) for \( i = -s, \ldots, s \) such that \( d_i = 2 \). From each vertex which is not leftmost in its row we draw an arrow to the vertex on its left.

For example for \( d = (2, 2, 2, 0, 2, 2, 2) \), we have \( D(d) \) as below.

\[ \begin{align*}
5 & \rightarrow 3 \rightarrow 1 \rightarrow 2 \rightarrow 4 \rightarrow 6 \\
6 & \rightarrow 4 \rightarrow 2 \rightarrow 1 \rightarrow 3 \rightarrow 5
\end{align*} \]

For \( d = (2, 0, 2, 0, 0, 2, 0, 2) \) we have

\[ \begin{align*}
3 & \rightarrow 1 \rightarrow 2 \rightarrow 4 \\
4 & \rightarrow 2 \rightarrow 1 \rightarrow 3
\end{align*} \]

In this case we have \( \rho = \sigma \) and the partition given by the Jordan normal form of \( x \) is \((\rho, \rho)\).

**Case 4.** \( d_0 = 0 \) and \( d \) has both entries 1 and 2.

To construct \( D(d) \) we draw vertices at the points \((i, 1)\) and \((i, -1)\) for \( i > 0 \) such that \( d_i \neq 0 \); furthermore, for \( i > 0 \) with \( d_i = 2 \), we draw vertices at \((i, -1)\) and \((-i, 1)\). We let \( l, m > 0 \) be minimal such that \( d_l \neq 0 \) and \( d_m = 2 \) (allowing \( l = m \)). From each vertex that
is not leftmost in its row and is not at \((l, 1)\) or \((m, -1)\), we draw an arrow to the vertex on its left. We draw additional arrows from \((l, 1)\) to \((-l, -1)\) and from \((m, -1)\) to \((-m, 1)\).

We illustrate \(D(d)\) in three examples: For \(d = (1, 1, 2, 2, 0, 2, 2, 1, 1)\) we have \(l = m = 1\) and \(D(d)\) as below

\[
\begin{align*}
&3 \rightarrow 1 \\
&6 \rightarrow 5 \rightarrow 4 \rightarrow 2 \rightarrow 1 \rightarrow 3 \\
&2 \rightarrow 4 \rightarrow 5 \rightarrow 6
\end{align*}
\]

for \(d = (2, 2, 1, 1, 0, 1, 1, 2, 2)\) we have \(l = 1, m = 3\) and \(D(d)\) as below

\[
\begin{align*}
&5 \rightarrow 3 \\
&6 \rightarrow 4 \rightarrow 2 \rightarrow 1 \\
&2 \rightarrow 4 \rightarrow 6 \\
&1 \rightarrow 3 \rightarrow 5
\end{align*}
\]

and for \(d = (1, 0, 1, 2, 1, 0, 0, 0, 1, 2, 1, 0, 1)\) we have \(l = 2, m = 3\) and \(D(d)\) as below

\[
\begin{align*}
&5 \rightarrow 4 \rightarrow 3 \rightarrow 1 \\
&2 \rightarrow 1 \rightarrow 3 \rightarrow 4 \rightarrow 5 \\
&3 \rightarrow 2
\end{align*}
\]

It is easy to see that the Jordan normal form of \(x\) is given by the partition \((\rho, \sigma)\).

### 4.2. General case.

Analogously to the orthogonal groups, we can now build up Richardson elements for arbitrary subgroups \(P(d)\) of \(G\). Given a (proper) symplectic dimension vector \(d\), we decompose it as a sum of dimension vectors \(d^j\) with entries at most 2. Then we build up a diagram \(D(d)\) from the diagrams \(D(d^j)\), which defines the Richardson element \(x\) as in the previous subsection from the formula (4.1). After describing the details of the construction we illustrate it with examples.

**Definition 4.1.** (i) First we explain how to make the decomposition \(d = d^0 + d^1 + \cdots + d^m\).

We initially set \(d^0 = 0\). Suppose we have defined \(d^0, d^1, \ldots, d^j\) and consider \(c^j = d - \sum_{i=0}^{j-1} d^i\). We define \(d^j\) by considering cases:

**Case A.** \(c^j_0 > 0\). Then we define \(d^j\) by

\[
d^j_i = \begin{cases} 
0 & \text{if } c^j_i = 0 \\
1 & \text{if } c^j_i = 1 \\
2 & \text{if } c^j_i \geq 2
\end{cases}
\]

**Case B.** \(c^j_0 = 0\) and all nonzero entries of \(c^j\) are 2 or greater. Then we define \(d^j\) by

\[
d^j_i = \begin{cases} 
0 & \text{if } c^j_i = 0 \\
2 & \text{if } c^j_i \geq 2
\end{cases}
\]
Case C. \( c_0^j = 0 \) and \( c^j \) has an entry equal to 1 and a positive even entry. Then we let \( a > 0 \) be the least positive even entry of \( c^j \) and define \( d^j \) by

\[
d_i^j = \begin{cases} 
0 & \text{if } c_i^j = 0 \\
1 & \text{if } c_i^j \in \{1, 3, \ldots, a - 1\} \\
2 & \text{if } c_i^j \geq a
\end{cases}
\]

Case D. \( c_0^j = 0 \) and all nonzero entries of \( c^j \) are odd. In this case we must have \( d^0 = 0 \) and we redefine it by

\[
d_i^0 = \begin{cases} 
0 & \text{if } c_i^j = 0 \\
1 & \text{if } c_i^j \geq 1
\end{cases}
\]

Then we update \( c^j \).

We continue until \( c^j = 0 \).

(ii) Now the diagram \( D(d) \) is constructed as follows. If \( d_0 \neq 0 \), then we draw the diagram \( D(d^0) \) (with unlabelled vertices); if \( d^0 = 0 \) then we do nothing. Suppose we have taken care of \( d^0, d^1, d^2, \ldots, d^{j-1} \). Then we insert the diagram \( D(d^j) \) (with unlabelled vertices) stretched in the vertical direction, so that a vertex \((i, \pm 1)\) is moved to \((i, \pm j)\) and diagonal arrows are stretched accordingly.

Label \( D(d) \) with the numbers \( \pm 1, \ldots, \pm n \) increasing from left to right and bottom to top. Then define \( x \) from the formula (4.1)

Examples 4.2. (1) Let \( d = (3, 4, 2, 4, 3) \). Then \( d^0 = 0 \), \( d^1 = (2, 2, 2, 2) \) and \( d^2 = (1, 2, 0, 2, 1) \). The diagram \( D(d) \) is shown below.

```
  7  3  1  4  7
  \downarrow \downarrow \downarrow \downarrow \downarrow
  5  3  1  3  6
  \downarrow \downarrow \downarrow \downarrow \downarrow
  6  3  1  2

(2) Let \( d = (2, 3, 2, 2, 3, 2) \). Then we have \( d^0 = (0, 1, 0, 0, 0, 1, 0) \) and \( d^1 = (2, 2, 2, 2, 2, 2) \). The diagram \( D(d) \) is as illustrated below.

  7  4  2  1  3  6  8
  \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow
  5
  \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow
  8  6  3  2  4  7
```

(3) Let \( d = (3, 1, 6, 1, 1, 2, 1, 1, 6, 1, 3) \). Then \( d^0 = 0, d^1 = (2, 1, 2, 1, 1, 2, 1, 1, 2), d^2 = (1, 0, 2, 0, 0, 0, 0, 2, 0, 1) \) and \( d_3 = (0, 0, 2, 0, 0, 0, 0, 2, 0, 0) \). The diagram \( D(d) \) is shown below.

(4) Let \( d = (5, 3, 5, 0, 5, 3, 5) \). Then \( d^0 = (1, 1, 1, 0, 1, 1, 1), d^1 = (2, 2, 2, 0, 2, 2, 2) \) and \( d^2 = (2, 0, 2, 0, 2, 0, 2) \). The diagram \( D(d) \) is shown below.

We now state a theorem saying that the Richardson elements constructed as in Definition 3.1 are indeed Richardson elements. The proof of Theorem 4.3 is completely analogous to the proof of the Theorem 3.3; it is based on the fact that the closure order for nilpotent orbits is given by the dominance order on partitions as mentioned in §2.2. Therefore, we omit the details.

**Theorem 4.3.** The element \( x \) defined from \( D(d) \) is a Richardson element for \( P(d) \).
As mentioned after the statement of Theorem 1.1, we now discuss the type of the Richardson element $x$. Let $\Gamma$ be the subset of $\Phi$ such that $x = \sum_{\alpha \in \Gamma} e_{\alpha}$; the elements of $\Gamma$ correspond to either single arrows in $D(d)$ that pass through the origin, or pairs of arrows in $D(d)$ that do not pass through the origin. One can check the type of $x$ given below is correct, and that it has rank equal to the size of $\Gamma$ as for the orthogonal case in §3.2. Therefore, we omit the details.

For $j = 1, \ldots, m$, we let $\sigma^j$ be the number of entries 2 in $d^j$ and let $\rho^j = |d^j| - \sigma^j$. We define $\delta^j = 0$ if $d^j_0 = 0$ or all nonzero entries of $d^j$ are 2; and $\delta^j = 1$ otherwise. We let $I_d = \{ j \mid \rho^j - \delta^j = \sigma^j + \delta^j \}$, $J_d = \{ j \mid \rho^j - 1 > \sigma^j - 1 \}$ and $\eta = \frac{1}{2}(|d^0| + \sum_{j \in J_d} (\rho^j + \sigma^j))$. Then the type of the orbit of $x$ is

$$C_{\eta} + \sum_{j \in I_d} A_{\rho^j - \delta^j - 1},$$

where by convention $C_1 = A_1$, but with a long root and therefore the corresponding Levi subgroup is in a different conjugacy class to an $A_1$ Levi subgroup.

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