A UNIVERSAL ALGEBRAIC APPROACH TO RACK COVERINGS

MARCO BONATTO, DAVID STANOVSKÝ

ABSTRACT. We study rack coverings from a universal algebraic viewpoint and we prove that they can be understood using the notion of strongly abelian congruence. We investigate and characterize several particular classes of coverings as central and abelian coverings and coverings preserving the displacement group. We give a new characterization of simply connected quandles and we show that the categorical notion of normal extension coincides with the notion of central covering. We answer several questions from the papers of Clark, Saito and Vendramin [CS17] and [CSV16] about identities preserved by quandle coverings.

INTRODUCTION

Racks and quandles are binary algebraic structures related to knot invariants, solutions to the Yang-Baxter equation and Hopf algebras. In the present paper we study rack coverings from an universal algebraic viewpoint, as a continuation of [BS19] in which we started to put extensions of racks into the universal algebraic framework of tame congruence theory [HM88] and commutator theory [FM87].

Coverings, also called extensions by constant cocycles, are one of the most important type of extensions. The primary reason is that the constant cocycles provide a wide class of powerful invariants of knots [CJK +03, CSV16, Eis03], but the construction is important also from the algebraic perspective. For example, it is easy to see that every quandle is a covering of a conjugation quandle, using a Cayley-like representation. There have been several attempts to build a comprehensive theory of extensions, see, for instance, [AG03, CENS03] and [Eis14] in which a theory of quandle coverings has been developed in analogy with the covering theory of topological spaces using a categorical language. For instance simply connected quandles, defined in [Eis14] in analogy with topology, turned out to be the connected quandles for which every covering is trivial (in Theorem 3.2 we provide an alternative characterization of such quandles).

From a categorical viewpoint, in [Eve14] it has also been shown that quandle coverings are equivalent to central extensions with respect to the adjunction between the category of quandles and the category of projection quandles.

The present paper is partly motivated by the questions raised in [CS17] and [CSV16]. Indeed, one of our goals is to contribute to the question to what extend coverings preserve identities. The question was raised by Clark, Saito and Vendramin in [CS17] and [CSV16]. They focused on the subclass of abelian coverings, and on a special kind of identities called inner identities, determined by composition of inner mappings (or translations, in our terminology). We extend some of their results to arbitrary coverings and arbitrary identities, solving some of the open problems posted in [CS17, CSV16]. The main result is that every covering of a connected n-symmetric quandle is again n-symmetric, see Theorem 4.12.

Nevertheless, the first goal of the present note is to point out a connection to the tame congruence theory of universal algebra: coverings are precisely the extensions that are strongly abelian (in other words, of unary type in their sense of [HM88]). We show that strongly abelian congruences for left-quasigroups coincide with congruences contained in the Cayley kernel in Proposition 5.1 (i.e. coverings in the case of racks). We also characterize strongly solvable racks by identities in Theorem 5.2, namely they are exactly the multipermutational racks in the sense of [JPZ19]. This result actually applies to the wider setting of Cayley varieties of left-quasigroups, including other binary algebras defined in the context of the study of the solutions to the Yang-Baxter equation (e.g. cycle sets). Strongly solvable algebras of this type are indeed related to a particular class of solutions, namely multipermutation solutions.

We also prove in Theorem 5.4 that every strongly solvable rack is nilpotent in the sense of [FM87] (this does not hold for arbitrary algebraic structures).
Central coverings are coverings given by central congruences (in the sense of [EMS7]). Using the commutator theory developed in [BS19] we show a characterization of such type of coverings in Proposition 5.3 and in Proposition 5.4 for the connected ones. In [DAEM18] the normal extensions with respect to the adjunction between the category of quandles and the category of projection quandles have been characterized. We show that they correspond to central coverings in Theorem 5.11.

A special family of central coverings is given by abelian extensions as defined in [CSV16]. We rename them as abelian coverings and we characterize them in Proposition 5.3 and in Proposition 5.6 we deal with the connected case.

The paper is organized as follows. In Section 1 we recall the necessary definitions and basic facts on racks and quandles. In Section 2.1 we introduce the Cayley kernel, in 2.2 the cocycle representations for coverings of racks and in 2.3 we provide the characterization of connected coverings preserving the displacement group. In Section 3 we give a new characterization of simply connected quandles.

Section 4 is about coverings and identities. In Section 4.2 we provide several counterexamples showing that identities are often not preserved. Section 4.3 contains a notable exception to this phenomenon concerning the symmetric laws.

In Section 5.1 we explain the characterization of coverings in terms of the tame congruence theory and we describe central coverings in 5.2. Section 6 is dedicated to abelian coverings.

1. Terminology and basic facts

1.1. Racks and quandles. A rack is a left distributive left-quasigroup, i.e. a binary algebraic structure $Q = (Q, \ast, \setminus)$ such that

\begin{align*}
(1) & \quad x \ast (x \setminus y) = x \setminus (x \ast y) = y \\
(2) & \quad x \ast (y \ast z) = (x \ast y) \ast (x \ast z)
\end{align*}

for every $x, y, z \in Q$. A quandle is an idempotent rack, i.e. it satisfies the identity $x \ast x = x$ for every $x \in Q$. The mappings $L_x : Q \to Q, y \mapsto x \ast y$, will be called (left) translations. The axioms (1) and (2) are equivalent to stating that all (left) translations are automorphisms of $Q$.

Example 1.1. A rack $Q$ is called permutation rack if the operation does not depend on the first argument, i.e., the rack operation is $a \ast b = f(b)$ where $f$ is a permutation of $Q$. If $Q$ is a quandle then $f$ is the identity on $Q$ and $Q$ is called projection quandle. If $|Q| = 1$ then $Q$ is called trivial.

Example 1.2. Let $G$ be a group and $f$ its automorphism and $H \leq \text{Fix}(f)$. We will denote $Q(G, H, f)$ the quandle $(G/H, \ast, \setminus)$ with the operation defined by $aH \ast bH = af(a^{-1}b)H$, and call it a coset quandle. If $H$ is the trivial group, $Q$ is called principal over the group $G$ and denoted by $Q(G, f)$. If $G$ is abelian $Q$ is called affine. In such case the operation is defined as $a \ast b = (1 - f)(a) + f(b)$ and we denoted this quandle as $\text{Aff}(G, f)$.

Two important permutation groups are associated to every quandle: the (left) multiplication group, generated by all (left) translations,

$$\text{LMlt}(Q) = \langle L_a : a \in Q \rangle \leq \text{Aut}(Q),$$

and its subgroup, the displacement group, defined by

$$\text{Dis}(Q) = \langle L_a L_b^{-1} : a, b \in Q \rangle.$$

A rack is called connected if its left multiplication group acts transitively on it. If $Q$ is a quandle the orbits of $\text{LMlt}(Q)$ and of $\text{Dis}(Q)$ are the same, hence $Q$ is connected if $\text{Dis}(Q)$ is transitive.

We will denote by $\text{Dis}(Q)_a$, the point-wise stabilizer of $a$ in $\text{Dis}(Q)_a$ and $\hat{L}_a$ be the inner automorphism of $L_a$ for every $a \in Q$.

Connected quandles can be represented as coset quandles over their displacement group.

Proposition 1.3. [HSV16] Theorem 4.1] A quandle $Q$ is connected if and only if $Q \cong Q(\text{Dis}(Q), \text{Dis}(Q)_a, \hat{L}_a)$ for every $a \in Q$.

Proposition 1.4. [Bon19] Proposition 2.1] A connected quandle $Q$ is principal if and only if $\text{Dis}(Q)$ is regular on $Q$, i.e. $Q \cong Q(\text{Dis}(Q), \hat{L}_a)$. 

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Remark 1.5. Accordingly to Lemma 3.11 of [Bia15], the action of the transvection group of a coset quandle $Q = Q(G, H, f)$ is given by the left action of $[G, f] = \langle gf(y)^{-1}, g \in G \rangle$ on the set $G/H$. If $Q$ is connected then $\text{Dis}(Q) = [\text{Dis}(Q), L_\alpha]$ and

$$\text{Core}_{\text{Dis}(Q)}(\text{Dis}(Q)_\alpha) = \bigcap_{g \in \text{Dis}(Q)} \text{Dis}(Q)_{g(\alpha)} = \bigcap_{b \in Q} \text{Dis}(Q)_b = 1$$

since the point-wise stabilizers in $\text{Dis}(Q)$ are all conjugate.

1.2. Congruences and subgroups. A congruence of a rack $Q$ is an equivalence relation on $Q$ invariant with respect to the operations $*$. Let $f : Q \to R$ be a rack homomorphism. The kernel, $\ker(f) = \{(x, y) \in Q^2 : f(x) = f(y)\}$ is a congruence of $Q$. By virtue of the first isomorphism theorem, homomorphic images of a rack (i.e. images of surjective homomorphisms) and congruence are essentially the same thing. The congruence lattice of $Q$ will be denoted by $\text{Con}(Q)$, blocks of a congruence $\alpha$ by $[a]_\alpha$ (omitting the subscript whenever there is no risk of confusion) and the correspondent factor rack by $Q/\alpha$. In particular, if $Q/\alpha$ is connected then the blocks of $\alpha$ have all the same cardinality and $\alpha$ is said to be uniform. Since the homomorphic image of a connected rack is connected, the congruences of connected racks are uniform. If $Q$ is a quandle, then the $\alpha$-blocks are subquandles and so, if $Q$ is also connected then the $\alpha$-blocks are isomorphic subquandles [Bia15, Proposition 2.5].

As we already noticed in [BS19], for every congruence $\alpha$ of a rack $Q$, we have a corresponding group homomorphism $\pi_\alpha : \text{LMlt}(Q) \to \text{LMlt}(Q/\alpha)$, defined by $L_\alpha \mapsto L_{[a]_\alpha}$. Observe that, for every $h \in \text{LMlt}(Q)$ and $b \in Q$, we have

$$\pi_\alpha(h)([b]_\alpha) = [h(b)]_\alpha.$$

Moreover, the mapping $\pi_\alpha$ restricts and corestricts to the displacements groups. We will denote the kernel of $\pi_\alpha$ as $\text{LMlt}^\alpha$ and the kernel of its restriction to the displacements groups as $\text{Dis}^\alpha$. Note that

$$\text{LMlt}^\alpha = \{h \in \text{LMlt}(Q) : h(a) \alpha a \text{ for every } a \in Q\},$$

$$\text{Dis}^\alpha = \{h \in \text{Dis}(Q) : h(a) \alpha a \text{ for every } a \in Q\}.$$ 

Let $a \in Q$, we define the block stabilizer of $a$ by $\text{LMlt}(Q)_{[a]_\alpha} = \pi_\alpha^{-1}(\text{LMlt}(Q/\alpha)_{[a]_\alpha}) = \{h \in \text{LMlt}(Q) : h(a) \alpha a\}$. The point-wise stabilizer $\text{LMlt}(Q)_a$ and the kernel $\text{LMlt}^\alpha$ are contained in the block stabilizer (similar inclusions holds for the subgroups $\text{Dis}(Q)_a$, $\text{Dis}^\alpha$ and $\text{Dis}(Q)_{[a]_\alpha} = \pi_\alpha^{-1}(\text{Dis}(Q/\alpha)_{[a]_\alpha})$).

Assume that $Q/\alpha$ is connected. Then for every $a_0, a \in Q$ there exists $h \in \text{LMlt}(Q)$ with $h([a_0]) = h([a])$ and so

$$\text{LMlt}^\alpha = \bigcap_{[a], \alpha \in Q/\alpha} \text{LMlt}(Q)_{[a]_\alpha} = \bigcap_{h \in \text{LMlt}(Q)} h^{-1}\text{LMlt}(Q)_{[a]_\alpha}h = \text{Core}_{\text{LMlt}(Q)}(\text{LMlt}(Q)_{[a]_\alpha}).$$

The following proposition gives a criterion for the connectedness of a rack in terms of the action of the block stabilizers.

Proposition 1.6. [Bia15 Proposition 1.3] Let $Q$ be a rack (resp. a quandle) and $\alpha \in \text{Con}(Q)$. Then $Q$ is connected if and only if $Q/\alpha$ is connected and $\text{LMlt}(Q)_{[a]_\alpha}$ (resp. $\text{Dis}(Q)_{[a]_\alpha}$) is transitive on $[a]_\alpha$ for every $a \in Q$.

In [BS19] we investigated the interplay between congruences and normal subgroups of the left multiplication group. For every congruence $\alpha$ of a rack $Q$ we can define the displacement group relative to $\alpha$ as

$$\text{Dis}_\alpha = \langle L_\alpha L_\alpha^{-1} : a \alpha b \rangle.$$ 

If $\alpha = 1_Q$ we recover the definition of the displacement group of $Q$. For a subgroup $N \leq \text{Sym}(S)$ we denote by $a^N$ the orbit of $a \in S$ under the action of $N$. For every normal subgroup $N$ of $\text{LMlt}(Q)$ we can define two congruences: $\mathcal{O}_N$ as the orbit decomposition of $Q$ with respect to the action of $N$ (i.e. $[a]_{\mathcal{O}_N} = a^N$ for every $a \in Q$) and

$$\text{con}_N = \{(a, b) \in Q \times Q : L_\alpha L_\alpha^{-1} \in N\}.$$ 

In particular, the inclusion $\mathcal{O}_N \leq \text{con}_N$ holds.

Abelianness and centrality for congruences of general algebras can be defined using the notion of commutator of congruences in the sense of [FMS7]. As for groups, a quandle $Q$ is nilpotent of length $n$ if there exist a chain of congruences

$$0_Q = \alpha_0 \leq \alpha_1 \leq \ldots \leq \alpha_n = 1_Q$$

such that $\alpha_i/\alpha_{i-1}$ is central in $Q/\alpha_i$ and that the center of $A$ is the biggest central congruence of $A$.
In [BS19] we develop the commutator theory for racks and it turned out that such properties of congruences are controlled by group theoretical properties of the relative displacement groups. To this end we need to introduce the following equivalence relation:

\[ a \sigma_Q b \] if and only if \( \text{Dis}(Q)_a = \text{Dis}(Q)_b \).

Here we collect the main results concerning nilpotence for quandles we are using in the paper.

**Proposition 1.7.** [BS19, Theorem 1.1, Proposition 5.9] Let \( Q \) be a rack and let \( \alpha \in \text{Con}(Q) \). Then \( \alpha \) is central if and only if \( \text{Dis}_\alpha \) is central in \( \text{Dis}(Q) \) and \( \alpha \leq \sigma_Q \). In particular, the center of \( Q \) is \( \mathcal{Q} = \text{Con}_K(\text{Dis}(Q)) \cap \sigma_Q \).

**Proposition 1.8.** [BS19, Lemma 6.2] A rack \( Q \) is nilpotent if and only if \( \text{Dis}(Q) \) is nilpotent. In particular, if \( \text{Dis}(Q) \) is nilpotent of length \( n \), then \( Q \) is nilpotent of length at most \( n + 1 \).

1.3. Terms and identities. A term \( t = t(x_1, \ldots, x_n) \) is a well-formed formal expression using the variables \( x_1, \ldots, x_n \) and the rack operations \( \ast, \setminus \). We will often omit parentheses, assuming implicitly the right parenthesizing; for example, \( x \ast y \ast z \setminus u \ast v \) will stand for \( x \ast (y \ast (z \setminus (u \ast v))) \). We also use juxtaposition for terms involving just \( \ast \), e.g. \( xy \) will stand for \( x \ast (x \ast y) \).

Formally, an identity is a pair of terms, to be written as \( t = s \). Two terms \( t, s \) are called equivalent in a class \( K \) if the identity \( t = s \) holds in every structure in \( K \). The following fact is well known.

**Proposition 1.9.** [BS19, Proposition 4.1] In racks, every term \( t(x_1, \ldots, x_n) \) is equivalent to a term of the form

\[ z_1 \circ_1 z_2 \circ_2 \ldots \circ_{n-1} z_m, \]

where \( z_1, \ldots, z_m \in \{x_1, \ldots, x_n\} \) and \( \circ_1, \ldots, \circ_m \in \{\ast, \setminus\} \).

Extending the definition from [CS17], an identity is called inner if it has the form

\[ z_1 \circ_1 z_2 \circ_2 \ldots \circ_m y = y, \]

where \( z_1, \ldots, z_m \) are selected arbitrarily from a set of variables \( x_1, \ldots, x_n \), excluding \( y \), and \( \circ_1, \ldots, \circ_m \in \{\ast, \setminus\} \). The symmetric laws are a particular example: a quandle is called \( n \)-symmetric if it satisfies the identity

\[ \underbrace{x \ast x \ast \ldots \ast x}_n \ast y = y. \]

Another example is mediality. It is usually defined as the identity \( (x \ast y) \ast (u \ast v) = (x \ast u) \ast (y \ast v) \), but it is easily proved to be equivalent to abelianess of the displacement group, which can be written as an inner identity \( x \ast y \setminus u \ast v \setminus y \ast x \ast v \setminus u \setminus z = z \).

A left-quasigroup \( Q \) is called \( n \)-multipermutational if the composition of any \( n \) right translations is a constant mapping, i.e., if the expression \((\ldots((u \ast x_1) \ast x_2)\ldots) \ast x_{n-1}) \ast x_n\) does not depend on the choice of \( u \), i.e., if \( Q \) satisfies the following identity

\[ ((\ldots((u \ast x_1) \ast x_2)\ldots) \ast x_{n-1}) \ast x_n = (\ldots((v \ast x_1) \ast x_2)\ldots) \ast x_{n-1}) \ast x_n \]

for every \( u, v, x_1, \ldots, x_n \).

Another important set of identities are the reductive laws, i.e. the identities

\[ (((x \ast y) \ast y)\ldots) \ast y = y \]

for \( n \in \mathbb{N} \). For medial quandles, reductivity is equivalent to \( \mathcal{Q} \), but not so in general (see [JPSZD15] for a discussion of reductive medial quandles, and [PR98] for the general phenomenon of reductivity). It is easy to check that \( 2 \)-multipermutability implies mediality, but there exist non-medial \( 3 \)-reductive quandles. For example, the quandle \( Q_{m,n} \) of [Fis14, Example 1.6] is \( 2 \)-multipermutational.

2. Coverings

2.1. The Cayley kernel. The Cayley representation is the mapping \( \Lambda_Q : Q \rightarrow \text{Sym}(Q), x \mapsto L_x \). Analogously to groups, \( \Lambda_Q \) is a rack homomorphism, with respect to the conjugation operation on \( \text{Sym}(Q) \), but, unlike for groups, \( \Lambda_Q \) is not necessarily injective. Its kernel,

\[ \lambda_Q = \{(x, y) \in Q^2 : \Lambda_Q(x) = \Lambda_Q(y)\} = \{(x, y) \in Q^2 : L_x = L_y\}, \]

will be called the Cayley kernel. Racks with trivial Cayley kernel are quandles and they are called faithful.

Note that for left quasigroups we can define the equivalence relation \( \Lambda_Q \) in the same fashion as for racks, but in general this is not a congruence. We say that a variety of left quasigroups \( \mathcal{V} \) is a Cayley variety
if the relation $\lambda_Q$ is a congruence for every $Q \in V$. Examples of such varieties are indeed the variety of racks and the variety of cycle set defined in [Run95] as the left quasigroups arising from involutive solutions of Yang-Baxter equation. For such algebraic structures the analogous of the congruence $\lambda_Q$ has been defined under the name of retraction relation in [ESS99].

Following [Eis14] we say that a covering homomorphism of racks is any surjective homomorphism whose kernel is a subcongruence of the Cayley kernel. That is, $f$ is a covering homomorphism if $f(x) = f(y)$ implies $L_x = L_y$ for every $x, y$. From now on we will deal with surjective homomorphisms through congruences, so we will say that $Q$ is a covering of $Q/\alpha$ whenever $\alpha \leq \lambda_Q$. It is immediate to see that $Q$ is a covering of $Q/\alpha$ if and only if $\text{Dis}_\alpha = 1$.

**Example 2.1.** Every rack is a covering of a conjugation quandle: consider the Cayley homomorphism $\pi$.

The blocks of $\lambda_Q$ with respect to the action of $\text{LMlt}(Q)$ as $\lambda_Q$ is a congruence. It turns out that they are also blocks with respect the action of $\text{Aut}(Q)$ and the canonical homomorphism $\pi_{\lambda_Q}$ defined as in [8] can be extended to a homomorphism of groups between the automorphism groups of $Q$ and $Q/\lambda_Q$.

**Proposition 2.2.** Let $Q$ be a rack. Then the mapping

$$\pi_{\lambda_Q} : \text{Aut}(Q) \to \text{Aut}(Q/\lambda_Q)$$

defined as in [9] is a well defined group homomorphism and $\ker(\pi_{\lambda_Q}) = C_{\text{Aut}(Q)}(\text{LMlt}(Q))$.

**Proof.** Let $h \in \text{Aut}(Q)$. If $a \lambda_Q b$, i.e. $L_a = L_b$ then also $L_{h(a)} = L_{h(b)}$ and $h(a) \lambda_Q h(b)$. Hence the mapping $\pi_{\lambda_Q}$ defined in [8] can be extended to the automorphism group of $Q$.

In particular, if $\text{Aut}(Q/\lambda_Q)$ is transitive over $Q/\lambda_Q$ (i.e. $Q$ is homogeneous) then $\lambda_Q$ is a uniform congruence. Indeed, the transitive action of the automorphism group of $Q/\lambda_Q$ provides a bijection between any two different blocks of $\lambda_Q$.

**Corollary 2.3.** [ESS14 Proposition 2.49] Let $Q$ be a rack. Then $\text{LMlt}^{\lambda_Q} = Z(\text{LMlt}(Q))$.

Using Corollary [ESS14] we can prove the following group theoretical corollary.

**Corollary 2.4.** Let $G$ be a group and $f \in \text{Aut}(G)$ such that $G = [G, f]$. Then $\text{Core}_G(\text{Fix}(f)) \leq Z(G)$.

**Proof.** The quandle $Q = Q(G, f)$ is a principal connected quandle since $G = [G, f] \cong \text{Dis}(Q)$. It is easy to see that $a \lambda_Q b$ if and only if $a^{-1}b \in \text{Fix}(f)$. Therefore $\text{Fix}(f) = \text{Dis}(Q)_{[1]}$ and so $\text{Core}(\text{Fix}(f)) = \text{Dis}^{\lambda_Q} \leq Z(G)$.

According to Corollary [ESS14] $\text{LMlt}^\alpha$ is central whenever $\alpha \leq \lambda_Q$. The converse fails, even under the assumption that $Q$ is connected. For example, if $Q$ is the trivial quandle, then $E$ is a covering of $Q$ if and only if it is a projection quandle, while $\text{LMlt}(E)$ is a central extension of $\text{LMlt}(Q)$ if and only if it is an abelian group. However, for example, $E = \text{Aff}(\mathbb{Z}_4, -1)$ is not a projection quandle, but $\text{LMlt}(E) = \langle(0, 2), (1, 3)\rangle$ is an abelian group.

For a rack $Q$ and a subset $X \subseteq Q$ we denote by $Sg(X)$ the subrack generated by $X$, i.e. the smallest subrack of $Q$ containing $X$. We define the equivalence relation $m_Q$ by setting

$$a \equiv b \text{ if and only } Sg(a) = Sg(b).$$

Note that $Sg(a) = \{L_a^k(a) : k \in \mathbb{N}\}$ because of left-distributivity.

**Proposition 2.5.** Let $Q$ be a rack. Then $m_Q$ is the smallest congruence with idempotent factor and $[a]_{m_Q} = Sg(a)$ for every $a \in Q$. Moreover, $m_Q \leq \lambda_Q$ and the mapping

$$\pi_{m_Q} : \text{Aut}(Q) \to \text{Aut}(Q/m_Q)$$

defined as in [9] is a well defined group homomorphism.

**Proof.** If $a \equiv c$ then $c \in Sg(e) = Sg(a)$. On the other hand, if $c = L_a^k(a) \in Sg(a)$, then $L_a = L_L^k(a) = L_a^k L_a L_a^{-k} = L_a$ and so $m_Q \leq \lambda_Q$. Moreover, $a = L_a^{-k} \in L_a^{-k}$, i.e. $Sg(c) = Sg(a)$ and therefore $[a]_{m_Q} = Sg(a)$. Let $a \equiv c \equiv L_a^k(a)$ and $b \equiv d \equiv L_a^{-k}(b)$ for some $m, k \in \mathbb{Z}$. Then we have

$$L_{c \equiv d}^\pm L_{a \equiv b} \equiv L_{L_a^k \equiv L_a^{-k}}^m(b) \equiv L_{L_a^k \equiv L_a^{-k}}^m L_a^\pm(b),$$

so $L_{c \equiv d}^\pm L_{a \equiv b}^\pm(b)$ and $m_Q$ is a congruence. In particular $Q/m_Q$ is a quandle.
If \( Q/\alpha \) is a quandle, then \([a * a]_\alpha = [a \backslash a]_\alpha = [a]_\alpha\), i.e. \( \text{Sg}(a) = [a]_{\mathfrak{m}_Q} \leq [a]_\alpha\).

Let \( h \in \text{Aut}Q \). Then \( h(L^k_\alpha(a)) = L^k_{h(a)}(h(a)) \) for every \( a \in Q \). Therefore if \( a \mathfrak{m}_Q b \) then \( h(a) \mathfrak{m}_Q h(b) \) and so the mapping \( \pi_{\mathfrak{m}_Q} \) is well defined. □

From the fact that the blocks of \( \mathfrak{m}_Q \) are \([a]_{\mathfrak{m}_Q} = \{L^k_\alpha(a) : k \in \mathbb{Z}\}\) it easy to deduce the following corollary.

**Corollary 2.6.** A rack \( Q \) is connected if and only if \( Q/\mathfrak{m}_Q \) is connected.

2.2. Rack cocycles. Let \( Q \) be a left-quasigroup, \( A \) a set, and \( \theta : Q^2 \to \text{Sym}(A) \) a mapping into the symmetric group over \( A \). Define a new operation on the set \( E = Q \times A \) by

\[
(x, a) * (y, b) = (x * y, \theta_{x,y}(b)).
\]

The resulting left-quasigroup \( (E, *) \) denoted by \( Q \times_\theta A \) is a rack if and only if \( Q \) is a rack and \( \theta \) satisfies

\[
\theta_{x,y,z} \circ \theta_{y,z} = \theta_{x+y,z} \circ \theta_{x,z},
\]

for every \( x, y, z \in Q \) (here \( \circ \) stands for composition of permutations). The rack \( E \) is a quandle if and only if \( Q \) is a quandle and \( \theta_{x,a} = 1 \) for every \( a \in Q \), too. The equation \( \theta \) is called cocycle condition and if a mapping \( \theta \) for which it holds is called a constant cocycle on \( Q \).

The rack \( E \) is called the covering of \( Q \) over \( \theta \) and denoted by \( E = Q \times_{\theta} A \). Indeed, the projection \( f : E \to Q \), \( (x, a) \mapsto x \) is a covering homomorphism whose kernel is contained in the Cayley kernel of \( E \). The converse also holds: if \( f : E \to Q \) is a rack homomorphism such that \( \ker(f) \) is a uniform congruence (this is guaranteed whenever \( Q \) is connected), then there exists a constant cocycle \( \theta \) such that \( E \) is isomorphic to the covering of \( Q \) over \( \theta \), see [AG03] Proposition 2.11. In particular, if \( \{h_{[x]} : [x] \to A : [x] \in Q/\alpha \} \) is a family of bijections indexed by the blocks of \( \alpha \) and \( \{x_0 \in [x] : [x] \in Q/\alpha \} \) is a set of representatives of the blocks of \( \alpha \) then the mapping \( \theta : X \times X \to \text{Sym}(A) \) defined by

\[
\theta_{[x],[y]} = h_{[x+y]}L_{x_0}h_{[y]}^{-1}
\]

is a rack cocycle and the mapping

\[
Q \to Q \times_{\theta} A, \quad x \mapsto ([x], h_{[x]}(x))
\]

is an isomorphism of racks.

Note that the constant mapping

\[
1 : Q \times Q \to \text{Sym}(A), \quad (a, b) \mapsto 1
\]

is a cocycle for every rack \( Q \) and every set \( A \). The rack \( Q \times_1 A \) is the direct product between \( Q \) and the projection quandle over \( A \) and it is called a trivial covering of \( Q \).

The set of all constant cocycles with coefficients in \( \text{Sym}(A) \) is denoted by \( Z^2(Q, A) \). Two cocycles \( \theta \) and \( \nu \) are called cohomologous if there exists a mapping \( \gamma : Q \to \text{Sym}(A) \) such that

\[
\nu_{x,y} \circ \gamma_y = \gamma_{x+y} \circ \theta_{x,y}
\]

for every \( x, y \in Q \). The factor set obtained by \( Z^2(Q, A) \) with respect to the equivalence of being cohomologous is denoted by \( H^2(Q, A) \).

Cohomologous cocycles leads to isomorphic racks [AG98R] but the converse is not true. Nevertheless, in some cases the isomorphism problem is strictly related to the cohomology classes of cocycles.

**Lemma 2.7.** Let \( Q \) be a rack, \( \theta, \varepsilon \in Z^2(Q, A) \), \( E = Q \times_{\theta} A \) and \( E' = Q \times_{\varepsilon} A \). If \( E/\lambda_E = E'/\lambda_{E'} = Q \) then \( E \cong E' \) if and only if there exists \( g \in \text{Aut}(Q) \) such that \( \theta \circ (g \times g) \) are cohomologous.

**Proof.** Let \( h : Q \to \text{Sym}(A) \) be a map such that \( \varepsilon_{x,y}h_y = h_{x+y}\theta_{g(x),g(y)} \) for every \( x, y \in Q \). Then the mapping \( h(x, a) = (g(x), h_x(a)) \) is an isomorphism.

Assume that \( h \) is an isomorphism between \( E \) and \( E' \). Then \( h(x, a) = (\pi_{\lambda_E}(h)(x), h_x(a)) \) for every \( x \in Q \) and every \( a \in A \), where \( h_x \in \text{Sym}(S) \). Let \( g = \pi_{\lambda_E}(h) \), then

\[
\begin{align*}
\theta(x, a) * (y, b) &= h_x * y, \theta_{x,y}(b)) = (g(x) * y, h_{x+y}\theta_{g(x),g(y)}(b)) \\
\theta(x, a) * (y, b) &= (g(x), h_x(a)) * (g(y), h_y(b)) = (g(x) * g(y), \varepsilon_{g(x),g(y)}h_y(b))
\end{align*}
\]

for every \( x, y \in Q \) and \( a, b \in A \), i.e. there exists a mapping \( h : Q \to \text{Sym}(A) \) such that \( \theta_{g(x),g(y)}h_y = h_{x+y}\varepsilon_{x,y} \) for every \( x, y \in Q \). Hence \( \theta \) and \( \varepsilon \circ (g \times g) \) are cohomologous. □

**Corollary 2.8.** Let \( Q \) be a faithful quandle and \( \theta, \varepsilon \in Z^2(Q, A) \). Then \( Q \times_{\theta} A \cong Q \times_{\varepsilon} A \) if and only if there exists \( g \in \text{Aut}(Q) \) such that \( \theta \) and \( \varepsilon \circ (g \times g) \) are cohomologous.
Consider the following particular type of cocycles. Let \( A \) be endowed with an abelian group operation \(+\), and assume that all permutations \( \theta_{x,y} \) are translations of the group \((A,+)\). Then we can identify the permutations and the respective group elements, and redefine the cocycle as \( \theta : Q^2 \to A \), resulting in the operation

\[
(x, a) * (y, b) = (x * y, b + \theta_{x,y})
\]
on the set \( E = Q \times A \). The cocycle condition then reads

\[
\theta_{x,y+z} + \theta_{y,z} = \theta_{x+y,x+z} + \theta_{x,z},
\]
and \( \theta_{x,x} = 0 \) for every \( x, y, z \in Q \) (here \( + \) stands for addition in \( A \), which corresponds to composition of the corresponding translations). In the literature coverings the construction above is called called \textit{abelian extension} of \( Q \) by \( A \), see [CSV16] [CS17]. Since they are a particular class of coverings we prefer to call them \textit{abelian coverings} and we refer to this cocycles as \textit{abelian cocycles}.

### 2.3. Quandle coverings preserving the displacement group.

A construction of quandle coverings based on the coset quandle construction has been introduced in [BV18] Proposition 2.13 and it is a source of examples and counterexamples through all the paper.

**Lemma 2.9.** [BV18] Proposition 2.13 Let \( G \) be a group, \( H_1 \leq H_2 \leq \text{Fix}(f), Q_{H_1} = Q(G, H_1, f) \) and

\[
p : Q_{H_1} \to Q_{H_2}, \quad aH_1 \mapsto aH_2.
\]

Then \( p \) is a surjective quandle morphism and \( Q_{H_1} \) is a covering of \( Q_{H_2} \).

Let \( Q \) be a rack and \( \alpha \) be one of its congruence. If \( \pi_\alpha \) is an isomorphism then \( \text{Dis}_\alpha \leq \text{Dis}^\alpha = 1 \) and so \( Q \) is a covering of \( Q/\alpha \) and they have isomorphic displacement groups. For connected quandles we can prove that such coverings have a particular form.

**Proposition 2.10.** Let \( Q \) be a connected quandle and \( \alpha \) be its congruence. The following are equivalent:

(i) \( \pi_\alpha \) is a group isomorphism.

(ii) \( Q \cong (Q/\alpha)_H = Q(\text{Dis}(Q/\alpha), H, \tilde{L}_{[\alpha]}) \) for some \( H \leq \text{Dis}(Q/\alpha)_{[\alpha]} \).

**Proof.** (i) \( \Rightarrow \) (ii) Let \( h \in \text{Dis}(Q)_{[\alpha]} \), then \( \pi_\alpha(h)([a]) = [h(a)] = [a] \), hence \( H = \pi_\alpha(\text{Dis}(Q)_{\alpha}) \leq \text{Dis}(Q/\alpha)_{[\alpha]} \).

Therefore the mapping

\[
Q(\text{Dis}(Q), \text{Dis}(Q)_{\alpha}, \tilde{L}_{\alpha}) \to Q(\text{Dis}(Q/\alpha), H, \tilde{L}_{[\alpha]}), \quad h\text{Dis}(Q)_{\alpha} \mapsto \pi_\alpha(h)H
\]
is a well defined isomorphism of quandles.

(ii) \( \Rightarrow \) (i) Let \( Q_H = Q(\text{Dis}(Q/\alpha), H, \tilde{L}_{[\alpha]}) \) for some \( H \leq \text{Dis}(Q/\alpha)_{[\alpha]} \). According to Remark 2.9, \( Q_H \) is connected since the action of \( \text{Dis}(Q_H) \) is the canonical left action of \( \text{Dis}(Q/\alpha) = [\text{Dis}(Q/\alpha), \tilde{L}_{[\alpha]}] \) over the set of cosets \( \text{Dis}(Q/\alpha)/H \). By Lemma 2.9 the mapping

\[
p : Q_H \to Q(\text{Dis}(Q/\alpha), \text{Dis}(Q/\alpha)_{[\alpha]}, \tilde{L}_{[\alpha]}), \quad bH \mapsto b\text{Dis}(Q/\alpha)_{[\alpha]}
\]
is a surjective quandle homomorphism and \( Q_H \) is a covering of \( Q/\alpha \). Moreover \( \pi_\alpha \) is an isomorphism if and only if \( \pi_{\text{ker}(p)} \) is an isomorphism. The action of \( \pi_{\text{ker}(p)}(h) \) is given by the left action of some \( t_h \in \text{Dis}(Q/\alpha) \), i.e.

\[
\pi_{\text{ker}(p)}(h)(b\text{Dis}(Q/\alpha)_{[\alpha]}) = t_h b\text{Dis}(Q/\alpha)_{[\alpha]}
\]
for every \( b \in \text{Dis}(Q/\alpha) \). So \( h \in \text{Dis}^\alpha \) if and only if \( t_h \in \text{Core}_{\text{Dis}(Q/\alpha)}(\text{Dis}(Q/\alpha)_{[\alpha]} = 1 \) (see Remark 2.9 again). Therefore \( h = 1 \) and \( \pi_{\text{ker}(p)} \) is a group isomorphism.

If \( Q \) is a finite connected quandle, then \( \text{Dis}(Q) \) and \( \text{Dis}(Q/\alpha) \) are finite groups and so item (i) of Proposition 2.10 can be substitute by the condition \( \text{Dis}(Q) \cong \text{Dis}(Q/\alpha) \). Indeed, since \( \pi_\alpha \) is a surjective morphism, if \( \text{Dis}(Q) \cong \text{Dis}(Q/\alpha) \) then \( \pi_\alpha \) is also injective. Therefore if \( Q \) is a finite connected quandle then its connected quandle coverings with the same displacement group are \( \{ Q_H : H \leq \text{Dis}(Q)_{\alpha} \} \).

**Proposition 2.11.** Let \( Q \) be a connected quandle. Then \( Q_H/\lambda_{Q_H} \cong Q(\text{Dis}(Q), \text{Fix}(\tilde{L}_\alpha), \tilde{L}_\alpha) \) for every \( H \leq \text{Dis}(Q)_{\alpha} \).
Proof. Let $G = \text{Dis}(Q)$, $f = \bar{L}_a$ and $Q_H = Q(G, H, f)$ for $H \leq \text{Dis}(Q)_a$. It is easy to see that $L_{bH} = L_{bH}$ if and only if $bf(b)^{-1} \in e f(c)^{-1} \text{Core}_G(H)$. According to Remark 1.3, $\text{Core}_G(H) = 1$, and then $c^{-1}b \in \text{Fix}(f)$. Since $H \leq \text{Dis}(Q)_a \leq \text{Fix}(f)$, the mapping $Q_H = \langle Q(\text{Dis}(Q), H, f) \rangle \hookrightarrow \langle Q(\text{Dis}(Q), \text{Fix}(f), f) \rangle$, $bH \mapsto b\text{Fix}(f)$, is a well defined quandle homomorphism which kernel is $\lambda_{Q_H}$. \hfill \Box

3. SIMPLY CONNECTED QUANTLES

A natural problem about quandle coverings is to characterize quandles for which every covering is trivial (i.e. isomorphic to the direct product with a projection quandle). In [Eis14] this problem has been tackle using a categorical approach with a particular focus on the Adjoint group of a quandle [Eis14] Definition 2.18. The Adjoint group of a quandle $Q$ is defined as

$$\text{Adj}(Q) = \langle Q \mid e_x e_y e_x^{-1} = e_{x+y}, x, y \in Q \rangle.$$  

This group is also called Enveloping group in [GnHV11] and Structure group in the framework of the solutions of the Yang Baxter equation [ESS99].

According to [Eis14], there exists a group homomorphism $\varepsilon : \text{Adj}(Q) \hookrightarrow \mathbb{Z}$ mapping every generator to 1 and $\text{Adj}(Q) \cong \text{Adj}(Q)^0 \times \mathbb{Z}$ where $\text{Adj}(Q)^0 = \ker(\varepsilon)$. The map $\Lambda_Q$ factors through $\text{Adj}(Q)$: indeed there exists a surjective group homomorphism $\Lambda_Q : \text{Adj}(Q) \twoheadrightarrow \text{LMlt}(Q)$ such that the following diagram is commutative

$$\begin{array}{ccc}
Q & \xrightarrow{\lambda_Q} & \text{Adj}(Q) \\
\downarrow & & \downarrow \\
\text{LMlt}(Q) & & \Lambda_Q
\end{array}$$

where $\iota$ maps every element of $Q$ to the correspondent generator of $\text{Adj}(Q)$. In this way we obtain an action of $\text{Adj}(Q)$ on $Q$ as $g \cdot a = \Lambda_Q(g)(a)$ for every $g \in \text{Adj}(Q)$ and $a \in Q$. In particular $\Lambda_Q(\text{Adj}(Q)^0) = \text{Dis}(Q)$ and $g \in \text{Adj}(Q)^0$ if and only if $\Lambda_Q(g) \in \text{Dis}(Q)_a$. With abuse of notation we denote with the same symbol the homomorphism $\Lambda_Q$ and its restriction to $\text{Adj}(Q)^0$, whose image is the displacement group of $Q$.

A quandle $Q$ is called simply connected if it is connected and $\text{Adj}(Q)^0_a = 1$ for every $a \in Q$ [Eis14] Definition 5.14. In [Eis14] it has been shown that simply connected quandles are the connected quandles for which every covering is trivial. The following proposition is a reformulation of [Eis14] Proposition 5.15.

Proposition 3.1. Let $Q$ be a connected quandle. The following are equivalent:

(i) $Q$ is simply connected.

(ii) $H^2(Q, A) = 1$ for every set $A$.

(iii) Every covering of $Q$ is isomorphic to a trivial covering.

An alternative characterization of simply connected quandles can be given in terms of the relation between $\text{Adj}(Q)$ and $\text{Dis}(Q)$.

Theorem 3.2. Let $Q$ be a connected quandle $Q$. Then the following are equivalent:

(i) $Q$ is simply connected.

(ii) $Q$ is principal and $\Lambda_Q$ is an isomorphism.

Proof. The inclusion $\ker(\Lambda_Q) \leq \text{Adj}(Q)^0_a$ holds for any quandle $Q$ and every $a \in Q$. According to Proposition 1.3 $Q$ is principal if and only if $\text{Dis}(Q)_a = 1$ for every $a \in Q$. Therefore $Q$ is principal if and only if $\text{Adj}(Q)^0 = \ker(\Lambda_Q)$, i.e. $\text{Adj}(Q)^0 = \ker(\Lambda_Q)$ for every $a \in Q$.

(i) $\Rightarrow$ (ii) Assume that $Q$ is simply connected. By Proposition 2.10 $Q_1 = \langle \text{Dis}(Q), \bar{L}_a \rangle$ is a connected covering of $Q$. Hence, $Q_1 \cong Q \times P$ where $P$ is a projection quandle. Therefore $|P| = |\text{Dis}(Q)_a| = 1$. Since $Q$ is principal then $\ker(\Lambda_Q) = \text{Adj}(Q)^0_a = 1$, then $\Lambda_Q$ is an isomorphism.

(ii) $\Rightarrow$ (i) If $Q$ is a principal connected quandle and $\Lambda_Q$ is an isomorphism, then $\ker(\Lambda_Q) = \text{Adj}(Q)^0_a = 1$ and so $Q$ is simply connected. \hfill \Box

The conditions in Theorem 3.2(ii) are independent as witnessed by some examples in the RIG library of GAP [RIG]; indeed for $Q = \text{SmallQuandle}(6,1)$ the mapping $\Lambda_Q$ is an isomorphism but $Q$ is not principal and $\text{SmallQuandle}(25,i), 1 \leq i \leq 5$ are affine but $\Lambda_Q$ is not an isomorphism.
Corollary 3.3. Let $Q$ be a simply connected quandle then $\text{Adj}(Q) \cong \text{Dis}(Q) \times \mathbb{Z}$.

The converse of the Corollary holds for finite connected quandles. Indeed if $Q$ is finite and connected then $\text{Adj}(Q)^0$ is also finite [GnHV11 Lemma 2.19]. The mapping $\Lambda_Q$ is surjective and then also injective since $|\text{Dis}(Q)| = |\text{Adj}(Q)^0|$. An infinite counterexample is given in [ES14 Example 1.25].

Connected quandles with a cyclic displacement group and connected quandles with doubly transitive displacement group are simply connected [BV18] and simply connected quandles of size $p^2$ have been classified in [GJ17]. Table 1 collects all the other simply connected quandles up to size 47 which do not fall in these families (the data have been computed using Theorem 3.2 and the labels are taken from the RIG library of GAP [RIG]).

| Size | SmallQuandle(Size, - ) |
|------|------------------------|
| 8    | 1                      |
| 24   | 1, 2, 8, 24, 25        |
| 27   | 1, 6, 14, 27, 28, 29, 30, 31, 32, 33, 34 |
| 40   | 4, 5, 6, 27, 28, 29, 30, 31, 32 |
| 45   | 38, 39, 40, 41, 42, 43 |

Simply connected quandles need not to be faithful, as witnessed by some of the quandles in Table 1.

4. Coverings and identities

4.1. When a covering preserves an identity? Every term $t = t(x_1, \ldots, x_n, y)$, where $y$ is the rightmost variable, can be written as $t_1 \circ t_2 \circ \ldots \circ t_m \circ y$ for certain terms $t_1, \ldots, t_n$ and $\circ_i \in \{*, \cdot\}$. We define a formal expression $\Theta_t = \Theta_t(x_1, \ldots, x_n, y)$ by

$$\Theta_t = \theta(t_1, t_2, \ldots, t_m, y) \circ \theta(t_1, t_2, \ldots, t_m, y) \circ \ldots \circ \theta(t_m, t_m, y).$$

If $t = y$, we define $\Theta_t = 1$. For a particular quandle $Q$, cocycle $\theta$, and choice of $x_1, \ldots, x_n, y$ from $Q$, we will treat the expression $\Theta_t$ as composition of the respective values of $\theta$. (This construction generalizes [CS17 Definition 4.2] to more general terms and to non-abelian cocycles.)

Example 4.1. For $t = (x\ldots xy)$, we have $t_1 = \ldots t_n = x$ and

$$\Theta_t = \theta(x, x, \ldots, x) \circ \theta(x, x, \ldots, x) \circ \ldots \circ \theta(x, x, y) \circ \theta(x, y).$$

For $t = (yx(yvy)$, we have $t_1 = x, t_2 = y$ and $\Theta_t = \theta(x, x, y) \circ \theta(v, y)$.

Lemma 4.2. Let $t$ be a term in variables $x_1, \ldots, x_n, y$ where $y$ is the rightmost variable. Let $Q$ be a quandle, $\theta$ a constant cocycle, and consider the covering $E$ of $Q$ over $\theta$. Then

$$t((u_1, a_1), \ldots, (u_n, a_n), (v, b)) = (t(u_1, \ldots, u_n, v), \Theta_t(u_1, \ldots, u_n, v)(b)).$$

Proof. Straightforward.

The following fact describes when a covering satisfies a quandle identity $t = s$ where both terms $t, s$ have the same rightmost variable, such as mediality or $n$-symmetry. It generalizes [CS17 Theorem 4.2(ii)], which addressed the special case of abelian coverings and inner identities.

Proposition 4.3. Let $t = s$ be any quandle identity in variables $x_1, \ldots, x_n, y$ where both terms $t, s$ have the same rightmost variable $y$. Let $Q$ be a quandle, $\theta$ a constant cocycle, and consider the covering $E$ of $Q$ over $\theta$. Then $E$ satisfies the identity $t = s$ if and only if $Q$ satisfies the identity $t = s$ and $\Theta_t = \Theta_s$ for any substitution of the elements of $Q$ for the variables $x_1, \ldots, x_n, y$.

Proof. Follows immediately from the lemma.

Example 4.4. Assume that $Q$ is $n$-symmetric. Then a covering over $\theta$ is $n$-symmetric if and only if

$$\Theta_{x\ldots xy} = \theta(x, x, \ldots, x) \circ \theta(x, x, \ldots, x) \circ \ldots \circ \theta(x, x, y) \circ \theta(x, y) = 1$$

for every $x, y \in Q$. 

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Example 4.5. Assume that $Q$ is medial. Then an covering over $\theta$ is medial if and only if
\[ \Theta(xu)(vy) = \theta(xu, vy) \circ \theta(v, y) = \theta(xv, uy) \circ \theta(u, y) = \Theta(xv)(uy) \]
for every $x, y, u, v \in Q$.

Remark 4.6. We note that every coverings (given by a non-trivial congruence) never satisfy any identity $t = s$ where the rightmost variables are different (for instance, commutativity), for a very simple reason: the blocks of the Cayley kernel are projection quandles, and they fail such identities. Therefore, we exclude such identities from our study.

4.2. Negative examples. For quandles that are not connected, one cannot expect that coverings preserve any kind of identities. Here is an example that even 2-symmetry is not preserved, the order of translations in the covering can exceed any finite bound. This example answers [CSV16, Question 8.8] negatively.

Example 4.7. Let $Q$ be the quandle defined by the following multiplication table:

|   | 0   | 1   | 2   |
|---|-----|-----|-----|
| 0 | 0   | 2   | 1   |
| 1 | 0   | 1   | 2   |
| 2 | 0   | 1   | 2   |

Clearly, $Q$ is 2-symmetric. Let $A$ be any abelian group, fix $a \in A$, and define $\theta : Q^2 \rightarrow A$ by $\theta(0, 2) = a$ and $\theta(x, y) = 0$ for all pairs $(x, y) \neq (0, 2)$. It is straightforward to verify that $\theta$ is an abelian cocycle, for any parameter $a \in A$. Consider the covering of $Q$ over $\theta$. The order of the translation $L_{(0, 0)}$ is $n = 2\text{ord}(a)$, twice the order of $a$ in $A$, since
\[ (1, 0) \mapsto (2, 0) \mapsto (1, a) \mapsto (2, a) \mapsto (1, 2a) \mapsto \ldots \mapsto (1, (n - 1)a) \mapsto (2, (n - 1)a) \mapsto (1, 0). \]
Therefore, the covering is not $m$-symmetric for any $m < n$. (It is easy to check that the covering is actually $n$-symmetric.)

At the other extreme, we have simply connected quandle. Indeed, the trivial coverings (i.e., direct products with projection quandles) preserve any identity with the same rightmost variable. For example, affine quandles over cyclic groups of prime order [Gra04, Lemma 5.1] or more generally connected affine quandle over a cyclic group [BV18, Theorem 1.1].

From now on, we will focus on general connected quandles. First, on the negative side, let us observe that abelian coverings do not necessarily preserve even some of the simplest identities: for example, mediality, a short identity in two variables, and a short inner identity.

Example 4.8. Let $Q = \text{Aff}(\mathbb{Z}_2, f)$ be the connected quandle of order 4. It is medial and it satisfies the identities $xyxy = y$ and $xyxzy = z$ (the latter is the "ababab" inner identity of [CS17]). But it has a connected abelian covering over the group $\mathbb{Z}_2$, namely the non-affine connected quandle of order 8, which fails each of the three identities, and also fails $xyxy = y$. (The information can be collected from various calculations of [CS17, CSV16], or one can verify the claim directly using the explicit construction of the 8-element quandle in [HSV16, Example 8.6].)

Mediality is not preserved by any finite connected coverings. Indeed, a finite connected medial quandle is a quasigroup [Sta15, Proposition 2.1].

Remark 4.9. Let $E$ be a covering of $Q$. Whenever $Q$ satisfies an identity as $s(x_1, \ldots, x_n) = t(y_1, \ldots, y_m)$, then $E$ satisfies
\[ s(x_1, \ldots, x_n) * z = t(y_1, \ldots, y_m) * z. \]
In particular, every covering of an $n$-multipermutational quandle is $(n + 1)$-multipermutational.

4.3. Symmetric laws. On the positive side, all coverings of connected quandles preserve the symmetric laws. This solves [CS17, Conjecture 5.2].

Lemma 4.10. Let $Q$ be a connected quandle and $\alpha \leq \lambda_Q$. If $Q/\alpha$ satisfies the inner identity
\[ L_{x_1}^{k_1} \ldots L_{x_n}^{k_n}(x) = x \]
and for every $a_1, \ldots, a_n \in Q$ there exists $b$ such that $L_{a_1}^{k_1} \ldots L_{a_n}^{k_n}(b) = b$ then $Q$ satisfies (11).
Proof. According to Proposition 2.23, the extension $\pi_\alpha : \text{LMlt}(Q) \to \text{LMlt}(Q/\alpha)$ is central. We have $\pi_\alpha(L_{a_1}^{k_1} \ldots L_{a_n}^{k_n}) = L_{\pi_\alpha(a_1)}^{k_1} \ldots L_{\pi_\alpha(a_n)}^{k_n} = 1$ for every $a_1, \ldots, a_n \in Q$, because $Q/\alpha$ satisfies (11), and thus for every $a_1, \ldots, a_n \in Q$ the automorphism $h = L_{a_1}^{k_1} \ldots L_{a_n}^{k_n} \in Z(\text{LMlt}(Q)) \cap \text{LMlt}_{Q_b}$ for some $b \in Q$. Therefore $h = 1$, since the center of a transitive group is semiregular. Therefore $Q$ satisfies (11). \( \square \)

Corollary 4.11. Every connected covering of an $n$-symmetric quandle is $n$-symmetric.

Proof. Assume that $Q$ is a covering of $Q/\alpha$ and that $Q/\alpha$ is $n$-symmetric. Since $L_{a_n}^n \in \text{LMlt}(Q)_a$ for every $a \in Q$, then we can apply Theorem 4.10 to the identity $L_n^a(y) = y$.

Actually, Corollary 4.11 can be slightly generalized as follows.

Theorem 4.12. Every quandle covering of a connected $n$-symmetric quandle is $n$-symmetric.

Proof. Assume that $Q$ is a covering of the connected $n$-symmetric quandle $Q/\alpha$. Then $L_{a_n}^n \in \ker(\pi_{\lambda_Q}) \leq Z(\text{LMlt}(Q))$. Since $Q/\lambda_Q$ is connected, for every $a, b \in Q$ there exists $h \in \text{LMlt}(Q)$ mapping the block of $a$ to the block of $b$ and in particular $L_{\pi(h(a))} = L_b$. Therefore

$$L_{a_n}^n = L_{\pi_\alpha(h(a))}^n = hL_{a_n}^n h^{-1} = L_{\alpha}^n,$$

since $L_{a_n}^n \in Z(\text{LMlt}(Q))$. Therefore $L_{a_n}^n(a) = L_{\alpha}^n(a) = a$ for every $a, b \in Q$, i.e. $Q$ is $n$-symmetric. \( \square \)

We will say that a quandle is strictly $n$-symmetric if it is $n$-symmetric, but not $m$-symmetric for any $m < n$.

Corollary 4.13. Let $Q$ be a strictly $n$-symmetric quandle and assume that the factor $Q/\lambda_Q$ is connected. Then $Q$ is strictly $n$-symmetric.

Corollary 4.14. Let $Q$ be a connected quandle. Then $Q$ is $n$-symmetric if and only if $\widetilde{L}_a \in \text{Aut}(\text{Dis}(Q))$ has order $n$.

Proof. The connected covering $Q_1 = Q(\text{Dis}(Q), \widetilde{L}_a)$ of $Q$ is $n$-symmetric if and only if the order of $\widetilde{L}_a$ is $n$ and according to Proposition 2.11, $Q/\lambda_Q \cong Q_1/\lambda_{Q_1}$. By Theorem 4.12 and Corollary 4.13, we obtain that $Q$ is $n$-symmetric if and only if $Q_1$ is $n$-symmetric. \( \square \)

5. A universal algebraic approach to coverings

5.1. Strongly abelian congruences. Let $\alpha \geq \beta$ be congruences of an algebraic structure $A$. Following [HMSS], we say that $\alpha$ is strongly abelian over $\beta$, if for every term $t(x, y_1, \ldots, y_n)$, every pair $a, b \in A$ and all tuples $\beta$ such that $a_1, a, b, c, \beta$ for every $i$,

\[(12) \quad t(a, a_1, \ldots, a_n) \beta t(b, a_1, \ldots, a_n) \quad \text{implies} \quad t(a, c_1, \ldots, c_n) \beta t(b, c_1, \ldots, c_n).
\]

A congruence is called strongly abelian if it is strongly abelian over the smallest congruence, $0_A$. Indeed, $\alpha$ is strongly abelian over $\beta$ in $A$ if and only if $\alpha/\beta$ is strongly abelian in $A/\beta$.

An algebraic structure $A$ is called strongly abelian, if its largest congruence, $1_A$, is strongly abelian. Projection quandles, or more generally, permutation racks are strongly abelian, since every term depends only on one variable. Conversely, permutation racks are the only examples: considering the term $t(x, y) = x * y$, we have $t(u, u' \backslash v) = t(v, v \backslash v)$, and strong abelianness gives $t(u, w) = t(v, w)$, i.e., $u * w = v * w$, for every $u, v, w$. This observation can be generalized to congruences. It is easy to show that it is enough to check (12) for every term $t$ in which the first variable occurs only once (indeed, we can replace every occurrence one-by-one, see [SV13], Lemma 4.1] for a formal proof).

Proposition 5.1. Let $Q$ be a left-quasigroup and $\alpha$ one of its congruence. The following are equivalent:

(i) $\alpha$ is strongly abelian.
(ii) $\alpha \leq \lambda_Q$.

In particular, the factor of a strongly abelian congruence is strongly abelian.

Proof. (i) $\Rightarrow$ (ii) We want to prove that $a \sim b$ implies $L_a = L_b$, that is, $a * c = b * c$ for every $c \in Q$. Consider the term $t(x, y) = x * y$. We have $a \sim b$, hence $a * c = b * c$, hence $c \alpha a \backslash (b * c)$. Applying strong abeliannes to $t(a, a \backslash (b * c)) = t(b, c)$, we obtain $t(a, c) = t(b, c)$.

(ii) $\Leftarrow$ (i) Using Proposition 1.24 we can consider a term $t(x, y)$ of the form $z_1 \circ_1 z_2 \circ_2 \ldots z_{m-1} \circ_m z_m$ where $z_1, \ldots, z_m \in \{y_1, \ldots, y_n, x\}$ and $\circ_1, \ldots, \circ_m \in \{*, \\}$.

Assume that $t(u, a) = t(v, b)$ for some $u \in v$ and $a, b \in A$. We need to consider two cases.

Case $x = z_m$: since all arguments are pairwise $\lambda_Q$-related, we have $\varphi(u) = t(u, a) = t(v, b) = \varphi(v)$ for some $\varphi \in \text{LMlt}(Q)$. Hence $u = v$, and $t(u, c) = t(v, c)$ holds trivially.
Case $x = z_i$ for $i \neq m$: similarly, $t(u, \hat{e}) = \varphi(c_i) = t(v, \hat{e})$ for some $\varphi \in \text{LMLt}(Q)$, since the parameters of the left translations are pairwise $\lambda_Q$-related.

Let $\beta \leq \alpha \leq \lambda_Q$. Then $[a]_{\beta} \ast [c]_{\beta} = [a \ast c]_{\beta} = [b]_{\beta} \ast [c]_{\beta}$ whenever $a \ast b$, i.e. $\alpha/\beta \leq \lambda_Q/\beta$. Hence, the factor of a strongly abelian congruence is strongly abelian. □

An algebraic structure is called strongly solvable of length $n$, if it possesses congruences $0_A = \alpha_0 \leq \alpha_1 \leq \ldots \leq \alpha_n = 1_A$ such that $\alpha_i+1$ is strongly abelian over $\alpha_i$, for every $i$. Observe that the identity (4) is equivalent to

$$(\ldots(u \ast x_1) \ast x_2) \ldots) \ast x_{n-1} \lambda_Q (\ldots((v \ast x_1) \ast x_2) \ldots) \ast x_{n-1}$$

for every $x_1, \ldots, x_{n-1} \in Q$. We will use this observation in the proof of Theorem 5.2.

**Theorem 5.2.** Let $Q$ be a left quasigroup in a Cayley variety. Then $Q$ is strongly solvable of length $n$ if and only if it is $n$-multipermutational.

**Proof.** We proceed by induction on $n$. For $n = 1$, the theorem says that a rack is strongly abelian if and only if it is 1-multipermutational, i.e., a permutation rack, which is a special case of Proposition 5.1 for $\alpha = 1_Q$. Now assume that the theorem holds for all $k < n$.

$(\Rightarrow)$ Assume that $Q$ is strongly solvable of length $n$, witnessed by the chain $0_Q = \alpha_0 \leq \alpha_1 \leq \ldots \leq \alpha_n = 1_Q$. Then $Q/\alpha_1$ is strongly solvable of length $n-1$, witnessed by the chain $0_Q/\alpha_1 = \alpha_1/\alpha_1 \leq \ldots \leq \alpha_n/\alpha_1 = 1_Q/\alpha_1$. By the induction assumption, $Q/\alpha_1$ is $(n-1)$-multipermutational, that is,

$$(\ldots((u \ast x_1) \ast x_2) \ldots) \ast x_{n-1} \alpha_1 (\ldots((v \ast x_1) \ast x_2) \ldots) \ast x_{n-1}$$

holds for all $u, v, x_1, x_2, \ldots, x_{n-1} \in Q$. Since $\alpha_1$ is strongly abelian, we have $\alpha_1 \leq \lambda_Q$ by Proposition 5.1 hence

$$(\ldots((u \ast x_1) \ast x_2) \ldots) \ast x_{n-1} \ast y = (\ldots((v \ast x_1) \ast x_2) \ldots) \ast x_{n-1} \ast y$$

for every $y \in Q$, and thus $Q$ is $n$-multipermutational.

$(\Leftarrow)$ Assume that $Q$ is $n$-multipermutational. Then

$$(\ldots((u \ast x_1) \ast x_2) \ldots) \ast x_{n-1} \lambda_Q (\ldots((v \ast x_1) \ast x_2) \ldots) \ast x_{n-1}$$

for every $u, v, x_1, x_2, \ldots, x_{n-1} \in Q$, hence $Q/\lambda_Q$ is $(n-1)$-multipermutational. By the induction assumption, $Q/\lambda_Q$ is strongly solvable of length $n-1$, witnessed by a chain $0_Q/\lambda_Q = \lambda_Q/\lambda_Q \leq \alpha_1/\lambda_Q \leq \ldots \leq \alpha_n/\lambda_Q = 1_Q/\lambda_Q$. Then $0_Q \leq \lambda_Q \leq \alpha_1 \leq \ldots \leq \alpha_n = 1_Q$ is a strongly solvable chain for $Q$. □

**Corollary 5.3.** Connected $n$-multipermutational quandles are trivial. In particular, the only simple $n$-multipermutational quandle is the projection quandle of size 2.

**Proof.** If $n = 1$, then $Q$ is connected and projection, therefore trivial. If $n > 1$ then $Q/\lambda_Q$ is connected and $n-1$-multipermutational. By induction $Q/\lambda_Q$ is trivial and so $Q$ is connected and projection, then trivial. □

Corollary 5.3 applies to any Cayley variety of idempotent left-quasigroups.

In general, strongly abelianess implies abelianess. On the other hand there is no mutual implication between strongly solvability and nilpotence. The commutator theory for racks can be used to prove that strongly solvable racks are indeed nilpotent.

**Theorem 5.4.** Every $n$-strongly solvable rack is nilpotent of length at most $n$.

**Proof.** If $Q$ is strongly abelian then it is abelian. Let $n \geq 2$. Then $Q$ is $n$-multipermutational. According to [JPZ10 Theorem 4.2] the left multiplication group of $n$-multipermutational racks is nilpotent of length at most $n-1$ and then so it is $\text{Dis}(Q)$. Then we can apply Proposition L.3 and so $Q$ is nilpotent of length at most $n$. □

The bound on nilpotency length given in Theorem 5.4 is not optimal. Indeed every medial $n$-multipermutational quandle is nilpotent of length 2 [BST10 Proposition 5.13].
5.2. Central coverings. In this section we want to characterize congruences which are central and strongly abelian at the same time. A covering $Q$ of $Q/\alpha$ is called central if $\alpha$ is a central congruence.

**Proposition 5.5.** Let $Q$ be a rack and $\alpha \leq \lambda_Q$. The following are equivalent:

(i) $Q$ is a central covering of $Q/\alpha$.
(ii) $\text{Dis}(Q)_{[a]}$ is semiregular on $[a]_\alpha$ for every $a \in Q$.

**Proof.** The relative displacement group is trivial. Applying Proposition 1.7 we have that $\alpha$ is central if and only if $\text{Dis}(Q)$ is $\alpha$-semiregular, i.e. $\text{Dis}(Q)_{\alpha} = \text{Dis}(Q)_{b}$ whenever $a \alpha b$, see Proposition 1.7. If $h(a) = a$, then $h(b) \alpha a$ for every $b \alpha a$, i.e. $h \in \text{Dis}(Q)_{[a]}$. So $\text{Dis}(Q)$ is $\alpha$-semiregular if and only if $\text{Dis}(Q)_{[a]}$ is semiregular on $[a]_\alpha$. □

**Corollary 5.6.** Let $Q$ be a connected quandle and $\alpha \leq \lambda_Q$. Then $Q$ is a central covering of $Q/\alpha$ if and only if $\text{Dis}(Q)_{[a]} \leq \text{Dis}(Q)_{[a]}$ for every $a \in Q$.

**Proof.** According to Proposition 1.6 $[a]_\alpha = a^{\text{Dis}(Q)_{[a]}}$ and $\text{Dis}(Q)_{h(a)} = h\text{Dis}(Q)_{a}h^{-1}$ for every $h \in \text{Dis}(Q)$. Hence $\text{Dis}(Q)_{h(a)} = \text{Dis}(Q)_{a}$ for every $h \in \text{Dis}(Q)_{[a]}$ if and only if $\text{Dis}(Q)_{a}$ is normal in $\text{Dis}(Q)_{[a]}$. □

Every rack is a central covering of a quandle.

**Corollary 5.7.** Let $Q$ be a rack. Then $Q$ is a central covering of the quandle $Q/m_Q$.

**Proof.** Assume that $h \in \text{Dis}(Q)_{a}$. Since $h$ is an automorphism of $Q$ then it fixes the subrack generated by $a$, i.e. $h(b) = b$ for every $b \in [a]_{m_Q}$. □

Recall that a rack is semiregular if the displacement group is semiregular [Bon19, Section 1.3]. For such racks every strongly abelian congruence is central.

**Lemma 5.8.** Let $Q$ be a semiregular rack and $\alpha \leq \lambda_Q$. Then $\alpha$ and $\text{con}_{\text{Dis}^\alpha}$ are central congruences.

**Proof.** The rack $Q$ is semiregular, then $\sigma_Q \cong 1_Q$. Therefore $\text{con}_{\text{Dis}^\alpha} \cong \text{con}_{\text{Dis}(Q)} = \zeta_Q$, since $\sigma_Q = 1_Q$ and $\text{Dis}^\alpha \cong Z(\text{LMlt}(Q)) \cap \text{Dis}(Q) \cong Z(\text{Dis}(Q))$. Therefore $\alpha \cong \text{con}_{\text{Dis}^\alpha}$, and so $\alpha$ is central. □

Lemma 5.8 applies in particular to principal quandles.

The following example shows that the factor of a central congruence is not central in general (we use the construction in Lemma 2.9).

**Example 5.9.** Let $Q = Q(\text{Dis}(Q), \text{Dis}(Q)_{a}, \tilde{L}_a)$ be a connected quandle, $H_1 \leq H_2 \leq \text{Dis}(Q)_{a}$ and $p_i : Q_1 \rightarrow Q_{H_i}, \quad g \mapsto gH_i$.

Then $Q_{H_1} \cong Q_1/\ker(p_1)$, $Q_{H_2} \cong Q_1/\ker(p_2)$. According to to Corollary 5.4 $Q_1$ is a central covering of both $Q_{H_i}$ and $Q_{H_2}$. i.e. $\ker(p_i)$ is a central congruence for $i = 1, 2$. Let $\alpha = \ker(p_2)/\ker(p_1)$ be the factor congruence of $\ker(p_2)$ in $Q_{H_1}$. Then $Q_{H_1} \cong Q_{H_1}/\alpha, \quad H_1 = \text{Dis}(Q_{H_2})_{H_1}$ and $\text{Dis}(Q_{H_1})_{[a]} = H_2$. Using again Corollary 5.4 we have that $Q_{H_1}$ is a central covering of $Q_{H_2}$ if and only if $H_1 \leq H_2$. If $H_1$ is not normal in $H_2$, then $Q_{H_1}$ is not a central covering of $Q_{H_2}$. Therefore the congruence $\alpha$ is a factor congruence of a central congruence, but it is not central itself.

In [Eve14, DAEh18] the categorical construction of central extension and normal extension with respect to the adjunction between the category of quandles and the category of projection quandles have been investigated (we remark that in this context the notion of central extension is different from the one used elsewhere for quandles, e.g. in [BS19]). Central extensions for quandles have been proved to be coverings in [Eve13, Theorem 2], and a characterization of normal quandle extensions is the following.

**Theorem 5.10.** [DAEh18] Proposition 3.2 Let $Q$ be a quandle. Then $Q$ is a normal extension of $Q/\alpha$ if and only if $Q$ the following implication holds

\begin{equation}
\text{if } L_{a_1}^{k_1} \ldots L_{a_n}^{k_n} (a_{n+1}) = a_{n+1} \Rightarrow L_{a'_1}^{k_1} \ldots L_{a'_n}^{k_n} (a'_{n+1}) = a'_{n+1}
\end{equation}

whenever $a, a'_i$ for $1 \leq i \leq n + 1$.

The notion of normal extension coincides with the notion of central covering.

**Theorem 5.11.** Let $Q$ be a quandle. Then $Q$ is a normal extension of $Q/\alpha$ if and only if $Q$ is a central covering of $Q/\alpha$. 

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Proof. \((\Rightarrow)\) Every normal extension is central \cite{DAEM18} and so \(L_a = L_{a'}\) whenever \(a \alpha a'\). So we can write the property \cite{Eis13} as

\[ h = L_{a_1}^{k_1} \ldots L_{a_n}^{k_n} \in \text{LMlt}(Q)_a \Rightarrow h|_{[a]_a} = 1. \]

which is nothing but the \(\alpha\)-semiregularity of \(\text{LMlt}(Q)\). Therefore \(\text{Dis}(Q)\) is \(\alpha\)-semiregular too and so \(\alpha\) is a central congruence.

\((\Leftarrow)\) Assume that \(\text{Dis}(Q)\) is \(\alpha\)-semiregular and let \(h = gL_a^k \in \text{LMlt}(Q)_a\) for some \(k \in \mathbb{Z}\) and \(g \in \text{Dis}(Q)_a\). If \(a \alpha b\) we have that \(h(b) = gL_a^k(b) = g(b) = b\), since \(\text{Dis}(Q)\) is \(\alpha\)-semiregular. Therefore \cite{Eis14} holds and so does \cite{Eis13}, i.e. \(Q\) is a normal extension of \(Q/\alpha\).

\[ \Box \]

6. Abelian coverings

Abelian coverings provide examples of central coverings. Nevertheless, the converse implication fails (see Example \cite{Eis17}).

Lemma 6.1. Every abelian covering of racks is a central covering.

Proof. Let \(Q = Q/\alpha \times_\theta A\). We need to check that if \(h((a, s)) = (a, s)\) then \(h((a, t)) = (a, t)\) for every \(t \in A\). Therefore if

\[ h(a, s) = (a, s + \Theta_{h,a}) = (a, s) \]

then \(\Theta_{h,a} = 0\) and it does not depend on \(s\). Hence \(h(a, t) = (a, t + \Theta_{h,a}) = (a, t)\).

In this section we characterize coverings which can be realized using abelian cocycles. Universal algebra does not seem to provide a good concept. Nevertheless a characterization of such coverings can be given in terms of the automorphism group.

Remark 6.2. Let \(G\) be a group acting regularly on a set \(Q\) and \(e \in Q\). We define a group structure on \(Q\) using the regular action of \(A\) on \(Q\). Indeed let \(a, b \in Q\) with \(a = f(e)\) we define \(a \cdot b = f(b)\). The operation is well defined since if \(g(e) = f(e)\) then \(g = f\). If \(a = f(e), b = g(e)\) and \(c = h(e)\) we have

\[ a \cdot (b \cdot c) = f(b \cdot c) = f(g(c)) = f(g(h(e))) = (f \circ g)(h(e)) = (f(g(e)) \cdot h(e)) = (a \cdot b) \cdot c \]

\[ f^{-1}(a) \cdot a = a \cdot f^{-1}(a) = f^{-1}(f(e)) = e \]

\[ e \cdot a = a \cdot e = f(e) = a. \]

Therefore \((Q, \cdot, \cdot)\) is a group and \(e\) is the unit. If \(G\) is abelian then so it is \((Q, \cdot, \cdot)\), indeed \(a \cdot b = f(g(e)) = g(f(e)) = b \cdot a\).

Proposition 6.3. Let \(Q\) be a rack and \(\alpha \leq \lambda_Q\). The following are equivalent:

(i) \(Q\) is an abelian covering of \(Q/\alpha\).

(ii) There exists a abelian subgroup \(A \leq \text{Aut}(Q)\) such that \(A_a = 1\) and \([a]_a = a^\Lambda\) for every \(a \in Q\).

Proof. \((i) \Rightarrow (ii)\) Let \(Q = Q/\alpha \times_\theta A\). The action of \(A\) on \(Q\) defined as \(\rho(h)([a], s) = ([a], s + h)\) is an action of \(A\) by automorphism on \(Q\). Indeed

\[ \rho(h)([a], s) \cdot ([b], t) = \rho(h)([a * [b], t + \Theta_{h,[b]}]) = ([a * [b], t + \Theta_{h,[b]}] + h) \]

\[ \rho(h)([a], s) \cdot \rho(h)([b], t) = ([a], s + h) \cdot ([b], t + h) = ([a * [b], t + \Theta_{h,[b]}] + h). \]

Clearly we have \(A_{(a,s)} = 1\) and \(([a],s) = (a,s)^A\) for every \((a,s) \in Q\).

\((ii) \Rightarrow (i)\) Let \(\{e_{[b]} : [b] \in Q/\alpha\}\) be a set of representatives of the blocks of \(\alpha\). We can endow the blocks of \(\alpha\) with an abelian group structure using the regular action of \(A\) as explained in Remark \cite{Eis15} taking \(e_b\) as the unit of \([b]\). The mapping \(h_{[b]}^{-1} : A \rightarrow [b]\) defined as \(h_{[b]}^{-1}(s) = s(e_{[b]})\) is a well-defined group isomorphism.

Let \(\beta_{[a], [b]}(s) = h_{[a] + [b]}L_{e_{[a]}}h_{[b]}^{-1}\) for every \([a], [b] \in Q/\alpha\) and \(s \in A\) as in \cite{Eis15}. Then we have that

\[ \beta_{[a], [b]}(s) = h_{[a] + [b]}L_{e_{[a]}}s(e_{[b]}) = h_{[a] + [b]}(sL_{e_{[a]}}(e_{[b]})) = h_{[a] + [b]}(s(e_{[a]} * e_{[b]})) \]

because \(sL_{e_{[a]}} = L_{e_{[a]}}s\). The element \(e_{[a]} * e_{[b]}\) can be written as \(\theta_{[a],[b]}(e_{[a]} * e_{[b]})\) for a unique element \(\theta_{[a],[b]} \in A\). So,

\[ \beta_{[a], [b]}(s) = h_{[a] + [b]}(s(e_{[a]} * e_{[b]})) = h_{[a] + [b]}(s(e_{[a] * e_{[b]}}) + \Theta_{[a],[b]}(e_{[a]} * e_{[b]})) = s + \Theta_{[a],[b]}. \]

Therefore, \(Q \cong Q/\alpha \times_\theta A\) is an abelian covering of \(Q/\alpha\).

\[ \Box \]

Proposition \cite{Eis15} shows that connected abelian coverings are exactly the Galois coverings as defined in \cite{Eis13} Definition 4.12 for which the group of the deck transformation is abelian.

Every connected rack is an abelian covering of a quandle.
Corollary 6.4. Let $Q$ be a connected rack. Then $Q$ is an abelian covering of $Q/m_Q$. In particular, $Q \cong Q/m_Q \times A$ where $A$ is a cyclic group.

Proof. The factor $Q/m_Q$ is a connected quandle, so $m_Q$ is a uniform congruence and $Q \cong Q/m_Q \times A$. Let \( \{(x, s, s) : x \in Q/m_Q \} \) be a set of representatives of the blocks of $m_Q$. The mapping $h(x, s) = L_{(x, s, s)}(x, s)$ is an automorphism of $Q$ and the action of $H = \langle h \rangle$ is semiregular and the orbits are the blocks of $m_Q$ (indeed $[(x, s, s)]_{m_Q} = \{L_{(x, s, s)}^k(x, s, s) : k \in \mathbb{Z}\}$). Hence we can apply Proposition 6.3 and so $Q \cong Q/m_Q \times h$ is an abelian covering of $Q/m_Q$. \hfill \Box

Another consequence of Proposition 6.3 is the following adaptation of [EG14, Proposition 3.1] to the context of coverings.

Corollary 6.5. Let $Q$ be a connected rack, $\alpha \leq \lambda_Q$ and $\beta = \mathcal{O}_{\text{LMlt}}\alpha$. Then $Q$ is an abelian covering of $Q/\beta$ and $\text{LMlt}(Q/\beta) \cong \text{LMlt}(Q/\alpha)$.

The statement of the corollary can be represented by the following diagrams

\[
\begin{array}{ccc}
Q & \xrightarrow{\pi_\beta} & \text{LMlt}(Q) \\
\downarrow & & \downarrow \pi_\alpha \\
Q/\alpha & \xrightarrow{\phi} & \text{LMlt}(Q/\alpha)
\end{array}
\]

where $\phi$ is an isomorphism and $Q$ is an abelian covering of $Q/\beta$.

Proof. The subgroup $H = \text{LMlt}^\alpha$ is contained in $Z(\text{LMlt}(Q)) \cap \text{Aut}^\alpha$. As $H$ is a central subgroup of a transitive group then it is semiregular. So $H$ is abelian and it has coincides with the blocks of $\beta$, i.e. $H_a = [a]_\beta$ and $H_a = 1$ for every $a \in Q$. Therefore we can apply Proposition 6.3 and so $Q$ is an abelian covering of $Q/\beta$.

Since $\beta \leq \alpha$ then $\text{LMlt}^\beta \leq \text{LMlt}^\alpha$. If $h \in \text{LMlt}^\alpha$ then clearly $h(a)h(a)$ for every $a \in Q$, therefore $\text{LMlt}^\alpha \leq \text{LMlt}^\beta$. Thus, $\text{LMlt}(Q/\alpha) \cong \text{LMlt}(Q/\beta)$. \hfill \Box

For quandle Corollary 6.5 follows directly from [BS19, Proposition 7.8]. In the connected case we can characterize abelian coverings in terms of the properties of the block stabilizer.

Proposition 6.6. Let $Q$ be a connected rack and $\alpha \leq \lambda_Q$. The following are equivalent:

(i) $Q$ is an abelian covering of $Q/\alpha$.

(ii) $\text{LMlt}(Q)[\alpha][\alpha]$ is abelian for every $a \in Q$.

Proof. (i) $\Rightarrow$ (ii) Let $h \in \text{LMlt}(Q)[\alpha]$. Then $h(a, s) = (a, s + \Theta_{h,a})$, where $\Theta_{h,a}$ does not depend on $s$. Hence: $gh(a, s) = (a, s + \Theta_{h,a} + \Theta_{g,a}) = hg(a, s)$, for every $h, g \in \text{LMlt}(Q)[\alpha]$. Therefore $\text{LMlt}(Q)[\alpha][\alpha]$ is abelian for every $a \in Q$.

(ii) $\Rightarrow$ (i) Let $a \in Q$. The group $A = \text{LMlt}(Q)[\alpha][\alpha]$ is regular on $[a]$. Indeed it is transitive by virtue of Proposition 1.6 and semiregular since it is abelian. So we can define the abelian group $\{(a) + \} as in Remark 6.2 with $a$ as the unit. Since $Q$ is connected for every $[b]$ there exists $h[0] \in \text{LMlt}(Q)$ such that $h[0]: [b] \rightarrow [a]$ is a bijection. Define $\beta_{[b],[c]}(s) = h_{[b,c]}L_b h_{c}^{-1}(s)$ for every $s \in [a]$ and $\theta_{[b],[c]} = \beta_{[b],[c]}(a)$. Let $s = k(a) \in [a]$ where $k \in A$ is the unique element which maps $a$ to $s$. Then $s + \theta_{[b],[c]} = kh_{[b,c]}L_b h_{c}^{-1}(a) = h_{[b,c]}L_b h_{c}^{-1}k(a) = h_{[b,c]}L_b h_{c}^{-1}k(s) = k(s) = \beta_{[b],[c]}(a)$ where we used that $h_{[b,c]}L_b h_{c}^{-1}[a] \in A$. Therefore $Q$ is an abelian covering of $Q/\alpha$.

Note that the implication (i) $\Rightarrow$ (ii) of Proposition 6.6 holds also for non connected quandles.

Using 6.6 we can provide examples of central coverings which are not abelian.

Example 6.7. Let $Q = Q(\text{Dis}(Q), \text{Dis}(Q)_\alpha, L_a)$ be a connected quandle, $Q_1 = Q(\text{Dis}(Q), \hat{L}_a)$ and $p : Q_1 \rightarrow Q$ the quandle homomorphism defined by $b \rightarrow b\text{Dis}(Q)_\alpha$. The quandle $Q_1$ is a connected central covering of $Q$ by virtue of Corollary 5.6. In this case the stabilizer of the block of 1 with respect to $\ker(p)$ is $\text{Dis}(Q)_\alpha$. Hence if $\text{Dis}(Q)_\alpha$ is not abelian $Q$ is not an abelian covering of $Q/\alpha$ by virtue of Proposition 6.6 (as already showed in [CDS17, Lemma B.6]).

The following theorem characterizes connected central coverings of principal quandles.
Theorem 6.8. Let \( Q \) be a connected quandle and \( \alpha \leq \lambda_Q \). The following are equivalent:

(i) \( Q/\alpha \) is principal.

(ii) \( Q \) is principal and \( \text{Dis}^\alpha \) is transitive on each block of \( \alpha \).

In particular, if (i) holds then \( Q \) is an abelian covering of \( Q/\alpha \).

Proof. (i) \( \Rightarrow \) (ii) According to [Bon19] Proposition 2.9] if \( Q/\alpha \) is principal, then \( \text{Dis}(Q)_{[\alpha]} = \text{Dis}^\alpha \) and it is transitive on each block of \( \alpha \) according to Proposition 1.4. Then \( \text{Dis}(Q)_{[\alpha]} \leq \text{Dis}^\alpha \leq Z(\text{Dis}(Q)) \) and so \( \text{Dis}^\alpha \) is regular on each block of \( \alpha \). Thus \( \text{Dis}(Q)_{[\alpha]} = 1 \) and so \( Q \) is principal by Proposition 1.4.

(ii) \( \Rightarrow \) (i) Both the subgroups \( \text{Dis}^\alpha \) and \( \text{Dis}(Q)_{[\alpha]} \) are regular on each block of \( \alpha \) since \( Q \) is connected and principal (see Proposition 1.4 again). Therefore \( \text{Dis}^\alpha = \text{Dis}(Q)_{[\alpha]} \) and so \( Q/\alpha \) is principal by virtue of [Bon19] Proposition 2.9].

If (i) holds then \( \text{LMlt}^\alpha = \text{LMlt}(Q)_{[\alpha]} \leq Z(\text{LMlt}(Q)) \) is abelian. So we can apply Proposition 6.6 and then \( Q \) is an abelian covering of \( Q/\alpha \). \( \square \)

Theorem 6.8 applies in particular to affine quandles. Principal quandles which are not covering of principal quandles exist: actually every connected quandle \( Q \) have a principal covering \( Q(\text{Dis}(Q), L\alpha) \) (see Lemma 2.3) and Example 6.7.

Example 6.9. Let \( A \) be an abelian group and \( f \in \text{Aut}(A) \) such that \( 1 - f \) is surjective but not injective. Then \( Q = \text{Aff}(A, f) \) is connected but not faithful and \( 1 - f \) is a quandle homomorphism whose kernel is \( \lambda_Q \). Hence we have an infinite chain of connected abelian coverings

\[ \ldots \to Q_{[\alpha]} \to Q_{[\alpha]} \to Q_{[\alpha]} \to \ldots \]

in which at each step we take the factor with respect to \( \lambda_Q \). This example answers in a positive way to [CSV16] Question 8.7]. Note that \( \text{Dis}(Q/\lambda_Q) \cong \text{Dis}(Q) \) but \( \text{Dis}^{\lambda_Q} = \text{Fix}(f) \), hence \( \pi_{\lambda_Q} \) is not an isomorphism.

For any connected group \( A \) (i.e. \( nA = A \)) with non trivial \( n \)-torsion (i.e. \( \{a \in A : na = 0 \} \neq 0 \)) and define \( Q = \text{Aff}(A, n + 1) \). Concrete examples are given by \( \text{Aff}(\mathbb{Z}_{p^n}, 1 + p) \) for every prime \( p \) or \( \text{Aff}(\mathbb{Z}/n, n) \) for every \( n \).

References

[AG03] Nicolás Andruskiewitsch and Matías Graña, From racks to pointed Hopf algebras, Adv. Math. 178 (2003), no. 2, 177–243. MR 1994219 (2004i:16046)

[BB19] Giuliano Bianco and Marco Bonatto, On connected quandles of prime power order, arXiv e-prints (2019), arXiv:1904.12801.

[Bia15] Giuliano Bianco, On the transsection group of a rack, Ph.D. thesis, Università degli studi di Ferrara, 2015.

[Bon19] Marco Bonatto, Principal and doubly homogeneous quandles, arXiv e-prints (2019), arXiv:1904.13388.

[BS19] Marco Bonatto and David Stanovský, Commutator theory for racks and quandles, arXiv e-prints (2019), arXiv:1902.08980.

[BV18] Marco Bonatto and Petr Vojtěchovský, Simply connected latin quandles, J. Knot Theory Ramifications 27 (2018), no. 11, 1843026, 22. MR 3868935

[CDS17] W. Edwin Clark, Larry A. Dunning, and Masahico Saito, Computations of quandle 2-cocycle knot invariants without explicit 2-cocycles, J. Knot Theory Ramifications 26 (2017), no. 7, 1750035, 22. MR 3608904

[CENS03] J. Scott Carter, Mohamed Elhamdadi, Marina Appiou Nikiforou, and Masahico Saito, Extensions of quandles and cocycle knot invariants, J. Knot Theory Ramifications 12 (2003), no. 6, 725–738. MR 2008876

[CJK+03] J. Scott Carter, Daniel Jelsovsky, Seiichi Kamada, Laurel Langford, and Masahico Saito, Quandle cohomology and state-sum invariants of knotted curves and surfaces, Trans. Amer. Math. Soc. 355 (2003), no. 10, 3947–3989. MR 1990571

[CS17] W. Edwin Clark and Masahico Saito, Quandle identities and homology, Knots, links, spatial graphs, and algebraic invariants, Contemp. Math., vol. 689, Amer. Math. Soc., Providence, RI, 2017, pp. 23–35. MR 3656320

[CSV16] W. Edwin Clark, Masahico Saito, and Leandro Vendramini, Quandle coloring and cocycle invariants of composite knots and abelian extensions, J. Knot Theory Ramifications 25 (2016), no. 5, 1650024, 34. MR 3488311

[DAEM18] Mathieu Duckerts-Antoine, Valérian Even, and Andrea Montoli, How to centralize and normalize quandle extensions, J. Knot Theory Ramifications 27 (2018), no. 2, 1850020, 23. MR 3770469

[EGL14] Valérain Even and Marino Gran, On factorization systems for surjective quandle homomorphisms, J. Knot Theory Ramifications 23 (2014), no. 11, 1450060, 15. MR 3293045

[Eis03] Michael Eisermann, Homological characterization of the unknot, J. Pure Appl. Algebra 177 (2003), no. 2, 131–157. MR 1954330

[Eis14] Valérain Even, Quandle coverings and their Galois correspondence, Fund. Math. 225 (2014), no. 1, 103–168. MR 3295568

[ESS99] Pavel Etingof, Travis Schedler, and Alexandre Soloviev, Set-theoretical solutions to the quantum Yang-Baxter equation, Duke Math. J. 100 (1999), no. 2, 169–209. MR 1722951 (2001c:16076)

[Eve14] Valérain Even, A Galois-theoretic approach to the covering theory of quandles, Appl. Categ. Structures 22 (2014), no. 5-6, 817–831. MR 3275277
[FM87] Ralph Freese and Ralph McKenzie, Commutator theory for congruence modular varieties, London Mathematical Society Lecture Note Series, vol. 125, Cambridge University Press, Cambridge, 1987. MR 909290

[GIV17] Agustín García Iglesias and Leandro Vendramin, An explicit description of the second cohomology group of a quandle, Math. Z. 286 (2017), no. 3-4, 1041–1068. MR 3671570

[GnHV11] M. Graña, I. Heckenberger, and L. Vendramin, Nichols algebras of group type with many quadratic relations, Adv. Math. 227 (2011), no. 5, 1956–1989. MR 2803792

[Gra04] Matías Graña, Indecomposable racks of order $p^2$, Beiträge Algebra Geom. 45 (2004), no. 2, 665–676. MR 2093034 (2005k:57025)

[HM88] David Hobby and Ralph McKenzie, The structure of finite algebras, Contemporary Mathematics, vol. 76, American Mathematical Society, Providence, RI, 1988. MR 958685

[HSV16] Alexander Hulpke, David Stanovský, and Petr Vojtěchovský, Connected quandles and transitive groups, J. Pure Appl. Algebra 220 (2016), no. 2, 735–758. MR 3399387

[JPSZD15] Přemysl Jedlička, Agata Pilitowska, David Stanovský, and Anna Zamojska-Dzienio, The structure of medial quandles, J. Algebra 443 (2015), 300–334. MR 3400403

[JPZ19] Přemysl Jedlička, Agata Pilitowska, and Anna Zamojska-Dzienio, Multipermutation distributive solutions of Yang-Baxter equation have nilpotent permutation groups, arXiv e-prints (2019), arXiv:1906.03960.

[PR98] A. Pilitowska and A. Romanowska, Reductive modes, Period. Math. Hungar. 36 (1998), no. 1, 67–78. MR 1684506

[RIG] Matías Graña and Leandro Vendramin, Rig, a GAP package for racks, quandles and Nichols algebras.

[Rum05] Wolfgang Rump, A decomposition theorem for square-free unitary solutions of the quantum Yang-Baxter equation, Adv. Math. 193 (2005), no. 1, 40–55. MR 2132760

[Sta15] David Stanovský, A guide to self-distributive quasigroups, or Latin quandles, Quasigroups Related Systems 23 (2015), no. 1, 91–128. MR 3553113

[SV14] David Stanovský and Petr Vojtěchovský, Commutator theory for loops, J. Algebra 399 (2014), 290–322. MR 3144590

(Bonatto) IMAS–CONICET and Universidad de Buenos Aires, Pabellón 1, Ciudad Universitaria, 1428, Buenos Aires, Argentina
E-mail address: marco.bonatto.87@gmail.com

(Stanovský) Department of Algebra, Faculty of Mathematics and Physics, Charles University, Prague, Czech Republic
E-mail address: stanovsk@karlin.mff.cuni.cz