On the kernel of the theta operator mod $p$

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Abstract

We construct many examples of level one Siegel modular forms in the kernel of theta operators mod $p$ by using theta series attached to positive definite quadratic forms.

1 Introduction

Ramanujan’s $\theta$ operator is a familiar topic in the theory of elliptic modular forms, defined by

$$f = \sum a(n)e^{2\pi inz} \mapsto \theta(f) = \frac{1}{2\pi i}f' = \sum na(n)e^{2\pi inz}$$

For Siegel modular forms of degree $n$, the Fourier expansion runs over positive semidefinite half-integral matrices of size $n$ and we can define several analogues of the Ramanujan $\theta$ operator: For $1 \leq r \leq n$ we may introduce

$$F = \sum_T a(T)e^{2\pi i \text{tr}(TZ)} \mapsto \Theta^{[r]}(F) := \frac{1}{(2\pi i)^r} \left( \frac{\partial}{\partial z} \right)^{[r]} F = \sum_T T^{[r]}a(T)e^{2\pi i \text{tr}(TZ)},$$

where, for a matrix $A$ of size $n$, we denote by $A^{[r]}$ the matrix of all the determinants of its submatrices of size $r$ and $\partial_{ij} := \frac{1}{2}(1 + \delta_{ij})\frac{\partial}{\partial z_{ij}}$.

In general, $\Theta^{[r]}(F)$ is no longer a modular form, but it is a modular form mod $p$ (even a $p$-adic modular form), vector-valued if $r < n$, see [8].

Our aim in the present paper is to explore the existence and explicit construction of Siegel modular forms which are in the kernel of such $\Theta$-operators mod $p$. Obvious candidates for such modular forms are theta series $\vartheta^p_S$

$$\vartheta^p_S(Z) = \sum_{X \in \mathbb{Z}^{(n,n)}} e^{2\pi i \text{tr}(X^{\top}SXZ)},$$

attached to positive quadratic forms $S$ of rank $n$ and level being a positive power of $p$; we will also consider variants of this involving a harmonic polynomial. Looking at the Fourier expansion, evidently such theta series are in the kernel of $\Theta^{[n]}$ mod $p$. On the other hand, these theta series are not of level
one and one has to do a level change to level one. This method only works
for even degree \( n \), because we otherwise enter into the realm of modular forms
of half-integral weight, but we shall exhibit a somewhat weaker variant of our
method also for the case of odd \( n \). On the other hand, for even degree, our
method provides plenty of examples for level one forms \( F \) (of weight in an
arb-\( \Theta^{[j]}(F) \equiv 0 \pmod{p} \) and
\( \Theta^{[j-1]}(F) \not\equiv 0 \pmod{p} \). Here \( j \) is almost arbitrary, the only obstruction comes
from the arithmetic of quadratic forms, which puts some constraint on \( (n, j, p) \).
We have to make an important comment on what we mean by “explicit con-
struktion” here: The kernel of \( \Theta^{[j]} \pmod{p} \) is a notion which depends only on
modular forms \( \pmod{p} \), therefore the weight of the constructed modular form
is only of interest \( \pmod{(p-1)} \). On the other hand, one is also interested in
explicit small weights for which we can get modular forms in the kernel \( \pmod{p} \).
In this paper we address both versions of explicit construction, we will call
them “weak construction” and “strong construction” respectively; in most cases
our “strong construction” also gives the smallest possible weight, which is called
“filtration” in the work of Serre and Swinnerton-Dyer, see [11] for details. In
the final section we also show that some of the known examples of congruences
for degree two Siegel modular forms can be explained by our methods.
Finally we remark that most of our results are formulated for odd primes only.
The reader interested in \( p = 2 \) may adjust some of our results and methods to \( p = 2 \).

2 Preliminaries

2.1 Siegel modular forms

For standard facts about Siegel modular forms we refer to [1][14][19]. The group
\( \text{Sp}(n, \mathbb{R}) \) acts on the upper half space \( \mathbb{H}_n \) in the usual way. For an integer \( k \), a
function \( f : \mathbb{H}_n \to \mathbb{C} \) and \( M = \begin{pmatrix} AB \\ CD \end{pmatrix} \) we define the slash operator by

\[
(f \mid_k M)(Z) := \det(CZ + D)^{-k} f((AZ + B)(CZ + D)^{-1}).
\]

For a congruence subgroup \( \Gamma \) of \( \text{Sp}(n, \mathbb{Z}) \) and a character \( \chi \) of \( \Gamma \) we denote by
\( M_k(\Gamma, \chi) \) the space of Siegel modular forms for \( \Gamma \) of weight \( k \) and character \( \chi \)
and \( S_k(\Gamma, \chi) \) the subspace consisting of cusp forms. If \( \chi \) is trivial, we just omit
it. We will mainly be concerned with congruence subgroups of type

\[
\Gamma_0^n(N) := \left\{ M = \begin{pmatrix} AB \\ CD \end{pmatrix} \mid C \equiv 0 \pmod{N} \right\}
\]

and with groups arising from these by conjugation within \( \text{Sp}(n, \mathbb{Z}) \). If \( N = 1 \)
we just write \( \Gamma^n \) instead of \( \Gamma_0^n(1) \). The only characters of \( \Gamma_0^n(N) \) occurring are
those arising from Dirichlet characters \( \pmod{N} \) in the usual way (i.e. \( \chi(M) = \)}
$\chi(\det D))$, the most important one will be the quadratic character

$$\chi_p(*) := \left(\frac{(-1)^{\frac{p-1}{2}}}{*}\right)$$

for an odd prime $p$. If $f$ is an element of $M_k(\Gamma)$ for an arbitrary $\Gamma$, then $f$ has a Fourier expansion

$$f(Z) = \sum_T a(T; f)e^{2\pi i tr(TZ)}$$

where $T$ runs over positive semidefinite rational symmetric matrices with bounded denominator. In particular, for $\Gamma = \Gamma_0^0(N)$, $T$ runs over positive semidefinite matrices in

$$\Lambda_n := \{T = (t_{ij}) \in \text{Sym}_n(\mathbb{Q}) \mid t_{ii}, 2t_{ij} \in \mathbb{Z}\}.$$

For a subring $R$ of $\mathbb{C}$ we denote by $M_k(\Gamma)(R)$ the set of $f \in M_k(\Gamma)$ whose Fourier coefficients lie in $R$.

### 2.2 Traces

We need the explicit form of the trace map

$$\text{Tr} : \begin{cases} M_k(\Gamma_0^0(p)) \rightarrow M_k(\Gamma^n) \\ f \mapsto \sum_\gamma f |_{k \gamma} \end{cases}$$

where $\gamma$ runs over $\Gamma_0^0(p) \setminus \Gamma^n$, see also [10]. To obtain an explicit set of representatives for these cosets we start from a Bruhat decomposition over the finite field $\mathbb{F}_p$:

$$\text{Sp}(n, \mathbb{F}_p) = \bigcup_{j=0}^{n} P(\mathbb{F}_p) \cdot \omega_j \cdot P(\mathbb{F}_p),$$

where $P \subset \text{Sp}(n, \mathbb{F}_p)$ denotes the Siegel parabolic defined by $C = 0$ and for $0 \leq j \leq n$ we put

$$\omega_j = \begin{pmatrix} 0_j & 0 & -1_j & 0 \\ 0 & 1_{n-j} & 0 & 0_{n-j} \\ 1_j & 0 & 0_j & 0 \\ 0 & 0_{n-j} & 0 & 1_{n-j} \end{pmatrix}.$$  

Using the Levi decomposition $P = MN$ with Levi factor

$$M = M_n = \left\{ m(A) = \begin{pmatrix} A & 0 \\ 0 & \nu A^{-1} \end{pmatrix} \mid A \in GL_n(\mathbb{F}_p) \right\}$$

and unipotent radical

$$N = \left\{ n(B) = \begin{pmatrix} 1_n & B \\ 0 & 1_n \end{pmatrix} \mid B \in M_n(\mathbb{F}_p) \text{ symmetric} \right\}$$
we easily see that
\[
\{ \omega_j \cdot n(B_j) \cdot m(A) \mid B_j \in M_j(\mathbb{F}_p) \text{ symmetric}, A \in P_{n,j}(\mathbb{F}_p) \setminus GL_n(\mathbb{F}_p) \}
\]
is a complete set of representatives for the cosets \(P(\mathbb{F}_p) \setminus P(\mathbb{F}_p) \cdot \omega_j \cdot P(\mathbb{F}_p)\).
Here \(M_j\) is naturally embedded into \(M_n\) by \(B_j \mapsto \begin{pmatrix} B_j & 0 \\ 0 & 0_{n-j} \end{pmatrix}\) and \(P_{n,j}\)
is a standard parabolic subgroup of \(GL_n\) defined by \(0^{(n-j,j)}\) being the lower
left corner of \(g\). We tacitly identify the matrices above with corresponding representatives
with entries in \(\mathbb{Z}\) and obtain sets of representatives for \(\Gamma_0^2(N) \setminus \Gamma^n\).
We analyse the contribution of fixed \(j\) to the trace of a given \(f \in M_k(\Gamma_0^2(p))\):
The function \(f \mid_k \omega_j\) is itself a modular form for the group conjugate to \(\Gamma_0^2(p)\)
by \(\omega_j\), it has a Fourier expansion
\[
f \mid_k \omega_j(Z) = \sum_{T=(t_{1m})} a_j(T; f) e^{2\pi i \text{tr}(TZ)},
\]
where the \(t_{1m}\) are integral or semi-integral except for the \(t_{1m}\) in the upper left
block of size \(j\) in \(T\), where \(p\) may occur in the denominator. Then an elementary
calculation using orthogonality of exponential sums shows that
\[
\sum_{B_j} (f \mid_k \omega_j) \mid_k n(B_j)(Z) = p^{\frac{j(j+1)}{2}} \sum_{T \in \Lambda_n} a_j(T; f) e^{2\pi i \text{tr}(TZ)}.
\]
The result of the action of the matrices \(m(A)\) is :
\[
\sum_{B_j, A} f \mid_k (\omega_j \cdot n(B_j) \cdot m(A)) = p^{\frac{j(j+1)}{2}} \sum_{T \in \Lambda_n} b_j(T; f) e^{2\pi i \text{tr}(TZ)}
\]
with
\[
b_j(T; f) = \sum_A a_j (A^{-1} T^t A^{-1}; f).
\]
We can therefore write the contribution of a fixed \(j\) to the trace as
\[
p^{\frac{j(j+1)}{2}} f \mid_k \omega_j \mid \tilde{U}_j(p),
\]
where \(\tilde{U}_j(p)\) is an operator which maps a Fourier series to a new Fourier series,
where the new coefficients are certain finite sums of the Fourier coefficients in
the series we started from. Most of time the exact shape of this operator will
not be important for us. At some point however we have to consider the actions
of \(n(B_j)\) and \(m(A)\) separately and we split the operator accordingly into two
pieces as
\[
\tilde{U}_j(p) = \tilde{U}_j^0(p) \circ D_j(p). \tag{2.1}
\]
We just mention the extreme cases: For \(j = 0\) the operator \(\tilde{U}_0(p)\) is just the
identity and \(\tilde{U}_n(p)\) is quite similar to the usual \(U(p)\)-operator:
\[
\tilde{U}_n(p) : \sum_{T \in \frac{1}{n} \Lambda_n} a(T) e^{2\pi i \text{tr}(TZ)} \mapsto \sum_{T \in \Lambda_n} a(T) e^{2\pi i \text{tr}(TZ)}.
\]
Using this terminology we decompose the trace into \( n + 1 \) pieces:

**Proposition 2.1.** For \( f \in M_k(\Gamma_0^n(p)) \)

\[
\operatorname{Tr}(f) = \sum_{j=0}^{n} Y_j
\]

with

\[
Y_j := \frac{p^{\frac{j(j+1)}{2}}}{2} (f \mid_k \omega_j) \mid \tilde{U}_j(p).
\]

**Remark 2.2.** It should be clear that this expression for the trace has an analogue for the more general case of taking the trace from \( \Gamma_0^n(NR) \) to \( \Gamma_0^n(N) \) if \( N \) and \( R \) are coprime and \( R \) is squarefree (see e.g. [10]).

### 2.3 Congruences

For a prime number \( p \) we denote by \( \nu_p \) the normalized additive valuation on \( \mathbb{Q} \) (i.e. \( \nu_p(p) = 1 \)). For a Siegel modular form \( f \in M_k(\Gamma)(\mathbb{Q}) \) with Fourier expansion \( f(Z) = \sum_T a(T; f) e^{2\pi i \operatorname{tr}(TZ)} \) we define

\[
\nu_p(f) := \min \{ \nu_p(a(T; f)) \mid T \geq 0 \}.
\]

Note that this minimum is well defined because the Fourier coefficients of \( f \) have bounded denominators. We also remark that \( \nu_p(f) \) makes sense not only for modular forms with rational Fourier coefficients but also for the general case \( f \in M_k(\Gamma)(\mathbb{C}) \) by tacitly extending the valuation to the field generated by all Fourier coefficients. For two modular forms \( f \) and \( g \) we define

\[
f \equiv g \pmod{p} :\iff \nu_p(f - g) \geq 1 + \nu_p(f).
\]

We finally remark that in this setting \( \nu_p(f \mid_k \gamma) \) also makes sense for arbitrary \( \gamma \in \Gamma^n \). In particular, for \( f \in M_k(\Gamma_0^n(p), \chi) \) we may consider \( \nu_p(f \mid_k \omega_j) \); the Fourier expansions of \( f \mid_k \omega_j \) may be viewed as “expansion of \( f \) in the cusp \( \omega_j \)”. (Strictly speaking, we should consider the double coset \( \Gamma_0^n(p) \cdot \omega_j \cdot \mathbb{P}(\mathbb{Z}) \) as a cusp for \( \Gamma_0^n(p) \); by abuse of language we will call the \( \omega_j \) “the cusps for \( \Gamma_0^n(p) \)” .) For basic results concerning fields generated by Fourier coefficients and boundedness of denominators we refer to [29]. We will however use these notions in the sequel only for theta series, where such properties are accessible in a much more elementary way.

### 2.4 Lattices and theta series

For an even integral positive definite matrix \( S \) of size \( m = 2k \) we define the degree \( n \) theta series in the usual way:

\[
\vartheta^n_S(Z) := \sum_{X \in \mathbb{Z}(m,n)} e^{\pi i tr(S[X]Z)} \quad (Z \in \mathbb{H}_n).
\]
where $S[X] = X^tSXX$. We will freely switch between the languages of matrices $S$ and corresponding lattices $L$ and we write sometimes $\vartheta^n(L)$ instead of $\vartheta^n_S$. For the transformation properties of such theta series see e.g. [1]. Following [6] a lattice $L$ will be called $p$-special, if it has an isometry of order $p$ with no fixed point in $L \setminus \{0\}$. The theta series of such a lattice automatically satisfies

$$\vartheta^n(L) \equiv 1 \pmod{p}.$$ 

There are many such lattices, see [7]:

**Proposition 2.3.** Let $p$ be an odd prime, then there are $p$-special (positive definite, even) lattices of rank $p - 1$, level $p$ and determinant $p^t$ for all $1 \leq t \leq p - 2$.

Linear combinations of theta series for such lattices provide modular forms with convenient congruence properties in all cusps, see Theorem 2 and Theorem 2’ in [7] (and also the erratum at the end of this paper).

In the case $m = n$, we consider the series

$$\vartheta^n_{S,\det}(Z) := \sum_{X \in \mathbb{Z}^{n,n}} \det X \cdot e^{\pi i \text{tr}(S[X]Z)}, \quad (Z \in \mathbb{H}_n).$$

It is known that the series becomes a modular form of weight $1 + \frac{n}{2}$ and vanishes identically if and only if there exists a matrix $U \in M_n(\mathbb{Z})$ such that

$$S[U] = S, \quad \det U = -1,$$

(e.g. cf. [14]).

As in [22], the theta series of this type allows us to construct cusp forms with accessible properties.

### 2.5 Theta series in the cusps $\omega_j$

We need precise information about theta series in the cusps $\omega_j$, in particular about the denominators, which occur in the Fourier expansions in the cusps. To quote the results from [12] [10], it is more convenient to use the geometric notation here: Let $(V,q)$ be a positive definite quadratic space over $\mathbb{Q}$ with attached bilinear form $B(x,y) = q(x + y) - q(x) - q(y)$ and let $L$ be an even lattice on $V$ (i.e. $q(L) \subseteq \mathbb{Z}$ of level $p$, which means $q(L^\perp) \cdot \mathbb{Z} = \frac{1}{p}Z$, where $L^\perp$ denotes the dual of $L$.) After fixing a basis of $V$ we may identify $V$ with $\mathbb{Q}^m$. The associated Gram matrix with respect to this basis will be denoted by $S$; for $x = (x_1, \ldots, x_n) \in V^n$ we then define the $n \times n$ matrix $q(x)$ by

$q(x)_{ij} = \frac{1}{2}B(x_i, x_j)$. Let $P : \mathbb{C}^{(m,n)} \rightarrow \mathbb{C}$ be a pluriharmonic polynomial with $P(XA) = \det(A)^n P(X)$ for $A \in GL(n, \mathbb{C})$. In this language, with

$$\vartheta^n(P, L, Z) := \sum_{x \in L^n} P(x) e^{2\pi i \text{tr}(q(x)Z)},$$

$$\vartheta^n_{S,\det}(Z) := \sum_{X \in \mathbb{Z}^{n,n}} \det X \cdot e^{\pi i \text{tr}(S[X]Z)}, \quad (Z \in \mathbb{H}_n).$$
and
\[
\vartheta^{(n-j,i)}(P, L, L^\sharp, Z) := \sum_{x \in L^{n-j} \times (L^\sharp)^j} P(x) e^{2\pi i \text{tr}(q(x) Z)}
\]
we get for all \(j\)
\[
\vartheta^n(P, L, Z) |_{\varpi + \nu} \omega_j = (\gamma_p s_p(V))^j \det(L)^{-j} \vartheta^{(n-j,i)}(P, L, L^\sharp, Z).
\]

Here \(s_p(V)\) is the Hasse-Witt invariant at \(p\) (as normalized in [28]) and \(\gamma_p\) depends only on \(\det(L) \cdot (Q_p^\times)^2\), in particular it is one, if \(\det(L)\) is a square.

There are obvious generalizations of this, if \(L\) is of more general level.

3 mod \(p\) kernel of theta operator

We are interested in constructing Siegel modular forms \(F\) of level one, scalar-valued, such that \(F \not\equiv 0 \mod p\) and \(\Theta[j](F) \equiv 0 \mod p\) for some \(j > 0\). Thanks to the results on level changing in [8] it is sufficient to construct such modular forms for groups \(\Gamma_0(pN)\), possibly with nebentypus character \(\chi_p\).

3.1 Some generalities

Remark 3.1. If \(\Theta^{[r]}(F) \equiv 0 \mod (p)\), then \(\Theta^{[r']}(F) \equiv 0 \mod (p)\) for all \(r' \geq r\).

Remark 3.2. Suppose that \(F = \sum a(T; F) e^{2\pi i \text{tr}(TZ)} \in M_k(\Gamma^n)\) is mod \(p\) singular of rank \(l\), i.e. \(a(T; F) \equiv 0 \mod (p)\) for all \(T\) with \(\text{rank}(T) > l\) and \(a(T_0; F) \not\equiv 0 \mod (p)\) for some \(T_0\) of rank \(l\), then \(F\) automatically satisfies \(\Theta^{[j]}(F) \equiv 0 \mod (p)\) for all \(j > l\). Such modular forms were investigated in [5], they always satisfy
\[
2k - l \equiv 0 \pmod {p - 1}.
\]

3.2 Weak Constructions

Let \(S\) be an even integral positive definite matrix of size \(n\), with \(n\) even. We assume that the rank of \(S\) over the finite field \(\mathbb{F}_p\) is \(r < n\) and that the level of \(S\) is a power of \(p\), denoted by \(p^l\). The determinant of \(S\) can then be written as \(p^d\). Then \(\vartheta^n_S \in M_{2d}(\Gamma_0^0(p^l), \chi_p^d)\) satisfies
\[
\vartheta^n_S \not\equiv 0 \pmod p,
\]
\[
\Theta^{[j]}(\vartheta^n_S) \equiv 0 \pmod p \quad (j > r).
\]

If moreover the order of \(\text{Aut}_\mathbb{Z}(S)\) is coprime to \(p\), then we have the stronger property
\[
\Theta^{[r]}(\vartheta^n_S) \not\equiv 0 \pmod p,
\]
because the coefficient of this Fourier series at \(S\) is just \(\#\text{Aut}_\mathbb{Z}(S) \cdot S^{[r]}\).

Now we assume in addition that \(S\) has no automorphism of determinant \(-1\) and that the order of \(\text{Aut}_\mathbb{Z}(S)\) is coprime to \(p\). Then \(\vartheta^n_{S,\det} \in S_{2d+1}(\Gamma_0^0(p^l), \chi_p^d)\) and

\[7\]
it also satisfies the congruences above.

**Conclusion:** Under the assumption, that quadratic forms $S$ with the properties above exist, we get Siegel modular forms (cusp forms respectively) $F$ of level one and with weight congruent to $\frac{n^2}{2} + d \cdot \frac{p-1}{2}$ (mod $(p-1)$) (resp. $\frac{n^2}{2} + 1 + d \cdot \frac{p-1}{2}$) respectively satisfying the congruences above.

Note that in general the actual weight of the level one form (using \[8\] or similar techniques) will be much larger.

**Remark 3.3.** The arithmetic of quadratic forms tells us, under which conditions on $(n, p, r, d)$ such forms $S$ exist or not. We also note that an inspection of the Fourier expansion shows that a set of $h$ pairwise inequivalent quadratic forms $S_i$ with the properties above gives a set of $h$ pairwise linearly independent modular forms mod $p$ in the kernel of $\Theta^{[n]}$, provided that none of the $S_i$ has an integral automorphism of order $p$.

**Remark 3.4.** There is a simpler construction, which covers more weights modulo $p$ and produces modular forms in the simultaneous kernel mod $p$ for all $\Theta$-operators: Suppose that $f \in M_k(\Gamma_0^n(\mathbb{Z}_p))$ satisfies $\nu_p(f) = 0$. Then $f(pz)$ is congruent mod $p$ to a modular form $F$ of level one which satisfies $\Theta^{[j]}(F) \equiv 0 \mod p$ for all $j \geq 1$. An analogous statement holds true if $f$ has nontrivial nebentypus $\chi_p$.

The remark above is quite useful: We recall from \[14\] that 

$$\Theta^{[r]}(f \cdot g) = \Theta^{[r]}(f) \cdot g + \cdots,$$

where $\cdots$ is an integral linear combination of products of entries of $\Theta^{[d]}f$ and of entries of $\Theta^{[j]}g$ ($i+j=r$). Combining the conclusion above with the remark 3.3. and keeping in mind that there always exist (integral, cuspidal) modular forms of level one and sufficiently large weight $k$, provided that $kn$ is even, we obtain elements in the kernel of $\Theta^{[j]}$ for almost all weights, if $n$ is even:

**Corollary 3.5.** Under the same assumption on the existence of quadratic form $S$ as above we obtain the existence of modular forms (resp. cusp forms) $F$ of level one and weight congruent mod $p-1$ to

$$\frac{n}{2} + d \cdot \frac{p-1}{2} + k \quad \text{(resp. } \frac{n}{2} + 1 + d \cdot \frac{p-1}{2} + k\text{)},$$

which satisfy the congruence above. Here $k$ is now arbitrary mod $p-1$.

### 3.3 A variant, in particular for odd degree

For $n$ arbitrary we choose an even integer $m$ with $m > n$ and a positive definite even integral quadratic form of size $m$ with rank$_p(S) = m' < n$ and level$(S)$ a power of $p$. Then $\vartheta^n_S$ is in the kernel of $\Theta^{[n]}$ mod $p$. This construction gives weaker results than before, because a lot of linear dependencies among the theta series may arise here (over $\mathbb{Q}$ or over $\mathbb{F}_p$).

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\[1\]This case appeared in a discussion with S. Takemori.
4 Strong Constructions

**Theorem 4.1.** For \( n \equiv 0 \mod 4 \) and all primes \( p \geq n + 3 \) there exists \( F \in M_{\frac{n}{2} + p - 1}(\Gamma^n) \), \( F \not\equiv 0 \pmod{p} \) such that \( \Theta^{[n-1]}(F) \equiv \Theta^{[n]}(F) \equiv 0 \pmod{p} \). If the prime \( p \) satisfies \( p \equiv 1 \pmod{4} \), the condition \( p \geq n \) is sufficient.

**Theorem 4.2.** For \( n \equiv 2 \pmod{4} \) (if \( p \equiv 3 \pmod{4} \)) or \( 4 \mid n \) (if \( p \equiv 1 \pmod{4} \)) and all primes \( p \geq 2n + 3 \) there exists \( F \in M_{\frac{n}{2} + p - 1}(\Gamma^n) \), \( F \not\equiv 0 \pmod{p} \) such that \( \Theta^{[n]}(F) \equiv 0 \pmod{p} \).

**Remark 4.3.** A similar result, using Eisenstein series was shown by the third author [25]. Our method of proof provides examples beyond Eisenstein series.

The proofs of these theorems are quite similar, they rely on results from [7]; we mainly have to assure the existence of appropriate even integral quadratic forms of rank \( n \). In all cases, we have just to consider \( n \mod 8 \), because we may add even unimodular quadratic forms as orthogonal summands if necessary.

To prove Theorem 4.1 we first observe, that there exist even integral quadratic forms \( S \) in dimension \( n \equiv 0 \mod 4 \) with \( \det(S) = p^2 \), level \( p \) and with rank \( n - 2 \) over \( \mathbb{F}_p \): For \( n = 4 \) we may choose a lattice corresponding to a maximal order in the definite rational quaternion algebra ramified only at \( p \). For \( n = 8 \) we recall that locally at \( p \) the even unimodular lattice \( E_8 \) corresponds to an orthogonal sum of hyperbolic planes. The requested quadratic form is then shown to exist by scaling one of the hyperbolic planes by the factor \( p \). The theta series attached to such \( S \) satisfies

\[
\nu_p(\vartheta^n_S | \frac{1}{2} \omega_j) = -j
\]

for all \( j \). Theorem 4 from [7] asserts the existence of \( F \in M_{\frac{n}{2} + p - 1}(\Gamma^n) \) with \( \vartheta^n_S \equiv F \pmod{p} \).

Concerning the second theorem, even integral quadratic forms \( S \) of level \( p \) and determinant \( p \) are known to exist in the dimensions mentioned in the theorem: For \( p \equiv 3 \pmod{4} \) and \( n = 2 \) we may chose the binary form \((x, y) \mapsto x^2 + xy + \frac{p+1}{4}y^2\). For \( p \equiv 3 \pmod{4} \) and \( n = 6 \) we consider the orthogonal sum of a binary form of determinant \( p \) and a quaternary form of determinant \( p^2 \). A maximal overlattice then has the requested property (see the next section for the notion of maximality). For \( p \equiv 1 \pmod{4} \) and \( n = 4 \) the existence is shown e.g. in [27, p.350]. For \( n = 8 \) we may take the orthogonal sum of quaternary forms of determinant \( p \) and \( p^2 \) and go to a maximal overlattice. For a local-global proof of existence in the case \( p \equiv 1 \pmod{4} \) see [30, lemme 19].

We may then construct \( F \) from \( \vartheta^n_S \) by

\[
F := \text{Tr}(\vartheta^n_S \cdot \mathcal{E}),
\]

provided that we can find \( \mathcal{E} \in M_{\frac{n-1}{2}}(\Gamma_0^1(p), \chi_p)(\mathbb{Z}) \) with \( \mathcal{E} \equiv 1 \pmod{p} \) and

\[
\nu_p(\mathcal{E} | \omega_j) \geq -\frac{j^2}{2} + 1 \quad (j \geq 1).
\]
Then all the $Y_j$ in the trace above are congruent zero mod $p$ except for $Y_0$. The existence of such $E$ is asserted by (the corrected version of) Theorem 2, [7] (see, Erratum, Theorem 2’).

**Remark 4.4.** Unfortunately, the procedure above does not work for the theta series $\vartheta_{S, \det}^p$; it breaks down because we get an additional factor $p$ in the denominators of the $Y_j$. This is the main reason for a different approach in the next section.

5 Strong construction with harmonic polynomial

5.1 Motivation

We want to explain the numerical examples for degree 2 (see the last section) by a refined version of our construction, in particular, we aim at

**Proposition 5.1.** Let $p$ be a prime congruent $3 \mod 4$, let $S$ be a binary quadratic form of discriminant $-p$ (without improper automorphism). Then there exists $F \in S_{2+3, \frac{p-1}{2}}(\Gamma^2)$ such that

$$F \equiv \vartheta_{S, \det}^2 \pmod{p}.$$  

This will be proved in a more general framework, using a variant of the calculus of traces. Our method strongly relies on the arithmetic of maximal lattices; we recall that a lattice $L$ in a quadratic space $(V, q)$ with $q(L) \subset \mathbb{Z}$ is called maximal, iff the only overlattice $L'$ of $L$ with $q(L') \subset \mathbb{Z}$ is $L$ itself. For basic properties of maximal lattices we refer to [9]. In particular, even integral positive definite matrices $S$ with squarefree determinant correspond to maximal lattices. The proposition from above is just a special case of the following general result

**Theorem 5.2.** Let $n$ be an even positive integer and $p$ a prime with $p \equiv 3 \mod 4$ satisfying $n + p - 1 \equiv 0 \mod 4$. Let $S$ be an even positive definite quadratic form of rank $n$ and with $\det(S) = p$. Then there exists $F \in S_{\frac{n}{2} + 1, \frac{n-1}{2}}(\Gamma^n)$ such that

$$F \equiv \vartheta_{S, \det}^n \pmod{p}.$$  

**Remark 5.3.** A similar result should hold for primes $p \equiv 1 \mod 4$, but our method seems not to be applicable in this case.

5.2 The case $n + p - 1 \equiv 4 \pmod{8}$, $p \equiv 3 \mod 4$

We start from the following situation (with $n$ and $p$ as above).

$S$ is a lattice of level and determinant $p$ with even rank $n$. The existence of such $S$ is guaranteed, because of $p \equiv 3 \pmod{4}$ and $n \equiv 2 \pmod{4}$.

Furthermore, let $L$ be a $p$-special lattice of rank $p-1$, level and determinant $p$, we may e.g. take the root lattice $A_{p-1}$, see [6].
We observe that $S \perp A_{p-1}$ is maximal (of determinant $p^2$) provided that $n+p-1 \equiv 4 \pmod{8}$ (in this case there is no even unimodular lattice; we also note that for $n+p-1 \equiv 0 \pmod{8}$ there is no maximal lattice of determinant $p^2$). Under these assumptions we consider

$$p \cdot \text{Tr} \left( \vartheta^n_{S, \det} \cdot \vartheta^n(L) \vartheta^n(L) \right) = \sum_{j=0}^{n} Y_j.$$

We can view this as the trace of a theta series attached to the quadratic form

$$Q := S \perp L \perp L \perp L$$

of rank $2k = n+3(p-1)$ and with determinant $p^4$. The transformation properties of such theta series give

$$\vartheta^n(Q) \mid_{k \omega_j} = s_p(V)^j p^{-2j} \sum_{X,R_1,R_2,R_3} \det((X^{(n-j)}, S^{-1}X^{(j)})) \exp(2\pi i \text{tr} L \cdot Z)$$

where

$$L = S[X^{(n-j)}, S^{-1}X^{(j)}] + L[R_1^{(n-j)}, L^{-1}R_1^{(j)}] + L[R_2^{(n-j)}, L^{-1}R_2^{(j)}] + L[R_3^{(n-j)}, L^{-1}R_3^{(j)}].$$

Furthermore, $s_p(V)$ is the Witt-invariant of the quadratic space underlying the lattice $Q$, and we decompose $X \in \mathbb{Z}^{(n,n)}$ and $R_i \in \mathbb{Z}^{(p-1,n)}$ into components with $n-j$ and $j$ columns.

Evidently we have $\nu_p(Y_0) = 1$ and from the transformation formula above we obtain for $j \geq 1$

$$\nu_p(Y_j) \geq 1 + \frac{j(j+1)}{2} - 2j - 1$$

The last summand $-1$ comes from the harmonic polynomial.

For $j \geq 4$ this is strictly positive and only for $1 \leq j \leq 3$ need further investigation. Actually, thanks to the arithmetic of maximal lattices, stronger integrality properties will be shown: All contributions $Y_j$ will be integral and only $j = 1$ and $j = 2$ may possibly be nonzero mod $p$.

We analyse the contributions of $\vartheta^n(Q) \mid_{\omega_j U_j(p)}$ separately for fixed $j \geq 1$.

First we recall the decomposition of the Hecke operator $U_j(p)$:

$$U_j(p) = U_j^0(p) \circ D_j(p)$$

(cf. §2.2, §2.1).

Furthermore we recall that $L$ is $p$-special, therefore we can write

$$\vartheta^n(L) \mid_{p-1} \omega_j \sim p^{-\frac{j}{2}} (1 + \vartheta^n(L)^0),$$
where $\vartheta_j^n(L)^0$ is a Fourier series with $\vartheta_j^n(L)^0 \equiv 0 \pmod{p}$. Therefore,

$$(\vartheta^n(L) \mid \omega_j)^3 \sim p^{-\frac{3j}{2}}(1 + 3\vartheta_j^n(L)^0 + p^2 \cdot W_j)$$ (5.1)

where $W_j$ is a Fourier series with integral Fourier coefficients and $\sim$ means equality up to a $p$-adic unit as factor.

According to (5.1), $(\vartheta_j^n(S, \det \cdot \vartheta_j^n(L))^3 \mid \omega_j) \sim p^{-2j} \sum X \det(X^{(n-j)}, S^{-1}X^{(j)}) \exp(2\pi i \text{tr}(S[X^{(n-j)}, S^{-1}X^{(j)}]Z)) \mid U_j^0(p)$. The maximality of $S$ implies that $X^{(j)} = S \cdot \bar{X}$ with $\bar{X} \in \mathbb{Z}^{(n-j)}$. Therefore this part just equals

$$p^{-2j} \vartheta_{S, \det}^n.$$

The second part is

$$3p^{-2j} \sum_{X,Y} \det(X^{(n-j)}, S^{-1}X^{(j)}) \cdot \exp(2\pi i \text{tr}(S[X^{(n-j)}, S^{-1}X^{(j)}]Z + L[Y^{(n-j)}, L^{-1}Y^{(j)}]Z)).$$

Here $Y \in \mathbb{Z}^{(p-1,n)}$ satisfies the additional condition $Y \neq 0$. We use that the lattice $S \perp L$ is maximal and we obtain by the same reasoning as before (keeping in mind that $L$ is $p$-special) that this contribution equals (up to a unit)

$$p^{-2j} \vartheta_{S, \det}^n \cdot \sum_{Y \neq 0} \exp(2\pi i \text{tr}(L[Y]Z)) = p^{-2j+1}W,$$

where $W$ is a Fourier series with integral Fourier coefficients.

Finally the last contribution will have its Fourier coefficients in

$$p^{-2j+1} \cdot \mathbb{Z}$$

because of the factor $p^2$ in front of $W_j$; note that the harmonic polynomial may bring in an additional $p$ in the denominator.

So far we have ignored the contribution of the operator $D_j(p)$; actually, in the first two contributions, thanks to the maximality of $S$ and $S \perp L$ the result after applying $U_j^0(p)$ is already invariant under $GL(n, \mathbb{Z}) \hookrightarrow Sp(n, \mathbb{Z})$ and only
the number \( d(j) \) of left cosets enters defining \( D_j(p) \) really matters; this is a group index

\[
d(j) = [GL(n, \mathbb{F}_p) : P_{n,j}(\mathbb{F}_p)] = \prod_{i=1}^{j} \frac{p^{j+i} - 1}{p^i - 1}
\]

see e.g. [21], p.73, in particular it is congruent 1 mod \( p \).

Now we collect everything into a precise formula for \( Y_j \) (\( 1 \leq j \leq n \)):

\[
Y_j = p^{1+ \frac{j(j+1)}{2}} s_p(V)^j \times \{ d(j) \cdot \vartheta^n_{S, \det} + 3(\ast) + (\ast\ast) \}
\]

where (\ast) and (\ast\ast) are Fourier series with coefficients divisible by \( p \).

The \( p \)-power in front of \( Y_j \) is (for \( 0 \leq j \leq n \))

\[
p^1, \quad p^0, \quad p^0, \quad p^1, \quad \cdots
\]

where the \( \cdots \) are divisible by \( p \). Therefore only \( j = 1 \) and \( j = 2 \) is relevant.

We summarize our calculation in the formula

\[
p \cdot \text{Tr}(\vartheta^n_{S, \det} \cdot \vartheta^n(L)^3) \equiv (s_p(V) + s_p(V)^2) \vartheta^n_{S, \det} \pmod{p}. \tag{5.2}
\]

Concerning \( s_p(V) \), we remark that the dimension of \( V \) is \( n + 3(p-1) \); then \( s_p(V) = 1 \) iff \( n + 3(p-1) \equiv 0 \pmod{8} \); this is assured for \( p \equiv 3 \pmod{4} \) under the assumption \( n + p - 1 \equiv 4 \pmod{8} \).

This proves Theorem 3 for the case in question.

### 5.3 Variant: The case \( n + p - 1 \equiv 0 \pmod{8} \)

In this case, there are no maximal lattices of determinant \( p^2 \) of rank \( n + p - 1 \); therefore we take an auxiliary prime \( q \) different from \( p \) and we consider \( q \cdot S \perp L \).

This quadratic form is of determinant \( q^n \cdot p^2 \) and of rank \( n + p - 1 \). Its Witt invariant at \( q \) can be computed as

\[
s_q(q \cdot S \perp L) = s_q(S) \cdot \left( \frac{-1}{q} \right)^{\frac{n(n-1)}{2}} \cdot \left( \frac{p}{q} \right)^{n-1} = \left( \frac{-p}{q} \right)
\]

We choose \( q \) such that

\[
\left( \frac{-p}{q} \right) = -1 \tag{5.3}
\]

holds. In view of \( s_{\infty}(q \cdot S \perp L) = 1 \) this implies that \( s_p(q \cdot S \perp L) = -1 \), in particular both \( q \cdot S \) and \( q \cdot S \perp L \) are maximal at \( p \).

Then we can use essentially the calculation above to find a modular form \( F \) of level \( q \) and weight \( \frac{n}{2} + 1 + 3 \cdot \frac{n-1}{2} \) such that \( F \equiv \vartheta^n_{S, \det} \pmod{p} \). We have to mention one subtle point here: The quadratic space \( V' \) in question is now given by \( qS \perp L \perp L \perp L \) with Witt invariant

\[
s_q(V') = -1, \quad s_{\infty}(V') = -1
\]
and therefore (by the product formula) \( s_p(V') = 1 \). This is necessary for the nonvanishing mod \( p \) of the analogue of (5.2). The lemma below then assures the existence of a modular form \( \tilde{F} \) of level one and weight \( \frac{n}{2} + 1 + 3 \cdot \frac{p-1}{2} \) such that
\[
\tilde{F} \equiv \vartheta_{S,\det}^n \pmod{p}.
\]

**Lemma 5.4.** Let \( p \) and \( q \) be different primes with \( \prod_{i=1}^{n}(1 + q^i) \) coprime to \( p \). Let \( f \in M_k(\Gamma_0^n(p), \chi_p)(\mathbb{Z}(p)) \) be given and assume that there exists \( G \in M_l(\Gamma_0^n(q))(\mathbb{Z}(p)) \) for some weight \( l \) such that
\[
g \equiv G \pmod{p},
\]
where the modular form \( g \) of level \( pq \) is given by \( g(Z) := f(qZ) \). Then there exists a level one form \( F \) of weight \( l \) such that
\[
f \equiv F \pmod{p}.
\]

**Proof.** We choose \( E \in M_{l-k}(\Gamma_0^n(p), \chi_p) \) such that \( E \equiv 1 \pmod{p} \).

Then
\[
g \cdot E(qZ) - G = p \cdot H,
\]
where \( H \in M_l(\Gamma_0^n(pq)) \) has \( p \)-integral coefficients. We apply the operator \( U(q) \) to obtain
\[
f \cdot E = G \mid U(q) + p \cdot H \mid U(q)
\]
Now we take the trace from \( \Gamma_0^n(pq) \) to \( \Gamma_0^n(p) \) on both sides. On the left side it just means that we multiply the function by the index \( [\Gamma_0^n(pq) : \Gamma_0^n(p)] = \prod_{i=1}^{n}(1 + q^i) \), which is coprime to \( p \); the formula for the index can be found e.g. in [18]. On the right side we observe that the trace does not affect the \( p \)-integrality, in particular, the \( H \)-part plays no role mod \( p \). This follows from the \( q \)-expansion principle, see e.g. [15, Theorem 2]. Moreover, to compute the trace of \( G \mid U(q) \), we observe that this trace has level one, because \( G \mid U(q) \) has level \( q \).

At the end, \( f \) is congruent mod \( p \) to a level one form of weight \( l \).

\( \square \)

**Remark 5.5.** Note that it is possible to choose the prime \( q \) in such a way that both the condition of the lemma and (5.3) are satisfied.

**Remark 5.6.** It should be possible to prove the statement above concerning the \( p \)-integrality of a trace from \( \Gamma_0^n(pq) \) to \( \Gamma_0^n(p) \) by a more elementary method for our special case (using properties of theta series), avoiding the \( q \)-expansion principle.

### 5.4 Some special properties

We cannot expect that all elements in the kernel mod \( p \) for the theta operator arise by theta series as in the sections above. In fact, our construction gives modular forms with amusing additional congruence properties:
For a positive integer $d$ and a modular form $f \in M_k(\Gamma_0^0(N), \chi)$ with Fourier expansion $f(Z) = \sum_T a(T; f)e^{2\pi i Tr(TZ)}$ we define

$$a_d(f) := \sum_T \frac{1}{\epsilon^+(T)} a(T; f)$$

where $T$ runs over representatives of the $SL(n, \mathbb{Z})$-equivalence classes of elements $T$ in $\Lambda_n$ with $T > 0$ and $\frac{1}{2} \det(2T) = d$ and

$$\epsilon^+(T) := \sharp\{U \in SL(n, \mathbb{Z}) \mid T[U] = T\}.$$  

These numbers appear naturally in the Koecher-Maaß Dirichlet series attached to $f$, see [23]:

$$KM(f, s) := \sum_d a_d(f)d^{-s}$$

For $f = \vartheta^n_{2S}$ and $f = \vartheta^n_{2S, \text{det}}$ with half integral $S$ of size $n$ the Koecher-Maaß Dirichlet series are very special:

$$KM(\vartheta^n_{2S}, s) = \det(2S)^{-s} \sum_X |\det(X)|^{-2s}$$

$$KM(\vartheta^n_{2S, \text{det}}, s) = \det(2S)^{-2s} \sum_X |\det(X)|^{-2s} = 0$$

Here $X$ runs over all non-degenerate integral matrices in $\mathbb{Z}^{(n,n)}$ modulo the action of $SL(n, \mathbb{Z})$.

**Proposition 5.7.** Let $p$ be a prime, $n$ even, and $S_1, \ldots, S_h$ half integral positive definite matrices of size $n$ with $\det(2S_i) = p$ for all $i$; we consider the modular form

$$f := \sum_i a_i \vartheta^n_{2S_i}$$

with $a_i \in \mathbb{Z}(p)$. Let $F$ be a modular form of level one with $F \equiv f \pmod p$. Then

$$a_d(F) \equiv 0 \pmod p$$

holds for all $d$ provided that $\sum_i a_i \equiv 0 \pmod p$.

**Proposition 5.8.** Let $p$ be a prime, $n$ even, and $S$ a half integral positive definite matrix of size $n$ with $\det(2S) = p$; we consider the level one modular form $F$ with $F \equiv \vartheta^n_{2S, \text{det}} \pmod p$. Then we have for all $d$

$$a_d(F) \equiv 0 \pmod p.$$  

**6 Degree 2 case**

Explicit examples of degree 2 modular forms in the kernel of the theta operator $\pmod p$ were known before our construction:
Let $\Delta_{12}$ denote the Ramanujan delta function and $[\Delta_{12}]$ the corresponding Klingen-type Eisenstein series of degree 2. In [3], the first author showed that

$$\Theta([\Delta_{12}]) \equiv 0 \pmod{23}.$$ 

Then Mizumoto [24] showed that the Klingen-type Eisenstein series $[\Delta_{16}]$ attached to a weight 16 elliptic cusp form satisfies the congruence

$$\Theta([\Delta_{16}]) \equiv 0 \pmod{31}.$$ 

The second and the third author and Kikuta found the congruence

$$\Theta(X_{35}) \equiv 0 \pmod{23},$$

where $X_{35}$ is Igusa’s cusp form of weight 35 (cf. [17]). They also predicted the existence of a form $X_{47}$ of odd weight 47 satisfying

$$\Theta(X_{47}) \equiv 0 \pmod{31}.$$ 

In fact, such modular form was constructed in [20] by using Igusa’s generators. In subsection 6.3 we will show that these examples can all be explained by our constructions, using appropriate binary quadratic forms.

### 6.1 Estimation of dimensions

In this section, we consider the dimension of the kernel of theta operator over $\mathbb{F}_p$. Let $B_D$ be the set of positive-definite, integral binary quadratic forms with discriminant $D$, i.e.,

$$B_D := \left\{ S = \left( \begin{array}{c} a & b/2 \\ b/2 & c \end{array} \right) \mid a, b, c \in \mathbb{Z}, S > 0, b^2 - 4ac = D \right\}.$$

In the following, we mainly treat the case $D < 0$. We denote the class number by $h(D) := \sharp(B_D/\text{SL}(2, \mathbb{Z}))$.

**Proposition 6.1.** Let $p$ be a prime number with $p \equiv 3 \pmod{4}$ and $p > 3$. Then

(i) $\dim_{\mathbb{F}_p}(\overline{\vartheta}_2 S \mid S \in B_{-p})_{\mathbb{F}_p} = \frac{h(-p)+1}{2},$

(ii) $\dim_{\mathbb{F}_p}(\overline{\vartheta}_2 S,\det \mid S \in B_{-p})_{\mathbb{F}_p} = \frac{h(-p)-1}{2},$

where $\overline{\vartheta}_2 S$ (resp. $\overline{\vartheta}_2 S,\det$) is the Fourier coefficientwise reduction modulo $p$ of $\vartheta_2 S$ (resp. $\vartheta_2 S,\det$).

**Remark 6.2.** For a modular form $F$, $F$ is considered as a formal power series over $\mathbb{F}_p$ via Fourier expansion (see [26]).
Proof. (i) Since the number of prime divisor dividing \( D = -p \) is just one, \( B_{-p}/\text{SL}(2, \mathbb{Z}) \) has a unique ambiguous class (cf. [31], p.112, Korollar). We fix a representative \( S_{p,0} \) of the ambiguous class and take a set of representatives of \( B_{-p}/\text{SL}(2, \mathbb{Z}) \) as

\[
\{ S_{p,0}, S_{p,i}, \tilde{S}_{p,i} \mid 1 \leq i \leq \frac{h(-p)-1}{2} \},
\]

where \( S_{p,i} \) and \( \tilde{S}_{p,i} \) are \( \text{GL}(2, \mathbb{Z}) \)-equivalent.

We consider the Fourier expansion of \( \vartheta_{Z}^2 S_{p,i} \):

\[
\vartheta_{Z}^2 S_{p,i}(Z) = \sum a(T; \vartheta_{Z}^2 S_{p,i}) e^{2\pi i \text{tr}(TZ)}.
\]

Then we have

\[
a(S_{p,j}; \vartheta_{Z}^2 S_{p,i}) = \# \{ U \in \text{GL}(2, \mathbb{Z}) \mid S_{p,i}[U] = S_{p,j} \} = \begin{cases} 4 & \text{if } i = j = 0, \\ 2 & \text{if } i = j > 0, \\ 0 & \text{otherwise.} \end{cases}
\]

This implies that

\[
\vartheta_{Z}^2_{S_{p,i}} \quad (1 \leq i \leq \frac{h(-p)-1}{2})
\]

are linearly independent over \( \mathbb{F}_p \).

(ii) It should be noted that \( \vartheta_{2S_{p,\text{det}}} \) is non-trivial if and only if \( S \) has no proper automorphisms. A similar argument in (i) shows that

\[
\vartheta_{S_{p,i,\text{det}}} \quad (1 \leq i \leq \frac{h(-p)-1}{2})
\]

are linearly independent over \( \mathbb{F}_p \). This proves (ii).

For a prime number \( p \), we define the following \( \mathbb{F}_p \)-vector spaces:

\[
\tilde{V}_{k,p} := \{ \tilde{f} \mid f \in M_k(\Gamma^2)(\mathbb{Z}_p), \Theta(f) \equiv 0 \pmod{p} \},
\]

\[
\tilde{V}_{k,p}^{\text{cusp}} := \{ \tilde{f} \mid f \in S_k(\Gamma^2)(\mathbb{Z}_p), \Theta(f) \equiv 0 \pmod{p} \}.
\]

Remark 6.3. As we noted in Remark 6.2, these spaces are considered as subspaces of certain vector space consisting of formal power series over \( \mathbb{F}_p \) (cf. [26]).

Considering Theorem 5.2, we can get the following estimates:

Corollary 6.4. Under the same assumption in Proposition 6.1 we have

(i) \( \dim_{\mathbb{F}_p} \tilde{V}_{k+1,p}^{\text{cusp}} \geq \frac{h(-p)+1}{2} \),

(ii) \( \dim_{\mathbb{F}_p} \tilde{V}_{k+1,p}^{\text{cusp}} \geq \frac{h(-p)-1}{2} \).
Example 6.5. We consider the case \( p = 23 \). In this case we have \( h(-23) = 3 \).

Under the previous notation, we can take a set of representatives of \( B_{-23}/SL(2, \mathbb{Z}) \) as

\[
\left\{ S_{23,0} = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 6 \end{pmatrix}, \quad S_{23,1} = \begin{pmatrix} 2 & 1/2 \\ 1/2 & 3 \end{pmatrix}, \quad \overline{S}_{23,1} = \begin{pmatrix} 2 & -1/2 \\ -1/2 & 3 \end{pmatrix} \right\}.
\]

In this case, we have \( \overline{V}_{12,23} = \langle \overline{\vartheta}^2_{2S_{23,0}}, \overline{\vartheta}^2_{2S_{23,1}} \rangle_{\mathbb{F}_{23}} \), namely we have

\[
\dim_{\mathbb{F}_{23}} \overline{V}_{12,23} = 2.
\]

We also obtain

\[
\overline{V}_{12,23}^{\text{cusp}} = \langle \overline{\vartheta}^2_{2S_{23,1}, \det} \rangle_{\mathbb{F}_{23}}.
\]

6.2 Average of Fourier coefficients

The congruence \( \Theta([\Delta_{12}]) \equiv 0 \pmod{23} \) was proved by the first author in \([3]\). In this paper, he remarked that the “average” of the Fourier coefficients is divisible by 23 as below. This phenomenon can be explained by the result obtained in Proposition 5.7.

The congruences given below are proved by checking them numerically for finitely many Fourier coefficients, using a “Sturm bound” from \([13]\).

The case \( p = 23 \):

We use the notation above. We have the congruence

\[
[\Delta_{12}] \equiv 12(\vartheta^2_{2S_{23,0}} - \vartheta^2_{2S_{23,1}}) \pmod{23}.
\]

By Proposition 5.7 we obtain

\[
a_d([\Delta_{12}]) \equiv 0 \pmod{23}.
\]

In the case of cusp forms, the following congruence holds

\[
X_{35} \equiv 12\vartheta^2_{2S_{23,1}, \det} \pmod{23}, \quad \text{and} \quad a_d(X_{35}) \equiv 0 \pmod{23},
\]

(Proposition 5.8).

The case \( p = 31 \):

In this case, we have \( h(-31) = 3 \).

\[
[\Delta_{16}] \equiv 16(\vartheta^2_{2S_{31,0}} - \vartheta^2_{2S_{31,1}}) \pmod{31}.
\]

This implies

\[
a_d([\Delta_{16}]) \equiv 0 \pmod{31}.
\]
In the case of cusp forms, we have
\[ X_{47} \equiv 16\vartheta_2^{2S_{47,0}} \mod 31, \quad \text{and} \quad a_d(X_{47}) \equiv 0 \mod 31, \]
where \( X_{47} \) is the cusp form introduced before.

**The case \( p = 47 \):**

This case is more interesting than the above two cases because \( h(-47) = 5 \) and \( \dim_{\mathbb{C}} S_{24}(\Gamma^1) = 2 \). We consider a couple of Klingen Eisenstein series

\[ \left[ E_4^3 \Delta_{12} \right] \quad \text{and} \quad \left[ \Delta_{12}^2 \right], \]

where \( E_4 \) is the normalized degree one Eisenstein series of weight 4. The following congruence relations hold:

\[ \left[ E_4^3 \Delta_{12} \right] \equiv 24(\vartheta_2^{2S_{47,0}} - 9\vartheta_2^{2S_{47,1}} + 8\vartheta_2^{2S_{47,2}}) \mod 47, \]
\[ \left[ \Delta_{12}^2 \right] \equiv 24(\vartheta_2^{2S_{47,1}} - \vartheta_2^{2S_{47,2}}) \mod 47. \]

Consequently

\[ a_d([E_4^3 \Delta_{12}]) \equiv a_d([\Delta_{12}^2]) \equiv 0 \mod 47. \]

**Erratum**

to our paper \[7\]

The formula in theorem 2 is incorrect, here is a correct version (Theorem 5 should then also be modified accordingly).

**Theorem 2’:** Assume that \( p \geq 2n + 3 \). Then there exists a modular form \( h \) of weight \( \frac{p^2 - 1}{2} \), level \( p \) and nebentypus \( \chi_p \) such that

\[ h \equiv 1 \mod p, \]
\[ \nu_p(h \mid \omega_j) \geq -\frac{j^2}{2} + 1 \quad (1 \leq j \leq n). \]

**Proof.** We write down an explicit linear combination of theta series with the requested property: Let \( L_j \) denote a \( p \)-special lattice of rank \( p-1 \) and discriminant \( 2j + 1 \) with \( 0 \leq j \leq n \). We put \( a_0 = 1 \) and for \( j \geq 1 \):

\[ a_j := (-1)^j \cdot p^\frac{j^2 + j}{2} \]

We define

\[ h := \sum_{j=0}^n a_j \vartheta^n(L_j) \]
Evidently, we have $h \equiv 1 \pmod{p}$.

We analyse the Fourier expansion of $h$ in all cusps $\omega_i$ with $1 \leq i \leq n$:

$$a_j \vartheta^n (L_j) \mid \omega_i = (-1)^j p^{-\frac{(i^2+1)}{2}} (-1)^j \cdot p^{\frac{j^2}{2}} (1 + \cdots).$$

We first check that for $j < i - 1$ and for $j > i$ the value of $\nu_p(a_j \vartheta^n (L_j) \mid \omega_i)$ is larger or equal to $-\frac{i^2}{2} + 1$:

The case $j = 0, i \geq 2$ is clear because $-\frac{i^2}{2} \geq -\frac{(i - 1)^2}{2} + 1$. For $j \geq 1$ our claim is equivalent to $-i(2j + 1) + j^2 + j \geq -i^2 + 2$.

To consider the case $j < i - 1$ we put $i = j + 1 + t$ with $t \geq 1$. We have to check that

$$-2(j + 1 + t) - (j + 1 + t) + (j + 1 + t)^2 + (j + 1 + t) - 2 \geq 0$$

This expression equals $t^2 + t - 2$.

Now we consider the second case, i.e. $j = i + t$ with $t > 0$: We have to check that

$$-2(i + t) - i + (i + t)^2 + i + t + i^2 - 2 \geq 0$$

This expression equals $t^2 + t - 2$.

It remains to consider the crucial cases $j = i - 1$ and $j = i$. In both cases we seem to pick up a denominator $p^{-\frac{i^2}{2}}$ when applying $\omega_i$ to $a_{i-1} \vartheta^n (L_{i-1})$ and to $a_i \vartheta^n (L_i)$. The key point is to analyze the constant term of

$$(a_{i-1} \vartheta^n (L_{i-1}) + a_i \vartheta^n (L_i)) \mid \omega_i. \quad (6.1)$$

This is equal to

$$(-1)^{i+i-1} p^{-\frac{(i^2-i-(i-1)^2+i-1)}{2}} + p^{-\frac{i(i^2+1)+i^2+i}{2}} = 0$$

The nonconstant parts of the Fourier expansion of (6.1) satisfy an additional congruence mod $p$ because the $L_i$ are special lattices.

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