LINEARIZED INTERNAL FUNCTIONALS FOR ANISOTROPIC
CONDUCTIVITIES

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Abstract. This paper concerns the reconstruction of an anisotropic conductivity tensor in an elliptic second-order equation from knowledge of the so-called power density functionals. This problem finds applications in several coupled-physics medical imaging modalities such as ultrasound modulated electrical impedance tomography and impedance-acoustic tomography.

We consider the linearization of the nonlinear hybrid inverse problem. We find sufficient conditions for the linearized problem, a system of partial differential equations, to be elliptic and for the system to be injective. Such conditions are found to hold for a lesser number of measurements than those required in recently established explicit reconstruction procedures for the nonlinear problem.

1. Introduction. In the context of hybrid medical imaging methods, a physical coupling between a high-contrast modality (e.g. Electrical Impedance Tomography, Optical Tomography) and a high-resolution modality (e.g. acoustic waves, Magnetic Resonance Imaging) is used in order to benefit from the advantages of both. Without this coupling, the high-contrast modality, usually modeled by an inverse problem involving the reconstruction of the constitutive parameter of an elliptic PDE from knowledge of boundary functionals, results in a mathematically severely ill-posed problem and suffers from poor resolution. The analysis of this coupling usually involves a two-step inversion procedure where the high-resolution modality provides internal functionals, from which we reconstruct the parameters of the elliptic equation, thus leading to improved resolution [2, 5, 14, 17, 22].

A problem that has received a lot of attention recently concerns the reconstruction of the conductivity tensor $\gamma$ in the elliptic equation

$$\nabla \cdot (\gamma \nabla u) = 0 \quad (X), \quad u|_{\partial X} = g,$$

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from knowledge of internal power density measurements of the form $\nabla u \cdot \gamma \nabla v$, where $u$ and $v$ both solve (1) with possibly different boundary conditions. This problem is motivated by a coupling between electrical impedance imaging and ultrasound imaging and also finds applications in thermo-acoustic imaging.

Explicit reconstruction procedures for the above non-linear problem have been established in [9, 6, 20, 19, 17], successively in the 2D, 3D, and nD isotropic case, and then in the 2D and nD anisotropic case. In these articles, the number of functionals may be quite large. The analyses in [17] were recently summarized and pushed further in [18]. If one decomposes $\gamma$ into the product of a scalar function $\tau = (\det \gamma)^{\frac{1}{n}}$ and a scaled anisotropic structure $\hat{\gamma}$ such that $\det \hat{\gamma} = 1$, the latter reference establishes explicit reconstruction formulas for both quantities with Lipschitz stability for $\tau$ in $W^{1,\infty}$, and involving the loss of one derivative for $\hat{\gamma}$.

In the isotropic case, several works study the above problem in the presence of a lesser number of functionals. The case of one functional is addressed in [4], whereas numerical simulations show good results with two functionals in dimension $n = 2$ [1, 12]. Theoretical and numerical analyses of the linearized inverse problem are considered in [15, 16]. The stabilizing nature of a class of internal functionals containing the power densities is demonstrated in [16] using a micro-local analysis of the linearized inverse problem. The above inverse problem is recast as a system of nonlinear partial differential equations in [3] and its linearization analyzed by means of theories of elliptic systems of equations. It is shown in the latter reference that $n + 1$ functionals, where $n$ is spatial dimension, is sufficient to reconstruct a scalar coefficient $\gamma$ with elliptic regularity, i.e., with no loss of derivatives, from power density measurements. This was confirmed by two-dimensional simulations in [7]. All known explicit reconstruction procedures require knowledge of a larger number of internal functionals.

In the present work, we study the linearized version of this inverse problem in the anisotropic case, i.e. we write an expansion of the form $\gamma^\varepsilon = \gamma_0 + \varepsilon \gamma$ with $\gamma_0$ known and $\varepsilon \ll 1$, and study the reconstructibility of $\gamma$ from linearized power densities (LPD). We first proceed by supporting the perturbation $\gamma$ away from the boundary $\partial X$ and analyze microlocally the symbol of the linearized functionals, and show that, as in [16], a large enough number of functionals allows us to construct a left-parametrix and set up a Fredholm inversion. The main difference between the isotropic and anisotropic settings is that the anisotropic part of the conductivity is reconstructed with a loss of one derivative. Such a loss of a derivative is optimal since our estimates are elliptic in nature. It is reminiscent of results obtained for a similar problem in [8].

Secondly, we show how the explicit inversion approach presented in [17, 18] carries through linearization, thus allowing for reconstruction of fully anisotropic tensors supported up to the boundary of $X$. In this case, we derive reconstruction formulas that require a smaller number of power densities than in the non-linear case, giving possible room for improvement in the non-linear inversion algorithms.

For additional information on hybrid inverse problems in other areas of (mostly medical) imaging, we refer the reader to, e.g., [2, 5, 21, 22].

2. Statement of the main results. Consider the conductivity equation (1), where $X \subset \mathbb{R}^n$ is open, bounded and connected with $n \geq 2$, and where $\gamma^\varepsilon$ is a uniformly elliptic conductivity tensor over $X$. 
We set boundary conditions \((g_1, \ldots, g_m)\) and call \(u_i^\varepsilon\) the unique solution to (1) with \(u_i^\varepsilon|_{\partial X} = g_i, 1 \leq i \leq m\) and conductivity \(\gamma^\varepsilon\). We consider the measurement functionals

\[
H_{ij} : \gamma^\varepsilon \mapsto H_{ij}^\varepsilon(\gamma^\varepsilon) = \nabla u_i^\varepsilon \cdot \gamma^\varepsilon \nabla u_j^\varepsilon(x), \quad 1 \leq i, j \leq m, \quad x \in X. \tag{2}
\]

Considering an expansion of the form \(\gamma^\varepsilon = \gamma_0 + \varepsilon \gamma\), where the background conductivity \(\gamma_0\) is known, uniformly elliptic and \(\varepsilon\) so small that the total \(\gamma^\varepsilon\) remains uniformly elliptic, we first look for the Fréchet derivative of (2) with respect to \(\gamma\) at \(\gamma_0\). Expanding the solutions \(u_i^\varepsilon\) accordingly as

\[
u_i^\varepsilon = u_i + \varepsilon v_i + O(\varepsilon^2), \quad 1 \leq i \leq m,
\]

the PDE (1) at orders \(O(\varepsilon^0)\) and \(O(\varepsilon^1)\) gives rise to two relations

\[-\nabla \cdot (\gamma_0 \nabla u_i) = 0 \quad (X), \quad u_i|_{\partial X} = g_i, \tag{3}
\]

\[-\nabla \cdot (\gamma_0 \nabla v_i) = \nabla \cdot (\gamma \nabla u_i) \quad (X), \quad v_i|_{\partial X} = 0. \tag{4}
\]

The measurements then look like

\[
H_{ij}^\varepsilon = \nabla u_i \cdot \gamma_0 \nabla u_j + \varepsilon (\nabla u_i \cdot \gamma \nabla u_j + \nabla u_i \cdot \gamma_0 \nabla v_j + \nabla u_j \cdot \gamma_0 \nabla v_i) + O(\varepsilon^2). \tag{5}
\]

Therefore, the component \(dH_{ij}\) of the Fréchet derivative of \(H\) at \(\gamma_0\) is

\[
dH_{ij}(\gamma) = \nabla u_i \cdot \gamma \nabla u_j + \nabla u_i \cdot \gamma_0 \nabla v_j + \nabla u_j \cdot \gamma_0 \nabla v_i, \quad x \in X, \tag{6}
\]

where the \(v_i\)'s are linear functions in \(\gamma\) according to (4).

In both subsequent approaches, reconstruction formulas are established under the following two assumptions about the behavior of solutions related to the conductivity of reference \(\gamma_0\). The first hypothesis deals with having a basis of gradients of solutions of (3) over a certain subset \(\Omega \subseteq X\).

**Hypothesis 2.1.** For an open set \(\Omega \subseteq X\), there exist \((g_1, \ldots, g_n) \in H^\frac{n}{2}(\partial X)^n\) such that the corresponding solutions \((u_1, \ldots, u_n)\) of (3) with boundary condition \(u_i|_{\partial X} = g_i\) \((1 \leq i \leq n)\) satisfy

\[
\inf_{x \in \Omega} \det(\nabla u_1, \ldots, \nabla u_n) \geq c_0 > 0.
\]

Once Hypothesis 2.1 is satisfied, any additional solution \(u_{n+1}\) of (3) gives rise to a \(n \times n\) matrix

\[
Z = [Z_1 \ldots Z_n], \quad \text{where} \quad Z_i := \frac{\nabla \det(\nabla u_1, \ldots, \nabla u_n)}{\det(\nabla u_1, \ldots, \nabla u_n)} \quad (\text{as defined in (7)}).
\]

As seen in [17, 18], such matrices can be computed from the power densities \(\{\nabla u_i \cdot \gamma_0 \nabla u_j\}_{i,j=1}^{n+1}\) and help impose orthogonality conditions on the anisotropic part of \(\gamma_0\). Once enough such conditions are obtained by considering enough additional solutions, then the anisotropy is reconstructed explicitly via a generalization of the usual cross-product defined in three dimensions. In the linearized setting, we find that one additional solution such that \(Z\) has full rank is enough to reconstruct the linear perturbation \(\gamma\). We thus formulate our second crucial assumption here:

**Hypothesis 2.2.** Assume that Hypothesis 2.1 holds over some fixed \(\Omega \subseteq X\). There exists \(g_{n+1} \in H^\frac{n}{2}(\partial X)\) such that the solution \(u_{n+1}\) of (3) with boundary condition \(u_{n+1}|_{\partial X} = g_{n+1}\) has a full-rank matrix \(Z\) (as defined in (7)) over \(\Omega\).
Remark 1 (Case $\gamma_0$ constant). In the case where $\gamma_0$ is constant, then it is straightforward to see that $g_i = x_i |_{\partial X}$ ($1 \leq i \leq n$) fulfill Hypothesis 2.1 over $X$. Moreover, if $Q = \{y_{ij}\}_{i,j=1}^{n}$ denotes an invertible constant matrix such that $Q : \gamma_0 = 0$, then the boundary condition $g_{n+1} := \frac{1}{2}q_{ij}x_i x_j |_{\partial X}$ fulfills Hypothesis 2.2, since we have $Q = I$.

Throughout the paper, we use for (real-valued) square matrices $A$ and $B$ the contraction notation $A : B = \text{tr} AB^T = \sum_{i,j} A_{ij}B_{ij}$, with $B^T$ the transpose matrix of $A$.

Remark 2. In the treatment of the non-linear case [6, 20, 17, 18], it has been pointed out that Hypothesis 2.1 may not be systematically satisfied globally in dimension $n \geq 3$. A more general hypothesis to consider would come from picking a larger family (of cardinality $> n$) of solutions whose gradients have maximal rank throughout $X$. While this additional technical point would not alter qualitatively the present reconstruction algorithms, it would add complexity in notation which the authors decided to avoid; see also [8].

2.1. Past work and heuristics for the linearization. In the reconstruction approach developed in [19, 17, 18] for the non-linear problem, it was shown that not every part of the conductivity was reconstructed with the same stability. Namely, consider the decomposition of the tensor $\gamma'$ into the product of a scalar function $\tau = (\det \gamma')^\frac{1}{2}$ and a scaled anisotropic structure $\tilde{\gamma}'$ with $\det \tilde{\gamma}' = 1$. The following results were then established. Starting from $n$ solutions whose gradients form a basis of $\mathbb{R}^n$ over a subset $\Omega \subset X$, it was shown that under knowledge of a $W^{1,\infty}(X)$ anisotropic structure $\tilde{\gamma}'$, the scalar function $\log \det \gamma'$ was uniquely and Lipschitz-stably reconstructible in $W^{1,\infty}(\Omega)$ from $W^{1,\infty}$ power densities. Additionally, if one added a finite number of solutions $u_{n+1}, \ldots, u_{n+l}$ such that the family of matrices $Z_{(1)}, \ldots, Z_{(l)}$ defined as in (7) imposed enough orthogonality constraints on $\tilde{\gamma}'$, then the latter was explicitly reconstructible over $\Omega$ from the mutual power densities of $(u_1, \ldots, u_{n+l})$. The latter reconstruction was stable in $L^\infty$ for power densities in $W^{1,\infty}$ norm, thus it involved the loss of one derivative.

Passing to the linearized setting now (recall $\gamma' = \gamma_0 + \varepsilon \gamma$), and anticipating that one scalar quantity may be more stably reconstructible than the others, this quantity should be the linearized version of $\log \det \gamma'$. Standard calculations yield

$$
\log \det (\gamma_0 + \varepsilon \gamma) = \log \det \gamma_0 + \log \det (\varepsilon_n + \varepsilon \gamma_0^{-1} \gamma) = \log \det \gamma_0 + \varepsilon \text{tr} (\gamma_0^{-1} \gamma) + O(\varepsilon^2),
$$

and thus the quantity that should be stably reconstructible is $\text{tr} (\gamma_0^{-1} \gamma)$. The linearization of the product decomposition $(\tau, \tilde{\gamma}')$ above is now a spherical-deviatoric one of the form

$$
\gamma = \frac{1}{n} \text{tr} (\gamma_0^{-1} \gamma) \gamma_0 + \gamma_d, \quad \gamma_d := \gamma_0 (\gamma_0^{-1} \gamma)^{\text{dev}},
$$

where $\text{dev}$ is the linear projection onto the hyperplane of traceless matrices $A^{\text{dev}} := A - \frac{\text{tr} A}{n} I_n$.

2.2. Microlocal inversion. The above inverse problem in (4)-(6) may be seen as a system of partial differential equations for $(\gamma, \{v_j\})$. This is the point of view considered in [3]. However, $\{v_j\}$ may be calculated from (4) and the expression plugged back into (6). This allows us to recast $dH$ as a linear operator for $\gamma$, which is smaller than the original linear system for $(\gamma, \{v_j\})$, but which is no longer differential and rather pseudo-differential. The objective in this section is to show,
following earlier work in the isotropic case in [16], that such an operator is elliptic under appropriate conditions.

We first fix $\Omega' \subset X$ and assume that $\text{supp} \gamma \subset \Omega'$, so that integrals of the form $\int_{\mathbb{R}^n} e^{i\xi \cdot x} p(x, \xi) \hat{\gamma}(\xi) d\xi$ are well-defined, with $p(x, \xi)$ a matrix-valued symbol whose entries are polynomials in $\xi$ (see [11, p.267]) and where the hat denotes the Fourier Transform $\hat{\gamma}(\xi) = \int_{\mathbb{R}^n} e^{-i\xi \cdot x} \gamma(x) \, dx$. We also assume that $\gamma_0 \in C^\infty(\Omega')$ and can be extended smoothly by $\gamma_0 = 1_n$ outside $\Omega'$. As pointed out in [16], in order to treat this problem microlocally, one must introduce cutoff versions of the $dH_{ij}$ operators, which in turn extend to pseudo-differential operators (ΨDO) on $\mathbb{R}^n$. Namely, if $\Omega''$ is a domain satisfying $\Omega' \subset \subset \Omega'' \subset X$ and $\chi_1$ is a smooth function supported in $X$ which is identically equal to 1 on a neighborhood of $\Omega''$, the operator $\gamma \mapsto \chi_1 dH_{ij}(\chi_1 \gamma)$ can be made a ΨDO upon considering $L_0 = -\nabla \cdot (\gamma_0 \nabla)$ as a second-order operator on $\mathbb{R}^n$ and using standard pseudo-differential parametrices to invert it [13]. We will therefore not distinguish the operators $dH_{ij}$ from their pseudo-differential counterparts. The task of this section is then to determine conditions under which a given collection of such functions becomes an elliptic operator of $\gamma$ over $\Omega'$.

Using relations (4) and (6), we aim at writing the operator $dH_{ij}$ in the following form

$$dH_{ij}(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\xi \cdot (x-y)} M_{ij}(x, \xi) : \gamma(y) \, d\xi \, dy,$$

with symbol $M_{ij}(x, \xi)$ (pseudo-differential terminology is recalled in Sec. 3.1). We first compute the main terms in the symbol expansion of $dH_{ij}$ (call this expansion $M_{ij} = M_{ij}|_0 + M_{ij}|_{-1} + O(|\xi|^{-2})$ with $M_{ij}|_p$ homogeneous of degree $p$ in $\xi$). From these expressions, we then directly deduce microlocal properties on the corresponding operators.

The first lemma shows that the principal symbols $M_{ij}|_0$ can never fully invert for $\gamma$, no matter how many solutions $u_i$ we pick. When Hypothesis 2.1 is satisfied, then the characteristic directions of the principal symbols $\{M_{ij}(x, \xi)\}_{1 \leq i,j \leq n}$ reduce to a $n-1$-dimensional subspace of $S_n(\mathbb{R})$. Here and below, we recall that the colon $\cdot$ denotes the inner product $A : B = \text{tr} (AB^T)$ for $(A, B) \in S_n(\mathbb{R})$ and $\otimes$ denotes the symmetric outer product $U \otimes V = \frac{1}{2}(U \otimes V + V \otimes U)$ for $U, V \in \mathbb{R}^n$.

**Lemma 2.3.**  
(i) For any $i, j$ and $x \in X$, the symbol $M_{ij}|_0$ satisfies

$$M_{ij}|_0 : (\gamma_0 \xi \cdot \eta) = 0, \text{ for all } \eta \in S^{n-1} \text{ satisfying } \eta \cdot \xi = 0. \quad (10)$$

(ii) Suppose that Hypothesis 2.1 holds over some $\Omega \subseteq X$. Then for any $x \in \Omega$, if $P \in S_n(\mathbb{R})$ is such that

$$M_{ij}|_0 : P = 0, \quad 1 \leq i \leq j \leq n,$$

then $P$ is of the form $P = \gamma_0 \xi \cdot \eta$ for some vector $\eta$ satisfying $\eta \cdot \xi = 0$.

Since an arbitrary number of zero-th order symbols can never be elliptic with respect to $\gamma$, we then consider the next term in the symbol expansion of $dH_{ij}$. We must also add one solution $u_{n+1}$ to the initial collection, exhibiting appropriate behavior, i.e. satisfying Hypothesis 2.2. The collection of functionals we consider below is thus of the form

$$dH := \{dH_{ij} \mid 1 \leq i \leq n, \ i \leq j \leq n+1\},$$

and emanates from $n+1$ solutions $(u_1, \ldots, u_{n+1})$ of (3) satisfying Hypotheses 2.1 and 2.2.
In order to formulate the result, we assume to construct a family of unit vector fields  
\[ \hat{\xi}_0(x, \xi) := \hat{A}_0(x)\xi, \hat{\xi}_1(x, \xi), \ldots, \hat{\xi}_{n-1}(x, \xi), \]
homogeneous of degree zero in \( \xi \), smooth in \( x \) and everywhere orthonormal. We then define the family of scalar elliptic zeroth-order \( \PsiDO \) operators 
\[ T : \gamma \mapsto T\gamma = \{ T_{pq}\gamma \}_{0 \leq p \leq q \leq n-1}, \]
where 
\[ T_{pq}\gamma(x) := (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\xi \cdot x} A_0^{-1}\xi_p \odot \xi_q A_0^{-1} : \hat{\gamma}(\xi) \, d\xi, \quad 1 \leq p \leq q \leq n, \]  
which can be thought of as a microlocal change of basis after which the operator \( dH(\gamma) \) becomes both diagonal and elliptic. Indeed, we verify (see section 3.4) that for any \( k \geq 1 \) and \( \gamma \) sufficiently regular, we have 
\[ \| \gamma \|_{H^k(\Omega')} \leq C \| T\gamma \|_{H^k(\Omega')} + C_2 \| \gamma \|_{L^2(\Omega')} \leq C_3 \| \gamma \|_{H^k(\Omega')} \]  
The above estimates come from standard result on pseudo-differential operators [13]. The presence of the constant \( C_3 \) indicates that \( T \) can be inverted microlocally, but may not injective.

Composing the measurements \( dH_{ij} \) with appropriate scalar \( \PsiDO \) of order 0 and 1, we are then able to recover each component of the operator (13). The well-chosen “parametrices” are made possible by the fact that the collection of symbols \( M_{ij}[0 + M_{ij}]_{-1} \) becomes elliptic over \( \Omega' \) when Hypotheses 2.1 and 2.2 are satisfied. Rather than using the full collection of measurements \( dH \) (12), we will consider the smaller collection \( \{ dH_{ij} \}_{1 \leq i, j \leq n} \) augmented with the \( n \) measurement operators 
\[ L_i(\gamma) = \sum_{j=1}^{n} \mu_j \, dH_{ij}(\gamma) + \mu \, dH_{i,n+1}(\gamma), \quad 1 \leq i \leq n, \]  
where \( (\mu_1, \ldots, \mu_n, \mu)(x) \), known from the measurements \( \{ H_{ij} \}_{i,j=1}^{n+1} \), are the coefficients in the relation of linear dependence 
\[ \mu_1 \nabla u_1 + \cdots + \mu_n \nabla u_n + \mu \nabla u_{n+1} = 0. \]
We also define the operator \( L_0^\perp \in \Psi^1 \) with principal symbol \( -\mu \| A_0\xi \| \). Our conclusions may be formulated as follows:

**Proposition 1.** Let the measurements \( dH \) defined in (12) satisfy Hypotheses 2.1 and 2.2.

(i) For \( (\alpha, \beta) = (0, 0) \) and \( 1 \leq \alpha \leq \beta \leq n - 1 \), there exist \( \{ Q_{\alpha\beta ij} \}_{1 \leq i, j \leq n} \in \Psi^0 \) such that 
\[ \sum_{1 \leq i, j \leq n} Q_{\alpha\beta ij} \circ dH_{ij} = T_{\alpha\beta} \mod \Psi^{-1}. \]  
(ii) For any \( 1 \leq \alpha \leq n - 1 \), there exist \( \{ B_{\alpha i} \}_{1 \leq i \leq n} \in \Psi^0 \) such that the following relation holds 
\[ L_0^\perp \circ B_{\alpha i} \circ L_i - R_\alpha \circ R = T_{0\alpha} \mod \Psi^{-1}, \]  
where the remainder \( R_\alpha \circ R \) can be expressed as a zeroth-order linear combination of the components \( T_{0\alpha} \) and \( \{ T_{pq}\} \) reconstructed in (i).
The presence of the $L_k^2$ term in part (ii) of Prop. 1 accounts for the loss of one derivative in the inversion process. From Prop. 1, we can then obtain stability estimates of the form
\[ \|T_{0 \gamma} \|_{H^{k+1}(\Omega')} + \sum_{1 \leq p \leq q \leq n-1} \|T_{pq} \gamma \|_{H^{k+1}(\Omega')} + \sum_{1 \leq p \leq n-1} \|T_{0p} \gamma \|_{H^{k}(\Omega')} \leq C\|dH\|_{H^{k}(\Omega')} + C_2\|\gamma\|_{L^2(\Omega')} \] \hspace{1cm} (18)

The above stability estimate holds for $k = 0$ using the results of Proposition 1 and in fact for any $k \geq 0$ updating by standard methods (not detailed here [13]) the parametrices in (16) and (17) to inversions modulo operators in $\Psi^{-k}$ (i.e., classical $\Psi$DO of order $-k$ [13]) provided that the coefficients $(\gamma_n, \{u_j\})$ are sufficiently smooth. The presence of the constant $C_2$ indicates that the reconstruction of $\gamma$ may be performed up to the existence of a finite dimensional kernel as an application of the Fredholm theory as in [16].

Equation (18) means that some components of $\gamma$ are reconstructed with a loss of one derivative while other components are reconstructed with no loss. The latter components are those that can be spanned by the components $T_{0 \gamma}$ and $\{T_{\alpha \beta}\}_{1 \leq \alpha, \beta \leq n-1}$. Some algebra shows that the only such linear combination is $\sum_{i=0}^{n-1} T_{ii} \gamma$, which, using the fact that $\sum_{i=0}^{n-1} \hat{\xi}_i \otimes \hat{\xi}_i = I_n$, can be computed as
\[ \sum_{i=0}^{n-1} T_{ii} \gamma = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i x \cdot \xi} A_0^{-1} I_n A_0^{-1} : \hat{\gamma}(\xi) \, d\xi = \gamma_0^{-1} : \left( (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i x \cdot \xi} \hat{\gamma}(\xi) \, d\xi \right) = \text{tr} \left( \gamma_0^{-1} \gamma \right), \]
confirming the heuristics of Sec. 2.1. It can be shown that all other components of $\gamma$ (i.e. any part of $\gamma_d$ in (8)) are, to some extent, spanned by the components $T_{0 \gamma}$, and as such cannot be reconstructed with better stability than the loss of one derivative in light of (18). Combining the above results with (14), we arrive at the main stability result of the paper:
\[ \|\text{tr} \left( \gamma_0^{-1} \gamma \right)\|_{H^k(\Omega')} + \|\gamma_d\|_{H^{k-1}(\Omega')} \leq C\|dH\|_{H^k(\Omega')} + C_2\|\gamma\|_{L^2(\Omega')}. \] \hspace{1cm} (19)

Such an estimate holds for any $k \geq 1$.

The above estimate holds with $C_2 = 0$ when $\gamma \mapsto dH(\gamma)$ is an injective (linear) operator. Injectivity cannot be verified by microlocal arguments since all inversions are performed up to smoothing operators; see [3] in the isotropic setting. In the next section, we obtain an injectivity result, which allows us to set $C_2 = 0$ in the above expression. However, the above stability estimate (19) is essentially optimal. An optimal estimate, which follows from the above and the equations for $(\gamma, \{v_j\})$ is the following:
\[ \|M_{[0]} \gamma\|_{H^k(\Omega')} + \|\gamma\|_{H^{k-1}(\Omega')} \leq C\|dH\|_{H^k(\Omega')} + C_2\|\gamma\|_{L^2(\Omega')} \leq C'\left( \|M_{[0]} \gamma\|_{H^k(\Omega')} + \|\gamma\|_{H^{k-1}(\Omega')} \right). \]

The left-hand-side inequality is a direct consequence of (19) and the expression of $dH$. The right-hand side is a direct consequence of the expression of $dH$. The above estimate is clearly optimal. The operator $M_{[0]}$ is of order 0. If it were elliptic, then $\gamma$ would be reconstructed with no loss of derivative. However, $M_{[0]}$ is not elliptic and the loss of ellipticity is precisely accounted for by the results in Lemma 2.3. As
we discussed above, it turns out that the only spatial coefficient controlled by $M_{[0]} \gamma$ is $\text{tr} \left( \gamma_0^{-1} \gamma \right)$, and hence (19).

2.3. Explicit inversion: Now, allowing $\gamma$ to be supported up to the boundary, we present a variation of the non-linear resolution technique used in [17, 18]. First considering $n$ solutions generated by boundary conditions fulfilling Hypothesis 2.1, we establish an expression for $\gamma$ in terms of the remaining unknowns $(v_1, \ldots, v_n)$:

$$
\gamma = \gamma_0 \left( [\nabla U]^T dH^{-1} [\nabla U] - [\nabla V]^T dH^{-1} [\nabla U]^T - [\nabla U]^T dH^{-1} [\nabla V]^T \right) \gamma_0, \quad (20)
$$

where $[\nabla U]$ and $[\nabla V]$ denote $n \times n$ matrices whose $j$-th columns are $\nabla u_j$ and $\nabla v_j$, respectively, and where $H = \{H_{ij}\}_{i,j=1}^n$ and $dH = \{dH_{ij}\}_{i,j=1}^n$. In particular we find from (20) the relation

$$
\text{tr} \left( \gamma_0^{-1} \gamma \right) = \text{tr} \left( H^{-1} dH \right) - 2\text{tr} M, \quad M := ([\nabla U][\nabla U]^T)^{-1}.
$$

Plugging (20) back into the second equation in (1) for $1 \leq i \leq n$, one can deduce a gradient equation for the quantity $\text{tr} \left( \gamma_0^{-1} \gamma \right)$ which in turn allows to reconstruct $\text{tr} \left( \gamma_0^{-1} \gamma \right)$ in a Lipschitz-stable manner with respect to the LPD $\{dH_{ij}\}_{i,j=1}^n$ (i.e. without loss of derivative).

Now turning to the full reconstruction of $\gamma$, we consider an additional solution $u_{n+1}$ generated by a boundary condition fulfilling Hyp. 2.2. The following proposition then establishes how to reconstruct $(v_1, \ldots, v_n)$ from $dH$:

**Proposition 2.** Assume that $(g_1, \ldots, g_{n+1})$ fulfill Hypotheses 2.1 and 2.2 over $X$ and consider the linearized power densities $dH = \{dH_{ij} : 1 \leq i \leq j \leq n + 1, i \neq n + 1\}$. Then the solutions $(v_1, \ldots, v_n)$ satisfy a strongly coupled elliptic system of the form

$$
-\nabla \cdot (\gamma_0 \nabla v_i) + W_{ij} \cdot \nabla v_j = f_i(dH, \nabla(dH)) \quad (X), \quad v_i|_{\partial X} = 0, \quad 1 \leq i \leq n, \quad (22)
$$

where the vector fields $W_{ij}$ are known and only depend on the behavior of $\gamma_0$, $Z$ and $u_1, \ldots, u_n$, and where the functionals $f_i$ are linear in the data $dH_{ij}$.

When the vector fields $W_{ij}$ are bounded, system (22) satisfies a Fredholm alternative from which we deduce that if (22) with a trivial right-hand side admits no non-trivial solution, then $(v_1, \ldots, v_n)$ is uniquely reconstructed from (22). We can then reconstruct $\gamma$ from (20).

**Remark 3** (Case $\gamma_0$ constant). In the case where $\gamma_0$ is constant, choosing solutions as in Remark 1, one arrives at a system of the form (22) where $W_{ij} = 0$ if $i \neq j$, so that the system is decoupled and clearly injective.

The conclusive theorem for the explicit inversion is thus given by

**Theorem 2.4.** Assume that $(g_1, \ldots, g_{n+1})$ fulfill Hypotheses 2.1 and 2.2 over $X$ and consider the linearized power densities $dH = \{dH_{ij} : 1 \leq i \leq j \leq n + 1, i \neq n + 1\}$. Assume further that the system (22) with trivial right-hand sides has no non-trivial solution. Then $\gamma$ is uniquely determined by $dH$ and we have the following stability estimate

$$
\|\text{tr} \left( \gamma_0^{-1} \gamma \right)\|_{H^1(X)} + \|\gamma\|_{L^2(X)} \leq C \|dH\|_{H^1(X)}, \quad (23)
$$
2.4. Outline. We cover the microlocal inversion in Sec. 3. Linear algebraic and pseudo-differential preliminaries are given in Sec. 3.1. The leading-order symbols of order 0 of the LPD functionals are computed in Sec. 3.2 and a proof of Lemma 2.3 is given. The symbols of order −1 are then computed in 3.3 and the proof Proposition 1 is given in Sec. 3.4. We then treat the explicit inversion in Sec. 4. Starting with some preliminaries in Sec. 4.1, we derive some crucial relations in Sections 4.2 and 4.3, before proving Proposition 2 and Theorem 2.4 in Sec. 4.4.

3. Microlocal inversion.

3.1. Preliminaries.

Linear algebra. In the following, we consider the $n \times n$ matrices $M_n(\mathbb{R})$ with the inner product structure

$$A : B = \text{tr} \,(AB^T) = \sum_{i,j=1}^{n} A_{ij} B_{ij},$$  \hspace{1cm} (24)

for which $M_n(\mathbb{R})$ admits the orthogonal decomposition $A_n(\mathbb{R}) \oplus S_n(\mathbb{R})$. For two vectors $U = (u_1, \ldots, u_n)^T$ and $V = (v_1, \ldots, v_n)^T$ in $\mathbb{R}^n$ we denote by $U \otimes V$ the matrix with entries $\{u_i v_j\}_{i,j=1}^{n}$, and we also define the symmetrized outer product

$$U \oslash V := \frac{1}{2}(U \otimes V + V \otimes U).$$  \hspace{1cm} (25)

With $\cdot$ denoting the standard dot product on $\mathbb{R}^n$, we have the following identities

$$2U \oslash V : X \oslash Y = (U \cdot X)(V \cdot Y) + (U \cdot Y)(V \cdot X), \quad U, V, X, Y \in \mathbb{R}^n,$$  \hspace{1cm} (26)

$$U \cdot MU = M : U \oslash U = M : U \oslash U, \quad U \in \mathbb{R}^n, M \in M_n(\mathbb{R}).$$  \hspace{1cm} (27)

Pseudo-differential calculus. Recall that we denote the set of symbols of order $m$ on $X$ by $S^m(X)$, which is the space of functions $p \in C^\infty(X \times \mathbb{R}^n)$ such that for all multi-indices $\alpha$ and $\beta$ and every compact set $K \subset X$ there is a constant $C_{\alpha, \beta, K}$ such that

$$\sup_{x \in K} |D_x^\alpha D_\xi^\beta p(x, \xi)| \leq C_{\alpha, \beta, K}(1 + |\xi|)^{m - |\alpha|}$$

We denote the operator $p(x, D)$ as

$$p(x, D)\gamma(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} p(x, \xi)\hat{\gamma}(\xi) d\xi$$

and the set of pseudo-differential operators (ΨDO) of order $m$ on $X$ by $\Psi^m(X)$, where

$$\Psi^m(X) = \{p(x, D) : p \in S^m(X)\}.$$  \hspace{1cm} (28)

Suppose $\{m_j\}_0^\infty$ is strictly decreasing and $\lim m_j = -\infty$, and suppose $p_j \in S^{m_k}(X)$ for each $j$. We denote an asymptotic expansion of the symbol $p \in S^{m_0}(X)$ as

$$p - \sum_{j<k} p_j \in S^{m_k}(X), \quad \text{for all } k > 0.$$  \hspace{1cm} (29)

Given two ΨDO $P$ and $Q$ with respective symbols $\sigma_P$ and $\sigma_Q$ and orders $d_P$ and $d_Q$, we will make repetitive use of the symbol expansion of the product operator $QP \equiv Q \circ P$ (see [11, Theorem (8.37)]) for instance)

$$\sigma_{QP}(x, \xi) \sim \sigma_Q \sigma_P + \frac{1}{\ell} \nabla_\xi \sigma_Q \cdot \nabla_x \sigma_P + O(|\xi|^{d_Q + d_P - 2}),$$  \hspace{1cm} (30)
Thus equation \( \Psi K \) of \( \Psi DO \) based on formula (2.3) of Guilleman-Bal Chenxi Guo and François Monard allows to claim that the symbol \( Q \) remains valid. In the next derivations, some operators have matrix-valued principal symbols. However, we will only compose them with operators with scalar symbols, so that the above calculus remains valid.

### 3.2. Symbol calculus for the LPD, properties of \( M_{ij} \) and proof of Lemma 2.3.

Writing \( v_i(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \hat{v}_i(\xi) \, d\xi \) and \( \gamma(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \hat{\gamma}(\xi) \, d\xi \) (understood in the componentwise sense), we have

\[
\sigma_{L_0 v_i} := -\nabla \cdot (\gamma_0 \nabla v_i) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \left( \xi \cdot \gamma_0 \xi - \nu(\nabla \cdot \gamma_0) \right) \hat{v}_i(\xi) \, d\xi,
\]

\[
P_i \gamma \equiv \nabla \cdot (\gamma \nabla u_i) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \left( \nu \xi \odot \nabla u_i + \nabla^2 u_i \right) : \hat{\gamma}(\xi) \, d\xi.
\]

Thus equation (4) reads \( L_0 v_i = P_i \gamma \), where the operators \( L_0 := -\nabla \cdot (\gamma_0 \nabla) \) and \( P_i \) have respective symbols

\[
\sigma_{L_0} = l_2 + l_1, \quad l_2 := \xi \cdot \gamma_0 \xi \in S^2 \quad \text{and} \quad l_1 := -\nu(\nabla \cdot \gamma_0) \cdot \xi \in S^1,
\]

(30)

\[
\sigma_{P_i} = p_{i,1} + p_{i,0}, \quad p_{i,1} := \nu \xi \odot \nabla u_i \in (S^1)^{n \times n} \quad \text{and} \quad p_{i,0} := \nabla^2 u_i \in (S^0)^{n \times n}.
\]

(31)

For \( Y \) a smooth vector field, we will also need in the sequel to express the operator \( Y \cdot \nabla \) as \( \Psi DO \), the symbol of which is denoted \( \sigma_{Y \cdot \nabla} = \sigma_{X \cdot \nabla} \mid_{1} := \nu \xi \cdot Y \).

We now write \( dH_{ij} \) as a \( \Psi DO \) of \( \gamma \) with symbol \( M_{ij} \) as in (9). \( dH_{ij} \) belongs to \( \Psi^0(X) \) and we will compute in this paper the first two terms in the expansion of \( M_{ij} \) (call them \( M_{ij} \) and \( M_{ij} \mid_{-1} \)), which in turn relies on constructing parametrices of \( L_0 \) of increasing order and doing some computations on symbols of products of \( \Psi DO \) based on formula (28). If \( Q \) is a parametrix of \( L_0 \) modulo \( \Psi^{-m} \), i.e. \( K \equiv Q L_0 - I d \in \Psi^{-m} \), then straightforward computations based on the relation \( L_0 v_i = P_i \gamma \) yield the following relation

\[
dH_{ij}(\gamma) = \gamma \cdot \nabla u_i \odot \nabla u_j + (\gamma_0 \nabla u_i \cdot \nabla) \circ Q \circ P_j \gamma + (\gamma_0 \nabla u_j \cdot \nabla) \circ Q \circ P_i \gamma + K_{ij} \gamma,
\]

(32)

where

\[
K_{ij} := (\gamma_0 \nabla u_i \cdot \nabla) \circ K L_0^{-1} P_j + (\gamma_0 \nabla u_j \cdot \nabla) \circ K L_0^{-1} P_i.
\]

(33)

For any \( i \), \( L_0^{-1} P_i \) denotes the operator \( \gamma \mapsto v_i \) where \( v_i \) solves (4), and standard elliptic theory allows to claim that \( L_0^{-1} P_i \) smoothes by one derivative so that the error operator \( K_{ij} \) defined in (33) smoothes by \( m \) derivatives. In particular, upon computing a parametrix \( Q \) of \( L_0 \) modulo \( \Psi^{-m} \), the first three terms in (32) are enough to construct the principal part of the symbol \( M_{ij} \) modulo \( \Psi^{-m} \).

Computation of \( M_{ij} \). In light of the last remark, we first compute a parametrix \( Q \) of \( L_0 \) modulo \( \Psi^{-1} \), that is, since \( L_0 \in \Psi^2 \), we look for a principal symbol of the form \( \sigma_Q = q_{-2} + O(|\xi|^{-3}) \). Clearly, we easily obtain \( q_{-2} = l_2^{-1} = (\xi \cdot \gamma_0 \xi)^{-1} \). In
where we have used \( \hat{\xi} \), thus compute the next term in the symbol expansion of the operators \( n \) with respective dimensions \( 1 \), \( M \), so that 

\[ \eta \text{ and } (\eta, \xi) \text{ dependent } n \text{-} 1 \text{-dimensional subspace of symmetric matrices } \gamma_0 \xi \circ \xi. \]

\( M_{ij} \) admits a somewhat more symmetric expression if pre- and post-multiplied by \( A_0 \), the unique positive squareroot of \( \gamma_0 \), so that we may write,

\[ M_{ij}(x, \xi) = A_0^{-1} \left( V_i \circ V_j - (\hat{\xi}_0 \cdot V_i)\hat{\xi}_0 \circ V_j - (\hat{\xi}_0 \cdot V_j)\hat{\xi}_0 \circ V_i \right) A_0^{-1}, \tag{34} \]

where we have defined \( \hat{\xi}_0 \equiv A_0 \xi \) and \( x := |x|^{-1}x \) for any \( x \in \mathbb{R}^n \). We will then show that enough symbols of the form \( \hat{\xi}_0 \cdot \hat{\xi}_0 \) proportional to \( \gamma_0 \), the unique positive squareroot of \( \gamma_0 \), as well as \( \hat{\xi}_0 \) and \( \hat{\xi}_0 \) are orthogonal to the \( (\xi, \xi) \text{ orthogonal to the } (\eta, \eta) \text{ } \)

Proof of Lemma 2.3. Let \( \eta \) such that \( \eta \cdot \xi = 0 \), and denote \( \eta' \equiv A_0^{-1} \eta \) so that \( \eta' \cdot \xi_0 = 0 \). Then using identity (26) and (34), we get

\[ 2||\xi_0||^{-1}M_{ij}(x, \xi) : \gamma_0 \xi \circ \eta \cdots \]

\[ = 2 \left[ V_i \circ V_j - (\hat{\xi}_0 \cdot V_i)\hat{\xi}_0 \circ V_j - (\hat{\xi}_0 \cdot V_j)\hat{\xi}_0 \circ V_i \right] \hat{\xi}_0 \cdot \eta' \]

\[ = (V_i \cdot \hat{\xi}_0)(V_j \cdot \eta') + (V_i \cdot \eta')(V_j \cdot \hat{\xi}_0) \]

\[ - (\hat{\xi}_0 \cdot V_i)(\hat{\xi}_0 \cdot \hat{\xi}_0)(V_j \cdot \eta') - (\hat{\xi}_0 \cdot V_j)(\hat{\xi}_0 \cdot \eta')(V_j \cdot \hat{\xi}_0) \]

\[ - (\hat{\xi}_0 \cdot V_j)(\hat{\xi}_0 \cdot \hat{\xi}_0)(V_i \cdot \eta') - (\hat{\xi}_0 \cdot V_j)(\hat{\xi}_0 \cdot \eta')(\hat{\xi}_0 \cdot \hat{\xi}_0) \]

\[ = 0, \]

where we have used \( \hat{\xi}_0 \cdot \hat{\xi}_0 = 1 \) and \( \hat{\xi}_0 \cdot \eta' = 0 \), thus \( (i) \) holds.

Proof of \( (ii) \): Recall that

\[ M_{ij}(x, \xi) : P = \left[ V_i \circ V_j - (\hat{\xi}_0 \cdot V_i)\hat{\xi}_0 \circ V_j - (\hat{\xi}_0 \cdot V_j)\hat{\xi}_0 \circ V_i \right] : A_0^{-1}PA_0^{-1}. \]

We write \( S_n(\mathbb{R}) \) as the direct orthogonal sum of three spaces:

\[ S_n(\mathbb{R}) = \left[ \mathbb{R} \hat{\xi}_0 \circ \hat{\xi}_0 \right] \oplus \left[ \{\hat{\xi}_0\}^\perp \circ \{\hat{\xi}_0\}^\perp \right] \oplus \left[ \hat{\xi}_0 \circ \hat{\xi}_0 \right], \tag{35} \]

with respective dimensions \( 1 \), \( n(n-1)/2 \) and \( n-1 \). Decomposing \( A_0^{-1}PA_0^{-1} \) uniquely into this sum, we write \( A_0^{-1}PA_0^{-1} = P_1 + P_2 + P_3 \). Direct calculations then show that

\[ M_{ij}(x, \xi) : A_0^{-1}PA_0^{-1} = V_i \circ V_j : (-P_1 + P_2), \quad 1 \leq i \leq j \leq n. \]

Since \( \{V_i\}_{i=1}^n \) is a basis of \( \mathbb{R}^n \), \( \{V_i \circ V_j\}_{1 \leq i \leq j \leq n} \) is a basis of \( S_n(\mathbb{R}) \) and thus (11) implies that

\[ -P_1 + P_2 = 0, \quad \text{i.e.} \quad P_1 = P_2 = 0. \]

Therefore \( P = A_0P_3A_0 \) with \( P_3 = \hat{\xi}_0 \circ \eta' \) for some \( \eta' \cdot \hat{\xi}_0 = 0 \), so \( P = \gamma_0 \xi \circ \eta \\ with \eta \proportional to A_0 \eta', \i.e. such that \eta \cdot \xi = 0 \), thus the proof is complete.

In other words, all symbols of order zero \( M_{ij}\) are orthogonal to the \( (x, \xi)-dependent \) \( (x, \xi)-dimensional subspace of symmetric matrices \( \gamma_0 \xi \circ \xi \). One must thus compute the next term in the symbol expansion of the operators \( dH_{ij} \), i.e. \( M_{ij} \). We will then show that enough symbols of the form \( M_{ij}(x, \xi) \) will suffice to span the entire space \( S_n(\mathbb{R}) \) for every \( x \in \Omega' \) and \( \xi \in \mathbb{S}^1 \), so that the corresponding family of operators is elliptic as a function of \( \gamma \).
3.3. Computation of $M_{ij}|_{-1}$. As the previous section explained, the principal symbols $M_{ij}|_{0}$ can never span $S_n^i(\mathbb{R})$. Therefore, we compute the next term $M_{ij}|_{-1}$ in their symbol expansion. We must first construct a parametrix $Q$ of $L_0$ modulo $\Psi^{-2}$, i.e. of the form
\[
\sigma_Q = q_{-2} + q_{-3} + \mathcal{O}(\langle \xi \rangle^{-4}), \quad q_i \in S^i.
\] (36)

**Lemma 3.1.** The symbols $q_{-2}$ and $q_{-3}$ defined in (36) have respective expressions
\[
q_{-2} = l_2^{-1} = (\xi \cdot \gamma_0 \xi)^{-1},
\]
\[
q_{-3} = l_2^{-3} \xi p \xi q \xi_j \left( [\gamma_0]_{pq} \partial_{x_i} [\gamma_0]_{ij} - 2 [\gamma_0]_{ij} \partial_{x_i} [\gamma_0]_{pq} \right).
\] (37) (38)

**Proof of Lemma 3.1.** Using formula (28) with $(Q,P) = (Q,L_0)$, and using the expansions of $\sigma_Q$ and $\sigma_{L_0}$, we get
\[
\sigma_{Q_{L_0}} \sim q_{-2} l_2 + (q_{-2} l_1 + q_{-3} l_2 + \frac{1}{\iota} \nabla \xi q_{-2} \cdot \nabla x l_2) + \mathcal{O}(\langle \xi \rangle^{-2}),
\]
In order to match the expansion $1 + 0 + \mathcal{O}(\langle \xi \rangle^{-2})$, the expansion above must satisfy, for large $\xi$,
\[
q_{-2} l_2 = 1 \quad \text{and} \quad q_{-2} l_1 + q_{-3} l_2 + \frac{1}{\iota} \nabla \xi q_{-2} \cdot \nabla x l_2 = 0,
\]
that is, $q_{-2} = l_2^{-1} = (\xi \cdot \gamma_0 \xi)^{-1}$ and
\[
q_{-3} = l_2^{-1} \left( -q_{-2} l_1 - \frac{1}{\iota} \nabla \xi q_{-2} \cdot \nabla x l_2 \right) = l_2^{-3} (-l_2 l_1 - \iota \nabla \xi l_2 \cdot \nabla x l_2).
\]

Now, we easily have $\nabla \xi l_2 = 2 \gamma_0 \xi$ and $\nabla x l_2 = \partial_{x_i} [\gamma_0]_{pq} \xi_p \xi_q e_i$, where $e_1, \ldots, e_n$ is the natural basis of $\mathbb{R}^n$. We thus deduce the expression of $q_{-3}$
\[
q_{-3} = l_2^{-3} \iota \left( [\gamma_0]_{pq} \xi_p \xi_q \partial_{x_i} [\gamma_0]_{ij} \xi_j - 2 [\gamma_0]_{ij} \xi_j \partial_{x_i} [\gamma_0]_{pq} \xi_p \xi_q \right),
\]
from which (38) holds. $q_{-3}$ is clearly in $S^{-3}$ from this expression, since $l_2^{-3}$ is of order $-6$. The proof is complete.

We now give the expression of $M_{ij}|_{-1}$ (or rather, that of $A_0 M_{ij}|_{-1} A_0$).

**Proposition 3** (Expression of $A_0 M_{ij}|_{-1} A_0$). The symbol $A_0 M_{ij}|_{-1} A_0$ admits the following expression for any $(i,j)$
\[
A_0 M_{ij}|_{-1}(x,\xi) A_0 = \iota \| \xi_0 \|^{-1} \left( (\xi_0 \cdot V_i) (H_i - 2 \xi_0 \otimes H_i \xi_0) + \xi_0 \otimes H_i V_i \right) + \iota \| \xi_0 \|^{-1} \left( (\xi_0 \cdot V_j) (H_j - 2 \xi_0 \otimes H_j \xi_0) + \xi_0 \otimes H_j V_i \right) + \iota \| \xi_0 \|^{-1} \left( V_j \cdot G(x,\xi) (\xi_0 \cdot V_i) + V_i \cdot G(x,\xi) (\xi_0 \cdot V_j) \right),
\] (39)

where we have defined $V_i := A_0 \nabla u_i$, $H_i := A_0 \nabla^2 u_i A_0$, as well as the vector field
\[
G(x,\xi) := \| \xi_0 \|^2 (\iota q_{-3} \xi_0 + A_0 \nabla x q_{-2}) \in (S^0)^n.
\] (40)

**Proof of Prop. (39).** Assume $Q$ is a parametrix of $L_0$ modulo $\Psi^{-2}$ and consider formula (32). Since the term $\gamma : \nabla u_i \otimes \nabla u_j$ is of order zero, the computation of $M_{ij}|_{-1}$ consists in computing the second term in the symbol expansion of $R_i \circ Q \circ P_j$, and the same term with $i,j$ permuted, where we denote $R_i := \gamma_0 \nabla u_i \cdot \nabla$ with symbol $r_{i,1} = \iota \gamma_0 \nabla u_i \cdot \xi$. Plugging $\sigma_{R_i} = r_{i,1}$, $\sigma_Q = q_{-2} + q_{-3}$ and $\sigma_{P_i} = p_{i,1} + p_{i,0}$ into
and keeping only the terms that are homogeneous of degree −1 in $\xi$, we arrive at the expression

$$\sigma_{R_i,QP_i}|_{-1} = r_{i,1}(q-3p_{j,1} + q-2p_{j,0}) + \frac{1}{\xi}(p_{j,1}\nabla_{\xi}r_{i,1} \cdot \nabla_{x}q - 2 + \nabla_{\xi}(q-2r_{i,1}) \cdot \nabla_{x}p_{j,1}).$$

(41)

Note that the multiplications commute because the symbols of $Q$ and $R_i$ are scalar, while that of $P_j$ is matrix-valued. Since $M_{ij}|_{-1} = \sigma_{R_i,QP_i}|_{-1} + \sigma_{R_j,QP_j}|_{-1}$, equation (39) will be proved when we show that

$$A_0 \sigma_{R_i,QP_i}|_{-1} A_0 = \iota\|\xi_0\|^{-1} \left((\hat{\xi}_0 \cdot V_i)(H_j - 2\hat{c}_0 \circ H_j)\hat{c}_0 + \hat{\xi}_0 \circ H_j V_i + V_i \cdot G(x,\xi)(\hat{\xi}_0 \circ V_j)\right).$$

(42)

**Proof of (42).** Starting from (41), plugging the expression $r_{i,1} = \iota(V_i \cdot \xi_0)$, using the identity

$$\nabla_{\xi}(q-2r_{i,1}) \cdot \nabla_{x}p_{j,1} = \iota \xi \circ (\nabla^2 u_j \nabla_{\xi}(q-2r_{i,1})), $$

and pre- and post-multiplying by $A_0$ yields the relation

$$A_0 \sigma_{R_i,QP_i}|_{-1} A_0 = \iota(V_i \cdot \xi_0)(q-3u_0 \circ V_j + q-2H_j) + (V_i \cdot A_0 \nabla_{x}q - 2)\iota\xi_0 \circ V_j + \xi_0 \circ H_j A_0^{-1}\nabla_{\xi}(q-2r_{i,1}).$$

(43)

Gathering the first and third terms recombines into $\iota\|\xi_0\|^{-1}V_i \cdot G(\xi_0 \circ V_j)$ (the last term of (42)). On to the second and fourth terms, we first compute

$$A_0^{-1}\nabla_{\xi}(r_{i,1}q - 2) = \iota(V_i \cdot \xi_0)(-\|\xi_0\|^{-2})2\xi_0 + \|\xi_0\|^{-2}\iota V_i = \iota\|\xi_0\|^{-2}(V_i - 2(V_i \cdot \hat{\xi}_0)\hat{c}_0).$$

Using this calculation, the second and fourth terms in (43) recombine into

$$\iota\|\xi_0\|^{-1} \left((\hat{\xi}_0 \cdot V_i)(H_j - 2\hat{c}_0 \circ H_j)\hat{c}_0 + \hat{\xi}_0 \circ H_j V_i\right),$$

thus the argument is complete. \hfill $\square$

3.4. **Proof of Proposition 1.** **Preliminaries:** By virtue of Hypothesis 2.1, $\nabla u_{n+1}$ may be decomposed into the basis $\nabla u_1, \ldots, \nabla u_n$ by means of scalars $\mu_1, \ldots, \mu_n, \mu$ such that

$$\sum_{i=1}^{n} \mu_i \frac{\mu_0}{\mu} \nabla u_i + \nabla u_{n+1} = 0.$$ 

(44)

As seen in [17, 18], the coefficients $\mu_1, \ldots, \mu_{n+1}$ are directly computable from the power densities $\{\nabla u_i, \gamma_0 \nabla u_j\}_{1 \leq i \leq j \leq n+1}$ and on the other hand, we have the relation

$$\frac{\mu_i}{\mu} = \frac{\det(\nabla u_1, \ldots, \nabla u_{n+1}, \ldots, \nabla u_n)}{\det(\nabla u_1, \ldots, \nabla u_m)}, \quad 1 \leq i \leq n,$$

thus $\nabla\mu = Z_i$ as defined in (7) for $1 \leq i \leq n$. In the next proofs, we will use the following

**Lemma 3.2.** **Under hypotheses 2.1 and 2.2, the following matrix-valued function**

$$\mathbb{M} := \mu_i H_i + \mu H_{n+1}$$

(45)

**is symmetric and uniformly invertible.**
Proof. Symmetry of $\mathcal{M}$ is obvious by definition. Taking gradient of (44), we arrive at
\[ \sum_{i=1}^{n} Z_i \otimes \nabla u_i + \frac{\mu_i}{\mu} \nabla^2 u_i + \nabla^2 u_{n+1} = 0. \]

Pre- and post-multiplying by $A_0$, we deduce that
\[ \mathcal{M} = \mu_i \mathbb{H}_i + \mu \mathbb{H}_{n+1} = -\mu A_0 Z_i \otimes V_i = -\mu A_0 Z V^T, \]
where $V := [V_1, \ldots, V_n]$. The proof is complete since Hyp. 2.1 ensures that $\mu$ never vanishes and $V$ is uniformly invertible, and Hyp. 2.2 ensures that $Z$ is uniformly invertible.

The $T_{pq}$ operators. Proof of Prop. 1: As advertised in Sec. 2.2, because of the algebraic form of the symbols of the linearized power density operators, it is convenient for inversion purposes to define the microlocal change of basis $T \gamma = \{T_{pq} \gamma\}_{1 \leq p, q \leq n}$ as in (13), i.e.
\[ T_{pq} \gamma(x) := (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\xi \cdot x} A^{-1}_0 A_{0}^{-1} \xi_p \otimes \xi_q A^{-1}_0 : \gamma(\xi) \ d\xi. \]

To convince ourselves that this collection forms a microlocally invertible operator of $\gamma$, let us introduce the zero-th order $\Psi$DOs $P_{ijpq}$ with scalar principal symbol $\sigma_{P_{ijpq}} := (e_i \cdot A_0 \xi_p) (e_j \cdot A_0 \xi_q)$ for $1 \leq i, j, p, q \leq n$. Then for any $1 \leq i \leq j \leq n$, the composition of operators $\sum_{p,q=1}^{n} P_{ijpq} \circ T_{pq}$ has principal symbol (repeated indices are summed over)
\[ (e_i \cdot A_0 \xi_p) (e_j \cdot A_0 \xi_q) A^{-1}_0 \xi_p \otimes \xi_q A^{-1}_0 = (e_i \cdot A_0 \xi_p) A^{-1}_0 \xi_p \otimes (e_j \cdot A_0 \xi_q) A^{-1}_0 \xi_q = e_i \otimes e_j, \]
where we have used the following property, true for any smooth vector field $V$:
\[ V = (V \cdot A_0 \xi_p) A^{-1}_0 \xi_p. \]

Thus for any $1 \leq i \leq j \leq n$, the composition $\sum_{p,q=1}^{n} P_{ijpq} \circ T_{pq}$ recovers $\gamma_{ij} = \gamma : e_i \otimes e_j$ up to a regularization term. This in particular justifies the estimates (14) and the subsequent inversion procedure. We are now ready to prove Proposition 1.

Proof of Proposition 1. From the fact that $(V_1, \ldots, V_n)$ is a basis at every point and given their dotproducts $H_{ij} = V_i \cdot V_j$, we have the following formula, true for every vector field $W$:
\[ W = H^{pq}(W \cdot V_p)V_q. \quad (46) \]

Proof of (i): Reconstruction of the components $T_{00} \gamma$ and $\{T_{n\beta} \gamma\}_{1 \leq n \leq \beta \leq n-1}$. We work with $M_{ij} := A_0 M_{ij} |_{0} = V_i \otimes V_j - (\xi_0 \cdot V_i) \xi_0 \otimes V_j - (\xi_0 \cdot V_j) \xi_0 \otimes V_i$. Using (46) with $W \equiv \xi_\alpha$, straightforward computations yield
\[ \sum_{i,j,p,q} H^{ij}(\xi_\alpha \cdot V_q)H^{pq}(\xi_\beta \cdot V_p)M_{ij} = \hat{\xi}_\alpha \otimes \hat{\xi}_\beta - (\xi_0 \cdot \hat{\xi}_\alpha) \xi_0 \otimes \hat{\xi}_\beta - (\xi_0 \cdot \hat{\xi}_\beta) \xi_0 \otimes \hat{\xi}_\alpha = \begin{cases} \hat{\xi}_0 \otimes \hat{\xi}_0 & \text{if} \quad \alpha = \beta = 0, \\ 0 & \text{if} \quad 0 = \alpha \neq \beta, \\ \hat{\xi}_\alpha \otimes \hat{\xi}_\beta & \text{if} \quad \alpha \neq 0, \beta \neq 0. \end{cases} \]
which means that upon defining $Q_{\alpha \beta i j} \in \Psi^0$ with scalar principal symbols
\[
\sigma_{Q_{\alpha \beta i j}} := - \sum_{p,q} H^{qj}(\hat{\xi}_0 \cdot V_q) H^{pi}(\hat{\xi}_0 \cdot V_p),
\]
\[
\sigma_{Q_{\alpha i j}} := \sum_{p,q} H^{qj}(\hat{\xi}_\alpha \cdot V_q) H^{pi}(\hat{\xi}_\beta \cdot V_p), \quad 1 \leq \alpha \leq \beta \leq n - 1,
\]
relation (16) is satisfied in the sense of operators since the previous calculation amounts to computing the principal symbol of the composition of operators in (16).

**Proof of (ii): Reconstruction of the components** \{T_{0\alpha \gamma}\}_{1 \leq \alpha \leq n - 1}. It remains

to construct appropriate operators that will map $dH(\gamma)$ to the components $T_{0\alpha \gamma}$ for $1 \leq \alpha \leq n - 1$, which is where the additional measurements $dH_{i,n+1}$ come into play. Let $(\mu_1, \ldots, \mu_n, \mu)$ as in (44) and construct the \(\Psi\)DO \{\(L_i(\gamma)\)\}_{i=1} as in (15). It is easy to see that, since the \(\mu_i\) are only functions of \(x\), the terms of fixed homogeneity in the symbol expansion of \(L_i\) satisfy
\[
\sigma_{L_i}|_k = \mu_j M_{i|k} + \mu M_{i,n+1}|k, \quad k = 0, -1, -2, \ldots.
\]
Then from equation (34) and relation (44), we deduce that \(\sigma_{L_i}|_0 = 0\), so that \(L_i \in \Psi^{-1}\). Moreover, using equation (39) together with relation (44), we deduce that
\[
\hat{\sigma}_{L_i}|_{-1} = A_0 \sigma_{L_i}|_{-1} A_0 = \|\xi_0\|^{-1} \left( (\hat{\xi}_0 \cdot V_i)(\hat{M} - 2\hat{\xi}_0 \circ \hat{M}_0) + \hat{\xi}_0 \circ V_i \hat{M}_i \right)
\]
is now the principal symbol of \(L_i\). Using relation (46) with \(W \equiv \hat{M}^{-1}\hat{\xi}_\alpha\), the symmetry of \(\hat{M}\) and multiplying by \(\hat{M}_i\), we have the relation
\[
\hat{\xi}_\alpha = H^{pi}(\hat{\xi}_\alpha \cdot \hat{M}^{-1}V_p) MV_i.
\]
Using this relation, we deduce the following calculation, for $1 \leq \alpha \leq n - 1
\[
H^{pi}(\hat{\xi}_\alpha \cdot \hat{M}^{-1}V_p) \hat{\sigma}_{L_i}|_{-1} = \|\xi_0\|^{-1} \left( (\hat{\xi}_0 \cdot \hat{M}^{-1}\hat{\xi}_\alpha)(\hat{M} - 2\hat{\xi}_0 \circ \hat{M}_0) + \hat{\xi}_0 \circ \hat{\xi}_\alpha \right).
\]
(47)

While the second term gives us the missing components $T_{0\alpha \gamma}$, we claim that the
first one is spanned by $\hat{\xi}_0 \circ \hat{\xi}_0$ and $(\hat{\xi}_\alpha \circ \hat{\xi}_\beta)_{1 \leq \alpha \leq \beta \leq n - 1}$. Indeed we have
\[
(\hat{M} - 2\hat{\xi}_0 \circ \hat{M}_0) : (\hat{\xi}_0 \circ \hat{\xi}_0) = 0, \quad 1 \leq \alpha \leq n - 1,
\]
\[
(\hat{M} - 2\hat{\xi}_0 \circ \hat{M}_0) : (\hat{\xi}_0 \circ \hat{\xi}_\alpha) = -\hat{\xi}_0 \cdot \hat{M}_0 \hat{\xi}_0,
\]
\[
(\hat{M} - 2\hat{\xi}_0 \circ \hat{M}_0) : (\hat{\xi}_\alpha \circ \hat{\xi}_0) = -\hat{\xi}_\alpha \cdot \hat{M}_0 \hat{\xi}_0,
\]
\[
(\hat{M} - 2\hat{\xi}_0 \circ \hat{M}_0) : (\hat{\xi}_\alpha \circ \hat{\xi}_\beta) = \hat{\xi}_\alpha \cdot \hat{M}_0 \hat{\xi}_\beta, \quad 1 \leq \alpha \leq \beta \leq n - 1,
\]
so we deduce that
\[
\hat{M} - 2\hat{\xi}_0 \circ \hat{M}_0 = - (\hat{\xi}_0 \cdot \hat{M}_0 \hat{\xi}_0) \hat{\xi}_0 \circ \hat{\xi}_0 + \sum_{1 \leq \alpha, \beta \leq n - 1} (\hat{\xi}_\alpha \cdot \hat{M}_0 \hat{\xi}_\beta) \hat{\xi}_0 \circ \hat{\xi}_\beta.
\]
(48)

In light of these algebraic calculations, we now build the parametrices. Let
$L_\phi^{\gamma} \in \Psi^1$, $B_\alpha \in \Psi^0$, $R \in (\Psi^0)^{n \times n}$, $R_\alpha \in \Psi^0$ and $R_{\alpha \beta} \in \Psi^0$ the \(\Psi\)DOs with respective principal symbols
\[
\sigma_{L_\phi^{\gamma}} = -\|\xi_0\|, \quad \sigma_{B_\alpha} = H^{pi}(\hat{\xi}_\alpha \cdot \hat{M}^{-1}V_p), \quad \sigma_R = \hat{M} - 2\hat{\xi}_0 \circ \hat{M}_0, \\
\sigma_{R_\alpha} = \hat{\xi}_0 \cdot \hat{M}^{-1}\hat{\xi}_\alpha, \quad \sigma_{R_{\alpha \beta}} = \hat{\xi}_\alpha \cdot \hat{M}_0 \hat{\xi}_\beta.
Then the relation (47) implies (17) at the principal symbol level. The operator $R$ can indeed be expressed as the following zero-th order linear combination of the components $T_{00}$ and $\{T_{\alpha\beta}\}_{1 \leq \alpha, \beta \leq n-1}$:

$$R = -R_{00}T_{00} + \sum_{1 \leq \alpha, \beta \leq n-1} R_{\alpha\beta}T_{\alpha\beta}$$

$$= \sum_{i,j=1}^{n} \left( -R_{00}Q_{00ij} + \sum_{1 \leq \alpha, \beta \leq n-1} R_{\alpha\beta}Q_{\alpha\beta ij} \right) \circ dH_{ij} \mod \Psi^{-1},$$

so that the left-hand side of (17) is expressed as a post-processing of measurement operators $dH_{ij}$ only. The proof is complete. \hfill \qed

4. Explicit inversion.

4.1. Preliminaries and notation. For a matrix $A$ with columns $A_1, \ldots, A_n$ and $(e_1, \ldots, e_n)$ the canonical basis, one has the following representation

$$A = \sum_{j=1}^{n} A_j \otimes e_j \quad \text{and} \quad A^T = \sum_{j=1}^{n} e_j \otimes A_j.$$  

More generally, for two matrices $A = [A_1|\ldots|A_n]$ and $B = [B_1|\ldots|B_n]$, we have the relation

$$\sum_{j=1}^{n} A_j \otimes B_j = AB^T.$$  

Finally, for $A$ a matrix and $V = [V_1|\ldots|V_n]$, the sum $A_{ij}V_j$ is nothing but the $i$-th column of the matrix $VA^T$.

4.2. Derivation of (20) from Hypothesis 2.1: Let us start from $n$ solutions $(u_1, \ldots, u_n)$ fulfilling Hypothesis 2.1, and let $(v_1, \ldots, v_n)$ the corresponding solutions of (4). We also denote $[\nabla U] := [\nabla u_1|\ldots|\nabla u_n]$ and $[\nabla V]$ similarly. We first mention that for any vector field $V$, we have the following formulas

$$V = H^{pq}(V \cdot \gamma_0 \nabla u_p) \nabla u_q = H^{pq}(V \cdot \nabla u_p)\gamma_0 \nabla u_q, \quad (49)$$

which also amounts to the following matrix relations

$$H^{pq}(\nabla u_p \otimes \nabla u_q)\gamma_0 = H^{pq}\gamma_0(\nabla u_p \otimes \nabla u_q) = I_n, \quad (50)$$

From the relation

$$dH_{ij} = (\gamma \nabla u_i + \gamma_0 \nabla v_i) \cdot \nabla u_j + \gamma_0 \nabla v_j \cdot \nabla u_i, \quad 1 \leq i, j \leq n,$$

we deduce, using (49),

$$\gamma \nabla u_i + \gamma_0 \nabla v_i = H^{pq}(dH_{ip} - \gamma_0 \nabla v_p \cdot \nabla u_i)\gamma_0 \nabla u_q, \quad 1 \leq i \leq n. \quad (51)$$

The previous equation allows us express $\gamma$ in terms of the remaining unknowns $(v_1, \ldots, v_n)$. Indeed, taking the tensor product of (51) with $H^{ij}\gamma_0 \nabla u_j$ and summing over $i$ yields

$$\gamma + \gamma_0 \nabla v_i \otimes \nabla u_j \gamma_0 H^{ij} = H^{pq}(dH_{ip} - \gamma_0 \nabla v_p \cdot \nabla u_i) (\gamma_0 \nabla u_q \otimes \nabla u_j \gamma_0 H^{ij})$$

$$= dH_{ip}\gamma_0(H^{pq} \nabla u_q \otimes H^{ij} \nabla u_j)\gamma_0 - \gamma_0 \nabla u_q \otimes \nabla v_p \gamma_0 H^{pq},$$
where we have used the identity (49) in the last right-hand side. We may rewrite this as
\[
\gamma = \gamma_0 \left( dH_{ip}(H^{pq}\nabla u_q \otimes H^{ij} \nabla u_j) - 2H^{ij} \nabla v_i \otimes \nabla u_j \right) \gamma_0.
\] (52)
One may notice that the above expression is indeed a symmetric matrix. In matrix notation, using the preliminaries, we arrive at the expression (20).

4.3. Algebraic equations obtained by considering additional solutions:
Let us now add another solution $u_{n+1}$ with corresponding solution $v_{n+1}$ at order $O(\varepsilon)$. By virtue of Hypothesis 2.1, as in section 3.4, $\nabla u_{n+1}$ may be expressed in the basis $(\nabla u_1, \ldots, \nabla u_n)$ as
\[
\sum_{i=1}^{n} \frac{\mu_i}{\mu} \nabla u_i + \nabla u_{n+1} = 0,
\] (53)
where the coefficients $\mu_i$ can be expressed as ratios of determinants, or equivalently, computable from the power densities at order $\varepsilon^0$, see [17, Appendix A.3]. For $1 \leq i \leq n$, we define $Z_i := \nabla(\mu^{-1}\mu_i)$, and notice that we have the following two algebraic relations
\[
\sum_{i=1}^{n} Z_i \cdot \gamma_0 \nabla u_i = 0 \quad \text{and} \quad \sum_{i=1}^{n} Z_i^i \cdot du_i = 0.
\] (54)
The first one is obtained after applying the operator $\nabla \cdot (\gamma_0 \cdot \cdot)$ to (53) and the second one is obtained after applying an exterior derivative to (53).

Moving on to the study of the corresponding $v_{n+1}$ solution, we write
\[
dH_{n+1,j} + \frac{\mu_i}{\mu} dH_{ij} = \left( \nabla v_{n+1} + \frac{\mu_i}{\mu} \nabla v_i \right) \cdot \gamma_0 \nabla u_j, \quad 1 \leq j \leq n,
\]
where we have cancelled sums of the form (53). Using the identity (49), we deduce that
\[
\nabla v_{n+1} + (\mu^{-1}\mu_i) \nabla v_i = H^{pq} \left( dH_{n+1,p} + (\mu^{-1}\mu_i)dH_{ip} \right) \nabla u_q.
\] (55)
Taking exterior derivative of the previous relation yields
\[
Z_i^i \cdot du_i = d \left( H^{pq} \left( dH_{n+1,p} + (\mu^{-1}\mu_i)dH_{ip} \right) \right) \cdot du_q.
\] (56)
We now apply $\nabla \cdot (\gamma_0 \cdot \cdot)$ to (55), the left-hand side becomes
\[
\nabla \cdot (\gamma_0(\nabla v_{n+1} + (\mu^{-1}\mu_i)\nabla v_i)) \ldots
\]
\[
= \nabla \cdot (\gamma_0 \nabla v_{n+1}) + Z_i \cdot \gamma_0 \nabla v_i + (\mu^{-1}\mu_i) \nabla \cdot (\gamma_0 \nabla v_i)
\]
\[
= -\nabla \cdot (\gamma \nabla u_{n+1}) + Z_i \cdot \gamma_0 \nabla v_i - (\mu^{-1}\mu_i) \nabla \cdot (\gamma \nabla v_i)
\]
\[
= Z_i \cdot \gamma_0 \nabla v_i - \nabla \cdot (\gamma(\nabla u_{n+1} + (\mu^{-1}\mu_i)\nabla u_i)) + Z_i \cdot \gamma \nabla u_i
\]
\[
= Z_i \cdot (\gamma_0 \nabla v_i + \gamma \nabla u_i),
\]
thus we arrive at the equation
\[
Z_i \cdot (\gamma_0 \nabla v_i + \gamma \nabla u_i) = \nabla \left( H^{pq} \left( dH_{n+1,p} + (\mu^{-1}\mu_i)dH_{ip} \right) \right) \cdot \gamma_0 \nabla u_q =: Y_q \cdot \gamma_0 \nabla u_q,
\]
where the vector fields
\[
Y_q := \nabla \left( H^{pq} \left( dH_{n+1,p} + (\mu^{-1}\mu_i)dH_{ip} \right) \right), \quad 1 \leq q \leq n,
\] (57)
are known from the data $dH$. Combining the latter equation with (51), we obtain
\[
(Z_i \cdot \gamma_0 \nabla u_q)H^{pq} \left( dH_{ip} - \gamma_0 \nabla v_p \cdot \nabla u_i \right) = Y_q \cdot \gamma_0 \nabla u_q,
\]
which we recast as

\[(Z_i \cdot \gamma_0 \nabla u_q)H^{pq}(\gamma_0 \nabla v_p \cdot \nabla u_i) = (Z_i \cdot \gamma_0 \nabla u_q)H^{pq} dH_{ip} - Y_q \cdot \gamma_0 \nabla u_q.\]

The left-hand side can be considerably simplified by noticing that the second equation of (54) implies \[\nabla U Z^T = Z [\nabla U]^T.\] With this fact in mind, the left-hand side looks like \(X_p \cdot \nabla v_p\), where we compute

\[X_p = H^{pq} \gamma_0 \nabla u_i \otimes Z_i \gamma_0 \nabla u_q = \gamma_0 \nabla U Z^T \gamma_0 \nabla U^{-1} e_p \]

\[= \gamma_0 Z [\nabla U]^T [\nabla U]^{-T} e_p = \gamma_0 Z_p.\]

Finally, we obtain the more compact equation

\[\sum_{p=1}^{n} \gamma_0 Z_p \cdot \nabla v_p = f, \quad \text{where} \quad f := (H^{pq} dH_{ip} Z_i - Y_q) \cdot \gamma_0 \nabla u_q, \quad (58)\]

with \(Y_q\) given in (57).

**Remark 4** (On algebraic inversion). In equations (56) and (58), the only unknown is the matrix \([\nabla V] := [\nabla v_1, \ldots, \nabla v_n]\). Equations (56) and (58) give us the projection of that matrix onto the space \(Z A_n(\mathbb{R})\) and onto the line \(\mathbb{R} \gamma_0 Z\) respectively. As in the non-linear case [17, 18], we expect that a rich enough set of such equations provided by a certain number of additional solutions \((u_{n+1}, \ldots, u_{n+l})\) leads to a pointwise, algebraic reconstruction of \([\nabla V]\), however we do not follow that route here.

### 4.4. Proof of Proposition 2 and Theorem 2.4.

We now show that provided that we use one additional solution \(u_{n+1}\) (on top of the basis \((u_1, \ldots, u_n)\)) such that the matrix \(Z\) is of full rank, then we can reconstruct \((v_1, \ldots, v_n)\) via a strongly coupled elliptic system of the form (22), after which we can reconstruct \(\gamma\) from \((\nabla v_1, \ldots, \nabla v_n)\) by formula (20). We now show how to derive this elliptic system.

**Proof of Proposition 2.** According to Hypothesis 2.2, the matrix \(Z = [Z_1| \ldots |Z_n]\) has full rank and we recall the important equations

\[\sum_{p=1}^{n} \gamma_0 Z_p \cdot \nabla v_p = f \quad \text{and} \quad \sum_{i=1}^{n} Z^i_\gamma \wedge dv_i = \omega, \quad \text{where} \quad (59)\]

\[\omega = Y_\gamma \wedge du_q, \quad Y_q := \nabla (H^{pq} (dH_{n+1,p} + (\mu^{-1} \mu_i) dH_{ip})), \quad (60)\]

and where \(f\) is given in (58). Assuming that \(Z\) has full rank, the family \((Z_1, \ldots, Z_n)\) is a frame with dotproducts defined as \(\Xi_{ij} = Z_i \cdot Z_j\), and in this case we define its dual frame \(Z^*_\gamma := \Xi^{ij} Z^*_j\) for \(1 \leq i \leq n\), such that \(Z^*_i \cdot Z^*_j = \delta_{ij}\), i.e. with \(Z^*\) the matrix with columns \(Z^*_i\), we have the relation \(Z^* = Z^{-T}\). The second equation of (59) may be rewritten as

\[Z^*_q \cdot \nabla v_p - Z^*_p \cdot \nabla v_q = \omega(Z^*_p, Z^*_q), \quad 1 \leq p, q \leq n. \quad (61)\]

Applying the differential operator \(Z^*_i \cdot \nabla\) to the first equation of (59), we obtain

\[\sum_{p=1}^{n} (Z^*_i \cdot \nabla)(\gamma_0 Z_p \cdot \nabla) v_p = (Z^*_i \cdot \nabla)f. \quad (62)\]
Using (61), we may rewrite the left-hand side of (62) as
\[
(Z_i^* \cdot \nabla)(\gamma_0 Z_p \cdot \nabla)v_p = [Z_i^*, \gamma_0 Z_p] \cdot \nabla v_p + (\gamma_0 Z_p \cdot \nabla)(Z_i^* \cdot \nabla)v_p
= [Z_i^*, \gamma_0 Z_p] \cdot \nabla v_p + (\gamma_0 Z_p \cdot \nabla)(Z_i^* \cdot \nabla)v_i \ldots
+ (\gamma_0 Z_p \cdot \nabla)(\omega(Z_p^*, Z_i^*))
\]
where we have introduced the Lie bracket of two vector fields, which may be written in the Euclidean connection
\[
[X, Y] := (X \cdot \nabla)Y - (Y \cdot \nabla)X.
\]
Plugging the last calculation into (62) (repeated indices are summed over)
\[
(\gamma_0 Z_p \cdot \nabla)(Z_p^* \cdot \nabla)v_i + [Z_i^*, \gamma_0 Z_p] \cdot \nabla v_p = (\gamma_0 Z_p \cdot \nabla)f - (\gamma_0 Z_p \cdot \nabla)(\omega(Z_p^*, Z_i^*)).
\]
(64)

We now look more closely at the principal part of this equation. The first term may be written as
\[
\gamma_0 Z_p \otimes Z_i^* : \nabla^2 v_i + ((\gamma_0 Z_p \cdot \nabla)Z_i^*) \cdot \nabla v_i = \gamma_0 : \nabla^2 v_i + ((\gamma_0 Z_p \cdot \nabla)Z_i^*) \cdot \nabla v_i,
\]
where we have used that $Z_p \otimes Z_i^* = I_n$. We thus obtain a strongly coupled elliptic system of the form (22), where
\[
W_{ij} := (\nabla \cdot \gamma_0 - ((\gamma_0 Z_p \cdot \nabla)Z_i^*)) \delta_{ij} - [Z_i^*, \gamma_0 Z_j], \quad 1 \leq i, j \leq n,
\]
(65)
\[
f_i := -Z_i^* \cdot \nabla f + (\gamma_0 Z_p \cdot \nabla)(\omega(Z_p^*, Z_i^*)), \quad 1 \leq i \leq n.
\]
(66)

This concludes the proof.

In order to assess the properties of system (22), we recast it as an integral equation as follows: Let us call $L_0 := -\nabla \cdot (\gamma_0 \nabla)$, and define $L_0^{-1} : H^{-1}(X) \ni f \mapsto u \in H^1_0(X)$, where $u$ is the unique solution to the equation
\[
-\nabla \cdot (\gamma_0 \nabla u) = f \quad (X), \quad u|_{\partial X} = 0.
\]
(67)

By the Lax-Milgram theorem (see e.g. [10]), one can establish that such solutions satisfy an estimate of the form $\|u\|_{H^1_0(X)} \leq C\|f\|_{H^{-1}(X)}$, where $C$ only depends on $X$ and the constant of ellipticity of $\gamma_0$, thus $L_0^{-1} : H^{-1}(X) \rightarrow H^1_0(X)$ is continuous, and by Rellich imbedding (i.e. the fact that the injection $L^2 \rightarrow H^{-1}$ is compact), $L_0^{-1} : L^2(X) \rightarrow H^1_0(X)$ is compact.

Applying the operator $L_0^{-1}$ to (22), we arrive at the integral system
\[
v_i + \sum_{j=1}^{n} L_0^{-1}(W_{ij} \cdot \nabla v_j) = h_i := L_0^{-1}f_i \quad (X), \quad 1 \leq i \leq n,
\]
(68)

where it is easy to establish that for $1 \leq i, j \leq n$, the operator
\[
P_{ij} : H^1_0(X) \ni v \mapsto P_{ij}v := L_0^{-1}(W_{ij} \cdot \nabla v) \in H^1_0(X)
\]
is compact whenever the vector fields $W_{ij}$ are bounded. In vector notation, if we define the vector space $\mathcal{H} = (H^1_0(X))^n$, $\mathbf{v} = (v_1, \ldots, v_n)$, $\mathbf{h} = (h_1, \ldots, h_n)$ and for $\mathbf{v} \in \mathcal{H}$,
\[
\mathbf{P} \mathbf{v} := (P_{1j}v_j, P_{2j}v_j, \ldots, P_{nj}v_j) \in \mathcal{H},
\]
(70)

we have that $\mathbf{P} : \mathcal{H} \rightarrow \mathcal{H}$ is a compact linear operator, and the system (22) is reduced to the following Fredholm (integral) equation
\[
(I + \mathbf{P})\mathbf{v} = \mathbf{h}.
\]
(71)
Injectivity and stability. Equation (71) satisfies a Fredholm alternative. In particular, if $-1$ is not an eigenvalue of $\mathbf{P}$, (71) admits a unique solution $v \in \mathcal{H}$ (injectivity), $(I + \mathbf{P})^{-1} : \mathcal{H} \rightarrow \mathcal{H}$ is well-defined and continuous and $v$ satisfies the estimate

$$
\|v\|_{\mathcal{H}} \leq \|(I + \mathbf{P})^{-1}\|_{\mathcal{L}(\mathcal{H})}\|h\|_{\mathcal{H}},
$$

from which we deduce stability below. In the statement of Theorem 2.4, the fact that "system (22) with trivial right-hand sides admits no non-trivial solution" precisely means that $-1$ is not an eigenvalue of the operator $\mathbf{P}$.

**Remark 5** (Injectivity when $\gamma_0$ is constant). When $\gamma_0$ is constant, constructing $(u_1, \ldots, u_{n+1})$ as in Remark 1 yields $Z = Q$ a constant matrix. In particular, the commutators $[Z^*_i, \gamma_0 Z_j]$ vanish in the expression (65) of $W_{ij}$. Thus system (22) is decoupled and clearly injective. By continuity, we also obtain that (22) is injective for $\gamma_0$ (not necessarily scalar) sufficiently close to a constant.

We now prove Theorem 2.4.

**Proof of Theorem 2.4.** Starting from the integral version (71) of the elliptic system (22) in the case where $-1 \notin \text{sp}(\mathbf{P})$, then the Fredholm alternative implies (72). In order to translate inequality (72) into a stability statement, we must bound $h$ in terms of the measurements $\{dH_{ij}\}$. We have for $1 \leq i \leq n$,

$$
\|h_i\|_{H^1_0(X)} \leq \|L_0^{-1}\|_{\mathcal{L}(H^{-1}_0, H^1)} \|f_i\|_{H^{-1}(X)},
$$

and since $f_i$ expressed in (66) involves the $dH_{ij}$ and their derivatives up to second order, if we assume all other multiplicative coefficients to be uniformly bounded, we obtain an estimate of the form

$$
\|h_i\|_{H^1_0(X)} \leq C \|dH\|_{H^1_0(X)}, \quad \text{where} \quad \|dH\|_{H^1_0(X)} := \sum_{1 \leq i \leq n, 1 \leq j \leq n+1} \|dH_{ij}\|_{H^1_0(X)},
$$

thus we obtain in the end, an estimate of the form

$$
\|\mathbf{v}\|_{H^1_0(X)} \leq C \|dH\|_{H^1_0(X)}. \quad (73)
$$

Once $\mathbf{v}$ is reconstructed, we can reconstruct $\gamma$ uniquely from $dH$ and $[\nabla V]$ using formula (20), with the stability estimate

$$
\|\nabla \gamma\|_{L^2_0(X)} \leq C \|dH\|_{H^1_0(X)}. \quad (74)
$$

**Regaining one derivative back on tr $(\gamma_0^{-1}\gamma)$:** In order to see that $\text{tr} \ (\gamma_0^{-1}\gamma)$ satisfies a gradient equation that improves the stability of its reconstruction, the quickest way is to linearize [18, Equation (7)] derived in the non-linear case, which reads as follows:

$$
\nabla \log \det \gamma^\varepsilon = \nabla \log \det H^\varepsilon + 2 \left( (\nabla (H^\varepsilon)^{jl}) \cdot \gamma^\varepsilon \nabla u_i^\varepsilon \right) \nabla u_j^\varepsilon,
$$

where $H^\varepsilon$ is the $n \times n$ matrix of power densities $H_{ij}^\varepsilon = \nabla u_i^\varepsilon \cdot \gamma^\varepsilon \nabla u_j^\varepsilon$ and $(H^\varepsilon)^{jl}$ is the $(j, l)$-th entry of $(H^\varepsilon)^{-1}$. Plugging the expansions $\gamma^\varepsilon = \gamma_0 + \varepsilon \gamma, \ u_i^\varepsilon = u_i + \varepsilon v_i, \ H_{ij}^\varepsilon = H_{ij} + \varepsilon dH_{ij}$, and using the fact that

$$
(H^\varepsilon)^{jl} = H^{jl} - \varepsilon (H^{-1} dH H^{-1})^{jl} + \mathcal{O}(\varepsilon^2),
$$

Note that the operator $\mathbf{P}$ defined in (70) depends only on $\gamma_0$ and the solutions $u_i$, so that the injectivity properties depend on the $\gamma_0$ around which we pose the problem, in particular, whether one can fulfill hypotheses 2.1 and 2.2.
the linearized equation at $O(\varepsilon)$ reads
\[
\frac{1}{2} \nabla \text{tr} \left( \gamma_0^{-1} \gamma \right) = \frac{1}{2} \nabla \text{tr} \left( H^{-1} dH \right) + (\nabla H^{-1} \cdot \gamma_0 \nabla u_l) \nabla v_j + (\nabla H^{-1} \cdot \gamma_0 \nabla v_l) \nabla u_j + (\nabla H^{-1} \gamma \nabla u_l) \nabla u_j - (\nabla H^{-1} dH H^{-1}) \nabla u_l \nabla u_j.
\]
From this equation, and using the stability estimates (73) and (74), it is straightforward to establish the estimate
\[
\| \text{tr} \left( \gamma_0^{-1} \gamma \right) \|_{H^1(X)} \leq C \| dH \|_{H^1(X)}^2,
\]
and thus the proof is complete. \qed

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