Prophet Secretary for Combinatorial Auctions and Matroids

Soheil Ehsani† MohammadTaghi Hajiaghayi† Thomas Kesselheim‡ Sahil Singla§

November 1, 2017

Abstract

The secretary and the prophet inequality problems are central to the field of Stopping Theory. Recently, there has been a lot of work in generalizing these models to multiple items because of their applications in mechanism design. The most important of these generalizations are to matroids and to combinatorial auctions (extends bipartite matching). Kleinberg-Weinberg [KW12] and Feldman et al. [FGL15] show that for adversarial arrival order of random variables the optimal prophet inequalities give a $\frac{1}{2}$-approximation. For many settings, however, it’s conceivable that the arrival order is chosen uniformly at random, akin to the secretary problem. For such a random arrival model, we improve upon the $\frac{1}{2}$-approximation and obtain $(1 - 1/e)$-approximation prophet inequalities for both matroids and combinatorial auctions. This also gives improvements to the results of Yan [Yan11] and Esfandiari et al. [EHLM17] who worked in the special cases where we can fully control the arrival order or when there is only a single item.

Our techniques are threshold based. We convert our discrete problem into a continuous setting and then give a generic template on how to dynamically adjust these thresholds to lower bound the expected total welfare.

---

*Part of this work was done while the authors were visiting the Simons Institute for the Theory of Computing.
†Department of Computer Science, University of Maryland, College Park, MD 20742 USA. Email: {ehsani,hajiagha}@cs.umd.edu. Supported in part by NSF CAREER award CCF-1053605, NSF BIGDATA grant IIS-1546108, NSF AF:Medium grant CCF-1161365, DARPA GRAPHS/AFOSR grant FA9550-12-1-0423, and another DARPA SIMPLEX grant.
‡Department of Computer Science, TU Dortmund, 44221 Dortmund, Germany. Email: thomas.kesselheim@tu-dortmund.de.
§Computer Science Department, Carnegie Mellon University, Pittsburgh, PA 15213, USA. Email: ssingla@cmu.edu. Supported in part by a CMU Presidential Fellowship and NSF awards CCF-1319811, CCF-1536002, and CCF-1617790
1 Introduction

Suppose there is a sequence of \( n \) buyers arriving with different values to your single item. On arrival a buyer offers a take-it-or-leave-it value for your item. How should you decide which buyer to assign the item to in order to maximize the value. There are two popular models in the field of Stopping Theory to study this problem: the secretary and the prophet inequality models. In the secretary model we assume no prior knowledge about the buyer values but the buyers arrive in a uniformly random order [Dyn63]. Meanwhile, in the prophet inequality model we assume stochastic knowledge about the buyer values but the arrival order of the buyers is chosen by an adversary [KS78, KS77]. Since the two models complement each other, both have been widely studied in the fields of mechanism design and combinatorial optimization (see related work).

These models assume that either the buyer values or the buyer arrival order is chosen by an adversary. In practice, however, it is often conceivable that there is no adversary acting against you. Can we design better strategies in such settings? The prophet secretary model introduced in [EHLM17] is a natural way to consider such a process where we assume both a stochastic knowledge about buyer values and that the buyers arrive in a uniformly random order. The goal is to design a strategy that maximizes expected accepted value, where the expectation is over the random arrival order, the stochastic buyer values, and also any internal randomness of the strategy.

In this paper, we consider generalizations of the above problem to combinatorial settings. Suppose the buyers correspond to elements of a matroid\(^1\) and we are allowed to accept any independent set in this matroid rather than only a single buyer. The buyers again arrive and offer take-it-or-leave-it value for being accepted. In the prophet inequality model, a surprising result of Kleinberg-Weinberg [KW12] gives a \( 1/2 \)-approximation strategy to this problem, i.e., the value of their strategy, in expectation, is at least half of the value of the expected offline optimum that selects the best set of buyers in hindsight. Simple examples show that for adversarial arrival one cannot improve this factor. On the other hand, if we are also allowed to control the arrival order of the buyers, Yan [Yan11] gives a \( 1−1/e \approx 0.63 \)-approximation strategy. But what if the arrival order is neither adversarial and nor in your control. In particular, can we beat the \( 1/2 \)-approximation for a uniformly random arrival order?

**Matroid Prophet Secretary Problem (MPS):** Given a matroid \( \mathcal{M} = ([n], \mathcal{I}) \) on \( n \) buyers (elements) and independent probability distributions on their values, suppose the outcome buyer values are revealed in a uniformly random order. Whenever a buyer value is revealed, the problem is to immediately and irrevocably decide whether to select the buyer. The goal is to maximize the sum of values of the selected buyers, while ensuring that they are always feasible in \( \mathcal{I} \).

Besides being a natural problem that relates two important Stopping Theory models, MPS is also interesting because of its applications in mechanism design. Often while designing mechanisms, we have to balance between maximizing revenue/welfare and the simplicity of the mechanism. While there exist optimal mechanisms such as VCG or Myerson’s mechanism, they are impractical in real markets [AM06, Rot07]. On the other hand, simple Sequentially Posted Pricing mechanisms, where we offer take-it-or-leave-it prices to buyers, are known to give good approximations to optimal mechanisms. This is because the problem gets reduced to designing a prophet inequality [CHMS10, Yan11, Ala14, KW12, FGL15].

Esfandiari et al. [EHLM17] study MPS in the special case of a rank 1 matroid and give a \( (1−1/e) \)-approximation algorithm. For general matroids, as in the original models of [CHMS10, Yan11, Ala14, KW12, FGL15].

\(^1\)A matroid \( \mathcal{M} \) consists of a ground set \( [n] = \{1, 2, \ldots, n\} \) and a non-empty downward-closed set system \( \mathcal{I} \subseteq 2^{[n]} \) satisfying the matroid exchange axiom: for all pairs of sets \( I, J \in \mathcal{I} \) such that \( |I| < |J| \), there exists an element \( x \in J \) such that \( I \cup \{x\} \in \mathcal{I} \). Elements of \( \mathcal{I} \) are called independent sets.
Yan11, KW12, it was unclear prior to the work of this paper whether beating the factor of $1/2$ is possible. In Section 4 we prove the following result.

**Theorem 1.** There exists a $(1 - 1/e)$-approximation algorithm to MPS.

Note that the approximation in this theorem as well as the following ones compare to the expected optimal offline solution for the particular outcomes of the distributions. That is, in the case of matroids, we have $E[\text{Alg}] \geq (1 - 1/e) \cdot E[\max_{I \subseteq \mathcal{I}} \sum_{i \in I} v_i]$, where $v_i$ is the value of buyer $i$.

Next, let us consider a combinatorial auctions setting. Suppose there are $n$ buyers that take combinatorial valuations (say, submodular) for $m$ indivisible items from $n$ independent probability distributions. The problem is to decide how to allocate the items to the buyers, while trying to maximize the welfare—the sum of valuations of all the buyers. Feldman et al. [FGL15] show that for XOS $(a$ generalization of submodular) valuations there exist static prices for items that gets a $1/2$-approximation for buyers arriving in an adversarial order. Since this factor cannot be improved for adversarial arrival, this leaves an important open question if we can design better algorithms when the arrival order can be controlled. Or ideally, we want to beat $1/2$ even when the arrival order cannot be controlled but is chosen uniformly at random.

**Combinatorial Auctions Prophet Secretary Problem (CAPS):** Suppose $n$ buyers take XOS valuations for $m$ items from $n$ independent probability distributions. The outcome buyer valuations are revealed in a uniformly random order. Whenever a buyer valuation is revealed, the problem is to immediately and irrevocably assign a subset of the remaining items to the buyer. The goal is maximize the sum of the valuations of all the buyers for their assigned subset of items.

In Section 3.2 we improve the online approximation result of [FGL15] for random order.

**Theorem 2.** There exists a $(1 - 1/e)$-approximation algorithm to CAPS.

Given access to demand and XOS oracles for stochastic utilities of different buyers, the algorithm in Theorem 2 can be made efficient. This is interesting because it matches the best possible $(1 - 1/e)$-approximation for XOS-welfare maximization in the offline setting [DNS10, Fei09].

A desirable property in the design of an economically viable mechanism is incentive-compatibility. In particular, a buyer is more likely to make decisions about their allocations based on their own personal incentives rather than to accept a given allocation that might optimize the social welfare but not the individuals’ profit. For the important case of unit-demand buyers (aka bipartite matching), in Section 3.1 we extend Theorem 2 to additionally obtain this property.

**Theorem 3.** For bipartite matchings, when buyers arrive in a uniformly random order, there exists an incentive-compatible mechanism based on dynamic prices that gives a $(1 - 1/e)$-approximation to the optimal welfare.

For this result, we require unit-demand buyers. This is because for general XOS functions shifting buyers to earlier arrivals can change the availability of items arbitrarily. For unit-demand functions, we show that this effect is bounded.

Finally, in Section 5 we conclude by showing that for the single-item case one can obtain a $(1 - 1/e)$-approximation even by using static prices, and that nothing better is possible.

---

2It’s not known if $1 - 1/e$ is tight for MPS. In fact, it’s even open if one can beat $1 - 1/e$ for a single item [AEE+17].

3A function $v: 2^M \rightarrow \mathbb{R}$ is an XOS function if there exists a collection of additive functions $A_1, \ldots, A_k$ such that for every $S \subseteq M$ we have $v(S) = \max_{1 \leq i \leq k} A_i(S)$. 
1.1 Our Techniques

In this section we discuss our three main ideas for a combinatorial auction. In this setting, our algorithm is threshold based, which means that we set dynamic prices to the items and allow a buyer to purchase a set of items only if her value is more than the price of that set. This allows us to view total value as the sum of utility of the buyers and the total generated revenue. Although powerful, dynamic prices often lead to involved calculations and become difficult to analyze beyond a single item setting [EHLM17, AEE+17]. To overcome this issue, we convert our discrete problem into a continuous setting. This is possible because a random permutation of buyers can be viewed as each buyer arriving at a time chosen uniformly at random between 0 and 1. The benefit of such a transformation is that the arrival times are independent, which keeps correlations manageable. Besides, it allows us to use tools from integral calculus such as integration by parts.

Our algorithm for combinatorial auctions sets a base price $b_j$ for every item $j$ based on its contribution to the expected offline optimum $E[\text{OPT}]$. Our approach is to define two time varying continuous functions: discount and residual. The discount function $\alpha(t) : [0, 1] \rightarrow [0, 1]$ is chosen such that the price of an unsold item $j$ at time $t$ is exactly $\alpha(t) \cdot b_j$. We define a residual function $r(t) : [0, 1] \rightarrow \mathbb{R}_{\geq 0}$ that intuitively denotes the expected value remaining in the instance at time $t$. Hence, $r(0) = E[\text{OPT}]$ and $r(1) = 0$. Computing $r(t)$ is difficult for a combinatorial auction since it depends on several random variables. However, assuming that we know $r(t)$, we use application specific techniques to compute lower bounds on both the expected revenue and the expected utility in terms of the functions $r(t)$ and $\alpha(t)$.

Finally, although we do not know $r(t)$, we can choose the function $\alpha(t)$ in a way that allows us to simplify the sum of expected revenue and utility, without ever computing $r(t)$ explicitly. This step exploits properties of the exponential function for integration (see Lemma 6).

1.2 Related Work

Starting with the works of Krengel-Sucheston [KS78, KS77] and Dynkin [Dyn63], there has been a long line of research on both prophet inequalities and secretary problems. One of the first generalizations is the multiple-choice prophet inequalities [Ken85, K86, Ker86] in which we are allowed to pick $k$ items and the goal is to maximize their sum. Alaei [Ala14] gives an almost tight $(1 - 1/\sqrt{k + 3})$-approximation algorithm for this problem (the lower bound is due to [HKS07]). Similarly, the multiple-choice secretary problem was first studied by Hajiaghayi et al. [IIKP04], and Kleinberg [Kle05] gives a $(1 - O(\sqrt{1/k}))$-approximation algorithm.

The research investigating the relation between prophet inequalities and online auctions is initiated in [HKS07, CHMS10]. This lead to several interesting follow up works for matroids [Yan11, KW12] and matchings [AHL12]. Meanwhile, the connection between secretary problems and online auctions is first explored in Hajiaghayi et al. [HKP04]. Its generalization to matroids is considered in [BIK07, Lac14, FSZ15] and to matchings in [GM08, KP09, MY11, KMT11, KRTV13, GS17].

Secretary problems and prophet inequalities have also been studied beyond a matroid/matching. For the intersection of $p$ matroids, Kleinberg and Weinberg [KW12] give an $O(p)$-approximation prophet inequality. Dütting and Kleinberg [DK15] extend this result to polymatroids. Feldman et al. [FGL15] study the generalizations to combinatorial auctions. Later, Dütting et al. [DFKL17] give a general framework to prove such prophet inequalities. Submodular variants of the secretary problem have been considered in [BHZ10, GRST10, FZ15, KMZ15]. Prophet and secretary problems have also been studied for many classical combinatorial problems (see e.g., [Mey01, GGLS08, GHK+14, DEH+15, DEH+17]). Rubinstein [Rub16] and Rubinstein-Singla [RS17] consider these problems for arbitrary downward-closed constraints.
In the prophet secretary model, Esfandiari et al. [EHL17] give a $(1 - 1/e)$-approximation in the special case of a single item. Going beyond $1 - 1/e$ has been challenging. Only recently, Abolhasani et al. [AEE17] and Correa et al. [CFH17] improve this factor for the single item i.i.d. setting. Extending this result to non-identical items or to matroids are interesting open problems.

# 2 Our Approach using a Residual

In this section, we define a residual and discuss how it can be used to design an approximation algorithm for a prophet secretary problem. Suppose there are $n$ requests that arrive at times $(T_i)_{i \in [n]}$ drawn i.i.d. from the uniform distribution in $[0, 1]$. These requests correspond to buyers of a combinatorial auction or to elements of a matroid.

Whenever a request arrives, we have to decide if and how to serve it. Depending on how we serve request $i$, say $x_i$, we gain a certain value $v_i(x_i)$. Our task is to maximize the sum of values over all requests $\sum_{i=1}^n v_i(x_i)$. Our algorithm Alg includes a time-dependent payment component. The payment that request $i$ has to make is the product of a time-dependent discount function $\alpha(t)$ and a base price $b(x_i)$. The base price depends on the allocation up to this point and how much the new choice limits other allocations in the future. However, it does not depend on $t$, the time that has passed up to this point. If request $i$ has to pay $p_i(x_i, T_i) = \alpha(T_i, b(x_i))$ for our decision $x_i$, then its utility is given by $u_i = v_i(x_i) - p_i(x_i, T_i)$. We write $\text{Utility} = \sum_{i=1}^n u_i$ for the sum of utilities and $\text{Revenue} = \sum_{i=1}^n p_i(x_i, T_i)$ for the sum of payments. The value achieved by Alg equals $\text{Utility} + \text{Revenue}$.

Next we define a residual function that has the interpretation of “expected remaining value in the instance at time $t$”. In Lemma 6 we show that the existence of a residual function for Alg suffices to give a $(1 - 1/e)$-approximation prophet secretary.

**Definition 4 (Residual).** Consider a prophet secretary problem with expected offline value $\mathbb{E}[\text{OPT}]$. For any algorithm Alg based on a differentiable discount function $\alpha(t) : [0, 1] \to [0, 1]$, a differentiable function $r(t) : [0, 1] \to \mathbb{R}_{\geq 0}$ is called a residual if it satisfies the following three conditions for every choice of $\alpha$.

\begin{align*}
    r(0) &= \mathbb{E}[\text{OPT}] \quad (1a) \\
    \mathbb{E}[\text{Revenue}] &\geq - \int_{t=0}^{1} \alpha(t) \cdot r'(t) \cdot dt \quad (1b) \\
    \mathbb{E}[\text{Utility}] &\geq \int_{t=0}^{1} (1 - \alpha(t)) \cdot r(t) \cdot dt. \quad (1c)
\end{align*}

We would like to remark here that this definition is similar in spirit to balanced thresholds [KW12] and balanced prices [DFKL17]. However, it is different because we have to take into account the random arrivals.

As an illustration of Definition 4, consider the case of a single item. That is, we are presented a sequence of $n$ real numbers and may select only up to one of them (previously studied in [EHL17]).

**Example 5 (Single Item).** Suppose buyer $i \in [n]$ arrives with random value $v_i$ at time $T_i$ chosen uniformly at random between 0 and 1. Define $b = \mathbb{E}[\max_i v_i]$ as the base price of the single item. A buyer arriving at time $t$ is offered the item at price $\alpha(t) \cdot b$, and she accepts the offer if and only if $v_i \geq \alpha(t) \cdot b$. We show that $r(t) = \text{Pr}[\text{item not sold before } t] \cdot b$ is a residual function.

By definition, (1a) holds trivially. To see that (1b) holds, observe that the increase in revenue from time $t$ to time $t + \epsilon$ is approximately $\alpha(t) \cdot b$ if the item is allocated during this time, and is 0
otherwise. That is, the expected increase in revenue is approximately $\alpha(t)(r(t) - r(t + \epsilon))$. Taking the limit for $\epsilon \to 0$ then implies (1b), i.e., $\mathbb{E}[\text{Revenue}] = -\int_{t=0}^{1} \alpha(t) r'(t) dt$.

For (1c), we consider the expected utility of a buyer $i$ conditioning on her arriving at time $t$

$$\mathbb{E}[u_i \mid T_i = t] = \mathbb{E}[1_{\text{item not sold before } t \cdot (v_i - \alpha(t) \cdot b)^+ \mid T_i = t]$$

$$= \mathbb{Pr}[\text{item not sold before } t \mid T_i = t] \cdot \mathbb{E}[(v_i - \alpha(t) \cdot b)^+]$$

Here we use that the event the item is sold before $t$ does not depend on $v_i$ because buyer $i$ only arrives at time $t$. The expectation in turn only depends on $v_i$. It is also important to observe that $\mathbb{Pr}[\text{item not sold before } t \mid T_i = t] \geq \mathbb{Pr}[\text{item not sold before } t]$. Next, we take the sum over all buyers $i$ and use that $\mathbb{E}[\sum_{i=1}^{n} (v_i - \alpha(t) \cdot b)^+] \geq \mathbb{E}[\max_{i} (v_i - \alpha(t) \cdot b)] = \mathbb{E}[\max_{i} v_i] - \alpha(t) \cdot b = (1 - \alpha(t)) \cdot b$ to get

$$\sum_{i=1}^{n} \mathbb{E}[u_i \mid T_i = t] \geq \mathbb{Pr}[\text{item not sold before } t] \cdot (1 - \alpha(t)) \cdot b = (1 - \alpha(t)) \cdot r(t).$$

This implies

$$\mathbb{E}[\text{Utility}] = \sum_{i=1}^{n} \int_{t=0}^{1} \mathbb{E}[u_i \mid T_i = t] dt = \int_{t=0}^{1} \sum_{i=1}^{n} \mathbb{E}[u_i \mid T_i = t] dt \geq \int_{t=0}^{1} (1 - \alpha(t)) \cdot r(t) \cdot dt.$$

We now use the properties of a residual function to design a $(1 - 1/e)$-approximation algorithm. To this end, we choose $\alpha(t)$ in a manner that makes the sum of the expected revenue and buyers’ utilities independent of $r(t)$. This allows us to compute expected welfare, even though we cannot compute $r(t)$ directly.

**Lemma 6.** For a prophet secretary problem, if there exists a residual function $r(t)$ for algorithm Alg as defined in Definition 4, then setting $\alpha(t) = 1 - e^{t-1}$ gives a $(1 - 1/e)$-approximation.

**Proof.** To further simplify Eq. (1b), we observe that applying integration by parts gives

$$\int r'(t) \cdot \alpha(t) \cdot dt = r(t) \cdot \alpha(t) - \int r(t) \alpha'(t) \cdot dt.$$ 

So in combination

$$\mathbb{E}[\text{Revenue}] \geq - \left( [r(t) \cdot \alpha(t)]_{t=0}^{1} - \int_{t=0}^{1} r(t) \cdot \alpha'(t) \cdot dt \right).$$

Now adding (2) and (1c) gives,

$$\mathbb{E}[\text{Alg}] = \mathbb{E}[\text{Utility}] + \mathbb{E}[\text{Revenue}]$$

$$\geq \int_{t=0}^{1} r(t) \cdot (1 - \alpha(t)) \cdot dt - [r(t)\alpha(t)]_{t=0}^{1} + \int_{t=0}^{1} r(t)\alpha'(t) \cdot dt$$

$$= \int_{t=0}^{1} r(t) \cdot (1 - \alpha(t) + \alpha'(t)) \cdot dt - [r(t)\alpha(t)]_{t=0}^{1}.$$
all $t$, this integral becomes independent of $r(t)$ and simplifies to 0. In particular, let $\alpha(t) = 1 - e^{t-1}$. This gives,

$$\mathbb{E}[\text{Alg}] \geq - [r(t) \cdot \alpha(t)]_{t=0}^{1}$$

$$= \left(1 - \frac{1}{e}\right) r(0)$$

$$= \left(1 - \frac{1}{e}\right) \mathbb{E}[\text{OPT}] .$$

\[\square\]

### 3 Prophet Secretary for Combinatorial Auctions

Let $N$ denote a set of $n$ buyers and $M$ denote the set of $m$ indivisible items. Suppose buyer $i$ arrives at a time $T_i$ chosen uniformly at random between 0 and 1. Let $v_i: 2^M \rightarrow \mathbb{R}_{\geq 0}$ (similarly $\hat{v}_i$) denote the random combinatorial valuation function of buyer $i$. In order to ensure polynomial running times, we assume that the distribution of $v_i$ has a polynomial support $\{v_i^1, v_i^2, \ldots, \}$, where $\sum_k \Pr[v_i = v_i^k] = 1$. Note that this assumption only simplifies notation. If we only have sample access to the distributions, then we can replace $\{v_i^1, v_i^2, \ldots, \}$ by an appropriate number of samples. Within our proofs, we will use $\hat{v}$ to denote an independent, fresh sample from the distribution.

By $T$ and $v$ (similarly $\hat{v}$) we denote the vector of all the buyer arrival times and valuations, respectively. Also, let $v_{-i}$ (similarly $\hat{v}_{-i}$) denote valuations of all buyers except buyer $i$. For the special case of single items, we let $v_{ij}$ denote $v_i({j})$. Let $q_j(t)$ denote the probability that item $j$ has not been sold before time $t$, where the probability is over valuations $v$, arrival times $T$, and any randomness of the algorithm.

#### 3.1 Bipartite Matching

In the bipartite matching setting all buyers are unit-demand, i.e. $v_i(S) = \max_{j \in S} v_{ij}$. We can therefore assume that no buyer buys more than one item. We restate our result.

**Theorem 3.** For bipartite matchings, when buyers arrive in a uniformly random order, there exists an incentive-compatible mechanism based on dynamic prices that gives a $(1 - 1/e)$-approximation to the optimal welfare.

To define prices of items, let base price $b_j$ denote the expected value of the buyer that buys item $j$ in the offline welfare maximizing allocation (maximum weight matching). Now consider an algorithm that prices item $j$ at $\alpha(t) \cdot b_j$ at time $t$ and allows the incoming buyer to pick any of the unsold items; here $\alpha(t)$ is a continuous differentiable discount function.

Consider the function $r(t) = \sum_j q_j(t) \cdot b_j$. Clearly, $r(0) = \mathbb{E}[\text{OPT}]$. Using the following Lemma 7 and Claim 8, we prove that $r$ is a residual function for our algorithm. Since the algorithm is clearly incentive-compatible, Lemma 6 implies Theorem 3.

**Lemma 7.** We can lower bound the total expected utility by

$$\mathbb{E}_v, T[\text{Utility}] \geq \sum_j \int_{t=0}^{1} q_j(t) \cdot (1 - \alpha(t)) \cdot b_j \cdot dt. \quad (3)$$

6
Proof. Since buyer \(i\) arriving at time \(t\) can pick any of the unsold items, we have

\[
\mathbb{E}_{\mathbf{v},\mathbf{T}}[u_i | T_i = t] = \mathbb{E}_{\mathbf{v}} \left[ \max_j \mathbf{1}_{j \text{ not sold before } t} \cdot (v_{i,j} - \alpha(t) \cdot b_j)^+ | T_i = t \right].
\]

One particular choice of buyer \(i\) is to choose item \(OPT_i(v_i, \hat{v}_{-i})\) if it is still available, and no item otherwise. This gives us a lower bound of

\[
\mathbb{E}_{\mathbf{v},\mathbf{T}}[u_i | T_i = t] \geq \mathbb{E}_{\mathbf{v}} \left[ \mathbf{1}_{OPT_i(v_i, \hat{v}_{-i}) \text{ not sold before } t} \cdot (v_{i,OPT_i(v_i, \hat{v}_{-i})} - \alpha(t) \cdot b_j)^+ | T_i = t \right]
= \sum_j \mathbb{E}_{\mathbf{v},\hat{v}} \left[ \mathbf{1}_{j \text{ not sold before } t} \cdot \mathbf{1}_{j=OPT_i(v_i, \hat{v}_{-i})} \cdot (v_{i,j} - \alpha(t) \cdot b_j)^+ | T_i = t \right].
\]

Note that in the product, the fact whether \(j\) is sold before \(t\) only depends on \(v_{-i}\) and the arrival times of the other buyers. It does not depend on \(v_i\) or \(\hat{v}\). The remaining terms, in contrast, only depend on \(v_i\) and \(\hat{v}_{-i}\). Therefore, we can use independence to split up the expectation and get

\[
\sum_j \Pr[j \text{ not sold before } t | T_i = t] \cdot \mathbb{E}_{v_i,\hat{v}_{-i}} \left[ \mathbf{1}_{j=OPT_i(v_i, \hat{v}_{-i})} \cdot (v_{i,j} - \alpha(t) \cdot b_j)^+ | T_i = t \right].
\]

Next, we use that \(\Pr[j \text{ not sold before } t | T_i = t] \geq q_j(t)\) by Lemma 9 and that \(v_i\) and \(\hat{v}_i\) are identically distributed. Therefore, we can swap their roles inside the expectation. Overall, this gives us

\[
\mathbb{E}_{\mathbf{v},\mathbf{T}}[u_i | T_i = t] \geq \sum_j q_j(t) \cdot \mathbb{E}_{\mathbf{v}} \left[ \mathbf{1}_{j=OPT_i(\hat{v})} \cdot (\hat{v}_{i,j} - \alpha(t) \cdot b_j)^+ \right]. \tag{4}
\]

Next, observe that \(\mathbb{E}_v[\sum_i \mathbf{1}_{j=OPT_i(\hat{v})} \cdot \hat{v}_{i,j}] = b_j\) by the definition of \(b_j\). Therefore, using linearity of expectation, summing up (4) over all buyers \(i\) gives us

\[
\sum_i \mathbb{E}_{\mathbf{v},\mathbf{T}}[u_i | T_i = t] \geq \sum_j q_j(t) \cdot (1 - \alpha(t)) \cdot b_j.
\]

Now, taking the expectation over \(t\), we get

\[
\mathbb{E}_{\mathbf{v},\mathbf{T}} \left[ \sum_i u_i \right] = \sum_i \int_{t=0}^1 \mathbb{E}_{\mathbf{v},\mathbf{T}}[u_i | T_i = t] \cdot dt
= \int_{t=0}^1 \sum_i \mathbb{E}_{\mathbf{v},\mathbf{T}}[u_i | T_i = t] \cdot dt
\geq \int_{t=0}^1 \sum_j q_j(t) \cdot (1 - \alpha(t)) \cdot b_j \cdot dt
= \sum_j \int_{t=0}^1 q_j(t) \cdot (1 - \alpha(t)) \cdot b_j \cdot dt.
\]

We next give a bound on the revenue generated by our algorithm.

\(\square\)
Claim 8. We can bound the total expected revenue by

\[
E_{v,T}[\text{Revenue}] = - \sum_j \int_{t=0}^1 q_j'(t) \alpha(t) \cdot b_j \cdot dt. \tag{5}
\]

Proof. Since \(-q_j'(t)dt\) is the probability that item \(j\) is bought between \(t\) and \(t + dt\) (note \(q_j(t)\) is decreasing in \(t\)), we have

\[
E[\text{Revenue}] = - \sum_j \int_{t=0}^1 q_j'(t) \alpha(t) \cdot b_j \cdot dt.
\]

Clearly, this is trivial, for multiple items and other combinatorial valuations it does not necessarily hold.

Finally, we prove the missing lemma that removes the conditioning on the arrival time.

Lemma 9. We have

\[
E_{v,T} \left[ \Pr_T[j \text{ not sold before } t \mid T_i = t] \right] \geq q_j(t).
\]

The idea is that if buyers arrive earlier in the process, this only \textit{reduces} the available items. It can never happen that such a change makes an item available at a later point. For a single item this is trivial, for multiple items and other combinatorial valuations it does not necessarily hold.

Proof. Consider the execution of our algorithm on two sequences that only differ in the arrival time of buyer \(i\). To this end, let \(v\) be arbitrary values and \(T\) be arbitrary arrival times. Let \(A_T\) be the set of items that are sold before time \(t'\) on the sequence defined by \(v\) and \(T\). Furthermore, let \(B_T\) be the set of items sold before time \(t\) if we replace \(T_i\) by \(t\). Ties are broken in the same way in both sequences.

We claim that \(B_{t'} \subseteq A_{t'}\) for all \(t' \leq t\).

To this end, we observe that by definition \(B_{t'} = A_{t'}\) for \(t' \leq \min\{T_i, t\}\) because the two sequences are identical before \(\min\{T_i, t\}\). This already shows the claim for \(T_i \geq t\). Otherwise, assume that there is some \(t' \leq t\) for which \(B_{t'} \not\subseteq A_{t'}\). Let \(t_{inf}\) be the infimum among these \(t'\). It has to hold that some buyer \(i'\) arrives at time \(t_{inf}\) and buys item \(j_A \not\in A_{t_{inf}}\) in the original sequence and \(j_B \not\in B_{t_{inf}}\) in the modified sequence. Furthermore, we now have to have \(B_{t_{inf}} \not\subseteq A_{t_{inf}}\) because \(t_{inf}\) was defined to be the infimum of all \(t'\) for which \(B_{t'} \subseteq A_{t'}\) is not fulfilled. Therefore, \(j_A \not\in B_{t_{inf}}\). Additionally, \(j_B \not\in A_{t_{inf}}\). The reason is that for any \(t' < t_{inf}\) before the next arrival \(B_{t'} = B_{t_{inf}} \cup \{j_B\}\).

Overall this means that in both sequences at time \(t_{inf}\) buyer \(i'\) has the choice between \(j_B\) and \(j_A\). As his values are identical and ties are broken the same way, it has to hold that \(j_B = j_A\), which then contradicts that \(B_{t_{inf}} \not\subseteq A_{t_{inf}}\).

Taking the expectation over both \(v\) and \(T\), we get

\[
\Pr_{T,v}[j \not\in A_t] \leq \Pr_{T,v}[j \not\in B_t].
\]

This implies the Lemma 9 because

\[
\Pr_{T,v}[j \text{ not sold before } t | T_i = t] = \Pr_{T,v}[j \not\in A_t]
\]

\[
\Pr_{T,v}[j \text{ not sold before } t | T_i = t] = \Pr_{T,v}[j \not\in B_t].
\]

\(\square\)
3.2 XOS Combinatorial Auctions

In this section we prove our main result (restated below) for combinatorial auctions.

**Theorem 2.** There exists a \((1 - 1/e)\)-approximation algorithm to CAPS.

Recollect that the random valuation \(v_i\) of every buyer \(i\) has a polynomial support. We can therefore write the following expectation-version of the configuration LP, which gives us an upper bound on the expected offline social welfare.

\[
\begin{align*}
\max & \quad \sum_i \sum_k \sum_S v_i^k(S) \cdot x_{i,S}^k \\
\text{s.t.} & \quad \sum_i \sum_k \sum_{S:j \in S} x_{i,S}^k = 1 \quad \text{for all } j \in M \\
\sum_S x_{i,S}^k &= \Pr[v_i = v_i^k] \quad \text{for all } i,k \\
\end{align*}
\]

The above configuration LP can be solved with a polynomial number of calls to demand oracles of buyer valuations (see [DNS10]). Since all functions \(v_i^k\) are XOS, there exist additive supporting valuations; that is, there exist numbers \(v_{i,j}^{k,S} \geq 0\) s.t. \(v_{i,j}^{k,S} = 0\) for \(j \not\in S\), \(\sum_{j \in S} v_{i,j}^{k,S} = v_i^k(S)\), and \(\sum_{j \in S'} v_{i,j}^{k,S} \leq v_i^k(S')\) for all \(S'\). Before describing our algorithm, we define a base price for every item.

**Definition 10.** The base price \(b_j\) of every item \(j \in M\) is \(\sum_{i,k} \sum_{S : j \in S} v_{i,j}^{k,S} x_{i,S}^k\).

Since \(\sum_S x_{i,S}^k = \Pr[v_i = v_i^k]\), consider an algorithm that on arrival of buyer \(i\) with valuation \(v_i^k\) draws an independent random set \(S\) with probability \(x_{i,S}^k / \Pr[v_i = v_i^k]\). Let \(S_i^*\) denote this drawn set. This distribution also satisfies that for every item \(j\),

\[
\sum_i \mathbb{E}_{v_i,S_i^*} [1_{j \in S_i^*} \cdot v_{i,j}^{k,S_i^*}] = \sum_{i,k} \Pr[v_i = v_i^k] \cdot \sum_{S : j \in S} \frac{x_{i,S}^k}{\Pr[v_i = v_i^k]} \cdot v_{i,j}^{k,S_i^*} = b_j. \tag{6}
\]

Now consider the supporting additive valuation for \(S_i^*\) in the XOS valuation function \(v_i^k\) of buyer \(i\). This can be found using the XOS oracle for \(v_i^k\) [DNS10]. Our algorithm assigns her every item \(j\) that has not been allocated so far and for which \(v_{i,j}^{k,S_i^*} \geq \alpha(t) \cdot b_j\), where \(\alpha(t)\) is a continuous differentiable function of \(t\). Note that since we do not allow buyer \(i\) to choose items outside set \(S_i^*\), the mechanism defined by this algorithm need not be incentive compatible.

Consider the function \(r(t) = \sum_j q_j(t) \cdot b_j\), where again \(q_j(t)\) denotes the probability that item \(j\) has not been sold before time \(t\). Clearly, \(r(0) = OPT\). Using the following Lemma 11 and Claim 12, we prove that \(r\) is a residual function for our algorithm. Hence, Lemma 6 implies Theorem 2.

**Lemma 11.** The expected utility of the above algorithm is lower bounded by

\[
\mathbb{E}_{v,T}[Utility] \geq \sum_j \int_{t=0}^{1} q_j(t) \cdot (1 - \alpha(t)) \cdot b_j \cdot dt. \tag{7}
\]

**Proof.** Given that buyer \(i\) arrives at \(t\) and only buys item \(j\) if \(v_{i,j}^{k,S_i^*} \geq \alpha(t) \cdot b_j\), her utility is

\[
\mathbb{E}_{v,T}[u_i \mid T_i = t] = \sum_j \mathbb{E}_{v,T,S_i^*} [1_{j \text{ not sold by } t} \cdot 1_{j \in S_i^*} \cdot (v_{i,j}^{k,S_i^*} - \alpha(t) \cdot b_j)^+] \mid T_i = t
\]
Using the fact that whether \( j \) is sold before \( t \) only depends on \( v_{-i} \) and \( T \), and not on \( v_i \) or \( S^*_i \),

\[
E_{v,T}[u_i \mid T_i = t] = \sum_j \Pr_{v_{-i},T}[j \text{ not sold by } t \mid T_i = t] \cdot E_{v_i,S^*_i}[1_{j \in S^*_i} \cdot (v_{i,j}^{k,S^*_i} - \alpha(t) \cdot b_j)^+] .
\]

Now, observe that in our algorithm every buyer \( i \) independently decides which set of items \( S^*_i \) it will attempt to buy. Crucially, the probability of an item \( j \) being sold by time \( t \) can only increase if more buyers arrive before \( t \). Therefore,

\[
\Pr_{v_{-i},T}[j \text{ not sold by } t \mid T_i = t] \geq \Pr_{v,T}[j \text{ not sold by } t] = q_j(t) .
\]

Thus, we get

\[
E_{v,T}[u_i \mid T_i = t] \geq \sum_j q_j(t) \cdot \Pr_{v_i,S^*_i}[1_{j \in S^*_i} \cdot (v_{i,j}^{k,S^*_i} - \alpha(t) \cdot b_j)^+] .
\]

Finally, recollect from Eq. (6) that \( \sum_i E_{v_i,S^*_i}[1_{j \in S^*_i} \cdot v_{i,j}^{k,S^*_i}] = b_j \). Moreover,

\[
\sum_i E_{v_i,S^*_i}[1_{j \in S^*_i}] = \sum_i \Pr[v_i = v_i^k] \cdot \sum_{S : j \in S} x_{i,S}^{k} = 1 .
\]

Hence, by linearity of expectation

\[
\sum_i E[u_i \mid T_i = t] \geq \sum_j q_j(t) \cdot (1 - \alpha(t)) \cdot b_j .
\]

We next give a bound on the revenue generated by our algorithm.

**Claim 12.** We can bound the total expected revenue by

\[
E_{v,T}[Revenue] = -\sum_j \int_{t=0}^1 q_j'(t) \alpha(t) \cdot b_j \cdot dt . \tag{8}
\]

**Proof.** Since \(-q_j'(t)dt\) is the probability that item \( j \) is bought between \( t \) and \( t + dt \) (note \( q_j(t) \) is decreasing in \( t \)), we have

\[
E[Revenue] = -\sum_j \int_{t=0}^1 q_j'(t) \alpha(t) \cdot b_j \cdot dt .
\]

\(\square\)
4 Prophet Secretary for Matroids

Let \( v_i \) denote the random value of the \( i \)'th buyer (element) and let \( \hat{v}_i \) denote another independent draw from the value distribution of the \( i \)'th buyer. The problem is to select a subset \( I \) of the buyers that form a feasible set in matroid \( \mathcal{M} \), while trying to maximize \( \sum_{i \in I} v_i \). We restate our main result for the matroid setting.

**Theorem 1.** There exists a \((1 - 1/e)\)-approximation algorithm to MPS.

We need the following notation to describe our algorithm.

**Definition 13.** For a given vector \( \hat{v} \) of values of \( n \) items and \( A \subseteq \{n\} \), we define the following:

- Let \( \text{Opt}(\hat{v} \mid A) \subseteq \{n\} \setminus A \) denote the optimal solution set in the contracted matroid \( \mathcal{M}/A \).
- Let \( R(A, \hat{v}) := \sum_{i \in \text{Opt}(\hat{v} \mid A)} \hat{v}_i \) denote the remaining value after selecting set \( A \).

We next define a base price of for every buyer \( i \).

**Definition 14.** Let \( A \) denote the independent set of buyers that have been accepted till now.

- Let \( b_i(A, \hat{v}) := R(A, \hat{v}) - R(A \cup \{i\}, \hat{v}) \) denote a threshold for buyer \( i \).
- Let \( b_i(A) := \mathbb{E}[b_i(A, \hat{v})] \) denote the base price for buyer \( i \).

Starting with \( A_0 = \emptyset \), let \( A_t \) denote the set of accepted buyers before time \( t \). This is a random variable that depends on the values \( v \) and arrival times \( T \). Suppose a buyer \( i \) arrives at time \( t \), then our algorithm selects \( i \) iff both \( v_i > \alpha(t) \cdot b_i(A_t) \) and selecting \( i \) is feasible in \( \mathcal{M} \).

Consider the function \( r(t) := \mathbb{E}_{v, T}[R(A_t, \hat{v})] \), where \( A_t \) is a function of \( v \) and \( T \). Clearly, \( r(0) = \mathbb{E}[\text{OPT}] \). Using the following Lemma 16 and Claim 15, we prove that \( r \) is a residual function. Hence, Lemma 6 implies Theorem 1.

**Claim 15.**

\[
\mathbb{E}_{v, T}[\text{Revenue}] = -\int_{t=0}^{1} \alpha(t) \cdot r'(t) dt.
\]

**Proof.** Consider the time from \( t \) to \( t + \epsilon \) for some \( t \in [0, 1] \), \( \epsilon > 0 \). Let us fix the arrival times \( T \) and values \( v \) of all elements. This also fixes the sets \( (A_t)_{t \in [0,1]} \). Let \( i_1, \ldots, i_k \) be the arrivals between \( t \) and \( t + \epsilon \) that get accepted in this order. Note that it is also possible that \( k = 0 \). The revenue obtained between \( t \) and \( t + \epsilon \) is now given as

\[
\text{Revenue} \leq t + \epsilon - \text{Revenue} \leq t = \sum_{j=1}^{k} \alpha(t_{i_j}) b_{i_j}(A_{t_{i_j}})
\]

\[
= \sum_{j=1}^{k} \alpha(t_{i_j}) \mathbb{E}[R(A_t \cup \{i_1, \ldots, i_{j-1}\}, \hat{v}) - R(A_t \cup \{i_1, \ldots, i_j\}, \hat{v})]
\]

\[
\geq \alpha(t + \epsilon) \mathbb{E}[R(A_t, \hat{v}) - R(A_{t+\epsilon}, \hat{v})].
\]

Taking the expectation over \( v \) and \( T \), we get by linearity of expectation

\[
\mathbb{E}_v, T[\text{Revenue} \leq t + \epsilon] - \mathbb{E}_v, T[\text{Revenue} \leq t] \geq \alpha(t + \epsilon)(r(t) - r(t + \epsilon)).
\]
By the same argument, we also have
\[ \mathbb{E}_{\mathbf{v}, \mathbf{T}}[\text{Revenue}_{ \leq t + \epsilon}] - \mathbb{E}_{\mathbf{v}, \mathbf{T}}[\text{Revenue}_{ \leq t}] \leq \alpha(t)(r(t) - r(t + \epsilon)). \]

In combination, we get that
\[ \frac{d}{dt} \mathbb{E}_{\mathbf{v}, \mathbf{T}}[\text{Revenue}_{ \leq t}] = -\alpha(t)r'(t), \]
which implies the claim. \( \square \)

**Lemma 16.**
\[ \mathbb{E}_{\mathbf{v}, \mathbf{T}}[\text{Utility}] \geq \int_{t=0}^{1} (1 - \alpha(t)) \cdot r(t) \, dt. \]

**Proof.** The utility of buyer \( i \) arriving at time \( t \) is given by
\[ \mathbb{E}_{\mathbf{v}, \mathbf{T}}[u_i \mid T_i = t] = \mathbb{E}_{\mathbf{v}, \mathbf{T}_{-i}} \left[ (v_i - \alpha(t) \cdot b_i(A_t))^+ \cdot 1_{i \notin \text{Span}(A_t)} \mid T_i = t \right]. \]

Observe that \( A_t \) does not depend on \( v_i \) if \( T_i = t \) because it includes only the acceptances before \( t \). It does not depend on \( \hat{v}_i \) either, as \( \hat{v}_i \) is only used for analysis purposes and not known to the algorithm. Since \( v_i \) and \( \hat{v}_i \) are identically distributed, we can also write
\[ \mathbb{E}_{\mathbf{v}, \mathbf{T}}[u_i \mid T_i = t] = \mathbb{E}_{\mathbf{v}, \mathbf{T}_{-i}} \left[ (\hat{v}_i - \alpha(t) \cdot b_i(A_t))^+ \cdot 1_{i \notin \text{Span}(A_t)} \mid T_i = t \right]. \quad (9) \]

Now observe that buyer \( i \) can belong to \( \text{Opt}(\hat{v} \mid A_t) \) only if it’s not already in \( \text{Span}(A_t) \), which implies \( 1_{i \in \text{Span}(A_t)} \geq 1_{i \in \text{Opt}(\hat{v} \mid A_t)} \). Using this and removing non-negativity, we get
\[ \mathbb{E}_{\mathbf{v}, \mathbf{T}}[u_i \mid T_i = t] \geq \mathbb{E}_{\mathbf{v}, \mathbf{T}_{-i}} \left[ (\hat{v}_i - \alpha(t) \cdot b_i(A_t)) \cdot 1_{i \in \text{Opt}(\hat{v} \mid A_t)} \mid T_i = t \right]. \]

Now we use Lemma 17 to remove the conditioning on buyer \( i \) arriving at time \( t \) as this gives a valid lower bound on expected utility,
\[ \mathbb{E}_{\mathbf{v}, \mathbf{T}}[u_i \mid T_i = t] \geq \mathbb{E}_{\mathbf{v}, \mathbf{T}} \left[ (\hat{v}_i - \alpha(t) \cdot b_i(A_t)) \cdot 1_{i \in \text{Opt}(\hat{v} \mid A_t)} \right]. \quad (10) \]

We can now lower bound sum of buyers’ utilities using Eq. (10) to get
\[
\mathbb{E}_{\mathbf{v}, \mathbf{T}}[\text{Utility}] = \sum_i \int_{t=0}^{1} \mathbb{E}_{\mathbf{v}, \mathbf{T}}[u_i \mid T_i = t] \cdot dt \\
\geq \sum_i \int_{t=0}^{1} \mathbb{E}_{\mathbf{v}, \mathbf{T}} \left[ (\hat{v}_i - \alpha(t) \cdot b_i(A_t)) \cdot 1_{i \in \text{Opt}(\hat{v} \mid A_t)} \right] \cdot dt.
\]

By moving the sum over buyers inside the integrals, we get
\[
\mathbb{E}_{\mathbf{v}, \mathbf{T}}[\text{Utility}] \geq \int_{t=0}^{1} \mathbb{E}_{\mathbf{v}, \mathbf{T}} \left[ \sum_i (\hat{v}_i - \alpha(t) \cdot b_i(A_t)) \cdot 1_{i \in \text{Opt}(\hat{v} \mid A_t)} \right] \cdot dt \\
= \int_{t=0}^{1} \mathbb{E}_{\mathbf{v}, \mathbf{T}} \left[ R(A_t, \hat{v}) - \alpha(t) \cdot \sum_{i \in \text{Opt}(\hat{v} \mid A_t)} b_i(A_t) \right] \cdot dt.
\]

Finally, using Lemma 18 for \( V = \text{Opt}(\hat{v} \mid A_t) \), we get
\[ \mathbb{E}_{\mathbf{v}, \mathbf{T}}[\text{Utility}] \geq \int_{t=0}^{1} \mathbb{E}_{\mathbf{v}, \mathbf{T}} [(1 - \alpha(t)) \cdot R(A_t, \hat{v})] \cdot dt. \]
Finally, we prove the missing lemma that removes the conditioning on item $i$ arriving at $t$.

**Lemma 17.** For any $i$, any time $t$, and any fixed $v, \hat{v}$, we have

$$\mathbb{E}_{T \sim i} \left[ (\hat{v}_i - \alpha(t) \cdot b_i(A_t)) \cdot 1_{i \in \text{Opt}(\hat{v} \mid A_t)} \mid T_i = t \right] \geq \mathbb{E}_T \left[ (\hat{v}_i - \alpha(t) \cdot b_i(A_t)) \cdot 1_{i \in \text{Opt}(\hat{v} \mid A_t)} \right].$$

*Proof.* We prove the lemma for any fixed $T \sim i$. Suppose we draw a uniformly random $T_i \in [0, 1]$. Observe that if $T_i \geq t$ then we have equality in the above equation because set $A_t$ is the same both with and without $i$. This is also the case when $T_i < t$ but $i$ is not selected into $A_t$. Finally, when $T_i < t$ and $i \in A_t$ we have $1_{i \in \text{Opt}(\hat{v} \mid A_t)} = 0$ in the presence of item $i$ (i.e., RHS of lemma), making the inequality trivially true.

**Lemma 18.** For any fixed $v, T$, time $t$, and set of elements $V$ that is independent in the matroid $\mathcal{M} / A_t$, we have

$$\sum_{i \in V} b_i(A_t) \leq \mathbb{E}_{\hat{v}} [R(A_t, \hat{v})].$$

*Proof.* By definition

$$\sum_{i \in V} b_i(A_t) = \mathbb{E}_{\hat{v}} \left[ \sum_{i \in V} (R(A_t, \hat{v}) - R(A_t \cup \{i\}, \hat{v})) \right].$$

Fix the values $\hat{v}$ arbitrarily, we also have

$$\sum_{i \in V} (R(A_t, \hat{v}) - R(A_t \cup \{i\}, \hat{v})) \leq R(A_t, \hat{v}).$$

This follows from the fact that $R(A_t, \hat{v}) - R(A_t \cup \{i\}, \hat{v})$ are the respective critical values of the greedy algorithm on $\mathcal{M} / A_t$ with values $\hat{v}$. Therefore, the bound follows from Lemma 3.2 in [LB10]. An alternative proof is given as Proposition 2 in [KW12] while in our case the first inequality can be skipped and the remaining steps can be followed replacing $A$ by $A_t$.

Taking the expectation over $\hat{v}$, the claim follows. 

## 5 Fixed Threshold Algorithms

In this section we discuss the powers and limitations of *Fixed-Threshold Algorithms (FTAs)* for single item prophet secretary. In an FTA we set a fixed threshold for the item at the beginning of the process and then assign it to the first buyer whose valuation exceeds the threshold. The motivation to study FTAs comes from their simplicity, transparency, and fairness in the design of a posted price mechanism (see e.g., [FGL15]).

In Section 5.1 we give a $(1 - 1/e)$-approximation FTA for single-item prophet secretary. This seemingly contradicts earlier impossibility results (e.g., [FGL15, EHL17]). However, as we show, these impossibility results do not hold in case of continuous distributions or equivalently randomized tie-breaking. Next, in Section 5.2 we present an upper bound for FTAs. In particular, we show that there is no FTA, even for identical distributions, with an approximation factor better than $1 - 1/e$. This indicates the tightness of our algorithm for prophet secretary. Furthermore, in Appendix A we generalize these single item ideas to present an alternate $(1 - 1/e)$-approximation algorithm for bipartite matching prophet secretary.
5.1 Single Item Prophet Secretary

Our analysis in this section is based on discrete arrival times for the buyers. In particular, we assume that the buyers arrive based on an initially unknown permutation \( \pi \), and at every time \( i \) the values of \( \pi(i) \) and \( v_{\pi(i)} \) are revealed in an online fashion. We prove the following result.

**Theorem 19.** There exists a \((1-1/e)\)-approximation FTA for prophet secretary.

**Proof.** Without loss of generality, we assume that all distributions have a finite expectation and a continuous CDF\(^4\). As two extreme selections for the threshold, if we set \( \tau \) to zero then the FTA selects the first item, and if we set it to infinity then no item will be selected. Therefore, the assumption for the continuity of the distribution function allows us to select a threshold \( \tau \) such that the FTA reaches the end of the sequence with an exact probability of \( 1/e \). This means all of drawn values are below \( \tau \) with probability \( 1/e \). In the remainder, we show that the FTA based on this choice of \( \tau \) lead to a \((1-1/e)\)-approximation algorithm.

Let OPT denote the maximum of all \( v \)'s and Alg be a random variable that indicates the value selected by the algorithm, or is zero if no item is selected. The goal is to show

\[
\mathbb{E}[\text{Alg}] \geq \left(1 - \frac{1}{e}\right) \mathbb{E}[\text{OPT}].
\]

We have \( \mathbb{E}[\text{Alg}] = \mathbb{E}[\text{Revenue}] + \mathbb{E}[\text{Utility}] \). Due to the definition of \( \tau \) the algorithm makes a selection with probability \( 1 - 1/e \), therefore \( \mathbb{E}[\text{Revenue}] = (1 - \frac{1}{e}) \tau \). In the remainder, we complete the proof by showing a lower bound on \( \mathbb{E}[\text{Utility}] \) based on the expectation of OPT when it is greater than or equal to \( \tau \).

Let us first define some notations. We use \( q(j) \) to denote the probability that the algorithm does not pick any of the first \( j \) item, i.e. \( q(j) := \Pr[\max_{1 \leq k \leq j} \{v_{\pi(k)}\} < \tau] \). We use \( q_{-i}(j) \) as the probability of the algorithm not picking any of the first \( j - 1 \) items conditioned on the event that item \( i \) appears at position \( j \). We use the following lemma to lower bound \( q_{-i}(j) \) with \( q(j) \).

**Lemma 20.** For any item \( 1 \leq i \leq n \) and any position \( 1 \leq j \leq n \), we have \( q_{-i}(j) \geq q(j) \).

This lemma appeared in [EHLM17]. We present a simplified proof here for completeness.

**Proof.** Note that \( q(j) \) has two sources of randomness, one for the choices of \( \pi \) and one for the valuations of \( v \)'s. The lemma can be proven by carefully analyzing the former, i.e. by considering whether item \( i \) appears among the first \( j-1 \) items or not. More precisely,

\[
q(j) = \Pr[\pi^{-1}(i) < j] \Pr \left[ \max_{1 \leq k \leq j} \{v_{\pi(k)}\} < \tau \middle| \pi^{-1}(i) < j \right] + \Pr[\pi^{-1}(i) \geq j] \Pr \left[ \max_{1 \leq k \leq j} \{v_{\pi(k)}\} < \tau \middle| \pi^{-1}(i) \geq j \right]. \tag{11}
\]

For \( \Pr \left[ \max_{1 \leq k \leq j} \{v_{\pi(k)}\} < \tau \middle| \pi^{-1}(i) < j \right] \) in (11) we have choices of \( j-1 \) items other than \( i \) and require all of their values to be below \( \tau \). Therefore this probability is no more than \( q_{-i}(j) \). Similarly for \( \Pr \left[ \max_{1 \leq k \leq j} \{v_{\pi(k)}\} < \tau \middle| \pi^{-1}(i) \geq j \right] \) in (12) we have choices of at least \( j-1 \) items other than \( i \) and again require all of their values to be below \( \tau \). Hence, this term is at most \( q_{-i}(j) \) too, and we have

\[
q(j) \leq \Pr[\pi^{-1}(i) < j]q_{-i}(j) + \Pr[\pi^{-1}(i) \geq j]q_{-i}(j) = q_{-i}(j),
\]

\(^4\)This assumption is without loss because the actual CDF can be approximated with arbitrary precision by a continuous function. This approximation corresponds to a randomized tie-breaking in case of pointmasses.
which completes the proof.

Now for the utility we have

\[
\mathbb{E}[\text{Utility}] = \sum_{i=1}^{n} \mathbb{E}[u_i] \\
= \sum_{i=1}^{n} \sum_{j=1}^{n} \Pr(\pi(j) = i) \ q_{-i}(j) \ \mathbb{E}[v_i \cdot 1_{v_i \geq \tau}] \\
= \sum_{i=1}^{n} \sum_{j=1}^{n} q_{-i}(j) \ \mathbb{E}[v_i \cdot 1_{v_i \geq \tau}] \frac{1}{n}.
\]

By applying Lemma 20 we get

\[
\mathbb{E}[\text{Utility}] \geq \sum_{i=1}^{n} \sum_{j=1}^{n} q(j) \ \mathbb{E}[v_i \cdot 1_{v_i \geq \tau}] \frac{1}{n} \\
= \sum_{i=1}^{n} \mathbb{E}[v_i \cdot 1_{v_i \geq \tau}] \sum_{j=1}^{n} q(j) \frac{1}{n} \\
= \mathbb{E}\left[ \sum_{i=1}^{n} v_i \cdot 1_{v_i \geq \tau} \right] \sum_{j=1}^{n} q(j) \frac{1}{n} \\
\geq \mathbb{E}[\text{OPT} \cdot 1_{\text{OPT} \geq \tau}] \sum_{j=1}^{n} q(j) \frac{1}{n}. \quad (14)
\]

To complete the proof of the theorem we need to show that the sum in Inequality (14) is at least \(1 - 1/e\). Define \(p(i) := \Pr[v_i < \tau]\). For every \(j\) we have

\[
q(j) = \Pr\left[ \max_{1 \leq k \leq j} \{v_{\pi(k)}\} < \tau \right] \\
= \mathbb{E}_\pi \left[ \prod_{k=1}^{j} p(\pi(k)) \right] \\
= \mathbb{E}_\pi \left[ \exp\left( \sum_{k=1}^{j} \ln p(\pi(k)) \right) \right] \\
\text{exp is convex} \\
\geq \exp\left( \mathbb{E}_\pi \left[ \sum_{k=1}^{j} \ln p(\pi(k)) \right] \right) \\
= \exp\left( \frac{j}{n} \sum_{k=1}^{n} \ln p(k) \right) \\
= \exp\left( -\frac{j}{n} \right).
\]

The last inequality holds because our choice of \(\tau\) results in \(q(n) = \prod_{i=1}^{n} p(i) = 1/e\). Therefore we have

\[
\sum_{j=1}^{n} q(j) \frac{1}{n} \geq \sum_{j=1}^{n} \exp\left( -\frac{j}{n} \right) \frac{1}{n} \geq \int_{0}^{1} \exp(-x) dx = 1 - \frac{1}{e}.
\]

15
This gives us $\mathbb{E}[\text{Utility}] \geq (1 - 1/e) \mathbb{E}[\text{OPT} \cdot 1_{\text{OPT} \geq \tau}]$. Now we are ready to wrap up the proof. We have

$$
\mathbb{E}[\text{Alg}] = \mathbb{E}[\text{Revenue}] + \mathbb{E}[\text{Utility}]
\geq \left(1 - \frac{1}{e}\right) \mathbb{E}[\text{OPT} \cdot 1_{\text{OPT} \geq \tau}]
\geq \left(1 - \frac{1}{e}\right) (\mathbb{E}[\text{OPT} \cdot 1_{\text{OPT} < \tau}] + \mathbb{E}[\text{OPT} \cdot 1_{\text{OPT} \geq \tau}])
= \left(1 - \frac{1}{e}\right) \mathbb{E}[\text{OPT}],
$$

which completes the proof.

\[\square\]

### 5.2 Impossibility for IID Prophet Inequalities

In the following we prove an impossibility result for FTAs for single item prophet secretary. We show this impossibility even for the special case of iid items. For every $n$, we give a common distribution $D$ for every item such that no FTA can achieve an approximation factor better than $1 - 1/e$. This also implies the tightness of the algorithm discussed in Section 5.1.

**Theorem 21.** Any FTA for iid prophet inequality is at most $(1 - \frac{1}{e} + O(\frac{1}{n}))$-approximation.

**Proof.** We prove the theorem by giving a hard input instance for every $n$ as follows: every $v_i$ is $n/(e - 1)$ with probability $1/n^2$ and is $(e - 2)/(e - 1)$ otherwise. The expected maximum value of these $n$ items is

$$
\mathbb{E}[\text{OPT}] = \left(1 - \frac{1}{n^2}\right)^n \cdot \frac{e - 2}{e - 1} + \left(1 - \left(1 - \frac{1}{n^2}\right)^n\right) \frac{n}{e - 1}
= 1 - O\left(\frac{1}{n}\right).
$$

In this instance, if $\tau < (e - 2)/(e - 1)$ then the algorithm selects the first item, and if $(e - 2)/(e - 1) < \tau \leq n/(e - 1)$ then the algorithm can only select $n/(e - 1)$. In these cases the approximation factor can be at most $(e - 2)/(e - 1) \approx 0.58$.

Now, note that the CDF of this input distribution is not continuous. Reshaping a discrete distribution function into a continuous one, however, does not change the approximation factor because for example in the above instance we only need a very slight change at the point $(e - 2)/(e - 1)$ of the CDF. This change gives us a randomness when $\tau = (e - 2)/(e - 1)$, which is equivalent to flipping a random coin and skipping every item with some probability $p \leq 1 - 1/n^2$ if the drawn value is $(e - 2)/(e - 1)$. With this assumption we have

$$
\mathbb{E}[\text{Alg}] = \sum_{i=1}^{n} p^i \mathbb{E}[v_i \cdot 1_{v_i \geq \tau}]
= \frac{1 - p^n}{1 - p} \mathbb{E}[v_i \cdot 1_{v_i \geq \tau}]
= \frac{1 - p^n}{1 - p} \left(\left(1 - \frac{1}{n^2} - p\right) \frac{e - 2}{e - 1} + \frac{1}{n^2} \frac{n}{e - 1}\right)
< \frac{1 - p^n}{e - 1} \left(e - 2 + \frac{1}{n(1 - p)}\right).
$$

(15)
To complete the proof, it suffices to show that the right hand side of Inequality (15) is at most $1 - 1/e + O(1/n)$. To this end, we try to maximize this term based on parameter $c$ where $p = 1 - c/n$. We can rewrite the right hand side of the inequality as

$$\frac{1 - (1 - \frac{c}{n})^n}{e - 1} \left( e - 2 + \frac{1}{c} \right).$$

If $c \in \Theta(n)$ then this term is at most $(e - 2 + \Theta(1/n))/(e - 1) \approx 0.41 + O(1/n)$ which is below $1 - 1/e$ for sufficiently large $n$. Otherwise $c/n \ll 1$ and we can approximate $(1 - c/n)^n$ as $e^{-c} + O(1/n)$. This upper bounds Inequality (15) by $(1 - e^{-c})(e - 2 + 1/c)/(e - 1) + O(1/n)$, where the first term is independent of $n$ and is at most $1 - 1/e$ for different constants $c$; thereby completing the proof.

We would like to note that the continuity of the CDF of the input distributions is a useful and natural property that can be used by an FTA. This is because making this assumption allows us to design a $(1 - 1/e)$-approximation algorithm, as shown by Theorem 19, but not assuming this puts a barrier of $1/2$ for any FTA, which is shown by [EHLM17]5. For example, in the above instance the approximation factor without assuming continuity would be at most $(e - 2)/(e - 1) \approx 0.58$, which is below the $1 - 1/e \approx 0.63$ claim of Theorem 19. This contradiction is because without this assumption on the input distribution the algorithm could not set $\tau$ in a way that the probability of selecting an items became exactly $1 - 1/e$.

Acknowledgments. We thank a number of colleagues for useful discussions. In particular, we are grateful to Matt Weinberg, Bobby Kleinberg, Anupam Gupta, and Hossein Esfandiari.

References

[AEEl+17] Melika Abolhassani, Soheil Esfandiari, Hossein Esfandiari, MohammadTaghi HajiAghayi, Robert Kleinberg, and Brendan Lucier. Beating 1-1/e for ordered prophets. In Proceedings of the 49th Annual ACM SIGACT Symposium on Theory of Computing, pages 61–71. ACM, 2017.

[AHL12] Saeed Alaei, MohammadTaghi Hajiaghayi, and Vahid Liaghat. Online prophet-inequality matching with applications to ad allocation. In Proceedings of the 13th ACM Conference on Electronic Commerce, pages 18–35. ACM, 2012.

[Ala14] Saeed Alaei. Bayesian combinatorial auctions: Expanding single buyer mechanisms to many buyers. SIAM Journal on Computing, 43(2):930–972, 2014.

[AM06] Lawrence M Ausubel and Paul Milgrom. The lovely but lonely vickrey auction. Combinatorial auctions, 17:22–26, 2006.

[BHZ10] MohammadHossein Bateni, MohammadTaghi Hajiaghayi, and Morteza Zadimoghaddam. Submodular secretary problem and extensions. In Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques, pages 39–52. Springer, 2010.

[BIK07] Moshe Babaioff, Nicole Immorlica, and Robert Kleinberg. Matroids, secretary problems, and online mechanisms. In Proceedings of the eighteenth annual ACM-SIAM symposium on Discrete algorithms, pages 434–443. Society for Industrial and Applied Mathematics, 2007.

5Their hardness instance contains different distributions, hence does not necessarily apply to iid prophet inequality.
[CFH+17] José Correa, Patricio Foncea, Ruben Hoeksma, Tim Oosterwik, and Tjark Vredeveld. Posted price mechanisms for a random stream of customers. In Proceedings of the 2017 ACM Conference on Economics and Computation, pages 169–186. ACM, 2017.

[CHMS10] Shuchi Chawla, Jason D. Hartline, David L. Malec, and Balasubramanian Sivan. Multi-parameter mechanism design and sequential posted pricing. In Proceedings of the 42nd ACM Symposium on Theory of Computing, STOC 2010, Cambridge, Massachusetts, USA, 5-8 June 2010, pages 311–320, 2010.

[DEH+15] Sina Dehghani, Soheil Ehsani, MohammadTaghi Hajiaghayi, Vahid Liaghat, and Saeed Seddighin. Online survivable network design and prophets. 2015.

[DEH+17] Sina Dehghani, Soheil Ehsani, MohammadTaghi Hajiaghayi, Vahid Liaghat, and Saeed Seddighin. Stochastic k-Server: How Should Uber Work? In ICALP 2017, 2017.

[DFKL17] Paul Dütting, Michal Feldman, Thomas Kesselheim, and Brendan Lucier. Prophet inequalities made easy: Stochastic optimization by pricing non-stochastic inputs. In IEEE 58th Annual Symposium on Foundations of Computer Science, FOCS 2017, Berkeley, CA, USA, 14-17 October, 2017, pages 540–551, 2017.

[DK15] Paul Dütting and Robert Kleinberg. Polymatroid prophet inequalities. In Algorithms-ESA 2015, pages 437–449. Springer, 2015.

[DNS10] Shahar Dobzinski, Noam Nisan, and Michael Schapira. Approximation algorithms for combinatorial auctions with complement-free bidders. Mathematics of Operations Research, 35(1):1–13, 2010.

[Dyn63] Eugene B Dynkin. The optimum choice of the instant for stopping a markov process. In Soviet Math. Dokl, volume 4, 1963.

[EHLM17] Hossein Esfandiari, MohammadTaghi Hajiaghayi, Vahid Liaghat, and Morteza Moneimizadeh. Prophet secretary. SIAM J. Discrete Math., 31(3):1685–1701, 2017.

[Fei09] Uriel Feige. On maximizing welfare when utility functions are subadditive. SIAM Journal on Computing, 39(1):122–142, 2009.

[FGL15] Michal Feldman, Nick Gravin, and Brendan Lucier. Combinatorial auctions via posted prices. In Proceedings of the Twenty-Sixth Annual ACM-SIAM Symposium on Discrete Algorithms, pages 123–135. Society for Industrial and Applied Mathematics, 2015.

[FSZ15] Moran Feldman, Ola Svensson, and Rico Zenklusen. A simple O (log log (rank))-competitive algorithm for the matroid secretary problem. In Proceedings of the Twenty-Sixth Annual ACM-SIAM Symposium on Discrete Algorithms, pages 1189–1201, 2015.

[FZ15] Moran Feldman and Rico Zenklusen. The submodular secretary problem goes linear. In Foundations of Computer Science (FOCS), 2015 IEEE 56th Annual Symposium on, pages 486–505. IEEE, 2015.

[GGLS08] Naveen Garg, Anupam Gupta, Stefano Leonardi, and Piotr Sankowski. Stochastic analyses for online combinatorial optimization problems. In Proceedings of the nineteenth annual ACM-SIAM symposium on Discrete algorithms, pages 942–951. Society for Industrial and Applied Mathematics, 2008.
[GHK+14] Oliver Göbel, Martin Hoefer, Thomas Kesselheim, Thomas Schleiden, and Berthold Vöcking. Online independent set beyond the worst-case: Secretaries, prophets, and periods. In *International Colloquium on Automata, Languages, and Programming*, pages 508–519. Springer, 2014.

[GM08] Gagan Goel and Aranyak Mehta. Online budgeted matching in random input models with applications to adwords. In *Proceedings of the Nineteenth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 982–991, 2008.

[GRST10] Anupam Gupta, Aaron Roth, Grant Schoenebeck, and Kunal Talwar. Constrained non-monotone submodular maximization: Offline and secretary algorithms. In *International Workshop on Internet and Network Economics*, pages 246–257. Springer, 2010.

[GS17] Guru Prashanth Guruganesh and Sahil Singla. Online matroid intersection: Beating half for random arrival. In *International Conference on Integer Programming and Combinatorial Optimization*, pages 241–253. Springer, 2017.

[HKP04] Mohammad Taghi Hajiaghayi, Robert Kleinberg, and David C Parkes. Adaptive limited-supply online auctions. In *Proceedings of the 5th ACM conference on Electronic commerce*, pages 71–80. ACM, 2004.

[HKS07] Mohammad Taghi Hajiaghayi, Robert Kleinberg, and Tuomas Sandholm. Automated online mechanism design and prophet inequalities. In *AAAI*, volume 7, pages 58–65, 2007.

[K+85] DP Kennedy et al. Optimal stopping of independent random variables and maximizing prophets. *The Annals of Probability*, 13(2):566–571, 1985.

[Ken87] DP Kennedy. Prophet-type inequalities for multi-choice optimal stopping. *Stochastic Processes and their Applications*, 24(1):77–88, 1987.

[Ker86] Robert P Kertz. Comparison of optimal value and constrained maxima expectations for independent random variables. *Advances in applied probability*, 18(02):311–340, 1986.

[Kle05] Robert Kleinberg. A multiple-choice secretary algorithm with applications to online auctions. In *Proceedings of the sixteenth annual ACM-SIAM symposium on Discrete algorithms*, pages 630–631. Society for Industrial and Applied Mathematics, 2005.

[KMT11] Chinmay Karande, Aranyak Mehta, and Pushkar Tripathi. Online bipartite matching with unknown distributions. In *Proceedings of the forty-third annual ACM symposium on Theory of computing*, pages 587–596. ACM, 2011.

[KMZ15] Nitish Korula, Vahab Mirrokni, and Morteza Zadimoghaddam. Online submodular welfare maximization: Greedy beats 1/2 in random order. In *Proceedings of the forty-seventh annual ACM symposium on Theory of computing*, pages 889–898. ACM, 2015.

[KP09] Nitish Korula and Martin Pál. Algorithms for secretary problems on graphs and hypergraphs. In *International Colloquium on Automata, Languages and Programming*, pages 508–520. Springer, 2009.

[KRTV13] Thomas Kesselheim, Klaus Radke, Andreas Tönnis, and Berthold Vöcking. An optimal online algorithm for weighted bipartite matching and extensions to combinatorial auctions. In *European Symposium on Algorithms*, pages 589–600. Springer, 2013.
Extension of FTAs to Bipartite Matchings

Here we study the set of algorithms that use $m$ fixed thresholds $\tau_1, \ldots, \tau_m$ for the items and have a recommendation strategy which at the arrival of every buyer offers her an item at its fixed price. In particular, when buyer $i$ arrives the algorithm recommends an unsold item $k$ to her and she accepts to buy it if $v_{i,k} \geq \tau_k$, i.e. her valuation for the item is greater than or equal to its price.

**Theorem 22.** For every instance of matching prophet secretary there exists a sequence of fixed thresholds $\tau_1, \ldots, \tau_k$ and a randomized algorithm which is $(1 - 1/e)$-approximation in expectation.
Proof. Our general approach is to extend the methods we have for single item FTA’s to an algorithm for matchings. This generalization is similar to a reduction from matchings to single items. However, there are some details that we have to consider. We first show a stronger claim than the statement of Theorem 19 which holds for a specific inputs class of prophet secretary. Then we propose a randomized algorithm which exploits the single item algorithm for that class in order to find such fixed thresholds that lead to a \((1 - 1/e)\)-approximation algorithm for matchings.

We note that the analysis of Theorem 19 indicates we can find a single threshold such that every item will be seen with probability at least \(1 - 1/e\). More precisely, the analysis shows

\[
\sum_{j=1}^{n} q(j) \geq 1 - 1/e,
\]

and consequently Inequality (13) shows the algorithm gets the same approximation factor from the utility of every buyer, i.e. \(\mathbb{E}[\text{Utility}] \geq (1 - 1/e) \sum_{i=1}^{n} \mathbb{E}[v_i \cdot 1_{v_i \geq \tau}]\). Now, if an input instance guarantees \(\sum_{i=1}^{n} \Pr[v_i > 0] \leq 1\) then we will have

\[
\mathbb{E}[\text{Revenue}] = \left(1 - \frac{1}{e}\right) \tau \\
\geq \left(1 - \frac{1}{e}\right) \tau \sum_{i=1}^{n} \Pr[v_i > 0] \\
\geq \left(1 - \frac{1}{e}\right) \sum_{i=1}^{n} \mathbb{E}[v_i \cdot 1_{v_i < \tau}] .
\]

This results in

\[
\mathbb{E}[\text{Alg}] = \mathbb{E}[\text{Revenue}] + \mathbb{E}[\text{Utility}] \\
\geq \left(1 - \frac{1}{e}\right) \sum_{i=1}^{n} (\mathbb{E}[v_i \cdot 1_{v_i < \tau}] + \mathbb{E}[v_i \cdot 1_{v_i \geq \tau}]) \\
= \left(1 - \frac{1}{e}\right) \sum_{i=1}^{n} \mathbb{E}[v_i] .
\]

The following claim formally states the above result.

Claim 23. If the input of prophet secretary guarantees \(\sum_{i=1}^{n} \Pr[v_i > 0] \leq 1\), then there exists an FTA such that

\[
\mathbb{E}[\text{Alg}] \geq \left(1 - \frac{1}{e}\right) \sum_{i=1}^{n} \mathbb{E}[v_i] .
\]

Now we demonstrate how the matching problem reduces to the instances describe above. Let us assume we already know thresholds \(\tau_1, \ldots, \tau_m\) for the \(m\) items. Upon the arrival of buyer \(i\) and realizing \(v_i\) we use the following algorithm to recommend an item \(k\) to buyer \(i\). We first calculate probabilities \(p_1, \ldots, p_m\) where \(p_k(v_i) := \Pr_{B \in \mathcal{M}}[(i, k) \in \mathcal{M}(B)]\) and \(\mathcal{M}(B)\) is the maximum matching of a bipartite graph \(B\). These are in fact the probabilities of each of those edges belonging to the maximum matching. Then, by drawing a random number \(r \in [0, 1]\) we select a candidate item \(k\) if \(\sum_{l=1}^{k-1} p_l < r \leq \sum_{l=1}^{k} p_l\). In this way, we dependently select a candidate such that every \(k\) becomes selected with probability \(p_k\). Note that the algorithm might sometimes select none of the items, in which cases there will be no candidate. Finally we recommend item \(k\) to buyer \(i\) if the item is still unsold, and she buys it if \(v_{i,k} \geq \tau_k\).

The above method for candidate selection has a close relationship with the optimum solution. To put it into perspective, let us define a new distribution \(\hat{D}_i : \mathbb{R}^m \rightarrow [0, 1]\) for every buyer \(i\).

\(^6\)WLOG we can assume it is unique for every graph.
This distribution is supposed to show the valuations of \( i \) on the items when they are selected as candidates. In other words, for every vector \( x = (x_1, \ldots, x_m) \) in which at most one of \( x_k \)'s is non-zero we have \( \Pr_{\hat{v}_i \sim \hat{D}_i}[\hat{v}_i = x] := \mathbb{E}_{v_i \sim D_i}[1_{v_i,k = x_k} \cdot 1_k \text{ is a candidate}] \). Equivalently, \( \hat{D}_i \) can be interpreted as the distribution of the value of the edge incident to \( i \) in the maximum matching. This is true because we select a candidate with the probability that it belongs to the maximum matching. Therefore:

\[
\mathbb{E}[\text{OPT}] = \mathbb{E}_{v \sim D}[M(v)] = \sum_{i=1}^{n} \mathbb{E}_{\hat{v}_i \sim \hat{D}_i}[\sum_{k=1}^{m} \hat{v}_{i,k}].
\] (16)

Now we reduce the problem to the single item case. By looking at a scenario of the problem from the viewpoint of item \( k \) we notice that the whole scenario and the algorithm run equivalent to the single item case. This item observes the buyers in a random order such that the valuation of buyer \( i \) comes from \( \hat{D}_i \). These scenarios occur in parallel for all the items, because no two items are offered to buyers at the same time. In addition, every item \( k \) is offered to a buyer with an overall probability of

\[
\sum_{i=1}^{n} \Pr_{\hat{v}_i \sim \hat{D}_i}[\hat{v}_{i,k} > 0] = \sum_{i=1}^{n} \mathbb{E}_{v_i, v_{-i}}[1_{v_{i,k} > 0} \cdot 1_{(i,k) \in M(v_i \cup v_{-i})}] = \Pr_{v \sim D}[k \text{ is matched}] \leq 1.
\]

Now we can use the result of Claim 23. The right hand side of Equality (16) can be written as \( \sum_{k=1}^{m} \sum_{i=1}^{n} \mathbb{E}_{\hat{v}_i \sim \hat{D}_i}[\hat{v}_{i,k}] \). The claim states that there exists a threshold \( \tau_k \) such that the FTA achieves at least \((1 - 1/e)\) \( \sum_{i=1}^{n} \mathbb{E}_{\hat{v}_i \sim \hat{D}_i}[\hat{v}_{i,k}] \) for every item \( k \). Therefore our algorithm is \((1 - 1/e)\)-approximation for the matching of all items.