Quantitative analysis of passive systems interconnected on graphs

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Abstract: This paper addresses quantitative performance analysis of an interconnected passive system. The passivity property, which provides a unified and abstracted description of dynamical systems, plays an important role for qualitative stability analysis of interconnected large-scale systems. In this paper, quantitative performance is further evaluated for the interconnected passive system. To this end, a performance-characterizing parameter is integrated into the conventional passivity. Then, by using the parameter, the $L_2$-gain of the entire interconnected system is evaluated. Further assuming that the interconnection rule is described by a graph, more detailed performance analysis and its reinforcement via network expansion are studied.

Key Words: passivity, interconnected systems, $L_2$-gain analysis, graph theory

1. Introduction

Passivity is a property of dynamical systems, which can provide a more abstracted description of the systems than standard models such as the state-space equations. It is known from the passivity theorem [23] that the interconnection of any passive systems inherits the individual passivity property if they are connected in the negative feedback. On the basis of the theorem, the passivity has been utilized for qualitative stability analysis of interconnected dynamical systems [15].

In the last decade, the passivity-based analysis is further applied to various control problems such as synchronization and cooperative control of multi-agent systems [2, 4, 8, 9, 11, 24]. Detailed models expressing agents are not required in the analysis and synthesis. Therefore, the passivity-based approach is compatible with such large-scale multi-agent systems, which cannot be modeled accurately. There have been studied qualitative stability analysis and stabilization of the entire systems. However, quantitative performance analysis has not been well studied. This paper addresses the quantitative performance analysis of interconnected passive systems.

In the previous work by the authors [13, 21], performance analysis of general interconnection of
passive systems is studied. In addition, it is shown that a class of the interconnection rules reinforces the $L_2$-gain performance of the entire system compared with disconnected subsystems. The results in [13] are refined in this paper. Then, a graph is considered in the interconnection rule and more detailed $L_2$-gain analysis is given.

Notation: The symbols $L_2$ and $L_{2c}$ denote the $L_2$-space and the extended $L_2$-space, respectively. For a causal and $L_2$-stable system $\Sigma$, the symbol $\|\Sigma\|_{L_2}$ denotes the $L_2$-gain:

$$\|\Sigma\|_{L_2} := \sup_{r \in L_2 \setminus \{0\}} \frac{\|r\|_{L_2}}{\|r\|_{L_2}},$$

where $r$ and $z$ are the input and output of $\Sigma$, respectively. For $L_{2c}$ signals $f$ and $g$, the symbol $(f, g)_T$ denotes the inner product:

$$(f, g)_T := \int_0^T f^\top(\tau)g(\tau)d\tau.$$ 

The symbol $1_k$ denotes a column vector in $\mathbb{R}^k$ where every element is equal to one (this is said to be an all-ones vector). For a positive-semidefinite matrix $X$, the symbols $\lambda_{\max}(X)$, $\lambda_{\min}(X)$, and $\lambda_2(X)$ denote the largest, smallest, and second smallest eigenvalues of $X$, respectively.

2. Preliminaries: system description and definitions

2.1 System description

We consider an interconnected dynamical system $\Sigma_{NW}$ illustrated in Fig. 1. The system $\Sigma_{NW}$ is composed of $N$ subsystems $\Sigma_i$, $i \in \{1, 2, \ldots, N\}$. Each of them is described as

$$\Sigma_i : y_i = \bar{\Sigma}_i u_i,$$

where $\bar{\Sigma}_i : L_{2c} \to L_{2c}$ is a causal operator, and $y_i \in \mathbb{R}^{m_i}$ and $u_i \in \mathbb{R}^{m_i}$ denote the output and input of $\Sigma_i$, respectively. For example, suppose that $\Sigma_i$ is a linear time-invariant dynamical system. Then, letting $\bar{\Sigma}_i(s)$ be the transfer function representation, we can describe $\Sigma_i$ as $y_i(s) = \bar{\Sigma}_i(s)u_i(s)$, where $y_i(s)$ and $u_i(s)$ are the Laplace transformation of $y_i$ and $u_i$, respectively.

Letting $m := \sum_{i=1}^N m_i$, we define $u := [u_1^\top \ u_2^\top \ \cdots \ u_N^\top]^\top \in \mathbb{R}^m$ and $y := [y_1^\top \ y_2^\top \ \cdots \ y_N^\top]^\top \in \mathbb{R}^m$. Then, the interconnection between subsystems $\Sigma_i$, $i \in \{1, 2, \ldots, N\}$ with the exogenous input $w \in \mathbb{R}^p$ is described as

$$u = -Ly + Ew,$$  \hspace{1cm} (1)

where $L \in \mathbb{R}^{m \times m}$ and $E \in \mathbb{R}^{m \times p}$. Furthermore, the regulated output $z \in \mathbb{R}^p$ is given by

$$z = E^\top y.$$  \hspace{1cm} (2)

The output is utilized for the performance evaluation of the entire system $\Sigma_{NW}$. We consider that $E$ is of full rank.

In this paper, we assume that $\Sigma_{NW}$ is well-posed, i.e., for any $w \in L_{2c}$, internal signals $y$ and $u$ uniquely exist and belong to $L_{2c}$. Then, letting $\bar{\Sigma}_{NW} : L_{2c} \to L_{2c}$ be a causal operator, we describe $\Sigma_{NW}$ as

$$\Sigma_{NW} : \ z = \bar{\Sigma}_{NW} w.$$  

2.2 Passivity for quantitative analysis

In this paper, we aim to analyze $\Sigma_{NW}$ using no detailed models of $\Sigma_i$, $i \in \{1, 2, \ldots, N\}$. To this end, abstracted model-sets are defined to describe the systems. Then, we consider that $\Sigma_i$ belongs to one of the model-sets to quantitatively analyze $\Sigma_{NW}$.

We define passivity to describe dynamical systems.
Definition 1 (see e.g. [7]) Consider a dynamical system $\Sigma$ with input $r \in \mathbb{R}^p$ and output $v \in \mathbb{R}^p$. Then, $\Sigma$ is said to be passive if there exists a constant $\beta \in \mathbb{R}$ such that for any input $r \in L_{2e}$ and its corresponding output $v$, it holds that
\begin{equation}
\langle r, v \rangle_T \geq \beta
\end{equation}
for all $T \in \mathbb{R}_+$. The condition (3) can be used to characterize the set of dynamical systems having the passivity property. The passivity enables model-set-based analysis of interconnected systems. Consider two passive subsystems. Then, the passivity theorem [23] states that their negative feedback system inherits the passivity property from the subsystems. We can further show the stability of the feedback system with some additional condition. As illustrated in this example, no precise model of the subsystem is required for the qualitative analysis of the feedback system. The analysis can be extended to stability analysis and stabilization of more general interconnected systems [3, 10, 12, 15] and consensus and cooperative control problems of multi-agent systems [2, 8, 11, 24].

In this paper, we integrate parameters to the passivity to enable quantitative analysis of interconnected systems. To this end, we define quantitative passivity property as follows.

Definition 2 Let $Q \in \mathbb{R}^{p \times p}$ be a symmetric matrix. Consider a dynamical system $\Sigma$ with input $r \in \mathbb{R}^p$ and output $v \in \mathbb{R}^p$. Then, $\Sigma$ is said to be $Q$-passive if there exists $\beta \in \mathbb{R}$ such that for any input $r \in L_{2e}$ and its corresponding output $v$, it holds that
\begin{equation}
\langle r, v \rangle_T - \frac{1}{2} \langle v, Qv \rangle_T \geq \beta
\end{equation}
for all $T \in \mathbb{R}_+$. This performance-integrated passivity gives a model-set characterized with a matrix-parameter. To emphasize this and show the dependency of the model-set on the matrix, we give the following notation.

Notation 1 For a symmetric matrix $Q$, the symbol $\mathcal{S}_p(Q)$ denotes the set of all the $Q$-passive systems.

For $Q > 0$, $\Sigma \in \mathcal{S}_p(Q)$ is said to be output-strictly passive. In addition, according to the notion in [19], it can be said that $\Sigma \in \mathcal{S}_p(Q)$ with $Q > 0$ ($Q \not\succ 0$) represents a system with the excess (shortage) of passivity.

It is further shown that any system in $\mathcal{S}_p(Q)$ with $Q > 0$ has a bounded $L_2$-gain, which is evaluated as follows.

Proposition 1 (see e.g. [13]) Let $Q > 0$. Then, any $\Sigma \in \mathcal{S}_p(Q)$ is $L_2$-stable and satisfies
\begin{equation}
\|\Sigma\|_{L_2} \leq \gamma(Q),
\end{equation}
where $\gamma(Q)$ is defined as
\begin{equation}
\gamma(Q) := 2\lambda_{\min}^{-1}(Q).
\end{equation}
The proposition provides a performance evaluation of any $\Sigma \in S_p(Q)$. We present the following definition for the model-set $S_p(Q)$.

**Definition 3** Consider $S_p(Q)$ with $Q > 0$. Then, the value of $\gamma(Q)$ is said to be the **proveable $L_2$-gain** of $S_p(Q)$.

In the following discussion, the proveable $L_2$-gain is a performance criterion for any dynamical system in the model-set $S_p(Q)$.

In this paper, we perform model-set-based quantitative analysis of $\Sigma_{NW}$: First, the subsystems are described as $\Sigma_i \in S_p(Q_i), i \in \{1, 2, \ldots, N\}$. Then, performance-characterizing $Q_{NW}$ is evaluated with $Q_i, i \in \{1, 2, \ldots, N\}$ such that $\Sigma_{NW} \in S_p(Q_{NW})$ holds. Finally, the problem of **performance reinforcement** via interconnection is addressed: We find a class of the connection matrices $L$ and $E$ such that the proveable $L_2$-gain of $S_p(Q_{NW})$ is strictly less than that for the disconnected case, i.e., $L = 0$.

**Remark 1** (Comparison with related concepts and analysis) Parameters are integrated into the passivity to quantify the passivity property in the literature [1, 5, 16, 18, 19, 22]. For example, $\gamma$-passivity [18], excess or shortage of the passivity [19], and passivity indices [1, 5, 22] are introduced. Then, they are utilized for the qualitative stability analysis of feedback and more general interconnected systems. Furthermore, passivity degradation caused by model approximation such as discretization or quantization, is quantitatively evaluated for single systems in e.g. [16, 22]. This paper is devoted to **quantitative** performance analysis of general interconnected systems. Then, interconnection rules for **performance reinforcement** of the entire systems are studied.

## 3. Quantitative analysis of interconnected passive system

The quantitative passivity of the interconnected system $\Sigma_{NW}$ is studied in this section. In the following sections, $\Sigma_{NW}$ is denoted by $\Sigma_{NW}(L, E)$ to explicitly signify the dependency on the connection matrices $L$ and $E$. For example, $\Sigma_{NW}(0, E)$ represents that the subsystems are completely disconnected each other. Then, using a specified pair $(L, E)$, quantitative passivity and performance of $\Sigma_{NW}(L, E)$ are evaluated.

### 3.1 General analysis on passivity of interconnected system

Consider the following description of subsystems and connection rule:

\[
\Sigma_i \in S_p(Q_i), \quad i \in \{1, 2, \ldots, N\}
\]

holds for some symmetric $Q_i \in \mathbb{R}^{m_i \times m_i}, i \in \{1, 2, \ldots, N\}$. Letting $Q \in \mathbb{R}^{m \times m}$ as

\[
Q := \text{diag}(Q_1, Q_2, \ldots, Q_N),
\]

we have the following proposition.

**Proposition 2** Suppose that

\[
L^T + L + Q > 0
\]

holds. Then, $\Sigma_{NW}(L, E) \in S_p(Q_{NW})$, where $Q_{NW} \in \mathbb{R}^{p \times p}$ is given by

\[
Q_{NW} = (E^T(L^T + L + Q)^{-1}E)^{-1}.
\]

In addition, the proveable $L_2$-gain of $S_p(Q_{NW})$ is given by

\[
\gamma(Q_{NW}) = 2\lambda_{\text{max}}(E^T(L^T + L + Q)^{-1}E).
\]
**Proof of Proposition 2:** We will show that for any input \( w \in L_{2e} \) to \( \Sigma_{NW}(L, E) \), there exists \( \beta_{NW} \in \mathbb{R} \) such that

\[
\langle w, z \rangle_T - \frac{1}{2} \langle z, Q_{NW} z \rangle_T \geq \beta_{NW}
\]

or equivalently

\[
\left\langle \begin{bmatrix} w \\ z \end{bmatrix}, \begin{bmatrix} 0 & I_p \\ I_p & -Q_{NW} \end{bmatrix} \begin{bmatrix} w \\ z \end{bmatrix} \right\rangle_T \geq 2\beta_{NW}
\]

holds for all \( T \in \mathbb{R}_+ \). Since \( z = E^Ty \), this (12) is equivalent to

\[
\left\langle \begin{bmatrix} w \\ y \end{bmatrix}, \begin{bmatrix} 0 & E^T \\ E & -E Q_{NW} E^T \end{bmatrix} \begin{bmatrix} w \\ y \end{bmatrix} \right\rangle_T \geq 2\beta_{NW}.
\]

Since \( \Sigma_{NW}(L, E) \) is well-posed, we see that \( u_i \in L_{2e} \) holds in each \( \Sigma_i \) if \( w \in L_{2e} \). Then,

\[
\left\langle \begin{bmatrix} u_i \\ y_i \end{bmatrix}, \begin{bmatrix} 0 & I_{m_i} \\ I_{m_i} & -Q_i \end{bmatrix} \begin{bmatrix} u_i \\ y_i \end{bmatrix} \right\rangle_T \geq \beta_i
\]

holds for all \( T \in \mathbb{R}_+ \). From (8), it follows that

\[
\sum_{i=1}^N \left\langle \begin{bmatrix} u_i \\ y_i \end{bmatrix}, \begin{bmatrix} 0 & I_{m_i} \\ I_{m_i} & -Q_i \end{bmatrix} \begin{bmatrix} u_i \\ y_i \end{bmatrix} \right\rangle_T = \sum_{i=1}^N \beta_i
\]

holds. From the connection rule (1), we have

\[
\begin{bmatrix} u \\ y \end{bmatrix}^T \begin{bmatrix} 0 & I_m \\ I_m & -Q \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} w \\ y \end{bmatrix}^T \begin{bmatrix} 0 & E^T \\ E & -(L^T + L + Q) \end{bmatrix} \begin{bmatrix} w \\ y \end{bmatrix}
\]

Noting that (9) holds, we see that \( E^T(L^T + L + Q)^{-1}E \) is positive. By Schur complement [6], we show that

\[
\begin{bmatrix} L^T + L + Q & E \\ E^T & E^T(L^T + L + Q)^{-1}E \end{bmatrix} \geq 0
\]

and consequently that

\[
L^T + L + Q \geq E(E^T(L^T + L + Q)^{-1}E)^{-1}E^T = EQ_{NW}E^T
\]

holds. From (14), (15), and (16), it follows that (13) holds for \( \beta_{NW} = \frac{1}{2} \sum_{i=1}^N \beta_i \).

Finally, from the definition (6), we see that

\[
\gamma(Q_{NW}) = 2\lambda_{\min}^{-1}((E^T(L^T + L + Q)^{-1}E)^{-1}).
\]

holds. Noting that \( \lambda_{\min}^{-1}(M^{-1}) = \lambda_{\max}(M) \) for any positive-definite matrix \( M \), we show that (11) holds. This completes the proof of Proposition 2. (QED)

Proposition 2 implies that \( \Sigma_{NW}(L, E) \) is output-strictly passive, in other words, it has the excess of passivity, since \( Q_{NW} \) is positive. Note that the proposition does not require the passivity in \( \Sigma_i, i \in \{1, 2, \ldots, N\} \) as long as the condition (9) holds. The shortage of passivity in some of \( \Sigma_i, i \in \{1, 2, \ldots, N\} \) can be compensated by the interconnection \( L \). Furthermore, we consider a sufficient condition for (9). Suppose that \( L^T + L \) is positive-semidefinite and that \( Q \) is positive-definite, i.e., every \( \Sigma_i \) is output-strictly passive. Then, (9) holds, and we see that \( \Sigma_{NW}(L, E) \) inherits the strict passivity from \( \Sigma_i, i \in \{1, 2, \ldots, N\} \).

In addition to the qualitative passivity analysis, the performance evaluation of \( \Sigma_{NW}(L, E) \) is also given in Proposition 2. In this paper, the performance of \( \Sigma_{NW}(L, E) \) is evaluated by the value of \( \gamma(Q_{NW}) \) given in (11).
3.2 Toward performance improvement

In Proposition 2, quantitative passivity analysis of the interconnected system is given. We further specialize the connection rule \((L, E)\) that improves the performance of \(\Sigma_{NW}(L, E)\) compared with that of \(\Sigma_{NW}(0, E)\). To this end, it is assumed that \(Q_i, i \in \{1, 2, \ldots, N\}\) are positive-definite to satisfy \(\gamma(Q_i) = 2\lambda_{\text{min}}(Q_i) = 2\lambda_{\text{max}}(Q_i^{-1})\), which is the performance criterion of each subsystem \(\Sigma_i\). Letting \(Q_0 \in \mathbb{R}^{p \times p}\) be

\[
Q_0 := (E^TQ^{-1}E)^{-1},
\]

which is equivalent to \(Q_{NW}\) of (10) when \(L + L^T = 0\). Then, we see that \(\Sigma_{NW}(0, E) \in S_p(Q_0)\) and \(\gamma(Q_0) = 2\lambda_{\text{max}}(E^TQ^{-1}E)\) holds, which is the performance criterion of \(\Sigma_{NW}(0, E)\). A problem is formulated as follows.

**Problem 1** Find a class of the interconnection rules \(L\) such that

\[
\gamma(Q_{NW}) < \gamma(Q_0)
\]

holds, which is called a problem of performance reinforcement via interconnection in this paper.

**Remark 2** A simplistic example for the performance reinforcement is given in this remark. Let us consider a special connection rule \((L, E)\) to further understand Proposition 2 and to address the performance reinforcement problem. Suppose here that

\[
L^T + L \geq \kappa I_m
\]

holds for some positive constant \(\kappa\). From Proposition 2, we see that \(\Sigma_{NW}(L, E) \in S_p(Q_{NW})\) holds and that \((L + L^T + Q)^{-1} \leq (\kappa I_m + Q)^{-1} < Q^{-1}\) holds. Then,

\[
\gamma(Q_{NW}) \leq 2\lambda_{\text{max}}(E^T(\kappa I_m + Q)^{-1}) < 2\lambda_{\text{max}}(E^TQ^{-1}E) = \gamma(Q_0)
\]

holds from (11), which implies (18) holds. The inequality (18) implies that the interconnection by \(L\) improves the performance for disturbance attenuation in the entire system. In addition, we see that the performance is gradually improved with the increase of the value of \(\kappa\). Such performance reinforcement via interconnection of passive subsystems has been seen in practical control problems and have been known intuitively. For example, consider passivity-based robotic control problems [7, 20]. In the problems, it is known that high gain in a passive controller is effective for reinforcing the performance of a passive plant. Although it is in the sense of a performance limit, the reinforcement is mathematically shown by the evaluation \(\gamma(Q_{NW})\) in this paper.

In Remark 2, it is shown and stated that the performance of \(\Sigma_{NW}(L, E)\) is reinforced via the special interconnection. We relax the assumption (19) to derive a more general condition for the performance reinforcement.

3.3 Performance reinforcement in interconnected passive system

We impose an assumption on \(L\) to derive a condition for performance reinforcement to satisfy (18). As a preliminary, an evaluation of \(\gamma(Q_{NW})\) is given in the following lemma.

**Lemma 1** Consider that \(\Sigma_i \in S_p(Q_i), i \in \{1, 2, \ldots, N\}\) for some positive-definite \(Q_i, i \in \{1, 2, \ldots, N\}\). Further, suppose that (9) holds. Then, \(\Sigma_{NW}(L, E) \in S_p(Q_{NW})\), and

\[
\gamma(Q_{NW}) \leq \gamma(Q_0) - \rho
\]

holds for a constant \(\rho \in \mathbb{R}\):

\[
\rho = 2v_0^T(E^TQ^{-1}F(S + F^TQ^{-1}F)^{-1}F^TQ^{-1}E)v_0,
\]

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where $v_0 \in \mathbb{R}^p$ is the normalized eigenvector corresponding to $\lambda_{\text{max}}(E^TQ^{-1}E)$, $S \in \mathbb{R}^{s \times s}$ is a diagonal matrix with diagonal entries 1 or $-1$, and $F \in \mathbb{R}^{n \times s}$ is an arbitrary matrix of full column rank satisfying

$$L^T + L = FSF^T.$$  \hspace{1cm} (21)

**Proof of Lemma 1:** From (11) in Proposition 2, $\gamma(Q_{NW})$ satisfies

$$\gamma(Q_{NW}) = 2\lambda_{\text{max}}(E^T(FSF^T + Q)^{-1}E).$$ \hspace{1cm} (22)

By applying the matrix inversion lemma to the matrix in the right-hand side of the equality in (22) and noting $S = S^{-1}$, we have

$$E^T(FSF^T + Q)^{-1}E = E^TQ^{-1}E - E^TQ^{-1}F(S + F^TQ^{-1}F)^{-1}F^TQ^{-1}E.$$ \hspace{1cm} (23)

Note that the first term of the right-hand side is positive-definite and that $\gamma(Q_0) = 2\lambda_{\text{max}}(E^TQ^{-1}E)$. It follows that from (22) and (23), (20) holds. This completes the proof of Lemma 1. (QED)

A condition for the performance reinforcement is given as follows. We suppose that $L^T + L$ is positive-semidefinite and that $\rho$ is positive if and only if (24) holds. We summarize the analysis for the performance reinforcement to derive the following theorem.

**Proposition 3** Consider that $\Sigma_i \in S_p(Q_i)$, $i \in \{1,2,\ldots,N\}$ for some positive-definite $Q_i$, $i \in \{1,2,\ldots,N\}$. Further suppose that $L^T + L$ is positive-semidefinite and (24) holds. Then, (18) holds.

**Remark 3** Proposition 3 gives the condition for the performance reinforcement of $\Sigma_{NW}(L,E)$. To further understand the condition described by (24), we consider a simple case: Each subsystem $\Sigma_i$ and the entire system $\Sigma_{NW}(L,E)$ are SISO. We suppose that $m_i = p = 1$ and $Q_i = qI_N$. Then, the condition (24) is reduced to $E^T(L^T + L)E \neq 0$, which implies that the vector $E$ is not orthogonal to Image $(L^T + L)$. As long as they are not orthogonal, any interconnection improves the performance of the entire system. Intuitively, if they are orthogonal, the influence of the exogenous input $w$ on the entire system cannot be modified. In this sense, we see that (24) for this simplified case is reasonable and necessary for the improvement.

4. Quantitative analysis of interconnected passive system on graph

4.1 Interconnection associated with undirected graph

The class of subsystems $\Sigma_i$, $i \in \{1,2,\ldots,N\}$ and interconnection matrix $L$ are more specialized than the setup in Section 3. In the same way as Remark 3, we consider SISO subsystems $\Sigma_i$ and SISO entire system $\Sigma_{NW}$ in this section. Let $q$ be a positive constant that characterizes the one common model-set $S_p(q)$ for every $\Sigma_i$, $i \in \{1,2,\ldots,N\}$. In other words, we suppose that $\Sigma_i \in S_p(q)$, $i \in \{1,2,\ldots,N\}$. It should be emphasized here that $\Sigma_{NW}(L,E)$ can still be a heterogeneous network. The dynamics of $\Sigma_i$, $i \in \{1,2,\ldots,N\}$ can be different each other, while they are described by the same model-set $S_p(q)$. We further suppose that $L = L_g$ is a Laplacian matrix that is associated with a strongly connected undirected graph $[14]$.

The following theorem presents a performance evaluation of $\Sigma_{NW}(L_g,E)$ and a condition for its reinforcement.
**Theorem 1** Suppose that $\Sigma_i \in \mathcal{S}_p(q)$, $i \in \{1, 2, \ldots, N\}$ hold for some $q > 0$. Then, letting $Q_{NWg} := (E^T(2L_g + qI_N)^{-1}E)^{-1}$, it holds that $\Sigma_{NWg}(L_g; E) \in \mathcal{S}_p(Q_{NWg})$, and

$$\gamma(Q_{NWg}) \leq \gamma(Q_0) - \frac{4\lambda_2(L_g)}{q^2 + 2q\lambda_2(L_g)}E^T(I_N - \frac{1}{N}I_N)E. \quad (25)$$

In addition, $\gamma(Q_{NWg}) < \gamma(Q_0)$ holds if $E \notin \text{Image } 1_N$.

**Proof of Theorem 1:** Note that the Laplacian matrix $L_g$ has the simple zero eigenvalue if its associated graph is strongly connected. In addition, the eigenvector that corresponds to the zero eigenvalue is given by $1_N$. Then, letting $R \in \mathbb{R}^{N \times (N-1)}$ be any arbitrary matrix satisfying $R^T 1_N = 0$ and $R^T R = I_{N-1}$, we have

$$L_g^T + L_g = 2L_g = 2RR^T. \quad (26)$$

where $\Gamma \in \mathbb{R}^{(N-1) \times (N-1)}$ is given by

$$\Gamma := \text{diag}(\lambda_{\max}(L_g), \ldots, \lambda_2(L_g)). \quad (27)$$

It follows that (21) holds for $F = \sqrt{2}R\Gamma^{1/2}$. From Lemma 1, it holds that

$$\gamma(Q_{NWg}) \leq \gamma(Q_0) - \frac{4q^{-2}E^T \Gamma^{1/2}(I_N + 2q^{-1}\Gamma)^{-1}\Gamma^{1/2}R^T E}{q}$$

$$= \gamma(Q_0) - \frac{4q^{-1}E^T R\text{diag}\left(\frac{\lambda_{\max}(L_g)}{q + 2\lambda_{\max}(L_g)}, \ldots, \frac{\lambda_2(L_g)}{q + 2\lambda_2(L_g)}\right)R^T E}{q}$$

$$\leq \gamma(Q_0) - \frac{4\lambda_2(L_g)}{q^2 + 2q\lambda_2(L_g)}E^T RR^T E$$

From the equality $I_N = RR^T + \frac{1}{N}1_N 1_N^T$, we see that (25) holds. In addition, if $E \notin \text{Image } 1_N$, it holds that

$$E^T(I_N - \frac{1}{N}1_N 1_N^T)E > 0.$$ 

This completes the proof of Theorem 1. (QED)

In (25) of Theorem 1, the performance of $\Sigma_{NWg}(L_g; E)$ is evaluated in more details than the evaluation in Lemma 1. The evaluation in the theorem does not requires to find a matrix $F$ or the vector $v_0$, while it requires the second smallest eigenvalue $\lambda_2(L_g)$. It is known that the eigenvalue $\lambda_2(L_g)$ is equivalent to the algebraic connectivity of the graph [14]. Furthermore, we note that the performance improvement depends on not only the value of $\lambda_2(L_g)$ but also the direction of $E$. Consider the extreme case: $\lambda_2(L_g) \to \infty$, to further study the role of $E$. Noting that $\gamma(Q_0)$ is expressed as

$$\gamma(Q_0) = \frac{2q}{q^2 + 2q\lambda_2(L_g)}E^T,$$

the inequality (25) is reduced to

$$\gamma(Q_{NWg}) \leq \frac{2q}{q^2 + 2q\lambda_2(L_g)}E^T 1_N 1_N^T E$$

if $\lambda_2(L_g) \to \infty$. Even in the extreme case, the actual $L_2$-gain of $\Sigma_{NWg}(L_g; E)$ becomes zero if $E$ and $1_N$ are orthogonal to each other. From this analysis, we expect that the larger the algebraic connectivity is, the less the $L_2$-gain of $\Sigma_{NWg}(L_g; E)$ is.

**4.2 Gradual performance reinforcement in expanding interconnected system: an example**

The analysis in Theorem 1 is applied to an illustrative system. For simplicity of quantitative analysis, we consider $E = E_1$: 192
This implies that only $\Sigma_1$ is directly affected by the exogenous input $w$. In addition, it follows that $z = y_1$, i.e., the output of $\Sigma_1$ becomes that of $\Sigma_{NW}(L_g, E_1)$. Then, the dynamics of $\Sigma_{NW}(0, E_1)$ are completely the same as those of $\Sigma_1$. In this subsection, interconnection is more specialized than the Laplacian matrix $L_g$, and the passivity and $L_2$-gain performance of $\Sigma_{NW}(L_g, E_1)$ are evaluated more precisely than Theorem 1.

The interconnection matrix $L$ is defined as a diffusive coupling $L_{DC} \in \mathbb{R}^{N \times N}$ [17] as follows:

$$ L_{DC} := \sigma \begin{bmatrix} N - 1 & -1 & \cdots & -1 \\ -1 & N - 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & \cdots & -1 & N - 1 \end{bmatrix}, $$ (29)

where $\sigma > 0$ represents the common connection strength. This $L_{DC}$ has the simple zero eigenvalue and its corresponding eigenvector is $1_N$. The interconnection $L_{DC}$ with the input matrix $E_1$ implies that $u_i, i \in \{1, 2, \ldots, N\}$ are described as

$$ u_1 = w - \sum_{j \neq 1} \sigma (y_1 - y_j), $$

$$ u_i = - \sum_{j \neq i} \sigma (y_i - y_j), \quad i \in \{2, 3, \ldots, N\}. $$

The performance bound of $\Sigma_{NW}(L_{DC}, E_1)$ is quantitatively evaluated. Since $L_{DC}$ can be expressed as

$$ L_{DC} = \sigma N I_N - \sigma 1_N 1_N^T, $$

it follows that

$$ \lambda_2(L_{DC}) = \sigma N > 0. $$ (30)

Therefore, Theorem 1 is applicable to $\Sigma_{NW}(L_{DC}, E_1)$.

**Corollary 1** Suppose that $\Sigma_i \in \mathcal{S}_p(q), i \in \{1, 2, \ldots, N\}$ for some $q > 0$. Then, letting $Q_{DC} := (E_1^T (2L_{DC} + qI_N)^{-1} E_1)^{-1}$, it holds that $\Sigma_{NW}(L_{DC}, E_1) \in \mathcal{S}_p(Q_{DC})$, and

$$ \gamma(Q_{DC}) \leq \frac{2(\sigma + q)}{q(2\sigma N + q)}. $$ (31)

**Proof of Corollary 1:** From Theorem 1 and (30), it follows that

$$ \gamma(Q_{DC}) \leq \gamma(Q_0) - \frac{4\sigma N}{2\sigma N q + q^2} E_1^T (I_N - \frac{1}{N} 1_N 1_N^T) E_1. $$ (32)

We see that $E_1^T E_1 = 1$, $E_1^T 1_N 1_N^T E_1 = 1$, and

$$ \gamma(Q_0) = \frac{2}{q} E_1^T E_1 = \frac{2}{q}. $$

Then, the inequality in (32) is reduced to (31). This completes the proof of Corollary 1. (QED)

Corollary 1 shows that the $L_2$-gain performance of $\Sigma_{NW}(L_{DC}, E_1)$ is gradually improved via the scale-expansion. The provable $L_2$-gain $\gamma(Q_{DC})$ of $\mathcal{S}_p(Q_{DC})$ gradually decreases with increase of the number of connected subsystems, i.e., the value of $N$. Ultimately, if $N \to \infty$, the actual $L_2$-gain of $\Sigma_{NW}(L_{DC}, E_1)$ becomes zero.
5. Conclusion

The performance-integrated passivity was defined for dynamical systems. Then, interconnection of the passive systems was expressed with the parameter transition, and the $L_2$-gain performance of the entire system was evaluated. Furthermore, an undirected graph was considered in the interconnection rule to derive more detailed performance evaluation.

The authors expect that the proposed passivity-based quantitative analysis is applicable to many practical systems such as power systems, bio-molecular systems, and so on. In them, the performance of the entire large-scale systems is evaluated by using only local property of subsystems with guaranteeing the stability of the entire system.

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