LOCALLY ONE-TO-ONE HARMONIC FUNCTIONS WITH STARLIKE ANALYTIC PART

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Abstract. Let $L_H$ denote the set of all normalized locally one-to-one and sense-preserving harmonic functions in the unit disc $\Delta$. It is well-known that every complex-valued harmonic function in the unit disc $\Delta$ can be uniquely represented as $f = h + \overline{g}$, where $h$ and $g$ are analytic in $\Delta$. In particular the decomposition formula holds true for functions of the class $L_H$. For a fixed analytic function $h$, an interesting problem arises - to describe all functions $g$, such that $f$ belongs to $L_H$.

The case when $f \in L_H$ and $h$ is the identity mapping was considered in [3]. More general results are given in [4], where $f \in L_H$ and $h$ is a convex analytic mapping. The focus of our present research is to characterize the set of all functions $f \in L_H$ having starlike analytic part $h$. In this paper, we provide coefficient, distortion and growth estimates of $g$. We also give growth and Jacobian estimates of $f$.

1. Introduction

A complex-valued harmonic function $f$ in the open unit disc $\Delta \subset \mathbb{C}$ can be uniquely represented as

$$f = h + \overline{g},$$

where $h$ and $g$ are analytic in $\Delta$ with $g(0) = 0$. Hence, $f$ is uniquely determined by coefficients of the following power series expansions

$$h(z) = \sum_{n=0}^{\infty} a_n z^n, \quad g(z) = \sum_{n=1}^{\infty} b_n z^n, \quad z \in \Delta,$$

where $a_n \in \mathbb{C}$, $n = 0, 1, 2, \cdots$ and $b_n \in \mathbb{C}$, $n = 1, 2, 3, \cdots$.

Such a function $f$, not identically constant, is said to be sense-preserving in $\Delta$ if and only if it satisfies the equation

$$g' = \omega h',$$

where $\omega$ is analytic in $\Delta$ with $|\omega(z)| < 1$ for all $z \in \Delta$. The function $\omega$ is called the second complex dilatation of $f$, and it is closely related to the Jacobian of $f$ defined as follows

$$J_f(z) := |h'(z)|^2 - |g'(z)|^2, \quad z \in \Delta.$$
Recall that a necessary and sufficient condition for \( f \) to be locally one-to-one and sense-preserving in \( \Delta \) is \( J_f(z) > 0, \ z \in \Delta \). This is an immediate consequence of Lewy’s theorem (see [3]). Observe that if \( J_f(z) > 0 \) then \( |h'(z)| > 0 \) and hence \( g'(z)/h'(z) \) is well defined for every \( z \in \Delta \). Thus the dilatation \( \omega \) of a locally univalent and sense-preserving function \( f \) in \( \Delta \) can be expressed as

\[
\omega(z) = \frac{g'(z)}{h'(z)}, \quad z \in \Delta.
\]

(1.5)

Let \( L_H \) denote the set of all locally one-to-one and sense-preserving harmonic functions \( f \) in \( \Delta \) satisfying (1.1), such that \( h(0) = 0 \) and \( h'(0) = 1 \) (\( g(0) = 0 \) by the uniqueness of (1.1)). Note that the known family \( S_H \) introduced in [1] by Clunie and Sheil-Small is a subset of \( L_H \), in fact, \( S_H \) consists of all one-to-one functions in \( L_H \).

The main idea of our research is to characterize subclasses of \( L_H \) defined by some additional geometric conditions on \( h \). The case when \( h \) is the identity mapping was studied in [3]. The paper [4] was devoted to the case when \( h \) is a convex analytic mapping. In this paper we consider functions \( f \in L_H \) having starlike analytic part \( h \).

**Definition 1.1.** Let \( \alpha \in [0, 1) \). We define the class \( \tilde{L}_H^\alpha \) of all \( f \in L_H \), such that \( |b_1| = \alpha \) and \( h \) is a starlike analytic function, where \( b_1 \) is taken from the power series expansion (1.2) of \( f \). Additionally, we define the class

\[
\tilde{L}_H := \bigcup_{\alpha \in [0, 1)} \tilde{L}_H^\alpha.
\]

**Remark 1.2.** Note that the estimate \( |b_1| < 1 \) holds for all \( f \in L_H \) (see [3], p. 79), in particular, this explains why we have taken \( \alpha \in [0, 1) \) in Definition 1.1.

**Example 1.3.** For a fixed \( \zeta \in \Delta \) consider \( f_\zeta := k + \frac{g_\zeta}{1 + \zeta} \) in \( \Delta \), where

\[
k(z) := \frac{z}{(1-z)^2}, \quad z \in \Delta
\]

and

\[
g_\zeta(z) := \left( \frac{1-\zeta}{1+\zeta} \right)^2 \log \frac{1+\zeta}{1-z} + \frac{z}{(1-z)^2} - \left( \frac{1-\zeta}{1+\zeta} \right) \frac{2z}{1-z}, \quad z \in \Delta.
\]

Straightforward calculation leads to the formula for the dilatation \( \omega_\zeta \) of \( f_\zeta \), i.e.

\[
\omega_\zeta(z) := \frac{z + \zeta}{1 + \zeta z}, \quad z \in \Delta,
\]

which ensures that \( f_\zeta \) is locally one-to-one and sense-preserving in \( \Delta \). Clearly, \( k(0) = 0, \ k'(0) = 1 \) and \( g_\zeta(0) = 0 \), hence \( f_\zeta \in L_H \). Finally, \( k \) is well-known starlike function (Koebe function) and one can easily check that \( g_\zeta'(0) = \zeta \), therefore \( f_\zeta \in \tilde{L}_H^\alpha \) with \( \alpha := |\zeta| \).

2. Results

At the beginning of this section we present a connection, discovered by us, between \( \tilde{L}_H^\alpha \) and the class of normalized harmonic mappings with convex analytic part (introduced in [4]). It does not seem to be surprising in view of the classical result concerning convex and starlike analytic functions due to J. W. Alexander.
Theorem 2.1. If \( f \in \hat{L}_H^\alpha \), then \( F := H + \bar{G} \) belongs to \( S_H \), \( H \) is a convex analytic function and \( |G'(0)| = \alpha \), where \( H \) and \( G \) satisfy the conditions \( h(z) = zH'(z) \) and \( g(z) = zG'(z) \), \( z \in \Delta \) with the normalization \( H(0) = 0 \) and \( G(0) = 0 \).

Proof. Let \( f \in \hat{L}_H^\alpha \). By definition of \( \hat{L}_H^\alpha \), \( h \) is a normalized starlike function. Hence, by Alexander’s theorem, the function \( H \) is a normalized convex function, namely, \( H(0) = 0 \) and \( H'(0) = \lim_{z \to 0} h(z)/z = h'(0) = 1 \). Moreover, \( G'(0) = \lim_{z \to 0} g(z)/z = g'(0) \) and \( |G'(0)| = \alpha \). In particular, \( |G'(0)| < |H'(0)| \). Now, observe that for all \( z \in \Delta \setminus \{0\} \) the interval \([0, h(z)]\) is a subset of \( h(\Delta) \) since \( h \) is a starlike function. Hence

\[
(2.1) \quad |g(z)| = |g \circ h^{-1}(h(z))| \leq \int_0^{h(z)} \left| \frac{d(g \circ h^{-1})}{d\zeta}(\zeta) \right| |d\zeta| < \int_0^{h(z)} |d\zeta| = |h(z)|.
\]

Again, by definition of \( \hat{L}_H^\alpha \), \( f \) is a sense-preserving harmonic function which together with (2.1) gives

\[
(2.2) \quad |G'(z)| = \lim_{z \to \zeta} \frac{|g(\zeta)|}{|\zeta|} < \lim_{z \to \zeta} \left| \frac{h(\zeta)}{\zeta} \right| = |H'(z)|, \quad z \in \Delta \setminus \{0\}.
\]

It shows that \( F \) is a locally univalent and sense-preserving harmonic function in \( \Delta \). Finally, appealing to \([4\text{ Corollary 2.3}]\), we conclude that \( F \in S_H \). \( \square \)

For \( f \in \hat{L}_H^\alpha \), the classical theory of univalent functions says (see e.g. \([7\text{, 8}]\))

\[
(2.3) \quad |a_n| \leq n, \quad n = 2, 3, 4, \ldots.
\]

The following theorem gives an estimate of the coefficient \( b_n \).

Theorem 2.2. If \( f \in \hat{L}_H^\alpha \), then we have

\[
(2.4) \quad |b_n| \leq \alpha + \sqrt{(n - \alpha^2)(n - 1)}, \quad n = 3, 4, 5, \ldots.
\]

Proof. If \( f \in \hat{L}_H^\alpha \), then by Theorem 2.1 the function \( F := H + \bar{G} \) belongs to \( S_H \), \( H \) is a convex analytic function and \( |G'(0)| = \alpha \), where \( H \) and \( G \) satisfy the conditions \( h(z) = zH'(z) \) and \( g(z) = zG'(z) \) for all \( z \in \Delta \) with the normalization \( H(0) = 0 \) and \( G(0) = 0 \). Clearly \( G \) can be expanded in a power series, say

\[
G(z) = \sum_{n=1}^{\infty} B_n z^n, \quad z \in \Delta,
\]

where \( B_n \in \mathbb{C}, \ n = 1, 2, 3, \ldots \). Using this expansion together with the expansion (1.2) of \( g \) and the formula \( g(z) = zG'(z) \), we obtain \( b_n = nB_n, \ n = 1, 2, 3, \ldots \). Next, by applying \([4\text{ Theorem 3.1}]\) we have

\[
(2.4) \quad |b_n| \leq \frac{n}{n} \alpha + \sqrt{(n - \alpha^2)(n - 1)}, \quad n = 2, 3, 4, \ldots,
\]

which gives (2.4). To improve the estimate in the case \( n = 2 \), consider the function

\[
F(z) := \frac{\omega(z) - \omega(0)}{1 - \omega(0)\omega(z)}, \quad z \in \Delta,
\]
where \( \omega \) is the dilatation of \( f \). Since \( \omega \) is analytic in \( \Delta \) it has a power series expansion, say

\[
\omega(z) = \sum_{n=0}^{\infty} c_n z^n, \quad z \in \Delta,
\]

where \( c_n \in \mathbb{C}, n = 0, 1, 2, \ldots \) and \(|c_0| = |\omega(0)| = |g'(0)| = |b_1| = \alpha \). Recall that \(|\omega(z)| < 1\) for all \( z \in \Delta \). Hence we can apply the Schwarz lemma to \( F \) and obtain

\[
|F'(0)| \leq 1,
\]

which yields

\[
|c_1| = |\omega'(0)| \leq 1 - |c_0|^2.
\]

(2.5)

On the other hand, the formula (1.5) gives

\[
2b_2 = 2a_2 c_0 + c_1,
\]

which together with (2.2) and (2.5) leads to the estimate

\[
2|b_2| \leq 4|c_0| + 1 - |c_0|^2.
\]

Since this is equivalent to (2.3), the proof is completed.

We have the following immediate corollary from Theorem 2.2.

**Corollary 2.3.** If \( f \in \hat{L}_H \) then

\[
|b_n| < n, \quad n = 2, 3, 4, \ldots
\]

Recall that by definition the analytic part \( h \) of \( f \in \hat{L}_H \) is starlike. Hence, it is known that

\[
\frac{1 - |z|}{(1 + |z|)^3} \leq |h'(z)| \leq \frac{1 + |z|}{(1 - |z|)^3}, \quad z \in \Delta.
\]

(2.6)

Our next result is the distortion estimate of the anti-analytic part \( g \) of \( f \in \hat{L}_H^0 \).

**Theorem 2.4.** If \( f \in \hat{L}_H^0 \), then

\[
|g'(z)| \geq \begin{cases} 
\frac{(\alpha - |z|)(1 - |z|)}{(1 - \alpha |z|)(1 + |z|)^3}, & |z| < \alpha, \\
0, & |z| \geq \alpha,
\end{cases}
\]

where \( z \in \Delta \)

\[
|g'(z)| \leq \frac{(\alpha + |z|)(1 + |z|)}{(1 + \alpha |z|)(1 - |z|)^3}, \quad z \in \Delta.
\]

(2.8)

**Proof.** Let \( \omega \) of the form (1.5) be the dilatation of \( f \in \hat{L}_H^0 \) and let \( g'(0) = \alpha e^{i\varphi} \).

Then the function

\[
\Omega(z) = \frac{e^{-i\varphi} \omega(z) - \alpha}{1 - \alpha e^{-i\varphi} \omega(z)}, \quad z \in \Delta
\]

satisfies the assumptions of the Schwarz lemma, which gives

\[
|e^{-i\varphi} \omega(z) - \alpha| \leq |z| |1 - \alpha e^{-i\varphi} \omega(z)|, \quad z \in \Delta.
\]

Equivalently we can write

\[
|e^{-i\varphi} \omega(z) - \alpha(1 - |z|^2)| \leq \frac{(1 - \alpha^2)|z|}{1 - \alpha^2 |z|^2}, \quad z \in \Delta
\]

and the equality holds only for the functions satisfying

\[
\omega(z) = e^{i\varphi} \frac{e^{i\varphi} z + \alpha}{1 + \alpha e^{i\varphi} z}, \quad z \in \Delta.
\]
where \( \psi \in \mathbb{R} \). Hence, by the triangle inequality we have

\[
\frac{\alpha - |z|}{1 - \alpha |z|} \leq |\omega(z)| \leq \frac{\alpha + |z|}{1 + \alpha |z|}, \quad z \in \Delta.
\]

Finally, applying the estimate (2.9) together with (2.6) to the identity (1.3) we obtain (2.7) and (2.8), so the proof is completed.

\[
\square
\]

**Corollary 2.5.** If \( f \in \tilde{L}_H \), then

\[
|g'(z)| \leq \frac{1 + |z|}{(1 - |z|)^3}, \quad z \in \Delta.
\]

Using the distortion estimates we can easily deduce the following Jacobian estimates of \( f \).

**Theorem 2.6.** If \( f \in \tilde{L}^\alpha_H \), then

\[
\frac{(1 - \alpha^2)(1 - |z|^2)(1 - |z|)^2}{(1 + \alpha |z|)^2(1 + |z|)^6} \leq J_f(z) \leq \begin{cases} 
\frac{(1 - \alpha^2)(1 + |z|)^3}{(1 - \alpha |z|)^2(1 - |z|)^6}, & |z| < \alpha, \\
\frac{(1 + |z|)^2}{(1 - |z|)^6}, & |z| \geq \alpha,
\end{cases}
\]

where \( z \in \Delta \).

**Proof.** Observe that if \( f \in \tilde{L}^\alpha_H \), then \( h' \) does not vanish in \( \Delta \) and we can write the Jacobian of \( f \) given by (1.4) in the form

\[
J_f(z) = |h'(z)|^2 \frac{(1 - |\omega(z)|^2)}{(1 + |z|)^2}, \quad z \in \Delta,
\]

where \( \omega \) is the dilatation of \( f \). By applying (2.6) and (2.9) to the above formula we obtain

\[
\frac{(1 - \alpha^2)(1 - |z|^2)(1 - |z|)^2}{(1 + \alpha |z|)^2(1 + |z|)^6} \leq J_f(z) \leq \begin{cases} 
\frac{(1 - \alpha^2)(1 - |z|^2)}{(1 - \alpha |z|)^2}, & |z| \leq \alpha, \\
\frac{(1 + |z|)^2}{(1 - |z|)^6}, & |z| \geq \alpha,
\end{cases}
\]

and the proof is completed. Note that these estimates can also be deduced from a more general result given in [2].

The growth estimate of the analytic part \( h \) of \( f \in \tilde{L}_H \) is known to be of the form

\[
\frac{|z|}{(1 + |z|)^2} \leq |h(z)| \leq \frac{|z|}{(1 - |z|)^2}, \quad z \in \Delta.
\]

In the following theorem we give the growth estimate of the anti-analytic part \( g \).

**Theorem 2.7.** If \( f \in \tilde{L}^\alpha_H \), then

\[
|g(z)| \geq \begin{cases} 
\frac{|z|(|\alpha - |z|)}{(1 - \alpha |z|)(1 + |z|)^2}, & |z| < \alpha, \\
0, & |z| \geq \alpha,
\end{cases}
\]

where \( z \in \Delta \) and

\[
|g(z)| \leq \frac{(1 - \alpha)}{(1 + \alpha)} \log \frac{1 + \alpha |z|}{1 - |z|} + \frac{|z|}{1 - |z|^2} < \frac{1 - \alpha}{1 + \alpha} \frac{2|z|}{1 - |z|} \leq \frac{|z|(|\alpha + |z|)}{(1 + \alpha |z|)(1 - |z|)^2}, \quad z \in \Delta.
\]
Proof. If \( f \in L_H^\nu \), then by Theorem 2.1 the function \( F := H + \nabla \) belongs to \( S_H^A \), \( H \) is a convex analytic function and \( |G'(0)| = \alpha \), where \( H \) and \( G \) satisfy the conditions \( h(z) = zH'(z) \) and \( g(z) = zG'(z) \) for all \( z \in \Delta \) with the normalization \( H(0) = 0 \) and \( G(0) = 0 \). Hence, by applying [3] Theorem 3.5 together with \( g(z) = zG'(z) \) we have
\[
\frac{\alpha - |z|}{(1 - \alpha|z|)(1 + |z|)^2} \leq |g(z)| \leq \frac{\alpha + |z|}{(1 + \alpha|z|)(1 - |z|)^2}, \quad z \in \Delta.
\]
To prove the first inequality in (2.12) we estimate the integral of \( g' \) along \( \gamma := [0, z] \) by applying (2.8), i.e.
\[
|g(z)| = \left| \int_\gamma g'(\zeta) \, d\zeta \right| \leq \int_\gamma |g'(\zeta)||d\zeta| \leq \int_0^{|z|} \frac{(\alpha + \rho)(1 + \rho)}{(1 + \alpha\rho)(1 - \rho)^3} \, d\rho
\]
\[
= \frac{(1 - \alpha)^2}{1 + \alpha} \log \frac{1 + \alpha|z|}{1 - |z|} + \frac{|z|}{1 - |z|^2} - \frac{(1 + \alpha)}{1 + \alpha} \frac{2|z|}{1 - |z|^2}, \quad z \in \Delta.
\]
Finally, observe that the function \( f_\epsilon \) defined in Example 1.3 with suitably chosen \( \zeta \in \Delta \) shows that the first inequality in (2.12) is best possible, which completes the proof.

Corollary 2.8. If \( f \in L_H \), then
\[
|g(z)| \leq \frac{|z|}{(1 - |z|)^2}, \quad z \in \Delta.
\]

Now, we can deduce the growth estimate of \( f \).

Theorem 2.9. If \( f \in L_H^\nu \), then
\begin{equation}
(2.13)
|f(z)| \geq \frac{(1 - \alpha)|z|(1 - |z|)}{(1 + \alpha|z|)(1 + |z|)^2}, \quad z \in \Delta.
\end{equation}
and
\begin{equation}
(2.14)
|f(z)| \leq \frac{1 - \alpha}{1 + \alpha} \log \frac{1 + \alpha|z|}{1 - |z|} + \frac{2|z|}{1 - |z|^2} - \frac{(1 + \alpha)}{1 + \alpha} \frac{2|z|}{1 - |z|^2}
\leq \frac{(1 + \alpha)|z|(1 + |z|)}{(1 + \alpha|z|)(1 - |z|)^2}, \quad z \in \Delta.
\end{equation}

Proof. If \( f \in L_H^\nu \), then by Theorem 2.1 the function \( F := H + \nabla \) belongs to \( S_H^A \), \( H \) is a convex analytic function and \( |G'(0)| = \alpha \), where \( H \) and \( G \) satisfy the conditions \( h(z) = zH'(z) \) and \( g(z) = zG'(z) \) for all \( z \in \Delta \) with the normalization \( H(0) = 0 \) and \( G(0) = 0 \). Hence, by applying inequality (2.10) and (2.9), which also holds true for the dilatation of \( F \) (see [4] the proof of Theorem 3.5) we have
\[
|f(z)| = |h(z) + G(z)| \geq |h(z)| - |g(z)| = |h(z)| \left( 1 - \frac{|g(z)|}{|h(z)|} \right)
\]
\[
= |h(z)| \left( 1 - \frac{|G'(z)|}{|H'(z)|} \right) \geq \frac{|z|}{1 + |z|^2} \left( 1 - \frac{\alpha + |z|}{1 + \alpha|z|} \right), \quad z \in \Delta.
\]
This proves the inequality (2.13). To prove (2.14) we use the triangle inequality \(|f(z)| \leq |h(z)| + |g(z)|\), which together with (2.10) and (2.12) leads to (2.14) so the proof is completed. \( \square \)
Corollary 2.10. If $f \in \tilde{L}_H$ then
$$|f(z)| \leq \frac{2|z|}{(1-|z|)^2}, \quad z \in \Delta.$$  

Remark 2.11. The estimates (2.4), (2.11) and (2.13) are probably not precise.

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