Value of Information Systems in Routing Games

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Abstract

We study a routing game in an environment with multiple heterogeneous information systems and an uncertain state that affects edge costs of a congested network. Each information system sends a noisy signal about the state to its subscribed traveler population. Travelers make route choices based on their private beliefs about the state and other populations' signals. The question then arises, "How does the presence of asymmetric and incomplete information affect the travelers' equilibrium route choices and costs?" We develop a systematic approach to characterize the equilibrium structure, and determine the effect of population sizes on the relative value of information (i.e. difference in expected traveler costs) between any two populations. This effect can be evaluated using a population-specific size threshold. One population enjoys a strictly positive value of information in comparison to the other if and only if its size is below the corresponding threshold. We also consider the situation when travelers may choose an information system based on its value, and characterize the set of equilibrium adoption rates delineating the sizes of subscribed traveler populations. The resulting routing strategies are such that all travelers face an identical expected cost, and no traveler has the incentive to change her subscription.

Index Term: Nonatomic games, Transportation technology, Information systems.

1 Introduction

Travelers are increasingly relying on traffic navigation services to make their route choice decisions. In the past decade, numerous services have come to the forefront, including Waze/Google maps, Apple maps, INRIX, etc. These Traffic Information Systems (TISs) provide their subscribers with costless information about the uncertain network condition (state), which is typically influenced by several exogenous factors such as weather, incidents, and road conditions. The information provided by TIS can be especially useful in making travel decisions when a change in state corresponds to changes in travel times.

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of multiple edges of the network. Experiential evidence suggests that the accuracy levels of TISs are less than perfect, and exhibit heterogeneities due to the inherent technological differences in data collection and analysis approaches. Moreover, travelers may use different TISs or choose not to use them at all, depending on factors such as marketing, usability, and availability. Therefore, we can reasonably expect that travelers face an environment of asymmetric and incomplete information about the network state.

Importantly, information heterogeneity can directly influence the travelers’ route choice decisions, and the resulting congestion externalities. For example, travelers who are informed by their TIS that a certain route has an incident and take a detour may not only reduce their own travel time, but also benefit the uninformed travelers by shifting traffic away from the affected route. However, if too many travelers take the detour, then this alternate route will also start getting congested, limiting the benefits of information. Thus, the question arises as to how information heterogeneity impacts the travelers route choice and costs.

In this article, we develop a game-theoretic approach to study this question. We consider a routing game in which the travelers are privately informed about the network state by their respective TIS, and choose strategies based on their beliefs about the state and other travelers’ behavior. Our approach enables a systematic study of equilibrium structure and travel costs for this routing game. The game is played on a general network by multiple traveler populations that are heterogeneous in the access and accuracy of state information. Specifically, we provide a complete characterization of how the population size vector impacts the equilibrium structure and the relative value of information faced by travelers subscribed to one TIS in comparison to the ones subscribed to another TIS.

Furthermore, we study the situation in which travelers can choose their TIS subscription based on the relative values of different available TISs. In general, the adoption rate of one TIS (i.e. the fraction of travelers subscribing to it) depends on a variety of factors such as TIS accuracies, uncertainty in network states, and cost parameters. Thus, determining the equilibrium adoption rates is a non-trivial yet practically-relevant problem. We provide a rather straightforward characterization of the set of equilibrium adoption rates. Particularly, we show that this set is comprised of the population size vectors for which all travelers face an identical expected cost in equilibrium (i.e. social fairness is achieved), and no traveler has an incentive to change her TIS subscription (i.e. stability is achieved).

1.1 Our Model and Contributions

We model the traffic routing problem in an asymmetric and incomplete information environment as a Bayesian routing game. We consider a general traffic network with a single origin-destination pair, and an uncertain state that is realized from a finite set according to a prior probability distribution. The cost function (travel time) of each edge in the network is increasing in the aggregate traffic load on that edge. Moreover, the edge costs are state-dependent in that the state can affect them in various ways. All travelers have identical preferences, and the total demand is inelastic. There are multiple heterogeneous TISs, each sending a noisy signal of the state to its subscribed traveler population. The
signal sent by each TIS is privately known to only its subscribers, but the joint distribution of the state and all signals is known by all travelers (common knowledge). This joint distribution is the common prior of the game, and each population’s private belief of the state and other populations’ private signals is derived from it. Our information environment is general in that we do not impose any structural assumption on the common prior. The signals of different TISs can be correlated or independent, conditional on the state. Also, one TIS can be more accurate than another in some states, but less in the other states. Therefore, we do not assume that the TISs are ordered according to the accuracy of their signals.

We use Bayesian Wardrop Equilibrium (BWE) as the solution concept of our game. In a BWE, all populations assign demand on routes with the smallest expected cost based on their private beliefs. In fact, our game is a weighted potential game, i.e. the set of BWE is the optimal solution set of a convex optimization problem that minimizes the weighted potential function over the set of feasible routing strategies of traveler populations (Theorem 1). This property establishes the essential uniqueness of BWE, i.e. the equilibrium edge load vector is unique. However, the strategy-based optimization problem is not directly useful for analyzing how the set of equilibrium strategies and population costs depend on the population size vector. To address this issue, we provide a characterization of the set of feasible route flows (i.e. route flows induced by feasible strategy profiles) as a convex polytope (Proposition 1), and show that this set is the optimal solution set of another convex optimization problem (Proposition 2). The optimal value of this flow-based formulation is the value of weighted potential function in equilibrium.

The flow-based formulation enables us to analyze the sensitivity of equilibrium structure with respect to perturbations in the size vector. The constraints in this formulation include: basic route feasibility constraints, a set of size-independent equality constraints, and a set of size-dependent inequality constraints. The equality constraints ensure that any shift in route flows resulting from a change of signal received by one population is independent of the signals received by other populations. Each inequality constraint corresponds to a single population, and ensures that the maximum extent to which the received signals impact its equilibrium routing behavior is limited by the population’s size. Hence, we refer to these inequality constraints as information impact constraints (IICs). Consequently, the effects of perturbation in the size vector on the equilibrium structure can be studied by evaluating the tightness of the IICs corresponding to the perturbed populations at the optimum of the flow-based formulation.

In particular, Theorem 2 describes how the qualitative properties of equilibrium route flows change under perturbations in the sizes of any two populations, with sizes of all other populations being fixed (i.e. directional perturbations of the size vector). Among the two perturbed populations, we say that a population is the “minor population” if its size is smaller than a certain (population-specific) threshold. The corresponding IIC is tight in equilibrium, i.e. the impact of information on the minor population is fully attained. These population-specific thresholds depend on the common prior distribution as well as sizes of all other populations, and each threshold can be computed by solving a linear program. Based on the two thresholds, we can distinguish three qualitatively distinct equilibrium regimes: In the two side regimes exactly one population assumes the
minority role, on the other hand, in the middle regime neither population is minor.

We can apply Theorem 2 to analyze the sensitivity of the equilibrium value of the weighted potential function under directional perturbation of population sizes (Proposition 3). In the middle regime, perturbing the relative sizes does not change the equilibrium value of the potential function. On the other hand, in the two side regimes the value of the potential function monotonically decreases as the size of the minor population increases. Thanks to the essential uniqueness of BWE, the equilibrium edge load vector does not change with this directional perturbation if and only if the size vector falls in the middle regime.

These results allow us to compare the expected cost in equilibrium faced by travelers in any pair of populations. In particular, we can evaluate how this cost difference — which we call the relative value of information — changes with pairwise size perturbations. By using the results on sensitivity analysis of general convex optimization problems (Fiacco 1984, Rockafellar 1984), we show that the relative value of information is proportional to the derivative of the equilibrium value of potential function in the direction of perturbation. Combining this observation with Proposition 3, we obtain that the minor population faces a lower cost relative to the other population in the two side regimes; whereas both populations face identical costs in the middle regime. Importantly, the relative value of information is non-increasing in the size of its subscribed population (Theorem 3). This result is based on the intuition that an individual traveler faces higher congestion externality when more travelers have access to the same information, and hence make their route choices according to the same strategy. Thus, an increase in the size of minor population decreases the relative imbalance in congestion externality, thereby reducing the advantage enjoyed by its travelers over the other population.

Our results can be easily specialized to a simpler information environment in which one of the populations does not have an access to TIS (uninformed population). In this case, we obtain that the equilibrium cost of the uninformed population is no less than that of any other population regardless of the size vector. That is, having access to a TIS always leads to a non-negative relative value of information in comparison to being uninformed (Proposition 4).

We also extend our approach of pairwise comparison of populations to study how the equilibrium outcome depends on population sizes in general. In particular, we characterize a non-empty set of size vectors, where all the IICs can be dropped from the flow-based optimization problem without changing its optimal value. The equilibrium edge load vector is size-independent in this set (Proposition 5). Furthermore, Theorem 4 shows that this set is comprised of all size vectors such that, in equilibrium, all travelers face identical costs (socially fair) and no travelers have incentive to change TIS subscription (stable). We say that such an outcome exhibits the stable social fairness property.

Theorem 4 is useful for evaluating the equilibrium adoption rate of each TIS when travelers have the flexibility to choose their TIS subscription. Particularly, we consider a two-stage game, in which travelers first choose their TIS, and then play the Bayesian routing game for the size vector induced from the first stage. A subgame perfect equilibrium (SPE) of this game requires that travelers have no incentive to change their respective TIS subscriptions, and that the routing strategies are chosen according to BWE of the
subgame. Hence, any TIS with a positive adoption rate must incur the lowest expected equilibrium cost among all TIS. We conclude that the set of equilibrium adoption rates in SPE is indeed the set of size vectors for which the induced equilibrium outcome satisfies the stable social fairness property.

Finally, we also identify a sufficient condition on the edge cost functions, under which there is no inefficiency caused by selfish routing. When this condition is satisfied, we obtain that equilibrium efficiency can be achieved in the stable socially fair set of size vectors (Proposition 6).

1.2 Related Work

*Congestion games.* Well-known results in classical congestion games include their equivalence with potential games (Rosenthal [1973], Monderer and Shapley [1996], Sandholm [2001], and Sorin and Wan [2016]), analysis of network formation games as congestion games (Gopalakrishnan et al. [2014], and Tardos and Wexler [2007]), and equilibrium inefficiency (Roughgarden and Tardos 2004, Koutsoupias and Papadimitriou 1999, Correa et al. 2007, Acemoglu and Ozdaglar 2007, and Nikolova and Stier-Moses 2014). Some models of congestion games in asymmetric information environments have been also reported. For example, Heumen et al. 1996 and Facchini et al. 1997 showed that for Bayesian congestion games with atomic players, a pure Nash equilibrium exists when the game has a common prior, but may not exist otherwise. Milchtaich 1996, and Mavromichalis et al. 2007 have studied congestion games with player-specific cost functions. These games can model both heterogeneous private information and heterogeneous preferences. However, the existence of a potential function or even a pure Nash equilibrium is not guaranteed. Since our game has non-atomic traveler populations, the existence of a pure equilibrium is guaranteed. Furthermore, in our game, the heterogeneity in the expected costs among populations arises only due to heterogeneous private beliefs, which are derived from a common prior. This feature makes our game a weighted potential game Sandholm [2001].

*Effect of Traffic Information Systems.* Prior work has studied the effects of TIS on travelers’ departure time choices (Arnott et al. 1991, and Khan and Amin 2018), and on their route choices (Ben-Akiva et al. 1991, and Ben-Akiva et al. 1996). In particular, Mahmassani and Jayakrishnan 1991 conducted a simulation to study the effect of real-time information on the performance of a congested traffic corridor, and concluded that as more travelers receive information, the informed travelers gradually start facing higher costs and their relative value of being informed diminishes. Our analysis, when applied to the game with one uninformed population and other more informed ones, also leads to similar conclusions. Our results are more general because they are applicable to routing games with multiple heterogeneously informed populations with arbitrary TIS accuracies.

Another related work is by Acemoglu et al. 2018, which studies a congestion game where travelers have different information sets about the available edges (routes). The authors identify a sufficient and necessary condition on the network topology under which receiving additional information does not increase the traveler costs. While their work focuses on heterogeneous information about the network structure, we use a Bayesian
approach to model the information heterogeneity resulting from the differences in TIS access and accuracy.

Value of Information. In a classical paper, Blackwell et al. [1951] showed that for a single decision maker, more informative signal always results in higher expected utility. The Blackwell’s criterion for the comparison of information has been refined by Athey and Levin [2017] and Persico [2000] for a class of decision making problems. In game-theoretic settings, it is generally difficult to determine whether the value of information in equilibrium for individual players and/or society is positive, zero or negative (see Hirshleifer [1971], and Haenfler [2002]). However, the value of information is guaranteed to be positive when certain conditions are satisfied; see for example Neyman [1991], Bassan et al. [2003], Gossner and Mertens [2001], and Lehrer and Rosenberg [2006]. Since travelers are non-atomic players in our game, the relative value of information between any two TISs is equivalent to the value of information for an individual traveler when her subscription changes unilaterally. We give precise conditions on the population sizes under which the value of information in our Bayesian routing game is positive, zero, or negative.

Equilibrium Adoption of Traffic Information Systems. In our model, the value of information to TIS users (as well as non-users) depends on the size of the population with access to information. Naturally, the notion of adoption rate becomes relevant, since if it is possible for the non-users of a particular TIS to receive a greater value by adopting it, then they will continue to do so until there is no longer a positive relative value in adopting that TIS. Yin et al. [2003] explore market adoption for the environment with single TIS provider. However, currently travelers have the choice of different competing TISs that vary in their accuracy and other technological features. Thus, it is important to consider not only the choice of adopting a TIS, but also which TIS the travelers choose to follow in making their route choices. Our results provide a way to incorporate both the adoption rate of a specific TIS as well as the effects of multiple competing TISs.

The paper is organized as follows: In Section 2 we motivate our analysis using a simple routing game. Section 3 introduces our Bayesian routing game model; and in Section 4 we show that this game is a weighted potential game; this property leads to characterizations of the set of equilibrium strategy profiles and the set of route flows. In Section 5 we analyze how the equilibrium structure and the relative value of information between any two populations change with their sizes. In Section 6, we characterize equilibrium TIS adoption rates for the case when travelers have the flexibility to choose or switch their subscription. Concluding remarks are drawn in Section 7.

The complete proofs, along with supplementary results, are provided in the appendix.

2 Motivating Example

In this section, we motivate our analysis using a simple game of two asymmetrically informed traveler populations routing over a network of two parallel routes, denoted $r_1$ and $r_2$. The network state $s$ belongs to the set $S = \{a, n\}$, where the state $a$ represents an incident condition on $r_1$, and the state $n$ represents the nominal condition. The state $a$ occurs with probability $p \in (0, 1)$. The network faces a unit size of demand ($D = 1$),
which is comprised of two traveler populations: population 1 with size $\lambda^1$ and population 2 with size $\lambda^2 = 1 - \lambda^1$. Each population $i \in \mathcal{I} = \{1, 2\}$ receives a signal $t^i$ of the state from its TIS. Thus, the signal space of population $i$ is $\mathcal{T}^i = \{a, n\}$. Assume for simplicity that population 1 receives the correct state with probability 1 (i.e. complete information), and population 2 receives signal $a$ or $n$ with probability 0.5, independent of the state (i.e. no information). This information structure is common knowledge.

Let $q^i(t^i)$ denote the traffic demand assigned to route $r_1$ by population $i$ when receiving signal $t^i$; the remaining demand $(\lambda^i - q^i(t^i))$ is assigned to route $r_2$. Since the signal $t^2$ is independent of the state, we have $q^2(a) = q^2(n) \triangleq q^2$. A feasible demand assignment must satisfy the constraints: $0 \leq q^2(t^1) \leq \lambda^1$ and $0 \leq q^2 \leq \lambda^2$. For this example, we can represent a routing strategy profile as $q^1 = (q^1(a), q^1(n), q^2)$. We denote the aggregate route flow on $r$ as $f_r$. Again, for simplicity, consider that the cost of each route is an affine function of the route flow, and both routes have identical free-flow travel time. That is, the cost function of $r_1$ is $c_1^a(f_1) = \alpha^a f_1 + b$ in state $a$, and $c_1^n(f_1) = \alpha^n f_1 + b$ in state $n$; the cost function of $r_2$ is $c_2(f_2) = \alpha_2 f_2 + b$. Furthermore, we assume that $\alpha^a < \alpha_2 < \alpha^n$. Since population 1 has complete information, its travelers know the exact cost function in both states. However, since population 2 travelers are uninformed, they make their route choices based on the expected cost of each route, evaluated according to the prior distribution of states.

This routing game with heterogeneously informed traveler populations admits a Bayesian Wardrop equilibrium, as discussed in Section 2. Let $q^* = (q^*_1(a), q^*_1(n), q^*_2)$ denote an equilibrium strategy profile. Each population, given the signal it receives, can either assign all its demand on one of the two routes, or splits on both routes. Thus, there are $3^2 = 27$ possible cases. Our results can be used to study how the equilibrium strategies and route flows change as the size of a population varies from 0 to 1. Detailed analysis for this simple routing game and some interesting variants are available in [Wu et al. 2017]. Specifically, we find that there exists a threshold size of population 1, $0 < \lambda^1 = \alpha_2 \left(\frac{1}{\alpha^a + \alpha_2} - \frac{1}{\alpha^n + \alpha_2} \right) < 1$, such that the qualitative structure of equilibrium routing strategies is different based on whether $\lambda^1 \in [0, \lambda^1)$ or $\lambda^1 \in [\lambda^1, 1]$.

In the first regime, i.e. when $\lambda^1 \in [0, \lambda^1)$, the game admits a unique equilibrium: $q^*_1(a) = 0$, $q^*_1(n) = \lambda^1$, and $q^*_2 = \frac{\alpha_2}{\alpha_1 + \alpha_2} - \lambda^1 (1 - p) \frac{\alpha^a}{\alpha_1 + \alpha_2}$, where $\alpha_1 = p \alpha^a + (1 - p) \alpha^n$. This equilibrium regime corresponds to the following outcome: in state $n$ (resp. state $a$), population 1 assigns all its demand on route $r_1$ (resp. route $r_2$), and population 2 splits its demand on both routes. The induced equilibrium flow on route $r_1$ is given by $f^*_1 = q^*_2$ if $t^1 = a$, and $f^*_1 = \lambda^1 + q^*_2$ if $t^1 = n$.

On the other hand, in the second regime, i.e. when $\lambda^1 \in [\lambda^1, 1]$, the equilibrium set may not be singleton, and can be represented as follows: $q^*_1(a) = \chi$, $q^*_1(n) = \lambda^1 + \chi$, and $q^*_2 = \frac{\alpha_2}{\alpha_1 + \alpha_2} - \chi$, where $max \{0, \lambda^1 - \frac{\alpha^a}{\alpha_1 + \alpha_2} \} \leq \chi \leq min \left\{ \frac{\alpha_2}{\alpha_1 + \alpha_2}, \lambda^1 - \lambda^1 \right\}$. Thus, the equilibrium set is a one-dimensional interval for $\lambda^1 \in (\lambda^1, 1)$, and a singleton set for $\lambda^1 = \lambda^1$ or $\lambda^1 = 1$. In this regime, both populations face identical expected route costs in equilibrium. Consequently, each population splits its demand on both routes. Moreover, the equilibrium route flow on each route is unique and independent of $\lambda^1$: $f^*_1 = \frac{\alpha_2}{\alpha_1 + \alpha_2}$ if
$t^1 = a$, and $f_1^* = \frac{\alpha_2}{\alpha_1 + \alpha_2}$ if $t^1 = n$.

Notice that when $\lambda^1 \in [0, \lambda^1)$, we have $q_1^*(n) - q_1^*(a) = \lambda^1$, i.e. population 1 shifts all its demand to $r_2$ when receiving the signal about the accident on $r_1$. However, if $\lambda^1 \in [\lambda^1, 1]$, we have $q_1^*(n) - q_1^*(a) = \lambda^1 < \lambda^1$, i.e. the change in the received signal only influences a part of travelers in population 1. One can say that the information impacts the entire demand of population 1 in the first regime, but not in the second regime.

For any feasible $\lambda = (\lambda^1, \lambda^2)$, we can calculate the equilibrium population costs, denoted $C^*(\lambda)$. If $\lambda^1 \in [0, \lambda^1)$, since population 1 takes $r_1$ in state $n$ and $r_2$ in state $a$, we can write $C_1^*(\lambda) = p \cdot c_2(1 - q_2^*) + (1 - p) \cdot c_1(n) \cdot q_1^* + q_2^*$. Population 2 uses both routes, and thus $C_2^*(\lambda) = p \cdot c_2(1 - q_2^*) + (1 - p) \cdot c_2(1 - \lambda^1 - q_2^*)$. It is easy to check that $C_2^*(\lambda) + (1 - \lambda^1) > 0$, i.e. when the state information is only available to a small fraction of travelers, the informed travelers have an advantage over the uninformed ones. On the other hand, if the size of informed population exceeds the threshold $\lambda^1$, then $C_1^*(\lambda) = C_2^*(\lambda)$. In this case, all travelers face identical cost in equilibrium, no uninformed traveler has the incentive to become informed (and vice versa).

Finally, the equilibrium average cost is simply the average expected cost in equilibrium, i.e. $C^*(\lambda) = \frac{1}{n} C_1^*(\lambda) + \frac{\lambda^2}{n} C_2^*(\lambda)$. We can check that $C^*(\lambda)$ monotonically decreases with $\lambda^1$ in the first regime, and attains a constant (minimum) value in the second regime. Thus, increasing the size of informed population decreases the equilibrium average cost but only when it is below $\lambda^1$.

We illustrate the aforementioned results in Fig. 1 using the following parameters: $\alpha_1^n = 1$, $\alpha_1^a = 3$, $\alpha_2 = 2$, $b = 20$, and $p = 0.2$. The costs are normalized by the socially optimal cost, denoted $C^{so}$, which is the minimum cost achievable by a social planner with complete information of the state.

3 Model

3.1 Environment

To generalize the simple routing game in Section 2, we consider a transportation network modeled as a directed graph with a single origin-destination pair. Let $E$ denote the set of edges and $R$ denote the set of routes. The finite set of network states, denoted $S$, represents the set of possible network conditions, such as incidents, weather, etc. The network state, denoted $s$, is randomly drawn by a fictitious player “Nature” from $S$ according to a distribution $\theta \in \Delta(S)$, which determines the prior probability of each state. For any edge $e \in E$ and state $s \in S$, the state-dependent edge cost function $c_e^s(\cdot)$ is a positive, increasing, and differentiable function of the load through the edge $e$. Note that the state can impact the edge costs in various ways.

The network serves a set of non-atomic travelers with a fixed total demand $D$. We assume that each traveler is subscribed exclusively to one of the TIS in the set $I = \{1, \cdots, I\}$. We refer to the set of travelers subscribed to the TIS $i \in I$ as population $i$. All travelers within a population receive an identical signal from their TIS. Let $\lambda^i$ denote the ratio of population $i$'s size and the total demand $D$. We also consider degenerate
The size vector $\lambda$ is considered as given in our analysis of equilibrium structure and costs (Sections 4 and 5). In Section 6 we consider a more general situation where $\lambda$ results from the travelers' TIS subscription choices.

Each TIS $i \in \mathcal{I}$ sends a noisy signal $t^i$ of the state to population $i$. The signal received by each population determines its type (private information). We assume that the type space of population $i$ is a finite set, denoted as $\mathcal{T}^i$. Note that the type spaces $\mathcal{T}^i$ and the state space $\mathcal{S}$ need not be of the same size. Let $t \triangleq (t^1, t^2, \ldots, t^I)$ denote a type profile, i.e. vector of signals received by the traveler populations; thus, $t \in \mathcal{T} \triangleq \prod_{i \in \mathcal{I}} \mathcal{T}^i$. The joint probability distribution of the state $s$ and the vector of signals $t$ is denoted $\pi \in \Delta(\mathcal{S} \times \mathcal{T})$, and it is the common prior of the game. The marginal distribution of $\pi$ on states is consistent with the common prior, i.e. $\sum_{t \in \mathcal{T}} \pi(s, t) = \theta(s)$ for all $s \in \mathcal{S}$.
The conditional probability of type profiles \( t \) on the state \( s \) is given by \( p(t|s) = \frac{\pi(s,t)}{\theta(s)} \); i.e. the joint distribution of signals received by the populations when the network state is \( s \). In our modeling environment, the signals of different TIS can be correlated, conditional on the state. Each population \( i \) generates a belief about the state \( s \) and the other populations’ types \( t^{-i} \) based on the signal received from the information system \( i \in \mathcal{I} \). We denote the population \( i \)'s belief as \( \beta_i(s, t^{-i} | t^i) \in \Delta(S \times T^{-i}) \).

The routing strategy of each population \( i \in \mathcal{I} \) is a function of its type, denoted as \( q_i(t^i) = (q_{ir}^i(t^i))_{r \in \mathcal{R}} \). One way to describe the generation of routing strategies is that each TIS \( i \in \mathcal{I} \) sends a noisy signal \( t^i \) of the state to its subscribed population, and the individual route choices of non-atomic travelers results in an aggregate routing strategy \( q_i(t^i) \).

An alternative viewpoint is that \( q_i(t^i) \) is a direct result of strategy route recommendations sent by each TIS to its subscribed population. That is, each TIS \( i \in \mathcal{I} \) routes travelers in population \( i \) according to the function \( q_i(t^i) \). For our purpose, these two viewpoints are equivalent in that given any population \( i \in \mathcal{I} \), and any type \( t^i \in T^i \), the demand of travelers on route \( r \in \mathcal{R} \) is \( q_{ir}^i(t^i) \).

We say that a routing strategy profile \( q \overset{\Delta}{=} (q_i)_{i \in \mathcal{I}} \) is feasible if it satisfies the following constraints:

\[
\begin{align*}
\sum_{r \in \mathcal{R}} q_{ir}^i(t^i) &= \lambda^i D, \quad \forall t^i \in T^i, \quad \forall i \in \mathcal{I}, \\
q_{ir}^i(t^i) &\geq 0, \quad \forall r \in \mathcal{R}, \quad \forall t^i \in T^i, \quad \forall i \in \mathcal{I}.
\end{align*}
\]

For a given size vector \( \lambda \), let \( Q^i(\lambda) \) denote the set of all feasible strategies of population \( i \). From (2a)-(2b), we know that the set of feasible strategy profiles \( Q(\lambda) \overset{\Delta}{=} \prod_{i \in \mathcal{I}} Q^i(\lambda) \) is a convex polytope.

### 3.2 Bayesian Routing Game

The Bayesian routing game for a fixed size vector \( \lambda \) can be defined as \( \Gamma(\lambda) \overset{\Delta}{=} (\mathcal{I}, \mathcal{S}, \mathcal{T}, Q(\lambda), \mathcal{C}, \beta) \), where

- \( \mathcal{I} \): Set of populations, \( \mathcal{I} = \{1, 2, \ldots, I\} \)
- \( \mathcal{S} \): Set of states with prior distribution \( \theta \in \Delta(S) \)
- \( \mathcal{T} = \prod_{i \in \mathcal{I}} T^i \): Set of population type profiles with element \( t = (t^i)_{i \in \mathcal{I}} \in \mathcal{T} \)
- \( Q(\lambda) = \prod_{i \in \mathcal{I}} Q^i(\lambda) \): Set of feasible strategy profiles for a given size vector \( \lambda \), with element \( q = (q_i)_{i \in \mathcal{I}} \in Q(\lambda) \)
- \( \mathcal{C} = \{c_e(\cdot)\}_{e \in \mathcal{E}, s \in \mathcal{S}} \): Set of state-dependent edge cost functions
- \( \beta = (\beta^i)_{i \in \mathcal{I}} \), \( \beta^i \overset{\Delta}{=} \Delta(S \times T^{-i}) \) is the population \( i \)'s belief on state \( s \) and other populations’ types \( t^{-i} \).
All parameters including the common prior $\pi$ are common knowledge, except that populations privately receive signals about the network state from their respective TIS. The game is played as shown in Fig. 2.

For any $i \in \mathcal{I}$ and $t^i \in \mathcal{T}^i$, the interim belief of population $i$ is derived from the common prior:

$$\beta^i(s, t^{-i}|t^i) = \frac{\pi(s, t^i, t^{-i})}{\Pr(t^i)}, \quad \forall s \in \mathcal{S}, \forall t^{-i} \in \mathcal{T}^{-i},$$

(3)

where $\Pr(t^i) = \sum_{s \in \mathcal{S}} \sum_{t^{-i} \in \mathcal{T}^{-i}} \pi(s, t^i, t^{-i})$. For a strategy profile $q \in \mathcal{Q}(\lambda)$, the induced route flow is denoted $f \triangleq (f_r(t))_{r \in \mathcal{R}, t \in \mathcal{T}}$, where $f_r(t)$ is the aggregate flow assigned to the route $r \in \mathcal{R}$ by populations with type profile $t$, i.e.

$$f_r(t) = \sum_{i \in \mathcal{I}} q^i_r(t^i), \quad \forall r \in \mathcal{R}, \forall t \in \mathcal{T}.$$  

(4)

Note that the dependence of $f$ on $q$ is implicit and is dropped for notational convenience.

Again, for the strategy profile $q \in \mathcal{Q}(\lambda)$, we denote the induced edge load as $w \triangleq (w_e(t))_{e \in \mathcal{E}, t \in \mathcal{T}}$, where $w_e(t)$ is the aggregate load on the edge $e$ assigned by populations with type profile $t$:

$$w_e(t) = \sum_{r \in \mathcal{R}} \sum_{i \in \mathcal{I}} q^i_r(t^i) \mathcal{E} \sum_{r \in \mathcal{R}} f_r(t), \quad \forall e \in \mathcal{E}, \forall t \in \mathcal{T}.$$  

(5)

The corresponding cost of edge $e \in \mathcal{E}$ in state $s \in \mathcal{S}$ is $c^e_e(w_e(t))$. Then, the cost of route $r \in \mathcal{R}$ in state $s \in \mathcal{S}$ can be obtained as: $c^e_r(q(t)) = \sum_{e \in \mathcal{E}} c^e_e(w_e(t))$. Finally, the expected cost of route $r$ for population $i \in \mathcal{I}$ can be expressed as follows:

$$\mathbb{E}[c_r(q)|t^i] = \sum_{s \in \mathcal{S}} \sum_{t^{-i} \in \mathcal{T}^{-i}} \sum_{e \in \mathcal{E}} \beta^i(s, t^{-i}|t^i) c^e_e(w_e(t^i, t^{-i}))$$

$$\mathbb{E}[c_r(q)|t^i] \sum_{s \in \mathcal{S}} \sum_{t^{-i} \in \mathcal{T}^{-i}} \sum_{e \in \mathcal{E}} \frac{\pi(s, t^i, t^{-i})}{\Pr(t^i)} c^e_e(w_e(t^i, t^{-i})), \forall r \in \mathcal{R}, \forall t^i \in \mathcal{T}^i, \forall i \in \mathcal{I},$$  

(6)

where $w_e(t^i, t^{-i})$ is given by (5).

The equilibrium concept for our game $\Gamma(\lambda)$ is Bayesian Wardrop equilibrium (BWE). A strategy profile $q^* \in \mathcal{Q}(\lambda)$ is a BWE if for any $i \in \mathcal{I}$ and any $t^i \in \mathcal{T}^i$:

$$\forall r \in \mathcal{R}, \quad q^*_r(t^i) > 0 \Rightarrow \mathbb{E}[c_r(q^*)|t^i] \leq \mathbb{E}[c_r(q^*)|t^i], \forall r^i \in \mathcal{R}.$$  

(7)
That is, in a BWE, each population \( i \) with type \( t^i \) assigns its demand only on routes that have the smallest expected cost based on its interim belief \( \beta(s, t^{-i}|t^i) \).

We define the equilibrium population cost, denoted \( C^is(\lambda) \), as the expected cost incurred by a traveler of a given population across all types and network states in equilibrium: \( C^is(\lambda) \triangleq \frac{1}{\lambda D} \sum_{t^i \in T} \Pr(t^i) \sum_{r \in R} E[c_r(q^*)|t^i]q^is_r(t^i) \). In fact, from (7), we can write:

\[
C^is(\lambda) \triangleq \frac{1}{\lambda D} \sum_{t^i \in T} \Pr(t^i) \left( \sum_{r \in R} q^is_r(t^i) \right) \min_{r \in R} E[c_r(q^*)|t^i] \triangleq \sum_{t^i \in T} \Pr(t^i) \min_{r \in R} E[c_r(q^*)|t^i].
\]  

(8)

Note that \( \lambda_i = 0 \) is a degenerate case for population \( i \) as its size approaches 0. In this case, the cost \( C^is(\lambda) \) can be viewed as the expected cost faced by an individual (non-atomic) traveler who subscribes to the TIS \( i \).

Finally, the equilibrium average cost, denoted \( C^*(\lambda) \), is the average cost incurred by a traveler of any population across all network states in equilibrium:

\[
C^*(\lambda) \triangleq \frac{1}{D} \sum_{i \in \mathcal{I}} \lambda_i C^is(\lambda) = \frac{1}{D} \sum_{i \in \mathcal{I}} \sum_{t^i \in T} \Pr(t^i) \sum_{r \in R} E[c_r(q^*)|t^i]q^is_r(t^i).
\]  

(9)

4 Equilibrium Characterization

In this section, we show that the game \( \Gamma(\lambda) \) is a weighted potential game. This property enables us to express the sets of equilibrium strategy profiles and route flows as optimal solution sets of certain convex optimization problems.

4.1 Equilibrium Strategy Profiles

Following [Sandholm 2001], the game \( \Gamma(\lambda) \) is a weighted potential game if there exists a continuously differentiable function \( \Phi : Q(\lambda) \to \mathbb{R} \) and a set of positive, type-specific weights \( \{\gamma(t^i)\}_{t^i \in T^i, i \in \mathcal{I}} \) such that:

\[
\frac{\partial \Phi(q(t^i))}{\partial q^i_r(t^i)} = \gamma(t^i)E[c_r(q)|t^i], \quad \forall r \in \mathcal{R}, \quad \forall t^i \in T^i, \quad \forall i \in \mathcal{I}.
\]  

(10)

Indeed, the game \( \Gamma(\lambda) \) is a weighted potential game with the function \( \Phi \) defined as follows:

\[
\Phi(q) \triangleq \sum_{s \in \mathcal{S}} \sum_{v \in \mathcal{E}} \sum_{t \in T} \pi(s, t) \int_0^{\sum_{r \in \mathcal{R}} \sum_{i \in \mathcal{I}} q^i_r(t^i)} c^s_v(z) dz,
\]  

(11)

and the positive type-specific weights are \( \gamma(t^i) \triangleq \Pr(t^i) \) for any \( t^i \in T^i \) and \( i \in \mathcal{I} \), see Lemma A.1.
Using (4) and (5), Φ can be equivalently expressed as a function of the route flow $f$ or the edge load $w$ induced by a strategy profile $q \in Q(\lambda)$:

$$
\hat{\Phi}(f) \triangleq \sum_{s \in S} \sum_{e \in E} \sum_{t \in T} \pi(s, t) \int_0^{\sum_{r \in R} f_r(t)} c_e(z) dz \quad (12)
$$

$$
\hat{\Phi}(w) \triangleq \sum_{s \in S} \sum_{e \in E} \sum_{t \in T} \pi(s, t) \int_0^{w_e(t)} c_e(z) dz. \quad (13)
$$

Thus, for any feasible strategy profile $q \in Q(\lambda)$, we can write $\Phi(q) \equiv \hat{\Phi}(f) \equiv \hat{\Phi}(w)$, where $f$ and $w$ are the route flow and edge loads induced by the strategy profile $q$. Moreover, $\hat{\Phi}(w)$ is twice continuously differentiable and strictly convex in $w$, see Lemma A.2.

Our first result provides a characterization of the set of equilibrium strategy profiles:

**Theorem 1.** A strategy profile $q \in Q(\lambda)$ is a BWE if and only if it is an optimal solution of the following convex optimization problem:

$$
\min \Phi(q) \\
\text{s.t.} \quad q \in Q(\lambda),
$$

where $Q(\lambda)$ is the set of feasible strategy profiles. The equilibrium edge load vector $w^*(\lambda)$ is unique.

The existence of BWE follows directly from Theorem 1. For any size vector $\lambda$, we denote the set of BWE for the game $\Gamma(\lambda)$ as $Q^*(\lambda)$. Importantly, since the equilibrium edge load $w^*(\lambda)$ is unique, the equilibrium population cost $C^*(\lambda)$ for each population $i \in I$ and the equilibrium average cost $C^*(\lambda)$ in (9) must also be unique for any $\lambda$. Thus, the equilibria of $\Gamma(\lambda)$ can be viewed as essentially unique. We denote the optimal value of (OPT-Q) as $\Psi(\lambda)$.

The Lagrangian of (OPT-Q) that we use in proving Theorem 1 is given as follows:

$$
L(q, \mu, \nu, \lambda) = \Phi(q) + \sum_{i \in I} \sum_{t^i \in T^i} \mu^{t^i} \left( \lambda^i D - \sum_{r \in R} q^i_r(t^i) \right) - \sum_{r \in R} \sum_{i \in I} \sum_{t^i \in T^i} \nu_r^{t^i} q^i_r(t^i), \quad (14)
$$

where $\mu = (\mu^{t^i})_{t^i \in T^i, i \in I}$ and $\nu = (\nu_r^{t^i})_{r \in R, t^i \in T^i, i \in I}$ are Lagrange multipliers associated with the constraints (2a) and (2b), respectively. In fact, we show in Lemma A.4 that for any BWE $q^* \in Q^*(\lambda)$, the optimal Lagrange multipliers $\mu^*$ and $\nu^*$ in (14) associated with $q^*$ are unique, and can be written as follows:

$$
\mu_r^{t^i} = \min_{r \in R} \Pr(t^i) E[c_r(q^*)] t^i, \quad \forall t^i \in T^i, \quad \forall i \in I \quad (15a)
$$

$$
\nu_r^{t^i} = \Pr(t^i) E[c_r(q^*)] t^i - \mu_r^{t^i}, \quad \forall r \in R, \quad \forall t^i \in T^i, \quad \forall i \in I. \quad (15b)
$$

This result follows from the fact that (OPT-Q) satisfies the Linear Independence Constraint Qualification (LICQ) condition (Wachsmuth [2013]), which ensures the uniqueness of Lagrange multipliers at the optimum of (OPT-Q); see Lemma A.3.
Equations (15a) - (15b) connect the expected route costs for each type \( t^i \) in equilibrium with the Lagrange multipliers \( \mu^{t^i} \) and \( \nu^{t^i} \) at the optimum of (OPT-\( Q^\star \)), and will be used in Section 5 for studying the relative ordering of equilibrium population costs.

4.2 Equilibrium Route Flows

Our main question of interest is how the set of BWE \( Q^\star(\lambda) \), i.e. optimal solution set of (OPT-\( Q^\star \)), and more importantly, the equilibrium edge load \( w^\star(\lambda) \), change with the perturbations in the size vector \( \lambda \). However, characterizing the effect of \( \lambda \) directly from (OPT-\( Q^\star \)) is not so straightforward. Recall that in the simple routing game in Section 2, the effects of perturbations in \( \lambda \) on the equilibrium route flow are relatively easier to describe in comparison to the effects on the set of equilibrium strategy profiles, because the equilibrium route flow remains fixed in a certain range of \( \lambda \), whereas the set of equilibrium strategy profiles do not. Thus, our approach involves first studying how \( \lambda \) effects the set of equilibrium route flows. We show two results in this regard: (i) The set of feasible route flows and the set of feasible strategy profiles that induces a particular route flow can be both expressed as polytopes (Proposition 1); (ii) The set of equilibrium route flows is the optimal solution set of a convex optimization problem (Proposition 2). These results enable us to evaluate how the equilibrium edge load and population costs change with perturbations in \( \lambda \).

Let us start by introducing the following set of route flows:

\[
\mathcal{F}(\lambda) \triangleq \{ f \in \mathbb{R}^{|\mathcal{R}| \times |\mathcal{T}|} | f \text{ satisfies (17a)-(17d)} \},
\]

where the constraints are given by:

\[
f_{r}(t^{i}, t^{-i}) - f_{r}(\tilde{t}^{i}, t^{-i}) = f_{r}(t^{i}, \tilde{t}^{-i}) - f_{r}(\tilde{t}^{i}, \tilde{t}^{-i}), \quad \forall r \in \mathcal{R}, \forall t^{i}, \tilde{t}^{i} \in \mathcal{T}^{i}, \text{ and } \forall t^{-i}, \tilde{t}^{-i} \in \mathcal{T}^{-i}, \forall i \in \mathcal{I},
\]

\[
\sum_{r \in \mathcal{R}} f_{r}(t) = D, \quad \forall t \in \mathcal{T}, \quad \tag{17a}
\]

\[
f_{r}(t) \geq 0, \quad \forall r \in \mathcal{R}, \forall t \in \mathcal{T}, \quad \tag{17b}
\]

\[
D - \sum_{r \in \mathcal{R}} \min_{\tilde{t} \in \mathcal{T}^{i}} f_{r}(t^{i}, t^{-i}) \leq \lambda^{i} D, \quad \forall t^{-i} \in \mathcal{T}^{-i}, \forall i \in \mathcal{I}. \quad \tag{17c}
\]

The constraints (17a)-(17c) do not depend on the size vector \( \lambda \) and can be understood as follows: (17a) captures the fact that the change in the flow through any route resulting from change in the type of population \( i \in \mathcal{I} \) does not depend on the particular types of the remaining populations; (17b) ensures that all the demand \( D \) is routed through the network; and (17c) guarantees that the demand assigned to any route is nonnegative.

On the other hand, the constraints in (17d) depend on the size vector \( \lambda \), wherein the size of each population \( i \in \mathcal{I}, \lambda^{i} \), appears linearly in the constraint corresponding to that population. To further interpret (17d), we define an “impact of information” metric for any given population as the maximum extent to which the signal received from its TIS can influence the routing behavior of travelers within the population. Specifically, for any
strategy profile $q \in \mathcal{Q}(\lambda)$ and population $i \in \mathcal{I}$, we define the impact of information on population $i \in \mathcal{I}$ as follows:

$$J^i(q) \triangleq \lambda^i D - \sum_{r \in \mathcal{R}} \min_{t^i \in \mathcal{T}^i} q^i_r(t^i). \quad (18)$$

Using (2a), we can re-write (18) as:

$$J^i(q) = \sum_{r \in \mathcal{R}} \max_{t^i \in \mathcal{T}^i} \left( f_r(t^i, \hat{t}^i) - f_r(\hat{t}^i, \hat{t}^i) \right) \quad (17a)$$

$$\hat{J}^i(f) \triangleq J^i(q) \quad (4)$$

$$\hat{J}^i(f) \leq \lambda^i D, \quad \forall i \in \mathcal{I}. \quad (IIC)$$

These constraints ensure that the impact of signals on any population’s strategy is bounded by its size. We will refer to them as information impact constraints (IIC). We use (IIC) and (17d) interchangeably, and refer the constraint in (IIC) corresponding to population $i \in \mathcal{I}$ as (IIC$_i$). Also, it is easy to see that for each $i \in \mathcal{I}$, (IIC$_i$) can be written as a set of affine inequalities:

$$D - \sum_{r \in \mathcal{R}} f_r(t^i_r, \hat{t}^i) \leq \lambda^i D, \quad \forall t^i_1 \in \mathcal{T}^i, \ldots, \forall t^i_{|R|} \in \mathcal{T}^i. \quad (21)$$

Thus, $\mathcal{F}(\lambda)$, as defined in (16), is a convex polytope. The following proposition relates the set of feasible strategy profiles and the induced route flows; see Fig. 3.

Figure 3: Map between $\mathcal{Q}(\lambda)$ and $\mathcal{F}(\lambda)$: $\mathcal{F}(\lambda)$ is the image of $\mathcal{Q}(\lambda)$ under the linear transformation (4). The pre-image of any $f \in \mathcal{F}(\lambda)$ is given by the set of feasible strategy profiles satisfying (22) and (23a)-(23c).
Proposition 1. The set of feasible route flows is the convex polytope $F(\lambda)$. Furthermore, for a given route flow $f \in F(\lambda)$, any feasible strategy profile $q = (q^i_r(t^i))_{r \in R, t^i \in T^i, i \in I} \in Q(\lambda)$ that induces $f$ can be expressed as:

$$q^i_r(t^i) = f_r(t^i, \hat{t}^i) - \hat{f}_r(\hat{t}^i) + \chi^i_r, \forall r \in R, \forall t^i \in T^i, \forall i \in I,$$

where $\hat{t} = (\hat{t}^i)_{i \in I}$ is any type profile in $T$, and $(\chi^i_r)_{r \in R, i \in I}$ is an $|R| \times |I|$-dimensional vector satisfying the following constraints:

$$\sum_{r \in R} \chi^i_r = \lambda^i D, \forall i \in I,$$

$$\sum_{i \in I} \chi^i_r = f_r(\hat{t}), \forall r \in R,$$

$$\chi^i_r \geq \max_{t^i \in T^i} (f_r(\hat{t}^i, \hat{t}^i) - f_r(t^i, \hat{t}^i)),$$

$\forall r \in R, \forall i \in I.$

The next proposition provides a characterization of the set of equilibrium route flows, and is analogous to Theorem 1 which characterizes the set of equilibrium strategy profiles.

Proposition 2. A feasible route flow $f \in F(\lambda)$ is an equilibrium route flow if and only if $f$ is an optimal solution of the following convex optimization problem:

$$\min \Phi(f)$$

$$s.t. \ f \in F(\lambda),$$

(OPT-$F$)

where $\Phi(f)$ is given by (12), and $F(\lambda)$ is the set of feasible route flow vectors, as defined by (16).

We denote the set of equilibrium route flows $f^*$ in the game $\Gamma(\lambda)$ as $F^*(\lambda)$. From Theorem 1 and (11)-(13), we know that for any size vector $\lambda$, and any $q^* \in Q^*(\lambda)$, $f^* \in F^*(\lambda)$, we have

$$\Psi(\lambda) = \Phi(q^*) = \Phi(f^*) = \Phi(w^*(\lambda)).$$

The Propositions 1 and 2 form the basis of our analysis of how the perturbations of size vector effects the equilibrium structure and population costs.

5 Pairwise Comparison of Populations

In this section, we first analyze the effects of perturbations in the relative sizes of any two populations on the equilibrium structure. Next, we study how the cost difference between any two populations depends on the population sizes.
5.1 Equilibrium Regimes

To study the effects of perturbations in the relative sizes of any two populations, we employ the notion of directional perturbation of size vector $\lambda$. In particular, for any two populations $i$ and $j$, we consider the $|\mathcal{I}|$-dimensional direction vector:

$$z_{ij}^\Delta = (\ldots 0 \ldots, 1_{i-th}, \ldots 0 \ldots, -1_{j-th}, \ldots 0 \ldots).$$

When $\lambda$ is perturbed in the direction of $z_{ij}^\Delta$, the size of population $i$ (resp. population $j$) increases (resp. decreases), and the sizes of the remaining populations do not change.

For any size vector $\lambda$ and any two populations $i$ and $j$, let the vector of the remaining populations' sizes be denoted $\lambda^{-ij} \overset{\Delta}{=} (\lambda^k)_{k \in \mathcal{I}\setminus\{i,j\}}$. The total size of the remaining populations is $|\lambda^{-ij}| \overset{\Delta}{=} \sum_{k \in \mathcal{I}\setminus\{i,j\}} \lambda^k$. To avoid triviality in pairwise comparison, we only consider the case when the sizes of both populations are strictly positive so that $\lambda^{-ij}$ satisfies the constraint $|\lambda^{-ij}| < 1$, and the range of the perturbations in the population $i$'s size is $(0, 1 - |\lambda^{-ij}|)$. We denote the set of admissible $\lambda^{-ij}$ as $\Lambda^{-ij}$.

Now consider an optimization problem that is similar to $\text{(OPT-}\mathcal{F})$, except that the two constraints in the $\text{(IIC)}$ set corresponding to the populations $i$ and $j$ are replaced by a single constraint:

$$\min \quad \hat{\Phi}(f)$$

s.t. $$\text{(17a), (17b), (17c), (IIC)\setminus\{i,j\}, (IIC)_{i,j}},$$

where the constraints $\text{(IIC)\setminus\{i,j\}}$ indicate that all but $\text{(IIC)}$ and $\text{(IIC)}_{i,j}$ from the original set $\text{(IIC)}$ are included, and the constraint $\text{(IIC)_{i,j}}$ is defined as follows:

$$\hat{J}^i(f) + \hat{J}^j(f) \leq (1 - |\lambda^{-ij}|) D. \quad \text{(IIC)_{i,j}}$$

The constraint $\text{(IIC)_{i,j}}$ ensures that the total impact of information on population $i$ and $j$ does not exceed their total demand. We denote the set of optimal solutions for $\text{(OPT-}\mathcal{F})^{-ij}$ as $\mathcal{F}^{ij,\dagger}$. Analogously to Theorem 1, we can show that any $f^{ij,\dagger} \in \mathcal{F}^{ij,\dagger}$ induces a unique edge load $w^{ij,\dagger}$, which can be obtained by (5); see Lemma B.1. Then, the optimal solution set of $\text{(OPT-}\mathcal{F})^{-ij}$ can be written as the following polytope:

$$\mathcal{F}^{ij,\dagger} = \left\{ f \mid f \text{ satisfies (17a), (17b), (17c), (IIC)\setminus\{i,j\}, and (IIC)_{i,j}}, \right\}. \quad (25)$$

Note that both $\mathcal{F}^{ij,\dagger}$ and $w^{ij,\dagger}$ depend on $\lambda^{-ij}$ but do not depend on $\lambda^i$ or $\lambda^j$.

Before proceeding further, we need to define two thresholds for the size of one of the two perturbed populations (say, population $i$):

$$\lambda^i \overset{\Delta}{=} \frac{1}{D} \min_{f^{ij,\dagger} \in \mathcal{F}^{ij,\dagger}} \left\{ \hat{J}^i(f^{ij,\dagger}) \right\}, \quad (26a)$$

$$\bar{\lambda}^i \overset{\Delta}{=} \frac{1}{D} \max_{f^{ij,\dagger} \in \mathcal{F}^{ij,\dagger}} \left\{ (1 - |\lambda^{-ij}|) D - \hat{J}^i(f^{ij,\dagger}) \right\}, \quad (26b)$$
where $\lambda_i^{j,(f^{ij,:})}$ and $\lambda_j^{j,(f^{ij,:})}$ are the impact of information metrics for the population $i$ and $j$, respectively. It is easy to check that the thresholds $\lambda^i$ and $\lambda^j$ satisfy $0 \leq \lambda^i \leq \lambda^j \leq 1 - |\lambda^{-ij}|$ (see Lemma B.2.), and that both (26a)-(26b) can be expressed as linear programming problems, see (B.7)-(B.8). These two thresholds play a crucial role in our subsequent analysis.

We are now ready to introduce the equilibrium regimes that are induced by the directional perturbation of size vector $\lambda$ in the direction $z^{ij}$, i.e. when the populations $i$ and $j$ undergo relative change in their sizes with fixed $\lambda^{-ij} \in \Lambda^{-ij}$. These regimes are defined by the following sets:

$$\Lambda_1^{ij} \doteq \{ (\lambda^i, \lambda^j, \lambda^{-ij}) \mid \lambda^i \in (0, \lambda^j) \}, \quad (27a)$$

$$\Lambda_2^{ij} \doteq \{ (\lambda^i, \lambda^j, \lambda^{-ij}) \mid \lambda^i \in [\lambda^j, \lambda^j] \setminus \{0, 1 - |\lambda^{-ij}|\} \}, \quad (27b)$$

$$\Lambda_3^{ij} \doteq \{ (\lambda^i, \lambda^j, \lambda^{-ij}) \mid \lambda^i \in (\lambda^j, 1 - |\lambda^{-ij}|) \}. \quad (27c)$$

We say that the population $i$ (resp. population $j$) is a “minor population” in regime $\Lambda_1^{ij}$ (resp. regime $\Lambda_3^{ij}$) because $\lambda^i < \lambda^j$ (resp. $\lambda^j < 1 - |\lambda^{-ij}| - \lambda^j$). Moreover, neither population is minor in regime $\Lambda_2^{ij}$. In each of the three regimes, $\lambda^i$ is in the admissible range $(0, 1 - |\lambda^{-ij}|)$. Note that degenerate situations are possible. In particular, if either one or both of the thresholds $\lambda^i$ and $\lambda^j$ take values in the set $\{0, 1 - |\lambda^{-ij}|\}$, then from the regime definitions (27a)-(27c), the number of regimes are reduced to two (for example, in the the simple routing game in Section 2) or even one regime (see Example B.2). The following theorem describes the properties of equilibrium route flows in the regimes that can be induced by directional perturbations in the size vector $\lambda$.

**Theorem 2.** For any two populations $i, j \in I$, and any given $\lambda^{-ij} \in \Lambda^{-ij}$, the set of equilibrium route flows $F^*(\lambda)$ when $\lambda$ is in regime $\Lambda_1^{ij}$ or regime $\Lambda_3^{ij}$ can be expressed as follows:

$$F^*(\lambda) = \left\{ \arg \min \Phi(f) \mid \begin{array}{l} \text{s.t. } (17a), (17b), (17c), (IIC) \text{ and } (IIC) \setminus \{j\} \quad \text{if } \lambda \in \Lambda_1^{ij} \\ \text{s.t. } (17a), (17b), (17c), (IIC) \text{ and } (IIC) \setminus \{i\} \quad \text{if } \lambda \in \Lambda_3^{ij} \end{array} \right\} \quad (28)$$

In regime $\Lambda_1^{ij}$ (resp. regime $\Lambda_3^{ij}$), the constraint (IIC) (resp. (IIC)) is tight in equilibrium. Additionally, in regime $\Lambda_2^{ij}$, we have $F^*(\lambda) \subseteq F^{ij,\dagger}$.

Essentially, this result is based on how the impact of information on each perturbed population compares with its size; i.e. whether or not the constraint (IIC) (resp. (IIC)) for the population $i$ (resp. population $j$) is tight in equilibrium. Moreover, since the impact of information metric summed over both populations cannot exceed their total demand, (IIC) must be satisfied in all regimes. Below we discuss the qualitative properties of equilibrium structure in the three regimes.

In the first side regime $\Lambda_1^{ij}$, the constraint (IIC) is tight at optimum of (OPT-F). This implies that the impact of information extends to the entire demand of the minor population $i$. In fact, the threshold $\lambda^i$ is the largest size of population $i$ for which the
impact of information on itself is fully attained. We can argue similarly for the other side regime \( \Lambda_{ij}^3 \), where population \( j \) is the minor population; i.e. \( \text{III} \) is tight at optimum of \( \text{OPT-} \mathcal{F} \) and \( (1 - |\lambda^{-ij}| - \hat{x}) \) is the largest size of population \( j \) such that the impact of information on itself is fully attained.

In contrast to the two side regimes, in the middle regime \( \Lambda_{ij}^3 \), the sizes of both populations \( i \) and \( j \) are above the threshold sizes \( \hat{x} \) and \( (1 - |\lambda^{-ij}| - \hat{x}) \), respectively. In this regime, we can replace the constraints \( \text{III} \) and \( \text{III} \) in the optimization problem \( \text{OPT-} \mathcal{F} \) by \( \text{III} \) without changing its optimal value, i.e. the optimal value of \( \text{OPT-} \mathcal{F} \) is equal to \( \Psi(\lambda) \). However, since the set \( \mathcal{F} \) (as defined in \( 25 \)) contains all route flows that attain the optimal value \( \Psi(\lambda) \) but may not necessarily satisfy the constraints \( \text{III} \) and \( \text{III} \), the equilibrium route flow set \( \mathcal{F}^*(\lambda) \) must be a subset of \( \mathcal{F} \).

An implication of Theorem 2 is that in the two side regimes \( \Lambda_{ij}^1 \) and \( \Lambda_{ij}^2 \), any equilibrium strategy profile cannot be in the interior of the feasible set \( \mathcal{Q}(\lambda) \). To see this, recall from Proposition 1 that given an equilibrium route flow \( f^* \in \mathcal{F}^*(\lambda) \), any equilibrium strategy profile \( q^* \) that induces \( f^* \) can be expressed as equation \( 22 \). Then from Theorem 2 based on the tightness of the constraint \( \text{III} \) or \( \text{III} \), we can conclude that:

In regime \( \Lambda_{ij}^1 \):
\[
q^*_r(t^i) = f^*_r(t^i, \hat{t}^{-i}) - \min_{\hat{t}^i \in \mathcal{T}} f^*_r(\hat{t}^i, \hat{t}^{-i}), \quad \forall r \in \mathcal{R}, \quad \forall t^i \in \mathcal{T}.
\]  
(29a)

In regime \( \Lambda_{ij}^2 \):
\[
q^*_r(t^j) = f^*_r(t^j, \hat{t}^{-j}) - \min_{\hat{t}^{-j} \in \mathcal{T}^{-j}} f^*_r(\hat{t}^i, \hat{t}^{-j}), \quad \forall r \in \mathcal{R}, \quad \forall t^j \in \mathcal{T}.
\]  
(29b)

where \( \hat{t}^{-i} \) (resp. \( \hat{t}^{-j} \)) is any vector of types in \( \mathcal{T}^{-i} \) (resp. \( \mathcal{T}^{-j} \)). Consequently, for each route \( r \in \mathcal{R} \), there exists at least one \( t^i \in \mathcal{T}^i \) (resp. \( t^j \in \mathcal{T}^j \)) such that \( q^*_r(t^i) = 0 \) (resp. \( q^*_r(t^j) = 0 \)) when the size vector \( \lambda \) is in the regime \( \Lambda_{ij}^1 \) (resp. \( \Lambda_{ij}^2 \)). So we obtain that \( q^* \) is not in the interior of the feasible set of strategy profiles \( \mathcal{Q}(\lambda) \) when \( \lambda \) lies in one of the two side regimes. Furthermore, this result implies that in each side regime \( \Lambda_{ij}^1 \) or \( \Lambda_{ij}^2 \), given any route \( r \in \mathcal{R} \), there exists at least one TIS signal for the corresponding minor population that triggers the population to shift its demand completely away from the route \( r \).

The general expressions of equilibrium routing strategies in \( 29a \)–\( 29b \) can be specialized to routing games with two heterogeneously informed populations and a parallel-route network. Such a game admits a unique equilibrium strategy profile in the regimes \( \Lambda_{ij}^2 \) and \( \Lambda_{ij}^3 \), see Corollary B.1.

Finally, thanks to Theorem 2, we can analyze the monotonicity of the value of potential function at equilibrium \( \Psi(\lambda) \) with respect to perturbations of \( \lambda \) in the direction \( z^{ij} \).

**Proposition 3.** For any two populations \( i, j \in \mathcal{I} \), and any given \( \lambda^{-ij} \in \Lambda^{-ij} \), the function \( \Psi(\lambda) \) exhibits the following properties under directional perturbations of \( \lambda \) along the direction \( z^{ij} \):

- **In regime \( \Lambda_{ij}^1 \):** \( \Psi(\lambda) \) monotonically decreases
- **In regime \( \Lambda_{ij}^2 \):** \( \Psi(\lambda) \) does not change
- **In regime \( \Lambda_{ij}^3 \):** \( \Psi(\lambda) \) monotonically increases.
Furthermore, the equilibrium edge load vector \( w^*(\lambda) = w^{ij;\dagger} \) if and only if \( \lambda \in \Lambda_{2}^{ij} \).

Following Theorem 2 in the side regime \( \Lambda_{1}^{ij} \) (resp. \( \Lambda_{2}^{ij} \)), the set of route flows which satisfy the constraints of the optimization problem in (28) increases (resp. decreases) as \( \lambda \) is perturbed in the direction \( z^{ij} \). Thus, the value of the potential function in equilibrium, \( \Psi(\lambda) \), is non-increasing (resp. non-decreasing) in the direction \( z^{ij} \). In fact, since the constraint (IIC) (resp. (IIC)) is tight in equilibrium, one can argue that \( \Psi(\lambda) \) strictly decreases (resp. increases) in the direction \( z^{ij} \). In contrast, in the middle regime \( \Lambda_{2}^{ij} \), since \( F^*(\lambda) \subseteq F^{ij;\dagger} \), we can conclude that \( w^*(\lambda) = w^{ij;\dagger} \). Therefore, \( \Psi(\lambda) = \hat{\Phi}(w^{ij;\dagger}) \), which does not change when \( \lambda \) is perturbed in the direction \( z^{ij} \).

The necessary and sufficient condition for the invariance of \( w^*(\lambda) \) under relative perturbations in the sizes of any two populations is a direct consequence of the monotonicity of \( \Psi(\lambda) \) and the uniqueness of \( w^*(\lambda) \). Proposition 3 is useful in determining the relative ordering of population costs in equilibrium, as discussed next.

5.2 Relative Value of Information

We now study the difference between the equilibrium costs of any two populations under perturbations in their relative sizes. (Recall that equilibrium population costs in (8) are unique, thanks to Theorem 1.) For any two populations \( i, j \in \mathcal{I} \) and size vector \( \lambda \), we define the relative value of information as follows:

\[
V^{ij*}(\lambda) \triangleq C^{ij*}(\lambda) - C^{ii*}(\lambda).
\]

Thus, \( V^{ij*}(\lambda) \) is the expected travel cost saving that a traveler in population \( i \) enjoys over a traveler in population \( j \). Equivalently, \( V^{ij*}(\lambda) \) is the expected reduction in the cost faced by an individual traveler when her subscription unilaterally changes from TIS \( i \) to TIS \( j \), while the TIS subscriptions of all other travelers remain unchanged. In particular, for a given size vector \( \lambda \), if \( V^{ij*}(\lambda) > 0 \) (resp. \( V^{ij*}(\lambda) < 0 \)), a population \( i \) traveler faces lower (resp. higher) cost in equilibrium than a population \( j \) traveler. Consequently, TIS \( i \) can be viewed as a relatively more valuable (resp. less valuable) information system than TIS \( j \). Similarly, if \( V^{ij*}(\lambda) = 0 \), TIS \( i \) is said to be as valuable as TIS \( j \).

It turns out that, for any given size vector \( \lambda \), \( V^{ij*}(\lambda) \) is closely related to the sensitivity of \( \Psi(\lambda) \) (i.e. the value of the potential function in equilibrium) to the perturbation in the relative sizes of populations \( i \) and \( j \). In particular, Lemma B.5 shows that \( \Psi(\lambda) \) is convex in \( \lambda \) and is directionally differentiable. Furthermore, for any \( i, j \in \mathcal{I} \), \( V^{ij*}(\lambda) \) can be expressed as follows:

\[
V^{ij*}(\lambda) = -\frac{1}{D} \nabla_{z^{ij}} \Psi(\lambda),
\]

where \( \nabla_{z^{ij}} \Psi(\lambda) \triangleq \lim_{\epsilon \to 0^+} \frac{\Psi(\lambda + \epsilon z^{ij}) - \Psi(\lambda)}{\epsilon} \) is the derivative of \( \Psi(\lambda) \) in the direction \( z^{ij} \). The proof of Lemma B.5 involves applying the results on sensitivity analysis of convex optimization problems, as summarized in Lemmas B.3 and B.4. For a detailed background on these technical results, we refer the reader to Fiacco [2009], Fiacco and Kyparisis [1986],
and Rockafellar [1984]. By applying these results, we obtain that the derivative $\nabla_{zij} \Psi(\lambda)$ is well-defined, and can be expressed in terms of $\mu^*$, i.e. the Lagrange multiplies at the optimum of (OPT-Q). Since $\mu^*$ is unique, and can be written in (15a) (see Lemma A.4), we can conclude that equation (31) holds. Our next theorem provides the qualitative structure of relative value of information in the three regimes (27a)-(27c).

**Theorem 3.** For any two populations $i, j \in \mathcal{I}$, and any $\lambda^{-ij} \in \Lambda^{-ij}$, the relative value of information $V_{ij}^*(\lambda)$ satisfies the following properties:

- In regime $\Lambda_1^{ij}$, $V_{ij}^*(\lambda) > 0$
- In regime $\Lambda_2^{ij}$, $V_{ij}^*(\lambda) = 0$
- In regime $\Lambda_3^{ij}$, $V_{ij}^*(\lambda) < 0$.

Furthermore, $V_{ij}^*(\lambda)$ is non-increasing in the direction $z_{ij}$.

In other words, one population has advantage over another population if and only if it is the minor population of the two. This result directly follows from Lemma B.5 and Proposition 3. The intuition behind the change in the relative value of information in different regimes is based on the tightness of (IIC) or (IIC) in equilibrium. Recall that the same argument is used in Theorem 2 for characterizing the set of equilibrium route flows in different regimes. In particular, for the two side regimes, the constraint in (IIC) corresponding to the minor population is tight because the information impacts its entire demand. As a result, in equilibrium, the travelers in the minor population do not choose the routes with a high expected cost based on the signal they receive from their TIS; however, the travelers in the other population may still choose these routes. On the other hand, in the middle regime, neither population has an advantage over the other one because the information only partially impacts each population’s demand, and as a result, both populations route their demand in a manner such that they face identical cost in equilibrium.

Additionally, since $V_{ij}^*(\lambda)$ is non-increasing in the direction $z_{ij}$, we can conclude that the travel cost saving that population $i$ travelers enjoy over the population $j$ is the highest when population $i$ has few travelers. Intuitively, in each side regime, the travelers in the larger (non-minor) population face a higher congestion externality relative to the travelers in the minor population, because all travelers within a population are routed according to the same strategy. Naturally, the difference in the equilibrium costs due to the imbalance in congestion externality decreases as the size of the minor population increases, and in fact reduces to zero in the middle regime.

Furthermore, given any two populations $i, j \in \mathcal{I}$ and the sizes of all other populations $\lambda^{-ij}$ being fixed, Theorem 3 provides a computational approach to compare the equilibrium costs of populations $i$ and $j$ for the full range of $\lambda^i \in (0, 1 - |\lambda^{-ij}|)$ without explicit computation of BWE or equilibrium route flows for each $\lambda^i$. This approach can be summarized as follows: (i) Solve (OPT-F) to obtain an optimal solution $f_{ij}^\dagger$; (ii) Compute $w_{ij}^\dagger$ by plugging $f_{ij}^\dagger$ into (5); (iii) Obtain $\lambda^i$ and $\bar{\lambda}^i$ by solving (26a)-(26b); and (iv) Find
the relative ordering of equilibrium costs of population \( i \) and \( j \) by checking which of the three possible regimes the size vector \( \lambda \) belongs to.

Finally, we can specialize Theorem 3 to analyze situations when a population does not have an access to a TIS, or chooses not to use it at all. Formally, we say that a population \( j \in \mathcal{I} \) is uninformed if its type is independent with the network state and other populations’ types:

\[
\Pr(t^j|s,t^{-j}) = \Pr(t^j), \quad \forall t^{-j} \in \mathcal{T}^{-j}, \quad \forall t^j \in \mathcal{T}^j, \quad \forall s \in \mathcal{S}.
\]

(32)

Following (3), the uninformed population \( j \)’s interim belief can be written as follows:

\[
\beta^j(s,t^{-j}|t^j) \triangleq \frac{\pi(s,t^j,t^{-j})}{\Pr(t^j)} \frac{\Pr(t^j|s,t^{-j}) \cdot \Pr(s,t^{-j})}{\Pr(t^j)} = \sum_{t \in \mathcal{T}^j} \pi(s,t^{-j},t^j), \quad \forall t^{-j} \in \mathcal{T}^{-j}, \quad \forall s \in \mathcal{S}.
\]

(33)

That is, the interim belief \( \beta^j(s,t^{-j}|t^j) \) is identical for any signal \( t^j \in \mathcal{T}^j \) received by population \( j \), and is equal to the marginal probability of \( (s,t^{-j}) \) calculated from the common prior \( \pi \). Therefore, the uninformed population has no further information besides the common knowledge. Our next result shows that the equilibrium cost of the uninformed travelers is no less than the cost faced by travelers in any other population.

**Proposition 4.** Consider the game \( \Gamma(\lambda) \) in which population \( j \) is uninformed. Then, for any size vector \( \lambda \), the equilibrium cost of population \( j \)’s travelers \( C^{j*}(\lambda) \) satisfies the following conditions:

\[
C^{j*}(\lambda) \begin{cases} 
> C^{i*}(\lambda), & \text{if } \lambda^j \in (0, \lambda^i) \\
= C^{i*}(\lambda), & \text{if } \lambda^j \in [\lambda^i, 1 - |\lambda^{-ij}|),
\end{cases}
\]

where the population \( i \) is any other population (i.e. \( i \in \mathcal{I} \setminus \{j\} \)).

Indeed, if population \( j \) is uninformed, we can argue that its equilibrium routing strategy \( q^{j*}(t^j) \) must be identical for any \( t^j \in \mathcal{T}^j \). Consequently, from (18), the impact of information metric for the population \( j \) is \( \tilde{J}^j(q^{j*}) = 0 \), and perturbing the relative sizes of population \( j \) and any other population \( i \in \mathcal{I} \setminus \{j\} \) never results in a regime in which population \( j \) is the minor population. Applying Theorem 3, we can conclude that the equilibrium cost of population \( j \) cannot be less than that of any other population.

**Example 1.** To illustrate the results on equilibrium structure and relative value of information, we consider a game with two populations on two parallel routes \( (r_1 \text{ and } r_2) \) with following parameters: \( \theta(a) = 0.2, D = 10, c^a_1(f_1) = f_1 + 15, c^a_2(f_1) = 3f_1 + 15, c_2(f_2) = 2f_2 + 20 \). Types \( t^1 \) and \( t^2 \) are independent conditional on the state, i.e. \( \Pr(t^1,t^2|s) = \Pr(t^1|s) \cdot \Pr(t^2|s) \). Population 1 has 0.8 chance of getting accurate information of the state, and population 2 has 0.6 chance, i.e. \( \Pr(t^1 = s|s) = 0.8 \), and \( \Pr(t^2 = s|s) = 0.6 \). We illustrate the value of the potential function in equilibrium, equilibrium route flows and costs in Fig. 4.
In this example, population 1 travelers receive more accurate state information than population 2 travelers. However, population 1 faces a higher cost than population 2 when its size is sufficiently large, i.e., when \( \lambda \) is in regime \( \Lambda_{12} \); see Figure 4c. This is due to the fact that in regime \( \Lambda_{12} \), the population 1’s advantage of receiving more accurate information is dominated by the congestion externality it faces due to its relatively large size, in comparison to population 2.

Additionally, we discuss three illustrative examples in the e-companion. Example B.1 shows that the regime \( \Lambda_{12} \) can be empty even when population 2 is not an uninformed population. Thus, an uninformed population \( j \) is sufficient but not necessary for \( \bar{\lambda}^i = 1 - |\lambda^{-ij}| \). In Example B.2, we present a situation when only single regime exists in equilibrium. Finally, in Example B.3, we consider the two-route game with two TIS of same accuracy, i.e., travelers in these populations have an identical chance of receiving correct state information. We compare two extreme cases for this game: (i) Types \( t^1 \) and \( t^2 \) are perfectly correlated, i.e., \( t^1 = t^2 \); and (ii) Types \( t^1 \) and \( t^2 \) are independent conditional on the state. This example illustrates how the correlation among received signals (or lack thereof) effects the equilibrium structure.

Our results so far focus on how equilibrium properties and population costs change with the directional perturbation of the size vector \( \lambda \). We emphasize that given any \( i, j \in \mathcal{I} \), the thresholds \( \lambda^i' \) and \( \lambda^j' \), as defined in (26a)-(26b), depend on the sizes of the remaining populations \( \lambda^{-ij} \), and the populations’ interim beliefs \( (\beta^i_{r})_{r \in \mathcal{I}} \) derived from the common prior \( \pi \). Importantly, the qualitative structure of the equilibrium regimes resulting from perturbations in the sizes of any two populations is applicable for any size vector \( \lambda \) and any common prior. The main property that drives these results is that the equilibrium regimes only depend on whether or not the impact of information on each population is fully attained.

6 Equilibrium Outcome: General Results

In this section, we extend our approach of pairwise comparison of populations to study how the equilibrium outcome depends on population sizes in general, and more broadly
how travelers choose TIS. In Section 6.1, we characterize a non-empty set of size vectors for which the equilibrium edge load vector is size-independent. In Section 6.2, we study the equilibrium adoption rates of TISs when travelers can choose their TIS subscription. Finally, in Section 6.3, we identify a condition on the cost functions under which there is no inefficiency due to selfish routing of heterogeneously informed traveler populations.

6.1 Size-Independence of Edge Load Vector

Our analysis in Section 5 showed that if perturbations in the relative sizes of any two populations $i, j \in I$ induce a middle regime $\Lambda_{ij}^2$, then the equilibrium outcome in this regime is independent of the sizes of the perturbed populations $i$ and $j$. A natural question to ask is whether this result can be generalized; i.e., can we find a set of size vectors for which the equilibrium edge load does not depend on the size of any population? It turns out that we can answer this question in the affirmative.

We now explicitly characterize the set of size vectors, denoted $\Lambda^\dagger$, for which the edge load is size-independent. Using (24), we can argue that the optimal value of $\text{OPT-}F$ must be identical for all $\lambda \in \Lambda^\dagger$. Since $\text{OPT-}F$ is a convex optimization problem, and the (IIC) constraints are the only size-dependent constraints, we can equivalently view $\Lambda^\dagger$ as the set of size vectors for which all the IICs can be dropped from $\text{OPT-}F$ without changing its optimal value. Hence, for any $\lambda \in \Lambda^\dagger$, the optimal value of $\text{OPT-}F$ is identical to that of the following convex optimization problem:

$$\min \limits_{\hat{\Phi}(f)} \quad \text{s.t. } (17a), (17b) \text{ and } (17c).$$

Let us denote the optimal solution set of (35) as $F^\dagger$. Analogous to Theorem 1, one can argue that any optimal solution $f^\dagger \in F^\dagger$ induces a unique edge load $w^\dagger$, obtained from (5). Thus, $F^\dagger$ can be written as the convex polytope:

$$F^\dagger = \{ f \mid \text{if satisfies } (17a), (17b) \text{ and } (17c), \sum_{i \in \mathbb{E}} f_i(t) = w_e^\dagger(t), \forall e \in \mathbb{E}, \forall t \in T \}.$$

Furthermore, since any route flow in the set $F^\dagger$ satisfies the constraints (17a)-(17c) – but not necessarily (IIC) constraints – and also attains the optimal value of $\text{OPT-}F$, we must have that for any $\lambda \in \Lambda^\dagger$, $F^\dagger(\lambda) \subset F^\dagger$. Therefore, for each $\lambda \in \Lambda^\dagger$, one must have that the feasibility constraints (1a)-(1b) are satisfied; and additionally there exists a $f^\dagger \in F^\dagger$ that is an equilibrium route flow, i.e. at least one $f^\dagger \in F^\dagger$ satisfies the (IIC) constraints corresponding to $\lambda$. Thus, we can write:

$$\Lambda^\dagger \triangleq \{ \lambda \mid \lambda \text{ satisfies } (1a) - (1b), \text{ and } \exists f^\dagger \in F^\dagger \text{ s.t. } \hat{J}_i(f^\dagger) \leq \lambda^i D, \forall i \in I \}$$

Before presenting our next result, we state two properties of $\Lambda^\dagger$: Firstly, $\Lambda^\dagger$ is a non-empty convex polytope (Lemma C.1). Secondly, $\Lambda^\dagger$ is the set of size vectors that attain the minimum value of the weighted potential function in equilibrium, i.e. $\Lambda^\dagger = \arg \min_\lambda \Psi(\lambda)$ (Lemma C.2).
Proposition 5. The equilibrium edge load vector $w^*(\lambda)$ is size-independent, and is equal to $w^\dagger$ if and only if $\lambda \in \Lambda^\dagger$.

We have already argued that $\lambda \in \Lambda^\dagger$ is sufficient to achieve the size independence of the equilibrium edge load vector: If $\lambda \in \Lambda^\dagger$, then $F^* (\lambda) \subseteq F^\dagger$ and since any $f^\dagger \in F^\dagger$ induces an identical edge load $w^\dagger$, we have $w^* (\lambda) = w^\dagger$. The necessity can be argued by contradiction: If $\lambda \notin \Lambda^\dagger$, we know from Lemma C.2 that $\Psi(\lambda)$ must be larger than $\Phi(w^\dagger)$. Hence, $w^*(\lambda)$ cannot be equal to $w^\dagger$.

6.2 Adoption Rates under Choice of TIS

We now argue that our results on the relative value of information (Section 5), and the size independence of the equilibrium edge load vector (Section 6.1) can be applied to analyze the stable social fairness of the equilibrium outcome as well as travelers’ choice of TIS subscription. We motivate the definition of stable social fairness by first introducing the notion of social fairness.

We define a socially fair outcome as a situation in which all travelers face identical expected costs in equilibrium, regardless of their TIS subscriptions. Note that social fairness refers to the equality of expected costs faced by travelers in non-degenerate populations. Indeed, in the degenerate case when the size of travelers subscribed to a particular TIS approaches zero, the corresponding population has no effect on the travelers’ expected costs in equilibrium. Thus, in a socially fair outcome, a degenerate population may have a different expected cost than the expected cost faced by non-degenerate populations.

Furthermore, we say that an equilibrium outcome of the routing game is stable socially fair if the expected cost faced by any non-degenerate population is socially fair and lowest among all populations, i.e., the cost faced by any degenerate population is no less than the cost faced by a non-degenerate population:

$$\lambda^i > 0, \Rightarrow C^i* (\lambda) = \min_{j \in I} C^j* (\lambda), \forall i \in I.$$  \hspace{1cm} (38)

Therefore, no traveler has an ex ante incentive to switch her subscription of TIS. Recall from Section 3 that in our model, each traveler is subscribed to a single TIS, and the vector $\lambda$ represents the resulting population size vector. This modeling feature reflects the limitation of individual travelers in accessing more than one TIS or in switching to another TIS. However, if the equilibrium outcome corresponding to a given size vector $\lambda$ is stable, any traveler’s incentive to remain subscribed to her TIS is not affected by the presence of other TIS, or the equilibrium costs faced by other travelers.

We note that if all populations are non-degenerate, then social fairness is equivalent to stable social fairness. However, if some populations are degenerate, then a socially fair equilibrium outcome may not be stable socially fair. For example, consider the simple routing game in Section 2 for the case when $\lambda^1 = 0$ (i.e., all travelers belong to the uninformed population 2). Social fairness is naturally achieved in this case since all travelers are homogeneously informed. However, this outcome is not stable socially fair because any informed traveler would face a strictly lower expected cost in equilibrium,
and thus all uninformed travelers have an incentive to switch to TIS 1 if such a choice were made available to them.

Our next theorem shows that all size vectors $\lambda \in \Lambda^\dagger$ achieve stable social fairness in equilibrium.

**Theorem 4.** An equilibrium outcome of $\Gamma(\lambda)$ is stable socially fair if and only if $\lambda \in \Lambda^\dagger$.

The proof of this result utilizes two facts: (i) $\Psi(\lambda)$ is a convex function, and for any $i, j \in I$, its derivative in the direction $z^{ij}$ is proportional to the difference in expected equilibrium costs faced by the travelers in populations $i$ and $j$; see Lemma B.5. (ii) $\Lambda^\dagger = \arg\min_\lambda \Psi(\lambda)$; see Lemma C.2. For any $\lambda \in \Lambda^\dagger$, we can check that for each pair $i, j \in I$, if both $\lambda^i$ and $\lambda^j$ are positive, then the directional derivative of $\Psi(\lambda)$ in the direction $z^{ij}$ must be zero. Therefore, both populations $i$ and $j$ face identical costs in equilibrium. However, for the limiting case when $\lambda^i$ approaches 0 but $\lambda^j > 0$, the derivative $\Psi(\lambda)$ is non-negative in the direction $z^{ij}$, and thus the expected cost of a population $i$ traveler is no less than that faced by travelers in population $j$.

On the other hand, for any feasible $\lambda \notin \Lambda^\dagger$, we can argue that there must exist a direction $z^{ij}$ in which the directional derivative of $\Psi(\lambda)$ is negative, because $\Psi(\lambda)$ is convex and $\lambda$ does not minimize $\Psi(\lambda)$. If both $\lambda^i$ and $\lambda^j$ are positive, then the costs of populations $i$ and $j$ are different, which violates the social fairness condition. However, when $\lambda^i$ approaches zero and $\lambda^j > 0$, the cost of a population $i$ traveler is smaller than that of the population $j$ travelers, which is not stable socially fair.

Therefore, stable social fairness is attained if and only if $\lambda \in \Lambda^\dagger$.

We now consider the situation where travelers can choose to subscribe to any TIS. We model this problem as a two-stage game $\tilde{\Gamma}$, defined as follows:

**Stage 1:** Travelers choose to subscribe to one TIS from the given set $I$. The induced size vector is $\lambda = (\lambda^i)_{i \in I}$, where $\lambda^i$ is the fraction of travelers who choose TIS $i$.

**Stage 2:** Given the size vector $\lambda$, travelers in each population make route choices based on the signal received from their TIS, i.e. travelers play the subgame $\Gamma(\lambda)$.

Note that the size vector $\lambda$ in the game $\tilde{\Gamma}$ is determined by the travelers’ TIS choices in the first stage, as opposed to being a parameter in our analysis so far. A strategy profile of the game $\tilde{\Gamma}$ is $(\lambda, q)$, where a feasible $\lambda$ satisfies (1a) - (1b), and $q \in Q(\lambda)$ is a feasible routing strategy profile.

A strategy profile, denoted $(\lambda^\dagger, q^\dagger)$, is a subgame perfect equilibrium (SPE) of the game $\tilde{\Gamma}$, if it satisfies two conditions: Firstly, the strategy $q^\dagger \in Q^*(\lambda)$ is a BWE in the subgame $\Gamma(\lambda)$ for each feasible $\lambda$, i.e. $q^\dagger$ satisfies (7). Secondly, no traveler has the incentive to change her TIS subscription in the first stage, i.e. $\lambda^\dagger$ satisfies (38).

We note that the notion of stable social fairness is equivalent to the definition of SPE in the context of this two stage game $\tilde{\Gamma}$. Hence, our results (Theorem 1 and Theorem 4) provide a characterization of the set of SPE in the game $\tilde{\Gamma}$, i.e. any $\lambda^\dagger \in \Lambda^\dagger$ and any $q^\dagger \in Q^*(\lambda)$ constitute a SPE of the game. On the other hand, if the size vector $\lambda \notin \Lambda^\dagger$, then from Theorem 4, we know that (38) is violated, and hence such a $\lambda$ cannot be induced by travelers’ TIS subscription choices in SPE.

Our characterization of the set $\Lambda^\dagger$ can be used to assess the adoption rate of each TIS when travelers can choose from multiple available information systems. Indeed, for any
\( \lambda^i \in \Lambda^i \), the support set of \( \lambda^i \), denoted \( \mathcal{I}(\lambda^i) \triangleq \{ i \in \mathcal{I} | \lambda^i > 0 \} \), represents the set of TIS chosen by travelers in SPE, and the value \( \lambda^i \) is the equilibrium adoption rate of each TIS \( i \in \mathcal{I} \). Note that since \( \Lambda^i \) is not necessarily a singleton set, the adoption rate of each TIS in equilibrium is not unique. However, since \( \Lambda^i \) is a convex set, the equilibrium adoption rate of each TIS \( i \) is in a continuous range, denoted \([\lambda^i_{\min}, \lambda^i_{\max}]\), where \( \lambda^i_{\min} = \min_{\lambda^i \in \Lambda^i} \lambda^i \) (resp. \( \lambda^i_{\max} = \max_{\lambda^i \in \Lambda^i} \lambda^i \)) is the minimum (resp. maximum) equilibrium adoption rate.

Furthermore, since the set \( \Lambda^i \) is determined by the heterogeneous information environment created by all TIS, the equilibrium adoption rate of each TIS \( i \) is not only determined by the accuracy of its signal (i.e. the distribution of the type \( t^i \) given state \( s \)), but is also related to the accuracy of other TIS, and the possible correlations between signals of different TIS.

Finally, Theorem 4 can be used to assess whether or not a set of TIS can induce a heterogeneous information environment. In particular, if \( |\mathcal{I}(\lambda^i)| = 1 \), then all travelers choose to subscribe to a single TIS even though multiple TIS are available. Thus, the resulting information environment is homogeneous; see Example B.2 where \( \lambda^1 = 1 \) and \( \lambda^2 = 0 \) is the only size vector chosen in equilibrium. However, if \( |\mathcal{I}(\lambda^i)| > 1 \), then more than one TIS are chosen, i.e., the heterogeneous information environment is sustained. This is the case in Example 4 where the set \( \Lambda^i \) is equal to \( \Lambda^i_2 \) and both TIS are chosen in equilibrium. Moreover, if TIS \( i \notin \mathcal{I}(\lambda^i) \) for any \( \lambda^i \in \Lambda^i \), then this TIS is redundant in that it is not chosen in equilibrium even if it is available to the travelers.

### 6.3 Equilibrium Inefficiency

Finally, we relate our work to the well-known result of Roughgarden [2012] on inefficiency due to selfish routing. We first define a benchmark to compare to the equilibrium average cost as defined in (39). Given any feasible strategy profile \( q \in \mathcal{Q}(\lambda) \), we denote the average cost of all travelers as \( C(q) \triangleq \frac{1}{D} \sum_{i \in \mathcal{I}} \sum_{t^i \in \mathcal{T}^i} \Pr(t^i) \sum_{r \in \mathcal{R}} \mathbb{E}[c_r(q) | t^i] q_i(r | t^i) \). For any feasible size vector \( \lambda \), we define \( C^{opt}(\lambda) \) as the minimum average cost over all feasible strategy profiles:

\[
C^{opt}(\lambda) \triangleq \min_{q \in \mathcal{Q}(\lambda)} C(q). \tag{39}
\]

Let \( q^{opt}(\lambda) \) denote a feasible strategy that attains the minimum average cost. It is clear that \( C^{opt}(\lambda) \) is a trivial lower bound of \( C^*(\lambda) \) for any size vector \( \lambda \).

The cost \( C^{opt}(\lambda) \) can be seen as the average cost minimized by an “informationally constrained” social planner who routes the demand of each population \( i \in \mathcal{I} \) according to \( q^{opt} \) based on the population’s private signal, but not knowing the network state or signals received by the other populations. Therefore, the benchmark \( C^{opt}(\lambda) \) is the minimum average cost in the same asymmetric information environment as the game \( \Gamma(\lambda) \), and the inefficiency caused by the selfish routing can be evaluated by the ratio \( C^*(\lambda)/C^{opt}(\lambda) \). By straightforward extension of the well-known result in Roughgarden and Tardos [2004] on the worst-case inefficiency in complete information congestion games, we obtain a tight upper bound of \( C^*(\lambda)/C^{opt}(\lambda) \) in our information environment for any size vector \( \lambda \), see Proposition C.1.
Note that one can consider socially optimal cost, denoted $C^{so}$, as an alternative benchmark to $C^{opt}(\lambda)$. As defined in (C.20), $C^{so}$ is the minimum average cost achieved by a fully informed social planner, i.e. the one who knows the true state and seeks to minimize the average cost over all nature states. Clearly, given any size vector $\lambda$, $C^{so} \leq C^{opt}(\lambda) \leq C^*(\lambda)$. This ratio $C^*(\lambda)/C^{so}$ represents the combined loss of efficiency due to selfish routing as well as information incompleteness about the network state. In general, the ratio is hard to analyze due to the complexity of heterogeneous information environment; see Roughgarden [2012] for conditions under which this ratio can be analyzed. However, since our focus is on evaluating the inefficiency due to selfish routing, we simply adopt $C^{opt}(\lambda)$ as our benchmark.

The next proposition provides a condition on the edge cost functions for which there is no inefficiency due to selfish routing, and consequently, one achieves a stable socially fair, and efficient outcome when $\lambda \in \Lambda^\dagger$.

**Proposition 6.** For any $s \in S$ and any $e \in E$, if the edge cost functions in game $\Gamma(\lambda)$ can be written as $c^s_e(w_e) = h^s_e(w_e) + b^s_e$, where $h^s_e$ is a homogeneous function with degree $k > 1$, and $\{b^s_e\}_{e \in E, s \in S}$ satisfies $\sum_{e \in E} b^s_e = b^s$, for any $s \in S$, and any $r \in R$, then $\frac{C^*(\lambda)}{C^{opt}(\lambda)} = 1$ for any size vector $\lambda$. Furthermore, the minimum of the equilibrium average cost $C^*(\lambda)$ is achieved in the set $\Lambda^\dagger$, i.e. $\Lambda^\dagger = \arg \min_\lambda C^*(\lambda)$.

The sufficient condition for this result restricts the edge cost functions to be comprised of a homogeneous function and a constant; in addition, the free flow travel times of all routes must be the same, although they can still be state-dependent. Under this condition, the average cost of all travelers $C(q)$ can be written as an affine transformation of the weighted potential function $\Phi(q)$. From Theorem 1 we know that any BWE $q^* \in Q^*(\lambda)$ minimizes the potential function $\Phi(q)$, and thus minimizes $C(q)$ as well. Hence, $\frac{C^*(\lambda)}{C^{opt}(\lambda)} = 1$ for any size vector $\lambda$. This result also implies that $C^*(\lambda)$ in (9) is a linear transformation of the value of the potential function in equilibrium, $\Psi(\lambda)$. Recalling that $\Lambda^\dagger = \arg \min_\lambda \Psi(\lambda)$ from Lemma C.2, we can conclude that the routing game attains the minimum of the equilibrium average cost for any $\lambda \in \Lambda^\dagger$.

**Example 2.** Fig. 5a illustrates how $C^*(\lambda)$ and $C^{opt}(\lambda)$ vary with $\lambda$ for the routing game considered in Example 1. In Fig. 5a. The size vector $\lambda$ that minimizes $C^*(\lambda)$ lies in regime $\Lambda^\dagger_1$. Fig. 5b presents the same plot for the game in which $c^1_1(f_1) = f_1 + 20$, $c^a_1(f_1) = 3f_1 + 20$; all other parameters remain unchanged. The sufficient condition in Proposition 6 is satisfied in this case, so there is no inefficiency due to selfish routing.
7 Concluding Remarks

In this article, we study the equilibrium route choices and costs in a heterogeneous information environment, in which each population receives a private signal from its traffic information system (TIS). Each population maintains a belief about the unknown network state and about the signals received by other traveler populations. We focus on analyzing the equilibrium structure under perturbations of population sizes, the relative value of information between any pair of populations, as well as the equilibrium adoption rates when travelers can choose their TIS subscription.

The main ideas behind our analysis approach are: (i) Identification of qualitatively distinct equilibrium regimes based on whether or not the impact of information is fully attained; (ii) Sensitivity analysis of the weighted potential function in equilibrium with respect to the population size vector; and (iii) Characterization of the set of size vectors such that stable social fairness is attained so that travelers have no incentive to change their TIS subscription. Our approach can be easily extended to games where the edge costs are non-decreasing (rather than strictly increasing) functions of the edge loads. In particular, such a game still admits a weighted potential function, although now the essential uniqueness only applies to the equilibrium edge cost vector, rather than the edge load vector. The qualitative properties of equilibrium structure, results about the relative ordering of population costs, and conditions for stable social fairness can be extended as well. However, the characterization of regime thresholds in this case is more complicated from a computational viewpoint due to the non-uniqueness of edge load vector.

A future research question of interest is to analyze how the travelers’ expected cost and TIS adoption rates change when one or more TIS providers make technological changes to their service (for example, improving accuracy levels), or when a new TIS is introduced. Addressing this problem would involve applying our results to evaluate the value of information for each traveler population as well as the adoption rates in a stable socially fair...
outcome under the new information environment, and comparing them with that of the current environment.

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A Proofs of statements in Section 4

Lemma A.1. Game $\Gamma(\lambda)$ is a weighted potential game with the potential function $\Phi$ given by (11), and the positive type-specific weight is $\gamma(t^i) = \Pr(t^i)$ for any $t^i \in T^i$ and any $i \in I$.

Proof of Lemma A.1. First note that $\Phi(q)$, as defined in (11), is a continuous and differentiable function of the strategy profile $q$. To show that $\Phi(q)$ is a weighted potential function of $\Gamma(\lambda)$, we write the first order derivative of $\Phi(q)$ with respect to $q^i_t(t^i)$:

$$\frac{\partial \Phi(q)}{\partial q^i_t(t^i)} = \sum_{s \in S} \sum_{t-i \in T-i} \pi(s, t^i, t^{-i}) \sum_{e \in r} c^s_e(w_e(t^i, t^{-i}))$$

Thus, $\Phi(q)$ satisfies (10) with $\gamma(t^i) = \Pr(t^i)$ for any $t^i \in T^i$ and any $i \in I$. □

Lemma A.2. $\tilde{\Phi}(w)$ is twice continuously differentiable and strictly convex in $w$.

Proof of Lemma A.2. Since each $c^s_e(w_e(t))$ is differentiable in $w_e(t)$, we know that $\tilde{\Phi}(w)$ is twice differentiable with respect to $w$. The first order partial derivative of $\tilde{\Phi}(w)$ with respect to $w_e(t)$ can be written as: $\frac{\partial \tilde{\Phi}(w)}{\partial w_e(t)} = \sum_{s \in S} \pi(s, t) c^s_e(w_e(t))$ for any $e \in E$, and any $t \in T$. Also, the second order derivative of $\tilde{\Phi}(w)$ can be written as follows:

$$\frac{\partial^2 \tilde{\Phi}(w)}{\partial w_e(t) \partial w_{e'}(t')} = \begin{cases} \sum_{s \in S} \pi(s, t) \frac{\partial c^s_e(w_e(t))}{\partial w_e(t)}, & \text{if } e = e' \text{ and } t = t', \\ 0, & \text{otherwise}, \end{cases} \forall e, e' \in E, \forall t, t' \in T.$$

Since for any $e \in E$ and any $s \in S$, $c^s_e(w_e(t))$ is increasing in $w_e(t)$, we have $\sum_{s \in S} \pi(s, t) \frac{\partial c^s_e(w_e(t))}{\partial w_e(t)} > 0$. Thus, the Hessian matrix of $\tilde{\Phi}(w)$ has positive elements on the diagonal and 0 in all other entries, i.e. it is positive definite. Therefore, $\tilde{\Phi}(w)$ is strictly convex in $w$. □

Proof of Theorem 4. We first show that any minimum of (OPT-Q) is a BWE. For any optimal solution $q$, there must exist $\mu$ and $\nu$ such that $(q, \mu, \nu)$ satisfies the following Karush-Kuhn-Tucker (KKT) conditions:

$$\frac{\partial L}{\partial q^i_t(t^i)} = \frac{\partial \Phi}{\partial q^i_t(t^i)} - \mu^{t^i} - \nu^{t^i}_r = 0, \quad \forall r \in R, \forall t^i \in T^i, \forall i \in I,$$  \hspace{1cm} (KKT.1)

$$\nu^{t^i}_r q^i_t(t^i) = 0, \quad \forall r \in R, \forall t^i \in T^i, \forall i \in I,$$  \hspace{1cm} (KKT.2)

$$\nu^{t^i}_r \geq 0, \quad \forall r \in R, \forall t^i \in T^i, \forall i \in I.$$  \hspace{1cm} (KKT.3)

Using (A.1) and (KKT.1), we have $\frac{\partial \Phi(q)}{\partial q^i_t(t^i)} = \Pr(t^i)E[c_r(q)|t^i] = \mu^{t^i} + \nu^{t^i}_r$ for any $r \in R$, and $t^i \in T^i$, $i \in I$. From (KKT.2), we see that for any $r \in R$, and $t^i \in T^i$, $i \in I$, if $q^i_t(t^i) > 0$, the corresponding Lagrange multiplier $\nu^{t^i}_r = 0$, and $\Pr(t^i)E[c_r(q)|t^i] = \mu^{t^i}$. 33
However, if \( q_r^i(t^i) = 0 \), then \( \Pr(t^i)\mathbb{E}[c_r(q)|t^i] = \mu_r^i + \nu_r^i \geq \mu_r^i \). Thus, for any \( r \in \mathcal{R} \), and \( t^i \in \mathcal{T}^i \), \( i \in \mathcal{I} \):
\[
q_r^i(t^i) > 0 \quad \Rightarrow \quad \Pr(t^i)\mathbb{E}[c_r(q)|t^i] = \mu_r^i \leq \mu_r^i + \nu_r^i \equiv \Pr(t^i)\mathbb{E}[c_r(q)|t^i], \; \forall r' \in \mathcal{R}.
\]

From (7), we conclude that an optimal solution of (OPT-Q) is a BWE.

Next, we show that any BWE \( q^* \) of the game \( \Gamma(\lambda) \) is an optimal solution of (OPT-Q).

Consider a pair of Lagrange multipliers \( \bar{\mu} \) (resp. \( \bar{\nu} \)) corresponding to the constraints (4a) (resp. (4b)), where \( \bar{\mu}^i = \min_{r \in \mathcal{R}} \Pr(t^i)\mathbb{E}[c_r(q^*)|t^i] \) and \( \bar{\nu}^i = \Pr(t^i)\mathbb{E}[c_r(q^*)|t^i] - \bar{\mu}^i \). We can easily check that (KKT.1) and (KKT.3) are satisfied by \( (q^*, \bar{\mu}, \bar{\nu}) \). Since \( q^* \) is a BWE, we know from (7) that for a route \( r \in \mathcal{R} \), and \( t^i \in \mathcal{T}^i \), \( i \in \mathcal{I} \), if \( q_r^i(t^i) > 0 \), then
\[
\mathbb{E}[c_r(q^*)|t^i] = \min_{r \in \mathcal{R}} \mathbb{E}[c_r(q^*)|t^i] \text{ and consequently } \bar{\nu}^i = \Pr(t^i)\mathbb{E}[c_r(q^*)|t^i] - \bar{\mu}^i = 0.
\]
This implies that (KKT.2) is also satisfied by \( (q^*, \bar{\mu}, \bar{\nu}) \). Noting that \( \Phi(q) \equiv \tilde{\Phi}(w) \), where the induced edge load \( w \) is linear in \( q \) (see (5)), and that \( \tilde{\Phi}(w) \) is strictly convex in \( w \) (Lemma A.2), we conclude that \( \Phi(q) \) is a convex function of \( q \). Furthermore, since \( Q(\lambda) \) is a convex polytope, (OPT-Q) is a convex problem. Thus, KKT conditions are also sufficient for optimality, and any BWE \( q^* \) is an optimal solution of (OPT-Q).

Finally, for any \( \lambda \), we can use equations (5) and (13) to re-express (OPT-Q) as an optimization problem whose solution gives an equilibrium edge load \( w^*(\lambda) \):
\[
\begin{align*}
\min_{w, q} \quad & \tilde{\Phi}(w) \\
\text{s.t.} \quad & w_e(t) = \sum_{r \in \mathcal{R}} \sum_{i \in \mathcal{I}} q_r^i(t^i), \quad \forall t \in \mathcal{T}, \quad \forall e \in \mathcal{E}, \\
& q \in Q(\lambda).
\end{align*}
\]
Clearly, the feasible set of the above problem is a convex polytope. From Lemma A.2, \( \tilde{\Phi}(w) \) is strictly convex in \( w \). Therefore, the equilibrium edge load \( w^*(\lambda) \) is unique. \( \square \)

**Lemma A.3.** (Theorem 2 in Wachsmuth [2013]) The Lagrange multipliers \( \mu^* \) and \( \nu^* \) associated with any \( q^* \in Q^*(\lambda) \) at the optimum of (OPT-Q) are unique if and only if the LICQ condition is satisfied in that the gradients of the set of tight constraints in (2a)–(2b) at the optimum are linearly independent.

**Lemma A.4.** The Lagrange multipliers \( \mu^* \) and \( \nu^* \) at the optimum of (OPT-Q) are unique, and can be written in (15a)–(15b).

**Proof of Lemma A.4** Let the set of constraints that are tight at optimum of (OPT-Q) in (2b) be denoted as \( \mathcal{B} \). Assume for the sake of contradiction that LICQ does not hold, i.e. the set of equality constraints (2a) and the elements in the set \( \mathcal{B} \) are linearly dependent. Now, note that the constraint sets (2a) and (2b) are each comprised of linearly independent affine functions. Hence, there must exist a type \( \bar{t} \) such that the gradient of the corresponding equality constraint (i.e. \( \sum_{r \in \mathcal{R}} q_r^{i*}(\bar{t}) = \lambda^D \)) is linearly dependent with the elements in the set \( \mathcal{B} \), which implies that \( q_r^{i*}(\bar{t}) = 0, \forall r \in \mathcal{R} \). However, this violates the equality constraint in (2a) as \( \sum_{r \in \mathcal{R}} q_r^{i*}(\bar{t}) = \lambda^D \neq 0 \), and we arrive at a contradiction.
Since LICQ holds, for any equilibrium strategy profile \( q^* \in Q^*(\lambda) \), the corresponding \( \mu^* \) and \( \nu^* \) must be unique. Following the proof of Theorem 1, we conclude that for any \( q^* \in Q^*(\lambda) \), \( (q^*, \mu^*, \nu^*) \) satisfies the KKT conditions, and \( \mu^{r*} \) and \( \nu_r^{r*} \) can be written as (15a) and (15b), respectively.

Finally, noting that the equilibrium edge load is unique (Theorem 1), \( \mu^* \) and \( \nu^* \) in (15a)-(15b) are thus unique in equilibrium. \( \square \)

**Proof of Proposition 1.** We proceed in three steps: In Step I, we show that any feasible route flow satisfies constraints (17a)-(17d). In Step II, we show that for any \( f \in \mathcal{F}(\lambda) \) (i.e. \( f \) satisfies constraints (17a)-(17d)), the set of feasible strategy profiles \( q \) that induce it can be expressed as (22). Finally, in Step III, we prove that such set of \( q \) in (22) is non-empty. Step II and Step III together show that any \( f \in \mathcal{F}(\lambda) \) can be induced by at least one feasible strategy, and thus is a feasible route flow.

**Step I:** We show that any \( q \in Q(\lambda) \) induces a route flow \( f \) that satisfies (17a)-(17d). From (4), we obtain that \( f \) satisfies (17a):

\[
\sum_{t \in T} f_r(t, t^{-i}) = q^i_r(t^i) + \sum_{j \in I \setminus \{i\}} q^j_r(t^i) - q^i_r(t^\hat{t}) - \sum_{j \in I \setminus \{i\}} q^j_r(t^\hat{t})
\]

\[
= q^i_r(t^i) + \sum_{j \in I \setminus \{i\}} q^j_r(t^\hat{t}) - q^i_r(t^\hat{t}) - \sum_{j \in I \setminus \{i\}} q^j_r(t^\hat{t})
\]

\[
= f_r(t^i, t^{-i}) - f_r(t^\hat{t}, t^{-i}), \quad \forall t^i, t^\hat{t} \in T^i, \quad \forall t^{-i}, t^\hat{t} \in T^{-i}, \text{ and } \forall i \in I.
\]

From (2a) and (2b), we can directly conclude that \( f \) must also satisfy (17b) and (17c). Additionally, we can write:

\[
D - \sum_{r \in R} \min_{t \in T} f_r(t^i, t^{-i}) = D - \sum_{r \in R} \left( \sum_{j \in I \setminus \{i\}} q^j_r(t^i) - \sum_{t \in T} \min_{t \in T^i} q^i_r(t^i) \right)
\]

\[
\leq D - \sum_{j \in I \setminus \{i\}} \lambda^j D - \sum_{t \in T^i} \min_{t \in T^i} q^i_r(t^i)
\]

\[
\leq \lambda^i D - \sum_{t \in T^i} \min_{t \in T^i} q^i_r(t^i) \leq \lambda^i D, \quad \forall t^{-i} \in T^{-i}, \quad \forall i \in I.
\]

Therefore, \( f \) satisfies (17d). Thus, any feasible route flow must satisfy (17a)-(17d).

**Step II:** Next, we show that for any route flow \( f \in \mathcal{F}(\lambda) \) (i.e. \( f \) that satisfies constraints (17a)-(17d)), the set of feasible strategies that induce \( f \) can be characterized by (22). For any route \( r \in R \), the linear system of equations (4) has \( \prod_{i \in I} |T^i| \) equations in \( \sum_{i \in I} |T^i| \) variables. Note that for any given \( \hat{t} = (t^i)_{i \in I} \in T \), the following equations are linearly independent:

\[
\sum_{i \in I} q^i_r(t^i) = f_r(\hat{t}),
\]

\[
q^i_r(t^i) + \sum_{j \in I \setminus \{i\}} q^j_r(t^i) = f_r(t^i, t^{-i}), \quad \forall t^i \in T^i \setminus \{t^i\}, \quad \forall i \in I.
\]

(A.3)
We then show that given any \( t \in \mathcal{T} \), \( \sum_{i \in \mathcal{I}} q^i_r(t^i) = f_r(t) \) is a linear combination of the equations in (A.3). Following (17a), we can write:

\[
\sum_{i \in \mathcal{I}} f_r(t^i, \hat{t}^{-i}) - (|\mathcal{I}| - 1) f_r(\hat{t}) = f_r(t^1, \hat{t}^{-1}) + f_r(t^2, \hat{t}^{-2}) + \sum_{i=3}^{|\mathcal{I}|} f_r(t^i, \hat{t}^{-i}) - (|\mathcal{I}| - 1) f_r(\hat{t})
\]

\[
= f_r(t^1, t^2, \hat{t}^{-1-2}) + \sum_{i=3}^{|\mathcal{I}|} f_r(t^i, \hat{t}^{-i}) - (|\mathcal{I}| - 2) f_r(\hat{t}),
\]

where \( \hat{t}^{-1-2} \) is the vector \((\hat{t}^3, \ldots, \hat{t}^{|\mathcal{I}|})\). We apply the same procedure iteratively for another \(|\mathcal{I}| - 2 \) times, and obtain:

\[
\sum_{i \in \mathcal{I}} f_r(t^i, \hat{t}^{-i}) - (|\mathcal{I}| - 1) f_r(\hat{t}) = f_r(t), \quad \forall t \in \mathcal{T}.
\]

(A.4)

Now for any \( r \in \mathcal{R} \) and \( t \in \mathcal{T} \), we can write:

\[
\sum_{i \in \mathcal{I}} q^i_r(t^i) = \sum_{i \in \mathcal{I}} \left( q^i_r(t^i) + \sum_{j \in \mathcal{I} \setminus \{i\}} q^j_r(t^j) \right) - (|\mathcal{I}| - 1) \sum_{i \in \mathcal{I}} q^i_r(t^i)
\]

\[
= \sum_{i \in \mathcal{I}} f_r(t^i, \hat{t}^{-i}) - (|\mathcal{I}| - 1) f_r(\hat{t}).
\]

Thus, for any \( r \in \mathcal{R} \), the linear system (4) is comprised of \( \sum_{i \in \mathcal{I}} |\mathcal{T}^i| \) variables, and any constraint can indeed be expressed as a linear combination of \( \sum_{i \in \mathcal{I}} |\mathcal{T}^i| - |\mathcal{I}| + 1 \) independent equations in (A.3). From the rank-nullity theorem, we conclude that the dimension of null space of this linear map is \(|\mathcal{I}| - 1\). Then, for any \( r \in \mathcal{R} \), any \( i \in \mathcal{I} \), setting \( q^i_r(t^i) = \chi^i_r \), any solution of (4) can be expressed as (22), where \( \hat{t} \in \mathcal{T} \) is an arbitrary type profile. Additionally, \( \sum_{i \in \mathcal{I}} \chi^i_r = \sum_{i \in \mathcal{I}} q^i_r(\hat{t}) = f_r(\hat{t}) \). Thus, \( \chi \) satisfies (23b), i.e. for each \( r \in \mathcal{R} \), out of the \( |\mathcal{I}| \) variables in \( \{\chi^i_r\}_{i \in \mathcal{I}} \), \(|\mathcal{I}| - 1 \) are free, and the remaining one is obtained from (23b). We can conclude that the strategy profile \( q \) as defined in (22) induces the route flow \( f \). It remains to be shown that if \( q \) is a feasible strategy profile, \( \chi \) must satisfy (23a) and (23c) as well. Since \( q \) satisfies (2a), we obtain that \( \chi \) satisfies (23a):

\[
\chi^i_r \stackrel{(2a)}{=} \sum_{r \in \mathcal{R}} q^i_r(t^i) \stackrel{(22)}{=} \sum_{r \in \mathcal{R}} \left( f_r(t^i, \hat{t}^{-i}) - f_r(\hat{t}^i, \hat{t}^{-i}) + \chi^i_r \right) \stackrel{(17a)}{=} \sum_{r \in \mathcal{R}} \chi^i_r, \quad \forall i \in \mathcal{I}.
\]

Additionally, from (2b), \( 0 \leq q^i_r(t^i) \stackrel{(22)}{=} f_r(t^i, \hat{t}^{-i}) - f_r(\hat{t}^i, \hat{t}^{-i}) + \chi^i_r \) for any \( r \in \mathcal{R} \) and any \( t^i \in \mathcal{T}^i \). Thus, \( \chi^i_r \geq \max_{t^i \in \mathcal{T}^i} \{ f_r(\hat{t}^i, \hat{t}^{-i}) - f_r(t^i, \hat{t}^{-i}) \} \), i.e. \( \chi \) satisfies (23c).
Step III: Finally, we show that the set of $\chi$ satisfying (23) is non-empty, i.e., any $f \in \mathcal{F}(\lambda)$ can be induced by at least one feasible strategy profile $q$. Consider any $f \in \mathcal{F}(\lambda)$, we explicitly construct the following $\chi$, and show that such $\chi$ satisfies (23):

$$\chi^i_r = \gamma_r \cdot \left( \lambda^i D - \sum_{r \in \mathcal{R}} \max_{\hat{t} \in T^i} (f_r(\hat{t}) - f_r(t^i, \hat{t}^{-i})) \right) + \max_{\hat{t} \in T^i} (f_r(\hat{t}) - f_r(t^i, \hat{t}^{-i})), \ \forall r \in \mathcal{R}, \ \forall i \in \mathcal{I},$$

\[(A.5)\]

where $\hat{t}$ is any arbitrary type profile, and

$$\gamma_r = \begin{cases} \frac{f_r(\hat{t}) - \sum_{i \in \mathcal{T}} \max_{\hat{t} \in T^i} (f_r(\hat{t}) - f_r(t^i, \hat{t}^{-i}))}{\sum_{r \in \mathcal{R}} [f_r(\hat{t}) - \sum_{i \in \mathcal{T}} \max_{\hat{t} \in T^i} (f_r(\hat{t}) - f_r(t^i, \hat{t}^{-i}))]} & \text{if } \sum_{r \in \mathcal{R}} [f_r(\hat{t}) - \sum_{i \in \mathcal{T}} \max_{\hat{t} \in T^i} (f_r(\hat{t}) - f_r(t^i, \hat{t}^{-i}))] \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

First, we check that the $(\chi^i_r)_{r \in \mathcal{R}, i \in \mathcal{I}}$ as defined in (A.5) satisfies (23c). Note that $\gamma_r \geq 0$. To see this, since for any $t \in T$, $\sum_{i \in \mathcal{T}} f_r(t^i, \hat{t}^{-i}) - (|\mathcal{T}| - 1)f_r(\hat{t}) \geq 0$, we know that $\min_{t \in T} \sum_{i \in \mathcal{T}} f_r(t^i, \hat{t}^{-i}) - (|\mathcal{T}| - 1)f_r(\hat{t}) = \min_{t \in T} f_r(t) \geq 0$. Thus, for any $r \in \mathcal{R}$, we obtain:

$$f_r(\hat{t}) - \sum_{i \in \mathcal{T}} \max_{\hat{t} \in T^i} (f_r(\hat{t}) - f_r(t^i, \hat{t}^{-i})) = (1 - |\mathcal{T}|) f_r(\hat{t}) + \sum_{i \in \mathcal{T}} \min_{t \in T^i} f_r(t^i, \hat{t}^{-i})$$

$$= \min_{t \in T} \sum_{i \in \mathcal{T}} f_r(t^i, \hat{t}^{-i}) - (|\mathcal{T}| - 1)f_r(\hat{t}) = \min_{t \in T} f_r(t) \geq 0. \quad (A.6)$$

Hence, we can conclude that $\gamma_r \geq 0$.

Next, we can write:

$$\lambda^i D - \sum_{r \in \mathcal{R}} \max_{\hat{t} \in T^i} (f_r(\hat{t}) - f_r(t^i, \hat{t}^{-i})) \geq 0 \quad (A.7d)$$

Using the above inequality, we obtain that $\chi^i_r$ as considered in (A.5) satisfies (23c).

Second, we check that $\chi^i_r$ satisfies (23a). If $\sum_{r \in \mathcal{R}} [f_r(\hat{t}) - \sum_{i \in \mathcal{T}} \max_{\hat{t} \in T^i} (f_r(\hat{t}) - f_r(t^i, \hat{t}^{-i}))] > 0$, we have:

$$\sum_{r \in \mathcal{R}} \gamma_r \cdot \left( \lambda^i D - \sum_{r \in \mathcal{R}} \max_{\hat{t} \in T^i} (f_r(\hat{t}) - f_r(t^i, \hat{t}^{-i})) \right) = \left( \lambda^i D - \sum_{r \in \mathcal{R}} \max_{\hat{t} \in T^i} (f_r(\hat{t}) - f_r(t^i, \hat{t}^{-i})) \right) \geq 0 \quad (A.7d)$$

On the other hand, if $\sum_{r \in \mathcal{R}} [f_r(\hat{t}) - \sum_{i \in \mathcal{T}} \max_{\hat{t} \in T^i} (f_r(\hat{t}) - f_r(t^i, \hat{t}^{-i}))] = 0$, we obtain that:

$$0 = \sum_{r \in \mathcal{R}} \left[ f_r(\hat{t}) - \sum_{i \in \mathcal{T}} \max_{\hat{t} \in T^i} (f_r(\hat{t}) - f_r(t^i, \hat{t}^{-i})) \right] \geq D - \sum_{i \in \mathcal{T}} \left( \sum_{r \in \mathcal{R}} \max_{\hat{t} \in T^i} (f_r(\hat{t}) - f_r(t^i, \hat{t}^{-i})) \right) \quad (A.7d)$$

$$\geq D - \sum_{i \in \mathcal{T}} \lambda^i D = 0,$$
which implies that for any \( i \in I \), \( \sum_{r \in R} \max_{t^i \in T^i} \left( f_r(\hat{t}) - f_r(t^i, \hat{t}^{-i}) \right) = \lambda^i D \). Since in this case, \( \gamma_r = 0 \), we can conclude that \( \sum_{r \in R} \chi^i_r = \sum_{r \in R} \max_{t^i \in T^i} \left( f_r(\hat{t}) - f_r(t^i, \hat{t}^{-i}) \right) = \lambda^i D \), i.e. \( \chi \) satisfies (23a).

Finally, \( \chi^i_r \) also satisfies (23b). If \( \sum_{r \in R} \left[ f_r(\hat{t}) - \sum_{i \in I} \max_{t^i \in T^i} \left( f_r(\hat{t}) - f_r(t^i, \hat{t}^{-i}) \right) \right] > 0 \), we have:

\[
\sum_{i \in I} \chi^i_r = \gamma_r \cdot \sum_{i \in I} \left( \lambda^i D - \sum_{r \in R} \max_{t^i \in T^i} \left( f_r(\hat{t}) - f_r(t^i, \hat{t}^{-i}) \right) \right) + \sum_{i \in I} \max_{t^i \in T^i} \left( f_r(\hat{t}) - f_r(t^i, \hat{t}^{-i}) \right)
\]

(23a)

\[
\gamma_r \cdot \left( D - \sum_{i \in I} \max_{t^i \in T^i} \left( f_r(\hat{t}) - f_r(t^i, \hat{t}^{-i}) \right) \right) + \sum_{i \in I} \max_{t^i \in T^i} \left( f_r(\hat{t}) - f_r(t^i, \hat{t}^{-i}) \right)
\]

(17b)

\[
f_r(\hat{t}) - \sum_{i \in I} \max_{t^i \in T^i} \left( f_r(\hat{t}) - f_r(t^i, \hat{t}^{-i}) \right) + \sum_{i \in I} \max_{t^i \in T^i} \left( f_r(\hat{t}) - f_r(t^i, \hat{t}^{-i}) \right) = f_r(\hat{t})
\]

If \( \sum_{r \in R} \left[ f_r(\hat{t}) - \sum_{i \in I} \max_{t^i \in T^i} \left( f_r(\hat{t}) - f_r(t^i, \hat{t}^{-i}) \right) \right] = 0 \), then we have:

\[
0 = \sum_{r \in R} \left[ f_r(\hat{t}) - \sum_{i \in I} \max_{t^i \in T^i} \left( f_r(\hat{t}) - f_r(t^i, \hat{t}^{-i}) \right) \right] = \min_{t \in T} f_r(t) \geq 0,
\]

which implies that for any \( r \in R \), \( \min_{t \in T} f_r(t) = 0 \). In this case, \( \gamma_r = 0 \), and thus \( \sum_{i \in I} \chi^i_r = \sum_{i \in I} \max_{t^i \in T^i} \left( f_r(\hat{t}) - f_r(t^i, \hat{t}^{-i}) \right) \) satisfies (23b), i.e. the set of \( \chi \) satisfying (23) is non-empty. We already showed in Step II that \( q \) as defined in (22), with parameter \( \chi \) satisfying (23) is a feasible strategy profile, and \( q \) induces \( f \). Therefore, if \( f \) satisfies (17a)-(17d), there exists a feasible \( q \) that induces \( f \), i.e. any \( f \in \mathcal{F}(\lambda) \) is a feasible route flow.

In summary, we have shown that any feasible route flow satisfies (17) (Step I); For any \( f \) that satisfies (17), the set of feasible strategy profiles that induce \( f \) can be written in (22)-(23) (Step II); Such set is non-empty, and hence \( f \) is feasible (Step III). We can thus conclude that the set of feasible route flows is \( \mathcal{F}(\lambda) \), and the set of feasible strategy profiles, which induce \( f \) is characterized by (22)-(23).

**Proof of Proposition 2.** From Proposition 1, we know that the set of feasible route flows is the set \( \mathcal{F}(\lambda) \) characterized by (17a)-(17d). Additionally, the weighted potential function in (11) can be equivalently written as a function of \( f \) in (12). Therefore, the minimum of \( (\text{OPT-J}) \) is equal to that in \( (\text{OPT-Q}) \), and the set of optimal solutions is the set of equilibrium route flows.

**B Proof of statements in Section 5**

**Lemma B.1.** The route flows \( f^{i,j} \in \mathcal{F}^{i,j} \) induce a unique edge load \( w^{i,j} \).
Proof of Lemma B.1: Following (5) and (13), any edge load \( w_{ij}^{\dagger} \) induced by route flows in \( F_{ij}^{\dagger} \) (which we defined as optimal solution set of \( \text{OPT-F}_{ij}^{\dagger} \)) is an optimal solution of the following optimization problem:

\[
\begin{align*}
\min_w & \quad \Phi(w), \\
\text{s.t.} & \quad w_e(t) = \sum_{r \in \mathcal{R}} f_r(t), \quad \forall t \in \mathcal{T}, \quad \forall e \in \mathcal{E}, \\
& \quad f \text{ satisfies } (17a), (17b), (17c), (\text{IIC}) \{i, j\}, (\text{IIC}_{ij}).
\end{align*}
\]

The constraints (17a), (17b), (17c) are linear constraints. Following from (21), constraints \( \text{IIC} \{i, j\}, (\text{IIC}_{ij}) \) are each equivalent to a set of linear constraints. Additionally, \( w \) is a linear function of \( f \), thus the feasible set of \( w \) in this optimization problem must also be a convex polytope. From Lemma A.2, \( \Phi(w) \) is a strictly convex function in \( w \). Therefore, the optimal solution \( w_{ij}^{\dagger} \) is unique. \( \square \)

Lemma B.2. \( 0 \leq \lambda^i \leq \bar{\lambda}^i \leq 1 - |\lambda^{ij}| \).

Proof of Lemma B.2: First, we show that both thresholds \( \lambda^i \) and \( \bar{\lambda}^i \) belong to the interval \([0, 1 - |\lambda^{ij}|] \). Since \( \lambda^i \) is attainable on the set \( F_{ij}^{\dagger} \), there exists \( f_{ij}^{\dagger} \in F_{ij}^{\dagger} \) such that:

\[
\lambda^i = \frac{1}{D} \tilde{J}^i(f_{ij}^{\dagger}) \geq \frac{1}{D} \left( D - \sum_{r \in \mathcal{R}} \min_{t' \in \mathcal{T}_i} \tilde{f}_{ij}^{\dagger}(t', \hat{t} - ^i) \right) \geq 0.
\]

Similarly, we can check that \( \bar{\lambda}^i \leq 1 - |\lambda^{ij}| \).

Additionally, we know for any \( f_{ij}^{\dagger} \in F_{ij}^{\dagger} \):

\[
\bar{\lambda}^i \geq \frac{1}{D} \left\{ (1 - |\lambda^{ij}|) D - \tilde{J}^i(f_{ij}^{\dagger}) \right\} \geq \lambda^i.
\]

Therefore, \( 0 \leq \lambda^i \leq \bar{\lambda}^i \leq 1 - |\lambda^{ij}| \). \( \square \)

For any two populations \( i, j \in \mathcal{I} \), we can re-write (26a) for computing the threshold \( \lambda^i \) as follows:

\[
\begin{align*}
\min_y & \quad y \\
\text{s.t.} & \quad D - \sum_{r \in \mathcal{R}} f_r(t'^i, \hat{t} - ^i) \leq y \cdot D, \quad \forall t'^i_1 \in \mathcal{T}^i, \ldots, \forall t'^i_{|\mathcal{R}|} \in \mathcal{T}^i, \\
& \quad f_{ij}^{\dagger} \in F_{ij}^{\dagger},
\end{align*}
\]

where \( F_{ij}^{\dagger} \) is the polytope defined in (25). Therefore, (B.7) is a linear programming. Analogously, the threshold \( \bar{\lambda}^i \) defined in (26b) is the optimal value of the following linear programming:

\[
\begin{align*}
\max_y & \quad y \\
\text{s.t.} & \quad -|\lambda^{ij}| D + \sum_{r \in \mathcal{R}} f_r(t'^j, \hat{t} - ^j) \geq y \cdot D, \quad \forall t'^j_1 \in \mathcal{T}^j, \ldots, \forall t'^j_{|\mathcal{R}|} \in \mathcal{T}^j, \\
& \quad f_{ij}^{\dagger} \in F_{ij}^{\dagger}.
\end{align*}
\]
Proof of Theorem 2 [Regime $\Lambda_{ij}^1$]: First, we show by contradiction that the constraint (IIC) is tight for any equilibrium route flow. Assume that for a given $\lambda \in \Lambda_{ij}^1$, there exists an equilibrium route flow $f^*$ such that (IIC) is not tight. From Proposition 2, we know that $f^*$ is an optimal solution of (OPT-$\mathcal{F}$). Since (OPT-$\mathcal{F}$) is a convex optimization problem, $f^*$ is still a minimizer of $\hat{\Phi}(f)$ if we drop the constraint (IIC). Additionally, the constraints (IIC) and (IIC) implies that $f^*$ must also satisfy (IIC). Thus, such $f^*$ is an optimal solution of the following problem:

$$\min_f \hat{\Phi}(f)$$

s.t. $(17a)$, $(17b)$, (IIC), and $(\text{IIC})\setminus\{i\}$.

Moreover, the threshold $\tilde{\lambda}^i$ defined in (26b) is attained by a route flow, say $\tilde{f}^{ij,\dagger}$, in the set $\mathcal{F}^{ij,\dagger}$. Thus, we can write:

$$1 - |\lambda - \tilde{\lambda}^i| - \frac{1}{D} \tilde{J}^i(\tilde{f}^{ij,\dagger}) = \tilde{\lambda}^i \geq \frac{\lambda^i}{2} \geq \lambda_i.$$

Rearranging, we obtain: $\frac{1}{D} \tilde{J}^i(\tilde{f}^{ij,\dagger}) < 1 - |\lambda - \tilde{\lambda}^i| - \lambda^i = \lambda^i$, and so such $\tilde{f}^{ij,\dagger}$ also satisfies (IIC). Since $\tilde{f}^{ij,\dagger}$ is an optimal solution of (25), which minimizes the same objective function as (B.9) but without the constraint (IIC), we thus know that $\tilde{f}^{ij,\dagger}$ is also an optimal solution in (B.9). Since the induced edge load is unique, we can conclude that the edge load induced by $\tilde{f}^{ij,\dagger}$ must be identical to that induced by $\tilde{f}^{ij,\dagger}$, which is $w^{ij,\dagger}$. Then, from (25), we have $f^* \in \mathcal{F}^{ij,\dagger}$. Therefore, from (26a), we can write $\lambda^i \leq \frac{1}{D} \tilde{J}^i(f^*)$. Since we assumed that (IIC) is not binding in equilibrium, we obtain: $\frac{1}{D} \tilde{J}^i(f^*) < \lambda^i \leq \frac{1}{D} \tilde{J}^i(f^*)$, which is a contradiction. Thus, (IIC) must be tight in equilibrium for any $\lambda$ in regime $\Lambda_{ij}^1$.

Finally, following the tightness of (IIC) at optimum of (OPT-$\mathcal{F}$), by rearranging the constraint (IIC) in (28), we have:

$$\tilde{J}^i(f^*) \leq (1 - |\lambda - \tilde{\lambda}^i|) D - \tilde{J}^i(f^*) = \lambda^i D.$$  

Thus, (IIC) is guaranteed to hold in Regime $\Lambda_{ij}^1$ given the constraint (IIC) and the fact that (IIC) is tight at the optimum of (OPT-$\mathcal{F}$). Hence, (IIC) can be dropped in (OPT-$\mathcal{F}$) without changing the optimal solution set.

[Regime $\Lambda_{ij}^2$]: Analogous to the proof given for regime $\Lambda_{ij}^1$, we can argue that constraint (IIC) is tight in any equilibrium for any $\lambda$ in regime $\Lambda_{ij}^2$. By imposing constraint (IIC), (IIC) can be dropped from the constraint set in (OPT-$\mathcal{F}$) without changing the optimal solution set.

[Regime $\Lambda_{ij}^3$]: To study this regime, we need two additional thresholds $\underline{\lambda}^i$ and $\bar{\lambda}^i$ which we define as follows:

$$\underline{\lambda}^i \triangleq \frac{1}{D} \max_{f^{ij,\dagger} \in \mathcal{F}^{ij,\dagger}} \left\{ \tilde{J}^i(f^{ij,\dagger}) \right\},$$  \hspace{1cm} (B.11a)

$$\bar{\lambda}^i \triangleq \frac{1}{D} \min_{f^{ij,\dagger} \in \mathcal{F}^{ij,\dagger}} \left\{ (1 - |\lambda - \tilde{\lambda}^i|) D - \tilde{J}^i(f^{ij,\dagger}) \right\}.$$  \hspace{1cm} (B.11b)
From (26a) and (26b), we can check that $\Delta^i \leq \bar{\Delta}^i$, and $\tilde{\lambda}^i \leq \bar{\lambda}^i$.

For any $\lambda^i \in [\bar{\Delta}^i, \bar{\Delta}^i]$, we argue that $\mathcal{F}^*(\lambda) \subseteq \mathcal{F}^{f,j}_i$. Since the set $\mathcal{F}^{f,j}_i$ as defined by (25) is a bounded polytope, and $\Delta^i$ (resp. $\bar{\Delta}^i$) is the minimum (resp. maximum) value of the continuous function $\hat{f}^j_i(t^j)$ on $\mathcal{F}^{f,j}_i$, we know from the mean value theorem that there exists a $\tilde{f}^{ij}_i \in \mathcal{F}^{f,j}_i$ satisfying $\lambda^i = \frac{1}{B} \hat{f}^j_i(\tilde{f}^{ij}_i)$. Similar to the argument in (B.10), such $\tilde{f}^{ij}_i$ also satisfies constraint $\{\text{III}\}$. Therefore, $\tilde{f}^{ij}_i$ satisfies all the constraints in (17), and minimizes $\hat{f}(f)$. So $\tilde{f}^{ij}_i$ is an equilibrium route flow, which implies that $\mathcal{F}^*(\lambda) \cap \mathcal{F}^{f,j}_i \neq \emptyset$. Since the equilibrium edge load vector is unique, and the edge load induced by $\tilde{f}^{ij}_i$ satisfies the constraint $\{\text{IIC}\}$, we can argue that for any $\lambda^i \in [\bar{\Delta}^i, \bar{\Delta}^i]$, $\mathcal{F}^*(\lambda) \subseteq \mathcal{F}^{f,j}_i$.

To prove that $\mathcal{F}^*(\lambda) \subseteq \mathcal{F}^{f,j}_i$ for any $\lambda$ in regime $\Lambda^i_j$, we need to argue two cases $\bar{\lambda}^i \geq \tilde{\lambda}^i$ and $\bar{\lambda}^i < \tilde{\lambda}^i$ separately:

Case 1: $\lambda^i \geq \tilde{\lambda}^i$. Since $[\bar{\lambda}^i, \tilde{\lambda}^i] \subseteq [\bar{\lambda}^i, \bar{\lambda}^i] \cup [\tilde{\lambda}^i, \bar{\lambda}^i]$, thus $\mathcal{F}^*(\lambda) \subseteq \mathcal{F}^{f,j}_i$ for any $\lambda$ in regime $\Lambda^i_j$.

Case 2. $\lambda^i < \tilde{\lambda}^i$: Consider any $\lambda^i \in (\bar{\lambda}^i, \tilde{\lambda}^i)$, we can check that any $f^{ij}_i \in \mathcal{F}^{f,j}_i$ satisfies the constraint $\{\text{III}\}$: $\frac{1}{B} \hat{f}^j_i(f^{ij}_i) \leq \lambda^i < \lambda^i$. Additionally, since $\lambda^i < \tilde{\lambda}^i \leq 1 - |\lambda^{-i}| - \frac{1}{B} \hat{f}^j_i(f^{ij}_i)$, we know that $\frac{1}{B} \hat{f}^j_i(f^{ij}_i) < 1 - |\lambda^{-i}| - \lambda^i = \bar{\lambda}^i$, i.e. $f^{ij}_i$ also satisfies the constraint $\{\text{IIIC}\}$. Therefore, any $f^{ij}_i \in \mathcal{F}^{f,j}_i$ is an equilibrium route flow, i.e. $\mathcal{F}^*(\lambda) = \mathcal{F}^{f,j}_i$ for any $\lambda^i \in (\bar{\lambda}^i, \tilde{\lambda}^i)$. Combined with the fact that $\mathcal{F}^*(\lambda) \subseteq \mathcal{F}^{f,j}_i$ for any $\lambda \in [\bar{\lambda}^i, \bar{\lambda}^i] \cup [\tilde{\lambda}^i, \bar{\lambda}^i]$, we can conclude that $\mathcal{F}^*(\lambda) \subseteq \mathcal{F}^{f,j}_i$ for any $\lambda$ in regime $\Lambda^i_j$.

**Corollary B.1.** If the game $\Gamma(\lambda)$ has a parallel-route network, then the equilibrium route flow $f^*$ is unique. Moreover, if there are two populations, then the equilibrium strategy profile is unique in regime $\Lambda^i_j$ or $\Lambda_2^i$, and can be written as follows:

In regime $\Lambda_1^i$:

$$q^{1*}_r(t^1) = f^{1*}_r(t^1_l, t^2), \quad \forall r \in R, \quad \forall t^1 \in T^1,$$

$$q^{2*}_r(t^2) = \min_{\hat{t}^1 \in T^1} f^{1*}_r(\hat{t}^1, t^2), \quad \forall r \in R, \quad \forall t^2 \in T^2,$$  

In regime $\Lambda_3^i$:

$$q^{1*}_r(t^1) = \min_{\bar{t}^1 \in T^1} f^{1*}_r(t^1, \bar{t}^2), \quad \forall r \in R, \quad \forall t^1 \in T^1,$$

$$q^{2*}_r(t^2) = f^{2*}_r(t^1, \bar{t}^2) = \min_{\hat{t}^2 \in T^2} f^{2*}_r(\hat{t}^1, \hat{t}^2), \quad \forall r \in R, \quad \forall t^2 \in T^2,$$

where $(\hat{t}^1, \hat{t}^2)$ is any type profile.

**Proof of Corollary B.1.** Given a parallel route network, we immediately obtain the uniqueness of $f^*$ from Theorem 1. Then from Proposition 1 any strategy profile that can induce $f^*$ can be expressed as in (22). In regime $\Lambda_2^i$, we know from Theorem 2 that the constraint $\{\text{IIIC}\}$ is tight in equilibrium. Therefore, from (23a) and (23c), we obtain:

$$\lambda^i D \sum_{r \in R} \lambda^i \geq \sum_{r \in R} \max_{t^1 \in T^1} \left( f^{1*}_r(\hat{t}^1, \hat{t}^2) - f^{1*}_r(t^1, \hat{t}^2) \right)$$

where $\hat{f}(f) = \lambda^i D$. 

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Thus, $(23c)$ is tight for any $r \in R$, i.e. $\chi^i_1 = \max_{t^i \in T^i} \left( f^i_r(\widehat{t}^i, \widehat{t}^j) - f^i_r(t^i, \widehat{t}^j) \right)$. Additionally, from $(23a)$, $\chi^j_2 = \min_{t^j \in T^j} f^j_r(t^i, \widehat{t}^j)$. Thus, $\chi$ as defined in $(23)$ is unique. Following $(22)$, we can obtain the unique $q^*$ as defined in $(B.12a)-(B.12b)$. Analogously, we can argue that the equilibrium strategy profile is also unique in regime $\Lambda^2_3$, and is written as in $(B.12c)-(B.12d)$.

Proof of Proposition 3. [Regime $\Lambda^{ij}_1$]: Consider any population size vector $\lambda \in \Lambda^{ij}_1$, there exists a sufficiently small $\epsilon > 0$ such that $\lambda' = \lambda + \epsilon z^{ij} \in \Lambda^{ij}_1$, i.e. $\lambda' = \lambda + \epsilon > \lambda$, $\lambda'' = \lambda - \epsilon < \lambda$, and the sizes of all other populations remain unchanged. Consider any equilibrium route flow $f^*(\lambda) \in F^*(\lambda)$ and any $f^*(\lambda') \in F^*(\lambda')$. We know from Theorem 2 that constraint (IIC) is tight in equilibrium, and thus $f^*(\lambda)$ and $f^*(\lambda')$ satisfy: $\frac{1}{\partial} \hat{J}^i(f^*(\lambda)) = \lambda'' < \lambda' = \frac{1}{\partial} \hat{J}^i(f^*(\lambda'))$. Consequently, any equilibrium route flow $f^*(\lambda)$ for size vector $\lambda$ is in the feasible domain of $(28)$ for size vector $\lambda'$, but $f^*(\lambda) \notin F^*(\lambda')$, because $\hat{J}^i(f^*(\lambda)) = \lambda'' < \lambda'$, i.e. the constraint (IIC) is satisfied, but not tight. Since $f^*(\lambda') \notin F^*(\lambda')$, we must have $\Psi(\lambda') = \hat{\Phi}(f^*(\lambda')) < \Phi(f^*(\lambda)) = \Psi(\lambda)$.

Additionally, from $(24)$, we know that $\Psi(\lambda') = \hat{\Phi}(w^*(\lambda')) < \hat{\Phi}(w^*(\lambda)) = \Psi(\lambda)$. Thus, the unique equilibrium edge load $w^*(\lambda)$ necessarily changes in the direction $z^{ij}$ in regime $\Lambda^{ij}_1$.

[Regime $\Lambda^{ij}_2$]: From Theorem 2, $F^*(\lambda) \subseteq F^{ij}_{i,j}$ for any $\lambda \in \Lambda^{ij}_2$. Since the equilibrium edge load is unique, we know $w^*(\lambda) = w^{ij}_{i,j}$. From $(24)$, we can conclude that $\Psi(\lambda) = \hat{\Phi}(w^{ij}_{i,j})$. Thus, $\Psi(\lambda)$ as well as $w^*(\lambda)$ remain fixed in regime $\Lambda^{ij}_2$.

[Regime $\Lambda^{ij}_3$]: Following similar argument in regime $\Lambda^{ij}_1$, we conclude that $\Psi(\lambda)$ monotonically increases in the direction $z^{ij}$ in regime $\Lambda^{ij}_3$. As a result, $w^*(\lambda)$ changes when $\lambda$ is perturbed in the direction $z^{ij}$ in regime $\Lambda^{ij}_3$. $\square$

Lemma B.3. (Corollary 2.2 in Fiacco and Kyparisis [1986], page 102) The value of the potential function in equilibrium, $\Psi(\lambda)$, is convex with respect to $\lambda$ if in $(\text{OPT-Q})$, $\Phi(q)$ is convex in $q$, and the constraints are affine in $q$ and $\lambda$.

Lemma B.4. (Proposition 6 in Fiacco [2009], page 3469) If in $(\text{OPT-Q})$, the objective function $\Phi(q)$ is convex and continuously differentiable in $q$ and $\lambda$, and additionally the set of BWE $q^*$ and the set of Lagrange multipliers $\mu^*$, $\nu^*$ are nonempty and bounded, then $\Psi(\lambda)$ is continuous and directionally differentiable in $\lambda$. Furthermore, for any given $i,j \in I$, the directional derivative of $\Psi(\lambda)$ in the direction $z^{ij}$ can be expressed as follows:

$$\nabla_{z^{ij}} \Psi(\lambda) = \min_{q^* \in Q^*(\lambda)} \max_{(\mu^*,\nu^*)} \nabla_\lambda L(q^*, \mu^*, \nu^*, \lambda) z^{ij}, \tag{B.13}$$

where $M(q^*)$ and $N(q^*)$ are the sets of Lagrange multipliers $\mu^*$ and $\nu^*$ in $(14)$ associated with the BWE $q^* \in Q^*(\lambda)$.

Lemma B.5. The value of the weighted potential function in equilibrium, $\Psi(\lambda)$ as defined in $(24)$, is convex and directionally differentiable in $\lambda$. For any $i,j \in I$, the derivative of $\Psi(\lambda)$ in the direction $z^{ij}$, $\nabla_{z^{ij}} \Psi(\lambda) = -D \cdot V^{ij}(\lambda)$. 

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Proof of Lemma B.3 Since in (OPT-Q), the weighted potential function \( \Phi(q) \) is convex in \( q \), and the constraints (2a)-2b) are affine in \( q \) and \( \lambda \), from Lemma B.3 we know that the optimal value of the potential function \( \Psi(\lambda) \) is convex in \( \lambda \).

We next check that the conditions in Lemma B.4 are satisfied in (OPT-Q): (1) The potential function \( \Phi(q) \) is continuously differentiable in \( q \), and constraints (2a)-2b) are linear in \( q \) and \( \lambda \); (2) The optimal solution set \( Q^*(\lambda) \) is non-empty and bounded (Theorem 1). The Lagrange multipliers at the optimum of (OPT-Q) are unique, and bounded (Lemma A.4). Therefore, from Lemma B.4, we know that \( \Psi(\lambda) \) is differentiable in the direction \( z^{ij} \), and \( \nabla_{z^{ij}} \Psi(\lambda) \) can be expressed as:

\[
\nabla_{z^{ij}} \Psi(\lambda) = \left( \sum_{t^i \in T^i} \mu^{*t^i} - \sum_{t^j \in T^j} \mu^{*t^j} \right) D
\]

\[
= \left( \sum_{t^i \in T^i} \min_{t^i \in T^i} \Pr(t^i)E[c_r(q^*)|t^i] - \sum_{t^j \in T^j} \min_{t^j \in T^j} \Pr(t^j)E[c_r(q^*)|t^j] \right) D
\]

\[
= (C^{i*}(\lambda) - C^{j*}(\lambda)) D - V^{ij*}(\lambda) \cdot D. \tag{B.14}
\]

Proof of Theorem 3. We prove (i) and (ii) in sequence:

(i): We know from Proposition 3 that in direction \( z^{ij} \), \( \Psi(\lambda) \) decreases in regime \( \Lambda_1^{ij} \), does not change in regime \( \Lambda_2^{ij} \) and increases in regime \( \Lambda_3^{ij} \). Following Lemma B.5, we directly obtain that \( V^{ij*}(\lambda) > 0 \) in regime \( \Lambda_1^{ij} \), \( V^{ij*}(\lambda) = 0 \) in regime \( \Lambda_2^{ij} \), and \( V^{ij*}(\lambda) < 0 \) in regime \( \Lambda_3^{ij} \).

(ii): We have shown in Lemma B.5 that \( \Psi(\lambda) \) is convex in \( \lambda \). Hence, for any \( i, j \in I \), the directional derivative \( \nabla_{z^{ij}} \Psi(\lambda) \) is non-decreasing in direction \( z^{ij} \). From (31), we can conclude that \( V^{ij*}(\lambda) \) is non-increasing in direction \( z^{ij} \).

Proof of Proposition 4. Since the interim belief of population \( j \), \( \beta^j(s, t^{-j}|b^j) \) in (33) is independent with type \( t^j \), the equilibrium strategy of the uninformed population \( q^{xy}(t^j) \) must be identical across all \( t^j \in T^j \). Following (18) and (19), the impact of information metric \( J^j(q^*) = \hat{J}^j(f^*) = 0 \) for any \( q^* \in Q^*(\lambda), f^* \in F^*(\lambda) \) and any \( \lambda \). For the sake of contradiction, we assume that the regime \( \Lambda_3^{ij} \) is non-empty. From Theorem 2, we know that the constraint (IIC) must be tight in equilibrium when \( \lambda \) is in regime \( \Lambda_3^{ij} \). However, since \( \hat{J}^j(f^*) = 0 \) for any \( \lambda \), the constraint (IIC) is tight only when \( \lambda^j = 0 \), i.e. \( \lambda^j = 1 - |\lambda^{-ij}| \). This implies that the regime \( \Lambda_3^{ij} \) is indeed empty. Thus, there are
at most two regimes $\Lambda_1^ij$ and $\Lambda_2^ij$. Following Proposition 3, we can conclude that $C^i*$ and $C^j*$ satisfy (34).

**Example B.1.** We consider the game with two populations on two parallel routes ($r_1$ and $r_2$) with the following cost functions: $c^1_s(f_1) = f_1 + 15$, $c^2_s(f_1) = 3f_1 + 15$, $c_2(f_2) = 20f_2 + 30$. The prior distribution $\theta$, the total demand $D$, and the information environment are the same as that in Example 1. Although both populations receive the accurate signal of the state with positive probability, we have $\bar{\lambda}^1 = 1$ as the impact of information on population 2 is zero. Since the free flow travel time on $r_2$ is much higher than that on $r_1$, population 2 travelers exclusively uses $r_1$ regardless of the received signal, see Fig. 6.

![Equilibrium population costs](image)

**Example B.2.** Consider a game with two populations on two parallel routes ($r_1$ and $r_2$). There are two states: $\{s_1, s_2\}$, each state is realized with probability 0.5. The cost functions are: $c^1_s(f_1) = f_1 + 10$, $c^2_s(f_1) = f_1 + 1$, $c^1_s(f_2) = f_2 + 1$, $c^2_s(f_2) = f_2 + 10$. Population 1 is completely informed, and population 2 is uninformed. The total demand is $D = 1$. We now obtain that $\lambda^1 = \bar{\lambda}^1 = 1$; thus, regimes $\Lambda_2^{12}$ and $\Lambda_3^{12}$ are empty sets, and population 1 has strictly lower expected cost than population 2 for any $\lambda^1 \in (0, 1)$, see Fig. 7.

**Example B.3.** We consider the game with two populations on two parallel routes ($r_1$ and $r_2$) with the same cost functions, prior distribution $\theta$ and total demand $D$ as that in Example 1. Both populations 1 and 2 have 0.75 chance of getting accurate information about the state, i.e. $\Pr(t^i = s|s) = 0.75$ for any $i \in I$ and any $s \in S$. In Fig. 8, we illustrate the equilibrium population costs in two cases: (i) Types $t^1$ and $t^2$ are perfectly correlated, i.e. $t^1 = t^2$; (ii) Types $t^1$ and $t^2$ are independent conditional on the state, i.e. $\Pr(t^1, t^2|s) = \Pr(t^1|s) \cdot \Pr(t^2|s)$. 

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Note that case (i) can be viewed as a single-population game. This is because when \( t_1 \) and \( t_2 \) are perfectly correlated, there is no information asymmetry among travelers. Thus, \( \lambda_1 \) has no impact on the equilibrium outcome, and \( \lambda_1 = 0, \hat{\lambda}_1 = 1 \).

However, case (ii) is not equivalent to a single-population game. Although both populations have identical chance of getting accurate information about the state, there is information heterogeneity among travelers of the two populations, i.e. travelers in one population do not know the signals received by travelers in the other population, and thus the equilibrium outcome changes with the size \( \lambda_1 \).
C Proof of Statements in Section 6

Lemma C.1. The set $\Lambda^\dagger$ is a non-empty convex polytope.

Proof of Lemma C.1 We first show that $\Lambda^\dagger$ is a non-empty set. For any $f^\dagger \in F^\dagger$, we consider a size vector $\lambda$ defined as follows:

$$\lambda^i = \frac{1}{D} \hat{J}(f^\dagger) + \frac{1}{|\mathcal{I}|} \left(1 - \sum_{k \in \mathcal{I}} \frac{1}{D} \hat{k}(f^\dagger)\right), \quad \forall i \in \mathcal{I}. \quad (C.15)$$

It is easy to check that $\lambda$ so defined satisfies $(1a)$. Additionally, we can check that:

$$\sum_{k \in \mathcal{I}} \frac{1}{D} \hat{k}(f^\dagger) \overset{[19]}{=} \frac{1}{D} \sum_{k \in \mathcal{I}} \left(D - \sum_{r \in \mathcal{R}} \min_{t^k \in \mathcal{T}^k} f^\dagger_r(t^k, \hat{\tau}^k)\right) \overset{[17b]}{=} \frac{1}{D} \sum_{r \in \mathcal{R}} \sum_{k \in \mathcal{I}} \left(f^\dagger_r(\hat{\tau}^k) - \min_{t^k \in \mathcal{T}^k} f^\dagger_r(t^k, \hat{\tau}^k)\right) \overset{[A.4]}{=} \frac{1}{D} \sum_{r \in \mathcal{R}} \max_{t \in \mathcal{T}} (f^\dagger_r(t) - f^\dagger_r(\hat{\tau}^k)) \overset{[17b]}{\leq} \frac{1}{D} \sum_{r \in \mathcal{R}} f^\dagger_r(\hat{\tau}^k) \overset{[17b]}{=} 1.$$

Thus, $\lambda$ in $(C.15)$ satisfies $(\text{IIC})$ for any $i \in \mathcal{I}$. From $(19)$, it is also clear that $\hat{J}(f^\dagger) \geq 0$ for any $i \in \mathcal{I}$. Thus $\lambda^i$ satisfies $(1b)$. Hence, $\lambda$ that we constructed satisfies the conditions in $(37)$. We can conclude that the set $\Lambda^\dagger$ is non-empty.

We next prove that the set $\Lambda^\dagger$ is a convex polytope. In fact, the definition in $(37)$ can be equivalently written as follows:

$$\Lambda^\dagger = \text{Proj}_\lambda \left\{(\lambda, f) \left| \hat{J}(f) \leq \lambda^i D, \quad \forall i \in \mathcal{I}, \quad f \in F^\dagger, \quad \lambda \text{ satisfies } (1a) - (1b) \right. \right\}, \quad (C.16)$$

where the operator $\text{Proj}_\lambda(\cdot)$ gives the orthogonal projection of the vector $(\lambda, f)$ onto the sub-space of $\lambda$. To see this we can check that any feasible $\lambda$ belongs to $\Lambda^\dagger$ (as defined in $(37)$) if and only if we can find a route flow $f^\dagger \in F^\dagger$ such that the pair $(\lambda, f^\dagger)$ satisfies the constraints in $(C.16)$.

We know that the set $F^\dagger$ is a convex polytope, and constraints $(1a)$-$(1b)$ are linear constraints. Thus, the set of $(\lambda, f)$ satisfying constraints in $(C.16)$ is a convex polytope, and the orthogonal projection image $\Lambda^\dagger$ is also a convex polytope. \qed

Lemma C.2. For any $\lambda \in \Lambda^\dagger$, the set of equilibrium route flows $F^\ast(\lambda) \subseteq F^\dagger$. Furthermore, the set $\Lambda^\dagger$ attains the minimum of $\Psi(\lambda)$, i.e. $\Lambda^\dagger = \arg \min_\lambda \Psi(\lambda)$.

Proof of Lemma C.2 First, we prove that for any $\lambda \in \Lambda^\dagger$, $F^\ast(\lambda) \subseteq F^\dagger$. From the definition of $\Lambda^\dagger$ in $(37)$, we know that for any $\lambda \in \Lambda^\dagger$, there exists at least one route flow $f^\dagger \in F^\dagger$ satisfying the constraints in $(\text{OPT-}\mathcal{F})$, and hence such $f^\dagger$ is a feasible solution of the optimization problem $(\text{OPT-}\mathcal{F})$; thus $\Phi(f^\dagger) \geq \Psi(\lambda)$. Additionally, since $f^\dagger$ is an optimal solution of $(35)$, which has the same objective function as $(\text{OPT-}\mathcal{F})$ but without
the constraints (IIC), we conclude that $\Phi(f^\dagger) \leq \Psi(\lambda)$ for any feasible $\lambda$ (including $\lambda \in \Lambda^\dagger$). Thus, $\Psi(\lambda) = \Phi(f^\dagger)$, and $f^\dagger$ is an equilibrium route flow. Analogous to the argument in proof of Theorem 1, the equilibrium edge load equals to $w^\dagger$. Since the set $\mathcal{F}^\dagger$ in (36) contains all route flows such that the induced edge load is $w^\dagger$, we can conclude that the set of equilibrium route flow $\mathcal{F}^*(\lambda) \subseteq \mathcal{F}^\dagger$ for any $\lambda \in \Lambda^\dagger$.

Now, we prove that $\Lambda^\dagger = \arg \min_{\lambda} \Psi(\lambda)$. We have argued in the first part of the proof that for any $f^\dagger \in \mathcal{F}^\dagger$, $\hat{\Phi}(f^\dagger) \leq \Psi(\lambda)$ for any feasible $\lambda$; and since for any $\lambda \in \Lambda^\dagger$, $\mathcal{F}^*(\lambda) \subseteq \mathcal{F}^\dagger$, we have $\Psi(\lambda) = \hat{\Phi}(f^\dagger)$. Therefore, $\hat{\Phi}(f^\dagger) = \min_{\lambda} \Psi(\lambda)$, and $\Lambda^\dagger \subseteq \arg \min_{\lambda} \Psi(\lambda)$. Additionally, for any $\lambda \in \arg \min_{\lambda} \Psi(\lambda)$, we have $\Psi(\lambda) = \min_{\lambda} \Psi(\lambda) = \hat{\Phi}(f^\dagger)$. Since $\mathcal{F}^\dagger$ includes all route flows that satisfies (17a)-(17c) and attains the minimum value of $\Psi(\lambda)$, any equilibrium route flow $f^* \in \mathcal{F}^*(\lambda)$ for $\lambda \in \arg \min_{\lambda} \Psi(\lambda)$ must be in $\mathcal{F}^\dagger$. Hence, such $\lambda$ must also be in $\Lambda^\dagger$ defined in (37), i.e. $\arg \min_{\lambda} \Psi(\lambda) \subseteq \Lambda^\dagger$. We can therefore conclude that $\Lambda^\dagger = \arg \min_{\lambda} \Psi(\lambda)$. \hfill $\Box$

**Proof of Proposition 5.** Following Lemma (C.2), for any $\lambda \in \Lambda^\dagger$, $\mathcal{F}^*(\lambda) \subseteq \mathcal{F}^\dagger$. Therefore, the unique equilibrium edge load is $w^\dagger$, which does not depend on $\lambda$. Additionally, for any feasible $\lambda \notin \Lambda^\dagger$, again from Lemma C.2, $\Psi(\lambda) > \hat{\Phi}(w^\dagger)$. Thus, $w^*(\lambda) \neq w^\dagger$. \hfill $\Box$

**Proof of Theorem 4.** Firstly, we show for any $\lambda \in \Lambda^\dagger$, all travelers have identical costs in equilibrium. From Lemma C.2, we know that for any $\lambda \in \Lambda^\dagger$, the value of the weighted potential function $\Psi(\lambda) = \Phi(w^\dagger) = \min_{\lambda} \Psi(\lambda)$. Consider any $i, j \in \mathcal{I}$ such that $\lambda^i > 0$ and $\lambda^j > 0$, the directional derivative of $\Psi(\lambda)$ in the direction $z^{ij}, \nabla_{z^{ij}} \Psi(\lambda)$, must be 0. Otherwise, $\Psi(\lambda)$ strictly decreases in the direction $z^{ij}$ (resp. $z^{ji}$) if $\nabla_{z^{ij}} \Psi(\lambda) < 0$ (resp. if $\nabla_{z^{ij}} \Psi(\lambda) > 0$), which contradicts the fact that $\Lambda^\dagger = \arg \min_{\lambda} \Psi(\lambda)$. From (31), we know that $C^{ii^*}(\lambda) = C^{jj^*}(\lambda)$. Therefore, any two populations with positive size have identical costs in equilibrium. We can thus conclude that the equilibrium costs faced by all travelers are identical.

If any $\lambda \in \Lambda^\dagger$ satisfies $\lambda^i > 0$ for all $i \in \mathcal{I}$, then the first step in our proof is sufficient to show that the set $\Lambda^\dagger$ attains stable social fairness. Otherwise, for any $\lambda \in \Lambda^\dagger$, and any degenerate population $i \in \{\mathcal{I}|\lambda^i = 0\}$, we need to show that $C^{ii^*}(\lambda) \geq C^{jj^*}(\lambda)$, where $\lambda^j > 0$. Since $\Lambda^\dagger = \arg \min_{\lambda} \Psi(\lambda)$ (Lemma C.2), we know that $\Psi(\lambda)$ must be non-decreasing in the direction $z^{ij}$. Thus, we obtain: $\nabla_{z^{ij}} \Psi(\lambda) \geq \{C^{ii^*}(\lambda) - C^{jj^*}(\lambda)\} D \geq 0$. Thus, $C^{ii^*}(\lambda) \geq C^{jj^*}(\lambda)$. The first and the second steps together show that any $\lambda \in \Lambda^\dagger$ satisfies (38), which implies that stable social fairness is attained in the set $\Lambda^\dagger$.

Finally, we show that any feasible $\lambda \notin \Lambda^\dagger$ cannot achieve stable social fairness. Since $\Lambda^\dagger = \arg \min_{\lambda} \Psi(\lambda)$ (Lemma C.2), for any $\lambda \notin \Lambda^\dagger$, we can claim that there must exist a direction $z^{ij}$ such that $\Psi(\lambda)$ decreases in the direction $z^{ij}$, $\nabla_{z^{ij}} \Psi(\lambda) < 0$. Otherwise, $\lambda$ is a local minimum of $\Psi(\lambda)$, and since $\Psi(\lambda)$ is convex in $\lambda$ (Lemma B.5), $\lambda$ is a global minimum, which contradicts the fact that $\lambda \notin \Lambda^\dagger$. For such a direction $z^{ij}$, there are two possible cases: (1) $\lambda^i > 0$ and $\lambda^j > 0$. In this case, from (31), $C^{ii^*}(\lambda) \neq C^{jj^*}(\lambda)$, and thus travelers do not have identical costs in equilibrium. (2) $\lambda^i = 0$ and $\lambda^j > 0$. In this case, from (31), we must have $\nabla_{z^{ij}} \Psi(\lambda) = \{C^{ii^*}(\lambda) - C^{jj^*}(\lambda)\} D < 0$. Therefore, $C^{jj^*}(\lambda) > C^{ii^*}(\lambda)$, which implies that travelers in population $j$ has incentive to change subscription to TIS $i$. To sum up, in either case, $\lambda$ cannot attain stable social fairness. \hfill $\Box$
Following Roughgarden and Tardos [2004], we define the Pigou bound $\kappa(\mathcal{C})$ as:

$$\kappa(\mathcal{C}) = \sup_{c \in \mathcal{C}} \sup_{x \geq 0, y \geq 0} \frac{y \cdot c(y)}{x \cdot c(x) + (y - x) \cdot c(y)},$$  \hspace{1cm} (C.17)

where $\mathcal{C}$ is the set of edge cost functions $\{c_e(w_e)\}_{e \in \mathcal{E}, s \in \mathcal{S}}$. We find that the inefficiency in equilibrium is upper bounded by $\kappa(\mathcal{C})$.

**Proposition C.1.** For any given $\lambda$, we have $\frac{C^*(\lambda)}{C_{\text{opt}}(\lambda)} \leq \kappa(\mathcal{C})$, and the bound is tight. If the edge cost functions $c_e^*(w_e)$ are affine functions for all $e \in \mathcal{E}, s \in \mathcal{S}$, we have $\frac{C^*(\lambda)}{C_{\text{opt}}(\lambda)} \leq \frac{4}{3}$.

**Proof of Proposition C.1.** From the definition of BWE in [7], we obtain:

$$\sum_{r \in \mathcal{R}} q_r^{i*}(t^i) \mathbb{E}[c_r(q^*)|t^i] = \lambda \Delta \cdot \min_{r \in \mathcal{R}} \mathbb{E}[c_r(q^*)|t^i]$$

$$\leq \sum_{r \in \mathcal{R}} q_r^{opti}(t^i) \mathbb{E}[c_r(q^*)|t^i], \quad \forall t^i \in \mathcal{T}, \forall i \in \mathcal{I}. \hspace{1cm} (C.18)$$

Define $f^{opt}$ as the route flow vector induced by an optimal average cost strategy $q^{opt}$, and $w^{opt}$ as the induced edge load vector. Following [C.18], we have:

$$\sum_{s \in \mathcal{S}} \sum_{e \in \mathcal{E}} \sum_{t \in \mathcal{T}} \pi(s, t, w_e^*(t)) c_e^*(w_e^*(t)) \leq \sum_{i \in \mathcal{I}} \sum_{t \in \mathcal{T}} \Pr(t^i) \left( \sum_{r \in \mathcal{R}} q_r^{opti}(t^i) \mathbb{E}[c_r(q^*)|t^i] \right)$$

$$\leq \sum_{i \in \mathcal{I}} \sum_{t \in \mathcal{T}} \Pr(t^i) \left( \sum_{r \in \mathcal{R}} q_r^{opti}(t^i) \mathbb{E}[c_r(q^*)|t^i] \right) \sum_{s \in \mathcal{S}} \sum_{e \in \mathcal{E}} \sum_{t \in \mathcal{T}} \pi(s, t, w_e^{opt}(t)) c_e^*(w_e^*(t)). \hspace{1cm} (C.19)$$

This leads to:

$$C^{opt}(\lambda) = \sum_{s \in \mathcal{S}} \sum_{e \in \mathcal{E}} \sum_{t \in \mathcal{T}} \pi(s, t, w_e^*(t)) w_e^{opt}(t)$$

$$= \sum_{s \in \mathcal{S}} \sum_{e \in \mathcal{E}} \sum_{t \in \mathcal{T}} \pi(s, t, w_e^*(t)) w_e^*(t) \cdot \left( w_e^{opt}(t) c_e^*(w_e^{opt}(t)) + (w_e^*(t) - w_e^{opt}(t)) c_e^*(w_e^*(t)) \right)$$

$$+ \sum_{s \in \mathcal{S}} \sum_{e \in \mathcal{E}} \sum_{t \in \mathcal{T}} \pi(s, t, w_e^{opt}(t) - w_e^*(t)) c_e^*(w_e^*(t))$$

$$\geq \left( \sum_{s \in \mathcal{S}} \sum_{e \in \mathcal{E}} \sum_{t \in \mathcal{T}} \pi(s, t, c_e^*(w_e^*(t))w_e^*(t)) \right) \cdot \left[ \min_{s \in \mathcal{S}, e \in \mathcal{E}, t \in \mathcal{T}} \frac{w_e^{opt}(t)c_e^*(w_e^{opt}(t)) + (w_e^*(t) - w_e^{opt}(t)) c_e^*(w_e^*(t))}{c_e^*(w_e^*(t)) w_e^*(t)} \right]$$

$$\geq \frac{C^*(\lambda)}{\kappa(\mathcal{C})}. \hspace{1cm} (C.19)$$

Thus, $\frac{C^*(\lambda)}{C_{\text{opt}}(\lambda)} \leq \kappa(\mathcal{C})$. Consider affine cost functions, for any $e \in \mathcal{E}$ and $s \in \mathcal{S}$, assuming $c_e^*(w_e) = \alpha_e^* w_e + \beta_e^*$, we have:

$$w_e^{opt}(t) (c_e^*(w_e^*(t)) - c_e^*(w_e^{opt}(t))) = \alpha_e^* w_e^{opt}(t) (w_e^*(t) - w_e^{opt}(t)) \leq \frac{1}{4} \alpha_e^* w_e^*(t)^2 \leq \frac{1}{4} w_e^*(t) c_e^*(w_e^*(t)).$$
Thus, we obtain:

\[
C^*(\lambda) = \sum_{s \in S} \sum_{e \in E} \sum_{t \in T} \pi(s, t) w^*_e(t) c^*_e(w^*_e(t)) \quad \overset{(C.10)}{\leq} \quad \sum_{s \in S} \sum_{e \in E} \sum_{t \in T} \pi(s, t) w^e(t) c^*_e(w^*_e(t)) \\
= \sum_{s \in S} \sum_{e \in E} \sum_{t \in T} \pi(s, t) w^e(t) c^*_e(w^*_e(t)) + \sum_{s \in S} \sum_{e \in E} \sum_{t \in T} \pi(s, t) w^e(t) (c^*_e(w^*_e(t)) - c^*_e(w^*_e(t))) \\
\leq \sum_{s \in S} \sum_{e \in E} \sum_{t \in T} \pi(s, t) w^e(t) c^*_e(w^*_e(t)) + \frac{1}{4} \sum_{s \in S} \sum_{e \in E} \sum_{t \in T} \pi(s, t) w^*_e(t) c^*_e(w^*_e(t)) \\
= C^{opt}(\lambda) + \frac{1}{4} C^*(\lambda).
\]

Therefore, \( \frac{C^*(\lambda)}{C^{opt}(\lambda)} \leq \frac{4}{3} \). Since the game with one population, one state is a special case of our game, and the bound \( \kappa(C) \) is tight, we can conclude that the bound is tight as well in \( \Gamma(\lambda) \).

The social optimal cost \( C^{so} \) is defined as follows:

\[
C^{so} \overset{\Delta}{=} \min_f \sum_{s \in S} \sum_{e \in E} c^*_e \left( \sum_{r \in R} f^s_r \right), \\
\text{s.t.} \quad \sum_{r \in R} f^s_r = D, \quad \forall s \in S, \quad (C.20) \\
f^s_r \geq 0, \quad \forall r \in R, \quad \forall s \in S,
\]

**Proof of Proposition 6.** We first show that \( \frac{C^*(\lambda)}{C^{opt}(\lambda)} = 1 \). For any feasible strategy profile \( q \in Q \), \( C(q) \) can be written as:

\[
C(q) = \frac{1}{D} \sum_{i \in I} \sum_{t^i \in T^i} \Pr(t^i) \sum_{r \in R} \mathbb{E}[c_r(q)|t^i]q^i_r(t^i) \overset{(C.11)}{=} \frac{1}{D} \sum_{s \in S} \sum_{e \in E} \sum_{t \in T} \pi(s, t) c^s_e(w^s_e(t)) \cdot w^s_e(t) \\
\overset{(C.12)}{=} \frac{1}{D} \sum_{s \in S} \sum_{e \in E} \sum_{t \in T} \pi(s, t) \int_0^{\sum_{r \in R} \sum_{i \in I} q^i_r(t^i)} \frac{d \left( c^s_e(x) \cdot x \right)}{dx} \, dx \\
= \frac{1}{D} \sum_{s \in S} \sum_{e \in E} \sum_{t \in T} \pi(s, t) \int_0^{\sum_{r \in R} \sum_{i \in I} q^i_r(t^i)} \left( c^s_e(x) + x \cdot \frac{dc^s_e(x)}{dx} \right) \, dx.
\]

Assume that the condition of the edge cost functions in Proposition 6 is satisfied. Since \( h^s_e(\cdot) \) is a k-th order homogeneous function of \( w_e \), we know from Euler’s homogeneous function theorem that \( x \cdot \frac{dh^s_e(x)}{dx} = k \cdot h^s_e(x) \). Therefore, \( c^s_e(x) + x \cdot \frac{dc^s_e(x)}{dx} = h^s_e(x) + b^s_e + x \cdot \)
\[
\frac{\partial h^*_e(x)}{\partial x} = (k + 1) \cdot h^*_e(x) + b^*_e. \]
The cost \( C(q) \) can be further expressed as:

\[
C(q) = \frac{1}{D} \sum_{e \in E} \sum_{t \in T} \sum_{s \in S} \pi(s, t) \int_0^{\sum_{r \in \mathcal{E}} \sum_{i \in \mathcal{I}} q^r(t)} (k + 1) \cdot h^*_e(x) dx + \frac{1}{D} \sum_{e \in E} \sum_{t \in T} \sum_{s \in S} \pi(s, t) \int_0^{\sum_{r \in \mathcal{E}} \sum_{i \in \mathcal{I}} q^r(t)} b^*_e dx
\]
\[
= \frac{k + 1}{D} \sum_{e \in E} \sum_{t \in T} \sum_{s \in S} \pi(s, t) \int_0^{\sum_{r \in \mathcal{E}} \sum_{i \in \mathcal{I}} q^r(t)} c^*_e(x) dx - \frac{k}{D} \sum_{e \in E} \sum_{t \in T} \sum_{s \in S} \pi(s, t) b^*_e w_e(t)
\]
Moreover, since \( \sum_{e \in r} b^*_e = b^* \) for any \( s \in S \), and any \( r \in \mathcal{R} \), we obtain:

\[
C(q) = \frac{(k + 1)}{D} \Phi(q) - \frac{k}{D} \sum_{s \in S} \sum_{t \in T} \pi(s, t) \cdot \left( \sum_{r \in \mathcal{R}} b^*_e \cdot f_r(t) \right)
\]
\[
\overset{(C.21)}{=} \frac{(k + 1)}{D} \Phi(q) - k \cdot \left( \sum_{s \in S} \left( \sum_{t \in T} \pi(s, t) \right) \cdot b^* \right) = \frac{(k + 1)}{D} \Phi(q) - k \cdot \left( \sum_{s \in S} \theta(s) \cdot b^* \right). \tag{C.21}
\]

Since the equilibrium strategy profile \( q^* \) minimizes \( \Phi(q) \), \( q^* \) also minimizes \( C(q) \). Thus, \( C^*(\lambda) = C^\text{opt}(\lambda) = 1 \).

Additionally, from (C.21), we obtain that:

\[
C^*(\lambda) = C^\text{opt}(\lambda) = \min_{q \in \mathcal{Q}(\lambda)} C(q) \overset{\text{C.21}}{=} \frac{(k + 1)}{D} \Psi(\lambda) - k \cdot \left( \sum_{s \in S} \theta(s) \cdot b^* \right).
\]

Therefore, \( C^*(\lambda) \) is an affine function of \( \Psi(\lambda) \). From Lemma C.2, we know that \( \Lambda^\dagger = \arg \min_{\lambda \in \Lambda} \Psi(\lambda) \). Consequently, \( \Lambda^\dagger = \arg \min_{\lambda \in \Lambda} C^*(\lambda) \). \qed