ON THE ESSENTIAL AND DISCRETE SPECTRUM OF A MODEL OPERATOR RELATED TO THREE-PARTICLE DISCRETE SCHröDINGER OPERATORS

SERGIO ALBEVERIO$^{1,2,3}$, SAIDAKHMAT N. LAKAEV$^{4,5}$, RAMIZA KH. DJUMANOVA$^{5}$

ABSTRACT. A model operator $H$ corresponding to a three-particle discrete Schrödinger operator on a lattice $\mathbb{Z}^3$ is studied. The essential spectrum is described via the spectrum of two Friedrichs models with parameters $h_\alpha(p)$, $\alpha = 1, 2$, $p \in T^3 = (-\pi, \pi]^3$. The following results are proven:

1) The operator $H$ has a finite number of eigenvalues lying below the bottom of the essential spectrum in any of the following cases: (i) both operators $h_\alpha(0)$, $\alpha = 1, 2$, have a zero eigenvalue; (ii) either $h_1(0)$ or $h_2(0)$ has a zero eigenvalue.

2) The operator $H$ has infinitely many eigenvalues lying below the bottom and accumulating at the bottom of the essential spectrum, if both operators $h_\alpha(0)$, $\alpha = 1, 2$, have a zero energy resonance.

Subject Classification: Primary: 81Q10, Secondary: 35P20, 47N50

Key words and phrases: Friedrichs model, eigenvalues, Efimov effect, Faddeev-Newton type integral equation, essential spectrum, Hilbert-Schmidt operators, infinitely many eigenvalues.

1. INTRODUCTION

One of the remarkable results in the spectral analysis for continuous and discrete three-particle Schrödinger operators is the Efimov effect: if in a system of three-particles, interacting by means of short-range pair potentials, none of the three two-particle subsystems has bound states with negative energy, but at least two of them have a resonance with zero energy, then this three-particle system has an infinite number of three-particle bound states with negative energy, accumulating at zero.

This effect was first discovered by Efimov [6]. Since then this problem has been studied in many works [1, 2, 4, 5, 11, 20, 23, 24, 25, 26]. A rigorous mathematical proof of the existence of Efimov’s effect was originally carried out by Yafaev in [26] and then in [20, 24, 23, 25].

In models of solid physics [7, 8, 18, 19, 21, 28] and also in lattice quantum field theory [17] discrete Schrödinger operators are considered, which are lattice analogues of the three-particle Schrödinger operator in a continuous space. The presence of Efimov’s effect for these operators was proved in [3, 12, 14, 16].

In [3] a system of three arbitrary quantum particles on the three-dimensional lattice $\mathbb{Z}^3$ interacting via zero-range pair attractive potentials has been considered.

Let us denote by $\tau_{\text{ess}}(K)$ the bottom of essential spectrum of the three-particle discrete Schrödinger operator $H(K)$, $K \in T^3$, and by $N(K, z)$ the number of eigenvalues below $z \leq \tau_{\text{ess}}(K)$.

Let us shortly recall the main results of [3]:

Date: November 20, 2018.
(i) For the number $N(0, z)$ the limit result
\[
\lim_{z \to 0} \frac{N(0, z)}{|\log |z||} = U_0, \ (0 < U_0 < \infty)
\]
holds.

(ii) For any $K \in U_0^0(0) = \{p \in \mathbb{T}^3 : 0 < |p| < \delta\}$ the number $N(K, \tau_{ess}(K))$ is finite and the following limit result
\[
\lim_{|K| \to 0} \frac{N(K, 0)}{|\log |K||} = 2U_0
\]
holds (see [3] for details).

In the present paper a model operator $H$ corresponding to the three-particle Schrödinger operator on the lattice $\mathbb{Z}^3$ acting in the Hilbert space $L_2((\mathbb{T}^3)^2)$ is considered. Here the role of the two-particle discrete Schrödinger operators is played by a family of Friedrichs models with parameters $h_{\alpha}(p), \ \alpha = 1, 2, \ p \in \mathbb{T}^3$.

We precisely describe the location and structure of the essential spectrum of $H$ via the spectrum of $h_{\alpha}(p), \ \alpha = 1, 2, \ p \in \mathbb{T}^3$.

Furthermore under some natural conditions on the family of the operators $h_{\alpha}(p), \ \alpha = 1, 2, \ p \in \mathbb{T}^3$, we obtain the following results:

(a) The operator $H$ has a finite number of eigenvalues lying below the bottom of the essential spectrum in the following two cases: (i) both operators $h_{\alpha}(0), \ \alpha = 1, 2$, have a zero eigenvalue; (ii) either $h_1(0)$ or $h_2(0)$ has a zero eigenvalue.

(b) The operator $H$ has infinitely many eigenvalues lying below the bottom and accumulating at the bottom of the essential spectrum, if the operators $h_{\alpha}(0), \ \alpha = 1, 2$, have a zero energy resonance.

Moreover for the number $N(z)$ of eigenvalues of $H$ lying below $z < 0$ the following limit exists
\[
\lim_{z \to 0} \frac{N(z)}{|\log |z||} = U_0 \ (0 < U_0 < \infty).
\]

We remark that the assertion (b) is similar to the case of the three-particle continuous and discrete Schrödinger operators and the assertion (a) is surprising and similar assertions has not been proved for the three-particle Schrödinger operators on $\mathbb{R}^3$ and $\mathbb{Z}^3$.

The plan of this paper is as follows:

Section 1 is an introduction to the whole work. In section 2 the model operator $H$ is introduced in the Hilbert space $L_2((\mathbb{T}^3)^2)$ and the main results of the present paper are formulated. In Section 3 we study some spectral properties of $h_{\alpha}(p), \ \alpha = 1, 2, \ p \in \mathbb{T}^3$.

In section 4 we obtain an analogue of the Faddeev-Newton type integral equation for the eigenfunctions of the model operator and precisely describe the location and the structure of the essential spectrum of $H$ (Theorem 2.9). In this section we prove an analogue of the Birman-Schwinger principle for $H$ and the part (i) of Theorem 2.9. In section 6 we prove the part (ii) of Theorem 2.9. Some technical material is collected in Appendices A, B.

Throughout the present paper we adopt the following conventions: For each $\delta > 0$ the notation $U_\delta(0) = \{p \in \mathbb{T}^3 : |p| < \delta\}$ stands for a $\delta$-neighborhood of the origin.

The subscript $\alpha$ (and also $\beta$) always is equal to 1 or 2 and $\alpha \neq \beta$ and $\mathbb{T}^3$ denotes the three-dimensional torus, the cube $(-\pi, \pi]^3$ with appropriately identified sides. Throughout the paper the torus $\mathbb{T}^3$ will always be considered as an abelian group with respect to the addition and multiplication by real numbers regarded as operations on $\mathbb{R}^3$ modulo $(2\pi \mathbb{Z})^3$. 


Remark 2.6. There are a function $\alpha$ such that the assertion holds.

Remark 2.3. Let Hypotheses 2.1 and 2.2 be fulfilled and $f = (f_1, f_2), f_\alpha \in L_2(\Omega), \alpha = 1, 2.$

Remark 2.5. We have given twice continuously differentiable function at the point $(0, 0)$ is finite and hence we can define continuous function on $H_0.$

Remark 2.4. Throughout this paper we assume the following additional hypothesis.

Hypothesis 2.1. The real-analytic function $u(p, q)$ on $(T^3)^2$ is even with respect to $(p, q),$ has a unique non-degenerate zero minimum at the point $(0, 0) \in (T^3)^2$ and there exists a positive definite matrix $U$ and real numbers $l, l_1, l_2 (l_1, l_2 > 0, l \neq 0)$ such that
\[
\left( \frac{\partial^2 u(0,0)}{\partial p^{(i)} \partial p^{(j)}} \right)_{i,j=1}^3 = l_1 U, \quad \left( \frac{\partial^2 u(0,0)}{\partial q^{(i)} \partial q^{(j)}} \right)_{i,j=1}^3 = l_2 U.
\]

Hypothesis 2.2. The real-analytic function $\varphi_\alpha(p), \alpha = 1, 2,$ is either even or odd on $T^3.

Set
\[u_p^{(1)}(q) = u(q, p), \quad u_p^{(2)}(q) = u(p, q).
\]

By Hypotheses 2.1 and 2.2 the integral
\[\int_{T^3} \frac{\varphi_\alpha^2(t) dt}{u_p^{(\alpha)}(t)}
\]
is finite and hence we can define continuous function on $T^3$, which will be denotes $\Lambda_\alpha(p)$.

Remark 2.3. Since the function $u(p, q)$ has a unique non degenerate minimum at the point $(0, 0) \in (T^3)^2$ the function $\Lambda_\alpha(p)$ is positive. In particular, if $\varphi_\alpha(0) = 0$ then $\Lambda_\alpha(p)$ is a twice continuously differentiable function at the point $p = 0$ (see proof of Lemma 2.7).

Hypothesis 2.4. (i) For any $p \in T^3$, $p \neq 0$ the function $\Lambda_\alpha(\cdot)$ satisfies $\Lambda_\alpha(p) < \Lambda_\alpha(0).$
(ii) If $\varphi_\alpha(0) = 0,$ then $\Lambda_\alpha(p)$ has a non-degenerate maximum at $p = 0.$

Remark 2.5. Let Hypotheses 2.1 and 2.2 be fulfilled and $\varphi_\alpha(0) \neq 0.$ Then it is easy to show that the assertion (i) of Hypothesis 2.4 is fulfilled, that is, the inequality $\Lambda_\alpha(p) < \Lambda_\alpha(0)$ holds for all sufficiently small nonzero $p \in T^3$ (see Corollary 2.7).

Remark 2.6. There are a function $u(p, q)$ and either even or odd functions $\varphi_\alpha(p), \alpha = 1, 2,$ so that Hypothesis 2.4 is fulfilled (see Appendix B).
Set
\[ \mu_{a}^{0} = \left( \int_{\mathbb{T}^{3}} \varphi_{a}(t)(u_{a}^{0}(t))^{-1} dt \right)^{-1} \quad \text{and} \quad M = \max_{\mu, q \in \mathbb{T}^{3}} u(p, q). \]

To study spectral properties of the operator \( H \) we introduce the following two families of bounded self-adjoint operators (Friedrichs model) \( \{ h_{\alpha}(p), \alpha = 1, 2, p \in \mathbb{T}^{3} \} \), acting in \( L_2(\mathbb{T}^{3}) \) by
\[ h_{\alpha}(p) = h_{\alpha}^{0}(p) - \mu_{a} v_{\alpha}, \]
where
\[ (h_{\alpha}^{0}(p)f)(q) = u_{p}^{(\alpha)}(q)f(q), \quad f \in L_2(\mathbb{T}^{3}), \]
\[ (v_{\alpha}f)(q) = \varphi_{a}(q) \int_{\mathbb{T}^{3}} \varphi_{a}(t)f(t)dt, \quad f \in L_2(\mathbb{T}^{3}). \]

For the definition of Friedrichs model and the study spectrum and resonances in this model see [9, 10, 13, 27].

Let \( \sigma_{d}(h_{\alpha}(p)) \) be the discrete spectrum of \( h_{\alpha}(p), p \in \mathbb{T}^{3} \), and
\[ a_{\alpha} = \inf \cup_{p \in \mathbb{T}^{3}} \sigma_{d}(h_{\alpha}(p)), \quad b_{\alpha} = \sup \cup_{p \in \mathbb{T}^{3}} \sigma_{d}(h_{\alpha}(p)), \quad \alpha = 1, 2. \]

The main results of the present paper are as follows:

**Theorem 2.7.** Assume Hypotheses 2.1 and 2.4 are fulfilled.
(i) Let \( \mu_{a} > \mu_{a}^{\text{max}}, \alpha = 1, 2 \), then
\[ \sigma_{\text{ess}}(H) = [a_{1}, b_{1}] \cup [a_{2}, b_{2}] \cup [0, M] \quad \text{and} \quad b_{\alpha} < 0, \alpha = 1, 2. \]
(ii) Let \( \mu_{a}^{\text{max}} \geq \mu_{a} > \mu_{a}^{0}, \alpha = 1, 2 \), then
\[ \sigma_{\text{ess}}(H) = [a, M] \quad \text{and} \quad a = \min\{a_{1}, a_{2}\} < 0. \]
(iii) Let \( \mu_{a}^{0} \geq \mu_{a} > 0, \alpha = 1, 2 \), then
\[ \sigma_{\text{ess}}(H) = [0, M]. \]

Let \( C(\mathbb{T}^{3}) \) be the Banach space of continuous functions on \( \mathbb{T}^{3} \).

**Definition 2.8.** Let Hypotheses 2.1 be fulfilled. The operator \( h_{\alpha}(0), \alpha = 1, 2 \), is said to have a zero energy resonance if the number 1 is an eigenvalue of the integral operator given by
\[ (G_{\alpha}\psi_{\alpha})(q) = \mu_{a} \varphi_{a}(q) \int_{\mathbb{T}^{3}} \frac{\varphi_{a}(t)\psi_{a}(t)dt}{u_{a}^{0}(t)}, \quad \psi_{\alpha} \in C(\mathbb{T}^{3}) \]
and \( \varphi_{a}(0) \neq 0 \).

**Theorem 2.9.** Assume Hypotheses 2.1, 2.2 and 2.4 are fulfilled and \( \mu_{a} = \mu_{a}^{0}, \alpha = 1, 2 \).
(i) Let either \( \varphi_{1}(0) = \varphi_{2}(0) = 0 \) or \( \varphi_{1}(0) = 0, \varphi_{2}(0) \neq 0 \) or \( \varphi_{1}(0) \neq 0, \varphi_{2}(0) = 0 \). Then the operator \( H \) has a finite number of eigenvalues outside of the essential spectrum.
(ii) Let \( \varphi_{\alpha}(0) \neq 0 \) for all \( \alpha = 1, 2 \). Then the discrete spectrum of \( H \) is infinite and the function \( N(z) \) obeys the relation
\[ \lim_{z \to 0} \frac{N(z)}{|\log |z||} = \lambda_{0} \quad (0 < \lambda_{0} < \infty). \]

**Remark 2.10.** The constant \( \lambda_{0} \) does not depend on the functions \( \varphi_{\alpha}(p), \alpha = 1, 2 \), and is given as a positive function depending only on the ratios \( \frac{\mu_{a}}{T}, \alpha = 1, 2 \).
Remark 2.11. The conditions $\mu_\alpha = \mu_\alpha^0$ and $\varphi_\alpha(0) \neq 0$, $\alpha = 1, 2$, (resp. $\varphi_\alpha(0) = 0$) means that the operator $h_\alpha(p)$ has a zero-energy resonance (resp. zero eigenvalue) (see Lemma 3.2 (resp. Lemma 3.3)).

Remark 2.12. Clearly, the infinite cardinality of the negative discrete spectrum of $H$ follows automatically from the positivity of $U_0$.

Remark 2.13. We note that the assumptions for the functions $u$ and $\varphi_i$, $i = 1, 2$, are far from the precise, but we will not develop this point here.

3. Spectral properties of the operator $h_\alpha(p)$

In this section we study some spectral properties of the operator $h_\alpha(p)$, $p \in \mathbb{T}^3$ given by (2.3).

The perturbation $v_\alpha$ of the multiplication operator $h_\alpha^0(p)$ is a one-dimensional self-adjoint integral operator. Therefore in accordance with invariance of the absolutely continuous spectrum under trace class perturbations the absolutely continuous spectrum of the operator $h_\alpha(p)$ fills the following interval on the real axis:

$$\sigma_{ac}(h_\alpha(p)) = [m_\alpha(p), M_\alpha(p)],$$

where the numbers $m_\alpha(p)$ and $M_\alpha(p)$ are defined by

$$m_\alpha(p) = \min_{q \in \mathbb{T}^3} u_\alpha(q), \quad M_\alpha(p) = \max_{q \in \mathbb{T}^3} u_\alpha(q).$$

Let $\mathbb{C}$ be the field of complex numbers. For any $p \in \mathbb{T}^3$ and $z \in \mathbb{C} \setminus \sigma_{ac}(h_\alpha(p))$ we define the function (the Fredholm determinant associated with the operator $h_\alpha(p)$)

$$\Delta_{\mu_\alpha}(p, z) = 1 - \mu_\alpha \int_{\mathbb{T}^3} \frac{\varphi_\alpha(t)dt}{u_\alpha(t) - z}.$$

Note that $\Delta_{\mu_\alpha}(p, z)$ is real-analytic in $\mathbb{T}^3 \times (\mathbb{C} \setminus \sigma_{ac}(h_\alpha(p)))$.

Denote by $r_\alpha^0(p, z) = (h_\alpha^0(p) - z)^{-1}$ the resolvent of the operator $h_\alpha^0(p)$, that is, the multiplication operator by the function $(u_\alpha(t) - z)^{-1}$.

Lemma 3.1. For all $\mu_\alpha > 0$ and $p \in \mathbb{T}^3$ the following statements are equivalent:

(i) The operator $h_\alpha(p)$ has an eigenvalue $z \in \mathbb{C} \setminus \sigma_{ac}(h_\alpha(p))$ below the bottom of the continuous spectrum.

(ii) $\Delta_{\mu_\alpha}(p, z) = 0$, $z \in \mathbb{C} \setminus \sigma_{ac}(h_\alpha(p))$.

(iii) $\Delta_{\mu_\alpha}(p, z') < 0$ for some $z' \leq m_\alpha(p)$.

Proof. The number $z \in \mathbb{C} \setminus \sigma_{ac}(h_\alpha(p))$ is an eigenvalue of $h_\alpha(p)$ if and only if (by the Birman-Schwinger principle) $\lambda = 1$ is an eigenvalue of the operator

$$G_{\mu_\alpha}(p, z) = \mu_\alpha \Delta_{\mu_\alpha}(p, z)v_\alpha^\frac{1}{2}.$$
where
\[ \Lambda_\alpha(p, z) = \int_{T^3} \frac{\varphi_\alpha^2(t)}{u_0^{(\alpha)}(t) - z} \, dt. \]

According to Fredholm’s theorem the number \( \lambda = 1 \) is an eigenvalue for the operator \( G_{\mu_\alpha}(p, z) \) if and only if
\[ 1 - \mu_\alpha \Lambda_\alpha(p, z) = 0, \quad \text{that is,} \quad \Delta_\mu_\alpha(p, z) = 0. \]
The equivalence of (i) and (ii) is proven.

Now we prove the equivalence of (ii) and (iii). Let \( \Delta_{\mu_\alpha}(p, z_0) = 0 \) for some \( z_0 \in \mathbb{C} \setminus \sigma_{ac}(h_\alpha(p)) \). The operator \( h_\alpha(p) \) is self-adjoint and (i) and (ii) is equivalent hence \( z_0 \) is a real number. Since \( \Delta_{\mu_\alpha}(p, z) > 1 \) for all \( z > M_\alpha(p) \), we have \( z_0 \in (-\infty, m_\alpha(p)) \). Since for any \( p \in T^3 \) the function \( \Delta_{\mu_\alpha}(p, z) \) is decreasing in \( z \in (-\infty, m_\alpha(p)) \) we have \( \Delta_{\mu_\alpha}(p, z') < \Delta_{\mu_\alpha}(p, z_0) = 0 \) for some \( z_0 < z' < m_\alpha(p) \).

Now we suppose that \( \Delta_{\mu_\alpha}(p, z') < 0 \) for some \( z' \leq m_\alpha(p) \). Since for any \( p \in T^3 \)
\[ \lim_{z \to -\infty} \Delta_{\mu_\alpha}(p, z) = 1 \]
and \( \Delta_{\mu_\alpha}(p, z) \) is continuous in \( z \in (-\infty, m_\alpha(p)) \), we obtain that there exists \( z_0 \in (-\infty, z') \) such that \( \Delta_{\mu_\alpha}(p, z_0) = 0 \). This completes the proof. \( \square \)

Since the function \( \Delta_{\mu_\alpha}(0, \cdot) \) is decreasing on \( (-\infty, 0) \) and the function \( u_0^{(\alpha)}(q) \) has a unique non-degenerate minimum at \( q = 0 \) (see Lemma A.1) by dominated convergence there exist the finite limit
\[ \Delta_{\mu_\alpha}(0, 0) = \lim_{z \to -\infty} \Delta_{\mu_\alpha}(0, z). \]

**Lemma 3.2.** Let Hypotheses be fulfilled. The following statements are equivalent:
(i) the operator \( h_\alpha(0) \) has a zero energy resonance.
(ii) \( \varphi_\alpha(0) \neq 0 \) and \( \Delta_{\mu_\alpha}(0, 0) = 0 \).
(iii) \( \varphi_\alpha(0) \neq 0 \) and \( \mu_\alpha = \mu_0 \).

**Proof.** Let the operator \( h_\alpha(0) \) have a zero energy resonance for some \( \mu_\alpha > 0 \). Then by Definition 2.8 the equation
\[ \psi_\alpha(q) = \mu_\alpha \varphi_\alpha(q) \int_{T^3} \frac{\varphi_\alpha(t)\psi_\alpha(t)dt}{u_0^{(\alpha)}(t)}, \quad \psi_\alpha \in C(T^3) \]
has a simple solution \( \psi_\alpha(q) \) in \( C(T^3) \).

One can check that this solution is equal to the function \( \varphi_\alpha(q) \) (up to a constant factor). Therefore we see that
\[ \varphi_\alpha(q) = \mu_\alpha \varphi_\alpha(q) \int_{T^3} \frac{\varphi_\alpha^2(t)dt}{u_0^{(\alpha)}(t)} \]
and hence
\[ \Delta_{\mu_\alpha}(0, 0) = 1 - \mu_\alpha \int_{T^3} \frac{\varphi_\alpha^2(t)dt}{u_0^{(\alpha)}(t)} = 0 \]
and so \( \mu_\alpha = \mu_0 \).

Let for some \( \mu_\alpha > 0 \) the equality
\[ \Delta_{\mu_\alpha}(0, 0) = 1 - \mu_\alpha \int_{T^3} \frac{\varphi_\alpha^2(t)dt}{u_0^{(\alpha)}(t)} \]
hold and consequently \( \mu_0 = \mu_0^0 \). Then only the function \( \varphi_{\alpha}(q) \in C(\mathbb{T}^3) \) is a solution of the equation

\[
\psi_{\alpha}(q) = \mu_0^{\alpha} \varphi_{\alpha}(q) \int_{\mathbb{T}^3} \frac{\varphi_{\alpha}(t) \psi_{\alpha}(t) dt}{u_{0}^{\alpha}(t)},
\]

that is, the operator \( h_{\alpha}(0) \) has a zero energy resonance. \( \Box \)

Lemma 3.3. Let Hypotheses 2.1 be fulfilled. The following statements are equivalent:

(i) the operator \( h_{\alpha}(0) \) has a zero eigenvalue.

(ii) \( \varphi_{\alpha}(0) = 0 \) and \( \Delta_{\mu_0}(0,0) = 0 \).

(iii) \( \varphi_{\alpha}(0) = 0 \) and \( \mu_0 = \mu_0^0 \).

Proof. Suppose \( f \in L_2(\mathbb{T}^3) \) is an eigenfunction of the operator \( h_{\alpha}(0) \) associated with the zero eigenvalue. Then \( f \) satisfies the equation

\[
(3.1) \quad u_{0}^{(\alpha)}(q)f(q) - \mu_0^{\alpha} \varphi_{\alpha}(q) \int_{\mathbb{T}^3} \varphi_{\alpha}(t)f(t) dt = 0.
\]

From (3.1) we find that \( f \), except for an arbitrary factor, is given by

\[
(3.2) \quad f(q) = \frac{\varphi_{\alpha}(q)}{u_{0}^{(\alpha)}(q)},
\]

and from (3.1) we derive the equality \( \Delta_{\mu_0}(0,0) = 0 \) and so \( \mu_0 = \mu_0^0 \).

Since the function \( u_{0}^{(\alpha)}(q) \) has a non-degenerate minimum at the point \( q = 0 \) (see Lemma A.1) from (3.2) we have \( \varphi_{\alpha}(0) = 0 \).

Substituting the expression (3.2) for the \( f \) we the (3.1) we get the equality

\[
\varphi_{\alpha}(q) = \mu_0^{\alpha} \varphi_{\alpha}(q) \int_{\mathbb{T}^3} \frac{\varphi_{\alpha}^2(t) dt}{u_{0}^{(\alpha)}(t)}.
\]

Hence \( \Delta_{\mu_0}(0,0) = 0 \) and so \( \mu_0 = \mu_0^0 \).

Let \( \varphi_{\alpha}(0) = 0 \) and \( \Delta_{\mu_0}(0,0) = 0 \) then \( \mu_0 = \mu_0^0 \) and the function

\[
(3.3) \quad f(q) = \frac{\varphi_{\alpha}(q)}{u_{0}^{(\alpha)}(q)}
\]

obeys the equation

\[
(3.4) \quad h_{\alpha}(0)f = 0
\]

and

\[
(3.5) \quad f \in L_2(\mathbb{T}^3).
\]

Indeed, the functions \( u_{0}^{(\alpha)}(q) \) and \( \varphi_{\alpha}(q) \) are analytic on \( \mathbb{T}^3 \) and the function \( u_{0}^{(\alpha)}(q) \) has a unique non-degenerate minimum at the origin, hence

\[
(3.6) \quad f(q) = \frac{\varphi(q)}{u_{0}^{(\alpha)}(q)} \in L_2(\mathbb{T}^3)
\]

if and only if \( \varphi_{\alpha}(0) = 0 \). \( \Box \)

Set

\[
\mu_{\alpha}^{max} = \max_{p \in \mathbb{T}^3} \Lambda_{\alpha}^{-1}(p,0).
\]
Lemma 3.4. Let Hypotheses 2.4 and 2.4 be fulfilled.

(i) Let \( \mu_\alpha > \mu_\alpha^{\text{max}} \). Then for any \( p \in \mathbb{T}^3 \) the operator \( h_\alpha(p) \) has a unique negative eigenvalue.

(ii) Let \( \mu_\alpha^{\text{max}} \geq \mu_\alpha > \mu_\alpha^0 \). Then there exists a non empty open set \( D_{\mu_\alpha} \subset \mathbb{T}^3 \) such that for any \( p \in D_{\mu_\alpha} \) the operator \( h_\alpha(p) \) has a unique negative eigenvalue and for \( p \in \mathbb{T}^3 \setminus D_{\mu_\alpha} \) the operator \( h_\alpha(p) \) has no negative eigenvalues.

(iii) Let \( 0 < \mu_\alpha \leq \mu_\alpha^0 \). Then for any \( p \in \mathbb{T}^3 \) the operator \( h_\alpha(p) \) has no negative eigenvalues.

Proof. (i) Let \( \mu_\alpha > \mu_\alpha^{\text{max}} \). Since \( \mathbb{T}^3 \) is a compact set and the function \( \Lambda_\alpha^{-1}(p,0) \) is the continuous on \( \mathbb{T}^3 \) by the definition of \( \mu_\alpha^{\text{max}} \) for all \( p \in \mathbb{T}^3 \) we get the inequalities

\[
\Lambda_\alpha^{-1}(p,0) \leq \mu_\alpha^{\text{max}} < \mu_\alpha.
\]

So we have

\[
\Delta_{\mu_\alpha}(p,0) < 0.
\]

Hence by Lemma 3.1 the operator \( h_\alpha(p), p \in \mathbb{T}^3 \) has a unique negative eigenvalue.

(ii) Let \( \mu_\alpha^{\text{max}} \geq \mu_\alpha > \mu_\alpha^0 \). By Hypothesis 2.4 for all \( p \in \mathbb{T}^3 \), \( p \neq 0 \) the inequality

\[
\Delta_{\mu_\alpha}(p,0) = 1 - \mu_\alpha^0 \Lambda_\alpha(p,0) > 0
\]

holds, that is,

\[
\Lambda_\alpha^{-1}(p,0) > \mu_\alpha^0.
\]

Let us introduce the notation

\[
D_{\mu_\alpha} = \{ p \in \mathbb{T}^3 : \Lambda_\alpha^{-1}(p,0) < \mu_\alpha \}.
\]

Since the function \( \Lambda_\alpha^{-1}(\cdot,0) \) is the continuous on \( \mathbb{T}^3 \) and \( \Lambda_\alpha^{-1}(0,0) = \mu_\alpha^0 \) we have that \( D_{\mu_\alpha} \neq \mathbb{T}^3 \) is a non void open set.

Thus we have

\[\Delta_{\mu_\alpha}(p,0) < 0 \quad \text{for all} \quad p \in D_{\mu_\alpha}.\]

Hence by Lemma 3.1 the operator \( h_\alpha(p), p \in D_{\mu_\alpha} \) has a unique negative eigenvalue.

For all \( p \in \mathbb{T}^3 \setminus D_{\mu_\alpha} \) we have

\[
\Lambda_\alpha^{-1}(p,0) \geq \mu_\alpha, \quad \text{that is,} \quad \Delta_{\mu_\alpha}(p,0) \geq 0.
\]

For any \( p \in \mathbb{T}^3 \) the function \( \Delta_{\mu_\alpha}(p,z) \) is decreasing in \( z \in (-\infty, m_\alpha(p)) \) and \( \lim_{z \to -\infty} \Delta_{\mu_\alpha}(p,z) = 1 \), hence from 3.4 for any \( z < 0 \) and \( p \in \mathbb{T}^3 \setminus D_{\mu_\alpha} \) we have \( \Delta_{\mu_\alpha}(p,z) > \Delta_{\mu_\alpha}(p,0) \geq 0 \). Then by Lemma 3.1 for \( p \in \mathbb{T}^3 \setminus D_{\mu_\alpha} \) the operator \( h_\alpha(p) \) has no negative eigenvalues.

(iii) Let \( 0 < \mu_\alpha \leq \mu_\alpha^0 \). Then by Hypotheses 2.4 we have

\[\Delta_{\mu_\alpha}(p,0) > 0 \quad \text{for all} \quad p \in \mathbb{T}^3.\]

For any \( p \in \mathbb{T}^3 \) the function \( \Delta_{\mu_\alpha}(p,z) \) is decreasing in \( z \in (-\infty, m_\alpha(p)) \) and

\[\lim_{z \to -\infty} \Delta_{\mu_\alpha}(p,z) = 1\]

and we have \( \Delta_{\mu_\alpha}(p,z) > \Delta_{\mu_\alpha}(p,0), z < 0 \). Then by Lemma 3.1 for all \( p \in \mathbb{T}^3 \) the operator \( h_\alpha(p) \) has no negative eigenvalues. \(\Box\)

The following decomposition plays a crucial role for the proof of the asymptotics 2.4.
Lemma 3.5. Assume Hypothesis 2.7 and 2.8 are fulfilled. Then for any \( p \in U_\delta(0) \), \( \delta > 0 \) sufficiently small, and \( z \leq 0 \) the following decomposition holds:

\[
\Delta_{\mu_\alpha}(p, z) = \Delta_{\mu_\alpha}(0, 0) + \frac{4\sqrt{2}\pi^2 \mu_\alpha \varphi_\alpha^2(0)}{l_3^2 \det(U)^{\frac{1}{2}}} \sqrt{m_\alpha(p)} - z + \Delta^{(02)}_{\mu_\alpha}(m_\alpha(p) - z) + \Delta^{(20)}_{\mu_\alpha}(p, z),
\]

where \( \Delta^{(02)}_{\mu_\alpha}(m_\alpha(p) - z) \) (resp. \( \Delta^{(20)}_{\mu_\alpha}(p, z) \)) is a function behaving like \( O(m_\alpha(p) - z) \) (resp. \( O(|p|^2) \)) as \( |m_\alpha(p) - z| \rightarrow 0 \) (resp. \( p \rightarrow 0 \)).

Proof. Let

\[
U_\alpha(p, q) = u_p^{(\alpha)}(q + q_\alpha(p)) - m_\alpha(p),
\]

where \( q_\alpha(p) \in T^3 \) is an analytic function in \( p \in U_\delta(0) \) (see Lemma A.1) and is the non-degenerate minimum point of the function \( u_p^{(\alpha)}(q) \) for any \( p \in U_\delta(0) \).

We define the function \( \Delta_{\mu_\alpha}(p, w) \) on \( U_\delta(0) \times \mathbb{C}_+ \) by

\[
\tilde{\Delta}_{\mu_\alpha}(p, w) = \Delta_{\mu_\alpha}(p, m_\alpha(p) - w^2),
\]

where \( \mathbb{C}_+ = \{ z \in \mathbb{C} : \text{Re} z > 0 \} \). Using (3.6) the function \( \tilde{\Delta}_{\mu_\alpha}(p, w) \) is represented as

\[
\tilde{\Delta}_{\mu_\alpha}(p, w) = 1 - \mu_\alpha \int_{\mathbb{R}^3} \varphi_\alpha^2(q + q_\alpha(p)) \frac{U_\alpha(p, q)}{U_\alpha(p, q) + w^2} dq.
\]

Let \( V_\delta(0) \) be a complex \( \delta \)-neighborhood of the point \( w = 0 \in \mathbb{C} \). Denote by \( \Delta^{*}_{\mu_\alpha}(p, w) \) the analytic continuation of the function \( \Delta_{\mu_\alpha}(p, w) \) to the region \( U_\delta(0) \times (\mathbb{C}_+ \cup V_\delta(0)) \). Since the functions \( |\varphi_\alpha(q)| \) and \( U_\alpha(p, q) \) are even (see Lemma A.2) we have that \( \Delta^{*}_{\mu_\alpha}(p, w) \) is even in \( p \in U_\delta(0) \).

Therefore

\[
\Delta^{*}_{\mu_\alpha}(p, w) = \Delta^{*}_{\mu_\alpha}(0, w) + \tilde{\Delta}^{(20)}_{\mu_\alpha}(p, w),
\]

where \( \tilde{\Delta}^{(20)}_{\mu_\alpha}(p, w) = O(|p|^2) \) uniformly in \( w \in \mathbb{C}_+ \) as \( p \rightarrow 0 \) (see also [15]). A Taylor series expansion gives

\[
\Delta^{*}_{\mu_\alpha}(0, w) = \Delta^{*}_{\mu_\alpha}(0, 0) + \tilde{\Delta}^{(01)}_{\mu_\alpha}(0, 0)w + \tilde{\Delta}^{(02)}_{\mu_\alpha}(0, 0)w^2,
\]

where \( \tilde{\Delta}^{(02)}_{\mu_\alpha}(0, 0) = O(1) \) as \( w \rightarrow 0 \).

A simple computation shows that

\[
\frac{\partial \Delta^{*}_{\mu_\alpha}(0, 0)}{\partial w} = \tilde{\Delta}^{(01)}_{\mu_\alpha}(0, 0) = \frac{4\sqrt{2}\pi^2 \mu_\alpha \varphi_\alpha^2(0)}{l_3^2 \det(U)^{\frac{1}{2}}}.
\]

The representations (3.7), (3.8) and the equality (3.9) give (3.5). \( \square \)

Corollary 3.6. Assume Hypothesis 2.7 and 2.8 are fulfilled and let \( h_\alpha(0) \) have a zero energy resonance. Then there exists a number \( \delta > 0 \) so that for any \( p \in U_\delta(0) \) and \( z \leq 0 \) the following decomposition holds

\[
\Delta_{\mu_\alpha}(p, z) = \frac{4\sqrt{2}\pi^2 \mu_\alpha \varphi_\alpha^2(0)}{l_3^2 \det(U)^{\frac{1}{2}}} \sqrt{m_\alpha(p)} - z + \Delta^{(02)}_{\mu_\alpha}(m_\alpha(p) - z) + \Delta^{(20)}_{\mu_\alpha}(p, z),
\]

where the functions \( \Delta^{(02)}_{\mu_\alpha}(m_\alpha(p) - z) \) and \( \Delta^{(20)}_{\mu_\alpha}(p, z) \) are the same as in Lemma 3.5.
Lemma 3.9. Assume Hypothesis 2.1 and 2.4 are fulfilled and let the operator and 

Proof. The proof of Corollary 3.6 immediately follows from decomposition (3.5) and the equality \( \Delta_{\mu_0}(0,0) = 0 \). \( \square \)

Corollary 3.7. Assume Hypothesis 2.1 and 2.2 are fulfilled and \( \varphi_\alpha(0) \neq 0 \). Then there exists \( \delta > 0 \) such that for any \( p \in U_\delta(0), p \neq 0 \)

(3.11) \[ \Delta_{\mu_0^\alpha}(p,0) > 0, \quad \text{that is,} \quad \Lambda_\alpha(p) < \Lambda_\alpha(0), \]

where \( \Lambda_\alpha(p) \) is defined using the integral (2.2).

Proof. Let \( \varphi_\alpha(0) \neq 0 \), then by Corollary 3.6 and the asymptotics (see (A.1))

(3.12) \[ m_\alpha(p) = \frac{l_1l_2 - l^2}{2l_\alpha}(Up,p) + O(p^4) \quad \text{as} \quad p \to 0 \]

we get

\[
\frac{4\sqrt{2\pi^2\mu_\alpha(p)^2}}{l_\alpha^2 \det(U)^{2.5}} \sqrt{m_\alpha(p)} > |\Delta_{\mu_0^\alpha}(0, p)|
\]

for \( p \in U_\delta(0), p \neq 0, \delta \) sufficiently small. This inequality gives (3.11). \( \square \)

Lemma 3.8. Assume Hypothesis 2.1 and 2.4 are fulfilled and the operator \( h_\alpha(0) \) has a zero-energy resonance. Then there exist positive numbers \( c, C \) and \( \delta \) such that

(3.13) \[ c|p| \leq \Delta_{\mu_0^\alpha}(p,0) \leq C|p| \quad \text{for any} \quad p \in U_\delta(0)
\]

and

(3.14) \[ \Delta_{\mu_0^\alpha}(p,0) \geq c \quad \text{for any} \quad p \in \mathbb{T}^3 \setminus U_\delta(0).
\]

Proof. From (3.11) and (3.12) we get (3.13) for some positive numbers \( c, C \).

By Hypothesis 2.4 we get \( \Delta_{\mu_0^\alpha}(p,0) > 0, p \neq 0 \). Since \( \Delta_{\mu_0^\alpha}(p,0) \) is continuous on \( \mathbb{T}^3 \) and \( \Delta_{\mu_0^\alpha}(0,0) = 0 \) we have (3.14) for some \( c > 0 \). \( \square \)

Lemma 3.9. Assume Hypothesis 2.1 and 2.4 are fulfilled and let the operator \( h_\alpha(0) \) have a zero eigenvalue, then there exist numbers \( \delta > 0 \) and \( c > 0 \) so that the following inequalities hold

\[
|\Delta_{\mu_0^\alpha}(p,0)| \geq c|p| \quad \text{for all} \quad p \in U_\delta(0),
\]

\[
|\Delta_{\mu_0^\alpha}(p,0)| \geq c \quad \text{for all} \quad p \in \mathbb{T}^3 \setminus U_\delta(0).
\]

Proof. Let the operator \( h_\alpha(0) \) have a zero eigenvalue. By Lemma 3.2 we have \( \mu_\alpha = \mu_0^\alpha \), \( \varphi_\alpha(0) = 0 \) and \( \Delta_{\mu_0^\alpha}(0,0) = 0 \). By Hypothesis 2.4 the function \( \Delta_{\mu_0^\alpha}(p,0) = 1 - \mu_0^\alpha \Lambda_\alpha(p) \) has a unique non-degenerate minimum at \( p = 0 \). Then there exist positive numbers \( \delta \) and \( c \) such that the statement of the lemma is fulfilled. \( \square \)

4. The essential spectrum of the operator \( H \)

In this section we introduce a multiplication operator perturbed by a partial integral operator and prove Theorem 2.7.

We consider the operator \( H_\alpha \) acting on the Hilbert space \( L_2(\mathbb{T}^3)^2 \) as

\[ H_\alpha = H_0 - \mu_\alpha V_\alpha, \quad \alpha = 1, 2. \]

The operator \( H_1 \) (resp. \( H_2 \)) commutes with all multiplication operators by functions \( w_1(q) \) (resp. \( w_2(p) \)) on \( L_2(\mathbb{T}^3)^2 \).
Therefore the decomposition of the space $L_2((\mathbb{T}^3)^2)$ into the direct integral

$$L_2((\mathbb{T}^3)^2) = \bigoplus_{p \in \mathbb{T}^3} L_2((\mathbb{T}^3))dp$$

yields for the operator $H_\alpha$ the decomposition into the direct integral

(4.1)$$H_\alpha = \bigoplus_{p \in \mathbb{T}^3} h_\alpha(p)dp.$$

Here the fiber operators $h_\alpha(p)$, $p \in \mathbb{T}^3$, are defined by (2.3).

4.1. The spectrum of the operators $H_\alpha$. The representation (4.1) of the operator $H_\alpha$ and the theorem on decomposable operators (see [22]) imply the following lemma:

Lemma 4.1. For the spectrum $\sigma(H_\alpha)$ of the $H_\alpha$ the equality

$$\sigma(H_\alpha) = \bigcup_{p \in \mathbb{T}^3} \sigma_d(h_\alpha(p)) \cup [0, M]$$

holds.

Set

(4.2)$$\sigma_{\text{two}}(H_\alpha) = \bigcup_{p \in \mathbb{T}^3} \sigma_d(h_\alpha(p)).$$

Now we precisely describe the location and structure of the spectrum of $H_\alpha$.

Lemma 4.2. Assume Hypotheses [2.7] is fulfilled.

(i) Let $\mu_\alpha > \mu^{\text{max}}_\alpha$, then

$$\sigma(H_\alpha) = [a_\alpha, b_\alpha] \cup [0, M] \quad \text{and} \quad b_\alpha < 0.$$

(ii) Let $\mu^{\text{max}}_\alpha \geq \mu_\alpha > \mu^0_\alpha$, then

$$\sigma(H_\alpha) = [a_\alpha, M] \quad \text{and} \quad a_\alpha < 0.$$

(iii) Let $\mu^0_\alpha \geq \mu_\alpha > 0$, then

$$\sigma(H_\alpha) = [0, M].$$

Proof. (i) Let $\mu_\alpha > \mu^{\text{max}}_\alpha$. Then by Lemma [3.3] for all $p \in \mathbb{T}^3$ the operator $h_\alpha(p)$, $\alpha = 1, 2$, has a unique negative eigenvalue $z_\alpha(p) < m_\alpha(p)$.

By Hypotheses [2.4] and [2.2] and Lemma [3.1] $z_\alpha : p \in \mathbb{T}^3 \to z_\alpha(p)$ is a real analytic function on $\mathbb{T}^3$.

Therefore $\text{Im}z_\alpha$ is a connected closed subset of $(-\infty, 0)$, that is, $\text{Im}z_\alpha = [a_\alpha, b_\alpha]$ and $b_\alpha < 0$ hence $\sigma_{\text{two}}(H_\alpha) = [a, b]$.

(ii) Let $\mu^{\text{max}}_\alpha \geq \mu_\alpha > \mu^0_\alpha$. Then by Lemma [3.4] there exists a non void open set $D_{\mu_\alpha}$ (see [4.4]) such that for any $p \in D_{\mu_\alpha}$ the operator $h_\alpha(p)$ has a unique negative eigenvalue $z_\alpha(p)$.

Since for any $p \in \mathbb{T}^3$ the operator $h_\alpha(p)$ is bounded and $\mathbb{T}^3$ is compact set there exist positive number $C$ such that $\sup_{p \in \mathbb{T}^3} ||h_\alpha(p)|| \leq C$ and for any $p \in \mathbb{T}^3$ we have

$$\sigma(h_\alpha(p)) \subset [-C, C].$$

For any $q \not\in \partial D_{\mu_\alpha} = \{p \in \mathbb{T}^3 : \Delta_{\mu_\alpha}(p, 0) = 0\}$ there exist $\{p_n\} \subset D_{\mu_\alpha}$ such that $p_n \to q$ as $n \to \infty$. Set $z_n = z(p_n)$. By Lemma [3.4] we have $\{z_n\} \subset [-C, 0]$ and without loss of a generality one may assume that $z_n \to z_0$ as $n \to \infty$. 

□
The function $\Delta_{\mu_\alpha}(p, z)$ is continuous in $\mathbb{T}^3 \times (-\infty, 0]$ and hence
\[
0 = \lim_{n \to \infty} \Delta(p_n, z_n) = \Delta(q, z_0) .
\]

Since for any $p \in \mathbb{T}^3$ the function $\Delta_{\mu_\alpha}(p, \cdot)$ is decreasing in $(-\infty, m_w(p)]$ and $p \in \partial D_{\mu_\alpha}$ we can see that $\Delta_{\mu_\alpha}(p, z_0) = 0$ if and only if $z_0 = 0$.

For any $p \in \partial D_{\mu_\alpha}$ we define
\[
z_\alpha(p) = \lim_{q \to p, q \in D_{\mu_\alpha}} z_\alpha(q) = 0 .
\]

Since the function $z_\alpha(p)$ is continuous on the compact set $D_{\mu_\alpha} \cup \partial D_{\mu_\alpha}$ and $z_\alpha(p) = 0$, $p \in \partial D_{\mu_\alpha}$ we conclude that
\[
\text{Im} z_\alpha = [a_\alpha, 0], \quad a_\alpha < 0 .
\]

Hence the set
\[
\{ z \in \sigma_{\text{two}}(H_\alpha) : z \leq 0 \} = \bigcup_{p \in \mathbb{T}^3} \sigma_d(h_\alpha(p)) \cap (-\infty, 0]
\]
coincides with the set $\text{Im} z_\alpha = [a_\alpha, 0]$.

Then Lemma 4.1 and (4.2) complete the proof of $(ii)$.

$(iii)$ Let $\rho^0_\alpha \geq \mu_\alpha > 0$. Then by Lemma 3.4 for all $p \in \mathbb{T}^3$ the operator $h_\alpha(p)$ has no negative eigenvalue.

Hence we have
\[
\sigma(H_\alpha) = [0, M] .
\]
\[
\square
\]

4.2. The Faddeev type integral equation. In this section we derive a Faddeev type system of integral equations and prove an analogue of the Birman-Schwinger principle.

Denote by $R_\alpha(z)$, $\alpha = 1, 2$, resp. $R_0(z)$ the resolvent operator of $H_\alpha$, $\alpha = 1, 2$, resp. $H_0$.

Let $W_\alpha(z), \alpha = 1, 2$, be the operators on $L_2((\mathbb{T}^3)^2)$ defined as
\[
W_\alpha(z) = I + \mu_\alpha V_\alpha^z R_0(z) V_\alpha^z, \quad z \in \rho(H_\alpha) ,
\]
where $I$ is the identity operator on $L_2((\mathbb{T}^3)^2)$ and $\rho(H_\alpha) = \mathbb{C} \setminus \sigma(H_\alpha)$ is the set of all regular points of the operator $H_\alpha$.

One can check that
\[
W_\alpha(z) = (I - \mu_\alpha V_\alpha^z R_0(z) V_\alpha^z)^{-1}, \quad \alpha = 1, 2.
\]

Let $A(z)$, $z \in \mathbb{C} \setminus (\sigma(H_1) \cup \sigma(H_2))$ be the operator on $L_2^2((\mathbb{T}^3)^2)$ with the entries
\[
A_{\alpha\alpha}(z) = 0 ,
\]
\[
A_{\alpha\beta}(z) = \sqrt{\mu_\alpha \mu_\beta} W_\alpha(z) V_\beta^z R_0(z) V_\beta^z , \quad \alpha \neq \beta, \quad \alpha, \beta = 1, 2 .
\]

The following theorem is an analogue of the result for the three-particle discrete Schrödinger operators with the zero-range interactions and may be proven in a way similar to the one in [14].

**Theorem 4.3.** The number $z \in \mathbb{C} \setminus (\sigma(H_1) \cup \sigma(H_2))$ is an eigenvalue of the operator $H$ if only if the number $1$ is an eigenvalue of $A(z)$. Moreover the eigenvalues $z$ and $1$ have the same multiplicities.
Proof. Let $z \in \mathbb{C} \setminus (\sigma(H_1) \cup \sigma(H_2))$ be the eigenvalue of $H$, that is, the equation
\begin{equation}
(4.3) \quad f = R_0(z) \sum_{\alpha=1}^{2} \mu_\alpha V_\alpha f, \quad f \in L_2((\mathbb{T}^3)^2)
\end{equation}
has a nontrivial solutions.

Multiplying (4.3) from the left by $\sqrt{\mu_\alpha} V_\alpha^* f$ and setting $\varphi_\alpha = \sqrt{\mu_\alpha} V_\alpha^* f$ we have the following system of two equations
\begin{equation}
(4.4) \quad \varphi_\alpha = \sum_{\beta=1}^{2} \sqrt{\mu_\alpha \mu_\beta} V_\alpha^* R_0(z) V_\beta^* \varphi_\beta, \quad \alpha = 1, 2,
\end{equation}
\begin{equation}
\varphi = (\varphi_1, \varphi_2) \in L_2^2((\mathbb{T}^3)^2)
\end{equation}
and this system of equations has a nontrivial solution if and only if the system of equations (4.3) has a nontrivial solution and the linear subspaces of solutions of (4.3) and (4.4) have the same dimensions.

Since the operator $W_\alpha(z) = (I - \mu_\alpha V_\alpha^* R_0(z) V_\alpha^*)^{-1}$, where $I$ is identity operator on $L_2((\mathbb{T}^3)^2)$, is invertible the system of equations (4.4) is equivalent to the following system of two equations
\begin{equation}
\varphi_\alpha = \sqrt{\mu_\alpha \mu_\beta} W_\alpha(z) V_\alpha^* R_0(z) V_\beta^* \varphi_\beta, \quad \alpha \neq \beta, \alpha, \beta = 1, 2,
\end{equation}
that is,
\begin{equation}
A(z)\varphi = \varphi, \quad \varphi = (\varphi_1, \varphi_2) \in L_2^2((\mathbb{T}^3)^2).
\end{equation}

\[\square\]

Let $T(z), z \in \mathbb{C} \setminus (\sigma(H_1) \cup \sigma(H_2))$ be the operator on $L_2^2(\mathbb{T}^3)$ with the entries
\begin{equation}
T_{11}(z) = T_{22}(z) = 0,
\end{equation}
\begin{equation}
(T_{12}(z)u)(q) = \frac{\sqrt{\mu_1 \mu_2}}{\Delta_{t_1}(z)} \int_{\mathbb{T}^3} \frac{\varphi_1(t) \varphi_2(q)}{u(t, q) - z} w(t) dt,
\end{equation}
\begin{equation}
(T_{21}(z)u)(p) = \frac{\sqrt{\mu_1 \mu_2}}{\Delta_{t_2}(z)} \int_{\mathbb{T}^3} \frac{\varphi_1(p) \varphi_2(t)}{u(p, t) - z} w(t) dt.
\end{equation}

Let $\Phi = \text{diag}\{\Phi_1, \Phi_2\} : L_2^2((\mathbb{T}^3)^2) \rightarrow L_2^2(\mathbb{T}^3)$ be the operator with the entries
\begin{equation}
(\Phi_1 f)(q) = \frac{1}{||\varphi_1||} \int_{\mathbb{T}^3} \varphi_1(t) f(t, q) dt, \quad (\Phi_2 f)(p) = \frac{1}{||\varphi_2||} \int_{\mathbb{T}^3} \varphi_2(t) f(p, t) dt,
\end{equation}
\begin{equation}
f \in L_2((\mathbb{T}^3)^2)
\end{equation}
and $\Phi^* = \text{diag}\{\Phi_1^*, \Phi_2^*\}$ its adjoint.

Proposition 4.4. Let $T_1, T_2$ be bounded operators. If $z \neq 0$ is an eigenvalue of $T_1 T_2$ then $z$ is an eigenvalue for $T_2 T_1$ as well with the same algebraic and geometric multiplicities.

This proposition is well known and its proof is omitted.

Lemma 4.5. For each $z \in \mathbb{C} \setminus (\sigma(H_1) \cup \sigma(H_2))$ the following equality
\begin{equation}
A(z) = \Phi^* T(z) \Phi
\end{equation}
holds and the nonzero eigenvalues of the operators $A(z)$ and $T(z)$ coincide and have the same algebraic and geometric multiplicities.
Proof. One can easily check that the equalities
\[(4.7)\quad (V_1^{1/2}f)(p,q) = \varphi_1(p)(\Phi_1f)(q), \quad (V_2^{1/2}f)(p,q) = \varphi_2(q)(\Phi_2f)(p)\]
hold. The equalities (4.7) imply the equality (4.6).

Therefore the inclusion
\[\sigma(\Phi_1f) \cup \sigma(\Phi_2f) \subset \sigma(I)\]
is discrete. Thus
\[\det(I - T(z)) = 0.\]
□

Now we establish the location of the essential spectrum of \(H\). For any \(z \in \mathbb{C} \setminus (\sigma(H_1) \cup \sigma(H_2))\) the kernels of the operators \(T_{\alpha\beta}(z), \alpha, \beta = 1, 2\), are continuous functions on \((\mathbb{T}^3)^2\). Therefore the Fredholm determinant \(\det(I - T(z))\) of the operator \(I - T(z)\), where \(I\) is the identity operator on \(L^2(\mathbb{T}^3)\), exists and is a real-analytic function on \(\mathbb{C} \setminus (\sigma(H_1) \cup \sigma(H_2))\). According to the Fredholm theorem the following lemma holds.

Lemma 4.6. The number \(z \in \mathbb{C} \setminus (\sigma(H_1) \cup \sigma(H_2))\) is an eigenvalue of the operator \(H\) if and only if
\[\det(I - T(z)) = 0.\]
□

Theorem 4.7. For the essential spectrum of the operator \(H\) the inequality
\[\sigma_{ess}(H) = \cup_{\alpha=1}^2 \cup_{p \in \mathbb{T}^3} \sigma(h_\alpha(p)) \cup [0, M]\]
holds.

Proof. By the definition of the essential spectrum, it is easy to show that \(\sigma(H_1) \cup \sigma(H_2) \subset \sigma_{ess}(H)\). Since the function \(\det(I - T(z))\) is analytic in \(\mathbb{C} \setminus (\sigma(H_1) \cup \sigma(H_2))\) by Lemma 4.6 we conclude that the set
\[\sigma(H) \setminus (\sigma(H_1) \cup \sigma(H_2)) = \{z \in \mathbb{C} \setminus (\sigma(H_1) \cup \sigma(H_2)) : \det(I - T(z)) = 0\}\]
is discrete. Thus
\[\sigma(H) \setminus (\sigma(H_1) \cup \sigma(H_2)) \subset \sigma(H) \setminus \sigma_{ess}(H).\]
Therefore the inclusion \(\sigma_{ess}(H) \subset (\sigma(H_1) \cup \sigma(H_2))\) holds. □

Proof of Theorem 2.7. The proof of Theorem 2.7 follows from Theorem 4.7 and Lemma 4.2.

Theorem 4.8. The operator \(H = H_0 - V\) has no eigenvalue lying on the right hand side of the essential spectrum \(\sigma_{ess}(H)\).

Proof. Since \(V = \mu_1V_1 + \mu_2V_2\) is a positive operator we have that the operator \(H = H_0 - V\) has no eigenvalues larger than \(M\). □

4.3. Birman-Schwinger principle. We recall that \(\tau_{ess}(H)\) denotes the bottom of the essential spectrum and \(N(z)\) the number of eigenvalues of \(H\) lying below \(z \leq \tau_{ess}(H)\).

Let \(A(z), \quad z < \tau_{ess}(H)\) be the operator on \(L^2(\mathbb{T}^3)^2\) with the entries
\[A_{\alpha\beta}(z) = 0,\]
\[A_{\alpha\beta}(z) = \sqrt{\mu_1 \mu_2} W_{\alpha\beta}^1(z) V_\alpha^1 R_\alpha(z) V_\beta^1 W_{\beta\beta}^1(z), \quad \alpha \neq \beta, \quad \alpha, \beta = 1, 2.\]

For a bounded self-adjoint operator \(B\), we define \(n(\lambda, B)\) as
\[n(\lambda, B) = \sup\{\text{dim} F : (Bu, u) > \lambda, \quad u \in F, \quad ||u|| = 1\}.\]
\( n(\lambda, B) \) is equal to the infinity if \( \lambda \) is in essential spectrum of \( B \) and if \( n(\lambda, B) \) is finite, it is equal to the number of the eigenvalues of \( B \) bigger than \( \lambda \). By the definition of \( N(z) \) we have
\[
N(z) = n(-z, -H), -z > -\tau_{ess}(H).
\]
The following lemma is a realization of well known Birman-Schwinger principle for the operator \( H \) (see \textbf{[3, 23, 25]}).

\textbf{Lemma 4.9.} The operator \( A(z) \) is compact and continuous in \( z < \tau_{ess}(H) \) and
\[
N(z) = n(1, A(z)).
\]

\textbf{Proof.} This Lemma is deduced by the arguments of \textbf{[23]}. Since \( H = H_0 + V \), and \( H_0 - zI \) is positive and invertible for \( z < \tau_{ess}(H) \), one has \( u \in L_2((\mathbb{T}^3)^2) \) and \((H - zI)u, u) < 0 \) if and only if \((R_0^2(z)V R_0^2(z) - I)v, v) > 0 \) and \( v = (H_0 - z)^{-1}u \).

It follows that \( N(z) = n(1, R_0^2(z)V R_0^2(z)) \). We have the following decomposition:
\[
R_0^2(z)V R_0^2(z) = Z^* Z,
\]
with \( Z : L_2((\mathbb{T}^3)^2) \to L_2((\mathbb{T}^3)^2) \) defined by
\[
Z = (\sqrt{\mu_1} R_0^2(z)V_1^Z, \sqrt{\mu_2} R_0^2(z)V_2^Z).
\]
Then by Proposition \textbf{4.2} we get
\[
n(\lambda, Z^* Z) = n(1, ZZ^*).
\]
Consequently,
\[
N(z) = n(1, ZZ^*).
\]
Since \( z < \tau_{ess}(H) \) the operator \( I - \mu_\alpha V_\alpha R_0(z) V_\alpha^* \), \( \alpha = 1, 2 \), is invertible, and
\[
(I - \mu_\alpha V_\alpha R_0(z) V_\alpha^*)^{-1/2} = W_\alpha^2(z) > 0, \alpha = 1, 2.
\]

Let \( \mathbb{I} \) be the identity operator on \( L_2((\mathbb{T}^3)^2) \). A direct calculations shows that \( y \in L_2((\mathbb{T}^3)^2) \) and \((Z Z^* - 1)y, y) > 0 \) if and only if \((A(z) - \mathbb{I})f, f) > 0 \) and \( y_\alpha = (I - \mu_\alpha V_\alpha R_0(z) V_\alpha^*)^{1/2} f_\alpha, \alpha = 1, 2 \), and that \( n(1, Z Z^*) = n(1, A(z)) \).

In our analysis of the spectrum of \( H \) the crucial role is played by the compact integral operator \( T(z), z < \tau_{ess}(H) \) in the space \( L_2((\mathbb{T}^3)^2) \) with the entries
\[
T_{11}(z) = T_{22}(z) = 0,
\]
\[
(T_{12}(z)\omega)(q) = \sqrt{\mu_1 \mu_2} \int_{\mathbb{T}^3} \frac{\xi_1(t) \xi_2(q)}{\sqrt{\Delta_{\mu_1}(q, z)\Delta_{\mu_2}(t, z)(u(t, q) - z)}} \omega(t) dt,
\]
\[
(T_{21}(z)\omega)(p) = \sqrt{\mu_1 \mu_2} \int_{\mathbb{T}^3} \frac{\xi_1(t) \xi_2(p)}{\sqrt{\Delta_{\mu_1}(p, z)\Delta_{\mu_2}(t, z)(u(p, t) - z)}} \omega(t) dt, \quad w \in L_2(\mathbb{T}^3).
\]

Using the equality \textbf{(4.7)} one may verify that
\[
A(z) = \Phi^* T(z) \Phi,
\]
where the operator \( \Phi \) is defined in \textbf{[45]}.

Since the operator \( \Phi^* T(z) \) resp. \( \Phi \) is a bounded in \( L_2((\mathbb{T}^3)^2) \) resp. \( L_2((\mathbb{T}^3)^2) \), by Proposition \textbf{4.4} and the equality \( \Phi \Phi^* T(z) = T(z) \) we have \( n(1, A(z)) = n(1, T(z)) \).
5. The Finiteness of the Number of Eigenvalues of the Operator $H$.

We start the proof of (i) of Theorem 2.9 with one elementary lemma.

**Lemma 5.1.** Assume Hypotheses 2.7, 2.2, and 2.4 are fulfilled and let $\mu_\alpha = \mu_\alpha^0$ for all $\alpha = 1, 2$, and either $\varphi_1(0) = 0$, $\varphi_2(0) \neq 0$ or $\varphi_1(0) \neq 0$, $\varphi_2(0) = 0$ or $\varphi_1(0) = 0$, $\varphi_2(0) = 0$. Then the operator $T(z)$ belongs to the Hilbert-Schmidt class and is continuous from the left up to $z = 0$.

**Proof.** We prove Lemma 5.1 in the case $\mu_\alpha = \mu_\alpha^0$, $\alpha = 1, 2$, and $\varphi_1(0) = 0$, $\varphi_2(0) \neq 0$ (the other cases are handled in a similar way).

Since the function $\varphi_1(p)$ is analytic and $\varphi_1(0) = 0$ we have $|\varphi_1(p)| \leq C|p|$ for some $C > 0$. By virtue of Lemmas 5.8, 5.9 and 1.3 the kernel of the operator $T_{12}(z)$, $z \leq 0$ is estimated by

$$C\left(\frac{\chi_1(p)}{|p|} + 1\right)\left(\frac{|q|\chi_1(p)\chi_3(q)}{p^2 + q^2} + 1\right)\left(\frac{\chi_3(q)}{|q|^2} + 1\right),$$

where $\chi_1(p)$ is the characteristic function of $U_1(0)$. Since the latter function is square integrable on $(T^3)^2$ we have that the operators $T_{12}(z)$ and $T_{21}(z) = T_{12}^*(z)$ are Hilbert-Schmidt operator. The kernel function of $T_{\alpha,\beta}(z)$ is continuous in $p, q \in T^3$, $z < 0$ and square integrable on $(T^3)^2$ as $z \leq 0$. Then by the dominated convergence theorem the operator $T_{\alpha,\beta}(z)$ is continuous from the left up to $z = 0$. \hfill \Box

We are now ready for the **Proof of (i) of Theorem 2.9**. Let the conditions of Theorem 2.9 be fulfilled. By Lemmas 4.9 we have

$$N(z) = n(1, T(z)) \text{ as } z < 0$$

and by Lemma 5.1 for any $\gamma \in (0, 1)$ the number $n(1 - \gamma, T(0))$ is finite. Then for all $z < 0$ and $\gamma \in (0, 1)$ we have

$$N(z) = n(1, T(z)) \leq n(1 - \gamma, T(0)) + n(\gamma, T(z) - T(0)).$$

This relation can be easily obtained by use of the Weyl inequality

$$n(\lambda_1 + \lambda_2, A_1 + A_2) \leq n(\lambda_1, A_1) + n(\lambda_2, A_2)$$

for sum of compact operators $A_1$ and $A_2$ and for any positive numbers $\lambda_1$ and $\lambda_2$. Since $T(z)$ is continuous from the left up to $z = 0$, we obtain

$$\lim_{z \to 0} N(z) = N(0) \leq n(1 - \gamma, T(0)) \text{ for all } \gamma \in (0, 1).$$

Thus

$$N(0) \leq n(1 - \gamma, T(0)) < \infty.$$ 

The latter inequality proves the assertion (i) of Theorem 2.9. \hfill \Box

6. Asymptotics for the Number of Eigenvalues of the Operator $H$.

In this section we shall closely follow A. Sobolev’s method 23 to derive the asymptotics for the number of eigenvalues of $H$.

We shall first establish the asymptotics of $n(1, T(z))$ as $z \to -0$. Then assertion (ii) of Theorem 2.9 will be deduced by a perturbation argument based on the following lemma.
Lemma 6.1. Let $A(z) = A_0(z) + A_1(z)$, where $A_0(z)$ ($A_1(z)$) is compact and continuous in $z < 0$ ($z \leq 0$). Assume that for some function $f(\cdot)$, $f(z) \to 0$, $z \to -0$ the limit

$$\lim_{z \to -0} f(z)n(\lambda, A_0(z)) = l(\lambda),$$

exists and is continuous in $\lambda > 0$. Then the same limit exists for $A(z)$ and

$$\lim_{z \to -0} f(z)n(\lambda, A(z)) = l(\lambda).$$

For the proof of Lemma 6.1 see Lemma 4.9 of [23].

By Hypothesis 2.1 we get

$$u(p, q) = \frac{1}{2}(l_1(U, p, q) + l_2(U, q, q)) + |p|^4 + |q|^4$$

as $p, q \to 0$, and by the (3.12) and Corollary 3.6 for any sufficiently small negative $z$ we get

$$\Delta_{\mu_0}^0(p, z) = \frac{4\pi^2 \mu_0^0 \delta(0)}{\ell^3} \begin{bmatrix} n_\alpha(U, p, p) - 2z|\delta| + O(|p|^2 + |z|) \end{bmatrix}$$

as $p, z \to 0$.

where

$$n_\alpha = (l_1l_2 - l^2)/l_\beta.$$

Let $T(\delta; |z|)$ be an operator on $L^2(\mathbb{T}^3)$ with the entries

$T_{11}(\delta; |z|) = T_{22}(\delta; |z|) = 0,$

$T_{\alpha\beta}(\delta; |z|)w(p) = d_0 \int_{\mathbb{T}^3} \delta(p) \delta(q) (n_\alpha(U, p, p) + 2|z|)^{-1/4} (n_\beta(U, q, q) + 2|z|)^{-1/4} w(q)dq$,

$w \in L_2(\mathbb{T}^3)$, $\alpha \neq \beta$, $\alpha, \beta = 1, 2$,

where $\delta(\cdot)$ is the characteristic function of $\tilde{U}(0) = \{p \in \mathbb{T}^3 : |U(p, \cdot)^{1/2} p| < \delta\}$ and

$$d_0 = \frac{\det U^{1/2}}{2\pi^2 |l_1|^2 l_2^2}.$$

The main technical point to apply Lemma 6.1 is the following

Lemma 6.2. Assume Hypotheses 2.1, 2.2, and 2.4 are fulfilled and let $\mu_\alpha = \mu_0^\alpha$ for all $\alpha = 1, 2$. The operator $T(z) - T(\delta; |z|)$ belongs to the Hilbert-Schmidt class and is continuous in $z \leq 0$.

Proof. Applying the asymptotics (6.1), (6.2) and Lemmas 3.8 and A.3 one can estimate the kernel of the operator $T_{\alpha\beta}(z) - T_{\alpha\beta}(\delta; |z|)$ by

$$C[(p^2 + q^2)^{-1} + |p|^{-2/3} (p^2 + q^2)^{-1} + |q|^{-2/3} (p^2 + q^2)^{-1} + 1]$$

and hence the operator $T_{\alpha\beta}(z) - T_{\alpha\beta}(\delta; |z|)$ belongs to the Hilbert-Schmidt class for all $z \leq 0$. In combination with the continuity of the kernel of the operator in $z < 0$ this gives the continuity of $T(z) - T(\delta; |z|)$ in $z \leq 0$.

Let us now recall some results from [23], which are important in our work. Let $S_T : L_2([0, r) \times \sigma^{(2)}) \to L_2([0, r) \times \sigma^{(2)})$ be the integral operator with the kernel

$$S_{\alpha\beta}(y, t) = \frac{1}{(2\pi)^{-2} \cosh(y + r_{\alpha\beta}) + s_{\alpha\beta} t}$$

and
\[ u_{\alpha \beta} = u_{\beta \alpha} = \left( \frac{1}{l_1 l_2} \right)^{\frac{1}{2}}, r_{\alpha \beta} = \frac{1}{2} \log \frac{l_1}{l_2}, s_{\alpha \beta} = s_{\beta \alpha} = \frac{l}{\sqrt{l_1 l_2}}, \alpha \neq \beta, \alpha, \beta = 1, 2, \]
\[ r = 1/2 |\log |z||, y = x - x', t = \langle \xi, \eta \rangle, \xi, \eta \in \mathbb{S}^2, \sigma = L_2(\mathbb{S}^2), \mathbb{S}^2 \text{ being unit sphere in } \mathbb{R}^3. \]

Let $\hat{S}(y), y \in \mathbb{R}$ be the integral operator on $\sigma^{(2)}$ whose kernel depends on the scalar product $t = \langle \xi, \eta \rangle$ of the arguments $\xi, \eta \in \mathbb{S}^2$ and has the form
\[ \hat{S}_{\alpha \alpha}(y) = 0, \quad \hat{S}_{\alpha \beta}(y) = \left( 2 \pi \right)^{-2} \frac{u_{\alpha \beta} e^{ir_{\alpha \beta} y} \sinh[y(\arccoss_{\alpha \beta} t)]}{\sqrt{1 - s_{\alpha \beta}^2} \sinh(\pi x)}. \]

For $\lambda > 0$, define
\[ U(\lambda) = (4 \pi)^{-1} \int_{-\infty}^{+\infty} n(\lambda, \hat{S}(y)) dy \]
and denote $U_0 = U(1)$.

The following lemma can be proved in the same way as Theorem 4.5 in [24].

**Lemma 6.3.** The following equality
\[ \lim_{r \to \infty} \frac{1}{2} r^{-1} n(\lambda, S_r) = U(\lambda) \]
holds.

The following theorem is basic for the proof of the asymptotics (2.4).

**Theorem 6.4.** The equality
\[ \lim_{|z| \to 0} \frac{n(1, T(z))}{|\log |z||} = \lim_{r \to \infty} \frac{1}{2} r^{-1} n(1, S_r) \]
holds.

**Remark 6.5.** Since $U(\cdot)$ is continuous in $\lambda$, according to Lemma 6.1 any perturbations of the operator $A_0(z)$ defined in Lemma 6.7 which are compact and continuous up to $z = 0$ do not contribute to the asymptotics (2.4). During the proof of Theorem 6.4 we use this fact without further comments.

**Proof of Theorem 6.4.** As in Lemma 6.2 it can be shown that $T(z) - T(\delta; |z|)$, $z \leq 0$, defines a compact operator continuous in $z \leq 0$ and it does not contribute to the asymptotics (2.4).

The space of vector-functions $w = (w_1, w_2)$ with coordinates having support in $\hat{U}_\delta(0)$ is an invariant subspace for the operator $T(\delta; |z|)$.

Let $\hat{T}_0(\delta; |z|)$ be the restriction of the operator $T(\delta; |z|)$ to the subspace $L_2^{(2)}(\hat{U}_\delta(0))$. One verifies that the operator $\hat{T}_0(\delta; |z|)$ is unitarily equivalent to the following operator $T_0(\delta; |z|)$ in $L_2^{(2)}(\hat{U}_\delta(0))$ with the entries
\[ T_{11}^{(0)}(\delta; |z|) = T_{22}^{(0)}(\delta; |z|) = 0, \]
\[ (T_{\alpha \beta}^{(0)}(\delta; |z|) w)(p) = d_1 \int_{U_\delta(0)} \frac{(n_\alpha p^2 + 2|z|)^{-1/4}(n_\beta q^2 + 2|z|)^{-1/4}}{l_\alpha p^2 + 2l(p, q) + l_\beta q^2 + 2|z|} w(q) dq, \]
\[ w \in L_2(U_\delta(0)), \quad \alpha \neq \beta, \quad \alpha, \beta = 1, 2, \]
where
\[ d_1 = (2\pi^2)^{-1} l_1^{3/4} l_2^{3/4}. \]

Here the equivalence is performed by the unitary dilation
\[ Y = \text{diag}(Y_1, Y_2) : L^2(U_0(0)) \rightarrow L^2(U_0(0)), \quad (Y_\alpha f)(p) = f(U_0^{-1}p), \alpha = 1, 2. \]

The operator \( T_0(\delta; |z|) \) is unitary equivalent to the operator \( T_1(\delta; |z|) \) with entries
\[ T_{11}^{(1)}(\delta; |z|) = T_{22}^{(1)}(\delta; |z|) = 0, \]
\[ (T_{\alpha\beta}^{(1)}(\delta; |z|)w)(p) = d_1 \int_{U_i(0)} \frac{(n_\alpha p^2 + 2|z|)^{-1/4}(n_\beta q^2 + 2|z|)^{-1/4}}{l_\alpha p^2 + 2l(p, q) + l_\beta q^2 + 2} w(q) dq, \]
acting in \( L^2(U_r(0)), U_r(0) = \{p \in \mathbb{R}^3 : |p| < r\}, r = |z|^{-\frac{2}{3}}. \)

The equivalence is performed by the unitary dilation
\[ B_r = \text{diag}(B_r, B_r) : L^2(U_0(0)) \rightarrow L^2(U_r(0)), \quad (B_r f)(p) = (\frac{p}{r})^{-3/2} f(\frac{\delta}{r} p). \]

Further, we may replace
\[ (n_\alpha p^2 + 2)^{-1/4}, (n_\beta q^2 + 2)^{-1/4} \quad \text{and} \quad l_\alpha p^2 + 2l(p, q) + l_\beta q^2 + 2 \]
by
\[ (n_\alpha p^2)^{-1/4}(1 - \chi_1(p)), (n_\beta q^2)^{-1/4}(1 - \chi_1(q)) \quad \text{and} \quad l_\alpha p^2 + 2l(p, q) + l_\beta q^2, \]
respectively, since the error will be a Hilbert-Schmidt operator continuous up to \( z = 0. \)

Then we get the operator \( T_2(r) \) in \( L^2(U_r(0) \setminus U_1(0)) \) with entries
\[ T_{11}^{(2)}(r) = T_{22}^{(2)}(r) = 0, \]
\[ (T_{\alpha\beta}^{(2)}(r)w)(p) = (n_1 n_2)^{-\frac{1}{2}} d_1 \int_{U_0(0) \setminus U_1(0)} \frac{|p|^{-1/2}|q|^{-1/2}}{l_\alpha p^2 + 2l(p, q) + l_\beta q^2} w(q) dq, \]
acting in \( L^2(U_r(0) \setminus U_1(0)), r = |z|^{-\frac{2}{3}}. \)

The operator \( T_2(r) \) is unitarily equivalent to the integral operator \( S_r \) with entries \((6.3). \)

The equivalence is performed by the unitary operator
\[ M = \text{diag}\{M, M\} : L^2(U_r(0) \setminus U_1(0)) \rightarrow L^2((0, r) \times \sigma^{(2)}), \]
where \((M f)(x, w) = e^{3\pi/2} f(e^{\pi} w), x \in (0, r), w \in S^2. \]

Proof of (ii) of Theorem 2.9 Similarly to (23) we can show that
\[ (6.4) \quad \mathcal{U}_0 = U(1) \geq \frac{1}{4\pi} \int_{-\infty}^{+\infty} n(1, \mathcal{S}_0(y)) dy \geq \frac{1}{4\pi} \min \{y : 2u_{12} e^{-\frac{\pi y}{2}} > 1\}, \]

where \( \mathcal{S}_0(y), y \in \mathbb{R} \) is the \( 2 \times 2 \) matrices with the entries
\[ \hat{S}_{0\alpha\beta}(y) = \frac{u_{\alpha\beta} e^{i\pi\alpha y} \sinh(y \arcsin s_{\alpha\beta})}{s_{\alpha\beta} y \cosh(\frac{\pi}{4})} \]
in the subspace of the harmonics of degree zero (see (23)).

The positivity of \( \mathcal{U}_0 \) follows from the fact that \( u_{12} > 1 \) and \( \hat{S}_{0\alpha\beta}(0) > 1 \) and the continuity of \( \hat{S}_{0\alpha\beta}(y). \)

Taking into account the inequality (6.4) and Lemmas 4.3, 6.4, 6.3 we complete the proof of (ii) of Theorem 2.9.
Appendix A. Some properties of the function $u_p^{(\alpha)}(q)$

Lemma A.1. Let Hypothesis 2.1 be fulfilled. Then there exists a $\delta$-neighborhood $U_\delta(0) \subset T^3$ of the point $p = 0$ and an analytic function $q_\alpha(p)$ defined on $U_\delta(0)$ such that:

(i) for any $p \in U_\delta(0)$ the point $q_\alpha(p)$ is a unique non-degenerate minimum of the function $u_p^{(\alpha)}(q)$ and

$$q_\alpha(p) = -\frac{l}{l_\beta} p + O(|p|^3) \text{ as } p \to 0.$$ 

(ii) The function $m_\alpha(p) = u_p(q_\alpha(p))$ is analytic in $U_\delta(0)$ and satisfies

$$m_\alpha(p) = \frac{l_1l_2 - l^2}{2l_\beta}(Up,p) + O(|p|^4) \text{ as } p \to 0, \quad \alpha \neq \beta, \alpha, \beta = 1, 2.$$ 

Proof. (i) By Hypothesis 2.1 we obtain

$$u_0^{(\alpha)}(q) > u_0^{(\alpha)}(0), \quad q \neq 0$$

and

$$\left(\frac{\partial^2 u_0^{(\alpha)}(0)}{\partial q^{(i)} \partial q^{(j)}}\right)_{i,j=1}^3 = l_\alpha U.$$ 

Since $U$ is a positive definite matrix the function $u_0^{(\alpha)}(q)$, $\alpha = 1, 2$, has a unique non-degenerate minimum at $q = 0$, the gradient $\nabla u_0^{(\alpha)}(q)$ is equal to zero at the point $q = 0$.

Now we apply the implicit function theorem to the equation

$$\nabla u_p^{(\alpha)}(q) = 0, \quad p, q \in T^3.$$ 

Then there exists a $\delta$-neighborhood $U_\delta(0)$ of the point $p = 0$ and a vector function $q_\alpha(p)$ defined and analytic in $U_\delta(0)$ and for all $p \in U_\delta(0)$ the identity $\nabla u_p^{(\alpha)}(q_\alpha(p)) = 0$ holds.

Denote by $B(p)$ the matrix of the second order partial derivatives of the function $u_p^{(\alpha)}(q)$ at the point $q_\alpha(p)$. The matrix $B(0) = l_\alpha U$ is positive and $B(p)$ is continuous in $U_\delta(0)$ and hence for any $p \in U_\delta(0)$ the matrix $B(p)$ is positive definite. Thus $q_\alpha(p)$, $p \in U_\delta(0)$ is the unique non-degenerate minimum point of $u_p^{(\alpha)}(q)$.

The non-degenerate minimum point $q_\alpha(p)$ is an odd function in $p \in U_\delta(0)$.

Indeed, since $u(p, q)$ is even we get $u_p^{(\alpha)}(-q) = u_p^{(\alpha)}(q)$, and we obtain

$$u_{p}^{(\alpha)}(-q_\alpha(p)) = m_\alpha(p) = m_\alpha(-p) = u_p^{(\alpha)}(q_\alpha(-p)).$$

Since for all $p \in U_\delta(0)$ the point $q_\alpha(p)$ is the unique non-degenerate minimum of $u_p^{(\alpha)}(q)$ we have

$$q_\alpha(-p) = -q_\alpha(p).$$

By Hypothesis 2.1 and the Taylor expansion we get

$$(A.2) \quad u_p^{(\alpha)}(q - \frac{l}{l_\beta} p) = \frac{l_\beta}{2}(Uq,p) + \frac{l_1l_2 - l^2}{2l_\beta}(Up,p) + O(|q|^4 + |p|^4) \text{ as } q, p \to 0.$$ 

Since $u_p^{(\alpha)}(q_\alpha(p)) = \min_{q \in U_\delta(0)} u_p^{(\alpha)}(q)$ and $q_\alpha(p)$ is odd we have

$$\frac{l_\beta}{2}(Uq_\alpha(p) + \frac{l}{l_\beta} p)^2 + \frac{l_1l_2 - l^2}{2l_\beta}(Up,p) + O(|p|^4) \leq \frac{l_1l_2 - l^2}{2l_\beta}(Up,p) + O(|p|^4) \text{ as } p \to 0.$$ 

that is,
\[ l_2 (U + \frac{1}{l_3} p) \leq O(|p|^4) \quad \text{as} \quad p \in U_\delta (0). \]

This inequality is not valid if \( q_\alpha (p) \) has the asymptotics \( q_\alpha (p) + \frac{1}{l_3} p = O(|p|) \) as \( p \to 0 \). Since \( q_\alpha (p) \) is an odd analytic function, we have \( q_\alpha (p) + \frac{1}{l_3} p = O(|p|^3) \) as \( p \to 0 \).

(ii) Since the functions \( u(p, q), p, q \in \mathbb{T}^3 \) and \( q_\alpha (p), p \in U_\delta (0) \) are analytic, we have that the function \( m_\alpha (p) = u_p (q_\alpha (p)) \) is also analytic on \( p \in U_\delta (0) \).

By \( q_\alpha (p) = -\frac{1}{l_3} p + O(|p|^3) \), \( p \to 0 \) and (A.2) we get the asymptotics (A.1). \( \Box \)

Denote by \( U_\alpha (p, q) \) the function defined on \( U_\delta (0) \times \mathbb{T}^3 \) as
\[ U_\alpha (p, q) = u_p^{(\alpha)} (q + q_\alpha (p)) - m_\alpha (p), \]
where the function \( q_\alpha (p), p \in U_\delta (0) \) is defined in Lemma A.1.

**Lemma A.2.** Let Hypothesis 2.7 be fulfilled. For any \( p, q \in U_\delta (0) \) the function \( U_\alpha (p, q) \) is even and represented as
\[ U_\alpha (p, q) = \frac{l_3}{2} (Uq, q) + h(p, q), \]
where \( h(p, q) \) satisfies \( h(p, q) = h(-p, -q) \) and
\[ (A.3) \quad h(p, q) = O(|p|^3 |q|^3) + O(|q|^4) \quad \text{as} \quad |p|, |q| \to 0. \]

**Proof.** Since the functions \( m_\alpha (p), q_\alpha (p) \) are analytic in \( p \in U_\delta (0) \) and \( u(p, q) \) is real analytic on \( (\mathbb{T}^3)^2 \) we have that the function \( U_\alpha (p, q) \) is analytic in \( (p, q) \in U_\delta (0) \times \mathbb{T}^3 \).

Using the representation (A.2) and Lemma A.1 we have
\[ (A.4) \quad U_\alpha (p, q) - \frac{l_3}{2} (Uq, q) = h(p, q), \quad h(p, q) = O(|p|^4 + |q|^4) \quad \text{as} \quad p, q \to 0. \]

From the equality \( u_n^{(\alpha)} (-q) = u_n^{(\alpha)} (q) \) and oddness of \( q_\alpha (p) \) we obtain evenness of \( U_\alpha (p, q) \), that is, \( U_\alpha (p, q) = U_\alpha (-p, -q) \) for all \( p \in U_\delta (0), q \in \mathbb{T}^3 \). Then we have \( h(p, q) = h(-p, -q) \).

By Lemma A.1 for all \( p \in U_\delta (0) \) the point \( q_\alpha (p) \) is the minimum point of \( u_n^{(\alpha)} (q) \) we get from (A.4) that \( h(p, 0) = 0 \) and \( \forall h(p, 0) = \left( \frac{\partial h(p, 0)}{\partial q_1}, \ldots, \frac{\partial h(p, 0)}{\partial q_3} \right) = 0 \).

Since \( h(p, q) \) is analytic the Taylor expansion around the point \((0, 0)\) gives (A.3). \( \Box \)

**Lemma A.3.** Let Hypotheses 2.7 be fulfilled. Then there exist numbers \( C_1, C_2, C_3 > 0 \) and \( \delta > 0 \) such that the following inequalities
\[ (i) \quad C_1 (|p|^2 + |q|^2) \leq u(p, q) \leq C_2 (|p|^2 + |q|^2) \quad \text{for all} \quad p, q \in U_\delta (0), \]
\[ (ii) \quad u(p, q) \geq C_3 \quad \text{for all} \quad (p, q) \notin U_\delta (0) \times U_\delta (0) \]
hold.

**Proof.** By Hypothesis 2.1 the point \((0, 0)\) is a unique non-degenerated minimum point of \( u(p, q) \). Then by (A.2) there exist positive numbers \( C_1, C_2, C_3 \) and a \( \delta \)-neighborhood of \( p = 0 \in \mathbb{T}^3 \) so that (i) and (ii) hold true. \( \Box \)
Lemma B.1. Let
\[ u(p, q) = \sum_{i=1}^{3} \left( 3 - \cos^2(i) - \cos(p(i) - q(i)) \right), \]
and either
\[ \varphi_\alpha(p) = a_\alpha^{(0)} + \sum_{i=1}^{3} a_\alpha^{(i)} \cos p(i), \quad a_\alpha^{(j)} \in \mathbb{R}, \quad j = 0, 1, 2, 3 \]
or
\[ \varphi_\alpha(p) = a_\alpha \sum_{i=1}^{3} \sin p(i), \quad a_\alpha \in \mathbb{R}. \]
Then Hypotheses 2.1, 2.2 and 2.4 are fulfilled.

Proof. It is easy to see that Hypotheses 2.1, 2.2 and 2.4 are fulfilled.

We prove that Hypothesis 2.4 is fulfilled. Since \( u(p, q) \) and \(|\varphi_\alpha(p)|\) are even the function \( \Lambda_\alpha(p) \) is also even.

Then we get
\[ \Lambda_\alpha(p) - \Lambda_\alpha(0) = \int_{\mathbb{T}^3} \frac{2u_0^{(\alpha)}(t) - (u_p^{(\alpha)}(t) + u_{-p}^{(\alpha)}(t))}{4u_p^{(\alpha)}(t)u_{-p}^{(\alpha)}(t)u_0^{(\alpha)}(t)} [u_p^{(\alpha)}(t) + u_{-p}^{(\alpha)}(t)] \varphi_\alpha^2(t) dt - \frac{1}{4} \int_{\mathbb{T}^3} \frac{(u_p^{(\alpha)}(t) - u_{-p}^{(\alpha)}(t))^2}{u_p^{(\alpha)}(t)u_{-p}^{(\alpha)}(t)u_0^{(\alpha)}(t)} \varphi_\alpha^2(t) dt. \]

From the equality
\[ u_0^{(\alpha)}(t) - \frac{u_p^{(\alpha)}(t) + u_{-p}^{(\alpha)}(-t)}{2} = \sum_{j=1}^{3} (\cos p^{(j)} - 1)(1 + \cos t^{(j)}) \]
and (B.3) we get for all \( p \in \mathbb{T}^3, p \neq 0 \) the inequality
\[ \Lambda_\alpha(p) - \Lambda_\alpha(0) \leq \int_{\mathbb{T}^3} \frac{\sum_{j=1}^{3} (\cos p^{(j)} - 1)(1 + \cos t^{(j)}) \varphi_\alpha^2(t) dt}{u_p^{(\alpha)}(t)u_{-p}^{(\alpha)}(-t)u_0^{(\alpha)}(t)} < 0, \]
that is, the assertion (i) of Hypothesis 2.4 holds.

Since for any \( p, q \in \mathbb{T}^3, p \neq 0 \) the inequality \( u_p^{(\alpha)}(q) > 0 \) holds for any nonzero \( p \in \mathbb{T}^3 \) and \( \varphi_\alpha(0) = 0 \) the integrals
\[ \lambda_{ij}^{(1)}(p) = \int_{\mathbb{T}^3} \frac{\partial^2}{\partial p^{(i)} \partial p^{(j)}} u_p^{(\alpha)}(t) \varphi_\alpha^2(t) dt \]
and
\[ \lambda_{ij}^{(2)}(p) = 2 \int_{\mathbb{T}^3} \frac{\partial}{\partial p^{(i)}} u_p^{(\alpha)}(t) \frac{\partial^2}{\partial p^{(j)} \partial p^{(i)}} u_p^{(\alpha)}(t) \varphi_\alpha^2(t) dt, \quad i, j = 1, 2, 3, \]
are finite and hence are bounded continuous functions on \( \mathbb{T}^3 \).
Then the function $\Lambda_\alpha(p)$ is a twice differentiable function on $\mathbb{T}^3$ and
\[
\frac{\partial^2 \Lambda_\alpha(p)}{\partial p^{(i)} \partial p^{(j)}} = -\lambda^{(1)}_{ij}(p) + \lambda^{(2)}_{ij}(p), \quad i, j = 1, 2, 3.
\]

Since
\[
\frac{\partial}{\partial p^{(i)}} u^{(\alpha)}_0(t) = \sin t^{(i)}, \quad \frac{\partial^2}{\partial p^{(i)} \partial p^{(i)}} u^{(\alpha)}_0(t) = 1 + \cos t^{(i)},
\]
\[
\frac{\partial^2}{\partial p^{(i)} \partial p^{(j)}} u^{(\alpha)}_0(t) = 0, \quad i \neq j, \ i, j = 1, 2, 3,
\]
we get
\[
\frac{\partial^2 \Lambda_\alpha(0)}{\partial p^{(i)} \partial p^{(i)}} = -2 \int_{\mathbb{T}^3} \frac{\sum_{s=1, s\neq i}^3 (1 - \cos t^{(s)}) (1 + \cos t^{(i)}) \varphi^2_\alpha(t) dt}{(u^{(\alpha)}_0(t))^3},
\]
\[
\frac{\partial^2 \Lambda_\alpha(0)}{\partial p^{(i)} \partial p^{(j)}} = 2 \int_{\mathbb{T}^3} \frac{\sin t^{(i)} \sin t^{(j)} \varphi^2_\alpha(t) dt}{(u^{(\alpha)}_0(t))^3}, \quad i \neq j, \ i, j = 1, 2, 3.
\]

If $\varphi_\alpha(p)$ is defined by (B.1), then from the last two equalities we get
\[
(B.4) \quad \frac{\partial^2 \Lambda_\alpha(0)}{\partial p^{(i)} \partial p^{(i)}} < 0, \quad \frac{\partial^2 \Lambda_\alpha(0)}{\partial p^{(i)} \partial p^{(j)}} = 0, \ i \neq j, \ i, j = 1, 2, 3.
\]

If $\varphi_\alpha(p)$ is defined by (B.2), then
\[
(B.5) \quad \frac{\partial^2 \Lambda_\alpha(0)}{\partial p^{(i)} \partial p^{(i)}} = \frac{\partial^2 \Lambda_\alpha(0)}{\partial p^{(1)} \partial p^{(1)}} = \frac{\partial^2 \Lambda_\alpha(0)}{\partial p^{(2)} \partial p^{(2)}} = \frac{\partial^2 \Lambda_\alpha(0)}{\partial p^{(3)} \partial p^{(3)}} < 0,
\]
\[
\frac{\partial^2 \Lambda_\alpha(0)}{\partial p^{(i)} \partial p^{(j)}} = \frac{\partial^2 \Lambda_\alpha(0)}{\partial p^{(1)} \partial p^{(2)}} = \frac{\partial^2 \Lambda_\alpha(0)}{\partial p^{(2)} \partial p^{(3)}} > 0,
\]
\[
\frac{\partial^2 \Lambda_\alpha(0)}{\partial p^{(i)} \partial p^{(j)}} + 2 \frac{\partial^2 \Lambda_\alpha(0)}{\partial p^{(i)} \partial p^{(j)}} < 0, \quad i \neq j, \ i, j = 1, 2, 3.
\]

Using (B.4) (resp. (B.5)) we get that the matrix of the second order partial derivatives of the function $\Lambda_\alpha(p)$ at the point $p = 0$ is negative definite.

Thus the function $\Lambda_\alpha(p)$ has a non-degenerate maximum at the point $p = 0$. □

**Acknowledgement** The authors would like to thank Prof. R.A. Minlos and Dr. Z.I.Muminov for several helpful discussions about the results of the paper. This work was supported by the DFG 436 USB 113/4 Project and the Fundamental Science Foundation of Uzbekistan. S.N.Lakaev gratefully acknowledge the hospitality of the Institute of Applied Mathematics and of the IZKS of the University of Bonn.

**References**

[1] S. Albeverio, R. Høegh-Krohn, and T. T. Wu: A class of exactly solvable three–body quantum mechanical problems and universal low energy behavior, Phys. Lett. A 83 (1971), 105-109.
[2] S.Albeverio, S.N.Lakaev, K.A.Makarov: The Efimov Effect and an Extended Szego-Kac Limit Theorem, Letters in Math.Phys.V.43(1998),pp.73-85.
[3] S. Albeverio, S.N.Lakaev, Z.I. Muminov: Schrödinger operators on lattices. The Efimov effect and discrete spectrum asymptotics. Ann. Henri Poincaré. S 743–772 (2004).
[4] R.D.Amado and J.V.Noble: Efimov effect; A new pathology of three-particle Systems.II,Phys.Lett.B.35, No.1,25-27,(1971); Phys.Lett.D.5. No.8, (1972), 1992-2002.
[5] G. F. Dell’Antonio, R. Figari, A. Teta: Hamiltonians for systems of N particles interacting through point interactions. Ann. Inst. H. Poincaré Phys. Théor. 60 (1994), no. 3, 253-290.
[6] V. Efimov: Energy levels of three resonantly interacting particles, Nucl. Phys. A 210 (1973), 157–158.
[7] G. M. Graf and D. Schenker: 2-magnon scattering in the Heisenberg model, Ann. Inst. H. Poincaré Phys. Théor. 67 91–107 (1997).
[8] Faria da Veiga P. A., Ioriatti L., and O’Carroll M.: Energy-momentum spectrum of some two-particle lattice Schrödinger Hamiltonians. Phys. Rev. E (3) 66, 016130, 9 pp. (2002).
[9] K. O. Friedrichs: Perturbation of spectra in Hilbert space. 1965, American Mathematical Society Providence, Rhode Island.
[10] L. D. Faddeev: On a model of Friedrichs in the theory of perturbations of the continuous spectrum. (Russian) Trudy Mat. Inst. Steklov 73 1964 292–313.
[11] L. D. Faddeev and S. P. Merkuriev: Quantum scattering theory for several particle systems. Kluwer Academic Publishers, 1993.
[12] S.N. Lakaev: On an infinite number of three-particle bound states of a system of quantum lattice particles, Theor. and Math. Phys. 91 (1992), No. 1, 362–372.
[13] S.N. Lakaev: The structure of resonances of the generalized Friedrichs model. (Russian) Funktsional. Anal. i Prilozhen. 17 (1983), no. 4, 88–89.
[14] S.N. Lakaev: The Efimov’s Effect of a system of Three Identical Quantum lattice Particles, Funkcionalnii analiz i ego priloj. , Theor. Math. Phys. 91 (1992), No. 1, 362-372.
[15] S.N. Lakaev and J.I. Abdullaev: The spectral properties of the three-particle difference Schrödinger operator, Funct. Anal. Appl. 33 (1999), No. 2, 84-88.
[16] V.A.Malishew and R.A.Minlos: Linear infinite-particle operators. Translations of Mathematical Monographs, 143. American Mathematical Society, Providence, RI, 1995.
[17] D.C.Mattis: The few-body problem on lattice, Rev. Modern Phys. 58 (1986), No. 2, 361-379
[18] A.I. Mogilner: Hamiltonians of solid state physics at few-particle discrete Schrödinger operators: problems and results. Advances in Sov. Math., 5 (1991), 139-194
[19] Yu.N. Ovchinnikov and I.M. Sigal: Number of bound states of three-particle systems and Efimov’s effect, Ann. Physics, 123 (1989), 274-295
[20] M. Reed and B. Simon: Methods of modern mathematical physics. III: Scattering theory. Academic Press, N.Y., 1979.
[21] M. Reed and B. Simon: Methods of modern mathematical physics. IV: Analysis of Operators. Academic Press, N.Y., 1979.
[22] A.V. Sobolev: The Efimov effect. Discrete spectrum asymptotics, Commun. Math. Phys. 156 (1993), 127–168.
[23] H. Tamura: The Efimov effect of three-body Schrödinger operator, J. Funct. Anal. 95 (1991), 433–459.
[24] H. Tamura: Asymptotics for the number of negative eigenvalues of three-body Schrödinger operators with Efimov effect. Spectral and scattering theory and applications, 311–322, Adv. Stud. Pure Math., 23. Math. Soc. Japan, Tokyo, 1994.
[25] D. R. Yafaev: On the theory of the discrete spectrum of the three-particle Schrödinger operator, Math. USSR-Sb. 23 (1974), 535-559.
[26] D. R. Yafaev: Mathematical scattering theory: general theory. Translations of mathematical Monographs, 105. American Mathematical society, Providence 1992.
[27] D. R. Yafaev: Scattering theory: Some old and new problems, Lecture Notes in Mathematics, 1735. Springer-Verlag, Berlin, 2000, 169 pp.

1 INSTITUT FÜR ANGEWANDTE MATHEMATIK, UNIVERSITÄT BONN, WEGELERSTR. 6, D-53115 BONN (GERMANY)
2 SFB 611, BONN, BIBOS, BIELEFELD - BONN;
3 CERFIM, LOCARNO AND ACC. ARCH.USI (SWITZERLAND) E-MAIL: ALBEVERIO@UNI.BONN.DE
4 SAMARKAND STATE UNIVERSITY, UNIVERSITY BOULEVARD 15, 703004, SAMARKAND (UZBEKISTAN) E-MAIL: SLAKAEV@MAIL.RU
5 SAMARKAND DIVISION OF ACADEMY OF SCIENCES OF UZBEKISTAN (UZBEKISTAN)