Online Decision Making with Nonconvex Local and Convex Global Constraints

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Abstract

We study the online decision making problem (ODMP) as a natural generalization of online linear programming. In ODMP, a single decision maker undertakes a sequence of decisions over \(T\) time steps. At each time step, the decision maker makes a locally feasible decision based on information available up to that point. The objective is to maximize the accumulated reward while satisfying some convex global constraints called goal constraints. The decision made at each step results in an \(m\)-dimensional vector that represents the contribution of this local decision to the goal constraints. In the online setting, these goal constraints are soft constraints that can be violated moderately. To handle potential nonconvexity and nonlinearity in ODMP, we propose a Fenchel dual-based online algorithm. At each time step, the algorithm requires solving a potentially nonconvex optimization problem over the local feasible set and a convex optimization problem over the goal set. Under certain stochastic input models, we show that the algorithm achieves \(O(\sqrt{mT})\) goal constraint violation deterministically, and \(\tilde{O}(\sqrt{mT})\) regret in expected reward. Numerical experiments on an online knapsack problem and an assortment optimization problem are conducted to demonstrate the potential of our proposed online algorithm.

1 Introduction

In this paper, we consider a broad class of online decision making problems (ODMPs) that includes online linear programming (OLP) and online resource allocation problems \cite{2, 9, 13, 17, 19, 25, 28, 37} as special cases. In ODMP, the decision maker has to make a sequence of irrevocable decisions over \(T\) time steps without observing the future, where each decision is associated with a reward. The decision maker tries to maximize the accumulated reward while meeting some long-term goals.

The offline version of the ODMP considered in this paper can be formulated as follows:

\[
\begin{align*}
    z^* &= \max_{r, x, y} \sum_{t=1}^{T} r^t \\
    \text{s.t. } & (r^t, x^t, y^t) \in \Omega^t, \quad t = 1, \ldots, T, \\
    & \sum_{t=1}^{T} y^t \in T\Psi.
\end{align*}
\]

In this formulation, there is a set of local decision variables \((r^t, x^t, y^t)\) associated with each time step \(t \in T := \{1, \ldots, T\}\), where the variable \(r^t \in \mathbb{R}\) stands for the contribution of the local decisions associated with time step \(t\) to the objective function. Constraints (1c) are the global constraints
that aggregate the impact of the local decisions and enforce them to jointly meet some long-term goals, and $T\Psi$ denotes the set $\{Ty : y \in \Psi\}$. The goal set $\Psi$ as well as the impact variables $y^t$ for all $t \in \mathcal{T}$ have dimension $m$, whereas the variables $x^t$ and consequently the sets $\Omega^t$ might have different dimensions for different $t \in \mathcal{T}$. As variables $x^t$ do not appear in the objective or in the global goal constraints, one may consider variables $x^t$ as auxiliary variables for the purpose of modeling. Set $\Psi$ is assumed to be a closed convex set, while sets $\Omega^t$ are assumed to be compact (closed and bounded). Throughout the paper, we assume that $m \leq T$. This formulation can be viewed as an extended formulation for separable optimization problems of the form

$$\max_{r,x,y} \sum_{t=1}^{T} h^t(x^t)$$

s.t. $x^t \in X^t$, $t = 1, \ldots, T$,

$$\sum_{t=1}^{T} g^t(x^t) \in T\Psi$$

obtained by setting $r^t = h^t(x^t)$ and $y^t = g^t(x^t)$, where functions $h^t$ and $g^t$ are not necessarily convex. The local constraint set $\Omega^t$ in (1b) above therefore captures the relationships among decision $x^t$, reward $r^t$ and impact $y^t$ at time step $t$, allowing for flexible modeling of both discreteness and nonlinearity.

In the online setting, the decision maker is given a feasible set $\Omega^t$ at each time step $t \in \mathcal{T}$, and they make a local decision $(r^t, x^t, y^t) \in \Omega^t$ whose result is a reward of $r^t$ and an impact vector of $y^t$ on the goal constraints for period $t$. When making the decision at time step $t$, the decision maker has the knowledge of the previous (locally feasible) decisions and their accumulated contribution to the objective function and goal constraints. At time $t$, the decision maker does not have access to $\Omega^\tau$ for $\tau > t$.

In our ODMP framework, we assume that the local constraints $(r^t, x^t, y^t) \in \Omega^t$ are hard constraints that must be satisfied for each $t \in \mathcal{T}$. On the other hand, the global goal constraints (1c), which require the average impact $\frac{1}{T} \sum_{t=1}^{T} y^t$ to be in $\Psi$, are treated as soft constraints that could possibly be violated at the end of the time horizon. This assumption is necessary in our general ODMP setting as one has to make sequential decisions without full knowledge of the problem and consequently cannot guarantee feasibility of the overall decisions.

As the long-term goal constraints are soft constraints that can be violated, we employ the following two metrics to evaluate the quality of a given solution $(\hat{r}^t, \hat{x}^t, \hat{y}^t)_{t=1}^{T}$:

(i) **Reward** = $\sum_{t=1}^{T} \hat{r}^t$, which measures the accumulated reward.

(ii) **GoalVio** = $\text{dist}_2(\sum_{t=1}^{T} \hat{y}^t, T\Psi)$, which measures the final deviation from the goals. Here, $\text{dist}_2(\cdot, \cdot)$ denotes the Euclidean distance from a point to a nonempty closed set.

### 1.1 Practical Motivation: Fairness over Time

While the goal constraints are typically used to model budget constraints in most current applications, our more general goal set $\Psi$ can help model different constraints. In particular, one can consider a setting where each entry of $y^t$ represents the impact of the decision made at time step $t$ on a particular stakeholder, and the goal constraints (1c) represent some long-term goal of fairness over time [29]. For example, in the vehicle routing problem, the decision maker might want to balance the workload of different vehicle drivers [34] in which case $\Psi$ may represent the set of fair workload distributions. In assortment optimization, each product may need to be shown with
certain minimum frequency in the long term in which case Ψ may represent the set of fair frequencies.

1.2 Main Contributions

The paper’s contributions are summarized as follows.

• We propose a simple primal-dual algorithm for solving ODMPs. At each time step $t$, the proposed algorithm solves an optimization problem over the local feasible set $\Omega^t$, and then a convex optimization problem over $\Psi$. Our algorithm is particularly compatible with mixed-integer local subproblems: when $\Omega^t$ is a mixed-integer set, the local nonconvex optimization problem at time step $t$ becomes a mixed-integer program (MIP) with only local decision variables, which in many cases can be efficiently solved using modern MIP techniques or off-the-shelf MIP solvers (e.g., [21]). From the dual perspective, the algorithm can be interpreted as an online gradient descent algorithm for a particular dual multiplier learning problem. For people of independent interests, the proposed algorithm can also be used as an approximation algorithm for solving loosely coupled large-scale offline optimization problems of the form (1).

• We show that, under some mild assumptions, the algorithm deterministically ensures a sublinear $O(\sqrt{mT})$ goal violation (i.e., $\text{GoalVio}$). When comparing against the reward of the optimal offline solution, the algorithm achieves $O(\sqrt{mT})$ regret (i.e., $\text{Reward} - z^*$) in expectation under the uniform random permutation model (specified in Section 3.1). We also demonstrate that the results can be generalized to some grouped random permutation models if the grouping of time steps is “almost even” (see Section 3.3).

• We test the proposed algorithm on online knapsack problems and online assortment problems to show some practical tradeoffs that should be taken into consideration when implementing the algorithm in practice.

1.3 Related Work

Online optimization problems have been receiving significant attention in recent years, with various well-known problems falling under this category. These include online bipartite matching [24], online routing [4], single-choice [15] and multiple-choice [26] secretary problems, online advertising [36], online knapsack [8] and OLP [13]. Most of the early studies focus on worst-case analysis. More recently, the focus has shifted towards less pessimistic stochastic settings where the random permutation model is used [20]. For example, in the context of online bipartite matching [18], it is shown that a competitive ratio of $1 - 1/e$ can be achieved by the greedy algorithm under the uniform random permutation model, which is better than the pessimistic worst-case competitive ratio of 1/2.

One of the most relevant and well-studied special cases of ODMP is the standard (multiple-choice packing) OLP. In OLP, the goal set is defined by $\Psi = \{y : y \leq d\}$ with $d > 0$, and the local feasible set $\Omega^t$ takes the form

$$\Omega^t = \{ (r^t, x^t, y^t) : r^t = (\alpha^t)^\top x^t, x^t \in \Delta, y^t = A^t x^t \},$$

(2)

where $A^t$ and $\alpha^t$ are nonnegative and $\Delta$ is the standard simplex $\{x \geq 0 : \sum_j x_j \leq 1\}$ representing a multiple-choice setting (with a void choice of $x = 0$). Some extensions of [2] involve mixed packing and covering constraints as well as convex objective functions [5, 6, 42].
There has been a stream of work on OLP under the random permutation model \cite{2, 17, 19, 25, 28, 37}. Existing works \cite{1, 2, 19, 25, 37} in OLP have also shown that if every entry of the vector $T d$ is sufficiently large, then there exist OLP algorithms achieving $1 - \epsilon$ competitive ratio in expected reward with respect to the optimal offline solution. We do not employ competitive ratio analysis in this paper but instead use regret-like metrics following \cite{1} and \cite{28} because the optimal reward $z^*$ in general ODMPs may even be negative, making the competitive ratio inapplicable in general.

Note that in most papers addressing online resource allocation, the metric $\text{GoalVio}$ is often overlooked. This is because obtaining feasible solutions for resource allocation problems is often easy. For example, in packing OLP, even if $\sum_{t=1}^{\tau} \hat{y}^t \not\leq \tau d$ at time step $\tau$, it is easy to “recover from failure” by choosing void decisions, i.e., $x^t = 0$, for all $t \geq \tau + 1$ as long as $\sum_{t=1}^{\tau} \hat{y}^t \leq T d$. Unfortunately, due to our general goal set and the absence of void decisions, this does not necessarily hold for general ODMPs, and therefore, it is necessary to consider $\text{GoalVio}$ in our case. We refer to \cite{1, 19} for examples where only almost feasible solutions can be guaranteed when the problem has general constraints other than packing constraints.

The formulation settings of ODMP in this paper are most similar to the settings of online stochastic convex programming \cite{1}, where a general convex long-term constraint is considered. In comparison, our approach is more practical in the sense that it does not rely on the estimation of a parameter $Z$ that represents the trade-off between the offline optimal objective value and the goal violation. Even though estimating such a parameter $Z$ can be easy for OLP, it is difficult in general especially in cases when sets $\Omega^t$ are nonconvex. \cite{1} propose to exactly solve a scaled version (with $t$ time steps) of the partially convexified problem \cite{1} at time points $t = 1, 2, 4, \ldots, 2^{\lceil \log(T) \rceil - 1}$ to estimate $Z$ in general. This can be particularly hard for problems with nonconvex local feasible sets $\Omega^t$ as convexification of these sets and exactly solving large-scale optimization problems can be computationally hard in general, hindering the practical application of the algorithms by \cite{1} on nonconvex ODMPs. Our method is instead easy to implement, requires solving only local problems, and leads to a deterministic bound on $\text{GoalVio}$ and a stronger guarantee on the $\text{Reward}$ regret (getting rid of a factor of $Z$), under slightly stronger assumptions on $\Psi$ and $(\Omega^t)_{t=1}^T$.

Our approach is motivated by the fast primal-dual approaches recently developed for OLPs \cite{28} and online resource allocation problems \cite{9}. One particularly useful property of these fast primal-dual approaches is that they only require solving local problems rather than scaled problems, which is of high value because current computational approaches for dealing with many real-world decision making problems have limited scalability. By considering a general dual multipliers learning problem, we extend these existing approaches to deal with more general ODMPs where (i) void decisions may not exist (we relax this assumption by requiring strong $\Psi$-feasibility for $\text{conv}(\Omega^t)$, see Section \ref{sec:approach}, which is a weaker assumption), and (ii) goal constraints may be general convex constraints rather than packing constraints.

## 2 The Dual Multipliers Learning Problem

In this section, we present an algorithm for the offline ODMP \cite{1} with $(\Omega^t)_{t=1}^T$ fixed. We analyze the performance of this algorithm in the online setting in the next section. Fenchel duality is a major tool we use for deriving the proposed algorithm. A main motivation for considering the dual of \cite{1} is the fact that duality gap of a separable nonconvex optimization problem relatively diminishes as the number of separable terms increases \cite{3} \cite{11}. In this section, we show how ODMP \cite{1} is connected to a dual multiplier learning problem and derive some results from the dual perspective.
2.1 Basic Assumptions and Fenchel Duality

We first state several assumptions on sets $\Omega^t$ for $t \in T$ and $\Psi$. 

**A1.** Problem \( (1) \) is feasible. Sets \((\Omega^t)_{t=1}^T \) are compact. Set $\Psi \subseteq \mathbb{R}^m$ is *Motzkin decomposable*, i.e., there exists a compact convex set $Q$ and a closed convex cone $C$ such that $\Psi = Q + C$. Here, $Q + C$ denotes the Minkowski sum of sets $Q$ and $C$.

**A2.** There exist constants $d_y, d_r \in \mathbb{R}^+$ and $d > 0$ such that for all $t \in T$,

- **A2(a).** $\max_{v \in Q} \|y^t - v\|_\infty \leq d_y$ for all $(r^t, x^t, y^t) \in \Omega^t$, and
- **A2(b).** there exists $(\tilde{r}^t, \tilde{x}^t, \tilde{y}^t) \in \text{conv}(\Omega^t)$ satisfying $\tilde{r}^t \geq \max_{(r^t, x^t, y^t) \in \Omega^t} r^t - d_r, \{\tilde{y}^t\} + dB_m \subseteq \Psi$ with $B_m$ denoting the $\ell_2$-norm unit ball in $\mathbb{R}^m$.

Note that assumption **A2** implies that set $\Psi$ is a full-dimensional set with diameter at least $d$. For simplicity, we treat $d_y, d_r$ as $O(1)$ constants and $d$ as a $\Omega(1)$ constant (independent of $m$ and $T$) in this paper. One can easily translate our results to obtain bounds in terms of these parameters. Also note that one can always rescale $(y^t)_{t \in T}$ and $(r^t)_{t \in T}$ variables so that $d_y = d_r = 1$ while the ratio between $d_y$ and $d$ would not change under rescaling, and our analysis will show that the performance guarantee of the proposed algorithm degrades as $d$ decreases.

Assumption **A2** implies the existence of *strongly $\Psi$-feasible* solutions $\tilde{y}^t \in \text{proj}_{\Psi, \text{conv}(\Omega^t)}$ (i.e., $\tilde{y}$ lying in the interior of $\Psi$, see Figure 1). It is a generalization of the existence of void decisions in OLPs (since $0$ is strictly contained in $\Psi = \{y : y \leq d\}$ for $d > 0$). Note that $(\tilde{r}^t, \tilde{x}^t, \tilde{y}^t)$ can be picked in $\text{conv}(\Omega^t)$ rather than $\Omega^t$. This offers us significant flexibility in formulating a sequential decision-making problem as an ODMP. For instance, when \( (1) \) is a MIP and \( (1c) \) represents a fairness-over-time constraint \[30\], assumption **A2** requires the existence of a *strongly $\Psi$-feasible but fractional* solution at each time step, rather than a *strongly $\Psi$-feasible and integer* solution. The next example shows that **A2** is usually not a strong assumption for fairness-oriented applications.

**Example 1** (Online Fair Assignment). *The classical assignment problem seeks to find the most profitable assignment of tasks to agents. Consider the fair sequential (uncapcitated) assignment problem of the following form:*

\[
\begin{align*}
\max_{x} & \quad \sum_{t=1}^{T} \sum_{i=1}^{m} \sum_{j=1}^{n_t} q_{ij}^t x_{ij}^t \quad & (3a) \\
\text{s.t.} & \quad \sum_{i=1}^{m} x_{ij}^t = 1, & t \in T, \; j = 1, \ldots, n_t, \quad (3b) \\
& \quad x_{ij}^t \in \{0, 1\}, & t \in T, \; i = 1, \ldots, m, \; j = 1, \ldots, n_t, \quad (3c)
\end{align*}
\]
Here \( n_t \) denotes the number of tasks received at time step \( t \). At each time step \( t \), for each agent \( i \), \( q^t_{ij} \geq 0 \) and \( w^t_{ij} \geq 0 \) denote the profit and the workload of task \( j \) (if assigned to agent \( i \)), respectively. Constraints (3d) state that the total workload of different agents should be distributed fairly in the long term where the set \( \Psi \) defines the set of “fair” workload distributions.

When \((q^t, w^t)_{t=1}^T\) arrive online, one may formulate (3) as an ODMP by introducing variables \( r^t \) and \( y^t \) and defining

\[
\Omega^t = \left\{(r^t, x^t, y^t) : r^t = \sum_{i=1}^{m} \sum_{j=1}^{n_t} q^t_{ij} x^t_{ij}, \quad y^t = \left(\sum_{j=1}^{n_t} w^t_{ij} x^t_{ij}\right)_{i=1}^{m}, \quad \sum_{i=1}^{m} x^t_{ij} = 1 \text{ for } j = 1, \ldots, n_t, \quad x^t \in \{0, 1\}^{m \times n_t}\right\}.
\]

Suppose set \( \Psi \) is defined by a max-min gap, i.e.,

\[
\Psi = \left\{y : \max_i y_i - \min_i y_i \leq \rho\right\}
\]

for some positive constant \( \rho \). Then one can verify that assumptions \( A1 \) and \( A2 \) hold. In particular, even though a fair assignment may not exist in \( \Omega^t \) for a single time step \( t \) (i.e., \( \text{proj}_y \Omega^t \cap \Psi = \emptyset \)), one can always find a “perfectly fair” assignment from \( \text{conv}(\Omega^t) \) (i.e., \( (\tilde{r}^t, \tilde{x}^t, \tilde{y}^t) \in \text{conv}(\Omega^t) \) satisfying \( \max_i \tilde{y}^t_i = \min_i \tilde{y}^t_i \)). See Appendix A for details.

We apply Fenchel duality [41] to the following relaxation of (1) obtained by individually, i.e., for each \( t \), convexifying the local feasible sets \( \Omega^t \) as

\[
z^R := \max_{r,x,y} \left\{ \sum_{t=1}^{T} r^t : (r^t, x^t, y^t) \in \text{conv}(\Omega^t) \right\},
\]

where \( \text{conv}(\Omega^t) \) denotes the convex hull of \( \Omega^t \). We can reformulate (4) as follows:

\[
z^R = \max_{y^1, \ldots, y^T} \sum_{t=1}^{T} f^t(y^t) - \delta_{T\Psi}\left(\sum_{t=1}^{T} y^t\right),
\]

where for each \( t \in T \), the function \( f^t : \mathbb{R}^m \to \mathbb{R} \cup \{\pm \infty\} \) is a concave function defined as

\[
f^t(y^t) := \max_{r^t, x^t} \left\{ r^t : (r^t, x^t, y^t) \in \text{conv}(\Omega^t) \right\},
\]

and \( \delta_{T\Psi} \) is the indicator function of \( T\Psi \) defined as

\[
\delta_{T\Psi}(y) = \begin{cases} 0 & \text{if } y \in T\Psi, \\ +\infty & \text{otherwise.} \end{cases}
\]
Then, the Fenchel dual of (5) becomes

$$z^F = \inf_{p} T h_{\Psi}(p) - \sum_{t=1}^{T} (f^t)^*(p),$$

(6)

where $h_{\Psi}$ is the support function of $\Psi$ defined by

$$h_{\Psi}(p) = \sup_{v} \{ p^\top v : v \in \Psi \},$$

and for $t \in T$, the conjugate $(f^t)^*$ of $f^t$ is defined as

$$(f^t)^*(p) := \min_{r^t, x^t, y^t} \{ p^\top y^t - r^t : (r^t, x^t, y^t) \in \text{conv}(\Omega^t) \}$$

$$= \min_{r^t, x^t, y^t} \{ p^\top y^t - r^t : (r^t, x^t, y^t) \in \Omega^t \}.$$ 

We next present a strong duality result under assumptions $A1$ and $A2$.

**Lemma 1.** Suppose assumptions $A1$ and $A2$ hold. Then, $z^R = z^F$. Moreover, there exist $(\bar{y}_t^T)_{t=1}^{T} \in \prod_{t=1}^{T} \text{conv}(\Omega^t)$ and $p^* \in \mathbb{R}^m$ such that $(\bar{y}_t^T)_{t=1}^{T}$ optimizes (5) and $p^*$ optimizes (6).

**Proof.** Note that by definition of $(f^t)^*$ and compactness of $\Omega^t$ from assumption $A1$

$$\text{dom} \left( \sum_{t=1}^{T} (f^t)^* \right) = \mathbb{R}^m \supseteq \text{dom}(T h_{\Psi}),$$

where $\text{dom}(\cdot)$ denotes the effective domain of a function. By [41, Corollary 31.2.1], it suffices to show that there exists $(y_t^T)_{t=1}^{T} \in \text{ri}(\text{proj}_Y(\prod_{t=1}^{T} \text{conv}(\Omega^t)))$ such that $(y_t^T)_{t=1}^{T} \in \text{ri}(T \Psi)$, where $\text{ri}(\cdot)$ denotes the relative interior of a set. By assumption $A2[b]$ we have

$$\sum_{t=1}^{T} \tilde{y}_t^T + \frac{T d}{2} B_m \subseteq \text{ri}(T \Psi).$$

(7)

Pick $(\tilde{y}_t^T)_{t=1}^{T} \in \text{ri}(\text{proj}_Y(\prod_{t=1}^{T} \text{conv}(\Omega^t)))$. Then we have

$$(y_\lambda^T)_{t=1}^{T} := (1 - \lambda)(\tilde{y}_t^T)_{t=1}^{T} + \lambda (y_t^T)_{t=1}^{T} \in \text{ri}(\text{proj}_Y(\prod_{t=1}^{T} \text{conv}(\Omega^t)))$$

for all $\lambda \in (0, 1]$ by [41, Theorem 6.1], and $\sum_{t=1}^{T} y_\lambda^t \in \text{ri}(T \Psi)$ for some small enough $\lambda > 0$ due to (7). The conclusion then follows. 

2.2 Learning the Dual Multipliers

Finding the optimal dual multiplier $p^*$ for the Fenchel dual problem (6) is nontrivial due to its large size and the potential nonconvexity of $(\Omega^t)_{t=1}^{T}$. However, one can connect the dual problem to an online convex optimization (OCO) problem [2]. Specifically, for all $t \in T$, define a convex cost function $\bar{z}^t(p) = h_{\Psi}(p) - (f^t)^*(p)$. Note that $\bar{z}^t(p) = +\infty$ if $p \notin \mathcal{C}^o$ by assumption $A1$. The Fenchel dual problem (5) is equivalent to $\inf_{p \in \mathcal{C}^o} \sum_{t=1}^{T} \bar{z}^t(p)$. The OCO problem associated with
Algorithm 1 A Fenchel Dual-Based Algorithm for ODMP

1: Initialize $p^1 = 0$
2: for $t = 1, \ldots, T$ do
3:   Solve the following problem:
4:   
$$\left(\hat{r}^t, \hat{x}^t, \hat{y}^t\right) \in \arg\max \left\{ r^t - (p^t)\top y^t : (r^t, x^t, y^t) \in \Omega^t \right\}$$
5:   Solve $\hat{v}^t \in \arg\max_{v \in Q}(p^t)\top v$
6:   Set $p^{t+1} = \text{proj}_{C^o}(p^t - \eta^t(\hat{v}^t - \hat{y}^t))$
7: end for

the Fenchel dual is defined as follows: at each time step $t$, one picks $p^t$ using information observed up to time step $t - 1$ and suffers a loss defined by $\tilde{z}^t(p^t)$. The objective of the OCO problem is to sequentially pick $p^t$ in a way that minimizes the following regret:

$$\text{DualRegret} = \sum_{t=1}^{T} \tilde{z}^t(p^t) - \inf_{p \in C^o} \sum_{t=1}^{T} \tilde{z}^t(p) = \sum_{t=1}^{T} \tilde{z}^t(p^t) - z^R.$$ 

One of the simplest algorithms for solving OCO problems is online gradient descent (OGD) [43]. The OGD algorithm applies a (sub)gradient descent step, with respect to the current cost function, to update the decision at each time step. For our dual multiplier learning problem, OGD translates to Algorithm 1. Indeed, given solution $(\hat{r}^t, \hat{x}^t, \hat{y}^t) \in \arg\max \left\{ r^t - (p^t)\top y^t : (r^t, x^t, y^t) \in \Omega^t \right\}$ and $\hat{v}^t \in \arg\max\{(p^t)\top v : v \in \Phi\}$, we have $\hat{v}^t - \hat{y}^t \in \partial \tilde{z}^t(p^t)$, where $\partial \tilde{z}^t(p^t)$ denotes the subdifferential of $\tilde{z}^t$ at $p^t$. In step 5 of Algorithm 1, $C^o$ denotes the polar cone of $C$, i.e., $C^o = \{u : u\top v \leq 0 \text{ for all } v \in C\}$, $\eta^t$ is the algorithm stepsize at time step $t$, and $\text{proj}_{C^o}(\cdot)$ denotes the projection from a point onto the nonempty closed convex cone $C^o$. Throughout the paper, we assume that an optimization oracle which can efficiently solve local optimization problems [8], for example, a MIP solver in the case when [8] is a MIP, is available. Algorithm 1 is also our proposed algorithm for solving ODMPs under specific stochastic input models, and its primal performance will be discussed in Section 3.

2.3 Non-Motzkin Decomposable $\Psi$

It is possible to generalize Algorithm 1 to deal with cases when we have a general convex set $\Psi$ that is not necessarily Motzkin decomposable. In particular, assumption A2(a) implies that there exists a compact convex set $Y$ such that $\text{proj}_Y \Omega^t \subseteq Y$ for all $t \in T$. Then one must have $\sum_{t=1}^{T} y^t \in TY$. Therefore, it is equivalent to replace constraint (1c) in the ODMP by $\sum_{t=1}^{T} y^t \in T\Psi$ where $\Psi := \Psi \cap Y$. If such $Y$ is known at the beginning, then one can replace $\Psi$ by $\Psi$ in Algorithm 1 in which case $Q = \Psi$ and $C^o = \mathbb{R}^m$, to obtain an OMDP algorithm that works for non-Motzkin decomposable $\Psi$. We denote such an algorithm by Algorithm 1. However, note that the algorithm after this replacement is not necessarily equivalent to the original algorithm even in the case when $\Psi$ is Motzkin decomposable. In particular, it is possible that one obtains a $\hat{v}^t$ with a large norm in line 5 of Algorithm 1 since $Y$ is potentially a large set and so is $\Psi \cap Y$, in which case $p^t$ changes significantly from iteration to iteration. Despite that we can generate theoretical guarantees for Algorithm 1 similar to the ones we can have for Algorithm 1 under proper assumptions, the difference in their empirical performance can be large. We will make a brief comparison of their numerical performance in Section 4.
2.4 Deterministic Bounds on the Dual Side

We first show some deterministic bounds for dual solutions generated by Algorithm 1. The following lemma bounds the \(l_2\)-norm of the dual multipliers when the stepsizes \(\eta^t\) are small enough, which is crucial for deriving later results. Similar bounds are derived for the primal-dual OLP algorithm by [28].

**Lemma 2.** Suppose assumptions [A1] and [A2] hold and let \(p^*\) be an optimal solution to (6). Then,

\[
\|p^*\|_2 \leq \frac{d_r}{d} = O(1).
\]

Moreover, if \(0 \leq \eta^t \leq \frac{1}{m}\) for all \(t\), then Algorithm 1 produces \((p^t)_{t=1}^{T+1}\) with

\[
\max_t \|p^t\|_2 \leq \frac{d_y^2 + 2d_r}{2d} + \frac{d_y}{\sqrt{m}} = O(1).
\]

**Proof.** Let \((\hat{r}^t, \hat{x}^t, \hat{y}^t)_{t=1}^{T}\) and \(p^*\) be such that Lemma 1 is satisfied, and \((\tilde{r}^t, \tilde{x}^t, \tilde{y}^t)_{t=1}^{T}\) be such that assumption [A2] holds. Then we have

\[
\sum_{t=1}^{T} (\tilde{r}^t + \|p^*\|_2) \leq \sum_{t=1}^{T} (\hat{r}^t + h_\Psi(p^*) - (p^*)^\top \hat{y}^t) \\
\leq \sum_{t=1}^{T} (h_\Psi(p^*) - (f^t)^*(p^*)) \\
= \sum_{t=1}^{T} \hat{r}^t \leq \sum_{t=1}^{T} (\tilde{r}^t + d_r),
\]

where the first inequality holds since \(T\|p^*\|_2 \leq \sum_{t=1}^{T} h_\Psi(p^*) - (p^*)^\top \hat{y}^t\) by the first inequality of Lemma 9 (in Appendix B) with \(p = p^*\), the second inequality holds due to the definition of \((f^t)^*\), and the last inequality holds by assumption [A2]. Therefore, \(\|p^*\|_2 \leq \frac{d_r}{d}\). Similarly, for all \(t\),

\[
\tilde{r}^t + \|p^t\|_2 \leq \hat{r}^t + h_\Psi(p^t) - (p^t)^\top \hat{y}^t \leq \hat{r}^t + h_\Psi(p^t) - (p^t)^\top \hat{y}^t \leq \hat{r}^t + d_r + (p^t)^\top (\hat{y}^t - \tilde{y}^t),
\]

where the second inequality holds due to optimality of \((\hat{r}^t, \hat{x}^t, \hat{y}^t)\) in line 3 of Algorithm 1, and the third inequality holds due to assumption [A2] and the optimality of \(\hat{y}^t\) in line 4 of Algorithm 1. By rearranging the second inequality in Lemma 9 (in Appendix B) with \(p = 0\), we have

\[
\|p^{t+1}\|_2^2 \leq \|p^t\|_2^2 + (\eta^t)^2 ma^2_y + 2\eta^t (p^t)^\top (\hat{y}^t - \tilde{y}^t).
\]

Combining (9) and (10), we have

\[
\|p^{t+1}\|_2^2 \leq \|p^t\|_2^2 + (\eta^t)^2 ma^2_y + 2\eta^t (d_r - \|p^t\|_2).\]

Suppose \(0 \leq \eta^t \leq \frac{1}{m}\). We have \(\|p^{t+1}\|_2 \leq \|p^t\|_2\) when \(\|p^t\|_2 \geq \frac{d^2 + 2d_r}{2d}\). On the other hand, when \(\|p^t\|_2 \leq \frac{d^2 + 2d_r}{2d}\), we have \(\|p^{t+1}\|_2 \leq \|p^t\|_2 + \eta^t \|\hat{y}^t - \tilde{y}^t\|_2 \leq \frac{d^2 + 2d_r}{2d} + \frac{d_y}{\sqrt{m}}\). Since \(p^1 = 0\), the conclusion follows from induction on \(\|p^t\|_2\).

We next present deterministic bounds on **DualRegret** and **GoalVio** when specific stepsizes...
are used.

**Theorem 3.** Suppose assumptions \( A1 \) and \( A2 \) hold and \( \eta^t = \min\{\frac{1}{m}, \frac{1}{\sqrt{mt}}\} \) for all \( t \). Then, Algorithm \( \mathbb{D} \) achieves \( \text{DualRegret} \leq O(\sqrt{mT}) \) and \( \text{GoalVio} \leq O(\sqrt{mT}) \).

**Proof.** Let \( \mathbf{p}^* \) be an optimal dual multiplier that satisfies Lemma \( \mathbb{A} \). For \( t \geq 1 \), by the second inequality of Lemma \( \mathbb{B} \) (in Appendix \( \mathbb{B} \)) with \( \mathbf{p} = \mathbf{p}^* \), we have

\[
(p^t - p^*)^T (v^t - \hat{y}^t) \leq \|p^t - p^*\|_2^2 - \|p^{t+1} - p^*\|_2^2 + \frac{\eta^t}{2} m d_y.
\]

(11)

By optimality of \( (\hat{t}^t, \hat{x}^t, \hat{y}^t) \) in line 3 of Algorithm \( \mathbb{A} \),

\[
\hat{z}^t(p^t) - \hat{z}^t(p^*) \leq (\hat{t}^t - (\hat{p}^t)^T \hat{y}^t + h_{\Psi}(p^t)) - (\hat{t}^t - (\hat{p}^t)^T \hat{y}^t + h_{\Psi}(p^*)) \leq (\hat{p}^t - p^*)^T (\hat{v}^t - \hat{y}^t).
\]

Together with (11) and boundedness results of dual multipliers in Lemma \( \mathbb{C} \), we have

\[
\text{DualRegret} = \sum_{t=1}^{T} (\hat{z}^t(p^t) - \hat{z}^t(p^*))
\]

\[
\leq \sum_{t=1}^{T} (p^t - p^*)^T (v^t - \hat{y}^t)
\]

\[
\leq \sum_{t=1}^{T} \|p^t - p^*\|_2^2 - \|p^{t+1} - p^*\|_2^2 + \sum_{t=1}^{T} \frac{\eta^t}{2} m d_y
\]

\[
\leq \|p^*\|_2^2 + \sum_{t=2}^{T} (\frac{1}{\eta^t} - \frac{1}{\eta^{t-1}}) \|p^t - p^*\|_2^2 + \sum_{t=1}^{T} \frac{\eta^t}{2} m d_y
\]

\[
\leq O(m) + O(\sqrt{mT}) = O(\sqrt{mT}).
\]

On the other hand, by Lemma \( \mathbb{E} \) (in Appendix \( \mathbb{B} \)), there exists \( \hat{u}^t \in C \) such that \( p^{t+1} = p^t - \eta^t (v^t - \hat{y}^t) - \hat{u}^t \) for \( t = 1, \ldots, T \). Therefore,

\[
\sum_{t=1}^{T} \hat{y}^t - \sum_{t=1}^{T} \left( \hat{v}^t + \frac{\hat{u}^t}{\eta^t} \right) = \sum_{t=1}^{T} \frac{p^{t+1} - p^t}{\eta^t} = \sum_{t=2}^{T} \left( \frac{1}{\eta^{t-1}} - \frac{1}{\eta^t} \right) p^t + \frac{1}{\eta^t} p^{T+1}.
\]

Note that \( \hat{v}^t + \frac{\hat{u}^t}{\eta^t} \in Q + C = \Psi \) for all \( t = 1, \ldots, T \). Then by Lemma \( \mathbb{D} \) we have

\[
\text{GoalVio} = \text{dist}_2 \left( \sum_{t=1}^{T} \hat{y}^t, T\Psi \right)
\]

\[
\leq \left\| \sum_{t=1}^{T} \hat{y}^t - \sum_{t=1}^{T} \left( \hat{v}^t + \frac{\hat{u}^t}{\eta^t} \right) \right\|_2
\]

\[
\leq \sum_{t=2}^{T} \left( \frac{1}{\eta^{t-1}} - \frac{1}{\eta^t} \right) \|p^t\|_2 + \frac{1}{\eta^t} \|p^{T+1}\|_2
\]
\[
\sum_{t=2}^{T} O\left(\frac{\sqrt{m}}{\sqrt{t}}\right) + O(\sqrt{mT}) = O(\sqrt{mT}).
\]

3 Primal Analysis for Algorithm 1

We have shown that dual multipliers generated by Algorithm 1 achieve a sublinear dual regret. In this section, we show that a strong primal guarantee can also be achieved by Algorithm 1 under some stochastic input models if we simply implement the primal solutions \((\hat{r}^t, \hat{x}^t, \hat{y}^t)\) obtained in line 3 of Algorithm 1 at each time step \(t\).

3.1 Stochastic Input Models

When comparing against the dynamic optimal decisions in hindsight, it is impossible to derive algorithms with a sublinear worst-case regret in reward even for OLP [17]. We adopt the convention in OLP for the underlying uncertainty by assuming that \((\Omega_t^t)_{t=1}^T\) follows a random permutation model. Specifically, there exist \(T\) (potentially unknown and adversarially chosen) deterministic feasible sets \(Z^1, \ldots, Z^T\), and the local constraint set \(\Omega^t\) observed at each time step \(t\) satisfies \(\Omega^t = Z^{\pi(t)}\) for some permutation function \(\pi\) of the set \(T\). Due to the symmetry in formulation (1), the optimal objective value \(z^*\) of (1) is invariant to the permutation \(\pi\). We consider two random permutation models, namely the uniform random permutation model and the grouped random permutation model. The uniform random permutation model, also known as the random-order model, is widely studied in the OLP context. The model assumes that the permutation function \(\pi\) is sampled from all possible \(T!\) permutations of \(T\) with equal probability. It is known to be more general than the IID model in which each \(\Omega^t\) is independently sampled from the same (potentially unknown) distribution. Due to practical concerns, we also consider the grouped random permutation model, which generalizes the uniform random permutation model by assuming that the set of time steps \(T\) is partitioned into \(K\) groups \((T^k)_{k=1}^K\). For each group \(T^k\), \((\Omega^t)_{t \in T^k}\) is a uniform random permutation of feasible sets \((Z^t)_{t \in T^k}\). The grouped random permutation model is also more general than the grouped IID model in which for \(t \in T^k\), each \(\Omega^t\) is independently sampled from a distribution associated with the group \(T^k\).

3.2 Under the Uniform Random Permutation Model

We let \(P_\pi\) and \(E_\pi\) denote the probability measure and expectation with respect to the uniform random permutation \(\pi\), respectively. As a benchmark, we consider the optimal objective value \(z^R\) of the partial convexification (4). Note that by definition \(z^R\) is at least as large as \(z^*\). Therefore, any lower bound result we derive for \(\text{Reward}\) when comparing against \(z^R\) implies a bound in terms of \(z^*\). We next establish an expected reward bound under the uniform random permutation model.

**Theorem 4.** Suppose assumptions \(A1\) and \(A2\) hold and \(\eta^t = \min\{\frac{1}{m}, \frac{1}{\sqrt{mt}}\}\) for all \(t\). Under the uniform random permutation model, Algorithm 1 achieves

\[
E_\pi[\text{Reward}] \geq z^R - O(\sqrt{m \log m} \sqrt{T}).
\]

**Proof.** Let \(p^*\) and \((\hat{r}^t, \hat{x}^t, \hat{y}^t)_{t=1}^T\) be such that Lemma 2 holds for the deterministic ODMP problem (1) with \((\Omega^t)_{t=1}^T = (Z^t)_{t=1}^T\). Then under the uniform random permutation model, with \((\Omega^t)_{t=1}^T = (Z^t)_{t=1}^T\).
$(Z^{\pi(t)})_{t=1}^T$ and $\pi$ being a uniform random permutation of $T$, we have

$$\text{Reward} = \sum_{t=1}^T (p_t)^\top \hat{y}^t + \sum_{t=1}^T (r_t - (p_t)^\top \hat{y}^t)$$

$$= \sum_{t=1}^T \left( (p_t)^\top \hat{y}^t + \max \left\{ r_t - (p_t)^\top y^t : (r_t, x^t, y^t) \in \Omega^t \right\} \right)$$

$$= \sum_{t=1}^T \left( (p_t)^\top \hat{y}^t + \max \left\{ r_t - (p_t)^\top y^t : (r_t, x^t, y^t) \in \text{conv}(\Omega) \right\} \right)$$

$$\geq \sum_{t=1}^T \pi(t) + \sum_{t=1}^T (p_t)^\top (\hat{y}^t - y^{\pi(t)})$$

$$= z^R + \sum_{t=1}^T (p_t)^\top (\hat{y}^t - \hat{v}^t) + \sum_{t=1}^T (p_t)^\top (\hat{v}^t - E_\pi[\hat{y}^{\pi(t)}]) + \sum_{t=1}^T (p_t)^\top (E_\pi[\hat{y}^{\pi(t)}] - \hat{y}^{\pi(t)}).$$

Here the second equality is due to the optimality of $(\hat{r}^t, \hat{x}^t, \hat{y}^t)$, and the first inequality holds since $(\pi, \hat{x}^t, \hat{y}^t) \in \text{conv}(\Omega)$ from Lemma 1.

We first bound $\sum_{t=1}^T (p_t)^\top (\hat{y}^t - \hat{v}^t) + \sum_{t=1}^T (p_t)^\top (\hat{v}^t - E_\pi[\hat{y}^{\pi(t)}])$. By rearranging the second inequality of Lemma 2 in Appendix 2 with $p = 0$, we have

$$\left( p_t \right)^\top (\tilde{y}^t - \hat{v}^t) \geq \frac{\left\| p_t \right\|_2^2 + \left\| p_t \right\|_2^3 - \frac{\eta}{2} m d_y^2}{2 \eta^t}.$$ 

Therefore, similar to inequality [12], we have

$$\sum_{t=1}^T \left( p_t \right)^\top (\hat{y}^t - \hat{v}^t) \geq - \sum_{t=2}^T \left( \frac{1}{2 \eta^t} - \frac{1}{2 \eta^{t-1}} \right) \left\| p_t \right\|_2^2 - \sum_{t=1}^T \frac{\eta^t}{2} m d_y^2 = -O(\sqrt{mT}).$$

Also note that $E_\pi[\hat{y}^{\pi(t)}] = T^{-1} \sum_{\pi=1}^T \hat{y}^t \in \Psi$ for all $t$ since $\pi$ is a uniform random permutation of $T$. It implies that $(p_t)^\top (\hat{v}^t - E_\pi[\hat{y}^{\pi(t)}]) = h_\Psi(p_t) - (p_t)^\top E_\pi[\hat{y}^{\pi(t)}] \geq 0$ for all $t$ by the definition of $h_\Psi$. Therefore,

$$\text{Reward} \geq z^R - O(\sqrt{mT}) + \sum_{t=1}^T (p_t)^\top (E_\pi[\hat{y}^{\pi(t)}] - \hat{y}^{\pi(t)}).$$

Let $\mathcal{F}^{t-1}$ be the sigma algebra generated by the random events up to time step $t - 1$. Next we bound the term $(p_t)^\top (E_\pi[\hat{y}^{\pi(t)}] - \hat{y}^{\pi(t)})$ conditioned on $\mathcal{F}^{t-1},$

$$E_\pi[(p_t)^\top (E_\pi[\hat{y}^{\pi(t)}] - \hat{y}^{\pi(t)}) | \mathcal{F}^{t-1}] = (p_t)^\top \left( E_\pi[\hat{y}^{\pi(t)}] - E_\pi[\hat{y}^{\pi(t)}] | \mathcal{F}^{t-1} \right)$$

$$= (p_t)^\top \left( E_\pi[\hat{y}^{\pi(t)}] - \frac{1}{T - t + 1} \sum_{\tau=t}^T \hat{y}^{\pi(\tau)} \right)$$

$$\geq - \left\| p_t \right\|_2 \left\| E_\pi[\hat{y}^{\pi(t)}] - \frac{1}{T - t + 1} \sum_{\tau=t}^T \hat{y}^{\pi(\tau)} \right\|_2^2.$$
where the first equality holds since \( p_i^{(t)} \) is determined based on \((\Omega^{(t)})_{\tau=1}^{T-1}\) (i.e., \( \mathcal{F}^{t-1} \)), the second equality is due to the fact that the conditional expectation of \( \bar{y}^{\pi(t)} \) is a sample from \( \{\bar{y}^{\pi(t)}\}_{t=1}^{T} \) with equal probability since \( \pi \) is a uniform random permutation.

As \( \pi \) is uniformly chosen at random, \((\sigma(t) = \pi(T - t + 1))_{t=1}^{T}\) is also a uniform random permutation of \( \mathcal{T} \). By Hoeffding’s inequality for sampling without replacement [23], for all \( \epsilon > 0 \) and \( i \in \{1, \ldots, m\} \) we have

\[
P_{\pi} \left( \left\| \left( \mathbb{E}_{\pi}[\bar{y}^{\pi(t)}] - \frac{1}{T-t+1} \sum_{\tau=t}^{T} \bar{y}^{\pi(\tau)} \right) \right\|_i > \epsilon \right) \leq 2 \exp \left( - \frac{(T-t+1)\epsilon^2}{2d_y^2} \right).
\]

In other words, for all \( \rho \in (0,1) \), with probability at least \( 1 - \rho/m \),

\[
\left\| \left( \mathbb{E}_{\pi}[\bar{y}^{\pi(t)}] - \frac{1}{T-t+1} \sum_{\tau=t}^{T} \bar{y}^{\pi(\tau)} \right) \right\| \leq \sqrt{ \frac{2d_y^2 \log(2m/\rho)}{T-t+1}}.
\]

Then by taking the union bound over \( i \in \{1, \ldots, m\} \), with probability at least \( 1 - \rho \),

\[
\left\| \mathbb{E}_{\pi}[\bar{y}^{\pi(t)}] - \frac{1}{T-t+1} \sum_{\tau=t}^{T} \bar{y}^{\pi(\tau)} \right\|_2 \leq \sqrt{ \frac{2md_y^2 \log(2m/\rho)}{T-t+1}} = O \left( \sqrt{ \frac{m \log m}{T-t+1} } + \sqrt{ \frac{m \log(2/\rho)}{T-t+1} } \right).
\]

(14)

By Lemma [2] and integrating the quantile function, we have

\[
\mathbb{E}_{\pi} \left[ (p_i^{(t)})^\top (\mathbb{E}_{\pi}[\bar{y}^{\pi(t)}] - \bar{y}^{\pi(t)}) \right] = \mathbb{E}_{\pi} \left[ \mathbb{E}_{\pi} \left[ (p_i^{(t)})^\top (\mathbb{E}_{\pi}[\bar{y}^{\pi(t)}] - \bar{y}^{\pi(t)}) | \mathcal{F}^{t-1} \right] \right]
\]

\[
\geq - \mathbb{E}_{\pi} \left[ \left\| p_i^\top \right\|_2 \left\| \mathbb{E}_{\pi}[\bar{y}^{\pi(t)}] - \frac{1}{T-t+1} \sum_{\tau=t}^{T} \bar{y}^{\pi(\tau)} \right\|_2 \right]
\]

\[
\geq - \int_0^1 O \left( \sqrt{ \frac{m \log m}{T-t+1} } + \sqrt{ \frac{m \log(2/\rho)}{T-t+1} } \right) d\rho
\]

\[
= - O \left( \sqrt{ \frac{m \log m}{T-t+1} } \right).
\]

(15)

It then follows that

\[
\mathbb{E}[\text{Reward}] \geq z^R + \sum_{t=1}^{T} \mathbb{E} \left[ (p_i^{(t)})^\top (\mathbb{E}_{\pi}[\bar{y}^{\pi(t)}] - \bar{y}^{\pi(t)}) \right] - O(\sqrt{mT}) \geq z^R - O(\sqrt{m \log m \sqrt{T}}).
\]

\[
\mathbb{E}[\text{Reward}] \geq z^R + \sum_{t=1}^{T} \mathbb{E} \left[ (p_i^{(t)})^\top (\mathbb{E}_{\pi}[\bar{y}^{\pi(t)}] - \bar{y}^{\pi(t)}) \right] - O(\sqrt{mT}) \geq z^R - O(\sqrt{m \log m \sqrt{T}}).
\]

The previous results can be easily generalized to cases when we instead use step sizes \( \eta_i^{(t)} = \min \{ \gamma m, \gamma \sqrt{mt} \} \) for all \( t \), where \( \gamma > 0 \) is a parameter chosen by the decision maker. We present these results in terms of reward and goal violation up to some time step \( \tau \in \{m, m+1, \ldots, T\} \). Omitted proofs can be found in Appendix [C].

**Corollary 5.** Let \( \gamma \) be an algorithmic choice. Suppose assumptions [A1] and [A2] hold and \( \eta_i^{(t)} = \min \{ \gamma m, \gamma \sqrt{mt} \} \).
\[ \min \{ \frac{\tau}{m}, \frac{\tau}{\sqrt{mt}} \} \text{ for all } t. \] Under the uniform random permutation model, Algorithm $1$ achieves

\[ \text{GoalVio}_\tau / \tau := \text{dist}_2 \left( \sum_{t=1}^{\tau} \frac{\Psi^t}{\tau}, \Psi \right) \leq O \left( (1 + 1/\gamma) \sqrt{m/\tau} \right) \]

and

\[ \mathbb{E}_\pi[\text{Reward}_\tau / \tau] := \mathbb{E}_\pi \left[ \sum_{t=1}^{\tau} r^t / \tau \right] \geq z^R / T - \tilde{O} \left( (1 + \gamma) \sqrt{m/\tau} \right) \]

for all $\tau \in \{m, m + 1, \ldots, T\}$.

Corollary 6 shows that the stepsize parameter $\gamma$ controls the tradeoff between GoalVio and Reward. When $\gamma$ is close to 0 and $t$ is small, $p^t$ is close to 0, causing (8) to prioritize optimizing the reward $r^t$. In this case, Algorithm $1$ typically achieves a better Reward but a worse GoalVio. Conversely, when $\gamma$ is large, Algorithm $1$ tends to be conservative by penalizing more goal constraint violations. In particular, if the scale of $d_x$ is very different from that of $d_y$, then there is a significant imbalance between the algorithm’s emphases on GoalVio and Reward, and in principal one might set $\gamma$ around $d_x / d_y$ to balance GoalVio and the regret of Reward.

Next, we extend the result of Theorem $4$ to a high probability bound.

**Corollary 6.** Suppose assumptions $A1$ and $A2$ hold and $\eta^t = \min \{ \frac{1}{m}, \frac{1}{\sqrt{mt}} \}$ for all $t$. Under the uniform random permutation model, for all $\rho \in (0, 1]$, with probability at least $1 - \rho$, Algorithm $7$ achieves $\text{Reward} \geq z^R - O(\sqrt{m \log m \sqrt{T}} + \sqrt{m T \log(T/\rho)})$.

It is worth mentioning that, with stepsizes $\eta^t = \min \{ \frac{1}{m}, \frac{1}{\sqrt{mt}} \}$ for all $t$, the algorithm does not require knowing the number of time steps $T$, and therefore works in cases when $T$ is not initially known. If $T$ is known, it is possible to apply a variant of Algorithm $1$ with a restart-at-$T/2$ strategy similar to the one used by [19] to get a slightly stronger high probability bound than the one in Corollary 6 (reducing $\log(T/\rho)$ to $\log(1/\rho)$) using the maximal inequality by [17].

### 3.3 Under the Grouped Random Permutation Model

In a more realistic setting, $(\Omega^t)_{t=1}^T$ may not follow a uniform random permutation of a single family of feasible sets. For example, if the ODMP problem requires the decision maker to make a decision every day, then weekday problems may be significantly different from weekend problems. We generalize the classic uniform random permutation model to what we call the grouped random permutation model. Formally speaking, we consider the case when the set of time steps $T = \{1, \ldots, T\}$ is partitioned into $K$ groups $(\mathcal{T}^k)_{k=1}^K$, and for each group $\mathcal{T}^k$, $(\Omega^t)_{t \in \mathcal{T}^k}$ is a uniform random permutation of feasible sets $(Z^t)_{t \in \mathcal{T}^k}$. The grouped random permutation model reduces to the uniform random permutation model when $K = 1$.

**Example 2** (Half-Half Partition). Consider an online budget planing problem where at each time step an item with certain weight and reward would arrive. Assume there are two phases of item arrivals, $T^1 = \{1, \ldots, T/2\}$ and $T^2 = \{T/2 + 1, \ldots, T\}$. Items arriving in phase 1 all have weight 2 and reward 1. Assume the decision maker has an average weight budget of 1 for each item arrival and has to decide whether to accept it or not when an item arrives. If items arriving in phase 2 all have weight 0 and reward 0, then the optimal offline decisions would be to accept all phase 1 items, yielding reward $T/2$. On the other side, if items arriving in phase 2 all have weight 2 and reward 2, then the optimal offline decisions would be to reject all phase 1 items and accept all phase 2
Table 1: Unevenness Measure of Different Partitions

| Partition            | \((T^k)^K_{k=1}\)                           | \(W\)                  |
|----------------------|-------------------------------------------|------------------------|
| Weekday-Weekend      | \(T^1 = \bigcup_{\tau=1}^5 \{ t \in T : t \equiv \tau \mod 7 \}, T^2 = T \setminus T^1\) | \(\Theta(\sqrt{mT})\) |
| Half-Half            | \(T^1 = \{ t \in T : t \leq T/2 \}, T^2 = T \setminus T^1\) | \(\Theta(\sqrt{mT^{3/2}})\) |
| K-Periodic           | \(T^k = \{ t \in T : t \equiv k \mod K \}, k = 1, \ldots, K\) | \(\Theta(K\sqrt{mT})\) |

items, yielding reward \(T\). Assuming the decision maker knows nothing about phase 2 in phase 1, no online policy can guarantee sublinear reward regret in the worst case.

Example 2 implies that to have a small (sublinear) regret in reward, the partition \((T^k)^K_{k=1}\) cannot be arbitrary as the decision maker cannot see into the future and can be biased by the observations made so far. We show that our results generalize to this more general grouped random permutation setting if \(T\) is partitioned “almost evenly”.

**Definition 1.** Let \(\bar{\mu}\) denote the uniform distribution over \(T\), and let \(\mu^k\) denote the uniform distribution over \(T^k\) for \(k = 1, \ldots, K\). For \(k = 1, \ldots, K\), we define the (1-)Wasserstein distance \(w^k\) between discrete distributions \(\bar{\mu}\) and \(\mu^k\) as the optimal objective value of the following optimal transportation linear program [38]:

\[
w^k := \min_{q \in \mathbb{R}^{T_k \times T}} \sum_{i \in T^k} \sum_{j \in T} d_{ij} q_{ij}
\]

\[
s.t. \quad \sum_{i \in T^k} q_{ij} = 1, \quad j \in T, \quad \sum_{j \in T} q_{ij} = 1 / |T^k|, \quad i \in T^k.
\]

(16)

We use the stepsizes to define the distance \(d_{ij}\) between two time steps \(i\) and \(j\). Specifically, we define

\[
d_{ij} := \begin{cases} 
\sum_{t=1}^{j-1} \eta^t & \text{if } i < j, \\
\sum_{t=j}^{i-1} \eta^t & \text{otherwise}.
\end{cases}
\]

We use the weighted Wasserstein distance sum \(W := \sum_{k=1}^K m|T^k|w^k\) to measure the unevenness of the partition \((T^k)^K_{k=1}\).

**Example 3.** Let \(\eta^t = \min\{\frac{1}{m^2}, \frac{1}{\sqrt{mt}}\}\) for all \(t\). Then for some special partitions of \(T\), the unevenness measure \(W\) is listed in Table 1. Note that weekday-weekend and \(K\)-periodic partitions (with \(K = o(\sqrt{T})\)) have a smaller unevenness measure \(W\) (sublinear in \(T\)) than the half-half partition does (\(W\) not sublinear in \(T\)).

We show in the next result that Algorithm 1 yields a small reward gap under the grouped random permutation model if the partition of \(T\) is “almost even”.

**Theorem 7.** Suppose assumptions [A1] and [A2] hold and \(\eta^t = \min\{\frac{1}{m^2}, \frac{1}{\sqrt{mt}}\}\) for all \(t\). Under the
grouped random permutation model with groups \((T^k)_{k=1}^K\), Algorithm \ref{alg:main} achieves

\[
\mathbb{E}_\pi[\text{Reward}] \geq z^R - O\left( W + \sqrt{m \log m} \sum_{k=1}^K \sqrt{|T^k|} \right).
\]

**Proof.** Let \(p^*\) and \((\hat{x}^t, \hat{y}^t)_{t=1}^T\) be such that Lemma \ref{lem:reward-bound} holds for the deterministic ODMP problem \((\mathcal{P})\) with \((\Omega^t)_{t=1}^T = (Z^t)_{t=1}^T\). Let \(p\) denote the grouped random permutation of \((Z^t)_{t=1}^T\) that such that \((\Omega^t)_{t=1}^T = (Z^{\pi(t)})_{t=1}^T\). Following arguments similar to the proof of Theorem \ref{thm:main}, we have the following deterministic bound

\[
\text{Reward} \geq z^R + \sum_{t=1}^T (p^t)\top (\hat{v}^t - \mathbb{E}_\pi[\bar{y}^{\pi(t)}])
+ \sum_{t=1}^T (p^t)\top (\mathbb{E}_\pi[\bar{y}^{\pi(t)}] - \bar{y}^{\pi(t)}) - O(\sqrt{mT}).
\] (17)

Under the grouped random permutation model, we have \(\mathbb{E}_\pi[\bar{y}^{\pi(t)}] = |T^k|^{-1} \sum_{\tau \in T^k} \bar{y}^\tau\) for all \(t \in T^k\) and \(k = 1, \ldots, K\). Let \(\bar{p} = T^{-1} \sum_{t=1}^T p^t\). Since \(T^{-1} \sum_{\tau \in T} \bar{y}^\tau \in \Psi\), then \((p^t)\top \hat{v}^t = h_\Psi(p^t) \geq (p^t)\top (T^{-1} \sum_{\tau \in T} \bar{y}^\tau)\). Fix an arbitrary \(\hat{v} \in Q\). We have

\[
\sum_{t=1}^T (p^t)\top (\hat{v}^t - \mathbb{E}_\pi[\bar{y}^{\pi(t)}])
\geq (\sum_{t=1}^T p^t)\top \left( T^{-1} \sum_{\tau \in T} \bar{y}^\tau \right) - \sum_{k=1}^K \left( \sum_{t \in T^k} p^t \right)\top \left( |T^k|^{-1} \sum_{\tau \in T^k} \bar{y}^\tau \right)
\geq \sum_{k=1}^K \left( |T^k| \bar{p} - \sum_{t \in T^k} p^t \right)\top \left( |T^k|^{-1} \sum_{\tau \in T^k} \bar{y}^\tau \right)
\geq - \sum_{k=1}^K |T^k| \left\| \bar{p} - |T^k|^{-1} \sum_{t \in T^k} p^t \right\|_2 \cdot O(\sqrt{m})
\geq - \sum_{k=1}^K |T^k| \left\| T^{-1} \sum_{t=1}^T p^t - |T^k|^{-1} \sum_{t \in T^k} p^t \right\|_2 \cdot O(\sqrt{m}),
\] (18)

where the second inequality holds since \(\left\| \bar{y}^\tau - \bar{v} \right\|_2 \leq \sqrt{m}d_y = O(\sqrt{m})\) by assumption \[A2\]. Note that for \(1 \leq i < j \leq T\), we have \(\|p^i - p^j\|_2 \leq \sum_{t=i}^{j-1} \eta^t \|\hat{v}^t - \hat{y}^t\|_2 = \sum_{t=i}^{j-1} \eta^t \cdot O(\sqrt{m})\). Let \(q^*\) denote an optimal solution of \((16)\). It then follows that

\[
\left\| T^{-1} \sum_{t=1}^T p^t - |T^k|^{-1} \sum_{t \in T^k} p^t \right\|_2
\]
\[ = \left\| \sum_{t \in T} \sum_{i \in T^k} q_i^t p^t - \sum_{t \in T^k} \sum_{j \in T} q_j^t p^t \right\| \]
\[ = \left\| \sum_{i \in T^k} \sum_{j \in T} q_{ij}^t (p^i - p^j) \right\| \]
\[ \leq \sum_{(i,j) \in T^k \times T; i < j} q_{ij}^t \sum_{t=1}^{j-1} \eta^t \cdot O(\sqrt{m}) + \sum_{(i,j) \in T^k \times T; j < i} q_{ij}^t \sum_{t=j}^{i-1} \eta^t \cdot O(\sqrt{m}) \]
\[ \leq \left( \sum_{i \in T^k} \sum_{j \in T} d_{ij}^* q_{ij}^* \right) \cdot O(\sqrt{m}) = w^k \cdot O(\sqrt{m}). \quad (19) \]

Therefore, combining (18) and (19), we have
\[ \sum_{t=1}^{T} (p^t)^\top (\hat{v}^t - E_\pi[\bar{y}^\pi(t)]) \geq -\sum_{k=1}^{K} |T^k| w^k \cdot O(m) = -O(W). \quad (20) \]

Similar to (15), for \( k = 1, \ldots, K \), we have
\[ \sum_{t \in T^k} E_\pi[(p^t)^\top (E_\pi[\bar{y}^\pi(t)] - \bar{y}^\pi(t))] \geq -\sum_{i=1}^{|T^k|} O\left(\sqrt{\frac{m \log m}{i}}\right) = -\sqrt{m \log m} \sqrt{|T^k|}. \quad (21) \]

The conclusion follows by combining (17), (20) and (21).

Corollary 6 also generalizes to the grouped random permutation setting.

**Corollary 8.** Suppose assumptions \( A1 \) and \( A2 \) hold and \( \eta^t = \min\{\frac{1}{m}, \frac{1}{\sqrt{m t}}\} \) for all \( t \). Under the grouped random permutation model with groups \( (T^k)_{k=1}^{K} \), for all \( \rho \in (0,1] \), with probability at least \( 1 - \rho \) Algorithm 1 achieves Reward \( \geq z^R - O(W + \sqrt{m \log m} + \sqrt{m \log(T/\rho)} \sum_{k=1}^{K} \sqrt{|T^k|}). \)

### 4 Numerical Experiments

To provide insight into the potential of our method, we conduct some experiments applying Algorithm 1 to a class of online knapsack problems and a class of assortment optimization problems.

#### 4.1 Online Knapsack with Fairness-over-Time Constraints

We first consider an online knapsack problem with fairness-over-time constraints (OKP-FOT). In OKP-FOT, at each time step, multiple agents request space to place their items in a knapsack. The decision maker faces a 0-1 knapsack problem at each time step and has to sequentially make decisions to maximize the total profit, subject to certain long-term fairness constraints regarding several stakeholders’ utilities. In particular, at each time step \( t \), the decision maker makes a decision \( \mathbf{x}^t \in \{0,1\}^n \) about how to pick from a set of \( n \) items, with weights \( \mathbf{w}^t = (w_j^t)_{j=1}^n \) and profits \( \mathbf{o}^t = (o_j^t)_{j=1}^n \), subject to a weight capacity constraint \( (\mathbf{w}^t)^\top \mathbf{x}^t \leq W^t \). In the pursuit of fairness, the decision maker aims for the average utilities of the agents to be within a predefined fairness-defining set \( \Psi \) in the long run. Specifically, the offline version of the OKP-FOT problem
Figure 2: Reward\(_t/t\) (left) and GoalViol\(_t/t\) (right) obtained by Algorithm 1 for OKP-FOT

The optimal solution takes the following form:

\[
    z^* = \max_x \sum_{t=1}^{T} (o^t)^\top x^t \\
    \text{s.t.} \quad (w^t)^\top x^t \leq W^t, \quad x^t \in \{0,1\}^n, \quad t \in T, \\
    \sum_{t=1}^{T} u^t(x^t) \in T\Psi.
\]

Note that in this case, \(\Omega^t = \{(o^t)^\top x^t, x^t, u^t(x^t)\} : (w^t)^\top x^t \leq W^t, x^t \in \{0,1\}^n\) for each \(t \in T\). We assume function \(u^t : \mathbb{R}^n \to \mathbb{R}^m\) defines the additive utility gains of \(m\) agents at time step \(t\). Regarding the definition of \(\Psi\), we consider the following utilitarian fairness settings. We assume that the utility gains of the \(m\) stakeholders at time step \(t\) are modeled by \(u^t(x^t) = U^t x^t\) for some \(U^t \in \mathbb{R}_{++}^{m \times n}\), and the fairness-defining set takes the form \(\Psi = \{v \in \mathbb{R}^m : \max_{i=1}^{m} v_i - \min_{i=1}^{m} v_i \leq \rho\}\) for some \(\rho > 0\). We also assume in OKP-FOT that the reward represents the utilitarian welfare, i.e., \((o^t)^\top x^t = 1^\top U^t x^t\) for all \(t\), where \(1\) denotes the vector of all ones. Note that OKP-FOT differs significantly from the traditional online knapsack problem (OKP) in online resource allocation in that the traditional OKP only requires making a binary decision at each time step and the long-term constraint in the traditional OKP is a knapsack budget constraint [33].

It is easy to verify that assumptions A1 and A2 hold for OKP-FOT with proper \(\rho\) and \(\tau\). We set \(n = 50, m = 10\) and \(T = 10,000\). The parameters \((w^t)^{T}_{t=1}\) and \((o^t)^{T}_{t=1}\) are generated so that each item’s profit is positively correlated with its weight, leading to more realistic but computationally difficult local knapsack problems [39]. We generate 100 instances by permuting the order of the local knapsack problems. Details about the generation of the test instances are presented in Appendix D.

We apply Algorithm 1 with stepsize \(\eta^t = \min\{\frac{\tau}{\gamma}, \frac{\gamma}{\sqrt{mt}}\}\), varying \(\gamma \in \{0.01, 0.1, 1, 10, 100\}\). The subproblem (8) in Algorithm 1 is a knapsack problem as \(u^t(x^t)\) is linear in \(x^t\).

In Figure 2, we plot the empirical means of GoalViol\(_t/t\) and Reward\(_t/t\) (defined in Corollary 5) over the 100 test instances as well as their ranges (in lighter colors). We observe that the ranges of GoalViol\(_t/t\) and Reward\(_t/t\) deviate less from the mean as \(t\) becomes larger. Larger \(\gamma\) leads to smaller GoalViol\(_t/t\) and Reward\(_t/t\) as expected from Corollary 5. Therefore, the “optimal” choice of the stepsize parameter \(\gamma\) depends on the preference of the decision maker. Larger \(\gamma\) leads to faster convergence of GoalViol\(_t/t\) to 0 but lower Reward\(_t/t\). A good choice of \(\gamma\) potentially
leads to small violation of the goal constraints without sacrificing much in reward. For example, a good choice of $\gamma$ for our OKP-FOT test instances could be $\gamma = 0.1$, which suggests that some tuning of $\gamma$ might be necessary when applying Algorithm 1 in practice.

Although the trend of convergence of $\text{Reward}_t/t$ is evident on Figures 2 for most values of $\gamma$, it is hard to visualize its convergence rate since $z^R$ is unknown. On the other hand, we present the log-log plot for $\text{GoalVio}_t/t$ (averaged over the 100 test instances) in Figure 3. We observe that $\text{GoalVio}_t/t$ converges to 0 roughly at the rate $O(1/\sqrt{t})$ (subject to numerical errors) in our numerical experiments for each choice of the $\gamma$ value.

As mentioned in Section 2.3, it is possible to apply a variant of Algorithm 1 (i.e., Algorithm $1'$ in Section 2.3) by assuming that there exists $Y \subseteq \mathbb{R}^m$ such that $\text{proj}_Y \Omega^t \subseteq Y$ and replace $\Psi$ by $\Psi \cap Y$ in Algorithm 1. However, Algorithm 1' can be less stable when $Y$ is large. We plot in Figure 4 the performance of this variant with $Y = [-1000, 20000]^m$, in particular, to compare it with the performance of Algorithm 1 in Figure 2. We observe that with such a large $Y$ (having a large $d_y$ associated with $\Psi \cap Y$), the convergence of $\text{Reward}_t/t$ is not evident for $\gamma \geq 0.1$. At the same time, the convergence of $\text{GoalVio}_t/t$ is much slower for $\gamma \geq 0.1$, demonstrating the instability of this variant of Algorithm 1' when $Y$ is large.

### 4.2 Assortment Optimization with Visibility and Cardinality Constraints

In this section, we show how Algorithm 1 can be used to approximately solve the online version of the Assortment Optimization problem with visibility and cardinality constraints under the Mixed Multinomial Logit model.

Assortment Optimization (AO) is a classical problem in revenue management where a decision maker selects a different subset (assortment) of products from among $m$ substitutable products to display for each customer. Each customer purchases at most one of the products offered to them according to their preferences. If a customer purchases product $i$, the decision maker receives a revenue $\gamma_i > 0$ and the objective for the decision maker is to maximize their total revenue. In

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1. A $O(1/\sqrt{t})$ sequence would correspond to a curve with slope less than $-1/2$ in a log-log plot.
practice, the decision maker often has to take into consideration some additional constraints such as visibility and cardinality constraints [10]. Visibility constraints (also known as fairness constraints [31]) require that each product $i$ has to be shown with frequency at least $f_i \in [0,1)$ while the cardinality constraints require that the assortment shown to each customer can have at most $s$ products.

A particular model that is often used to capture the customer preferences is the Multinomial Logit (MNL) model [32]. In the MNL model, it is assumed that all customers have the (same) preference weights $(v_i)_{i=1}^m$ for the products and given an assortment $S \subseteq \{1, \ldots, m\}$, they purchase product $i$ with probability $v_i \cdot (1 + \sum_{j \in S} v_j)^{-1}$. A generalization of the classical MNL model is the Mixed Multinomial Logit (MMNL) model [33] where it is assumed that there are $K \geq 1$ customer types and if a customer is of type $k \in \{1, \ldots, K\}$ then their preference weight for product $i$ is $v_i^k > 0$ for $i \in \{1, \ldots, m\}$.

It is shown by [10] that AO (under the classical MNL model) with visibility constraints can be solved in polynomial time whereas AO with visibility and cardinality constraints is strongly NP-hard. In this section, we show how Algorithm 1 can approximately solve the online AO problem with visibility and cardinality constraints (AOVC) under the MMNL model. We consider the MMNL model in AOVC since its online setting (with unknown customer distribution) makes more sense than the MNL model.

Before we discuss the online model, we first present a binary fractional programming formulation for the offline problem. Assume that there are a total of $T$ customers and $T_k$ of them are of type $k$ with $\sum_{k=1}^K T_k = T$. We use the index set $\mathcal{T} = \{1, \ldots, T\}$ for the customers and a partition $(\mathcal{T}^k)_{k=1}^K$ of $\mathcal{T}$ with $|\mathcal{T}^k| = T_k$ to identify customers of different types. With this notation, the offline version of the AO with visibility and cardinality constraints becomes

$$z^* = \max_x \sum_{k=1}^K \sum_{t \in \mathcal{T}^k} \frac{\sum_{i=1}^m \gamma_i v_i^k x_i^t}{1 + \sum_{i=1}^m v_i^k x_i^t} \quad \text{(22a)}$$

s.t. $\sum_{i=1}^m x_i^t \leq s$, \quad $t \in \mathcal{T}$, \quad \text{(22b)}$

\sum_{t=1}^T x_i^t \geq f_i T$, \quad $i = 1, \ldots, m$, \quad \text{(22c)}
where variable \( x^t_i \) denotes if product \( i \) is chosen for the assortment shown to customer \( t \) and the objective is to maximize the expected revenue. In particular, it can be viewed as an offline OMDP \((1)\) by defining

\[
\Omega^t = \left\{ \left( \frac{\sum_{i=1}^m \gamma_i v^k_i x^t_i}{1 + \sum_{i=1}^m v^k_i x^t_i}, x^t, x^t \right) : \sum_{i=1}^m x^t_i \leq s, x^t \in \{0,1\}^m \right\}, \quad t \in T^k,
\]

and

\[
\Psi = \{ y : f_i \leq y_i \leq 1, \ i = 1, \ldots, m \}.
\]

It is also easy to verify that assumptions \( A1 \) and \( A2 \) hold if \( s - \sum_{i=1}^m f_i \geq \Omega(m) \), i.e., if the visibility frequency requirements do not consume almost all the capacity of the assortment, which is in general a weak assumption. Throughout this section, we assume that this condition holds.

In the online setting, we assume that the total number of customers (i.e., \( T \)), the partition \((T^k)_{k=1}^K\), and the number of customers belonging to each type \( k \) (i.e., \( T^k \)) are unknown. When a customer arrives, the decision maker observes the type of the customer and chooses an assortment for the customer without any knowledge about the future arrivals. We assume the arriving order of the \( T \) customers follows some random permutation model (as defined in Section 3.1) with

\[
Z^t = \left\{ \left( \frac{\sum_{i=1}^m \gamma_i v^k_i x^t_i}{1 + \sum_{i=1}^m v^k_i x^t_i}, x^t, x^t \right) : \sum_{i=1}^m x^t_i \leq s, x^t \in \{0,1\}^m \right\}, \quad t \in T^k.
\]

Independent of the random permutation model, Theorem 3 implies that Algorithm 1 always achieves \( O(\sqrt{mT}) \) visibility constraint violation for the online version of AOVC. Moreover, when customer arrivals follow the uniform random permutation model or the grouped random permutation model, we also have the Reward guarantees developed in Sections 3.2 and 3.3 for Algorithm 1.

Regarding the implementation of Algorithm 1 at time step \( t \), the subproblem (8) in Algorithm 1 for AOVC takes the form

\[
\max_{x^t} \sum_{i=1}^m \frac{\gamma_i v^k_i x^t_i}{1 + \sum_{i=1}^m v^k_i x^t_i} - \sum_{i=1}^m p^t_i x^t_i 
\]

s.t.

\[
\sum_{i=1}^m x^t_i \leq s, \ x^t \in \{0,1\}^m,
\]

where \( k \) denotes the type of customer \( t \). Problem (23) can be viewed as a cardinality constrained revenue management problem under a mixture of independent demand and multinomial logit models, and is in general NP-hard [14]. Therefore, we reformulate it as a mixed-integer linear program using standard techniques [27] and solve it by Gurobi.

Regarding the test instances, we take the MMNL model from [16], which is estimated from real data provided by Expedia. The MMNL model consists of 16 customer types (i.e., \( K = 16 \)), and the preference weight of each product for each customer type is estimated using 8 features associated with each product. The original data set has hundreds of thousands of products (representing different hotels options queried by all customers). To create instances of reasonable sizes, we

\[\text{Note that, for solving the offline version (1), one can always achieve the Reward guarantees in Sections 3.2 by first randomly permuting the arriving orders of the customers and then applying Algorithm 1 to the permuted problem.}\]
sample 40 products (i.e., \( m = 40 \)) randomly from the data set with 50% no-purchase rate from \([16]\), and set \( s = 10 \) as assortment cardinality for each customer.

We apply Algorithm 1 to AOVC with stepsize \( \eta^t = \min\{\frac{\gamma}{m}, \frac{\gamma}{\sqrt{mt}}\} \), varying \( \gamma \in \{1, 100, 10000\} \). Unlike OKP-FOT, we do not use \( \gamma < 1 \) here since \( d_r \) is orders of magnitude larger than \( d_y \) for the AOVC instances. We consider two types of input models:

**S(tochastic).** We assume that the arriving order of customers from different customer types is uniformly randomly permuted. We generate 100 such instances by applying the uniform random permutation with different random seeds.

**B(atched).** We assume that customers of the same type arrive together, i.e., customers of type 1 all arrive first, then customers of type 2 arrive, so on and so forth.

Figure 5: \( \text{Reward}_t/t \) (left) and \( \text{GoalVio}_t/t \) (right) obtained by Algorithm 1 for AOVC

In Figure 5, we plot the performance of Algorithm 1 on AOVC. We use solid lines to plot empirical means of \( \text{Reward}_t/t \) and \( \text{GoalVio}_t/t \) (defined in Corollary 5) obtained by Algorithm 1 over the 100 instances generated using the S(tochastic) input model (and their ranges are in lighter colors). We use dashed lines to plot \( \text{Reward}_t/t \) and \( \text{GoalVio}_t/t \) obtained by Algorithm 1 on the instances generated using the B(atched) input model. We observe that larger stepsizes lead to smaller \( \text{GoalVio}_t/t \) and lower \( \text{Reward}_t/t \), which is in alignment with Corollary 5. In the case of having batched input, \( \text{GoalVio}_t/t \) and \( \text{Reward}_t/t \) are less smooth than the random input case. At the same time, \( \text{GoalVio}_t/t \) converges to 0 as \( t \) increases in both cases, in alignment with Theorem 3 (which is independent of the input model). On the other hand, we observe that the total average reward \( \text{Reward}_T/T \) is lower if the input is batched rather than stochastic. This is not surprising since the theoretical guarantees on reward we developed in Section 3 only hold under stochastic input models.

5 Conclusions

We propose an algorithm for solving online decision making problems with nonconvex local and convex global constraints. Its performance is analyzed under specific stochastic input models. We also conduct experiments to evaluate the empirical performance of the algorithm on a synthetic online knapsack problem and a more realistic assortment optimization problem.
Since our algorithm involves solving potentially nonconvex local problems, which would likely be its computational bottleneck, an interesting research direction is to investigate whether any nontrivial performance guarantees can be established when optimization tolerances are allowed in the solution of local problems \[8\]. Additionally, it would be worthwhile to explore how the algorithm can be enhanced by leveraging historical data.

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A Verifying Assumptions for Fair Online Assignment

Here we verify assumptions [A1] and [A2] for Example 1 with $\Psi = \{y : \max_i y_i - \min_i y_i \leq \rho\}$. Note that assumption [A1] naturally holds with $Q = \{y : y \geq 0, y \leq \rho\}$ and $C = \{\lambda \in \mathbb{R}\}$. Denote $\max_i \sum_{j=1}^{n_i} q_{ij}^t$ by $q_{\text{max}}$ (representing maximum total profit in a single time step) and $\max_i \sum_{j=1}^{n_i} w_{ij}^t$ by $w_{\text{max}}$ (representing maximum workload of a single agent in a single time step).

We next verify assumption [A2]. In particular, we next show that there exists $(\tilde{r}^t, \tilde{x}^t, \tilde{y}^t) \in \text{conv}(\Omega^t)$ such that $\max_i \tilde{y}_i^t = \min_i \tilde{y}_i^t$ for all $t$. Fix $t \in T$. Note that for each $i \in \{1, 2, \ldots, m\}$, it is feasible to have agent $i$ finishing all the tasks in which case the workload of agent $i$ is $\tilde{w}_i^t := \sum_{j=1}^{n_i} w_{ij}^t$ and the workload of other agents is 0. Denote this particular solution by $(\hat{r}^t, \hat{x}^t, \hat{y}^t, \tilde{x}^t, \tilde{y}^t)$ for each $i \in \{1, 2, \ldots, m\}$. Without loss of generality, we assume $\hat{w}_i^t > 0$ for each agent $i$ (otherwise it is perfectly fair to assign all tasks to the agent $i$ with $\hat{w}_i^t = 0$). Let

$$\lambda_i := \frac{(\hat{w}_i^t)^{-1}}{\sum_{k=1}^{m} (\hat{w}_k^t)^{-1}}, \quad i = 1, \ldots, m.$$ 

Obviously, we have $\lambda_i \geq 0$ for all $i$, $\sum_{i=1}^{m} \lambda_i = 1$. Define $(\tilde{r}^t, \tilde{x}^t, \tilde{y}^t) = \sum_{i=1}^{m} \lambda_i (\hat{r}^t, \hat{x}^t, \hat{y}^t)$ for all $t$, and $\tilde{w}_i^t = (\sum_{i=1}^{m} (\hat{w}_i^t)^{-1})^{-1}$. Then $(\tilde{r}^t, \tilde{x}^t, \tilde{y}^t) \in \text{conv}(\Omega^t)$ and $\max_i \tilde{y}_i^t = \min_i \tilde{y}_i^t = \tilde{w}_i^t$, in which case assumption [A2] holds with $d_y = w_{\text{max}}, d_r = q_{\text{max}}$ and $\underline{d} = \rho/2$.

B Some Other Lemmas

Lemma 9. Suppose assumptions [A1] and [A2] hold. For all $p \in \mathcal{C}^o$ and $t = 1, \ldots, T$, we have

1. $\|p\|_2 \leq \eta \psi(p) - p^\top \tilde{y}_t$, and
2. $(p^t - p)^\top (\hat{v}_t - \tilde{y}_t) \leq \frac{\|p^t - p\|^2 + \|p^{t+1} - p\|^2}{2\eta^t} + \frac{\eta^t}{2} m d_y^2$

in Algorithm 1.

Proof. Since $\tilde{y}_t + dB_m \subseteq \Psi$ by assumption [A2], we have $h_\psi(p) \geq h_{\{\tilde{y}_t\} + dB_m}(p) = p^\top \tilde{y}_t + \|p\|_2 \underline{d}$. The first conclusion then follows. For the second conclusion, note that $p \in \mathcal{C}^o$ and $p^{t+1} = \text{proj}_{C^o}(p^t - \eta t \hat{v}_t - \tilde{y}_t)$ in Algorithm 1; it follows that

$$\|p^{t+1} - p\|^2 = \|p^t - \eta t (\hat{v}_t - \tilde{y}_t) - p\|^2 = \|p^t - p\|^2 + \eta t^2 \|\hat{v}_t - \tilde{y}_t\|^2 - 2\eta t(p^t - p)^\top (\hat{v}_t - \tilde{y}_t).$$

By rearranging the inequality and applying assumption [A2] we have

$$\begin{align*}
(p^t - p)^\top (\hat{v}_t - \tilde{y}_t) & \leq \frac{\|p^t - p\|^2 + \|p^{t+1} - p\|^2}{2\eta^t} + \frac{\eta^t}{2} \|\hat{v}_t - \tilde{y}_t\|^2 \\
& \leq \frac{\|p^t - p\|^2 + \|p^{t+1} - p\|^2}{2\eta^t} + \frac{\eta^t}{2} m d_y^2.
\end{align*}$$

Lemma 10. Suppose assumption [A1] holds. Then for $t = 1, \ldots, T$, we have $p^{t+1} = p^t - \eta t (\hat{v}_t - \tilde{y}_t) - \hat{u}^t$ for some $\hat{u}^t \in \mathcal{C}$ in Algorithm 1.
Proof. Let \( \tilde{u}^t := p^t - \eta_t (\tilde{v}^t - \tilde{y}^t) - p^{t+1} \) and \( p^{t+1/2} := p^t - \eta_t (\tilde{v}^t - \tilde{y}^t) \). Then \( p^{t+1} = \text{proj}_{C^o}(p^{t+1/2}) \) and \( \tilde{u}^t = p^{t+1/2} - p^{t+1} = p^{t+1/2} - \text{proj}_{C^o}(p^{t+1/2}) \). By [22] Chapter III Theorem 3.2.3, we have \( \tilde{u}^t = p^{t+1/2} - \text{proj}_{C^o}(p^{t+1/2}) \in C^o \). Since \( C \) is a nonempty closed convex cone, we have \( C^o = C \) by [22] Chapter III Proposition 4.2.7. The conclusion then follows.

\[ \square \]

C Omitted Proofs

Proof of Corollary 6. Similar to Lemma 2, one can show that \( \|p^t\|_2 \leq O(1+\gamma) \) if \( \max_t \eta_t \leq \gamma/m \) for all \( t \). Note that \( \mathbb{E}_\pi[\sum_{t=1}^T \tilde{r}^\pi(t)] = (\tau/T) \sum_{t=1}^T \tilde{r}^t = \tau z^R/T \) under the uniform random permutation model. The conclusion follows from arguments similar to the proofs of Theorem 3 and Theorem 4.

Proof. Proof of Corollary 6. Replacing \( \rho \) by \( \rho/2T \) in (14) and taking the union bound over \( t \), with probability at least \( 1 - \rho/2 \) we have

\[
\sum_{t=1}^T (p^t)^\top \left( \mathbb{E}_\pi[\tilde{y}^\pi(t)] - \mathbb{E}_\pi[\tilde{y}^\pi(t)\mid \mathcal{F}^{t-1}] \right)
\geq - \sum_{t=1}^T \|p^t\|_2 \left\| \mathbb{E}_\pi[\tilde{y}^\pi(t)] - \frac{1}{T-t+1} \sum_{\tau=t}^T \tilde{y}^\pi(\tau) \right\|_2
\geq - O(\sqrt{m \log m \sqrt{T} + \sqrt{Tm} \log(T/\rho)).
\] (24)

Now define random variables \( Z^0_0 = 0 \) and \( Z^t = (p^t)^\top (\mathbb{E}_\pi[\tilde{y}^\pi(t)\mid \mathcal{F}^{t-1}] - \tilde{y}^\pi(t)) \) for \( t \in T \). Note that \( \mathbb{E}_\pi[Z^t] = 0 \). Therefore, \( \sum_{\tau=t}^T Z^t \) is a martingale with respect to the filtration \( (\mathcal{F}^t)_{t=0}^T \). Also note that \( \|Z^t\| \leq 2\sqrt{md}_y \|p^t\|_2 = O(\sqrt{m}) \) for all \( t \). By Azuma-Hoeffding inequality [7], for all \( \epsilon > 0 \) we have

\[
\mathbb{P} \left( \sum_{t=1}^T Z_t \leq -\epsilon \right) \leq \exp \left( \frac{-\epsilon^2}{O(mT)} \right).
\]

Let \( \epsilon = \sqrt{\log(2/\rho)O(mT)} = O(\sqrt{mT \log(1/\rho)}) \). Then with probability at least \( 1 - \rho/2 \) we have

\[
\sum_{t=1}^T (p^t)^\top (\mathbb{E}_\pi[\tilde{y}^\pi(t)\mid \mathcal{F}^{t-1}] - \tilde{y}^\pi(t)) = \sum_{t=1}^T Z_t \geq - O(\sqrt{mT \log(1/\rho)}). \] (25)

Taking the union bound of (24) and (25), the conclusion then follows from (13).

\[ \square \]

Proof of Corollary 8. Note that by the proof of Theorem 7, inequality \( \sum_{t=1}^T (p^t)^\top (v^t - \mathbb{E}_\pi[\tilde{y}^\pi(t)]) \geq -O(W) \) holds deterministically for any grouped random permutation \( \pi \). Following arguments similar to the proof of Corollary 6 with probability at least \( 1 - \rho/2 \) we have

\[
\sum_{t=1}^T (p^t)^\top (\mathbb{E}_\pi[\tilde{y}^\pi(t)] - \mathbb{E}_\pi[\tilde{y}^\pi(t)\mid \mathcal{F}^{t-1}] = \sum_{k=1}^K \sum_{t \in T^k} (p^t)^\top (\mathbb{E}_\pi[\tilde{y}^\pi(t)] - \mathbb{E}_\pi[\tilde{y}^\pi(t)\mid \mathcal{F}^{t-1}] \)
\]
\[ \geq -O\left( \sqrt{m \log m + m \log(T/\rho)} \sum_{k=1}^{K} \sqrt{T^k} \right), \]  
(26)

and with probability at least \(1 - \rho/2\) we have

\[ \sum_{t=1}^{T} (p^t)\top \left( \mathbb{E}_{\pi} [\tilde{y}^\pi(t)|\mathcal{F}^{t-1}] - \tilde{y}^\pi(t) \right) \geq -O(\sqrt{mT \log(1/\rho)}). \]  
(27)

The conclusion follows by taking the union bound of (26) and (27).

D Generation of OKP-FOT Test Instances

For OKP-FOT, we generate a base instance, which then leads to 100 test instances by randomly permuting the arrival order of \(\Omega^t\) in the base instance. The base instance for OKP-FOT is generated following the schemes described below:

- \(w^t_j \sim \text{Uniform}(1, 1000), j = 1, \ldots, n, t \in T.\)
- \(W^t = 0.3 \cdot \sum_{j=1}^{n} w^t_j, t \in T.\)
- \(U^t_{ij} \sim \text{Uniform}(w^t_j - 20i, w^t_j + 40i), i = 1, \ldots, m, j = 1, \ldots, n, t \in T.\)
- \(\rho = 100.\)