Toric hyperkahler manifolds with quaternionic Kahler bases and supergravity solutions

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Abstract

In the present work some examples of toric hyperkahler metrics in eight dimensions are constructed. First it is described how the Calderbank-Pedersen metrics arise as a consequence of the Joyce description of selfdual structures in four dimensions, the Jones-Tod correspondence and a result due to Tod and Przanowski. It is also shown that any quaternionic Kahler metric with $T^2$ isometry is locally isometric to a Calderbank-Pedersen one. The Swann construction of hyperkahler metrics in eight dimensions is applied to them to find hyperkahler examples with $U(1) \times U(1)$ isometry. The connection with the Pedersen-Poon toric hyperkahler metrics is explained and it is shown that there is a class of solutions of the generalized monopole equation related to eigenfunctions of certain linear equation. This hyperkahler examples are lifted to solutions of the D=11 supergravity and type IIA and IIB backgrounds are found by use of dualities. As before, all the description is achieved in terms of a single eigenfunction $F$. Some explicit $F$ are found, together with the Toda structure corresponding to the trajectories of the Killing vectors of the Calderbank-Pedersen bases.

1. Introduction

The relevance of hypergeometry in field theory has been made manifest during the last twenty years. For example the moduli space of magnetic monopoles [1] or the moduli space of Yang-Mills instantons in flat space [2] are hyperkahler manifolds. The relation between hyperkahler spaces and hypermultiplets of field theories in D=4 with N=2 rigid supersymmetries has been pointed out in [3], [4] and [5] and it was shown that when the supersymmetry is made local the hypermultiplets couple to supergravity and the resulting target space is a quaternionic Kahler manifold [6]. Many other modern applications of this subject to supersymmetric theories in D=4 can be found in [7]-[12] and references therein.

Quaternionic geometry is deeply related to gravity theories in different dimensions, and to superstring and M-theories. This is because quaternionic Kahler metrics are always Euclidean vacuum Einstein with cosmological constant $\Lambda$ and in the limit $\Lambda \to 0$ it is obtained an hyperkahler metric. Four dimensional quaternionic metrics can be extended to examples of special holonomy [13], which are internal spaces of supergravity theories preserving some amount of supersymmetries. Moreover compact $G_2$-holonomy spaces with orbifold singularities are believed to arise as a quotient of a conical hyperkahler manifold in D=8 by one of its isometries.

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Quaternionic manifolds also characterize the hypermultiplet geometry of classical and perturbative moduli spaces of type II strings compactified on a Calabi-Yau manifold [15].

Hypergeometry is also an active tool in modern mathematics. Quaternionic Kahler and hyperkahler spaces are $D=4n$ dimensional and constitute the special case of the Berger list with holonomies included in $Sp(n) \times Sp(1)$ and $Sp(n)$ respectively [16]. Some of their properties has been investigated for instance in [17]-[19] but they are not completely classified at the present.

One of the latest achievements in the subject is the hyperkahler quotient, developed in [4] and [20] and providing a way to construct hyperkahler manifolds of a given dimension taking the quotient of a higher dimensional hyperkahler one by certain group generating triholomorphic isometries. A sort of inverse method is due to Swann [21] who shows how a quaternionic Kahler metric in $D=4n$ can be extended to a quaternionic Kahler and hyperkahler examples in $D=4(n+1)$. The Swann construction was applied recently to construct hyperkahler cones in [22], relevant in theories with $N=2$ rigid supersymmetries, and to construct certain scalar manifolds in M-theory on a Calabi-Yau threefold in the vicinity of a flop transition [23].

The present work is mainly focused in the construction of eight dimensional hyperkahler metrics with two commuting $U(1)$ isometries (usually called toric hyperkahler) and the Swann extension is crucial to do this. The reason is that the quaternionic Kahler metrics in $D=4$ with two commuting $U(1)$ isometries has been locally completely classified by Calderbank and Pedersen [34] in terms of solutions $F$ of a simple linear second order equation, namely

$$F_{\rho\rho} + F_{\eta\eta} = \frac{3F}{4\rho^2}.$$  

Such four dimensional metrics will be extended by the Swann construction to hyperkahler ones in $D=8$ and it will be seen that the $T^2$ isometry is preserved in this extension, therefore the result is toric hyperkahler. As in the Calderbank-Pedersen case, all the description is achieved in terms of the linear equation given above, which make this picture very simple.

There exist a physical motivation to construct toric hyperkahler examples. They arise naturally in the M-theory context as solutions corresponding to multiple intersecting branes [24], but their range of applications is of course, not limited to this case. For instance, the moduli space of scattering of well separated BPS monopoles or well separated dyons due to a $(p,q)$ string in a D-3 brane are toric hyperkahler manifolds [26]-[28]. The metric of the moduli space of the $k=1$ $SU(n)$ periodic instantons (or calorons) has been shown to be toric hyperkahler [29]. Applications related to intersections in domain walls can be found in [30] and [31] and in [25], there has been studied solitons in a (2+1)-dimensional sigma model with a toric hyperkahler target space preserving 1/2 of the supersymmetries and their realization in M-theory.

It will be of interest to compare the results presented here with the Pedersen-Poon description [32] of toric hyperkahler spaces, which is the most suitable for physical purposes. They statement is that for every of such spaces there is a coordinate system in which they locally takes the generalized Gibbons-Hawking anzatz

$$\bar{g} = U_{ij} dx^i \cdot dx^j + U^{ij}(dt_i + A_i)(dt_j + A_j),$$

in terms of solutions of the generalized monopole equations, namely, a pair $(U_{ij}, A_i)$ satisfying

$$F_{x_i x_j} = \epsilon_{\mu\nu\lambda} \nabla_{x_\lambda} U_j,$$

$$\nabla_{x_\lambda} U_j = \nabla_{x_\lambda} U_i.$$
It will be shown that this statement is true for the metrics presented here and therefore it is again checked that they are toric hyperkahler. As a consequence a family of solutions of the Pedersen-Poon monopole equation are found in term of the eigenfunctions $F$ presented above.

To finish we recall that the Calderbank-Pedersen spaces are related to Einstein-Weyl structures by the Jones-Tod correspondence [53], which states that for any four dimensional self-dual space with at least one isometry the space of the trajectories of the Killing vector is an Einstein-Weyl space with a Toda structure defined over it. This statement applies for the Calderbank-Pedersen spaces. Einstein-Weyl structures are described by the continuum limit of the Toda equation [38]

$$(e^u)_{zz} + u_{xx} = 0,$$

and the Jones-Tod correspondence gives a map between this equation and the corresponding for $F$. This fact is of interest because gives a recipe to find solutions of a non-linear equation (the Toda one) by solving a linear one. This correspondence is crucial to find the Einstein representatives among the conformal structures with self-dual Weyl tensor with at least one isometry.

The organization of the present work is as follows: in section 2 there are described the Joyce spaces, which are the most general self-dual conformal structures with two surface orthogonal commuting Killing vectors. The underlying Toda structure of the Joyce spaces corresponding to the trajectories of its Killing vectors is found in section 3 by use of the Jones-Tod correspondence. In section 4 the quaternionic Kahler examples among them are found, that is, the Calderbank-Pedersen metrics. In section 5 the Swann construction is applied to them to find hyperkahler examples with two commuting triholomorphic isometries. The relation with the Pedersen-Poon metrics is explained in section 6 and it is find a class of solutions of the Pedersen-Poon system in terms of an eigenfunction $F$. Such form is the most suitable for physical purposes. As an application it is shown in section 7 that the hyperkahler metrics of section 4 and 5 can be extended to different supergravity solutions by use of dualities. In section 8 the Jones-Tod correspondence is used to generate some implicit and explicit solutions of the equations mentioned above. Section 9 contains the conclusions.

2. Self-dual structures with two commuting isometries

In four dimensions to say that a manifold is quaternionic Kahler is equivalent to say that is Einstein with self-dual Weyl tensor. For this reason in order to classify the toric quaternionic manifolds it is needed to classify the self-dual structures with two commuting isometries in D=4. Fortunately, there exists and complete classification of them made by Joyce if the Killing vectors are surface orthogonal [33]. The demand of $U(1) \times U(1)$ isometry and selfduality is very restrictive and in consequence all the description is made in terms of solutions of a linear system of differential equations. This section is intended to describe the Joyce classification in the most simple way as possible, and the other two are devoted to show which metrics among them are Einstein and thus toric quaternionic Kahler.

It should be reminded that for an Euclidean space in D=4 the rotation group $SO(4)$ is locally isomorphic to $SU(2) \times SU(2)$ and therefore the Weyl tensor $W$ decomposes as $W = W_+ \oplus W_-$ where the components $W_\pm$ corresponds to one of the $SU(2)$ groups. $W$ is by definition the conformally invariant part of the Riemann tensor, this means that is unchanged under an scaling $g \to \Omega^2 g$. A conformal structure $[g]$ is defined as the family of metrics obtained from $g$
by conformal transformations. If \( W_\perp = 0 \) for a given \( g \) of \([g]\) then \( g \) is called selfdual and, by conformal invariance, \([g]\) will be a selfdual structure.

Let us focus in spaces \( M \) with two commuting \( U(1) \) isometries. The manifolds in consideration are then of the form \( M = N \times T^2 \) where \( N \) is a Riemann surface, and \( T^2 = U(1) \times U(1) \) is the two dimensional flat torus. We will denote as \((\theta, \varphi)\) the periodic angles parameterizing \( T^2 \). Consider an structure \([g]\) over \( M \) with representatives \( g \) that locally takes the Gowdy form

\[
g = g_{ab}dx^a dx^b + g_{\alpha\beta}dx^\alpha dx^\beta. \tag{2.1}
\]

The latin indices \( a, b \) corresponds to vectors on \( N \) and the greek indices \( \alpha, \beta \) to vectors on \( T^2 \). Both \( g_{ab} \) and \( g_{\alpha\beta} \) are supposed to be independent of \( x^{\alpha} = (\theta, \varphi) \). It is seen that the Killing vectors are \( \partial/\partial \theta \) and \( \partial/\partial \varphi \) and the level surfaces of constant \( \theta \) and \( \varphi \) are orthogonal to both Killing fields.

By Gauss theorem there exists a local scale transformation \( g \rightarrow \Omega^2 g \) which reduce (2.1) to

\[
g = d\rho^2 + d\eta^2 + \tilde{g}_{\alpha\beta}dx^\alpha dx^\beta. \tag{2.2}
\]

Because selfduality is a property of \([g]\) rather than \( g \) there is not loss of generality in consider the anzatz (2.2) instead of (2.1). Define the basis \((e_1, e_2)\) such that

\[
\tilde{g}_{\alpha\beta}dx^\alpha dx^\beta = e_1^2 + e_2^2. 
\]

There exist a linear transformation \( T \) connecting the basis \((\rho d\theta, \rho d\varphi)\) with \((e_1, e_2)\), which we will write as

\[
T = \begin{pmatrix} A_0 & A_1 \\ B_0 & B_1 \end{pmatrix},
\]

where \( A_i \) and \( B_i \) are certain functions of \((\rho, \eta)\). By calculating \( T^{-1} \) it is seen that the angular part of \( g \) can be expressed as

\[
\tilde{g}_{\alpha\beta}dx^\alpha dx^\beta = \frac{(\rho A_0 d\theta - \rho B_0 d\varphi)^2 + (\rho A_1 d\theta - \rho B_1 d\varphi)^2}{(A_0 B_1 - A_1 B_0)^2}. \tag{2.3}
\]

The advantage of this form is that the selfduality condition is equivalent to a system of linear equations. Imposing the condition \( W_\perp = 0 \) for (2.2) gives the following proposition [33]:

**Proposition 1** Any selfdual \( g \) with two commuting killing vectors \( \partial/\partial \theta \) and \( \partial/\partial \varphi \) over \( M = N \times T^2 \) is locally conformal to a selfdual metric \( g_j \) of the form

\[
g_j = (A_0 B_1 - A_1 B_0) \frac{d\rho^2 + d\eta^2}{\rho^2} + \frac{(A_0 d\theta - B_0 d\varphi)^2 + (A_1 d\theta - B_1 d\varphi)^2}{(A_0 B_1 - A_1 B_0)}, \tag{2.4}
\]

where the functions \( A_i \) satisfies

\[
(A_0)_\rho + (A_1)_\eta = A_0/\rho, \tag{2.5}
\]

\[
(A_0)_\eta - (A_1)_\rho = 0, \tag{2.6}
\]

and the same equations holds for \( B_i \).

Equations (2.5) and (2.6) are equivalent to the condition \( W_\perp = 0 \). The Joyce metrics (2.4) are obtained by introducing (2.3) in (2.2) and making a conformal rescaling with a factor.
\[(A_0 B_1 - A_1 B_0)/\rho^2.\] Such form is more convenient in order to find the Einstein metrics among the Joyce ones. Therefore the problem to find toric selfdual structures in D=4 has been reduced to solve a linear system for \(A^i\) and independently for \(B^i\). The original proof of proposition 1 has been obtained in a rather different way than here; it is based on a method discussed in Appendix B.

It should be noted that (2.6) implies that

\[A_0 = G_\rho; \quad A_1 = G_\eta,\] (2.7)

for certain potential function \(G\). Then (2.5) implies that \(G_{\rho\rho} + G_{\eta\eta} = G_\rho/\rho\). Conversely (2.5) implies that

\[A_0 = -\rho V_\eta; \quad A_1 = \rho V_\rho,\] (2.8)

and (2.6) gives the Ward monopole equation [38]

\[V_{\eta\eta} + \rho^{-1}(\rho V_\rho)_\rho = 0.\] (2.9)

which has been proven to describe hyperkahler metrics with two commuting isometries. The relations

\[G_\rho = -\rho V_\eta = A_0; \quad G_\eta = \rho V_\rho = A_1.\] (2.10)

constitute a Backlund transformation allowing to find a monopole \(V\) starting with a known \(G\) or viceversa. The functions \(B_i\) can be also expressed in terms of another potential functions \(G'\) and \(V'\) satisfying the same equations than \(V\) and \(G\).

3. The Toda structure

This section presents some results of key importance in order to recognize the Einstein metrics among the Joyce ones. But before to present them in more detail we should state some important properties about Einstein-Weyl structures. We recall from Appendix A that a 3-dimensional Einstein-Weyl structure is an structure \([h]\) characterized by a representative \(h\) of the form (1.111) and a connection \(D\) preserving \([h]\), namely

\[h = e^u(dx^2 + dy^2) + dz^2, \quad D_a h_{bc} = \omega_a h_{bc} \quad \omega = -u_z dz.\] (3.11)

The function \(u\) satisfies the \(SU(\infty)\) Toda equation

\[(e^u)_{zz} + u_{yy} + u_{xx} = 0.\] (3.12)

With this result it is possible to enunciate the Jones-Tod correspondence [53] contained in the following proposition:

Proposition 2 a) Consider an Einstein-Weyl structure \([h]\) in D=3 and a representative \(h\). Then the four dimensional metric

\[g = Uh^+ (dt + A)^2 \] (3.13)

is selfdual with one Killing vector \(\partial_t\) if the pair of functions \((U, A)\) satisfies the generalized monopole equation

\[dA = \ast_h(dU - U\omega).\] (3.14)
The Hodge star $\ast_h$ is taken with respect to $h_{ij}$ and $\omega$ is defined in terms of the Toda solution by the third (3.11).

b) Conversely if a given $g$ is selfdual and has one conformal Killing vector $K^a$ then a conformal transformation can be performed in order that $K^a$ becomes a Killing vector $\partial_t$ and there exists a system of coordinates in which $g$ takes the form (3.13), being $h$ a representative of an Einstein-Weyl structure. The factor $\omega$ will be obtained in this case through (3.14).

From formula (3.11) it is seen that

$$dU - \omega U = U_x dx + U_y dy + (U_z + u_z U)dz = U_x dx + U_y dy + e^{-u}(e^u U)_z dz$$

and (3.14) is then explicitly

$$dA = \ast_h (dU - \omega U) = U_x dz \wedge dy + U_y dx \wedge dz + (e^u U)_z dy \wedge dx.$$ 

Therefore the integrability condition for the existence of $A$ is

$$(Ue^u)_{zz} + U_{yy} + U_{xx} = 0.$$ \hspace{1cm} (3.15)

In other words the Jones-Tod result states that for every four dimensional selfdual space with at least one isometry, the space of trajectories of the Killing vector define an Einstein-Weyl structure in 3-dimensions, and conversely every 3-dimensional Einstein-Weyl structure is the space of trajectories of a Killing field of a four dimensional selfdual space. This result applies for the Joyce spaces (2.4) as long as they have two isometries. The Jones-Tod correspondence has been originally obtained by use of minitwistor theory. But the advantage to reduce the Joyce metrics to the form (3.13) is that the following theorem [35]-[36] can be applied to find the Einstein representatives:

**Proposition 3** Any selfdual Einstein metric $g$ with one Killing vector in $D=4$ there exist a system of coordinates $(x, y, z, t)$ for which takes the form

$$g = \frac{1}{z^2}[U(e^u(dx^2 + dy^2) + dz^2) + \frac{1}{U}(dt + A)^2].$$ \hspace{1cm} (3.16)

The functions $(U, A, u)$ are independent of the variable $t$ and satisfies

$$(e^u)_{zz} + u_{yy} + u_{xx} = 0,$$ \hspace{1cm} (3.17)

$$dA = U_x dz \wedge dy + U_y dx \wedge dz + (Ue^u)_z dy \wedge dx,$$ \hspace{1cm} (3.18)

$$U = 2 - zu_z.$$ \hspace{1cm} (3.19)

Conversely, any solution of (3.17), (3.17) and (3.19) define by (3.16) a selfdual Einstein metric.

It is easily seen that if the condition (3.19) is relaxed then proposition 3 reduces to the proposition 2 up to an scaling by $1/z^2$. Then (3.19) is the condition to be satisfied in order to have an Einstein metric. It is sufficient because it can be checked that the integrability condition

$$(Ue^u)_{zz} + U_{yy} + U_{xx} = 0,$$

is always satisfied for $U = 2 - zu_z$. In other words, every $SU(\infty)$ Toda solution define a selfdual metric by (3.16). Then the problem to find the Einstein metrics among the Joyce ones is to
reduce them to the form (3.13) and then to apply (3.19). The result will be an extra relation between the functions \(A_i\) and \(B_i\) and the resulting metrics will be toric quaternionic Kahler.

The first task is to find a new coordinate system \((x, y, z, t)\) for the Joyce metrics (2.4) defined in terms of the old one \((\rho, \eta, \theta, \varphi)\) for which they are expressed as

\[
g = [U(e^u(dx^2 + dy^2) + dz^2) + \frac{1}{U}(dt + A)^2].
\]

according to (3.13). To do this it is needed to write (2.4) as

\[
g_{ij} = \frac{A_0B_1 - A_1B_0}{\rho^2(A_0^2 + A_1^2)}((A_0^2 + A_1^2)(d\rho^2 + d\eta^2) + \rho^2 d\varphi^2) + \frac{A_0^2 + A_1^2}{A_0B_1 - A_1B_0}(d\theta - \frac{(A_0B_0 + A_1B_1)d\varphi}{A_0^2 + A_1^2})^2
\]

and is seen that after rescaling by \(\rho\) and identifying \(t = \theta\) that it takes the form (3.13) with a metric \(h\) and a monopole \((U, A)\) given by

\[
h = (A_0^2 + A_1^2)(d\rho^2 + d\eta^2) + \rho^2 d\varphi^2, \quad U = \frac{A_1B_0 - A_0B_1}{\rho(A_0^2 + A_1^2)},
\]

\[
A = -\frac{(A_0B_0 + A_1B_1)}{A_0^2 + A_1^2}d\varphi.
\]

The factor \(\omega\) can be calculated through

\[
dA = *h(dU - U\omega)\]

and is

\[
\omega = -\frac{2A_0}{\rho(A_0^2 + A_1^2)}dG; \quad dG = -\rho V_\eta d\rho + \rho V_\eta d\eta.
\]

The next problem to find the coordinates \((x, y, z)\) for which (3.22) takes the form (3.11). The relation \(\omega = -u_x dz\) and (3.24) suggests that \(dz = dG\) and therefore \(G = z\) up to a translation. Indeed, the other possible differential constructed with \(V\) is

\[
dV = V_\rho d\rho + V_\eta d\eta,
\]

and it can be easily checked that

\[
dG^2 + \rho^2 dV^2 = \rho^2(V_\rho^2 + V_\eta^2)(d\rho^2 + d\eta^2) = (A_0^2 + A_1^2)(d\rho^2 + d\eta^2),
\]

where in the last step formula (2.8) has been used. From the last expression is seen that (3.22) is

\[
h = \rho^2(dV^2 + d\varphi^2) + dG^2.
\]

Comparision between (3.25) and (3.11) shows that a solution \(u(x, z)\) of the continuum Toda equation is defined by the identifications

\[
e^u = \rho^2, \quad x = V, \quad y = \varphi, \quad z = G.
\]

The solution \(u\) is independent of \(y\) is due to the presence of the other isometry, which is also a symmetry of \(h\). Formula (3.26) defines the coordinate system that we were looking for.

At first sight (3.26) relates the solutions of the axially symmetric Toda equation with two solutions \(V\) and \(G\) of two different linear differential equations. But they are related by a Backlund transformation and it can be directly checked that if \(V\) is a Ward monopole, then \(W\) such that \(W_\eta = V\) is also a Ward monopole and it follows that \(G = \rho W_\rho\). Inserting the expressions
in terms of $W$ in (3.26) and changing the notation replacing $W$ by $V$ by convenience gives the following proposition [38]:

**Proposition 4** Any solution $V$ of the equation $V_{\eta\eta} + \rho^{-1}(\rho V_\rho)_\rho = 0$ defines locally the coordinate system $(x, z)$

\[ x = V_\eta, \quad z = \rho V_\rho, \tag{3.27} \]

in terms of $(\rho, \eta)$ and conversely (3.27) defines implicitly $(\rho, \eta)$ as functions of $(x, t)$. Then the function $u(x, z) = \log(\rho^2)$ is a solution of the axially symmetric Toda equation

\[ (e^u)_{zz} + u_{xx} = 0. \tag{3.28} \]

This procedure can be inverted in order to find a Ward monopole $V$ starting with a given Toda solution.

Proposition 4 gives a method to find solutions of a non linear equation (the continuum Toda one) starting with a solution of a linear one (the Ward equation). But it is difficult in practice to find explicit solutions of (3.28) and usually proposition 4 gives implicit solutions.

An important detail is that the Toda structure (3.22) and the Toda solution $u$ are completely determined just in terms of $A_i$. Only the monopole $(U, A, \omega)$ depends on both $A_i$ and $B_i$, which are not related in any way.

### 4. Quaternionic-Kahler metrics with $U(1) \times U(1)$ isometry

It is of special interest to determine which $g$ among the Joyce metrics (2.4) are Einstein; in four dimensions selfdual Einstein spaces are quaternionic-Kahler [18]. This will be performed by applying the Einstein condition (3.19) to (2.4) and the result is the Calderbank and Pedersen metrics [34].

However it has been shown in the previous section that (2.4), (2.5) and (2.6) describe all the toric selfdual metrics with surface orthogonal Killing vectors, but there are examples that admit $T^2$ actions for which surface orthogonality do not hold, even locally (see [33] pag. 534). Nevertheless the Killing vectors of a selfdual metric with $U(1) \times U(1)$ isometry are surface orthogonal if it is Einstein [34] and this implies that the Calderbank-Pedersen metrics are the most general toric quaternionic-Kahler ones. This statement do not hold in the hyperkahler limit, in which the scalar curvature tends to zero.

For Joyce spaces the relation $\omega = -u_z dz$ and (3.24) gives

\[ u_z = \frac{A_0}{\rho(A_0^2 + A_1^2)}, \quad 2 - u_z = \frac{\rho(A_0^2 + A_1^2) - GA_0}{\rho(A_0^2 + A_1^2)}. \]

Then the insertion of the expression for $U$ (3.23) in terms of $A_i$ and $B_i$ into the Einstein condition $U = 2 - zu_z$ gives

\[ A_1B_0 - A_0B_1 = \rho(A_0^2 + A_1^2) - GA_0. \]

Thus $B_0 = \rho A_1 + \xi_0$ and $B_1 = G - A_0 + \xi_1$ with $A_1\xi_0 = A_0\xi_1$. The functions $\xi_i$ are determined by asking $B_i$ to satisfy the Joyce system (2.5) and (2.6), the result is $\xi_0 = -\eta A_0$ and $\xi_1 = -\eta A_1$. Therefore the metric $g_{ij}/\rho z^2$ is Einstein if and only if

\[ A_0 = G_\rho; \quad A_1 = G_\eta \tag{4.29} \]
\[
B_0 = \eta G_\rho - \rho G_\eta; \quad B_1 = \rho G_\rho + \eta G_\eta - G,
\]
which is the Calderbank-Pedersen solution. Defining \( G = \sqrt{\rho} F \) it follows that \( F \) satisfies
\[
F_{\rho\rho} + F_{\eta\eta} = \frac{3F}{4\rho^2}.
\]

Then inserting (4.29) and (4.30) expressed in terms of \( F \) into \( g_j/\rho z^2 \) and making the identification \( z = G \) gives the following proposition [34]:

**Proposition 5** For any Einstein-metric with selfdual Weyl tensor and nonzero scalar curvature possessing two linearly independent commuting Killing fields there exists a coordinate system in which the metric \( g \) has locally the form
\[
ds^2 = \frac{F^2 - 4\rho^2(F_\rho^2 + F_\eta^2)}{4F^2} \, d\rho^2 + d\eta^2
\]
\[
+ \frac{[(F - 2\rho F_\rho)\alpha - 2\rho F_\eta\beta]^2 + [(F + 2\rho F_\rho)\beta - 2\rho F_\eta\alpha]^2}{F^2[F^2 - 4\rho^2(F_\rho^2 + F_\eta^2)]},
\]
where \( \alpha = \sqrt{\rho}d\theta \) and \( \beta = (d\varphi + \eta d\theta)/\sqrt{\rho} \) and \( F(\rho, \eta) \) is a solution of the equation
\[
F_{\rho\rho} + F_{\eta\eta} = \frac{3F}{4\rho^2}.
\]

on some open subset of the half-space \( \rho > 0 \). On the open set defined by \( F^2 > 4\rho^2(F_\rho^2 + F_\eta^2) \) the metric \( g \) has positive scalar curvature, whereas \( F^2 < 4\rho^2(F_\rho^2 + F_\eta^2) \) - \( g \) is selfdual with negative scalar curvature.

The Einstein condition \( R_{ij} = \kappa g_{ij} \) is not invariant under scale transformations, so Proposition 5 gives all the quaternionic-Kahler metrics with \( T^2 \) isometry up to a constant multiple. The problem to find them is reduced to find an \( F \) satisfying the linear equation (4.32), that is, an eigenfunction of the hyperbolic laplacian with eigenvalue \( 3/4 \).

The equation for \( V \) (2.9) and has solutions of the form
\[
V_1(\rho, \eta) = W(\eta, i\rho) + c.c, \quad V_2(\rho, \eta) = W(i\eta, \rho) + c.c
\]
\[
W(\eta, \rho) = \frac{1}{2\pi} \int_0^{2\pi} H(\rho \text{sen}(\theta) + \eta) d\theta
\]
where \( H(z) \) is an arbitrary function of one variable [38]. The Backlund relations (2.10) define \( V \) in terms of \( G \), and consequently in terms of \( F \), and viceversa. For instance, non trivial eigenfunctions \( F \) can be constructed selecting an arbitrary \( H(z) \), performing the integration (4.33) and finding \( G \) through (2.10), then \( F = G/\sqrt{\rho} \). In the same way an axially symmetric Toda solution \( u \) can be constructed starting with an arbitrary \( H(z) \) by using proposition 3.

### 5. Toric hyperkahler geometry in eight dimensions

Four dimensional quaternionic-Kahler manifolds can be used as base spaces to construct \( G_2 \) holonomy manifolds and 8 dimensional hyperkahler ones, by use of the Bryant-Salamon [13]
and Swann [21] constructions respectively. Both types of manifolds can be extended to different supergravity solutions by use of dualities. The aim of the following two sections is to construct the hyperkahler metrics corresponding to the class (4.31) by means to the Swann extension. As it will be clear, the two $U(1)$ isometries of the Calderbank-Pedersen metrics are extended to the resulting hyperkahler ones and are triholomorphic, which means that preserve the complex structures defined over them. The presence of the Killing vectors is of importance when dealing with compactification because II supergravity backgrounds can be found starting with 11 supergravity solutions by reduction along the isometries. This point will be discussed in section 7 in more detail.

5.1 Properties of quaternionic Kahler manifolds

Before to present the Swann construction it is convenient to review certain properties of quaternionic manifolds [18]. Consider a Riemannian space $M$ of real dimension $4n$ endowed with a metric 

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu$$

and a set of three almost complex structures $J^i$ with $i = 1, 2, 3$ satisfying the quaternionic algebra 

$$J^i \cdot J^j = -\delta_{ij} + \epsilon_{ijk}J^k,$$ \hspace{1cm} (5.34)

and for which the metric $g$ satisfies $g(J^iX, J^jY) = g(X, Y)$ for any $X, Y$ in $T_x M$. A metric for which the last condition holds is known as quaternionic hermitian and it follows that $J^i_{\beta\alpha} = -J^i_{\alpha\beta}$. Any combination $C$ of the form 

$$C = a^i J^i, \quad a^i a^i = 1$$

will be an almost complex structure too, so $M$ has a family of almost complex structures parameterized by the space $S^2$ of unit imaginary quaternions. From the three almost complex structures (5.34) one can define an $SU(2)$ "gauge field"

$$\tilde{A}^i_\mu = \omega^{mn}_\mu J^i_{mn},$$ \hspace{1cm} (5.35)

and consequently an $SU(2)$-curvature 

$$F^i = d\tilde{A}^i + \epsilon_{ijk}\tilde{A}^j \wedge \tilde{A}^k,$$

where $\omega^{mn}$ is the antiselfdual part of the spin connection of $g$. Also it is possible to generalize the Kahler form corresponding to complex manifolds to an hyperkahler triplet $\Omega$, defined by 

$$\Omega^i = e^m \wedge J^i_{mn} e^n.$$ \hspace{1cm} (5.36)

Then the manifold $M$ is quaternionic Kahler if 

$$F^i = \kappa \Omega^i,$$ \hspace{1cm} (5.37)

holds, being $\kappa$ the scalar curvature of $M$.

If (5.37) is satisfied then the usual Bianchi identities of gauge theories implies that 

$$\nabla_\alpha \Omega^i = d\Omega^i + \epsilon_{ijk} \tilde{A}^j \wedge \Omega^k = 0.$$ \hspace{1cm} (5.38)
The relation (5.38) shows that the hyperkähler form of every quaternionic Kahler space is covariantly closed with respect to the connection $\tilde{\nabla}$. In the hyperkähler limit $\kappa \to 0$ and (5.37) shows that $A$ is a pure gauge and can be reduced to zero. Then

$$d\Omega^i = 0,$$

that is the hyperkähler triplet of an hyperkähler manifold is closed.

It can be shown [18] that any quaternionic metric is an Einstein space with curvature $\kappa$ and

$$R_{mn} = 3\kappa g_{mn}.$$

In four dimensions a quaternionic-Kahler metric is an Einstein metric with self-dual Weyl tensor. The holonomy $H \subseteq Sp(1) \times Sp(n)$. In the hyperkähler limit it is Ricci-flat and the holonomy is reduced to $H \subseteq Sp(n)$.

In four dimensions we can select a self-dual complex structure $(J_{ab} = -\epsilon_{abcd}J_{cd}/2)$ and the components of $\Omega^i$ and the $SU(2)$ gauge field $A_\mu$ will be given explicitly by

$$\Omega^1 = e^0 \wedge e^3 - e^1 \wedge e^2, \quad \Omega^2 = e^0 \wedge e^2 + e^3 \wedge e^1, \quad \Omega^3 = -e^0 \wedge e^1 + e^2 \wedge e^3,$$

(5.40)

$$A^1 = \omega_{\mu}^{03} - \omega_{\mu}^{12}, \quad A^2 = \omega_{\mu}^{02} + \omega_{\mu}^{31}, \quad A^3 = -\omega_{\mu}^{01} + \omega_{\mu}^{23}.$$  

(5.41)

### 5.2 The Swann extension

In order to construct toric hyperkähler metrics in eight dimensions it is convenient to introduce the quaternionic notation used in $SU(2)$ gauge theory. A metric in D=4 will be written as $g = e\overline{e} = |e|^2$ where the quaternionic valued einbein $e$ is $e = e_0 + e_iJ^i$ and $\overline{e}$ is its quaternionic conjugated. In general for two pure quaternionic 1-forms

$$\mu = \mu_0 + \mu_1J^1 + \mu_2J^2 + \mu_3J^3, \quad \nu = \nu_0 + \nu_1J^1 + \nu_2J^2 + \nu_3J^3$$

the quaternionic wedge product is defined as

$$\mu \wedge \nu = (\mu_0 \wedge \nu_1 - \mu_2 \wedge \nu_3)J^1 + (\mu_0 \wedge \nu_2 - \mu_3 \wedge \nu_1)J^2 + (\mu_0 \wedge \nu_3 - \mu_1 \wedge \nu_2)J^3. \quad (5.42)$$

$$+\mu_0 \wedge \nu_0 + \mu_1 \wedge \nu_1 + \mu_2 \wedge \nu_2 + \mu_3 \wedge \nu_3,$$

and in particular

$$\overline{e} \wedge \mu = (\mu_0 \wedge e_1 - \mu_2 \wedge e_3)J^1 + (\mu_0 \wedge e_2 - \mu_3 \wedge e_1)J^2 + (\mu_0 \wedge e_3 - \mu_1 \wedge e_2)J^3 \quad (5.43)$$

pure quaternionic components. Using (5.43) the formulas (5.40) and (5.41) can be expressed more compactly as

$$\Omega = e \wedge \overline{e}, \quad A = A^iJ^i.$$  

(5.44)

Formula (5.44) can be easily generalized to higher dimensions, for instance, a metric $g$ in eight dimensions can be expressed as $g = e_1\overline{e}_1 + e_2\overline{e}_2$ with two quaternion einbeins $e_1$ and $e_2$ and then

$$\Omega = e_1 \wedge \overline{e}_1 + e_2 \wedge \overline{e}_2.$$  

(5.45)

The quaternionic expression of the relations (5.38) and (5.37) in D=4n is expressed with the help of this notation as

$$d\Omega - \tilde{A} \wedge \Omega + \Omega \wedge \tilde{A} = 0, \quad d\tilde{A} - \tilde{A} \wedge \tilde{A} + \kappa \Omega = 0.$$  

(5.46)
It is important to present the Swann construction to express $\Omega$ and $d\Omega$ entirely in terms of $A$ and its derivatives. This is easily achieved introducing the second (5.46) into the first to give

$$\kappa d\Omega + \tilde{A} \wedge d\tilde{A} - d\tilde{A} \wedge \tilde{A}, \quad d\tilde{A} - \tilde{A} \wedge \tilde{A} + \kappa \Omega = 0.$$  (5.47)

Formulas (5.47) are the desired result. They are very useful in order to extend a quaternionic Kahler metric $g = e e$ in $D = 4$ with local coordinates $(x_1, \ldots, x_4)$ to an hyperkahler one $\overline{g}$ in $D = 8$ parameterized by $(x_1, \ldots, x_4, q)$, being $q = q_0 + q_i J^i$ a quaternionic coordinate. To achieve this task first it should be noted that the quaternionic form $\Xi$ given by

$$\Xi = dq \wedge (\kappa \Omega + d\tilde{A} - \tilde{A} \wedge \tilde{A}) \overline{\eta} + q(\kappa \Omega + d\tilde{A} - \tilde{A} \wedge \tilde{A}) \wedge d\overline{\eta},$$

is identically zero by (5.47). By another side it is possible to express $\Xi$ as a differential $\Xi = d\Phi$, where

$$\Phi = \kappa q \Omega \eta + (dq + \tilde{A}q) \wedge (dq + \tilde{A}q).$$  (5.48)

The condition $\Xi = 0$ then implies that $\Phi$ is closed. By (5.43) it follows that $\Phi$ is a pure quaternion and therefore is a candidate to be the hyperkahler triplet of an hyperkahler metric $\overline{g}$. From (5.45) it is seen that the einbein of $\overline{g}$ should be $e_1 = qe$ and $e_2 = dq + q\tilde{A}$. Then the hyperkahler metric $\overline{g} = e_1 \overline{e}_1 + e_2 \overline{e}_2$ with (5.48) as hyperkahler form is

$$\overline{g} = \kappa |q|^2 g + |dq + q\tilde{A}|^2.$$

Therefore we have obtained the Swann theorem in D=8, namely [21]:

**Proposition 6** If a four dimensional quaternionic Kahler metric $g$ is given, then the eight dimensional metric

$$\overline{g} = \kappa |q|^2 g + |dq + q\tilde{A}|^2$$  (5.49)

is hyperkahler. The coordinate $q = q_0 + q_1 J^1 + q_2 J^2 + q_3 J^3$ takes quaternionic values and the "SU(2) gauge field" $\tilde{A}$ is defined by (5.35)

Proposition 6 provides an extension to four to eight dimensions but its converse is not necessarily true, that is, not every eight dimensional hyperkahler metric can be expressed as (5.49) with a quaternionic Kahler base $g$. A counterexample will be given in the next section.

Clearly proposition 4 applies to the Calderbank-Pedersen metrics (4.31) and, by use of (5.49), it follows immediately a family of toric hyperkahler spaces for every solution of the equation (4.32). In order to find them one should calculate the expressions (5.40) and (5.41) corresponding to (4.31), the result is [34]

$$\Omega^1 = \frac{1}{F^2}(F^2 - 4\rho^2(F^2 + F^2_\alpha))(\frac{d\rho \wedge d\eta}{\rho^2} + \alpha \wedge \beta), \quad \Omega^2 = \frac{1}{F^2}(\rho F_\eta \beta + (\rho F_\alpha + \frac{1}{2} F)\alpha) \wedge \frac{d\rho}{\rho},$$

$$\Omega^3 = \frac{1}{F^2}(\rho F_\eta \alpha - (\rho F_\alpha + \frac{1}{2} F)\beta) \wedge \frac{d\eta}{\rho},$$  (5.50)

$$\overline{A}^1 = \frac{1}{F}(\frac{1}{2} F + \rho F_\rho \frac{d\eta}{\rho} - \rho F_\eta \frac{d\rho}{\rho}), \quad \overline{A}^2 = -\frac{\alpha}{F}, \quad \overline{A}^3 = \frac{\beta}{F}.$$  (5.51)
The explicit form of the resulting $\mathcal{g}$ is

$$\mathcal{g} = \kappa |q|^2 g_{cp} + (dq_o - q_i \tilde{A}^i)^2 + (dq_i + q_0 \tilde{A}^i + \epsilon_{ijk} q_k \tilde{A}^j)^2$$

(5.52)

where $g_{cp}$ is (4.31) and $\tilde{A}^i$ is given by (5.51). Also from (5.51) and (5.50) follows an explicit expression for (5.48) A new coordinate system for this metrics, more suitable for physical applications, will be found in the next section.

6. Connection with the Pedersen-Poon metrics

If the Calderbank-Pedersen metrics (4.31) are used as base spaces in the Swann construction (5.49) the resulting metrics are (5.52) and the two isometries corresponding to the Killing vectors $\partial/\partial \theta$ and $\partial/\partial \phi$ are preserved in this extension and are triholomorphic, which means that

$$\mathcal{L}_{\partial/\partial \phi} J^i = 0, \quad \mathcal{L}_{\partial/\partial \theta} J^i = 0.$$

Therefore the result is a toric hyperkahler metric in eight dimensions. But the hyperkahler metrics in $D=4n$ with $n$ commuting triholomorphic $U(1)$ isometries has been completely classified locally by Pedersen and Poon in terms of the generalized Gibbons-Hawking anzatz. Their eight dimensional statement is [32]:

**Proposition 7** For any hyperkahler metric in $D = 8$ with two commuting triholomorphic $U(1)$ isometries there exists a coordinate system in which takes the form

$$\mathcal{g} = U_{ij} dx^i \cdot dx^j + U_{ij}(dt_i + A_i)(dt_j + A_j),$$

(6.53)

where $(U_{ij}, A_i)$ are solutions of the generalized monopole equation

$$F_{x^i, x^j} = \epsilon_{\mu \nu \lambda} \nabla_{x^\mu} U_j,$$

$$\nabla_{x^\mu} U_j = \nabla_{x^\lambda} U_i,$$

(6.54)

and the coordinates $(x_1^i, x_2^i)$ with $i = 1, 2, 3$ are the momentum maps of the triholomorphic vector fields $\partial/\partial \theta$ and $\partial/\partial \phi$.

The Gibbons-Hawking form (6.53) is the most appropriated to discuss supergravity solutions and for this reason it will be instructive to check that (5.52) can be reduced to the form (6.53). But it is convenient first to explain why $(x_1^i, x_2^i)$ are the momentum maps of the isometries. In general momentum maps are related to a compact Lie group $G$ acting over an hyperkahler manifold $M$ by triholomorphic Killing vectors $X$, i.e, satisfying

$$\mathcal{L}_X J^k = 0.$$

The last condition implies that $X$ preserves the Kahler-form $\Phi_k$, that is

$$\mathcal{L}_X \Phi_k = 0 = i_X d\Phi_k + d(i_X \Phi_k).$$
Here $i_X\Phi_k$ denotes the contraction of $X$ with the Kahler forms. By supposition $M$ is hyperkahler, then $d\Phi_k = 0$ and

$$d(i_X\Phi_k) = 0.$$  

This implies that $i_X\Phi_k$ are a differential. The momentum maps $x_k^X$ are defined by

$$dx_k^X = i_X\Phi_k. \quad (6.55)$$

In the Pedersen-Poon case the isometries are $\partial/\partial \theta$ and $\partial/\partial \varphi$ and the hyperkahler form corresponding to (6.53) is \[\Phi_k = (d\theta_i + A_i)dx_k^i - U_{ij}(dx^i \wedge dx^j)_k\]

where it should be identified $t_1 = \theta$ and $t_2 = \varphi$. From the last expression it follows that

$$dx_k^\theta = i_\theta \Phi_k, \quad dx_k^\varphi = i_\varphi \Phi_k$$

and therefore $(x_1^\theta, x_2^\varphi)$ are the momentum maps of the isometries.

With this fact in mind it is possible to find the momentum map system $(x_1^\theta, x_2^\varphi)$ for (5.52). The contraction of $\partial/\partial \theta$ with the hyperkahler form (5.48) gives

$$dx_1^\theta = \frac{1}{\sqrt{\rho F}}(2q_0 dq_2 + 2q_2 dq_0 - 2q_1 dq_3 - 2q_3 dq_1 - \left(\frac{1}{2\rho} + \frac{F}{F}\right)d\rho - \frac{F}{F} d\eta),$$

$$dx_2^\theta = \frac{1}{\sqrt{\rho F}}(2q_2 dq_3 + 2q_3 dq_2 - 2q_0 dq_1 - 2q_1 dq_0 - \left(\frac{1}{2\rho} + \frac{F}{F}\right)d\rho - \frac{F}{F} d\eta),$$

$$dx_3^\theta = \frac{1}{\sqrt{\rho F}}(2q_0 dq_0 + 2q_1 dq_1 - 2q_2 dq_2 + 2q_3 dq_3 - \left(\frac{1}{2\rho} + \frac{F}{F}\right)d\rho - \frac{F}{F} d\eta).$$

The last expressions can be integrated to obtain

$$x_1^\theta = \frac{2(q_0 q_2 + q_1 q_3)}{\sqrt{\rho F}}, \quad x_2^\theta = \frac{2(q_2 q_3 - q_0 q_1)}{\sqrt{\rho F}}, \quad x_3^\theta = \frac{q_0^2 - q_1^2 - q_2^2 + q_3^2}{\sqrt{\rho F}}. \quad (6.56)$$

Similarly for $\partial/\partial \varphi$ it is found

$$x_1^\varphi = \eta x_1^\theta + \frac{2\sqrt{\rho}(q_1 q_2 - q_0 q_3)}{F}, \quad x_2^\varphi = \eta x_2^\theta + \frac{\sqrt{\rho}(q_0^2 - q_1^2 + q_2^2 - q_3^2)}{F},$$

$$x_3^\varphi = \eta x_3^\theta + \frac{2\sqrt{\rho}(q_0 q_1 + q_2 q_3)}{F}, \quad (6.57)$$

in accordance with [34].

The next step is to determine the matrix $U_{ij}$ for (5.52). This is easily found noticing that from (6.53) it follows that

$$U^{\alpha\beta} = \mathcal{F}(\frac{\partial}{\partial t^\alpha}, \frac{\partial}{\partial t^\beta}), \quad U^{\alpha\beta} A_j = \mathcal{F}(\frac{\partial}{\partial t^\alpha}, \cdot). \quad (6.58)$$

Then introducing the expression for (5.52) into the first (6.58) gives

$$U^{\alpha\beta} = \frac{|q|^2}{F(\frac{1}{4}F^2 - \rho^2(F_\rho^2 + F_\eta^2))} \left( \frac{\frac{1}{2}F - \rho F_\rho}{\frac{1}{2}F + \rho F_\rho}, -\frac{\rho F_\eta}{\frac{1}{2}F + \rho F_\rho}, -\frac{1}{2}F - \rho F_\rho \right), \quad (6.59)$$
with inverse
\[ U_{ij} = \frac{F}{|q|^2} \left( \frac{1}{2} F + \rho F_\rho + \frac{\rho F_\eta}{F} \right) \left( \frac{1}{2} F - \rho F_\rho \right). \]  
(6.60)

To find \( A_i \) one should obtain from (5.49) and (4.31) that
\[
\mathcal{g}(\frac{\partial}{\partial \theta}, \cdot) = \frac{1}{F} \left[ (q_2 \sqrt{\rho} + q_3 \eta \sqrt{\eta}) (2q_1 A^1 - dq_0) + 2(q_2 \eta \sqrt{\rho} + q_3 \sqrt{\rho})(dq_1 + q_0 A^1) - 2(q_0 \sqrt{\rho} + q_1 \eta \sqrt{\eta})(dq_2 + q_3 A^1) \right. \\
\left. + 2(q_0 \eta \sqrt{\rho} - q_1 \sqrt{\rho})(dq_3 - q_2 A^1) \right],
\]
\[
\mathcal{g}(\frac{\partial}{\partial \varphi}, \cdot) = \frac{1}{F} \left[ \frac{q_3}{\sqrt{\rho}} (2q_1 A^1 - dq_0) + \frac{2q_2}{\sqrt{\rho}} (dq_1 + q_0 A^1) - \frac{2q_1}{\sqrt{\rho}} (dq_2 + q_3 A^1) + \frac{2q_0}{\sqrt{\rho}} (dq_3 - q_2 A^1) \right],
\]
where \( A^1 \) is given in (5.51) in terms of \( F \). Then from the second (6.58) it is obtained
\[
A_1 = \frac{F}{|q|^2} \left[ \frac{F}{2} + \rho F_\rho \mathcal{g}(\frac{\partial}{\partial \theta}, \cdot) + \rho F_\eta \mathcal{g}(\frac{\partial}{\partial \varphi}, \cdot) \right],
\]
(6.61)
\[
A_2 = \frac{F}{|q|^2} \left[ \rho F_\eta \mathcal{g}(\frac{\partial}{\partial \theta}, \cdot) + \left( \frac{F}{2} - \rho F_\rho \right) \mathcal{g}(\frac{\partial}{\partial \varphi}, \cdot) \right].
\]
(6.62)

Therefore we have reduced (5.52) with this data to the form (6.53). Formulas (6.60), (6.61) and (6.62) define a class of solutions of the Pedersen-Poon equations (6.54) and an hyperkähler metric (6.53) simply described in terms of an unknown function \( F \) satisfying (4.32). We must recall however that this simplicity is just apparent because \( U_{ij} \) depends explicitly on the coordinates \((\rho, \eta, |q|^2)\), which depends implicitly on the momentum maps \((x^i_\varphi, x^i_\theta)\) by (6.56) and (6.57). Therefore \( U_{ij} \) is given only as an implicit function of the momentum maps.

For physical applications it is important to find solutions which in this limit tends to
\[
\mathcal{g} = U_{ij}^\infty dx^i \cdot dx^j + U_{ij}^\infty dt_i dt_j,
\]
(6.63)
for a constant invertible matrix \( U_{ij}^\infty \) [24]. Formulas (6.56) and (6.57) shows that the asymptotic limit \( x_\theta \to \infty \) or \( x_\varphi \to \infty \) corresponds to \( q \to \infty \) or \( \sqrt{\rho} F \to 0 \). In consequence from (6.60) and (6.59) it follows that \( U_{ij}^\infty \to \infty \) and \( U_{ij} \to 0 \) asymptotically, which is not the desired result. This problem can be evaded defining a new metric (6.53) with
\[
\overline{U}_{ij} = U_{ij} + U_{ij}^\infty,
\]
(6.64)
and with the same one-forms (6.61) and (6.62) and coordinate system (6.56) and (6.57). Clearly to add this constant do not affect the solution and this data is again a solution of the Pedersen-Poon equation (6.54) for which \( \overline{U}_{ij} \to U_{ij}^\infty \) and \( \overline{U}^\infty_{ij} \to U_{ij}^\infty \). Explicitly we have
\[
\overline{U}_{ij} = \frac{F}{|q|^2} \left( \frac{1}{2} F + \rho F_\rho + \frac{U_{ij}^\infty |q|^2}{F} \right) \left( \frac{1}{2} F - \rho F_\rho \right),
\]
(6.65)
with inverse
\[
\overline{U}^{ij} = \frac{1}{\det(U_{ij})} \left( \frac{1}{2} F - \rho F_\rho + \frac{U_{ij}^\infty |q|^2}{F} \right) \left( -\rho F_\eta + \frac{U_{ij}^\infty |q|^2}{F} \right),
\]
(6.66)
This modified metric is more suitable for physical purposes, but do not correspond to a Calderbank-Pedersen base and this show that the converse of the Swann theorem is not necessarily true.
7. Supergravity solutions related to hyperkahler manifolds

The hyperkahler spaces defined by (6.56), (6.57), (6.60), (6.61) and (6.62) can be extended to 11-dimensional supergravity solutions and to IIA and IIB backgrounds by use of dualities. This section present them following mainly [24], more details can be found there and in references therein.

The hyperkahler solutions obtained in the previous section can be lifted to D=11 supergravity solutions with vanishing fermion fields and $F_{\mu
u\alpha\beta}$. Such solutions are of the form

$$ds^2 = ds^2(E^{2,1}) + U_{ij}dx^i \cdot dx^j + U^{ij}(dt_i + A_i)(dt_j + A_j),$$  

(7.67)

and admits the action of a torus. Because the fields are invariant under the action of the Killing vectors a solution of the IIA supergravity can be found by reduction along one of the isometries, say $\partial/\partial \phi$. The Kaluza Klein anzatz is

$$ds^2 = e^{-\frac{3}{4} \varphi(x)} g_{\mu\nu}(x)dx^\mu dx^\nu + e^{\frac{3}{2} \varphi(x)}(dy + C_\mu(x)dx^\mu)^2,$$

(7.68)

$$A_{11} = A(x) + B(x) \wedge dy.$$  

(7.69)

The field $A_{11}$ is the 3-form potential and $x^\mu$ are the coordinates of the D=10 spacetime. The $NS \otimes NS$ sector is $(\phi, g_{\mu\nu}, B_{\mu\nu})$ and the $R \otimes R$ sector is $(C_\mu, A_{\mu\nu\rho})$. After reduction the nonvanishing fields are

$$ds_{10}^2 = (\frac{U_{11}}{detU})^2[ds^2(E^{2,1}) + U_{ij}dx^i \cdot dx^j] + (\frac{1}{U_{11}detU})^{1/2}(d\theta + A_1)^2,$$

(7.70)

$$\phi = \frac{3}{4} \log(U_{11}) - \frac{3}{4} \log(detU),$$

(7.71)

$$C = A_2 - \frac{U_{12}}{U_{11}}(d\theta + A_1).$$

(7.72)

All the quantities were independent of $\varphi$ and survived as Killing spinors of the reduced theory.

The field $\phi$ is independent of $\theta$ and $C$ satisfies $\mathcal{L}_k C = 0$ and one can use T-duality rules to construct a IIB supergravity solution

$$ds^2 = [g_{mn} - g_{\theta\theta}^{-1}(g_{m\theta}g_{n\theta} - B_{m\theta}B_{n\theta})]dx^m dx^n + 2g_{\theta\theta}^{-1}B_{m\theta}d\theta dx^n + g_{\theta\theta}^{-1}d\theta^2$$

(7.73)

$$\tilde{B} = \frac{1}{2} dx^m \wedge dx^n[B_{mn} + 2g_{\theta\theta}^{-1}(g_{m\theta}B_{n\theta})] + g_{\theta\theta}^{-1}g_{\theta m} d\theta \wedge dx^m$$

(7.74)

$$\tilde{\phi} = \phi - \log(g_{\theta\theta})$$

(7.75)

where the tilde indicates the transformed fields. The restrictions

$$B = 0, \quad i_k A = 0,$$

gives the T-dual fields

$$l = C_\theta,$$

$$B' = [C_{mn} - (g_{\theta\theta})^{-1}C_{\theta m}]dx^m \wedge d\theta,$$

(7.76)

$$i_k D = A,$$
where $l$ is the IIB pseudoscalar, $B'$ is the Ramond-Ramond 2-form potential and $D$ is the IIB 4-form potential. The non vanishing IIB fields resulting from the application of the T-duality are

$$ds^2_{10} = (detU)^{3/4}[(detU)^{-1}ds^2(E^{2,1}) + (detU)^{-1}U_{ij}dx^i \cdot dx^j + d\theta^2],$$  \hspace{1cm} (7.77)

$$B_i = A_i \wedge d\theta,$$  \hspace{1cm} (7.78)

$$\tau = -\frac{U_{12}}{U_{11}} + i\sqrt{\det U} U_{11},$$  \hspace{1cm} (7.79)

where

$$\tau = l + ie^{-\phi_B}, \quad B_1 = B, \quad B_2 = B',$$

and $ds^2_{10}$ is the Einstein frame metric satisfying

$$ds^2_{10} = e^{-\phi_B/2}ds^2_{11B}.$$  

More examples can be obtained by reducing (7.67) along one of the space directions $E^{2,1}$ and it is obtained the IIA solution

$$ds^2 = ds^2(E^{1,1}) + U_{ij}dx^i \cdot dx^j + U^{ij}(dt_i + A_i)(dt_j + A_j),$$  \hspace{1cm} (7.80)

with the other fields equal to zero. After T-dualizing in both angular directions it is obtained

$$ds^2 = ds^2(E^{1,1}) + U_{ij}dX^i \cdot dX^j,$$  \hspace{1cm} (7.81)

$$B = A_i \wedge dt_i,$$  

$$\phi = \frac{1}{2} \log(detU),$$

where $X^i = x^i, t^i$. This solution can be lifted to a D=11 supergravity solution

$$ds^2_{11} = (detU)^{-3/2}[(detU)^{-1}ds^2(E^{1,1}) + (detU)^{-1}U_{ij}dx^i \cdot dX^j + d\theta^2],$$  \hspace{1cm} (7.82)

$$F = F_i \wedge dt_i \wedge d\theta.$$  \hspace{1cm} (7.83)

It is possible to generalize (7.67) to include a non vanishing 4-form $F$. The result is the membrane solution

$$ds^2 = H^{-2/3}ds^2(E^{2,1}) + H^{1/3}[U_{ij}dx^i \cdot dx^j + U^{ij}(dt_i + A_i)(dt_j + A_j)],$$  \hspace{1cm} (7.84)

$$F = \pm \omega(E^{2,1}) \wedge dH^{-1},$$  \hspace{1cm} (7.85)

where $H$ is an harmonic function on the hyperkahler manifold, i.e, satisfies

$$U^{ij} \partial_i \cdot \partial_j H = 0.$$  

We have seen in the last section that every entry of $U_{ij}$ is an harmonic function and so such $H$ can be generated with an hyperbolic eigenfunction $F$. After reduction along $\varphi$ it is found the following IIB solution

$$ds^2_{10} = (detU)^{3/4}H^{1/2}[H^{-1}(detU)^{-1}ds^2(E^{2,1}) + (detU)^{-1}U_{ij}dx^i \cdot dx^j + H^{-1}d\theta^2],$$  \hspace{1cm} (7.86)

$$B_i = A_i \wedge d\theta,$$  \hspace{1cm} (7.87)
\[\tau = -\frac{U_{12}}{U_{11}} + i \frac{\sqrt{\text{det} U}}{U_{11}},\]  
\[i_k D = \pm \omega(E^{1,1}) \wedge dH^{-1}.\]  

If instead (7.84) is dimensionally reduced along a flat direction it is obtained the IIA solution

\[ds^2 = ds^2(E^{1,1}) + U_{ij} dx^i \cdot dx^j + U_{ij} (dt_i + A_i)(dt_j + A_j),\]  
\[B = \omega(E^{1,1}) H^{-1},\]  
\[\phi = -\frac{1}{2} \log(H).\]  

A double dualization gives a new IIA solution

\[ds^2 = H^{-1} ds^2(E^{1,1}) + U_{ij} dX^i \cdot dX^j,\]  
\[B_i = A_i \wedge d\phi^i + \omega(E^{1,1}) H^{-1},\]  
\[\phi = \frac{1}{2} \log(\text{det} U) - \frac{1}{2} \log(H).\]  

The lifting to eleven dimensions gives

\[ds_{11}^2 = H^{1/3}(\text{det} U)^{2/3} [H^{-1}(\text{det} U)^{-1} ds^2(E^{1,1}) + (\text{det} U)^{-1} U_{ij} dX^i \cdot dX^j + H^{-1} d\theta^2],\]  
\[F = (F_i \wedge dt_i + \omega(E^{1,1}) \wedge dH^{-1}) \wedge d\theta.\]

All the backgrounds presented in this section can be constructed with a single F satisfying (4.32), but the dependence on \((x_\theta, x_\varphi)\) remains implicit.

### 8. Explicit and implicit solutions

In this section some particular solutions of the linear equations for F and V will be constructed together with their Toda counterparts.

The Ward equation \(V_{\eta\eta} + \rho^{-1}(\rho V_\rho)_\rho = 0\) can be solved by separation of variables. The solutions obtained in this way are

\[V_0 = (A + B\eta) \log(\rho), \quad V_+ = (C\cos(\omega\eta) + D\sin(\omega\eta))(EI_0(\omega \rho) + FK_0(\omega \rho)),\]  
\[V_- = (G\cosh(\omega\eta) + H\sinh(\omega\eta))(MJ_0(\omega \rho) + NY_0(\omega \rho)),\]  

where \(\omega, A, B, ..., N\) are constants, \(J_0(\omega \rho)\) and \(Y_0(\omega \rho)\) are Bessel functions of first and second kind (or Newmann functions) and \(I_0(\omega \rho)\) and \(K_0(\omega \rho)\) are modified Bessel functions of first and second kind (or MacDonald functions). From (2.10) it follows that

\[G_\rho = -\rho V_\eta; \quad G_\eta = \rho V_\rho\]

and \(F = G/\sqrt{\rho}\) define separated solutions \(F\) of (4.32) given by

\[F_0 = (A\rho^{3/2} + B\rho^{-1/2})(C\eta + D),\]  
\[F_+ = \rho^{1/2}(EI_1(\omega \rho) + FK_1(\omega \rho))(G\sin(\omega\eta) + H\cos(\omega\eta)),\]  

\[(8.99)\]
\[ F_- = \rho^{1/2}(NJ_1(\omega\rho) + MY_1(\omega\rho))(V\sinh(\omega\eta) + U\cosh(\omega\eta)). \]

This type of solutions has been used in [43] to construct certain \( G_2 \) holonomy examples. One way to obtain non factorized solutions is to take the continuum limit of (8.98) and (8.99) by selecting \( A, ..., U, V \) as functions of \( \omega \) and integrating in terms of this variable.

The task of finding non separated solutions can be achieved selecting an arbitrary complex function \( H(z) \) in (4.33) and solving (2.10). The powers \( H(z) = z^n \) can be integrated out giving polynomial solutions. For instance selecting \( H(z) = z^3 \) gives

\[ V = 3\eta\rho^2 - 2\eta^3; \quad F = \frac{3}{4}\rho^{3/2}(4\eta^2 - \rho^2). \]

and a Toda solution holds by defining the coordinate system \((x, z)\)

\[ x = 3\rho^2 - 6\eta^2, \quad z = 6\eta\rho^2 \]

and \( u(x, t) = \log(\rho^2) \). Even in this simple case the coordinates \((\rho, \eta)\) are given implicitly by the relations

\[ \rho^2 = \frac{z}{6\eta}, \quad 6\eta^3 + 2\eta x - z = 0. \]

For \( H(z) = z^5 \) it is obtained

\[ V = \eta^5 - 5\eta^3 \rho^2 + \frac{15\eta^4}{8}; \quad F = \frac{15\eta^2\rho^{7/2}}{4} - \frac{5\eta^4\rho^{3/2}}{2} - \frac{15\rho^{11/2}}{48}. \]

The powers \( H(z) = z^n \log(z) \) can be also integrated explicit to give more complicated expressions including logarithmic terms. Simple separated solutions has been used to study loop corrections to the universal dilaton supermultiplet for type IIA strings on a Calabi-Yau manifold [52].

One important class of non separated solutions are the \( m \)-pole ones, investigated in [46] and [34]. They give rise to the toric quaternionic Kahler metrics that are complete, compact and admitting only orbifold singularities. Therefore the \( G_2 \) holonomy manifolds constructed with them as basis are appropriated to discuss the appearance of non abelian gauge groups and chiral matter by M-theory compactifications [14], [44] and [45]. The basic eigenfunctions \( F \) of (4.32) from which this solutions are constructed are

\[ F(\rho, \eta, y) = \frac{\sqrt{(\rho)^2 + (\eta - y)^2}}{\sqrt{\rho}} \tag{8.100} \]

where the parameter \( y \) takes arbitrary values. Using the Backlund transformation it is found the basic monopole

\[ V(\eta, \rho, y) = -\log[\eta - y + \sqrt{\rho^2 + (\eta - y)^2}]. \tag{8.101} \]

The 2-pole solutions are given by

\[ F_1 = \frac{1 + \sqrt{\rho^2 + \eta^2}}{\sqrt{\rho}}; \quad F_2 = \frac{\sqrt{(\rho)^2 + (\eta + 1)^2}}{\sqrt{\rho}} - \frac{\sqrt{(\rho)^2 + (\eta - 1)^2}}{\sqrt{\rho}}, \]

The first one gives rise to the spherical metric and the second to the hyperbolic metric

\[ ds^2 = (1 - r_1^2 - r_2^2)^{-2}(dr_1^2 + dr_2^2 + r_1^2d\theta_1^2 + r_2^2d\theta_2^2). \]
The relation between the coordinates \((r_1, r_2)\) and \((\rho, \eta)\) can be extracted from the relation

\[
(r_1 + ir_2)^2 = \frac{\eta - 1 + i\rho}{\eta + 1 + i\rho}.
\]

The general "3-pole" solution is

\[
F = \frac{1}{\sqrt{\rho}} + \frac{b + c/m}{2} \frac{\sqrt{\rho^2 + (\eta + m)^2}}{\sqrt{\rho}} + \frac{b - c/m}{2} \frac{\sqrt{\rho^2 + (\eta - m)^2}}{\sqrt{\rho}}.
\]

By definition \(-m^2 = \pm 1\), which means that \(m\) can be imaginary or real. The corresponding solutions are denominated type I and type II respectively and encode many well known examples like the Bergmann metric on \(CH^2\), the Eguchi-Hanson metrics, the Bianchi VIII metrics, the bi-axial Bianchi IX metric and the Fubbini-Study metric on \(CP^2\) [34]. There are also included to some quaternionic Kahler extensions of hyperkahler metrics with two centers and \(U(1) \times U(1)\) isometry [47]. 3-pole solutions have many physical applications. They have been considered in [48] to construct type IIA solutions that can be interpreted as intersecting 6-branes. Moreover in [49], the N=2 gauged supergravity coupled to the universal hypermultiplet with a quaternionic geometry corresponding to the 3-pole solution has been considered, and it has been studied the possibility to obtain the de Sitter vacua.

The general \(m\)-pole solution is of the form

\[
F(\rho, \eta) = \sum_{k=0}^{\infty} \frac{\sqrt{a_k^2 \rho^2 + (a_k \eta - b_k)^2}}{\sqrt{\rho}},
\]

for some real moduli \((a_k, b_k)\) and the duality group \(SL(2, \mathbb{R})\) acts over them [34]. The corresponding monopole \(V\) is

\[
V(\rho, \eta) = -\sum_{k=0}^{\infty} \text{Log}[a_k \eta - b_k + \sqrt{a_k^2 \rho^2 + (a_k \eta - b_k)^2}]
\]

and it can be checked that it satisfies (2.9).

Another solution with application to string theories is [50],[51] and [49]

\[
F(\rho, \eta) \sim f_{3/2}(\tau, \bar{\tau}) = \sum_{(p,n)\neq (0,0)} \frac{\tau_2^{3/2}}{|p + n\tau|^3}
\]

where \(\tau = \tau_1 + i\tau_2 = \eta + i\rho\) and \(f_{3/2}(\tau, \bar{\tau})\) is defined by the Eisenstein series. Solution (8.102) is invariant under the \(SL(2, \mathbb{Z})\) duality

\[
\tau \rightarrow \frac{a\tau + b}{c\tau + d}, \quad ad - bc = 1, \quad (a, b, c, d) \in \mathbb{Z}
\]

and describes the D-instanton corrections of the Universal Hypermultiplet moduli space preserving some \(U(1) \times U(1)\) symmetry, which are originated by Calabi-Yau wrapped two branes.

Every solution \(V\) presented here define implicitly a solution of the continuum Toda equation by Proposition 5 and every \(F\) define completely a toric quaternionic Kahler metric by proposition 3, a toric 8 dimensional hyperkahler metric by (6.56), (6.57), (6.60), (6.61), (6.62) and (6.53), and different supergravities solutions by the results of section 6.
9. Conclusions

In the present work some important facts about toric quaternionic Kahler geometries in D=4 and toric hyperkahler geometries in D=8 has been presented and in particular, it has been shown that the Swann construction provides a valid link between them. The fundamental reason is that if a quaternionic Kahler space with torus isometry is used as a base in this construction, then the two commuting isometries are preserved and are triholomorphic. As a result the Swann extension of a toric quaternionic kahler space in D=4 is an hyperkahler one with $T^2$ symmetry in D=8. Because toric quaternionic spaces in D=4 are described in terms of a simple linear equation (the Calderbank-Pedersen theorem), the resulting eight dimensional spaces are also described in this manner. This result can be compared with the Pedersen-Poon theorem that states that for toric hyperkahler metrics there exists a coordinate system for which they take the generalized Gibbons-Hawking anzatz. This system is related to the momentum maps of the isometries, and to solutions of a generalized monopole equation. As a consequence of this comparison there are found solutions for the generalized Gibbons-Hawking anzatz just in terms of solutions the Calderbank-Pedersen equation, as was shown in section 6. The simplicity of these solutions is just apparent, because the dependence of the Gibbons-Hawking metric in terms of the momentum maps is given implicitly.

It should be remarked that there is not reason to state, in principle, that all the eightdimensional toric hyperkahler spaces arise by the construction presented here. We have presented only a subfamily among them in this paper. Such spaces were lifted to 11-dimensional and IIA and IIB supergravity solutions by use of dualities, and all the description is made as before in terms of a single F but with implicit dependence on the momentum maps.

The Calderbank-Pedersen description can be viewed as a consequence of the Joyce classification of selfdual structures in four dimensions, together with a result due to Tod and Przanowski. Section 2 was intended to give the most simple presentation of the Joyce spaces as possible. The Calderbank-Pedersen metrics are the Einstein representatives among the Joyce families and are the most general local form of a quaternionic Kahler metric in D=4 with $U(1) \times U(1)$ isometry. The conditions arising from the demand of $T^2$ isometry together with the selfduality of the Weyl tensor are very restrictive and as a consequence, all description is achieved only in terms of the Joyce system, which is linear. The Einstein condition reduces this system to a linear eigenvalue problem, that is, to find certain eigenfunctions $F$ of the laplacian of the two dimensional hyperbolic plane.

This results presented in this paper can have different applications. For instance proposition 3 describes the most general local form of a selfdual Einstein metric with one isometry, in terms of the Toda equation. They are the most general quaternionic Kahler ones with one Killing vector and can be extended to $G_2$ holonomy metrics by use of the Bryant-Salamon construction [13], that are asymptotically a cone over a weak Calabi-Yau manifold. Such 7-dimensional metrics can be lifted to 11-supergravity backgrounds and preserves 1/8 of the supersymmetries after compactification. The presence of one Killing vector allows a type IIA interpretation after Kaluza-Klein reduction along the isometry.

It will be also useful to analyze which are the symmetries of the solutions of the Pedersen-Poon equations presented in this work. In the well known brane solutions [24] the dependence in terms of the momentum maps is explicit, and the symmetries of this solution related to momentum map system are directly seen. But the solutions presented here, although they have the same asymptotic limit, have not such explicit form and an interesting problem is to find their symmetries.
10. Acknowledgments

We were benefited with mail correspondence with D.Joyce that made us clear certain features about his work. We give thanks to A.Isaev, T. Mohaupt and S.Ketov for discussions about the subject and its applications and to D.Mladenov, B.Dimitrov, T.Pilling and A.Oskin for many useful conversations. O.P.S thanks specially to Luis Masperi for his permanent and unconditional support during his mandate at the CLAF (Centro Latinoamericano de Fisica).

A Generalities about Einstein-Weyl structures

The Einstein-Weyl structures are a generalization of the ordinary Einstein equation for which exists a twistor correspondence [37]. The Einstein equations are generalized in this picture in order to include invariance under coordinate rotations plus dilatations. In this section some important facts about them are sketched following [37] and [53].

It is known that for a given space \( \mathbb{W} \) endowed with a metric \( g_{ab} \) the Levi-Civita connection \( \nabla \) is uniquely defined by

\[
\nabla g = 0, \quad T(\nabla) = 0
\]

where \( T(\nabla) \) is the torsion. A Weyl-structure is defined by the manifold \( \mathbb{W} \) together with:

(a) A class of conformal metrics \([g]\), whose elements are related by the conformal rescaling (or gauge transformation)

\[
g_{ab} \rightarrow \Omega^2 g_{ab},
\]

(1.104)

together with \( SO(n) \)-coordinate transformations. \( \Omega^2 \) is an smooth, positive real function over \( \mathbb{W} \).

(b) A torsion-free connection \( D_a \) which acts over a representative \( g_{ab} \) of the conformal class \([g]\) as

\[
D_a g_{bc} = \omega_a g_{bc},
\]

(1.105)

for certain functions \( \omega_a \) defining an one form \( \omega \). Then it is said that \( D \) preserves \([g]\).

The conformal group in \( n \)-dimensions is \( CO(n) = R^+ \times SO(n) \) and includes rotations plus general scale transformations. The structure \([g]\) is called \( CO(n) \)-structure over \( \mathbb{W} \). The connection \( D \) is uniquely determined by (1.105) in terms of \( \omega \) and \( g \). This can be seen in a coordinate basis \( \partial_k \) in which the system (1.105) takes the form

\[
\eta_{ab,c} = g_{lb} \Gamma_{ac}^l + g_{al} \Gamma_{bc}^l + \omega_a g_{bc},
\]

where the symbols \( \Gamma_{jk}^i \) denotes the connection coefficients of \( D \), which are symmetric in the lower indices by the torsionless condition. Thus a series of steps analogous to those needed to determine the Levi-Civita connection shows that \( \Gamma_{jk}^i \) is

\[
\Gamma_{jk}^i = \Gamma_{ij}^k + \gamma_{jk}^i
\]

(1.106)

where \( \Gamma_{ik}^j \) are the Christofel symbols and the add \( \gamma_{jk}^i \) is

\[
2\gamma_{jk}^i = (\delta_j^i \omega_k + \delta_k^i \omega_j + g_{jk} \omega^i).
\]

(1.107)
The form $\omega$ is not invariant under (1.108), its transformation law can be obtained from (1.105) and (1.107) and is

$$\omega_a \rightarrow \omega_a + 2 \partial_a \log(\Omega).$$

(1.108)

It follows from (1.107) and (1.108) that the Levi-Civita of any $g$ of $[g]$ preserves the conformal structure.

If for a Weyl-structure the symmetric part of the Ricci tensor $\tilde{R}_{(ij)}$ constructed with $D_i$ satisfies

$$\tilde{R}_{(ij)} = \Lambda g_{ij}$$

(1.109)

for certain $\Lambda$, then will be called Einstein-Weyl. If in addition the antisymmetric part of $\tilde{R}_{[ij]}$ vanish there exists a gauge in which (1.109) reduces to the vacuum Einstein equation with cosmological constant $\Lambda$. To see this it is needed to calculate

$$\tilde{R} = D_X D_Y - D_Y D_X - D_{[X,Y]}$$

using the formula (1.106) for $D$. The relation $CO(4) = R_+ \times SO(4)$ decompose $\tilde{R}$ into a real component $R_0$ and into an $SO(4)$-component $R$ that is equal to the curvature tensor constructed with $\nabla$. After contracting indices it is obtained

$$\tilde{R}_{ij} = R_{ij} + \nabla_i \omega_j - \frac{1}{2} \nabla_i \omega_j - \frac{1}{4} \omega_i \omega_j + g_{ij} \left( \frac{1}{2} \nabla_k \omega_k + \frac{1}{4} \omega_k \omega^k \right),$$

where $R_{ij}$ is the Ricci tensor found with $\nabla$. The antisymmetric part is originated by the $R_0$ component and is determined in terms of $\omega$ as

$$\tilde{R}_{[ij]} = \frac{3}{2} \nabla_{[i} \omega_{j]}.$$

(1.110)

If (1.110) is zero, then $\omega$ is the gradient of certain function $\Psi$ over $W$. The conformal rescaling (1.108) with $\Omega = -e^\Psi$ set $\omega = 0$. This reduce the symmetric part

$$\tilde{R}_{(ij)} = R_{ij} - \frac{1}{2} \nabla_{(i} \omega_{j)} - \frac{1}{4} \omega_i \omega_j$$

to $R_{ij}$ and (1.109) is the Einstein equation with $\Lambda$, thus $[g]$ contains an Einstein metric.

In 3 dimensions Einstein-Weyl structures are related to the solutions of the continuum limit of the Toda equation. If the anzatz for the metric

$$h = e^u (dx^2 + dy^2) + dz^2,$$

(1.111)

is introduced in the Einstein equation (1.109) then it follows that $u$ should satisfy [38]

$$(e^u)_{zz} + u_{yy} + u_{xx} = 0,$$

(1.112)

$$\omega = -u_z dz.$$

(1.113)

The non linear equation (1.112) is known as the $SU(\infty)$ Toda equation, and is seen (1.113) that $\omega$ is entirely determined in terms of $u$. This equation is integrable, but not many explicit solutions are known.
B The Joyce description of selfdual structures

One way to construct selfdual structures in four dimensions is to use the definition, i.e., to take an anzatz for a metric tensor $g$, find the Levi-Civita connection and solve the system of equations corresponding to $W_- = 0$, then $[g]$ will be selfdual. By another side it seems more natural to describe an structure $[g]$ in terms of a connection $D$ preserving it like (1.105) than in terms of the Levi-Civita one. This was done by Joyce who considered which conditions should satisfy the curvature and the torsion of a connection $D$ preserving $[g]$ in order to insure selfduality [33]. Some important results have been successfully reformulated in this context, but the novelty is that it gives rise to the classification of all 4 dimensional self-dual structures with two commuting isometries that are surface orthogonal.

The first question that arise is if it is possible to express $W$ in terms of $D$ in order to impose $W_- = 0$ as a condition on $D$. It is well known that the irreducible components of the Riemann tensor under the action of $SO(4)$ are the scalar curvature $S$, is the traceless part of the Ricci tensor $R_{ij}$ and the two components $W_\pm$ of the Weyl tensor [39]-[40]. But this result is valid only for the curvature constructed with the $SO(4)$ Levi-Civita connection $\nabla$. If instead a torsionless $CO(4)$-connection $D$ preserving $[g]$ is considered, the relation $CO(4) = R_+ \times SO(4)$ splits $D$ into a real valued connection and an $SO(4)$ connection and therefore the corresponding curvature $\tilde{R}$ is divided into an $SO(4)$ component $R(D)$ and into an $R$-component $R_0(D)$. The irreducible parts are in this case 6 and among them there are two, denoted as $W_\pm(D)$, which are equal to $W_\pm$ (see for instance Appendix A of [54]). Therefore $W$ can be described in terms of a torsionless $D$ preserving $[g]$.

If $D$ is supposed to have torsion then $\tilde{R}$ has 10 irreducible parts [33] and $W_\pm(D)$ is different from $W_\pm$, even in the limit $\omega \to 0$. Nevertheless, a careful analysis shows that if $T(D)$ is selfdual, then $W_-(D) = W_-$. From this discussion holds the following important result [33]:

**Proposition 8** If for a conformal class $[g]$ it exists a connection preserving $D$ for which

$$T_- (D) = W_- (D) = 0, \quad (2.114)$$

then $[g]$ is selfdual.

Conditions (2.114) shows that selfdual $[g]$ can be characterized in terms of a connection $D$ preserving it if the torsion is selfdual. Proposition 8 is a powerful method to construct selfdual families $[g]$, although it is not the most general one.

The Ashtekar-Jacobson-Smolin description of selfdual Einstein spaces arise as a simple application of Proposition 8. Consider a manifold $M$ and four vector fields $e^a$ forming an oriented basis for $TM$ at each point, and the metric $g$ constructed with $e^a$. The parallelizing connection $D$ satisfies $De^a = 0$, in this basis the Christofel symbols are all zero and so $R(D) = 0$. For this reason the condition $W^-(D) = 0$ is trivially satisfied. $D$ preserves the metric $g$, and the class $[g]$ of $g$. The torsion is $T(e^a, e^b) = De^a e^b - D e^a e^b = [e^a, e^b] = - [e^a, e^b]$ and has the anti-selfdual components

$$T(e^1, e^2) - T(e^3, e^4), \quad T(e^4, e^1) - T(e^2, e^3), \quad T(e^1, e^4) - T(e^2, e^3),$$

thus $T^-(D) = 0$ if and only if

$$[e^1, e^2] - [e^3, e^4] = 0, \quad [e^1, e^3] - [e^2, e^4] = 0, \quad [e^1, e^4] - [e^2, e^3] = 0. \quad (2.115)$$
By Proposition 8 the equations (2.115) defines a selfdual structure \([g]\); it is known as the Ashtekar-Jacobson-Smolin formulation of the selfdual Einstein equations [41]. In particular selecting \(e^j = f_j \partial/\partial x_1 + \partial/\partial x_j\) it is obtained

\[
\frac{\partial f_1}{\partial x_2} - \frac{\partial f_3}{\partial x_4} + \frac{\partial f_4}{\partial x_3} = 0, \quad \frac{\partial f_1}{\partial x_3} - \frac{\partial f_4}{\partial x_2} + \frac{\partial f_2}{\partial x_4} = 0, \quad \frac{\partial f_1}{\partial x_4} - \frac{\partial f_2}{\partial x_3} + \frac{\partial f_3}{\partial x_2} = 0.
\]

If the metric has the Killing vector \(\partial/\partial x^1\), then the functions \(f_i\) are independent of \(x_1\) and this system reduces to

\[
\nabla U = \nabla \times \omega
\]

where we have defined \(U = f_1\) and \(\omega = (f_2, f_3, f_4)\). The corresponding metric is

\[
g = V^{-1}(dt - \omega)^2 + V dx \cdot dx,
\]

(2.117)

It has been proved that (2.117) describes all four dimensional selfdual examples that are Ricci-flat with a triholomorphic \(U(1)\)-isometry, they are known as the Gibbons-Hawking metrics [42].

To finish it should be mentioned than the Jones-Tod correspondence and the Joyce description of selfdual structures with \(U(1) \times U(1)\) isometry arise as a consequence of proposition 8 but the task to find the connection \(D\) is more difficult. The reader interested in details can look at the original reference [33].

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