Non-Cooperative Rational Interactive Proofs *

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Abstract

Interactive-proof games model the scenario where an honest party interacts with powerful but strategic provers, to elicit from them the correct answer to a computational question. Interactive proofs are increasingly used as a framework to design protocols for computation outsourcing.

Existing interactive-proof games largely fall into two categories: either as games of cooperation such as multi-prover interactive proofs and cooperative rational proofs, where the provers work together as a team; or as games of conflict such as refereed games, where the provers directly compete with each other in a zero-sum game. Neither of these extremes truly capture the strategic nature of service providers in outsourcing applications. How to design and analyze non-cooperative interactive proofs is an important open problem.

In this paper, we introduce a mechanism-design approach to define a multi-prover interactive-proof model in which the provers are rational and non-cooperative—they act to maximize their expected utility given others’ strategies. We define a strong notion of backwards induction as our solution concept to analyze the resulting extensive-form game with imperfect information.

We fully characterize the complexity of our proof system under different utility gap guarantees. (At a high level, a utility gap of $u$ means that the protocol is robust against provers that may not care about a utility loss of $1/u$.) We show, for example, that the power of non-cooperative rational interactive proofs with a polynomial utility gap is exactly equal to the complexity class $\text{P}^\text{NEXP}$.

1 Introduction

Game theory has played a central role in analyzing the conflict and cooperation in interactive proof games. These games model the scenario where an honest party interacts with powerful but strategic agents, to elicit from them the correct answer to a computational question. The extensive study of these games over decades has fueled our understanding of important complexity classes (e.g., [1,15,24,26,28,42]). From a modern perspective, these games capture the essence of computation outsourcing—the honest party is a client outsourcing his computation to powerful rational service providers in exchange for money.

In this paper, we consider a natural type of interactive-proof game. For the moment, let us call our client Arthur. Arthur hires a service provider Merlin to solve a computational problem for him, and hires a second service provider Megan to cross-check Merlin’s answer. Arthur wants the game (and associated payments) to be designed such that if Merlin gives the correct answer,

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Megan agrees with him; however, if Merlin cheats and gives a wrong answer, Megan is incentivized to contradict him, informing Arthur of Merlin’s dishonesty. This means that Merlin and Megan are not purely cooperative nor purely competitive. Each is simply a rational agent who wants to maximize their own utility.

This is a mechanism design problem—how can Arthur incentivize non-cooperative rational agents (Merlin and Megan) to give truthful answers to his questions, helping him solve a computational problem? This problem is the focus of our paper.

Structure of the game. We borrow the structure and terminology of interactive proofs [3,7,33], as was done in previous work on rational proofs [1,2,13,14,19,21,34,35] and refereed games [18,24,26,28,40,45]. We call Arthur the verifier and assume that he is computationally bounded (he may be probabilistic, but must run in polynomial time). Arthur’s coin flips are treated as Nature moves in the game. We call Merlin and Megan the provers; they have unbounded computational power.

The verifier exchanges messages with the provers in order to determine the answer to a decision problem. The exchange proceeds in rounds: in a round, either a verifier sends a message to all provers or receives a response from each. The provers cannot observe the messages exchanged between the verifier and other provers.

At the end, the verifier gives a payment to each prover. Our goal is to design protocols and payments such that, under an appropriate solution concept of the resulting game, the provers’ best strategies lead the verifier to the correct answer.

The interactive protocols described above form an extensive-form game of imperfect information. To analyze them, we essentially use a strong notion of backward induction as our solution concept. We refine it further by eliminating strategies that are weakly dominated on “subgames” within the entire game. We define the solution concept formally in Section 2.2.

Comparison to previous work. The model of our games is based on interactive proof systems [3,33], in which a verifier exchanges messages with untrustworthy provers and at the end either accepts or rejects their claim. Interactive proofs guarantee that, roughly speaking, the verifier accepts a truthful claim with probability at least 2/3 (completeness) and no strategy of the provers can make the verifier accept a false claim with probability more than 1/3 (soundness).

The study of interactive proofs has found extensive applications in both theory and practice. Classical results on IPs have led us to better understand complexity classes through characterizations such as IP = PSPACE [12,18] and MIP = NEXP [1,25,29], and later led to the important area of probabilistically checkable proofs [49]. More recently, the study of IPs has resulted in extremely efficient (e.g., near linear or even logarithmic time) protocols for delegation of computation [8,10,17,32,46]. Such super-efficient IPs have brought theory closer to practice, resulting in “nearly practical” systems (e.g., see [9,15,50,53]).

Indeed, interactive proofs are not only a fundamental theoretical concept but an indispensable framework to design efficient computation-outsourcing protocols.

Existing interactive-proof games Interactive-proof systems with multiple provers have largely been studied as games that fall into two categories: either as games of cooperation such as MIP [7], cooperative multi-prover rational proofs (MRIP) [20], and variants [1,12,29,32,38], where the provers work together to convince the verifier of their joint claim; or as games of conflict such as refereed games [16,18,24,26,28,39], where the provers directly compete with each other to convince the verifier of their conflicting claims.
Both of these categories have limitations. In a game of cooperation, provers cannot be leveraged directly against each other. That is, the verifier cannot directly ask one prover if another prover is lying. On the other hand, in a game of conflict, such as refereed games, one prover must “win” the zero-sum game. Thus, such games need to assume that at least one prover—who must be the winning prover in a correct protocol—can be trusted to always tell the truth. Despite their limitations, both models have proved to be fundamental constructs to understand and characterize important complexity classes \[4, 18, 20, 24, 28\], and to design efficient computation outsourcing protocols \[8, 9, 16, 17, 32\].

1.1 Contributions and Results

In this paper, we introduce a new interactive-proof game, non-cooperative rational interactive proofs (ncRIP). This model generalizes multi-prover rational proofs \[19–21\].

Solution concept for ncRIP  We define a refinement of sequential equilibrium \[41\], strong sequential equilibrium (SSE), that essentially says that players’ beliefs about the histories that led them to an unreachable information set should be irrelevant to their best response. From a mechanism-design perspective, we want to design the protocols and payments that allow this strong guarantee to hold—letting the players’ best responses be unaffected by their beliefs.¹

Finally, we eliminate SSE strategies that are suboptimal within “subgames” by defining and enforcing a backward-induction-compatible notion of dominance. Roughly speaking, we say a protocol is a ncRIP if there exists a strategy profile of the provers that is a dominant SSE among the subforms of the extensive form game, and under this strategy the provers’ lead the verifier to the correct answer. We define the model formally in Section 2.

Utility gap for non-cooperative provers  Utility gap is a fundamental concept for rational proofs \[20, 21, 34\] which is analogous to soundness gap in interactive proofs. It measures how robust a protocol is against the provers’ possible deviations from the desired strategy.

This notion is straightforward to define for cooperative rational protocols—they have a utility gap of \(u\) if the total expected payment decreases by \(1/u\) whenever the provers report the wrong answer. In non-cooperative protocols, however, it is not a priori clear how to define such a payment loss or to choose which prover should incur the loss. A payment loss solely imposed on the total payment may not prevent some provers from deviating, and a loss solely imposed on the provers’ final payments may not prevent them from deviating within subgames.

We define a meaningful notion of utility gap for ncRIP that is naturally incorporated in a backward-induction-compatible way to the dominant SSE concept.

Tight characterizations of ncRIP classes  In this paper, we completely characterize the power of non-cooperative rational proofs under different utility-gap guarantees.

We construct ncRIP protocols with constant, polynomial, and exponential utility gaps for powerful complexity classes, demonstrating the strength of our solution concept. Our protocols are simple and intuitive (requiring only a few careful tweaks from their cooperative counterparts), and are thus easy to explain and implement. However, proving their correctness involves analyzing the extensive-game (including subtleties in the incentives and beliefs of each player at each round) to show that the protocol meets the strong solution-concept and utility-gap requirements.

¹We believe that SSE is of independent interest as a solution concept for designing extensive-form mechanisms (e.g. \[23, 31, 51\]). In Section 6, we prove important properties of SSE that may prove useful in future studies.
We then prove tight upper bounds for all three ncRIP classes. Proving tight upper bounds is the most technically challenging part of the paper. We prove the upper bounds by simulating the decisions of the verifier and provers with a Turing Machine. However, there are several obstacles to attain the correct bounds. For example, the polynomial randomness of the verifier can induce an exponential-sized game tree, which is too large to be verified by the polynomial-time machine in Theorems 1 and 2. Furthermore, an NEXP oracle cannot verify whether a strategy profile is a dominant SSE. The key lemma that helps us overcome these challenges is the pruning lemma (Lemma 14). At a high level, it shows that we can prune the nature moves of the verifier in the resulting game tree, while preserving the dominant-SSE and utility-gap guarantees.

Our results are summarized in Figure 1, where we use $O(1)$-ncRIP, $\text{poly}(n)$-ncRIP and $\exp(n)$-ncRIP to denote ncRIP classes with constant, polynomial and exponential utility gaps respectively. The notations are analogous for MRIP [19] (the cooperative variant). We characterize ncRIP classes via oracle Turing machines. In particular, $\text{P}^{\text{NEXP}[O(1)]}$ is the class of languages decided by a polynomial-time Turing machine that makes $O(1)$ queries to an NEXP oracle, and $\text{EXP}^{\text{poly-NEXP}}$ is the class decided by an exponential-time Turing machine with polynomial-length queries to an NEXP oracle.

| Theorem 1. $O(1)$-ncRIP = $\text{P}^{\text{NEXP}[O(1)]}$ | Corollary 4. $O(1)$-ncRIP = $O(1)$-MRIP |
|-----------------------------------------------|-----------------------------------------|
| Theorem 2. $\text{poly}(n)$-ncRIP = $\text{P}^{\text{NEXP}}$ | Corollary 5. $\text{poly}(n)$-ncRIP $\supseteq$ $\text{poly}(n)$-MRIP |
| Theorem 3. $\text{exp}(n)$-ncRIP = $\text{EXP}^{\text{poly-NEXP}}$ | Corollary 6. $\text{exp}(n)$-ncRIP = $\text{exp}(n)$-MRIP |

Figure 1: Summary of our results.

Power of non-cooperative vs. cooperative and competitive provers Interestingly, in the case of constant and exponential utility gap, the power of ncRIP and MRIP coincide. This can be explained by the power of adaptive versus non-adaptive queries in oracle Turing machines.

Indeed, our results reveal the main difference between non-cooperative and cooperative provers: the former can be used to handle adaptive oracle queries, the latter cannot (see [19,20]). Intuitively, this makes sense—cooperative provers may collude across adaptive queries, answering some of them incorrectly to gain on future queries. On the other hand, non-cooperativeness allows us to treat the subgame involving the oracle queries as a separate game from the rest.

Our results also show that non-cooperative provers are more powerful than competing provers. Feige and Kilian [24] proved that the power of refereed games with imperfect information and perfect recall is equal to EXP.

2 Non-Cooperative Rational Interactive Proofs

In this section we introduce the model for ncRIP.

Notation. First, we review the structure of ncRIP protocols and related notation; this is largely the same as [20].

The decision problem being solved by an interactive proof is modeled as whether a given string $x$ is in language $L$. An interactive protocol is a pair $(V, \vec{P})$, where $V$ is the verifier, $\vec{P} = (P_1, \ldots, P_{p(n)})$ is the vector of $p(n)$ provers, where $p(n)$ is polynomial in $n = |x|$. The verifier runs in polynomial time and flips private coins. Each $P_i$ is computationally unbounded. The verifier
and provers are given the input $x$. Similar to classical multi-prover interactive proofs, the verifier can communicate with each prover privately, but no two provers can communicate with each other once the protocol begins.

In a round, either each prover sends a message to $V$, or $V$ sends a message to each prover, and these two cases alternate. The length of each message $\ell(n)$, and the number of rounds $k(n)$ are both polynomial in $n$. The final transcript $\bar{m}$ of the protocol is a random variable depending on $r$, the random string used by $V$. At the end of the communication, the verifier computes an answer bit $c \in \{0, 1\}$ for the membership of $x$ in $L$ based on $x$, $r$, and $\bar{m}$. $V$ also computes a payment vector $\bar{R} = (R_1, R_2, \ldots, R_{p(n)})$, where $R_i$ is the payment given to $P_i$, $R_i \in [-1, 1]$, and the total \(\sum_{i=1}^{p(n)} R_i \in [-1, 1]\) as well. The protocol and the payment function $\bar{R}$ are public knowledge.

Each prover $P_i$'s strategy at round $j$ maps the transcript seen at the beginning of round $j$ to the message he sends in that round. Let $s_i = (s_{i1}, \ldots, s_{ik(n)})$ be the strategy of prover $P_i$, and $s = (s_1, \ldots, s_{p(n)})$ be the strategy profile of the provers. Given input $x$, and strategy profile $s$, let $u_k(x, s, (V, \bar{P}))$ denote the expected payment of prover $P_k$ in the protocol $(V, \bar{P})$ based on randomness $r$, input $x$ and $s$; if $(V, \bar{P})$ is clear from context, we shorten this to $u_k(x, s)$ or $u_k(s)$.

The protocol forms an extensive-form game with imperfect information which we describe in the next section. The protocol and payments should be designed such that the provers are incentivized to reach an equilibrium that leads $V$ to the correct answer bit $c$. We formalize the solution concept in Section 2.2.

### 2.1 Extensive-form Games and nRIP

We describe the underlying extensive-form game resulting from nRIP protocols in this section. For details on extensive-form games, we refer to the textbook by Osborne and Rubinstein [44].

In a protocol $(V, \bar{P})$ with input $x$, the set of provers $\bar{P} = (P_1, \ldots, P_{p(n)})$ are the players. $V$ is not a player of the game—the deterministic moves of $V$ form the structure of the game tree and the randomized moves of $V$ are treated as Nature moves.

A history $h$ of the game is a sequence of actions taken by the players, written $h = (a^1, a^2, \ldots, a^K)$ for some actions $a^1, \ldots, a^K$. The set of histories (including $\phi$, the empty history corresponding to the root) is denoted by $H$. Note that every prefix of $h = (a^1, a^2, \ldots, a^K) \in H$ must also be a valid history, that is, $(a^1, a^2, \ldots, a^L) \in H$ for any $L < K$.

A history $h = (a^1, \ldots, a^K)$ is terminal if it corresponds to a leaf in the game tree—there is no $K + 1$ such that $(a^1, \ldots, a^K, a^{K+1}) \in H$—and non-terminal otherwise.

Let $Z(h)$ denote the player whose turn it is to act following a non-terminal history $h$—note that even though in an nRIP protocol more than one prover may send a message to the verifier in a round, without loss of generality we can increase the number of rounds such that only a single prover acts in each round. Let $A(h)$ denote the set of actions available to the acting player at a non-terminal history $h$: that is, $A(h) = \{a : (h, a) \in H\}$. If $Z(h)$ is Nature, then $A(h)$ is the set of possible coin flips and messages of the verifier following $h$; otherwise $A(h)$ is the set of possible messages that $Z(h)$ may send to the verifier. For each terminal history $h$, the utility of a player $i$ following $h$, $u_i(h)$, is the payment $R_i$ computed by the verifier given $x$ and $h$.

As the verifier’s coins are private and the verifier exchanges private messages with each of the provers, an nRIP protocol forms an extensive-form game of imperfect information.

An information set $I_i$ of a player $P_i$ is a subset of all possible histories $h$ with $Z(h) = P_i$, and represents all the information that the player knows when acting in one of the decision nodes.

\[^2\text{Negative payments are used to reflect punishment. The individual payments and the total payment can be shifted and scaled to lie in } [0, 1].\]
in $I_i$. That is, when a decision node in $I_i$ is reached, $P_i$ knows that $I_i$ has been reached but does not know exactly which node he is at. The set of actions available to player $i$ at every decision node in a particular information set is the same, i.e., $A(h) = A(h')$ for all $h, h' \in I_i$.

Let $A(I_i)$ denote the set of available actions at an information set $I_i$. The set of all information sets of $P_i$ forms a partition of the set $\{h \in H : Z(h) = P_i\}$, and let $I_i$ to denote this partition, referred to as the information partition of $P_i$. In terms of the protocol, $I_i$ is in a one-to-one correspondence with the set of possible message sequences $(m_{i1}, \ldots, m_{ij})$ seen by $P_i$, where $j \in \{1, \ldots, p(n)\}$ and $P_i$ is acting in round $j$.

A **pure strategy** $s_i$ of a player $P_i$ in an extensive-form game is a function that assigns an action in $A(I_i)$ to each information set $I_i \in \mathcal{I}_i$. A **behavioral strategy** $\beta_i$ of $P_i$ is a collection $(\beta_i(I_i))_{I_i \in \mathcal{I}_i}$ of independent probability measures, where $\beta_i(I_i)$ is a probability measure over the action set $A(I_i)$. A behavioral strategy $\beta_i$ is completely mixed if each $\beta_i(I_i)$ assigns a positive probability to every action in $A(I_i)$.

In this paper, the provers are deterministic and thus we only consider pure strategies. However, the solution concept introduced in this paper applies to behavioral strategies as well.

A player $i$'s utility under a strategy profile $s$, $u_i(s)$, is his expected utility over the distribution of histories induced by $s$ and the verifier’s randomness.

The provers are computationally unbounded and never “forget” anything and thus the corresponding extensive-form game has perfect recall. That is, for any two histories $h$ and $h'$ in the same information set $I_i$ of a player $P_i$, $h$ and $h'$ pass the same sequence of information sets to player $P_i$. Furthermore, for any information set in this sequence, player $P_i$ took the same action in $h$ and $h'$. This holds in any ncRIP protocol since all histories of prover $P_i$ in the same information set $I_i$ at round $j$ correspond to the sequence of messages $(m_{i1}, \ldots, m_{ij})$ seen by $P_i$ up to round $j$.

### 2.2 Solution concept for ncRIP

We want the solution concept for ncRIP to satisfy a strong notion of backward induction \[44\], a standard criterion applied to extensive-form games based on the common knowledge of rationality. Backwards induction refers to the condition of being “sequentially rational” in an extensive-form game, that is, each player must play his best response at each node where he has to move, even if his rationality implies that such a node will not be reached.

If an interactive protocol forms an extensive-form game of perfect information, it is easy to formalize this condition. A strategy $s$ is sequentially rational or satisfies backward induction, if for every player $i$ and every decision node of $i$, conditioned on reaching the decision node, $s_i$ is a best response to $s_{-i}$, that is, $u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i})$ for any strategy $s'_i$ of prover $i$. In other words, $s$ induces a best response at every subgame.

In a game of imperfect information, the decision nodes corresponding to a player’s turn are partitioned into information sets, where the player is unable to distinguish between the possible histories within an information set. To reason about sequential rationality we need a probability distribution $u_I$ on each information set $I$, so as to determine the players’ expected utility conditioned on reaching $I$ and thus their best response at $I$. The probability distribution $\mu_I$ is referred to as the player’s beliefs about the potential histories leading to $I$.

Given a strategy profile $s$, beliefs $u_I$ at reachable information sets (reached with non-zero probability under $s$) are derived from $s$ using Bayes’ rule; this is a standard derivation used in

\[A\] A subgame is a subtree that can be treated as a separate well-defined game. In a perfect-information game, every node starts a new subgame. “Backward induction” and “subgame-perfect equilibrium” are used interchangeably in the literature \[31\].
most solution concepts for extensive-form games [14]. We sometimes write \( \mu_i^s \) to emphasize that the beliefs depend on \( s \).

Past work has introduced a variety of methods for defining the beliefs \( u_i^s \) at unreachable information sets \( I \) (i.e., information sets reached with probability zero under \( s \)); see e.g. [22,11,13,47]. The most well-known is sequential equilibrium [11], which demands an explicit system of beliefs that satisfies a (somewhat artificial) consistency condition. Other equilibria, like trembling hand [47], reason implicitly about beliefs at unreachable information sets by assigning a negligible probability with which the player’s hand “trembles,” and reaches an otherwise-unreachable information set. Further refinements of these take the structure and payoffs of the game into account [5,22,43].

The treatment of beliefs at unreachable information sets in these solution concepts is often focused on ensuring that they can be used to analyze every extensive-form game. From a mechanism-design perspective, our focus is different—we want to design mechanisms in such a way that they admit much stronger equilibrium requirements, even if such an equilibrium cannot be used to analyze every game.

At a high-level, we want the players’ beliefs to be irrelevant in determining their best response at unreachable information sets. We call this notion strong sequential rationality. A strategy profile \( s \) is strongly sequentially rational if for every information set \( I \), conditioned on reaching \( I \), \( s_i \) is a best response to \( s_{-i} \) with respect to \( \mu_i^s \), where

- \( \mu_i^s \) is derived using Bayes’s if \( I \) is reachable under \( s \), and
- \( \mu_i^s \) is any arbitrary probability distribution if \( I \) is unreachable under \( s \).

In Section 6 we show that this requirement is equivalent to saying that, at an unreachable information set \( I \), \( s_i \) must be a best response to \( s_{-i} \) conditioned on reaching each history \( h \in I \). In other words, at an unreachable information set \( I \), each player must have a single action that is the best response to every possible history in \( I \). We say a strategy profile is a strong sequential equilibrium (SSE) if it satisfies strong sequential rationality.

We refine our solution concept further to eliminate strategies that are weakly dominated within “subgames” of the entire game. This is crucial to deal with equilibrium selection, in particular, because the players’ cannot unilaterally deviate out of a suboptimal equilibrium. We say an SSE \( s \) weakly dominates another SSE \( s' \) if, for any player \( i \), \( u_i(s) \geq u_i(s') \). A strategy \( s \) is weakly dominant if it dominates all SSEs. Next we eliminate SSEs that are weakly dominated in subgames of the entire game. We use the generalized notion of subgames, called subforms, defined by Kreps and Wilson [11] for extensive-form games with imperfect information.

To review the definition of subforms, we need further notation. Let \( H \) be the set of histories of the game. Recall that a history is a sequence \((a^1, \ldots, a^K)\) of actions taken by the players. For histories \( h, h' \in H \), we say \( h \) has \( h' \) as a prefix if there exists some sequence of actions \( b^1, \ldots, b^L \) (possibly empty) such that \( h = (h', b^1, \ldots, b^L) \). For a history \( h \in H \), let \( I(h) \) be the unique information set containing \( h \).

For an information set \( I \), let \( H_I \) be the set of all histories following \( I \), that is, \( H_I \) is the set of all histories \( h \in H \) such that \( h \) has a prefix in \( I \). We say that \( H_I \) is a subform rooted at \( I \) if for every information set \( I' \) such that \( I' \cap H_I \neq \emptyset \), it holds that \( I' \subseteq H_I \). Roughly speaking, a subform \( H_I \) “completely contains” all histories of the information sets following \( I \), so there is no information asymmetry between the players acting within \( H_I \).

Thus, given a strategy profile, the subform \( H_I \) together with the probability distribution \( \mu_I^s \) on \( I \), can be treated as a well-defined game.

We say an SSE \( s \) weakly dominates SSE \( s' \) on a subform \( H_I \) if, for any player \( j \) acting in \( H_I \), the expected utility of \( j \) under \( s_I \) in the game \((H_I, \mu_I^s)\) is greater than or equal to their utility
under $s'_i$ in the game $(H_1, \mu^*_i)$.

We eliminate weakly dominated strategies by imposing this dominance condition in a backward-induction-compatible way on the subforms as follows.

**Definition 7** (Dominant Strong Sequential Equilibrium). A strategy profile $s$ is a dominant strong sequential equilibrium if $s$ is an SSE and

- for every subform $H_1$ of height 1: $s$ weakly dominates $s'$ on $H_1$ for any SSE $s'$
- for every subform $H_1$ subgame of height > 1: $s$ weakly dominates $s'$ on $H_1$ for any SSE $s'$ that is a dominant SSE in all subforms of height at most $h - 1$.

We are ready to define non-cooperative rational interactive proofs.

**Definition 8** (Non-Cooperative Rational Interactive Proof). Fix an arbitrary string $x$ and language $L$. An interactive protocol $(V, \bar{P})$ is a non-cooperative rational interactive proof (ncRIP) protocol for $L$ if there exists a strategy profile $s$ of the provers that is a dominant SSE in the resulting extensive-form game, and under any dominant SSE, the answer bit $c$ output by the verifier is correct (i.e., $c = 1$ iff $x \in L$) with probability 1, where the probability is taken over the verifier’s randomness.

### 2.3 Utility Gap in ncRIP Protocols

In game theory, players are assumed to be perfectly rational and “sensitive” to arbitrarily small utility losses. In reality, some provers may not care about small losses. Such provers may not have sufficient incentive to reach a dominant SSE, and could end up leading the verifier to the wrong answer. To design ncRIP protocols that are robust against such “insensitive” provers, we define the notion of utility gap.

Informally, a utility gap of $u$ means that if a strategy profile $s$ leads the verifier to the wrong answer, there must exist a subform, such that some provers must lose at least a $1/u$ amount in their final individual payments (compared to their optimal strategy in that subform). As a consequence, these provers will not deviate to $s$, as long as they care about $1/u$ payment losses. We formalize this notion below. (We say a subform $H_1$ is reachable under $s$ if the information set $I$ is reached under $s$ with non-zero probability.)

**Definition 9** (Utility Gap). Let $(V, \bar{P})$ be an ncRIP protocol for a language $L$ and $s^*$ be a dominant SSE of the resulting game. The protocol $(V, \bar{P})$ has an $\alpha(n)$-utility gap or $\alpha(n)$-gap, if for any strategy profile $s'$ under which the answer bit $c'$ is wrong, there exists a subform $H_1$ reachable under $s'$, and a prover $P_j$ acting in $H_1$ who has deviated from $s^*$ such that

$$u_j(x, (s'_-I, s'_I), (V, \bar{P})) - u_j(x, (s'_-I, s'_I), (V, \bar{P})) > 1/\alpha(n),$$

where $s'_-I$ denotes the strategy profile $s'$ outside subform $H_1$, that is, $s'_-I = s' \setminus s'_I$.

The class of languages that have an ncRIP protocol with constant, polynomial and exponential utility gap, are denoted by $O(1)$-ncRIP, $\text{poly}(n)$-ncRIP, and $\text{exp}(n)$-ncRIP respectively.

Note that $\alpha(n)$ gap corresponds to a payment loss of $1/\alpha(n)$, so an exponential utility gap is the weakest guarantee.

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These classes are formally defined by taking the union over languages with $\alpha(n)$ utility gap, for every $\alpha(n)$ that is constant, polynomial and exponential in $n$ respectively.
3 Lower Bounds: ncRIP Protocols with Utility Gap

In this section, we give an \(O(1)\)-utility gap ncRIP protocol for the class \(\text{NEXP}\) and use it to give an \(O(\alpha(n))\)-utility gap ncRIP protocol for the class \(\text{P}^{\text{NEXP}[\alpha(n)]}\). Setting \(\alpha(n)\) to be a constant or polynomial in \(n\) gives us \(\text{P}^{\text{NEXP}[O(1)]} \subseteq \text{O}(1)\)-ncRIP and \(\text{P}^{\text{NEXP}} \subseteq \text{poly}(n)\)-ncRIP respectively.

**A constant-gap ncRIP protocol for \(\text{NEXP}\)** The ncRIP protocol for any language in \(\text{NEXP}\) is in Figure 2. The protocol uses the 2-prover 1-round MIP for \(\text{NEXP}\) [25] as a blackbox. The protocol in Figure 2 essentially forces the non-cooperative provers to coordinate by giving them identical payments. As a result, it is almost identical to the MRIP protocol for \(\text{NEXP}\) [20].

While the payment scheme is simple, in the analysis we have to open up the black-box MIP. In particular, if \(P_1\) sends \(c = 0\) in round \(1\) all the information sets of \(P_1\) and \(P_2\) in round \(3\) become unreachable. To show that an SSE exists, we show that the provers have a best response at these unreachable sets, which is argued based on the messages exchanged in the MIP protocol.

**Lemma 10.** Any language \(L \in \text{NEXP}\) has a 2-prover 3-round 6/5-gap ncRIP protocol.

*Proof.* The ncRIP protocol for any language \(L \in \text{NEXP}\) is given in Figure 2.

We show that there exists a strategy profile \(s = (s_1, s_2)\) of provers \(P_1\) and \(P_2\) respectively that is a dominant SSE of the game tree corresponding to the protocol \((V, P_1, P_2)\) and under any dominant SSE, the answer bit \(c = 1\) if and only if \(x \in L\).

In the protocol, if \(c = 0\), no player acts. If \(c = 1\), the verifier executes the 1-round blackbox MIP protocol with \(P_1\) and \(P_2\). To exhibit a strategy that is a best response for \(P_1\) and \(P_2\) on their information sets at step 3, we look at the messages the verifier sends to each prover in the classic MIP protocol. In the MIP protocol, the verifier sends \(P_1\) a tuple of message pairs \(\vec{m}_1 = ((q_1, x_1), \ldots, (q_m, x_m))\) where \(m\) is a polynomial in \(n\) and \(V\) sends \(P_2\) a tuple of random messages \(\vec{m}_2 = (y_1, \ldots, y_m)\). \(P_1\) sends back a polynomial \(P(t)\) and \(P_2\) sends back the value of the polynomial \(P(t)\) for \(t\) satisfying \(q_j + tx_j = y_j\). The verifier rejects if their answers are inconsistent.

To analyze the SSE strategy, without loss of generality, suppose \(P_1\) moves last in the MIP protocol. Any information set \(I_1\) of \(P_1\) at step 3 is characterized by the message \(\vec{m}_1\) he receives. The decision nodes in \(I_1\) correspond to each possible message \(\vec{m}_2\) that \(P_2\) could have received.

Because the \(V\) gives the largest payment when the MIP protocol accepts, given \(P_2\)’s strategy, if any information set \(I_1\) of \(P_1\) is reached under \(s\) then \(P_1\)’s best response at \(I_1\) is to maximize the acceptance-probability of the MIP protocol given his beliefs on \(I_1\). Similarly, given \(P_2\)’s strategy, if any information set \(I_1\) of \(P_1\) is unreachable under \(s\) then \(P_1\)’s best response at \(I_1\) for every decision node in \(I_1\) is the following: given \(\vec{m}_1 = ((q_1, x_1), \ldots, (q_m, x_m))\), respond with a polynomial \(P(t)\) such that \(P(t)\)’s value at all \(t\) coincides with \(P_2\)’s reply on all \(y_j\) where \(q_j + tx_j = y_j\).

Given \(P_1\)’s strategy of committing to a polynomial \(P(t)\) that matches \(P_2\) on all values of \(t\), \(P_2\)’ best response at any information set \(I_2\) (reachable or unreachable under \(s\)) at step 3 at every decision node in \(I_2\) is to answer the tuple of queries \((y_1, \ldots, y_m)\) so as to maximize the acceptance probability of the MIP protocol. The verifier’s move at step 3 is the root of a non-trivial subform. Conditioned on step 3 being reached, any dominant SSE at this subform corresponds to a strategy profile \(s\) that is an SSE, which when restricted to this subform, maximizes the acceptance probability of the MIP protocol. Under any such dominant SSE, we show that \(P_1\)’s best response at step 3 is to send the correct answer bit.

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5 It is also possible to give a scoring-rule based ncRIP protocol for \(\text{NEXP}\), similar to MRIP [20]. However, such a protocol has an exponential utility gap.
Suppose \( x \in L \). If \( P_1 \) sends \( c = 0 \), then \( R_1 = 1/2 \) with probability 1. On the other hand, if \( P_1 \) sends \( c = 1 \), by the soundness condition of the MIP protocol, the acceptance probability is 1, leading to \( R_1 = 1 \). Thus for \( x \in L \), \( s \) is a dominant SSE iff \( P_1 \) sends \( c = 1 \).

Suppose \( x \not\in L \). If \( P_1 \) reports \( c = 0 \), then \( R_1 = 1/2 \) with probability 1. On the other hand if \( P_1 \) reports \( c = 1 \), then by the soundness condition of the MIP protocol, the maximum acceptance probability is 1/3 leading to \( R_1 = 1 \). The protocol rejects with probability at least 2/3 leading to \( R_1 = -1 \). Thus, \( P_1 \)'s expected payment for misreporting the answer bit is at most \( R_1 = -1/3 \).

Thus for \( x \not\in L \), \( s \) is a dominant SSE iff \( P_1 \) sends \( c = 0 \).

Thus, under \( s \) which is a dominant SSE, \( c = 1 \) if and only if \( x \in L \).

Furthermore, the payment incurred by the provers when the answer bit sent in the first round is incorrect is at least \( 5/6 \) for both provers and thus the protocol has constant utility gap. \( \square \)

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For any input \( x \) and language \( L \in \text{NEXP} \), the protocol \((V, P_1, P_2)\) for \( L \) is:

1. \( P_1 \) sends a bit \( c \) to \( V \). \( V \) outputs \( c \) at the end of the protocol.
2. If \( c = 0 \), then the protocol ends and the payments are \( R_1 = R_2 = 1/2 \).
3. Otherwise, \( V \) runs the classic 2-prover 1-round MIP protocol for \( \text{NEXP} \) \([25]\) with \( P_1 \) and \( P_2 \) to prove if \( x \in L \). If the MIP protocol accepts then \( R_1 = 1, R_2 = 1 \); else, \( R_1 = -1, R_2 = -1 \).

![Figure 2: A simple \( O(1) \)-utility gap ncRIP protocol for \( \text{NEXP} \).](image)

An \( O(\alpha(n)) \)-gap ncRIP protocol for \( \text{P}^{\text{NEXP}[\alpha(n)]} \) Using the above \( \text{NEXP} \) protocol as a subroutine, we give an ncRIP protocol with \( O(\alpha(n)) \)-utility gap for the class \( \text{P}^{\text{NEXP}[\alpha(n)]} \). This protocol works for any function \( \alpha(n) \) which (1) is a positive integer for all \( n \), (2) is upper-bounded by a polynomial in \( n \), and (3) is polynomial-time computable\(^6\).

The ncRIP protocol for any \( L \in \text{P}^{\text{NEXP}[\alpha(n)]} \) is in Figure 3. It is fairly intuitive—\( V \) simulates the polynomial-time machine directly, and uses the ncRIP protocol for \( \text{NEXP} \) for the oracle queries.

For any input \( x \) of length \( n \), the protocol \((V, \bar{P})\) works as follows.

1. \( P_1 \) sends \((c, c_1, \ldots, c_{\alpha(n)}) \in \{0,1\}^{\alpha(n)+1} \) to \( V \). \( V \) outputs \( c \) at the end of the protocol.
2. \( V \) simulates \( M \) on \( x \) using the bits \( c_1, \ldots, c_{\alpha(n)} \) as answers to \( \text{NEXP} \) queries \( \phi_1, \ldots, \phi_{\alpha(n)} \) generated by \( M \) respectively. If \( M \) accepts and \( c = 0 \) or \( M \) rejects and \( c = 1 \), then the protocol ends and \( R_1 = -1, R_2 = R_3 = 0 \).
3. \( V \) picks a random index \( i' \) from \( \{1, \ldots, \alpha(n)\} \) and sends \((i', \phi_{i'})\) to \( P_2 \) and \( P_3 \).
4. \( V \) runs the 2-prover 3-round \( O(1) \)-gap ncRIP protocol for \( \text{NEXP} \) (Figure 2) with \( P_2 \) and \( P_3 \) on \( \phi_i \). \( P_2 \) and \( P_3 \) get payments \( R_2 \) and \( R_3 \) based on the protocol. Let \( c_{i'}^* \) be the answer bit in the \( \text{NEXP} \) protocol. If \( c_{i'}^* \neq c_{i'} \), then \( R_1 = 0 \); otherwise \( R_1 = 1 \).

![Figure 3: An \( O(\alpha(n)) \)-utility gap ncRIP protocol for \( \text{P}^{\text{NEXP}[\alpha(n)]} \).](image)

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\(^6\)For Theorem 1 and Theorem 2 \( \alpha(n) \) need only be a constant or polynomial in \( n \). However, Lemma 11 holds for all \( \alpha(n) \)'s that are polynomial-time computable (given \( 1^n \)) and polynomially bounded, such as \( \log n, \sqrt{n} \), etc.
We first argue the correctness of this protocol at a high-level and then present the formal proof. Under any strategy of $P_1$, the resulting NEXP queries in the protocol in Figure 3 are the roots of non-trivial subforms. Which of these subforms are reachable under a strategy profile $s$ is determined solely by the strategy of $P_1$. However, because weak dominance is imposed on all subforms in a bottom-up fashion, $P_2$ and $P_3$ must play their optimal strategy in these subforms regardless of their reachability—and therefore, they must play optimally for any strategy of $P_1$. (This is one example of why ruling out weakly-dominated strategies in subforms in the definition of dominant SSEs is crucial to arguing correctness.) From the correctness of the NEXP protocol in Figure 2 we know that the optimal strategy of $P_2$ and $P_3$ is to compute the NEXP queries correctly. Given that the best response of $P_2$ and $P_3$ is to solve the NEXP queries correctly, and given that $V$ randomly verifies 1 out of $\alpha(n)$ queries, $P_1$ must commit to correct answer bits in the first round, or risk losing a $O(1/\alpha(n))$ amount from his expected payment.

If $P_1$ gives the correct answer bits in step 1, but $P_2$ or $P_3$ deviate within a subform corresponding to an NEXP query $\phi_q$, then with probability $1/\alpha(n)$, $V$ simulates the protocol in Figure 3 on $\phi_q$, in which case they lose a constant amount of their expected payment.

**Lemma 11.** Any language $L \in \text{P}^{\text{NEXP}[\alpha(n)]}$ has a 3-prover 5-round ncRIP protocol that has a utility gap of $6/(5\alpha(n))$.

**Proof.** Consider any language $L \in \text{P}^{\text{NEXP}[\alpha(n)]}$. Let $M$ be a polynomial-time Turing machine deciding $L$, with access to an oracle $O$ for an NEXP language.

The ncRIP protocol for $L$ is given in Figure 3.

Let $s_1, s_2, s_3$ denote the strategy used by $P_1$, $P_2$ and $P_3$ for the protocol in Figure 3 and $s = (s_1, s_2, s_3)$. First, note that regardless of $s_2$ and $s_3$, $P_1$’s best response at step 1 is to send the bits $c, c_1, \ldots, c_{\alpha(n)}$ such that the verification in step 2 goes through. In particular, if $s_1$ is such that the output of $M$ on input $x$, using $c_1, \ldots, c_{\alpha(n)}$ as answers to NEXP queries $\phi_1, \ldots, \phi_{\alpha(n)}$ is consistent with $c$, then $P_1$ gets $R_1 \geq 0$. Meanwhile, if the verification in step 2 fails then $R = -1$. Thus, under any SSE $s$, the answer bits $c_1, \ldots, c_{\alpha(n)}$ sent by $P_1$ must be consistent with the computation of $M$ on $x$ and the final the answer bit $c$, regardless of $s_2$ and $s_3$.

We now argue using backward induction. Each random index $i'$ chosen by $V$ in step 3 together with $\phi_{i'}$ starts a subform. In particular, since $P_2$ and $P_3$ both know $(i', \phi_{i'})$, all their information sets starting from step 4 are completely disjoint from information sets reached under a different index and NEXP query. By Lemma 10 there exists a dominant SSE $s$ on each such subform simulating an NEXP query, and under any dominant SSE, $s_2$ and $s_3$ are such that $c_{i'}^*$ is the correct answer to the NEXP query.

Moving up the tree, the next subform is induced by $V$’s nature move at step 8 assigning a probability to each subsequent subform. Since under any dominant SSE, the expected payments of $P_2$ and $P_3$ (conditioned on reaching these subforms) are maximized, the overall expected payments under $V$’s nature move at step 8 is also maximized.

We move up a further level in the tree to the root. We show that $P_1$’s best response at step 1 is to send the correct answer bits, given that under any dominant SSE $s$:

- $P_2$ and $P_3$ answer each NEXP query $\phi_{i'}$ determined by $s_1$ and index $i'$ correctly, and
- the verification in step 2 goes through (i.e. $P$ does not set $R_1 = -1$) under $s_1$.

Suppose $s_1$ is such that there exists an NEXP query where $P_1$ lies. Let $k$ be the first NEXP query index such that $c_k$ is not the correct answer to query $\phi_k$, where $1 \leq k \leq \alpha(n)$. In particular, the instance $\phi_k$ is evaluated correctly (by running $M$ on $x$ using the correct answers to previous queries, $c_1, \ldots, c_{k-1}$) but the answer $c_k$ is not evaluated correctly based on $\phi_k$. Then with probability $1/\alpha(n)$, $V$ picks $k$ in step 3 and crosschecks the $c_k$ with $c_{i'}^*$, in which case the verification fails and $R_1 = 0$. Thus, $P_1$’s expected payment is at most $1 - 1/\alpha(n)$. If $P_1$ answers all NEXP queries correctly, since
the verification in step 2 goes through, \( P_1 \) gets \( R_1 = 1 \) with probability 1. Thus, \( c, c_1, \ldots, c_{\alpha(n)} \) are correct under any dominant SSE \( s \), and \( c = 0 \) if and only if \( x \in L \).

Now, we show that protocol \((V, \vec{P})\) has \( O(\alpha(n)) \) utility gap. Let \( s^* \) be a dominant SSE of the game resulting from \((V, \vec{P})\). Suppose \( s' \) is such that the answer bit \( c' \) under \( s' \) is incorrect. We go “bottom-up” in the game tree and exhibit a subform \( H_1 \) (reachable under \( s' \)) such that some prover acting in that subform loses \( O(1/\alpha(n)) \) compared to the strategy where \( s^*_1 \) is played on \( H_1 \), keeping the rest of the strategy fixed.

First, consider all the \( \text{NEXP} \) queries at step 4 that start subforms. Suppose there exists a query \( \phi_k \) committed under \( s^*_1 \), for \( 1 \leq k \leq \alpha(n) \), such that \( c_{k*} \) is the wrong answer to \( \phi_k \). By Lemma 10, both \( P_2 \) and \( P_3 \) lose a constant amount (5/6 in particular) from their expected payment (conditioned on reaching this subform) compared to the dominant SSE strategy profile \( s^*_\phi_k \) which reports the correct answer to \( \phi_k \). Since \( V \) chooses \( \phi_k \) with probability \( 1/\alpha(n) \), \( P_2 \) and \( P_3 \) can gain \( O(1/\alpha(n)) \) in their overall expected payment by deviating to strategy profile \( s^*_\phi_k \), at the subform corresponding to \((k, \phi_k)\) keeping \( s'_{\phi_k} \) fixed. Specifically,

\[
\mu_i \left( x, r, (s'_{\phi_k}, s^*_\phi_k), (V, \vec{P}) \right) - \mu_i \left( x, r, (s'_{\phi_k}, s^*_\phi_k), (V, \vec{P}) \right) > \frac{1}{\alpha(n)} \left( \frac{5}{6} \right), \quad \text{for } i \in \{2, 3\}.
\]

Finally, suppose \( P_2 \) and \( P_3 \) answer all \( \text{NEXP} \) queries (reachable under \( s' \)) correctly. Then, \( P_1 \) loses at least \( 1/\alpha(n) \) at the subform at the root—the entire game. Since the answer bit \( c' \) under \( s' \) is incorrect, either step 2 fails or \( P_1 \) lies on some \( \text{NEXP} \) query. In the first case, \( P_1 \) gets \(-1\) with probability 1 compared to an expected payment of 1 under \( s^* \). In the second case, \( P_1 \) gets caught in step 4 with probability \( 1/\alpha(n) \), and gets an expected payment of at most \( 1 - 1/\alpha(n) \), losing at least \( 1/\alpha(n) \) compared to \( s^* \).

Thus, the protocol \((V, \vec{P})\) is an ncRIP protocol for \( P^{\text{NEXP}|O(\alpha(n))} \) and has \( O(\alpha(n)) \) utility gap. \( \square \)

**Exponential utility gap** We show how to simulate a general MRIP protocol \((V, \vec{P})\) with \( p(n) \) provers and \( k(n) \) rounds for a language \( L \) using a 2-prover 3-round ncRIP protocol \((V', P'_1, P'_2)\) with exponential-utility gap. (The protocol \((V', P'_1, P'_2)\) is in Figure 4)

Essentially, \( V' \) gives all the randomness of \( V \) to \( P'_1 \) and asks for the entire transcript and uses \( P'_2 \) to commit to a single prover’s message, and cross-checks their answers. However, we don’t want \( P'_1 \) who has access to all the randomness to dictate what information sets of \( P'_2 \) are reachable. Because the ncRIP protocol only needs an exponential utility gap, \( V' \) asks one prover a totally random question (independent of \( P'_1 \)), and with exponentially small probability this random message is exactly the message \( V' \) intended to check. This protocol shows why exponential gap guarantees do not lead to meaningful protocols—a verifier that asks random questions can still extract honest behavior from rational provers through the exponentially small changes in expected payments.

**Lemma 12.** Any MRIP protocol can be simulated using a 2-prover 3-round ncRIP protocol with \( O(1/2^{n^k}) \)-utility gap, for some constant \( k \), where \( n \) is the length of the input.

**Proof.** Without loss of generality, let each message in the protocol be of length \( \ell(n) \) for any input of length \( n \), where \( \ell(n) \) is a polynomial in \( n \). We shift and rescale the payment function of \( V \), so that the payment is always in \([0, 1]\), and the expected payment is strictly greater than 0 under the provers’ best strategy profile.

We simulate \((V, \vec{P})\) using an ncRIP protocol \((V', (P'_1, P'_2))\), given in Figure 4.

Let \( s^*_1 \) and \( s^*_2 \) denote the strategy of the provers \( P'_1 \) and \( P'_2 \) respectively and \( s' = (s^*_1, s^*_2) \). Since \( P'_2 \) is queried only once and about a single message in Step 4 any strategy \( s^*_2 \) of \( P'_2 \) de facto commits to a strategy profile for the provers in \((V, \vec{P})\).
Given an input $x$ of length $n$, and an MRIP protocol $(V, \vec{P})$, the ncRIP protocol $(V', \vec{P}')$ is:

1. $P'_1$ sends the round 1 messages $m_{11}, \ldots, m_{p(n)1}$ of $(V, \vec{P})$ to $V'$. $V'$ outputs $c$, the first bit of $m_{11}$, at the end of the protocol.

2. $V'$ selects a prover index $i \in \{1, \ldots, p(n)\}$ and a random round $j \in \{1, \ldots, k(n)\}$.

   Then, $V'$ generates a random string $\vec{m}_{ij}$ of length $(j - 1)\ell(n)$.

3. $V'$ sends $(i, j, \vec{m}_{ij})$ to $P'_2$. $P'_2$ simulates $P_1$ on round $j$, and sends the message $m'_{ij}$ to $V'$.

4. $V'$ generates all the randomness $r$ used by $V$ and sends it to $P'_1$.

5. $P'_1$ uses $r$ to simulate the protocol $(V, \vec{P})$, and sends the resulting transcript $\vec{m}$ to $V'$.

6. If $\vec{m}_{ij} \neq (m_{11}, \ldots, m_{ij-1})$, where $m_{ij}$ denotes prover $P_i$’s message in round $j$ according to $\vec{m}$ sent by $P_1$, then the protocol ends and $R'_1 = R'_2 = 0$.

7. Otherwise, if $m_{ij} \neq m'_{ij}$, then $R'_1 = R'_2 = -1$.

8. Else, $V'$ computes the payment $R$ in $(V, \vec{P})$ using $x$, $r$ and $\vec{m}$, and sets $R'_1 = 0$, $R'_2 = R$.

Figure 4: Simulating any MRIP using an ncRIP protocol with exponential utility gap.

We analyze the game tree of the protocol $(V', \vec{P}')$ bottom-up.

The last move is by $P'_1$ sending the entire transcript $\vec{m}$ at step 8. Any information set $I'_1$ of $P'_1$ is characterized by the randomness $r$ received by $P'_1$ in step 4 and all information sets are reachable under any $s'$. The decision nodes in $I'_1$ correspond to different strings $\vec{m}_{ij}$ that $P'_2$ could have been asked in step 2. Given $s'_2$, the best response of $P'_1$ at any information set $I'_1$, for any beliefs at $I'_1$, is to match the transcript committed by $P'_2$ and make the verification in step 7 go through. Suppose there exists a prover index $i$ and round $j$ such that the message $m_{ij}$ in $\vec{m}$ that is inconsistent with the corresponding message $m'_{ij}$ committed under $s'_2$. With probability $\frac{1}{p(n)k(n)}$, the random string $\vec{m}_{ij}$ generated by $V'$ in step 2 is equal to $(m_{11}, \ldots, m_{ij-1})$, otherwise the protocol ends with $R'_1 = 0$. With probability at least $\frac{1}{1 + \frac{1}{2(p(n))}}$, $V'$ chooses $(i, j)$ in step 2 and queries $P'_2$ for $m'_{ij}$ and $R'_1 = 1$. If $(i, j)$ is not chosen then $R'_1 = 0$. Thus, $P'_1$ expected payment at $I'_1$ is at most

$$\sum_{i \leq p(n), 1 \leq j \leq k(n)} \frac{1}{2(p(n)k(n))} \cdot \frac{1}{p(n)k(n)} \cdot \left(\mathbb{I}_{m_{ij} \neq m'_{ij}} \cdot (-1) + \mathbb{I}_{m_{ij} = m'_{ij}} \cdot 0\right) < 0.$$ 

On the other hand, matching $s'_2$ on all messages leads to an expected payment of 0 at $I'_1$ for $P'_1$.

Given that $P'_1$ best response is to make the verifier in step 7 go through for every randomness $r$, we analyze $P'_2$ move at step 5. Any information set $I'_2$ of $P'_2$ is characterized by the random string $\vec{m}_{ij}$ received by $P'_2$ in step 2 and all information sets are reachable under any $s'$. The decision nodes in $I'_2$ correspond to different random strings $r$ that $P'_1$ could have been asked in step 2. The best response of $P'_2$ at any information set $I'_2$, for any beliefs at $I'_2$, is to commit to the correct strategy profile $s$ of the provers $\vec{P}$. Suppose $P'_2$ commits to a strategy profile $s'$ such that the answer bit under $s'$ is wrong. With probability $\frac{1}{2(p(n)k(n))}$, the random string $\vec{m}_{ij}$ generated by $V'$ in step 2 matches $(m_{11}, \ldots, m_{ij-1})$, otherwise the protocol ends with $R'_2 = 0$. If it matches, then $P'_2$ expected payment is determined by the expected payment that $\hat{s}$ gets in $(V, \vec{P})$ given $x$ and randomness $r$, which is strictly less than the expected payment under the strategy profile $s$ which commits to the correct answer bit (by correctness of the original MRIP protocol). That is,

$$\sum_{1 \leq j \leq k(n)} \frac{1}{k(n)} \cdot \frac{1}{2(p(n)k(n))} \cdot u_{(V, \vec{P})}(x, \hat{s}) < \sum_{1 \leq j \leq k(n)} \frac{1}{k(n)} \cdot \frac{1}{2(p(n)k(n))} \cdot u_{(V, \vec{P})}(x, s).$$

Thus, given that $s'_1$ matches $s'_2$ for every randomness $r$, the best response by $P'_2$ is to commit to a
strategy profile $s'_2 = s$ that maximizes the total expected payment of the original protocol $(V, \vec{P})$ and thus has the correct answer bit.

There are no non-trivial subform in the game. Any weakly-dominant SSE is a dominant SSE, under which both $P'_1$ and $P'_2$ maximize their expected payments—$P'_1$ matches $P'_2$ on all messages and $P'_2$ commits to the correct strategy profile $s$. Thus, the protocol $(V, \vec{P})$ is correct. 

4 Upper Bounds: ncRIP Protocols with Utility Gap

In this section, we prove matching upper bounds on the classes of ncRIP protocols with constant and polynomial utility gaps. In particular, we show that any language in $O(1)$-ncRIP (or poly$(n)$-ncRIP) can be decided by a polynomial-time Turing machine with a constant (resp. polynomial) number of queries to an NEXP oracle.

To simulate an ncRIP protocol, we need to find a strategy profile “close enough” to the dominant SSE so that the answer bit is still correct, i.e. a strategy profile that satisfies the utility-gap guarantee. We formalize this restatement of Definition 9 below.

Observation 13. Given input $x$ and an ncRIP protocol $(V, \vec{P})$ with $\alpha(n)$-utility gap, let $s$ be a strategy profile such that for all reachable subforms $H_I$ and all provers $P_j$ acting in $H_I$,

$$u_j(x, r, (V, \vec{P}), (s_{-I}, s^*_I)) - u_j(x, r, (V, \vec{P}), (s_{-I}, s_I)) < \frac{1}{\alpha(n)},$$

where $s^*$ is a dominant SSE. Then, the answer bit $c$ under $s$ must be correct.

There are several challenges involved in finding a strategy profile satisfying Observation 13. First, the size of the game tree of any ncRIP protocol—small gap notwithstanding—can be exponential in $n$. Even if the polynomial-time machine considers a single strategy profile $s$ at a time, since $V$ can flip polynomially many coins, the part of the tree “in play”—the number of decision nodes reached with positive probability under $s$—can be exponential in $n$.

The second (and related) challenge is that of verifying whether a strategy profile is a dominant SSE. While the NEXP oracle can guess and verify an SSE, it cannot directly help with dominant SSEs. The polynomial-time machine must check using backward induction if an SSE is dominant on all its reachable subforms, which can again be exponential in $n$.

Finally, the polynomial-time machine needs to search through the exponentially large strategy-profile space in an efficient way to find one which leads to the correct answer.

In the remainder of the section we address these challenges. In Lemma 14 we show that we can prune the game tree, resolving the first two challenges. Then in Lemmas 18 and 19 we show how to efficiently search through the strategy-profile space.

Pruning Nature moves in ncRIP protocols We now give our main technical lemma for the upper bound, which shows that we can limit ourselves to examining protocols with bounded game trees without loss of generality.

Recall that a verifier’s coin flips in an ncRIP protocol represent Nature moves in the resulting game. The problem is that a polynomial-time verifier can result in Nature moves that impose nonzero probabilities over exponentially many outcomes.

We prune the Nature moves of a verifier so that a polynomial-time Turing machine simulating an $\alpha(n)$-utility-gap protocol can traverse the game tree reachable under a given $s$. This pruning operation takes exponential time (linear in the size of the game tree), and can be performed by the NEXP oracle.
Lemma 14 (Pruning Lemma). Let $L \in \alpha(n)$-ncRIP and let $(V, \bar{P})$ be an ncRIP protocol for $L$ with $\alpha(n)$ utility gap and $p(n)$ provers. Given an input $x$ and a strategy $s$, the protocol $(V, \bar{P})$ can be transformed in exponential time to a new protocol $(V', \bar{P})$, where

- the probability distribution on the outcomes imposed by the Nature moves of $V'$ for input $x$ has $O(\alpha(n))$ support,
- if $s$ is a dominant SSE of $(V, \bar{P})$, then $s$ induces a dominant SSE in $(V', \bar{P})$,
- $|u_j(x, s, (V, \bar{P}))-u_j(x, s, (V', \bar{P}))| < 1/(4\alpha(n))$ for all $j \in \{1, \ldots, p(n)\}$, and
- the utility gap guarantee is preserved, that is, if the answer bit under $s$ is wrong, then there exists a subform $H_1$ in the game $(V', \bar{P})$ (reachable under $s$) and a prover $P_j$ acting at $H_1$, such that $P_j$ loses a $1/(2\alpha(n))$ amount in his expected payment compared to a strategy profile where $s_1$ (induced by $s$ on $H_1$) is replaced by $s_1^*$ (the dominant SSE on $H_1$), keeping the strategy profile outside $H_1$, $s_{-1}$, fixed.

We prove Lemma 14 in several parts. First, given an input $x$ and a strategy $s$ of the provers, we show how to transform any verifier $V$ that imposes a probability distribution over outcomes with exponential support into a verifier $V'$ that imposes a probability distribution with $O(\alpha(n))$ support.

Let $(V, \bar{P})$ use $p(n)$ provers and let the running time of $V$ be $n^k$ for some constant $k$. There can be at most $2^nn^k$ different payments that $V$ can generate for a particular prover given input $x$. Given $x$ and $s$, fix a prover index $j \in \{1, \ldots, p(n)\}$. Let $R_1, R_2, \ldots, R_m$ be the payments generated by $V$ on $s$ for $P_j$. Let $V$’s randomness assign probability distribution $\mu = (p_1, p_2, \ldots, p_m)$ to $R_1, R_2, \ldots, R_m$ respectively. Then, the expected payment of $P_j$ under $s$ is $u_j(x, s, (V, \bar{P})) = \sum_{i=1}^{m} p_i R_i$.

Recall that $u_j(x, s, (V, \bar{P})) \in [-1, 1]$ for all $1 \leq j \leq p(n)$. For each prover $P_j$, divide the interval $[-1, 1]$ into $4\alpha(n)$ intervals, each of length $1/(2\alpha(n))$. In other words, prover $P_j$’s $i$th interval is $[i/2\alpha(n), (i+1)/2\alpha(n)]$ for each $i \in \{-2\alpha(n), \ldots, 2\alpha(n) - 1\}$.

We round the possible payments for $P_j$ to a representative of the their corresponding interval. Specifically, we map each payment $R_i$ to $r_j$ as described in Equation 1. There are potentially exponentially many different payments $R_i$, and only polynomially many different payments $r_j$, so several $R_i$ must map to the same $r_j$. Let $T_j = \{i : R_i \text{ maps to } r_j\}$. Let $T = \bigcup_j \{T_j\}$. Thus the total number of distinct $r_j$ is $8\alpha(n)$, so $|T| = O(\alpha(n))$. Let $S : \{1, \ldots, m\} \rightarrow T$ such that $S(i) = T_j$ if and only if $i \in T_j$.

For each $T_j \in T$, let $f(T_j)$ denote a unique index in the set $T_j$. Without loss of generality, let $f(T_j)$ be the lowest index in $T_j$. We define a new probability distribution $\mu' = (p'_1, \ldots, p'_h)$ over the payments $R_1, \ldots, R_h$ respectively, given by Equation 2. In particular, for every $T_j \in T$, assign $R_{f(T_j)}$ probability $\sum_{k \in T_j} p_k$ and for every other index $\ell \in T_j$, $\ell \neq f(T_j)$, assign $R_\ell$ probability 0.

Define $V'$ as a polynomial-time verifier that simulates all deterministic computation of $V$. For a fixed input $x$, $V'$ imposes a probability distribution $\mu'$ with $O(\alpha(n))$ support for any probability distribution $\mu$ imposed by $V$. For other inputs, $V'$ simulates $V$ without any modification.

\footnote{To include 1 as a possible payment, interval $2\alpha(n) - 1$ should be closed on both sides; we ignore this for simplicity.}
Note that given input $x$, a strategy profile $s$ and the protocol $(V, \vec{P})$, transforming the distribution $\mu$ to $\mu'$ takes time linear in the size of the game tree, and thus exponential in $n$. (This means that an NEXP oracle, given $x$, can guess a particular $s$ and perform the transformation.)

The remainder of the proof of Lemma 14 consists of the following three claims.

First, we show that if the strategy profile $s$ is a dominant SSE of $(V, \vec{P})$, then $s$ restricted to the pruned game tree of $(V', \vec{P})$ imposes a dominant SSE on $(V', \vec{P})$ as well.

**Claim 15.** Any dominant SSE $s$ of the game formed by $(V, \vec{P})$ induces a dominant SSE in the game formed by $(V', \vec{P})$.

**Proof.** By contradiction, suppose $s$ is not an SSE of $(V', \vec{P})$. Then there exists an information set $I = \{h_1, \ldots, h_m\}$, such that, conditioned on reaching $I$, the prover acting at $I$ can improve his expected payment by deviating (given his belief $u_I'$ at $I$ if $I$ is reachable under $s$ and for any belief he may hold at $I$ if $I$ is unreachable under $s$).

We split into two cases: $I$ is either reachable or unreachable under $s$.

By construction, if $I$ is reachable under $s$ in $(V', \vec{P})$, then $I$ must also be reachable under $s$ in $(V, \vec{P})$. Let $\mu'_I = (p'_1, \ldots, p'_m)$, where $p'_i$ is the probability assigned to $h_i$ and the support of $\mu'_I$ is $O(\alpha(n))$. Let $R_1, \ldots, R_m$ be the payments that the player acting on $I$ gets under $s$ conditioned on reaching $h_1, \ldots, h_m$ respectively. Similarly, let $R'_1, \ldots, R'_m$ be the payments conditioned on reaching $h_1, \ldots, h_m$ respectively under the strategy to which the player at $I$ deviates from $s$. Then, $\sum_{i=1}^m p'_i R'_i > \sum_{i=1}^m p_i R_i$. Let $\mu_I = (p_1, \ldots, p_m)$ be the beliefs on $I$ under $s$ in $(V, \vec{P})$. We use the relationship between the distributions $\mu'_I$ and $\mu_I$, to show that such a deviation in $(V', \vec{P})$ would imply a deviation in $(V, \vec{P})$. In particular, mapping $\mu'_I$ back to $\mu_I$, using Equation 2 we get:

\[
\sum_{i=1}^m \left( \mathbb{I}_{i=f(S(i))} \cdot \sum_{k \in S(i)} p_k \right) R'_i > \sum_{i=1}^m \left( \mathbb{I}_{i=f(S(i))} \cdot \sum_{k \in S(i)} p_k \right) R_i
\]

\[
\sum_{i=1}^m \left( \mathbb{I}_{i=f(S(i))} \cdot \sum_{k \in S(i)} p_k \right) \cdot \min_{k \in S(i)} R_k' > \sum_{i=1}^m \left( \mathbb{I}_{i=f(S(i))} \cdot \sum_{k \in S(i)} p_k \right) \cdot \max_{k \in S(i)} R_k
\]  \hspace{1cm} (3)

\[
\sum_{i=1}^m \left( \mathbb{I}_{i=f(S(i))} \cdot \sum_{k \in S(i)} p_k R_k' \right) > \sum_{i=1}^m \left( \mathbb{I}_{i=f(S(i))} \cdot \sum_{k \in S(i)} p_k R_k \right)
\]

\[
\sum_{i=1}^m p_i R'_i > \sum_{i=1}^m p_i R_i
\]  \hspace{1cm} (4)

Inequality 3 holds because $R'_{f(S(i))} > R'_{f(S(i))}$, and so the two payments lie in different intervals in the mapping (Equation 1). Thus the minimum payment in the interval of $R'_{f(S(i))}$ will be greater than the maximum payment in the interval of $R_{f(S(i))}$. Finally, Inequality 4 contradicts the fact that $s$ was an SSE in $(V, \vec{P})$, achieving a contradiction for the case of reachable information sets.

For unreachable information sets the argument is easy. If $I$ is unreachable under $s$ in $(V', \vec{P})$, then $I$ must be unreachable under $s$ in $(V, \vec{P})$. If the action of prover acting at $I$ is not his best response in $(V', \vec{P})$ for some history $h \in I$ then, it contradicts the fact that $s$ is an SSE of $(V, \vec{P})$.

Now, suppose $s$ is not a dominant SSE of $(V', \vec{P})$. Then there exists a subgame $H_I$ of height $k$ such that $s$ is dominant on all subgames following $H_I$ of height $k$ but not weakly-dominant at $H_I$ (among SSE’s that are dominant at all subforms following $H_I$). Let $s^*$ be dominant on $H_I$, then the expected payment of at least one prover $P_j$ is better under $s^*$, while everyone else does just as well (given the beliefs at $I$ derived using Bayes’ rule if $I$ is reachable under $s$ or given any beliefs if $I$ is unreachable under $s$). Writing out the expression of expected payment of $P_j$ conditioned
on reaching $H_I$ and “unfolding” the probability distribution back to the original game, we get a contradiction that $s$ could not have been a dominant SSE of the original game, as the same strategy $s^*$ would give $P_j$ a better expected payment at $H_I$ while doing as well for other provers. The proof is similar to the above and we omit the details. □

The following claim states that for a given $s$, the expected payments of the provers under $(V, \bar{P})$ and under $(V', \bar{P})$ are not too far off. This claim is one of the bullet points in Lemma 14 and will be used to prove Claim 17.

Claim 16. For all $j \in \{1, \ldots, p(n)\}$, $|u_j(x, s, (V, \bar{P})) - u_j(x, s, (V', \bar{P}))| < 1/(4\alpha(n))$.

Proof. Given input $x$ and strategy profile $s$, fix a prover $P_j$. Let $V$ generate payments $R_1, R_2, \ldots, R_m$ under $s$ for $P_j$, and assign the probability distribution $\mu = (p_1, p_2, \ldots, p_m)$ on $R_1, R_2, \ldots, R_m$ respectively. Using Equations (1) and (2) we compare $P_j$’s expected payment:

$$|u_j(x, s, (V, \bar{P})) - u_j(x, s, (V', \bar{P}))| = \left| \sum_{i=1}^{m} p_i R_i - \sum_{i=1}^{m} p_i R_{f(T_j)} \right|$$

$$= \sum_{T_j \in T} \sum_{k \in T_j} p_k \left| R_{f(T_j)} - R_i \right| < \sum_{T_j \in T} p_i \left( \frac{1}{4\alpha(n)} \right) = \left( \sum_{i=1}^{m} p_i \right) \frac{1}{4\alpha(n)} = \frac{1}{4\alpha(n)}$$

□

To complete the proof of Lemma 14 we show that $(V', \bar{P})$ preserves utility gap guarantees.

Claim 17. Given input $x$, if the answer bit under $s$ is wrong, then there exists a subform $H_I$ reachable under $s$ in $(V', \bar{P})$ and $P_j$ acting at $H_I$, such that $P_j$’s expected payment under $s$ is $\frac{1}{2\alpha(n)}$ less than his expected payment under $(s_{-I}, s^*_I)$, where $s^*_I$ is a dominant SSE on $H_I$.

Proof. Consider a strategy profile $s^*$ that is a dominant SSE in the game tree of $(V, \bar{P})$. Since $s$ gives the wrong answer bit, from the $\alpha(n)$-utility gap guarantee of $(V, \bar{P})$ and Definition 9 there exists a subform $H_I$ reachable under $s$, such that a prover $P_j$ acting in $H_I$ loses $1/\alpha(n)$ in his expected payment under $s$ compared to the strategy profile $(s_{-I}, s^*_I)$. That is,

$$u_j(x, (s_{-I}, s^*_I), (V, \bar{P})) - u_j(x, (s_{-I}, s_I), (V, \bar{P})) > \frac{1}{\alpha(n)}.$$  \hspace{1cm} (5)

Using Claim 15 $s^*$ also induces a dominant SSE in the game tree of $(V', \bar{P})$. And since $H_I$ is reachable under $s$ in $(V, \bar{P})$, it is reachable under $s$ in $(V', \bar{P})$ as well. We show that:

$$u_j(x, (s_{-I}, s^*_I), (V', \bar{P})) - u_j(x, (s_{-I}, s_I), (V', \bar{P})) > \frac{1}{2\alpha(n)}.$$  \hspace{1cm} (6)

Using Claim 16 prover $P_j$’s expected payments in the two protocols under $s$ and $s^*$ follow:

$$|u_j(x, (s_{-I}, s^*_I), (V, \bar{P})) - u_j(x, (s_{-I}, s_I), (V', \bar{P}))| < \frac{1}{4\alpha(n)}$$  \hspace{1cm} (7)

$$|u_j(x, (s_{-I}, s_I), (V, \bar{P})) - u_j(x, (s_{-I}, s_I), (V', \bar{P}))| < \frac{1}{4\alpha(n)}$$  \hspace{1cm} (8)

There are four cases depending on the sign of the left hand side of Inequalities (7) and (8). We show that Claim 17 holds for one of the cases and omit the details of the others, which are similar.
Suppose the left hand side of both inequalities is positive, that is, \( u_j(x, (s_{-I}, s_I^*), (V', \tilde{P})) > u_j(x, (s_{-I}, s_I^*), (V', \tilde{P})) \) and \( u_j(x, (s_{-I}, s_I), (V, \tilde{P})) > u_j(x, (s_{-I}, s_I), (V', \tilde{P})) \). Then
\[
    u_j(x, (s_{-I}, s_I^*), (V', \tilde{P})) - u_j(x, (s_{-I}, s_I), (V', \tilde{P})) > \left( u_j(x, (s_{-I}, s_I^*), (V, \tilde{P})) - \frac{1}{4\alpha(n)} \right) - u_j(s', x, (V', \tilde{P})) \\
    > \left( u_j(x, (s_{-I}, s_I^*), (V, \tilde{P})) + \frac{1}{\alpha(n)} \right) - \frac{1}{4\alpha(n)} - u_j(x, (s_{-I}, s_I), (V', \tilde{P})) > \frac{3}{4\alpha(n)}. \quad \Box
\]

Using Lemma 18, given an \( O(\alpha(n)) \)-gap ncRIP protocol (where \( \alpha(n) \) is constant or polynomial), a polynomial-time oracle Turing machine can use its NEXP oracle to guess a strategy profile \( s \), prune the verifier’s Nature moves, and report the resulting \( O(\alpha(n)) \)-support distribution bit-by-bit. Thus, it can simulate the new distribution and find the decision nodes that are reachable under \( s \).

**Searching through the strategy-profile space efficiently** The next question is: how should the polynomial-time Turing machine navigate the potential strategy-profile space (in polynomial time) to find the strategy profile that satisfies Observation 13? To cut down on the search space, we invoke a recurring idea: divide each prover’s expected payment interval \([-1, 1]\), evenly into \( 8\alpha(n) \) subintervals of length \( 1/(4\alpha(n)) \), and consider subinterval profiles (a tuple of subintervals, one for each prover).

**Lemma 18.** Given an input \( x \) and an ncRIP protocol \((V, \tilde{P})\) with \( \alpha(n) \)-utility gap, consider a subinterval profile \((L_1, \ldots, L_{p(n)})\), where each \( L_i = [k/(4\alpha), (k+1)/(4\alpha + 1)] \) denotes a subinterval of prover \( P_i \) in \([-1, 1]\), for some \( k \in \{-2\alpha(n), \ldots, 2\alpha(n) - 1\}\). Let \( s \) be an SSE that has an expected payment profile \( \tilde{u}(x, s) \) such that \( u_i(x, s) \in L_i \) for all \( 1 \leq i \leq p(n) \), and \( s \) does not satisfy Observation 13. Then the expected payment profile \( \tilde{u}(x, s^*) \) under a dominant SSE \( s^* \) cannot lie in the same subinterval profile, that is, there exists a prover index \( j \) such that \( u_j(x, s^*) \notin L_j \).

**Proof.** Since \( s \) does not satisfy Observation 13 there exists a reachable subform \( H_I \) and prover \( P_j \) acting on \( H_I \) such that the following holds. Without loss of generality, let \( \mu_j(s, x) \in L_k \).
\[
    u_j(x, (s_{-I}, s_I^*), (V, \tilde{P})) - u_j(x, (s_{-I}, s_I), (V, \tilde{P})) > \frac{1}{\alpha(n)} \\
    u_j(x, s^*, (V, \tilde{P})) > \frac{1}{\alpha(n)} + \frac{k}{4\alpha(n)} \implies u_j(x, s^*, (V, \tilde{P})) \notin L_k \quad \Box
\]

Using Lemma 18, if the polynomial-time Turing machine is able to test any SSE \( s \) with \( \tilde{u}(x, s) \) in a subinterval profile, for all subinterval profiles, then it is guaranteed to find one that satisfies Observation 13. This is because a dominant SSE of an ncRIP protocol is guaranteed to exist and its expected payment profile must belong to some subinterval profile.

However, there are still \( O(\alpha(n)) \) subintervals for each prover, and thus \( O(\alpha(n)^p(n)) \) total subinterval profiles. A polynomial-time machine cannot test SSEs for each of them.

To reduce the search space further, we show that it is sufficient to consider subintervals of the total expected payment rather than individual and test an SSE \( s \) for each of them. Recall that a SSE \( s \) is weakly dominant if for any player \( i \) and SSE \( s' \), \( u_i(s) \geq u_i(s') \).

**Lemma 19.** If a weakly-dominant SSE exists, then a strategy profile \( s \) is a weakly-dominant SSE if and only if \( s \) is an SSE and \( s \) maximizes the sum of utilities of all players among all SSEs.

We are now ready to prove the upper bound for ncRIP classes with constant, polynomial, and exponential utility gap.
**Constant utility gap** Using Lemma 14 and Lemma 19, simulating a constant-gap protocol using a $P^{\text{NEXP}[O(1)]}$ machine $M$ is straightforward. We give a high-level overview below.

There are at most $O(1)$ subforms that are reachable under any strategy profile $s$, and the total expected payment of the provers conditioned on reaching these subforms will be in one of the $O(1)$ subintervals. Thus, there are $O(1)$ combinations of total expected payments on all subforms (including the whole game). $M$ queries its NEXP oracle whether there exists an SSE that achieves that combination of total expected payments on those subforms, for all combinations.

**Lemma 20.** $O(1)-\text{ncRIP} \subseteq P^{\text{NEXP}[O(1)]}$. 

**Proof.** Given any $L \in \alpha(n)-\text{ncRIP}$, let $(V, \tilde{P})$ be the MRIP protocol with $\alpha(n)$ utility gap for $L$, where $\alpha(n)$ is a constant.

Given an input $x$ of length $n$, consider the following deterministic polynomial-time oracle Turing machine $M$ with access to an oracle $O$ for an NEXP language. Similar to the proof of Lemma 22, $M$ divides $[-1,1]$ into $8\alpha(n)$ intervals, each of length $1/4\alpha(n)$. In other words, the $i$th interval is $[i/4\alpha(n), (i+1)/4\alpha(n))$ for each $i \in \{-4\alpha(n), \ldots, 4\alpha(n)-1\}$. 

Using Lemma 14 under a given input $x$ and strategy profile $s$, there are at most $8\alpha(n)$ subforms are reached under any $s$ in the modified game. Total expected payment of provers acting within any subform (conditioned on reaching the subform) must lie in any one of the $8\alpha(n)$ intervals in $[-1,1]$. Thus overall, there are $O(\alpha(n)^n)$ combinations of total expected payments over subforms, which is still $O(1)$. Let $(u, u_1, \ldots, u_k)$ be a tuple of total expected payments, where $k = 8\alpha(n)$, the maximum number of subforms reachable under any $s$, and $u$ represents the total expected payment of the whole game, whereas $u_1$ represents total expected payment of the provers acting in subform $I_j$ (conditioned on reaching $I_j$).

For each combination $(u, u_1, \ldots, u_k)$, $M$ queries $O$: does there exist a strategy profile that is an SSE and the total expected payments over reachable subforms under $s$ and $O(\alpha(n))$ support $\text{Nature moves imposed by Lemma 14}$ is $(u, u_1, \ldots, u_k)$ (conditioned on reaching the subforms)? Among the queries to which the oracle’s answer is “yes”, $M$ finds the combination that achieves maximum total expected payment for all subforms. Such a combination is guaranteed to exist because $(V, \tilde{P})$ is an ncRIP protocol, and a dominant SSE of the game exists. 

**Remark 21.** The polynomial-time oracle Turing machine in Lemma 24 can issue all its queries non-adaptively. That is, $\alpha(n)-\text{ncRIP} \subseteq P^{\text{NEXP}[O(1)]}$. Furthermore in Section 4, we show that $O(1)-\text{ncRIP} \subseteq P^{\text{NEXP}[O(1)]}$. Indeed, the two classes are equal: $P^{\text{NEXP}[O(1)]} = P^{\text{NEXP}[O(1)]}$. Since $O(1)-\text{MRIP} = P^{\text{NEXP}[O(1)]}$, this shows that cooperative provers are as powerful as non-cooperative provers under constant utility-gap guarantees, and we obtain Corollary 4.

**Polynomial utility gap** Next, we prove the upper bound of the case of polynomial utility gap. We note that the simple strategy of querying all possible payment combinations as in Lemma 20 does not work (there are $O(\alpha(n)^{\alpha(n)})$ total combinations).

To simulate a polynomial-utility gap ncRIP protocol $(V, \tilde{P})$, using a $P^{\text{NEXP}}$ machine $M$, we put to use all the structure we have established in this section.

For each of the $O(\alpha(n))$ total payment subintervals of the interval $[-1,1]$ that correspond to an SSE, $M$ does a recursive search to find an exact total expected payment $u(x, s)$ that is generated by an SSE. (We can restrict ourselves to $O(\alpha(n))$ oracle queries due to Lemma 19.) In particular, $M$ queries the NEXP oracle: does there exist an SSE with total expected payment in the first half of

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8To include 1 as a possible reward, interval $4\alpha(n) - 1$ should be closed on both sides; we ignore this for simplicity.
the $i$th interval? If the answer is yes then $M$ recurses on the first half of the $i$th interval; $M$ does not need to search the second half by Lemma \ref{lem:subintervals}. Otherwise (if the answer is no) then $M$ recurses on the second half. Thus, in polynomial time and with polynomial queries, $M$ can find an exact $u(x,s)$ for an SSE $s$ in the subinterval using the power of its adaptive queries.

Next, $M$ simulates the protocol $(V, \bar{P})$ with the help of the oracle, under the SSE $s$ for a given subinterval. Lemma \ref{lem:subintervals} is crucial for $M$ to simulate the verifier’s moves, because $V$ in general can induce exponential-size distributions. $M$ traverses the tree reachable under $s$ “top-down” using the oracle to learn the pruned distributions and provers’ moves. Finally, $M$ goes “bottom-up” to test whether $s$ satisfies Observation \ref{obs:subintervals} on all its reachable subgames.

**Lemma 22.** $\text{poly}(n)\text{-ncRIP} \subseteq \text{pNEXP}$.

**Proof.** Given any $L \in \text{poly}(n)\text{-ncRIP}$, let $(V, \bar{P})$ be the ncRIP protocol with $\alpha(n)$ utility gap for $L$, where $\alpha(n) = n^k$ for some constant $k$.

Given an input $x$ of length $n$, consider the following deterministic polynomial-time oracle Turing machine $M$ with access to an oracle $O$ for an NEXP language. $M$ divides $[-1,1]$ into $8\alpha(n)$ intervals, each of length $1/4\alpha(n)$. In other words, the $i$th interval is $[i/4\alpha(n), (i+1)/4\alpha(n))$ for each $i \in \{-4\alpha(n), \ldots, 4\alpha(n) - 1\}$.

For each interval $[i/4\alpha(n), (i+1)/4\alpha(n))$, $M$ makes the following queries to $O$: does there exist a strategy profile $s$ that is an SSE and the sum of expected payments of all provers $u(x,s)$ is in the $i$th interval? Let $L$ denote the set of intervals for which the answer to the query is “yes”.

For each interval $[\ell/4\alpha(n), (\ell+1)/4\alpha(n)) \in L$, $M$ queries $O$: does there exist a strategy profile $s$ that is an SSE and the sum of expected payments of all provers $u(x,s)$ is in the first half of the $\ell$th interval? If the answer is “yes”, then $M$ recurses on the first half, else $M$ recurses on the second half of the interval. In polynomial time and polynomial queries, $M$ can find the exact total expected payment $u(x,s,(V,\bar{P}))$ in the interval that is generated by an SSE. $M$ asks further queries to figure out the exact payment profile under such an SSE. For $k \in \{1, \ldots, p(n)\}$, where $p(n)$ is the total number of provers in $(V,\bar{P})$, and for each $j \in \{1, \ldots, n^k\}$, where $n^k$ is the running time of $V$ ($k'$ is a constant), $M$ asks the following queries adaptively: under an SSE where $\sum_{i=1}^{p(n)} \mu_i(x,s) = u(x,s)$, what is the $j$th bit in the expected payment $\mu_k(x,s)$ of prover $P_k$, given and the first $j-1$ bits of $\mu_k(x,s)$ and $\mu_1(x,s), \ldots, \mu_{k-1}(x,s)$? In $O(n^{k^2}p(n))$ queries, $M$ can figure out the exact payment profile $\bar{u}(x,s) = (\mu_1(x,s), \ldots, \mu_k(x,s))$ under an SSE $s$, such that the total expected payment is in the $\ell$th interval.

$M$ now verifies whether the SSE corresponding to the payment profile $\bar{u}(x,s)$ satisfies the condition of Observation \ref{obs:subintervals}. $M$ proceeds in two phases: first, $M$ wants to go “top-down” figuring out what part of the game tree is being played under $s$ on input $x$, using the oracle to simulate the provers and the verifier. Then, it goes “bottom-up” in the tree being played under $s$, to check whether all subforms are “$(1/\alpha(n))$-close” to the dominant strategy at that subform.

**Top-down phase.** Let $k(n)$ be the total number of rounds in $(V,\bar{P})$. Note that $k(n)$ is polynomial in $n$. Let $m_{ij}$ denote the message sent by prover $P_j$ at round $j$. Then, for each round $j$ and each prover $i$ where $1 \leq j \leq k(n)$ and $1 \leq k \leq p(n)$, $M$ first asks the oracle to give the “pruned” $O(\alpha(n))$ support distribution imposed by the Nature move of $V$ at round $j$ bit by bit as follows: “under an SSE where the expected payment profile is $\bar{u}(x,s)$, what is the $r$th bit of the distribution imposed by $V'$ using $V$ and Lemma \ref{lem:pruning}?” This requires a polynomial number of bits (and therefore queries) because the distribution is polynomial sized. The pruned distribution preserves the dominant SSE and changes the utility gap by only a factor 2 (this factor does not affect the proof as our intervals

\footnote{To include 1 as a possible reward, interval $4\alpha(n) - 1$ should be closed on both sides; we ignore this for simplicity.}
are scaled down to handle it). Given this distribution, $M$ simulates $V$ on the support of the distribution to figure out the messages that $V$ sends to the provers in round $j$. In particular, $M$ does not have access to random bits, so instead it simulates every action of $V$ in the support. To simulate the provers at round $j$, $M$ similarly queries $O$ bit by bit: “under an SSE where the expected payment profile is $\tilde{u}(x, s)$, what is the $r$th bit of the message sent by $P_k$”. Thus, after simulating the moves of $V$ and $P$ under $s$, $M$ has sketched out the $O(\alpha(n))$ size part of the game tree being played under $s$ corresponding to $\tilde{u}(x, s)$.

**Bottom-up phase.** Given the $O(\alpha(n))$ nodes of the game tree under play, $M$ can mark out the subforms reachable under $s$ corresponding to $\tilde{u}(x, s)$. Going from the last level up, for each subform $H_I$ reachable under $s$, $M$ uses the oracle to figure out which payment interval the expected payments of the weakly-dominant SSE on $H_I$ lie in (given the expected weakly-dominant SSE payments on the reachable subforms verified so far), until it finds a subform that violates the condition of Observation 13.

In particular, for each subform $H_I$ of height $k$, let $\tilde{u}(x, s, I')$ denote the tuple of total expected payments under $s$ on all subforms $H_I'$ of height $< k$ following $I$ (conditioned on reaching $I$) verified so far. $M$ divides the interval $[-1, 1]$ into $8\alpha(n)$ intervals of size $\alpha(n)/4$ as before and for each interval queries the oracle $O$: does there exist a strategy profile $s_I$ on subgame $H_I$ that is an SSE and the sum of expected payments of all provers $u(x, s, I)$ is in the $x$th interval, and gets a total expected payments on subforms $H_{I'}$ of height $< k$ following $I$ equal to $\tilde{u}(x, s, I')$.

Then, $M$ finds the maximum interval $[i/4\alpha(n), (i+1)/4\alpha(n)]$ among the intervals for which the oracle says yes. By Lemma 19 the weakly-dominant SSE $s_I^{\text{max}}$ at $H_I$ also lies in the $i$th interval. Using the probability $p_I$ assigned by $H_I$ ($M$ knows the distribution imposed by all “pruned” Nature moves), $M$ checks whether the total expected payment of weakly-dominant SSE $s_I^{\text{max}}$ is in the same interval as the sum of expected payments of provers in $Z_I$ under $s$. If it is not, then $s$ fails the test and $M$ continues to the next interval in $L$. Otherwise, $M$ continues to the next reachable subform.

If $s$ passes the test for all subforms (including at the root), then by Observation 13 the answer bit under $s$ is correct. $M$’s final query to $O$ is: “under an SSE where the expected payment profile is $\tilde{u}(x, s)$, what is the answer bit $c$? If $c = 1$, then $M$ accepts $x$, otherwise $M$ rejects $x$.

$M$ is guaranteed to find a payment profile $\tilde{u}(x, s)$ (and thus a strategy profile $s$) that passes the test. Since $(V, \tilde{P})$ is an ncRIP protocol for $L$, there exists a dominant SSE $s^*$ in some interval in $L$. By Observation 13 if a strategy profile $s'$ fails the test, the dominant SSE can not get a total expected payment in the same interval as $s'$. Thus, we can rule out intervals by checking any SSE with total expected payment in that interval. Since a dominant SSE $s^*$ exists, $M$ must eventually find an interval, where the corresponding SSE passes the test.

To complete the proof, we note that (a) $M$ runs in polynomial time, (b) each query to the oracle is polynomial, and, (c) the oracle queries can be answered in non-deterministic exponential time.

First, (a) holds because each top-down and bottom-up phase is executed $O(\alpha(n))$ times and each of the phases take polynomial time. In the top-down phase, $M$ simulates the protocol on strategy $s$ using the oracle while restricting the verifier’s Nature moves to be of $O(\alpha(n))$ support. Thus this phase takes polynomial time. For the bottom-up phase, $M$ finds weakly-dominant SSEs at each reachable subforms under $s$. Since there are at most $O(\alpha(n))$ subforms and at most $O(\alpha(n))$ interval queries for each subform, the bottom-up phase takes time polynomial in $n$.

Second, (b) holds each oracle query involves a total expected payment $\tilde{u}(x, s)$ or an interval of

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10 $M$ does not need to send the total expected payments of the subforms at lower levels. Instead, $M$ can just send the total expected payment $u(x, s)$ at the root and ask $O$ to guess $s$ as well. An NEXP can verify if one SSE weakly dominates another. This observation is crucial in extending this proof to exponential utility gap.
size $\alpha(n)/2$, both of which can be generated by $V$ and hence are polynomial in $n$.

To prove (c), it is sufficient to show that an NEXP machine can guess a strategy profile and verify if it is an SSE and if it gets expected payments in a certain interval. Since the transcript of any ncRIP protocol is polynomial in $n$, a strategy profile $s$ of the provers can be represented in exponential bits, and thus $O$ can guess such an $s$. Now given $s$ and the protocol $(V, \vec{F})$, by Lemma 24 it is possible to verify whether $s$ is an SSE of the game in time linear in the size of the game tree, and thus exponential in $n$. Furthermore, it can compute the expected payments of the provers under $s$ in exponential time as well, which is sufficient to answer all the queries made by $M$.

Exponential utility gap We conclude by giving a tight upper bound on the class of ncRIP protocols with exponential utility gaps. The proof follows immediately from that of Lemma 22. In fact, it is simpler as the exponential-time Turing machine is powerful enough to (a) simulate $V$’s Nature moves directly, and (b) test all possible payment profiles. Thus, in the case of exponential utility gap, we do not need Lemma 14 or the notion of subintervals.

Lemma 23. $\text{ncRIP} \subseteq \text{EXP}^{\text{poly}} - \text{NEXP}$.

Remark 24. Since $\text{EXP}^{\text{poly}} - \text{NEXP} \subseteq \text{EXP}^{\text{NP}} - \text{NEXP} = \text{EXP}^{\text{NP}}$, and $\text{EXP}^{\text{NP}} \subseteq \text{MRIP}$ [20], Lemma 23 shows that $\text{exp}(n) - \text{ncRIP} \subseteq \text{exp}(n) - \text{MRIP}$ and using Lemma 12 we get that in general the two classes coincide. In other words, non-cooperative rational proofs are as powerful as cooperative multi-prover rational proofs under exponential utility gap and we obtain Corollary 6.

5 Additional Related Work

Rational Proofs The model of single-prover rational interactive proofs (RIP) was introduced by Azar and Micali [1], who used scoring rules as the main tool to construct simple and efficient RIP protocols. In a follow-up work [2], they extended this work to design super-efficient rational proofs that have sublinear verification and computation complexity. Guo et al. present rational arguments for a computationally bounded prover and a sublinear verifier in [34], and construct rational arguments for all languages in $\text{P}$ [35]. Campanelli and Rosario [13] study sequentially composable rational proofs and rational proofs for space bounded computations [14], while Zhang and Blanton [54] design protocols to outsource matrix multiplications to a rational cloud.

The model of multi-prover (cooperative) rational interactive proofs (MRIP) was introduced by Chen et al. [20]. In this model, the provers work together to maximize their total payment. They show that the class equals $\text{EXP}^{\text{NP}}$ under exponential utility gap and $\text{P}^{\text{NP}}$ under polynomial utility gap. In the full version [19], they show that MRIP under constant utility gap is equal to $\text{P}^{\text{NP}[O(1)]}$. In follow-up work [21], the authors scale down the power of the verifier and design super-efficient MRIP protocols with strong utility-gap guarantees.

Game-Theoretic Characterization of Complexity Classes Game-theoretic characterization of complexity classes has been largely studied in the form of refereed games [18,24,26,28,40,45]. Chandra and Stockmeyer [18] show that any language in $\text{PSPACE}$ is refereable by a game of perfect information. Feige and Kilian [24] show that the class of imperfect information, perfect recall refereed games is exactly $\text{EXP}$. Feigenbaum, Koller and Shor [28] show that if provers are allowed to have imperfect recall (essentially acting as oracles), refereed games can simulate $\text{EXP}^{\text{NP}}$. 

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6 Properties of Strong Sequential Equilibrium

In this section, we prove several important properties of strong sequential equilibrium, which make it a good candidate solution concept in designing extensive-form mechanisms.

Strong sequential equilibrium admits a sequential equilibrium We first show that, given a strategy profile $s$ that is a strong sequential equilibrium (thus does not rely on a belief system), we can construct a belief system $\mu$ such that the pair $(s, \mu)$ forms a sequential equilibrium.

**Lemma 25.** For any strategy profile $s$ that is a strong sequential equilibrium, there exists a belief system $\mu$ such that $(s, \mu)$ is a sequential equilibrium.

**Proof.** The sequential-rationality requirement will follow easily from the definition of SSE. To prove that $s$ admits a sequential equilibrium, the key is to pair it with a consistent belief system; see Section 2 for definition. Indeed, we construct a belief system $\mu$ and show that, there exists a sequence of pairs $(s^\varepsilon, \mu^\varepsilon)_{\varepsilon \to 0}$ which converges to $(s, \mu)$, as $\varepsilon$ goes to 0, where each $s^\varepsilon$ is a profile of completely mixed behavioral strategies and each $\mu^\varepsilon$ is the belief system derived from $s^\varepsilon$ using Bayes’ rule.

Recall that a strategy profile $s$ defines a probability distribution over the actions available to a player at an information set where he acts. That is, for each information set $I_i$ of a player $i$, $s_i(I_i)$ is a probability distribution over $A(I_i)$, the set of actions available to player $i$ at $I_i$. In particular, if $A(I_i) = \{a_1, \ldots, a_k\}$, then $s_i(I_i) = (p_i(a_1), \ldots, p_i(a_k))$ where $p_i(a_k)$ is the probability that player $i$ chooses action $a_k$ at $I_i$. Let $A^+(I_i)$ and $A^0(I_i)$ be the set of actions at information set $I_i$ which player $i$ chooses with positive probability and zero probability respectively; that is, $A^+(I_i) = \{a_k \in A(I_i) \mid p_i(a_k) > 0\}$ and $A^0(I_i) = A(I_i) \setminus A^+(I_i)$. For any $\varepsilon \in (0,1)$, we define $s_i^\varepsilon$ for player $i$ at information set $I_i$ as follows: if $A^0(I_i) = \emptyset$ then $s_i^\varepsilon(I_i) = s_i(I_i)$; otherwise,

$$s_i^\varepsilon(I_i)(a_k) = \begin{cases} (1 - \varepsilon) \cdot p_i(a_k) & \text{for each } a_k \in A^+(I_i); \\ \frac{\varepsilon}{|A(I_i)|} & \text{for each } a_k \in A^0(I_i). \end{cases}$$

By construction, $s_i^\varepsilon(I_i)$ is a valid probability distribution over $I_i$ and is completely mixed, that is, assigns a positive probability to every action in $I_i$. Indeed, because $\sum_{k=1}^K p_i(a_k) = \sum_{a_k \in A^+(I_i)} p_i(a_k) = 1$, when $A^0(I_i) \neq \emptyset$ we have $\sum_{a_k \in A^0(I_i)} s_i^\varepsilon(I_i)(a_k) = \sum_{a_k \in A^+(I_i)} (1 - \varepsilon) p_i(a_k) + \varepsilon = 1$. It is easy to see that $s_i^\varepsilon$ converges to $s_i$ when $\varepsilon \to 0$.

Given the strategy profile $s^\varepsilon$, to define $\mu_i^\varepsilon$, the belief system of a player $i$, consider an arbitrary information set $I_i$ where player $i$ acts. The probability that a particular history $h = (a^1, \ldots, a^K) \in I_i$ occurs can be derived from $s^\varepsilon$ as follows. For any history $h' = (a^1, \ldots, a^w)$ with $0 \leq w \leq K - 1$, recall that $Z(h')$ is the player acting following history $h'$. For any action $a \in A(h')$, let $s_{Z(h')}(h')(a)$ denote the probability assigned by $s^\varepsilon_{Z(h')}$ to action $a$ at history $h'$ (i.e., at the information set containing $h'$). We have

$$\Pr \{h \text{ occurs under } s^\varepsilon\} = \prod_{w=0}^{K-1} s_{Z(a^1,\ldots,a^w)}^\varepsilon(a^1,\ldots,a^w)(a^{w+1}) = c_h \varepsilon^{e_h} (1 - \varepsilon)^{f_h},$$

where $c_h, e_h$ and $f_h$ are positive constants depending on $s$ and $h$, but not on $\varepsilon$. In particular, letting $S^0$ be the set of actions $a^{w+1}$ in $h$ that are assigned zero probability by $s_{Z(h')}$ at history
follows from the definition of SSE. Thus (distribution derived from s)
player i's lim
µ
µ
ε
ε
I
I
I
I
At unreachable information sets
• At reachable information sets
We now give an equivalent definition of SSE, which says that a player's strategy at an unreachable
holds at that set, his action should be a best response to that belief and the other players' strategies.
The notion of strong sequential equilibrium
Alternate definition of strong sequential equilibrium
Lemma 27. For any strategy profile s, any player i and information set I_i of i that is not reached
with positive probability under s, conditional on I_i being reached, s_i is a best response to s_{-i} with
respect to all possible beliefs that player i may hold at I_i if and only if for every history h ∈ I_i, s_i
is a best response to s_{-i} given i's belief that he is at h with probability 1.
Proof. The “only if” part is immediate, because for any history \( h \in I_i \), “at \( h \) with probability 1 (and any other history with probability 0)” is a specific belief that \( i \) may hold at \( I_i \).

The “if” part is also easy to show. Suppose that \( s_i \) is a best response to \( s_{-i} \) conditional on every history \( h \in I_i \) (i.e., at \( h \) with probability 1). To show that \( s_i \) is a best response to \( s_{-i} \) conditional on all possible beliefs player \( i \) may hold at information set \( I_i \), arbitrarily fix a belief \( \mu_i(I_i) \) over \( I_i \) and a strategy \( s_i' \). Let \( I_i = \{h_1, h_2, \ldots, h_m\} \) and \( \mu_i(I_i) = (\mu_i(I_i)(h_1), \mu_i(I_i)(h_2), \ldots, \mu_i(I_i)(h_m)) \), where \( \mu_i(I_i)(h_k) \) is the probability with which player \( i \) believes that history \( h_k \) occurs conditional on \( I_i \) being reached. Then, player \( i \)'s expected utilities under \( s_i \) and \( s_i' \) respectively, conditioned on \( I_i, \mu_i(I_i) \) and \( s_{-i}, \) are

\[
\begin{align*}
&u_i(s_i, s_{-i} | \mu_i(I_i)) = \sum_{k=1}^{m} \mu_i(I_i)(h_k) \cdot u_i(s_i, s_{-i} | h_k) \quad \text{and} \quad u_i(s_i', s_{-i} | \mu_i(I_i)) = \sum_{k=1}^{m} \mu_i(I_i)(h_k) \cdot u_i(s_i', s_{-i} | h_k),
\end{align*}
\]

where \( u_i(s_i, s_{-i} | h_k) \) is player \( i \)'s utility under \((s_i, s_{-i})\), conditioned on history \( h_k \) being reached at \( I_i \). Since \( s_i \) is a best response to \( s_{-i} \) at every \( h_k \in I_i \), we have \( u_i(s_i, s_{-i} | h_k) \geq u_i(s_i', s_{-i} | h_k) \\forall k \in \{1, \ldots, m\} \). Thus \( u_i(s_i, s_{-i} | \mu_i(I_i)) \geq u_i(s_i', s_{-i} | \mu_i(I_i)) \) and the “if” part holds.

One-shot deviation for strong sequential equilibrium Informally, the one-shot deviation principle says that a player cannot change his action at a single information set (without changing the rest of his strategy) and improve his expected reward.

In the context of sequential equilibrium, it is well known that given a consistent belief system \( \mu, (s, \mu) \) is a sequential equilibrium if and only if the one-shot deviation principle holds, that is, no player \( i \) has an information set \( I_i \) at which a change in \( s_i(I_i) \)—holding the remaining of \( s_i \) fixed—increases his expected utility conditional on reaching \( I_i \) [37, 44].

Since strong sequential equilibrium does not require artificial notion of beliefs for unreachable information sets, we define a stronger notion of one-shot deviation at those information sets—for every decision node (i.e., history) in an unreachable information set of player \( i \), there does not exist a one-shot deviation at that node which improves player \( i \)'s utility conditional on that node being reached. Note that at reachable information sets, both the definition and proof of the one-shot deviation condition for SSE are exactly the same as in SE [37].

Lemma 28 (One-shot deviation for strong sequential equilibrium). For any strategy profile \( s, s \) is a strong sequential equilibrium if and only if it satisfies the following one-shot deviation principle: For every player \( i \) and every information set \( I_i \) of \( i \),

- If \( I_i \) is reachable under \( s \): there does not exist a change in \( s_i(I_i) \) (holding the rest of \( s_i \) fixed) that increases player \( i \)'s expected utility conditional on reaching \( I_i \), given his belief at \( I_i \) derived using Bayes’ rule.
- If \( I_i \) is unreachable under \( s \): for every history \( h \in I_i \), there does not exist a change in \( s_i(I_i) \) (holding the rest of \( s_i \) fixed) that increases player \( i \)'s expected utility conditional on reaching \( h \).

Proof. The “only if” part follows immediately from Definition [26] and the fact that a one-shot deviation results in a different strategy for the deviating player. We now prove the “if” part, that is, if \( s \) satisfies the one-shot deviation principle then it is a strong sequential equilibrium.

Reachable information sets. First of all, similar to the proof of Lemma [25] we can construct a belief system \( \mu \) such that \( s \) and \( \mu \) are consistent. Indeed, the construction of \( \mu \) only depends on the actions taken by \( s \) and does not depend on the utilities induced by \( s \) at all. Since \( s \) satisfies the one-shot deviation principle at every reachable information set and at every history in each unreachable information set, it is not hard to see that \( s \) satisfies the one-shot deviation principle with respect to \( \mu \). Thus \((s, \mu)\) is a sequential equilibrium. Accordingly, for any player \( i \) and information set \( I_i \)
of \( i \) that is reachable by \( s, s_i \) is a best response to \( s_{-i} \) conditional on \( \mu_i(I_i) \) (which is derived from \( s \) using Bayes’ rule at \( I_i \)), as desired by the definition of SSE.

**Unreachable information sets.** Next, we use backward induction to show that, for any player \( i \), information set \( I_i \) of \( i \) that is unreachable by \( s \), and history \( h \in I_i \), \( s_i \) is a best response to \( s_{-i} \) conditional on reaching \( h \). To begin with, if \( h \) is of height 1 then this immediately holds: indeed, the strategy induced by \( s_i \) following \( h \) is exactly the action \( s_i(I_i) \), thus the one-shot deviation principle implies that \( s_i \) is a best response to \( s_{-i} \) at \( h \).

Now, arbitrarily fix a player \( i \), information set \( I_i \) of \( i \) unreachable by \( s \), and a history \( h \in I_i \) of height larger than 1. By induction, assume that for any information set \( I_i' \) of \( i \) unreachable by \( s \), and history \( h' \in I_i' \) of height smaller than that of \( h \), \( s_i \) is a best response to \( s_{-i} \) at \( h' \). For the sake of contradiction, suppose player \( i \) can deviate to strategy \( s_i' \) and increase his utility conditional on reaching \( h \), that is,

\[
u_i(s_i', s_{-i}|h) > u_i(s_i, s_{-i}|h).
\]

If \( s_i'(I_i) = s_i(I_i) \), consider the first history \( h' \) following \( h \) where player \( i \) acts and \( s_i' \) differs from \( s_i \). As \( h \) is unreachable by \( s \), \( h' \) is unreachable by \( s \) as well. However, the height of \( h' \) is smaller than that of \( h \) and \( u_i(s_i', s_{-i}|h') = u_i(s_i', s_{-i}|h) > u_i(s_i, s_{-i}|h) = u_i(s_i, s_{-i}|h') \), contradicting the inductive hypothesis. Thus we have

\[s_i'(I_i) \neq s_i(I_i).\]

If \( s_i' \) is the same as \( s_i \) at all the histories following \( (h, s_i'(I_i)) \) where player \( i \) acts, then the one-shot deviation principle is violated. Accordingly, there must exist a history following \( (h, s_i'(I_i)) \), where player \( i \) acts and \( s_i' \) differ from \( s_i \). Letting \( h' \) be the first such history, we have that the height of \( h' \) is smaller than that of \( h \). Since \( h' \) is unreachable by \( s \), by the inductive hypothesis we have that \( s_i \) is a best response to \( s_{-i} \) at \( h' \). Thus \( u_i(s_i, s_{-i}|h') \geq u_i(s_i', s_{-i}|h') \). As \( u_i(s_i', s_{-i}|h') = u_i(s_i', s_{-i}|h) > u_i(s_i, s_{-i}|h) \), we have

\[u_i(s_i, s_{-i}|h') > u_i(s_i, s_{-i}|h).
\]

Let strategy \( s_i'' \) be such that, it follows \( s_i \) till history \( h \), then follows action \( s_i'(I_i) \), then follows \( s_i' \) (and \( s_i \) as well, because they are the same after \( (h, s_i'(I_i)) \) and before \( h' \)) till history \( h' \), and then follows \( s_i \) for the rest. Note that \( s_i'' \) can be obtained from \( s_i \) by a one-shot deviation from \( s_i(I_i) \) to \( s_i'(I_i) \). However,

\[u_i(s_i'', s_{-i}|h) = u_i(s_i', s_{-i}|h') = u_i(s_i, s_{-i}|h') > u_i(s_i, s_{-i}|h),
\]

contradicting the one-shot deviation principle. Therefore \( s_i \) is a best response to \( s_{-i} \) conditional on reaching \( h \), as desired.

Combining everything together, by Definition 26 \( s \) is an SSE and Lemma 28 holds.

**Verifying strong sequential equilibrium** Given an extensive-form game with arbitrary number of players, it is possible to decide whether a pair \((s, \mu)\) is a sequential equilibrium in time polynomial in the size of the game tree 30.

However, if only a strategy profile \( s \) is given, then it is NP-hard to decide whether \( s \) is part of an SE (that is, whether there exists a belief system \( \mu \) such that \((s, \mu)\) is an SE) 36. As strong sequential equilibrium does not rely on belief systems, we prove the following.

**Lemma 29.** Given an extensive-form game and a strategy profile \( s \) of the players, deciding whether \( s \) is a SSE of the game can be done in time polynomial in the size of the game tree.
Proof. First of all, we can traverse the game tree in polynomial time, mark each information set whether it is reachable by $s$ or not, and compute, for each player $i$ and each reachable information set $I_i$ of $i$, the belief $\mu_i(I_i)$ derived from $s$ using Bayes’ rule. Next, we apply the one-shot deviation principle following Lemma 28.

To do so, we start from the bottom level of the tree and proceed up. For every player $i$ and every information set $I_i$ of $i$, if $I_i$ is unreachable under $s$, then we go through each $h \in I_i$ and each $a \in A(I_i)$, and check if changing $s_i(I_i)$ to $a$ improves $i$’s utility conditional on reaching $h$. If so then $s$ is not an SSE. If $I_i$ is reachable under $s$, then we go through every $a \in A(I_i)$, and check if changing $s_i(I_i)$ to $a$ improves $i$’s expected utility conditional on $I_i$ and $\mu_i(I_i)$. If so then again $s$ is not an SSE. If all the checks above pass, then $s$ is an SSE.

Since this procedure goes through each decision node of the game tree at most once, and since it takes polynomial time to compute player $i$’s (expected) utility under $s$ following a decision node (or an information set), deciding whether $s$ is an SSE takes polynomial time in the size of the tree. \qed

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