GLOBAL REGULARITY FOR THE 3D AXISYMMETRIC MHD EQUATIONS WITH HORIZONTAL DISSIPATION AND VERTICAL MAGNETIC DIFFUSION

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ABSTRACT. Whether or not classical solutions of the 3D incompressible MHD equations with full dissipation and magnetic diffusion can develop finite-time singularities is a long standing open problem of fluid dynamics and PDE theory. In this paper, we investigate the Cauchy problem for the 3D axisymmetric MHD equations with horizontal dissipation and vertical magnetic diffusion. We get a unique global smooth solution under the assumption that \( u_\theta \) and \( b_r \) are trivial. In absence of some viscosities, there is no smoothing effect on the derivatives of that direction. However, we take full advantage of the structures of MHD system to make up this shortcoming.

1. Introduction. In this paper, we consider the 3D axisymmetric incompressible Magneto-hydrodynamics (MHD) equations

\[
\begin{align*}
 u_t - \nu_x \partial_{xx} u - \nu_y \partial_{yy} u - \nu_z \partial_{zz} u + u \cdot \nabla u &= -\nabla (p + \frac{1}{2} |b|^2) + b \cdot \nabla b, \\
 b_t - \kappa_x \partial_{xx} b - \kappa_y \partial_{yy} b - \kappa_z \partial_{zz} b + u \cdot \nabla b &= b \cdot \nabla u, \\
 \nabla \cdot u &= \nabla \cdot b = 0,
\end{align*}
\]

where \( u, b \) represent the fluid velocity field and the magnetic field respectively, \( p \) is a scalar pressure and \( \nu_x, \nu_y, \nu_z, \kappa_x, \kappa_y, \kappa_z \) are nonnegative real parameters. In the following context, for simplicity, we denote \( \pi = p + \frac{1}{2} |b|^2 \). Moreover, if \( b = 0 \), this system becomes the incompressible Navier-Stokes equations.
Magneto-hydrodynamics (MHD) equations are to study the behavior of an electrically-conducting fluids. Examples of such fluids include plasmas, liquid metals, salt water, etc. The field of MHD was initiated by Hannes Alfvén, for which he received the Nobel Prize in Physics in 1970. However, the mathematical theory on MHD is still very little known until today.

Let us recall the developments of global regularity for MHD equations in brief. In the two-dimensional case, the constants \( \nu_z \) and \( \kappa_z \) become zero. For this case, when \( \nu_x, \nu_y, \kappa_x \) and \( \kappa_y \) are positive constants, The local well-posedness of (1.1)–(1.3) was established in [6] and [14] where the authors also proved the global well-posedness. If all the parameters are zero, (1.1)–(1.3) becomes inviscid and the global regularity problem appears to be out of reach.

Nevertheless, in the past years, the intermediate cases when some of the parameters are positive have also attracted considerable attention and big progress has been made recently. To be specific, when \( \nu_x > 0, \nu_y = 0, \kappa_x = 0, \kappa_y > 0 \) or \( \nu_x = 0, \nu_y > 0, \kappa_x > 0, \kappa_y = 0 \), Cao and Wu established the global regularity in [2]. But for the case when \( \nu_y = \nu_y = 0, \kappa_x > 0, \kappa_y > 0 \), only partial answer was obtained in [2] and [10]. The authors got global \( H^1 \) weak solutions, however, the uniqueness of such weak solutions and global \( H^2 \) estimates remain unknown. Many attempts have also been made on the MHD equations with only dissipation, namely (1.1)–(1.3) with \( \nu_x > 0, \nu_y > 0, \kappa_x = \kappa_y = 0 \) or \( \nu_x = \nu_y = 0, \kappa_x > 0, \kappa_y > 0 \). Recently, Lin, Xu and Zhang [12] made an progress on the global well-posedness of classical solutions for this case under the assumption that the initial velocity field and the displacement of the magnetic field from a non-zero constant is sufficiently small in appropriate Sobolev spaces. In spite of this, the problem of global regularity for these cases remains open.

For the three dimensional case, it is well known that global regularity of incompressible Navier-Stokes equations is still widely open even in the axially-symmetric case. If the swirl component of the velocity field \( u_\theta \) is trivial, Ladyzhenskaya [9], Ukhovskii and Yudovich [15] proved that weak solutions are regular for all time (see also [11]) independently. For MHD system, under the assumption that \( u_\theta, b_r, \) and \( b_z \) are trivial, Lei in [10] showed that there exits a unique global solution if the initial data is smooth enough recently.

Recently, Miao and Zheng [13] established the global regularity for the 3D axisymmetric Boussinesq system with partial viscosities and without swirl. Motivated by [10] and [13], we consider the following MHD system with partial viscosities:

\[
\begin{align*}
    u_t - \Delta u + u \cdot \nabla u &= -\nabla \pi + b \cdot \nabla b, \\
    b_t - \partial_z b + u \cdot \nabla b &= b \cdot \nabla u, \\
    \nabla \cdot u &= \nabla \cdot b = 0.
\end{align*}
\]

(1.4) (1.5) (1.6)

We will prove the global regularity for the 3D axisymmetric MHD system of (1.4)–(1.6) with \( u_\theta \) and \( b_z \) being trivial in the cylindrical coordinate systems. It should be noted that in [10] the 3D axisymmetric MHD system was studied with the full dissipations on the velocity and magnetic fields and with \( u_\theta, b_r \) and \( b_z \) being trivial. To obtain the existence and uniqueness of regular solution, we will make the \( L^\infty([0, \infty); H^2) \) estimates of the velocity and magnetic fields. To this end, we will apply for the equations satisfied by the vorticity \( \nabla \times u \) and the current of magnetic fields \( \nabla \times b \). New difficulties will be encountered in this paper. First, because of lacking smoothing effect from the full dissipation terms, the system is
degenerate along some directions and this leads to more difficulties in the a priori estimates. Second, since \( b_z \) is not zero, the current of the magnetic fields will have three non-trivial components \( j_r, j_\theta \) and \( j_z \) and hence this deduces that the estimates of \((u, b)\) in \( L^\infty([0, \infty); H^2) \) become much more difficult due to the strong coupling in the nonlinearities. However, we notice that, since \( b_r \) is trivial, it follows from the incompressible condition which is

\[
\partial_z b_z = -\partial_r b_r - \frac{b_r}{r} = 0
\]

that \( b_z \) does not depend on \( z \)-variable. This is one of advantages and will play an important role in our high-order estimates.

Now we are in the position to state the main results of this article.

**Theorem 1.1.** Let \((u_0, b_0) \in H^2\) are axisymmetric divergence free vector fields such that \( u^\theta_0 = b^\theta_0 = 0 \) and \( \nabla b_0 \in L^\infty \). Then the system (1.4)–(1.6) with the initial data \((u_0, b_0)\) has a unique global classical solution \((u, b)\) satisfying

\[
(u, b) \in L^\infty([0, \infty); H^2), \quad (\nabla_h w, \partial_z j) \in L^\infty([0, \infty); H^1),
\]

where \( u^\theta_0 = u_\theta(x, 0), \ b^\theta_0 = b_r(x, 0), \ w = \nabla \times u \) and \( j = \nabla \times b \).

This paper is organized as follows. In section 2, we introduce some notations and technical lemmas used for our estimates in the following sections. In section 3, we will concentrate on doing \( H^2 \) estimates. Section 4 is devoted to proof of the main theorem.

2. Preliminary.

2.1. Notations. In this section, we introduce some definitions and notations for the axisymmetric solutions.

**Definition 2.1.** A vector field \( u(x, t) \) is called axisymmetric if it can be written as

\[
u(x, t) = u_r(r, z, t)e_r + u_\theta(r, z, t)e_\theta + u_z(r, z, t)e_z
\]

in the cylindrical coordinate systems, where

\[
e_r = (\cos \theta, \sin \theta, 0), \ e_\theta = (-\sin \theta, \cos \theta, 0), \ e_z = (0, 0, 1).
\]

**Remark.** In the rest of this article, for convenience, we denote \( u_r(r, z, t) \) as \( u_r \) and others are similar.

In the cylindrical coordinate systems, the gradient operator \( \nabla \) is given by

\[
\nabla = e_r \partial_r + \frac{1}{r} e_\theta \partial_\theta + e_z \partial_z.
\]

Thus, some simple calculations can lead to

\[
\nabla \cdot u = \partial_r u_r + \frac{u_r}{r} + \frac{1}{r} \partial_\theta u_\theta + \partial_z u_z,
\]

and

\[
\Delta = \partial_r^2 + \frac{1}{r} \partial_r + \frac{1}{r^2} \partial_\theta^2 + \partial_z^2.
\]

Moreover, under the assumptions that \( u_\theta \) and \( b_r \) are trivial and the incompressible condition, one can get that

\[
\partial_z b_z = -\partial_r b_r - \frac{b_r}{r} = 0.
\]
Therefore, the system (1.4)–(1.6) can be rewritten as

\[ \partial_t u_r - (\Delta_h - \frac{1}{r^2})u_r + u \cdot \nabla u_r = -\partial_r \pi - \frac{b^2}{r}, \quad (2.7) \]
\[ \partial_t u_z - \Delta_h u_z + u \cdot \nabla u_z = -\partial_z \pi, \quad (2.8) \]
\[ \partial_t b_\theta - \partial_{zz} b_\theta + u \cdot \nabla b_\theta = \frac{u_r b_\theta}{r}, \quad (2.9) \]
\[ \partial_t b_z + u \cdot \nabla b_z = b_z \partial_z u_z, \quad (2.10) \]
\[ \partial_r u_r + \frac{u_r}{r} + \partial_z u_z = 0, \quad (2.11) \]

Now we deduce the equations of the vorticity and current. In the cylindrical coordinate systems, \( w = \nabla \times u \) can be written as

\[ w(x, t) = w_r(r, z, t)e_r + w_\theta(r, z, t)e_\theta + w_z(r, z, t)e_z, \]

where

\[ w_r = -\partial_z u_\theta, \quad w_\theta = \partial_r u_z - \partial_z u_r, \quad w_z = \frac{1}{r} \partial_r (ru_\theta). \]

Thus, set \( j = \nabla \times b \), one can rewrite the equations of vorticity and current as

\[ \partial_t w_\theta - (\Delta_h - \frac{1}{r^2})w_\theta + u \cdot \nabla w_\theta = \frac{u_r w_\theta}{r} + b_z \partial_z j_\theta - j \cdot \nabla b_\theta + \frac{b_\theta j_r}{r}, \]
\[ \partial_t j_r - \partial_{zz} j_r + u \cdot \nabla j_r = \partial_z u \cdot \nabla b_\theta - \frac{b_\theta}{r} \partial_z u_r - \frac{u_r}{r} \partial_z b_\theta, \]
\[ \partial_t j_\theta + u \cdot \nabla j_\theta = b_z \partial_z w_\theta + \partial_r u \cdot \nabla b_z - \partial_r b_z \partial_z u_z, \]
\[ \partial_t j_z - \partial_{zz} j_z + u \cdot \nabla j_z = u \cdot \nabla \left( \frac{b_\theta}{r} \right) + \frac{1}{r} \partial_r (u_r b_\theta) - \frac{1}{r} \partial_r (ru_\theta) \cdot \nabla b_\theta. \]

By use of the condition that \( \partial_z b_z = 0 \), it follows that

\[ b_z \partial_z j_\theta - j \cdot \nabla b_\theta + \frac{b_\theta j_r}{r} \]
\[ = -b_z \partial_r \partial_z b_z + \partial_r b_\theta \partial_z b_\theta - \partial_z b_\theta \partial_r b_\theta - \frac{\partial_z b_\theta^2}{r} \]
\[ = -\frac{\partial_z b_\theta^2}{r}, \]

and

\[ \partial_r u \cdot \nabla b_z - \partial_z u_z \partial_r b_z \]
\[ = \partial_r u_r \partial_r b_z - \partial_z u_z \partial_r b_z + \partial_r u_z \partial_z b_z \]
\[ = (\partial_r u_r - \partial_z u_z) \partial_r b_z, \]

therefore the equations of vorticity and current can be written as

\[ \partial_t w_\theta - (\Delta - \frac{1}{r^2})w_\theta + u \cdot \nabla w_\theta = \frac{u_r w_\theta}{r} - \frac{\partial_z b_\theta^2}{r}, \quad (2.12) \]
\[ \partial_t j_r - \partial_{zz} j_r + u \cdot \nabla j_r = \frac{u_r}{r} j_r + \partial_z u \cdot \nabla b_\theta - \frac{b_\theta}{r} \partial_z u_r, \quad (2.13) \]
\[ \partial_t j_\theta + u \cdot \nabla j_\theta = b_z \partial_z w_\theta + (\partial_r u_r - \partial_z u_z) \partial_r b_z, \quad (2.14) \]
\begin{equation}
\frac{\partial_t u_j}{z} - \frac{\partial_{zz} u_j}{z} + u \cdot \nabla j_z = u \cdot \nabla (\frac{b_\theta}{r}) + \frac{1}{r} \partial_r (u_r b_\theta) - \frac{1}{r} \partial_r (ru) \cdot \nabla b_\theta. \tag{2.15}
\end{equation}

2.2. **Axisymmetric estimates.** In this subsection, we present some estimates in the axisymmetric case.

**Lemma 2.2.** Suppose that $u = u(r, z) \in H^1(\mathbb{R}^3)$ be an axisymmetric field with zero divergence, then there holds
\begin{equation}
\| \nabla \tilde{u} \|_{L^p} \leq C_0 \| w \theta \|_{L^p}, \quad \forall p \in (1, \infty),
\end{equation}
where $\tilde{u} = u_r e_r + u_z e_z$ and $C_0$ is an absolute constant.

**Proof.** See [4]. □

**Lemma 2.3.** Suppose that $u = u(r, z) \in H^1(\mathbb{R}^3)$ be an axisymmetric field with zero divergence, then there holds
\begin{equation}
\left\| \frac{u_r}{r} \right\|_{L^\infty} \leq C_0 \left\| \frac{w \theta}{r} \right\|_{L^\frac{2}{3}} \left\| \nabla_h \left( \frac{w \theta}{r} \right) \right\|_{L^2},
\end{equation}
where $C_0$ is an absolute constant.

**Proof.** See [13]. □

The following estimates and proofs can be found in [10, 13] and we present them here for completeness.

**Lemma 2.4.** Suppose that $u = u(r, z) \in H^1(\mathbb{R}^3)$ be an axisymmetric field with zero divergence, then there holds
\begin{equation}
\left\| \partial_z \left( \frac{u_r}{r} \right) \right\|_{L^p} \leq C_0 \left\| \frac{w \theta}{r} \right\|_{L^p}, \quad \forall p \in (1, \infty),
\end{equation}
where $C_0$ is an absolute constant.

**Proof.** Similar to [8], by incompressible constraint, one can set the angular stream function $\phi_\theta$ such that
\[-(\partial_{rr} + \frac{3}{r} \partial_r + \partial_{zz}) \frac{\phi_\theta}{r} = \frac{w \theta}{r},
\]
with
\[u_r = -\partial_z \phi_\theta, \quad u_z = \frac{1}{r} \partial_r (r \phi_\theta).
\]

Since $\partial_{rr} + \frac{3}{r} \partial_r + \partial_{zz}$ can be viewed as a five-dimension Laplace operator in the axisymmetric form, we can write $\frac{\phi_\theta}{r}$ as
\[\frac{\phi_\theta}{r} = (-\Delta_y)^{-1} \frac{w \theta}{r}.
\]

Moreover, in the 3D axisymmetric case,
\[\nabla = e_r \partial_r + \frac{1}{r} e_\theta \partial_\theta + e_z \partial_z,
\]
thus some simple calculations give that
\begin{equation}
\nabla^2 \frac{\phi_\theta}{r} = \nabla [(e_r \partial_r + \frac{1}{r} e_\theta \partial_\theta + e_z \partial_z) \frac{\phi_\theta}{r}]
\]
\[= \nabla (e_r \partial_r \frac{\phi_\theta}{r}) + \nabla (e_z \partial_z \frac{\phi_\theta}{r})
\]
\[= (e_r \partial_r + \frac{1}{r} e_\theta \partial_\theta + e_z \partial_z)(e_r \partial_r \frac{\phi_\theta}{r}) + \nabla \partial_z \frac{\phi_\theta}{r} \otimes e_z
\]
\[= \partial_{rr} \frac{\phi_\theta}{r} e_r \otimes e_r + \frac{1}{r} \partial_r \frac{\phi_\theta}{r} e_\theta \otimes e_\theta + \partial_{zz} \frac{\phi_\theta}{r} e_z \otimes e_z.
\]
\[ + \partial_{zr} \frac{\phi_{\theta}}{r} (e_z \otimes e_r + e_r \otimes e_z). \]

which implies
\[ |\nabla^2 \frac{\phi_{\theta}}{r}|^2 \approx |\partial_{rr} \frac{\phi_{\theta}}{r}|^2 + |1 \frac{1}{r} \partial_r \frac{\phi_{\theta}}{r}|^2 + |1 \frac{1}{r} \partial_z \frac{\phi_{\theta}}{r}|^2 + |\partial_{zz} \frac{\phi_{\theta}}{r}|^2. \]

Here \( \approx \) means equivalence. Similarly, in the 5D axisymmetric case, we also have
\[ |\nabla^2 \frac{\phi_{\theta}}{r}|^2 \approx |\partial_{rr} \frac{\phi_{\theta}}{r}|^2 + |1 \frac{1}{r} \partial_r \frac{\phi_{\theta}}{r}|^2 + |1 \frac{1}{r} \partial_z \frac{\phi_{\theta}}{r}|^2 + |\partial_{zz} \frac{\phi_{\theta}}{r}|^2. \]

Thus we have
\[ \int_{\mathbb{R}^3} |\nabla^2 \frac{\phi_{\theta}}{r}|^p dx \approx \int_{-\infty}^{\infty} \int_{0}^{\infty} (|\partial_{rr} \frac{\phi_{\theta}}{r}|^p + |1 \frac{1}{r} \partial_r \frac{\phi_{\theta}}{r}|^p + |1 \frac{1}{r} \partial_z \frac{\phi_{\theta}}{r}|^p) r dr dz \]
\[ \approx \int_{-\infty}^{\infty} \int_{0}^{\infty} |\nabla^2 \frac{\phi_{\theta}}{r}|^p w(r) r^3 dr dz \]
\[ \approx \int_{\mathbb{R}^5} |\nabla^2 (-\Delta)^{-1} \frac{w_{\theta}}{r}|^p w(r) dy, \]

where \( w(r) = \frac{1}{r^2} \) is a weight function.

Consequently, for \( 1 < p < \infty \), by Lemma 2 in [8], the inequality
\[ \| \partial_z \left( \frac{u_x}{r} \right) \|^p_{L^p} \leq \int_{\mathbb{R}^5} |\nabla^2 (-\Delta)^{-1} \frac{w_{\theta}}{r}|^p w(r) dy \]
\[ \leq \int_{\mathbb{R}^5} \left| \frac{w_{\theta}}{r} \right|^p w(r) dy \]
\[ \leq \int_{\mathbb{R}^3} \left| \frac{w_{\theta}}{r} \right|^p dx \]

holds. \( \square \)

**Lemma 2.5.** Suppose that \( u = u(r, z) \in H^1(\mathbb{R}^3) \) be an axisymmetric field with zero divergence, then there holds
\[ \| \frac{u_x}{r} \|^p_{L^{\frac{3p}{p-1}}} \leq C_0 \| \frac{w_{\theta}}{r} \|^p_{L^p}, \quad \forall p \in (1, 3), \quad (2.19) \]

where \( C_0 \) is an absolute constant.

**Proof.** It is clear that
\[ \frac{u_x}{r} = \partial_z (-\Delta)^{-1} \left( \frac{w_{\theta}}{r} \right) - \frac{1}{r} \partial_r (-\Delta)^{-1} \left( \frac{w_{\theta}}{r} \right), \]

thus by Riesz theorem, imbedding theorem and Proposition 2.9 in [7], \( \forall p \in (1, 3) \), one can reach that
\[ \| \frac{u_x}{r} \|^p_{L^{\frac{3p}{p-1}}} \leq C_0 \| \partial_z (-\Delta)^{-1} \left( \frac{w_{\theta}}{r} \right) \|^p_{L^{\frac{3p}{p-1}}} \]
\[ \leq C_0 \| \frac{w_{\theta}}{r} \|^p_{L^p}. \]

\( \square \)
2.3. Partial derivative estimates. Now we list some inequalities needed later, the proof of which can be found in [5] or [13].

Lemma 2.6. Let \( u \in H^1(\mathbb{R}^3) \) be an vector field, then there holds
\[
\| u \|_{L^\infty} \leq C_0 \| \nabla u \|_{L^2}^{\frac{1}{2}} \| \nabla \nabla u \|_{L^2}^{\frac{1}{2}},
\]  
(2.20)
where \( C_0 \) is an absolute constant.

Lemma 2.7. Let \( f, g, h \) be smooth functions in \( \mathbb{R}^3 \), then there exists an absolute constant \( C_0 \) such that the following inequality
\[
\int_{\mathbb{R}^3} fgh dx dy dz \leq C_0 \| f \|_{L^{2(p-1)}(\mathbb{R}^3)} \| \partial_x f \|_{L^2(\mathbb{R}^3)} \| g \|_{L^2(\mathbb{R}^3)} \| \partial_x g \|_{L^2(\mathbb{R}^3)} \| h \|_{L^2(\mathbb{R}^3)} \| \partial_x h \|_{L^2(\mathbb{R}^3)},
\]
holds for \( p \in [2, \infty] \).

Lemma 2.8. Let \( f, g, h \) be smooth functions in \( \mathbb{R}^3 \), then
\[
\int_{\mathbb{R}^3} fgh dx dy dz \leq C_0 \| f \|_{L^2(\mathbb{R}^3)} \| \partial_x f \|_{L^2(\mathbb{R}^3)} \| g \|_{L^2(\mathbb{R}^3)} \| \partial_x g \|_{L^2(\mathbb{R}^3)} \| h \|_{L^2(\mathbb{R}^3)} \| \partial_x h \|_{L^2(\mathbb{R}^3)},
\]
(2.22)
where \( C_0 \) is an absolute constant.

Lemma 2.9. Let \( f, g, h \) be smooth functions in \( \mathbb{R}^3 \), then
\[
\int_{\mathbb{R}^3} fgh dx dy dz \leq C_0 \| f \|_{L^2(\mathbb{R}^3)} \| \partial_x f \|_{L^2(\mathbb{R}^3)} \| g \|_{L^2(\mathbb{R}^3)} \| \partial_x g \|_{L^2(\mathbb{R}^3)} \| h \|_{L^2(\mathbb{R}^3)} \| \partial_x h \|_{L^2(\mathbb{R}^3)},
\]
(2.23)
where \( C_0 \) is an absolute constant.

Proof. We can get the conclusion by setting \( p = 4 \) in Lemma 2.7.
\[ \square \]

3. A priori estimates.

3.1. \( H^1 \) estimates. In this subsection, we intend to get \( H^1 \) estimates of \((u, b)\). The following is the usual energy estimates:

Proposition 1. If \((u, b)\) solves the system (2.7)-(2.11) with \( u = u_x e_r + u_z e_z \) and \( b = b_y e_0 + b_z e_z \), there holds
\[
\| u \|_{L^2}^2 + \| b \|_{L^2}^2 + \int_0^T (\| \nabla u \|_{L^2}^2 + \| \nabla b \|_{L^2}^2) dt \leq C_1,
\]
(3.24)
where the constant \( C_1 \) depends only on \( \| u_0 \|_{L^2} \) and \( \| b_0 \|_{L^2} \).

To achieve \( H^1 \) estimates, we first get the estimates of \( \| \frac{\partial u}{r} \|_{L^\infty} \) and \( \| \frac{\partial b}{r} \|_{L^\infty(\mathbb{R}^3)} \).

Lemma 3.1. If \((u, b)\) solves the system (1.4)-(1.6) with \( u = u_x e_r + u_z e_z \) and \( b = b_y e_0 + b_z e_z \), it will hold that
\[
\| \frac{\partial b}{r} \|_{L^p} \leq C_2, \; \forall p \in (1, \infty],
\]
(3.25)
where the constant \( C_2 \) depends only on \( \| \frac{\partial u}{r} \|_{L^p} \).

Proof. Considering that \( b_y \) satisfies the following equation
\[
\partial_t b_y - \partial_{zz} b_y + u \cdot \nabla b_y = \frac{u_x b_y}{r},
\]
one can easily deduce that the quantity \( \Omega = \frac{b_y}{r} \) solves
\[
\partial_t \Omega + u \cdot \nabla \Omega - \partial_{zz} \Omega = 0.
\]
It follows that
\[ ||\Omega(t)||_{L^p} \leq ||\Omega_0||_{L^p}, \forall p \in (1, \infty), \]
letting \( p \to \infty \), then there holds that
\[ ||\Omega(t)||_{L^\infty} \leq ||\Omega_0||_{L^\infty}. \]
Thus, we have derived
\[ ||\frac{b_\theta}{r}||_{L^p} \leq ||\frac{b_\theta}{r}||_{L^p}, \forall p \in (1, \infty]. \]

\[ \square \]

**Lemma 3.2.** If \((u, b)\) solves the system (1.4)–(1.6) with \( u = u_r e_r + u_z e_z \) and \( b = b_\theta e_\theta + b_z e_z \), then the following estimate holds
\[ \left\| \frac{w_\theta}{r} \right\|_{L^2}^2 + \left\| \frac{b_\theta}{r} \right\|_{L^2}^2 + \int_0^T \left( \left\| \nabla_h \left( \frac{w_\theta}{r} \right) \right\|_{L^2}^2 + \left\| \partial_z \left( \frac{b_\theta}{r} \right) \right\|_{L^2}^2 \right) dt \leq C_3, \quad (3.26) \]
where the constant \( C_3 \) depends only on \( ||\frac{w_\theta}{r}||_{L^2} \) and \( ||\frac{b_\theta}{r}||_{L^2 \cap L^\infty} \).

**Proof.** Let \( \Gamma = \frac{w_\theta}{r} \), then \((\Gamma, \Omega)\) solves the following system
\[ \begin{cases} \partial_t \Gamma + u \cdot \nabla \Gamma = \left( \Delta_h + \frac{2}{r} \partial_r \right) \Gamma - \partial_z \Omega^2, \\ \partial_\Omega + u \cdot \nabla \Omega = \partial_z \Omega. \end{cases} \quad (3.27) \]
Taking inner product of (3.27) with \( \Gamma \) and \( \Omega \) and integrating on \( \mathbb{R}^3 \), we have
\[ \begin{align*} \frac{1}{2} \frac{d}{dt} \left( \| \Gamma \|_{L^2}^2 + \| \Omega \|_{L^2}^2 \right) &+ \| \nabla_h \Gamma \|_{L^2}^2 + \| \partial_z \Omega \|_{L^2}^2 \\ &\leq -\int_{\mathbb{R}^3} \nabla_\Omega \Omega^2 dx \\ &\leq \| \Gamma \|_{L^2} \| \Omega \|_{L^\infty} \| \partial_z \Omega \|_{L^2} \\ &\leq \frac{1}{2} \| \partial_z \Omega \|_{L^2}^2 + C \| \Gamma \|_{L^2}^2 \| \Omega \|_{L^\infty}^2, \end{align*} \]
by use of Gronwall inequality, one can reach that
\[ \| \Gamma \|_{L^2}^2 + \| \Omega \|_{L^2}^2 + \int_0^T \left( \| \nabla_h \Gamma \|_{L^2}^2 + \| \partial_z \Omega \|_{L^2}^2 \right) dt \leq C \left( \| \Gamma_0 \|_{L^2}, \| \Omega_0 \|_{L^2 \cap L^\infty} \right). \]
\[ \square \]

**Lemma 3.3.** If \((u, b)\) solves the system (1.4)–(1.6) with \( u = u_r e_r + u_z e_z \) and \( b = b_\theta e_\theta + b_z e_z \), there holds
\[ ||b_\theta||_{L^\infty} \leq C_4, \quad (3.28) \]
where the constant \( C_4 \) depends only on \( T, ||\frac{w_\theta}{r}||_{L^2}, ||\frac{b_\theta}{r}||_{L^2 \cap L^\infty} \) and \( ||b_\theta||_{L^\infty} \).

**Proof.** By Lemma 2.3 and Lemma 3.2, one can get that
\[ \begin{align*} \int_0^T \| \frac{u_r}{r} \|_{L^\infty} dt &\leq C \sup_{0 \leq t \leq T} \| \frac{u_\theta}{r} \|_{L^2}^\frac{1}{2} \int_0^T \| \nabla_h \left( \frac{w_\theta}{r} \right) \|_{L^2}^\frac{1}{2} dt \\ &\leq C \left( \| \frac{u_\theta}{r} \|_{L^2}, \| \frac{b_\theta}{r} \|_{L^2 \cap L^\infty} \right). \end{align*} \]
\[ \leq C(C_3, T). \] (3.29)

In addition, as \( b_\theta \) satisfies the following equation
\[ \partial_t b_\theta - \partial_{zz} b_\theta + u \cdot \nabla b_\theta = \frac{u_r b_\theta}{r}. \]

For \( p > 1 \), taking inner product with \( |b_\theta|^{p-2} b_\theta \) and integrating on \( \mathbb{R}^3 \), we finally obtain
\[ \frac{d}{dt} \| b_\theta \|_{L^p} \leq \| \frac{u_r}{r} \|_{L^\infty} \| b_\theta \|_{L^p}. \]

It follows from Gronwall inequality that
\[ \| b_\theta \|_{L^p} \leq \| b_\theta \|_{L^p} e^{\int_0^T \| \frac{u_r}{r} \|_{L^\infty} \, dt}. \]

Letting \( p \to \infty \), then one can derive that
\[ \| b_\theta \|_{L^\infty} \leq \| b_\theta \|_{L^\infty} e^{\int_0^T \| \frac{u_r}{r} \|_{L^\infty} \, dt} \]
\[ \leq C(T, \| \frac{u_\theta}{r} \|_{L^2}, \| b_\theta \|_{L^2}, \| b_\theta \|_{L^1 \cap L^\infty}, \| b_\theta \|_{L^\infty}). \]

The proof of the lemma is finished. \( \square \)

The \( H^1 \) estimates of \( (u, b) \) are as follows.

**Proposition 2.** If \( (u, b) \) solves the system (1.4)–(1.6) with \( u = u_r e_r + u_z e_z \), \( b = b_\theta e_\theta + b_z e_z \) and denote \( w = \nabla \times u = w_\theta e_\theta \), \( j = \nabla \times b = j_r e_r + j_\theta e_\theta + j_z e_z \), then one can derive that
\[ \| w \|_{L^2}^2 + \| j \|_{L^2}^2 + \int_0^T (\| \nabla_h w \|_{L^2}^2 + \| \partial_z j \|_{L^2}^2) \, dt \leq C_5, \] (3.30)

where the constant \( C_5 \) depends only on \( T, \| \frac{u_\theta}{r} \|_{L^2}, \| \frac{b_\theta}{r} \|_{L^\infty}, \| b_\theta \|_{L^\infty}, \| w_\theta \|_{L^2} \) and \( \| j_0 \|_{L^2} \).

**Proof.** Taking inner product of (2.12) with \( w_\theta \) and integrating on \( \mathbb{R}^3 \) lead to
\[ \frac{1}{2} \frac{d}{dt} \| w_\theta \|_{L^2}^2 + \| \nabla_h w_\theta \|_{L^2}^2 + \| \frac{w_\theta}{r} \|_{L^2}^2 = \int_{\mathbb{R}^3} \frac{w_\theta^2 u_r}{r} \, dx - \int_{\mathbb{R}^3} \frac{w_\theta \partial_z b_\theta^2}{r} \, dx. \]

Using Lemma 2.9, one has
\[ \frac{1}{2} \frac{d}{dt} \| w_\theta \|_{L^2}^2 + \| \nabla_h w_\theta \|_{L^2}^2 + \| \frac{w_\theta}{r} \|_{L^2}^2 \]
\[ \leq \| \frac{u_r}{r} \|_{L^\infty} \| \frac{1}{2} \nabla (\frac{u_r}{r}) \|_{L^2}^2 \| \nabla_h w_\theta \|_{L^2}^2 + \| \frac{w_\theta}{r} \|_{L^\infty} \| \partial_z b_\theta \|_{L^2} \| w_\theta \|_{L^2} \]
\[ \leq \| \frac{u_\theta}{r} \|_{L^2} \| w_\theta \|_{L^2}^2 + \| \nabla_h w_\theta \|_{L^2}^2 + \| \partial_z b_\theta \|_{L^2}^2 + \| \frac{b_\theta}{r} \|_{L^\infty} \| w_\theta \|_{L^2}^2 \]
\[ \leq \frac{1}{2} \| \nabla_h w_\theta \|_{L^2}^2 + C \frac{1}{2} \| w_\theta \|_{L^2}^2 + \| \partial_z b_\theta \|_{L^2}^2 + \| \frac{b_\theta}{r} \|_{L^\infty} \| w_\theta \|_{L^2}^2. \]

Thus combing with Gronwall inequality, it follows that
\[ \| w_\theta \|_{L^2}^2 + \int_0^T (\| \nabla_h w_\theta \|_{L^2}^2 + \| \frac{w_\theta}{r} \|_{L^2}^2) \, dt. \]
\[
\leq C\|w_0\|_{L^2}^2 e^{\int_0^T (\frac{\|w\|_{L^2}^2}{2} + \frac{b_0^2}{2}) dt} + \int_0^T \|\partial_z b_0\|_{L^2}^2 dt \\
\leq C(T, ||w_0||_{L^2}, C_1, C_3).
\]

Taking inner product of (2.13)–(2.15) with \(j_r, j_\theta, j_z\) and integrating on \(\mathbb{R}^3\) respectively, it is easy to get

\[
\frac{1}{2} \frac{d}{dt} (||j_r||_{L^2}^2 + ||j_\theta||_{L^2}^2 + ||j_z||_{L^2}^2) + ||\partial_z j_r||_{L^2}^2 + ||\partial_z j_\theta||_{L^2}^2 + ||\partial_z j_z||_{L^2}^2 = 2 \int_{\mathbb{R}^3} \partial_z u_r \partial_r b_0 j_r dx - \int_{\mathbb{R}^3} (2\partial_z u_z + \partial_r u_r) j_r, j_r dx + \int_{\mathbb{R}^3} b_z \partial_z w_0 j_\theta dx \\
+ \int_{\mathbb{R}^3} (\partial_r u_r - \partial_z u_z) \partial_r b_z j_\theta dx + \int_{\mathbb{R}^3} (\partial_r u_z - \partial_z u_r) j_r j_r dx \\
+ \int_{\mathbb{R}^3} u_r \partial_r (\frac{b_0}{r}) j_z dx + \int_{\mathbb{R}^3} (\frac{b_0}{r} - \partial_r b_0) \partial_r u_r j_z dx \\
= I + II + III + IV + V + VI + VII.
\]

In the following context, we do the estimate for each term respectively. By use of Lemma 2.6, one can reach that

\[
I = 2 \int_{\mathbb{R}^3} \partial_z u_r \partial_r b_0 j_r dx \\
= -2 \int_{\mathbb{R}^3} u_r \partial_z \partial_r b_0 j_r dx - 2 \int_{\mathbb{R}^3} u_r \partial_t b_0 \partial_z j_r dx \\
\leq \|u_r\|_{L^\infty} (||j_r||_{L^2}^2 ||\partial_z j_z||_{L^2} + ||j_z||_{L^2} ||\partial_z j_r||_{L^2}) \\
\leq \frac{1}{10} (||\partial_z j_z||_{L^2}^2 + ||\partial_z j_r||_{L^2}^2) + C \|u_r\|_{L^\infty} (||j_r||_{L^2}^2 + ||j_z||_{L^2}^2) \\
\leq \frac{1}{10} (||\partial_z j_z||_{L^2}^2 + ||\partial_z j_r||_{L^2}^2) + C \|\nabla u_r\|_{L^2} \|\nabla_h \nabla u_r\|_{L^2} (||j_z||_{L^2}^2 + ||j_r||_{L^2}^2) \\
\leq \frac{1}{10} (||\partial_z j_z||_{L^2}^2 + ||\partial_z j_r||_{L^2}^2) + C (||w_\theta||_{L^2}^2 + ||\nabla_h w_\theta||_{L^2}^2) (||j_z||_{L^2}^2 + ||j_r||_{L^2}^2),
\]

\[
II = - \int_{\mathbb{R}^3} (\partial_z u_z - \frac{u_r}{r}) j_r j_r dx \\
= -2 \int_{\mathbb{R}^3} u_z \partial_z j_r j_r dx + \int_{\mathbb{R}^3} \frac{u_r}{r} j_r j_r dx \\
\leq C \|u_z\|_{L^\infty} ||j_r||_{L^2} ||\partial_z j_r||_{L^2} + \|\frac{u_r}{r}\|_{L^\infty} \|j_z||_{L^2}^2 \\
\leq \frac{1}{10} ||\partial_z j_r||_{L^2}^2 + C (||w_\theta||_{L^2}^2 + ||\nabla_h w_\theta||_{L^2}^2 + \|\frac{u_r}{r}\|_{L^\infty}) ||j_r||_{L^2}^2,
\]

\[
III + IV = - \int_{\mathbb{R}^3} \partial_z b_z w_\theta j_\theta dx - \int_{\mathbb{R}^3} b_z w_\theta \partial_z j_\theta dx + \int_{\mathbb{R}^3} (\partial_r u_r - \partial_z u_z) \partial_r b_z j_\theta dx \\
= \int_{\mathbb{R}^3} \frac{u_r}{r} j_\theta^2 dx + 2 \int_{\mathbb{R}^3} \partial_z u_z j_\theta^2 dx \\
= \int_{\mathbb{R}^3} \frac{u_r}{r} j_\theta^2 dx - 4 \int_{\mathbb{R}^3} u_z j_\theta \partial_z j_\theta dx \\
\leq \|\frac{u_r}{r}\|_{L^\infty} ||j_\theta||_{L^2}^2 + C \|u_z\|_{L^\infty} ||j_\theta||_{L^2} \|\partial_z j_\theta||_{L^2}
\]
Thus, summing up all the estimates I - VII, there holds that

$$\|j_r\|^2_{L^2} + \|j_o\|^2_{L^2} + \|j_z\|^2_{L^2} + \int_0^T (\|\partial_z j_r\|^2_{L^2} + \|\partial_z j_z\|^2_{L^2}) dt$$

$$\leq C \|j_0\|^2_{L^2} e^{\int_0^T (\|w_0\|^2_{L^2} + \|\nabla w_0\|^2_{L^2} + \|\nabla w_0\|^2_{L^\infty} + \|b_0\|^2_{L^\infty}) dt}$$

$$+ \int_0^T (\|\nabla w_0\|^2_{L^2} + \|b_0\|^2_{L^\infty} + \|w_0\|^2_{L^2} + \|b_0\|^2_{L^\infty}) dt$$
where the constant $C(3.32)$, one can derive that

$$2.9, \text{ one can reach that}$$

Lemma 3.4.

$$\parallel \nabla H_j \parallel_{L^2} + \int_0^T (\parallel \partial_\tau w \parallel_{L^2}^2 + \parallel \partial_\tau j \parallel_{L^2}^2) dt$$

$$\leq C(T, \parallel w_0 \parallel_{L^2}, \parallel j_0 \parallel_{L^2}, C_1, C_3, C_4).$$

(3.32)

Since $w = w_0 e_\theta$ and $j = j_\tau e_r + j_\theta e_\theta + j_\tau e_z$, combining the estimates (3.31) and (3.32), one can derive that

$$\parallel w \parallel_{L^2}^2 + \parallel j \parallel_{L^2}^2 + \int_0^T (\parallel \partial_\tau w \parallel_{L^2}^2 + \parallel \partial_\tau j \parallel_{L^2}^2) dt$$

$$\leq C(T, \parallel w_0 \parallel_{L^2}, \parallel j_0 \parallel_{L^2}, C_1, C_3, C_4).$$

3.2. $H^2$ estimates. This subsection is devoted to getting $H^2$ estimates of $(u, b)$, to begin with, we do estimate of $\parallel \nabla \left( \frac{b_\theta}{r} \right) \parallel_{L^\infty([0,T], L^2)}$.

Lemma 3.4. If $(u, b)$ solves the system (1.4)–(1.6) with $u = u_r e_r + u_z e_z$ and $b = b_\theta e_\theta + b_z e_z$, it holds that

$$\parallel \nabla \left( \frac{b_\theta}{r} \right) \parallel_{L^2} + \int_0^T \parallel \nabla \partial_\tau \left( \frac{b_\theta}{r} \right) \parallel_{L^2}^2 dt \leq C_6,$$

where the constant $C_6$ depends only on $T, \parallel w_0 \parallel_{L^2}, \parallel b_\theta \parallel_{L^\infty}, \parallel \nabla \left( \frac{b_\theta}{r} \right) \parallel_{L^2}, \parallel w_0 \parallel_{L^2}$ and $\parallel j_0 \parallel_{L^2}$.

Proof. Since $\Omega = \frac{b_\theta}{r}$ solves

$$\partial_\tau \Omega + u \cdot \nabla \Omega = \partial_\tau \Omega.$$

(3.34)

Taking inner product of (3.34) with $-\Delta \Omega$ and integrating on $\mathbb{R}^3$ yields

$$\frac{1}{2} \frac{d}{dt} (\parallel \nabla \Omega \parallel_{L^2}^2 + \parallel \partial_\tau \Omega \parallel_{L^2}^2) + \parallel \nabla \partial_\tau \Omega \parallel_{L^2}^2$$

$$= \int_{\mathbb{R}^3} u \cdot \nabla \Omega \cdot \Delta \Omega dx + \int_{\mathbb{R}^3} u \cdot \nabla \Omega \cdot \partial_\tau \Omega dx$$

$$= \int_{\mathbb{R}^3} (u_r \partial_r + u_z \partial_z) \Omega \partial_r (r \partial_r \Omega) dr dz + \int_{\mathbb{R}^3} (u_r \partial_r + u_z \partial_z) \Omega \partial_z \Omega dx$$

$$= -\int_{\mathbb{R}^3} \partial_r u_r \partial_r \Omega \partial_r \Omega dx - \int_{\mathbb{R}^3} \partial_z u_z \partial_z \Omega \partial_z \Omega dx - \int_{\mathbb{R}^3} \partial_z u_r \partial_r \Omega \partial_z \Omega dx$$

$$- \int_{\mathbb{R}^3} \partial_r u_z \partial_r \Omega \partial_z \Omega dx$$

$$= \int_{\mathbb{R}^3} \partial_z u_r \partial_r \Omega \partial_z \Omega dx + \int_{\mathbb{R}^3} \frac{u_r}{r} \partial_r \Omega \partial_r \Omega dx - \int_{\mathbb{R}^3} \partial_z u_r \partial_r \Omega \partial_z \Omega dx$$

$$- \int_{\mathbb{R}^3} \partial_r u_z \partial_r \Omega \partial_z \Omega dx - \int_{\mathbb{R}^3} \partial_z u_z \partial_r \Omega \partial_z \Omega dx$$

$$= -2 \int_{\mathbb{R}^3} u_z \partial_\tau \Omega \partial_z \Omega dx + \int_{\mathbb{R}^3} \frac{u_r}{r} \partial_r \Omega \partial_r \Omega dx - \int_{\mathbb{R}^3} \partial_z u_z \partial_\tau \Omega \partial_\tau \Omega dx$$

$$- \int_{\mathbb{R}^3} \partial_z u_r \partial_\tau \Omega \partial_r \Omega dx - \int_{\mathbb{R}^3} \partial_r u_z \partial_\tau \Omega \partial_z \Omega dx$$

$$= I + II + III + IV + V.$$

Now, we estimate the terms $I$–$V$ respectively. Making use of Lemma 2.6–Lemma 2.9, one can reach that

$$I + II = -2 \int_{\mathbb{R}^3} u_z \partial_\tau \Omega \partial_z \Omega dx + \int_{\mathbb{R}^3} \frac{u_r}{r} \partial_r \Omega \partial_r \Omega dx$$

$$= -2 \int_{\mathbb{R}^3} u_z \partial_\tau \Omega \partial_z \Omega dx.$$
By (2.12),

Proof.

Thus, summing up all above estimates and making use of Gronwall inequality, there will hold that

\[
\|\nabla_h \Omega\|_{L^2}^2 + \|\partial_z \Omega\|_{L^2}^2 + \int_0^T (\|\nabla_h \partial_z \Omega\|_{L^2}^2 + \|\partial_z \Omega\|_{L^2}^2) \, dt 
\leq C(T, \|\nabla (b_0)\|_{L^2}, C_5).
\]

We continue to pursue \( H^2 \) estimates.

**Proposition 3.** If \((u, b)\) solves the system (1.4)–(1.6) with \(u = u_r e_r + u_z e_z\), \(b = b_\theta e_\theta + b_z e_z\) and denote \(w = \nabla \times u = w_\theta e_\theta\), \(j = \nabla \times b = j_r e_r + j_\theta e_\theta + j_z e_z\), then there holds that

\[
\|\nabla w\|_{L^2}^2 + \|\nabla j\|_{L^2}^2 + \int_0^T (\|\nabla_h \partial_z w\|_{L^2}^2 + \|\nabla^2 w\|_{L^2}^2) \, dt 
\leq C_7,
\]

where the constant \(C_7\) depends only on \(T, \|w_\theta\|_{L^2}, \|b_0\|_{L^2 \cap L^\infty}, \|b_0\|_{L^\infty}, \|w_\theta\|_{H^1}\), and \(\|j_0\|_{H^1}\).

**Proof.** By (2.12), \(w = w_\theta e_\theta\) solves

\[
\partial_t w - \Delta_h w + u \cdot \nabla w = \frac{u_r w}{r} - \frac{\partial_z b_\theta^2}{r} e_\theta.
\]
Taking inner product of this equation with \(-\Delta w\) and integrating on \(\mathbb{R}^3\), we find
\[
\frac{1}{2} \frac{d}{dt} (\|\nabla_h w\|_{L^2}^2 + \|\partial_r w\|_{L^2}^2) + (\|\nabla_h \partial_r w\|_{L^2}^2 + \|\nabla_h \partial_z w\|_{L^2}^2)
= \int_{\mathbb{R}^3} (u \cdot \nabla w - \frac{u_r}{r} w) \cdot \Delta w dx + \int_{\mathbb{R}^3} \frac{\partial_r b_0^2}{r} \cdot \Delta w dx
= I + II.
\]
Before estimating term I, we rewrite it as follows.
\[
I = \int_{\mathbb{R}^3} (u_r \partial_r + u_z \partial_z) \cdot \partial_r (r \partial_r w)drdz + \int_{\mathbb{R}^3} (u_r \partial_r + u_z \partial_z)w \cdot \partial_z w dx
- \int_{\mathbb{R}^3} \frac{u_r w}{r} \cdot \partial_r (r \partial_r w)drdz - \int_{\mathbb{R}^3} \frac{u_r w}{r} \cdot \partial_z w dx
= \int_{\mathbb{R}^3} (\partial_r u_r \partial_r w \cdot \partial_r w + \partial_r u_z \partial_z w \cdot \partial_z w)dx
+ \int_{\mathbb{R}^3} (\partial_z u_r \partial_r w \cdot \partial_z w + \partial_z u_z \partial_z w \cdot \partial_z w)dx
+ \int_{\mathbb{R}^3} \partial_r u_r \frac{w}{r} \cdot \partial_r w dx + \int_{\mathbb{R}^3} \frac{u_r}{r} \partial_r w \cdot \partial_r w dx + \int_{\mathbb{R}^3} \frac{\partial_z u_r}{r} w \cdot \partial_r w dx
+ \int_{\mathbb{R}^3} \frac{u_r}{r} \partial_z w \cdot \partial_z w dx
= \{1\} + \{2\} + \{3\} + \{4\} + \{5\} + \{6\}
\]
Similarly, by Lemma 2.6–Lemma 2.9, one can reach that
\[
\{1\} + \{2\} \leq \|\partial_r u_r\|_{L^2}^2 \|\nabla_h \partial_r u_r\|_{L^2}^2 \|\partial_r w\|_{L^2}^2 \|\nabla_h \partial_r w\|_{L^2}^2 \|\partial_z \partial_r w\|_{L^2}^2
+ \|\partial_r u_z\|_{L^2}^2 \|\nabla_h \partial_z u_z\|_{L^2}^2 \|\partial_z w\|_{L^2}^2 \|\nabla_h \partial_z w\|_{L^2}^2 \|\partial_z \partial_r w\|_{L^2}^2
+ \|\partial_r u_z\|_{L^2}^2 \|\nabla_h \partial_z u_z\|_{L^2}^2 \|\partial_z w\|_{L^2}^2 \|\nabla_h \partial_z w\|_{L^2}^2 \|\partial_z \partial_r w\|_{L^2}^2
+ \|\partial_z u_z\|_{L^2}^2 \|\nabla_h \partial_z u_z\|_{L^2}^2 \|\partial_z w\|_{L^2}^2 \|\nabla_h \partial_z w\|_{L^2}^2 \|\partial_z \partial_r w\|_{L^2}^2
+ \|\partial_z u_z\|_{L^2}^2 \|\nabla_h \partial_z u_z\|_{L^2}^2 \|\partial_z w\|_{L^2}^2 \|\nabla_h \partial_z w\|_{L^2}^2
\leq \|w_0\|_{L^2}^2 \|\nabla_h w_0\|_{L^2}^2 \|\partial_r w\|_{L^2}^2 \|\nabla_h \partial_r w\|_{L^2}^2 \|\nabla_h w\|_{L^2}
+ \|w_0\|_{L^2}^2 \|\partial_r w_0\|_{L^2}^2 \|\partial_z w\|_{L^2}^2 \|\nabla_h \partial_z w\|_{L^2}
+ 2\|w_0\|_{L^2}^2 \|\nabla_h w_0\|_{L^2}^2 \|\partial_r w\|_{L^2}^2 \|\partial_z w\|_{L^2}^2 \|\nabla_h \partial_z w\|_{L^2}
\leq \frac{1}{10} (\|\nabla_h \partial_r w\|_{L^2}^2 + \|\nabla_h w\|_{L^2}^2)
+ (\|w_0\|_{L^2}^2 + \|\nabla_h w_0\|_{L^2}^2) (\|\partial_r w\|_{L^2}^2 + \|\partial_z w\|_{L^2}^2),
\]
\[
\{3\} + \{4\} \leq \|\partial_r u_r\|_{L^2}^2 \|\nabla_h \partial_r u_r\|_{L^2}^2 \|\partial_r w\|_{L^2}^2 \|\nabla_h \partial_r w\|_{L^2}^2 \|\partial_z \partial_r w\|_{L^2}^2
+ \|\frac{u_r}{r}\|_{L^\infty} \|\partial_r w\|_{L^2}^2
\leq \frac{1}{10} (\|\nabla_h \partial_z w\|_{L^2}^2 + \|\nabla_h w\|_{L^2}^2)
+ (\|w_0\|_{L^2}^2 + \|\nabla_h w_0\|_{L^2}^2 + \|\frac{u_r}{r}\|_{L^\infty}) \|\nabla_h w\|_{L^2}^2,
\]
\[
\{5\} + \{6\} \leq C \|\partial_z (\frac{u_r}{r})\|_{L^2} \|w\|_{L^2}^2 \|\partial_z w\|_{L^2}^2 \|\partial_z \partial_r w\|_{L^2}^2 \|\nabla_h \partial_z w\|_{L^2}^2
\]
Combining with Gronwall inequality, it is clear that
\[ +\|\frac{u_r}{r}\|_{L^\infty} \|\partial_z w\|_{L^2}^2 \leq C\|\frac{w_y}{r}\|_{L^2} \|\nabla_h w\|_{L^2}^\frac{3}{2} \|\partial_z w\|_{L^2} \|\nabla_h \partial_z w\|_{L^2}^\frac{1}{2} + \|\frac{u_r}{r}\|_{L^\infty} \|\partial_z w\|_{L^2}^2 + C\|\frac{w_y}{r}\|_{L^2} \|\nabla_h \partial_z w\|_{L^2}^\frac{1}{2} \leq \frac{1}{10} \|\nabla_h \partial_z w\|_{L^2}^2 + C\|\frac{w_y}{r}\|_{L^2}^\frac{4}{3} (\|\nabla_h w\|_{L^2}^2 + \|\partial_z w\|_{L^2}^2) \]

Adding up all above estimates \{1\} to \{8\},
\[ \{7\} \leq \|\frac{b_y}{r}\|_{L^\infty} \|\partial_z w\|_{L^2} \|\Delta_h w\|_{L^2} \leq \frac{1}{10} \|\nabla_h w\|_{L^2}^2 + C\|\frac{b_y}{r}\|_{L^\infty} \|\partial_z w\|_{L^2}^2, \]

and
\[ \{8\} = \int_{\mathbb{R}^3} \frac{\partial_z b_y}{r} \partial_z b_y e_\theta \cdot \partial_z w \, dx + \int_{\mathbb{R}^3} \frac{b_y \partial_z b_y}{r} e_\theta \cdot \partial_z w \, dx \leq \|\partial_z (\frac{b_y}{r})\|_{L^2} \|\nabla_h \partial_z (\frac{b_y}{r})\|_{L^2} \|\partial_z w\|_{L^2} \|\nabla_h \partial_z w\|_{L^2} \]

Making use of Lemma 2.8 and 3.1, one has
\[ \{7\} \leq \frac{1}{10} \|\nabla_h w\|_{L^2}^2 + C\|\frac{b_y}{r}\|_{L^\infty} \|\partial_z w\|_{L^2}^2, \]

and
\[ \{8\} = \int_{\mathbb{R}^3} \frac{\partial_z b_y}{r} \partial_z b_y e_\theta \cdot \partial_z w \, dx + \int_{\mathbb{R}^3} \frac{b_y \partial_z b_y}{r} e_\theta \cdot \partial_z w \, dx \leq \|\partial_z (\frac{b_y}{r})\|_{L^2} \|\nabla_h \partial_z (\frac{b_y}{r})\|_{L^2} \|\partial_z w\|_{L^2} \|\nabla_h \partial_z w\|_{L^2} \]

Adding up all above estimates \{1\} to \{8\} yields
\[ \frac{1}{2} \frac{d}{dt} \|\nabla w\|_{L^2}^2 + (\|\nabla_h \partial_z w\|_{L^2}^2 + \|\nabla_h w\|_{L^2}^2) \leq C(\|w_y\|_{L^2}^2 + \|\nabla_h w\|_{L^2}^2 + \|\frac{u_r}{r}\|_{L^\infty} + \|\frac{w_y}{r}\|_{L^2}^\frac{2}{3} + \|\frac{b_y}{r}\|_{L^\infty}^2 + \|\partial_z w\|_{L^2}^2) \|\nabla w\|_{L^2}^2 \]

Combining with Gronwall inequality, it is clear that
\[ \|\nabla w\|_{L^2}^2 + \int_0^T (\|\nabla_h \partial_z w\|_{L^2}^2 + \|\nabla_h w\|_{L^2}^2) \, dt \]
\[ I = \int_{\mathbb{R}^3} (u_r \partial_r + u_z \partial_z) j \cdot \partial_r (r \partial_r j) dr dz + \int_{\mathbb{R}^3} u \cdot \nabla j \cdot \partial_{zz} j dx \]

Similarly, by (2.13)–(2.15), \( j = j_r e_r + j_\theta e_\theta + j_z e_z \) solves
\[ \partial_t j - \partial_{zz} j + u \cdot \nabla j = a + b + c, \]

where
\[ a = [\partial_z u_r, \partial_r b_\theta + \partial_r u_z \partial_z b_\theta - \frac{1}{r} \partial_z (u_r b_\theta)] e_r, \]
\[ b = [b_z \partial_z w_\theta + \frac{u_r}{r} j_\theta + \partial_z u_z j_\theta] e_\theta, \]

and
\[ c = [u \cdot \nabla \left( \frac{b_\theta}{r} \right) + \frac{1}{r} \partial_r (u_r b_\theta) - \frac{1}{r} \partial_r (ru) \cdot \nabla b_\theta] e_z. \]

Taking inner product of this equation with \(-\Delta j\) and integrating on \(\mathbb{R}^3\), it follows that
\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \|\nabla j\|_{L^2}^2 + & \|\nabla h \partial_z j\|_{L^2}^2 + \|\partial_{zz} j\|_{L^2}^2 \\
= & \int_{\mathbb{R}^3} u \cdot \nabla j \cdot \Delta j dx - \int_{\mathbb{R}^3} a \cdot \Delta j dx - \int_{\mathbb{R}^3} b \cdot \Delta j dx - \int_{\mathbb{R}^3} c \cdot \Delta j dx \\
= & I + II + III + IV.
\end{align*}
\]

Using Lemma 2.6–Lemma 2.9, one has
\[
I = \int_{\mathbb{R}^3} (u_r \partial_r + u_z \partial_z) j \cdot \partial_r (r \partial_r j) dr dz + \int_{\mathbb{R}^3} u \cdot \nabla j \cdot \partial_{zz} j dx
\]

\[
\leq \frac{1}{4} \left( \|\nabla h \partial_z j\|_{L^2}^2 + \|\partial_{zz} j\|_{L^2}^2 \right)
+ C(\|w_\theta\|_{L^2}^2 + \|\nabla h w_\theta\|_{L^2}^2 + \|\frac{u_r}{r}\|_{L^\infty}) \|\nabla j\|_{L^2}^2,
\]

\[ (3.36) \]
and

\[
II = -\int_{-\infty}^{\infty} \int_{0}^{\infty} \left[ \partial_z u_r (\partial_r b_\theta - \frac{b_\theta}{r}) - \partial_z u_z j_r \right] \partial_r (r \partial_r j_r) \, dr \, dz \\
- \int_{-\infty}^{\infty} \int_{0}^{\infty} u_r j_r \partial_r j_r \, dr \, dz + \int_{\mathbb{R}^3} \partial_z u_r (\partial_r b_\theta - \frac{b_\theta}{r}) j_r \, dx \\
- \int_{\mathbb{R}^3} (\partial_z u_z - \frac{u_r}{r}) j_r \, dx - \int_{\mathbb{R}^3} \partial_z u_r (\partial_r b_\theta - \frac{b_\theta}{r}) \partial_z j_r \, dx \\
+ \int_{\mathbb{R}^3} (2 \partial_z u_z + \partial_z u_r) j_z \partial z j_z \, dx
\]

\[
= \int_{\mathbb{R}^3} \partial_z [\partial_z u_r (\partial_r b_\theta - \frac{b_\theta}{r}) - \partial_z u_z j_r] \partial_r j_z \, dx + \int_{\mathbb{R}^3} \frac{u_r}{r} \partial_z j_z \, dx \\
+ \int_{\mathbb{R}^3} \partial_z u_r \frac{j_r}{r} \partial_z j_r \, dx - \int_{\mathbb{R}^3} \frac{u_r}{r} \partial_z (\partial_r b_\theta - \frac{b_\theta}{r}) j_r \, dx \\
- \int_{\mathbb{R}^3} \frac{u_r}{r} \partial_z (\partial_r b_\theta - \frac{b_\theta}{r}) \frac{j_r}{r} \, dx + 2 \int_{\mathbb{R}^3} u_z \partial_z j_r \frac{j_r}{r} \, dx + \int_{\mathbb{R}^3} \frac{u_r}{r} \frac{j_r}{r} \, dx \\
- \int_{\mathbb{R}^3} [\partial_z u_r (\partial_r b_\theta - \frac{b_\theta}{r}) - (2 \partial_z u_z + \partial_z u_r) j_z] \partial z j_z \, dx
\]

\[
= \int_{\mathbb{R}^3} \partial_z [\partial_z u_r (\partial_r b_\theta - \frac{b_\theta}{r}) - \partial_z u_z j_r] \partial_r j_z \, dx \\
+ \int_{\mathbb{R}^3} \partial_z u_r (\partial_r b_\theta - \frac{b_\theta}{r}) - (2 \partial_z u_z + \partial_z u_r) j_z \partial z j_z \, dx \\
- \int_{\mathbb{R}^3} \frac{u_r}{r} \partial_z (\partial_r b_\theta - \frac{b_\theta}{r}) \frac{j_r}{r} + (\partial_r b_\theta - \frac{b_\theta}{r}) \frac{j_r}{r} \, dx \\
+ \int_{\mathbb{R}^3} \frac{u_r}{r} \partial_z j_z \partial_z j_r \, dx + \int_{\mathbb{R}^3} \frac{u_r}{r} \frac{j_r}{r} \, dx \\
+ 2 \int_{\mathbb{R}^3} u_z \partial_z j_r \frac{j_r}{r} \, dx \\
\leq \frac{1}{4} \left( \| \partial_z j_r \|_{L^2}^2 + \| \partial_z j_z \|_{L^2}^2 \right) \leq \frac{1}{4} \left( \| \partial_z j_r \|_{L^2}^2 + \| \partial_z j_z \|_{L^2}^2 \right)
\]

Noting that \( \partial_z j_\theta = -\partial_r \partial_z b_z = 0 \), one can deduce that

\[
III = -\int_{\mathbb{R}^3} u_r \frac{j_\theta}{r} \left( \Delta - \frac{1}{r^2} \right) j_\theta \, dx - \int_{\mathbb{R}^3} b_z \partial_z w_\theta \left( \Delta - \frac{1}{r^2} \right) j_\theta \, dx
\]
As for the last term $IV$, we split it into three parts and do estimates separately,

$$IV = \int_{\mathbb{R}^3} [u \cdot \nabla (\frac{b_0}{r}) + \frac{1}{r} \partial_r (u_r b_0) - \frac{1}{r} \partial_r (ru) \cdot \nabla b_0] (\partial_{rr} + \frac{1}{r} \partial_r + \partial_{zz}) j_0 dx$$

\[\leq \|u\|_{L^\infty} \|\nabla j\|_{L^2}^2.\]

1) \[= \int_{\mathbb{R}^3} [r \partial_{rr} u r \partial_r (\frac{b_0}{r}) - u_r \partial_r (\frac{b_0}{r})] j_0 dx\]

\[= \int_{\mathbb{R}^3} \partial_{rr} u_r (\partial_r b_0 - \frac{b_0}{r}) j_r dx + 2 \int_{\mathbb{R}^3} \frac{u_r}{r} [\partial_r b_0 + \frac{b_0}{r^2} - \frac{2}{r} \partial_r b_0] j_r dx\]

\[\leq \|\nabla u\|_{L^2}^2 \|\nabla_j b\|_{L^2}^2 \|\nabla_j \partial_{zz} j\|_{L^2}^2 + \|\frac{u_r}{r}\|_{L^\infty} \|\nabla_j\|_{L^2}^2 + \|\frac{b_0}{r}\|_{L^\infty} \|\nabla_j\|_{L^2}^2\]

2) \[= \int_{\mathbb{R}^3} u_r \partial_r (\frac{b_0}{r}) j_r dx - \int_{\mathbb{R}^3} \partial_r u_r (\partial_r b_0 - \frac{b_0}{r}) j_r dx\]

\[\leq \|u\|_{L^\infty} \|\nabla_j\|_{L^2} + \|\partial_r u_r\|_{L^2} \|\partial_r j_r\|_{L^2} \|\nabla_h j\|_{L^2}^2 \|\partial_{zz} j\|_{L^2}\]

\[\leq \frac{1}{4} \|\nabla_h \partial_{zz} j\|_{L^2}^2 + C(\|\nabla_j\|_{L^2}^2 + \|\nabla w_\theta\|_{L^2}^2) \|\nabla_j\|_{L^2}^2 + \|\nabla w_\theta\|_{L^2}^2,\]

3) \[= \int_{\mathbb{R}^3} \partial_r u_z j_r \partial_r (r \partial_r j_r) dz + \int_{\mathbb{R}^3} \partial_r u_z j_r \partial_{zz} j_r dx\]

\[\leq \|\partial_r u_z\|_{L^2} \|\nabla_h \partial_{rr} u_{\theta z}\|_{L^2} \|\partial_r j_r\|_{L^2} \|\nabla_h j_r\|_{L^2} \|\partial_{zz} j\|_{L^2} \|\partial_{zz} j\|_{L^2} \|\nabla_{h j_r}\|_{L^2} \|\partial_{zz} j\|_{L^2} \|\partial_{zz} j\|_{L^2}\]
\[ \|\nabla \times \varphi\|^2_{L^2} + \|\nabla \times \psi\|^2_{L^2} \leq C(\|\nabla \varphi\|^2_{L^2} + \|\nabla \psi\|^2_{L^2}) \]

Adding up all above estimates, there holds that

\[ \frac{1}{2} \frac{d}{dt} \|\nabla \varphi\|^2_{L^2} + \|\nabla \psi\|^2_{L^2} \leq C(\|\nabla \varphi\|^2_{L^2} + \|\nabla \psi\|^2_{L^2}) \]

Combining with Gr"onwall inequality, one can achieve that

\[ \|\nabla \varphi\|^2_{L^2} + \int_0^T (\|\nabla \psi\|^2_{L^2} + \|\nabla \varphi\|^2_{L^2}) dt \leq C(\|\nabla \varphi\|^2_{L^2} + \|\nabla \psi\|^2_{L^2} + \|\nabla \varphi\|^2_{L^2} + \|\nabla \psi\|^2_{L^2}). \]

Thus by (3.36) and (3.37), we can reach the conclusion. \qed

4. Proof of Theorem 1.1.

4.1. \( L^1 \rightarrow L^\infty \) estimates.

Proposition 4. If \( (u, b) \) solves the system (1.4)–(1.6) with \( u = u_r e_r + u_z e_z \) and \( b = b_\theta e_\theta + b_z e_z \), then there holds that

\[ \int_0^T (\|\nabla u\|_{L^\infty} + \|\nabla b\|_{L^\infty}) dt \leq C_8, \]

where the constant \( C_8 \) depends only on \( T, \|b_\theta\|_{L^\infty}, \|w_0\|_{H^1} \) and \( \|j_0\|_{H^1} \).

Proof. Step 1. To prove

\[ \int_0^T \|\nabla u\|_{L^\infty} dt \leq C. \]

Since \( u \) solves

\[ \partial_t u - \Delta u + u \cdot \nabla u = -\nabla \pi + \frac{b_\theta^2}{\rho}, \]

and

\[ u \cdot \nabla u = (\nabla \times u) \times u - \frac{1}{2} \nabla |u|^2, \]

then it follows that \( \nabla \times u \) solves

\[ \partial_t (\nabla \times u) - \Delta (\nabla \times u) = \nabla \times (\nabla \times u \times u) + \partial_z \left( \frac{b_\theta^2}{\rho} \right) e_\theta. \] (4.39)

On one side, Propositions 1, 2 and 3 tell us that

\[ \nabla \times u \in L^\infty([0, T], L^6), \quad u \in L^\infty([0, T], L^2). \]
Moreover $\text{div}u = 0$, thus by imbedding inequality, one can get that
\[
\|u\|_{L^\infty([0,T],L^\infty)} \leq \|u\|_{L^\infty([0,T],L^2)} + \|\nabla \times u\|_{L^\infty([0,T],L^6)} \\
\leq C,
\]
which implies that
\[
\|(\nabla \times u) \times u\|_{L^1([0,T],L^6)} \leq C. \tag{4.40}
\]

On the other side, Lemma 3.1–Lemma 3.3 give that
\[
\frac{b_2^2}{r} \in L^\infty([0, T], L^2) \cap L^\infty([0, T], L^\infty) \\
\rightarrow L^1([0, T], L^6). \tag{4.41}
\]
Therefore, making use of estimate (4.40), (4.41) and regular estimates of the velocity in the horizontal direction for (4.39) (see Lemma 3.4 in [13]), one has
\[
\|\nabla_h \nabla \times u\|_{L^1([0,T],L^6)} \leq C,
\]
which implies that
\[
\|\partial_r u_r\|_{L^1([0,T],L^\infty)} + \|\partial_r u_z\|_{L^1([0,T],L^\infty)} \leq C.
\]
It follows from Proposition 3 that
\[
\|\partial_z u_r\|_{L^1([0,T],L^\infty)} \leq C \int_0^T \|\partial_z \nabla u_r\|_2^\frac{3}{2} \|\nabla_h \partial_z \nabla u_r\|_2^\frac{3}{2} \, dt \\
\leq C \|\partial_z w\|_2^\frac{3}{2} \|\nabla_h \partial_z w\|_{L^1([0,T],L^2)} \\
\leq C. \tag{4.42}
\]

**Step 2.** To deduce
\[
\int_0^T \|\nabla b\|_{L^\infty} \, dt \leq C.
\]
By (2.7)–(2.11), one has
\[
\begin{cases}
\partial_t \nabla b_\theta - \partial_{zz} \nabla b_\theta + u \cdot \nabla \nabla b_\theta = -\nabla u \cdot \nabla b_\theta + \frac{u_r}{r} \nabla b_\theta + (\nabla u_r - \frac{u_r}{r} b_\theta), \\
\partial_t \nabla b_z + u \cdot \nabla \nabla b_z - b_z \partial_z \nabla u_z = \partial_z u_z \nabla b_z - \nabla u \cdot \nabla b_z.
\end{cases}
\]
Then for any $p > 1$, taking inner product with $|\nabla b_\theta|^{p-2} \nabla b_\theta$, $|\nabla b_z|^{p-2} \nabla b_z$ and integrating on $\mathbb{R}^3$ respectively, it isn’t hard to derive that
\[
\frac{d}{dt} \left( \|\nabla b_\theta\|_{L^p}^p + \|\nabla b_z\|_{L^p}^p \right) \leq \|\nabla u\|_{L^\infty} \left( \|\nabla b_\theta\|_{L^p} + \|\nabla b_z\|_{L^p} \right) \\
+ \|\nabla u\|_{L^\infty} \frac{b_\theta}{r} \|L^\infty.}
\]
By Gronwall inequality, there holds that
\[
\| \nabla b_\theta \|_{L^p} + \| \nabla b_z \|_{L^p} \leq \| \nabla b_0 \|_{L^p} e^{\int_0^T \| \nabla u \|_{L^\infty} dt} + \int_0^T \| \nabla u \|_{L^\infty} \| \frac{b_\theta}{r} \|_{L^\infty} dt.
\]
Letting \( p \to \infty \), this gives that
\[
\| \nabla b \|_{L^\infty} \leq \| \nabla b_0 \|_{L^\infty} e^{\int_0^T \| \nabla u \|_{L^\infty} dt} + \int_0^T \| \nabla u \|_{L^\infty} \| \frac{b_\theta r}{L^\infty} dt,
\]
and together with estimates (4.42) and Lemma 3.1, it can be reached that
\[
\| \nabla b \|_{L^\infty} \leq C.
\]
The proof of this proposition is finished. \( \square \)

4.2. **Proof of Theorem 1.1.** By Propositions 1–3, the proof can be achieved through a parabolic regularization process. Let \( \delta > 0 \) be a small parameter and consider a family solutions \((u_\delta, b_\delta)\) satisfying the regularized system
\[
\begin{align*}
\partial_t u_\delta - \delta \Delta_h u_\delta + u_\delta \cdot \nabla u_\delta &= -\nabla \pi_\delta + b_\delta \cdot \nabla b_\delta, \\
\partial_t b_\delta - \delta \partial_z b_\delta + u_\delta \cdot \nabla b_\delta &= b_\delta \cdot \nabla u_\delta, \\
\nabla \cdot u_\delta &= \nabla \cdot b_\delta = 0, \\
u_\delta(x, 0) &= \phi_\delta * u_0, \quad b_\delta(x, 0) = \phi_\delta * b_0,
\end{align*}
\]
where \( \phi_\delta \) is a standard mollifier.

Since \( u_\delta(x, 0) \) and \( b_\delta(x, 0) \) are smooth, the standard theory on the 3D viscous MHD equations ensures that (4.43)–(4.46) has a unique global smooth solution \((u_\delta, b_\delta)\) which obeys the a priori bounds in Propositions 1–3 uniformly in \( \delta \). By standard compactness arguments and Lions-Aubin Lemma, we can show that this family \((u_\delta, b_\delta)\) converges to \((u, b)\) which satisfies in turn our initial problem and obeys the bounds in Propositions 1–3.

Using
\[
\int_0^T (\| \nabla u \|_{L^\infty} + \| \nabla b \|_{L^\infty}) dt < \infty,
\]
the uniqueness can be proved by the standard method and we omit the details here (see [16, 1] for instance). The proof of the theorem is finished. \( \square \)

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