In this article we propose a new fractional derivative without singular kernel. We consider the potential application for modeling the steady heat-conduction problem. The analytical solution of the fractional-order heat flow is also obtained by means of the Laplace transform.

Key words: heat conduction, steady heat flow, analytical solution, Laplace transform, fractional derivative without singular kernel

Introduction

Fractional derivatives with singular kernel [1], namely, the Riemann-Liouville [2-3], Caputo [4-5] and other derivatives (see [6-8] and the references therein), have nowadays a wide application in the field of heat-transfer engineering.

More recently, the fractional Caputo-Fabrizio derivative operator without singular kernel was given as (see [1,9-12]):

\[ CF \mathcal{D}_{x}^{(\nu)} T(x) = \mathcal{I}(\nu) \int_{0}^{x} \exp \left( -\frac{\nu}{1-\nu} (x-\lambda) \right) T^{(1)}(\lambda) d\lambda, \quad (1) \]

where \( \mathcal{I}(\nu) \) is a normalization constant depending on \( \nu \) \((0 < \nu < 1)\).

Following Eq. (1), Losada and Nieto suggested the new fractional Caputo-Fabrizio derivative operator [10-12]

\[ CF^{\#} \mathcal{D}_{x}^{(\nu)} T(x) = \frac{1}{1-\nu} \int_{0}^{x} \exp \left( -\frac{\nu}{1-\nu} (x-\lambda) \right) T^{(1)}(\lambda) d\lambda, \quad (2) \]

where \( \nu \) \((0 < \nu < 1)\) is a real number and \( \mathcal{I}(\nu) = 2/(2-\nu) \).

Eqs. (1) and (2) represent an extension of the Caputo fractional derivative with singular kernel. However, an analog of the Riemann-Liouville fractional
derivative with singular kernel has not yet been formulated. The main aim of the article is to propose a new fractional derivative without singular kernel, which is an extension of the Riemann-Liouville fractional derivative with singular kernel, and to study its application in the modeling of the fractional-order heat flow.

In this line of thought, the structure of the article is as follows. Section 2 presents a new fractional derivative without singular kernel. Section 3 discusses its application to the steady fractional-order steady heat flow in the heat-conduction problem. Finally, Section 4 outlines the conclusions.

Mathematical tools

The Riemann-Liouville fractional derivative of fractional order \( \nu \) of the function \( T(x) \) is defined as [1]

\[
RLD^{(\nu)}_{a^+} T(x) = \frac{1}{\Gamma(1-\nu)} \frac{d}{dx} \int_{a}^{x} T(\lambda) (x-\lambda)^{\nu} d\lambda, \tag{3}
\]

where \( a \leq x \) and \( \nu (0 < \nu < 1) \) is a real number.

Replacing the function \( \frac{1}{(x-\lambda)^\nu} \Gamma(1+\nu) \) by the function \( \Re(\nu) \exp\left(-\frac{\nu}{1-\nu} (x-\lambda)\right) \) / \((1-\nu)\), we obtain a new fractional derivative given by:

\[
D^{(\nu)}_{a^+} T(x) = \frac{\Re(\nu)}{1-\nu} \frac{d}{dx} \int_{a}^{x} \exp\left(-\frac{\nu}{1-\nu} (x-\lambda)\right) T(\lambda) d\lambda, \tag{4}
\]

where \( a \leq x \), \( \nu (0 < \nu < 1) \) is a real number and \( \Re(\nu) \) is a normalization function depending on \( \nu \) such that \( \Re(0) = \Re(1) = 1 \).

Taking \( \psi = 1/\nu - 1 \), with \( 0 < \psi < +\infty \), Eq. (4) can be rewritten as:

\[
D^{(\psi)}_{a^+} T(x) = \Re(\psi) \frac{d}{dx} \int_{a}^{x} \Pi(\lambda) T(\lambda) d\lambda, \tag{5}
\]

where \( \Re(\psi) = (\psi + 1)\Re(1/(\psi + 1)) \) and \( \Pi(\lambda) = \exp\left(- (x-\lambda)/\psi\right)/\psi \).

With the help of the following approximation to the identity \([9,13]\]

\[
\lim_{\psi \to 0} \Pi(\lambda) = \delta(x-\lambda), \tag{6}
\]

where \( \nu \to 1 \) (or \( \psi \to 0 \)), Eq. (4) becomes

\[
\lim_{\nu \to 1} D^{(\nu)}_{a^+} T(x) = \lim_{\psi \to 0} \Re(\psi) \frac{d}{dx} \int_{a}^{x} \Pi(\lambda) T(\lambda) d\lambda = T^{(1)}(x). \tag{7}
\]

When \( \nu \to 0 \) (or \( \psi \to +\infty \)), Eq. (4) can be written as

\[
\lim_{\nu \to 0} D^{(\nu)}_{a^+} T(x) = \lim_{\nu \to 0} \Re(\nu) \frac{d}{dx} \int_{a}^{x} \exp\left(-\frac{\nu}{1-\nu} (x-\lambda)\right) T(\lambda) d\lambda = T(x). \tag{8}
\]
Taking the Laplace transform of the new fractional derivative without singular kernel for the parameter \( a = 0 \), we have

\[
L \left( D_0^{(\nu)} T(x) \right) = \frac{\Re(\nu)}{\nu(1-\nu)} s T(s),
\]

where \( L(\xi(x)) := \int_0^\infty \exp(-sx) \xi(x) \, dx = \xi(s) \) represents the Laplace transform of the function \( \xi(x) \) (see [14]).

We now consider

\[
T(s) = \left( \frac{\nu}{\Re(\nu)s} + \frac{1-\nu}{\Re(\nu)} \right) \Xi(s),
\]

where \( D_0^{(\nu)} T(x) = \Xi(x) \) and \( L(\Xi(x)) = \Xi(x) \).

Taking the inverse Laplace transform of Eq. (10) we obtain

\[
T(x) = \frac{1-\nu}{\Re(\nu)} \Xi(x) + \frac{\nu}{\Re(\nu)} \int_0^x \Xi(x) \, dx, \quad x > 0, \quad 0 < \nu < 1.
\]

If \( 0 < \nu < 1 \) and \( \Re(\nu) = 1 \), then Eq. (11) and Eq. (13) can be written as

\[
* D_a^{(\nu)} T(x) = \frac{1}{1-\nu} \frac{d}{dx} \int_a^x \exp\left( -\frac{\nu}{1-\nu} (x-\lambda) \right) T(\lambda) \, d\lambda,
\]

and

\[
T(x) = (1-\nu) \Xi(x) + \nu \int_0^x \Xi(x) \, dx, \quad x > 0, \quad 0 < \nu < 1,
\]

respectively.

**Modelling the fractional-order steady heat flow**

The fractional-order Fourier law in one-dimension case is suggested as:

\[
KD_0^{(\nu)} T(x) = -H(x),
\]

where \( K \) is the thermal conductivity of the material and \( H(x) \) represents the heat flux density.

The heat flow of the fractional-order heat conduction is presented as

\[
H(x) = g,
\]

where \( g \) is the heat flow (a constant) of the material.

By submitting Eq. (13) into Eq. (14) and taking the Laplace transform, it results:

\[
\frac{\Re(\nu)}{\nu + (1-\nu)s} T(s) = -\frac{g}{K},
\]

which leads to

\[
T(s) = \frac{-g(\nu + (1-\nu)s)}{K\Re(\nu)s}.
\]
Taking the inverse Laplace transform of Eq. (17), we obtain

$$T(x) = -C \left( \frac{g \nu x}{K \Re(\nu)} + \frac{g (1 - \nu)}{K \Re(\nu)} \right),$$

(18)

where $C$ is a constant depending on the initial value $T(x)$.

The corresponding graphs with different orders $\nu = \{0.3, 0.6, 1\}$ are shown in Figure 1.

**Figure 1.** The plots of $T(x)$ with the parameters $\nu = \{0.3, 0.6, 1\}$, $C = -1$, $g = 2$, $K = 3$ and $\Re(\nu) = 1$.

**Conclusions**

In this work a new fractional-order operator without singular kernel, which is an analog of the Riemann-Liouville fractional derivative with singular kernel, was proposed for the first time. An illustrative example for modelling the fractional-order steady heat flow was given and the analytical solution for the governing equation involving the fractional derivative without singular kernel was discussed.
References

1. Yang, X. J, Baleanu, D, Srivastava, H. M., Local Fractional Integral Transforms and Their Applications, Academic Press, New York, 2015

2. Hristov J, et al., Thermal impedance estimations by semi-derivatives and semi-integrals: 1-D semi-infinite cases, Thermal Science, 17 (2013), 2, pp.581-589

3. Povstenko Y. Z., Fractional radial heat conduction in an infinite medium with a cylindrical cavity and associated thermal stresses, Mechanics Research Communications, 37(2010), 4, pp.436-40

4. Hussein, E. M., Fractional order thermoelastic problem for an infinitely long solid circular cylinder, Journal of Thermal Stresses, 38 (2015), 2, pp.133-45

5. Wei, S., et al., Implicit local radial basis function method for solving two-dimensional time fractional diffusion equations, Thermal Science, 19 (2015), S1, pp.59-67

6. Zhao, D., et al., Some fractal heat-transfer problems with local fractional calculus, Thermal Science, 19 (2015), 5, pp.1867-1871

7. Yang, X. J, et al., A new numerical technique for solving the local fractional diffusion equation: Two-dimensional extended differential transform approach, Applied Mathematics and Computation, 274 (2016), pp.143-151

8. Jafari, H., et al., A decomposition method for solving diffusion equations via local fractional time derivative, Thermal Science, 19 (2015), S1, pp.123-9

9. Caputo, M., et al., A new definition of fractional derivative without singular Kernel, Progress in Fractional Differentiation and Applications, 1 (2015), 2, pp.73-85

10. Lozada, J. et al., Properties of a new fractional derivative without singular kernel, Progress in Fractional Differentiation and Applications, 2015(1), 2, pp.87-92

11. Atangana, A., On the new fractional derivative and application to nonlinear Fisher’s reaction–diffusion equation, Applied Mathematics and Computation, 273(2016), pp.948-956

12. Alsaedi, A., et al., Fractional electrical circuits, Advances in Mechanical Engineering, 7 (2015),12, pp.1-7

13. Stein, E., Weiss, G., Introduction to Fourier Analysis on Euclidean Spaces, Princeton University Press, 1971

14. Debnath, L., Bhatta, D., Integral Transforms and Their Applications, CRC Press, 2014.