CONSTRUCTION OF MULTI-BUBBLE SOLUTIONS FOR THE CRITICAL GKDVEQUATION

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Abstract. We prove the existence of solutions of the mass critical generalized Korteweg–de Vries equation \( \partial_t u + \partial_x (\partial_{xx} u + u^5) = 0 \) containing an arbitrary number \( K \geq 2 \) of blow up bubbles, for any choice of sign and scaling parameters: for any \( \ell_1 > \ell_2 > \cdots > \ell_K > 0 \) and \( \epsilon_1, \ldots, \epsilon_K \in \{\pm 1\} \), there exists an \( H^1 \) solution \( u(t) \) of the equation such that

\[
\lim_{t \downarrow 0} \left( \sum_{k=1}^{K} \frac{\epsilon_k}{\lambda_k^2(t)} Q \left( \frac{\cdot - x_k(t)}{\lambda_k(t)} \right) \right) = 0 \quad \text{in} \quad H^1
\]

with \( \lambda_k(t) \sim \ell_k t \) and \( x_k(t) \sim -\ell_k^2 t^{-1} \) as \( t \downarrow 0 \). The construction uses and extends techniques developed mainly in [18] and [22, 23, 24]. Due to strong interactions between the bubbles, it also relies decisively on the sharp properties of the minimal mass blow up solution (single bubble case) proved in [3].

1. Introduction

1.1. Main result. We consider the mass critical generalized Korteweg–de Vries equation

\[
(gKdV) \quad \begin{cases}
\partial_t u + \partial_x (\partial_{xx} u + u^5) = 0, & (t,x) \in [0,T) \times \mathbb{R}, \\
u(0,x) = u_0(x), & x \in \mathbb{R}.
\end{cases}
\]

We first recall a few well-known facts about this equation. The Cauchy problem is locally well-posed in the energy space \( H^1 \) from [10, 11, 12]: for a given \( u_0 \in H^1 \), there exists a unique (in a suitable functional space) maximal solution \( u \) of (1.1) in \( C([0,T), H^1) \) and

\[
T < +\infty \quad \text{implies} \quad \lim_{t \uparrow T} \|\partial_x u(t)\|_{L^2} = +\infty.
\]

Moreover, such \( H^1 \) solutions satisfy the conservation of mass and energy

\[
\|u(t)\|_{L^2}^2 = \int u^2(t) = \|u_0\|_{L^2}^2, \quad E(u(t)) = \frac{1}{2} \int (\partial_x u)^2(t) - \frac{1}{6} \int u^6(t) = E(u_0).
\]

Equation (1.1) is invariant under scaling and translation: if \( u \) is a solution of (1.1) then \( u_{\lambda, x_0} \), defined by

\[
u_{\lambda, x_0}(t,x) = \lambda^\frac{3}{2} u(\lambda^3 t, \lambda x + x_0), \quad \lambda > 0, \ x_0 \in \mathbb{R},
\]

is also a solution of (1.1).

Recall that equation (1.1) admits solutions of the form \( u(t,x) = Q(x - t) \), called solitons, where \( Q \) is the ground state

\[
Q(x) = \left( \frac{3}{\cosh(2x)} \right)^{\frac{1}{2}}, \quad Q'' + Q^5 = Q,
\]
which attains the best constant in the following sharp Gagliardo–Nirenberg inequality, as proved in [40]:

\[ \forall v \in H^1, \quad \int v^6 \leq 3 \int (\partial_x v)^2 \left( \frac{\int v^2}{Q^2} \right)^2. \]

It is well-known that the conservation of mass and energy, the above inequality, and the blow up criterion (1.2) ensure that \( H^1 \) initial data with subcritical mass, \( i.e. \| u_0 \|_{L^2} < \| Q \|_{L^2} \), generate global in time and bounded solutions.

Concerning the mass threshold, it was proved in [23] that there exists a unique (up to invariances) blow up solution \( S \) of (1.1) with \( \| S(t) \|_{L^2} = \| Q \|_{L^2} \), called the minimal mass blow up solution. Fixing by convention the blow up time as \( 0, 22, 23, 24 \) and references therein). The

\[ \text{Theorem 1.1.} \] Let \( K \geq 2, \ell_1 > \ell_2 > \cdots > \ell_K > 0 \text{ and } \epsilon_1, \ldots, \epsilon_K \in \{ \pm 1 \}. \text{ Then there exist} \ T_0 > 0 \text{ and a solution} \ u \in C((0, T_0], H^1) \text{ to (1.1) such that, for all} \ t \in (0, T_0], \]

\[ \left\| u(t) - \sum_{k=1}^{K} \frac{\epsilon_k}{\lambda_k(t)} Q \left( \frac{-x_k(t)}{\lambda_k(t)} \right) \right\|_{H^1} \lesssim t^{\frac{1}{42}}, \]

(1.3)

where \( \lambda_k(t) \) and \( x_k(t) \) satisfy, for all \( 1 \leq k \leq K \),

\[ |\lambda_k(t) - \ell_k t| \lesssim t^{\frac{21}{42}}, \quad |x_k(t) + \ell_k^{-2} t^{-1}| \lesssim t^{-\frac{21}{42}}. \]

Moreover, the values of the mass and the energy of \( u(t) \) are \( \| u(t) \|_{L^2}^2 = K \| Q \|_{L^2}^2 \) and

\[ E(u(t)) = \frac{1}{16} \| Q \|_{L^1}^2 \sum_{k=1}^{K} \ell_k \left( 1 + 2 \sum_{j=1}^{k-1} \epsilon_k \epsilon_j \sqrt{\frac{\ell_k}{\ell_j}} \right) > 0. \]

Note that the blow up points \( x_k(t) \) go to infinity as \( t \downarrow 0 \), as in all previous blow up results for the mass critical (gKdV) equation (see [22, 23, 24] and references therein). The assumption that the values of \( \ell_k \) are all different implies some decoupling between the bubbles. In particular, the solution \( u(t) \) given by Theorem 1.1 satisfies \( \| u(t) \|_{L^2}^2 = K \| Q \|_{L^2}^2 \), which is expected to be the minimal amount of mass so that blow up occurs at \( K \) different points.

However, because of the slow decay of \( S(t, x) \) for \( x \leq -\frac{1}{2} - 1 \) (see (1.6) below), the bubbles strongly interact close to the blow up time, as shown by the value of the energy of \( u(t) \) given in Theorem 1.1. Indeed, if on the one hand the first part of the energy \( \frac{1}{16} \| Q \|_{L^1}^2 \sum_{k=1}^{K} \ell_k \) is due to the bubbles themselves (recall that \( E(S(t)) = \frac{1}{16} \| Q \|_{L^1}^2 \) from [3]), on the other hand the additional terms \( \sum_{j<k} \epsilon_k \epsilon_j \sqrt{\frac{\ell_k}{\ell_j}} \) are due to interactions.
These interactions need to be carefully computed to construct the solution \( u(t) \) and this is why the present paper relies decisively on the sharp properties of the minimal mass blow up solution \( S \) derived in [3], as well as on some additional technical arguments from [24]. We recall relevant previous results in the next section before commenting on the proof of Theorem 1.1 in Section 1.3. We also briefly review some results on multi-bubble blow up for other nonlinear models in Section 1.4.

1.2. Previous results on minimal mass blow up for critical \((gKdV)\). We first recall the main result in [23].

**Theorem 1.2** (Existence and uniqueness of the minimal mass blow up solution [23]).

(i) Existence. There exist a solution \( S \in \mathcal{C}((0, +\infty), H^1) \) to (1.1) and universal constants \( c_0 \in \mathbb{R}, C_0 > 0 \) such that \( \|S(t)\|_{L^2} = \|Q\|_{L^2} \) for all \( t > 0 \) and

\[
\|\partial_x S(t)\|_{L^2} \sim \frac{C_0}{t} \quad \text{as} \quad t \downarrow 0,
\]

\[
S(t) - \frac{1}{t^2} Q \left( \frac{-1}{t} + c_0 \right) \rightarrow 0 \quad \text{in} \quad L^2 \quad \text{as} \quad t \downarrow 0. \tag{1.4}
\]

(ii) Uniqueness. Let \( u_0 \in H^1 \) with \( \|u_0\|_{L^2} = \|Q\|_{L^2} \) and assume that the corresponding solution \( u(t) \) to (1.1) blows up in finite or infinite time. Then, \( u \equiv S \) up to the symmetries of the flow.

In [23], it is also proved that \( S(t, x) \) has exponential decay in space for \( x \geq -\frac{1}{t} \), but the behavior of \( S(t, x) \) for \( x \leq -\frac{1}{t} \) is not studied, and the convergence result (1.4) is limited to the \( L^2 \) norm. In view of their objective to eventually prove Theorem 1.1, the authors of the present paper established in [3] the following sharp time and space asymptotics for \( S(t) \) close to the blow up time.

**Theorem 1.3** (Time asymptotics [3]). Let \( S \) and \( c_0 \) be defined as in (i) of Theorem 1.2. Then there exist functions \( \{Q_k\}_{k \geq 0} \subset S(\mathbb{R}) \) (with \( Q_0 = Q \)) such that the following holds.

For all \( m \geq 0 \), there exists \( T_0 > 0 \) such that, for all \( t \in (0, T_0] \),

\[
\left\| \partial_x^m S(t) - \sum_{k=0}^{m} \frac{1}{t^{2k+m+2k}} Q_k^{(m-k)} \left( \frac{-1}{t} + c_0 \right) \right\|_{L^2} \lesssim t^{1+m}. \tag{1.5}
\]

**Theorem 1.4** (Space asymptotics [3]). Let \( S \) be defined as in (i) of Theorem 1.2 and \( m \geq 0 \). Then there exists \( T_0 > 0 \) such that the following hold.

(i) Pointwise asymptotics on the left. For all \( t \in (0, T_0] \), for all \( x \leq -\frac{1}{t} - 1 \),

\[
|S(t, x) + \frac{1}{2} \|Q\|_{L^1} |x|^{-\frac{3}{2}}| \lesssim |x|^{-\frac{3}{2} - \frac{1}{2m}}, \tag{1.6}
\]

\[
|\partial_x^m S(t, x)| \lesssim |x|^{-\frac{3}{2} - m}. \tag{1.7}
\]

(ii) Pointwise bounds on the right. There exists \( \gamma_m > 0 \) such that, for all \( t \in (0, T_0] \), for all \( x \in \mathbb{R} \),

\[
|\partial_x^m S(t, x)| \lesssim \frac{1}{t^{\frac{3}{2} + m}} \exp \left( -\gamma_m \frac{x + \frac{1}{t}}{t} \right). \tag{1.8}
\]
As a corollary of Theorems 1.3 and 1.4, we also obtain time estimates in exponential weighted spaces for $S(t)$, stated and proved in Appendix A.

Let us now comment on the main consequences of the above results on the construction of multi-bubbles. First, Theorem 1.3 means that $S(t)$ is smooth and that its behavior as $t \downarrow 0$ is completely understood in any Sobolev norm, which makes it a good candidate as the building brick to construct multi-bubble solutions.

Second, while the decay of $S(t,x)$ for $x \geq -\frac{1}{t}$ is exponential, which means weak interactions on the other bubbles on its right, the asymptotic behavior of $S(t,x)$ for $x \leq -\frac{1}{t} - 1$ is like $|x|^{-\frac{3}{2}}$. Surprisingly, this asymptotic behavior does not translate like the bubble, but it rather describes a fixed explicit power-like tail, which interacts with the other bubbles on the left of $S(t,x)$. Such a power-like perturbation can be considered as a strong interaction compared to the usual exponential interactions between solitons.

Finally, also observe that the convergence of $u(t)$ to the sum of bubbles in $H^1$ in (1.3) is of order $t^{-\frac{26}{23}}$ as $t \downarrow 0$. The reason why we cannot improve this convergence result are the possible fluctuations of order $|x|^{-\frac{3}{2} - \frac{1}{2t}}$ of the tail of $S(t,x)$ around the explicit tail $-\frac{1}{t} \|Q\|_{L^1} |x|^{-\frac{3}{2}}$ given by (1.6). Of course, the exponent $\frac{1}{2t}$ is not optimal in [3], but it seems difficult to obtain a significantly better estimate. Another observation to complement Theorem 1.1 is that $u(t)$ exhibits an explicit tail made of the sum of the tails of each modulated version of $S(t)$ (see Remark 3.5 at the end of this paper).

### 1.3. Outline of the proof of Theorem 1.1.

Since we anticipate an explicit blow up rate, it is natural to use rescaled variables — see (2.6) and (2.7). Though not absolutely necessary, this change of variables allows us to reduce to the case of bounded solitons in large time at the cost of an additional scaling term in the equation.

Next, we introduce an approximate solution to the multi-bubble problem whose main term is a sum of $K$ rescaled and modulated versions of $S(t)$. As discussed above, the interactions between the bubbles are strong because of the tail on the left of the building brick $S(t,x)$. However, continuing the formal discussion of Section 1.4 in [24], we notice that a tail of the form $c|x|^{-\frac{3}{2}}$, with any $c \in \mathbb{R}$, is compatible with the blow up rate $t^{-1}$. Thanks to the precise space asymptotics in Theorem 1.4 and following the technique developed in [24], we improve the ansatz by adding terms taking into account the leading order of the interactions of the bubbles with such fixed tails. Constructed in this way, the approximate solution is sharp enough to compute the energy of the multi-bubble solution given in Theorem 1.1.

The full ansatz $V$ is introduced in Section 2.2 and studied in Lemmas 2.4 and 2.6. Then, it only remains to construct a solution close to $V$ using the by now standard (see references in the next section) strategy of defining a sequence of backwards solutions satisfying uniform estimates close to $V$. To derive such uniform estimates, we study both the evolution of the modulation parameters and the remaining infinite dimensional part (denoted $\varepsilon$ in the standard decomposition result Lemma 2.7). To control $\varepsilon$, we use a modification of the mixed energy–virial functional introduced in [22] for one bubble. Due to the presence of $K$ bubbles, we need to run it recursively on each soliton. At this point, the argument is reminiscent from the construction of bounded multi-solitons in [2, 5, 17, 25]. Finally, some instability directions have to be controlled by adjusting the initial parameters of the approximate solution, e.g. as in [2, 5] for the supercritical (gKdV) equation.
1.4. Previous results for related models. We start by recalling early results on minimal mass blow up for the mass critical nonlinear Schrödinger equation (in dimension \(N \geq 1\)),

\[
\text{(NLS) } i\partial_t u + \Delta u + |u|^{\frac{4}{N}} u = 0, \quad (t, x) \in [0, T) \times \mathbb{R}^N.
\]

For (NLS), it is well-known (see e.g. [1]) that the pseudo conformal symmetry generates an explicit minimal mass blow up solution

\[
S_{\text{NLS}}(t, x) = \frac{1}{t^\frac{N}{2}} e^{-i|\frac{x}{t}|^2} Q_{\text{NLS}} \left( \frac{x}{t} \right),
\]

defined for all \(t > 0\) and blowing up as \(t \downarrow 0\), where \(Q_{\text{NLS}}\) is the unique radial ground state of (NLS), solution to

\[
\Delta Q_{\text{NLS}} - Q_{\text{NLS}} + |Q_{\text{NLS}}|^{\frac{4}{N}} Q_{\text{NLS}} = 0, \quad Q_{\text{NLS}} > 0, \quad Q_{\text{NLS}} \in H^1(\mathbb{R}^N).
\]

Using the pseudo conformal symmetry, it was proved by Merle [30] that \(S_{\text{NLS}}\) is the unique (up to the symmetries of the equation) minimal mass blow up solution of (NLS) in the energy space. The situation is thus similar to the one for (gKdV), though much more explicit and simple due to the pseudo conformal symmetry.

The first result on the construction of solutions of a dispersive PDE blowing up at \(K\) given points in \(\mathbb{R}^N\) is given in another pioneering paper of Merle [28] for the mass critical (NLS). In this paper, the solution is obtained as the limit of backwards solutions containing \(K\) focusing bubbles \(S_{\text{NLS}}\) and satisfying uniform estimates. This strategy has been widely used and extended to other constructions of multi-bubble solutions, both in regular or singular regimes — see [2, 5, 17, 22, 39] notably. Since the blow up solution is obtained by gluing (rescaled and translated) solutions \(S_{\text{NLS}}(t)\), it has the so-called conformal blow up rate \(t^{-1}\). As a consequence of the exponential decay of \(S_{\text{NLS}}(t)\), the interactions between the bubbles are exponentially small in \(t^{-1}\). This construction has been extended to (NLS) in bounded sets with Dirichlet boundary conditions in [8].

Recall also that, for (NLS), the stable blow up is not the conformal one, but the so-called log-log blow up, whose rate is \(\sqrt{\frac{1}{t} \log |\log t|}\). This blow up behavior has been studied thoughtfully by Merle and Raphael [32, 33, 34] (see also references therein) in a neighborhood of \(Q_{\text{NLS}}\). Multiple point log-log blow up solutions and log-log blow up in a bounded set were studied in [7] and [38].

For the mass critical (NLS), we also mention the construction in [27] of a solution blowing up strictly faster than the conformal blow up rate using strong interactions between several colliding blow up bubbles.

For the (gKdV) equation in the slightly supercritical case, which reads

\[
\partial_t u + \partial_x (\partial_{xx} u + u^p) = 0, \quad (t, x) \in [0, T) \times \mathbb{R},
\]

with \(5 < p < 5 + \alpha\) and \(\alpha > 0\) small, Lan [15, 16] constructed solutions blowing up at \(K\) given points in the self-similar regime, i.e. with blow up rate \(t^{-\frac{4}{p-5}}\). The idea to construct explicit self-similar blow up for slightly supercritical models originates from [35] for the (NLS) model. In contrast to [35], where self-similar solutions are built using the soliton, [15, 16] take advantage of the exact self-similar profiles constructed by Koch [14] for \(p > 5\) close to 5. As for the present paper, some technical tools in [15, 16], e.g. the mixed energy–virial functional, are taken from [18, 22]. However, the context and the difficulties of the construction are different in the self-similar blow up regime, since the blow up points are finite and the interactions between the solitons are controlled as perturbation terms (see the outline of the proof in [16]).
For a numerical study of blow up for the critical and supercritical (gKdV) equations, we refer to [13] and to the references therein.

Note that, for the $L^2$ critical modified Benjamin–Ono equation, a minimal mass blow up solution was recently constructed in [26], following [3] and [23] as well as more specific previous works on Benjamin–Ono type equations (see references in [26]).

For the semilinear wave equation in the energy subcritical case, we refer to [6, 36, 37], where multi-soliton profiles are relevant in the refined study of the behavior of solutions at a blow up point. We also refer to the construction by Jendrej [9] of radial two-bubbles for the energy critical wave equation in large dimensions. In this work, the solution is global in one direction of time and one bubble stays bounded. On the top of this standing bubble, a second bubble concentrates with a specific rate as time goes to infinity.

In the parabolic context, a similar result of blow up at $K$ points for the energy subcritical nonlinear heat equation was proved in the early work [29], using crucially continuity properties of blow up by perturbation of the data for this equation. We also refer the reader to the recent result [4] where, in the case of the energy critical heat equation in a bounded set with zero Dirichlet condition, blow up in infinite time at $K$ given points in the domain is obtained, provided a particular property related to the Green’s function is satisfied. This construction is another example of strong interactions, both between the bubbles and the boundary condition and between the bubbles themselves.

1.5. Notation. For $f, g \in L^2(\mathbb{R})$, their $L^2$ scalar product is denoted as

$$\langle f, g \rangle = \int_{\mathbb{R}} f(x)g(x) \, dx.$$  

The Schwartz space $S$ is classically defined as

$$S = S(\mathbb{R}) = \{ f \in C^\infty(\mathbb{R}, \mathbb{R}) \mid \forall j \in \mathbb{N}, \forall \alpha \in \mathbb{N}, \exists C_{j,\alpha} \geq 0, \forall y \in \mathbb{R}, |y^\alpha f^{(j)}(y)| \leq C_{j,\alpha} \}.$$

For $f \in S$ and $k \geq 0$, we use the notation $f^{(-k)}$ defined by induction as $f^{(0)} = f$ and

$$f^{(-k-1)}(x) = -\int_x^{+\infty} f^{(-k)}(y) \, dy.$$  \hspace{1cm} (1.9)

Let the generator of $L^2$ scaling be

$$\Lambda f = \frac{1}{2} f + y \partial_y f.$$

For brevity, $\sum_j$ and $\sum_k$ denote $\sum_{j=1}^K$ and $\sum_{k=1}^K$ respectively. For a given $k \geq 1$, $\sum_{j<k}$ denotes $\sum_{j=1}^{k-1}$ when $k \geq 2$, and 0 when $k = 1$.

All numbers $C$ appearing in inequalities are real positive constants (with respect to the context), which may change in each step of an inequality.

Finally, for $N \geq 1$, we denote by $B_N$ (resp. $S_N$) the closed unit ball (resp. the unit sphere) of $\mathbb{R}^N$ for the euclidian norm.

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2. Decomposition of the solution

2.1. Properties of the linearized operator. Let the functional space \( \mathcal{Y} \) be defined by
\[
\mathcal{Y} = \{ f \in C^\infty(\mathbb{R}, \mathbb{R}) \mid \forall j \in \mathbb{N}, \exists C_j, r_j \geq 0, \forall y \in \mathbb{R}, |f^{(j)}(y)| \leq C_j(1 + |y|)^r e^{-|y|} \}
\]
and \( L \) be the linearized operator close to \( Q \) given by
\[
Lf = -\partial_{yy} f + f - 5Q^4 f.
\]

We recall without proof the following properties of \( L \). The properties (i)–(iii) are taken e.g. from [19, 41], while (iv) is proved in [22] and (v) in [24].

Lemma 2.1 (Properties of \( L \)). The self-adjoint operator \( L \) defined on \( L^2 \) satisfies:

(i) Kernel: \( \ker L = \{ aQ' ; a \in \mathbb{R} \} \).

(ii) Scaling: \( L(\Lambda Q) = -2Q \).

(iii) Coercivity: there exists \( \kappa_0 > 0 \) such that, for all \( f \in H^1 \),
\[
\langle Lf, f \rangle \geq \kappa_0 \| f \|_{H^1}^2 - \frac{1}{\kappa_0} \left( \langle f, Q \rangle^2 + \langle f, \Lambda Q \rangle^2 + \langle f, y\Lambda Q \rangle^2 \right). \tag{2.1}
\]

(iv) There exists a unique function \( P \) such that \( P' \in \mathcal{Y} \) and
\[
(LP)' = \Lambda Q, \quad (P, Q) = \frac{1}{16} \| Q \|_{H^1}^2 > 0, \quad (P, Q') = 0, \tag{2.2}
\]
\[
\forall y > 0, \quad |P(y)| + \left| P(-y) - \frac{1}{2} \| Q \|_{L^1} \right| \lesssim e^{-y/2}. \tag{2.3}
\]

(v) There exists a unique even function \( R \in \mathcal{Y} \) such that
\[
LR = 5Q^4, \quad (Q, R) = -\frac{3}{4} \| Q \|_{L^1}. \tag{2.4}
\]

We recall that the function \( P \) plays a crucial role in [3, 22, 23, 24] for understanding the blow up dynamics. In particular, it appears naturally in the definition of \( Q_1 \), exhibited in Theorem 1.3. Indeed, from (3.47) in [3], there exist \( \lambda_0, c_1 \in \mathbb{R} \) such that
\[
Q_1 = -P' - \lambda_0 (\Lambda Q)' + c_1 Q''. \tag{2.5}
\]

2.2. Approximate multi-soliton of the rescaled flow. Let \( u(t, x) \) be any solution of (1.1) defined for \( t > 0 \) and \( x \in \mathbb{R} \). For \( s < 0 \) and \( y \in \mathbb{R} \), we consider the rescaled version of \( u \) defined by
\[
\tilde{u}(s, y) = \frac{1}{(-2s)^{\frac{3}{2}}} u(t, x) \quad \text{with} \quad t = \frac{1}{\sqrt{-2s}} \quad \text{and} \quad x = \frac{y}{\sqrt{-2s}}, \tag{2.6}
\]
or, equivalently,
\[
u(t, x) = \frac{1}{t^2} \tilde{u}(s, y) \quad \text{with} \quad s = -\frac{1}{2t^2} \quad \text{and} \quad y = \frac{x}{t}.
\]

In these new variables, \( \tilde{u} \) is continuous with values in \( H^1 \) and is solution of the equation
\[
\partial_x v + \frac{1}{2s} \Lambda v + \partial_y (\partial_{yy} v + v^5) = 0. \tag{2.7}
\]
Conversely, by a solution \( v \) of (2.7), we mean \( v = \tilde{u} \) where \( u \) is an \( H^1 \) solution of (1.1) in the sense of [11, 12].
Let $S$ be the minimal mass solution of (1.1) defined in Theorem 1.2. Let $t_0 > 0$ be the minimum of the $T_0$ given by Theorems 1.3 and 1.4 applied with any $m$ such that $0 \leq m \leq 20$, and let $s_0 = -\frac{1}{2t_0}$. We define $\tilde{S}(s, y)$ from $S(t, x)$ as above, so that $\tilde{S}$ satisfies (2.7) on $(-\infty, s_0]$.

As in Theorem 1.1, let $K \geq 2$, $\ell_1 > \ell_2 > \cdots > \ell_K > 0$ and $\epsilon_1, \ldots, \epsilon_K \in \{\pm 1\}$. We look for an approximate solution $V$ of (2.7) under the form of a sum of $K$ rescaled bubbles $\tilde{S}$ plus correction terms and modulation.

Let $s_1 < 2\ell_1^2s_0 < 0$ with $|s_1| \gg 1$ to be fixed later, and let $I \subset (-\infty, s_1)$ be a compact interval. For all $1 \leq k \leq K$, we consider $C^1$ functions $\mu_k$ and $y_k$ defined on $I$ to be determined later, and define $\tau_k$ by

$$d\tau_k = \mu_k^{-3}. \quad (2.8)$$

In view of proving Proposition 3.1, which implies Theorem 1.1 by rescaling (see Section 3.8), we assume that $\mu_k$, $\tau_k$ and $y_k$ satisfy, for some $0 < \alpha < 1$, for all $s \in I$,

$$|\tilde{\mu}_k(s)| \leq \alpha, \quad |\tilde{\tau}_k(s)| \leq \alpha, \quad |\tilde{y}_k(s)| \leq \alpha,$$

with $\tilde{\mu}_k(s) = \frac{\mu_k(s)}{\bar{s}_k} - 1$, $\tilde{\tau}_k(s) = \frac{\tau_k(s)}{s^2\bar{s}_k^2} - 1$, $\tilde{y}_k(s) = \frac{y_k(s)}{2s\bar{s}_k^2} - 1$. \quad (2.9)

For all $1 \leq k \leq K$, for all $s \in I$ and all $y \in \mathbb{R}$, we set

$$W_k(s, y) = \epsilon_k\mu_k^{-\frac{1}{2}}(s)\tilde{S}\left(\tau_k(s), \frac{y-z_k(s)}{\mu_k(s)}\right) \quad \text{with} \quad z_k = y_k + \mu_k\left(-2\tau_k + c_0 - \frac{c_1}{2\tau_k}\right),$$

and similarly, letting $\tilde{\mu}_k = \mu_k\left(1 + \frac{1}{\bar{s}_k}\right)^{-1}$,

$$Q_k(s, y) = \epsilon_k\tilde{\mu}_k^{-\frac{1}{2}}(s)Q\left(\frac{y-y_k(s)}{\tilde{\mu}_k(s)}\right), \quad R_k(s, y) = \epsilon_k\tilde{\mu}_k^{-\frac{1}{2}}(s)R\left(\frac{y-y_k(s)}{\tilde{\mu}_k(s)}\right),$$

where $R$ and $P$ are defined in Lemma 2.1, $\lambda_0$ and $c_1$ are defined in (2.5), and $c_0$ is defined in Theorem 1.2. As a consequence of (2.9), we observe that

$$|z_k(s)| \lesssim |\tilde{\mu}_k(s)| + |\tilde{\tau}_k(s)| + |\tilde{y}_k(s)| + |s|^{-1} \lesssim \alpha, \quad \text{with} \quad \tilde{z}_k(s) = \frac{z_k(s)}{s},$$

(2.10)

since, from the definition of $\tilde{z}_k$,

$$\tilde{z}_k = \frac{z_k}{s} = 2\ell_k^{-2}(1 + \bar{y}_k) - 2\ell_k^{-2}(1 + \bar{\mu}_k)(1 + \bar{\tau}_k) + O(|s|^{-1})$$

$$= O(|\bar{y}_k| + |\bar{\mu}_k| + |\bar{\tau}_k| + |s|^{-1}).$$

Similarly, directly from the definition of $\tilde{\mu}_k$ and (2.9), we find

$$|\tilde{\mu}_k(s) - \mu_k(s)| \lesssim |s|^{-1} \quad \text{and} \quad \left|\frac{\tilde{\mu}_k(s)}{\mu_k(s)} - \frac{\mu_k(s)}{\mu_k(s)}\right| \lesssim |s|^{-2}. \quad (2.11)$$

As in [3, 22], we proceed to a simple localization of $\tilde{P}_k$ to avoid some artificial growth at $-\infty$. Let

$$\gamma = \min_{1 \leq k \leq K - 1} \left\{ \frac{1}{4\ell_k} \left( \frac{1}{\ell_{k+1}^2} - \frac{1}{\ell_k^2} \right) \right\} > 0.$$
Let $\chi \in C^\infty(\mathbb{R}, \mathbb{R})$ be such that $0 \leq \chi \leq 1$, $\chi' \geq 0$ on $\mathbb{R}$, $\chi \equiv 1$ on $[-\gamma, +\infty)$ and $\chi \equiv 0$ on $(-\infty, -2\gamma]$. We define

$$P_k(s, y) = \tilde{P}_k(s, y)\chi\left(\frac{y - y_k(s)}{\tilde{\mu}_k(s)}|s|^{-\gamma}\right).$$

Setting for notational purposes $y_{K+1}(s) = y_K(s) - 6\gamma \ell_k|s|$, we prove the following result.

**Lemma 2.2** (Estimates on $P_k$). There exists $\alpha > 0$ small such that the following hold.

For all $m \geq 1$, for all $1 \leq k \leq K$, for all $s \in I$ and for all $y \in \mathbb{R}$,

$$\left\{\begin{aligned}
|P_k(y)| + |(y - y_k)\partial_y^m P_k(y)| &\lesssim e^{-\frac{|y - y_k|}{2\rho_k}} + \frac{1}{2}|y_{k+1} + y_k| < y < y_k(y), \\
|\partial_y^m P_k(y)| &\lesssim e^{-\frac{|y - y_k|}{2\rho_k}} + |s|^{-m} \frac{1}{2}|y_{k+1} + y_k| < y < y_k - \frac{\ell_k\gamma}{2}|s|(y).
\end{aligned}\right. \tag{2.12}$$

In particular, for all $m \geq 1$, for all $1 \leq k \leq K$ and for all $s \in I$,

$$\|(\cdot - y_k)P_k\|_{H^m} + \|P_k\|_{L^2} \lesssim |s|^{\frac{m}{2}} \quad \text{and} \quad \|\partial_y^m P_k\|_{L^2} \lesssim 1. \tag{2.13}$$

**Proof.** From (2.9) and the definitions of $\gamma$ and $\chi$, we first claim that, for all $1 \leq k \leq K$,

$$\chi\left(\frac{y - y_k(s)}{\tilde{\mu}_k(s)}|s|^{-\gamma}\right) = \begin{cases} 0 & \text{for all } y \leq \frac{1}{2}[y_{k+1}(s) + y_k(s)], \\ 1 & \text{for all } y \geq y_k(s) - \frac{\ell_k\gamma}{2}|s|. \end{cases} \tag{2.14}$$

Indeed, if $\chi\left(\frac{y - y_k(s)}{\tilde{\mu}_k(s)}|s|^{-\gamma}\right) < 1$ then $\frac{y - y_k(s)}{\tilde{\mu}_k(s)}|s|^{-\gamma} < -\gamma$ and so, since $\tilde{\mu}_k(s) \geq \frac{\ell_k}{2}$ from (2.9) and (2.11) by taking $\alpha \leq \frac{1}{2}$,

$$y < -\gamma|s|\tilde{\mu}_k(s) + y_k(s) \leq y_k(s) - \frac{\ell_k\gamma}{2}|s|.$$ 

Similarly, if $\chi\left(\frac{y - y_k(s)}{\tilde{\mu}_k(s)}|s|^{-\gamma}\right) > 0$ then $\frac{y - y_k(s)}{\tilde{\mu}_k(s)}|s|^{-\gamma} > -2\gamma$, and so

$$y > -2\gamma|s|\tilde{\mu}_k(s) + y_k(s) \geq -2(1 + \alpha)|s| \left(\gamma \ell_k + \ell_k^{-2}\right).$$

Thus, for $1 \leq k \leq K - 1$,

$$y > \frac{1}{2}(1 + \alpha)|s| \left(\ell_k^{-2} + 3\ell_k^{-2}\right) \geq \frac{1}{2}(1 + \alpha)|s| \left(2\ell_k^{-2} + 2\ell_k^{-2} - 4\ell_k\gamma\right).$$

But, from (2.9), we have

$$-4(1 + \alpha)|s|\ell_k^{-2} \leq y_{k+1}(s) + y_k(s) \leq -2(1 - \alpha)|s|(\ell_k^{-2} + \ell_k^{-2}),$$

and so

$$y > \frac{1}{2}[y_{k+1}(s) + y_k(s)] \left(\frac{1 + \alpha}{1 - \alpha} - \ell_k^3\gamma\right) \geq \frac{1}{2}[y_{k+1}(s) + y_k(s)],$$

by taking $\alpha > 0$ small enough, namely $\alpha \leq \frac{\rho}{2\ell_k^2}$ with $\rho = \ell_k^3\gamma > 0$. For $k = K$, we find similarly, from (2.9) and the definition of $y_{K+1}$ above,

$$y > -2(1 + \alpha)\ell_k^{-2}|s|(1 + \rho) \geq \ell_k^{-2}|s|(2\alpha - 2 - 3\rho) \geq y_K(s) - 3\gamma \ell_k|s| = \frac{1}{2}[y_{K+1}(s) + y_K(s)]$$

by taking again $\alpha > 0$ small enough, namely $\alpha \leq \frac{\rho}{2(\ell_k + 2)}$.

Thus, the claim (2.14) is proved, and we deduce directly (2.12) from this claim, the definition of $P_k$ and the properties of $P$ as recalled in Lemma 2.1. Finally, estimates (2.13) are obtained as direct consequences of (2.12), which concludes the proof of Lemma 2.2. \qed
In order to handle the growth of \( \tilde{P}_k \) at \(-\infty\), we also introduce the local norms

\[
\|f\|_{L^\infty_k} = \sup_{y \in \mathbb{R}} |f(y)e^{-\frac{|y-y_k|}{10\tau_k}}| \quad \text{and} \quad \|f\|_{L^2_k}^2 = \int f^2(y)e^{-\frac{|y-y_k|}{10\tau_k}} dy.
\]

We may now prove the following key result for our analysis, which translates in terms of \( W_k \) the properties satisfied by \( S \) given in Theorems 1.3 and 1.4 and in Corollary A.1.

**Lemma 2.3** (Estimates on \( W_k \)). Let \( 1 \leq k \leq K \) and \( 1 \leq m \leq 20 \). Then the function \( W_k \) satisfies the following, for all \( s \in I \).

(i) Sobolev norms estimates:

\[
\|W_k - Q_k\|_{L^2} \lesssim |s|^{-\frac{1}{2}}, \quad \|W_k - Q_k\|_{H^m} \lesssim |s|^{-1}, \tag{2.15}
\]

and, in particular,

\[
\|W_k - Q_k\|_{L^\infty} \lesssim |s|^{-\frac{1}{2}}. \tag{2.16}
\]

More precisely,

\[
\|W_k - Q_k - \frac{1}{2\tau_k} \tilde{P}_k\|_{H^m} \lesssim s^{-\frac{5}{2}}. \tag{2.17}
\]

(ii) Exponential weighted estimate:

\[
\|W_k - Q_k - \frac{1}{2\tau_k} \tilde{P}_k\|_{L^\infty_k} \lesssim |s|^{-2}. \tag{2.18}
\]

(iii) Pointwise asymptotics on the left: for all \( y \leq y_k - |s|^\frac{1}{4} \),

\[
|W_k(s, y) + \frac{\epsilon_k}{2}\|Q\|_{L^{1,\mu_k}} \sqrt{2\tau_k}|y - z_k|^{-\frac{1}{2}}| \lesssim |s|^{\frac{1}{4}} |y - z_k|^{-\frac{3}{2} - \frac{1}{4}}, \tag{2.19}
\]

\[
|
\partial_{y}^m W_k(s, y)| \lesssim |s|^{\frac{1}{2}} |y - z_k|^{-\frac{5}{2} - m}. \tag{2.20}
\]

(iv) Pointwise bounds on the right: there exists \( \rho_{m-1} > 0 \) such that, for all \( y \in \mathbb{R} \),

\[
|
\partial_{y}^{m-1} W_k(s, y)| \lesssim \exp \left[-\rho_{m-1} \left(\frac{y - y_k}{\mu_k}\right)\right]. \tag{2.21}
\]

(v) Polynomial weighted estimates:

\[
\| (\cdot - y_k)(W_k - Q_k) \|_{H^m} \lesssim |s|^{-\frac{1}{2}}. \tag{2.22}
\]

(vi) Equation:

\[
\partial_s W_k + \frac{1}{2s} \Lambda W_k + \partial_y \left( \partial_{y} W_k + W_k^3 \right) = \tilde{m}_k^0 \cdot \vec{M}_k W_k, \tag{2.23}
\]

with \( \vec{M}_k = \left(-\Lambda_k \right)^\top \), \( \Lambda_k = \frac{1}{2} + (y - y_k) \partial_y \) and

\[
\tilde{m}_k^0 = \left(\begin{array}{c}
\frac{\dot{\mu}_k}{\mu_k} + \frac{1}{2\mu_k^2 \tau_k} - \frac{1}{2s} \\
\dot{y}_k - \frac{1}{2s} - \frac{2c_0}{2\mu_k^2 \tau_k} + \frac{3c_1}{4\mu_k^2 \tau_k}
\end{array}\right).
\]
Proof. (i) First note that (2.9) and (2.17) imply directly the second estimate in (2.15) since $P' \in \mathcal{Y}$ from Lemma 2.1. Note also that (2.16) is a direct consequence of (2.15) from the standard inequality $\|f\|_{L^2} \leq \|f\|_{L^2} \|\partial_y f\|_{L^2}$, valid for any $f \in H^1(\mathbb{R})$. Thus, we just have to prove the first estimate in (2.15) and (2.17).

To prove the first estimate in (2.15), we apply (1.5) with $m = 0$ and obtain, for all $t \in (0, t_0]$, 
\[
\left\| S(t) - \frac{1}{t^{\frac{k}{2}}} Q \left( \frac{\cdot + \frac{1}{t}}{t} + c_0 \right) \right\|_{L^2} \lesssim t.
\]
Thus, applying the change of variables (2.6), we get, for all $s \leq s_0$,
\[
\left\| \tilde{S}(s) - Q (\cdot - 2s + c_0) \right\|_{L^2} \lesssim |s|^{-\frac{1}{2}}
\]
and so, from the definition of $W_k$, for all $s \in I$,
\[
\left\| W_k - \epsilon_k \bar{\mu}_k^{-\frac{1}{2}} Q \left( \frac{\cdot - \frac{y_k}{\bar{\mu}_k} + c_1}{2\tau_k} \right) \right\|_{L^2} \lesssim |s|^{-\frac{1}{2}}.
\]
But, from (2.11) and the a priori estimate (2.9) on $\tau_k$, we have
\[
\left\| \bar{\mu}_k^{-\frac{1}{2}} Q \left( \frac{\cdot - \frac{c_1}{2\tau_k}}{\bar{\mu}_k} \right) - \tilde{\mu}_k^{-\frac{1}{2}} Q \left( \frac{\cdot - \frac{c_1}{\bar{\mu}_k}}{\bar{\mu}_k} \right) \right\|_{L^2} \lesssim |s|^{-1}.
\]
We deduce the first estimate in (2.15) from the two above estimates and the definition of $Q_k$.

To prove (2.17), we proceed similarly and first get from (1.5), applied with any $1 \leq m \leq 20$, for all $t \in (0, t_0)$,
\[
\left\| \partial_x^m S(t) - \frac{1}{t^{\frac{k}{2} + m}} Q^{(m)} \left( \frac{\cdot + \frac{1}{t}}{t} + c_0 \right) - \frac{1}{t^{\frac{k}{2} + m - 2}} Q^{(m-1)} \left( \frac{\cdot + \frac{1}{t}}{t} + c_0 \right) \right\|_{L^2} \lesssim t^{3-m}.
\]
Note that we may obtain a sharper estimate in the case $m \geq 2$, but the above one will be enough for our purpose. Thus, applying the change of variables (2.6), we get, for all $s \leq s_0$,
\[
\left\| \partial_y^m \tilde{S}(s) - Q^{(m)} (-2s + c_0) + \frac{1}{2s} Q^{(m-1)} (-2s + c_0) \right\|_{L^2} \lesssim |s|^{-\frac{3}{2}}
\]
or equivalently, from the expression (2.5) of $Q_1$,
\[
\left\| \tilde{S}(s) - \left[ Q + \frac{1}{2s} P + \lambda_0 \frac{1}{2s} \Lambda Q - \frac{c_1}{2s} Q' \right] (-2s + c_0) \right\|_{\dot{H}^1} \lesssim |s|^{-\frac{3}{2}}.
\]
Noticing that, from a Taylor expansion with remainder of integral form, for all $j \geq 0$,
\[
\left\| \left( 1 + \frac{\lambda_0}{2s} \right)^{\frac{1}{2}} Q \left[ \left( 1 + \frac{\lambda_0}{2s} \right) \left( - \frac{c_1}{2s} \right) \right] - \left( Q + \frac{\lambda_0}{2s} \Lambda Q - \frac{c_1}{2s} Q' \right) \right\|_{\dot{H}^j} \lesssim |s|^{-2},
\]
we obtain
\[
\left\| \tilde{S}(s) - \left( 1 + \frac{\lambda_0}{2s} \right)^{\frac{1}{2}} \left( Q + \frac{1}{2s} P \right) \left[ \left( 1 + \frac{\lambda_0}{2s} \right) \left( -2s + c_0 - \frac{c_1}{2s} \right) \right] \right\|_{\dot{H}^m} \lesssim |s|^{-\frac{3}{2}}.
\]
From the definitions of $\tilde{\mu}_k$ and $W_k$, it gives, for all $s \in I$,
\[
\left\| W_k - \epsilon_k \tilde{\mu}_k^{-\frac{1}{2}} \left( Q + \frac{1}{2\tau_k} P \right) \left( \cdot - \frac{y_k}{\tilde{\mu}_k} \right) \right\|_{\dot{H}^m} \lesssim |s|^{-\frac{3}{2}},
\]
then (2.17) from the definitions of $Q_k$ and $\tilde{P}_k$. 

(ii) The proof of (2.18) follows closely the one of (2.17) above. From (A.2) applied with \( B = 10, M = 1 \) and \( m = 0,1 \), we first obtain, for all \( (0,t_0) \),
\[
\left\| \left[ S(t) - \frac{1}{t^{\frac{1}{2}}} \left( Q + t^2 Q_1^{(-1)} \right) \left( \frac{\lambda_0}{2s} (\cdot - 2s + c_0) \right) e^{-\frac{|\cdot - 2s + c_0|}{10}} \right] \right\|_{L^\infty} \lesssim t^{\frac{3}{2}},
\]
where \( Q_1^{(-1)} = - P - \lambda_0 \Lambda Q + c_1 Q' \) from (1.9), (2.3) and (2.5), and so, by (2.6),
\[
\left\| \left[ S(s) - \left( Q + \frac{1}{2s} P + \lambda_0 \Lambda Q - \frac{c_1}{2s} Q' \right) (\cdot - 2s + c_0) \right] e^{-\frac{|\cdot - 2s + c_0|}{10}} \right\|_{L^\infty} \lesssim |s|^{-2}.
\]
Using (2.24) for \( j = 0,1 \), we get, for all \( s \leq s_0 \),
\[
\left\| \left[ S(s) - \left( 1 + \frac{\lambda_0}{2s} \right) \left( Q + \frac{1}{2s} P \right) \left( \frac{\cdot - y_k}{\mu_k} \right) \right] e^{-\frac{|\cdot - y_k|}{10 \mu_k}} \right\|_{L^\infty} \lesssim |s|^{-2},
\]
From the definitions of \( \mu_k \) and \( W_k \), it gives, for all \( s \in I \),
\[
\left\| \left[ W_k - \epsilon_k \frac{1}{\mu_k} \left( Q + \frac{1}{2s} P \right) \left( \frac{\cdot - y_k}{\mu_k} \right) \right] e^{-\frac{|\cdot - y_k|}{10 \mu_k}} \right\|_{L^\infty} \lesssim |s|^{-2},
\]
then (2.18) from the definitions of \( Q_k, \tilde{P}_k \) and \( L_k^\infty \).

(iii) From (1.6), (1.7) and (2.6), we obtain, for all \( y \leq 2s - \sqrt{2s} \),
\[
\left| \hat{S}(s,y) + \frac{1}{2} \|Q\|_{L^1} \sqrt{2s} |y|^{-\frac{3}{4}} \right| \lesssim |s|^{-\frac{1}{2} + \frac{1}{4}} |y|^{-\frac{3}{4}} \quad \text{and} \quad |\partial_y^m \hat{S}(s,y)| \lesssim |s|^\frac{1}{2} |y|^{-\frac{3}{4} - m}.
\]
We deduce (2.19) and (2.20) from these two estimates, the \( a \) \( \text{priori} \) estimates (2.9) and the definition of \( W_k \).

(iv) From (1.8) and (2.6), we obtain, for all \( s \leq s_0 \) and all \( y \in \mathbb{R} \),
\[
|\partial_y^{-1} \hat{S}(s,y)| \lesssim \exp \left[ - \gamma_{m-1} (y - 2s) \right],
\]
which gives, from the definition of \( W_k \) and (2.11), for all \( s \in I \) and all \( y \geq y_k \),
\[
|\partial_y^{-1} W_k(s,y)| \lesssim \exp \left[ - \gamma_{m-1} \left( \frac{y - y_k}{\mu_k} \right) \right] \lesssim \exp \left[ \frac{2}{3} \gamma_{m-1} \left( \frac{y - y_k}{\mu_k} \right) \right],
\]
then (2.21) by letting \( \rho_{m-1} = \frac{2}{3} \gamma_{m-1} > 0 \). Note that (2.21) holds also obviously for \( y \leq y_k \) since \( \| \partial_y^{m-1} W_k \|_{L^\infty} \lesssim 1 \) from (2.15).

(v) To prove (2.22), we first notice that \( \| \partial_y^{m-1}(W_k - Q_k) \|_{L^2} \lesssim |s|^{-\frac{1}{2}} \) from (2.15). Then we decompose \( \| (\cdot - y_k) \partial_y^m (W_k - Q_k) \|_{L^2} \) on the three regions \( y < 2y_k, 2y_k \leq y \leq 0 \) and \( y > 0 \). Indeed, using first (2.20) and the exponential decay of \( Q \), we obtain
\[
\| (\cdot - y_k) \partial_y^m (W_k - Q_k) \|_{L^2(y < 2y_k)} \lesssim |s| \int_{y < 2y_k} (y_k - y)^2 (\frac{y_k}{\mu_k} - y)^{-2m} dy + \int_{y < 2y_k} (y_k - y)^2 e^{\frac{2(y - y_k)}{\mu_k}} dy \lesssim |s| |y_k|^{-2} + e^{y_k/\mu_k} \lesssim |s|^{-1},
\]
from (2.9), (2.10) and the fact that we assume \( m \geq 1 \). Next, from (2.17), we get
\[
\| (\cdot - y_k) \partial_y^m (W_k - Q_k) \|_{L^2(2y_k \leq y \leq 0)} \lesssim |s| \left\| W_k - Q_k - \frac{1}{2\tau_k} \tilde{P}_k \right\|_{H^n} + |s|^{-1} \| (\cdot - y_k) \partial_y^m \tilde{P}_k \|_{L^2} \lesssim |s|^{-\frac{1}{2}},
\]
using also the exponential decay of $P' \in \mathcal{V}$. Finally, from (2.21) and the exponential decay of $Q$, we obtain, with $\rho_m' = \min(\rho_m, 1) > 0$,

$$\|(-yk)\partial_y^m (W_k - Q_k)\|_{L^2(y > 0)}^2 \lesssim \int_{y > 0} e^{-\rho_m' \left( \frac{y}{\mu_k} \right)} dy \lesssim e^{\rho_m' y_k / \mu_k} \lesssim |s|^{-10}.$$  

Gathering the above estimates, we obtain (2.22).

(vi) First, note that $\epsilon_k \tilde{S}$ satisfies (2.7) and

$$\epsilon_k \tilde{S}(\tau_k(s), y) = \mu_k^2 (s) W_k(s, \mu_k(s)y + z_k(s)).$$

Using (2.8), we compute

$$\epsilon_k \partial_y \tilde{S}(\tau_k, y) = \mu_k^2 \partial_y \left[ \epsilon_k \tilde{S}(\tau_k, y) \right] = \mu_k^2 \left[ \frac{\tilde{\mu}_k W_k}{2} + \partial_y W_k + \tilde{\mu}_k y \partial_y W_k + \dot{z}_k \partial_y W_k \right] (s, \mu_k y + z_k)$$

and

$$\frac{\epsilon_k}{2} \tilde{S}(\tau_k, y) = \mu_k^2 \left[ \frac{1}{2 \tau_k} \frac{W_k}{2} + \mu_k y \partial_y W_k \right] (s, \mu_k y + z_k)$$

then

$$\epsilon_k \partial_y (\partial_{yy} \tilde{S} + \tilde{S}^5)(\tau_k, y) = \mu_k^2 \left[ \partial_y \left( \partial_{yy} W_k + W_k^5 \right) \right] (s, \mu_k y + z_k).$$

Thus, summing the above terms and using the definition of $z_k$, $W_k$ satisfies the equation (2.23), which concludes the proof of Lemma 2.3. \hfill \Box

For $1 \leq k \leq K$, we consider additional $C^1$ functions $r_k$, $d_k$ and $a_k$ defined on $I$ by

$$r_k(s) = \epsilon_k \tilde{\mu}_k^2 (s) \sum_{j \neq k} W_j(s, y_k(s)), \quad d_k(s) = \epsilon_k \tilde{\mu}_k^2 (s) \sum_{j \neq k} \partial_y W_j(s, y_k(s)),$$

and $a_k$ to be determined later. We assume that $a_k$ satisfies, for all $s \in I$,

$$|a_k(s)| \lesssim |s|^{-1},$$

and we observe that, from (2.9) and (2.19)–(2.21),

$$|r_k(s)| \lesssim |s|^{-1}, \quad |d_k(s)| \lesssim |s|^{-2}.\quad (2.27)$$

We will prove more precise asymptotics on $r_k$ and $d_k$ in Lemma 2.5 below.

Finally, for all $s \in I$ and $y \in \mathbb{R}$, let

$$V_k(s, y) = W_k(s, y) + r_k(s)R_k(s, y) + a_k(s)P_k(s, y)$$

and define

$$\mathbf{W} = \sum_{k=1}^K W_k \quad \text{and} \quad \mathbf{V} = \sum_{k=1}^K V_k = \mathbf{W} + \sum_{k=1}^K (r_k R_k + a_k P_k).$$
Note that, by (2.13), (2.15)–(2.16), (2.22) and (2.26)–(2.27), we have, for all $1 \leq k \leq K$, all $1 \leq m \leq 19$ and all $s \in I$,

\[
\begin{align*}
&\left\| (\cdot - y_k)(V_k - Q_k) \right\|_{H^m} + \left\| V_k - Q_k \right\|_{L^2} \lesssim |s|^{-\frac{3}{2}}, \\
&\left\| V_k - Q_k \right\|_{L^\infty} \lesssim |s|^{-\frac{1}{2}}, \\
&\left\| V_k - Q_k \right\|_{H^m} \lesssim |s|^{-1}, \\
&\left\| \partial_y^{m-1} (V_k - Q_k) \right\|_{L^\infty} \lesssim |s|^{-1},
\end{align*}
\]  

(2.28)

and, in particular, $\left\| \partial_y^{m-1} V \right\|_{L^2} + \left\| \partial_y^{m-1} V \right\|_{L^\infty} \lesssim 1$.

In the next lemma, we prove that such an ansatz $V$ is indeed an approximate solution of (2.7), in a precise sense. In Lemma 2.6 below, we also estimate the mass and the energy of $V$, relying on the sharp estimates of Lemma 2.5.

**Lemma 2.4 (Approximate rescaled multi-soliton).** The error of the flow (2.7) at $V$, defined as

\[
\mathcal{E}_V = \partial_s V + \frac{1}{2s} \Lambda V + \partial_y (\partial_y V + V^5),
\]

decomposes as

\[
\mathcal{E}_V = \sum_j \tilde{m}_j \cdot \tilde{M}_j V_j + \sum_j (\dot{r}_j R_j + \dot{a}_j P_j) + \Psi
\]

(2.29)

where, for all $1 \leq j \leq K$,

\[
\tilde{m}_j = \left( \begin{array}{c} m_{j,1} \\ m_{j,2} \end{array} \right) = \left( \begin{array}{c} \hat{\mu}_j \frac{1}{\mu_j} + \frac{1}{2\mu_j^3 \tau_j} - \frac{1}{2s} \mu_j^3 \tau_j + \frac{a_j}{\mu_j^3} \\ \gamma_j - \frac{1}{\mu_j} - \frac{c_0}{2\mu_j^3 \tau_j} + \frac{3c_1}{4\mu_j^3 \tau_j} \end{array} \right) = \tilde{m}_j^0 + \left( \begin{array}{c} a_j \\ \mu_j^3 \end{array} \right),
\]

(2.30)

and $\tilde{m}_j^0$ and $\tilde{M}_j$ are defined in (vi) of Lemma 2.3. Moreover, for all $s \in I$,

\[
\| \Psi \|_{H^2} \lesssim |s|^{-\frac{3}{2}} + |s|^{-\frac{1}{2}} \sum_j |a_j|,
\]

(2.31)

and, for any $1 \leq k \leq K$,

\[
\| \Psi \|_{H^2(y > y_k, |s|^{\frac{1}{4}})} \lesssim |s|^{-\frac{13}{8}} + |s|^{-\frac{1}{2}} \sum_{j < k} |a_j|, \quad \| \Psi \|_{L^\infty} \lesssim |s|^{-2},
\]

(2.32)

\[
\langle \Psi, Q_k \rangle - \Omega_k \lesssim |s|^{-\frac{5}{4}},
\]

(2.33)

with

\[
\Omega_k(s) = \frac{\| Q \|_{L^1}^2}{8 \mu_k^3(s)} \left[ a_k(s) + a_k^2(s) \right] + \frac{\| Q \|_{L^1}^2}{\mu_k^3(s)} \delta_k(s).
\]

**Proof. Equation of $V$.** We insert the definition of $V = \sum_j (W_j + r_j R_j + a_j P_j)$ in $\mathcal{E}_V$ and, using the equation (2.23) satisfied by $W_j$, we obtain

\[
\mathcal{E}_V = \sum_j \tilde{m}_j \cdot \tilde{M}_j W_j + \sum_j (\dot{r}_j R_j + \dot{a}_j P_j) + |s|^{-1} \sum_j a_j Z_j
\]

\[
- \sum_j \frac{\dot{\mu}_j}{\mu_j} \Lambda_j (r_j R_j + a_j P_j) - \sum_j \gamma_j \partial_y (r_j R_j + a_j P_j) + \frac{1}{2s} \sum_j \Lambda (r_j R_j + a_j P_j)
\]

\[
+ \sum_j \partial_y y (r_j R_j + a_j P_j) + \partial_y \left( V^5 - \sum_j W_j^5 \right),
\]
where we have denoted

\[
Z_j(s, y) = \left( \frac{y - y_j(s)}{\tilde{\mu}_j(s)} \right) \varepsilon_j \tilde{\mu}_j^{-\frac{2}{3}}(s) P \left( \frac{y - y_j(s)}{\tilde{\mu}_j(s)} \right) \chi' \left( \frac{y - y_j(s)}{\tilde{\mu}_j(s)} |s|^{-1} \right). \tag{2.34}
\]

Since \( \frac{\dot{\mu}_j}{\mu_j} = \frac{\dot{\mu}_j}{\mu_j} + \frac{\lambda_0}{2\mu_j^3 \tau_j} \left( 1 + \frac{\lambda_0}{2\tau_j} \right)^{-1} \) from the definition of \( \tilde{\mu}_j \), we may decompose \( \mathcal{E}_V \) as

\[
\mathcal{E}_V = \sum_j \tilde{m}_j \cdot \tilde{M}_j V_j + \sum_j (\dot{r}_j R_j + \dot{a}_j P_j) + \Psi,
\]

where we have set

\[
\Psi = |s|^{-1} \sum_j a_j Z_j + \sum_j \left[ \frac{1}{2\mu_j^3 \tau_j} + \frac{a_j}{\mu_j^3} - \frac{\lambda_0}{2\mu_j^3 \tau_j} \left( 1 + \frac{\lambda_0}{2\tau_j} \right)^{-1} \right] \Lambda_j (r_j R_j + a_j P_j)
+ \sum_j \left[ -\frac{c_0}{2\mu_j^3 \tau_j} + \frac{3c_1}{4\mu_j^3 \tau_j^2} \right] \partial_y (r_j R_j + a_j P_j)
+ \sum_j a_j \mu_j^{-3} \Lambda_j W_j + \sum_j a_j \partial_y \left( \partial_{yy} P_j - \mu_j^{-2} P_j + 5Q_j^4 P_j \right)
+ \sum_j r_j \partial_y \left( \partial_{yy} R_j - \mu_j^{-2} R_j + 5Q_j^4 R_j \right) + \partial_y \left[ V^5 - \sum_j W_j^5 - 5 \sum_j Q_j^4 (r_j R_j + a_j P_j) \right]
= (\Psi^I + \Psi^{II}) + \Psi^{III} + \Psi^{IV} + \Psi^V.
\]

**Estimate of \( \Psi^I \).** We rely on (2.3) and (2.14) to estimate \( Z_j \). For instance, we first note that \( \| Z_j \|_{H^2} \lesssim |s|^\frac{1}{4} \). Thus,

\[
\| \Psi^I \|_{H^2} \lesssim |s|^\frac{1}{4} \sum_j |a_j|.
\]

Moreover, we observe that, for \( j \geq k \), \( Z_j(y) = 0 \) for \( y > y_k - |s|^\frac{1}{4} \), and so

\[
\| \Psi^I \|_{H^2(y > y_k - |s|^\frac{1}{4})} \lesssim |s|^{\frac{1}{2}} \sum_{j < k} |a_j|.
\]

Finally, we note that \( \| Z_j \|_{L^\infty_k} \lesssim |s|^{-10} \) for all \( 1 < j < K \). Thus, using also the exponential decay of \( Q \), we get

\[
\| \langle \Psi^I, Q_k \rangle \| \lesssim \| \Psi^I \|_{L^\infty_k} \lesssim |s|^{-12}.
\]

**Estimate of \( \Psi^{II} \).** First, from (2.9) and (2.26), we notice that

\[
\left| \frac{1}{2\mu_j^3 \tau_j} + \frac{a_j}{\mu_j^3} \right| \lesssim |s|^{-1} \quad \text{and} \quad \left| \frac{\lambda_0}{2\mu_j^3 \tau_j^2} \left( 1 + \frac{\lambda_0}{2\tau_j} \right)^{-1} \right| \lesssim |s|^{-2}.
\]

Second, from the exponential decay of \( R \in \mathcal{Y} \) and (2.13), we have \( \| \Lambda_j R_j \|_{H^2} \lesssim 1 \) and \( \| \Lambda_j P_j \|_{H^2} \lesssim |s|^\frac{1}{4} \). Thus, using also (2.27), we find

\[
\| \Psi^{II} \|_{H^2} \lesssim |s|^{-2} + |s|^{-\frac{1}{4}} \sum_j |a_j|.
\]
Moreover, since $\|\Lambda_j P_j\|_{H^2(y > y_k - |x|^{1/4})} \lesssim |s|^{1/4}$ for $j \geq k$ from (2.12), we have

$$
\|\Psi^II\|_{H^2(y > y_k - |x|^{1/4})} \lesssim |s|^{-2} + |s|^{-2+1/8} + |s|^{-1/2} \sum_{j < k} |a_j| \lesssim |s|^{-1/8} + |s|^{-1/2} \sum_{j < k} |a_j|.
$$

Similarly, since $\|\Lambda_j P_j\|_{L^\infty_k} \lesssim |s|^{-10}$ for $j \neq k$ and $\|\Lambda_k P_k\|_{L^\infty_k} \lesssim 1$, we find

$$
\|\Psi^II\|_{L^\infty_k} \lesssim |s|^{-2}.
$$

Finally, projecting on $Q_k$, observing that $\langle \Lambda_k R_k, Q_k \rangle = \langle AR, Q \rangle$ and, from the properties of $\chi$, $\langle \Lambda_k P_k, Q_k \rangle = \langle AP, Q \rangle + O(|s|^{-10})$, we obtain, using also (2.14),

$$
\left|\langle \Psi^II, Q_k \rangle - \left(\frac{r_k}{2\mu^2_k \tau_k} + \frac{a_k r_k}{\mu^2_k}\right) \langle AR, Q \rangle - \left(\frac{a_k}{2\mu^2_k \tau_k} + \frac{a_k^2}{\mu^2_k}\right) \langle AP, Q \rangle\right| \lesssim |s|^{-3}.
$$

**Estimate of $\Psi^III$.** We proceed as in the estimate of $\Psi^II$ except that, from Lemma 2.1, $\langle R_k, \partial_y Q_k \rangle = \tilde{\mu}_k^{-1}(R, Q') = 0$ by parity, and

$$
\langle P_k, \partial_y Q_k \rangle = \langle \bar{P}_k, \partial_y Q_k \rangle + O(|s|^{-10}) = \tilde{\mu}_k^{-1} \langle P, Q' \rangle + O(|s|^{-10}) = O(|s|^{-10}).
$$

Since we also have $\|\partial_y P_j\|_{H^2} + \|\partial_y R_j\|_{H^2} \lesssim 1$ from (2.13), we obtain 

$$
\|\Psi^III\|_{H^2} \lesssim |s|^{-2} \quad \text{and} \quad |\langle \Psi^III, Q_k \rangle| \lesssim |s|^{-12}.
$$

**Estimate of $\Psi^IV$.** For $(l, m) \in \mathbb{N}^2$, set

$$
P_j^{(l,m)}(s, y) = \epsilon_j \tilde{\mu}_j^{-2+(l^2+m^2)/2} \left(\frac{y - y_j(s)}{\tilde{\mu}_j(s)}\right)^m \chi(m) \left(\frac{y - y_j(s)}{\tilde{\mu}_j(s)}\right)^{-1}.
$$

Note for instance that $P_j^{(0,0)} = P_j$. From the relation $(LP)' = \Lambda Q$ in (2.2), we find

$$
\partial_y \left(\partial_{yy} \bar{P}_j - \tilde{\mu}_j^{-2} \bar{P}_j + 5Q^4_j P_j\right) = -\tilde{\mu}_j^{-3} \Lambda_j Q_j,
$$

and so

$$
\partial_y \left(\partial_{yy} P_j - \mu_j^{-2} P_j + 5Q^4_j P_j\right)
$$

$$
= -\tilde{\mu}_j^{-3} \Lambda_j Q_j \chi \left(\frac{-y_j(s)}{\tilde{\mu}_j(s)}|s|^{-1}\right) + \tilde{\mu}_j \left(\tilde{\mu}_j^{-2} - \mu_j^{-2}\right) P_j^{(1,0)} - \mu_j^{-2} |s|^{-1} P_j^{(0,1)}
$$

$$
+ 5|s|^{-1} Q^4_j P_j^{(0,1)} + 3|s|^{-1} P_j^{(2,1)} + 3|s|^{-2} P_j^{(1,2)} + |s|^{-3} P_j^{(0,3)}.
$$

Thus,

$$
\Psi^IV = \sum_j a_j \mu_j^{-3} \Lambda_j (W_j - Q_j) + \sum_j a_j \tilde{\mu}_j^{-2} \Lambda_j Q_j \left[1 - \chi \left(\frac{-y_j(s)}{\tilde{\mu}_j(s)}|s|^{-1}\right)\right]
$$

$$
+ \sum_j a_j (\mu_j^{-3} - \tilde{\mu}_j^{-3}) \Lambda_j Q_j + \sum_j a_j \tilde{\mu}_j (\tilde{\mu}_j^{-2} - \mu_j^{-2}) P_j^{(1,0)} - \sum_j a_j \mu_j^{-2} |s|^{-1} P_j^{(0,1)}
$$

$$
+ \sum_j a_j |s|^{-1} \left[5Q^4_j P_j^{(0,1)} + 3P_j^{(2,1)} + 3|s|^{-1} P_j^{(1,2)} + |s|^{-2} P_j^{(0,3)}\right]
$$

$$
= \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4 + \Sigma_5 + \Sigma_6.
$$
By the properties of $P$ and $\chi$, we have
\[
\|Q_j^4 P_j^{(0,1)}\|_{H^2} + \|P_j^{(2,1)}\|_{H^2} + \|P_j^{(1,2)}\|_{H^2} \lesssim |s|^{-10},
\]
\[
\|\Lambda_j Q_j\|_{H^2} + \|P_j^{(1,0)}\|_{H^2} \lesssim 1, \quad \|P_j^{(0,1)}\|_{H^2} + \|P_j^{(0,3)}\|_{H^2} \lesssim |s|^{\frac{1}{2}}.
\]
Thus, using also (2.11) and (2.26), we obtain
\[
\|\Sigma_3\|_{H^2} + \|\Sigma_4\|_{H^2} \lesssim |s|^{-2}, \quad \|\Sigma_5\|_{H^2} \lesssim |s|^{-\frac{5}{2}} \sum_j |a_j| \quad \text{and} \quad \|\Sigma_6\|_{H^2} \lesssim |s|^{-\frac{7}{2}}.
\]
By (2.14) and the exponential decay of $Q$, we also have $\|\Sigma_2\|_{H^2} \lesssim |s|^{-10}$. Now, from (2.15) and (2.22), we notice that
\[
\|\Lambda_j (W_j - Q_j)\|_{H^2} \lesssim |s|^{-\frac{1}{2}}, \quad \text{and so} \quad \|\Sigma_1\|_{H^2} \lesssim |s|^{-\frac{7}{2}} \sum_j |a_j|.
\]
Thus, gathering the previous estimates, we have obtained
\[
\|\Psi^4\|_{H^2} \lesssim |s|^{-2} + |s|^{-\frac{7}{2}} \sum_j |a_j|.
\]
Moreover, we observe as before from (2.14) that, for $j \geq k$, $P_j^{(0,1)}(y) = 0$ for $y > y_k - |s|^{\frac{1}{2}}$, and so
\[
\|\Sigma_5\|_{H^2(y>y_k-|s|^{\frac{1}{2}})} \lesssim |s|^{-\frac{5}{2}} \sum_{j<k} |a_j|.
\]
Next, we claim that, for all $j \geq k$,
\[
\|\Lambda_j (W_j - Q_j)\|_{H^2(y>y_k-|s|^{\frac{1}{2}})} \lesssim |s|^{-\frac{2}{3}}.
\]
Note that it is enough to prove the claim for $j = k$ since $y_j \leq y_k$. To do so, we first notice that
\[
\|W_k - Q_k\|_{H^m(y>y_k-|s|^{\frac{1}{2}})} \lesssim |s|^{-1}
\]
for $m = 1, 2$ from (2.15). Second, using (2.16) and (2.21), we estimate
\[
\|W_k - Q_k\|_{L^2(y>y_k-|s|^{\frac{1}{2}})} = \|W_k - Q_k\|_{L^2(y_k-|s|^{\frac{1}{2}} < y < y_k + |s|^{\frac{1}{2}})} + \|W_k - Q_k\|_{L^2(y_k + |s|^{\frac{1}{2}})}
\]
\[
\lesssim \|W_k - Q_k\|_{L^\infty} \lesssim |s|^{\frac{1}{2}} \mu_k + e^{-2\rho_0 |s|^{\frac{1}{2}}/\mu_k} + e^{-2|s|^{\frac{1}{2}}/\mu_k} \lesssim |s|^{-\frac{3}{4}}.
\]
Next, to estimate $\|(-y_k)\partial_y^m (W_k - Q_k)\|_{L^2(y>y_k-|s|^{\frac{1}{2}})}$ for $m = 1, 2, 3$, we proceed similarly and find, using (2.15) and again (2.21),
\[
\|(-y_k)\partial_y^m (W_k - Q_k)\|_{L^2(y>y_k-|s|^{\frac{1}{2}})} \lesssim |s|^{\frac{1}{2}} \|W_k - Q_k\|_{H^m} + e^{-\rho_0 |s|^{\frac{1}{2}}/\mu_k} + e^{-|s|^{\frac{1}{2}}/\mu_k} \lesssim |s|^{-\frac{3}{2}}.
\]
Thus the claim is proved and, together with the above estimates, it gives
\[
\|\Psi^4\|_{H^2(y>y_k-|s|^{\frac{1}{2}})} \lesssim |s|^{-\frac{13}{8}} + |s|^{-\frac{1}{2}} \sum_{j<k} |a_j|.
\]
Now, we control $\|\Sigma_1\|_{L^\infty_k}$ and $\|\Sigma_5\|_{L^\infty_k}$. First, by (2.14), we find $\|P_j^{(0,1)}\|_{L^\infty_k} \lesssim |s|^{-10}$ for all $1 \leq j \leq K$, and so
\[
\|\Sigma_5\|_{L^\infty_k} \lesssim |s|^{-12}.
\]
Moreover, as before, we have \( \|\Lambda_k(W_k - Q_k)\|_{L^\infty_k} \lesssim |s|^{-1} \). But, for \( j \neq k \), by (2.9), (2.10) and (2.19)–(2.21), we have \( \|\Lambda_j(W_j - Q_j)\|_{L^\infty_k} \lesssim |s|^{-10} \). Thus, using (2.26), we obtain \( \|\Sigma_1\|_{L^\infty_k} \lesssim |s|^{-2} \), which proves, together with the above estimates,

\[
\|\Psi^V\|_{L^\infty_k} \lesssim |s|^{-2}.
\]

Finally, we look at the projection on \( Q_k \). Note that, from (2.18),

\[
\langle \Lambda_k(W_k - Q_k), Q_k \rangle = -\langle W_k - Q_k, \Lambda_k Q_k \rangle = -\frac{1}{2\tau_k} \langle \tilde{P}_k, \Lambda_k Q_k \rangle + O(|s|^{-2})
\]

\[
= -\frac{1}{2\tau_k} \langle P, \Lambda Q \rangle + O(|s|^{-2}) = \frac{1}{2\tau_k} \langle \Lambda P, Q \rangle + O(|s|^{-2}).
\]

Moreover, as before, we have \( \|\langle \Lambda_j(W_j - Q_j), Q_k \rangle\| \lesssim \|\Lambda_j(W_j - Q_j)\|_{L^\infty_k} \lesssim |s|^{-10} \) for \( j \neq k \), and similarly \( \|\langle \Sigma_5, Q_k \rangle\| \lesssim \|\Sigma_5\|_{L^\infty_k} \lesssim |s|^{-12} \). Also, since \( \langle \Lambda_k Q_k, Q_k \rangle = \langle \Lambda Q, Q \rangle = 0 \) and \( \langle P_{k}^{(1,0)}, Q_k \rangle = -\tilde{\mu}_k^{-1} \langle P, Q' \rangle + O(|s|^{-10}) = O(|s|^{-10}) \) from (2.2), we find \( \|\langle \Sigma_3 + \Sigma_4, Q_k \rangle\| \lesssim |s|^{-12} \). To conclude for this term, we have obtained, using also (2.26),

\[
\left| \langle \Psi^V, Q_k \rangle - \frac{a_k}{2\mu_k \tau_k} \langle \Lambda P, Q \rangle \right| \lesssim |s|^{-3}.
\]

**Decomposition of \( \Psi^V \).** From the relation \( LR = 5Q^4 \) in (2.4), we find

\[
\partial_y R_j - \tilde{\mu}_j^{-2} R_j + 5Q_j^4 R_j = -5\epsilon_j \tilde{\mu}_j^{-\frac{5}{2}} Q_j^4,
\]

and so

\[
\Psi^V = \partial_y \left[ V^5 - \sum_j W_j^5 - 5 \sum_j Q_j^4 (r_j R_j + a_j P_j) + \sum_j r_j (\tilde{\mu}_j^{-2} - \mu_j^{-2}) R_j - 5 \sum_j r_j \epsilon_j \tilde{\mu}_j^{-\frac{5}{2}} Q_j^4 \right].
\]

Thus, we may further decompose \( \Psi^V \) as

\[
\Psi^V = \Psi^{VI} + \Psi^{VII} + \Psi^{VIII},
\]

with

\[
\Psi^{VI} = \partial_y \left[ V^5 - W^5 - 5 \sum_j Q_j^4 (r_j R_j + a_j P_j) \right],
\]

\[
\Psi^{VII} = \partial_y \left[ W^5 - \sum_j W_j^5 - 5 \sum_j r_j \epsilon_j \tilde{\mu}_j^{-\frac{5}{2}} Q_j^4 \right],
\]

\[
\Psi^{VIII} = \partial_y \left[ \sum_j r_j (\tilde{\mu}_j^{-2} - \mu_j^{-2}) R_j \right].
\]

**Estimate of \( \Psi^{VI} \).** A binomial expansion first gives

\[
V^5 - W^5 = \left[ V + (V - W) \right]^5 - W^5 = \sum_{i=1}^{5} \binom{5}{i} W^{5-i}(V - W)^i.
\]
We estimate each term of the sum separately, and we recall that $V - W = \sum_j (r_j R_j + a_j P_j)$. First, for $i = 2$, by (2.15) and (2.26)–(2.27),

$$
\|W^3(V - W)^2\|_{H^3} \lesssim \|W\|_{H^3}^3 \sum_{p=0}^3 \|\partial_y^p(V - W)\|_{L^\infty} \lesssim \sum_j (r_j^2 + a_j^2) \lesssim |s|^{-2}.
$$

Similarly, for $i = 3, 4$,

$$
\|W^{5-i}(V - W)^i\|_{H^3} \lesssim \|W\|_{H^3}^5 \sum_{p=0}^3 \|\partial_y^p(V - W)\|_{L^\infty} \lesssim |s|^{-i}.
$$

And, for $i = 5$, using also (2.13),

$$
\|(V - W)^5\|_{H^3} \lesssim \|V - W\|_{H^3} \sum_{j=1}^K (\|\partial_y(V - W)\|_{H^3}^4 \lesssim |s|^{-\frac{1}{2}}|s|^{-4} = |s|^{-\frac{3}{2}}.
$$

To estimate the term corresponding to $i = 1$,

$$
W^4(V - W) = \left(\sum_j W_j\right)^4 \left(\sum_j (r_j R_j + a_j P_j)\right) = \sum_{j_1, \ldots, j_5} (r_{j_1} R_{j_1} + a_{j_1} P_{j_1}) \prod_{l=2}^5 W_{j_l},
$$

we rely on the following claim. Let $2 \leq p \leq 5$. Let $j_1, \ldots, j_p \in \{1, \ldots, K\}$ with $j_1 \neq j_i$ for $2 \leq l \leq p$. Then, from the decay properties of $W_{j_l}, R_{j_l}$ and $P_{j_l}$, we have

$$
\left\|R_{j_1} W_{j_1}^{5-p} \prod_{l=2}^p W_{j_l}\right\|_{H^3} \lesssim |s|^{p+1} \quad \text{and} \quad \left\|P_{j_1} W_{j_1}^{5-p} \prod_{l=2}^p W_{j_l}\right\|_{H^3} \lesssim |s|^{-p+\frac{3}{2}}.
$$

Indeed, if $j_1 > j_2, \ldots, j_p$, we first find, decomposing on the two regions $y < y_{j_1} + |s|^{\frac{1}{4}}$ and $y > y_{j_1} + |s|^{\frac{1}{4}}$, and using (2.19),

$$
\left\|R_{j_1} W_{j_1}^{5-p} \prod_{l=2}^p W_{j_l}\right\|_{L^2}^2 \lesssim |s|^{p-1} \int_{y < y_{j_1} + |s|^{\frac{1}{4}}} \left( e^{-\frac{|y-y_{j_1}|}{\rho_{j_1}} \prod_{l=2}^p |y - z_{j_l}|^{-3}} dy + e^{-|s|^{\frac{1}{4}}/\bar{\mu}_{j_1}}\right)
\lesssim |s|^{p-1} |s|^{-3(p-1)} \int e^{-\frac{|y-y_{j_1}|}{\rho_{j_1}} dy + |s|^{-10} \lesssim |s|^{-2(p-1)}.
$$

Note that such an estimate also holds in $\dot{H}^m$ for $m = 1, 2, 3$ from (2.20). In the case where there exists $2 \leq l \leq p$ such that $j_l > j_1$, we find, decomposing on the regions $y < y_{j_1} - |s|^{\frac{1}{4}}$ and $y > y_{j_1} - |s|^{\frac{1}{4}}$, and using (2.21),

$$
\left\|R_{j_1} W_{j_1}^{5-p} \prod_{l=2}^p W_{j_l}\right\|_{H^3} \lesssim e^{-|s|^{\frac{1}{4}}/\bar{\mu}_{j_1}} + e^{-\rho_{j_1}|s|/\bar{\mu}_{j_1}} \lesssim |s|^{-10}.
$$

Proceeding similarly with $P_{j_l}$ and (2.12), the claim is proved and we obtain, as a consequence,

$$
\left\|W^4(V - W) - \sum_j Q^4_j (r_j R_j + a_j P_j) - \sum_j (W_j^4 - Q_j^4) (r_j R_j + a_j P_j)\right\|_{H^3} \lesssim |s|^{-2} + |s|^{-\frac{1}{4}} \sum_j |a_j|.
$$
But we also observe that, using (2.15) and (2.16),
\[
\left\| (W_j^4 - Q_j^4)(r_j R_j + a_j P_j) \right\|_{H^3} \\
\lesssim \left( \left\| W_j \right\|_{H^3}^3 + \left\| Q_j \right\|_{H^3}^3 \right) \left( \sum_{p=0}^{3} \left\| \partial_y^p (r_j R_j + a_j P_j) \right\|_{L^\infty} \right) \left( \sum_{p=0}^{3} \left\| \partial_y^p (W_j - Q_j) \right\|_{L^\infty} \right) \\
\lesssim (|r_j| + |a_j|) |s|^{-\frac{3}{4}} \lesssim |s|^{-\frac{7}{4}}.
\]
Gathering the previous estimates, we have obtained
\[
\| \Psi^{VI} \|_{H^2} \lesssim |s|^{-\frac{7}{4}} + |s|^{-\frac{7}{4}} \sum_j |a_j|,
\]
but also the more precise estimates
\[
\left\| \Psi^{VI} - \partial_y \left[ 20 \sum_j a_j P_j W_j^3 \sum_{j \neq j} W_j \right] \right\|_{H^2} \lesssim |s|^{-\frac{7}{4}} \tag{2.35}
\]
and
\[
\left\| \Psi^{VI} - \partial_y \left[ 5 \sum_j (W_j^4 - Q_j^4) (r_j R_j + a_j P_j) + 10 W^3 (V - W)^2 \\
+ 20 \sum_j (r_j R_j + a_j P_j) W_j^3 \sum_{j \neq j} W_j \right] \right\|_{H^2} \lesssim |s|^{-\frac{7}{4}} \tag{2.36}
\]

For \( j \geq k \), we note that \( \| P_j \|_{H^3(y > y_k - |s|^{\frac{1}{4}})} \lesssim |s|^{\frac{1}{4}} \) from (2.12), and thus, using (2.35),
\[
\| \Psi^{VI} \|_{H^2(y > y_k - |s|^{\frac{1}{4}})} \lesssim |s|^{-\frac{7}{4}} + |s|^{-2 + \frac{1}{4}} + |s|^{-\frac{7}{4}} \sum_{j < k} |a_j| \lesssim |s|^{-\frac{7}{4}} + |s|^{-\frac{7}{4}} \sum_{j < k} |a_j|.
\]

To control \( \| \Psi^{VI} \|_{L^\infty_k} \), we rely on (2.36). First, as observed before,
\[
\| \partial_y [W^3 (V - W)^2] \|_{L^\infty_k} \lesssim \| W^3 (V - W)^2 \|_{H^2} \lesssim |s|^{-2}.
\]
Next, for \( j \neq k \), it follows from similar arguments as before that
\[
\left\| \partial_y \left[ (W_j^4 - Q_j^4)(r_j R_j + a_j P_j) \right] \right\|_{L^\infty_k} + \left\| \partial_y \left[ (r_j R_j + a_j P_j) W_j^3 \sum_{j \neq j} W_j \right] \right\|_{L^\infty_k} \lesssim |s|^{-10}.
\]
Finally, by (2.15), (2.18) and \( |a_k| + |r_k| \lesssim |s|^{-1} \),
\[
\left\| \partial_y \left[ (W_k^4 - Q_k^4)(r_k R_k + a_k P_k) \right] \right\|_{L^\infty_k} \lesssim |s|^{-2},
\]
and, by (2.19)–(2.21),
\[
\left\| \partial_y \left[ (r_k R_k + a_k P_k) W_k^3 \sum_{k \neq k} W_k \right] \right\|_{L^\infty_k} \lesssim |s|^{-2},
\]
which proves
\[
\| \Psi^{VI} \|_{L^\infty_k} \lesssim |s|^{-2}.
\]
Concerning the projection on $Q_k$, we note from (2.16) and (2.36) that

$$\left| \langle \Psi^{VI}, Q_k \rangle + 5 \sum_j \langle (W^4_j - Q^4_j)(r_j R_j + a_j P_j), \partial_y Q_k \rangle + 10 \langle W^3(V - W)^2, \partial_y Q_k \rangle \right| + 20 \sum_j \langle (r_j R_j + a_j P_j)Q^3_j \sum_{j \neq j} W_{j2}, \partial_y Q_k \rangle \lesssim |s|^{-\frac{3}{2}}.$$ 

Since all the terms corresponding to $j \neq k$ in this estimate are controlled by $|s|^{-10}$, and since we find similarly as above

$$\left| \langle W^3(V - W)^2 - W^3_k(r_k R_k + a_k P_k)^2, \partial_y Q_k \rangle \right| \lesssim |s|^{-3},$$

we obtain, using again (2.16),

$$\left| \langle \Psi^{VI}, Q_k \rangle + 5 \langle (W^4_k - Q^4_k)(r_k R_k + a_k P_k), \partial_y Q_k \rangle + 10 \langle Q^3_k(r_k R_k + a_k P_k)^2, \partial_y Q_k \rangle \right| + 20 \langle (r_k R_k + a_k P_k)Q^3_k \sum_{k \neq k} W_{k2}, \partial_y Q_k \rangle \lesssim |s|^{-\frac{5}{2}}.$$ 

By (2.11) and (2.18), we have

$$5 \langle (W^4_k - Q^4_k)(r_k R_k + a_k P_k), \partial_y Q_k \rangle = 10 \frac{r_k}{\tau_k} \langle Q^3_k \tilde{P}_k R_k, \partial_y Q_k \rangle + 10 \frac{a_k}{\tau_k} \langle Q^3_k \tilde{P}_k P_k, \partial_y Q_k \rangle + O(|s|^{-\frac{3}{2}})$$

$$= 10 \frac{r_k}{\mu_k \tau_k} \langle Q^3 P R, Q' \rangle + 10 \frac{a_k}{\mu_k \tau_k} \langle Q^3 P^2, Q' \rangle + O(|s|^{-\frac{3}{2}}).$$

Moreover, since $\langle Q^3_k R_k^2, \partial_y Q_k \rangle = \tilde{\mu}_k^{-3} \langle Q^3 R^2, Q' \rangle = 0$ by parity, we have

$$10 \langle Q^3_k(r_k R_k + a_k P_k)^2, \partial_y Q_k \rangle = 20 r_k a_k \langle Q^3_k R_k P_k, \partial_y Q_k \rangle + 10 a_k^2 \langle Q^3_k P^2, \partial_y Q_k \rangle$$

$$= 20 \frac{r_k a_k}{\mu_k \tau_k} \langle Q^3 R P, Q' \rangle + 10 \frac{a_k^2}{\mu_k} \langle Q^3 P^2, Q' \rangle + O(|s|^{-10}).$$

Finally, noticing by (2.20) that, for all $i \geq 1$,

$$\left| Q_k \sum_{k \neq k} \left[ W_{k2}(s) - W_{k2}(s, y_k(s)) \right] \right|_{L_\infty} \lesssim |s|^{-2}, \quad (2.37)$$

then, by the definition (2.25) of $r_k$ and the cancellation $\langle R_k Q^3_k, \partial_y Q_k \rangle = \epsilon_k \tilde{\mu}_k^{-\frac{3}{2}} \langle R Q^3, Q' \rangle = 0$ obtained again by parity, we find

$$20 \langle (r_k R_k + a_k P_k)Q^3_k \sum_{k \neq k} W_{k2}, \partial_y Q_k \rangle = 20 \epsilon_k a_k r_k \tilde{\mu}_k^{-\frac{3}{2}} \langle P_k Q^3_k, \partial_y Q_k \rangle + O(|s|^{-3})$$

$$= 20 \frac{a_k r_k}{\mu_k^3} \langle P Q^3, Q' \rangle + O(|s|^{-3}).$$
Therefore, we have obtained
\[
\left| (\Psi^{VI}, Q_k) + 10 \frac{r_k}{\mu_k r_k} \langle Q^3 P R, Q' \rangle + 10 \frac{a_k}{\mu_k r_k} \langle Q^3 P^2, Q' \rangle + 10 \frac{a_k^2}{\mu_k^2} \langle Q^3 P^2, Q' \rangle \\
+ 20 \frac{a_k r_k}{\mu_k} \langle Q^3 R P, Q' \rangle + 20 \frac{a_k r_k}{\mu_k} \langle P Q^3, Q' \rangle \right| \lesssim |s|^{-\frac{3}{2}}.
\]

**Estimate of \( \Psi^{VII} \).** Using the definition (2.25) of \( r_j \), we decompose \( \Psi^{VII} \) as
\[
\Psi^{VII} = 5 \partial_y \left[ \sum_j \left( W_j^4 - Q_j^4 \right) \sum_{j_1 \neq j} W_{j_1} \right] \\
+ 5 \partial_y \left[ \sum_j Q_j^4 \sum_{j_1 \neq j} \left( W_{j_1}(s) - W_{j_1}(s, y_j(s)) \right) \right] \\
+ \partial_y \left[ \left( \sum_j W_j \right)^5 - \sum_j W_j^5 - 5 \sum_j W_j^4 \sum_{j_1 \neq j} W_{j_1} \right].
\]

As before, using (2.15)–(2.16) and (2.19)–(2.21), we have
\[
\left\| \left( W_j^4 - Q_j^4 \right) \sum_{j_1 \neq j} W_{j_1} \right\|_{H^3} \lesssim |s|^{-\frac{3}{2}} \quad \text{and} \quad \left\| Q_j^4 \sum_{j_1 \neq j} \left( W_{j_1}(s) - W_{j_1}(s, y_j(s)) \right) \right\|_{H^3} \lesssim |s|^{-2}.
\]

Using also (2.18), we get
\[
\left\| \left( W_j^4 - Q_j^4 \right) \sum_{j_1 \neq j} W_{j_1} \right\|_{L^\infty_x} \lesssim |s|^{-2}.
\]

Projecting on \( Q_k \), proceeding as before, we find
\[
5 \left\langle \partial_y \left[ \left( W_k^4 - Q_k^4 \right) \sum_{k_1 \neq k} W_{k_1} \right], Q_k \right\rangle = -10 \frac{r_k}{\epsilon_k \mu_k} \frac{1}{2} Q_k^2 \delta_y Q_k + O(|s|^{-\frac{5}{2}})
\]
\[
= -10 \frac{r_k}{\mu_k} \langle Q^3 P, Q' \rangle + O(|s|^{-\frac{5}{2}}).
\]

Moreover, from (2.11), the definition (2.25) of \( d_k \) and the relation \( Q'' + \bar{Q}^5 = Q \), we find
\[
5 \left\langle \partial_y \left[ Q_k^4 \sum_{k_1 \neq k} \left( W_{k_1}(s) - W_{k_1}(s, y_k(s)) \right) \right], Q_k \right\rangle = -5 \epsilon_k d_k \frac{1}{2} \sum_{k_1 \neq k} y_k^4 Q_{k_1}^4 \delta_y Q_{k_1} + O(|s|^{-3})
\]
\[
= -5 \frac{d_k}{\mu_k^3} \int y Q^4(y) Q'(y) dy + O(|s|^{-3}) = \frac{d_k}{\mu_k^3} \int Q^5(y) dy + O(|s|^{-3}) = \frac{d_k}{\mu_k^3} \|Q\|_{L^1} + O(|s|^{-3}).
\]

To estimate the last term in \( \Psi^{VII} \), we use the decay properties of \( W_j \) and obtain as before
\[
\left\| \left( \sum_j W_j \right)^5 - \sum_j W_j^5 - 5 \sum_j W_j^4 \sum_{j_1 \neq j} W_{j_1} \right\|_{H^3} \lesssim |s|^{-2},
\]
and similarly, using also (2.16),
\[
\left\| \left( \sum_j W_j \right)^5 - \sum_j W_j^5 - 5 \sum_j W_j^4 \sum_{j_1 \neq j} W_{j_1} - 10 \sum_j Q_j^3 \sum_{j_1, j_2 \neq j} W_{j_1} W_{j_2} \right\|_{H^3} \lesssim |s|^{-\frac{11}{4}}. \tag{2.38}
\]
But, since \( \langle Q_k^3, \partial_y Q_k \rangle = \tilde{\mu}_k^2 \langle Q^3, Q' \rangle = 0 \) by parity, we find, from the definition (2.25) of \( r_k \),
\[
\langle Q_k^3 \sum_{k_1, k_2 \neq k} W_{k_1} W_{k_2}, \partial_y Q_k \rangle \\
= \sum_{k_1, k_2 \neq k} \int Q_k^3(y) \left[ W_{k_1}(s, y_k(s)) + \left( W_{k_1}(s, y) - W_{k_1}(s, y_k(s)) \right) \right] \\
\times \left[ W_{k_2}(s, y_k(s)) + \left( W_{k_2}(s, y) - W_{k_2}(s, y_k(s)) \right) \right] \partial_y Q_k(y) \, dy \\
= 2 \epsilon_i \tilde{\mu}_k^2 r_k \sum_{k' \neq k} \int Q_k^3(y) \left[ W_{k'}(s, y) - W_{k'}(s, y_k(s)) \right] \partial_y Q_k(y) \, dy \\
+ \sum_{k_1, k_2 \neq k} Q_k^3(y) \left( W_{k_1}(s, y) - W_{k_1}(s, y_k(s)) \right) \left( W_{k_2}(s, y) - W_{k_2}(s, y_k(s)) \right) \partial_y Q_k(y) \, dy
\]
and so, from (2.27) and (2.37),
\[
\left\| \left( \sum_{k_1, k_2 \neq k} W_{k_1} W_{k_2}, \partial_y Q_k \right) \right\| \lesssim |r_k| \sum_{k' \neq k} \left\| Q_k^3 \left[ W_{k'}(s) - W_{k'}(s, y_k(s)) \right] \right\|_{L^\infty} \\
+ \sum_{k' \neq k} \left\| Q_k \left[ W_{k'}(s) - W_{k'}(s, y_k(s)) \right] \right\|_{L^\infty}^2 \\
\lesssim |s|^{-1} |s|^{-2} + |s|^{-1} \lesssim |s|^{-3}.
\]
Thus, from (2.38), we finally deduce
\[
\left\| \left( \sum_j W_j \right)^5 - \sum_j W_j^5 - 5 \sum_j W_j^4 \sum_{j_1 \neq j} W_{j_1}, \partial_y Q_k \right\| \lesssim |s|^{-\frac{11}{4}}.
\]
Therefore, we have obtained
\[
\| \Psi^{\text{VIII}} \|_{H^2} \lesssim |s|^{-\frac{7}{4}}, \quad \| \Psi^{\text{VIII}} \|_{L^\infty} \lesssim |s|^{-2}, \\
\| \langle \Psi^{\text{VIII}}, Q_k \rangle + 10 \frac{T_k}{\mu_k r_k} \langle Q^3, P, Q' \rangle - \frac{d_k}{\mu_k^2} \| Q \|_{L^1} \rangle \| \lesssim |s|^{-\frac{7}{2}}.
\]

**Estimate of \( \Psi^{\text{VIII}} \).** We estimate \( \Psi^{\text{VIII}} \) as \( \Psi^{\text{III}} \), noticing that \( \| R_j \|_{H^3} \lesssim 1, \| r_j \| \lesssim |s|^{-1} \) from (2.27), \( |\tilde{\mu}_j - \mu_j| \lesssim |s|^{-1} \) from (2.11), \( \langle R_k, \partial_y Q_k \rangle = \tilde{\mu}_k^{-1} \langle R, Q' \rangle = 0 \) by parity and \( |\langle R_j, \partial_y Q_k \rangle| \lesssim |s|^{-10} \) for \( j \neq k \), and thus obtain
\[
\| \Psi^{\text{VIII}} \|_{H^2} \lesssim |s|^{-2}, \quad |\langle \Psi^{\text{VIII}}, Q_k \rangle| \lesssim |s|^{-12}.
\]

**Conclusion.** Gathering the estimates in \( H^2 \) of \( \Psi^1, \ldots, \Psi^{\text{VIII}} \) above, we obtain (2.31). Then, using also the additional estimates in \( H^2(y > y_k - |s|^{\frac{1}{4}}) \) and \( L^\infty_k \) when necessary, we get (2.32).
Finally, gathering the projections of $\Psi^1, \ldots, \Psi^{\text{VIII}}$ on $Q_k$ yields (2.33), with
\[
\Omega_k = \left( \frac{r_k}{2\mu_k^3 \tau_k} + \frac{a_k r_k}{\mu_k^2} \right) \left( \langle \Lambda R, Q \rangle - 20\langle Q^3 PR, Q' \rangle - 20\langle PQ^3, Q' \rangle \right) + \frac{a_k}{\mu_k^3} \frac{r_k}{\mu_k^2} \left( \langle \Lambda P, Q \rangle - 10\langle Q^3 P^2, Q' \rangle \right) + \frac{d_k}{\mu_k^3} \|Q\|_{L^1}.
\]

But, from [22] and [24], we recall the identities
\[
\langle \Lambda P, Q \rangle - 10\langle Q^3 P^2, Q' \rangle = \frac{1}{8} \|Q\|_{L^1}^2, \quad \text{and} \quad \langle \Lambda R, Q \rangle - 20\langle Q^3 PR, Q' \rangle - 20\langle PQ^3, Q' \rangle = 0,
\]
and so
\[
\Omega_k = \frac{1}{8} \|Q\|_{L^1}^2 \left( \frac{a_k}{\mu_k^3 \tau_k} + \frac{a_k}{\mu_k^2} \right) + \frac{d_k}{\mu_k^3} \|Q\|_{L^1},
\]
which concludes the proof of Lemma 2.4.

Next, we give precise asymptotics on $\hat{r}_k$ and $r_k$.

**Lemma 2.5.** (i) Asymptotics of $\hat{r}_k$: for all $1 \leq k \leq K$, for all $s \in I$,
\[
\left| \frac{d_k}{\mu_k^3} - \left( \hat{r}_k + \frac{r_k}{4\mu_k^3 \tau_k} - \frac{a_k r_k}{2\mu_k^3} \right) \right| \lesssim |s|^{-3} + |s|^{-1} |\vec{m}_k| + |s|^{-1} \sum_{j<k} |\vec{m}_j^0| + |s|^{-10} \sum_{j>k} |\vec{m}_j^0|, \quad (2.39)
\]
and, in particular,
\[
|\hat{r}_k| \lesssim |s|^{-2} + |s|^{-1} \sum_j |\vec{m}_j|.
\]

(ii) Asymptotics of $r_k$: for all $1 \leq k \leq K$, for all $s \in I$,
\[
|r_k(s) - \frac{\|Q\|_{L^1}}{4s} \ell_k^2 \theta_k \left( 1 + \frac{1}{2} \frac{\mu_k}{2\mu_k} - \frac{3}{2} \frac{\mu_k}{2\mu_k} \right)\right| \lesssim |s|^{-1} \left( |s|^{-2} \frac{\|Q\|_{L^1}}{4s} \ell_k^2 \theta_k \right) + \sum_{j<k} (|\vec{m}_j| + |\vec{m}_j| + |\vec{m}_j|), \quad (2.41)
\]
where the constant $\theta_k \in \mathbb{R}$ is defined by
\[
\theta_k = \sum_{j<k} \epsilon_k \epsilon_j \sqrt{\frac{\ell_k}{\ell_j}}. \quad (2.42)
\]

**Proof.** (i) By the definition (2.25) of $r_k$, we have
\[
\hat{r}_k = \frac{1}{2\mu_k^3} \mu_k^3 r_k + \epsilon_k \frac{1}{2\mu_k^3} \sum_{j \neq k} \partial_s W_j(s, y_k) + \epsilon_k \frac{1}{2\mu_k^3} \mu_k^3 \sum_{j \neq k} \partial_y W_j(s, y_k).
\]
First, from (2.11) and the definition (2.30) of $m_{k,1}$, we find
\[
\frac{\hat{r}_k}{\hat{r}_k} = \frac{\hat{r}_k}{\hat{r}_k} + O(|s|^{-2}) = - \frac{1}{2\mu_k^3} + \frac{1}{2s} - \frac{a_k}{\mu_k^3} + O(|s|^{-2}) + O(|\vec{m}_k|).
\]
Thus, since $|r_k| \lesssim |s|^{-1}$, by (2.27),
\[
\frac{1}{2\mu_k^3} r_k = - \frac{r_k}{4\mu_k^3 \tau_k} + \frac{r_k}{4s} - \frac{a_k r_k}{2\mu_k^3} + O(|s|^{-3}) + O(|s|^{-1} |\vec{m}_k|).
\]
Second, by (2.23), for all \( j \neq k \),
\[
\partial_s W_j = -\frac{1}{2s} \Delta W_j - \partial_y \left( \partial_{yy} W_j + W^5_j \right) + \vec{m}_j \cdot \vec{M}_j W_j.
\]
By (2.9)–(2.10) and (2.19)–(2.20), we estimate, for \( j < k \),
\[
|\partial_{yy} W_j(s, y_k)| \lesssim |s|^{-4}, \quad |\partial_y (W^5_j(s, y_k))| \lesssim |s|^{-6}, \quad |\vec{M}_j W_j(s, y_k)| \lesssim |s|^{-1},
\]
and, for \( j > k \), using (2.21),
\[
|y_k \partial_y W_j(s, y_k)| + |W_j(s, y_k)| + |\partial_{yy} W_j(s, y_k)| + |\partial_y (W^5_j(s, y_k)) + |\vec{M}_j W_j(s, y_k)| \lesssim |s|^{-10}.
\]
Thus, for \( j < k \),
\[
\partial_s W_j(s, y_k) = -\frac{1}{4s} W_j(s, y_k) - \frac{y_k}{2s} \partial_y W_j(s, y_k) + O(|s|^{-4}) + O(|s|^{-1} |\vec{m}_j|),
\]
and, for \( j > k \),
\[
|\partial_s W_j(s, y_k)| \lesssim |s|^{-10} + |s|^{-1} |\vec{m}_j|.
\]
Therefore, from (2.25),
\[
\epsilon_k \mu_k \sum_{j \neq k} \partial_s W_j(s, y_k) = -\frac{r_k}{4s} y_k \mu_k^{-1} d_k + O(|s|^{-4}) + O \left( |s|^{-1} \sum_{j < k} |\vec{m}_j| \right) + O \left( |s|^{-10} \sum_{j > k} |\vec{m}_j| \right).
\]
Third, from (2.11), (2.27) and (2.30),
\[
\epsilon_k \mu_k \sum_{j \neq k} \partial_y W_j(s, y_k) = \mu_k^{-1} y_k d_k = \mu_k^{-1} \left( \frac{y_k}{2s} + \frac{1}{\mu_k^2} \right) d_k + O(|s|^{-3}) + O(|s|^{-2} |\vec{m}_k|)
\]
\[
= \frac{y_k}{2s} \mu_k^{-1} d_k + \frac{d_k}{\mu_k^2} + O(|s|^{-3}) + O(|s|^{-2} |\vec{m}_k|).
\]
Gathering the above estimates, we find (2.39). Finally, using the \textit{a priori} estimates (2.9) and (2.26)–(2.27), we directly deduce (2.40) from (2.39).

(ii) From the definition (2.25) of \( r_k \), the \textit{a priori} estimates (2.9)–(2.10), and estimates (2.11), (2.19) and (2.21), we find
\[
r_k = -\frac{1}{2} \epsilon_k \mu_k \sum_{j < k} \epsilon_j \mu_j \sqrt{-2 \tau_j} |y_k - z_j|^{-\frac{3}{2}} + O \left( |s|^{-\frac{1}{2}} \sum_{j < k} |y_k - z_j|^{-\frac{3}{2} - \frac{1}{2}} \right) + O(|s|^{-10})
\]
\[
= -\frac{1}{2} \epsilon_k \mu_k \sqrt{|y_k|^{-\frac{3}{2}}} \sum_{j < k} \epsilon_j \mu_j \sqrt{-2 \tau_j} + O \left( |s|^{-1} \sum_{j < k} |z_j| \right) + O(|s|^{-1 - \frac{1}{2}})
\]
\[
= -\frac{1}{2} \epsilon_k \mu_k \sqrt{|y_k|^{-\frac{3}{2}}} \sum_{j < k} \epsilon_j \ell_j^{-\frac{3}{2}} + O \left( |s|^{-1} \sum_{j < k} (|z_j| + |\mu_j| + |\tau_j|) \right) + O(|s|^{-1 - \frac{1}{2}}).
\]
From (2.9), we have \( \mu_k = \ell_k (1 + \bar{\mu}_k) \) and \( |y_k| = -2s \ell_k^{-2} (1 + \bar{y}_k) \), thus a Taylor expansion gives
\[
\mu_k = \ell_k^{-\frac{3}{2}} \left[ 1 + \frac{1}{2} \bar{\mu}_k + O(|\bar{\mu}_k|^2) \right] \quad \text{and} \quad |y_k|^{-\frac{3}{2}} = (-2s)^{-\frac{3}{2}} \ell_k^{-\frac{3}{2}} \left[ 1 - \frac{3}{2} \bar{y}_k + O(|\bar{y}_k|^2) \right],
\]
which leads to (2.41) together with the above estimate of \( r_k \) and (2.10).

Finally, we give sharp asymptotics on the mass and the energy of the approximate rescaled multi-soliton \( \vec{V} \).
Lemma 2.6. (i) Mass of $V$: for all $s \in I$,
\[
\left| \| V(s) \|_{L^2}^2 - K \| Q \|_{L^2}^2 \right| \lesssim \sum_k |a_k| + |s| \sum_k a_k^2 + |s|^{-1} \sum_k (|\bar{\mu}_k| + |\bar{\tau}_k| + |\bar{\gamma}_k|) + |s|^{-1 - \frac{4}{7}}. \tag{2.43}
\]

(ii) Energy of $V$: for all $1 \leq k \leq K$, for all $s \in I$, let the local energy $e_k$ be defined by
\[
e_k(s) = \frac{s}{\mu_k^2(s)} \left[ a_k(s) + \frac{1}{2\tau_k(s)} + 4r_k(s) \right]. \tag{2.44}\]

Then, for all $s \in I$,
\[
\left| E(V(s)) + \frac{\| Q \|_{L^1}^2}{16s} \sum_k e_k(s) \right| \lesssim |s|^{-\frac{4}{7}} \tag{2.45}
\]
and, in particular,
\[
\left| E(V(s)) + \frac{\| Q \|_{L^1}^2}{32s} \sum_k \ell_k(1 + 2\theta_k) \right| \lesssim \sum_k |a_k| + |s|^{-1} \sum_k (|\bar{\mu}_k| + |\bar{\tau}_k| + |\bar{\gamma}_k|) + |s|^{-1 - \frac{4}{7}}. \tag{2.46}\]

Proof. (i) First, expanding $V = \sum_k (W_k + r_k R_k + a_k P_k)$, using (2.27) and $\| P_k \|_{L^2} \lesssim |s|^{\frac{1}{2}}$ from (2.13), we find
\[
\int V^2 = \sum_k \sum_j \int (W_k W_j + 2r_j W_k R_j + 2a_j W_k P_j) + O \left( |s| \sum_k a_k^2 \right) + O(|s|^{-2}).
\]

Thus, distinguishing the cases $j = k$ and $j \neq k$, expanding $W_k = Q_k + (W_k - Q_k)$ and using estimates (2.15), (2.16) and (2.18),
\[
\int V^2 = \sum_k \int \left( W_k^2 + 2r_k Q_k R_k \right) + \sum_k \sum_{j \neq k} \int W_k W_j + O \left( \sum_k |a_k| \right) + O \left( |s| \sum_k a_k^2 \right) + O(|s|^{-1 - \frac{4}{7}}).
\]

Now note that, by the definitions of $W_k$ and $\bar{S}$, and by the value $\| S(t) \|_{L^2} = \| Q \|_{L^2}$, we have
\[
\| W_k \|_{L^2} = \| \bar{S}(\tau_k) \|_{L^2} = \left\| S \left( \frac{1}{\sqrt{1 - 2\tau_k}} \right) \right\|_{L^2} = \| Q \|_{L^2}.
\]

Thus, using also $\langle Q_k, R_k \rangle = \langle Q, R \rangle = -\frac{3}{4} \| Q \|_{L^1}$, from (2.4), we obtain
\[
\| V \|_{L^2}^2 = K \| Q \|_{L^2}^2 + \sum_k \left( 2 \sum_{j < k} \int W_k W_j - 3 \| Q \|_{L^1} r_k \right)
\]
\[
\quad + O \left( \sum_k |a_k| \right) + O \left( |s| \sum_k a_k^2 \right) + O(|s|^{-1 - \frac{3}{4}}).
\]

Finally, we claim that, for all $1 \leq k \leq K$,
\[
\left| \sum_{j < k} \int W_k W_j - 3 \frac{\| Q \|_{L^1}^2}{16s} \ell_k^3 \theta_k \right| \lesssim |s|^{-1} \sum_{j < k} (|\bar{\mu}_j| + |\bar{\tau}_j| + |\bar{\gamma}_j|) + |s|^{-1 - \frac{4}{7}}, \tag{2.47}
\]

which gives (2.43), together with (2.41) and the calculation above.
To prove (2.47), we estimate $\langle W_k, W_j \rangle$ with $j < k$ on the three regions $y < y_k - |s|^{\frac{4}{7}}$, $|y - y_k| \leq |s|^{\frac{4}{7}}$ and $y > y_k + |s|^{\frac{4}{7}}$. First, using (2.19) then (2.9)–(2.10), we obtain
\[
\int_{y < y_k - |s|^{\frac{4}{7}}} W_k W_j = \frac{1}{2} \|Q\|_{L^1}^2 \epsilon_k \epsilon_j \mu_k \mu_j \sqrt{\tau_k \tau_j} \int_{y < y_k - |s|^{\frac{4}{7}}} |y - z_k|^{-\frac{3}{2}} |y - z_j|^{-\frac{3}{2}} dy + O(|s|^{1 - \frac{1}{12}})
\]
\[
= \frac{1}{4} \|Q\|_{L^1}^2 \epsilon_k \epsilon_j \mu_k \mu_j \sqrt{\tau_k \tau_j} |y_k|^{-2} + O \left(|s|^{-1} (|\tilde{z}_j| + |\tilde{z}_k|)\right) + O(|s|^{1 - \frac{1}{12}})
\]
\[
= -\frac{\|Q\|_{L^1}^2}{16s} \epsilon_k \epsilon_j \ell_k \ell_j^{-\frac{1}{2}} + O \left(|s|^{-1} (|\tilde{\mu}_j| + |\tilde{\mu}_k| + |\tilde{\tau}_j| + |\tilde{\tau}_k| + |\tilde{y}_j| + |\tilde{y}_k|)\right) + O(|s|^{1 - \frac{1}{12}}).
\]
Second, we decompose the next term as
\[
\int_{|y - y_k| \leq |s|^{\frac{4}{7}}} W_k W_j = \int_{|y - y_k| \leq |s|^{\frac{4}{7}}} (W_k - Q_k)W_j + \int_{|y - y_k| \leq |s|^{\frac{4}{7}}} Q_k (W_j - W_j(y_k)) + W_j(y_k) \int_{|y - y_k| \leq |s|^{\frac{4}{7}}} Q_k = A_{j,k} + B_{j,k} + C_{j,k}.
\]
By the Cauchy–Schwarz inequality, (2.15) and again (2.19), we find
\[
|A_{j,k}| \lesssim |s|^{-1} \int_{|y - y_k| \leq |s|^{\frac{4}{7}}} |W_k - Q_k| \lesssim |s|^{-1} |s|^{\frac{4}{7}} \|W_k - Q_k\|_{L^2} \lesssim |s|^{-\frac{1}{2}}.
\]
Next, similarly to (2.37), we obtain $|B_{j,k}| \lesssim |s|^{-2}$. To estimate $C_{j,k}$, we use the exponential decay of $Q$ and (2.11) to get
\[
\int_{|y - y_k| \leq |s|^{\frac{4}{7}}} Q_k(y) dy = \epsilon_k \tilde{\mu}_k \int_{|z| \leq \tilde{\mu}_k |s|^{\frac{4}{7}}} Q(z) dz
\]
\[
= \epsilon_k \tilde{\mu}_k \|Q\|_{L^1} + O(|s|^{-10}) = \epsilon_k \tilde{\mu}_k \|Q\|_{L^1} + O(|s|^{-1}),
\]
and again (2.19) then (2.9)–(2.10) to obtain as before
\[
W_j(y_k) = -\frac{\epsilon_j}{2} \|Q\|_{L^1} \mu_j \sqrt{-2 \tau_j |y_k - z_j|^{-\frac{3}{2}}} + O(|s|^{1 - \frac{1}{12}})
\]
\[
= -\frac{\epsilon_j}{2} \|Q\|_{L^1} \ell_j \sqrt{-2 \ell_j |s|^{\frac{3}{7}} (-2s \ell_j^2)^{-\frac{3}{2}}} + O \left(|s|^{-1} (|\tilde{z}_j| + |\tilde{\mu}_j| + |\tilde{\tau}_j| + |\tilde{y}_j|)\right) + O(|s|^{1 - \frac{1}{12}})
\]
\[
= \frac{\epsilon_j}{4s} \|Q\|_{L^1} \ell_j \ell_j^{-\frac{1}{2}} + O \left(|s|^{-1} (|\tilde{\mu}_j| + |\tilde{\tau}_j| + |\tilde{y}_j|)\right) + O(|s|^{1 - \frac{1}{12}}).
\]
Thus,
\[
C_{j,k} = \frac{\|Q\|_{L^1}^2}{4s} \epsilon_k \epsilon_j \ell_k \ell_j^{-\frac{1}{2}} + O \left(|s|^{-1} (|\tilde{\mu}_j| + |\tilde{\tau}_j| + |\tilde{y}_j|)\right) + O(|s|^{1 - \frac{1}{12}}).
\]
Finally, from (2.21), we observe that
\[
\int_{y > y_k + |s|^{\frac{4}{7}}} W_k W_j \lesssim |s|^{-10}.
\]
Thus, gathering the previous estimates, we have obtained
\[
\int W_k W_j - \frac{3}{16s} \|Q\|_{L^1}^2 \epsilon_k \epsilon_j \ell_k \ell_j^{-\frac{1}{2}} \lesssim |s|^{-1} (|\tilde{\mu}_j| + |\tilde{\tau}_j| + |\tilde{y}_j|) + |s|^{-1 - \frac{1}{12}}
\]
and so, summing over $j < k$ and using the definition (2.42) of $\theta_k$, we obtain (2.47), which concludes the proof of (2.43).
(ii) To compute the gradient term in the energy of $V$, we proceed as in (i). Indeed, expanding first $V = \sum_k (W_k + r_k R_k + a_k P_k)$, using (2.26)–(2.27) and $\| \partial_y P_k \|_{L^2} \lesssim 1$ from (2.13), we find similarly

$$\frac{1}{2} \int (\partial_y V)^2 = \sum_k \sum_j \int \left( \frac{1}{2} \partial_y W_k \partial_y W_j + r_j \partial_y W_k \partial_y R_j + a_j \partial_y W_k \partial_y P_j \right) + O(|s|^{-2}).$$

Now note that, because of the stronger decay (2.20) on the left of $\partial_y W_k$ than the decay (2.19) on the left of $W_k$, we may estimate simply, for $j < k$,

$$\left| \int \partial_y W_k \partial_y W_j \right| \lesssim \| \partial_y W_k \|_{L^2} \| \partial_y W_j \|_{L^2(y < y_k + |s|^{1/4})} + \| \partial_y W_j \|_{L^2} \| \partial_y W_k \|_{L^2(y > y_k + |s|^{1/4})} \lesssim |s|^{-\frac{3}{2}}.$$

Thus, distinguishing again the cases $j = k$ and $j \neq k$, expanding $W_k = Q_k + (W_k - Q_k)$, integrating by parts and using estimates (2.19) and (2.21), we obtain

$$\frac{1}{2} \int (\partial_y V)^2 = \sum_k \left[ \frac{1}{2} \int (\partial_y W_k)^2 - r_k \partial_y Q_k R_k - a_k \partial_y Q_k P_k \right] + O(|s|^{-\frac{3}{2}}).$$

To compute the nonlinear term in the energy of $V$, we follow the estimates of $\Psi^{VI}$ and $\Psi^{VII}$ in the proof of Lemma 2.4. Indeed, expanding $V = W + (V - W)$, we first decompose this term as

$$\frac{1}{6} \int V^6 = \frac{1}{6} \sum_{i=0}^{6} \binom{6}{i} \int V^{6-i} (V - W)^i = \frac{1}{6} \int W^6 + \int W^5 (V - W) + O(|s|^{-2}),$$

since, for all $2 \leq i \leq 6$, using $\| V - W \|_{L^\infty} \lesssim |s|^{-1}$ and $\| V - W \|_{L^2} \lesssim |s|^{-\frac{3}{2}},$

$$\left| \int W^{6-i} (V - W)^i \right| \lesssim |s|^{-2}.$$

Now we expand $W = \sum_k W_k$ to decompose the nonlinear term as

$$\frac{1}{6} \int V^6 = \frac{1}{6} \sum_k \int W_k^6 + \sum_{k \neq k} \int W_k^5 W_j + \frac{1}{6} \int \left[ \left( \sum_k W_k \right)^5 - \sum_k W_k^6 - 6 \sum_k W_k^5 \sum_j W_j \right]$$

$$+ \int \left( \sum_k W_k^5 \right) (V - W) + \int \left[ \left( \sum_k W_k \right)^5 - \sum_k W_k^5 \right] (V - W) + O(|s|^{-2}).$$

As before, using (2.19) and (2.21), we observe that

$$\int \left| \left( \sum_k W_k \right)^6 - \sum_k W_k^6 - 6 \sum_k W_k^5 \sum_j W_j \right| \lesssim \sum_{1 \leq k_6 \leq \cdots \leq k_1 \leq K} \int \prod_{j=1}^{6} |W_{k_j}|$$

$$\lesssim \sum_{1 \leq k_6 \leq k_5 \leq k_2 \leq k_1 \leq K} \int_{y < y_{k_1} + |s|^{1/4}} |W_{k_1}||W_{k_2}||W_{k_5}||W_{k_6}| + \sum_{k_1} \int_{y > y_{k_1} + |s|^{1/4}} |W_{k_1}|$$

$$\lesssim |s|^{-2} \sum_{1 \leq k_2 \leq k_1 \leq K} \| W_{k_1} \|_{L^2} \| W_{k_2} \|_{L^2} + |s|^{-10} \lesssim |s|^{-2},$$

and similarly

$$\int \left| \left( \sum_k W_k \right)^5 - \sum_k W_k^5 \right| \lesssim |s|^{-1}. $$
Thus, using also $\| V - W \|_{L^\infty} \lesssim |s|^{-1}$, we obtain
\[
\frac{1}{6} \int V^6 = \frac{1}{6} \sum_k \int W_k^6 + \sum_{k,j \neq k} \int W_k^5 W_j + \int \left( \sum_k W_k^5 \right) (V - W) + O(|s|^{-2}).
\]

Next, expanding $V - W = \sum_k (r_k R_k + a_k P_k)$ and $W_k = Q_k + (W_k - Q_k)$, using (2.15)–(2.16) and (2.18), we obtain
\[
\int \left( \sum_k W_k^5 \right) (V - W) = \sum_k \int Q_k^5 (r_k R_k + a_k P_k) + O(|s|^{-2}).
\]

Gathering the above estimates and using the identity $\partial_{yy} Q_k + Q_k^5 = \tilde{\mu}_k^{-2} Q_k$, we have obtained
\[
E(V) = \sum_k E(W_k) - \sum_k \int \tilde{\mu}_k^{-2} (r_k Q_k R_k + a_k Q_k P_k) - \sum_{k,j \neq k} \int W_k^5 W_j + O(|s|^{-2}).
\]

Now note that, by the definitions of $W_k$ and $\tilde{S}$, and by the value $E(S(t)) = \frac{||Q||_{L^1}^2}{16}$, we have
\[
E(W_k) = \mu_k^{-2} E(\tilde{S}(\tau_k)) = \frac{\mu_k^{-2}}{2\tau_k} E \left( S \left( \frac{1}{\sqrt{-2\tau_k}} \right) \right) = - \frac{||Q||_{L^1}^2}{16} \frac{1}{2\mu_k^2 \tau_k}.
\]

Thus, using also (2.11) and the identities $\langle Q_k, R_k \rangle = \langle Q, R \rangle = -\frac{3}{4} ||Q||_{L^1}$ from (2.4) and
\[
\langle Q_k, P_k \rangle = \langle Q, P \rangle + O(|s|^{-10}) = \frac{||Q||_{L^1}^2}{16} + O(|s|^{-10})
\]
from (2.2), we find
\[
E(V) = - \frac{||Q||_{L^1}^2}{16} \sum_k \frac{1}{2\mu_k^2 \tau_k} + \frac{3||Q||_{L^1}^2}{4} \sum_k \frac{r_k}{\mu_k^2} - \frac{||Q||_{L^1}^2}{16} \sum_k \frac{a_k}{\mu_k^2} - \sum_{k,j \neq k} \int W_k^5 W_j + O(|s|^{-\frac{7}{4}}).
\]

Finally, we claim that, for all $1 \leq k \leq K$,
\[
\left| \sum_{j \neq k} \int W_k^5 W_j \right| \lesssim |s|^{-\frac{7}{4}},
\]
so that, from the definition (2.44) of $e_k$, we obtain
\[
E(V) = - \frac{||Q||_{L^1}^2}{16} \sum_k \frac{1}{\mu_k^2} \left( \frac{1}{2\tau_k} + \frac{4r_k}{||Q||_{L^1}} + a_k \right) + O(|s|^{-\frac{7}{4}}) = - \frac{||Q||_{L^1}^2}{16s} \sum_k e_k + O(|s|^{-\frac{7}{4}}).
\]

Thus (2.45) is proved, and (2.46) is obtained as a direct consequence by inserting (2.9) and (2.41) into (2.45).

We conclude by proving the claim (2.48). First note that, by (2.19) and (2.21), for $j \neq k$,
\[
\left| \int_{y > y_k + |s|^\frac{7}{4}} W_k^5 W_j \right| \lesssim |s|^{-10} \quad \text{and} \quad \left| \int_{y < y_k - |s|^\frac{7}{4}} W_k^5 W_j \right| \lesssim |s|^{-4},
\]

since $|W_k(y)|^4 \lesssim |s|^{-4}$ for $y < y_k - |s|^\frac{3}{4}$. Next, for $j \neq k$ and $|y - y_k| \lesssim |s|^\frac{3}{4}$, we have $|W_j(y)| \lesssim |s|^{-1}$ from (2.19) and (2.21) again, and thus, using also (2.16),
\[
\left| \int_{|y - y_k| \leq |s|^\frac{3}{4}} (W_k^5 - Q_k^5) W_j \right| \lesssim |s|^{-\frac{7}{4}}.
\]
Moreover, by (2.11), (2.25), (2.27) and the exponential decay of $Q$, we get similarly as before
\[
\sum_{j \neq k} \int_{|y-y_k| \leq |s|\frac{3}{4}} Q^2_k W_j = \sum_{j \neq k} W_j(y_k) \int_{|y-y_k| \leq |s|\frac{3}{4}} Q^2_k(y) \, dy + O(|s|^{-2})
\]
\[
= r_k \mu_k^{-2} \int_{|z| \leq |\tilde{\mu}| \frac{3}{4}} Q^5(z) \, dz + O(|s|^{-2})
\]
\[
= r_k \tilde{\mu}_k^{-2} \int Q^5 + O(|s|^{-2}) = r_k \mu_k^{-2} \|Q\|_{L^1} + O(|s|^{-2}).
\]
Combining these estimates, we obtain (2.48), which concludes the proof of Lemma 2.6. \qed

2.3. Modulation around the approximate multi-soliton. We want to construct solutions $v$ of (2.7) of the form
\[
v(s, y) = V(s, y) + \varepsilon(s, y), \quad \text{with} \quad \|\varepsilon(s)\|_{H^1} \leq |s|^{-\frac{1}{2}}.
\]  (2.49)

From the definition of $E_V$ in Lemma 2.4, the equation of $\varepsilon$ reads
\[
\partial_s \varepsilon + \frac{1}{2} \Lambda \varepsilon + \partial_y \left( \partial_y \varepsilon + (V + \varepsilon)^5 - V^5 \right) + E_V = 0.
\]  (2.50)

First, for a suitable solution $v$ of (2.7), we construct in the next lemma a time-dependent parameter vector
\[
\Gamma = (\tau_1, \mu_1, y_1, a_1, \ldots, \tau_K, \mu_K, y_K, a_K)^T \in ((-\infty, 0) \times (0, +\infty) \times \mathbb{R} \times \mathbb{R})^K
\]

Then there exist $0 < \bar{s} < |s_1|$ and a unique $C^1$ function $\Gamma$ defined on $[s^in, s^in + \bar{s}]$ and taking values into $((-\infty, 0) \times (0, +\infty) \times \mathbb{R} \times \mathbb{R})^K$ such that $\Gamma(s^in) = \Gamma^in$ and, on $[s^in, s^in + \bar{s}]$, $\Gamma$ and $\varepsilon = v - V[\Gamma]$ satisfy (2.8) and (2.51). Moreover, $\Gamma$ and $\varepsilon$ also satisfy (2.9), (2.26) and (2.49).

Remark 2.8. The strategy of the proof of Lemma 2.7 below is to write the relation (2.8) and the orthogonality conditions (2.51) as a non-autonomous differential system $\dot{\Gamma} = A(s, \Gamma)$ satisfied by $\Gamma$ and to apply the Cauchy–Lipschitz theorem. More precisely, we prove that the function $A$ is continuous and Lipschitz in $\Gamma$ on $[s^in, s^in + \bar{s}] \times D_\Gamma$, where $\bar{s} > 0$ is small enough and $D_\Gamma$ is the compact set of $\Gamma$ satisfying (2.9) and (2.26).
Proof of Lemma 2.7. To start, we observe that the relation (2.8) may be rewritten as
\[ L^\tau \Gamma = F^\tau_k \] with \( L^\tau_k = (0, \ldots, 0, 1, 0, \ldots, 0) \) and \( F^\tau_k = \mu_k^{-3} \), where the nonzero coefficient in \( L^\tau_k \) is located at the position \( 4(k-1) + 1 \). Note that \( F^\tau_k \) is locally Lipschitz in \( \Gamma \) from (2.9).

Next, we compute \( \frac{d}{ds} \langle \varepsilon, \Lambda_k Q_k \rangle = (\partial_s \varepsilon, \Lambda_k Q_k) + \langle \varepsilon, \partial_s (\Lambda_k Q_k) \rangle \). Note that \( Q_k \) satisfies
\[ \partial_s Q_k = -\frac{\dot{\mu}_k}{\mu_k} \Lambda_k Q_k - \dot{y}_k \partial_y (\Lambda_k Q_k), \] (2.52)
and in particular \( \partial_s (\Lambda_k Q_k) = -\frac{\dot{\mu}_k}{\mu_k} \Lambda_k^2 Q_k - \dot{y}_k \partial_y (\Lambda_k Q_k) \). Thus, using the equation (2.50) of \( \varepsilon \) and the expression (2.29) of \( \mathcal{E}_\mathcal{V} \), we find
\[ \frac{d}{ds} \langle \varepsilon, \Lambda_k Q_k \rangle = -\frac{1}{2s} \langle \Lambda \varepsilon, \Lambda_k Q_k \rangle - \langle \partial_y \left( \partial_{yy} \varepsilon + (\mathbf{V} + \varepsilon)^5 - \mathbf{V}^5 \right), \Lambda_k Q_k \rangle - \langle \Psi, \Lambda_k Q_k \rangle \]
\[ - \sum_j \langle \bar{m}_j \cdot \bar{M}_j V_j, \Lambda_k Q_k \rangle - \sum_j \bar{r}_j \langle R_j, \Lambda_k Q_k \rangle - \sum_j \bar{a}_j \langle P_j, \Lambda_k Q_k \rangle \]
\[ - \frac{\dot{\mu}_k}{\mu_k} \langle \varepsilon, \Lambda_k^2 Q_k \rangle - \dot{y}_k \langle \varepsilon, \partial_y (\Lambda_k Q_k) \rangle. \] (2.53)

The first term in the right-hand side of (2.53) rewrites
\[ -\frac{1}{2s} \langle \Lambda \varepsilon, \Lambda_k Q_k \rangle = \frac{1}{2s} \langle v, \Lambda \Lambda_k Q_k \rangle - \frac{1}{2s} \langle \mathbf{V}, \Lambda \Lambda_k Q_k \rangle. \]
Note that \( v \) is continuous in \( L^2 \) as a function of \( s \), and \( \Lambda \Lambda_k Q_k(\Gamma) \) is locally Lipschitz in \( L^2 \) as a function of \( \Gamma \), so \( \frac{1}{2s} \langle v, \Lambda \Lambda_k Q_k \rangle \) is continuous in \( s \) and locally Lipschitz in \( \Gamma \). For the second term, we only have to check that \( \mathbf{V}(\Gamma) = \sum_j V_j(\Gamma) \) is locally Lipschitz in \( L^2 \) as a function of \( \Gamma \). For the regularity in \( \tau_j \) of \( W_j \), it suffices to use the fact that \( S \) satisfies (1.1) and the regularity of \( S \) from Theorem 1.3. The regularity in \( \mu_j \) and \( y_j \) of \( W_j \) follows from the decay property of the derivative of \( W_j \) from Lemma 2.3. The other terms in \( V_j \), i.e., \( R_j, P_j \), and \( a_j P_j \), are clearly locally Lipschitz in \( L^2 \) as functions of \( \Gamma \) (again the regularity of \( r_j \) follows from the properties of \( S \) in Theorem 1.3).

The second term in the right-hand side of (2.53) rewrites
\[ - \langle \partial_y \left( \partial_{yy} \varepsilon + (\mathbf{V} + \varepsilon)^5 - \mathbf{V}^5 \right), \Lambda_k Q_k \rangle = \langle v - \mathbf{V}, \partial_{yyy}(\Lambda_k Q_k) \rangle + \langle v^5 - \mathbf{V}^5, \partial_y (\Lambda_k Q_k) \rangle \]
and the regularity of these terms is obtained as before. Next, observe that the regularity of the term \( \langle \Psi, \Lambda_k Q_k \rangle \) follows from the explicit expression of \( \Psi \) in the proof of Lemma 2.4 and similar arguments. The regularity of all the other terms in the right-hand side of (2.53) is proved similarly, and we will not comment on it further.

Now note that, from (2.18), we have
\[ \langle \Lambda_k W_k, \Lambda_k Q_k \rangle = \langle \Lambda_k Q_k, \Lambda_k Q_k \rangle + \eta = \| \Lambda Q \|^2_{L^2} + \eta \]
and, using also \( \langle Q', \Lambda Q \rangle = 0 \) by parity,
\[ \langle \partial_y W_k, \Lambda_k Q_k \rangle = \langle \partial_y Q_k, \Lambda_k Q_k \rangle + \eta = \eta, \]
where the notation \( \eta \) denotes various functions which are locally Lipschitz in \( \Gamma \) and small for \( |s_1| \) large and \( \alpha \) small. Thus, we find
\[ -\langle \bar{m}_k \cdot \bar{M}_k V_k, \Lambda_k Q_k \rangle = \frac{\dot{\mu}_k}{\mu_k} \| \Lambda Q \|^2_{L^2} + \dot{\mu}_k \eta + \dot{y}_k \eta + \eta. \]
Moreover, for \( j \neq k \), with similar notation, since \( \langle \Lambda_j W_j, \Lambda_k Q_k \rangle = \eta \) and \( \langle \partial_s W_j, \Lambda_k Q_k \rangle = \eta \),
\[
-\langle \tilde{m}_j \cdot \tilde{M}_j V_j, \Lambda_k Q_k \rangle = \tilde{\mu}_j \eta + \tilde{y}_j \eta + \eta.
\]
Next, using the computations in the proof of Lemma 2.5 and recalling the identity
\[
\frac{\dot{\mu}_j}{\mu_j} = \frac{\dot{\mu}_j}{\mu_j} + \frac{\lambda_0}{2\mu_j^2} \left( 1 + \frac{\lambda_0}{2\tau_j} \right)^{-1},
\]
we have also \( \dot{\tau}_j = \dot{\mu}_j \eta + \dot{y}_j \eta + \eta \) for all \( 1 \leq j \leq K \). For the next term, we simply note that \( \langle P_k, \Lambda_k Q_k \rangle = \langle P, \Lambda Q \rangle + \eta \) and \( \langle P_j, \Lambda_k Q_k \rangle = \eta \) for \( j \neq k \). Finally, we find similarly
\[
\frac{\dot{\mu}_k}{\mu_k} \langle \varepsilon, \Lambda_k^2 Q_k \rangle = \dot{\mu}_k \eta + \eta \quad \text{and} \quad \dot{y}_k \langle \varepsilon, \partial_s (\Lambda_k Q_k) \rangle = \dot{y}_k \eta
\]
where, replacing \( \varepsilon = v - V \), the functions \( \eta \) above depend again continuously on \( s \) through the function \( v \), are locally Lipschitz in \( \Gamma \) and are small for \( |s_1| \) large and \( \alpha \) small.

In the computations of these mass, the relation \( \frac{d}{ds} \langle \varepsilon, \Lambda_k Q_k \rangle = 0 \) rewrites from (2.53) as
\[
L^{\mu_k} \hat{\Gamma} = F^{\mu_k} \quad \text{with} \quad L^{\mu_k} = L_0^{\mu_k} + L_0^{\mu_k}
\]
and \( L_0^{\mu_k} = (0, \ldots, 0, \ell^{-1}_k \| \Lambda Q \|^2_{L^2}, 0, \langle P, \Lambda Q \rangle, 0, \ldots, 0) \),
where the nonzero coefficients in \( L_0^{\mu_k} \) are located at the positions \( 4(k-1) + 2 \) and \( 4k \), and where \( L_0^{\mu_k} \) are locally Lipschitz in \( \Gamma \) and continuous in \( s \), and \( \| L_0^{\mu_k} \| \ll 1 \) for \( |s| \) large and \( \alpha \) small.

Similarly, using \( \langle Q', y \Lambda Q \rangle = \frac{1}{2} \| yQ \|^2_{L^2}, \langle P, Q \rangle = \frac{1}{4} \| Q \|^2_{L^4} \) and \( \langle \Lambda Q, y \Lambda Q \rangle = \langle \Lambda Q, Q \rangle = \langle Q', Q \rangle = 0 \), we check that the other two orthogonality conditions in (2.51) rewrite as
\[
L^{y_k} \hat{\Gamma} = F^{y_k} \quad \text{with} \quad L^{y_k} = L_0^{y_k} + L_0^{y_k}
\]
and \( L_0^{y_k} = (0, \ldots, 0, \frac{1}{2} \| yQ \|^2_{L^2}, \langle P, y \Lambda Q \rangle, 0, \ldots, 0) \),
where the nonzero coefficients in \( L_0^{y_k} \) are located at the positions \( 4(k-1) + 3 \) and \( 4k \), the nonzero coefficient in \( L_0^{y_k} \) is located at the position \( 4k \), and where \( L_0^{y_k}, F^{y_k}, L_0^{a_k}, F^{a_k} \) are locally Lipschitz in \( \Gamma \) and continuous in \( s \), and \( \| L_0^{y_k} \| + \| L_0^{a_k} \| \ll 1 \) for \( |s| \) large and \( \alpha \) small.

We refer to (2.60) and (2.62) in the proof of Lemma 2.9 below for more details about the computation of \( \frac{d}{ds} \langle \varepsilon, (-y_k) \Lambda K Q_k \rangle \) and \( \frac{d}{ds} \langle \varepsilon, Q_k \rangle \).

Denoting by \( \mathbf{D} \) the \((4K) \times (4K)\) matrix formed by the line vectors \( L^{\mu_1}, L^{\mu_2}, L^{\mu_3}, L^{a_1}, \ldots, L^{a_{4K}} \), and denoting \( \mathbf{F} = (F^{\mu_1}, F^{\mu_2}, F^{\mu_3}, F^{a_1}, \ldots, F^{a_{4K}}, F^{\mu_{4K}}, F^{\mu_{4K}}, F^{a_{4K}})^T \), we are reduced to solve the differential system \( \mathbf{D} \hat{\Gamma} = \mathbf{F} \). Note that \( \mathbf{D} \) is a perturbation of the block matrix
\[
\mathbf{D}_0 = \begin{pmatrix}
D_0^1 & 0 & \cdots & 0 \\
0 & D_0^2 & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & 0 & \cdots & D_0^{4K}
\end{pmatrix}, \quad \text{with} \quad D_0^k = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \ell^{-1}_k \| \Lambda Q \|^2_{L^2} & 0 & 0 \\
0 & 0 & \frac{1}{2} \| yQ \|^2_{L^2} & 0 \\
0 & 0 & 0 & \frac{1}{16} \| Q \|^2_{L^4}
\end{pmatrix},
\]
in the sense that \( \mathbf{D} = \mathbf{D}_0 + \mathbf{D}_\eta \) with \( \| \mathbf{D}_\eta \| \ll 1 \) for \( |s| \) large and \( \alpha \) small, and so \( \mathbf{D} \) is invertible, with bounded inverse. The system thus rewrites \( \hat{\mathbf{D}} = \mathbf{D}^{-1} \mathbf{F} \) and the existence and uniqueness of a \( C^1 \) solution \( \hat{\Gamma} \) satisfying (2.8) and (2.51) on some time interval \([s_0, s_0 + \bar{s}]\) with \( \bar{s} > 0 \) follows from the Cauchy–Lipschitz theorem. By the assumptions at \( s_0 \) and continuity arguments, possibly reducing \( \bar{s} \), (2.9), (2.26) and (2.49) hold on \([s_0, s_0 + \bar{s}]\).
**Lemma 2.9** (Modulation equations). Assume that the hypotheses of Lemma 2.7 hold on $I$. Then, for all $1 \leq k \leq K$ and for all $s \in I$,

$$|\tilde{m}_k| \lesssim \|\varepsilon\|_{L^2_k}^2 + \sum_j \|\varepsilon\|_{L^2_k}^2 + |s|^{-2},$$

(2.54)

$$|\dot{a}_k| \lesssim \sum_j \|\varepsilon\|_{L^2_k}^2 + |s|^{-2},$$

(2.55)

$$|\dot{r}_k| \lesssim |s|^{-1} \sum_j \|\varepsilon\|_{L^2_k}^2 + |s|^{-2}.$$  

(2.56)

More precisely, denoting

$$A_k(s, y) = \frac{1}{\|\Lambda Q\|_{L^2_k}^2} \left[ \partial_{yyy}(\Lambda_k Q_k) - \mu_k^{-2} \partial_y(\Lambda_k Q_k) + 5Q_k^2 \partial_y(\Lambda_k Q_k) \right],$$

there holds

$$\left| \frac{\dot{\mu}_k}{\mu_k} + \frac{1}{2\mu_k^2 \tau_k} - \frac{1}{2s} + \frac{a_k}{\mu_k} + \langle \varepsilon, A_k \rangle \right| \lesssim \sum_j \|\varepsilon\|_{L^2_k}^2 + |s|^{-2}.$$  

(2.57)

Finally, there holds

$$|\dot{\varepsilon}_k| \lesssim \sum_j \|\varepsilon\|_{L^2_k}^2 + |s| \sum_j \|\varepsilon\|_{L^2_k}^2 + \sum_{j < k} |a_j| + |s|^{-\frac{2}{3}}.$$  

(2.58)

**Proof.** Computation of $\frac{d}{ds} \langle \varepsilon, \Lambda_k Q_k \rangle$. We continue the calculation started in the above proof of Lemma 2.7 and rewrite (2.53), after integrating by parts,

$$\frac{d}{ds} \langle \varepsilon, \Lambda_k Q_k \rangle = -\left( \frac{\dot{\mu}_k}{\mu_k} - \frac{1}{2s} \right) \langle \varepsilon, \Lambda_k^2 Q_k \rangle - \left( \dot{y}_k - \frac{y_k}{2s} - \frac{1}{\mu_k} \right) \langle \varepsilon, \partial_y(\Lambda_k Q_k) \rangle$$

$$+ \langle \partial_{yy} \varepsilon - \mu_k^{-2} \varepsilon + (V + \varepsilon^5 - V^5, \partial_y(\Lambda_k Q_k)) - \sum_j (\tilde{m}_j \cdot \tilde{M}_j V_j, \Lambda_k Q_k) - \sum_j \dot{r}_j (R_j, \Lambda_k Q_k) - \sum_j \dot{a}_j \langle P_j, \Lambda_k Q_k \rangle - \langle \Psi, \Lambda_k Q_k \rangle. $$

For the first term, we note that, from (2.11) and (2.30),

$$-\left( \frac{\dot{\mu}_k}{\mu_k} - \frac{1}{2s} \right) \langle \varepsilon, \Lambda_k^2 Q_k \rangle = -\left( \frac{\dot{\mu}_k}{\mu_k} - \frac{1}{2s} \right) \langle \varepsilon, \Lambda_k^2 Q_k \rangle + O \left( |s|^{-2} \|\varepsilon\|_{L^2_k} \right)$$

$$= -m_{k,1} \langle \varepsilon, \Lambda_k^2 Q_k \rangle + O \left( |s|^{-1} \|\varepsilon\|_{L^2_k} \right)$$

$$= O \left( \|\tilde{m}_k\| \|\varepsilon\|_{L^2_k} \right) + O \left( |s|^{-1} \|\varepsilon\|_{L^2_k} \right)$$

and similarly

$$-\left( \frac{\dot{y}_k}{2s} - \frac{1}{\mu_k^2} \right) \langle \varepsilon, \partial_y(\Lambda_k Q_k) \rangle = O \left( \|\tilde{m}_k\| \|\varepsilon\|_{L^2_k} \right) + O \left( |s|^{-1} \|\varepsilon\|_{L^2_k} \right).$$

Next, from the definition of $A_k$ and integration by parts, we find

$$\langle \partial_{yy} \varepsilon - \mu_k^{-2} \varepsilon + (V + \varepsilon^5 - V^5, \partial_y(\Lambda_k Q_k) \rangle$$

$$= \|\Lambda Q\|_{L^2_k}^2 \langle \varepsilon, A_k \rangle + \langle (V + \varepsilon^5 - V^5, 5Q_k^2 - 5Q_k^2 \varepsilon, \partial_y(\Lambda_k Q_k) \rangle$$

$$= \|\Lambda Q\|_{L^2_k}^2 \langle \varepsilon, A_k \rangle + O \left( |s|^{-1} \|\varepsilon\|_{L^2_k} \right) + O \left( \|\varepsilon\|_{L^2_k}^2 \right).$$
Indeed, since
\[
\left| (V + \varepsilon)^5 - V^5 - 5Q_k^2 \varepsilon \right| \leq \left| (V + \varepsilon)^5 - V^5 - \left( (Q_k + \varepsilon)^5 - Q_k^5 \right) \right| + \left| (Q_k + \varepsilon)^5 - Q_k^5 - 5Q_k^2 \varepsilon \right|
\]
\[
\lesssim |V - Q_k| |\varepsilon| + |\varepsilon|^2 \lesssim |V_k - Q_k||\varepsilon| + \sum_{j \neq k} |V_j||\varepsilon| + |\varepsilon|^2,
\]
we have, using (2.18), (2.19) and (2.21),
\[
\left| \left\langle (V + \varepsilon)^5 - V^5 - 5Q_k^2 \varepsilon, \partial_y (\Lambda_k Q_k) \right\rangle \right| \lesssim |s|^{-1} \|\varepsilon\|_{L_k^2} + \|\varepsilon\|_{L_k^2}^2.
\]
Moreover, since \( \langle \Lambda_k W_k, \Lambda_k Q_k \rangle = \langle \Lambda_k Q_k, \Lambda_k Q_k \rangle + O(|s|^{-1}) = \|\Lambda Q\|_{L^2}^2 + O(|s|^{-1}) \) from (2.18), and \( \langle Q', \Lambda Q \rangle = 0 \) by parity, we have
\[
\langle \tilde{m}_k \cdot \tilde{M}_k V_k, \Lambda_k Q_k \rangle = -m_{k,1} \left[ \|\Lambda Q\|_{L^2}^2 + O(|s|^{-1}) \right] + O(|s|^{-1} |\tilde{m}_k|).\]
Note also that, for \( j \neq k \),
\[
\left| \langle \tilde{M}_j V_j, \Lambda_k Q_k \rangle \right| + \left| \langle R_j, \Lambda_k Q_k \rangle \right| + \left| \langle P_j, \Lambda_k Q_k \rangle \right| \lesssim |s|^{-10}.
\]
Thus, using \( \|\langle \Psi, \Lambda_k Q_k \rangle\| \gtrsim \|\Psi\|_{L_k^\infty} \lesssim |s|^{-2} \) from (2.32), we find
\[
\frac{d}{ds} \langle \varepsilon, \Lambda_k Q_k \rangle = m_{k,1} \left[ \|\Lambda Q\|_{L^2}^2 + O(|s|^{-1}) \right] + \|\Lambda Q\|_{L^2}^2 \langle \varepsilon, A_k \rangle + O(|s|^{-1} |\tilde{m}_k|) + O(|\tilde{m}_k||\varepsilon|_{L_k^2})
\]
\[
+ O(|\varepsilon|_{L_k^2}^2) + O(|\tilde{r}_k|) + O(|\tilde{a}_k|) + O \left( |s|^{-1} \sum_{j \neq k} |\tilde{m}_j| + |\tilde{a}_j| + |\tilde{r}_j| \right) + O(|s|^{-2}).
\]
Finally, we use (2.40) to estimate \( \tilde{r}_k \) and \( \tilde{r}_j \) for \( j \neq k \), and simplify the last expression as
\[
\frac{d}{ds} \langle \varepsilon, \Lambda_k Q_k \rangle = m_{k,1} \left[ \|\Lambda Q\|_{L^2}^2 + O(|s|^{-1}) \right] + \|\Lambda Q\|_{L^2}^2 \langle \varepsilon, A_k \rangle + O(|\tilde{m}_k||\varepsilon|_{L_k^2})
\]
\[
+ O(|\varepsilon|_{L_k^2}^2) + O \left( \sum_j |\tilde{a}_j| \right) + O \left( |s|^{-1} \sum_j |\tilde{m}_j| \right) + O(|s|^{-2}). \tag{2.59}
\]

**Computation of** \( \frac{d}{ds} \langle \varepsilon, (\cdot - y_k) \Lambda_k Q_k \rangle \). Using \( \langle Q', y \Lambda Q \rangle = \frac{1}{2} \|y Q\|_{L^2}^2 \) and \( \langle \Lambda Q, y \Lambda Q \rangle = 0 \) by parity, we find with similar computation
\[
\frac{d}{ds} \langle \varepsilon, (\cdot - y_k) \Lambda_k Q_k \rangle = m_{k,2} \left[ \frac{1}{2} \|y Q\|_{L^2}^2 + O(|s|^{-1}) \right] + O(|\tilde{m}_k||\varepsilon|_{L_k^2})
\]
\[
+ O(|\varepsilon|_{L_k^2}^2) + O \left( \sum_j |\tilde{a}_j| \right) + O \left( |s|^{-1} \sum_j |\tilde{m}_j| \right) + O(|s|^{-2}). \tag{2.60}
\]

**Computation of** \( \frac{d}{ds} \langle \varepsilon, Q_k \rangle \). From the equation (2.50) of \( \varepsilon \), the expression (2.29) of \( E_V \) and (2.52), we find
\[
\frac{d}{ds} \langle \varepsilon, Q_k \rangle = -\frac{1}{2s} \langle \Lambda \varepsilon, Q_k \rangle - \left\langle \partial_y \left( \partial_{yy} \varepsilon + (V + \varepsilon)^5 - V^5 \right), Q_k \right\rangle - \sum_j \langle \tilde{m}_j \cdot \tilde{M}_j V_j, Q_k \rangle
\]
\[
- \sum_j \tilde{r}_j \langle R_j, Q_k \rangle - \sum_j \tilde{a}_j \langle P_j, Q_k \rangle - \langle \Psi, Q_k \rangle - \frac{\tilde{M}_k}{\mu_k} \langle \varepsilon, \Lambda_k Q_k \rangle - \tilde{y}_k \langle \varepsilon, \partial_y Q_k \rangle.
\]
Note again that, for \( j \neq k \),

\[
|\langle \tilde{M}_j V_j, Q_k \rangle| + |\langle R_j, Q_k \rangle| + |\langle P_j, Q_k \rangle| \lesssim |s|^{-10}.
\]

Moreover, from (2.2) and (2.4),

\[
\langle P_k, Q_k \rangle = \frac{1}{16} \|Q\|_{L^1}^2 + O(|s|^{-10}), \quad \langle R_k, Q_k \rangle = -\frac{3}{4} \|Q\|_{L^1},
\]

and

\[
\langle \Lambda_k Q_k, Q_k \rangle = \langle \partial_y Q_k, Q_k \rangle = 0.
\]

Thus, after integration by parts,

\[
\frac{d}{ds} \langle \varepsilon, Q_k \rangle = \langle \partial_{yy} \varepsilon - \tilde{\mu}_k^{-2} \varepsilon + (V + \varepsilon)^5 - V^5, \partial_y Q_k \rangle + \frac{3}{4} \|Q\|_{L^1} \dot{r}_k - \frac{1}{16} \|Q\|_{L^1}^2 \dot{\alpha}_k \\
- \langle \Psi, Q_k \rangle - \left( \frac{\dot{\mu}_k}{\tilde{\mu}_k} - \frac{1}{2s} \right) \langle \varepsilon, \Lambda_k Q_k \rangle - \left( \frac{\dot{y}_k}{y_k} - \frac{1}{\tilde{\mu}_k^2} \right) \langle \varepsilon, \partial_y Q_k \rangle \\
+ O(|s|^{-1} |\tilde{m}_k|) + O \left( |s|^{-10} \sum_{j \neq k} (|\tilde{m}_j| + |\dot{\alpha}_j| + |\dot{r}_j|) \right) + O(|s|^{-10} |\dot{\alpha}_k|).
\]

But, from the cancellation \( LQ' = 0 \) and the definition of \( Q_k \), we have

\[
\partial_{yy} Q_k - \tilde{\mu}_k^{-2} \partial_y Q_k + 5Q_k^4 \partial_y Q_k = 0. \tag{2.61}
\]

Thus, from (2.11) and again integration by parts, we obtain as before

\[
\frac{d}{ds} \langle \varepsilon, Q_k \rangle = \langle (V + \varepsilon)^5 - V^5, \partial_y Q_k \rangle + \frac{3}{4} \|Q\|_{L^1} \dot{r}_k - \frac{1}{16} \|Q\|_{L^1}^2 \dot{\alpha}_k \\
- \langle \Psi, Q_k \rangle + O \left( |\tilde{m}_k| \|\varepsilon\|_{L^2_k} \right) + O \left( |s|^{-1} \|\varepsilon\|_{L^2_k} \right) \\
+ O(|s|^{-1} |\tilde{m}_k|) + O \left( |s|^{-10} \sum_{j} (|\tilde{m}_j| + |\dot{\alpha}_j| + |\dot{r}_j|) \right),
\]

and also

\[
\left| \langle (V + \varepsilon)^5 - V^5, \partial_y Q_k \rangle \right| \lesssim |s|^{-1} \|\varepsilon\|_{L^2_k} + \|\varepsilon\|_{L^2_k}^2.
\]

Therefore, using (2.33), (2.39) and (2.40), we obtain

\[
\frac{d}{ds} \langle \varepsilon, Q_k \rangle = -\frac{1}{16} \|Q\|_{L^1}^2 \left( \dot{\alpha}_k + \frac{2\alpha_k}{\mu_k^4 \tau_k} + \frac{2\alpha_k^2}{\mu_k^4} \right) - \frac{1}{4} \|Q\|_{L^1} \left( \dot{r}_k + \frac{r_k}{\mu_k^4 \tau_k} + \frac{2\alpha_k r_k}{\mu_k^4} \right) \\
+ O(|s|^{-1} \|\varepsilon\|_{L^2_k}) + O(|\tilde{m}_k| \|\varepsilon\|_{L^2_k}) + O(\|\varepsilon\|_{L^2_k}^2) \\
+ O \left( |s|^{-1} \sum_{j < k} |a_j| \right) + O \left( |s|^{-10} \sum_{j} |\dot{\alpha}_j| \right) \\
+ O \left( |s|^{-1} \sum_{j < k} |\tilde{m}_j| \right) + O \left( |s|^{-10} \sum_{j > k} |\tilde{m}_j| \right) + O(|s|^{-\frac{5}{2}}). \tag{2.62}
\]
Proof of (2.54)–(2.57). We assume now that the three orthogonality conditions (2.51) are satisfied. First, we give a simplified bound on $|\dot{a}_k|$ deduced from $\frac{d}{ds}(\varepsilon, Q_k) = 0$ in (2.62). Using also (2.40), we find

$$|\dot{a}_k| \lesssim |s|^{-2} + |\vec{m}_k||\varepsilon||L^2_k| + ||\varepsilon||^2_{L^2_k} + |s|^{-1} \sum_j (|\dot{a}_j| + |\vec{m}_j|).$$

Summing over $1 \leq k \leq K$ and taking $|s_1|$ large enough, we obtain

$$\sum_k |\dot{a}_k| \lesssim |s|^{-2} \sum_j |\vec{m}_j||\varepsilon||L^2_j| + \sum_j ||\varepsilon||^2_{L^2_j} + |s|^{-1} \sum_j |\vec{m}_j|,$$

and so, inserting this estimate in the above bound on $|\dot{a}_k|$, we get

$$|\dot{a}_k| \lesssim |s|^{-2} + |\vec{m}_k||\varepsilon||L^2_k| + \sum_j ||\varepsilon||^2_{L^2_j} + |s|^{-1} \sum_j |\vec{m}_j|.$$

Now, we insert this estimate on $|\dot{a}_k|$ into (2.59) and (2.60) and we obtain, using (2.51),

$$|\vec{m}_k| \lesssim ||\varepsilon||L^2_k| + \sum_j |\vec{m}_j||\varepsilon||L^2_j| + \sum_j ||\varepsilon||^2_{L^2_j} + |s|^{-1} \sum_j |\vec{m}_j| + |s|^{-2}.$$

Summing as before over $1 \leq k \leq K$, using (2.49) and taking $|s_1|$ large enough, we find

$$\sum_k |\vec{m}_k| \lesssim \sum_j ||\varepsilon||L^2_j| + |s|^{-2},$$

and so (2.54). Inserting this estimate into the above simplified bound on $|\dot{a}_k|$ and into (2.40), we obtain respectively (2.55) and (2.56). Finally, inserting (2.54) into (2.59), we obtain (2.57).

Proof of (2.58). Directly from the definition (2.44) of $\varepsilon_k$, we first compute

$$\dot{\varepsilon}_k = \frac{1}{\mu^2_k} \left[ \left( a_k + \frac{1}{2\tau_k} + \frac{4r_k}{\|Q\|L^1} \right) - 2s \frac{\mu^2_k}{\mu^2_k} \left( a_k + \frac{1}{2\tau_k} + \frac{4r_k}{\|Q\|L^1} \right) + s \left( \dot{a}_k - \frac{\dot{\tau}_k}{2\tau_k} + \frac{4\dot{r}_k}{\|Q\|L^1} \right) \right].$$

Using (2.8) and the definition (2.30) of $m_{k,1}$, we find

$$\dot{\varepsilon}_k = \frac{s}{\mu^2_k} \left[ \left( \dot{a}_k + \frac{2a_k}{\mu^3_k \tau_k} + \frac{2a^2_k}{\mu^3_k} \right) + \frac{4}{\|Q\|L^1} \left( \dot{\tau}_k + \frac{r_k}{\mu^3_k \tau_k} + \frac{2a_k r_k}{\mu^3_k} \right) \right] + O(|\vec{m}_k|).$$

But, inserting estimates (2.54)–(2.56) into (2.62), and using from (2.51) the orthogonality condition $\frac{d}{ds}(\varepsilon, Q_k) = 0$, we also find

$$\left\| \frac{1}{16} \|Q\|_{L^1} \left( \dot{a}_k + \frac{2a_k}{\mu^3_k \tau_k} + \frac{2a^2_k}{\mu^3_k} \right) + \frac{1}{4} \|Q\|_{L^1} \left( \dot{\tau}_k + \frac{r_k}{\mu^3_k \tau_k} + \frac{2a_k r_k}{\mu^3_k} \right) \right\| \lesssim \sum_{j<k} ||\varepsilon||_{L^2_j} + \sum_j ||\varepsilon||^2_{L^2_j} + |s|^{-1} \sum_{j<k} |a_j| + |s|^{-\frac{3}{2}}.$$

Gathering the two last estimates and (2.54), we obtain (2.58), which concludes the proof of Lemma 2.9.

Next, we prove more precise estimates on the modulation parameters, using the notation introduced in (9).
Lemma 2.10 (Refined modulation equations). Assume that the hypotheses of Lemma 2.7 hold on \( I \). Let \( f_k = \tilde{\mu}_k + \tilde{y}_k \). Then, for all \( 1 \leq k \leq K \) and for all \( s \in I \),
\[
\left| \dot{f}_k + \frac{1}{2} (1 + 3\theta_k) \frac{f_k}{s} + (1 + \tilde{\mu}_k) \langle \varepsilon, A_k \rangle \right| \lesssim |s|^{-1} \left| e_k - \ell_k \left( \frac{1}{2} + \theta_k \right) \right| + \sum_j \| \varepsilon \|_{L_j^2}^2 + |s|^{-1} \left( |s|^{-\frac{1}{2}} + |\tilde{\mu}_k|^2 + |\tilde{y}_k|^2 + \sum_{j<k} (|\tilde{\mu}_j| + |\tilde{\tau}_j| + |\tilde{y}_j|) \right) \quad (2.63)
\]
and
\[
\left| \dot{\tilde{y}}_k - \frac{\tilde{y}_k}{2s} + \frac{f_k}{s} \right| \lesssim \sum_j \| \varepsilon \|_{L_j^2}^2 + |s|^{-1} |\tilde{\mu}_k|^2 + |s|^{-2}. \quad (2.64)
\]

Proof. We first prove the two estimates
\[
\left| \dot{\tilde{\mu}}_k - \frac{\tilde{\mu}_k}{2s} + \frac{1}{s} \left( \frac{e_k}{\ell_k} - \frac{1}{2} - \theta_k \right) + \frac{3\theta_k}{2s} (\tilde{\mu}_k + \tilde{y}_k) + (1 + \tilde{\mu}_k) \langle \varepsilon, A_k \rangle \right| \lesssim \sum_j \| \varepsilon \|_{L_j^2}^2 + |s|^{-1} \left( |s|^{-\frac{1}{2}} + |\tilde{\mu}_k|^2 + |\tilde{y}_k|^2 + \sum_{j<k} (|\tilde{\mu}_j| + |\tilde{\tau}_j| + |\tilde{y}_j|) \right) \quad (2.65)
\]
and
\[
\left| \dot{\tilde{\mu}}_k - \frac{\tilde{\mu}_k}{2s} + \frac{\mu_k}{s} \right| \lesssim \sum_j \| \varepsilon \|_{L_j^2}^2 + |s|^{-1} |\mu_k|^2 + |s|^{-2}. \quad (2.66)
\]
Indeed, multiplying (2.57) by \( \mu_k \), replacing
\[
\frac{a_k}{\mu_k^2} + \frac{1}{2\mu_k^2 \tau_k} = \frac{e_k}{s} - \frac{4r_k}{\mu_k^2} \| Q \|_{L^1}
\]
from (2.44) and \( r_k \) by its asymptotics (2.41), we find
\[
\left| \dot{\mu}_k + \frac{e_k}{s} - \frac{\mu_k}{2s} + \mu_k \langle \varepsilon, A_k \rangle - \frac{\ell_k^2 \theta_k}{\mu_k^2 s} \left( 1 + \frac{1}{2} \mu_k - \frac{3}{2} \tilde{y}_k \right) \right| \lesssim \sum_j \| \varepsilon \|_{L_j^2}^2 + |s|^{-1} \left( |s|^{-\frac{1}{2}} + |\mu_k|^2 + |\tilde{y}_k|^2 + \sum_{j<k} (|\tilde{\mu}_j| + |\tilde{\tau}_j| + |\tilde{y}_j|) \right),
\]
and so (2.65) after expanding \( \mu_k = \ell_k (1 + \tilde{\mu}_k) \) and \( \mu_k^{-2} = \ell_k^{-2} \left[ 1 - 2\tilde{\mu}_k + O(|\tilde{\mu}_k|^2) \right] \) in the left-hand side. Similarly, from \( \dot{y}_k = 2\ell_k^{-2} s (1 + \tilde{y}_k) \), we get \( \dot{\tilde{y}}_k = 2\ell_k^{-2} (1 + \tilde{y}_k) + 2\ell_k^{-2} s \tilde{y}_k \), which gives, inserted into the second line of (2.54),
\[
|2s \dot{\tilde{y}}_k + \tilde{y}_k + 2\mu_k| \lesssim |\mu_k|^2 + \| \varepsilon \|_{L^2}^2 + \sum_j \| \varepsilon \|_{L_j^2}^2 + |s|^{-1}.
\]
Dividing the last estimate by \( 2s \), we get (2.66). Finally, adding (2.65) and (2.66) yields (2.63), and replacing \( \dot{\mu}_k = f_k - \tilde{y}_k \) into (2.66) yields (2.64).

Finally, as in the previous works [18, 22, 31] related to blow up for (gKdV), we need some \( L^1 \) type estimates on \( \varepsilon \). Technically, we prove a priori estimates on the quantity \( \langle \varepsilon, P_k \rangle \) and its time variation in the following lemma.
Lemma 2.11. Assume that the hypotheses of Lemma 2.7 hold on $I$. Then, for all $1 \leq k \leq K$ and for all $s \in I$,

$$|\langle \varepsilon, P_k \rangle| \lesssim |s|^\frac{1}{2} \|\varepsilon\|_{L^2(y>y_{k+1})},$$

(2.67)

and, for some constant $c > 0$,

$$\left| \frac{d}{ds} \langle \varepsilon, P_k \rangle - cs \dot{a}_k \right| \lesssim |s|^{-\frac{3}{2}} \|\varepsilon\|_{L^2(y>y_{k+1})} + \sum_{j \leq k} \|\varepsilon\|_{L^2_j} + \sum_{j \leq k} |a_j| + \sum_j \|\varepsilon\|^2_{L^2_j} + |s|^{-\frac{5}{2}}.$$  

(2.68)

**Proof.** To prove (2.67), we note that, from (2.14) and the definition of $P_k$, we have $P_k(y) = 0$ for $y \leq \frac{1}{2}(y_{k+1} + y_k)$. Thus, using also $\|P_k\|_{L^2} \lesssim |s|^\frac{1}{2}$ from (2.13),

$$|\langle \varepsilon, P_k \rangle| \lesssim \|P_k\|_{L^2} \|\varepsilon\|_{L^2(y>y_{k+1})} \lesssim |s|^\frac{1}{2} \|\varepsilon\|_{L^2(y>y_{k+1})}.$$ 

To prove (2.68), we proceed as in the proof of Lemma 2.9. First note that

$$\partial_s P_k = -\frac{\mu_k}{\mu} \Lambda_k P_k - \dot{y}_k \partial_y P_k + |s|^{-1} Z_k,$$

where $Z_k$ is defined by (2.34) and satisfies, from (2.9) and (2.14),

$$|Z_k(y)| \lesssim \frac{1}{2}(y_{k+1} + y_k) < y_k - \frac{\theta_k^2}{2}|s|(y).$$

Thus, using the equation (2.50) of $\varepsilon$, the decomposition (2.29) of $E_V$ and integrations by parts, we find

$$\frac{d}{ds} \langle \varepsilon, P_k \rangle = -\left( \frac{\mu_k}{\mu} - \frac{1}{2s} \right) \langle \varepsilon, \Lambda_k P_k \rangle - \left( \dot{y}_k - \frac{y_k}{2s} \right) \langle \varepsilon, \partial_y P_k \rangle$$

$$+ |s|^{-1} \langle \varepsilon, Z_k \rangle + \langle \varepsilon, \partial_{yy} P_k \rangle + \langle (V + \varepsilon)^5 - V^5, \partial_y P_k \rangle$$

$$- \sum_j \langle \tilde{m}_j, \tilde{N}_j V_j, P_k \rangle - \sum_j \dot{r}_j \langle R_j, P_k \rangle - \sum_j \dot{a}_j \langle P_j, P_k \rangle - \langle \Psi, P_k \rangle.$$ 

For the first term, from (2.11), (2.30), (2.49) and (2.54), we note that

$$\left| \frac{\mu_k}{\mu} - \frac{1}{2s} \right| \lesssim \left| \frac{\mu_k}{\mu} - \frac{1}{2s} \right| + |s|^{-2} \lesssim |\tilde{m}_k| + |s|^{-1} \lesssim \|\varepsilon\|_{L^2_k} + |s|^{-1}.$$ 

Thus, using also (2.12), we find

$$\left| \left( \frac{\mu_k}{\mu} - \frac{1}{2s} \right) \langle \varepsilon, \Lambda_k P_k \rangle \right| \lesssim \left( \|\varepsilon\|_{L^2_k} + |s|^{-1} \right) \left( \|\varepsilon\|_{L^2_k} + |s|^{-\frac{1}{2}} \|\varepsilon\|_{L^2(y>y_{k+1})} \right)$$

$$\lesssim \|\varepsilon\|_{L^2_k} + |s|^{-\frac{1}{2}} \|\varepsilon\|_{L^2(y>y_{k+1})}.$$ 

Similarly, we also find

$$\left| \left( \dot{y}_k - \frac{y_k}{2s} \right) \langle \varepsilon, \partial_y P_k \rangle \right| \lesssim \int |\varepsilon||\partial_y P_k| \lesssim \|\varepsilon\|_{L^2_k} + |s|^{-\frac{1}{2}} \|\varepsilon\|_{L^2(y>y_{k+1})}$$

and

$$|s|^{-1} \langle \varepsilon, Z_k \rangle \lesssim |s|^{-\frac{1}{2}} \|\varepsilon\|_{L^2(y>y_{k+1})}.$$ 

Using again (2.12), we find

$$|\langle \varepsilon, \partial_{yy} P_k \rangle| \lesssim \|\varepsilon\|_{L^2_k} + |s|^{-\frac{1}{2}} \|\varepsilon\|_{L^2(y>y_{k+1})}.$$ 


and, since \( \|\mathbf{V}\|_{L^\infty} + \|\varepsilon\|_{L^\infty} \lesssim 1 \),
\[
\left| \langle (\mathbf{V} + \varepsilon)^5 - \mathbf{V}^5, \partial_y P_k \rangle \right| \lesssim \|\varepsilon\|_{L^2_{x,y}} + |s|^{-\frac{1}{2}} \|\varepsilon\|_{L^2(y > y_{k+1})}.
\]

For the next term, we first note that, from (2.3), (2.9) and (2.14), for \( j \neq k \),
\[
\begin{align*}
\{ |\langle \Lambda_j R_j, P_k \rangle| &+ |\langle \partial_y R_j, P_k \rangle| + |\langle \Lambda_j P_j, P_k \rangle| + |\langle \partial_y P_j, P_k \rangle| \} \lesssim |s|^{-10}, \\
\{ |\langle \Lambda_k R_k, P_k \rangle| + |\langle \partial_y R_k, P_k \rangle| \} &\lesssim 1 \quad \text{and} \quad \langle \Lambda_k P_k, P_k \rangle = \langle \partial_y P_k, P_k \rangle = 0.
\end{align*}
\]

Then, for \( j > k \), from (2.14) and (2.21), we find
\[
|\langle \Lambda_j W_j, P_k \rangle| + |\langle \partial_y W_j, P_k \rangle| \lesssim |s|^{-10}.
\]

Finally, for \( j \leq k \), using (2.13), (2.15) and (2.22), we have
\[
|\langle \Lambda_j W_j, P_k \rangle| \lesssim |\langle \Lambda_j (W_j - Q_j), P_k \rangle| + |\langle \Lambda_j Q_j, P_k \rangle| \lesssim \|\Lambda_j (W_j - Q_j)\|_{L^2} \|P_k\|_{L^2} + \|\Lambda_j Q_j\|_{L^1} \lesssim 1
\]
and, using (2.15) for \( m = 1 \) and \( \langle P, Q' \rangle = 0 \), we find similarly \( |\langle \partial_y W_j, P_k \rangle| \lesssim |s|^{-\frac{1}{2}} \). Thus, gathering the previous estimates, using (2.54) and then (2.49), we obtain
\[
\left| \sum_j \langle \bar{m}_j \cdot \tilde{M}_j V_j, P_k \rangle \right| \lesssim \sum_{j \leq k} |\bar{m}_j| + |s|^{-10} \sum_{j > k} |\bar{m}_j| \lesssim \sum_{j \leq k} \|\varepsilon\|_{L^2_j} + \sum_j \|\varepsilon\|_{L^2_j}^2 + |s|^{-2}.
\]

Next, by (2.56) and the decay properties of \( R_j \), we find
\[
\left| \sum_j \tilde{r}_j \langle R_j, P_k \rangle \right| \lesssim \sum_j |\tilde{r}_j| \lesssim |s|^{-1} \sum_j \|\varepsilon\|_{L^2_j} + |s|^{-2} \lesssim |s|^{-\frac{3}{2}}.
\]

For the next term, we first note that, by (2.3) and (2.14), we have \( |\langle P_j, P_k \rangle| \lesssim |s|^{-10} \) for \( j \neq k \). For \( j = k \), we compute
\[
\int P_k^2 = \tilde{\mu}_k^{-1} \int P^2 \left( \frac{y - y_k}{\mu_k} \right) \chi^2 \left( \frac{y - y_k}{\mu_k} |s|^{-1} \right) dy = |s| \int P^2(|s| z) \chi^2(z) dz.
\]

But, again from (2.3), we find
\[
|s| \int_{z > 0} P^2(|s| z) \chi^2(z) dz \lesssim |s| \int_{z > 0} e^{-|s| z} dz \lesssim 1
\]
and
\[
|s| \int_{z < 0} P^2(|s| z) \chi^2(z) dz - \frac{1}{4} \|Q\|_{L^1}^2 \int_{z < 0} \chi^2(z) dz \lesssim |s| \int_{z < 0} e^{|s| z / 2} dz \lesssim 1.
\]

Thus, denoting \( c = \frac{1}{4} \|Q\|_{L^1}^2 \int_{z < 0} \chi^2(z) dz > 0 \), we have found \( |\langle P_k, P_k \rangle + cs| \lesssim 1 \) and so, using (2.49) and (2.55),
\[
\left| \sum_j \tilde{a}_j \langle P_j, P_k \rangle + cs\tilde{a}_k \right| \lesssim |\tilde{a}_k| + |s|^{-11} \lesssim \sum_j \|\varepsilon\|_{L^2_j}^2 + |s|^{-2}.
\]

Finally, to control the source term \( \langle \Psi, P_k \rangle \), we use (2.13), (2.14) and (2.32), and thus obtain, for \( k < K \),
\[
|\langle \Psi, P_k \rangle| \lesssim \|P_k\|_{L^2} \|\Psi\|_{L^2(y > \frac{1}{2} (y_{k+1} + y_k))} \lesssim |s|^{-\frac{1}{2}} \|\Psi\|_{L^2(y > y_{k+1} - |s|^{1/2})} \lesssim |s|^{-\frac{3}{2}} + \sum_{j \leq k} |a_j|.
\]
Similarly, for \( k = K \), using (2.31), we find
\[
|\langle \Psi, P_K \rangle| \lesssim \|P_K\|_{L^2} \|\Psi\|_{L^2} \lesssim |s|^{-\frac{1}{2}} + \sum_j |a_j|.
\]
Gathering the above estimates, we get (2.68), which concludes the proof of Lemma 2.11. \( \square \)

2.4. Weak \( H^1 \) stability of the decomposition. The next lemma shows that the decomposition of Lemma 2.7 is stable by weak \( H^1 \) limit. To prove such a result, we rely on the weak \( H^1 \) stability of the flow of (1.1), as stated in [3, Lemma 2.10] and proved e.g. in [18].

**Lemma 2.12.** Let \((v_{0,n})\) be a sequence of \( H^1 \) functions such that
\[
\forall s \in I, \quad v_{0,n}(s) \to v(s) \quad \text{in } H^1 \text{ weak as } n \to +\infty,
\]
and
\[
\forall s \in I, \quad \varepsilon_n(s) \to \varepsilon(s) \text{ in } H^1 \text{ weak, } \Gamma_n(s) \to \Gamma(s) \text{ as } n \to +\infty,
\]
where \( \Gamma \) and \( \varepsilon = v - V[\Gamma] \) satisfy (2.8), (2.9), (2.26), (2.49) and (2.51) on \( I \).

**Proof.** Let \( u_n \) and \( u \) be the solutions of (1.1) defined from the change of variables (2.6) such that \( v_n = \tilde{u}_n \) and \( v = \tilde{u} \) respectively. Let \( T_i = \sqrt{-2s_i} \) for \( i = 0, 1, 2 \). Then \( u_n(T_0) \to u(T_0) \) in \( H^1 \) weak as \( n \to +\infty \) and, since \( v_n \) satisfies the assumptions of Lemma 2.7, there exists \( C_1 > 0 \) such that \( \max_{t \in [T_1, T_2]} \|u_n(t)\|_{H^1} \leq C_1 \). We deduce from the \( H^1 \) weak stability of the flow of (1.1) that, for all \( t \in [T_1, T_2], u_n(t) \to u(t) \) in \( H^1 \) weak and so, from (2.6), \( v_n(s) \to v(s) \) in \( H^1 \) weak for any \( s \in I \).

Next, it follows from (2.9), (2.26) and (2.8), (2.54), (2.55) that \( |\Gamma_n| + |\dot{\Gamma}_n| \leq C \) on \( I \) for some \( C > 0 \). Thus, by Ascoli’s theorem, up to the extraction of a subsequence, there exists a continuous function \( \Gamma \) satisfying (2.9) and (2.26) such that \( \Gamma_n \to \Gamma \) uniformly on \( I \) as \( n \to +\infty \). By the definition of \( V[\Gamma] \), it follows that \( V[\Gamma_n] \to V[\Gamma] \) in \( H^1(\mathbb{R}) \) as \( n \to +\infty \). The weak \( H^1 \) convergence of \( \varepsilon_n \) to \( \varepsilon \) then follows. Writing the integral form of the differential system satisfied by \( \Gamma_n \) as stated in Remark 2.8 and passing to the limit, we obtain that \( \Gamma \) is a \( C^1 \) function of time which also satisfies (2.8) on \( I \).

By weak convergence, \( \langle \varepsilon, \Lambda_k Q_k \rangle \), \( \langle \varepsilon, (-y_k)\Lambda_k Q_k \rangle \) and \( \langle \varepsilon, Q_k \rangle \) are constant functions of time and thus (2.51) holds for \( \Gamma \) and \( \varepsilon \). By weak convergence again, the function \( v \) satisfies the assumptions of Lemma 2.7 on \( I \), and the uniqueness statement proves that \( \Gamma \) and \( \varepsilon \) correspond to the decomposition of \( v \) from Lemma 2.7. It follows that the above limits hold for the whole sequence. \( \square \)

3. Proof of Theorem 1.1

The proof of Theorem 1.1 is mainly based on the following proposition, written with the notation of Section 2, i.e. in the rescaled variables (2.6)–(2.7) and using the approximate solution \( V(s) = V[\Gamma(s)] \).
Proposition 3.1. There exist $S_0 < -1$ with $|S_0|$ large, an $H^1$ solution $v$ of (2.7) and a $C^1$ function $\Gamma = (\tau_1, \mu_1, y_1, a_1, \ldots, \tau_K, \mu_K, y_K, a_K)^T$ defined on $(-\infty, S_0]$ such that, defining $\varepsilon$ by

$$v(s) = V[\Gamma(s)] + \varepsilon(s),$$

for all $1 \leq k \leq K$, for all $s \leq S_0$,

$$\begin{aligned}
|\mu_k(s)\ell_k - 1| + |\tau_k(s) - 1| + |y_k(s) - 1| \leq |s|^{-\frac{1}{2}},
\end{aligned}$$

(3.1)

and

$$\langle \varepsilon(s), \Lambda_k Q_k(s) \rangle = \langle \varepsilon(s), (\cdot - y_k)\Lambda_k Q_k(s) \rangle = \langle \varepsilon(s), Q_k(s) \rangle = 0.$$

(3.2)

The proof of Proposition 3.1 is by compactness. For $n > 1$ large, we let $S_n = -n$, and we construct a solution $v_n$ of (2.7) with a well-prepared data $v_0(S_n)$. Proposition 3.2 below shows uniform estimates on $v_n$ on a time interval $[S_n, S_0]$, where $S_0 < -1$ and $|S_0| > 2|s_1|$ is large enough but independent of $n$. In Section 3.8, we obtain $v_n(S_0) \rightarrow v_0$ passing to the weak limit in $n$ (up to a subsequence). Then, from the uniform estimates and Lemma 2.12, the initial data $v(S_0) = v_0$ provides the desired solution $v$ of (2.7).

The proof of Theorem 1.1 is also given in Section 3.8, as a straightforward consequence of Proposition 3.1 and the change of variables (2.6).

3.1. Formal discussion on the system of parameters. We discuss formally how to derive estimates on the various parameters on a time interval $[S_n, S_0]$ from Lemmas 2.9 and 2.10, in particular from (2.58), (2.63) and (2.64). We refer to Sections 3.6 and 3.7 below for rigorous arguments. Suppose that we already know from an induction argument that, for all $j < k$,

$$|\tilde{\mu}_j + \tilde{\tau}_j| + |\tilde{y}_j| \leq |s|^{-\delta_{k-1}}, \quad |a_j| \leq |s|^{-1-\delta_{k-1}},$$

for some small $\delta_{k-1} > 0$. Then, from (2.58) and neglecting $\varepsilon$, we obtain $|\dot{e}_k| \lesssim |s|^{-1-\delta_{k-1}}$ and so, by integration and assuming $c_k(S_n) = \ell_k \left(\frac{1}{2} + \theta_k\right)$, we find $|c_k - \ell_k \left(\frac{1}{2} + \theta_k\right)| \lesssim |s|^{-\delta_{k-1}}$. Next, from (2.63), inserting the above information on $c_k$, taking $\delta_{k-1} \lesssim \frac{1}{12}$ and neglecting for the moment the higher order terms $|\tilde{\mu}_k|^2$ and $|\tilde{y}_k|^2$, we get

$$|f_k + \frac{1}{2}(1 + 3\theta_k)\frac{f_k}{s}| \lesssim |s|^{-1-\delta_{k-1}}.$$

The general behavior of functions $f_k$ satisfying such a differential inequality depends on the value of the parameter $\frac{1}{2}(1 + 3\theta_k)$. Indeed, the last inequality is equivalent to

$$\left|\frac{d}{ds} \left[ (-s)^{\frac{1}{2}(1+3\theta_k)} f_k \right] \right| \lesssim |s|^{-1-\delta_{k-1}+\frac{1}{2}(1+3\theta_k)},$$

If $\frac{1}{2}(1 + 3\theta_k) < \delta_{k-1}$, then it is clear by direct integration on $[S_n, s]$ that any data $f_k(S_n)$ such that $|f_k(S_n)| \lesssim |S_n|^{-\delta_{k-1}}$ leads to $|f_k(s)| \lesssim |s|^{-\delta_{k-1}}$. On the contrary, if $\frac{1}{2}(1 + 3\theta_k) > \delta_{k-1}$, we have to choose the data $f_k(S_n)$ carefully to obtain the same estimate on $f_k(s)$ and avoid the instability. At the technical level, we will need to treat such indices $k$ by a global topological argument. For the sake of simplicity, we will choose the values of $\delta_k$ so that we avoid the remaining case, i.e. when $\frac{1}{2}(1 + 3\theta_k) = \delta_{k-1}$. Next, from (2.64), we get similarly

$$\left|\frac{\ddot{y}_k}{2s} - \frac{f_k}{s} \right| \lesssim \frac{f_k}{s} + |s|^{-2} \lesssim |s|^{-1-\delta_{k-1}}, \quad \text{which rewrites as} \quad \left|\frac{d}{ds} \left[ (-s)^{-\frac{1}{2}} \frac{\ddot{y}_k}{s} \right] \right| \lesssim |s|^{-1-\delta_{k-1}-\frac{1}{2}}.$$
Here, we note that any sufficiently small data \( \tilde{y}_k(S_n) \) leads to the estimate \( |\tilde{y}_k(s)| \lessapprox |s|^{-\delta_k-1} \), and so \( |\tilde{\mu}_k(s)| \lessapprox |s|^{-\delta_k-1} \) since \( f_k = \tilde{\mu}_k + \tilde{y}_k \). However, from (2.8), since we obtain
\[
\left| \frac{\dot{\tau}_k + \tau_k}{s} \right| \lessapprox |s|^{-1} |\tilde{\mu}_k| \lessapprox |s|^{-1-\delta_k-1},
\]
which rewrites as
\[
\left| \frac{d}{ds}(s\tau_k) \right| \lessapprox |s|^{-\delta_k-1},
\]
we deduce that \( \tau_k \) is an unstable parameter for all \( 1 \leq k \leq K \), and so the data \( \tau_k(S_n) \) will also have to be included in the global topological argument. Finally, the parameter \( \delta \) is deduced that
\[
\bar{\mu}_k(s) = \| \bar{Q} \|_{L^1},
\]
where \( \xi \) and in particular
\[
\bar{\mu}_k(s) = |S_n|^{-\delta_k} \xi_k
\]
for each \( 1 \leq k \leq K \). Indeed, for technical reasons, it will be convenient to have several orders of smallness in \( |s|^{-1} \) for each \( 1 \leq k \leq K \).

3.2. The bootstrap setting. From the formal discussion in the previous section, we define \( V_n(S_n) = V[\Gamma_n](S_n) \) as in Section 2.2, for parameters \( \{\mu_k(S_n)\}_k, \{\tau_k(S_n)\}_k, \{y_k(S_n)\}_k \) and \( \{e_k(S_n)\}_k \) chosen as
\[
\begin{align*}
\mu_k(S_n) &= \ell_k \quad \text{for } k \in K^-, \\
\mu_k(S_n) &= \ell_k \left( 1 + |S_n|^{-\delta_+^+} \xi_k \right) \quad \text{for } k \in K^+, \\
\tau_k(S_n) &= S_n \ell_k^3 \left( 1 + |S_n|^{-\delta_k} \zeta_k \right), \\
y_k(S_n) &= 2S_n \ell_k^{-2}, \\
a_k(S_n) &= \frac{\ell_k \mu_k^2(S_n)}{S_n} \left( \frac{1}{2} + \theta_k \right) - \frac{1}{2} \tau_k(S_n) - \frac{4r_k(S_n)}{\|Q\|_{L^1}},
\end{align*}
\]
where \( \xi = \{\xi_k\}_{k \in K} \) and \( \zeta = \{\zeta_k\}_{1 \leq k \leq K} \), satisfying \( (\xi, \zeta) \in B_{d+K} \), will be chosen later.

We claim the following consequences of these choices of initial parameters:
\[
\begin{align*}
\bar{\mu}_k(S_n) &= 0 \quad \text{for } k \in K^-, \\
\bar{\mu}_k(S_n) &= |S_n|^{-\delta_k^+} \xi_k \quad \text{for } k \in K^+, \\
\bar{\tau}_k(S_n) &= |S_n|^{-\delta_k} \zeta_k, \\
\bar{y}_k(S_n) &= 0, \\
e_k(S_n) &= \ell_k \left( \frac{1}{2} + \theta_k \right), \\
|a_k(S_n)| &\leq \frac{1}{2} |S_n|^{-1-\delta_k^-},
\end{align*}
\]
and in particular \( |\bar{\mu}_k(S_n)| \lessapprox |S_n|^{-\delta_k^+} \) for any \( 1 \leq k \leq K \). Indeed, the values of \( \bar{\mu}_k(S_n), \bar{\tau}_k(S_n) \) and \( \bar{y}_k(S_n) \) are straightforward consequences of their definition in (2.9). The value of \( e_k(S_n) \) is a direct consequence of the choice of \( a_k(S_n) \) and the definition (2.44) of \( e_k \).
Finally, by the definition of $a_k(S_n)$, we observe that, for $k \in \mathcal{K}^+$,
\[
a_k(S_n) = \frac{\ell_k^3 (1 + |S_n|^{-\delta_k^+} \xi_k)^2}{S_n} \left( \frac{1}{2} + \theta_k \right) - \frac{\ell_k^3}{2S_n} (1 + |S_n|^{-\delta_k^+} \xi_k)^{-1} - \frac{4r_k(S_n)}{\|Q\|_{L^1}}.
\]

Moreover, by (2.41), and since $2\delta_k^+ > \delta_{k-1}$ and $\frac{1}{42} > \delta_{k-1}$ from (3.3), we obtain
\[
|r_k(S_n) - \frac{\|Q\|_{L^1}}{4S_n} \ell_k^3 \theta_k \left( \frac{1}{2} + \frac{1}{2} \bar{\mu}_k(S_n) \right)| \lesssim |S_n|^{-1-\delta_{k-1}}
\]
and thus
\[
\frac{|4r_k(S_n) - \ell_k^3 \theta_k S_n|}{\|Q\|_{L^1}} \lesssim |S_n|^{-1-\delta_k^+}.
\]

It follows that $|a_k(S_n)| \lesssim |S_n|^{-1-\delta_k}$ and thus, for $n$ large enough, $|a_k(S_n)| \lesssim \frac{1}{2} |S_n|^{-1-\delta_k}$. Noticing that the proof is the same for $k \in \mathcal{K}^-$, it concludes the proof of (3.4).

Let $v_n$ be the solution of (2.7) with initial data $v_n(S_n) = V_n(S_n)$. From Lemma 2.7 and (3.4), assuming $n$ large enough, we may decompose $v_n$ as
\[
v_n(s, y) = V_n(s, y) + \varepsilon_n(s, y)
\]
on $[S_n, S_n + \bar{s}_n]$ for some $\bar{s}_n > 0$, where $V_n = V[\Gamma_n]$ is defined as in Section 2.2 with parameters $\{\mu_k(s)\}_k, \{\tau_k(s)\}_k, \{y_k(s)\}_k, \{a_k(s)\}_k$ and $\varepsilon_n$ satisfies the orthogonality conditions (2.5). Note that, for simplicity of notation, we do not specify the dependency in $n$ of the parameters $\mu_k, \tau_k, y_k$ and $a_k$. Finally, in view of applying (2.63), we denote
\[
g_k(s) = \bar{\mu}_k(s) + \bar{y}_k(s) + \int_{S_n}^s (1 + \bar{\mu}_k(\tau)) (\varepsilon_n(\tau), A_k(\tau)) \, d\tau.
\]
Note also that $\varepsilon_n(S_n) \equiv 0$ and $g_k(S_n) = \bar{\mu}_k(S_n)$ for all $1 \leq k \leq K$.

We work with the following bootstrap estimates, for $1 \leq k \leq K$:
\[
\left\{ \begin{array}{l}
\|\varepsilon_n(s)\|_{H^1} \leq |s|^{-\frac{1}{2}-\delta_{k+1}}, \\
\|\varepsilon_n(s)\|_{H^1(y > y_k)} \leq |s|^{-\frac{1}{2}-\delta_k^+}, \\
\int_{S_n}^s |\tau|^{1+\frac{1}{4}} \left( \|\partial_y \varepsilon_n(\tau)\|_{L^2_k}^2 + \|\varepsilon_n(\tau)\|_{L^2_k}^2 \right) \, d\tau \leq |s|^{-2\delta_k^+ + \frac{1}{4}}, \\
|\bar{\mu}_k(s)| + |\bar{y}_k(s)| \leq |s|^{-\delta_k}, \\
|a_k(s)| \leq |s|^{-1-\delta_k},
\end{array} \right.
\]
and
\[
\sum_{k \in \mathcal{K}^+} \left| |s|^{\delta_k^+} g_k(s) \right|^2 + \sum_{k=1}^K \left[ |s|^{\delta_k} \bar{\tau}_k(s) \right]^2 \leq 1.
\]

From the application of Lemma 2.7 above and continuity arguments, these bootstrap estimates are satisfied on $[S_n, S_n + \bar{s}_n]$, for a possibly smaller $\bar{s}_n > 0$. Now let $S_0 < -1$ with $|S_0|$ large to be fixed later. Assuming $|S_0|$ large enough so that $|S_0|^{-\delta_{K+1}} \leq \min \left( \frac{1}{2}, \frac{\alpha}{2} \right)$ with $\alpha$ given by Lemma 2.7, we may set
\[
S^*_n(\xi, \zeta) = \sup \{ S_n \leq \tau \leq S_0 \text{ such that (3.5) and (3.6) are satisfied for all } s \in [S_n, \tau] \}.
\]

When there is no risk of confusion, we will denote $S^*_n(\xi, \zeta)$ simply by $S^*_n$. 

Note for future reference that, as a direct consequence of Lemma 2.11, (3.3) and (3.5), we have, for $|S_0|$ large enough, for all $1 \leq k \leq K$, for all $s \in [S_n, S^*_n]$, \( |\langle \varepsilon_n, P_k \rangle| \lesssim |s|^{-\delta_k^{+1}} \) (3.7) and we estimate their time variation in the following lemma.

Note also that, from the orthogonality conditions (2.51) satisfied by $\varepsilon_n$ and the identity $\varepsilon_n(S_n) = 0$, we obtain by integration, for all $1 \leq k \leq K$, for all $s \in [S_n, S^*_n]$, \( |\langle \varepsilon_n, P_k \rangle - cs\hat{a}_k| \lesssim \sum_{j \leq k} ||\varepsilon_n||_{L^j_s} + |s|^{-1-\delta_k^{+1}} \). (3.8)

Now we prove that, for all $n$ large, there exists at least one choice of $(\xi, \zeta)$ such that the bootstrap estimates (3.5)–(3.6) hold up to some time $S_0$, with $|S_0|$ large, independent of $n$.

**Proposition 3.2.** There exists $S_0 < -1$, independent of $n$, such that, for all $n$ large enough, there exists $(\xi, \zeta) \in B_{d+K}$ such that $S^*_n(\xi, \zeta) = S_0$.

The proof of Proposition 3.2 is given in Sections 3.3 to 3.7. We argue by contradiction, assuming that $S^*_n(\xi, \zeta) \in [S_n, S^*_0]$ for all $(\xi, \zeta) \in B_{d+K}$. The way to reach a contradiction is first to strictly improve all estimates in (3.5) (provided that $|S_0|$ is large enough, independently of $n$), and second to use a topological obstruction on a certain continuous map related to the definition of $S^*_n(\xi, \zeta)$, the bootstrap (3.6), the definition (2.8) of $\tau_k$ for all $1 \leq k \leq K$, and the differential inequality (2.63) for $k \in K^+$.\[3.3.\text{Monotonicity of global energies.}\] We define the mass related to $\varepsilon$ as

\[ G_n = \int \varepsilon_n^2 + 2 \sum_k a_k \langle \varepsilon_n, P_k \rangle + c|s| \sum_k a_k^2, \]

where $c > 0$ is defined in Lemma 2.11, the energy related to $\varepsilon$ as

\[ H_n = \frac{1}{2} \int (\partial_y \varepsilon_n)^2 - \frac{1}{6} \int \left[ (V_n + \varepsilon_n)^6 - V_n^6 - 6V_n^5 \varepsilon_n \right], \]

and we estimate their time variation in the following lemma.

**Lemma 3.3.** There exists $C > 0$ such that, for all $s \in [S_n, S^*_n]$, \( \frac{d}{ds} (|s|G_n) \leq C|s| \left( \sum_j ||\varepsilon_n||_{L^j_s}^2 + |s|^{-2-\delta_K^{-}} \right) \) (3.10) and

\[ \frac{d}{ds} (|s|H_n) \leq C|s| \left( \sum_j ||\varepsilon_n||_{L^j_s}^2 + |s|^{-1}||\varepsilon_n||_{L^2}^2 + |s|^{-2-2\delta_K} \right). \] (3.11)

**Proof.** We denote $\varepsilon_n$, $V_n$, $G_n$ and $H_n$ simply by $\varepsilon$, $V$, $G$ and $H$. For brevity, we also use the notation $f_x = \partial_x f$ and $f_y = \partial_y f$.

**Proof of (3.10).** We first compute

\[ \frac{1}{|s|} \frac{d}{ds} (|s|G) = -\frac{1}{|s|} \left( \int \varepsilon_n^2 + 2 \sum_k a_k \langle \varepsilon_n, P_k \rangle + c|s| \sum_k a_k^2 + 2 \int \varepsilon_n \varepsilon \right. + 2 \sum_k \dot{a}_k \langle \varepsilon_n, P_k \rangle + 2 \sum_k a_k \frac{d}{ds} \langle \varepsilon_n, P_k \rangle + 2c|s| \sum_k \dot{a}_k a_k - c \sum_k a_k^2, \]

\[ + \sum_k a_k \frac{d}{ds} \langle \varepsilon_n, P_k \rangle + 2 \sum_k a_k \frac{d}{ds} \langle \varepsilon_n, P_k \rangle + 2c|s| \sum_k \dot{a}_k a_k - c \sum_k a_k^2, \]

\[ + \sum_k a_k \frac{d}{ds} \langle \varepsilon_n, P_k \rangle + 2 \sum_k a_k \frac{d}{ds} \langle \varepsilon_n, P_k \rangle + 2c|s| \sum_k \dot{a}_k a_k - c \sum_k a_k^2, \]
which gives
\begin{align*}
\frac{1}{|s|} \frac{d}{ds} (|s| G) &\leq -\frac{1}{|s|} \int \varepsilon^2 + 2 \sum_k a_k \frac{d}{ds} \langle \varepsilon, P_k \rangle + 2c |s| \sum_k \dot{a}_k a_k \\
&+ C |s|^{-1} \sum_k |a_k(\varepsilon, P_k)| + 2 \int \varepsilon \dot{s} \varepsilon + 2 \sum_k \dot{a}_k(\varepsilon, P_k).
\end{align*}

Then, by estimates (3.8) and $|a_k| \lesssim |s|^{-1-\delta_k}$ from (3.5), we find
\begin{align*}
|a_k \frac{d}{ds} \langle \varepsilon, P_k \rangle + c |s| \dot{a}_k a_k| &\lesssim |s|^{-1-\delta_k} \left( \sum_{j \leq k} \|s\|_{L^2_j}^2 + |s|^{-1-\delta^+_k} \right) \\
&\lesssim \sum_{j \leq k} \|s\|_{L^2_j}^2 + |s|^{-2-\delta_k-\delta^+_k+1},
\end{align*}
(3.12)

Similarly, from (3.7), we find
\begin{align*}
|s|^{-1} |a_k(\varepsilon, P_k)| &\lesssim |s|^{-2-\delta_k-\delta^+_k+1}.
\end{align*}
(3.13)

Next, using the equation (2.50) satisfied by $\varepsilon$, the expression (2.29) of $\mathcal{E}_V$, and integration by parts (recall that $\int (\Lambda \varepsilon) \varepsilon = 0$), we find
\begin{align*}
\int \varepsilon \dot{s} + \sum_k \dot{a}_k \varepsilon, P_k = \int \left[ (V + \varepsilon)^5 - V^5 \right] \varepsilon_y - \int \left( \sum_k \tilde{m}_k \cdot \tilde{M}_k V_k + \sum_k \dot{R}_k \right) \varepsilon.
\end{align*}

For the first term, integrating by parts, we have
\begin{align*}
\int \left[ (V + \varepsilon)^5 - V^5 \right] \varepsilon_y = -\int \left[ (V + \varepsilon)^5 - V^5 - 5V^4 \varepsilon \right] V_y
\end{align*}

and, by (2.28) and (2.49), we obtain
\begin{align*}
\left| \int \left[ (V + \varepsilon)^5 - V^5 - 5V^4 \varepsilon \right] V_y \right| &\lesssim \|V_y\|_{L^\infty} \left( \int |\varepsilon|^5 + \int |V|^3 e^2 \right) \\
&\lesssim \|e\|_{H^1} + \|e\|_{L^2_j}^2 \sum_j \|V_j - Q_j\|_{L^\infty}^2 + \sum_j \|e\|_{L^2_j}^2 \lesssim |s|^{-\frac{5}{2}} + |s|^{-\frac{11}{4}} + \sum_j \|e\|_{L^2_j}^2.
\end{align*}

Second, again by (2.28), we note that
\begin{align*}
\left| \int (\tilde{M}_k V_k) \varepsilon \right| &\lesssim \int |\tilde{M}_k (V_k - Q_k)| \varepsilon + \int |\tilde{M}_k Q_k| \varepsilon |\lesssim |s|^{-\frac{1}{2}} \|e\|_{L^2} + \|e\|_{L^2_k}.
\end{align*}

Thus, by (2.54) and (3.5),
\begin{align*}
\left| \int (\tilde{m}_k \cdot \tilde{M}_k V_k) e \right| &\lesssim \left( \|e\|_{L^2_k} + \sum_j \|e\|_{L^2_j}^2 + |s|^{-2} \right) \left( |s|^{-\frac{5}{4}} \|e\|_{L^2} + \|e\|_{L^2_k} \right) \\
&\lesssim C \sum_j \|e\|_{L^2_j}^2 + \frac{1}{8K |s|} \|e\|_{L^2}^2 + C |s|^{-4}.
\end{align*}

Third, by (2.56) and the exponential decay of $R \in \mathcal{V}$,
\begin{align*}
\left| \int \dot{R}_k \varepsilon \right| &\lesssim \left( |s|^{-1} \sum_j \|e\|_{L^2_j} + |s|^{-2} \right) \|e\|_{L^2_k} \lesssim \sum_j \|e\|_{L^2_j}^2 + |s|^{-4}.
\end{align*}
Finally, by (2.31) and (3.5), we find
\[ \left| \int \Psi \varepsilon \right| \leq \| \Psi \|_{L^2} \| \varepsilon \|_{L^2} \lesssim |s|^{-\frac{3}{2} - \delta_K} \| \varepsilon \|_{L^2} \lesssim C|s|^{-2 - 2\delta_K} + \frac{1}{8|s|} \| \varepsilon \|_{L^2}^2. \]

Gathering the above estimates, we have proved (3.10).

**Proof of (3.11).** We proceed similarly, and first compute
\[
\frac{1}{|s|} \frac{d}{ds} (|s|H) = -\frac{1}{2|s|} \int \varepsilon_y^2 + \frac{1}{6|s|} \int \left[ (V + \varepsilon)^6 - V^6 - 6V^5 \varepsilon \right] - \int \varepsilon_y \left[ (V + \varepsilon)^5 - V^5 \right] - \int V_s \left[ (V + \varepsilon)^5 - V^5 - 5V^4 \varepsilon \right].
\]

Using the equation (2.50) of \( \varepsilon \) and the definition of \( \mathcal{E}_V \) in Lemma 2.4, we obtain
\[
\frac{1}{|s|} \frac{d}{ds} (|s|H) \approx \frac{1}{2s} \int \varepsilon_y^2 - \frac{1}{6s} \int \left[ (V + \varepsilon)^6 - V^6 - 6V^5 \varepsilon \right] + \frac{1}{2s} \int (\Lambda \varepsilon) \varepsilon_{yy}
+ \frac{1}{2s} \int \Lambda \varepsilon \left[ (V + \varepsilon)^5 - V^5 \right] + \frac{1}{2s} \int \Lambda V \left[ (V + \varepsilon)^5 - V^5 - 5V^4 \varepsilon \right]
+ \int \partial_y (V_{yy} + V^5) \left[ (V + \varepsilon)^5 - V^5 - 5V^4 \varepsilon \right] + \int \mathcal{E}_V (\varepsilon_{yy} + 5V^4 \varepsilon).
\]

But, from integrations by parts, we get \( \int (\Lambda \varepsilon) \varepsilon_{yy} = -\int \varepsilon_y^2 \) and
\[ \int \Lambda \varepsilon \left[ (V + \varepsilon)^5 - V^5 \right] = \frac{1}{3} \int \left[ (V + \varepsilon)^6 - V^6 - 6V^5 \varepsilon \right] - \int \Lambda V \left[ (V + \varepsilon)^5 - V^5 - 5V^4 \varepsilon \right]. \]

Thus, we have obtained the simplified expression
\[
\frac{1}{|s|} \frac{d}{ds} (|s|H) = \int \left( V_{yyy} + 5V_y V^4 \right) \left[ (V + \varepsilon)^5 - V^5 - 5V^4 \varepsilon \right] + \int \left( \partial_{yy} \mathcal{E}_V + 5V^4 \mathcal{E}_V \right) \varepsilon.
\]

For the first term, we obtain as before
\[ \left| \int \left( V_{yyy} + 5V_y V^4 \right) \left[ (V + \varepsilon)^5 - V^5 - 5V^4 \varepsilon \right] \right| \lesssim \int \left( |\varepsilon|^5 + |V|^3 \varepsilon^2 \right) \lesssim |s|^{-\frac{3}{2}} + \sum_j \| \varepsilon \|_{L^2_j}^2. \]

For the second term, we use the expression (2.29) of \( \mathcal{E}_V \) and get
\[
\int \left( \partial_{yy} \mathcal{E}_V + 5V^4 \mathcal{E}_V \right) \varepsilon = \sum_k \int \tilde{m}_k \cdot \left( \partial_{yy} \tilde{M}_k V + 5V^4 \tilde{M}_k V_k \right) \varepsilon + \sum_k \tilde{r}_k \int \left( \partial_{yy} R_k + 5V^4 R_k \right) \varepsilon
+ \sum_k \tilde{a}_k \int \left( \partial_{yy} P_k + 5V^4 P_k \right) \varepsilon + \int \left( \Psi_{yy} + 5V^4 \Psi \right) \varepsilon
= h_1 + h_2 + h_3 + h_4.
\]

To estimate these four terms, we follow the proof of (3.10) above, and obtain similarly
\[
|h_1| \lesssim \sum_k \| \tilde{m}_k \| \left( |s|^{-\frac{3}{2}} \| \varepsilon \|_{L^2} + \| \varepsilon \|_{L^2}^2 \right) \lesssim \sum_j \| \varepsilon \|_{L^2_j}^2 + |s|^{-1} \| \varepsilon \|_{L^2}^2 + |s|^{-4},
|h_2| \lesssim \sum_k \| \tilde{r}_k \| \| \varepsilon \|_{L^2_k} \lesssim \sum_j \| \varepsilon \|_{L^2_j}^2 + |s|^{-4},
|h_3| \lesssim \| \Psi \|_{H^2} \| \varepsilon \|_{L^2} \lesssim |s|^{-\frac{3}{2} - \delta_K} \| \varepsilon \|_{L^2} \lesssim |s|^{-1} \| \varepsilon \|_{L^2}^2 + |s|^{-2 - 2\delta_K}.\]
Finally, to estimate $h_3$, we use (2.49) and (2.55) to obtain
\[
|h_3| \lesssim \sum_k |\dot{a}_k| \left( \left\| \partial_y y P_k \right\|_{L^2} + \left\| P_k \right\|_{L^\infty} \left\| V \right\|_{L^2}^3 \right) \left\| \varepsilon \right\|_{L^2}
\]
\[
\lesssim \left\| \varepsilon \right\|_{L^2} \left( \sum_k |\dot{a}_k| \right) \lesssim |s|^{-\frac{3}{2}} \left( \sum_j \left\| \varepsilon \right\|_{L^2}^2 + |s|^{-2} \right) \lesssim \sum_j \left\| \varepsilon \right\|_{L^2}^2 + |s|^{-\frac{3}{2}}.
\]
Gathering the above estimates, we obtain (3.11), which concludes the proof of Lemma 3.3. \qed

3.4. Monotonicity of local energies. Let $\psi, \varphi \in C^\infty(\mathbb{R}, \mathbb{R})$ be nondecreasing functions such that
\[
\psi(y) = \begin{cases} 
  e^y & \text{for } y < -1, \\
  1 & \text{for } y > -\frac{1}{2},
\end{cases}
\]
and
\[
\varphi(y) = \begin{cases} 
  e^y & \text{for } y < -1, \\
  y + 1 & \text{for } -\frac{1}{2} < y < \frac{1}{2}, \\
  2 - e^{-y} & \text{for } y > 1.
\end{cases}
\]

We note that such functions satisfy $\frac{1}{3} e^y \leq \psi(y) \leq 3 e^y$ and $\frac{1}{3} e^y \leq \varphi(y) \leq 3 e^y$ for $y < 0$, and $\frac{1}{3} \varphi(y) \leq \psi(y) \leq 3 \varphi(y)$ for all $y \in \mathbb{R}$. Moreover, we may choose the function $\varphi$ such that $\frac{1}{4} \leq \varphi'(y) \leq 1$ for $y \in [-1, 1]$ and so $\frac{1}{3} e^{-|y|} \leq \varphi'(y) \leq 3 e^{-|y|}$ for all $y \in \mathbb{R}$.

For $B > 100(1 + \ell_1)$ large to be chosen later and $1 \leq k \leq K$, we define
\[
\psi_k(s, y) = \psi \left( \frac{y - y_k(s)}{B} \right) \quad \text{and} \quad \varphi_k(s, y) = \varphi \left( \frac{y - y_k(s)}{B} \right).
\]
Note that, directly from the definitions of $\psi$ and $\varphi$, we have, for all $y \in \mathbb{R},$
\[
\begin{cases} 
  \frac{1}{3} e^{-\frac{1}{3}|y - y_k|} \leq B \partial_y \varphi_k(y) \leq 3 e^{-\frac{1}{3}|y - y_k|} \leq 9 \varphi_k(y), \\
  \frac{1}{2} \varphi_k(y) \leq \psi_k(y) \leq 3 \varphi_k(y), \\
  \partial_y \psi_k(y) + B^2 |\partial_{yy} \psi_k(y)| + B^2 |\partial_{yy} \varphi_k(y)| \lesssim \partial_y \varphi_k(y),
\end{cases}
\]
and, for all $y < y_k,$
\[
\frac{1}{2} e^{\frac{1}{3}(y - y_k)} \lesssim \psi_k(y) \leq 3 e^{\frac{1}{3}(y - y_k)} \quad \text{and} \quad \frac{1}{3} e^{\frac{1}{3}(y - y_k)} \leq \varphi_k(y) \leq 3 e^{\frac{1}{3}(y - y_k)}.
\]

In particular, from the definition of the $L^2_k$ norm in Section 2.2, we have the control
\[
\left\| \varepsilon_n \right\|_{L^2_k}^2 \lesssim B \int \varepsilon_n^2 \partial_y \varphi_k.
\]
Similarly as in [3, 22], we define now the mixed energy–virial functional
\[
\mathcal{F}_{k,n} = \int (\partial_y \varepsilon_n)^2 \psi_k + \mu k^{-2} \int \varepsilon_n^2 \varphi_k - \frac{1}{3} \int \left[ (V_n + \varepsilon_n)^6 - V_n^6 - 6 V_n^5 \varepsilon_n \right] \psi_k,
\]
and we estimate it, as long as its time variation, in the following lemma.

**Lemma 3.4.** There exist $B > 100$, $C > 0$ and $\kappa > 0$ such that, for all $1 \leq k \leq K$ and all $s \in [S_n, S_n^0]$, the following hold.

(i) Almost coercivity of $\mathcal{F}_{k,n}$:
\[
\mathcal{F}_{k,n} \geq \kappa \left[ \int (\partial_y \varepsilon_n)^2 \psi_k + \int \varepsilon_n^2 \varphi_k \right] - |s|^{-10}.
\]
(ii) Time variation of $\mathcal{F}_{k,n}$:

$$
\frac{d}{ds} \left\{ |s|^{1+\frac{\alpha}{2}} \left[ \mathcal{F}_{k,n} + 4\mu_k^{-2} \sum_{j<k} a_j (\varepsilon_n, P_j) + 2c\mu_k^{-2} |s| \sum_{j<k} a_j^2 \right] + \kappa |s|^{1+\frac{\alpha}{2}} \left[ \| \partial_y \varepsilon_n \|_{L^2_x}^2 + \| \varepsilon_n \|_{L^2_x}^2 \right] \right\} 
\leq C |s|^{1+\frac{\alpha}{2}} \left[ \sum_{j<k} \| \varepsilon_n \|_{L^2_x}^2 + |s|^{-\frac{\alpha}{2}} \sum_j \| \varepsilon_n \|_{L^2_x}^2 + |s|^{-2-\delta_k-\delta_k-1} \right]. \tag{3.18}
$$

Proof. We denote again $\varepsilon_n$, $V_n$ and $\mathcal{F}_{k,n}$ simply by $\varepsilon$, $V$ and $\mathcal{F}_k$. As in the proof of Lemma 3.3, we also use the notation $f_s = \partial_s f$ and $f_y = \partial_y f$.

(i) To prove (3.17), we rely on the coercivity of the linearized energy (2.1), together with the choice of orthogonality conditions (3.9) and standard localization arguments (we refer to the proof of Lemma 4 in [25] for similar arguments). We first decompose $\mathcal{F}_k$ as

$$
\mathcal{F}_k = \int \left[ \varepsilon_y^2 \psi_k + \tilde{\mu}_k^{-2} \varepsilon^2 \varphi_k - 5\varepsilon^2 \left( \sum_{j \leq k} Q_j^4 \right) \psi_k \right] + (\mu_k^{-2} - \tilde{\mu}_k^{-2}) \int \varepsilon^2 \varphi_k
$$

$$
- 5 \int \left( V^4 - \sum_{j \leq k} Q_j^4 \right) \varepsilon^2 \psi_k - \frac{1}{3} \int \left[ (V + \varepsilon)^6 - V^6 - 6V^5\varepsilon - 15V^4\varepsilon^2 \right] \psi_k.
$$

To estimate the first term, we claim that, for some $\bar{\kappa} > 0$ and for $B$ large enough,

$$
\int \left[ \varepsilon_y^2 \psi_k + \tilde{\mu}_k^{-2} \varepsilon^2 \varphi_k - 5\varepsilon^2 \left( \sum_{j \leq k} Q_j^4 \right) \psi_k \right] \geq \bar{\kappa} \int \left( \varepsilon_y^2 \psi_k + \varepsilon^2 \varphi_k \right).
$$

Proceeding as in the Appendix A of [20] (see also Claim (29) in the proof of Lemma 4 in [25]), we recall without proof a localized version of (2.1). Consider a smooth even function $\Phi$ such that $\Phi' \leq 0$ on $[0, +\infty)$, $\Phi \equiv 1$ on $[0, 1]$, $\Phi(x) = e^{-x}$ for $x \in [2, +\infty)$ and $e^{-x} \leq \Phi(x) \leq 3e^{-x}$ for $x \in [0, +\infty)$. For $D > 0$, let $\Phi_D(x) = \Phi\left(\frac{x}{D}\right)$. Then there exists $0 < \kappa_1 < 1$ such that, for $D$ large enough, for all $\eta \in H^1(\mathbb{R})$ such that $\langle \eta, Q \rangle = \langle \eta, \Lambda Q \rangle = \langle \eta, y\Lambda Q \rangle = 0$, there holds

$$
(1 - \kappa_1) \int \left( \eta_x^2 + \eta^2 \right) \Phi_D \geq 5 \int \eta^2 Q^4 \Phi_D.
$$

For $1 \leq j \leq k$, let $\Phi_j(s, y) = \Phi_D(x)$ and define $\eta_j$ such that $\varepsilon(s, y) = \tilde{\mu}_j^{-2} \eta_j(s, x)$, with $x = \tilde{\mu}_j^{-1}(y - y_j)$. Then, since $\langle \eta_j, Q \rangle = \langle \eta_j, \Lambda Q \rangle = \langle \eta_j, y\Lambda Q \rangle = 0$ by (3.9), we may apply the above estimate to $\eta_j$ and obtain

$$
(1 - \kappa_1) \int \left( \varepsilon_y^2 + \tilde{\mu}_j^{-2} \varepsilon^2 \right) \Phi_j \geq 5 \int \varepsilon^2 Q^4 \Phi_j.
$$

Moreover, we have $\tilde{\mu}_j^{-2} \leq \mu_k^{-2}$ for $1 \leq j \leq k$, and thus, summing over $1 \leq j \leq k$, we find

$$
(1 - \kappa_1) \int \left( \varepsilon_y^2 + \mu_k^{-2} \varepsilon^2 \right) \left( \sum_{j \leq k} \Phi_j \right) \geq 5 \int \varepsilon^2 \left( \sum_{j \leq k} Q_j^4 \Phi_j \right).
$$

But, by the definitions of $\psi$, $\varphi$ and $\Phi$, we have

$$
\sum_{j \leq k} \Phi_j \leq (1 + |s|^{-10})\psi_k \quad \text{and} \quad \sum_{j \leq k} \Phi_j \leq (1 + |s|^{-10})\varphi_k,
$$

and
and, using moreover the exponential decay of $Q$,
\[
\sum_{j \leq k} Q_j^4 \Phi_j \geq \left( \sum_{j \leq k} Q_j^4 \right) \psi_k - \left( |s|^{-10} + B^{-10} + D^{-10} \right) \varphi_k.
\]
Thus, for $D$, $B$ and $|S_0|$ large enough, we deduce that
\[
\left(1 - \frac{\bar{\kappa}^1}{2}\right) \int \left( \varepsilon_j^2 \psi_k + \bar{\mu}_k^{-2} \varepsilon^2 \varphi_k \right) \geq 5 \int \varepsilon^2 \left( \sum_{j \leq k} Q_j^4 \right) \psi_k,
\]
which proves the claim with $\bar{\kappa} = \frac{\kappa}{2} > 0$.

To estimate the second term, we use (2.11) to obtain, for $|S_0|$ large enough,
\[
\left| (\mu_k^{-2} - \bar{\mu}_k^{-2}) \int \varepsilon^2 \varphi_k \right| \lesssim |s|^{-1} \int \varepsilon^2 \varphi_k \leq \frac{\bar{\kappa}}{4} \int \varepsilon^2 \varphi_k.
\]

To estimate the third term, we use (2.28) and (3.14) to obtain
\[
5 \left| \int \left( V^4 - \sum_{j \leq k} Q_j^4 \right) \varepsilon^2 \psi_k \right| \lesssim |s|^{-\frac{1}{4}} \int \varepsilon^2 \varphi_k + \sum_{j > k} \int Q_j^4 \varepsilon^2 \psi_k.
\]
Moreover, for $j > k$, we have, from (2.49), (3.14)–(3.15) and the exponential decay of $Q$,
\[
\int Q_j^4 \varepsilon^2 \psi_k \lesssim \int_{y < y_k - |s|^{\frac{1}{4}}} \varepsilon^2 \psi_k + \int_{y > y_k - |s|^{\frac{1}{4}}} Q_j^4 \varepsilon^2 \varphi_k \lesssim |s|^{-11} + |s|^{-10} \int \varepsilon^2 \varphi_k.
\]
Thus, for $|S_0|$ large enough,
\[
5 \left| \int \left( V^4 - \sum_{j \leq k} Q_j^4 \right) \varepsilon^2 \psi_k \right| \lesssim |s|^{-\frac{1}{4}} \int \varepsilon^2 \varphi_k + |s|^{-11} \leq \frac{\bar{\kappa}}{4} \int \varepsilon^2 \varphi_k + |s|^{-10}.
\]
Finally, the nonlinear term is estimated as
\[
\frac{1}{3} \left| \int \left( [V + \varepsilon]^6 - 6V^5 \varepsilon - 15V^4 \varepsilon^2 \right) \psi_k \right| \lesssim \|\varepsilon\|_{L^\infty} \int \varepsilon^2 \varphi_k \leq \frac{\bar{\kappa}}{4} \int \varepsilon^2 \varphi_k,
\]
by choosing again $|S_0|$ large enough. Gathering the above estimates and letting $\kappa = \frac{\bar{\kappa}}{4} > 0$, we obtain (3.17).

(ii) The proof is partly similar to the proof of Lemma 3.4 in [3] (see also Proposition 3.1 in [22]). We first compute
\[
- \frac{1}{|s|^{1+\frac{1}{4}}} \frac{d}{ds} \left( |s|^{1+\frac{1}{4}} \left[ F_k + 4\mu_k^{-2} \sum_{j < k} a_j \langle \varepsilon, P_j \rangle + 2c\mu_k^{-2} |s| \sum_{j < k} a_j^2 \right] \right) = \frac{dF_k}{ds} + 4\mu_k^{-2} \sum_{j < k} \dot{a}_j \langle \varepsilon, P_j \rangle
\]
\[
+ 4\mu_k^{-2} \left( \sum_{j < k} \int_{d8} \langle \varepsilon, P_j \rangle + c |s| \sum_{j < k} \dot{a}_j a_j \right) - 4\dot{\mu}_k \mu_k^{-3} \left( 2 \sum_{j < k} a_j \langle \varepsilon, P_j \rangle + c |s| \sum_{j < k} a_j^2 \right)
\]
\[
- 2c\mu_k^{-2} \sum_{j < k} a_j^2 - \left( 1 + \frac{1}{43} \right) \frac{1}{|s|} \left[ F_k + 4\mu_k^{-2} \sum_{j < k} a_j \langle \varepsilon, P_j \rangle + 2c\mu_k^{-2} |s| \sum_{j < k} a_j^2 \right].
\]
Now note that, by combining (2.30), (2.49) and (2.54), we have $|a_k| \lesssim \|\varepsilon\|_{L^2_k} + |s|^{-1}$. Thus, using also (3.12) and (3.13), and the estimate on $|a_j|$ from (3.5), we find
\[
\frac{1}{|s|^{1+\frac{1}{2}}} \frac{d}{ds} \left( |s|^{1+\frac{1}{2}} \left[ F_k + 4\mu_k^2 \sum_{j<k} a_j (\varepsilon, P_j) + 2\delta \mu_k^2 |s| \sum_j |a_j|^2 \right] \right) \leq - \left( 1 + \frac{1}{43} \right) \frac{F_k}{|s|} + \frac{dF_k}{ds} + 4\mu_k^{-2} \sum_{j<k} \tilde{a}_j (\varepsilon, P_j) + C \left\| \varepsilon \right\|_{L^2_k}^2 + C \sum_{j<k} \|\varepsilon\|_{L^2_j}^2 + C \left| |s|^{-1-\delta_k-1-\delta^+} \|\varepsilon\|_{L^2_k}^{1+\delta^+} \right|^2.
\]

Next, denoting $G_k(\varepsilon) = \left\{ -\varepsilon_y \partial_y \psi_k - \varepsilon_{yy} \psi_k + \mu_k^{-2} \varepsilon \varphi_k - \left( (\mathbf{V} + \varepsilon)^5 - \mathbf{V}^5 \right) \psi_k \right\}$, an integration by parts gives, since $\partial_s \psi_k = -\dot{y}_k \partial_y \psi_k$ and $\partial_s \varphi_k = -\dot{y}_k \partial_y \varphi_k$,
\[
\frac{dF_k}{ds} = -2 \int (\mathbf{V}_s + \dot{y}_k \mathbf{V}_y) \left[ (\mathbf{V} + \varepsilon)^5 - \mathbf{V}^5 - 5\mathbf{V}^4 \varepsilon \right] \psi_k + 2 \int (\varepsilon_s + \dot{y}_k \varepsilon) G_k(\varepsilon) - 2\mu_k \mu_k^{-3} \int \varepsilon^2 \varphi_k.
\]

Therefore, using the equation (2.50) satisfied by $\varepsilon$, the identity $\Lambda_k = \frac{1}{2} + (y - y_k) \partial_y = \Lambda - y_k \partial_y$, and the definition of $\tilde{\mathbf{V}}$ in Lemma 2.1, we find
\[
- \left( 1 + \frac{1}{43} \right) \frac{F_k}{|s|} + \frac{dF_k}{ds} + 4\mu_k^{-2} \sum_{j<k} \tilde{a}_j (\varepsilon, P_j) = f_1 + f_2 + f_3 + f_4
\]

where, denoting $\tilde{G}_k(\varepsilon) = \left\{ -\varepsilon_y \partial_y \psi_k - \varepsilon_{yy} \psi_k + \mu_k^{-2} \varepsilon \varphi_k - 5\mathbf{V}^4 \varepsilon \psi_k \right\}$, we have set
\[
f_1 = -2 \int \partial_y \left[ \varepsilon_{yy} - \mu_k^{-2} \varepsilon + (\mathbf{V} + \varepsilon)^5 - \mathbf{V}^5 \right] G_k(\varepsilon),
\]
\[
f_2 = - \left( 1 + \frac{1}{43} \right) \frac{F_k}{|s|} + \frac{1}{|s|} \int (\Lambda_k \varepsilon) G_k(\varepsilon) - \frac{1}{|s|} \int (\Lambda_k \mathbf{V}) \left[ (\mathbf{V} + \varepsilon)^5 - \mathbf{V}^5 - 5\mathbf{V}^4 \varepsilon \right] \psi_k,
\]
\[
f_3 = -2 \int \varepsilon \tilde{G}_k(\varepsilon) + 4\mu_k^{-2} \sum_{j<k} \tilde{a}_j (\varepsilon, P_j) - 2\mu_k \mu_k^{-3} \int \varepsilon^2 \varphi_k,
\]
\[
f_4 = 2 \int \partial_y \left[ \varepsilon_{yy} + \mathbf{V}^5 - \left( \frac{y_k}{2s} - \frac{1}{\mu_k^2} \right) \mathbf{V} \right] \left[ (\mathbf{V} + \varepsilon)^5 - \mathbf{V}^5 - 5\mathbf{V}^4 \varepsilon \right] \psi_k + 2 \left( \frac{y_k}{2s} - \frac{1}{\mu_k^2} \right) \int \varepsilon \varphi_k.
\]

We now estimate these four terms separately.

**Control of $f_1$.** From multiple integrations by parts, we may rewrite $f_1$ as
\[
f_1 = - \int \left[ 3\varepsilon_{yy}^2 \partial_y \psi_k + \varepsilon_{yy}^2 \left( 3\mu_k^{-2} \partial_y \varphi_k + \mu_k^{-2} \partial_y \psi_k - \partial_y \psi_k \right) + \varepsilon_{yy}^2 \left( \mu_k^{-4} \partial_y \varphi_k - \mu_k^{-2} \partial_{yy} \psi_k \right) \right] - \frac{1}{3} \mu_k^{-2} \int \left[ (\mathbf{V} + \varepsilon)^6 - \mathbf{V}^6 - 6\mathbf{V}^5 \varepsilon - 6 \left( (\mathbf{V} + \varepsilon)^5 - \mathbf{V}^5 \right) \varepsilon \right] \left( \partial_y \varphi_k - \partial_y \psi_k \right)
\]
\[
- 2\mu_k^{-2} \int \varepsilon \left( \mathbf{V} + \varepsilon \right)^5 - \mathbf{V}^5 - 5\mathbf{V}^4 \varepsilon \right] \left( \varphi_k - \psi_k \right)
\]
\[
+ 10 \int \varepsilon \left[ \varepsilon_y \left( \mathbf{V} + \varepsilon \right)^4 - \mathbf{V}^4 \right] + \varepsilon_y \left( \mathbf{V} + \varepsilon \right)^4 \right] \partial_y \psi_k
\]
\[
- \int \left\{ \left[ -\varepsilon_{yy} + \mu_k^{-2} \varepsilon - \left( (\mathbf{V} + \varepsilon)^5 - \mathbf{V}^5 \right) \right]^2 - \left[ -\varepsilon_{yy} + \mu_k^{-2} \varepsilon \right]^2 \right\} \partial_y \psi_k = f_1^- + f_1^+ + f_1^+,
where $f_1^\prec$, $f_1^\sim$ and $f_1^\succ$ respectively correspond to integration on $y - y_k < -\frac{B}{2}$, $|y - y_k| \leq \frac{B}{2}$ and $y - y_k > \frac{B}{2}$.

**Estimate of $f_1^\prec$.** From (2.9) and (3.14)–(3.15), taking $B$ large enough (depending on $\ell_k$), we find

\[
 f_1^\prec \leq -3 \int_{y - y_k < -\frac{B}{2}} \varepsilon_{yy}^2 \partial_y \psi_k - \frac{1}{2} \mu_k^{-2} \int_{y - y_k < -\frac{B}{2}} (\varepsilon_y^2 + \mu_k^{-2} \varepsilon^2) \partial_y \varphi_k
 + C \int_{y - y_k < -\frac{B}{2}} (c^6 + V^4 \varepsilon^2) \partial_y \varphi_k + CB \int_{y - y_k < -\frac{B}{2}} |V_y|(|\varepsilon|^5 + |V|^3 \varepsilon^2) \partial_y \varphi_k
 + C \int_{y - y_k < -\frac{B}{2}} |\varepsilon_y| \left[ |V_y| \left( |V|^3 |\varepsilon| + \varepsilon^4 \right) + |\varepsilon_x| (V^4 + \varepsilon^4) \right] \partial_y \psi_k
 + C \int_{y - y_k < -\frac{B}{2}} (V^4 |\varepsilon| + |\varepsilon|^5) \left( |\varepsilon_{yy}| + |\varepsilon| + V^4 |\varepsilon| + |\varepsilon|^5 \right) \partial_y \psi_k.
\]

Using (2.28) we find, for $-|s|^\frac{1}{4} < y - y_k < -\frac{B}{2}$,

\[
 |V(y)| \lesssim e^{-\frac{B}{|\varepsilon|}} + |s|^{-\frac{1}{4}} \quad \text{and} \quad |V_y(y)| \lesssim e^{-\frac{B}{|\varepsilon|}} + |s|^{-1}.
\]

Moreover, for $y - y_k < -|s|^\frac{1}{4}$, we have $\partial_y \psi_k(y) + \partial_y \varphi_k(y) \lesssim e^{-\frac{B}{|\varepsilon|}}|s|^\frac{1}{4}$ from (3.14). Therefore,

\[
 f_1^\prec \leq -3 \int_{y - y_k < -\frac{B}{2}} \varepsilon_{yy}^2 \partial_y \psi_k - \frac{1}{2} \mu_k^{-2} \int_{y - y_k < -\frac{B}{2}} (\varepsilon_y^2 + \mu_k^{-2} \varepsilon^2) \partial_y \varphi_k
 + C_0 B \left( \|\varepsilon\|_{L^\infty}^2 + e^{-\frac{B}{|\varepsilon|}} + |s|^{-3} \right) \mu_k^{-2} \int_{|s|^\frac{1}{4} < y - y_k < -\frac{B}{2}} (\varepsilon_y^2 + \mu_k^{-2} \varepsilon^2) \partial_y \varphi_k
 + C_1 \left( \|\varepsilon\|_{L^\infty}^4 + e^{-\frac{B}{|\varepsilon|}} + |s|^{-3} \right) \int_{|s|^\frac{1}{4} < y - y_k < -\frac{B}{2}} \varepsilon_{yy} \partial_y \psi_k
 + C_2 |s|^{-10} \int_{|s|^\frac{1}{4} < y - y_k < -\frac{B}{2}} \varepsilon_{yy} \partial_y \psi_k + C|s|^{-10}
\]

for some constants $C_0, C_1, C_2 > 0$. We choose $B$ large enough, and then $|S_0|$ large enough (depending on $B$), so that $C_2 |s|^{-10} \lesssim 1$,

\[
 C_1 \left( \|\varepsilon\|_{L^\infty}^4 + e^{-\frac{B}{|\varepsilon|}} + |s|^{-3} \right) \leq 1 \quad \text{and} \quad C_0 B \left( \|\varepsilon\|_{L^\infty}^2 + e^{-\frac{B}{|\varepsilon|}} + |s|^{-3} \right) \leq \frac{1}{4}.
\]

Hence, we obtain

\[
 f_1^\prec \leq -\frac{1}{4} \mu_k^{-2} \int_{y - y_k < -\frac{B}{2}} \left( \varepsilon_y^2 + \mu_k^{-2} \varepsilon^2 \right) \partial_y \varphi_k + C|s|^{-10}.
\]

**Estimate of $f_1^\sim$.** For $y - y_k > \frac{B}{2}$, we have $\psi_k(y) = 1$ and so $\partial_y \psi_k(y) = \partial_{yy} \psi_k(y) = 0$. Thus,

\[
 f_1^\sim = -\int_{y - y_k > \frac{B}{2}} \left[ 3 \mu_k^{-2} \varepsilon_y^2 \partial_y \varphi_k + \mu_k^{-4} \varepsilon^2 \left( \partial_y \varphi_k - \mu_k^2 \partial_{yy} \varphi_k \right) \right]
 - \frac{1}{3} \mu_k^{-2} \int_{y - y_k > \frac{B}{2}} \left[ (V + \varepsilon)^6 - V^6 - 6V^5 \varepsilon - 6 \left( (V + \varepsilon)^5 - V^5 \right) \varepsilon \right] \partial_y \varphi_k
 - 2 \mu_k^{-2} \int_{y - y_k > \frac{B}{2}} V_y \left[ (V + \varepsilon)^5 - V^5 - 5V^4 \varepsilon \right] (\varphi_k - 1).
\]
Using (3.14) and $\|V_y\|_{L^\infty} \lesssim 1$, and taking $B$ large enough as before, we find

$$f_1^\prec \leq -\frac{1}{2} \mu_k^2 \int_{|y-y_k| > \frac{B}{2}} \left( \varepsilon_y^2 + \mu_k^{-2} \varepsilon^2 \right) \partial_y \varphi_k + C \int_{|y-y_k| > \frac{B}{2}} (\varepsilon_y + V^4 \varepsilon^2) \partial_y \varphi_k$$

$$+ C \int_{|y-y_k| > \frac{B}{2}} \left( |\varepsilon|^5 + |V|^3 \varepsilon^2 \right).$$

For $\frac{B}{2} < |y-y_k| < |s|^\frac{2}{3}$, by (2.28) and the exponential decay of $Q$, we find $|V(y)| \lesssim e^{-\frac{B}{2} y + |s|^{\frac{1}{3}}}$. For $y - y_k > 12 |s|^{\frac{1}{3}}$, by (3.14), we have $\partial_y \varphi_k(y) \lesssim e^{-\frac{B}{2} |s|^{\frac{1}{3}}}$. Thus, we obtain

$$f_1^\prec \leq -\frac{1}{2} \mu_k^{-2} \int_{|y-y_k| > \frac{B}{2}} \left( \varepsilon_y^2 + \mu_k^{-2} \varepsilon^2 \right) \partial_y \varphi_k + C |s|^{-10} + C \varepsilon_{\bar{H}} + C \int_{|y-y_k| > \frac{B}{2}} |V|^3 \varepsilon^2$$

$$+ C \left( \varepsilon_{\bar{H}}^4 + e^{-\frac{B}{2} y} + |s|^{-3} \right) \int_{\frac{B}{2} < |y-y_k| < |s|^\frac{2}{3}} \varepsilon^2 \partial_y \varphi_k.$$

Then, using also (2.49), we find as before, for $B$ large enough and $|S_0|$ large enough,

$$f_1^\prec \leq -\frac{1}{4} \mu_k^{-2} \int_{|y-y_k| > \frac{B}{2}} \left( \varepsilon_y^2 + \mu_k^{-2} \varepsilon^2 \right) \partial_y \varphi_k + C \int_{|y-y_k| > \frac{B}{2}} |V|^3 \varepsilon^2 + C |s|^{-\frac{5}{2}}.$$

But, by (2.28) and the exponential decay of $Q$, we have

$$\left\| V - \sum_{j \leq k} Q_j \right\|_{L^\infty(y-y_k > \frac{B}{2})} \lesssim \sum_{j > k} \|Q_j\|_{L^\infty(y-y_k > \frac{B}{2})} + \sum_j \|V_j - Q_j\|_{L^\infty} \lesssim |s|^{-\frac{1}{2}}.$$

And since $\int \left| Q_j \right|^3 \varepsilon^2 \lesssim \varepsilon_{L^2}^2$ and $\int_{|y-y_k| > \frac{B}{2}} \left| Q_k \right|^3 \varepsilon^2 \lesssim e^{-\frac{B}{2} y} \varepsilon_{L^2}^2 \lesssim B^{-2} \varepsilon_{L^2}^2$, we find

$$\int_{|y-y_k| > \frac{B}{2}} |V|^3 \varepsilon^2 \lesssim \sum_{j < k} \varepsilon_{L^2}^2 + B^{-2} \varepsilon_{L^2}^2 + |s|^{-\frac{1}{2}} \varepsilon_{L^2}^2.$$

In conclusion for this term, we have

$$f_1^\prec \leq -\frac{1}{4} \mu_k^{-2} \int_{|y-y_k| > \frac{B}{2}} \left( \varepsilon_y^2 + \mu_k^{-2} \varepsilon^2 \right) \partial_y \varphi_k + C \sum_{j < k} \varepsilon_{L^2}^2 + CB^{-2} \varepsilon_{L^2}^2 + C |s|^{-\frac{5}{2}}.$$

**Estimate of $f_1^\succ$.** For $|y - y_k| \leq \frac{B}{2}$, we still have $\psi_k(y) = 1$ and $\partial_y \psi_k(y) = \partial_y \psi_k(y) = 0$. Moreover, $\varphi_k(y) = \frac{y - y_k}{B} + 2$, $\partial_y \varphi_k = \frac{1}{B}$ and $\partial_y \psi_k(y) = 0$. Thus, we may rewrite $f_1^\succ$ as

$$B f_1^\succ = -\mu_k^{-2} \int_{|y-y_k| \leq \frac{B}{2}} \left( 3 \varepsilon_y^2 + \mu_k^{-2} \varepsilon^2 \right)$$

$$- \frac{1}{3} \mu_k^{-2} \int_{|y-y_k| \leq \frac{B}{2}} \left[ (V + \varepsilon)^6 - V^6 - 6V^5 \varepsilon - 6 \left( (V + \varepsilon)^5 - V^5 \right) \varepsilon \right]$$

$$- 2 \mu_k^{-2} \int_{|y-y_k| \leq \frac{B}{2}} (y - y_k) V_y \left[ (V + \varepsilon)^5 - V^5 - 5V^4 \varepsilon \right]$$

$$= -\mu_k^{-2} \int_{|y-y_k| \leq \frac{B}{2}} \left( 3 \varepsilon_y^2 + \mu_k^{-2} \varepsilon^2 - 5Q^4_k \varepsilon^2 + 20(y - y_k) \partial_y Q_k Q_k^3 \varepsilon^2 \right) + \mu_k^{-2} R_{\text{vir}}(\varepsilon),$$
with
\[ R_{V\mu}(\varepsilon) = (\bar{\mu}_k^{-2} - \mu_k^{-2}) \int_{|y-y_k| \leq \frac{B}{2}} \varepsilon^2 + \frac{1}{3} \int_{|y-y_k| \leq \frac{B}{2}} (40V^3 + 45V^2\varepsilon + 24V\varepsilon^2 + 5\varepsilon^3) \varepsilon^3 \\
+ 5 \int_{|y-y_k| \leq \frac{B}{2}} \left[ (V^4 - Q_k^4) - 4(y - y_k) (V_y V^3 - \partial_y Q_k Q_k^3) \right] \varepsilon^2 \\
- 2 \int_{|y-y_k| \leq \frac{B}{2}} (y - y_k) V_y (10V^2 + 5V \varepsilon + \varepsilon^2) \varepsilon^3. \]

With the change of space variable \( x = \bar{\mu}_k^{-1}(y - y_k) \), and letting \( \varepsilon(s, y) = \bar{\mu}_k^{-\frac{1}{2}} \eta(s, x) \), we find
\[ \int_{|y-y_k| \leq \frac{B}{2}} \left( 3\varepsilon_y^2 + \bar{\mu}_k^{-2} \varepsilon^2 - 5Q_k^4 \varepsilon^2 + 20(y - y_k) \partial_y Q_k Q_k^3 \varepsilon^2 \right) \]
\[ = \bar{\mu}_k^{-2} \int_{|x| \leq \bar{\mu}_k^{-1} \frac{B}{2}} (3\eta_x^2 + \eta^2 - 5Q^4 \eta^2 + 20xQ^3 \eta^2) \]
and, by (3.9),
\[ \langle \eta, \Lambda Q \rangle = \langle \eta, y\Lambda Q \rangle = \langle \eta, Q \rangle = 0. \]
But, from Lemma 3.4 in [22] (see also Proposition 4 in [18]), these orthogonality conditions ensure that, for \( B \) large enough (here depending on \( \ell_k \)) and for some \( \kappa_0 > 0 \),
\[ \int_{|x| \leq \bar{\mu}_k^{-1} \frac{B}{2}} (3\eta_x^2 + \eta^2 - 5Q^4 \eta^2 + 20xQ^3 \eta^2) \geq \kappa_0 \int_{|x| \leq \bar{\mu}_k^{-1} \frac{B}{2}} (\eta_x^2 + \eta^2) - B^{-1} \int \eta^2 e^{-|x|/2}. \]
Thus, going back to the original space variable, we find
\[ \int_{|y-y_k| \leq \frac{B}{2}} \left( 3\varepsilon_y^2 + \bar{\mu}_k^{-2} \varepsilon^2 - 5Q_k^4 \varepsilon^2 + 20(y - y_k) \partial_y Q_k Q_k^3 \varepsilon^2 \right) \]
\[ \geq \kappa_0 \int_{|y-y_k| \leq \frac{B}{2}} (\varepsilon_y^2 + \bar{\mu}_k^{-2} \varepsilon^2) - B^{-1} \bar{\mu}_k^{-2} \varepsilon^2 \]
Now, we estimate \( R_{V\mu}(\varepsilon) \). By (2.28) and the exponential decay of \( Q \), we observe that
\[ \|V - Q_k\|_{L^\infty(|y-y_k| \leq \frac{B}{2})} \lesssim \sum_j \|V_j - Q_j\|_{L^\infty} + \sum_{j \neq k} \|Q_j\|_{L^\infty(|y-y_k| \leq \frac{B}{2})} \lesssim |s|^{-\frac{3}{4}} \]
and, similarly, \( \|\partial_y (V - Q_k)\|_{L^\infty(|y-y_k| \leq \frac{B}{2})} \lesssim |s|^{-\frac{1}{4}} \). Using also (2.11) and \( \|\varepsilon\|_{L^\infty} \lesssim |s|^{-\frac{1}{4}} \) from (2.49), we thus find
\[ |R_{V\mu}(\varepsilon)| \lesssim B |s|^{-\frac{3}{4}} \int_{|y-y_k| \leq \frac{B}{2}} \varepsilon^2. \]
In conclusion for this term, we have obtained, for \( |S_0| \) large enough (depending on \( B \)),
\[ f_1^\sim \leq -\frac{\kappa_0}{2} \mu_k^{-2} \int_{|y-y_k| \leq \frac{B}{2}} (\varepsilon_y^2 + \mu_k^{-2} \varepsilon^2) \partial_y \varphi_k + CB^{-2} \varepsilon^2 \]
\[ \text{Estimate of } f_1. \text{ Gathering the above estimates of } f_1^<, f_1^\sim \text{ and } f_1^>, \text{ using (3.16), assuming } \kappa_0 < \frac{1}{4} \text{ and } B \text{ large enough, and finally letting } \kappa_1 = \frac{\kappa_0}{4} > 0, \text{ we obtain} \]
\[ f_1 \leq -\kappa_1 \mu_k^{-2} \int (\varepsilon_y^2 + \mu_k^{-2} \varepsilon^2) \partial_y \varphi_k + C \sum_{j < k} \|\varepsilon\|_{L^2_j}^2 + C |s|^{-\frac{3}{4}}. \]
Control of $f_2$. Using integrations by parts, we find the identities
\[ - \int (\Lambda_k \varepsilon) \partial_y (\psi_k \varepsilon_y) = \int \varepsilon_y^2 \psi_k - \frac{1}{2} \int (y - y_k) \varepsilon_y^2 \partial_y \psi_k, \]
and
\[ \int (\Lambda_k \varepsilon) \varepsilon \varphi_k = - \frac{1}{2} \int (y - y_k) \varepsilon_y^2 \partial_y \varphi_k, \]
and
\[ - \int (\Lambda_k \varepsilon) [(V + \varepsilon)^5 - V^5] \psi_k = \frac{1}{6} \int [(y - y_k) \partial_y \psi_k - 2 \psi_k] [(V + \varepsilon)^6 - V^6 - 6V^5 \varepsilon] \]
\[ + \int (\Lambda_k V) [(V + \varepsilon)^5 - V^5 - 5V^4 \varepsilon] \psi_k. \]
Thus, from the definition of $F_k$, we may rewrite $f_2$ as
\[ f_2 = - \frac{F_k}{43|s|} - \frac{\mu_k^{-2}}{|s|} \int \varepsilon_y^2 \varphi_k - \frac{1}{2} |s|^{-1} \int (y - y_k) \varepsilon_y^2 \partial_y \psi_k - \frac{1}{2} \mu_k^{-2} |s|^{-1} \int (y - y_k) \varepsilon_y^2 \partial_y \varphi_k \]
\[ + 1 \frac{6}{|s|} \int (y - y_k) [(V + \varepsilon)^6 - V^6 - 6V^5 \varepsilon] \partial_y \psi_k, \]
and so, using (3.14) and (3.17),
\[ f_2 \leq - \frac{\mu_k^{-2}}{|s|} \int \varepsilon_y^2 \varphi_k + C |s|^{-1} \int |y - y_k| \left( \varepsilon_y^2 + \mu_k^{-2} \varepsilon^2 \right) \partial_y \varphi_k + C |s|^{-10}. \]
From (3.14) again, if $|y - y_k| \geq |s|^\frac{1}{2}$, then $|y - y_k| \partial_y \varphi_k(y) \leq e^{-\frac{1}{2}m|s|^\frac{1}{4}}$, and so, using also (2.49),
\[ |s|^{-1} \int |y - y_k| \left( \varepsilon_y^2 + \mu_k^{-2} \varepsilon^2 \right) \partial_y \varphi_k \leq |s|^{-1} \int \left( \varepsilon_y^2 + \mu_k^{-2} \varepsilon^2 \right) \partial_y \varphi_k + |s|^{-10}. \]
In conclusion for this term, we have obtained, for $|S_0|$ large enough,
\[ f_2 \leq - \frac{\mu_k^{-2}}{|s|} \int \varepsilon_y^2 \varphi_k + \frac{\kappa_1}{10} \mu_k^{-2} \int \left( \varepsilon_y^2 + \mu_k^{-2} \varepsilon^2 \right) \partial_y \varphi_k + C |s|^{-10}. \]
Control of $f_3$. From the expression (2.29) of $\mathcal{E}_V$, we may rewrite $f_3$ as
\[ \frac{1}{2} f_3 = \sum_j m_{j,1} \int (\Lambda_j V_j) \tilde{G}_k(\varepsilon) + \sum_j m_{j,2} \int (\partial_y V_j) \tilde{G}_k(\varepsilon) - \sum_j \tilde{r}_j \int R_j \tilde{G}_k(\varepsilon) \]
\[ + \left( 2 \mu_k^{-2} \sum_{j < k} \bar{a}_j \langle \varepsilon, P_j \rangle - \sum_j \bar{a}_j \int P_j \tilde{G}_k(\varepsilon) \right) - \int \Psi \tilde{G}_k(\varepsilon) - \tilde{\mu}_k \tilde{\mu}_k^{-3} \int \varepsilon_y^2 \varphi_k \]
\[ = f_{3,1} + f_{3,2} + f_{3,3} + f_{3,4} + f_{3,5} + f_{3,6}. \]
Estimate of $f_{3,1}$. By integration by parts, we have
\[ \int (\Lambda_j V_j) \tilde{G}_k(\varepsilon) = - \int \varepsilon \left[ \partial_y (\Lambda_j V_j) \psi_k - \mu_k^2 (\Lambda_j V_j) \varphi_k + 5V^4 (\Lambda_j V_j) \psi_k + \partial_y (\Lambda_j V_j) \partial_y \psi_k \right]. \]
First, by $R \in \mathcal{Y}$, (2.12) and (2.21), we note that, for all $1 \leq j \leq K$ and all $0 \leq m \leq 19$, $|\partial_y^m V_j(y)| \lesssim e^{-\rho_1 m (y - y_j)}$ for $y > y_j$. In particular, for $j > k$, using (3.15) and the estimate $|m_{j,1}| \lesssim ||\varepsilon||_{L^2} \lesssim |s|^{-\frac{1}{2}}$ from (2.49) and (2.54), we find
\[ |m_{j,1} \int (\Lambda_j V_j) \tilde{G}_k(\varepsilon)| \lesssim |s|^{-10}. \]
Next, we focus on the case \( j = k \). First, by (2.28), we obtain
\[
\int_{|y - y_k| < |s|^{\frac{1}{4}}} |A_k V_k - A_k Q_k|^2 \lesssim \left( \|V_k - Q_k\|_{L^\infty} + |s|^{\frac{1}{2}} \|\partial_y (V_k - Q_k)\|_{L^\infty} \right) |s|^{\frac{1}{8}} \lesssim |s|^{-\frac{5}{8}}.
\]
Moreover, by the decay properties of \( V_k \) and \( Q_k \) for \( y > y_k \) and by (3.15), we have
\[
\int_{|y - y_k| > |s|^{\frac{1}{4}}} |A_k V_k - A_k Q_k|^2 \varphi_k \lesssim |s|^{-10} \quad \text{and so} \quad \int |A_k V_k - A_k Q_k|^2 \varphi_k \lesssim |s|^{-\frac{7}{8}}.
\]
Similarly, we get \( \int |\partial_{yy} (A_k V_k) - \partial_{yy} (A_k Q_k)|^2 \psi_k \lesssim |s|^{-\frac{5}{8}} \) and also
\[
\int |V^4 (A_k V_k) - Q_k^4 (A_k Q_k)|^2 \psi_k + \int |\partial_y (A_k V_k) - \partial_y (A_k Q_k)|^2 \partial_y \psi_k \lesssim |s|^{-\frac{7}{8}}.
\]
We also observe that, by the definitions of \( \varphi_k \) and \( \psi_k \) and the exponential decay of \( Q \),
\[
\int |A_k Q_k| |\varphi_k - \left( 1 + \frac{y - y_k}{B} \right)|^2 + \int \left( |\partial_{yy} (A_k Q_k)| + |Q_k^4 (A_k Q_k)| \right) |\psi_k| - 1|^2 + \int |\partial_y (A_k Q_k)| |\partial_y \psi_k|^2 \lesssim e^{-\frac{\mu}{4k}}.
\]
Therefore, by the Cauchy–Schwarz inequality,
\[
\left| \int (A_k V_k) \tilde{G}_k (\varepsilon) - \int \varepsilon \left[ \partial_{yy} (A_k Q_k) - \mu_k^{-2} \left( 1 + \frac{y - y_k}{B} \right) (A_k Q_k) + 5Q_k^4 (A_k Q_k) \right] \right| \lesssim |s|^{-\frac{5}{8}} \left( \int \varepsilon^2 \varphi_k \right)^{\frac{1}{2}} + e^{-\frac{\mu}{4k}} |\varepsilon|_{L_k^2}.
\]
Now note that, since \( L (\Lambda Q) = -2Q \) from Lemma 2.1, we have the identity
\[
\partial_{yy} (A_k Q_k) - \tilde{\mu}_k^{-2} (A_k Q_k) + 5Q_k^4 (A_k Q_k) = 2\tilde{\mu}_k^{-2} Q_k.
\]
Thus, from the three orthogonality conditions (3.9), the second term in the last estimate cancels and we find
\[
\left| \int (A_k V_k) \tilde{G}_k (\varepsilon) \right| \lesssim |s|^{-\frac{5}{8}} \left( \int \varepsilon^2 \varphi_k \right)^{\frac{1}{2}} + e^{-\frac{\mu}{4k}} |\varepsilon|_{L_k^2}.
\]
Using also (2.49) and (2.54), it follows that
\[
|m_{k,1} \int (A_k V_k) \tilde{G}_k (\varepsilon)| \lesssim \left[ \|\varepsilon\|_{L_k^2} + \sum_{j'} \|\varepsilon\|_{L_{j'}^2}^2 + |s|^{-2} \right] \left[ |s|^{-\frac{5}{8}} \left( \int \varepsilon^2 \varphi_k \right)^{\frac{1}{2}} + e^{-\frac{\mu}{4k}} |\varepsilon|_{L_k^2} \right]
\lesssim \left( e^{-\frac{\mu}{4k}} + |s|^{-\frac{5}{8}} \right) \|\varepsilon\|_{L_k^2}^2 + |s|^{-\frac{5}{8}} \int \varepsilon^2 \varphi_k + |s|^{-\frac{1}{8}} \sum_{j'} \|\varepsilon\|_{L_{j'}^2}^2 + |s|^{-4}.
\]
Finally, for \( j < k \), we observe that, by (2.28), \( \|A_j V_j - A_j Q_j\|_{L^2} \lesssim |s|^{-\frac{5}{8}} \), and similarly
\[
\|\partial_{yy} (A_j V_j) - \partial_{yy} (A_j Q_j)\|_{L^2} + \|V^4 (A_j V_j) - Q_k^4 (A_j Q_j)\|_{L^2} + \|\partial_y (A_j V_j) \partial_y \psi_k\|_{L^2} \lesssim |s|^{-\frac{7}{8}}.
\]
Proceeding as before, using (3.14), we obtain in this case
\[
\left| \int (A_j V_j) \tilde{G}_k (\varepsilon) \right| \lesssim |s|^{-\frac{5}{8}} \left( \int \varepsilon^2 \varphi_k \right)^{\frac{1}{2}} + \|\varepsilon\|_{L_j^2}.
\]
By (2.49) and (2.54), it follows that
\[
|m_{j,1} \int (\Lambda_j V_j) \bar{G}_k(\varepsilon)| \lesssim \left[ \|\varepsilon\|_{L^2_j}^2 + \sum_{j'} \|\varepsilon\|_{L^2_{j'}}^2 + |s|^{-2} \right] \left[ |s|^{-\frac{1}{2}} \left( \int \varepsilon^2 \varphi_k \right)^{\frac{1}{2}} + \|\varepsilon\|_{L^2_j} \right]
\leq C \|\varepsilon\|_{L^2_{j_k}}^2 + \frac{\mu_k^{-2}}{10K|s|} \int \varepsilon^2 \varphi_k + C|s|^{-\frac{1}{2}} \sum_{j'} \|\varepsilon\|_{L^2_{j'}}^2 + C|s|^{-4}.
\]

Therefore we have obtained, for $B$ and $|S_0|$ large enough, and using (3.16),
\[
|f_{3,1}| \leq \frac{k_1}{10} \mu_k^{-4} \int \varepsilon^2 \partial_y \varphi_k + \frac{2\mu_k^{-2}}{10|s|} \int \varepsilon^2 \varphi_k + C|s|^{-\frac{1}{2}} \sum_j \|\varepsilon\|_{L^2_j}^2 + C \sum_{j<k} \|\varepsilon\|_{L^2_j}^2 + C|s|^{-4}.
\]

\textit{Estimate of $f_{3,2}$}. By integration by parts, we have
\[
\int \partial_y(V_j) \bar{G}_k(\varepsilon) = -\int \varepsilon \left( \partial_{y_{jk}} V_j \psi_k - \mu_k^{-2} \partial_y V_j \varphi_k + 5V^4 \partial_y V_j \psi_k + \partial_{y_{jk}} V_j \partial_y \psi_k \right).
\]

We use similar arguments as for $f_{3,1}$, and obtain the same estimates in the cases $j > k$ and $j < k$. In the case $j = k$, we also obtain similarly, using (2.28),
\[
\left| \int \partial_y(V_k) \bar{G}_k(\varepsilon) - \int \varepsilon \left( \partial_{y_{yy}} Q_k - \mu_k^{-2} \left( 1 + \frac{y}{B} \right) \partial_y Q_k + 5Q_k \partial_y Q_k \right) \right| \lesssim |s|^{-\frac{1}{2}} \left( \int \varepsilon^2 \varphi_k \right)^{\frac{1}{2}} + e^{-\frac{B}{\mu_k}} \|\varepsilon\|_{L^2_k}.
\]

Using again (3.9), we find $\langle \varepsilon, \cdot_{y_k} \partial_y Q_k \rangle = \langle \varepsilon, \Lambda_k Q_k \rangle - \frac{1}{2} \langle \varepsilon, Q_k \rangle = 0$. Thus, from the cancellation (2.61) and the estimate (2.11), we have
\[
\left| \int \partial_y(V_k) \bar{G}_k(\varepsilon) \right| \lesssim |s|^{-\frac{1}{2}} \left( \int \varepsilon^2 \varphi_k \right)^{\frac{1}{2}} + \left( e^{-\frac{B}{\mu_k}} + |s|^{-1} \right) \|\varepsilon\|_{L^2_k}.
\]

Therefore we also obtain, for $B$ and $|S_0|$ large enough,
\[
|f_{3,2}| \leq \frac{k_1}{10} \mu_k^{-4} \int \varepsilon^2 \partial_y \varphi_k + \frac{2\mu_k^{-2}}{10|s|} \int \varepsilon^2 \varphi_k + C|s|^{-\frac{1}{2}} \sum_j \|\varepsilon\|_{L^2_j}^2 + C \sum_{j<k} \|\varepsilon\|_{L^2_j}^2 + C|s|^{-4}.
\]

\textit{Estimate of $f_{3,3}$}. By integration by parts, we have
\[
- \int R_j \bar{G}_k(\varepsilon) = \int \varepsilon \left( \partial_{y_{yy}} R_j \psi_k + \partial_y R_j \partial_y \psi_k + 5V^4 R_j \psi_k - \mu_k^{-2} R_j \varphi_k \right).
\]

Since $R \in \mathcal{Y}$, we have $\|R_j\|_{H^2} \lesssim 1$ for any $1 \leq j \leq K$ and so we find, using also $\|V\|_{L^\infty} \lesssim 1$ and (2.49),
\[
\left| \int R_j \bar{G}_k(\varepsilon) \right| \lesssim \|R_j\|_{H^2} \|\varepsilon\|_{L^2} \lesssim |s|^{-\frac{1}{2}}.
\]

Thus, using (2.56), we obtain as before
\[
|f_{3,3}| \lesssim |s|^{-\frac{1}{2}} \sum_j |\tilde{r}_j| \lesssim |s|^{-\frac{1}{2}} \sum_j \|\varepsilon\|_{L^2_j}^2 + |s|^{-\frac{3}{2}}.
\]

\textit{Estimate of $f_{3,4}$}. By integration by parts, we have
\[
- \int P_j \bar{G}_k(\varepsilon) = \int \varepsilon \left( \partial_{y_{yy}} P_j \psi_k + \partial_y P_j \partial_y \psi_k + 5V^4 P_j \psi_k - \mu_k^{-2} P_j \varphi_k \right).
\]
From (2.13), \( \| \partial_y^m P_j \|_{L^2} \lesssim 1 \) for \( m = 1, 2 \) and so, by (2.49), for any \( 1 \leq j \leq K \),
\[
\left| \int \varepsilon (\partial_y P_j \psi_k + \partial_y P_j \partial_y \psi_k) \right| \lesssim \| \varepsilon \|_{L^2} \lesssim |s|^{-\frac{1}{2}}.
\]
From (2.28), we have \( \| \mathbf{V} \|_{L^2} + \| \mathbf{V} \|_{L^\infty} \lesssim 1 \). Thus, for any \( 1 \leq j \leq K \),
\[
\left| \int \mathbf{V}^4 \nu P_j \psi_k \right| \lesssim \int \mathbf{V}^4 \varepsilon |\varepsilon| \lesssim \| \mathbf{V} \|^3_{L^\infty} \| \mathbf{V} \|_{L^2} \| \varepsilon \|_{L^2} \lesssim \| \varepsilon \|_{L^2} \lesssim |s|^{-\frac{1}{2}}.
\]
Next, for \( j \geq k \), by (2.12) and \( \varphi_k(y) \leq e^{\frac{1}{2} y(y - y_k)} \) for \( y < y_k \) from (3.15), we have \( \| P_j \varphi_k \|_{L^2} \lesssim 1 \), and so
\[
|\langle \varepsilon, P_j \varphi_k \rangle| \lesssim \| \varepsilon \|_{L^2} \| P_j \varphi_k \|_{L^2} \lesssim \| \varepsilon \|_{L^2} \lesssim |s|^{-\frac{1}{2}}.
\]
In contrast, for \( j < k \), by (2.12) and \( \varphi_k(y) = 2 e^{-\frac{1}{8} y(y - y_k)} \) for \( y > y_k + B \) from the definition of \( \varphi \), we have \( \| P_j (\varphi_k - 2) \|_{L^2} \lesssim |s|^{-10} \), and so
\[
|2 \langle \varepsilon, P_j \rangle - \langle \varepsilon, P_j \varphi_k \rangle| \lesssim \| \varepsilon \|_{L^2} \| P_j (\varphi_k - 2) \|_{L^2} \lesssim |s|^{-10}.
\]
Using also (2.55), it follows from these estimates that
\[
|f_{3,4}| \lesssim |s|^{-\frac{1}{2}} \sum_j |a_j| \lesssim |s|^{-\frac{1}{2}} \sum_j \| \varepsilon \|_{L^2_j}^2 + |s|^{-\frac{1}{2}}.
\]
**Estimate of** \( f_{3,5} \). By integration by parts, we have
\[
- \int \mathbf{V} \mathbf{G}_k (\varepsilon) = \int \varepsilon \left( \mathbf{V} \nu_{yy} \psi_k + \mathbf{V} \nu \partial_y \psi_k + 5 \mathbf{V}^4 \nu \psi_k - \mu_k^{-2} \mathbf{V} \varphi_k \right).
\]
From (3.14), we have \( \psi_k + \partial_y \psi_k \lesssim \varphi_k \) on \( \mathbb{R} \) and \( \varphi_k(y) \lesssim e^{\frac{1}{2} y(y - y_k)} \) for \( y < y_k \). Thus, using also \( \| \mathbf{V} \|_{L^\infty} \lesssim 1 \) from (2.28), we find
\[
|f_{3,5}| \lesssim \| \mathbf{V} \|_{H^2(y > y_k - |s|^{-\frac{1}{4}})} \left( \int \varepsilon^2 \varphi_k \right)^{\frac{1}{2}} + e^{-\frac{1}{2} |s|^{-\frac{1}{4}}} \| \mathbf{V} \|_{H^2} \| \varepsilon \|_{L^2}.
\]
From (2.31), (2.32) and (3.5), we obtain the estimate
\[
|f_{3,5}| \lesssim \left( |s|^{-\frac{1}{10}} + |s|^{-\frac{3}{2} - \delta_k - 1} \right) \left( \int \varepsilon^2 \varphi_k \right)^{\frac{1}{2}} + |s|^{-10} \lesssim \frac{\mu_k^{-2}}{10 |s|} \int \varepsilon^2 \varphi_k + C |s|^{-2 - 2\delta_k - 1}.
\]
**Estimate of** \( f_{3,6} \). We first decompose \( f_{3,6} \) as
\[
- f_{3,6} = \left[ \left( \frac{\mu_k}{\mu_k} + \frac{1}{2 \mu_k^3 \tau_k} - \frac{1}{2} \frac{a_k}{\mu_k^2} \right) - \left( \frac{1}{2 \mu_k^3 \tau_k} - \frac{1}{2} \frac{a_k}{\mu_k^2} \right) \right] \mu_k^{-2} \int \varepsilon^2 \varphi_k.
\]
Thus, using on the one hand the definition (2.30) and the estimate (2.54) of \( m_{k,1} \), and on the other hand the estimates \( \left| \frac{1}{2 \mu_k^3 \tau_k} - \frac{1}{2} \right| \lesssim |s|^{-1 - \delta_k} \) and \( |a_k| \lesssim |s|^{-1 - \delta_k} \) from (3.5)–(3.6), we find
\[
|f_{3,6}| \lesssim \left( \| \varepsilon \|_{L^2_k}^2 + \sum_j \| \varepsilon \|^2_{L^2_j} + |s|^{-1 - \delta_k} \right) \mu_k^{-2} \int \varepsilon^2 \varphi_k
\]
\[
\lesssim \| \varepsilon \|^2_{L^2_k} \| \varepsilon \|_{L^2_k} + \| \varepsilon \|^2_{L^2_k} \sum_j \| \varepsilon \|^2_{L^2_j} + |s|^{-1 - \delta_k} \mu_k^{-2} \int \varepsilon^2 \varphi_k.
\]
Using the estimate on \( \| \varepsilon \|_{L^2} \) from (3.5), we obtain
\[
\| \varepsilon \|^2_{L^2} \| \varepsilon \|^2_{L^2_k} \lesssim |s|^{-1 - 2\delta_k + 1} \| \varepsilon \|_{L^2_k} \lesssim |s|^{-\delta_k + 1} \| \varepsilon \|^2_{L^2_k} + |s|^{-2 - 3\delta_k + 1},
\]
Thus, we may rewrite
\[ |f_{3,6}| \lesssim |s|^{-\delta_k+1} \mu_k^{-4} \int \varepsilon^2 \partial_y \varphi_k + |s|^{-2-2\delta_k-1} + |s|^{-1} \sum_j \|\varepsilon\|_{L^2}^2 + |s|^{-1-\delta_k} \mu_k^{-2} \int \varepsilon^2 \varphi_k. \]

Thus, taking $|S_0|$ large enough, we obtain
\[ |f_{3,6}| \lesssim \frac{\kappa_1}{10} \mu_k^{-4} \int \varepsilon^2 \partial_y \varphi_k + \frac{\mu_k^{-2}}{10|s|} \int \varepsilon^2 \varphi_k + C|s|^{-1} \sum_j \|\varepsilon\|_{L^2}^2 + C|s|^{-2-2\delta_k-1}. \]

**Estimate of $f_3$.** Gathering the above estimates of $f_{3,1}, \ldots, f_{3,6}$, we obtain
\[ |f_3| \lesssim \frac{3\kappa_1}{10} \mu_k^{-4} \int \varepsilon^2 \partial_y \varphi_k + \frac{6\mu_k^{-2}}{10|s|} \int \varepsilon^2 \varphi_k + C|s|^{-\frac{1}{2}} \sum_j \|\varepsilon\|_{L^2}^2 + C \sum_{j<k} \|\varepsilon\|_{L^2}^2 + C|s|^{-2-2\delta_k-1}. \]

**Control of $f_4$.** By integration by parts, we have
\[
\int \varepsilon G_k(\varepsilon) = -\frac{1}{2} \int \varepsilon \partial_y \psi_k - \frac{1}{2} \mu_k^{-2} \int \varepsilon \partial_y \varphi_k + \frac{1}{6} \int \left[ (V + \varepsilon)^6 - V^6 - 6V^5 \varepsilon \right] \partial_y \psi_k \\
+ \int V_y \left[ (V + \varepsilon)^5 - V^5 - 5V^4 \varepsilon \right] \psi_k.
\]

Thus, we may rewrite $f_4$ as
\[
f_4 = -\left( \frac{\dot{y}_k - y_k}{2s} - \frac{1}{\mu_k} \right) \int \left\{ \varepsilon^2 \partial_y \psi_k + \mu_k^{-2} \varepsilon^2 \partial_y \varphi_k - \frac{1}{3} \left[ (V + \varepsilon)^6 - V^6 - 6V^5 \varepsilon \right] \partial_y \psi_k \right\} \\
+ 2 \int \left[ V_{yy} + 5V_y V^4 - \mu_k^{-2} V_y \right] \left[ (V + \varepsilon)^5 - V^5 - 5V^4 \varepsilon \right] \psi_k = f_{4,1} + f_{4,2}.
\]

**Estimate of $f_{4,1}$.** We first note that, from the definition (2.30) and the estimate (2.54) of $m_{k,2}$, and from (2.49),
\[
\left| \frac{\dot{y}_k - y_k}{2s} - \frac{1}{\mu_k} \right| \lesssim |\bar{m}_k| + |s|^{-1} \lesssim \|\varepsilon\|_{L^2} + |s|^{-1} \lesssim |s|^{-\frac{1}{2}}.
\]

Thus, using (3.14) and taking $|S_0|$ large enough, we obtain
\[ |f_{4,1}| \lesssim |s|^{-\frac{1}{2}} \left( \varepsilon^2 + \mu_k^{-2} \varepsilon^2 \right) \partial_y \varphi_k \lesssim \frac{\kappa_1}{10} \mu_k^{-2} \int \left( \varepsilon^2 + \mu_k^{-2} \varepsilon^2 \right) \partial_y \varphi_k.
\]

**Estimate of $f_{4,2}$.** We decompose $f_{4,2}$ as $f_{4,2} = f_{4,2}^C + f_{4,2}^C + f_{4,2}^C$, where $f_{4,2}^C$, $f_{4,2}^C$, and $f_{4,2}^C$ respectively correspond to integration on $y - y_k < -\frac{3B}{2}$, $|y - y_k| \leq \frac{B}{2}$, and $y - y_k > \frac{B}{2}$, and we follow the above calculation done for the estimate of $f_1$.

First, to estimate $f_{4,2}^C$, we recall that $\|V\|_{L^\infty} \lesssim 1$ from (2.28), and so, using also (3.14) and (3.15),
\[ |f_{4,2}^C| \lesssim B \int_{y-y_k<\frac{-3B}{2}} \left( |V_{yy}| + |V_y| \right) \left( |\varepsilon|^5 + |V|^3 \varepsilon^2 \right) \partial_y \varphi_k.
\]

Thus, we may estimate this term as the one similar in $f_1^C$ and find, taking $B$ large enough and then $|S_0|$ large enough,
\[ |f_{4,2}^C| \lesssim \frac{\kappa_1}{10} \mu_k^{-2} \int \varepsilon^2 \partial_y \varphi_k + C|s|^{-10}.
\]
Second, to estimate $f_{4,2}^+$, we recall that $\|\partial_y^m V\|_{L^\infty} \lesssim 1$ for $m = 1, 3$ from (2.28) and so, proceeding as in the estimate of $f_1^+$, we find

$$|f_{4,2}^+| \lesssim \int_{y-y_k > \frac{d}{4}} \left( |\varepsilon|^5 + |V|^3 \varepsilon^2 \right) \lesssim \sum_{j < k} \|\varepsilon\|^2_{L^2_j} + B^{-2} \|\varepsilon\|^2_{L^2_k} + |s|^{-\frac{3}{2}}.$$

Thus, using (3.16) and taking $B$ large enough, we find

$$|f_{4,2}^+| \leq \frac{\kappa_1}{10} \mu_k^{-4} \int \varepsilon^2 \partial_y \varphi_k + C \sum_{j < k} \|\varepsilon\|^2_{L^2_j} + C |s|^{-\frac{3}{2}}.$$

Third, to estimate $f_{4,2}^-$, we find as in the estimate of $f_1^-$ that, for $m = 1, 3$,

$$\|V - Q_k\|_{L^\infty([y-y_k] \leq \frac{d}{4})} \lesssim |s|^{-\frac{3}{2}} \quad \text{and} \quad \|\partial_y^m (V - Q_k)\|_{L^\infty([y-y_k] \leq \frac{d}{4})} \lesssim |s|^{-1}.$$

Thus, using also (2.11), the cancellation (2.61) and the identity $\partial_y \varphi_k(y) = \frac{1}{B}$ for $|y-y_k| \leq \frac{d}{4}$, we find

$$|f_{4,2}^-| \lesssim \int_{|y-y_k| < \frac{d}{4}} \left| V_{yy} + 5V_y V^4 - \mu_k^{-2} V_{yy} \right| \varepsilon^2 \lesssim |s|^{-\frac{3}{2}} \int_{|y-y_k| < \frac{d}{4}} \varepsilon^2 \lesssim B |s|^{-\frac{3}{4}} \int \varepsilon^2 \partial_y \varphi_k.$$

Finally, gathering the estimates of $f_{4,2}^+, f_{4,2}^-$ and $f_{4,2}^+$, and taking $|S_0|$ large enough, we have

$$|f_{4,2}| \leq \frac{3\kappa_1}{10} \mu_k^{-4} \int \varepsilon^2 \partial_y \varphi_k + C \sum_{j < k} \|\varepsilon\|^2_{L^2_j} + C |s|^{-\frac{3}{2}}.$$

**Estimate of $f_4$.** Gathering the above estimates of $f_{4,1}$ and $f_{4,2}$, we obtain

$$|f_4| \leq \frac{4\kappa_1}{10} \mu_k^{-2} \int \left( \varepsilon_y^2 + \mu_k^{-2} \varepsilon^2 \right) \partial_y \varphi_k + C \sum_{j < k} \|\varepsilon\|^2_{L^2_j} + C |s|^{-\frac{3}{2}}.$$

**Conclusion.** Gathering the above estimates of $f_1$, $f_2$, $f_3$ and $f_4$, we have obtained

$$f_1 + f_2 + f_3 + f_4 \leq -\frac{\kappa_1}{5} \mu_k^{-2} \int \left( \varepsilon_y^2 + \mu_k^{-2} \varepsilon^2 \right) \partial_y \varphi_k - \frac{2\mu_k^{-2}}{5|s|} \int \varepsilon^2 \varphi_k
+C \sum_{j < k} \|\varepsilon\|^2_{L^2_j} + C |s|^{-\frac{3}{2}} \sum_j \|\varepsilon\|^2_{L^2_j} + C |s|^{-2-\delta_k - \delta_k^*}.$$

Thus, taking $|S_0|$ large enough, we have

$$\frac{1}{|s|^{1+\frac{3}{2}}} \frac{d}{ds} \left\{ |s|^{1+\frac{3}{2}} \left[ F_k + 4 \mu_k^{-2} \sum_{j < k} a_j (\varepsilon, P_j) + 2c\mu_k^{-2} |s| \sum_{j < k} a_j^2 \right] \right\}
\leq -\frac{\kappa_1}{10} \mu_k^{-2} \int \left( \varepsilon_y^2 + \mu_k^{-2} \varepsilon^2 \right) \partial_y \varphi_k + C \sum_{j < k} \|\varepsilon\|^2_{L^2_j} + C |s|^{-\frac{3}{2}} \sum_j \|\varepsilon\|^2_{L^2_j} + C |s|^{-2-\delta_k - \delta_k^*},$$

which proves (3.18) and concludes the proof of Lemma 3.4. \qed
3.5. **Local and global estimates on** $\varepsilon_n$. We claim the following estimates on $\varepsilon_n$, strictly improving the first two lines of (3.5) for $|S_0|$ large enough:

$$
\|\varepsilon_n(s)\|^2_{H^1} \lesssim |s|^{-1-\delta_{K,-}\delta_{K+1}^+} \lesssim \frac{1}{4}|s|^{-1-2\delta_{K+1}^+},
$$

(3.19)

$$
\|\varepsilon_n(s)\|^2_{H^1(y>y_K)} \lesssim |s|^{-1-\delta_{K,-1}^+} \lesssim \frac{1}{4}|s|^{-1-2\delta_{K}^+},
$$

(3.20)

$$
\int_{S_n} |\tau|^{1+\frac{4}{n}} \left( (\|\partial_y\varepsilon_n(\tau)\|^2_{L^2_y} + \|\varepsilon_n(\tau)\|^2_{L^2_y}) \right) d\tau \lesssim |s|^{-\delta_{K,-1}^+} \lesssim \frac{1}{2}|s|^{-2\delta_{K}^+}.\tag{3.21}
$$

Note that, as a consequence of (3.21), by the Cauchy–Schwarz inequality,

$$
\int_{S_n} \|\varepsilon_n(\tau)\|^2_{L^2_y} d\tau \lesssim |s|^{-\frac{1}{36}} \left( \int_{S_n} |\tau|^{1+\frac{4}{n}} \|\varepsilon_n(\tau)\|^2_{L^2_y} d\tau \right)^{\frac{1}{2}} \lesssim |s|^{-\frac{1}{2}(\delta_{K,-1}^+ + \delta_{K}^+)}.\tag{3.22}
$$

**Proof of (3.19).** As before, we denote $\varepsilon_n$ and $V_n$ simply by $\varepsilon$ and $V$. First, we prove the bound on $\|\varepsilon(s)\|_{L^2}$. Integrating (3.10) on $[S_n, s]$, using $\int_{S_n} |\tau| \|\varepsilon(\tau)\|^2_{L^2_y} d\tau \leq |s|^{-2\delta_{K}^+}$ from (3.5) and finally (3.4), we get

$$
\|\varepsilon\|^2_{L^2} + 2 \sum_k a_k(\varepsilon, P_k) + c|s| \sum_k a_k^2
$$

$$
\leq C|s|^{-1} \sum_j \int_{S_n} |\tau| \|\varepsilon(\tau)\|^2_{L^2_y} d\tau + C|s|^{-1-\delta_{K,-}\delta_{K+1}^+} + c|s|^{-1}|S_n|^2 \sum_k a_k^2(S_n)
$$

$$
\leq C|s|^{-1-2\delta_{K}^+} + C|s|^{-1-\delta_{K,-}\delta_{K+1}^+} + C|s|^{-1-2\delta_{K}} \leq C|s|^{-1-\delta_{K,-}\delta_{K+1}^+}.
$$

Using (3.7) and the bound on $|a_k|$ from (3.5), we obtain $\|\varepsilon\|^2_{L^2} \lesssim |s|^{-1-\delta_{K,-}\delta_{K+1}^+}$. Now, we may prove the bound on $\|\varepsilon(s)\|_{H^1}$. Integrating (3.11) on $[S_n, s]$, using the bound on $\|\varepsilon\|_{L^2}$ found above, we also get

$$
\|\varepsilon(s)\|^2_{H^1} \lesssim |s|^{-1} \sum_j \int_{S_n} |\tau| \|\varepsilon(\tau)\|^2_{L^2_y} d\tau + \int \left| (V + \varepsilon)^6 - V^6 - 6V^5\varepsilon \right|
$$

$$
\lesssim |s|^{-1-\delta_{K}^+} + |s|^{-1-\delta_{K,-}\delta_{K+1}^+} + \|\varepsilon\|^2_{L^2} \lesssim |s|^{-1-\delta_{K,-}\delta_{K+1}^+}.
$$

**Proof of (3.20) and (3.21).** We denote $\varepsilon_n$, $V_n$ and $\mathcal{F}_{k,n}$ by $\varepsilon$, $V$ and $\mathcal{F}_k$. Integrating (3.18) on $[S_n, s]$, using from (3.5) the estimates

$$
\int_{S_n} |\tau|^{1+\frac{4}{n}} \|\varepsilon(\tau)\|^2_{L^2_y} d\tau \leq |s|^{-2\delta_{K}^+ + \frac{1}{4}}, \quad \text{for } j < k,
$$

$$
\int_{S_n} |\tau|^{\frac{2}{n}} \|\varepsilon(\tau)\|^2_{L^2_y} d\tau \leq |s|^{-\frac{1}{2} - 2\delta_{K}^+ + \frac{1}{4}}, \quad \text{for all } 1 \leq j \leq K,
$$

we find, using also the estimate $a_j^2(S_n) \lesssim |S_n|^{-2-2\delta_{K,-1}^+}$ for $j < k$ from (3.4),

$$
|s|^{1+\frac{4}{n}} \left( \mathcal{F}_k + 4\mu_{k}^{-2} \sum_{j<k} a_j(\varepsilon, P_j) + 2c\mu_{k}^{-2} |s| \sum_{j<k} a_j^2 \right) + \kappa \int_{S_n} |\tau|^{1+\frac{4}{n}} \left( (\|\partial_y\varepsilon\|^2_{L^2_y} + \|\varepsilon\|^2_{L^2_y}) \right) d\tau
$$

$$
\lesssim |s|^{-2\delta_{K,-1}^+ + \frac{1}{4}} + |s|^{-\frac{1}{2} - 2\delta_{K}^+ + \frac{1}{4}} + |s|^{-\delta_{K} - \delta_{K+1}^+} + |s|^{-2\delta_{K}^+ + \frac{1}{4}} \lesssim |s|^{-\delta_{K} - \delta_{K+1}^+ + \frac{1}{4}}.
$$
Moreover, using (3.7) and the bound on $|a_j|$ from (3.5), we have
\[
\sum_{j < k} |a_j \langle \varepsilon, P_j \rangle| \lesssim |s|^{-1-\delta_k^+-\delta_{k-1}^-}.
\]

Thus, using also (3.17) and the definitions of $\psi_k$ and $\varphi_k$, we obtain
\[
\|\varepsilon\|_{H^1(y > y_k)}^2 \lesssim \int (\partial_y \varepsilon)^2 \psi_k + \int \varepsilon^2 \varphi_k \lesssim |s|^{-1-\delta_k^+-\delta_{k-1}^-}
\]
and
\[
\int_{S_n} |\tau|^{1+\frac{1}{2\delta}} \left( \|\partial_y \varepsilon(\tau)\|_{L^2_k}^2 + \|\varepsilon(\tau)\|_{L^2_k}^2 \right) d\tau \lesssim |s|^{-\delta_k^+-\delta_{k-1}^+ + \frac{1}{2\delta}}.
\]

3.6. Parameters estimates. Now, we close the parameters estimates in (3.5) on the time interval $[S_n, S_n^*]$. Technically, even if the parameters $e_k$ and $f_k$ do not appear in the bootstrap (3.5), their estimates are necessary to handle the other parameters.

Estimate of $e_k$. Integrating (2.58) on $[S_n, s]$, using (3.22) for the terms in $\|\varepsilon_n\|_{L^2_j}$ for $j \leq k$ and (3.5) for the terms in $\|\varepsilon_n\|_{L^2_j}$ and $a_j$, we obtain, using also (3.3),
\[
|e_k(s) - e_k(S_n)| \lesssim \sum_{j < k} \int_{S_n} \|\varepsilon_n(\tau)\|_{L^2_j}^2 d\tau + \sum_j \int_{S_n} |\tau| \|\varepsilon_n(\tau)\|_{L^2_j}^2 d\tau + \sum_{j < k} \int_{S_n} |a_j(\tau)| d\tau + |s|^{-\frac{1}{2}}
\]
\[
\lesssim |s|^{-\frac{1}{4}(\delta_{k-1}^- + \delta_k^+)} + |s|^{-2\delta_k^+} + |s|^{-\delta_{k-1}^-} + |s|^{-\frac{1}{2}} \lesssim |s|^{-\frac{1}{4}(\delta_{k-1}^- + \delta_k^+)}.
\]

Thus, the value of $e_k(S_n)$ being given in (3.4), we find
\[
\left| e_k(s) - \ell_k \left( \frac{1}{2} + \theta_k \right) \right| \lesssim |s|^{-\frac{1}{4}(\delta_{k-1}^- + \delta_k^+)}.
\]  

Estimate of $f_k$. We first rewrite (2.63) in terms of
\[
g_k(s) = f_k(s) + \int_{S_n} (1 + \bar{\mu}_k(\tau)) \langle \varepsilon_n(\tau), A_k(\tau) \rangle d\tau,
\]
as
\[
\left| \dot{g}_k + \frac{1}{2} (1 + 3\delta_k) \frac{g_k}{s} \right| \lesssim |s|^{-1} \left| \int_{S_n} (1 + \bar{\mu}_k(\tau)) \langle \varepsilon_n(\tau), A_k(\tau) \rangle d\tau \right| + \sum_j \|\varepsilon\|_{L^2_j}^2
\]
\[
+ |s|^{-1} \left( |s|^{-\frac{1}{2\delta}} + |\bar{\mu}_k|^2 + |\bar{g}_k|^2 + |e_k - \ell_k \left( \frac{1}{2} + \theta_k \right) | + \sum_{j < k} (|\bar{\mu}_j| + |\bar{r}_j| + |\bar{y}_j|) \right).
\]

Using the decay properties of $A_k$, (3.5)–(3.6) and (2.3), we obtain
\[
\left| \dot{g}_k + \frac{1}{2} (1 + 3\delta_k) \frac{g_k}{s} \right| \lesssim |s|^{-1} \int_{S_n} \|\varepsilon_n(\tau)\|_{L^2_k}^2 d\tau + |s|^{-1-2\delta_{k+1}^+}
\]
\[
+ |s|^{-1} \left( |s|^{-\frac{1}{2\delta}} + |s|^{-2\delta_k^+} + |s|^{-\frac{1}{4}(\delta_{k-1}^- + \delta_k^+)} + |s|^{-\delta_{k-1}^-} \right)
\]
and so, using also (3.22) and $2\delta_k > 2\delta_{k+1}^+ > \frac{1}{2\delta} > \delta_{k-1}^- > \delta_{k-1}^+ > \delta_k^+$ from (3.3),
\[
\left| \dot{g}_k + \frac{1}{2} (1 + 3\delta_k) \frac{g_k}{s} \right| \lesssim |s|^{-1-\frac{1}{4}(\delta_{k-1}^- + \delta_k^+)}.
\]  


Following the discussion in Section 3.1, we consider now separately the cases \( k \in \mathcal{K}^- \) and \( k \in \mathcal{K}^+ \). First, let \( k \in \mathcal{K}^- \) (if \( \mathcal{K}^- \) is empty, we just skip this case). Then (3.24) rewrites as

\[
\left| \frac{d}{ds} \left[ (-s)^{\frac{1}{2}}(1+3\theta_0)g_k \right] \right| \lesssim |s|^{-\frac{1}{2}(\delta_{k-1}^+-\delta_k^+ subtitle)| (1+3\theta_0)g_k |.\]

Since \(-\frac{1}{2}(\delta_{k-1}^-+\delta_k^+) + \frac{1}{2}(1+3\theta_0) < -\delta_k^- + \delta_{K+1} < 0 \) for \( k \in \mathcal{K}^- \), integrating on \([S_n, s] \), using \( g_k(S_n) = 0 \) from (3.4), we get \( |g_k(s)| \lesssim |s|^{-\frac{1}{2}(\delta_{k-1}^-+\delta_k^+)} \). Thus, by (3.22), for |\( S_0 \) large enough,

\[
|f_k(s)| \lesssim |g_k(s)| + \int_{S_n}^s \|\varepsilon_n(\tau)\|_{L^2_\delta} d\tau \lesssim |s|^{-\frac{1}{2}(\delta_{k-1}^-+\delta_k^+)} \lesssim |s|^{\frac{1}{2}k}. \tag{3.25}
\]

Now, let \( k \in \mathcal{K}^+ \). We use (3.6) to obtain directly \( |g_k(s)| \lesssim |s|^{-\delta_k^+} \). As above we obtain, for |\( S_0 \) large enough again, \( |f_k(s)| \lesssim |s|^{-\delta_k^+} + C|s|^{-\frac{1}{2}(\delta_{k-1}^-+\delta_k^+)} \lesssim 2|s|^{-\delta_k^+}. \)

In conclusion, for all \( 1 \leqslant k \leqslant K \), we have

\[
|f_k(s)| \lesssim 2|s|^{-\delta_k^+}. \tag{3.25}
\]

**Estimate of \( \bar{y}_k \).** Note that (2.64) rewrites as

\[
\left| \frac{d}{ds} \left[ (-s)^{-\frac{1}{2}}\bar{y}_k \right] \right| \lesssim |s|^{-\frac{1}{2}}|f_k| + |s|^{-\frac{1}{2}\sum_j \|\varepsilon\|^2_{L^2_\delta} + |s|^{-\frac{1}{2}|\bar{\mu}|^2 + |s|^{-\frac{1}{2}}. \]

Thus, using (3.5) and (3.25), we obtain, since \( 2\delta_k^+ > 2\delta_{K+1}^+ > \delta_0 > \delta_k^+ \) from (3.3),

\[
\left| \frac{d}{ds} \left[ (-s)^{-\frac{1}{2}}\bar{y}_k \right] \right| \lesssim |s|^{-\frac{1}{2}-\delta_k^+} + |s|^{-\frac{1}{2}-2\delta_{K+1}^+} + |s|^{-\frac{1}{2}-2\delta_k} \lesssim |s|^{-\frac{1}{2}-\delta_k^+}. \]

Integrating on \([S_n, s] \), using \( \bar{y}_k(S_n) = 0 \) from (3.4), we obtain, for |\( S_0 \) large enough,

\[
|\bar{y}_k(s)| \lesssim |s|^-\delta_k^+ \lesssim \frac{1}{2}|s|^{-\delta_k}. \tag{3.26}
\]

**Estimate of \( \bar{\mu}_k \).** From (3.25) and (3.26), since \( f_k = \bar{\mu}_k + \bar{y}_k \), we obtain also, for |\( S_0 \) large enough,

\[
|\bar{\mu}_k(s)| \lesssim |s|^{-\delta_k^+} \lesssim \frac{1}{2}|s|^{-\delta_k}. \tag{3.27}
\]

**Estimate of \( a_k \).** We extract \( a_k \) from the definition (2.44) of the local energy \( e_k \) as

\[
a_k = \frac{\bar{\mu}_k^2}{s} e_k - \frac{1}{2\tau_k} - \frac{4r_k}{\|Q\|_{L^1}}.
\]

But, by (2.41) and (3.5)–(3.6), we find

\[
\left| \frac{4r_k}{\|Q\|_{L^1}} - \frac{\ell^3_k \theta_k}{s} \right| \lesssim |s|^{-1-\delta_k}. \]

Thus, from (3.23) and again (3.5)–(3.6), we have

\[
a_k = \frac{\ell^3_k}{s} (1 + \bar{\mu}_k)^2 \left( \frac{1}{2} + \theta_k \right) - \frac{\ell^3_k}{2s} (1 + \tau_k)^{-1} - \frac{\ell^3_k \theta_k}{s} + O\left(|s|^{-1-\delta_k}\right) = O\left(|s|^{-1-\delta_k}\right)
\]

and so, for |\( S_0 \) large enough,

\[
|a_k| \lesssim |s|^{-1-\delta_k} \lesssim \frac{1}{2}|s|^{-1-\delta_k}. \]

In conclusion, we have strictly improved all the estimates in (3.5).
3.7. Topological obstruction. Since the estimates in (3.5) have been strictly improved on \([S_n, S_n^*]\) and since we have assumed \(S_n^* < S_0\), by continuity, we know that the bootstrap estimate (3.6) has to be saturated at \(s = S_n^*\) and, for some \(S_{n^*}^* \in (S_n^*, S_0)\) close enough to \(S_n^*\), the bootstrap estimates (3.5) hold on \([S_n, S_{n^*}^*]\).

Thus, for \(s \in [S_n, S_{n^*}^*]\), we may consider
\[
\mathcal{N}(s) = \sum_{k \in K^+} \left[|s|^\delta_k g_k(s) \right]^2 + \sum_{k=1}^{K} \left[|s|^\delta_k \tilde{\tau}_k(s) \right]^2
\]
and note that \(\mathcal{N}(S_n^*) = 1\). By continuity again, we choose \(S_{n^*}^* \) possibly closer to \(S_n^*\) so that \(\mathcal{N}(s) \leq 2\) for all \(s \in [S_n, S_{n^*}^*]\). Now we claim that, for all \(s^* \in [S_n, S_{n^*}^*]\) such that \(\mathcal{N}(s^*) = 1\),
\[
\frac{d\mathcal{N}}{ds}(s^*) > 0. \tag{3.28}
\]
Indeed, we first compute
\[
\frac{1}{2} \frac{d\mathcal{N}}{ds} = \sum_{k \in K^+} \left( -\delta_k^+ |s|^{2\delta_k^+ - 1} g_k^2 + |s|^{2\delta_k^+} \dot{g}_k g_k \right) + \sum_{k} \left( -\delta_k^- |s|^{2\delta_k^- - 1} \dot{\tilde{\tau}}_k + |s|^{2\delta_k^-} \ddot{\tilde{\tau}}_k \tilde{\tau}_k \right).
\]
But from (3.24) we have, for all \(1 \leq k \leq K\),
\[
\dot{g}_k = \frac{1}{2} (1 + 3\theta_k) \frac{g_k}{s} + O \left( |s|^{-1 + \frac{1}{2}(\delta_{k-1}^- + \delta_k^+)} \right).
\]
Moreover, from (2.8) and (3.27),
\[
\ddot{\tilde{\tau}}_k = \frac{\delta_k^3}{s} \left( \frac{\tau_k}{s} - \frac{\tau_k}{s^2} \right) = \frac{\delta_k^3}{s} \left( \mu_k^{-3} - \ell_k^{-3} (1 + \bar{\tau}_k) \right)
\]
\[
= -|s|^{-1} \left[ (1 + \bar{\mu}_k)^{-3} - (1 + \bar{\tau}_k) \right] = |s|^{-1} \tilde{\tau}_k + O \left( |s|^{-1 - \delta_k^+} \right).
\]
Inserting these estimates in the above formula of \(\frac{d\mathcal{N}}{ds}\), we obtain
\[
\frac{1}{2} \frac{d\mathcal{N}}{ds} = \sum_{k \in K^+} \left[ -\delta_k^+ + \frac{1}{2} (1 + 3\theta_k) \right] |s|^{2\delta_k^+ - 1} g_k^2 + O \left( |s|^{-1 + \frac{1}{2}(\delta_{k-1}^- + \delta_k^+) |g_k|} \right)
\]
\[
+ \sum_{k} \left( -\delta_k^- + 1 \right) |s|^{2\delta_k^- - 1} \dot{\tilde{\tau}}_k + O \left( |s|^{-1 - \delta_k^-} \right)
\]
and so, since \(\mathcal{N}(s) \leq 2\) for \(s \in [S_n, S_{n^*}^*]\),
\[
\frac{1}{2} \frac{d\mathcal{N}}{ds} = \sum_{k \in K^+} \left[ -\delta_k^+ + \frac{1}{2} (1 + 3\theta_k) \right] |s|^{2\delta_k^+ - 1} g_k^2 + \sum_{k} \left( -\delta_k^- + 1 \right) |s|^{2\delta_k^- - 1} \dot{\tilde{\tau}}_k
\]
\[
+ O \left( |s|^{-1 - \frac{1}{2}(\delta_{k-1}^- + \delta_k^+)} \right) + O \left( |s|^{-1 - (\delta_k^- - \delta_k^+)} \right).
\]
Now we recall that \(\frac{1}{2} (1 + 3\theta_k) \geq \delta_0 > \delta_k^+ \geq \delta_k^- \) for \(k \in K^+\), from (3.3) and the definition of \(K^+\), and similarly \(1 > \frac{1}{2\delta} \geq \delta_0 > \delta_k^+ > \delta_k\) for all \(1 \leq k \leq K\). Thus,
\[
\frac{1}{2} \frac{d\mathcal{N}}{ds} \geq (\delta_0 - \delta_k^+) |s|^{-1} \mathcal{N} - C|s|^{-1 - \frac{1}{2}(\delta_{k-1}^- - \delta_k^+)} - C|s|^{-1 - (\delta_k^- - \delta_k)}.
\]
Since \(\mathcal{N}(s^*) = 1\), taking \(|S_0|\) large enough, we obtain
\[
\frac{1}{2} \frac{d\mathcal{N}}{ds}(s^*) \geq (\delta_0 - \delta_k^+) |s^*|^{-1} - C|s^*|^{-1 - \frac{1}{2}(\delta_{k-1}^- - \delta_k^+)} - C|s^*|^{-1 - (\delta_k^- - \delta_k)} \geq \frac{1}{2} (\delta_0 - \delta_k^+) |S_n|^{-1} > 0,
\]
and (3.28) is proved.
Finally, we follow classical arguments as in [2, 5] to conclude. We first note that the map \((\xi, \zeta) \mapsto S^*_n(\xi, \zeta)\) is continuous. Indeed, if \(\epsilon > 0\) is given, since \(\frac{d}{d\tau}(S^*_n) > 0\) from (3.28), \(\mathcal{N}\) is strictly increasing on \([S^*_n - \epsilon, S^*_n + \epsilon]\) and there exists \(\delta > 0\) such that \(\mathcal{N}(S^*_n + \epsilon) > 1 + \delta\) and \(\mathcal{N}(S^*_n - \epsilon) < 1 - \delta\). But, from the transversality property (3.28), we may choose \(\delta > 0\) possibly smaller so that \(\mathcal{N}(s) > 1 + \delta\) for \(s \in [S^*_n + \epsilon, S^*_n]\) and \(\mathcal{N}(s) < 1 - \delta\) for \(s \in [S_n, S^*_n - \epsilon]\). Now note that, by continuity of the flow, there exists \(\eta > 0\) such that if \(||(\xi, \zeta) - (\xi, \zeta)|| < \eta\) then \(|\mathcal{N}(s) - \mathcal{N}(s)| < \delta/2\) for all \(s \in [S_n, S^*_n]\). Thus, from the definition of \(S^*_n\), we may conclude that \(|S^*_n(\xi, \zeta) - S^*_n(\xi, \zeta)| < \epsilon\) whenever \(||(\xi, \zeta) - (\xi, \zeta)|| < \eta\), which proves that \((\xi, \zeta) \mapsto S^*_n(\xi, \zeta)\) is indeed continuous.

In particular, the map
\[
\mathcal{M} : \mathcal{B}_{d+K} \to \mathcal{S}_{d+K}
\]
\[
(\xi, \zeta) \mapsto \left(\{|S^*_n|^{\delta_k} g_k(S^*_n)\}_{k \in K^+}, \{|S^*_n|^{\delta_k} \tau_k(S^*_n)\}_{1 \leq k < K}\right)
\]
is also continuous. Moreover, if \((\xi, \zeta) \in \mathcal{S}_{d+K}\) then \(\mathcal{N}(S_n) = 1\) from (3.4), thus \(S^*_n = S_n\) from (3.28) and the definition of \(S^*_n\), and so \(\mathcal{M}(\xi, \zeta) = (\xi, \zeta)\) again from (3.4). In other words, \(\mathcal{M}\) restricted to \(\mathcal{S}_{d+K}\) is the identity, and the existence of such a map is contradictory with Brouwer’s fixed-point theorem. Therefore, Proposition 3.2 is proved.

### 3.8. Conclusion.

**End of the proof of Proposition 3.1.** By Proposition 3.2, we may consider a sequence \((v_n)\) of solutions of (2.7) as in Section 3.2, such that their decomposition \(v_n = V[\Gamma_n] + \varepsilon_n\) satisfies the uniform estimates (3.5)–(3.6) and the orthogonality conditions (3.9) on some time interval \([S_n, S_0]\). In particular, it follows from these estimates that \(||v_n(S_0)||_{H^1} \leq C\) for some \(C > 0\), so there exists \(v_0 \in H^1\) such that, up to a subsequence, \(v_n(S_0) \rightharpoonup v_0\) in \(H^1\) weak, and we consider the solution \(v\) of (2.7) such that \(v(S_0) = v_0\).

Let any \(S < S_0\), and consider \(n\) large enough so that \(S_n < S\). Then, applying Lemma 2.12 on \([S, S_0]\), we obtain the existence of a \(C^1\) function \(\Gamma\) such that \(\Gamma\) and \(\varepsilon\) defined by \(\varepsilon = v - V[\Gamma]\) satisfy the estimates (3.5)–(3.6) on \([S, S_0]\), and also (3.9) since \(\varepsilon_n(s) \rightharpoonup \varepsilon(s)\) for all \(s \in [S, S_0]\). Since \(S < S_0\) is arbitrary, the function \(\Gamma\) and \(\varepsilon\) satisfy the estimates (3.1) and the orthogonality conditions (3.2) on \((1-\infty, S_0]\), which concludes the proof of Proposition 3.1. \(\square\)

**Proof of Theorem 1.1.** We consider the solution \(v\) of (2.7) obtained in Proposition 3.1. First note that, from (3.1), we have \(||v - V||_{H^1} = ||\varepsilon||_{H^1} \leq |s|^{-\frac{1}{2} - \frac{1}{43}}\) for all \(s \leq S_0\). Moreover, by the definition of \(V\) and (2.28), we recall that \(||V - \sum_k Q_k||_{L^2} \lesssim |s|^{-\frac{1}{2}}\) and \(||V - \sum_k Q_k||_{H^1} \lesssim |s|^{-1}\). By the triangle inequality, it follows that
\[
||v - \sum_k Q_k||_{L^2} \lesssim |s|^{-\frac{1}{2}} \quad \text{and} \quad ||v - \sum_k Q_k||_{H^1} \lesssim |s|^{-\frac{1}{2} - \frac{1}{43}}.
\]

Now we let \(T_0 = \frac{1}{\sqrt{2s_0}}\) and consider the solution \(u\) of (1.1) defined on \((0, T_0]\) by \(v = \tilde{u}\) from the change of variables (2.6). For \(0 < t \leq T_0\), set
\[
\lambda_k(t) = t \bar{\mu}_k(s), \quad x_k(t) = ty_k(s), \quad s = -\frac{1}{2t^2}.
\]

Then, by (2.11), (3.1) and possibly taking a smaller \(T_0 > 0\),
\[
|\lambda_k(t) - \ell_k t| \lesssim t^{1 + \frac{1}{43}} \lesssim t^{1 + \frac{1}{43}}, \quad |x_k(t) + \ell_k^{-2} t^{-1}| \lesssim t^{-1 + \frac{1}{43}} \lesssim t^{-1 + \frac{1}{43}}.
\]
Moreover, from the above estimates on $v$, we obtain
\[
\left\| u(t) - \sum_k \epsilon_k \lambda_k^{-\frac{1}{2}}(t)Q \left( \frac{\cdot - x_k(t)}{\lambda_k(t)} \right) \right\|_{L^2} \lesssim t,
\]
and thus (1.3) by possibly taking a smaller $T_0 > 0$.

Now, we compute the mass and the energy of the solution $u(t)$. First, by Lemma 2.6 and (3.1), for any $s \leq S_0$, we note that
\[
\| V(s) \|^2_{L^2} - K\| Q \|^2_{L^2} \lesssim |s|^{-\frac{1}{6}}
\]
and
\[
E(V(s)) + \| Q \|^2_{L^2} + \sum_k \ell_k(1 + 2\theta_k) \lesssim |s|^{-\frac{1}{6}}.
\]

By (3.1) again, it first follows that
\[
\| v(s) \|^2_{L^2} \lesssim |v(s) - V(s)|_{L^2} + \| V(s) \|^2_{L^2} - K\| Q \|^2_{L^2} \lesssim |s|^{-\frac{1}{2} - \frac{2}{3\lambda}}.
\]

Moreover, from (2.6), we have $\| u(t) \|^2_{L^2} = \| v(s) \|^2_{L^2}$ for $s = -\frac{1}{2\lambda}$. Thus, since the mass of $u(t)$ is constant, passing to the limit $s \to -\infty$ in the last estimate, we obtain $\| u(t) \|^2_{L^2} = K\| Q \|^2_{L^2}$.

To estimate similarly $E(v(s)) = E(V(s) + \epsilon(s))$, we first note that, by the definition of the energy and (3.1),
\[
E(V + \epsilon) - E(V) - \int (\partial_y V \partial_y \epsilon - V^5 \epsilon) \lesssim \| \epsilon \|^2_{H^1} \lesssim |s|^{-\frac{1}{2} - \frac{1}{3\lambda}}.
\]

Moreover, by (2.28) and (3.1), we have
\[
\left| \int (\partial_y V \partial_y \epsilon - V^5 \epsilon) - \sum_k \int (\partial_y Q_k \partial_y \epsilon - Q_k^5 \epsilon) \right| \lesssim \left( \| V - \sum Q_k \|_{H^1} + \| V - \sum Q_k \|_{L^\infty} + \left( \sum Q_k \right)^5 - \sum Q_k^5 \right) \| \epsilon \|_{H^1} \lesssim |s|^{-\frac{1}{2} - \frac{1}{4\lambda}}.
\]

But, by integration by parts and (3.2), for all $1 \leq k \leq K$, we also have the cancellation
\[
\int (\partial_y Q_k \partial_y \epsilon - Q_k^5 \epsilon) = -\int (\partial_y Q_k + Q_k^5) \epsilon = -\tilde{\mu}_k^{-2} \langle Q_k, \epsilon \rangle = 0.
\]

Thus, we have proved $|E(V + \epsilon) - E(V)| \lesssim |s|^{-\frac{1}{2} - \frac{2}{3\lambda}}$ and so
\[
E(v(s)) + \| Q \|^2_{L^2} + \sum_k \ell_k(1 + 2\theta_k) \lesssim |s|^{-\frac{1}{2} - \frac{1}{3\lambda}}.
\]

Moreover, from (2.6), we have $E(u(t)) = -2sE(v(s))$ for $s = -\frac{1}{2\lambda}$. Thus, since $E(u(t))$ is constant, multiplying the last estimate by $2|s|$ and passing to the limit $s \to -\infty$, we obtain
\[
E(u(t)) = \frac{\| Q \|^2_{L^2}}{16} \sum_k \ell_k (1 + 2\theta_k).
\]
We finally note that $E(u(t)) > 0$. Indeed, using the definition (2.42) of $\theta_k$ and rewriting
\[
\frac{16}{\|Q\|_{L^1}^2} E(u(t)) = \sum_{k=1}^{K} \ell_k^2 \left( \frac{1}{\ell_k} + 2 \frac{\epsilon_k}{\sqrt{\ell_k}} \sum_{j=1}^{k-1} \frac{\epsilon_j}{\sqrt{\ell_j}} \right) = \sum_{k=1}^{K} \ell_k^2 \left[ \left( \sum_{j=1}^{k} \frac{\epsilon_j}{\sqrt{\ell_j}} \right)^2 - \left( \sum_{j=1}^{k-1} \frac{\epsilon_j}{\sqrt{\ell_j}} \right)^2 \right],
\]
we obtain, after integration by parts and since $\ell_k > \ell_{k+1}$ for all $1 \leq k \leq K - 1$,
\[
\frac{16}{\|Q\|_{L^1}^2} E(u(t)) = \sum_{k=1}^{K-1} \left( (\ell_k^2 - \ell_{k+1}^2) \left( \sum_{j=1}^{k} \frac{\epsilon_j}{\sqrt{\ell_j}} \right)^2 + \ell_k^2 \left( \sum_{j=1}^{K-1} \frac{\epsilon_j}{\sqrt{\ell_j}} \right)^2 \right) > 0.
\]
This finishes the proof of Theorem 1.1. \qed

**Remark 3.5.** We prove that the solution $u(t,x)$ of (1.1) constructed in Theorem 1.1 satisfies an estimate as $x \to -\infty$ similar to, but weaker than, the estimate (1.6) for $S(t,x)$ in Theorem 1.4.

Indeed, since $\|v(s) - V(s)\|_{H^1} \leq |s|^{-\frac{1}{2} - \frac{1}{43}}$ by (3.1), we find $\|v(s) - \sum_k W_k(s)\|_{L^2} \lesssim |s|^{-\frac{1}{2} - \frac{1}{43}}$ by the estimates $|\tau_k(s)| \lesssim |s|^{-1}$ from (2.27), $\|R_k\|_{L^2} \lesssim 1$, $|a_k(s)| \lesssim |s|^{-1 - \frac{1}{43}}$ from (3.1) and $\|P_k\|_{L^2} \lesssim |s|^{\frac{1}{43}}$ from (2.13), valid for all $1 \leq k \leq K$. Therefore, from the definition of $W_k$ and the change of variables (2.6),
\[
\left\| u(t) - \sum_k \epsilon_k \frac{1}{2}(t) S \left( \frac{\rho_k(t)}{\lambda_k(t)} \right) \right\|_{L^2} \lesssim t^{1 + \frac{1}{43}} \tag{3.29}
\]
with
\[
\rho_k(t) = \frac{1}{\sqrt{-2\tau_k(s)}}, \quad \lambda_k(t) = t \mu_k(s) \sqrt{-2\tau_k(s)}, \quad \tilde{x}_k(t) = tz_k(s), \quad s = -\frac{1}{2t^2}.
\]
Moreover, from (3.1), we have $|\rho_k(t) - \ell_k^2 t| \lesssim t^{1 + \frac{1}{43}}$, $|\lambda_k(t) - \ell_k^2 | \lesssim t^{\frac{7}{43}}$ and $|\tilde{x}_k(t)| \lesssim t^{-1 + \frac{1}{43}}$.

Using (1.6), a first interesting consequence of (3.29) is the following $L^2$ version of (1.6) on the solution $u(t)$:
\[
\left\| u(t) + \left( \frac{1}{2} \left\| Q \right\|_{L^1} \sum_k \frac{\epsilon_k}{\sqrt{\ell_k}} \right) \cdot \left| \cdot \right|_{L^2} \right\|_{L^2(x < -2\ell_k^2 t^{-1})} \lesssim t^{1 + \frac{1}{43}}.
\]
It means that an explicit tail, sum of the tails of each rescaled version of $S(t)$, is visible in the asymptotic behavior of $u(t,x)$ for $x \to -\infty$.

Finally, using the estimate $\left\| \cdot \left| \cdot \right|_{L^2(x < -2\ell_k^2 t^{-1})} \right\| \gtrsim t$, the exponential decay of $Q$ and the $L^2$ estimate on $u(t)$ given in the above proof of Theorem 1.1, we obtain the sharp control
\[
1 \lesssim \left\| u(t) - \sum_k \epsilon_k \lambda_k^{-\frac{1}{2}}(t) Q \left( \frac{\cdot - x_k(t)}{\lambda_k(t)} \right) \right\|_{L^2} \lesssim t.
\]
Thus, as a second interesting consequence of (3.29), we may notice that, in $L^2$ norm, $u(t)$ is closer to the sum of $K$ modulated versions of $S$ than to the corresponding sum of pure modulated solitons $Q$. 

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Appendix A

As a corollary of Theorems 1.3 and 1.4, we prove in this appendix the following time estimates in exponential weighted spaces, which are used to prove (2.18). Note that we use the notation (1.9) for the antiderivative.

**Corollary A.1** (Time asymptotics in weighted spaces). With the notation of Theorem 1.3, there exists $B_0 > 0$ such that, for all $B \geq B_0$, $M \geq 0$, and all $t \in (0, T_0]$,

$$
\left\| \frac{\partial^m}{\partial x^m} S(t) - \sum_{k=0}^{M} \frac{1}{t^{x+m-2k}} Q_k^{(m-k)} \left( \frac{s + \frac{1}{2}}{t} + c_0 \right) \right\|_{L^2} \lesssim t^{2M+2-m}. \tag{A.1}
$$

**Remark A.2.** A weaker but useful estimate may be directly deduced from (A.1). Indeed, we observe that, for all $B > 0$, for all $M \geq 0$, for all $t \in (0, T_0]$,

$$
\left\| \frac{\partial^m}{\partial x^m} S(t) - \sum_{k=0}^{M} \frac{1}{t^{x+m-2k}} Q_k^{(m-k)} \left( \frac{s + \frac{1}{2}}{t} + c_0 \right) e^{-\frac{|s+1/2|}{m}} \right\|_{L^2} \lesssim t^{2M+2-m}. \tag{A.2}
$$

Note that this estimate becomes weaker as $B$ gets smaller, and thus is valid for all $B > 0$ without restriction. Note also that this estimate could be directly deduced from (1.5) for $M \leq m-1$, and thus is only interesting for $M \geq m$. In particular, we observe that (A.2) is sharper than (1.5) in the limit case $M = m$, where both estimates are valid.

To prove Corollary A.1, we first need the following technical lemma.

**Lemma A.3.** Let $p \geq 1$. Assume that $f \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}) \cap \dot{H}^p(\mathbb{R})$ satisfies, for all $0 \leq \ell \leq p$, for all $x > 0$,

$$
|f^{(\ell)}(x)| \lesssim e^{-x}. \tag{A.3}
$$

Then, for all $0 < \kappa < 1$, all $0 < \sigma < 1 - \kappa$, and all $x \in \mathbb{R}$,

$$
|f(x)| e^{\kappa x} \lesssim \|f^{(p)}\|_{L^2} + \|f^{(p)}\|_{L^2}. \tag{A.4}
$$

**Proof.** By the Taylor formula with integral remainder term, we have, for all $x, y \in \mathbb{R}$,

$$
f(x) = f(y) + (x - y)f'(y) + \cdots + \frac{(x - y)^{p-1}}{(p-1)!} f^{(p-1)}(y) + \int_y^x \frac{f^{(p)}(z)}{(p-1)!} (x - z)^{p-1} \, dz.
$$

By (A.3), passing to the limit as $y \to +\infty$, we obtain, for all $x \in \mathbb{R}$,

$$
f(x) = -\int_x^{+\infty} \frac{f^{(p)}(z)}{(p-1)!} (x - z)^{p-1} \, dz.
$$

For $x \geq 0$, we deduce, from (A.3) and Hölder’s inequality,

$$
|f(x)| e^{\kappa x} \lesssim \int_x^{+\infty} z^{p-1} e^{\kappa z} |f^{(p)}(z)| \, dz \lesssim \int_x^{+\infty} [z^{p-1} e^{\kappa z} e^{-(1-\sigma)z}] |e^z f^{(p)}(z)|^{1-\sigma} |f^{(p)}(z)|^{\sigma} \, dz \lesssim \|f^{(p)}\|_{L^2}^\sigma.
$$
For $x < 0$, we obtain similarly
\[
|f(x)| e^{\kappa x} \lesssim e^{\kappa x} \int_0^{\infty} (|x|^{p-1} + z^{p-1}) |f^{(p)}(z)| dz + e^{\kappa x} |x|^{p-1} \int_0^0 |f^{(p)}(z)| dz \\
\lesssim |x|^{p-1} e^{\kappa x} \int_0^{\infty} e^{-(1-\sigma)z} |e^z f^{(p)}(z)|^{1-\sigma} |f^{(p)}(z)|^{\sigma} dz \\
+ \int_0^{\infty} [z^{p-1} e^{-(1-\sigma)z}] |e^z f^{(p)}(z)|^{1-\sigma} |f^{(p)}(z)|^{\sigma} dz + e^{\kappa x} |x|^{p-1+\frac{1}{2}} \|f^{(p)}\|_{L^2} \\
\lesssim \|f^{(p)}\|_{L^2}^{\sigma} + \|f^{(p)}\|_{L^2},
\]
which concludes the proof of Lemma A.3.

\[\square\]

Proof of Corollary A.1. Let $B > 0$, $m \geq 0$, $M \geq 0$ and $t \in (0, T_0)$. We prove (A.1) as a consequence of the time estimate (1.5) and the space estimate (1.8), that we rewrite in a rescaled version so that we may apply Lemma A.3.

For this purpose, we let $L \in \mathbb{N}$ to be fixed later, we denote
\[
\tilde{\gamma}_0 = \min_{0 \leq \ell \leq L} \gamma_\ell > 0, \quad \tilde{\gamma} = \min \left\{ \tilde{\gamma}_0, \frac{1}{2} \right\}, \quad A = \tilde{\gamma}^{-1} \geq 2,
\]
where $\gamma_\ell$ is defined in (1.8), and we consider the change of variables
\[
y = \tilde{\gamma} \left( \frac{x + \frac{t}{\ell} + c_0}{t} \right) \text{ or, equivalently, } x = Aty - c_0 t - \frac{1}{\ell}.
\]

First note that the rescaled version of (A.1) that we want to prove rewrites $\|g_{A/B}\|_{L^2} \lesssim t^{2M+2}$ where, for $r > 0$, $g_r(y) = g(y) e^{r y}$ and
\[
g(y) = t^{\frac{1}{2}+m} \partial_x^m S \left( t, Aty - c_0 t - \frac{1}{t} \right) - \sum_{k=0}^M t^{2k} Q_k^{(m-k)}(Ay).
\]

Next, from (1.5), we obtain, for all $0 \leq \ell \leq L$,
\[
\left\| t^{\frac{1}{2}+\ell} \partial_x^\ell S \left( t, At \cdot c_0 t - \frac{1}{t} \right) \right\|_{L^2} \lesssim t^{1+2\ell}.
\]

And, from (1.8), we get, for all $0 \leq \ell \leq L$, for all $y > 0$,
\[
\left| t^{\frac{1}{2}+\ell} \partial_x^\ell S \left( t, Aty - c_0 t - \frac{1}{t} \right) \right| \lesssim e^{-\gamma_\ell Ay} \lesssim e^{-y}.
\]

Now note that, since $Q_k \in \mathcal{Y}$ for all $k \geq 0$ from the proof of Theorem 1.3 in [3], we have $|Q_k^{(n)}(y)| \lesssim e^{-y/2}$ for all $n \in \mathbb{N}$ and $y > 0$. Since moreover $A \geq 2$ and since we use the convention of antiderivative (1.9), we obtain, for all $y > 0$ and all $n \in \mathbb{Z}$,
\[
|Q_k^{(n)}(Ay)| \lesssim e^{-y}.
\]

Thus, we may apply Lemma A.3 with $p = 2M + 2$ and
\[
f(y) = t^{\frac{1}{2}+m} \partial_x^m S \left( t, Aty - c_0 t - \frac{1}{t} \right) - \sum_{k=0}^{2M+1} t^{2k} Q_k^{(m-k)}(Ay).
\]

Indeed, letting $L = m + p$, we have $f \in C^\infty(\mathbb{R}, \mathbb{R})$ and $f$ satisfies (A.3) from the above space estimates. Moreover, applying (A.5) with $\ell = L$, we get $\|f^{(p)}\|_{L^2} \lesssim t^{4M+4}$. 

For all \( 0 < \kappa \leq \frac{1}{4} \), we take \( \sigma = 1 - 2\kappa \geq \frac{1}{2} \), so that we have \( 0 < \sigma < 1 - \kappa \) and we obtain, from (A.4), for all \( y \in \mathbb{R} \),

\[
|f(y)|e^{\kappa y} \leq \|f(p)\|_{L^2}^\sigma + \|f(p)\|_{L^2} \lesssim e^{\sigma(4M+4)} + t^{4M+4} \lesssim t^{2M+2}.
\]

In particular, we obtain \( \|g_\kappa\|_{L^\infty} \lesssim t^{2M+2} \) for all \( 0 < \kappa \leq \frac{1}{4} \). To conclude, we estimate

\[
\|g_\kappa\|_{L^2} = \int g^2(y)e^{2\kappa y} \, dy = \int_{-\infty}^0 g^2(y)e^{2\kappa y} \, dy + \int_{0}^{4\kappa} g^2(y)e^{2\kappa y} \, dy
\]

\[
\lesssim \|g_{\kappa/2}\|_{L^\infty}^2 \int_{-\infty}^0 e^{\kappa y} \, dy + \|g_{3\kappa/2}\|_{L^\infty}^2 \int_{0}^{4\kappa} e^{-\kappa y} \, dy \lesssim \|g_{\kappa/2}\|_{L^\infty}^2 + \|g_{3\kappa/2}\|_{L^\infty}^2,
\]

and so we obtain \( \|g_{A/B}\|_{L^2} \lesssim t^{2M+2} \) as expected, provided that \( \kappa = \frac{A}{B} \) satisfies \( \frac{2\kappa}{3} \leq \frac{1}{4} \), which rewrites \( B \geq B_0 \) with \( B_0 = 6A \).

\[
\square
\]

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