Ballistic Transport in Superconducting Weak Links in a Microwave Field

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Nonequilibrium effects and their impact on a charge transport in superconducting ballistic weak links biased by an ac voltage are investigated within the framework of the Keldysh technique. We demonstrate that the microwave field destroys the phase coherence during the multiple Andreev reflection cycle and leads to the effective cooling of subgap quasiparticles accelerated due to multiple Andreev reflection. For small bias voltages this effect results in a strong suppression of both the excess current and the conductance of the weak link. In the opposite limit of large bias voltages the excess current remains unaffected. We also demonstrate that a simple Boltzmann kinetic approach becomes inadequate if an ac voltage bias is applied to the weak link.

\[ 74.40.+k, 74.50.+r, 74.80.Fp \]

Multiple Andreev reflection leads to excitation of quasiparticles in voltage biased superconducting weak links. As a result the quasiparticle distribution function for such systems is driven out of equilibrium already at small voltages. In the case of a time independent voltage bias charge transport in ballistic superconducting weak links in the presence of such nonequilibrium effects has been studied by Octavio et al. within the framework of a simple classical Boltzmann kinetic equation. This so-called OTBK model was then widely used in a large number of experimental as well as theoretical works.

Although the OTBK model provides a transparent physical picture of multiple Andreev reflection and dissipative charge transport in superconducting weak links it remained unclear if (and/or under which conditions) this model is sufficient to describe quantum nonequilibrium effects in such systems. More recently a rigorous theory of charge transport in ballistic superconductor–normal metal–superconductor (SNS) structures in the presence of a constant voltage bias has been developed by means of the Keldysh technique. In the absence of inelastic relaxation the result for a time-averaged dissipative current across the system obtained in exactly coincides with that derived from the OTBK model thus providing a formal justification for the OTBK results.

How far can one go applying the OTBK model to various nonequilibrium effects in superconducting weak links? Is the agreement between the results and specific for ballistic weak links biased by a dc voltage or further generalization of the OTBK model (see e.g. ) is possible?

In this Letter we will study nonequilibrium effects in superconducting microconstrictions in the presence of a time-dependent voltage within the framework of the Keldysh technique. We shall determine the distribution function and obtain the current voltage characteristics (CVC) of the weak links. We will demonstrate that photon absorption and emission processes in the weak link (in combination with multiple Andreev reflection) may have a strong impact on the transport properties of the system leading to effective heating or cooling of subgap quasiparticles and to a strong suppression of the current in the limit of small bias voltages. The latter effect may be of interest for applications of SNS structures as microwave detectors. We will furthermore argue that such nonequilibrium effects cannot be adequately described within the (generalized) OTBK model.

The model. Nonequilibrium effects in inhomogeneous superconductors are conveniently described by means of quasi-classical Greens functions in Keldysh–Nambu space. 

\[ \hbar \partial_t \vec{\nabla} \hat{G} + \hat{\sigma}_3 \hbar \partial_t \hat{G} + \hbar \partial_t \hat{G} \hat{\sigma}_3 + [\hat{K}, \hat{G}] = 0, \] (1)

where \( \hat{G} \) is a 2 × 2 matrix in the Keldysh space consisting of retarded, advanced and Keldysh Greens functions \( \hat{G}_{R,A} \) and \( \hat{G}_{K} \). The latter are in turn 2 × 2 matrices in Nambu space. \( \hat{G} \) obeys the normalization condition \( \hat{G} \hat{G}^\dagger = 1 \delta(t-t') \) where the ”\( \sigma \)” product indicates integration over the internal time variable. The same integration is also implied in the commutator \( [\hat{K}, \hat{G}] \) where \( \hat{K}(\vec{r}, t, t') = \hat{H}(\vec{r}, t) \delta(t-t') + i \hat{\Sigma}(\vec{r}, t, t') \) with \( \hat{H}(\vec{r}, t) = i[U(\vec{r}, t) 1 - \Delta(\vec{r}, t)] \). Here, \( U \) is the scalar potential, and \( \Delta \) is the off-diagonal pair potential.

We consider a standard model of a short superconducting microconstriction (or bridge): two superconducting bulks are connected directly or via a small piece of a normal metal of a typical size much smaller than the superconducting coherence length \( \xi_0 \) (see e.g. ). We will assume that the external voltage \( V(t) \) applied to the system drops at the junction, and the electric field does not penetrate into superconducting electrodes. The superconducting phase difference across the junction is then defined in a standard way \( \varphi(t) = 2eV(t)/\hbar \).

Nonequilibrium distribution function. We follow Ref. and construct the Green functions by solving Eqs. in each superconductor and matching these solutions continuously at the contact interface. The quantum kinetic properties of the system are completely described by the Keldysh Green function \( \hat{G}_K \) which can be expressed as \( \hat{G}_K = \hat{G}_R \circ \hat{h} - \hat{h} \hat{G}_A \) with \( \hat{h} = 1 - 2f - 2\hat{f} \hat{\sigma}_3 \). With \( \hat{G}_R, \hat{G}_A \) and \( \hat{G}_K \) known we have constructed \( f \) and \( \hat{f} \). The Fourier transforms of the latter with respect to time difference represent the ”longitudinal” and ”transversal” components of the distribution function describing respectively energy and charge modes. \( \hat{\sigma}_3, \sigma_3 \) is the Pauli matrix. In equilibrium \( f \) is equal to the Fermi distribution function \( f_0 \) and \( \hat{f} = 0 \). A striking feature of ballistic SNS bridges is that inside the N-metal the function \( f \) is equal to zero even if \( f \) is driven out of equilibrium due to multiple Andreev reflection in the presence of an externally applied bias.
time dependent. In a Fourier representation 

\[ f(E) = \sum_{n=0}^{\infty} f_{\pm,n}(E) \]  

As the latter component cannot be described by means of a classical Boltzmann equation we conclude that the condition \( \tilde{f} = 0 \) is an important prerequisite for the validity of the OTBK model.

In the limit of a constant voltage \( V(t) = \tilde{V} \) applied to the system the distribution function \( f \) does not depend on time and is given by

\[ f^{(0)}_{\pm,n}(E) = \prod_{j=0}^{n-1} \mathcal{A}(E \mp leV) \left[ 1 - \mathcal{A}(E \mp neV) \right] f_0(E \mp ne\tilde{V}). \]  

Here \( \mathcal{A}(E) \equiv |\gamma_R(E)|^2 \) is the Andreev reflection probability where (neglecting inelastic scattering) \( \gamma_R(E) = (E - \sqrt{E^2 - \Delta^2})/\Delta \). "+" and "−" label quasiparticles with momentum in and opposite to the direction of the current flow. Due to multiple Andreev reflection in the presence of electric field "+" quasiparticles are accelerated and the distribution function (3) for energies within the gap increases, i.e. the effect of heating of such quasiparticles takes place. It is accompanied by a symmetric effect of cooling of "−" quasiparticles which distribution function for subgap energies is suppressed.

Note that in the absence of inelastic relaxation (which causes deviations of \( \mathcal{A}(E) \) from its BCS value) the expression (3) exactly coincides with that obtained by OTBK (4).

This coincidence is by no means surprising: electrons and holes suffer no scattering in a clean N-metal and obviously can be described by the classical Boltzmann equation. Taking into account Andreev reflection at NS interfaces by imposing proper boundary conditions (4) one arrives at the results equivalent to those obtained within the general quantum kinetic analysis (5).

The situation changes in the presence of an additional microwave field \( V(t) = \tilde{V} + \tilde{V} \cos \omega t \). In this case electrons and holes moving in the N-metal can absorb and emit photons. Obviously such processes cannot be correctly described by the Boltzmann equation which does not contain information about off-diagonal elements of the density matrix. Here we elaborate the quantum kinetic analysis (5) and evaluate the distribution function \( f_{\pm}(E, t) \) which now becomes time dependent. In a Fourier representation

\[ f_{\pm}(E, t) = \sum_{\kappa = -\infty}^{\infty} f_{\pm,\kappa}(E) \exp(\pm i\kappa \omega t) \]  

we find

\[ f_{\pm,\kappa}(E) = \left[ 1 - \mathcal{A}(E) \right] f_0(E) \delta_{\kappa,0} \]  

and

\[ f_{\pm,\kappa}(E) = \sum_{n=0}^{\infty} m_{\kappa,n}(E, \pm k\hbar \omega). \]  

Here the terms containing the coefficients

\[ m_{\kappa,n}(E) = (\pm i)^{n-\kappa} J_{\kappa}(\gamma_R(E) \gamma_R(E + \kappa \hbar \omega))^n \]  

take care about all possible photon absorption and emission processes (the probability amplitude for \( k \)-photon processes is given by the Bessel function \( J_\kappa(2\sqrt{E} / \hbar \omega) \)).

Eqs. (3) and (4) selfconsistently determine the quasiparticle distribution function for a weak link in the presence of an AC voltage. This is one of the main results of the present paper. It is important to point out that terms with \( \kappa \neq 0 \) contain information about the phase shift of electron and hole amplitudes due to photon absorption and emission and cannot be recovered from the Boltzmann equation analysis (5).

We have solved Eqs. (3) and (4) selfconsistently for various values of \( \omega, \tilde{V}, \tilde{V} \) and \( T \). The results for the time independent part of the distribution function \( f_{\pm,\kappa=0}(E) \) are shown in Fig. 1.

Our numerical analysis captures all essential features which can be summarized as follows.

In the case \( \tilde{V} = 0 \) (Fig. 1a)) the microwave field excites out of the Fermi surface. At frequencies \( \hbar \omega < 2\Delta \) and small microwave amplitudes only single photon processes are important and we find an increase of \( f_{+}(E) = f_{-}(E) = f(E) \) at energies \( -\Delta < E < 0 \) and a decrease at \( 0 < E < +\Delta \). On top of that \( f(E) \) exhibits a peak at \( |E| = \Delta \). This behavior is a consequence of inelastic quasiparticle scattering into (and out of) the bound states in the short constriction, \( E_{\pm \kappa} = \pm \Delta \cos(\varphi/2) \), due to photon absorption. The small alternating phase difference \( \varphi(t) \) leads on the one hand to a certain smearing of the bound state level around \( E_{\pm \kappa} = \pm \Delta \). On the other hand photon absorption of quasiparticles in the \( E_0 \) state yields an emptying of the latter. Symmetrically, quasiparticles with energies \( E < -\Delta \) absorb a photon and fill up the \( E_0 \) state. It is interesting that due to this resonance effect the subgap distribution function may be tilted to the right which corresponds to an effective cooling. For frequencies \( \hbar \omega > 2\Delta \) the sign of the change in the distribution function, \( \Delta f_\kappa \), is altered and we find an effective heating. In this case quasiparticles with \( E \leq -\Delta \) may be scattered into the energy range \( E > 0 \). In addition we note here (without showing a figure) that also for \( \hbar \omega < 2\Delta \) the sign of \( \Delta f_\kappa \) may be changed if the microwave amplitude \( \tilde{V} \) is large so that multi photon processes become important.

For nonzero \( \tilde{V} \) we observe a nontrivial combination of two effects: acceleration of quasiparticles due to multiple Andreev reflection (cf. Eq. (5)) and their excitation by a microwave field. The former effect depends on the momentum direction whereas the latter is sensitive to the sign of the energy \( E \). As a result the structure of the distribution function for subgap energies turns out to be quite complicated. This function is
shown in Fig. 1b) (solid lines) for two different frequencies of the microwave field. For small \( \omega \) (see curve with \( \omega = 0.5\Delta \)) we observe a relative cooling of the “+” branch with respect to the case \( \bar{V} = 0 \) (the latter curve is depicted by dashed-dotted lines). The physical reason for this effect is clear: photon absorption and emission processes break the multiple Andreev reflection cycle thus preventing from further acceleration of Green function \( \hat{G} \). The time independent component of this current we find the Shapiro peaks disappear and Eq. (6) is solved by the T=0 solution (emission) of a photon in the weak link leads to a shift of the microwave amplitude. For \( \bar{V} \) between “+” and “−” only a few Andreev reflections (since the current \( \bar{I} \) is proportional to \( \frac{dE}{d\bar{V}} \)) deviate from those obtained here. On the other hand, for larger voltages \( \bar{V} > 2\Delta \) only one Andreev reflection is possible and the microwave field effect less important. For a wide interval of voltages we observe additional subharmonic gap structures with the period \( \hbar\omega \) (Fig. 1b) or satellite structures with \( \delta E = \pm \hbar\omega \) in the vicinity of the usual subharmonic gap structures (figures not shown).

As we already pointed out the problem has been recently investigated by Zimmerman and Keck [13] within the classical Boltzmann kinetic equation approach extending the OTBK analysis [14] to the ac voltage situation. This approach essentially deals only with diagonal elements of the density matrix and contains no information about off-diagonal phase sensitive terms. Our results reduce to those of Ref. [13] provided these terms (\( m_{\alpha,\kappa'} \) with \( \kappa \neq 0 \) in Eq. (3)) are neglected. At low voltages these terms are significant, and the results [13] (dashed lines in Fig. 1) considerably (and for \( \hbar\omega < 2\Delta \) even qualitatively) deviate from those obtained here. On the other hand for larger \( \bar{V} \sim \Delta \) only a few Andreev reflections are possible, off-diagonal elements play a minor role and the agreement between our results and those of Ref. [1] is better.

**Current-voltage characteristics.** The current across the weak link is determined by the expression for the Keldysh Green function \( \hat{G}^{K} \) in a standard way (see e.g. [15]). For the time independent component of this current we find

\[
I = \bar{V}/R_0 + I_{\text{exc}}(\omega, \bar{V}, \bar{V}) + I_{\text{Shapiro}}(\omega, \bar{V}, \bar{V}),
\]

where \( R_0 \) is the Sharvin resistance of the junction, \( I_{\text{exc}} \) is the additional current due to multiple Andreev reflection and \( I_{\text{Shapiro}} = \sum_{k,n} I_{k,n} \delta(2ne\bar{V} - \hbar\omega) \) represents Shapiro peaks at the discrete constant voltages \( \bar{V} = (k/2n)\hbar\omega/e \). At low \( T \) the latter includes subharmonics (\( n > 1 \)) because the current phase relation strongly deviates from a standard sinusoidal form [13]. Below we shall focus our attention on the microwave field effect on the excess current \( I_{\text{exc}} = 2 \int dE I_{\text{exc}}(E) \), where similarly to Eq. (3) \( I_{\text{exc}} \) is defined selfconsistently by

\[
I_{\text{exc}}(E) = \sum_{k=-\infty}^{\infty} \sum_{k'=\infty}^{\infty} m_{\kappa,\kappa'}(+,k,E)
\times \{ i_0(E - e\bar{V} - k\hbar\omega) \delta_{\kappa,+} + I_{\kappa'}(E - e\bar{V} - k\hbar\omega) \},
\]

with \( i_0(E) = (1/2eR_0)[A(E) - 1]\tanh(E/2k_BT) \). For \( \bar{V} \to 0 \) the Shapiro peaks disappear and Eq. (6) is solved by the expression \( I_{\text{exc}}(E) = \delta_{\kappa,0} \sum_{n=1}^{\infty} I_0(E) \) with

\[
I_0(E) = \frac{1}{2eR_0} \prod_{t=0}^{n-1} (A(E - ne\bar{V}) - 1) \tanh \left( \frac{E - ne\bar{V}}{2k_BT} \right).
\]

**FIG. 2.** CVC in the limit of constant voltage bias and in the presence of an additional microwave field for two different frequencies. Shapiro peaks are not shown.

The I-V curves calculated numerically from Eqs. (3) and (4) for different \( \omega \) are presented in Fig. 2 (Shapiro peaks are not shown). For large voltages \( \bar{V} \gg \Delta, \hbar\omega \) one observes no influence of the microwave field on the excess current. With the aid of Eq. (4) one can also demonstrate analytically that this result remains valid for any microwave amplitude \( \bar{V} \).

The physical reason for this result is transparent. Absorption (emission) of a photon in the weak link leads to a shift of the energy of quasiparticles by the value \( \hbar\omega \). If only one Andreev reflection takes place (i.e. \( \bar{V} \gg \Delta \)) this shift is unique for all quasiparticles and drops out from the expression for the current. The same argument holds for multiphoton processes if the total energy shift is smaller than \( \bar{V} \).

In contrast to the above limit, for smaller voltages \( \bar{V} < 2\Delta \) we find a considerable suppression of the excess current \( I_{\text{exc}} \). The effect becomes particularly pronounced at large frequencies \( \omega > 2\Delta \) (see Fig. 2). This result is a direct consequence of microwave induced cooling and heating of subgap quasiparticles with opposite momenta: the difference between “+” and “−” distribution functions decreases (see Fig. 1) and so does \( I_{\text{exc}} \) (since the current \( I \) is proportional to \( \int dE f_+(E - e\bar{V}/2) - f_-(E + e\bar{V}/2) \)).

To study this effect further let us calculate the low voltage conductance \( G = 1/R_0 + 2 \int dE G_{\alpha,\alpha}(E) \) as a function of the microwave amplitude. For \( \bar{V} \to 0 \) one can derive from Eq. (7)

\[
G_{\alpha}(E) = \sum_{k=-\infty}^{\infty} \sum_{k'=\infty}^{\infty} m_{\alpha,\alpha'}(s = +1, k, E)
\times \{ g_{0,\alpha'}(E - k\hbar\omega) + G_{\alpha'}(E - k\hbar\omega) \},
\]

where \( g_{0,\alpha}(E) = -e d[i_0(E) \delta_{\kappa,0} + I_0(E = 0, E)]/dE \).

For \( \bar{V} = 0 \) the solution of Eq. (8) reads

\[
G_{\alpha}(E) = (1/2R_0)[A(E) - 1]\tanh(E/2k_BT) dE \delta_{\kappa,0}.
\]

and we reproduce the conductance peak \( G - 1/R_0 \sim \hbar\omega = 0.5\Delta \).
sensitive terms photons. These processes destroy the phase coherence within the whole multiple Andreev reflection cycle. In the presence of weak links the conductance is suppressed down to its normal state value $1/R_0$.

**Discussion.** Our analysis of the microwave-induced effects in superconducting weak links has led to a transparent physical picture which can be summarized as follows. At low dc voltages subgap quasiparticles suffer multiple Andreev reflection increasing their energy by the value $e\tilde{V}$ after each traverse across the weak link. Quasiparticle states with the momentum direction in (opposite to) that of the current become overpopulated (underpopulated) and the low voltage conductance increases. The phase coherence is preserved during the whole multiple Andreev reflection cycle. In the presence of an ac field subgap quasiparticles can also absorb and emit photons. These processes destroy the phase coherence within a multiple Andreev reflection cycle and prevent quasiparticles from further acceleration or, equivalently, lead to relative *cooling* (heating) of $+$ (−) subgap quasiparticles in comparison to the case $\tilde{V} = 0$ (see Fig. 1). As a result both the excess current and the system conductance at low voltages can be strongly suppressed (Figs. 2,3). The suppression increases with the energy of absorbed and emitted photons $\hbar\omega$ and the amplitude of the microwave field $\tilde{V}$ in which case many-photon processes gain importance. This property makes it possible to use SNS structures as spectral microwave detectors. For larger voltages $e\tilde{V} \gtrsim \Delta$ subgap quasiparticles are accelerated and leave the weak link already after a few Andreev reflection events. In this case the effect of an ac voltage becomes less pronounced. In the limit $e\tilde{V} \gg \Delta$ only one Andreev reflection takes place and the I-V curve is not sensitive to an ac field.

In this paper we discussed the behavior of *voltage biased* weak links in which case the low voltage “foot” structure of CVC is caused by multiple Andreev reflection. Although CVC of *current biased* weak links shows a similar structure, in the latter case the “foot” is due to a different physical reason — the dc Josephson effect. In many experiments the CVC of weak links is measured in the regime intermediate between the voltage and current biased limits. Therefore it may be quite difficult to judge which of the above physical reasons could actually explain experimental results, in particular for short superconducting constrictions in which case both the dc Josephson current and the low voltage excess current are of the same order and have the same temperature dependence.

The results obtained here suggest that a clear distinction between these two mechanisms can be easily made in the presence of a microwave field. Indeed in a microwave field (and at not very low $T$) stimulation of the Josephson critical current takes place in superconducting microbridges and SNS junctions whereas the low voltage excess current and the conductance become strongly suppressed. We believe that these opposite trends can be easily identified experimentally.

In conclusion, making use of the Keldysh technique we developed a microscopic theory of nonequilibrium effects in superconducting weak links in the presence of an external microwave field and demonstrated that these effects may have a dramatic impact on charge transport in such structures.

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![Graph](image)

**FIG. 3.** Zero-bias-conductance $G(\tilde{V})$ normalized to $G(\tilde{V} = 0) = 15/R_0$ versus amplitude $\tilde{V}$ of the applied microwave field. Dashed curves: $G(\tilde{V})$ calculated without phase sensitive terms $m_{\kappa,\kappa'}$, $\kappa \neq 0$.

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