Pre-thermal Phases of Matter Protected by Time-Translation Symmetry

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In a periodically driven (Floquet) system, there is the possibility for new phases of matter, not present in stationary systems, protected by discrete time-translation symmetry. This includes topological phases protected in part by time-translation symmetry, as well as phases distinguished by the spontaneous breaking of this symmetry, dubbed “Floquet time crystals”. We show that such phases of matter can exist in the pre-thermal regime of periodically-driven systems, which exists generically for sufficiently large drive frequency, thereby eliminating the need for integrability or strong quenched disorder that limited previous constructions. We prove a theorem that states that such a pre-thermal regime persists until times that are nearly exponentially-long in the ratio of certain couplings to the drive frequency. By similar techniques, we can also construct stationary systems which spontaneously break continuous time-translation symmetry. We argue furthermore that for driven systems coupled to a cold bath, the pre-thermal regime could potentially persist to infinite time.

I. INTRODUCTION

Much of condensed matter physics revolves around determining which distinct phases of matter can exist as equilibrium states of physical systems. Within a phase, the properties of the system vary continuously as external parameters are varied, while different phases are separated by phase transitions, at which the properties change abruptly. An extremely rich set of observed phases can be characterized by symmetry. The best known example is spontaneous symmetry-breaking, as a result of which the equilibrium state of the system is less symmetrical than the Hamiltonian. More recently, a set of uniquely quantum phases—symmetry-protected topological (SPT) phases [1–19], including topological insulators [20, 21], and symmetry-enriched topological (SET) phases [22–25]—has been discovered. These phases, while symmetric, manifest the symmetry in subtly anomalous ways, and are distinct only as long as the symmetry is preserved. We can collectively refer to these three classes of phases as symmetry-protected phases of matter.

Thus far, the concept of symmetry-protected phases of matter has not been as successful in describing systems away from equilibrium. Recently, however, it was realized that certain periodically-driven “Floquet” systems can exhibit distinct phases, akin to those of equilibrium systems [26]. In this paper, we show that there is, in fact, a very general set of non-equilibrium conditions under which such phases can arise, due to a remarkable phenomenon called "pre-thermalization". In Floquet systems, pre-thermalization occurs when a time-dependent change of basis removes all but a small residual time-dependence from the Hamiltonian, and thus allows the properties of the system to be mapped approximately onto those of a system in thermal equilibrium. The residual time-dependence is nearly exponentially-small in a large parameter $\alpha$ of the original Hamiltonian of the system. One can then talk about a “pre-thermal regime” in which the system reaches a thermal equilibrium state with respect to the approximate effective time-independent Hamiltonian that results from neglecting the small residual time dependence. In this regime, the system can exhibit phases and phase transitions analogous to those seen in thermal equilibrium, such as symmetry-protected phases. Nevertheless, in the original non-rotating frame, the system remains very far from thermal equilibrium with respect to the instantaneous Hamiltonian at any given time. After the characteristic time $t_*$, which is nearly exponentially-long in the large parameter $\alpha$, other physics (related the residual time-dependence) takes over.

In this paper, we show that pre-thermal systems can also exhibit phases of matter that cannot exist in thermal equilibrium. These novel phases can also be understood as symmetry-protected phases but of a variety that cannot occur in thermal equilibrium: these phases are protected by discrete time-translation symmetry. While these include topological phases protected by time-translation symmetry [20–24], perhaps the most dramatic of these are “time crystals” that spontaneously break time-translation symmetry. The idea of time crystals that spontaneously break continuous time-translation symmetry was first proposed by Wilczek and Shapere [25, 26], but finding a satisfactory equilibrium model has proven difficult and some no-go theorems exist [27–31]. In this paper, we construct pre-thermal “Floquet time crystals”, which spontaneously break the discrete time-translation symmetry of periodically-driven systems [19]. Floquet time crystals are the focus of this paper, but as a by-product of our analysis, we also find pre-thermal – i.e. non-equilibrium – time crystals that spontaneously break continuous time-translation symmetry. We also construct SPT and SET phases protected by discrete time-translation symmetry.

1 For an alternative view of such systems that focuses on other symmetries of the discrete time-translation operator, see Refs. 20 [41] and 21

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Periodically-driven systems have long been considered an unlikely place to find interesting phases of matter and phase transitions since generic driven closed systems will heat up to infinite temperature. It has been known that the heating problem can be avoided if the system is integrable or if the system has sufficiently strong quenched disorder that it undergoes many-body localization (MBL)\cite{64,65,66,67}. However, integrability relies on fine-tuning, and MBL requires the system to be completely decoupled from the environment \cite{66,72}. Furthermore, the disorder must be sufficiently strong, which may be difficult to realize in an experiment but does not constitute fine-tuning.

The central result of this paper is therefore to show that pre-thermalization makes it possible for non-equilibrium phases protected by time-translation symmetry to occur in more generic non-equilibrium systems without the need for fine-tuning, strong disorder, or complete decoupling from the environment. Remarkably, these non-equilibrium phases and phase transitions, which have have no direct analogues in thermal equilibrium, have a mathematical formulation that is identical to that of equilibrium phases, though with a different physical interpretation. Since MBL is not a requirement, it is conceivable that pre-thermal-time-translation protected phases could survive the presence of coupling to an environment. In fact, we will discuss a plausible scenario by which these phases can actually be stabilized by coupling to a sufficiently cold thermal bath, such that the system remains in the pre-thermal regime even at infinite time.

The structure of the paper will be as follows. In Section II, we state our main technical result. In Section III, we apply this to construct prethermal Floquet time crystals which spontaneously break discrete time-translation symmetry. In Section IV, we show that a continuous time-translation symmetry can also be spontaneously broken in the pre-thermal regime for a system with a time-independent Hamiltonian. In Section V, we outline how our methods can also be applied to construct SPT and SET phases protected by time-translating symmetry. In Section VII, we discuss what we expect to happen for non-isolated systems coupled to a cold thermal bath. Finally, we discuss implications and interpretations in Section VII.

II. PRE-THERMALIZATION RESULTS

The simplest incarnation of pre-thermalization occurs in periodically-driven systems when the driving frequency \(\nu\) is much larger than all of the local energy scales of the instantaneous Hamiltonian \cite{73,77} (see also Refs.\cite{73,77} for numerical results). The key technical result of our paper will be a theorem generalizing these results to other regimes in which the driving frequency is not greater than all the local scales of the Hamiltonian. Thus, it is not clear how to understand the different phases of matter in this picture.

In order to proceed further, we need a new approach. In this paper, we develop a new formalism that
analyzes $U(T,0)$ itself rather than $U(NT,0)$, allowing the effects of time-translation symmetry to be seen in a transparent way. Our central tool is a theorem that we will prove, substantially generalizing those of Abanin et al.\cite{74}. A more precise version of our theorem will be given momentarily, and the proof will be given in Appendix A; the theorem essentially states that there exists a time-independent local unitary rotation $U$ such that $U_T \approx U_T = U^\dagger (X e^{-iDT}) U$, where $X = U_0(T,0)$ is the time evolution of $H_0$ over one time cycle, and $D$ is a quasi-local Hamiltonian that commutes with $X$. The dynamics at stroboscopic times are well-approximated by $U_T$ for times $t \ll t_*$, where $t_* = e^{O(1/\lambda T[\log(1/\lambda T)]^3)}$. This result combines ideas in Ref.\cite{74} about (1) the high-frequency limit of driven systems and (2) approximate symmetries in systems with a large separation of scales. Recall that, in the high-frequency limit of a driven system with a large separation of scales, $H_0$ can be approximated by the Floquet operator: the Floquet cycle of the size of the local terms whose sum makes up a Hamiltonian; the subscript $n$ parametrizes the extent to which the norm suppresses the weight of operators with larger spatial support. An explicit definition of the norm is given in Appendix A. The theorem states the following.

**Theorem 1.** Consider a periodically-driven system with Floquet operator:

$$U_j = T \exp \left( -i \int_0^T H(t) dt \right)$$

where $H(t) = H_0(t) + V(t)$, and $X \equiv U_0(0,T)$ satisfies $X^N = 1$ for some integer $N$. We assume that $H_0(t)$ can be written as a sum $H_0(t) = \sum_i h_i(t)$ of terms acting only on single sites $i$. Define $\lambda \equiv \|V\|_1$. Assume that

$$\lambda T \leq \frac{\gamma \kappa_1^2}{N+3}, \quad \gamma \approx 0.14.$$  \hfill (4)

Then there exists a (time-independent) unitary $U$ such that

$$UU_j U^\dagger = X T \exp \left( -i \int_0^T [D + E + V(t)] dt \right)$$  \hfill (5)

where $D$ is local and $[D,X] = 0$; $D, E$ are independent of time; and

$$\|V\|_{n_*} \leq \lambda \left( \frac{1}{2} \right)^{n_*},$$  \hfill (6)

$$\|E\|_{n_*} \leq \lambda \left( \frac{1}{2} \right)^{n_*}.$$  \hfill (7)

The exponent $n_*$ is given by

$$n_* = \frac{\lambda_0 \lambda}{[1 + \log(\lambda_0/\lambda)]^2}, \quad \lambda_0 = \frac{(\kappa_1)^2}{72(N+3)(N+4)T}$$  \hfill (8)

Furthermore,

$$\|D - \nabla\|_{n_*} \leq \mu (\lambda^2/\lambda_0), \quad \mu \approx 2.9,$$  \hfill (9)

where

$$\nabla = \frac{1}{NT} \int_0^{NT} V^{\text{int}}(t) dt$$

$$= \left. \frac{1}{N} \sum_{k=0}^{N-1} X^{-k} \left( \frac{1}{T} \int_0^T V^{\text{int}}(t) dt \right) X^k. \right)$$  \hfill (10)

The proof is given in Appendix A. The statement of the theorem makes use of a number $\kappa_1$. It is chosen so that $\|H\|_1$ is finite; the details are given when the norm is given in Appendix A.

Unpacking the theorem a bit in order to make contact with the discussion above, we see that it states that there is a time-independent unitary operator $U$ that transforms the Floquet operator into the form $X e^{-iDT}$ with $[D,X] = 0$ and local $D$, up to corrections that are exponentially small in $n_* \sim 1/(\lambda T[\log(1/\lambda T)]^3)$. These ‘error
terms” fall into two categories: time-independent terms that do not commute with $X$, which are grouped into $E$; and time-dependent terms, which are grouped into $V(t)$. Both types of corrections are exponentially-small in $n_*$. Since they are exponentially-small \(\|E\|_{n_*} \|V\|_{n_*} \sim (1/2)^n_*\), these terms do not affect the evolution of the system until exponentially-long times, $t_* \sim e^{Cn_*}$ (for some constant $C$). It is not possible to find a time-independent unitary transformation that exactly transforms the Floquet operator into the form $X e^{-iDT}$ because the system must, eventually, heat up to infinite temperature and the true Floquet eigenstates are infinite-temperature states, not the eigenstates of an operator of the form $X e^{-iDT}$ with local $D$. In the interim, however, the approximate Floquet operator $X e^{-iDT}$ leads to Floquet time crystal behavior, as we will discuss in the next Section.

The proof of Theorem 1 constructs $U$ and $D$ through a recursive procedure, which combines elements of the proofs of pre-thermalization in driven and undriven systems given by Abanin et al. [74]

In the case of pre-thermal undriven systems, the theorem we need has essentially already been given in Ref. [74] but we will restate the result in a form analogous with Theorem [1] which entails some slightly different bounds (however, they are easily derivable using the techniques of Ref. [74].

**Theorem 2.** Consider a time-independent Hamiltonian $H$ of the form

$$H = -uL + V,$$

where $e^{2\pi i L} = 1$. We assume that $L$ can be written as a sum $L = \sum_i L_i$ of terms acting only on single sites $i$. Define $\lambda \equiv \|V\|_1$, and assume that

$$\lambda/a \leq \gamma \kappa_1^2, \quad \gamma \approx 0.14.$$  \hfill (12)

Then there exists a local unitary transformation $U$ such that

$$\bar{U}HL\bar{U}^\dagger = -uL + D + \tilde{V}$$

where $[L, D] = 0$ and $\tilde{V}$ satisfies

$$\|\tilde{V}\|_{n_*} \leq \lambda \left( \frac{1}{2} \right)^{n_*}$$

where

$$n_* = \frac{\lambda_0/\lambda}{1 + \log(\lambda_0/\lambda)} \kappa_1^2, \quad \lambda_0 = -\frac{au}{144}.$$ \hfill (15)

Furthermore,

$$\|D - \langle V \rangle\|_{n_*} \leq \mu(\lambda^2/\lambda_0), \quad \mu \approx 2.9.$$ \hfill (16)

Here, we have defined, following Ref. [74] the symmetrized operator $\langle V \rangle$ according to

$$\langle V \rangle = \int_0^{2\pi} \frac{d\theta}{2\pi} e^{iL\theta} \tilde{V} e^{-iL\theta}$$ \hfill (17)

which, by construction, satisfies $[L, \langle V \rangle] = 0$.

### III. Pre-thermalized Floquet Time Crystals

#### A. Basic Picture

The results of the previous section give us the tools that we need to construct a model which is a Floquet time crystal in the pre-thermalized regime. Our approach is reminiscent of Ref. [43] where the Floquet-MBL time crystals of Ref. [43] were reinterpreted in terms of a spontaneously broken “emergent” $Z_2$ symmetry. Here, “emergent” refers to the fact that the symmetry is in some sense hidden – its form depends on the parameters on the Hamiltonian in a manner that is not a priori known. Furthermore, it is not a symmetry of the Hamiltonian, but is a symmetry of the Floquet operator.

In particular, suppose that we have a model where we can set $X = \prod_i \sigma_i^z$. (Thus $N = 2$). We then have $U_t = U_t(\langle X e^{-iDT} \rangle) U_t$, where the quasi-local Hamiltonian $D$ by construction respects the Ising symmetry generated by $X$. This Ising symmetry corresponds to an approximate “emergent” symmetry $UXU^\dagger$ of $U_t$ (“emergent” for the reason stated above and approximate because it is an exact symmetry of $U_t$, not $U_t$, and therefore is approximately conserved for times $t \ll t_*$.) Suppose that $D$ spontaneously breaks the symmetry $X$ below some finite critical temperature $\tau_c$. For example, working in two dimensions or higher, we could have $D = -J \sum_{\langle i,j \rangle} \sigma_i^z \sigma_j^z$ plus additional smaller terms of strength which break integrability. We will be interested in the regime where the heating time $t_* \gg t_{\text{pre-thermal}}$, where $t_{\text{pre-thermal}}$ is the thermalization time of $D$.

Now consider the time evolution $|\psi(t)\rangle$, starting from a given short-range correlated state $|\psi(0)\rangle$. We also define the rotated states $|\tilde{\psi}(t)\rangle = U|\psi(t)\rangle$. At stroboscopic times $t = nT$, we find that $|\tilde{\psi}(nT)\rangle = (X e^{-iDT})^n|\tilde{\psi}(0)\rangle$.

Since $(X e^{-iDT})^2 = e^{2iDT}$, we see that at even multiples of the period, $t = 2nT$, the time evolution of $|\tilde{\psi}(t)\rangle$ is described by the time-independent Hamiltonian $D$. Thus, we expect that, after the time $t_{\text{pre-thermal}}$, the system appears to be in a thermal state of $D$ at temperature $\tau$. Thus, $|\tilde{\psi}(2nT)\rangle \langle \tilde{\psi}(2nT) | \approx \tilde{\rho}$, where $\tilde{\rho}$ is a thermal density matrix for $D$ at some temperature $\tau$, and the approximate equality means that the expectation values of local observables are approximately the same. Note that for $\tau < \tau_c$, the Ising symmetry of $D$ is spontaneously broken and $\tilde{\rho}$ must either select a nonzero value for the order parameter $M_{2n} = \langle \sigma_i^z \rangle$ or have long-range correlations. The latter case is impossible given our initial state, as long-range correlations cannot be generated in finite time. Then, at odd times $t = (2n + 1)T$, we have

$$|\tilde{\psi}((2n+1)T)\rangle \langle \tilde{\psi}((2n+1)T) | \approx (X e^{-iDT})\tilde{\rho}(e^{iDT}X)$$ \hfill (18)

$$= X\tilde{\rho}X$$ \hfill (19)

(since $\tilde{\rho}$ commutes with $D$.) Therefore, at odd times, the
order parameter
\[ M_{2n+1} = \langle \sigma_i^z \rangle_{X \neq X} = -M_{2n}. \] (20)

Thus, the state of the system at odd times is different from the state at even times, and time translation by \( T \) is spontaneously broken to time translation by 2\( T \).

The above analysis took place in the frame rotated by \( \mathcal{U} \). However, we can also consider the expectation values of operators in the original frame, for example \( \langle \psi(t) | \sigma_i^z | \psi(t) \rangle = \langle \bar{\psi}(t) | \mathcal{U} | \sigma_i^z | \mathcal{U}^\dagger | \psi(t) \rangle \). The rotation \( \mathcal{U} \) is close to the identity in the regime where the heating time is large\(^2\) so \( \sigma_i^z \) has large overlap with \( \mathcal{U} \sigma_i^z \mathcal{U}^\dagger \) and therefore will display fractional frequency oscillations. We recall that the condition for fractional frequency oscillations in the pre-thermalized regime is that (a) \( D \) must spontaneously break the Ising symmetry \( X \) up to a finite critical temperature \( \tau_c \); and (b) the energy density with respect to \( D \) of \( | \psi(0) \rangle \) must correspond to a temperature \( \tau < \tau_c \). In Figure 1, we show the expected behavior at low temperatures \( \tau \) and contrast it with the expected behavior in a system which is not a time crystal in the pre-thermal regime.

**B. Example: periodically-driven Ising spins**

Let us now consider a concrete model which realizes the behavior described above. We consider an Ising ferromagnet, with a longitudinal field applied to break the Ising symmetry explicitly, and driven at high frequency by a very strong transverse field. Thus, we take
\[ H(t) = H_0(t) + V, \] (21)
where
\[ H_0(t) = - \sum_i h \sigma_i^z \] (22)
\[ V = - J \sum_{(i,j)} \sigma_i^z \sigma_j^z - h^x \sum_i \sigma_i^x, \] (23)
and we choose the driving profile such that
\[ \int_0^T h^x(t) dt = \frac{\pi}{2}, \] (24)
ensuring that the “unperturbed” Floquet operator \( U_0 \) implements a \( \pi \) pulse, \( X = \prod_i \sigma_i^z \), and we can set \( N = 2 \). (If the driving does not exactly implement a \( \pi \) pulse, this is not a significant problem since we can just incorporate the difference into \( V \).) This implies that \( h^x \sim 1/T \), and we assume that \( h^z \ll J \ll 1/T \).

Then by the results of Section II (with \( J \) playing the role of \( \lambda \) here), we find a quasi-local Hamiltonian \( D = V + \frac{J}{T} O((JT)^2) \), where
\[ V = \frac{1}{2T} \int_0^{2T} V_{int}(t) dt. \] (25)

In particular, in the case where the \( \pi \) pulse acts instantaneously, so that
\[ h^x(t) = \frac{\pi}{2} \sum_{k=-\infty}^\infty \delta(t - kT), \] (26)
we find that
\[ V = -J \sum_{(i,j)} \sigma_i^z \sigma_j^z \] (27)
(this Hamiltonian is integrable, but in general the higher order corrections to \( D \) will destroy integrability.) More generally, if the delta function is smeared out so that the \( \pi \) pulse acts over a time window \( \delta \), the corrections from Eq. (27) will be at most of order \( \sim J \delta / T \). Therefore, so long as \( \delta \ll T \), then in two dimensions or higher, the Hamiltonian \( D \) will indeed spontaneously break the Ising symmetry up to some finite temperature \( \tau_c \), and we will observe the time-crystal behavior described above.

**C. Field Theory of the Pre-Thermal Floquet Time Crystal State**

The universal behavior of a pre-thermal Floquet time crystal state can be encapsulated in a field theory. For the sake of concreteness, we derive this theory from the model analyzed in the previous section. The Floquet operator can be written, up to nearly exponential accuracy, as:
\[ U_t \approx \mathcal{U}( X e^{-iDT}) \mathcal{U}^\dagger \] (28)
Consequently, the transition amplitude from an initial state \( | \psi_i \rangle \) at time \( t_0 \) to a final state \( | \psi_f \rangle \) at time \( t_0 + mT \) can be written in the following form, provided \( t_{pre-thermal} < t_0 < t_0 + mT < t_* \): \[ \langle \psi_f | (U_t)^m | \psi_i \rangle = \langle \tilde{\psi}_f | \mathcal{U}( X e^{-iDT})^m \mathcal{U}^\dagger | \tilde{\psi}_i \rangle \] (29)
where \( | \tilde{\psi}_i \rangle \equiv U^\dagger | \psi_i \rangle \) and \( | \tilde{\psi}_f \rangle \equiv X^m U^\dagger | \psi_f \rangle \); recall that \( X^m \) is 1 or \( X \) for, respectively, \( m \) even or odd.

The second line of Eq. (29) is just the transition amplitude for the quantum transverse field Ising model in \((d+1)\)-dimensional spacetime, with \( d \geq 2 \). The model has nearest-neighbor interaction \( 27 \) together with higher-order terms that are present in the full expression for \( D \). Hence, it can be represented by the standard functional

\(^2\) Specifically, it follows from the construction of \( \mathcal{U} \) that \( \mathcal{U} = 1 + O(\lambda T) \), and \( \lambda T \ll 1 \) is the regime where the heating time is large.
The expected time dependence of $\langle \sigma_z^x \rangle$ at stroboscopic times, starting from a state which is low-temperature with respect to $U\mathcal{D}U\dagger$ (for example, for a state with all spins polarized in the $z$ direction), in (a) the pre-thermal time crystal phase, and (b) the non-time crystal pre-thermal phase.

The rotated Floquet operator $U\mathcal{D}U\dagger$ has an approximate $\mathbb{Z}_2$ symmetry generated by the operator $X$ since $U\mathcal{D}U\dagger \approx X e^{-i DT}$ and $[D, X] = 0$. Hence, $U\mathcal{D}U\dagger$ commutes with the (unrotated) Floquet operator $U_i$. It is not a microscopic symmetry in the conventional sense, since $U\mathcal{D}U\dagger$ does not commute with the time-dependent Hamiltonian $H(t)$, except for special fine-tuned points in the Floquet time crystal phase. However, since it commutes with the Floquet operator, it is a symmetry of the continuum-limit field theory (30). (See Ref. 45 for a discussion of Floquet time crystals in the MBL context that focuses on such symmetries, sometimes called “emergent symmetries”.) Within the field theory (30), this symmetry acts according to $\varphi \rightarrow -\varphi$, i.e. it acts in precisely the same way as time-translation by a single period. Again, this is analogous to the case of an Ising anti-ferromagnet, but with the time-translation taking the place of spatial translation. Thus, it is possible to view the symmetry-breaking pattern as $\mathbb{Z}_{TTS} \times \mathbb{Z}_2 \rightarrow \mathbb{Z}$. The unbroken $\mathbb{Z}$ symmetry is generated by the combination of time-translation by one period and the action of $U\mathcal{D}U\dagger$. However, there is an important difference between a Floquet time crystal and an Ising antiferromagnet. In the latter case, it is possible to explicitly break the Ising symmetry without breaking translational symmetry (e.g. with a uniform longitudinal magnetic field) and vice versa (e.g. with a spatially-oscillating exchange coupling). In a Floquet time crystal, this is not possible because there is always a $\mathbb{Z}_2$ symmetry $U\mathcal{D}U\dagger$ regardless of what small perturbation (compared to the drive frequency) is added to the Hamiltonian. The only way to explicitly prevent the system from having a $\mathbb{Z}_2$ symmetry is to explicitly break the time-translation symmetry. Suppose the Floquet operator is $UX e^{-i DT} U\dagger$. When a weak perturbation with period $2T$ is added, the Flo-
cated relationship to the microscopic degrees of freedom.

This functional integral is computed with boundary conditions on $\varphi$ at $t = t_0$ and $t_0 + mT$. Time-ordered correlation functions can be computed by inserting operators between the factors of $U_t$. However, if we are interested in equal-time correlation functions (at stroboscopic times $t = kT$),

$$\langle \psi | \hat{O}(x, kT) \hat{O}(0, kT) | \psi \rangle \equiv \langle \psi | (U_t)^{-k} \hat{O}(x, 0) \hat{O}(0, 0) (U_t)^k | \psi \rangle$$

(31)

then we can make use of the fact that the system rapidly pre-thermalizes to replace $(U_t)^k | \psi \rangle$ by a thermal state:

$$\langle \psi | (U_t)^{-k} \hat{O}(x, 0) \hat{O}(0, 0) (U_t)^k | \psi \rangle = \text{tr}(e^{-\beta D} \hat{O}(x) \hat{O}(0))$$

(32)

where $\beta$ is determined by $\text{tr}(e^{-\beta D} D) = \langle \psi | D | \psi \rangle$. The latter has an imaginary-time functional integral representation:

$$\text{tr}(e^{-\beta D} \hat{O}(x) \hat{O}(0)) = \int D\varphi e^{-\int d^d x \tau [\frac{1}{2} K(\partial_x \varphi)^2 + \frac{\omega}{2} K(\nabla \varphi)^2 + U(\varphi)]}$$

(33)

This equation expresses equal-time correlation functions in a pre-thermal Floquet time crystal in terms of the standard imaginary-time functional integral for the Ising model but with the understanding that the field $\varphi$ in the functional integral is related to the Ising spins in the manner noted above.

In order to compute unequal-time correlation functions, it is convenient to use the Schwinger-Keldysh formalism [81, 82] (see Ref. [83] for a modern review). This can be done by following the logic that led from the first line of Eq. (29) to the second and thence to Eq. (30). This will be presented in detail elsewhere [84].

We close this subsection by noting that the advantage of the field theory formulation of a pre-thermal Floquet time crystal is the salience of the similarity with the equilibrium Ising model; for instance, it is clear that the transition out of the Floquet time crystal (e.g. as a function of the energy of the initial state) in the pre-thermal regime is an ordinary Ising phase transition. The disadvantage is that it is difficult to connect it to measurable properties in a quantitative way because the field $\varphi$ has a complicated relationship to the microscopic degrees of freedom.

D. Relation to formal definitions of time crystals

In the above discussion, we have implicitly been adopting an “operational” definition of time-crystal: it is a system in which, for physically reasonable initial states, the system displays oscillations at a frequency other than the drive frequency forever (or at least, in the pre-thermal case, for a nearly exponentially long time.) This is a perfectly reasonable definition of time crystal, but it has the disadvantage of obscuring the analogies with spontaneous breaking of other symmetries, which tends not to be defined in this way. (Although in fact it could be; for example, an “operational” definition of spontaneously broken Ising symmetry, say, would be a system in which the symmetry-breaking order parameter does not decay with time for physically reasonable initial states [85].) It was for this reason that in Ref. [43] we introduced a formal definition of time-translation symmetry-breaking in MBL systems in terms of eigenstates (two equivalent formulations of which we called TTSB-1 and TTSB-2.)

The definitions TTSB-1 and TTSB-2 of Ref. [43] are natural generalizations of the notion of “eigenstate order” used to define spontaneous breaking of other symmetries in MBL [86, 87]. On the other hand they, like the notion of eigenstate order in general, are not really appropriate outside of the MBL context. In this subsection, we will review the usual formal definitions of spontaneous symmetry breaking in equilibrium. Then we will show how they can be extended in a natural way to time-translation symmetries, and that these extended versions are satisfied by the pre-thermal Floquet time crystals constructed above.

Let us first forget about time-translation symmetry, and consider a time-independent Hamiltonian $H$ with an Ising symmetry generated by $X$. Let $\rho$ be a steady state of the Hamiltonian; that is, it is invariant under the time evolution generated by $H$. (Here, we work in the thermodynamic limit, so by $\rho$ we really mean a function which maps local observables to their expectation values; that is, we define a state in the C*-algebra sense [87].) Generically, we expect $\rho$ to be essentially a thermal state. If the symmetry is spontaneously broken, then $\rho$ can obey the cluster decomposition (i.e. its correlations can be short-ranged), or it can be invariant under the symmetry $X$, but not both. That is, any state invariant under the symmetry decomposes as $\rho = \frac{1}{2}(\rho_1 + \rho_2)$, where $\rho_1$ and $\rho_2$ have opposite values of the Ising order parameter, and are mapped into each other under $X$. Thus, a formal definition of spontaneously broken Ising symmetry can be given as follows. We call a symmetry-invariant steady state $\rho$ state an “extremal symmetry-respecting state” if there do not exist states $\rho_1$ and $\rho_2$ such that $\rho = p\rho_1 + (1 - p)\rho_2$ for some $p \in (0,1)$, where $\rho_1$ and $\rho_2$ are symmetry-invariant steady states. We say the Ising symmetry is spontaneously broken if extremal symmetry-invariant steady states do not satisfy the cluster decomposition. Similar statements can be made for Floquet systems, where by “steady state” we now mean...
a state that returns to itself after one time cycle.

We can now state the natural generalization to time-translation symmetry. For time-translation symmetry, “symmetry-invariant” and “steady state” actually mean the same thing. So we say that time-translation symmetry is spontaneously broken if extremal steady states do not satisfy the cluster decomposition. This is similar to our definition TTSB-2 from Ref. 43 (but not exactly the same, since TTSB-2 was expressed in terms of eigenstates, rather than extremal steady states in an infinite system), so we call it “TTSB-2’.” We note that TTSB-2’ implies that any short-range correlated state \( \rho \), i.e., a state \( \rho \) which satisfies the cluster decomposition, must not be an extremal steady state. Non-extremal states never satisfy the cluster decomposition, so we conclude that short-range correlated states must not be steady states at all, so they cannot simply return to themselves after one time cycle. (This is similar to, but again not identical with, TTSB-1 in Ref. 43.)

We note that, for clean systems, the only steady state of the Floquet operator \( \hat{U}_t \) is believed to be the infinite temperature state \( |0\> \) which always obeys the cluster property, and hence time translation symmetry is not broken spontaneously. This does not contradict our previous results, since we already saw that time translation symmetry is only spontaneously broken in the pre-thermal regime, not at infinitely long times. Instead, we should examine the steady states of the approximate Floquet operator \( \hat{U}_t \) which describes the dynamics in the pre-thermal regime. We recall that, after a unitary change of basis, \( \hat{U}_t = X e^{-i DT} \), where \( D \) commutes with \( X \) and spontaneously breaks the Ising symmetry generated by \( X \) (for temperatures \( \tau < \tau_c \)). Hence \( \hat{U}_t^2 = e^{-2i DT} \). Any steady state \( \rho \) of \( \hat{U}_t \) must be a steady state of \( \hat{U}_t^2 \), which implies that its energy density corresponds to a temperature \( \tau < \tau_c \) that it must be of the form \( \rho = \rho_{SB} + (1 - t)X \rho_{SB} X \), where \( \rho_{SB} \) is an Ising symmetry-breaking state of temperature \( \tau \) for the Hamiltonian \( D \). Hence, we see (since \( \rho_{SB} \) is invariant under \( e^{-i DT} \) that \( \hat{U}_t \) ) that it must be of the form \( \rho = \rho_{SB} + (1 - t)X \rho_{SB} X \). So if \( \rho \) is a steady state of \( \hat{U}_t \) and not just \( \hat{U}_t^2 \), we must have \( t = 1/2 \). But then the state \( \rho \) clearly violates the cluster property. Hence, time translation is spontaneously broken.

IV. SPONTANEOUSLY-BROKEN CONTINUOUS TIME-TRANSLATION SYMMETRY IN THE PRE-HERMAL REGIME

A. Basic Picture

The pre-thermalized Floquet time crystals discussed above have a natural analog in undriven systems with continuous time translation symmetry. Suppose we have a time-independent Hamiltonian

\[
H = -uL + V ,
\]

where the eigenvalues of \( L \) are integers; in other words, for time \( T = 2\pi/u \), the condition \( e^{i n u LT} = 1 \) holds for all \( n \in \mathbb{Z} \). We also assume that \( L \) is a sum of local terms of local strength \( O(1) \); and \( V \) is a local Hamiltonian of local strength \( \lambda \ll u \). Then by Theorem 3.1 of Ref. 74, restated in Theorem 2 in Section II, there exists a local unitary \( U \) such that \( U H U^\dagger = -u L + D + \hat{V} \) such that \( [D, L] = 0 \) and the local strength of \( \hat{V} \) is \( \sim \lambda e^{-O((\log \lambda T)^5/|\lambda T|^3)} \). As noted in Theorem 2 in Section II, the first term in the explicit iterative construction of \( D \) in Ref. 74 is \( D = \langle V \rangle + \frac{1}{2} O(\lambda T)^2 \), where

\[
\langle V \rangle \equiv \frac{1}{2\pi} \int_0^{2\pi} d\theta e^{i L \theta} V e^{-i L \theta}.
\]

As a result of this theorem, such a system has an approximate U(1) symmetry generated by \( U^t L U \) that is explicitly broken only by nearly exponentially-small terms. Consequently, \( U^t L U \) is conserved by the dynamics of \( H \) for times \( t \ll t_* = O((\log \lambda T)^3/|\lambda T|^3) \). We will call the Hamiltonian \( -u L + D \) the “pre-thermal” Hamiltonian, since it governs the dynamics of the system for times short compared to \( t_* \). We will assume that we have added a constant to the Hamiltonian such that \( L \) is positive-definite; this will allow us to abuse terminology a little by referring to the expectation value of \( L \) as the “particle number”, in order to make analogies with well-known properties of Bose gases, in which the generator of the U(1) symmetry is the particle number operator. In this vein, we will call \( u \) the electric potential, in analogy with (negatively) charged superfluids.

We will further suppose that \( L \) is neither integrable nor many-body localized, so that the dynamics of \( L \) will cause an arbitrary initial state \( |\psi_0\> \) with non-zero energy density and non-zero \( \langle \psi_0 | L | \psi_0 \rangle \) to rapidly thermalize on some short (compared to \( t_* \) ) time scale \( t_{\text{pre-thermal}} \sim \lambda^{-1} \). The resulting thermalized state can be characterized by the expectation values of \( L \) and \( D \), both of which will be the same as in the initial state, since energy and particle number are conserved. Equivalently, the thermalized state can be characterized by its temperature \( \beta \) (defined with respect to \( D \) ) and effective chemical potential \( \mu \). In other words, all local correlation functions of local operators can be computed with respect to the density matrix \( \rho = e^{-\beta(D-\mu L)} \). The chemical potential \( \mu \) has been introduced to enforce the condition \( \text{tr}(\rho L) = \langle \psi_0 | L | \psi_0 \rangle \).

Now suppose that we choose \( V \) such that \( D \) spontaneously breaks the U(1) symmetry in some range of temperature \( 1/\beta \) and chemical potential \( \mu \). Suppose, further, that we prepare the system in a short-range correlated initial state \( |\psi_0\> \) such that the energy density (and hence, its temperature) is sufficiently low, and the number density sufficiently high, so that the corresponding thermalized state spontaneously breaks the U(1) symmetry generated by \( L \). Then, the preceding statement must be slightly revised: all local correlation functions of local operators can be computed with respect to the
density matrix $\rho = e^{-\beta(D-\mu L-\epsilon X)}$ for some $X$ satisfying $[X,L] \neq 0$. The limit $\epsilon \to 0$ is taken after the thermodynamic limit is taken; the direction of the infinitesimal symmetry-breaking field $X$ is determined by the initial state. To avoid clutter, we will not explicitly write the $\epsilon X$ in the next paragraph, but it is understood.

Consider an operator $\Phi$ that satisfies $[L, \Phi] = \Phi$. For example, if we interpret $L$ as the particle number, we can take $\Phi$ to be the particle creation operator.) Its expectation value at time $t$ can take $\Phi$ to be the particle creation operator. Its value must be independent of time. Hence, so long as $\beta V$ is determined by the initial state approaches a thermal state of the full Hamiltonian $-\mu L + D + V$. Since $V$ is small, this is approximately the same as a thermal state of $-uL + D$. However, because $V$ is not exactly zero, the particle number is not conserved and in equilibrium the system chooses the particle number that minimizes its free energy, which corresponds to the “electrochemical potential” being zero, $\mu - u = 0$. Since this corresponds to zero frequency of oscillations, it follows that no oscillations are observed at infinite time.

The above discussion is essentially the logic that was discussed in Refs.\cite{11, 42} and \cite{38} where it was pointed out that a superfluid at non-zero chemical potential is a time crystal as a result of the well-known time-dependence of the order parameter \cite{89}. However, there is an important difference: the U(1) symmetry is not a symmetry of the Hamiltonian of the problem and, therefore, does not require fine-tuning but, instead, emerges in the $u \to \infty$ limit, thereby evading the criticism \cite{41, 42, 90, 92} that the phase winds in the ground state only if the U(1) symmetry is exact.

B. Example: XY Ferromagnet in a Large Perpendicular Field

Consider the concrete example of a spin-1/2 system in three spatial dimensions, with Hamiltonian

$$
H = -\hbar^2 \sum_i S_i^x - \hbar^2 \sum_i S_i^y - \sum_{i,j} [J_{ij} S_i^x S_j^x + J_{ij} S_i^y S_j^y + J_{ij} S_i^z S_j^z],
$$

(37)

We take $L = S_i^z = \sum_i S_i^z$, and the longitudinal magnetic field $\hbar^2$ plays the role of $u$ in the preceding section. We take $J_{ij}$ and $J_{ij}^z$ to vanish except for nearest neighbors, for which $J_{ij}^x = J + \delta J$, $J_{ij}^y = J_y + \delta J$, and $J_{ij}^z = J_z$. (We do not assume $\delta J \ll J$.) The local scale of $V$ is given by $\lambda = \max(J + \delta J, \hbar^2)$, so that the condition $\lambda \ll T^{-1} \sim \hbar^2$ is satisfied if $J + \delta J, \hbar^2 \ll \hbar^2$. In this case, $D$ is (to first order) the Hamiltonian of an XY ferromagnet:

$$
D = - \sum_{(i,j)} \left[ J(S_i^x S_j^y + S_i^y S_j^x) + J^2 S_i^z S_j^z \right] + \frac{1}{T} O(\lambda/\hbar^2)^2.
$$

(38)

Then, starting from a short-range correlated state with appropriate values of energy and $\langle S^z \rangle$, we expect that time evolution governed by $D$ causes the system to “pre-thermalize” into a symmetry-breaking state with some value of the order parameter $\langle S^z \rangle = n_0 e^{i\phi}$. According to the preceding discussion, the order parameter will then rotate in time with angular frequency $\omega = \mu - \hbar^2$ (where $\mu \lesssim \lambda$ is determined by the initial value of $\langle S^z \rangle$) for times short compared to the thermalization time $t_\star$.

Note, however, that we have assumed that the system is completely isolated. If the system is not isolated, then the periodic rotation of the order parameter will cause the system to emit radiation, and this radiation will cause the system to decay to its true ground state \cite{37, 93}.

C. Field Theory of Pre-Thermal Continuous-TTSB Time Crystal

For simplicity we will give only the imaginary-time field theory for equal-time correlation functions deep within the pre-thermal regime; the Schwinger-Keldysh functional integral for unequal-time correlation functions, with nearly exponentially-small thermalization effects taken into account, will be discussed elsewhere \cite{84}. Introducing the field $\phi \sim (S_z + i S_y) e^{i(\mu - u)t}$, we apply Eq. (36) to the XY ferromagnet of the previous section, thereby obtaining the effective action:

$$
S_{\text{eff}} = \int d^3 x d\tau \left[ \phi^* \partial_t \phi - \mu \phi^* \phi + g(\phi^* \phi)^2 + \ldots \right]
$$

(39)

The ... represents higher-order terms. The U(1) symmetry generated by $S^z$ acts according $\phi \to e^{i\theta} \phi$. Time-translation symmetry acts according to $\phi(t) \to
\[ e^{i(a-u)a} \phi(t+a) \] for any \( a \). Thus, when \( \phi \) develops an expectation value, both symmetries are broken and a combination of them is preserved according to the symmetry-breaking pattern \( \mathbb{R} \times_{\text{TTS}} U(1) \rightarrow \mathbb{R} \), where the unbroken \( \mathbb{R} \) is generated by a gauge transformation by \( \theta \) and a time-translation \( t \rightarrow t + \frac{\theta}{a} \).

From the mathematical equivalence of Eq. (39) to the effective field theory of a neutral superfluid, we see that (1) in 2D, there is a quasi-long-range-ordered phase – an ‘algebraic time crystal’ – for initial state energies below a Kosterlitz-Thouless transition; (2) the TTGS phase transition in 3D is in the ordinary XY universality class in 3D; (3) the 3D time crystal phase has Goldstone boson excitations. If we write \( \phi(x,t) = \sqrt{\frac{i}{2a} + \delta \rho(x,t)} e^{i\theta(x,t)} \), and integrate out the gapped field \( \delta \rho(x,t) \), then the effective action for the gapless Goldstone boson \( \theta(x,t) \) is of the form discussed in Ref. [31].

V. PRE-THERMALIZED FLOQUET TOPOLOGICAL PHASES

We can also apply our general results of Section 11 to Floquet symmetry-enriched (SPT) and symmetry-enriched (SET) topological phases, even those which don’t exist in stationary systems. (We will henceforth use the abbreviation SxT to refer to either SPT or SET phases.)

As was argued in Refs. [31] and [32] any such phase protected by symmetry \( G \) is analogous to a topological phase of a stationary system protected by symmetry \( \mathbb{Z} \times G \), where the extra \( \mathbb{Z} \) corresponds to the time translation symmetry. Here the product is semi-direct for anti-unitary symmetries and direct for unitary symmetries. For simplicity, here we will consider only unitary symmetries. Similar arguments can be made for anti-unitary symmetries.

We will consider the class of phases which can still be realized when the \( \mathbb{Z} \) is refined to \( \mathbb{Z}_N \). That is, the analogous stationary phase can be protected by a unitary representation \( W(g) \) of the group \( \tilde{G} = \mathbb{Z}_N \times G \). Then, in applying the general result of Section 11 we will choose \( H_0(t) \) such that its time evolution over one time cycle is equal to \( X \equiv W(T) \), where \( T \) is the generator of \( \mathbb{Z}_N \). Then it follows that, for a generic perturbation \( V \) of small enough local strength \( \lambda \), there exists a local unitary rotation \( U \) (commuting with all the symmetries of \( U_i \)) such that \( U_i \approx \tilde{U}_i \), where \( \tilde{U}_i = UXe^{-iDTU^\dagger} \). Thus, for initial state mean energies \( \langle H(t) \rangle \), for all \( g \in G \), \( W(g)\,H(t)\,W(g)^{-1} \). By inspection of the explicit construction for \( U \) and \( D \) (see Appendix A), it is easy to see that in this case \( U \) is a symmetry-respecting local unitary with respect to \( W(g) \), and \( D \) commutes with \( W(g) \). That is, the rotation by \( U \) preserves the existing symmetry \( G \) as well as revealing a new \( \mathbb{Z}_N \) symmetry generated by \( X \) (which in the original frame was “hidden”).

Therefore, we can choose \( D \) to be a Hamiltonian whose ground state is in the stationary SxT phase protected by \( \mathbb{Z}_N \times G \). It follows (by the same arguments discussed in Ref. [31] for the MBL case) that the ground state \( D \) will display the desired Floquet-SxT order under the time evolution generated by \( \tilde{U}U\tilde{U} = Xe^{-iDT} \). Furthermore, since Floquet-SxT order is invariant under symmetry-respecting local unitaries, the ground state of \( UDU^\dagger \) will display the desired Floquet-SxT order under \( U_i \).

We note, however, that topological order, in contrast to symmetry-breaking order, does not exist at nonzero temperature (in clean systems, for spatial dimensions \( d < 4 \)). Thus, for initial state mean energies \( \langle H(t) \rangle \), for all \( g \in G \), \( W(g)\,H(t)\,W(g)^{-1} \). By inspection of the explicit construction for \( U \) and \( D \) (see Appendix A), it is easy to see that in this case \( U \) is a symmetry-respecting local unitary with respect to \( W(g) \), and \( D \) commutes with \( W(g) \). That is, the rotation by \( U \) preserves the existing symmetry \( G \) as well as revealing a new \( \mathbb{Z}_N \) symmetry generated by \( X \) (which in the original frame was “hidden”).

VI. OPEN SYSTEMS

So far, we have considered only isolated systems. In practice, of course, some coupling to the environment will always be present. One can also consider the effect of classical noise, for example some time-dependent randomness in the parameters of the drive, so that successive time steps do not implement exactly the same time evolution. The Floquet-MBL time crystals of Ref. [33] are not expected to remain robust in such setups, since MBL will be destroyed. Since some amount of coupling to the environment is inevitable in realistic setups, this limits the timescales over which one could expect to observe Floquet-MBL time crystals experimentally.

However, the situation could be quite different for the pre-thermal time crystals of this work. A complete treatment is beyond the scope of the present work, so in this section we will confine ourselves to stating one very interesting hypothesis: Floquet case time-crystals can actually be stabilized in open systems so that the oscillations actually continue forever for any initial state (in
VII. DISCUSSION

In this paper, we have described how phases protected by time-translation symmetry can be observed in the pre-thermal regime of driven and undriven quantum systems. This greatly increases the set of experimental systems in which such phases can be observed, since, as opposed to previous proposals, we do not require many-body localization to robustly prevent the system from heating to infinite temperature. While many-body localization has been observed in experiments [94–96], the ideas put forward in this paper significantly reduce experimental requirements as strong disorder is not required.

Our Theorem 1 implies that the time-translation-protected behavior (for example, the fractional-frequency oscillations in the Floquet time crystal) can be observed to nearly exponentially-late times, provided that the drive frequency is sufficiently high. However, the rigorous bound given in the theorem – which requires a drive frequency \( \sim 10^3 \) times larger than the local couplings in the time-dependent Hamiltonian – may not be tight. Therefore, it would be interesting to check numerically whether (in the Floquet time crystal case, say) long-lived oscillations are observed in systems with drive frequency only moderately larger than the local couplings. This may be challenging in small systems, in which there isn’t a large separation of energy scales between the local coupling strength and the width of the many-body spectrum (which the frequency should certainly not exceed). In one-dimensional systems, oscillations will not be observed to exponentially-long (in the drive frequency) times, but will have a finite correlation time for any non-zero energy density initial state. However, there will be a universal quantum critical regime in which the correlation time will be the inverse effective temperature.

Although naive application of Theorem 1 suggests that the ideal situation is the one in which the drive frequency becomes infinitely large, in practice very high-frequency driving will tend to excite high energy modes that were ignored in constructing the model lattice Hamiltonian. For example, if the model Hamiltonian describes electrons moving in a periodic potential in the tight-binding approximation, high frequency driving would excite higher orbitals that were excluded. Thus, the driving frequency \( \Omega \) needs to be much greater than the local energy scales of the degrees of freedom included in the model Hamiltonian (except for one particular coupling, as discussed in Section III), but also much less than the local energy scales of the degrees of freedom not included. (One cannot simply include all degrees of freedom in the model Hamiltonian, because then the norm of local terms would be unbounded, and Theorem 1 would not apply.)

In the case of undriven systems, we have shown that continuous time-translation symmetry breaking can similarly occur on nearly exponentially-long time intervals even without any fine-tuning of the Hamiltonian, provided that there is a large separation of scales in the Hamiltonian. We show how in certain cases this can be described in terms of approximate Goldstone bosons associated with the spontaneously-broken time-translation symmetry.

Our analysis relied on the construction of hidden approximate symmetries that are present in a pre-thermal regime. The analogous symmetries in MBL systems, where they are exact, were elucidated in the interesting work of von Keyserlingk et al. [45]. In the time-translation protected phases discussed here, the symmetry generated by the operator \( U \mid \hat{X} U \) is enslaved to time-translation symmetry.
translation symmetry since, in the absence of fine-tuning, such a symmetry exists only if time-translation symmetry is present. (That is, if we add fields to the Hamiltonian that are periodic with period $nT$ and not period $T$, then the hidden symmetry no longer exists.) Moreover, this symmetry is broken if and only if time-translation symmetry is broken. (Similar statements hold in the MBL case[45].) In the Floquet time crystal case, the hidden symmetry generated by $U(t)U^*$ acts on the order parameter at stroboscopic times in the same way as time-translation by $T$ (a single period of the drive), and therefore it does not constrain correlation functions any more than they already are by time-translation symmetry. The same observation holds for the approximate symmetry generated by $L_z$ in the undriven case.

However, there are systems in which time crystal behavior actually does “piggyback” off another broken symmetry. This does require fine-tuning, since it is necessary to ensure that the system possesses the “primary” symmetry, but such tuning may be physically natural (e.g. helium atoms have a very long lifetime, leading to a $U(1)$ symmetry). The broken symmetry allows a many-body system to effectively become a few-body system. Thus, time crystal behavior can occur in such systems for the same reason that oscillations can persist in few-body systems. Oscillating Bose condensates (e.g. the AC Josephson effect and the model of Ref. 97) can, thus be viewed as fine-tuned time crystals. They are not stable to arbitrary time-translation symmetry-respecting perturbations; a perturbation that breaks the “primary” symmetry will cause the oscillations to decay. Indeed, most few-body systems are actually many-body systems in which a spontaneously-broken symmetry approximately decouples a few degrees of freedom. A pendulum is a system of $10^{23}$ atoms that can be treated as a single rigid body due to spontaneously-broken spatial translational symmetry: its oscillations owe their persistence to this broken symmetry, which decouples the center-of-mass position from the other degrees of freedom.

With the need for MBL obviated by prethermalization, we have opened up the possibility of time-translation protected phases in open systems, in which MBL is impossible [44][22]. In fact, since the results of Appendix 1 show that TTSB can occur in non-thermal states, it is possible for the coupling to a cold bath to counteract the heating effect that would otherwise bring an end to the pre-thermal state at time $t_e$. This raises the possibility of time-translation protected phases that survive to infinite times in non-equilibrium steady states; the construction of such states is an interesting avenue for future work.

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Note added: After the submission of this paper, two experimental papers (J. Zhang et al., arXiv:1609.08684 and S. Choi et al., arXiv:1610.08057) have appeared with evidence consistent with the observation of a Floquet time crystal. We note that the J. Zhang et al. paper implements disorder by addressing each ion sequentially. A pre-thermal version of this experiment would not need disorder, thereby sidestepping this bottleneck standing in the way of experiments on larger systems. The Choi et al. paper occurs in a system that is unlikely to be many-body localize, and therefore occurs during a slow approach to equilibrium. This is unlikely to correspond to a prethermal regime, but the approximate short-time form of the time evolution entailed in our Theorem 1 might still be relevant to understanding the results.

Appendix A: Rigorous proof of pre-thermalization results

a. Definition of the norm

Let’s suppose, for the sake of concreteness, that we have a spin system with a local time-dependent Hamiltonian of the form:

$$H(t) = \sum_{i,j} J_{ij}^\beta(t) S_i^\alpha S_j^\beta + \sum_{i,j,k} K_{ij,j,k}^\alpha\beta\gamma(t) S_i^\alpha S_j^\beta S_k^\gamma + \ldots$$

$$= \sum_{p} \sum_{p-tuples} A_{i_1,\ldots,i_p}$$

(A1)

Here $\alpha = x,y,z$ are the components of the spins, and $i,j,k$ are lattice sites. In the first line, we have explicitly written the 2-site and 3-site terms; the . . . represents terms up to $n$-site terms, for some finite $n$. It is assumed that these interactions have finite range $r \geq n$ such that all of the sites in a $k$-site term are within distance $r$. In the second line, we have re-expressed the Hamiltonian in a more generic form in terms of $p$-site terms $A_{i_1,\ldots,i_p}$ with $i_1 \neq \ldots \neq i_p$. To avoid clutter, we have not explicitly denoted the $r$-dependence of $A_{i_1,\ldots,i_p}$. We define the local instantaneous norm $\|A_{i_1,\ldots,i_p}\|_n$ according to

$$\|A_{i_1,\ldots,i_p}\|_n^{\text{inst}} \equiv e^{\kappa_n t} \|A_{i_1,\ldots,i_p}\|$$

(A2)

where $\|A_{i_1,\ldots,i_p}\|$ is the operator norm of $A_{i_1,\ldots,i_p}$ at a given instant of time $t$ and

$$\kappa_n \equiv \kappa_1/[1 + \ln n].$$

(A3)

We make this choice of $n$-dependence of $\kappa_n$, following Ref. 74 for reasons that will be clear later. We then av-
verage the instantaneous norm over one cycle of the drive:
\[ \|A_{i_1,...,i_p}\|_n \equiv \frac{1}{T} \int_0^T dt \|A_{i_1,...,i_p}\|_{\text{inst}} \quad (A4) \]

It is only in this step that we differ from Abanin et al. [24], who consider the supremum over \( t \) rather than the average. In analyzing the Floquet operator, i.e. the evolution due to \( H \) at stroboscopic times, it is the total effect of \( H \), which is determined by its integral over a cycle, that concerns us. Error terms that act over a very short time, even if they are relatively strong, have little effect on the Floquet operator so long as their norm, as defined above, is small. Finally, we define the global time-averaged norm of the Hamiltonian \( H \):
\[ \|H\|_n \equiv \sup_j \sum_p \sum_k [\sum \delta_{j,ik}] \|A_{i_1,...,i_p}\|_n \quad (A5) \]

The term in square braces restricts the sum to \( p \)-tuples that contain the site \( j \).

b. More technical statement of Theorem 1

Theorem 1 stated above will follow from the following slightly more technical formulation. For notational simplicity we work in units with \( T = 1 \).

**Theorem 1**. Consider a periodically-driven system with Floquet operator:
\[ U_j = T \exp \left(-i \int_0^T [H_0(t) + V(t)] dt \right), \quad (A6) \]
where \( X \equiv T \exp \left(-i \int_0^T H_0(t) \right) \) satisfies \( X^N = 1 \) for some integer \( N \), and we assume that \( H_0 \) can be written as a sum \( H_0(t) = \sum_i h_i(t) \) of terms acting on single sites \( i \). Define \( \lambda \equiv \|V\|_1 \). Then there exists a sequence of quasi-local \( A_n \) such that, defining \( U_n = e^{-i A_n} \ldots e^{-i A_1} \), we have
\[ U_n U_j U_n^{\dagger} = X T \exp \left(-i \int_0^1 [D_n + E_n + V_n(t)] dt \right), \quad (A7) \]
where \( [D_n, X] = 0; D_n, E_n \) are independent of time; and
\[ \|V_n\|_n, \|E_n\|_n \leq 2K_n \lambda^n, \quad (A8) \]
\[ \|A_n\|_n \leq (N+1)K_n \lambda^n, \quad (A9) \]
\[ \|D_n - D_n-1\|_n \leq K_n \lambda^n, \quad (A10) \]
where we have defined \( \lambda \equiv \|V\|_1 \), and
\[ K_n = C^{n-1} \prod_{k=1}^{n-1} m(k), \quad C = 2(N+3)(N+4), \]
\[ m(n) = \frac{18}{\kappa_{n+1}(\kappa_n - \kappa_{n+1})} \quad (A11) \]

These bounds hold provided that \( n \leq n_* \), with
\[ n_* = \frac{\lambda_0/\lambda}{\left(1 + \log\left(\lambda_0/\lambda\right)\right)^3}, \quad \lambda_0 = (36C)^{-1} \quad (A12) \]
and provided that
\[ \lambda \leq \frac{\mu}{N + 3}, \quad \mu \approx 0.07. \quad (A13) \]

Theorem 1 follows from Theorem 1', because \( n_* \) is chosen such that \( n \leq n_* \) implies \( Cm(n) \leq 1/3 \). It then follows that \( K_{n+1} \lambda^{n+1}/(K_n \lambda^n) = Cm(n) \lambda \leq 2/3 \), and hence that \( K_n \lambda^n \leq \lambda/2^{n-1} \). Moreover, we obtain Eq. (10) by summing Eq. (A10), from which we see that \( \|D_n - D_1\|_n \leq \sum_{k=2}^{\infty} K_k \lambda^k \leq K_2 \lambda^2 \sum_{k=2}^{\infty} \left(\frac{1}{2}\right)^{k-2} = 2K_2 \lambda^2 = 2Cm(1) \lambda^2 \approx 2.9 \lambda^2/\lambda_0 \). (Here we use the fact that \( \| \cdot \|_{n+1} \leq \| \cdot \|_n \)).

In the next subsections, we will give a proof of Theorem 1'.

c. Iterative construction

The idea is to construct the \( D_n, V_n, E_n, A_n \) discussed above iteratively. That is, suppose that at the \( n \)-th step, we have
\[ U_n U_i U_n^{\dagger} = U_i^{(n)} = X T \exp \left(-i \int_0^1 \mathcal{H}_n(t) dt \right), \quad (A14) \]
where \( \mathcal{H}_n(t) = F_n + V_n(t) \), with \( F_n = \int_0^T \mathcal{H}_n(t) dt \) time-independent. We will choose to separate the time-independent piece \( F_n \) according to \( F_n = D_n + E_n \), where \( D_n = \langle F_n \rangle \), and we have defined the symmetrization
\[ \langle O \rangle = \frac{1}{N} \sum_{k=0}^{N-1} X^{-k} O X^k. \quad (A15) \]
In particular, this implies that \( [D_n, X] = 0 \) and \( \langle D_n \rangle = D_n \), and therefore \( \langle E_n \rangle = \langle F_n \rangle - \langle D_n \rangle = D_n - D_n = 0 \).

We will now introduce a local unitary \( A_i = e^{-i A_i} \), which we use to rotate the Floquet operator \( U_i^{(n)} \), giving a new Floquet operator
\[ U_i^{(n+1)} = A_n U_i^{(n)} A_n^{\dagger} = X T \exp \left(-i \int_0^1 \mathcal{H}_{n+1}(t) dt \right), \quad (A16) \]
The ultimate goal, decomposing \( \mathcal{H}_{n+1}(t) = D_{n+1} + E_{n+1} + V_{n+1}(t) \) as before, is to ensure that the residual error terms \( E_{n+1} \) and \( V_{n+1} \) are much smaller than \( E_n \) and \( V_n \). This goal is achieved in two separate steps. The first step ensures that \( E_{n+1} \) is small (that is, the time-independent part of \( \mathcal{H}_{n+1}(t) \) nearly commutes with \( X \), and the second step ensures that \( V_{n+1} \) is small.

**Step One**. This step proceeds similarly to the recursion relation of Abanin et al. [24] for the time-independent case (Section 5.4 of Ref. [24]). There the recursion relation
was designed to make the Hamiltonian commute with its zero-th order version. This is analogous to our present goal of making the Floquet operator commute with $X$. Here, we adapt the analysis of Ref. \cite{abalin2018} to the Floquet case.

We observe that

$$U_{i}^{(n+1)} = A_{n} U_{i}^{(n)} A_{n}^{\dagger},$$

$$= X \left[ X^\dagger A_{n} X \times T \exp \left( -i \int_{0}^{1} \mathcal{H}_{n}(t) dt \right) \times A_{n}^\dagger \right],$$

$$= X \left[ e^{-X^\dagger i A_{n} X} \times T \exp \left( -i \int_{0}^{1} \mathcal{H}_{n}(t) dt \right) e^{i A_{n}} \right],$$

$$= X \times T \exp \left( -i \int_{0}^{1} \mathcal{H}_{n}'(t) dt \right),$$

where

$$\mathcal{H}_{n}'(t) = \begin{cases} \frac{1}{\alpha} (-A_{n}) & 0 \leq t \leq a \\ \frac{1}{1-2a} \mathcal{H}_{n} \left( \frac{t-a}{1-2a} \right) & a \leq t \leq (1-a), \\ \frac{1}{X^\dagger A_{n} X} & (1-a) \leq t \leq 1, \end{cases}$$

(A17)

(A18)

(A19)

(A20)

(for some constant $\alpha \in [0, 1/2]$ which can be chosen arbitrarily.) Let us decompose $\mathcal{H}_{n}'(t) = D_{n}' + V_{n}'(t)$, where $D_{n}' = \frac{1}{\alpha} \int_{0}^{1} \mathcal{H}_{n}'(t) dt$. Our goal will be to ensure that the time-independent part $D_{n}'$ commutes with $X$. It turns out this can actually be achieved exactly, and in particular we can choose $A_{n}$ such that $D_{n}' = D_{n}$.

To this end, we first observe that

$$D_{n}' = D_{n} + E_{n} + X^\dagger A_{n} X - A_{n}. \quad (A21)$$

(A22)

We now claim that $D_{n}' = D_{n}$ if we choose

$$A_{n} := \frac{1}{N} \sum_{k=0}^{N-1} \sum_{p=0}^{k} E_{n}^{(p)}, \quad E_{n}^{(p)} = X^{-p} EXP. \quad (A23)$$

(A24)

(A25)

(A26)

(A27)

To see this, note that, by construction,

$$X^\dagger A_{n} X - A_{n} = \frac{1}{N} \sum_{k=0}^{N-1} \sum_{p=0}^{k} \left[ E_{n}^{(p+1)} - E_{n}^{(p)} \right]$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} \left[ E_{n}^{(k+1)} - E_{n}^{(k)} \right]$$

$$= -E_{n} + (E_{n})$$

$$= -E_{n}, \quad (A28)$$

since $(E_{n}) = 0$.

Step Two.- The next step is now to find a new time-dependent Hamiltonian $\mathcal{H}_{n+1}(t)$ which gives the same unitary evolution as $H_{n}'(t)$ over the time interval $[0, 1]$, while making the time-dependent part smaller. That is, making the decomposition $\mathcal{H}_{n+1}(t) = D_{n+1} + E_{n+1} + V_{n+1}(t)$ as before, the goal is to make $V_{n+1}$ small. In fact, this is precisely the problem already considered by Abalin et al.\cite{abalin2018}, and we can use the procedure described in Section 4.1 of that paper.

One might worry whether Step Two undoes the good work done by Step One. That is, does making $V_{n+1}$ small come at the cost of making $E_{n+1}$ larger again? However, this turns out not to be a problem, as the bounds we derive below will make clear.

$$d(n) = \|D_{n}\|_{n}, \quad v(n) = \|V_{n}\|_{n}, \quad v'(n) = \|V_{n}'\|_{n}, \quad e(n) = \|E_{n}\|_{n}, \quad \delta d(n) = \|D_{n+1} - D_{n}\|_{n+1}.$$  \quad (A28)

First of all, from Eq. (A23) we have a bound on $A_{n}$:

$$\|A_{n}\|_{n} \leq \frac{N+1}{2} e(n). \quad (A29)$$

From Eq. (A21) we observe that

$$V_{n}'(t) = \begin{cases} \frac{1}{\alpha} (-A_{n}) - D_{n} & 0 \leq t \leq a \\ \frac{1}{1-2a} \left[ 2aD_{n} + E_{n} + V_{n} \left( \frac{t-a}{1-2a} \right) \right] & a \leq t \leq (1-a), \\ \frac{1}{X^\dagger A_{n} X} - D_{n} & (1-a) \leq t \leq 1, \end{cases}$$

(A30)

and hence

$$v'(n) \leq 2\|A_{n}\|_{n} + \|E_{n}\|_{n} + \|V_{n}\|_{n} + 4a\|D_{n}\|_{n}.$$  \quad (A31)

Hence, we can send $a \to 0$ to give (using Eq. (A29))

$$v'(n) \leq (N+2)e(n) + v(n). \quad (A32)$$

Then, as our construction of $\mathcal{H}_{n+1}$ from $\mathcal{H}_{n}'$ is the one described in Section 4.1 of Abalin et al. we can use their bounds

$$\|D_{n+1} + E_{n+1} - D_{n}\|_{n+1} \leq \epsilon_{n}/2 \quad (A33)$$

$$v(n+1) \leq \epsilon_{n} \quad (A34)$$

where

$$\epsilon_{n} = m(n)v'(n)(d(n) + 2v'(n)), \quad (A35)$$

$$m(n) = \frac{18}{(\kappa_{n+1} - \kappa_{n+1})\kappa_{n+1}}. \quad (A36)$$

These bounds hold provided that

$$3v'(n) \leq \kappa_{n} - \kappa_{n+1}.$$

(A37)

Since $D_{n+1} - D_{n} = (D_{n+1} + E_{n+1} - D_{n})$, we see that

$$\delta d(n) \leq \|D_{n+1} + E_{n+1} - D_{n}\|_{n+1} \leq \epsilon_{n}/2$$  \quad (A38)

and

$$e(n+1) \leq \|D_{n+1} + E_{n+1} - D_{n}\|_{n+1} + \|D_{n+1} - D_{n}\|_{n+1} \leq \epsilon_{n} \quad (A39)$$
e. Proof of Theorem 1' by induction

The idea now is to apply the bounds of the previous subsection recursively to give bounds expressed in terms of the original Floquet operator,

\[
U_i = U_i^{(1)} = T \exp \left( -i \int_0^1 [H_0(t) + V(t)] \right) \tag{A40}
\]

and in particular the quantity \( \lambda = \|V_{\text{int}}\|_1 = \|V\|_1 \). First of all, we write \( \mathcal{H}_1(t) \equiv V_{\text{int}}(t) = F_1 + V_1(t) \), where \( F_1 = \int_0^1 V_{\text{int}}(t) dt \), and then separate \( F_1 = D_1 + E_1 \), where \( D_1 = \langle F_1 \rangle \). We note that \( \|F_1\|_1 \leq \lambda \), which implies that \( \langle 1 \rangle \leq \|V_{\text{int}}\|_1 + \|F_1\|_1 \leq 2\lambda \), and \( d(1) \leq \lambda \). In turn this gives \( e(1) \leq \|D_1\|_1 + \|F_1\|_1 \leq 2\lambda \).

Now we proceed by induction. Suppose that we have some \( n \) such that, for all \( 1 \leq k \leq n \), we have

\[
e(k), v(k) \leq 2K_k \lambda^k, \tag{A42}
\]

and for all \( 1 \leq k < n 
\]

\[
\delta d(k) \leq K_{k+1} \lambda^{k+1} \tag{A43}
\]

where the coefficients \( K_k \) satisfy \( K_{k+1}/K_k \leq 1/2\lambda \). (The preceding discussion shows that this induction condition is satisfied for \( n = 1 \) with \( K_1 = 1 \).)

Then from Eq. \( \text{A32} \) we find that

\[
v'(n) \leq 2c_N K_n \lambda^n, \quad c_N = N + 3, \tag{A44}
\]

and hence

\[
\epsilon_n \leq m(n) 2c_N K_n \lambda^n (d(n) + 2c_N K_n \lambda^n). \tag{A45}
\]

We note that the triangle inequality and the fact that \( \|\cdot\|_n \) decreases with \( n \) ensures that \( d(n+1) - d(n) \leq \delta d(n) \). Hence we can bound \( d(n) \) by

\[
d(n) \leq d(1) + \sum_{k=1}^{n-1} \delta d(k) \tag{A46}
\]

\[
\leq \lambda + \sum_{k=1}^{n-1} K_{k+1} \lambda^{k+1} \tag{A47}
\]

\[
= \sum_{k=1}^n K_k \lambda^k \tag{A48}
\]

\[
\leq \sum_{k=1}^n \lambda \left( \frac{1}{2} \right)^{k-1} \tag{A49}
\]

\[
\leq 2\lambda \tag{A50}
\]

In Eq. \( \text{A49} \), we used the inequality \( K_{k+1}/K_k \leq 1/(2\lambda) \). This same inequality also ensures that \( K_n \lambda^n \leq \lambda \), so inserting into Eq. \( \text{A45} \) gives

\[
\epsilon_n \leq m(n) 2c_N K_n (2 + 2c_N) \lambda^{n+1} \approx 2Cm(n) K_n \lambda^{n+1} \equiv K_{n+1} \lambda^{n+1}. \tag{A51}
\]

Here we chose

\[
K_{n+1} = Cm(n) K_n, \quad C = 2c_N(1 + c_N). \tag{A52}
\]

Next we need to examine the conditions under which Eq. \( \text{A37} \) holds. Given the bounds on \( v'(n) \) and using the inequality \( K_n \lambda^n \leq \lambda(1/2)^n \), it is sufficient to demand that

\[
3c_N(1/2)^n - 1 \leq \kappa_{n+1} - \kappa_n, \tag{A53}
\]

or in other words

\[
\lambda \leq \frac{1}{3c_N} \max_{n \in \mathbb{N}} \left[ 2(n-1)(\kappa_{n+1} - \kappa_n) \right] = \frac{1}{3c_N} (\kappa_2 - \kappa_1) \approx \frac{0.14\kappa_1}{N + 3}. \tag{A54}
\]

Provided that Eq. \( \text{A54} \) holds, we then find that

\[
\delta d(n), v(n+1)/2, e(n+1)/2 \leq K_{n+1} \lambda^{n+1}. \tag{A55}
\]

Therefore, we can continue the induction provided that \( K_{n+1}/K_n \leq 1/2\lambda \). Since \( K_{n+1}/K_n = Cm(n) \), this is true provided that \( n \leq n_* \). This completes the proof of Theorem 1'.

Appendix B: Proof of phase-winding when a U(1) symmetry is spontaneously broken

Here we intend to prove the claim made in Section IV A above that the expectation value

\[
\text{Tr}(\rho_X e^{i\mathcal{H} t} \Phi e^{-i\mathcal{H} t}) \equiv g_X(t) \tag{B1}
\]

must be independent of time \( t \), where we have defined \( K = D - \mu L \) and \( \rho_X \equiv \lim_{t \to \infty} \frac{1}{2\pi} e^{-\beta(K+X)} \). The idea is to make a connection with results of Ref. \( \text{[42]} \) however, these were expressed in terms of \textit{two-point} correlation functions, and also did not have the \( eX \) term in the definition of the density matrix. To make a connection, we assume that the symmetric density matrix \( \rho = \frac{1}{2} e^{-\beta K} \) can be recovered by symmetrizing a symmetry-breaking state,

\[
\rho = \frac{1}{2\pi} \int_0^{2\pi} e^{-i\theta L} \rho_X e^{i\theta L} d\theta, \tag{B2}
\]

and that the symmetry-breaking state \( \rho_X \) is short-range correlated. Now we calculate the two-point correlation function (where \( \Phi(x) \) and \( \Phi(y) \) are two operators acting at different spatial locations \( x \) and \( y \)
where we used the fact that $L$ and $K$ commute and that $e^{i\theta L}e^{-i\theta L} = e^{i\theta}. In the last line we sent $|x - y| \to \infty$ and used the assumption that $\rho_X$ has short-range correlations.

Now, the theorem of Ref. [42] rigorously proves that the function $f(t)$ must be independent of time. Hence, unless $g_X(0) = 0$, we conclude that $g_X(t)$ must be independent of time. (If $g_X(0) = 0$ but $g_X(t)$ is not independent of time then there must be some $t$ such that $g_X(t) \neq 0$. Then we can just relabel the time-coordinate so that $g_X(0) \neq 0$ and repeat the argument.)

\[ f(t) = \text{Tr}[\rho e^{itK}\Phi(x)e^{itK}\Phi(y)] \]
\[ = \frac{1}{2\pi} \int_0^{2\pi} d\theta \text{Tr}[e^{-i\theta L}\rho_X e^{i\theta L}e^{itK}\Phi(x)e^{-itK}\Phi(y)] \]
\[ = \frac{1}{2\pi} \int_0^{2\pi} d\theta \text{Tr}[\rho_X e^{i\theta K}\{e^{i\theta L}\Phi(x)e^{-i\theta L}\}e^{-itK}\{e^{i\theta L}\Phi(y)e^{-i\theta L}\}] \]
\[ = \text{Tr}[\rho_X\{e^{-itK}\Phi(x)e^{itK}\Phi(y)\}] \]
\[ = g_X(t)[g_X(0)]^*, \]

\[ \text{(B3)} \]
\[ \text{(B4)} \]
\[ \text{(B5)} \]
\[ \text{(B6)} \]
\[ \text{(B7)} \]

\[ \text{Appendix C: Open systems} \]

In this section, we will elaborate on our hypothesis for open systems introduced in Section [VI] above, namely that in a large class of systems the steady state will have low energy. First we need to clarify what we mean by “energy” and “steady state” in the Floquet context. Let $H_S(t)$ be the time-evolution of the system alone (not taking to account the coupling to the environment.) We define the Floquet operator $U_t = T \exp\left(-i \int_0^T H_S(t)dt\right)$. Recall that in the regime discussed in Section [III] where $\lambda T \ll 1$, we can write $H_S(t) = H_X(t) + V(t)$. Here $V(t)$ is a very weak residual perturbation, and $H_X(t)$ is such that, if we define the approximate Floquet operator by $\tilde{U}_t = T \exp\left(-i \int_0^T H_X(t)\right)$, then it can be expressed, following a local unitary time-independent change of basis (which we will here set to 1 for notational simplicity), as $\tilde{U}_t = X e^{-iDT}$, where $X^2 = 1$ and $D$ is a quasi-local Hamiltonian $D$ that commutes with $X$. In particular, we have $\tilde{U}_t^2 = e^{-2iDT}$. This implies that we can make a time-dependent local unitary change of basis $W(t)$, periodic with period $2T$ and satisfying $W(0) = 1$, such that the transformed Hamiltonian, which is related to $H_S(t)$ according to

\[ \tilde{H}' = WH_S W^\dagger + i[\partial_t W] W^\dagger, \]

\[ \text{(C1)} \]
is time-independent and equal to $D$. Therefore, in this new reference frame, it is clear that we should refer to the expectation value of $D$ as “energy”. We emphasize that we have not gotten rid of the time-dependence completely: even in the new reference frame the residual driving term $V(t)$, as well as any couplings to the environment, will still be time-dependent. (Due to the time-dependent change of basis, the latter will gain a time-dependence even if it was originally time-independent.)

The steady state is now determined by some balance between the residual periodic driving $V(t)$, the classical noise, and the coupling to the environment. We leave a detailed analysis of this open system process for future work[43], but we expect that in a suitable regime the energy-density of the steady state will be low. We will now explain why this implies oscillations (which are observed in the original reference frame, not the rotating one defined above.)

Consider a short-range correlated steady state $\rho$ whose energy density with respect to $D$ is small. Recall that in Section [III] we argued that if $\rho$ is a thermal state it must spontaneously break the symmetry generated by $X$, and it follows that under $\tilde{U}_t$ it oscillates at twice the drive frequency. Of course, for an open system the steady-state need not be thermal, and time evolution of the open system is not exactly given by $\tilde{U}_t$. However, as we prove in Appendix [D] even non-thermal states must fail to be invariant under the symmetry $X$ if their energy density with respect to $D$ is sufficiently small, provided that they satisfy a physically reasonable “thermalizability” condition. Moreover, if $\lambda T \ll 1$ (so that we can approximate $\tilde{U}_t \approx X$), and the coupling to the environment sufficiently weak, then the resulting state after one time period is approximately given by $X \rho X^\dagger$, which by the preceding discussion is not the same as $\rho$. (We make this argument more precise in Section [D].) Thus, provided that the energy of the steady-state is sufficiently small, it does not return to itself after one time period, and oscillations with period $2T$ will be observed.

Generic baths will destroy continuous-time time crystals. The difference with the discrete-time case is the existence of an extra variable characterizing thermal states of $D$: namely, the chemical potential $\mu$. This extra vari-

\[ \text{\footnote{For one study of steady states of many-body Floquet systems coupled to a bath, see Ref. [95]}} \]
able is needed because of the presence of the hidden $U(1)$ symmetry in the continuous-time regime. (There is no analogous variable when the hidden symmetry is discrete.) Thus, one certainly cannot make any statement that all low-energy states of $D$ oscillate, because, in particular, a thermal state of $D$ in which the electrochemical potential $\mu - u = 0$ does not oscillate. A coupling to a generic bath will not preserve the hidden $U(1)$ symmetry, and thus to the extent that the steady state of an open system process is close to a thermal state of $D$, we in fact expect it to have $\mu - u = 0$, since this corresponds to minimizing the free energy.

In principle, one could fine-tune the bath so that it respects the symmetry. This would allow the time crystal to survive, but is clearly contrived. One might wonder whether the bath itself could also pre-thermalize: if we try, and thus to the extent that the steady state of an open system process is close to a thermal state of $A$, we have $\|\rho - X^k\rho X^{-k}\|_A \geq \gamma$ for all $0 < k < N$.

Now let $\rho$ be any state (not necessarily thermal) such that the energy density $\epsilon \equiv \langle D \rangle_\rho / V < \epsilon_c$ (with $V$ the volume of the system.) We assume the following thermalizability condition, which roughly states that $\rho$ can thermalize when time-evolved under $D$. More precisely:

**Assumption 1** (Spontaneous symmetry-breaking). There exists some finite region $A$ and some $\gamma > 0$, such that, for any short-range correlated thermal state $\rho_\tau$ with energy density $\epsilon < \epsilon_c$, we have $\|\rho_\tau - X^k\rho_\tau X^{-k}\|_A \geq \gamma$ for all $0 < k < N$.

From Assumptions 1 and 2 we derive the following lemma, which quantifies the sense in which the state $\rho$ must break the symmetry.

**Lemma 1.** There exists a finite region $A'$ such that $\|\rho - X^k\rho X^{-k}\|_{A'} \geq 3\gamma / 4$.

**Proof.** From the triangle inequality it follows that

\[
\|\rho(t_1) - X^k\rho(t_1)X^{-k}\|_A \\
\geq \|\rho - X^k\rho X^{-k}\|_A - \|\rho(t_1) - X^k\rho(t_1)X^{-k} - (\rho - X^k\rho X^{-k})\|_A \\
\geq \gamma - 2\gamma / 8 \\
= 3\gamma / 4.
\]

Using the characterization of the trace norm as

\[
\|\rho\|_1 = \sup_{\hat{o} : \|\hat{o}\| = 1} |\langle \hat{o} \rangle_{\rho}|, \tag{D6}
\]

it follows that there exists an operator $\hat{o}_A$ supported on $A$, with $\|\hat{o}_A\| = 1$, such that $|\langle X^{-k}\hat{o}_A X^k - \hat{o}_A \rangle_{\rho(t_1)}| \geq 3\gamma / 4$. Now, since $D$ is quasi-local, it must obey a Lieb-Robinson bound [99, 100], which implies that there exists a local operator $\hat{O}_{A'}$ supported on a finite region $A'$ such that $|\hat{o}(t_1) - \hat{O}_{A'}| \leq \gamma / 8$, where $\hat{o}(t_1) = e^{iDt_1, \hat{o}} e^{-iDt_1}$.

Hence we see that

\[
|\langle X^{-k}\hat{O}_{A'} X^k - \hat{O}_{A'} \rangle_{\rho}| \\
\geq -\gamma / 4 + |\langle X^{-k}\hat{o}_A(t_1)X^k - \hat{o}_A \rangle_{\rho(t_1)}| \\
= -\gamma / 4 + |\langle X^{-k}\hat{o}_A X^k - \hat{o}_A \rangle_{\rho(t_1)}| \\
\geq -\gamma / 4 + 3\gamma / 4, \tag{D9}
\]

To get to line Eq. (D9), we used the fact that $X$ and $D$ commute. The lemma follows.

Now consider a system which in isolation would evolve under a time-dependent Hamiltonian $H(t)$, which is periodic with period $T$. We assume that $H(t)$ exhibits
the pre-thermalization phenomena discussed in the main text. That is, we assume that the Floquet operator can be approximated according to $U_t \approx \tilde{U}_t = X e^{-iDT}$, where $D$ is quasi-local and commutes with $X$, and where $U_t$ is close to $\tilde{U}_t$ in the sense that

$$\|U_t^\dagger O_A' U_t - \tilde{U}_t^\dagger O_A' \tilde{U}_t\| \leq \frac{\gamma}{8} \|O_A\| \quad (D12)$$

for any operator $O_A$ supported on $A'$.

Let $\rho_{\text{open}}(t)$ be the reduced state of the system (tracing out the bath) at time $t$, taking into account the system-bath coupling, and we assume that $\rho_{\text{open}}(0) \equiv \rho$ satisfies Assumption 2 above. We assume the coupling to the bath is sufficiently weak, in the following sense:

**Assumption 3** (Weak coupling). For any time $0 \leq t \leq T$, we have $\|\rho_{\text{int}}^\dagger(t) - \rho\|_{A'} \leq \gamma/8$.

Here we defined the interaction picture state $\rho_{\text{int}}^\dagger(t) = U(0,t)^{-1} \rho_{\text{open}}(t) U(0,t)$, where $U(0,t)$ is the time evolution generated by $H(t)$. If we were to set the coupling to the bath to zero then the state $\rho_{\text{open}}^\dagger(t)$ would be constant in time, so Assumption 3 corresponds to weak coupling. Finally, we will assume that the strength of $DT$ is small enough so that

**Assumption 4.** For any observable $O_{A'}$ supported on $A'$, we have

$$\|e^{-iDT} O_{A'} e^{iDT} - O_{A'}\| \leq \frac{\gamma}{8} \|O_{A'}\| \quad (D13)$$

This will always be true in the regime of interest, $\lambda T \ll 1$ (where $\lambda$ is as defined in Section II), because $\|D\|_{\ast}$ is $O(\lambda)$ [see Eq. (9) in Theorem I].

From the above assumptions we can now derive our main result:

**Theorem 3.**

$$\|\rho_{\text{open}}(T) - \rho\|_{A'} \geq \gamma/8. \quad (D14)$$

**Proof.**

$$\|\rho_{\text{open}}(T) - \rho\|_{A'} = \|U_{\text{open}}^\dagger(T) U_{\text{int}}^\dagger - \rho\|_{A'} \geq -\gamma/8 + \|\rho_{\text{int}}(T) e^{iDT} - X^\dagger \rho X\|_{A'} \geq -\gamma/8 - \gamma/8 + \|\rho_{\text{open}}(T) - \rho\|_{A'} \geq -\gamma/8 - \gamma/8 + \|\rho - X^\dagger \rho X\|_{A'} \geq -\gamma/8 - \gamma/8 + \gamma/2. \quad (D21)$$

$$= \gamma/8. \quad (D22)$$

In other words, the state of the open system at times $t = T$ and $t = 0$ are locally distinguishable. That is, for the stated assumptions, the state of the system does not synchronize with the drive and time translation symmetry is spontaneously broken.

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