Some Generalizations of the Hellinger Theorem for Second Order Difference Equations with Matrix Elements

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Abstract

We obtain several generalizations the Hellinger theorem about $l^2$ solutions of difference equations: instead of second order equations and $l^2$-solutions, we consider second-order equations with matrix coefficients and their solutions in $l^p$, $1 \leq p \leq \infty$. In particular it is shown that for a certain class of symmetric difference equations an analog of this theorem holds for $1 \leq p \leq 2$, but it does not hold for $p > 2$.

1 Introduction

In the study of the spectral properties of infinite Jacoby matrices and analytic properties of continued $J$-fractions, a significant place belongs to the result established by E. Hellinger [1, 2, 3]:

**Theorem.** Suppose that for some $z = z_0 \in \mathbb{C}$, any solution $u = u(z) = (u_i(z))_{i=0}^{\infty}$ of the infinite system of the difference equations

$$a_{i-1}u_{i-1} + b_i u_i + a_i u_{i+1} = z u_i, \quad i \geq 1,$$

$$a_i, b_i \in \mathbb{C}, \quad a_i \neq 0,$$

satisfies the condition $\sum_{i=0}^{\infty} |u(z_0)|^2 < \infty$ (and therefore, belongs to the space $l^2$). Then, for any $z \in \mathbb{C}$ and $M > 0$, the series $\sum_{i=0}^{\infty} |u(z)|^2$ converges uniformly for $|z - z_0| < M$.

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This theorem was applied to the study of the essential spectrum of second order difference operators in \[4\]. It should be observed that since \( z_0 \) is an arbitrary complex number, the Hellinger theorem makes the study of the deficiency indices of symmetric second order difference operators more simple than a similar investigation for differential Sturm-Liouville operators. For some classes of difference operators of an arbitrary order, an extension of the Hellinger theorem was obtained in \[5\] (with the space \( l^2 \) replaced by \( l^p, 1 \leq p \leq \infty \)). The goal of this paper is to find a similar extension for the second order difference equations with matrix coefficients. These issues are essential in the analysis of properties of continued fractions with matrix (or operator) elements \[6, 7, 11\].

2 Preliminaries

As mentioned above, there is a connection between the Hellinger theorem and some spectral properties of the infinite Jacoby matrices. Instead of a Jacoby matrix, here we consider the infinite three-diagonal matrix \( A = (A_{i,j})_{i,j=0}^{\infty} \)

\[
A = \begin{pmatrix}
A_{0,0} & A_{0,1} & 0 & O & \cdots \\
A_{1,0} & A_{1,1} & A_{1,2} & O & \cdots \\
O & A_{2,1} & A_{2,2} & A_{2,3} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix},
\]

where \( A_{i,j} \) is a square matrix of order \( n \) whose elements are complex numbers; \( O \) is a zero matrix of order \( n \). Also assume that \( A_{i+1,i}, A_{i,i+1}, i \geq 0 \) are invertible. Then \( A \) generates a linear operator in the space \( l^2_n \) of sequences \( u = (u_0, u_1, \ldots) \), where the vector column \( u_j \in \mathbb{C}^n \) with inner product \((u, v) = \sum_{j=0}^{\infty} v_j^* u_j\). For this operator we will keep the same notation \( A \).

To the matrix \( A \) we assign second-order finite-difference equations in the matrices \( Y_j, Y_j^+ \) of order \( n \):

\[
l(Y)_j \equiv A_{j,j-1} Y_{j-1} + A_{j,j} Y_j + A_{j,j+1} Y_{j+1} = z Y_j, \quad (1)
\]

\[
l^+(Y^+)_j \equiv Y_{j-1}^+ A_{j-1,j} + Y_j^+ A_{j,j} + Y_{j+1}^+ A_{j+1,j} = z Y_j^+, \quad (2)
\]

\[j \geq 0, \; z \in \mathbb{C}; \quad A_{0,-1} = A_{-1,0} = -E,\]

where \( E \) is a unit matrix.
Denote by $P(z) = \{P_j(z)\}_{j=-1}^{\infty}$, $Q(z) = \{Q_j(z)\}_{j=-1}^{\infty}$, $P^+(z) = \{P_j^+(z)\}_{j=-1}^{\infty}$, $Q^+(z) = \{Q_j^+(z)\}_{j=-1}^{\infty}$ the solutions of (1) and (2) respectively, satisfying the initial conditions

$$P_{-1}(z) = P_{-1}^+(z) = Q_0(z) = Q_0^+(z) = E;$$
$$P_0(z) = P_0^+(z) = Q_{-1}(z) = Q_{-1}^+(z) = O;$$

These solutions are matrix polynomials in $z$; $Q(z), Q^+(z)$ and $P(z), P^+(z)$ are analogs of the polynomials of the first and the second kind for scalar Jacobi matrices [8]. They play an important role in the spectral analysis of the corresponding operator $A$. For example, if $A$ is bounded, then for any $z$ from its resolvent set

$$\lim_{m \to \infty} \|Q_m(z)\|_m > 1, \quad \lim_{m \to \infty} \|Q_+^m(z)\|_m > 1,$$ see [9] for more details. The following identities, which can be verified by induction on $j \geq 0$, are valid [9]:

$$P_jQ_j^+ - Q_jP_j^+ = O; \quad P_{j+1}Q_j^+ - Q_{j+1}P_j^+ = A_{j,j+1}^{-1}; \quad Q_jP_j^+ - P_jQ_{j+1}^+ = A_{j+1,j}^{-1};$$

$$Q_{j+1}^+A_{j+1,j}Q_j - Q_j^+A_{j,j+1}Q_{j+1} = P_{j+1}^+A_{j+1,j}P_j - P_j^+A_{j,j+1}P_{j+1} = O;$$
$$P_{j+1}^+A_{j+1,j}Q_j - P_j^+A_{j,j+1}Q_{j+1} = Q_j^+A_{j,j+1}P_{j+1} - Q_{j+1}^+A_{j+1,j}P_j = E,$$ (3)

(4)

(the parameter $z$ is omitted for convenience of notation).

Now let $F_j, \; j \geq 0$ be an arbitrary sequence of $n \times n$ matrices. Assuming that $P(z), Q(z)$ and $P^+(z), Q^+(z)$ are known, we solve the inhomogeneous equations

$$l(U)_j - zU_j = F_j,$$ 
$$l^+(U^+_j) - zU^+_j = F_j,$$ 
$$j \geq 0, \; z \in \mathbb{C}$$

by variation of constants on setting

$$U_j(z) = Q_j(z)C_j^1 + P_j(z)C_j^2,$$
$$U_j^+(z) = C_j^{1+}Q_j^+(z) + C_j^{2+}P_j^+(z).$$
Lemma 1. For the matrices $C^1_j$, $C^2_j$, $C^{1+}_j$, $C^{2+}_j$ the following recursive representations are valid:

\[ C^1_j = C^1_k - \sum_{i=k}^{j-1} P^+_i(z) F_i, \quad (9) \]
\[ C^2_j = C^2_k + \sum_{i=k}^{j-1} Q^+_i(z) F_i; \quad (10) \]
\[ C^{1+}_j = C^{1+}_k - \sum_{i=k}^{j-1} F_i P_i(z), \quad (11) \]
\[ C^{2+}_j = C^{2+}_k + \sum_{i=k}^{j-1} F_i Q_i(z); \quad (12) \]

\[ k = 0, 1, \ldots \quad j = k + 1, k + 2, \ldots \]

Also, for \( j = -1, 0 \)

\[ U_j(z) = Q_j(z) C^1_0 + P_j(z) C^2_0; \quad U^+_j(z) = C^1_0 + Q^+_j(z) + C^{2+}_0 + P^+_j(z), \]

where $C^1_0, C^2_0, C^{1+}_0, C^{2+}_0$ - arbitrary constant matrices.

Proof. For \( j \geq 0 \) consider the system

\[
\begin{cases}
Q_j(z) \Delta C^1_{j+1} + P_j(z) \Delta C^2_{j+1} = 0, \\
A_{j,j+1}(Q_{j+1}(z)) \Delta C^1_{j+1} + P_{j+1}(z) \Delta C^2_{j+1} = F_j
\end{cases}
\]

where $\Delta C^i_{j+1} = C^i_{j+1} - C^i_j, i = 1, 2$. One can check by direct substitution that if the matrices $C^i_j$ are chosen in this way, then the sequence $U_j(z)$ defined by (7) is a solution of (5). Let us show that this system has a unique solution. Indeed, multiplying the first equation of the system on the left by $P^+_j(z) A_{j+1,j}$ and the second equation by $-P^+_j(z)$ and summing the resulting equations, we obtain

\[
(P^+_j(z) A_{j+1,j} Q_j(z) - P^+_j(z) A_{j,j+1} Q_{j+1}(z)) \Delta C^1_{j+1} + \\
+(P^+_j(z) A_{j+1,j} P_j(z) - P^+_j(z) A_{j,j+1} P_{j+1}(z)) \Delta C^2_{j+1} = -P^+_j(z) F_j
\]

Taking into account the identities (4), we find that

\[ \Delta C^1_{j+1} = -P^+_j(z) F_j, \quad (13) \]
Using similar arguments, we can show that
\[ \Delta C_{j+1}^2 = Q_j^+ F_j. \]  
(14)
Hence the above system has a unique solution. Summing (13) and (14) by \( i = k, k + 1, \ldots, j - 1 \) we finally obtain (9) and (10). The formulas (11)-(12) can be obtained in a similar manner by using the system
\[
\begin{cases}
\Delta C_{j+1}^1 Q_j^+(z) + \Delta C_{j+1}^2 P_j^+(z) = 0, \\
(\Delta C_{j+1}^1 Q_{j+1}^+(z) + \Delta C_{j+1}^2 P_{j+1}^+(z))A_{j+1,j} = F_j
\end{cases}
\]
with respect to \( \Delta C_{j+1}^{i,j} = C_{j+1}^{i,j} - C_{j}^{i,j}, i = 1, 2. \)

3 Main results

Now for \( 1 \leq p \leq \infty \) consider the Banach spaces \( l_p^n \) of sequences \( u = (u_{-1}, u_0, u_1, \ldots) \), such that the vector column \( u_j \in \mathbb{C}^n \), with the norm \( \|u\|_p = (\sum_j |u_j|^p)^{1/p} < \infty, 1 \leq p < \infty \), where \( |\cdot| \) is a certain vector norm. For the case \( p = \infty \|u\|_\infty = \text{sup}_j |u_j| \).

Alongside with (1)-(2), consider the equations in the vectors \( u_j, v_j \in \mathbb{C}^n \):
\[
\begin{align*}
l(u)_j &\equiv A_{j,j-1} u_{j-1} + A_{j,j} u_j + A_{j,j+1} u_{j+1} = z u_j, \quad (15) \\
l^+(v^*)_j &\equiv v^*_{j-1} A_{j-1,j} + v^*_j A_{j,j} + v^*_{j+1} A_{j+1,j} = z v^*_j, \quad (16)
\end{align*}
\]
where \( j \geq 0, z \in \mathbb{C} \).

Since the polynomials \( P(z), Q(z) \) and \( P^+(z), Q^+(z) \) form the fundamental systems of solutions of the equations (1) and (2) respectively, one can easily see that if all the solutions of (15) belong to the space \( l^p_n \), then \( M^p_k(z) \to \infty \) as \( k \to \infty \), where
\[
M^p_k(z) \equiv \max \left\{ \left( \sum_{j=k}^{\infty} \|P_j(z)\|^p \right)^{1/p}, \left( \sum_{j=k}^{\infty} \|Q_j(z)\|^p \right)^{1/p} \right\}
\]
(17)
and \( \|\cdot\| \) is a matrix norm. Similarly, if all the solutions of (16) belong to the space \( l^q_n \), \( 1 \leq q < \infty \), then \( M^{q+}_k(z) \to \infty \) as \( k \to \infty \), where
\[
M^{q+}_k(z) \equiv \max \left\{ \left( \sum_{j=k}^{\infty} \|P^+_j(z)\|^q \right)^{1/q}, \left( \sum_{j=k}^{\infty} \|Q^+_j(z)\|^q \right)^{1/q} \right\}
\]
(18)
Theorem 1. If all solutions of (13) are in $l^p_n$, $1 \leq p \leq \infty$ for some $z = z_0 \in \mathbb{C}$ and all solutions of the equation (16) are in $l^q_n$, $1/q + 1/p = 1$, then for any $z \in \mathbb{C}$ all solutions of (13) and (16) are in $l^p_n$ and $l^q_n$ respectively.

Proof. First consider the case $1 < p < \infty$. The equation (11) can be written in the form

\[ l(Y(z))_j - z_0Y_j(z) = (z - z_0)Y_j(z), \quad j \geq 0. \]

This equation can be reduced to (5) by setting $F_j = (z - z_0)Y_j$. Substituting (9) and (10) into (7) we obtain the following representation for $Y_j(z)$:

\[
Y_j(z) = Q_j(z_0)C_k^1 + P_j(z_0)C_k^2 + \\
+ (z - z_0) \sum_{i=k}^{j-1} (P_j(z_0)Q_i^+(z_0) - Q_j(Z_0)P_i^+(z_0)) Y_i(z), \tag{19}
\]

\[ j = k + 1, k + 2, \ldots \quad k = 0, 1, \ldots. \]

Consider the latter sum in the above equation. Using the matrix norm properties and applying the Hölder inequality, we find

\[
\| \sum_{i=k}^{j-1} (P_j(z_0)Q_i^+(z_0) - Q_j(Z_0)P_i^+(z_0)) Y_i(z) \| \leq \\
\leq \| P_j(z_0) \| \| \sum_{i=k}^{j-1} \| Q_i^+(z_0) \| \| Y_i(z) \| + \| Q_j(z_0) \| \| \sum_{i=k}^{j-1} \| P_i^+(z_0) \| \| Y_i(z) \| \leq \\
\leq (\| P_j(z_0) \| + \| Q_j(z_0) \|) M_k^{q^+}(z_0) \left( \sum_{i=k}^{j-1} \| Y_i(z) \|^p \right)^{1/p},
\]

where $M_k^{q^+}(z_0)$ is defined by (18). Set $N_{k,j}^p = \left( \sum_{i=k}^{j-1} \| Y_i(z) \|^p \right)^{1/p}$ then for $Y_j(z)$ we get

\[
\| Y_j(z) \| \leq (C_k + |z - z_0| M_k^{q^+}(z_0) N_{k,j}^p)(\| P_j(z_0) \| + \| Q_j(z_0) \|), \quad j = k+1, k+2, \ldots.
\]

where $C_k = \max C_k^1, C_k^2$. Obviously, the above inequality also holds when $j = k$. Therefore, for $i = k, k+1, \ldots, j - 1$ we have:

\[
\| Y_i(z) \| \leq (C_k + |z - z_0| M_k^{q^+}(z_0) N_{k,j}^p)(\| P_i(z_0) \| + \| Q_i(z_0) \|).
\]
Now we raise both sides of these inequalities to the $p$-th power and perform the summation from $i = k$ to $j - 1$. Then, extracting the $p$-th root from the both sides and applying the Minkowski inequality, we finally get

$$N_{k,j}^p \leq 2C_k M_k^p(z_0) + 2|z - z_0| M_k^{q,+}(z_0) M_k^p(z_0) N_{k,j}^p.$$ 

From the assumption of the theorem it follows that both $M_k^p(z_0)$ and $M_k^{q,+}(z_0)$ tend to zero as $k \to \infty$. Thus we can find an index $k_0$ such that for $k \geq k_0$

$$|z - z_0| M_k^{q,+}(z_0) M_k^p(z_0) \leq \frac{1}{4},$$

and therefore

$$N_{k,j}^p \leq 4C_k M_k^p(z_0).$$

The right-hand side of the above inequality is independent of $j$, and therefore, $N_{k,j}^p$ has a limit as $j \to \infty$. Since \{Y_i(z)\} is an arbitrary sequence, it implies that all the solutions of (15) belong to the space $l^p_n$. For the equation (2) written in the form

$$l^+(Y^+(z))_j - z_0 Y^+_j(z) = (z - z_0) Y^+_j(z), \quad j \geq 0,$$

we obtain the following representation for $Y^+_j(z)$ by substituting (11)-(12) into (8):

$$Y^+_j(z) = C_1^1 J^+_j(z_0) + C_2^1 J^+_j(z_0) +$$

$$+(z - z_0) \sum_{i=k}^{j-1} Y^+_i(z) \left(Q_i(z_0) P^+_j(z_0) - P_i(z_0) Q^+_j(z_0)\right),$$

where

$$j = k + 1, k + 2, \ldots \quad k = 0, 1, \ldots.$$

By applying to this formula the same arguments as to (19), we find that all the solutions of (16) belong to the space $l^q_n$.

The case $p = 1(q = \infty)$ is considered separately on the basis of similar arguments applied to (19) and (20). Here instead of $M_k^{q,+}(z_0)$ one can take

$$M^+(z_0) = \max \{\sup_j \|P^+_j(z_0)\|, \sup_j \|Q^+_j(z_0)\|\},$$

and $M_k^p(z_0) = M_k^1(z_0)$ is same as above.

A closer examination of the above proof allows one to establish the following generalization of the result obtained.
Theorem 2. If all solutions of the equations
\[ l(u)_j = 0_n, \quad l^+(v)_j = 0^*_n, \quad j \geq 0, \]
where the zero vector column $0_n \in \mathbb{C}^n$ belong to the spaces $l^p_n$ and $l^q_n$ respectively, where $1/p + 1/q = 1$, then this is also true for the solutions of perturbed equations
\[ l(u)_j = F_j u_j, \quad l^+(v)_j = v_j^* G_j, \quad j \geq 0, \]
where $F_j$ and $G_j \in \mathbb{C}^{n \times n}$, and the conditions
\[ \sup_{j \geq 0} \|F_j\| < \infty; \quad \sup_{j \geq 0} \|G_j\| < \infty \]
are held.

Note that because of the embedding $l^p_n \supset l^q_n$ for $p_1 < p_2$ we have that if $1 \leq p \leq 2$ and the condition of the Theorem 1 is fulfilled, then all the solutions of (15) and (16) are in $l^p$ for any $z \in \mathbb{C}$.

Now consider the matrix $A$ in the symmetric case
\[ A_{j,j} = A^*_{j,j}, \quad A_{j+1,j} = A_{j,j+1} > 0, \quad (\text{here } * \text{ denotes Hermitian conjugation}) \]
so $A$ is a Jacobi matrix. In this case we have $P^+(z) = P^*(z)$ and $Q^+(z) = Q^*(z)$ for $z \in \mathbb{R}$ (and the equation (16) is a conjugate to (15)). In view of the above, we get the following result:

Theorem 3. If all solutions of the equation (15) with matrix coefficients satisfying (21) for some $z = z_0 \in \mathbb{R}$ belong to the space $l^p_n$, $1 \leq p \leq 2$, then this is also true for any $z \in \mathbb{C}$.

For the case $p = 2$ this theorem was proved in [10] (Theorem 1). Now consider the case $p > 2$. As an example, take the following matrix $A$:
\[ A_{j,j} = O, \quad A_{j+1,j} = A_{j,j+1} = (j + 1)E, \quad j \geq 0, \]
where $O$ and $E$ are zero and unit matrices of the second order. Then for the corresponding equation (15) where $z = 0$, all its solutions are in $l^{2+\epsilon}_2$ for any $\epsilon > 0$ (one easily find by direct calculation of $P_n(0)$ and $Q_n(0)$ that both $\|P_n(0)\|$ and $\|Q_n(0)\|$ $\sim n^{-1/2}$ as $n \rightarrow \infty$, and therefore the condition (17) holds in this case). However if we take $z = i$ (the imaginary unit) or $z = -i$, then there exist the solutions of (15) which belong to $l^\infty_2$, but not tend to
zero as $j \to \infty$. Note that for a scalar Jacobi matrix case a similar result was obtained in [5] by using the grouping in block approach offered in [12]. Here we can apply similar arguments. Thus we are coming to the following conclusion:

**Theorem 4.** The Theorem 3 is not valid for $p > 2$.

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