_INTRODUCTION AND PRELIMINARIES

Consider the closed ball $\bar{B}(C,R)$ in the space $\mathbb{R}^3$ with center $C = (a, b, c) \in \mathbb{R}^3$ and radius $R > 0$ defined as

$$\bar{B}(C,R) = \{(x, y, z) \in \mathbb{R}^3 | (x-a)^2 + (y-b)^2 + (z-c)^2 \leq R^2 \}.$$ 

Also consider $\sigma(C,R)$ as the boundary (the surface) of $\bar{B}(C,R)$, i.e.,

$$\sigma(C,R) = \{(x, y, z) \in \mathbb{R}^3 | (x-a)^2 + (y-b)^2 + (z-c)^2 = R^2 \}.$$ 

The following result has been proved in [1], which is the Hermite–Hadamard’s inequality for convex functions defined on closed ball $\bar{B}(C,R)$.

**Theorem 1.1** Let $f: \bar{B}(C,R) \rightarrow \mathbb{R}$ be a convex mapping on the ball $\bar{B}(C,R)$. Then we have the inequality

$$f(C) \leq \frac{1}{v(\bar{B}(C,R))} \iiint_{\bar{B}(C,R)} f(x,y,z) \, dx \, dy \, dz \leq \frac{1}{\sigma(\bar{B}(C,R))} \int_{\sigma(B(C,R))} f(x,y,z) \, d\sigma,$$

(1)

where $v(\bar{B}(C,R)) = \frac{4\pi R^3}{3}$ and $\sigma(\bar{B}(C,R)) = \frac{4\pi R^2}{3}$. 

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The main purpose of this paper is estimating two bounds $B_1$ and $B_2$ such that
\[
\left| \frac{1}{4\pi R^3} \iiint_{B(C,R)} f(x,y,z) \, dV - f(C) \right| \leq B_1, \tag{2}
\]
and
\[
\left| \frac{1}{4\pi R^3} \int_{f(C,R)} f(x,y,z) \, d\sigma - \frac{1}{4\pi R^3} \iiint_{B(C,R)} f(x,y,z) \, dV \right| \leq B_2. \tag{3}
\]
Depending on the properties of the function $f$ and the radius $R$, different values will be obtained for $B_1$ and $B_2$.

We call (2) a mid-point type inequality due to the following result obtained in [2] and interpretation of Fig. 1.

**Theorem 1.2** Let $f : I^o \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on $I^o$, $a, b \in I^o$ with $a < b$. If $|f'|$ is convex on $[a, b]$, then we have
\[
\left| \int_a^b f(x) \, dx - (b - a)f\left(\frac{a + b}{2}\right) \right| \leq \frac{1}{8}(b - a)^2 \left(|f'(a)| + |f'(b)|\right). \tag{4}
\]
According to (4), we have an estimate for the difference between the area under the graph of $f$, i.e., $\int_a^b f(x) \, dx$, and the area of rectangle $abcd$, i.e., $(b - a)f\left(\frac{a + b}{2}\right)$ (see Fig. 1).

Also we call (3) a trapezoid type inequality due to the following result and Fig. 2.
Theorem 1.3 ([3]) Let \( f : I^* \subseteq \mathbb{R} \rightarrow \mathbb{R} \) be a differentiable mapping on \( I^* \), \( a, b \in I^* \) with \( a < b \). If \( |f'| \) is convex on \( [a, b] \), then the following inequality holds:

\[
\left| \int_a^b f(x) \, dx - (b - a) \frac{f(a) + f(b)}{2} \right| \leq \frac{1}{8} (b - a)^2 \left( |f'(a)| + |f'(b)| \right). \tag{5}
\]

According to (5), we can estimate the difference between the area of trapezoid \( abcd \), i.e., \( (b - a) \frac{f(a) + f(b)}{2} \), and the area under the graph of \( f \) (see Fig. 2).

Note that to obtain (4) and (5), the absolute values of the derivative of \( f \) at boundary points of interval \([a, b]\) play a fundamental role. For more results about Hermite–Hadamard’s inequality, we refer an interested reader to [4–18] and the references therein.

Before presenting our main results, here we obtain a new representation of (1) and also give a reverse type theorem.

If we consider a convex function \( f : \tilde{D}(a, b, R) \rightarrow \mathbb{R} \) and the change of coordinates

\[
\mathcal{T} : \tilde{D}((a, b), R) \times [0, 1] \rightarrow \tilde{B}(C, R),
\]

\[
\mathcal{T}(x, y, \lambda) = (x, y, (2\lambda - 1) \sqrt{R^2 - x^2 - y^2}),
\]

where \( \tilde{D}((a, b), R) \) is a closed disk centered at the point \((a, b)\) having radius \( R > 0 \), then we obtain

\[
\int_{\tilde{B}(C, R)} \int_{\tilde{B}(C, R)} \int_{\tilde{B}(C, R)} f(x, y, z) \, dV
\]

\[
= 2 \int_{-R}^{R} \int_{\sqrt{R^2 - x^2}}^{-\sqrt{R^2 - x^2}} \int_{0}^{1} \left( 1 - \lambda \right) f \left( x, y, \sqrt{R^2 - x^2 - y^2} \right) + \lambda \left( x, y, \sqrt{R^2 - x^2 - y^2} \right) \sqrt{R^2 - x^2 - y^2} \, d\lambda \, dy \, dx
\]

\[
\leq 2 \int_{-R}^{R} \int_{\sqrt{R^2 - x^2}}^{-\sqrt{R^2 - x^2}} \int_{0}^{1} \left( 1 - \lambda \right) f \left( x, y, \sqrt{R^2 - x^2 - y^2} \right) \sqrt{R^2 - x^2 - y^2} \, d\lambda \, dy \, dx
\]

\[
+ 2 \int_{-R}^{R} \int_{\sqrt{R^2 - x^2}}^{-\sqrt{R^2 - x^2}} \int_{0}^{1} \lambda f \left( x, y, \sqrt{R^2 - x^2 - y^2} \right) \sqrt{R^2 - x^2 - y^2} \, d\lambda \, dy \, dx
\]

\[
= \int_{-R}^{R} \int_{\sqrt{R^2 - x^2}}^{-\sqrt{R^2 - x^2}} f \left( x, y, \sqrt{R^2 - x^2 - y^2} \right) \sqrt{R^2 - x^2 - y^2} \, dy \, dx
\]

\[
+ \int_{-R}^{R} \int_{\sqrt{R^2 - x^2}}^{-\sqrt{R^2 - x^2}} f \left( x, y, \sqrt{R^2 - x^2 - y^2} \right) \sqrt{R^2 - x^2 - y^2} \, dy \, dx. \tag{7}
\]

Choosing \( z = \sqrt{R^2 - x^2 - y^2} \) in the latter integrals, the fact that \( \sqrt{1 + \left( \frac{ay}{R} \right)^2 + \left( \frac{bx}{R} \right)^2} = \frac{R}{\sqrt{R^2 - x^2 - y^2}} = \frac{R}{z} \), and using the surface integral formula for \( \sigma(C, R) \) imply that

\[
\int_{\tilde{B}(C, R)} \int_{\tilde{B}(C, R)} \int_{\tilde{B}(C, R)} f(x, y, z) \, dV \leq \frac{1}{R} \int_{\sigma(C, R)} f(x, y, z) \, z^2 \, d\sigma. \tag{8}
\]

Inequality (8) gives another representation for (1).

In a special case for a convex function \( f : \tilde{D}(a, b, R) \rightarrow \mathbb{R} \) we get

\[
\int_{\tilde{D}(a, b, R)} f(x, y) \, dA \leq \frac{1}{R} \int_{\sigma((a, b), R)} f(x, y) \, y^2 \, d\sigma.
\]
Now for a reverse type result, consider a continuous function \( f \) defined on a convex subset \( V \subset \mathbb{R}^3 \) such that (8) holds for all closed balls included in \( V \). Then \( f \) is convex on \( V \) because otherwise there would exist \( x, y \in V \) and \( \lambda \in (0, 1) \) such that

\[
\begin{align*}
\lambda f(x) + (1 - \lambda)f(y) &< f(\lambda x + (1 - \lambda)y) \quad \text{(9)}
\end{align*}
\]

Since \( f \) is continuous on \( V \), we can find \( R > 0 \) and a point \( z = (\bar{a}, \bar{b}, \bar{c}) \) in a convex combination of \( x \) and \( y \) such that (9) holds on the whole of \( \bar{B}((\bar{a}, \bar{b}, \bar{c}), R) \subset V \). So by the change of coordinates (6) and structure presented in (7) for \( \bar{D}((\bar{a}, \bar{b}), R) \) and \( \bar{B}((\bar{a}, \bar{b}, \bar{c}), R) \), we obtain

\[
\int_{\bar{B}((\bar{a}, \bar{b}, \bar{c}), R)} f(x, y, z) dV > \frac{1}{R} \int_{\sigma((\bar{a}, \bar{b}, \bar{c}), R)} f(x, y, z)^2 d\sigma,
\]

which is a contradiction and this proves the convexity of \( f \) on \( V \).

In the following sections we consider convex, Lipschitz, and bounded functions to obtain some trapezoid and mid-point type inequalities on a closed ball. We use the spherical coordinates in calculating the integrals.

### 2 Convex functions

In this section we obtain trapezoid and mid-point type inequalities for the case that the partial derivative absolute values of a considered function with respect to the radius in spherical coordinates is convex. We need the following lemma.

**Lemma 2.1** For an integrable function \( f : \bar{B}(C, R) \to \mathbb{R} \), we have

\[
\begin{align*}
\int_{\bar{B}(C, R)} f(x, y, z) dV &= \int_{0}^{2\pi} \int_{0}^{\pi} \int_{0}^{R} f(a + \rho \cos \theta \sin \varphi, b + \rho \sin \theta \sin \varphi, c + \rho \cos \varphi) \\
&\quad \times \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta \
\end{align*}
\]

and

\[
\begin{align*}
\int_{\sigma((C, R))} f(x, y, z) d\sigma &= \int_{0}^{2\pi} \int_{0}^{\pi} \int_{0}^{R} f(a + R \cos \theta \sin \varphi, b + R \sin \theta \sin \varphi, c + R \cos \varphi) R^2 \sin \varphi \, d\rho \, d\varphi \, d\theta \\
&= \frac{3}{R} \int_{0}^{2\pi} \int_{0}^{\pi} \int_{0}^{R} f(a + R \cos \theta \sin \varphi, b + R \sin \theta \sin \varphi, c + R \cos \varphi) \\
&\quad \times \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta.
\end{align*}
\]

**Proof** Consider the spherical transformation

\[
\begin{align*}
x(\rho, \varphi, \theta) &= a + \rho \cos \theta \sin \varphi, \\
y(\rho, \varphi, \theta) &= b + \rho \sin \theta \sin \varphi, \quad \rho \in [0, R], \varphi \in [0, \pi], \theta \in [0, 2\pi]. \\
z(\rho, \varphi, \theta) &= c + \rho \cos \varphi,
\end{align*}
\]
It is obvious that the Jacobian of this transformation is \( f = \rho^2 \sin \varphi \). So we have (10).

For (11), consider the curve \( \eta : [0, \pi] \times [0, 2\pi] \rightarrow \mathbb{R}^3 \) defined by

\[
\eta(\theta, \varphi) = \begin{cases} 
  x(\varphi, \theta) = a + R \cos \theta \sin \varphi, \\
  y(\varphi, \theta) = b + R \sin \theta \sin \varphi, \\
  z(\varphi, \theta) = c + R \cos \varphi,
\end{cases}
\]

\( \varphi \in [0, \pi], \theta \in [0, 2\pi] \).

It is clear that \( \eta([0, \pi] \times [0, 2\pi]) = \sigma(C, R) \) and then by integrating with respect to the surface (arc length) we get

\[
\iint_{\sigma(C,R)} f(x, y, z) \, d\sigma = \iint_{\sigma(C,R)} f(\eta) \, d\eta
\]

\[
= \int_0^{2\pi} \int_0^\pi f(x(\varphi, \theta), y(\varphi, \theta), z(\varphi, \theta))
\times \left( \left[ \frac{\partial x(\varphi, \theta)}{\partial \theta} \right]^2 + \left[ \frac{\partial y(\varphi, \theta)}{\partial \theta} \right]^2 + \left[ \frac{\partial z(\varphi, \theta)}{\partial \theta} \right]^2 \right) \frac{1}{2} d\varphi d\theta
\times \left( \left[ \frac{\partial x(\varphi, \theta)}{\partial \varphi} \right]^2 + \left[ \frac{\partial y(\varphi, \theta)}{\partial \varphi} \right]^2 + \left[ \frac{\partial z(\varphi, \theta)}{\partial \varphi} \right]^2 \right) \frac{1}{2} d\varphi d\theta
\]

\[
= \int_0^{2\pi} \int_0^\pi f(a + R \cos \theta \sin \varphi, b + R \sin \theta \sin \varphi, c + R \cos \varphi) R^2 \sin \varphi \, d\varphi d\theta
\]

\[
= \frac{3}{R} \int_0^{2\pi} \int_0^\pi f(a + R \cos \theta \sin \varphi, b + R \sin \theta \sin \varphi, c + R \cos \varphi) \rho^2 \sin \varphi \, d\rho d\varphi d\theta.
\]

This proves (11). \( \square \)

The following is a sharp trapezoid type inequality related to (1), where we consider a function with convex partial derivative (with respect to the radius \( \rho \)) absolute values defined on \( \bar{B}(C, R) \).

**Theorem 2.2** For \( \mathcal{V} \subset \mathbb{R}^3 \), suppose that \( \bar{B}(C, R) \subset \mathcal{V}^\circ \) where \( \mathcal{V}^\circ \) is the interior of \( \mathcal{V} \). Consider \( f : \bar{B}(C, R) \rightarrow \mathbb{R} \) which has continuous partial derivatives with respect to the variables \( \rho, \varphi, \) and \( \theta \) on \( \bar{B}(C, R) \) in spherical coordinates. If \( |\frac{\partial f}{\partial \rho}| \) is convex on \( \bar{B}(C, R) \), then

\[
\left| \frac{1}{4\pi R^2} \int_{\sigma(C,R)} f(x, y, z) \, d\sigma - \frac{1}{4\pi R^2} \iiint_{\bar{B}(C,R)} f(x, y, z) \, dV \right|
\]

\[
\leq \frac{1}{16\pi R} \int_{\sigma(C,R)} |\frac{\partial f}{\partial \rho}|(x, y, z) \, d\sigma.
\]

(12)

Furthermore, inequality (12) is sharp.

**Proof** For fixed \( \varphi \in [0, \pi] \) and \( \theta \in [0, 2\pi] \) and arbitrary \( \rho \in [0, R] \), since

\[
\left( \left[ \frac{\partial x(\rho, \varphi, \theta)}{\partial \rho} \right]^2 + \left[ \frac{\partial y(\rho, \varphi, \theta)}{\partial \rho} \right]^2 + \left[ \frac{\partial z(\rho, \varphi, \theta)}{\partial \rho} \right]^2 \right)^\frac{1}{2} = 1,
\]
by integration by parts we have

$$\int_0^R \frac{\partial f}{\partial \rho}(a + \rho \cos \theta \sin \varphi, b + \rho \sin \theta \sin \varphi, c + \rho \cos \varphi) \rho^3 \sin \varphi \, d\rho$$

$$= R^3 f(a + R \cos \theta \sin \varphi, b + R \sin \theta \sin \varphi, c + R \cos \varphi)$$

$$- 3 \int_0^R f(a + \rho \cos \theta \sin \varphi, b + \rho \sin \theta \sin \varphi, c + \rho \cos \varphi) \rho^2 \sin \varphi \, d\rho.$$  \hspace{1cm} (13)

So integrating with respect to \( \varphi \in [0, \pi] \) and \( \theta \in [0, 2\pi] \) in (13), along with (10) and (11) obtained in Lemma 2.1 and the convexity of \( |\frac{\partial f}{\partial \rho}| \) on \( \overline{B}(C, R) \), implies that

$$\left| R \iiint_{\sigma(C,R)} f(x,y,z) \, d\sigma - 3 \iiint_{\overline{B}(C,R)} f(x,y,z) \, dV \right|$$

$$\leq \int_0^{2\pi} \int_0^\pi \int_0^R \left( 1 - \frac{\rho}{R} \right) \left| \frac{\partial f}{\partial \rho} \right| (a, b, c) \sin \varphi \, d\rho \, d\varphi \, d\theta$$

$$+ \int_0^{2\pi} \int_0^\pi \int_0^R \rho^3 \left( 1 - \frac{\rho}{R} \right) \left| \frac{\partial f}{\partial \rho} \right| (a + R \cos \theta \sin \varphi, b + R \sin \theta \sin \varphi, c + R \cos \varphi) \sin \varphi \, d\rho \, d\varphi \, d\theta$$

$$= \frac{\pi R^4}{5} \left| \frac{\partial f}{\partial \rho} \right| (a, b, c) + \frac{R^2}{5} \iiint_{\sigma(C,R)} \left| \frac{\partial f}{\partial \rho} \right| (x,y,z) \, d\sigma. \hspace{1cm} (14)$$

By considering the left-hand side of (1) for \( |\frac{\partial f}{\partial \rho}| \) and applying it in (14), we have

$$\left| R \iiint_{\sigma(C,R)} f(x,y,z) \, d\sigma - 3 \iiint_{\overline{B}(C,R)} f(x,y,z) \, dV \right|$$

$$\leq \frac{R^2}{20} \iiint_{\sigma(C,R)} \left| \frac{\partial f}{\partial \rho} \right| (x,y,z) \, d\sigma + \frac{R^2}{5} \iiint_{\sigma(C,R)} \left| \frac{\partial f}{\partial \rho} \right| (x,y,z) \, d\sigma$$

$$= \frac{R^2}{4} \iiint_{\sigma(C,R)} \left| \frac{\partial f}{\partial \rho} \right| (x,y,z) \, d\sigma. \hspace{1cm} (15)$$

By dividing (15) with \( 4\pi R^3 \), we obtain the desired result (12).

To show the sharpness of (12), consider the function \( f : \overline{B}(C, R) \to \mathbb{R} \) defined as

$$f(x,y,z) = R - \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}.$$
Using spherical coordinates, we have 
\[ f(\rho, \phi, \theta) = R - \rho, \text{ for } \rho \in [0, R], \phi \in [0, \pi] \text{ and } \theta \in [0, 2\pi]. \]
With some calculations we obtain that
\[
\frac{1}{\frac{4}{3}\pi R^3} \iiint_{\bar{B}(C, R)} f(x, y, z) \, dV = \frac{1}{\frac{4}{3}\pi R^3} \int_0^{2\pi} \int_0^\pi \int_0^R (R - \rho)^2 \sin \phi \, d\rho \, d\phi \, d\theta = \frac{R}{4},
\]
and
\[
\iiint_{\sigma(C, R)} f(x, y, z) \, d\sigma = 0.
\]
On the other hand, since \( |\frac{\partial f}{\partial \rho}| = 1 \),
\[
\frac{1}{16\pi R} \int_{\sigma(C, R)} |\frac{\partial f}{\partial \rho}|(x, y, z) \, d\sigma = \frac{R}{4}.
\]
From (16) and (17) we have the sharpness of (12).

Now we obtain the midpoint type inequality related to (1), where the partial derivative absolute value of considered function defined on \( \bar{B}(C, R) \) is convex.

**Theorem 2.3** Suppose that \( \bar{B}(C, R) \subset V^o \), where \( V \subset \mathbb{R}^3 \). Consider \( f : \bar{B}(C, R) \to \mathbb{R} \) which has continuous partial derivatives with respect to the variables \( \rho, \phi, \text{ and } \theta \) on \( \bar{B}(C, R) \) in spherical coordinates. If \( |\frac{\partial f}{\partial \rho}| \) is convex on \( \bar{B}(C, R) \), then
\[
\left| \frac{1}{\frac{4}{3}\pi R^3} \iiint_{\bar{B}(C, R)} f(x, y, z) \, dV - f(C) \right| \leq \frac{5}{16\pi R} \int_{\sigma(C, R)} |\frac{\partial f}{\partial \rho}|(x, y, z) \, d\sigma .
\]

**Proof** Similar to the proof of Theorem 2.2, for fixed \( \phi \in [0, \pi] \) and \( \theta \in [0, 2\pi] \), we have
\[
\int_0^R \frac{\partial f}{\partial \rho} (a + \rho \cos \theta \sin \phi, b + \rho \sin \theta \cos \phi, c + \rho \cos \phi) \sin \phi \, d\rho
\]
\[
= f(a + R \cos \theta \sin \phi, b + R \sin \theta \sin \phi, c + R \cos \phi) \sin \phi - f(C) \sin \phi.
\]
Integration with respect to the variables \( \phi \in [0, \pi] \) and \( \theta \in [0, 2\pi] \) in (19) implies that
\[
\int_0^{2\pi} \int_0^\pi \int_0^R \frac{\partial f}{\partial \rho} (a + \rho \cos \theta \sin \phi, b + \rho \sin \theta \cos \phi, c + \rho \cos \phi) \sin \phi \, d\rho \, d\phi \, d\theta
\]
\[
= \frac{1}{R^2} \iint_{\sigma(C, R)} f(x, y, z) \, d\sigma - 4\pi f(C).
\]
So from the convexity of \( |\frac{\partial f}{\partial \rho}| \) we get
\[
\left| \frac{1}{4\pi R^2} \iint_{\sigma(C, R)} f(x, y, z) \, d\sigma - f(C) \right|
\]
\[
\leq \frac{1}{4\pi} \int_0^{2\pi} \int_0^\pi \int_0^8 \frac{\partial f}{\partial \rho} (a + \rho \cos \theta \sin \varphi, b + \rho \sin \theta \sin \varphi, c + \rho \cos \varphi) \sin \varphi \, d\rho \, d\varphi \, d\theta
\]
\[
- \frac{1}{4\pi} \int_0^{2\pi} \int_0^\pi \int_0^8 \left(1 - \frac{\rho}{R}\right) \frac{\partial f}{\partial \rho} |(C) \sin \varphi \, d\rho \, d\varphi \, d\theta
\]
\[
+ \frac{1}{4\pi} \int_0^{2\pi} \int_0^\pi \int_0^8 \frac{\rho}{R} \frac{\partial f}{\partial \rho} (a + R \cos \theta \sin \varphi, b + R \sin \theta \sin \varphi, c + R \cos \varphi)
\]
\[
\times \sin \varphi \, d\rho \, d\varphi \, d\theta
\]
\[
= \frac{R}{2} \frac{\partial f}{\partial \rho} |(C) + \frac{1}{8\pi R} \int_{\sigma(C,R)} \frac{\partial f}{\partial \rho} |(x,y,z) \, d\sigma. \tag{20}
\]

It follows from triangle inequality, (20), (12) and (1) (for \(|\frac{\partial f}{\partial \rho}\)|) that

\[
\frac{1}{\pi R^2} \iiint_{B(C,R)} f(x, y, z) \, dV - f(C)
\]
\[
\leq \frac{1}{16\pi R} \int_{\sigma(C,R)} \frac{\partial f}{\partial \rho} |(x,y,z) \, d\sigma + \frac{R}{2} \frac{\partial f}{\partial \rho} |(C) + \frac{1}{8\pi R} \int_{\sigma(C,R)} \frac{\partial f}{\partial \rho} |(x,y,z) \, d\sigma
\]
\[
\leq \frac{3}{16\pi R} \int_{\sigma(C,R)} \frac{\partial f}{\partial \rho} |(x,y,z) \, d\sigma + \frac{1}{8\pi R} \int_{\sigma(C,R)} \frac{\partial f}{\partial \rho} |(x,y,z) \, d\sigma
\]
\[
= \frac{5}{16\pi R} \int_{\sigma(C,R)} \frac{\partial f}{\partial \rho} |(x,y,z) \, d\sigma,
\]

which implies the desired result. \(\square\)

**Corollary 2.4** ([17]) Consider a set \(I \subset \mathbb{R}^2\) with \(D(C,R) \subset I^2\). Suppose that the mapping \(f : D(C,R) \to \mathbb{R}\) has continuous partial derivatives in the disk \(D(C,R)\) with respect to the variables \(r\) and \(\theta\) in polar coordinates. If for any constant \(\theta \in [0, 2\pi]\), the function \(|\frac{\partial f}{\partial r}|\) is convex with respect to the variable \(r\) on \([0, R]\) then

\[
\frac{1}{2\pi R} \int_{\sigma(C,R)} f(y) \, dl(y) - \frac{1}{\pi R^2} \iint_{D(C,R)} f(x,y) \, dx \, dy \leq \frac{1}{6\pi} \int_{\sigma(C,R)} \frac{\partial f}{\partial r} |(y) \, dl(y),
\]
\[
\frac{1}{\pi R^2} \iint_{D(C,R)} f(x,y) \, dx \, dy - f(C) \leq \frac{2}{3\pi} \int_{\sigma(C,R)} \frac{\partial f}{\partial r} |(y) \, dl(y).
\]

**Remark 2.5** In the proof of Theorem 2.3, we can find the following inequality:

\[
\frac{1}{4\pi R^2} \iint_{\sigma(C,R)} f(x,y,z) \, d\sigma - f(C)
\]
\[
\leq \frac{R}{2} \frac{\partial f}{\partial \rho} |(C) + \frac{1}{8\pi R} \int_{\sigma(C,R)} \frac{\partial f}{\partial \rho} |(x,y,z) \, d\sigma. \tag{21}
\]

Although (18) is not sharp, if we consider \(f(x,y,z) = \sqrt{x^2 + y^2 + z^2}\) for \(x,y,z \in \bar{B}(C,R)\), we will find that inequality (21) is sharp.
Remark 2.6 If we drop out the convexity condition of $\frac{\partial f}{\partial \rho}$ in Theorems 2.2, 2.3, and consider the condition

$$\left\| \frac{\partial f}{\partial \rho} \right\|_{\infty B(C,R)} = \sup_{w \in B(C,R)} \left| f(w) \right| < \infty,$$

instead of that, then we get the following Ostrowski type inequalities (see [19, 20]) on a closed ball:

$$\left| \frac{1}{4\pi R^2} \int_{\sigma(C,R)} f(x,y,z) d\sigma - \frac{1}{4\pi} \int_{B(C,R)} f(x,y,z) dV \right| \leq \frac{R \left\| \frac{\partial f}{\partial \rho} \right\|_{\infty B(C,R)}}{4},$$

and

$$\left| \frac{1}{4\pi} \int_{B(C,R)} f(x,y,z) dV - f(C) \right| \leq R \left\| \frac{\partial f}{\partial \rho} \right\|_{B(C,R)}.$$

3 Lipschitz functions

In this section we consider Lipschitz functions with respect to the Euclidian norm to obtain some trapezoid and mid-point type inequalities on $B(C,R)$.

Definition 3.1 ([21]) A function $f : \mathcal{V} \subset \mathbb{R}^3 \rightarrow \mathbb{R}$ is said to satisfy a Lipschitz condition (briefly, $f$ is $\mathcal{L}$-Lipschitz) on $\mathcal{V}$ with respect to a norm $\| \cdot \|$, if there exists a constant $\mathcal{L} > 0$ such that

$$|f(x) - f(y)| \leq \mathcal{L} \|x - y\|,$$

for any $x, y \in \mathcal{V}$.

If $f : \mathcal{B}(C,R)$ is Lipschitz with respect to the Euclidian norm with the constant $\mathcal{L} > 0$, then for any $x = (a + \rho_1 \cos \theta_1 \sin \varphi_1, b + \rho_1 \sin \theta_1 \sin \varphi_1, c + \rho_1 \cos \varphi_1)$ and $y = (a + \rho_2 \cos \theta_2 \sin \varphi_2, b + \rho_2 \sin \theta_2 \sin \varphi_2, c + \rho_2 \cos \varphi_2)$, with some calculations we obtain that

$$|f(x) - f(y)| \leq \mathcal{L} \sqrt{\rho_1^2 + \rho_2^2 - 2\rho_1 \rho_2 M(\varphi_1, \varphi_2, \theta_1, \theta_2)},$$

where $M(\varphi_1, \varphi_2, \theta_1, \theta_2) = [\sin \varphi_1 \sin \varphi_2 \cos(\theta_1 - \theta_2) + \cos \varphi_1 \cos \varphi_2]$, $\rho_1, \rho_2 \in [0,R]$, $\theta_1, \theta_2 \in [0,2\pi]$ and $\varphi_1, \varphi_2 \in [0,\pi]$. Also it is obvious that if $f : \mathcal{V} \subset \mathbb{R}^3 \rightarrow \mathbb{R}$ is Lipschitz with a constant $\mathcal{L} > 0$ on $\mathcal{V}$, then it is continuous and so integrable on $\mathcal{V}$. We need the following result.

Lemma 3.2 For any $\varphi_i \in [0,\pi]$ and $\theta_i \in [0,2\pi]$ ($i \in \{1,2\}$) we have

$$-1 \leq \cos(\varphi_1 + \varphi_2) \leq M(\varphi_1, \varphi_2, \theta_1, \theta_2) \leq \cos(\varphi_1 - \varphi_2) \leq 1.$$

Proof For any $\theta_1, \theta_2 \in [0,2\pi]$ it is obvious that $\cos(\theta_1 - \theta_2) \leq 1$. On the other hand, since for any $\varphi_1, \varphi_2 \in [0,\pi]$, $\sin \varphi_1 \sin \varphi_2$ is nonnegative,

$$\sin \varphi_1 \sin \varphi_2 \cos(\theta_1 - \theta_2) \leq \sin \varphi_1 \sin \varphi_2.$$
So

\[ M(\varphi_1, \varphi_2, \theta_1, \theta_2) = \sin \varphi_1 \sin \varphi_2 \cos(\theta_1 - \theta_2) + \cos \varphi_1 \cos \varphi_2 \leq \sin \varphi_1 \sin \varphi_2 + \cos \varphi_1 \cos \varphi_2 = \cos(\varphi_1 - \varphi_2) \leq 1. \]

Similarly, we can prove that \( M(\varphi_1, \varphi_2, \theta_1, \theta_2) \geq \cos(\varphi_1 + \varphi_2) \geq -1. \]

The following trapezoid type inequality related to (1) for \( \mathcal{L} \)-Lipschitz functions on \( \mathcal{B}(C, R) \) holds.

**Theorem 3.3** Let \( f : \mathcal{B}(C, R) \to \mathbb{R} \) be an \( \mathcal{L} \)-Lipschitz function. Then

\[
\left| \frac{1}{4\pi R^2} \int_{\sigma(C,R)} f(x, y, z) \, d\sigma - \frac{1}{4\pi R^3} \iiint_{\mathcal{B}(C,R)} f(x, y, z) \, dV \right| \leq \frac{\mathcal{L} R}{4}. \tag{22}
\]

Inequality (22) is sharp.

**Proof** Since \( f \) is Lipschitz with constant \( \mathcal{L} > 0 \) on \( \mathcal{B}(C, R) \), we get

\[
\left| \int_0^{2\pi} \int_0^\pi \int_0^R f(a + \rho \cos \theta \sin \varphi, b + \rho \sin \theta \cos \varphi, c + \rho \cos \varphi) \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta \\
- \int_0^{2\pi} \int_0^\pi \int_0^R f(a + R \cos \theta \sin \varphi, b + R \sin \theta \cos \varphi, c + R \cos \varphi) \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta \right|
\]

\[
\leq \mathcal{L} \int_0^{2\pi} \int_0^\pi \int_0^R \| (\rho - R) \cos \theta \sin \varphi, (\rho - R) \sin \theta \cos \varphi, (\rho - R) \cos \varphi \| \sin \varphi \, d\rho \, d\varphi \, d\theta
\]

\[
\leq \mathcal{L} \int_0^{\pi} \int_0^\pi \int_0^R (R - \rho)^2 \sin \varphi \, d\rho \, d\varphi \, d\theta = \frac{\mathcal{L} \pi R^4}{3}. \tag{23}
\]

Now by replacing (10) and (11) in (23) and then dividing the result by \( \frac{4\pi R^3}{} \), we deduce the desired result.

To prove the sharpness of (22), consider the function \( f : \mathcal{B}(C, R) \to \mathbb{R} \) defined by

\[
f(a + \rho \cos \theta \sin \varphi, b + \rho \sin \theta \cos \varphi, c + \rho \cos \varphi) = \mathcal{L}(R - \rho),
\]

for \( \mathcal{L} > 0 \), \( \rho \in [0, R] \), \( \varphi \in [0, \pi] \), and \( \theta \in [0, 2\pi] \). The function \( f \) is Lipschitz with constant \( \mathcal{L} \). Consider \( x = (a + \rho_1 \cos \theta_1 \sin \varphi_1, b + \rho_1 \sin \theta_1 \sin \varphi_1, c + \rho_1 \cos \varphi_1) \) and \( y = (a + \rho_2 \cos \theta_2 \sin \varphi_2, b + \rho_2 \sin \theta_2 \sin \varphi_2, c + \rho_2 \cos \varphi_2) \), for \( \rho_1, \rho_2 \in [0, R] \), \( \varphi_1, \varphi_2 \in [0, \pi] \), \( \theta_1, \theta_2 \in [0, 2\pi] \). Then by Lemma 3.2 we have

\[
|f(x) - f(y)|
\]

\[
= \left| f(a + \rho_1 \cos \theta_1 \sin \varphi_1, b + \rho_1 \sin \theta_1 \sin \varphi_1, c + \rho_1 \cos \varphi_1) \\
- f(a + \rho_2 \cos \theta_2 \sin \varphi_2, b + \rho_2 \sin \theta_2 \sin \varphi_2, c + \rho_2 \cos \varphi_2) \right|
\]

\[
= \mathcal{L} |\rho_2 - \rho_1| \leq \mathcal{L} \sqrt{\rho_1^2 + \rho_2^2 - 2\rho_1 \rho_2} \leq \mathcal{L} \sqrt{\rho_1^2 + \rho_2^2} \leq \mathcal{L} \sqrt{\rho_1^2 + \rho_2^2 - 2\rho_1 \rho_2 M(\varphi_1, \varphi_2, \theta_1, \theta_2)}
\]
It is not hard to see that \( f(a + \rho \cos \theta \sin \varphi, b + \rho \sin \theta \cos \varphi, c + \rho \cos \varphi) \geq 0 \) for all \( 0 \leq \rho \leq R \), \( 0 \leq \varphi \leq \pi \), and \( 0 \leq \theta \leq 2\pi \). Also for the case \( \rho = R \), we have \( f(a + R \cos \theta \sin \varphi, b + R \sin \theta \cos \varphi, c + R \cos \varphi) = 0 \). So we have

\[
\begin{align*}
\left| \frac{1}{4\pi R^2} \iiint_{\partial B} f(x, y, z) \, d\sigma - \frac{1}{4\pi R^3} \iiint_{B} f(x, y, z) \, dV \right| &= \frac{1}{4\pi R^3} \iiint_{B} f(x, y, z) \, dV \\
&= \frac{\mathcal{L}}{4\pi R^3} \int_0^{2\pi} \int_0^{\pi} \int_0^R (R - \rho) \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta = \frac{LR}{4}.
\end{align*}
\]

For \( \mathcal{L} \)-Lipschitz functions we can obtain a mid-point type inequality as follows:

**Theorem 3.4** Let \( f : \bar{B}(C, R) \rightarrow \mathbb{R} \) be an \( \mathcal{L} \)-Lipschitz function. Then

\[
\left| \frac{1}{4\pi R^3} \iiint_{B(C,R)} f(x, y, z) \, dV - f(C) \right| \leq \frac{3LR}{4}.
\] (24)

Inequality (24) is sharp.

**Proof** Since the function \( f \) is \( \mathcal{L} \)-Lipschitz on \( \bar{B}(C, R) \), we have

\[
\begin{align*}
|f(a + \rho \cos \theta \sin \varphi, b + \rho \sin \theta \cos \varphi, c + \rho \cos \varphi) - f(C)| &\leq \mathcal{L} \| (\rho \cos \theta \sin \varphi, \rho \sin \theta \cos \varphi, \rho \cos \varphi) \| = \mathcal{L} \rho,
\end{align*}
\]

for all \( \rho \in [0, R] \), \( \varphi \in [0, \pi] \), and \( \theta \in [0, 2\pi] \). It follows that

\[
\begin{align*}
&\left| \int_0^{2\pi} \int_0^{\pi} \int_0^R f(a + \rho \cos \theta \sin \varphi, b + \rho \sin \theta \cos \varphi, c + \rho \cos \varphi) \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta \\
&\quad - \int_0^{2\pi} \int_0^{\pi} \int_0^R f(a, b, c) \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta \right| \\
&\leq \int_0^{2\pi} \int_0^{\pi} \int_0^R \left| f(a + \rho \cos \theta \sin \varphi, b + \rho \sin \theta \cos \varphi, c + \rho \cos \varphi) - f(C) \right| \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta \\
&\quad \times \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta \\
&\leq \mathcal{L} \int_0^{2\pi} \int_0^{\pi} \int_0^R \rho^3 \sin \varphi \, d\rho \, d\varphi \, d\theta = \mathcal{L} \pi R^4.
\end{align*}
\]

So we obtain that

\[
\left| \iiint_{B(C,R)} f(x, y, z) \, dV - \frac{4}{3} \pi R^3 f(C) \right| \leq \mathcal{L} \pi R^4,
\]

which implies the desired result.
Now consider the function $f : \bar{B}(C, R) \to \mathbb{R}$ defined by

$$f(a + \rho \cos \theta \sin \varphi, b + \rho \sin \theta \cos \varphi, c + \rho \cos \varphi) = \mathcal{L} \rho,$$

for $\mathcal{L} > 0$, $0 \leq \rho \leq R$, $0 \leq \varphi \leq \pi$, and $0 \leq \theta \leq 2\pi$. It is obvious that $f(C) = 0$. By a similar method used in the proof of Theorem 3.3, the function $f$ is $\mathcal{L}$-Lipschitz. So we have

$$\left| \frac{1}{3 \pi R^3} \iiint_{B(C, R)} f(x, y, z) dV - f(C) \right| = \frac{\mathcal{L}}{3 \pi R^3} \int_0^{2\pi} \int_0^R \rho^3 \sin \varphi d\rho d\varphi d\theta = \frac{3 \mathcal{L} R}{4},$$

showing that inequality (24) is sharp. \hfill \Box

**Remark 3.5** Consider an open set $V \subset \mathbb{R}^3$ including $\bar{B}(C, R)$. For convex function $f$ defined on $V$, from Theorem D of Sect. 41 in [21] we have that $f$ is $\mathcal{L}$-Lipschitz on $\bar{B}(C, R)$ and so from inequalities (22) and (24), along with inequality (1), we get the following results:

$$0 \leq \frac{1}{4 \pi R^2} \int_{\pi(C, R)} f(x, y, z) d\sigma - \frac{1}{3 \pi R^3} \iiint_{B(C, R)} f(x, y, z) dV \leq \frac{\mathcal{L} R}{3},$$

and

$$0 \leq \frac{1}{3 \pi R^3} \iiint_{\bar{B}(C, R)} f(x, y, z) dV - f(C) \leq \frac{2 \mathcal{L} R}{3}.$$

In the following, as an example we obtain a Lipschitz constant $\mathcal{L}$ for a real-valued function defined on a closed ball in $\mathbb{R}^3$.

**Example 3.6** Consider $W = f(x, y, z) = (x - a)^n + (y - b)^n + (z - c)^n$, $n \in \mathbb{N}$, $(x, y, z) \in \bar{B}(C, R)$. To find a Lipschitz constant for $f$, we will do some calculations as follows. For $A, B \in \bar{B}(C, R)$, consider the path $\psi : [0, 1] \to \bar{B}(C, R)$ from $B$ to $A$ in $\bar{B}(C, R)$ as

$$\psi(t) = tA + (1 - t)B,$$

for $t \in [0, 1]$. Now using the fundamental theorem of calculus, we obtain that

$$|f(A) - f(B)| = |f(\psi(1)) - f(\psi(0))| = \left| \int_0^1 \frac{df(\psi(t))}{dt} dt \right|.$$

On the other hand, from the chain rule for differentiation, we get

$$\frac{df(\psi(t))}{dt} = \nabla f(\psi(t)) \cdot \frac{d\psi}{dt} = \nabla f(\psi(t))(A - B),$$

where $\nabla f$ is the gradient vector of $f$. So using the Euclidean norm $\| \cdot \|$, we obtain

$$\left| \int_0^1 \frac{df(\psi(t))}{dt} dt \right| = \left| \int_0^1 \nabla f(\psi(t))(A - B) dt \right| \leq \|A - B\| \int_0^1 \| \nabla f(\psi(t)) \| dt$$

$$\leq \|A - B\| \sup_{u \in \bar{B}(C, R)} \| \nabla f(u) \|,$$
which implies
\[
|f(A) - f(B)| \leq \|A - B\| \sup_{u \in B(C,R)} \|\nabla f(u)\|.
\]
This shows that \( L = \sup_{u \in B(C,R)} \|\nabla f(u)\| \) (if it exists) is a Lipschitz constant for \( f \). Now for any \( w = (x,y,z) \in \bar{B}(C,R) \), we have
\[
\nabla f(w) = n((x-a)^{n-1}, (y-b)^{n-1}, (z-c)^{n-1}),
\]
and then
\[
\|\nabla f(u)\| = n\sqrt{((x-a)^{n-1})^2 + ((y-b)^{n-1})^2 + ((z-c)^{n-1})^2}
\leq n\sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}^{n-1} = nR^{n-1}.
\]
So we can choose \( L = \sup_{u \in B(C,R)} \|\nabla f(u)\| = nR^{n-1} \) as a Lipschitz constant for \( f \) on \( \bar{B}(C,R) \).

Using the above example, we have the following result:

**Example 3.7** For \( n \in \mathbb{N} \setminus \{1\} \), consider the function \( f(\rho,\varphi,\theta) = (x_0 - \rho)^n + (y_0 - \rho)^n + (z_0 - \rho)^n \)
defined on \( \bar{B}(x_0, y_0, z_0, R) \) such that \( x_0, y_0, z_0 > 0, 0 < R \leq \min\{x_0, y_0, z_0\} \) and \( 0 \leq \rho \leq R \). It follows that
\[
\nabla \left( \frac{\partial f}{\partial \rho} \right)(\rho,\varphi,\theta) = n(n-1)(x_0^0 - \rho)^{n-2} + (y_0^0 - \rho)^{n-2} + (z_0^0 - \rho)^{n-2}, 0, 0),
\]
and then
\[
L = n(n-1)(x_0^{n-2} + y_0^{n-2} + z_0^{n-2}),
\]
is a Lipschitz constant for \( \nabla (\frac{\partial f}{\partial \rho}) \). On the other hand, it is not hard to prove that
\[
\left| \frac{1}{4\pi R^2} \int_{\partial B(C,R)} f(x,y,z) \, d\sigma - f(C) \right| \leq \frac{LR^2}{2}.
\]
(25)
So by (25), we have the following numerical inequality:
\[
\left| (x_0^n - (y_0^n - R)^n + (z_0^n - R)^n) - (x_0^n + y_0^n + z_0^n) \right|
\leq \frac{n(n-1)(x_0^{n-2} + y_0^{n-2} + z_0^{n-2})R^2}{2}.
\]

**Remark 3.8** For any function \( f : \bar{B}(C,R) \to \mathbb{R} \), we can apply the structure mentioned in the above example to obtain a Lipschitz constant \( L = \sup_{x \in \bar{B}(C,R)} \|\nabla f(x)\| \) with respect to the Euclidian norm \( \| \cdot \| \), provided that the gradient vector of \( f \) exists everywhere in \( \bar{B}(C,R) \) and also \( L < \infty \).

**Remark 3.9** In Theorems 3.3 and 3.4, if we consider that \( \frac{\partial f}{\partial \rho} : \bar{B}(C,R) \to \mathbb{R} \) is \( L \)-Lipschitz and \( f : \bar{B}(C,R) \to \mathbb{R} \) is integrable, then by (13) and (19) we can obtain (the details are
omitted)

\[
\left| \frac{1}{4\pi R^2} \iint_{\sigma(C,R)} f(x, y, z) \, d\sigma - \frac{1}{\frac{1}{2}\pi R^3} \iiint_{B(C,R)} f(x, y, z) \, dV \right| \leq \frac{1}{5} \mathcal{L} R^2,
\]

and

\[
\left| \frac{1}{\frac{1}{2}\pi R^3} \iiint_{B(C,R)} f(x, y, z) \, dV - f(C) \right| \leq \frac{7}{10} \mathcal{L} R^2.
\]

4 Bounded functions

In the last section we investigate trapezoid and mid-point type inequalities where considered functions are bounded.

**Theorem 4.1** Suppose that \( V \subset \mathbb{R}^3, \bar{B}(C, R) \subset V^o \) and \( f: \bar{B}(C, R) \to \mathbb{R} \) has continuous partial derivatives with respect to the variables \( \rho, \varphi, \) and \( \theta \) on \( \bar{B}(C, R) \) in spherical coordinates. If \( \frac{\partial f}{\partial \rho} \) is bounded on \( \bar{B}(C, R) \), then

\[
\left| \frac{1}{4\pi R^2} \iint_{\sigma(C,R)} f(x, y, z) \, d\sigma - \frac{1}{\frac{1}{2}\pi R^3} \iiint_{B(C,R)} f(x, y, z) \, dV \right| \leq \left( \frac{\mathcal{U}_B - \mathcal{L}_B + |\mathcal{L}_B + \mathcal{U}_B|}{8} \right) R,
\]

where \( \mathcal{L}_B \) and \( \mathcal{U}_B \) are lower and upper bounds of \( \frac{\partial f}{\partial \rho} \) on \( \bar{B}(C, R) \), respectively.

**Proof** Consider \( \mathcal{U}_B \) and \( \mathcal{L}_B \) as the upper and lower bounds of an arbitrary function \( g \) defined on a set \( V \subset \mathbb{R}^3 \), respectively. Then for all \( x, y, z \in V \), we have

\[
\mathcal{L}_B - \frac{\mathcal{L}_B + \mathcal{U}_B}{2} \leq g(x, y, z) - \frac{\mathcal{L}_B + \mathcal{U}_B}{2} \leq \mathcal{U}_B - \frac{\mathcal{L}_B + \mathcal{U}_B}{2},
\]

which implies that

\[
\left| g(x, y, z) - \frac{\mathcal{L}_B + \mathcal{U}_B}{2} \right| \leq \frac{\mathcal{U}_B - \mathcal{L}_B}{2}, \tag{27}
\]

for all \( x, y, z \in V \). On the other hand, from (13) we get

\[
\begin{align*}
&\int_0^{2\pi} \int_0^\pi \int_0^R \frac{\partial f}{\partial \rho} (a + \rho \cos \theta \sin \varphi, b + \rho \sin \theta \sin \varphi, c + \rho \cos \varphi) \rho^3 \sin \varphi \, d\rho \, d\varphi \, d\theta \\
&\quad - \int_0^{2\pi} \int_0^\pi \int_0^R \frac{\mathcal{L}_B + \mathcal{U}_B}{2} \rho^3 \sin \varphi \, d\rho \, d\varphi \, d\theta \\
&\quad = \int_0^{2\pi} \int_0^\pi R^3 f(a + R \cos \theta \sin \varphi, b + R \sin \theta \sin \varphi, c + R \cos \varphi) \, d\varphi \, d\theta \\
&\quad - 3 \int_0^{2\pi} \int_0^\pi \int_0^R f(a + \rho \cos \theta \sin \varphi, b + \rho \sin \theta \sin \varphi, c + \rho \cos \varphi) \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta \\
&\quad - \pi R^4 \frac{\mathcal{L}_B + \mathcal{U}_B}{2}.
\end{align*}
\]
Now if in (27) we consider \( g = \frac{\partial f}{\partial \rho} \), \( V = \overline{B}(C, R) \), and utilize Lemma 2.1, then we obtain that

\[
R \left| \int_{\bar{B}(C, R)} f(x, y, z) \, d\sigma - 3 \int_{\bar{B}(C, R)} f(x, y, z) \, dV - \pi R^4 \frac{L_B + 4U_B}{2} \right|
\]

\[
\leq \int_0^{2\pi} \int_0^\pi \int_0^R \frac{\partial f}{\partial \rho} (a + \rho \cos \theta \sin \varphi, b + \rho \sin \theta \sin \varphi, c + \rho \cos \varphi) - \frac{L_B + 4U_B}{2} \times \rho^3 \sin \varphi \, d\rho \, d\varphi \, d\theta
\]

\[
\leq \int_0^{2\pi} \int_0^\pi \int_0^R \left| \frac{\partial f}{\partial \rho} \right| \rho^3 \sin \varphi \, d\rho \, d\varphi \, d\theta = \frac{4U_B - L_B}{2} \pi R^4.
\]

Finally, by the use of the triangle inequality and dividing the result by \( 4\pi R^3 \), we obtain inequality (26).

**Theorem 4.2** Suppose that \( V \subset \mathbb{R}^3, \overline{B}(C, R) \subset V^0 \) and \( f : \overline{B}(C, R) \to \mathbb{R} \) has continuous partial derivatives with respect to the variables \( \rho, \varphi, \) and \( \theta \) on \( \overline{B}(C, R) \) in spherical coordinates. If \( \frac{\partial f}{\partial \rho} \) is bounded on \( \overline{B}(C, R) \), then

\[
\left| \frac{1}{4\pi R^2} \int_{\sigma(C, R)} f(x, y, z) \, d\sigma - f(C) \right| \leq \left( \frac{U_B - L_B + |L_B + 4U_B|}{2} \right) R,
\]

(28)

where \( L_B \) and \( U_B \) are lower and upper bounds of \( \frac{\partial f}{\partial \rho} \) on \( \overline{B}(C, R) \), respectively.

**Proof** Consider \( L_B \) and \( U_B \) as the upper and lower bounds of \( \frac{\partial f}{\partial \rho} \). By (19), the following relations hold:

\[
\int_0^{2\pi} \int_0^\pi \int_0^R \frac{\partial f}{\partial \rho} (a + \rho \cos \theta \sin \varphi, b + \rho \sin \theta \sin \varphi, c + \rho \cos \varphi) - \frac{L_B + 4U_B}{2} \times \sin \varphi \, d\rho \, d\varphi \, d\theta
\]

\[
= \int_0^{2\pi} \int_0^\pi \int_0^R f(a + R \cos \theta \sin \varphi, b + R \sin \theta \sin \varphi, c + R \cos \varphi) \sin \varphi \, d\rho \, d\varphi \, d\theta
\]

\[
- \int_0^{2\pi} \int_0^\pi f(C) \sin \varphi \, d\varphi \, d\theta - \frac{L_B + 4U_B}{2} R \int_0^{2\pi} \int_0^\pi \sin \varphi \, d\varphi \, d\theta
\]

\[
= \frac{1}{R^2} \int_{\sigma(C, R)} f(x, y, z) \, d\sigma - 4\pi f(C) - 2\pi (L_B + U_B) R.
\]

This implies that

\[
\left| \frac{1}{4\pi R^2} \int_{\sigma(C, R)} f(x, y, z) \, d\sigma - f(C) - \frac{L_B + 4U_B}{2} R \right|
\]

\[
\leq \frac{1}{4\pi} \int_0^{2\pi} \int_0^\pi \int_0^R \left| \frac{\partial f}{\partial \rho} (a + \rho \cos \theta \sin \varphi, b + \rho \sin \theta \sin \varphi, c + \rho \cos \varphi) - \frac{L_B + 4U_B}{2} \right| \times \sin \varphi \, d\rho \, d\varphi \, d\theta = \frac{U_B - L_B}{2} R.
\]
Finally, by using the triangle inequality, we get
\[
\left| \frac{1}{4\pi R^2} \iint_{\sigma(C,R)} f(x,y,z) \, d\sigma - f(C) \right| \leq \left( \frac{U_B - L_B + |L_B + U_B|}{2} \right) R.
\]
\[\square\]

**Remark 4.3** If \( f : \overline{B}(C,R) \to \mathbb{R} \) is a convex function and bounded from above on \( \overline{B}(C,R) \) (\( U_B \) exists), then \( f \) is bounded on \( \overline{B}(C,R) \) because for an arbitrary \( X \in \overline{B}(0,R) \) and \( C = \frac{1}{2}(X + C) + \frac{1}{2}(-X + C) \), from the convexity of \( f \) we have \( 2f(C) - f(-X + C) \leq f(X + C) \). This implies that \( 2f(C) - U_B \leq f(X + C) \) where \( X + C \) and \( -X + C \) belong to \( \overline{B}(C,R) \). Now it is enough to set \( L_B = 2f(C) - U_B \).

So if \( \frac{\partial f}{\partial \rho} : \overline{B}(C,R) \to \mathbb{R} \) is convex and bounded from above, then by (26), (28), and (1), the following inequalities hold:
\[
0 \leq \frac{1}{4\pi R^2} \iint_{\sigma(C,R)} f(x,y,z) \, d\sigma - \frac{1}{8\pi R^3} \iiint_{\overline{B}(C,R)} f(x,y,z) \, dV \leq \left( \frac{U_B - L_B + |L_B + U_B|}{8} \right) R,
\]
and
\[
0 \leq \frac{1}{4\pi R^2} \iint_{\overline{B}(C,R)} f(x,y,z) \, dV - f(C) \leq \left( \frac{U_B - L_B + |L_B + U_B|}{2} \right) R,
\]
where \( L_B \) and \( U_B \) are lower and upper bounds of \( \frac{\partial f}{\partial \rho} \) on \( \overline{B}(C,R) \), respectively.

**Acknowledgements**
The author is grateful to the referee(s) and the editor for their valuable comments and suggestions.

**Funding**
Not applicable.

**Availability of data and materials**
Not applicable.

**Competing interests**
The author declares that he has no competing interests.

**Authors’ contributions**
The author read and approved the final manuscript.

**Publisher’s Note**
Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

**Received: 7 November 2019 Accepted: 14 April 2020 Published online: 28 April 2020**

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