Thermodynamics of Evolving Lorentzian Wormhole at Apparent and Event Horizons

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Abstract: We have investigated the non-static Lorentzian Wormhole model in presence of anisotropic pressure. We have presented some exact solutions of Einstein equations for anisotropic pressure case. Introducing two EoS parameters we have shown that these solutions give very rich dynamics of the universe yielding to the different expansion history of it in the r-direction and in the T-direction. The corresponding explicit forms of the shape function $b(r)$ is presented. We have shown that the Einstein’s field equations and unified first law are equivalent for the dynamical wormhole model. The first law of thermodynamics has been derived by using the Unified first law. The physical quantities including surface gravity and the temperature are derived for the wormhole. Here we have obtained all the results without any choice of the shape function. The validity of generalized second law (GSL) of thermodynamics has been examined at apparent and event horizons for the evolving Lorentzian wormhole.

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I. INTRODUCTION

There are a number of similarities between black-hole physics and thermodynamics. Most striking is the similarity in the behaviors of black-hole area and of entropy: Both quantities tend to increase irreversibly. Employing the concepts of information theory to the black hole physics, Bekenstein introduced the concept of black-hole entropy as the measure of information about a black hole interior which is inaccessible to an exterior observer. Moreover dimensional considerations indicated that the black-hole entropy is equal to the ratio of the black-hole area to the square of the Planck length times a dimensionless constant of order unity \cite{1}. Later on he deduced the generalized second law (GSL) of thermodynamics which stated that the combined entropy of black horizon and ‘common entropy’ does not decrease \cite{2}. Numerous approaches have been utilized to prove the GSL \cite{3}, while this law has found numerous applications in cosmology \cite{5}. The connection between gravity and thermodynamics was extended to cosmological horizons with repulsive cosmological constant \cite{4}. Hawking showed \cite{7} that black holes emit thermal radiation corresponding to a temperature proportional to surface gravity and entropy proportional to the horizon area \((S \propto A/4)\). This entropy-area relation was also proved via other approaches in \cite{8}. Birrel & Davies also confirmed the thermal nature of the emitted radiation for the massless Thirring model of a self-interacting fermion field in a curved two-dimensional background spacetime \cite{6}. The horizon temperature and entropy obey a simple differential relationship \(dE = TdS\), called the first law of black hole thermodynamics \cite{9}, where \(E\) is the energy. Wald deduced that black hole thermodynamics is nothing more than ordinary thermodynamics applied to a self-gravitating quantum system \cite{10}. Unruh and Wald described how energy from a black hole can be mined under the Bekenstein entropy-energy ratio \cite{11}. Li & Liu pointed out that the Unruh-Wald conclusion does not hold because Hawking radiation near the horizon is not thermal \cite{12}. Zurek & Thorne showed that entropy of a rotating, charged black hole is equal to the logarithm of the number of quantum mechanically distinct ways that the hole could have been made \cite{13}. Visser showed that Hawking radiation can occur in physical situations in which the laws of black hole mechanics do not apply, and in physical situations in which the notion of black hole entropy does not even make any sense \cite{14}. In recent years, the phenomenon of Hawking radiation is also studied in the frameworks of string theory and loop quantum gravity \cite{15}. Another significant development was made by Jacobson \cite{16} by deriving Einstein field equations from the proportionality of entropy to the horizon area together with the fundamental relation \(\delta Q = TdS\), where \(\delta Q\) and \(T\) are the energy flux and Unruh temperature seen by an accelerated observer just inside the horizon.
Padmanabhan [17] made the major development by launching a general formalism for the spherically symmetric black hole spacetimes to understand the thermodynamics of horizons and showed that the Einstein field equations evaluated at event horizon can be expressed in the form, $TdS = dE + pdV$, of thermodynamics. Later on Padmanabhan et al and others [18, 19] studied this approach for more general spacetime geometries and in various gravity theories. In the cosmological setup, Cai and his collaborators [20–24] made the major development by showing that the Einstein field equations evaluated at the apparent horizon can also be expressed as $TdS = dE + WdV$ in various theories of gravity. This connection between gravity and thermodynamics has also been extended in the braneworld cosmology [25]. More recently, using Clausius relation $\delta Q = TdS$, to the apparent horizon of a FRW universe, Cai et al are able to derive the modified Friedman equation by employing quantum corrected area-entropy formula [26]. All these calculations indicate that the thermal interpretation of gravity is to be generic, so we have to investigate this relation for a more general spacetimes.

In this work, we employ the metric of an evolving Lorentzian wormhole [33] and aim to show that the Einstein field equations and the Unified first law are equivalent. We have shown that the isotropic pressure for non-static wormhole generates the standard FRW model. The non-static wormhole exits only for anisotropic pressure. The previous works of Jamil et al [27], Farook et al [28] and Rahaman et al [29] have some computational errors for wormhole thermodynamics in presence of isotropic pressure. In this work, we have corrected these assigning with anisotropic pressure in the field equations. To evaluate the thermodynamical quantities, we use the apparent and event horizons of the evolving wormhole.

The plan of the paper as follows. In section II, we write down the field equations and energy conservation equation for the evolving wormhole. In section III, we study the wormhole thermodynamics using first law of thermodynamics and the entropy-area law. The conclusion is presented at the end of the work.

II. EVOLVING LORENTZIAN WORMHOLE

A wormhole consists of a tunnel of trapped surfaces between two mouths, defined as temporal outer trapping horizons with opposite senses, in mutual causal contact [30]. In static cases, the mouths coincide as the throat of a Morris-Thorne (MT) wormhole. To keep the wormhole’s throat open, an exotic fluid violating the null energy condition is required [31]. The zeroth, first and second laws are derived for wormholes are derived in [32]. A simple generalization of Morris-
Thorne wormhole to the time dependent background is given by the evolving Lorentzian wormhole

\[ ds^2 = -e^{2\Phi(t,r)} dt^2 + a^2(t) \left[ \frac{dr^2}{1 - \frac{b(r)}{r}} + r^2 d\Omega_2^2 \right]. \] (1)

Here \( d\Omega_2^2 \equiv d\theta^2 + \sin^2 \theta d\phi^2 \) is the line element of two dimensional unit sphere, \( b(r) \) and \( \Phi(t,r) \) are the shape and potential functions respectively and \( a(t) \) is the scale factor of the universe. It is clear from the metric (1) that if both \( b(r) \to kr^3 \), and \( \Phi(t,r) \to \text{constant} \), the above metric reduces to the FRW metric. Furthermore, when \( a(t) \to \text{constant} \) and \( \Phi(t,r) \to \Phi(r) \), it turns out the static MT wormhole [30]. If one takes \( a(t) = e^{\chi t} \), the metric (1) represents an inflating Lorentzian wormhole [34], where the arbitrary constant \( \chi \) can be fixed by taking it a cosmological constant \( \Lambda \).

Now consider the components of energy-momentum tensor [33]

\[ T^t_t = -\rho(t,r), \quad T^r_r = p_r(t,r), \quad T^\theta_\theta = T^\phi_\phi = p_T(t,r), \] (2)

where \( \rho(t,r), p_r(t,r) \) and \( p_T(t,r) \) are the energy density, radial pressure and tangential pressure respectively. If \( p_T = p_r \) then the pressure will be isotropic otherwise anisotropic. From the Einstein’s equation \( G_{\mu\nu} = 8\pi G T_{\mu\nu} \), we get the following field equations: [33]

\[ 3e^{-2\Phi} \dot{H}^2 + \frac{b'}{a^2 r^2} = 8\pi G \rho, \] (3)

\[ -e^{-2\Phi}(2\dot{H} + 3H^2) + 2e^{-2\Phi} \dot{\Phi}H - \frac{b}{a^2 r^3} = 8\pi G p_r, \] (4)

\[ -e^{-2\Phi}(2\dot{H} + 3H^2) + 2e^{-2\Phi} \dot{\Phi}H + \frac{b - r b'}{2a^2 r^3} = 8\pi G p_T, \] (5)

\[ 2\Phi' H = 0, \] (6)

where \( H = \frac{\dot{a}}{a} \) is the Hubble parameter and dot and dash refer to derivative w.r.t. \( t \) and \( r \) respectively. Now the equation (6) implies \( \Phi' = 0 \) i.e., \( \Phi(t,r) = \Phi(t) \) i.e., \( \Phi \) is a function of time only. So without any loss of generality, by rescaling the time coordinate we can set \( \Phi = 0 \). So the field equations (3) - (5) reduces to

\[ 3H^2 + \frac{b'}{a^2 r^2} = 8\pi G \rho, \] (7)

\[ 2\dot{H} + 3H^2 + \frac{b}{a^2 r^3} = -8\pi G p_r, \] (8)
\[ 2\dot{H} + 3H^2 - \frac{b - rb'}{2a^2r^3} = -8\pi Gp_T. \]  

(9)

From equations (8) and (9) we obtain the following form,

\[ \frac{rb' - 3b}{2a^2r^3} = 8\pi G(p_r - p_T). \]  

(10)

Now from the energy conservation equation \( T^\mu_{\nu} = 0 \), we have

\[ \dot{\rho} + H(3\rho + p_r + 2p_T) = 0, \]  

(11)

\[ 2(p_T - p_r) = rp_r'. \]  

(12)

For isotropic pressure \( p_T = p_r \) and so from (12) we get, \( p_r' = 0 \) i.e., \( p_r \) is function of time \( t \) only. In this case, from (10), we find \( b(r) = kr^3 \) and hence the metric (1) reduces to FRW metric. In this case (7) and (12) imply \( p_T \) and \( \rho \) are functions of \( t \) only. Since we want to study the wormhole model with pressure depending on both variables \( t \) and \( r \), so we must consider only anisotropic pressures, thus requiring \( p_r \neq p_T \). One of interesting consequences of considering anisotropic pressures is (as we see below) different dynamics of the universe in \( r \)-direction and in \( T \)-direction. To demonstrate this phenomenon let us consider the simple power-law solution: \( a = a_0 t^n \). Then \( H = nt^{-1} \). Substituting these expressions into (7)-(9) we get (below we assume \( 8\pi G = 1 \))

\[ \rho = \frac{3n^2}{t^2} + \frac{b'}{a_0^2t^{2n}r^2}, \]  

(13)

\[ p_r = \frac{n(2 - 3n)}{t^2} - \frac{b}{a_0^2t^{2n}r^3}, \]  

(14)

\[ p_T = \frac{n(2 - 3n)}{t^2} + \frac{b - rb'}{2a_0^2t^{2n}r^2}. \]  

(15)

Now let us introduce separate two EoS parameters for the \( r \)-direction and for the \( T \)-direction as

\[ \omega_r = \frac{p_r}{\rho}, \quad \omega_T = \frac{p_T}{\rho}. \]  

(16)

Then for the solutions (13)-(15) we obtain

\[ \omega_r = \frac{n(2 - 3n)a_0^2r^3t^{2n-2} - b}{r[3n^2a_0^2r^2t^{2n-2} + b']}, \]  

(17)

\[ \omega_T = \frac{2n(2 - 3n)a_0^2r^3t^{2n-2} + (b - rb')}{2r[3n^2a_0^2r^2t^{2n-2} + b']}. \]  

(18)

In these formulas we have one arbitrary function \( b(r) \) and two constant parameters \( n \) and \( a_0 \). To find the explicit form of unknown \( b(r) \), as example, we assume that in the \( r \)-direction we have an
accelerated expansion so that we can put \( \omega_r = -1 \). Then from (17) we determine the unknown function \( b(r) \) as

\[
b(r) = r[C - na_0^2 r^2 t^{2n-2}], \tag{19}
\]

where \( C = \text{constant} \). To eliminate the dependence of this function of \( t \), we put \( n = 1 \). Then finally we get

\[
b(r) = r[C - a_0^2 r^2]. \tag{20}
\]

So for the EoS parameters we get

\[
\begin{align*}
\omega_r &= -1, \tag{21} \\
\omega_T &= 0 \tag{22}
\end{align*}
\]

that corresponds the accelerated expansion of the universe in \( r \)-direction and dust matter dominated case in the \( T \)-direction.

ii) Our next example is also the power-law solution but with \( \omega_r = \text{const} = \omega_{r0} \). Then for the density of energy, pressures and EoS parameters we get the same expressions as in the previous case. To find \( b(r) \) we use again (17) and get the expression

\[
b(r) = C r^{-\frac{1}{\omega_{r0}}} + \frac{2n - 3(1 + \omega_{r0})n^2}{1 + 3\omega_{r0}} a_0^2 r^3 t^{2n-2}, \tag{23}
\]

where \( C = \text{constant} \). To eliminate the \( t \) dependence of \( b \) we again put \( n = 1 \). Then finally we get

\[
b(r) = C r^{-\frac{1}{\omega_{r0}}} - a_0^2 r^3, \tag{24}
\]

In our case the formulas (17)-(18) become

\[
\begin{align*}
\omega_r &= -\frac{a_0^2 r^3 b}{r[3a_0^2 r^2 + b]}, \tag{25} \\
\omega_T &= \frac{b - rb' - 2a_0^2 r^3}{2r[3a_0^2 r^2 + b]}. \tag{26}
\end{align*}
\]

So from these formulas and (24) finally we obtain

\[
\begin{align*}
\omega_r &= \omega_{r0}, \tag{27} \\
\omega_T &= -\frac{1 + \omega_{r0}}{2r}. \tag{28}
\end{align*}
\]

Consider particular cases. 1) Let \( \omega_{r0} = 1/3 \) that is radiation. Then

\[
\begin{align*}
\omega_r &= 1/3, \tag{29} \\
\omega_T &= -\frac{2}{3r}. \tag{30}
\end{align*}
\]
2) Let $\omega_{r_0} < -1$ that is phantom matter. Then

$$\omega_r < -1,$$  \hspace{1cm} (31)

$$\omega_T > 0.$$  \hspace{1cm} (32)

This means that in the $r$-direction we have the radiation dominated dynamics but in the $T$-direction more complicated one. For $r = r_0 = 3/2$ we have the transition point from the phantom to the quintessence case.

3) Let $\omega_{r_0} > 1$ that is ekpyrotic matter. Then

$$\omega_r > 1,$$  \hspace{1cm} (33)

$$\omega_T < -\frac{1}{r}.$$  \hspace{1cm} (34)

It is interesting to note that in this case we have the ekpyrotic matter in $r$-direction but phantom in $T$-direction if $r < 1$. So that $r = 1$ is a transition point.

III. WORMHOLE THERMODYNAMICS

Thermal properties of wormholes have been studied in the literature. Hong & Kim constructed the wormhole’s entropy and Hawking temperature by exploiting Unruh effects and proposed a possibility of negative temperature originated from exotic matter distribution of the wormhole [35]. In [36], the authors have shown that the Einstein field equations can be rewritten as a similar form of the first law of thermodynamics at the dynamical trapping horizon for the (2+1)-dimensional evolving wormhole spacetime. In [37], the authors studied the generalized second law of thermodynamics at the apparent horizon of the evolving wormhole. In [38], the authors studied the validity of the generalized second law of thermodynamics by assuming the logarithmic correction to the horizon entropy of an evolving wormhole. In [39], the author has shown the validity of the generalize second law for a Euclidean wormhole.

We quote the laws of wormhole thermodynamics [40], as our later use “First law: The change in the gravitational energy of a wormhole equals the sum of the energy removed from the wormhole plus the work done in the wormhole. Second law: The entropy of a dynamical wormhole is given by its surface area which always increases. Third law: It is impossible to reach the absolute zero for surface gravity by any dynamical process.”

We consider the metric in the following form [20]

$$ds^2 = h_{ij}dx^idx^j + r^2d\Omega_2^2, \hspace{1cm} i, j = 0, 1$$  \hspace{1cm} (35)
where, \( h_{ij} = \left( -1, a^2 \left( 1 - \frac{b(r)}{r} \right)^{-1} \right) \). Now write, \( \tilde{r} = ar \). From this we get, \( \dot{\tilde{r}} = \tilde{r} H \). The unified first law is defined by \[ A \Psi + W dV, \] (36)

where

\[ A = 4\pi \tilde{r}^2, \] (37)

is the area and the volume \( V \) is defined by

\[ V = \frac{4}{3} \pi \tilde{r}^3. \] (38)

The unified first law (14) expresses the gradient of the active gravitational energy \( E \) according to the Einstein equation, divided into energy-supply and work terms. The first term on the right hand side could be interpreted as an energy supply term, i.e., this term produces a change in the energy of the spacetime due to the energy flux \( \Psi \) generated by the surrounding material (which generates this geometry). The second term \( W \) behaves like a work term, something like the work that the matter content must do to support this configuration \[ 40, 41 \].

The work density function is given by

\[ W = -\frac{1}{2} h_{ij} T_{ij} = \frac{1}{2} \left( \rho - p_r \right). \] (39)

The energy-supply vector is given by

\[ \Psi_i = h^{\lambda \gamma} T_{i \lambda} \partial_j (\tilde{r}) + W \partial_i (\tilde{r}) = \left( -\frac{1}{2} (\rho + p_r) \tilde{r} H, \frac{1}{2} (\rho + p_r) a \right). \] (40)

So we have

\[ \Psi = \Psi_i dx^i = \frac{1}{2} (\rho + p_r) (-\tilde{r} H dt + adr). \] (41)

The energy inside the surface is given by

\[ E = \frac{4\pi \tilde{r}}{8\pi G} \left( 1 - h^{ij} \partial_i \tilde{r} \partial_j \tilde{r} \right) = \frac{\tilde{r}}{2G} \left( \tilde{r}^2 H^2 + \frac{b}{r} \right). \] (42)

Now we get

\[ A \Psi + W dV = -4\pi \tilde{r}^3 H p_v dt + 4\pi a \tilde{r}^2 p_r dr. \] (43)

From (20), we get

\[ dE = \frac{\tilde{r} H}{2G} \left[ \tilde{r}^2 (2\dot{H} + 3H^2) + \frac{b}{r} \right] dt + \frac{1}{2G} \left[ 3a \tilde{r}^2 H^2 + ab' \right] dr. \] (44)
Using (21) and (22) and the unified first law (14), on comparing the coefficients of $dt$ and $dr$, we can directly obtained the field equations (7) and (8). Also using the conservation equation (11), the last field equation (9) can be obtained.

Now the Gibb’s law of thermodynamics states that

$$T_I dS_I = \frac{1}{3}(p_r + 2p_T)dV + d(\rho V), \quad (45)$$

where $S_I$ is the entropy within the horizon and assume the average pressure inside the horizon. The variation of internal entropy is obtained as

$$T_h dS_I = \frac{4\pi \tilde{r}_h^2}{3}(3\rho + p_r + 2p_T + \frac{\tilde{r}_h \rho'}{a})(d\tilde{r}_h - H\tilde{r}_h dt). \quad (46)$$

In the following subsection, the first law of thermodynamics will be derived using unified first law. Then the GSL will be examined for apparent and event horizons of the wormhole using first law of thermodynamics.

### A. Using First Law (of Thermodynamics)

We know that heat is one of the form of energy. Therefore, the heat flow $\delta Q$ through the horizon is just the amount of energy crossing it during the time interval $dt$. That is, $\delta Q = -dE$ is the change of the energy inside the horizon. So from equation (14) and (21) we have the amount of the energy crossing on the horizon as

$$-dE_h = 4\pi \tilde{r}_h^3 H p_r dt - 4\pi \tilde{r}_h^2 \rho(d\tilde{r}_h - H\tilde{r}_h dt) = 4\pi \tilde{r}_h^3 H(\rho + p_r)dt - 4\pi \tilde{r}_h^2 \rho d\tilde{r}_h. \quad (47)$$

From this, we see that there is no effect of density and tangential pressure on the horizon. The first law of thermodynamics (Clausius relation) on the horizon is defined as follows:

$$T_h dS_h = dQ = -dE_h. \quad (48)$$

From these equations, the variation of entropy on the horizon is given by

$$T_h dS_h = 4\pi \tilde{r}_h^3 H(\rho + p_r)dt - 4\pi \tilde{r}_h^2 \rho d\tilde{r}_h. \quad (49)$$

From (24) and (27), we obtain the variation of total entropy as

$$T_h \dot{S}_{total} = \frac{4\pi \tilde{r}_h^2}{3}(p_r + 2p_T + \frac{\tilde{r}_h \rho'}{a})\dot{\tilde{r}}_h + \frac{8\pi \tilde{r}_h^3 H}{3}(2p_r - 2p_T - \frac{\tilde{r}_h \rho'}{a}), \quad (50)$$
which becomes

\[ T_h \dot{S}_{total} = \frac{1}{6G} \left[ -3\tilde{r}_h^2(2\dot{H} + 3H^2) - 3b'(\tilde{r}_h/a) + \frac{b''(\tilde{r}_h/a)}{a\tilde{r}_h} \right] \dot{\tilde{r}}_h \]

\[ + \frac{H}{6G} \left[ \{\tilde{r}b(\tilde{r}_h/a) - 3ab(\tilde{r}_h/a)\} - \frac{2\tilde{r}_h}{a} \{\tilde{r}h b''(\tilde{r}_h/a) - 2ab'(\tilde{r}_h/a)\} \right]. \]  

(51)

Now we shall analyze the apparent and event horizons for wormhole and find out the radius on both horizons and investigate the GSL of thermodynamics in general way.

### 1. Apparent Horizon

The dynamical apparent horizon \( \tilde{r}_A \), a marginally trapped surface with vanishing expansion, is determined by the relation

\[ [h^{ij}\partial_i\tilde{r}\partial_j\tilde{r}], \tilde{r}=\tilde{r}_A = 0, \]  

(52)

i.e.,

\[ H^2\tilde{r}_A^2 = 1 - \frac{ab(\tilde{r}_A/a)}{\tilde{r}_A}. \]  

(53)

Taking derivative, we obtain,

\[ \dot{\tilde{r}}_A = \frac{H\tilde{r}_A\{\tilde{r}A b'(\tilde{r}_A/a) - ab(\tilde{r}_A/a) - 2\dot{H}\tilde{r}_A^3\}}{\{\tilde{r}A b'(\tilde{r}_A/a) - ab(\tilde{r}_A/a) + 2H^2\tilde{r}_A^3\}}. \]  

(54)

From (29), we obtain the rate of change of total entropy for apparent horizon as

\[ T_A \dot{S}_{total} = \frac{H\tilde{r}_A\{\tilde{r}A b'(\tilde{r}_A/a) - ab(\tilde{r}_A/a) - 2\dot{H}\tilde{r}_A^3\}}{6G\{\tilde{r}A b'(\tilde{r}_A/a) - ab(\tilde{r}_A/a) + 2H^2\tilde{r}_A^3\}} \left[ -3\tilde{r}_A^2(2\dot{H} + 3H^2) - 3b'(\tilde{r}_A/a) + \frac{b''(\tilde{r}_A/a)}{a\tilde{r}_A} \right] \]

\[ + \frac{H}{6G} \left[ \{\tilde{r}b(\tilde{r}_A/a) - 3ab(\tilde{r}_A/a)\} - \frac{2\tilde{r}_A}{a} \{\tilde{r}A b''(\tilde{r}_A/a) - 2ab'(\tilde{r}_A/a)\} \right]. \]  

(55)

The GSL for the apparent horizon will be satisfied if the r.h.s of the above expression is non-negative.

### 2. Event Horizon

Event horizon radius \( \tilde{r}_E \) can be found from the relation (i.e., \( ds^2 = 0 = d\Omega_2^2 \))

\[ \dot{\tilde{r}}_E = \tilde{r}_E H - \sqrt{1 - \frac{ab(\tilde{r}_E/a)}{\tilde{r}_E}}, \]  

(56)
or

\[
\int_{0}^{\tilde{r}_E} \frac{dr}{\sqrt{1 - \frac{b(r)}{r}}} = \int_{t}^{\infty} \frac{dt}{a}.
\]  

(57)

From (29), we obtain the rate of change of total entropy for event horizon as

\[
T_E \dot{S}_{total} = \frac{1}{6G} \left[ -3\tilde{r}_E^2 (2\dot{H} + 3H^2) - 3b'(\tilde{r}_E/a) + \frac{b''(\tilde{r}_E/a)}{a\tilde{r}_E} \right] \left( \tilde{r}_E H - \sqrt{1 - \frac{ab(\tilde{r}_E/a)}{\tilde{r}_E}} \right)
+ \frac{H}{6G} \left\{ \tilde{r}b(\tilde{r}_E/a) - 3ab(\tilde{r}_E/a) \} - \frac{2\tilde{r}_E}{a} \left\{ \tilde{r}_E b''(\tilde{r}_E/a) - 2ab'(\tilde{r}_E/a) \right\} \right].
\]

(58)

If the above expression is non-negative, we can say that the GSL is valid for event horizon.

In the following subsection, we shall consider the area law of thermodynamics i.e., the entropy on the horizon is proportional to the area of the spherical horizon surface. Then the GSL will be examined for apparent and event horizons of the wormhole using area law of thermodynamics.

**B. Using Area Law (of Thermodynamics)**

Now we shall analyze the apparent and event horizons for wormhole and find out the radius on both horizons and investigate the GSL of thermodynamics in general way.

1. **Apparent Horizon**

The surface gravity is defined as

\[
\kappa = \frac{1}{2\sqrt{-h}} \partial_i(\sqrt{-h} \ h^{ij} \partial_j \tilde{r}).
\]

(59)

Here \( h = det(h_{ij}) \). The dynamical apparent horizon radius \( \tilde{r}_A \) is given in equation (31). So we get the surface gravity on the apparent horizon:

\[
\kappa = \frac{1}{2} \tilde{r}_A (\dot{H} + 2H^2) + \frac{1}{4\tilde{r}_A} \left[ ab(\tilde{r}_A/a) - \tilde{r}_A b'(\tilde{r}_A/a) \right].
\]

(60)

Now the apparent horizon temperature is

\[
T_A = \frac{\kappa}{2\pi} = -\frac{1}{4\pi} \tilde{r}_A (\dot{H} + 2H^2) + \frac{1}{8\pi \tilde{r}_A^2} \left[ ab(\tilde{r}_A/a) - \tilde{r}_A b'(\tilde{r}_A/a) \right].
\]

(61)
Since the area of the wormhole horizon is \( A = 4\pi \tilde{r}_A^2 \), so one can relate the entropy with the surface area of the apparent horizon (area law) through \( S_A = A/4G \). Therefore we have

\[
S_A = \frac{\pi \tilde{r}_A^2}{G},
\]

so that

\[
dS_A = \frac{2\pi \tilde{r}_A d\tilde{r}_A}{G}.
\]

Using (24), (39) and (41) we have

\[
T_A \dot{S}_{\text{total}} = T_A (\dot{S}_I + \dot{S}_A) = \frac{4\pi \tilde{r}_A^2}{3} (3\rho + p_r + 2p_T + \frac{\tilde{r}_A^4'}{a})(\dot{\tilde{r}}_A - H\tilde{r}_A) + \frac{2\pi \tilde{r}_A T_A \dot{\tilde{r}}_A}{G},
\]

which can be written as

\[
T_A \dot{S}_{\text{total}} = \frac{H(H^2 + \dot{H})\tilde{r}_A^4}{3G\{\tilde{r}_A b' (\tilde{r}_A/a) - ab(\tilde{r}_A/a) + 2H^2\tilde{r}_A^3\}} \left[ 3\dot{H} \tilde{r}_A^2 - \frac{b''(\tilde{r}_A/a)}{a\tilde{r}_A} \right]
\]

\[
+ \frac{H}{2G} \left[ ab(\tilde{r}_A/a) - \tilde{r}_A b' (\tilde{r}_A/a) - 2\tilde{r}_A^3(\dot{H} + 2H^2) \right] \left\{ \frac{\tilde{r}_A^4 b' (\tilde{r}_A/a) - ab(\tilde{r}_A/a) - 2\tilde{r}_A^3}{\tilde{r}_A^4 b' (\tilde{r}_A/a) - ab(\tilde{r}_A/a) + 2H^2\tilde{r}_A^3} \right\},
\]

The GSL for the apparent horizon will be satisfied if the r.h.s of the above expression is non-negative.

2. Event Horizon

We obtain the rate of change of total entropy for event horizon as

\[
T_E \dot{S}_{\text{total}} = T_E (\dot{S}_I + \dot{S}_E) = \frac{4\pi \tilde{r}_E^2}{3} (3\rho + p_r + 2p_T + \frac{\tilde{r}_E^4'}{a})(\dot{\tilde{r}}_E - H\tilde{r}_E) + \frac{2\pi \tilde{r}_E T_E \dot{\tilde{r}}_E}{G},
\]

which can be written as

\[
T_E \dot{S}_{\text{total}} = \frac{1}{6G} \left[ 6\tilde{r}_E^2 \dot{H} - \frac{b''(\tilde{r}_E/a)}{a\tilde{r}_E} \right] \sqrt{1 - \frac{ab(\tilde{r}_E/a)}{\tilde{r}_E}}
\]

\[
+ \frac{H}{2G} \left[ ab(\tilde{r}_E/a) - \tilde{r}_E b'(\tilde{r}_E/a) - 2\tilde{r}_E^3(\dot{H} + 2H^2) \right] \left[ \tilde{r}_E H - \sqrt{1 - \frac{ab(\tilde{r}_E/a)}{\tilde{r}_E}} \right].
\]

If the above expression is non-negative, we can say that the GSL is valid for event horizon.
IV. CONCLUSIONS

We have studied the time-dependent Lorentzian Wormhole model in presence of anisotropic (i.e., radial and tangential) pressure. The density and pressure are considered in both $t$ and $r$ dependent. For isotropic pressure, the radial pressure transforms to function of time only. In this case, we obtain the shape function in the form $b_r = kr^3$, so the the model reduces to the standard FRW model. We have shown that the Einstein’s field equations and unified first law are equivalent for the dynamical wormhole model. We have presented some exact solutions of Einstein equations for anisotropic pressure case. Introducing two EoS parameters we have shown that these solutions give very rich dynamics of the universe yielding to the different expansion history of it in the $r$ - direction and in the $T$ - direction. The corresponding explicit forms of the shape function $b(r)$ is presented. The first law of thermodynamics has been derived by using the Unified first law in presence of anisotropic pressure. The physical quantities including surface gravity ($\kappa$) and the equilibrium temperature ($T$) are derived for the wormhole model. Here we have obtained all the results like entropy on the horizons, variation of internal and horizon entropies in general way without any choice of the shape function. Finally, the validity of generalized second law (GSL) of thermodynamics has been examined at apparent and event horizons by considering first law and area law of thermodynamics for the evolving Lorentzian wormhole.

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