Cohomology of generalised configuration spaces of points on $\mathbb{R}^r$

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April 20, 2020

Abstract. We compute the cohomology ring of a generalised type of configuration space of points in $\mathbb{R}^r$. This configuration space is indexed by a graph. In the case the graph is complete the result is known and it is due to Arnold and Cohen. However, our computations give a generalisation to any graph and an alternative proof of the classical result. Moreover, we show that there are deletion-contraction short exact sequences for this cohomology rings.

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1 Introduction

The configuration space of $n$ points in $\mathbb{R}^r$, that we denote as $\text{Conf}_r(n)$, is defined as

$$\text{Conf}_r(n) = \{(x_1, \ldots, x_n) \in \mathbb{R}^{rn}; x_i \neq x_j \text{ for } i \neq j, 0 < i, j \leq n\}.$$ 

Its cohomology ring has been computed by Arnold [1] in the case $r = 2$ and Cohen [3] for $r \geq 3$ and it is given by the following quotient of graded rings

$$H^*(\text{Conf}_r(n)) = \mathbb{Z}[e_{m,i}]/\sim$$

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where $0 < i, j \leq n$, $\mathbb{Z}[e_{\alpha_{i,j}}]$ is the free commutative graded algebra generated by $e_{\alpha_{i,j}}$ of degree $r - 1$ and $\sim$ are the relations

- $e_{\alpha_{i,j}} = (-1)^r e_{\alpha_{j,i}}$
- $e_{\alpha_{i,j}}^2 = 0$ if $r$ is odd
- $e_{\alpha_{a,b}} e_{\alpha_{b,c}} + e_{\alpha_{b,c}} e_{\alpha_{c,a}} + e_{\alpha_{a,c}} e_{\alpha_{a,b}} = 0$

In this paper we will study a generalisation of the definition of the configuration spaces $\text{Conf}_r(n)$ to configuration spaces depending on a graph. These were defined by Eastwood and Huggett [4], and also described in [2]. Given a graph $\Gamma$, we denote the configuration space of points in $\mathbb{R}^r$ depending on a graph by $\text{Conf}_r(\Gamma)$. Let $\alpha_{i,j}$ denote an edge in $\Gamma$ between the vertices $v_i$ and $v_j$, the generalised configuration space of points in $\mathbb{R}^r$ is defined as

$$\text{Conf}_r(\Gamma) = \{(x_1, \ldots, x_n) \in \mathbb{R}^{rn}; x_i \neq x_j \text{ if } \alpha_{i,j} \text{ is an edge in } \Gamma\}.$$ 

The main result in the article is provided by the computation of the cohomology of $\text{Conf}_r(\Gamma)$ for any graph $\Gamma$. The cohomology of $\text{Conf}_r(\Gamma)$ is given by a commutative graded ring that depends on the parity of the integer $r$. Let $\mathbb{Z}[e_{\alpha_{i,j}}]$ be the free commutative graded algebra generated by $e_{\alpha_{i,j}}$ of degree $r - 1$, where $\alpha_{i,j}$ is an edge in $\Gamma$ between the vertices $i$ and $j$ oriented from $i$ to $j$. Let $w$ be a circuit in $\Gamma$, that is graph consisting of an ordered set of edges $w_1, w_2, \ldots, w_k$ and vertices $v_1(w), \ldots, v_k(w), v_{k+1}(w) = v_1(w)$ such that $v_i(w), v_{i+1}(w)$ are the two vertices incident to $w_i$. We denote by $e_w = e_{v_1,2} \cdot e_{v_2,3} \cdot \ldots \cdot e_{v_k,1}$ the product of the generators corresponding to the edges in the cycle $w$. We prove that the cohomology ring is given by the following quotient of graded rings

$$H^*(\text{Conf}_r(\Gamma)) = \mathbb{Z}[e_{\alpha_{i,j}}]/\sim$$

where $\sim$ are the relations

- $e_{\alpha_{i,j}} = (-1)^r e_{\alpha_{j,i}}$
- $e_{\alpha_{i,j}}^2 = 0$ if $r$ is odd
- $A(w) = \sum_i (-1)^{i(r-1)} e_{v_{i,2}} \cdots e_{v_{i,j}} \cdots e_{v_{i,j}} = 0$ for every circuit $w$ in $\Gamma$.

We call the relations $A(w)$ generalised Arnold relations. Moreover, let $\Gamma \setminus a$ denote the graph obtained from $\Gamma$ by deleting the edge $\alpha$, and $\Gamma/\alpha$ the graph obtained by contracting the edge $\alpha$. There is a deletion-contraction short exact sequence in cohomology

$$0 \to H^*(\text{Conf}_r(\Gamma \setminus a)) \to H^*(\text{Conf}_r(\Gamma)) \to H^{*-r+1}(\text{Conf}_r(\Gamma/\alpha)) \to 0.$$ 

Our approach is as follows. In section 2 we define a graded commutative ring depending on a graph $\Gamma$. The definition is purely algebraic. There are
two cases, depending on whether the generators are in even or odd degrees. We establish a deletion-contraction exact sequence for these rings. Such long exact deletion-contraction sequences are well known in graph cohomology, but our main algebraic result is that these long exact sequences actually break up into short exact sequences.

In section 3 we show that the cohomology rings of the graph indexed configuration space of points in an open discs are given by the rings defined in the previous section. The method is to first establish deletion-contraction long exact sequences of cohomology of generalized configuration spaces, then to show that the algebraic exact sequences of the previous paragraph maps to these short exact sequences by surjections, and conclude that these surjections are actually isomorphisms by induction over the number of edges in the graphs. This method also gives an alternative approach to the computation of the cohomology of not generalized configuration spaces.

The results here presented are part of the second author’s Ph.D. thesis Graph complexes and cohomology of configuration spaces, supervised by the first author.

2 Algebraic description of $R^r(\Gamma)$

In this section we will describe a graded commutative ring $R^r(\Gamma)$. If $r$ is even, $\Gamma$ can be any not oriented graph. If $r$ is odd, we demand that each edge of $\Gamma$ come with an orientation, that is an ordering of its two adjacent vertices. In both cases we will assume that $\Gamma$ does not have loops, but we do allow multiple edges.

We introduce some notation. A circuit in a graph consists of ordered sets of edges $w_1, w_2, \ldots, w_k$ and vertices $v_1(w), \ldots, v_k(w), v_{k+1}(w) = v_1(w)$ such that $v_i(w), v_{i+1}(w)$ are the two vertices incident to $w_i$. In case $r$ is odd, the circuit $w$ comes with an additional signs $\epsilon_i(w)$. The given orientation of the edge $w_i$ determines an order of the pair of vertices $v_i(w), v_{i+1}(w)$, If in this order $v_i < v_{i+1}$, then $\epsilon_i(w) = 1$, else $\epsilon_i(w) = -1$.

For a circuit $w$ of length $l(w)$, we denote by $e_w = w_1 \cdot w_2 \cdots w_l(w)$ the product of the generators corresponding to the edges in the circuit $w$. If one changes the order of $w$ by a cyclic permutation to obtain a new circuit $w'$, one changes $e_w$ by at most by a sign: $e_w = \pm e_{w'}$.

For $r$ and $\Gamma$ we will define a graded commutative algebra $\Lambda^r(\Gamma)$, and an ideal $I^r(\Gamma)$ in this algebra. The precise definitions depend on whether $r$ is even or odd.

**Definition 2.1.** If $r$ is even, the algebra $\Lambda^r(\Gamma)$ is the free graded commutative algebra over the integers with one generator $e_\alpha$ in degree $r - 1$ for each edge $\alpha \in E(\Gamma)$. If $r$ is odd, $\Lambda^r(\Gamma)$ is the quotient of the graded commutative algebra over the integers with one generator $e_\alpha$ for each edge $\alpha \in E(\Gamma)$ by the relations $e_\alpha^2 = 0$. In this case, $\Lambda^r(\Gamma)$ is actually commutative.
circuit we define its Arnold class:

\[ A(w) = \begin{cases} 
\sum_i (-1)^i w_1 \cdots \hat{w}_i \cdots w_l(w) & \text{if } r \text{ is even,} \\
\sum_i \epsilon_i(w) w_1 \cdots \hat{w}_i \cdots w_l(w) & \text{if } r \text{ is odd.}
\end{cases} \]

In case \( w \) consists of a single edge, which then has to be loop, this will be interpreted as \( A(w) = 1 \). Let the generalized Arnold ideal \( I^r(\Gamma) \) be the ideal of \( \Lambda^r(\Gamma) \) generated by the Arnold classes. Finally, let \( R^r(\Gamma) \) be the quotient ring \( \Lambda^r(\Gamma)/I^r(\Gamma) \).

A map of graphs \( f : \Gamma \to \Gamma' \) induces a map \( f^R : \Lambda^r(\Gamma) \to \Lambda^r(\Gamma') \) which preserves the Arnold classes, so it also induces a map of rings \( f^R : R^r(\Gamma) \to R^r(\Gamma') \).

We can usually assume that \( \Gamma \) has no multiple edges, since the following lemma holds.

**Lemma 2.2.** Let \( \Gamma \) be a graph and \( e \) an edge of \( \Gamma \) such that there exists a different edge \( e' \) incident to the same vertices as \( e \). Let \( \Gamma' = \Gamma \setminus e \). Then, the rings \( R^r(\Gamma) \) and \( R^r(\Gamma') \) are isomorphic.

**Proof.** There is an inclusion of graphs \( i : \Gamma' \to \Gamma \), and a left inverse \( p \) to \( i \), such that \( p(e) = e' \). In \( R^r(\Gamma) \) we have the generalised Arnold relation \( e' - e = 0 \), so that \( e' = e \in R^r(\Gamma) \). It follows easily that the induced maps \( i_* : R^r(\Gamma') \to R^r(\Gamma) \) and \( p_* : R^r(\Gamma) \to R^r(\Gamma') \) are inverse isomorphisms. \( \square \)

### 2.1 Deletion-contraction short exact sequence for \( R^r(\Gamma) \)

Let \( \Gamma \) be a graph and \( \alpha \in E(\Gamma) \). We will mainly be interested in graphs without loops and multiple edges, but it is convenient not to exclude these cases, in order to be able to formulate certain induction arguments in a smooth way. We can delete the edge \( \alpha \) from the graph \( \Gamma \) to obtain the graph \( \iota_\alpha : \Gamma \setminus \alpha \subset \Gamma \). We can also contract the edge \( \alpha \) to obtain a graph \( \Gamma/\alpha \). The graph \( \Gamma/\alpha \) might have multiple edges, but it does not have loops.

There is a map \( p_\alpha : \Gamma \to \Gamma/\alpha \) which identifies the two vertices incident to \( \alpha \). There are induced maps of graded algebras \( i^r_\alpha : \Lambda^r[\Gamma \setminus \alpha] \to \Lambda^r(\Gamma) \) and \( p^r_\alpha : \Lambda^r[\Gamma] \to \Lambda^r[\Gamma/\alpha] \).

If \( \beta \in E(\Gamma \setminus \alpha) \) we alternatively denote its image \( i_\alpha(\beta) \in E(\Gamma) \) by \( \beta \). If \( \gamma \in E(\Gamma) \), we alternatively denote its image in \( p_\alpha(\gamma) \in E(\Gamma/\alpha) \) by \( [\gamma] \).

Let \( \Lambda[e_\alpha] \) be the exterior algebra on the generator \( e_\alpha \) of degree \( r - 1 \).

**Definition 2.3.** We consider the following two ring homomorphisms:

\[ i_\alpha : \Lambda^r[\Gamma/\alpha] \to \Lambda^r(\Gamma/\alpha) \otimes \Lambda^r[e_\alpha] \quad \psi_\alpha : \Lambda^r(\Gamma) \to \Lambda^r(\Gamma/\alpha) \otimes \Lambda^r[e_\alpha] \]

\[ i_\alpha(e_{[\eta]}) = e_{[\eta]} \otimes 1 \quad \psi_\alpha(e_{[\eta]}) = \begin{cases} 
1 \otimes e_{[\alpha]} & \text{if } \eta \neq \alpha, \\
e_{[\eta]} \otimes 1 & \text{if } \eta = \alpha.
\end{cases} \]
Consider the generalized Arnold classes in $\Lambda^r(\Gamma/\alpha) \otimes \Lambda^r[e_\alpha]$. We define the Arnold ideal in $\Lambda^r(\Gamma/\alpha) \otimes \Lambda^r[e_\alpha]$ to be the ideal generated by the generalized Arnold classes in $\Lambda^r(\Gamma/\alpha) \subset \Lambda^r(\Gamma/\alpha) \otimes \Lambda^r[e_\alpha]$. If $\Gamma'$ is the graph obtained from $\Gamma/\alpha$ by adding a single vertex and a single edge connecting the new vertex to $\alpha$, there is an obvious isomorphism preserving Arnold ideals between this ring and $\Lambda(\Gamma')$.

**Lemma 2.4.** The following diagram of ring maps is commutative and the maps preserve the generalized Arnold ideals.

\[
\begin{array}{ccc}
\Lambda^r(\Gamma \setminus \alpha) & \xrightarrow{p^\alpha_{\Lambda} \circ i^\Lambda_\alpha} & \Lambda^r(\Gamma/\alpha) \\
\downarrow i^\Lambda_\alpha & & \downarrow i^\Lambda_\alpha \\
\Lambda^r(\Gamma) & \xrightarrow{\psi^\Lambda_\alpha} & \Lambda^r(\Gamma/\alpha) \otimes \Lambda^r[e_\alpha]
\end{array}
\]

**Proof.** The diagram commutes since if $e_\eta \in E[\Gamma \setminus \alpha]$ is a generator both paths to the lower right square takes it to $[\beta] \otimes 1$. $i^\Lambda_\alpha$ and $p^\alpha_{\Lambda}$ takes an Arnold class to an Arnold class, since they are induced by maps of graphs. The map $i^\Lambda_\alpha$ obviously preserves the Arnold elements, so we only need to check that $\psi^\Lambda_\alpha$ does. Let $w$ be a circuit in $\Gamma$. We have to prove that $\psi^\Lambda_\alpha(A(w))$ is contained in the ideal generated by $A(u) \otimes 1$ for $u$ a circuit in $\Gamma/\alpha$. There are three cases. If $w_i \neq \alpha$ for all $i$, then $u = p^\Lambda_{\alpha}(w)$ is a circuit in $\Gamma/\alpha$ such that $p^\Lambda_{\alpha}(A(w)) = A(u)$, and we are done. If $\alpha$ occurs more than once in $w$, then $A(w) = 0$, and we are done again. If finally $e_i = \alpha$ for a unique $i$, then $u = [w_1][w_2] \ldots [\hat{w}_i] \ldots [w_l(w)]$ is a circuit in $\Gamma/\alpha$ such that $\psi^\Lambda_\alpha(A(w)) = A(u)$, and all our work is done. \qed

It follows from lemma 2.4 that we get an induced commutative diagram of ring maps

\[
\begin{array}{ccc}
R^r(\Gamma \setminus \alpha) & \xrightarrow{p^R_{\alpha} \circ i^R_\alpha} & R^r(\Gamma/\alpha) \\
\downarrow i^R_\alpha & & \downarrow i^R_\alpha \\
R^r(\Gamma) & \xrightarrow{\psi^R_\alpha} & R^r(\Gamma/\alpha) \otimes \Lambda^r[e_\alpha]
\end{array}
\]

**Theorem 2.5.** The above diagram is a pullback diagram. The map $i^R_\alpha : R^r(\Gamma \setminus \alpha) \to R^r(\Gamma/\alpha)$ is injective, and the map $\psi^R_\alpha : R^r(\Gamma) \to R^r(\Gamma/\alpha) \otimes \Lambda^r[e_\alpha]$ is surjective.

We will prove this theorem in the next subsection. As consequences we have

**Corollary 2.6.** Suppose that $\Gamma'$ is a subgraph of $\Gamma$ such that $V(\Gamma') = V(\Gamma)$. The map induced by inclusion $i_* : R^r(\Gamma') \to R^r(\Gamma)$ is injective.
Corollary 2.7. For every $\alpha \in E(\Gamma)$ there is a short exact sequence of Abelian groups

$$0 \to R^r(\Gamma \setminus \alpha)_k \to R^r(\Gamma)_k \to R^r(\Gamma/\alpha)_{k-r+1} \to 0$$

where the indices $k$ and $k - r + 1$ denote the grading in the ring.

2.2 Proof of Theorem 2.5

Let $\Lambda^r[\Gamma/\alpha] \otimes e_{\alpha}$ denote the ideal in $\Lambda^r[\Gamma/\alpha] \otimes \Lambda^r[e_{\alpha}]$ generated by $e_{\alpha}$. As an Abelian group, $\Lambda^r[\Gamma/\alpha] \otimes \Lambda^r[e_{\alpha}]$ is the direct sum of the image of the injective map $i_{\alpha}^\Lambda$ and the ideal $\Lambda^r[\Gamma/\alpha] \otimes e_{\alpha}$. Let $\pi : \Lambda^r[\Gamma/\alpha] \otimes \Lambda[e_{\alpha}] \to \Lambda^r[\Gamma/\alpha] \otimes e_{\alpha}$ be the projection, and define

$$g_{\alpha}^\Lambda = \pi \circ \psi_{\alpha}^\Lambda : \Lambda[\Gamma] \to \Lambda^r[\Gamma/\alpha] \otimes e_{\alpha}.$$ 

Since $\psi_{\alpha}^\Lambda$ and $\pi$ preserve the ideal generated by the Arnold classes, so does $g_{\alpha}^\Lambda$. We obtain a restricted map $g_{\alpha}^I : I(\Gamma) \to I(\Gamma/\alpha) \otimes e_{\alpha}$ and a quotient map $g_{\alpha}^R : R^r(\Gamma) \to R^r(\Gamma/\alpha) \otimes e_{\alpha}$. We also immediately obtain a commutative diagram:

\[
\begin{array}{ccccccccc}
0 & 0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & I^r(\Gamma \setminus \alpha) & \Lambda^r[\Gamma \setminus \alpha] & R^r(\Gamma \setminus \alpha) & 0 \\
\downarrow & i_{\alpha}^I & \downarrow i_{\alpha}^\Lambda & \downarrow g_{\alpha} & \downarrow 0 \\
0 & I^r(\Gamma) & \Lambda^r[\Gamma] & R^r(\Gamma) & 0 \\
\downarrow & g_{\alpha}^I & \downarrow g_{\alpha}^\Lambda & \downarrow g_{\alpha}^R & \downarrow 0 \\
0 & I^r(\Gamma/\alpha) \otimes e_{\alpha} & \Lambda^r[\Gamma/\alpha] \otimes e_{\alpha} & R^r(\Gamma/\alpha) \otimes e_{\alpha} & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & 0 & 0 & 0
\end{array}
\]

The rows of this diagram are short exact by definition. Most of this subsection will go into proving that the three columns are short exact.

Lemma 2.8. The middle column in diagram [1] is exact.

**Proof.** The inclusion $i_{\alpha}^I$ is clearly injective by its definition.

An element in $\Lambda[\Gamma]$ can be uniquely written as $x + ye_{\alpha}$ where $x, y$ are products of edges different from $\alpha$, that is $x, y$ are both in the image of $i_{\alpha}^I$. The image of $i_{\alpha}^I$ are the classes for which $y = 0$. Since

$$g_{\alpha}^\Lambda(x + ye_{\alpha}) = \pi(x \otimes 1 + y \otimes e_{\alpha}) = y \otimes e_{\alpha},$$

This completes the proof.
It follows that the kernel of \( g^\Lambda_{\alpha} \) also consists of the classes for which \( y = 0 \). This proves exactness at \( \Lambda'(\Gamma) \). Finally, the map \( p^\Lambda_{\alpha} \circ \iota^\Lambda_{\alpha} : \Lambda'(\Gamma \setminus \alpha) \to \Lambda'(\Gamma/\alpha) \) is an isomorphism, so that for any class \( x \otimes e_\alpha \in \Lambda(\Gamma/\alpha) \otimes e_\alpha \) we can find \( \bar{x} \in \Lambda'(\Gamma \setminus \alpha) \) such that \( p^\Lambda_{\alpha} \iota^\Lambda_{\alpha} \bar{x} = x \) and

\[
g^\Lambda_{\alpha}(\iota^\Lambda_{\alpha}(\bar{x})e_\alpha) = \pi(\psi^\Lambda_{\alpha}(\iota^\Lambda_{\alpha}(\bar{x})e_\alpha)) = \pi(p^\Lambda_{\alpha} \iota^\Lambda_{\alpha} \bar{x} \otimes e_\alpha) = \pi(x \otimes e_\alpha) = x \otimes e_\alpha.
\]

It follows that \( g^\Lambda_{\alpha} \) is surjective. \( \square \)

**Lemma 2.9.** The map \( g^\alpha_{\lambda} \) in diagram 7 is surjective.

**Proof.** The map \( g^\alpha_{\lambda} \) is a map of \( \Lambda'(\Gamma \setminus \alpha) \)-modules since

\[
g^\alpha_{\lambda}((x)(x_2 + y_2 e_\alpha)) = g^\alpha_{\lambda}(x_1 x_2 + (x_1 y_2)e_\alpha) = x_1 y_2 \otimes e_\alpha = x_1 g^\alpha_{\lambda}(x_2 + y_2 \alpha)
\]

This means that it is sufficient to prove that each element of a set of generators for \( I(\Gamma/\alpha) \otimes e_\alpha \) as \( \Lambda(\Gamma \setminus \alpha) \)-module is in the image of \( g^\alpha_{\lambda} \). Note that the map \( x \mapsto x \otimes e_\alpha \) defines an isomorphism of \( \Lambda(\Gamma \setminus \alpha) \) modules \( I(\Gamma/\alpha) \to I(\Gamma/\alpha) \otimes e_\alpha \). It follows that if we define

\[
\bar{A}(w) = A(w) \otimes e_\alpha
\]

for circuits \( w \) in \( \Gamma/\alpha \), then the classes \( \bar{A}(w) \) form a set of generators for \( I(\Gamma/\alpha) \otimes e_\alpha \). We conclude that it suffices to show that for each circuit \( w \) in \( \Gamma/\alpha \), the class \( \bar{A}(w) \) is in the image of \( g^\alpha_{\lambda} \).

Let \( v_1, v_2 \in V(\Gamma) = V(\Gamma \setminus \alpha) \) be the vertices incident to \( \alpha \), and \( v \in V(\Gamma/\alpha) \) the vertex given by collapsing \( v_1 \) and \( v_2 \). The vertex \( v \) might be incident to some of the edges \( [w_i] \in E(\Gamma/\alpha) \). Since we are assuming that \( \Gamma \) has no multiple edges or loops, \( \Gamma/\alpha \) also has no loops, although it might have double edges. We can decompose the circuit \( w \) as a composition of circuits \( w^{(i)} \), each starting and ending with the vertex \( v \), such that

\[
w^{(1)}_1, w^{(1)}_2, \ldots, w^{(1)}_{l(w^{(1)})}, w^{(2)}_1, \ldots, w^{(2)}_{l(w^{(2)})}, \ldots, w^{(k)}_1, \ldots, w^{(k)}_{l(w^{(k)})}
\]

is a cyclic reordering of \( w_1 \ldots w_{l(w)} \). A cyclic reordering will at most flip the sign of \( A(w) \) so we get that

\[
A(w) = \sum_i \pm e_{w^{(i)}} e_{w^{(i+1)}} \cdots e_{w^{(i-1)}} A(w^{(i)}) e_{w^{(i+1)}} \cdots e_{w^{(k)}}
\]

This reduces the lemma further to the case when at most two edges of \( w \) are incident to \( v \).

Let the circuit be \( \{w_1, w_2, \ldots, w_{l(w)}\} \). Since the map \( p \circ i : E(\Gamma \setminus \alpha) \to E(\Gamma/\alpha) \) is a bijection, each edge \( w_i \in E(\Gamma/\alpha) \) is the image of some unique \( w'_i \in E(\Gamma \setminus \alpha) \). If the edges \( w'_1, \ldots, w'_{l(w)} \) form a circuit \( w' \) in \( \Gamma \setminus \alpha \), then \( A(w) = g^\alpha_{\lambda}(w') \), and we are finished here.
If the edges $w'_1, \ldots, w'_l(w)$ do not form a circuit, this is because there is an $i$ so that $w_i$ and $w_{i+1}$ are adjacent to $v_1$ and $v_2$ (in either order). Since $\alpha$ is an edge incident to the vertices $v_1,v_2$, we can form the circuit $w''$ to be circuit $w'_1, \ldots w'_i, \alpha, w'_{i+1} \ldots w'_l(w)$. Then $g^I_\alpha(w'') = \pm A(w)$, and the proof is complete.

**Corollary 2.10.** The sequence

$$R^r(\Gamma \setminus \alpha) \xrightarrow{i^R_\alpha} R^r(\Gamma) \xrightarrow{g^R_\alpha} R^r(\Gamma / \alpha) \to 0$$

is exact.

**Proof.** The map $g^R_\alpha$ is surjective since $g^\Lambda_\alpha$ is surjective. That the composite $g^R_\alpha i^R_\alpha$ is trivial follows from a simple diagram chase, using that the composite in the middle column is trivial, and that the quotient map $\Lambda[\Gamma \setminus \alpha] \to R^r(\Gamma \setminus \alpha)$ is surjective. The only thing left to check is that $\text{im}(i^R_\alpha) = \ker(g^R_\alpha)$.

The columns of the diagram are chain complexes, so that the diagram defines a short exact sequence of chain complexes. By lemma 2.8 the homology of the middle column vanishes. Using the long exact sequence of a short exact sequence of chain complexes, we see that the quotient $\ker(g^R_\alpha) / \text{im}(i^R_\alpha)$ is isomorphic to the cokernel of the map $g^I_\alpha$. According to lemma 2.9 this cokernel is trivial.

In preparation for the proof of theorem 2.5, we need a lemma.

**Lemma 2.11.** Let $\alpha, \beta$ be two different edges of $\Gamma$. The following diagram is commutative

$$
\begin{array}{c}
R^r(\Gamma \setminus \alpha) \xrightarrow{g^R_\beta} R^r(\Gamma \setminus \alpha / \beta) \otimes e_\beta \\
\downarrow i^R_\alpha \quad \downarrow i^R_\alpha \otimes \text{Id} \\
R^r(\Gamma) \xrightarrow{g^R_\beta} R^r(\Gamma / \beta) \otimes e_\beta
\end{array}
$$

**Proof.** In the formulation of the lemma we have tacitely and legitimately identified the graph $(\Gamma \setminus \alpha)/\beta$ with the graph $(\Gamma / \beta) \setminus \alpha$. We first note the commutativity of the diagram

$$
\begin{array}{c}
\Lambda^r(\Gamma \setminus \alpha) \xrightarrow{\psi^\Lambda_\beta} \Lambda^r(\Gamma \setminus \alpha / \beta) \otimes \Lambda^r[e_\beta] \\
\downarrow i^R_\alpha \quad \downarrow i^R_\alpha \otimes \text{Id} \\
\Lambda^r(\Gamma) \xrightarrow{\psi^\Lambda_\beta} \Lambda^r(\Gamma / \beta) \otimes \Lambda^r[e_\beta]
\end{array}
$$

Since $\psi^\Lambda_\beta$ and $i^R_\alpha$ are ring maps, it suffices to check this on generators $e_\eta$, which is trivial to do. Applying the projection $\pi$, we obtain that the following
diagram is commutative:

\[
\begin{array}{ccc}
\Lambda^r(\Gamma \setminus \alpha) & \xrightarrow{g^R_\beta} & \Lambda^r(\Gamma \setminus \alpha/\beta) \otimes e_\beta \\
\downarrow i^R_\alpha & & \downarrow \delta_\alpha \otimes \text{Id} \\
\Lambda^r(\Gamma) & \xrightarrow{g^R_\beta} & \Lambda^r(\Gamma/\beta) \otimes e_\beta
\end{array}
\]

(4)

The statement of the lemma follows from that there is a surjective map from diagram 4 to diagram 2.

Note that if Γ does not have any loops, the ring map \(c : \Lambda[\Gamma] \to \mathbb{Z}\), \(c(1) = 1\) and \(c(e_\alpha) = 0\) for all edges \(\alpha\) in \(\Gamma\) factors over \(R^r(\Gamma)\). This is not the case if Γ has a loop \(\alpha\) because the Arnold relation corresponding to the circuit consisting of the single edge \(\alpha\) is not mapped to 0 by \(c\). It follows that if Γ has no loops, the canonical map \(\mathbb{Z} \to R^r(\Gamma)\) is a split inclusion, with left inverse the map \(\eta\) that maps each \(e_\alpha\) to 0. We say that a graph Γ satisfies (*) if both of the following two statements are true.

- If Γ has no loops, for each \(\alpha \in E(\Gamma)\) the map \(i_\alpha : R^r(\Gamma \setminus \alpha) \to R^r(\Gamma)\) is injective.
- If Γ does not have loops or multiple edges and if \(x \in R^r(\Gamma)\) and \(x \neq \mathbb{Z}\), there exists a \(\beta \in E(\Gamma)\) such that \(g^R_\beta(x) \neq 0\).

**Lemma 2.12.** Every graph Γ satisfies (*).

**Proof.** We will argue by induction on the number of edges of Γ. The graph with one vertex and no edges satisfies (*) for trivial reasons.

The induction hypothesis is that every graph with at most \(n - 1\) edges satisfies (*). Let Γ be a graph with \(n\) edges. We need to show that Γ satisfies (*).

We first show that \(i_\alpha^R : R^r(\Gamma \setminus \alpha) \to R^r(\Gamma)\) is injective. Using lemma 2.2 we easily reduce to the case that Γ has no multiple edges. The map \(i_\alpha\) preserves the direct sum decomposition \(R^r(\Gamma) \cong \mathbb{Z} \oplus \ker(\eta)\), so it suffices to show that if \(\alpha \in E(\Gamma)\), \(x \in R^r(\Gamma) \setminus \mathbb{Z}\), then \(i_\alpha^R(x) \neq 0\).

Since \(\Gamma \setminus \alpha\) has no multiple edges and satisfies (*) by assumption, there is an edge \(\beta \in E(\Gamma \setminus \alpha)\) such that \(g^R_\beta(x) \neq 0 \in R^r(\Gamma \setminus \alpha/\beta)\). Since \(\Gamma/\beta\) has no loops, it satisfies (*), \(i_\beta^R g^R_\beta(x) \neq 0\). Now apply lemma 2.11 to prove that \(i_\alpha^R(x) \neq 0 \in R^r(\Gamma)\) as required.

We finally need to prove that if Γ has no multiple edges, \(x \in R^r(\Gamma) \setminus \mathbb{Z}\) and \(g^R_\eta(x) = 0\) for all \(\eta \in E(\Gamma)\), then \(x = 0\). Pick any \(\alpha \in E(\Gamma)\). Since \(g^R_\alpha(x) = 0\), by lemma 2.10 there is an \(y \in R^r(\Gamma \setminus \alpha)\) such that \(i^R_\alpha(y) = x\). Using lemma 2.11 again, we see that for any \(\beta \in E(\gamma \setminus \alpha)\):

\[(i_\alpha^R \otimes \text{Id})g^R_\beta(y) = g^R_\beta i_\alpha^R(y) = g^R_\beta(x) = 0\]
Because $\Gamma/\beta$ satisfies $(*),$ and because $(\Gamma \setminus \alpha)/\beta$ either equals $\Gamma/\beta$ or $(\Gamma/\beta) \setminus \alpha$ the map $i_R^\alpha \otimes \text{Id} : R^r(\Gamma \setminus \alpha/\beta) \to R^r(\gamma/\beta)$ is injective, so that $g^R_\beta(y) = 0.$ Because this is true for every $\beta \in E(\Gamma \setminus \alpha),$ and $\Gamma \setminus \alpha$ also satisfies $(*),$ it follows that $y = 0$ so that $x = 0.$

We sum up in

**Theorem 2.13.** The columns of diagram [2] are short exact.

**Proof.** The middle column is exact by lemma [2.8]. The right hand column is exact by lemma [2.10] and lemma [2.12]. The exactness of the left hand column follows from this by the nine-lemma (or by simple diagram chase).

**Proof of theorem [2.5]** The injectivity follows from lemma [2.12]. The map $\psi^R_\alpha$ is surjective and the map $i^R_\alpha$ is injective, so it suffices to show that if $\psi^R_\alpha(x) \in \text{im}(i^R_\alpha),$ then $x \in \text{im}(i^R_\alpha).$ But $\psi^R_\alpha(x) \in \text{im}(i^R_\alpha)$ if and only if $g^R_\alpha(x) = 0,$ so the theorem follows from the exactness of the right column in diagram [1].

### 3 Deletion-contraction in the space $\text{Conf}_r(\Gamma)$

In this section we will prove that there is an isomorphism between the ring $R^r(\Gamma)$ defined in the previous section and the cohomology ring of $\text{Conf}_r(\Gamma).$ Moreover, Lemma [3.4] provides the existence of a short exact sequence of the form

$$0 \to H^*(\text{Conf}_r(\Gamma \setminus e)) \to H^*(\text{Conf}_r(\Gamma)) \to H^{*+r+1}(\text{Conf}_r(\Gamma/e)) \to 0.$$

The first step is to describe the deletion-contraction long exact sequence that occurs for configuration spaces.

#### 3.1 The long exact sequence

We will prove the following theorem.

**Theorem 3.1.** There is a long exact sequence in cohomology

$$\cdots \to H^*(\text{Conf}_r(\Gamma \setminus e)) \to H^*(\text{Conf}_r(\Gamma)) \to H^{*+r+1}(\text{Conf}_r(\Gamma/e)) \to \cdots$$

Before we turn to the proof, we make a few preliminary observations.

Let $e$ be an edge in $\Gamma$ between the vertices $a$ and $b.$ The space $\text{Conf}_r(\Gamma)$ is an open subspace of $\text{Conf}_r(\Gamma \setminus e).$ The complement

$$A_e(\Gamma) = \text{Conf}_r(\Gamma \setminus e) - \text{Conf}_r(\Gamma)$$

is a closed subspace in $\text{Conf}_r(\Gamma \setminus e)$ and

$$A_e(\Gamma) = \{(x_1, \ldots, x_n) \in \mathbb{R}^n; x_i \neq x_j \text{ if } \alpha_{i,j} \in E(\Gamma) \setminus \{e\} \text{ while } x_a = x_b\}.$$
There is a canonical homeomorphism between $A_e(\Gamma)$ and $\text{Conf}_r(\Gamma/e)$ sending $(x_1, \ldots, x_n)$ to $(x_1, \ldots, x_a, \hat{x}_b, \ldots, x_n)$. Let $m_{a,b}(x) > 0$ be the minimum of the numbers $|x_a - x_c|$ such that $c \neq b$, but $c$ is connected by an edge to $a$. This number will be independent of $x_b$. We define an open neighborhood $V_e(\Gamma)$ of $A_e(\Gamma)$ in $\text{Conf}_r(\Gamma \setminus e)$ in the following way

$$V_e(\Gamma) = \{ x = (x_1, \ldots, x_n) \in \text{Conf}_r(\Gamma \setminus e); |x_a - x_b| < \frac{1}{2}m_{a,b}(x) \}$$

**Lemma 3.2.** $\text{Conf}_r(\Gamma) \cap V_e$ is homotopy equivalent to $S^{r-1} \times \text{Conf}_r(\Gamma/e)$.

**Proof.** $\text{Conf}_r(\Gamma) \cap V_e$ is the space

$$\{ (x_1, \ldots, x_n) \in \text{Conf}_r(\Gamma): 0 < |x_a - x_b| < \frac{1}{2}m_{a,b}(x) \}$$

We define the maps

$$f : \text{Conf}_r(\Gamma) \cap V_e \to S^{r-1} \times \text{Conf}_r(\Gamma/e)$$

by

$$f((x_1, \ldots, x_n)) = \left( \frac{x_a - x_b}{|x_a - x_b|}, (x_1, \ldots, x_a, \ldots, \hat{x}_b, \ldots, x_n) \right)$$

and

$$g : S^{r-1} \times \text{Conf}_r(\Gamma/e) \to \text{Conf}_r(\Gamma) \cap V_e$$

by

$$g(y, (x_1, \ldots, x_n)) = (x_1, \ldots, x_a, \ldots, x_a + m_{a,b}(x)y, \ldots, x_n)$$

Now $gf$ is clearly homotopic to the identity and $fg$ equals the identity. □

**Proof of theorem 3.1.** We have two open subspaces $\text{Conf}_r(\Gamma)$ and $V_e(\Gamma)$ of $\text{Conf}_r(\Gamma \setminus e)$ such that $\text{Conf}_r(\Gamma) \cup V_e(\Gamma) = \text{Conf}_r(\Gamma \setminus e)$. There is a pushout diagram

$$\begin{array}{ccc}
V_e(\Gamma) \cap \text{Conf}_r(\Gamma) & \longrightarrow & V_e(\Gamma) \\
\downarrow & & \downarrow \\
\text{Conf}_r(\Gamma) & \longrightarrow & \text{Conf}_r(\Gamma \setminus e)
\end{array}$$

We obtain a Mayer-Vietoris long exact sequence in cohomology

$$\cdots \longrightarrow H^*(\text{Conf}_r(\Gamma \setminus e)) \xrightarrow{\phi^*} H^*(\text{Conf}_r(\Gamma)) \oplus H^*(V_e(\Gamma)) \xrightarrow{\psi^*} H^*(\text{Conf}_r(\Gamma) \cap V_e(\Gamma)) \xrightarrow{\delta^*} H^{*+1}(\text{Conf}_r(\Gamma \setminus e)) \longrightarrow \cdots$$

where $\phi$ is the map assigning to each cohomology class $x$ its restrictions $(x|_{\text{Conf}_r(\Gamma)}, x|_{V_e(\Gamma)})$ and $\psi(x,y) = x - y$. 

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We notice that $V_e(\Gamma)$ is homotopy equivalent to $\text{Conf}_r(\Gamma/e)$ and by Lemma 3.2 $\text{Conf}_r(\Gamma) \cap V_e$ is homotopy equivalent to $S^{r-1} \times \text{Conf}_r(\Gamma/e)$. Let $[\mu]$ denote the fundamental class of $S^{r-1}$. By the Kunneth formula, we can rewrite the long exact sequence as

$$\cdots \rightarrow H^*(\text{Conf}_r(\Gamma \setminus e)) \rightarrow H^*(\text{Conf}_r(\Gamma)) \oplus H^*(\text{Conf}_r(\Gamma/e)) \rightarrow \bigoplus_{k+l=*} H^k(S^{r-1}) \otimes H^l(\text{Conf}_r(\Gamma/e)) \rightarrow H^{*+1}(\text{Conf}_r(\Gamma \setminus e)) \rightarrow \cdots$$

This implies the existence of the long exact sequence

$$\cdots \rightarrow H^*(\text{Conf}_r(\Gamma \setminus e)) \rightarrow H^*(\text{Conf}_r(\Gamma)) \rightarrow [\mu]H^*(\text{Conf}_r(\Gamma/e)) \rightarrow H^{*+1}(\text{Conf}_r(\Gamma \setminus e)) \rightarrow \cdots$$

Finally, using the isomorphism $[\mu]H^{*+1}(\text{Conf}_r(\Gamma/e)) \cong H^{*+1}(\text{Conf}_r(\Gamma/e))$ we have the deletion-contraction long exact sequence for generalised configuration spaces:

$$\cdots \rightarrow H^*(\text{Conf}_r(\Gamma \setminus e)) \rightarrow H^*(\text{Conf}_r(\Gamma)) \rightarrow H^{*+1}(\text{Conf}_r(\Gamma/e)) \rightarrow H^{*+1}(\text{Conf}_r(\Gamma \setminus e)) \rightarrow \cdots$$

3.2 The map from $R^r(\Gamma)$.

Let $\Gamma$ be a graph and $r$ a natural number. For any edge $e = e(v_1, v_2) \in E(\Gamma)$, ordered by that $v_1 < v_2$, there is a map

$$p_e : \text{Conf}_r(\Gamma) \rightarrow S^{r-1}$$

defined by

$$p_e(x) \mapsto \frac{x_{v_2} - x_{v_1}}{|x_{v_2} - x_{v_1}|} \in S^{r-1} \subset \mathbb{R}^r \setminus \{0\}.$$ 

If all edges in $\Gamma$ have an orientation, we can combine these maps to a map

$$p(\Gamma) : \text{Conf}_r(\Gamma) \rightarrow (S^{r-1})^{E(\Gamma)}.$$ 

We choose a standard generator $[S^{r-1}] \in H^{r-1}(S^{r-1})$. After choosing a total order of the edges, we can identify $H^*((S^{r-1})^{E(\Gamma)})$ with the ring $\Lambda[E(\Gamma)]$. If $r$ is even, this identification depends of the order of the edges, but not on the orientation of the edges. If $r$ is odd, the identification depends on the orientation of the edges, but not of the order of the edges. In both cases, two different choices differ by an isomorphism.

**Definition 3.3.** Let $r$ be an even number, the maps $p_r(\Gamma)$ induce ring homeomorphisms

$$p_r(\Gamma)^* : \Lambda[\Gamma] \rightarrow H^*(\text{Conf}_r(\Gamma)); \quad p_r(\Gamma)(e) = p_e^*([S^{r-1}]).$$
Lemma 3.4. The map $p_r(\Gamma)^*$ is surjective. There is a short exact sequence

$$0 \to H^*(\text{Conf}(\Gamma \setminus e)) \to H^*(\text{Conf}_r(\Gamma)) \to H^{*-\gamma+1}(\text{Conf}_r(\Gamma/e)) \to 0.$$  

Proof. We prove it by induction on the number of edges in $\Gamma$. The lemma is true if $\Gamma$ has one edge. Now we suppose the result true for graphs with $n-1$ edges. Chose one edge $\alpha \in E(\Gamma)$.

If we have made choices of orientation of edges and order of edges for $\Gamma$, we can make compatible choices for $\Gamma \setminus \alpha$ respectively $\Gamma/\alpha$, so that the maps $i : E(\Gamma \setminus \alpha) \to E(\Gamma)$ respectively $p : E(\Gamma) \to E(\Gamma/\alpha)$ preserve the orientations and orders of the edges. Assume that we have made such compatible choices.

We have a commutative diagram

$$\begin{array}{cccccc}
0 & \longrightarrow & \Lambda[\Gamma \setminus \alpha] & \stackrel{i_\alpha^*}{\longrightarrow} & \Lambda[\Gamma] & \stackrel{p_\alpha^*}{\longrightarrow} & \Lambda[\Gamma/\alpha] & \longrightarrow & 0 \\
& & \downarrow p_r(\Gamma/\alpha)^* & & \downarrow p_r(\Gamma)^* & & \downarrow p_r(\Gamma/\alpha)^* & & \\
\cdots & \longrightarrow & H^*(\text{Conf}_r(\Gamma \setminus \alpha)) & \stackrel{\phi^*}{\longrightarrow} & H^*(\text{Conf}_r(\Gamma)) & \longrightarrow & H^{*-\gamma+1}(\text{Conf}_r(\Gamma/\alpha)) & \longrightarrow & \cdots
\end{array}$$

The first and last vertical maps are surjective by the induction hypothesis and $p_\alpha^*$ is also surjective by lemma 2.8. By the commutativity of the diagram $p_r(\Gamma/e) \circ p_\alpha^* = \phi^* \circ p_r(\Gamma)$. Moreover $p_r(\Gamma/e) \circ p_\alpha^*$ is surjective since it is the composition of surjective maps. It follows that $\phi^*$ is surjective, so that the long exact sequence at the bottom row breaks up into short exact sequences. Therefore the diagram above is a map of short exact sequences, and it follows by the five lemma that the middle vertical map is surjective. \qed

Lemma 3.5. The map $p_r(\Gamma)^*$ maps elements in the ideal generated by the generalised Arnold relations to 0.

Proof. It suffices to show that if $w$ is a circuit in $\Gamma$, and if $A(w) \in \Lambda[\Gamma]$ is the corresponding Arnold element, then $p_r(\Gamma)^*(A(w)) = 0$. Let $C_n$ be a cyclic graph with vertices $v_1, v_2, \ldots, v_n$ and edges $c_i = e(v_i, v_{i+1})$ for $i \leq n-1$ together with $c_n = e(v_n, v_1)$. The edges of $C_n$ form a circuit $c^n$. There is a map of graphs $f : C_{\text{conf}}(w) \to \Gamma$ which maps the edge $c^n_i \in E(C_n)$ to $w_i \in E(\Gamma)$. This map induces $f^\Lambda_* : \Lambda[C_n] \to \Lambda[\Gamma]$ and $(f_{\text{conf}})^* : H^*(\text{Conf}_r(C_n)) \to H^*(\text{Conf}_r(\Gamma))$. By naturality, there is a commutative diagram:

$$\begin{array}{ccc}
\Lambda^*(C_n) & \stackrel{f^\Lambda_*}{\longrightarrow} & \Lambda^*(\Gamma) \\
\downarrow p_r(C_n)^* & & \downarrow p_r(\Gamma)^* \\
H^*(\text{Conf}_r(C_n)) & \stackrel{(f_{\text{conf}})^*}{\longrightarrow} & H^*(\text{Conf}_r(\Gamma))
\end{array}$$

Using that $A(w) = f^\Lambda_*(A(c^n))$, it follows from this diagram that it suffices to show that for every $n$, the Arnold class $A(c^n)$ is in the kernel of the map $p_r(C_n)^*$. 

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In order to prove the lemma, we investigate the kernel of the map of classes in degree \((n-1)(r-1)\). We will prove inductively that the kernel of \(p_r(C_n)^s : (\Lambda[1,C_r])^{(n-1)(r-1)} \to H^{(n-1)(r-1)}(Conf_r(C_r))\) is generated by the Arnold element \(A(c^n)\). Note that the group \((\Lambda(C_n))^{(n-1)(r-1)} \cong \mathbb{Z}^n\) is generated by the classes \(c_1^n, c_2^n, \ldots, c_n^n\).

We make a preliminary remark. Let \(I_n\) be the linear graph \(C_n \setminus c_n\). Then

\[
Conf_r(I_n) = \{(x_1, \ldots, x_n) \in \mathbb{R}^{kr} : x_1 \neq x_2, \ldots, x_{n-1} \neq x_n\}
\]

The map \(p(I_n)\) is a homotopy equivalence, and \(p_r(I_n)^s\) an isomorphism. In particular \(H^{(n-1)(r-1)}(Conf_r(I_n)) \cong \mathbb{Z}\), generated by the class \(p_r(I_r)^s(c_1^n c_2^n \cdot \cdots c_n^n)\).

Let \(n = 2\). In this case \(C_2\) has a double edge, so that \(Conf_r(I_2) \to Conf_r(C_2)\) is a homeomorphism, \(A(c_2) = \pm c_1 \pm c_2\) and

\[
p_r(C_2)^s : (\Lambda[2]/\{A(c^2) = 0\})^{(r-1)} \cong H^{(r-1)}(Conf_r(C_2)),
\]

so that we have an induction start.

Let \(n \geq 3\), and assume the induction hypothesis for \(n-1\). Let \(c_i\) be any edge of \(C_n\), so that \(C_n \setminus c_i\) is isomorphic to \(I_n\). Let \(m = (n-1)(r-1)\). There is a surjective map of short exact sequences

\[
\Lambda[C_n \setminus c_i]^{(m)} \xrightarrow{i^*_i} \Lambda[C_n]^{(m)} \xrightarrow{p^*_i} \Lambda[C_n/c_i]^{(m)} \xrightarrow{\partial^*_i} H^{(m)}(Conf_r(C_n \setminus c_i)) \xrightarrow{\phi^*_i} H^{(m)}(Conf_r(C_n)) \xrightarrow{\psi^*_i} H^{(m)}(Conf_r(C_n/c_i))
\]

Because the rows are short exact, we have an induced exact sequence of kernels and cokernels:

\[
\ker(p_r(C_n \setminus c_i)^s) \xrightarrow{i^*_i} \ker(p_r(C_n)^s) \xrightarrow{p^*_i} \ker(p_r(C_n/c_i)^s) \xrightarrow{\partial} \coker(p_r(C_n \setminus c_i)^s)
\]

The map \(p_r(C_n \setminus c_i)^s\) is an isomorphism by the preliminary remark. Therefore \(p^*_i\) restricts to an isomorphism \(\ker(p_r(C_n)^s) \to \ker(p_r(C_n/c_i)^s) \cong \mathbb{Z}\). Notice also that \(p^*_i(A(c^n)) = A(c^{n-1})\), where we in the notation have identified \(C_n/c_i\) with \(C_{n-1}\). In order to complete the proof, we only need to show that \(A(c^n) \in \ker(p_r(C_n)^s)\).

By the diagram and the inductive assumption, \(p_r(C_n)^s(A(c^n))\) is in the image of the map \(\phi^*_i\), so there is an \(x \in \Lambda[C_n \setminus c_i]\) such that \(p_r(C_n)^s(A(c^n) - i^*_i(x)) \in \ker(p_r(C_n)^s)\). We conclude: For every \(i\), \(1 \leq i \leq n\) there is a number \(n_i\) such that

\[
p_r(C_n)^s(A(c^n) - i^*_i(n_1 c_1^n \cdots c_i^n \cdots c_n^n)) = 0
\]
We need to show that $n_i = 0$. To prove this, we pick $j \neq i$.

$$p_r(C_n/c_i^n)^* p_c^A(c^n)(A(c^n) - i_c^n (n_i c_1 \cdots \hat{c}_i \cdots c_n))$$

$$= p_r(C_n/c_i^n)^* ((A(c^{n-1}) - n_i c_1 \cdots \hat{c}_i \cdots c_j \cdots c_n)$$

$$= -p_r(C_n/c_i^n)^* (n_i c_1 \cdots \hat{c}_i \cdots c_j \cdots c_n).$$

Since by the inductive assumption the kernel of $p_r(C_n/c_i^n)^*$ is the subgroup generated by $A(c^{n-1})$, and since $n_i c_1 \cdots \hat{c}_i \cdots c_j \cdots c_n$ is only in this subgroup if $n_i = 0$, we have proved

$$p_c^A(i_c^n (n_i c_1 \cdots \hat{c}_i \cdots c_n)) = c_1 c_2 \cdots \hat{c}_i \cdots \hat{c}_j \cdots c_n$$

is not in the subgroup of $\Lambda[C_n/c_j]$ generated by $A(c_{n-1})$, since $n - 1 > 1$. It follows from the induction hypothesis that $n_i = 0$, so that $p_r(C_n)^* (A(c^n)) = 0$. This finishes the proof.

**Corollary 3.6.** The map $p_r(\Gamma)^*: \Lambda[\Gamma] \to H^*(\text{Conf}_r(\Gamma))$ factors uniquely over the map $\lambda_r: R^r(\Gamma) \to H^*(\text{Conf}_r(\Gamma))$.

**Theorem 3.7.** There is a isomorphism of graded commutative rings

$$\lambda_r: R^r(\Gamma) \to H^*(\text{Conf}_r(\Gamma)).$$

**Proof.** We prove the theorem by induction on the number of edges in the graph. Assume that the lemma is true for all graphs with $n-1$ or fewer edges. Let $\Gamma$ be graph with $n$ edges. If $\Gamma$ has multiple edges, the lemma follows from the induction hypothesis and lemma 2.2. Consider the following map of short exact sequences:

$$
\begin{array}{ccccccc}
0 & \longrightarrow & R^r(\Gamma \setminus e) & \longrightarrow & R^r(\Gamma) & \longrightarrow & R^r(\Gamma/e) & \longrightarrow & 0 \\
\uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
0 & \longrightarrow & H^*(\text{Conf}_r(\Gamma \setminus e)) & \longrightarrow & H^*(\text{Conf}_r(\Gamma)) & \longrightarrow & H^*(\text{Conf}_r(\Gamma/e)) & \longrightarrow & 0 \\
\end{array}
$$

By the five lemma if follows that the middle map is also an isomorphism.

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