Trajectories of a quadratic differential related to a quasi-exactly solvable sextic oscillator

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Abstract
In this paper, we discuss the existence of solution (as Cauchy transform of a signed measure) of a particular algebraic quadratic equation of the form:
\[ zC^2(z) - \left( z^2 + \frac{\gamma}{2} z \right) C(z) + \left( z + \frac{\delta}{3} \right)^2 = 0. \]
This problem remains to describe the critical graph of a related meromorphic quadratic differential; in particular, we discuss the existence of finite critical trajectories of this quadratic differential.

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1 Introduction
This paper is a continuation of a collection of works about applications of the theory of quadratic differentials in quantum theory. In fact, quadratic differentials have provided an important tool in the asymptotic study of some solutions of algebraic equations. In quantum theory, trajectories of some quadratic differentials have crucial role in the WKB analysis.

We consider the eigenvalue problem
\[ -y'' + V_{s,m}(x)y = \lambda y, \] (1)
where the potential \( V_{s,m}(x) \) is given by:
\[ V_{s,m}(x) = \frac{(4s - 1)(4s - 3)}{x^2} + (x^6 - (4s + 4m - 2)x^2). \]
This problem was studied by A.Turbiner [1],[5]. For \( s \in \left[ \frac{1}{4}, \frac{3}{4} \right] \), there is an attractive centrifugal term. For \( s \in \left] -\infty, \frac{1}{4} \right[ \cup \left[ \frac{3}{4}, +\infty \right[ \), the centrifugal term is repulsive. For \( s = \frac{1}{4} \) or \( \frac{3}{4} \), the centrifugal core term disappears leaving a sextic oscillator

\[
V_{p,m}(x) = x^6 - (4m + p)x^2; \quad p \in \{-1, 1\}
\]

We will study the case \( s = \frac{1}{4} \) (so \( p = -1 \)), with a perturbed potential

\[
V_{-1,m+1}(x) = V_m(x) = x^6 + \gamma m^{1/2}x^4 + \left( \frac{\gamma^2 m}{4} - (4m + 3) \right)x^2,
\]

with boundray condition \( y(\pm \infty) = 0 \). This problem is exactly solvable: that means that for every \( \gamma \) there exist \( m + 1 \) eigenfunctions of the form

\[
\phi(x) = q_m(x)e^{-\frac{x^4}{4} - \frac{\gamma m^{1/2}}{4}x^2},
\]

where \( q_m \) is an even-parity polynomial of degree \( 2m \), corresponding to \( m + 1 \) eigenvalues \( \lambda_m \); see [4]. Note that in the case \( p = 1 \) we have the same form of eigenfunctions with an odd-parity polynomials; for more details see again [4].

Substituting in the Schrödinger equation (1), we find that

\[
-q''_m(x) + (2x^3 + \gamma m^{1/2}x)q'_m(x) - (4mx^2 - \frac{\gamma m^{1/2}}{2})q_m(x) = \lambda_m q_m(x).
\]

The differential operator

\[
K = -\frac{d^2}{dx^2} + (2x^3 + \gamma m^{1/2}x)\frac{d}{dx} - (4mx^2 - \frac{\gamma m^{1/2}}{2})
\]

preserves the \((m + 1)\)-dimensional linear space of all even polynomials of degree \( \leq 2m \). Using \( z = x^2 \), we find that

\[
-4zq''_m(z) + (4z^2 + 2\gamma m^{1/2}z - 2)q'_m(z) - (4mz - \frac{\gamma m^{1/2}}{2})q_m(z) = \lambda_m q_m(z). \tag{2}
\]

Our main goal is the study of the asymptotic root-counting measure \( \vartheta_m \) of an appropriate re-scaled polynomial sequence \( \{ Q_m(z) = q_m(m^2 z) \} \). Let \( \nu_m \) be the normalized root-counting measure of the sequence \( (q_m) \). The Cauchy transform \( C_{\nu_m} \) and \( C_{\vartheta_m} \) of respectively \( \nu_m \) and \( \vartheta_m \) satisfies the following equations :

\[
4zC_{\nu_m}^2(z) - (4z^2 + 2\gamma m^{1/2}z - 2)C_{\nu_m}(z) + \left( \frac{4mz - \frac{\gamma m^{1/2}}{2} + \lambda_m}{m} \right) + 4zC_{\nu_m}'(z) = 0.
\]
4zm^{1-\varepsilon}C_{\vartheta_m}^2(z) - (4m^{\varepsilon}z^2 + 2\gamma m^{1/2}z - 2m^{-\varepsilon})C_{\vartheta_m}(z) + \frac{(4m^{1+\varepsilon}z - \frac{\gamma m^{1/2}}{2} + \lambda_m)}{m} + 4m^{-\varepsilon}zC'_{\vartheta_m}(z) = 0.

For \(\varepsilon = 1/2\), we get

\[-4zC_{\vartheta_m}^2(z) + (4z^2 + 2\gamma z - \frac{2}{m})C_{\vartheta_m}(z) - (4z - \frac{\gamma}{2m} + \frac{\lambda_m}{m^{3/2}}) - 4z \frac{C'_{\vartheta_m}(z)}{m} = 0.\]  

(3)

It was shown in [6], that the sequence \((\lambda_m/m^{4/3})\) is bounded. By the Helly selection Theorem, we may assume that

\[\lim_{m\to\infty} \left(\frac{\lambda_m}{m^{4/3}}\right) = \delta,\]

and then, there exists a compactly-supported positive measure \(\nu\) such that

\[\lim_{m\to\infty} \vartheta_m = \nu, \quad \lim_{m\to\infty} C_{\vartheta_m} = C_\nu = C.\]

Finally, taking the limits in (3), we obtain the algebraic equation :

\[zC^2(z) - (z^2 + \frac{\gamma}{2} z)C(z) + (z + \frac{\delta}{4}) = 0.\]  

(4)

In this paper, we discuss the existence of solutions of equation (4) as Cauchy transform of compactly-supported signed measure. Section 3, we make the connection between this a algebraic equation and a particular quadratic differential. In section 2, we describe the critical graph of the related quadratic differential in the Riemann sphere \(\mathbb{C}\), more precisely, we discuss the number of its finite critical trajectories.

2 A quadratic differential

Below, we describe the critical graphs of the the family of quadratic differentials

\[\omega_q = -\frac{q(z)}{z}dz^2,\]  

(5)

where \(q\) is a monic polynomial of degree 3. We begin our investigation by some immediate observations from the theory of quadratic differentials. For more details, we refer the reader to [22],[23].

Recall that finite critical points of a given meromorphic quadratic differential \(-Q(z)dz^2\) on the Riemann sphere \(\mathbb{C}\) are its zeros and simple poles;
poles of order 2 or greater then 1 called infinite critical points. All other points of \( \mathbb{C} \) are called regular points.

Horizontal trajectories (or just trajectories) of the quadratic differential are the zero loci of the equation

\[-Q(z) dz^2 > 0,\]
or equivalently

\[\Re \int^{z} \sqrt{Q(t)} dt = \text{const.} \quad (6)\]

If \( z(t), t \in \mathbb{R} \) is a horizontal trajectory, then the function

\[t \mapsto \Im \int^{t} \sqrt{Q(z(u))} z'(u) du\]
is monotone.

The vertical (or, orthogonal) trajectories are obtained by replacing \( \Im \) by \( \Re \) in equation \((6)\). The horizontal and vertical trajectories produce two pairwise orthogonal foliations of the Riemann sphere \( \mathbb{C} \).

A trajectory passing through a critical point is called critical trajectory. In particular, if it starts and ends at a finite critical point, it is called finite critical trajectory or short trajectory, otherwise, we call it an infinite critical trajectory. A short trajectory is called unbroken if it does not pass through any finite critical points except its two endpoints. The closure of the set of finite and infinite critical trajectories is called the critical graph.

A necessary condition for the existence of a short trajectory connecting finite critical points is the existence of a Jordan arc \( \gamma \) connecting them, such that

\[\Re \int_{\gamma} \sqrt{Q(t)} dt = 0. \quad (7)\]

However, this condition is sufficient in general; see counter-example in \[19\].

The local structure of the trajectories is as follow :

- At any regular point, horizontal (resp. vertical) trajectories look locally as simple analytic arcs passing through this point, and through every regular point passes a uniquely determined horizontal (resp. vertical) trajectory; these horizontal and vertical trajectories are locally orthogonal at this point.

- From each zero of multiplicity \( r \), there emanate \( r + 2 \) critical trajectories spacing under equal angle \( 2\pi / (r + 2) \).
• At a simple pole, there emanates exactly one horizontal trajectory.

• At the pole of order $r > 2$, there are $r - 2$ asymptotic directions (called critical directions) spacing under equal angle $2\pi / (r - 2)$ and a neighborhood $\mathcal{U}$, such that each trajectory entering $\mathcal{U}$ stays in $\mathcal{U}$ and tends to the pole in one of the critical directions. See Figure 1.

Figure 1: Structure of the trajectories near a simple zero (left), a simple pole (center), and a pole of order 6 (right).

A very helpful tool that will be used in our investigation is the Teichmüller lemma (see [22, Theorem 14.1]).

**Definition 1** A domain in $\mathbb{C}$ bounded only by segments of horizontal and/or vertical trajectories of $\varpi_q$ (and their endpoints) is called $\varpi_q$-polygon.

**Lemma 2 (Teichmüller)** Let $\Omega$ be a $\varpi_q$-polygon, and let $z_j$ be the critical points on the boundary $\partial \Omega$ of $\Omega$, and let $\theta_j$ be the corresponding interior angles with vertices at $z_j$, respectively. Then

$$\sum \left(1 - \frac{(n_j + 2)\theta_j}{2\pi}\right) = 2 + \sum m_i, \quad \text{(8)}$$

where $n_j$ are the multiplicities of $z_j = 1$, and $m_i$ the multiplicities of critical points inside $\Omega$.

We will focus on the case where

$$q(z) = q_a(z) = (z - 1)(z - a)(z - \overline{a}),$$
with
\[ a \in \mathbb{C}^+ = \{a \in \mathbb{C} \mid \Im (a) > 0\}. \]

We have the following immediate observations:

- The finite critical points of \( \varpi_q \) are \( 1, a, \overline{a} \) as simple zeros, and \( -1 \) as a simple pole.
- With the parametrization \( u = 1/z \), we get
  \[ \varpi_q (u) = \left( -\frac{1}{u^6} + \mathcal{O}\left( \frac{1}{u^5}\right) \right) du^2, \ u \to 0, \]
  thus, infinity is an infinite critical point of \( \varpi_q \), as a pole of order 6.
- Since \( \infty \) is the only infinite critical point of \( \varpi_q \), any critical trajectory which is not finite diverges to \( \infty \) following one of the 4 directions:
  \[ D_k = \left\{ z \in \mathbb{C} \mid \arg (z) = (2k + 1)\frac{\pi}{4} \right\}, k = 0, 1, 2, 3. \]
  The same thing happen to the orthogonal trajectories at \( \infty \), but the critical directions are:
  \[ D_k^\perp = \left\{ z \in \mathbb{C} : \arg (z) = \frac{k\pi}{2} \right\}; k = 0, 1, 2, 3. \]
  Observe that if two trajectories diverge to \( \infty \) in a same direction \( D_k \), then there exists a neighborhood \( V \) of \( \infty \), such that any orthogonal trajectory traversing \( D_k \) in \( V \), must traverse these two trajectories.
- Since the quadratic differential \( \varpi_q \) has two poles, Jenkins Three-pole Theorem (see [22, Theorem 15.2]) asserts that the situation of the so-called recurrent trajectory (whose closure might be dense in some domain in \( \mathbb{C} \)) cannot happen.

**Lemma 3** Two critical trajectories of \( \varpi_q \) emanating from the same zero cannot diverge to \( \infty \) in the same direction.

**Lemma 4** For any Jordan arc \( \gamma \) connecting \( a \) and \( \overline{a} \) in \( \mathbb{C} \setminus [0, 1] \) we have
\[ \Re \int_{\gamma} \sqrt{\frac{(z-1)(z-a)(z-\overline{a})}{z}} dz = 0, \]
and then, condition [7] is fulfilled.
We consider the set
\[ \Sigma = \left\{ z \in \mathbb{C} : \Re \int_0^z \sqrt{\frac{(t-1)(t-z)(t-\bar{z})}{t}} \, dt = 0 \right\} \]

**Lemma 5** The set \( \Sigma \) is symmetric with respect to the real axis, and it is formed by 3 Jordan arcs:

- the segment \([0, 1]\);
- two curves \( \Sigma^\pm \) emerging from \( z = 1 \), and diverging respectively to infinity in \( \mathbb{C}^\pm \).

![Figure 2: Approximate plot of the curve \( \Sigma \)](image)

We give here the behavior of \( \Sigma \) at \( z = 1 \) and at the infinity:

**Lemma 6** The following results hold:

\[
\lim_{z \to \infty, z \in \Sigma \cap \mathbb{C}^+} \arg(z) = \frac{\pi}{2},
\]
\[
\lim_{z \to 1, z \in \Sigma \cap \mathbb{C}^+} \arg(z) = \frac{\pi}{3}.
\]

From Lemma 5 \( \Sigma \) splits \( \mathbb{C} \) into 2 connected domains:

- \( \Omega_1 \) limited by \( \Sigma^\pm \) and containing \( z = 2 \);
- \( \Omega_2 = \mathbb{C} \setminus (\Omega_1 \cup \Sigma^\pm \cup [0, 1]) \). See Figure 2.
Proposition 7 For any complex number $a$, the quadratic differential $\varpi_q$ has:

- two short trajectories if $a \in \Omega_1$: the segment $[0,1]$ and another one connecting $a$ and $\pi$ in $\Omega_1$; see Figure 3.
- three short trajectories if $a \in \Sigma^\pm$: the segment $[0,1]$ and two others connecting $z = 1$ with $a$ and $\pi$; see Figure 4.
- two short trajectories if $a \in \Omega_2$: the segment $[0,1]$ and another one connecting $a$ and $\pi$ in $\Omega_2$; see Figure 5.

Figure 3: Critical graphs when $a \in \Omega_1$, here $a = 1.6+2i$ (left) and $a = 1.8+2i$ (right).

Remark 8 The case when $q = \prod_{i=1}^3 (z - a_i) \in \mathbb{R}[X]$, with real zeros is quite simple; if the zeros are simple: $a_1 < a_2 < a_3$, then the segments $[a_1,a_2]$ and $[a_3,a_4]$ are two short trajectories of $\varpi_q$. See Figure 6.
3 Connection with the algebraic equation

The Cauchy transform \( C_\nu \) of a compactly supported Borelian complex measure \( \nu \) is defined in \( \mathbb{C} \setminus \text{supp}(\nu) \) by:

\[
C_\nu(z) = \int_C \frac{d\nu(t)}{z-t}.
\]

It satisfies

\[
C_\nu(z) = \frac{\nu(C)}{z} + O(z^{-2}), \quad z \to \infty,
\]

and the inversion formula (which should be understood in the distributions sense): \[\nu = \frac{1}{\pi} \frac{\partial C_\nu}{\partial z}.\]

In particular, the normalized root-counting measure \( \nu_n = \nu(P_n) \) of a complex polynomial \( P_n \) of a complex polynomial \( P_n \) of degree \( n \) is defined in \( \mathbb{C} \) by:

\[
\nu_n = \frac{1}{n} \sum_{P_n(a)=0} \delta_a, \quad \text{(each zero is counted with its multiplicity)};
\]

the Cauchy transform of \( \nu_n \) is:

\[
C_{\nu_n}(z) = \int_C \frac{d\nu_n(t)}{z-t} = \frac{P_n'(z)}{nP_n(z)}; P_n(z) \neq 0.
\]
Figure 5: Critical graphs when $a \in \Omega_2$, here $a = 0.5 + 2i$ (left) and $a = 2i$ (right).

Let us come back to the algebraic equation (4). We are seeking a compactly-supported signed measure $\nu$ such that, its Cauchy transform $C_\nu$ satisfies almost everywhere in $\mathbb{C}$ equation (4).

With the choice of the square root of the discriminant

$$
\Delta (z) = \frac{z}{4} \left( 4z^3 + 4\gamma z^2 + (\gamma^2 - 16) z - 16\delta \right)
$$

of the quadratic equation (4) (as a quadratic equation) with condition

$$
\sqrt{\Delta(z)} \sim z^2, z \to \infty,
$$

it is easy to check that independently of the complex numbers $\gamma$ and $\delta$, we have:

$$
C(z) = \frac{2z^2 + \gamma z - 2\sqrt{\Delta(z)}}{4z} = \frac{1}{z} + \mathcal{O}(z^{-2}), z \to \infty,
$$

which let us be hopeful for the existence of the measure $\nu$.

The following Lemma gives a sufficient condition on a solution of (4) to be the Cauchy transform of some compactly supported measure in $\mathbb{C}$ :

**Lemma 9 ([32, comp. Th. 1.2, Ch. II])** Suppose $f \in L^1_{\text{loc}}(\mathbb{C})$ and that $f(z) \to 0$ as $z \to \infty$ and let $\mu$ be a compactly-supported measure in $\mathbb{C}$ such
that
\[
\mu = \frac{1}{\pi} \frac{\partial f}{\partial z}
\]
in the sense of distributions. Then \( f(z) = C \mu(z) \) almost everywhere in \( \mathbb{C} \).

The following Proposition gives a necessary condition the existence of measures \( \nu \):

**Proposition 10** Let us consider the quadratic differential
\[
- \frac{\Delta(z)}{z^2} \, dz^2.
\]
(9)

If the signed measure \( \nu \) exists, then, the quadratic differential (9) has two short trajectories, and, the support of \( \nu \) coincides with these short trajectories. In particular, if \( \Delta(z) \) is a real polynomial, then the problem of finding the measure \( \nu \) is solved.

4 Proofs

**Proof of Lemma [3]** Suppose that \( \gamma_1 \) and \( \gamma_2 \) are two such trajectories emanating from the zero \( a \) or 1, spacing with angle \( \theta \in \{2\pi/3, 4\pi/3\} \). Consider
the $\varpi_q$-polygon with edges $\gamma_1$ and $\gamma_2$, and vertices $z_j$, and infinity. The right side of (8) can take only the values 0 or $-1$, while the left side is at list 2; a contradiction. (Observe the $\varpi_{q\tau}$-polygon cannot contain the pole $z = 0$, otherwise it contains $z = 1$ and, again we get a contradiction with (8).)

Proof of Lemma 4. Since \( \frac{q(t)}{t} \) is a real rational fraction, then

\[
\sqrt{\frac{q(t)}{t}} = \sqrt{\frac{q(t)}{t}}, \quad t \neq 0,
\]

(10)

and we get, after the change of variable $u = \frac{t}{7}$ in second integral :

\[
\Re \left( \int_1^z \sqrt{\frac{q(t)}{t}} \, dt \right) = \Re \left( \int_1^z \sqrt{\frac{q(t)}{t}} \, dt - \int_1^z \sqrt{\frac{q(t)}{t}} \, dt \right)
\]

\[
= \Re \left( \int_1^z \sqrt{\frac{q(t)}{t}} \, dt - \int_1^z \sqrt{\frac{q(t)}{t}} \, dt \right)
\]

\[
= \Re \left( 2i \Im \left( \int_1^z \sqrt{\frac{q(t)}{t}} \, dt \right) \right)
\]

\[= 0.
\]

Let us give a necessary condition to get two short trajectories joining two different pairs of finite critical points of $\varpi_q$ in the general case :

\[
\frac{q(z)}{z} = \frac{z^3 + \alpha z^2 + \beta z + \gamma}{z} = \frac{(z-a)(z-b)(z-c)}{z}, \quad a, b, c \in \mathbb{C}.
\]

Consider two disjoint oriented Jordan arcs $\gamma_1$ and $\gamma_2$ connecting two distinct pairs of zeros. We define the single-valued function $\sqrt{\frac{q(z)}{z}}$ in $\mathbb{C} \setminus (\gamma_1 \cup \gamma_2)$ with condition $\sqrt{\frac{q(z)}{z}} \sim z, z \to \infty$. For $s \in \gamma_1 \cup \gamma_2$, we denote by $\left( \sqrt{p(s)} \right)_+$ and $\left( \sqrt{p(s)} \right)_-$ the limits from the + and − sides, respectively. (As usual, the + side of an oriented curve lies to the left and the − side lies to the right, if one traverses the curve according to its orientation.)

From the Laurent expansion at $\infty$ of $\sqrt{q(z)}$ :

\[
\sqrt{\frac{q(z)}{z}} = z + \frac{\alpha}{2} - \left( \frac{\alpha^2 - 4\beta}{8z} \right) + O(z^{-2}),
\]

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we deduce the residue

\[ \text{res}_{\infty} \left( \sqrt{\frac{q(z)}{z}} \right) = \frac{1}{8} \left( \alpha^2 - 4\beta \right). \]

Let

\[ I = \int_{\gamma_1} \left( \sqrt{\frac{q(s)}{s}} \right)_+ \, ds + \int_{\gamma_2} \left( \sqrt{\frac{q(s)}{s}} \right)_+ \, ds. \]

Since

\[ \left( \sqrt{\frac{q(s)}{s}} \right)_+ = -\left( \sqrt{\frac{q(s)}{s}} \right)_-, \quad s \in \gamma_1 \cup \gamma_2, \]

we have

\[ 2I = \int_{\gamma_1 \cup \gamma_2} \left[ \left( \sqrt{\frac{q(s)}{s}} \right)_+ - \left( \sqrt{\frac{q(s)}{s}} \right)_- \right] \, ds = \oint_{\Gamma_q} \sqrt{\frac{q(z)}{z}} \, dz, \]

where \( \Gamma_q \) is a closed contours encircling the curves \( \gamma_1 \) and \( \gamma_2 \). After the contour deformation, we pick up the residue at \( z = \infty \), and we get

\[ I = \frac{1}{2} \oint_{\Gamma_q} \sqrt{\frac{q(z)}{z}} \, dz = \pm i \pi \text{res}_{\infty} \left( \sqrt{\frac{q(z)}{z}} \right) = \pm \frac{\pi i}{8} \left( \alpha^2 - 4\beta \right) \]

and the necessary condition is

\[ \Im \left( \alpha^2 - 4\beta \right) = 0, \]

which is satisfied for the case when \( q \) is real. ■

**Proof of Lemma 5.** It is clear that \( \Sigma \cap \mathbb{R} = [0, 1] \). The fact that \( \Sigma \) is symmetric with respect to the real axis follows from the observation \( 10 \). In order to prove that \( \Sigma \) is a curve, we consider the real functions \( F \) and \( G \)
defined for \((x, y) \in \mathbb{C}_+\) by:

\[
F(x, y) = \Re \left( \int_0^x \sqrt{\frac{(u - (x + iy))(u - (x - iy))(u - 1)}{u}} \, du \right)
\]

\[
= \Re \left( \int_0^x \sqrt{\frac{(u - x)^2 + y^2}{u}} \, du \right);
\]

\[
G(x, y) = \Re \left( \int_x^{x+iy} \sqrt{\frac{(u - (x + iy))(u - (x - iy))(u - 1)}{u}} \, du \right)
\]

\[
= -\int_0^1 y^2 \sqrt{1 - t^2} \Im \sqrt{1 - \frac{1}{x + ity}} \, dt.
\]

Observe that

\[
\Sigma = \{ (x, y) \in \mathbb{R}^2 \mid (F + G)(x, y) = 0 \}.
\]

We prove first that \(\Sigma \setminus [0, 1] \subset \{ z \in \mathbb{C} \mid \Re z > 1 \}. \) If \(x \leq 1\) and \(y > 0\), then, it is obvious that, \(F(x, y) = 0\). By the other hand, we have for \(0 < t \leq 1\) :

\[
0 < \arg (x + ity) < \arg (x - 1 + ity) < \pi
\]

\[
\Rightarrow 0 < \arg \left(1 - \frac{1}{x + ity}\right) < \pi \Rightarrow \arg \sqrt{1 - \frac{1}{x + ity}} \in \left]0, \frac{\pi}{2}\right[
\]

\[
\Rightarrow \Im \sqrt{1 - \frac{1}{x + ity}} > 0 \Rightarrow G(x, y) < 0.
\]

Hence, \((F + G)(x, y) < 0\) which proves that \((x, y) \notin \Sigma.\)

Let us prove now that \(\Sigma\) is a curve in the set

\[
\Pi = \{ (x, y) ; x > 1, y > 0 \}.
\]

We have

\[
\frac{\partial F}{\partial x}(x, y) = \sqrt{\frac{y^2(x - 1)}{x}} + \int_1^x \frac{(x - u)(u - 1)}{\sqrt{(u - x)^2 + y^2}(u - 1) u} \, dt > 0.
\]

By the other hand, with \(u_t = x + ity, t \in [0, 1]\), we get

\[
\frac{\partial G}{\partial x}(x, y) = \frac{\partial}{\partial x} \left[ \Re \left( \int_0^1 iy^2 \sqrt{1 - t^2} \sqrt{1 - \frac{1}{u_t}} \, dt \right) \right]
\]

\[
= -\int_0^1 y^2 \sqrt{1 - t^2} \Im \left( \frac{1}{u_t \sqrt{1 - \frac{1}{u_t}}} \right) \, dt
\]

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It is sufficient to prove that,

$$\forall t \in [0, 1], \Im \left( \frac{1}{u_t^2 \sqrt{1 - \frac{1}{u_t}}} \right) \leq 0,$$

which is equivalent to prove that,

$$\forall t \in [0, 1], \arg \left( \frac{1}{u_t^2 \sqrt{1 - \frac{1}{u_t}}} \right) \in [\pi, 2\pi[,$$

where the argument is taken in $[0, 2\pi[$. It follows from (11) that for any $t \in [0, 1]$ :

$$\arg \left( \frac{1}{u_t^2 \sqrt{1 - \frac{1}{u_t}}} \right) = 2\pi - \left( \frac{3}{2} \arg (u_t) + \frac{1}{2} \arg (u_t - 1) \right) \in ]\pi, 2\pi[.$$

We deduce that for any $0 \leq t \leq 1$, $\Im \left( \frac{1}{u_t^2 \sqrt{1 - \frac{1}{u_t}}} \right) \leq 0$, and then $\frac{\partial G}{\partial x} (x, y) \geq 0$. Finally, we just proved that

$$\frac{\partial (F + G)}{\partial x} (x, y) \neq 0, (x, y) \in \Sigma \cap \Pi.$$

We conclude that the set $\Sigma$ is a curve in $\mathbb{C}$ by applying the Implicit Function Theorem to the function $F + G$. ■

**Proof of Lemma 6.** Let us put $z = re^{ix} \in \Sigma, r > 1, x \in [0, \frac{\pi}{2}]$. With the change of variable $t = se^{ix}$, we get

$$\Re \left( e^{2ix} \int_0^1 \frac{1}{\sqrt{s - \frac{1}{r} e^{-ix} (s - 1) (s - e^{-2ix})}} ds \right) = 0.$$

Taking the limits when $r \to \infty$, we get

$$0 = \Re \int_0^1 e^{2ix} \sqrt{(s - 1) (s - e^{-2ix})}. \ (12)$$

We see that $x \neq 0$; suppose that $x \neq \frac{\pi}{2}$. With the change of variable $t = au + \beta$, where

$$\beta = \frac{1 + e^{-2ix}}{2}, \alpha = \frac{1 - e^{-2ix}}{2},$$

...
\[ (12) \text{ gives } \]
\[ 0 = \Re \left( \int_{\cot x}^{i} \sqrt{u^2 + 1} du \right) = \Re \left( \int_{\cot x}^{0} \sqrt{u^2 + 1} du + \int_{0}^{i} \sqrt{u^2 + 1} du \right) \]
\[ = \Re \left( \int_{\cot x}^{0} \sqrt{u^2 + 1} du \right) > 0; \]
a contradiction.

The Laurent serie of \( \sqrt{(t-1)(t-z)(t-z)} \) when \( t \to 1 \) is:
\[ \sqrt{(t-1)(t-z)(t-z)} t = |z - 1| \sqrt{t - 1} + o \left( (t - 1)^{\frac{1}{2}} \right). \]
We conclude that
\[ 0 = \lim_{z \to 1, z \in \Sigma^+} \Re \int_{1}^{z} \frac{(t-1)(t-z)(t-z)}{t} dt = 2 \cdot \frac{2}{3} |z - 1| \Re (z - 1)^{\frac{3}{2}}, \]
and then
\[ \arg (z - 1)^{\frac{3}{2}} \equiv \frac{\pi}{2} \text{ mod } (\pi), \]
which finishes the proof. □

**Proof of Proposition 7.** It is clear that the segment \([0, 1]\) is always a short trajectory of \( \varpi_q \). If \( a \notin \Sigma \), then, from \((7)\) there is no short trajectory connecting \( a \) to 0 or 1. From Lemma \([3]\) at most two critical trajectories emanating from \( a \) can diverge to \( \infty \) in the upper half-plane \( \mathbb{C}^+ \). By consideration of symmetry with respect to the real axis, at list one critical trajectory emanating from \( a \) meets a critical trajectory emanating from \( 0 \), at some point \( b \in \mathbb{R} \setminus [0, 1] \). Since \( b \) cannot be a zero of the quadratic differential \( \varpi_q \), we conclude that these two critical trajectories form a short one.

If \( a \in \Sigma \), and no short trajectory connecting \( a \) to \( z = 1 \), then, there exist two critical trajectories \( \gamma_a \) and \( \gamma_1 \) emanating respectively from \( a \) and 1 and diverging to infinity in a same direction \( D_k \). From the behaviour of orthogonal trajectories at \( \infty \), we can take an orthogonal trajectory \( \sigma \) that hits \( \gamma_1 \) and \( \gamma_a \) respectively in two points \( b \) and \( c \) (there are infinitely many such orthogonal trajectories \( \sigma \) ). We consider a path \( \gamma \) connecting \( z = 1 \) and \( a \) formed by the part of \( \gamma_1 \) from \( z = 1 \) to \( b \), the part of \( \sigma \) from \( b \) to \( c \), and the part of \( \gamma_a \) from \( c \) to \( a \). Then
\[ \Re \int_{\gamma} \sqrt{p(t)} dt = \Re \int_{1}^{b} \sqrt{p(t)} dt + \Re \int_{b}^{c} \sqrt{p(t)} dt + \Re \int_{c}^{a} \sqrt{p(t)} dt \]
\[ = \Re \int_{b}^{c} \sqrt{p(t)} dt \neq 0, \]
which contradicts the fact \( a \in \Sigma \). ■

**Proof of Proposition 10** The fact that the support of \( \nu \) is formed by horizontal trajectories of the quadratic differential \( (9) \) is classic and it is based on the so-called Plemelj-Sokhotsky Formula. For more details, we refer the reader to [25], [27], [29], [24]...

Since the Cauchy transform is a single-valued function in \( \mathbb{C} \backslash \text{supp}(\nu) \), then, the support of \( \nu \) should include all the singular points (finite critical points) of the quadratic differential \( (9) \). But, the horizontal trajectories that contain all finite critical points are exactly short trajectories. The measure \( \nu \) is absolutely continuous with respect to the linear Lebesgue measure, and it is given on its support (with an adequate orientation) by the expression:

\[
d\nu(t) = \frac{1}{8i\pi} \left( \frac{\sqrt{\Delta(t)}}{t} \right)^+ dt.
\]

It is easy to check that the Cauchy transform of \( \nu \) satisfies (4), indeed:

\[
C_{\nu}(z) = \frac{1}{2i\pi} \int \frac{\sqrt{\Delta(t)}}{4(z-t)}^+ dt = \frac{1}{4i\pi} \oint \frac{\sqrt{\Delta(t)}}{4(z-t)} dt
\]

\[
= \frac{1}{4} \left( \text{res}_z \sqrt{\Delta(t)} - \text{res}_\infty \sqrt{\Delta(t)} \right)
\]

\[
= -\sqrt{\frac{\Delta(z)}{z}} + \frac{\gamma + 2z}{4} = -\sqrt{z\Delta(z) + \gamma z + 2z^2} + \frac{\gamma + 2z}{4z},
\]

where the path of integration in the first integral is formed by the two short trajectories, and, in the second integral is a closed contour including the two short trajectories and far away from \( z \). ■

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**References**

[1] A. Turbiner, Sov. Phys., J. E. T. P. 67, 230 (1988).
[2] C. Bender and G. V. Dunne, Quasi-Exactly Solvable Systems and Orthogonal Polynomials, Journal of Mathematical Physics 37, 6 (1996)

[3] Kohei Iwaki, Tomoki Nakanishi; Exact WKB analysis and cluster algebras. Journal of Physics A Mathematical and Theoretical 47(47) · January 2014.

[4] A. Varos. The return of quartic oscillator. The complex WKB method.” Annales de l’I.H.P. Physique théorique 39.3 (1983): 211-338. <http://eudml.org/doc/76217>.

[5] Carl M Bender and Stefan Boettcherz. Quasi-exactly solvable quartic potential. J. Phys. A: Math. Gen. 31 (1998) L273–L277

[6] Carl M Bender. Introduction to PT -Symmetric Quantum Theory. Contemporary Physics 46(4) · February 2005

[7] Boris Shapiro and Milos Tater. On Spectral Asymptotics Of Quasi-Exactly Solvable Quartic And Yablonskii-Vorob’ev Polynomials,https://arxiv.org/abs/1412.3026.

[8] A. Eremenko and A. Gabrielov. Quasi-exactly solvable quartic: elementary integrals and asymptotics. 2011 J. Phys. A: Math. Theor. 44 312001

[9] C. M. Bender and A. Turbiner. Analytic continuation of eigenvalue problems. Phys. Lett. A, 173:442–446, 1993.

[10] C. M. Bender and E. J. Weniger, Numerical evidence that the perturbation expansion for a non-Hermitian PT -symmetric Hamiltonian is Stieltjes. J. Math. Phys., 42:2167–2183, 2001.

[11] E. Caliceti, S. Graffi and M. Maioli, Perturbation theory of odd anharmonic oscillators. Comm. Math.Phys., 75:51–66, 1980.

[12] E. Delabaere and F. Pham. Eigenvalues of complex Hamiltonians with PT-symmetry I, II. Phys. Lett.A, 250:25–32, 1998.

[13] E. Delabaere and D. T. Trinh. Spectral analysis of the complex cubic oscillator. J. Phys. A: Math. Gen.,33:8771–8796, 2000.

[14] P. Dorey, C. Dunning and R. Tateo. Spectral equivalences, Bethe ansatz equations, and reality properties in PT -symmetric quantum mechanics. J. Phys. A: Math. Gen,34:5679–5704, 2001.
[15] Boris Shapiro and Miloš Tater, Asymptotics and Monodromy of the Algebraic Spectrum of Quasi-Exactly Solvable Sextic Oscillator.

[16] M. J. Atia, Andrei Martínez-Finkelshtein, Pedro Martínez-Gonzalez, and F. Thabet, Quadratic differentials and asymptotics of Laguerre polynomials with varying complex parameters, J. Math. Anal. Appl. 416 (2014).

[17] M. J. Atia and F. Thabet, Quadratic differentials \[ A (z - a) (z - b) \frac{dz^2}{(z - c)^2} \] and algebraic Cauchy transform, Czech.Math.J., 66 (141) (2016), 351–363.

[18] Andrei Martínez-Finkelshtein, Pedro Martínez-Gonzalez, and Faouzi Thabet, Trajectories of quadratic differentials for Jacobi polynomials with complex parameters, Comp.Meth.Funct.Theory.

[19] F. Thabet, On the existence of finite critical trajectories in a family of quadratic differentials, Bull. Aust. Math. Soc. 94 (2016), 80–91.

[20] Boris Shapiro and Miloš Tater, On special asymptotics of quasi-exactly solvable quartic and Yablonskii-Vorob’ev polynomials.

[21] A. Martínez-Finkelshtein, E. A. Rakhmanov, Critical measures, quadratic differentials, and weak limits of zeros of Stieltjes polynomials, Commun. Math. Phys. vol. 302 (2011) 53-111.

[22] K. Strebel, Quadratic differentials, Vol. 5 of Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], Springer-Verlag, Berlin, 1984.

[23] J. A. Jenkins, Univalent functions and conformal mapping, Ergebnisse der Mathematik und ihrer Grenzgebiete. Neue Folge, Heft 18. Reihe: Moderne Funktionentheorie, Springer-Verlag, Berlin, 1958.

[24] T. Bergkvist and H. Rullgård, On polynomial eigenfunctions for a class of differential operators. Math. Res. Lett. 9 (2002), 153-171.

[25] A. Martínez-Finkelshtein, E. A. Rakhmanov, Critical measures, quadratic differentials, and weak limits of zeros of Stieltjes polynomials, Commun. Math. Phys. vol. 302 (2011) 53-111.

[26] Boris Shapiro, Kouichi Takemura, and Miloš Tater, On spectral polynomials of the Heun equation. II.
[27] Igor E. Pritsker. How to find a measure from its potential. Computational Methods and Function Theory, Volume 8 (2008), No.2, 597-614. Funktionentheorie, Springer-Verlag, Berlin, 1958. Zbl 0083.29606, MR0096806.

[28] J.-E. Bjork, J. Borcea, R. Bőgvad, Subharmonic Configurations and Algebraic Cauchy Transforms of Probability Measures. Notions of Positivity and the Geometry of Polynomials Trends in Mathematics 2011, pp 39-62.

[29] Rikard Bőgvad and Boris Shapiro, On motherbody measures and algebraic Cauchy transform

[30] Boris Shapiro, Kouichi Takemura, and Miloš Tater, On Spectral Polynomials of the Heun Equation.II.

[31] Yuliy Baryshnikov, On stokes Setsin New developments in singularity theory (Cambridge,2000)”, 65-86, NAT O Sci. Ser. II Math. Phys. Chem., Vol. 21, Kluwer Acad. Publ., Dordrecht,(2001).

[32] J.B. Garnett, Analytic capacity and measure, LNM 297, Springer-Verlag, 1972, 138 pp.

[33] A. Turbiner, A, Ushveridze, Spectral singularities and the quasi exactly solvable problem,Phys. Lett. 126 A (1987), 181–183.

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