Seeking connections between wormholes, gravastars, and black holes via noncommutative geometry

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Abstract
Noncommutative geometry, an offshoot of string theory, replaces point-like objects by smeared objects. The resulting uncertainty may cause a black hole to be observationally indistinguishable from a traversable wormhole, while the latter, in turn, may become observationally indistinguishable from a gravastar. The same noncommutative-geometry background allows the theoretical construction of thin-shell wormholes from gravastars and may even serve as a model for dark energy.

Keywords: Noncommutative geometry; wormholes; gravastars.

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1 Introduction
The purpose of this paper is to examine the effect of noncommutative geometry on several diverse phenomena. These include traversable wormholes, gravastars, and thin-shell wormholes from gravastars. The noncommutative-geometry background itself may even serve as a model for dark energy.

Noncommutative geometry, an offshoot of string theory, replaces point-like objects by smeared objects [1, 2, 3]. As a result, spacetime can be encoded in the commutator $[x^\mu, x^\nu] = i\theta^{\mu\nu}$, where $\theta^{\mu\nu}$ is an antisymmetric matrix that determines the fundamental cell discretization of spacetime in the same way that Planck’s constant discretizes phase space [2]. An efficient way to model the smearing effect, discussed in Refs. [4, 5, 6], is to assume that the energy density of the static and spherically symmetric and particle-like gravitational source is given by

$$\rho(r) = \frac{\mu\sqrt{\beta}}{\pi^2(r^2 + \beta)^2},$$

(1)

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which can be interpreted to mean that the gravitational source causes the mass \( \mu \) of a particle to be diffused throughout the region of linear dimension \( \sqrt{\beta} \) due to the uncertainty; \( \sqrt{\beta} \) has units of length. Eq. (1) leads to the mass distribution

\[
\int_0^r 4\pi(r')^2 \rho(r') dr' = \frac{2M}{\pi} \left( \tan^{-1} \frac{r}{\sqrt{\beta}} - \frac{r\sqrt{\beta}}{r^2 + \beta} \right),
\]

(2)

where \( M \) is now the total mass of the source.

According to Ref. [2], noncommutative geometry is an intrinsic property of spacetime and does not depend on any particular features such as curvature and can therefore be extremely useful. This usefulness is further emphasized in Ref. [2]: there seems to be a need to modify the four-dimensional Einstein action to incorporate the noncommutative effects. It is shown, however, that the effects of noncommutativity can be taken into account by keeping the standard form of the Einstein tensor on the left-hand side of the field equations and inserting the modified energy-momentum tensor as a source on the right-hand side. This leads to the conclusion, also discussed in Ref. [7], that the length scales can be macroscopic. This scale allows the determination of the mass distribution in Eq. (2), as well as the application to macroscopic objects such as wormholes.

To illustrate an important aspect of noncommutative geometry, let us start with the Schwarzschild line element

\[
d s^2 = - \left( 1 - \frac{2M}{r} \right) dt^2 + \frac{dr^2}{1 - 2M/r} + r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2).
\]

(3)

Now denote the right-hand side of Eq. (2) by \( M_\beta(r) \). Used in Eq. (3), we get

\[
d s^2 = - \left( 1 - \frac{2M_\beta(r)}{r} \right) dt^2 + \frac{dr^2}{1 - 2M_\beta(r)/r} + r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2).
\]

(4)

By letting \( \beta \to 0 \), we recover the Schwarzschild line element. However, since

\[
\lim_{r\to 0} \frac{M_\beta(r)}{r} = 0,
\]

there is no curvature singularity, a consequence of having replaced the point-like object at \( r = 0 \) by a smeared object. The result is a regular black hole.

Noncommutative geometry is by no means the only modified gravitational theory to have had a profound impact on the above topics. For example, on the subject of wormholes, Lobo and Oliveira [5] investigated such structures in the context of \( f(R) \) modified gravity, while noncommutative-geometry inspired wormholes in \( f(R) \) gravity are analyzed in Ref. [9]. Modified gravitational theories also play a key role in the study of compact stellar objects, including gravastars. In particular, the effect of an electromagnetic field on isotropic gravastar models in \( f(R, T) \) gravity is discussed by Z. Yousaf; included among its features is a stability analysis. (See Ref. [10] and references therein.) Moreover, self-gravitating spherical stars evolving in the presence of an imperfect fluid in \( f(R, T) \) gravity may exhibit an irregular energy density whose causes are investigated in Ref. [11]. The influence of the same modified gravitational theory on the dynamics of radiating spherical
fluids is presented in Ref. [12]. Other noteworthy references involving \( f(G, T) \) modified gravity are Refs. [13, 14].

Wormhole solutions in \( f(T) \) modified gravity, again with a noncommutative-geometry background, are studied in Ref. [15], while galactic-halo wormholes in \( f(T) \) gravity are discussed in Ref. [16].

2 The approximate metric

Consider the line element

\[
    ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{dr^2}{1 - m(r)/r} + r^2(d\theta^2 + \sin^2\theta d\phi^2).
\]  

To accommodate this metric, the effective mass \( m(r) \), again determined by integration, must have the form

\[
    m(r) = \frac{4M}{\pi}\left(\tan^{-1}\frac{r}{\sqrt{\beta}} - \frac{r\sqrt{\beta}}{r^2 + \beta}\right) + C_\beta,
\]

where \( C_\beta > 0 \) is a set of arbitrary constants such that \( \lim_{\beta \to 0} C_\beta = 0 \). The reason is that

\[
    \lim_{\beta \to 0} m(r) = 2M,
\]

thereby recovering the Schwarzschild line element. The meaning of \( m(r) \) in Eq. (5) now becomes clear: it is a smeared mass that, seen from a distance, is simply \( 2M \). So line element (5) approximates the Schwarzschild metric (3).

Before continuing, let us list the Einstein field equations using the more general metric from Ref. [17]:

\[
    ds^2 = -e^{2\Phi(r)}dt^2 + \frac{dr^2}{1 - m(r)/r} + r^2(d\theta^2 + \sin^2\theta d\phi^2).
\]

The equations are

\[
    \rho(r) = \frac{m'}{8\pi r^2},
\]

\[
    p_r(r) = \frac{1}{8\pi} \left[ -\frac{m}{r^3} + 2\left(1 - \frac{m}{r}\right)\Phi'\right],
\]

\[
    p_t(r) = \frac{1}{8\pi} \left(1 - \frac{m}{r}\right) \left[ \Phi'' - \frac{m' r - m}{2r(r - m)}\Phi' + \frac{\Phi'}{r} - \frac{m' r - m}{2r^2(r - m)} \right].
\]

So from Eq. (9) and Eq. (11), it follows that

\[
    0 < m'(r) < 1,
\]

since \( \rho > 0 \).
2.1 Wormholes

Wormholes are handles or tunnels in spacetime that are able to link widely separated regions of our Universe or entirely different universes. Morris and Thorne [18] proposed the following static and spherically symmetric line element for the wormhole spacetime:

\[ ds^2 = -e^{2\Phi(r)}dt^2 + \frac{dr^2}{1 - b(r)/r} + r^2(d\theta^2 + \sin^2\theta \, d\phi^2), \]  

(13)

using units in which \( c = G = 1 \). Here \( b = b(r) \) is called the shape function and \( \Phi = \Phi(r) \) is called the redshift function, which must be everywhere finite to avoid an event horizon. For the shape function we must have \( b(r_{th}) = r_{th} \), where \( r = r_{th} \) is the radius of the throat of the wormhole. An important requirement is the flare-out condition at the throat:

\[ b'(r_{th}) < 1, \]  

while \( b(r) < r \) near the throat. The flare-out condition can only be met by violating the null energy condition, which states that for the energy-momentum tensor \( T_{\alpha\beta} \),

\[ T_{\alpha\beta}k^\alpha k^\beta \geq 0 \]  

for all null vectors \( k^\alpha \). (Here \( T^t_t = -\rho \) is the energy density, \( T^r_r = p_r \) the radial pressure, and \( T^\theta_\theta = T^\phi_\phi = p_\perp \) the lateral pressure.) For example, if the outgoing null vector \((1,1,0,0)\) yields \( \rho + p_r < 0 \), the condition is violated. Matter that violates the null energy condition is called “exotic” in Ref. [18].

Instead of a throat, the Schwarzschild black hole has an event horizon at \( r = 2M \), making it a very different structure. On the other hand, their topological similarities have suggested a deeper connection: according to Hayward [19], if enough exotic matter is pumped into a black hole, it becomes a traversable wormhole. In this paper, we will use noncommutative geometry to show that the characteristic uncertainty could make the black hole observationally indistinguishable from a wormhole. Such a structure is sometimes called a black-hole mimicker [20, 21, 22, 23].

Since the function

\[ f(r) = \frac{4M}{\pi} \left( \tan^{-1} \frac{r}{\sqrt{\beta}} - \frac{r\sqrt{\beta}}{r^2 + \beta} \right), \]  

(14)

approaches \( 2M \) from below, \( C_\beta \) needs to be large enough so that \( m(r) \) has the form in Fig. 1. Thus \( m(r) \) is a translation of \( f(r) \) in the vertical direction. Here \( r = r_0 \) is the radius of the event horizon. Since the slope of \( m(r) \) is strictly less than unity, \( m(r) \) crosses the 45-degree line at some \( r = r_1 \). As a result,

\[ m(r_1) = r_1 \quad \text{and} \quad 0 < m'(r_1) < 1, \]  

\( r_1 \) being the radius of the throat, thereby satisfying all the requirements of the shape function of a Morris-Thorne wormhole. For \( r \geq r_1 \), the redshift function \( \Phi = \Phi(r) \) can be determined from \( e^{2\Phi(r)} = 1 - 2M/r \), i.e., \( \Phi(r) = \frac{1}{2} \ln (1 - 2M/r) \), leading to

\[ \Phi'(r_1) = \frac{M}{r_1^2} \frac{1}{1 - 2M/r_1}. \]  

(15)

So if \( r_1 \) is sufficiently close to \( r_0 \), the tidal forces become so large that the wormhole becomes observationally indistinguishable from a black hole, at least in the short run. (Eventually, the Hawking radiation would have an observable effect.)
Figure 1: $r = r_0$ is the radius of the event horizon and $r = r_1$ is the radius of the throat.

According to Ref. [2], there is no need to change the Einstein tensor in the field equations since the noncommutative effects can be implemented by modifying only the stress-energy tensor. So the length scales can be macroscopic. As a result, the constant $C_\beta$ can be large and still lead to a valid solution. To be consistent with observation, however, $C_\beta$ is likely to be confined to a narrow range with $r_1$ close to $r_0$, making the wormhole a black-hole mimicker.

An alternative interpretation, also based on noncommutative geometry, is discussed in the next subsection.

### 2.2 Gravastars

Returning to Subsection 2.1, a traversable wormhole results from an appropriate choice of $C_\beta$ within a narrow range and with $r_1$ close to $r_0$. The noncommutative-geometry background suggests another interpretation that remains valid for larger $r$. First we observe that decreasing $\beta$ in Eq. (14) has the same effect as increasing $r$, i.e., $\lim_{\beta \to 0} f(r) = \lim_{r \to \infty} f(r) = 2M$. So for sufficiently small $\beta$, $f(r)$ will converge to $2M$ so rapidly that $f(r)$ is approximately constant (thus $m'(r_1) \approx 0$). So while this may not exclude a wormhole with a throat at $r = r_1$, the structure would become observationally indistinguishable from an ordinary spherical shell of thickness $\delta$, which could be described as $\delta = r_2 - r_1$ for some $r_2 > r_1$, and becoming extremely dense whenever $M$ is sufficiently large. Since such a shell would consist of ordinary matter, its equation of state becomes $p(r) = \rho(r)$. (For the exterior region $r > r_2$, we have $p(r) = \rho(r) = 0$.) Such an extremely dense shell
would ordinarily collapse, but, given our noncommutative-geometry background, we can return to Ref. [2] to note that the relationship between the radial pressure and energy density is given by

\[ p_r(r) = -\rho(r). \]  \hspace{1cm} (16)

The reason is that the source is a self-gravitating droplet of anisotropic fluid of density \( \rho \), and the radial pressure is needed to prevent a collapse to a matter point. So far, then, the properties of the gravastar are correlated with our noncommutative-geometry background. However, having been reduced to an ordinary spherical volume, we would expect the interior pressure to be isotropic. Indeed, from Eq. (1), we have

\[ p_r = -\rho = -\frac{\mu \sqrt{\beta}}{\pi^2 (r^2 + \beta)^2}; \]  \hspace{1cm} (17)

now, according to Ref. [2], the lateral pressure \( p_{\perp} \) is given by

\[ p_{\perp} = -\rho - \frac{r \partial \rho}{2 \partial r} = -\frac{\mu \sqrt{\beta}}{\pi^2 (r^2 + \beta)^2} + \frac{2\mu r^2 \sqrt{\beta}}{\pi^2 (r^2 + \beta)^3}. \]  \hspace{1cm} (18)

For larger \( r \), this reduces to

\[ p_{\perp} = -\frac{\mu \sqrt{\beta}}{\pi^2 (r^2 + \beta)^2} \]  \hspace{1cm} (19)

in agreement with Eq. (17). Thanks to the noncommutative-geometry background, we can therefore take the equation of state for the interior to be

\[ p(r) = -\rho(r). \]  \hspace{1cm} (20)

Eq. (20) makes sense in another way: it describes a de Sitter vacuum that counteracts the extreme tension from the dense shell.

Taken together, these results can serve as a model of a gravastar, as described in Refs. [24, 25]. In other words, the interior volume contains an isotropic de Sitter vacuum, while the exterior is characterized by a Schwarzschild geometry, separated by a thin shell of stiff matter, i.e., there are three regions with the following respective equations of state:

1. interior: \( 0 \leq r < r_1 \), \( p = -\rho \);
2. shell: \( r_1 < r < r_2 \), \( p = +\rho \);
3. exterior: \( r > r_2 \), \( p = \rho = 0 \).

We conclude that a noncommutative-geometry wormhole may be observationally indistinguishable from a gravastar and could therefore be called a gravastar-mimicker.

Remark: While the properties of the gravastar can be accounted for in the context of noncommutative geometry, these properties have analogues in other theoretical approaches. An example is the electromagnetic field discussed in Ref. [10]. More precisely, just as the electromagnetic charge affects the pressure-density profile, the proper length of the thin shell and its energy content, the noncommutative-geometry background produces the interior isotropic pressure, as well as an extremely dense shell.
3 Additional aspects

So far we have concentrated on certain observational effects of a noncommutative-geometry background caused by the uncertainty, to which we may now add another, somewhat simpler, wormhole model. Two other aspects, also discussed in this section, are concerned with thin-shell wormholes from gravastars, as well as a possible model for dark energy.

3.1 A special wormhole

Consider the metric

$$ds^2 = -\left(1 - \frac{f(r)}{r}\right)dt^2 + \frac{dr^2}{1 - 2M/r} + r^2(d\theta^2 + \sin^2\theta
d\phi^2),$$

(21)

where $f(r)$ is defined in Eq. (14). As before, if $\beta \to 0$, we recover the Schwarzschild line element. Now recall from Fig. 1 that for $\beta > 0$, $f(r) < 2M$ in the vicinity of $r = r_0$. In particular, $f(r_0) < 2M$. It follows that Eq. (21) represents a wormhole with $b(r) = 2M$; $r = r_0$ is the radius of the throat. Since $f(r_0)$ is close to $2M$, this wormhole is also a black-hole mimicker.

3.2 Thin-shell wormholes from gravastars

In this section we will consider the theoretical construction of thin-shell wormholes using the standard cut-and-paste technique originally due to Visser [26]. The difference is that instead of a Schwarzschild black hole, we employ a gravastar combined with a noncommutative-geometry background. Accordingly, following Ref. [26], we start with two copies of a gravastar and remove from the interior of each the four-dimensional region

$$\Omega^\pm = \{r \leq a \mid a < r_g\},$$

(22)

where $r = r_g$ is the inside radius of the stiff-matter shell of the gravastar. Now identify the time-like hypersurfaces

$$\partial\Omega^\pm = \{r = a \mid a < r_g\}.$$

(23)

The resulting manifold is geodesically complete and possesses two regions connected by a throat.

Since the spherical surface $r = a$ is infinitely thin, $p_r = 0$. From the equation of state $p_r = -\rho$, we also have $\rho = 0$. So for the null vector $(1, 0, 0, 1)$, we have from Eq. (19),

$$\rho + p_\perp = 0 - \frac{\mu \sqrt{\beta}}{\pi^2(r^2 + \beta)^2} < 0,$$

(24)

so that the null energy condition is violated, thereby fulfilling a key requirement for maintaining a wormhole.

The question is whether such a wormhole can be stable. The de Sitter vacuum inside the gravastar prevents a collapse of the thin shell. Normally, however, the same de Sitter vacuum would cause the thin shell to explode, but this outcome is prevented by the stiff shell of the gravastar.
By contrast, thin-shell wormholes from Schwarzschild black holes are unstable unless the velocity of sound exceeds the velocity of light.

Remark: Thin-shell wormholes from compact stellar objects are discussed in Ref. [27].

3.3 Dark energy

Up to now our primary concern has been wormholes and gravastars, but our noncommutative geometry background can also be applied to a more general cosmological setting. Here we would be dealing with a perfect fluid with the barotropic equation of state $p = \omega \rho$, where $\omega$ is a constant. Now, by Eq. (20), $\omega = -1$, which corresponds to Einstein's cosmological constant, still considered to be the best model for dark energy [28]. The reason is that dark energy is vacuum energy with the observed cosmological constant $(1.35 \pm 0.15) \times 10^{-123}$ in Planck units.

These observations are consistent with the accelerated expansion $\ddot{a}(t) > 0$, referring to the Friedmann equation

$$\frac{\ddot{a}(t)}{a(t)} = -\frac{4\pi}{3}(\rho + 3p).$$

By Eq. (20), $p = -\rho$; so it follows directly that

$$-\frac{4\pi}{3}(\rho - 3\rho) > 0.$$

4 Conclusion

Noncommutative geometry, an offshoot of string theory, replaces point-like objects by smeared objects, thereby introducing a level of uncertainty into spacetime. As a result of this uncertainty, a black hole may become observationally indistinguishable from a wormhole. The wormhole, in turn, may become observationally indistinguishable from a gravastar. It is also shown that the same noncommutative-geometry background allows the theoretical construction of thin-shell wormholes from gravastars and may even serve as a model for dark energy.

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