LOCAL LARGE DEVIATION PRINCIPLE, LARGE DEVIATION PRINCIPLE AND INFORMATION THEORY FOR THE SIGNAL -TO- INTERFERENCE -PLUS- NOISE RATIO GRAPH MODELS

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Abstract. Given devices space \( D \), an intensity measure \( \lambda m \in (0, \infty) \), a transition kernel \( Q \) from the space \( D \) to positive real numbers \( \mathbb{R}_+ \), a path-loss function (which depends on the Euclidean distance between the devices and a positive constant \( \alpha \)), we define a Marked Poisson Point process (MPPP). For a given MPPP and technical constants \( \tau_{\lambda}, \gamma_{\lambda} : (0, \infty) \to (0, \infty) \), we define a Marked Signal-to-Interference and Noise Ratio (SINR) graph, and associate with it two empirical measures; the empirical marked measure and the empirical connectivity measure.

For a class of marked SINR graphs, we prove a joint large deviation principle (LDP) for these empirical measures, with speed \( \lambda \) in the \( \tau \)-topology. From the joint large deviation principle for the empirical marked measure and the empirical connectivity measure, we obtain an Asymptotic Equipartition Property (AEP) for network structured data modelled as a marked SINR graph. Specifically, we show that for large dense marked SINR graph one require approximately about \( \lambda^2 H(Q \times Q)/\log 2 \) bits to transmit the information contained in the network with high probability, where \( H(Q \times Q) \) is a properly defined entropy for the exponential transition kernel with parameter \( c \).

Further, we prove a local large deviation principle (LLDP) for the class of marked SINR graphs on \( D \), where \( \lambda[\tau_{\lambda}(a) \gamma_{\lambda}(a) + \lambda \tau_{\lambda}(b) \gamma_{\lambda}(b)] \to \beta(a, b), \ a, b \in (0, \infty) \), with speed \( \lambda \) from a spectral potential point. From the LLDP we derive a conditional LDP for the marked SINR graphs.

Note that, while the joint LDP is established in the \( \tau \)-topology, the LLDP assume no topological restriction on the space of marked SINR graphs. Observe also that all our rate functions are expressed in terms of the relative entropy or the kullback action or divergence function of the marked SINR on the devices space \( D \).

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1. Introduction and Background

Wireless ad-hoc and sensor networks have been the topic of much recent research. Now, with the introduction of 5th generation (5G) cellular systems, several techniques; including advanced multiple access technology, massive-MIMO, full-duplex, advanced modulation and coding schemes (MCSs), and simultaneous wireless information and power transfer (SWIPT) will constitute the next phase in global telecommunication standard, see Luo et al. [17]. 5G, a type of communication which is based on parallel processing hardware and artificial intelligence, will play a key role in wireless networks of the next generation, see Bangerter et al. [5]. Furthermore, the process of 5G usages will come along with unprecedented and exigent requirement of which connectivity is a vital cornerstone.

In telecommunication, wireless network comprises of a number of nodes which connect over a wireless channel. See Gupta and Kumar [14]. The Signal-to-Inference-Plus-Noise Ratio (SINR) determines whether a given pair of nodes can communicate with each other at a given time. Connectivity occurs in wireless network, if two nodes communicate, possibly via intermediate nodes and also, the information transport capacity of the network, See Ganesh and Torrisi [12]. In addition, network connectivity is related to various layers, components, and metrics of wireless communication systems; however, one vital performance indicator that strongly affects other metrics as well is the signal-to-interference-plus-noise-ratio (SINR). See, Oehmann et al. [19].

The SINR is of key significant to the analysis and design of wireless networks. In the process of addressing the additional requirement imposed on wireless communication, in particular, a higher availability of a highly accurate modeling of the SINR is required. Grönkvist and Hansson [13] works on SINR model rely on the assumption that nodes are uniformly distributed in the plane. On the contrast, the complexity of solution paves way for computational efficiency See, example, Behzad and Rubin [6].

More so, the SINR model can be made a complex model such that each transmission is given a power and then assumes a distance-dependent path loss. A transmission is deemed to be successful if the SINR is more than some specified threshold. See, Amdrews & Dinitz [2]. In contrast, a lot of recent work has shown that packets are successfully received only when SINR exceeds a given threshold, and assumes that packet reception rate (PRR) is zero below this threshold. See example, Santi et al. [20]. Further study of the SINR graph model has shown that an SINR model of interference is a more realistic model of interference than the protocol model of interference: a receiver node receives a packet so long as the signal to interference plus noise ratio is above a certain threshold. See, Bakshi et al. [4]. Furthermore, Manesh and Kaabouch [18] stated that SINR is successful if the desired receiver surpasses the threshold. This enables the transmitted signal to be decoded with satisfactory root error probability.

The fundamental concept of SINR model determine as transceiver design on communication system that considers interference as noise. [2] examine a set of transmitter receiver pairs located in the plane with each having an associated SINR requirement; and satisfies as many of the requirements as possible. In all communication systems, noise generated by circuit component in the receiver is a source of signal interruption. The ratio of the signal power to noise power is termed as SINR. The SINR is a vital indicator of communication link quality. See Jeske and Sampath [15]. In the article [20] the wireless link scheduling problem under a graded version of the SINR interference model is revisited. Indeed, the article defines wireless link scheduling problem under the graded SINR model,
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where they impose an additional constraint on the minimum quality of the usable links.

Li et al. [10] examined the statistical distribution of the SINR for the Minimum Mean Square Error (MMSE) receiver in multiple-input multiple output wireless communication. Their study decomposed SINR model into two independent random variables; the first part has an exact gamma distribution and the second part was shown to converge in distribution to a Normal distribution and approximate by Generalized Gamma. Also, AIAlammouri et al. [1] examined the SINR and throughput of dense cellular network with stretched exponential path loss. It was established (in the article) that the area spectral efficiency, which assumes an adaptive SINR threshold, is non-decreasing with the base station density and converges to a constant for high densities.

An accurate SINR estimation provides for both a more efficient system and a higher user perceived quality of service.

In this paper, we prove the local large deviation and large deviation principles of the Signal-To-Noise and Interference Ratio graph model (SINR). In this sequel we introduce a Marked Poisson Point Process (MPPP) and the marked SINR graph model. For a class of the marked SINR graph, we define the empirical marked measure and the empirical connectivity measure. Then, we prove a joint Large Deviation Principle (LDP) for the empirical marked measure and the empirical connectivity measure of the marked SINR graph model, with speed $\lambda$ and $\tau$-topology. From the joint large deviation principle, we obtain an Asymptotic Equipartition Property (AEP) for network structured data modelled as an SINR graph. See, example, Doku-Amponsah [9] for a generalized version of the AEP for wireless sensor networks.

Further, we prove an LLDP for the SINR graph and deduce weak variant of LDP for the SINR graph models from a spectral potential point. To be specific about this approach, given an empirical marked measure $\omega$, we define the so-called spectral potential $U_{RD}(\omega, \cdot)$, for the marked SINR graph process, where $R^D$ is a properly defined constant function which depends on the device locations and the marks. And we show that the Kullback action or the Kullback action or the divergence function $I_{\omega}(\pi)$, with respect to the empirical connectivity measure $\pi$, is the legendre dual of the spectral potential. See, example Doku-Amponsah [7] for similar results for the critical multitype Galton-Watson process.

2. Statement of Results

2.1 The Marked SINR Model for Telecommunication Networks.

Fix a dimension $d \in \mathbb{N}$ and a measureable set $D \subset \mathbb{R}^d$ with respect to the Borel-Sigma algebra $\mathcal{B}(\mathbb{R}^d)$. Denote by $m$ the Lebesgue measure on $\mathbb{R}^d$. Given an intensity measure, $\lambda m : D \rightarrow [0, 1]$, a probability kernel $Q$ from $D$ to $\mathbb{R}_+$, path loss function $\ell(r) = r^{-\alpha}$, (where $\alpha \in (0, \infty)$), and technical constants $\tau_\lambda, \gamma_\lambda : (0, \infty) \rightarrow (0, \infty)$ we define the marked SINR Graph as follows:

- We pick $X = (X_i)_{i \in I}$ a Poisson Point Process (PPP) with intensity measure $\lambda m : D \rightarrow [0, 1]$.
- Given $X$, we assign each $X_i$ a mark $\sigma(X_i) = \sigma_i$ independently according to the transition kernel $Q(\cdot, X_i)$.
- For any two marked points $((X_i, \sigma_i), (X_j, \sigma_j))$ we connect an edge iff 

$$\text{SINR}(X_i, X_j, X) \geq \tau_\lambda(\sigma_j) \quad \text{and} \quad \text{SINR}(X_j, X_i, X) \geq \tau_\lambda(\sigma_i),$$

where 

$$\text{SINR}(X_j, X_i, X) = \frac{\sigma_i \ell(||X_i - X_j||)}{N_0 + \gamma_\lambda(\sigma_j) \sum_{i \in \Gamma \setminus \{j\}} \sigma_i \ell(||X_i - X_j||)}$$
We consider $X^\lambda(\mu, Q, \ell) = \{(X_i, \sigma_i), i \in I\}$ under the joint law of the Marked PPP and the graph. We shall interpret $X^\lambda$ as a marked SINR graph and $(X_i, \sigma_i) := X_i^\lambda$ the mark of site $i$. We write

$$S(D) = \bigcup_{x \in D} \{ x : |x \cap A| < \infty, \text{ for any bounded } A \subset D \},$$

(2.1)

where $|A|$ denotes the Cardinality of the set $A$. We write $\mathcal{X} = S(D \times \mathbb{R}_+)$ and by $\mathcal{M}(\mathcal{X})$ we denote the space of positive measures on the space $\mathcal{X}$ equipped with $\tau-$ topology. Henceforth, we shall refer to $\mathcal{X}$ as locally finite subset of the set $D \times \mathbb{R}_+$.

For any SINR graph $X^\lambda$ we define a probability measure, the empirical mark measure, $L_1^\lambda \in \mathcal{M}(\mathcal{X})$, by

$$L_1^\lambda([x, \sigma_x]) := \frac{1}{\lambda} \sum_{i \in I} \delta_{X_i^\lambda}([x, \sigma_x])$$

and a symmetric finite measure, the empirical pair measure $L_2^\lambda \in \mathcal{M}(\mathcal{X} \times \mathcal{X})$, by

$$L_2^\lambda([x, \sigma_x], [y, \sigma_y]) := \frac{1}{\lambda^2} \sum_{(i,j) \in E} \left[ \delta_{(X_i^\lambda, X_j^\lambda)}([x, \sigma_x], [y, \sigma_y]) + \delta_{(X_j^\lambda, X_i^\lambda)}([x, \sigma_x], [y, \sigma_y]) \right].$$

Note that the total mass $\|L_2^\lambda\|$ of the empirical marked measure is 1 and total mass of the empirical pair measure is $2|E|/\lambda^2$. Observe that, $\mathcal{M}(\mathcal{X}) \times \mathcal{M}(\mathcal{X} \times \mathcal{X})$ is a closed subset of $\mathcal{M}(\mathcal{X}) \times \mathcal{M}(D \times \mathbb{R}_+ \times D \times \mathbb{R}_+)$ and

$$\mathbb{P}\left\{ (L_1^\lambda, L_2^\lambda) \in \mathcal{M}(\mathcal{X}) \times \mathcal{M}(\mathcal{X} \times \mathcal{X}) \right\} = 1.$$

Hence, in view of [11, Lemma 4.1.5] it is sufficient to establish Joint LDP for $(L_1^\lambda, L_2^\lambda)$ in the space $\mathcal{M}(\mathcal{X}) \times \mathcal{M}(\mathcal{X} \times \mathcal{X})$. The first theorem in this section, Theorem [2.1] is the LDP for the empirical marked measure of the SINR graph models in the space $\mathcal{M}(\mathcal{X})$.

**Theorem 2.1.** Suppose $X^\lambda$ is an SINR graph with intensity measure $\lambda m : D \rightarrow [0, 1]$ and a marked probability kernel $Q$ from $D$ to $\mathbb{R}_+$, and path loss function $\ell(r) = r^{-\alpha}$, for $\alpha > 0$. Then, as $\lambda \to \infty$, $L_1^\lambda$ satisfies an LDP in the space $\mathcal{M}(\mathcal{X})$ with good rate function

$$I_1(\omega) = \begin{cases} H(\omega | m \otimes Q), & \text{if } \|\omega\| = 1 \\ \infty, & \text{otherwise.} \end{cases}$$

We write $R^D([x, \sigma_x], [y, \sigma_y]) := \lim_{\lambda \to \infty} \lambda R^D_\lambda([x, \sigma_x], [y, \sigma_y])$, where

$$R^D_\lambda([x, \sigma_x], [y, \sigma_y]) = \int_D \left[ \frac{\tau_\lambda(\sigma_x)\gamma_\lambda(\sigma_x)}{\tau_\lambda(\sigma_x)\gamma_\lambda(\sigma_x) + \|x-y\|^{\alpha}} + \frac{\tau_\lambda(\sigma_y)\gamma_\lambda(\sigma_y)}{\tau_\lambda(\sigma_y)\gamma_\lambda(\sigma_y) + \|x-y\|^{\alpha}} \right] dz.$$

The next theorem, Theorem [2.2] is a conditional LDP for the empirical connectivity measure given the empirical marked measure, and joint LDP for the empirical marked measure and empirical connectivity measure of the SINR graph model.

**Theorem 2.2.** Suppose $X^\lambda$ is an SINR graph with intensity measure $\lambda m : D \rightarrow [0, 1]$ and a marked probability kernel $Q$ from $D$ to $\mathbb{R}_+$, and path loss function $\ell(r) = r^{-\alpha}$, for $\alpha > 0$. Let $Q$ be the exponential distribution with parameter $c$.

(i) Then, as $\lambda \to \infty$, conditional on the event $L_1^\lambda = \omega$, $L_2^\lambda$ satisfies an LDP in the space $\mathcal{M}(\mathcal{X} \times \mathcal{X})$ with speed $\lambda$ and good rate function

$$I_\omega(\pi) = \begin{cases} 0, & \text{if } \pi = e^{-R^D} \omega \otimes \omega \\ \infty, & \text{otherwise.} \end{cases}$$

(2.2)
Theorem 2.3. Suppose $X^\lambda$ is an SINR graph with intensity measure $\lambda m : D \to \mathbb{R}_+$ and a marked probability kernel $Q$ from $D$ to $\mathbb{R}_+$ and path loss function $\ell(r) = r^{-\alpha}$, for $\alpha > 0$. Assume $\lambda \tau(\lambda(\sigma_x) + \tau(\sigma_y)) \to \beta(\sigma_x, \sigma_y)$, for $x \in D$ and $\sigma_x, \sigma_y \in \mathbb{R}_+$, then we have

$$R^D([x, \sigma_x], [y, \sigma_y]) = q_\alpha \beta(\sigma_x, \sigma_y) ||y - x||^\alpha,$$

where $q_\alpha := \int_D ||z||^{-\alpha} dz < \infty$. Note, $\sigma_x$ and $\sigma_y$ are iid with common exponential distribution $Q$, with parameter $c$ and define the so-called Shannon Entropy $H$ by

$$H(Q \times Q) = - \int_X \int_X \left[ e^{-q_\alpha \beta(a,b) ||y-x||^\alpha} \log \left( \frac{e^{-q_\alpha \beta(a,b) ||y-x||^\alpha}}{1 - e^{-q_\alpha \beta(a,b) ||y-x||^\alpha}} \right) + \log(1 - e^{-q_\alpha \beta(a,b) ||y-x||^\alpha}) \right] Q(da) dx Q(db) dy.$$

The next theorem, Theorem 2.3, is the Asymptotic Equipartition Theorem or the Shannon-McMillan-Breiman Theorem for the class of SINR graphs.

Theorem 2.3. Suppose $X^\lambda$ is an SINR graph with intensity measure $\lambda m : D \to \mathbb{R}_+$ and a marked probability kernel $Q$ from $D$ to $\mathbb{R}_+$ and path loss function $\ell(r) = r^{-\alpha}$, for $\alpha > 0$. Assume $\lambda \tau(\lambda(\sigma_x) + \tau(\sigma_y)) \to \beta(\sigma_x, \sigma_y)$, for $x \in D$ and $\sigma_x, \sigma_y \in \mathbb{R}_+$. Let $Q$ be the exponential distribution with parameter $c$. Then,

$$\lim_{\lambda \to \infty} - \frac{1}{\lambda^2} \log P(X^\lambda) = H(Q \times Q), \quad \text{with high probability.}$$

Remark 1. Theorem 2.3 can be interpreted as follows: In order to code or transmit the information contain in a large telecommunication network modelled as SINR graph model, one require with high probability, approximately $\lambda^2 H(Q \times Q)/\log 2$ bits.

Let $G_\mathcal{P}$ be the set of all marked SINR graphs with intensity measure $\lambda m$, where $\lambda > 0$. For $\omega \in \mathcal{M}(\mathcal{X})$, we denote by $P_\omega = P\{ \cdot \mid L^\lambda_1 = \omega \}$ and write

$$\mathcal{M}_\omega = \left\{ \nu \in \mathcal{M}(\mathcal{X} \times \mathcal{X}) : \|\nu\| = \int_X e^{-q_\alpha \beta(a,b) ||y-x||^\alpha} \omega(dx, da) \omega(dy, db) \right\}.$$

Observe that, in this case the rate function $I_\omega(\pi)$ is given by

$$I_\omega(\pi) = \begin{cases} 0, & \text{if } \pi = e^{-R^D} \omega \otimes \omega \\ \infty, & \text{otherwise,} \end{cases} \quad (2.4)$$

where

$$R^D([x, \sigma_x], [y, \sigma_y]) = q_\alpha \beta(\sigma_x, \sigma_y) ||y - x||^\alpha.$$
**Theorem 2.4.** Suppose $X^\lambda$ is an SINR graph with intensity measure $\lambda m : D \to [0, 1]$ and a marked probability kernel $Q$ from $D$ to $\mathbb{R}_+$ and path loss function $\ell(r) = r^{-\alpha}$, for $\alpha > 0$. Assume $\lambda \left[ \tau_\lambda(a) \gamma_\lambda(a) + \tau_\lambda(b) \gamma_\lambda(b) \right] \to \beta(a, b)$, for all $a, b \in \mathbb{R}_+$. Let $Q$ be the exponential distribution with parameter $c$. Then,

- for any functional $\nu \in M_\omega$ and a number $\varepsilon > 0$, there exists a weak neighbourhood $B_\nu$ such that
  $$\mathbb{P}_\omega \left\{ X^\lambda \in G_P \mid L_2^\lambda \in B_\nu \right\} \leq e^{-\lambda \omega(\pi) - \lambda \varepsilon}.$$
- for any $\nu \in M_\omega$, a number $\varepsilon > 0$ and a fine neighbourhood $B_\nu$, we have the estimate:
  $$\mathbb{P}_\omega \left\{ X^\lambda \in G_P \mid L_2^\lambda \in B_\nu \right\} \geq e^{-\lambda \omega(\pi) + \lambda \varepsilon}.$$

The last result, Corollary 2.5, is the LDP for for the SINR graph model without any topological restriction on the space $G_P$.

**Corollary 2.5.** Suppose $X^\lambda$ is an SINR graph with intensity measure $\lambda m : D \to [0, 1]$ and a marked probability kernel $Q$ from $D$ to $\mathbb{R}_+$ and path loss function $\ell(r) = r^{-\alpha}$, for $\alpha > 0$. Assume $\lambda \left[ \tau_\lambda(a) \gamma_\lambda(a) + \tau_\lambda(b) \gamma_\lambda(b) \right] \to \beta(a, b)$, for all $a, b \in \mathbb{R}_+$. Let $Q$ be the exponential distribution with parameter $c$.

- Let $F$ be closed subset $M_\omega$. Then we have
  $$\limsup_{\lambda \to \infty} \frac{1}{\lambda} \log \mathbb{P}_\omega \left\{ X^\lambda \in G_P \mid L_2^\lambda \in F \right\} \leq - \inf_{\pi \in F} I_\omega(\pi).$$
- Let $O$ be open subset $M_\omega$. Then we have
  $$\liminf_{\lambda \to \infty} \frac{1}{\lambda} \log \mathbb{P}_\omega \left\{ X^\lambda \in G_P \mid L_2^\lambda \in O \right\} \geq - \inf_{\pi \in O} I_\omega(\pi).$$

**Remark 2.** We observe from Corollary 2.5 that
$$\lim_{\lambda \to \infty} \mathbb{P}_\omega \left\{ X^\lambda \in G_P \mid L_2^\lambda = e^{-R_D} \omega \otimes \omega \right\} = 1.$$

### 3. Proof of Theorem 2.1 by Method of Types

Let $A_1, \ldots, A_n$ be decomposition of $D \times \mathbb{R}_+ \subset \mathbb{R}^d \times \mathbb{R}_+$. We shall assume henceforth that $n < \lambda$ and note by the locally finite property of the MPPP that we have
$$\sum_{i=1}^{n} \log \left[ \frac{e^{-\lambda m \otimes Q(A_i)} [\lambda m \otimes Q(A_i)]^{\lambda \omega(A_i)}}{\lambda \omega(A_i)!} \right] \leq \log P(L_1^\lambda = \omega) \leq \sum_{i=1}^{n} \log \left[ \frac{e^{-\lambda m \otimes Q(A_i)} [\lambda m \otimes Q(A_i)]^{\lambda \omega(A_i)}}{\lambda \omega(A_i)!} \right] + \eta_n,$$where $\lim_{n \to \infty, \lambda \to \infty} \eta_n(\lambda, A_1, \ldots, A_n) = 0$. The proof of Lemma below will use the refined Stirling’s formula
$$(2\pi)^{\frac{1}{2}} \lambda^{\frac{1}{2}} e^{-\lambda^{1/2} + 1/(12\lambda + 1)} < \lambda! < (2\pi)^{\frac{1}{2}} \lambda^{\frac{1}{2}} e^{-\lambda^{1/2} + 1/(12\lambda)}.$$

**Lemma 3.1.** Suppose $X^\lambda$ is a marked PPP in a compact set $D \times \mathbb{R}_+$ with intensity measure $\lambda m \otimes Q$ such that $m$ is absolutely continuous measure on $D$. Then,
$$e^{-\lambda H(\omega(n) \mid m^{(n)} \otimes Q^{(n)})} + \theta_1(\lambda) \leq \mathbb{P} \{ L_1^\lambda = \omega \} \leq e^{-\lambda H(\omega(n) \mid m^{(n)} \otimes Q^{(n)})} + \theta_2(\lambda),$$where $\omega(n)$ and $m^{(n)} \otimes Q^{(n)}$ are the coarsest projections of $\omega$ and $m \otimes Q$ on the decomposition $(A_1, \ldots, A_n).$
Proof. For large $\lambda$, we have that
\[
\log P(L_1^1 = \omega) \leq \sum \{-\lambda m \otimes Q(A_j)\} - \log[(2\pi)^{\frac{1}{2}}(\lambda m(A_j))^{\frac{\omega(A_j)}{2}} \exp^{-\frac{1}{2}\lambda \omega(A_j)}]
\]
\[
+ \frac{1}{12(\lambda \omega(A_j) + 1)} + \lambda \omega(A_j) \log[\lambda m \otimes Q(A_j)] + \eta_n(\lambda, A_1, ..., A_n)
\]
\[
\log P(L_1^\lambda = \omega) \leq \sum \{-\lambda m \otimes Q(A_j)\} - \frac{1}{2} \log(2\pi) - [(\lambda \omega(A_j)) + \frac{1}{2}] \log[(\lambda \omega(A_j)]
\]
\[
+ (\lambda \omega(A_j \times \Gamma_j)) + \frac{1}{12(\lambda \omega(A_j) + 1)} + \lambda \omega(A_j) \log[\lambda m \otimes Q(A_j)] + \eta_n(\lambda, A_1, ..., A_n)
\]
\[
\log P(L_1^\lambda = \omega) \leq \sum \{-\lambda m \otimes Q(A_j)\} - \lambda \omega(A_j) \log[\frac{\lambda m \otimes Q(A_j)}{\omega(A_j)}]
\]
\[
- \frac{1}{2} \log[\lambda m(A_j)] - \frac{1}{12[\lambda \omega(A_j) + 1]} - \frac{1}{2} \log(2\pi) \} + \eta_n(\lambda, A_1, ..., A_n)
\]
\[
\log P(L_1^\lambda = \omega) \leq \sum \{-\lambda[m \otimes Q(A_j) - \omega(A_j)] - \lambda \omega(A_j) \log[\frac{\lambda m \otimes Q(A_j)}{\omega(A_j)}]
\]
\[
- \lambda \frac{[\log[\lambda \omega(A_j)]}{2}\lambda - \frac{1}{12\lambda^2 \lambda \omega(A_j) + \lambda} + \frac{\log(2\pi)}{2\lambda}\}
\]
\[
+ \eta_n(\lambda, A_1, ..., A_n)
\]
We choose $\theta_2(\lambda)$ as
\[
\theta_2(\lambda) = \frac{\log[\lambda \omega(A_j)]}{2\lambda} - \frac{1}{12\lambda^2 \omega(A_j) + \lambda} + \frac{\log(2\pi)}{2\lambda} + \eta_n(\lambda, A_1, ..., A_n)
\]
and observe that
\[
\lim_{\lambda \to \infty} \theta_2(\lambda) = \lim_{\lambda \to \infty} \left[\frac{\log[\lambda \omega(A_j)]}{2\lambda} - \frac{1}{12\lambda^2 \omega(A_j) + \lambda} + \frac{\log(2\pi)}{2\lambda} + \eta_n(\lambda, A_1, ..., A_n)\right] = \lim_{\lambda \to \infty} \frac{1}{\lambda}\eta_n(\lambda, A_1, ..., A_n)
\]
which proves the upper bound in the Lemma

For large $\lambda$, we have the lower bound
\[
\log P(L_1^\lambda = \omega) \geq \sum_{j=1}^{n} \{-\lambda m \otimes Q(A_j)\} - \log[(2\pi)^{\frac{1}{2}}(\lambda m(A_j))^{\frac{\omega(A_j)}{2}} \exp^{-\frac{1}{2}\lambda \omega(A_j)}]
\]
\[
+ \frac{1}{12(\lambda \omega(A_j) + 1)} + \lambda \omega(A_j) \log[\lambda m \otimes Q(A_j)]
\]
We choose $\theta_1(\lambda)$ as

$$\theta_1(\lambda) = \frac{\log(\lambda \omega(A_j))}{2 \lambda} - \frac{1}{12 \lambda^2 \omega(A_j)} + \frac{\log(2\pi)}{2 \lambda},$$

and observe that

$$\lim_{\lambda \to \infty} \theta_1(\lambda) = \lim_{\lambda \to \infty} \left[ \frac{\log(\lambda \omega(A_j))}{2 \lambda} - \frac{1}{12 \lambda^2 \omega(A_j)} + \frac{\log(2\pi)}{2 \lambda} \right] = 0.$$ 

This proves the lower bound of Lemma 3.1.

\[\square\]

**Lemma 3.2.** Suppose $X^\lambda$ is an SINR graph with intensity measure $\lambda m : D \to [0, 1]$ and a marked probability kernel $Q$ from $D$ to $\mathbb{R}_+$ and path loss function $\ell(r) = r^{-\alpha}$, for $\alpha > 0$. Then, for large $\lambda$ we have

$$|I| \leq 2\lambda \text{ almost surely.}$$

*Proof.* Note that $|I|$ is expressible as $|I| = \sum_{k=1}^{m} I_k$, where $I_1, I_2, I_3, ..., I_m$ are iid poisson random variables each with mean $\lambda/m$ and variance $\lambda/m$. Observe that $I_k \leq a := Vol(D)$, for all $k = 1, 2, 3, ..., m$ and hence, by applying the Bennett’s inequality to the sequence $I_1, I_2, I_3, ..., I_m$; we have that

$$\mathbb{P}\left\{ |I| - \mathbb{E}|I| > \lambda \right\} \leq \exp\left\{ -\frac{\lambda^2}{6} h(a) \right\},$$

where $Vol(D)$ means the Volume of the geometry space $D$ and $h(u) = (1 + u) \log(1 + u) - u$. Now, we use equation 3.1 to obtain

$$\mathbb{P}\left\{ |I| \leq \mathbb{E}|I| + \lambda \right\} \geq 1 - \exp\left\{ -\frac{\lambda^2}{6} h(a) \right\}$$

which gives

$$\lim_{\lambda \to \infty} \mathbb{P}\left\{ |I| \leq 2\lambda \right\} \geq 1.$$ 

This ends the proof of the Lemma.

\[\square\]

Let $\mathcal{M}_\lambda(\mathcal{X}) := \{ \omega \in \mathcal{M}(\mathcal{X}) : \lambda \omega(a) \in \mathbb{N} \text{ for all } a \in \mathcal{X} \}$ and let $F$ be a subset of $\mathcal{M}(\mathcal{X})$. We write $\beta_n := \max(|\mathcal{X} \cap A_1|, |\mathcal{X} \cap A_2|, ..., |\mathcal{X} \cap A_n|)$ and note that $|\mathcal{X} \cap A_i| < \infty$, for all $i = 1, 2, 3, ..., n$, by construction. We use Lemma 3 and Lemma 3.2 to obtain

$$\begin{align*}
(1 + 2\lambda)^{-n \beta_n} e^{-\lambda \inf_{\omega \in F \cap \mathcal{M}_\lambda(\mathcal{X})} H(\omega^{(n)} \mid m^{(n)} \otimes Q^{(n)}) + \theta_2(\lambda)} &\leq \sum_{\omega \in F \cap \mathcal{M}_\lambda(\mathcal{X})} e^{-\lambda H(\omega^{(n)} \mid m^{(n)} \otimes Q^{(n)}) + \theta_1(\lambda)} \\
&\leq \mathbb{P}\left\{ I_1^\lambda \in F \right\} \\
&\leq \sum_{\omega \in cl(F) \cap \mathcal{M}_\lambda(\mathcal{X})} e^{-\lambda H(\omega^{(n)} \mid m^{(n)} \otimes Q^{(n)}) + \theta_2(\lambda)} \\
&\leq (1 + 2\lambda)^{n \beta_n} e^{-\lambda \inf_{\omega \in cl(F) \cap \mathcal{M}_\lambda(\mathcal{X})} H(\omega^{(n)} \mid m^{(n)} \otimes Q^{(n)}) + \theta_2(\lambda)},
\end{align*}$$

where $\omega^{(n)}$ and $m^{(n)} \otimes Q^{(n)}$ are the coarsening projections of $\omega$ and $m \otimes Q$ on the decomposition $(A_1, ..., A_n)$. 
Taking limit as $\lambda \to \infty$ we have that
\[
\liminf_{\lambda \to \infty} \left\{ -\inf_{\{\omega \in F^n \cap \mathcal{M}_\lambda(\mathcal{X})\}} H\left(\omega^{(n)} \mid m^{(n)} \otimes Q^{(n)}\right) \right\} \leq \lim_{\lambda \to \infty} \frac{1}{\lambda} \log \mathbb{P}\left\{ L_1^\lambda \in F \right\} \\
\leq \limsup_{\lambda \to \infty} \left\{ -\inf_{\{\omega \in F^n \cap \mathcal{M}_\lambda(\mathcal{X})\}} H\left(\omega^{(n)} \mid m^{(n)} \otimes Q^{(n)}\right) \right\},
\]
Now we observe that $cl(F) \cap \mathcal{M}_\lambda(\mathcal{X}) \subseteq cl(F)$ for all $\lambda \in \mathbb{R}_+$ and hence we have
\[
\limsup_{\lambda \to \infty} \left\{ -\inf_{\{\omega \in F^n \cap \mathcal{M}_\lambda(\mathcal{X})\}} H\left(\omega^{(n)} \mid m^{(n)} \otimes Q^{(n)}\right) \right\} \leq -\inf_{\{\omega \in F^n\}} H\left(\omega^{(n)} \mid m^{(n)} \otimes Q^{(n)}\right).
\]
Using similar arguments as [11 Page 17] we obtain
\[
\liminf_{\lambda \to \infty} \left\{ -\inf_{\{\omega \in F^n \cap \mathcal{M}_\lambda(\mathcal{X})\}} H\left(\omega^{(n)} \mid m^{(n)} \otimes Q^{(n)}\right) \right\} \geq -\inf_{\{\omega \in F^n\}} H\left(\omega^{(n)} \mid m^{(n)} \otimes Q^{(n)}\right)
\]
Therefore, we have
\[
-\inf_{\{\omega \in F^n\}} H\left(\omega^{(n)} \mid m^{(n)} \otimes Q^{(n)}\right) \leq \lim_{\lambda \to \infty} \frac{1}{\lambda} \log \mathbb{P}\left\{ L_1^\lambda \in F \right\} \leq -\inf_{\{\omega \in F^n\}} H\left(\omega^{(n)} \mid m^{(n)} \otimes Q^{(n)}\right),
\]
where $\omega^{(n)}$ and $m^{(n)} \otimes Q^{(n)}$ are the coarsening projections of $\omega$ and $m \otimes Q$ on the decomposition $(A_1, ..., A_n)$. Now taking limit as $n \to \infty$ we have
\[
-\inf_{\{\omega \in F^n\}} H\left(\omega \mid m \otimes Q\right) \leq \lim_{\lambda \to \infty} \frac{1}{\lambda} \log \mathbb{P}\left\{ L_1^\lambda \in F \right\} \leq -\inf_{\{\omega \in F^n\}} H\left(\omega \mid m \otimes Q\right),
\]
which proves the Theorem 2.1

4. Proof of Theorem 2.2 by Gartner-Ellis Theorem and the Method of Mixing

Let $A_1, ..., A_n$ be the decomposition of the space $D \times \mathbb{R}_+$. Note that, for every $(x, y) \in A_i, i = 1, 2, 3, ..., n$, $\lambda L^2(x, y)$ given $\lambda L^2(x) = \lambda \omega$, is binomial with parameters $\lambda^2 \omega(x) w(y)/2$ and $p_\lambda(x, y)$. Let $K$ be the exponential distribution with parameter $c$ and recall that

4.1 Proof of Theorem 2.2(i) by Gartner-Ellis Theorem

\[
R^D_\lambda([x, \sigma_x], [y, \sigma_y]) = \int_D \left[ \frac{\tau_\lambda(\sigma_x)\gamma_\lambda(\sigma_x)}{\tau_\lambda(\sigma_x)\gamma_\lambda(\sigma_x) + (\|\sigma_x - \sigma_y\|^2)^{\alpha}} + \frac{\tau_\lambda(\sigma_y)\gamma_\lambda(\sigma_y)}{\tau_\lambda(\sigma_y)\gamma_\lambda(\sigma_y) + (\|\sigma_y - \sigma_x\|^2)^{\alpha}} \right] dz.
\]

Lemma 4.1. Suppose $X^\lambda$ is an SINR graph with intensity measure $\lambda m : D \to [0, 1]$ and a marked probability kernel $Q$ from $D$ to $\mathbb{R}_+$ and path loss function $\ell(r) = r^{-\alpha}$, for $\alpha > 0$. Then,

\[
p_\lambda([x, \sigma_x], [y, \sigma_y]) = e^{-\lambda R^D_\lambda([x, \sigma_x], [y, \sigma_y])} \quad \text{and} \quad \lim_{\lambda \to \infty} \lambda R^D_\lambda([x, \sigma_x], [y, \sigma_y]) = R^D([x, \sigma_x], [y, \sigma_y]).
\].
Proof. **Calculation of Connectivity Probability by the Laplace Transform:** We note that the Signal-Interference and Noise Ratio is given as

\[ \text{SINR}(\tilde{X}_j, \tilde{X}_i, \tilde{X}) = \frac{\sigma_i \ell(\|X_i - X_j\|)}{N_0 + \gamma \lambda(\sigma_j) \sum_{i \in I - \{j\}} \sigma_i \ell(\|X_i - X_j\|)} \]

and the total interference is defined as

\[ I_{X,\sigma}(Y) = \sum_{i \in I} \sigma_i I_i, \]

where \( I_i = \ell(\|X_i - X_j\|). \)

The probability that \( \tilde{X}_i = (y, \sigma_y) \) and \( \tilde{X}_j = (y, \sigma_y) \) are connected.

\[ P(\tilde{X}_j, \tilde{X}_i) = P \left[ \frac{\sigma_i \ell(\|X_i - X_j\|)}{N_0 + \gamma \lambda(\sigma_j)} \geq \tau \lambda(\sigma_j) \right] P \left[ \frac{\sigma_i \ell(\|X_j - X_i\|)}{\sum_{j \in I - \{i\}} \sigma_i \ell(\|X_j - X_i\|)} \geq \tau \lambda(\sigma_i) \right] \]

Now we have that

\[ P(\tilde{X}_j, \tilde{X}_i) = P \left[ \sigma_i \ell(\|X_j - X_i\|) \geq \left( N_0 + \gamma \lambda(\sigma_j) \sum_{i \in I - \{j\}} \sigma_i \ell(\|X_j - X_i\|) \right) \tau \lambda(\sigma_j) \right] \]

Let \( X_i = y, X_j = x \) and \( I_{x,\sigma}(y) = \sum_{j \in I} \ell(\|X_j - y\|) \)

\[ P_{\lambda}(x, x_y) = \int_0^\infty P\left( \sigma \geq \frac{\tau \lambda(\sigma_y)}{\ell(\|y - x\|)} \right) P\left( N_0 + \gamma \lambda(\sigma_y) I_{x,\sigma}(y) \in ds \right) \]

Assuming that \( \sigma \) follow exponential distribution (c) we have

\[ P_{\lambda}(x, x_y) = \int_0^\infty \int_0^\infty e^{-\frac{c \tau \lambda(\sigma_y)s}{\ell(\|y - x\|)}} P\left( N_0 + \gamma \lambda(\sigma_y) I_{x,\sigma}(y) \in ds \right) \]

Using Laplace Transform gives

\[ P_{\lambda}(x, x_y) = \left[ L_{N_0 + \gamma \lambda(\sigma_y) I_{y,\sigma}} \left( \frac{c \tau \lambda(\sigma_y)s}{\ell(\|y - x\|)} \right) \right] \times \left[ L_{N_0 + \gamma \lambda(\sigma_x) I_{x,\sigma}} \left( \frac{c \tau \lambda(\sigma_x)s}{\ell(\|x - y\|)} \right) \right] \]

Since the exterior noise and interference are independent

\[ P_{\lambda}(x, x_y) = \left[ L_{N_0} \left( \frac{c \tau \lambda(\sigma_y)s}{\ell(\|y - x\|)} \right) L_{I_{y,\sigma}} \left( \frac{c \tau \lambda(\sigma_y)\gamma \lambda(\sigma_y)}{\ell(\|y - x\|)} \right) \right] \times \left[ L_{N_0} \left( \frac{c \tau \lambda(\sigma_x)s}{\ell(\|x - y\|)} \right) L_{I_{x,\sigma}} \left( \frac{c \tau \lambda(\sigma_x)\gamma \lambda(\sigma_x)}{\ell(\|x - y\|)} \right) \right] \]

Assuming there is no external noise

\[ P_{\lambda}(x, x_y) = \left[ L_{I_{y,\sigma}} \left( \frac{c \tau \lambda(\sigma_y)\gamma \lambda(\sigma_y)}{\ell(\|y - x\|)} \right) \right] \times \left[ L_{I_{x,\sigma}} \left( \frac{c \tau \lambda(\sigma_x)\gamma \lambda(\sigma_x)}{\ell(\|x - y\|)} \right) \right] \]

Hence, by symmetry, we have that

\[ P_{\lambda}(x, x_y) = P_{\lambda}(x, x_y) = \left[ L_{I_{y,\sigma}} \left( \frac{c \tau \lambda(\sigma_y)\gamma \lambda(\sigma_y)}{\ell(\|y - x\|)} \right) \right] \times \left[ L_{I_{x,\sigma}} \left( \frac{c \tau \lambda(\sigma_x)\gamma \lambda(\sigma_x)}{\ell(\|y - x\|)} \right) \right] \]
Note that
\[ L_{I(x,\sigma)}(s) = \mathbb{E}(e^{-sI(x,\sigma)}), \text{for } s = \frac{c\tau_\lambda(\sigma_\gamma)\gamma_\lambda(\sigma_\gamma)}{\ell(||x-y||)}. \]
\[ L_{I(x,\sigma)}(s) = \exp \left\{ \int_D \int_0^\infty [e^{-s\sigma\ell(||z||)} - 1] Q(d\sigma, x) \mu(dz) \right\} \]

Let \( \mu(dz) = \lambda dz \) and recall that the battery is assumed to be \( Q(d\sigma, x) = ce^{-c\sigma} \ |
\[ L_{I(x,\sigma)}(s) = \exp \left\{ \int_D \int_0^\infty [e^{-s\sigma\ell(||z||)} - 1] ce^{-c\sigma} d\sigma \lambda dz \right\} \]
\[ L_{I(x,\sigma)}(s) = \exp \left\{ \lambda \int_D \int_0^\infty [ce^{-s\sigma\ell(||z||)} - c] ce^{-c\sigma} d\sigma \lambda dz \right\} \]
\[ L_{I(x,\sigma)}(s) = \exp \left\{ \lambda \int_D \int_0^\infty e^{-\sigma[s\ell(||z||)]} - \int_0^\infty ce^{-c\sigma} d\sigma \lambda dz \right\} \]
\[ L_{I(x,\sigma)}(s) = \exp \left\{ \lambda \int_D \int_0^\infty e^{-\sigma[s\ell(||z||)]} - \int_0^\infty ce^{-c\sigma} d\sigma \lambda dz \right\} \]
\[ p_\lambda([x, \sigma_x], [y, \sigma_y]) = \exp \left\{ -\lambda \int_D \int_0^\infty \frac{\sigma_x}{\ell(||x-y||)} d\sigma \right\} + \frac{t}{\ell(||x-y||)} \int_0^\infty d\sigma \lambda dz \]

Using \( \ell(r) = r^{-\alpha} \) we obtain the expression
\[ p_\lambda([x, \sigma_x], [y, \sigma_y]) = \exp \left\{ -\lambda \int_D \int_0^\infty \frac{\sigma_x}{\ell(||x-y||)} d\sigma \right\} + \frac{t}{\ell(||x-y||)} \int_0^\infty d\sigma \lambda dz \]

We write
\[ R^D_\lambda([x, \sigma_x], [y, \sigma_y]) = \int_D \left[ \frac{\tau_\lambda(\sigma_x)\gamma_\lambda(\sigma_x)}{\ell(||x-y||)} + \frac{\tau_\lambda(\sigma_y)\gamma_\lambda(\sigma_y)}{\ell(||y-x||)} \right] d\sigma \]
and observe that we have
\[ p_\lambda([x, \sigma_x], [y, \sigma_y]) = e^{-\lambda R^D_\lambda([x, \sigma_x], [y, \sigma_y])} \]
and Therefore, we have
\[ \lim_{\lambda \to \infty} \lambda R^D_\lambda([x, \sigma_x], [y, \sigma_y]) = R^D([x, \sigma_x], [y, \sigma_y]) \]
which completes the proof of Lemma 4.1.

\[ \square \]

Computation of the log moment generation function
Lemma 4.2. Suppose $X^\lambda$ is an SINR graph with intensity measure $\lambda L e(x) : D \to [0, 1]$ and a marked probability kernel $Q$ from $D$ to $\mathbb{R}_+\mu$ and path loss function $\ell(r) = r^{-\alpha}$, for $\alpha > 0$, conditional on the event $L_1^\lambda = \omega$. Let $g : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ be bounded function. Then,

$$\lim_{\lambda \to \infty} \frac{1}{\lambda} \log \mathbb{E}\left\{ e^{\lambda(g, L_1^\lambda)/2} \mid L_1^\lambda = \omega \right\} = \frac{1}{2} \lim_{n \to \infty} \left[ \sum_{j=1}^{\infty} \sum_{i=1}^{\gamma} \int_{x \in A_i} \int_{y \in A_j} g(x, y)e^{-R_D(x, y)}\omega(dx)\omega(dy) \right]$$

$$= \frac{1}{2} \int_{\mathcal{X}} \int_{\mathcal{X}} g(x, y)e^{-R_D(x, y)}\omega(dx)\omega(dy).$$

**Proof.** Now we observe that

$$\mathbb{E}\left\{ e^{\int \lambda g(x, y)L_2^\lambda(dx, dy)/2} \mid L_1^\lambda = \omega \right\} = \mathbb{E}\left\{ \prod_{x \in D} \prod_{y \in D} e^{g(x, y)\lambda L_2^\lambda(dx, dy)/2} \right\}$$

$$= \prod_{i=1}^{\infty} \prod_{j=1}^{\infty} \prod_{x \in A_i, y \in A_j} \mathbb{E}\left\{ e^{g(x, y)\lambda^2 L_2^\lambda(dx, dy)/2} \right\}$$

Hence by from Lemma 4.1 we have

$$\log \left\{ e^{\lambda(g, L_2^\lambda)/2} \mid L_1^\lambda = \omega \right\} = \sum_{j=1}^{\infty} \sum_{i=1}^{\gamma} \int_{A_i} \int_{A_j} \log \left[ 1 - p_\lambda(x, y) + p_\lambda(x, y)e^{g(x, y)/\lambda} \right] \lambda^2 \omega \otimes \omega(dx, dy)/2$$

By Euler’s Formula, see example [10] pp. 1998, we have

$$\frac{1}{\lambda} \log \mathbb{E}\left\{ e^{\lambda(g, L_2^\lambda)/2} \mid L_1^\lambda = \omega \right\} = \frac{1}{\lambda} \sum_{j=1}^{\infty} \sum_{i=1}^{\gamma} \int_{A_i} \int_{A_j} \log \left[ 1 + \frac{g(x, y)}{\lambda} - p_\lambda(x, y) + o(\lambda^2) \right] \lambda^2 \omega \otimes \omega(dx, dy)/2$$

$$\frac{1}{\lambda} \log \mathbb{E}\left\{ e^{\lambda(g, L_2^\lambda)/2} \mid L_1^\lambda = \omega \right\} = \lim_{\lambda \to \infty} \sum_{j=1}^{\infty} \sum_{i=1}^{\gamma} \int_{A_i} \int_{A_j} \log \left[ 1 + \frac{g(x, y)}{\lambda} - p_\lambda(x, y) + o(\lambda^2) \right] \lambda^2 \omega \otimes \omega(dx, dy)/2$$

$$\lim_{\lambda \to \infty} \frac{1}{\lambda} \log \mathbb{E}\left\{ e^{\lambda(g, L_2^\lambda)/2} \mid L_1^\lambda = \omega \right\} = \frac{1}{2} \sum_{j=1}^{\infty} \sum_{i=1}^{\gamma} \int_{A_i} \int_{A_j} \log \left[ e^{g(x, y)e^{-R_D(x, y)}} \right] \omega \otimes \omega(dx, dy)$$

$$\lim_{\lambda \to \infty} \frac{1}{\lambda} \log \mathbb{E}\left\{ e^{\lambda(g, L_2^\lambda)/2} \mid \omega \right\} = \frac{1}{2} \sum_{j=1}^{\infty} \sum_{i=1}^{\gamma} \int_{A_i} \int_{A_j} g(x, y)e^{-R_D(x, y)}\omega \otimes \omega(dx, dy)$$

$$= \frac{1}{2} \int_{\mathcal{X}} \int_{\mathcal{X}} g(x, y)e^{-R_D(x, y)}\omega \otimes \omega(dx, dy)$$

Hence, by Lemma 4.2 and the Gartner-Ellis theorem, $L_2^\lambda$ conditional on $L_1^\lambda = \omega$ obey a large deviation principle with speed and rate function

$$I_\omega(\pi) = \frac{1}{2} \sup_g \left\{ \int_{\mathcal{X}} \int_{\mathcal{X}} g(x, y)\pi(dx, dy) - \int_{\mathcal{X}} \int_{\mathcal{X}} g(x, y)e^{R_D(x, y)}\omega \otimes \omega(dx, dy) \right\}$$

which clearly reduces to the rate function given by
\[
I_\omega(\pi) = \begin{cases} 
0 & \text{if } \pi = e^{-RD} \omega \otimes \omega \\
\infty & \text{otherwise.}
\end{cases} \tag{4.1}
\]

### 4.2 Proof of Theorem 2.1(ii) by Method of Mixtures.

For any \( \lambda \in \mathbb{R}_+ \) we define
\[
\mathcal{M}_\lambda(\mathcal{X}) := \{ \omega \in \mathcal{M}(\mathcal{X}) : \lambda \omega(a) \in \mathbb{N} \text{ for all } a \in \mathcal{X} \},
\]
\[
\mathcal{M}_\lambda(\mathcal{X} \times \mathcal{X}) := \{ \pi \in \mathcal{M}(\mathcal{X} \times \mathcal{X}) : \lambda \pi(a,b) \in \mathbb{N}, \text{ for all } a, b \in \mathcal{X} \times \mathcal{X} \}.
\]
We denote by \( \Theta_{\lambda} := \mathcal{M}_\lambda(\mathcal{X}) \) and \( \Theta := \mathcal{M}(\mathcal{X}) \). With
\[
P_{\omega,\lambda}(\eta_\lambda) := \mathbb{P}\{ L_1^\lambda = \eta_\lambda \mid L_2^\lambda = \omega_\lambda \},
\]
\[
P(\lambda) := \mathbb{P}\{ L_1^\lambda = \omega_\lambda \}
\]
the joint distribution of \( L_1^\lambda \) and \( L_2^\lambda \) is the mixture of \( P_{\omega,\lambda} \) with \( P(\lambda) \) defined as
\[
d\hat{P}^\lambda(\omega_\lambda, \eta_\lambda) := dP_{\omega,\lambda}(\eta_\lambda) dP(\lambda).
\]

The following lemmas ensure the validity of large deviation principles for the mixtures and for the goodness of the rate function if individual large deviation principles are known. See for example, [10, Page 30] and the references therein. We observe that the family of measures \( (P^\lambda : \lambda \in (0, \infty)) \) is exponentially tight on \( \Theta \).

**Lemma 4.3.** The family of measures \( (\hat{P}^\lambda : \lambda \in \mathbb{R}_+) \) is exponentially tight on \( \Theta \times \mathcal{M}(\mathcal{X} \times \mathcal{X}) \).

**Proof.** Let \( \eta > \min_{a,b} R^D(a,b) > 0 \) and \( t = 1 - (1 - e^{-1})e^{-\eta} \). Then, we use Chebyshev’s inequality and Lemma 4.2 to obtain (for sufficiently large \( \lambda \)),
\[
\mathbb{P}\{ |E| \geq \lambda^2 t \} \leq e^{-\lambda^2 t} \mathbb{E}\{ |E| \} \leq e^{-\lambda^2 t} \sum_{i=0}^{\infty} \sum_{k=0}^{i} e^k \binom{i}{k} (e^{-\eta})^k (1 - e^{-\eta})^i - k e^{-\lambda \lambda^i} \leq e^{-\lambda^2 t} e^{-\lambda e^{t \lambda}}.
\]

Given \( N \in \mathbb{N} \) we choose \( N > q \) and observe that for sufficiently large \( \lambda \) we have
\[
\mathbb{P}\{ |E| \geq \lambda^2 N \} \leq e^{-\lambda^2 q}.
\]
Therefore, we have
\[
\mathbb{P}\{ \|L_2^\lambda\| \geq \lambda^2 N/2 \} \leq e^{-\lambda^2 q/2},
\]
which establishes Lemma 4.3.

\[\square\]

Define the function \( I : \Theta \times \mathcal{M}(\mathcal{X} \times \mathcal{X}) \to [0, \infty] \), by
\[
I(\omega, \pi) = \begin{cases} 
H(\omega \mid m \otimes Q), & \text{if } \pi = e^{-RD} \omega \otimes \omega \\
\infty & \text{otherwise.}
\end{cases} \tag{4.3}
\]

**Lemma 4.4.** \( I \) is lower semi-continuous.
Proof. Let \((\omega, \pi) \in \Theta \times \mathcal{M}(\mathcal{X} \times \mathcal{X})\) and observe that \(\pi = e^{-R^D} \omega \otimes \omega\) is closed condition. Further, we note that the relative entropy, \(H\left(\omega \mid m \otimes Q\right)\), is a lower semi-continuous function on the space \(\Theta \times \mathcal{M}(\mathcal{X} \times \mathcal{X})\). As \(I\) is a function of a relative entropy, we conclude that \(I\) is lower semi-continuous.

\[\square\]

Using [3, Theorem 5(b)] together with the two previous lemmas and the large deviation principles we have established Theorem 2.1 and Theorem 2.2(i) ensure that under \((\bar{P}^\lambda)\) the random variables \((L_1^\lambda, L_2^\lambda)\) satisfy a large deviation principle on \(\mathcal{M}(\mathcal{X} \times \mathcal{X})\) with good rate function \(I\) which ends the proof of Theorem 2.2(ii).

5. Proof of Theorem 2.3 by Large Deviation Technique

5.1 Proof of Theorem 2.3 We begin the proof of the asymptotic equipartition property, by first establishing a weak law of large numbers for the empirical mark measure and the empirical pair measure of the SINR graph.

Lemma 5.1. Suppose \(X^\lambda\) is an SINR graph with intensity measure \(\lambda m : D \rightarrow [0,1]\) and a marked probability kernel \(Q\) from \(D\) to \(\mathbb{R}_+\) and path loss function \(\ell(r) = r^{-\alpha}\), for \(\alpha > 0\). Assume \(\lambda [\tau_\lambda(a)\gamma_\lambda(a) + \tau_\lambda(b)\gamma_\lambda(b)] \rightarrow \beta(a,b) \in (0,\infty),\) for all \(a, b \in \mathbb{R}_+\).

Let \(Q\) be the exponential distribution with parameter \(c\). Then, for \(\varepsilon > 0\) we have

\[
\lim_{\lambda \rightarrow \infty} \mathbb{P}\left\{ \sup_{(x,\sigma_x) \in \mathcal{X}} \left| L_1^\lambda(x,\sigma_x) - m \otimes Q(x,\sigma_x) \right| > \varepsilon \right\} = 0
\]

and

\[
\lim_{\lambda \rightarrow \infty} \mathbb{P}\left\{ \sup_{([x,\sigma_x],[y,\sigma_y]) \in \mathcal{X} \times \mathcal{X}} \left| L_2^\lambda([x,\sigma_x],[y,\sigma_y]) - e^{-R^D} m \otimes Q \times m \otimes Q([x,\sigma_x],[y,\sigma_y]) \right| > \varepsilon \right\} = 0
\]

Proof. Let

\[F_1 = \left\{ \omega : \sup_{(x,\sigma_x) \in \mathcal{X}} |\omega(x,\sigma_x) - m \otimes Q(x,\sigma_x)| > \varepsilon \right\},\]

\[F_2 = \left\{ \omega : \sup_{([x,\sigma_x],[y,\sigma_y]) \in \mathcal{X} \times \mathcal{X}} |\omega([x,\sigma_x],[y,\sigma_y]) - e^{-R^D} m \otimes Q \times m \otimes Q([x,\sigma_x],[y,\sigma_y])| > \varepsilon \right\}\]

and \(F_3 = F_1 \cup F_2\). Now, observe from Theorem 2.1 that

\[
\lim_{\lambda \rightarrow \infty} \frac{1}{\lambda} \log \mathbb{P}\left\{ (L_1^\lambda, L_2^\lambda) \in F_3^c \right\} \leq - \inf_{(\omega,\varpi) \in F_3^c} I(\omega, \varpi).
\]

It suffices for us to show that \(I\) is strictly positive. Suppose there is a sequence \((\omega_n, \varpi_n) \rightarrow (\omega, \varpi)\) such that \(I(\omega_n, \omega_n) \downarrow I(\omega, \varpi) = 0\). This implies \(\omega = m \otimes Q\) and \(\varpi = e^{-R^D} m \otimes Q \times m \otimes Q\) which contradicts \((\omega, \varpi) \in F_3^c\). This ends the proof of the Lemma.

Now, the distribution of the marked PPP \(P(x) = \mathbb{P}\{X^\lambda = x\}\) is given by

\[
P_\lambda(x) = \prod_{i=1}^I |\mu \otimes Q(x_i,\sigma_i)| \prod_{(i,j) \in E} e^{-\lambda R^D([x_i,\sigma_i],[y_j,\sigma_j])} \prod_{(i,j) \in E} (1 - e^{-\lambda R^D([x_i,\sigma_i],[y_j,\sigma_j])}) \prod_{i=1}^I (1 - e^{-\lambda R^D([x_i,\sigma_i],[y,\sigma_i])})
\]
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\[-\frac{1}{\lambda} \log P_\lambda(x) = \frac{1}{\lambda} \left\langle - \log \mu \otimes Q, L_1^\lambda \right\rangle + \left\langle - \log \left( \frac{e^{\lambda R_D^1}}{1-e^{-\lambda R_D^1}} \right), L_2^\lambda \right\rangle + \left\langle - \log \left(1-e^{-\lambda R_D^1} \right), L_1^\lambda \otimes L_1^\lambda \right\rangle \]

Notice, \( \lim_{\lambda \to \infty} \lambda R_D^1 = R_D \), \( \lim_{\lambda \to \infty} \frac{1}{\lambda} \left\langle - \log \mu \otimes Q, L_1^\lambda \right\rangle = 0 \).

Using, Lemma 5.1 we have
\[ \lim_{\lambda \to \infty} \left\langle - \log \left( e^{-\lambda R_D^1} \right), L_2^\lambda \right\rangle = \left\langle - \log \left( e^{-R_D^1} \right), \mu \otimes Q \times m \otimes Q \right\rangle \]
\[ \lim_{\lambda \to \infty} \left\langle - \log \left(1-e^{-\lambda R_D^1} \right), L_1^\lambda \otimes L_1^\lambda \right\rangle = \left\langle - \log \left(1-e^{-R_D^1} \right), \mu \otimes Q \otimes m \otimes Q \right\rangle, \]
which concludes the proof of Theorem 2.3.

### 6. Proof of Theorem 2.4 and Corollary 2.5

For \( \omega \in \mathcal{P}(\mathcal{X}) \) we define the spectral potential of the marked SINR graph \((X^\lambda)\) conditional on the event \( \{L_1^\lambda = \omega\} \), \( U_Q(g, \omega) \) as

\[ U_Q(g, \omega) = \left\langle g, e^{-R_D^1} \omega \otimes \omega \right\rangle. \]  

(6.1)

The following remarkable properties holds for \( U_Q \):

- (i) It is finite on \( C(\omega) := \{ g \in \mathcal{X} : \mathbb{E}^{U_Q(g, \omega)} < \infty \} \)
- (ii) It is monotone.
- (iii) It is additively homogeneous.
- (iv) It is convex in \( g \).

For \( \pi \in \mathcal{M}(\mathcal{X} \times \mathcal{X}) \), we observe that \( L_\omega(\pi) \) is the Kullback action of the marked SINR graph \( X^\lambda \).

**Lemma 6.1.** The following hold for the Kullback action or divergence function \( L_\omega(\pi) \):

- \( L_\omega(\pi) = \sup_{g \in C} \{ \langle g, \pi \rangle - \langle g, e^{-R_D^1} \omega \otimes \omega \rangle \} \)
- The function \( L_\omega(\pi) \) is convex and lower semi-continuous on the space \( \mathcal{M}(\mathcal{X} \times \mathcal{X}) \).
- For any real \( \alpha \), the set \( \{ \pi \in \mathcal{M}(\mathcal{X} \times \mathcal{X}) : L_\omega(\pi) \leq \alpha \} \) is weakly compact.

The proof of Lemma 6.1 is omitted from the article. Interested readers may refer to [8] for similar proof for empirical measures of the Typed Random Graph Processes, and/or the references therein for proof of the lemma for empirical measures on measurable spaces.

Now note from Lemma 6.1 for any \( \varepsilon > 0 \), there exists a function \( g \in \mathcal{M}(\mathcal{X} \times \mathcal{X}) \) such that
\[ L_\omega(\pi) - \frac{\varepsilon}{2} < \langle g, \pi \rangle - U_Q(g, \omega). \]

Define the probability distribution \( P_\omega \) by
\[
P_\omega(x) = \prod_{(i,j) \in E} e^{g(x_i,x_j)} \prod_{(i,j) \in E} e^{h_\lambda(x_i,x_j)},
\]

where

\[
h_\lambda(x,y) = \lambda \log \left( 1 - e^{-\lambda R_D^P(x,y)} + e^{-\lambda R_D^P(x,y) + g(x,y)/\lambda} \right)
\]

Then, observe that

\[
\frac{dP_\omega}{d\tilde{P}_\omega}(x) = \prod_{(i,j) \in E} e^{-g(x_i,x_j)/\lambda} \prod_{(i,j) \in E} e^{-h_\lambda(x_i,x_j)/\lambda}
\]

where

\[
h_\lambda(x,y) = \lambda \log \left( 1 - e^{-\lambda R_D^P(x,y)} + e^{-\lambda R_D^P(x,y) + g(x,y)/\lambda} \right)
\]

Now, we define the neighbourhood of \( \nu \), \( B_\nu \) by

\[
B_\nu := \{ \pi \in M(X \times X) : \omega_\nu(1/2) - \omega_\nu(1/2) \} = \{ g \in L^2(\nu) : \langle g, \nu \rangle \}
\]

Observe, under the condition \( L^2(\nu) \in B_\nu \) we have

\[
\frac{dP_\omega}{d\tilde{P}_\omega}(x) = e^{-\lambda \log (1 - e^{-\lambda R_D^P(x,y)} + e^{-\lambda R_D^P(x,y) + g(x,y)/\lambda})}
\]

Hence, we have

\[
P_\omega \left\{ x^\lambda \in \mathcal{F} \mid L^2(\nu) \in B_\nu \right\} \leq \int \mathbb{1}_{\{ L^2(\nu) \in B_\nu \}} d\tilde{P}_\omega(x^\lambda) \leq \int e^{-\lambda L^2(\nu) - \lambda \epsilon} d\tilde{P}_\omega(x^\lambda) \leq e^{-\lambda L^2(\nu) - \lambda \epsilon}.
\]

Observe that \( L^2(\nu) = \infty \) implies Theorem 2.4 (ii), hence it sufficient for us to establish it for a probability measure of the form \( \nu = e^{-R_D^P} \otimes \omega \), where \( g = 1 \) and for \( L^2(\nu) = 0 \). Fix any number \( \epsilon > 0 \) and any neighbourhood \( B_\nu \subset M(X \times X) \). Now define the sequence of sets

\[
\mathcal{G}^\lambda = \left\{ y \in \mathcal{F} : L^2(y) \in B_\nu, \langle g, L^2(y) \rangle - \langle g, \nu \rangle \leq \frac{\epsilon}{2} \right\}.
\]

Note that for all \( y \in \mathcal{G}^\lambda \) we have

\[
\frac{dP_\omega}{d\tilde{P}_\omega}(y) > e^{-\left(\frac{\epsilon}{2} g, \nu \right) + U_Q(g, \omega) + \lambda \epsilon} > e^{\lambda \epsilon}.
\]

This yields

\[
P_\omega(\mathcal{G}^\lambda) = \int_{\mathcal{G}^\lambda} dP_\omega(y) \geq \int e^{-\left(\frac{\epsilon}{2} g, \nu \right) + U_Q(g, \omega) + \lambda \epsilon} d\tilde{P}_\omega(y) \geq e^{\lambda \epsilon} \tilde{P}_\omega(\mathcal{G}^\lambda).
\]

Using the law of large numbers, we have that \( \lim_{\lambda \to \infty} \tilde{P}_\omega(\mathcal{G}^\lambda) = 1 \). This completes of the Theorem.

**Proof of Corollary 2.5**

We observe that, by Lemma 4.3 the law of empirical connectivity measure is exponentially tight. Henceforth, without loss of generality we can assume that the set \( F \) in Theorem 2.2(ii) above is relatively compact. If we choose any \( \epsilon > 0 \); then for each functional \( \nu \in F \) we can find a weak
neighbourhood such that the estimate of Theorem 2.1(i) above holds. From all these neighbourhoods, we choose a finite cover of $G_P$ and sum up over the estimate in Theorem 2.1(i) above to obtain
\[
\limsup_{\lambda \to \infty} \frac{1}{\lambda} \log P_{\omega} \{ X^\lambda \in G_P \mid L_2^\lambda \in F \} \leq - \inf_{\pi \in F} I_{\omega}(\pi) + \varepsilon.
\]
Since $\varepsilon$ was arbitrarily chosen and the lower bound in Theorem 2.1(ii) is implies the lower bound in Theorem 2.2(i) we have the required results which completes the proof.

References

[1] AlAmmouri, A., Andrews, J. G., and Baccelli, F. (2017). Sinr and throughput of dense cellular networks with stretched exponential path loss. *IEEE Transactions on Wireless Communications*, 17(2):1147–1160.

[2] Andrews, M. and Dinitz, M. (2009). Maximizing capacity in arbitrary wireless networks in the sinr model: Complexity and game theory. In *IEEE INFOCOM 2009*, pages 1332–1340. IEEE.

[3] Biggins, J. D.(2004). Large deviations for mixtures In *Electron. Comm. Probab.*9 6071

[4] Bakshi, M., Jaumard, B., and Narayanan, L. (2017). Optimal aggregated convergecast scheduling with an sinr interference model. In *2017 IEEE 13th International Conference on Wireless and Mobile Computing, Networking and Communications (WiMob)*, pages 1–8. IEEE.

[5] Bangerter, B., Talwar, S., Arefi, R., and Stewart, K. (2014). Networks and devices for the 5g era. *IEEE Communications Magazine*, 52(2):90–96.

[6] Behzad, A. and Rubin, I. (2003). On the performance of graph-based scheduling algorithms for packet radio networks. In *GLOBECOM'03. IEEE Global Telecommunications Conference (IEEE Cat. No. 03CH37489)*, volume 6, pages 3432–3436. IEEE.

[7] Doku-Amponsah, K. (2017). Local large deviations: Mcmillian theorem for multitype galton-watson processes. *Far East J. of Mathematical Sciences (FJMS)* 102(10), pp.2307-2319.

[8] Doku-Amponsah, K. and Moeters. P. (2010). Large deviation principle for empirical measures of coloured random graphs. *Ann. Appl. Prob. 20(6),1089-2021.

[9] Doku-Amponsah, K. (2017). Local Large deviation: A McMillian Theorem for Coloured Random Graph Processes *Journal of Mathematics and Statistics* 13(4) (2017) 347-352.

[10] Doku-Amponsah, K. (2019). Lossy asymptotic equipartition property for geometric networked data structures. *Journal of Information and Optimization Sciences, 40:6, 1211-1219.

[11] Doku-Amponsah, K. and Moeters. P. (2010). Large deviation principle for empirical measures of coloured random graphs. *Ann. Appl. Prob. 20(6),1089-2021.

[12] Dembo, A. J. and Zetouni, O. (1998). Large deviation Techniques and Applications. *Springer, New, York.*

[13] Ganesh, A. J. and Torrisi, G. L. (2008). Large deviations of the interference in a wireless communication model. *IEEE Transactions on Information Theory*, 54(8):3505–3517.

[14] Grönkvist, J. and Hansson, A. (2001). Comparison between graph-based and interference-based stdma scheduling. In *Proceedings of the 2nd ACM international symposium on Mobile ad hoc networking & computing*, pages 255–258. ACM.

[15] Gupta, P. and Kumar, P. R. (2000). The capacity of wireless networks. *IEEE Transactions on information theory, 46(2):388–404.

[16] Jeske, D. R. and Sampath, A. (2004). Signal-to-interference-plus-noise ratio estimation for wireless communication systems: Methods and analysis. *Naval Research Logistics (NRL)*, 51(5):720–740.
[16] Li, P., Paul, D., Narasimhan, R., and Cioffi, J. (2006). On the distribution of sinr for the mmse mimo receiver and performance analysis. *IEEE Transactions on Information Theory*, 52(1):271–286.

[17] Luo, Y., Shi, Z., Bu, F., and Xiong, J. (2019). Joint optimization of area spectral efficiency and energy efficiency for two-tier heterogeneous ultra-dense networks. *IEEE Access*, 7:12073–12086.

[18] Manesh, M. R. and Kaabouch, N. (2017). Interference modeling in cognitive radio networks: A survey. *arXiv preprint arXiv:1707.09391*.

[19] Oehmann, D., Awada, A., Viering, I., Simsek, M., and Fettweis, G. P. (2015). Sinr model with best server association for high availability studies of wireless networks. *IEEE Wireless Communications Letters*, 5(1):60–63.

[20] Santi, P., Maheshwari, R., Resta, G., Das, S., and Blough, D. M. (2009). Wireless link scheduling under a graded sinr interference model. In *Proceedings of the 2nd ACM international workshop on Foundations of wireless ad hoc and sensor networking and computing*, pages 3–12. ACM.