\((\kappa, \mu, \nu = \text{const.})\)-CONTACT METRIC MANIFOLDS WITH \(\xi(I_M) = 0\)

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Abstract. We give a local classification of \((\kappa, \mu, \nu = \text{const.})\)-contact metric manifold \((M, \phi, \xi, \eta, g)\) with \(\kappa < 1\) which satisfies the condition "the Boeckx invariant function \(I_M = \frac{1}{\sqrt{1 - \kappa}}\) is constant along the integral curves of the characteristic vector field \(\xi\)."

1. Introduction

It is well known that there exist contact Riemannian manifolds \((M^{2n+1}, \phi, \xi, \eta, g)\) for which the curvature tensor \(R\) in the direction of characteristic vector field \(\xi\) satisfies \(R(X, Y)\xi = 0\), for all \(X, Y \in \Gamma(TM)\). For example, the tangent sphere bundle of a flat Riemannian manifold carries such a structure. In [2] Blair studied for the first time the class of contact metric manifolds satisfying above condition. If one applies a \(D_\alpha\)-homothetic deformation on \(M^{2n+1}\) with \(R(X, Y)\xi = 0\), one can find a new class of contact metric manifolds satisfying

\[(1.1) \quad R(X, Y)\xi = \kappa (\eta(\eta Y)X - \eta(X)Y) + \mu (\eta(\eta Y)hX - \eta(X)hY) ,\]

for some constants \(\kappa\) and \(\mu\), where \(2h\) denotes the Lie derivative of the structure tensor \(\phi\) with respect to characteristic vector field \(\xi\). A contact metric manifold belonging to this class is called \((\kappa, \mu)\)-contact metric manifold. This new class of Riemannian manifolds was introduced in [4] as a natural generalization both of \(R(X, Y)\xi = 0\) and the Sasakian condition \(R(X, Y)\xi = \eta(\eta Y)X - \eta(X)Y\). Nowadays contact \((\kappa, \mu)\)-manifolds are considered a very important topic in contact Riemannian geometry. In fact in despite of the technical appearance of the definition, there are good reasons for studying \((\kappa, \mu)\)-spaces. The first is that, in the non-Sasakian case (that is for \(\kappa \neq 1\)), the condition \((1.1)\) determines the curvature tensor field completely; next, \((\kappa, \mu)\)-spaces provide non-trivial examples of some remarkable classes of contact Riemannian manifolds, like CR-integrable contact metric manifolds ([14]), \(H\)-contact manifolds ([12]), harmonic contact metric manifolds ([15]), or contact Riemannian manifolds with \(\eta\)-parallel tensor ([6]); moreover, a local classification is known ([7]) and while the values of \(\kappa\) and \(\mu\) change, the form of \((1.1)\) is invariant under \(D_\alpha\)-homothetic deformations [4]. Finally, there are also non-trivial examples of \((\kappa, \mu)\)-contact metric manifolds, the most important being the unit tangent sphere bundle of a Riemannian manifold of constant sectional curvature with the usual contact metric structure.

Date: 13 June.

1991 Mathematics Subject Classification. [2000] Primary 53D10, 53C15 Secondary 53C25.

Key words and phrases. Contact metric manifold, \((\kappa, \mu, \nu)\)-contact metric manifold, nullity distributions.
In [7] Boeckx provided a local classification of non-Sasakian \((\kappa, \mu)\)-contact metric manifolds with respect to the number
\[
I_M = \frac{1 - \mu}{\sqrt{1 - \kappa}},
\]
which is an invariant of a \((\kappa, \mu)\)-contact metric manifold up to \(\mathcal{D}_\alpha\)-homothetic deformations.

Koufogiorgos and Tsichlias [8] proved the existence of a new class of 3-dimensional contact metric manifolds which are called generalized \((\kappa, \mu)\)-contact metric manifolds. Such a manifold satisfies the
\[
\kappa, \mu \text{ are non constant smooth functions on } M.
\]
Moreover, it is showed in [8] that if \(n > 1\), then \(\kappa\) and \(\mu\) are necessarily constant.

In [11] the condition (1.1) is generalized as
\[
R(X, Y)\xi = \kappa (\eta (Y) X - \eta (X) Y) + \mu (\eta (Y) hX - \eta (X) hY) + v (\eta (Y) \phi hX - \eta (X) \phi hY),
\]
where \(\kappa, \mu\) and \(v\) are non constant smooth functions on \(M\). If the curvature tensor field of the Levi-Civita connection on \(M\) satisfies (1.3), we say \((M, \phi, \xi, \eta, g)\) is a \((\kappa, \mu, v)\)-contact metric manifolds. Also, it is proved that, for dimensions greater than three, such manifolds are reduced to \((\kappa, \mu)\)-contact metric manifolds whereas, in three dimensions, \((\kappa, \mu, v)\)-contact metric manifolds.

Koufogiorgos and Tsichlias [10] gave a local classification of a non-Sasakian generalized \((\kappa, \mu)\)-contact metric manifold which satisfies the condition “the function \(\mu\) is constant along the integral curves of the characteristic vector field \(\xi\), i.e. \(\xi(\mu) = 0\).” One can easily prove that this condition is equivalent to \(\xi(I_M) = 0\) for a non-Sasakian generalized \((\kappa, \mu)\)-contact metric manifold. This has been our motivation for studying non-Sasakian \((\kappa, \mu, v)\)-contact metric manifolds with \(\xi(I_M) = 0\). We can prove that \(\xi(I_M) = 0\) satisfies the condition \(\xi(\mu) = v(\mu - 2)\). Moreover, the converse is also true.

The paper is organized as follows. Section 2 contains some necessary background on contact metric manifolds. In Section 3, we give some result concerning \((\kappa, \mu, v)\)-contact metric manifolds. In the last section, we locally classify \((\kappa, \mu, v = \text{const.})\)-contact metric manifold with \(\xi(I_M) = 0\). All manifolds are assumed to be connected.

2. Preliminaries

A differentiable manifold \(M\) of dimension \(2n + 1\) is said to be a contact manifold if it carries a global 1-form \(\eta\) such that \(\eta \wedge (d\eta)^n \neq 0\). It is well known that then there exists a unique vector field \(\xi\) (called the Reeb vector field) such that \(\eta(\xi) = 1\) and \(d\eta(\xi, \cdot) = 0\). Any contact manifold \((M, \eta)\) admits a Riemannian metric \(g\) and a \((1, 1)\)-tensor field \(\phi\) such that
\[
\phi^2 = -I + \eta \otimes \xi, \quad \phi \xi = 0, \quad \eta(X) = g(X, \xi)
\]
\[
g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad g(X, \phi Y) = d\eta(X, Y),
\]
for any vector field \(X\) and \(Y\) on \(M\). Define an operator \(h\) by \(h = \frac{1}{2}L_\xi \phi\), where \(L\) denotes Lie differentiation. The tensor field \(h\) vanishes identically if and only if the vector field \(\xi\) is Killing and in this case the contact metric manifold is said to be K-contact. It is well known that \(h\) and \(\phi h\) are symmetric operators, \(h\) anti-commutes with \(\phi\)
\[
\phi h + h\phi = 0, \quad h\xi = 0, \quad \eta \circ h = 0, \quad tr h = tr \phi h = 0,
\]
where \( tr h \) denotes the trace of \( h \). Since \( h \) anti-commutes with \( \phi \), if \( X \) is an eigenvector of \( h \) corresponding to the eigenvalue \( \lambda \) then \( \phi X \) is also an eigenvector of \( h \) corresponding to the eigenvalue \(-\lambda\) \[13\]. Moreover, for any contact manifold \( M \), the following is satisfied

\[
\nabla_X \xi = -\phi X - \phi h X
\]

where \( \nabla \) is the Riemannian connection of \( g \). If a contact metric manifold \( M \) is normal (i.e., \( N_\phi + 2d\eta \otimes \xi = 0 \)), where \( N_\phi \) denotes the Nijenhuis tensor formed with \( \phi \), then \( M \) is called a Sasakian manifold. Equivalently, a contact metric manifold is Sasakian if and only if \( R(X,Y)\xi = \eta(Y)X - \eta(X)Y \). Moreover, any Sasakian manifold is \( K \)-contact and in 3-dimension the converse also holds \[1\].

As a generalization of both \( R(X,Y)\xi = 0 \) and the Sasakian case consider

\[
R(X,Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY)
\]

for constants \( \kappa \) and \( \mu \). This condition is called \((\kappa, \mu)\)-nullity condition. This kind of manifold is called \((\kappa, \mu)\)-contact metric manifold which was introduced and deeply studied by Blair, Koufogiorgos and Papantoniou in \[4\].

The standard contact metric structure on the tangent sphere bundle \( T_1M \) satisfies the \((\kappa, \mu)\)-nullity condition if and only if the base manifold \( M \) is of constant curvature. In particular if \( M \) has constant curvature \( c \), then \( \kappa = c(2 - c) \) and \( \mu = -2c \).

Given a non-Sasakian \((\kappa, \mu)\)-contact metric manifold \( M \), Boeckx \[2\] introduced an invariant \( I_M := \frac{1}{\sqrt{1-c}} \), and proved that two non-Sasakian \((\kappa, \mu)\)-contact metric manifolds \((M_1, \phi_1, \xi_1, \eta_1, g_1)\) and \((M_2, \phi_2, \xi_2, \eta_2, g_2)\) are locally isometric as contact metric manifolds if and only if \( I_{M_1} = I_{M_2} \). Then the invariant \( I_M \) was used by Boeckx for providing a full classification of \((\kappa, \mu)\)-contact metric manifolds.

By a generalized \((\kappa, \mu)\)-contact metric manifold we mean a 3-dimensional contact metric manifold such that it satisfies \[(2.5)\], where \( \kappa, \mu \) are smooth non-constant functions on \( M \). A manifold of this class was studied by Koufogiorgos and Tsichlias in \[8\], \[9\] and \[10\]. A recent generalization of the \((\kappa, \mu)\)-contact metric manifold is given following definition.

**Definition 1 \[11\].** A \((\kappa, \mu, \nu)\)-contact metric manifold is a contact metric manifold \((M^{2n+1}, \phi, \xi, \eta, g)\) on which the Riemannian curvature tensor satisfies for every \( X, Y \in \Gamma(TM) \) the condition

\[
R(X,Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY) + \nu(\eta(Y)\phi hX - \eta(X)\phi hY),
\]

where \( \kappa, \mu, \nu \) are smooth functions on \( M \).

A contact metric manifold whose characteristic vector field \( \xi \) is a harmonic vector field is called an \( H \)-contact manifold. Moreover, in \[12\] Perrone proved that \( \xi \) is a harmonic vector field if and only if \( \xi \) is an eigenvector of the Ricci operator. In \[11\] Koufogiorgos, Markellos and Papantoniou characterized the 3-dimensional \( H \)-contact metric manifolds in \((\kappa, \mu, \nu)\)-contact metric manifolds. In particular, they proved following Theorem.

**Theorem 1 \[11\].** Let \((M^{2n+1}, \phi, \xi, \eta, g)\) be a 3-dimensional contact metric manifold. If \( M \) is a \((\kappa, \mu, \nu)\)-contact metric manifold, then \( M \) is an \( H \)-contact metric manifold. Conversely, if \( M \) is a 3-dimensional \( H \)-contact metric manifold, then \( M \) is a \((\kappa, \mu, \nu)\)-contact metric manifold on an everywhere open and dense subset of \( M \).
It is proved that for a \((\kappa, \mu, \upsilon)\)-contact metric manifold \(M\) of dimension greater than 3, the functions \(\kappa, \mu\) are constants and \(\upsilon\) is the zero function \([11]\).

Given a contact metric structure \((M^{2n+1}, \phi, \xi, \eta, g)\), consider the deformed structure

\[
\bar{\eta} = \alpha \eta, \quad \bar{\xi} = \frac{1}{\alpha} \xi, \quad \bar{\phi} = \phi, \quad \bar{g} = \alpha g + \alpha(\alpha - 1)\eta \otimes \eta,
\]

where \(\alpha\) is a positive constant. This deformation is called \(D_{\alpha}\)-homothetic deformation \([14]\). It is well known that \((M^{2n+1}, \bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})\) is also a contact metric manifold. By the direct computations we easily see that the tensor \(\bar{h}\) and the curvature tensor transform in the following manner \([4]\);

\[
\bar{h} = \frac{1}{\alpha} h
\]

and

\[
\alpha \bar{R}(X, Y)\bar{\xi} = R(X, Y)\xi + (\alpha - 1)^2(\eta(Y)X - \eta(X)Y) - (\alpha - 1)((\nabla_X \phi)Y - (\nabla_Y \phi)X + \eta(X)(Y + hY) - \eta(Y)(X + hX)),
\]

for any \(X, Y \in \Gamma(TM)\). Moreover, it is well known (\([4]\) or \([14]\)) that every 3-dimensional contact metric manifold satisfies

\[
(\nabla_X \phi Y) = g(X + hX, Y)\xi - \eta(Y)(X + hX).
\]

Using \((2.9)\) and \((2.10)\), we obtain that

\[
\alpha \bar{R}(X, Y)\bar{\xi} = \frac{\kappa + \alpha^2 - 1}{\alpha^2}(\eta(Y)X - \eta(X)Y) + \frac{\mu + 2(\alpha - 1)}{\alpha}(\eta(Y)\bar{h}X - \eta(X)\bar{h}Y)
\]

\[
+ \frac{\upsilon}{\alpha}(\eta(Y)\bar{h}X - \eta(X)\bar{h}Y)
\]

for any \(X, Y \in \Gamma(TM)\). Thus \((M^{2n+1}, \bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})\) is a \((\bar{\kappa}, \bar{\mu}, \bar{\upsilon})\)-contact metric manifold with

\[
\bar{\kappa} = \frac{\kappa + \alpha^2 - 1}{\alpha^2}, \quad \bar{\mu} = \frac{\mu + 2(\alpha - 1)}{\alpha}, \quad \bar{\upsilon} = \frac{\upsilon}{\alpha}.
\]

### 3. \((\kappa, \mu, \upsilon)\)-Contact Metric Manifolds

In this section, we will give some basic results of \((\kappa, \mu, \upsilon)\)-contact metric manifolds.

**Lemma 1 (\([11]\)).** The following relations are satisfied on any \((\kappa, \mu, \upsilon)\)-contact metric manifold \((M^3, \phi, \xi, \eta, g)\).

\[
(3.1) \quad h^2 = (\kappa - 1)\phi^2, \quad \kappa = \frac{T r l}{2} \leq 1,
\]

\[
(3.2) \quad \xi(\kappa) = 2\upsilon(\kappa - 1),
\]

\[
(3.3) \quad Q\xi = 2\kappa\xi,
\]

\[
(3.4) \quad Q = \left(\frac{\tau}{2} - \kappa\right)I + \left(-\frac{\tau}{2} + 3\kappa\right)\eta \otimes \xi + \mu h + \upsilon \phi h, \quad \kappa < 1
\]

where \(Q\) is the Ricci operator of \(M\), \(\tau\) denotes scalar curvature of \(M\) and \(l = R(., \xi)\xi\).
Lemma 2. Let $(M, \phi, \xi, \eta, g)$ be a $(\kappa, \mu, \nu)$-contact metric manifold. Then, for any point $P \in M$, with $\kappa(P) < 1$ there exist a neighbourhood $U$ of $P$ and an $h$-frame on $U$, i.e. orthonormal vector fields $\xi, X, \phi X$, defined on $U$, such that

\begin{equation}
\tag{3.5}
hX = \lambda X, \quad h\phi X = -\lambda \phi X, \quad h\xi = 0, \quad \lambda = \sqrt{1 - \kappa}
\end{equation}

at any point $q \in U$. Moreover, setting $A = X\lambda, B = \phi X\lambda$ and $C = X\nu, D = \phi X\nu$ on $U$ the following formulas are true :

\begin{align}
\nabla_{\xi} X &= -(\lambda + 1)\phi X, \quad \nabla_{\phi X} \xi = (1 - \lambda)X, \\
\nabla_{\xi} X &= -\frac{\mu}{2} \phi X, \quad \nabla_{\phi} \phi X \xi = \frac{\mu}{2} X, \\
\nabla_{X} X &= \frac{B}{2\lambda} \phi X, \quad \nabla_{X} (\phi X) X = \frac{A}{2\lambda} X, \\
\nabla_{\phi X} X &= -\frac{A}{2\lambda} \phi X + (\lambda - 1)\xi, \quad \nabla_{X} \phi X = -\frac{B}{2\lambda} X + (\lambda + 1)\xi, \\
\n[\xi, X] &= (1 + \lambda - \frac{\mu}{2}) \phi X, \quad [\xi, \phi X] = (\lambda - 1 + \frac{\mu}{2}) X, \\
\n[X, \phi X] &= -\frac{B}{2\lambda} X + A \phi X + 2\xi, \\
\h grad\mu + \phi h grad\nu &= grad\kappa - \xi(\kappa)\xi, \\
\nX\mu &= -2A - D, \\
\phi X\nu &= 2B + C, \\
\xi(A) &= (1 + \lambda - \frac{\mu}{2})B + \nu A + \lambda C, \\
\xi(B) &= (\lambda - 1 + \frac{\mu}{2}) A + \nu B + \lambda D.
\end{align}

Proof. The proofs of (3.6) – (3.11) are given in [8] and [9]. In order to prove (3.13), we will use well known formula

\begin{equation}
\frac{1}{2} grad \tau = \sum_{i=1}^{3} (\nabla_{X_{i}} Q) X_{i},
\end{equation}

where $\{X_{1} = \xi, X_{2} = X, X_{3} = \phi X\}$. Using (3.4) and (2.4), since $trh = trh\phi = 0$, we have

\begin{align}
\sum_{i=1}^{3} (\nabla_{X_{i}} Q) X_{i} &= \sum_{i=1}^{3} X_{i}(\frac{\tau}{2} - \kappa) + \sum_{i=1}^{3} (X_{i}(\mu) hX_{i} + X_{i}(v) \phi hX_{i}) \\
&+ \mu \sum_{i=1}^{3} (\nabla_{X_{i}} h) X_{i} + \nu \sum_{i=1}^{3} (\nabla_{X_{i}} \phi h) X_{i} + \xi(\frac{-\tau}{2} + 3\kappa)\xi \\
&= \frac{1}{2} grad\tau - grad\kappa + h grad\mu + \phi h grad\nu + \xi(\frac{-\tau}{2} + 3\kappa)\xi \\
&+ \mu \sum_{i=1}^{3} (\nabla_{X_{i}} h) X_{i} + \nu \sum_{i=1}^{3} (\nabla_{X_{i}} \phi h) X_{i}
\end{align}
From the relations (3.7), (3.8) and (3.9), we obtain \( \sum_{i=1}^{3} (\nabla X_i h) X_i = 0 \) and \( \sum_{i=1}^{3} (\nabla X_i \phi h) X_i = 2\lambda^2 \xi \). Using the last relations in (3.17), one has

\[
\frac{1}{2} \text{grad} \tau = \frac{1}{2} \text{grad} \tau - \text{grad} \kappa + h \text{grad} \mu + \phi h \text{grad} \nu + \xi (-\frac{\tau}{2} + 3\kappa) \xi + 2\lambda^2 \nu \xi
\]

that is

\[
\xi (\kappa) \xi - \text{grad} \kappa + h \text{grad} \mu + \phi h \text{grad} \nu + \xi (-\frac{\tau}{2} + 3\kappa) \xi + 2\lambda^2 \nu \xi = 0.
\]

Since the vector field \( \xi (\kappa) \xi - \text{grad} \kappa + h \text{grad} \mu + \phi h \text{grad} \nu \) is orthogonal to \( \xi \). So, we get (3.12). The equations (3.13) and (3.14) are immediate consequences of (3.12).

By virtue of (3.2) and (3.10), we have

\[
\xi (A) = \xi X \lambda = [\xi, X] \lambda + X \xi \lambda = (1 + \lambda - \frac{\mu}{2}) \phi X \lambda + \lambda X \nu + \nu X \lambda
\]

\[
= (1 + \lambda - \frac{\mu}{2}) B + C \lambda + vA.
\]

Similarly, the equation (3.10) is proved. \( \square \)

4. \((\kappa, \mu, \nu = \text{const.})\)-CONTACT METRIC MANIFOLDS WITH \( \xi (I_M) = 0 \)

Koufogiorgos and Tsichlias [10] gave a local classification of a non-Sasakian generalized \((\kappa, \mu)\)-contact metric manifold which satisfies the condition \( \xi (\mu) = 0 \). We recall the (1.2). We can easily prove that \( \xi (\mu) = 0 \) if and only if \( \xi (I_M) = 0 \). Now, we assume that \((M, \phi, \xi, \eta, g)\) be a non-Sasakian \((\kappa, \mu, \nu)\)-contact metric manifold. Using (3.2), we can easily obtain that \( \xi (I_M) = 0 \) if and only if \( \xi (\mu) = \nu (\mu - 2) \). This case is also our motivation. If \( \nu = 0 \), we have classification which is given in [10]. Because of this fact we assume that \( \nu \neq 0 \). Let us concentrate that the value \( \nu \) is constant. Under this assumption, we will give a local classification of \((\kappa, \mu, \nu = \text{const.})\)-contact metric manifold with \( \kappa < 1 \) satisfying the condition \( \xi (I_M) = 0 \) in the following Theorem.

**Theorem 2** (Main Theorem). Let \((M, \phi, \xi, \eta, g)\) be a non-Sasakian \((\kappa, \mu, \nu = \text{const.})\)-contact metric manifold and \( \xi (I_M) = 0 \), where \( \nu = \text{const.} \neq 0 \). Then

1) At any point of \( M \), precisely one of the following relations is valid: \( \mu = 2(1 + \sqrt{1 - \kappa}) \), or \( \mu = 2(1 - \sqrt{1 - \kappa}) \)

2) At any point \( P \in M \) there exists a chart \((U, (x, y, z))\) with \( P \in U \subseteq M \), such that

i) the functions \( \kappa, \mu \) depend only on the variables \( x, z \).

ii) if \( \mu = 2(1 + \sqrt{1 - \kappa}) \), (resp. \( \mu = 2(1 - \sqrt{1 - \kappa}) \)), the tensor fields \( \eta, \xi, \phi, g, h \) are given by the relations,

\[
\xi = \frac{\partial}{\partial x}, \quad \eta = dx - adz
\]

\[
g = \begin{pmatrix}
1 & 0 & -a \\
0 & 1 & -b \\
-a & -b & 1 + a^2 + b^2
\end{pmatrix}
\quad \text{resp.} \quad g = \begin{pmatrix}
1 & 0 & -a \\
0 & 1 & -b \\
-a & -b & 1 + a^2 + b^2
\end{pmatrix}
\]

\[
\phi = \begin{pmatrix}
0 & a & -ab \\
0 & b & -1 - b^2 \\
0 & 1 & -b
\end{pmatrix}
\quad \text{resp.} \quad \phi = \begin{pmatrix}
0 & -a & ab \\
0 & b & 1 + b^2 \\
0 & -1 & b
\end{pmatrix}
\]

\[
h = \begin{pmatrix}
0 & 0 & -a \lambda \\
0 & \lambda & -2 \lambda b \\
0 & 0 & -\lambda
\end{pmatrix}
\quad \text{resp.} \quad h = \begin{pmatrix}
0 & 0 & a \lambda \\
0 & -\lambda & 2 \lambda b \\
0 & 0 & \lambda
\end{pmatrix}
\]
with respect to the basis \( \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \), where \( a = 2y + f(z) \) (resp. \( a = -2y + f(z) \)), \( b = -\frac{\partial}{\partial z} - y f(z) + \frac{\partial}{\partial y} r(z) + \frac{\partial}{\partial z} s(z) \) (resp. \( b = \frac{\partial}{\partial z} - y f(z) + \frac{\partial}{\partial y} r(z) + \frac{\partial}{\partial z} s(z) \)), \( \lambda = \lambda(x, z) = r(z) e^{\nu x} \) and \( f(z), r(z), s(z) \) are arbitrary smooth functions of \( z \).

Before the proof of the main theorem, we will give a Lemma which contains some necessary relations to prove the main theorem.

**Lemma 3.** Let \((M, \phi, \xi, \eta, g)\) be a non-Sasakian \((\kappa, \mu, \nu = \text{const.})\)-contact metric manifold. The following formulas are valid.

- (4.1) \( \xi(A) = (1 + \lambda - \frac{\mu}{2})B + \nu A \),
- (4.2) \( \xi(B) = (\lambda - 1 + \frac{\mu}{2})A + \nu B \),
- (4.3) \( X\mu = -2A \),
- (4.4) \( \phi X\mu = 2B \),
- (4.5) \( [\xi, \phi \text{grad} \lambda] = \nu (A\phi X - BX) \).

**Proof.** Using (3.15), (3.16) and constant of \( \nu \), we have the relations (4.1) and (4.2). From (3.13) and (3.14), we obtain (4.3) and (4.4). By (3.2) and (2.1), we have

\[
[\xi, \phi \text{grad} \lambda] = [\xi, A\phi X] - B[\xi, X] = v(A\phi X - BX).
\]

**Proof of the Main Theorem:** Let \( \{\xi, X, \phi X\} \) be an \( h \)-frame, such that \( hX = \lambda X \), \( h\phi X = -\lambda \phi X \), \( \lambda = \sqrt{1 - \kappa} \)

in an appropriate neighborhood of an arbitrary point of \( M \). Using the hypothesis \( \xi(I_M) = 0 \) (i.e. \( \xi(\mu) = \nu(\mu - 2) \)) and (4.3), (4.4), we have the following relations,

- (4.7) \( (\phi \text{grad} \lambda)\mu = 4AB \),
- (4.8) \( [\xi, \phi \text{grad} \lambda] \mu = 4\nu AB \),
- (4.9) \( \xi(AB) = 2\nu AB \),
- (4.10) \( A\xi B + B\xi A = 2ABv \),
- (4.11) \( A^2(\lambda - 1 + \frac{\mu}{2}) + B^2(1 + \lambda - \frac{\mu}{2}) = 0 \).

Differentiating the relation (4.11) with respect to \( \xi \) and using the relations (3.2), \( \xi(\mu) = \nu(\mu - 2) \), (4.11) and (4.2) we can successively obtain

- (4.12) \( (1 + \lambda - \frac{\mu}{2})(\lambda - 1 + \frac{\mu}{2})AB = 0 \).

We distinguish following cases:
Case I) $M_1 = \{ P \in M \mid A(P) = 0, B(P) = 0 \}$, or
Case II) $M_2 = \{ P \in M \mid A(P) = 0, B(P) \neq 0 \}$, or
Case III) $M_3 = \{ P \in M \mid A(P) \neq 0, B(P) = 0 \}$, or
Case IV) $M_4 = \{ P \in M \mid (1 + \lambda - \frac{B}{2})(P) = 0, (\lambda - 1 + \frac{B}{2})(P) = 0 \}$, or
Case V) $M_5 = \{ P \in M \mid (1 + \lambda - \frac{B}{2})(P) = 0, (\lambda - 1 + \frac{B}{2})(P) \neq 0 \}$, or
Case VI) $M_6 = \{ P \in M \mid (1 + \lambda - \frac{B}{2})(P) \neq 0, (\lambda - 1 + \frac{B}{2})(P) = 0 \}$.

Firstly we will examine the Case I and the Case IV. We assume that the Case I is true. In this case, by (3.11) and (3.2), we get $\lambda(P) = 0$. Since $\n \neq 0$, we obtain that $\lambda(P) = 0$. This requires that $\kappa(P) = 1$ which is contradiction with $\kappa(P) < 1$. Let us suppose that the Case IV is valid. But in this situation, we have $\lambda(P) = 0$, or equivalently $\kappa(P) = 1$, which is impossible by the assumption of the main theorem.

Secondly we consider the Case II. From the formula (4.11), we find $(1 + \lambda - \frac{B}{2})(P) = 0$, or equivalently $(1 + \lambda - \frac{B}{2})(P) = 0$. From (4.17) we find $\frac{\partial}{\partial \lambda} \varphi X = 2\lambda X$, or equivalently $\kappa = 2(1 + \lambda)$. Similarly, regarding the Case VI we obtain $\mu = 2(1 - \lambda) = 2(1 - \sqrt{1 - \kappa})$. Therefore, (1) is proved. Now, we will examine the cases $\mu = 2(1 + \sqrt{1 - \kappa})$ and $\mu = 2(1 - \sqrt{1 - \kappa})$.

Case V: $\mu = 2(1 + \sqrt{1 - \kappa})$.

Let $P \in M$ and $\{ \xi, X, \phi X \}$ be an $h$-frame on a neighborhood $U$ of $P$. Using the assumption $\mu = 2(1 + \sqrt{1 - \kappa})$ and (4.11) we obtain $A = 0$ and thus the relations (3.10) and (3.11) reduce to

\begin{align*}
(4.13) \quad [\xi, X] &= 0, \\
(4.14) \quad [\xi, \phi X] &= 2\lambda X, \\
(4.15) \quad [X, \phi X] &= -\frac{B}{2\lambda} X + 2\xi.
\end{align*}

Since $[\xi, X] = 0$, the distribution which is spanned by $\xi$ and $X$ is integrable and so for any $q \in V$ there exist a chart $(V, (x, y, z))$ at $P \in V \subset U$, such that

\begin{equation}
(4.16) \quad \xi = \frac{\partial}{\partial x}, \quad X = \frac{\partial}{\partial y}, \quad \phi X = a \frac{\partial}{\partial x} + b \frac{\partial}{\partial y} + c \frac{\partial}{\partial z},
\end{equation}

where $a$, $b$ and $c$ are smooth functions on $V$. Since $\xi$, $X$ and $\phi X$ are linearly independent we have $c \neq 0$ at any point of $V$. By using (4.16), (3.2) and $A = 0$ we obtain

\begin{equation}
(4.17) \quad \frac{\partial \lambda}{\partial x} = \nu \lambda \quad \text{and} \quad \frac{\partial \lambda}{\partial y} = 0.
\end{equation}

From (4.17) we find

\begin{equation}
(4.18) \quad \lambda = r(z)e^{\nu x},
\end{equation}

where $r(z)$ is smooth function of $z$ defined on $V$. By using (4.14), (4.15) and (4.16) we have following partial differential equations:

\begin{align*}
(4.19) \quad &\frac{\partial a}{\partial x} = 0, \quad \frac{\partial b}{\partial x} = 2\lambda, \quad \frac{\partial c}{\partial x} = 0, \\
(4.20) \quad &\frac{\partial a}{\partial y} = 2, \quad \frac{\partial b}{\partial y} = -\frac{B}{2\lambda}, \quad \frac{\partial c}{\partial y} = 0.
\end{align*}
From $\frac{\partial}{\partial x} = \frac{\partial}{\partial y} = 0$ it follows that $c = c(z)$ and because of the fact that $c \neq 0$, we can assume that $c = 1$ through a reparametrization of the variable $z$. For the sake of simplicity we will continue to use the same coordinates $(x, y, z)$, taking into account that $c = 1$ in the relations that we have occured. From $\frac{\partial a}{\partial x} = 0, \frac{\partial a}{\partial y} = 2$ we obtain

$$a = a(x, y, z) = 2y + f(z),$$

where $f(z)$ is smooth function of $z$ defined on $V$. Differentiating $\lambda$ with respect to $\phi X$ and using (4.17) we have

$$(4.21) \quad B = [(2y + f(z))\nu \tau (z) + \tau'(z)]e^{ux},$$

where $\tau'(z) = \frac{\partial r}{\partial z}$. By using the relations $\frac{\partial h}{\partial y} = 2\lambda, \frac{\partial h}{\partial y} = -\frac{\partial}{\partial x}$ and (4.18) we get

$$b = -\frac{y}{2} (yv + v f(z) + \frac{\tau'(z)}{\tau(z)} + \frac{2}{v} \tau(z)e^{ux} + s(z),$$

where $s(z)$ is smooth function of $z$ defined on $V$. We will calculate the tensor fields $\eta, \phi, g$ and $h$ with respect to the basis $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$. For the components $g_{ij}$ of the Riemannian metric, using (4.16) we have

$$g_{11} = g\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right) = g(\xi, \xi) = 1, \quad g_{22} = g\left(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}\right) = g(X, X) = 1,$$

$$g_{12} = g_{21} = g\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = 0,$$

$$g_{13} = g_{31} = g\left(\frac{\partial}{\partial x}, \phi X - a \frac{\partial}{\partial x} - b \frac{\partial}{\partial y}\right) = g(\xi, \phi X) - ag_{11} = -a,$$

$$g_{23} = g_{32} = g\left(\frac{\partial}{\partial y}, \phi X - a \frac{\partial}{\partial x} - b \frac{\partial}{\partial y}\right) = g(X, \phi X) - bg_{12} - bg_{22} = -b,$$

$$1 = g(\phi X, \phi X) = g_{33} = a^2 + b^2 + g_{33} + 2abg_{12} + 2ag_{13} + 2bg_{23} = a^2 + b^2 + g_{33} - 2a^2 - 2b^2 = g_{33} - a^2 - b^2,$$

from which we obtain $g_{33} = 1 + a^2 + b^2$.

The components of the tensor field $\phi$ are immediate consequences of

$$\phi(\xi) = \phi\left(\frac{\partial}{\partial x}\right) = 0, \quad \phi\left(\frac{\partial}{\partial y}\right) = \phi X = a \frac{\partial}{\partial x} + b \frac{\partial}{\partial y} + \frac{\partial}{\partial z},$$

$$\phi\left(\frac{\partial}{\partial z}\right) = \phi(\phi X - a \frac{\partial}{\partial x} - b \frac{\partial}{\partial y}) = \phi^2 X - a\phi\left(\frac{\partial}{\partial x}\right) - b\phi\left(\frac{\partial}{\partial y}\right) = -X - b(a \frac{\partial}{\partial x} + b \frac{\partial}{\partial y} + \frac{\partial}{\partial z}),$$

$$= -\frac{\partial}{\partial y} - ab \frac{\partial}{\partial x} - b^2 \frac{\partial}{\partial y} - b \frac{\partial}{\partial z} = -ab \frac{\partial}{\partial x} - (1 + b^2) \frac{\partial}{\partial y} - b \frac{\partial}{\partial z}.$$
Now we calculate the components of the tensor field \( h \) with respect to the basis \( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \). 

\[
\begin{align*}
\frac{h(\xi)}{\partial x} &= h(\frac{\partial}{\partial x}) = 0, \quad \frac{h(\phi X)}{\partial y} = \lambda \frac{\partial}{\partial y}, \\
\frac{h(\phi X - a \frac{\partial}{\partial x} - b \frac{\partial}{\partial y})}{\partial z} &= -\lambda \phi X - b \lambda \frac{\partial}{\partial y} \\
\frac{h(\phi X - a \frac{\partial}{\partial x} - b \frac{\partial}{\partial y})}{\partial z} &= -\lambda(a \frac{\partial}{\partial x} + b \frac{\partial}{\partial y} + c \frac{\partial}{\partial z}) - b \lambda \frac{\partial}{\partial y}.
\end{align*}
\]

Thus the proof of the Case V is completed.

Case VI): \( \mu = 2(1 - \sqrt{1 - \kappa}) \).

As in the Case V, we consider an \( h \)-frame \( \{ \xi, X, \phi X \} \). Using the assumption \( \mu = 2(1 - \sqrt{1 - \kappa}) \) and (4.11) we obtain \( B = 0 \) and thus the relation (3.10) is written as

\[
\begin{align*}
[\xi, X] &= 2\lambda \phi X, \\
[\xi, \phi X] &= 0, \\
[X, \phi X] &= A \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}.
\end{align*}
\]

Because of (4.23) we find that there is a chart \((V', (x, y, z))\) such that

\[
\begin{align*}
\xi &= \frac{\partial}{\partial x}, \quad \phi X = \frac{\partial}{\partial y}
\end{align*}
\]

on \( V' \). We put

\[
X = a \frac{\partial}{\partial x} + b \frac{\partial}{\partial y} + c \frac{\partial}{\partial z},
\]

where \( a, b, c \) are smooth functions defined on \( V' \). As in the Case V, we can directly calculate the tensor fields \( \eta, \phi, g \) and \( h \) with respect to the basis \( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \). This completes the proof of the main theorem. \( \square \)

In the following Theorem, we will locally construct \((\kappa, \mu, \nu = \text{const.} \neq 0)\)-contact metric manifolds with \( \kappa < 1 \) and \( \xi(I_M) = 0 \).

**Theorem 3.** Let \( \kappa : I = I_1 \times I_2 \subset \mathbb{R}^2 \to \mathbb{R} \) be a smooth function defined on open subset \( I \) of \( \mathbb{R}^2 \), such that \( \kappa(x, z) = 1 - (r(z)e^{\nu(z)})^2 < 1 \) for any \((x, z) \in I, r(z) \) is a smooth function on open interval \( I_2 \) and \( \nu \) is constant different from zero. Then we can construct two families of non-Sasakian \((\kappa_i, \mu_i, \nu)\)-manifolds \((M_i, \phi_i, \xi_i, \eta_i, g_i), i = 1, 2, \) in the set \( M = I \times \mathbb{R} \subset \mathbb{R}^3 \), so that for any \( P(x, z, y) \in M, \) the following are valid.

\[
\begin{align*}
\kappa_1(P) = \kappa_2(P) = \kappa(x, z), \quad \mu_1(P) = 2(1 + \sqrt{1 - \kappa(x, z)}) \quad \text{and} \quad \mu_2(P) = 2(1 - \sqrt{1 - \kappa(x, z)})
\end{align*}
\]

Each family is determined by two arbitrary smooth functions of two variables.
(\kappa, \mu, \nu = \text{const.})-CONTACT METRIC MANIFOLDS WITH \( \xi(I_M) = 0 \)

**Proof.** We put \( \lambda(x, z) = \sqrt{1 - \kappa(x, z)} = r(z)e^{\nu z} > 0 \) and consider on \( M \) the linearly independent vector fields

\[
\xi_1 = \frac{\partial}{\partial x}, \quad X_1 = \frac{\partial}{\partial y} \quad \text{and} \quad Y_1 = a \frac{\partial}{\partial x} + b \frac{\partial}{\partial y} + \frac{\partial}{\partial z},
\]

where \( a(x, y, z) = 2y + f(z), \quad b(x, y, z) = -\frac{f(yu + vzf(z) + r'(z))}{r(z)} + \frac{2}{r(z)}e^{\nu z} + s(z), \)

\( f(z), r(z), s(z) \) are arbitrary smooth functions of \( z \). The structure tensor fields \( \eta_1, g_1, \phi_1 \) are defined by \( \eta_1 = dx - (2y + f(z))dz, \quad g_1 = \begin{pmatrix} 1 & 0 & -a \\ 0 & 1 & -b \\ -a & -b & 1 + a^2 + b^2 \end{pmatrix} \) and \( \phi_1 = \begin{pmatrix} 0 & a & -ab \\ 0 & b & -1 - b^2 \\ 0 & 1 & -b \end{pmatrix} \), respectively. From (4.25), we can easily obtain

\[
\begin{align*}
[\xi_1, X_1] &= 0, \\
[X_1, Y_1] &= -\frac{[(2y + f(z))\nu z + r'(z)]e^{\nu z}}{2\lambda(x, z)}X_1 + 2\xi_1.
\end{align*}
\]

Since \( \eta_1 \wedge d\eta_1 = -2dx \wedge dy \wedge dz \neq 0 \) everywhere on \( M \), we decide that \( \eta_1 \) is a contact form. By using just defined \( g_1 \) and \( \phi_1 \), we find \( g_1 = g(\cdot, \xi_1), \quad \phi_1 X_1 = Y_1, \quad \phi_1 Y_1 = -X_1, \quad \phi_1 \xi_1 = 0 \) and \( d\eta_1(Z, W) = g_1(Z, \phi_1 W), \quad g_1(\phi_1 Z, \phi_1 W) = g_1(Z, W) - \eta_1(Z)\eta_1(W) \) for any \( Z, W \in \Gamma(M) \). From the well known Koszul’s formula and (4.24), we obtain

\[
\begin{align*}
\nabla_{X_1}\xi_1 &= -(\lambda(x, z) + 1)Y_1, \\
\nabla_{Y_1}\xi_1 &= (1 - \lambda(x, z))X_1, \\
\nabla\xi_1, X_1 &= -(1 + \lambda(x, z))Y_1, \\
\nabla\xi_1, Y_1 &= (1 + \lambda(x, z))X_1, \\
\n\nabla_{X_1}X_1 &= \frac{[(2y + f(z))\nu z + r'(z)]e^{\nu z}}{2\lambda(x, z)}Y_1, \\
\n\nabla_{Y_1}Y_1 &= 0.
\end{align*}
\]

\[
\begin{align*}
\nabla_{Y_1}X_1 &= (\lambda(x, z) - 1)\xi_1, \\
\nabla_{X_1}Y_1 &= -\frac{[(2y + f(z))\nu z + r'(z)]e^{\nu z}}{2\lambda(x, z)}X_1 + (\lambda(x, z) + 1)\xi_1,
\end{align*}
\]

\( h_1 \phi_1 X_1 = -\lambda(x, z)\phi_1 X_1 \) and \( h_1 X_1 = \lambda(x, z)X_1 \), where \( \nabla \) is Levi-Civita connection of \( g_1 \). By using the relations (4.28)-(4.32) we obtain

\[
\begin{align*}
R(X_1, \xi_1)\xi_1 &= \kappa_1 X_1 + \mu_1 h_1 X_1 + \nu \phi_1 h_1 X_1, \\
R(Y_1, \xi_1)\xi_1 &= \kappa_1 Y_1 + \mu_1 h_1 Y_1 + \nu \phi_1 h_1 Y_1, \\
R(X_1, Y_1)\xi_1 &= 0.
\end{align*}
\]

From the above relations and by virtue of the linearity of the curvature tensor \( R \), we conclude that

\[
R(Z, W)\xi_1 = (\kappa_1 I + \mu_1 h_1 + \nu \phi_1 h_1)(\eta_1(Z)W - \eta_1(W)Z)
\]

for any \( Z, W \in \Gamma(M) \), i.e. \( (M, \phi_1, \xi_1, \eta_1, g_1) \) is \( (\kappa_1, \mu_1, \nu = \text{const.}) \) contact metric manifold with \( \xi(I_M) = 0 \) and thus the construction of the first family is completed. For the second construction, we consider the vector fields

\[
\xi_2 = \frac{\partial}{\partial x}, \quad Y_2 = \frac{\partial}{\partial y},
\]
(4.34) \[ X_2 = (-2y + f(z)) \frac{\partial}{\partial x} + \frac{y^2}{2} \frac{\partial}{\partial y} - \frac{y f(z)}{2} - \frac{y r'(z)}{2 r(z)} + \frac{2}{v} r(z) e^{\nu x} + s(z) \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \]

and define the tensor fields \( \eta_2, g_2, \phi_2, h_2 \) as follows:

\[ \eta_2 = dx - (-2y + f(z))dz \]

\[ g_2 = \begin{pmatrix} 1 & 0 & -a \\ 0 & 1 & -b \\ -a & -b & 1 + a^2 + b^2 \end{pmatrix}, \quad \phi_2 = \begin{pmatrix} 0 & -a & ab \\ 0 & -b & 1 + b^2 \\ 0 & -1 & b \end{pmatrix}, \]

\[ h_2 = \begin{pmatrix} 0 & 0 & a\lambda_2 \\ 0 & -\lambda_2 & 2\lambda_2b \\ 0 & 0 & \lambda_2 \end{pmatrix} \]

with respect to the basis \( \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \), where \( a = -2y + f(z), b = \left( \frac{y^2}{2} - y \frac{f(z)}{2} - \frac{y r'(z)}{2 r(z)} + \frac{2}{v} r(z) e^{\nu x} + s(z) \right) \). As in first construction, we say that \( (M, \phi_2, \xi_2, \eta_2, g_2) \) is \( (\kappa_2, \mu_2, \nu = \text{const.}) \)-contact metric manifold with \( \xi(I_M) = 0 \), where \( \kappa_2(x, y, z) = \kappa_2(x, z) = r(z) e^{\nu x}, \mu_2(x, y, z) = 2(1 - \sqrt{\kappa_2(x, z)}) \). This completes the proof of the Theorem. □

The Ricci operator \( Q \) was given in the relation (3.4) for any \( (\kappa, \mu, \nu) \)-contact metric manifold \( (M^3, \phi, \xi, \eta, g) \). If we carefully look at this relation, the scalar curvature \( \tau \) is not obvious. Now, we will give the scalar curvature \( \tau \) respect to \( \kappa, \mu \) and \( \nu \) for \( (\kappa, \mu, \nu = \text{const.}) \)-contact metric manifold.

**Theorem 4.** Let \( (M, \phi, \xi, \eta, g) \) be a non Sasakian \( (\kappa, \mu, \nu = \text{const.}) \)-contact metric manifold. Then,

\[ \Delta \lambda = X(A) + \phi X(B) + \nu^2 \lambda - \frac{1}{2\lambda} (A^2 + B^2) \]

and

\[ \tau = \frac{1}{\lambda} (\Delta \lambda - \nu^2 \lambda) - \frac{1}{\lambda^2} \| \text{grad}\lambda \|^2 + 2(\kappa - \mu), \]

where \( \Delta \lambda \) is Laplacian of \( \lambda \).

**Proof.** Using the definition of the Laplacian and together by Lemma 1, we have

\[ \Delta \lambda = XX(\lambda) + \phi X \phi X(\lambda) + \xi \xi(\lambda) - (\nabla_X X)\lambda - (\nabla_{\phi X} \phi X)\lambda - (\nabla_{\xi} \xi)\lambda = X(A) + \phi X(B) + \nu^2 \lambda - \frac{1}{2\lambda} (A^2 + B^2). \]
For the computing scalar curvature \( \tau \) of \( M \), we will use (3.6)-(3.9). Defining the curvature tensor \( R \), we obtain

\[
R(X, \phi X)\phi X = \nabla_X \nabla_{\phi X} \phi X - \nabla_{\phi X} \nabla_X \phi X - \nabla_{[X, \phi X]} \phi X
\]

\[
= \nabla_X \left( \frac{A}{2\lambda} X \right) - \nabla_{\phi X} \left( \frac{B}{2\lambda} X + (1 + \lambda) \xi \right) - \nabla_{-\frac{A}{2\lambda} X + \frac{1}{2} \phi X + \xi} \phi X
\]

\[
= X \left( \frac{A}{2\lambda} \right) X + \frac{A}{2\lambda} \nabla_X X + \phi X \left( \frac{B}{2\lambda} \right) X + \frac{B}{2\lambda} \nabla_{\phi X} X
\]

\[
- \phi X(\lambda) \xi - (1 + \lambda) \nabla_{\phi X} \xi + \frac{B}{2\lambda} \nabla_X \phi X - \frac{A}{2\lambda} \nabla_X X - 2 \nabla_\xi \phi X
\]

\[
= X \left( \frac{A}{2\lambda} \right) X + \frac{A}{2\lambda} B \phi X + \phi X \left( \frac{B}{2\lambda} \right) X
\]

\[
+ B \left( \frac{A}{2\lambda} \right) \phi X + (1 + \lambda) \xi - \frac{A}{2\lambda} \left( \frac{A}{2\lambda} \right) X - 2 \left( \frac{\mu}{2} \right) X
\]

\[
= \left[ X \left( \frac{A}{2\lambda} \right) + \phi X \left( \frac{B}{2\lambda} \right) - \frac{B^2}{4\lambda^2} - \frac{A^2}{4\lambda^2} + (\lambda^2 - 1) - \mu \right] X
\]

\[
= \left[ \frac{1}{2} \left( \frac{X(A)\lambda - A^2}{\lambda^2} + \phi X(B)\lambda - B^2 \right) - \frac{1}{4\lambda^2} (A^2 + B^2) + (\lambda^2 - 1) - \mu \right] X
\]

\[
= \left[ \frac{1}{2} \left( X(A) + \phi X(B) \right) - \frac{1}{2\lambda^2} (A^2 + B^2) - \frac{1}{4\lambda^2} (A^2 + B^2) + (\lambda^2 - 1) - \mu \right] X
\]

\[
= \left[ \frac{1}{2\lambda} \left( \Delta \lambda - \nu^2 \lambda \right) - \frac{1}{2\lambda^2} \parallel \text{grad} \lambda \parallel^2 - (\kappa + \mu) \right] X
\]

and thus

\[
g(R(X, \phi X)\phi X, X) = \frac{1}{2\lambda} (\Delta \lambda - \nu^2 \lambda) - \frac{1}{2\lambda^2} \parallel \text{grad} \lambda \parallel^2 - (\kappa + \mu)
\]

By definition of scalar curvature, i.e. \( \tau = Tr Q = g(QX, X) + g(Q\phi X, \phi X) + g(Q\xi, \xi) \), and using (3.3), we have

\[
\tau = 2g(R(X, \phi X)\phi X, X) + 2g(Q\xi, \xi)
\]

\[
= \frac{1}{\lambda} (\Delta \lambda - \nu^2 \lambda) - \frac{1}{\lambda^2} \parallel \text{grad} \lambda \parallel^2 - 2(\kappa + \mu) + 4\kappa
\]

\[
= \frac{1}{\lambda} (\Delta \lambda - \nu^2 \lambda) - \frac{1}{\lambda^2} \parallel \text{grad} \lambda \parallel^2 + 2(\kappa - \mu).
\]

Thus the proof of the Theorem is completed. \(\square\)

**Remark 1.** Let us suppose that \( \mu = 2 \). By (4.3) and (4.4), we have \( X(\lambda) = \phi X(\lambda) = 0 \). Using this relation in (7.11), we obtain \( [X, \phi X] = 2\xi \). But this relation says that \( [X, \phi X](\lambda) = 0 = 2\xi(\lambda) = 2\nu\lambda \). This is a contradiction with \( \lambda \neq 0 \) and \( \nu = \text{const.} \neq 0 \). Because of this fact, we did not consider this case.
References

[1] Blair D.E., Contact manifolds in Riemannian geometry, Lectures Notes in Mathematics 509, 1976, Springer-Verlag, Berlin, 146p.
[2] Blair D.E., Two remarks on contact metric structures, Tohoku Math. J. 28 (1976), 373-379.
[3] Blair D.E., and Ogiue, K., Positively Curved Integral Submanifolds of a Contact Distribution, Illinois J.Math. 19, (1975), 628–631.
[4] Blair D.E., Koufogiorgos T., Papantoniou B.J., Contact metric manifolds satisfying a nullity condition, Israel J. Math. 91 (1995), 189–214.
[5] Boeckx, E., A class of locally $\phi$–symmetric contact metric spaces, Arch. Math. 72 (1999), 466-472
[6] Boeckx, E., Cho, T. J., $\eta$-parallel contact metric spaces, Different. Geom. Appl. 22 (2005), 275-285
[7] Boeckx, E., A full classification of contact metric $(\kappa, \mu)$-spaces, Illinois J. Math. 44 (2000), 212–219.
[8] Koufogiorgos T. and Tsichlias C., On the existence a new class of contact metric manifolds, Canad. Math. Bull. Vol 43 (2000), 440–447.
[9] Koufogiorgos T. and Tsichlias C., Generalized $(\kappa, \mu)$-contact metric manifolds with $\|\text{grad} \kappa\|=\text{constant}$, J.Geom. 78 (2003), 83-91.
[10] Koufogiorgos T. and Tsichlias C., Generalized $(\kappa, \mu)$-contact metric manifolds $\xi(\mu) = 0$, Tokyo J. Math Vol 31 (2008), 39-57.
[11] Koufogiorgos T., Markellos M., and Papantoni V., The harmonicity of the Reeb vector field on contact metric $3$-manifolds, Pacific J. Vol 234 (2008), 325-344.
[12] Perrone, D., Contact metric manifolds whose characteristic vector field is a harmonic vector field Differ.Geom. Appl. 20 (20049, 367-378.
[13] Tanno S., Ricci Curvatures of Contact Riemannian Manifolds, Tôhoku Math. J. 40 (1988), 441-448.
[14] Tanno S., Variational problems on contact manifolds, Trans. Amer. Math. Soc. 314 (1989), 349–379.
[15] Vergara-Diaz, E. and Wood C.M., Harmonic contact metric structures, Geom. Dedicata 123 (2006), 131–151.
[16] Yano K., Kon M., Structures on manifolds, World Scientific, 1984, 508p.

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