Algebraic dynamics in O*-algebras: a perturbative approach

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Abstract
In this paper the problem of recovering an algebraic dynamics in a perturbative approach is discussed. The mathematical environment in which the physical problem is considered is that of algebras of unbounded operators endowed with the quasi-uniform topology. After some remarks on the domain of the perturbation, conditions are given for the dynamics to exist as the limit of a net of regularized linear maps.
1 Introduction

In the so-called algebraic approach to quantum systems, one of the basic problems to solve consists in the rigorous definition of the algebraic dynamics, i.e. the time evolution of observables and/or states. For instance, in quantum statistical mechanics or in quantum field theory one tries to recover the dynamics by performing a certain limit of the strictly local dynamics. However, this can be successfully done only for few models and under quite strong topological assumptions (see, for instance, [1] and references therein).

In many physical models the use of local observables corresponds, roughly speaking, to the introduction of some cut-off (and to its successive removal) and this is in a sense a general and frequently used procedure, see [2, 3, 4] for conservative and [5, 6] for dissipative systems.

Introducing a cut-off means that in the description of some physical system, we know a regularized hamiltonian $H_L$, where $L$ is a certain parameter closely depending on the nature of the system under consideration. We assume that $H_L$ is a bounded self-adjoint operator in the Hilbert space $\mathcal{H}$ of the physical system.

There are several possible situations of some interest. Among these we will consider the following ones:

a) $H_L$ converges to an operator $H$

This is apparently the simplest situation. Of course we should specify the sense in which the convergence is understood. But for the moment, we want only focus on the possible problems that arise.

For each fixed $L$, we know the solution of the dynamical problem, i.e., we know the solution of the Heisenberg equation

$$\frac{id\alpha^t_L(A)}{dt} = [H_L, \alpha^t_L(A)].$$

This solution, $\alpha^t_L(A) = e^{iH_Lt}Ae^{-iH_Lt}$, would give the cut-offed dynamics of the system. Then it make sense to ask the question as to whether $\alpha^t_L(A)$ converges, possibly in the same sense as $H_L$ converges to $H$, to the solution $\alpha^t(A)$ of the Heisenberg equation

$$\frac{id\alpha^t(A)}{dt} = [H, \alpha^t(A)].$$

It is worth stressing that even though $H$ is a well defined self-adjoint operator, it is in general, unbounded. For this reason, while the right hand side of
Eqn. (1) is perfectly meaningful, the need of clarifying the right hand side of Eqn. (2) is always in order since $H$ is certainly not everywhere defined in $\mathcal{H}$.

Of course, the analysis of the convergence of $\alpha^t_L(A)$ to $\alpha^t(A)$ (in this case where $\alpha^t(A)$ is explicitly known) is significant only for deciding the accuracy of the approximation of $\alpha^t(A)$ with $\alpha^t_L(A)$.

b) $H_L$ does not converge

In this case, the situation becomes more difficult and a series of questions arise whose answer is highly non trivial.

As a first step, one could begin with considering the derivations

$$\delta_L(A) = i[H_L, A]$$

that give, at infinitesimal level, the dynamics of the system.

The first question, of course, is if these derivations converge, in a certain sense, to a derivation $\delta$ and which properties this derivation $\delta$ enjoys. For instance, is it a spatial derivation? (i.e, is there a symmetric operator $H$ that implements, at least in a generalized sense, the derivation? [7])

Further, can this derivation be integrated to some automorphisms group of the operator algebra we are dealing with? Or, conversely, since $\delta_L$ can be integrated without any problem, $\alpha^t_L(A) = e^{iH_Lt}Ae^{-iH_Lt}$, what can be said about the limit of $\alpha^t_L$? And how are these two problems related?

These questions are well-known not to admit an easy general solution.

In this paper we will be mainly concerned with situation a) above, while we will only make few comments on the more difficult situation b) which will be considered in more details in a future paper.

Our basic assumptions is that the hamiltonian $H$ of the system can be expressed in the form

$$H = H_0 + B;$$

in other words, our approach is tentatively perturbative: indeed, we suppose that we have full knowledge of the unperturbed system whose hamiltonian is $H_0$. In other words, given $H$ we can extract what we call a free hamiltonian $H_0$, which we know in all details, and consider $B := H - H_0$ as a perturbation of $H_0$ itself.

As we have already said, handling with unbounded operators poses a problem of domain for the algebra generated by the powers of the hamiltonian $H_0$. The natural choice is to take the set of $C^\infty$-vectors of $H_0$. Once a perturbation $B$ is introduced, it is natural to ask ourselves in which sense the corresponding domain for $H$ is related to that of $H_0$.
This is the main subject of Section 2, where we start with the assumption that \( \mathcal{D}^\infty(H) = \mathcal{D}^\infty(H_0) \) and derive some properties of the corresponding \textit{quasi-uniform} topologies that the two operators define.

Then we give, in a quite general way, conditions on two self-adjoint operators \( H_0 \) and \( H \) for \( \mathcal{D}^\infty(H) \) and \( \mathcal{D}^\infty(H_0) \) to coincide.

In Section 3, we come back to the problem of describing the dynamics of the perturbed system as limit of a cut-offed dynamics. In other words, we introduce a regularized hamiltonian \( H_L = Q^0_L H Q^0_L \) where the \( Q^0_L \)'s are certain spectral projection of the unperturbed hamiltonian \( H_0 \) and we look for conditions under which the unitary group generated by \( H_L \) converges to that generated by \( H \). A class of examples fitting our hypotheses is also given.

The main scope of the paper is to try and construct a mathematical environment where this kind of problems can be successful treated and also to develop techniques that could be adapted for the study of the more relevant case b) outlined above. It is worth stressing that this is a rather common situation in physics (think of mean-field models or systems with ultra-violet cutoff [2, 3]) and a perturbative approach should also be considered for the derivations that describe the system at infinitesimal level. A short discussion on this point is made in Section 4.

2 The mathematical framework

We begin this Section with summarizing some known facts on unbounded operator algebras and their topological properties. We refer to [4, 8, 9, 10] for full details.

Let \( \mathcal{D} \) be a dense domain in Hilbert space \( \mathcal{H} \); with \( \mathcal{L}^\dagger(\mathcal{D}) \) we denote the set of all weakly continuous endomorphisms of \( \mathcal{D} \). Then to each operator \( A \in \mathcal{L}^\dagger(\mathcal{D}) \) we can associate an operator \( A^\dagger \in \mathcal{L}^\dagger(\mathcal{D}) \) with \( A^\dagger = A^* \upharpoonright \mathcal{D} \) where \( A^* \) is the usual Hilbert adjoint of \( A \). Then \( \mathcal{L}^\dagger(\mathcal{D}) \), under the usual operations and the involution \( \dagger \), is a \( * \)-algebra of unbounded operators or, simply, an \( O^* \)-algebra.

Let now \( S \) be a selfadjoint operator in \( \mathcal{H} \) and

\[
\mathcal{D} := \mathcal{D}^\infty(S) = \bigcap_{n \geq 1} D(S^n).
\]

Then \( \mathcal{D} \) endowed with the topology \( t_S \) of \( \mathcal{D}^\infty(S) \) defined by the set of seminorms

\[
f \mapsto \| S^n f \|, \quad n = 0, 1, \ldots
\]
or, equivalently
\[ f \mapsto \|(1 + S^2)^{1/2} f\|, \quad n = 0, 1, \ldots \]
is a reflexive Fréchet space and the topology \( t_S \) is equivalent to the topology \( t_{\mathcal{L}^1(D)} \) defined on \( D \) by the set of seminorms
\[ f \mapsto \|Af\|, \quad A \in \mathcal{L}^\dagger(D). \]

In the \(*\)-algebra \( \mathcal{L}^\dagger(D) \) several topologies can be defined. For the purposes of this paper we will only need the quasi-uniform topology defined on \( \mathcal{L}^\dagger(D) \) in the following way. Put
\[ \|A\|^N,B = \sup_{\phi \in N} \|BA\phi\|, \quad B \in \mathcal{L}^\dagger(D), \quad N \text{ bounded in } D[t_{\mathcal{L}^1(D)}]. \]

Then, the quasi-uniform topology, \( \tau^D_\ast \) on \( \mathcal{L}^\dagger(D) \) is defined by the set of seminorms:
\[ A \in \mathcal{L}^\dagger(D) \mapsto \max\{\|A\|^N,B, \|A^\dagger\|^N,B\}. \]

In the case where \( D = D^\infty(S) \), the quasi-uniform topology on \( \mathcal{L}^\dagger(D) \) can be described in an easier way.

Indeed, let \( \mathcal{F} \) denote the class of all positive, bounded and continuous functions \( f(x) \) on \( \mathbb{R}_+ \), which are decreasing faster than any inverse power of \( x \), i.e., \( \sup_{x \in \mathbb{R}_+} x^k f(x) < \infty, \ k = 0, 1, \ldots \)

Then, if we put
\[ S_f = \{f(S)\phi; \phi \in D, \|\phi\| = 1\} \]
for \( f \in \mathcal{F} \), the family \( \{S_f\}_{f \in \mathcal{F}} \) is a basis for the bounded sets of \( D[t_S] \).

In practice this means that, for each \( t_S \)-bounded set \( N \) in \( D \), there exists an \( S_f \) such that \( N \subset S_f \).

This fact easily implies that the quasi-uniform topology, \( \tau^D_\ast \) on \( \mathcal{L}^\dagger(D) \) can be, equivalently, defined by the set of seminorms:
\[ \mathcal{L}^\dagger(D) \in A \mapsto \|A\|_s^{f,k} = \max\{\|S^k Af(S)\|, \|f(S)AS^k\|\} \]
\[ f \in \mathcal{F}, \ k \in \mathbb{N} \cup \{0\} \]
(3)

where the norm on the right hand side of (3) is the usual norm in \( B(H) \).

The \(*\)-algebra \( \mathcal{L}^\dagger(D)[\tau^D_\ast] \) is, in this case, a complete locally convex \(*\)-algebra, i.e. the involution and the right- and left-multiplications are continuous.

**Remark** – When estimating seminorms of type (3) we will often consider only the term \( \|f(S)AS^k\| \); this is exactly what is needed when \( A = A^\dagger \).

In the general case, any \( A \in \mathcal{L}^\dagger(D) \) is a linear combination of symmetric
elements and so, as far as only estimates are concerned, the arguments go usually through.

We can now consider more concrete situations. To begin with, we consider the simplest possible example in which a physical system is described by a Hamiltonian \( H_0 \) that mathematically is a self-adjoint operator; we assume \( H_0 \geq 1 \); then \( H_0 \) has a spectral decomposition

\[
H_0 = \int_1^\infty \lambda dE(\lambda).
\]

We put, for \( L \geq 1 \)

\[
Q_L^0 = \int_1^L dE(\lambda)
\]

and define the regularized Hamiltonian by:

\[
H_L = Q_L^0 H_0 Q_L^0.
\]

Then if \( \mathcal{D} = \mathcal{D}^\infty(H_0) \) it turns out that the operators \( Q_L^0 \) and \( H_L \) are bounded operators in \( \mathcal{B}(\mathcal{H}) \) which belong to \( \mathcal{L}^\dagger(\mathcal{D}) \) (the \( Q_L^0 \)'s are indeed projectors) and they commute with each other and with \( H_0 \).

This makes quite easy to prove the following convergence properties (in what follows the topology \( \tau_\ast^\mathcal{D} \) is that defined in Eqn. (3) with \( S \) replaced by \( H_0 \)):

(c1) \( H_L \to H_0 \) with respect to the topology \( \tau_\ast^\mathcal{D} \)

(c2) \( e^{itH_L} \to e^{itH_0} \) with respect to the topology \( \tau_\ast^\mathcal{D} \)

(c3) For each \( A \in \mathcal{L}^\dagger(\mathcal{D}) \), \( e^{itH_L} A e^{-itH_L} \to e^{itH_0} A e^{-itH_0} \)

All these statements can be derived from Lemma 2.2 below.

The next step consists in considering a Hamiltonian

\[
H = H_0 + B
\]

where \( B \) is regarded as a perturbation of the operator \( H_0 \). We suppose that the cut-off is determined by \( H_0 \), i.e., we assume that

\[
H_L = Q_L^0(H_0 + B)Q_L^0 = H_0Q_L^0 + Q_L^0BQ_L^0
\]

where \( Q_L^0 \) is defined as in Eqn. (4) by the spectral family \( E(\cdot) \) of \( H_0 \). The r.h.s. is well defined since \( Q_L^0 A Q_L^0 \) is bounded for any \( A \in \mathcal{L}^\dagger(\mathcal{D}) \).

Clearly (6) must be read as a formal expression unless the domains of the involved operators are specified. To be more definite, we make the following assumptions:
(a) $D = D^\infty(H_0)$
(b) $D(H_0) \subseteq D(B)$ and $H = H_0 + B$ is self-adjoint on $D(H_0)$
(c) $D^\infty(H_0) = D^\infty(H)$

Under these assumptions, we have:

**Lemma 2.1**
(1) The topologies $t_{H_0}$ and $t_H$ are equivalent on $D$;
(2) the topologies on $L^1(D)$ defined respectively by the set of seminorms

\[ L^1(D) \in A \mapsto \max\{\|H_0^k A f(H_0)\|, \|f(H_0) A H_0^k\|\} \quad f \in \mathcal{F}, \quad k \in \mathbb{N} \]

and

\[ L^1(D) \in A \mapsto \max\{\|H A f(H)\|, \|f(H) A H^k\|\} \quad f \in \mathcal{F}, \quad k \in \mathbb{N} \]

are equivalent

**Proof** – The statement (1) follows by taking into account that $H$ is continuous with respect to $t_{H_0}$ and $H_0$ is continuous with respect to $t_H$, according to the fact that the domain is reflexive.

The statement (2) follows from (1), since the family of $t_H$-bounded subsets of $D$ and the family of $t_{H_0}$-bounded subsets coincide.

By the previous Lemma, the topology $\tau_*^D$, can be described, following the convenience, via the seminorms in $H$ or by those in $H_0$. Now, we can prove the following

**Lemma 2.2** For each $X \in L^1(D)$, $X = \tau_*^D - \lim_{L \to \infty} Q_L^0 X Q_L^0$

**Proof** – First, notice that, for $\ell \in \mathbb{N}^+$, we have

\[ \|H_0^{-\ell}(I - Q_L^0)\phi\|^2 = \int_L^\infty \frac{1}{\lambda^{2\ell}} d(E(\lambda)\phi, \phi) \leq \frac{1}{\lambda^{2\ell}} \|\phi\|^2, \quad \forall \phi \in D \]

and so

\[ \|H_0^{-\ell}(I - Q_L^0)\| \to 0 \text{ as } L \to \infty. \]

Let now $f \in \mathcal{F}$ and $k \in \mathbb{N}$; then we have:

\[ \|f(H_0)(B - Q_L^0 B Q_L^0)H_0^k\| \]
\[ \leq \|f(H_0) B H_0^k (I - Q_L^0)\| + \|f(H_0) (1 - Q_L^0) B H_0^k Q_L^0\| \]
\[ = \sup_{\|\phi\| = \|\psi\| = 1} |< H_0^{-\ell}(1 - Q_L^0)\phi, H_0^{k+\ell}B^+ f(H_0)\psi>| | \]
\[ + \sup_{\|\phi\| = \|\psi\| = 1} |< f(H_0) H_0^k B H_0^k Q_L^0\phi, H_0^{-\ell}(1 - Q_L^0)\psi>| | \]
\[ \leq \|H_0^{-\ell}(1 - Q_L^0)\| \|H_0^{k+\ell}B^+ f(H_0)\| + \|H_0^k f(H_0) B H_0^k\| \|H_0^{-\ell}(1 - Q_L^0)\| \to 0 \]
Incidentally, this lemma gives a proof of (c1) and (c2) above. The proof of (c3) requires the use of a triangular inequality, of (c2) and of the commutation rule \( [H_0, H_L] = 0 \).

Taking into account the separate continuity of the multiplication and the previous lemma, we have:

**Corollary 2.3** \( \delta_L(A) := i[A, H_L] \) converges to \( \delta(A) := i[A, H] \) with respect to the topology \( \tau^{D}_* \).

Going back to our assumptions on the domains, it is apparent that conditions (b) and (c) given above are quite strong. It is natural to ask the question under which conditions on \( B \) they are indeed satisfied.

### 2.1 The domain

Our starting point is an operator

\[ H = H_0 + B \]

under the assumption that the *perturbation* \( B \) is a symmetric operator and \( D(B) \supseteq D(H_0) \). In general \( H \) may fail to be self-adjoint, unless \( B \) is \( H_0 \)-bounded in the sense that there exist two real numbers \( a, b \) such that

\[ \| B\phi \| \leq a\| H_0\phi \| + b\| \phi \|, \quad \forall \phi \in D(H_0). \tag{7} \]

If the inf of the numbers \( a \) for which (7) holds (the so called *relative bound*) is smaller than 1, then the Kato-Rellich theorem \[11\] states that \( H \) is self-adjoint and essentially self-adjoint on any core of \( H_0 \). This is clearly always true if \( B \) is bounded: in this case the relative bound is 0. In conclusion, the Kato-Rellich theorem gives a sufficient condition for (b) to be satisfied.

Let us now focus our attention on condition (c). We first discuss some examples.

**Example 1** –To begin with, we stress the fact that the conditions of the Kato-Rellich theorem are not sufficient to imply that \( D^\infty(H) = D^\infty(H_0) \). This can be seen explicitly with a simple example. Indeed, let us consider the case where \( B = P_f \) with \( f \in \mathcal{H} \setminus D(H_0) \) and \( P_f \) the projection onto the one-dimensional subspace generated by \( f \). It is quite simple to prove that, in this case:

\[ D((H_0 + P_f)^2) \cap D(H_0^2) = D(H_0) \cap \{ f \}^\perp. \]
This equality implies that neither $D^\infty(H_0 + P_f)$ is a subset of $D^\infty(H_0)$ nor the contrary. So, in this example, $D^\infty(H)$ and $D^\infty(H_0)$ do not compare.

**Example 2**—Let $p$ and $q$ be the operators in $L^2(\mathbb{R})$ defined by:

$$
(pf)(x) = if'(x), \quad f \in W^{1,2}(\mathbb{R})
$$

$$
(qf)(x) = xf(x), \quad f \in \mathcal{F}W^{1,2}(\mathbb{R})
$$

where $\mathcal{F}$ denotes the Fourier transform. Let us consider

$$
H_0 = p^2 + q^2
$$

then, as is known, $H_0$ is an essentially self-adjoint operator on $\mathcal{S}(\mathbb{R})$ and this domain is exactly $D^\infty(H_0)$.

Let us now take as $B$ the operator $-q^2$, then

$$
D^\infty(H) = \{f \in C^\infty(\mathbb{R}) : f^{(k)} \in L^2(\mathbb{R}), \forall k \in \mathbb{N}\}.
$$

Thus, in this case $D^\infty(H) \supset D^\infty(H_0)$.

In order to construct an example where the opposite inclusion hold, we start by taking $H_0 = p^2$ and $B = q^2$. In this case,

$$
D^\infty(H) = \mathcal{S}(\mathbb{R}) \subset \{f \in C^\infty(\mathbb{R}) : f^{(k)} \in L^2(\mathbb{R}), \forall k \in \mathbb{N}\} = D^\infty(H_0).
$$

These examples show that all situations are possible, when comparing $D^\infty(H)$ and $D^\infty(H_0)$.

For shortness, we will call $B$ a *KR-perturbation* if it satisfies the assumption of the Kato-Rellich theorem. Before going forth, we give the following

**Proposition 2.4** Let $A$ and $B$ two selfadjoint operators in Hilbert space $\mathcal{H}$. Then

$$
D^\infty(A) = D^\infty(B)
$$

if, and only if, the following two conditions hold:

(i) for each $k \in \mathbb{N}$ there exists $\ell \in \mathbb{N}$ such that $D(B^\ell) \subseteq D(A^k)$;

(ii) for each $h \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $D(A^m) \subseteq D(B^h)$. 

Proof – We put $D = D^\infty(A) = D^\infty(B)$. Because of Lemma 2.1, the topologies $t_A$ and $t_B$ are equivalent. Without loss of generality we assume that $A \geq 0$, $B \geq 0$; this makes the usual families of seminorms defining the two topologies directed. This implies that for each $k \in \mathbb{N}$ there exist $\ell \in \mathbb{N}$ and $C_k > 0$:

$$\|A^k \phi\| \leq C_k \|B^\ell \phi\|, \quad \phi \in D.$$ 

But $D$ is a core for any power of $B$, therefore for each $f \in D(B^\ell)$ there exists a sequence $(f_n)$ of elements of $D$ such that $f_n \to f$ and $(B^\ell f_n)$ is convergent. Then we get

$$\|A^k (f_n - f_m)\| \leq C_k \|B^\ell (f_n - f_m)\| \to 0$$

and therefore $f \in D(A^k)$.

The proof of (ii) is similar.

Let us now assume that (i) and (ii) hold. For any $k \in \mathbb{N}$ we put

$$\ell_k = \min \{\ell \in \mathbb{N} : D(B^\ell) \subset D(A^k)\}.$$ 

Then we have:

$$D^\infty(B) \subseteq \bigcap_{k=1}^\infty D(B^{\ell_k}) \subset \bigcap_{k=1}^\infty D(A^k) = D^\infty(A).$$

In similar way the converse inclusion can be proven. □

Example – The previous proposition easily implies the following well-known fact:

$$D^\infty(A^k) = D^\infty(A), \quad \forall k \in \mathbb{N}.$$ 

since, (i) and (ii) hold, as is readily seen.

Proposition 2.5 Let $A \geq 1$ and $B \geq 1$. Then if

$$D^\infty(A) = D^\infty(B)$$

the following two conditions hold:

(i) for each $k \in \mathbb{N}$ there exist $\ell \in \mathbb{N}$ such that $A^k B^{-\ell}$ is bounded;

(ii) for each $h \in \mathbb{N}$ there exist $m \in \mathbb{N}$ such that $B^h A^{-m}$ is bounded.

Conversely, if $D^\infty(A) \cap D^\infty(B)$ contains a common core $D_0$ for all powers of $A$ and $B$ and both (i) and (ii) hold, then

$$D^\infty(A) = D^\infty(B)$$
Proposition 2.6 Let $B$ be a KR-perturbation and assume $B : \mathcal{D}^\infty(H_0) \to \mathcal{D}^\infty(H_0)$. Then $\mathcal{D}^\infty(H_0) \subseteq \mathcal{D}^\infty(H)$. Moreover, if the families of seminorms are directed,

$$\forall k, s \in \mathbb{N} \ \exists \ell \in \mathbb{N}, C_k > 0 : \|H_0^s H_k^\ell \phi\| \leq C_k \|H_0^\ell \phi\|, \quad \forall \phi \in \mathcal{D}^\infty(H_0).$$

Proof – In this case, $\mathcal{D}^\infty(H_0)$ is left invariant by $H$; but $\mathcal{D}^\infty(H)$ is the largest domain with this property. Therefore $\mathcal{D}^\infty(H_0) \subseteq \mathcal{D}^\infty(H)$. The given inequality follows easily from the continuity of $H_k$ in $\mathcal{D}^\infty(H_0)$. □

Remark – The above inequality also says that $t_{H_0}$ is, in general, finer than $t_H$.

In order to get the equality of the two domains some stronger condition on $B$ must be added. We have, indeed:

Proposition 2.7 Let $B$ be a perturbation of $H_0$ such that $H := H_0 + B$ is selfadjoint on $\mathcal{D}(H) = \mathcal{D}(H_0)$. In order that

$$\mathcal{D}^\infty(H) = \mathcal{D}^\infty(H_0)$$

it is necessary and sufficient that the following conditions hold:
(i) \( B : \mathcal{D}^\infty(H_0) \rightarrow \mathcal{D}^\infty(H_0) \);

(ii) \( H \) is essentially self-adjoint in \( \mathcal{D}^\infty(H_0) \);

(iii) the topologies \( t_{H_0} \) and \( t_H \) are equivalent on \( \mathcal{D}^\infty(H_0) \)

**Proof** – The necessity of (i) is obvious. As for (ii), it is well-known that \( \mathcal{D}^\infty(H) \) is a core for \( H \) (and for all its powers). The necessity of (iii) follows from (1) in Lemma 2.1.

We now prove the sufficiency.

First, by Proposition 2.6 and (i) it follows that \( \mathcal{D}^\infty(H_0) \subseteq \mathcal{D}^\infty(H) \) and since \( H \) is essentially self-adjoint in \( \mathcal{D}^\infty(H_0) \),

\[ \mathcal{D}^\infty(\Pi|_{\mathcal{D}^\infty(H_0)}) = \mathcal{D}^\infty(H). \]

But as is well known, the domain on the left hand side is the completion of \( \mathcal{D}^\infty(H_0) \) in the topology \( t_{H_0} \). The equivalence of \( t_{H_0} \) and \( t_H \) implies that \( \mathcal{D}^\infty(H_0) \) is complete under \( t_{H_0} \) and so the statement is proved. \( \square \)

**Remark** – If \( B \) is bounded, then \( H = H_0 + B \) is automatically essentially self-adjoint in \( \mathcal{D}^\infty(H_0) \)

**Example** – Let \( H_0 = p^2 + q^2 \); then \( \mathcal{D}^\infty(H_0) = \mathcal{S}(\mathbb{R}) \). Let \( B = \alpha q \) with \( \alpha \in \mathbb{R} \).

Then it is easily seen that \( H = p^2 + q^2 + \alpha q \) leaves \( \mathcal{S}(\mathbb{R}) \) invariant.

Since \( H = p^2 + (q - \beta)^2 + \beta^2 \) with \( \beta = \alpha/2 \), it is clear that \( \mathcal{S}(\mathbb{R}) \) is a domain of essential self-adjointness for \( H \).

The equivalence of the topologies \( t_{H_0} \) and \( t_H \) can be proven with easy estimates of the respective seminorms. Thus Proposition 2.7 leads us to conclude that \( \mathcal{D}^\infty(H_0) = \mathcal{D}^\infty(H) \).

As a consequence of Proposition 2.7, we consider now the special case of a perturbation weakly commuting with \( H_0 \).

Let \( \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H}) \) denote the space of all closable operators \( A \) in \( \mathcal{H} \) such that \( D(A) = \mathcal{D}, D(A^*) \subset \mathcal{D} \). As for \( \mathcal{L}^\dagger(\mathcal{D}) \), we put \( A^\dagger = A^*|_{\mathcal{D}} \).

Now, if \( A \) is a \( \dagger \)-invariant subset of \( \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H}) \), the weak unbounded commutant \( \mathcal{A}_\sigma' \) of \( A \) is defined as

\[ \mathcal{A}_\sigma' = \{ Y \in \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H}) :< X f, Y^\dagger g > =< Y f, X^\dagger g >, \forall f, g \in D; \forall X \in A \}. \]

If \( T \) is a self-adjoint operator in \( \mathcal{H} \), we can consider the \( \mathcal{O}^* \)-algebra \( \mathcal{P}(T) \) generated by \( T \) on \( \mathcal{D}^\infty(T) \). It is well-known [10] that \( \mathcal{P}(T)'_\sigma = \{ T \}'_\sigma \). Furthermore, any \( Y \in \{ T \}'_\sigma \) leaves \( \mathcal{D}^\infty(T) \) invariant. We now apply these facts to our situation:
Corollary 2.8 Let $B$ be a perturbation of $H_0$. Assume that $B$ satisfies the conditions:

(i) $<H_0 f, Bg> = <Bf, H_0 g>$, $\forall f, g \in \mathcal{D}^{\infty}(H_0)$

(ii) $H$ is essentially self-adjoint in $\mathcal{D}^{\infty}(H_0)$

(iii) $\|H_0 f\| \leq \|H f\|$, $\forall f \in \mathcal{D}^{\infty}(H_0)$

then $\mathcal{D}^{\infty}(H_0) = \mathcal{D}^{\infty}(H)$.

Proof – Condition (i) implies that $B$ leaves $\mathcal{D}^{\infty}(H_0)$ invariant; therefore $H$ is $t_{H_0}$-continuous (together with all its powers). So it remains to check that $t_H$ is finer than $t_{H_0}$ in order to apply Proposition 2.7.

We will prove by induction that

$$\|H^n_0 f\| \leq \|H^n f\|, \quad \forall f \in \mathcal{D}^{\infty}(H_0).$$

The case $n = 1$ is exactly condition (iii). Now we assume the statement true for $n - 1$. Then we get:

$$\|H^n_0 f\| = \|H_0(H^{n-1}_0 f)\|$$

$$\leq \|H(H^{n-1}_0 f)\|$$

$$= \|H^{n-1}_0 H f\| \leq \|H^n f\|, \quad \forall f \in \mathcal{D}^{\infty}(H_0).$$

since, by (i), $H_0$ and $H$ commute (algebraically) on $\mathcal{D}^{\infty}(H_0)$.

3 Dynamical aspects

We now come back to the dynamical problem posed at beginning of the paper concerning the perturbative situation and again we will consider the case where $H$ exists. So far, we were able to prove the convergence of the dynamics only at infinitesimal level (Corollary 2.3). The problem of the convergence of $\alpha_L^t(A)$ to $\alpha^t(A)$ is not completely solved neither in the simple case we are dealing with. Of course, given $H$ and its spectral projections $Q_L$ as seen in the previous Section, we can always prove, setting $H_L = Q_LHQ_L$, that $e^{iH_LT}Ae^{-iH_LT}$ converges to $e^{iHt}Ae^{-iHt}$ for any $A$ in $\mathcal{L}^\dagger(\mathcal{D})$. What makes here the difference (and this is the spirit of the whole paper), is that we are defining the cut-offed hamiltonian $H_L = Q^0_LHQ^0_L$ via the spectral projections of the unperturbed hamiltonian $H_0$. This is of practical interest since only in very few instances (finite discrete systems, harmonic
oscillators, hydrogen atoms,...) the spectral projections of $H$ are known. On the other hand, $H_0$ can be chosen with a sufficient freedom to guarantee the knowledge of the $Q_0^L$.

With this in mind, we consider the problem of finding conditions under which $e^{iH_L t}$ converges, with respect to the topology $\tau_s^D$, to $e^{iH t}$.

To this aim, we define the operator function

$$g_L(t) = e^{iH_L t} - e^{iH t} = i \int_0^t e^{iH_L (t-t')} (H_L - H) e^{iH t'} dt',$$

the latter equality being got by solving the equation

$$\dot{g}_L(t) = iH_L g_L(t) + i(H_L - H)e^{iH t}$$

which comes directly from the definition of $g_L(t)$. Easy estimates allow to state the following

**Lemma 3.1** For each $k \in \mathbb{N}$ there exists $s \in \mathbb{N}$ such that

$$\lim_{L \to \infty} \|H_0^{-s} (H_L - H) H_k^0\| = 0.$$ 

then we have

**Proposition 3.2** If there exists $T > 0$ such that, for each $f \in F$, $s \in \mathbb{N} \cup \{0\}$ there exists $M = M(T, f, s)$ such that

$$\int_0^t \|f(H_0) e^{iH_L (t-t')} H_0^s\| dt' < M, \quad t \in [0, T]$$

then

$$\tau_s^D - \lim_{L \to \infty} g_L(t) = 0.$$ 

**Proof** – We have indeed:

$$\|g_L(t)\|_{f^k} \leq \int_0^t \|f(H) e^{iH_L(t-t')} (H_L - H)e^{-iH t'} H_k^0\| dt'$$

$$= \int_0^t \|f(H) e^{iH_L(t-t')} (H_L - H) H_k\| dt'$$

$$\leq C \int_0^t \|f_1(H_0) e^{iH_L(t-t')} (H_L - H) H_0^{k_1}\| dt'$$

$$\leq C \int_0^t \|f_1(H_0) e^{iH_L(t-t')} H_0^{s}\| dt' \cdot \|H_0^{-s}(H_L - H)H_0^{k_1}\|$$
for suitable $C > 0$, $f_1 \in \mathcal{F}$ and $k_1 \in \mathbb{N}$ and with $s$ chosen, correspondingly to $k_1$ so that Lemma 3.1 can be used. □

This proposition implies that the Schrödinger dynamics can be defined. The analysis of the Heisenberg dynamics is more complicated and will not be considered here.

The assumptions of Proposition 3.1 are indeed quite strong. They are, of course, verified if the perturbation $B$ commutes with $H_0$. But this is, clearly, a trivial situation. We will now discuss a non-trivial example where they are satisfied.

**Example** – Let $H_0 = a^\dagger a$ and $B = a^n$, $n$ being an integer larger than 1. The conditions on the domains of the operators discussed in Section 2 are satisfied, as it is more easily seen working in the configuration space, so that our procedure can be applied. Here $Q_L^0 = \Pi_0^0 + \Pi_1^0 + \Pi_2^0 + \ldots + \Pi_L^0$, where $\Pi_i^0$ is the projection operator of $H_0$, $H_0 = \sum_{i=0}^{\infty} i\Pi_i^0$. Using the algebraic rules discussed in [12], and, in particular the commutation rules $Q_L^0 a = aQ_L^0 + 1$ and $\Pi_i^0 a = a\Pi_i^0 + 1$, we find that $H_i = Q_L^0 HQ_L^0 = HQ_L^0$.

It is a straightforward computation now to check that for any $f \in \mathcal{F}$ and for any natural $s$, $\|f(H)e^{iH_L \tau}H^s\| = \|f(H)(H + (e^{iH\tau} - 1)a^nP_{L,n})^s\|$, where we have defined the following orthogonal projection operator

$$P_{L,n}^0 = \Pi_{L+1}^0 + \Pi_{L+2}^0 + \ldots + \Pi_{L+n}^0 = Q_{L+n}^0 - Q_L^0.$$  

These seminorm can be estimated for each value of $s$ and it is not difficult to check that they are bounded by a constant which depends on $f$, $s$ and $n$ (obviously) but not on $L$ and $\tau$. Therefore the main hypothesis of Proposition 3.1 is verified and so the Schrödinger dynamics can be defined. We give the estimate of the above seminorm here only in the easiest non trivial case, $s = 1$. The extension to $s > 1$ only increases the length of the computation but does not affect the result.

$$\|f(H)(H + (e^{iH\tau} - 1)a^nP_{L,n})\| \leq \|f(H)H\| + \|f(H)(e^{iH\tau} - 1)a^n\||P_{L,n}^0\| \leq \|f(H)H\| + 2\|f(H)a^n\|,$$

which is bounded and independent of both $L$ and $\tau$.

The same strategy can also be applied to the more general situation when $B$ is any given polynomial in $a$ and $a^\dagger$.

In order to find more cases in which Proposition 3.2 can be applied, we begin with the following
Lemma 3.3  For each $f \in \mathcal{F}$ and for each $k, \ell \in \mathbb{N}$ we have:

$$\lim_{L \to \infty} \|f(H)H_L^{\ell} - H^k\| = 0$$

Proof – We proceed by induction on $\ell$.
For $\ell = 1$ the statement follows immediately by the equivalence of the topologies and from Lemma 2.2.

Now,

$$\|f(H)(H_L^{\ell+1} - H^{\ell+1})H^k\| \leq \|f(H)H_L(H_L^{\ell} - H^{\ell})H^k\| + \|f(H)(H_L - H)H^{\ell+1}\|$$

and the second term on the r.h.s. goes to 0 because we have just proved the induction for $\ell = 1$.

The first term of the rhs can easily be estimated, making once more use of the equivalence of the topologies, by a term of the kind

$$C\|f_1(H_0)(H_L^{\ell} - H^{\ell})H^k\|$$

which goes to zero again because of the induction. □

Proposition 3.4  If there exists $m \in \mathbb{N}$ such that $[H_L, H]_m = 0$ then

$$\int_0^t \|f(H_0)e^{iH_L(t-t')}H_0^s\| dt' < \infty, \quad t \in \mathbb{R}^+$$

for each $f \in \mathcal{F}$ and for each $s \in \mathbb{N} \cup \{0\}$

Proof – By the assumption, we have:

$$e^{iH_L(t-t')}He^{-iH_L(t-t')} = H + i(t - t')[H_L, H] + \cdots \frac{i^m}{m!}[H_L, H]^m. \quad (8)$$

Now, using the equivalence between the topologies produced by $H_0$ and $H$, it is easy to see that:

$$\|f(H_0)e^{iH_L(t-t')}H_0^s\| \leq C\|f_1(H)(e^{iH_L(t-t')}He^{-iH_L(t-t')}\)^s\|.$$ 

Inserting (8) on the r.h.s. and making use of Lemma 3.3 we finally get the estimate:

$$\|f(H_0)e^{iH_L(t-t')}H_0^s\| \leq C\|f_1(H)H^s\|,$$

and this easily imply the statement. □
Clearly, even if the conditions given above for the $\tau^D$-convergence of $e^{iH_L t}$ to $e^{iHt}$ occur, the convergence of $\alpha^t_L(A)$ to $\alpha^t(A)$ is not guaranteed. For this reason we conclude this Section by outlining a different possible approach.

Assume that, for each $L \in \mathbb{R}$ there exists a one-parameter family $\beta^t_L(A)$ of linear maps of $\mathcal{L}^\dagger(D)$ (not necessarily an automorphisms group) such that, for each $f \in \mathcal{F}$ and $k \in \mathbb{N}$,

$$
\|f(H) (\beta^t_L(A) - \alpha^t_L(A)) H^k\| \to 0 \quad \text{as } L \to \infty
$$

Clearly the convergence of $\beta^t_L(A)$ to $\alpha^t(A)$ would directly lead to the solution of the dynamical problem. We want to stress that $\beta^t_L$ could be rather unusual and, therefore, it should be only considered as a technical tool.

In general, however, the possibility of finding a good definition for the $\beta^t_L$’s that allows (9) to hold, is quite difficult and the lesson of the previous discussion on the convergence of $e^{iH_L t}$ is that strong assumptions must be imposed in order to get it.

A weaker condition on the $\beta^t_L$’s, whose content of information is, nevertheless, non-empty, would consists in requiring, instead of (9), that

(a) $\beta^t_L(A)$ converges to $\alpha^t(A)$, for each $A \in \mathcal{L}^\dagger(D)$ together with all time derivatives. This means that an Heisenberg dynamics $e^{iHt}(\cdot)e^{-iHt}$ can be recovered.

(b) As for the Schrödinger dynamics, that is for the dynamics in the space of vectors, we ask $\beta^t_L$ of being in same way (to be specified further) generated by a family of bounded operators which, when applied to any $\Psi \in \mathcal{D}$, is $t_H$-convergent together with all time derivatives.

This happens, for instance, in the case where $H$ exists, if we define

$$
\beta^t_L(A) = V^t_L AV^{-t}_L.
$$

where $V^t_L := Q^0_L e^{iHt} Q^0_L$ and the $Q^0_L$’s are the projection of $H_0$. Under these assumptions, $V_L$ is a well-defined bounded operator of $\mathcal{L}^\dagger(D)$, and the following Proposition holds:

**Proposition 3.5** In the above conditions, the following statements hold:

(i) $t_H - \lim_{L \to \infty} V^t_L \psi = \psi(t) := e^{iHt} \psi$, $\forall \psi \in \mathcal{D}$

(ii) $\tau^D - \lim_{L \to \infty} V^t_L = e^{iHt}$

(iii) $\tau^D - \lim_{L \to \infty} \beta^t_L(A) = \alpha^t(A) := e^{iHt} A e^{-iHt}$, $\forall A \in \mathcal{L}^\dagger(D)$
and, more generally:

\( (i') \quad t_H - \lim_{L \to \infty} \frac{d^n}{dt^n} V^t_L \psi = \frac{d^n}{dt^n} \psi(t), \quad \forall \psi \in D, \quad n \in \mathbb{N} \cup \{0\} \)

\( (iii') \quad \tau^D - \lim_{L \to \infty} \frac{d^n}{dt^n} \beta^t_L(A) = \frac{d^n}{dt^n} \alpha^t(A), \quad \forall A \in \mathcal{L}^\dagger(D), \quad n \in \mathbb{N} \cup \{0\} \)

The proof of this Proposition follows from the equivalence between the topologies generated by \( H \) and \( H_0 \), proved in Lemma 2.1.

This approach, which is only one of the possible strategies when \( H \) exists, could be of a certain interest for situations when the dynamics can only be obtained via a net of operators \( H_L = H_0 + B_L \), \( H_0 \) being the free Hamiltonian and \( B_L \) being a regularized perturbation. In this case the approach to the thermodynamical limit could involve the family of bounded operators

\[
V^t_{LM} := Q^0_L e^{iH_M t} Q^0_L, \quad \text{and one can try to extend the above results. A further analysis on this subject is currently work in progress.}
\]

4 Outcome and possible developments

In this paper we have analyzed a possible approach to define an algebraic dynamics when a free hamiltonian \( H_0 \) is perturbed by an operator \( B \) which essentially leaves the domain of all the powers of \( H_0 \) invariant.

What is still missing, how we discussed in the Introduction, is the analysis of the situation where the definition of the dynamics is not straightforward since it should follow from a net of operators \( \{H_L\} \) whose limit does not exist in any physical topology \([2, 3]\). In this case a possible approach can be made in terms of derivations, for instance, in the way explained below.

Let us suppose that to a free spatial derivation \( \delta_0(\cdot) = i[H_0, \cdot] \) a perturbation term \( \delta_P \) is added, so that

\[
\delta(A) = \delta_0(A) + \delta_P(A), \quad A \in \mathcal{L}^\dagger(D).
\]

In this case we define \( \eta_L(A) = Q^0_L \delta(A) Q^0_L \), with \( A \in \mathcal{L}^\dagger(D) \) and \( Q^0_L \) as in the previous Sections. It is easy to see that \( \eta_L \) is not in general a derivation because the Leibniz rule may fail. Let \( \Delta_L \) be a map on \( \mathcal{L}^\dagger(D) \) which has the property that \( \delta_L = \eta_L + \Delta_L \) satisfies the Leibniz rule together with the other properties of a derivation. Of course, this map is not unique since, for instance, we can always add a commutator \( i[H', \cdot] \) to \( \Delta_L \), with any self-adjoint operator \( H' \), without affecting the properties of a derivation (we should only care about domain problems in choosing \( H'! \)). From a physical point of view it is reasonable to expect that \( \Delta_L \) can be chosen in such a
way that $\|\Delta_L(A)\|_{f,k} \to 0$ with $L$ since we would like to recover the original derivation $\delta$ after removing of the cutoff and we know from Corollary 2.3 that $\|\eta_L(A) - \delta(A)\|_{f,k} \to 0$. If also $\delta_P$ is spatial, then is not difficult to give an explicit expression for $\Delta_L$ and to check that the requirements above are satisfied. In this case in fact

$$\Delta_L(A) = \{Q^0_L H_0, [Q^0_L, A]\} + Q^0_L B [Q^0_L, A] + [Q^0_L, A] B Q^0_L,$$

where $\{X,Y\} = XY + YX$.

Once we have introduced $\delta_L$ the next step is to find conditions for this map to be spatial. The related operator $H_L$, which we expect to be of the form $Q^0_L(H_0 + B)Q^0_L$, for a suitable self-adjoint operator $B$, can be used to perform the same analysis as that discussed in the previous Section.

Of course this is by no means the only possibility of approaching this problem, but is the one which is closer to our previous analysis, and in this perspective, is particularly relevant for us. We hope to discuss this problem in full details in a future paper.

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