Finite Chains with Quantum Affine Symmetries

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Abstract

We consider an extension of the ($t$–$U$) Hubbard model taking into account new interactions between the numbers of up and down electrons. We confine ourselves to a one-dimensional open chain with $L$ sites ($4^L$ states) and derive the effective Hamiltonian in the strong repulsion (large $U$) regime. This Hamiltonian acts on $3^L$ states. We show that the spectrum of the latter Hamiltonian (not the degeneracies) coincides with the spectrum of the anisotropic Heisenberg chain (XXZ model) in the presence of a $Z$ field ($2^L$ states). The wave functions of the $3^L$-state system are obtained explicitly from those of the $2^L$-state system, and the degeneracies can be understood in terms of irreducible representations of $U_q(\hat{sl}(2))$. 

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1 Introduction

The one-dimensional Hubbard model, which depends on two parameters \((U\) and \(t)\), has received a lot of attention due to its possible application (in its 2-d version) to describe high-\(T_c\) oxide materials [1]. The large \(U\) limit of this system was discussed in Refs. [2].

In this paper we consider the following Hamiltonian:

\[
H = H_0 + H_1 ,
\]

where

\[
H_0 = \sum_{i=1}^{L-1} \left\{ - \sum_{\sigma=\uparrow,\downarrow} \left( c_i^{\dagger,\sigma} c_{i+1,\sigma} - c_{i,\sigma} c_{i+1,\sigma}^{\dagger} \right) + \frac{U}{2t} (n_{i,\uparrow} n_{i,\downarrow} + n_{i+1,\uparrow} n_{i+1,\downarrow}) \right. \\
+ \frac{V}{t} (n_{i,\uparrow} - n_{i,\downarrow})^2 (n_{i+1,\uparrow} - n_{i+1,\downarrow})^2 \\
+ \frac{W}{2t} \left[ (n_{i,\uparrow} - n_{i,\downarrow})^2 + (n_{i+1,\uparrow} - n_{i+1,\downarrow})^2 \right] \\
+ \frac{A}{2t} \left[ (n_{i,\uparrow} - n_{i,\downarrow})^2 - (n_{i+1,\uparrow} - n_{i+1,\downarrow})^2 \right] + B \right\} 
\]

and

\[
H_1 = \frac{G}{2t} \sum_{i=1}^{L} (n_{i,\uparrow} - n_{i,\downarrow}).
\]

Here \(c_{i,\sigma}^{\dagger}, c_{i,\sigma}\) and \(n_{i,\sigma} = c_{i,\sigma}^{\dagger} c_{i,\sigma}\) (\(\sigma = \uparrow, \downarrow\)) are creation, annihilation and occupation number fermionic operators. In the Hubbard model one has the hopping term (with coefficient \(-1\)), the repulsion term (with coefficient \(\frac{U}{2t}\)) and the external field (1.2b) (coefficient \(\frac{G}{2t}\)). We have generalized this Hamiltonian by adding a quartic interaction term (coefficient \(\frac{V}{t}\)) and an external field type term (coefficient \(\frac{W}{2t}\)). The term proportional to \(\frac{A}{2t}\) corresponds to a boundary term; it does not change the thermodynamical properties of the system but, as we shall see, plays an important role in its symmetry properties. For similar reasons we have added the constant \(\frac{B}{t}\), which will be shown
to be useful for the identification of representations of some Hecke algebras. We have no phenomenological reasons to introduce the supplementary terms in the Hamiltonian but, as will be seen, the phase structure of the physical system thus obtained is much richer.

If, as in the $t$–$J$ model [1], we take the large $U/t$ limit keeping $G_{2t} = g, V_{2t} = v, W_{2t} = w, A_{2t} = a$ and $B_{2t} = b$ fixed, as shown in Appendix A, we obtain instead of a Hamiltonian acting on $4^L$ states, an effective Hamiltonian acting on $3^L$ states since the remaining states get an infinite energy and drop out of the spectrum. This effective Hamiltonian reads

$$H' = H'_0 + H'_1,$$  \hspace{1cm} (1.3)

where

$$H'_0 = \sum_{i=1}^{L-1} U_i$$  \hspace{1cm} (1.4)

and

$$U_i = - \left( \varrho^- \varrho^+_{i+1} + \varrho^+ \varrho^-_{i+1} + \tau^+ \tau^-_{i+1} + \tau^- \tau^+_{i+1} \right)$$

$$+ \ v \varepsilon^0 \varepsilon^0_{i+1} + w (\varepsilon^0_i + \varepsilon^0_{i+1})$$

$$+ \ a (\varepsilon^0_i - \varepsilon^0_{i+1}) + b$$  \hspace{1cm} (1.5)

and

$$H'_1 = - g \sum_{i=1}^{L} \varepsilon^z_i.$$  \hspace{1cm} (1.6)

We use here the notation of Ref. [3]:

$$\varrho^+ = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \varrho^- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \tau^+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\tau^- = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \varepsilon^0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \varepsilon^z = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$
\[ \varepsilon^+ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \varepsilon^- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \] (1.7)

Obviously:

\[ [H'_0, S^z] = [H'_0, S^+] = [H'_0, S^-] = [H'_0, S^0] = 0, \] (1.8)

where

\[ S^z = \frac{1}{2} \sum_{i=1}^{L} \varepsilon^z_i; \quad S^\pm = \sum_{i=1}^{L} \varepsilon^\pm_i; \quad S^0 = \sum_{i=1}^{L} \varepsilon^0_i. \] (1.9)

Thus \( H'_0 \) is \( SU(2) \otimes U(1) \) invariant (generators \( S^\pm, S^z \) and \( S^0 \)). As will be seen later, the symmetry algebra is much larger, namely \( H'_0 \) is invariant under \( U_q(sl(2)) \otimes U(1) \) for any value of \( q \! \). Special cases of the quantum chain (1.4) were already obtained in a mathematical context. In Ref. [4] it was noticed that if

\[ v = 0, \ w + a = q, \ w - a = q^{-1}, \ b = 0 \] (1.10)

the \( U_j \)'s are generators of the Hecke algebra. We notice that if we take:

\[ v = 0, \ w = 1, \ a = 0 \] (1.11)

we obtain a special case of the one-parameter-dependent \( U_q(sl(2))(q^3 = 1) \) invariant quantum chain of Ref. [3]. In this model one uses nilpotent representations. The same quantum chain plays an important role in a physical application [5]. If one considers the one-dimensional problem in which two types of molecules, say A and B, diffuse in a lattice (vacancies represent the third state of the problem) and undergo chemical reactions, the time evolution operator that appears in the master equation is given, for certain rates, by the Hamiltonian (1.4). We stress that in order to obtain Eq. (1.4) the molecules A and B have to play a symmetric role.

It is the aim of this paper to study the spectrum and eigenfunctions of the Hamiltonian \( H'_0 \). If this problem is solved, since \( H'_1 \) commutes with \( H'_0 \), the spectrum of \( H' \) is trivially obtained for finite chains. The calculation in the thermodynamical limit of the spectrum and correlation functions in the presence of \( H'_1 \) has to be done, as usual, with care, since the ground state of the system is dependent on the coupling constant \( g \) appearing in Eq. (1.6).
As will be shown in the present paper, the spectrum and wave functions of the Hamiltonian $H'_0$ can be obtained from those of another Hamiltonian, namely the XXZ spin $\frac{1}{2}$ Heisenberg chain in the presence of a $Z$ field, which is integrable. The last Hamiltonian depends (leaving aside the boundary terms) on the anisotropy $\Delta$ and the strength $h$ of the $Z$ field. These two parameters are related to the two parameters $v$ and $w$ in Eq. (1.5). It turns out that the energy levels of the Heisenberg chain ($2^L$ states) and those of the chain given by Eq. (1.4) ($3^L$ states) are identical. The degeneracies of the $3^L$ states chain depend on those of the $2^L$ chain, which in turn depend on the values of $\Delta$, $h$ and boundary terms.

The paper is organized as follows. In Sec. 2 we shortly review the phase diagram of the Heisenberg chain since the Hamiltonian (1.4) has the same phase diagram. We also remind the reader of the known symmetries of the model (quantum group symmetries) and point to an amusing new symmetry, which has as consequence the degeneracy of just two special energy levels for any finite length (see also Appendix B). We then rewrite the Heisenberg model in a different basis and show how to get other quantum chains with the same spectrum, the Hamiltonian (1.4) being the simplest case.

In Sec. 3 we give a simple construction that allows, starting from an eigenfunction of the Heisenberg chain corresponding to a given eigenvalue, to get all the corresponding eigenfunctions of the $3^L$ states chain with the same energy.

In Sec. 4 we look at this problem from a different point of view. Starting from the observation that $H'_0$ is invariant under the $SU(2)$ given by the generators $S^\pm$ and $S^z$, we try to $q$-deform the chain and look for a Hamiltonian that commutes with the known generators $\tilde{S}^\pm$ and $S^z$ of the quantum algebra $U_q(sl(2))$ (for $q = 1, \tilde{S}^\pm$ coincides with $S^\pm$). Surprisingly enough, the chain stays $q$-independent although the generators $\tilde{S}^\pm$ are (using the coproduct rules) $q$-dependent. This implies that the chain is invariant under $U_q(sl(2))$ for any $q$, in particular it is invariant under the affine $U_q(\hat{sl}(2))$ algebra for any $q$. In $U_q(\hat{sl}(2))$ one combines $U_q(sl(2))$ with $U_{q^{-1}}(sl(2))$. This allows us to understand the degeneracies of the (1.4) chain in terms of finite-dimensional irreducible representations of $U_q(\hat{sl}(2))$.

Starting from this observation, in Sec. 5 (see also Appendix C) we ask under which circumstances a $3^L$ chain can be invariant under $U_q(\hat{sl}(2))$, if we take a spin 0 and a spin $\frac{1}{2}$ on each site. The answer we get is again the
Hamiltonian (1.4). We warn the reader that, as opposed to the first sections, the last two (4 and 5) are rather mathematical in character.

Finally, in Sec. 6 we describe the degeneracies of the $3^L$ chain if the underlying Heisenberg chain has supplementary symmetries as described in Sec. 2 and Appendix B. Our conclusions are presented in Sec. 7.

2 A procedure to obtain quantum chains with the same spectrum as the Heisenberg chain

The spin $\frac{1}{2}$ anisotropic Heisenberg chain is defined by the Hamiltonian

$$H'' = \sum_{i=1}^{L-1} V_i ,$$

(2.1)

where

$$V_i = -\frac{1}{2} \left[ \sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \Delta \sigma_i^z \sigma_{i+1}^z + h(\sigma_i^z + \sigma_{i+1}^z) + \alpha(\sigma_i^z - \sigma_{i+1}^z) + \beta \right]$$

and $\sigma^x, \sigma^y$ and $\sigma^z$ are Pauli matrices. As is well known, this chain is integrable with the help of the Bethe ansatz. If $\Delta, h$ and $\alpha$ are arbitrary, there are no degeneracies in the spectrum. Obviously,

$$[I^z, H''] = 0 ,$$

(2.3)

where

$$I^z = \frac{1}{2} \sum_{i=1}^{L} \sigma_i^z .$$

(2.4)

One obtains higher symmetries in three cases:

Case 1a

$$\Delta = 0, \ h + \alpha = -q, \ h - \alpha = -q^{-1}, \ \beta = -(q - q^{-1}) .$$

(2.5)

In this case the Hamiltonian is $U_q(sl(1/1))$-symmetric [7] and its diagonalized form can be expressed [8] through fermionic occupation numbers:

$$H = \varepsilon_0 n_0 + \sum_{k=1}^{L-1} \left( q + q^{-1} - 2 \cos \left( \frac{\pi k}{L} \right) \right) n_k ,$$

(2.6)
where $\varepsilon_0 = 0$ is a zero fermionic mode and the $n_i$'s ($i = 0, \ldots, L - 1$) take the values 0 and 1. Barring accidental degeneracies, each energy level of the Hamiltonian (2.6) is twice degenerate due to the zero fermionic mode as long as $q$ is real. If $q$ is a root of unity, additional degeneracies occur (see Ref. [8]). The $V_i$'s defined by Eqs. (2.2), (2.5) are the generators of the Hecke algebra:

$$V_i V_{i\pm 1} - V_{i\pm 1} V_i = V_{i\pm 1} V_{i\pm 1} - V_{i\pm 1}$$

$$[V_i, V_j] = 0, \quad |i - j| \geq 2$$

$$V_i^2 = (q + q^{-1}) V_i \quad (i = 1, \ldots, L - 1)$$

(2.7)

corresponding to the quotient [9]:

$$(V_i V_{i+2}) V_{i+1} (q + q^{-1} - V_i) (q + q^{-1} - V_{i+2}) = 0 \quad (i = 1, 2, \ldots, L - 3). \quad (2.8)$$

Case 1b

$$\Delta = \frac{q + q^{-1}}{2}, \quad h = 0, \quad \alpha = \frac{q - q^{-1}}{2}, \quad \beta = -\frac{q + q^{-1}}{4}.$$  \quad (2.9)

The Hamiltonian is $U_q(sl(2))$-symmetric [10], the $V_i$'s are again the generators of the Hecke algebra (2.7) but now correspond to the Temperley-Lieb quotient:

$$V_i V_{i\pm 1} V_i = V_i.$$ \quad (2.10)

Cases 2 and 3

$$(\Delta \pm h)^2 = 1 + \alpha^2.$$ \quad (2.11)

This symmetry is a bit unusual. It just says that the unique energy level with $I_z = \frac{L}{2}$ has the same energy as one level with $I_z = \frac{L}{2} - 1$. Alternatively the level with $I_z = -\frac{L}{2}$ has the same energy as one level with $I_z = -\frac{L}{2} + 1$. For details see Appendix B.

At this point we will make a change of basis in the Hamiltonian (2.1) suggested by the quantum chain formulation of diffusion-reaction processes done in Ref. [5]. For an $N$-state model ($N = 2$ for the Heisenberg chain, $N = 3$ for the Hamiltonian (1.4)) we will denote the states by $m = 0, 1, \ldots, N - 1$. The state $m = 0$ will play a special role. In the space of $N \times N$ matrices we take the usual basis of matrices $E^{mn}(m, n = 0, 1, \ldots, N - 1)$. The matrix $E^{mn}$ has as only non-vanishing element the one in the $m^{th}$ row and $n^{th}$
column, and its value is 1. In this basis the Pauli matrix $\sigma^z$ can be written as
\[ \sigma^z = E^{00} - E^{11} = 1 - 2E^{11}. \] (2.12)
In the new basis the $V_i$'s of Eq. (2.2) have the following expression:
\[ V_i = -\left(E_i^{01}E_{i+1}^{10} + E_i^{10}E_{i+1}^{01}\right) + vE_i^{11}E_{i+1}^{11} + w\left(E_i^{11} + E_{i+1}^{11}\right) + a\left(E_i^{11} - E_{i+1}^{11}\right) + b, \] (2.13)
where
\[ v = -2\Delta, \quad w = \Delta + h, \quad a = \alpha, \quad b = -\left(\frac{\Delta + \beta}{2} + h\right). \] (2.14)

In order to construct an $N$-state system with the same spectrum, we write non-diagonal terms that do not mix the states $m = 1, 2, \ldots, N - 1$ and diagonal terms where one permutes the states "m" (the state "0" excluded). This takes us to the following Hamiltonian:
\[ H = \sum_{i=1}^{L-1} V_i, \] (2.15)
where
\[ V_i = -\sum_{m=1}^{N-1} \left(E_i^{0m}E_{i+1}^{m0} + E_i^{m0}E_{i+1}^{0m}\right) + v\varepsilon_i^0\varepsilon_{i+1}^0 + w\left(\varepsilon_i^0 + \varepsilon_{i+1}^0\right) + a\left(\varepsilon_i^0 - \varepsilon_{i+1}^0\right) + b \] (2.16)
and
\[ \varepsilon^0 = \sum_{m=1}^{N-1} E^{mm}. \] (2.17)
In particular, if $N = 3$, we recover the $U_j$'s of Eq. (1.5). Since the states "0" play a special role it is useful to introduce an operator $Z$, which counts them:
\[ Z = \sum_{i=1}^{L} E_i^{00} = L - \sum_{i=1}^{L} \varepsilon_i^0, \] (2.18)
its eigenvalues $z$ are 0, 1, \ldots, $L$. In particular if $N = 2$:
\[ Z = \frac{L}{2} + I^z, \] (2.19)
i.e., it coincides with the magnetization.

In Sec. 3 we will show, for \( N \geq 2 \), that the spectrum of the Hamiltonian (2.15) is independent of the number of states \( N \). One corollary of this observation (which has also been checked by direct calculation) is that the \( V_i \)'s of Eq. (2.16), if the conditions (2.5) or (2.9) are satisfied, also satisfy the Hecke algebra (2.7) and the corresponding quotient relations (2.8) or (2.10). We have to stress that all these results are valid for open chains only. The chains with periodic boundary conditions have spectra which depend on the number of states \( N \) (see Sec. 3).

At this point it is useful to remind the reader of the phase diagram [11] of the Heisenberg chain, which is shown in Fig. 1. For \( h = 0 \), the system is massive with a ferromagnetic ground state if \( \Delta > 1 \), massless and conformally invariant if \( -1 \leq \Delta \leq 1 \), and again massive with an antiferromagnetic ground state if \( \Delta < -1 \). For a given \( h \), the system is massive ferromagnetic if \( \Delta > 1 - h \), then undergoes a Pokrovsky-Talapov [12] transition at \( \Delta = \Delta_{PT} = 1 - h \), is in a massless incommensurate phase for \( \Delta_c < \Delta < 1 - h \), and reaches again the massive antiferromagnetic phase at \( \Delta < \Delta_c \). The \( \Delta_c = \Delta_c(h) \) curve is obtained by solving the equation

\[
h = 2 \sinh \theta \prod_{j=1}^{\infty} \left( \frac{1 - e^{j\theta}}{1 + e^{j\theta}} \right)^2 , \quad \Delta_c = - \cosh \theta .
\]  

Notice that the \( \Delta = \Delta_{PT} \) line corresponds to the special symmetry related to the condition (2.11) for \( \alpha = 0 \). The same phase diagram is valid through the translation given by Eq. (2.14) for all the \( N \)-state systems described by Eqs. (2.15) and (2.16), and therefore for the infinite repulsion limit of the extended Hubbard model. In this case, in order to interpret the phase diagram, it is better to work with a fixed value of the electron nearest-neighbour interaction \( (v) \) and to vary by changing the fugacity \( (w) \). Phase (III) corresponds to a phase where the fermions and holes are physically separated. In phase (I) there are no fermions in the ground state, and in phase (II) we have fermions and holes mixed.
3 Energy levels and wave functions of the $N$-state model

In this section, we write the wave functions of the Hamiltonian (2.15), (2.16) in terms of those of the Hamiltonian (2.1), (2.13), and deduce from this that their spectra coincide. The degeneracies are also explained in terms of the degeneracies of the $N = 2$ system (see also Sec. 6).

Since the Hamiltonian (2.1) commutes with the operator $Z$ (2.18), we can choose its eigenstates among the eigenstates of $Z$, i.e., with definite value for the number of states "0" (and hence for the number of states "1"). The Hamiltonian (2.15) also commutes with $Z$, and moreover it commutes also with all the $\sum_i E_{i}^{mm}$ for $m = 1, \ldots, N - 1$. All the charges "m" are then conserved. It is actually easy to check that the eigenstates of (2.1) are also eigenstates of (2.15). They define a set of $2^L$ eigenstates of (2.15) that involve only the states "0" and "1". So we already know that the spectrum of (2.1) is included in that of (2.15).

We will now introduce a set of operators that commute with the Hamiltonian (2.15), and then use them to construct all the eigenstates of (2.15) by action on the previously defined eigenstates that involve only the states "0" and "1".

Let us define

$$J_{(r,s)}^{nm} ; \quad n, m = 1, \ldots, N - 1; \quad r = 1, \ldots, L; \quad s = 1, \ldots, r$$

(3.1)

as the operator with the following properties:

a) $J_{(r,s)}^{nm} Z = Z J_{(r,s)}^{nm} = (L - r) J_{(r,s)}^{nm}$, i.e., $J_{(r,s)}^{nm}$ vanishes on states with a number of "0" different from $L - r$.

b) $J_{(r,s)}^{nm}$ acts as $E_{s}^{nm}$ on the $s^{th}$ site on which the state is different from "0". (Note that $s \leq r$ and that we count from the left to the right.)

For instance,

$$J_{(4,3)}^{32} |201201\rangle = |201301\rangle$$

$$J_{(r,s)}^{nm} |201201\rangle = 0 \text{ if } r \neq 4 .$$

(3.2)

We then have

$$\sum_{r=1}^{L} \sum_{s=1}^{r} \sum_{m=1}^{N-1} \frac{1}{r} J_{(r,s)}^{nm} = 1 - J_{(0)} .$$

(3.3)
where \( J_{(0)} \) is the projector on the state \( z = L \), i.e., with only "0". By straightforward calculations one can show that the Hamiltonian (2.15) commutes with all the \( J_{(r,s)}^{nm} \) \( (r = 1, \cdots, L; \ s = 1, \cdots, r; \ n, m = 1, \cdots, N - 1) \) and \( J_{(0)} \). This property is only valid for the open chain.

Consider now an eigenstate \( \psi \) of the Hamiltonian (2.15), involving only the states "0" and "1", and let \( r \) be the eigenvalue of \( L - Z \) on it. Then the wave functions defined by

\[
\prod_{s=1}^{r} J_{(r,s)}^{n_s,1} \psi
\]

are eigenstates of the Hamiltonian (2.15) with the same energy as \( \psi \). Here the \( n_s \) take the set of values 1, \cdots, \( N - 1 \) for \( s = 1, \cdots, r \). The set of these wave functions with \( (n_1, \cdots, n_r) \in \{1, \cdots, N - 1\}^r \) is linearly independent.

So each \( \psi \) eigenstate of (2.1) leads to \( (N - 1)^r \) eigenstates of (2.15). The number of \( \psi \)'s such that \( (L - Z)\psi = r\psi \) being \( \binom{L}{r} \), we get a total of

\[
\sum_{r=0}^{L} \binom{L}{r} (N - 1)^r = N^L
\]

eigenstates. Thus all the eigenstates are given by Eq. (3.4) and the spectrum of (2.15) is identical with that of (2.1). This construction also provides the degeneracies of eigenvalues of (2.15) in terms of those of (2.1): each state of a multiplet of (2.1) provides \( (N - 1)^r \) states of the corresponding multiplet of (2.15). We conclude this section with the following observations:

a) This construction does not work for periodic chains. (The \( J_{(r,s)}^{nm} \) do not commute with the Hamiltonian in this case.)

b) As explained in the following sections in the example of \( N = 3 \) the Hamiltonian (2.15) is in fact invariant under \( U_q(sl(N - 1)) \) for any \( q \). To each site is associated the sum of the trivial ("0" state) and the fundamental ("1" to "\( N - 1 \)" states) representations. The previous construction can be seen as a consequence of the fact that the iterated tensor products of the fundamental representation of \( U_q(sl(N - 1)) \) are irreducible [13] (See Appendix C).

c) The \( N \)-state generalization of the Hamiltonian with the constraints (2.5) has the diagonal form given by Eq. (2.6) where now \( n_k \)
1, \ldots, L - 1) takes the values one (one time) and zero (\(N - 1\) times). This is a generalization of the fermionic occupation number.

4 Invariance of the Hamiltonian under \(U_q(sl(2))\) for all \(q\)

In the present and in the subsequent sections, we will discuss the relationship between the spectra of the 2-state Hamiltonian (2.1), (2.13) and the 3-state Hamiltonian (2.15), (2.16) from a different point of view, namely, the \(U_q(sl(2))\)-invariance of the latter. As indicated, we choose \(N = 3\), although our discussion holds in the general case, too. As we first noticed by numerical diagonalizations, the spectrum of the 3-state Hamiltonian \(H\) is highly degenerate. This suggests that the symmetry of \(H\) should be much higher than the obvious \(SU(2) \otimes U(1)\) invariance mentioned in the introduction. Visibly, this invariance is generated at one site by the matrices \(\varepsilon^\pm, \varepsilon^z, \varepsilon^0\) specified in (1.7). Thus the 3-dimensional state space at one site carries the direct sum of a singlet and a doublet representation of \(SU(2)\) (corresponding to the indices 0 and 1, 2, respectively).

To gain some hints towards the symmetry of \(H\) we sat about to construct a \(q\)-analogue of this Hamiltonian. This is easily done, once one has noticed that the doublets \((t^\pm)\) and \((t'^\pm)\) defined by

\[
\begin{align*}
t^+ &= E^{10}, & t^- &= E^{20} \\
t'^+ &= E^{02}, & t'^- &= -E^{01}
\end{align*}
\]

are tensor operators of \(SU(2)\) and that, up to the normalization,

\[
\begin{align*}
(t \otimes t')_0 &= -E^{10} \otimes E^{01} - E^{20} \otimes E^{02} \\
(t' \otimes t)_0 &= E^{02} \otimes E^{20} + E^{01} \otimes E^{10}
\end{align*}
\]

are the scalar parts of the tensor product of \(t\) and \(t'\) and of \(t'\) and \(t\), respectively. Notice that in Eq. (1.5) we put \(\tau^- = t^+\), \(\rho^- = t^-\), \(\rho^+ = t'^+\) and \(\tau^+ = -t'^-\).

To \(q\)-deform this structure we follow Ref. [14]. We recall some of the definitions. Let \(q\) be a non-zero complex number with \(q^2 \neq 1\). Then \(U_q(sl(2))\) is
the universal algebra (associative, with unit element) with generators $e, f, k^{\pm1}$ and relations
\begin{align*}
kk^{-1} &= k^{-1}k = 1 \\
kek^{-1} &= qe, \quad kfk^{-1} = q^{-1}f \quad \text{(4.5)} \\
ef - fe &= \frac{k^2 - k^{-2}}{q - q^{-1}} \quad \text{(4.7)}
\end{align*}
($e$ and $f$ correspond to the generators $\tilde{S}^{\pm}$ described in Sec. 1). It is converted into a Hopf algebra by introducing the coproduct $\Delta$ through
\begin{align*}
\Delta(e) &= e \otimes k^{-1} + k \otimes e \\
\Delta(f) &= f \otimes k^{-1} + k \otimes f \quad \text{(4.8)} \\
\Delta(k^{\pm1}) &= k^{\pm1} \otimes k^{\mp1},
\end{align*}
the co-unit $\varepsilon$ through
\begin{align*}
\varepsilon(e) = \varepsilon(f) &= 0 \\
\varepsilon(k^{\pm1}) &= 1, \quad \text{(4.9)}
\end{align*}
and the antipode $S$ through
\begin{align*}
S(e) &= -q^{-1}e \\
S(f) &= -qf \quad \text{(4.10)} \\
S(k^{\pm1}) &= k^{\mp1}.
\end{align*}
It is well known that the spin $\frac{1}{2}$ representation of $U_q(sl(2))$ is undeformed (and, of course, so is the spin 0 representation). Thus the relevant representation of $U_q(sl(2))$ at one site is undeformed as well and is given by
\begin{align*}
e \to E^{12}, \quad f \to E^{21}, \quad k \to E^{00} + q^{1/2}E^{11} + q^{-1/2}E^{22}. \quad \text{(4.11)}
\end{align*}
Consequently, the deformation of the representation of $U_q(sl(2))$ in the state space of $H$ is introduced by the non-trivial coproduct.

The $q$-deformed tensor operators ($t_q^{\pm}$) and ($t_q'$) are easily found to be
\begin{align*}
t_q^+ &= t^+ = E^{10}, \quad t_q^- = t^- = E^{20} \quad \text{(4.12)}
\end{align*}
\[ t'^+_q = t'^+ = E^{02}, \quad t'^-_q = qt'^- = -qE^{01}. \] (4.13)

Using the techniques of Ref. [14], we can then calculate the \( q \)-deformed versions of Eqs. (4.3), (4.4). The surprising result is that with an adequate normalization

\[ (t_q \otimes t'_q)_0 = (t \otimes t')_0 \] (4.14)
\[ (t'_q \otimes t_q)_0 = (t' \otimes t)_0 \] (4.15)

(the underlining of \( \otimes \) is to indicate that the tensor product is constructed according to the rules for tensor operators over quasitriangular Hopf algebras). Thus the scalar operators in Eqs. (4.3), (4.4) are undeformed. This is not completely trivial since, again with a suitable normalization, we find

\[ (t_q \otimes t_q)_0 = E^{10} \otimes E^{20} - qE^{20} \otimes E^{10} \] (4.16)
\[ (t'_q \otimes t'_q)_0 = E^{01} \otimes E^{02} - qE^{02} \otimes E^{01}, \] (4.17)

thus, these scalar operators are deformed.

Since the remaining terms of \( H \) are obviously \( U_q(\hat{sl}(2)) \)-invariant, we conclude that the undeformed Hamiltonian \( H \) is \( U_q(\hat{sl}(2)) \)-invariant, for all \( q \). In view of the well-known homomorphism of the \( q \)-deformed affine algebra \( U_q(\hat{sl}(2)) \) onto \( U_q(sl(2)) \) (see the subsequent section), it comes to the same to say that \( H \) is even \( U_q(\hat{sl}(2)) \)-invariant, for all \( q \).

5 Construction of \( U_q(\hat{sl}(2)) \)-invariant Hamiltonians

The upshot of the foregoing section was that the Hamiltonian \( H \) discussed there is \( U_q(\hat{sl}(2)) \)-invariant. In the present section we are going to rederive this result by means of a more systematic approach. We shall start from a class of Hamiltonians generalizing \( H \) and derive the conditions under which they are \( U_q(\hat{sl}(2)) \)-invariant for a fixed value \( q' \) of \( q \). This will lead us back to the Hamiltonian \( H \).

To begin with, let us recall the definition of the Hopf algebra \( U_q(\hat{sl}(2)) \). With \( q \) chosen as above, this is the universal algebra (associative, with unit element) generated by \( e_i, f_i, k_i^{\pm 1}; \ i \in \{1, 2\} \), with relations \((i, j \in \{1, 2\})\):

\[ k_i k_i^{-1} = k_i^{-1} k_i = 1 , \quad k_i k_j = k_j k_i \] (5.1)
\[ k_i e_j k_i^{-1} = q^{\frac{1}{2} a_{ij}} e_j, \quad k_i f_j k_i^{-1} = q^{-\frac{1}{2} a_{ij}} f_j \] (5.2)

\[ e_i f_j - f_j e_i = \delta_{ij} \frac{k_i^2 - k_i^{-2}}{q - q^{-1}} \] (5.3)

\[ e_i^3 e_j - [3]_q e_i^2 e_j e_i + [3]_q e_i e_j e_i^2 - e_j e_i^3 = 0 \text{, } i \neq j \] (5.4)

\[ f_i^3 f_j - [3]_q f_i^2 f_j f_i + [3]_q f_i f_j f_i^2 - f_j f_i^3 = 0 \text{, } i \neq j, \] (5.5)

where

\[ (a_{ij}) = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \] (5.6)

is the Cartan matrix of the affine Lie algebra \( \hat{sl}(2) \) and where, for any integer \( n \), we define

\[ [n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}. \] (5.7)

The Hopf algebra structure is defined by introducing the coproduct \( \Delta \) through

\[
\begin{align*}
\Delta(e_i) &= e_i \otimes k_i^{-1} + k_i \otimes e_i \\
\Delta(f_i) &= f_i \otimes k_i^{-1} + k_i \otimes f_i \\
\Delta(k_i^{\pm1}) &= k_i^{\pm1} \otimes k_i^{\pm1},
\end{align*}
\] (5.8)

the co-unit \( \varepsilon \) through

\[
\begin{align*}
\varepsilon(e_i) &= \varepsilon(f_i) = 0 \\
\varepsilon(k_i^{\pm1}) &= 1,
\end{align*}
\] (5.9)

and the antipode \( S \) through

\[
\begin{align*}
S(e_i) &= -q^{-1} e_i \\
S(f_i) &= -q f_i \\
S(k_i^{\pm1}) &= k_i^{\mp1}.
\end{align*}
\] (5.10)
For our purposes it is important to recall that for any non-zero complex number \( c \) there exists a unique algebra homomorphism \( g \) of \( U_q(\widehat{sl}(2)) \) onto \( U_q(sl(2)) \) such that

\[
\begin{align*}
g(e_1) &= e, & g(e_0) &= cf, \\
g(f_1) &= f, & g(f_0) &= c^{-1}e, \\
g(k_1) &= k, & g(k_0) &= k^{-1}.
\end{align*}
\]

Consequently, if \( \omega \) is any representation of \( U_q(sl(2)) \) in a vector space \( W \), then \( \omega \circ g \) is a representation of \( U_q(\widehat{sl}(2)) \) in that vector space. The coproduct of \( U_q(\widehat{sl}(2)) \) can then be used to construct the \( L \)-th tensorial power of \( \omega \circ g \). We stress that this representation is different from that obtained by composing the \( L \)-th tensorial power of \( \omega \) with \( g \) [for \( g \) is not a coalgebra homomorphism, i.e., in general we have \( \Delta g(a) \neq (g \otimes g)\Delta(a) \) for \( a \in U_q(\widehat{sl}(2)) \) and hence, for example, \( (\omega \otimes \omega)\Delta g(a) \neq (\omega g \otimes \omega g)\Delta(a) \)]. Visibly, the representation \( \omega \circ g \) depends on \( c \). In the following we shall choose \( c = 1 \).

After these preliminaries we proceed to our main topic. Let \( A \) be any finite index set not containing the numbers 1 and 2, and let \( n \) be the number of its elements. Later in this section we shall choose \( A = \{0\} \). In the following, the Greek indices \( \alpha, \beta, \gamma, \delta \) run through \( A \cup \{1, 2\} \), and the Latin indices \( a, b, c, d \) through \( A \).

We consider the 2-sites Hamiltonian

\[
H_{12} = \sum c_{\alpha \beta \gamma \delta} E^{\alpha \beta} \otimes E^{\gamma \delta},
\]

where the \( c_{\alpha \beta \gamma \delta} \) are arbitrary complex numbers, the \( E^{\alpha \beta} \) are the basic \((n + 2) \times (n + 2)\) matrices operating on \( W = \mathbb{C}^{n+2} \) (\( \mathbb{C} \) denotes the field of complex numbers), and where \( H_{12} \) is an operator in \( W \otimes W \). We choose a non-zero complex number \( q' \) with \( q'^2 \neq 1 \) (later on we shall assume that \( q' \) is not a root of unity). According to the foregoing discussion there exists a unique representation \( \rho \) of \( U_{q'}(\widehat{sl}(2)) \) in \( W \) such that

\[
\begin{align*}
\rho(e_1) &= \rho(f_0) = E^{12}, \\
\rho(f_1) &= \rho(e_0) = E^{21}, \\
\rho(k_1) &= \rho(k_0^{-1}) = K,
\end{align*}
\]

with

\[
K = \sum_a E^{aa} + q'^{\frac{1}{2}} E^{11} + q'^{-\frac{1}{2}} E^{22}.
\]

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Using the coproduct of $U_q(\hat{sl}(2))$ we obtain the tensorial square $(\rho \otimes \rho) \circ \Delta$ of $\rho$, i.e., the following representation of $U_q'(\hat{sl}(2))$ in $W \otimes W$:

\begin{align*}
(\rho \otimes \rho)\Delta(\epsilon_1) &= E^{12} \otimes K^{-1} + K \otimes E^{12} \\
(\rho \otimes \rho)\Delta(f_0) &= E^{12} \otimes K + K^{-1} \otimes E^{12} \\
(\rho \otimes \rho)\Delta(f_1) &= E^{21} \otimes K^{-1} + K \otimes E^{21} \\
(\rho \otimes \rho)\Delta(e_0) &= E^{21} \otimes K + K^{-1} \otimes E^{21} \\
(\rho \otimes \rho)\Delta(k_0) &= K \otimes K \\
(\rho \otimes \rho)\Delta(k_1) &= K^{-1} \otimes K^{-1}.
\end{align*}

One can show that $H_{12}$ commutes with $(\rho \otimes \rho)\Delta(\epsilon_j)$ and $(\rho \otimes \rho)\Delta(f_j)$; $j \in \{1, 2\}$, if and only if it takes the following special form:

\begin{equation}
H_{12} = \sum_{abcd} c_{abcd} E^{ab} \otimes E^{cd} + \sum_{ab} c_{ab11} E^{ab} \otimes (E^{11} + E^{22}) + \sum_{cd} c_{11cd} (E^{11} + E^{22}) \otimes E^{cd} + \sum_{ad} c_{a11d} (E^{a1} \otimes E^{1d} + E^{a2} \otimes E^{2d}) + \sum_{bc} c_{1bcl} (E^{1b} \otimes E^{c1} + E^{2b} \otimes E^{c2}) + c_{1111} (E^{11} + E^{22}) \otimes (E^{11} + E^{22}).
\end{equation}

We stress that the parameters appearing in this Hamiltonian are arbitrary complex numbers. Thus, no $q$'-dependence has been introduced by our requirement for $U_q'(\hat{sl}(2))$-invariance. Consequently, the Hamiltonian (5.16) is $U_q(\hat{sl}(2))$-invariant for all $q$.

It is easy to see that $H_{12}$ has the following more general symmetries. Define

\begin{align*}
D &= q_0 \sum_a E^{aa} + q_1 E^{11} + q_2 E^{22} \quad (5.17) \\
D' &= q_0 \sum_a E^{aa} + q'_1 E^{11} + q'_2 E^{22}, \quad (5.18)
\end{align*}

where $q_0, q_1, q'_1, q_2, q'_2$ are arbitrary complex numbers. Then $H_{12}$ commutes with

\begin{align*}
E^{12} \otimes D + D' \otimes E^{12}, \quad E^{21} \otimes D + D' \otimes E^{21}, \quad D \otimes D, \quad (5.19)
\end{align*}
and, of course, also with $D' \otimes D'$. Obviously, this implies that the corresponding $L$-sites Hamiltonian

$$H = \sum_{i=1}^{L-1} H_{i,i+1} \quad (5.20)$$

commutes with the corresponding symmetry operators in $W^\otimes L$, i.e., with

$$\sum_i D'(i-1) \otimes E^{12} \otimes D^\otimes(L-i),$$

$$\sum_i D'(i-1) \otimes E^{21} \otimes D^\otimes(L-i),$$

$$D^\otimes L, \quad D'^\otimes L \quad (5.21)$$

The relevance will be illustrated in the following example. We shall restrict our attention to the case $A = \{0\}$; furthermore, we shall assume that $c_{0110}$ and $c_{1001}$ are different from 0. In that case, we may even assume that

$$c_{0110} = c_{1001} = -1, \quad (5.22)$$

for up to the overall normalization of $H$ this can be achieved by a similarity transformation in the state space $W^\otimes L$ (recall that $W = C^{n+2} = C^3$), which preserves the special form of $H$. With (5.22), $H$ takes the form (2.15), (2.16), with

$$v = c_{1111} - c_{1100} - c_{0011} + c_{0000}$$

$$w = \frac{1}{2}(c_{1100} + c_{0011}) - c_{0000}$$

$$a = \frac{1}{2}(c_{1100} - c_{0011})$$

$$b = c_{0000} \quad (5.23)$$

Thus, choosing $A = \{0\}$ and assuming that (5.22) is satisfied, the $U_q(sl(2))$-invariant Hamiltonians of the form (5.12) are exactly the Hamiltonians introduced in (2.15), (2.16).
It is easy to see what the symmetry transformations (5.21) imply for the spectral decomposition of $H$. Let $v_0, v_1, v_2$ denote the canonical basis vectors of $W = \mathbb{C}^3$. For any subset $J \subset \{1, 2, \cdots, L\}$, let $V_J$ denote the subspace of

$$V = W \otimes L,$$  

(5.24)

which is spanned by the tensors $v_{\alpha_1} \otimes \cdots \otimes v_{\alpha_L}$, with $\alpha_i \in \{1, 2\}$ if $i \in J$, and $\alpha_i = 0$ otherwise. Obviously, $V$ is the direct sum of the subspaces $V_J$ and

$$\dim V_J = 2^r \quad \text{with} \quad r = \sharp J.$$  

(5.25)

Furthermore, the $V_J$ are invariant under the transformations (5.21), and it can be shown that these transformations act irreducibly on $V_J$.

The latter result can also be derived as follows. Visibly, $V_J$ carries the $r^\text{th}$ tensorial power of the elementary spin $\frac{1}{2}$ representation of $U_{q'}(s\mathfrak{l}(2))$. But if $q'$ is not a root of unity, it is known [13] that this representation is irreducible.

The upshot is that $V = W \otimes L$ decomposes into irreducible $2^r$-dimensional subspaces, with $r \in \{0, 1, \cdots, L\}$, and that exactly $\binom{L}{r}$ of these subspaces have that dimension. This implies that the eigenspaces of $H$ also decompose into irreducible subspaces of these dimensions.

To be more precise, let $r \in \{0, 1, \cdots, L\}$ and let

$$V_r = \bigoplus_{\sharp J = r} V_J.$$  

(5.26)

Thus $V_r$ is the eigenspace of the operator $Z$ (defined in Eq. (2.18)) corresponding to the eigenvalue $L - r$. We stress that for $r$ fixed, the irreducible representations acting in the $V_J$ with $\sharp J = r$ are equivalent.

Furthermore, for $a \in \{1, 2\}$ we define

$$V^a = (Cv_0 + Cv_a)^\otimes L,$$  

(5.27)

i.e., $V^a$ is spanned by the tensors of the form $x_1 \otimes \cdots \otimes x_L$, with $x_i \in \{v_0, v_a\}$ for all $i$. Finally, let

$$V^a_r = V^a \cap V_r.$$  

(5.28)

Obviously, we have

$$V^a = \bigoplus_{r=0}^{L} V^a_r.$$  

(5.29)
Furthermore, it is easy to see that $H$ maps the spaces $V^a$ and $V_r$ into themselves, hence also the spaces $V^a_r$. Thus, according to the foregoing discussion, we only have to construct the spectral decomposition of $H$ in the subspaces $V^a_r$ for one value of $a$, say $a = 1$. The action of the symmetry transformations will then yield the corresponding decomposition of $V_r$, hence of $V$. In particular, if the multiplicity of an eigenvalue of $H$ in $V^1_r$ is equal to $m$, then the multiplicity of that eigenvalue of $H$ in $V_r$ is equal to $m \cdot 2^r$.

Summarizing the preceding discussion we see that by means of the symmetry transformations the spectral decomposition of $H$ in $V$ has been reduced to that of $H$ in one of the subspaces $V^a$, i.e., to that of the 2-state Hamiltonian (2.1), (2.13). This is in perfect agreement with the results of Sec. 3. In particular, the existence of the symmetries $J_{(r,s)}^{nm}$ described in Sec. 3 follows directly from the results of the present section. This will be explained in Appendix C.

6 Multiplets of the $3^L$ chain when the underlying Heisenberg chain has supplementary symmetries

Consider the $3^L$ chain obtained with the construction of Sec. 5 from the $2^L$ chain. We use the notation
\[ d_1 = w - a , \quad d_2 = w + a , \quad d_3 = v + 2w \quad (6.1) \]
for the diagonal parameters. As explained in Appendix B, we have to consider four cases for the values of $d_1, d_2, d_3$.

Case 1a
\[ d_1 = q^{-1} , \quad d_2 = q , \quad d_3 = q + q^{-1} \quad (6.2) \]
The underlying $2^L$ chain then has the $U_q(sl(1/1))$ symmetry, and its states build doublets corresponding to adjacent eigenvalues $r$ and $r - 1$ of $L - Z$. In the $3^L$ chain, the two vectors of each doublet are highest-weight vectors of multiplets for the symmetry $U'_q(sl(2))$, with dimensions $2^r$ and $2^{r-1}$. These multiplets are gathered in a higher multiplet of dimension $2^r + 2^{r-1} = 3 \cdot 2^{r-1}$. There are \( \binom{L-1}{r-1} \) multiplets of this dimension. In the fermion picture, \( \binom{L-1}{r-1} \)
is the number of states with \( r - 1 \) fermions (among \( L - 1 \) fermions with non-zero energy).

Case 1b

\[
d_1 = q^{-1} , \quad d_2 = q , \quad d_3 = 0 . \tag{6.3}
\]

The underlying \( 2^L \) chain has \( U_q(sl(2)) \) symmetry, and the states are gathered into multiplets corresponding to the decomposition of \( (\text{Spin}_{\frac{1}{2}})^{\otimes L} \) with the fusion rules of \( U_q(sl(2)) \). We shall consider only the case where \( q \) is not a root of unity. [The case where it is would easily follow from the decomposition of \( (\text{Spin}_{\frac{1}{2}})^{\otimes L} \) into irreducible and indecomposable representations.] The multiplets of the \( 2^L \) chain collect states with different eigenvalues of \( L - Z \) as follows:

\[
\begin{align*}
\{L, L - 1, \cdots, 0\} & , \quad [L + 1] , \quad (1) \\
\{L - 1, L - 2, \cdots, 1\} & , \quad [L - 1] , \quad (L - 1) \\
& \vdots \\
\{r, r - 1, \cdots, L - r\} & , \quad [2r - L + 1] , \quad L_{(L - r + 1)} - L_{(L - r - 1)} (L - 1)
\end{align*}
\]

with \( 2r \geq L \). We have put in braces the eigenvalues of \( L - Z \). The brackets indicate the dimension of this \( U_q(sl(2)) \)-multiplet, whereas the parentheses indicates the number of multiplets of this type.

For the \( 3^L \) chain, each state of these \( U_q(sl(2)) \)-multiplets becomes a highest-weight vector of a \( U_q'(sl(2)) \)-multiplet. We then have larger multiplets of dimensions

\[
\begin{align*}
2^L + 2^{L-1} + \cdots + 2^0 & = 2^{L+1} - 1 , \quad (1) \\
2^{L-1} + 2^{L-2} + \cdots + 2^1 & = 2^L - 2 , \quad (L - 1) \\
& \vdots \\
2^r + 2^{r-1} + \cdots + 2^{L-r} & = 2^{r+1} - 2^{L-r} , \quad L_{(L - r + 1)} - L_{(L - r - 1)} (L - 1)
\end{align*}
\]

where again, in the parenthesis, we indicate the numbers of these multiplets.

Case 2

\[
d_1d_2 = 1 , \quad (d_1 - d_3)(d_2 - d_3) \neq 1 . \tag{6.4}
\]

As shown in Appendix B, the supplementary symmetry builds a unique 2-dimensional multiplet of the \( 2^L \) chain with the state \( v_0^{\otimes L} \) [for the notation,
see the lines preceding Eq. (5.24)] and a state with the eigenvalue of $Z$ equal to $L - 1$. In the $3^L$ chain, these two states become highest-weight vectors of $U_q'(sl(2))$-multiplets of dimension 1 and 2, respectively. So we finally get a unique multiplet of dimension three, while the other $U_q'(sl(2))$-multiplets are not combined.

**Case 3**

$$d_1d_2 \neq 1 \quad , \quad (d_1 - d_3)(d_2 - d_3) = 1 . \quad (6.5)$$

The two states that build a doublet in the $2^L$ chain have now $Z$-eigenvalues 0 and 1. In the $3^L$ chain problem, these states are highest-weight vectors of $U_q'(sl(2))$-multiplets of dimensions $2^L$ and $2^{L-1}$. We then have a multiplet of dimension $3 \cdot 2^{L-1}$. All the other states remain in multiplets of dimension $2^r \ (0 \leq r < L)$.

## 7 Conclusions

We have started our paper with a physically interesting one-dimensional system of fermions with an interaction given by three coupling constants $\frac{U}{2\pi} \frac{V}{2\pi}$ and $\frac{W}{2\pi}$. The phase diagram of this system is unknown. One rigorous result of our investigation is that in the strong repulsion limit, the phase diagram coincides with the one of the Heisenberg chain in a magnetic field (see Fig. 1). The Eq. (2.14) of Sec. 2 describes this mapping. We also give an interpretation of the phase diagram of Fig. 1, which is usually understood in the language of magnetic systems, in terms of hard-fermion physics (see Appendix A). Our work has gone beyond this investigation. On the one hand in Sec. 2 we have obtained $N$-state models whose spectra coincide with those of the Heisenberg model (the $N = 3$ case being relevant to the fermionic problem). The eigenfunctions of these Hamiltonians can be obtained from those of the $N = 2$ case using the construction given in Sec. 3 (see also Appendix C). On the other hand we have noticed (see Sec. 4) that for $N = 3$ the multiplet structure of the chains is related to finite-dimensional representations of the affine algebra $U_q(sl(2))$, the chains being invariant under this algebra for any $q$. In Sec. 5 we give the conditions under which an $(n+2)$-state Hamiltonian is invariant under $U_q(sl(2))$ for any $q$ and find a new set of Hamiltonians whose spectra are no longer given by the Heisenberg
model. Finally, we would like to notice that for the Heisenberg model we have found a new symmetry valid for finite chains when the parameters verify the relations (2.11). This symmetry is discussed in Appendix B. In particular, this symmetry applies to the Pokrovsky-Talapov line.
Appendix A  From 4-state fermionic to 3-state magnetic spin chains

In this appendix we obtain the Hamiltonian (1.3)–(1.5) as the $U \to \infty$ limit of the fermionic Hamiltonian (1.1), (1.2). Since in this limit double occupancy of fermions is forbidden (it costs an infinite energy to put fermions at the same site) we can start our derivation by constructing a Fock space where, in a given lattice point, we may have no fermions ("0"), a fermion with up spin ("↑") or down spin ("↓"). Due to the fermionic nature of the operators appearing in (1.3)–(1.5) we choose an order and construct directly a representation in the $3^L$-dimensional Fock space. The chosen order is illustrated in the following two examples

\begin{align}
| \uparrow, \uparrow, \downarrow, 0, \uparrow, \ldots, \uparrow \rangle &= c_{1, \uparrow}^+ c_{2, \uparrow}^+ c_{3, \uparrow}^+ \ldots c_{L, \uparrow}^+ |0, 0, 0, \ldots, 0\rangle \\
|0, \uparrow, \downarrow, 0, \uparrow, \ldots, \rangle &= c_{2, \uparrow}^+ c_{3, \downarrow}^+ c_{5, \uparrow}^+ \ldots |0, 0, 0, \ldots, 0\rangle ,
\end{align}

(A.1)

where

\begin{equation}
\begin{aligned}
c_{i, \sigma}|0, 0, 0, \ldots\rangle &= 0, & i = 1, 2, \ldots, L; & \sigma = \uparrow, \downarrow .
\end{aligned}
\end{equation}

(A.2)

With this convention we can represent the Hamiltonian (1.1), (1.2) as

\begin{equation}
H'_0 = \sum_{i=1}^{L-1} 1_1 \otimes \ldots \otimes 1_{i-1} \otimes \left[ \begin{array}{cccc}
d_0 & : & : \\
-1 & d_1 & : \\
& -1 & & \\
& & -1 & \\
& & & \\
\vdots & d_2 & : \\
& d_3 & : \\
& & d_3 & \\
& & & \\
& & & \\
-1 & d_2 & : \\
& d_3 & : \\
& & d_3 & i,i+1
\end{array} \right] \otimes \ldots \otimes 1_L \quad \text{(A.3)}
\end{equation}
\[ H'_i = \frac{G}{2t} \sum_{i=1}^{L-1} \mathbf{1}_1 \otimes \ldots \otimes \mathbf{1}_{i-1} \otimes \left[ \begin{array}{cccc} 0 & : & : & \\ 1 & : & : & \\ -1 & : & : & \\ \ldots & \ldots & \ldots & \ldots \\ : & 1 & : & : \\ : & 2 & : & : \\ : & 0 & : & : \\ \ldots & \ldots & \ldots & \ldots \\ : & : & : & -2 \end{array} \right]_{i,i+1} \otimes \ldots \otimes \mathbf{1}_L, \quad (A.4) \]

where \( d_0 = 0, d_1 = \frac{W}{2t} - \frac{A}{2t}, d_2 = \frac{W}{2t} + \frac{A}{2t}, d_3 = \frac{V}{t} + \frac{W}{t} \) and \( \mathbf{1}_i \) is a 3 \times 3 unit matrix attached at site \( i \). The elements not shown explicitly in the above matrices have zero value.

This Hamiltonian can promptly be rewritten in terms of the matrices introduced in (1.7)

\[ H'_0 = \sum_{i=1}^{L-1} \left\{ -\left( e_i^+ e_{i+1}^+ + e_{i+1}^- e_i^- + \tau_i^+ \tau_{i+1}^- + \tau_i^- \tau_{i+1}^+ \right) + \frac{V}{t} \varepsilon_i^0 \varepsilon_{i+1}^0 + \frac{W}{2t} \left( \varepsilon_i^0 + \varepsilon_{i+1}^0 \right) + \frac{A}{2t} \left( \varepsilon_i^0 - \varepsilon_{i+1}^0 \right) + b \right\} \quad (A.5) \]

\[ H'_1 = -\frac{G}{2t} \sum_{i=1}^{L} \varepsilon_i^z. \quad (A.6) \]

The reader has already noticed that our proof of the connection between the Hamiltonians (1.3)–(1.5) and (1.1) and (1.2) applies to open chains only. In the case of a periodic chain one has to split the sectors of the fermionic Fock space according to the parity of the fermionic number. Sectors with an odd (even) number of fermions will correspond to the Hamiltonian (1.3)–(1.5) with periodic (antiperiodic) boundary condition.
Appendix B  New symmetries of the Heisenberg model

Let us consider the $2^L$ state Hamiltonian $H = \sum_{j=1}^{L-1} V_j$ with $V_j$ given by Eq. (2.13); $V_j$ can also be written

$$V_j = b + \begin{pmatrix} 0 & \vdots & \vdots \\ d_1 & -1 & \\
\vdots & \ddots & \vdots \\ -1 & d_2 & \vdots \\
\vdots & & d_3 \end{pmatrix}$$  \hspace{1cm} (B.1)$$

with $d_1 = w - a$, $d_2 = w + a$, $d_3 = v + 2w$.

We look for symmetries of $H$ of the form

$$\tilde{e} = \sum_{j=1}^{L} D^{\otimes(j-1)} \otimes E^{01} \otimes D^{\otimes(L-j)} ,$$  \hspace{1cm} (B.2)$$

where $D$ and $D'$ are diagonal matrices and $D^{\otimes(j-1)} = D \otimes D \otimes \ldots \otimes D$ ($j - 1$ times).

If $H$ commutes with $\tilde{e}$ given by (B.2), it will also commute with $\tilde{f}$, its transpose, given by

$$\tilde{f} = \sum_{j=1}^{L} D'^{\otimes(j-1)} \otimes E^{10} \otimes D^{\otimes(L-j)} .$$  \hspace{1cm} (B.3)$$

The existence of these symmetries depends on the values of $d_1, d_2, d_3$. Let us write

$$D = \begin{pmatrix} l_0 & 0 \\ 0 & l_1 \end{pmatrix} , \quad D' = \begin{pmatrix} l'_0 & 0 \\ 0 & l'_1 \end{pmatrix} .$$  \hspace{1cm} (B.4)$$

Then the conditions on $l_i, l'_i, d_1, d_2, d_3$ for $[H, \tilde{e}] = 0$ are

$$d_1 l_0' - l_0 = 0 , \quad l_0' - d_2 l_0 = 0$$

$$\quad (d_1 - d_3) l_1 - l_1' = 0 , \quad l_1 - (d_2 - d_3) l_1' = 0 .$$  \hspace{1cm} (B.5)$$
The solvability of these equations depends on the values of $d_1, d_2, d_3$, and we have to consider four cases:

**Case 1**

\[ d_1 d_2 = 1 \quad , \quad (d_1 - d_3)(d_2 - d_3) = 1 \]  \hspace{1cm} (B.6)

i.e., either

\[ d_1 = q^{-1} \quad , \quad d_2 = q \quad , \quad d_3 = q + q^{-1} \]  \hspace{1cm} (B.7a)

or

\[ d_1 = q^{-1} \quad , \quad d_2 = q \quad , \quad d_3 = 0 \]  \hspace{1cm} (B.7b)

These cases correspond to the conditions (2.5) and (2.9), respectively, and the symmetries generated by $\tilde{e}$ and $\tilde{f}$ can be identified with $U_{-q}(sl(1/1))$ and $U_q(sl(2))$, respectively.

**Case 2**

\[ d_1 d_2 = 1 \quad , \quad (d_1 - d_3)(d_2 - d_3) \neq 1 \]  \hspace{1cm} (B.8)

Then

\[ D = \begin{pmatrix} l_0 & 0 \\ 0 & 0 \end{pmatrix} , \quad D' = \begin{pmatrix} d_2 l_0 & 0 \\ 0 & 0 \end{pmatrix} . \]  \hspace{1cm} (B.9)

Dropping an overall factor $l_0^{L-1}$, we have

\[ \tilde{e} = \sum_{j=1}^{L} d_2^{-1}(E^{00})^{\otimes(j-1)} \otimes E^{01} \otimes (E^{00})^{\otimes(L-j)} \]  \hspace{1cm} (B.10)

\[ \tilde{f} = \sum_{j=1}^{L} d_2^{-1}(E^{00})^{\otimes(j-1)} \otimes E^{10} \otimes (E^{00})^{\otimes(L-j)} . \]  \hspace{1cm} (B.11)

We then see that all the states are cancelled by $\tilde{f}$, except the state with charges “0” only. In fact, $\tilde{e}$ and $\tilde{f}$ connect the states $|0\ldots0\rangle$ and $\tilde{f}|0\ldots0\rangle$ in a doublet, and all the remaining states are singlets.

Two levels are then degenerate, the others being non-degenerate. This degeneracy of two levels only has also been checked numerically.

**Case 3**

\[ d_1 d_2 \neq 1 \quad , \quad (d_1 - d_3)(d_2 - d_3) = 1 \]  \hspace{1cm} (B.12)
The situation is similar to the previous case and we can thus write

\[ \tilde{e} = \sum_{j=1}^{L} (d_1 - d_3)^{j-1} \left( E^{11} \right)^{\otimes(j-1)} \otimes E^{01} \otimes (E^{11})^{\otimes(L-j)} \]  

(B.13)

\[ \tilde{f} = \sum_{j=1}^{L} (d_1 - d_3)^{j-1} \left( E^{11} \right)^{\otimes(j-1)} \otimes E^{10} \otimes (E^{11})^{\otimes(L-j)} \]  

(B.14)

Now \( \tilde{e} \) vanishes on all the states but \(|1\ldots1\rangle\), and \{\(|1\ldots1\rangle, \tilde{e}|1\ldots1\rangle\} \) build a doublet. All the other states are singlets for \( \tilde{e}, \tilde{f} \). Again, only two levels coincide.

**Case 4**

\[ d_1 d_2 \neq 1 \quad \text{and} \quad (d_1 - d_3)(d_2 - d_3) \neq 1 \]  

(B.15)

Then \( \tilde{e} = \tilde{f} = 0 \); we get no further symmetry.

The symmetries that exist for the \( 2^L \) state Hamiltonian in the cases (1), (2), and (3) are combined with the other symmetries \([U_q(sl(N))\) in the \((N+1)^L \) state case\]. In particular, when two states build a doublet of the \( 2^L \) state Hamiltonian, the representations of \( U_q(sl(N)) \) associated with these states in the \((N+1)^L \)-state case are also combined into higher multiplets. In the case (2), we get a \( 1+N \)-dimensional multiplet, together with \( N^r \)-dimensional multiplets. In the case (3), we get a \( N^L + N^{L-1} \)-dimensional multiplet, together with \( N^r \)-dimensional multiplets (see Sec. 6).

**Appendix C  The symmetries \( J_{(r,s)}^{mn} \) reconsidered**

In the present appendix we want to show that the existence of the symmetries \( J_{(r,s)}^{mn} \) described in Sec. 3 follows from the results of Sec. 5 by using the standard representation theory of associative algebras. In doing so we use the notation introduced in Sec. 5.

Let \( \rho^L \) be the \( L^{th} \) tensorial power of the representation \( \rho \) of \( U_q(sl(2)) \) as specified in Eq. (5.13) (we simplify the notation and write \( q \) instead of \( q' \)). Thus \( \rho^L \) acts in the state space \( V = W^{\otimes L} \) of the \( L \)-sites chain, and we know
that $V$ decomposes into the direct sum of the invariant irreducible subspaces $V_J, J \subset \{1, 2, \ldots, L\}$. Obviously, if $J, J' \subset \{1, 2, \ldots, L\}$ and if $\sharp J$ and $\sharp J'$ are different, the representations given by $V_J$ and $V_{J'}$ are inequivalent. On the other hand, if $\sharp J = \sharp J' = r$, these representations are equivalent. More precisely, if we forget about the tensorial factors $v_0$ (which can be done by use of canonical $U_q(\widehat{sl(2)})$-module isomorphisms) they simply coincide with the $r^{th}$ tensorial power of the elementary 2-dimensional representation of $U_q(sl(2))$, acting in $Cv_1 + Cv_2)^\otimes r$.

Now the representation theory of associative algebras tells us the following. Choose $r \in \{0, 1, \ldots, L\}$ and let $A$ be an arbitrary linear operator in $(Cv_1 + Cv_2)^\otimes r$. Then there exists an element $a \in U_q(\widehat{sl(2)})$ such that $\rho^L(a)$ acts on $V_J$ as $A$ if $\sharp J = r$, and as 0 if $\sharp J \neq r$.

Of course, all these operators $\rho^L(a)$ commute with the Hamiltonian $H$; moreover, it is easy to construct the operator $\rho^L(a)$ explicitly. Without loss of generality, we may assume that

$$A = A_1 \otimes \ldots \otimes A_r,$$  \hspace{1cm} (C.1)

where $A_1, \ldots, A_r$ are arbitrary $2 \times 2$ matrices, with matrix indices in $\{1, 2\}$. Let $\bar{A}_s$ be the $3 \times 3$ matrix obtained from $A_s$ by adding a zeroth row and column consisting of zeros only. Define the operator $J_r(A_1, \ldots, A_r)$, acting in $V$, as follows:

$$J_r(A_1, \ldots, A_r) = \sum E^{00} \otimes \ldots \otimes \bar{A}_1 \otimes \ldots \otimes \bar{A}_2 \otimes \ldots \otimes \bar{A}_r \otimes \ldots \otimes E^{00},$$  \hspace{1cm} (C.2)

where the sum is extended over all tensor products of the matrices $\bar{A}_1, \ldots, \bar{A}_r$ and $L - r$ matrices $E^{00}$, with the $\bar{A}_1, \ldots, \bar{A}_r$ appearing in their given order. Then $J_r(A_1, \ldots, A_r)$ is the operator acting on $V_J$ as $A_1 \otimes \ldots \otimes A_r$ if $\sharp J = r$ and as 0 if $\sharp J \neq r$. Thus we know that the operator $J_r(A_1, \ldots, A_r)$ commutes with the Hamiltonian $H$.

We note that if $r' \in \{0, 1, \ldots, L\}$ and if $A'_1, \ldots, A'_{r'}$ is a second family of $2 \times 2$ matrices, we have

$$J_r(A_1, \ldots, A_r)J_{r'}(A'_1, \ldots, A'_{r'}) = J_r(A_1A'_1, \ldots, A_rA'_{r'}) \quad \text{if} \quad r = r'$$

$$= 0 \quad \text{if} \quad r \neq r'.$$  \hspace{1cm} (C.3)

Moreover, our results show that the operator algebra $\rho^L(U_q(\widehat{sl(2)}))$ coincides with the linear span of the set of all operators $J_r(A_1, \ldots, A_r)$. In particular,
the operators \( \rho^L(e_i) \), \( \rho^L(f_i) \), \( \rho^L(k_i) \); \( i \in \{1, 2\} \), can be written in this form. For example, by definition of \( \rho^L \), we have

\[
\rho^L(e_1) = \sum_{r=s=1}^{L} J_r(k, \ldots, k, E^{12}, k^{-1}, \ldots, k^{-1}),
\]

where \( E^{12} \) is in position \( s \) and \( k \) denotes the \( 2 \times 2 \) matrix

\[
k = q^{1/2} E^{11} + q^{-1/2} E^{22}.
\]

On the other hand, it is easy to see that

\[
J^m_{(r,s)} = J_r(1, \ldots, E^{mn}, \ldots, 1),
\]

where, now, \( E^{mn} \) is in position \( s \) and 1 denotes the \( 2 \times 2 \) unit matrix. In view of Eq. (C.3), this implies that the operators \( J^m_{(r,s)} \) and \( J_0 \) generate the associative operator algebra \( \rho^L(U_q(sl(2))) \).

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