Nambu-Eliashberg theory for multi-scale quantum criticality: Application to ferromagnetic quantum criticality in the surface of three dimensional topological insulators

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We develop an Eliashberg theory for multi-scale quantum criticality, considering ferromagnetic quantum criticality in the surface of three dimensional topological insulators. Although an analysis based on the random phase approximation has been performed for multi-scale quantum criticality, an extension to an Eliashberg framework was claimed to be far from triviality in respect that the self-energy correction beyond the random phase approximation, which originates from scattering with \( z = 3 \) longitudinal fluctuations, changes the dynamical exponent \( z = 2 \) in the transverse mode, explicitly demonstrated in nematic quantum criticality. A novel ingredient of the present study is to introduce an anomalous self-energy associated with the spin-flip channel. Such an anomalous self-energy turns out to be essential for self-consistency of the Eliashberg framework in the multi-scale quantum critical point because this off diagonal self-energy cancels the normal self-energy exactly in the low energy limit, preserving the dynamics of both \( z = 3 \) longitudinal and \( z = 2 \) transverse modes. This multi-scale quantum criticality is consistent with that in a perturbative analysis for the nematic quantum critical point, where a vertex correction in the fermion bubble diagram cancels a singular contribution due to the self-energy correction, maintaining the \( z = 2 \) transverse mode. We also claim that this off diagonal self-energy gives rise to an artificial electric field in the energy-momentum space in addition to the Berry curvature. We discuss the role of such an anomalous self-energy in the anomalous Hall conductivity.

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I. INTRODUCTION

Investigation on the role of the momentum-space Berry curvature has driven an interesting direction of research in modern condensed matter physics, covering weak antilocalization, anomalous charge and spin Hall effects, and topological insulators [1–3]. A recent trend is to study interplay between the topological band structure and interaction, suggesting that topological excitations such as skyrmions and vortices carry fermion quantum numbers of electric charge or spin via the associated topological term [4].

Quantum criticality has also driven an important direction of research in modern condensed matter physics, particularly focusing on the nature of non-Fermi liquid physics and mechanism of superconductivity out of the non-Fermi liquid state [5]. Combined with the topological band structure, critical degrees of freedom given by topological excitations in the duality picture will carry fermion quantum numbers, thus a novel type of non-Fermi liquid physics may arise.

In this paper we study ferromagnetic quantum criticality in the surface of three dimensional topological insulators from the side of a disordered phase. A recent analysis based on the random phase approximation (RPA) has shown a multi-scale quantum critical point, where longitudinal spin fluctuations are described by the dynamical critical exponent \( z = 3 \) while transverse modes are given by \( z = 2 \) [6]. Although scaling can be demonstrated within this analysis, stability of the RPA dynamics is not guaranteed beyond this approximation. In order to describe scaling near quantum criticality, one should go beyond the perturbative analysis, introducing quantum corrections fully self-consistently. An Eliashberg theory is desirable, where self-energy corrections are taken into account fully self-consistently in the one loop level [7]. The scaling expression of the free energy can be derived within the Eliashberg approximation, where critical exponents are determined [8]. Unfortunately, an extension to an Eliashberg framework was claimed to be far from triviality in multi-scale quantum criticality because the self-energy correction, which originates from scattering with \( z = 3 \) critical fluctuations, turns out to change the dynamical exponent \( z = 2 \) in the transverse mode, explicitly demonstrated in nematic quantum criticality [9].

In this study we construct an Eliashberg theory for multi-scale quantum criticality, considering ferromagnetic quantum criticality in the topological surface. A novel ingredient is to introduce an anomalous self-energy associated with the spin-flip channel. Such an anomalous self-energy turns out to be essential for self-consistency of the Eliashberg framework because this off diagonal self-energy cancels the normal self-energy exactly in the low energy limit, preserving the dynamics of both \( z = 3 \) and \( z = 2 \) critical modes. This multi-scale quantum criticality is consistent with that in a perturbative analysis of the nematic quantum critical point, where a vertex correction in the fermion bubble diagram cancels a singular contribution due to the self-energy correction, maintain-
II. HERZ-MORIYA-MILLIS THEORY IN THE SURFACE OF THE TOPOLOGICAL INSULATOR

A. Model

We start from an effective field theory for ferromagnetic quantum criticality in the surface of topological insulators

$$Z = \int D\psi_i D\phi_i e^{-\int_0^\beta d\tau \int d^2r L},$$

$$L = \bar{\psi}((\partial_\tau - \mu)\delta_{a\alpha'} + iv\mathbf{\nabla} \cdot (\sigma \times \nabla)_{a\alpha'})\psi_{a'},$$

$$+|\partial_\tau \phi_i|^2 + V_0^2|\nabla \phi_i|^2 + m^2|\phi_i|^2 + V(|\phi_i|)$$

$$+g\bar{\psi}_\alpha \sigma_{\alpha\alpha'} \psi_{\alpha'}$$,

(1)

where $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ represents an electron field in the Nambu-spinor representation and $\phi = \phi_x \hat{x} + \phi_y \hat{y}$ corresponds to a spin excitation. We consider the case of an XY spin for simplicity, which can be generalized into the case of O(3) spin. $v_F$ is the Dirac velocity and $\mu$ is the chemical potential for surface electrons. $\sigma$ is the Pauli matrix, associated with the spin state. $\nu_b$ is the spin-fluctuation velocity and $m_b^2$ is the inverse of correlation-length for spin fluctuations, corresponding to an XY ordered state when $m_b^2 < 0$ and a quantum disordered phase when $m_b^2 > 0$. $V(|\phi_i|)$ is an effective potential for spin fluctuations, where an easy plane anisotropy allows the XY spin dynamics only. $g$ is an effective coupling constant between electrons and spin excitations, where tuning $g$ leads the spin correlation-length to diverge, resulting in quantum criticality. The quantum critical point between the XY ordered and disordered phases is defined by the vanishing effective mass for spin fluctuations, tuned by the coupling parameter $g$.

This effective field theory would be realized when an insulating ferromagnet lies on the surface of topological insulators. In this case ferromagnetic spin fluctuations denoted by $\phi_i$ describe those in the insulating ferromagnet. Then, the coupling parameter $g$ represents the strength of the couplings between spins in the ferromagnet and itinerant electrons on the topological surface. Another similar setting is that ferromagnetism will be induced by doped magnetic impurities. It was demonstrated that Ruderman-Kittel-Kasuya-Yosida interactions between doped magnetic impurities are ferromagnetic. Through this exchange mechanism, magnetic atoms are expected to form a ferromagnetically ordered film, deposited uniformly on the surface of a topological insulator. In this case a possible ferromagnetic transition may be realized, controlling the distance of doped impurities in the regular impurity pattern.

We see two important features in dynamics of electrons on the topological surface state. The first is that spin dynamics is quenched to the orbital motion of surface electrons. As a result, one finds that the role of an external magnetic field differs from that in graphene, where the pseudo-spin quantum number is constrained to the orbital motion. In particular, one of the authors could reveal that the Kondo effect and Friedel oscillation around the magnetic impurity should be modified in the presence of an external magnetic field, compared with the graphene structure. The second aspect is more fundamental, that is, only an odd number of Dirac fermions is allowed to appear in the topological surface state. As a result, quantum anomaly is realized, giving rise to nontrivial topological properties of the system. For example, a vortex in the XY ordered state will carry an electric charge, giving rise to an anomalous Hall effect, different from the "conventional" anomalous Hall effect in conventional ferromagnets. It should be noted that this phenomenon does not occur in usual condensed matter systems because the quantum anomaly is often cancelled by the so called fermion doubling effect.

Recently, interaction effects have been investigated in the graphene structure, proposing an interesting phase diagram based on the quantum Monte Carlo simulation. Immediately, such a phase diagram was interpreted in various analytical scenarios, where the singlet-channel interaction was emphasized. On the other hand, we focus on the role of spin-flip scattering, allowed by interactions in the spin-triplet channel. We introduce an anomalous electron self-energy associated with the spin-flip scattering, and discuss its possible implication in connection with the topological nature of the surface state.

B. Eliashberg approximation

One can perform an Eliashberg approximation for the effective Lagrangian [Eq. (1)] at the quantum critical point ($m_b^2 = 0$), where both fermion and boson self-energy corrections are determined fully self-consistently in the one-loop level. Considering the cumulant expansion up to the second order, we can find an effective action.
\[ S_{QC} = \int d^3x \int d^3x' \left( \psi_\alpha^\dagger(x) \left( (\partial_\tau - \mu)\delta_{\alpha\alpha'} + i v_f \hat{z} \cdot (\sigma \times \nabla)_{\alpha\alpha'} \right) \delta(x-x') + \Sigma_{\alpha\alpha'}(x-x') \psi_{\alpha'}(x') \right) 
- N_\sigma \Sigma_{\alpha\alpha'}(x-x') G_{\alpha\alpha'}(x-x') \right) 
+ \int d^3x \int d^3x' \left( \phi_i(x) \left[ (-\partial_\tau^2 - v_f^2 \nabla^2) \delta(x-x') \delta_{ij} + \Pi_{ij}(x-x') \phi_j(x') \right. \right. 
- \Pi_{ij}(x-x') D_{ij}(x-x') \right) 
+ \frac{g^2}{2} \int d^3x \int d^3x' D_{ij}(x-x') \text{tr} \left( \sigma_i G(x-x') \sigma_j G(x'-x) \right). \] (2)

\[ \Sigma_{\alpha\alpha'}(x-x') \text{ and } G_{\alpha\alpha'}(x-x') \text{ are the electron self-energy and Green's function, respectively, where } \alpha \text{ and } \alpha' \text{ represent spin states, } \uparrow \text{ and } \downarrow. \] \[ \Pi_{ij}(x-x') \text{ and } D_{ij}(x-x') \text{ are the spin-fluctuation self-energy and Green's function, respectively, where } i \text{ and } j \text{ express spin directions, } x \text{ and } y. \] All repeated indices are summed. The last term is called the Luttinger-Ward functional \[16\], which is the key ingredient for an interaction effect obtained in the Eliashberg approximation.

Integrating over electrons and spin fluctuations, we obtain an effective free energy as a functional for self-energies

\[ F_{LW}[\Sigma(i\omega), \Pi(q, i\Omega)] = -\frac{N_\sigma}{\beta} \sum_{i\omega} \int \frac{d^2k}{(2\pi)^2} \text{tr}_{\alpha\alpha'} \left\{ \ln[\left| G^{-1}(k, i\omega) \right| + \Sigma(k, i\omega) G(k, i\omega)] \right\} 
+ \frac{1}{\beta} \sum_{i\omega} \int \frac{d^2q}{(2\pi)^2} \text{tr}_{ij} \left\{ \ln[\left| D^{-1}(q, i\Omega) \right| - \Pi(q, i\Omega) D(q, i\Omega)] \right\} 
+ N_\sigma \frac{g^2}{2} \sum_{i\omega} \int \frac{d^2q}{(2\pi)^2} \frac{1}{\beta} \sum_{i\omega} \int \frac{d^2k}{(2\pi)^2} D_{ij}(q, i\Omega) \text{tr}_{\alpha\alpha'} \left( \sigma_i G(k + q, i\omega + i\Omega) \sigma_j G(k, i\omega) \right), \] (3)

where

\[ G(k, i\omega) = \left( g^{-1}(k, i\omega) - \Sigma(k, i\omega) \right)^{-1}, \]

\[ D(q, i\Omega) = \left( \left| \Omega^2 + v_b^2 q^2 \right| I + \Pi(q, i\Omega) \right)^{-1} \] (4)

are fully renormalized propagators of electrons and spin fluctuations, respectively.

\[ g(k, i\omega) = \frac{(i\omega + \mu)I + v_f \epsilon_{ij} k_i \sigma_j}{(i\omega + \mu)^2 - v_b^2 k^2} \]

is the bare propagator of electrons, where \( \epsilon_{ij} \) is the two dimensional antisymmetric tensor with \( i,j = x,y \). \( N_\sigma \) represents the spin degeneracy, in our case \( N_\sigma = 2 \). \( \text{tr}_{\alpha\alpha'} \) and \( \text{tr}_{ij} \) mean trace for spin states and spin directions, respectively.

Minimizing the free energy functional with respect to both self-energy corrections, we obtain fully self-consistent coupled Eliashberg equations for electron and boson self-energies

\[ \Pi_{ij}(q, i\Omega) = N_\sigma \frac{g^2}{2} \sum_{i\omega} \int \frac{d^2k}{(2\pi)^2} \text{tr}_{\alpha\alpha'} \left( \sigma_i G(k + q, i\omega + i\Omega) \sigma_j G(k, i\omega) \right), \]

\[ \Sigma(k, i\omega) = \frac{g^2}{\beta} \sum_{i\omega} \int \frac{d^2q}{(2\pi)^2} D_{ij}(q, i\Omega) \sigma_i G(k + q, i\omega + i\Omega) \sigma_j. \] (5)

As discussed before, the triplet interaction channel gives rise to the spin-flip scattering, corresponding to an off-diagonal term in the electron self-energy matrix. The normal Eliashberg self-energy is proposed to depend on
frequency only because the momentum dependence is regular and the singular behavior can be shown from the frequency dependence \[2, 17\]. On the other hand, the momentum dependence for the anomalous self-energy has not been clarified yet. We propose the following ansatz for the electron self-energy

\[
\Sigma(k, i\omega) = \Sigma(i\omega)I + \Phi(i\omega)c_{ij}\hat{k}_i\sigma_j,
\]

(6)

where the momentum dependence for the off-diagonal self-energy is given by the same dependence as the topological band structure. In other words, the off-diagonal self-energy is added to the dispersion relation as the normal self-energy in the conventional system. \(\hat{k} = k/|k|\) is the unit vector.

\[
\phi = \left(C_p g^2 \frac{|\Omega|}{v_f|q|} + v_f^2|q|^2 \right)\phi_i(q, i\Omega)P_{ij}\phi_j(-q, -i\Omega) + \left(C_T g^2 \frac{\Omega^2}{v_f^2|q|^2} + v_T^2|q|^2 \right)\phi_i(q, i\Omega)(\delta_{ij} - P_{ij})\phi_j(-q, -i\Omega)\quad (7)
\]

where

\[
P_{ij} = \frac{q_iq_j}{|q|^2}
\]

is the projection operator to satisfy \(P_{ik}P_{kj} = P_{ij}\). \(C_{p(T)}\) is a constant of the order of 1 and \(v_{L(T)}\) is the renormalized velocity for spin fluctuations, where \(L(T)\) represents the longitudinal (transverse) mode. An important feature for spin dynamics is multi-scale quantum criticality, where the longitudinal spin dynamics is given by the dynamical exponent \(z = 3\) while the transverse one is described by \(z = 2\). The multi-scale quantum criticality can be interpreted as follows. The longitudinal spin dynamics is not involved with the spin-flip scattering, described by the \(z = 3\) ferromagnetic quantum criticality. On the other hand, the transverse dynamics is associated with the spin-flip process. In the topological surface state this process appears with finite momentum transfer, similar to the process with antiferromagnetic fluctuations. As a result, the transverse spin dynamics is described by \(z = 2\). We note that the self-consistent Eliashberg approximation gives essentially the same result as the random phase approximation \[6\].

C. Polarization function

Inserting the self-energy expression into the Green’s function, we obtain

\[
G(k, i\omega) = \frac{[i\omega + \mu - \Sigma(i\omega)]I + [v_f|k| + \Phi(i\omega)]c_{ij}\hat{k}_i\sigma_j}{[i\omega + \mu - \Sigma(i\omega)]^2 - [v_f|k| + \Phi(i\omega)]^2}
\]

Based on this expression, we can evaluate the boson self-energy, given by the fermion polarization bubble. A detailed procedure is shown in appendix A.

It turns out that spin dynamics at the \(XY\) ferromagnetic quantum critical point is given by the following effective Lagrangian.

Our novel point is that the anomalous self-energy correction is exactly the same as the normal one, given by

D. Electron self-energy

The spin-fluctuation propagator is given by

\[
D_{ij}(q, i\Omega) = \frac{P_{ij}}{C_p g^2 \frac{|\Omega|}{v_f|q|} + v_f^2|q|^2} + \frac{\delta_{ij} - P_{ij}}{C_T g^2 \frac{\Omega^2}{v_f^2|q|^2} + v_T^2|q|^2}
\]

\[
\equiv D_L(q, i\Omega)P_{ij} + D_T(q, i\Omega)(\delta_{ij} - P_{ij}),
\]

(8)
where the last approximate equality is given by the $z = 3$ longitudinal mode $D_L(q,i\Omega)$. This equivalence is at the heart of the self-consistency in the Nambu-Eliashberg framework. As discussed in the introduction, the self-energy correction in the fermion bubble diagram changes the $z = 2$ dynamical exponent in the transverse mode. Actually, one can see this effect from Eqs. (A9) and (A13) in appendix A. If the anomalous self-energy is not introduced, the normal self-energy contribution dominates over the bare frequency, modifying the dynamical exponent of the transverse mode from $z = 2$ to $z = 12/5$. However, introduction of the anomalous self-energy cancels the normal self-energy exactly in the low energy limit, recovering the RPA result $\frac{5}{3}$.

It is interesting to compare the mechanism of cancellation with the perturbative analysis in nematic quantum criticality, where next leading corrections given by electron self-energy and Maki-Thompson vertex corrections are introduced for dynamics of critical bosonic modes $\frac{5}{3}$. Since the fermion self-energy is driven by $z = 3$ critical fluctuations, the self-energy correction in the transverse mode, corresponding to the $z = 2$ dynamics in the RPA level, gives rise to an additional singular correction, changing the dynamical critical exponent from $z = 2$ to $z = 12/5$. This weird result is cured by the Maki-Thompson vertex correction, where the singular correction of the self-energy is cancelled by the singular vertex correction exactly, which is a fundamental cancellation based on the Ward identity.

The self-consistency of the Nambu-Eliashberg theory is far from triviality for multi-scale quantum criticality, and this effective field theory is certainly beyond the simple Eliashberg description in respect that vertex corrections are introduced in some sense. However, it turns out that critical exponents for thermodynamics is completely the same as the conventional Eliashberg theory without an anomalous self-energy correction $\frac{10}{3}$.

\begin{equation}
\Sigma(i\omega) = -\Phi(i\omega) \approx -i g^2 \frac{1}{2 \beta} \sum_{\Omega} dq \left( D_L(q,i\Omega) + D_R(q,i\Omega) \right) \frac{\text{sgn}(\omega + \Omega)}{\sqrt{\omega + \Omega}^2 + v f q^2} \propto |\omega|^{2/3},
\end{equation}

\[ (9) \]

E. Stability of the Eliashberg framework

The Eliashberg theory is non-perturbative in respect that quantum corrections are introduced fully self-consistently in the one-loop level, consistent with one-loop renormalization group analysis. Indeed, the scaling expression of the free energy was derived explicitly, based on the Eliashberg approximation $\frac{5}{3}$. In this respect the Eliashberg theory for quantum criticality implies the theory with critical exponents given by the Eliashberg approximation, satisfying the scaling theory.

One cautious person may criticize the Eliashberg approximation, neglecting vertex corrections for self-energies. Recently, it has been clarified that two dimensional Fermi surface problems are still strongly interacting even in the large-$N$ limit $\frac{19}{24}$, implying that vertex corrections should be incorporated. An important question is whether these vertex corrections give rise to novel critical exponents beyond the Eliashberg theory. Several perturbative analysis demonstrated that although ladder-type vertex corrections do not change critical exponents of the Eliashberg theory, Aslamasov-Larkin corrections result in modifications for such critical exponents $\frac{21}{22}$. If this is a general feature beyond this level of approximation, various quantum critical phenomena $\frac{25}{28}$ should be reconsidered because novel anomalous exponents in fermion self-energy corrections are expected to cause various novel critical exponents for thermodynamics, transport, and etc. However, considering our recent experiences on comparison with various experiments in heavy fermion quantum criticality, we are surprised at the fact that critical exponents based on the Eliashberg theory explain thermodynamics $\frac{20}{27}$, both electrical and thermal transport coefficients $\frac{27}{24}$, uniform spin susceptibility $\frac{28}{29}$, and Seebeck effect $\frac{29}{24}$ quite well.

Recently, one of the authors investigated the role of vertex corrections non-perturbatively, summing vertex corrections up to an infinite order $\frac{19}{24}$. It turns out that particular vertex corrections given by ladder diagrams do not change Eliashberg critical exponents at all $\frac{24}{24}$, consistent with the perturbative analysis. This was performed in a fully self-consistent way, resorting to the Ward identity. In contrast with the previous perturbative analysis, Aslamasov-Larkin corrections were shown not to modify the Eliashberg dynamics $\frac{19}{19}$. Resorting to the analogy with superconductivity, where the superconducting instability described by the Aslamasov-Larkin vertex correction is reformulated by the anomalous self-energy in the Eliashberg framework of the Nambu spinor representation $\frac{30}{30}$, we claimed that the off-diagonal self-energy associated with the $2k_F$ particle-hole channel incorporates the same (Aslamasov-Larkin) class of quantum corrections in the Nambu spinor representation. We evaluated an anomalous pairing self-energy in the Nambu-Eliashberg approximation, which vanishes at zero energy but displays the same power law dependence for frequency as the normal Eliashberg self-energy. As a result, even the pairing self-energy correction does not modify the Eliashberg dynamics without the Nambu spinor representation, resulting in essentially the same scaling physics for thermodynamics at quantum criticality. However, we admit that this issue should be investigated more sincerely, resorting to a direct summation for
such a class of quantum corrections.

III. ARTIFICIAL ELECTRIC FIELD IN THE ENERGY-MOMENTUM SPACE

An essential question is on the role of the anomalous self-energy in transport. Considering that the off-diagonal self-energy has the same momen
tum dependence as the kinetic-energy term, it is natural to examine its role in the Hall conductivity.

An effective equation of motion for quasi-particle dynamics was proposed as follows [10]

\[ \frac{dR}{dt} = v + (B - E \times v) \times \frac{dk}{dt}, \]
\[ \frac{dk}{dt} = -E + B \times \frac{dR}{dt}, \]

(10)
describing the wave-packet dynamics in the semi-classical level. \( R \) represents the center coordinate of the wave-packet and \( k \) expresses the Bloch wave vector. \( E \) and \( B \) are external electric and magnetic fields while \( \mathcal{E} \) and \( \mathcal{B} \) are artificial electric and magnetic fields in the energy and momentum space. \( B \) is usually referred as the Berry curvature. \( v = \frac{\partial E}{\partial k} \) is the group velocity, where \( E_k \) is the quasi-particle energy dispersion. A key ingredient is that interactions give rise to an artificial electric field in the energy-momentum space, modifying the quasi-particle dynamics according to the Maxwell equa

tion. As a result, the Hall conductivity should be corrected in the following way [10]

\[ \sigma_{xy} = \frac{e^2}{h} \int \frac{d^2k}{(2\pi)^2} f(E_k) \left( \mathcal{B}_z - [v_y \mathcal{E}_x - v_x \mathcal{E}_y] \right), \]

(11)

where the Berry curvature is shifted by \( \mathcal{E} \times v \) due to interactions. \( f(E_k) \) is the Fermi-Dirac distribution function. In this section we calculate this quantity with an introduction of the anomalous self-energy correction.

We consider the following effective Hamiltonian \( H_{eff} = N(\mathbf{k}, \omega) \cdot \sigma \), which incorporates the effect of interactions through the self-energy. The pseudospin vector is given by

\[ N(\mathbf{k}, \omega) = \mathbf{d}(\mathbf{k}) + \Delta \mathbf{d}(\omega), \]
\[ \mathbf{d}(\mathbf{k}) = -v_f k_y \mathbf{\hat{a}}_x + v_f k_x \mathbf{\hat{a}}_y + d_z(\mathbf{k}) \mathbf{\hat{a}}_z, \]
\[ \Delta \mathbf{d}(\omega) = -\Phi(\omega)(k_y/k) \mathbf{\hat{a}}_x + \Phi(\omega)(k_x/k) \mathbf{\hat{a}}_y + \Delta d_z(\omega) \mathbf{\hat{a}}_z, \]

(12)

where the \( \mathbf{d}(\mathbf{k}) \) vector is nothing but the bare dispersion and \( \Delta \mathbf{d}(\omega) \) is associated with the anomalous self-energy correction. \( d_z(\mathbf{k}) \) is introduced for time reversal symmetry breaking as the z-directional magnetic field, set to be \( d_z(\mathbf{k}) \rightarrow m \). \( \Delta d_z(\omega) \) is also introduced for consistency. \( k = \sqrt{k_x^2 + k_y^2} \) is the amplitude of the momentum.

The Berry curvature and artificial electric field are expressed in terms of the pseudospin vector

\[ B_z = \frac{1}{2} \partial_{k_x} \mathbf{\hat{N}} \times \partial_{k_y} \mathbf{\hat{N}} \cdot \mathbf{\hat{N}}, \]
\[ \mathcal{E}_x = \frac{1}{2} \partial_{k_y} \mathbf{\hat{N}} \times \partial_{k_x} \mathbf{\hat{N}} \cdot \mathbf{\hat{N}}, \]
\[ \mathcal{E}_y = \frac{1}{2} \partial_{k_x} \mathbf{\hat{N}} \times \partial_{k_y} \mathbf{\hat{N}} \cdot \mathbf{\hat{N}}, \]

(13)

where \( \mathbf{\hat{N}} = N/|N| \) is the unit vector.

One can understand this expression based on another equivalent representation, where the field strength is expressed by the gauge field. Resorting to the CP\( ^1 \) representation [31]

\[ N \cdot \sigma = |N| U \sigma_3 U^\dagger, \]

(14)

where \( U \) is an SU(2) matrix field to describe a direction of the pseudospin vector, we can define an SU(2) gauge connection

\[ A_\mu = -i [\partial_\mu U^\dagger] U. \]

(15)

This naturally leads to both Berry curvature and artificial electric field

\[ B_z = \partial_{k_z} A_y - \partial_{k_y} A_x, \]
\[ \mathcal{E}_x = \partial_{k_x} A_y - \partial_{k_y} A_x, \]
\[ \mathcal{E}_y = \partial_{k_x} A_y - \partial_{k_y} A_x, \]

(16)

where the U(1) projected component of the SU(2) gauge field is given by \( A_\mu = A^{(3)}_\mu \).

Inserting the pseudospin vector Eq. (12) into Eq. (13), we obtain both Berry curvature and artificial electric field

\[ B_z \approx \frac{1}{2} \left[ \frac{|v_f + \Phi(\omega)/k|^2 [m + \Delta d_z(\omega)]}{(|v_f + \Phi(\omega)/k|^2 + [m + \Delta d_z(\omega)]^2)^{3/2}} \right], \]
\[ \mathcal{E}_x \approx \frac{1}{2} k_x F, \quad \mathcal{E}_y \approx \frac{1}{2} k_y F, \]

where \( F \) is given by

\[ F = -\frac{[\partial_\omega \Phi(\omega)(1/k)[v_f + \Phi(\omega)/k]][m + \Delta d_z(\omega)]}{\left( |v_f + \Phi(\omega)/k|^2 + [m + \Delta d_z(\omega)]^2 \right)^{3/2}} + \frac{|v_f + \Phi(\omega)/k|^2 [\partial_\omega \Delta d_z(\omega)]}{\left( |v_f + \Phi(\omega)/k|^2 + [m + \Delta d_z(\omega)]^2 \right)^{3/2}}. \]

(16)

Considering the group velocity

\[ v_y = k_y \nu, \quad v_x = k_x \nu, \]

with

\[ \nu = \frac{1}{k} \frac{|v_f + \Phi(\omega)| v_f}{\sqrt{|v_f + \Phi(\omega)|^2 + [m + \Delta d_z(\omega)]^2}}, \]

\[ \nu \approx \frac{1}{k} \frac{|v_f + \Phi(\omega)| v_f}{|v_f + \Phi(\omega)|}. \]
we find that the contribution of the electric field $\mathbf{E}$ to the shift of $\mathcal{B}$ in Eq. (11) vanishes identically

$$v_y \mathcal{E}_x - v_x \mathcal{E}_y = 0.$$ 

On the other hand, the Berry curvature should be modified in the presence of the self-energy correction. As a result, we obtain the following expression for the Hall conductivity

$$\sigma_{xy} = \frac{e^2}{2\hbar} \int_0^\infty \frac{dk}{k} \frac{[v_f k + \Phi(\omega)]^2 [m + \Delta d_z(\omega)]}{[v_f k + \Phi(\omega)]^2 + [m + \Delta d_z(\omega)]^2} = \frac{e^2}{2\hbar} \left( F[k_H \to \infty] - F[k_L \to 0] \right), \quad (17)$$

where

$$F[k_H \to \infty] = \frac{[m + \Delta d_z(\omega)]\Phi(\omega)}{[m + \Delta d_z(\omega)]^2 + \Phi^2(\omega)} + \frac{[m + \Delta d_z(\omega)]\Phi^2(\omega)}{\Phi^2(\omega) + [m + \Delta d_z(\omega)]^2} \ln \frac{1}{\Phi(\omega) + \sqrt{\Phi^2(\omega) + [m + \Delta d_z(\omega)]^2}},$$

$$F[k_L \to 0] = \frac{[m + \Delta d_z(\omega)]^2 [m + \Delta d_z(\omega)]\Phi^2(\omega)}{\Phi^2(\omega) + [m + \Delta d_z(\omega)]^2} + \frac{[m + \Delta d_z(\omega)]^2 [m + \Delta d_z(\omega)]\Phi^2(\omega) + \sqrt{\Phi^2(\omega) + [m + \Delta d_z(\omega)]^2}}{\Phi^2(\omega) + [m + \Delta d_z(\omega)]^2} \ln \frac{k_L}{\Phi(\omega) + \sqrt{\Phi^2(\omega) + [m + \Delta d_z(\omega)]^2}}.$$

It is straightforward to see that the noninteracting case in $\Phi(\omega) = 0$ recovers the well known result, that is, the half Hall conductivity $\frac{5}{2}$, where $F[k_H \to \infty] = 0$ and $F[k_L \to 0] = -\text{sgn}(m)$.

An interesting point in this expression is that the Hall conductivity diverges in the $k_L \to 0$ limit, seen from the log term. Although we do not understand the reason for this divergence, it implies that we should incorporate vertex corrections even for the intrinsic topological effect in the Hall conductivity when interactions are introduced. If we take into account vertex corrections, this log divergence may be re-summed to result in the common power-law behavior with an exponent, which vanishes in the $k_L \to 0$ limit.

We suspect that the underlying mechanism for this divergence may be charged vortices. As discussed before, the parity anomaly assigns an electric charge to a vortex excitation $\frac{13}{13}$. From the ordered side in the duality picture, such vortex excitations are well defined particles, thus taken into account with electrons on an equal footing. An interplay between electrons and vortices would be described by the mutual Chern-Simons theory $\frac{5}{5}$, where their mutual statistics is guaranteed by the mutual Chern-Simons term. It is not clear at all how such an interplay affects transport, in particular, the Hall conductivity. It will be an interesting future direction to investigate this problem in the duality picture because topological excitations carry nontrivial fermion quantum numbers in this situation.

IV. DISCUSSION AND SUMMARY

In this study we constructed an Eliashberg framework for the Hertz-Moriya-Millis theory $\frac{3}{3}$, describing ferromagnetic quantum criticality in the surface state of three dimensional topological insulators. An idea is to introduce an anomalous self-energy correction beyond the previous study $\frac{10}{10}$, resulting from spin-flip scattering. We could show that our ansatz for the anomalous self-energy allows a set of fully self-consistent solutions for electrons and spin fluctuations.

The spin-fluctuation dynamics was shown to coincide with the solution based on the random phase approximation, where both $z = 3$ and $z = 2$ quantum critical dynamics arise $\frac{4}{4}$. The $z = 3$ quantum criticality describes the longitudinal spin dynamics, where only forward scattering contributions result in the $z = 3$ Landau damping, while the $z = 2$ quantum criticality for transverse spin fluctuations originates from the spin-flip scattering, involved with the finite-momentum transfer due to the spin-orbital quenching.

This multi-scale quantum criticality was shown to allow the self-consistent Nambu-Eliashberg solution for the electron dynamics. The normal self-energy turns out to follow the $z = 3$ quantum criticality because such critical fluctuations give rise to more singular contributions than the $z = 2$ transverse mode. We found that the anomalous self-energy shows exactly the same frequency dependence as the normal self-energy although the underlying mechanism is not completely clear. The equivalence between
the normal and anomalous self-energies with a different sign is at the heart of the self-consistency in the Nambu-Eliashberg theory for multi-scale quantum criticality. In other words, the simple Eliashberg theory without the off diagonal self-energy cannot be self-consistent for multi-scale quantum criticality. We argued that the mechanism of cancellation between the normal and anomalous self-energies in the fermion bubble diagram may be rooted in the Ward identity, comparing our result with the perturbative analysis for nematic quantum criticality, where the singular self-energy correction is cancelled by the vertex correction based on the Ward identity [3].

We demonstrated that such an off-diagonal self-energy gives rise to an artificial electric field in the energy-momentum space beyond the static Berry curvature from the topological band structure. In particular, we calculated the anomalous Hall conductivity in terms of both Berry curvature and artificial electric field. Although the contribution from the artificial electric field vanishes identically, we could find that the Berry curvature is modified due to the anomalous self-energy correction, resulting in the log-divergence for the Hall conductivity. We argued that the log-divergence may disappear if vertex corrections are re-summed up to an infinite order, which is expected to turn the log-divergence into the power-law dependence, vanishing algebraically. We suggested an idea that this phenomenology may be related with charged vortices in the duality picture, where the electron quantum number of the vortex results from the quantum anomaly, an essential feature of the topological surface state.

Before closing, we discuss the stability of the Eliashberg approximation in another aspect, considering an interesting recent study for nematic quantum criticality [18]. The nematic quantum critical point is well known to show multi-scale quantum criticality, where critical dynamics of nematic fluctuations is governed by both critical fluctuations dominantly, scattering with $z = 3$ critical fluctuations. We argued that the mechanism of cancellation between the normal and anomalous self-energies with a different sign is at the heart of the self-consistency in the Nambu-Eliashberg theory for multi-scale quantum criticality. In other words, the simple Eliashberg theory without the off diagonal self-energy cannot be self-consistent for multi-scale quantum criticality. We argued that the mechanism of cancellation between the normal and anomalous self-energies in the fermion bubble diagram may be rooted in the Ward identity, comparing our result with the perturbative analysis for nematic quantum criticality, where the singular self-energy correction is cancelled by the vertex correction based on the Ward identity [3].

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Before closing, we discuss the stability of the Eliashberg approximation in another aspect, considering an interesting recent study for nematic quantum criticality [18]. The nematic quantum critical point is well known to show multi-scale quantum criticality, where critical dynamics of nematic fluctuations is governed by both $z = 3$ and $z = 2$. An important notice is that although the self-energy correction results from scattering with $z = 3$ critical fluctuations dominantly, scattering with $z = 2$ critical modes gives rise to the logarithmic singularity for the quasi-particle residue $D_0$, which means that vertex corrections should be introduced up to an infinite order. Based on a complicated renormalization group argument, the authors proposed an interesting expression for the electron Green’s function

$$G(k, i\omega) \propto \frac{D_0^{-\eta}}{|i\omega + \Sigma(i\omega) - \epsilon_k|^{1-\eta}}.$$  

$D_0$ has a constant of an energy unit proportional to the Fermi energy, and $\Sigma(i\omega) \propto |\omega|^{2/3}$ is the $z = 3$ self-energy. $\epsilon_k$ is the band dispersion. It should be noted that the anomalous exponent $\eta$ results from scattering with $z = 2$ critical fluctuations.

In the present system a significant quantity is the off diagonal self-energy. Although such an anomalous self-energy will not be allowed in nematic quantum criticality due to the absence of the spin-orbit coupling, it may play an important role for the exponent $\eta$ in ferromagnetic quantum criticality of the topological surface. A more complete framework would be to introduce vertex corrections fully self-consistently with self-energy corrections in the Nambu-spinor representation, where the off-diagonal self-energy contribution is also included.

In summary, quantum criticality with the topological band structure is expected to open a novel direction of research in respect that the interplay between interaction and topology may cause unexpected non-Fermi liquid physics. We found an anomalous behavior in the intrinsic topological contribution of the Hall coefficient, the source of which is the energy-momentum-space field strength beyond the static Berry curvature as a result of the interplay between quantum criticality and topological structure.

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Appendix A: Polarization function

In appendix A we derive the self-energy correction for spin fluctuations, where the longitudinal spin dynamics is given by $z = 3$ while the transverse one is described by $z = 2$. Inserting the electron Green’s function with the ansatz for the electron self-energy matrix into the Eliashberg equation for the boson self-energy, we obtain

\begin{equation}
\Pi_{ij}(q, i\Omega) = \frac{q^2}{2} \int \frac{d^2k}{(2\pi)^2} \text{tr} \left\{ \sigma_i \frac{[i\omega + \Sigma(i\omega) + \mu - \Sigma(i\omega + i\Omega)]I + [v_f|k + q ] + \Phi(i\omega + i\Omega)\epsilon_{nm}(\hat{k}_n + \hat{q}_n)\sigma_m}{[i\omega + i\Omega + \mu - \Sigma(i\omega + i\Omega)]^2 - [v_f|k + q ] + \Phi(i\omega + i\Omega)} \right\} \sigma_j \frac{[i\omega + \mu - \Sigma(i\omega)]I + [v_f|k ] + \Phi(i\omega)\epsilon_{m'n'}\hat{k}_n\sigma_{m'}}{[i\omega + \mu - \Sigma(i\omega)]^2 - [v_f|k ] + \Phi(i\omega)} \right\},
\end{equation}

(A1)
where the following approximation for the momentum
\[
\hat{k}_n + \hat{q}_n = \frac{k_n + q_n}{|k + q|} \approx \frac{k_n}{|k_f|} + \frac{q_n}{|k_f|}
\]  
(A2)
is utilized.

Resorting to identities for Pauli matrix products
\[
\begin{align*}
\sigma_i \sigma_j &= \delta_{ij} I + i \epsilon_{ij} \sigma_i, \\
\sigma_i \sigma_m \sigma_j &= \delta_{im} \sigma_j + i \epsilon_{imj} I - (\delta_{ij} \delta_{ms} - \delta_{is} \delta_{mj}) \sigma_s, \\
\sigma_i \sigma_m \sigma_j \sigma_{m'} &= \delta_{im} \left( \delta_{jm'} I + i \epsilon_{jm'v} \sigma_v \right) + i \epsilon_{imj} \sigma_{m'} - (\delta_{ij} \delta_{ms} - \delta_{is} \delta_{mj}) \left( \delta_{sm'} I + i \epsilon_{sm'v} \sigma_v \right)
\end{align*}
\]  
(A3)
and performing the decomposition for the denominator, we can rewrite Eq. (A1) as follows
\[
\Pi_{ij}(q, i\Omega) = N_o g^2 \left( \frac{1}{\beta} \int \frac{d^2 k}{(2\pi)^2} \right) \frac{1}{2 |i\omega + \mu - \Sigma(i\omega + i\Omega)|^2} \frac{1}{2 |i\omega + \mu - \Sigma(i\omega)|^2}
\]
\[
\left\{ G[i\omega + i\Omega, v_f | k + q] + \Phi(i\omega + i\Omega)]G[i\omega, v_f | k] + \Phi(i\omega)] + G[i\omega + i\Omega, v_f | k - q] - \Phi(i\omega + i\Omega)]G[i\omega, v_f | k + q] + \Phi(i\omega)] \right\}
\]
\[
+ G[i\omega + i\Omega, v_f | k + q] + \Phi(i\omega + i\Omega)]G[i\omega, v_f | k] - \Phi(i\omega)] + G[i\omega + i\Omega, v_f | k - q] - \Phi(i\omega + i\Omega)]G[i\omega, v_f | k + q] - \Phi(i\omega)]
\]
\[
+ G[i\omega + i\Omega, v_f | k + q] + \Phi(i\omega + i\Omega)]G[i\omega, v_f | k] - \Phi(i\omega)] + G[i\omega + i\Omega, v_f | k - q] - \Phi(i\omega + i\Omega)]G[i\omega, v_f | k + q] - \Phi(i\omega)]
\]
\[
+ N_o g^2 \left( \frac{1}{\beta} \int \frac{d^2 k}{(2\pi)^2} \right) \{ - (\hat{k} + \hat{q}) \cdot \hat{k} + \frac{\epsilon_{in} \epsilon_{in'} (\hat{k}_n + \hat{q}_n) \hat{k}_{n'} + \epsilon_{jn} \epsilon_{jn'} (\hat{k}_n + \hat{q}_n) \hat{k}_{n'} \} \}
\]
\[
\left\{ G[i\omega + i\Omega, v_f | k + q] + \Phi(i\omega + i\Omega)]G[i\omega, v_f | k] + \Phi(i\omega)] - G[i\omega + i\Omega, v_f | k - q] - \Phi(i\omega + i\Omega)]G[i\omega, v_f | k] - \Phi(i\omega)]
\]
\[
- G[i\omega + i\Omega, v_f | k + q] + \Phi(i\omega + i\Omega)]G[i\omega, v_f | k] - \Phi(i\omega)] + G[i\omega + i\Omega, v_f | k - q] - \Phi(i\omega + i\Omega)]G[i\omega, v_f | k] - \Phi(i\omega)] \}
\]  
(A4)
where \( \epsilon_{ij} \) is an antisymmetric tensor with \( i, j = x, y, \) and
\[
G[i\omega, \chi] = \frac{1}{i\omega + \mu - \Sigma(i\omega) - \chi}
\]
We rearrange the above expression as
\[
\Pi_{ij}(q, i\Omega) = N_o g^2 \left( \frac{1}{\beta} \int \frac{d^2 k}{(2\pi)^2} \right) \delta_{ij}
\]
\[
\left\{ G[i\omega + i\Omega, v_f | k + q] + \Phi(i\omega + i\Omega)]G[i\omega, v_f | k] + \Phi(i\omega)] + G[i\omega + i\Omega, v_f | k - q] - \Phi(i\omega + i\Omega)]G[i\omega, v_f | k + q] - \Phi(i\omega)] \right\}
\]
\[
N_o g^2 \left( \frac{1}{\beta} \int \frac{d^2 k}{(2\pi)^2} \right) \{ - (\hat{k} + \hat{q}) \cdot \hat{k} + \frac{\epsilon_{in} \epsilon_{in'} (\hat{k}_n + \hat{q}_n) \hat{k}_{n'} + \epsilon_{jn} \epsilon_{jn'} (\hat{k}_n + \hat{q}_n) \hat{k}_{n'} \} \}
\]
\[
\left\{ G[i\omega + i\Omega, v_f | k + q] + \Phi(i\omega + i\Omega)]G[i\omega, v_f | k] + \Phi(i\omega)] - G[i\omega + i\Omega, v_f | k - q] - \Phi(i\omega + i\Omega)]G[i\omega, v_f | k] - \Phi(i\omega)]
\]
\[
- G[i\omega + i\Omega, v_f | k + q] + \Phi(i\omega + i\Omega)]G[i\omega, v_f | k] - \Phi(i\omega)] + G[i\omega + i\Omega, v_f | k - q] - \Phi(i\omega + i\Omega)]G[i\omega, v_f | k] - \Phi(i\omega)] \}
\]  
(A5)
where the frequency term in the numerator is reorganized.

It is convenient for the momentum integration to linearize the band dispersion near the chemical potential
\[
\mu - v_f |k + q| \approx -v_f \hat{k}_f \cdot (k - k_f) - v_f \hat{k}_f \cdot q,
\]  
(A6)
where the Fermi momentum \( k_f = |k_f| \hat{k}_f \) is defined from \( \mu - v_f |k_f| = 0 \).
Inserting the linearized band into Eq. (A5) and performing the decomposition for the denominator, we obtain

\[ \Pi_{ij}(q, i\Omega) \approx N_\sigma \frac{g^2}{4} \frac{1}{\beta} \sum_{i\omega} \int \frac{d^2k}{(2\pi)^2} \delta_{ij} \frac{1}{i\Omega - [\Sigma(i\omega + i\Omega) - \Sigma(i\omega)] - [\Phi(i\omega + i\Omega) - \Phi(i\omega)] - v_f \hat{k}_f \cdot \mathbf{q}} \]

\[ + N_\sigma \frac{g^2}{4} \frac{1}{\beta} \sum_{i\omega} \int \frac{d^2k}{(2\pi)^2} \left\{ \frac{1}{i\omega - \Sigma(i\omega) - \Phi(i\omega)} - \frac{1}{i\omega + \Sigma(i\omega) + i\Omega - \Phi(i\omega + i\Omega)} \right\} - v_f \hat{k}_f \cdot (k - k_f) - v_f \hat{k}_f \cdot \mathbf{q} \]

\[ + N_\sigma \frac{g^2}{4} \frac{1}{\beta} \sum_{i\omega} \int \frac{d^2k}{(2\pi)^2} \left\{ \frac{1}{i\omega - \Sigma(i\omega) - \Phi(i\omega)} - \frac{1}{i\omega + \Sigma(i\omega) + i\Omega - \Phi(i\omega + i\Omega)} \right\} - v_f \hat{k}_f \cdot (k - k_f) - v_f \hat{k}_f \cdot \mathbf{q} \]

where only dominant contributions are selected, which means that all terms with the chemical potential in the denominator are safely neglected.

Before evaluating integrals, we summarize each component for the spin-fluctuation self-energy as

\[ \Pi_{xx}(q, i\Omega) = N_\sigma \frac{g^2}{4} \frac{1}{\beta} \sum_{i\omega} \int \frac{d^2k}{(2\pi)^2} \frac{1}{i\Omega - [\Sigma(i\omega + i\Omega) - \Sigma(i\omega)] - [\Phi(i\omega + i\Omega) - \Phi(i\omega)] - v_f \hat{k}_f \cdot \mathbf{q}} \]

\[ + N_\sigma \frac{g^2}{4} \frac{1}{\beta} \sum_{i\omega} \int \frac{d^2k}{(2\pi)^2} \left\{ \frac{1}{i\omega - \Sigma(i\omega) - \Phi(i\omega)} - \frac{1}{i\omega + \Sigma(i\omega) + i\Omega - \Phi(i\omega + i\Omega)} \right\} - v_f \hat{k}_f \cdot (k - k_f) - v_f \hat{k}_f \cdot \mathbf{q} \]

\[ \Pi_{yy}(q, i\Omega) = N_\sigma \frac{g^2}{4} \frac{1}{\beta} \sum_{i\omega} \int \frac{d^2k}{(2\pi)^2} \frac{1}{i\Omega - [\Sigma(i\omega + i\Omega) - \Sigma(i\omega)] - [\Phi(i\omega + i\Omega) - \Phi(i\omega)] - v_f \hat{k}_f \cdot \mathbf{q}} \]

\[ + N_\sigma \frac{g^2}{4} \frac{1}{\beta} \sum_{i\omega} \int \frac{d^2k}{(2\pi)^2} \left\{ \frac{1}{i\omega - \Sigma(i\omega) - \Phi(i\omega)} - \frac{1}{i\omega + \Sigma(i\omega) + i\Omega - \Phi(i\omega + i\Omega)} \right\} - v_f \hat{k}_f \cdot (k - k_f) - v_f \hat{k}_f \cdot \mathbf{q} \]

\[ \Pi_{xy}(q, i\Omega) = \Pi_{yx}(q, i\Omega) = -N_\sigma \frac{g^2}{4} \frac{1}{\beta} \sum_{i\omega} \int \frac{d^2k}{(2\pi)^2} \frac{2\hat{k}_x \hat{k}_y + (\hat{q}_x \hat{k}_y + \hat{q}_y \hat{k}_x)}{i\Omega - [\Sigma(i\omega + i\Omega) - \Sigma(i\omega)] - [\Phi(i\omega + i\Omega) - \Phi(i\omega)] - v_f \hat{k}_f \cdot \mathbf{q}} \]

First, we consider the diagonal component of the boson self-energy. It is convenient for the momentum integration to take the angular coordinate

\[ \Pi_{xx}(q, i\Omega) = N_\sigma \frac{g^2}{4v_f(2\pi)^2} \frac{1}{\beta} \sum_{i\omega} \int_0^{2\pi} d\theta \int_{-\infty}^{\infty} d\omega \frac{2\sin^2 \theta}{i\Omega - [\Sigma(i\omega + i\Omega) - \Sigma(i\omega)] - [\Phi(i\omega + i\Omega) - \Phi(i\omega)] - v_f q_x \cos \theta - v_f q_y \sin \theta} \]

\[ + N_\sigma \frac{g^2}{4v_f(2\pi)^2} \int_0^{2\pi} d\omega \frac{2\sin^2 \theta [\text{sgn}(\omega) - \text{sgn}(\omega + \Omega)]}{i\Omega - [\Sigma(i\omega + i\Omega) - \Sigma(i\omega)] - [\Phi(i\omega + i\Omega) - \Phi(i\omega)] - v_f q_x \cos \theta - v_f q_y \sin \theta} \]

where we considered the zero temperature limit in the last equality.
The angular integration is given by
\[
\int_0^{2\pi} d\theta \frac{\sin^2 \theta}{i\Omega' - v_f q_x \cos \theta - v_f q_y \sin \theta} = 2\pi i \left( \frac{1}{\sqrt{\Omega'^2 + v_y'^2 q_y^2}} + \frac{\Omega'^2}{v_y'^2 q_y^2} \sqrt{\Omega'^2 + v_y'^2 q_y^2} \right) \frac{q_y^2}{q^2} - 2\pi i \left( \frac{\Omega'}{v_f q^2} + \frac{\Omega'^2}{v_f q^2} \sqrt{\Omega'^2 + v_y'^2 q_y^2} \right) \frac{q_y^2}{q^2},
\] (A10)
where
\[
\Omega' = \Omega + i[\Sigma(i\omega + i\Omega) - \Sigma(i\omega)] + i[\Phi(i\omega + i\Omega) - \Phi(i\omega)]
\]
is an effective frequency with both normal and anomalous self-energies.

Using this angular integration, we find that both \( z = 3 \) and \( z = 2 \) critical dynamics appear in the following way
\[
\Pi_{xx}(q, i\Omega) \approx N_\sigma \frac{g^2}{8\pi v_f} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left[ sgn(\omega) - sgn(\omega + \Omega) \right]
\]
\[
\left\{ \left( \frac{1}{\sqrt{\Omega'^2 + v_y'^2 q_y^2}} + \frac{\Omega'^2}{v_y'^2 q_y^2} \sqrt{\Omega'^2 + v_y'^2 q_y^2} \right) \frac{q_y^2}{q^2} - \left( \frac{\Omega}{v_f q^2} + \frac{\Omega^2}{v_f q^2} \sqrt{\Omega'^2 + v_y'^2 q_y^2} \right) \frac{q_y^2}{q^2} \right\}
\]
\[
\approx N_\sigma \frac{g^2}{8\pi v_f} \left( \frac{\Omega}{v_f q^2} + \frac{\Omega^2}{v_f q^2} \right) \frac{q_y^2}{q^2} - N_\sigma \frac{g^2}{8\pi v_f} \left( \frac{\Omega}{v_f q^2} + \frac{\Omega^2}{v_f q^2} \right) \frac{q_y^2}{q^2}.
\] (A11)

A key point in our derivation is \( \Omega' \approx \Omega \) in the low energy limit. In other words, the normal self-energy is cancelled by the anomalous self-energy exactly in the low energy limit, explicitly shown in appendix B. If we do not introduce the anomalous self-energy correction, the normal self-energy dominates over the bare frequency, changing the dynamical exponent from \( z = 2 \) to \( z = 12/5 \) for the transverse mode, consistent with the perturbative analysis in nematic quantum criticality [9].

The other diagonal boson self-energy can be obtained in the similar way
\[
\Pi_{yy}(q, i\Omega) \approx N_\sigma \frac{g^2}{8\pi v_f} \left( \frac{\Omega}{v_f q^2} + \frac{\Omega^2}{v_f q^2} \right) \frac{q_y^2}{q^2} - N_\sigma \frac{g^2}{8\pi v_f} \left( \frac{\Omega}{v_f q^2} + \frac{\Omega^2}{v_f q^2} \right) \frac{q_y^2}{q^2}.
\] (A12)

Rewriting the off-diagonal boson self-energy in the angular coordinate
\[
\Pi_{xy}(q, i\Omega) = -N_\sigma \frac{g^2}{4(2\pi)^2} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{0}^{2\pi} d\theta \int_{-\infty}^{\infty} d\epsilon \epsilon (i\Omega - \{\Sigma(i\omega + i\Omega) - \Sigma(i\omega)\} - [\Phi(i\omega + i\Omega) - \Phi(i\omega)] - v_f q_x \cos \theta - v_f q_y \sin \theta)
\]
\[
\left\{ \frac{1}{i\Omega - \{\Sigma(i\omega + i\Omega) - \Sigma(i\omega)\} - [\Phi(i\omega + i\Omega) - \Phi(i\omega)] - v_f q_x \cos \theta - v_f q_y \sin \theta} \right\}
\]
\[
\int_{\epsilon}^{\infty} d\epsilon \approx \frac{g^2}{8\pi v_f} \left( \frac{\Omega}{v_f q^2} + \frac{\Omega^2}{v_f q^2} \right) \frac{q_y^2}{q^2}.
\] (A13)

It is straightforward to rewrite Eqs. (A11), (A12), and (A14) with an introduction of the longitudinal projection operator \( P_{ij} \). Then, we find the \( z = 3 \) critical dynamics for the longitudinal mode and the \( z = 2 \) critical dynamics for the transverse mode, given by Eq. (7).

**Appendix B: Electron self-energy**

In appendix B we derive an electron self-energy. In particular, we show that the ansatz of Eq. (6) allows the fully self-consistent Eliashberg solution, where the off-diagonal electron self-energy turns out to be the same as the normal self-energy. Inserting the electron Green’s function into the Eliashberg equation for the electron self-energy, we obtain
\[
\Sigma(k, i\omega) = g^2 \frac{1}{\beta} \sum_{i\Omega} \int \frac{d^2 q}{(2\pi)^2} D_{ij}(q, i\Omega) \sigma_i \left[ i\omega + i\Omega + \mu - \Sigma(i\omega + i\Omega) I + v_f |k + q| + \Phi(i\omega + i\Omega) \right] \epsilon_{nm}(\hat{k}_n + \hat{q}_m) \sigma_m \sigma_j.
\] (B1)
Considering Pauli matrix identities Eq. (A3) and taking the decomposition for the denominator, we can separate the electron self-energy matrix into two parts

\[ \Sigma(k, i\omega) = \Sigma_1(k, i\omega) + \Sigma_2(k, i\omega), \tag{B2} \]

where each part is given by

\[
\Sigma_1(k, i\omega) = \frac{g^2}{2} \sum_{\Omega} \int \frac{d^2q}{(2\pi)^2} \left( D_L(q, i\Omega) + D_T(q, i\Omega) \right) \left\{ \frac{1}{i\omega + i\Omega + \Sigma(i\omega + i\Omega) - v_f |k + q| - \Phi(i\omega + i\Omega)} + \frac{1}{i\omega + i\Omega + \Sigma(i\omega + i\Omega) + v_f |k + q| + \Phi(i\omega + i\Omega)} \right\} \]

\[
\begin{align*}
- \frac{g^2}{2} \sum_{\Omega} \int \frac{d^2q}{(2\pi)^2} \left( \frac{1}{i\omega + i\Omega + \Sigma(i\omega + i\Omega) - v_f |k + q| - \Phi(i\omega + i\Omega)} \right) \left( D_L(q, i\Omega) + D_T(q, i\Omega) \right) \left\{ \frac{1}{i\omega + i\Omega + \Sigma(i\omega + i\Omega) + v_f |k + q| + \Phi(i\omega + i\Omega)} \right\} \right)
\end{align*}
\]

\[
\Sigma_2(k, i\omega) = \frac{g^2}{2} \sum_{\Omega} \int \frac{d^2q}{(2\pi)^2} \left\{ \left( \frac{D_L(q, i\Omega)q_y^2}{q^2} + D_T(q, i\Omega)q_z^2 \right) k_y + q_y \right\} \left( \frac{1}{i\omega + i\Omega + \Sigma(i\omega + i\Omega) - v_f |k + q| - \Phi(i\omega + i\Omega)} \right) \left( \frac{1}{i\omega + i\Omega + \Sigma(i\omega + i\Omega) + v_f |k + q| + \Phi(i\omega + i\Omega)} \right)
\]

respectively. We will see that \( \Sigma_2(k, i\omega) \) is irrelevant at low energies, compared with \( \Sigma_1(k, i\omega) \).

First, we consider \( \Sigma_1(k, i\omega) \). Performing the linearization near the chemical potential [Eq. (A6)] and taking only dominant contributions near the Fermi surface, we obtain the following expression

\[
\Sigma_1(k, i\omega) \approx \frac{g^2}{2} \sum_{\Omega} \int \frac{d^2q}{(2\pi)^2} \left( \frac{D_L(q, i\Omega) + D_T(q, i\Omega)}{i\omega + i\Omega - \Sigma(i\omega + i\Omega) - \Phi(i\omega + i\Omega) - v_f |k_f| \cdot (k - k_f) - v_f k_f \cdot q} \right) \]

\[
\begin{align*}
- \frac{g^2}{2} \sum_{\Omega} \int \frac{d^2q}{(2\pi)^2} \left( \frac{D_L(q, i\Omega) + D_T(q, i\Omega)}{i\omega + i\Omega - \Sigma(i\omega + i\Omega) - \Phi(i\omega + i\Omega) - v_f |k_f| \cdot (k - k_f) - v_f k_f \cdot q} \right)
\end{align*}
\]

Introducing the polar coordinate

\[
q = q(\cos \theta \hat{x} + \sin \theta \hat{y}), \quad k_f = \cos \phi \hat{x} + \sin \phi \hat{y}, \tag{B5}
\]

we can rewrite \( \Sigma_1(k, i\omega) \) as \( \Sigma_1(k, i\omega) \approx \Sigma(i\omega) I + \Phi(i\omega) \epsilon_i k_i \sigma_j + \Delta \Sigma_1(i\omega, k_F) \), where

\[
\Delta \Sigma_1(i\omega, k_F) = - \frac{g^2}{2} \sum_{\Omega} \int_0^\infty dq \int_0^{2\pi} d\theta \left( \frac{q}{k_F} \cos(\theta + \phi) \sigma_y - \frac{q}{k_F} \sin(\theta + \phi) \sigma_x \right) D_L(q, i\Omega) + D_T(q, i\Omega) \]

\[
\frac{1}{i\omega + i\Omega - \Sigma(i\omega + i\Omega) - \Phi(i\omega + i\Omega) - v_f q \cos \theta} \tag{B6}
\]

can be regarded as a correction for the Eliashberg self-energies, \( \Sigma(i\omega) \) and \( \Phi(i\omega) \) given by Eq. (9).
It is straightforward to perform the angular integral, reaching the following expression

\[
\Delta \Sigma_1(i\omega, k_F) = \frac{g^2}{2v_f k_F} \sum_{\sigma} \int_0^\infty dq q k_F \left( D_L(q, i\Omega) + D_T(q, i\Omega) \right) (\hat{k}_x \sigma_y - \hat{k}_y \sigma_x) \\
- \frac{g^2}{2v_f k_F} \sum_{\sigma} \int_0^\infty dq q k_F \left( D_L(q, i\Omega) + D_T(q, i\Omega) \right) \frac{\omega + \Omega + i\Sigma(i\omega + i\Omega) - i\Phi(i\omega + i\Omega) \text{sgn}(\omega + \Omega) \sqrt{\omega + \Omega + i\Sigma(i\omega + i\Omega) + i\Phi(i\omega + i\Omega)^2 + (vfq)^2}}{\sqrt{\omega + \Omega + i\Sigma(i\omega + i\Omega) + i\Phi(i\omega + i\Omega)^2 + (vfq)^2}} (\hat{k}_x \sigma_y - \hat{k}_y \sigma_x) \\
- \frac{g^2}{2v_f k_F} \sum_{\sigma} \int_0^\infty dq q k_F \left( D_L(q, i\Omega) + D_T(q, i\Omega) \right) \ln \left( \frac{\omega + \Omega - \Sigma(i\omega + i\Omega) - \Phi(i\omega + i\Omega) - v_f q}{\omega + \Omega - \Sigma(i\omega + i\Omega) - \Phi(i\omega + i\Omega) + v_f q} \right) (\hat{k}_x \sigma_x + \hat{k}_y \sigma_y).
\]

(B7)

Comparing this expression with the Eliashberg self-energies, we find that $\Delta \Sigma_1(i\omega, k_F)$ is irrelevant at low energies due to the fact that the scaling dimension of the argument itself in the integral expression is higher than that of the Eliashberg solution in addition to the additional momentum factor $q/k_F$. As a result, we obtain

\[
\Sigma_1(i\omega, k_F) = \Sigma(i\omega) \mathbf{I} + \Phi(i\omega) (\hat{k}_x \sigma_y - \hat{k}_y \sigma_x),
\]

\[
\Sigma(i\omega) = -\Phi(i\omega) = -\frac{g^2}{2} \sum_{\sigma} \int_0^\infty dq q k_F \left( D_L(q, i\Omega) + D_T(q, i\Omega) \right) \frac{\text{sgn}(\omega + \Omega)}{\sqrt{\omega + \Omega + i\Sigma(i\omega + i\Omega) + i\Phi(i\omega + i\Omega)^2 + (vfq)^2}}.
\]

(B8)

Next, we show that $\Sigma_2(i\omega, k_F)$ is irrelevant at low energies. Performing the linearization near the chemical potential [Eq. (A6)] and taking only dominant contributions near the Fermi surface, we obtain the following expression

\[
\Sigma_2(i\omega, k_F) = g^2 \sum_{\sigma} \sum_{\sigma'} \int \frac{d^2 q}{(2\pi)^2} \left\{ -\left( D_L(q, i\Omega) \frac{q^2}{q^2} + D_T(q, i\Omega) \frac{q^2}{q^2} \right) \hat{k}_y \sigma_x + \left( D_L(q, i\Omega) \frac{q^2}{q^2} + D_T(q, i\Omega) \frac{q^2}{q^2} \right) \hat{k}_x \sigma_y \right\} \\
\int_0^\infty \frac{d\Omega}{(2\pi)^2} \left\{ \frac{1}{\omega + i\Omega - \Sigma(i\omega + i\Omega) - \Phi(i\omega + i\Omega) - v_f \hat{k}_f \cdot q} + \frac{g^2}{2} \sum_{\sigma} \int \frac{d^2 q}{(2\pi)^2} \left( D_L(q, i\Omega) \frac{q^2}{q^2} + D_T(q, i\Omega) \frac{q^2}{q^2} \right) \frac{q_y \sigma_x + \left( D_L(q, i\Omega) \frac{q^2}{q^2} + D_T(q, i\Omega) \frac{q^2}{q^2} \right) \frac{q_x}{k_F} \sigma_y \right\} \\
\int_0^\infty \frac{d\Omega}{(2\pi)^2} \left\{ \frac{1}{\omega + i\Omega - \Sigma(i\omega + i\Omega) - \Phi(i\omega + i\Omega) - v_f \hat{k}_f \cdot q} + \frac{g^2}{2} \sum_{\sigma} \int \frac{d^2 q}{(2\pi)^2} \left( D_L(q, i\Omega) \frac{q^2}{q^2} + D_T(q, i\Omega) \frac{q^2}{q^2} \right) \frac{q_x \sigma_x + \left( D_L(q, i\Omega) \frac{q^2}{q^2} + D_T(q, i\Omega) \frac{q^2}{q^2} \right) \frac{q_y}{k_F} \sigma_y \right\} \\
\int_0^\infty \frac{d\Omega}{(2\pi)^2} \left\{ \frac{1}{\omega + i\Omega - \Sigma(i\omega + i\Omega) - \Phi(i\omega + i\Omega) - v_f \hat{k}_f \cdot q} \right\}.
\]

(B9)
Introducing the polar coordinate of Eq. (B5), we rewrite the above expression as follows

\[
\Sigma_\parallel(i\omega, k_F) = \frac{g^2}{4\pi^2\beta} \sum_{i\Omega} \int_0^\infty dq \int_0^{2\pi} d\theta \left\{ -\left( D_L(q, i\Omega) \cos^2 \theta + D_T(q, i\Omega) \sin^2 \theta \right) \sin \phi \sigma_x + i\omega + i\Omega - \Phi(i\omega + i\Omega) - v_f \cos(\theta - \phi) \right\} \\
\frac{1}{\cos \theta \sin \theta} \\
\frac{\int_0^{2\pi} d\theta \left( D_L(q, i\Omega) \cos^2 \theta + D_T(q, i\Omega) \sin^2 \theta \right)}{\cos \theta \sin \theta} \left( \frac{q}{k_F} \cos \theta \sigma_y - \frac{q}{k_F} \sin \theta \sigma_y \right)
\]

Performing the angular integration, one can see irrelevance of this contribution, basically resulting from higher order momentum integrals. We prove that our ansatz is fully self-consistent in the Nambu-Eliashberg approximation.

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