Quantum information processing, science of - The theoretical, experimental and technological areas covering the use of quantum mechanics for communication and computation. Quantum information processing includes investigations in quantum information theory, quantum communication, quantum computation, quantum algorithms and their complexity, and quantum control. The science of quantum information processing is a highly interdisciplinary field. In the context of mathematics it is stimulating research in pure mathematics (e.g. coding theory, *-algebras, quantum topology) as well as requiring and providing many opportunities for applied mathematics.

The science of quantum information processing emerged from the recognition that usable notions of information need to be physically implementable. In the 1960s and 1970s researchers such as R. Landauer, C. Bennett, C. Helstrom and A. Holevo realized that the laws of physics give rise to fundamental constraints on the ability to implement and manipulate information. Landauer repeatedly stated that “information is physical”, providing impetus to the idea that it should be possible to found theories of information on the laws of physics. This is in contrast to the introspective approach which led to the basic definitions of computer science and information theory as formulated by A. Church, A. Turing, C. Shannon and others in the first half of the 20th century.

Early work in studying the physical foundations of information focused on the effects of energy limitations and the need for dissipating heat in computation and communication. Beginning with S. Wiesner’s work on applications of quantum mechanics to cryptography in the late 1960s, it was realized that there may be intrinsic advantages to using quantum physics in information processing. Quantum cryptography and quantum communication in general were soon established as interesting and non-trivial extensions of classical communication based on bits. That quantum mechanics may be used to improve the efficiency of algorithms was first realized when attempts at simulating quantum mechanical systems resulted in exponentially complex algorithms compared to the physical resources associated with the system simulated. In the 1980s, P. Benioff and R. Feynman introduced the idea of a quantum computer for efficiently implementing quantum physics simulations. Models of quantum computers were developed by D. Deutsch, leading to the formulation of artificial problems that could be solved more efficiently by quantum than by classical computers. The advantages of quantum computers became widely recognized when P. Shor (1994) discovered...
that they can be used to efficiently factor large numbers — a problem believed to be hard for classi-
cical deterministic or probabilistic computation and whose difficulty underlies the security of widely
used public key encryption methods. Subsequent work established principles of quantum error-
correction to ensure that quantum information processing was robustly implementable. See [5, 2]
for introductions to quantum information processing and a quantum mechanics tutorial.

In the context of quantum information theory, information in the sense of C. Shannon is referred
to as classical information. The fundamental unit of classical information is the bit, which can be
understood as an ideal system in one of two states or configurations, usually denoted by 0 and 1.
The fundamental units of quantum information are qubits (short for “quantum bits”), whose states
are identified with all “unit superpositions” of the classical states. It is common practice to use
the bra-ket conventions for denoting states. In these conventions, the classical configurations are
denoted by \( |0\rangle \) and \( |1\rangle \), and superpositions are formal sums \( \alpha |0\rangle + \beta |1\rangle \), where \( \alpha \) and \( \beta \) are complex
numbers satisfying \( |\alpha|^2 + |\beta|^2 = 1 \). The states \( |0\rangle \) and \( |1\rangle \) represent a standard orthonormal basis
of a two-dimensional Hilbert space. Their superpositions are unit vectors in this space. The state
space associated with \( n > 1 \) qubits is formally the tensor product of the Hilbert spaces of each
qubit. This state space can also be obtained as an extension of the state space of \( n \) classical bits
by identifying the classical configurations with a standard orthonormal basis of a \( 2^n \) dimensional
Hilbert space.

Access to qubit states is based on the postulates of quantum mechanics with the additional
restriction that they are local in the sense that elementary operations apply to one or two qubits at a
time. Most operations can be expressed in terms of standard measurements of a qubit and two-qubit
quantum gates. The standard qubit measurement has the effect of randomly projecting the state
of the qubit onto one of its classical states; this state is an output of the measurement (accessible
for use in a classical computer if desired). For example, using the tensor product representation
of the state space of several qubits, a measurement of the first qubit is associated with the two
projection operators \( P^{(1)}_0 = P_0 \otimes I \otimes \ldots \) and \( 1 - P^{(1)}_0 \), where \( P_0|0\rangle = |0\rangle \) and \( P_0|1\rangle = 0 \). If \( \psi \) is
the initial state of the qubits, then the measurement outcome is 0 with probability \( p_0 = ||P_0\psi||^2 \),
in which case the new state is \( P_0\psi/p_0 \), and the outcome is 1 with probability \( 1 - p_0 = ||P_1\psi||^2 \)
with new state \( P_1\psi/(1 - p_0) \). This is a special case of a von Neumann measurement. A general
two-qubit quantum gate is associated with a unitary operator \( U \) acting on the state space of two
qubits. Thus \( U \) may be represented by a \( 4 \times 4 \) unitary matrix in the standard basis of two qubits.
The quantum gate may be applied to any two chosen qubits. For example, if the state of \( n \) qubits
is \( \psi \) and the gate is applied to the first two qubits, then the new state is given by \( (U \otimes I \otimes \ldots)\psi \).
Another important operation of quantum information processing is preparation of the $|0\rangle$ state of a qubit, which can be implemented in terms of a measurement and subsequent applications of a gate depending on the outcome.

Most problems of theoretical quantum information processing can be cast in terms of the elementary operations above, restrictions on how they can be used and an accounting of the physical resources or cost associated with implementing the operations. Since classical information processing may be viewed as a special case of quantum information processing, problems of classical information theory and computation are generalized and greatly enriched by the availability of quantum superpositions. The two main problem areas of theoretical quantum information processing are quantum computation and quantum communication.

In studies of quantum computation (cf. quantum computation) one investigates how the availability of qubits can be used to improve the efficiency of algorithmic problem solving. Resources counted include the number of quantum gates applied and the number of qubits accessed. This can be done by defining and investigating various types of quantum automata, most prominently quantum Turing machines, and studying their behavior using approaches borrowed from the classical theory of automata and languages. It is convenient to combine classical and quantum automata, for example by allowing a classical computer access to qubits as defined above, and then investigating the complexity of algorithms by counting both classical and quantum resources, thus obtaining trade-offs between the two.

Most of the complexity classes for classical computation have analogues for quantum computation, and an important research area is concerned with establishing relationships between these complexity classes. Corresponding to the classical class $\mathbf{P}$ of polynomially decidable languages is the class of languages decidable in bounded error quantum polynomial time, $\mathbf{BQP}$. While it is believed that $\mathbf{P}$ is properly contained in $\mathbf{BQP}$, whether this is so is at present an open problem. $\mathbf{BQP}$ is known to be contained in the class $\mathbf{P}^{\#\mathbf{P}}$ (languages decidable in classical polynomial time given access to an oracle for computing the permanent of 0-1 matrices), but the relationship of $\mathbf{BQP}$ to the important class of nondeterministic polynomial time languages $\mathbf{NP}$ is not known.

In quantum communication one considers the situation where two or more entities with access to local qubits can make use of both classical and quantum (communication) channels for exchanging information. The basic operations now include the ability to send classical bits and the ability to send quantum bits. There are two main areas of investigation in quantum communication. The first aims at determining the advantages of quantum communication for solving classically posed communication problems with applications to cryptography and to distributed
computation. The second is concerned with establishing relationships between different types of communication resources, particularly with respect to noisy quantum channels, thus generalizing classical communication theory.

Early investigations of quantum channels focused on using them for transmitting classical information by encoding a source of information (cf. information, source of) with uses of a quantum channel (cf. quantum communication channel). The central result of these investigations is A. Holevo’s bound (1973) on the amount of classical information that can be conveyed through a quantum channel. Asymptotic achievability of the bound (using block coding of the information source) was shown in the closing years of the twentieth century. With some technical caveats, the bound and its achievability form a quantum information-theoretic analogue of Shannon’s capacity theorem for classical communication channels.

Quantum cryptography, distributed quantum computation and quantum memory require transmitting (or storing) quantum states. As a result it is of great interest to understand how one can communicate quantum information through quantum channels. In this case, the source of information is replaced by a source of quantum states, which are to be transmitted through the channel with high fidelity. As in the classical case, the state is encoded before transmission and decoded afterwards. There are many measures of fidelity which may be used to evaluate the quality of the transmission protocol. They are chosen so that a good fidelity value implies that with high probability, quantum information processing tasks behave the same using the original or the transmitted states. A commonly used fidelity measure is the Bures-Uhlmann fidelity, which is an extension of the Hilbert space norm to probability distributions of states (represented by density operators). In most cases, asymptotic properties of quantum channels do not depend on the details of the fidelity measure adopted.

To improve the reliability of transmission over a noisy quantum channel, one uses quantum error-correcting codes to encode a state generated by the quantum information source with multiple uses of the channel. The theory of quantum codes can be viewed as an extension of classical coding theory. Concepts such as minimum distance and its relationship to error-correction generalize to quantum codes. Many results from the classical theory, including some linear programming upper bounds and the Gilbert-Varshamov lower bounds on the achievable rates of classical codes have their analogues for quantum codes. In the classical theory, linear codes are particularly useful and play a special role. In the quantum theory, this role is played by the stabilizer or additive quantum codes, which are in one-to-one correspondence with self-dual (with respect to a specific symplectic inner product) classical GF₂-linear codes over GF₄ (cf. finite fields).
The capacity of a quantum channel with respect to encoding with quantum codes is not as well understood as the capacity for transmission of classical information. The exact capacity is known only for a few special classes of quantum channels. Although there are information theoretic upper bounds, they depend on the number of channel instances, and whether or not they can be achieved is an open problem. A further complication is that the capacity of quantum channels depends on whether one-way or two-way classical communication may be used to restore the transmitted quantum information [1].

The above examples illustrate the fact that there are many different types of information utilized in quantum information theory, making it a richer subject than classical information theory. Another physical resource whose properties appear to be best described by information-theoretic means is quantum entanglement. A quantum state of more than one quantum system (e.g. two qubits) is said to be entangled if the state cannot be factorized as a product of states of the individual quantum systems. Entanglement is believed to play a crucial role in quantum information processing, as demonstrated by its enabling role in effects such as quantum key distribution, superdense coding, quantum teleportation, and quantum error-correction. Beginning in 1995 an enormous amount of effort has been devoted to understanding the principles governing the behavior of entanglement. This has resulted in the discovery of connections between quantum entanglement and classical information theory, the theory of positive maps [3] and majorization [4].

The investigation of quantum channel capacity, entanglement, and many other areas of quantum information processing involves various quantum generalizations of the notion of entropy, most notably the von Neumann entropy. The von Neumann entropy is defined as $H(\rho) = \text{tr} \rho \log_2(\rho)$ for density operators $\rho$ ($\rho$ is positive Hermitian and of trace 1). It has many (but not all) of the properties of the classical information function $H(\cdot)$ (cf. information, amount of). Understanding these properties has been crucial to the development of quantum information processing (see [3, 8, 7] for reviews). Probably the most powerful known result about the von Neumann entropy is the strong subadditivity inequality. Many of the bounds on quantum communication follow as easy corollaries of strong subadditivity. Whether still more powerful entropic inequalities exist is not known.

An important property of both classical and quantum information is that although it is intended to be physically realizable, it is abstractly defined and therefore independent of the details of a physical realization. It is generally believed that qubits encapsulate everything that is finitely realizable using accessible physics. This belief implies that any information processing implemented by available physical systems using resources appropriate for those systems can be implemented as
efficiently (with at most polynomial overhead) using qubits. It is noteworthy that there is presently no proof that information processing based on quantum field theory (cf. quantum field theory) is not more efficient than information processing with qubits. Furthermore, the as-yet unresolved problem of combining quantum mechanics with general relativity in a theory of quantum gravity prevents a fully satisfactory analysis of the information processing power afforded by fundamental physical laws.

Much effort in the science of quantum information processing is being expended on developing and testing the technology required for implementing it. An important task in this direction is to establish that quantum information processing can be implemented robustly in the presence of noise. At first it was believed that this was not possible. Arguments against the robustness of quantum information were based on the apparent relationship to analogue computation (due to the continuity of the amplitudes in the superpositions of configurations) and the fact that it seemed difficult to observe quantum superpositions in nature (due to the rapid loss of phase relationships called decoherence). However, the work on quantum error-correcting codes rapidly led to the realization that provided the physical noise behaves locally and is not too large, it is at least in principle possible to process quantum information fault tolerantly. Research in how to process quantum information reliably continues; the main problems is improving the estimates on the maximum amount of tolerable noise for general models of quantum noise and for the types of noise expected in specific physical systems. Other issues include the need to take into consideration restrictions imposed by possible architectures and interconnection networks.

There are many physical systems that can potentially be used for quantum information processing [7]. An active area of investigation involves determining the general mathematical features of quantum mechanics required for implementing quantum information. More closely tied to existing experimental techniques are studies of specific physical systems. In the context of communication, optical systems are likely to play an important role, while for computation there are proposals for using electrons or nuclei in solid state, ions or atoms in electromagnetic traps, excitations of superconductive devices etc. In all of these, important theoretical issues arise. These issues include how to optimally use the available means for controlling the quantum systems (quantum control), how to best realize quantum information (possibly indirectly), what architectures can be implemented, how to translate abstract sequences of quantum gates to physical control actions, how to interface the system with optics for communication, refining the theoretical models for how the system is affected by noise and thermodynamic effects, and how to reduce the effects of noise.
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