Upper and lower bound theorems for graph-associahedra.

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Abstract

From the paper of the first author it follows that upper and lower bounds for \(\gamma\)-vector of a
simple polytope imply the bounds for its \(g\), \(h\) and \(f\)-vectors. In the paper of the second author it
was obtained unimprovable upper and lower bounds for \(\gamma\)-vectors of flag nestohedra, particularly,
Gal’s conjecture was proved for this case. In the present paper we obtain unimprovable upper and
lower bounds for \(\gamma\)-vectors (consequently, for \(g\), \(h\) and \(f\)-vectors) of graph-associahedra and some its
important subclasses. We use the constructions that for an \((n-1)\)-dimensional graph-associahedron
\(P_{n-1}\) give the \(n\)-dimensional graph-associahedron \(P_{n+1}\) that is obtained from the cylinder \(P_{n-1} \times I\)
by sequential shaving some facets of its bases. We show that the well-known series of polytopes
(associahedra, cyclohedra, permutohedra and stellohedra) can be derived by these constructions.
As a corollary we obtain inductive formulas for \(\gamma\) and \(h\) vectors of the mentioned series. These
formulas communicate the method of differential equations developed by the first author with the
method of shavings developed by the second author.

1 Introduction

Simple polytopes play important role in toric geometry and topology (see [BR]). The classical problem
of upper and lower bounds for \(h\)-vectors of \(n\)-dimensional simple polytopes with fixed number of facets
is solved in [Ba1], [Ba2] and [Mc].

Nowadays there appeared an important subclass of simple polytopes - Delzant polytopes. For every
Delzant polytope \(P^n\) there exists a Hamiltonian toric manifold \(M^{2n}\) such that \(P^n\) is the image of the
moment map (see [CS], [D]). Davis-Januszkiewicz theorem (see [DJ]) states that odd Betti numbers
\(b_{2i-1}(M^{2n})\) are zero and even Betti numbers \(b_{2i}(M^{2n})\) are equal to components \(h_i(P^n)\) of the \(h\)-vector
of \(P^n\). So, the problem of upper and lower bounds for \(h\)-vectors of Delzant polytopes become actual,
because its solution gives upper and lower bounds for Betti numbers of Hamiltonian toric manifolds.

Feichtner and Sturmfels (see [FS]) and Postnikov (see [P]) showed that the Minkowski sum of some set
of regular simplices is a simple polytope if this set satisfies certain combinatorial conditions identifying
it as a building set. The resulting family of simple polytopes was called nestohedra in [PRW] because of
their connection to nested sets considered by De Concini and Procesi (see [DP]) in the context of subspace
arrangements. Note that from results of [FS] directly follows that nestohedra are Delzant polytopes.
Special cases of building sets are vertex sets of connected subgraphs in a given graph: the corresponding
nestohedra called graph-associahedra by Carr and Devadoss were first studied in [CD], [DJS], [P], [TL], [Ze].

The main goal of this paper is to establish upper and lower bounds for \(f\), \(g\), \(h\)- and \(\gamma\)-vectors of
graph-associahedra and some its important subclasses.

From [FM] we know that if \(B_1 \subseteq B_2\) for connected building sets, then \(P_{B_2}\) is obtained from \(P_{B_1}\) by
sequential shaving some faces, consequently, \(h_i(P_{B_1}) \leq h_i(P_{B_2})\). Therefore, \(h_i(\Delta^n) \leq h_i(P_B) \leq h_i(Pe^n)\)
for every \(n\)-dimensional nestohedron \(P_B\) and these bounds are unimprovable.

In the combinatorics of simple polytopes especially interested is \(\gamma\)-vector. Using [Bu1] and definitions
of \(g\), \(h\)- and \(f\)-vectors one can prove that componentwise inequality \(\gamma(P_1) \leq \gamma(P_2)\) for simple \(n\)-polytopes
\(P_1\) and \(P_2\) implies componentwise inequalities: \(g(P_1) \leq g(P_2)\), \(h(P_1) \leq h(P_2)\), \(f(P_1) \leq f(P_2)\).

Gal’s conjecture (see [G]) states that flag simple polytopes have nonnegative \(\gamma\)-vectors. In [Ba2]
it was described realization of the associahedron as a polytope obtained from the standard cube by
shaving faces of codimension 2. The main result of [V1], [V2] is that every flag nestohedron has such a
realization. As a corollary it was derived that unimprovable bounds for \(\gamma\)-vectors of flag nestohedra are
\(\gamma(I^n)\) and \(\gamma(Pe^n)\). That includes Gal’s conjecture for flag nestohedra, since \(\gamma_i(I^n) = 0, i > 0\).
There are remarkable series of graph-associahedra corresponding to series of graphs: associahedra \( As^n \) (path graphs), cyclohedra \( Cy^n \) (cyclic graphs), permutohedra \( Pe^n \) (complete graphs) and stellohedra \( St^n \) (star graphs). Using these series we obtain the main result of the paper:

**Theorem.** There are following unimprovable bounds:

1) \( \gamma_i(As^n) \leq \gamma_i(P_{\Gamma_{n+1}}) \leq \gamma_i(Pe^n) \) for any connected graph \( \Gamma_{n+1} \) on \([n+1]\);
2) \( \gamma(Cy^n) \leq \gamma_i(P_{\Gamma_{n+1}}) \leq \gamma_i(Pe^n) \) for any Hamiltonian graph \( \Gamma_{n+1} \) on \([n+1]\);
3) \( \gamma_i(As^n) \leq \gamma_i(P_{\Gamma_{n+1}}) \leq \gamma_i(St^n) \) for any tree \( \Gamma_{n+1} \) on \([n+1]\).

The last part was predicted in [PRW, Conjecture 14.1], where it was calculated \(\gamma\)-vectors of trees on 7 nodes and it was noticed that more branched and forked trees give polytopes with higher \(\gamma\)-vectors.

We use the constructions that for an \((n-1)\)-dimensional graph-associahedron \(P_{\Gamma_n}\) produce the \(n\)-dimensional graph-associahedron \(P_{\Gamma_{n+1}}\) that is obtained from the cylinder \(P_{\Gamma_n} \times I\) by sequential shaving some facets of its bases. We show that the mentioned series of polytopes (associahedra, cyclohedra, permutohedra and stellohedra) can be derived by these constructions. As a corollary we obtain inductive formulas for \(\gamma\) and \(h\)-vectors of the above series. These formulas communicate the method of differential equations developed in [Bu1] with the method of shavings developed in [V1, V2].

## 2 Face polynomials

The convex \(n\)-dimensional polytope \(P\) is called simple if its every vertex belongs to exactly \(n\) facets. Let \(f_i\) be the number of \(i\)-dimensional faces of an \(n\)-dimensional polytope \(P\). The vector \((f_0, \ldots, f_n)\) is called the \(f\)-vector of \(P\). The \(F\)-polynomial of \(P\) is defined by:

\[
F(P)(\alpha, t) = a^n + f_{n-1}a^{n-1}t + \cdots + f_1\alpha t^{n-1} + f_0 t^n.
\]

The \(h\)-vector and \(H\)-polynomial of \(P\) are defined by:

\[
H(P)(\alpha, t) = h_0 a^n + h_1 a^{n-1}t + \cdots + h_{n-1} \alpha t^{n-1} + h_n t^n = F(P)(\alpha - t, t).
\]

The \(g\)-vector of a simple polytope \(P\) is the vector \((g_0, g_1, \ldots, g(\frac{n}{2}))\), where \(g_0 = 1\), \(g_i = h_i - h_{i-1}, i > 0\). The Dehn-Sommerville equations (see [Z1]) state that \(H(P)\) is symmetric for any simple polytope. Therefore, it can be represented as a polynomial of \(a = \alpha + t\) and \(b = \alpha t\):

\[
H(P) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \gamma_i (\alpha t)^i (a + t)^{n-2i}.
\]

The \(\gamma\)-vector of \(P\) is the vector \((\gamma_0, \gamma_1, \ldots, \gamma(\frac{n}{2}))\). The \(\gamma\)-polynomial of \(P\) is defined by:

\[
\gamma(P)(\tau) = \gamma_0 + \gamma_1 \tau + \cdots + \gamma(\frac{n}{2}) \tau^{\lfloor \frac{n}{2} \rfloor}.
\]

**Lemma 1.** Let \(\gamma_i(P_1) \leq \gamma_i(P_2), i = 0, \ldots, \lfloor \frac{n}{2} \rfloor,\) where \(P_1\) and \(P_2\) are simple \(n\)-polytopes, then

1) \(g_i(P_1) \leq g_i(P_2)\);
2) \(h_i(P_1) \leq h_i(P_2)\);
3) \(f_i(P_1) \leq f_i(P_2)\).

**Proof.** The following formula for simple \(n\)-polytopes (see [Bu1]) implies part 1).

\[
g_i(P) = (n-2i+1) \sum_{j=0}^{i} \frac{1}{n-i-j+1} \binom{n-2j}{i-j} \gamma_j(P).
\]

Next formulas derived from definitions of \(g\) and \(h\)-vectors show that 1) implies 2) and 2) implies 3).

\[
h_i(P) = \sum_{j=0}^{i} g_j(P); \quad f_i(P) = \sum_{j=i}^{n} \binom{j}{i} h_{n-j}(P).
\]
3 Nestohedra and graph-associahedra

In this section we state well-known facts about nestohedra. They can be found in [ES], [P], [Ze].

Notation. By $[n]$ and $[i,j]$ denote the sets $\{1,\ldots,n\}$ and $\{i,\ldots,j\}$.

Definition 1. A collection $B$ of nonempty subsets of $[n+1]$ is called a building set on $[n+1]$ if the following conditions hold:

1) If $S_1, S_2 \in B$ and $S_1 \cap S_2 \neq \emptyset$, then $S_1 \cup S_2 \in B$;

2) $\{i\} \in B$ for every $i \in [n+1]$.

The building set $B$ is connected if $[n+1] \in B$.

The restriction of the building set $B$ to $S \in B$ is the following building set on $|[S]|$:

$$B|_S = \{S' \in B : S' \subseteq S\}.$$  

The contraction of the building set $B$ along $S \in B$ is the following building set on $[n+1] - |S|$:

$$B/S = \{S' \subseteq [n+1] \setminus S : S' \in B \text{ or } S' \cup S \in B\} = \{S' \setminus S, S' \in B\}.$$  

Definition 2. Let $\Gamma$ be a graph with no loops or multiple edges on the node set $[n+1]$. The graphical building set $B(\Gamma)$ is the collection of nonempty subsets $S \subseteq [n+1]$ such that the induced subgraph $\Gamma|_S$ on the node set $S$ is connected.

Remark 1. Building set $B(\Gamma)$ is connected if and only if $\Gamma$ is connected.

Remark 2. Let $\Gamma$ be a connected graph on $[n+1]$ and $S \in B(\Gamma)$, then $B|_S$ and $B/S$ are both graphical building sets corresponding to connected graphs $\Gamma|_S$ and $\Gamma/S$.

Let $M_1$ and $M_2$ be subsets of $\mathbb{R}^n$. The Minkowski sum of $M_1$ and $M_2$ is the following subset of $\mathbb{R}^n$:

$$M_1 + M_2 = \{x \in \mathbb{R}^n : x = x_1 + x_2, x_1 \in M_1, x_2 \in M_2\}.$$  

If $M_1$ and $M_2$ are convex polytopes, then so is $M_1 + M_2$.

Definition 3. Let $e_i$ be the endpoints of the basis vectors of $\mathbb{R}^{n+1}$. Define the nestohedron $P_B$ corresponding to the building set $B$ as following

$$P_B = \sum_{S \in B} \Delta^S, \text{ where } \Delta^S = \text{conv}\{e_i, i \in S\}.$$  

If $B(\Gamma)$ is a graphical building set, then $P_\Gamma = P_{B(\Gamma)}$ is called a graph-associahedron.

Example. Here we especially interested by the following series of graph-associahedra:

- Let $L_{n+1}$ be the path graph on $[n+1]$, then the polytope $P_{L_{n+1}}$ is called associahedron (Stasheff polytope) and denoted by $As^n$;

- Let $C_{n+1}$ be the cyclic graph on $[n+1]$, then the polytope $P_{C_{n+1}}$ is called cyclohedron (Bott-Taubes polytope) and denoted by $Cy^n$;

- Let $K_{n+1}$ be the complete graph on $[n+1]$, then the polytope $P_{K_{n+1}}$ is called permutohedron and denoted by $Pe^n$;

- Let $K_{1,n}$ be the complete bipartite graph on $[n+1]$, then the polytope $P_{K_{1,n}}$ is called stellohedron and denoted by $St^n$.

The simple $n$-polytope $P \subset \mathbb{R}^n$ is called a Delzant polytope if for every its vertex $p$ there exist integer vectors parallel to the edges meeting at $p$ and forming a $Z$ basis of $Z^n \subset \mathbb{R}^n$. 

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Proposition 1. Let $B$ be a connected building set on $[n+1]$. Then, $\dim P_B = n$ and $P_B$ can be realized as a Delzant polytope. Particularly, every nestohedron is simple.

The convex polytope $P$ is called flag if any collection of its pairwise intersecting facets has a nonempty intersection.

Proposition 2. Every graph-associahedron is flag.

To understand the combinatorics of $P_B$ we need the following statement.

Proposition 3. Let $B$ be a connected building set on $[n+1]$. Then, elements $S$ of $B \setminus [n+1]$ are in bijection with facets (denoted by $F_S$) of $P_B$, which are combinatorially equivalent to $P_{B|S} \times P_{B/S}$. Facets $F_{S_1}, \ldots, F_{S_k}$ have a nonempty intersection if and only if the following conditions hold:

1) For any $S_i, S_j$ we have $S_i \subset S_j$, or $S_i \supset S_j$, or $S_i \cap S_j = \emptyset$;
2) For any $S_{i_1}, \ldots, S_{i_p}$ such that $S_{i_j} \cap S_{i_k} = \emptyset$ we have $S_{i_1} \sqcup \ldots \sqcup S_{i_p} \notin B$.

Notation. Eventually, we will write ”facet $S$” meaning ”facet $F_S$”.

4 Technique

4.1 The $\gamma$-vectors of flag nestohedra

We need the next results from [V1, V2].

Theorem 1. The $\gamma$-vector of any flag nestohedron has nonnegative entries.

Theorem 2. If $B_1$ and $B_2$ are connected building sets on $[n+1]$, $B_1 \subseteq B_2$, and $P_{B_1}$ are flag, then

$$\gamma_i(P_{B_1}) \leq \gamma_i(P_{B_2}).$$

4.2 Shavings

Here we describe used machinery from [V1, V2], which is based on [FM].

Construction (Decomposition of $S \in B_1$ by elements of $B_0$). Let $B_0$ and $B_1$ be connected building sets on $[n+1]$, $B_0 \subset B_1$, and $S \in B_1$. Let us call the decomposition of $S$ by elements of $B_0$ the representation $S = S_1 \sqcup \ldots \sqcup S_k, S_j \in B_0$ such that $k$ is minimal among such disjoint representations.

It is easy to check that the decomposition exists and is unique.

The next statement can be extracted from [FM] Theorem 4.2 and also the direct proof in accepted terms is given in [V1] Lemma 5.

Theorem 3. Let $B_0$ and $B_1$ be connected building sets on $[n+1]$ and $B_0 \subset B_1$. The set $B_1$ is partially ordered by inclusion. Let us number all the elements of $B_1 \setminus B_0$ by indexes $i$ in such a way that $i \leq i'$ provided $S_i \supseteq S_{i'}$.

Then, $P_{B_1}$ is obtained from $P_{B_0}$ by sequential shaving faces $G^i = \bigcap_{j=1}^{k_i} F_{S_j}$ that correspond to decompositions $S^i = S_1^i \sqcup \ldots \sqcup S_{k_i}^i \in B_1 \setminus B_0$ starting from $i = 1$ (i.e. in reverse inclusion order).

According to [V1, V2], if $P_{B_0}$ and $P_{B_1}$ are flag, then we can change the order of shavings (compare to the last theorem) in such a way that only faces of codimension 2 will be shaved off. This type of shavings increases the $\gamma$-vector in case of flag nestohedra.

Proposition 4. (cf. [V1, Proposition 6]) Let the polytope $Q$ be obtained from the simple polytope $P$ by shaving the face $G$ of codimension 2, then

1) $\gamma(Q) = \gamma(P) + \tau \gamma(G)$;
2) $H(Q) = H(P) + \alpha H(G)$. 

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4.3 Substitution of building sets

Construction (N. Erokhovets). Let $B, B_1, \ldots, B_{n+1}$ be connected building sets on $[n+1], [k_1], \ldots, [k_{n+1}]$. Define the connected building set $B(B_1, \ldots, B_{n+1})$ on $[k_1] \cup \ldots \cup [k_{n+1}] = [k_1 + \ldots + k_{n+1}]$ consisting of the elements $S_i \in B_i$ and $\bigcup_{i \in S} [k_i]$, where $S \in B$.

Proposition 5 (N. Erokhovets). Let $B, B_1, \ldots, B_{n+1}$ be connected building sets on $[n+1], [k_1], \ldots, [k_{n+1}]$, and $B' = B(B_1, \ldots, B_{n+1})$. Then $P_{B'}$ is combinatorially equivalent to $P_B \times P_{B_1} \times \cdots \times P_{B_{n+1}}$ and the following mapping $\varphi$: $(B \setminus [n+1]) \bigcup_{i=1}^{n+1} (B_i \setminus [k_i]) \rightarrow B' \setminus [k_1 + \ldots + k_{n+1}]$ defines the facet correspondence.

$$\varphi(S) = \begin{cases} S & \text{if } S \in B_i; \\ \bigcup_{i \in S} [k_i] & \text{if } S \in B. \end{cases}$$

Example. Let $B, B_1, B_2$ be building sets $\{\{1\}, \{2\}, \{1, 2\}\}$, which correspond to the interval $J$. Let us show what is $B(B_1, B_2)$. We substitute $B_1 = \{\{1\}, \{2\}, \{1, 2\}\}$ as $a$ and $B_2 = \{\{3\}, \{4\}, \{3, 4\}\}$ as $b$ to the building set $\{a, b, \{a, b\}\}$ and obtain the building set $B'$ on $[4]$ consisting of $\{1, 2\}, \{3, 4\}, [4]$. Here we reordered $B, B_1, B_2$ to make them not intersecting.

The facet correspondence is:

- $\{1\} \in B_1 \rightarrow \{1\} \in B'$;
- $\{2\} \in B_1 \rightarrow \{2\} \in B'$;
- $\{3\} \in B_2 \rightarrow \{3\} \in B'$;
- $\{4\} \in B_2 \rightarrow \{4\} \in B'$;
- $\{a\} \in B_2 \rightarrow \{1, 2\} \in B'$;
- $\{b\} \in B_2 \rightarrow \{3, 4\} \in B'$.

Notation. If, for example, building sets $B_1, \ldots, B_n$ are $\{1\}, \ldots, \{n\}$, then we will write $B(1, 2, \ldots, n, B_{n+1})$ instead of $B(\{1\}, \{2\}, \ldots, \{n\}, B_{n+1})$ to simplify the notations.

5 Inductive formulas

Let $J$ be the building set $\{\{1\}, \{2\}, \{1, 2\}\}$, which corresponds to the interval $P_J = I$.

Construction 1. Let $\Gamma_n$ be a connected graph on $[n]$ and $V \subseteq [n]$ induces the complete subgraph of $\Gamma_n$. Set $\Gamma_{n+1} = \Gamma_n \cup \{n+1\} \cup \{n+1, V\}$, i.e. the vertexes $V$ are adjacent to the new vertex $\{n+1\}$.

According to proposition 5 the building set $B_1 = J(B(\Gamma_n), n+1) = B(\Gamma_n) \cup \{n+1\} \cup [n+1]$ corresponds to $P_{B_1} = P_{\Gamma_n} \times I$: the bottom and top bases of $P_{\Gamma_n} \times I$ are $[n], \{n+1\} \in B_1$; the side facets are $S \in B(\Gamma_n) \setminus \{n\} \subseteq B_1$. Thus, side facetas $S$ are naturally identified with facets of $P_{\Gamma_n}$.

$B(\Gamma_{n+1}) \setminus B_1 = \{S \cup \{n+1\}, S \in S\},$ where $S = \{S \in B(\Gamma_n) \setminus \{n\}: S \cap V \neq \emptyset\}$. By Theorem 4 $P_{\Gamma_{n+1}}$ is obtained from $P_{\Gamma_n} \times I$ by shaving intersections of the top base $F_{\Gamma_{n+1}}$ with the side facets $F_S$ for $S \in S$. Since the top base doesn’t change after shaving its facets, the shaved off faces are exactly $P_{\Gamma_n|S} \times P_{\Gamma_n/S}, S \in S$. By proposition 4 we have:

$$\gamma(P_{\Gamma_{n+1}}) = \gamma(P_{\Gamma_n}) + \tau \sum_{S \in S} \gamma(P_{\Gamma_n|S})\gamma(P_{\Gamma_n/S});$$

$$H(P_{\Gamma_{n+1}}) = (\alpha + t) H(P_{\Gamma_n}) + \alpha t \sum_{S \in S} H(P_{\Gamma_n|S})H(P_{\Gamma_n/S}).$$

We required that $V$ induces the complete subgraph of $\Gamma_n$, because in this case every element of $B(\Gamma_{n+1}) \setminus B_1$ has decomposition consisting of two elements $(S \cup \{n+1\},$ where $\{n+1\}, S \in B_1)$ and we know the combinatorial type of shaved off faces, which have codimension 2.
5.1 Associahedra

Let us apply the construction \[ \Pi \] to the path graph \( L_n \). Here \( V = \{ n \} \) and \( S = \{ [i, n], i = 2, \ldots, n \} \). Therefore, the shaved off faces are \( As^{i-1} \times As^{n-i-1}, i = 1, \ldots, n - 1 \) and we obtain inductive formulas:

\[
\gamma(As^n) = \gamma(As^{n-1}) + \tau \sum_{i=1}^{n-1} \gamma(As^{i-1}) \gamma(As^{n-i-1});
\]

\[
H(As^n) = (\alpha + t)H(As^{n-1}) + \alpha t \sum_{i=1}^{n-1} H(As^{i-1}) H(As^{n-i-1}).
\]

The inductive formulas for associahedra are equivalent to the equations:

\[
\gamma_{As}(x) = 1 + x\gamma_{As}(x) + \tau x^2 \gamma_{As}^2(x), \quad \text{where} \quad \gamma_{As}(x) = \sum_{n=0}^{\infty} \gamma(As^n) x^n;
\]

\[
H_{As}(x) = (1 + \alpha x H_{As}(x))(1 + tx H_{As}(x)), \quad \text{where} \quad H_{As}(x) = \sum_{n=0}^{\infty} H(As^n) x^n.
\]

The last equation is equivalent to:

\[
\frac{x H_{As}(x)}{(1 + \alpha x H_{As}(x))(1 + tx H_{As}(x))} = x.
\]

Set \( U(x) = x H_{As}(x) \). Then, \( U(0) = 0, U'(0) = 1 \) and

\[
\frac{U}{(1 + \alpha U)(1 + tU)} = x.
\]

Applying the classical Lagrange Inversion Formula we obtain:

\[
U(x) = -\frac{1}{2\pi i} \oint_{|z|=\varepsilon} \ln \left[ 1 - \frac{x}{z} (1 + \alpha z)(1 + tz) \right] dz = \sum_{n=1}^{\infty} \left( \frac{1}{2\pi i} \oint_{|z|=\varepsilon} \left[ \frac{(1 + \alpha z)^n (1 + tz)^n}{z^n} \right] dz \right) \frac{x^n}{n} = \sum_{n=1}^{\infty} \left( \sum_{i+j=n-1} \binom{n}{i} \binom{n}{j} \alpha^i t^j \right) \frac{x^n}{n}.
\]

Therefore,

\[
H(As^n) = \frac{1}{n+1} \sum_{i+j=n} \binom{n+1}{i} \binom{n+1}{j} \alpha^i t^j = \frac{1}{n+1} \sum_{i=0}^{n} \binom{n+1}{i} \binom{n+1}{i+1} \alpha^{n-i} t^i.
\]

Lemma 2. For every connected graph \( \Gamma_{n+1} \) on \([n+1]\) we have \( \gamma_i(P_{\Gamma_{n+1}}) \geq \gamma_i(As^n) \).

Proof. Notice, that it is enough to prove the lemma for trees. Indeed, for every connected graph \( \Gamma \) there exists a tree \( T \subseteq \Gamma \) on the same nodes. Then, \( B(T) \subseteq B(\Gamma) \) and we can apply Theorem 2.

For \( n = 1 \) there is nothing to prove.

Assume that the lemma holds for \( m \leq n \). Let \( \Gamma_{n+1} \) be a tree on \([n+1]\). Without loss of generality, assume that \([n+1]\) is adjacent only to \([n]\). Then, we can use the construction \[ \Pi \] putting \( \Gamma_n = \Gamma_{n+1} \setminus \{n+1\} \) and \( V = \{n\} \). For every \( i \in [1, n-1] \) there exists a connected subgraph of \( \Gamma_n \) on \( i \) vertexes containing \([n]\), i.e. there exists \( S \in \mathcal{S} \); \(|S| = i\). Therefore, comparing \[ \Pi \] to \[ \Pi \] and using the inductive assumption and remark \[ \Pi \] we obtain the lemma.
5.2 Cyclohedra

Let $C_{n+1}$ be a cyclic graph on $[n+1]$. We apply the construction different from the construction \[ \text{[1]} \]
According to proposition \[ \text{[5]} \] the building set $B_1 = B(C_n)(1, \ldots, n-1, J(n, n+1))$ corresponds to $P_{B_1} = Cy^{n-1} \times I$: the bottom and top bases of $P_{B_1} \times I$ are $\{n\}, \{n+1\} \in B_1$; the side facets are $S \in B_1 \setminus \{n+1\}$ such that $\{n, n+1\}$ is either contained in $S$ or doesn’t intersect $S$. Side facets $S$ are identified with facets of $Cy^{n-1} = P_{B_1}$ by collapsing $\{n, n+1\}$ to the point $\{n\}$.

$B(C_{n+1}) \setminus B_1 = \{S \cup \{n\}, S \in S_n\} \cup \{S \cup \{n+1\}, S \in S_{n+1}\},$ where $S_n = \{i, n-1, i = 1, \ldots, n-1\}$ and $S_{n+1} = \{[1, i], i = 1, \ldots, n-1\}$. By Theorem \[ \text{[10]} \] $Cy^n$ is obtained from $Cy^{n-1} \times I$ by shaving intersections of the bottom base $F_1 \{n\}$ with the side facets $F_S$ for $S \in S_n$ and intersections of the top base $F_{i[n+1]}$ with the side faces $F_S$ for $S \in S_{n+1}$. Since bases don’t change after shaving their facets, the shaved off faces are exactly $P_{C_n[S \times P_{C_n}\{S\}} \in S_n \cup S_{n+1}$, which are $As^{i-1} \times Cy^{n-i-1}, i = 1, \ldots, n-1$ in the top and the same type faces in the bottom. By proposition \[ \text{[1]} \] we obtain inductive formulas:

\[
\gamma(Cy^n) = \gamma(Cy^{n-1}) + 2\tau \sum_{i=1}^{n-1} \gamma(As^{i-1})\gamma(Cy^{n-i-1}); \quad (5)
\]
\[
H(Cy^n) = (\alpha + t)H(Cy^{n-1}) + 2at \sum_{i=1}^{n-1} H(As^{i-1})H(Cy^{n-i-1}). \quad (6)
\]

The inductive formulas for cyclohedra are equivalent to the equations:

\[
\gamma_{Cy}(x) = 1 + x\gamma_{Cy}(x) + \tau x^2\gamma_{As}(x)\gamma_{Cy}(x), \quad \text{where} \quad \gamma_{Cy}(x) = \sum_{n=0}^{\infty} \gamma(Cy^n)x^n;
\]
\[
H_{Cy}(x) = 1 + (\alpha + t)xH_{Cy}(x) + 2atx^2H_{As}(x)H_{Cy}(x), \quad \text{where} \quad H_{Cy}(x) = \sum_{n=0}^{\infty} H(Cy^n)x^n.
\]

Set $V(x) = xH_{Cy}(x)$. Then,

\[
\frac{V}{1 + (\alpha + t)V + 2atUV} = x.
\]

Therefore,

\[
\frac{U}{(1 + \alpha U)(1 + tU)} = \frac{V}{1 + (\alpha + t)V + 2atUV}.
\]

And we obtain:

\[
V = \frac{U}{1 - atU^2}.
\]

5.3 Permutohedra

Let us apply the construction \[ \text{[1]} \] to the complete graph $K_n$. Here $V = [n]$ and $S = 2^{[n]} \setminus \emptyset, [n]$. Therefore, we shave off $\binom{n}{i}$ faces of the type $Pe^{i-1} \times Pe^{n-i-1}, i = 1, \ldots, n-1$ and obtain inductive formulas:

\[
\gamma(Pe^n) = \gamma(Pe^{n-1}) + \tau \sum_{i=1}^{n-1} \binom{n}{i}\gamma(Pe^{i-1})\gamma(Pe^{n-i-1}); \quad (7)
\]
\[
H(Pe^n) = (\alpha + t)H(Pe^{n-1}) + at \sum_{i=1}^{n-1} \binom{n}{i}H(Pe^{i-1})H(Pe^{n-i-1}). \quad (8)
\]
The inductive formulas for permutohedra are equivalent to the differential equations:

\[
\frac{d\gamma_{Pe}(x)}{dx} = 1 + \gamma_{Pe}(x) + \tau \gamma_{Pe}(x)^2, \quad \text{where} \quad \gamma_{Pe}(x) = \sum_{n=0}^{\infty} \gamma(Pe^n) \frac{x^{n+1}}{(n+1)!};
\]

\[
\frac{dH_{Pe}(x)}{dx} = (1 + \alpha H_{Pe}(x))(1 + tH_{Pe}(x)), \quad \text{where} \quad H_{Pe}(x) = \sum_{n=0}^{\infty} H(Pe^n) \frac{x^{n+1}}{(n+1)!}.
\]

One can explicitly solve the last equation and obtain:

\[H_{Pe}(x) = \frac{e^{\alpha x} - e^{tx}}{\alpha e^{tx} - te^{\alpha x}}.\]

Let \(A(n, k) = |\{\sigma \in \text{Sym}(n) : \text{des}(\sigma) = k\}|,\) then

\[H(Pe^n) = \sum_{i=0}^{n} A(n + 1, k) \alpha^k t^{n-k}.\]

### 5.4 Stellohedra

Let us apply the construction \(\Box \) to the complete bipartite graph \(K_{1,n-1}\) or \((n-1)\)-star. Here \(V = \{1\}\) and \(S = \{\{1\} \cup S, S \subseteq [2,n]\}\). Therefore, we shave off \({n-1 \choose i-1}\) faces of the type \(St^{i-1} \times Pe^{n-i-1}\), \(i = 1, \ldots, n-1\) and obtain inductive formulas:

\[\gamma(St^n) = \gamma(St^{n-1}) + \tau \sum_{i=1}^{n-1} {n-1 \choose i-1} \gamma(St^{i-1}) \gamma(Pe^{n-i-1});\]  

(9) \[H(St^n) = (\alpha + t)H(St^{n-1}) + \alpha t \sum_{i=1}^{n-1} {n-1 \choose i-1} H(St^{i-1}) H(Pe^{n-i-1}).\]  

(10)

The inductive formulas for stellohedra are equivalent to the differential equations:

\[
\frac{d\gamma_{St}(x)}{dx} = \gamma_{St}(x)(1 + \tau x \gamma_{Pe}(x)), \quad \text{where} \quad \gamma_{St}(x) = \sum_{n=0}^{\infty} \gamma(St^n) \frac{x^n}{n!};
\]

\[
\frac{dH_{St}(x)}{dx} = H_{St}(x)(\alpha + t + \alpha txH_{Pe}(x)), \quad \text{where} \quad H_{St}(x) = \sum_{n=0}^{\infty} H(St^n) \frac{x^n}{n!}.
\]

**Lemma 3.** For every tree \(\Gamma_{n+1}\) on \([n+1]\) we have \(\gamma_i(P_{\Gamma_{n+1}}) \leq \gamma_i(St^n)\).

**Proof.** For \(n = 1\) there is nothing to prove.

Assume that the lemma holds for \(m \leq n\). Let \(\Gamma_{n+1}\) be a tree on \([n+1]\). Without loss of generality, assume that \([n+1]\) is adjacent only to \(\{n\}\). Then, we can use the construction \(\Box \) putting \(\Gamma_n = \Gamma_{n+1} \setminus \{n+1\}\) and \(V = \{n\}\). For every \(i \in [1, n-1]\) there are no more than \({n-1 \choose i-1}\) elements \(S \in S; |S| = i\) and for each such \(S\) we have \(\gamma(P_{\Gamma_n[S]} \gamma(P_{\Gamma_{n[S]} \leq \gamma(St^{i-1}) \gamma(Pe^{n-i-1})}\). Therefore, comparing (11) to (9) and using the inductive assumption we obtain the lemma.

### 6 Bounds of face polynomials

**Definition 4.** The graph \(\Gamma\) is called Hamiltonian if it contains a Hamiltonian cycle, i.e. a closed loop that visits each vertex of \(\Gamma\) exactly once.

Summarizing Lemmas 1-3 and Theorems 1-2 we obtain the following results:
**Theorem 4.** For any flag $n$-dimensional nestohedron $P_B$ we have:

1) $\gamma_i(I^n) \leq \gamma_i(P_B) \leq \gamma_i(Pe^n)$;
2) $g_i(I^n) \leq g_i(P_B) \leq g_i(Pe^n)$;
3) $h_i(I^n) \leq h_i(P_B) \leq h_i(Pe^n)$;
4) $f_i(I^n) \leq f_i(P_B) \leq f_i(Pe^n)$.

**Theorem 5.** For any connected graph $\Gamma_{n+1}$ on $[n+1]$ we have:

1) $\gamma_i(As^n) \leq \gamma_i(P_{\Gamma_{n+1}}) \leq \gamma_i(Pe^n)$;
2) $g_i(As^n) \leq g_i(P_{\Gamma_{n+1}}) \leq g_i(Pe^n)$;
3) $h_i(As^n) \leq h_i(P_{\Gamma_{n+1}}) \leq h_i(Pe^n)$;
4) $f_i(As^n) \leq f_i(P_{\Gamma_{n+1}}) \leq f_i(Pe^n)$.

**Theorem 6.** For any Hamiltonian graph $\Gamma_{n+1}$ on $[n+1]$ we have:

1) $\gamma_i(Cy^n) \leq \gamma_i(P_{\Gamma_{n+1}}) \leq \gamma_i(Pe^n)$;
2) $g_i(Cy^n) \leq g_i(P_{\Gamma_{n+1}}) \leq g_i(Pe^n)$;
3) $h_i(Cy^n) \leq h_i(P_{\Gamma_{n+1}}) \leq h_i(Pe^n)$;
4) $f_i(Cy^n) \leq f_i(P_{\Gamma_{n+1}}) \leq f_i(Pe^n)$.

**Theorem 7.** For any tree $\Gamma_{n+1}$ on $[n+1]$ we have:

1) $\gamma_i(As^n) \leq \gamma_i(P_{\Gamma_{n+1}}) \leq \gamma_i(St^n)$;
2) $g_i(As^n) \leq g_i(P_{\Gamma_{n+1}}) \leq g_i(St^n)$;
3) $h_i(As^n) \leq h_i(P_{\Gamma_{n+1}}) \leq h_i(St^n)$;
4) $f_i(As^n) \leq f_i(P_{\Gamma_{n+1}}) \leq f_i(St^n)$.

**Proof of Theorem 4.** Since $\Gamma_{n+1}$ is Hamiltonian, there exists a cyclic subgraph $C_{n+1} \subseteq \Gamma_{n+1}$. Therefore, $B(C_{n+1}) \subseteq B(\Gamma_{n+1})$ and Theorem 2 allows to finish the proof.

The mentioned bounds can be written explicitly using results about $f$-, $h$-, $g$- and $\gamma$-vectors of the considered series (cf. [PRW] and [Bu1]):

$$h_i(I^n) = \binom{n}{i}; \quad h_i(As^n) = \frac{1}{n+1} \binom{n+1}{i+1} \binom{n+1}{i+1}; \quad h_i(Cy^n) = \binom{n}{i}^2;$$

$$h_i(Pe^n) = A(n+1, i); \quad h_i(St^n) = \sum_{k=1}^{n} \binom{n}{k} A(k, i-1), i > 0;$$

$$\gamma_i(I^n) = 0, i > 0; \quad \gamma_i(As^n) = \frac{1}{i+1} \binom{2i}{i} \binom{n}{2i}; \quad \gamma_i(Cy^n) = \binom{n}{i, i, n-2i}.$$
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