Renormalons from eight loop expansion of the gluon condensate in lattice gauge theory

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Abstract

We use a numerical method to obtain the weak coupling perturbative coefficients of local operators with lattice regularization. Such a method allows us to extend the perturbative expansions obtained so far by analytical Feynman diagrams calculations. In $SU(3)$ lattice gauge theory in four dimensions we compute the first eight coefficients of the expectation value of the Wilson loop on the elementary plaquette which is related to the gluon condensate. The computed eight coefficients grow with the order much faster than predicted by the presence of the infrared renormalon associated to the dimension of the gluon condensate. However the renormalon behaviour for large order is quite well reproduced if one considers the expansion coefficients in a new coupling related to the lattice coupling by large perturbative corrections. This is expected since the lattice and continuum $\Lambda$ scales differ by almost two orders of magnitude.
1 Introduction

In asymptotically free theories, such as Yang-Mills gauge theories in four dimensions, the
short distance distributions are reliably approximated by the perturbative expansion in
$\alpha_s(Q)$, the running coupling at the large momentum scale $Q$. However, even in this case the
perturbative expansions are affected \cite{1}-\cite{4} by ambiguities due to infrared (IR) renormalons,
i.e. singularities on the integration contour in the Borel variable conjugate to $\alpha_s(Q)$. The
origin of these singularities is based on the renormalization group properties. Since in computing
short distance distributions also small momenta $k$ are involved in loop integrations,
one has singularities due to the Landau pole in the running coupling $\alpha_s(k)$ at $k \ll Q$.

The IR renormalons in the expectation value of composite operators have been studied
in \cite{5} in the case of the $O(N)$ non-linear sigma model for $N \to \infty$. One finds that the related
ambiguity is absorbed by non-perturbative contributions present in the various terms of the
operator product expansion. A similar mechanism should also hold in the Yang-Mills case
and prescriptions for resolving the ambiguities have been proposed \cite{3, 4}.

Perturbative expansions of short distance distributions in QCD, such as the total
$e^+e^-$-hadronic cross section and total decay widths, are known \cite{6} up to three loops in the $\overline{\text{MS}}$
renormalization prescription. It is difficult to expect that one may reliably explore the
properties of the IR renormalons on the basis of these not very long expansions.

Recently we have proposed \cite{7} a numerical method to obtain the weak coupling pertur-
bative expansion of local observables in pure Yang-Mills gauge theory in four dimensions in
the lattice regularization. The method is based on the idea of taking the weak coupling expansion
in the Langevin stochastic formulation of the theory \cite{8} and solving numerically the
truncated set of equations corresponding to a given perturbative order. By this method we
hoped to be able to obtain perturbative expansions longer than the ones obtain by analytical
methods \cite{6, 9}.

In this paper we present the results obtained in $SU(3)$ lattice gauge theory in four di-
ensions for the first eight coefficients of the expectation value of the “plaquette variable”,
i.e. the Wilson loop on the elementary plaquette of lattice size $a$. The first four coefficients
have been presented in Ref. \cite{7}. On the basis of these new results we try to study the asymp-
totic behaviour of the coefficients for large orders and we try to analyze the IR renormalon
singularities.

We consider the standard $SU(3)$ gauge lattice action

$$S[U] = -\frac{\beta}{6} \sum_P \text{Tr} \left( U_P + U_P^\dagger \right), \quad \frac{\beta}{6} = \frac{1}{4\pi\alpha_s}, \quad (1)$$

where $\alpha_s$ is the usual coupling at the lattice scale. The sum extends to all plaquettes $P$
in a hypercubic lattice in four dimensions, $U_P$ is the plaquette gauge field obtained from
the link variable $U_\mu(x) = \exp \{ A_\mu(x)/\sqrt{\beta} \}$. The observable we consider is the dimensionless
expectation value of the plaquette

$$W_{4}^{\text{lat}}(\beta) \equiv 1 - \frac{1}{2} \langle \text{Tr} U_P \rangle. \quad (2)$$
and we compute the coefficients $c_n^{\text{lat}}$ of the weak coupling expansion

$$W_4^{\text{lat}}(\beta) \sim \sum_{n=1} c_n^{\text{lat}} \beta^{-n},$$

(3)

The physical interest of $W_4(\beta)$ is that, in the operator product expansion of this observable, the first non trivial term is the gluon condensate $\langle \alpha_s F^2 \rangle$ which has dimension four. This quantity is renormalization group invariant and therefore its contribution to $W_4(\beta)$ is proportional to $(a\Lambda)^4$, where $\Lambda$ is the Yang-Mills scale given, at two loops, by

$$(a\Lambda)^2 = C \left( \frac{\beta}{6b_0} \right)^{b_1/b_0^2} e^{-\beta/6b_0}, \quad b_0 = 11/(4\pi)^2, \quad b_1 = 102/(4\pi)^4.$$  

(4)

There is no perturbative expansion for the quantity $a\Lambda$ and therefore also for the physical condensate $\langle \alpha_s F^2 \rangle$. The only perturbative contribution in (3) is given by the first trivial term in the operator product expansion of $W_4^{\text{lat}}(\beta)$, i.e. the unit operator. The coefficients $c_n^{\text{lat}}$ are obtained from quartically divergent Feynman diagrams in which the lattice size $a$ plays the role of the UV cutoff. The IR renormalons are present in this perturbative expansion since the momenta along the loops are integrated down to zero.

The result of our calculation is that the first eight coefficients $c_n^{\text{lat}}$ grow much faster then expected from the conventional analysis of IR renormalons. For this comparison one uses the expansion

$$W_4^{\text{ren}}(\beta) = \sum_{n=1} c_n^{\text{ren}} \beta^{-n}$$

(5)

with $c_n^{\text{ren}}$ determined by the presence of an IR renormalon singularity associated to the gluon condensate, i.e. of dimension four. We find then that $c_n^{\text{lat}}$ grow with $n$ much faster than $c_n^{\text{ren}}$ for $n \leq 8$. This feature was already observed in the four loop calculation [7] but now it is much more clear. Such a fast growth could be due to large subleading contributions in $n$. Indeed, as well known [10], the lattice and continuum Λ scales differ by almost two order of magnitude so that the perturbative relation between the lattice and continuum couplings

$$\beta_{\text{cont}} = \beta_{\text{latt}} - r + O(\beta_{\text{latt}}^{-1}),$$

(6)

involves the large correction $r = 12b_0 \ln(\Lambda/\pi \Lambda_{\text{latt}})$. This suggests to compare the lattice expansion $W_4^{\text{lat}}(\beta_{\text{latt}})$ with the renormalon expansion $W_4^{\text{ren}}(\beta_{\text{cont}})$ in eq. (4) where the two couplings are related by (3). We find that for $r \sim 2.4$ the coefficients of $\beta_{\text{latt}}^{-n}$ in the expansion of $W_4^{\text{ren}}(\beta_{\text{cont}})$ agree quite well with the coefficients $c_n^{\text{lat}}$ for $n \geq 4$. Such a large value for $r$ agrees with the values obtained [10] in usual continuum renormalization schemes although it appears somehow larger than the one of MS renormalization.

In Sect. 2 we briefly recall the method for obtaining numerically the perturbative coefficients $c_n^{\text{LGT}}$ for the SU(3) lattice gauge theory in four dimension. In Sect. 3 we present our results and discuss the statistical significance. In Sect. 4 we recall the asymptotic behaviour of the coefficients $c_n^{\text{LGT}}$ deduced from the usual IR renormalon analysis by using the one and two loop running coupling. In Sect. 5 we discuss the behaviour of the eight computed coefficients in comparison with the one predicted by the IR renormalon analysis including the two loop correction of the running coupling. Sect. 6 contains some final comments. We include in the Appendix a conjecture, not plausible on the light of the present analysis, that the fast growth of the lattice coefficients $c_n^{\text{lat}}$ could be due, at least partially, to additional infrared singularities in the Feynman diagrams.
2 Weak coupling expansion on the computer

Let us briefly recall the essential ingredients of our method for calculating the weak coupling expansion in the lattice regularization. First we describe the method for a scalar field; then we proceed to the case of $SU(N)$ lattice gauge theory and its gauge fixing problem. (For a recent review about Langevin simulations see e.g. [11]).

The Langevin method consists in the simulation of a stochastic dynamical system having the field configuration space as its state space. Time evolution is dictated by the general equation

$$\frac{\partial \phi(x,t)}{\partial t} = - \frac{\partial S[\phi]}{\partial \phi(x,t)} + \eta(x,t), \quad (7)$$

where $\phi$ is the field, $S[\phi]$ the action and $\eta$ a Gaussian random noise satisfying the normalization

$$\langle \eta(x,t)\eta(x',t') \rangle = \delta(x-x')\delta(t-t'). \quad (8)$$

As a matter of fact, stochastic dynamics is devised in such a way that time averages along a trajectory converge to averages with respect to the Gibbs measure

$$\frac{1}{T} \int_0^T dt \langle O[\phi(t)] \rangle_\eta \to \frac{1}{Z} \int D\phi \ O[\phi] \ e^{-S[\phi]}. \quad (9)$$

where the suffix $\eta$ denotes an average over the Gaussian noise.

In order to implement Eq. 7 on a computer, one can take $t$ discrete with a time step $dt = \varepsilon$ [12]:

$$\phi(x, n+1) = \phi(x, n) - f(x, n), \quad (10)$$

where

$$f(x, n) = \varepsilon \frac{\partial S}{\partial \phi(x, n)} + \sqrt{\varepsilon} \eta(x, n) \quad (11)$$

and now $\eta$ is normalized by:

$$\langle \eta(x, n)\eta(x', n') \rangle = \delta_{xx'}\delta_{nn'}. \quad (12)$$

In this discrete form, Langevin equation is affected by a systematic error, which makes it necessary to extrapolate the results at $\varepsilon \to 0$ or to devise some higher order approximation to the continuum stochastic equation. In this paper we adhere to the simple recipe which consists in performing the simulation at several values of $\varepsilon$ with a linear fitting to $\varepsilon = 0$.

Let us now assume that the action is given by a free part (say $S_0 = \frac{1}{2} \sum_x \phi(x)(-\Delta + m_0^2)\phi(x)$) plus an interaction term $g \sum_x \phi^4$; the solution to Langevin’s equation will then depend parametrically on the coupling constant $g$; we insert the formal expansion

$$\phi(x, t, g) \sim \sum_{k \geq 0} g^k \phi^{(k)}(x, t) \quad (13)$$

into the Langevin evolution equation to obtain a hierarchy of stochastic evolution steps of the kind

$$\phi^{(0)}(x, n+1) = \phi^{(0)}(x, n) - \varepsilon(-\Delta + m_0^2)\phi^{(0)}(x, n) + \sqrt{\varepsilon}\eta(x, n)$$
\[ \begin{align*}
\phi^{(1)}(x, n + 1) &= \phi^{(1)}(x, n) - \varepsilon (-\Delta + m_0^2) \phi^{(1)}(x, n) - \varepsilon \left( \phi^{(0)}(x, n) \right)^3 \\
\phi^{(2)}(x, n + 1) &= \phi^{(2)}(x, n) - \varepsilon (-\Delta + m_0^2) \phi^{(2)}(x, n) - \varepsilon \left( \phi^{(0)}(x, n) \right)^2 \phi^{(1)}(x, n)
\end{align*} \]

... 

If we want to measure the expansion in \( g \) of any given observable \( O[\phi] \) we have just to insert the formal expansion (13) and take averages on the random process. For instance the expansion for the energy density will be given by 

\[ \langle E[\phi] \rangle = \langle \left( \nabla \phi(x) \right)^2 \rangle \sim \langle \left( \nabla \phi^{(0)}(x) \right)^2 \rangle + 2g \langle \nabla \phi^{(0)} \cdot \nabla \phi^{(1)} \rangle + g^2 \left( \langle \left( \nabla \phi^{(1)} \right)^2 \rangle - 2 \nabla \phi^{(0)} \cdot \nabla \phi^{(2)} \right) + \cdots \]

The idea is very general and it can be applied to spin systems or to lattice gauge models.

In \( SU(3) \) lattice gauge theory the action is given by Eq. (14) defined on a four dimensional periodic lattice. The link gauge variables \( U_\mu(x) \) are \( SU(N) \) matrices labelled by the vector index \( \mu = 1, \ldots, 4 \) and by the site \( x \) of the lattice. Each configuration is then described by 36 complex numbers at each lattice site. In this case the Langevin evolution step in discrete time \( \varepsilon \) is given (see Ref [13]) 

\[ U_\mu(x, n + 1) = e^{-F_\mu(x, n)} U_\mu(x, n), \]

where 

\[ F_\mu(x) = \frac{\varepsilon \beta}{4N} \left[ \sum_{U_P \supset U_\mu} \left( U_P - U_P^\dagger \right) - \frac{1}{N} \sum_{U_P \supset U_\mu} \text{Tr} \left( U_P - U_P^\dagger \right) \right] + \sqrt{\varepsilon} H_\mu(x) \]

and \( H_\mu(x) \) is a traceless antihermitian matrix extracted at random independently at each step from a Gaussian ensemble with normalization given by 

\[ \langle H_{ik}(x, n) H_{lm}(x', n') \rangle_H = \delta_{il} \delta_{km} - \frac{1}{N} \delta_{ik} \delta_{lm} \delta_{xx'} \delta_{nn'}. \]

In the weak coupling limit we can parameterize the link variables in terms of the usual gauge potentials \( A_\mu \), namely we set 

\[ U_\mu(x) = e^{A_\mu(x)} / \sqrt{\beta}, \quad A_\mu(x)^\dagger = -A_\mu(x), \quad \text{Tr} \ A_\mu = 0. \]

It follows that Langevin’s equation takes the form: 

\[ e^{A_\mu(x, n + 1)} / \sqrt{\beta} = e^{-F_\mu(x, n)} e^{A_\mu(x, n)} / \sqrt{\beta}, \]

where \( n \) is Langevin’s discrete time. As in the scalar case the trajectories \( A_\mu(x, n) \) will depend parametrically on \( \beta \); we expand them in a formal series in powers of \( \beta^{-1/2} \)

\[ A_\mu(x) = \sum_{k \geq 0} \beta^{-k/2} A_\mu^{(k)}(x). \]

and we rescale the time step \( \varepsilon = \tau / \beta \) in such a way that Langevin’s algorithm can be reformulated in terms of the fields \( A_\mu^{(k)} \). The formulae rapidly get rather cumbersome but they can easily be handled by a symbolic language or defining convenient structures as it is natural within the ZZ language of APE.
Figure 1: The eight independent time averages for each of the observables yielding an estimator of $c_1^{LGT}, \ldots, c_6^{LGT}$ at $\tau = 0.01$.

We have reported the calculation of the average plaquette up to fourth loop ($\beta^{-4}$) in [7]. It was shown that in order to obtain a stable process one has to impose some kind of gauge fixing; this fact seems quite natural in view of the infinite number of zero modes which one has to deal with in the expansion around a given classical vacuum. A gauge fixing procedure for Langevin quantization was proposed long ago by Zwanziger [14] and it was implemented on the lattice in [15]. The idea consists in introducing a new source in Langevin’s equation which does not modify the asymptotic stationary distribution $\exp\{-S\}$ but it acts as an attractor towards the Landau gauge manifold defined by $\nabla \cdot A = 0$. The implementation on the lattice consists in a gauge transformation which is executed after each Langevin step:

$$U_\mu(x) \rightarrow e^{w[U_\mu]}U_\mu(x) e^{-w[U_\mu(x+\mu)]},$$

(20)

with

$$w[U_\mu(x)] = \lambda \sum_\mu \left( \Delta_{-\mu} \left[ U_\mu(x) - U_\mu(x - \mu) \right] \right)_{tr}, \quad \Delta_{-\mu}U_\nu(x) \equiv U_\nu(x) - U_\nu(x - \mu),$$

(21)

where the suffix “tr” denotes the traceless part of the matrix and $\lambda$ is a free parameter which we choose proportional to the time step $\tau$.

3 Eight loop results for the plaquette

The algorithmic setup presented in the last section lends itself to the possibility of performing high order weak coupling expansions; needless to say, since the calculation to $n$-th loop involves $2n - 1$ auxiliary fields $A^{(0)}, A^{(1)}, \ldots, A^{(2n-2)}$, it tends to be quite demanding in terms of computer memory and cpu time. The results reported here were obtained on an array processor with 64 units of the APE family. They required approximately 600 hours (roughly equivalent to 5000 hours on a single cpu of a Cray YMP) for the final run, which consisted of 24 independent Langevin histories of an average 4000 steps. Since the finite size effects on the single plaquette expectation value are rather small, we work on a $8^4$ lattice with periodic boundary conditions which fits just fine on the memory of a single APE board, thus obtaining eight statistically independent Langevin histories at each run. There were three independent runs at $\tau = .02, .015, .01$ from which we extrapolate at the desired $\tau = 0$ limit. We report in Fig. 1 the signal we get for $\tau = 0.01$. Since the higher loop observables depend on the lowest orders, we chose to start measuring the various observables in cascade, in order to let the system gradually thermalize.

The statistical errors are computed by taking the standard deviation divided by square root of the number of statistically independent samples, which we estimate to be around 64. This is a conservative estimate based on the presence of eight independent Langevin trajectories assuming an autocorrelation time of 300 steps. We plan to have a reliable measure of autocorrelation in the near future, when longer Langevin histories will be available. Fig. 2 reports the process of extrapolation, together with the central value and the estimated statistical error for the first eight expansion coefficients.
| n  | 1     | 2     | 3     | 4     | 5     | 6     | 7     | 8     |
|----|-------|-------|-------|-------|-------|-------|-------|-------|
| $c_{\text{Langevin}}$ | 1.998(1) | 1.218(1) | 2.940(5) | 9.28(2) | 34.0(2) | 134.9(9) | 563(5) | 2488(29) |
| $c_{\text{analytic}}$ | 2 | 1.218(7) | 2.9602 | — | — | — | — | — |

Table 1: The expansion coefficients $c_n^{LGT}$ obtained by the stochastic method and the known coefficients from Ref.(9).

Figure 2: The extrapolation at $\tau = 0$ of the first eight perturbative coefficients $c_n^{LGT}$ on the $8^4$ lattice.

We end this section by a remark about the finite size corrections. To get a rough estimate on them we performed a series of runs on a very small lattice ($L^4$ with $L = 4$) and on the largest feasible lattice on our APE ($L = 12$). We observe an increase in the values of $c_n(L)$ with $L$ which is more sensible for higher loops. Essentially is appears rather reasonable that finite size corrections become important as $n/L \sim 1$. The overall picture seems to agree with the analysis of Ref.[9], but a deeper analysis would admittedly be desirable.

## 4 IR renormalon analysis

We recall here the usual conventional IR renormalon analysis. The Feynman diagrams of the plaquette expectation value $W_4(\beta)$ are quadratically divergent. In general the perturbative contribution $W_{2\sigma}(\beta)$ to the expectation of a composite operator which has UV divergences of degree $2\sigma$ (for the plaquette $2\sigma = 4$) has the form

$$W_{2\sigma}(\beta) = \int_0^{Q^2} \frac{dk^2}{k^2} \left( \frac{k^2}{Q^2} \right)^\sigma f(k), \quad \frac{6}{\beta} = 4\pi\alpha_s(Q),$$

where $Q$ is of the order of the inverse of lattice size, $Q \sim 1/a$, which plays the rôle of the UV cutoff. The function $f(k)$ is dimensionless, renormalization group invariant and to first order is given by the one gluon exchange diagram, i.e. is proportional to $\alpha_s(Q)$. In QED the contributions from the resummation of fermion loop diagrams [1] gives $f(k) = \alpha(k)$, the running coupling in QED. The important simplification is that, from Ward identities, the vertex and wave function corrections cancel. In QCD the situation is different. Actually to reconstruct $\alpha_s(k)$ one needs also vertex and wave function corrections which do not cancel. So in QCD one needs to consider more diagrams and there are no simple criteria to select the important ones. By taking into account that $f(k)$ is renormalization group invariant one assumes that also in Yang-Mills gauge theory

$$f(k) = \alpha_s(k).$$

In this case the Landau pole in $\alpha_s(k)$ is at a small value $k \ll Q$ and therefore the integral in (22) diverges. The perturbative expansion of $W_{2\sigma}(\beta)$ in $\alpha_s(Q)$ is obtained by expanding $\alpha_s(k)$. The Landau pole disappears and each coefficient is finite. The divergence of (22) at small $k$ is reflected on the fact that the coefficients grow fast and the expansion diverges. A
A nice framework for discussing this feature is given by the Borel transform. Introducing the (Borel) variable $z$

$$z \equiv z_0 \left(1 - \frac{\alpha_s(Q)}{\alpha_s(k)}\right), \quad z_0 = \frac{\sigma}{6b_0}, \quad (24)$$

and using (14) the integrand of (22) can be written

$$\frac{dk^2}{k^2} \left(\frac{k^2}{Q^2}\right)^\sigma \alpha_s(k) = \frac{4\pi b_0}{\beta(\alpha_s(k))} \frac{6}{4\pi} \int_0^\infty dz e^{-\beta z} \left(1 - \frac{z}{z_0}\right)^{-1-\gamma}, \quad \gamma = \sigma \frac{b_1}{b_0}, \quad (25)$$

where $\beta(\alpha_s)$ is the beta function

$$\frac{\beta(\alpha_s(k))}{4\pi b_0 \alpha_s(k)^2} = 1 + 4\pi b_0 \frac{b_1}{b_0} \alpha_s(k) + \cdots \quad (26)$$

We have now to consider the range of integration for $z$ corresponding to $0 < k < Q$. Beyond the Landau pole at $z = z_0$ the mapping (24) becomes ambiguous. However at two loops we have

$$z = \frac{4\pi \sigma}{6} \alpha_s(Q) \left\{1 + 4\pi \frac{b_1}{b_0} \alpha_s(Q)\right\} \ln \frac{Q^2}{k^2} + \cdots,$$

thus we assume the range $0 < z < \infty$. Neglecting the corrections in (26) we obtain the Borel representation

$$W_{\text{ren}}^{2\sigma}(\beta) = \int_0^{Q^2} \frac{dk^2}{k^2} \left(\frac{k^2}{Q^2}\right)^\sigma \alpha_s(k) = \frac{6}{4\pi} \int_0^\infty dz e^{-\beta z} \left(1 - \frac{z}{z_0}\right)^{-1-\gamma}, \quad (27)$$

with the perturbative coefficients given by

$$W_{\text{ren}}^{2\sigma}(\beta) = \sum_{n=1}^{\infty} c_n^{\text{ren}}(2\sigma, b_1) \beta^{-n}, \quad c_n^{\text{ren}}(2\sigma, b_1) = C \frac{\Gamma(\gamma + n)}{z_0^{\gamma+n}}, \quad \gamma = \sigma \frac{b_1}{b_0}, \quad z_0 = \frac{\sigma}{6b_0}. \quad (28)$$

where $C$ is a numerical constant. A correction in the integrand in (22) proportional to $\alpha_s(k)$, as given for instance by taking into account the two loop contribution of the beta function in (26), simply modifies the normalization constant $C$. The IR renormalon singularity on the integration contour at $z = z_0$ corresponds to the Landau pole of $\alpha_s(k)$. For the plaquette we have

$$2\sigma = 4, \quad z_0 = \frac{1}{3b_0} = 4.785... \quad (29)$$

To give a meaning to the representation (27) one has to give a prescription to specify the way the integration contour avoids the singularity $z = z_0$. The various prescriptions differ by contributions proportional to the residue at the pole. The ambiguity in (27) is then given by

$$\Delta W_s(\beta) \sim e^{-\beta z_0} = e^{-\frac{\beta \sigma}{6b_0}} \sim (a\Lambda)^{2\sigma}. \quad (30)$$

For the plaquette $2\sigma = 4$ this ambiguity is of the same order as the condensate ($\langle \alpha_s F^2 \rangle \sim \Lambda^4$), i.e. the first non trivial term in the operator product expansion of the plaquette. This suggests that the IR renormalon ambiguity should be absorbed by non perturbative contributions coming from the various terms of the operator product expansion.
5 Renormalon and the numerical coefficients

In Fig. 3 we plot the eight coefficients $c_{\text{lat}}^n$ of the perturbative expansion of the plaquette $W_4^{\text{lat}}(\beta)$ in (3) obtained from the numerical method reported in Sect. 3. We plot also the coefficients $c_{\text{ren}}^n$ for the IR renormalon in (28) with $2\sigma = 4$ and $z_0$ is given in (29). The normalization constant $C$ is fixed in such a way that $c_{\text{ren}}^8 = c_{\text{lat}}^8$. From the comparison we conclude that the growth of the coefficients $c_{\text{lat}}^n$ is much stronger than the one of $c_{\text{ren}}^n (4, b_1)$.

A simple explanation for the discrepancy observed between $c_{\text{ren}}^n$ and $c_{\text{lat}}^n$ is that the coupling $\beta = \beta_{\text{latt}}$ in the lattice action (1) and the coupling suitable for the analysis of the IR renormalons in (22) are related by large perturbative corrections. One expects that the continuum coupling $\beta_{\text{cont}}$ is the suitable coupling for the renormalon analysis. As well known in the Yang-Mills case the $\Lambda$ scales in the lattice and continuum renormalization schemes are quite different

$$\Lambda_{\text{cont}} = K \Lambda_{\text{latt}},$$

with large values for $K$. In the $\overline{\text{MS}}$ scheme one has $K = 28.8$. If the coupling $\alpha_s(Q)$ is defined by the static potential between quark-antiquark pair or by the point vertex one finds the constants $K = 46.1$ or $K = 69.4$ respectively. This relation can be converted in terms of a relation between the lattice and continuum scale. From (4) and (31) one finds the relation between the two couplings where $r = 1.85$, $r = 2.25$ and $r = 2.6$ in the renormalization scheme in which the coupling is defined in $\overline{\text{MS}}$, by the static potential or by the three point vertex respectively.

In order to analyze the renormalon behaviour of the computed lattice coefficients $c_{\text{lat}}^n$ we compare the lattice expansion (3) in the coupling $\beta_{\text{latt}}^{-1}$ with the renormalon expansion (28) in the coupling $\beta_{\text{cont}}^{-1}$

$$W_{2\sigma}^{\text{ren}}(\beta_{\text{cont}}) = \sum_{n=1} c_{\text{ren}}^n (2\sigma, b_1) \beta_{\text{cont}}^{-n} = \sum_{n=1} C_{\text{ren}}^n (2\sigma, b_1) \beta_{\text{latt}}^{-n},$$

where we assume the following relation between the two couplings

$$\beta_{\text{cont}} = \beta_{\text{latt}} - r - \frac{r'}{\beta_{\text{latt}}},$$

so that $C_{\text{ren}}^n (2\sigma, b_1) = c_{\text{ren}}^n (2\sigma, b_1)$ for $r = r' = 0$. The parameter $r'$ takes contributions from three loops. We have $r' = 6r_1/b_0 + 36\delta b_2/b_0$ with $\delta b_2$ the difference between the three loop beta function coefficients in the lattice and continuum renormalization schemes.

In Fig. 3 we plot the new coefficients $C_{\text{ren}}^n (4, b_1)$ obtained by fitting the two parameters $r$ and $r'$. We find $r = 2.41$ and $\delta b_2 = .12$. For $n > 4$ we have a very good agreement between $c_{\text{lat}} (4, b_1)$ and $C_{\text{ren}}^n (4, b_1)$.

The value of $r = 2.41$ corresponds to the scale constant in (31) given by $K = 56.1$ which is between the ones obtained by defining $\alpha_s(Q)$ by the static potential or by the three vector coupling and smaller than the one in the $\overline{\text{MS}}$ scheme.

\footnote{Notice that this value is remarkably close to Lepage and Makenzie’s (10) taking into account the effective momentum $q^* \sim 2.8$ proper to $\alpha_V$.}
Figure 3: The values of $c_{\text{lat}}^n$, ($n = 1 \ldots, 8$) obtained by the stochastic method together with renormalon coefficients $c_{\text{ren}}(4, b_1)$ and $\bar{c}_{\text{ren}}(4, b_1, r)$ with $r = 2.41$.

Since the value of $r$ is large, the relation between $c_{\text{lat}}^n$ and $C_{\text{ren}}^n$ involves large cancelations. This implies that the results are sensible to $r$, as is rather evident from Fig 3.

We have studied also the dependence on the two loop corrections by setting $b_1 = 0$ in the coefficients $C_{\text{ren}}^n(4, b_1 = 0)$. In order to obtain the fit with the lattice coefficients for $n > 4$ we find $r = 4.13$ rather larger than the two loop value given before.

6 Final considerations

We used a numerical method to obtain on a computer long perturbative expansion in four dimensional Yang-Mills theory with a lattice regularization. The expansion parameter is the lattice coupling $\beta_{\text{latt}} = 2\pi\alpha_s/3$ entering in the Wilson action (1) at the lattice size $a$. The large order behaviour of the plaquette perturbative coefficients $c_{\text{lat}}^n$ agrees for $n > 4$ with the presence of a IR renormalon at (29) with $2\sigma = 4$ corresponding to the dimension of the gluon condensate $\langle \alpha_s F^2 \rangle$. However to obtain such a result one has to consider large corrections in the perturbative relation (1) between the lattice coupling $\beta_{\text{latt}}$ and the continuum coupling $\beta_{\text{cont}}$ in which one discuss the renormalons. Such a large perturbative correction ($r = 2.41$) is expected since, as well known [10], the $\Lambda$ scales for the lattice and continuum regularizations differ almost by two orders of magnitude. It should be noticed that since $c_{\text{lat}}^n$ grow fast with $n$ and $r$ is large the coefficients in the continuum regularization are obtained in terms of the lattice ones by large cancelations.

Since there are large perturbative corrections between lattice and continuum couplings it may be important to know the perturbative relation in (1) with many more coefficients. This could be attempted for instance by computing perturbatively by our method the static quark-antiquark potential.

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All the numerical results presented here have been obtained on a APE computer; we are indebted to Nicola Cabibbo, Franco Marzano and the APE group for providing us with this facility.

Appendix

As discussed in the text the plausible explanation of the quite fast growth of the lattice coefficients $c_{\text{lat}}^n$ is the large perturbative corrections between the lattice and continuum couplings (1). However we would like to add an alternative explanation. The fast growth could be explained by an effective value of the dimension $2\sigma$ much smaller than the actual value $2\sigma = 4$. Indeed for the effective value of $2\sigma = 1$ the coefficients $c_{\text{ren}}(1, b_1)$ in (28) fit...
the numerical values of $c_n^\text{lat}$ for $n \gtrsim 3 - 4$. To explain this smaller effective dimension suppose that, as suggested by perturbative QCD studies (see [16, 17]), the argument of the running coupling $\alpha_s$ is modified by infrared corrections of Feynman diagrams in the Minkowski form so that the integrand $f(k)$ is (22) is given by

$$f(k) = \int_0^1 dx \alpha_s(xk).$$

(34)

The perturbative contributions of $f(k)$ in the expansion parameter $\alpha_s(k)$ are finite and given by integrations over $\ln^n x$. Resumming the expansion in $\alpha_s(k)$ one has a Landau pole in the variable $xk$ instead of $k$. By assuming for the integrand in (22) the expression in (34) one finds that the renormalon contribution in (22) is

$$W_{2\sigma}^{\text{ren}}(\beta) = \int_0^{Q^2} \frac{dk^2}{k^2} \left( \frac{k^2}{Q^2} \right)^\sigma \int_0^1 dx \alpha_s(xk) = \frac{1}{2\sigma - 1} \int_0^{Q^2} \frac{dq^2}{q^2} \alpha_s(q) \left\{ \left( \frac{q^2}{Q^2} \right)^{\frac{1}{2}} - \left( \frac{q^2}{Q^2} \right)^\sigma \right\}.$$  

(35)

The first term, which gives the dominant contribution, is just the expression in (22) with dimension $2\sigma \to 1$ with the Borel singularity at the universal value independent of the dimension $\sigma$

$$2\sigma_{\text{eff}} = 1, \quad z_{\text{eff}}^0 = \frac{1}{12b_0} = 1.196 \ldots$$

(36)

This Landau singularity would correspond to a perturbative ambiguity given by

$$\Delta W_2(\beta) \sim e^{-\beta z_0^{\text{eff}}} \simeq (a\Lambda).$$

(37)

This large non perturbative corrections has been observed for quantities such as the average trust [18].

The prove of the conjecture (34) would need a difficult analysis since one should study contributions in Feynman diagrams of terms with infrared singularities which are integrable. If the contributions in (34) are present one should care about the mechanism to resolve the perturbative ambiguity in (37). There are no operators to cure the ambiguity associated to dimension $2\sigma_{\text{eff}} = 1$. There is no contradiction with the analysis in the $O(N)$ non-linear planar sigma model in two-dimensional [3] in which the position of the renormalon is actually related to the real dimension $2\sigma$. In two dimensions the structure of Feynman diagram singularities is completely different from the case of Yang-Mills theory in four dimensions especially if one uses the Minkowski form. The only connection between these models is the sign of the beta function, i.e. the $\alpha_s^n(Q) \ln^n k$ resummation.

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