Double Slip Effects and Heat Transfer Characteristics for Channel Transport of Engine Oil With Titanium and Aluminum Alloy Nanoparticles: A Fractional Study

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This work was supported in part by the Center of Excellence in Theoretical and Computational Science (TaCS-CoE), King Mongkut’s University of Technology Thonburi (KMUTT), and in part by the Thailand Science Research and Innovation (TSRI) Basic Research Fund: Fiscal Year 2021 under Project 64A306000005. The work of Asifa and Talha Anwar was supported by the Petchra Pra Jom Klao Ph.D. Research Scholarship under Grant 25/2563 and Grant 14/2562.

ABSTRACT The core purpose of this study is the formulation of a fractional model to anticipate the improvement in heat transfer potential of a particular diathermal oil i.e. engine oil under thermal radiative flux. The magnetohydrodynamic (MHD) freely convectional transport of two different types of engine oil based nanofluids comprised of Titanium (Ti6Al4V) and Aluminum (AA7075) alloy nanoparticles is considered in a vertical channel frame. In addition, the channel is assumed to be embedded in a permeable media and slip effects are observed at both ends. The transmutation of the governed model from classical to fractional environment is achieved by operating the Atangana-Baleanu derivative. To procure solutions of the proposed fractional model, Laplace transform is employed with an adequate choice of some unit-free quantities. Numerical simulations are performed and outcomes are conveyed through graphical illustrations to discuss the contribution of considered alloy nanoparticles in flow mechanism and thermal behavior of engine oil. It is reported that Ti6Al4V is more effective to enhance the thermal efficiency of engine oil as compared to AA7075. It is claimed that there is an augmentation of 32.50% in the heat transfer rate of engine oil due to Ti6Al4V, which is almost twice the improvement in heat transfer rate provided by AA7075. Furthermore, the slip parameters lead to expedite the channel flow of engine oil. This study culminates that Ti6Al4V and AA7075 significantly improve the lubrication and cooling characteristics of engine oil.

INDEX TERMS Double slip effects, fractional study, engine oil, alloy nanoparticles, heat transfer rate.

I. INTRODUCTION

In this modern era, where objects are reducing in size with enhanced practical features, nanotechnology is an exponentially progressing field, which is known as the engineering of atomic-scale functional systems. The existence of nanotechnology can be realized in multiple industrial and daily life problems, for instance, the use of silver nanoparticles as antibacterial agents, repairing and improving biological tissues for their proper functioning, and precise transportation of drugs in the body [1], [2]. Thanks to nanotechnology, computers are inexpensive, fast, and can store large-scale data sets due to improved storage capacity. The traditional materials are replaced with nanomaterials like nanopillars to combat the production charges of solar cells. Nanotechnology has enabled synthetic chemistry to develop any desired...
structure of small molecules, and these techniques are employed to manufacture a wide range of effective chemicals like pharmaceuticals and polymers. Further applications involve manufacturing of cars with nanomaterials to reduce the number of required metals and to improve fuel performance, infusion of bandages with nanoparticles for quick healing of cuts, and preparation of transparent nanoparticles based sunscreens [3]. One of the fundamental units of nanotechnology is nanofluid that provides solutions to several heat transfer problems. In recent times, an intriguing subject of major concern for several sub-branches of engineering and science is the cooling of channels, ducts, pipes, and other involved geometries by dint of conventional heat transfer. Usually, some conventional fluids such as engine oil, water, and kerosene oil are considered for heating and cooling purposes to manage thermal efficiency of advanced technology industries with technical challenges of improving the cooling performance of thermal systems and to minimize the surface area. The basic research challenges for thermal management of high-tech industries can be categorized into three types: (1) discovering appropriate geometry or optimal functioning of cooling devices (2) minimizing characteristic length (3) improving heat transfer characteristics of coolants. Choi’s idea of suspending different nanoparticles in conventional fluids makes an effective contribution to address the last challenge [4].

Nanofluids usually incorporate oxide, ceramic, and metallic particles, and so far, various one and two-step techniques are proposed to prepare nanofluids [5]. From many engineering applications, conventional fluids are being replaced with nanofluids because there exist certain deficiencies in the thermal and physical characteristics of these fluids. On the other end, the newly developed class of nanofluids is considered as an advanced generation of heat transferring fluids owing to their enhanced thermal features. The widespread applications of nanofluids in industrial and biomedical fields include drug delivery, detection of image flaws, electronic cooling, cancer treatment, determination and separation of bio-molecules, solar collectors, biosensors, and so forth [6]–[8]. Scientific research on nanofluids has been flourishing for the last two decades to chew over current and projected goals, utilities, and challenges. Lacabazzi et al. [9] used the idea of phonon liquid theory to investigate various mechanisms such as clustering, mass difference scattering, and layering, which influence the thermal conductance of a nanofluid. By performing theoretical and experimental analyses they reported that mass difference scattering is the fundamental mechanism that leads to intensively diminish the thermal conductivity of observed nanofluid. Milanesi et al. [10] examined CuO and Cu nanoparticles to explain variances between their thermal conductivities by means of a layering mechanism. The numerical inspection unfolded that water molecules form shell-like layers around Cu, which elaborates the greater thermal conductivity of Cu as equated to the thermal conductivity of CuO. They experimentally verified this result as well.

Saffarian et al. [11] communicated that in addition to a choice of suitable nanofluids, a change in flow direction is a supplementary technique to enhance the heat transfer coefficient for flat plate solar collectors. They considered the flow of CuO-water and Al2O3-water nanofluids through spiral, wavy, and U-shaped pipes of uniform lengths and concluded a rise of 78% in heat transfer with the selection of CuO-water and wavy geometry. Sreenivasulu et al. [12] contributed a computational model with numerical results to explore the impacts of axial slip and Lorentzian force on nanofluid flow past an extending surface. In experimental work, Sun et al. [13] measured the convectional heat transfer coefficient for migration of Fe3O4-water nanofluid through a circular tube subject to the influence of a magnetic field. The noteworthy role of gold nanoparticles in double-diffusive transport of blood inside an asymmetric channel under Hall effects is graphically evaluated by Asha and Sunitha [14]. Reddy et al. [15] dissected heat transmission properties of kerosene and methanol based ferrofluids for 3D flow along an extending boundary while deliberating the radiation in nonlinear form. Some studies imparting useful results about nanofluids are listed in [16]–[23].

Another term to specifically represent the oil based nanofluid is known as Nanolubricant [24]. An all-inclusive review of literature eventuates that a limited number of studies have been conducted for nanolubricants [25]–[27]. Several industrial processes demand the preparation of diathermal oil based nanofluids with enhanced heat transport capacity to effectively serve machine and engine lubrication roles. Diathermal oils have a wide variety of practical utilities, for example, they are used as cooling agents to minimize the temperature of electric transformers. Moreover, the property of diathermal oils to do not easily adhere to other materials makes them potential lubricants for many engineering exercises. In internal combustion engines of multiple machines such as lawnmowers, vehicles, and gas turbines, surfaces of the parts moving over each other strike, and the consequent friction consumes a notable proportion of power by producing heat from kinetic energy. It may also damage or deform the concerned parts, which results in poor efficiency and the engine’s devaluation. All these ramifications can be avoided through proper lubrication of the engine utilizing diathermal oils. Furthermore, engine longevity, reduced wear (material’s deformation at inter-facial surfaces), decreased power wastage, and minimized consumption of fuel are supplementary advantages. To control the temperature of the engine, a principal task of diathermal oils is to transfer away heat produced due to the movement of the engine’s parts [28]. Hence, the usage of nanoparticles to upgrade physical and thermal attributes of diathermal oils will surely ameliorate the thermal management propensity of these oils. Consequently, heat transfer challenges in highly sensitive systems can be encountered in a better and effective way. Some investigations on nanolubricants are documented in [29]–[32].

For the last two decades, it has been witnessing that modeling trends to explain real-world situations are shifting from
classical calculus to fractional calculus. The idea of fractional calculus was initiated by Leibniz that what would be the best physical description of $d^n/dx^n$ when $n$ is replaced with 1/2. After this, numerous mathematicians and researchers began working in this direction and proposed several non-integer order derivatives coupled with particular features and complications. Since in the formulation of a non-integer order derivative, a kernel is integrated with the usual derivative so, the goal of propounding multiple such derivatives was the characterization of different kernels to come up with the most effective but practical one. Based on the power-law kernel, Caputo and Riemann-Liouville are the most popular and extensively operated fractional derivatives but, earlier indicates kernel singularity issues while differentiation of constant functions does not lead to zero value in case of the former fractional derivative [33]. These deficiencies narrowed down the functionality of the aforementioned derivatives however, Caputo and Fabrizio proffered a novel non-integer order derivative to circumvent evident shortcomings of previously proposed derivatives with the assistance of an exponential kernel [34]. This derivative gathered significant appreciation from the scientific community however, after few studies, researchers reported the locality problems of the respective kernel, which means that this derivative is not applicable for a suitable explanation of memory effects. To overcome this drawback, a fractional derivative was recently developed by Atangana and Baleanu, which embodies a non-local and non-singular kernel [35].

Recent theoretical and experimental analyses report that mathematical modeling of physical mechanisms performed on the basis of fractional approaches is more accurate, flexible to variations in initial data, and provide a better explanation. Therefore, widespread implementations of fractional derivatives can be viewed in many areas such as demography, economics, quantum theory, bioengineering, rheology, medicines, control theory, and electromagnetism. Iqbal et al. [36] proposed a nonlinear fractional model with a structure-preserving numerical scheme to explore the dynamics of HIV/AIDS outbreak. The crucial role of fractional calculus in the evolution of mathematical economics is comprehensively explained by Tarasov [37]. Based on fractional modeling, Caputo and Cametti [38] presented their results to discuss issues and to date progress on drug permeation model. Asifa et al. [39] modified some existing models in terms of fractional derivatives to conduct a detailed dynamical analysis of COVID-19, and to describe the effectiveness of various precautionary measures such as lockdown and quarantine in controlling the expansion of decease. When it comes to fluid dynamics, specific knowledge about the rheological nature of a particular nanofluid determines its efficacy to encounter heat transfer challenges. The fractional derivative is a useful tool to study and analyze certain rheological attributes of nanofluids. Some newly investigated results regarding applications of fractional derivatives on nanofluids can be viewed from [40]–[43].

The literature survey about nanofluids highlights that Al₂O₃, Cu, Ag, and CuO are the most commonly studied nanoparticles from the last twenty years. However, Ti₆Al₄V and AA7075 alloy nanoparticles have abundant significant applications because of their efficacious thermal and physical attributes. For instance, biological surgeries, preparation of plastics and soaps, aerospace structures, bone plate operations, fabrication of micro-sensors, and heat treatments involve these nanoparticles. The core motivation of this article is the development of a fractional model to predict the augmentation in heat transfer performance of engine oil due to the suspension of the aforementioned alloy nanoparticles. In addition to MHD and radiative thermal flux, slip effects on both ends of the flow enclosing channel are taken into account. The non-integer order model is formulated via the application of the Atangana-Baleanu derivative and solved by dint of Laplace transform. Graphical representations of computed solutions are provided to serve various tasks, for instance, comparison of fractional and ordinary models, verification of solutions, and to analyze the control of concomitant parameters. Furthermore, numerical results for heat transfer rate and shear stress are reported in tabular form to gain a clear understanding of these physical phenomena.

II. STATEMENT AND MODELING OF THE PROBLEM

In this study, the impacts of Titanium and Aluminum alloy nanoparticles on the thermal efficiency of engine oil are investigated during its flow through two vertical plates nested in a permeable medium and separated by a distance $L$. The host fluid contains spherical-shaped nanoparticles, and magnetic field and thermal radiative flux are operating normally to the direction of flow. Moreover, slip effects on both plates of the micro-channel are considered, and a thermal state of equilibrium is assumed between engine oil and nanoparticles. Initially, the temperature of the system is $T_L$, and it is considered that channel plates and the nanofluid exhibit no motion at time $\tau = 0$. Later, for $\tau > 0$, free convection takes place, and the temperature of the right plate enhances from $T_L$ to $T_W$, and nanofluid starts to flow along with the $\tilde{x}$-axis. Based on the supposition of the infinitely long channel, only time $\tau$ and spatial component $\tilde{y}$ appear as independent variables in primary equations. The geometrical representation of the stated problem is shown in Fig. 1. Keeping view of Rosseland [44] and Boussinesq [45] approximations, the governing model to indicate flow and heat transfer processes is presented as

$$
\dot{\rho}_{nf} \left\{ \frac{\partial \tilde{w}(\tilde{y}, \tilde{\tau})}{\partial \tilde{\tau}} + \beta_b \tilde{w}(\tilde{y}, \tilde{\tau}) \right\} = \tilde{\mu}_{nf} \frac{\partial^2 \tilde{w}(\tilde{y}, \tilde{\tau})}{\partial \tilde{y}^2} + g \left( \tilde{\rho}_{nf} \right) \tilde{T}(\tilde{y}, \tilde{\tau}) - g \left( \tilde{\rho}_{nf} \right) \tilde{T}_L \tilde{\rho}_{nf} \frac{\phi_p}{\kappa_1} \tilde{w}(\tilde{y}, \tilde{\tau}) - \beta_0^2 \rho_{nf} \tilde{w}(\tilde{y}, \tilde{\tau}),
$$

(1)
where \( \kappa_1 \) indicates permeability of the medium, \( \kappa_2 \) is Rossland absorption coefficient, \( \beta_0 \) denotes magnetic strength, \( \beta_b \) is Brinkman fluid parameter, \( \sigma_0 \) is Stefan-Boltzmann constant, and \( \Phi_p \) denotes porosity of the medium. The initial conditions, slip flow conditions on the channel edges, and temperature boundary conditions are given as

\[
\begin{align*}
\tilde{\tau} = 0 : & \quad \tilde{T}(\tilde{y}, \tilde{\tau}) = \tilde{T}_L, \quad \tilde{w}(\tilde{y}, \tilde{\tau}) = 0, \quad \text{for } \tilde{y} \in (0, L), \\
\tilde{y} = 0 : & \quad \tilde{T}(\tilde{y}, \tilde{\tau}) = \tilde{T}_L, \quad \tilde{w}(\tilde{y}, \tilde{\tau}) = a_1 \frac{\partial \tilde{w}(\tilde{y}, \tilde{\tau})}{\partial \tilde{y}} = 0, \\
\tilde{y} = L : & \quad \tilde{T}(\tilde{y}, \tilde{\tau}) = \tilde{T}_W, \quad \tilde{w}(\tilde{y}, \tilde{\tau}) = a_2 \frac{\partial \tilde{w}(\tilde{y}, \tilde{\tau})}{\partial \tilde{y}} = 0.
\end{align*}
\]

### A. PHYSICAL AND THERMAL FEATURES OF NANOFUID

The mathematical relations to efficiently anticipate the thermo-physical features of nanofluid presented in Eqs. (1) and (2) such as heat-conduction potential \( \kappa_{nf} \), viscosity \( \mu_{nf} \), heat capacitance \( \tilde{\rho} \tilde{C}_p_{nf} \), density \( \tilde{\rho}_{nf} \), electrical conductivity \( \tilde{\sigma}_{nf} \), and thermal expansion \( \tilde{\beta} \tilde{\beta}_{nf} \) are described as [46]

\[
\begin{align*}
\kappa_{nf} &= \kappa_e \left[ \frac{2(K_e - (\kappa_e - \kappa_{an})) \Phi + \kappa_{an}}{2K_e + (\kappa_e - \kappa_{an}) \Phi + \kappa_{an}} \right], \\
\mu_{nf} &= \mu_e \left( 1 - \Phi \right)^{\frac{1}{2}} , \\
\tilde{\rho} \tilde{C}_p_{nf} &= (1 - \Phi) \tilde{\rho} \tilde{C}_p + \Phi \tilde{\rho} \tilde{C}_p_{an} , \\
\tilde{\rho}_{nf} &= (1 - \Phi) \tilde{\rho}_e + \Phi \tilde{\rho}_{an} , \\
\tilde{\sigma}_{nf} &= \tilde{\sigma}_e + 3 \Phi \left( \frac{\tilde{s}_{nf}}{\tilde{\sigma}_e} - 1 \right) , \\
\tilde{\beta} \tilde{\beta}_{nf} &= \tilde{\beta} \tilde{\beta}_e + \Phi \tilde{\beta} \tilde{\beta}_{an} .
\end{align*}
\]

where subscripts \( e, an, \text{ and } nf \) are introduced to express the properties of engine oil, alloy nanoparticles, and the resulted nanofluid, respectively. The volume proportion of solid alloys in the host fluid engine oil is denoted with \( \Phi \). Furthermore, Table 1 imparts numerical inputs for pertinent physical features of engine oil and alloy nanoparticles used during the simulation process.

### III. DEVELOPMENT OF UNIT-FREE FRACTIONAL MODEL

To construct the unit-free version of our formulated model, the following unit-less quantities are incorporated in Eqs. (1)-(5)

\[
\begin{align*}
\tilde{w} &= \frac{\tilde{w}}{\Omega} , \quad \tilde{y} = \frac{\tilde{y}}{L} , \quad \Theta = \frac{\tilde{T} - \tilde{T}_L}{\tilde{T}_W - \tilde{T}_L} , \\
\tau &= \frac{\nu_e}{L} , \quad a_1 = \frac{a_1^\ast}{L} , \quad a_2 = \frac{a_2^\ast}{L} .
\end{align*}
\]

The subsequent unit-free form of principal equations with initial and boundary conditions is acquired as

\[
\begin{align*}
&b_3 \left\{ \frac{\partial \tilde{w}(\eta, \tau)}{\partial \tau} + \beta \tilde{w}(\eta, \tau) \right\} = b_4 \frac{\partial^2 \tilde{w}(\eta, \tau)}{\partial \eta^2}, \\
&- \left( Mb_5 + \frac{1}{K_p} b_4 \right) \tilde{w}(\eta, \tau) + b_6 G_r \Theta(\eta, \tau), \\
&b_1 P_r \frac{\partial \Theta(\eta, \tau)}{\partial \tau} = b_2 \frac{\partial^2 \Theta(\eta, \tau)}{\partial \eta^2} + R_d \frac{\partial^2 \Theta(\eta, \tau)}{\partial \eta^2}, \\
&\tau = 0 : \quad \Theta(\eta, \tau) = 0, \quad \tilde{w}(\eta, \tau) = 0, \quad \text{for } \eta \in (0, 1), \\
&\eta = 0 : \quad \Theta(\eta, \tau) = 0, \quad \tilde{w}(\eta, \tau) = -a_1 \frac{\partial \tilde{w}(\eta, \tau)}{\partial \eta} = 0, \\
&\eta = 1 : \quad \Theta(\eta, \tau) = 1, \quad \tilde{w}(\eta, \tau) + a_2 \frac{\partial \tilde{w}(\eta, \tau)}{\partial \eta} = 0.
\end{align*}
\]

where \( \Omega \) is characteristic velocity, and \( a_1 \) and \( a_2 \) are slip lengths. The mathematical relations for unit-free parameters and volume proportion dependent constants appearing in Eqs. (13)-(17) are displayed in Table 2. In recent times, several studies reveal that in comparison to standard derivatives, fractional derivatives are more efficient to describe flow and heat transfer phenomena because an adequate modification of involved fractional parameter leads to an excellent agreement between theoretical results and experimental information, which assures the authenticity of theoretical outcomes. Fractional derivatives provide a more suitable explanation of memory effects and successfully express various viscoelastic characteristics. Moreover, fractional derivatives are useful to achieve the augmentation purpose of heat transfer rate. Keeping a view of the practical effectiveness of fractional derivatives, the next important task of this section is the transmutation of unit-free velocity and temperature equations from classical setting to fractional setting. For this purpose,
TABLE 1. Physical and thermal features of Ti6Al4V, AA7075, and engine oil [47], [48].

| Properties          | Units          | Symbols | Ti6Al4V | AA7075 | Engine oil |
|---------------------|----------------|---------|---------|---------|------------|
| Specific heat       | J/(kg·K)      | $\tilde{C}_p$ | 0.56    | 960     | 2048       |
| Density             | kg/m³         | $\tilde{\rho}$ | 4420    | 2810    | 863        |
| Thermal conductance | W/(m·K)       | $\tilde{\kappa}$ | 7.2     | 173     | 0.1404     |
| Electric conductance| S/m           | $\tilde{\sigma}$ | $5.8 \times 10^5$ | $26.77 \times 10^6$ | $55 \times 10^{-6}$ |
| Thermal expansion   | 1/K           | $\tilde{\beta} \times 10^{-5}$ | 28.17   | 29.64   | 70         |

TABLE 2. Unit-free parameters and volume proportion dependent constants appearing in Eqs. (13)-(17).

| Parameters          | Symbols | Mathematical relations |
|---------------------|---------|------------------------|
| Grashof number      | $G_r$   | $\frac{\alpha L^3(T_{w0} - T_s)}{\rho \tilde{v}_c \kappa}$ |
| Porosity            | $K_p$   | $\frac{\alpha L^2}{\rho \kappa}$ |
| Brinkman            | $\beta$ | $\frac{\beta \alpha L^2}{\rho \kappa}$ |
| Radiation           | $R_d$   | $\left\{ \frac{16 \sigma_0}{3 \alpha_x} \right\}_{\alpha_x}$ |
| Magnetic            | $M$     | $\left( \frac{\alpha_p}{\rho \kappa} \right) \frac{\beta \alpha L^2}{\rho \kappa}$ |
| Prandtl number      | $P_r$   | $\left( \frac{\mu \alpha_p}{\kappa} \right)_{\alpha_x}$ |

Volume proportion dependent constants

\[ b_1 \left[ \frac{\alpha L^3(T_{w0} - T_s)}{\rho \tilde{v}_c \kappa} \right] \left[ w(\eta, \tau) \right] + \beta w(\eta, \tau) = b_4 \frac{\partial^2 w(\eta, \tau)}{\partial \eta^2} - \left( M \frac{\alpha_k}{K_p} + \frac{1}{K_p} \right) w(\eta, \tau) + b_5 G_c \Theta(\eta, \tau), \]
\[ b_1 P_r \left[ \frac{\alpha L^3(T_{w0} - T_s)}{\rho \tilde{v}_c \kappa} \right] \left[ w(\eta, \tau) \right] = (b_2 + R_d) \frac{\partial^2 \Theta(\eta, \tau)}{\partial \eta^2}, \]

where the Caputo sense definition of Atangana-Baleanu (AB) derivative for a function $h(\eta, \tau)$ is provided as [35]

\[ AB \frac{\partial^\sigma}{\partial \tau^\sigma} \left[ h(\eta, \tau) \right] = \begin{cases} \frac{1}{\Gamma(1-\sigma)} \int_0^\tau \frac{\partial h(\eta, p)}{\partial \tau} \left( \frac{p}{\Gamma(1-\sigma)} \right)^{\sigma - 1} dp, & \text{for the first part,} \\ \frac{\partial h(\eta, \tau)}{\partial \tau}, & \text{for the second part,} \end{cases} \]

where $\sigma \in (0, 1)$ for the first part and $\sigma = 1$ for the second part of the above-defined piece-wise function.
IV. SOLUTIONS OF THE FRACTIONAL MODEL

In this section, the solutions of Eqs. (18) and (19) are developed under conditions expressed in Eqs. (15)-(17). The Laplace transform (LT) [49] method is applied to reduce fractional order PDEs to ODEs, and later these ODEs are solved through some conventional technique. As the principal equations are comprised of AB derivative, so before applying LT to these equations, it is significant to provide the LT of AB derivative

\[ \mathcal{L}\left\{ \text{AB} \right\} \left\{ h(\eta, \tau) \right\} = \frac{p^\sigma \mathcal{L}\left\{ h(\eta, \tau) \right\} - p^{\sigma-1} h(\eta, 0)}{\sigma + p^\sigma (1 - \sigma)} \]

where \( \text{LT} \) of function \( h(\eta, \tau) \) is defined as

\[ \mathcal{L}\left\{ h(\eta, \tau) \right\} = \int_0^\infty e^{-pt} h(\eta, \tau) d\tau. \]  

A. TEMPERATURE FIELD

In the presence of Eqs. (15), (20), and (21), implementation of LT yields the following form of the temperature equation (19)

\[ \frac{b_1 P_r}{R_d + b_2} \left( \frac{1}{1 - \sigma} \times \frac{p^\sigma}{\sigma + p^\sigma} \right) \tilde{\Theta}(\eta, p) = \frac{d^2 \tilde{\Theta}(\eta, p)}{d\eta^2}, \]

\[ \frac{m_2 p^\sigma}{m_1 + p^\sigma} \tilde{\Theta}(\eta, p) = \frac{d^2 \tilde{\Theta}(\eta, p)}{d\eta^2}. \]  

The Laplace domain form of pertinent boundary conditions is investigated as

\[ \tilde{\Theta}(\eta, p) = 0, \quad \text{for} \quad \eta = 0, \]

\[ \tilde{\Theta}(\eta, p) = \frac{1}{p}, \quad \text{for} \quad \eta = 1. \]  

Here \( \tilde{\Theta}(\eta, p) \) refers to the Laplace domain temperature field, and other constants are defined as

\[ b_7 = \frac{b_1 P_r}{R_d + b_2}, \quad m_0 = \frac{1}{1 - \sigma}, \quad m_1 = m_0 \sigma, \quad m_2 = m_0 b_7. \]

The general solution of second-order ODE (22) is secured as

\[ \tilde{\Theta}(\eta, p) = C_1 \exp \left( \eta \sqrt{\frac{m_2 p^\sigma}{m_1 + p^\sigma}} \right) + C_2 \exp \left( -\eta \sqrt{\frac{m_2 p^\sigma}{m_1 + p^\sigma}} \right). \]  

After deducing the constants \( C_1 \) and \( C_2 \) with the aid of Eq. (23), the compact form of temperature function is provided as

\[ \tilde{\Theta}(\eta, p) = \frac{\sinh \left( \eta \sqrt{\frac{m_2 p^\sigma}{m_1 + p^\sigma}} \right)}{p \sinh \left( \sqrt{\frac{m_2 p^\sigma}{m_1 + p^\sigma}} \right)}. \]  

The series form representation of the above equation is

\[ \tilde{\Theta}(\eta, p) = \frac{1}{p} \sum_{j=0}^{\infty} \exp \left( \sqrt{\frac{m_2 p^\sigma}{m_1 + p^\sigma}} (1 - \eta + 2j) \right) 
- \frac{1}{p} \sum_{j=0}^{\infty} \exp \left( \sqrt{\frac{m_2 p^\sigma}{m_1 + p^\sigma}} (1 + \eta + 2j) \right). \]  

Writing the above equation in the following way

\[ \tilde{\Theta}(\eta, p) = \frac{1}{p^{1-\sigma}} \left\{ f_1(\eta, p) - f_2(\eta, p) \right\} \]  

where

\[ f_1(\eta, p) = \frac{1}{p^{\sigma}} \sum_{j=0}^{\infty} \exp \left( \sqrt{\frac{m_2 p^\sigma}{m_1 + p^\sigma}} (1 - \eta + 2j) \right), \]

\[ f_2(\eta, p) = \frac{1}{p^{\sigma}} \sum_{j=0}^{\infty} \exp \left( \sqrt{\frac{m_2 p^\sigma}{m_1 + p^\sigma}} (1 + \eta + 2j) \right). \]  

On transforming back Eq. (27) in the primary domain \((\eta, \tau)\) through Laplace inversion, the final form of temperature function is furnished as

\[ \Theta(\eta, \tau) = \frac{1}{\tau^{\sigma} \Gamma(1 - \sigma)} \ast \left\{ f_1(\eta, \tau) - f_2(\eta, \tau) \right\}. \]  

where \( \ast \) refers to the convolution product, and \( f_1(\eta, \tau) \) and \( f_2(\eta, \tau) \) are given below

\[ f_1(\eta, \tau) = \frac{1}{\pi} \int_0^\infty \int_0^\infty \left\{ vq^{\sigma} \sin(\pi \sigma) \right\} h_1(\eta, \tau) \times \exp \left( -vq^{\sigma} \cos(\pi \sigma) - \tau q \right) dq dv, \]

\[ f_2(\eta, \tau) = \frac{1}{\pi} \int_0^\infty \int_0^\infty \left\{ vq^{\sigma} \sin(\pi \sigma) \right\} h_2(\eta, \tau) \times \exp \left( -vq^{\sigma} \cos(\pi \sigma) - \tau q \right) dq dv, \]

with

\[ h_1(\eta, \tau) = \sum_{j=0}^{\infty} \left\{ 1 - \frac{2m_2}{\pi} \int_0^\infty \left\{ \exp \left( -\frac{m_1 u^2 \tau}{u^2 + m_2} \right) \right\} \sin \left( (1 - \eta + 2j) u \right) \frac{du}{u^2 + m_2} \right\}, \]

\[ h_2(\eta, \tau) = \sum_{j=0}^{\infty} \left\{ 1 - \frac{2m_2}{\pi} \int_0^\infty \left\{ \exp \left( -\frac{m_1 u^2 \tau}{u^2 + m_2} \right) \right\} \sin \left( (1 + \eta + 2j) u \right) \frac{du}{u^2 + m_2} \right\}. \]
B. VELOCITY FIELD

In the presence of Eqs. (15), (20), and (21), application of LT yields the following form of the flow equation (18)

\[ b_3 \left\{ \frac{m_0p^2 \tilde{w}(\eta, p)}{m_1 + p^2} + \beta \tilde{w}(\eta, p) \right\} = b_4 \frac{d^2 \tilde{w}(\eta, p)}{d \eta^2} + \frac{d^2 \tilde{w}(\eta, p)}{d \eta^2} \left( \tilde{w}(\eta, p) \right) \]

Substitution of Eq. (25) in Eq. (35) and minor simplification gives

\[ \delta_1(p) \tilde{w}(\eta, p) = \frac{d^2 \tilde{w}(\eta, p)}{d \eta^2} + \delta_2(p) \sinh \left( \eta \sqrt{\frac{m_2p^2}{m_1 + p^2}} \right) \]  

(36)

where

\[ b_8 = M b_5 + \frac{1}{K_p} b_4, \]
\[ b_9 = b_8 + b_3 m_0, \]
\[ \delta_1(p) = \frac{m_3 + b_9 p^2}{m_4 + b_4 p^2}, \]
\[ \delta_2(p) = b_4 \sinh \left( \sqrt{\frac{m_2p^2}{m_1 + p^2}} \right) \].

The respective Laplace domain flow conditions are as follows

\[ \tilde{w}(\eta, 0) - \frac{\partial \tilde{w}(\eta, p)}{\partial \eta} = 0, \quad \text{for} \quad \eta = 0, \]
\[ \tilde{w}(\eta, 0) + \frac{\partial \tilde{w}(\eta, p)}{\partial \eta} = 0, \quad \text{for} \quad \eta = 1. \]  

(37)

Making use of Eq. (37), the solution of second-order ODE (36) is procured as

\[ \tilde{w}(\eta, p) = \mathcal{N}_1(p) \exp \left( \eta \sqrt{\delta_1(p)} \right) + \mathcal{N}_2(p) \exp \left( -\eta \sqrt{\delta_1(p)} \right) - \delta_3(p) \sinh \left( \eta \sqrt{\frac{m_2p^2}{m_1 + p^2}} \right) \]  

(38)

where

\[ \mathcal{N}_1(p) = \frac{a_1 \delta_3(p) \sqrt{m_2p^2}}{\sqrt{\delta_1(p) a_1} - 1} - \mathcal{N}_2(p) \mathcal{G}_1(p), \]
\[ \mathcal{N}_2(p) = \frac{d_1(p) + d_2(p)}{e_2(p) - e_1(p)}, \quad \mathcal{G}_1(p) = \frac{1 + \sqrt{\delta_1(p) a_1}}{1 - \sqrt{\delta_1(p) a_1}}, \]
\[ \delta_3(p) = \frac{m_2 - \delta_1(p) p^2 - m_1 \delta_1(p)}{m_1 + p^2}, \]
\[ d_1(p) = \mathcal{G}_2(p) \left\{ a_1 \delta_3(p) \sqrt{m_2p^2} \right\} \exp \left( \sqrt{\delta_1(p)} \right), \]
\[ d_2(p) = \delta_3(p) \left\{ \sinh \left( \sqrt{\frac{m_2p^2}{m_1 + p^2}} \right) + \delta_4(p) \right\} \mathcal{G}_3(p), \]
\[ e_1(p) = \mathcal{G}_2(p) \mathcal{G}_4(p) \exp \left( \sqrt{\delta_1(p)} \right), \]
\[ e_2(p) = \mathcal{G}_3(p) \mathcal{G}_5(p) \exp \left( -\sqrt{\delta_1(p)} \right), \]
\[ \delta_4(p) = \sqrt{\frac{m_2p^2}{m_1 + p^2}} \cosh \left( \sqrt{\frac{m_2p^2}{m_1 + p^2}} \right), \]
\[ \Gamma_2(p) = 1 + \sqrt{\delta_1(p) a_2}, \quad \Gamma_3(p) = 1 - \sqrt{\delta_1(p) a_1}, \]
\[ \Gamma_4(p) = 1 + \sqrt{\delta_1(p) a_1}, \quad \Gamma_5(p) = 1 - \sqrt{\delta_1(p) a_2}. \]

It can be noticed that Eq. (38) contains multiple protracted and convoluted combinations of Laplace frequency “p”, due to which application of analytic Laplace inversion is not possible. The preferred alternative means to effectively encounter these types of complications is the utilization of numerical Laplace inversion techniques. In recent times, Siddique et al. [50], Asjad et al. [51], and Tahir et al. [52] employed such techniques to solve fractional order flow models. For the present case, Stehfest’s [53] numerical technique is adopted during the simulation process to transmute back the velocity solution in the real-time domain. To ensure the specificity of the adopted numerical technique, the current velocity solution is also determined with the help of Durbin’s [54] and Zakian’s [55] numerical techniques. The formulation of Stehfest’s [53] formula to evaluate the Laplace inverse of h(η, p) is

\[ h(\eta, \tau) = \frac{\ln(2)}{\tau} \sum_{i=1}^{2^m} M_i h_i \left( \eta, \frac{\ln(2)}{\tau} \right), \quad \text{where} \]
\[ M_i = B_0 \sum_{l=\left[ \frac{\ln(2)}{\tau} \right]}^{\min(i, m)} \left[ \frac{l^m}{(m-l)!(l-1)!(l-1)!} \right], \]
\[ \text{and} \quad B_0 = (-1)^{i+m}. \]

To numerically invert back a function, formulas proposed by Durbin [54] and Zakian [55] are respectively defined as

\[ h(\eta, \tau) = \frac{e^{\pi \tau}}{T} \left[ -\frac{1}{2} \Re \{ h(\eta, \gamma) \} + \sum_{j=0}^{\infty} \Re \{ h(\eta, \gamma + \frac{\pi j}{T}) \} \cos \left( \frac{\pi j \tau}{T} \right) - \sum_{j=0}^{\infty} \Im \{ h(\eta, \gamma + \frac{\pi j}{T}) \} \sin \left( \frac{\pi j \tau}{T} \right) \right]. \]
\[ h(\eta, \tau) = \frac{2}{\tau} \sum_{j=1}^{5} \Re \{ \chi_j h(\eta, \frac{\theta_j}{\tau}) \}. \]

C. SHEAR STRESS AND HEAT TRANSFER RATE

For the current model, expressions for the coefficient of skin friction and Nusselt number to forecast shear stress and rate of heat transfer at the left edge of the channel are respectively determined as

\[ C_f = \frac{S}{2} \left( \frac{\bar{v}}{\bar{u}} \right), \quad Nu = \frac{Q}{\Omega \left( \bar{T}_W - \bar{T}_L \right)} \left( \frac{\bar{v}}{\bar{k}} \right). \]  

(39)
where \( S \) and \( Q \) are defined as

\[
S = \tilde{\mu}_{nf} \frac{d\bar{w}(\tilde{\eta}, \tilde{\tau})}{d\tilde{\eta}} \bigg|_{\tilde{\eta} = 0} = \left\{ b_2 \Omega^2 \left( \frac{\tilde{\mu}}{\tilde{v}} \right) \right\} \frac{d\bar{w}(\eta, \tau)}{d\eta} \bigg|_{\eta = 0},
\]

\[
Q = -\tilde{k}_{nf} \frac{d\bar{T}(\tilde{\eta}, \tilde{\tau})}{d\tilde{\eta}} \bigg|_{\tilde{\eta} = 0} = - \left\{ b_2 \Omega \left( \tilde{T}_w - \tilde{T}_l \right) \left( \frac{\tilde{k}}{\tilde{v}} \right) \right\} \frac{d\bar{\Theta}(\eta, \tau)}{d\eta} \bigg|_{\eta = 0}.
\]

V. RESULTS AND DISCUSSION

This work aims to anticipate advancement in the heat transfer capacity of engine oil (EO) due to the addition of Ti6Al4V and AA7075 alloy nanoparticles. The geometrical setting of this problem involves an upward channel that is nested in a permeable media, and it encounters heat radiative flux and magnetic force while slip effects are imposed at both ends. The flow phenomenon and heat transfer process are modeled and explained by operating the fractional derivative proposed by Atangana and Baleanu so-called Atangana-Baleanu derivative. Introducing some unit-free quantities and treating corresponding unit-free versions of modeled equations, the exact energy function is derived via Laplace transform. However, the transformed velocity function contains various ravel expressions so, it is estimated in the real-time domain by utilizing Stehfest’s Laplace inversion method. For the cause of certification, the velocity function is also approximated through Durbin’s and Zakian’s Laplace inversion methods, and outcomes are compared graphically. In this section, ramifications of altering the strength of accompanying parameters on flow behavior and thermal performance of engine oil based nanofluids are investigated and interpreted through multiple plots. Besides, a detailed tabular inspection of shear stress and heat transfer rate is performed. All tables and figures are developed for both Ti6Al4V and AA7075 alloy nanoparticles for an extensive and in-depth comparative analysis.

Figures 2 and 3 are developed to observe the flow and thermal behaviors of considered nanofluids for fractional and classical models. These figures contain velocity and energy profiles of EO-Ti6Al4V and EO-AA7075 nanofluids for two distinct values of time \( \tau \). It is followed from these figures that the behavior of velocity and energy functions is similar for the alteration of the fractional parameter (\( \sigma \)). More specifically, as \( \sigma \) increases for a short duration of time, the flow is decelerated and temperature indicates a decaying profile. In this case, both momentum and energy boundary layers are attenuated for \( \sigma = 1 \), which shows that the fractional model produces dominant profiles for a small value of \( \tau \). However, for a long time duration \( \tau = 2.0 \), contrary trends of velocity and energy functions are perceived and higher corresponding profiles are witnessed for \( \sigma = 1 \). The time exerts reverse impacts on the thickness of momentum and thermal boundary layers for short and long intervals, which lead to such outcomes. Moreover, it is noticed from figure 2 that the flow is more smooth for higher values of time. For lower and higher values of \( \tau \), the crucial role of radiative heat flux in the development of energy and velocity distributions is analyzed with aid of figures 4 and 5, respectively. It is probed that trends exhibited by energy and flow profiles for augmented power of radiation waves are identical. More precisely stated, growth in the inputs of \( R_d \) leads to raising both curves of energy and velocity as depicted in figures 4 and 5. From the physical point of view, divergence of thermal radiation flux escalates corresponding to large values of \( R_d \), and the rate of radiative heat transmission at channel boundary is expedited. Mathematically, for fixed parameters \( \tilde{k}_{nf} \) and \( \tilde{G}_0 \), \( \kappa_2 \) diminishes (Eq. (2)), which allows the nanofluid to absorb heat energy and cool down the conduit. A consequent upsurge is witnessed in nanofluid temperature due to the aforementioned physical mechanism. Figure 4 also reveals that radiation effects are more pronounced when \( R_d \) possesses small values. Now, switching the discussion towards flow phenomenon, it is being discussed that why considered nanofluids flow with greater speed when thermal radiation is active. The supply of additional heat energy excites nanofluid particles, and a process of rapid collisions among particles takes place as a result. In consequence, a cohesive attraction between particles minimizes due to breakage of bonds, which restricts them from the provision of significant resistive forces. These physical arguments describe the vital contribution of \( R_d \) as a flow assisting parameter. The correspondence between nanofluid
flow patterns and slip parameter ($a_1$) is demonstrated through figure 6 for two values of $\tau$. Since $a_1$ is a slip parameter at the left channel wall, therefore it can be seen that it only affects the flow on the left end, and there is no disturbance in the flow field at the right end. It is also depicted that the curve representing the velocity field rises as parameter $a_1$ increases because the slip flow is accelerated as the corresponding slip length expands. Furthermore, maximum velocity of flow is observed near the right plate because of double slip effects. 

Grashof number ($G_r$) is a principal parameter that arises in heat transfer processes due to temperature gradient. The primary force that supports freely convective flow is buoyancy force and it occurs because of convection currents whereas, the viscous force is the main opposition force, which decelerates the flow. $G_r$ is used to measure these forces such that it is defined as the ratio of former and later force. It is clear from the definition that amplified values of $G_r$ indicate that inner friction loses its strength whereas, gravitational force becomes sufficiently strong such that it overpowers the impacts of viscous force. Due to this reason, the flow of nanofluid upsurges and increasing velocity function is witnessed in figure 7. The response of the velocity field for variation in the Brinkman parameter ($\beta$) is reported in figure 8. It is detected that enlargement in $\beta$ retards the flow of nanofluid. In the physical sense, $\beta$ possesses a direct relationship with drag force so, enlargement in $\beta$ corresponds to the higher strength of drag force, which results in deceleration of flow. Furthermore, EO-Ti$_6$Al$_4$V nanofluid exhibits a rapid flow as equated to the flow of EO-AA7075 nanofluid due to its particular physical features. Figure 9 is drawn to investigate how the disturbance in imposed magnetic force controls the flow phenomenon. For this purpose, the magnetic parameter ($M$) is varied from 0.0 to 5.6 while fixing all the other connected parameters. It is visualized from the respective figure that nanofluid exhibits maximum flow speed in the absence of magnetic effects, whereas the strong magnetic force restraints the flow. In the physical sense, the magnetic field generates the Lorentz force that operates in the reverse direction and offers dominant resistance to nanofluid motion. Due to vigorous magnetic effects, the flow assisting forces become unable to encounter the influence of Lorentz force, so consequently, a declining flow pattern is witnessed. Figure 10 presents the flow patterns to investigate the impacts of rising the volume proportion $\Phi$ in host fluid from 1% to 4%. The figure communicates that the flow profile goes through a significant decay due to this rise in $\Phi$. Physically, the suspension of nanoparticles enhances the viscosity of engine oil, which leads to raising its boiling point. Generally, possession of a high boiling point is a key characteristic of a good lubricant therefore it is significant to mention that alloy nanoparticles improve lubrication properties of engine oil. In addition, EO-AA7075 nanofluid is more viscous than EO-Ti$_6$Al$_4$V nanofluid.
The impact of second slip parameter ($a_2$) on flow behavior is highlighted in figure 11. The flow profile in figure 11 is similar to that in figure 6, but it is relatively more stable in the current case. Since parameter $a_2$ is involved in the boundary condition at the right channel wall, it only disturbs the flow at the right end. As time passes, there is a comparatively more smooth increment in the flow profile, as compared to parameter $a_1$, and the flow curve converges to a single point at the left end, which depicts that effects of $a_2$ are dispersed before reaching the nanofluid layer attached to the left channel wall. In the modern era, the transportation of fluids through permeable media is a matter of prevalent interest and it has originated a separate area of research. The applications of porous flows are found in inkjet printers, energy storage, disposal technologies of nuclear and industrial wastes, and so forth. In this regard, the effective control of permeable media on nanofluid transport inside a vertical channel is explored through figure 12. It is found that the presence of a permeable media causes an upsurge in the velocity field of both EO-Ti$_6$Al$_4$V and EO-AA7075 nanofluids. The salient physical logic supporting this response is an expansion in the size of voids (pores) due to the increment of the porosity parameter ($K_p$). Correspondingly, the area of these voids increases, and they enable additional amount of nanofluid to transport through them. Furthermore, resistive forces become weak due to a rise in the value of $K_p$, and the usual flow of nanofluid faces a sudden push. These are the main reasons that justify the accelerated flow of nanofluid. The illustrated flow patterns show complete accordance with this explanation. The purpose of validating the velocity solution derived via Stehfest’s method is achieved by comparing respective numerical outcomes with those estimated by making use of Durbin’s and Zakian’s methods. Subject to two different values of $Gr$, this comparison is portrayed in figure 13, where figure 13(a) is drawn for EO-Ti$_6$Al$_4$V nanofluid and figure 13(b) is produced for EO-AA7075 nanofluid. It can be witnessed that
the graphs of the three methods are overlapping for both cases so, it is patently claimed that the velocity function evaluated utilizing Stehfest’s method is valid hence reliable. Figures 14(a) and 15(a) are sketched to showcase the three dimensional view of flow patterns and temperature distribution for EO-Ti$_{6}$Al$_{4}$V nanofluid. Similarly, for EO-AA7075 nanofluid, a three dimensional representation of velocity and thermal profiles is displayed in figures 14(b) and 15(b), respectively. From these figures, the increasing behaviors of flow and thermal profiles under transient effects (variation of $\tau$) can be easily observed. Moreover, these profiles clearly highlight that solutions computed for velocity and energy equations satisfy the pertinent initial and boundary conditions, which is a basic requirement to secure the accuracy of solutions.

Table 3 is furnished to follow up the signification of Ti$_{6}$Al$_{4}$V and AA7075 in the thermal performance of engine oil, based on the variation of their volume proportion. In this work, Hamilton and Crosser model [46] is employed for precise anticipation of thermal conductivity of EO-Ti$_{6}$Al$_{4}$V.
TABLE 3. Variation in temperature distribution against $\Phi$.

| $\Phi$ | $\tau$ | $\sigma$ | $R_d$ | Temperature |
|-------|--------|----------|-------|-------------|
| 0.00  | 2.0    | 0.6      | 2.0   | 0.2436      |
| 0.01  | 2.0    | 0.6      | 2.0   | 0.4025      |
| 0.02  | 2.0    | 0.6      | 2.0   | 0.4062      |
| 0.03  | 2.0    | 0.6      | 2.0   | 0.4100      |
| 0.04  | 2.0    | 0.6      | 2.0   | 0.4138      |

TABLE 4. Percentage augmentation in heat transfer rate due to Titanium and Aluminum alloy nanoparticles.

| $\Phi$ | $\tau$ | $\sigma$ | $R_d$ | $N_{\text{TiAl4V}}$ | $N_{\text{AA7075}}$ |
|-------|--------|----------|-------|----------------------|----------------------|
| 0.00  | 2.0    | 0.6      | 2.0   | 0.0686               | -                     |
| 0.01  | 2.0    | 0.6      | 2.0   | 0.0736               | 7.29                 |
| 0.02  | 2.0    | 0.6      | 2.0   | 0.0790               | 15.16                |
| 0.03  | 2.0    | 0.6      | 2.0   | 0.0848               | 23.62                |
| 0.04  | 2.0    | 0.6      | 2.0   | 0.0909               | 32.50                |

TABLE 5. Variation in heat transfer rate due to relevant parameters.

| $\Phi$ | $\tau$ | $\sigma$ | $R_d$ | $N_u$ |
|-------|--------|----------|-------|-------|
| 0.04  | 1.5    | 0.6      | 2.0   | 0.0709 |
| 0.04  | 1.8    | 0.6      | 2.0   | 0.0830 |
| 0.04  | 2.5    | 0.6      | 2.0   | 0.1101 |
| 0.04  | 2.0    | 0.2      | 2.0   | 0.0537 |
| 0.04  | 2.0    | 0.5      | 2.0   | 0.0796 |
| 0.04  | 2.0    | 0.8      | 2.0   | 0.1111 |
| 0.04  | 2.0    | 0.6      | 0.5   | 0.0160 |
| 0.04  | 2.0    | 0.6      | 1.5   | 0.0616 |
| 0.04  | 2.0    | 0.6      | 2.5   | 0.1222 |

and EO-AA7075 nanofluids. The table shows that for $\Phi = 0$, the value of temperature function is same for both cases because, for this condition, there are no nanoparticles present in the host fluid. As $\Phi$ starts increasing, an enlargement in temperature values of both nanofluids is evident, which is the perfect computational display of the physical and thermal behavior of alloy nanoparticles. These results delineate that effective thermal characteristics of nanoparticles boost up the thermal conductivity of engine oil and as a consequence, it becomes an efficient heat conductor. In fluid dynamics, the rate of heat transfer at the liquid-solid interface is quantified in terms of Nusselt number ($N_u$). Tables 4 and 5 are exhibited to highlight the influence of Ti$_6$Al$_4$V and AA7075 alloy nanoparticles and other parameters of interest on $N_u$. Table 4 communicates an improvement of 32.50% in $N_u$ for suspension of Ti$_6$Al$_4$V in engine oil whereas, this improvement is 17.34% in the case of AA7075 when volume concentration of these nanoparticles is enhanced from 0.01 to 0.04. On the basis of this comparison, it is claimed that Ti$_6$Al$_4$V is more effective than AA7075 when the prime target is the rapid cooling rate of a conduit. These results describe noteworthy implications of nanoparticles in ameliorating the efficiency of engine oil for lubrication and cooling processes. Table 5 reports that $N_u$ is a growing function of $\tau$, $R_d$, and $\sigma$. Physically, raising the values of $R_d$ disturbs the Rosseland approximation due to which a
sufficient increase in the dominance of temperature gradient occurs. This allows the boundary to release the heat, which results in the augmentation of $N_u$. Here, it is significant to highlight that the fractional model is more appropriate in comparison to the conventional model for achieving better control on the heat transfer rate. Another important physical quantity is boundary shear stress, which is usually quantified in the form of skin friction coefficient ($C_f$). To have some basic information about shear stress is necessary for various engineering applications however, it is a commonly recognized fact that enhancement in shear stress leads to multiple technical issues. Table 6 is created for the under observation model to discuss possible actions of controlling the shear stress by anticipating the impacts of several embedded parameters on it. It is evaluated that choosing a small value of the fractional parameter $\sigma$ and taking dominant magnetic

| $\eta$ | $\sigma = 0.1$ | $\sigma = 0.4$ | $\sigma = 0.7$ | $\sigma = 1.0$ |
|--------|----------------|----------------|----------------|----------------|
|        | $T_{\text{Al}_{2}O_{3}}$ | AA7075 | $T_{\text{Al}_{2}O_{3}}$ | AA7075 | $T_{\text{Al}_{2}O_{3}}$ | AA7075 | $T_{\text{Al}_{2}O_{3}}$ | AA7075 |
| 0.0    | 0.0000          | 0.0000         | 0.0000          | 0.0000         | 0.0000          | 0.0000         | 0.0000          | 0.0000         | 0.1079 |
| 0.1    | 0.0047          | 0.0041         | 0.0065          | 0.0057         | 0.0095          | 0.0084         | 0.0102          | 0.0084         |
| 0.2    | 0.0108          | 0.0095         | 0.0147          | 0.0131         | 0.0217          | 0.0193         | 0.0256          | 0.0216         |
| 0.3    | 0.0203          | 0.0181         | 0.0270          | 0.0243         | 0.0396          | 0.0357         | 0.0518          | 0.0452         |
| 0.4    | 0.0361          | 0.0327         | 0.0467          | 0.0426         | 0.0676          | 0.0618         | 0.0958          | 0.0863         |
| 0.5    | 0.0632          | 0.0582         | 0.0789          | 0.0731         | 0.1117          | 0.1039         | 0.1653          | 0.1530         |
| 0.6    | 0.1100          | 0.1030         | 0.1323          | 0.1244         | 0.1808          | 0.1709         | 0.2670          | 0.2529         |
| 0.7    | 0.1911          | 0.1819         | 0.2206          | 0.2107         | 0.2869          | 0.2755         | 0.4052          | 0.3912         |
| 0.8    | 0.3319          | 0.3210         | 0.3664          | 0.3554         | 0.4459          | 0.4345         | 0.5789          | 0.5676         |
| 0.9    | 0.5761          | 0.5667         | 0.6064          | 0.5973         | 0.6769          | 0.6687         | 0.7814          | 0.7750         |
| 1.0    | 1.0000          | 1.0000         | 1.0000          | 1.0000         | 1.0000          | 1.0000         | 1.0000          | 1.0000         |
TABLE 8. Detailed numerical analysis of fractional parameter’s impact on flow patterns.

| $\eta$ | $\sigma = 0.1$ | $\sigma = 0.4$ | $\sigma = 0.7$ | $\sigma = 1.0$ |
|--------|----------------|----------------|----------------|---------------|
|        | $T_{16}Al_4V$ | $AA7075$       | $T_{16}Al_4V$ | $AA7075$       |
| 0.0    | 0.0011         | 0.0010         | 0.0013         | 0.0012         |
| 0.1    | 0.0120         | 0.0112         | 0.0143         | 0.0133         |
| 0.2    | 0.0231         | 0.0216         | 0.0275         | 0.0256         |
| 0.3    | 0.0346         | 0.0323         | 0.0409         | 0.0382         |
| 0.4    | 0.0463         | 0.0432         | 0.0545         | 0.0508         |
| 0.5    | 0.0579         | 0.0541         | 0.0675         | 0.0630         |
| 0.6    | 0.0683         | 0.0638         | 0.0787         | 0.0736         |
| 0.7    | 0.0752         | 0.0703         | 0.0855         | 0.0799         |
| 0.8    | 0.0741         | 0.0694         | 0.0828         | 0.0775         |
| 0.9    | 0.0566         | 0.0530         | 0.0620         | 0.0581         |
| 1.0    | 0.0073         | 0.0069         | 0.0078         | 0.0074         |

FIGURE 13. Comparison of velocity distribution for three different Laplace inversion methods.

FIGURE 14. A three dimensional view of flow patterns.

effects into account can efficiently minimize the shear stress. Similarly, this task can be accomplished when buoyancy force generated due to free convection does not exert significant effects and voids of permeable media possess a small radius.

However, it is important to mention that certain physical and thermal features will be compromised due to restrictive choices so, for practical purposes, a clear and effective selection of parameters should be made to attain all the desired
targets. Tables 7 and 8 are prepared to numerically analyze the control of \( \sigma \) on temperature distribution and flow patterns respectively. For this purpose, eleven different values of the spatial variable \( \eta \) and four different values of \( \sigma \) are considered, as presented in both tables. The aim of these tables is to deeply observe the thermal and flow performance of EO-Ti\(_6\)Al\(_4\)V and EO-AA7075 nanofluids. It is seen that the velocity and temperature functions of both nanofluids follow identical trends for varying inputs of \( \sigma \). In other words, they keep increasing when \( \sigma \) expands from 0.1 to 1.0 and gain their maximum values for \( \sigma = 1.0 \).

VI. CONCLUSION

The theoretical analysis reported in this article is conducted to propose a fractional model in the form of Atangana-Baleanu derivative for demonstrating flow and heat transfer phenomena of engine oil based nanofluids. The comparative roles of Ti\(_6\)Al\(_4\)V and AA7075 alloy nanoparticles to improve the thermal and physical properties of engine oil are studied. An infinitely long upward channel embedded in a permeable media enclosing nanofluid flow with slip effects at both edges is considered for this problem. The supplementary impacts of radiative flux and magnetic force are encountered with the assistance of Rosseland approximation and set of Maxwell equations respectively. The fractional order exact solution of the energy equation is procured by Laplace transform while due to the presence of intricate expressions in the Laplace domain solution of the velocity field, it is converted to the real-time domain by utilizing Stehfest’s numerical technique. To verify the accuracy and reliability of the evaluated velocity function, its graphical comparison is drawn with results of Durbin’s and Zakian’s numerical methods. Several graphical illustrations are constructed to analyze the influences of Ti\(_6\)Al\(_4\)V and AA7075 on the material and thermal attributes of engine oil for both fractional and classical frameworks. Two important quantities of physical significance named skin friction coefficient and Nusselt number are delineated in tabular form for an in-depth investigation of shear stress and heat transfer rate. Based on tabular and graphical information, the main results of this fractional study are summarized below.

- Ti\(_6\)Al\(_4\)V is more productive as compared to AA7075 in terms of enhancing the thermal features of engine oil.
- The viscosity of engine oil based nanofluid is greater than ordinary engine oil due to the structure and particular material characteristics of dispersed alloy nanoparticles. This leads to augment the boiling point of nanofluid which results in better thermal stability and increased heat carrying ability.
- Fluids with greater viscosity possess low freezing points. The suspension of nanoparticles restricts the freezing point of engine oil, which improves its effectiveness as a lubricant.
- There is a rise of 32.50% in the heat transfer rate of engine oil due to Ti\(_6\)Al\(_4\)V, which is almost twice the improvement in heat transfer rate provided by AA7075, that is 17.49%.
- A fractional model is more efficient than a conventional model to control heat transfer and flow phenomena.
- Velocity and temperature curves of Ti\(_6\)Al\(_4\)V are always superior to AA7075.
- For a small-time duration, energy and flow functions indicate decreasing profiles subject to variation in \( \sigma \). But for higher values of time, this trend reverses, and both functions attain their maximum values for \( \sigma = 1.0 \).
- The speed of nanofluid flow is an increasing function of slip lengths \( a_1 \) and \( a_2 \).
- The use of permeable media is helpful to facilitate the flow, whereas magnetic effects attenuate the boundary layer of velocity.
- Shear stress is minimum for AA7075 as equated to that of Ti\(_6\)Al\(_4\)V. The strong magnetic field effectively restricts the shear stress, whereas \( \sigma \), \( K_p \), and \( G_r \) contribute to intensely increase the strength of shear stress.
- Numerical simulations performed for Stehfest’s, Durbin’s, and Zakian’s approximation methods express perfect agreement between outcomes of velocity solutions.
- There is a momentous contribution of heat radiative flux in nanofluid flow and heat transfer processes.
CONFLICTS OF INTEREST
The authors declare that there is no competing interest.

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