AN APPLICATION OF LATTICE POINTS COUNTING TO SHRINKING TARGET PROBLEMS

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Abstract. We apply lattice points counting results to solve a shrinking target problem in the setting of discrete time geodesic flows on hyperbolic manifolds of finite volume.

1. Introduction. Let $(X, \mu)$ be a probability space and $T : X \to X$ a measure-preserving transformation. For a sequence of measurable sets $B_n \subset X$, consider the set

$$\limsup_{n} T^{-n} B_n = \cap_{m=1}^{\infty} \cup_{n=m}^{\infty} T^{-n} B_n$$

of points $x \in X$ such that $T^n x \in B_n$ for infinitely many $n \in \mathbb{N}$. The Borel-Cantelli Lemma implies that if $\sum_{n=1}^{\infty} \mu(B_n)$ is finite, then $\mu(\limsup_n T^{-n} B_n) = 0$. The (converse) divergence case requires additional assumptions on the sets $B_n$. The classical Borel-Cantelli Lemma would imply that the measure of $\limsup_n T^{-n} B_n$ is full if the sets $T^{-n} B_n$ are pairwise independent, an assumption which is hard to establish for deterministic dynamical systems.

In many cases however a milder version of independence can be verified, still implying the full measure of the limsup set. Such results are usually referred to as dynamical Borel-Cantelli Lemmas. In many applications the family of sets $\{B_n\}$ is nested, and thus can be viewed as a ‘shrinking target’, hence the terminology ‘Shrinking Target Problems’. For example, if $\{B_n\}$ are shrinking balls centered at a point $p \in X$, a dynamical Borel-Cantelli Lemma can be thought of as a quantitative way to express density of trajectories of a generic point of $X$ at this fixed point $p$. Starting from the work of Phillip [15], there have been many results of this flavor. For example Sullivan [17] proved a Borel-Cantelli type theorem for cusp neighborhoods in hyperbolic manifolds of finite volume (here $p = \infty$), and the first named author with Margulis [11] extended the result of Sullivan to non-compact Riemannian symmetric spaces. See also [2, 5, 6, 8, 9] for more references, and [1] for a nice survey of the area.

One particular example of a shrinking target property can be found in a paper by Maucourant [13]. He considered nested balls in hyperbolic manifolds (quotients of hyperbolic manifolds).
the $n$-dimensional hyperbolic space $\mathbb{H}^n$) of finite volume, and proved the following theorem:

**Theorem 1.1.** Let $V$ be a finite volume hyperbolic manifold of real dimension $n$, $T^1V$ the unit tangent bundle of $V$, $\pi: T^1V \to V$ the canonical projection, $(\phi_t)_{t \in \mathbb{R}}$ the geodesic flow on $T^1V$, $\mu$ the Liouville measure on $T^1V$, and $d$ the Riemannian distance on $V$. Let $(B_t)_{t \geq 0}$ be a decreasing family of closed balls in $V$ (with respect to the metric $d$) of radius $(r_t)_t \geq 0$. Then for $\mu$-almost every $v$ in $T^1V$, the set $\{t \geq 0 : \pi(\phi^t v) \in B_t\}$ is bounded provided

$$\int_0^\infty r_t^{n-1} dt$$

(1.1)

converges, and is unbounded if (1.1) diverges.

Note that Maucourant’s theorem holds for the continuous-time geodesic flow on $T^1V$. Now suppose that one replaces the continuous family $(B_t)_{t \geq 0}$ by a sequence $(B_t)_{t \in \mathbb{N}}$, and instead of the continuous geodesic flow considers the $h$-step discrete geodesic flow $(\phi_{ht})_{t \in \mathbb{N}}$ for fixed $h \in \mathbb{R}_+$. The goal of this work is to provide additional argument needed to prove the Borel-Cantelli property, assuming some restrictions on the sequence $(B_t)$.

One of the ingredients in Maucourant’s proof is a counting result for the number of lattice points inside balls in $\mathbb{H}^n$. To address a discrete time analogue of Theorem 1.1 we use more refined lattice point counting results, namely an error term estimate for the number of lattice points in large balls in $\mathbb{H}^n$.

We use the following notation throughout the paper: for two non-negative functions $f$ and $g$, the notation $f(x) \ll g(x)$ means $f(x) \leq Cg(x)$ where $C > 0$ is a constant independent of $x$.

Here is a special case of our main result:

**Theorem 1.2.** Let $V$ be as in Theorem 1.1, and let $(B_t)_{t \in \mathbb{N}}$ be a decreasing family of closed balls in $V$ centered at $p_0 \in V$ of radius $r_t$. Fix $h > 0$ and let $(\phi_{ht})_{t \in \mathbb{N}}$ be the $h$-step discrete geodesic flow. Then for $\mu$-almost every $v$ in $T^1V$, the set

$$\{t \in \mathbb{N} : \pi(\phi^t v) \in B_t\}$$

(1.2)

is finite provided the sum

$$\sum_{t \in \mathbb{N}} r_t^n$$

(1.3)

converges. Also, if one assumes that (1.3) diverges and, in addition, that

$$\frac{-\ln r_t}{r_t} \ll t \text{ for large enough } t,$$

(1.4)

then for $\mu$-almost every $v$ in $T^1V$, the set (1.2) is infinite.

That is, in the terminology of [4], the sequence $(B_t)$ is a Borel-Cantelli sequence. Note that the difference in exponents in (1.1) and (1.3) is due to the fact that Theorem 1.1, unlike Theorem 1.2, deals with a continuous time setting.

It is well known that the geodesic flow on $T^1V$ as above has exponential decay of correlations, see e.g. [14, 16]. For systems with exponential mixing similar dynamical Borel-Cantelli Lemmas have been established before. For example, it follows from [9, Theorem 4.1] that the set (1.2) will be infinite provided

$$\mu(B_t) \gg \frac{\ln t}{t}.$$  

(1.5)
or, equivalently, \( r_t \gg (\ln t)^{1/n} \). This shows that the restriction (1.4) is weaker than the one coming from [9, Theorem 4.1]. For example, take \( r_t = \frac{C}{t^\alpha} \), where \( \alpha \leq \frac{1}{n} \). Then (1.3) diverges, and one can write

\[
-\ln r_t \leq -\ln \frac{C}{r_t} = \frac{1}{C} (\ln C + \alpha \ln t) t^{\alpha} \ll t
\]

when \( t \) is large enough, therefore (1.4) is satisfied. Note that in the ‘critical exponent’ case \( \alpha = \frac{1}{n} \) condition (1.5) fails to hold, thus the methods of [9] are not powerful enough to treat this case. The same also works for \( r_t = \frac{C}{t^{\alpha/(\ln t)^\beta}} \) where \( 0 < \beta \leq 1 \): one has

\[
-\ln r_t \leq -\ln \frac{C}{t^{\alpha/(\ln t)^\beta}} = \frac{1}{C} t^{1/(\ln t)^\beta} \left( \frac{1}{n} \ln t + \beta \ln(\ln t) - \ln C \right) \ll t
\]

for large enough \( t \).

We derive Theorem 1.2 from a more general statement, Theorem 1.3, which involves a technical condition (1.6) weaker than (1.4):

**Theorem 1.3.** Let \( V \) be as above, and let \( (B_t)_{t \in \mathbb{N}} \) and \( h \) be as in Theorem 1.2. Then for \( \mu \)-almost every \( v \in T^1 V \), the set (1.2) is finite provided the sum (1.3) converges. Also there exist \( C_1, C_2 > 0 \) such that if (1.3) diverges and, in addition, that

\[
\sum_{t=1}^s r_t^{n-1} \ll \sum_{t=1}^s r_t^n \quad \text{when } s \text{ is large enough},
\]

(1.6) then for \( \mu \)-almost every \( v \in T^1 V \), the set (1.2) is infinite.

In the next section we will reduce Theorem 1.3 to a certain \( L^2 \) bound, Theorem 2.2, which will be verified in §3, and in §4 we will deduce Theorem 1.2 from Theorem 1.3.

2. **Reduction to Theorem 2.2.** First note that for the divergence case of Theorem 1.3 without loss of generality one can assume that \( r_t \to 0 \) when \( t \to \infty \): indeed, if \( (r_t) \) is bounded from below by a positive constant, then the ergodicity of the geodesic flow implies that

\[
\pi(\varphi^{ht}v) \in B_t \text{ infinitely often for } \mu\text{-a.e. } v \in T^1 V.
\]

(2.1)

Furthermore, for a fixed \( R > 0 \) we can assume that \( r_t \leq R \) for all \( t \in \mathbb{N} \). Indeed, if the theorem is proved under that assumption, then applying it to the family \( \{B_t : t \geq t_0 \} \) where \( t_0 \) is such that \( r_t \leq R \) when \( t \geq t_0 \), we still recover condition (2.1). This \( R \) will be fixed later, see (3.4).

Our proof follows Maucourant’s approach in [13]. Let us first introduce some terminology. Let \( F = (f_t)_{t \in \mathbb{N}} \) be a family of measurable functions on a probability space \( (X, \mu) \). We call \( F \) decreasing if \( f_s(x) \leq f_t(x) \) for any \( x \in X \) whenever \( s \geq t \). Also let us write

\[
S_T[F](x) \overset{\text{def}}{=} \sum_{t=1}^T f_t(\varphi^{ht}x), \quad I_T[F] \overset{\text{def}}{=} \sum_{t=1}^T \int_X f_t \, d\mu.
\]

We are going to use the following proposition from Maucourant’s paper:
Proposition 2.1. ([13, Proposition 1]) Let $F = \{f_t\}_{t \in \mathbb{N}}$ be a decreasing family of non-negative measurable functions on $X$ such that $f_t \in L^2(X, \mu)$ for all $t$. Assume that $\lim_{T \to \infty} I_T[F] = \infty$, and that $S_T[F]/I_T[F]$ is bounded in $L^2$-norm as $T \to \infty$.

Then, as $T \to \infty$, $S_T[F]/I_T[F]$ converges to 1 weakly in $L^2(X, \mu)$, and for $\mu$-almost every $x$ in $X$ one has

$$\limsup_{T \to +\infty} \frac{S_T[F](x)}{I_T[F]} \geq 1. \quad (2.2)$$

We note that the above proposition was stated in [13] for the case of a continuous family of functions, but it is immediate to deduce a discrete version. To prove Theorem 1.3, we will apply Proposition 2.1 to the family of characteristic functions of Theorems 1.2 and 1.3 immediately follows from the Borel-Cantelli Lemma, (2.2) of Proposition 2.1 implies that the set (1.2) is infinite. Since the convergence of every $x$ in $X$ one has

$$\limsup_{T \to +\infty} \frac{S_T[F](x)}{I_T[F]} \geq 1. \quad (2.2)$$

Proof of Theorem 2.2. Let $F = \{f_t\}_{t \in \mathbb{N}}$ be as in (2.3). Then there exist $C_1, C_2 > 0$ such that if $I_T[F]$ diverges when $T$ goes to $\infty$ and condition (1.6) holds, then the $L^2$-norm of $S_T[F]/I_T[F]$ is bounded for all $T \geq 1$.

3. Proof of Theorem 2.2. To prove Theorem 2.2, following the same methodology as in [13], we will apply a result on counting lattice points stated below (Theorem 3.3) together with a measure estimate for the space of discrete geodesics (Theorem 3.7).

3.1. Counting lattice points. Write $T^n V = \Gamma \backslash G$, where $\Gamma$ is a lattice in $G = \text{SO}(n, 1)$, the isometry group of $V = \mathbb{H}^n$. Choose a lift $\tilde{p}_0 \in \mathbb{H}^n$ of $p_0$ and for $r > 0$ and $i \in \mathbb{N},$ let us denote

$$\hat{\Gamma}_i(r) \overset{\text{def}}{=} \{\gamma \in \Gamma : d(\tilde{p}_0, \gamma \tilde{p}_0) \in (hi - r, hi + r]\}.$$  

Then

$$\# \hat{\Gamma}_i(r) = \#(\Gamma \cap D_{hi+r}) - \#(\Gamma \cap D_{hi-r}),$$

where

$$D_t = \{g \in G : d(gp_0, \tilde{p}_0) \leq t\}.$$  

An estimate for $\# \hat{\Gamma}_i(r)$ would follow from a reasonable estimate for the error term in the asymptotics of the size of $\Gamma \cap D_t$ for large $t$. Such estimates are due to Huber [10] for $n = 2$ and to Selberg for the general case, see [12], and also [3, 7] for more recent results of this flavor. Denote by $m_G$ the Haar measure on $G$ which locally projects onto $\mu$. The following is a consequence of [12, Theorem 1]:

**Theorem 3.1.** There exist constants $0 < q < 1$ and $t_1, c_1 > 0$ such that

$$|\#(\Gamma \cap D_t) - m_G(D_t)| \leq c_1 m_G(D_t)^q,$$

for all $t > t_1$.  

An important property of the family $\{D_i\}$ is so-called Hölder well-roundedness, see [7]. In particular the following is true:

**Proposition 3.2.** There exist $t_2, c_2, c_3 > 0$ such that:

(i) For any $\varepsilon < 1$ and $t > t_2$, we have that

$$m_G(D_{t+\varepsilon}) - m_G(D_{t-\varepsilon}) \leq c_2 \varepsilon m_G(D_{t-\varepsilon}).$$  \hfill (3.1)

(ii) For any $t > 0$,

$$m_G(D_t) \leq c_3 e^{(n-1)t}.$$  \hfill (3.2)

From the two statements above one can easily derive the following estimate:

**Theorem 3.3.** There exist constants $c_4, c_5$ with the following property: if $0 < r < 1$ and $i \in \mathbb{N}$ are such that

$$h_i \geq \max(-c_4 \ln r, r + t_0),$$

where $t_0 = \max(t_1, t_2)$, then

$$\#\hat{\Gamma}_i(r) \leq c_5 r e^{(n-1)hi}.$$

**Proof of Theorem 3.3.** Applying Theorem 3.1 for all $i$ with $h_i - r > t_0$, we get that

$$\#(\Gamma \cap D_{h_i+r}) \leq m_G(D_{h_i+r}) + c_1 m_G(D_{h_i+r})^q$$

and

$$\#(\Gamma \cap D_{h_i-r}) > m_G(D_{h_i-r}) - c_1 m_G(D_{h_i-r})^q.$$

Therefore, by (3.1) and (3.2), we have:

$$\#\{\gamma \in \Gamma : d(\gamma p_0, \gamma p_0) \in [h_i - r, h_i + r]\}
\leq \#(\Gamma \cap D_{h_i+r}) - \#(\Gamma \cap D_{h_i-r})
\leq m_G(D_{h_i+r}) + c_1 m_G(D_{h_i+r})^q - m_G(D_{h_i-r}) + c_1 m_G(D_{h_i-r})^q
\leq m_G(D_{h_i+r}) - m_G(D_{h_i-r}) + c_1 (m_G(D_{h_i+r})^q + m_G(D_{h_i-r})^q)
\ll \text{r} m_G(D_{h_i-r}) + (m_G(D_{h_i+r})^q + m_G(D_{h_i-r})^q)
\ll \text{r} e^{(n-1)(h_i-r) + e(n-1)q(h_i+r) + e(n-1)q(h_i-r)}
\leq \text{r} e^{(n-1)hi} \left(1 + \frac{e^{-(n-1)(1-q)hi}}{r} \left(e(n-1)qr + e^{-(n-1)qr}\right)\right).$$

Since $q < 1$ and $r < 1$, we have

$$e^{(n-1)qr} + e^{-(n-1)qr} < 2e^{n-1},$$

and clearly $\frac{1}{r} e^{-(n-1)(1-q)hi} \leq 1$ whenever $h_i \geq -\frac{\ln r}{(1-q)(n-1)}$. Summarizing the above, if

$$h_i \geq \max\left(-\frac{1}{(1-q)(n-1)} \ln r, r + t_0\right),$$

then $\#\hat{\Gamma}_i(r) \ll r e^{(n-1)hi}$. \hfill $\square$
3.2. The space of discrete geodesics on $\mathbb{H}^n$. In this section we will state measure estimates for spaces of geodesics on $\mathbb{H}^n$.

**Definition 3.4.** We will write $\mathcal{G}$ as the space of oriented, unpointed continuous geodesics on $\mathbb{H}^n$. Using the fact that $T^1\mathbb{H}^n$ can be written as $\mathcal{G} \times \mathbb{R}$, we can define a measure $\nu$ on $\mathcal{G}$ by $\mu = \nu \times dt$, where $\mu$ is the Liouville measure on $T^1\mathbb{H}^n$.

Then we will describe a similar definition for discrete geodesic flows. Namely:

**Definition 3.5.** For fixed $h > 0$, $\mathcal{G}_h$ is the space of all $h$-step discrete geodesic trajectories: $\{\varphi^{ht} : t \in \mathbb{Z}\}$. That is $\mathcal{G}_h = \mathcal{G} \times S_h$ where $S_h$ is $[0, h]$ with 0 and $h$ identified. In addition, since we can write $T^1\mathbb{H}^n = \mathcal{G}_h \times \mathbb{Z}h$, then we can define the measure $m$ on $\mathcal{G}_h$ by $m = \nu \otimes \lambda$, where $\nu$ is the measure on $\mathcal{G}$ defined above and $\lambda$ the Lebesgue measure on $S_h$. Furthermore, the measure $\mu$ on the unit tangent bundle $T^1\mathbb{H}^n$ becomes the product of the measure $m$ on $\mathcal{G}_h$ with the counting measure on $\mathbb{Z}h$.

In [13], Maucourant considered the space of continuous geodesics, and estimated the probability that a random geodesic visits two fixed balls in $V$ as follows:

**Theorem 3.6.** [13, Lemma 4] There exists a constant $c_6 > 0$ such that, for any two balls in $\mathbb{H}^n$ of respective centers and radii $(o_1, r_1)$, $(o_2, r_2)$ that satisfy $r_1, r_2 < 1$, and $d(o_1, o_2) > 2$, the $\nu$-measure of continuous geodesics meeting those two balls is less than

$$c_6 r_1^{n-1} r_2^{n-1} e^{-(n-1)d(o_1, o_2)}.$$

Here is a similar estimate for discrete geodesics on $T^1\mathbb{H}^n$:

**Theorem 3.7.** Consider two balls in $\mathbb{H}^n$ with respective centers and radii $(o_1, r_1)$, $(o_2, r_2)$ that satisfy $r_1 < 1$, $r_2 < 1$, and $d(o_1, o_2) > 2$. Also assume that $h > 2 \min(r_1, r_2)$. Then the $\mu$-measure of the $h$-step geodesics which intersect those two balls is less than

$$\begin{cases} 2c_6 r_1^{n-1} r_2^{n-1} e^{-(n-1)d(\min(r_1, r_2), r_2)} & \text{if } \text{dist}(d, h\mathbb{Z}) \leq 2 \max(r_1, r_2) \\ 0 & \text{otherwise,} \end{cases}$$

where $c_6$ is as in Theorem 3.6.

**Proof.** An $h$-step geodesic will fail to intersect both balls if for any $k$ we have

$$|d - kh| > 2 \max(r_1, r_2);$$

in this case the measure we are to estimate is zero. So only if there is an integer $k$ such that (3.3) fails, can the $h$-step geodesic meet those balls. Using Theorem 3.6 and the fact that the space of discrete geodesics is $\mathcal{G} \times S_h$ with measure $m = \nu \otimes dh$, one can notice that the measure of such geodesics is bounded by $2c_6 \min(r_1, r_2) r_1^{n-1} r_2^{n-1} e^{-(n-1)d}$. \qed

3.3. A bound for the $L^2$-norm of $S_T[F]$. Recall that for $t \in \mathbb{N}$ we defined $f_t$ to be the characteristic function of $B_t$, which is a ball centered at $p_0 \in V = \Gamma \setminus \mathbb{H}^n$ of radius $r_t$, see (2.3), and considered the family of functions $F = (f_t)_{t \in \mathbb{N}}$ on $T^1V$. Also we have chosen a lift $\tilde{p}_0 \in \mathbb{H}^n$ of $p_0$. Now define $\tilde{B}_t$ to be a ball in $\mathbb{H}^n$ centered at $\tilde{p}_0$ of radius $r_t$, and let $g_t$ be the characteristic function of $\tilde{B}_t$. Thus, the lift $\tilde{f}_t$ of $f_t$ to $T^1\tilde{V}$ satisfies

$$\tilde{f}_t = \sum_{\gamma \in \Gamma} g_t \circ \gamma.$$
Fix a fundamental domain $D$ of $\mathbb{H}^n$ for $\Gamma$ containing $\tilde{p}_0$, and define
\[ i_V(\tilde{p}_0) \overset{\text{def}}{=} \sup \{ r \in \mathbb{R} : B(\tilde{p}_0, r) \subset D \}. \]

Also define
\[ R \overset{\text{def}}{=} \min \{ i_V(\tilde{p}_0)/4, 1, h \}, \tag{3.4} \]
and, for $i \in \mathbb{Z}_+$,
\[ \Gamma_i \overset{\text{def}}{=} \left\{ \gamma \in \Gamma : d(\tilde{p}_0, \gamma \tilde{p}_0) \in \left[ hi - \frac{h}{2}, hi + \frac{h}{2} \right] \right\}. \]

**Theorem 3.8.** Let $D \subset \mathbb{H}^n$ be a fundamental domain for $\Gamma$ such that $D$ contains the ball of center $\tilde{p}_0$ and of radius $3R$. Then for all $T \in \mathbb{N}$,
\[ \int_{T^1 V} S_T[F](v)^2 \, d\mu(v) \leq 2 \sum_{s=1}^{T} \sum_{i=1}^{[\frac{s}{2R}]} \sum_{\gamma \in \Gamma_i} \int_{T^1 D} g_s(v) g_t(\gamma \phi^{h(s-t)} v) \, d\tilde{\mu}(v). \]

**Proof.** For fixed $T \in \mathbb{N}$ and $v \in T^1 V$, we know that
\[ S_T[F](v)^2 = \left( \sum_{s=1}^{T} f_s(\phi^{h} v) \right) \left( \sum_{s=1}^{T} f_s(\phi^{h_s} v) \right) = 2 \sum_{s=1}^{T} \sum_{t \leq s} f_s(\phi^{h} v) f_s(\phi^{h_s} v). \]

Now we can integrate $S_T[F](v)^2$ over $T^1 V$ and make a change of variable $w = \phi^{h_s} v$. Since $\phi^{h_s}$ preserves the measure, we have the following:
\[ \int_{T^1 V} S_T[F](v)^2 \, d\mu(v) \leq 2 \sum_{s=1}^{T} \sum_{t \leq s} \int_{T^1 V} f_s(w) f_s(\phi^{h(t-s)} w) \, d\mu(w). \]

By the fact that $\tilde{f}_t$ is the lift of $f_t$, we obtain that
\[ \int_{T^1 V} S_T[F](v)^2 \, d\mu(v) \leq 2 \sum_{s=1}^{T} \sum_{t \leq s} \int_{T^1 \mathbb{H}^n} \tilde{f}_s(w) \tilde{f}_s(\phi^{h(t-s)} w) \, d\tilde{\mu}(w). \]

Since $\tilde{f}_t = \sum_{\gamma \in \Gamma} g_t \circ \gamma$, we can write
\[ \int_{T^1 V} S_T[F](v)^2 \, d\mu(v) \leq 2 \sum_{s=1}^{T} \sum_{\gamma \in \Gamma} \left( \sum_{\gamma \in \Gamma} g_s(\gamma w) \right) \left( \sum_{\gamma \in \Gamma} g_t(\gamma \phi^{h(t-s)} w) \right) \, d\tilde{\mu}(w). \]

Recall that $D$ is the fundamental domain of $\mathbb{H}^n$ for $\Gamma$. This insures that for all $w$ in $T^1 D$, in the sum $\sum_{\gamma \in \Gamma} g_s(\gamma w)$, all terms but the one corresponding to $\gamma = \text{id}$ are zero. So we have
\[ \int_{T^1 V} S_T[F](v)^2 \, d\mu(v) \leq 2 \sum_{s=1}^{T} \sum_{\gamma \in \Gamma} \sum_{t \leq s} \int_{T^1 D} g_s(w) g_t(\gamma \phi^{h(t-s)} w) \, d\tilde{\mu}(w). \]

Making another change of variables $v = -w$, where $-w$ means the point in $T^1 D$ with the same projection as $w$ and the tangent vector pointing in the opposite direction, we deduce that
\[ \int_{T^1 V} S_T[F](v)^2 \, d\mu(v) \leq 2 \sum_{s=1}^{T} \sum_{\gamma \in \Gamma} \sum_{t=1}^{s} g_s(v) g_t(\gamma \phi^{h(s-t)} v) \, d\tilde{\mu}(v). \]
For fixed \( v \in T^1 D \), we know that \( g_s(v)g_t(\gamma \phi^{h(s-t)}v) \) is zero when \( \pi(v) \notin \tilde{B}_s \) or \( \pi(\phi^{h(s-t)}v) \notin \gamma^{-1} \tilde{B}_t \), which implies that \( g_s(v)g_t(\gamma \phi^{h(s-t)}v) \) vanishes when
\[
|h(s-t) - d(\tilde{p}_0, \gamma^{-1} \tilde{p}_0)| > 2r_t + 2r_s.
\]
Since we know that \( 2r_t + 2r_s < 4R \), we can conclude that \( g_s(v)g_t(\gamma \phi^{h(s-t)}v) \) vanishes when \( t \) is outside of the interval
\[
\left[ s - \frac{d(\tilde{p}_0, \gamma^{-1} \tilde{p}_0)}{h}, \frac{4R}{h}, s - \frac{d(\tilde{p}_0, \gamma^{-1} \tilde{p}_0)}{h} + \frac{4R}{h} \right].
\]
Therefore, for any \( v \in T^1 V \) and any \( s \in \mathbb{N} \),
\[
\#\left\{ 1 \leq t \leq s : g_s(v)g_t(\gamma \phi^{h(s-t)}v) = 1 \right\} \leq \frac{8R}{h}. \tag{3.5}
\]
Furthermore, the integral is zero if \( |d(\tilde{p}_0, \gamma^{-1} \tilde{p}_0) - hi| > 2R > r_t + r_s \) for all \( i \). Hence this integral vanishes when \( |hi - h(s-t)| > 6R \), i.e. when
\[
hi - h(s-t) > 6R \quad \text{or} \quad hi - h(s-t) < -6R.
\]
In particular, we see that the quantity \( g_s(v)g_t(\gamma \phi^{h(s-t)}v) \) is zero if
\[
hi > hs + 6R > h(s-t) + 6R, \quad \text{i.e.} \quad i > s + \frac{6R}{h}.
\]

By the above fact and the fact that the union of all \( \Gamma_i \) is \( \Gamma \), we have
\[
\int_{T^1 V} S_T[\mathcal{F}](v)^2 \, d\mu(v) \leq 2 \sum_{s=1}^{T} \sum_{i \geq 0} \sum_{\gamma \in \Gamma_i} \int_{T^1 D} g_s(v)g_t(\gamma \phi^{h(s-t)}v) \, d\tilde{\mu}(v).
\]
\[
= 2 \sum_{s=1}^{T} \sum_{i=0}^{[s+\frac{6R}{h}]} \sum_{\gamma \in \Gamma_i} \int_{T^1 D} \sum_{t=1}^{s} g_s(v)g_t(\gamma \phi^{h(s-t)}v) \, d\tilde{\mu}(v).
\]
\[\square\]

Now let us define
\[
e_R = \frac{6R + 2}{h}
\]
and split the estimate of Theorem 3.8 into two parts:
\[
\int_{T^1 V} S_T[\mathcal{F}](v)^2 \, d\mu(v) \leq 2 \sum_{s=1}^{T} \sum_{i=1}^{[e_R]} \sum_{\gamma \in \Gamma_i} \int_{T^1 D} \sum_{t=1}^{s} g_s(v)g_t(\gamma \phi^{h(s-t)}v) \, d\tilde{\mu}(v)
\]
\[
+ 2 \sum_{s=1}^{T} \sum_{i=[e_R]}^{[s+\frac{6R}{h}]} \sum_{\gamma \in \Gamma_i} \int_{T^1 D} \sum_{t=1}^{s} g_s(v)g_t(\gamma \phi^{h(s-t)}v) \, d\tilde{\mu}(v). \tag{3.6}
\]

3.4. A bound on the first part of (3.6). It is not hard to estimate the first part.

**Theorem 3.9.** There is constant \( c_7 \), only depending on \( R \) and \( h \), such that for all \( T \in \mathbb{N} \)
\[
\sum_{s=1}^{T} \sum_{i=0}^{[e_R]} \sum_{\gamma \in \Gamma_i} \int_{T^1 D} \sum_{t=1}^{s} g_s(v)g_t(\phi^{h(s-t)}v) \, d\tilde{\mu}(v) \leq c_7 \sum_{t=1}^{T} r_t^n.
\]
Proof. Observing that $\bigcup_{i=3}^{c_R} \Gamma_i$ is a finite set, we write $N$ as its cardinal. Moreover, using (3.5), we get that
\[
\sum_{t=1}^{s} g_s(v)g_t(\gamma \phi^{h(s-t)}v) \leq \frac{8R}{h}.
\] (3.7)

In addition, we notice the following facts:
- if $g_s(v) = 0$, then $g_s(v)g_t(\gamma \phi^{s-t}v)$ in the left side vanishes;
- if $g_s(v) = 1$, then $g_s(v)g_t(\gamma \phi^{s-t}v)$ is at most 1.

Therefore, (3.7) is equivalent to the following:
\[
\sum_{t=1}^{s} g_s(v)g_t(\gamma \phi^{h(s-t)}v) \leq \frac{8R}{h} g_s(v).
\]

This allows us to write
\[
\sum_{s=1}^{T} \sum_{i=0}^{[c_R]} \sum_{\gamma \in \Gamma_i} \int_{T^1D} \sum_{t=1}^{s} g_s(v)g_t(\gamma \phi^{h(s-t)}v) d\mu(v) \leq \frac{8NR}{h} \sum_{s=1}^{T} \int_{T^1D} g_s(v) d\mu(v).
\]

Since $\int_{T^1D} g_s(v) d\mu(v)$ is equivalent to $r_s^n$, up to a multiplicative constant, there exists some positive constant $c_7$, depending only on $R$ and $h$, such that
\[
\sum_{s=1}^{T} \sum_{i=0}^{[c_R]} \sum_{\gamma \in \Gamma_i} \int_{T^1D} \sum_{t=1}^{s} g_s(v)g_t(\gamma \phi^{h(s-t)}v) d\mu(v) \leq c_7 \sum_{t=1}^{T} r_t^n.
\]

3.5. A bound on the second part of (3.6).

Theorem 3.10. There exist constants $c_8$ and $c_9$, only depending on $R$, such that
\[
\sum_{s=1}^{T} \sum_{i=[s\frac{R}{h}]}^{[s+\frac{9R}{h}]} \sum_{\gamma \in \Gamma_i} \int_{T^1D} \sum_{t=1}^{s} g_s(v)g_t(\gamma \phi^{h(s-t)}v) d\mu(v)
\]
\[
\leq c_8 \sum_{s=1}^{T} r_s^n \sum_{t=[s+\frac{4}{h} \ln r_s - \frac{6R}{h}-2]}^{[s-10R+4]} r_t^{n-1} + c_9 \sum_{s=1}^{T} r_s^n \sum_{t=1}^{T} r_t^n,
\]

where $c_4$ is as in Theorem 3.3.

Proof. Let us fix $s$ and produce an upper bound on
\[
\int_{T^1D} \sum_{t=1}^{s} g_s(v)g_t(\gamma \phi^{h(s-t)}v) d\mu(v)
\]

This requires the following observations:
1. (3.7) tells us that $\sum_{t=1}^{s} g_s(v)g_t(\gamma \phi^{h(s-t)}v) \leq \frac{8R}{h}$ for any $s$ and $v$.
2. We know that $|d(\tilde{p}_0, \gamma^{-1}\tilde{p}_0) - hi| < 2R$, i.e.
\[
hi - 2R < d(\tilde{p}_0, \gamma^{-1}\tilde{p}_0) < hi + 2R.
\]

Therefore, $i \geq c_R \geq \frac{6R+2}{h}$ implies that
\[
d(\tilde{p}_0, \gamma^{-1}\tilde{p}_0) > hi - 2R > 6R + 2 - 2R > 2.
\]

Hence, we know that the distance between the centers of $B(\tilde{p}_0, r_s)$ and $B(\gamma^{-1}\tilde{p}_0, r_t)$ is greater than 2. Thus by Theorem 3.7, the measure $m$ of the set of
discrete geodesics intersecting both $B(p_0, r_s)$ and $B(\gamma^{-1}p_0, r_t)$ is bounded by $2c_dq^{\gamma^{-1}r_s(r_t^{-1}-e^{-(n-1)(hi+1)})}r_s$.

3. Moreover, $D$ contains the ball of center $\tilde{p}_0$ with radius $3R$. So we know that for fixed $v$, $\# \{z \in \mathbb{Z}h : g_t(\phi^{h+1}z) > 0\} \leq \frac{3R}{h}$.

4. In addition, notice that $g_s(v)g_t(\gamma \phi^{h(s-t)}v)$ is not zero only if $|hi - h(s-t)| < 6R$.

This implies that

$$s - i - \frac{6R}{h} < t < s - i + \frac{6R}{h}.$$ 

Now since $(r_i)$ is decreasing, for all $i \geq c_R = \frac{6R+2}{h}$, we have that

$$\int_{T^1D} \sum_{i=1}^{s} g_s(v)g_t(\gamma \phi^{h(s-t)}v) \, d\mu(v) \ll r_s^{n^{n-1}}e^{-(n-1)hi}.$$ 

Therefore, for all $i \geq c_R$, we obtain that

$$\sum_{i=1}^{s} \int_{T^1D} \sum_{s \in \Gamma_i} g_s(v)g_t(\gamma \phi^{h(s-t)}v) \, d\mu(v) \ll N_i r_s^{n^{n-1}}e^{-(n-1)hi},$$

where $N_i$ is the number of elements of $\Gamma_i$ such that the integrated function is not zero. Now we can consider the sum over all $s$ and $i \geq c_R$:

$$\sum_{s=1}^{T} \sum_{i=1}^{s} \sum_{s \in \Gamma_i} \int_{T^1D} \sum_{t=1}^{s} g_s(v)g_t(\gamma \phi^{h(s-t)}v) \, d\mu(v)$$

$$\ll \sum_{s=1}^{T} \sum_{i=1}^{s} r_s^{n^{n-1}}e^{-(n-1)hi} N_i.$$ 

Our goal now is to estimate $N_i$. Recall Theorem 3.3, which allows us to estimate $\#\Gamma_i(r)$ when $hi \geq \max(-c_4 \ln r, r + t_0)$ for some constants $c_4, t_0 > 0$. We will take $r = r_{s-i-\frac{6n}{h}-1}$. Indeed, since $(r_i)$ is decreasing and we have assumed that $r_s \leq R < 1$, it follows that

$$-\ln r_s \geq -\ln r_{s-i-\frac{6n}{h}-1}$$

and $r_{s-i-\frac{6n}{h}-1} < R$.

Now let us define

$$V_s \overset{\text{def}}{=} \max\left(-c_4 \ln r_s, \frac{1 + t_0}{h}\right) + c_R. \quad (3.8)$$

Then $i \geq V_s$ implies that

$$hi \geq \max\left(-c_4 \ln r_{s-i-\frac{6n}{h}-1}, r_{s-i-\frac{6n}{h}-1} + t_0\right).$$

Meanwhile, we also know that $g_s(v)g_t(\gamma \phi^{h(s-t)}v)$ is not zero only if $d(\tilde{p}_0, \gamma^{-1}\tilde{p}_0) \in [ih - r_{s-i-\frac{6n}{h}-1}, ih + r_{s-i-\frac{6n}{h}-1}]$, where $i$ is such that $\gamma^{-1} \in \Gamma_i$. Therefore

$$N_i \leq \# \{ \gamma : d(\tilde{p}_0, \gamma \tilde{p}_0) \in [hi - r_{s-i-\frac{6n}{h}-1}, hi + r_{s-i-\frac{6n}{h}-1}] \}$$

$$\leq c_0 e^{(n-1)(hi+1)} r_{s-i-\frac{6n}{h}-1}.$$
when \( i \geq V_s \).

By applying the fact that \((r_t)\) is decreasing, we have the following:

\[
\sum_{s=1}^{T} \sum_{i=\lceil [V_s] \rceil}^{T} r_s^n r_{s-i}^{n-1} e^{-(n-1)(hi-1)} N_i \ll \sum_{s=1}^{T} r_s^n \sum_{t=1}^{s} r_t^{n-1}.
\]

When \( i < V_s \), we will use the counting lattice point estimate (Theorem 3.1) to conclude that \( N_i \ll e^{(n-1)hi} \). Recalling the definition of \( V_s \), see (3.8), we know that the assumption \( i < -\frac{h}{R} \ln r_s \) implies that \( i < V_s \). Meanwhile, since \((r_t)\) is decreasing, we have that

\[
\sum_{s=1}^{T} \sum_{i=\lceil [V_s] \rceil}^{T} r_s^n r_{s-i}^{n-1} e^{-(n-1)(hi-1)} N_i \ll \sum_{s=1}^{T} r_s^n \sum_{t=1}^{\lceil [V_s] \rceil} r_t^{n-1}.
\]

Putting it all together, we conclude that

\[
\sum_{s=1}^{T} \sum_{i=\lceil [V_s] \rceil}^{T} \int_{D} \int_{1}^{s} g_s(v) g_t(\gamma \phi^{k(s-t)}v) d\tilde{\mu}(v) \leq c_8 \sum_{s=1}^{T} r_s^n \sum_{t=1}^{\lceil [V_s] \rceil} r_t^{n-1} + c_9 \sum_{s=1}^{T} r_s^n \sum_{t=1}^{s} r_t^{n-1}.
\]

3.6. **Completion of the proof of Theorem 2.2.**

**Proof.** Recall that so far we have

\[
\int_{T^4 V} S_T[F](v)^2 d\mu(v) \leq c_7 \sum_{s=1}^{T} r_s^n + c_8 \sum_{s=1}^{T} r_s^n \sum_{t=1}^{s} r_t^{n-1} + c_9 \sum_{s=1}^{T} r_s^n \sum_{t=1}^{s} r_t^{n-1}.
\]

Now let us take \( C_1 = c_4/h, C_2 = 2 + 6R/h \), and let us assume (1.6), i.e. that there exist \( C, s_0 \) such that

\[
\sum_{t=1}^{s} r_t^{n-1} \leq C \sum_{i=1}^{s} r_i^n \quad \text{when } s \geq s_0.
\]

Then we can write

\[
\int_{T^4 V} S_T[F](v)^2 d\mu(v) \leq c_7 \sum_{s=1}^{T} r_s^n + c_8 \sum_{s=1}^{T} r_s^n \sum_{t=1}^{s} r_t^{n-1} + c_9 \sum_{s=1}^{T} r_s^n \sum_{t=1}^{s} r_t^{n-1} + C \cdot c_8 \sum_{s=s_0}^{T} r_s^n \sum_{t=1}^{s} r_t^{n} + c_9 \sum_{s=1}^{T} r_s^n \sum_{t=1}^{s} r_t^{n} + c_{14}
\]

\[
\leq c_7 \sum_{s=1}^{T} r_s^n + c_{13} \sum_{s=1}^{T} r_s^n \sum_{t=1}^{s} r_t^{n} + c_{14}
\]

\[\]
Since $\int_{T^1} I_T[F]^2 d\mu(v)$ is equivalent, up to a multiplicative constant, to $\sum_{s=1}^T r_s^n \sum_{t=1}^s r_t^n$, and with the assumption that $\int_{T^1} I_T[F]^2 d\mu(v) \to \infty$, one can easily conclude that $\frac{\int_{T^1} I_T[F]^2 d\mu(v)}{\int_{T^1} I_T[F]^2 d\mu(v)}$ is bounded in $L^2$-norm. \[\blacksquare\]

4. Proof of Theorem 1.2. Recall that we are given a non-increasing sequence $r_t$, which tends to 0 as $t \to \infty$ and such that $\sum_0^\infty r_t^n = \infty$ and in addition satisfying (1.4), that is, for some $C_0, s_0 > 0$ it holds that

\[
- \ln r_s \leq C_0 s \quad \text{when} \quad s \geq s_0.
\]  

(4.1)

We need to show that this sequence satisfies condition (1.6). This will be an easy consequence of the following lemma:

**Lemma 4.1.** Under the above assumptions, for any $C_1, C_2 > 0$ there exist $C_3, T > 0$ such that

\[
\sum_{s=\max(s+C_1 \ln r_s-C_2)}^s r_t^{n-1} \leq C_3 \sum_{t=1}^s r_t^n \quad \text{when} \quad s \geq T.
\]

**Proof.** By (4.1),

\[
- C_1 \ln r_s \leq C_1 C_0 s r_s \quad \text{when} \quad s \geq s_0.
\]  

(4.2)

Take $s_1$ such that $- \ln r_{s_1} \geq \frac{C_2}{C_1}$ when $s > s_1$. This and (4.2) imply that

\[
C_2 \leq C_1 C_0 s r_s
\]

and

\[
2C_1 C_0 s r_s \geq -C_1 \ln r_s + C_2.
\]  

(4.3)

Since $r_s$ is non-increasing, (4.3) implies that

\[
2C_1 C_0 s r_s \geq \sum_{s=\max(s+C_1 \ln r_s-C_2)}^s r_t^{n-1} \geq -C_1 \ln r_s + C_2
\]

when $s > \max(s_0, s_1)$. Due to the fact that $0 < r_s < r_{\max(s+C_1 \ln r_s-C_2)} < 1$, we have that

\[
2C_1 C_0 s r_s \geq \sum_{s=\max(s+C_1 \ln r_s-C_2)}^s r_t^{n-1} \geq \left(1 - C_1 \ln r_s + C_2\right) \left(1 + r_{\max(s+C_1 \ln r_s-C_2)} - r_s\right);
\]

thus

\[
2C_1 C_0 s r_s \geq \left(1 - C_1 \ln r_s + C_2\right) \left(1 + r_{\max(s+C_1 \ln r_s-C_2)} - r_s\right)
\]

\[
\geq \left(- C_1 \ln r_s + C_2\right) \left(1 - r_s\right) + \left(- C_1 \ln r_s + C_2\right) r_{\max(s+C_1 \ln r_s-C_2)}.
\]

Therefore, when $s > \max(s_0, s_1)$, we obtain that

\[
\left(2C_1 C_0 s + C_1 \ln r_s - C_2\right) r_{\max(s+C_1 \ln r_s-C_2)} \geq \left(- C_1 \ln r_s + C_2\right) \left(1 - r_s\right).
\]  

(4.4)

Now take $s_2 > 0$ such that $r_s < \frac{1}{4C_1 C_0}$ when $s > s_2$, and let $T \defeq \max(s_0, s_1, s_2)$. Then (4.3) implies that, when $s > T$,

\[
s \geq 2C_1 C_0 \left(- C_1 \ln r_s + C_2\right).
\]

Thus, by adding $2C_1 C_0 s + (2C_1 C_0 + 1)(C_1 \ln r_s - C_2)$ to both sides, we conclude that, when $s > T$,

\[
(2C_1 C_0 + 1)(s + C_1 \ln r_s - C_2) \geq 2C_1 C_0 s + C_1 \ln r_s - C_2.
\]

Now let us define $C_3 \defeq 2C_1 C_0 + 1$. 


Then we have that, when $s > T$

$$C_3(s + C_1 \ln r_s - C_2) \geq 2C_1C_0s + C_1 \ln r_s - C_2.$$ 

which, in view of (4.4), implies

$$C_3(s + C_1 \ln r_s - C_2)r_{[s+C_1 \ln r_s-C_2]} \geq (-C_1 \ln r_s + C_2)(1 - r_s).$$

Since $r_{[s+C_1 \ln r_s-C_2]} > 0$, the above inequality implies that, when $s \geq T$,

$$C_3(s + C_1 \ln r_s - C_2)r^n_{[s+C_1 \ln r_s-C_2]} \geq (-C_1 \ln r_s + C_2)(1 - r_s)r^{n-1}_{[s+C_1 \ln r_s-C_2]}.$$  (4.5)

On the other hand, since $r_s$ is non-increasing, one will notice that

$$C_3(s + C_1 \ln r_s - C_2)r^n_{[s+C_1 \ln r_s-C_2]} \leq C_3 \sum_{t=1}^{[s+C_1 \ln r_s-C_2]} r^n_t,$$

and

$$\sum_{t=[s+C_1 \ln r_s-C_2]}^{s} (r^n_t - r^n_{t-1}) = \sum_{t=[s+C_1 \ln r_s-C_2]}^{s} (1 - r_t)r^{n-1}_t$$

$$\leq (-C_1 \ln r_s + C_2)(1 - r_s)r^{n-1}_{[s+C_1 \ln r_s-C_2]}.$$ 

Therefore, by (4.5), we have that, when $s \geq T$,

$$\sum_{t=[s+C_1 \ln r_s-C_2]}^{s} (r^n_t - r^n_{t-1}) \leq C_3 \sum_{t=1}^{[s+C_1 \ln r_s-C_2]} r^n_t,$$

and hence

$$\sum_{t=[s+C_1 \ln r_s-C_2]}^{s} r^{n-1}_t \leq C_3 \sum_{t=1}^{[s+C_1 \ln r_s-C_2]} r^n_t + \sum_{t=[s+C_1 \ln r_s-C_2]}^{s} r^n_t \leq C_3 \sum_{t=1}^{s} r^n_t.$$

This shows that (1.4) implies (1.6), and finishes the proof of Theorem 1.2.  \(\square\)

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