Research Article
The Sigma Coindex of Graph Operations

Yasar Nacaroglu

Department of Mathematics, Kahramanmaras Sutcu Imam University, 46100, Kahramanmaras, Turkey
Correspondence should be addressed to Yasar Nacaroglu; yasarnacaroglu@ksu.edu.tr

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The sigma coindex is defined as the sum of the squares of the differences between the degrees of all nonadjacent vertex pairs. In this paper, we propose some mathematical properties of the sigma coindex. Later, we present precise results for the sigma coindices of various graph operations such as tensor product, Cartesian product, lexicographic product, disjunction, strong product, union, join, and corona product.

1. Introduction

Let $G$ be a simple graph with a vertex set $V(G)$ and edge set $E(G)$, where $|V(G)| = n_G$ and $|E(G)| = m_G$. The degree of a vertex $g$ in $G$, denoted by $\deg_G(g)$, is defined as the number of incident edges to it. The complement of a $G$ graph, denoted by $\overline{G}$, is the graph with the same vertex set. Here, any two vertices $g_1$ and $g_2$ are adjacent if and only if they are not adjacent in $G$. The number of edges of the graph $G$ is denoted by $m_G$, where $m_G = \binom{n_G}{2} - m_G$. For other undefined notations and terminology from graph theory, the readers are referred to [1].

Chemical graph theory is the field of study of mathematical chemistry in relation to chemical graphs. The basic idea here is to reveal the properties of molecules using the information corresponding to chemical graphics. For this, topological indices are among the most used tools. Topological indices are constant numbers that reveal the structure of the graph. These constant numbers are used in the modeling of molecules in chemistry and biology. Until today, many topology indices have been defined and used as a tool in QSAR/QSPR studies.

The oldest degree-based topological indices are the first and second Zagreb indices being considered in [2]. These indices are defined as follows:

\[ M_1(G) = \sum_{g \in V(G)} \deg_G^2(g) = \sum_{g_1, g_2 \in E(G)} [\deg_G(g_1) + \deg_G(g_2)], \]
\[ M_2(G) = \sum_{g_1, g_2 \in E(G)} \deg_G(g_1)\deg_G(g_2). \]  
(1)

The first and second Zagreb coindices are defined as [3]

\[ \overline{M}_1(G) = \sum_{g_1, g_2 \in E(G)} [\deg_G(g_1) + \deg_G(g_2)], \]
\[ \overline{M}_2(G) = \sum_{g_1, g_2 \in E(G)} \deg_G(g_1)\deg_G(g_2). \]  
(2)

respectively.

The forgotten topological index was introduced by Furtula and Gutman [4], $F(G)$, as the sum of cubes of vertex degrees:

\[ F(G) = \sum_{g \in V(G)} \deg_G^3(g) = \sum_{g_1, g_2 \in E(G)} [\deg_G^2(g_1) + \deg_G^2(g_2)]. \]  
(3)

The forgotten coindex of a graph $G$ is introduced as follows [5]:

\[ F(G) = \sum_{g_1, g_2 \in E(G)} [\deg_G^2(g_1) + \deg_G^2(g_2)]. \]  
(4)

The hyper-Zagreb index was first introduced by [6]. This index is defined as follows:
The hyper-Zagreb coindex was introduced by Veylaki et al. [7], as are defined as follows:
\[
H\overline{M}(G) = \sum_{g_1, g_2 \in E(G)} (d_G(g_1) + d_G(g_2))^2.
\]  
(5)

The sigma index of a graph is defined as [8]
\[
\sigma(G) = \sum_{g_1, g_2 \in E(G)} (d_G(g_1) - d_G(g_2))^2 = F(G) - 2M_2(G).
\]  
(6)

Graph operations are an important subject of graph theory. Many complex graphs can be obtained by applying graph operations to simpler graphs. Until today, many studies have been done on graph operations. In [10–12], the algebraic properties of tensor, lexicographic, and Cartesian products of monogenic semigroup graphs were presented. Azari [13] put forward some results on the eccentric connectivity coindex of several graph operations. In [14], the upper bounds on the multiplicative Zagreb indices of graph operations were given. Nacaroglu et al. [15] gave some bounds on the multiplicative Zagreb coindices of graph operations. Acioglu et al. [16] presented formulae for omega invariant of some graph operations. In [17], F index of different corona products of two given graphs was calculated. Das et al. [18] examined the Harary index of graph operations. We refer the reader to [19] for more properties and applications of graph products.

In this study, we will calculate the sigma coindex of two graphs under some graph products as corona, join, union, lexicographic product, disjunction, tensor product, Cartesian product, and strong product.

2. Some Properties of Sigma Coindex

All operations examined in this section are binary. Therefore, we will consider graphs of \(G\) and \(H\) as two finite and simple graphs. Let us examine the sigma coindices of some special graphs before moving on to the basic results.

Proposition 1. \(\overline{\sigma}(G) = \sigma(\overline{G})\).

Proof. From definition of the sigma coindex, we have
\[
\overline{\sigma}(G) = \sum_{g_1, g_2 \in E(G)} (d_G(g_1) - d_G(g_2))^2
= \sum_{g_1, g_2 \in E(G)} \left( (n_G - 1 - d_G(g_1)) - (n_G - 1 - d_G(g_2)) \right)^2
= \sum_{g_1, g_2 \in E(G)} (d_G^2(g_1) - d_G^2(g_2))
= \sigma(\overline{G}),
\]  
(7)
as required.

Proposition 2. \(\overline{\sigma}(\overline{G}) = \sigma(G)\).

Proof. From definition of the sigma coindex, we have
\[
\sigma(G) = \sum_{g_1, g_2 \in E(G)} (d_G(g_1) - d_G(g_2))^2
= \sum_{g_1, g_2 \in E(G)} \left( (n_G - 1 - d_G(g_1)) - (n_G - 1 - d_G(g_2)) \right)^2
= \sum_{g_1, g_2 \in E(G)} (d_G^2(g_1) - d_G^2(g_2))
= \sigma(\overline{G}).
\]  
(8)
as required.

Proposition 3. \(\overline{\sigma}(G) = n_G M_1(G) - \sigma(G) - 4m^2_G\).

Proof. The proof follows by the expression \(\overline{\sigma}(G) = n_G M_1(G) - \sigma(G) - 4m^2_G\), from Lemma 4 of [20].

By the following proposition, we can give the sigma coindices of the complete graphs, star graphs, cycles, and path graphs.

Proposition 4
\[
\begin{align*}
\overline{\sigma}(K_n) &= 0, \\
\overline{\sigma}(S_n) &= 0, \\
\overline{\sigma}(C_n) &= 0, \\
\overline{\sigma}(P_n) &= 2(n - 3).
\end{align*}
\]  
(9)

3. Sigma Coindex under Graph Operations

In this section, we give some formulae for the sigma coindices of some graph operations as union, join, corona product, tensor product, Cartesian product, lexicographic product, and strong product.

The tensor product of graphs \(G\) and \(H\), denoted as \(G \otimes H\), is the graph with \(V(G \otimes H) = V(G) \times V(H)\). The vertices \((g_1, h_1)\) and \((g_2, h_2)\) are adjacent if and only if \(g_1\) is adjacent to \(g_2\) in \(G\) and \(h_1\) is adjacent to \(h_2\) in \(H\). Also we know that \(d_{G \otimes H}(g_1, h_1) = d_G(g_1)d_H(h_1)\) and \(m(G \otimes H) = 2m(G)m(H)\).
Let us first formulate the sigma index of the tensor products of any two graphs as shown in Theorem 1.

**Theorem 1.** Let \( \Gamma = G \otimes H \). Then
\[
\sigma(\Gamma) = F(G)F(H) - 4M_2(G)M_2(H).
\]

(12)

The proof follows from the expressions \( M_2(G \otimes H) = 2M_2(G)M_2(H) \) in Theorem 2.1 of [21], \( F(G \otimes H) = F(G)F(H) \) in Theorem 7 of [22], and (7).

Now we can express the sigma coindex under the tensor product of any two graphs using Theorem 1.

**Theorem 2.** Let \( \Gamma = G \otimes H \). Then,
\[
\overline{\sigma}(\Gamma) = n_Gh_H M_1(G)M_1(H) + 4M_2(G)M_2(H) - F(G)F(H) - 16m_Gm_H^2.
\]

(13)

**Proof.** From the expression \( \sigma(\Gamma) + \overline{\sigma}(\Gamma) = n \Gamma M_1(\Gamma) - 4m_1^2 \) in Lemma 4 of [20], we get
\[
\overline{\sigma}(\Gamma) = n_Gh_H M_1(G \otimes H) - \sigma(G \otimes H) - 4m_G^2h_H. \tag{14}
\]

The proof follows from the expressions \( M_1(G_1 \otimes G_2) = M_1(G_1)M_1(G_2) \) in Theorem 2.1 of [21] and Theorem 1.

**Example 1.** Using Theorem 2, we have
\begin{enumerate}
\item \( \sigma(P_n \otimes P_m) = 8(n + m)(nm + 2) - 16(n + m)^2 + 4nm + 44 \),
\item \( \sigma(P_n \otimes C_m) = 8m(nm - 2m - 2) \),
\item \( \overline{\sigma}(P_n \otimes K_m) = m(m - 1)^2(2nm - 6m + 2) \).
\end{enumerate}

The Cartesian product of \( G \) and \( H \), denoted by \( G \times H \), is the graph with the vertex set \( V(G) \times V(H) \). The vertices \( (g_1, h_1) \) and \( (g_2, h_2) \) are adjacent if and only if \( g_1 = g_2 \) and \( h_1 \neq h_2 \) or \( h_1 = h_2 \) and \( g_1 \neq g_2 \) in \( G \) and \( H \). Also we know that \( m_{G \times H} = m_Gm_H + m_G + m_H \) and \( d_{G \times H}(g_1, h_1) = d_G(g_1) + d_H(h_1) \), respectively.

**Theorem 3.** Let \( \Gamma = G \times H \) be the Cartesian product of two graphs \( G \) and \( H \). Then
\[
\overline{\sigma}(\Gamma) = n_G^2 \sigma_1(H) + n_H^2 \sigma_1(G) - n_G \sigma(H) - n_H \sigma(G).
\]

(15)

**Proof.** In Theorem 1 of [23] and Theorem 1 of [9], the following formulae are given, respectively:
\[
M_1(G \times H) = n_HM_1(G) + n_GM_1(H) + 8m_Gm_H,
\]

(16)
\[
\sigma(G \times H) = n_H\sigma(G) + n_G\sigma(H),
\]

(17)
respectively. So the proof is completed by applying (8), (16), and (17) in Proposition 3.

**Example 2.** By using Theorem 3, we have
\begin{enumerate}
\item \( \sigma(P_n \times P_m) = 2n^2m + 2nm^2 - 4n^2 - 4m^2 - 2n - 2m \),
\item \( \overline{\sigma}(P_n \times C_m) = 2nm^2 - 4m^2 - 2m \).
\end{enumerate}

The lexicographic product \( \Gamma = G[H] \) of graphs \( G \) and \( H \) is a graph with the vertex set \( V(G) \times V(H) \). Any two vertices \( (g_1, h_1) \) and \( (g_2, h_2) \) are adjacent in \( G[H] \) if and only if either \( g_1 \) is adjacent to \( g_2 \) in \( G \) or \( g_1 = g_2 \) and \( h_1 \) is adjacent to \( h_2 \) in \( H \). Also we know that \( m_{G[H]} = n_Gm_H + m_Gm_H \) and \( d_{G[H]}(g_1, h_1) = n_Hd_G(g_1) + d_H(h_1) \).

We need sigma index to calculate the sigma coindex of the lexicographic product of two graphs. But Theorem 2 in [9] is not true. The correct statement is shown in Theorem 4.

**Theorem 4.** Let \( \Gamma = G[H] \) be the lexicographic product of two graphs \( G \) and \( H \). Then
\[
\sigma(\Gamma) = n_G^2 \sigma(G) + n_H \sigma(H) + 2m_Gn_HM_1(H) - 8m_G^2m_H^2.
\]

(18)

In Theorem 5, we present the sigma coindex of the lexicographic product of the two graphs depending on some topological indices of these graphs.

**Theorem 5.** Let \( \Gamma = G[H] \) be the lexicographic product of two graphs \( G \) and \( H \). Then
\[
\overline{\sigma}(\Gamma) = n_G^2 \overline{\sigma}(G) + (n_H^2 - 2m_G)\sigma_1(H) - n_H \sigma(G).
\]

(19)

**Proof.** From Proposition 3, we have
\[
\overline{\sigma}(\Gamma) = n_H \overline{\sigma}(G) - \sigma(\Gamma) - 4m_G^2.
\]

(20)

In Theorem 3 of [23], the following formula is given as
\[
M_1(\Gamma) = n_H^2M_1(G) + n_GM_1(H) + 8n_Hm_Gm_H.
\]

(21)

By applying (18) and (21) in (20), we get
\[
\overline{\sigma}(\Gamma) = n_H^2(n_GM_1(G) - \sigma(G)) + (n_H^2 - 2m_Gn_H)M_1(H)
\]
\[
- n_G \sigma(H) - 4(n_G^2m_H^2 - 2m_Gm_H^2 + n_H^4m_G^2).
\]

(22)

Also we have
\[
\sigma_1(G) = \sigma(G) + \overline{\sigma}(G) = n_GM_1(G) - 4m_G^2.
\]

(23)

So the proof is completed by applying (23) in (22).

**Example 3.** The sigma coindices of the fence graph \( P_n[P_m] \) and the closed fence graph \( C_n[P_m] \) are given as follows:
\begin{enumerate}
\item \( \overline{\sigma}(P_n[P_m]) = 2nm^4 + 2n^2m^2 - 6m^2 - 4nm - 4m^2 - 6n + 8 \),
\item \( \overline{\sigma}(C_n[P_m]) = 2n^4m^2 + 2nm^2 + 6n - 4n^2 \).
\end{enumerate}

The disjunction product of \( G \) and \( H \), denoted by \( G \lor H \), is a graph with the vertex set \( V(G) \times V(H) \). The vertices \( (g_1, h_1) \) and \( (g_2, h_2) \) are adjacent iff either \( g_1 \) is adjacent to
$g_2$ in $G$; or $h_1$ is adjacent to $h_2$ in $H$. Also we know that $m_{G\vee H} = n_G^2 m_H + n_H^2 n_G - 2 n_G n_H$ and $d_{G\vee H} (g_1, h_1) = n_H d_G (g_1) + n_G d_H (h_1) - d_G (g_1) d_H (h_1)$.

**Theorem 6.** Let $\Gamma = G \vee H$ be the disjunctive product of the graphs $G$ and $H$. Then

$$
\sigma(\Gamma) = \left[ n_G^3 - 4 n_G m_H + M_1 (H) + 2 n_H^2 m_H - 2 n_H M_1 (H) + \frac{1}{2} \overline{HM} (H) \right] \sigma(G)
+ \left[ n_G^3 - 4 n_G m_G + M_1 (G) + 2 n_G^2 m_G - 2 n_G M_1 (G) + \frac{1}{2} \overline{HM} (G) \right] \sigma(H).
$$

**Proof.** From definition of the disjunctive product and the sigma coindex, we have

$$
\sigma(\Gamma) = \sum_{x,y \in V(\Gamma)} [d_\Gamma (x) - d_\Gamma (y)]^2
= \sum_{g_1 \in V(G)} \sum_{h_1 \in V(H)} (d_{G\vee H} (g_1, h_1) - d_{G\vee H} (g_1, h_1))^2
+ \sum_{g_1 \in V(G)} \sum_{h_1 \in V(H)} (d_{G\vee H} (g_1, h_1) - d_{G\vee H} (g_2, h_1))^2
+ \sum_{g_1 \in V(G)} \sum_{h_1 \in V(H)} (d_{G\vee H} (g_1, h_1) - d_{G\vee H} (g_2, h_2))^2.
$$

In other world, we have

$$
\sigma(\Gamma) = \sum_{g_1 \in V(G)} \sum_{h_1 \in V(H)} (n_G - d_G (g_1))^2 (d_H (h_1) - d_H (h_2))^2
+ \sum_{h_1 \in V(H)} \sum_{g_1, g_2 \in E(G)} (n_H - d_H (h_1))^2 (d_G (g_1) - d_G (g_2))^2
+ \sum_{g_1, g_2 \in E(G)} \sum_{h_1, h_2 \in V(H)} \left[ (n_H - \frac{1}{2} d_H (h_1) + d_H (h_2)) (d_G (g_1) - d_G (g_2)) + (n_G - \frac{1}{2} d_G (g_1) + d_G (g_2)) (d_H (h_1) - d_H (h_2)) \right]^2.
$$

The proof is completed by applying (1), (2), (6), and (8) in (26).

Let $G$ and $H$ be two graphs. The strong product of $G$ and $H$ is a graph with the vertex set of $V(G) \times V(H)$, denoted by $G \ast H$. The vertices $(g_1, h_1)$ and $(g_2, h_2)$ are adjacent if either $g_1 = g_2$ and $h_1 h_2 \in E(H)$; or $h_1 = h_2$ and $g_1 g_2 \in E(G)$; or

$$g_1 g_2 \in E(G) \text{ and } h_1 h_2 \in E(H).$$

Also we know that $m(G \ast H) = n(G)m(H) + n(G)n(H) + 2m(G)m(H)$ and $d_{G \ast H} (g_1, h_1) = d_G (g_1) + d_H (h_1) + d_G (g_2) d_H (h_2)$.

**Theorem 7.** Let $\Gamma = G \ast H$ be the strong product of two graphs $G$ and $H$. Then

$$
\sigma(\Gamma) = \left[ \frac{1}{2} M_1 (H) + 4 m_H n_H + 2 m_H^2 + n_H^2 \right] \sigma(G) + \left[ \frac{1}{2} M_1 (G) + 4 m_G n_G + 2 m_G^2 + n_G^2 \right] \sigma(H).
$$
Proof. From definition of the strong product and the sigma coincident, we have

$$
\sigma(G \ast H) = \sum_{(g_1, h_1)(g_2, h_2) \in E(G \ast H)} \left[ d_{G \ast H}(g_1, h_1) - d_{G \ast H}(g_1, h_2) \right]^2.
$$

(28)

The sum can be split into five parts:

$$
\begin{align*}
S_1 &= \sum_{g_1, g_2 \in E(G)} \sum_{h_1, h_2 \notin E(H)} \left[ d_{G \ast H}(g_1, h_1) - d_{G \ast H}(g_1, h_2) \right]^2 \\
S_2 &= \sum_{g_1, g_2 \notin E(G)} \sum_{h_1, h_2 \in E(H)} \left[ d_{G \ast H}(g_1, h_1) - d_{G \ast H}(g_2, h_1) \right]^2
\end{align*}
$$

(29)

which we denote by $S_1, S_2, S_3, S_4$, and $S_5$, respectively. We have

$$
S_1 = \sum_{g_1, g_2 \in E(G)} \sum_{h_1, h_2 \notin E(H)} \left[ d_{G \ast H}(g_1, h_1) - d_{G \ast H}(g_1, h_2) \right]^2
= \sum_{g_1, g_2 \in E(G)} \sum_{h_1, h_2 \notin E(H)} (d_G(g_1) + 1)^2 (d_H(h_1) - d_H(h_2))^2
= \left( M_1(G) + 4m_G + n_G \right) \sigma(H).
$$

(30)

Similarly, we get

$$
S_2 = \sum_{g_1, g_2 \notin E(G)} \sum_{h_1, h_2 \in E(H)} \left[ d_{G \ast H}(g_1, h_1) - d_{G \ast H}(g_2, h_1) \right]^2
= \left( M_1(H) + 4m_H + n_H \right) \sigma(G).
$$

(31)

On the contrary, we have

$$
S_3 = \sum_{g_1, g_2 \in E(G)} \sum_{h_1, h_2 \notin E(H)} \left[ d_{G \ast H}(g_1, h_1) - d_{G \ast H}(g_2, h_2) \right]^2
$$

$$
= \sum_{g_1, g_2 \in E(G)} \sum_{h_1, h_2 \notin E(H)} \left[ 1 + \frac{1}{2} (d_H(h_1) + d_H(h_2)) \right] (d_G(g_1) - d_G(g_2))^2
+ \left[ 1 + \frac{1}{2} (d_H(h_1) + d_H(h_2)) \right] (d_H(h_1) - d_H(h_2))^2
$$

$$
= \left[ 2m_H + 2M_1(H) + \frac{1}{2} HM(G) \right] \sigma(G) + \left[ 2m_G + 2M_1(G) + \frac{1}{2} HM(G) \right] \sigma(H).
$$

(32)

Finally, similar to (32), we get

$$
S_4 = \sum_{g_1, g_2 \notin E(G)} \sum_{h_1, h_2 \in E(H)} \left[ d_{G \ast H}(g_1, h_1) - d_{G \ast H}(g_2, h_2) \right]^2
$$

(33)

$$
= \left[ 2m_G + 2M_1(G) + \frac{1}{2} HM(G) \right] \sigma(H) + \left[ 2m_H + 2M_1(H) + \frac{1}{2} HM(G) \right] \sigma(G).
$$

$$
S_5 = \sum_{g_1, g_2 \notin E(G)} \sum_{h_1, h_2 \in E(H)} \left[ d_{G \ast H}(g_1, h_1) - d_{G \ast H}(g_2, h_2) \right]^2
$$

(34)

$$
= \left[ 2m_H + 2M_1(H) + \frac{1}{2} HM(G) \right] \sigma(G) + \left[ 2m_G + 2M_1(G) + \frac{1}{2} HM(G) \right] \sigma(H).
$$

The proof is completed by using (30)–(34) and the relations $HM(G) + HM(G) = \left( n_G - 2 \right) M_1(G) + 4m_G^2$ in Theorem 2.1 of [25], $M_1(G) + M_1(G) = 2m_G(n_G - 1)$ in Theorem 1 of [26].
As an application of Theorem 7, we give below the sigma coinices of \( P_\sigma P_m \) and \( P_\sigma C_m \).

**Example 4**

\[
P_\sigma P_m = 6(n + m)(3nm - 6n - 6m - 1) + 28nm + 140,
\]

\[
P_\sigma C_m = 18m^2 + 18nm^2 - 108m.
\]

(35)

Let \( \Gamma \) be vertex-disjoint graphs. Then, the join of two graphs \( G \) and \( H \), denoted \( G \cup H \), is the graph with vertex set \( V(G) \cup V(H) \) and edge set \( E(G) \cup E(H) \).

**Theorem 8.** Let \( \Gamma = G \cup H \). Then

\[
\sigma(\Gamma) = \sigma(G) + \sigma(H) + n_H M_1(G) + n_G M_1(H) - 8m_G m_H.
\]

(36)

Proof. From the definition of the sigma coindex of a graph, we have

\[
\sigma(\Gamma) = \sum_{xy \notin E(\Gamma)} (d_\Gamma(x) - d_\Gamma(y))^2
\]

\[
= \sum_{g, h \notin E(\Gamma)} (d_G(g_1) - d_G(g_2))^2 + \sum_{h, h_1 \notin E(H)} (d_H(h_1) - d_H(h_2))^2
\]

\[
+ \sum_{g \in V(G)} \sum_{h \in V(H)} (d_G(g) - d_H(h))^2
\]

\[
= \sigma(G) + \sigma(H) + \sum_{g \in V(G)} \sum_{h \in V(H)} (d_G^2(g) + d_H^2(h) - 2d_G(g)d_H(h))
\]

\[
= \sigma(G) + \sigma(H) + n_H M_1(G) + n_G M_1(H) - 8m_G m_H.
\]

\[\square\]

**Example 5.** \( \sigma(K_{n_1} \cup K_{n_2}) = n_1n_2(n_1 - n_2)^2 \).

Let \( G \) and \( H \) be vertex-disjoint graphs. Then the join, \( G + H \), of \( G \) and \( H \), is the supergraph of \( G \cup H \) in which each vertex of \( G \) is adjacent to every vertex of \( H \). The join of two graphs is also known as their sum. Thus, for example, the complete bipartite graph is \( K_n + K_m = K_{nm} \). The degree of a vertex \( x \) of \( G + H \) is defined by

\[
d_{G+H}(x) = \begin{cases} d_G(x) + n_H, & \text{if } x \in V(G), \\ d_H(x) + n_G, & \text{if } x \in V(H). \end{cases}
\]

(38)

**Theorem 10.** Let \( \Gamma = G + H \). Then

\[
\sigma(\Gamma) = \sigma(G) + \sigma(H).
\]

(39)

Proof. From the definition of the sigma coindex of a graph, we have

\[
\sigma(\Gamma) = \sum_{xy \notin E(\Gamma)} (d_\Gamma(x) - d_\Gamma(y))^2
\]

\[
= \sum_{g_1, g_2 \notin E(G)} (d_G(g_1) + n_H - d_G(g_2) - n_H)^2
\]

\[
+ \sum_{h_1, h_2 \notin E(H)} (d_H(h_1) + n_G - d_H(h_2) - n_G)^2
\]

\[
= \sigma(G) + \sigma(H).
\]

(40)
\[ K = n_H M_1 (G) + n_G M_1 (H) + 4 m_G n_H (n_H - 1) \]
\[ - 4 m_H n_G (n_H - 1) - 8 n_G m_H + n_G n_H (n_H - 1)^2. \]

(44)

Proof. From definition of the sigma coindex, we have

\[
\sigma (G) = \sum_{x \notin E} \left( d_x (x) - d_x (y) \right)^2
\]
\[
+ \sum_{g_1, g_2 \in E (G)} \left( (d_G (g_1) + n_H) - (d_G (g_2) + n_H) \right)^2
\]
\[
+ n_G \sum_{h_i, h_2 \in E (H)} \left( (d_H (h_i) + 1) - (d_H (h_2) + 1) \right)^2
\]
\[
+ (n_G - 1) \sum_{g_1 \in V (G)} \sum_{h_i \in V (H)} \left( (d_G (g_1) + n_H) - (d_H (h_i) + 1) \right)^2
\]
\[
+ \left( \frac{n_G}{2} \right) \sum_{h_i, h_2 \in V (H)} \left( (d_H (h_i) + 1) - (d_H (h_2) - 1) \right)^2
\]
\[
= \sigma (G) + n_G \sigma (H) + 2 \left( \frac{n_G}{2} \right) \sigma_H (H) + (n_G - 1) K.
\]

Let \( k_1, k_2, \ldots, k_n \) be nonnegative integers. The thorn graph of the graph \( G \), denoted by \( G^k \), is a graph obtained by attaching \( k_i \) new vertices of degree one to the vertex \( g_i \) of the graph \( G \), \( i = 1, 2, \ldots, n \) (see [29]). If \( k_1 = k_2 = \cdots = k_n = k \), then \( G^{*k} \equiv G \ast K_k \), where \( K_k \) is the complement of a complete graph \( K_k \).

Corollary 2. \( \bar{\sigma} (G^{*k}) = \bar{\sigma} (G) + k (n - 1) M_1 (G) + k (k - 1) (n - 1) [4 m + n (k - 1)]. \)

4. Conclusions

In this paper, we have presented the exact formulae for the sigma coindices of graphs under some graph operations. We have also applied these results to some special graph types. However, there are also graph products that are not presented here. This remains as an open problem.

Data Availability

No data were used to support the study.

Conflicts of Interest

The author declares no conflicts of interest.

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