HAUSDORFF DIMENSION OF THE LARGE VALUES OF WEYL SUMS

CHANGHAO CHEN AND IGOR E. SHPARLINSKI

ABSTRACT. The authors have recently obtained a lower bound of the Hausdorff dimension of the sets of vectors \((x_1, \ldots, x_d) \in [0, 1)^d\) with large Weyl sums, namely of vectors for which

\[
\left| \sum_{n=1}^{N} \exp \left( 2\pi i (x_1 n + \ldots + x_d n^d) \right) \right| \geq N^\alpha
\]

for infinitely many integers \(N \geq 1\). Here we obtain an upper bound for the Hausdorff dimension of these exceptional sets.

1. Introduction

1.1. Motivation and background. For an integer \(d \geq 2\), let \(T_d = (\mathbb{R}/\mathbb{Z})^d\) be the \(d\)-dimensional unit torus.

For a vector \(x = (x_1, \ldots, x_d) \in T_d\) and \(N \in \mathbb{N}\), we consider the exponential sums

\[
S_d(x; N) = \sum_{n=1}^{N} e \left( x_1 n + \ldots + x_d n^d \right),
\]

which are commonly called Weyl sums, where throughout the paper we denote \(e(x) = \exp(2\pi i x)\).

The authors [2, Appendix A] have show that for almost all \(x \in T_d\) (with respect to Lebesgue measure) one has

\[
|S_d(x; N)| \leq N^{1/2+o(1)} \text{ as } N \to \infty,
\]

see also [3, Corollary 1.3] for a different proof. It is very natural to conjecture that the exponent \(1/2\) is the best possible value, however there seems to be no results in this direction.

For integer \(d \geq 2\) and \(0 < \alpha < 1\) our main object is defined as

\[
\mathcal{E}_{d, \alpha} = \{ x \in T_d : |S_d(x; N)| \geq N^\alpha \text{ for infinitely many } N \in \mathbb{N} \}.
\]

We can restate the bound (1.1) in the following way: for any \(\alpha \in (1/2, 1)\) the set \(\mathcal{E}_{d, \alpha}\) is of Lebesgue measure zero. Here we are mostly
interested in the structure of the sets $E_{d, \alpha}$, and for convenience we call the set $E_{\alpha, d}$ the exceptional set for any integer $d \geq 2$ and each $0 < \alpha < 1$.

The authors [2] show that in terms of the Baire categories and Hausdorff dimension the exceptional sets $E_{d, \alpha}$ are quite massive. By [2, Theorem 1.3], for each $0 < \alpha < 1$ and integer $d \geq 2$ the set $T_d \setminus E_{d, \alpha}$ is of the first Baire category. Alternatively, this is equivalent to the statement that the complement $T_d \setminus \Xi_d$ to the set

\[(1.2) \quad \Xi_d = \{ x \in T_d : \forall \varepsilon > 0, |S_d(x; N)| \geq N^{1-\varepsilon} \text{ for infinitely many } N \in \mathbb{N} \}, \]

is of first category, see [2] for more details and reference therein. For the Hausdorff dimension it is shown in [2, Theorem 1.5] that for any $d \geq 2$ and $0 < \alpha < 1$ one has

\[(1.3) \quad \dim E_{d, \alpha} \geq \xi(d, \alpha) > 0 \]

with some explicit constant $\xi(d, \alpha)$.

We remark that the authors [3, Corollary 1.9] have obtained a non-trivial upper bound for the Hausdorff dimension of $E_{d, \alpha}$ for some $\alpha$, however the bounds there are not fully explicit and not cover the whole range $1/2 < \alpha < 1$.

Here we obtain the nontrivial upper bound of $\dim E_{d, \alpha}$ for all $1/2 < \alpha < 1$ and $d \geq 2$.

On the other hand, we note that we do not have any plausible conjecture about the exact value of the Hausdorff dimension of $E_{d, \alpha}$.

### 1.2. Main results

For $A \subseteq \mathbb{R}^d$, the $s$-dimension Hausdorff measure of $A$ is defined as

\[ \mathcal{H}^s(A) = \lim_{\delta \to 0} \mathcal{H}_\delta^s(A), \]

where

\[ \mathcal{H}_\delta^s(A) = \inf \left\{ \sum_{i=1}^{\infty} (\text{diam} U_i)^s : A \subseteq \bigcup_{i=1}^{\infty} U_i \text{ and diam } U_i \leq \delta, i \in \mathbb{N} \right\}. \]

The Hausdorff dimension of $A$ is defined as

\[ \dim A = \inf \{ s > 0 : \mathcal{H}^s(A) = 0 \} = \sup \{ s > 0 : \mathcal{H}^s(A) = \infty \}. \]

We refer to [4] for more details and properties of Hausdorff dimension.

For integer $d \geq 2$ and $0 < \alpha < 1$ denote

\[(1.4) \quad u(d, \alpha) = \min_{k=0, \ldots, d-1} \frac{(2d^2 + 4d)(1 - \alpha) + k(k+1)}{4 - 2\alpha + 2k}. \]
Theorem 1.1. For any integer $d \geq 2$ and $0 < \alpha < 1$ we have

$$\dim \mathcal{E}_{d,\alpha} \leq u(d, \alpha).$$

For $d \geq 2$ and any $1/2 < \alpha < 1$ an elementary calculation gives that $u(d, \alpha) < d$. In fact by taking $k = d - 1$ in (1.4) we derive

$$u(d, \alpha) \leq d - \frac{d(d + 1)(2\alpha - 1)}{2(d + 1 - \alpha)}.$$

Thus, we have

Corollary 1.2. For any integer $d \geq 2$ and any $1/2 < \alpha < 1$ we have

$$\dim \mathcal{E}_{d,\alpha} < d.$$

Furthermore taking, for example, $k = 0$ in (1.4) we obtain

$$\dim \mathcal{E}_{d,\alpha} \leq u(d, \alpha) \leq \frac{(2d^2 + 4d)(1 - \alpha)}{4 - 2\alpha}.$$

We note that although lower bound (1.3) and the upper bound of Theorem 1.1 are of very different magnitude with respect to $d$, however for $\alpha \to 1$ they give the same rate convergency to zero of order $1 - \alpha$. Namely the explicit formula for $\xi(d, \alpha)$ from [2] and the formula (1.4) yield

$$c_1(d) \leq \liminf_{\alpha \to 1} (1 - \alpha)^{-1} \dim \mathcal{E}_{d,\alpha} \leq \limsup_{\alpha \to 1} (1 - \alpha)^{-1} \dim \mathcal{E}_{d,\alpha} \leq c_2(d)$$

for two positive constants $c_1(d), c_2(d)$ depending only on $d$. In fact for $d = 2$ we have

$$c_1(2) = 3 \quad \text{and} \quad c_2(2) = 8,$$

while for $d \geq 3$ we have

$$c_1(d) = \max_{\nu=1,\ldots,d} \min \left\{ \frac{1}{\nu}, \frac{2}{2d - \nu} \right\} \quad \text{and} \quad c_2(d) = d^2 + 2d.$$

In particular, we have

Corollary 1.3. For any integer $d \geq 2$, if $\alpha \to 1$ then $\dim \mathcal{E}_{d,\alpha} \to 0$.

From the definition of $\Xi_d$, see (1.2), we have $\Xi_d \subseteq \mathcal{E}_{d,\alpha}$ for any $0 < \alpha < 1$. Therefore

Corollary 1.4. For any integer $d \geq 2$, we have $\dim \Xi_d = 0$. 
2. Preliminaries

2.1. Notation and conventions. Throughout the paper, the notation $U = O(V)$, $U \ll V$ and $V \gg U$ are equivalent to $|U| \leq c|V|$ for some positive constant $c$, which throughout the paper may depend on the degree $d$ and occasionally on the small real positive parameters $\varepsilon$ and $\delta$.

For any quantity $V > 1$ we write $U = V^{o(1)}$ (as $V \to \infty$) to indicate a function of $V$ which satisfies $|U| \leq V^\varepsilon$ for any $\varepsilon > 0$, provided $V$ that is large enough.

We use $\# \mathcal{X}$ to denote the cardinality of set $\mathcal{X}$.

We always identify $T_d$ with half-open unit cube $[0, 1)^d$, in particular we naturally associate Euclidean norm $\|x\|$ with points $x \in T_d$.

We say that some property holds for almost all $x \in T_d$ if it holds for a set $\mathcal{X} \subseteq T_d$ of Lebesgue measure $\lambda(\mathcal{X}) = 1$.

We always keep the subscript $d$ in notations for our main objects of interest such as $E_{\alpha,d}$, $S_d(x; N)$ and $T_d$, but sometimes suppress it in auxiliary quantities.

2.2. Mean value theorems. The Vinogradov mean value theorem in the currently known form, due to Bourgain, Demeter and Guth [1] for $d \geq 4$ and Wooley [5] for $d = 3$, asserts that,

$$\int_{T_d} |S_d(x; N)|^{2s(d)} dx \leq N^{s(d)+o(1)},$$

where $s(d) = d(d+1)/2$.

Using the orthogonality of exponential functions the value of the integral

$$\int_{T_d} |S_d(x; N)|^{2s(d)} dx$$

counts the number of integer vector $(n_1, \ldots, n_{2s(d)})$ such that

$$1 \leq n_1, \ldots, n_{2s(d)} \leq N$$

and for each $j = 1, \ldots, d$ one has

$$n_1^j + \ldots + n_{s(d)}^j = n_{s(d)+1}^j + \ldots + n_{2s(d)}^j.$$

An important fact is that if $(n_1, \ldots, n_{2s(d)})$ is satisfies the equation (2.1) for all $j = 1, \ldots, d$, then for any real $a$ the translated vector

$$(n_1 + a, \ldots, n_{2s(d)} + a)$$

also satisfies the equation (2.1) for all $j = 1, \ldots, d$. Using this translation invariance and again applying orthogonality of exponential functions we obtain the following result for sums over short intervals. For
any integers \(L, N \geq 1\) we have the upper bound
\[
\int_{T_d} |S_d(x; N + L) - S_d(x; N)|^{2s(d)}\, dx \leq (NL)^{o(1)} L^{s(d)},
\]
as \(NL \to \infty\).

2.3. Distribution of large values of exponential sums. We use the following two results from [3].

For \(u \in \mathbb{R}^d\) and \(\zeta = (\zeta_1, \ldots, \zeta_d)\) with \(\zeta_j > 0, j = 1, \ldots, d\), we define the \(d\)-dimensional rectangle (or box) with center \(u\) and side lengths \(2\zeta\) by
\[
R(u, \zeta) = [u_1 - \zeta_1, u_1 + \zeta_1) \times \ldots \times [u_d - \zeta_d, u_d + \zeta_d).
\]

By [3, Lemmas 2.6] we have:

**Lemma 2.1.** Let \(0 < \alpha < 1\) and \(\varepsilon > 0\) be a small parameter. For each \(j = 1, \ldots, d\) let
\[
\zeta_j = 1/ \left[ N^{j+1+\varepsilon-\alpha} \right].
\]
We divide \(T_d\) into
\[
U = \prod_{j=1}^{d} \zeta_j^{-1}
\]
boxes of the type
\[
[n_1 \zeta_1, (n_1 + 1) \zeta_1) \times \ldots \times [n_d \zeta_d, (n_d + 1) \zeta_d),
\]
where \(n_j = 0, \ldots, 1/\zeta_j - 1, j = 1, \ldots, d\). Let \(R\) be the collection of these boxes, and
\[
\tilde{R} = \{ R \in R : \exists x \in R \text{ with } |S_d(x; N)| \geq N^{\alpha} \}.
\]
Then one has
\[
\#\tilde{R} \leq UN^{s(d)(1-2\alpha)+o(1)}.
\]

Note that the above bound of \(\#\tilde{R}\) is nontrivial when \(1/2 < \alpha < 1\).

We have the following bound on the amount of certain boxes which admit values for the difference \(|S_d(x; L) - S_d(x; K)|\), see [3, Lemmas 2.8].

**Lemma 2.2.** Let \(0 < \alpha < 1\) and \(\varepsilon > 0\) be a small parameter. Let \(\rho > 1\) and \(N_i = \lfloor i^\rho \rfloor\), \(i \in \mathbb{N}\). Let \(N_i \leq K < L \leq N_{i+1}\) for some \(i \in \mathbb{N}\). For each \(j = 1, \ldots, d\) let
\[
\eta_j = 1/ \left[ N^{j+\varepsilon-\alpha}(L - K) \right].
\]
We divide the \(T_d\) into
\[
W = \prod_{j=1}^{d} \eta_j^{-1}
\]
boxes in the same way as in the Lemma 2.1. Denote $Q$ the collection of these boxes. Then one has

$$\#\{R \in Q : \exists x \in R \text{ with } |S_d(x;L) - S_d(x;K)| \geq N_i^\alpha \} \lesssim W(L - K)^{s(d)} N_i^{-2\alpha s(d) + o(1)}.$$

2.4. Iterated construction. We adapt the construction in the [3, Proof of Theorem 1.2] to our setting.

For each $i \in \mathbb{N}$ let $N_i = i^\rho$ and denote $L_i = N_{i+1} - N_i$.

Note that we have

$$i^{\rho - 1} \ll L_i \ll i^{\rho - 1}.$$

Let $M$ be a large integer number to be determined later. We choose $\vartheta_i \in (0, 1)$ as the largest real number with

$$1/\vartheta_i \in \mathbb{N} \quad \text{and} \quad \vartheta_i^M L_i \leq N_i^\alpha. \tag{2.2}$$

Let $\mathcal{I} = [a, b] \subseteq [N_i, N_{i+1}]$ and let $I = b - a$. We divide $\mathcal{I}$ into $1/\vartheta_i$ intervals with the equal length $\vartheta_i I$. For each $v = 1, \ldots, 1/\vartheta_i$ let

$$A_{i,\mathcal{I}}(v) = \{x \in T_d : |S_d(x; a + v\vartheta_i I) - S_d(x; a)| \geq N_i^\alpha \},$$

and

$$A_{i,\mathcal{I}} = \bigcup_{v=1}^{1/\vartheta_i} A_{i,\mathcal{I}}(v).$$

Remark 2.3. Clearly, $a + v\vartheta_i I$ may not be an integer for some $v = 1, \ldots, 1/\vartheta_i$. In this case we replace it with the closest integer. Under this modification we still get $1/\vartheta_i$ intervals and each of them has the length $\vartheta_i I + O(1)$, which does not effect our result. In the following and throughout the paper we use this convention, and thus to simplify the exposition we always treat $a + v\vartheta_i I$ as an integer for each $v = 1, \ldots, 1/\vartheta_i$.

Now we divide interval $[N_i, N_{i+1}]$ into $1/\vartheta_i$ intervals with the equal length $\vartheta_i L_i$. Let $\mathcal{D}_1$ be the collection of these $1/\vartheta_i$ intervals. Precisely

$$\mathcal{D}_1 = \{[N_i + v\vartheta_i L_i, N_i + v\vartheta_i L_i + \vartheta_i L_i] : 0 \leq v < 1/\vartheta_i \}.$$

For each interval of $\mathcal{D}_1$ we do this process again, and let $\mathcal{D}_2$ be the collections of all these intervals. Now $\mathcal{D}_2$ has $(1/\vartheta_i)^2$ intervals with equal length $\vartheta_i^2 L_i$. We continue this process until $M$ steps.

For $1 \leq m \leq M - 1$ we can write

$$\mathcal{D}_m = \{[N_i + v\vartheta^m_i L_i, N_i + v\vartheta^m_i L_i + \vartheta^m_i L_i] : 0 \leq v < (1/\vartheta_i)^m \}.$$
We may write $D_0 = \{ [N_i, N_{i+1}] \}$ for convenience. Note that each interval of $D_m$ has length $\vartheta_i^m L_i$. For each $1 \leq m \leq M$ we define

$$B_{i,m} = \bigcup_{I \in D_{m-1}} A_{i,I}$$

and

$$B_m = \bigcap_{k=1}^\infty \bigcup_{i=k}^\infty B_{i,m}.$$ 

Specially we define

$$B_0 = \bigcap_{k=1}^\infty \bigcup_{i=k}^\infty \{ x \in T_d : |S_d(x; N_i)| \geq N_i^\alpha \}.$$

By applying the similar arguments as in [2, Proof of Theorem 1.2] we derive the following.

**Lemma 2.4.** Let $0 < \alpha < 1$ and $d \geq 2$. Using above notation for any small $\epsilon > 0$ we have

$$\mathcal{E}_{d, \alpha + \epsilon} \subseteq \bigcup_{m=0}^M B_m.$$ 

**Proof.** First observe that it is sufficient to show that for any $x \notin B_m$ for each $m = 0, 1, \ldots M$ one has

$$|S_d(x; N)| \ll N^\alpha.$$ 

Now let $x \notin B_i$ for each $i = 0, 1, \ldots, M$. Then there exists $i_x \in \mathbb{N}$ such that for each $i \geq i_x$ one has

$$|S_d(x; N_i)| \leq N_i^{\alpha},$$

and for each $m = 1, \ldots M$ one has

$$x \notin B_{i,m}.$$ 

Applying $x \notin B_{i,1}$ and (2.3) we conclude that for any $j = 1, \ldots, 1/\vartheta_i$ one has

$$|S_d(x; N_i + j \vartheta_i L_i)| \leq |S_d(x; N_i + j \vartheta_i L_i) - S_d(x; N_i)| + |S_d(x; N_i)|$$

$$\ll N_i^{\alpha}.$$ 

By using this process, finally we obtain that for each $1 \leq j \leq (1/\vartheta_i)^M$ one has

$$S_d(x; N_i + j \vartheta_i^M L_i) \ll N_i^{\alpha}.$$ 

Now for any $N \geq N_{i_x}$ there exists $i \geq i_x$ and $1 \leq j \leq (1/\vartheta_i)^M$ such that

$$N_i + j \vartheta_i^M L_i \leq N \leq N_i + (j + 1) \vartheta_i^M L_i.$$
Combining (2.4) with (2.2) and the trivial bound \(|e(x)| \leq 1\) we obtain
\[
|S_d(x; N)| \leq |S_d(x; N) - S_d(x; N_i + j\vartheta_i^M L_i)| \\
+ |S_d(x; N_i + j\vartheta_i^M L_i)|
\]
\[
\ll N_i^\alpha,
\]
which gives the result. \(\square\)

2.5. Covering the sets of large Weyl sums by a family of rectangles.

For each \(m = 0, \ldots, M\) we intend to cover \(B_m\) by a family of rectangles. We will use the notation from Subsections 2.3, 2.4.

We formulate the Lemmas 2.1, 2.2 in the following ways for the convenience of our applications. From Lemma 2.1 we have

**Corollary 2.5.** Let \(0 < \alpha < 1\) and \(\varepsilon > 0\) be a small parameter. For large enough \(i \in \mathbb{N}\) we have
\[
\{x \in T_d : |S_d(x; N)| \geq N_i^\alpha\} \subseteq \bigcup_{\mathcal{R} \in \mathcal{R}_0(i)} \mathcal{R}
\]
where each \(\mathcal{R}\) of \(\mathcal{R}_0(i)\) has side length \(\zeta = (\zeta_1, \ldots, \zeta_d)\) such that
\[
\zeta_j = 1 / \left[N_i^{s(d) + 1 + \varepsilon - \alpha}\right], \quad j = 1, \ldots, d,
\]
and furthermore
\[
\# \mathcal{R}_0(i) \ll N_i^{s(d) - 2\alpha s(d)} \prod_{j=1}^d \zeta_j^{-1} \ll N_i^{s(d)(1 - \alpha) + d(1 - \alpha) + d\varepsilon + o(1)}.
\]

From Lemma 2.2 we obtain

**Corollary 2.6.** Let \(0 < \alpha < 1\) and \(\varepsilon > 0\) be a small parameter. Let \(i \in \mathbb{N}, m = 1, \ldots, M\) and \(I \in D_{m-1}(i)\), then for each \(v = 1, \ldots, 1/\vartheta_i\) we have
\[
\bigcup_{I \in D_{m-1}} A_{i,I}(v) \subseteq \bigcup_{\mathcal{R} \in \mathcal{R}_m(i,v)} \mathcal{R}
\]
where each \(\mathcal{R}\) of \(\mathcal{R}_m(i, v)\) has side length \(\eta(v) = (\eta_1(v), \ldots, \eta_d(v))\) such that
\[
\eta_j(v) = 1 / \left[N_i^{j+\varepsilon - \alpha} v\vartheta_i^m L_i\right], \quad j = 1, \ldots, d,
\]
and
\[
\# \mathcal{R}_m(i, v) \leq \# D_{m-1}(v\vartheta_i^m L_i)^{s(d)} N_i^{-2\alpha s(d)} \prod_{j=1}^d \eta_j(v)^{-1}
\]
\[
\ll N_i^{s(d) - 2\alpha s(d) - d\alpha + d\varepsilon + s(d) + s(d) + d + s(d) + m d + s(d) + m + 1}.
\]
Let $\mathcal{R}_m(i)$ be the collection of boxes of $\mathcal{R}_m(i, v)$, $v = 1, \ldots, \vartheta_i - 1$, that is,
$$
\mathcal{R}_m(i) = \{ \mathcal{R} : \mathcal{R} \in \mathcal{R}_m(i, v), v = 1, \ldots, \vartheta_i - 1 \}.
$$
Then we have
$$
\mathcal{B}_m \subseteq \bigcap_{k=1}^{\infty} \bigcup_{i=k}^{\infty} \bigcup_{\mathcal{R} \in \mathcal{R}_m(i)} \mathcal{R}.
$$

3. Proof of Theorem 1.1

We start from some auxiliary results. Firstly we adapt the definition of the singular value function from [4, Chapter 9] to the following.

**Definition 3.1.** Let $\mathcal{R} \subseteq \mathbb{R}^d$ be a rectangle with side lengths $r_1 \geq \ldots \geq r_d$.

For $0 < t \leq d$ we set
$$
\varphi_{0,t}(\mathcal{R}) = r_1^t,
$$
and for $k = 1, \ldots, d-1$ we define
$$
\varphi_{k,t}(\mathcal{R}) = r_1 \cdots r_{k+1}^t r_k^{t-k}.
$$

Note that for a rectangle $\mathcal{R} \subseteq \mathbb{R}^2$ with side length $r_1 \geq r_2$ we have
$$
\varphi_{k,t}(\mathcal{R}) = \begin{cases} r_1^t & k = 0 \\ r_1 r_2^{t-1} & k = 1. \end{cases}
$$

**Remark 3.2.** The notation $\varphi_{k,t}(\mathcal{R})$ roughly means that we can cover the rectangle $\mathcal{R}$ by about (up to a constant factor)
$$
\frac{r_1}{r_{k+1}} \cdots \frac{r_k}{r_{k+1}}
$$
balls of radius $r_{k+1}$, and hence this leads to the term
$$
\varphi_{k,t}(\mathcal{R}) = \frac{r_1}{r_{k+1}} \cdots \frac{r_k}{r_{k+1}} r_k^t
$$
in the expression for the Hausdorff measure with the parameter $t$ (again up a constant factor which does not affect our results).

The following easy inequality shows the importance of the function $\varphi_{k,t}(\mathcal{R})$.

**Lemma 3.3.** Using the above notation, for each $m = 0, \ldots, M$ we have
$$
\dim \mathcal{B}_m \leq \inf \left\{ t > 0 : \sum_{i=1}^{\infty} \sum_{\mathcal{R} \in \mathcal{R}_m(i)} \varphi_{k,t}(\mathcal{R}) < \infty, k = 0, \ldots, d-1 \right\}.
$$
Proof. From Corollaries 2.5 and 2.6 for each $m = 0, \ldots, M$ we have
\[
B_m \subseteq \bigcap_{q=1}^{\infty} \bigcup_{i=q}^{\infty} \bigcup_{R \in \mathcal{R}_m(i)}^{} R.
\]
Let $k = 0, \ldots, d - 1$ and $t > 0$ such that
\[
(3.1) \quad \sum_{i=1}^{\infty} \sum_{R \in \mathcal{R}_m(i)}^{} \varphi_{k,t}(R) < \infty.
\]
Note that for any $\delta > 0$ there exists $q_\delta$ such that for any $i \geq q_\delta$ and any $R \in \mathcal{R}_m(i)$ one has
\[
\text{diam}(R) \leq \delta.
\]
Furthermore for any $q \in \mathbb{N}$ the set $\bigcup_{i=q}^{\infty} \bigcup_{R \in \mathcal{R}_m(i)}^{} R$ is a cover of $B_m$. From the definition of the Hausdorff measure and the Remark 3.2 for $\varphi_{k,t}(R)$ we obtain
\[
\mathcal{H}_t^\delta(B_m) \ll \sum_{R \in \mathcal{R}_m(i)}^{} \varphi_{k,t}(R) \ll \sum_{i=1}^{\infty} \sum_{R \in \mathcal{R}_m(i)}^{} \varphi_{k,t}(R),
\]
where the implied constant does not depend on $\delta$. Combining this with (3.1) we obtain
\[
\mathcal{H}_t^\delta(B_m) < \infty,
\]
and hence $\dim B_m \leq t$. We finish the proof by the arbitrary choice of $t > 0$ and $k = 0, \ldots, d - 1$. \qed

We also use the following countable stability property of Hausdorff dimension, see [4, Section 2.2] for a short proof and other basic properties of Hausdorff dimension. Let $\mathcal{A}_1, \mathcal{A}_2, \ldots$ be a sequence subsets of $\mathcal{R}^d$ then
\[
(3.2) \quad \dim \bigcup_{i \in \mathbb{N}}^{} \mathcal{A}_i = \sup_{i \in \mathbb{N}} \dim \mathcal{A}_i.
\]
Now we turn to the proof of Theorem 1.1. We consider the upper bound of \( \dim B_0 \) first. For \( k = 1, \ldots, d - 1 \) and \( 0 < t \leq d \) we have

\[
\sum_{\mathcal{R} \in \mathcal{R}_0(i)} \varphi_{k,t}(\mathcal{R}) = \# \mathcal{R}_0(i) \zeta_{k+1}^{t-k} \prod_{j=1}^{k} \zeta_j \\
\leq N_i^{2s(d)(1-\alpha)+d(1-\alpha)+d\varepsilon+o(1)} \\
\times \left( N_i^{\alpha-1-\varepsilon-(k+1)} \right)^{t-k} \prod_{j=1}^{k} N_i^{\alpha-j-1-\varepsilon} \\
\leq N_i^{2s(d)(1-\alpha)+d(1-\alpha)+d\varepsilon+(t-k)(\alpha-k-2-\varepsilon)+k(\alpha-1-\varepsilon)-s(k)+o(1)}.
\]

Here and in the following we denote

\[ s(k) = \frac{k(k+1)}{2}. \]

We remark that this bound also holds for the case \( k = 0 \), which we take \( k = 0 \) and \( s(k) = 0 \) in the last line of (3.3). To be precise for \( k = 0 \) we have

\[
\sum_{\mathcal{R} \in \mathcal{R}_0(i)} \varphi_{0,t}(\mathcal{R}) \leq N_i^{2s(d)(1-\alpha)+d(1-\alpha)+d\varepsilon+t(\alpha-2-\varepsilon)+o(1)}.
\]

Applying Lemma 3.3 we conclude that

\[ \dim B_0 \leq t \]

provided that the parameters \( \alpha, \rho, k, t \) satisfy the following further condition

\[ \rho(2s(d)(1-\alpha)+d(1-\alpha)+(t-k)(\alpha-k-2)+k(\alpha-1)-s(k)) < -1, \]

which becomes

\[
t > \frac{2s(d)(1-\alpha)+d(1-\alpha)+s(k)+1/\rho}{k+2-\alpha}.
\]
Now we turn to the upper bound of \( \dim B_m \) with \( m = 1, \ldots, M \). For \( k = 1, \ldots, d - 1 \), \( 0 < t \leq d \) and \( m = 1, \ldots, M \) we have

\[
\sum_{R \in \mathcal{R}_m(i)} \varphi_{k,t}(\mathcal{R}) \leq \sum_{v=1}^{1/\vartheta_i-1} \sum_{R \in \mathcal{R}_m(i,v)} \varphi_{k,t}(\mathcal{R}) \\
\leq \sum_{v=1}^{1/\vartheta_i-1} \# \mathcal{R}_i(i,v) \eta_{k+1}(v)^{t-k} \prod_{j=1}^{k} \eta_j(v) \\
\leq \sum_{v=1}^{1/\vartheta_i-1} \# \mathcal{R}_i(i,v) \left( N_i^{\alpha-(k+1)-\varepsilon} \right)^{t-k} \prod_{j=1}^{k} N_i^{\alpha-j \varepsilon} \eta_{i}^{-m} L_i^{-1} \eta_{i}^{-m} \\
\leq \sum_{v=1}^{1/\vartheta_i-1} \# \mathcal{R}_i(i,v) \left( N_i^{\alpha-(k+1)-\varepsilon} \right)^{t-k} \eta_{i}^{-m} L_i^{-1} \eta_{i}^{-m} \\
\leq \sum_{v=1}^{1/\vartheta_i-1} \# \mathcal{R}_i(i,v) N_i^{t(\alpha-k-1+\varepsilon)+s(k)} L_i^{-t} \eta_{i}^{-mt} \\
\leq N_i^{s(d)-2\alpha s(d)-d \alpha+d \varepsilon+2(t(\alpha-k-1+\varepsilon)+s(k)+o(1))} \\
\times L_i^{d+s(d)-t} \eta_{i}^{n(d+s-1-t)-d+s+t}.
\]  

(3.5)

We remark that this bound also holds for the case \( k = 0 \), which we take \( k = 0 \) and \( s(k) = 0 \) in the last inequality of (3.5).

Observe that for each \( m \geq 1 \) we have

\[
\eta_{i}^{-m} L_i^{-t} \eta_{i}^{-m} \leq 1/\vartheta_i.
\]

Let \( M = \lfloor \rho / \varepsilon \rfloor \). Then by (2.2) we have the following trivial inequality

\[
1/\vartheta_i \leq \left( \frac{L_i}{N_i^{\alpha}} \right)^{1/M} \leq N_i^{1/M} \approx \varepsilon.
\]

Combining this and taking into account that \( \varepsilon \) can be chosen arbitrary small, we conclude that for any \( m = 1, \ldots, M \) one has

\[
\dim B_m \leq t
\]

provided the parameters satisfy the condition

\[
\rho(s(d)-2\alpha s(d)-d \alpha+2(t(\alpha-k-1)+s(k))+(\rho-1)(d+s(d)-t) < -1,
\]

which becomes

\[
t > \frac{s(d)-2\alpha s(d)-d \alpha+s(k)+(1-1/\rho)(d+s(d))+1/\rho}{k+2-\alpha-1/\rho}.
\]

(3.6)
Applying conditions (3.4) and (3.6), by choosing large enough \( \rho \) we obtain that for any \( m = 0, \ldots, M \) on has

\[
\dim B_m \leq t
\]

provided that

\[
t > \frac{2s(d)(1-\alpha) + d(1-\alpha) + s(k)}{k + 2 - \alpha}.
\]

It follows that for any small \( \tau > 0 \) there exists large enough \( \rho \) (and sufficient small \( \varepsilon > 0 \) in the above arguments) such that

\[
(3.7) \quad \max_{m=0,\ldots,M} \dim B_m \leq \frac{2s(d)(1-\alpha) + d(1-\alpha) + s(k)}{k + 2 - \alpha} + \tau.
\]

By the monotonicity property of the Hausdorff dimension we see from Lemma 2.4 that for any \( \epsilon > 0 \) one has

\[
\dim \mathcal{E}_{d,\alpha+\epsilon} \leq \dim \bigcup_{m=0,\ldots,M} B_m.
\]

Combining with the countable stability property of the Hausdorff dimension (3.2) and (3.7) we derive

\[
\dim \mathcal{E}_{d,\alpha+\epsilon} \leq \frac{2s(d)(1-\alpha) + d(1-\alpha) + s(k)}{k + 2 - \alpha} + \tau.
\]

By taking \( \gamma = \alpha + \epsilon \) we obtain

\[
\dim \mathcal{E}_{d,\gamma} \leq \frac{2s(d)(1-\gamma + \epsilon) + d(1-\gamma + \epsilon) + s(k)}{k + 2 - \gamma + \epsilon} + \tau.
\]

Since this holds for any small \( \epsilon > 0 \) and \( \tau > 0 \), this gives the desired bound.

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**References**

[1] J. Bourgain, C. Demeter and L. Guth, ‘Proof of the main conjecture in Vinogradov’s mean value theorem for degrees higher than three’, *Ann. Math.*, **184** (2016), 633–682.

[2] C. Chen and I. E. Shparlinski, ‘On large values of Weyl sums’, *Preprint*, 2019, available at https://arxiv.org/abs/1811.04971.

[3] C. Chen and I. E. Shparlinski, ‘New bounds of Weyl sums’, *Preprint*, 2019, available at https://arxiv.org/abs/1903.07330.

[4] K. J. Falconer, *Fractal geometry: Mathematical foundations and applications*, John Wiley, 2nd Ed., 2003.

[5] T. D. Wooley, ‘The cubic case of the main conjecture in Vinogradov’s mean value theorem’, *Adv. in Math.*, **294** (2016), 532–561.
Department of Pure Mathematics, University of New South Wales, Sydney, NSW 2052, Australia

E-mail address: changhao.chenm@gmail.com

Department of Pure Mathematics, University of New South Wales, Sydney, NSW 2052, Australia

E-mail address: igor.shparlinski@unsw.edu.au