Curvature and topology dependency of the cosmological spectra

Ali A. Asgari, Amir H. Abbassi, Jafar Khodagholizadeh

Department of Physics, School of Sciences, Tarbiat Modares University, P.O.Box 14155-4838, Tehran, Iran
Farhangian University, P.O. Box 11876-13311, Tehran, Iran
E-mail: asgari.phy@iran.ir, ahabbasi@modares.ac.ir, gholizadeh@ipm.ir

Abstract. In this article we investigate dependency of the adiabatic and entropy spectral indices of the cosmological perturbations on the geometry and topology of the background universe. Our discussion includes the post-inflationary universe i.e. radiation-dust mixture era. For this purpose, we first extract an explicit equation describing evolution of the comoving curvature perturbation in the FLRW universe with arbitrary spatial sectional curvature. We may perceive when \( K \neq 0 \), curvature scale would be as significant as the perturbations scales to recognize the behavior of the spectral indices. We also focus on the entropy perturbation in order to extract behavior of the isocurvature spectral index in terms of the curvature index and time. Our analysis shows that spectra of curvature and entropy perturbations in sub-horizon scales could be function of topology. Moreover, an accurate analysis makes clear that time-average of isocurvature index in case \( K = 0 \) is about zero, so that imprint of entropy perturbation in time duration may be negligible. We also consider evolution of the cosmological indices for super-curvature modes in the case \( K = -1 \). In the most results dependency to curvature, initial conditions and scale modes are thoroughly vivid.
1 Introduction

In 1980s inflation theory was proposed by A. Guth as a solution for some long-lasting problems in classical cosmology[1]. However, it was revealed very soon the inflation could explain the origin of the CMB anisotropy and structure formation[2, 3]. As it has been cleared, the quantum fluctuations of a single scalar field in the inflationary epoch (inflaton) are yielded to the classical perturbations which are ascertained as the origin of CMB anisotropy and large structures in the universe[4, 5]. The classical perturbations in cosmology could be described by gauge-invariant random fields such as comoving curvature perturbation \( R \) which is associated with the primordial scalar spectrum as well as the conformal factor of the spatial section of the universe[5–7]. Evolution of \( R \) at the inflationary era associated with spatially flat FLRW universe may be described by the Mukhanov-Sasaki equation [8, 9] which results in a nearly scale-invariant spectrum[2, 6]. The Mukhanov-Sasaki equation has been generalized to include the entire history of the universe[10]. It could be significant because elucidates evolution of the curvature spectrum during the time. Although observations are consistent with a spatially flat FLRW universe as the background geometry, the current bounds on \( \Omega_K \) doesn’t deny the possibility that the universe is a slightly curved[11–13]. So the Mukhanov-Sasaki equation may be generalized for the inflationary models with positive curvature index leads into the resolution of running number problem[14]. In this paper we are going to examine adiabatic and entropy spectra in terms of curvature and topology of the spatial section of the background spacetime. For this purpose, we first acquire a curvature-dependent equation for the evolution of \( R \) satisfactory for different perturbative scales. Additionally, a discussion of the associated spectral index evolution after inflation in terms of the sectional curvature and time presented. We consider evolution of \( R \) and its Spectrum for the super-curvature perturbations in case \( K = -1 \) too.

The outline of this paper is as follow. In section 2, we derive an explicit equation shows dependency of \( R \) on curvature and rescaled time which is authentic for all history areas of the universe. Then, we find special solutions of derived equation. In section 4, we focus on the universe containing matter and radiation and study numerical solutions of the \( R \)-evolution equation accompanied with the Kodama-Sasaki equation. For this goal, two different initial conditions would be considered. Moreover, we inquire into curvature and topology dependency of the entropy perturbation as the time passes. Finally, the behavior of the spectra in super-curvature modes will be discussed.
2 Generalized Mukhanov-Sasaki equation

The line element of the universe in the comoving quasi-Cartesian coordinates may be written as [7]

\[ ds^2 = a^2 \left\{ - (1 + E) d\tau^2 + 2 \partial_i F d\tau dx^i + [(1 + A) \bar{g}_{ij} + \mathcal{H}_{ij} B] dx^i dx^j \right\}, \]  

(2.1)

where \( \mathcal{H}_{ij} = \nabla_i \nabla_j \) is the \textit{covariant Hessian operator} and \( \bar{g}_{ij} = \delta_{ij} + K \frac{x_i x_j}{1 - K x_i x_j} \) in which \( K (= 0, \pm 1) \) shows curvature index of spatial slices of the universe. Furthermore, \( E, F, \) and \( B \) are scalar random fields describe departure from homogeneity and isotropy and are considered as first order perturbations. Such universe can be split as \( \mathbb{R} \times \mathcal{M} \) where \( \mathcal{M} \) is compact if \( K = +1 \) and non-compact if \( K = 0 \) or \(-1 \) (Indeed topology of the universe is \( \mathbb{R} \times \mathcal{M} \)).

On the other hand, the energy-momentum of the cosmic fluid may be decomposed as [7]

\[ T_{00} = a^2 \left[ \bar{\rho} (1 + E) + \delta \rho \right], \]  

(2.2)

\[ T_{0i} = a^2 \left[ \bar{\rho} \partial_i F - \left( \frac{\bar{\rho} + \bar{p}}{a} \right) \partial_i (\delta u) \right], \]  

(2.3)

\[ T_{ij} = a^2 \left[ \bar{p} (1 + A) \bar{g}_{ij} + \delta \bar{p} \bar{g}_{ij} + \mathcal{H}_{ij} \left( \bar{p} B + \Pi^S \right) \right], \]  

(2.4)

where \( \delta u \) and \( \Pi^S \) are the scalar velocity potential and scalar anisotropy inertia of the cosmic fluid, respectively. \( \Pi^S \) measures deviation of the cosmic fluid from perfectness. Moreover, \( \rho = \bar{\rho} + \delta \rho \) and \( p = \bar{p} + \delta p \) are energy density and pressure of the cosmic fluid, respectively (Bar over every quantity shows its unperturbed value). One may combine the perturbative scalars to construct gauge-invariant quantities [15]

\[ \Psi = \frac{A}{2} - \mathcal{H} \sigma, \]  

(2.5)

\[ \Phi = \frac{E}{2} + \mathcal{H} \sigma + \sigma', \]  

(2.6)

\[ R = \frac{A}{2} + H \delta u, \]  

(2.7)

\[ \zeta = \frac{A}{2} - \mathcal{H} \frac{\delta \rho}{\bar{\rho}}, \]  

(2.8)

\[ \Delta = \mathcal{H} \frac{\delta \rho}{\bar{\rho}} + H \delta u, \]  

(2.9)

\[ \Gamma = \delta p - c_s^2 \delta \rho, \]  

(2.10)

with \( \sigma = F - \frac{1}{2} B' \) (shear potential of the cosmic fluid) and \( \mathcal{H} = Ha \) (comoving Hubble parameter). Here the prime symbol stands for derivative with respect to \( \tau \) and \( c_s^2 = \frac{d \bar{p}}{d \bar{\rho}} \) is the adiabatic sound speed in the cosmic fluid. Now letâ€™s focus on \( R \) (comoving curvature perturbation) which according to perturbative forms of the Friedmann equations as well as energy-momentum conservation law depends on the other gauge-invariant potentials

\[ R = \frac{-2 \mathcal{H}^2 + \mathcal{H}' - K}{\mathcal{H}^2 - \mathcal{H}' + K} \Psi - \frac{\mathcal{H}}{\mathcal{H}^2 - \mathcal{H}' + K} \Psi' + \frac{8 \pi G H a^2}{\mathcal{H}^2 - \mathcal{H}' + K} \Pi^S, \]  

(2.11)

\[ R' = \frac{- \mathcal{H} c_s^2}{\mathcal{H}^2 - \mathcal{H}' + K} \nabla^2 \Psi - \frac{4 \pi G H a^2}{\mathcal{H}^2 - \mathcal{H}' + K} (\Gamma + \nabla^2 \Pi^S) \]  

\[ \quad - \frac{K}{\mathcal{H}^2 - \mathcal{H}' + K} \left[ \Psi' + \mathcal{H} (1 + 3 c_s^2) \Psi \right]. \]  

(2.12)
Here $\Box^2$ is the Laplace-Beltrami operator associated with spatial folia in the universe which is defined as contracted Hessian operator with $\tilde{g}_{ij}$

$$\Box^2 = \tilde{g}^{ij} H_{ij}.$$ By taking these two equations to the Fourier space i.e. using Fourier transformation on $M$ (spatial slice of the universe) one can rewrite equations (2.11) and (2.12) as [7, 16, 17]

$$R_q = -2H^2 + H'H + K \Psi_q - \frac{H}{H^2 - H' + K} \Psi'_q + \frac{8\pi G H^2 a^2}{H^2 - H' + K} \Pi^S_q,$$ (2.13)

$$R_q' = \frac{Hc_s^2(q^2 - K)}{H^2 - H' + K} \Psi_q - \frac{4\pi G H a^2}{H^2 - H' + K} \left[ \Gamma_q - (q^2 - K) \Pi^S_q \right]$$

$$- \frac{K}{H^2 - H' + K} \left[ \Psi'_q + H \left( 1 + 3c_s^2 \right) \Psi_q \right].$$ (2.14)

Here Fourier transform can be thought of as decomposition in terms of eigen functions of the Laplace-Beltrami operator [7, 16]. Note the index $q$ stands for the Fourier transforms and refers to the perturbation scale. Additionally, $q$ is non-negative and continuous for $K = 0$ and $-1$ while natural (discrete) for case $K = +1$ as a direct consequence of compactness of $M$. By a tedious calculation it is possible to combine equations (2.13) and (2.14) to derive an explicit equation in terms of $R_q$

$$R_q'' + 2 \frac{D_q'}{D_q} R_q' + \left[ K \left( \frac{D_q'}{D_q} - \frac{\mathcal{A}'}{\mathcal{A}} \right) + c_s^2 (q^2 - 4K) - K \right] R_q =$$

$$4\pi G a^2 \left[ \left( \frac{D_q'}{D_q} - 4H^2 - H' - K \right) \frac{\mathcal{C}_q}{\mathcal{A}} - \frac{\mathcal{C}_q'}{\mathcal{A}} + 2 \frac{\mathcal{B}_q}{\mathcal{A}} \Pi^S_q \right],$$ (2.15)

where

$$\mathcal{A} = H^2 - H' + K,$$ (2.16)

$$\mathcal{B}_q = c_s^2 (q^2 - 4K) H^2 + K \mathcal{A},$$ (2.17)

$$\mathcal{C}_q = \Gamma_q - (q^2 - 3K) \Pi^S_q,$$ (2.18)

$$D_q = a \sqrt{\frac{\mathcal{A}}{\mathcal{B}_q}}.$$ (2.19)

Equation (2.15) is the most general equation which specifies time evolution of $R_q$ responsible not only for the inflation duration and spatially flat universe but also for all other epochs in the history of the universe with arbitrary curvature index. So we refer equation (2.15) as the generalized Mukhanov-Sasaki equation.

### 3 Special solutions

As cited before, there are new observational evidences indicate the sectional curvature of the universe is not zero [12, 13]. On the other hand, if $K \neq 0$ the curvature scale $|K| = |\Omega_K (t)|$ would be significant like the perturbation mode scales. The curvature scale may be expressed as the sectional curvature of the spatial slices of the universe $\mathcal{K}_s$ because $\Omega_K (t) = -\frac{K_s(t)}{\Pi^S(t)}$. Now let’s focus on the universe filled by pure dust, so

$$\Pi^S = 0 \quad , \quad \Gamma = 0 \quad , \quad c_s^2 = 0.$$
Consequently, equation (2.15) reduces to

\[ R''_q + 2\mathcal{H}R'_q - R_q = 0, \tag{3.1} \]

where \( \mathcal{H} = \sqrt{K} \cot \left( \frac{\sqrt{K}}{2} \tau \right) \). The solution of equation (3.1) is

\[ R_q \propto -\frac{2}{\tau} \cos \left( \frac{\sqrt{K}}{2} \tau \right) + \frac{1}{\sqrt{K}} \sin \left( \frac{\sqrt{K}}{2} \tau \right). \tag{3.2} \]

Note that equation (3.1) has another solution which is singular at \( \tau = 0 \), so we condone it. The behavior of \( R_q \) for \( K = 0, \pm 1 \) is depicted in figure 1. Obviously \( R_q \) is independent of perturbation scales in spite of curvature scale, i.e. in epochs in which \( |\Omega_K(t)| \ll 1 \) equation (3.1) results in \( R_q = \text{const} \) regardless of \( K \).

Conversely, in a radiated-dominated universe

\[ \Pi^S = 0 \quad , \quad \Gamma = 0 \quad , \quad c_s^2 = \frac{1}{3}. \]

So equation (2.15) can be written as

\[
R''_q + \frac{2\sqrt{K} (q^2 + 2K) \cot \sqrt{K} \tau \left( 1 + \cot^2 \sqrt{K} \tau \right)}{6K + (q^2 + 2K) \cot^2 \sqrt{K} \tau} R'_q + \left[ \frac{q^2 - K}{3} - \frac{2K (q^2 + 2K) \left( 1 + \cot^2 \sqrt{K} \tau \right)}{6K + (q^2 + 2K) \cot^2 \sqrt{K} \tau} \right] R_q = 0. \tag{3.3} \]

For deep inside the horizon modes \( (q \gg \mathcal{H}) \) \( R_q \) varies as

\[ R_q \propto \left( -\frac{q}{2\sqrt{3}}, \frac{q}{2\sqrt{3}}, -\frac{1}{2} \cos^2 \sqrt{K} \tau \right), \]

Figure 1. Evolution of \( R_q \) in a universe constructed from dust for \( K = 0, \pm 1 \). Obviously, \( R_q \) is independent of perturbation scales.
which $2F_1$ is the hypergeometric function. Now let’s turn to the inflation era. Inflation may be treated as a perfect fluid for which

$$\Pi^S = 0 \quad , \quad \Gamma = 0 \quad , \quad c_s^2 = 1.$$  
So

$$R'_q + 2q'R'_q + \left[ K \left( \frac{\mathcal{P}_q'}{H\mathcal{P}_q'} - \frac{q'}{H\mathcal{P}_q'} \right) + q^2 - 5K \right] R_q = 0, \quad (3.4)$$

which is the Mukhanov-Sasaki equation for general $K$, has been derived in [18] for $K = +1$. We leave solutions of equation (3.4) for next works and now turn to investigation the evolution of $R_q$ in post-inflation epoch.

4 Evolution of $R_q$ in a more realized universe

Here we are going to investigate evolution of $R_q$ in a more concrete model of the universe in which the Cosmic fluid is a mixture of dust and radiation without any interaction. It means there is no energy and momentum transfer between them. This model was used by Seljak[19] to analyze the CMB anisotropy. As a comparison to the real universe neutrons and photons belong in radiation and CDM is a member of dust part. Obviously, $\bar{\rho} = \bar{\rho}_M + \bar{\rho}_R$ in which $\bar{\rho}_M \propto a^{-3}$ and $\bar{\rho}_R \propto a^{-4}$. One may define the normalized scale factor as

$$y = \frac{a}{a_{eq}} = \frac{\bar{\rho}_M}{\bar{\rho}_R}.$$  

Here $a_{eq}$ is scale factor in the time of matter-radiation equality. It is not hard to show for the composite cosmic fluid

$$\omega = \frac{1}{3(y+1)} \quad , \quad c_s^2 = \frac{4}{3(3y+4)} \quad , \quad \Pi^S = 0. \quad (4.1)$$

Besides, according to the Friedmann equation one can show

$$\mathcal{H} = \frac{y'}{y} = \sqrt{-2Ky^2 + (\mathcal{H}_{eq}^2 + K)(y+1)} \sqrt{2y}, \quad (4.2)$$

in which $\mathcal{H}_{eq}$ is the comoving Hubble parameter of matter-radiation equality. On the other hand, it can be shown that

$$\Gamma = -\bar{\rho}_Mc_s^2\mathcal{S}, \quad (4.3)$$

where $\mathcal{S} = \delta_M - \delta_R = \frac{\delta\rho_M}{\bar{\rho}_M} - \frac{3}{4}\frac{\delta\rho_R}{\bar{\rho}_R}$ is the entropy perturbation between matter and radiation. Consequently,

$$\Gamma_q = \frac{\mathcal{H}_{eq}^2 + K}{4\pi G \alpha_{eq}^2} \frac{S_q}{y^3(3y+4)}. \quad (4.4)$$
By substituting equations (4.1), (4.2), and (4.4) in equation (2.15) we find
\[
\mathcal{R}_q^{**} + \left\{ -\frac{8 (q^2 - 4K) (-4K y + \mathcal{H}_q^2 + K) + 18K (\mathcal{H}_q^2 + K) (3y + 4)}{8 (q^2 - 4K) [-2K y^2 + (\mathcal{H}_q^2 + K) (y + 1)] + 3K (\mathcal{H}_q^2 + K) (3y + 4)^2} + \frac{(\mathcal{H}_q^2 + K) (y + 2)}{2y [2K y^2 - (\mathcal{H}_q^2 + K) (y + 1)]} + \frac{3}{y} + \frac{6}{3 y + 4} \right\} \mathcal{R}_q + \left\{ -\frac{2}{y} \left[ \frac{2}{y} \mathcal{R}_q - 2K y^2 + (\mathcal{H}_q^2 + K) (y + 1) \right] \right\} \mathcal{R}_q
\]
\[
= \frac{4}{(3y + 4)^2} \left\{ -\frac{72K y^3 + 80K y^2 - (\mathcal{H}_q^2 + K) (3y^2 + 80y + 40)}{2y (3y + 4)} \right\} \mathcal{S}_q
\]
\[
- \frac{48}{(3y + 4)^2} \mathcal{S}_q + \frac{4}{(3y + 4)^2} \mathcal{S}_q^{**} \tag{4.5}
\]
Here "\( \star \)" stands for the partial derivative with respect to \( y \). It is clear from equation (4.5) that \( \mathcal{R}_q \)-evolution depends on \( \mathcal{S}_q \)-evolution directly. On the other hand, \( \mathcal{S}_q \) in a universe contained radiation and dust obey from the Kodama-Sasaki equation\[20\]
\[
\mathcal{S}_q'' + 3c_s^2 \mathcal{H} \mathcal{S}_q' - \frac{1}{3} (3c_s^2 - 1) (q^2 - K) \mathcal{S}_q = - (q^2 - K) \Delta_q, \tag{4.6}
\]
where \( \Delta_q \) depends on the Bardeen’s potential via the Poisson’s equation
\[
\Delta_q = \frac{q^2 - 4K}{3 \left( \mathcal{H}^2 - \mathcal{H}' + K \right)} \Psi_q. \tag{4.7}
\]
By combination of equations (2.13), (2.14), and (4.7) we have
\[
\Delta_q = \frac{q^2 - 4K}{3 \left[ \mathcal{H}^2 c_s^2 (q^2 - 4K) + K (\mathcal{H}^2 - \mathcal{H}' + K) \right]} \left\{ \mathcal{H} \mathcal{R}_q' - \mathcal{R}_q + \frac{4\pi G \mathcal{H}^2 a^2}{\mathcal{H}^2 - \mathcal{H}' + K} \left[ \Gamma_q - (q^2 - K) \Pi_q^s \right] \right\}. \tag{4.8}
\]
Now by inserting equation (4.8) into equation (4.6) and rewriting equation (4.6) in terms of \( y \) we find
\[
\mathcal{S}_q^{**} + \left\{ -\frac{1}{y} \left( \frac{\mathcal{H}_q^2 + K) (y + 2)}{4K y^2 - 2(\mathcal{H}_q^2 + K) (y + 1)} + \frac{4}{3 y + 4} + 1 \right) \mathcal{S}_q = \frac{2 (q^2 - K)}{2K y^2 - (\mathcal{H}_q^2 + K) (y + 1)} \times \right\}
\[
\left\{ \frac{y}{3 y + 4} \mathcal{S}_q + \frac{2 y (3 y + 4) (q^2 - 4K)}{8 (q^2 - 4K) [-2K y^2 + (\mathcal{H}_q^2 + K) (y + 1)] + 3K (\mathcal{H}_q^2 + K) (3y + 4)^2} \times \right\}
\[
\left[ (-2K y^2 + (\mathcal{H}_q^2 + K) (y + 1)) \left( \mathcal{R}_q' - \frac{4}{(3y + 4)^2} \mathcal{S}_q \right) - 2y \mathcal{R}_q \right]. \tag{4.9}
\]
Indeed, equations (4.5) and (4.9) are coupled and must be solved simultaneously. Besides, in early stage (more accurately at the end of inflation) remarkable part of perturbations are placed outside the horizon and the role of curvature may be insignificant\[21\] i.e.
\[
\frac{\left| K \right|}{\mathcal{H}^2} \ll 1 \quad \text{and} \quad \frac{a}{\mathcal{H}} \ll 1. \tag{4.10}
\]
Under these conditions in question system has two outstanding solutions
• Solution 1

\[
\begin{aligned}
S_q &= 0, \\
R_q &= \text{const}.
\end{aligned}
\]

• Solution 2

\[
\begin{aligned}
S_q &= \text{const}, \\
R_q &= \frac{y}{3y + 4} S_q = \frac{1}{3} (1 - 3c_s^2) S_q.
\end{aligned}
\]

These solutions are plausible around \( y = 0 \) so they specify the initial conditions for the system. Strictly speaking, the first solution which according to the inflationary theory may be written as\[6\]

\[
y \sim 0 : \begin{cases} 
S_q \sim 0, \\
R_q \sim Nq^{-2 + \frac{n_{s0}}{3}} \quad (N \simeq 10^{-5} \quad \text{and} \quad n_{s0} \simeq 0.96')
\end{cases}
\]

is referred as the \textit{adiabatic initial condition}. On the contrary, solution 2 is called the \textit{isocurvature initial condition} in accordance with the \textit{Liddle-Mazumdar model}[22] may be written as

\[
y \sim 0 : \begin{cases} 
S_q \sim Mq^{-2 + \frac{n_{iso0}}{3}} \quad (M \simeq 10^{-5} \quad \text{and} \quad n_{iso0} \simeq 4.43) \\
R_q \sim \frac{y}{3y + 4} S_q,
\end{cases}
\]

The system under consideration may be solved via numerical methods like as the Runge-Kutta 4th order. The results are represented in the following figures. Note all equations have been written in terms of \( \kappa = a_{\text{eq}}^2 K_s \) which is \textit{rescaled sectional curvature of spatial slices of the universe}. \( \kappa \) against \( K \) is a continuous variable and can be interpreted as a topological index carries general geometrical properties, namely if \( \kappa > 0 \), \( M \) topologically is equivalent to \( S^3 \) otherwise \( M \) is \( \mathbb{R}^3 \).

The spectral indices of the adiabatic and isocurvature perturbations may be defined respectively as

\[
n_s (q, \kappa, t) = 4 + \frac{q}{P_R} \frac{\partial P_R}{\partial q} = 4 + 2 \frac{q}{R_q} \frac{\partial R_q}{\partial q},
\]

\[
n_{iso} (q, \kappa, t) = 4 + \frac{q}{P_S} \frac{\partial P_S}{\partial q} = 4 + 2 \frac{q}{S_q} \frac{\partial S_q}{\partial q},
\]

where \( P_R \) and \( P_S \) are the spectra of \( R \) and \( S \) respectively. The diagrams of \( n_s \) and \( n_{iso} \) have been shown in figures 4, 5, 6, and 7 for adiabatic and isocurvature initial conditions. It is clear that \( n_s \) and \( n_{iso} \) depend severely on \( q \), so we may say that the spectral indices are running. From figures 4 and 5, it seems \( n_s \) and \( n_{iso} \) for sub-horizon modes subject to adiabatic initial
Figure 2. Three-dimensional surface plot for the comoving curvature perturbation $\mathcal{R}_q$ in a universe constructed from dust and radiation vs. rescaled sectional curvature $\kappa$ and normalized scale factor $y$ for the comoving wave number $q = 10^{-5}$ i.e. super-horizon scales (left) and $q = 10^{+5}$ namely severe sub-horizon modes (right) providing the adiabatic initial condition. We suppose $n_{iso} = 0.96$ and $N$, the amplitude of $\mathcal{R}_q$ at the end of inflation is roughly $10^{-5}$. It seems that the general behavior of $\mathcal{R}_q$ is independent of $q$. Notice that both $q$ and $\mathcal{R}_q$ are dimensionless.

Figure 3. Three-dimensional surface plot for the entropy perturbation $\mathcal{S}_q$ in a universe constructed from dust and radiation vs. rescaled sectional curvature $\kappa$ and normalized scale factor $y$ for the comoving wave number $q = 10^{+5}$ subject to the adiabatic initial condition (left) and isocurvature initial condition (right). We supposed the amplitude of $\mathcal{S}_q$ at the end of inflation is about $10^{-5}$ and $n_{iso} = 4.43$.

condition are sensitive to topology. It may to be true for $n_{iso}$ at super-horizon modes under entropic initial condition too. On the other hand, $\mathcal{R}_q$ is inversely proportional to absolute magnitude of the sectional curvature under adiabatic initial condition. It is also true for $\mathcal{S}_q$ at sub-horizon modes although the concavity of $\mathcal{S}_q$-graph changes at $\kappa = 0$ under isocurvature initial condition. Besides, at super-horizon modes subject to the adiabatic initial condition $n_s$ is minimized at $\kappa = 0$ nevertheless $n_{iso}$ has inflection point. Comparison of the same figure subject to different initial conditions can illustrate the significance of the early stage in the evolution of the universe.
5 Super-curvature modes

In case $K = -1$ (open universe) eigenvalues of the Laplace-Beltrami operator are

$$\kappa_q = -1 - q^2$$  \hspace{1cm} (5.1)

We may suppose $0 < q^2 < +\infty$ which results in sub-curvature modes. On the contrary, someone may suppose $-1 < q^2 < 0$ from which super-curvature modes come out. Super-curvature modes appear merely in the open universe and may be the solutions of odd phenomena observed in CMB anisotropy[13]. So we investigate evolution of $R_q$, $S_q$, $n_s$, and $n_{iso}$ for the super-curvature modes under different initial conditions separately. The results can be observed in figures 8-11.
6 Discussion and Conclusion

We have derived a neat equation (generalized Mukhanov-Sasaki equation) for the evolution of comoving curvature perturbation as a leading random field in the cosmological perturbation Theory. This equation obviously divulges dependency of the comoving curvature perturbation to the curvature index of the FLRW universe. We also considered the numerical solutions of this equation for the mixture of dust and radiation subject to adiabatic and isocurvature initial conditions. As we have seen, in question equation cannot be solved alone due to the presence of another random field i.e. entropy perturbation, so we are coerced to solve the Mukhanov-Sasaki equation accompanied by the Kodama-Sasaki equation simultaneously. We investigated the time evolution of the adiabatic and entropy spectral indices under different initial conditions too. It sounds both spectral indices for perturbations with scales deep inside the horizon under adiabatic initial condition are severely sensitive to topology-changing. Moreover, we found that $n_s$ decreases when the universe becomes more flat. It is also clear that in case $K = 0$, $n_s$ has increasing rate for super-horizon modes in early times regardless
Figure 8. Evolution of the comoving curvature perturbation $R_q$ in a universe constructed from dust and radiation for a typical super-curvature scale subject to adiabatic (left) and isocurvature (right) initial conditions.

Figure 9. The same as Figure 8, except for the entropy perturbation $S_q$.

of the initial conditions. Besides, the oscillating behavior of cosmological perturbations and their indices in case $K = +1$ for all scales is thoroughly clear. Finally, we examined evolution of $R_q$, $S_q$, and their spectra for the super-curvature perturbation for which $K = -1$.

References

[1] Guth A H, Inflationary universe: a possible solution to the horizon and flatness problems, Phys.Rev. D 23 (1981) 347.

[2] Guth A H and Pi So-Young, Fluctuations in the new inflationary universe, Phys.Rev.Lett 49 (1982) 1110.

[3] Bardeen J M, Steinhardt P J and Turner M H, Spontaneous creation of almost scale-free density perturbations in an inflationary universe, Phys.Rev. D 28 (1983) 679.

[4] Mukhanov V, Physical foundations of cosmology, Cambridge University Press (2005).
Figure 10. The same as Figure 8, except for the curvature spectral index $n_s$.

Figure 11. The same as Figure 8, except for the isocurvature spectral index $n_{iso}$.

[5] Lyth D H and Liddle A R, *The Primordial density perturbation: cosmology, inflation and the origin of structure*, Cambridge University Press (2009).

[6] Weinberg S, *Cosmology*, Oxford University Press (2008).

[7] Asgari A A, Abbassi A H and Khodaghilizadeh J, *On the perturbation theory in spatially closed background*, *Eur.Phys.J. C* 74 (2014) 2917.

[8] Mukhanov V, *Gravitational instability of the universe filled with a scalar field*, *JETP Lett* 41 (1985) 439.

[9] Sasaki M, *Large scale quantum fluctuations in the inflationary universe*, *Prog.Theor. Phys.* 76 (1986) 1036.

[10] Asgari A A and Abbassi A H, *Evolution of the spectral index after inflation*, *J. Cosmol. Astropart. Phys.* 09 (2014) 042.

[11] Ade P A R et al., *Planck 2013 results. XVI. Cosmological parameters*, *Astron. Astrophys.* 571 (2014) A16.

[12] Ooba J, Ratra B and Sugiyama N, *Planck 2015 Constraints on the non-flat $\Lambda$ CDM inflation model*, arXiv:1707.03452v1.
[13] Liddle A R and Cortés M, *Cosmic microwave background anomalies in an open universe*, Phys.Rev.Lett 111 (2013) 111302.

[14] Asgari A A and Abbassi A H, *Slow-roll inïnçationary scenario in the maximally extended background*, Eur.Phys.J. C 75 (2015) 544.

[15] Bardeen J M, *Gauge-invariant cosmological perturbations*, Phys.Rev. D 22 (1980) 1882.

[16] Lachiéze-Rey M, *Laplacian eigenmodes for the three-sphere*, J. Phys. A: Math. Gen. 37 (2004) 5625.

[17] Tomita K, *Tensor spherical and pseudo-spherical harmonics in four-dimensional spaces*, Prog.Theor. Phys. 68 (1982) 310.

[18] Asgari A A and Abbassi A H, *Scalar and tensor power spectra in a universe with positive curvature index and maximally symmetric spatial section*, Phys. Dark Univ. 7-8 (2015) 1.

[19] Seljak U, *A two-fluid approximation for calculating the cosmic microwave background anisotropies*, Astrophys. J. 435 (1994) L87.

[20] Kodama H and Sasaki M, *Cosmological perturbation theory*, Progr. Theor. Phys. Suppl 78 (1984) 1.

[21] Asgari A A and Abbassi A H, *Imprints of the intrinsic and exterior curvatures in cosmology*, arXiv:1712.09358.

[22] Liddle A R and Mazumdar A, *Perturbation amplitude in isocurvature inflation scenarios*, Phys.Rev. D 61 (2000) 123507.