Leonov’s Method of Nonstationary Stabilization in the Theory of Linear Control Systems

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Abstract. A brief review on stabilization problem for linear control systems is presented. The statements of Pyragas’ and Brockett’s problems for stabilization of linear control systems by time-delayed feedback control and time-varying static output feedback are given. Leonov’s algorithms of nonstationary stabilization for solutions of Pyragas’ and Brockett’s problems are presented. Necessary and sufficient conditions for the stabilizability of two- and three-dimensional systems are given.

Introduction

The problem of stabilization of dynamic systems is one of the most important subjects in control theory. In the last decades the flow of publications devoted to problems of linear control systems has increased. This topic remains in the focus of attention of numerous researchers. The increasing interest in the stabilization problems is motivated by both the needs of the practice of control and formulations of open problems by many well-known scientists. A large number of papers and books are devoted to the theory and practice of control system stabilization and related issues (see, for example, reviews [1-10]).

Three decades ago, in the control theory of dynamical systems a new direction has arisen: chaos stabilization. This direction was initiated by the work of E. Ott, C. Grebogi, and J. A. Yorke [11], in which on the example of a discrete Henon system the possibility of stabilizing one of the unstable periodic orbits contained in the strange attractor of the system was shown by computer modeling using “small” input control.

A new method of stabilization based on delayed feedback control was proposed in the pioneer work of K. Pyragas [12], which stimulated a lot of further research. The idea of Pyragas’ control method consists in constructing a system feedback such that the difference between the current state of the system and the state with a time delay equal to (or multiple of) the period of the stabilized periodic orbit with a certain constant gain is used. Since the issue of the paper [12] more than thousand papers have been devoted to stabilization of unstable equilibrium states and unstable periodic orbits embedded in strange attractors of various continuous and discrete dynamical systems by Pyragas’ method (see reviews [6-10]). Although the Pyragas’ method proved to be a very effective and powerful tool for stabilizing chaos, it was found that its capabilities are significantly limited. This circumstance was first indicated in the works of Ushio [13] (for discrete systems), Nakajima [14] and Just et al. [15] (for continuous systems). Later, similar results for other classes of systems were established in [16-20].
For continuous systems this limitation to which the Pyragas’ method is subjected is that no unstable periodic orbit (no unstable equilibrium) with an odd number of real Floquet multipliers greater than unity (odd number of positive eigenvalues of Jacoby matrix) can be stabilized by any choice of Pyragas’ delayed feedback control with a constant feedback gain. Now it is referred to as the “odd number limitation”. This problem has been intensively discussed in the literature in order to overcome it (see, for example, the review [7] and [21–29]). However the proposed algorithms are only approximate and based on the numerical – analytical studies. In the work of G.A. Leonov [30] an algorithm based on the strict mathematical means is proposed that allows overcoming the odd number limitation when stabilizing unstable periodic orbits in the chaotic continuous systems. The Leonov’ algorithm uses a non-stationary Pyragas’ delayed feedback control in which the control is designed by a periodic feedback gain. In [31] this algorithm is carried over to the discrete systems and is applied in [32] for stabilizing of unstable equilibria in continuous systems.

Another problem in control theory that stimulated a number of publications was the Brockett’s problem [33] on stabilization of linear time-invariant control systems by means of time-varying static output feedback. For solving of Brockett problem Leonov [34–37] proposed algorithms of low-frequency stabilization based on the construction the periodic feedback with feedback gain in the class of piecewise-constant matrices. In the scalar case when the input and output of the control system are scalars, the high-frequency stabilization algorithms are constructed in the class of the continuous periodic functions by Moreau and Aeyels [38–40]. In [41] a discrete analog of the Brockett problem is considered. Later general necessary and sufficient conditions for the existence of the classes of stabilizing matrices of the arbitrary form that solve the Brockett problem were obtained in [42].

The aim of this article is to present a brief review of some Leonov’s results relating to stabilization of linear control systems by non-stationary (time-varying) feedback.

Non-stationary stabilization. Pyragas’ Problem. Brockett’s Problem

Here we briefly consider the non-stationary stabilization of the unstable periodic orbits and unstable equilibria of dynamical control systems.

Pyragas’ Problem

From a mathematical point of view the Pyragas’ problem is formulated as follows. Let \( \xi(t) = \xi(t - T) \) be an orbitally unstable periodic trajectory (probably on a strange attractor) with period \( T > 0 \) of a system described by the differential equation

\[
\frac{dx}{dt} = f(x) + u
\]  

(1)

for \( u = 0 \). Here \( x \in \mathbb{R}^n \) is a state vector, \( u \in \mathbb{R}^n \) is an input (control) vector, and dot over \( x \) denotes the time derivative. Feedback of the form [12] is introduced into the system (1)

\[
u(t) = K[x(t) - x(t - \tau)], \quad \tau > 0,\]

(2)

where \( K \) is some real constant \( n \times n \) -matrix. As the result the closed system (1), (2) takes the form

\[
\frac{dx(t)}{dt} = f(x(t)) + K[x(t) - x(t - \tau)].
\]

(3)

In (3) \( K \) and \( \tau > 0 \) are the varied parameters. The problem is to find a matrix \( K \) and a number \( \tau > 0 \) such that the trajectory of the system (3) would be orbitally asymptotically stable.

In Leonov’s work [30] to overcome the above-mentioned odd number limitation a non-stationary feedback of the type of (2) with a periodic matrix \( K = K(t) \) and \( \tau = 2T \) is introduced, where the matrix \( K(t) \) is to be determined. In [30] it is proved that there exists a real periodic matrix \( K(t) \) with period \( 6T \) such that the feedback (2) with \( K = K(t) \) and \( \tau = 2T \) stabilizes the unstable periodic trajectory \( \xi(t) \) of system (1) related to the odd number limitation. In the paper [31] for the discrete systems a method for stabilization of unstable periodic solutions is proposed. It is based on the delayed feedback
with pulse periodic gain which is twice the period of an orbit being stabilized. Such an approach allows overcoming the odd number limitation imposed by stabilization with constant gain. This algorithm is applicable if the linearized system around the unstable orbit has any number of eigenvalues (Floquet multipliers) larger than unity. In [32] the algorithm proposed in [30] is applied to stabilization of unstable equilibria of the system (1). The stabilization is achieved by the periodic Pyragas’ feedback (2) in the class of piecewise-constant periodic matrices \( K = K(t) \).

**Brockett’s Problem**

In the book [33] R. Brockett formulated the following problem:

Consider a linear control system

\[
\frac{dx}{dt} = Ax + Bu, \quad y = Cx, \tag{4}
\]

where \( x \in \mathbb{R}^n \) is a state vector, \( u \in \mathbb{R}^m \) is an input (control) vector, \( y \in \mathbb{R}^l \) is an output vector, \( A, B \) and \( C \) are real constant matrices of dimensions \( n \times n, \ n \times m \) and \( l \times n \), respectively. Under what conditions does there exist a time-dependent matrix \( K(t) \) such that the system (4) would be stabilized by the feedback \( y(t)Ku(t) = 0 \), i.e., the closed system

\[
\frac{dx}{dt} = (A + BK(t)C)x \tag{5}
\]

would be asymptotically stable?

In the case of a constant matrix \( K(t) = K = \text{const} \), the problem of stabilization of the system (5) has been studied by numerous authors and continues to be in the focus of many researches (see, for example, reviews [1–5]). The Brockett problem requires the answer to the question: how much does the introduction of the *time-dependent matrices* in the feedback enlarge the possibilities of stationary stabilization?

The solution of the Brockett problem is given in works [34–37] in the class of the piecewise continuous periodic matrices with a sufficiently large period. In the scalar case the solution is presented in [38–40] in the class of continuous periodic functions with a sufficiently small period. In [42] a general algorithm for constructing a family of stabilizing matrices is proposed.

Below we present the results of the works [34–37] related to stabilization of two-and three-dimensional control systems. These are necessary and sufficient conditions for stabilizability of two- and three-dimensional systems by a periodic feedback.

Necessary and sufficient conditions that two-dimensional system of the type of (4), \( n = 2 \), \( y = c_1x_1 + x_2 \), \( x = (x_1, x_2) \) be stabilized by a non-stationary periodic feedback \( u = K(t)y \) is that at least one of the following conditions would be satisfied:

1) \( c_1 > 0 \) \ or \ 2) \( c_1 \leq 0, \ c_1a_2 < a_1 + c_1^2 \),

where \( a_1 \) and \( a_2 \) are coefficients of the characteristic polynomial \( \det(pI - A) = p^2 + a_2p + a_1 \) of the matrix \( A \). Note that stationary stabilization provides more "restrictive" conditions: 1) \( c_1 > 0 \) \ or \ 2) \( c_1 \leq 0, \ c_1a_2 < a_1 \).

Necessary and sufficient condition that three-dimensional system of the type of (4), \( n = 3 \), \( y = x_1 \), \( x = (x_1, x_2, x_3) \) be stabilized by the non-stationary periodic feedback \( u = K(t)y \) is that the condition \( a_3 > 0 \) should hold, where \( \det(pI - A) = p^3 + a_3p^2 + a_2p + a_1 \). In the case of stationary stabilization the corresponding conditions are the following: \( a_2 > 0, \ a_3 > 0 \).
If the outputs of the three-dimensional system of the type of (4), $n = 3$, are $y = x_2$ and $y = x_3$, the corresponding conditions for stabilizability by non-stationary periodic feedback $u = K(\hat{t})y$ are the following: $a_1 \neq 0$, $a_3 > 0$ and $a_1 < 0$, $a_2 > 0$, respectively. In the case of stationary stabilization we have the conditions: $a_1 > 0$, $a_3 > 0$ and $a_1 > 0$, $a_2 > 0$, respectively.

As is shown these conditions illustrate the advantages of non-stationary stabilization in comparison with stationary one very well. The results of works [34–40] are presented in detail in the paper [43] and monographs [44, 45].

**Conclusion**

In the paper a brief review on stabilization problem for linear control systems is presented. Stabilization of systems by non-stationary Pyragas’ time-delay feedback control and time-varying static output feedback are considered. The statements of Pyragas’ and Brockett’s problems for stabilization of control systems are given. Leonov’s algorithms of nonstationary stabilization for solutions of Pyragas’ and Brockett’s problems are presented. Necessary and sufficient conditions for the stabilizability of two- and three-dimensional systems are given. These conditions illustrate the advantages of non-stationary stabilization of linear control systems in comparison with stationary one very well.

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