Origins of Hydrodynamics for a Granular Gas

James W. Dufty\textsuperscript{1} and J. Javier Brey\textsuperscript{2}

\textsuperscript{1} Department of Physics, University of Florida, Gainesville, FL 32611, USA
dufty@phys.ufl.edu
\textsuperscript{2} Fisica Teorica, Universidad de Sevilla, Apartado de Correos 1065, E-41080
Sevilla, Spain brey@cica.es

Summary. The basis for a hydrodynamic description of granular gases is discussed for a low density gas of smooth, inelastic hard spheres. The more fundamental mesoscopic description is taken to be the nonlinear Boltzmann kinetic equation. Two arguments are presented in favor of a hydrodynamics for granular gases. The first one is the concept of a "normal" solution and its explicit approximate construction via the Chapman-Enskog method. The second is the demonstration of hydrodynamic modes in the spectrum of the generator for the dynamics of small spatial perturbations of the homogeneous reference state. In the first case, a derivation of the nonlinear hydrodynamic equations is given to Navier-Stokes order, with explicit expressions for the transport coefficients. The approach is formal and the context of the derivation is left unspecified, although internal mathematically consistency is demonstrated. The second method is more restricted, leading only to linearized hydrodynamics, but with the potential to define more sharply the context of hydrodynamics.

1 Introduction

The form and validity of a hydrodynamic description for granular gases remains a controversial subject [1]. The objective here is to formulate the conceptual and mathematical questions in terms of the underlying Boltzmann kinetic equation, and to describe some recent results providing support for hydrodynamics [2, 3, 4]. Two distinct approaches will be discussed. The first is a consideration of "constitutive equations" for a closure of the exact macroscopic balance equations for the average mass, energy (or temperature), and momentum (or flow velocity). These equations follow directly from the existence of a "normal" solution to the Boltzmann equation. The expected conditions for a normal solution, e.g., away from spatial or temporal boundary layers, characterize the context in which a hydrodynamic description can be expected. The explicit construction of an approximate normal solution under conditions of small spatial variation of the hydrodynamic fields is given by an extension of the Chapman-Enskog method [5]. It is emphasized that no
additional assumptions beyond the normal form and small gradients are required by the inelasticity of collisions. Differences from normal gases (e.g., the reference state, linear stability, form of the constitutive equations) are a consequence of the method, not a limitation on it. The resulting hydrodynamics comprises the Navier-Stokes approximation for a granular gas. Finally, it is emphasized that a breakdown of the latter does not imply the absence of a hydrodynamic description. Indeed, it is an interesting new challenge of granular gases to obtain constitutive equations for more general conditions [1, 6].

The second means to explore the existence and context of hydrodynamics is to consider small spatial perturbations of a strictly homogeneous state. This problem is somewhat simpler as it is governed by the linearized Boltzmann equation, and it is then sufficient to consider a single Fourier component. The spectrum of the generator for the dynamics determines all possible excitations, and it can be asked if the hydrodynamic excitations are among these. For consistency, the hydrodynamic excitations found in this way must be the same as those from the Navier-Stokes equations under the same conditions.

Recently, the eigenvalue problem for this generator has been considered [2, 3, 4]. There is a technical problem associated with the “cooling” of the reference homogeneous state, which is controlled by the introduction of suitable dimensionless variables. This entails a corresponding change in the form of the generator for the dynamics. The existence of hydrodynamic modes has been proved by the exact construction of the hydrodynamic eigenvalues and eigenfunctions in the long wavelength limit. Their form to Navier-Stokes order is then obtained by perturbation theory, and complete agreement with the results from the Chapman-Enskog method is found. The dominance of these hydrodynamic excitations at long times requires that their eigenvalues be the smallest ones and be isolated from the rest of the spectrum. Some comments are offered on why this is to be expected. Elsewhere in this volume [7] these expectations are confirmed for an exactly solvable kinetic model.

The objective of this brief overview is to clarify the concept of a hydrodynamic description in its most general sense, and to sharpen the sufficient conditions for its validity.

2 Boltzmann Equation and Balance Equations

Consider a system of \( N \) smooth hard spheres of mass \( m \) and diameter \( \sigma \) in a large volume \( V \). If the density \( n = N/V \) is sufficiently small, \( n\sigma^3 << 1 \), the phase space density, \( f(r, v, t) \), for the number of particles with position \( r \) and velocity \( v \) at time \( t \) is determined from the Boltzmann equation [8, 9, 10],

\[
\left( \frac{\partial}{\partial t} + v \cdot \nabla \right) f(r, v, t) = J[r, v|f(t)],
\]

where \( J \) is the inelastic Boltzmann collision operator. The most important properties following from this collision operator for the following discussion
are:

\[
\int d\mathbf{v} \left( \frac{1}{\frac{4}{3}m (\mathbf{v} - \mathbf{u})^2} \right) J [\mathbf{r}, \mathbf{v}, f(t)] = \begin{pmatrix}
0 \\
0 \\
-\frac{3}{2}n(\mathbf{r}, t)T(\mathbf{r}, t)\zeta(\mathbf{r}, t)
\end{pmatrix}.
\]

The two zeros on the right side of (2) follow from conservation of mass and momentum. The last term results from non-conservation of energy, where \(\zeta\) is the fractional rate of decrease in the energy due to the inelasticity,

\[
\zeta(\mathbf{r}, t) = \frac{(1 - \alpha^2)m\pi\sigma^2}{24n(\mathbf{r}, t)T(\mathbf{r}, t)} \int d\mathbf{v} g f(\mathbf{r}, \mathbf{v}, t, f(\mathbf{r}, \mathbf{v}_1, t). \]

Here \(g = \mathbf{v} - \mathbf{v}_1\) and \(\alpha\) is the restitution coefficient (\(\alpha = 1\) is the elastic limit). The “hydrodynamic fields” appearing in (2) and (3) are the density \(n(\mathbf{r}, t)\), the temperature \(T(\mathbf{r}, t)\), and the macroscopic flow velocity \(\mathbf{u}(\mathbf{r}, t)\). These are defined in terms of velocity moments of \(f(\mathbf{r}, \mathbf{v}, t)\) by

\[
\begin{pmatrix}
n(\mathbf{r}, t) \\
n(\mathbf{r}, t)\mathbf{u}(\mathbf{r}, t) \\
\frac{3}{4}n(\mathbf{r}, t)T(\mathbf{r}, t)
\end{pmatrix} = \int d\mathbf{v} \left( \frac{1}{\frac{4}{3}m (\mathbf{v} - \mathbf{u})^2} \right) f(\mathbf{r}, \mathbf{v}, t).
\]

The macroscopic balance equations for these fields follow directly from the Boltzmann equation using the properties (2),

\[
\begin{align*}
\partial_t n + \nabla \cdot (n\mathbf{u}) &= 0, \\
(\partial_t + \mathbf{u} \cdot \nabla) \mathbf{u}_i + (mn)^{-1} \nabla_j P_{ij} &= 0, \\
(\partial_t + \mathbf{u} \cdot \nabla + \zeta) T + \frac{2}{3n} (P_{ij} \nabla_j u_i + \nabla \cdot \mathbf{q}) &= 0.
\end{align*}
\]

The pressure tensor \(P_{ij}\) and heat flux \(\mathbf{q}\) are defined by

\[
\begin{align*}
P_{ij}(\mathbf{r}, t) &= \int d\mathbf{v} m(v_i - u_i)(v_j - u_j)f(\mathbf{r}, \mathbf{v}, t), \\
\mathbf{q}(\mathbf{r}, t) &= \int d\mathbf{v} \frac{m}{2}(\mathbf{v} - \mathbf{u})^2(\mathbf{v} - \mathbf{u})f(\mathbf{r}, \mathbf{v}, t).
\end{align*}
\]

In the above equations and in the following, summation over repeated indexes is implicit. For a normal gas with elastic collisions, Eqs. (5)-(9) are the local macroscopic conservation laws for mass, momentum, and energy. The only formal difference here is the appearance of the “cooling rate”, \(\zeta\), in the equation for the temperature. Moreover, all the explicit dependence on the restitution coefficient \(\alpha\) occurs in this cooling rate.

The macroscopic balance equations are the starting point for developing a hydrodynamic description of the gas. They are an exact consequence of the Boltzmann equation, but are of limited utility without further specification of the three unknown functions \(P_{ij}\), \(\mathbf{q}\), and \(\zeta\). Since these are specific functionals of \(f(\mathbf{r}, \mathbf{v}, t)\), the above requires a solution to the Boltzmann equation. The special class of solutions leading to hydrodynamics, normal solutions, are described in the next section.
3 Normal Solution, Constitutive Equations, and Hydrodynamics

A sufficient condition for a hydrodynamic description is the existence of a normal solution to the Boltzmann equation. To describe it, denote the hydrodynamic fields collectively by the set \( \{y_\alpha(r, t)\} \). Then, a normal solution is defined as one for which all space and time dependence of the distribution function is determined by the hydrodynamic fields, i.e.,

\[
f(r, v, t) = f[r, v, t | \{y_\alpha\}],
\]

such that

\[
\left( \frac{\partial}{\nabla} f[r, v, t | \{y_\alpha\}] \right) = \int dt' \int dr' \frac{\delta f(r, v, t | \{y_\alpha\})}{\delta y_\beta(r', t')} \left( \frac{\partial}{\nabla} y_\beta(r', t') \right).
\]

If such a solution can be found, its use in the functionals (3, (8), and (9), converts them to normal forms as well,

\[
\zeta = \zeta[r, t | \{y_\alpha\}], \quad P_{ij} = P_{ij}[r, t | \{y_\alpha\}], \quad q = q[r, t | \{y_\alpha\}].
\]

Such normal forms are called “constitutive” equations. Finally, use of these normal forms in the macroscopic balance equations gives a closed set of equations for the hydrodynamic fields:

\[
\partial_t y_\alpha(r, t) = N_\alpha[r, t | \{y_\beta\}],
\]

where \( N_\alpha[r, t | \{y_\beta\}] \) is, in general, a non-local, non-linear functional of the set of fields \( \{y_\beta\} \). Equations (13) are the hydrodynamic equations in their most general form, encompassing both rheological and viscoelastic non-linear transport. Such a general context is typically reserved for complex fluids (e.g., colloids, large molecular weight fluids) and is of little interest for normal gases. However, structurally simple granular gases can exhibit many of the non-Newtonian properties of complex fluids. Therefore, in discussing the applicability of a hydrodynamic description for granular gases, it is important to admit the possibility of a closed set of equations more complex than the local partial differential equations of the Navier-Stokes approximation (see below).

The construction of a hydrodynamic description is seen to be intimately connected to finding a normal solution to the Boltzmann equation. In fact the two problems comprise a single self-consistent problem. For a normal solution, the Boltzmann equation (1) becomes

\[
\int dt' \int dr' \frac{\delta f[r, v, t | \{y_\alpha\}]}{\delta y_\beta(r', t')} \left( N_\beta[r', t' | \{y_\gamma\}] + v \cdot \nabla' y_\beta(r', t') \right) = J[r, v | f].
\]
determination of the normal forms in (12), and hence \( N_\alpha \{ r, t | \{ y_\beta \} \} \). Finally, solution of Eq. (13), with suitable initial and boundary conditions, provides the explicit forms for the fields, and completes the normal solution. The existence and determination of this solution is the central problem for establishing a hydrodynamic description for both normal and granular gases. The concept of a normal solution and its use in the macroscopic balance equations makes no special reference to the possible inelasticity of collisions.

To clarify the conditions under which a normal solution could be expected, consider first a normal gas with elastic collisions in an initial non-equilibrium state with specified hydrodynamic fields \( \{ y_\alpha (r, t = 0) \} \), whose values vary smoothly across the system. In each small region (e.g., cells of 20-30 particles), the velocity distribution approaches a Maxwellian characterized by the hydrodynamic fields at its central point \( r \). Subsequently, exchange of mass, energy, and momentum tends to equilibrate these fields to uniform values (or to steady values if the system is driven). The first stage, approach to a universal form for the velocity distribution, occurs after a few collisions. This establishes the normal form of the solution, where the hydrodynamic fields and their gradients characterize the state. Deviations from the Maxwellian are due to fluxes of mass, momentum, and energy across the cells. These fluxes are proportional to the differences in values of the fields (i.e., to their spatial gradients). The second stage is the evolution of the distribution through the changing values of the fields, according to the hydrodynamic equations.

This two-stage evolution can be expected for granular gases as well. The initial velocity relaxation will not approach the local Maxwellian, but some other corresponding normal distribution determined from the inelastic Boltzmann equation (see below). Subsequently, the deviations from this normal distribution characterizing spatial inhomogeneities, will again be via the macroscopic balance equations for the granular gas.

### 3.1 Constructive Example: Navier-Stokes Approximation

It is easily verified that the normal solution cannot be simply a local function of the fields alone, i.e. \( f \neq f \{ \mathbf{v} | \{ y_\alpha \} \} \). However, the nonlocal functional dependence can be converted to a local function of the fields and all their gradients,

\[
f = f (\mathbf{v}, \{ y_\alpha (r, t) \}, \{ \nabla y_\alpha (r, t) \}, \{ \nabla \nabla y_\alpha (r, t) \}, \ldots).
\]

(15)

The time derivatives have not been included, since they can be expressed formally in terms of the gradients through the hydrodynamic equations (13), with the corresponding local form \( N_\alpha (\{ y_\beta (r, t) \}, \{ \nabla y_\beta (r, t) \}, \{ \nabla \nabla y_\beta (r, t) \}, \ldots) \). Such a representation is useful when some higher order gradients can be neglected. For example, if the initial and boundary conditions suggest nearly linear hydrodynamic profiles (such as for shear flow between parallel moving plates), the above can be estimated as \( f \sim f (\mathbf{v}, \{ y_\alpha (r, t) \}, \{ \nabla y_\alpha (r, t) \}) \). This approximation could still include large gradients and nonlinear transport.
A simpler case is that for which all the spatial gradients are small. More precisely, consider states for which the fractional changes in the hydrodynamic fields are small over a mean free path, i.e.,

\[ \ell \left| \nabla \alpha \left( \mathbf{r}, t \right) / y_\alpha \left( \mathbf{r}, t \right) \right| \sim \epsilon \ll 1 \] and\n
\[ \ell \left| \nabla^{n+1} y_\alpha \left( \mathbf{r}, t \right) / \nabla^n y_\alpha \left( \mathbf{r}, t \right) \right| \sim \epsilon \ll 1. \]

The parameter \( \epsilon \) is called the uniformity parameter. The normal solution can now be calculated perturbatively as an expansion in this small parameter. To first order in \( \epsilon \), it must be linear in the hydrodynamic gradients

\[
f(\mathbf{r}, v, t) \sim f^{(0)}(\mathbf{v}, \{y_\alpha \left( \mathbf{r}, t \right) \}) \left( 1 + \epsilon \mathbf{X}_\alpha \left[ \mathbf{v}, \{y_\gamma \left( \mathbf{r}, t \right) \} \right] \cdot \nabla y_\beta \left( \mathbf{r}, t \right) \right) .
\]

Before discussing the specific forms for \( f^{(0)} \) and \( \mathbf{X}_\alpha \), the normal forms for \( \zeta \left( \mathbf{r}, t, \right), P_{ij} \left( \mathbf{r}, t, \right), \) and \( q(\mathbf{r}, t, ) \) can be determined directly by use of (16) in the functionals (3), (8), and (9), and invoking the fluid symmetry,

\[
\zeta = \left( 1 - \alpha^2 \right) n \left( \mathbf{r}, t \right) \sigma^2 \left( \frac{T \left( \mathbf{r}, t \right)}{m} \right)^{1/2} \zeta ,
\]

\[
q = -\kappa \left[ T \left( \mathbf{r}, t \right) \right] \nabla T \left( \mathbf{r}, t \right) - \mu \left[ T \left( \mathbf{r}, t \right), n \left( \mathbf{r}, t \right) \right] \nabla n \left( \mathbf{r}, t \right) ,
\]

\[
P_{ij} = p \left( \mathbf{r}, t \right) \delta_{ij} - \eta \left[ T \left( \mathbf{r}, t \right) \right] \left[ \nabla_i u_j \left( \mathbf{r}, t \right) + \nabla_j u_i \left( \mathbf{r}, t \right) \right] - \frac{2}{3} \delta_{ij} \nabla \cdot \mathbf{u} \left( \mathbf{r}, t \right) .
\]

Here \( p = nT \) is the pressure, \( \zeta \) is a dimensionless constant determined from \( f^{(0)} \), and the transport coefficients \( \kappa, \mu, \) and \( \eta \) are linear functionals of the \( \mathbf{X}_\alpha \). Their exact expressions in terms of \( f^{(0)} \) and the \( \mathbf{X}_\alpha \) are given elsewhere \[5, 11\] and will not be repeated here. In the expression for the cooling rate (17), only the leading contribution (zeroth order in the gradients) is given. In principle, consistency would require to consider up to second order in the gradients, since this is the order of the terms associated with the heat flux and the pressure tensor. Nevertheless, contributions from the cooling rate of first order in the gradients of the fields are seen to vanish because of symmetry reasons, and those of second order have been shown to be negligible as compared with the corresponding contributions from Eqs. (18) and (19) \[5\].

The macroscopic balance equations of the last section together with (17)-(19) constitute the Navier-Stokes hydrodynamic equations. Their form is the same as for a normal fluid, except for the presence of the cooling rate \( \zeta \) and the new transport coefficient \( \mu \) in the equation for the heat flux. Of course, the values of the transport coefficients are different, depending on the value of the restitution coefficient \( \alpha \).

The leading term \( f^{(0)} \) in (16) is determined from the Boltzmann equation to order \( \epsilon^0 \),

\[
-\zeta T \frac{\partial f^{(0)}}{\partial T} = J \left[ \mathbf{r}, \mathbf{v} \right] f^{(0)} .
\]
Since \( f^{(0)} \) is normal, it follows from dimensional analysis that its dependence on temperature can occur only through normalization and a scaling of the velocity. Then (20) becomes:

\[
\frac{\zeta}{2} \frac{\partial}{\partial v} \left[ (v - u) f^{(0)} \right] = J \left[ v, v | f^{(0)} \right].
\]

This equation also arises when considering the Boltzmann equation for a spatially homogeneous state, the so-called homogeneous cooling state (HCS), as discussed in the next section. Here, it is determining the same velocity dependence, but scaled with respect to the exact hydrodynamic fields for the nonuniform, nonstationary state. Thus the solution here is more appropriately named the local HCS. It is important to note that this leading order “reference” state is determined by the Chapman-Enskog procedure itself, not by an arbitrary, independent choice.

Similarly, the first corrections, determined by \( X_\alpha \), are obtained from the Boltzmann equation to order \( \epsilon \), with the result:

\[
\left[ L X_\alpha - f^{(0)-1} \zeta T \frac{\partial}{\partial T} \left( f^{(0)} X_\alpha \right) \right] \cdot \nabla y_\alpha - X_T \cdot \nabla (\zeta T) = -\frac{\partial \ln f^{(0)}}{\partial y_\alpha} \left[ N^{(1)}_\alpha \right. \left. (r, t | \{ y_\beta \}) + v \cdot \nabla y_\alpha \right],
\]

where \( N^{(1)}_\alpha \) is the term of order \( \epsilon \) in the expansion of \( N_\alpha \), \( X_T \) is the coefficient of \( \nabla T \) in Eq. (16), and \( L \) is the linearized Boltzmann collision operator,

\[
LY(v) = - f^{(0)-1}(v) \int d\nu' \frac{\delta J}{\delta f^{(0)}(\nu')} f^{(0)}(v') Y(v'),
\]

for arbitrary \( Y(v) \). By equating coefficients of the hydrodynamic gradients in (22), the coefficients having independent scalar, vector, and tensor transformation properties, a set of linear inhomogeneous integral equations are obtained. The Fredholm conditions have been proved for these equations, so that existence of solutions is assured. The details are described in [5] and [11].

In summary, a normal solution to the Boltzmann equation up through first order in \( \epsilon \) is given by (16). This solution determines the constitutive equations in normal form also to this order. These constitutive equations together with the macroscopic balance equations, give approximate hydrodynamic equations which are the Navier-Stokes equations for a granular gas. The transport coefficients in these equations are also provided by the derivation.

The main point of the discussion in this section has been to show that hydrodynamic equations for granular gases have the same conceptual basis as for normal gases: 1) exact macroscopic balance equations for \( n, T, \) and \( u \), with exact expressions for the unknown fluxes (and the cooling rate) as functionals of the distribution function \( f \); 2) a rapid relaxation of \( f \) to a normal solution. These two ingredients are sufficient for the existence of a hydrodynamic description (closed, deterministic equations for the hydrodynamic fields).
construction of the normal solution is a difficult, unsolved problem in general. However, for states with small spatial gradients, the Chapman-Enskog perturbation expansion leads to an explicit approximate normal solution. No explicit reference to inelasticity occurs at this level. The detailed computation of the normal solution and characterization of the space and time scales for its validity does, of course, entail differences from that for normal gases. But these are technical issues that should be separated from the conceptual basis.

Property 2) above is the most uncertain component of the discussion. If a normal solution exists, what is its context? For molecular gases, the assumption is that it applies only outside certain “boundary layers”: a few mean free paths away from spatial boundaries, a few mean free times later than any imposed initial conditions, and away from “discontinuities” (e.g., shock fronts). Similar restrictions are expected to apply for granular gases, and it may be that these are stronger constraints in that case. In addition, there is a new time scale for granular gases set by the cooling rate. The hydrodynamic equations described here include the dynamics of cooling, so there are new questions regarding the separation of hydrodynamic from all other time scales. Monte Carlo and molecular dynamics simulations suggest that the separation remains valid even at strong dissipation, but it remains to demonstrate this directly from the Boltzmann equation. The next section provides a more restricted context in which this question can be posed clearly.

4 Linearized Boltzmann Equation and Hydrodynamic Modes

In this section, the existence of hydrodynamics for granular gases is considered from a different perspective, independent of the concept of a normal solution to the Boltzmann equation. The idea is to study the initial value problem for small perturbations of the homogeneous state of an isolated system. In this case, the Boltzmann equation can be linearized about the homogeneous reference state. For appropriate dimensionless variables, the general solution is characterized by a linear eigenvalue problem. The eigenvalues determine all modes of excitation in the gas due to the small perturbations. If hydrodynamics exists, the hydrodynamic modes must be among these excitations. These modes are defined in the long wavelength limit as being the same as those obtained from the exact linearized macroscopic equations. The reference homogeneous state for an isolated granular gas is taken of the form

\[ f_{\text{hcs}}(v, t) = nv_0^{-3}(t)f_{\text{hcs}}[v/v_0(t)], \quad (24) \]

with

\[ v_0(t) = \left( \frac{2T(t)}{m} \right)^{1/2}. \quad (25) \]

Use of this scaling form in the expression for the cooling rate (3), gives again the form (17), now as an exact result,
\[
\zeta(t) = \frac{(1 - \alpha)^2 \pi \sigma_n^2}{12} \left( \frac{2T(t)}{m} \right)^{1/2} \\
\times \int dv \int dv_1 |v - v_1|^{3} f_{hcs}^*(v) f_{hcs}^*(v_1).
\] (26)

The macroscopic balance equations for the homogeneous state reduce to
\[
\partial_t n = \partial_t u = 0, \quad \partial_t \ln T(t) = -\zeta(t).
\] (27)

Since the dependence of \(\zeta(t)\) on the temperature is known, the time dependence of the temperature can be calculated. Substitution of (24) into the Boltzmann equation gives the equation for \(f_{hcs}\). It is found to be the same as that for the local HCS distribution \(f^{(0)}\), given by (20) or (21) above. The difference here is that the hydrodynamic fields parameterizing this distribution are those for the homogeneous state, whereas in the Chapman-Enskog context they are the fields for a general inhomogeneous state. As already mentioned, the distribution \(f_{hcs}\) is referred to as the HCS solution. Its velocity dependence is the same as that of the local HCS solution.

In the following, it will be assumed that \(f_{hcs}\) is known. Consider an initial small spatial perturbation of the HCS (other examples might arise from small external forces or boundary conditions). Since the perturbation is small initially, it is assumed that the deviation of the distribution function from \(f_{hcs}\) remains small over the times of interest, such that the Boltzmann equation can be linearized. Consequently, it is sufficient to consider a single Fourier component,

\[
f(r, v, t) = f_{hcs}(v, t) \left[ 1 + e^{i \mathbf{k} \cdot \mathbf{r}} g(k, v, t) \right].
\] (28)

Substitution into the Boltzmann equation and retaining only linear order in \(g \) gives the linear Boltzmann equation,

\[
\partial_t + i \mathbf{k} \cdot \mathbf{v} + L f_{hcs}(v) \frac{\zeta}{2} \frac{\partial}{\partial v} (\mathbf{v} f_{hcs}) \left( g(k, v, t) = 0. \right)
\] (29)

Here, \(L\) is the linear operator defined in (23) evaluated at \(f^{(0)} \rightarrow f_{hcs}\).

The time dependence of the reference HCS leads to an explicit time dependence of \(L, \zeta, \) and \(f_{hcs}\). This can be removed by introducing the following dimensionless variables

\[
s = \int_0^t dt' \frac{v_0(t')}{\ell}, \quad v^* = \frac{v}{v_0(t)}, \quad k^* = k\ell, \quad (30)
\]

\[
g^*(k^*, v^*, s) = g(k, v, t), \quad \zeta^* = \frac{\ell \zeta}{v_0(t)}. \quad (31)
\]

where \(\ell = 1/n_\sigma^3\) is proportional to the mean free path and \(s\) is proportional to the average number of collisions per particle in the time interval \((0, t)\). The linearized Boltzmann equation then becomes...
\[
(\partial_s + i\mathbf{k}^* \cdot \mathbf{v}^* + \Lambda^*) g^* = 0, \quad (32)
\]

where the linear collision operator \(\Lambda^*\) is defined by
\[
\Lambda^* Y = L^* Y + \frac{Z^*}{2} f_{hcs}^{-1} \frac{\partial}{\partial v} \cdot (v f_{hcs}^* Y), \quad L^* = \ell v_0(t) L, \quad (33)
\]

It is seen that the operator \(\Lambda^*\) is not simply the linearized Boltzmann collision operator in dimensionless form, but it includes as well an operator that describes the effects of cooling explicitly.

Normalization of the distribution function requires only that \(f_{hcs}^* g^*\) be integrable. However, we make the stronger requirement that \(g^*\) be an element of a Hilbert space with scalar product
\[
(a, b) \equiv \int d\mathbf{v}^* f_{hcs}^* (\mathbf{v}^*) a^+(\mathbf{v}^*) b(\mathbf{v}^*), \quad (34)
\]

where \(a^+\) denotes the complex conjugated of \(a\). The formal solution to the linear Boltzmann equation is
\[
g^* (\mathbf{k}^*, \mathbf{v}^*, s) = \oint \frac{dz}{2\pi i} e^{-zs} \mathcal{R}(z) g^* (\mathbf{k}^*, \mathbf{v}^*, 0), \quad (35)
\]

\[
\mathcal{R}(z) = (z - i\mathbf{k}^* \cdot \mathbf{v}^* - \Lambda^*)^{-1}, \quad (36)
\]

where the contour encloses the entire spectrum of \(i\mathbf{k}^* \cdot \mathbf{v}^* + \Lambda^*\), both point and residual, counterclockwise. All linear excitations are determined from this spectrum. The hydrodynamic modes, whenever they exist, are part of the point spectrum so it is sufficient to study the eigenvalue problem
\[
(i\mathbf{k}^* \cdot \mathbf{v}^* + \Lambda^*) \psi_n (\mathbf{v}^*) = \lambda_n (\mathbf{k}^*) \psi_n (\mathbf{v}^*). \quad (37)
\]

The hydrodynamic modes are defined from the macroscopic balance equations in the following way. First, a normal form for the cooling rate, pressure tensor, and heat flux is assumed to give a closed set of equations (hydrodynamics). Next, these equations are linearized about the HCS state \((n, \mathbf{v} = 0, T(t))\). The deviations of the density, flow velocity, and temperature are then put in dimensionless form and given a Fourier representation,
\[
\{\delta y_\alpha (\mathbf{k}, s)\} \equiv \int d\mathbf{r} e^{i\mathbf{k} \cdot \mathbf{r}} \left\{ \frac{\delta n (\mathbf{r}, t)}{n}, \frac{\delta T (\mathbf{r}, t)}{T(t)}, \frac{\delta u (\mathbf{r}, t)}{v_0(t)} \right\}. \quad (38)
\]

The resulting linear hydrodynamic equations take the form
\[
\partial_s \delta y_\alpha (\mathbf{k}, s) + M_{\alpha\beta} (\mathbf{k}) \delta y_\beta (\mathbf{k}, s) = 0. \quad (39)
\]

The hydrodynamic modes are defined to be the eigenvalues of \(M_{\alpha\beta}(\mathbf{k})\). Interestingly, the expression for \(M_{\alpha\beta}(\mathbf{k})\) up through order \(k\) does not depend on the assumed forms for the pressure tensor and heat flux; only the fact that
the cooling rate is normal and dimensional analysis is required. Consequently, the corresponding eigenvalues can be considered as known exactly up through this order. They are found to be:

\[
\{ \lambda_n^{(h)}(k) \} = \left\{ 0, \frac{\zeta^*}{2}, -\frac{\zeta^*}{2}, -\frac{\zeta^*}{2}, -\frac{\zeta^*}{2} \right\} + \text{terms of order } k^2.
\] (40)

### 4.1 Existence of Hydrodynamic Modes

The search for hydrodynamic excitations in the spectrum of the operator \((ik^* \cdot v^* + \Lambda^*)\) can be performed by assuming they are analytic in \(k\) and looking first for the \(k = 0\) solutions of (32). The practical issue of constructing these modes at finite \(k\) is a separate issue from their existence and is discussed in the next subsection.

Consider again Eq. (21) for the HCS solution in terms of the nonlinear Boltzmann collision operator. Although it is an equation for the velocity dependence of the distribution function, it is parameterized by the hydrodynamic fields. Sequential differentiation with respect to these fields introduces the linearized Boltzmann collision operator,

\[
\frac{\partial}{\partial y_{\alpha}} \frac{\partial}{\partial v^*} \left[ (v - u) f_{\text{hcs}} \right] = \frac{\partial}{\partial y_{\alpha}} J \left[ v | f_{\text{hcs}} \right] = -f_{\text{hcs}}(v) L \frac{\partial \ln f_{\text{hcs}}}{\partial y_{\alpha}}.  
\] (41)

This gives five equations relating the operation of \(L\) on derivatives of the HCS solution to other derivatives of the same distribution. In detail, these terms can be rearranged to construct eigenvalue equations for \(\Lambda^*\). In this way five eigenfunctions and eigenvalues are directly identified [2, 3, 4]

\[
\{ \lambda_n(0) \} = \left\{ 0, \frac{\zeta^*}{2}, -\frac{\zeta^*}{2}, -\frac{\zeta^*}{2}, -\frac{\zeta^*}{2} \right\} .
\] (42)

\[
\{ \psi_n(0) \} = \left\{ 4 + \nu^* \cdot \frac{\partial \ln f_{\text{hcs}}}{\partial \nu^*}, -3 - \nu^* \cdot \frac{\partial \ln f_{\text{hcs}}}{\partial \nu^*}, -\partial \ln f_{\text{hcs}} \right\} .
\] (43)

Clearly, these are the hydrodynamic modes in the long wavelength limit defined by (40). This is therefore a direct confirmation of the existence of hydrodynamic excitations for a granular gas, independent of other derivations such as the Chapman-Enskog expansion.

### 4.2 Navier-Stokes Order Modes

Returning to the eigenvalue problem (37) at finite \(k\), the hydrodynamic modes can be calculated for small \(k\) by a perturbation expansion

\[
\lambda_n(k) = \lambda_n^{(0)} + k \lambda_n^{(1)} + k^2 \lambda_n^{(2)} + \cdots,
\] (44)

\[
\psi_n(k) = \psi_n^{(0)} + k \psi_n^{(1)} + k^2 \psi_n^{(2)} + \cdots.
\] (45)
with \( \lambda_n^{(0)} = \lambda_n(0) \) and \( \psi_n^{(0)} = \psi_n(0) \) given by (40) and (43), respectively.

The analysis is complicated by the three-fold degeneracy associated with the unperturbed eigenvalues \(-\zeta/2\). These are the eigenvalues of a vector, which can be decomposed into a longitudinal component along \( \hat{k}^* \) and two transverse components. The operator \( \hat{k}^* \cdot \mathbf{v}^* + \Lambda^* \) is invariant under rotations about \( \hat{k}^* \) so the Hilbert space can be decomposed into two invariant subspaces comprised of the rotationally invariant functions and their orthogonal complement. The transverse and longitudinal eigenfunctions lie in different subspaces. If \( \psi_3^{(0)} \) is taken to be the longitudinal component, the eigenvalue problem can be separated into two independent eigenvalue problems for \( n = 1, 2, 3 \) and for \( n = 4, 5 \), respectively. The eigenvalues for the first problem are all distinct, and therefore standard perturbation theory applies. The second problem still has a two-fold degeneracy that remains even at finite \( k \).

The fact that \( \hat{k}^* \cdot \mathbf{v}^* + \Lambda^* \) is not Hermitian is a second complication. The calculations are simplified by introducing a set that is biorthogonal to \( \{ \psi_n^{(0)} \} \),

\[
\phi_n^{(0)} \to \left\{ 1, \frac{v^2}{2} + \frac{1}{2} \hat{k}^* \cdot \mathbf{v}^*, \hat{\mathbf{e}}^{(i)} \cdot \mathbf{v}^* \right\}, \quad \left( \phi_n^{(0)}, \psi_n^{(0)} \right) = \delta_{nm},
\]

where \( \{ \hat{k}^*, \hat{\mathbf{e}}^{(i)}; i = 1, 2 \} \) are three pairwise orthogonal unit vectors. The determination of the eigenvalues and eigenvectors to order \( k^2 \) is then straightforward but lengthy and the details will be given elsewhere [12]. Here, only the results for the eigenvalues are quoted. First, all corrections to first order in \( k \) vanish as expected from the result (37). To order \( k^2 \) the coefficients are

\[
\lambda_1^{(2)} = \left( \hat{k} \cdot \mathbf{v}, \frac{\Lambda^* - \zeta^*}{2} \right) \psi_1^{(0)}(v),
\]

\[
\lambda_2^{(2)} = \left( \hat{k} \cdot \phi_2^{(0)}(v), \left( \frac{\Lambda^* - \zeta^*}{2} \right)^{-1} \hat{k} \cdot \mathbf{v} \psi_2^{(0)} \right) + Z \left[ f_{hcs}^{*}, \psi_2^{(0)} \right],
\]

\[
\lambda_3^{(2)} = \left( \left( \hat{k} \cdot \mathbf{v} \right)^2, \left( \frac{\Lambda^* + \zeta^*}{2} \right)^{-1} \hat{k} \cdot \psi_3^{(0)}(v) \right),
\]

\[
\lambda_{4,5}^{(2)} = \left( \hat{\mathbf{e}}^{(1,2)} \cdot \mathbf{v}, \left( \hat{k} \cdot \mathbf{v} \right), \left( \frac{\Lambda^* + \zeta^*}{2} \right)^{-1} \hat{k} \cdot \psi_{4,5}^{(0)}(v) \right).
\]

In the expression of \( \lambda_2^{(2)} \), \( Z \) is a rather involved functional of \( f_{hcs}^* \) and \( \psi_2^{(0)} \) whose explicit expression will be not given here [3]. It corresponds to the second order in the gradients contributions that were neglected upon writing expression (17) for the cooling rate in the Navier-Stokes approximation. The corresponding results obtained from the linearized Navier-Stokes equations are given in terms of the transport coefficients, which are defined there from the Chapman-Enskog method. It is possible to show that the results
(47) are exactly the same as those from the Chapman-Enskog method, and independent confirmation of the latter.

It has been stressed in section 3 that “hydrodynamics” should be understood in a more general context than the Navier-Stokes approximation. In the present section that means the existence of modes \( \{ \lambda_n(k) \} \) beyond quadratic order in \( k \). For elastic collisions, it has been shown that the perturbation expansion converges [13] so that all higher order terms are meaningful. However, the proof does not extend to granular gases. Instead, support for the existence of hydrodynamics beyond Navier-Stokes order can be obtained from realistic but simpler kinetic model equations. In reference [7] such a kinetic model is solved exactly (reduced to quadratures) to demonstrate the hydrodynamic modes in the spectrum of the general solution to the linear kinetic equation, and to display the equation for the dispersion relations determining \( \{ \lambda_n(k) \} \). In that case, complete information about the rest of the spectrum is available as well, so that the context of a hydrodynamic description can also be studied.

5 Summary and Outlook

The origin of hydrodynamics starts with the balance equations. These have an exact basis in statistical mechanics and are little more than a statement of conservation laws at the macroscopic level. Such equations are correct even for very complex fluids, solids, composites, or worse. The next step to obtain a closed set of equations for the fields (hydrodynamic, elastic, viscoelastic, other) is the point of conceptual and practical difficulty. The choice of constitutive equations is simple only for systems with well-defined symmetries and response characteristics: e.g., simple atomic fluids, elastic solids, liquid crystals. In these cases, the symmetries and time scales restrict the form of the constitutive equations and the resulting macroscopic equations have little uncertainty beyond the numerical values of the constants appearing in them.

Constitutive equations for complex fluids (e.g., polymers, colloids, foams) are more difficult to fix as neither symmetries nor characteristic space and time scales are easily determined. Surprisingly, granular gases are more like complex fluids than like simple atomic fluids, in spite of their simple structure. To a great extent this is due to the single feature of inelastic collisions. Corn starch flows like a simple gas under gravity, but responds to large shear as a solid. It is probably too ambitious to attempt to find constitutive equations encompassing both classes of phenomena. Here attention has been limited to conditions of low density gas flow, but even with this restriction it subsumes common conditions of both subsonic and supersonic flows for granular gases. Thus, it may be expected that simple constitutive relations may have more limited applicability than for normal gases.

In the first part of this presentation, the notion of constitutive equations has been tied to the existence of a normal solution to the Boltzmann equation. If a normal solution applies, a hydrodynamic description applies (albeit
with possibly complicated constitutive equations). The idea of a normal solution is suggested by a two stage process of evolution following preparation of the system. The first is a rapid velocity relaxation in each small cell of the gas to a distribution characterized by the hydrodynamic fields at each point, and also by the gradients of these fields, reflecting the fact that each cell is an open system. The second stage is an “equilibration” of the spatial variations in the fields from cell to cell, as enforced by the macroscopic balance equations. Important open questions at this point include: 1) what is the precise definition of a normal solution to the Boltzmann equation, i.e. how is the problem properly posed?, 2) is there a “universal” normal solution approached for a wide class of preparations?, 3) what is the time scale for approach to a normal solution and is it short as compared with the hydrodynamic time scales? The Chapman-Enskog perturbation expansion gives partial answers to these questions in a very restricted context. Regarding 1), the problem is well-posed with solutions assured at each order in the perturbation. The Chapman-Enskog normal solution is indeed a universal function of the fields and their gradients. Different hydrodynamic states result only from boundary and initial conditions determining the values of these fields. Regarding 2) and 3), the Chapman-Enskog method gives no information.

The second part of this presentation considers the linear response to small spatial perturbations of a homogeneous reference state. The question posed is whether the hydrodynamic excitations occur in the spectrum of the generator for this linear dynamics. The primary result reported here is that the hydrodynamic excitations do exist in the long wavelength limit (small $k$). Under the assumptions of analyticity in $k$ and convergence of a formal perturbation expansion, the hydrodynamic spectrum is established. This approach to hydrodynamics for a granular gas provides a somewhat sharper mathematical formulation of the questions: 1) what is the appropriate function space to study the spectrum of $(ik^* \cdot v^* + A^*)$? and, 2) is the hydrodynamic point spectrum found here isolated and smaller than all other parts of the spectrum?

Simple experiments on driven systems exhibit complex phenomena (e.g., symmetry breaking) that nevertheless can be explained by the correct Navier-Stokes hydrodynamics. Then, complex phenomena alone should not preclude the simplest Navier-Stokes hydrodynamic description. On the other hand, some steady state conditions are necessarily non-Newtonian (beyond Navier-Stokes). This should not be interpreted as a failure of hydrodynamics, but only as a challenge to explore the correct constitutive equations for these conditions.

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