The gravitational energy

M. Novello and E. Bittencourt
Instituto de Cosmologia Relatividade Astron۸sfica ICRA - CBPF
Rua Dr. Xavier Sigaud 150 - 22290-180 Rio de Janeiro - Brazil

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We present the expression $t_{\mu \nu}$ of the energy-momentum tensor of the gravitational field in the framework of the recent proposal of the Geometric Scalar theory of gravity (GSG). From the conservation of $t_{\mu \nu}$ it follows the dynamics of the gravitational field. As an example of this expression for $t_{\mu \nu}$ we calculate the gravitational energy of a compact object.

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I. INTRODUCTION

In spite of several attempts, the theory of general relativity (GR) is not able to exhibit a true energy-momentum tensor for the gravitational field. This is usually understood as the impossibility to localize the gravitational energy. To bypass this criticism, many proposals have been discussed in the literature suggesting that for the case of gravity the ambiguity of the localization of its energy should be necessarily treated in terms of pseudo-tensorial quantities as it was proposed by Einstein [7], Tolman [8], Landau and Lifshitz [9] and Møller [10]. The origin of such difficulty is due to the characteristic of this theory that deals with a dynamics imposed directly on the geometry. Such property led to the general acceptance that for gravity processes, the definition of energy should be transcended and considered as a fruitful and important concept limited to some special configurations like, for instance, in asymptotically flat geometries.

This situation is not present in those alternative theories in which the dynamics of the gravitational field is not directly imposed upon the metric structure. This is the case of the recent Geometric Scalar Gravity (GSG) [6] that combine the main idea of general relativity—that is, gravity is a geometrical phenomenon that should be treated as a modification of the metric of space-time—with the dynamics being imposed only indirectly to the geometry. The construction of a true energy-momentum tensor for the gravitational field in the GSG is the focus of this paper.

We shall present arguments to support the identification of a symmetric second order tensor to the energy-momentum tensor of the gravitational field. This will be made in the realm of the GSG in which the basic fundamental quantity that represents the gravitational field is the scalar field $\Phi$.

The characterization of the gravity as nothing but the geometry of space-time gave a beautiful interpretation of the equivalence principle. The fact that it is possible to annihilate locally the gravitational effects led to the general belief that the gravitational energy should not be associated to a true tensorial object but to a pseudo-tensor in order to be eliminated locally according to the equivalence principle.

The discovery of a pseudo-tensor to represent the non-localization of the gravitational energy was perfectly in accord with these main ideas that are at the basis of general relativity. Einstein for instance proposed to characterize the energy content in the gravitational field in terms of a super-potential $H_{\mu}^{[\nu \lambda]}$ such that one obtains the pseudo-tensor

$$T^\nu_{\mu} = \partial_\lambda H_{\mu}^{[\nu \lambda]},$$

where

$$H_{\mu}^{[\nu \lambda]} = \frac{1}{2\kappa} \tilde{g}_{\mu \nu} \partial_\alpha \left( \tilde{g}^{\nu \lambda} \tilde{g}^{\alpha \gamma} - \tilde{g}^{\nu \gamma} \tilde{g}^{\alpha \lambda} \right).$$

We note in passing that distinct from all other energy-momentum tensor of field theories—e.g., electromagnetic field—that contains only first order derivative of the field, the role of the energy-momentum tensor displayed in GR by the quantity $T^\nu_{\mu}$ contains second order derivatives of the basic variable $\tilde{g}^{\nu \lambda} = \sqrt{-\tilde{g}} g^{\nu \lambda}$.

Nothing similar in the Geometric Scalar Gravity that attributes a real tensor $t^\nu_{\mu}$ to the description of the energy content of the field and consequently cannot be locally made to vanish by any choice of coordinates. We emphasize that this property does not conflict with the Equivalence Principle, once the basic variable $\Phi$ interacts with matter only through the combination that generate the metric $q_{\mu \nu}$. The universality of the gravitational interaction of any kind of matter and energy only through this metric and its derivatives allows the possibility of local annihilation of the gravitational field by making to vanish the associated Christoffel symbol. This makes GSG in perfect agreement with the Equivalence Principle.

The adjective “geometric” in its name means that this theory accepts the hypothesis originally made in general relativity, that gravity is a geometrical phenomenon described by a Riemannian metric. In GR the ten components of the metric tensor are the basic variables of...
the theory (up to coordinate transformations). In GSG the metric tensor is determined by the derivatives of a fundamental independent physical quantity represented by the scalar field $\Phi$. In order to exhibit the similarities and differences between GSG and GR let us summarize their main properties.

**Basic properties of General Relativity:**

- The gravitational interaction is described by a second order tensor field $h_{\mu\nu}$;
- The field $h_{\mu\nu}$ satisfies a nonlinear dynamics;
- The theory satisfies the principle of general covariance which, in other words, means that GR is not a theory restricted to the realm of special relativity;
- All kinds of matter and energy interact with $h_{\mu\nu}$ only through the pseudo-Riemannian metric
  \[ g^{\mu\nu} = \eta^{\mu\nu} + h^{\mu\nu}; \]
- Test particles follow geodesics relative to the gravitational metric $g_{\mu\nu}$;
- $h_{\mu\nu}$ is related in a nontrivial way with the Newtonian potential $\Phi_N$;
- The background Minkowski metric is not observable. Matter and energy interact gravitationally only through the combination $\eta_{\mu\nu} + h_{\mu\nu}$ and its derivatives;
- Electromagnetic waves propagate along null geodesics relative to the metric $g_{\mu\nu}$.

- The contravariant definition of the metric in GSG, that is, $g_{\mu\nu}$, is also a binomial expression $q_{\mu\nu} = 1/\alpha \eta_{\mu\nu} - \beta \partial_{\mu} \Phi \partial_{\nu} \Phi$, is given by the infinite series 
  \[ q_{\mu\nu} = 1/\alpha \eta_{\mu\nu} - \beta \partial_{\mu} \Phi \partial_{\nu} \Phi. \]

**Basic properties of the Geometric Scalar Gravity:**

- The gravitational interaction is described by a scalar field $\Phi$;
- The field $\Phi$ satisfies a nonlinear dynamics;
- The theory satisfies the principle of general covariance which, in other words, means that GSG is not a theory restricted to the realm of special relativity;
- All kinds of matter and energy interact with $\Phi$ only through the pseudo-Riemannian metric
  \[ q^{\mu\nu} = \alpha \eta^{\mu\nu} + \beta w \partial^\mu \Phi \partial^\nu \Phi, \]
  where $\alpha$ and $\beta$ are functions of $\Phi$ and $w$ is a function of $\Phi$.
- Test particles follow geodesics relative to the gravitational metric $q_{\mu\nu}$;
- $\Phi$ is related in a nontrivial way with the Newtonian potential $\Phi_N$;
- The background Minkowski metric is not observable. Matter and energy interact gravitationally only through the combination
  \[ q_{\mu\nu} = 1/\alpha \eta_{\mu\nu} - \beta \partial_{\mu} \Phi \partial_{\nu} \Phi. \]

The parameters $\alpha$ and $\beta$ that are functionals of the scalar field $\Phi$ were specified in [7] by fixing the dynamics of the scalar field. We note that in both theories (GR and GSG) the auxiliary (Minkowski) metric $\eta^{\mu\nu}$ is unobservable because the gravitational field couples to matter only through the effective metric $q^{\mu\nu}$ in the case of GR, or to $q_{\mu\nu}$ in the case of GSG. Thus, in both theories a unique geometrical entity interacts with all forms of matter and energy and the geometry underlying all events is controlled by the gravitational phenomena.

From this postulate it follows immediately that the space-time geometry is an evolutionary process identified to the dynamics of the gravitational field. This beautiful hypothesis made by Einstein and that constitutes the true basis of general relativity is contained in each observation as a specific example of a geometry solving that dynamics. That is, the metric couples universally and minimally to all fields of the standard model by replacing everywhere the Minkowski metric $\eta_{\mu\nu}$ either by $g_{\mu\nu}$, in the case of GR, or by $q_{\mu\nu}$ in the case of GSG.

The origin of this departure from general relativity was explained in [7] where the expressions of $\alpha$ and $\beta$ were obtained from the analysis of the gravitational properties of planetary orbits yielding.
\[ \alpha = e^{-2\Phi}, \quad (3) \]
\[ \beta = \frac{1}{4}(e^{-2\Phi} - 1)(e^{-2\Phi} - 9) \quad (4) \]
and
\[ V = \frac{(e^{\Phi} - 3e^{\Phi})^2}{4}, \quad (5) \]
where \( V(\Phi) \) is the nonlinear potential of the lagrangian that is at the basis of GSG.

We shall prove that in the Geometric Scalar Gravity, the energy density of a star—let us call it \( \mathcal{E}_g \)—is obtained as a solution of the first-order differential equation
\[ \frac{d\mathcal{E}_g}{d\Phi} + Q(\Phi) \mathcal{E}_g + P(\Phi) = 0, \]
where the functions \( Q \) and \( P \) are given in terms of the scalar gravitational field. The gravitational energy density \( \mathcal{E}_g \) is defined as the 0 − 0 component of the true energy-momentum tensor \( \mathcal{T}^{\mu \nu}_g \). The simplest way to present this proposal is to use the description of GSG in the framework of a field theory suggested by Gupta [8], Deser [9], Grishchuk [10], Feynman [11] and others in the case of GR. We start this program by a short review of the correspondent scheme of general relativity.

II. THE FIELD THEORY DESCRIPTION OF GENERAL RELATIVITY

In the geometric formulation of general relativity the metric of space-time \( g_{\mu \nu} \) satisfies the equation
\[ G_{\mu \nu} \equiv R_{\mu \nu} - \frac{1}{2} R g_{\mu \nu} = -\kappa T_{\mu \nu}. \quad (6) \]

There is another form to present such dynamics using a description in terms of a field theory embedded in an auxiliary flat space-time taken a priori as non-observable [11].

In this formulation, one starts by defining a symmetric second order tensor \( h_{\mu \nu} \), which lies on the Minkowski background \( \eta_{\mu \nu} \). Then the hypothesis is made that matter and energy of any kind interacts gravitationally only through the combination\(^2\) given by equation
\[ g^{\mu \nu} = \eta^{\mu \nu} + h^{\mu \nu}. \quad (7) \]

This binomial form is an exact expression for the metric \( g^{\mu \nu} \). Consequently the covariant tensor \( g_{\mu \nu} \) is an infinite series:
\[ g_{\mu \nu} = \eta_{\mu \nu} - h_{\mu \nu} + h_{\mu \alpha} h^{\alpha \nu} + ... \quad (8) \]

There are two main postulates founding general relativity:
- The background Minkowski metric is not observable. Matter and energy interact gravitationally only through the combination \( \eta^{\mu \nu} + h^{\mu \nu} \) and its derivatives. Any test body in a gravitational field moves along a geodesic relative to the metric \( g_{\mu \nu} \);
- The dynamics of gravity is described by an equation relating the contracted curvature tensor \( R_{\mu \nu} \) to the stress-energy tensor of matter.

In the next section, these postulates will be applied in the derivation of the field theory formulation of GR.

III. INTERPRETING THE NON-LINEAR TERMS OF THE DYNAMICS OF THE GRAVITATIONAL FIELD \( g_{\mu \nu} \) OF GENERAL RELATIVITY IN TERMS OF EQUIVALENT “ENERGY-MOMENTUM REPRESENTATION”

Using equations (7) and (8) we re-write the dynamics \[ \] as an infinite series
\[ G^L_{\mu \nu} = -\kappa \left( T_{\mu \nu} + t_{\mu \nu}^{(1)} + t_{\mu \nu}^{(2)} + t_{\mu \nu}^{(3)} + ... \right), \quad (9) \]
where \( G^L_{\mu \nu} \) is the linear part of the Einstein tensor \( G_{\mu \nu} \) and the quantities \( t_{\mu \nu}^{(n)} \) for \( n = 1, 2, 3, ... \) contains non-linear terms of the Ricci tensor developed in series of order \( h^n \). The tensor of matter is divergence free
\[ T_{\mu \nu} : \theta = 0. \]

Writing in a compact form
\[ G^L_{\mu \nu} = -\kappa \left( T_{\mu \nu} + t_{\mu \nu} \right), \]
we note that although the quantity \( t_{\mu \nu} \) appears as the non-linear source of the gravitational field \( h_{\mu \nu} \) it should not be identified with the gravitational energy, once it follows from this dynamics that it is not a conserved tensor in the metric \( g_{\mu \nu} \), once its divergence does not vanish
\[ t^{\mu \nu} : \theta \neq 0, \]
which is the origin of the main difficulties to undertake this field-theoretical path to construct a well-grounded energy-momentum tensor for the gravitational field within the GR theory.

\(^2\) Let us note that Grishchuk et al. [11] made a different choice and they use the definition not in terms of a tensorial equation, but defining the pseudo-tensor terms as
\[ \sqrt{-g} g^{\mu \nu} = \sqrt{-\eta} (\eta^{\mu \nu} + h^{\mu \nu}). \]
IV. THE FIELD THEORY DESCRIPTION OF THE GEOMETRICAL SCALAR GRAVITY

The dynamics of the GSG as proposed in \[7\] is given by the equation

\[
\sqrt{V} \, \Box \Phi = \kappa \chi, \tag{10}
\]

where \( V \) is given by equation \[10\] and the d’Alembert operator \( \Box \) is defined in the curved metric

\[
q^{\mu\nu} = \alpha \eta^{\mu\nu} + \beta \frac{\partial^{\mu} \Phi \partial^{\nu} \Phi}{w},
\]

that is

\[
\Box \Phi \equiv \frac{1}{\sqrt{-q}} \partial \left( \sqrt{-q} \, \partial_{\nu} \Phi q^{\mu\nu} \right). \tag{11}
\]

The determinant \( q \) of \( q_{\mu\nu} \) is given by

\[
\sqrt{-q} = \frac{\sqrt{-\eta}}{\alpha^{3/2} \sqrt{\alpha + \beta}}. \tag{12}
\]

The source \( \chi \) of the gravitational field was obtained in \[7\] and its explicit expression is written as

\[
\chi = \frac{1}{2} \left( \frac{3 e^{2\Phi} + 1}{3 e^{2\Phi} - 1} \right) E - T - \nabla_{\lambda} C^{\lambda}. \tag{13}\]

The quantities \( E, T \) and \( C^{\lambda} \) are given in terms of the energy-momentum tensor

\[
T \equiv T^{\mu\nu} q_{\mu\nu}, \quad E \equiv \frac{T^{\mu\nu} \partial_{\mu} \Phi \partial_{\nu} \Phi}{\Omega}
\]

and

\[
C^{\lambda} = \frac{\beta}{\alpha \Omega} \left( E q^{\lambda\mu} - T^{\lambda\mu} \right) \partial_{\mu} \Phi,
\]

where

\[
\Omega \equiv \partial_{\mu} \Phi \partial_{\nu} \Phi q^{\mu\nu} = \alpha^{3} V w.
\]

The energy-momentum tensor of matter is defined in the standard way

\[
T_{\mu\nu} = \frac{2}{\sqrt{-q}} \frac{\delta \left( \sqrt{-q} L_{\text{mat}} \right)}{\delta q^{\mu\nu}}. \tag{14}
\]

Equation \[10\] describes the dynamics of the GSG in presence of matter. The source of the gravity expressed by the quantity \( \chi \) involves a non-trivial coupling between the gradient of the scalar field \( \partial_{\mu} \Phi \) and the energy-momentum tensor of the matter field \( T_{\mu\nu} \) and not uniquely its trace, as the previous unsuccessful scalar theories of gravity \[12\].

In order to express this theory along the same lines as it was made in the case of general relativity, let us describe the dynamics of GSG following similar steps, according to Feynman’s approach, and try to describe the dynamics of the scalar field \( \Phi \) in a flat space-time endowed with the metric \( \eta_{\mu\nu} \). We start with the Lagrangian

\[
L = V(\Phi) w. \tag{15}
\]

Variation of \( L \) with respect to \( \Phi \) yields

\[
\delta \int \sqrt{-\eta} L = \int \sqrt{-\eta} \left( V \Box_{M} \Phi + \frac{1}{2} \frac{dV}{d\Phi} w \right) \delta \Phi.
\]

In this expression \( \Box_{M} \) represents the d’Alembert operator in the auxiliary Minkowski metric, defined as

\[
\Box_{M} \Phi = \frac{1}{\sqrt{-\eta}} \partial_{\mu} \left( \sqrt{-\eta} q^{\mu\nu} \partial_{\nu} \Phi \right).
\]

The dynamics issued from this Lagrangian is given by

\[
V \Box_{M} \Phi + \frac{1}{2} V' w = 0, \tag{16}
\]

where \( V' \equiv dV/d\Phi \). Using the identity

\[
\Box \Phi = (\alpha + \beta) \left( \Box_{M} \Phi + \frac{1}{2} V' w \right), \tag{17}
\]

where \( \Box \) is given by \[11\], it follows that the dynamics described by \[15\] is equivalent to the form

\[
\Box \Phi = 0. \tag{18}
\]

At this point, the geometrical scalar gravity takes a step beyond and following similar lines as in GR states the fundamental hypothesis that all forms of matter and energy interact with the gravitational field \( \Phi \) only through the combination of metric \( q_{\mu\nu} \) and its derivatives in a covariant way. In other words, given the Lagrangian of matter \( L_{\text{mat}} \) in the special relativistic theory then, in order to couple this matter to the gravitational field, one has only to use the minimal coupling principle and substitute the unobserved metric \( \eta_{\mu\nu} \) by the gravitational one \( q_{\mu\nu} \). Thus, using this principle one obtains the equation of motion that drives the effects of matter in the generation of a gravitational field, that is equation \[13\]. The final step to complete the theory is to fix the dependence of \( V \) on the coefficients \( \alpha \) and \( \beta \) and its relation with \( \Phi \). In the quoted paper we obtained the values displayed above in Eqs. \[3\], \[4\] and \[5\].

Then, inserting these expressions in the matter action \[10\] one obtains the form of \( \chi \) given above in Eq. \[10\].

Let us remark that only the trace \( T \) and the projections of \( T_{\mu\nu} \) onto the gradient of \( \Phi \) appear in this expression. This means that only five components from the ten contained in the energy-momentum tensor of matter appear in the dynamics. Nothing similar in GR once in this theory all components of \( T_{\mu\nu} \) are involved in the dynamics.
V. INTERPRETING THE NON-LINEAR TERMS OF THE DYNAMICS OF THE GRAVITATIONAL FIELD $\Phi$ IN TERMS OF EQUIVALENT “ENERGY-MOMENTUM REPRESENTATION”

In this section we will finish the task of achieving in the Geometric Scalar Gravity the equivalent result of Gupta-Feynman representation of GR as a field theory description in an unobservable background endowed with a Minkowski metric.

From the bridge relation (17), we rewrite the dynamics of $\Phi$ in terms of the auxiliary flat space-time. The dynamics of $\Phi$ in (10) takes the form

$$\square_M \Phi = \frac{\kappa \chi_{\text{mat}}}{(\alpha + \beta) \sqrt{V}} - \frac{V'}{2V} w$$  \hspace{2cm} (19)$$

where $\chi_{\text{mat}}$ represents the source matter terms. Our task then is to construct an associated energy-momentum tensor $\Theta_{\mu\nu}$ such that using the expression (19), we can rewrite the non-linear term of the r.h.s. of the above equation as a source of the field or, in other words, to choose $\Theta_{\mu\nu}$ such that the $\chi$–term associated to this tensor reproduces precisely the non-linear term that is

$$\frac{\kappa \chi_\theta}{(\alpha + \beta) \sqrt{V}} = - \frac{V'}{2V} w.$$  

A first guess is almost univocally determined by setting

$$\Theta_{\mu\nu} = a \partial_\mu \Phi \partial_\nu \Phi + b q_{\mu\nu},$$  \hspace{2cm} (20)$$

where $a$ and $b$ may depend on $\Phi$ and on its derivative $\Omega$. Let us prove that this is indeed possible. From the definitions of $E, T$ and $C_{\lambda \theta}$ a direct calculation yields

$$E_\theta = a \Omega + b; \quad T_\theta = a \Omega + 4b \quad \text{and} \quad (C^{\lambda \theta})_\theta = 0.$$  

Using these values on the expression of $\chi$ we obtain

$$\chi_\theta = \frac{1}{2(3 - \alpha)} [2 a \Omega a + (5 \alpha - 9) b].$$  \hspace{2cm} (21)$$

The question now is to find values of $a$ and $b$ such that allows for the identification:

$$- \frac{w' V'}{2V} = \frac{\kappa \chi_\theta}{(\alpha + \beta) \sqrt{V}}.$$  

Developing both sides of this equation we obtain the unique condition relating $a$ and $b$, that is:

$$b = \frac{2 \Omega}{9 e^{2\Phi} - 5} \left( a - \sqrt{V} (1 - 9 e^{2\Phi}) \right)$$  \hspace{2cm} (22)$$

A particular solution of this relation is given by the values

$$a = -4 \sqrt{V} \quad \text{and} \quad b = 2 \sqrt{V} \Omega.$$  

(In the appendix [A] we explain the origin of such choice.)

This ends the proof that it is possible to describe the dynamics of GSG through a similar procedure as it was made by GR, that is we can rewrite the gravitational dynamics (10) under the form

$$\square_M \Phi = \frac{\chi_{\text{mat}} + \chi_\theta}{(\alpha + \beta) \sqrt{V}},$$  \hspace{2cm} (23)$$

which is the version for GSG of the expression (9) of general relativity.

The first part of our task is finished. Let us now turn to a more ambitious question, that is, such symmetric second order tensor $\Theta_{\mu\nu}$ can be associated to the gravitational energy?

In the absence of matter we should expect that the true energy-momentum tensor of the gravitational field $\mu_{\nu}$ should be conserved, that is, we should have

$$t_{\mu\nu} = 0$$

where the covariant derivative, of course, being taken in the observable gravitational metric $q_{\mu\nu}$. In the case of general relativity is evident that the quantity $t_{\mu\nu}$ introduced by Feynman and others cannot represent a conserved energy tensor. What about the quantity $\Theta_{\mu\nu}$ of GSG? A direct calculation yields

$$\Theta_{\mu\nu} = (\partial_\nu a \Phi^\nu + a \square \Phi) \Phi_\mu + \frac{a}{2} \partial_\mu \Omega + \partial_\mu b.$$  \hspace{2cm} (24)$$

Note that the indexes of $\Theta_{\mu\nu}$ must be lowered and raised with the gravitational metric $q^{\mu\nu}$. Using the particular values of $a$ and $b$ displayed above, it follows (setting $\square \Phi = 0$)

$$\Theta_{\mu\nu}^\nu = - \frac{w'}{\sqrt{V}} \Omega \partial_\mu \Phi.$$  \hspace{2cm} (25)$$

We have pointed out that not all components of the energy-momentum tensor enter in the dynamics of $\Phi$. This property allows us to ask the following question: is it possible to add to this $\Theta_{\mu\nu}$ another term—call it $\Delta_{\mu\nu}$—such that it does not change the dynamics of $\Phi$ and generate a conserved quantity? Let us show that this is indeed possible$^3$.

The extra tensor $\Delta_{\mu\nu}$

From what we have shown in the precedent sections, we realize that there is a freedom on the expression of the gravitational energy-momentum tensor that allows one to envisage the possibility to add to the tensor $\Theta_{\mu\nu}$ another extra term $\Delta_{\mu\nu}$ that satisfies the two conditions:

$^3$ This is nothing but the freedom to add a total derivative on the Lagrangian. See the Appendix [A] for more details.
conditions. The associated expression $\chi_m$ and look for values of $m$ and $n$ that satisfy the above conditions. The associated expression $\chi_\Delta$ defined in terms of the quantities $E, T$ and $C^\Delta$ gives

$$E_\Delta = m \Omega + n; \quad T_\Delta = m \Omega + 4n \quad \text{and} \quad C^\Delta_\Delta = 0.$$  

Thus, using equation (28) it follows that such extra term does not produce any modification on the dynamics of $\Phi$ that is, $\chi_\Delta = 0$ is

$$\frac{3e^{2\Phi} + 1}{3e^{2\Phi} - 1} E_\Delta - T_\Delta = 0,$$  

which is satisfied by imposing the relation

$$n = \frac{2 \Omega}{9 e^{2\Phi} - 5} m. \quad (27)$$

Although the tensor $\Delta_{\mu \nu}$ does not produce any modification on the dynamics of $\Phi$, it can help in the construction of a conserved quantity that we will call the energy-momentum tensor of the gravitational field:

$$t^g_{\mu \nu} \equiv \Theta_{\mu \nu} + \Delta_{\mu \nu}.$$  

Indeed, in order to obtain the dynamics from the conservation $t^g_{\mu \nu, \nu} = 0$ the condition

$$(a + m) \nu \Phi^\nu \Phi_\mu + \frac{1}{2} (a + m) \Omega_\mu + (b + n)_{, \mu} = 0, \quad (28)$$

must be satisfied where $a, b, m$ and $n$ are related by the two equations (22) and (27).

In the particular case of a static and spherically symmetric configuration admitting a time-like Killing vector where the gravitational field does not depend on time, the general expression

$$t^g_{\mu \nu} = (a + m) \Phi_\mu \Phi_\nu + (b + n) q_{\mu \nu}$$

provides the total energy of the gravitational field.

We note that an observer endowed with a four-velocity $v^\mu$ call density of energy $\mathcal{E}_g$ is given by the projection $\mathcal{E}_g = t_{\mu \nu} v^\mu v^\nu$. Choose the normalized four-vector $v^\mu = \sqrt{\alpha} \delta_0^\mu$ to obtain

$$\mathcal{E}_g = b + n, \quad (29)$$

where the density of energy is given by solutions of the equation (28) that in this case reduces to

$$\partial_\lambda \left( \frac{9 e^{2\Phi} - 5}{2 \Omega} \mathcal{E}_g + \frac{1}{\kappa} \sqrt{\lambda} (1 - 9 e^{2\Phi}) \right) \Phi_\lambda \Phi_\mu + \frac{1}{2} \left( \frac{9 e^{2\Phi} - 5}{2 \Omega} \mathcal{E}_g + \frac{1}{\kappa} \sqrt{1 - 9 e^{2\Phi}} \right) \Omega_\mu + \partial_\mu \mathcal{E}_g = 0.$$  

Let us now give a specific example of this formula for the case of a star.

VI. THE GRAVITATIONAL ENERGY OF A STAR

In the gravitational field of a spherically symmetric and static configuration was calculated. The gravitational metric associated to this configuration has the same expression as in the Schwarzschild solution of general relativity. This means that the geometry constructed with $q_{\mu \nu}$ in the case that $\Phi = \Phi(r)$ given by

$$ds^2 = \left(1 - \frac{r_H}{r}\right) dt^2 - \left(1 - \frac{r_H}{r}\right)^{-1} dr^2 - r^2 d\Omega^2 \quad (30)$$

is a solution of both dynamics, that is, the vacuum equations of general relativity $R_{\mu \nu} = 0$ and also of the geometric scalar theory, that is

$$\Box \Phi = 0, \quad (31)$$

where the covariant derivatives are taken, obviously, in the metric (30). The gravitational field, satisfying (31) is

$$\Phi = \frac{1}{2} \ln \left(1 - \frac{r_H}{r}\right). \quad (32)$$

Let us now evaluate the gravitational energy of a star according to the analysis presented in the previous section.

The first step is to know the value of tensor $t_{\mu \nu} = \Theta_{\mu \nu} + \Delta_{\mu \nu}$. In the present case a simplification appears due to the fact that the norm $\Omega$ can be written as an algebraic expression of the scalar field. Indeed, a direct calculation gives

$$\Omega = - \frac{1}{4 r_H^2} \frac{(\alpha - 1)^3}{\alpha^3},$$

where $\alpha = (1 - r_H/r)^{-1}$. Using this property into equation (28), we can evaluate the gravitational energy density of a star by the formula $\mathcal{E}_g = b + n$. From the condition of conservation (28), we have

$$\frac{d(a + m)}{d\alpha} \Omega + \frac{(a + m)}{2} \frac{d\Omega}{d\alpha} + \frac{d(b + n)}{d\alpha} = 0.$$  

Now, from the relation between $a + m$ and $b + n$ it follows

$$a + m = \left( \frac{9 - 5\alpha}{2 \Omega} \right) (b + n) + \frac{\sqrt{\lambda} (\alpha - 9)}{\kappa \alpha}.$$  

Using these results it implies that the gravitational energy density of a star satisfies the linear differential equation

$$\frac{d\mathcal{E}_g}{d\alpha} - \frac{5 \alpha + 3}{6 \alpha (\alpha - 1)} \mathcal{E}_g + P(\alpha) = 0, \quad (33)$$

where

$$P(\alpha) = \frac{1}{6 \kappa r_H^2} \frac{(\alpha - 1)^3 (7 \alpha^2 - 45 \alpha + 54)}{|\alpha - 3|^{11/2}}.$$
This equation can be integrated analytically and all details are presented in Appendix [13]. It is important to remark that we have to choose the integration constant in order to set $E_g$ equal to zero when $r$ goes to infinity.

From the conservation of $t^\mu{}^\nu$ it follows that

$$E_g = \int t^{\mu^0} k_\mu \sqrt{-q} d^3 x$$

is a constant, where $k_\mu = (1/\alpha, 0, 0, 0)$ is the time-like Killing vector of the Schwarzschild solution. Then, we obtain the gravitational energy $E_g$ for the static configuration [33] by integrating the energy density in the whole spacetime volume where the Killing vector is well defined, that is, from the event horizon $r_H$ up to the spatial infinity

$$E_g = 4\pi \int_{r_H}^{\infty} E_g(r) r^2 dr, \quad (34)$$

where $E_g$ is solution of equation [33]. However, this integral cannot be done analytically and, therefore, we appeal to numerical methods for the calculation of the total energy. Using the open-source SciPy© [13], we obtain $E_g > 0.8 M c^2$ as an estimation for the integral [34].

VII. CONCLUSION

The formula of $t^\mu{}^\nu$ proposed here is a consequence of the description of the gravitational field in terms of the unique scalar function $\Phi$. We used the field theoretical approach as it was done in the case of General Relativity by Feynman et al., once it seems that it is the most direct way to bypass the difficulties invoked by many authors based on the equivalence principle. We have shown that it is possible to define a true energy-momentum tensor $t^\mu{}^\nu$ whose conservation implies the dynamical equations of the field. In this paper we only start the program to examine the behavior of this quantity in stars. There remains the task to analyze further the complete properties of $t^\mu{}^\nu$, and to apply it in others configurations. This is a matter for future work.

Appendix A: Particular choice of $a$ and $b$ from the variational principle

Let us explain here the origin of the particular values of $a$ and $b$ proposed in the text to specify the tensor $\Theta_{\mu\nu}$.

Start by re-calling the action

$$I = \frac{1}{\kappa} \int \sqrt{-\eta} V w$$

The dynamics of $\Phi$ is obtained directly by varying $I$. However this procedure can be realized in two stages: varying the action with respect to the gravitational metric $q_{\mu\nu}$ it gives which components of the matter tensor will be taken into account in the dynamics, obtaining Eq. [13], and then, varying $q_{\mu\nu}$ with respect to $\Phi$ we get the dynamics [10].

Let us use the definition of the gravitational metric $g_{\mu\nu}$ and rewrite this action in the equivalent manner

$$I = \frac{1}{\kappa} \int \sqrt{-q} \sqrt{\eta} \partial_\mu \Phi \partial_\nu \Phi q^{\mu\nu} \quad (A1)$$

The dynamics of $\Phi$ is obtained by variation of $I$ with respect to arbitrary variations $\delta q$. In a first step we vary with respect to $\delta q^{\mu\nu}$. We thus obtain the intermediary variation

$$\delta I = \frac{1}{\kappa} \int \sqrt{-q} \sqrt{\eta} \left( \partial_\mu \Phi \partial_\nu \Phi - \frac{1}{2} \Omega q_{\mu\nu} \right) \delta q^{\mu\nu}.$$  

This suggests to define the associated $\Theta_{\mu\nu}$ using equation [14] up to a factor 2:

$$\Theta_{\mu\nu} = \frac{1}{\sqrt{-q}} \frac{\delta (\sqrt{-q} \sqrt{\eta} \partial_\mu \Phi \partial_\nu \Phi q^{\mu\nu})}{\delta q^{\mu\nu}}.$$  

It then follows

$$\Theta_{\mu\nu} = -4 \sqrt{\eta} \left( \partial_\nu \Phi \partial_\nu \Phi - \frac{1}{2} \Omega q_{\mu\nu} \right). \quad (A2)$$

This expression yields the values of $a$ and $b$ that we used in the original expression of $\Theta_{\mu\nu}$.

Appendix B: The energy density of a compact object

We shall give the steps to solve the differential equation [33] for the energy density. Due to the term $|\alpha - 3|$ in the function $P(\alpha)$, we separate $E_g$ in two regimes: $E_{g,1}$ for $1 < \alpha < 3$ and $E_{g,2}$ for $\alpha > 3$. The general solution of [33] is easily obtained if we do the change of variables $z = \sqrt{\alpha - 1}$. Therefore, for $0 < z < \sqrt{2}$, we have
\[ \mathcal{E}_{g,1}(z) = -\frac{1}{6Kr_H} \left[ -\frac{9z^2}{2(z^3+1)^2} + \frac{54z^2}{(z^3+1)^3} - \frac{6z^2}{9(z^3+1)} + \frac{7\sqrt{3}}{27} \arctan \left( \frac{-1+2z\sqrt{3}}{\sqrt{3}} \right) + \right. \\
\left. \frac{-2^{8/3}\sqrt{3}}{27} \arctan \left( \frac{1+2^{2/3}z}{\sqrt{3}} \right) - \frac{7}{27} \ln(1+z) - \frac{2^{8/3}}{27} \ln(2-2^{2/3}z) + \frac{7}{94} \ln(1-z+z^2) + \right. \\
\left. \frac{2^{8/3}}{27} \ln(2+2^{2/3}z+2^{1/3}z^2) - \frac{7\sqrt{3}}{27} \arctan \left( \frac{\sqrt{3}}{3} \right) + \frac{\sqrt{3}2^{8/3}}{27} \arctan \left( \frac{\sqrt{3}}{3} \right) \right] z^4(z^3+1)^{-1/2}. \]

where the constant of integration is chosen in such a way that \( \mathcal{E}_{g,1} \) vanishes when \( z \) goes to zero (which means \( r \to \infty \)). For \( z > \sqrt{2} \), we obtain that \( \mathcal{E}_{g,2} = -\mathcal{E}_{g,1} \). Finally, the total energy in given by

\[ E = 12\pi \int_0^{\sqrt{2}} \mathcal{E}_g(z) \frac{(z^3+1)^2}{z^{10}} \, dz = 12\pi \left( \int_0^{\sqrt{2}} \mathcal{E}_{g,1}(z) \frac{(z^3+1)^2}{z^{10}} \, dz + \int_{\sqrt{2}}^{\infty} \mathcal{E}_{g,2}(z) \frac{(z^3+1)^2}{z^{10}} \, dz \right). \]

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[12] We note that distinct from GR in the Scalar Gravity only five components of the energy-momentum of matter enter in the dynamics of the gravitational field. We used this property to re-write the dynamics of GSG in terms of a field theory in the auxiliary Minkowski background along the same lines as it was undertaken by Gupta, Feynmann and others in the realm of GR.
[13] Some parts of the numerical integration were also verified making use of Maple™ and Mathematica™.