CENTRAL EXTENSIONS OF LAX OPERATOR ALGEBRAS

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Introduction

- algebras of current type, only recently introduced
- generalisations of affine Lie algebras of Krichever-Novikov type – they are generalisations of classical affine Lie algebras
- related to integrable systems
- related to the moduli space of bundles over compact Riemann surfaces

**Goal:** Classify almost-graded central extensions of these algebras

**Joint work with Oleg Sheinman** (appeared in Russ. Math. Surveys 63(4), 727-766 (2008))
Geometric Set-Up

\( \Sigma \) a compact Riemann surface, \\
\( A = I \cup O \) disjoint union of finitely many points, \( I \) and \( O \) non-empty (here only \( I = \{ P_+ \} \) and \( O = \{ P_- \} \))

**Tyurin data:** \( n \cdot g \) points \( (n \in \mathbb{N}, \ g \text{ genus of } \Sigma) \)

\[
W := \{ \gamma_s \in \Sigma \setminus \{ P_+, P_- \} \mid s = 1, \ldots, ng \}.
\]

\( \gamma_s \mapsto \alpha_s \in \mathbb{C}^n, \quad T := \{ (\gamma_s, \alpha_s) \in \Sigma \times \mathbb{C}^n \mid s = 1, \ldots, ng \} \)

relation to the moduli space of semi-stable framed algebraic vector bundles of rank \( n \) and degree \( n \cdot g \)

fix local coordinates \( z_{\pm} \) at \( P_{\pm} \) and \( z_s \) at \( \gamma_s, s = 1, \ldots, ng \)
Algebraic Set-Up

$\mathfrak{g}$ be one of the matrix algebras $\mathfrak{gl}(n)$, $\mathfrak{sl}(n)$, $\mathfrak{so}(n)$, $\mathfrak{sp}(2n)$, or $\mathfrak{s}(n)$ (the algebra of scalar matrices)

Consider meromorphic functions (more precisely trivialisations of sections of a bundle)

$L : \Sigma \rightarrow \mathfrak{g},$

which are

1. holomorphic outside $W \cup \{P_+, P_-\},$
2. have at most poles of order one (resp. of order two for $\mathfrak{sp}(2n)$) at the points in $W,$
3. and fulfill certain conditions at $W$ depending on $T$ and $\mathfrak{g}.$
The singularities at $W$ are called weak singularities.

What are the additional properties? (Here only for $\text{gl}(n)$)

For $s = 1, \ldots, ng$ there exist $\beta_s \in \mathbb{C}^n$ and $\kappa_s \in \mathbb{C}$ such that we get the expansion at $\gamma_s \in W$

$$L(z_s) = \frac{L_{s,-1}}{z_s} + L_{s,0} + \sum_{k>0} L_{s,k} z_s^k$$

with

$$L_{s,-1} = \alpha_s \beta_s^t, \quad \text{tr}(L_{s,-1}) = t \beta_s \alpha_s = 0, \quad L_{s,0} \alpha_s = \kappa_s \alpha_s.$$

In particular, if $\alpha_s \neq 0$ $L_{s,-1}$ is a rank 1 matrix, and $\alpha_s$ is an eigenvector of $L_{s,0}$. 
\textbf{Algebraic Set-Up}

$\mathfrak{sl}(n)$ matrices are trace-less
$\mathfrak{s}(n)$ matrices are scalar matrices
$\mathfrak{so}(n)$ and $\mathfrak{l}(n)$ matrices of the corresponding type, with modified additional conditions.

\textbf{Theorem}

Under the pointwise matrix commutator these objects constitute a Lie algebra, denoted by $\overline{\mathfrak{g}}$ if the finite Lie algebra is denoted by $\mathfrak{g}$. 
If all $\alpha_s = 0$ classical KN current algebras.
If $g = 0$ then classical current algebras.

$\mathcal{A}$ associative algebra of meromorphic functions on $\Sigma$
holomorphic outside of $\mathcal{A}$

$\mathcal{L}$ Lie algebra of meromorphic vector fields on $\Sigma$
holomorphic outside of $\mathcal{A}$

classical KN current algebra:

$$\bar{g} = g \otimes \mathcal{A}, \quad [x \otimes f, y \otimes g] := [x, y] \otimes fg$$

$g = 0, \Sigma = \mathbb{P}^1(\mathbb{C}),$ points $0, \infty,$
$\mathcal{A} = \mathbb{C}[z, z^{-1}], \quad \bar{g} = g \otimes \mathbb{C}[z, z^{-1}].$
Almost-graded structure

Grading is important for infinite dimensional Lie algebras but a weaker concept almost-grading will do.

**Definition**

$\mathcal{V}$ an arbitrary Lie algebra is called almost-graded if

1. $\mathcal{V} = \bigoplus_{n \in \mathbb{Z}} \mathcal{V}_n$, dim $\mathcal{V}_n < \infty$ as vector space
2. There exists $L_1, L_2 \in \mathbb{Z}$ such that

$$[\mathcal{V}_n, \mathcal{V}_m] \subseteq \bigoplus_{h=n+m+L_1}^{n+m+L_2} \mathcal{V}_h, \quad \forall n, m$$

$\mathcal{A}, \mathcal{L}$, and the current algebras of KN type are almost-graded.
Theorem

\( \bar{g} \) is almost-graded, i.e. \( \bar{g} = \bigoplus \bar{g}_m \), \( \dim \bar{g}_m = \dim g \), and

\[
[\bar{g}_m, \bar{g}_n] \subseteq \bigoplus_{h=m+n} \bar{g}_h \]

The generic bound is \( M = g \), the genus of \( \Sigma \).

Given \( X \in \bar{g} \): there exists a unique \( X_m \in \bar{g}_m \) such that \( X_m = Xz^m_z + O(z^{m+1}_z) \).

Classical situation: we get the well-known grading
Goal: Construct and classify central extensions of the Lax operator algebras

Why: Needed by the applications, like regularisation, 2nd quantization, etc.

Mathematical back-ground: by regularisation we obtain only projective action of \( \bar{g} \), they correspond to linear actions of a central extension \( \hat{g} \)

Strictly speaking: from these application we need only central extensions of \( \bar{g} \) which allow to extend the almost-grading to \( \hat{g} \).
Central extensions

How are central extensions constructed?

\( \hat{g} = \tilde{g} \oplus \mathbb{C} t \) as vector space (\( t \) is the central element)

\[
[\hat{L}_1, \hat{L}_2] = [\tilde{L}_1, \tilde{L}_2] + \psi(L_1, L_2)t
\]

\( \hat{g} \) is a Lie algebra if and only if \( \psi \) is a Lie algebra 2-cocycle, i.e.

1. \( \psi \) is antisymmetric
2. \( \psi([L_1, L_2], L_3) + \psi([L_2, L_3], L_1) + \psi([L_3, L_1], L_2) = 0. \)

Two different central extensions are equivalent iff difference of the two 2-cocycles is a coboundary (\( \phi \) a linear form)

\[
\psi_1(L_1, L_2) - \psi_2(L_1, L_2) = \phi([L_1, L_2])
\]
Central extensions

Hence, we need 2-cocycles

For current type KN algebras: \((x, y \in g, g, h \in \mathcal{A})\)

\[ \psi(x \otimes g, y \otimes h) = \langle x, y \rangle \int_C gdh. \]

\(\langle ., . \rangle\) invariant symmetric bilinear form,
\(C\) a closed contour on \(\Sigma \setminus \mathcal{A}\).

For Lax operator algebras we do not have such a splitting
our functions are not really functions but sections,
before defining a differentiation we need to choose a connection.
The connection $\nabla^{\omega}$ is defined with the help of $\omega$

1. a $g$-valued meromorphic 1-form
2. holomorphic outside of $A$ and $W$
3. obey certain conditions at the weak singularity points:
   - points $\gamma_s \in W$ with $\alpha_s = 0$: $\omega$ is regular there
   - points $\gamma_s$ with $\alpha_s \neq 0$: the expansion

$$
\omega(z_s) = \left( \frac{\omega_{s,1}}{z_s} + \omega_{s,0} + \sum_{k \geq 1} \omega_{s,k} z_s^k \right) dz_s.
$$

there exist $\tilde{\beta}_s \in \mathbb{C}^n$ and $\tilde{\kappa}_s \in \mathbb{C}$ such that

$$
\omega_{s,-1} = \alpha_s^t \tilde{\beta}_s, \quad \omega_{s,0} \alpha_s = \tilde{\kappa}_s^t \alpha_s, \quad \text{tr}(\omega_{s,-1}) = t \tilde{\beta}_s \alpha_s = 1.
$$
Such $\omega$ exist we can choose an $\omega$ holomorphic at $P_+$

$$\nabla^{(\omega)} = d + [\omega, .]$$

covariant derivative

$$\nabla^{(\omega)}_e = dz(e) \frac{d}{dz} + [\omega(e), .], \quad e \in \mathcal{L}$$

**Theorem**

*The covariant derivative makes $\bar{\mathfrak{g}}$ to an almost-graded Lie module over $\mathcal{L}$.*
Define
\[ \gamma_{1,\omega,C}(L, L') = \frac{1}{2\pi i} \int_C \text{tr}(L \cdot \nabla^{(\omega)} L'), \quad L, L' \in \bar{g}, \]
and
\[ \gamma_{2,\omega,C}(L, L') = \frac{1}{2\pi i} \int_C \text{tr}(L) \cdot \text{tr}(\nabla^{(\omega)} L'), \quad L, L' \in \bar{g}. \]
Indeed these are cocycles \( \gamma_{2,\omega,C} \) does not depend on \( \omega \), vanishes for \( g \neq \mathfrak{gl}(n), \mathfrak{s}(n) \)
\( \gamma_{1,\omega,C} \) for different \( \omega \) are cohomologous
cocycles depend on the integration path
**Geometric Cocycles**

**Definition**

A cocycle $\gamma$ for $\bar{g}$ is called $\mathcal{L}$-invariant (with respect to $\omega$) if

$$\gamma(\nabla^{(\omega)}_e L, L') + \gamma(L, \nabla^{(\omega)}_e L') = 0, \quad \forall e \in \mathcal{L}, \quad \forall L, L' \in \bar{g}.$$  

**Definition**

A cocycle $\gamma$ for $\bar{g}$ is called local if there exists $M_1, M_2 \in \mathbb{Z}$ such that for all $n, m$

$$\gamma(\bar{g}_m, \bar{g}_n) \neq 0, \quad \implies \quad M_1 \leq n + m \leq M_2.$$  

Almost-grading can be extended to the central extension if and only if the defining cocycle is local.
For cohomology classes use the definition if one representative is of this type. **Warning:** not all elements in the class of certain type are of this type.

**Theorem**

The cocycles $\gamma_1, \omega, c$ and $\gamma_2, c$ are $\mathcal{L}$-invariant.

Locality is in general not true. Essentially different integration cycles yield essentially different 2-cocycle classes $\implies$ a lot of non-equivalent central extensions appear but, denote by $C_S$ an integration cycle separating the point in $I$ from the points in $O$. 
THEOREM

The cocycles $\gamma_1, \omega, C$ and $\gamma_2, C$ with integration over a separating cycle $C = C_S$ are local.

Question: is the opposite true?
Essentially uniqueness of almost-graded central extensions
Answer:
In the simple case: yes
In the $\mathfrak{gl}(n)$ case: we have to add $\mathcal{L}$-invariance and then obtain a two-dimensional family of central extensions.

$$\gamma_1, \omega := \gamma_1, \omega, C_S, \quad \gamma_2 := \gamma_2, C_S$$
Main Results

**Theorem**

If \( \mathfrak{g} \) is simple (i.e. \( \mathfrak{g} = \mathfrak{sl}(n), \mathfrak{so}(n), \mathfrak{sp}(2n) \)) then the space of local cohomology classes is one-dimensional. The space will be generated by the class of \( \gamma_1, \omega \). Every \( \mathcal{L} \)-invariant local cocycle is a scalar multiple of \( \gamma_1, \omega \).

**Theorem**

For \( \overline{\mathfrak{g}} = \overline{\mathfrak{gl}(n)} \) the space of local cohomology classes which are \( \mathcal{L} \)-invariant having been restricted to the scalar subalgebra is two-dimensional. The space will be generated by the classes of the cocycles \( \gamma_1, \omega \) and \( \gamma_2 \). Every \( \mathcal{L} \)-invariant local cocycle is a linear combination of \( \gamma_1, \omega \) and \( \gamma_2 \).
Some words on the proof

- start with local and \( \mathcal{L} \)-invariant cocycle
- use almost-graded structure to show that everything can be reduced to level zero – \( \gamma(L_n, L_m) \) is of level \( n + m \)
- reduced means: \( \gamma(L_n, L_m) = 0 \) if \( n + m > 0 \) and is fixed by knowing the values of \( \gamma \) at level zero
- Hence, we only have to show that at level zero it is of the required form.
- Now: we have to get rid of the condition of \( \mathcal{L} \)-invariance
- abelian part it is o.k. as there we put it into the requirements
Some words on the proof

- **simple part**: we show that in every class there is a $\mathcal{L}$-invariant representative.
- for this: consider the Chevalley generators of the finite-dimensional simple Lie algebra.
- use almost-gradedness inside of $\bar{g}$ and boundedness from above of the cocycle to make cohomologous changes - stay in the same class.
- show that for the modified cocycle everything depends on one cocycle value evaluated for a fixed pair of elements.
- hence, the cohomology space is at most one-dimensional.
- $\gamma_1,\omega$ is a local cocycle which is not a coboundary, hence it is a generator and the space is one-dimensional.
- but $\gamma_1,\omega$ is also $\mathcal{L}$-invariant.
- gives the proof.
Remark:

For the abelian part $\mathcal{L}$-invariance is really needed. Otherwise, uniqueness can never be true.

Coming from applications (e.g. regularisation of fermionic Fock space representations) $\mathcal{L}$-invariance of the defining 2-cocycle is very often automatic.

Reason is that the representation there is in fact a representation of an by the vector field augmented algebra.
First part (start with $\mathcal{L}$-invariance):

$e_p \in \mathcal{L}$, $e_p = z^{p+1}_+ \frac{d}{dz_+}$ of degree $p$

$L^r_m, L^s_n \in \bar{g}$ of degree $m$ and $n$

recall that for $X \in \mathfrak{g}$ we have a unique $X_m \in \bar{g}_m$ with $X_m = Xz^m_+ + O(z^{m+1}_+)$

almost-graded action:

$\nabla e_p L^r_m = mL^r_{p+m} + L'$ with $L'$ of higher order

the $\mathcal{L}$-invariance

$$\gamma(\nabla e_p L^r_m, L^s_n) + \gamma(L^r_m, \nabla e_p L^s_n) = 0$$

implies

$$m\gamma(L^r_{p+m}, L^s_n) + n\gamma(L^r_m, L^s_{n+p}) = \text{cocycle value at higher level}$$
In particular, for $p = 0$

$$(m + n)\gamma(L_m^r, L_n^s)$$

is of higher level

This shows that for $(m + n) \neq 0$ the value is given by higher level values.

Hence, everything reduces to level zero.

Further analysis shows

$$\gamma(L_0^r, L_0^s) = 0, \quad \gamma(L_n^r, L_{-n}^s) = n\gamma(L_1^r, L_{-1}^s) + \text{higher level}.$$  

Hence, $\gamma(L_1^r, L_{-1}^s)$ fixes everything.
Some more details

Given $\gamma$ consider the map

$$\psi_{\gamma} : g \times g \to \mathbb{C}, \quad \psi_{\gamma}(X, Y) = \gamma(X_1, Y_{-1}).$$

$\psi_{\gamma}$ is a symmetric, invariant bilinear form on $g$. For $g$ simple it is a multiple of the Cartan-Killing form.

For $\mathfrak{gl}(n) = \mathfrak{s}(n) \oplus \mathfrak{sl}(n)$ the cocycle splits.

As $\mathfrak{s}(n) \cong \mathcal{A}$ we can use an earlier result of mine on the uniqueness of $\mathcal{L}$-invariant cocycles for the abelian part.
for $\mathfrak{g}$ simple in every class there is an $\mathcal{L}$ invariant one:

$E^\alpha, E^{-\alpha}, H^\alpha$ Chevalley generators of $\mathfrak{g}$

Chevalley-Serre relations for the finite-dimensional $\mathfrak{g}$.

these structure equations are also structure equations in $\bar{\mathfrak{g}}$ modulo higher level terms (comes from the almost-graded structure)
by almost-gradedness and boundedness of the cocycle we can change the cocycle in a cohomologous way such that finally zero is an upper bound for nonvanishing level and that for the level $< 0$ everything is fixed by level zero

this modified cocycle is called normalized cocycle

at level zero all cocycle values (normalized) can be expressed in relation to the cocycle value $\gamma(H^{\alpha}_1, H^{\alpha}_{-1})$ for a single fixed simple root

hence up to coboundary all local cocycles are multiples of a single one