VECTOR COHERENT STATES ON CLIFFORD ALGEBRAS

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Abstract. The well-known canonical coherent states are expressed as an infinite series in powers of a complex number \( z \) and a positive sequence of real numbers \( \rho(m) = m! \). In this article, in analogy with the canonical coherent states, we present a class of vector coherent states by replacing the complex variable \( z \) by a real Clifford matrix. We also present another class of vector coherent states by simultaneously replacing \( z \) by a real Clifford matrix and \( \rho(m) \) by a real matrix. As examples, we present vector coherent states on quaternions and octonions with their real matrix representations.

1. Introduction

Coherent states (CS for short) form an overcomplete family of vectors in a Hilbert space. Typically, CS are quantum states and known to describe quantum phenomenon favorably. In particular, CS provide a mathematical tool to bring a close connection between classical and quantum formalisms. The conventional CS can be defined in a number of ways [1]. One way to define a set of CS is as follows.

Definition 1.1. Let \( \mathcal{H} \) be a Hilbert space with an orthonormal basis \( \{ \phi_m \}_{m=0}^\infty \) and let \( \mathbb{C} \) be the complex plane. For \( z \in D \), an open subset of \( \mathbb{C} \), the states

\[
|z\rangle = N(|z|)^{-1/2} \sum_{m=0}^\infty \frac{z^m}{\sqrt{\rho(m)}} \phi_m
\]

are said to form a set of CS if

(a) The states \( |z\rangle \) are normalized, that is, \( \langle z | z \rangle = 1 \),

(b) The states \( |z\rangle \) give a resolution of the identity, that is,

\[
\int_D |z\rangle W(|z|) \langle z | d\mu = I,
\]

where \( N(|z|) \) is a normalization factor, \( \{ \rho(m) \}_{m=0}^\infty \) is a sequence of nonzero positive real numbers, \( W(|z|) \) is a positive weight function, \( d\mu \) is an appropriately chosen measure on \( D \), and \( I \) is the identity operator on \( \mathcal{H} \).

In [8] vector coherent states (VCS for short) were presented in the form (1.1) by replacing the complex number \( z \) by an \( n \times n \) matrix \( Z = A(r)e^{i\Theta(k)} \) with the conditions

\[
A(r)A(r)^\dagger = A(r)^\dagger A(r),
\]

\[
\Theta(k)^\dagger = \Theta(k), \quad \text{and} \quad \Theta(k)^2 = I_n,
\]

\[
[A(r), \Theta(k)] = 0
\]
imposed on the $n \times n$ matrices $A(r)$ and $\Theta(k)$, where the superscript $\dagger$ stands for the complex conjugate transpose of a matrix, and $I_n$ is the $n \times n$ identity matrix. As an example, VCS were constructed from the complex representation of the quaternions. Further, these VCS, stemming from the complex matrix representation of quaternions, were analyzed to a certain extent.

In this article, we present VCS on real Clifford algebras by considering their real matrix representations. Here, we do not consider the matrix in the form $Z = A(r)e^{i\zeta}\Theta(k)$ and we do not impose any conditions on the matrices, but, in addition to the Clifford parameters, we introduce a new parameter $\theta$ whose role in the construction will be made clear later.

2. On a General Clifford Algebra

Let $X$ be a real linear space and $(X, f)$ be a real quadratic space. By this, we mean that

\[(2.1) \quad f = \langle \cdot | \cdot \rangle : X \times X \to \mathbb{C}, \quad (x, y) \mapsto f(x, y)\]

is the usual inner product on $X$. Now, the Clifford algebra of $(X, f)$ is a pair $(C(f), \Theta)$, where $C(f)$ is a $\mathbb{R}$-algebra and

\[(2.2) \quad \Theta : X \to C(f), \quad x \mapsto \Theta(x)\]

is a linear function such that

\[(2.3) \quad \Theta(x)\Theta(x)^T = \Theta(x)^T\Theta(x) = f(x, x)I_n = \|x\|^2I_n, \quad \forall \ x \in X,\]

where we have taken an $n \times n$ matrix representation of $C(f)$ by real matrices satisfying (2.3) and denoted these matrices by the same symbol as the elements of the algebra $C(f)$. For each $z \in S^1$, the unit circle, we define a new linear function $\Theta_z(x) = z\Theta(x)$. The new function satisfies the relation

\[(2.4) \quad \Theta_z(x)\Theta_z(x)^\dagger = \Theta_z(x)^\dagger\Theta_z(x) = f(x, x)I_n = \|x\|^2I_n, \quad \forall \ x \in X.\]

Therefore, for each $z \in S^1$, the pair $(C(f), \Theta_z)$ is again a Clifford algebra. Further, it can be noticed that, for any two different $z \in S^1$, the corresponding Clifford algebras are isomorphic.

Let $\chi^1, \ldots, \chi^n$ be an orthonormal basis of $\mathbb{C}^n$, and let $\{\phi_m\}_{m=0}^\infty$ be an orthonormal basis of an arbitrary Hilbert space $\mathfrak{H}$. With these considerations, we define on $\mathbb{C}^n \otimes \mathfrak{H}$ the states

\[(2.5) \quad |\Theta_z, j\rangle = |\Theta_z)^j = \mathcal{N}^{-1/2}(\|x\|) \sum_{m=0}^\infty \frac{\Theta_z(x)^m}{\rho(m)} \chi^j \otimes \phi_m, \quad j = 1, 2, \ldots, n,\]

where the normalization constant $\mathcal{N}$ and $\rho(m)$ have to be identified so that the states (2.5) satisfy the normalization condition and a resolution of the identity.

2.1. Normalization and Resolution of the Identity. From (2.4) we have that

\[(\Theta_z(x)^j)^m\Theta_z(x)^m = \|x\|^{2m}I_n.\]

Let us make the following identification

\[(2.6) \quad \| \cdot \| : X \to \mathbb{R}_+, \quad \|x\| \equiv t,\]

and set

\[(2.7) \quad x_m = \frac{\rho(m)}{\rho(m-1)}, \quad \forall \ m \geq 1.\]
We can then define
\[ x_m! = \rho(m), \quad m \geq 1 \]
\[ x_0! = 1. \]

Assuming that
\[ \lim_{m \to \infty} x_m = R, \]
the set of labels becomes
\[ D = \{(t, \theta) | 0 \leq t < L, \ 0 \leq \theta \leq 2\pi\}, \quad l = \sqrt{R}, \]
with the measure \( d\mu = tdtd\theta \), where \( d\theta \) is a measure on \( S^1 \), whose elements are parametrized as \( e^{i\theta}, \ \theta \in [0, 2\pi] \). We can now state the following

**Theorem 2.1.** The states in (2.5) are normalized in the sense that
\[ \sum_{j=1}^{n} \langle \Theta_z, j | \Theta_z, j \rangle = 1, \]
and they achieve a resolution of the identity
\[ \int_0^L \int_0^{2\pi} w(t) \sum_{j=1}^{n} |\Theta_z, j\rangle\langle \Theta_z, j | tdtd\theta = \mathbb{I}_n \otimes \mathbb{I}, \]
where
\[ N(t) = n \sum_{m=0}^{\infty} \frac{t^{2m}}{x_m!}, \]
\[ w(t) = \frac{N(t)}{2\pi} \lambda(t), \]
and \( \lambda(t) \) is to be chosen from the moment problem
\[ \int_0^L \lambda(t)t^{2n+1}dt = x_m!. \]

**Proof.** It is straightforward that, for \((t, \theta) \in D\), the normalization condition reads
\[ \sum_{j=1}^{n} \langle \Theta_z, j | \Theta_z, j \rangle = N(t)^{-1} \sum_{j=1}^{n} \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \frac{1}{\rho(m)\rho(l)} \langle \Theta_z(x)^{l} \Theta_z(x)^{m} | \chi^j \rangle \langle \phi_m, \phi_l \rangle \]
\[ = N(t)^{-1} \sum_{j=1}^{n} \sum_{m=0}^{\infty} \frac{1}{\rho(m)} t^{2m} \langle \chi^j | \chi^j \rangle = nN(t)^{-1} \sum_{m=0}^{\infty} \frac{t^{2m}}{\rho(m)} \]
\[ = 1, \]
that is,
\[ N(t) = n \sum_{m=0}^{\infty} \frac{t^{2m}}{x_m!}. \]
In the case, \( x_m = m \), for all \( m \), \( N(t) = n e^{t^2} \).
Let us turn now to the condition leading to a resolution of the identity. We have
that
\[
\int_0^L \int_0^{2\pi} w(t) \sum_{j=1}^n | \Theta_z, j \rangle \langle \Theta_z, j | \, tdtd\theta =
\]
\[
= \sum_{j=1}^n \sum_{m=0}^\infty \sum_{l=0}^\infty \int_0^L \int_0^{2\pi} \frac{w(t)}{N(t)\sqrt{\rho(m)\rho(l)}} | \Theta_z^m \chi_j^l \otimes \phi_m \rangle \langle \Theta_z^l \chi_j^l \otimes \phi_l | \, d\mu
\]
\[
= \sum_{m=0}^\infty \sum_{l=0}^\infty \int_0^L \int_0^{2\pi} \frac{w(t)}{N(t)\sqrt{\rho(m)\rho(l)}} \Theta_z^m \mathbb{I}_n \Theta_z^l \otimes | \phi_m \rangle \langle \phi_l | \, d\mu
\]
\[
= \sum_{m=0}^\infty \int_0^L \frac{2\pi w(t)}{N(t)\rho(m)} \Theta(x)^m \Theta(x)^l \otimes | \phi_m \rangle \langle \phi_l | \, dt
\]
\[
= \sum_{m=0}^\infty \int_0^L \frac{2\pi w(t)}{N(t)x_m!} t^{2m+1} dt \mathbb{I}_n \otimes | \phi_m \rangle \langle \phi_m |
\]
choosing \( w(t) = \frac{N(t)}{2\pi} \lambda(t) \), we obtain that
\[
\int_0^L \int_0^{2\pi} w(t) \sum_{j=1}^n | \Theta_z, j \rangle \langle \Theta_z, j | \, tdtd\theta =
\]
\[
= \sum_{m=0}^\infty \frac{1}{x_m!} \left[ \int_0^L \lambda(t) t^{2m+1} dt \right] \mathbb{I}_n \otimes | \phi_m \rangle \langle \phi_m |
\]
\[
= \sum_{m=0}^\infty \mathbb{I}_n \otimes | \phi_m \rangle \langle \phi_m |
\]
provided the function \( \lambda \) is such that
\[
\int_0^L \lambda(t) t^{2m+1} dt = x_m !.
\]
This condition is achievable: For example, when \( x_m = m \) (or \( \rho(m) = m! \)), we have \( L = \infty \), \( w(t) = n/\pi \), and the choice \( \lambda(t) = e^{-t^2} \), together with the substitution \( t^2 = r \), give the expected resolution of the identity through the equation
\[
\int_0^\infty e^{-r\rho(m+1)} dr = \Gamma(m + 1).
\]

Theorem 2.1 shows that the set of states in \( \mathbb{S} \) are VCS.

In the above construction we have introduced an additional complex number \( e^{i\theta} \) from \( S^1 \) to make our calculations easier. It can easily be noted that the new parameter intervenes in the picture only by bringing the double sum in the resolution
of the identity condition to a single sum through the identity

$$\int_0^{2\pi} e^{i(m-l)\theta} d\theta = \begin{cases} 0 & \text{if } m \neq l \\ 2\pi & \text{if } m = l. \end{cases}$$

Since the moment problem is in the form of a classical moment problem, \(\rho(m)\) can be chosen in several ways. For example, several moment problems were solved in [4].

3. Annihilation, Creation, and Number operators

Here, we briefly show that the approach adopted in [6] for the complex representation of quaternions can be applied to any Clifford algebra. We define the annihilation, creation and the number operator on the basis \(\{\phi_m\}\) by

$$a\phi_m = \sqrt{x_m}\phi_{m-1},$$

$$a^\dagger\phi_m = \sqrt{x_{m+1}}\phi_{m+1},$$

$$N'\phi_m = x_m\phi_m.$$  

The corresponding operators for the states \(|\Theta_z(x), j\rangle\) are

$$A = I_n \otimes a,$$

$$A^\dagger = I_n \otimes a^\dagger,$$

$$N = I_n \otimes N'.$$

The action of these operators is given by the same relations of [6]. Further, when \(x_m = m\) we have

$$|\Theta_z(x), j\rangle = \frac{1}{\sqrt{n}} e^{i\Theta_z(x) \otimes a^\dagger - \Theta_z(x)^\dagger \otimes a} x \otimes \phi_0.$$  

The proof is similar to the proof presented in [6].

Let us define the self-adjoint operators

$$\hat{q} = \frac{1}{\sqrt{2}}(a + a^\dagger), \quad \hat{p} = \frac{1}{i\sqrt{2}}(a - a^\dagger),$$

and the corresponding operators for the VCS as

$$Q = \frac{1}{\sqrt{2}}(A + A^\dagger), \quad P = \frac{1}{i\sqrt{2}}(A - A^\dagger).$$

When \(\rho(m) = m!\), the CS in (1.1) can be written as

$$|z\rangle = e^{z a^\dagger - \Theta_z} \phi_0 = e^{i\hat{p}q - \Theta_z} \phi_0, \quad z = \frac{1}{\sqrt{2}}(q - ip).$$

The operators \(\hat{q}, \hat{p}\), and \(I\) generate an irreducible representation of the Lie algebra \(\mathfrak{g}_{WH}\) of the Weyl-Heisenberg group \(G_{WH}\) on the Hilbert space \(\mathfrak{h}\). A unitary irreducible representation of \(G_{WH}\) on \(\mathfrak{h}\) is given by the operators \(U(\theta, q, p) = e^{i(\theta I + \hat{p}\hat{q} + \Theta_z)}\). Thus \(|z\rangle = U(0, q, p)\phi_0\). As it was given in [6] an exact analogue follows for the VCS, \(|\Theta_z(x), j\rangle\). The operators \(A, A^\dagger, N\) generate an algebra (under the commutator bracket) \(U_{osc}\), the so-called oscillator algebra. The nature of this algebra primarily depends on the choice of \(\rho(m)\).
4.1. Example 1: Quaternionic VCS. Here, we present quaternionic VCS with the real matrix representation of quaternions.

Let
\[ H = \{ q' = a_0 + a_1 i + a_2 j + a_3 k \mid i^2 = j^2 = k^2 = -1, \ ijk = -1, \ a_0, a_1, a_2, a_3 \in \mathbb{R} \} \]
be the real quaternion division algebra. It is known that \( H \) is algebraically isomorphic to the real matrix algebra
\[
M = \left\{ q' = \begin{pmatrix} a_0 & -a_1 & -a_2 & -a_3 \\ a_1 & a_0 & -a_3 & a_2 \\ a_2 & a_3 & a_0 & -a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{pmatrix} \mid a_0, a_1, a_2, a_3 \in \mathbb{R} \right\}.
\]

For detailed explanation see \cite{7} and the references therein.

For \( z = e^{i\theta} \in S^1 \) and \( q' \in M \), let \( q = z q' \). Then,
\[
qq^\dagger = q' q = (a_0^2 + a_1^2 + a_2^2 + a_3^2)I_4 = |q'|^2 I_4 = |q|^2 I_4,
\]
where \( |q'| \) is the norm of the quaternion \( q' \) and \( I_4 \) is the \( 4 \times 4 \) identity matrix. Thus, with the notations of the previous sections we have a set of VCS in the Hilbert space \( \mathbb{C}^4 \otimes \mathcal{H} \),
\[
|q, j\rangle = \frac{1}{2} e^{-|q|^2/2} \sum_{m=0}^{\infty} \frac{q^m}{\sqrt{m!}} \chi_j^m \otimes \phi_m, \quad j = 1, \ldots, 4.
\]

Here also, the choice \( \rho(m) = m! \) can be replaced by other choices as mentioned earlier, because the moment condition takes the form \( \text{(5.1)} \).

4.1.1. Quaternionic minimum uncertainty states. The eigenvalues of \( q \) are \( z_1 = a_0 + ib \) and \( z_2 = a_0 - ib \), each with multiplicity 2, where \( b = \sqrt{a_1^2 + a_2^2 + a_3^2} \). Let the orthonormal eigenvectors (they do exist) corresponding to \( z_1 \) and \( z_2 \) be \( \chi_j^{(1)} \), \( \chi_j^{(2)} \), \( j = 1, 2 \) respectively. Since
\[
q^m \chi_j^{(1)} = z_1^m \chi_j^{(1)}, \quad \text{and} \quad q^m \chi_j^{(2)} = z_2^m \chi_j^{(2)},
\]
the states
\[
|q, j, i\rangle = \frac{1}{2} e^{-|q|^2/2} \sum_{m=0}^{\infty} \frac{z_i^m}{\sqrt{m!}} \chi_j^{(i)} \otimes \phi_m, \quad j, i = 1, 2
\]
saturate the Heisenberg uncertainty relation
\[
\langle \Delta Q_L \rangle \langle \Delta P_L \rangle \geq \frac{1}{2}.
\]

4.1.2. The exponential form. As introduced in the general case, we can take the annihilation, creation, and number operators for the VCS as
\[
A = I_4 \otimes a, \quad A^\dagger = I_4 \otimes a^\dagger, \quad N = I_4 \otimes N'.
\]

With these operators we can write the VCS in the form
\[
|q, j\rangle = \frac{1}{2} e^{i q \otimes a^{\dagger} - q^* \otimes a} \chi_j^{\dagger} \otimes \phi_0, \quad j = 1, \ldots, 4.
\]

Once again, the proof is similar to the one presented in \cite{6}.
4.2. Example 2: Octonionic VCS. Let \( \mathbb{O} \) denotes the octonion algebra over the real number field \( \mathbb{R} \). In [4] it was shown that any \( a \in \mathbb{O} \) has a left matrix representation \( \omega(a) \) and a right matrix representation \( \nu(a) \), given respectively by

\[
\omega(a) = \begin{pmatrix}
0 & -a_1 & -a_2 & -a_3 & -a_4 & -a_5 & -a_6 & -a_7 \\
a_1 & 0 & -a_3 & -a_2 & -a_5 & a_4 & a_7 & -a_6 \\
a_2 & a_3 & 0 & a_0 & -a_1 & -a_6 & a_4 & a_5 \\
a_3 & -a_2 & a_1 & 0 & a_0 & -a_7 & a_6 & -a_5 \\
a_4 & a_5 & a_6 & a_7 & 0 & -a_1 & -a_2 & -a_3 \\
a_5 & -a_4 & a_7 & -a_6 & a_1 & a_0 & a_3 & -a_2 \\
a_6 & -a_7 & -a_4 & a_3 & a_2 & -a_3 & a_0 & a_1 \\
a_7 & a_6 & -a_5 & -a_4 & a_3 & a_2 & -a_1 & a_0
\end{pmatrix},
\]

and

\[
\nu(a) = \begin{pmatrix}
0 & -a_1 & -a_2 & -a_3 & -a_4 & -a_5 & -a_6 & -a_7 \\
a_1 & 0 & a_3 & -a_2 & a_5 & a_4 & -a_7 & a_6 \\
a_2 & -a_3 & a_0 & a_1 & a_6 & a_7 & -a_4 & -a_5 \\
a_3 & a_2 & -a_1 & a_0 & a_7 & -a_6 & a_5 & -a_4 \\
a_4 & -a_5 & -a_6 & -a_7 & a_0 & a_1 & a_2 & a_3 \\
a_5 & a_4 & -a_7 & a_6 & -a_1 & a_0 & -a_3 & a_2 \\
a_6 & a_7 & a_4 & -a_5 & -a_2 & a_3 & a_0 & -a_1 \\
a_7 & -a_6 & a_5 & a_4 & -a_3 & -a_2 & a_1 & a_0
\end{pmatrix},
\]

The relationship between the two representations is given by the equation

\[
\nu(a) = K_8 \omega(a)^T K_8,
\]

where \( K_8 = \text{diag}(K_4, I_4) \) is an orthogonal matrix, \( K_4 = \text{diag}(1, -1, -1, -1) \) being the metric of the space of Minkowski \( \mathbb{R}^4_{1,3} \).

Let \( z = e^{i\theta} \in S^1 \), and define

\[
\omega(a, z) = z \omega(a), \quad \nu(a, z) = z \nu(a).
\]

Then,

\[
\omega(a, z) \omega(a, z)^T = \omega(a, z)^T \omega(a, z) = \nu(a, z) \nu(a, z)^T = \nu(a, z)^T \nu(a, z)
\]

\[
= (a_0^2 + a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 + a_6^2 + a_7^2) \mathbb{I}_8
\]

Thus, we have two sets of VCS

\[
| \omega(a, z), j \rangle = \frac{1}{\sqrt{8}} e^{-\|a\|^2/2} \sum_{m=0}^{\infty} \frac{\omega(a, z)^m}{\sqrt{m!}} \chi^j \otimes \phi_m, \quad j = 1, \ldots, 8
\]

\[
| \nu(a, z), j \rangle = \frac{1}{\sqrt{8}} e^{-\|a\|^2/2} \sum_{m=0}^{\infty} \frac{\nu(a, z)^m}{\sqrt{m!}} \chi^j \otimes \phi_m \quad j = 1, \ldots, 8.
\]

In fact, using (4.10), the "right" VCS can be obtained from the "left" ones by the transform

\[
| \nu(a, z), j \rangle = K_8 | \omega(a, z), j \rangle \mathbb{I}_8, \quad a \in \mathbb{O}, \quad j = 1, \ldots, 8.
\]

Again, the moment condition takes the form (5.2). Thus, \( \rho(m) \) can be chosen in many ways. We keep stuck with the simplest choice \( \rho(m) = m! \).
4.2.1. The oscillator algebra. Now let us denote the corresponding annihilation, creation, and number operators as before and define accordingly, for the left representation,

\begin{equation}
A_L, A_L^\dagger, N_L,
\end{equation}

and, for the right representation,

\begin{equation}
A_R, A_R^\dagger, N_R.
\end{equation}

The action and commutation relations of these operators, and the corresponding algebras take the same form as in the previous sections. In principle, these two sets of operators generate two algebras \( U_{osc}^L \) and \( U_{osc}^R \), but they are the same.

4.2.2. Octonionic minimum uncertainty states. Let us denote the self-adjoint operators for the left representation by \( Q_L \) and \( P_L \). Let

\begin{equation}
b = \sqrt{a_1^2 + a_2^2 + \ldots + a_7^2}.
\end{equation}

Then, the eigenvalues of \( \omega(a, z) \) are \( z_1 = e^{i\theta}(a_0 + ib) \) and \( z_2 = e^{i\theta}(a_0 - ib) \), each with multiplicity 4. Let the corresponding normalized eigenvectors be \( \chi_j^{(1)}, \chi_j^{(2)} \), \( j = 1, \ldots, 4 \). For \( i = 1, 2 \), the states

\begin{equation}
| \omega(a, z), j, i \rangle = \frac{1}{\sqrt{8}} e^{-\|a\|^2/2} \sum_{m=0}^{\infty} \frac{\omega(a, z)^m}{\sqrt{m!}} \chi_j^{(i)} \otimes \phi_m,
\end{equation}

saturate the Heisenberg uncertainty relation (4.5).

A similar set of VCS can also be obtained for the right representation.

4.2.3. The exponential form. Here again the states can be written in the form

\begin{equation}
| \omega(a, z), j \rangle = \frac{1}{\sqrt{8}} e^{-\|a\|^2/2} \sum_{m=0}^{\infty} \frac{\omega(a, z)^m}{\sqrt{m!}} \chi_j \otimes \phi_m,
\end{equation}

and

\begin{equation}
| \nu(a, z), j \rangle = \frac{1}{\sqrt{8}} e^{-\|a\|^2/2} \sum_{m=0}^{\infty} \frac{z_j^m}{\sqrt{m!}} \chi_j \otimes \phi_m,
\end{equation}

saturate the Heisenberg uncertainty relation (4.5).

5. VCS with matrix moments

So far, we have obtained VCS by replacing the complex number \( z \) of (1.1) by a Clifford matrix and keeping the moments \( \rho(m) \), as usual, a positive sequence of real numbers. In this section, we attempt to replace both, \( z \) and \( \rho(m) \) by matrices, namely, the first by a Clifford matrix, and the second by a matrix \( R(m) \). This procedure foreshadows a more general picture to be analyzed in a forthcoming paper.

In order to do this we demand the matrix \( R(m) \) to satisfy the condition,

\begin{equation}
R(m)R(m)^\dagger = R(m)^\dagger R(m) = f(m)I_n,
\end{equation}

where \( n \) has to be chosen to match the size of the Clifford matrix. Now, we define our VCS as follows:

\begin{equation}
| Z, j \rangle = \mathcal{N}(|Z|)^{-1/2} \sum_{m=0}^{\infty} R(m)Z^m \chi_j \otimes \phi_m,
\end{equation}

where \( Z = a_1 + a_2 + \ldots + a_7 \).
Let us then look at the normalization and resolution of the identity conditions for the matrices of sections 4.1 and 4.2.

5.1. For the quaternions. Let $q \in \mathbb{H}$, $Z = q e^{i\theta}$, and, for $R(m)$, let us take, for example, for a fixed $x$,

$$
R(m) = \frac{1}{\sqrt{m!}} \begin{pmatrix}
I_2 \cos x & -I_2 \sin x \\
I_2 \sin x & I_2 \cos x
\end{pmatrix}.
$$

We have

$$
R(m)R(m)\dagger = R(m)\dagger R(m) = \frac{1}{m!} I_4.
$$

The normalization condition takes the form,

$$
\sum_{j=1}^{4} \langle Z, j | Z, j \rangle = 4N(|Z|) \sum_{m=0}^{\infty} \frac{|q|^{2m}}{m!} = 4N(|Z|) \frac{1}{e|q|^2} = 1,
$$

and leads to the normalization factor

$$
N(|Z|) = 4e|q|^2.
$$

On the other hand, under the identification (2.6) and using the measure

$$
d\mu = \frac{4|q|d|q|d\theta}{\pi},
$$

we have

$$
\sum_{j=1}^{4} \int_{0}^{2\pi} \int_{0}^{\infty} \langle Z, j | Z, j \rangle d\mu = \sum_{m=0}^{\infty} \sum_{l=0}^{m} \int_{0}^{2\pi} e^{i(m-l)\theta} R(m) |\chi^l\rangle \langle \chi^l | R(l)q\rangle \dagger \otimes |\phi_m\rangle \langle \phi_l | e^{-|q|^2} \frac{|q|d|q|d\theta}{\pi} \cdot
$$

$$
\sum_{m=0}^{\infty} 1\frac{1}{m!} \left[ \int_{0}^{\infty} e^{-|q|^2} |q|^{2m} \frac{2|q|d|q|}{\pi} \right] I_4 \otimes |\phi_m\rangle \langle \phi_l |
$$

$$
= I_4 \otimes I.
$$

5.2. For the octonions. Here we can take either $Z = \omega(a, z)$ or $Z = \nu(a, z)$, where $\omega(a, z)$ and $\nu(a, z)$ are as in section 4.2. For $R(m)$, one could take, for a fixed $x$,

$$
R(m) = \frac{1}{\sqrt{m!}} \begin{pmatrix}
I_4 \cos x & -I_4 \sin x \\
I_4 \sin x & I_4 \cos x
\end{pmatrix}.
$$

The rest of the details are similar to the case of quaternions.

Remark 5.1. Since $Z$ and $R(m)$ are matrices, in general, placing $R(m)$ on the right of $Z$ is different from placing it on the left of $Z$. In our case, by the properties of the Clifford matrices and by the assumption on $R(m)$ the construction can be carried out in either way without any obstacles. Further, in CS constructions the sequence $\rho(m)$ of (1.1) is taken to be a positive sequence, and, thereby, in getting
a resolution of the identity, we end up with a positive moment problem. When we replace $\rho(m)$ by a matrix $R(m)$, one could expect it to be a positive definite matrix. But in our construction we have used a non-positive definite matrix $R(m)$. It can be observed that, it doesn’t really matter which matrix we start with, but what does matter is that we finally end up with a moment problems with positive moments throughout in the construction.

**Remark 5.2.** In our knowledge, a physical system which can be described by quaternions or octonions is not known yet. There have been several attempts to describe relativistic physics in terms of quaternions or octonions (see [2] and the references therein), but without the expected success. Nevertheless, the VCS of the type presented here have been introduced recently [6], and quaternionic VCS are suspected to be useful in the description of the spin orbit interaction between a spinning electron and an external magnetic field.

6. CONCLUSION

We have presented a class of VCS using the real matrix representation of Clifford algebras. In [6], quaternionic VCS were presented without introducing the supplementary term $e^{i\theta}$, this is used in the construction developed in this paper only for technical purposes. Finally, we have introduced VCS with matrix moments, carrying out the construction for real matrix representations of quaternions and octonions. A wide-ranging study of VCS with matrix moments will emerge in a companion paper.

REFERENCES

[1] Ali, S.T., Antoine, J-P., Gazeau, J-P., *Coherent States, Wavelets and their Generalizations*, Springer, New York, (2000).

[2] Daboul, J., Delbourgo, R., “Matrix representation of octonions and generalizations,” J. Math. Phys, 40, (1999) 4134-4150.

[3] Klauder, J.R., Skagerstam, B.S., *Coherent States, Applications in Physics and Mathematical Physics*, World Scientific, Singapore, (1985).

[4] Klauder, J.R., Penson, K.A., Sixdeniers, J-M., ”Constructing coherent states through solutions of Steielljes and Hausdorff moment problems,” Phys. Rev. A, 64 013817.

[5] Pérélovom A.M., *Generalized Coherent States and their Applications*, Springer-Verlag, Berlin, (1986).

[6] Thirulogasanthar, K., Ali S.T., ”A class of vector coherent states defined over matrix domains,” Preprint, [math-ph/0305036](http://arxiv.org/abs/math-ph/0305036).

[7] Yongge Tian, ”Matrix representations of octonions and their applications,” Adv. Appl. Clifford Algebras 10, no. 1 (2000) 61-90.

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