DECAY OF SOLUTIONS OF WAVE-TYPE
PSEUDO-DIFFERENTIAL EQUATIONS OVER \( p \)-ADIC
FIELDS

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Abstract. We show that the solutions of \( p \)-adic pseudo-differential
equations of wave type have a decay similar to the solutions of classical
generalized wave equations.

1. Introduction

During the eighties several physical models using \( p \)-adic numbers were
proposed. Particularly various models of \( p \)-adic quantum mechanics \[14],
[16], [24], [25]. As a consequence of this fact, several new mathematical
problems emerged, among them, the study of \( p \)-adic pseudo-differential
equations \[11], [25]. In this paper we initiate the study of the decay of the
solutions of wave-type pseudo-differential equations over \( p \)-adic fields; these
equations were introduced by Kochubei \[12] in connection with the problem
of characterizing the \( p \)-adic wave functions using pseudo-differential operators. We show that the solutions of \( p \)-adic wave-type equations have a
decay similar to the solutions of classical generalized wave equations.

Let \( K \) be a \( p \)-adic field, i.e. a finite extension of \( \mathbb{Q}_p \). Let \( R_K \) be the
valuation ring of \( K \), \( P_K \) the maximal ideal of \( R_K \), and \( \overline{K} = R_K / P_K \)
the residue field of \( K \). Let \( \pi \) denote a fixed local parameter of \( R_K \). The
cardinality of \( \overline{K} \) is denoted by \( q \). For \( z \in K \), \( v(z) \in \mathbb{Z} \cup \{+\infty\} \) denotes the
valuation of \( z \), and \( |z|_K = q^{-v(z)} \). Let \( \mathbb{S}(K^n) \) denote the \( \mathbb{C} \)-vector space of
Schwartz-Bruhat functions over \( K^n \), the dual space \( \mathbb{S}'(K^n) \) is the space of
distributions over \( K^n \). Let \( \mathcal{F} \) denote the Fourier transform over \( \mathbb{S}(K^{n+1}) \).
The reader can consult any of the references \[9], [25], [20\] for an exposition
of the theory of distributions over \( p \)-adic fields.

This article aims to study the following initial value problem:

\[
\begin{align*}
(Hu)(x,t) &= 0, \quad x \in K^n, \quad t \in K \\
 u(x,0) &= f_0(x),
\end{align*}
\]

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where \( n \geq 1, f_0 (x) \in \mathcal{S}(K^n) \), and
\[
(H\Phi) (t, x) := \mathcal{F}^{-1}_{(\tau, \xi) \rightarrow (t, x)} \left( |\tau - \phi (\xi)|_K \mathcal{F}_{(t,x) \rightarrow (\tau, \xi)} \Phi \right), \quad \Phi \in \mathcal{S}(K^{n+1}),
\]
is a pseudo-differential operator with symbol \( |\tau - \phi (\xi)|_K \), where \( \phi (\xi) \) is a polynomial in \( K [\xi_1, \ldots, \xi_n] \) satisfying \( \phi(0) = 0 \). In the case in which \( \phi (\xi) = a_1 \xi_1^2 + \ldots + a_n \xi_n^2 \), \( H \) is called a Schrödinger-type pseudo-differential operator; this operator was introduced by Kochubei in [12]. For \( n = 1 \) the solution of (1.1) appears in the formalism of \( p \)-adic quantum mechanics as the wave function for the free particle [24]. The problem of characterizing the \( p \)-adic wave functions as solutions of some pseudo-differential equation remains open.

Let \( \Psi (\cdot) \) denote an additive character of \( K \) trivial on \( R_K \) but no on \( P_K^{-1} \). By passing to the Fourier transform in (1.1) one gets that
\[
|\tau - \phi (\xi)|_K \mathcal{F}_{(x,t) \rightarrow (\tau, \xi)} u = 0.
\]
Then any distribution \( g \) with \( g \) a distribution supported on \( \tau - \phi (\xi) = 0 \) is a solution. By taking
\[
g (\xi, \tau) = (\mathcal{F}_{x \rightarrow \xi} f_0) \delta (\tau - \phi (\xi)) ,
\]
where \( \delta \) is the Dirac distribution, one gets
\[
(1.2) \quad u(x,t) = \int_{K^n} \Psi \left( t\phi (\xi) + \sum_{i=1}^{n} x_i \xi_i \right) \left( \mathcal{F}_{x \rightarrow \xi} f_0 \right) (\xi) |d\xi| ,
\]
here \( |d\xi| \) is the Haar measure of \( K^n \) normalized so that \( vol (R_K^n) = 1 \).

In this paper we show that the decay of \( u(x,t) \) is completely similar to the decay of the solution of the following initial value problem:
\[
\begin{cases}
\frac{\partial u_{arch}(x,t)}{\partial t} = i\phi (D) u_{arch} (x,t) , & x \in \mathbb{R}^n , \ t \in \mathbb{R} \\
u_{arch} (x,0) = f_0 (x) ,
\end{cases}
(1.3)
\]
here \( \phi (D) \) is a pseudo-differential operator having symbol \( \phi (\xi) \). In this case
\[
(1.4) \quad u_{arch} (x,t) = \int_{\mathbb{R}^n} \exp 2\pi i \left( t\phi (\xi) + \sum_{i=1}^{n} x_i \xi_i \right) \left( \mathcal{F}_{x \rightarrow \xi} f_0 \right) (\xi) d\xi
\]
is the solution of the initial value problem (1.3). If \( \phi (\xi) = \xi_1^2 + \ldots + \xi_n^2 \), i.e. \( \phi (D) \) is the Laplacian, \( u_{arch} (x,t) \) satisfies
\[
(1.5) \quad \left\| u^{arch} (x,t) \right\|_{L^{2(n+2)/n}} \leq c \left\| f_0 \right\|_{L^2} ,
\]
(see [22]). If \( n = 1 \) and \( \phi (\xi) = \xi^2 \), \( u_{arch} (x,t) \) satisfies
\[
(1.6) \quad \left\| u^{arch} (x,t) \right\|_{L^8} \leq c \left\| f_0 \right\|_{L^2} ,
\]
(see [13]). We show that \( u(x,t) \) satisfies (1.5), if \( \phi (\xi) = \xi_1^2 + \ldots + \xi_n^2 \) (see Theorem [5.2]), and that \( u(x,t) \) satisfies (1.6), if \( \phi (\xi) = \xi_3^2 \) (see Theorem [5.2]).
For more general symbols we are able to describe the decay of \( u(x,t) \) in \( L^\sigma(K^{n+1}) \), however, in this case the index \( \sigma \) is not optimal (see Theorem 5.1). The proof is achieved by adapting standard techniques in PDEs and by using number-theoretic techniques for estimating exponential sums modulo \( \pi^m \). Indeed, like in the classical case the estimation of the decay rate can be reduced to the problem of estimating of the restriction of Fourier transforms to non-degenerate hypersurfaces [20]; we solve this problem (see Theorems 4.1, 4.2) by reducing it to the estimation of exponential sums modulo \( \pi^m \) (see Theorems 3.1, 3.2). These exponential sums are related to the Igusa zeta function for non-degenerate polynomials [6], [10], [28], [29]. More precisely, by using Igusa’s method, the estimation of these exponential sums can be reduced to the description of the poles of twisted local zeta functions [6], [25], [29].

The restriction of Fourier transforms in \( \mathbb{R}^n \) (see e.g. [20, Chap. VIII]) was first posed and partially solved by Stein [8]. This problem have been intensively studied during the last thirty years [2], [20], [22], [27]. Recently Mockenhaupt and Tao have studied the restriction problem in \( \mathbb{F}_q^n \) [15]. In this paper we initiate the study of the restriction problem in the non-archimedean field setting.

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2. The Non-archimedean Principle of the Stationary Phase

Given \( f(x) \in K[x], x = (x_1, \ldots, x_m) \), we denote by

\[
C_f(K) = \left\{ z \in K^m \mid \frac{\partial f}{\partial x_1}(z) = \cdots = \frac{\partial f}{\partial x_m}(z) = 0 \right\}
\]

the critical set of the mapping \( f : K^m \to K \). If \( f(x) \in R_K[x] \), we denote by \( \overline{f}(x) \) its reduction modulo \( \pi \), i.e. the polynomial obtained by reducing the coefficients of \( f(x) \) modulo \( \pi \).

Give a compact open set \( A \subset K^m \), we set

\[
E_A(z, f) = \int_A \Psi(zf(x)) \, |dx|,
\]

for \( z \in K \), where \( |dx| \) is the normalized Haar measure of \( K^m \). If \( A = R^m_K \) we use the simplified notation \( E(z, f) \) instead of \( E_A(z, f) \). If \( f(x) \in R_K[x] \), then

\[
E(z, f) = q^{-am} \sum_{x \mod \pi^n} \Psi(zf(x));
\]

thus \( E(z, f) \) is a generalized Gaussian sum.

**Lemma 2.1.** Let \( f(x) \in R_K[x], x = (x_1, \ldots, x_m) \), be a non-constant polynomial. Let \( A \) be the preimage of \( \overline{A} \subseteq \mathbb{F}_q^m \) under the canonical homomorphism \( R^m_K \to (R_K/P_K)^m \). If \( C_f(K) \cap A = \emptyset \), then there exists a constant \( I(f,A) \) such that

\[
E(z, f) = 0, \quad \text{for} \quad |z|_K > q^{2I(f,A)+1}.
\]
Proof. We define

\[ I(f,a) = \min_{1 \leq i \leq m} \left\{ v \left( \frac{\partial f}{\partial x_i} (a) \right) \right\}, \]

for any \( a \in A \), and

\[ I(f,A) = \sup_{a \in A} \{ I(f,a) \}. \]

Since \( A \) is compact and \( C_f(K) \cap A = \emptyset \), \( I(f,A) < \infty \).

We denote by \( a^* \) an equivalence class of \( \mathbb{R}^m_K \) modulo \( (P_I(f,A)+1)^m \), and by \( a \in \mathbb{R}^m_K \) a fixed representative of \( a^* \). By decomposing \( A \) into equivalence classes modulo \( (P_I(f,A)+1)^m \), one gets

\[ E(z,f) = \sum_{a^* \subseteq A} q^{-m(I(f,A)+1)} \int_{R_K^m} \Psi \left( z f \left( a + \pi^{I(f,A)+1} x \right) \right) \, |dx|. \]

Thus, it is sufficient to show that \( \int_{R_K^m} \Psi \left( z f \left( a + \pi^{I(f,A)+1} x \right) \right) \, |dx| = 0 \) for \( |z|_K > q^{2I(f,A)+1} \).

On the other hand, if \( a = (a_1, \ldots, a_m) \), then

\[ \frac{f \left( a + \pi^{I(f,A)+1} x \right) - f \left( a \right)}{\pi^{I(f,A)+1+\alpha_0}} \]

equals

\[ \sum_{i=1}^m \pi^{-\alpha_0} \frac{\partial f}{\partial x_i} (a) (x - a_i) + \pi^{I(f,A)+1-\alpha_0} \text{ (higher order terms)}, \]

where

\[ \alpha_0 = \min_i \left\{ v \left( \frac{\partial f}{\partial x_i} (a) \right) \right\}. \]

Therefore

\[ f \left( a + \pi^{I(f,A)+1} x \right) - f \left( a \right) = \pi^{I(f,A)+1+\alpha_0} \tilde{f}(x) \]

with \( \tilde{f}(x) \in R_K[x] \), and since \( C_f(K) \cap A = \emptyset \), there exists an \( i_0 \in \{1, \ldots, m\} \) such that

\[ \frac{\partial f}{\partial x_{i_0}} (\pi) \neq 0. \]

We put \( y = \Phi(x) = (\Phi_1(x), \ldots, \Phi_m(x)) \) where

\[ \Phi_i(x) = \begin{cases} \tilde{f}(x) & i = i_0 \\ x_i & i \neq i_0. \end{cases} \]

Since \( \Phi_1(x), \ldots, \Phi_m(x) \) are restricted power series and

\[ f \left( \frac{(y_1, \ldots, y_m)}{(x_1, \ldots, x_m)} \right) = \frac{\partial f}{\partial x_{i_0}} (\pi) \neq 0, \]
the non-archimedean implicit function theorem implies that $y = \Phi(x)$ gives
a measure-preserving map from $R_K^m$ to $R_K^m$ (see [10, Lemma 7.43]). Therefore
$$\int_{R_K^m} \Psi \left( zf \left( a + \pi^{I(f,A)+1} x \right) \right) |dx| =$$
$$\Psi \left( zf (a) \right) \int_{R_K^m} \Psi \left( z\pi^{I(f,A)+1+\alpha_0} y_{i_0} \right) |dy_{i_0}| = 0,$$
for $v(z) < - (I(f,A) + 1 + \alpha_0)$, i.e. for $|z|_K > q^{I(f,A)+1+\alpha_0}$, and a fortiori
$$\int_{R_K^m} \Psi \left( zf \left( a + \pi^{I(f,A)+1} x \right) \right) |dx| = 0,$$
for $|z|_K > q^{2I(f,A)+1}$ and any $a$. □

**Theorem 2.1.** Let $f(x) \in K[x]$, $x = (x_1, \ldots, x_m)$, be a non-constant polynomial. Let $B \subset K^m$ be a compact open set. If $C_f(K) \cap B = \emptyset$, then there exist a constant $c(f,B)$ such that
$$E_B(z, f) = 0, \quad |z|_K \geq c(f,B).$$

**Proof.** By taking a covering $\bigcup_i (y_i + (\pi^\alpha R_K)^m)$ of $B$, $E_B(z, f)$ can be expressed as linear combination of integrals of the form $E(z, f_i)$ with $f_i(x) \in K[x]$. After changing $z$ by $z \pi^\beta$, we may suppose that $f_i(x) \in R_K[x]$. By applying Lemma 2.1 we get that $E(z, f_i) = 0$, for $|z|_K > c_i$. Therefore
$$E_B(z, f) = 0, \quad |z|_K > \max_i c_i. \quad (2.3)$$

We note that the previous result implies that
$$E_B(z, f) = O(|z|_K^{-M}),$$
for any $M \geq 0$. This is the standard form of the principle of the stationary phase.

### 3. Local Zeta Functions and Exponential Sums

In this section we review some results about exponential sums and Newton polyhedra that will be used in the next section. For $x \in K$ we denote by $ac(x) = x\pi^{-v(x)}$ its angular component. Let $f(x) \in R_K[x]$, $x = (x_1, \ldots, x_m)$ be a non-constant polynomial, and $\chi : R_K^* \to \mathbb{C}^*$ a character of $R_K^*$, the group of units of $R_K$. We formally put $\chi(0) = 0$. To these data one associates the Igusa local zeta function,
$$Z(s, f, \chi) = \int_{R_K^m} \chi(acf(x)) |f(x)|_K^s \ |dx|, \quad s \in \mathbb{C},$$
for $Re(s) > 0$, where $|dx|$ denotes the normalized Haar measure of $K^n$. The Igusa local zeta function admits a meromorphic continuation to the complex plane as a rational function of $q^{-s}$. Furthermore, it is related to the number
of solutions of polynomial congruences modulo $\pi^m$ and exponential sums modulo $\pi^m$ [5], [10].

3.1. Exponential Sums Associated with Non-degenerate Polynomials. We set $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x \geq 0\}$. Let $f(x) = \sum_l a_l x^l \in K[x]$, $x = (x_1, \ldots, x_n)$ be a non-constant polynomial satisfying $f(0) = 0$. The set $\text{supp}(f) = \{l \in \mathbb{N}^m \mid a_l \neq 0\}$ is called the support of $f$. The Newton polyhedron $\Gamma(f)$ of $f$ is defined as the convex hull in $\mathbb{R}^m$ of the set

$$\bigcup_{l \in \text{supp}(f)} (l + \mathbb{R}_+).$$

We denote by $\langle \cdot, \cdot \rangle$ the usual inner product of $\mathbb{R}^m$, and identify $\mathbb{R}^m$ with its dual by means of it. We set

$$\langle a_\gamma, x \rangle = m(a_\gamma),$$

for the equation of the supporting hyperplane of a facet $\gamma$ (i.e. a face of codimension 1 of $\Gamma(f)$) with perpendicular vector $a_\gamma = (a_1, \ldots, a_n) \in \mathbb{N}^n \setminus \{0\}$, and $\sigma(a_\gamma) := \sum_i a_i$.

Definition 3.1. A polynomial $f(x) \in K[x]$ is called non-degenerate with respect to its Newton polyhedron $\Gamma(f)$, if it satisfies the following two properties: (i) $C_f(K) = \{0\} \subset \mathbb{N}^n$; (ii) for every proper face $\gamma \subset \Gamma(f)$, the critical set $C_{f_\gamma}(K)$ of $f_\gamma(x) := \sum_{i \in \gamma} a_i x^i$ satisfies $C_{f_\gamma}(K) \cap (K \setminus \{0\})^m = \emptyset$.

We note that the above definition is not standard because it requires that the origin be an isolated critical point (see e.g. [6], [7], [29]). The condition (ii) can be replaced by

$$(3.1) \quad \{x \in \mathbb{K}^m \mid f_\gamma(x) = 0\} \cap C_{f_\gamma}(K) \cap (K \setminus \{0\})^m = \emptyset.$$

If $K$ has characteristic $p > 0$, by using Euler’s identity, it can be verified that condition (ii) in the above definition is equivalent to (3.1), if $p$ does not divide the $m(a_\gamma) \neq 0$, for any facet $\gamma$.

In [29] the author showed that if $f$ is non-degenerate with respect $\Gamma(f)$, then the poles of $(1 - q^{-1-s}) Z(s, f, \chi_{\text{triv}})$ and $Z(s, f, \chi), \chi \neq \chi_{\text{triv}}$, have the form

$$s = -\frac{\sigma(a_\gamma)}{m(a_\gamma)} + \frac{2\pi i}{\log q m(a_\gamma)} k, \quad k \in \mathbb{Z},$$

for some facet $\gamma$ of $\Gamma(f)$ with perpendicular $a_\gamma$, and $m(a_\gamma) \neq 0$ (see [29 Theorem A, and Lemma 4.4]). Furthermore, if $\chi \neq \chi_{\text{triv}}$ and the order of $\chi$ does not divide any $m(a_\gamma) \neq 0$, where $\gamma$ is a facet of $\Gamma(f)$, then $Z(s, f, \chi)$ is a polynomial in $q^{-s}$, and its degree is bounded by a constant independent of $\chi$ (see [29 Theorem B]). These two results imply that for $|z|_K$ big enough $E(z, f)$ is a finite $\mathbb{C}$-linear combination of functions of the form

$$\chi(ac(z)) |z|_K^\lambda (\log(|z|_K))^\gamma,$$

with coefficients independent of $z$, and with $\lambda \in \mathbb{C}$ a pole of

$$(1 - q^{-1-s}) Z(s, f, \chi_{\text{triv}})$$

or of $Z(s, f, \chi), \chi \neq \chi_{\text{triv}}$. 

and $\gamma \in \mathbb{N}, \gamma \leq (\text{multiplicity of pole } \lambda) - 1$ (see [3, Corollary 1.4.5]). Moreover all poles $\lambda$ appear effectively in this linear combination. Therefore

$$(3.2) \quad |E(z, f)| \leq C |z|^{-\beta_f + \epsilon},$$

with $\epsilon > 0$, and

$$\beta_f := \min_{\tau} \left\{ \frac{\sigma(a_{\tau})}{m(a_{\tau})} \right\},$$

where $\tau$ runs through all facets of $\Gamma(f)$ satisfying $m(a_{\tau}) \neq 0$. The point

$$T_0 = (\beta_f^{-1}, ..., \beta_f^{-1}) \in \mathbb{Q}^m$$

is the intersection point of the boundary of the Newton polyhedron $\Gamma(f)$ with the diagonal $\Delta = \{(t, ..., t) \mid t \in \mathbb{R}\} \subset \mathbb{R}^m$. By combining estimation (3.2) and Theorem 2.1, we obtain the following result.

**Theorem 3.1.** Let $f(x) \in K[x]$ be non-degenerate with respect to its Newton polyhedron $\Gamma(f)$. Let $B \subset K^m$ a compact open subset. Then

$$|E_B(z, f)| \leq C |z|^{-\beta_f + \epsilon},$$

for any $\epsilon > 0$.

We have to mention that the previous result is known by the experts, however the author did not find a suitable reference for the purposes of this article. If $K$ has characteristic $p > 0$, the previous result is valid if $p$ does not divide the $m(a_{\tau}) \neq 0$ [29, Corollary 6.1].

### 3.2. Exponential Sums Associated with Quasi-homogeneous Polynomials.

**Definition 3.2.** Let $f(x) \in K[x], x = (x_1, \ldots, x_m)$ be a non-constant polynomial satisfying $f(0) = 0$. The polynomial $f(x)$ is called quasi-homogeneous of degree $d$ with respect $\alpha = (\alpha_1, \ldots, \alpha_m) \in (\mathbb{N} \setminus \{0\})^m$, if it satisfies

$$f(\lambda^{\alpha_1}x_1, \ldots, \lambda^{\alpha_m}x_m) = \lambda^d f(x),$$

for every $\lambda \in K$.

In addition, if $C_f(K)$ is the origin of $K^m$, then $f(x)$ is called a non-degenerate quasi-homogeneous polynomial.

The non-degenerate quasi-homogeneous polynomials are a subset of the non-degenerate polynomials with respect to the Newton polyhedron. For these type of polynomials the bound (3.2) can be improved:

$$(3.3) \quad |E(z, f)| \leq C |z|^{-\beta_f},$$

where $\beta_f = \frac{1}{d} \sum_{i=1}^m \alpha_i$. By using the techniques exposed in [28, Theorem 3.5], and [29, Lemma 2.4] follow that the poles of $(1 - q^{-1-s}) Z(s, f, \chi_{\text{triv}})$ and $Z(s, f, \chi), \chi \neq \chi_{\text{triv}}$, have the form

$$s = -\frac{\sigma(\alpha)}{d} + \frac{2\pi i}{\log q} \frac{k}{d}, k \in \mathbb{Z}.$$
Then by using the same reasoning as before, we obtain \( (3.3) \). This estimate and Theorem 2.1 imply the following result.

**Theorem 3.2.** Let \( f(x) \in K[x] \), \( x = (x_1, \ldots, x_m) \) be a non-degenerate quasi-homogeneous polynomial of degree \( d \) with respect to \( \alpha = (\alpha_1, \ldots, \alpha_m) \).

Let \( B \subset K^m \) be a compact open set. Then

\[
|E_B (z, f)| \leq C |z|^{-\beta} f.
\]

If \( K \) has characteristic \( p > 0 \), the above result is valid, if \( p \) does not divide \( \sigma (\alpha) \).

4. Fourier Transform of Measures Supported on Hypersurfaces

Let \( Y \) be a closed smooth submanifold of \( K^n \) of dimension \( n - 1 \). If

\[
I = \{ i_1, \ldots, i_{n-1} \} \text{ with } i_1 < i_2 < \ldots < i_{n-1}
\]

is a subset of \( \{ 1, \ldots, n \} \) we denote by \( \omega_I \) the differential form induced on \( Y \) by \( dx_{i_1} \wedge dx_{i_2} \wedge \ldots \wedge dx_{i_{n-1}} \), and by \( d\sigma_Y I \) the corresponding measure on \( Y \). The canonical measure of \( Y \) is defined as

\[
d\sigma_Y = \sup_I \{ d\sigma_Y I \}
\]

where \( I \) runs through all the subsets of form \((4.1)\). Given \( S \) a compact open subset of \( K^n \) with characteristic function \( \Theta_S \), we define \( d\mu_{Y,S} = d\mu_Y = \Theta_S d\sigma_Y \). The canonical measure \( d\mu_Y \) was introduced by Serre in [17]. The Fourier transform of \( d\mu_Y \) is defined as

\[
\hat{d\mu_Y}(\xi) = \int_Y \Psi (-[x,\xi]) d\mu_Y (x),
\]

where \( [x,y] := \sum_{i=1}^n x_i y_i \), with \( x, y \in K^n \). The analysis of the decay of \( |d\mu_Y(\xi)| \) as \( ||\xi||_K := \max_i \{|\xi_i|_K\} \) approaches infinity plays a central role in this paper. This analysis can be simplified taking into account the following facts. Any compact open set of \( K^n \) is a finite union of classes modulo \( \pi^e \), by taking \( e \) big enough, and taking into account that \( Y \cap y + (\pi^e R_K)^n \) is a hypersurface of the form

\[
\{ x \in y + (\pi^e R_K)^n | x_n = \phi (x_1, \ldots, x_{n-1}) \}
\]

with \( \phi \) an analytic function satisfying

\[
\phi (0) = \frac{\partial \phi}{\partial x_1} (0) = \ldots = \frac{\partial \phi}{\partial x_{n-1}} (0) = 0,
\]

(see [17] page 147), we may assume that \( Y \) is a hypersurface of the form \( x_n - \phi (x_1, \ldots, x_{n-1}) = 0 \), with \( \phi \) satisfying \((4.2)\). In this case \( d\sigma_Y (x) = |dx_1| \ldots |dx_{n-1}| \), the normalized Haar measure of \( K^{n-1} \).
Finally we want to mention that if \( X = \{ x \in K^n \mid f(x) = 0 \} \) is a hypersurface then

\[
\frac{dx_1 \ldots dx_{n-1}}{\left| \frac{\partial f}{\partial x_n} \right|_K}
\]

is a measure on a neighborhood of \( X \) provided that \( \left| \frac{\partial f}{\partial x_n} \right|_K \neq 0 \) (see \cite[Sec. 7.6]{10}). This measure is not intrinsic to \( X \), but if \( S \) is small enough, it coincides with \( d\mu_X = \Theta_S d\sigma_X \) for a polynomial of type \( f(x) = x_n - \phi(x_1, \ldots, x_{n-1}) \). The Serre measure allow us to define \( \hat{d\mu_Y}(\xi) \) intrinsically for an arbitrary submanifold \( Y \).

**Theorem 4.1.** Let \( \phi(x) \in R_K[x] \), \( x = (x_1, \ldots, x_{n-1}) \), be a non-constant polynomial such that \( C_\phi(K) = \{0\} \subset K^{n-1} \). Let \( d_j(\phi) \) be the degree of \( \phi \) with respect the variable \( x_j \), and let \( \beta_\phi := \max_j d_j(\phi) \). Let \( \Theta_S \) be the characteristic function of a compact open set \( S \), let \( Y = \{ x \in K^n \mid x_n = \phi(x_1, \ldots, x_{n-1}) \} \), and let \( d\mu_Y = \Theta_S d\sigma_Y \). Then

\[
\left| \hat{d\mu_Y}(\xi) \right| \leq C \| \xi \|^{-\beta_K},
\]

for \( 0 \leq \beta \leq \beta_\phi - \epsilon \), with \( \epsilon > 0 \).

**Proof.** By passing to a sufficiently fine covering we may suppose that

\[
\hat{d\mu_Y}(\xi) = \int_{(x_0 + \pi^a R_K)^{n-1}} \Psi \left( -\xi_n \phi(x) - [x, \xi'] \right) |dx|.
\]

By applying Theorem 6.1 of \cite{3}, we have

\[
\left| \hat{d\mu_Y}(\xi) \right| \leq C \left( \log_q \| \xi \|_K \right)^{n-1} \| \xi \|^{-\beta_\phi}_K,
\]

and then

\[
\left| \hat{d\mu_Y}(\xi) \right| \leq C \| \xi \|^{-\beta_\phi}_K, \text{ for } 0 \leq \beta \leq \beta_\phi - \epsilon, \epsilon > 0.
\]

It is important to mention that Cluckers’ Theorem 6.1 is established only for \( \mathbb{Q}_p \), however this result is valid for any \( p \)-adic field. Indeed, the proof of this result is based on a result of Chubarikov \cite[Lemma 3]{4} whose proof uses inductively an estimation for one-dimensional exponential sums due to I. M. Vinogradov (see e.g. \cite[Theorem 2.1]{1}). The proof of this last estimation as given in \cite{1} can be adapted to the case of \( p \)-adic fields easily using the notion of dilation as in \cite{28}.

The Cluckers’ result does not give an optimal decay rate, and then \( \beta_\phi \) is not optimal (see also \cite{30}).

**Remark 1.** If \( \phi(x) = \sum_{i=1}^{n-1} a_i x_i^2 \), then the phase of \( \hat{d\mu_Y}(\xi) \) around any critical point has the form \( \sum_{i=1}^{n-1} a_i x_i^2 \). By using Theorem 3.2 one verifies
that the decay rate around the point is \( \frac{n-1}{2} \), therefore Theorem 4.1 holds for \( 0 \leq \beta \leq \frac{n-1}{2} := \beta_\phi \). If \( n = 1 \) and \( \phi(x) = x^d, \ d > 1 \), the phase of \( \hat{d\mu}_Y(\xi) \) around a critical point can take the form \( x^f p(x), \ 2 \leq f \leq d, \ p(x) \neq 0 \) locally. By using the fact the real parts of the possible poles of the corresponding local zeta functions have the form \( \frac{1}{d}, \ 2 \leq f \leq d \), and Theorem 8.4.2 in [10], one verifies that Theorem 4.1 holds for \( 0 \leq \beta \leq \frac{1}{d} := \beta_\phi \).

In the case of real numbers the results described in the previous remark are well-known (see e.g. [20]).

4.1. Restriction of the Fourier Transform to Non-degenerate Hypersurfaces. Let \( X \) be a submanifold of \( K^n \) with \( d\sigma_X \) its canonical measure. We set \( d\mu_{Y,S} = \Theta_S d\sigma_Y \), where \( \Theta_S \) is the characteristic function of a compact open set \( S \) in \( K^n \). We say that the \( L^\rho \) restriction property is valid for \( X \) if there exists a \( \tau(\rho) \) so that

\[
\left( \int_X |\mathcal{F}g(\xi)|_K^{\frac{1}{\tau}} d\mu_{X,S}(\xi) \right)^{\frac{1}{\tau}} \leq C_{\tau,\rho}(S) \|g\|_{L^\rho}
\]

holds for each \( g \in \mathbb{S}(K^n) \) and any compact open set \( S \) of \( K^n \).

The restriction problem in \( \mathbb{R}^n \) (see e.g. [20] Chap. VIII) was first posed and partially solved by Stein [5]. This problem have been intensively studied during the last thirty years [2], [20], [22], [27]. Recently Mockenhaupt and Tao have studied the restriction problem in \( F_q^n \) [15]. In this paper we study the restriction problem in the non-archimedean field setting. More precisely, in the case in which \( X \) is a non-degenerate hypersurface and \( \tau = 2 \). The proof of the restriction property in the non-archimedean case uses the Lemma of interpolation of operators (see e.g. [20] Chap. IX) and the estimates for oscillatory integrals obtained in the previous section. The interpolation Lemma given in [20] Chap. IX] is valid in the non-archimedean case. For the sake of completeness we rewrite this lemma here.

Let \( \{U^z\} \) be a family of operators on the strip \( a \leq \text{Re}(z) \leq b \) defined by

\[
(U^z g)(x) = \int_{K^n} \mathfrak{R}_z(x,y) g(y) |dy|,
\]

where the kernels \( \mathfrak{R}_z(x,y) \) have a fixed compact support and are uniformly bounded for \( (x,y) \in K^n \times K^n \) and \( a \leq \text{Re}(z) \leq b \). We also assume that for each \( (x,y) \), the function \( \mathfrak{R}_z(x,y) \) is analytic in \( a < \text{Re}(z) < b \) and is continuous in the closure \( a \leq \text{Re}(z) \leq b \), and that

\[
\begin{align*}
\|U^z g\|_{L^\tau_0} &\leq M_0 \|g\|_{L^{\rho_0}}, \text{ when } \text{Re}(z) = a, \\
\|U^z g\|_{L^\tau_1} &\leq M_1 \|g\|_{L^{\rho_1}}, \text{ when } \text{Re}(z) = b;
\end{align*}
\]

here \( (\tau_i, \rho_i) \) are two pairs of given exponents with \( 1 \leq \tau_i, \rho_i \leq \infty \).
Lemma 4.1 (Interpolation Lemma \cite{[20]} Chap. IX). Under the above hypotheses,
\[ \left\| U^{(1-\theta)+b\theta} g \right\|_{L^\tau} \leq M_0^{1-\theta} M_1^\theta \left\| g \right\|_{L^\rho} \]
where \( 0 \leq \theta \leq 1, \frac{1}{\tau} = \frac{(1-\theta)}{\tau_0} + \frac{\theta}{\tau_1}, \) and \( \frac{1}{\rho} = \frac{(1-\theta)}{\rho_0} + \frac{\theta}{\rho_1}. \)

Theorem 4.2. Let \( \phi(x) \in R_K[x], x = (x_1, \ldots, x_{n-1}), \) be a non-constant polynomial such that \( C_\phi(K) = \{0\} \subset K^{n-1}. \) Let
\[ Y = \{ x \in K^n \mid x_n = \phi(x_1, \ldots, x_{n-1}) \} \]
with the measure \( d\mu_{Y,S} = \Theta_S \sigma_Y, \) where \( \Theta_S \) is the characteristic function of a compact open subset \( S \) of \( K^n. \) Then
\[ (4.4) \quad \left( \int_Y \left| Fg(\xi) \right|^2 d\mu_{Y,S}(\xi) \right)^\frac{1}{2} \leq C(Y) \left\| g \right\|_{L^\rho}, \]
holds for each \( 1 \leq \rho < \frac{2(1+\beta_\phi)}{2+\beta_\phi}. \)

Proof. We first note that
\[ \int_Y \left| Fg(\xi) \right|^2 d\mu_{Y,S}(\xi) = \int_Y Fg(\xi) \overline{Fg(\xi)} d\mu_{Y,S}(\xi) \]
\[ = \int_{K^n} (Tg)(x) \overline{g(x)} \left| dx \right| \]
(4.5) where \( (Tg)(x) = (g \ast \mathcal{R})(x) \) with
\[ \mathcal{R}(x) = \int_Y \Psi([x,\xi]) d\mu_{Y,S}(\xi) = \tilde{d\mu_{Y,S}}(-x). \]
The theorem follows from (4.5) by Hölder’s inequality if we show that
\[ \left\| T(g) \right\|_{L^{\rho_0}} \leq C \left\| g \right\|_{L^{\rho_0}} \]
where \( \rho_0' \) is the dual exponent of \( \rho_0. \) Now we define \( \mathcal{R}_\mathcal{Z}(x) \) as equal to
\[ \gamma(z) \int_{K^n} \Psi([x,\xi]) \left| \xi_n - \phi(\xi') \right|^{-1+\gamma} \eta(\xi_n - \phi(\xi')) \Theta_S(\xi', \phi(\xi')) \left| d\xi \right|, \]
where \( \gamma(z) = \left( \frac{1-q^{-z}}{1-q^{-1}}, z \in (1, \ldots, n-1), \right. \) \( \eta(\xi) \) is the characteristic function of the ball \( P_{\epsilon_0}^0, \) \( \epsilon_0 \geq 1, \) and \( \text{Re}(z) > 0. \) By making \( y = \xi_n - \phi(\xi') \) in the above integral we obtain
\[ \mathcal{R}_\mathcal{Z}(x) = \mathcal{Z}_\mathcal{Z}(x_n) \mathcal{R}(x) \]
with
\[ \mathcal{Z}_\mathcal{Z}(x_n) = \gamma(z) \int_{K^n} \Psi(x_n y) \left| y \right|^{-1+\gamma} \eta(y) \left| dy \right|, \text{ Re}(z) > 0. \]
On the other hand,
\[
\zeta_z(x_n) = \begin{cases} 
q^{-e_0 z}, & \text{if } |x_n|^K \leq q^{e_0}; \\
\left(\frac{1-q^{-e_0}-1}{1-q^{-e_0}}\right)|x_n|^{-z}, & \text{if } |x_n|^K > q^{e_0}; 
\end{cases}
\]
(for a similar calculation the reader can see [23, page 54]), then \(\zeta_z(x_n)\) has an analytic continuation to the complex plane as an entire function; also \(\zeta_0(x_n) = 1\), and \(|\zeta_z(x_n)| \leq c|x_n|^{-\Re(z)}\) where \(|x_n|^K \geq q^{e_0}\). Therefore \(\zeta_z(x_n)\) has an analytic continuation to an entire function satisfying the following properties:

(i) \(\mathfrak{R}_0(x) = \mathfrak{R}(x)\),
(ii) \(|\mathfrak{R}_{-\beta+i\gamma}(x)| \leq C\), for every \(x \in K^n\), \(\gamma \in \mathbb{R}\), and \(0 \leq \beta \leq \beta_\phi - \epsilon\), \(\epsilon > 0\),
(iii) \(|\mathcal{F}\mathfrak{R}_{1+i\gamma}(x)| \leq C\), for \(x \in K^n\), and \(\gamma \in \mathbb{R}\).

In fact (ii) follows from Theorem 4.1, and (iii) is an immediate consequence of the definition of \(\mathfrak{R}_z(x)\).

Now we consider the analytic family of operators \(T_z(g) = (g * \mathfrak{R}_z)(x)\). From (ii) one has
\[
\|T_{-\beta+i\gamma}(g)\|_{L^\infty} \leq C \|g\|_{L^1},
\]
for \(0 \leq \beta \leq \beta_\phi - \epsilon\), \(\epsilon > 0\), and \(\gamma \in \mathbb{R}\), and from (iii) and Plancherel’s Theorem one gets
\[
\|T_{1+i\gamma}(g)\|_{L^2} \leq C \|g\|_{L^2},
\]
for \(\gamma \in \mathbb{R}\). By applying the Interpolation Lemma with
\[
\theta = \frac{\beta}{1+\beta},
\]
we obtain
\[
\|T_0(g)\|_{L^\rho'} \leq C \|g\|_{L^\rho},
\]
with \(\rho'\) the dual exponent of \(\rho = \frac{2(1+\beta)}{2+\beta}\), and \(0 \leq \beta \leq \beta_\phi - \epsilon\), \(\epsilon > 0\). Therefore the previous estimate for \(\|T_0(g)\|_{L^\rho'}\) is valid for \(1 \leq \rho \leq \frac{2(1+\beta_\phi - \epsilon)}{2+\beta_\phi - \epsilon}\). \(\square\)

Our proof of Theorem 4.2 is strongly influenced by Stein’s proof for the restriction problem in the case of a smooth hypersurface in \(\mathbb{R}^n\) with non-zero Gaussian curvature [19].

5. Asymptotic Decay of Solutions of Wave-type Equations

Like in the classical case [22], the decay of the solutions of wave-type pseudo-differential equations can be deduced from the restriction theorem proved in the previous section, taking into account that the following two problems are completely equivalent if \(\frac{1}{\rho} + \frac{1}{\sigma} = 1\):
Problem 1. For which values of $\rho$, $1 \leq \rho < 2$, is it true that $f \in L^\rho(K^n)$ implies that $Ff$ has a well-defined restriction to $Y$ in $L^2(d\mu_{Y,s})$ with
\[
\left( \int_Y |Ff|^2 d\mu_{Y,s} \right)^{\frac{1}{2}} \leq C_\rho \|f\|_{L^\rho}.
\]

Problem 2. For which values of $\sigma$, $2 < \sigma \leq \infty$, is it true that the distribution $gd\mu_{Y,s}$ for each $g \in L^2(d\mu_{Y,s})$ has Fourier transform in $L^\sigma(K^n)$ with
\[
\|F(gd\mu_{Y,s})\|_{L^\sigma} \leq C_\sigma \left( \int_Y |g|^2 d\mu_{Y,s} \right)^{\frac{1}{2}}.
\]

5.1. Wave-type Equations with Non-degenerate Symbols.

**Theorem 5.1 (Main Result).** Let $\phi(\xi) \in R_K[\xi]$, $\xi = (\xi_1, \ldots, \xi_n)$, be a non-constant polynomial such that $C_\phi(K) = \{0\} \subset K^n$. Let $u(x, t)$ be the solution of the following initial value problem:

\[
\begin{aligned}
(Hu)(x, t) &= 0, \quad x \in K^n, \quad t \in K, \\
u(x, 0) &= f_0(x),
\end{aligned}
\]

where $f_0(x) \in S(K^n)$. Then
\[
\|u(x, t)\|_{L^\sigma(K^{n+1})} \leq A \|f_0(x)\|_{L^2(K^n)},
\]
for $\frac{2(1+\beta_{\phi})}{\beta_\phi} < \sigma \leq \infty$.

**Proof.** Since
\[
u(x, t) = \int_{K^n} \Psi (t\phi(\xi) + [x, \xi]) \mathcal{F}f_0(\xi) \, d\xi
\]
\[
= \int_Y \Psi ([x, \xi]) \mathcal{F}f_0(\xi) \, d\mu_{Y,s}(\xi),
\]
where $\xi = (\xi, \xi_{n+1}) \in K^{n+1}$, $x = (x, t) \in K^{n+1}$,
\[
Y = \{\xi \in K^{n+1} | \xi_{n+1} = \phi(\xi)\},
\]
and $d\mu_{Y,s} = \Theta_S d\sigma_Y$, with $\Theta_S$ the characteristic function of a compact open set $S$ containing the support of $\mathcal{F}f_0$. By applying Theorem 4.2 replacing $n$ with $n + 1$, and dualizing, one gets
\[
\|u(x, t)\|_{L^\sigma(K^{n+1})} \leq A \|f_0(x)\|_{L^2(K^n)},
\]
where \( \sigma = \frac{2(1+\beta)}{\beta} \) is the dual exponent of \( \rho \) in Theorem 4.2 and \( 0 \leq \beta < \beta_\phi \), therefore (5.2) is valid for \( \frac{2(1+\beta_\phi)}{\beta_\phi} < \sigma \leq \infty \). \( \square \)

5.2. Wave-type Equations with Homogeneous Symbols. In the cases \( \phi(\xi) = a_1\xi_1^2 + \ldots + a_n\xi_n^2 \) and \( n = 1 \), \( \phi(\xi) = \xi^d \) by using Remark 1 we have the following estimations for the solution of Cauchy problem (1).

**Theorem 5.2.** If \( \phi(\xi) = a_1\xi_1^2 + \ldots + a_n\xi_n^2 \), then
\[
\| u(x,t) \|_{L^2(2^{(n+1)}(K^{n+1}))} \leq C \| f_0(x) \|_{L^2(K^n)}.
\]

**Theorem 5.3.** If \( n = 1 \) and \( \phi(\xi) = \xi^d \), then
\[
\| u(x,t) \|_{L^2(4^{(d+1)}(K^2))} \leq C \| f_0(x) \|_{L^2(K)}.
\]

In particular if \( d = 3 \), then
\[
\| u(x,t) \|_{L^8(K^2)} \leq C \| f_0(x) \|_{L^2(K)}.
\]

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