External Electromagnetic Fields of a Slowly Rotating Magnetized Star with Gravitomagnetic Charge

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Abstract

We study Maxwell equations in the external background spacetime of a slowly rotating magnetized NUT star and find analytical solutions for the exterior electric fields after separating the equations of electric field into angular and radial parts in the lowest order approximation. The star is considered isolated and in vacuum, with dipolar magnetic field aligned with the axis of rotation. The contribution to the external electric field of star from the NUT charge is considered in detail.

Keywords General relativity; Kerr-Taub-NUT spacetime; Electromagnetic fields.

1 Introduction

The existence of strong electromagnetic fields is one of the most important features of rotating magnetized neutron stars observed as pulsars (Hewish et al. 1968) and magnetars (Duncan and Thompson 1992). It was shown starting pioneering paper of Deutsch (1955) that the electric field is induced due to the rotation of highly magnetized star. The general relativistic effect of dragging of inertial frames is a source of additional electric field around rotating magnetized relativistic stars (Muslimov and Harding 1997).

Muslimov and Tsygan 1992, Rezolla et al. 2001, Rezolla and Ahmedov 2004, Kojima et al. 2004). One of the exotic solution of the Einstein’s equation of general relativity is achieved by introducing an extra nontrivial parameter, the so-called gravitomagnetic monopole moment or NUT charge. The generalized solution describing spacetime of a localized stationary and axisymmetric object with nonvanishing gravitomagnetic charge is known as the Kerr-Taub-NUT, where NUT stands for Newman-Unti-Tamburino (Newman et al. 1963). In the presence of the NUT charge the spacetime loses its asymptotically flatness property and, in contrast to the Kerr spacetime, becomes asymptotically nonflat. One of the features of the spacetime with NUT charge is that the later has no curvature singularity, there are conical singularities on its axis of symmetry that result in the gravitomagnetic analogue of Diracs string quantization condition (see e.g., Misner 1963, Misner and Taub 1968). The conical singularities can be removed by imposing an appropriate periodicity condition on the time coordinate. However, this generates closed timelike curves in the spacetime that makes it hard to interpret the solution as a regular black hole. In an alternative interpretation, one may consider the conical singularities as the source of a physical string threading the spacetime along the axis of symmetry (Bonnor 1969). In spite of these undesired features, the NUT solution still serves as an attractive example of spacetimes with asymptotic non-flat structure for exploring various physical phenomena in general relativity.

At present there is no any observational evidence for the existence of gravitomagnetic monopole or magnetic mass. Therefore it is interesting to study the electromagnetic fields in NUT space with the aim to get new tool for studying new important general relativistic effects which are associated with nondiagonal components of the metric tensor and have no Newtonian ana-
logues (see, e.g. Nouri-Zonoz 2004; Kagramanova et al. 2008; Morozova and Ahmedov 2009, where solutions for electromagnetic waves and interferometry in spacetime with NUT parameter have been studied). Kerr-Taub-NUT spacetime with Maxwell and dilaton fields is also recently investigated in (Aliev et al. 2008). In our preceding papers (Morozova et al. 2008; Abduljabbarov et al. 2008) we have studied the magnetospheric structure and related effects in a plasma surrounding a rotating, magnetized neutron star and charged particle motion around compact objects in the presence of the NUT parameter. General relativistic effects associated with the gravitomagnetic multipole moment of a gravitational source have been investigated by Kagramanova and Ahmedov (2006) through the analysis of the motion of test particles and electromagnetic fields distribution in the spacetime around the nonrotating cylindrical NUT source. The collision of two particles with the different rest masses moving in the equatorial plane of a Kerr-Taub-NUT spacetime has been considered by Liu et al. (2011). The analytic solutions of Maxwell equations for infinitely long cylindrical conductors with nonvanishing electric charge and currents in the external background spacetime of a line gravitomagnetic monopole have been presented by Ahmedov and Fattoyev (2008). We look for separable solutions of Maxwell equations in the form

\[ \begin{align*}
B^\tau(r, \theta) &= F(r) \cos \theta , \\
B^\phi(r, \theta) &= G(r) \sin \theta , \\
B^\theta(r, \theta) &= 0 ,
\end{align*} \]

where functions \( F(r) \) and \( G(r) \) will account for the relativistic corrections due to a curved background spacetime. In the case of infinite conductivity and as far as the stationary magnetic field is concerned, the study of Maxwell equations in a slow rotation metric provides no additional information with respect to a non-rotating metric. The dependence from the frame dragging effects and gravitomagnetic charge is therefore expected to appear at \( O(\omega^2, l^2) \). The stationary vacuum magnetic field external to an aligned magnetized relativistic star is known and given by Ginzburg and Ozernoy (1963)

\[ B^\tau = -\frac{3\mu}{4M^3} \left[ \ln N^2 + \frac{2M}{r} \left( 1 + \frac{M}{r} \right) \right] \cos \theta , \]

\[ B^\phi = \frac{3\mu N}{4M^2r} \left[ \frac{r}{M} \ln N^2 + \frac{1}{N^2} + 1 \right] \sin \theta , \]

where \( \mu \) is magnetic dipole. The search for the form of the electric field is much more involved than for the magnetic field. However, hereafter we will make use of the insight gained in Rezolla et al. (2001a) as a guide and start the derivation of the solution by rewriting

2 Electromagnetic Field Equations Around Magnetized Star with NUT parameter

Our approach is based on the reasonable assumption that the metric of spacetime is known, i.e. neglecting the influence of the electromagnetic field on the gravitational one and finding analytical solutions of Maxwell equations on a given, fixed background. The next our approximation is in the specific form of the background metric in a spherical coordinate system \((ct, r, \theta, \phi)\) which we choose to be that of a stationary, axially symmetric system truncated at the first order in the angular velocity \( \Omega \) and in gravitomagnetic monopole moment \( l \) as (see, for example, Dadhich and Turakulov 2002; Bini et al. 2003)

\[ ds^2 = -N^2dt^2 + N^{-2}dr^2 + r^2d\theta^2 + r^2 \sin^2 \theta d\phi^2 - 2 \left[ \frac{\omega(r)}{r} \sin^2 \theta + 2N^2 \cos \theta \right] dt d\phi , \]

where parameter \( N \equiv (1 - 2M/r)^{1/2} \) and \( \omega(r) \equiv 2J/r^3 \) can be interpreted as the angular velocity of a free falling (inertial) frame and is also known as the Lense-Thirring angular velocity, \( J = I(M, R)\Omega \) is the total angular momentum of metric source with total mass \( M \). The nondiagonal component of the metric tensor is finite at the infinity: \( \lim_{r \to \infty} g_{03} = -2l \cos \theta \) which is meant that the metric is not asymptotically flat.

Here we will look for stationary solutions of the Maxwell equation, i.e. for solutions in which we assume that the magnetic moment of the star does not vary in time as a result of the infinite conductivity of the stellar interior (see for the details of the stellar model to Ahmedov and Fattoyev (2008)). We look for separable solutions of Maxwell equations in the form

\[ \begin{align*}
B^\tau(r, \theta) &= F(r) \cos \theta , \\
B^\phi(r, \theta) &= G(r) \sin \theta , \\
B^\theta(r, \theta) &= 0 ,
\end{align*} \]
vacuum Maxwell equations as

\[ \sin \theta (\omega r^2 B^\rho)_r + \frac{2 l \cos \theta}{\sin \theta} (N^2 B^\rho)_r + \omega N^{-1} r \left( \sin \theta B^\rho \right)_{,\rho} + \frac{2 l N}{r} \left( \cos \theta \sin \theta B^\rho \right)_{,\theta} = 0, \]

and which already indicate that the dragging of inertial frames with angular velocity \( \omega \) and gravitomagnetic charge, respectively. Using as a reference the solutions for slowly rotating magnetized sphere \( \text{Rezolla et al.} \ (2001a) \), we look for the simplest solutions of vacuum Maxwell equations in the form

\[ E^\rho = \left\{ \frac{15 \omega r^3}{16 M^2 c} C_3 \left[ (3 - \frac{2M}{r}) \ln N^2 + \frac{2M^2}{3r^2} - 4 \right] + \frac{2M}{r} + \frac{2M^2}{5r^2} \ln N^2 + \frac{4M^3}{5r^3} \right\} + \frac{\Omega}{6cR^2} C_1 C_2 \left[ \frac{3M^2}{2} - 4 \right] \]

\[ + \left( 3 - \frac{2M}{r} \right) \ln N^2 - 2 \left( \frac{M^2}{3r^2} N^2 \right) \right\} (3 \cos^2 \theta - 1) \mu + E^\rho_1, \]  

\[ E^\theta = -\left\{ \frac{45 \omega r^3}{16 M^3 c} N C_3 \left[ (1 - \frac{r}{M}) \ln N^2 - 2 - \frac{2M^2}{3r^2 N^2} \right] + \frac{4M^4}{15r^4 N^2} \left\} + \frac{\Omega}{2cR^2} C_1 C_2 N \left[ (1 - \frac{r}{M}) \ln N^2 \right] \]

\[ - 2 - \frac{2M^2}{3r^2 N^2} \right\} \mu \sin \theta + E^\rho_1. \]  

The values of the arbitrary constants \( C_1, C_2 \) and \( C_3 \) are found after imposing the continuity of the tangential electric field across the star surface \( \text{Rezolla et al.} \ (2001a) \)

\[ C_1 = -\frac{3R^3}{4M^3} \ln \left( 1 - \frac{2M}{R} \right) + \frac{2M}{R} \left( 1 + \frac{M}{R} \right), \]

\[ C_2 = \frac{M^2}{15R^2} C_2 \left[ \ln \left( \frac{N^2}{M^2} \right) - 2 - \frac{2M^2}{3R^2 N^2} \right]^{-1}, \]

\[ C_3 = \frac{2M^2}{15R^2} C_2 \left[ \ln \left( \frac{N^2}{M^2} \right) + \frac{2M}{R} \right], \]  

with \( N^2 \equiv N^2(r = R) = 1 - 2M/R \).

For unknown components \( E^\rho_1 \) and \( E^\theta \) of the electric field we have the following set of differential equations

\[ \sin \theta (r^2 E^\rho_1)_r + N^{-1} \left( \sin \theta E^\rho_1 \right)_{,\rho} = 0, \]

\[ \frac{3\mu}{M^2 r^2} \left[ \frac{\cos^2 \theta}{\sin \theta} \left( \ln N^2 + \frac{2M}{r} + \frac{2M^2}{r^2} \right) \right. \]

\[ \times \left( \frac{r}{M} \ln N^2 + \frac{1}{N^2} + 1 \right) \]  

\[ - E^\rho_{1,\rho} + \left( r N E^\rho_1 \right)_r = 0, \]  

derived from the Maxwell equations \( 5 \) and \( 6 \).

### 3 Electromagnetic Field due to nonvanishing Gravitomagnetic Charge

In this section we will try to find exact analytical solution for electromagnetic field of magnetized neutron star due to nonvanishing gravitomagnetic charge. Introducing new variables

\[ F_1(r, \theta) = r^2 E^\rho_1, \]

\[ F_2(r, \theta) = r N \sin \theta E^\rho_1, \]

\[ G(r, \theta) = \frac{3\mu}{M^2 r^2} \left[ \frac{\cos^2 \theta}{\sin \theta} \left( \ln N^2 + \frac{2M}{r} + \frac{2M^2}{r^2} \right) \right. \]

\[ \times \left( \frac{r}{M} \ln N^2 + \frac{1}{N^2} + 1 \right) \]  

and making some algebraic transformations one can write the equations \( 12 \) and \( 13 \) in more convenient way

\[ r^2 N^2 F_{1,rr} + F_{1,\theta\theta} + 2M F_{1,r} + \]

\[ \cot \theta F_{1,\theta} - \frac{r^2}{\sin \theta} (\sin \theta G), \theta = 0, \]  

\[ F_{2,\theta} = -N^2 F_{1,r} \sin \theta. \]  

The last term in the right hand side of the equation \( 15 \) in the linear in \( l \) approximation and neglecting \( l M/r^2 \ll 1 \) takes the following form

\[ \frac{r^2}{\sin \theta} (\sin \theta G), \theta \approx \frac{4\mu}{r^2} \cos \theta + O \left( \frac{M}{r^2} \right). \]  

Inserting these expansions into equation \( 16 \) one can get the set of equations to be solved including \( 16 \) and the following one:

\[ \xi (\xi - 1) F_{1,\xi\xi} + F_{1,\theta\theta} + F_{1,\xi} + \cot \theta F_{1,\theta} - \frac{\mu}{M^2 l^2} \cos \theta = 0, \]
which has been rewritten in new variable $\xi = r/2M$.

Solution of equation (18) can be easily presented in the more convenient form, which has been rewritten in new nondimensional variables

$$\xi(\xi - 1)f_\xi + f_\xi - 2f = \frac{\mu l}{M^2\xi^2},$$

through two separate functions of different variables $\xi$ and $\theta$ as $F_1(\xi, \theta) = f(\xi)\cos \theta$. The equation (19) can be solved analytically as

$$F_1(\xi, \theta) = \frac{\mu l \xi^2}{M^2} \left[ C_4 \left( \frac{1}{\xi} + \frac{1}{2\xi^2} \right) + \ln \left( \frac{\xi - 1}{\xi} \right) - \frac{1}{3\xi^3} \right] \cos \theta,$$

where $C_4$ is the integration constant.

According to the definition [14] the values of electric field $E^\varphi_l$ and $E^\theta_l$ produced by the effect of gravitomagnetic charge $l$ are

$$E^\varphi_l(r, \theta) = \frac{\mu l \xi^2}{Mr^3} \left[ C_4 \left( 1 + \frac{r}{M} \right) + \frac{r^2}{2M^2} \ln N^2 \right] \frac{r}{2M} - \frac{2}{3} \cos \theta,$$

$$E^\theta_l(\xi, \theta) = \frac{\mu l \xi^2}{Mr^3} \left[ \frac{1}{6} - \left( 1 + \frac{r}{M} + \frac{r^2}{M^2} N^2 \ln N \right) \right] N \cos 2\theta,$$

where $C_4$ can be easily found from boundary condition and has the following form:

$$C_4 = \left[ \frac{2M}{3R} \left( 1 - \frac{6M}{R} \right) - \frac{3R}{MN_R} \left( 1 + N^2_R \right) - \frac{3R^2}{M^2N_R} \ln N^2_R \right] \theta + \frac{R}{M} + \frac{R^2}{M^2N^2_R} \ln N_R \right]^{-1}.$$

For the models of neutron star in the present study we choose the following parameters $R = 10$ km and the polar surface field strength $B(\varphi = r/R = 1) = 10^{12}$G, period $T = 10^{-2}$ s. Following this, we plot the radial dependence of the exterior electric field for various values of $l$ as shown in Fig. 1. The enhancement of the exterior electric field at the surface of the relativistic star is given in Table 1 which varies for several times depending on parameter $l$ selected. As can be seen from the presented graphs in Fig. 1 an additional electric field produced by NUT parameter is becoming extremely important outside the star depending on the value of parameter $l$ selected and maybe more relevant to observational phenomenology from slowly rotating pulsars with low rotation period.

### 4 Conclusion

Here we have considered the external solution for electromagnetic fields of magnetized neutron star in the presence of nonvanishing NUT parameter. The central object considered as slowly rotating and magnetic moment of the latter is considered not changing by time. The dependence of the electromagnetic field from the NUT charge in analytical way is presented in this paper. In the previous works the upper limit for the gravitomagnetic charge has been obtained comparing astrophysical data with theoretical results as (the units of the gravitomagnetic charge are taken in units of stellar mass $M = 1.5M_\odot$) (i) $l \leq 0.006M$ from the gravitational microlensing [Rahvar and Habibi 2004], (ii) $l \leq 4.52M$ from the interferometry experiments on ultracold atoms [Morozova and Ahmedov 2009], (iii) and similar limit has been obtained from the experiments on Mach-Zehnder interferometer [Kagramanova et al 2008]. In the linear approximation in NUT parameter the external magnetic field depends only from mass of neutron star whereas exterior electric fields depend on the gravitomagnetic charge linearly. The detailed comparison of the astrophysical data from pulsars related the electromagnetic phenomena might give one more precise astrophysical limit for the NUT parameter in this framework.

### Acknowledgments

Authors thank the IUCAA for warm hospitality during their stay in Pune and AS-ICTP for the travel support. This research is supported in part by the UzFFR (projects 1-10 and 11-10) and projects FA-F2-F079 and FA-F2-F061 of the UzAS. ABJ thanks the TWAS for the Regular Associateship grant. AAA thanks the German Academic Exchange Service (DAAD) for financial support.
Fig. 1 Radial dependence of the radial (a) and azimuthal (b) components of the electric field of magnetized rotating star with nonvanishing NUT parameter for the different values of gravitomagnetic monopole momentum $l$. The units of the gravitomagnetic monopole momentum are taken in mass of the star with mass $M = 1.5 M_\odot$.

Table 1 The amplification of the magnitude of exterior electric field around the typical relativistic compact star with gravitomagnetic monopole momentum in the distance of $R$ from the surface of the star. The mass of star is $M = 1.5 M_\odot$, period $T = 10^{-2}$ s and the radius is $R = 10$ km. The values of NUT parameter $l$ are taken as usual in units of $M$, the values of electric field are taken in units $10^{13}$ V/m. For the comparison we also provide the values for the exterior electric field corresponding to the upper values of the NUT charge obtained by [Rahvar and Habibi (2004); Morozova and Ahmedov (2009)].

| $l$  | $0.006$ | $0.01$ | $0.02$ | $0.03$ | $0.05$ | $0.08$ | $0.1$ | $4.52$ |
|------|---------|--------|--------|--------|--------|-------|-------|-------|
|      | Rahvar and Habibi (2004) |        |        |        |        |       |       |       |
| $E$  | $0.977$ | $1.14$ | $1.61$ | $2.12$ | $3.18$ | $4.81$ | $5.90$ | $249.74$ |
References

Abdujabbarov, A. A., Ahmedov, B. J., Kagramanova, V. G.: Gen. Rel. Grav. 40, 2515 (2008)
Ahmedov, B. J., Fattoyev, F. J.: Phys. Rev. D 78, 047501 (2008)
Ahmedov, B. J., Khugaev, A. V., Rakhmatov, N. I.: Int. J. Mod. Phys. D 14, 687 (2005)
Aliev, A. N., Cebeci, H., Dereli, T: Phys. Rev. D 77, 124022 (2008)
Bini, D., Cherubini, C., Janzen, R.T., Mashhoon, B.: Class. Quantum Grav. 20, 457 (2003)
Bonnor, W. B.: Proc. Cambridge Philos. Soc. 66, 145 (1969)
Dadhich, N., Turakulov, Z.Ya.: Class. Quantum Grav. 19, 2765 (2002)
Deutsch, A. J.: Ann. Astrophys. 1, 1 (1955)
Duncan, R. C., Thompson C.: Astrophys. J. 392, L9 (1992)
Ginzburg, V. L., Ozernoy, L. M.: Sov. Phys. JETP 20, 689 (1965)
Hewish, A., Bell, S. J., Pilkington, J. D. H., Scott, P. F., Collins, R. A.: Nature 217, Issue 5130, 709 (1968)
Kagramanova, V., Ahmedov, B.: Gen. Rel. Gravit. 38, Issue 5, 823 (2006)
Kagramanova, V., Kunz, J., Lämmerzahl, C.: Class. Quantum Grav. 25, 105023 (2008)
Kojima, Y., Matsumaga, N., Okita, T: Mon. Not. R. Astron. Soc. 348, 1388 (2004)
Liu C., Chen S., Ding C., Jing J.: Phys. Lett. B 701, Issue 3, 285 (2011)

Misner, C.W.: J. Math. Phys. (N.Y.) 4, 924 (1963)
Misner, C.W., Taub, A. H.: Zh. Eksp. Teor. Fiz. 55, 223 (1968)
Morozova, V. S., Ahmedov, B. J., Kagramanova, V. G.: Astrophys. J. 684, 1359 (2008)
Morozova, V.S., Ahmedov, B.J.: Int. J. Mod. Phys. D 18, 107 (2009)
Muslimov, A., K. Harding, A.: Astrophys. J. 485, 735 (1997)
Muslimov, A., Tsygan, A. I.: Mon. Not. R. Astron. Soc. 255, 61 (1992)
Newman, E., Tamburino, L., Unti, T: J. Math. Phys. 4, 915 (1963)
Nouri-Zonoz, M.: Class. Quantum Grav. 21, 471 (2004)
Rahvar, S., Habibi, F.: Astroph. J., 610, 673 (2004)
Rezzolla, L., Ahmedov, B. J., Miller, J. C.: Mon. Not. R. Astron. Soc. 322, 723 (2001a); Erratum 338, 816(E) (2003)
Rezzolla, L., Ahmedov, B. J., Miller, J. C.: Found. Phys. 31, 1051 (2001b)
Rezzolla, L., Ahmedov, B. J.: Mon. Not. R. Astron. Soc. 352, 1161 (2004)