A GENERALIZED DOUGLAS-RACHFORD SPLITTING ALGORITHM FOR NONCONVEX OPTIMIZATION

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Abstract. In this paper, we propose a generalized Douglas-Rachford splitting method for a class of nonconvex optimization problem. A new merit function is constructed to establish the convergence of the whole sequence generated by the generalized Douglas-Rachford splitting method. We then apply the generalized Douglas-Rachford splitting method to two important classes of nonconvex optimization problems arising in data science: low rank matrix completion and logistic regression. Numerical results validate the effectiveness of our generalized Douglas-Rachford splitting method compared with some other classical methods.

Key words. Generalized Douglas-Rachford splitting method; nonconvex optimization problems; global convergence; low rank matrix completion; logistic regression

AMS subject classifications. 90C26, 90C30, 90C90,15A83,65K05

1. Introduction. In this paper we study the following general model:

\begin{equation}
\min_u f(u) + g(u),
\end{equation}

where \(f\) and \(g\) are proper closed possibly nonconvex functions. Many nonconvex optimization problems can be formulated into this form. It often appears in many modern applications such as machine learning, statistics learning and low rank matrix recovery. In these applications, the function \(f\) is usually data fitting term and the function \(g\) is some regularization term. To solve such problems, an important and powerful class of algorithms is the splitting methods, such as forward-backward splitting [4, 12], Douglas-Rachford splitting [10, 11], the alternating direction method of multipliers (ADMM) [38, 23] and its linearized versions [28], primal-dual algorithms [8, 35, 34] and (generalized) alternating projections algorithms [16, 17]. In this class of methods, the objective functions is split into simpler individuals which appear in the corresponding subproblems. In particular, one popular splitting method in the literature is the Douglas-Rachford (DR) splitting method.

The DR splitting method was originally introduced in [19] for finding numerical solutions of heat differential equations. Lions and Mercier [27] used this method to minimize the sum of two closed convex functions \(f(u) + g(u)\), by solving the optimality condition

\begin{equation}
0 \in \partial f(u) + \partial g(u).
\end{equation}

In recent years, the behavior of the DR splitting algorithm in the nonconvex cases has attracted a lot of attention. One reason is that the theoretical results in the nonconvex
case are far from complete and the other reason is that DR method has been successfully applied to many important practical nonconvex problems [5, 29, 30, 15, 10], such as the feasibility problem, which aims to find a point in the intersection of two closed sets. Hesse and Luke showed in [29] that the DR splitting method exhibits local linear convergence for an affine set and a super-regular set, see also [15] for the similar results of DR splitting method for two super-regular sets. Li and Pong [10] showed that DR splitting method can be applied to the nonconvex problem (1.1) under some assumptions for \( f \) and \( g \), and then used this nonconvex DR splitting method to find a point in the intersection of a closed convex set \( C \) and a general closed set \( D \). Very recently, Themelis and Patrinos [3] gave a unified method to prove the convergence for ADMM and DR splitting applied to nonconvex problems under less restrictive assumptions than [10].

In this paper, we propose a generalized Douglas-Rachford (GDR) splitting method to make it more effective than DR splitting method for many important nonconvex optimization problems arising in machine learning. We consider the problem (1.1) for \( f \) having a Lipschitz continuous gradient and \( g \) being a proper closed function. We show that, if the step-size parameter is smaller than a computable threshold and the sequence generated by the GDR splitting method has a cluster point, then it gives a stationary point to a Tykhonov regularization of problem (1.1). Remark that, for many important non-convex optimization problems, this Tykhonov regularization of problem (1.1) will be shown to be very efficient. We also present some sufficient conditions to guarantee the boundedness of the sequence generated by the GDR splitting, and so the existence of cluster point. Furthermore, we show the convergence of the whole sequence generated by the GDR splitting under the additional assumption that \( f \) and \( g \) are semi-algebraic.

The generalized Douglas-Rachford splitting method is applied to solve the low rank matrix completion. This problem appears in many areas of engineering and applied science such as machine learning [37], computer vision [7] and control [24]. Specifically, matrix completion problem is to recover an unknown matrix from a sampling of its entries. This problem is extraordinarily ill-posed due to the fewer samples than entries. Therefore, we have infinitely many completions and identifying the "correct" solution from these candidate solutions is apparently impossible. However, in many practical applications, the matrix we wish to recover has low rank or approximately low rank structure, such as the famous Netflix problem [33], the structure-from-motion problem in computer vision, multi-class learning in machine learning and so on. Under this assumption of low rank, the matrix completion problem becomes a feasibility problem, which also attracts a rapidly growing interest see, for example, [22, 25, 20] and references therein. When the GDR splitting method is applied to the low rank matrix completion, the results show that our GDR splitting method is more accurate than classical methods such as singular value projection (SVP) method [25] and singular value thresholding (SVT) method [22]. In addition, our algorithm outperforms the DR splitting method for this problem. Finally, we also apply our algorithm to a nonconvex optimization problem induced by logistic regression. Logistic regression is an important method borrowed by machine learning from the field of statistics. Nowadays, it has become a go-to method for binary classification problems. Besides machine learning, logistic regression is also used in other fields, such as most medical fields and social sciences, see, for example [32, 21]. We will see that for logistic regression problem, our algorithm is better than the DR splitting method in...
terms of accuracy, runtime and number of iterations.

The rest of this paper is structured as follows. In section 2, we give some notations and preliminary materials. In section 3, we introduce the generalized Douglas-Rachford splitting algorithm and establish the convergence of this algorithm for non-convex optimization problems where the objective function is the sum of a smooth function \( f \) with Lipschitz continuous gradient and a closed function \( g \). In section 4, as stated before, we demonstrate how the GDR splitting method can be applied well to solve two important classes of nonconvex optimization problems arising in machine learning: low rank matrix completion and logistic regression. In section 5, we give some concluding remarks.

2. Notation and preliminaries. In this section, we introduce our notations and state some basic concepts. We refer the reader to the textbooks [31, 14] for these basic knowledge.

Let \( \mathbb{R}^n \) denote the \( n \)-dimensional Euclidean space, \( \langle \cdot, \cdot \rangle \) denote the inner product and the induced norm by \( \| \cdot \| = \sqrt{\langle \cdot, \cdot \rangle} \). For an extended-real-valued function \( f : \mathbb{R}^n \to (-\infty, \infty] \), we say that \( f \) is proper if it is never \(-\infty\) and its domain, \( \text{dom} f := \{ x \in \mathbb{R}^n : f(x) < +\infty \} \), is nonempty. The function is called closed if it is proper and lower semicontinuous.

**Definition 2.1.** (limiting subdifferential) Let \( f \) be a proper function. The limiting subdifferential of \( f \) at \( x \in \text{dom} f \) is defined by

\[
\partial f(x) := \left\{ v \in \mathbb{R}^n : \exists x^t \to x, f(x^t) \to f(x), v^t \to v \text{ with } \liminf_{z \to x^t} \frac{f(z) - f(x^t) - \langle v^t, z - x^t \rangle}{\|z - x^t\|} \geq 0 \text{ for each } t \right\},
\]

If \( f \) is differentiable at \( x \), we have \( \partial f(x) = \{ \nabla f(x) \} \). If \( f \) is convex, we have

\[
\partial f(x) = \left\{ v \in \mathbb{R}^n : f(z) \geq f(x) + \langle v, z - x \rangle \text{ for any } z \in \mathbb{R}^n \right\},
\]

which is the classic definition of subdifferential in convex analysis. Moreover, we also have the following robustness property:

\[
\left\{ v \in \mathbb{R}^n : \exists x^t \to x, f(x^t) \to f(x), v^t \to v, v^t \in \partial f(x^t) \right\} \subseteq \partial f(x).
\]

A point \( x^* \) is a stationary point of a function \( f \) if \( 0 \in \partial f(x^*) \). \( x^* \) is a critical point of \( f \) if \( f \) is differentiable at \( x^* \) and \( \nabla f(x^*) = 0 \). A function is called to be coercive if \( \liminf_{\|x\| \to \infty} f(x) = \infty \). We say that a function \( f \) is a strongly convex function with modulus \( \sigma > 0 \) if \( f - \frac{\sigma}{2} \| \cdot \|^2 \) is a convex function.

For any \( \gamma > 0 \), the proximal mapping of \( f \) is defined by

\[
P_{\gamma f}(x) : x \to \arg\min_{y \in \mathbb{R}^n} \left\{ f(y) + \frac{1}{2\gamma} \| y - x \|^2 \right\},
\]
assuming that the arg min exists, where \(\rightarrow\) means a possibly set-valued mapping.

**Definition 2.2. (indicator function).** For a closed set \(S \subseteq \mathbb{R}^n\), its indicator function \(\delta_S\) is defined by

\[
\delta_S(x) = \begin{cases} 
0, & \text{if } x \in S, \\
+\infty, & \text{if } x \notin S.
\end{cases}
\]

**Definition 2.3. (real semialgebraic set).** A semi-algebraic set \(S \subseteq \mathbb{R}^n\) is a finite union of sets of the form

\[
\left\{ x \in \mathbb{R}^n : h_1(x) = \cdots = h_k(x) = 0, \ g_1(x) < 0, \ldots, g_l(x) < 0 \right\},
\]

where \(g_1, \ldots, g_l\) and \(h_1, \ldots, h_k\) are real polynomials.

**Definition 2.4. (real semialgebraic function).** A function \(F : \mathbb{R}^n \to \mathbb{R}\) is semi-algebraic if the set \(\{(x, F(x)) \in \mathbb{R}^{n+1} : x \in \mathbb{R}^n\}\) is semi-algebraic.

Remark that the semi-algebraic sets and semi-algebraic functions can be easily identified and contain a large number of possibly nonconvex functions arising in applications, such as see [13, 12, 18]. We will also use the following Kurdyka-Lojasiewicz (KL) property which holds in particular for semi-algebraic functions.

**Definition 2.5. (KL property and KL function).** The function \(F : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}\) has the Kurdyka-Lojasiewicz property at \(x^* \in \text{dom } \partial F\) if there exist \(\eta \in (0, \infty]\), a neighborhood \(U\) of \(x^*\), and a continuous concave function \(\varphi : [0, \eta) \to \mathbb{R}_+\) such that:

(i) \(\varphi(0) = 0, \ \varphi \in C^1((0, \eta)),\) and \(\varphi'(s) > 0\) for all \(s \in (0, \eta)\);

(ii) for all \(x \in U \cap [F(x^*) < F < F(x^*) + \eta]\) the Kurdyka-Lojasiewicz inequality holds, i.e.,

\[
\varphi'(F(x) - F(x^*)) \text{dist}(0, \partial F(x)) \geq 1.
\]

If the function \(F\) satisfies the Kurdyka-Lojasiewicz property at each point of \(\text{dom } \partial F\), it is called a KL function.

Remark 2.6. It follows from [13] that a proper closed semi-algebraic function always satisfies the KL property.

3. Generalized Douglas-Rachford splitting algorithm and its convergence. In this section, we consider the following optimization problem:

\[
\min_u f(u) + g(u),
\]

where \(f\) and \(g\) are proper closed possibly nonconvex functions. Problems in the form \((3.1)\) arise naturally in many areas of engineering and applied science. For example, many sparse learning problems has the form of \((3.1)\) where \(f\) is a loss function and \(g\) is a regularizer with the proximal mapping of \(g\) easy to compute. In the applications of this paper we will take \(g = \|\cdot\|_0\) as a regularizer that is apparently nonconvex and the proximal mapping of \(g\) is well-defined and easy to compute. Therefore, we make additionally the following assumption on \(g\) in this paper.

**Assumption 1** The function \(g\) is proper closed with a nonempty proximal mapping \(P_{\gamma g}(x)\) for any \(x\) and for \(\gamma > 0\).

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In addition, we notice that the Lipschitz differentiability of \( f \) is very important to the recent convergence analysis of the DR splitting method in the nonconvex settings, for instance in [10]. Therefore, we make the following assumption on \( f \) throughout this paper.

**Assumption 2** The function \( f \) has a Lipschitz continuous gradient whose Lipschitz continuity constant is bounded by \( L > 0 \).

**Remark 3.1.** Under the Assumption 2, we can always find \( l \in \mathbb{R} \) such that \( f + \frac{l}{2} \| \cdot \|^2 \) is convex, in particular, we can take \( l = L \).

Now we give the generalized Douglas-Rachford splitting method as follows:

**Algorithm 3.1** Generalized Douglas-Rachford Splitting Algorithm

**Step 0.** Choose a step-size \( \gamma > 0, \alpha \in (0, 2] \) and an initial point \( x^0 \).

**Step 1.** Set

\[
\begin{align*}
    u^{t+1} &\in \arg \min_u \left\{ f(u) + \frac{1}{2\gamma} \| u - x^t \|^2 \right\}, \\
    v^{t+1} &\in \arg \min_v \left\{ g(v) + \frac{1}{2\gamma} \| v - \alpha u^{t+1} + x^t \|^2 \right\}, \\
    x^{t+1} &= x^t + (v^{t+1} - u^{t+1}).
\end{align*}
\]

**Step 2.** If a termination criterion is not met, go to Step 1.

Remark that, the above algorithm (3.2) is the DR splitting algorithm when \( \alpha = 2 \).

It is known from [10] that the convergence analysis of the DR splitting method relies on the so called Douglas-Rachford merit function given by (see Definition 2 in [10])

\[
D_{\gamma}(y, z, x) := f(y) + g(z) - \frac{1}{2\gamma} \| y - z \|^2 + \frac{1}{\gamma} \langle x - y, z - y \rangle,
\]

which was motivated by the Douglas-Rachford envelope considered in [26] in the convex case. Inspired by these, we construct the following generalized Douglas-Rachford merit function:

\[
M_{\gamma}(u, v, x) := f(u) + g(v) - \frac{1}{2\gamma} \| u - v \|^2 + \frac{1}{\gamma} \langle x - (\alpha - 1)u, v - u \rangle + \frac{2 - \alpha}{2\gamma} \| u \|^2.
\]

Moreover, it is not hard to see that the merit function \( M_{\gamma} \) can be alternatively written as

\[
\begin{align*}
    M_{\gamma}(u, v, x) &= f(u) + g(v) + \frac{1}{2\gamma} \| \alpha u - v - x \|^2 - \frac{1}{2\gamma} \| x - (\alpha - 1)u \|^2 \\
    &\quad - \frac{1}{\gamma} \| u - v \|^2 + \frac{2 - \alpha}{2\gamma} \| u \|^2 \\
    &= f(u) + g(v) + \frac{1}{2\gamma} \left( \| x - u \|^2 - \| x - v \|^2 \right) + \frac{1}{\gamma} \langle (2 - \alpha)u, v - u \rangle \\
    &\quad + \frac{2 - \alpha}{2\gamma} \| u \|^2.
\end{align*}
\]
where the Equation (3.4a) follows from the elementary relation \( \langle a, b \rangle = \frac{1}{2} (\|a + b\|^2 - \|a\|^2 - \|b\|^2) \) applied with \( a = x - (\alpha - 1)u \) and \( b = v - u \) in (3.3), and the Equation (3.4b) follows from the relation \( \langle a, b \rangle = \frac{1}{2} (\|a\|^2 + \|b\|^2 - \|a - b\|^2) \) in (3.3) with \( a = x - u \) and \( b = v - u \). These equivalent relations will be made use of in our convergence analysis.

By using the optimal conditions and the subdifferential calculus rule [31] for \( u \) and \( v \)-updates in (3.2) we have

\[
\begin{align*}
0 &= \nabla f(u^{t+1}) + \frac{1}{\gamma}(u^{t+1} - x^t), \\
0 &\in \partial g(v^{t+1}) + \frac{1}{\gamma}(v^{t+1} - x^t - \alpha u^{t+1}).
\end{align*}
\]

By the optimal condition (3.5) and the Lipschitz continuity of \( \nabla f \), we can easily get the following lemma.

**Lemma 3.2.** Suppose function \( f \) has a Lipschitz continuous gradient whose Lipschitz continuity constant is bounded by \( L > 0 \). Then the sequence \( \{(u^t, v^t, x^t)\} \) generated by (3.2) satisfies

\[
\|x^t - x^{t-1}\| \leq (1 + \gamma L)\|u^{t+1} - u^t\|.
\]

Next, we state and prove a convergence theorem for the generalize Douglas-Rachford splitting algorithm (3.2). We point out that our proof mainly follows the similar line of arguments as [10]. However, some new difficulties occur in our proof. First, we will make use of the merit function (3.3) instead of the Douglas-Rachford merit function. Second, as shown in the upper estimate in (3.20), we will use \( \frac{2 - \alpha}{2\gamma} \|u\|^2 \) to offset the additional terms caused by our GDR splitting to establish the non-increasing property of the sequence \( \{M_\gamma(u^t, v^t, x^t)\} \).

**Theorem 3.3.** (Global subsequential convergence) Assume that \( 0 < \alpha \leq 2 \) and the parameter \( \gamma > 0 \) is chosen such that

\[
\frac{4 - \alpha}{2} (1 + \gamma L)^2 + \frac{8 - 2\alpha + 1}{2} \gamma l - \frac{1 + \alpha}{2} < 0.
\]

Then the sequence \( \{M_\gamma(u^t, v^t, x^t)\} \) is nonincreasing. Moreover, if a cluster point of the sequence \( \{(u^t, v^t, x^t)\} \) exists, then

\[
\lim_{t \to \infty} \|x^{t+1} - x^t\| = \lim_{t \to \infty} \|v^{t+1} - u^{t+1}\| = 0.
\]

Furthermore, for any cluster point \((u^*, v^*, x^*)\), we have \( u^* = v^* \), and

\[
\nabla f(v^*) + \partial g(v^*) + \frac{1}{\gamma}(2 - \alpha)v^*.
\]

**Remark 3.4.** Note that

\[
\lim_{\gamma \to 0, \alpha \to 2} \left[ \frac{4 - \alpha}{2} (1 + \gamma L)^2 + \frac{8 - 2\alpha + 1}{2} \gamma l - \frac{1 + \alpha}{2} \right] = -\frac{1}{2} < 0.
\]

Therefore, given \( l \in \mathbb{R} \) and \( L > 0 \), the condition (3.8) will be fulfilled if \( \gamma > 0 \) is sufficiently small and \( \alpha \) is sufficiently close to 2.
Substituting (3.16) to (3.12), we obtain
\[ H \leq \frac{1}{\gamma} \langle x^{t+1} - x^t, v^{t+1} - u^{t+1} \rangle \]
\[ = \frac{1}{\gamma} \| x^{t+1} - x^t \|^2. \]
Hence, we see further that
\[ \frac{1}{\gamma} \| x^{t+1} - x^t \|^2. \]

Secondly, employing (3.4a) and the fact that \( v^{t+1} \) is a minimizer, we have
\[ M_v(u^{t+1}, v^{t+1}, x^t) - M_v(u^{t+1}, v^t, x^t) \]
\[ = g(v^{t+1}) + \frac{1}{2\gamma} \| \alpha x^{t+1} - v^{t+1} - x^t \|^2 - \frac{1}{\gamma} \| x^{t+1} - v^{t+1} \|^2 \]
\[ - g(v^t) - \frac{1}{2\gamma} \| \alpha x^{t+1} - v^t - x^t \|^2 + \frac{1}{\gamma} \| x^{t+1} - v^t \|^2 \]
\[ \leq \frac{1}{\gamma} (\| x^{t+1} - v^{t+1} \|^2 - \| x^{t+1} - v^t \|^2) \]
\[ = \frac{1}{\gamma} (\| x^{t+1} - v^t \|^2 - \| x^{t+1} - x^t \|^2), \]
where the definition of \( x^{t+1} \) is used in the last equality. Note that from equation (3.5),
we have
\[ \nabla \left( f + \frac{1}{2} \| \cdot \|^2 \right) (u^{t+1}) = \frac{1}{\gamma} (x^t - u^{t+1}) + l u^{t+1}. \]
Recall that \( f + \frac{1}{2} \| \cdot \|^2 \) is convex function, by the monotonicity of the gradient of a convex function, we obtain that for any \( t \geq 1, \)
\[ \left( \frac{1}{\gamma} (x^t - u^{t+1}) + l u^{t+1} \right) - \left( \frac{1}{\gamma} (x^{t-1} - u^t) + l u^t \right) \geq 0, \]
which implies that
\[ \langle u^{t+1} - u^t, x^t - x^{t-1} \rangle \geq (1 - \gamma l) \| u^{t+1} - u^t \|^2. \]
Hence, we see further that
\[ \| u^{t+1} - v^t \|^2 = \| u^{t+1} - u^t + u^t - v^t \|^2 = \| u^{t+1} - u^t - (x^t - x^{t-1}) \|^2 \]
\[ \leq \| v^{t+1} - u^t \|^2 - 2(1 - \gamma l) \| u^{t+1} - u^t \|^2 + \| x^t - x^{t-1} \|^2 \]
\[ = (-1 + 2\gamma l) \| u^{t+1} - u^t \|^2 + \| x^t - x^{t-1} \|^2. \]
Substituting (3.16) to (3.12), we obtain
\[ M_v(u^{t+1}, v^{t+1}, x^t) - M_v(u^{t+1}, v^t, x^t) \]
\[ \leq - \frac{1}{\gamma} \| x^{t+1} - x^t \|^2 + \frac{1}{\gamma} \left[ (-1 + 2\gamma l) \| u^{t+1} - u^t \|^2 + \| x^t - x^{t-1} \|^2 \right]. \]
Finally, from (3.4b) we get that
\[ M_v(u^{t+1}, v^t, x^t) - M_v(u^t, v^t, x^t) \]
\[ = f(u^{t+1}) + \frac{1}{2\gamma} \| x^t - u^{t+1} \|^2 + \frac{1}{\gamma} \langle (2 - \alpha) u^{t+1}, v^t - u^{t+1} \rangle + \frac{2 - \alpha}{2\gamma} \| u^{t+1} \|^2 \]
\[ - f(u^t) - \frac{1}{2\gamma} \| x^t - u^t \|^2 - \frac{1}{\gamma} \langle (2 - \alpha) u^t, v^t - u^t \rangle - \frac{2 - \alpha}{2\gamma} \| u^t \|^2. \]
By the conditions (3.8) and $0 < \alpha \leq 2$ we have $l < \frac{1+\alpha}{\gamma - 2\alpha + 1} < \frac{1}{\gamma}$. Therefore, $f + \frac{1}{2\gamma}\|x^t - \|\|^2$ is a strongly convex function with modulus $\frac{1}{\gamma} - l$. This together with the definition of $u^{t+1}$ gives
\[
 f(u^{t+1}) + \frac{1}{2\gamma}\|x^t - u^{t+1}\|^2 - f(u^t) - \frac{1}{2\gamma}\|x^t - u^t\|^2
\]
(3.19)
\[
 \leq -\frac{1}{2} \left( \frac{1}{\gamma} - l \right) \|u^{t+1} - u^t\|^2.
\]
Hence, we have
\[
 \mathcal{M}_\gamma(u^{t+1}, v^t, x^t) - \mathcal{M}_\gamma(u^t, v^t, x^t) 
\]
(3.20)
\[
 \leq -\frac{1}{2} \left( \frac{1}{\gamma} - l \right) \|u^{t+1} - u^t\|^2 + \frac{2 - \alpha}{\gamma}\|u^{t+1} - u^t\|^2 + \frac{2 - \alpha}{\gamma}\|u^t - u^{t+1}\|^2 
\]
\[
 + \frac{2 - \alpha}{\gamma}\|u^t - u^{t+1}\|^2 + \frac{2 - \alpha}{\gamma}\|u^{t+1} - u^t\|^2 
\]
\[
 \leq -\frac{1}{2} \left( \frac{1}{\gamma} - l \right) \|u^{t+1} - u^t\|^2 + \frac{2 - \alpha}{\gamma}\|u^{t+1} - u^t\|^2 + \frac{2 - \alpha}{\gamma}\|u^t - u^{t+1}\|^2 
\]
\[
 + \frac{2 - \alpha}{\gamma}\|u^t - u^{t+1}\|^2 + \frac{2 - \alpha}{\gamma}\|u^{t+1} - u^t\|^2 
\]
\[
 \leq \left[ -\frac{1}{2} \left( \frac{1}{\gamma} - l \right) + \frac{2 - \alpha}{\gamma} \right] \|u^{t+1} - u^t\|^2 + \frac{2 - \alpha}{\gamma}\|u^t - u^{t+1}\|^2.
\]
where we have used the elementary fact $2\langle u^t, u^t - u^{t+1}\rangle + \|u^{t+1}\|^2 - \|u^t\|^2 = \|u^t - u^{t+1}\|^2$ in the last inequality. By using (3.16) and Lemma 3.2, we obtain further
\[
 \mathcal{M}_\gamma(u^{t+1}, v^t, x^t) - \mathcal{M}_\gamma(u^t, v^t, x^t) 
\]
(3.21)
\[
 \leq \left[ -\frac{1}{2} \left( \frac{1}{\gamma} - l \right) + \frac{2 - \alpha}{\gamma} \right] \|u^{t+1} - u^t\|^2 
\]
\[
 + \frac{2 - \alpha}{\gamma}\left[ (-1 + 2\gamma l)\|u^{t+1} - u^t\|^2 + \|x^t - x^{t-1}\|^2 \right] 
\]
\[
 \leq \left[ -\frac{1}{2} \left( \frac{1}{\gamma} - l \right) + \frac{2 - \alpha}{\gamma} + \frac{2 - \alpha}{\gamma}\left( -1 + 2\gamma l \right) + \frac{2 - \alpha}{\gamma}\left( 1 + \gamma L \right)^2 \right] \|u^{t+1} - u^t\|^2. 
\]
Summing up (3.11), (3.17) and (3.21) we obtain
\[
 \mathcal{M}_\gamma(u^{t+1}, v^{t+1}, x^{t+1}) - \mathcal{M}_\gamma(u^t, v^t, x^t) 
\]
(3.22)
\[
 \leq \frac{1}{\gamma} \left[ \frac{4 - \alpha}{2} (1 + \gamma L)^2 + \frac{8 - 2\alpha + 1}{\gamma} \frac{1 + \alpha}{2} \right] \|u^{t+1} - u^t\|^2 
\]
\[
 =: -A\|u^{t+1} - u^t\|^2.
\]
Notice the constant $A > 0$ by the choice of $\gamma$ and $\alpha$. Therefore, $\mathcal{M}_\gamma(u^t, v^t, x^t)_{t \geq 1}$ is nonincreasing.
Summing (3.22) from $t = 1$ to $N - 1 \geq 1$, we get

$$
(3.23) \quad \mathcal{M}_g(u^N, v^N, x^N) - \mathcal{M}_g(u^1, v^1, x^1) \leq -A \sum_{t=1}^{N} \|u^{t+1} - u^t\|^2.
$$

Therefore, if there exists a cluster point $(u^*, v^*, x^*)$ with a convergent subsequence $\lim_{j \to \infty}(u^{t_j}, v^{t_j}, x^{t_j}) = (u^*, v^*, x^*)$, since $\mathcal{M}_g$ is lower semi-continuity function and $f$, $g$ are both proper functions, then taking limit as $j \to \infty$ with $N = t_j$ in (3.23), we obtain

$$
(3.24) \quad -\infty < \mathcal{M}_g(u^*, v^*, x^*) - \mathcal{M}_g(u^1, v^1, x^1) \leq -A \sum_{t=1}^{\infty} \|u^{t+1} - u^t\|^2.
$$

This implies immediately that $\lim_{t \to \infty}\|u^{t+1} - u^t\|^2 = 0$. From Lemma 3.2 we conclude that (3.9) holds. Furthermore, using the third relation in (3.2), we get further that $\lim_{t \to \infty}\|v^{t+1} - v^t\| = 0$. Hence, if $(u^*, v^*, x^*)$ is a cluster point of $\{(u^t, v^t, x^t)\}_{t \geq 1}$, that is, the latter has a subsequence $\{(u^{t_j}, v^{t_j}, x^{t_j})\}$ fulfilling $\{(u^{t_j}, v^{t_j}, x^{t_j})\} \to (u^*, v^*, x^*)$ as $j \to \infty$, then

$$
(3.25) \quad \lim_{j \to \infty}(u^{t_j}, v^{t_j}, x^{t_j}) = \lim_{j \to \infty}(u^{t_j-1}, v^{t_j-1}, x^{t_j-1}) = (u^*, v^*, x^*).
$$

Since $v^t$ is a minimizer of the second relation in (3.2), we obtain

$$
(3.26) \quad g(v^t) + \frac{1}{2\gamma}\|\alpha u^t - v^t - x^{t-1}\|^2 \leq g(u^*) + \frac{1}{2\gamma}\|\alpha u^* - v^* - x^{t-1}\|^2.
$$

Taking limit along the convergent subsequence and using (3.25) yields

$$
(3.27) \quad \limsup_{j \to \infty} g(v^{t_j}) \leq g(v^*)
$$

On the other hand, we have $\liminf_{j \to \infty} g(v^{t_j}) \geq g(v^*)$ because of the lower semicontinuity of $g$. Hence

$$
(3.28) \quad \lim_{j \to \infty} g(v^{t_j}) = g(v^*).
$$

Summing (3.5) and (3.6) and then taking limit along the convergent subsequence $\{(u^{t_j}, v^{t_j}, x^{t_j})\}$, and applying (3.28), (3.9) and (2.3), the conclusion of the theorem follows immediately.

Furthermore, assume additionally that the functions $f$ and $g$ are semi-algebraic, we will show in the next theorem that, if the sequence generated by (3.2) has a cluster point, then it is actually convergent. Our argument is largely inspired by [10, 12] with suitable modifications.

**Theorem 3.5. (Global convergence of the whole sequence)** Suppose that $0 < \alpha \leq 2$ and the step-size parameter $\gamma > 0$ is chosen so that (3.8) holds. Suppose additionally that the functions $f$ and $g$ are semi-algebraic. If the sequence $\{(u^t, v^t, x^t)\}$ generated by (3.2) has a cluster point, then the whole sequence $\{(u^t, v^t, x^t)\}$ is convergent.

**Proof.** We split the proof into three steps.

**Step 1.** There exists $\tau > 0$ such that whenever $t \geq 1$,

$$
\text{dist}(0, \partial \mathcal{M}_g(u^t, v^t, x^t)) \leq \tau \|u^{t+1} - u^t\|.
$$
We first compute the subdifferential of $\mathcal{M}_\gamma$ at $(u^{t+1}, v^{t+1}, x^{t+1})$. It is not difficult to obtain that for any $t \geq 0$,
\[
\nabla_x \mathcal{M}_\gamma(u^{t+1}, v^{t+1}, x^{t+1}) = \frac{1}{\gamma} (v^{t+1} - u^{t+1}) = \frac{1}{\gamma} (x^{t+1} - x^t),
\]
\[
\nabla_u \mathcal{M}_\gamma(u^{t+1}, v^{t+1}, x^{t+1}) = \nabla f(u^{t+1}) + \frac{1}{\gamma} (v^{t+1} - x^{t+1}) + \frac{2 - \alpha}{\gamma} (v^{t+1} - u^{t+1}) = \frac{\alpha - 1}{\gamma} (x^t - x^{t+1}),
\]
where the first gradient follows by using the definition of $u^{t+1}$, while the second gradient follows by using (3.4b) and the relation (3.5). Moreover, for the subdifferential with respect to $v$, from (3.4b) we have
\[
\partial_v \mathcal{M}_\gamma(u^{t+1}, v^{t+1}, x^{t+1}) = \partial g(v^{t+1}) - \frac{1}{\gamma} (v^{t+1} - x^{t+1}) + \frac{2 - \alpha}{\gamma} u^{t+1}
\]
\[
= \partial g(v^{t+1}) + \frac{1}{\gamma} (v^{t+1} + x^t - \alpha u^{t+1}) + \frac{1}{\gamma} (2u^{t+1} - x^t - u^{t+1}) - \frac{1}{\gamma} (v^{t+1} - x^{t+1})
\]
\[
\geq - \frac{2}{\gamma} (v^{t+1} - u^{t+1}) + \frac{1}{\gamma} (x^{t+1} - x^t) = - \frac{1}{\gamma} (x^{t+1} - x^t),
\]
where the inclusion follows from the relation (3.6) and the last equality follows from the definition of $x^{t+1}$. The above relations together with Lemma 3.2 imply that there exists $\tau > 0$ (in particular, one may take $\tau = \frac{\alpha + 3}{\gamma}$) such that whenever $t \geq 1$, we have
\[
\text{dist}(0, \partial \mathcal{M}_\gamma(u^t, v^t, x^t)) \leq \tau \|u^{t+1} - u^t\|.
\]  
**Step 2.** The limit $\lim_{t \to \infty} \mathcal{M}_\gamma(u^t, v^t, x^t)$ exists and equals to $\mathcal{M}_\gamma(u^*, v^*, x^*)$ for any cluster point $(u^*, v^*, x^*)$ of sequence $\{(u^t, v^t, x^t)\}$.

It follows from (3.22) that there exists $A > 0$ such that
\[
\mathcal{M}_\gamma(u^t, v^t, x^t) - \mathcal{M}_\gamma(u^{t+1}, v^{t+1}, x^{t+1}) \geq A \|u^{t+1} - u^t\|^2.
\]
Hence, $\mathcal{M}_\gamma(u^t, v^t, x^t)$ is nonincreasing. Let $\{(u^{i_t}, v^{i_t}, x^{i_t})\}$ be a convergent subsequence which converges to $(u^*, v^*, x^*)$. Then, by the lower semicontinuity of $\mathcal{M}_\gamma$, we know that the sequence $\mathcal{M}_\gamma(u^{i_t}, v^{i_t}, x^{i_t})$ is bounded below. This together with the nonincreasing property of $\mathcal{M}_\gamma(u^t, v^t, x^t)$ implies that $\mathcal{M}_\gamma(u^t, v^t, x^t)$ is also bounded below. Therefore, $\lim_{t \to \infty} \mathcal{M}_\gamma(u^t, v^t, x^t) = \theta^*$ exists. We claim that $\theta^* = \mathcal{M}_\gamma(u^*, v^*, x^*)$.

Indeed, let $\{(u^{j_t}, v^{j_t}, x^{j_t})\}$ be any sequence that converges to $(u^*, v^*, x^*)$. Then by the lower semicontinuity, we have
\[
\liminf_{j \to \infty} \mathcal{M}_\gamma(u^{j_t}, v^{j_t}, x^{j_t}) \geq \mathcal{M}_\gamma(u^*, v^*, x^*).
\]
Moreover, similar to (3.25), (3.26) and (3.27), we also have
\[
\limsup_{j \to \infty} \mathcal{M}_\gamma(u^{j_t}, v^{j_t}, x^{j_t}) \leq \mathcal{M}_\gamma(u^*, v^*, x^*).
Now we easily get $\theta^* = M_\gamma(u^t, v^t, x^t)$, as claimed. Note that if $M_\gamma(u^{t}, v^{t}, x^{t}) = \theta^*$ for some $t_0 \geq 1$, then $M_\gamma(u^{t_0+k_0}, v^{t_0+k_0}, x^{t_0+k_0}) = M_\gamma(u^{t_0}, v^{t_0}, x^{t_0})$ for all $k \geq 0$ since the sequence is nonincreasing. Then from (3.30), we have $u^{t_0+k} = u^{t_0}$ for all $k \geq 0$. By (3.7), we see that $x^{t_0+k} = x^{t_0}$ for all $k \geq 0$. These together with the third relation in (3.2) show that we also have $v^{t_0+k} = v^{t_0+1}$ for all $k \geq 1$. Thus, the sequence $(u^t, v^t, x^t)$ remains constant starting with the $(t_0+1)$st iteration. Hence, the theorem holds trivially when this happens. Next we always assume that $M_\gamma(u^t, v^t, x^t) > \theta^*$ for any $t \geq 1$.

**Step 3.** $\{||u^{t+1} - u^t||\}$ is summable and end the proof. It follows from [13] and the semi-algebraic assumption that the function

$$(u, v, x) \rightarrow M_\gamma(u, v, x)$$

is a KL function. By the property of KL function (see Definition 2.5), there exist $\eta > 0$, a neighborhood $U$ of $(u^*, v^*, x^*)$ and a continuous concave function $\varphi : [0, \eta) \rightarrow \mathbb{R}_+$ such that for all $(u, v, x) \in U$ satisfying $\theta^* < M_\gamma(u, v, x) < \theta^* + \eta$, we have

$$\varphi'(M_\gamma(u, v, x) - \theta^*) \text{dist}(0, \partial M_\gamma(u, v, x)) \geq 1.$$ \hfill (3.31)

Since $U$ is an open set, take $\rho > 0$ such that

$$B_\rho := \{(u, v, x) : \|u - u^*\| < \rho, \|v - v^*\| < 2\rho, \|x - x^*\| < (2 + \gamma L)\rho\} \subseteq U$$

and set $B_\rho := \{u : \|u - u^*\| < \rho\}$. From (3.5), we can get

$$\|x^t - x^*\| \leq \|x^t - x^{t-1}\| + \|x^{t-1} - x^*\| \leq \|x^t - x^{t-1}\| + (1 + \gamma L)\|u^t - u^*\|.$$ 

By Theorem 3.3, there exists $N_0 \geq 1$ such that $\|x^t - x^{t-1}\| < \rho$ whenever $t \geq N_0$. Hence, it follows that $\|x^t - x^*\| < (2 + \gamma L)\rho$ whenever $u^t \in B_\rho$ and $t \geq N_0$. Applying the third relation in (3.2), we also have that whenever $u^t \in B_\rho$ and for $t \geq N_0$,

$$\|v^t - v^*\| \leq \|u^t - u^*\| + \|x^t - x^{t-1}\| < 2\rho.$$

Therefore, we obtain that if $u^t \in B_\rho$ and $t \geq N_0$, then $(u^t, v^t, x^t) \in B_\rho \subseteq U$. Now, by the facts that $(u^*, v^*, x^*)$ is a cluster point, that $M_\gamma(u^t, v^t, x^t) > \theta^*$ for every $t \geq 1$, and that $\lim_{t \rightarrow \infty} M_\gamma(u^t, v^t, x^t) = \theta^*$, we easily see that there exists $(u^{N}, v^{N}, x^{N})$ with $N \geq N_0$ such that

(i) $u^{N} \in B_\rho$ and $\theta^* < M_\gamma(u^{N}, v^{N}, x^{N}) < \theta^* + \eta$;

(ii) $\|u^{N} - u^*\| + \frac{\gamma}{4}\|\varphi(M_\gamma(u^{N}, v^{N}, x^{N}) - \theta)^*\| < \rho$.

Next, we prove that whenever $u^t \in B_\rho$ and $\theta^* < M_\gamma(u^t, v^t, x^t) < \theta^* + \eta$ for some $t \geq N_0$, we have

$$\|u^{t+1} - u^t\| \leq \frac{\rho}{\gamma} \left[\varphi(M_\gamma(u^t, v^t, x^t) - \theta^*) - \varphi(M_\gamma(u^{t+1}, v^{t+1}, x^{t+1}) - \theta^*)\right].$$ \hfill (3.32)

Recall that $\{M_\gamma(u^t, v^t, x^t)\}$ is non-increasing and $\varphi$ is increasing, (3.32) holds obviously if $u^t = u^{t+1}$. Without loss generality, we assume that $u^{t+1} \neq u^t$. Since $u^t \in B_\rho$ and $t \geq N_0$, we have $(u^t, v^t, x^t) \in B_\rho \subseteq U$. Hence, for $(u^t, v^t, x^t)$, (3.31) holds. Using
(3.29), (3.30), (3.31) and the concavity of $\varphi$, we obtain that for such $t$,
\[
\tau \|u^{t+1} - u^t\| \cdot \left[ \varphi(\mathcal{M}_\gamma(u^t, v^t, x^t) - \theta^*) - \varphi(\mathcal{M}_\gamma(u^{t+1}, v^{t+1}, x^{t+1}) - \theta^*) \right]
\geq \text{dist}(0, \partial\mathcal{M}_\gamma(u^t, v^t, x^t)) \cdot \left[ \varphi(\mathcal{M}_\gamma(u^t, v^t, x^t) - \theta^*) - \varphi(\mathcal{M}_\gamma(u^{t+1}, v^{t+1}, x^{t+1}) - \theta^*) \right]
\geq \text{dist}(0, \partial\mathcal{M}_\gamma(u^t, v^t, x^t)) \cdot \varphi'(\mathcal{M}_\gamma(u^t, v^t, x^t) - \theta^*)
\cdot \left[ \mathcal{M}_\gamma(u^t, v^t, x^t) - \mathcal{M}_\gamma(u^{t+1}, v^{t+1}, x^{t+1}) \right]
\geq A\|u^{t+1} - u^t\|^2.
\]

This implies that (3.32) holds immediately.

We next claim that $u^t \in B_\rho$ for all $t \geq N$. First, the claim is true whenever $t = N$ by construction. Now, suppose that the claim is true for $t = N, \ldots, N + k - 1$ for some $k \geq 1$, that is, $u^N, \ldots, u^{N+k-1} \in B_\rho$. Note that $\theta^* < \mathcal{M}(u^t, v^t, x^t) < \theta^* + \eta$ for all $t \geq N$ by the choice of $N$ and non-increase property of $\{\mathcal{M}_\gamma(u^t, v^t, x^t)\}$. Hence, (3.32) can be used for $t = N, \ldots, N + k - 1$. Thus, for $t = N + k$, we have
\[
\|u^{N+k} - u^*\| \leq \|u^N - u^*\| + \sum_{j=1}^{k} \|u^{N+j} - u^{N+j-1}\|
\leq \|u^N - u^*\| + \frac{\tau}{A} \sum_{j=1}^{k} \left[ \varphi(\mathcal{M}_\gamma(u^{N+j-1}, v^{N+j-1}, x^{N+j-1}) - \theta^*) - \varphi(\mathcal{M}_\gamma(u^N, v^N, x^N) - \theta^*) < \rho. \right.
\]

Hence, $u^{N+k} \in B_\rho$. By induction, we obtain that $u^t \in B_\rho$ for all $t \geq N$.

Note that we have shown that $u^t \in B_\rho$ and $\theta^* < \mathcal{M}_\gamma(u^t, v^t, x^t) < \theta^* + \eta$ for all $t \geq N$. Summing (3.32) from $t = N$ to $M$ and letting $M \to \infty$, we obtain
\[
\sum_{t=N}^{M} \|u^{t+1} - u^t\| \leq \frac{\tau}{A} \varphi(\mathcal{M}_\gamma(u^N, v^N, x^N) - \theta^*) < +\infty.
\]

This shows that $\{\|u^{t+1} - u^t\|\}$ is summable and hence the whole sequence $\{u^t\}$ converges to $u^*$. From this and (3.5) we also obtain the sequence $\{x^t\}$ is convergent. Finally, by the third relation in (3.2), the convergence of $\{v^t\}$ follows immediately. The proof is completed.

**Theorem 3.6. (Boundedness of the sequence generated from the GDR splitting method)** Let $\alpha \in (\frac{3}{4}, 2]$ and $\gamma$ satisfy (3.8). Suppose that both $f$ and $g$ are bounded from below, and that at least one of them is coercive. Then the sequence $\{(u^t, v^t, x^t)\}$ generated by (3.2) is bounded.

**Proof.** Since $f$ is bounded from below, that is, there exists $\zeta^* > -\infty$ such that for any $x$, we have
\[
\zeta^* \leq f \left( x - \frac{1}{L} \nabla f(x) \right)
\leq f(x) + \left( \nabla f(x), \left( x - \frac{1}{L} \nabla f(x) \right) - x \right) + \frac{L}{2} \left\| \left( x - \frac{1}{L} \nabla f(x) \right) - x \right\|^2
= f(x) - \frac{1}{2L} \left\| \nabla f(x) \right\|^2,
\]

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where the second inequality follows from the descent lemma in [14]. Furthermore, based on the choice of $\gamma$ and $\alpha$, we have from Theorem 3.3 that for all $t \geq 1$,

$$\mathcal{M}_\gamma(u^t, v^t, x^t) \leq \mathcal{M}_\gamma(u^1, x^1)$$

Using (3.4b), we have for $t \geq 1$ that

$$\mathcal{M}_\gamma(u^t, v^t, x^t) = f(u^t) + g(v^t) - \frac{1}{2\gamma}\|x^t - v^t\|^2 + \frac{1}{2\gamma}\|x^t - u^t\|^2 + \frac{1}{\gamma}(2 - \alpha)u^t - v^t + \frac{2 - \alpha}{2\gamma}\|u^t\|^2$$

$$= f(u^t) + g(v^t) - \frac{1}{2\gamma}\|x^{t-1} - u^t\|^2 + \frac{1}{2\gamma}\|x^t - u^t\|^2 + \frac{2 - \alpha}{\gamma}\langle u^t, v^t - u^t \rangle + \frac{2 - \alpha}{2\gamma}\|u^t\|^2,$$

where the last equality follows from the third relation in (3.2). Next, we see from (3.5) for any $t \geq 1$ that

$$0 = \nabla f(u^t) + \frac{1}{\gamma}(u^t - x^{t-1}),$$

which implies that $\|x^{t-1} - u^t\|^2 = \gamma^2\|\nabla f(u^t)\|^2$. Further, because of (3.8), we can choose $\alpha \sim 2, \mu \in (0, 1)$ such that $-(\frac{1}{2\gamma} + \frac{2 - \alpha}{\gamma})^2 \leq -\frac{1 - \mu}{2L}$. Combining these with (3.34) and (3.35), we obtain that

$$\mathcal{M}_\gamma(u^1, v^1, x^1) \geq \mathcal{M}_\gamma(u^t, v^t, x^t)$$

$$= f(u^t) + g(v^t) - \frac{1}{2\gamma}\|x^{t-1} - u^t\|^2 + \frac{1}{2\gamma}\|x^t - u^t\|^2 + \frac{2 - \alpha}{\gamma}\left[\langle u^t, v^t - u^t \rangle + \|u^t\|^2\right]$$

$$\geq f(u^t) + g(v^t) - \frac{1}{2\gamma}\|x^{t-1} - u^t\|^2 + \frac{1}{2\gamma}\|x^t - u^t\|^2 - \frac{2 - \alpha}{2\gamma}\|v^t - u^t\|^2$$

$$\geq f(u^t) + g(v^t) - \frac{1}{2\gamma}\left(\|x^{t-1} - u^t\|^2 - \|x^t - u^t\|^2\right) - \frac{2 - \alpha}{\gamma}\|x^t - u^t\|^2$$

$$= f(u^t) + g(v^t) - (\frac{1}{2\gamma} + \frac{2 - \alpha}{\gamma})\|x^{t-1} - u^t\|^2 + (\frac{1}{2\gamma} - \frac{2 - \alpha}{\gamma})\|x^t - u^t\|^2$$

$$\geq f(u^t) + g(u^t) - (\frac{1}{2\gamma} + \frac{2 - \alpha}{\gamma})\gamma^2 \|\nabla f(u^t)\|^2 + \frac{2\alpha - 3}{2\gamma}\|x^t - u^t\|^2$$

$$= \mu f(u^t) + (1 - \mu)f(u^t) - \frac{1 - \mu}{2L}\|\nabla f(u^t)\|^2 + \left[\frac{1 - \mu}{2L} - \frac{2 - \alpha}{\gamma}\gamma^2\right]\|\nabla f(u^t)\|^2$$

$$+ g(v^t) + \frac{2\alpha - 3}{2\gamma}\|x^t - u^t\|^2$$

$$\geq \mu f(u^t) + (1 - \mu)\zeta^* + \left[\frac{1 - \mu}{2L} - \frac{2 - \alpha}{\gamma}\gamma^2\right]\|\nabla f(u^t)\|^2$$

$$+ g(v^t) + \frac{2\alpha - 3}{2\gamma}\|x^t - u^t\|^2,$$

where the first inequality follows from the Cauchy inequality $ab \leq \frac{a^2}{2} + \frac{b^2}{2}$, the second inequality follows from the third relation in (3.2), the third inequality follows from
First, we assume that $g$ is coercive. Notice from (3.37) that $\{v^t\}, \{\nabla f(u^t)\}$ and $\{x^t - u^t\}$ are bounded. We also obtain from (3.36) that $\{u^t - x^{t-1}\}$ is bounded. This combine with the boundedness of $\{x^t - u^t\}$ shows that $\{x^t - x^{t-1}\}$ is also bounded. Using the third relation in (3.2), we derive that $\{v^t - u^t\}$ is bounded. Note that we have proved that $\{v^t\}$ is bounded, hence $\{u^t\}$ is also bounded. The boundedness of $\{x^t\}$ follows immediately from the boundedness of $\{x^t - u^t\}$.

Finally, suppose that $f$ is coercive. Then we know from (3.37) that $\{u^t\}$ and $\{x^t - u^t\}$ are bounded. This shows that $\{x^t\}$ is also bounded. We now obtain easily the boundedness of $\{v^t\}$ by the third relation in (3.2). The proof is completed. 

4. Numerical examples.

4.1. Low rank matrix completion. There is a rapidly growing interest in the recovery of an unknown low-rank or approximately low-rank matrix from very limited information. There are many classical methods to solve this problem, such as SVP [25] and SVT [22]. In the following we employ the generalized Douglas-Rachford splitting method to resolve this problem and compare our results to the previous ones. All the experiments are conducted in MATLAB using a desktop computer equipped with a 4.0GHz 8-core AMD processor and 16 GB memory. The singular value decomposition (SVD) involved in all the experiments uses PROPACK coming in a MATLAB version.

Simulation data. We generate $n \times n$ matrices of rank $r$ by sampling two $n \times r$ factors $M_L$ and $M_R$ independently, each having i.i.d. Gaussian entries, and setting $M = M_L M_R^*$ as suggested in [9]. The set of observed entries $\Omega$ is sampled uniformly at random among all sets of cardinality $m$. The sampling ratio is defined as $\rho := \frac{m}{nr}$. We wish to recover a matrix with lowest rank such that its entries are equal to those of $M$ on $\Omega$. For this purpose, we consider the following optimization problem

$$\text{(4.1)} \quad \arg\min_X \left\{ \frac{1}{2} \|P_\Omega(X) - P_\Omega(M)\|^2 + I_{C(r)}(X) \right\},$$

where $C(r) := \{X \mid \text{rank}(X) \leq r\}$, $I_{C(r)}(\cdot)$ denotes the indicator function of $C(r)$ and $P_\Omega$ is the orthogonal projector onto the span of matrices vanishing outside of $\Omega$ so that the $(i, j)$th component of $P_\Omega(X)$ is equal to $X_{ij}$ if $(i, j) \in \Omega$ and zero otherwise.

Applying the generalized Douglas-Rachford splitting method (3.2) to problem (4.1), we get the following algorithm

$$\left\{ \begin{array}{l}
u^{t+1} = \arg\min_U \left\{ \frac{1}{2} \|P_\Omega(U) - P_\Omega(M)\|^2 + \frac{1}{2\gamma} \|U - X^t\|^2 \right\}, \\
V^{t+1} = \arg\min_V \left\{ I_{C(r)}(V) + \frac{1}{2\gamma} \|\alpha U^{t+1} - X^t - V\|^2 \right\}, \\
X^{t+1} = X^t + (V^{t+1} - U^{t+1}). 
\end{array} \right.$$  

We see from the above algorithm that both subproblems in (4.2) can be solved easily. For the first subproblem, whose solution can be obtained by using the optimal conditions. On the other hand, it is well known that the second problem has an explicit solution, i.e., the projector from $V$ to the set $C(r)$. Thus, the solution of (4.2) may
be expressed as

\[
\begin{align*}
U_{t+1} &= \begin{cases} 
\frac{1}{1+\gamma} (X_{t}^i + \gamma M_{i,j}), & (i,j) \in \Omega, \\
X_{t}^i, & (i,j) \notin \Omega,
\end{cases} \\
V_{t+1} &= P_{C(\tau)} (\alpha U_{t+1} - V_{t}), \\
X_{t+1} &= X_{t} + (V_{t+1} - U_{t+1}).
\end{align*}
\]

(4.3)

We also recall the SVP [25] and the SVT [22] algorithms to solve problem (4.1) are as follows:

\[
\begin{align*}
Y_{t+1} &= X_{t} - \eta P_{\Omega}(P_{\Omega}(X_{t}) - b), \\
X_{t+1} &= P_{C(\tau)}(Y_{t+1}),
\end{align*}
\]

(4.4) \quad (SVP)

\[
\begin{align*}
Y_{t+1} &= \sum_{j=1}^{p} (\sigma_{j}^t - \tau)u_{j}^t v_{j}^t, \\
X_{t+1}^{ij} &= \begin{cases} 
0, & \text{if } (i,j) \notin \Omega, \\
X_{t}^{ij} + \delta (M_{ij} - Y_{t+1}^{ij}), & \text{if } (i,j) \in \Omega,
\end{cases}
\end{align*}
\]

(4.5) \quad (SVT)

where \(U_{t}, \Sigma_{t}, V_{t}\) are the singular value decomposition of the matrix \(Y_{t}\), and \(u_{j}^t, \sigma_{j}^t, v_{j}^t\) are corresponding singular vectors and singular value.

Next we will show some numerical experiments of our Algorithm (4.2) in the cases \(\alpha = 1.9, 1.8, 1.7, 1.6\), and compare with the results of SVT, SVP and classical DR splitting method (i.e., \(\alpha = 2\) in (4.2)). In all of these experiments, we use

\[
\|P_{\Omega}(X_{t} - M)\|_F < 1 \times 10^{-4}
\]

(4.6)

as a stopping criterion, where \(\| \cdot \|_F\) represents the Frobenius norm. For the SVT method, the parameters \(\tau = 5\alpha\) and \(\delta = 1.2p^{-1}\) are chosen as in [22]. For the SVP method, the parameter \(\eta\) is set to \(\eta = 0.3\). As pointed out in [10] for the DR splitting method on solving a nonconvex problem, if we choose directly a small \(\gamma < \gamma_{0}\) in Algorithm (4.2) where \(\gamma_{0}\) is the larger positive root of the quadratic equation of \(\gamma\) corresponding to (3.8), then the sequence generated tends to get stuck at stationary points that are not global minimizers. Therefore, for all the cases \(\alpha = 2, 1.9, 1.8, 1.7\) and 1.6 in Algorithm (4.2), the parameter \(\gamma\) is set to \(\gamma = t \times 1000\) inspired by [10]. All the parameters are chosen to guarantee the convergence and according to the lowest relative error as follows:

\[
\text{relative error} = \frac{\|X_{\text{opt}} - M\|_F}{\|M\|_F}.
\]

(4.7)

Specifically, for the low sampling ratio \(p = 0.08\) and the high sampling ratio \(p = 0.30\), we recover respectively the matrix of rank =10 and 30 in the different sizes \(n = 3000, 5000, 8000\) and 10000. Our computational results are displayed in the following. All of these quantities are averaged over five runs.

Table 1 compares the runtime and the number of iterations required by various methods to reach the stopping criterion (4.6) for rank= 10 and 30 in different sizes of matrix when \(p = 0.08\). Clearly, Our algorithms are substantially faster than the
### Table 1
Results of the runtime and number of iterations with $p = 0.08$

| rank | size | Average runtime(s) / iterations |
|------|------|--------------------------------|
|      |      | SVT   | SVP   | $\alpha = 2$ | $\alpha = 1.9$ | $\alpha = 1.8$ | $\alpha = 1.7$ | $\alpha = 1.6$ |
| rank=10 |      |       |       |             |             |             |             |             |
| 3000  |      | 41/92 | 176/618 | 76/223 | 33/104 | 21/65 | 19/55 | 28/83 |
| 5000  |      | 72/134 | 521/526 | 171/203 | 87/101 | 56/93 | 41/47 | 55/68 |
| 8000  |      | 97/174 | 443/191 | 221/99 | 137/61 | 101/45 | 137/61 |
| 10000 |      | 137/61 | 340/187 | 216/61 | 152/44 | 209/59 |
| rank=30 |      |       |       |             |             |             |             |             |
| 3000  |      | 102/167 | 555/658 | 169/297 | 79/122 | 40/80 | 60/106 | 80/148 |
| 5000  |      | 276/111 | 980/750 | 358/247 | 160/111 | 98/70 | 98/68 | 152/99 |
| 8000  |      | 501/86 | 1557/606 | 688/219 | 336/105 | 207/66 | 154/99 |
| 10000 |      | 912/78 | 2113/562 | 1052/210 | 352/104 | 308/65 | 237/78 |

### Table 2
Results of relative error with $p = 0.08$

| rank | size | Relative error (10^{-4}) |
|------|------|-------------------------|
|      |      | SVT   | SVP   | $\alpha = 2$ | $\alpha = 1.9$ | $\alpha = 1.8$ | $\alpha = 1.7$ | $\alpha = 1.6$ |
| rank=10 |      |       |       |             |             |             |             |             |
| 3000  |      | 1.38  | 1.44  | 1.38  | 1.19  | 1.07  | 1.19  | 1.25  |
| 5000  |      | 1.20  | 1.27  | 1.24  | 1.10  | 1.01  | 1.00  | 1.13  |
| 8000  |      | 1.17  | 1.18  | 1.17  | 1.02  | 1.00  | 0.98  | 1.04  |
| 10000 |      | 1.05  | 1.15  | 1.14  | 1.00  | 0.92  | 0.91  | 1.02  |
| rank=30 |      |       |       |             |             |             |             |             |
| 3000  |      | 1.85  | 1.68  | 1.86  | 1.46  | 1.12  | 1.43  | 1.48  |
| 5000  |      | 1.51  | 1.57  | 1.52  | 1.33  | 1.04  | 1.23  | 1.27  |
| 8000  |      | 1.30  | 1.39  | 1.33  | 1.20  | 1.00  | 1.13  | 1.15  |
| 10000 |      | 1.24  | 1.32  | 1.27  | 1.11  | 0.99  | 1.00  | 1.02  |

SVT, SVP and DR splitting methods. In particular, when $\alpha = 1.7$, our algorithm performs extremely fast in these experiments. We also see that the runtime of DR splitting method ($\alpha = 2$) is less than that of both SVT and SVP methods, our algorithms with $\alpha = 1.7$ and 1.8 only need almost one third of the running time of DR method, and our algorithms with $\alpha = 1.6$ and 1.9 also takes only half of the runtime of DR method. Moreover, from the Table 2, we know that the relative error of our algorithm when $\alpha = 1.7$ and 1.8 is also lower than the others. For $\alpha = 1.6$ and 1.9, the relative error of our algorithm is also smaller than that of SVT, SVP and DR methods.

### Table 3
Results of the runtime and number of iterations with $p = 0.30$

| rank | size | Average runtime(s) / iterations |
|------|------|--------------------------------|
|      |      | SVT   | SVP   | $\alpha = 2$ | $\alpha = 1.9$ | $\alpha = 1.8$ | $\alpha = 1.7$ | $\alpha = 1.6$ |
| rank=10 |      |       |       |             |             |             |             |             |
| 3000  |      | 119/55 | 46/114 | 30/54 | 23/42 | 11/33 | 14/28 | 11/22 |
| 5000  |      | 309/50 | 124/108 | 76/51 | 60/41 | 46/32 | 39/27 | 31/21 |
| 8000  |      | 726/46 | 258/105 | 185/50 | 150/41 | 116/32 | 102/27 | 79/21 |
| 10000 |      | 1313/45 | 462/104 | 279/49 | 233/41 | 181/32 | 156/27 | 125/21 |
| rank=30 |      |       |       |             |             |             |             |             |
| 3000  |      | 130/69 | 78/135 | 54/65 | 36/47 | 29/36 | 22/29 | 20/23 |
| 5000  |      | 561/60 | 190/121 | 108/57 | 84/44 | 68/35 | 55/28 | 47/22 |
| 8000  |      | 1623/65 | 581/152 | 259/54 | 201/42 | 164/33 | 154/28 | 132/22 |
| 10000 |      | 2450/54 | 649/111 | 446/52 | 353/42 | 284/33 | 243/27 | 203/22 |

Table 3 shows the runtime and the number of iterations required by various methods when $p = 0.30$. The Table 4 gives the results of relative error when $p = 0.30$. These results demonstrate that our algorithms are significantly better than other three methods in terms of runtime, number of iterations and relative error.
Table 3, when $\alpha$ decreases from 1.9 to 1.6, the runtime and the number of iterations required by our algorithm also gradually decrease, especially for $\alpha = 1.6$ and 1.7, whose runtime are far less than others. On the other hand, the Table 4 shows that our algorithms for $\alpha = 1.6$ and 1.7 are significantly more accurate than the SVT, SVP and DR splitting methods, while the algorithms with $\alpha = 1.8$ and 1.9 also have a lower relative error than the others. Finally, we also point out that when $\alpha \leq 1.5$, the experiments showed that our algorithms is not well, although it runs in less time, its relative error does not show good behavior.

**Real data** We now evaluate our algorithms on the Movie-Lens [1], which contains one million ratings for 3900 movies by 6040 users. Table 5 shows the RMSE (root mean square error) obtained by each method with varying rank $r$. For SVP, we take step size to be $\eta = \frac{1}{p\sqrt{t}}$ as in [25], where $t$ is the number of iterations. For the classical DR splitting and our algorithm, we choose the step size to be $\gamma = 10$. Since the rank of matrices obtained by SVT cannot be fixed, we here don’t consider SVT method. As shown in Table 5, our algorithms when $\alpha = 1.95$ and 1.90 perform better than both SVP and DR methods. Note that our algorithm with $\alpha = 1.95$ incurs a RMSE of 0.90 when rank $r = 5$. In contrast, SVP achieves a RMSE of 1.04 and DR splitting achieves a RMSE of 1.00 for the same rank. Remark that, SVP can achieve RMSE up to 0.90 but required solution with rank 25. We attribute the relatively poor performance of our algorithms when $\alpha = 1.8$ and 1.7 as compared with $\alpha = 1.95$ and 1.90 to the impact of regularization parameter in (3.10) on the algorithms.

**4.2. Logistic Regression.** In this section, we consider a common type of problem – logistic regression, which arises in the area of statistics and machine learning [32, 21]. In the following we state this problem in detail.
Given a set of training data \( \{(x_i, y_i) : i = 1, \ldots, N\} \), where \( x_i \in \mathbb{R}^n \) is input data and \( y_i \in \{1, -1\} \) is output data. Our purpose is to find a classification rule from the training data, such that when given a new input \( x \), to which we can assign a class \( y \) from \( \{1, -1\} \). To this end, we consider the following sparse logistic regression optimization problem

\[
\min_{(\beta, \omega) \in X} \frac{1}{N} \sum_{i=1}^{N} \log \left( 1 + \exp \left( -y_i (\beta^T x_i + \omega) \right) \right) + \lambda \|\beta\|_0,
\]

where \( X = [-10^{10}, 10^{10}]^n \times [-10^{10}, 10^{10}] \), \( N \) is the number of training data, \( \| \cdot \|_0 \) is the standard \( l_0 \) norm and \( \lambda > 0 \) is the regularization parameter. We perform this numerical experiment by using the real data set gisette which is taken from [6]. In this data set, the train set contains 6000 samples of 5000 dimensions, and the test set contains 1000 samples of 5000 dimensions.

We next apply the generalized Douglas-Rachford splitting method to solve problem (4.8). Thus, the GDR splitting method for (4.8) can be given as follows:

\[
\begin{align*}
\alpha^{t+1} &\in \arg \min_{\alpha} \left\{ \frac{1}{N} \sum_{i=1}^{N} \log \left( 1 + \exp \left( -y_i (\beta_1^T x_i + \omega) \right) \right) + \frac{1}{\gamma} \| \alpha \|_0 \right\}, \\
\beta^{t+1} &\in \arg \min_{\beta} \left\{ \lambda \| \beta \|_0 + \frac{1}{2\gamma} \| \alpha \beta^{t+1} - x^t - v^t \|_2^2 \right\}, \\
x^{t+1} &:= x^t + (\alpha^{t+1} - \beta^{t+1}).
\end{align*}
\]

Before discussing the numerical results, we analyze two subproblems of the above algorithm. Notice that we can use the nonlinear conjugate gradient method to solve the first subproblem easily. On the other hand, the second subproblem is equivalent to solving the following optimization problem

\[
\begin{align*}
\arg \min_{v \in \mathbb{R}^{n+1}} \left\{ \delta_X(v) + \lambda \| v \|_0 + \frac{1}{2\gamma} \| \alpha \beta^{t+1} - x^t - v \|_2^2 \right\},
\end{align*}
\]

where \( \delta_X \) is the indicator function of \( X \). Note that problem (4.10) has a separable structure since \( X \) is a box constraint. Hence, we only need to figure out the following simple one-dimensional problem

\[
\arg \min_{z \in \mathbb{R}} h(z) := \delta_{\tilde{X}}(z) + \lambda \| z \|_0 + \frac{1}{2\gamma}(z - c)^2,
\]

where \( \tilde{X} := \{ z \in \mathbb{R} : a \leq z \leq b \} \). In fact, it follows from [36] that the solutions of (4.11) can be written explicitly:

\[
\arg \min_{z \in \mathbb{R}} h(z) = \begin{cases} 
P_{\tilde{X}}(c), & \text{if } 0 \not\in \tilde{X}, \\
\mathcal{H}_{\sqrt{2\gamma}}(P_{\tilde{X}}(c)), & \text{if } 0 \in \tilde{X} \text{ and } c \in \tilde{X}, \\
\arg \min_{z \in \{0,b\}} h(z), & \text{if } 0 \in \tilde{X} \text{ and } c > b, \\
\arg \min_{z \in \{0,a\}} h(z), & \text{if } 0 \in \tilde{X} \text{ and } c < a,
\end{cases}
\]
where $P_{\tilde{X}}(c)$ is the projection operator defined on the set $\tilde{X}$ and $\mathcal{H}_{\sqrt{2}\lambda\gamma}(\cdot)$ is the hard thresholding operator defined by

\begin{equation}
\mathcal{H}_{\sqrt{2}\lambda\gamma}(z) = \begin{cases} 
\{z\}, & \text{if } |z| > \sqrt{2\lambda\gamma}, \\
\{0, z\}, & \text{if } |z| = \sqrt{2\lambda\gamma}, \\
\{0\}, & \text{if } |z| < \sqrt{2\lambda\gamma}.
\end{cases}
\end{equation}

We now give our experimental results. For this experiment, we set the stopping criteria to be

\begin{equation}
\|x_k - x_{k-1}\|_\infty < 5 \times 10^{-4},
\end{equation}

and the initial point is obtained by FISTA [2] for $l_1$ minimization (where the initial point is $\text{zeros}(n+1, 1)$, and stopping criteria is $\|x_k - x_{k-1}\|_\infty < 0.02$, the regularization parameter $\lambda = 0.001$). All the parameters are chosen according to accuracy. It is well known that when $\alpha = 2$, (4.9) becomes the classical DR splitting method. Specifically, we take $\alpha = 1.9$ and $1.8$ in our experiment respectively and choose the regularization parameter $\lambda = 0.00005$ for these different choices of $\alpha$. The results are listed in following table:

| $\alpha$ | Iteration number | Runtime(s) | Accuracy | Sparsity |
|----------|------------------|------------|----------|----------|
| 2        | 120              | 4286       | 0.9730   | 757      |
| 1.9      | 29               | 2319       | 0.9750   | 742      |
| 1.8      | 19               | 1681       | 0.9700   | 714      |

Table 6
Results of Logistic Regression.

We see from Table 6 that the results of generalized Douglas-Rachford splitting method is better than Douglas-Rachford splitting method. In particular, when $\alpha = 1.9$ and $1.8$, the iteration number and runtime of GDR splitting is much less than those of DR splitting, the sparsity of solutions obtained by GDR splitting is better. Moreover, we see from the Figure 1 that as the number of iteration steps increases, the accuracy of the test data obtained by GDR splitting method is always higher than that of DR splitting method. In Figure 1, we also see that the final accuracy of GDR splitting method is comparable to DR splitting method.

5. Concluding remarks. In this paper, we propose a generalized Douglas-Rachford splitting method for solving nonconvex optimization problems. By constructing the new generalized Douglas-Rachford merit function, we establish the global convergence of the generalized Douglas-Rachford splitting method when the parameters $\gamma$ and $\alpha$ satisfy (3.8) and the sequence generated has a cluster point. We also give sufficient conditions to guarantee the boundedness of the sequence generated by the proposed method and thus the existence of cluster points. Finally, we apply our generalized DR splitting method to two important classes of nonconvex optimization problems arising in machine learning: low rank matrix completion and logistic regression. The numerical experiments indicate that our generalized DR splitting method is significantly better than the classical method for low rank matrix completion in terms of runtime, number of iterations and relative error. The numerical experiments also show that our generalized DR splitting method outperforms the DR splitting method for some nonconvex optimization problems such as low rank matrix completion and logistic regression.
Fig. 1. Results of test accuracy during training

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