Black holes and stellar structures in $f(R)$-gravity

Mariafelicia De Laurentis and Salvatore Capozziello

*Dipartimento di Scienze Fisiche, Università di Napoli “Federico II”,
and
INFN Sezione di Napoli,
Compl. Univ. di Monte S. Angelo, Edificio G, Via Cinthia, I-80126, Napoli, Italy.
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In the last ten years, increasing attention has been devoted to Extended Theories of Gravity with the aim to understand several cosmological and astrophysical issues such as the today observed accelerated expansion of the universe and the presence of Dark Matter in self-gravitating structures. Some of these models assume modifications of General Relativity by adding higher order terms of curvature invariants like the Ricci scalar $R$, the Ricci tensor $R_{\mu\nu}$ and the Riemann tensors $R_{\mu\nu\lambda\sigma}$, or the presence of suitable scalar fields like the former Brans-Dicke theory. It is therefore natural to ask for black hole solutions in this context since, on the one hand, black holes signatures may be the test-bed to compare new models to the Einstein gravity; on the other hand, they may lead to rule out models which disagree with observations. Although black holes are one of the most striking predictions of General Relativity, they remain one of its least tested concepts. Electromagnetic observations allow indirectly to infer their existence, but direct evidences remains elusive. In the next decade, data coming from very long-baseline interferometry and gravitational wave detectors should allow to image and study black holes in detail. Such observations will test General Relativity in the non-linear and strong-field regimes where data are currently lacking. Testing strong-field features of General Relativity is of utmost importance to physics and astrophysics as a whole. This is because the black holes solutions, such as the Schwarzschild and Kerr metrics, enter several calculations, including accretion disk structure, gravitational lensing, cosmology and gravitational waves theory. These black hole solutions could indicate strong-field departures from General Relativity with deep implications for the still unknown fundamental theory of gravity. Beside the physical interest, black hole solutions represent a very active area for mathematical physics investigations. Here we review the problem of black holes in a particular class of Extended Theories of Gravity, the so called $f(R)$-gravity, discussing some resolution techniques, obtaining exact solutions and comparing results with standard General Relativity. Furthermore, we discuss the problems of hydrostatic equilibrium and stellar structure in the context of $f(R)$-gravity showing that new features could emerge. The observation of such features could both explain the physics of exotic self-gravitating objects and constitute a signature for Extended Theories of Gravity.

I. INTRODUCTION

The issue to extend General Relativity (GR) has recently become dramatically urgent due to the missing matter problem at all astrophysical scales and the accelerating behavior of cosmic fluid, detected by Super Novae Ia used as standard candles. Up to now, no final answer on new particles has been given at fundamental level so Dark Energy and Dark Matter constitute a puzzle to be solved in order to achieve a self-consistent picture of the observed Universe. $f(R)$-gravity, where $f(R)$ is a generic function of the Ricci scalar $R$, comes into the game as a straightforward extension of GR where further geometrical degrees of freedom are considered instead of searching for new material ingredients [1]. From an epistemological point of view, the action of gravity is not selected a priori, but it could be “reconstructed”, in principle, by matching consistently the observations [2][3]. This approach can be adopted considering any function of the curvature invariants as $R_{\mu\nu}R^{\mu\nu}$, $R\Box R$ and so on. Several of these extended models reproduce Solar System tests so they are not in conflict with GR experimental results but actually extend them enclosing new features that could be, in principle, observed [4][5].

From a genuine mathematical point of view, Extended Theories of Gravity pose the problem to recover the well-established results of GR as the initial value problem [6], the stability of solutions and, in particular, the issue of finding out new solutions. As it is well known, beside cosmological solutions, spherically and axially symmetric solutions play a fundamental role in several astrophysical problems ranging from black holes to active galactic nuclei. Extended gravities, to be consistent with results of GR, should comprise solutions like Schwarzschild and Kerr ones but could present, in general, new solutions of physical interest. Due to this reason, methods to find out exact and approximate solutions are particularly relevant in order to check if observations can be framed in Extended Theories of Gravity [7].

Recently, the interest in spherically symmetric solutions of $f(R)$-gravity is growing up. In [10], solutions in vacuum have been found considering relations among functions that define the spherical metric or imposing a constant Ricci
curvature scalar. The authors have reconstructed the form of some $f(R)$-models, discussing their physical relevance. In [11], the same authors have discussed static spherically symmetric solutions, in presence of perfect fluid matter, adopting the metric formalism. They have shown that a given matter distribution is not capable of globally determining the functional form of $f(R)$. Others authors have discussed in details the spherical symmetry of $f(R)$-gravity considering also the relations with the weak field limit. Exact solutions are obtained for constant Ricci curvature scalar and for Ricci scalar depending on the radial coordinate. In particular, it can be considered how to obtain results consistent with GR assuming the well-known post-Newtonian and post-Minkowskian limits as consistency checks [12].

As we will discuss below, a general method to find out rotating black hole solutions can be achieved by performing a complex coordinate transformation on the spherical black hole metrics. Since the discovery of the Kerr solution [13], many attempts have been made to find a physically reasonable interior matter distribution that may be considered as its source. For a review on these approaches see [14, 15]. Though much progress has been made, results have been generally disappointing. As far as we know, nobody has obtained a physically satisfactory interior solution. This seems surprising given the success of matching internal spherically symmetric solutions to the Schwarzschild metric. The problem is that the loss of a degree of symmetry makes the derivation of analytic results much more difficult. Severe restrictions are placed on the interior metric by maintaining that it must be joined smoothly to the external axially symmetric metric. Further restrictions are placed on the interior solutions to ensure that they correspond to physical objects.

Furthermore since the axially symmetric metric has no radiation field associated with it, its source should be also non-radiating. This places even further constraints on the structure of the interior solution [16]. Given the strenuous nature of these limiting conditions, it is not surprising to learn that no satisfactory solution to the problem of finding sources for the Kerr metric has been obtained. In general, the failure is due to internal structures whose physical properties are unknown. This shortcoming makes hard to find consistent boundary conditions.

Newman and Janis showed that it is possible to obtain rotating solutions (like the Kerr metric) by making an elementary complex transformation on the Schwarzschild solution [17]. This same method has been used to obtain a new stationary and axially symmetric solution known as the Kerr-Newman metric [18]. The Kerr-Newman space-time is associated to the exterior geometry of a rotating massive and charged black-hole. For a review on the Newman-Janis method to obtain both the Kerr and Kerr-Newman metrics see [19, 20].

By means of very elegant mathematical arguments, Schiffer et al. [21] have given a rigorous proof to show how the Kerr metric can be derived starting from a complex transformation on the Schwarzschild solution. We will not go into details of this demonstration, but point out that the proof relies on two main assumptions. The first is that metric belongs to the same algebraic class of the Kerr-Newman solution, namely the Kerr-Schild class [22]. The second assumption is that metric corresponds to an empty solution of the Einstein field equations [21]. It is clear, by the generation of the Kerr-Newman metric, that all the components of the stress-energy tensor need to be non-zero for the Newman-Janis method to be successful. Such a transformation can be extended to $f(R)$-gravity as discussed in [23].

On the other hand, the strong gravity regime is another way to check the viability of these theories [24]. In general the formation and the evolution of stars can be considered suitable test-beds for Extended Theories of Gravity. Considering the case of $f(R)$-gravity, divergences stemming from the functional form of $f(R)$ may prevent the existence of relativistic stars in these theories [25], but thanks to the chameleon mechanism, introduced by Khoury and Weltman [26], the possible problems jeopardizing the existence of these objects may be avoided [27]. Furthermore, there are also numerical solutions corresponding to static star configurations with strong gravitational fields [28] where the choice of the equation of state is crucial for the existence of solutions.

It is also important to stress that $f(R)$-gravity has interesting applications also in stellar astrophysics and could contribute to solve several puzzles related to observed peculiar objects (e.g. stars in the instability strips, protostars, etc. [29, 30]), structure and star formation [31, 32]. Furthermore some observed stellar systems are incompatible with the standard models of stellar structure. We refer to anomalous neutron stars, the so called ”magnetars” [33] with masses larger than their expected Volkoff mass. It seems that, on particular length scales, the gravitational force is larger or smaller than the corresponding GR value. For example, a modification of the Hilbert-Einstein Lagrangian, consisting of $R^2$ terms, enables a major attraction while a $R_{\alpha\beta}R^\alpha\beta$ term gives a repulsive contribution [34]. Understanding on which scales the modifications to GR are working or what is the weight of corrections to gravitational potential is a crucial point that could confirm or rule out these extended approaches to gravitational interaction.

This Chapter is organized as follow. In the Sec [III] we introduce the $f(R)$-gravity action, the field equations and give some general remarks on spherical symmetry. In Sec [III] a summary is given on the Noether Simmetry Approach [9]. This technique is extremely useful to find exact solutions. In particular, we find spherically symmetric black hole solutions for $f(R)$-gravity. In Sec [IV] we review the Newman-Janis method to obtain rotating solutions starting from spherically symmetric ones. The resulting metric is written in terms of two arbitrary functions. A further suitable coordinate transformation allows to write the metric in the so called Boyer-Lindquist coordinates.
Such a transformation makes the physical interpretation much clearer and reduces the amount of algebra required to calculate the metric properties. In Sec.\[X\] the Newman-Janis method is applied to a spherically symmetric exact solution, previously derived by the Noether Symmetry, and an axially symmetric exact solution is obtained. This result shows that the Newman-Janis method works also in $f(R)$-gravity. Physical applications of the result are also discussed.

In Sec.\[\text{V}\] we review the classical hydrostatic problem for stellar structures. In Sec.\[\text{VII}\] we derive the Newtonian limit of $f(R)$-gravity obtaining the modified Poisson equation. The modified Lane-Emden equation is obtained in Sec.\[\text{VIII}\] and its structure is compared with respect to the standard one. In Sec.\[\text{IX}\] we show how analytical solutions of standard Lane-Emden equation can be compared with those perturbatively obtained from $f(R)$-gravity. In order to apply the above results, in Section \[X\] the classical theory of gravitational collapse for dust-dominated systems is discussed. The difference between GR and $f(R)$-gravity are put in evidence, in particular the Jeans mass profiles with respect to the temperature. We report a catalogue of observed molecular clouds in order to compare the classical Jeans mass to the $f(R)$-one. Finally, in Section \[X\text{II}\] we discuss the results and draw conclusions.

II. SPHERICAL SYMMETRY IN $f(R)$-GRAVITY

Let us start by discussing exact solutions in $f(R)$-gravity with spherical symmetry. As we will see, a crucial role is played by the relation between the metric potentials and the Ricci scalar that can be regarded as a constraint assuming the form of a Bernoulli equation.

Let us consider an analytic function $f(R)$ of the Ricci scalar $R$ in four dimensions. The variational principle for this action is:

$$\delta \int d^4x \sqrt{-g} \left[ f(R) + \mathcal{X} \mathcal{L}_m \right] = 0$$

(1)

where $\mathcal{X} = \frac{8\pi G}{c^4}$, $\mathcal{L}_m$ is the standard matter Lagrangian and $g$ is the determinant of the metric\(^1\).

By varying with respect to the metric, we obtain the field equations\(^2\)

$$\begin{cases}
  H_{\mu\nu} = f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - f'(R)\gamma_{\mu\nu} + g_{\mu\nu}\Box f'(R) = \mathcal{X} T_{\mu\nu} \\
  H = g^{\rho\sigma} H_{\rho\sigma} = 3\Box f'(R) + f'(R)R - 2f(R) = \mathcal{X} T
\end{cases}$$

(2)

where $T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \delta \sqrt{-g \mathcal{L}_m} / \delta g^{\mu\nu}$ is the energy-momentum tensor of standard fluid matter and the second equation is the trace. The most general spherically symmetric solution can be written as follows:

$$ds^2 = m_1(t', r') dt'^2 + m_2(t', r') dr'^2 + m_3(t', r') dt' dr' + m_4(t', r') d\Omega,$$

(3)

where $m_i$ are functions of the radius $r'$ and of the time $t'$. $d\Omega$ is the solid angle. We can consider a coordinate transformation that maps the metric \([3]\) in a new one where the off-diagonal term vanishes and $m_4(t', r') = -r^2$, that is\(^3\):

\[1\] We are adopting the convention $c = 1$. The convention $R_{\mu\nu} = R^\rho_{\mu\rho\nu}$ for the Ricci tensor and $R^\alpha_{\beta\mu\nu} = \Gamma^\alpha_{\beta\nu,\mu} - ...$, for the Riemann tensor. Connections are Levi-Civita:

$$\Gamma^\alpha_{\beta\rho} = \frac{1}{2} g^{\rho\sigma} (g_{\alpha\sigma,\beta} + g_{\beta\sigma,\alpha} - g_{\alpha\beta,\sigma}).$$

The signature is $(+ - - -)$.

\[2\] It is possible to take into account also the Palatini approach in which the metric $g$ and the connection $\Gamma$ are considered independent variables (see for example \([30]\)). Here we will consider the Levi-Civita connection and will use the metric approach. See \([3, 34, 37]\) for a detailed comparison between the two pictures.

\[3\] This condition allows to obtain the standard definition of the circumference with radius $r$. 
\[ ds^2 = g_{tt}(t,r)dt^2 - g_{rr}(t,r)dr^2 - r^2 d\Omega. \]  

This expression can be considered, without loss of generality, as the most general definition of a spherically symmetric metric compatible with a pseudo-Riemannian manifold without torsion. Actually, by inserting this metric into the field Eqs. (76), one obtains:

\[
\begin{align*}
& \left\{ \begin{array}{l}
    f'(R) R_{\mu\nu} - \frac{1}{2} f(R) g_{\mu\nu} + \mathcal{H}_{\mu\nu} = \mathcal{X} T_{\mu\nu} \\
    f'(R) R - 2 f(R) + \mathcal{H} = \mathcal{X} T
\end{array} \right.
\end{align*}
\]

where the two quantities \( \mathcal{H}_{\mu\nu} \) and \( \mathcal{H} \) read:

\[
\begin{align*}
\mathcal{H}_{\mu\nu} &= - f''(R) \left\{ R_{,\mu\nu} - \Gamma^\tau_{\mu\nu} R_{,\tau} - \Gamma^\tau_{\mu\nu} R_{,\tau} - g_{\mu\nu} \left[ \left( g^{tt}_{,t} + \\
    + g^{tt} \ln \sqrt{-g} \right) R_{,t} + \left( g^{rr}_{,r} + g^{rr} \ln \sqrt{-g} \right) R_{,r} + \\
    + g^{tt} R_{,tt} + g^{rr} R_{,rr} \right] - f'''(R) \left[ R_{,\mu} R_{,\nu} - g_{\mu\nu} \left( g^{tt}_{,t}^2 + g^{rr} R_{,r}^2 \right) \right] \right\}, \\
\mathcal{H} &= g^{rr} \mathcal{H}_{\sigma\tau} = 3 f''(R) \left[ \left( g^{tt}_{,t} + g^{tt} \ln \sqrt{-g} \right) R_{,t} + \\
    + \left( g^{rr}_{,r} + g^{rr} \ln \sqrt{-g} \right) R_{,r} + g^{tt} R_{,tt} + g^{rr} R_{,rr} \right] + 3 f'''(R) \left[ g^{tt}_{,t}^2 + g^{rr} R_{,r}^2 \right].
\end{align*}
\]

Our task is now to find out exact spherically symmetric solutions.

In the case of time-independent metric, i.e., \( g_{tt} = a(r) \) and \( g_{rr} = b(r) \), the Ricci scalar can be recast as a Bernoulli equation of index two with respect to the metric potential \( b(r) \) (see [12] for details):

\[
b'(r) + \left\{ \frac{r^2 a'(r)^2 - 4 a(r)^2 - 2 r a(r) [2 a(r)' + r a(r)'']}{r a(r) [4 a(r) + r a'(r)]} \right\} b(r) + \\
+ \left\{ \frac{2 a(r)}{r} \left[ \frac{2 + r^2 R(r)}{4 a(r) + r a'(r)} \right] \right\} b(r)^2 = 0.
\]

where \( R = R(r) \) is the Ricci scalar. A general solution of (8) is:

\[
b(r) = \frac{\exp \left[ - \int dr h(r) \right]}{K + \int dr \left[ l(r) \exp \left[ - \int dr h(r) \right] \right]},
\]

where \( K \) is an integration constant while \( h(r) \) and \( l(r) \) are two functions that, according to Eq. (8), define the coefficients of the quadratic and the linear term with respect to \( b(r) \). We can fix \( l(r) = 0 \); this choice allows to find out solutions with a Ricci scalar scaling as \( \frac{2}{r^2} \) in term of the radial coordinate. On the other hand, it is not possible to have \( h(r) = 0 \) since, otherwise, we get imaginary solutions. A particular consideration deserves the limit \( r \to \infty \). In order to achieve a gravitational potential \( b(r) \) with the correct Minkowski limit, both \( h(r) \) and \( l(r) \) have to go to zero at infinity, provided that the quantity \( r^2 R(r) \) turns out to be constant: this result implies \( b'(r) = 0 \), and, finally, also the metric potential \( b(r) \) has a correct Minkowski limit.

In general, if we ask for the asymptotic flatness of the metric as a feature of the theory, the Ricci scalar has to evolve to infinity as \( r^{-n} \) with \( n \geq 2 \). Formally, it has to be:

\[
\lim_{r \to \infty} r^2 R(r) = r^{-n},
\]
with \( n \in \mathbb{N} \). Any other behavior of the Ricci scalar could affect the requirement to achieve a correct asymptotic flatness.

The case of constant curvature is equivalent to GR with a cosmological constant and the solution is time independent. This result is well known (see, for example, [39]) but we report, for the sake of completeness, some considerations related with it in order to deal with more general cases where a radial dependence for the Ricci scalar is supposed. If the scalar curvature is constant \( (R = R_0) \), field Eqs. (5), being \( \mathcal{H}_{\mu\nu} = 0 \), reduce to:

\[
\begin{align*}
  f'' R_{\mu\nu} - \frac{1}{2} f_0 g_{\mu\nu} &= \mathcal{X} T_{\mu\nu} \\
  f'_0 R_0 - 2 f_0 &= \mathcal{X} T
\end{align*}
\]  

(11)

where \( f(R_0) = f_0, f'(R_0) = f'_0 \). A general solution, when one considers a stress-energy tensor of perfect-fluid \( T_{\mu\nu} = (\rho + p) u_\mu u_\nu - pg_{\mu\nu}, \) is

\[
ds^2 = \left( 1 + \frac{k_1}{r} + \frac{q X \rho - \lambda}{3} r^2 \right) dt^2 - \frac{dr^2}{1 + \frac{k_1}{r} + \frac{q X \rho - \lambda}{3} r^2} - r^2 d\Omega .
\]  

(12)

when \( p = -\rho, \lambda = -\frac{f'_0}{f_0} \) and \( q^{-1} = f'_0 \). This result means that any \( f(R) \)-model, in the case of constant curvature, exhibits solutions with de Sitter-like behavior. This is one of the reasons why the dark energy issue can be addressed using these theories [1].

If \( f(R) \) is analytic, it is possible to write the series:

\[
f(R) = \Lambda + \Psi_0 R + \Psi(R) ,
\]  

(13)

where \( \Psi_0 \) is a coupling constant, \( \Lambda \) plays the role of the cosmological constant and \( \Psi(R) \) is a generic analytic function of \( R \) satisfying the condition

\[
\lim_{R \to 0} R^{-2} \Psi(R) = \Psi_1 ,
\]  

(14)

where \( \Psi_1 \) is a constant. If we neglect the cosmological constant \( \Lambda \) and \( \Psi_0 \) is set to zero, we obtain a new class of theories which, in the limit \( R \to 0 \), does not reproduce GR (from Eq. (13), we have \( \lim_{R \to 0} f(R) \sim R^2 \)). In such a case, analyzing the whole set of Eqs. (11), one can observe that both zero and constant \( \neq 0 \) curvature solutions are possible. In particular, if \( R = R_0 = 0 \) field equations are solved for any form of gravitational potential entering the spherically symmetric background, provided that the Bernoulli Eq. (8), relating these functions, is fulfilled for the particular case \( R(r) = 0 \). The solutions are thus defined by the relation

\[
b(r) = \frac{\exp\left[-\int dr h(r)\right]}{K + 4 \int \frac{dr a(r) \exp\left[-\int dr h(r)\right]}{\left[\frac{\left|a(r)\right|}{r} + ra'(r)\right]}} ,
\]  

(15)

being \( g_\Omega(t, r) = b(r) \) from Eq. (11). In [12], some examples of \( f(R) \)-models admitting solutions with constant \( \neq 0 \) or null scalar curvature are discussed.

### III. THE NOETHER SYMMETRY APPROACH

Besides spherically symmetric solutions with constant curvature scalar, also solutions with the Ricci scalar depending on radial coordinate \( r \) are possible in \( f(R) \)-gravity [12]. Furthermore, spherically symmetric solutions can be achieved starting from a point-like \( f(R) \)-Lagrangian [9]. Such a Lagrangian can be obtained by imposing the spherical symmetry directly in the action [11]. As a consequence, the infinite number of degrees of freedom of the original field theory will be reduced to a finite number. The technique is based on the choice of a suitable Lagrange multiplier defined by assuming the Ricci scalar, argument of the function \( f(R) \) in spherical symmetry.

Starting from the above considerations, a static spherically symmetric metric can be expressed as

\[
ds^2 = A(r) dt^2 - B(r) dr^2 - M(r) d\Omega ,
\]  

(16)
and then the point-like \( f(R) \) Lagrangian\(^4\) is

\[
\mathcal{L} = -\frac{A^{1/2}f'}{2MB^{1/2}} M'^2 - \frac{f'}{A^{1/2}B^{1/2}} A'M' - \frac{Mf''}{A^{1/2}B^{1/2}} A'R' + \frac{2A^{1/2}f''}{B^{1/2}} R'M' - A^{1/2}B^{1/2}/(2 + MR)f' - Mf, \tag{17}
\]

which is canonical since only the configuration variables and their first order derivatives with respect to the radial coordinate \( r \) are present. Details of calculations are in \cite{9}. Eq. (17) can be recast in a more compact form introducing the matrix representation:

\[
\mathcal{L} = \frac{1}{2} \dot{q}^t \hat{T} \dot{q} + V \tag{18}
\]

where \( q = (A, B, M, R) \) and \( q' = (A', B', M', R') \) are the generalized positions and velocities associated to \( \mathcal{L} \). It is easy to check the complete analogy between the field equation approach and point-like Lagrangian approach \cite{9}.

In order to find out solutions for the Lagrangian (17), we can search for symmetries related to cyclic variables and then reduce dynamics. This approach allows, in principle, to select \( f(R) \)-gravity models compatible with spherical symmetry. As a general remark, the Noether Theorem states that conserved quantities are related to the existence of cyclic variables into dynamics \cite{40-42}.

It is worth noticing that the Hessian determinant of Eq. (17), \( \left| \frac{\partial^2 \mathcal{L}}{\partial q \partial q'} \right| \), is zero. This result clearly depends on the absence of the generalized velocity \( B' \) into the point-like Lagrangian. As matter of fact, using a point-like Lagrangian approach implies that the metric variable \( B \) does not contributes to dynamics, but the equation of motion for \( B \) has to be considered as a further constraint equation. Then the Lagrangian (17) has three degrees of freedom and not four, as one should expect a priori.

Now, since the equation of motion describing the evolution of the metric potential \( B \) does not depend on its derivative, it can be explicitly solved in term of \( B \) as a function of the other coordinates:

\[
B = \frac{2M^2f'' A'R' + 2Mf'A'M' + 4AMf'' M'R' + Af'M'^2}{2AM/(2 + MR)f' - Mf}. \tag{19}
\]

By inserting Eq. (19) into the Lagrangian (17), we obtain a non-vanishing Hessian matrix removing the singular dynamics. The new Lagrangian reads\(^5\)

\[
\mathcal{L}^* = L^{1/2} \tag{20}
\]

with

\[
L = q'^t \hat{L} q' = \frac{[(2 + MR)f' - Mf]}{M}[2M^2f'' A'R' + 2MM'(f'A' + 2Af'R') + Af'M'^2].
\]

If one assumes the spherical symmetry, the role of the affine parameter is played by the coordinate radius \( r \). In this case, the configuration space is given by \( \mathcal{Q} = \{A, M, R\} \) and the tangent space by \( T\mathcal{Q} = \{A, A', M, M', R, R'\} \). On the other hand, according to the Noether Theorem, the existence of a symmetry for dynamics described by Lagrangian (17) implies a constant of motion. Let us apply the Lie derivative to the (17), we have\(^6\):

\[
L_x L = \alpha \cdot \nabla_q L + \alpha' \cdot \nabla_q L = q'^t \left[ \alpha \cdot \nabla_q \hat{L} + 2 \left( \nabla_q \alpha \right)^t \hat{L} \right] q', \tag{21}
\]

that vanishes if the functions \( \alpha \) satisfy the following system

\[
\alpha \cdot \nabla_q \hat{L} + 2(\nabla_q \alpha)^t \hat{L} = 0 \implies \alpha_i \frac{\partial \hat{L}_{km}}{\partial q_i} + 2 \frac{\partial \alpha_j}{\partial q_i} \hat{L}_{im} = 0. \tag{22}
\]

\(^4\) Obviously, the above choices are recovered for \( A(r) = a(r), B(r) = b(r), \) and \( M(r) = r^2 \). Here we deal with \( A, B, M \) as a set of coordinates in a configuration space.

\(^5\) Lowering the dimension of configuration space through the substitution (19) does not affect the dynamics since \( B \) is a non-evolving quantity. In fact, inserting Eq. (19) into the dynamical equations given by (17), they coincide with those derived by (17).

\(^6\) From now on, \( q \) indicates the vector \( \{A, M, R\} \).
Solving the system (22) means to find out the functions \( \alpha \), which assign the Noether vector \( \Omega \). However the system (22) implicitly depends on the form of \( f(R) \) and then, by solving it, we get also \( f(R) \)-models compatible with spherical symmetry. On the other hand, by choosing the \( f(R) \)-form, we can explicitly solve (22). As an example, one finds that the system (22) is satisfied if we choose

\[
f(R) = f_0 R^s, \quad \Omega = (\alpha_1, \alpha_2, \alpha_3) = \begin{pmatrix} (3-2s)kA, -kM, kR \end{pmatrix},
\]

with \( s \) a real number, \( k \) an integration constant and \( f_0 \) a dimensional coupling constant\(^7\). This means that, for any \( f(R) = R^s \), exists, at least, a Noether symmetry and a related constant of motion \( \Sigma_0 \):

\[
\Sigma_0 = \alpha_1 \cdot \nabla_q L = 2skMR^{2s-3}[2s + (s - 1)MR][(s - 2)RA' - (2s^2 - 3s + 1)AR'].
\]

A physical interpretation of \( \Sigma_0 \) is possible if one gives an interpretation of this quantity in GR, that means for \( f(R) = R \) and \( s = 1 \). In other words, the above procedure has to be applied to the Lagrangian of GR. We obtain the solution

\[
\Omega_{GR} = (-kA, kM).
\]

The functions \( A \) and \( M \) give the Schwarzschild solution, and then the constant of motion acquires the standard form

\[
\Sigma_0 = \frac{2GM}{c^2}.
\]

In other words, in the case of Einstein gravity, the Noether symmetry gives, as a conserved quantity, the Schwarzschild radius or the mass of the gravitating system. This result can be assumed as a consistency check.

In the general case, \( f(R) = R^s \), the Lagrangian (17) becomes

\[
L = \frac{sR^{2s-3}[2s + (s - 1)MR]}{M} [2(s-1)M^2A'R' + 2MRM'A' + 4(s-1)AMM'R' + ARM'^2],
\]

and the expression (19) for \( B \) is

\[
B = \frac{s[2(s-1)M^2A'R' + 2MRM'A' + 4(s-1)AMM'R' + ARM'^2]}{2AMR[2s + (s - 1)MR]}.
\]

As it can be easily checked, GR is recovered for \( s = 1 \).

Using the constant of motion (24), we solve in term of \( A \) and obtain

\[
A = R^{\frac{2s^2 - 9s + 5}{s-2}} \left\{ k_1 + \Sigma_0 \int R^{\frac{4s^2 - 9s + 5}{s-2}} \frac{dr}{2ks(s - 2)M[2s + (s - 1)MR]} \right\}
\]

for \( s \neq 2 \) and \( k_1 \) an integration constant. For \( s = 2 \), one finds

\[
A = -\frac{\Sigma_0}{12ks^2(4 + r^2R)RR'}.
\]

These relations allow to find out general black hole solutions for the field equations giving the dependence of the Ricci scalar on the radial coordinate \( r \). For example, a solution is found for

\[
s = 5/4, \quad M = r^2, \quad R = 5r^{-2},
\]

obtaining the spherically symmetric space-time

\[
ds^2 = (\alpha + \beta r)dt^2 - \frac{\beta r}{2\alpha + \beta r}dr^2 - r^2d\Omega,
\]

where \( \alpha \) is a combination of \( \Sigma_0 \) and \( k \) and \( \beta = k_1 \). In principle, the same procedure can be worked out any time Noether symmetries are identified. Our task is now to show how, from a spherically symmetric solution, one can generate an axially symmetric solution adopting the Newman-Janis procedure that works in GR. In general, the approach is not immediately straightforward since, as soon as \( f(R) \neq R \), we are dealing with fourth-order field equations which have, in principle, different existence theorems and boundary conditions. However, the existence of the Noether symmetry guarantees the consistency of the chosen \( f(R) \)-model with the field equations.

\(^7\) The dimensions are given by \( R^{1-s} \) in terms of the Ricci scalar. For the sake of simplicity, we will put \( f_0 = 1 \) in the forthcoming discussion.
IV. AXIAL SYMMETRY DERIVED FROM SPHERICAL SYMMETRY

We want to show now how it is possible to obtain an rotating solution starting from a spherically symmetric one adopting the method developed by Newman and Janis in GR. Such an algorithm can be applied to a static spherically symmetric metric considered as a "seed" metric. Let us recast the spherically symmetric metric \( \text{(31)} \) in the form

\[
{\text{ds}}^2 = e^{2\phi(r)} dt^2 - e^{2\lambda(r)} dr^2 - r^2 d\Omega,
\]

(32)

with \( g_{tt}(t, r) = e^{2\phi(r)} \) and \( g_{rr}(t, r) = e^{2\lambda(r)} \). Such a form is suitable for the considerations below. Following Newman and Janis, Eq. \( \text{(32)} \) can be written in the so-called Eddington–Finkelstein coordinates \((u, r, \theta, \phi)\), i.e. the \( g_{tt} \) component is eliminated by a change of coordinates and a cross term is introduced \[43\]. Specifically this is achieved by defining\[8\]

\[ l \]

The bar indicates the complex conjugation. At any point in space, the tetrad can be chosen in the following manner:

\[
\text{The matrix } (34) \text{ can be written in terms of a null tetrad as}
\]

\[
g^{\mu\nu} = l^\mu n^\nu + l^\nu n^\mu - m^\mu \bar{m}^\nu - \bar{m}^\mu m^\nu,
\]

(35)

where \( l^\mu, n^\mu, m^\mu \) and \( \bar{m}^\mu \) are the vectors satisfying the conditions

\[
l_\mu l^\mu = m_\mu m^\mu = n_\mu n^\mu = 0, \quad l_\mu n^\mu = -m_\mu \bar{m}^\mu = 1, \quad l_\mu m^\mu = n_\mu m^\mu = 0.
\]

(36)

The bar indicates the complex conjugation. At any point in space, the tetrad can be chosen in the following manner: \( l^\mu \) is the outward null vector tangent to the cone, \( n^\mu \) is the inward null vector pointing toward the origin, and \( m^\mu \) and \( \bar{m}^\mu \) are the vectors tangent to the two-dimensional sphere defined by constant \( r \) and \( u \). For the spacetime \([34]\), the tetrad null vectors can be

\[
\begin{align*}
l^\mu &= \delta^\mu_1 \\
n^\mu &= -\frac{1}{2} e^{-2\lambda(r)} \delta^\mu_2 + e^{-\lambda(r) - \phi(r)} \delta^\mu_0 \\
m^\mu &= \frac{1}{\sqrt{2r}} (\delta^\mu_2 + \frac{i}{\sin \theta} \delta^\mu_3) \\
\bar{m}^\mu &= \frac{1}{\sqrt{2r}} (\delta^\mu_2 - \frac{i}{\sin \theta} \delta^\mu_3)
\end{align*}
\]

(37)

Now we need to extend the set of coordinates \( x^\mu = (u, r, \theta, \phi) \) replacing the real radial coordinate by a complex variable. Then the tetrad null vectors become \[8\]

\[\text{It is worth noticing that a certain arbitrariness is present in the complexification process of the functions } \lambda \text{ and } \phi. \text{ Obviously, we have to obtain the metric } (34) \text{ as soon as } r = \bar{r}.\]
\[
\begin{align*}
\ell^\mu &= \delta_1^\mu \\
n^\mu &= -\frac{1}{2}e^{-2\lambda(r,\bar{r})}\delta_1^\mu + e^{-\lambda(r,\bar{r})-\phi(r,\bar{r})}\delta_0^\mu \\
m^\mu &= \frac{1}{\sqrt{2r}}(\delta_2^\mu + \frac{i}{\sin \theta}\delta_3^\mu) \\
\bar{m}^\mu &= \frac{1}{\sqrt{2r}}(\delta_2^\mu - \frac{i}{\sin \theta}\delta_3^\mu)
\end{align*}
\] (38)

A new metric is obtained by making a complex coordinates transformation

\[
x^\mu \to \tilde{x}^\mu = x^\mu + iy^\mu(x^\sigma),
\] (39)

where \(y^\mu(x^\sigma)\) are analytic functions of the real coordinates \(x^\sigma\), and simultaneously let the null tetrad vectors \(Z_a^\mu = (l^\mu, n^\mu, m^\mu, \bar{m}^\mu)\), with \(a = 1, 2, 3, 4\), undergo the transformation

\[
Z_a^\mu \to \tilde{Z}_a^\mu(\tilde{x}^\sigma, \bar{\tilde{x}}^\sigma) = Z_a^\mu \frac{\partial \tilde{x}^\mu}{\partial x^\rho}.
\] (40)

Obviously, one has to recover the old tetrads and metric as soon as \(\tilde{x}^\sigma = \bar{\tilde{x}}^\sigma\). In summary, the effect of the "tilde transformation" (39) is to generate a new metric whose components are (real) functions of complex variables, that is

\[
g_{\mu\nu} \to \tilde{g}_{\mu\nu} : \tilde{x} \times \bar{\tilde{x}} \mapsto \mathbb{R}
\] (41)

with

\[
\tilde{Z}_a^\mu(\tilde{x}^\sigma, \bar{\tilde{x}}^\sigma)|_{\tilde{x}=\tilde{x}} = Z_a^\mu(x^\sigma).
\] (42)

For our aims, we can make the choice

\[
\tilde{x}^\mu = x^\mu + ia(\delta_1^\mu - \delta_0^\mu)\cos \theta \to \begin{cases} 
\tilde{u} = u + ia \cos \theta \\
\tilde{r} = r - ia \cos \theta \\
\tilde{\theta} = \theta \\
\tilde{\phi} = \phi
\end{cases}
\] (43)

where \(a\) is constant and the tetrad null vectors (38), if we choose \(\tilde{r} = \bar{\tilde{r}}\), become

\[
\begin{align*}
\tilde{l}^\mu &= \delta_1^\mu \\
\tilde{n}^\mu &= -\frac{1}{2}e^{-2\lambda(\tilde{r},\bar{\tilde{r}})}\delta_1^\mu + e^{-\lambda(\tilde{r},\bar{\tilde{r}})-\phi(\tilde{r},\bar{\tilde{r}})}\delta_0^\mu \\
\tilde{m}^\mu &= \frac{1}{\sqrt{2(\tilde{r} - ia \cos \theta)}} \left[ ia(\delta_0^\mu - \delta_1^\mu) \sin \theta + \delta_2^\mu + \frac{i}{\sin \theta}\delta_3^\mu \right] \\
\bar{\tilde{m}}^\mu &= \frac{1}{\sqrt{2(\tilde{r} + ia \cos \theta)}} \left[-ia(\delta_0^\mu - \delta_1^\mu) \sin \theta + \delta_2^\mu - \frac{i}{\sin \theta}\delta_3^\mu \right]
\end{align*}
\] (44)

From the transformed null tetrad vectors, a new metric is recovered using (33). For the null tetrad vectors given by (44) and the transformation given by (43), the new metric, with coordinates \(\tilde{x}^\mu = (\tilde{u}, \tilde{r}, \tilde{\theta}, \tilde{\phi})\), is
\[ g^{\mu\nu} = \begin{pmatrix} -a^2 \sin^2 \theta -e^{-\lambda(\bar{r}, \theta)} - \phi(\bar{r}, \theta) + a^2 \sin^2 \theta & 0 & -a \sqrt{\Sigma} \\ . & -e^{-2\lambda(\bar{r}, \theta)} - \frac{a^2 \sin^2 \theta}{\Sigma^2} & 0 \\ . & . & -\frac{1}{\Sigma^2} \end{pmatrix} \]

where \( \Sigma = \sqrt{\bar{r}^2 + a^2 \cos^2 \theta} \). In the covariant form, the metric \( [45] \) is

\[
\begin{align*}
\text{ds}^2 &= e^{2\phi(\bar{r}, \theta)} d\bar{u}^2 + 2e^{\lambda(\bar{r}, \theta)} d\bar{u} d\bar{r} + 2 a e^{\phi(\bar{r}, \theta)} [e^{\lambda(\bar{r}, \theta)} - e^{\phi(\bar{r}, \theta)}] \sin^2 \theta d\theta d\phi + \\
&\quad -2 a e^{\phi(\bar{r}, \theta) + \lambda(\bar{r}, \theta)} \sin^2 \theta d\phi - \Sigma^2 a^2 \sin^2 \theta d\phi^2 + \\
&\quad - [\Sigma^2 + a^2 \sin^2 \theta e^{\phi(\bar{r}, \theta)} (2 e^{\lambda(\bar{r}, \theta)} - e^{\phi(\bar{r}, \theta)})] \sin^2 \theta d\phi^2
\end{align*}
\]  

(46)

Since the metric is symmetric, the dots in the matrix are used to indicate \( g^{\mu\nu} = g^{\nu\mu} \). The form of this metric gives the general result of the Newman-Janis algorithm starting from any spherically symmetric "seed" metric.

The metric given in Eq. \( [46] \) can be simplified by a further gauge transformation so that the only off-diagonal component is \( g_{\phi t} \). This procedure makes it easier to compare with the standard Boyer-Lindquist form of the Kerr metric \( [43] \) and to interpret physical properties such as the frame dragging. The coordinates \( \bar{u} \) and \( \phi \) can be redefined in such a way that the metric in the new coordinate system has the properties described above. More explicitly, if we define the coordinates in the following way

\[
d\bar{u} = dt + g(\bar{r}) d\bar{r} \quad \text{and} \quad d\phi = d\phi + h(\bar{r}) d\bar{r}
\]

(47)

where

\[
\begin{align*}
g(\bar{r}) &= \frac{e^{\lambda(\bar{r}, \theta)} (\Sigma^2 + a^2 \sin^2 \theta e^{\lambda(\bar{r}, \theta)} + \phi(\bar{r}, \theta))}{e^{\phi(\bar{r}, \theta)} (\Sigma^2 + a^2 \sin^2 \theta e^{\lambda(\bar{r}, \theta)})} \\
h(\bar{r}) &= -\frac{\Sigma^2 a^2 \sin^2 \theta e^{2\lambda(\bar{r}, \theta)}}{\Sigma^2 + a^2 \sin^2 \theta e^{2\lambda(\bar{r}, \theta)}}
\end{align*}
\]  

(48)

after some algebraic manipulations, one finds that, in \( (t, \bar{r}, \theta, \phi) \) coordinates system, the metric \( [46] \) becomes

\[
\begin{align*}
\text{ds}^2 &= e^{2\phi(\bar{r}, \theta)} dt^2 + ae^{\phi(\bar{r}, \theta)} [e^{\lambda(\bar{r}, \theta)} - e^{\phi(\bar{r}, \theta)}] \sin^2 \theta dt d\phi - \\
&\quad \frac{\Sigma^2}{(\Sigma^2 e^{-2\lambda(\bar{r}, \theta)} + a^2 \sin^2 \theta)} d\bar{r}^2 - \Sigma^2 dt^2 + \\
&\quad - [\Sigma^2 + a^2 \sin^2 \theta e^{\phi(\bar{r}, \theta)} (2 e^{\lambda(\bar{r}, \theta)} - e^{\phi(\bar{r}, \theta)})] \sin^2 \theta d\phi^2
\end{align*}
\]  

(49)

This metric represents the complete family of metrics that may be obtained by performing the Newman-Janis algorithm on any static spherically symmetric "seed" metric, written in Boyer-Lindquist type coordinates. The validity of these transformations requires the condition \( \Sigma^2 + a^2 \sin^2 \theta e^{2\lambda(\bar{r}, \theta)} \neq 0 \), where \( e^{2\lambda(\bar{r}, \theta)} > 0 \). Our task is now to show that such an approach can be used to derive axially symmetric solutions also in \( f(R) \)-gravity that, possibly, can be regarded as black hole solutions.

V. AXIALLY SYMMETRIC SOLUTIONS IN \( f(R) \)-GRAVITY

Starting from the above spherically symmetric solution \( [31] \), the metric tensor, written in the Eddington–Finkelstein coordinates \( (u, r, \theta, \phi) \) of the form \( [31] \) is

\[
\begin{pmatrix}
0 & \frac{1}{2r} & 0 & 0 \\
. & -2 - \frac{2a}{\bar{r}} & 0 & 0 \\
. & . & -1/r^2 & 0 \\
. & . & . & -1/(r^2 \sin^2 \theta)
\end{pmatrix}
\]

(50)
The complex tetrad null vectors (38) are now
\[
\begin{cases}
\ell^\mu = \delta^\mu_1 \\
n^\mu = -\left[1 + \frac{2}{\beta} \left(\frac{1}{2} + \frac{1}{2}\right)\right] \delta^\mu_1 + \frac{2}{\beta} \frac{1}{\sqrt{\Sigma}} \delta^\mu_0 \\
m^\mu = \frac{1}{\sqrt{2r}} (\delta^\mu_0 + \frac{1}{\sin \theta} \delta^\mu_3)
\end{cases}
\] (51)

By computing the complex coordinates transformation (43), the tetrad null vectors become
\[
\begin{cases}
\tilde{\ell}^\mu = \delta^\mu_1 \\
\tilde{n}^\mu = -\left[1 + \frac{2}{\beta} \Re\{r\} \right] \delta^\mu_1 + \frac{2}{\beta} \frac{1}{\sqrt{\Sigma}} \delta^\mu_0 \\
\tilde{m}^\mu = \frac{1}{\sqrt{2r} (\tilde{r} + ia \cos \theta)} \left[ia(\delta^\mu_0 - \delta^\mu_1) \sin \theta + \delta^\mu_2 + \frac{i}{\sin \theta} \delta^\mu_3\right]
\end{cases}
\] (52)

Now by performing the same procedure as in previous section, we derive an axially symmetric metric of the form (49) but starting from the spherically symmetric metric (31), that is
\[
\begin{align*}
&ds^2 = r(\alpha + \beta r) + a^2 \beta \cos^2 \theta du^2 + 2a(-2\alpha r - 2\beta \Sigma^2 + \sqrt{2\beta \Sigma^3/2}) \sin^2 \theta du d\phi + \\
&\quad -\frac{\beta \Sigma^2}{2\alpha r + \beta (a^2 + r^2 + \Sigma^2)} dr^2 - \Sigma^2 d\theta^2 + \\
&\quad -\left[\Sigma^2 - \frac{a^2 (\alpha r + \beta \Sigma^2 - \sqrt{2\beta \Sigma^3/2}) \sin^2 \theta}{\Sigma}\right] \sin^2 \theta d\phi^2
\end{align*}
\] (53)

It is worth noticing that the condition \(a = 0\) immediately gives the metric (31). This is nothing else but an example: the method is general and can be extended to any spherically symmetric solution derived in \(f(R)\)-gravity.

A. Physical applications: geodesics and orbits

Let us discuss now possible physical applications of the above results. We will take into account a freely falling particle moving in the space-time described by the metric (53). For our aims, we make explicit use of the Hamiltonian formalism. Given a metric \(g^\mu_\nu\), the motion along the geodesics is described by the Lagrangian
\[
\mathcal{L}(x^\mu, \dot{x}^\mu) = \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu,
\] (54)

where the overdot stands for derivative with respect to an affine parameter \(\lambda\) used to parametrize the curve. The Hamiltonian description is achieved by considering the canonical momenta and the Hamiltonian function
\[
p_\mu = \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = g^{\mu\nu} p_\nu p_\nu, \quad \mathcal{H} = p_\mu \dot{x}^\mu - \mathcal{L},
\] (55)

that results \(\mathcal{H} = \frac{1}{2} p_\mu p_\mu g^{\mu\nu}\). The advantage of the Hamiltonian formalism with respect to the Lagrangian one is that the resulting equations of motion do not contain any sign ambiguity coming from turning points in the orbits (see, for example, [44]). The Hamiltonian results explicitly independent of time and it is \(\mathcal{H} = -\frac{1}{2} m^2\), where the rest mass \(m\) is a constant (\(m = 0\) for photons). The geodesic equations are
\[
\frac{dx^\mu}{d\lambda} = \frac{\partial \mathcal{H}}{\partial p_\mu} = g^{\mu\nu} p_\nu = p^\mu,
\] (56)
\[
\frac{dp_\mu}{d\lambda} = \frac{\partial H}{\partial x_\mu} = -\frac{1}{2} \frac{\partial g^{\alpha\beta}}{\partial x_\mu} p_\alpha p_\beta = g^{\gamma\beta} \Gamma_{\mu\gamma}^\alpha p_\alpha p_\beta. \tag{57}
\]

In addition, since the Hamiltonian is independent of the affine parameter \(\lambda\), one can directly use the coordinate time as integration parameter. The problem is so reduced to solve six equations of motion. Using the above definitions, it is easy to achieve the reduced Hamiltonian (now linear in the momenta)

\[
H = -p_0 = \left[ p_i g^{0i} + \left( p_i g^{0i} \frac{m^2 + p_3 p_3}{g^{00}} \right)^{1/2} \right] \tag{58}
\]

with the equations of motion

\[
\frac{dx^i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial x^i}, \tag{59}
\]

that give the orbits. The method can be applied to the above solution \([53]\) by which the elements of the inverse metric can be easily obtained:

\[
g^{tt} = \left\{ 4 \Sigma^2 \left[ \Sigma^2 - \frac{a^2 \sin^2 \theta \left( r\alpha - \sqrt{2\sqrt{3} \Sigma^3/2 + \beta \Sigma^2} \right)}{\Sigma} \right] \right\} \times \left\{ a^2 \sin^2 \theta \left( 2r\alpha - \sqrt{2\sqrt{3} \Sigma^3/2 + 2\beta \Sigma^2} \right)^2 + 4\Sigma \left[ a^2 \beta \cos^2 \theta + r(\beta + \alpha) \right] \times \left( \Sigma^2 - \frac{a^2 \sin^2 \theta \left( r\alpha - \sqrt{2\sqrt{3} \Sigma^3/2 + \beta \Sigma^2} \right)}{\Sigma} \right) \right\}^{-1},
\]

\[
g^{rr} = -\frac{\beta (a^2 + r^2 + \Sigma^2) + 2r\alpha}{\beta \Sigma^2},
\]

\[
g^{\theta\theta} = -\frac{1}{\Sigma^2},
\]

\[
g^{\phi\phi} = \left\{ 2a\Sigma \left[ -2r\alpha + \sqrt{2\sqrt{3} \Sigma^3/2 - 2\beta \Sigma^2} \right] \right\} \times \left\{ a^2 \sin^2 \theta \left( 2r\alpha - \sqrt{2\sqrt{3} \Sigma^3/2 + 2\beta \Sigma^2} \right)^2 + 4\Sigma \left[ a^2 \beta \cos^2 \theta + r(\beta + \alpha) \right] \times \left( \Sigma^2 - \frac{a^2 \sin^2 \theta \left( r\alpha - \sqrt{2\sqrt{3} \Sigma^3/2 + \beta \Sigma^2} \right)}{\Sigma} \right) \right\}^{-1},
\]

\[
and the null ones \quad \quad g^{tr} = g^{t\theta} = g^{r\phi} = g^{\theta\phi} = 0. \tag{60}
\]

Let us consider the equatorial plane, \textit{i.e.} \(\theta = \frac{\pi}{2}, \dot{\theta} = 0\), and assume \(\alpha = 1\) and \(\beta = 2\). The reduced Hamiltonian \(H(r, \theta, \phi, p_r, p_\theta, p_\phi; t) = H\) can be written as
FIG. 1: Relative motion of the test particle with $m = 1$.

\[
H = \frac{2a p_\phi (-2r^3 + r^2 - 1)}{a^2 (-2(r-1)r^2 - 1) + r^5} + \left\{ \left( 4a^2 p_\phi^2 (-2r^3 + r^2 - 1)^2 + a^2 (-2(r-1)r^2 - 1 - r^5) (a^2 (r(2r-3)(2r+1) + 6) - 2) + (2r+1)r^4 \times \left( -\frac{p_\phi (2r+1)}{a^2 (r(2r-3)(2r+1) + 6) - 2} + (2r+1)r^4 \right) \right) \right\}^{1/2}. \tag{62}
\]

It is independent of $\phi$ (i.e. we are considering an azimuthally symmetric spacetime), and then the conjugate momentum $p_\phi$ is an integral of motion. From Eqs. (59), one can derive the coupled equations for $\{r, \theta, \phi, p_r, p_\theta\}$ and integrate them numerically (the expressions are very cumbersome and will not be reported here). To this goal, we have to specify the initial value of the position-momentum vector in the phase space. A Runge-Kutta method can be used to solve the differential equations. In Fig. 1, the relative trajectories are sketched.

VI. HYDROSTATIC EQUILIBRIUM OF STELLAR STRUCTURES

We have shown that it is possible to derive black hole spherically and axially symmetric solutions in $f(R)$-gravity. The physics of black holes contains a lot of unsolved problems. One of them is the question on the nature of the black holes as well as the nature of stellar objects presenting anomalous behaviors (e.g. the magnetars) that do not obey the standard dynamics of (relativistic and non-relativistic) stellar structures. For this purpose, it is necessary to understand the hydrostatic equilibrium of such objects, that is one of fundamental properties (or one of the basic assumptions) of self-gravitating systems. Here we start by describing the standard hydrostatic equilibrium assuming Newtonian (i.e. the GR weak field limit) and then we compare these results with the equilibrium derived from the weak field limit of $f(R)$-gravity.

The condition of hydrostatic equilibrium for stellar structures in Newtonian dynamics is achieved by considering the equation

\[
\frac{dp}{dr} = \frac{d\Phi}{dr} r^p, \tag{63}
\]
where $p$ is the pressure, $-\Phi$ is the gravitational potential, and $\rho$ is the density. Together with the above equation, the Poisson equation
\[
\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d\Phi}{dr} \right) = -4\pi G \rho,
\]
gives the gravitational potential as solution for a given matter density $\rho$. Since we are taking into account only static and stationary situations, here we consider only time-independent solutions. In general, the temperature $\tau$ appears in Eqs. (63) and (64) the density satisfies an equation of state of the form $\rho = \rho(p, \tau)$. In any case, we assume that there exists a polytropic relation between $p$ and $\rho$ of the form
\[
p = K \rho^\gamma,
\]
where $K$ and $\gamma$ are constant. Note that $\Phi > 0$ in the interior of the model since we define the gravitational potential as $-\Phi$. The polytropic constant $K$ is fixed and can be obtained as a combination of fundamental constants. However there are several realistic cases where $K$ is not fixed and another equation for its evolution is needed. The constant $\gamma$ is the polytropic exponent. Inserting the polytropic equation of state into Eq. (63), we obtain
\[
d\Phi/dr = \gamma K \rho^{\gamma-1} \frac{d\rho}{dr},
\]
For $\gamma \neq 1$, the above equation can be integrated giving
\[
\frac{\gamma K}{\gamma - 1} \rho^{\gamma-1} = \Phi \rightarrow \rho = \left[ \frac{\gamma - 1}{\gamma K} \right]^\frac{1}{\gamma - 1} A_n \Phi^n
\]
where we have chosen the integration constant to give $\Phi = 0$ at surface ($\rho = 0$). The constant $n$ is called the polytropic index and is defined as $n = \frac{1}{\gamma - 1}$. Inserting the relation into the Poisson equation, we obtain a differential equation for the gravitational potential
\[
d^2\Phi/dr^2 + 2 \frac{d\Phi}{r} = -4\pi G A_n \Phi^n.
\]
Let us define now the dimensionless variables
\[
\begin{cases}
z = |x| \sqrt{\frac{x A_n \Phi^n}{\Phi}} \\
w(z) = \frac{\Phi}{\Phi_c} = (\frac{r}{r_c})^{\frac{\gamma}{\gamma - 1}}
\end{cases}
\]
where the subscript $c$ refers to the center of the star and the relation between $\rho$ and $\Phi$ is given by Eq. (67). At the center ($r = 0$), we have $z = 0, \Phi = \Phi_c, \rho = \rho_c$ and therefore $w = 1$. Then Eq. (68) can be written
\[
\frac{d^2w}{dz^2} + 2 \frac{dw}{z} + w^n = 0
\]
This is the standard Lane-Emden equation describing the hydrostatic equilibrium of stellar structures in the Newtonian theory. Now we want to compare this standard result with the one coming from $f(R)$-gravity.

9 The radius $r$ is assumed as the spatial coordinate. It varies from $r = 0$ at the center to $r = \xi$ at the surface of the star.
VII. THE NEWTONIAN LIMIT OF $f(R)$-GRAVITY

In order to achieve the Newtonian limit of the theory the metric tensor $g_{\mu\nu}$ have to be approximated as follows

$$g_{\mu\nu} \sim \begin{pmatrix} 1 - 2\Phi(t, x) + O(4) & O(3) \\ O(3) & -\delta_{ij} + O(2) \end{pmatrix}$$

(71)

where $O(n)$ (with $n = \text{integer}$) denotes the order of the expansion (see [46] for details). The set of coordinates $x^\mu = (t, x^1, x^2, x^3)$. The Ricci scalar formally becomes

$$R \sim R^{(2)}(t, x) + O(4)$$

(72)

The $n$-th derivative of Ricci function can be developed as

$$f^n(R) \sim f^n(R^{(2)} + O(4)) \sim f^n(0) + f^{n+1}(0)R^{(2)} + O(4)$$

(73)

here $R^{(n)}$ denotes a quantity of order $O(n)$. From lowest order of field equations, we have $f^{(0)} = 0$ which trivially follows from the above assumption (71) for the metric. This means that the space-time is asymptotically Minkowskian and we are discarding a cosmological constant term in this analysis\(^{11}\). Field equations at $O(2)$-order, that is at Newtonian level, are

$$\begin{cases} R_{tt}^{(2)} = \frac{1}{2} f''(0) \Delta R^{(2)} = \mathcal{X} T_{tt}^{(0)} \\ -3f''(0) \Delta R^{(2)} - R^{(2)} = \mathcal{X} T^{(0)} \end{cases}$$

(74)

where $\Delta$ is the Laplacian in the flat space, $R_{tt}^{(2)} = -\Delta \Phi(t, x)$ and, for the sake of simplicity, we set $f'(0) = 1$. We recall that the energy-momentum tensor for a perfect fluid is

$$T_{\mu\nu} = (\epsilon + p) u_\mu u_\nu - p g_{\mu\nu}$$

(75)

where $p$ is the pressure and $\epsilon$ is the energy density. Being the pressure contribution negligible in the field equations in Newtonian approximation, we have

$$\begin{cases} \Delta \Phi + \frac{R^{(2)}}{2} + f''(0) \Delta R^{(2)} = -\mathcal{X} \rho \\ 3f''(0) \Delta R^{(2)} + R^{(2)} = -\mathcal{X} \rho \end{cases}$$

(76)

where $\rho$ is now the mass density\(^{12}\). We note that for $f''(0) = 0$ we have the standard Poisson equation: $\Delta \Phi = -4\pi G\rho$. This means that as soon as the second derivative of $f(R)$ is different from zero, deviations from the Newtonian limit of GR emerge.

The gravitational potential $-\Phi$, solution of Eqs. (76), has in general a Yukawa-like behavior depending on a characteristic length on which it evolves\(^{10}\). Then as it is evident the Gauss theorem is not valid\(^{13}\) since the force law is not $\propto |x|^{-2}$. The equivalence between a spherically symmetric distribution and point-like distribution is not valid and how the matter is distributed in the space is very important\(^{10-48}\).

\(^{10}\) The Greek index runs between 0 and 3; the Latin index between 1 and 3.

\(^{11}\) This assumption is quite natural since the contribution of a cosmological constant term is irrelevant at stellar level.

\(^{12}\) Generally it is $\epsilon = \rho c^2$.

\(^{13}\) It is worth noticing that also if the Gauss theorem does not hold, the Bianchi identities are always valid so the conservation laws are guaranteed.
Besides the Birkhoff theorem results modified at Newtonian level: the solution can be only factorized by a space-depending function and an arbitrary time-depending function \[46\]. Furthermore the correction to the gravitational potential is depending on the only first two derivatives of \(f(R)\) in \(R = 0\). This means that different analytical theories, from the third derivative perturbation terms on, admit the same Newtonian limit \[46, 47\].

Eqs. (76) can be considered the modified Poisson equation for \(f(R)\)-gravity. They do not depend on gauge condition choice \[48\]. We know that \(R^{(2)} \simeq \frac{1}{2} \nabla^2 g^{(2)}_{00} - \frac{1}{4} \nabla^2 g^{(2)}_{ii}\). 

Inserting in the above result the approximations \(71\) we obtain

\[
R^{(2)} \simeq \nabla^2 (\Phi - \Psi). \tag{78}
\]

Finally, we obtain the field equations

\[
\nabla^2 \Phi + \nabla^2 \Psi - 2 f''(0) \nabla^4 \Phi + 2 f''(0) \nabla^4 \Psi = 2 X \rho \tag{79}
\]

\[
\nabla^2 \Phi - \nabla^2 \Psi + 3 f''(0) \nabla^4 \Phi - 3 f''(0) \nabla^4 \Psi = -X \rho. \tag{80}
\]

By eliminating the higher-order terms, the standard Poisson equation is recovered. This last step, will be useful to calculate the Jeans instability.

**VIII. STELLAR HYDROSTATIC EQUILIBRIUM IN \(f(R)\)-GRAVITY**

From the Bianchi identity, satisfied by the field equations, we have

\[
T^\mu_\nu \mu = 0 \rightarrow \frac{\partial p}{\partial x^k} = -\frac{1}{2} (p + \epsilon) \frac{\partial \ln g_{tt}}{\partial x^k} \tag{81}
\]

If the dependence on the temperature \(\tau\) is negligible, i.e. \(p = \rho(p)\), this relation can be introduced into Eqs. (76), which become a system of three equations for \(p, \Phi\) and \(R^{(2)}\) and can be solved without the other structure equations.

Let us suppose that matter satisfies still a polytropic equation \(p = K \rho^\gamma\). If we introduce Eq. (67) into Eqs. (76) we obtain an integro-differential equation for the gravitational potential \(-\Phi\), that is

\[
\Delta \Phi(x) + \frac{2 \lambda A_n}{3} \Phi(x)^n = -\frac{m^2 \lambda A_n}{6} \int d^3x' G(x, x') \Phi(x')^n \tag{82}
\]

where \(G(x, x') = -\frac{1}{4\pi} \frac{e^{-m|x-x'|}}{|x-x'|}\) is the Green function of the field operator \(\Delta_x - m^2\) for systems with spherical symmetry and \(m^2 = -\frac{1}{3f''(0)}\) (for details see \[47, 48\]). The integro-differential nature of Eq. (82) is the proof of the non-viability of Gauss theorem for \(f(R)\)-gravity. Adopting again the dimensionless variables

\[
\begin{align*}
    z &= \frac{|x|}{\xi_0} \\
    w(z) &= \frac{\Phi}{\Phi_c}
\end{align*}
\]

where

\[
\xi_0 \doteq \sqrt{\frac{3}{2 \lambda A_n \Phi_c^{n-1}}} \tag{84}
\]

is a characteristic length linked to stellar radius \(\xi\). Eq. (82) becomes
\[
\frac{d^2 w(z)}{dz^2} + \frac{2}{z} \frac{dw(z)}{dz} + w(z)^n = \frac{m \xi_0}{8} \int_0^{\xi/\xi_0} dz' z' \left\{ e^{-m \xi_0 |z-z'|} - e^{-m \xi_0 |z+z'|} \right\} w(z')^n
\]

(85)

which is the modified Lané-Emden equation deduced from \( f(R) \)-gravity. Clearly the particular \( f(R) \)-model is specified by the parameters \( m \) and \( \xi_0 \). If \( m \to \infty \) (i.e. \( f(R) \to R \)), Eq. (85) becomes Eq. (70). We are only interested in solutions of Eq. (85) that are finite at the center, that is for \( z = 0 \). Since the center must be an equilibrium point, the gravitational acceleration \( |g| = -\frac{d \Phi}{dr} \propto \frac{dw}{dz} \) must vanish for \( w'(0) = 0 \). Let us assume we have solutions \( w(z) \) of Eq. (85) that fulfill the boundary conditions \( w(0) = 1 \) and \( w(\xi/\xi_0) = 0 \); then according to the choice (83), the radial distribution of density is given by

\[
\rho(|x|) = \rho_c w^n, \quad \rho_c = A_n \Phi_c^n
\]

and the pressure by

\[
p(|x|) = p_c w^{n+1}, \quad p_c = K \rho_c^\gamma
\]

(87)

For \( \gamma = 1 \) (or \( n = \infty \)) the integro-differential Eq. (85) is not correct. This means that the theory does not contain the case of isothermal sphere of ideal gas. In this case, the polytropic relation is \( p = K \rho \). Putting this relation into Eq. (81) we have

\[
\Phi(K) = \ln \rho - \ln \rho_c \to \rho = \rho_c e^{\Phi/K}
\]

(88)

where the constant of integration is chosen in such a way that the gravitational potential is zero at the center. If we introduce Eq. (88) into Eqs. (76), we have

\[
\Delta \Phi(x) + \frac{2 \chi \rho_c}{3} e^{\Phi(|x|)/K} = -\frac{m^2 \chi \rho_c}{6} \int d^3 x' G(x, x') e^{\Phi(|x'|)/K}
\]

(89)

Assuming the dimensionless variables \( z = |x| / \xi_1 \) and \( w(z) = \Phi/K \) where \( \xi_1 = \sqrt{3K/2\chi \rho_c} \), Eq. (89) becomes

\[
\frac{d^2 w(z)}{dz^2} + \frac{2}{z} \frac{dw(z)}{dz} + e^{w(z)} = \frac{m \xi_1}{8} \int_0^{\xi/\xi_1} dz' z' \left\{ e^{-m \xi_1 |z-z'|} - e^{-m \xi_1 |z+z'|} \right\} e^{w(z')}
\]

(90)

which is the modified "isothermal" Lané-Emden equation derived \( f(R) \)-gravity.

**IX. SOLUTIONS OF THE STANDARD AND MODIFIED LANÉ-EMDEN EQUATIONS**

The task is now to solve the modified Lané-Emden equation and compare its solutions to those of standard Newtonian theory. Only for three values of \( n \), the solutions of Eq. (70) have analytical expressions \( [45] \)

\[
\begin{align*}
  n = 0 & \quad \Rightarrow w^{(0)}_{GR}(z) = 1 - \frac{z^2}{6} \\
  n = 1 & \quad \Rightarrow w^{(1)}_{GR}(z) = \frac{\sin z}{z} \\
  n = 5 & \quad \Rightarrow w^{(5)}_{GR}(z) = \frac{1}{\sqrt{1 + \frac{z^2}{3}}}
\end{align*}
\]

(91)

We label these solution with \( GR \) since they agree with the Newtonian limit of GR. The surface of the polytrope of index \( n \) is defined by the value \( z = z^{(n)} \), where \( \rho = 0 \) and thus \( w = 0 \). For \( n = 0 \) and \( n = 1 \) the surface is reached
for a finite value of $z^{(n)}$. The case $n = 5$ yields a model of infinite radius. It can be shown that for $n < 5$ the radius of polytropic models is finite; for $n > 5$ they have infinite radius. From Eqs. (91) one finds $z^{(0)}_{GR} = \sqrt{6}$, $z^{(1)}_{GR} = \pi$ and $z^{(5)}_{GR} = \infty$. A general property of the solutions is that $z^{(n)}$ grows monotonically with the polytropic index $n$. In Fig. 2 we show the behavior of solutions $w^{(n)}_{GR}$ for $n = 0, 1, 5$. Apart from the three cases where analytic solutions are known, the classical Lane-Emden Eq. (70) has to be be solved numerically, considering with the expression

$$w^{(n)}_{GR}(z) = \sum_{i=0}^{\infty} a_i^{(n)} z^i$$

(92)

for the neighborhood of the center. Inserting Eq. (92) into Eq. (70) and by comparing coefficients one finds, at lowest orders, a classification of solutions by the index $n$, that is

$$w^{(n)}_{GR}(z) = 1 - \frac{z^2}{6} + \frac{n}{120} z^4 + \ldots$$

(93)

The case $\gamma = 5/3$ and $n = 3/2$ is the non-relativistic limit while the case $\gamma = 4/3$ and $n = 3$ is the relativistic limit of a completely degenerate gas.

Also for modified Lane-Emden Eq. (85), we have an exact solution for $n = 0$. In fact, it is straightforward to find out

$$w^{(0)}_{fR}(z) = 1 - \frac{z^2}{8} + \left(1 + m\xi\right) e^{-m\xi} \left[1 - \frac{\sinh m\xi_0 z}{m\xi_0 z}\right]$$

(94)

where the boundary conditions $w(0) = 1$ and $w'(0) = 0$ are satisfied. A comment on the GR limit (that is $f(R) \to R$) of solution (94) is necessary. In fact when we perform the limit $m \to \infty$, we do not recover exactly $w^{(0)}_{GR}(z)$. The difference is in the definition of quantity $\xi_0$. In $f(R)$-gravity we have the definition (84) while in GR it is $\xi_0 = \sqrt{\frac{2}{X A_n \Phi_c^2}}$, since in the first equation of (76), when we perform $f(R) \to R$, we have to eliminate the trace equation condition. In general, this means that the Newtonian limit and the Eddington parameterization of different relativistic theories of gravity cannot coincide with those of GR (see [49] for further details on this point).

The point $z^{(0)}_{fR}$ is calculated by imposing $w^{(0)}_{fR}(z^{(0)}_{fR}) = 0$ and by considering the Taylor expansion

$$\frac{\sinh m\xi_0 z}{m\xi_0 z} \sim 1 + \frac{1}{6} (m\xi_0 z)^2 + O(m\xi_0 z)^4$$

(95)

we obtain

$$z^{(0)}_{fR} = \frac{2\sqrt{6}}{\sqrt{3 + (1 + m\xi)e^{-m\xi}}}.$$}

Since the stellar radius $\xi$ is given by definition $\xi = \xi_0 z^{(0)}_{fR}$, we obtain the constraint

$$\xi = \sqrt{\frac{3\Phi_c}{2\pi G}} \frac{1}{\sqrt{1 + \frac{1 + m\xi}{3} e^{-m\xi}}}$$

(96)

By solving numerically the constraint\textsuperscript{14} Eq. (96), we find the modified expression of the radius $\xi$. If $m \to \infty$ we have the standard expression $\xi = \sqrt{\frac{3\Phi_c}{2\pi G}}$ valid for the Newtonian limit of GR. Besides, it is worth noticing that in the $f(R)$-gravity case, for $n = 0$, the radius is smaller than in GR. On the other hand, the gravitational potential $-\Phi$ gives rise to a deeper potential well than the corresponding Newtonian potential derived from GR [17].

In the case $n = 1$, Eq. (85) can be recast as follows

\textsuperscript{14} In principle, there is a solution for any value of $m$. 

\[
\frac{d^2 \tilde{w}(z)}{dz^2} + \tilde{w}(z) = \frac{m \xi_0}{8} \int_0^{\xi/\xi_0} dz' \left\{ e^{-m |\xi_0 z - z'|} - e^{-m |\xi_0 z + z'|} \right\} \tilde{w}(z')
\]

(97)

where \( \tilde{w} = z w \). If we consider the solution of GR as a small perturbation to the one of GR, we have

\[
\tilde{w}_{f(R)}^{(1)}(z) \sim \tilde{w}_{GR}^{(1)}(z) + e^{-m \xi} \Delta \tilde{w}_{f(R)}^{(1)}(z)
\]

(98)

The coefficient \( e^{-m \xi} < 1 \) is the parameter with respect to which we perturb Eq. (97). Besides these position ensure us that when \( m \to \infty \) the solution converge to something like \( \tilde{w}_{GR}^{(1)}(z) \). Substituting Eq. (98) in Eq. (97), we have

\[
\frac{d^2 \Delta \tilde{w}_{f(R)}^{(1)}(z)}{dz^2} + \Delta \tilde{w}_{f(R)}^{(1)}(z) = \frac{m \xi_0 e^{m \xi}}{8} \int_0^{\xi/\xi_0} dz' \left\{ e^{-m |\xi_0 z - z'|} - e^{-m |\xi_0 z + z'|} \right\} \tilde{w}_{GR}^{(1)}(z')
\]

(99)

and the solution is easily found

\[
w_{f(R)}^{(1)}(z) \sim \frac{\sin z}{z} \left\{ 1 + \frac{m^2 \xi^2}{8(1 + m^2 \xi^2)} \left[ 1 + \frac{2 e^{-m \xi}}{1 + m^2 \xi} \left( \cos \xi/\xi_0 + m \xi_0 \sin \xi/\xi_0 \right) \right] \right\}
\]

\[ - \frac{m^2 \xi_0^2}{8(1 + m^2 \xi_0^2)} \left[ \frac{2 e^{-m \xi}}{1 + m^2 \xi_0} \left( \cos \xi/\xi_0 + m \xi_0 \sin \xi/\xi_0 \right) \frac{\sinh m \xi_0 z}{m \xi_0 z} + \cos z \right]
\]

(100)

Also in this case, for \( m \to \infty \), we do not recover exactly \( w_{GR}^{(1)}(z) \). The reason is the same of previous \( n = 0 \) case [49]. Analytical solutions for other values of \( n \) are not available.

To conclude this section, we report the gravitational potential profile generated by a spherically symmetric source of uniform mass with radius \( \xi \). We can impose a mass density of the form

\[
\rho = \frac{3M}{4\pi \xi^3} \Theta(\xi - |x|)
\]

(101)

where \( \Theta \) is the Heaviside function and \( M \) is the mass [47, 48]. By solving field Eqs. inside the star and considering the boundary conditions \( w(0) = 1 \) and \( w'(0) = 0 \), we get

\[
w_{f(R)}(z) = \left[ \frac{3}{2 \xi} + \frac{1}{m^2 \xi^3} - \frac{e^{-m \xi(1 + m \xi)}}{m^2 \xi^3} \right]^{-1} \times
\]

\[ \times \left[ \frac{3}{2 \xi} + \frac{1}{m^2 \xi^3} - \frac{\xi_0^2 z^2}{2 \xi^4} - \frac{e^{-m \xi(1 + m \xi) \sinh m \xi_0 z}}{m^2 \xi^3} \frac{m \xi_0 z}{m \xi_0 z} \right]
\]

(102)

In the limit \( m \to \infty \), we recover the GR case \( w_{GR}(z) = 1 - \frac{\xi_0^2 z^2}{2 \xi^2} \). In Fig. 2 we show the behaviors of \( w_{f(R)}^{(0)}(z) \) and \( w_{f(R)}^{(1)}(z) \) with respect to the corresponding GR cases. Furthermore, we plot the potential generated by a uniform spherically symmetric mass distribution in GR and \( f(R) \)-gravity and the case \( w_{GR}^{(5)}(z) \).

X. EXAMPLES: DUST-DOMINATED SELF-GRAVITATING SYSTEMS

From the above equations of hydrostatic equilibrium in \( f(R) \)-gravity, one can study the formation and collapse of self-gravitating objects like stars and black holes. It is well known that the scenario involving the formation of cosmic structures (i.e. from stars up to galaxies and clusters of galaxies) occurs for gravitational instability. In the standard picture, the first objects that form are the dark matter halos, which aggregate in a hierarchical way due to gravitational collapse. If such systems reach a virial equilibrium, the collapse can stop otherwise it can indefinitely continue. Subsequently, the baryons are affected by the gravity of the halo potential wells: gas, of collisional nature,
converts the kinetic energy of the 'fall' into thermal energy and heat reaching the virial temperature. Subsequently, the radiation losses cause the cooling of the baryonic component, its condensation and subsequent formation of molecular clouds, finally of stars. The standard theory of gravitational collapse for dust dominated systems and can be compared to that for \( f(R) \)-gravity. The collapse of self-gravitational collisionless systems can be dealt with the introduction of coupled collisionless Boltzmann and Poisson equations (for details, see [50]):

\[
\frac{\partial f(\vec{r}, \vec{v}, t)}{\partial t} + (\vec{v} \cdot \nabla_{\vec{r}}) f(\vec{r}, \vec{v}, t) - \left( \nabla_{\vec{v}} \Phi_0 + \nabla_{\vec{v}} \right) f(\vec{r}, \vec{v}, t) = 0
\]

(103)

\[
\nabla^2 \Phi_1(\vec{r}, t) = 4\pi G \int f_1(\vec{r}, \vec{v}, t) d\vec{v},
\]

(104)

A self-gravitating system at equilibrium is described by a time-independent distribution function \( f_0(x, \vec{v}) \) and a potential \( \Phi_0(x) \) that are solutions of Eq. (103) and (104). Considering a small perturbation to this equilibrium:

\[
f(\vec{r}, \vec{v}, t) = f_0(\vec{r}, \vec{v}) + \epsilon f_1(\vec{r}, \vec{v}, t), \\
\Phi(\vec{r}, t) = \Phi_0(\vec{r}) + \epsilon \Phi_1(\vec{r}, t),
\]

(105, 106)

where \( \epsilon \ll 1 \) and by substituting in Eq. (103) and (104) and by linearizing, one obtains:

\[
\frac{\partial f_1(\vec{r}, \vec{v}, t)}{\partial t} + \vec{v} \cdot \frac{\partial f_1(\vec{r}, \vec{v}, t)}{\partial \vec{r}} - \nabla_{\vec{v}} \Phi_0(\vec{r}) \cdot \frac{\partial f_0(\vec{r}, \vec{v})}{\partial \vec{v}} - \nabla_{\vec{v}} \Phi_1(\vec{r}, \vec{v}, t) \cdot \frac{\partial f_1(\vec{r}, \vec{v}, t)}{\partial \vec{v}} = 0,
\]

(107)

\[
\nabla^2 \Phi_1(\vec{r}, t) = 4\pi G \int f_1(\vec{r}, \vec{v}, t) d\vec{v},
\]

(108)

Since the equilibrium state is assumed to be homogeneous and time-independent, one can set \( f_0(\vec{x}, \vec{v}, t) = f_0(\vec{v}) \), and the so-called Jeans "swindle" to set \( \Phi_0 = 0 \). In Fourier components, Eqs. (107) and (108) become:

\[
-i \omega f_1 + \vec{v} \cdot \left( i\vec{k} f_1 \right) - \left( i\vec{k} \Phi_0 \right) \cdot \frac{\partial f_0}{\partial \vec{v}} = 0,
\]

(109)

\[
-k^2 \Phi_1 = 4\pi G \int f_1 d\vec{v}.
\]

(110)
By combining these equations, the dispersion relation

\[ 1 + \frac{4\pi G}{k^2} \int \frac{\vec{k} \cdot \partial f_0}{\vec{v} \cdot \vec{k} - \omega} dv = 0; \]  

(111)

is obtained. In the case of stellar systems, by assuming a Maxwellian distribution function for \( f_0 \), we have

\[ f_0 = \frac{\rho_0}{(2\pi\sigma^2)^{\frac{3}{2}}} e^{-\frac{v^2}{2\sigma^2}}, \]

(112)

imposing that \( \vec{k} = (k, 0, 0) \) and substituting in Eq. (111), one gets:

\[ 1 - \frac{2\sqrt{2\pi G \rho_0}}{k\sigma^3} \int \frac{v_x e^{-2\sigma^2}}{kv_x - \omega} dv_x = 0. \]

(113)

By setting \( \omega = 0 \), the limit for instability is obtained:

\[ k^2(\omega = 0) = \frac{4\pi G \rho_0}{\sigma^2} = k_J^2, \]

(114)

by which it is possible to define the Jeans mass \( (M_J) \) as the mass originally contained within a sphere of diameter \( \lambda_J \):

\[ M_J = \frac{4\pi}{3} \rho_0 \left( \frac{1}{2} \lambda_J \right)^3, \]

(115)

where

\[ \lambda_J^2 = \frac{\pi \sigma^2}{G \rho_0} \]

(116)

is the Jeans length. Substituting Eq. (116) into Eq. (115), we recover

\[ M_J = \frac{\pi}{6} \left( \frac{1}{\rho_0} \left( \frac{\pi \sigma^2}{G} \right)^3 \right). \]

(117)

All perturbations with wavelengths \( \lambda > \lambda_J \) are unstable in the stellar system. In order to evaluate the integral in the dispersion relation for real and nonzero values of \( \omega \), the dispersion relation has to be rewritten as

\[ 1 - \frac{k^2}{k^2} W \left( \frac{\omega}{k\sigma} \right) = 0, \]

(118)

defining

\[ W \left( \frac{\omega}{k\sigma} \right) = \frac{1}{\sqrt{2\pi}} \int \frac{x^2}{x - Z} dx, \]

(119)

and setting \( \omega = i\omega_I \) and \( Re(W \left( \frac{\omega}{k\sigma} \right)) = 0 \). In order to study unstable modes (for details, see Appendix B in [50]) we replace the following identities

\[ \left\{ \begin{array}{l} \int_0^{\infty} \frac{x^2 e^{-x^2}}{x^2 + \beta^2} dx = \frac{1}{2} \sqrt{\pi} - \frac{1}{2} \pi \beta e^{\beta^2} \left[ 1 - \text{erf} \beta \right], \\ \text{erf} z = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt. \end{array} \right. \]

\[ k^2 = k_J^2 \left\{ 1 - \frac{\sqrt{\pi} \omega_I}{\sqrt{2k\sigma}} e^{\left( \frac{\omega_I}{2k\sigma} \right)^2} \left[ 1 - \text{erf} \left( \frac{\omega_I}{\sqrt{2k\sigma}} \right) \right] \right\}. \]

(120)

This is the standard dispersion relation describing the criterion to collapse for infinite homogeneous fluid and stellar systems [50].
XI. JEANS CRITERION FOR GRAVITATIONAL INSTABILITY IN \( f(R) \)-GRAVITY

Our task is now to check how the Jeans instability occurs in \( f(R) \)-gravity [32]. Let us approach the Jeans instability with the Poisson equations given by Eqs. (79) and (80) after assuming the collisionless Boltzmann equation:

\[
\frac{\partial f(\vec{r}, \vec{v}, t)}{\partial t} + (\vec{v} \cdot \nabla) f(\vec{r}, \vec{v}, t) - \left( \nabla \Phi \cdot \nabla \right) f(\vec{r}, \vec{v}, t) = 0.
\]

(121)

Then we have

\[
\nabla^2 (\Phi + \Psi) - 2\alpha \nabla^4 (\Phi - \Psi) = 16\pi G \int f(\vec{r}, \vec{v}, t) d\vec{v},
\]

(122)

\[
\nabla^2 (\Phi - \Psi) + 3\alpha \nabla^4 (\Phi - \Psi) = -8\pi G \int f(\vec{r}, \vec{v}, t) d\vec{v}.
\]

(123)

In the previous equations, we have replaced \( f''(0) \) with the greek letter \( \alpha \). As in standard case, we consider small perturbation to the equilibrium and linearize the equations. After we write equations in Fourier space so they became

\[
-i\omega f_1 + \vec{v} \cdot \left( i\vec{k} f_1 \right) - \left( i\vec{k} \Phi_1 \right) \cdot \frac{\partial f_0}{\partial \vec{v}} = 0,
\]

(124)

\[
-k^2 (\Phi_1 + \Psi_1) - 2\alpha k^4 (\Phi_1 - \Psi_1) = 16\pi G \int f_1 d\vec{v},
\]

(125)

\[
k^2 (\Phi_1 - \Psi_1) - 3\alpha k^4 (\Phi_1 - \Psi_1) = 8\pi G \int f_1 d\vec{v}.
\]

(126)

Combining Eqs. (125) and (126), we obtain a relation between \( \Phi_1 \) and \( \Psi_1 \),

\[
\Psi_1 = \frac{3}{1 - 4\alpha k^2} \Phi_1
\]

inserting this relation in Eq. (125) and combining it with Eq. (124), we obtain the dispersion relation

\[
1 - 4\pi G \frac{1 - 4\alpha k^2}{3\alpha k^4 - k^2} \int \left( \frac{\vec{k} \cdot \frac{\partial f_0}{\partial \vec{v}}}{\vec{v} \cdot \vec{k} - \omega} \right) d\vec{v} = 0.
\]

(127)

If we assume, as in standard case, that \( f_0 \) is given by (112) and \( \vec{k} = (k, 0, 0) \), one can write

\[
1 + 2\sqrt{2\pi} G \rho_0 \frac{1 - 4\alpha k^2}{3\alpha k^4 - k^2} \left[ \int k v_x e^{-\frac{v^2}{2\sigma^2}} \left( 1 - \text{erf} (x) \right) dv_x \right] = 0.
\]

(128)

By eliminating the higher-order terms (imposing \( \alpha = 0 \)), we obtain again the standard dispersion Eq. (111). In order to compute the integral in the dispersion relation (128), we consider the same approach used in the classical case, and finally we obtain:

\[
1 + G \frac{1 - 4\alpha k^2}{3\alpha k^4 - k^2} \left[ 1 - \sqrt{\pi} x e^{-x^2} (1 - \text{erf} (x)) \right] = 0,
\]

where \( x = \frac{\omega_i}{\sqrt{2k} \sigma} \) and \( G = \frac{4\pi G \rho_0}{\sigma^2} \). In order to evaluate Eq. (129) comparing it with the classical one, given by Eq. (111), it is very useful to normalize the equation to the classical Jeans length showed in Eq. (116), by fixing the parameter of \( f(R) \)-gravity, that is

\[
\alpha = -\frac{1}{k f_j^2} = -\frac{\sigma^2}{4\pi G \rho_0}.
\]

(130)
This parameterization is correct because the dimension $\alpha$ (an inverse of squared length) allows us to parametrize as in the standard case. Finally we write

$$\frac{3k^4}{k_j^2} + \frac{k^2}{k_j^2} = \left( \frac{4k^2}{k_j^2} + 1 \right) \left[ 1 - \sqrt{\pi x^2} (1 - \text{erf}[x]) \right] = 0. \quad (131)$$

The function is plotted in Fig. 3, where Eq. (129) and the standard dispersion (50) are confronted in order to see the difference between $f(R)$ and Newtonian gravity.

As shown in the Figure 3, the effects of a different theory of gravity changes the limit of instability. The limit is higher than the classical case and the curve has a greater slope. This fact is important because the mass limit value of interstellar clouds decreases changing the initial conditions to start the collapse. This feature could have dramatic effects for star and black hole formation.

XII. THE JEANS MASS LIMIT IN $f(R)$-GRAVITY

A numerical estimation of the $f(R)$-instability length in terms of the standard Newtonian one can be achieved. By solving numerically Eq. (131) with the condition $\omega = 0$, we obtain that the collapse occurs for

$$k^2 = 1.2637k_j^2. \quad (132)$$

However we can estimate also analytically the limit for the instability. In order to evaluate the Jeans mass limit in $f(R)$-gravity, we set $\omega = 0$ in Eq. (128) and then

$$3\sigma^2 \alpha k^4 - (16\pi G\rho_0 \alpha + \sigma^2) k^2 + 4\pi G\rho_0 = 0. \quad (133)$$

It is worth stressing that the additional condition $\alpha < 0$ discriminates the class of viable $f(R)$ models: in such a case we obtain stable cosmological solution and positively defined massive states (51). In other words, this condition selects the physically viable models allowing to solve Eq. (133) for real values of $k$. In particular, the above numerical solution can be recast as

$$k^2 = \frac{2}{3} \left( 3 + \sqrt{21} \right) \pi \frac{G\rho}{\sigma^2}. \quad (134)$$
The relation to the Newtonian value of the Jeans instability is

\[ k^2 = \frac{1}{6} \left( 3 + \sqrt{21} \right) k_J^2. \quad (135) \]

Now, we can define the new Jeans mass as:

\[ \tilde{M}_J = 6 \sqrt{\frac{6}{(3 + \sqrt{21})^3}} M_J, \quad (136) \]

that is proportional to the standard Newtonian value. We will confront this specific solutions with some observed structures.

Before this comparison, some considerations are in order. Star formation is one of the best settled problems of modern astrophysics. However, some shortcomings emerge as soon as one faces dynamics of diffuse gas evolving into stars and star formation in galactic environment. One can deal with the star formation problem in two ways: i) we can take into account the formation of individual stars and ii) we can discuss the formation of the whole star system starting from interstellar clouds [52]. To answer these problems it is very important to study the interstellar medium (ISM) and its properties. The ISM physical conditions in the galaxies change in a very wide range, from hot X-ray emitting plasma to cold molecular gas, so it is very complicated to classify the ISM by its properties. However, we can distinguish, in the first approximation, between [53–56]:

- **Diffuse Hydrogen Clouds.** The most powerful tool to measure the properties of these clouds is the 21cm line emission of HI. They are cold clouds so the temperature is in the range 10 ÷ 50 K, and their extension is up to 50 ÷ 100 kpc from galactic center.

- **Diffuse Molecular Clouds** are generally self-gravitating, magnetized, turbulent fluids systems, observed in sub-mm. The most of the molecular gas is H$_2$, and the rest is CO. Here, the conditions are very similar to the HI clouds but in this case, the cloud can be more massive. They have, typically, masses in the range 3 ÷ 100 M$_\odot$, temperature in 15 ÷ 50 K and particle density in (5 ÷ 50) × 10$^8$ m$^{-3}$.

- **Giant Molecular Clouds** are very large complexes of particles (dust and gas), in which the range of the masses is typically 10$^5$ ÷ 10$^6$ M$_\odot$ but they are very cold. The temperature is ~ 15K, and the number of particles is (1 ÷ 3) × 10$^8$ m$^{-3}$ [52, 57–59]. However, there exist also small molecular clouds with masses $M < 10^4 M_\odot$ [60]. They are the best sites for star formation, despite the mechanism of formation does not recover the star formation rate that would be 250 M$_\odot$ yr$^{-1}$ [57].

- **HII regions.** They are ISM regions with temperatures in the range 10$^3$ ÷ 10$^4$ K, emitting primarily in the radio and IR regions. At low frequencies, observations are associated to free-free electron transition (thermal Bremsstrahlung). Their densities range from over a million particles per cm$^3$ in the ultra-compact H II regions to only a few particles per cm$^3$ in the largest and most extended regions. This implies total masses between 10$^2$ and 10$^5$ M$_\odot$ [60].

- **Bok Globules** are dark clouds of dense cosmic dust and gas in which star formation sometimes takes place. Bok globules are found within H II regions, and typically have a mass of about 2 to 50 M$_\odot$ contained within a region of about a light year.

Using very general conditions [52, 60], we want to show the difference in the Jeans mass value between standard and f(R)-gravity. Let us take into account Eq. (117) and Eq. (136):

\[ M_J = \frac{\pi}{6} \sqrt{\frac{1}{\rho_0} \left( \frac{\pi \sigma^2}{G} \right)^3}, \quad (137) \]

in which $\rho_0$ is the ISM density and $\sigma$ is the velocity dispersion of particles due to the temperature. These two quantities are defined as

\[ \rho_0 = m_H n_H \mu, \quad \sigma^2 = \frac{k_B T}{m_H} \]

where $n_H$ is the number of particles measured in m$^{-3}$, $\mu$ is the mean molecular weight, $k_B$ is the Boltzmann constant and $m_H$ is the proton mass. By using these relations, we are able to compute the Jeans mass for interstellar clouds and to show the behavior of the Jeans mass with the temperature. Results are shown in Tab.I and Fig.I.
\[
\text{Subject} & \quad T \quad (\text{K}) & \quad n \quad (10^3 \text{m}^{-3}) & \quad \mu \quad (M_j) & \quad M_j \\
\hline
\text{Diffuse Hydrogen Clouds} & 50 & 5.0 & 1 & 795.13 \\
\text{Diffuse Molecular Clouds} & 30 & 50 & 2 & 82.63 \\
\text{Giant Molecular Clouds} & 15 & 1.0 & 2 & 206.58 \\
\text{Bok Globules} & 10 & 100 & 2 & 11.24 \\
\end{array}
\]

TABLE I: Jeans masses derived from Eq. (117) (Newtonian gravity) and (136) (f(R)-gravity).

FIG. 4: The \(M_J-T\) relation. Dashed-line indicates the Newtonian Jeans mass behavior with respect to the temperature. Continue-line indicates the same for \(f(R)\)-gravity Jeans mass.

By using Eq. (136) and by referring to the catalog of molecular clouds in Roman-Duval J. et al. [61], we have calculated the Jeans mass in the Newtonian and \(f(R)\) cases. Tab. II shows the results. In all cases we note a substantial difference between the classical and \(f(R)\) value. We can conclude that, in \(f(R)\)-scenario, molecular clouds become sites where star formation is strongly supported and more efficient.

XIII. DISCUSSION AND CONCLUSIONS

Black hole solutions can be find out, in principle, in any Extended Theory of Gravity. Here, we have shown that exact solutions can be achieved in \(f(R)\)-gravity starting from symmetry considerations. In particular, the existence of Noether symmetries allow to find exact solutions. Such solutions can be eventually classified as black holes.

In particular, we have shown that the Newman-Janis method, used to derive rotating black hole solutions in GR, works also in \(f(R)\)-gravity. In principle, it could be consistently applied any time a spherically symmetric solutions is derived. The method does not depend on the field equations but directly works on the solutions that, a posteriori, has to be checked to fulfill the field equations.

The key point of the method is to find out a suitable complex transformation which, from a physical viewpoint, corresponds to the fact that we are reducing the number of independent Killing vectors. From a mathematical viewpoint, it is useful since allows to overcome the problem of a direct search for axially symmetric solutions that, in \(f(R)\)-gravity, could be extremely cumbersome due to the fourth-order field equations.

We have to stress that the utility of generating techniques is not simply to obtain a new metric, but a metric of a new spacetime with specific properties as the transformation properties of the energy-momentum tensor and Killing vectors. In its original application, the Newman-Janis procedure transforms an Einstein-Maxwell solution (Reissner-Nordstrom) into another Einstein-Maxwell solution (Kerr-Newman). As a particular case (setting the charge to zero) it is possible to achieve the transformation between two vacuum solutions (Schwarzschild and Kerr). Also in case of \(f(R)\)-gravity, new features emerge by adopting such a technique. In particular, it is worth studying how certain features of spherically symmetic metrics, derived in \(f(R)\)-gravity, result transformed in the new axially symmetric solutions for black holes. For example, considering the \(f(R)\) spherically symmetric solution studied here, the Ricci scalar evolves as \(r^{-2}\) and then the asymptotic flatness is recovered. Let us consider now the axially symmetric metric.
| Subject                  | T  | n    | $M_J$ | $\dot{M}_J$ |
|-------------------------|----|------|-------|-------------|
|                         | (K) | ($10^8$ m$^{-3}$) | ($M_\odot$) | ($M_\odot$) |
| GRSMC G053.59+00.04     | 5.97 | 1.48 | 18.25 | 12.85       |
| GRSMC G049.49-00.41     | 6.48 | 1.54 | 21.32 | 15.00       |
| GRSMC G018.89-00.51     | 6.61 | 1.58 | 22.65 | 15.94       |
| GRSMC G030.49-00.36     | 7.05 | 1.66 | 22.81 | 16.06       |
| GRSMC G035.14-00.76     | 7.11 | 1.89 | 28.88 | 20.33       |
| GRSMC G034.24+00.14     | 7.15 | 2.04 | 29.61 | 22.56       |
| GRSMC G019.94-00.81     | 7.17 | 2.43 | 29.80 | 20.98       |
| GRSMC G038.94-00.46     | 7.35 | 2.61 | 30.27 | 22.01       |
| GRSMC G053.14+00.04     | 7.78 | 2.97 | 32.78 | 25.09       |
| GRSMC G049.39-00.26     | 7.90 | 2.81 | 35.64 | 25.09       |
| GRSMC G019.39-00.01     | 7.99 | 2.87 | 35.84 | 25.23       |
| GRSMC G034.74-00.66     | 8.27 | 3.04 | 36.94 | 26.00       |
| GRSMC G023.04-00.41     | 8.28 | 3.06 | 38.22 | 26.90       |
| GRSMC G018.69-00.06     | 8.30 | 3.62 | 40.34 | 28.40       |
| GRSMC G023.24-00.36     | 8.57 | 3.75 | 41.10 | 28.93       |
| GRSMC G019.89-00.56     | 8.64 | 3.87 | 41.82 | 29.44       |
| GRSMC G022.04+00.19     | 8.69 | 4.41 | 47.02 | 33.10       |
| GRSMC G018.89-00.66     | 8.79 | 4.46 | 47.73 | 33.60       |
| GRSMC G023.34-00.21     | 8.87 | 4.99 | 48.98 | 34.48       |
| GRSMC G034.99+00.34     | 8.90 | 5.74 | 50.44 | 35.50       |
| GRSMC G029.64-00.61     | 8.90 | 6.14 | 55.41 | 39.00       |
| GRSMC G018.94-00.26     | 9.16 | 6.16 | 55.64 | 39.16       |
| GRSMC G024.94-00.16     | 9.17 | 6.93 | 56.81 | 39.99       |
| GRSMC G025.19-00.26     | 9.72 | 7.11 | 58.21 | 40.97       |
| GRSMC G019.84-00.41     | 9.97 | 11.3 | 58.52 | 41.19       |

TABLE II: The name of molecular clouds, the particle number density, the excitation temperature and the value of Jeans mass are reported for Newtonian and $f(R)$ case, respectively. This table is only a part of the catalog of molecular clouds reported in [61]

achieved by the Newman-Janis method. The parameter $a \neq 0$ indicates that the spherical symmetry ($a = 0$) is broken. Such a parameter can be immediately related to the presence of an axis of symmetry and then to the fact that a Killing vector, related to the angle $\theta$, has been lost. To conclude, we can say that once the vacuum case is discussed, more general spherical metrics can be transformed in new axially symmetric metrics adopting more general techniques [16], that would allow us to have the best configurations to describe the black hole dynamics.

A part the mathematical interest in finding out new solutions, such results could have remarkable applications, at least as toy models, in the study of self-gravitating systems. We have shown that the stellar theory as well as the Jeans analysis of instability and collapse can be dramatically altered if one adopt $f(R)$-gravity instead of GR. The study has been performed starting from the Newtonian limit of $f(R)$-field equations. Since the field equations satisfy in any case the Bianchi identity, we can use the conservation law of energy-momentum tensor. In particular adopting a polytropic equation of state relating the mass density to the pressure, we derive the modified Lané-Emden equation and its solutions for $n = 0, 1$ which can be compared to the analogous solutions coming from the Newtonian limit of GR. When we consider the limit $f(R) \to R$, we obtain the standard hydrostatic equilibrium theory coming from GR. A peculiarity of $f(R)$-gravity is the non-viability of Gauss theorem and then the modified Lané-Emden equation is an integro-differential equation where the mass distribution plays a crucial role. Furthermore the correlation between two points in the star is given by a Yukawa-like term of the corresponding Green function.

These solutions have been matched with those coming from GR and the corresponding density radial profiles have been derived. In the case $n = 0$, we find an exact solution, while, for $n = 1$, we used a perturbative analysis with respect to the solution coming from GR. It is possible to demonstrate that density radial profiles coming from $f(R)$-gravity analytic models and close to those coming from GR are compatible. This result rules out some wrong claims in the literature stating that $f(R)$-gravity is not compatible with self-gravitating systems. Obviously the choice of the
free parameter of the theory has to be consistent with boundary conditions and then the solutions are parameterized by a suitable "wave-length" $m = \sqrt{\frac{1}{3f''(0)}}$ that should be experimentally fixed.

The next step is to derive self-consistent numerical solutions of modified Lané-Emden equation and build up realistic star models where further values of the polytropic index $\alpha$ and other physical parameters, e.g. temperature, opacity, transport of energy, are considered. Interesting cases are the non-relativistic limit ($n = 3/2$) and relativistic limit ($n = 3$) of completely degenerate gas. These models are a challenging task since, up to now, there is no self-consistent, final explanation for compact objects (e.g. neutron stars) with masses larger than Volkoff mass, while observational evidences widely indicate these objects \[33\]. In fact it is plausible that the gravity manifests itself on different characteristic lengths and also other contributions in the gravitational potential should be considered for these exotic objects. As we have seen above, the gravitational potential well results modified by higher-order corrections in the curvature. In particular, it is possible to show that if we put in the gravitational action other curvature invariants also repulsive contributions can emerge \[34, 48\]. These situations have to be seriously taken into account in order to address several issues of relativistic astrophysics that seem to be out of the explanation range of the standard theory.

For this purpose, we have analyzed the Jeans instability mechanism, adopted for star formation, considering the Newtonian approximation of $f(R)$-gravity. The related Boltzmann-Vlasov system leads to modified Poisson equations depending on the $f(R)$-model. In particular, considering Eqs. (79) and (80), it is possible to get a new dispersion relation \[129\] where instability criterion results modified (see also \[31\]). The leading parameter is $\alpha$, i.e. the second derivative of the specific $f(R)$-model. Standard Newtonian Jeans instability is immediately recovered for $\alpha = 0$ corresponding to the Hilbert-Einstein Lagrangian of GR. In Fig. 3 dispersion relations for Newtonian and a specific $f(R)$-model are numerically compared. The modified characteristic length van be given in terms of the classical one.

Both in the classical and in $f(R)$ analysis, the system damps the perturbation. This damping is not associated to the collisions because we neglect them in our treatment, but it is linked to the so called Landau damping \[50\].

A new condition for the gravitational instability is derived, showing unstable modes with faster growth rates. Finally we can observe the instability decrease in $f(R)$-gravity: such decrease is related to a larger Jeans length and then to a lower Jeans mass. We have also compared the behavior with the temperature of the Jeans mass for various types of interstellar molecular clouds (Fig. 4). In Tables I and II we show the results given by this new limit of the Jeans mass for a sample of giant molecular clouds. In our model the limit (in unit of mass) to start the collapse of an interstellar cloud is lower than the classical one advantaging the structure formation. Real solutions for the Jean mass can be achieved only for $\alpha < 0$ and this result is in agreement with cosmology \[51\]. In particular, the condition $\alpha < 0$ is essentials to have a well-formulated and well-posed Cauchy problem in $f(R)$-gravity \[51\]. Finally, it is worth noticing that the Newtonian value is an upper limit for the Jean mass coinciding with $f(R) = R$.

This analysis is intended to indicate the possibility to deal with ISM collapsing clouds under different assumptions about gravity. It is important to stress that we fully recover the standard collapse mechanisms but we could also describe proto-stellar systems that escape the standard collapse model. On the other hand, this is the first step to study star formation and physical black holes in alternative theories of gravity (see also \[29–32\]). From an observational point of view, reliable constraints can be achieved from a careful analysis of the proto-stellar phase taking into account magnetic fields, turbulence and collisions. Finally, addressing stellar systems by this approach could be an extremely important to test observationally $f(R)$-gravity.

Moreover, the approach developed here admits direct generalizations for other modified gravities, like non-local gravity, modified Gauss-Bonnet theory, string-inspired gravity, etc. In these cases, the constrained Poisson equation may be even more complicated due to the presence of extra scalar(s) in non-local or string-inspired gravity. Developing further this approach gives, in general, the possibility to confront the observable dynamics of astrophysical objects (like stars) with predictions of alternative gravities.

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