Spin and abelian electromagnetic duality on four-manifolds

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Abstract
We investigate the electromagnetic duality properties of an abelian gauge theory on a compact oriented four-manifold by analysing the behaviour of a generalised partition function under modular transformations of the dimensionless coupling constants. The true partition function is invariant under the full modular group but the generalised partition function exhibits more complicated behaviour depending on topological properties of the four-manifold concerned. It is already known that there may be “modular weights” which are linear combinations of the Euler number and Hirzebruch signature of the four-manifold. But sometimes the partition function transforms only under a subgroup of the modular group (the Hecke subgroup). In this case it is impossible to define real spinor wavefunctions on the four-manifold. But complex spinors are possible provided the background magnetic fluxes are appropriately fractional rather than integral. This gives rise to a second partition function which enables the full modular group to be realised by permuting the two partition functions, together with a third. Thus the full modular group is realised in all cases. The demonstration makes use of various constructions concerning integral lattices and theta functions that seem to be of intrinsic interest.

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1. Introduction

It is now widely accepted that electromagnetic duality provides a powerful and useful new principle which is valid in a large class of physically interesting quantum field theories in a Minkowski space-time of four dimensions. The prototype for the quantum version of this idea was furnished by a proposal of Montonen and one of the present authors, [Montonen and Olive 1977], who considered the context of a special sort of spontaneously broken $SU(2)$ gauge theory, later realised to be naturally supersymmetric [D'Adda, Di Vecchia and Horsley 1978, Witten and Olive 1978, Osborn 1979]. This can be regarded as a semi-realistic theory of unified particle interactions as it is the same sort of theory as the standard model even though it differs in some crucial respects. The manner in which duality is realised on particle states requires magnetically charged states arising as solitons solutions and, in addition, quantum bound state of these [Sen 1994].

This acceptance, which now extends to superstring theories, where solitons occur as higher branes, has been achieved despite the fact that no sort of proof has been found, even in the case of the $SU(2)$ gauge theory with the highest allowable degree of supersymmetry, $N = 4$. It is this situation for which most evidence has steadily accumulated that the idea is exactly true. This supporting evidence has been facilitated by the acquisition of new mathematical techniques that have enhanced our understanding of quantum field theory. For example, the Atiyah-Singer index theorem, which is related to the theory of axial anomalies, plays a ubiquitous role. A modified version of this theorem is crucial in determining both classical and quantum properties of self-dual monopole solutions.

Since the quantum electromagnetic duality transformations combine to form a group related to the modular group (or its generalisations) it is reasonable to expect some version of the theory of modular forms to become increasingly important. Conversely the idea of electromagnetic duality has led to breakthroughs in the classification theory of four-manifolds (playing the role of Euclidean space-times for twisted supersymmetric gauge theories).

It would be nice to have a simpler, toy model which, although less realistic physically, could, in compensation, be more tractable mathematically. Indeed, such a theory exists. It is simply free Maxwell theory with only putative couplings, not realised in practice. Thus particle states possess what could be called a “Cheshire cat” existence, that will be become clearer later. Such a theory would be too trivial on flat space time and, in order to obtain some worthwhile structure, the ambient space-time manifold has to be allowed to be fairly general. It is taken to be smooth, compact and oriented. Hence it is a four-manifold which obeys the topological symmetry known as Poincaré duality, and this will be a crucial ingredient establishing electromagnetic duality here. This toy model follows the original proposals by E Verlinde and E Witten in 1995.

What is interesting about this model is that much of the same sort of mathematical structure as mentioned above again comes into play. In particular there enter modular invariant theta functions of a rather general nature. The Atiyah-Singer index theorem plays a mysterious role interlocked with the modular group and its subgroups of index three and the nature of space-time and its possible spin structures. This is despite the significant differences between the two situations (no hint of supersymmetry, curved Riemannian space-time in one case, $N = 4$ supersymmetry, flat, non-compact Minkowski space-time in
the other). This points to the conclusion that electromagnetic duality is indeed a rather
general phenomenon.

Moreover it is likely that this picture extends to any space-time with Minkowski metric
and dimension which is a multiple of four. If this dimension is denoted $4k$ then the putative
accounting of Maxwell field strengths with charged particles is replaced by a coupling of a
$2k$-form field strength to $(2k - 2)$-branes.

In section 2 we start with the naive idea of electromagnetic duality as a classical
symmetry of the energy-momentum tensor of Maxwell theory in Minkowski space with
respect to rotations between the electric and magnetic fields. It is shown how this idea can
be extended to a larger group of $SL(2, \mathbb{R})$ transformations acting on the energy-momentum
tensor. It is explained how this leads in turn to consideration of the Feynman path integral,
of, rather surprisingly, the exponential of the Euclidean action. A special case of this
Euclidean path integral is the Minkowski space partition function.

The effect of the Dirac quantisation condition for magnetic fluxes exhibits some extra
subtleties in space-times of four dimensions, particularly when complex spinor wave functions are considered. This is reviewed in section 3. The effect is to break the continuous $SL(2, \mathbb{R})$ group to a discrete subgroup related to the modular group in a way that is explained in sections 4 and 5. It is noteworthy that nothing like the Zwanziger-Schwinger quantisation condition for dyonic charges plays any role.

Owing to the complicated topology of the four-manifold of space-time, the possible
magnetic fluxes are related to a lattice formed by the free part of the second homology
group. This lattice is unimodular by virtue of Poincaré duality and its even or oddness
properties are related to the presence or absence of spin structures on the space-time four
manifold. In fact four-manifolds can be separated into three distinct types whose properties
are described in section 3.

In section 4 properties of integral lattices are reviewed. Starting from an odd unimodular
lattice a general construction is given of an even integral lattice, leading sometimes
to new unimodular lattices which can be both odd or even. Furthermore, also in section
4 the relevance to the Dirac quantisation condition and the question of spin structures is
explained.

In section 5, following the arguments of E Verlinde and E Witten [1995], the “extended
partition functions” are evaluated explicitly using the Dirac quantisation conditions and
the semiclassical method. The results are proportional to generalised theta functions
associated with the unimodular lattice formed by the free part of the second homology group.
Particular attention is paid to the different possibilities afforded by the compatibility of
either scalar or spinor complex wave functions on four-manifolds of type II, i.e. when the
relevant lattice is odd.

In section 6 a more general construction is presented that associates theta functions
with any integral lattice, not necessarily unimodular, whether or not the scalar product is
positive definite. Such integral lattices are contained as subgroups of the lattices reciprocal
to them and so define a finite number of cosets, to each of which corresponds a theta
function. An action of the modular group is defined on these theta functions when the
lattice is even and of the Hecke subgroup when the lattice is odd. Careful analysis of the
self-consistency of this action furnishes a proof of “Milgram’s formula”, valid for any even
integral lattice. This expresses the signature of the lattice, mod eight, in terms of coset properties.

In section 7, the theta function construction of section 6 is applied to the even integral lattice associated with an odd unimodular lattice by the construction of section 4. The result is to associate up to four theta functions with a given odd unimodular lattice, though there are usually linear dependences. When the odd unimodular lattice corresponds to that associated with any type II four-manifold, two of these theta functions enter the distinct Maxwell partition functions for fluxes supporting either scalar or spinor complex wave functions, provided the electric charges they carry coincide. Although each partition function is individually covariant with respect to only a subgroup of $SL(2, \mathbb{Z})$ of index three, they are related to each other by the missing transformations. In this way the full $SL(2, \mathbb{Z})$ group of electromagnetic duality transformations is restored for space-time four-manifolds of type II.

2. Abelian gauge fields and electromagnetic duality

We shall usually be considering a single abelian Maxwell field strength described, in exterior calculus notation, by a closed two form $F$ on a space-time manifold $\mathcal{M}_4$ which is closed, compact, connected, smooth and oriented. Nevertheless much of the argument extends to higher dimensional space-times of the same type, (which will be denoted $\mathcal{M}_{4k}$), as long as their dimension is a multiple of four and the field-strength $F$ is a closed $2k$-form, that is a mid-form. It would then be a generalised Kalb-Ramond field, [Kalb and Ramond 1974], that could couple to the world-volume of a $2k-2$-brane.

For reasons that will become clear it is important to allow the space-time manifold to be topologically complicated. Sometimes we shall suppose that $\mathcal{M}_{4k}$ be such that it can be endowed with a Minkowski metric (that is with one time component) and this would require that its Euler number $\chi(\mathcal{M}_{4k})$ vanish. We may further require that this metric can always be “Wick rotated” to a Euclidean metric (with no time components) by an analytic continuation. This may impose further constraints on the topological properties of $\mathcal{M}_{4k}$.

With either sort of metric the Hodge star operation, $\ast$, can be defined, converting $p$-forms on $\mathcal{M}_{4k}$ to $4k-p$-forms. Acting on $2k$-forms, that is, mid-forms, the repeated action of the Hodge star operation yields

$$** = (-1)^t,$$

where $t$ equals the number of time components, that is, one or zero. So, in the Minkowski case, $\ast$ has eigenvalues $\pm i$ and hence no real eigenfunctions. It is this Minkowski situation, rather than the Euclidean one, in which electromagnetic duality can apply.

The basic idea of a resemblance between the parts of $F$ thought of as the electric and magnetic fields, $\mathbf{E}$ and $\mathbf{B}$, is very old and was reinforced by Maxwell’s discovery of his equations governing their behaviour in vacuo. “Duality rotations” between $\mathbf{E}$ and $\mathbf{B}$ provide a symmetry of the Minkowski energy density $(E^2 + B^2)/2$ (this particular expression applies when the space-time is flat), and more generally the complete energy momentum tensor. Moreover the rotations map between solutions of the equations even though the action does change.
This idea was extended in the context of supergravity theories by Gaillard and Zumino [1981], building on ideas of Cremmer and Julia [1979]. The following action can be defined on any of the space-times mentioned:

$$W = \frac{1}{2\tau_2} \int_{\mathcal{M}_4} F \wedge \hat{\tau} F,$$  \hspace{1cm} (2.2)

where, in Minkowski space,

$$\hat{\tau} = \tau_1 + \ast \tau_2 = \frac{\theta}{2\pi} + \ast \frac{2\pi \hbar}{q^2},$$  \hspace{1cm} (2.3)

so that this action, (2.2), is indeed real. In suitably extended supergravity theories $\tau$ will depend upon scalar fields related to the metric tensor by supersymmetry transformations. As far as this paper is concerned, there are no scalar fields and no supersymmetry. $\tau_1$ and $\tau_2$ are simply dimensionless quantities parametrising the theory. It will be convenient to combine them as real and imaginary parts of a single complex variable

$$\tau = \tau_1 + i\tau_2 = \frac{\theta}{2\pi} + i \frac{2\pi \hbar}{q^2}$$  \hspace{1cm} (2.4)

The conventional Maxwell term is $\frac{1}{2} \int F \wedge \ast F$. The other term, called the theta term, has no apparent effect classically as it affects neither the Euler-Lagrange equations, nor the value of the energy-momentum tensor. However it can affect the quantum phase when the topology of the background space-time is sufficiently non-trivial. In circumstances to be explained, $(\tau_2)^{-1} \int_{\mathcal{M}_4} F \wedge F$ is quantised so that the exponentiated quantum action, $\exp\left(\frac{i}{\hbar} W\right)$, depends upon the parameter $\theta$ in a periodic manner. But the dependence upon $\theta$ disappears altogether when the topology is too trivial, that is, when the second Betti number of $\mathcal{M}_4$ vanishes.

Following Gaillard and Zumino we consider the effect of the linear transformations:

$$\hat{\tau} F \rightarrow \hat{\tau}' F' = A\hat{\tau} F + BF$$  \hspace{1cm} (2.5a)

$$F \rightarrow F' = C\hat{\tau} F + DF,$$  \hspace{1cm} (2.5b)

where $A, B, C$ and $D$ are real constants.

Then the dimensionless complex coupling constant variable, $\tau$, (2.4) undergoes the fractional linear transformation

$$\tau \rightarrow \tau' = \frac{A\tau + B}{C\tau + D}.$$  \hspace{1cm} (2.6a)

while its imaginary part $\tau_2$ undergoes

$$\tau_2 \rightarrow \tau_2' = \frac{\tau_2(AD - BC)}{(C\tau + D)^2}.$$  \hspace{1cm} (2.6b)
The transformations (2.5) evidently provide symmetries of the two equations of motion \( df = 0 \) and \( d\hat{\tau}f = 0 \). Now we consider the effect on the symmetric energy momentum tensor \( T_{\mu\nu} \), obtained from (2.2) by variation of the metric and written in the Sugawara form,

\[
T_{\mu\nu} = \frac{1}{2}(F_{\mu\lambda}g^{\lambda\sigma}F_{\sigma\nu} + *F_{\mu\lambda}g^{\lambda\sigma}*F_{\sigma\nu}),
\]

(2.7)

where \( F = F_{\mu\nu}dx^\mu \wedge dx^\nu/2 \) and \( *F = *F_{\mu\nu}dx^\mu \wedge dx^\nu/2 \). The result is

\[
T_{\mu\nu} \rightarrow T'_{\mu\nu} = |C\tau + D|^2T_{\mu\nu}.
\]

If we restrict the transformations (2.5) to the subgroup leaving \( \tau \) unchanged in (2.6), evidently \( \tau_2 \) is also unchanged and so by (2.6b), the result is

\[
T_{\mu\nu} \rightarrow T'_{\mu\nu} = (AD - BC)T_{\mu\nu}.
\]

Hence the energy momentum tensor is invariant under the subgroup if

\[
AD - BC = 1
\]

(2.8)

and this yields a \( U(1) \) subgroup comprising the duality rotations previously mentioned. The symmetry of the energy momentum tensor (2.7) can be enlarged from this \( U(1) \) to the full \( SL(2, \mathbb{R}) \) if the transformations (2.5) are modified in the following natural way. Let us substitute for the physical field strengths \( F \) more geometrical quantities \( G \), by \( F = \hbar G/q \). \( G \) is more geometrical in the sense that its fluxes, unlike those of \( F \) are dimensionless. In terms of \( G \) the dimensionless action is given by

\[
\frac{W}{\hbar} = \frac{1}{4\pi} \int_{\mathcal{M}_4} G \wedge \hat{\tau}G
\]

and so is explicitly a function of \( \tau \). Now the energy momentum tensor (2.7) reads

\[
T_{\mu\nu} = \frac{\hbar\tau_2}{4\pi}(G_{\mu\lambda}g^{\lambda\sigma}G_{\sigma\nu} + *G_{\mu\lambda}g^{\lambda\sigma}*G_{\sigma\nu}).
\]

(2.7')

so that a dependence on \( \tau_2 \) is made explicit. Now substitute \( G \) for \( F \) in (2.5), thereby defining new duality transformations, still leading to (2.6). Under these transformations acting on both \( G \) and \( \tau \) in the energy momentum tensor \( T_{\mu\nu} \) in the form (2.7'), the preceding calculations show that \( T_{\mu\nu} \), is invariant providing only that (2.8) holds.

Thus now the dimensionless complex coupling \( \tau \) changes whilst preserving the positive nature of \( \tau_2 \). This is appropriate as \( \tau_2 \) is the inverse of the fine structure constant and hence intrinsically positive.

These transformations form the three dimensional non compact group \( SL(2, \mathbb{R}) \), or what is the same by a group theory isomorphism, the symplectic group \( Sp(2, \mathbb{R}) \).

The transformations also map between solutions of the free Maxwell equations and the key question will be to what extent these transformations continue to provide symmetries when there is a possibility of electrically charged particles (or branes) being present, subject
to the rules of quantum theory. Because it is the Minkowski energy, rather than the action, which is invariant classically, the natural quantity to consider in the quantum theory is the partition function constructed with this energy:

$$Z(\tau) = Tr \left( e^{-E(\tau)} \right). \quad (2.9)$$

It is this partition function that will be the candidate for quantum electromagnetic duality, just as it is the partition function that displays the Kramers-Wannier duality of the Ising model [Kramers and Wannier 1941]. Indeed it will be found that (2.9) is invariant under the transformations (2.6) provided they are restricted to a discrete subgroup, isomorphic to the modular group. The discreteness is a consequence of the Dirac quantisation condition that the magnetic fluxes have to satisfy in order to permit complex wave functions.

For rather general, nonlinear dynamical systems with a finite number of degrees of freedom and the property that the action includes only terms quadratic, linear and independent of velocities there is a Feynman path integral expression for the partition function (2.9).

$$Z(\tau) = \int \ldots \int \delta A e^{i\bar{h} W_{\text{EUCLIDEAN}}}.$$

The Euclidean action $W_{\text{EUCLIDEAN}}$ is obtained from the original action by what can be thought of as a “Wick rotation” whereby velocities are multiplied by i and time by $-i$. As a result, $iW_{\text{EUCLIDEAN}}$ has an imaginary part linear in velocities, and a real part that is negative definite if the original energy is positive. Consequently the path integral is highly convergent. Because of the trace in (2.9), the paths integrated over are closed paths traversed in configuration space in unit time with distinguished end points. This result is known as the Feynman-Kac formula [Feynman and Hibbs, Feynman].

The presence of a complex phase factor in (2.10) due to terms linear in velocity appears to contradict the manifest reality of the partition function as defined in (2.9). But this is illusory because the space of closed paths in configuration space that are integrated over possess a $Z_2$ symmetry with respect to the interchange of pairs of identical paths differing only in the sense of time evolution along the path. Under this interchange the two contributions to $\exp i\bar{h} W_{\text{EUCLIDEAN}}$ are related by complex conjugation. As a result the sum of these two complex contributions is indeed real.

Because it is quadratic in field strengths, something similar happens with the more complicated action (2.2) under consideration here. As a result of a similar argument, the Maxwell partition function can be expressed in the form (2.10) where now $W_{\text{EUCLIDEAN}}$ is obtained from $W$, (2.2), by a “Wick rotation” of the metric tensor. This tensor only enters the Maxwell term as the theta term is “topological” and hence independent of the metric, and so unaffected by the Wick rotation. The result is that $W_{\text{EUCLIDEAN}}$ is given by the same expression as before, (2.2), when it is understood that $\tau$ is replaced by

$$\tau_{\text{EUCLIDEAN}} = \tau + i*\tau_2 = \frac{\theta}{2\pi} + i * \frac{2\pi \hbar}{q^2}. \quad (2.11)$$

The metric dependence is encoded in the Hodge $*$ operator which, by (2.1), now has unit square and hence eigenvalues $\pm 1$. This has two consequences. One is that $iW_{\text{EUCLIDEAN}}$
is complex when the field strengths are real and that its real part is negative definite, thereby ensuring convergence of the integral over gauge potentials $A$ in (2.9). The other consequence is that if $*$ is regarded as imaginary in Minkowski space-time and real in Euclidean space, in view of its eigenvalues, then $\tau$ has the same complex structure in either case. In this sense it is unaffected by the Wick rotation. Accordingly the complex variable $\tau$ given by the expression (2.4) not involving the Hodge $*$ is equally relevant with either metric.

In evaluating this partition function the space-time four manifold has to be considered as $M_4 = S_1 \times M_3$, where “time” is the coordinate around the circle, (periodic because of the trace), and $M_3$ an appropriate section of $M_4$. If the metric on $S_1 \times M_3$ factorises correspondingly it is easy to see that the partition function will be again real because reversing the sense of time around this circle will effect a complex conjugation of the two quantum amplitude contributions.

It will turn out to be highly instructive to consider what, by abuse of terminology, is often also called a “partition function”. This expression is given by the path integral (2.10) but with the integral in the action being over the four manifold given by the full space-time $M_4$, instead of $S_1 \times M_3$. This path integral can be defined even for four manifolds with non-vanishing Euler number, that is ones for which a Minkowski metric is impossible. There is no reason for this new quantity to be real but it will turn out to have an interesting response to the electromagnetic duality transformations (2.6) (as pointed out by E Witten and E Verlinde). So these extended partition functions do have interesting mathematical properties as we shall see in more detail and it would be interesting to understand what, if any, physical significance they have. We shall henceforth refer to the real partition functions associated with $S_1 \times M_3$ as “strict partition functions”.

We shall see that both the Euler number, $\chi(M_4)$, and the Hirzebruch signature, $\eta(M_4)$, vanish for manifolds $S_1 \times M_3$. These are the two topological invariants of a $M_{4k}$ manifold that are “local” in the sense that they can be expressed as integrals of closed forms over the manifold. Linear combinations of these topological invariants, namely $(\chi \pm \eta)/2$, will specify in a precise way how the extended partition functions deviate from satisfying exact electromagnetic duality.

These conclusions will depend on the explicit evaluation of the functional integral expression (2.10) for the partition function on any $M_4$, and this is facilitated by taking account of another aspect of the quantum theory. If electrically charged particles are to be treated quantum mechanically, the background field strengths must satisfy certain Dirac flux quantisation conditions in order to allow the possible presence of complex wave functions for them. It is this that imposes a discrete structure that converts (2.10) to a sum rather than an integral, at least in the semiclassical approximation, which is very likely exact.

As we shall see, the resultant expression for the partition function (2.9) is proportional to an infinite sum forming a generalised sort of theta function associated with the lattice of homology classes of two-cycles in the space-time four-manifold $M_4$. As explained below, this lattice is what is known as the free part of $H_2(M_4, \mathbb{Z})$ and is intimately connected to the Dirac quantised fluxes.

If $M_4$ is orientable, smooth, closed and compact, its topological structure satisfies the
symmetry known as Poincaré duality. This implies that the following relation between the five Betti numbers of $M_4$:

$$b_0 = b_4, \quad b_1 = b_3.$$  \tag{2.12}

Hence the Euler number is given by

$$\chi(M_4) = 2(b_0 - b_1) + b_2.$$  \tag{2.13}

Furthermore, the aforementioned lattice, which has dimension $b_2$, is unimodular with respect to the scalar product furnished by the intersection number. It is this which is the origin of the covariance of (2.9) with respect to the $S$-transformation of electromagnetic duality:

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$  \tag{2.14}

sending $\tau$ to $-1/\tau$, a especially interesting example of (2.6). Before explaining this we must review the Dirac quantisation condition for fluxes in more detail, paying particular attention to the extra subtleties associated with spinning particles carrying electric charge.

3. The quantisation condition on four-manifolds

As already explained, we consider Maxwell theory in space-times that are compact, connected and oriented four-dimensional manifolds $M_4$, whether or not their Euler number vanishes. These spaces are particularly convenient because they satisfy Poincaré duality, which is a topological property closely related to electromagnetic duality. The main mathematical tool needed in understanding the implications of the global topology of $M_4$ on the duality properties of Maxwell theory is homology and cohomology theory. This discipline is described in many textbooks (see for example [Schwarz 1994]), and a short introduction to the relevant ideas was given in our earlier paper [M Alvarez and Olive 1999]. We shall follow the notations of the latter without full explanation.

The physical relevance of homology and cohomology theory is that it provides the natural mathematical language for the ideas of Faraday, Maxwell and, later, Dirac concerning electromagnetic theory. Important physical quantities are the (magnetic) fluxes of the field strength $F$ through a complete set of two-cycles $\Sigma_1, \ldots, \Sigma_{b_2}$ within the space-time four manifold $M_4$.

According to Poincaré’s lemma, the gauge potential $A$, satisfying $F = dA$, can be constructed by integration locally, in topologically trivial neighbourhoods of $M_4$, up to a gauge transformation. $A$ is needed to define the electromagnetic coupling to complex wave-functions of electrically charged particles. If the wave function is scalar, corresponding to a boson with charge $q_B$, quantum mechanical consistency of the patching procedure for it requires the fluxes to be quantised [Dirac 1931, Wu and Yang 1975, O Alvarez 1985]:

$$\frac{q_B}{2\pi\hbar} \int_{\Sigma} F = m(\Sigma) \in \mathbb{Z}$$  \tag{3.1}

By virtue of ordinary Stokes’ theorem and that the facts that $\Sigma$ and $F$ are both closed, the value of the flux is unchanged either if $\Sigma$ is replaced by $\Sigma + \partial \Pi$, with $\Pi$ a three-dimensional
chain, or if $F$ is replaced by $F + dB$. Also, integer linear combination of cycles that satisfy the quantisation condition $\Sigma$ also satisfy the same condition.

These statements can be summarised by saying that the cycles $\Sigma$ are free elements of the integer homology class $H_2(\mathcal{M}_4, \mathbb{Z})$, and $F$ is in a cohomology class $H^2(\mathcal{M}_4, \mathbb{Z})$. Now $H_2(\mathcal{M}_4, \mathbb{Z})$ is an abelian group (with respect to the natural addition operation) and it possesses a unique subgroup built of elements of finite order, called the torsion group $T_2(\mathcal{M}_4, \mathbb{Z})$. The quotient group

$$F_2(\mathcal{M}_4, \mathbb{Z}) \equiv H_2(\mathcal{M}_4, \mathbb{Z})/T_2(\mathcal{M}_4, \mathbb{Z}) \quad (3.2)$$

is “free” and consists of $b_2$ copies of the integers, where $b_2$ is the second Betti number. This is the same as saying that $F_2(\mathcal{M}_4, \mathbb{Z})$ is a lattice of dimension $b_2$.

A slightly more general version of (3.1) is the “quantum Stokes’ relation”:

$$e^{i \frac{2\pi}{h} \int_{\Sigma} F} = e^{i \frac{2\pi}{h} \int_{\partial\Sigma} A}. \quad (3.3)$$

Now $\Sigma$ is allowed to have a non-vanishing boundary $\partial\Sigma$ and hence be a two-chain rather than a two-cycle. In the limiting case when the boundary vanishes, (3.1) is recovered (and so is necessary for the validity of (3.3)). The quantity on the right hand side of (3.3) is Dirac’s path dependent phase factor [Dirac 1955]. It is well defined in the situation described even though the exponent is not. Such phase factors are relevant in several different contexts [Bohm-Aharonov, Wilson] and go by several other names (Wilson loop, $U(1)$ holonomy etc).

However, many electrically charged particles, such as the electron, also carry spin and, as a result (3.1) and (3.3) may have to be modified if the topology of space-time is sufficiently complicated. If the complex wave function to which $A$ couples is spinor rather than scalar, and the associated fermionic particle carries charge $q_F$, (3.3) is modified by the presence of a possible minus sign [M Alvarez and Olive 1999].

$$e^{i \frac{2\pi}{h} \int_{\Sigma} F} = (-1)^{w(\Sigma)} e^{i \frac{2\pi}{h} \int_{\partial\Sigma} A}, \quad (3.4)$$

at least for two-chains $\Sigma$ whose boundary is an even cycle:

$$\partial\Sigma = 2\alpha. \quad (3.5)$$

There are essentially two possibilities for this when $\Sigma$ is odd. Either $\alpha$ vanishes and $\Sigma$ is a closed surface, or not. If the latter, $\Sigma$ could be the real projective plane in two dimensions, and so not orientable. Notice that although $2\alpha$ is closed, $\alpha$ itself is not. Hence the one-cycle $\alpha$ is what is known as a torsion cycle.

The sign factor $(-1)^{w(\Sigma)}$ in (3.4) arises unambiguously in the procedure of patching together the neighbourhoods that make up $\Sigma$, precisely when $\Sigma$ satisfies (3.5), [M Alvarez and Olive 1999]. However, when $\Sigma$ is closed, $w(\Sigma)$ can be constructed independently as the integer specifying the self-intersection number of $\Sigma$ with itself. This is possible because $\Sigma$ is a two-cycle in a closed oriented four-manifold. The equivalence of the two notions (mod 2) can be deduced from the Atiyah-Singer index theorem on $\mathcal{M}_4$. When $\Sigma$ is neither closed
nor even, yet satisfies (3.5), it is not oriented and its self-intersection number can only be defined mod 2, and not absolutely. This still matches the previous definition, according to Wu’s formula. The important point is that the sign factor depends only on the topology of the background space-time.

Let us temporarily set the charges $q_B$ and $q_F$ equal. When the sign factor $(-1)^w(\Sigma)$ equals 1 equations (3.3) and (3.4) agree but when it equals $-1$ they appear to differ. However the discrepancy is illusory as the gauge potentials in each equation differ as they are constructed by gauge inequivalent patching procedures.

On the other hand, when we consider the limiting case in which $q_B$ and $q_F$ both vanish the two versions of “quantum Stokes”, (3.3) and (3.4), reduce to $1 = 1$ and $1 = (-1)^w(\Sigma)$, respectively. Now the second equation is manifestly a contradiction if the sign factor is negative. What this means is that a false assumption has been adopted, namely that it is possible to place an electrically neutral spinor wave function on $M_4$. Clearly this is forbidden if there is any $\Sigma$ satisfying (3.5) for which the sign factor $(-1)^w(\Sigma)$ is negative. Mathematicians are familiar with this phenomenon and say that such an $M_4$ lacks a “spin structure”. They recognise $w(\Sigma)$, as the Stiefel-Whitney class. It is an element of $H^2(M_4, \mathbb{Z}_2)$ and its nontriviality provides an obstruction to spin structures. See Appendix A of [Lawson and Michelsohn].

In our previous paper we found it convenient to separate all the four-manifolds under consideration into three types: I, II and III.

A four-manifold is of Type I if the sign factor $(-1)^w(\Sigma)$ is plus one in all cases (3.5). The flux quantisation condition reads

$$\frac{q_F}{2\pi \hbar} \int_{\Sigma} F = m(\Sigma) \in \mathbb{Z}. \quad (3.6)$$

This is the same as (3.1) with $q_F$ replacing $q_B$. It follows that all cycles have even self-intersection number.

The intersection numbers of pairs of two-cycles endow the lattice $F_2(M_4, \mathbb{Z})$ of free cycles with an integral scalar product. The Poincaré duality that applies to the four-manifolds under consideration implies that this scalar product is unimodular. Unimodular lattices fall naturally into two classes even, or odd. It follows from our remarks that the unimodular lattice $F_2(M_4, \mathbb{Z})$ is even for Type I manifolds.

A four-manifold is of Type II if the sign factor $(-1)^w(\Sigma)$ is minus one for at least one two-cycle $\Sigma$. Such a cycle has an odd self-intersection number and consequently the unimodular lattice $F_2(M_4, \mathbb{Z})$ is odd.

The flux quantisation implied by (3.4) reads

$$\frac{q_F}{2\pi \hbar} \int_{\Sigma} F = m(\Sigma) + \frac{w(\Sigma)}{2}; \quad m(\Sigma) \in \mathbb{Z}. \quad (3.7)$$

Thus, when $w(\Sigma)$ is odd, the flux is fractional rather than integral and, in particular, can never vanish, unlike the integral fluxes.

The remaining possibility, Type III, arises when the sign factor $(-1)^w(\Sigma)$ equals plus one for all cycles but minus for at least one of the open two-chains satisfying (3.5). As the self-intersection numbers of all cycles are even so is the unimodular lattice $F_2(M_4, \mathbb{Z})$. 

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Thus, in summary, type I manifolds are the only ones that support spin structures, that is electrically neutral spinor wave functions. All three types support \( \text{spin}_C \) structures, that is electrically charged complex spinor wave functions, provided the background fluxes satisfy the appropriate quantisation conditions.

These conditions imply that all fluxes are integral if the unimodular lattice \( F_2(M_4, \mathbb{Z}) \) is even, that is for types I and III, but that some fluxes at least must be fractional when the lattice is odd, that is for Type II manifolds. Standard examples of the three types of four-manifold are, for type I the torus, \( T^4 \), the sphere \( S^4 \) and \( K(3) \), for type II, the complex projective space, \( CP(2) \) and for type III, \( S^2 \times S^2/\mathbb{Z}_2 \).

4. Integral lattices

Unimodular lattices are a sort of integral lattice with special structural features that will play a role in the picture of flux quantisation that we have begun to explain. Here we pause to explain some of the relevant concepts.

A lattice \( \Lambda \) of dimension \( n \) is a discrete subgroup of \( \mathbb{R}^n \) defined as

\[
\Lambda = \left\{ \sum_{i=1}^{n} n_i e_i, \quad n_i \in \mathbb{Z} \right\}, \tag{4.1}
\]

where \( e_1, e_2, \ldots, e_n \) are elements of \( \Lambda \) spanning \( \mathbb{R}^n \) and so providing a basis for \( \Lambda \).

The vector space \( \mathbb{R}^n \) may be endowed with a real, symmetric scalar product, denoted \( x \cdot y \), which is nonsingular but not necessarily positive definite. If this scalar product has \( b^+ \) positive (negative) eigenvalues the signature \( \eta \) of \( \Lambda \) is

\[
\eta(\Lambda) = b^+ - b^-, \quad \text{where} \quad b^+ + b^- = n. \tag{4.2}
\]

For example, when \( F_2(M_4, \mathbb{Z}) \) is endowed with the scalar product given by the intersection numbers, its signature is known as the Hirzebruch signature of \( M_4 \), and denoted \( \eta(M_4) \).

Given a lattice and a scalar product we can define another lattice, known as the reciprocal lattice:

\[
\Lambda^* = \{ x \in \mathbb{R}^n : y \cdot x \in \mathbb{Z} \forall y \in \Lambda \}. \tag{4.3}
\]

Obviously \( \Lambda^{**} = \Lambda \). \( \Lambda \) is said to be integral if

\[
\Lambda \subseteq \Lambda^* \tag{4.4}
\]

as a subgroup. Then the quotient group is well-defined and abelian

\[
Z(\Lambda) = \Lambda^*/\Lambda. \tag{4.5}
\]

\( \Lambda \) is unimodular if its order, \( |Z(\Lambda)| \), is one. This is the same as saying that the lattices \( \Lambda \) and \( \Lambda^* \) coincide.

A convenient choice of fundamental domain in \( \Lambda \) consists of \( X = \sum_i x_i e_i \) with \( 0 \leq x_i \leq 1 \). The volume of this fundamental domain is

\[
V(\Lambda) = \sqrt{\det(e_i \cdot e_j)} = |\det(e_1, e_2, \ldots, e_n)|. \tag{4.6}
\]
A standard basis for the reciprocal lattice is the reciprocal basis $f_1, f_2, \ldots, f_n$ satisfying $e_i \cdot f_j = \delta_{ij}$. Then

$$V(\Lambda^*) V(\Lambda) = |\det(f_1, f_2, \ldots, f_n) \det(e_1, e_2, \ldots, e_n)| = |\det(e_i \cdot f_j)| = \det(\delta_{ij}) = 1.$$ But $|Z(\Lambda)|$ copies of the fundamental domain of $\Lambda^*$ make up a fundamental domain for $\Lambda$. Hence

$$V(\Lambda) = (V(\Lambda^*))^{-1} = \sqrt{|Z(\Lambda)|}. \quad (4.6)$$

Thus $\Lambda$ is unimodular if it is integral and the volume of its fundamental domain equals unity. This is what was used above.

In fact any choice of fundamental domain has the same volume. So changes of basis form the infinite discrete group $SL(n, \mathbb{Z})$.

If $x$ is an element of an integral lattice $x \cdot x$ is automatically an integer. If it is always an even integer $\Lambda$ is said to be even. Otherwise $\Lambda$ is odd.

Of course this applies to unimodular lattices but these possess an extra feature, the existence of elements called characteristic elements. $c \in \Lambda$ is a characteristic vector if

$$c \cdot x + x \cdot x \in 2\mathbb{Z} \quad \forall x \in \Lambda. \quad (4.7)$$

It is easy to establish that such quantities always exist and that there is an ambiguity of precisely mod $2\Lambda$. It follows that $c \cdot c$ is uniquely defined, mod 8 for any unimodular lattice. In fact it is known that

$$c \cdot c = \eta(\Lambda) + 8\mathbb{Z} \quad (4.8)$$

This will follow from our discussion of theta functions but a direct proof can be found in [Milnor and Husemoller]. The zero element is always a characteristic vector for an even unimodular lattice. Hence (4.8) implies that for such lattices the signature is a multiple of eight. A famous example is the $E_8$ root lattice.

The fact that an odd unimodular lattice $\Lambda$ possesses a characteristic vector $c$ will facilitate some constructions which will be relevant to the analysis of type II four manifolds. First:

$$\Lambda_{TOTAL} = \Lambda \cup (\Lambda + c/2) \quad (4.9)$$

defines a lattice. Obviously $\Lambda_{TOTAL}/\Lambda = \mathbb{Z}_2$, so that, by a slight extension of (4.6), $V(\Lambda_{TOTAL}) = 1/2$. Furthermore we can split

$$\Lambda = \Lambda_{EVEN} \cup \Lambda_{ODD} \quad (4.10)$$

where $\Lambda_{EVEN/ODD}$ consists of those elements of $\Lambda$ with even/odd squared length. Now $\Lambda/\Lambda_{EVEN} = \mathbb{Z}_2$ and so $V(\Lambda_{EVEN}) = 2$. Thus $c$ itself will be in $V(\Lambda_{EVEN/ODD})$ according as $c^2$ is even or odd, or according to (4.8), as the signature (4.2) is even or odd. It is easy to see that $\Lambda_{EVEN}$ and $\Lambda_{TOTAL}$ are a pair of reciprocal lattices. Furthermore we have

$$Z(\Lambda_{EVEN}) = \Lambda_{TOTAL}/\Lambda_{EVEN} = \begin{cases} \mathbb{Z}_4 & \text{if } c \in \Lambda_{ODD}; \\
\mathbb{Z}_2 \times \mathbb{Z}_2 & \text{if } c \in \Lambda_{EVEN}. \end{cases} \quad (4.11)$$
This follows because the quotient group is determined by the addition rules for the relevant cosets in the decomposition

$$\Lambda_{TOTAL} = \Lambda_{EVEN} \cup (\Lambda_{EVEN} + c/2) \cup \Lambda_{ODD} \cup (\Lambda_{ODD} + c/2)$$  \hspace{1cm} (4.12)

It follows from (4.11) that, when $c \in \Lambda_{EVEN}$,

$$\Lambda' \equiv \Lambda_{EVEN} \cup (\Lambda_{ODD} + c/2), \quad \Lambda'' \equiv \Lambda_{EVEN} \cup (\Lambda_{EVEN} + c/2)$$  \hspace{1cm} (4.13)

are both lattices and that $V(\Lambda') = V(\Lambda'') = V(\Lambda_{EVEN})/2 = 1$. Hence, when $\Lambda'$ and $\Lambda''$ are both integral i.e. when $c^2$ is a multiple of four, they are unimodular, being even or odd according as $c^2/4$ is. This is a powerful result as odd unimodular lattices can be found for all signatures, simply by considering products of integers (hypercubic lattices). In fact this is the simplest construction of an even unimodular lattice and it works precisely when the signature $\eta$ is a multiple of eight since (4.8) can be checked explicitly for the hypercubic lattices.

The lattice of free two-cycles $F_2(\mathcal{M}_4, \mathbb{Z})$, endowed with the scalar product furnished by the intersection number of pairs of two-cycles, was unimodular. It was odd when $\mathcal{M}_4$ was Type II and even otherwise. In the former case it must have a characteristic vector (4.7). This turns out to be related to the quantity $w(\Sigma)$ introduced via the quantum Stokes relation (3.3). Recall that

$$w(\Sigma) = I(\Sigma, \Sigma) \mod 2 \quad \forall \Sigma \in F_2(\mathcal{M}_4, \mathbb{Z}).$$  \hspace{1cm} (4.14)

Now expand $\Sigma$ in terms of a basis $\Sigma = \sum_{j=1}^{b_2} n^j \Sigma_j$ and introduce the notations

$$I(\Sigma_i, \Sigma_j) = (Q^{-1})_{ij}, \quad w(\Sigma_i) = w_i,$$  \hspace{1cm} (4.15)

where the matrix $Q$ and its inverse both have integer entries as it has determinant $\pm 1$. Now (4.14) reads

$$\sum_{i,j,k=1}^{b_2} n^i (Q^{-1})_{ij} Q^{jk} w_k = \sum_{i,j=1}^{b_2} n^i (Q^{-1})_{ij} n^j.$$  \hspace{1cm} (4.16)

This means that the characteristic vector of the lattice $F_2(\mathcal{M}_4, \mathbb{Z})$ can be represented by the two-cycle

$$\gamma = - \sum_{j,k=1}^{b_2} w_j Q^{jk} \Sigma_k.$$  \hspace{1cm} (4.16)

Actually this argument is the reverse of that used in our previous paper.

What will be more important is the lattice structure of the quantised fluxes through these two-cycles. A scalar product is provided by the theta term contribution to the action (2.2). This is due to a generalisation of the Riemann bilinear identity applicable to any pair of closed forms. Here it reads

$$\int_{\mathcal{M}_4} F \wedge F' = \sum_{i,j=1}^{b_2} \int_{\Sigma_i} F Q^{ij} \int_{\Sigma_j} F'.$$  \hspace{1cm} (4.17)
Considering first the quantised flux background necessary for $M_4$ to support a complex scalar wave function, it is natural to define field strengths $F^1, F^2, \ldots, F^{b_2}$ referring to a basis reciprocal to $\Sigma_i$:

$$\frac{q_B}{2\pi \hbar} \int_{\Sigma_i} F^j = \delta_i^j. \quad (4.18)$$

The general solution to the quantisation condition (3.1) is

$$F = \sum_{i=1}^{b_2} m_i F^i, \quad m_i \in \mathbb{Z} \quad (4.19)$$

(or something cohomologous). These de Rham cohomology classes form a lattice with unimodular scalar product given by

$$\left(\frac{q_B}{2\pi \hbar}\right)^2 \int_{M_4} F \wedge F = \sum_{i,j=1}^{b_2} m_i Q^{ij} m_j. \quad (4.20)$$

Thus the fluxes too form a unimodular lattice which is odd if $M_4$ is of Type II and even otherwise.

Now consider the quantised flux backgrounds necessary for $M_4$ to support a complex spinor wave function and define a basis similar to (4.18) but with $q_F$ replacing $q_B$. Then the general solution to the quantisation condition (3.6) is

$$F = \sum_{i=1}^{b_2} \left(m_i + \frac{w_i}{2}\right) F^i, \quad (4.21)$$

(or something cohomologous), with scalar product

$$\left(\frac{q_F}{2\pi \hbar}\right)^2 \int_{M_4} F \wedge F = \sum_{i,j=1}^{b_2} \left(m_i + \frac{w_i}{2}\right) Q^{ij} \left(m_j + \frac{w_j}{2}\right). \quad (4.22)$$

It follows from (4.16) that $\sum_i w_i F^i$ represents the characteristic vector for the unimodular flux lattice. Thus when $M_4$ is of Type I or III so that all $w_i$ vanish (mod 2), the fluxes lie on an even unimodular lattice. But when $M_4$ is of Type II the fluxes lie on an odd unimodular lattice displaced by half its characteristic vector, that is the non trivial coset of (4.9).

In particular, if $q_B$ and $q_F$ are equal, the choice between fluxes corresponding to the two terms in the decomposition (4.9) of $\Lambda_{TOT}$ depends on whether the complex wave function is scalar or spinor. Notice that according to (4.7) one half of the expression (4.22) differs from $c \cdot c/8$ by an integer. Then (4.8) would follow from the integrality of the index of the Dirac operator on $M_4$ using the version of the Atiyah-Singer index theorem quoted in [M Alvarez and Olive].
5. The Maxwell partition function and theta functions

We are now in a position to return to the evaluation of the partition function (2.9), (2.10) in terms of the lattice structures just described.

The basic idea of the semi-classical approximation to (2.10) is to expand the integrand about the stationary points of the exponent which is given in terms of the classical action (2.2), that is, solutions to the Euler-Lagrange equations, here the Maxwell equation $d \ast F = 0$. This together with $dF = 0$ means that classical solutions $F$ are harmonic two-forms on $\mathcal{M}_4$. Because the relevant metric on $\mathcal{M}_4$ is Euclidean, Hodge's theorem is valid and states that there really is one and only one harmonic two-form, i.e. stationary point, in each cohomology class. As we saw these classes are labelled by the $b_2$ magnetic integers $m_1, m_2, \ldots, m_{b_2}$. It is convenient now to suppose that the representative field strengths $F^1, F^2, \ldots, F^{b_2}$ satisfying (4.18) are these harmonic ones. Then the classical solutions in each class are given precisely by (4.19) or (4.21), as the case may be. The space of harmonic two-forms on $\mathcal{M}_4$ divides into a direct sum of two subspaces consisting of self-dual and anti-self-dual harmonic two-forms. The dimensions of these subspaces are the same numbers $b^+$ and $b^−$ previously defined in purely topological terms by means of the $F_2(\mathcal{M}_4, \mathbb{Z})$ intersection matrix $Q^{-1}$.

The result is a sum over stationary points, that is the points of the lattices described in the previous section. The contribution of each of these stationary points consists of the exponential of $\frac{i}{\hbar}$ times the Euclidean action $W_{EUCLIDEAN}$ evaluated at the stationary point, all multiplied by a determinantal factor $\Delta(\tau)$ formed by the Gaussian integral of the quadratic fluctuations about it as well as zero modes. According to the arguments of E Verlinde and Witten this determinantal factor is common to all the terms of the sum. Furthermore, Witten argued that it takes the form

$$\Delta(\tau) = Z_0(\tau_2)^{b_1-1}$$  \hspace{1cm} (5.1)

where $Z_0$ is independent of $\tau$ and $b_1$ denotes the first Betti number. $\tau_1$ and $\tau_2$ are as defined in (2.3) but with $q$ replaced by $q_B$ or $q_F$, as appropriate. Hence the partition function is given by this factor $\Delta$ times a sum over a lattice of an exponential of an expression quadratic in the coordinates of the lattice points. This is recognisable as a sort of theta function associated with the flux lattice which we shall examine in more detail. Because the action (2.2) is quadratic in field strengths the result of the procedure is exact and this is what makes this version of Maxwell theory mathematically tractable.

First we evaluate $\exp\left(\frac{i}{\hbar}W_{EUCLIDEAN}\right)$ at the stationary points (4.19) relevant to complex scalar wave functions on any four manifold and to complex spinor wave functions on four-manifolds of Types I and III. The contribution of the theta term to the exponent is evaluated using (4.20):

$$\frac{i\tau_1}{2\hbar \tau_2} \int_{\mathcal{M}_4} F \wedge F = i\pi \tau_1 m^T Q m.$$  \hspace{1cm} (5.2)

The evaluation of the Maxwell term relies on the fact that, if $F^i$ is harmonic, so is its dual $\ast F^i$. Hence there exists a matrix $G$ whereby

$$\ast F^i = G^{ij}(Q^{-1})_{jk} F^k.$$  \hspace{1cm} (5.3)
Because of (2.1) and the Euclidean nature of the metric
\[(GQ^{-1})^2 = 1.\]  \hspace{1cm} (5.4)
Hence
\[\frac{1}{2\hbar} \int_{M_4} F \wedge \ast F = -\pi \tau_2 m^T G m.\] \hspace{1cm} (5.5)
In the course of this argument it becomes clear that the $b_2 \times b_2$ matrix $G$ is symmetric and positive definite, reflecting the nature of the metric upon which it depends. Finally, evaluated at the classical solution (4.19),
\[e^{\frac{i}{\hbar} W_{EUCLIDEAN}} = e^{i \pi m^T (\Omega(\hat{\tau})) m}.\] \hspace{1cm} (5.6)
where
\[\Omega(\tau) = \tau_1 Q + i\tau_2 G,\] \hspace{1cm} (5.7)
Hence, for the backgrounds considered so far, the partition function
\[Z(\tau) = \Delta(\tau) \Theta(\Omega(\tau)),\] \hspace{1cm} (5.8)
where the richest structure resides in the theta function factor which has the general form
\[\Theta(\Omega) = \sum_{m \in \mathbb{Z}} e^{i \pi m^T \Omega m}.\] \hspace{1cm} (5.9)
The sum converges if the complex symmetric matrix $\Omega$ has imaginary part which is positive definite. This is certainly so in our case as both $\tau_2$ and $G$ are positive in view of their physical interpretations.

As a consequence of the Poisson summation formula, the theta function (5.9) obeys the property
\[\Theta(-\Omega^{-1}) = \sqrt{\det(-i\Omega)} \Theta(\Omega),\] \hspace{1cm} (5.10a)
where the positive sign of the root is understood. Furthermore
\[\Theta(\Omega) = \Theta(A\Omega A^T) = \Theta(\Omega + B)\] \hspace{1cm} (5.10b)
where $A \in GL(b_2, \mathbb{Z})$ and $B$ is a symmetric matrix with integer entries which are even on the diagonal. These symmetries generate a subgroup of $Sp(2b_2, \mathbb{Z})$ with finite index. We do not know whether this has any physical significance but it does contain a subgroup recognisable as consisting of discrete electromagnetic duality transformations acting on $\tau$ when we take account of some special properties possessed by the matrix (5.7), namely
\[\Omega(-1/\tau) = -Q\Omega(\tau)^{-1}Q, \quad \Omega(\tau + 1) = \Omega(\tau) + Q,\] \hspace{1cm} (5.11a)
and
\[\sqrt{\det(-i\Omega(\tau))} = e^{-\frac{2\pi}{8} \tau^b \tau^b/2},\] \hspace{1cm} (5.11b)
checked using (5.4) and the fact that \( GQ^{-1} \) has \( b^\pm \) eigenvalues \( \pm 1 \). As before, \( \eta = b^+ - b^- \) is the signature of \( Q \) and hence the Hirzebruch signature of \( \mathcal{M}_4 \).

Regarding the theta functions as functions of \( \tau \), and denoting them accordingly as \( \Theta(\tau) \), it follows from (5.10) and (5.11) that

\[
\Theta(-1/\tau) = e^{-\frac{2\pi i}{\tau^*} b^+/2 (\tau^*)^{-1/2}} \Theta(\tau) \tag{5.12a}
\]

\[
\Theta(\tau + 1) = \Theta(\tau) \quad \text{if } Q \text{ is even,} \tag{5.12b}
\]

and

\[
\Theta(\tau + 2) = \Theta(\tau) \quad \text{otherwise,} \tag{5.12c}
\]

We have already met the \( S \)-transformation (2.14). Together with

\[
T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \tag{5.13}
\]

sending \( \tau \rightarrow \tau + 1 \), it generates the discrete group \( Sp(2, \mathbb{Z}) \), a subgroup of the \( Sp(2, \mathbb{R}) \) duality group of section 2. On the other hand, \( S \) and \( T^2 \) generate a subgroup of \( Sp(2, \mathbb{Z}) \) called the Hecke group, \( \Gamma_\theta \), one of the three distinct subgroups of index three.

So, roughly speaking, the partition function associated with backgrounds of integrally quantised fluxes does manifest a property of electromagnetic duality in that it transforms simply under certain discrete subgroups of \( Sp(2, \mathbb{R}) \) acting on the dimensionless variable \( \tau \). This discrete subgroup is \( Sp(2, \mathbb{Z}) \) or \( \Gamma_\theta \) depending on whether the lattice of integrally quantised fluxes is even or odd. But it is precisely in the latter case that real spinor wave functions are forbidden.

The existence of complex spinor wave functions requires fractionally quantised fluxes, as explained above, and hence a different partition function. Repetition of the above calculation using (4.21) rather than (4.19) yields a similar result but with a different theta function, namely

\[
\Theta(\tau)_{w/2} = \sum_{m_i \in \mathbb{Z}} e^{i\pi (m + \frac{1}{4} w)^T \Omega(\tau)(m + \frac{1}{4} w)}, \tag{5.14}
\]

We shall show that this transforms nicely under the action of another subgroup of \( Sp(2, \mathbb{Z}) \), also of index three, like \( \Gamma_\theta \). Furthermore, if \( q_F = q_B \), (5.14) is related to the previous theta function, (5.9), by an element \( ST \) of \( Sp(2, \mathbb{Z}) \), outside the two subgroups. Thus there is a sense in which the action of the full modular group \( PSL(2, \mathbb{Z}) \equiv Sp(2, \mathbb{Z})/Z_2 \) of electromagnetic duality transformations can be realised taking into account the difference in the background of quantised fluxes needed to support complex scalar and spinor wave functions respectively on four-manifolds of type II.

In order to explain this in more detail we need more developments in formalism.

Notice that although \( \Theta(\Omega) \) is holomorphic in \( \Omega \), \( \Theta(\tau) \) is not holomorphic in \( \tau \) unless \( b^- \) vanishes. For then \( G = Q \) and \( \Omega(\tau) = \tau Q \). Similarly, if \( b^+ \) vanishes, \( \Omega(\tau) = \tau^* Q \).
6. Theta functions and integral lattices

In this section the theta function construction above is extended in a way that is similar to that described in the book [Green, Schwarz and Witten 1987] but which goes further. It seems to be novel and intrinsically interesting. A theta function is associated to each element of the group $\mathbb{Z}(\Lambda)$, (4.5), defined by an integral lattice $\Lambda$, whatever the signature of its scalar product. These $|\mathbb{Z}(\Lambda)|$ theta functions support an action of the group $Sp(2,\mathbb{Z})$ (i.e. $SL(2,\mathbb{Z})$) if $\Lambda$ is even, and its Hecke subgroup $\Gamma_\theta$ if it is odd (or, more properly, their metaplectic extensions).

Careful analysis of how the effect of the generators $S$ and $T$ of $Sp(2,\mathbb{Z})$ satisfy the relation $(ST)^3 = -I$ will yield “Milgram’s formula” expressing the signature (mod 8) in terms of the structure of $\mathbb{Z}(\Lambda)$, when $\Lambda$ is even. The construction (4.11) means that odd unimodular lattices can also be dealt with, leading to a proof of (4.8). The relevance of all this to electromagnetic duality will be explained in the following section.

We start with Poisson’s summation formula in the form:

$$\sum_{n_i \in \mathbb{Z}} f(x_i + n_i) = \sum_{m_i \in \mathbb{Z}} e^{2\pi i m_j x_j} \tilde{f}(m_i), \quad \text{where} \quad \tilde{f}(k) = \int d^nx e^{-2\pi ik_j x_j} f(x) \quad (6.1)$$

denotes the Fourier transform and the summation convention is understood. The sums and integrals converge if

$$f(x) = e^{\pi i x_j \Omega_{jm} x_m} \quad \text{so} \quad \tilde{f}(k) = \frac{1}{\sqrt{\det(-i\Omega)}} e^{-\pi i k_j (\Omega^{-1})_{jm} k_m}$$

and $\Omega$ is a complex symmetric matrix with positive definite imaginary part, as before. Hence

$$\sum_{n_i \in \mathbb{Z}} e^{\pi i (x_j + n_j) \Omega_{jm} (x_m + n_m)} = \frac{1}{\sqrt{\det(-i\Omega)}} \sum_{m_i \in \mathbb{Z}} e^{2\pi i m_j x_j} e^{-\pi i m_j \Omega^{-1}_{jk} m_k}.$$

Now suppose that, as in (4.1), $n_j$ are the coordinates of the point $l$ of the lattice $\Lambda$ with respect to the basis $e_i$, and that $m_j$ are the coordinates of the point $l^*$ of the reciprocal lattice $\Lambda^*$ with basis $f_j$. Thus

$$l = \sum_{j=1}^n n_j e_j, \quad l^* = \sum_{j=1}^n m_j f_j \quad \text{while} \quad X = \sum_{j=1}^n x_j e_j.$$

Then, if we denote

$$\hat{\Omega} = f_j \Omega_{jk} f_k^T, \quad \text{then} \quad (\hat{\Omega})^{-1} = e_j (\Omega^{-1})_{jk} e_k^T$$

and the Poisson summation formula reads

$$\sum_{l \in \Lambda} e^{\pi i (X+l) \cdot \hat{\Omega} (X+l)} = \frac{1}{\sqrt{\det(-i\Omega)}} \sum_{l^* \in \Lambda^*} e^{2\pi i l^* \cdot X} e^{-\pi i l^* \cdot (\hat{\Omega}^{-1}) \cdot l^*}. $$
Now suppose that the lattice $\Lambda$ is integral and that the $Z(\Lambda)$ coset decomposition of $\Lambda^*$ can be written
\begin{equation}
\Lambda^* = \Lambda \cup (\lambda_1 + \Lambda) \cup (\lambda_2 + \Lambda) \ldots \cup (\lambda_{|Z(\Lambda)|-1} + \Lambda), \tag{6.2}
\end{equation}
where $\lambda_\beta$ is a representative element of the $\beta$’th coset and $\lambda_0$ is understood to vanish. Choosing $X$ to be the $\alpha$’th of these representatives so $e^{2\pi i l^* \cdot X} = e^{2\pi i \lambda_\alpha \cdot \lambda_\beta}$ if $l^* \in \lambda_\beta + \Lambda$, we can rearrange the Poisson summation formula so that sums over $\Lambda$ occur on both sides:
\begin{equation}
\sum_{l \in \lambda_\alpha + \Lambda} e^{\pi i l \cdot \hat{\Omega} \cdot l} = \frac{1}{\sqrt{\det(-i\Omega)}} \sum_{\beta=0}^{Z(\Lambda)|-1} e^{2\pi i \lambda_\alpha \cdot \lambda_\beta} \sum_{l \in \lambda_\beta + \Lambda} e^{-\pi i l \cdot \hat{\Omega}^{-1} \cdot l}.
\end{equation}

Now consider $\Omega$ to have the special structure (5.7), where, as before, $Q_{ij} = e_i \cdot e_j$, but is now only integral and not necessarily unimodular. Then by the properties of the reciprocal basis
\begin{equation}
\hat{\Omega}(\tau) = \tau_1 I + i\tau_2 \hat{G} \quad \text{where} \quad \hat{G} = f_i G_{ij} f_j^T \quad \text{and} \quad \hat{G}^2 = I
\end{equation}
by virtue of equations (5.4) and (2.1). It follows that $\hat{\Omega}^{-1}(\tau) = \hat{\Omega}(1/\tau)$. Hence, defining the $|Z(\Lambda)|$ theta functions
\begin{equation}
\Theta_{\alpha}(\tau) = \sum_{l \in \lambda_\alpha + \Lambda} e^{\pi i l \cdot (\tau_1 + i\tau_2 \hat{G}) \cdot l}, \quad \alpha = 0, 1 \ldots |Z(\Lambda)| - 1, \tag{6.3}
\end{equation}
the Poisson summation formula now reads
\begin{equation}
\Theta_{\alpha}(\tau) = \tau^{-b^+/2}(\tau^*)^{-b^-/2} \frac{e^{2\pi i \eta/8}}{\sqrt{|Z(\Lambda)|}} \sum_{\beta=0}^{Z(\Lambda)|-1} e^{2\pi i \lambda_\alpha \cdot \lambda_\beta} \Theta_{\beta}(-1/\tau) \tag{6.4}
\end{equation}
where the determinant was evaluated by (5.11b), modified to take account of the extra factor $|\det Q| = |Z(\Lambda)|$, by (2.10), arising because $Q$ is no longer necessarily unimodular.

This is the action of the $S$-transformation (2.14). The response to $T$, (5.13), is simple when $\Lambda$ is even. Otherwise $T^2$ must be considered
\begin{equation}
\Theta_{\alpha}(\tau + 1) = e^{\pi i \lambda^2_{\alpha}} \Theta_{\alpha}(\tau) \quad \text{if} \quad \Lambda \text{ is even} \tag{6.5a}
\end{equation}
\begin{equation}
\Theta_{\alpha}(\tau + 2) = e^{2\pi i \lambda^2_{\alpha}} \Theta_{\alpha}(\tau) \quad \text{otherwise} \tag{6.5b}
\end{equation}
As mentioned earlier, the two matrices $S$, (2.14), and $T$, (5.13), generate $Sp(2, \mathbb{Z}) \equiv SL(2, \mathbb{Z})$, a discrete subgroup of the original duality group, $Sp(2, \mathbb{R})$, while $S$ and $T^2$ generate the Hecke subgroup, $\Gamma_\theta$. $S$ and $T$ are not independent since they satisfy the relations
\begin{equation}
S^2 = -I_2 = (ST)^3. \tag{6.6}
\end{equation}
Given the responses (6.4) and (6.5) of the theta functions, the relations (6.6) will have remarkable consequences that we now develop. First define the following action of the
general element \( B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) of \( Sp(2, \mathbb{R}) \) on a function \( f(\tau) \) of \( \tau_1 \) and \( \tau_2 \), (or equivalently \( \tau \) and \( \tau^* \)),

\[
\hat{A}_{k_1k_2}(B)f(\tau) = (c\tau + d)^{-k_1}(c\tau^* + d)^{-k_2}f \left( \frac{a\tau + b}{c\tau + d} \right).
\] (6.7)

This is meaningful if \( k_1 \) and \( k_2 \) are a pair of integers as will be supposed. It is also relevant for \( k \) and \( k \) to be half-integers. Treatment of this case requires the introduction of the metaplectic group to take account of the sign ambiguities that arise because of the square roots. In the interests of avoiding an overcumbersome notation this will not be done.

The virtue of the definition (6.7) is that this action satisfies the group property:

\[
\hat{A}_{k_1k_2}(B)\hat{A}_{k_1k_2}(B') = \hat{A}_{k_1k_2}(BB').
\] (6.8)

Given a column vector of such functions, \( f_1(\tau), f_2(\tau) \ldots f_N(\tau) \), we say, following [V Kac 1990], that they form a vector modular form if there exist matrices \( D_{\beta\alpha}(B) \) such that

\[
\hat{A}_{k_1k_2}(B)f_{\alpha}(\tau) = \sum_{\beta=1}^{N} f_{\beta}(\tau)D_{\beta\alpha}(B).
\] (6.9)

The integers \( k_1 \) and \( k_2 \) are then called the weights of the functions \( f \) with respect to whatever discrete subgroup of \( Sp(2, \mathbb{Z}) \) is considered.

It follows from (6.8) and (6.9) that, if the \( N \) functions \( f \) are linearly independent, then the matrices \( D \) represent the relevant group:

\[
D(B)D(B') = D(BB').
\] (6.10)

Furthermore

\[
D_{\alpha\beta}(I_2) = \delta_{\alpha\beta}, \quad D_{\alpha\beta}(-I_2) = (-1)^{k_1-k_2}\delta_{\alpha\beta}.
\] (6.11)

But the theta functions (6.3) are not necessarily linearly independent as they may be related by permutation matrices (which are real). For example,

\[
\Theta_\alpha(\tau)P_{\alpha\beta} = \Theta_\beta(\tau) \quad \text{where} \quad P_{\alpha\beta} = \delta_{\lambda_\alpha+\lambda_\beta,0},
\] (6.12)

and it is understood in the Kronecker delta function that the equality to zero is mod \( \Lambda \).

It follows from rearrangement of (6.4) and (6.5) that the \( |Z(\Lambda)| \) theta functions (6.3) constitute a vector modular form with weights \( (k_1, k_2) = (b^+/2, b^-/2) \) and that, for them,

\[
D_{\alpha\beta}(S) = e^{2\pi i\eta/8}e^{-2\pi i\lambda_\alpha \cdot \lambda_\beta}/|Z(\Lambda)|
\] (6.13)

\[
D_{\alpha\beta}(T) = e^{\pi i\lambda_\alpha^2}\delta_{\alpha\beta} \quad \text{if } \Lambda \text{ is even},
\] (6.14a)

\[
D_{\alpha\beta}(T^2) = e^{2\pi i\lambda_\alpha^2}\delta_{\alpha\beta} \quad \text{otherwise}.
\] (6.14b)

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It is important to notice that in view of the assumed properties of the lattice \( \Lambda \), these matrix elements are independent of the choice of representatives \( \lambda_\alpha \) of the cosets (6.2). Furthermore they are independent of the parameters in the \( b^+b^- \)-dimensional moduli space of the matrices \( G \).

The matrices (6.13) and (6.14) are unitary, obviously so for \( D(T) \) and \( D(T^2) \). The unitarity of \( D(S) \) follows from the orthogonality properties of the characters of the abelian group \( Z(\Lambda) \) which appear as the phases \( \exp(2\pi i \lambda_\alpha \cdot \lambda_\beta) \). Actually the unitarity of \( D(S) \) was used in the derivation of (6.13) from (6.4).

By inspection,

\[
D(S)^T = D(S), \quad D^*(S) = (-1)^{\eta/2} PD(S) = (-1)^{\eta/2} D(S) P
\]

where \( P \) is defined by (6.12). Hence

\[
D(S)^2 = D(S)D(S)^\dagger(-1)^{\eta/2}P = (-1)^{\eta/2}P = D(-I_2)P
\]

using unitarity and (6.11). As \( P \) equals unity on theta functions, (6.12), the first of the relations (6.6) is therefore checked on the theta functions (6.3). The other identity is more interesting. By (6.4) and (6.5),

\[
(D(S)D(T))_{\alpha\beta} = \Psi(\Lambda)e^{-\frac{2\pi i\eta}{8}2\lambda_\alpha \cdot \lambda_\beta - \lambda_\alpha^2 - \lambda_\beta^2},
\]

Rearranging the sum over \( \gamma \), this can be written in the form

\[
(D(S)D(T))_{\alpha\beta} = \frac{\Psi(\Lambda)e^{-\frac{2\pi i\eta}{8}}}{\sqrt{|Z(\Lambda)|}} e^{-\pi i(2\lambda_\alpha \cdot \lambda_\beta + \lambda_\alpha^2)}.
\]

This can be made proportional to \( (D(S)D(T))^2 \) by means of the matrix \( P \), which changes the sign of the vector \( \lambda_\beta \):

\[
(D(S)D(T)\dagger P)_{\alpha\beta} = (D^*(S)D^*(T)P)_{\beta\alpha} = \frac{1}{\sqrt{|Z(\Lambda)|}} e^{\frac{2\pi i\eta}{8}2\pi i\eta e^{-\pi i(2\lambda_\alpha \cdot \lambda_\beta + \lambda_\alpha^2)}}.
\]
Comparing the last equation with (6.16) we find that
\[(D(S)D(T))^2 = e^{-4\pi i n/8} \Psi (D(S)D(T))^\dagger P.\]

The matrix $D(S)D(T)$ is unitary, and this implies that
\[(D(S)D(T))^3 = e^{-4\pi i n/8} \Psi P.\]

But, by (6.11), $D(-I_2) = \exp(4\pi i n/8)$ and therefore the result we are looking for is
\[D(-I_2) (D(S)D(T))^3 = \Psi(\Lambda) P.\]

This has to be equal to the identity in order to satisfy the relation $(ST)^3 = -I_2$, (6.6). When acting on theta functions the matrix $P$ is the identity, and this implies that $\Psi(\Lambda)$, (6.17), must have a particular value, namely unity. Hence
\[
\frac{1}{\sqrt{|Z(\Lambda)|}} \sum_{\gamma=0}^{\lfloor Z(\Lambda) \rfloor - 1} e^{\pi i \lambda^2 \gamma} = e^{\frac{1}{8} 2\pi i n(\Lambda)}.
\]

This is Milgram’s formula, relating the signature of any even lattice to its structure, but so far only proven when $b^+$ and $b^-$ and hence their difference, $\eta$, the signature of $\Lambda$, are all even. But it is easy to check (6.18) explicitly for the even lattice $\Lambda$ given by $\sqrt{2} \mathbb{Z}$, so $Z(\Lambda) = Z_2$, for either sign in the natural scalar product. Furthermore, if $\Lambda_1 \oplus \Lambda_2$ is the even lattice which is the orthogonal sum of the two even lattices $\Lambda_1$ and $\Lambda_2$ it is easy to check that $\Psi(\Lambda_1 \oplus \Lambda_2) = \Psi(\Lambda_1) \Psi(\Lambda_2)$. Thus if two of the three lattices satisfies Milgram’s formula so does the third. Using this (6.18) can then be deduced for any even lattice.

In particular, Milgram’s formula, (6.18), implies that the signature of an even unimodular lattice is a multiple of eight as stated just after equation (4.8).
7. Theta functions and odd unimodular lattices

If the lattice $\Lambda$ is unimodular, the group $\mathbb{Z}(\Lambda)$ consists just of the unit element. Hence the previous construction (6.3) yields just one theta function and the matrices $D$ are simply phase factors, by unitarity, usually called multipliers.

If, in addition, $\Lambda$ is even, the signature is a multiple of eight, by Milgram’s formula (6.18). Hence the multipliers (6.13) and (6.14a) are trivial:

$$D(S) = D(T) = 1.$$  

If, instead, the unimodular $\Lambda$ is odd there is no restriction on its signature. Again there is just one theta function supporting transformations with respect to the Hecke subgroup, $\Gamma_\theta$, generated by $S$ and $T^2$ but with non-trivial multipliers

$$D(S) = e^{-2\pi i\eta/8} \quad \text{and} \quad D(T^2) = 1. \quad (7.1)$$

However, associated with an odd unimodular lattice $\Lambda$ is the even lattice $\Lambda_{EVEN}$ for which the group $\mathbb{Z}(\Lambda_{EVEN})$ consists of four elements, (4.11). The results of the previous section suggest that there are three more theta functions which, altogether, support an action under the full group $SL(2, \mathbb{Z})$ which we now investigate.

It is this construction that will be precisely relevant to the Maxwell partition functions associated with four-manifolds of Type II.

Since

$$SL(2, \mathbb{Z}) = \Gamma_\theta \cup T \Gamma_\theta \cup ST \Gamma_\theta, \quad (7.2)$$

$\Gamma_\theta$ is a subgroup of index three in $SL(2, \mathbb{Z})$ and so it is natural to expect that only three theta functions are needed for the complete action. Indeed this is true and the fourth theta function can be chosen to be orthogonal to the three whilst supporting an $SL(2, \mathbb{Z})$ action on its own (if it does not vanish), as we shall see.

The following vectors are chosen as representatives of the cosets in the decomposition (4.12) of $\Lambda_{TOTAL} = \Lambda_{EVEN}^*$:

$$0, \quad \lambda_v, \quad \lambda_s = c/2, \quad \lambda_t = \lambda_v + c/2, \quad (7.3)$$

where $\lambda_v$ is simply any element of $\Lambda_{ODD}$. Then the defining properties of the characteristic vector $c$ lead to the conclusion that

$$\frac{1}{2} \sum_{\gamma=0,v,s,t} e^{\pi i \lambda_\gamma^2} = \frac{(1 - 1 + 2e^{\pi i c^2/4})}{2} = e^{2\pi i c^2/8}$$

Using this, Milgram’s formula (6.18) reduces to (4.8) which is thereby proven.

The notation in (7.3) is the same as one customarily used for the decomposition of the weight lattice of the Lie algebra $so(2r)$. Corresponding to this $\Lambda$ is the hypercubic lattice $\mathbb{Z}^r$. This is indeed a special case and many of the properties familiar in that case will turn out to be true in much greater generality.
This is despite the fact that the theta functions associated with the odd unimodular Euclidean lattice $\mathbf{Z}^r$ take the form

$$\left(\sum_{n \in \mathbf{Z}} e^{\pi i \tau n^2}\right)^r$$

and no such factorisation occurs for more complicated odd unimodular lattices such as $\Gamma_{8n+4}$ given by the construction (4.13), starting from the hypercubic lattice $\Lambda = \mathbf{Z}^{8n+4}$, or the theta functions associated with $\mathbf{Z}^r$ with indefinite scalar product.

Corresponding to the decomposition (7.3) there are four theta functions, denoted, respectively $\Theta_0(\tau), \Theta_v(\tau), \Theta_s(\tau)$ and $\Theta_t(\tau)$. It is easy to evaluate the four-by-four matrices $D(S)$ and $D(T)$ given by (6.13), (6.14a) and (7.3) in terms of $\omega = e^{-2\pi i \eta(\Lambda)/8}$ as

$$D_{\alpha\beta}(S) = \frac{\omega}{2} e^{-2\pi i \lambda_\alpha \cdot \lambda_\beta} = \frac{\omega}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & \omega^2 & -\omega^2 \\ 1 & -1 & -\omega^2 & \omega^2 \end{pmatrix},$$

$$D(T) = \text{diag}(1, -1, \omega^{-1}, \omega^{-1}).$$

It immediately follows from the definition (6.9) with (7.4) and (7.5) that

$$\hat{A}(S) (\Theta_s - \Theta_t) = \omega^3 (\Theta_s - \Theta_t) \quad \hat{A}(T) (\Theta_s - \Theta_t) = \omega^{-1} (\Theta_s - \Theta_t)$$

Hence the linear combination $\Theta_s - \Theta_t$ supports an action of $SL(2, \mathbf{Z})$ with the multipliers indicated. Quite often $\Theta_s - \Theta_t$ vanishes, for example if the signature $\eta(\Lambda)$ is odd or if $\Lambda$ is Euclidean and hypercubic.

The sub-space orthogonal to $\Theta_s - \Theta_t$ is three dimensional and spanned by $\Theta_0, \Theta_v$ and $\Theta_s + \Theta_t$. Of course the theta function associated with the original odd unimodular lattice $\Lambda$ is

$$\Theta = \Theta_0 + \Theta_v$$

and it is this which supports the action of the Hecke subgroup $\Gamma_\theta$. It is convenient to define

$$\Theta^T(\tau) \equiv \hat{A}(T)\Theta = \Theta_0(\tau) - \Theta_v(\tau)$$

using (7.5), and

$$\Theta^{ST}(\tau) \equiv \hat{A}(S)\Theta^T = \hat{A}(ST)\Theta = \omega(\Theta_s(\tau) + \Theta_t(\tau)),$$

using (7.4). Then $\Theta, \Theta^T$ and $\Theta^{ST}$ form an alternative orthogonal basis for the three dimensional subspace in which the action of $S$ is now represented by permutation matrices.

It is this basis that is relevant to the distinct Maxwell partition functions arising when the space-time four-manifold is of Type II so that the unimodular lattice $F_2(\mathcal{M}_4, \mathbf{Z})$, (3.2), is unimodular and odd. The partition function for backgrounds supporting complex scalar wave functions is

$$Z_{\text{SCALAR}}(\tau) = \Delta(\tau)\Theta(\tau)$$
while the partition function for backgrounds supporting complex spinor wave functions is

\[ Z_{\text{SPINOR}}(\tau) = \Delta(\tau)\Theta_{\omega/2}(\tau) = \Delta(\tau)(\Theta_s(\tau) + \Theta_t(\tau)) = \omega^{-1}\Delta(\tau)\Theta^{ST}(\tau) \]

using (5.14) and assuming that the respective electric charges \( q_B \) and \( q_F \) are equal.

Of course \( \Delta(\tau)\Theta^T(\tau) \) is simply \( Z_{SCALAR} \) with the angle \( \theta \), (2.3), (2.4) increased by \( 2\pi \).

We have seen that the four dimensional space of theta functions associated with the even lattice, \( \Lambda_{\text{EVEN}} \) constructed from any odd unimodular lattice via (4.10) usually decomposes under the \( SL(2,\mathbb{Z}) \) action as \( 4 = 3 + 1 \), though sometimes, for example when the signature \( \eta(\Lambda) \) is odd, it is \( 4 = 3 + 0 \) as \( \Theta_s - \Theta_t \) vanishes. In fact when \( \eta(\Lambda) \) is a multiple of four there is a further decomposition \( 3 = 2 + 1 \) whose details depend on whether \( \eta(\Lambda)/4 \) is odd or even. This is because of the occurrence of the two new unimodular lattices (4.13) which are odd or even as \( \eta(\Lambda)/4 \) is. Associated with these are the theta functions \( \Theta_0 + \Theta_s \) and \( \Theta_0 + \Theta_t \) obeying

\[ \hat{A}(S)(\Theta_0 + \Theta_s) = \omega(\Theta_0 + \Theta_s), \]
\[ \hat{A}(S)(\Theta_0 + \Theta_t) = \omega(\Theta_0 + \Theta_t), \]

where, now, \( \omega = (-1)^{\eta(\Lambda)/4} \). Also

\[ \hat{A}(T)(\Theta_0 + \Theta_s) = \Theta_0 + \omega\Theta_s, \]
\[ \hat{A}(T)(\Theta_0 + \Theta_t) = \Theta_0 + \omega\Theta_t. \]

So, when \( \eta(\Lambda) \) is a multiple of eight, so \( \omega = 1 \), the three dimensional subspace contains the modular function

\[ 2\Theta_0 + \Theta_s + \Theta_t = \Theta + \Theta^T + \Theta^{ST}, \]

with multipliers \( D(S) = 1 = D(T) \). On the other hand when \( \eta(\Lambda) \in 8\mathbb{Z} + 4 \), so \( \omega = -1 \) it is

\[ 2\Theta_v - \Theta_s - \Theta_t = \Theta - \Theta^T + \Theta^{ST} \]

which has multipliers \( D(S) = D(T) = -1 \).
8. Discussion

We have seen that the response of the extended partition function $Z$ associated with a four-manifold $M_4$ depends on the topological properties of $M_4$. For example, if $M_4$ is of type I or III (according to the terminology of section 3), so that $F_2(M_4, \mathbb{Z})$ is an even unimodular lattice, there is just one partition function and it transforms under the full electromagnetic duality group $SL(2, \mathbb{Z})$ acting on the dimensionless electromagnetic coupling $\tau$, (2.4), by fractional linear transformations. If $M_4$ is of type II so that $F_2(M_4, \mathbb{Z})$ is an odd unimodular lattice there are three possible partition functions, two appropriate to fluxes supporting complex scalar wave functions and the third, complex spinor wave functions. Individually these transform under three distinct but conjugate subgroups of $SL(2, \mathbb{Z})$ of index three (one of which is the Hecke subgroup). The full electromagnetic duality $SL(2, \mathbb{Z})$ is realised by permutations of these three, as explained above.

In talking of transformations we allow for the possibility of non-zero “modular weights”. We saw after equation (6.12) that the modular weights of the theta functions were given by $b^{\pm}(M_4)/2$. Witten [95] argued that the prefactors due to the Van Vleck determinants (5.1) had modular weights both given by $(1 - b_1(M_4))/2$. Hence the total modular weights of $Z$ are

$$\frac{1 - b_1(M_4)}{2} + b^{\pm}(M_4) = \chi(M_4) \pm \eta(M_4)$$

using (2.13) and (4.2). These numbers possess several interesting properties. First they are integers (or maybe half-integers in which case we should properly talk of metaplectic versions of $SL(2, \mathbb{Z})$). Secondly they are rather special topological numbers possessing the property of “locality” in the sense that they can be expressed as integrals over $M_4$ of local densities.

Suppose space-time $M_4$ exhibits a discrete $Z_2$ symmetry with no fixed points. Then the quotient $M_4/Z_2$ obtained by identifying related points is also a four-manifold and it follows from the locality properties that

$$\chi(M_4/Z_2) = \frac{\chi(M_4)}{2}.$$  

(8.2)

If the quotient $M_4/Z_2$ is Poincaré dual, the same relation (8.2) holds for the Hirzebruch signature $\eta$ and hence for the modular weights (8.1).

If the partition function considered is a “strict” one, that is it is indeed a trace, (2.9), so that $M_4$ has the form $S^1 \times M_3$, then, as $S^1 \equiv S^1/Z_2$, $M_4 \equiv M_4/Z_2$, where the $Z_2$ relates diametrically opposite points on the circle. Consequently, by (8.2), $\chi$, $\eta$ and the modular weights (8.1) all vanish and the partition function is actually invariant, in agreement with initial expectation. It is easy to check by calculation that $F_2(S^1 \times M_3, \mathbb{Z})$ is an even lattice so that type II is ruled out.

In general, the extended partition functions described in section 2 as being associated with more general four-manifolds are only modular covariant as the weights (8.1) need not vanish. This result is clearly of interest, but it has to be admitted that it is unclear what its physical meaning is. One reason is that a choice has to be made in selecting the Euclidean metric on $M_4$. Fortunately the dependence on that choice is not too strong.
For example, if the metric is altered by a Weyl rescaling,

\[ g_{\mu\nu}(x) \rightarrow \lambda(x)^2 g_{\mu\nu}(x), \]  

then \( *F \) is unaltered if \( F \) itself is. Consequently the matrix \( G \) defined by (5.3) is unchanged as is the complete theta function. Perhaps this is related to the fact that when \( b^+ \) or \( b^- \) vanishes, then \( G = \pm Q \) and so all dependence on the metric disappears from the theta function (which is then holomorphic or anti-holomorphic in \( \tau \)). If \( b^+ \) and \( b^- \) both vanish the action (2.1) is independent of \( \theta \), as mentioned earlier. Now we see that the theta function is then simply a constant independent of \( \tau \).

Of course, if \( \mathcal{M}_4 \) supports a Minkowski metric, its Euler number, \( \chi \), vanishes and the modular weights (8.1) become equal and opposite.

We should like to emphasise that the breakdown of \( SL(2, \mathbb{R}) \) to its discrete subgroup \( SL(2, \mathbb{Z}) \) was a consequence simply of the Dirac quantisation condition without recourse to the Zwanziger-Schwinger condition, which, accordingly, seems less fundamental.

In this paper we have endeavoured to elucidate the conceptual and mathematical structure in what seems to be the simplest possible context for electromagnetic duality. It would be physically interesting to extend the work in many different ways, for example:

1) to consider a larger number of abelian gauge fields on \( \mathcal{M}_4 \).
2) to consider space-times \( \mathcal{M}_{4k} \) with \( 2k \)-form field strengths with potential couplings to \( (2k - 2) \)-branes, possibly spinning.
3) to consider space-times \( \mathcal{M}_{4k+2} \) with several \( 2k + 1 \)-form field strengths.
4) to consider open space-times with boundary.
5) to consider the effect of space-time topology on conventional superstring theories.

Undoubtedly yet more mathematical tools would have to brought into play.

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