FINITE RANK BERGMAN-TOEPLITZ AND BARGMANN-TOEPLITZ OPERATORS IN MANY DIMENSIONS

GRIGORI ROZENBLUM AND NIKOLAI SHIROKOV

Abstract. The recent theorem by D. Luecking that finite rank Toeplitz-Bergman operators must be generated by a measure consisting of finitely many point masses is carried over to the many-dimensional case.

1. Introduction and the main result

Toeplitz operators play an important role in many branches of analysis. For Toeplitz operators, as well as for Hankel operators of some types, the following cut-off property is often encountered: if the operator of a certain class is too 'small' then it must have a rather special form or even be zero. ‘Smallness’ is measured most often in the terms of compactness or membership in Schatten classes. One of such cut-off questions, about Toeplitz-Bergman and Toeplitz-Bargmann operators has been under discussion for rather long time. Suppose that such operator, with a certain weight measure, has finite rank. What can one say about the measure?

We identify $\mathbb{R}^{2d}$ and $\mathbb{C}^d$ in the standard way:

$\mathbb{C} \ni z_j = x_j + iy_j \sim (x_j, y_j) \in \mathbb{R}^2, j = 1, 2, \ldots, d; z = (z_1, \ldots, z_d) \in \mathbb{C}^d$.

Let $dm(z)$ be the Lebesgue measure in $\mathbb{C}^d$ and $d\gamma(z) = (2\pi)^{-d}e^{-|z|^2/2}dm(z)$.

In the space $L^2 = L^2(\mathbb{C}^d, \mu)$, the closed subspace $\mathcal{F}^2$ of entire analytic functions is considered. Denote by $P$ the orthogonal projection in $L^2$ onto $\mathcal{F}^2$. This is an integral operator with the kernel

$$P(z, w) = \exp(z\bar{w}/2).$$ (1.1)

Next, let $V$ be a bounded complex valued function on $\mathbb{C}^d$, referred to as the weight function in what follows. The Toeplitz operator we are interested in (we call it Toeplitz-Bargmann operator) is defined by

$$T^F_V : \mathcal{F}^2 \rightarrow \mathcal{F}^2, \ T_V : u \mapsto PVu.$$ (1.1)

Equivalently, the operator $T_V$ in $\mathcal{F}^2$ is defined in terms of its sesquilinear form:

$$(T_V^* u, v)_{\mathcal{F}^2} = \int_{\mathbb{C}} V(z)u(z)\overline{v(z)}d\gamma(z), \quad u, v \in \mathcal{F}^2.$$ (1.1)

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The operator $T^\mathcal{F}_\mu$ can be also expressed as an integral operator of the form

$$(T^\mathcal{F}_\mu u)(z) = \int_{\mathbb{C}^d} u(w)P(z, w)V(w)d\mu(w),$$

(1.3)

where $P(z, w)$ is the kernel (1.1). This latter expression opens the possibility to consider Toeplitz operators with a measure playing the role of the weight function. In $\mu$ is a complex regular Borel measure on $\mathbb{C}^d$, having a bounded support, we can consider the operator $T^\mathcal{F}_\mu$ defined by the relation

$$(T^\mathcal{F}_\mu u)(z) = \int_{\mathbb{C}^d} u(w)P(z, w)d\gamma(w).$$

(1.4)

The operator (1.4) takes the form (1.3) if the measure $\mu$ is absolutely continuous with respect to $\gamma$, with $V$ being the derivative $d\mu/d\gamma$.

Along with Toeplitz-Bargmann operators we consider Toeplitz-Bergman operators. Let $\Omega$ be a bounded connected domain in $\mathbb{C}^d$ and $B(z, w)$ be its Bergman kernel, the integral kernel of the projection $P_\Omega$ of the space $L^2(\Omega)$ onto the Bergman space $H^2(\Omega)$ consisting of holomorphic functions in $L^2(\Omega)$. For a complex regular Borel measure $\mu$ on $\Omega$, having compact support in $\Omega$, we consider the Toeplitz-Bergman operator $T^\Omega_\mu$

$$(T^\Omega_\mu u)(z) = \int B(z, w)u(z)d\mu(z), \ u \in H^2(\Omega).$$

(1.5)

The operator can be described by the quadratic form

$$(T^\Omega_\mu u, v) = \int u(z)v(z)d\mu(z)$$

(1.6)

Again, if the weight measure $\mu$ is absolutely continuous with respect to Lebesgue measure with derivative $V$, the operator, denoted now by $T^\mathcal{F}_V$, can be represented in the standard Toeplitz form

$$T^\mathcal{F}_V u = PVu.$$  

(1.7)

It is clear that if the measure $\mu$ is a linear combination of finitely many point masses, $\mu = \sum_{k=1}^N \lambda_k \delta(z - z_k)$, $\lambda_k \neq 0$ then the operators $T^\mathcal{F}_\mu$, $T^\mathcal{Q}_\mu$ have finite rank, with the range coinciding with the linear span of functions $P(z, z_k)$, resp., $B(z, z_k)$,

$$T^\mathcal{Q}_\mu u(z) = \sum B(z, w_k)\lambda_k u(w_k).$$

(1.8)

So, the natural question arises whether the converse is true: if the operators have finite rank does this imply that the measure consists only of a finite set of point masses. For absolutely continuous measures this would mean that finite rank Toeplitz operators must be zero.

The result in the case of rank zero is folklore. If the operator (Bargmann or Bergman) is zero than the measure should be zero, and it follows easily from Stone-Weierstrass theorem. In 1987 in [4] a proof
was proposed of the finite rank conjecture for $d = 1$, but it was seriously flawed. In 2002 an attempt to prove this conjecture, again for $d = 1$, was made by N. Das in [1], however, again, with incorrigible mistakes, see the review [2].

The authors became interested in the finite rank conjecture due to its relation to the study of the spectral properties of the perturbed Landau Hamiltonian. The unperturbed Landau operator describes the movement of a charged quantum particle confined to a plane under the action of a uniform magnetic field. This operator has spectrum consisting of Landau levels, infinitely degenerated eigenvalues placed at the points of an arithmetic progression. The corresponding eigenspaces are explicitly expressed via the Bargmann space. When the Landau operator is perturbed by a compactly supported (or fast decaying) electrostatic potential or magnetic field the Landau levels split into clusters of eigenvalues, having Landau levels as their only limit points. The distribution of perturbed eigenvalues in clusters is essentially governed by the spectrum of Toeplitz-Bargmann operators with weight function $V$ expressed in an explicit way in the terms of the perturbation. Many results in this direction have been obtained in [7], [6], [9], [8] and other publications. In particular, simple operator-theoretical arguments (see e.g. [7, Proposition 4.1]) show that the Landau level is, in fact, the accumulation point of a cluster if and only if $T^V_B$ has infinite rank. So, if $T^V_B$ has finite rank, the Landau level remains, even after the perturbation, being an isolated eigenvalue of infinite multiplicity. The affirmative answer to the finite rank conjecture would mean that under a non-zero perturbation the Landau levels necessarily split into infinite clusters.

The authors, together with A. Pushnitsky, spent some time in 2005, trying to prove the conjecture. Certain partial results were obtained, and a text was in preparation, when a beautiful proof by Dan Luecking [5] appeared. In that paper, in the case $d = 1$, the finite rank conjecture was proved for the operators $T^B_\mu$ and $T^\Omega_\mu$ without any extra conditions. Being quite impressed, we stopped our work.

A year later, the authors decided to return to the problem, in its multi-dimensional setting. Besides the natural curiosity, we were moved by fact that the proof in [5] used essentially the specifics of the case $d = 1$, while the higher-dimensional case is also of interest for applications, see, e.g., [9]. As a result we managed to carry over the result of [5] to the multi-dimensional case in its full generality.

**Theorem 1.1.** Suppose that for a certain finite measure $\mu$, compactly supported in $\mathbb{C}^d$ or in $\Omega$, the corresponding operator $T^F_\mu$; resp., $T^\Omega_\mu$ has finite rank $N$. Then the measure $\mu$ is a linear combination of $N$ point masses. In particular, if the measure $\mu$ is absolutely continuous with respect to Lebesgue measure then it is zero.
We should note that the condition of the measure to have compact support cannot be dropped altogether. In [3] an example of a non-trivial measure $\mu$ without compact support was constructed such that the corresponding operator $T^F_\mu$ is zero.

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2. General properties

The first simple observation, used in a slightly different form in [5], shows that the finite rank property implies the finite rank for a certain infinite matrix. For the given measure $\mu$ we consider the infinite matrix

$$A(\mu) = (a_{\alpha\beta}) : a_{\alpha\beta} = \int z^\alpha \bar{z}^\beta d\mu(z), \quad (2.1)$$

where $\alpha, \beta$ are multi-indices in $\mathbb{Z}^d_+$.

**Lemma 2.1.** Let the operator $T = T^F_\mu$ or $T = T^\Omega_\mu$ have finite rank. Then the matrix $A(\mu)$ has finite rank, $\text{rank}(A(\mu)) \leq \text{rank}(T)$

In fact, if we had $\text{rank}(A(\mu)) > N = \text{rank}(T)$, this would mean that for some $M$ the dimension of the range of the operator $T$ restricted to the finite-dimensional subspace consisting of polynomials of degree less than $M$ is greater than the dimension of the range of $T$ itself, which is impossible.

At the moment we cannot establish the converse to Lemma 2.1 in full generality. This result will follow from Theorem 1.1, as it will be explained later. However for some nice domains $\Omega$ we can prove the converse right now.

**Lemma 2.2.** Let the domain $\Omega$ be a polidisk in $\mathbb{C}^d$. Then $\text{rank}(A(\mu)) = \text{rank}(T^\Omega_\mu)$.

**Proof.** We must show that if $\text{rank}(A(\mu)) = M < \infty$ then $\text{rank}(T) \leq M$, $T = (T^\Omega_\mu)$. Suppose that $\text{rank}(T) > M$. Then there must exist $M + 1$ functions $u_1, \ldots, u_{M+1} \in H^2(\Omega)$ such that the $Tu_j$ are linearly independent. So the matrix $\tilde{A}(\mu)$ with entries

$$\tilde{a}_{j,k} = (Tu_j, u_k) = \int u_j(z)\overline{u_k(z)}d\mu(z), \quad 1 \leq j, k \leq M + 1, \quad (2.2)$$

is nonsingular. However in a polidisk any holomorphic function can be uniformly on compact sets approximated by polynomials, the starting segments of the Taylor series. If $p_j$ are polynomials approximating $u_j$, the matrix with elements $(Tp_j, p_k)$ has rank not greater than $M$ and
therefore singular, so it cannot approximate the nonsingular matrix $\tilde{A}(\mu)$. □

The same reasoning, using this approximation property and density of polynomials in the Bargmann space proves the similar statement for the Bargmann operators.

**Lemma 2.3.** For a measure $\mu$ with compact support $\text{rank}(A(\mu)) = \text{rank}(T_\mu^F)$.

Lemma 2.2 enables us to reduce the Bargmann case to the Bergman one. Suppose that for some measure $\mu$ the operator $T_\mu^F$ has finite rank. Then, by Lemma 2.1, the matrix $A(\mu)$ has finite rank. Consider a polidisk $\Omega$ containing the support of $\mu$ in a compact set inside. Since the matrix $A(\mu)$ is the same for operators $T_\mu^F$ and $T_\mu^\Omega$, it follows from Lemma 2.2 that $T_\mu^\Omega$ has finite rank.

So it is sufficient to prove Theorem 1.1 for the Bergman case only. We will suppress the superscript $\Omega$ further on.

Now we introduce a functional parameter. Let $g(z)$ be a bounded holomorphic function in $\Omega$. We consider the measure $\mu_g = |g|^2\mu$.

**Lemma 2.4.** Suppose that the operator $T_\mu$ has finite rank. Than the operator $T_{\mu_g}$ has finite rank as well and $\text{rank}(T_{\mu_g}) \leq \text{rank}(T_\mu)$. If the function $g(z)^{-1}$ is also bounded and holomorphic than $\text{rank}(T_\mu^\Omega) = \text{rank}(T_\mu^\Omega)$.

*Proof.* By considering the quadratic forms, we see that operator $u \mapsto gT_\mu^\Omega gu$ coincides with $T_{\mu_g}^\Omega$, the multiplication by a bounded analytical function, being a bounded operator in the Bergman space, cannot increase the rank, and therefore the first statement follows. The second statement is now obvious. □

3. **Proof of the main result**

We are going to prove Theorem 1.1 using induction in dimension. It will follow from a more general result.

**Proposition 3.1.** Let $\mu$ be a regular Borel measure with compact support in $\mathbb{C}^d$. Suppose that for any function $g(z)$, bounded and holomorphic in a fixed polidisk neighborhood of the support of $\mu$ such that $g(z)^{-1}$ is also bounded and holomorphic, the matrix $A(\mu_g)$ has rank $N$. Then the measure $\mu$ is a linear combination of $N$ point masses.

*Proof.* The base of induction is the result by D. Luecking in [5] for $d = 1$. In fact, it is even sufficient here to have the finite rank of the matrix $A(\mu)$ only, without the extra function $g$.

Now we perform the step of induction. Suppose that the statement is proved in dimension $d - 1$. Consider the $d$-dimensional case. We denote the co-ordinates in $\mathbb{C}^d$ by $z = (z_1, z')$, $z' \in \mathbb{C}^{d-1}$. For a bounded holomorphic function $g(z)$, $z \in D$, we denote $\mu_g = |g|^2\mu$. 
Let $\pi$ denote the projection, $\pi(z_1, z') = z'$. For the measure $\mu$ on $\mathbb{C}^d$ we will denote by $\nu = \pi_* \mu$ the induced measure on $\mathbb{C}^{d-1}$,

$$\nu(E) = \mu(\pi^{-1} E),$$

(3.1)

for Borel sets $E \subset \mathbb{C}^{d-1}$. If $\mu$ is a regular Borel measure, the same is true for $\nu$.

In the matrix $A(\mu_g)$ we consider the sub-matrix $A'(\mu_g)$ consisting of entries $a_{\alpha\beta}$ with multi-indices $\alpha, \beta$ having zero as their first component, $\alpha_1, \beta_1 = 0$. Then the element $a_{\alpha\beta}$ can be written as

$$a'_{\alpha', \beta'} = a_{(0, \alpha')(0, \beta')} = \int z^{\alpha'} \ov{z}^{\beta'} \mu_g(z) = \int z^{\alpha'} \ov{z}^{\beta'} \nu_g(z'),$$

(3.2)

the last equality expressing the fact that the integrand is independent of $z_1$. Since the matrix $A'(\mu_g)$ is a submatrix of the finite rank matrix $A(\mu_g)$, its rank is not greater than the rank of $A(\mu_g)$, and therefore

$$\text{rank } A'(\mu_g) \leq N,$$

(3.3)

for any function $g$ satisfying the conditions above. This means that the measure $\nu_g$ satisfies the conditions of the inductive hypothesis in dimension $d - 1$, and therefore the measure $\nu_g$ is a finite combination of point masses,

$$\nu_g = \sum_{j=1}^{M} \lambda_j \delta(z' - \zeta_j),$$

(3.4)

where $M \leq N$, coefficients $\lambda_j \neq 0$ and the points $\zeta_j \in \mathbb{C}^{d-1}$ may depend, generally, on the function $g$: $M = M(g)$, $\lambda_j = \lambda_j(g)$, $\zeta_j = \zeta_j(g)$.

We are going to show now that, at least locally, the points $\zeta_j$ do not depend on $g$.

The number $M(g)$, considered as a function of the set of functions $g$, attains its maximal value, $M_0 = \max_g M(g) \leq N$ at some function $g = g_0$. Without loss in generality, we can assume that already $g_0 = 1$ gives the extremal value of $M(g)$ (otherwise, we re-denote $\mu_{g_0}$ by $\mu$). For any $j$, $1 \leq j \leq M_0$, we consider the measure $\mu^j$ in $\mathbb{C}^d$ defined as $\mu^j(G) = \mu(G \cap \pi^{-1}\{\zeta_j(1)\})$, $G \subset \mathbb{C}^d$. So the measure $\mu^j$ is supported in the pre-image of the point $\zeta_j(1)$ under the projection $\pi$. Moreover, the coefficients $\lambda_j(1)$ in front of point masses $\delta(z' - \zeta_j(1))$ are equal to $\nu(\{\zeta_j(1)\}) = \mu^j(\pi^{-1}\{\zeta_j(1)\}) = \int 1d\mu^j(z)$. Now, for a function $g$ as above, we can express $\nu_g(\{\zeta_j(1)\})$ as

$$\nu_g(\{\zeta_j(1)\}) = \int |g(z)|^2 d\mu^j(z).$$

(3.5)

By (3.5), the quantities $\nu_g(\{\zeta_j(1)\})$ depend continuously on the function $g$ in the topology of uniform convergence on the support of the measure $\mu$. Thus, since they are not zero for $g = 1$, they are not zero for $g$ sufficiently close to 1 in the above topology, so, for such $g$, the point
masses at the point masses at $\zeta_j(1)$ are present in the measure $\nu_g$ with nonzero coefficients. However since there are $M_0$ points $\zeta_j(1)$, and $M_0$ is the maximal possible quantity of the point masses in $\nu_g$, this means that no more point masses appear. Thus, for functions $g$ sufficiently close to 1 in uniform norm on the support of $\mu$, the points $\zeta_j(g)$ do not actually depend on $g$, in other words, for such $g$ the measure $\nu_g$ is a sum of point masses placed at the points not depending on $g$.

Now we consider the measure $\tilde{\mu} = \mu - \sum \mu^j$, so $\tilde{\mu}(G) = 0$ for any Borel set $G \subset \cup \pi^{-1}(\{\zeta_j(1)\})$. Therefore $\pi_*\tilde{\mu}_g(\{\zeta_j(1)\}) = 0$ for any $g$. At the same time, since for $g$ close to 1 the support of $\nu(g)$ consists only of the points $\zeta_j(1)$, we have $\pi_*\tilde{\mu}_g(E) = 0$ for any Borel set $E \subset C^{d-1}$ not containing the points $\zeta_j(1)$. Taken together, these two properties mean that

$$\pi_*\tilde{\mu}_g = 0$$

for functions $g$ sufficiently close to 1. In particular, we have

$$\pi_*\tilde{\mu}_g(C^{d-1}) = \int |g(z)|^2 d\tilde{\mu}(z) = 0. \quad (3.7)$$

Now we extend the equality (3.7) from functions $g$ which are close to 1 to all functions $g$, analytical in the polidisk neighborhood of the support of $\mu$. In fact, for a given function $g$ set $g_\epsilon = 1 + \epsilon g$ and apply (3.7) for $g_\epsilon$, with any $\epsilon$ small enough. We obtain

$$\int (1 + 2\epsilon \Re g(z) + \epsilon^2 |g(z)|^2) d\tilde{\mu}(z) = 0. \quad (3.8)$$

Due to the arbitrariness of a small $\epsilon$, (3.8) implies (3.7).

Now we can show that the measure $\tilde{\mu}$ is, in fact, zero. Consider the algebra generated by functions $|g|^2$. This algebra, obviously, satisfies all conditions of Stone-Weierstrass theorem, therefore any function continuous on a compact set can be uniformly on this compact approximated by linear combinations of the functions of the form $|g|^2$. Therefore, the relation (3.7) implies that $\int f(z) d\tilde{\mu}(z) = 0$ for any continuous function, and thus

$$\tilde{\mu} = 0, \quad \mu = \sum \mu^j = \sum \mu(G \cap \pi^{-1}(\{\zeta_j(1)\})). \quad (3.9)$$

So, the support of the measure $\mu$ is contained in no more than $N$ complex planes $z' = \zeta_j(1)$, $z_1 \in C^1$, $j = 1, \ldots, M_0$.

Now we can repeat the same reasoning but considering the splitting of co-ordinates $z = (z'', z_d)$ and the corresponding projection $z \mapsto z''$. We obtain that the support of the measure $\mu$ is contained in no more than $N$ complex planes $z'' = \xi_k, z_d \in C^1, k \leq N$. Surely, the support of $\mu$ must lie in the intersection of these planes, which gives us no more than $N^2$ points $Q(j, k) : z' = \zeta_j, z'' = \xi_k$. Finally, to reduce the quantity of points in the support of the measure, we rotate the coordinates in $C^d$ by means of a unitary matrix to that in new co-ordinates $\omega = (\omega_1, \ldots, \omega_d = (\omega_1, \omega'))$ the points $Q(j, k)$ all have
different $\omega'$ co-ordinates. Repeating our reasoning for the third time, we obtain that the points $Q(j, k)$ lie on no more than $N$ complex planes $\omega' = \chi_l$, and since each of these planes contains no more than one of the points $H(j, k)$, this means that actually no more than $N$ points belong to the support of the measure. It is clear that the number of point masses is exactly $N$. If there were fewer of them, then the rank of the Toeplitz operator would be smaller than $N$. \hfill \square

Finally we can establish the converse to Lemma 2.1 for an arbitrary domain $\Omega$. It shows that the rank of the operator does not decrease if we restrict it to polynomials, even if polynomials are not dense in the space of holomorphic functions.

**Proposition 3.2.** Let $\Omega$ be an arbitrary domain in $\mathbb{C}^d$. Then $\text{rank}(A(\mu)) = \text{rank}(T_\Omega^\mu)$.

**Proof.** By Theorem 1.1 the measure $\mu$ consists of $N$ point masses, $N = \text{rank}(A(\mu))$, with nonzero coefficients. But now the relation (1.8) shows that the operator $T_\mu^\Omega$ has the same rank. \hfill \square

### References

[1] Das, N., Finite rank Toeplitz operators on the Bergman space. Bull. Calcutta Math. Soc. 94 (2002), no. 2, 113–120.

[2] Englis, M. Mathematical Reviews, Review 2003g:47047.

[3] Grudsky, S. M.; Vasilevski, N. L. Toeplitz operators on the Fock space: radial component effects. Integral Equations Operator Theory, 44 (2002), no. 1, 10–37.

[4] Luecking, D. Trace ideal criteria for Toeplitz operators. J. Funct. Anal. 73 (1987), no. 2, 345–368.

[5] Luecking, D. Finite rank Toeplitz operators on the Bergman space. To appear in Proc. Amer. Math. Soc.

[6] Melgaard, M., Rozenblum, G.: Eigenvalue asymptotics for weakly perturbed Dirac and Schrödinger operators with constant magnetic fields of full rank. Comm. Partial Differential Equations 28 (2003), 697–736.

[7] Raikov, G. Warzel, S., Quasi-classical versus non-classical spectral asymptotics for magnetic Schrödinger operators with decreasing electric potentials. Rev. Math. Phys. 14 (2002), 1051–1072.

[8] Rozenblum G., Sobolev, A.V. Discrete spectrum distribution of the Landau operator perturbed by an expanding electric potential, http://front.math.ucdavis.edu/0711.2158, to appear in Contemporary Mathematics, AMS.

[9] Rozenblum, G., Tashchiyan, G., On the spectrum of the perturbed Landau Hamiltonian. http://front.math.ucdavis.edu/math-ph/0605038 to appear in Comm. Part. Diff. Equat.
(G. Rozenblum) Department of Mathematics, Chalmers University of Technology, and Department of Mathematics University of Gothenburg, Chalmers Tvärgatan, 3, S-412 96 Gothenburg Sweden
E-mail address: grigori@math.chalmers.se

(N. Shirokov) Department of Mathematical Analysis, Faculty of Mathematics and Mechanics, St. Petersburg State University, 2, Bibliotechnaya pl., Petrodvorets, St.Petersburg, 198504, Russia
E-mail address: matan@math.spbu.ru