Super polyharmonic property and asymptotic behavior of solutions to the higher order Hardy-Hénon equation near isolated singularities

Xia Huang, Yuan Li, Hui Yang

December 6, 2022

Abstract

In this paper, we are devoted to studying the positive solutions of the following higher order Hardy-Hénon equation

\[-\Delta^m u = |x|^\alpha u^p \quad \text{in } B_1 \setminus \{0\} \subset \mathbb{R}^n\]

with an isolated singularity at the origin, where \(\alpha > -2m, m \geq 1\) is an integer and \(n > 2m\). For \(1 < p < \frac{n+2m}{n-2m}\), singularity and decay estimates of solutions will be given. For \(\frac{n+\alpha}{n-2m} < p < \frac{n+2m}{n-2m}\) with \(-2m < \alpha < 2m\), we show the super polyharmonic properties of solutions near the singularity, which are essential tools in the study of polyharmonic equation. Using these properties, a classification of isolated singularities of positive solutions is established for the fourth order case, i.e., \(m = 2\). Moreover, when \(m = 2\), \(\frac{n+\alpha}{n-4} < p < \frac{n+4+\alpha}{n-4}\) and \(p \neq \frac{n+4+2\alpha}{n-4}\) with \(-4 < \alpha \leq 0\), we obtain the precise behavior of solutions near the singularity, i.e., either \(x = 0\) is a removable singularity or

\[
\lim_{|x| \to 0} |x|^{\frac{4\alpha}{n-4}} u(x) = [A_0]^\frac{1}{p-1},
\]

where \(A_0 > 0\) is an exact constant.

Mathematics Subject Classification (2020): 35B40; 35B65; 35J30

Keywords: Higher order Hardy-Hénon equation; Super polyharmonic properties; Asymptotic behavior; Isolated singularity

1 Introduction

In this paper, we are interested in the positive singular solutions of the following higher order elliptic equation

\[-\Delta^m u(x) = |x|^\alpha u^p(x) \quad \text{in } B_1 \setminus \{0\} \subset \mathbb{R}^n, \quad (1.1)\]
where \( m \geq 2 \) is an integer, the punctured unit ball \( B_1 \setminus \{0\} \subset \mathbb{R}^n \) with \( n > 2m, 1 < p < \frac{n+2m}{n-2m} \) and \( \alpha > -2m \). Equation (1.1) with \( m = 1 \) is traditionally called the Hénon (resp. Hardy or Lane-Emden) equation if \( \alpha > 0 \) (resp. \( \alpha < 0 \) or \( \alpha = 0 \)). In the literature, equation (1.1) as a type of classical equations has been widely studied in the last decades, since it can be seen from various geometric and physical problems.

To tackle equation (1.1), various type of solutions were introduced and studied such as classical solutions, weak solutions, distributional solutions, singular solutions, etc. In this paper, we say that a function \( u \) is a solution to (1.1) if it belongs to \( C^{2m}(B_1 \setminus \{0\}) \) and satisfies the equation in the classical sense, and \( u \) is a distributional solution to (1.1) if

\[
u \in L^1_{\text{loc}}(B_1 \setminus \{0\}), \quad |x|^\alpha u \in L^1_{\text{loc}}(B_1 \setminus \{0\}),
\]

and (1.1) is satisfied in the sense of distributions, that is,

\[
\int_{B_1} u(-\Delta)^m \phi dx = \int_{B_1} |x|^\alpha u \phi dx \quad \forall \phi \in C^\infty_c(B_1 \setminus \{0\}).
\]

Meanwhile, let us present two important numbers: Sobolev type exponent \( P_S(m, \alpha) = \frac{n+2m+2\alpha}{n-2m} \) (\( = \infty \), if \( n = 2m \)) and Serrin type exponent \( P_C(m, \alpha) = \frac{n+\alpha}{n-2m} \) (\( = \infty \), if \( n = 2m \)), which play a vital role in the local behavior of solutions when \( p \) passes through them.

Now we recall some results of positive singular solutions to the second order equation, i.e., \( m = 1 \). In the autonomous case (i.e., \( \alpha = 0 \)), the understanding of asymptotic behavior of positive singular solutions is relatively complete. More precisely, it was studied by Lions [18] for \( 1 < p < \frac{n}{n-2} \) and by Gidas-Spruck [9] for \( \frac{n}{n-2} < p < \frac{n+2}{n-2} \). When \( p = \frac{n+2}{n-2} \), the critical equation \(-\Delta u = u^{\frac{n+2}{n-2}} \) is closely related to the classical Yamabe problem, and Caffarelli-Gidas-Spruck [3] showed that every positive singular solution \( u \) satisfies the following asymptotic radial symmetry

\[
u(x) = \bar{u}(|x|)(1 + O(|x|)) \quad \text{as} \ |x| \to 0,
\]

where \( \bar{u}(|x|) = \int_{S^{n-1}} u(r, \omega) d\omega \) is the spherical average of \( u \). Based on this result, they also established the precise asymptotic behavior of singular solutions. We may also refer to Korevaar-Mazzeo-Pacard-Schoen [12] for this critical case. When \( p > \frac{n+2}{n-2} \), the asymptotic behavior was studied by Bidaut-Véron and Véron in [2].

In the non-autonomous case, Li [13] proved the asymptotic radial symmetry of positive solutions to equation (1.1) with \(-2 < \alpha \leq 0 \) and \( 1 < p \leq \frac{n+2+\alpha}{n-2} \). When \(-2 < \alpha < 2 \), for \( 1 < p < \frac{n+\alpha}{n-2} \), Gidas-Spruck [9, Appendix A] obtained the asymptotic behavior and Zhang-Zhao [28] studied the existence of positive singular solutions; for \( p = \frac{n+\alpha}{n-2} \), Aviles [1] obtained that either \( x = 0 \) is a removable singularity or

\[
\lim_{|x| \to 0} |x|^{n-2}(\ln |x|)^{\frac{n-2}{\alpha+2}} u(x) = \left( \frac{n-2}{\sqrt{\alpha+2}} \right)^{n-2} \;
\]

for \( \frac{n+\alpha}{n-2} < p < \frac{n+2}{n-2} \) and \( p \neq \frac{n+2+\alpha}{n-2} \). Gidas-Spruck [9] showed that the singularity at \( x = 0 \) is removable or there holds

\[
\lim_{|x| \to 0} |x|^\frac{\alpha+2}{p-1} u(x) = \left[ \frac{(2+\alpha)(n-2)}{(p-1)^2} \left( p - \frac{n+\alpha}{n-2} \right) \right]^\frac{1}{p-1}.
\]
These mean that isolated singularities of positive solutions to the equation (1.1) with \( m = 1 \) have been very well understood.

However, as far as we know, for the general polyharmonic situation \( m \geq 2 \), isolated singularities of positive solutions to the equation (1.1) are less understood. One main difficulty for such higher order equations is the absence of the maximum principle. Recently, this problem is beginning to attract a lot of attention. First we state the results in the autonomous case, i.e., \( \alpha = 0 \). Jin-Xiong [11] proved sharp blow up rates and the asymptotic radial symmetry of positive solutions of (1.1) with \( p = \frac{n+2m}{n-2m} \) under the sign assumptions

\[
(-\Delta)^k u > 0 \quad \text{in } B_1 \setminus \{0\} \quad \text{for all } 1 \leq k \leq m - 1. \tag{1.2}
\]

These are also called the super polyharmonic properties and are essential tools in the study of polyharmonic equation. When \( m = 2 \) and \( \frac{n}{n-1} < p < \frac{n+2}{n-1} \), Yang [25] showed that either the singularity at \( x = 0 \) is removable or there holds

\[
\lim_{|x| \to 0} \frac{1}{|x|^{\frac{4}{p-1}}} u(x) = \left[ K_0(p, n) \right]^{\frac{1}{p-1}}
\]

for an exact constant \( K_0(p, n) > 0 \) without the sign assumption of \(-\Delta u\). For the supercritical case, the results of singular solutions are less known. Assuming that \( |x|^{\frac{1}{p-1}} u(x) \in L^\infty(B_1(0)) \), Wu [24] recently studied the classification of isolated singularities for \( \frac{n+4}{n-4} < p < \frac{n+4}{n-4} + \varepsilon \), where \( \varepsilon > 0 \) is a constant.

In the non-autonomous case \(-2m < \alpha \leq 0\), under the sign assumptions (1.2), Caristi-Mitidieri-Soranzo [4] obtained the local behavior of positive solutions for \( 1 < p < \frac{n+\alpha}{n-2m} \) and Li [15] proved that every positive solution of (1.1) is asymptotically radially symmetric near the origin for \( \frac{n+\alpha}{n-2m} < p \leq \frac{n+2m+2\alpha}{n-2m} \). Thus, it is natural to ask:

(1) Can the sign assumptions (1.2) be removed?

(2) Can we describe the precise asymptotic behavior of solutions to (1.1) near the singularity when \( \alpha \neq 0 \)?

In this paper, we focus on super polyharmonic properties and asymptotic behavior of positive solutions to the higher order Hardy-Hénon equation (1.1). In particular, when \(-2m < \alpha < 2m \) and \( \frac{n+\alpha}{n-2m} < p < \frac{n+2m}{n-2m} \), we would show that all the positive solutions of (1.1) with non-removable singularities satisfy the super polyharmonic properties (1.2) near the origin. Using this as a tool, we establish a Harnack inequality for singular solutions of (1.1), and further give a classification of isolated singularities in the fourth order case (i.e., \( m = 2 \)). Moreover, for \( m = 2 \) and \( -4 < \alpha \leq 0 \), we also obtain the precise asymptotic behavior of positive singular solutions of (1.1).

We denote \( B_r(x) \) as the open ball in \( \mathbb{R}^n \) with center \( x \) and radius \( r \), and also write \( B_r(0) \) as \( B_r \) for simplicity. First of all, we have the following singularity and decay estimates. Under the sign assumptions (1.2) and the condition \( 1 < \frac{n+\alpha}{n-2m} < p < \frac{n+2m}{n-2m} \), the singularity estimate in (1.3) has also been proved in [15].

**Theorem 1.1.** Let \( n > 2m, \alpha > -2m \) and \( 1 < p < \frac{n+2m}{n-2m} \). Then there exists a constant \( C = C(m, n, p, \alpha) > 0 \) such that the following two conclusions hold.
(i) Any nonnegative solution of (1.1) in \(B_1 \setminus \{0\}\) satisfies
\[
\sum_{i \leq 2m-1} |x|^{\frac{2m+i-2\alpha}{p-1}} |\nabla^i u(x)| \leq C \quad \text{for } x \in B_{1/2} \setminus \{0\}. \tag{1.3}
\]

(ii) Any nonnegative solution of (1.1) in \(B_1^c\) satisfies
\[
\sum_{i \leq 2m-1} |x|^{\frac{2m+i-2\alpha}{p-1}} |\nabla^i u(x)| \leq C \quad \text{for } x \in B_2^c. \tag{1.4}
\]

The origin \(x = 0\) is called a non-removable singularity of solution \(u\) of (1.1) if \(u(x)\) cannot be extended as a continuous function near 0. Using the above upper bound result, we further show that the super polyharmonic properties hold in some small punctured ball \(B_\tau \setminus \{0\}\). Remark that the similar super polyharmonic properties of equation (1.1) on the entire space \(\mathbb{R}^n\) (or, on \(\mathbb{R}^n \setminus \{0\}\)) have been widely studied and applied; see [6–8,17,19,23] and the references therein. However, to the best of our knowledge, such results for the local equation (1.1) are relatively few.

**Theorem 1.2.** Let \(n > 2m, -2m < \alpha < 2m\) and \(\frac{n+\alpha}{2m} < p < \frac{n+2m}{n-2m}\). Assume that \(u \in C^{2m}(B_1 \setminus \{0\})\) is a positive solution of (1.1) and \(x = 0\) is a non-removable singularity. Then
\[
(-\Delta)^k u > 0 \quad \text{for all } 1 \leq k \leq m - 1
\]
in some small punctured ball \(B_\tau \setminus \{0\}\) for some \(\tau > 0\).

In particular, Theorem 1.2 holds for the critical Hardy-Sobolev exponent \(p = \frac{n+2m+2\alpha}{n-2m}\) with \(-2m < \alpha < 0\). Moreover, using Theorem 1.2 the additional sign conditions (1.2) in [15] can be removed when showing asymptotic radial symmetry of positive solutions. Such super polyharmonic properties allow one to obtain an important Harnack inequality. By employing Harnack inequality and a monotonicity formula, we can establish the following classification of isolated singularities to the fourth order equation.

**Theorem 1.3.** Let \(n > 4, -4 < \alpha < 4\) and \(\frac{n+\alpha}{n-4} < p < \frac{n+4}{n-4}\). Assume that \(u \in C^4(B_1(0) \setminus \{0\})\) is a positive solution of (1.1) with \(m = 2\). Then either the singularity at \(x = 0\) is removable, or there exist two positive constants \(C_1\) and \(C_2\) such that
\[
C_1 |x|^{-\frac{4+\alpha}{p-1}} \leq u(x) \leq C_2 |x|^{-\frac{4+\alpha}{p-1}} \quad \text{for } x \in B_{1/2} \setminus \{0\}.
\]

Furthermore, we can show the precise asymptotic behavior of singular solutions to (1.1) for \(m = 2\) as follows. Remark that our proof of Theorem 1.4 does not depend on Theorems 1.2 and 1.3.

**Theorem 1.4.** Let \(n > 4, -4 < \alpha \leq 0, \frac{n+\alpha}{n-4} < p < \frac{n+4+\alpha}{n-4}\) and \(p \neq \frac{n+4+2\alpha}{n-4}\). Assume that \(u \in C^4(B_1(0) \setminus \{0\})\) is a nonnegative solution of (1.1) with \(m = 2\). Then either \(x = 0\) is a removable singularity or there holds
\[
\lim_{|x| \to 0} |x|^{\frac{4+\alpha}{p-1}} u(x) = A_0^{\frac{1}{p-1}},
\]
where
\[
A_0 := \frac{4+\alpha}{p-1} \left[ \left(\frac{4+\alpha}{p-1}\right)^3 - 2(n-4) \left(\frac{4+\alpha}{p-1}\right)^2 + (n^2 - 10n + 20) \left(\frac{4+\alpha}{p-1}\right) + 2(n-2)(n-4) \right].
\]
This paper is organized as follows. In Section 2, we recall some classical results which will be used in our arguments. In Section 3, we show the upper bound in Theorem 1.1, and establish the super polyharmonic properties in Theorem 1.2. In Section 4, we prove Theorems 1.3 and 1.4 for the fourth order equation.

2 Preliminaries

In this section, for the reader’s convenience, we recall some classical results which will be used in our arguments. Let us start with a classification result of polyharmonic equations in Wei-Xu [23].

**Proposition 2.1** ([23]). Suppose that \( u \) is a nonnegative classical solution of
\[
(-\Delta)^m u = u^p \quad \text{in } \mathbb{R}^n
\]
for \( 1 < p < \frac{n+2m}{n-2m} \). Then \( u \equiv 0 \) in \( \mathbb{R}^n \).

**Proposition 2.2** ([26]). Let \( m \geq 1 \) be an integer and \(-2m < \alpha \leq 0\). Suppose that \( u \in C(\mathbb{R}^n \setminus \{0\}) \) is a nonnegative distributional solution of
\[
(-\Delta)^m u = |x|^\alpha u^p \quad \text{in } \mathbb{R}^n \setminus \{0\}
\]
with \( \frac{n+\alpha}{n-2m} \leq p \leq \frac{n+2m+\alpha}{n-2m} \). For \( \alpha = 0 \) and \( p = \frac{n+2m}{n-2m} \), suppose also that \( u \) has a non-removable singularity at the origin. Then \( u \) is radially symmetric with respect to the origin.

It is worth pointing out that \( \frac{n+2m+\alpha}{n-2m} \geq \frac{n+2m+2\alpha}{n-2m} \) holds for \( \alpha \leq 0 \). For the supercritical case \( \frac{n+2m+2\alpha}{n-2m} \leq p \leq \frac{n+2m+\alpha}{n-2m} \), the radial symmetry result in Proposition 2.2 holds via the Kelvin transformation.

We also need the following doubly weighted Hardy-Littlewood-Sobolev inequality, which was showed by Stein-Weiss [22]. See also Lieb [16].

**Proposition 2.3** ([16, 22]). Let \( 0 < \lambda < n \), \( 1 < p \leq q < \infty \), \( 0 \leq \gamma \leq \frac{n}{p} \) (with \( \frac{1}{p} + \frac{1}{p'} = 1 \)), \( 0 \leq \beta < \frac{n}{q} \) and \( \frac{1}{p} + \frac{\lambda + \gamma + \beta}{n} = 1 + \frac{1}{q} \). Let the function
\[
V(x, y) = |x|^{-\beta} |x - y|^{-\lambda} |y|^{-\gamma}
\]
be an integral kernel on \( \mathbb{R}^n \). Then the map \( f \to V f := \int_{\mathbb{R}^n} V(\cdot, y)f(y)dy \) is bounded from \( L^p(\mathbb{R}^n) \) to \( L^q(\mathbb{R}^n) \).

Next we recall the doubling lemma from Poláčik-Quittner-Souplet [21], which is important in the study of singularity and decay estimates.

**Proposition 2.4** ([21]). Suppose that \( \emptyset \neq D \subset \Sigma \subset \mathbb{R}^n, \Sigma \) is closed and \( \Gamma = \Sigma \setminus D \). Let \( M : D \to (0, \infty) \) be bounded on compact subset of \( D \). If for a fixed constant \( k > 0 \), there exists \( y \in D \) such that
\[
M(y) \text{dist}(y, \Gamma) > 2k,
\]
then there exists \( x \in D \) such that
\[
M(x) \text{dist}(x, \Gamma) > 2k, \quad M(x) \geq M(y),
\]
and for all \( z \in D \cap B_{M^{-1}(x)}(x) \),
\[
M(z) \leq 2M(x).
\]
At the end of this section, we state an important regularity lifting result established by Chen-Li [5], which plays a key role in proving the removable singularity. Let \( V \) be a Hausdorff topological vector space. Suppose that there are two extend norms defined in \( V \),
\[
\| \cdot \|_X, \| \cdot \|_Y : V \to [0, \infty].
\]
Let
\[
X := \{ x \in V : \| x \|_X < \infty \} \quad \text{and} \quad Y := \{ y \in V : \| y \|_Y < \infty \}.
\]
Then we have

**Proposition 2.5** ([5]). Let \( T \) be a contraction map from \( X \) into itself and from \( Y \) into itself. Assume that \( f \in X \) and there exists a function \( g \in Z := X \cap Y \) such that \( f = Tf + g \) in \( X \). Then there holds \( f \in Z \).

### 3 Upper bound and super polyharmonic properties

To prove the upper bound in Theorem 1.1, we need the following estimate which is inspired by [20, 21].

**Lemma 3.1.** Let \( u \) be a nonnegative solution of
\[
(-\Delta)^m u(x) = c(x)u^p(x), \quad x \in B_1 \subset \mathbb{R}^n,
\]
where \( m \geq 1 \) is an integer, \( n > 2m \) and \( 1 < p < \frac{n+2m}{n-2m} \). Assume that \( c(x) \in C^\gamma(\overline{B_1}) \) for some \( \gamma \in (0, 1] \) and satisfies
\[
\| c \|_{C^\gamma(\overline{B_1})} \leq C_1, \quad c(x) \geq C_2, \quad \forall x \in B_1,
\]
for some constants \( C_1, C_2 > 0 \). Then there exists a constant \( C = C(\gamma, C_1, C_2, p, n, m) > 0 \) such that
\[
\sum_{i=0}^{2m-1} |\nabla^i u(x)|^{p-1}_{2m+i+p-1} \leq C(1 + \text{dist}^{-1}(x, \partial B_1)), \quad \forall x \in B_1.
\]

**Proof.** Suppose by contradiction that there exist sequences \( \{u_k\}, \{c_k\} \) verifying (3.1), (3.2) and a sequence of points \( \{y_k\} \) such that the functions \( M_k(y) := \sum_{i=0}^{2m-1} |\nabla^i u_k(y)|^{p-1}_{2m+i+p-1} \) satisfy
\[
M_k(y_k) > 2k(1 + \text{dist}^{-1}(y_k, \partial B_1)).
\]
Then by the doubling lemma (see Proposition 2.4), there exists another sequence \( \{x_k\} \) such that
\[
M_k(x_k) \geq M_k(y_k), \quad M_k(x_k) > 2k(1 + \text{dist}^{-1}(x_k, \partial B_1)),
\]
and
\[
M_k(z) \leq 2M_k(x_k) \quad \text{for} \quad |z - x_k| \leq kM_k^{-1}(x_k).
\]
Let \( \lambda_k := M_k^{-1}(x_k) \). Then
\[
\lambda_k \to 0 \quad \text{as} \quad k \to \infty
\]
due to \( M_k(x_k) \geq M_k(y_k) > 2k \).
Set
\[ v_k(y) = \lambda_k^{-\frac{2m}{p-1}} u(x_k + \lambda_k y) \quad \text{and} \quad \tilde{c}_k(y) = c_k(x_k + \lambda_k y). \]
Then we have
\[ \sum_{i=0}^{2m-1} |\nabla^i v_k(0)|^\frac{p-1}{2m+i} = 1 \]
and
\[ \sum_{i=0}^{2m-1} |\nabla^i v_k(y)|^\frac{p-1}{2m+i} \leq 2 \quad \text{for} \ |y| \leq k. \]
Moreover, \( v_k \) satisfies
\[ (-\Delta)^m v_k(y) = \tilde{c}_k(y) v_k(y), \quad |y| \leq k. \]

By (3.2), we get \( C_2 \leq \tilde{c}_k(y) \leq C_1 \) and
\[ |\tilde{c}_k(y) - \tilde{c}_k(z)| \leq C |\lambda_k(y - z)|^\gamma \leq C |y - z|^\gamma \quad \text{for} \ k \text{ large enough.} \quad (3.3) \]

Using Arzelà-Ascoli theorem and combining with (3.2) and (3.3), there exists a constant \( c_0 \geq C_2 > 0 \) such that \( \tilde{c}_k \to c_0 \) as \( k \to +\infty \). It follows from the standard elliptic estimates that, up to a subsequence, \( v_k \to v \) in \( C^{2m}_{\text{loc}}(\mathbb{R}^n) \) and \( v \geq 0 \) is a classical solution of
\[ (-\Delta)^m v = c_0 v^p \quad \text{in} \ \mathbb{R}^n \]
with \( \sum_{i=0}^{2m-1} |\nabla^i v(0)|^\frac{p-1}{2m+i} = 1 \). As \( 1 < p < \frac{n+2m}{n-2m} \), this contradicts the Liouville type result in Proposition 2.1. \( \square \)

**Proof of Theorem 1.1.** (i). Assume \( 0 < |x_0| < \frac{1}{2} \) and denote \( R = \frac{|x_0|}{2} \). Then we have that for any \( y \in B_1 \), there holds
\[ \frac{|x_0|}{2} < |x_0 + Ry| < \frac{3|x_0|}{2}. \]
Hence \( x_0 + Ry \in B_1 \). Let us define the function
\[ v(y) = R^{\frac{2m+n}{p-1}} u(x_0 + Ry), \quad y \in B_1. \]
Then \( v \) satisfies
\[ (-\Delta)^m v(y) = c(y) v^p(y), \]
where \( c(y) = |y + \frac{x_0}{R}|^{\alpha} \). For any \( y \in B_1 \), it is easy to see that
\[ 1 \leq |y + \frac{x_0}{R}| \leq 3. \]
This implies that for any \( y \in B_1 \),
\[ 1 \leq c(y) \leq 3^\alpha \quad \text{and} \quad |\nabla c(y)| \leq C(\alpha), \]
for some constant \( C(\alpha) \). It follows from Lemma 3.1 that
\[ \sum_{i=0}^{2m-1} |\nabla^i v(0)|^\frac{p-1}{2m+i} \leq C. \]
Since $x_0 \in B_{1/2} \setminus \{0\}$ is arbitrary, we obtain that

$$
2m-1 \sum_{i=0}^{2m-1} \left| x \right|^{2m+i} \left| \nabla^i u(x) \right| \leq C, \quad \forall x \in B_{1/2} \setminus \{0\}.
$$

This completes the proof of the first part of Theorem 1.1.

(ii). With the help of Lemma 3.1, we can follow the above procedure step by step to prove the second part of Theorem 1.1, the detail will be omitted.

Next we will show the super polyharmonic properties in Theorem 1.2 based on an integral representation of solutions, which is inspired by the work of Jin-Xiong [11]. Recall that the Green function of $-\Delta$ on the unit ball is given by

$$
G_1(x, y) = \frac{1}{(n-2)\omega_{n-1}} \left( |x - y|^{2-n} - \frac{x}{|x|} \cdot \frac{y}{|y|} \right),
$$

where $\omega_{n-1}$ is the surface area of the unit sphere in $\mathbb{R}^n$. Hence, for any $u \in C^2(B_1) \cap C(\overline{B_1})$,

$$
u(x) = \int_{B_1} G_1(x, y)(-\Delta)u(y)dy + \int_{\partial B_1} H_1(x, y)u(y)d\sigma,
$$

where

$$
H_1(x, y) = -\frac{\partial}{\partial \nu_y}G_1(x, y) = \frac{1 - |x|^2}{\omega_{n-1}|x-y|^n} \quad \text{for} \ x \in B_1, \ y \in \partial B_1.
$$

By induction, we have for $n > 2m$ and $u \in C^{2m}(B_1) \cap C^{2m-2}(\overline{B_1})$,

$$
u(x) = \int_{B_1} G_m(x, y)(-\Delta)^m u(y)dy + \sum_{i=1}^{m} \int_{\partial B_1} H_i(x, y)(-\Delta)^{i-1}u(y)d\sigma,
$$

where

$$
G_m(x, y) = \int_{B_1 \times \cdots \times B_1} G_1(x, y_1)G_1(y_1, y_2) \cdots G_1(y_{m-1}, y)m dy_1 \cdots dy_{m-1}
$$

and

$$
H_i(x, y) = \int_{B_1 \times \cdots \times B_1} G_1(x, y_1)G_1(y_1, y_2) \cdots G_1(y_{i-2}, y_{i-1})H_1(y_{i-1}, y)m dy_1 \cdots dy_{i-1}.
$$

for $2 \leq i \leq m$.

The following result is very similar to [11, Lemma 2.1], so the proof is omitted here.

**Lemma 3.2.** Let $u \in C^{2m}(\overline{B_1} \setminus \{0\})$ be a nonnegative solution of

$$
(-\Delta)^m u = |x|^\alpha u^p \quad \text{in} \ B_1 \setminus \{0\},
$$

where $p > \frac{n+\alpha}{n-2m}$ with $\alpha > -2m$. Then $|x|^\alpha u^p \in L^1(B_1)$ and

$$
u(x) = \int_{B_1} G_m(x, y)|y|^\alpha u^p(y)dy + \sum_{i=1}^{m} \int_{\partial B_1} H_i(x, y)(-\Delta)^{i-1}u(y)d\sigma \quad \text{for} \ x \in B_1 \setminus \{0\}.\]
Proof of Theorem 1.2. Without loss of generality, we may suppose that \( u \in C^{2m}(B_1 \setminus \{0\}) \). By Lemma 3.2, for \( k = 1, \ldots, m - 1 \) and \( x \in B_1 \setminus \{0\} \), we have

\[
(-\Delta)^k u(x) = \int_{B_1} G_{m-k}(x,y)|y|^\alpha u^p(y)dy + \sum_{i=k+1}^{m} \int_{\partial B_1} H_{i-k}(x,y)(-\Delta)^{i-1} u(y)d\sigma
\]

\[= C \int_{B_1} \frac{|y|^\alpha u^p(y)}{|x-y|^{n-2(m-k)}} dy + h(x),\]

where \( h(x) \) is a smooth function in \( B_{1/2} \) and \( C = C(n, m) > 0 \).

Claim: there holds

\[
\int_{B_{1/2}} \frac{|y|^\alpha u^p(y)}{|y|^{n+2-2m}} dy = +\infty.
\]

If the claim is true, then there exists \( \tau_1 \in (0, 1/8) \) small such that

\[
C \int_{B_{1/2} \setminus B_{\tau_1}} \frac{|y|^\alpha u^p(y)}{|y|^{n+2-2m}} dy \geq \|h\|_{L^\infty(B_{1/4})} + 1.
\]

Setting \( \tau = \frac{\tau_1}{2} > 0 \), then for \( x \in B_{\tau} \setminus \{0\} \) we have

\[
(-\Delta)^k u(x) = C \int_{B_1} \frac{|y|^\alpha u^p(y)}{|x-y|^{n-2(m-k)}} dy + h(x)
\]

\[
\geq C \int_{B_1 \setminus B_{\tau_1}} \frac{|y|^\alpha u^p(y)}{|x-y|^{n-2(m-k)}} dy + h(x)
\]

\[
\geq C \int_{B_1 \setminus B_{\tau_1}} \frac{|y|^\alpha u^p(y)}{|y|^{n-2(m-k)}} dy + h(x)
\]

\[
\geq C \int_{B_1 \setminus B_{\tau_1}} \frac{|y|^\alpha u^p(y)}{|y|^{n-2m+2}} dy + h(x) \geq 1.
\]

Hence, \((-\Delta)^k u\) is positive in the small punctured ball \( B_{\tau} \setminus \{0\} \).

Next, we show that the claim holds. If not, then this yields

\[
\int_{B_{1/2}} \frac{|y|^\alpha u^p(y)}{|y|^{n+2-2m}} dy < \infty.
\]

By Theorem 1.1, we have \( u(x) \leq C|x|^{-\frac{2m+\alpha}{p-1}} \) in \( B_{1/2} \). Meanwhile, as \( p > \frac{n+\alpha}{n-2m} \) and \(-2m < \alpha < 2m\), we have \( \frac{2m+\alpha}{p-1} < n - 2m \). This implies that there exists \( q_0 > \frac{n}{n-2m} \) such that \( \frac{2m+\alpha}{p-1} q_0 < n \). Therefore, we get

\[
\int_{B_{1/2}} y^{q_0}(y) dy \leq C \int_{B_{1/2}} |y|^{-\frac{2m+\alpha}{p-1} q_0} dy < \infty.
\]

That is, \( u \in L^q(B_{1/2}) \) for some \( q > \frac{n}{n-2m} \).

On the other hand, we want to show \( |y|^\alpha u^{p-1} \in L^\infty_{\text{loc}}(B_{1/2}) \). First we consider the case \( \frac{n+\alpha}{n-2m} \leq p < \frac{n+2m}{n-2m} \) with \( \alpha_+ := \max \{0, \alpha\} \), which gives \( (p-1)\frac{n}{2m} - p \geq 0 \). By Theorem 1.1, we obtain

\[
\int_{B_{1/2}} |y|^\alpha u^{p-1}(y) \frac{n}{2m} dy = \int_{B_{1/2}} |y|^\alpha u^p \left[ |y|^{\frac{m}{2m}} |y|^{-\alpha} u^{p-1} \right] \frac{n}{2m} dy
\]

\[
\leq \int_{B_{1/2}} |y|^\alpha u^p \left[ |y|^{\frac{m}{2m}} |y|^{-\alpha} u^{p-1} \right] \frac{n}{2m} dy.
\]
Moreover, note that
\[
\frac{2m + \alpha}{p - 1} \left[ (p - 1) \frac{n}{2m} - p \right] - (\frac{n}{2m} - 1) \alpha = \frac{(2m + \alpha) n}{2m} - \frac{p(2m + \alpha)}{p - 1} - \frac{\alpha(n - 2m)}{2m} = n - 2m - \frac{2m + \alpha}{p - 1} < n + 2 - 2m.
\]

Hence,
\[
\int_{B_{1/2}} |y|^\alpha u^{p-1}(y) \frac{n}{2m} dy \leq \int_{B_{1/2}} |y|^\alpha u^p(y) dy < \infty.
\]

Thus \(|\cdot|^\alpha u^{p-1} \in L^{\frac{n}{2m}}(B_{1/2})\).

Now we consider \(\frac{n + \alpha}{n - 2m} < p < \frac{n}{n - 2m}\) with \(-2m < \alpha < 0\). In this case, we have \((p - 1) \frac{n}{2m} < p\).

Denote \(s = \frac{p - 1}{p} \frac{n}{2m} > 1\). By Hölder inequality,
\[
\int_{B_{1/2}} |y|^\alpha u^{p-1}(y) \frac{n}{2m} dy = \int_{B_{1/2}} |y|^\alpha u^p(y) \frac{n}{2m} \left| y \right|^{\frac{2m - \alpha}{s}} dy \
\leq \left( \int_{B_{1/2}} |y|^{\alpha \frac{n}{2m} + \frac{n - 2m + 2 - \alpha}{s} s'} dy \right)^{\frac{1}{s'}} \times \left( \int_{B_{1/2}} |y|^\alpha u^p(y) \frac{n}{2m} dy \right)^{\frac{1}{s}},
\]
where \(\frac{1}{s'} + \frac{1}{s} = 1\). We only need to show
\[
\left( \alpha \frac{n}{2m} + \frac{n - 2m + 2 - \alpha}{s} \right) s' > -n. \tag{3.4}
\]

Since
\[
\left( \alpha \frac{n}{2m} + \frac{n - 2m + 2 - \alpha}{s} \right) s' = \frac{n}{2m} \frac{(p - 1)(n + 2 - 2m) + \alpha}{p - (p - 1) \frac{n}{2m}} \frac{s}{s - 1},
\]
the inequality (3.4) holds if and only if
\[
\frac{n}{2m} \frac{(p - 1)(n + 2 - 2m) + \alpha}{p - (p - 1) \frac{n}{2m}} > -n
\]
\[
\iff (p - 1)(n + 2 - 2m) + \alpha > -2m \left[ p - (p - 1) \frac{n}{2m} \right]
\]
\[
\iff 2p - 2 + 2m + \alpha > 0,
\]
which is true due to \(p > 1\) and \(\alpha > -2m\). Hence, for \(\frac{n + \alpha}{n - 2m} < p < \frac{n + 2m}{n - 2m}\) and \(-2m < \alpha < 2m\), we obtain that \(|\cdot|^\alpha u^{p-1} \in L^{\frac{n}{2m}}(B_{1/2})\).

It follows from the regularity result, Corollary 1.1 in Li [14], that \(u \in L^\infty(B_{1/2})\) and thus \(x = 0\) is removable. This contradicts the assumption. The proof of Theorem 1.2 is completed. \(\square\)
4 Classification and asymptotic behavior

In this section, we shall prove Theorems 1.3 and 1.4. The upper bound has been proved in Theorem 1.1. To show the lower bound in Theorem 1.3, an essential tool is a Harnack inequality for singular positive solutions. Such Harnack inequality is a consequence of the super polyharmonic properties in Theorem 1.2 and the upper bound in Theorem 1.1. First we recall the following Harnack inequality for regular solutions of an integral equation in Jin-Li-Xiong [10].

**Proposition 4.1** ([10]). For \( n \geq 1, 0 < \sigma < \frac{n}{2}, r > \frac{n}{n-2\sigma} \) and \( p > \frac{n}{2\sigma} \), let \( 0 \leq V \in L^p(B_3) \), \( 0 \leq h \in C^0(B_3) \) and \( 0 \leq u \in L^r(B_3) \) satisfy

\[
    u(x) = \int_{B_3} \frac{V(y)u(y)}{|x-y|^{n-2\sigma}}dy + h(x), \quad x \in B_2.
\]

If there exists a constant \( C_0 \geq 1 \) such that \( \max_{B_1} u \leq C_0 \min_{B_1} h \), then

\[
    \max_{B_1} u \leq C \min_{B_1} u,
\]

where \( C > 0 \) depends only on \( n, \sigma, C_0, p \) and an upper bound of \( \|V\|_{L^p(B_3)} \).

**Proposition 4.2.** Let \( n > 2m, -2m < \alpha < 2m \) and \( \frac{n+\alpha}{n-2m} \) \( < p < \frac{n+2m}{n-2m} \). Suppose that \( u \in C^{2m}(B_1 \setminus \{0\}) \) is a positive solution to (1.1) with a non-removable singularity at the origin. Then there exist two constants \( C > 0 \) and \( r_0 \in (0, 1/16) \) such that for all \( 0 < r < r_0 \), we have

\[
    \sup_{r/2 \leq |x| \leq 3r/2} u(x) \leq C \inf_{r/2 \leq |x| \leq 3r/2} u(x).
\]

**Proof.** By Lemma 3.2 and the same arguments in [11, Theorem 1.1] (see the beginning of the proof of Theorem 1.1 there), we can write, modulo a positive constant, that

\[
    u(x) = \int_{B_{r_0}} \frac{|y|^{\alpha}u^p(y)}{|x-y|^{n-2m}}dy + h(x) \quad \text{for } x \in B_{r_0} \setminus \{0\}, \quad (4.1)
\]

where \( r_0 \in (0, 1/4) \) is small and \( h \) is smooth in \( B_{r_0} \). Moreover, by using Theorem 1.2 we can obtain that \( h \) has a positive lower bound in \( B_{r_0} \). Replacing \( u(x) \) by \( \left( \frac{r}{r_0} \right)^{2n+\alpha} u\left( \frac{r}{r_0} x \right) \), we may consider the integral equation (4.1) in \( B_2 \setminus \{0\} \) for convenience. Namely,

\[
    u(x) = \int_{B_2} \frac{|y|^{\alpha}u^p(y)}{|x-y|^{n-2m}}dy + h(x) \quad \text{for } x \in B_2 \setminus \{0\}, \quad (4.2)
\]

where \( u \in C(B_2 \setminus \{0\}) \), \( |\cdot|^{\alpha}u^p \in L^1(B_2) \), and \( h \in C^\infty(B_2) \) is a positive function satisfying

\[
    |\nabla \ln h| \leq C_1 \quad \text{in } B_{3/2}. \quad (4.3)
\]

Let \( v(y) = r^{\frac{2m+\alpha}{p+1}} u(ry) \) with \( 0 < r < 1/2 \). Then we have, for \( y \in B_{2/r} \setminus \{0\} \),

\[
    v(y) = \int_{B_{2/r}} \frac{|z|^{\alpha}u^p(z)}{|y-z|^{n-2m}}dz + h_r(y),
\]
where \( h_r(y) = \frac{2m+\alpha}{r^{p-1}} h(y) \). By Theorem 1.1 (i), we know that \( u(x) \leq C |x|^{-\frac{2m+\alpha}{p-1}} \) and thus \( v(y) \leq C \) for \( y \in B_2 \setminus B_{1/10} \). For \( |x| = 1 \), define

\[
g_x(y) = \int_{B_2 \setminus B_{1/10}(x)} \frac{|z|^\alpha v^p(z)}{|y-z|^{2m-2m}} dz.
\]

Then for any \( y_1, y_2 \in B_{1/2}(x) \), we have

\[
g_x(y_1) = \int_{B_2 \setminus B_{1/10}(x)} \frac{|y_2 - z|^{n-2m}}{|y_1 - z|^{n-2m}} \frac{|z|^\alpha v^p(z)}{|y_2 - z|^{n-2m}} dz \leq C_{n,m} \int_{B_2 \setminus B_{1/10}(x)} \frac{|z|^\alpha v^p(z)}{|y_2 - z|^{n-2m}} dz = C_{n,m} g_x(y_2).
\]

Hence, \( g \) satisfies the Harnack inequality in \( B_{1/2}(x) \). By (4.3) we have that \( h \) also satisfies the Harnack inequality in \( B_{1/2}(x) \). Note that

\[
v(y) = \int_{B_{1/10}(x)} \frac{|z|^\alpha v^p(z)}{|y-z|^{n-2m}} dz + g_x(y) + h_r(y) \quad \text{in} \quad B_{1/2}(x).
\]

It follows from Proposition 4.1 that

\[
\sup_{B_{1/2}(x)} v \leq C \inf_{B_{1/2}(x)} v.
\]

A standard covering argument leads to

\[
\sup_{1/2 \leq |y| \leq 3/2} v(y) \leq C \inf_{1/2 \leq |y| \leq 3/2} v(y).
\]

By rescaling back to \( u \), we obtain

\[
\sup_{r/2 \leq |x| \leq 3r/2} u(x) \leq C \inf_{r/2 \leq |x| \leq 3r/2} u(x).
\]

This completes the proof. \(\square\)

Another important tool for proving Theorem 1.3 is a monotonicity formula, which will also be used in the proof of Theorem 1.4. Let \( u \) be a nonnegative solution of equation (1.1) with \( m = 2 \). Using the Emden-Fowler transformation to the equation (1.1), i.e., setting \( w(t, \theta) = |x|^{\frac{m+\alpha}{p-1}} u(|x|, \theta) \) with \( t = \ln |x| \) and \( \theta = \frac{x}{|x|} \), then the function \( w(t, \theta) \) satisfies

\[
\partial_t^4 w + A_3 \partial_t^3 w + A_2 \partial_t w + A_1 \partial w + A_0 w + \Delta^2 \theta w + A_4 \Delta^2 \theta_\theta w + 2 \partial_t \Delta_\theta w \Delta^2 w = w^p \quad \text{in} \ (-\infty, 0) \times S^{n-1},
\]

where \( \Delta_\theta \) is the Beltrami-Laplace operator on \( S^{n-1} \) and the coefficients \( A_i \) (\( i = 0, 1, \ldots, 4 \)) are given by

\[
A_0 = B^4 - 2(n - 4)B^3 + (n^2 - 10n + 20)B^2 + 2(n - 2)(n - 4)B,
A_1 = -4B^3 + 6(n - 4)B^2 - 2(n^2 - 10n + 20)B - 2(n - 2)(n - 4),
A_2 = 6B^2 - 6(n - 4)B + n^2 - 10n + 20,
A_3 = -4B + 2n - 8,
A_4 = 2B^2 - 2(n - 4)B - 2(n - 4),
\]
with
\[ B := \frac{4 + \alpha}{p - 1}. \]

Furthermore, we get that the coefficients \( A_0, A_1 \) and \( A_3 \) have the following properties.

**Lemma 4.1.** Let \( n \geq 5 \) and \( \alpha > -4 \).

(i) For \( \frac{n + \alpha}{n - 4} < p < \frac{n + 4 + 2\alpha}{n - 4} \), there holds
\[ A_0 > 0, \quad A_1 > 0, \quad A_3 < 0. \]

(ii) For \( p > \frac{n + 4 + 2\alpha}{n - 4} \), there holds
\[ A_0 > 0, \quad A_1 < 0, \quad A_3 > 0. \]

**Remark 4.1.** Note that for \( p = \frac{n + 4 + 2\alpha}{n - 4} \), equation (1.1) is critical and conformally invariant. At this moment, we can see that \( A_1 = A_3 = 0 \).

**Proof.** Direct calculations give that
\[ B \in \begin{cases} \left(\frac{n - 4}{2}, n - 4\right) & \text{for } \frac{n + \alpha}{n - 4} < p < \frac{n + 4 + 2\alpha}{n - 4}, \\ (0, \frac{n - 4}{2}) & \text{for } \frac{n + 4 + 2\alpha}{n - 4} < p. \end{cases} \]

Let
\[ f(s) = -4s^2 + 2n - 8. \]

Then for \( s > \frac{n - 4}{2} \) (resp. \( < \)), we have \( f(s) < 0 \) (resp. \( > \)). This implies that \( A_3 < 0 \) (resp. \( > \)) for \( \frac{n + \alpha}{n - 4} < p < \frac{n + 4 + 2\alpha}{n - 4} \) (resp. \( \frac{n + 4 + 2\alpha}{n - 4} < p \)).

To judge the sign of \( A_1 \), consider the function
\[ g(s) = -4s^3 + 6(n - 4)s^2 - 2(n^2 - 10n + 20)s - 2(n - 2)(n - 4). \]

Then
\[ g'(s) = -12s^2 + 12(n - 4)s - 2(n^2 - 10n + 20). \]

Two roots of \( g'(s) = 0 \) are given by \( s_1 = \frac{3(n - 4) + \sqrt{3(n - 2)^2 + 12}}{6} \) and \( s_2 = \frac{3(n - 4) - \sqrt{3(n - 2)^2 + 12}}{6} \). When \( n \geq 8 \), we have \( n^2 - 10n + 20 > 0 \) and this yields
\[ \frac{n - 4}{2} < s_1 < n - 4, \quad 0 < s_2 < \frac{n - 4}{2}. \]

When \( 5 \leq n \leq 7 \), we have \( n^2 - 10n + 20 < 0 \) and hence
\[ s_1 > n - 4, \quad s_2 < 0. \]

These imply that
\[ g(s) > \min \{ g\left(\frac{n - 4}{2}\right), g(n - 4)\} \quad \forall s \in \left(\frac{n - 4}{2}, n - 4\right). \]
and
\[ g(s) < \max\{g(0), g\left(\frac{n-4}{2}\right)\} \quad \forall s \in \left(0, \frac{n-4}{2}\right). \]

On the other hand, direct calculations give
\[ g(0) < 0, \quad g\left(\frac{n-4}{2}\right) = 0 \quad \text{and} \quad g(n-4) > 0. \]

Hence, we obtain that \( A_1 > 0 \) for \( \frac{n-4}{2} < B < n-4 \) and \( A_1 < 0 \) for \( 0 < B < \frac{n-4}{2} \).

Finally, let’s determine the sign of \( A_0 \). Since
\[ A_0 = B^4 - 2(n-4)B^3 + (n^2 - 10n + 20)B^2 + 2(n-2)(n-4)B \]
\[ = B[B^3 - 2(n-4)B^2 + (n^2 - 10n + 20)B + 2(n-2)(n-4)], \]
we consider the function
\[ h(s) = s^3 - 2(n-4)s^2 + (n^2 - 10n + 20)s + 2(n-2)(n-4) \]
for \( 0 < s < n-4 \). Notice that
\[ h'(s) = 3s^2 - 4(n-4)s + (n^2 - 10n + 20) \]
and \( h'(n-4) = 4 - 2n < 0 \). Then we have
\[ h(s) > \min\{h(0), h(n-4)\} \quad \forall s \in (0, n-4). \]

It is easy to check that
\[ h(0) > 0 \quad \text{and} \quad h(n-4) = 0. \]

This implies \( A_0 > 0 \) for all \( p > \frac{2+\alpha}{n-4}. \)

Next, we introduce an important energy function \( E(t, w) \) defined by
\[
E(t; w) := \int_{S^{n-1}} \partial_t^{(3)} w \partial_t w d\sigma - \frac{1}{2} \int_{S^{n-1}} \left[ (\partial_t w)^2 - 2A_3 \partial_t^2 w \partial_t w - A_2 (\partial_t w)^2 \right] d\sigma \\
+ \frac{1}{2} \int_{S^{n-1}} \left[ |\Delta_{\theta} w|^2 - A_4 |\nabla_{\theta} w|^2 \right] d\sigma + \frac{A_0}{2} \int_{S^{n-1}} w^2 d\sigma - \frac{1}{p+1} \int_{S^{n-1}} w^{p+1} d\sigma \\
- \int_{S^{n-1}} |\partial_t \nabla_{\theta} w|^2 d\sigma. \tag{4.5}
\]

Combining the equation (4.4) and the singularity estimate in Theorem 1.1, we obtain the following monotonicity formula for \( E(t; w) \) and the uniform boundedness for \( w \). The proofs are essentially the same as those in [25], so we omit them here.

**Proposition 4.3.** Suppose that \( n \geq 5, \alpha > -4 \) and \( w \) is a nonnegative \( C^1 \) solution of (4.4). Then \( E(t; w) \) is non-increasing (resp. non-decreasing) in \( t \in (-\infty, 0) \) for \( \frac{n+\alpha}{n-4} < p \leq \frac{n+4+2\alpha}{n-4} \) (resp. \( p > \frac{n+4+2\alpha}{n-4} \)). Furthermore, we have
\[
\frac{d}{dt} E(t; w) = A_3 \int_{S^{n-1}} \left[ (\partial_t w)^2 + |\partial_t \nabla_{\theta} w|^2 \right] d\sigma - A_1 \int_{S^{n-1}} (\partial_t w)^2 d\sigma. \tag{4.6}
\]
Proposition 4.4. Let \( w \) be a nonnegative \( C^4 \) solution of (4.4) with \( 1 < p < \frac{n+4}{n-4} \). Then \( w, \partial_tw, \partial_{tt}w, \partial_{ttt}w, \Delta_gw \) and \( |\nabla_gw| \) are uniformly bounded in \( (-\infty, -\ln 2) \times S^{n-1} \).

From the above two propositions, we know that \( \lim_{t \to -\infty} E(t; w) \) exists for \( \frac{n+\alpha}{n-4} < p < \frac{n+4}{n-4} \) and \(-4 < \alpha < 4\). Define
\[
\tilde{E}(r; u) := E(t; w)
\]
with \( t = \ln r \). Then we get
\[
\tilde{E}(0; u) := \lim_{r \to 0^+} \tilde{E}(r; u) = \lim_{t \to -\infty} E(t; w).
\]
For any \( \lambda > 0 \), define \( u^\lambda(x) = \lambda^{\frac{4+\alpha}{p-1}} u(\lambda x) \). Then the function \( u^\lambda(x) \) satisfies
\[
\Delta^2 u^\lambda(x) = |x|^\alpha (u^\lambda(x))^p \quad \text{in } B_{1/\lambda}(0) \setminus \{0\},
\]
and
\[
\tilde{E}(r; u^\lambda) = \tilde{E}(r; \lambda).
\]

Now, using the Harnack inequality in Proposition 4.2 and the monotonicity formula in Proposition 4.3, we show the following result which is important in establishing the lower bound of singular solutions.

Proposition 4.5. Let \( m = 2, -4 < \alpha < 4 \) and \( \frac{n+\alpha}{n-4} < p < \frac{n+4}{n-4} \). Suppose that \( u \in C^4(B_1 \setminus \{0\}) \) is a positive solution to (1.1) with a non-removable singularity at the origin. If
\[
\liminf_{|x| \to 0} |x|^\frac{4+\alpha}{p-1} u(x) = 0,
\]
then
\[
\lim_{|x| \to 0} |x|^\frac{4+\alpha}{p-1} u(x) = 0.
\]

Proof. We show the proof by discussing two cases separately: \( p \neq \frac{n+4+2\alpha}{n-4} \) and \( p = \frac{n+4+2\alpha}{n-4} \).

Case 1: For \( \frac{n+\alpha}{n-4} < p < \frac{n+4}{n-4} \) and \( p \neq \frac{n+4+2\alpha}{n-4} \). In this case, we need to use some arguments in Yang-Zou [27] and divide the proof into three steps.

Step 1. If \( \liminf_{|x| \to 0} |x|^\frac{4+\alpha}{p-1} u(x) = 0 \), then there holds \( \lim_{r \to 0^+} \tilde{E}(r; u) = 0 \). Since \( \liminf_{|x| \to 0} |x|^\frac{4+\alpha}{p-1} u(x) = 0 \), there exists a sequence of points \( \{x_i\} \) such that \( x_i \to 0 \) and
\[
|x_i|^\frac{4+\alpha}{p-1} u(x_i) \to 0 \quad \text{as } i \to \infty.
\]
Let \( r_i := |x_i| \). Using the Harnack inequality in Proposition 4.2, we have
\[
r_i^\frac{4+\alpha}{p-1} u(r_i e_1) \to 0 \quad \text{as } i \to \infty,
\]
where \( e_1 = (1, 0, \cdots, 0) \in \mathbb{R}^n \). Define \( v_i(x) = r_i^\frac{4+\alpha}{p-1} u(r_i x) \), then the function \( v_i(x) \) satisfies
\[
\Delta^2 v_i(x) = |x|^\alpha v_i^p(x) \quad x \in B_{2/r_i} \setminus \{0\}.
\]
It follows from Theorem 1.1 and Proposition 4.2 that \( v_i \) is locally uniformly bounded away from the origin. By elliptic estimates and Theorem 1.2 there exists \( v \in C^4(\mathbb{R}^n \setminus \{0\}) \) such that \( v_i \to v \) in \( C^4_{\text{loc}}(\mathbb{R}^n \setminus \{0\}) \) as \( i \to \infty \), and \( v \) satisfies

\[
\begin{cases}
\Delta^2 v = |x|^p v, & \text{in } \mathbb{R}^n \setminus \{0\}, \\
\Delta v \geq 0 & \text{in } \mathbb{R}^n \setminus \{0\}.
\end{cases}
\]

By (4.8), we have \( v(e_1) = 0 \). The maximum principle implies that \( v \equiv 0 \) in \( \mathbb{R}^n \setminus \{0\} \). On the other hand, since \( \bar{E}(r; u) \) is invariant under the scaling, we get

\[
\lim_{i \to \infty} \bar{E}(r_i; u) = \lim_{i \to \infty} \bar{E}(1; v_i) = \bar{E}(1; v) = 0.
\]

By the monotonicity in Proposition 4.3, we obtain

\[
\lim_{r \to 0^+} \bar{E}(r; u) = 0.
\]

**Step 2.** Let \( v \in C^4(\mathbb{R}^n \setminus \{0\}) \) be a nonnegative solution of \( \Delta^2 v = |x|^p v \) in \( \mathbb{R}^n \setminus \{0\} \). If \( \bar{E}(r; v) = 0 \) for \( r \in (0, +\infty) \), then \( v \equiv 0 \) in \( \mathbb{R}^n \setminus \{0\} \).

By the definition of \( \bar{E}(r; u) \) in (4.7), we have \( \bar{E}(t; w) = 0 \) for \( t \in (-\infty, \infty) \), where \( w(t, \theta) = |x|^{\frac{4}{p-1}} v(|x|, \theta) \) with \( t = \ln |x| \). This implies that \( \frac{d}{dt} \bar{E}(t; w) = 0 \). It follows from Proposition 4.3 and Lemma 4.1 that \( \partial_t w = 0 \) due to \( p \neq \frac{4+4+\alpha}{n-4} \). Therefore (4.4) can be reduced to

\[
A_0 w + \Delta^2 \theta w + A_4 \Delta \theta w = w^p.
\]

Integrating on \( S^{n-1} \), we get

\[
\int_{S^{n-1}} A_0 w^2 + |\Delta \theta w|^2 - A_4 |\nabla \theta w|^2 dS = \int_{S^{n-1}} w^{p+1} dS.
\]

On the other hand, \( E(t; w) = 0 \) implies

\[
\frac{1}{2} \int_{S^{n-1}} |\Delta \theta w|^2 - A_4 |\nabla \theta w|^2 dS + \frac{A_0}{2} \int_{S^{n-1}} w^2 dS - \frac{1}{p+1} \int_{S^{n-1}} w^{p+1} dS = 0.
\]

Therefore, we obtain

\[
(1 - \frac{2}{p+1}) \int_{S^{n-1}} w^{p+1} dS = 0.
\]

This means that \( w \equiv 0 \) on \( S^{n-1} \) and hence \( v \equiv 0 \) in \( \mathbb{R}^n \setminus \{0\} \).

**Step 3.** Define \( u^\lambda(x) = \lambda^{\frac{4}{p-1}} u(\lambda x) \) with \( \lambda > 0 \) small. It follows from Theorem 1.1 and Proposition 4.2 that \( u^\lambda \) is locally uniformly bounded away from the origin. Hence, there is a subsequence \( \{\lambda_i\} \) such that \( \lambda_i \to 0 \) and \( u^\lambda_i \to u^0 \in C^4_{\text{loc}}(\mathbb{R}^n \setminus \{0\}) \) as \( i \to \infty \). Moreover, \( u^0 \) is nonnegative and satisfies

\[
\Delta^2 u^0 = |x|^p u^0, \quad x \in \mathbb{R}^n \setminus \{0\}.
\]

The scaling invariance of \( \bar{E}(r; u) \) leads to

\[
\bar{E}(r; u^0) = \lim_{i \to \infty} \bar{E}(r; u^\lambda_i) = \lim_{i \to \infty} \bar{E}(r; u^\lambda_i = \lim_{r \to 0} \bar{E}(r; u) = 0 \quad \text{for all } 0 < r < \infty.
\]
By the conclusion in Step 2, we obtain \( u^0 \equiv 0 \). In particular,
\[
\lim_{\lambda \to 0} \lambda^{\frac{4+\alpha}{p-1}} u(\lambda x) = 0
\]
uniformly for \( x \in \partial B_1 \), which immediately implies
\[
\lim_{|x| \to 0} |x|^{\frac{4+\alpha}{p-1}} u(x) = 0.
\]

**Case 2:** For \( p = \frac{n+4+2\alpha}{n-4} \). Assume by contradiction that \( \limsup_{|x| \to 0} |x|^{\frac{4+\alpha}{p-1}} u(x) = C > 0 \). Then there exists a sequence of points \( \{x_i\} \) such that \( s_i := |x_i| \to 0 \) and \( u(x_i) \to \infty \) as \( i \to \infty \). By the Harnack inequality in Proposition 4.2, we have
\[
\inf_{\partial B_{s_i}} u \geq C^{-1} u(x_i) \to \infty.
\]

It follows from Theorem 1.2 that \(-\Delta u \geq 0 \) in \( B_{\tau} \setminus \{0\} \) for some \( \tau > 0 \). Using the maximum principle, we get
\[
\inf_{B_{s_i} \setminus B_{s_{i+1}}} u \geq \min \left\{ \inf_{\partial B_{s_i}} u, \inf_{\partial B_{s_{i+1}}} u \right\} \to \infty,
\]
and hence
\[
\liminf_{|x| \to 0} u(x) = \infty.
\]

Moreover, from the assumptions and Proposition 4.2, we can find \( r_i \to 0 \) such that \( r_i^{\frac{4+\alpha}{p-1}} \varphi(r_i) \to 0 \) as \( i \to \infty \), and \( r_i \) is a local minimum point of \( r_i^{\frac{4+\alpha}{p-1}} \varphi(r) \) with \( \varphi(r) = \frac{1}{|\partial B_r|} \int_{\partial B_r} u \, dS \). As in the proof of Proposition 4.2, we may suppose that \( u \) satisfies the integral equation
\[
u(x) = \int_{B_2} \frac{|y|^\alpha u^p(y)}{|x - y|^{n-4}} \, dy + h(x) \quad \text{for} \ x \in B_2 \setminus \{0\},
\]
where \( u \in C^4(B_2 \setminus \{0\}) \), \( |\cdot|^\alpha u^p \in L^1(B_2) \) and \( h \in C^\infty(B_2) \) is a positive function. Now we use some arguments from [11, Proposition 4.5]. Define
\[
\varphi_i(y) = \frac{u(r_i y)}{u(r_i e_1)}
\]
where \( e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^n \). Then \( \varphi_i \) satisfies
\[
\varphi_i(y) = \int_{B_{r_i} / r_i} \left( r_i^{\frac{4+\alpha}{p-1}} u(r_i e_1) \right)^{p-1} \left| z \right|^{\alpha} \varphi_i^p(z) \, dz \, \phi_i(y), \quad y \in B_{r_i} / r_i \setminus \{0\}, \tag{4.9}
\]
where \( \phi_i(y) = u(r_i e_1)^{-1} h(r_i y) \to 0 \) in \( C^4_{\text{loc}}(\mathbb{R}^n) \) thanks to \( u(r_i e_1) \to \infty \) as \( i \to \infty \).

It follows from the Harnack inequality in Proposition 4.2 that \( r_i^{\frac{4+\alpha}{p-1}} u(r_i e_1) \to 0 \) and \( \varphi_i \) is locally uniformly bounded in \( B_{r_i} \setminus \{0\} \). Hence
\[
\left( r_i^{\frac{4+\alpha}{p-1}} u(r_i e_1) \right)^{p-1} \left| z \right|^{\alpha} \varphi_i^p(z) \to 0 \quad \text{in} \ C_{\text{loc}}(\mathbb{R}^n \setminus \{0\}). \tag{4.10}
\]
For any $t > 1$, $0 < |y| < t$ and $0 < \varepsilon < \frac{|y|}{100}$, we have, after passing to a subsequence,

$$
\lim_{i \to \infty} \int_{B_t} \frac{\left( r_i^{p-1} u(r_i e_1) \right)^{p-1} |z|^{\alpha} \varphi_i^p(z)}{|y - z|^{n-4}} \, dz
= \lim_{i \to \infty} \int_{B_\varepsilon} \frac{\left( r_i^{p-1} u(r_i e_1) \right)^{p-1} |z|^{\alpha} \varphi_i^p(z)}{|y - z|^{n-4}} \, dz
= |y|^{4-n} (1 + O(\varepsilon)) \lim_{i \to \infty} \int_{B_\varepsilon} \left( r_i^{p-1} u(r_i e_1) \right)^{p-1} |z|^{\alpha} \varphi_i^p(z) \, dz.
$$

Sending $\varepsilon \to 0$, we obtain

$$
\lim_{i \to \infty} \int_{B_t} \frac{\left( r_i^{p-1} u(r_i e_1) \right)^{p-1} |z|^{\alpha} \varphi_i^p(z)}{|y - z|^{n-4}} \, dz = \frac{a}{|y|^{n-4}}
$$

for some constant $a \geq 0$. On the other hand, by the equation (4.9) we have

$$
\lim_{i \to \infty} \int_{B_{2/r_i} \setminus B_t} \frac{\left( r_i^{p-1} u(r_i e_1) \right)^{p-1} |z|^{\alpha} \varphi_i^p(z)}{|y - z|^{n-4}} \, dz \to f(y) \geq 0 \quad \text{in } C^4_{loc}(B_t)
$$

for some function $f \in C^4(B_t)$. We claim that $f$ is a constant function in $B_t$. Indeed, for any fixed large $R > 0$ and $y \in B_t$, it follows from (4.10) that

$$
\lim_{i \to \infty} \int_{t \leq |z| \leq R} \frac{\left( r_i^{p-1} u(r_i e_1) \right)^{p-1} |z|^{\alpha} \varphi_i^p(z)}{|y - z|^{n-4}} \, dz \to 0 \quad \text{as } i \to \infty.
$$

For any $y', y'' \in B_t$, we have

$$
\int_{B_{2/r_i} \setminus B_R} \frac{\left( r_i^{p-1} u(r_i e_1) \right)^{p-1} |z|^{\alpha} \varphi_i^p(z)}{|y' - z|^{n-4}} \, dz
\leq \left( \frac{R + t}{R - t} \right)^{n-4} \int_{B_{2/r_i} \setminus B_R} \frac{\left( r_i^{p-1} u(r_i e_1) \right)^{p-1} |z|^{\alpha} \varphi_i^p(z)}{|y'' - z|^{n-4}} \, dz.
$$

Therefore,

$$
f(y') \leq \left( \frac{R + t}{R - t} \right)^{n-4} f(y'').
$$

By sending $R \to \infty$, we get

$$
f(y) = f(0) \quad \text{for all } y \in B_t.
$$
Note that \( \varphi_i \) is locally uniformly bounded in \( C^5(B_{2/r_i} \setminus \{0\}) \). Thus, up to a subsequence, we have

\[
\varphi_i(y) \to \varphi(y) := \frac{a}{|y|^{n-2}} + f(0) \quad \text{in } C^4_{loc}(\mathbb{R}^n \setminus \{0\}) \quad \text{as } i \to \infty.
\]

Since \( \varphi_i(e_1) = 1 \) and \( \frac{d}{dr}(r^{n-1} \varphi_i(r)) \big|_{r=1} = 0 \), we have \( \varphi(e_1) = 1 \) and

\[
\frac{d}{dr}(r^{n-1} \varphi(r)) \big|_{r=1} = 0.
\]

These imply that

\[
a = \frac{4 + \alpha}{(p - 1)(n - 4)} = \frac{1}{2} \quad \text{and} \quad f(0) = 1 - a = \frac{1}{2}.
\]

Since \( |\nabla^k \varphi_i| \leq C \) near \( \partial B_1 \) and \( r_i^{\frac{4 + \alpha}{p - 1}} u(r_i e_1) = o(1) \), we obtain

\[
|\nabla^k u(x)| \leq C r_i^{-k} u(r_i e_1) = o(1) r_i^{-\frac{4 + \alpha}{p - 1} - k} \quad \text{for all } |x| = r_i, \ k = 0, 1, 2, 3.
\]

Set \( w(t, \theta) = |x|^{\frac{4 + \alpha}{p - 1}} u(|x|, \theta) \) with \( t = \ln |x| \) and \( \theta = \frac{x}{|x|} \). Then the function \( w \) satisfies (4.4) and

\[
w, w_t, w_{tt}, w_{ttt}, |\nabla \theta w|, \Delta_\theta w = o(1) \quad \text{for } t_i = \ln r_i.
\]

Let \( E(t; w) \) be defined as in (4.5). Then

\[
\lim_{t \to \infty} E(t_i; w) = 0.
\]

Using Proposition 4.3, we see that \( \frac{d}{dt} E(t; w) = 0 \) due to \( A_1 = A_3 = 0 \) in the case of \( p = \frac{n+4+2\alpha}{n-4} \). Hence

\[
E(t; w) = 0 \quad \text{for all } t \in (-\infty, \infty).
\]

Set \( \bar{\varphi}_i(t, \theta) := |x|^{\frac{4 + \alpha}{p - 1}} \varphi_i(|x|, \theta) \) with \( t = \ln |x| \). Then we have

\[
\bar{\varphi}_i(t, \theta) \to \bar{\varphi}(t, \theta) := \frac{1}{2} \left[ e^{(\frac{4 + \alpha}{p - 1} + n - 1)t} + e^{\frac{4 + \alpha}{p - 1}t} \right]
\]

and

\[
r_i^{\frac{4 + \alpha}{p - 1}} u(r_i e_1) \bar{\varphi}_i(t, \theta) = w(t + \ln r_i, \theta).
\]

Therefore,

\[
0 = E(t; w) = E(t + \ln r_i; w) = E(t; w(t + \ln r_i, \theta)) = E(t; r_i^{\frac{4 + \alpha}{p - 1}} u(r_i e_1) \bar{\varphi}_i(t, \theta)),
\]

that is,

\[
0 = \int_{S^{n-1}} (\bar{\varphi}_i(t, \theta))_{tt} (\bar{\varphi}_i(t, \theta))_t dS - \frac{1}{2} \int_{S^{n-1}} (1 - A_2) (\bar{\varphi}_i(t, \theta))_t^2 dS + \frac{A_0}{2} \int_{S^{n-1}} \bar{\varphi}_i^2(t, \theta) dS
+ \int_{S^{n-1}} A_3 (\bar{\varphi}_i(t, \theta))_{tt} (\bar{\varphi}_i(t, \theta))_t dS + \frac{1}{2} \int_{S^{n-1}} |\nabla_\theta \bar{\varphi}_i(t, \theta)|^2 - A_4 |\nabla_\theta \bar{\varphi}_i(t, \theta)|^2 dS
- \frac{1}{p + 1} \int_{S^{n-1}} \left( r_i^{\frac{4 + \alpha}{p - 1}} u(r_i e_1) \right)^{p-1} (\bar{\varphi}_i(t, \theta))^{p+1} dS - \int_{S^{n-1}} |\partial_\theta \nabla_\theta \bar{\varphi}_i(t, \theta)|^2 dS.
\]

19
Letting $i \to \infty$, we get
\[ 0 = \int_{S^{n-1}} \tilde{\varphi}_{tt} \tilde{\varphi}_t dS - \frac{1}{2} \int_{S^{n-1}} (1 - A_2) [\tilde{\varphi}_t]^2 dS + \int_{S^{n-1}} A_3 \tilde{\varphi}_t \tilde{\varphi}_t dS + \frac{A_0}{2} \int_{S^{n-1}} \tilde{\varphi}^2 dS. \]
Taking $t = 0$, we have
\[ \tilde{\varphi}_t|_{t=0} = \frac{1}{2} \left[ \left( \frac{4 + \alpha}{p - 1} + 4 - n \right) + \frac{4 + \alpha}{p - 1} \right] = 0 \]
and
\[ \tilde{\varphi}_{tt}|_{t=0} = \frac{1}{2} \left[ \left( \frac{4 + \alpha}{p - 1} + 4 - n \right)^3 + \left( \frac{4 + \alpha}{p - 1} \right)^3 \right] = 0. \]
Hence, we get
\[ 0 = \left[ \int_{S^{n-1}} \tilde{\varphi}_{tt} \tilde{\varphi}_t dS - \frac{1}{2} \int_{S^{n-1}} (1 - A_2) [\tilde{\varphi}_t]^2 dS + \int_{S^{n-1}} A_3 \tilde{\varphi}_t \tilde{\varphi}_t dS + \frac{A_0}{2} \int_{S^{n-1}} \tilde{\varphi}^2 dS \right] |_{t=0} = \frac{A_0}{2} |S^{n-1}| > 0. \]
This is a contradiction. The proof of Proposition 4.5 is completed.

To complete the proof of Theorem 1.3, we also need the following sufficient condition for removability of isolated singularities. This will also be used in the proof of Theorem 1.4.

**Proposition 4.6.** Let $m = 2$, $-4 < \alpha < 4$ and \( \frac{n+\alpha}{n-4} \) < \( \frac{n+4}{n-4} \). Suppose that \( u \in C^4(B_1 \setminus \{0\}) \) is a positive solution to (1.1). If
\[ \lim_{|x| \to 0} |x|^{\frac{4 + \alpha}{p - 1}} u(x) = 0, \quad (4.11) \]
then the singularity at $x = 0$ is removable, i.e., $u$ can be extended as a continuous function near the origin 0.

**Proof.** First, we claim that there holds
\[ \int_{B_{1/2}} u(x) |x|^{-n + \frac{4 + \alpha}{p - 1}} dx < \infty. \quad (4.12) \]
Set $\psi(x) = |x|^{-\gamma}$ with $\gamma = n - 4 - \frac{4 + \alpha}{p - 1} > 0$. Then
\[ \Delta^2 \psi(r) = \gamma(\gamma + 2)(\gamma - n + 2)(\gamma - n + 4)r^{-\gamma - 4}. \]
Denote
\[ \Lambda := \gamma(\gamma + 2)(\gamma - n + 2)(\gamma - n + 4). \]
Then we have $\Lambda > 0$ and
\[ \frac{\Delta^2 \psi}{\psi} = \Lambda |x|^{-4} \quad \text{in} \quad \mathbb{R}^n \setminus \{0\}. \]
For any small \( \varepsilon > 0 \), we define a smooth cut-off function as follows:

\[
\zeta_\varepsilon(x) = \begin{cases} 
1, & \varepsilon \leq |x| \leq \frac{1}{2}, \\
0, & |x| \leq \frac{\varepsilon}{2} \text{ or } |x| \geq \frac{3}{4}, 
\end{cases}
\]

\[
|\nabla^k \zeta_\varepsilon(x)| \leq C \varepsilon^{-k} \quad \text{for } \frac{\varepsilon}{2} \leq |x| \leq \varepsilon,
\]

and

\[
|\nabla^k \zeta_\varepsilon(x)| \leq C \left( \frac{1}{4} \right)^{-k} \quad \text{for } \frac{1}{2} \leq |x| \leq \frac{3}{4},
\]

where \( C > 0 \) is independent of \( \varepsilon \). Taking \( \zeta_\varepsilon(x) \psi(x) \) as a test function in (1.1), we have

\[
\int_{B_1} u \zeta_\varepsilon \psi \left( \Delta^2 \psi - |x|^\alpha u^{p-1} \right) \, dx = -\int_{B_1} u H(\zeta_\varepsilon, \psi) \, dx,
\]

where

\[
H(\zeta_\varepsilon, \psi) = 4 \nabla \zeta_\varepsilon \cdot \nabla \Delta \psi + 2 \Delta \zeta_\varepsilon \Delta \psi + 4 \sum_{i,j=1}^n (\zeta_\varepsilon)_{x_i x_j} \psi_{x_i x_j}
\]

\[
+ 4 \nabla \Delta \zeta_\varepsilon \cdot \nabla \psi + \psi \Delta^2 \zeta_\varepsilon.
\]

From Theorem 1.1, (4.13) and (4.14), we get that

\[
\left| \int_{B_1} u H(\zeta_\varepsilon, \psi) \, dx \right| \leq \int_{\left\{ \frac{1}{2} \leq |x| \leq \frac{3}{4} \right\}} u H(\zeta_\varepsilon, \psi) \, dx + \int_{\left\{ \frac{1}{2} \leq |x| \leq \frac{3}{4} \right\}} u H(\zeta_\varepsilon, \psi) \, dx
\]

\[
\leq C_1 + C_2 \varepsilon^{-\gamma -4} \varepsilon^\frac{-4 \alpha}{n} \leq C < \infty,
\]

where \( C > 0 \) is independent of \( \varepsilon \).

On the other hand, the assumption (4.11) implies that

\[
u^{p-1}(x) = o(|x|^{-(4 + \alpha)}) \quad \text{as } |x| \to 0.
\]

Therefore, we have

\[
\int_{B_1} u \zeta_\varepsilon |x|^{-\gamma -4} \, dx \leq C < \infty.
\]

Sending \( \varepsilon \to 0 \), the claim is proved.

Now, we consider the two cases \( -4 \leq \alpha \leq 0 \) and \( 0 \leq \alpha < 4 \) separately.

**Case 1**: \( -4 \leq \alpha \leq 0 \). Note that \( \left( \frac{p-1}{4 + \alpha} \right) > 1 \) due to \( p > \frac{n+\alpha}{n-1} \) and \( \alpha > -4 \). By Theorem 1.1 and (4.12) we have

\[
\int_{B_{1/2}} u \frac{(p-1)n}{4 + \alpha} \, dx \leq C \int_{B_{1/2}} u |x|^{-\frac{4 + \alpha}{p-1} \left[ \frac{(p-1)n}{4 + \alpha} - 1 \right]} \, dx
\]

\[
= C \int_{B_{1/2}} u |x|^{-n + \frac{4 + \alpha}{p-1}} \, dx < \infty.
\]
Define the function $v(x)$ as follows

$$v(x) := -u(x) + \int_{B_{1/2}} G(x, y)|y|^\alpha u^p(y) dy, \quad x \in B_{1/2}, \quad (4.16)$$

where $G(x, y)$ is the Green’s function of $\Delta^2$ in $B_{1/2}$ with homogeneous Dirichlet boundary conditions. Then there exists a positive constant $C_n$ such that

$$0 < G(x, y) \leq \Gamma(x, y) := C_n|x - y|^{4-n} \quad \text{for } x, y \in B_{1/2}, \ x \neq y.$$

A similar argument as in [25, Lemma 3.1] shows that $v$ satisfies

$$\Delta^2 v = 0 \quad \text{in } B_{1/2}$$

in the distributional sense. According to the interior regularity theory, we know that $v \in C(B_{1/2}).$

Set $a(x) := u^{p-1}(x).$ Then by $(4.15)$ we have $a \in L^{n\alpha/(n-4)}(B_{1/2}).$ For any large number $L > 0,$ let

$$a_L(x) = \begin{cases} a(x) & \text{if } |a(x)| \geq L, \\ 0 & \text{otherwise}, \end{cases}$$

and

$$a_M(x) = a(x) - a_L(x).$$

Then $(4.16)$ can be rewritten as

$$u(x) = \int_{B_{1/2}} G(x, y)|y|^\alpha u^p(y) dy - v(x)$$

$$= \int_{B_{1/4}} G(x, y)|y|^\alpha a_L(y) u(y) dy + \int_{B_{1/4}} G(x, y)|y|^\alpha a_M(y) u(y) dy$$

$$+ \int_{B_{1/2}\setminus B_{1/4}} G(x, y)|y|^\alpha u^p dy - v(x)$$

$$=: T_L u(x) + F_L u(x) + H(x) - v(x),$$

where $T_L$ is a linear operator defined as

$$T_L : w \rightarrow \int_{B_{1/4}} G(x, y)|y|^\alpha a_L(y) w(y) dy.$$

Notice that

$$|H(x)| \leq C \int_{\{ \frac{1}{4} \leq |y| \leq \frac{1}{2} \}} G(x, y) dy \leq C \int_{B_1} |y|^{4-n} dy \leq C \quad \text{for all } x \in B_{1/4}.$$

Hence $v, H \in L^\infty(B_{1/4}).$

We shall prove that $x = 0$ is a removable singularity. By Proposition 2.5 it is sufficient to show that $T_L$ is a contracting operator from $L^q(B_{1/4})$ to $L^q(B_{1/4})$ for $L$ large and $F_L \in L^q(B_{1/4})$ for any $q \in (\frac{n}{n-4}, \infty).$
For any \( q \in \left( \frac{n}{n+\alpha}, \infty \right) \), there exists \( r \in \left( \frac{n}{n+\alpha}, \frac{n}{4+\alpha} \right) \) such that (noticing that \( \frac{n}{n+\alpha} \geq 1 \) due to \(-4 < \alpha \leq 0\))

\[
\frac{1}{q} = \frac{1}{r} - \frac{4 + \alpha}{n}.
\]

By the doubly weighted Hardy-Littlewood-Sobolev inequality (see Proposition 2.3) and Hölder inequality, we obtain

\[
\| T_L w \|_{L^q(B_{1/4})} \leq C \left\| \int_{B_{1/4}} \cdot - y \right\|_{L^q(B_{1/4})} \leq C \left\| a_L w \right\|_{L^r(B_{1/4})} \leq C \left\| a_L \right\|_{L^{\frac{n}{4+\alpha}}(B_{1/4})} \left\| w \right\|_{L^q(B_{1/4})},
\]

where we used the fact \( 0 \leq -\alpha < \frac{n}{r} = n(1 - \frac{1}{q}) \) because of \( q > \frac{n}{n-\alpha} \). Since \( a \in L^{\frac{n}{4+\alpha}}(B_{1/2}) \), we have for \( L > 0 \) large enough that

\[
C \left\| a_L \right\|_{L^{\frac{n}{4+\alpha}}(B_{1/4})} \leq \frac{1}{2}.
\]

Therefore, \( T_L : L^q(B_{1/4}) \rightarrow L^q(B_{1/4}) \) is a contracting operator for large \( L \).

On the other hand, by (4.15) we know

\[
u \in L^r(B_{1/4}) \quad \text{for any} \quad 1 < r \leq \frac{(p-1)n}{4 + \alpha}.
\]

Note that \( \frac{(p-1)n}{4+\alpha} > \frac{n}{n+\alpha} \) due to \( p > \frac{n+\alpha}{n-4} \) and \( \alpha > -4 \). Since \( a_M \) is a bounded function, by the doubly weighted Hardy-Littlewood-Sobolev inequality we have

\[
\| F_L u \|_{L^q(B_{1/4})} \leq \| a_M u \|_{L^r(B_{1/4})} \leq C \| u \|_{L^r(B_{1/4})},
\]

where \( r \), \( q \) satisfy (4.17) and \( \frac{n}{n+\alpha} < r \leq \frac{(p-1)n}{4 + \alpha} \). It is easy to see that

\[
q = \frac{(p-1)n}{(4 + \alpha)(2 - p)} \quad \text{if} \quad r = \frac{(p-1)n}{4 + \alpha}.
\]

This implies that \( F_L u \in L^q(B_{1/4}) \) for

\[
\begin{cases}
1 < q < \infty & \text{if} \ p \geq 2, \\
1 < q \leq \frac{(p-1)n}{(4+\alpha)(2-p)} & \text{if} \ p < 2.
\end{cases}
\]

Using the Regularity Lifting Theorem in Section 2, we obtain \( u \in L^q(B_{1/4}) \) for

\[
\begin{cases}
1 < q < \infty & \text{if} \ p \geq 2, \\
1 < q \leq \frac{(p-1)n}{(4+\alpha)(2-p)} & \text{if} \ p < 2.
\end{cases}
\]

If \( u \in L^{\frac{(p-1)n}{(4+\alpha)(2-p)}}(B_{1/4}) \) (for \( p < 2 \)), then we have \( F_L u \in L^q(B_{1/4}) \) for

\[
\begin{cases}
1 < q < \infty & \text{if} \ p \geq \frac{3}{2}, \\
1 < q \leq \frac{(p-1)n}{(4+\alpha)(3-2p)} & \text{if} \ p < \frac{3}{2}.
\end{cases}
\]
Using the Regularity Lifting Theorem again, we obtain \( u \in L^q(B_{1/4}) \) for

\[
\begin{align*}
1 < q < \infty & \quad \text{if } p \geq \frac{3}{2}, \\
1 < q \leq \frac{(p-1)n}{(4+\alpha)(3-2p)} & \quad \text{if } p < \frac{3}{2}.
\end{align*}
\]

Continuing this process, a finite number of iterations gives

\[ u \in L^q(B_{1/4}) \text{ for any } 1 < q < \infty. \]

Therefore, we have by Hölder inequality,

\[
\int_{B_{1/4}} G(x,y)|y|^\alpha u^p(y)dy \in L^\infty(B_{1/4}).
\]

This implies \( u \in L^\infty(B_{1/4}) \) by using the integral equation (4.16). Combined with the fact \(| \cdot |^\alpha \in L^{q_0}(B_{1/2})\) for some \( q_0 > n/4 \), we easily obtain that \( x = 0 \) is a removable singularity.

Next, we discuss the case \( 0 \leq \alpha < 4 \). Since the method is similar to the above, we only give the details that need to be changed.

**Case 2:** \( 0 \leq \alpha < 4 \). In this case, we have \((p-1)\frac{n}{4} - 1 > 0\). By Theorem 1.1 and (4.12) we obtain

\[
\int_{B_{1/2}} (|x|^\alpha u^{p-1})^\frac{n}{4} dx \leq C \int_{B_{1/2}} u|x|^{\frac{n}{4} + \frac{4+\alpha}{p-1}[(p-1)\frac{n}{4} - 1]} dx
\]

\[
= C \int_{B_{1/2}} u|x|^{-n+\frac{4+\alpha}{p-1}}dx < \infty. \quad (4.18)
\]

Set \( b(x) := |x|^\alpha u^{p-1}(x) \). Then by (4.18) we have \( b \in L^{\frac{n}{4}}(B_{1/2}) \). For any large number \( L > 0 \), let

\[
b_L(x) = \begin{cases} b(x) & \text{if } |b(x)| \geq L, \\ 0 & \text{otherwise}, \end{cases}
\]

and

\[ b_M(x) = b(x) - b_L(x). \]

Similar to Case 1, we can obtain the following integral equation

\[
u(x) = T^b_L u(x) + F^b_L u(x) + H(x) - v(x),
\]

where \( v, H \in L^\infty(B_{1/4}), T^b_L \) is a linear operator given by

\[
T^b_L w(x) = \int_{B_{1/4}} G(x,y)b_L(y)w(y)dy,
\]

and

\[
F^b_L u(x) = \int_{B_{1/4}} G(x,y)b_M(y)u(y)dy.
\]

For any \( q \in (\frac{n}{n-4}, \infty) \), there exists \( r \in (1, \frac{4}{n}) \) such that

\[
\frac{1}{q} = \frac{1}{r} - \frac{4}{n}. \quad (4.19)
\]
By Hardy-Littlewood-Sobolev inequality and Hölder inequality, we obtain that $T^b_L : L^q(B_{1/4}) \to L^q(B_{1/4})$ is a contracting operator for large $L$.

Moreover, it is easy to verify that (4.15) still holds in this case. Namely,

\[ u \in L^r(B_{1/4}) \quad \text{for any } 1 < r \leq \frac{(p - 1)n}{4 + \alpha}. \]

By the boundedness of $b_M$ and Hardy-Littlewood-Sobolev inequality, we have $F^b_L u \in L^q(B_{1/4})$ for

\[
\begin{cases}
1 < q < \infty & \text{if } p \geq (8 + \alpha)/4, \\
1 < q \leq \frac{(p-1)n}{8+\alpha-4p} & \text{if } p < (8 + \alpha)/4.
\end{cases}
\]

Using the Regularity Lifting Theorem, we obtain $u \in L^q(B_{1/4})$ for

\[
\begin{cases}
1 < q < \infty & \text{if } p \geq (8 + \alpha)/4, \\
1 < q \leq \frac{(p-1)n}{8+\alpha-4p} & \text{if } p < (8 + \alpha)/4.
\end{cases}
\]

Similar to Case 1, a finite number of iterations yields $u \in L^q(B_{1/4})$ for any $1 < q < \infty$.

The rest is the same as Case 1, so we omit the details. The proof of Proposition 4.6 is completed.

**Proof of Theorem 1.3** It follows from Theorem 1.1, Propositions 4.5 and 4.6.

**Proof of Theorem 1.4.** Our proof is based on the monotonicity formula in Proposition 4.3, the radial symmetry in Proposition 2.2 and the removable singularity in Proposition 4.6.

Let $u \in C^4(B_{1}(0) \setminus \{0\})$ be a nonnegative solution of (1.1) with $m = 2$. We claim that either

\[
\lim_{|x| \to 0} |x|^\frac{4+\alpha}{p-1} u(x) = 0
\]

or

\[
\lim_{|x| \to 0} |x|^\frac{4+\alpha}{p-1} u(x) = A_0^{\frac{1}{p-1}}.
\]

For $\lambda > 0$, define $u^\lambda(x) = \lambda^\frac{4+\alpha}{p-1} u(\lambda x)$. It follows from Theorem 1.1 that $u^\lambda$ is uniformly bounded in $C^{4,\gamma}(K)$ on every compact set $K \subset B_{1/2\lambda} \setminus \{0\}$ for some $0 < \gamma < 1$. Therefore, there exists a nonnegative function $u^0 \in C^4(\mathbb{R}^n \setminus \{0\})$ such that, after passing to a subsequence of $\lambda \to 0$,

\[ u^\lambda \to u^0 \quad \text{in } C^4_{\text{loc}}(\mathbb{R}^n \setminus \{0\}) \quad \text{as } \lambda \to 0, \]

and $u^0$ satisfies

\[ \Delta^2 u^0 = |x|^\alpha (u^0)^p \quad \text{in } \mathbb{R}^n \setminus \{0\}. \]

It follows from Proposition 2.2 that $u^0$ is radially symmetric with respect to the origin. Moreover, by the scaling invariance of $E$, we have that for any $r > 0$

\[ \tilde{E}(r; u^0) = \lim_{\lambda \to 0} \tilde{E}(r; u^\lambda) = \lim_{\lambda \to 0} \tilde{E}(r\lambda; u) = \tilde{E}(0; u). \]
This implies that $\widetilde{E}(r; u^0)$ is a constant. Let

$$w^0(t) = |x|^{\frac{4 + \alpha}{p-1}} u^0(|x|) \quad \text{with} \ t = \ln |x|.$$  

Then the function $w^0$ satisfies

$$\partial_t (4) w^0 + A_3 \partial_t (3) w^0 + A_2 \partial_{tt} w^0 + A_1 \partial_t w^0 + A_0 w^0 = (w^0)^p$$  

(4.22) and

$$E(t; w^0) = \widetilde{E}(r; u^0) \equiv \text{const.}$$

Thus, we get

$$0 = \frac{d}{dt} E(t; w^0) = |S^{n-1}| \left[ A_3 (\partial_{tt} w^0)^2 - A_1 (\partial_t w^0)^2 \right] .$$

Since $A_3 < 0$, $A_1 > 0$ (or $A_3 > 0$, $A_1 < 0$), we obtain that $w^0$ is a constant. Furthermore, by the equation (4.22) we have

either $w^0 \equiv 0$ or $w^0 = A_0^{\frac{1}{p-1}}$.

This, combined with the existence of $\lim_{r \to 0^+} \widetilde{E}(r; u)$, we obtain that either

$$\lim_{|x| \to 0} |x|^{\frac{4 + \alpha}{p-1}} u(x) = 0$$

or

$$\lim_{|x| \to 0} |x|^{\frac{4 + \alpha}{p-1}} u(x) = A_0^{\frac{1}{p-1}}.$$  

Together with Proposition 4.6, we proved Theorem 1.4.  

Acknowledgements. X. Huang is partially supported by NSFC (No. 12271164); the Shanghai Frontier Research Center of Modern Analysis. Huang and Li are also supported in part by Science and Technology Commission of Shanghai Municipality (No. 22DZ2229014).

References

[1] P. Aviles, Local behavior of solutions of some elliptic equations, Comm. Math. Phys. 108 (1987), no. 2, 177-192.

[2] M. Bidaut-Véron, L. Véron, Nonlinear elliptic equations on compact Riemannian manifolds and asymptotics of Emden equations, Invent. Math. 106 (1991), no. 3, 489-539.

[3] L. A. Caffarelli, B. Gidas, J. Spruck, Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth, Comm. Pure Appl. Math. 42 (1989), no. 3, 271-297.

[4] G. Caristi, E. Mitidieri, R. Soranzo, Isolated singularities of polyharmonic equations. Dedicated to Prof. C. Vinti (Italian) (Perugia, 1996), Atti Sem. Mat. Fis. Univ. Modena 46 (1998), suppl., 257-294.

[5] W.X. Chen, C.M. Li, Methods on nonlinear elliptic equations, AIMS Series on Differential Equations Dynamical Systems 4 (2010).
[6] W.X. Chen, C.M. Li, Super polyharmonic property of solutions for PDE systems and its applications, Commun. Pure Appl. Anal. 12 (2013), no. 6, 2497-2514.

[7] W.X. Chen, C.M. Li, B. Ou, Classification of solutions for an integral equation, Comm. Pure Appl. Math. 59 (2006), no. 3, 330-343.

[8] W. Dai, G. Qin, Liouville type theorems for fractional and higher order Hénon-Hardy type equations via the method of scaling spheres, arXiv:1810.02752v7.

[9] B. Gidas and J. Spruck. Global and local behavior of positive solutions of nonlinear elliptic equations, Comm. Pure Appl. Math. 34 (1981), no. 4, 525-598.

[10] T.L. Jin, Y.Y. Li, J.G. Xiong, The Nirenberg problem and its generalizations: a unified approach, Math. Ann. 369 (2017), no. 1-2, 109-151.

[11] T.L. Jin, J.G. Xiong, Asymptotic symmetry and local behavior of solutions of higher order conformally invariant equations with isolated singularities, Ann. Inst. H. Poincaré Anal. Non Linéaire. 38 (2021), no. 4, 1167-1216.

[12] N. Korevaar, R. Mazzeo, F. Pacard, R. Schoen, Refined asymptotics for constant scalar curvature metrics with isolated singularities, Invent. Math. 135 (1999), no. 2, 233-272.

[13] C.M. Li, Local asymptotic symmetry of singular solutions to nonlinear elliptic equations, Invent. Math. 123 (1996), 221-231.

[14] Y.Y. Li, Remark on some conformally invariant integral equations: the method of moving spheres, J. Eur. Math. Soc. 6 (2004), 153-180.

[15] Y.M. Li, The local behavior of positive solutions for higher order equation with isolated singularities, Calc. Var. Partial Differential Equations 60 (2021), no. 6, Paper No. 201, 19 pp.

[16] E.H. Lieb, Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities, Ann. of Math. (2) 118 (1983), no. 2, 349-374.

[17] C.-S. Lin, A classification of solutions of a conformally invariant fourth order equation in \( \mathbb{R}^n \), Comment. Math. Helv. 73 (1998), 206-231.

[18] P.L. Lions, Isolated singularities in semilinear problems, J. Differential Equations 38 (1980), no. 3, 441-450.

[19] Q.A. Ngô, D. Ye, Existence and non-existence results for the higher order Hardy-Hénon equations revisited, J. Math. Pures Appl. 163 (2022), 265-298.

[20] Q. Phan, P. Souplet, Liouville-type theorems and bounds of solutions of Hardy-Hénon equations, J. Differential Equations 252 (2012), no. 3, 2544-2562.

[21] P. Poláčik, P. Quittner and P. Souplet, Singularity and decay estimates in superlinear problems via Liouville-type theorems, I. Elliptic equations and systems, Duke Math. J. 139 (2007), no. 3, 555-579.

[22] E.M. Stein, G. Weiss, Fractional integrals on \( n \)-dimensional Euclidean space, J. Math. Mech. 7 (1958), 503-514.

[23] J.C. Wei, X.W. Xu, Classification of solutions of higher order conformally invariant equations, Math. Ann. 313 (1999), no. 2, 207-228.

[24] K. Wu, Local behavior of positive solutions to a nonlinear biharmonic equation near isolated singularities, Nonlinear Anal. 214 (2022), Paper No. 112594, 15 pp.
[25] H. Yang, Asymptotic behavior of positive solutions to a nonlinear biharmonic equation near isolated singularities, Calc. Var. Partial Differential Equations. 59 (2020), no. 4, Paper No. 130, 21 pp.

[26] H. Yang, Liouville-type theorems, radial symmetry and integral representation of solutions to Hardy-Hénon equations involving higher order fractional Laplacians, arXiv:2109.09441.

[27] H. Yang, W.M. Zou, Sharp blow up estimates and precise asymptotic behavior of singular positive solutions to fractional Hardy-Hénon equations, J. Differential Equations 278 (2021), 393-429.

[28] Qi S. Zhang, Z. Zhao, Singular solutions of semilinear elliptic and parabolic equations, Math. Ann. 310 (1998), 777-794.

X. Huang
School of Mathematical Sciences and Shanghai Key Laboratory of PMMP
East China Normal University, Shanghai 200241, China
Email: xhuang@cpde.ecnu.edu.cn

Y. Li
School of Mathematical Sciences and Shanghai Key Laboratory of PMMP
East China Normal University, Shanghai 200241, China
Email: yli@math.ecnu.edu.cn

H. Yang
Department of Mathematics, The Hong Kong University of Science and Technology
Clear Water Bay, Kowloon, Hong Kong, China
Email: mahuiyang@ust.hk