Convex Equipartitions via Equivariant Obstruction Theory*

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Abstract

We describe a regular cell complex model for the configuration space \( F(\mathbb{R}^d, n) \). Based on this, we use equivariant obstruction theory to prove the prime power case of the conjecture by Nandakumar and Ramana Rao that every polygon can be partitioned into \( n \) convex parts of equal area and perimeter. For a generalization of the conjecture we get a complete answer: It holds if and only if \( n \) is a prime power.

1 Introduction

R. Nandakumar and N. Ramana Rao in June 2006 posed the following problem, which was characterized “interesting and annoyingly resistant” in [7]:

**Conjecture 1.1** (Nandakumar & Ramana Rao [33]). *Every convex polygon \( P \) in the plane can be partitioned into any prescribed number \( n \) of convex pieces that have equal area and equal perimeter.*

The conjecture was proved by Nandakumar & Ramana Rao [34] for \( n = 2 \), where it follows from the intermediate value theorem, that is, the Borsuk–Ulam theorem for maps \( f : S^1 \to S^0 \); the same authors also gave elementary arguments for the result for \( n = 2^k \). For \( n = 3 \) the conjecture was proven by Bárány, Blagojević & Szűcs [7] via advanced topological methods.

For general \( n \), it was noted by Karasev [29] and Hubard & Aronov [25] that the conjecture would follow from the non-existence of an \( \mathfrak{S}_n \)-equivariant map \( F(\mathbb{R}^2, n) \to S(W_n) \), where \( F(\mathbb{R}^2, n) \) denotes the configuration space of \( n \) labeled points in the plane, that is, the set of all real \( 2 \times n \) matrices with pairwise distinct columns, and \( W_n := \{ (x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 + \cdots + x_n = 0 \} \) is the set of all row vectors in \( \mathbb{R}^n \) with vanishing sum of coordinates. Both \( F(\mathbb{R}^2, n) \) and \( W_n \) have the obvious action of \( \mathfrak{S}_n \) by permuting columns. Similarly, for \( d \)-dimensional versions of the problem one would need the non-existence of \( \mathfrak{S}_n \)-equivariant maps \( F(\mathbb{R}^d, n) \to S(W_n^{\otimes d-1}) \).

First Karasev [29] and later Hubard & Aronov [25] proposed to show this non-existence for prime power \( n = p^k \) based on a cell-decomposition of the one-point compactification of the configuration space that appears in work by Fuks [21] and Vassiliev [40]. Thus they have to deal with twisted Euler classes on non-compact manifolds resp. on compactifications that are not manifolds. In this paper, we avoid these difficulties, and obtain the same results and more via equivariant obstruction theory. Indeed, we obtain an “if and only if” statement:

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Theorem 1.2. For \( d, n \geq 2 \), an \( S_n \)-equivariant map
\[
F(\mathbb{R}^d, n) \rightarrow S_n(W_n^{n,d-1})
\]
exists if and only if \( n \) is not a prime power.

For this, we describe in Section 3 a beautiful \( S_n \)-equivariant \((d-1)(n-1)\)-dimensional cell complex with \( n! \) vertices and \( n! \) facets that is an \( S_n \)-equivariant retract of \( F(\mathbb{R}^d, n) \). For \( d = 2 \) this complex is implicit in work by Deligne [18] and explicit in Salvetti’s work [36]. For \( d \geq 2 \) we obtain it by specialization of a construction due to Björner & Ziegler [11].

Theorem 1.2 in particular proves the original problem by Nandakumar & Ramana Rao for prime powers \( n \). However, we also get a much stronger theorem for the prime power case, which fails if \( n \) is not a prime power.

Theorem 1.3. Let \( d \geq 2 \) and let \( \mu \) be a continuous positive probability measure on \( \mathbb{R}^d \).

If \( n \) is a prime power, then for any \( d-1 \) continuous real functions \( f_1, \ldots, f_{d-1} \) on the space of convex polyhedra \( C \subset \mathbb{R}^d \) with non-empty interior there is a regular partition of \( \mathbb{R}^d \) into \( n \) convex regions \( C_1, \ldots, C_n \) that equiparts \( \mu \),
\[
\mu(C_1) = \cdots = \mu(C_n) = \frac{1}{n},
\]
and equalizes the functions \( f_1, \ldots, f_{d-1} \):
\[
f_i(C_1) = \cdots = f_i(C_n) \quad \text{for } 1 \leq i \leq d-1.
\]

If \( n \) is not a prime power, then this is false: For any given \( d \geq 2 \), any continuous positive probability measure \( \mu \) on \( \mathbb{R}^d \), and \( n \) not a prime power, there are continuous functions \( f_1, \ldots, f_{d-1} \) on the space of \( d \)-dimensional convex polyhedra with nonempty interior, such that no regular subdivision equiparts the measure \( \mu \) and equalizes the functions \( f_i \).

The “regular subdivisions” appearing in the statement of this theorem are partitions that arise from Monge–Kantorovich type optimal transport for a quadratic assignment function; see Villani [42, Chap. 2]. The same type of partitions are called “regular” in the set-up of Gelfand, Kapranov & Zelevinsky [22]. In Computational Geometry they are known as generalized Voronoi diagrams or power diagrams. The equivalence between partitions induced by optimal transport and those given by the domains of linearity of a minimum of \( n \) linear functions was proved by Aurenhammer [2], and certainly also by others; see Siersma & van Manen [37].

Theorem 1.3 is equivalent to Theorem 1.2 as regular equipartitions are parametrized by \( F(\mathbb{R}^d, n) \). This remarkable observation can be traced back to Minkowski [30, 31]. It was independently developed in Optimal Transport (see Villani [42, Chap. 2]) and in Computational Geometry (see Aurenhammer, Hoffmann & Aronov [3, 4]).

Our proof of Theorem 1.2 relies on equivariant obstruction theory as formulated by tom Dieck [39, Sec. II.3]. We happen to be in the fortunate situation where we have to deal only with primary obstructions, for mapping a cell complex of dimension \((d-1)(n-1)\) to a sphere of dimension \((d-1)(n-1)-1\). Moreover, the cell complex has a single orbit of maximal cells (“facets”), where the obstruction cocycle evaluates to 1. At the same time, we have \( n-1 \) orbits of ridges, of sizes \( \binom{n}{i} \). The proof is completed via the number theoretic fact about the \( n \)-th row of the Pascal triangle
\[
\gcd\{\binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n-1}\} = \begin{cases} p & \text{if } n \text{ is a prime power, } n = p^k, \\ 1 & \text{otherwise.} \end{cases}
\]

This may be derived from the work by Legendre and Kummer on prime factors in factorials and binomial coefficients, apparently was first proved by Ram [35] in 1909, extended by Joris et al. [26], and was later rediscovered by A. Granville (see Soulé [38] and Kaplan & Levy [28]).
2 From Partitioning Polygons to Equivariant Maps of Configuration Spaces

According to Gelfand et al. [22], a polyhedral subdivision is regular if it is given by the domains of linearity of a convex, piecewise-linear lifting function. See also [43, Lect. 4]. According to Aurenhammer [2], the regular partitions are exactly the ones obtained by optimal transport for quadratic distance functions.

The classical partitions of this form are known in Computational Geometry as Voronoi diagrams and in Number Theory as Dirichlet tessellations. They are obtained as \( V(x_1, \ldots, x_n) = (C_1, \ldots, C_n) \) via

\[
C_i := \{ x \in \mathbb{R}^d : \| x - x_i \| \leq \| x - x_j \| \text{ for } 1 \leq j \leq n \} \
= \{ x \in \mathbb{R}^d : \| x - x_i \|^2 - \| x \|^2 \leq \| x - x_j \|^2 - \| x \|^2 \text{ for } 1 \leq j \leq n \}.
\]

Thus \( C_i \) is the domain in \( \mathbb{R}^d \) where the linear (!) function \( f_i(x) = \| x - x_i \|^2 - \| x \|^2 \) yields the minimum among all functions \( f_j(x) \) of the same type, and thus the Voronoi diagram is given by the domains of linearity of the piecewise-linear convex function \( f(x) = \min \{ \| x - x_i \|^2 - \| x \|^2 : 1 \leq i \leq n \} \).

For generalized Voronoi diagrams or power diagrams we introduce additional real weights \( w_1, \ldots, w_n \in \mathbb{R} \), and set \( \mathcal{P}(x_1, \ldots, x_n; w_1, \ldots, w_n) = (P_1, \ldots, P_n) \) with

\[
P_i := \{ x \in \mathbb{R}^d : \| x - x_i \|^2 - w_i \leq \| x - x_j \|^2 - w_j \text{ for } 1 \leq j \leq n \}.
\]

As an additive constant does not change the partition, we may w.l.o.g. replace \( w_i \) by \( w_i - \frac{1}{n}(w_1 + \cdots + w_n) \), and thus assume that \( w_1 + \cdots + w_n = 0 \), that is, \( (w_1, \ldots, w_n) \in W_n \).

Again, the partition of space given by a generalized Voronoi diagram is a regular subdivision. However, now some of the parts \( P_i \) may be empty, and \( x_i \in P_i \) does not hold in general. We refer to Aurenhammer & Klein [5] and Siersma & van Manen [37] for further background.

The major part of the following theorem (except for the continuity, which of course is essential for us) was provided by Aurenhammer, Hoffmann & Aronov [3] in 1992. The 1998 journal version [4] of their paper noted the connection to optimal transport; thus, the proof can be done using optimal assignment and linear programming duality, as developed (for this purpose) by Kantorovich in 1939 [27], with the "double convexification" trick. For a modern exposition see Villani [42, Chap. 2]. Evans [19] provides a very nice and simple combinatorial treatment (which needs minor adjustments in the use of LP duality, as the dual programs [19] (1.15), (13.6)) should not have nonnegativity constraints \( u_i \geq 0, v_j \geq 0 \).

**Theorem 2.1** (Kantorovich 1939, etc.). Let \( \mu \) be a continuous probability measure on \( \mathbb{R}^d \). For any \( x_1, \ldots, x_n \in \mathbb{R}^d \) with \( x_i \neq x_j \) for all \( 1 \leq i < j \leq n \) there are weights \( w_1, \ldots, w_n \in \mathbb{R} \), \( w_1 + \cdots + w_n = 0 \), such that the power diagram \( \mathcal{P}(x_1, \ldots, x_n; w_1, \ldots, w_n) \) equiparts the measure \( \mu \). Moreover, the weights \( w_1, \ldots, w_n \)

- are unique,
- depend continuously on \( x_1, \ldots, x_n \), and
- can be characterized/computed via optimal assignment with respect to quadratic cost functions.

In short, this theorem states that the configuration space \( F(\mathbb{R}^d, n) \) of \( n \) distinct labeled points in \( \mathbb{R}^d \) parametrizes the regular \( n \)-equipartitions of any continuous measure \( \mu \). Thus any counterexample to Theorem 1.3 would yield an \( \mathcal{G}_n \)-equivariant continuous map

\[
F(\mathbb{R}^d, n) \rightarrow \mathcal{G}_n \ W_n^{\otimes d - 1} \setminus \{0\}.
\]

In the following we will show that, indeed, this map does not exist if \( n \) is a prime power, but that it does exist otherwise.
3 A Cell Complex Model for $F(\mathbb{R}^d, n)$

In order to apply obstruction theory to $F(\mathbb{R}^d, n)$, we specialize the cell complex models given for complements of real subspace arrangements by Björner & Ziegler [11]. For $d = 2$, the barycentric subdivision of this cell complex is treated implicitly in Deligne [18]. The cell complex for a general complement of a complexified hyperplane arrangements is developed in more detail by Salvetti [35] and thus known as the “Salvetti complex.”

These cell complex models are closely related to the cell complex structures described already in 1962 by Fox & Neuwirth, and later by Fuks for $\mathbb{S}^{2n} = \mathbb{R}^{2n} \cup \{\ast\} \supset F(\mathbb{R}^2, n)$. The analogous constructions for $F(\mathbb{R}^d, n)$ were done by Vassiliev [41]. See also de Concini & Salvetti and Basabe et al. [8], as well as very recently Guisti & Sinha [23] and Ayala & Hepworth [6].

To obtain our cell complex model, we first retract $F(\mathbb{R}^d, n)$ by $\pi_i := x_i - \frac{1}{n}(x_1 + \cdots + x_n)$ to

$$F(\mathbb{R}^d, n) := \{(x_1, \ldots, x_n) \in \mathbb{R}^{d \times n} : x_1 + \cdots + x_n = 0, x_i \neq x_j \text{ for } 1 \leq i < j \leq n\},$$

which is the complement of an essential arrangement of linear subspaces of codimension $d$ in the vector space $W_{n,d} = \{(x_1, \ldots, x_n) \in \mathbb{R}^{d \times n} : x_1 + \cdots + x_n = 0\}$ of dimension $d(n - 1)$.

An arrangement of linear subspaces is essential if the intersection of all the subspaces is the origin. All the subspaces in the arrangement have codimension $d$ and all their intersections have codimensions that are multiples of $d$, so this is a $d$-arrangement in the sense of Goresky & MacPherson [24 Sect. III.4].

To construct the Fox–Neuwirth stratification of this essential arrangement of linear subspaces (which, except for the ordering of the coordinates, is equivalent to the “$s^{(1)}$-stratification” of [11 Sect. 9.3]), we order the $n$ columns of the matrix $(x_1, \ldots, x_n) \in W_{n,d}$ in lexicographic order. That is, we find the permutation $\sigma = \sigma_1 \sigma_2 \cdots \sigma_n \in S_n$ such that for $1 \leq j < n$,

- either for some $i_j, 1 \leq i_j \leq d$, we have $x_{i_j, \sigma_j} < x_{i_j, \sigma_{j+1}}$ with $x_{i', \sigma_j} = x_{i', \sigma_{j+1}}$ for all $i' < i_j$,

- or $x_{\sigma_j} = x_{\sigma_{j+1}}$; in this case we set $i_j = d + 1$ and impose the extra condition that $\sigma_j < \sigma_{j+1}$.

Then we assign to $(x_1, \ldots, x_n)$ the combinatorial data

$$(\sigma_1 < i_1, \sigma_2 < i_2 \cdots < i_{n-1}, \sigma_n),$$

where $\sigma = \sigma_1 \sigma_2 \cdots \sigma_n \in S_n$ is a permutation and $i = (i_1, \ldots, i_{n-1}) \in \{1, \ldots, d, d + 1\}^{n-1}$ is a vector of coordinate indices.

Example 3.1. Let $d = 2$, $n = 8$: The 8 points $x_1, \ldots, x_8 \in \mathbb{R}^2$ in the left part of Figure 1 correspond to the permutation $\sigma = 38147652$. The combinatorial data for this configuration are $$(\sigma, i) = (3 < 8 < 1 < 2 < 4 < 7 < 6 < 5 < 2).$$

For the point configuration on the right we get the permutation $\sigma' = 31847652$ and the combinatorial data $(\sigma', i') = (3 < 2 < 8 < 4 < 1 < 7 < 6 < 5 < 2).$
All the points in $W_{n}^{\mathbb{R}^{d}}$ with the same combinatorial data $(\sigma, i)$ make up a stratum that we denote by $C(\sigma, i)$. This stratum is the relative interior of a polyhedral cone of codimension $(i_1 - 1) + \cdots + (i_{n-1} - 1)$, that is, of dimension $(d + 1)(n - 1) - (i_1 \cdots + i_{n-1})$ in $W_{n}^{\mathbb{R}^{d}}$.

**Example 3.2.** The stratum $C(\sigma, i) = (3<28<1<24<17<16<25<22)$ of Example 3.1 has dimension 10. It is

$$
\left\{ \left( \begin{array}{llll} x_{11} & x_{12} & \cdots & x_{18} \\
 x_{21} & x_{22} & \cdots & x_{28} 
\end{array} \right) \in W_{8}^{\mathbb{R}^{2} \times 8} : \begin{array}{llllllllllll}
 x_{13} = x_{18} < x_{11} = x_{14} < x_{17} < x_{16} = x_{15} = x_{12} \\
 x_{23} < x_{28} & x_{21} < x_{24} & x_{27} & x_{26} < x_{25} < x_{22} 
\end{array} \right\}.
$$

**Lemma 3.3.** The closure of each stratum $C(\sigma, i) \subset W_{n}^{\mathbb{R}^{d}}$ is a union of strata, namely of all strata $C(\sigma', i')$ such that

(*) If $\ldots \sigma_k \ldots < i_p' \ldots \sigma_{\ell} \ldots$ appear in this order in $(\sigma', i')$, then

either $\ldots \sigma_k \ldots < i_p \ldots \sigma_{\ell} \ldots$ appear in this order in $(\sigma, i)$ with $i_p \leq i_p'$,

or $\ldots \sigma_{\ell} \ldots < i_p \ldots \sigma_k \ldots$ appear in this order in $(\sigma, i)$ with $i_p < i_p'$.

**Proof.** Let $(x_1, \ldots, x_n)$ be a configuration of points that lies in the stratum $C(\sigma, i)$, which is a relatively open polyhedral cone characterized by equations and strict inequalities. The closure of the cone is given by the condition that the equations still hold, and the inequalities still hold weakly. Thus the smallest index of a coordinate in which two points $x_{\sigma_k}$ and $x_{\sigma_{\ell}}$ differ may not go down when moving to $x'_{\sigma_k}$ and $x'_{\sigma_{\ell}}$, and if it stays the same, then the order in this coordinate must be preserved.

Thus if $\ldots \sigma_k \ldots < i_p' \ldots \sigma_{\ell} \ldots$ appear in this order in $(\sigma', i')$ and $i_p'$ is not the smallest such index, then no condition is posed. If $i_p'$ is the smallest such index, then we either have the same in $(\sigma, i)$, or there is a smaller index $i_p$. If there is no smaller index, then we still require that $\ldots \sigma_k \ldots < i_p' \ldots \sigma_{\ell} \ldots$ appear in this order in $(\sigma, i)$. If there is a smaller index $i_p < i_p'$ between $\sigma_k$ and $\sigma_{\ell}$, then the order is arbitrary.

Condition (*) describes this.

**Example 3.4.** For the point configurations of Example 3.1, the 9-dimensional stratum $C(\sigma', i')$ lies in the boundary of the 10-dimensional stratum $C(\sigma, i)$.

**Definition 3.5.** For $d \geq 1, n \geq 2$ the set of pairs $(\sigma, i)$ with the partial order $(\sigma, i) \geq (\sigma', i')$ described by Condition (*) in Lemma 3.3 is denoted by $S(d,n)$. Its minimal element $(1<_{d+1}2<_{d+1} \ldots <_{d+1}n)$ is denoted by $\hat{0}$.

**Example 3.6.** The face poset of $S(2,3)$ is displayed in Figure 2. (Compare [31] Fig. 2.3.) Here black dots denote cells in the complement of the arrangement given by the codimension 2 subspaces “$x_i = x_j$” (that is, lying in the configuration space $F(\mathbb{R}^{2}, 3)$), while the white dots correspond to cells on the arrangement.

By Lemma 3.3, the intersection of the strata of the Fox–Neuwirth stratification of $W_{n}^{\mathbb{R}^{d}}$ with the unit sphere $S(W_{n}^{\mathbb{R}^{d}})$ yields a regular CW decomposition of this sphere of dimension $d(n - 1) - 1$, whose face poset is given by $S(d,n)$. (Here the stratum $\{0\}$ corresponds to $\hat{0} \in S(d,n)$; it corresponds to the empty cell in the cellular sphere.)

The union of the arrangement is composed of all the strata with at least one entry $<_{d+1}$, while the strata where all comparisons $<_{k}$ have $k \leq d$ lie in the complement. In particular, the link of the arrangement (its intersection with the unit sphere) is given by a subcomplex of the cell decomposition of the unit sphere in $W_{n}^{\mathbb{R}^{d}}$ induced by the stratification.

**Example 3.7.** For $d \geq 1, n = 2$ the CW decomposition of the unit sphere in $W_{2}^{\mathbb{R}^{d}} \equiv \mathbb{R}^{d}$ we obtain is the minimal, centrally-symmetric cell decomposition of the $(d-1)$-sphere that has two vertices, two edges, two 2-cells, etc.: The two $i$-cells are $C(1<_{d-1}2) \cap S(W_{2}^{\mathbb{R}^{d}})$ and $C(2<_{d-1}1) \cap S(W_{2}^{\mathbb{R}^{d}})$, for $0 \leq i < d$. Both $i$-cells lie in the boundary of both $j$-cells, for $0 \leq i < j < d$.  

5
Figure 2: The face poset $S(2, 3)$.
**Example 3.8.** For \( d = 1, n \geq 2 \) the stratification of \( W_n \) is given by the real arrangement of \( \binom{n}{2} \) hyperplanes \( x_i = x_j \) in \( W_n \). The CW decomposition of the unit sphere in \( W_n \) has \( 2^n - 2 \) vertices, indexed by \( (\sigma_1 < 2 < \cdots < 2 < \sigma_{k-1} < 2 < \cdots < 2 < \sigma_n) \) with \( 1 \leq k < n, \sigma_1 < \cdots < \sigma_{k-1} \) and \( \sigma_k < \cdots < \sigma_n \), and \( n! \) facets corresponding to the regions of the hyperplane arrangement, indexed by \( (\sigma_1 < 1 < \cdots < 1 < \sigma_n) \) for permutations \( \sigma \in \mathcal{S}_n \).

We are dealing with a partition (“stratification”) of a Euclidean space \( \mathbb{R}^m \) into a finite number of relative-open polyhedral cones such that the relative boundary of each cone \( C \) (i.e. its boundary in the affine hull) is a union of (necessarily: finitely many, lower-dimensional) cones of the stratification. Furthermore the stratification is *weakly essential* in the following sense: some of the strata may not be pointed (i.e., have \( \{0\} \) as a face of the closure), but \( \{0\} \) is a stratum that lies in the boundary of all other strata, which thus are proper subsets of their affine hulls. In this situation any selection of points \( v_C \) in the relative interiors of the strata \( C \) induces a PL-homeomorphism between the order complex of the face poset of the stratification to a starshaped PL-ball that is a neighborhood of the origin in \( \mathbb{R}^m \).

**Example 3.9.** Figure 3 shows a weakly essential stratification of \( \mathbb{R}^2 \), its face poset, and a realization of the order complex of its face poset.

![Figure 3: A stratification of \( \mathbb{R}^2 \), its face poset, and a realization of its order complex](image)

We now implement this construction for our specific stratification of \( W_n^{\otimes d} \).

**Lemma 3.10.** For \( d \geq 1, n \geq 2 \), in each stratum \( C(\sigma, i) \) a relative-interior point \( v(\sigma, i) = (x_1, \ldots, x_n) \in W_n^{\otimes d} \) is obtained as follows: Set

\[
\tilde{x}_{\sigma(1)} := 0, \quad \tilde{x}_{\sigma(j+1)} := \tilde{x}_{\sigma(j)} + e_{i_j},
\]

where \( e_i \) denotes the \( i \)th standard unit vector in \( \mathbb{R}^d \), with \( e_{d+1} := 0 \). The point \( v(\sigma, i) \) is the image of this \( \tilde{x}(\sigma, i) \) under the orthogonal projection map \( \nu \) from \( \mathbb{R}^{d \times n} \) to the subspace \( W_n^{\otimes d} \), which translates the barycenter of the configuration to the origin, given by \( x_j := \nu(\tilde{x}_j) = \tilde{x}_j - \frac{1}{n} (\tilde{x}_1 + \cdots + \tilde{x}_n) \).
These points \( v(\sigma, i) \) yield the vertices of a geometric realization of the order complex \( \Delta S(d, n) \) by a starshaped PL neighborhood \( sdB(d, n) \) of \( W_\infty^{\oplus d} \). Its boundary is a geometric realization of the barycentric subdivision of the cell decomposition of \( S(W_\infty^{\oplus d}) \) given by the Fox–Neuwirth stratification.

**Example 3.11.** For the data \( (\sigma, i) = (3 < 2 8 < 1 < 2 4 < 1 7 < 1 6 < 2 5 < 2 2) \) of the first configuration in Example 3.1, Lemma 3.10 yields the following configuration of Figure 4, which is then normalized by the map \( \nu \).

![Figure 4: The coordinates of Lemma 3.10 for the first configuration of Example 3.1](image)

**Example 3.12.** For \( n = 3, d = 1 \) the starshaped PL neighborhood \( sdB(1, 3) \) of \( W_3 \subset \mathbb{R}^3 \) has 13 vertices, some of whose coordinates according to Lemma 3.10 are indicated in Figure 5.

![Figure 5: Coordinates for the realization of \( sdB(1, 3) \) in \( W_3 \) according to Lemma 3.10](image)

In particular, the boundary of the star-shaped PL neighborhood \( sdB(d, n) \) is an \( \mathcal{S}_n \)-invariant PL sphere. The link of the arrangement is represented by an induced subcomplex of this sphere.
The dual cell complex of the sphere contains, as a subcomplex, a cellular model for the complement of the arrangement – that is, a simplicial complex that is a strong deformation of the configuration space $F(\mathbb{R}^d, n)$. The cell complex is regular in the sense that the attaching maps of cells do not make identifications on the boundary; in particular, the barycentric subdivision of any such complex is a simplicial complex, given by the order complex of the face poset (see e.g. Björner et al. [10 Sect. 4.7], Cooke & Finney [16], Munkres [32]).

**Theorem 3.13** (Cell complex model for $F(\mathbb{R}^d, n)$). Let $d \geq 1$ and $n \geq 2$. Then $F(\mathbb{R}^d, n)$ contains, as a strong deformation retract, a finite (and thus compact) regular CW complex $F(d, n)$ of dimension $(d-1)(n-1)$ with $n!$ vertices, $n!$ facets, and $(n-1)!n$ ridges.

The nonempty faces of $F(d, n)$ are indexed by the data of the form

$$(\sigma, i) := (\sigma_1 < i_1 \sigma_2 < i_2 \cdots < i_{n-1} \sigma_n),$$

where $\sigma = \sigma_1 \sigma_2 \ldots \sigma_n \in S_n$ is a permutation, and $i = (i_1, \ldots, i_{n-1}) \in \{1, \ldots, d\}^{n-1}$. Let $F(d, n)$ be the set of these strings.

The dimension of the cell $\mathcal{C}(\sigma, i)$ associated with $(\sigma, i)$ is $(i_1 + \cdots + i_{n-1}) - (n-1)$.

The inclusion relation between cells $\mathcal{C}(\sigma, i)$, and thus the partial order of the face poset $F(d, n)$, is as follows:

$$(\sigma, i) \preceq (\sigma', i')$$

holds if and only if

**(**) whenever $\sigma_1 \ldots < i'_p \ldots \sigma_l \ldots$ appear in this order in $(\sigma', \mathbf{1})$, then either $\sigma_1 \ldots < i_p \ldots \sigma_l \ldots$ appear in this order in $(\sigma, \mathbf{i})$ with $i_p \leq i'_p$, or $\sigma_1 \ldots < i_p \ldots \sigma_l \ldots$ appear in this order in $(\sigma', \mathbf{i})$ with $i_p < i'_p$.

The barycentric subdivision of the cell complex (that is, the order complex $\Delta F(d, n)$ of its face poset) has a geometric realization in $W^{\n\times \mathbb{R}^d}$ with vertices $v(\sigma, i) = (x_1, \ldots, x_n) \in W_n^{\n\times \mathbb{R}^d}$ placed according to

$$\tilde{x}_{\sigma(1)} := 0, \quad \tilde{x}_{\sigma(j+1)} := \tilde{x}_{\sigma(j)} + e_{i_j},$$

where $e_i$ denotes the $i$th standard unit vector in $\mathbb{R}^d$, followed by orthogonal projection $\mathbb{R}^{d \times n} \to W^{\n\times \mathbb{R}^d}$, given by $x_j := \tilde{x}_j = \frac{1}{n}(\tilde{x}_1 + \cdots + \tilde{x}_n)$. This geometric realization in $W_n^{\n\times \mathbb{R}^d} \subset \mathbb{R}^{d \times n}$ is an $S_n$-equivariant strong deformation retract of $F(\mathbb{R}^d, n)$.

The group $S_n$ acts on the poset $F(d, n)$ via $\pi \cdot (\sigma, i) = (\pi \sigma, i)$, and correspondingly by $\pi \cdot v(\sigma, i) = v(\pi \sigma, i)$ on the geometric realization $\text{sd} F(\mathbb{R}^d, n)$ with vertex set $\{v(\sigma, i) : (\sigma, i) \in F(d, n)\}$.

**Proof.** Orthogonal projection $S_n$-equivariantly retracts $F(\mathbb{R}^d, n)$ to $\overline{F}(\mathbb{R}^d, n) \subset W_n^{\n\times \mathbb{R}^d}$, and further radial projection $S_n$-equivariantly retracts this to a subset of the boundary of the starshaped neighborhood of the origin, $\partial \text{sd} B(d, n)$, which is a simplicial realization of the cell decomposition of the $(d(n-1) - 1)$-dimensional sphere $S(W_n^{\n\times \mathbb{R}^d})$ given by the Fox–Neuwirth stratification.

The same maps identify the link of the arrangement (that is, the intersection of its union with the unit sphere) with the induced subcomplex of $\partial \text{sd} B(d, n)$ on the vertices $v(\sigma, i)$ that have some index $i_j = d + 1$.

As the cell decomposition of the $(d(n-1) - 1)$-sphere $S(W_n^{\n\times \mathbb{R}^d})$ is PL, following Munkres [32 § 64] we can pass to the “Alexander dual” cell decomposition. Thus any nonempty cell $\mathcal{C}(\sigma, i)$ of dimension $(d+1)(n-1) - (i_1 + \cdots + i_n) - 1$ has an associated dual cell $\mathcal{C}^*(\sigma, i)$ of dimension $(d(n-1) - 1) - ((d+1)(n-1) - (i_1 + \cdots + i_{n-1}) - 1 - (n-1)$, and the inclusion relation is just the opposite of the one in the primal complex, as described in Lemma 3.3.
According to the Retraction Lemma [10, Lemma 4.7.27] [32, Lemma 70.1], the complement admits a strong deformation retraction to the subcomplex whose cells are indexed by all pairs \((\sigma, i)\) with all indices \(i_j \neq d + 1\). (This retraction is easy to describe in barycentric coordinates in the barycentric subdivision of the cell complex, that is, on the order complex of the face poset. The \(\mathfrak{S}_n\)-action restricts to the subcomplex, and the retraction is canonical, and thus \(\mathfrak{S}_n\)-equivariant.)

So we obtain a strong deformation retract of \(F(\mathbb{R}^d, n)\) that is a cell complex with cells indexed by \((\sigma, i)\) with all indices \(i_j \neq 0\). The barycentric subdivision of this cell complex is realized as a simplicial complex in \(W_n^{2d}\) as the induced subcomplex of \(\partial N(d, n)\) on the vertices \(v(\sigma, i)\) with all indices \(i_j \neq d + 1\). Thus also the \(\mathfrak{S}_n\)-action restricts to this simplicial model for the complement.

**Example 3.14.** For \(n = 2\) the cell complex \(F(d, 2)\) turns out to be the centrally-symmetric cell decomposition of \(S^{d-1}\) whose \((i - 1)\)-cells are given by \(c(1, i, 2)\) and \(c(2, i, 1)\) for \(1 \leq i \leq d\). (Compare to Example 3.7.)

**Example 3.15.** For \(d = 1\) the configuration space \(F(\mathbb{R}, n)\) is the complement of a real hyperplane arrangement in \(W_n\). The cell complex \(F(1, n)\) is 0-dimensional, given by the \(n!\) vertices \(v(\sigma_1 \prec_1 \sigma_2 \ldots \prec_1 \sigma_n) \in W_n\), one in each chamber of the arrangement. (Compare to Example 3.12.)

**Example 3.16.** For \(d = 2\) the configuration space \(F(\mathbb{R}^2, n) = F(\mathbb{C}, n)\) may be viewed as the complement of a complex hyperplane arrangement, known as the braid arrangement. This is the particular situation studied by Arnol’d [1], Deligne [18], and many others. The cell complex that we obtain is a particularly interesting instance of the cell complex constructed by Salvetti [36] for the complements of complexified hyperplane arrangements (see also [11]).

In this case we may simplify our notation a bit: The non-empty cells of the complex are indexed by \((\sigma_1 \prec_1 \ldots \prec_{i-1} \sigma_n)\) with \(i_j \in \{1, 2\}\). Thus we can identify our cells uniquely if we just write a bar instead of “\(\prec_1\)”, no bar for “\(\prec_2\)”. The inclusion of a cell into the boundary of a higher-dimensional cell is then represented in the partial order by (repeatedly) “removing a bar, and shuffling the two blocks that were separated by the bar”.

For example, for \(n = 3\) we obtain the face poset \(F(2, 3)\) displayed in Figure 6. It is obtained by taking the elements of the partial order \(S(2, 3)\) displayed in Example 3.6 that are marked by black dots, with labels simplified, and the partial order reversed.

![Figure 6: The poset F(2,3)](image-url)
We see that the cell complex $F(2, n)$ is particularly nice:

- The cell complex $F(2, n)$ is a regular cell complex of dimension $n - 1$.
- It has $n!$ facets, given by permutations $\sigma_1\sigma_2\ldots\sigma_n$, and $n!$ vertices, given by the barred permutations $\sigma_1[\sigma_2]\ldots[\sigma_n]$.
- It has $\binom{n-1}{i}n!$ cells of dimension $i$, given by permutations with $n - i - 1$ bars.
- The order relation is generated by the operation of “remove a bar, and merge the two adjacent blocks separated by the bar by a shuffle”.
- The maximal cells have the combinatorial structure of an $(n - 1)$-dimensional permutahedron. (Thus all cells have the combinatorial structure of simple polytopes, namely of products of permutahedra.)

Figure 7 indicates the structure of the cell complex $F(2, 3)$: This is a regular cell complex with $3! = 6$ vertices, $2 \cdot 3! = 12$ edges, and $3! = 6$ 2-cells, which are hexagons. The figure displays the six 2-cells shaded in separate drawings; for example, the 2-cell $\hat{c}(123)$ is bounded by the six edges $\hat{c}(1|23), \hat{c}(3|12), \hat{c}(23|1), \hat{c}(2|13), \hat{c}(12|3)$. In the complex $F(2, 3)$ each edge is contained in three of the six hexagon 2-cells. The six edges in the boundary of each 2-cell lie in two different orbits of the group $\mathfrak{S}_3$; for the 2-cell $\hat{c}(123)$ displayed in Figure 7 the three edges $\hat{c}(1|23), \hat{c}(3|12), \hat{c}(2|13)$ lie in one $\mathfrak{S}_3$-orbit, while $\hat{c}(13|2), \hat{c}(23|1), \hat{c}(12|3)$ lie in the other one.

Our drawing also specifies orientations of the cells that will be discussed in the next section (namely, the edges and 2-cells).

### 3.1 A small summary

We have constructed and described the following objects:

- $S(d, n)$: the face poset of the Fox–Neuwirth stratification of $W\oplus d_n$, with minimal element $\hat{0}$;
- $S(d, n)$: a regular cell complex homeomorphic to $S(W\oplus d_n) \cong S^{d(n-1)-1}$, a PL sphere; its face poset is $S(d, n) \setminus \{\hat{0}\}$;
- $B(d, n)$: a regular cell complex homeomorphic to $B(W\oplus d_n)$, a PL ball, given by $S(d, n)$ plus one additional $d(n - 1)$-cell;
- $sdB(d, n)$: the barycentric subdivision of $B(d, n)$: a simplicial complex, geometrically embedded as a starshaped neighborhood of the origin in $W\oplus d_n$.
- $F(d, n)$: the poset of all strata that lie in the complement of the arrangement (that is, have no $i_j = d + 1$), ordered by reversed inclusion; it is the subposet of $S(d, n)^{op}$ given by all $(\sigma, 1)$ without any index $i_j = d + 1$;
- $F(d, n)$: a regular cell complex of dimension $(d - 1)(n - 1)$; it is a subcomplex of the dual cell complex to $S(d, n)$; its poset of non-empty faces is $F(d, n)$.
- $sdF(d, n)$: the barycentric subdivision of $F(d, n)$, the order complex $\Delta F(d, n)$: a simplicial complex, geometrically embedded into the complement $F(\mathbb{R}^d, n) \subset W\oplus d_n$ as a subcomplex of the boundary of the ball $sdB(d, n)$.
Figure 7: The six 2-cells of $\mathcal{F}(2,3)$
4 Equivariant Obstruction Theory

Our task now is to prove that for any \( d \geq 1 \) and \( n \geq 2 \) an equivariant map

\[
\mathcal{F}(d, n) \to \mathfrak{S}_n \quad S(W_n^{\oplus d-1})
\]

exists if and only if \( n \) is not a prime power. Here

- \( \mathcal{F}(d, n) \) is a finite regular CW complex of dimension \( M := (d-1)(n-1) \) on which \( \mathfrak{S}_n \) acts freely, permuting the \( M \)-dimensional cells, as described in Theorem 3.13
- \( S(W_n^{\oplus d-1}) \) is the set of all \((d-1) \times n\) matrices of column sum 0 and sum of the squares of all entries equal to 1. This is a sphere of dimension \((d-1)(n-1)-1 = M-1 \), on which \( \mathfrak{S}_n \) acts by permutation of columns; this action is not free for \( n > 2 \), but it has no fixed points.

We proceed to apply equivariant obstruction theory, according to tom Dieck \[39\] Sect. II.3: Since

- \( \mathcal{F}(d, n) \) is a free \( \mathfrak{S}_n \)-cell complex of dimension \( M \),
- the dimension of the \( \mathfrak{S}_n \)-sphere \( S(W_n^{\oplus d-1}) \) is \( M-1 \), and
- \( S(W_n^{\oplus d-1}) \) is \((M-1)\)-simple and \((M-2)\)-connected,

the existence of an \( \mathfrak{S}_n \)-equivariant map is equivalent to the vanishing of the primary obstruction

\[
o = [c_f] \in H^M_{\mathfrak{S}_n}(\mathcal{F}(d, n); \pi_{M-1}(S(W_n^{\oplus d-1}))).
\]

Here \( c_f \) denotes the obstruction cocycle associated with a general position equivariant map \( f : \mathcal{F}(d, n) \to W_n^{\oplus d-1} \) (cf. [12, Def. 1.5, p. 2639]). Its values on the \( M \)-cells \( \check{c} \) are given by the degrees

\[
c_f(\check{c}) = \deg \left( r \circ f : \partial \check{c} \to W_n^{\oplus d-1} \setminus \{0\} \to S(W_n^{\oplus d-1}) \right),
\]

where \( r \) denotes the radial projection.

The Hurewicz isomorphism gives an isomorphism of the coefficient \( \mathfrak{S}_n \)-module with a homology group:

\[
\pi_{M-1}(S(W_n^{\oplus d-1})) \cong H_{M-1}(S(W_n^{\oplus d-1}); \mathbb{Z}) =: \mathbb{Z}.
\]

As an abelian group this \( \mathfrak{S}_n \)-module \( \mathbb{Z} = \langle \xi \rangle \) is isomorphic to \( \mathbb{Z} \). The action of the permutations \( \tau \in \mathfrak{S}_n \) on the module \( \mathbb{Z} \) is given by

\[
\tau \cdot \xi = (\text{sgn } \tau)^{d-1} \xi.
\]

Indeed, each transposition \( \tau_{ij} \) in \( \mathfrak{S}_n \) acts on \( W_n \) by reflection in the hyperplane \( x_i = x_j \). Thus the action of \( \mathfrak{S}_n \) on \( W_n \) reverses orientations according to the sign character, and the action on \( W_n^{\oplus d-1} \) is given by \( \text{sgn } d-1 \).

4.1 Computing the obstruction cocycle

We will now compute the equivariant obstruction cocycle \( c_f \) in the cellular cochain group

\[
C^M_{\mathfrak{S}_n}(\mathcal{F}(d, n); \mathbb{Z})
\]

and then show that \( c_f \) is a coboundary of an equivariant \((M-1)\)-cochain if and only if \( n \) is not a power of a prime.

To compute the obstruction cocycle, we use the \( \mathfrak{S}_n \)-equivariant linear projection map

\[
f : W_n^{\oplus d} \to W_n^{\oplus d-1}
\]

obtained by deleting the last row from any matrix \((y_1, \ldots, y_n) \in W_n^{\oplus d} \) of row sum 0. This map clearly commutes with the projection map \( \nu : \mathbb{R}^{d \times n} \to W_n^{\oplus d} \) which subtracts from each column the average of the columns.
Lemma 4.1. The linear map $f$ maps all $M$-cells of $sdF(d,n) \subset W_n^{sd}$ homeomorphically to the same starshaped neighborhood $sdB(d-1,n)$ of $0$ in $W_n^{sd}-1$.

The symmetric group $\mathfrak{S}_n$ acts on this neighborhood by homeomorphisms that reverse orientations according to $\text{sgn}^{d-1}$.

Thus the $M$-cells and the $(M-1)$-cells of the complex $F(d,n)$ can be oriented in such a way that the $\mathfrak{S}_n$-action on the $M$-cells and on the $(M-1)$-cells changes orientations according to $\text{sgn}^{d-1}$, while the boundary of any $M$-cell is the sum of all $(M-1)$-cells in its boundary with $+1$ coefficients.

With these orientations, the obstruction cocycle $c_f$ has the value $+1$ on all oriented $M$-cells of $F(d,n)$.

Proof. The $M$-cells of $F(d,n)$ are given as $\hat{c}(\sigma_1<\sigma_2<\ldots<\sigma_n)$ for some permutation $\sigma \in \mathfrak{S}_n$. The $(M-1)$-cells are given by $\hat{c}(\sigma_1<\ldots\sigma_{i-1}\sigma_{i+1}<\ldots<\sigma_n)$ for some $\sigma \in \mathfrak{S}_n$ and $1 \leq i < n$.

The vertices of the barycentric subdivision of the cell $\hat{c}(\sigma_1<\sigma_2<\ldots<\sigma_n)$ are exactly those vertices $v(\sigma'_1<\ldots<\sigma'_n)$ for which the letters $\sigma'_j$ that are separated only by "$<$"s appear in the same order in $\sigma$.

The projection $f$ deletes the last row, and thus maps the vertex $v(\sigma_1<\ldots<\sigma_{n-1})$ to the vertex with the same symbol $v(\sigma_1<\ldots<\sigma_{n-1})$ of $sdB(d-1,n)$, except for reordering of symbols separated only by $<$. Thus exactly the vertices which differ only by reordering letters $\sigma'_j$ that are separated only by "$<$"s are mapped by $f$ to the same vertex of $sdB(d-1,n)$.

Thus we get that each maximal cell is mapped homeomorphically to the starshaped neighborhood $sdB(d-1,n)$ of the origin in $W_n^{sd}-1$. In particular, the vertex $v(\sigma_1<\ldots<\sigma_n)$ is the only point of the $M$-cell $\hat{c}(\sigma_1<\sigma_2<\ldots<\sigma_n)$ that is mapped to $0 \in W_n^{sd}-1$.

We interpret $sdB(d-1,n)$ as the barycentric subdivision of a cellular $M$-ball $B(d-1,n)$ with one $M$-cell. The $(M-1)$-cells in its boundary are given by $\hat{c}(\sigma_1<\ldots<\sigma_{j-1}\sigma_{j+1}<\ldots<\sigma_n)$ with exactly one index $i_j = d-1$ and all other $i_k$'s equal to $d$. Clearly there are $2^n-2$ of those, as they are given by the proper nonempty subsets $\{\sigma_1,\ldots,\sigma_j\} \subset \{1,\ldots,n\}$.

As $f$ induces a surjective cellular map from $F(d,n)$ to $B(d-1,n)$ that is a homeomorphism restricted to each cell, we can proceed as follows. We fix an orientation of the one $M$-cell of $B(d-1,n)$. (This amounts to fixing an orientation of $W_n^{sd-1}$.) We define orientations of the $(M-1)$-cells in the boundary of the $M$-cell of $B(d-1,n)$ by demanding that the cellular boundary of the $M$-cell is given by the formal sum of all the $(M-1)$-cells with $+1$ coefficients. We then define the orientation of the $(M-1)$-cells of $F(d,n)$ by demanding that the map $f$ preserves orientation for all of them.

4.2 When is the obstruction cocycle a coboundary?

Lemma 4.2. Let $b$ be any equivariant $(M-1)$-dimensional integral equivariant cellular cochain on $F(d,n)$, then the value of its coboundary is

$$x_1\binom{n}{1} + x_1\binom{n}{2} + \ldots + x_{n-1}\binom{n}{n-1}$$

on all $M$-cells, for integers $x_1,\ldots,x_{n-1}$.

Proof. As $b$ needs to be equivariant, we get the condition that the values are constant on orbits. (Indeed, this is since the boundary operator does not introduce any signs, and the symmetric group acts by introducing the same signs $\text{sgn}^{d-1}$ both on the $(M-1)$-cells and on the $\mathfrak{S}_n$-module $Z$.)

Finally, the $\mathfrak{S}_n$-orbit of the $(M-1)$-cell $\hat{c}(\sigma_1<\ldots<\sigma_{d-1}\sigma_{d+1}<\ldots<\sigma_n)$ has size $\binom{n}{d}$. 

14
Proof of Theorem 1.2. By [39] Sect. II.3, pp. 119-120], the equivariant map exists if and only if the cohomology class \( \sigma = [c_f] \) vanishes, that is, if \( c_f \) is the coboundary \( c_f = \delta b \) of some \((M - 1)\)-dimensional equivariant cochain \( b \). We have seen in Lemmas 4.1 and 4.2 that this is the case if and only if

\[
x_1 \binom{n}{1} + x_2 \binom{n}{2} + \cdots + x_{n-1} \binom{n}{n-1} = 1
\]

has a solution in integers. By the Chinese remainder theorem, this happens if and only if the binomial coefficients do not have a non-trivial common factor. By Ram’s result, as quoted in the introduction and proved below, this holds if and only if \( n \) is not a prime power.

Proposition 4.3. For all \( n \in \mathbb{N} \) we have

\[
\gcd \left\{ \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n-1} \right\} = \begin{cases} p & \text{if } n \text{ is a prime power, } n = p^k \\ 1 & \text{otherwise} \end{cases}
\]

Proof. If \( n = p^k \), \( k \geq 1 \), then \( p \) appears only once as a factor of \( \binom{n}{n-1} \), and at least once in every other binomial coefficient considered. Since \( \binom{n}{1} = p^k \), we obtain \( \gcd \{ \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n-1} \} = p \).

If \( n \) is not a prime power, let \( n = \prod_{i=1}^{m} p_i^{k_i} \) for \( m > 1 \) and distinct primes \( p_i \). The number of times \( p_i \) appears as a factor in \( \binom{n}{n-1} \) is \( \frac{n}{(n-p_i)}! \sum_{j=1}^{k_i} p_i^{-j} \), so \( p_i \) does not divide \( \binom{n}{n-1} \). Hence \( \gcd \{ n, \binom{n}{p_1}, \binom{n}{p_2}, \ldots, \binom{n}{p_m} \} = 1 \) and thus \( \gcd \{ \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n-1} \} = 1 \).

In the case when \( n = 2 \) the equation (1) has no solution modulo 2, which implies that the top Stiefel–Whitney class of the bundle \( W_n \to F(\mathbb{R}^d, n) \times \mathfrak{S}_n \) is non-trivial. This fact was first proved by Cohen & Handel [15, Lemma 3.2, page 203] in the case \( d = 2 \), and by Chisholm [14, Lemma 3] in the case when \( d \) is a power of 2. For further application of this fact in the context of \( k \)-regular mappings of Euclidean spaces consult [13].

Two corollaries

Our calculation above implies a complete calculation for the top equivariant cohomology group of the cell complex \( F(d, n) \).

Corollary 4.4. For \( d, n \geq 2 \),

\[
H^M_{G_n}(F(d, n); \pi_{M-1}(S(W_n^{\mathbb{R}^d-1}))) = \langle [c_f] \rangle = \begin{cases} \mathbb{Z}/p & \text{if } n = p^k \text{ is a prime power,} \\ \{0\} & \text{otherwise} \end{cases}
\]

Indeed, our cell complex model has only one orbit of maximal cells, thus the group of equivariant \( M \)-dimensional cochains is isomorphic to \( \mathbb{Z} \). Our calculation shows that the subgroup of coboundaries is generated by the element \( \gcd \{ \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n-1} \} \).

Moreover, we note a stronger version of the non-existence part of Theorem 1.2

Corollary 4.5. For \( d \geq 2 \) and prime powers \( n = p^k \), there is no \( G \)-equivariant map

\[
F(\mathbb{R}^d, n) \to_G S(W_n^{\mathbb{R}^d-1})
\]

for any \( p \)-Sylow subgroup \( G \subset \mathfrak{S}_n \).

This follows from our proof for Theorem 1.2 using transfer; see [13]. Moreover, in [13] we will derive from equivariant cohomology calculations (the Fadell–Husseini index) that in the case when \( n = p \) is a prime, the equivariant map does not exist for the cyclic (Sylow) subgroup \( \mathbb{Z}_n \subset \mathfrak{S}_n \).
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When the first version of this paper was released on the arXiv, Dev Sinha pointed us to the recent preprints [23] and [6]. In response to this, we decided to change our notation to be in line with these papers, which continue the developments started by Fox & Neuwirth and Fuks.

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