More nonexistence results for symmetric pair coverings

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Abstract

A \((v, k, \lambda)\)-covering is a pair \((V, B)\), where \(V\) is a \(v\)-set of points and \(B\) is a collection of \(k\)-subsets of \(V\) (called blocks), such that every unordered pair of points in \(V\) is contained in at least \(\lambda\) blocks in \(B\). The excess of such a covering is the multigraph on vertex set \(V\) in which the edge between vertices \(x\) and \(y\) has multiplicity \(r_{xy} - \lambda\), where \(r_{xy}\) is the number of blocks which contain the pair \(\{x, y\}\). A covering is symmetric if it has the same number of blocks as points. Bryant et al. [4] adapted the determinant related arguments used in the proof of the Bruck-Ryser-Chowla theorem to establish the nonexistence of certain symmetric coverings with 2-regular excesses. Here, we adapt the arguments related to rational congruence of matrices and show that they imply the nonexistence of some cyclic symmetric coverings and of various symmetric coverings with specified excesses.

Keywords. pair covering, excess, Bruck-Ryser-Chowla theorem, rationally congruent matrices, Hasse-Minkowski invariant, almost difference set

Mathematics Subject Classifications: 05B40, 15A63

1 Introduction

Suppose \(V\) is a set of \(v\) points and \(B\) is a collection of \(k\)-subsets of \(V\), called blocks. The pair \((V, B)\) is a \((v, k, \lambda)\)-design or a \((v, k, \lambda)\)-covering if each pair of points of \(V\) occurs in exactly \(\lambda\) or at least \(\lambda\) blocks of \(B\), respectively. The number of blocks in a design is determined by \(v, k\) and \(\lambda\). In the case of coverings, one is usually interested in finding a covering with as few blocks as possible.

It is known that every non-trivial \((v, k, \lambda)\)-design has at least \(v\) blocks (see [6]), and consequently designs with exactly \(v\) blocks, called symmetric designs, are of particular interest. Many families of symmetric designs are known to exist, the most famous example being projective planes. One of the most celebrated results in the study of block designs is the Bruck-Ryser-Chowla theorem [3, 5] which establishes the nonexistence of certain symmetric \((v, k, \lambda)\)-designs.
The excess of a \((v, k, \lambda)\)-covering \((V, \mathcal{B})\) is the multigraph with vertex set \(V\) in which the multiplicity of the edge joining \(x\) and \(y\) is \(r_{xy} - \lambda\) where \(r_{xy}\) is the number of blocks in \(\mathcal{B}\) that contain both \(x\) and \(y\). For many parameter sets \((v, k, \lambda)\), a covering with a minimum number of blocks has an \(m\)-regular excess, where \(m < k - 1\). It has been shown in [2] and [4] that, barring some trivial exceptions, \((v, k, \lambda)\)-coverings with fewer blocks than points and 1- or 2-regular excesses do not exist (see [10] for a recent generalisation). In this paper, we study symmetric coverings with 2-regular excesses; that is, coverings with an equal number of points and blocks whose excess is 2-regular.

A \((v, k, \lambda)\)-design is a \((v, k, \lambda)\)-covering whose excess is empty, and hence the Bruck-Ryser-Chowla theorem [3, 5] can be viewed as establishing the nonexistence of certain symmetric \((v, k, \lambda)\)-coverings with empty excesses. Bose and Connor [2] were able to adapt the arguments used in the proof of the Bruck-Ryser-Chowla theorem to establish the nonexistence of certain symmetric \((v, k, \lambda)\)-coverings with 1-regular excesses. The case of 2-regular excesses is significantly more complicated because, for large \(v\), there are many non-isomorphic 2-regular multigraphs on \(v\) vertices. Nevertheless, Bryant et al. [4] were able to adapt some of these arguments (those concerning determinants) to the case of 2-regular excesses. In particular, they prove the following result

**Theorem 1.1.** [4] Let \(v, k, \lambda\) be positive integers such that \(\lambda \geq 1\) and \(3 \leq k < v\). If there exists a symmetric \((v, k, \lambda)\)-covering with 2-regular excess then

- when \(v\) is even either
  * \(\lambda\) is even and \((v, k, \lambda) = (\lambda + 4, \lambda + 2, \lambda)\), or
  * \(k - \lambda - 2\) is a perfect square and the excess has an odd number of cycles, or
  * \(k - \lambda + 2\) is a perfect square and the excess has an even number of cycles;

- when \(v\) is odd either
  * \(\lambda\) is odd and \((v, k, \lambda) = (\lambda + 4, \lambda + 2, \lambda)\), and
  * the excess has an odd number of cycles.

In [4], Bryant et al. comment that

Given the nature of the incidence matrices of the coverings we are considering, it seems very difficult to adapt the more advanced arguments from the proof of the Bruck-Ryser-Chowla Theorem.

Here, we investigate adapting these arguments (which employ Hasse-Minkowski invariants) to prove the nonexistence of certain symmetric coverings with 2-regular excesses. We establish new results and also outline some limitations to this approach. Our main findings are as follows.

- In Section 3 we develop an efficient way to calculate the Hasse-Minkowski invariant for the family of matrices of interest for this problem (see Lemmas 3.2 and 3.7).
In Section 4 we present computational results showing that our techniques can be used to rule out the existence of a variety of symmetric coverings with specified excesses. We do not find any parameter sets \((v, k, \lambda)\) for which our techniques completely rule out the existence of a symmetric \((v, k, \lambda)\)-covering, but we do find some for which our techniques show there does not exist a cyclic symmetric \((v, k, \lambda)\)-covering. This implies the nonexistence of certain interesting almost difference sets.

In Sections 5–8 we turn our attention to proving the nonexistence of families of symmetric coverings with excesses of specific forms. Sections 5, 6, 7 and 8 deal with coverings for which, respectively, the excess contains an odd number of cycles whose lengths are divisible by 4, the excess is a Hamilton cycle, the excess consists of cycles of uniform length, and the excess consists of only 2- and 3-cycles. In each case we prove a general result and exhibit an infinite family of symmetric coverings with specified excesses whose nonexistence is established by the result.

2 Preliminaries

In this section we give an outline of the approach we shall take to establishing the nonexistence of coverings. We first introduce some notation and concepts that we will require throughout the paper.

If a symmetric \((v, k, \lambda)\)-covering has a 2-regular excess, then by counting pairs of points we see that 
\[
\lambda v (v-1) + 2k = \frac{vk(k-1)}{\lambda} + 1
\]
and hence that 
\[
v = \frac{k(k-1)}{\lambda} - 2\lambda + 1.
\]
The previous equality also implies that each point in such a covering appears in 
\[
\lambda (v-1) + 2k - 1 = k
\]
blocks. Conversely, any \((v, k, \lambda)\)-covering with \(v = \frac{k(k-1)}{\lambda} - 2\lambda + 1\) and with a minimum number of blocks is necessarily symmetric and has 2-regular excess, provided that \(k \geq 4\).

If the excess of a symmetric covering on \(v\) points is 2-regular, then it is necessarily a vertex-disjoint union of cycles whose lengths add to \(v\). Note that here and throughout the paper we consider a pair of parallel edges to form a 2-cycle. We say that a 2-regular excess has cycle type \([c_1, \ldots, c_t]\) when it is the vertex-disjoint union of cycles of lengths \(c_1, \ldots, c_t\) with \(c_1 \leq \cdots \leq c_t\). We say that a cycle type \([c_1, \ldots, c_t]\) is \(v\)-feasible if \(c_1 \geq 2\) and \(c_1 + \cdots + c_t = v\). Occasionally we will use the shorthand exponential notation \(c^\ell\) to represent \(c_1, \ldots, c_\ell\) with \(c_1 = \cdots = c_\ell = c\).

For a prime \(p\), each positive integer \(n\) can be written uniquely as \(\bar{n}^\ell p^\alpha\) where \(\bar{n}\) and \(\alpha\) are integers such that \(\bar{n} \not\equiv 0 \pmod{p}\). We refer to \(\bar{n}^\ell p^\alpha\) as the \(p\)-factorisation of \(n\).

The determinant of a square matrix \(X\) is denoted by \(|X|\). If \(M_1, \ldots, M_t\) are square matrices then we denote by \(\text{diag}(M_1, \ldots, M_t)\) the block diagonal matrix with blocks \(M_1, \ldots, M_t\). When using this notation we sometimes abbreviate and use \(x\) to represent a \(1 \times 1\) matrix whose only entry is \(x\). For each positive integer \(n\), we denote the \(n \times n\) identity matrix by \(I_n\) and the \(n \times n\) all-ones matrix by \(J_n\).

Let \((V, \mathcal{B})\) be a \((v, k, \lambda)\)-covering (possibly a design) with \(b\) blocks and suppose we have ordered the elements of \(V\) and \(\mathcal{B}\). The incidence matrix \(A = (a_{xy})\) of \((V, \mathcal{B})\) is the \(v \times b\) matrix such that \(a_{xy} = 1\) if the \(x\)th point is in the \(y\)th block and \(a_{xy} = 0\) otherwise. The proof of the Bruck-Ryser-Chowla theorem observes that if \(A\) is the incidence matrix of a symmetric
(v, k, λ)-design, then $AA^T$ is equal to the $v \times v$ matrix $X = \text{diag}(k, \ldots, k) + \lambda J_v$. It follows that the determinant of $X$ is a perfect square and also that $X$ is rationally congruent to the identity matrix (rational congruence is defined later in this section - see Definition 2.4). In [3], a contradiction to one of these facts is obtained for certain parameter sets, thus establishing the nonexistence of a design with those parameters. Bryant et al. [4] have adapted the arguments relating to the determinant of $X$ to symmetric coverings with 2-regular excesses. Here we concentrate on the arguments concerning rational congruence.

We first establish the structure of the matrix $AA^T$ when $A$ is the incidence matrix of a symmetric covering with 2-regular excess. To do so we will use the following family of matrices.

**Definition 2.1.** For positive integers $a$ and $n$, where $n \geq 2$, we define a matrix $B_n(a)$ as follows.

$$B_2(a) = \begin{pmatrix} a & 2 \\ 2 & a \end{pmatrix}, \quad B_n(a) = \begin{pmatrix} a & 1 & 0 & 0 & \cdots & 1 \\ 1 & a & 1 & 0 & \cdots & 0 \\ 0 & 1 & a & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & a & 1 \\ 1 & 0 & \cdots & 0 & 1 & a \end{pmatrix} \text{ for } n \geq 3.$$  

Note that the matrices denoted by $B_n(a)$ here were denoted by $B_n'(a)$ in [4]. We make the change in order to keep our notation as clean as possible. Further, when there is no risk of confusion, we will sometimes abbreviate $B_n(a)$ to $B_n$.

**Definition 2.2.** For positive integers $v$, $k$ and $\lambda$ with $\lambda < k < v$, and any $v$-feasible cycle type $[c_1, \ldots, c_t]$ we define a $v \times v$ matrix

$$X_{(v,k,\lambda)}[c_1, \ldots, c_t] = \text{diag}(B_{c_1}(k-\lambda), B_{c_2}(k-\lambda), \ldots, B_{c_t}(k-\lambda)) + \lambda J_v.$$  

We sometimes abbreviate $X_{(v,k,\lambda)}[c_1, \ldots, c_t]$ to $X$.

**Proposition 2.3.** Let $v$, $k$ and $\lambda$ be positive integers such that $\lambda < k < v$. Suppose there exists a symmetric $(v,k,\lambda)$-covering $(V, B)$ whose excess has cycle type $[c_1, \ldots, c_t]$. Then $AA^T = X_{(v,k,\lambda)}[c_1, \ldots, c_t]$, where $A$ is the incidence matrix of $(V, B)$ for some appropriate ordering of $V$ and $B$. Consequently, $|X_{(v,k,\lambda)}[c_1, \ldots, c_t]|$ is a perfect square.

**Proof.** Order $V$ so that the first $c_1$ points are the vertices of a $c_1$-cycle in the excess, the next $c_2$ points are the vertices of a $c_2$-cycle in the excess, and so on. Within the vertex set of a cycle, order the points in any way such that consecutive points in the ordering are adjacent in the cycle. Order $B$ arbitrarily. For $x \in \{1, \ldots, v\}$, the entry in $x$th row and $y$th column of $AA^T$ is the number of blocks that contain the $x$th point, which we have seen is $k$. For distinct $x, y \in \{1, \ldots, v\}$, the entry in $x$th row and $y$th column of $AA^T$ is the number of blocks that contain both the $x$th and $y$th points, which is $\lambda + \mu(xy)$ where $\mu(xy)$ is the multiplicity of the edge $xy$ in the excess. It can now be seen that $AA^T$ has the required form.

Finally, $|X_{(v,k,\lambda)}[c_1, \ldots, c_t]| = |A||A^T| = |A|^2$ where $|A|$ is an integer because $A$ is a $(0,1)$-matrix.
Definition 2.4. Two square matrices $X$ and $Y$ of the same size with rational entries are 
rationally congruent, denoted $X \sim Y$, if there exists an invertible matrix $P$ with rational 
entries such that $X = P^TYP$.

It is shown in [4] that a symmetric $(v, k, \lambda)$-covering with 2-regular excess and $k - \lambda \leq 2$ 
exists if and only if $(v, k, \lambda) = (\lambda + 4, \lambda + 2, \lambda)$ (see the proof of Lemma 3.3 of [4]). So, in the 
remainder of this paper, we consider only parameter sets $(v, k, \lambda)$ such that $\lambda + 2 < k < v$. Our 
interest in rational congruence of matrices stems from the following observation.

Proposition 2.5. Let $v$, $k$ and $\lambda$ be positive integers such that $\lambda + 2 < k < v$. Suppose 
there exists a symmetric $(v, k, \lambda)$-covering whose excess has cycle type $[c_1, \ldots, c_t]$. Then 
$X_{(v,k,\lambda)}[c_1, \ldots, c_t] \sim I_v$.

Proof. Let $(V, B)$ be such a symmetric $(v, k, \lambda)$-covering. By Proposition 2.3,

$$AI_vA^T = AA^T = X_{(v,k,\lambda)}[c_1, \ldots, c_t],$$

where $A$ is the incidence matrix of $(V, B)$ for some appropriate ordering of $V$ and $B$. Clearly the 
entries of $A$ are rational. It follows from Lemma 2.1 of [4] that $|X_{(v,k,\lambda)}[c_1, \ldots, c_t]| \neq 0$ and hence 
that $A$ is invertible. So from the definition of rational congruence, $X_{(v,k,\lambda)}[c_1, \ldots, c_t] \sim I_v$. □

To establish that certain matrices $X_{(v,k,\lambda)}[c_1, \ldots, c_t]$ are not rationally congruent to $I_v$, we 
employ Hasse-Minkowski invariants. These are defined in terms of Hilbert symbols which, for 
our purposes, can be defined as follows. See [14] and [9, p. 121-122] for proofs that the definition 
given here is equivalent to the usual definition. Recall that $(\frac{a}{p})$ denotes the well-known Legendre 
symbol which, for a prime $p$ and an integer $a$ coprime to $p$, is given by $(\frac{a}{p}) = 1$ if $a$ is a quadratic 
residue modulo $p$ and $(\frac{a}{p}) = -1$ if $a$ is not a quadratic residue modulo $p$. We often employ 
basic properties of the Legendre symbol (see [1] for example).

Definition 2.6. For a prime $p$ and non-zero integers $a$ and $b$ with $p$-factorisations $\bar{a}p^\alpha$ and $\bar{b}p^\beta$
the Hilbert symbol $(a, b)_p$ can be defined by

$$ (a, b)_p = \begin{cases} 
\left(\frac{-1}{p}\right)^{\alpha\beta} \left(\frac{a}{p}\right)^{\beta} \left(\frac{b}{p}\right)^{\alpha}, & \text{if } p > 2; \\
\left(-1\right)^{(a-1)(b-1)/4+\alpha(\beta^2-1)/8+\beta(\bar{a}^2-1)/8}, & \text{if } p = 2.
\end{cases}$$

(2.1)

For non-zero integers $a$ and $b$, the Hilbert symbol $(a, b)_\infty = -1$ is equal to $-1$ if $a$ and $b$ are 
both negative and 1 otherwise.

From this definition it is easy to deduce some basic facts about Hilbert symbols that we will 
assume tacitly in the remainder of this paper. For any prime $p$ and non-zero integers $a$, $a'$ and 
b $b$ we have that $(a, b)_p = (b, a)_p$ and $(a, 1)_p = 1$. Moreover, if $p$ is an odd prime, $a \not\equiv 0 \pmod{p}$ 
and $a \equiv a' \pmod{p}$ then $(a, b)_p = (a', b)_p$. Finally, if $a, b \not\equiv 0 \pmod{p}$ and $p$ is an odd prime 
then $(a, b) = 1$.

For an $n \times n$ matrix $X$ with rational entries and $i \in \{1, \ldots, n\}$, the $i$th principal minor of 
$X$ is the $i \times i$ submatrix of $X$ formed by the entries that are in the first $i$ rows and the first $i$ 
columns of $X$. We say that $X$ is nondegenerate if its $i$th principal minor is invertible for each 
$i \in \{1, \ldots, n\}$.
Definition 2.7. Let $p$ be a prime or $\infty$, let $X$ be an $n \times n$ nondegenerate matrix with rational entries and, for $i \in \{1, \ldots, n\}$, let $X_i$ be the $i$th principal minor of $X$. Then the Hasse-Minkowski invariant of $X$ with respect to $p$, denoted $C_p(X)$, is either $+1$ or $-1$ according to

$$C_p(X) = (1, -|X_n|) \prod_{i=1}^{n-1} (|X_i|, -|X_{i+1}|).$$

For our purposes the critical property of Hasse-Minkowski invariants is as follows.

Theorem 2.8. [14] Let $X$ and $Y$ be nondegenerate square matrices with rational entries. Then $X \sim Y$ if and only if $C_p(X) = C_p(Y)$ for all primes $p$ and for $p = \infty$.

Lemma 2.9. Let $v$, $k$ and $\lambda$ be positive integers such that $\lambda + 2 < k < v$, and let $[c_1, \ldots, c_t]$ be a $v$-feasible cycle type. The matrix $X_{(v,k,\lambda)}[c_1, \ldots, c_t]$ is positive definite and thus nondegenerate.

Proof. Let $X = X_{(v,k,\lambda)}[c_1, \ldots, c_t]$, let $B = \text{diag}(B_{c_1}(k-\lambda), B_{c_1}(k-\lambda), \ldots, B_{c_t}(k-\lambda))$ and note that $X = B + \lambda J_v$. Now observe that in each row of $B$, the diagonal entry is $k - \lambda \geq 3$ and the sum of the absolute values of the non-diagonal entries is 2. Thus, by the Gershgorin circle theorem [11], every eigenvalue of $B$ is positive and hence $B$ is positive definite. Because $J_v$ is positive semi-definite, $X$ is positive definite. By Sylvester’s criterion [11], this implies that the determinant of every leading principal minor of $X$ is positive and hence that $X$ is nondegenerate.

By Lemma 2.9, we know that $C_p(X_{(v,k,\lambda)}[c_1, \ldots, c_t])$ exists for any parameter set such that $\lambda + 2 < k < v$. We shall use this fact tacitly from now on. The following lemma encapsulates the approach to establishing the nonexistence of coverings that we shall take in this paper.

Lemma 2.10. Let $v$, $k$ and $\lambda$ be positive integers such that $\lambda + 2 < k < v$, and let $[c_1, \ldots, c_t]$ be a $v$-feasible cycle type. There does not exist a symmetric $(v,k,\lambda)$-covering $(V,B)$ whose excess has cycle type $[c_1, \ldots, c_t]$ if either

$$C_p(X_{(v,k,\lambda)}[c_1, \ldots, c_t]) = +1 \quad \text{for some } p \in \{2, \infty\}; \text{ or}$$

$$C_p(X_{(v,k,\lambda)}[c_1, \ldots, c_t]) = -1 \quad \text{for some odd prime } p.$$

Proof. Suppose that $X_{(v,k,\lambda)}[c_1, \ldots, c_t]$ satisfies the hypotheses of the lemma. It is easy to see (for example, see [2]) that the definition of the Hasse-Minkowski invariant implies

$$C_p(I_v) = \begin{cases} -1, & \text{if } p \in \{2, \infty\} \\ +1, & \text{if } p \text{ is an odd prime.} \end{cases}$$

So $X_{(v,k,\lambda)}[c_1, \ldots, c_t] \sim I_v$ by Theorem 2.8. Thus, by Proposition 2.5 there does not exist a symmetric $(v,k,\lambda)$-covering whose excess has cycle type $[c_1, \ldots, c_t]$.

We conclude this section with some useful identities involving Hilbert symbols and Hasse-Minkowski invariants, which we shall use frequently in the paper. Sometimes we will not
reference them explicitly. For any non-zero integers \(a, b, s\) and \(t\), the following hold.

\[
\begin{align*}
(as^2, bt^2)_p &= (a, b)_p \\
(a_1a_2, b)_p &= (a_1, b)_p(a_2, b)_p \\
(a, -a)_p &= 1 \\
(a, a)_p &= (a, -1)_p \\
(a, b)_p &= (-ab, a + b)_p
\end{align*}
\] (2.3)–(2.7)

Equations (2.3)–(2.6) follow easily from our definition of Hilbert symbols (for example, see [7]) and (2.7) is proved in [2]. The following hold for any nondegenerate \(n \times n\) matrix \(X\) whose \((n-1)\)th principal minor is denoted \(X_{n-1}\) and any nondegenerate \(m \times m\) matrix \(Y\) (see [2] for proofs).

\[
\begin{align*}
C_p(X) &= C_p(X_{n-1})(|X|, -|X_{n-1}|)_p \\
C_p(\text{diag}(X, Y)) &= C_p(X)C_p(Y)(-1, -1)_p(|X|, |Y|)_p
\end{align*}
\] (2.8)–(2.9)

Moreover, since \((|X|, -|X_{n-1}|)_p = \pm 1\) by the definition of the Hilbert symbol, (2.8) can be rearranged into \(C_p(X_{n-1}) = C_p(X)(|X|, -|X_{n-1}|)_p\), which is a form we will often use.

## 3 Computing \(C_p(X)\) efficiently

It is possible to calculate the Hasse-Minkowski invariant of a matrix \(X_{(v,k,\lambda)}[c_1, \ldots, c_t]\) directly from Definition 2.7, but this becomes very slow for large \(v\) because it involves computing the determinant of an \(i \times i\) matrix for each \(i \in \{1, \ldots, v\}\). In this section we prove results that allow the Hasse-Minkowski invariant of matrices \(X_{(v,k,\lambda)}[c_1, \ldots, c_t]\) to be efficiently calculated. These results allow us to perform the computational investigations in Section 4 and they are also useful in proving our nonexistence results in Sections 5–8. We focus on the case where the determinant of our matrix is a perfect square, because otherwise the corresponding covering cannot exist by Proposition 2.3.

This section is organised as follows. In Lemma 3.2, we show that we can express \(C_p(X_{(v,k,\lambda)}[c_1, \ldots, c_t])\) in terms of the parameters of \(X_{(v,k,\lambda)}[c_1, \ldots, c_t]\) and the Hasse-Minkowski invariants of the matrices \(B_c(k - \lambda)\). Then we turn our attention to finding expressions for \(C_p(B_n(a))\) for positive integers \(n\) and \(a\). To aid us in this task we define matrices \(B^*_n(a)\), which are related to the matrices \(B_n(a)\), and a recursive sequence \(g_i(a)\) of polynomials in \(a\). In Lemma 3.4 we express \(C_p(B_n(a))\) in terms of the Hasse-Minkowski invariant and determinant of \(B^*_n(a)\) and then in Lemma 3.6 we express these two values in terms of the sequence \(g_i(a)\). These results allow us to prove Lemma 3.7 which gives \(C_p(B_n(a))\) in terms of the sequence \(g_i(a)\). Between them, Lemmas 3.2 and 3.7 give an efficient method of calculating \(C_p(X_{(v,k,\lambda)}[c_1, \ldots, c_t])\).

In the proofs of Lemmas 3.2, 3.4 and 3.6 we shall make use of the fact that if we perform a series of elementary row operations on a matrix followed by the corresponding series of elementary column operations, then the resulting matrix is rationally congruent to the original matrix. This follows from the definition of rational congruence because each elementary row operation can be represented as premultiplication by an elementary matrix \(M\) (which has
rational entries and is invertible) and the corresponding elementary column operation can be represented as postmultiplication by $M^T$. See [11] for details on elementary row operations and elementary matrices.

In [4] the determinant of the matrix $B_\lambda$ was found up to a square term.

**Lemma 3.1.** [4] Let $n$ and $a$ be positive integers such that $n \geq 2$ and $a > 2$. Then there exists a polynomial $h \in \mathbb{Z}[x]$ such that

$$|B_n(a)| = \begin{cases} (a + 2) \cdot h(a)^2, & \text{if } n \text{ is odd,} \\ (a^2 - 4) \cdot h(a)^2, & \text{if } n \text{ is even.} \end{cases}$$

**Lemma 3.2.** Let $\nu$, $k$ and $\lambda$ be positive integers such that $\lambda + 2 < k < \nu$ and let $p$ be a prime or $p = \infty$. Let $[c_1, \ldots, c_t]$ be a $\nu$-feasible cycle type and let $e = |\{i : c_i \text{ is even}\}|$. If $|X_{(\nu,k,\lambda)}[c_1, \ldots, c_t]|$ is a perfect square, then

$$C_p(X_{(\nu,k,\lambda)}[c_1, \ldots, c_t]) = f_p(k - \lambda, \lambda, t, e) \prod_{i=1}^{t} C_p(B_{c_i}(k - \lambda))$$

where

$$f_p(a, \lambda, t, e) = (-1, -1)^{t-1} (a + 2, -1)^{\nu} (a^2 - 4, -1) (a + 2, a^2 - 4) e^{(t-e)} (-\lambda, (a+2)^t(a-2)^e) p.$$ 

**Proof.** Let $X = X_{(\nu,k,\lambda)}[c_1, \ldots, c_t]$ and let $X'$ be the matrix $\text{diag}(X, -\lambda)$. Then

$$C_p(X) = C_p(X')(|X'|, -|X|)_p \quad \text{by rearranging (2.8)}$$

$$= C_p(X')(-\lambda |X|, -|X|)_p \quad \text{since } |X'| = -\lambda |X|$$

$$= C_p(X')(-\lambda, -1)_p \quad \text{by (2.3) since } |X| \text{ is square.}$$

Let $X''$ be the matrix obtained from $X'$ by adding the last row to all other rows and then adding the last column to all other columns. Note that the $v$th principal minor of $X''$ is

$$D = \text{diag}(B_{c_1}, B_{c_2}, \ldots, B_{c_t}).$$

Using the equation above, we have

$$C_p(X) = C_p(X'')(-\lambda, -1)_p \quad \text{since } X'' \sim X'$$

$$= C_p(D)(|X''|, -|D|)_p(-\lambda, -1)_p \quad \text{by (2.8)}$$

$$= C_p(D)(-\lambda, -|D|)_p(-\lambda, -1)_p \quad \text{since } |X''| = |X'| = -\lambda |X|$$

$$= C_p(D) \left( -\lambda, \prod_{i=1}^{t} |B_{c_i}|_p \right) \quad \text{by (2.4) and the definition of } D.$$ 

By repeatedly applying (2.9), we have

$$C_p(D) = \left( \prod_{i=1}^{t} C_p(B_{c_i}) \right) (-1, -1)^{t-1} \left( \prod_{1 \leq i < j \leq t} (|B_{c_i}|, |B_{c_j}|)_p \right).$$

Using Lemma 3.1, (2.6) and (2.4) we have

$$\prod_{1 \leq i < j \leq t} (|B_{c_i}|, |B_{c_j}|)_p = (a + 2, -1)^{\nu} (a^2 - 4, -1)^{t} (a + 2, a^2 - 4) e^{(t-e)}; \text{ and}$$

$$\left( -\lambda, \prod_{i=1}^{t} |B_{c_i}|_p \right) = (-\lambda, (a + 2)^t(a-2)^e)_p = (-\lambda, (a + 2)^t(a-2)^e)_p.$$ 

The result follows by substituting these last three equations into our expression for $C_p(X)$. 

8
Remark. When investigating the existence of symmetric \((v, k, \lambda)-\)coverings with 2-regular excesses for some fixed \((v, k, \lambda)\), Lemmas 2.10 and 3.2 can be viewed as operating in the following way. For each \(p\), we can (in principle) find the set

\[S_p = \{ n \in \{2, \ldots, v\} : C_p(B_n(k - \lambda)) = -1 \}.
\]

By combining Lemmas 2.10 and 3.2, we can then establish, for a given \(t\) and \(e\), that any excess of a symmetric \((v, k, \lambda)-\)covering that consists of \(e\) even cycles and \(t - e\) odd cycles either has an even number of cycles with lengths in \(S_p\) or has an odd number of cycles with lengths in \(S_p\). Which of these two results is established depends on whether \(p \in \{2, \infty\}\) and on the value of \(f_p(k - \lambda, \lambda, t, e)\) in Lemma 3.2. One interesting special case is when \(p\) is an odd prime that does not divide \(\lambda(a^2 - 4)\). Then \(f_p(k - \lambda, \lambda, t, e) = 1\) irrespective of the values of \(t\) and \(e\) and we can conclude that any excess of a symmetric \((v, k, \lambda)-\)covering has an even number of cycles with lengths in \(S_p\).

Next, we introduce a family of sparse square matrices \(B_n^*(a)\) which we later need in the computation of \(C_p(B_n(a))\). We will sometimes abbreviate \(B_n^*(a)\) to \(B_n^*\).

**Definition 3.3.** For positive integers \(a\) and \(n\), where \(n \geq 2\), we define a tridiagonal matrix \(B_n^*(a)\) as follows.

\[
B_2^*(a) = \begin{pmatrix} a - 1 & 1 \\ 1 & a - 1 \end{pmatrix}, \quad B_n^*(a) = \begin{pmatrix} a - 1 & 1 & 0 & 0 & \cdots & 0 \\ 1 & a & 1 & 0 & \cdots & 0 \\ 0 & 1 & a & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & a & 1 \\ 0 & 0 & \cdots & 0 & 1 & a - 1 \end{pmatrix}
\]

for \(n \geq 3\).

**Lemma 3.4.** Let \(a\) and \(n\) be positive integers such that \(a > 2\) and \(n \geq 2\), and let \(p\) be a prime or \(p = \infty\). Then,

\[C_p(B_n(a)) = C_p(B_n^*(a))(-(a + 2)(a - 2)^{n+1}, |B_n^*(a)|)_p.
\]

**Proof.** Let \(Y'\) be the matrix \(\text{diag}(B_n, -1)\). Then

\[
C_p(B_n) = C_p(Y')(|Y'|, -|B_n|)_p \quad \text{by rearranging (2.8)}
\]

\[
= C_p(Y')(-|B_n|, -1)_p \quad \text{by (2.6) since } |Y'| = -|B_n|.
\]

Let \(Y''\) be the matrix obtained from \(Y'\) by adding the last row to the first row and second-last row and then adding the last column to the first column and second-last column. Note that \(B_n^*\) is the \(n\)th principal minor of \(Y''\). Using the equation above, we have

\[
C_p(B_n) = C_p(Y'')(|-B_n|, -1)_p \quad \text{since } Y'' \sim Y'
\]

\[
= C_p(B_n^*)(|Y''|, -|B_n^*|)_p(-|B_n|, -1)_p \quad \text{by (2.8)}
\]

\[
= C_p(B_n^*)(-|B_n|, |B_n^*|)_p \quad \text{by (2.4) since } |Y''| = |Y'| = -|B_n|.
\]

The result now follows by applying Lemma 3.1 and (2.3).
Definition 3.5. For each positive integer \( n \), let \( g_n(a) \) be a polynomial in \( a \) defined by the recurrence

\[
g_1(a) = 1; \\
g_2(a) = a; \\
g_n(a) = ag_{n-1}(a) - g_{n-2}(a) \quad \text{for } n \geq 3.
\]

Note that \( g_n(a) \) is positive for all integers \( n \geq 1 \) and \( a \geq 2 \). We will sometimes abbreviate \( g_n(a) \) to \( g_n \). Below we give \( g_n(a) \) for \( n \in \{1, \ldots, 9\} \).

| \( n \) | \( g_n(a) \) |
|-----|-----|
| 1   | 1   |
| 2   | \( a \) |
| 3   | \( a^2 - 1 \) |
| 4   | \( a^3 - 2a \) |
| 5   | \( a^4 - 3a^2 + 1 \) |
| 6   | \( a^5 - 4a^3 + 3a \) |
| 7   | \( a^6 - 5a^4 + 6a^2 - 1 \) |
| 8   | \( a^7 - 6a^5 + 10a^3 - 4a \) |
| 9   | \( a^8 - 7a^6 + 15a^4 - 10a^2 + 1 \) |

Lemma 3.6. Let \( a \) be an integer such that \( a > 2 \) and let \( p \) be a prime or \( p = \infty \). Then

(a) \( |B_n^*(a)| = (a - 2)g_n(a) \) for each integer \( n \geq 2 \);

(b) \( C_p(B_n^*(a)) = C_p(B_n^*-(a))(-g_n(a), g_{n-1}(a))_p \) for each integer \( n \geq 3 \); and

(c) \( C_p(B_n^*(a)) = (-1, 2 - a)_p \sum_{i=2}^{n} (-g_i(a), g_{i-1}(a))_p \) for each integer \( n \geq 2 \).

Proof. Proof of (a). For each positive integer \( i \geq 2 \), let \( T_i \) be the \( i \times i \) tridiagonal matrix such that every entry of the lead diagonal of \( T_i \) is an \( a \) and every entry of the superdiagonal and subdiagonal is a 1. Using the well-known recursive expression for the determinant of a tridiagonal matrix (see [12] for example), we see that \( |T_i| = g_{i+1} \) for each positive integer \( i \).

When \( n = \{2, 3\} \), \( |B_n^*| \) is easily directly computed. Note that when \( n \geq 4 \), \( B_n^* \) can be obtained from \( T_n \) by adding the \( n \)-dimensional column vectors \((-1, 0, \ldots, 0)^T\) and \((0, \ldots, 0, -1)^T\) to the first and last columns, respectively. Thus, using the multilinearity of the determinant as a function of columns and simplifying, it can be deduced that

\[
|B_n^*| = |T_n| - 2|T_{n-1}| + |T_{n-2}|
= g_{n+1} - 2g_n + g_{n-1}
= (a - 2)g_n,
\]

where the last equality follows by substituting \( g_{n+1} = ag_n - g_{n-1} \). So (a) holds.

Proof of (b). Assume \( n \geq 3 \) and let \( Z' = \text{diag}(B_n^*, -1) \). Then

\[
C_p(B_n^*) = C_p(Z')(|Z'|, -|B_n^*|)_p \quad \text{by rearranging (2.8)}
= C_p(Z')(-|B_n^*|, -1)_p \quad \text{by (2.6) since } |Z'| = -|B_n^*|.
\]
Let $Z''$ be the matrix obtained from $Z'$ by adding the last row to the second-last row and third-last row and then adding the last column to the second-last column and third-last column. Note that $Z' = \text{diag}(B_{n-1}^*, a-2)$ is the $n$th principal minor of $Z''$. Using the equation above, we have

$$C_p(B_n^*) = C_p(Z'')(\lfloor |B_n^*|, -1\rfloor_p$$

$$= C_p(Z')(\lfloor |Z''|, -|Z'|\rfloor_p(\lfloor |B_n^*|, -1\rfloor_p$$

$$= C_p(Z')(\lfloor |B_n^*|, -|Z'\rfloor\rfloor_p(\lfloor |B_n^*|, -1\rfloor_p$$

$$= C_p(Z')(\lfloor |B_n^*|, (a-2)|B_{n-1}^*|\rfloor_p$$

$$= C_p(Z')(\lfloor |B_n^*|, (a-2)|B_{n-1}^*|\rfloor_p$$

$$= C_p(B_n^*)(g_{n-1}g_n, g_{n-1}p$$

$$= C_p(B_n^*)(g_{n-1}g_n, g_{n-1}p$$

$$= C_p(B_n^*)(g_{n-1}g_n, g_{n-1}p$$

Proof of (c). When $n = 2$, following the argument used in the proof of (b) establishes that $C_p(B_2^*) = (-1, 2-a)p(-g_2, g_1)p$. Then, when $n \geq 3$, the statement follows by repeatedly applying (b).

**Lemma 3.7.** Let $n$ and $a$ be integers such that $a > 2$ and $n \geq 2$, and let $p$ be a prime or $p = \infty$. Then

$$C_p(B_n(a)) = (-a + 2)(a-2)^{n+1}, -g_n(a))p\prod_{i=2}^{n} (-g_i(a), g_{i-1}(a))p.$$

**Proof.** Let $\Delta = (a+2)(a-2)^{n+1}$. Combining the results of Lemmas 3.4(a), 3.4(c) and 3.6 we have

$$C_p(B_n) = (-1, 2-a)p(-\Delta, (a-2)g_n)p\prod_{i=2}^{n} (-g_i, g_{i-1})p$$

$$= (-1, 2-a)p(-\Delta, 2-a)p(-\Delta, g_n)p\prod_{i=2}^{n} (-g_i, g_{i-1})p$$

$$= (\Delta, 2-a)p(-\Delta, -g_n)p\prod_{i=2}^{n} (-g_i, g_{i-1})p$$

The result now follows by observing that $(a^2 - 4, 2-a)p = (a + 2, 2-a)p$ by (2.4) and (2.5) and furthermore $(a + 2, 2-a)p = (-a + 2)(2-a), 4)p = 1$ by (2.7) and (2.3). 

Lemmas 3.2 and 3.7 allow us to compute $C_p(X_{\nu,k,\lambda}[c_1, \ldots, c_l])$ for any set of parameters. To apply Lemma 3.7 we need to recursively compute the value of $g_i$ for $1 \leq i \leq c_i$ which can be done in linear time in $c_i$. Then we immediately obtain $C_p(X_{\nu,k,\lambda}[c_1, \ldots, c_l])$ as a product of Hilbert symbols.

**Remark.** Let $v$, $k$ and $\lambda$ be positive integers such that $\lambda + 2 < k < v$, and let $[c_1, \ldots, c_l]$ be a $v$-feasible cycle type. Applying Lemma 2.10 with $p = \infty$, or with $p$ chosen to be a prime that does not divide any of $\lambda(a^2 - 4), g_1(k-\lambda), g_2(k-\lambda), \ldots, g_{c_i}(k-\lambda)$, will never rule out the existence
of a \((v, k, \lambda)\)-covering whose excess has cycle type \([c_1, \ldots, c_t]\). When \(p = \infty\), \(f_\infty(k - \lambda, \lambda, t, e) = (-1)^{t-1}\) in Lemma 3.2 and, by Lemma 3.7, \(C_\infty(B_n(k - \lambda)) = -1\) for any \(n \in \{2, \ldots, v\}\). When \(p\) is a prime that does not divide any of \(\lambda(a^2 - 4), g_1(k - \lambda), g_2(k - \lambda), \ldots, g_v(k - \lambda)\), \(f_p(k - \lambda, \lambda, t, e) = 1\) in Lemma 3.2 and, by Lemma 3.7, \(C_p(B_n(k - \lambda)) = 1\) for any \(n \in \{2, \ldots, v\}\). So in either case it can be seen from Lemma 3.2 that Lemma 2.10 tells us nothing. Since the choice \(p = \infty\) is never of any use, we do not consider it in the remainder of the paper.

4 Observations and general computational results

We begin this section by noting that it can be seen from Lemma 3.2 that for all parameter sets \((v, k, \lambda)\) with \(v \equiv 0 \pmod{4}\), there will exist cycle types such that Lemma 2.10 cannot rule out the existence of a \((v, k, \lambda)\)-covering whose excess has that cycle type. To see this consider a \(v\)-feasible cycle type \([c_1, \ldots, c_t]\) such that \(t \equiv 0 \pmod{4}\), \(|\{i : c_i \text{ is even}\}| \equiv 0 \pmod{4}\) and \(c_{2i-1} = c_{2i}\) for all \(i \in \{1, \ldots, \frac{t}{2}\}\) (for example, \([d^{v/d}]\) for any odd divisor \(d\) of \(v\)). From the last of these conditions it follows that \(\prod_{i=1}^t C_p(B_n(k - \lambda)) = 1\), and from the other two conditions we have that, in Lemma 3.2, \(f_p(a, \lambda, t, e) = 1\) for any odd prime \(p\) and \(f_p(a, \lambda, t, e) = -1\) for \(p = 2\). As an example, one can take \((v, k, \lambda) = (36, 9, 2)\) and cycle types \([3^{12}]\) or \([9^4]\).

In general, we do not expect that there are any parameter sets \((v, k, \lambda)\) for which Lemmas 2.10, 3.2 and 3.7 will completely rule out the existence of a \((v, k, \lambda)\)-covering with 2-regular excess. As we shall see however, for many parameter sets \((v, k, \lambda)\), these results can be used to establish that many cycle types are not realisable as the excess for a \((v, k, \lambda)\)-covering. We begin with a small example of this before moving on to a more general investigation.

Example 4.1. We consider symmetric \((11, 4, 1)\)-coverings. Such a covering necessarily has a 2-regular excess, and Theorem 1.1 implies that this excess has an odd number of cycles. So the possible cycle types of the excess of such a covering are as follows.

\[
[11] \ [2, 2, 7] \ [2, 3, 6] \ [2, 4, 5] \ [3, 3, 5] \ [3, 4, 4] \ [2, 2, 2, 3]
\]

Since the values of \(g_n(3)\) for \(n = 1, \ldots, 11\) are \(1, 3, 8, 21, 55, 144, 377, 987, 2584, 6765, 17711\), respectively, it can be computed that the only choices of \(p\) for which Lemma 2.10 may rule out \((11, 4, 1)\)-coverings with particular excesses are \(2, 3, 5, 7, 11, 13, 17, 19, 29, 41, 47, 89, 199\) (see the remark after Lemma 3.7). For each possible cycle type \([c_1, \ldots, c_t]\) and each choice of \(p\) we can apply Lemmas 3.2 and 3.7 to determine \(C_p(X_{(11,4,1)}[c_1, \ldots, c_t])\) and then determine whether Lemma 2.10 rules out the existence of a \((11, 4, 1)\)-covering whose excess has cycle type \([c_1, \ldots, c_t]\). Below we list for each cycle type all values of \(p\) for which this occurs.

| cycle type | \([11]\) | \([2, 2, 7]\) | \([2, 3, 6]\) | \([2, 4, 5]\) | \([3, 3, 5]\) | \([3, 4, 4]\) | \([2, 2, 2, 3]\) |
|-----------|---------|---------|---------|---------|---------|---------|---------|
| values of \(p\) | 5, 13 | 3, 5 | 2, 5 | 2, 5 |

Of the three possible cycle types that are not ruled out by Lemma 2.10, it transpires that two are realisable and one is not. A symmetric \((11, 4, 1)\)-covering with Hamilton cycle excess can be constructed from the block \([0, 1, 2, 5]\) under the permutation \((0, 1, \ldots, 10)\), and the following list of blocks forms a symmetric \((11, 4, 1)\)-covering with excess cycle type \([2, 3, 6]\).
We performed an exhaustive computer search to rule out the existence of a symmetric \((11, 4, 1)\)-covering with excess cycle type \([3, 3, 5]\).

Obviously for other parameter sets we can apply a similar procedure to attempt to rule out the existence of coverings whose excesses have certain cycle types. Our results on symmetric coverings with \(\lambda = 1\) and \(k \in \{4, 5, 6, 7, 8, 9\}\) are given in Table 1. Here it is infeasible to determine the list of relevant values of \(p\) as in Example 4.1 because finding the prime divisors of \(g_v(a)\) becomes increasingly difficult (for example, when \((v, k, \lambda) = (29, 6, 1)\), \(g_{29}(5) \approx 1.18 \times 10^{19}\)); instead we test each prime \(p < 10^3\).

| \((v, k, \lambda)\) | number of cycle types | number ruled out by Theorem 1.1 | number ruled out by Lemma 2.10 with \(p < 10^3\) | number which may exist |
|---------------------|-----------------------|---------------------------------|---------------------------------|-----------------------|
| \((11, 4, 1)\)      | 14                    | 7                               | 4                              | 3                     |
| \((19, 5, 1)\)      | 105                   | 52                              | 43                             | 10                    |
| \((29, 6, 1)\)      | 847                   | 423                             | 393                            | 31                    |
| \((41, 7, 1)\)      | 7245                  | 3621                            | 3376                           | 248                   |
| \((55, 8, 1)\)      | 65121                 | 32555                           | 30746                          | 1820                  |
| \((71, 9, 1)\)      | 609237                | 304604                          | 292475                         | 12158                 |

Table 1: Consequences of Theorem 1.1 and Lemma 2.10 for symmetric coverings with \(\lambda = 1\) and \(k < 10\).

A cyclic symmetric covering is one whose block set can be obtained by applying a cyclic permutation to a single block. A cyclic symmetric \((v, k, \lambda)\)-covering with 2-regular excess is equivalent to a \((v, k, \lambda, v-3)\)-almost difference set (see [13]). Such coverings necessarily have excesses consisting of a number (possibly one) of cycles of uniform length. Table 2 lists parameter sets \((v, k, \lambda)\) with \(v < 200\) for which we can use Lemmas 3.2, 3.7 and 2.10, choosing \(p < 10^3\), to computationally rule out the existence of a cyclic symmetric covering.

| \(v\) | \(k\) | \(\lambda\) | \(v\) | \(k\) | \(\lambda\) | \(v\) | \(k\) | \(\lambda\) | \(v\) | \(k\) | \(\lambda\) |
|-------|-------|------------|-------|-------|------------|-------|-------|------------|-------|-------|------------|
| 153   | 18    | 2          | 111   | 32    | 9          | 95    | 49    | 25         | 199   | 98    | 48         |
| 37    | 11    | 3          | 157   | 38    | 9          | 53    | 38    | 27         | 199   | 101   | 51         |
| 169   | 23    | 3          | 81    | 34    | 14         | 81    | 47    | 27         | 137   | 87    | 55         |
| 23    | 10    | 4          | \textbf{63} | 30  | 14         | \textbf{123} | 60   | \textbf{29} | 111   | 79    | 56         |
| 53    | 15    | 4          | 63    | 33    | 17         | 123   | 63    | 32         | 117   | 86    | 63         |
| \textbf{27} | \textbf{12} | \textbf{5} | 37    | 26    | 18         | \textbf{135} | \textbf{66} | \textbf{32} | 157   | 119   | 90         |
| 23    | 13    | 7          | 121   | 47    | 18         | 135   | 69    | 35         | 199   | 134   | 90         |
| 161   | 34    | 7          | 137   | 50    | 18         | \textbf{171} | \textbf{84} | \textbf{41} | 161   | 127   | 100        |
| 27    | 15    | 8          | 199   | 65    | 21         | 171   | 87    | 44         | 153   | 135   | 119        |
| 117   | 31    | 8          | \textbf{95} | \textbf{46} | \textbf{22} | 121   | 74    | 45         | 169   | 146   | 126        |

Table 2: Parameter sets \((v, k, \lambda)\) for which Lemma 2.10 rules out the existence of a cyclic symmetric covering.

An open problem posed in [13] is to find \((v, \frac{v-3}{2}, \frac{v-7}{4}, v - 3)\)-almost difference sets in \(\mathbb{Z}_v\) where \(v \equiv 3 \pmod{4}\) (these are of interest because they produce sequences with desirable
autocorrelation properties). Observe that the parameter sets in boldface in Table 2 establish the nonexistence of some \((v, \frac{v-3}{2}, \frac{v-7}{4}, v-3)\)-almost difference sets. Furthermore, using primes \(p < 10^3\), we can similarly rule out the existence of \((v, \frac{v-3}{2}, \frac{v-7}{4}, v-3)\)-almost difference sets for the following values of \(v\), where \(v < 800\) (the first eight of which are contained in Table 2).

\[
\begin{align*}
23, 27, 63, 95, 123, 135, 171, 199, 207, 215, 231, 243, 255, 267, \\
271, 307, 343, 351, 355, 363, 367, 371, 375, 399, 407, 411, 471, \\
495, 543, 555, 567, 651, 663, 671, 675, 699, 703, 711, 783
\end{align*}
\]

Even when Lemma 2.10 does not rule out the existence of a cyclic symmetric covering it may place restrictions on the possible cycle types of the excess. For example, from Lemmas 2.10, 3.2 and 3.7, using primes \(p < 10^3\), it follows that if there exists a cyclic symmetric \((51, 24, 11)\)-covering, then its excess can only be a Hamilton cycle. Similarly, if there exists a cyclic symmetric \((75, 36, 17)\)-covering, then its excess can only consist of 3 cycles of length 25.

Below we list the values of \(v \equiv 3 \pmod{4}\), where \(v < 800\), for which we can show (using \(p < 10^3\)) that if a cyclic symmetric \((v, \frac{v-3}{2}, \frac{v-7}{4})\)-covering does exist then it can only have a Hamilton cycle excess. In the context of almost difference sets, this means the difference repeated \(\lambda + 1\) times is relatively prime to \(v\). We exclude prime values of \(v\) from the list since the result is trivial in those cases.

\[
\begin{align*}
15, 51, 87, 111, 143, 159, 299, 303, 319, 335, 339, 415, 447, \\
511, 519, 535, 559, 591, 611, 635, 655, 687, 731, 767, 771
\end{align*}
\]

In the remaining sections of this paper we address some cases in which we can prove the nonexistence of coverings whose excesses have certain cycle types. In Section 5 we show that, for a parameter set \((v, k, \lambda)\), choosing a value of \(p\) dividing \(k - \lambda\) allows us to give a quite general restriction on what cycle types the excesses of symmetric \((v, k, \lambda)\)-coverings may have. In Sections 6 and 7, we concentrate on the case of coverings whose excess is a Hamilton cycle or a number of cycles of equal length. These cases are of particular interest because, as we have seen, any 2-regular excess of a cyclic symmetric covering is necessarily of one of these forms. Finally, in Section 8, we consider coverings whose excess is composed of 2-cycles and 3-cycles. Results of Bose and Connor (see [2]) already cover the case in which the excess is composed entirely of 2-cycles or entirely of 3-cycles.

## 5 Choices of \(p\) that divide \(k - \lambda\)

In this section we obtain a general result on the nonexistence of symmetric \((v, k, \lambda)\)-coverings with certain excesses by choosing values of \(p\) which divide \(k - \lambda\). We take advantage of the fact that, under this choice of \(p\), the \(p\)-factorisations of most of the terms \(g_i(k - \lambda)\) are well behaved.

**Theorem 5.1.** Let \(v, k\) and \(\lambda\) be positive integers such that \(k > \lambda + 2\) and \(v = \frac{k(k-1)-2}{\lambda} + 1\). For any prime \(p \equiv 3 \pmod{4}\) such that \(p\) does not divide \(\lambda\), and \(p\) has odd multiplicity in the prime factorisation of \(k - \lambda\), there does not exist a symmetric \((v, k, \lambda)\)-covering with 2-regular excess that contains an odd number of cycles with lengths divisible by 4 and no cycle of length divisible by 2\(p\).
It follows from (5.2) that
\[ n^2 \]
Note that for any positive integer \( i \) or \( -i \) is odd, and \( g_i \) is even. From Lemma 3.7,
\[ C_p(B_n) = \prod_{i=2}^{n} (-g_i, g_{i-1})_p. \] (5.2)

Observe that \((-a+2)(a-2)^n+1, -g_n) = 1\) for any \( n \) because \( p \) does not divide \( a+2, a-2, \) or \( g_n \) when \( n \) is odd, and \((-a+2)(a-2) \equiv 2^2 (mod p) \) when \( n \) is even. From Lemma 3.7,
\[ C_p(B_n) = \prod_{i=2}^{n} (-g_i, g_{i-1})_p. \] (5.2)

Note that for any positive integer \( i \) we have
\[
(-g_{2i}, g_{2i-1})_p (-g_{2i+1}, g_{2i})_p = (-g_{2i}, g_{2i-1})_p (g_{2i-1}, g_{2i})_p \quad \text{since} \quad g_{2i-1} \equiv -g_{2i+1} (mod p) \quad \text{by (5.1)}
\]
\[
= (-g_{2i}, g_{2i-1})_p \quad \text{by (2.4)}
\]
\[
= (-1, g_{2i-1})_p \quad \text{by (2.3)}
\]
\[
= 1 \quad \text{since} \quad g_{2i-1} \equiv \pm 1 (mod p) \quad \text{by (5.1)}.
\]

It follows from (5.2) that
\[ C_p(B_n) = \begin{cases} 
1, & \text{if } n \text{ is odd;} \\
(-g_n, g_{n-1})_p, & \text{if } n \text{ is even.} 
\end{cases} \]

For even \( n \), it follows from (5.1) that \( g_{n-1} \equiv (-1)^{n/2-1} (mod p) \) and, since \( n \) is not divisible by \( 2p \), that \( g_n = g_n p^\alpha \) for some integer \( g_n \) not divisible by \( p \). Thus, using (2.1), for even \( n \),
\[
(-g_n, g_{n-1})_p = \left( \frac{(-1)^{(n/2-1)}}{p} \right)^\alpha.
\]

It now follows that \( C_p(B_n) = -1 \) if and only if \( n \equiv 0 (mod 4) \) from basic properties of Legendre symbols (note that, from our hypotheses, \( p \equiv 3 (mod 4) \) and \( \alpha \) is odd).

It is easy to find infinite families of symmetric coverings with specified excesses whose existence is ruled out by Theorem 5.1. The following corollary, easily proved by setting \( p = 3 \) in Theorem 5.1, gives one example.

**Corollary 5.2.** If \( k \equiv 7, 31, 34, 58 (mod 72) \) and \( v = k(k-1) - 1 \), then there does not exist a symmetric \((v, k, 1)\)-covering with excess having cycle type \([2, 3, 4^{(v-5)/4}]\).
In Table 3 we list the parameters \((v, k, \lambda)\) where \(1 \leq \lambda \leq 2\) and \(\lambda + 2 < k < 30\) for which there exists a prime \(p \equiv 3 \pmod{4}\) that has odd multiplicity in the prime factorisation of \(k - \lambda\). For each parameter set, we uniformly at random sample 1000 distinct integer partitions of \(v\) which are \(v\)-feasible cycle types, or consider all such partitions if \(v\) is small. Of the cycle types not forbidden as excesses by Theorem 1.1, we list the proportion which are ruled out using Theorem 5.1.

| \(v\) | \(k\) | \(\lambda\) | \(p\) | proportion ruled out | \(v\) | \(k\) | \(\lambda\) | \(p\) | proportion ruled out |
|---|---|---|---|---|---|---|---|---|---|
| 11 | 4 | 1 | 3 | 0.143 | 10 | 5 | 2 | 3 | 0.167 |
| 41 | 7 | 1 | 3 | 0.206 | 28 | 8 | 2 | 3 | 0.312 |
| 55 | 8 | 1 | 7 | 0.422 | 36 | 9 | 2 | 7 | 0.392 |
| 131 | 12 | 1 | 11 | 0.412 | 78 | 13 | 2 | 11 | 0.442 |
| 155 | 13 | 1 | 3 | 0.097 | 91 | 14 | 2 | 3 | 0.143 |
| 209 | 15 | 1 | 7 | 0.264 | 120 | 16 | 2 | 7 | 0.351 |
| 239 | 16 | 1 | 3 | 0.036 | 136 | 17 | 2 | 3 | 0 |
| 379 | 20 | 1 | 19 | 0.458 | 210 | 21 | 2 | 19 | 0 |
| 461 | 22 | 1 | 3 | 0.021 | 253 | 23 | 2 | 3 | 0.0356 |
| 461 | 22 | 1 | 7 | 0.171 | 253 | 23 | 2 | 7 | 0.285 |
| 505 | 23 | 1 | 11 | 0.296 | 276 | 24 | 2 | 11 | 0 |
| 551 | 24 | 1 | 23 | 0.444 | 300 | 25 | 2 | 23 | 0.485 |
| 599 | 25 | 1 | 3 | 0.01 | 325 | 26 | 2 | 3 | 0.0179 |
| 755 | 28 | 1 | 3 | 0.00596 | 406 | 29 | 2 | 3 | 0.0207 |
| 811 | 29 | 1 | 7 | 0.0936 |

Table 3: Proportion of cycle types ruled out by Lemma 5.1 out of those which were not already ruled out by Theorem 1.1 from a uniform random sample of \(v\)-feasible cycle types.

### 6 Hamilton cycle excesses

In this section we investigate the existence of symmetric \((v, k, \lambda)\)-coverings whose excess is a Hamilton cycle. We start with some computational results. We compute \(C_p(X_{v,k,\lambda}[v])\) for \(1 \leq \lambda \leq 5\) and \(\lambda + 2 < k < 30\) and \(p < 10^4\). In our search space, there are 18 possible parameter sets \((v, k, \lambda)\) for a symmetric covering on even number of points \(v\); of these, 12 cases are ruled out by Theorem 1.1 and only 5 are ruled out by Lemma 2.10. On the other hand, there are 61 possible parameter sets \((v, k, \lambda)\) where \(v\) is odd; of these, none are ruled out by Theorem 1.1 and 26 are ruled out by Lemma 2.10. Consequently, we focus our attention on the case where \(v\) is odd.

Table 4 is a summary of parameters for symmetric coverings which cannot have a Hamilton cycle excess by Lemma 2.10 and which are not ruled out by Theorem 1.1. Although there does not appear to be an obvious pattern in the list of primes \(p\) which rule out the existence of coverings with Hamilton cycle excesses, we observe that values of \(p\) that are odd and divide \(k\) are often effective when \(\lambda = 2\); they are marked in boldface. Next, we generalise this pattern to investigate which cases can be ruled out with a prime \(p\) that divides \(k - \lambda + 2\).

The remainder of this section is organised as follows. For choices of \(p\) that divide \(a + 2\), Lemmas 6.1 and 6.2 give results about the behaviour of \(g_i(a) \pmod{p}\) and Lemma 6.3 finds
Table 4: Parameter sets \((v, k, \lambda)\) for which Lemma 2.10 rules out the existence of a symmetric covering with Hamilton cycle excess.

| \(v\) | \(k\) | \(\lambda\) | \(p\)     | \(v\) | \(k\) | \(\lambda\) | \(p\)     |
|-------|------|-------|---------|-------|------|-------|---------|
| 55    | 8    | 1     | 4,307   | 37    | 11   | 3     | 73       |
| 109   | 11   | 1     | 1307    | 169   | 23   | 3     | 337,2027 |
| 305   | 18   | 1     | 6709    | 271   | 29   | 3     | 3793     |
| 341   | 19   | 1     | 557,2417| 23    | 10   | 4     | 229      |
| 21    | 7    | 2     | 7,13    | 53    | 15   | 4     | 317      |
| 28    | 8    | 2     | 2,3     | 116   | 22   | 4     | 173,347  |
| 45    | 10   | 2     | 29,149  | 127   | 23   | 4     | 1777     |
| 55    | 11   | 2     | 11,109,197| 78    | 13   | 2     | 2,5      |
| 91    | 14   | 2     | 7,223   | 27    | 12   | 5     | 2,3,107  |
| 105   | 15   | 2     | 59,419,509| 93    | 22   | 5     | 991      |
| 153   | 18   | 2     | 5,71,101,2447,5303| 111   | 24   | 5     | 2,3      |
| 171   | 19   | 2     | 19,113,227,1367,4217,5813| 141   | 27   | 5     | 281      |
| 190   | 20   | 2     | 37,113,797| 163   | 29   | 5     | 2281     |
| 231   | 22   | 2     | 11,41   | 253   | 23   | 2     | 23,43    |
| 325   | 26   | 2     | 19,29,4549| 351   | 27   | 2     | 2,3,71,233,1637|
| 406   | 29   | 2     | 41,461  |        |      |       |          |

an expression for \(C_p(B_n(a))\). Lemmas 6.2 and 6.3 will also be used in Section 7. We then use these results, along with the technical Lemma 6.4, to prove Theorem 6.5 which establishes the nonexistence of a symmetric \((v, k, \lambda)\)-covering with Hamilton cycle excess for an infinite number of parameter sets.

**Lemma 6.1.** If \(p\) is a prime and \(a, s\) and \(\alpha\) are positive integers such that \(a + 2 = p^\alpha s\), then \(g_n(a) \equiv (-1)^{n+1}n \pmod{p^\alpha}\) for each positive integer \(n\).

**Proof.** Obviously \(a \equiv -2 \pmod{p^\alpha}\). Using this and the recursive definition of \(g_n\), the result follows easily by induction. 

**Lemma 6.2.** Let \(p\) be an odd prime and \(a\) and \(n\) be positive integers such that \(a > 2, a + 2 \equiv 0 \pmod{p}\), and \(n \equiv 0 \pmod{p}\). Let \(sp^\alpha\) and \(\bar{\eta}p^\delta\) be the \(p\)-factorisations of \(a + 2\) and \(n\) respectively. If \((p, \alpha) \neq (3, 1)\), then \(g_n(a) = \bar{g}p^\delta\) for some integer \(\bar{g} \equiv (-1)^{n+1}\bar{n} \pmod{p}\).

**Proof.** We show that \(\frac{a}{p}\) is an integer congruent to \((-1)^{n+1}\bar{n}\) modulo \(p\) which will suffice to prove the result. The value of \(g_n\) is defined by a second order recurrence relation. Solving this, we see that \(g_n = \frac{\zeta_1 - \zeta_2}{\zeta_1 - \zeta_2} \zeta_1 = \frac{1}{2}(a + \sqrt{a^2 - 4}), \zeta_2 = \frac{1}{2}(a - \sqrt{a^2 - 4})\). Let \(b = \sqrt{a^2 - 4}\).
Now,

\[ g_n = \frac{1}{2n^b} \left( (a + b)^n - (a - b)^n \right) \]

\[ = \frac{1}{2n^b} \left( \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i - \sum_{i=0}^{n} \binom{n}{i} a^{n-i} (-1)^i b^i \right) \]

\[ = \frac{1}{2n^b} \left( 2 \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \binom{n}{2i+1} a^{n-2i-1} b^{2i+1} \right) \]

\[ = \frac{1}{2n^b} \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \binom{n}{2i+1} a^{n-2i-1} (a^2 - 4)^i \]

\[ = \frac{1}{2n^b} \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} T_i \]

where, for each \( i \in \{0, \ldots, \lfloor \frac{n-1}{2} \rfloor \}, \)

\[ T_i = \left( \frac{n}{2i+1} \right) (sp^a - 2)^{n-2i-1} sp^a (sp^a - 4)^i. \]

Since \( n = \bar{m}p^\delta \), it is clear that \( T_0 \) is divisible by \( p^\delta \). For each \( i \in \{1, \ldots, \lfloor \frac{n-1}{2} \rfloor \}, \) we will show that \( T_i \) is divisible by \( p^{\delta+1} \).

Fix \( j \in \{1, \ldots, \lfloor \frac{n-1}{2} \rfloor \} \) and let \( \bar{m}p^\delta \) be the \( p \)-factorisation of \( 2j + 1 \). Since \( j, \alpha, \bar{m} \geq 1, (p, \alpha) \neq (3, 1) \) and \( j = \frac{m^{\delta-1}}{2} \), it is not difficult to see that \( \alpha j \geq j > \xi \). Note that \( T_j \) is divisible by \( p^\alpha \). If \( \xi > \delta \), then it can be seen that \( \alpha j > \xi \geq \delta + 1 \) and hence that \( T_j \) is divisible by \( p^{\delta+1} \). If \( \xi \leq \delta \), then \( \binom{n}{2j+1} = \binom{n}{2j+1} \binom{2j+1}{\frac{n}{2j+1}} = p^{\delta-\xi} \frac{n}{m} \binom{2j+1}{\frac{n}{2j+1}} \), and so \( \binom{n}{2j+1} \) is divisible by \( p^{\delta-\xi} \). So \( T_j \) is divisible by \( p^{\alpha j + \delta - \xi} \), \( \alpha j \geq \xi + 1 \), and \( T_j \) is divisible by \( p^{\delta+1} \).

So \( T_i \) is divisible by \( p^{\delta+1} \) for each \( i \in \{1, \ldots, \lfloor \frac{n-1}{2} \rfloor \} \) and \( T_0 \) is divisible by \( p^\delta \). It follows that \( \frac{2n}{p^\delta} \) is an integer and

\[ \frac{g_n}{p^\delta} \equiv \frac{1}{2n^b} \frac{T_0}{p^\delta} \equiv \frac{1}{2n^b} \bar{m} (sp^a - 2)^{n-1} \equiv \frac{1}{2n^b} \bar{m} (-2)^{n-1} \equiv (-1)^{n+1} \bar{m} \pmod{p}. \]

The result follows.

\[ \square \]

**Lemma 6.3.** Let \( p \) be an odd prime and \( a \) and \( n \) be positive integers such that \( a > 2 \) and \( a + 2 \equiv 0 \pmod{p} \). Then

(i) \( C_p(B_n(a)) = \left( \frac{(-1)^n}{p} g^n a^a \right) \); and

(ii) \( C_p(B_n(a)) = \left( \frac{(-1)^n}{p} \right) a \) when \( n \not\equiv 0 \pmod{p} \);

where \( sp^a \) and \( \bar{m}p^\delta \) are the \( p \)-factorisations of \( a + 2 \) and \( g_n(a) \) respectively.
Proof. Using Lemma 6.1, it is not difficult to see that (i) implies (ii), so it suffices to show that (i) holds. From Lemma 3.7, we have

\[ C_p(B_n) = (-a + 2)(a - 2)^{n+1}, -g_n) = \prod_{i=2}^{n} (-g_i, g_{i-1})p. \] (6.1)

Next, we find an expression for \( \prod_{i=2}^{n} (-g_i, g_{i-1})_p \).

If neither \( g_i \) nor \( g_{i-1} \) is divisible by \( p \), then \( (-g_i, g_{i-1})_p = 1 \). Thus, by Lemma 6.1,

\[ \prod_{i=2}^{n} (-g_i, g_{i-1})_p = \prod_{i \in S} (-g_i, g_{i-1})_p, \]

where \( S = \{ i \in \{2, \ldots, n \} : i \equiv 0, 1 \pmod{p} \} \).

For each integer \( j \equiv 0 \pmod{p} \), let \( g_jp^\beta \) be the \( p \)-factorisation of \( g_j \). Note that \( \beta_n = \beta \).

For each integer \( j \equiv 0 \pmod{p} \), it can be seen using Lemma 6.1 that both \(-g_{j+1} \) and \( g_{j-1} \) are congruent to \((-1)^{j+1} \) modulo \( p \) and hence, by (2.1), we have

\[ (-g_{j+1}, g_j)_p = (-g_j, g_{j-1})_p = \left( \frac{(-1)^{j+1}}{p} \right)^\beta. \]

Obviously this implies that \( (-g_{j+1}, g_j)_p(-g_j, g_{j-1})_p = 1 \) for each integer \( j \equiv 0 \pmod{p} \). Using these facts it can be seen that

\[ \prod_{i=2}^{n} (-g_i, g_{i-1})_p = \begin{cases} 1, & \text{if } n \not\equiv 0 \pmod{p}; \\ (-g_n, g_{n-1})_p = \left( \frac{(-1)^{n+1}}{p} \right)^\beta, & \text{if } n \equiv 0 \pmod{p}. \end{cases} \] (6.2)

The proof now splits into cases according to whether \( n \) is odd or even.

Case 1. Suppose that \( n \) is odd. Then, by (6.2), \( \prod_{i=2}^{n} (-g_i, g_{i-1})_p = 1. \) So, from (6.1),

\[ C_p(B_n) = (-a - 2, -g_n)_p \]
\[ = (-sp^\alpha, -g^p\beta)_p \]
\[ = (s, g^p\beta)_p(-p^\alpha, g)_p(-p^\alpha, -p^\beta)_p \] using (2.3)
\[ = \left( \frac{s}{p} \right)^\beta \left( \frac{g}{p} \right)^\alpha \left( \frac{-1}{p} \right)^{\alpha + \beta} \] by (2.4)

Using basic properties of Legendre symbols, the result follows.

Case 2. Suppose that \( n \) is even. Then, by (6.2), \( \prod_{i=2}^{n} (-g_i, g_{i-1})_p = \left( \frac{-1}{p} \right)^\beta. \) So, from (6.1),

\[ C_p(B_n) = ((a + 2)(2 - a), -g_n)_p \left( \frac{-1}{p} \right)^\beta \]
\[ = (sp^\alpha(4 - sp^\alpha), -g^p\beta)_p \left( \frac{-1}{p} \right)^\beta \]
\[ = (s(4 - sp^\alpha), -g^p\beta)_p(p^\alpha, g)_p(p^\alpha, -p^\beta)_p \left( \frac{-1}{p} \right)^\beta \] using (2.3)
\[ = \left( \frac{s}{p} \right)^\beta \left( \frac{g}{p} \right)^\alpha \left( \frac{-1}{p} \right)^{\alpha + \alpha} \left( \frac{-1}{p} \right)^\beta \] by (2.4)

Using basic properties of Legendre symbols, the result follows (note that \( 4 = 2^2 \)).
Lemma 6.4. Let $v$, $k$ and $\lambda$ be positive integers such that $k > \lambda + 2$ and $v = \frac{k(k-1)-2}{\lambda} + 1$, and let $p$ be an odd prime such that $k - \lambda + 2 \equiv 0 \pmod{p}$. Then

(i) $\lambda v = (\lambda - 2)^2 + sp^\alpha(sp^\alpha + 2\lambda - 5)$ where $sp^\alpha$ is the $p$-factorisation of $k - \lambda + 2$; and

(ii) $v \equiv 0 \pmod{p}$ if and only if $\lambda \equiv 2 \pmod{p}$.

Proof. Note that $\alpha \geq 1$. Because $v = \frac{k(k-1)-2}{\lambda} + 1$ and $k = sp^\alpha + \lambda - 2$, a straightforward calculation yields (i).

Suppose that $\lambda \equiv 2 \pmod{p}$. Then $p$ divides $\lambda - 2$ and $p$ does not divide $\lambda$. So it follows from (i) that $v \equiv 0 \pmod{p}$. Now suppose that $v \equiv 0 \pmod{p}$. Then it follows immediately from (i) that $p$ divides $(\lambda - 2)^2$ and hence that $\lambda \equiv 2 \pmod{p}$. \hfill \Box

Theorem 6.5. Let $v$, $k$ and $\lambda$ be positive integers such that $k > \lambda + 2$, $v = \frac{k(k-1)-2}{\lambda} + 1$ and $v$ is odd. There does not exist a symmetric $(v,k,\lambda)$-covering whose excess is a Hamilton cycle if there is a prime $p \equiv 3 \pmod{4}$ that divides both $v$ and $k - \lambda + 2$ and such that either

(i) $\alpha$ is odd, $(p,\alpha) \neq (3,1)$, and either $\lambda > 2$ and $\alpha < 2\gamma$ or $\lambda = 2$; or

(ii) $\lambda > 2$, $\alpha = 2\gamma$, and $\delta$ is odd;

where $sp^\alpha$ and $\bar{v}p^\delta$ are the $p$-factorisations of $k - \lambda + 2$ and $v$ respectively, and $\ell p^\gamma$ is the $p$-factorisation of $\lambda - 2$ if $\lambda > 2$.

Proof. Let $p$ be a prime satisfying the hypotheses of the lemma and let $sp^\alpha$ and $\bar{v}p^\delta$ be the $p$-factorisations of $k - \lambda + 2$ and $v$ respectively. Note that $\alpha, \delta \geq 1$. Let $X = X_{(v,k,\lambda)}[v]$. We may assume that $|X|$ is a perfect square for otherwise we are finished by Proposition 2.3. By Lemma 2.10 it suffices to show that $C_p(X) = -1$.

By Lemma 3.2, remembering that $v$ is odd, we have

$$C_p(X) = C_p(B_v(k-\lambda))(-\lambda, k-\lambda + 2)_p$$

$$= C_p(B_v(k-\lambda))(-\lambda, sp^\alpha)_p$$

$$= C_p(B_v(k-\lambda))\left(\frac{-\lambda}{p}\right)^\alpha,$$

where we used (2.1) and the fact that $p$ does not divide $\lambda$ to deduce the last equality. By Lemma 6.2, $g_v(k-\lambda) = \bar{v}p^\delta$ for some integer $\bar{v} \equiv \bar{v} \pmod{p}$ and thus, by Lemma 6.3(i) (noting that $(p,\alpha) \neq (3,1)$), we have

$$C_p(X) = \left(\frac{-1)^{\alpha+\alpha+\delta^\alpha}}{p}\right)\left(\frac{-\lambda}{p}\right)^\alpha = \left(\frac{(-1)^{\alpha+\alpha+\delta^\alpha}}{p}\right)^\alpha. \quad (6.3)$$

Case 1. Suppose that $\lambda = 2$, $\alpha$ is odd and $(p,\alpha) \neq (3,1)$. By Lemma 6.4(i), $2\bar{v}p^\delta = s(sp^\alpha - 1)p^\alpha$. So, since $s(sp^\alpha - 1) \equiv -s \pmod{p}$, it follows that $\delta = \alpha$ and $2\bar{v} \equiv -s \pmod{p}$. Then (6.3) implies $C_p(X) = \left(\frac{-1}{p}\right) = -1$ as required.

Case 2. Suppose that $\lambda > 2$. Let $\ell p^\gamma$ be the $p$-factorisation of $\lambda - 2$. Because $v = \bar{v}p^\delta$ and $\lambda = \ell p^\gamma + 2$, Lemma 6.4(i) implies that

$$\lambda \bar{v}p^\delta = (\ell p^\gamma)^2 + sp^\alpha(2(\ell p^\gamma + 2) - 5 + sp^\alpha)$$

$$\lambda \bar{v}p^{\delta-\alpha} = \ell^2 p^{2\gamma-\alpha} + s(2\ell p^\gamma + sp^\alpha - 1). \quad (6.4)$$
Recall that $\alpha \geq 1$ and note that $\lambda \equiv 2 (\mod p)$ by Lemma 6.4(ii), so $\gamma \geq 1$. The proof now splits into subcases according to whether the assumptions of (i) or (ii) hold.

**Case 2a.** Suppose further that $\alpha$ is odd, $\alpha < 2\gamma$ and $(p, \alpha) \neq (3, 1)$. Then the right hand side of (6.4) is an integer congruent to $-s \mod p$. So $\delta = \alpha$ and $\lambda \bar{v} \equiv -s (\mod p)$. Now, using (6.3), we have $C_p(X) = (\frac{-s^2}{p}) = (-\frac{1}{p}) = -1$ as required.

**Case 2b.** Suppose further that $\alpha = 2\gamma$ and $\delta$ is odd. Then the right hand side of (6.4) is an integer and, because $p$ does not divide $\lambda$, it follows that $\delta \geq \alpha + 1$. So $p$ divides the right hand side of (6.4) and it follows that $s \equiv \ell^2 (\mod p)$. Now, using (6.3), we have $C_p(X) = (\frac{-s^2}{p}) = (\frac{-\ell^2}{p}) = -1$ as required. 

**Remark.** It can be shown that Theorem 6.5 is close to the best result achievable via Lemma 2.10. Specifically, if $k > \lambda + 2$, $k - \lambda + 2 \equiv 0 (\mod p)$, $v = \frac{k(k-1)-2}{\lambda} + 1$ and $|X_(v,k,\lambda)[v]|$ is a perfect square, but the hypotheses of Theorem 6.5 do not hold (because $p \neq 3 (\mod 4)$ or $v \neq p (\mod 2p)$ or because (i) and (ii) fail), then $C_p(X_(v,k,\lambda)[v]) = 1$ unless $(p, \alpha) = (3, 1)$ and $s \equiv 1 (\mod 3)$. When $(p, \alpha) = (3, 1)$ and $s \equiv 1 (\mod 3)$, we have $C_3(X_(v,k,\lambda)[v]) = -1$ for some $v$ and $C_3(X_(v,k,\lambda)[v]) = 1$ for other $v$. In the interests of brevity we do not prove any of this here, however.

We give an example of an infinite family of parameter sets for which Theorem 6.5 rules out the existence of a symmetric covering with Hamilton cycle excess.

**Corollary 6.6.** Suppose $p$ is an odd prime, where $p \equiv 3 (\mod 4)$, and $\alpha$ is an odd positive integer, such that $(p, \alpha) \neq (3, 1)$. Then there does not exist a symmetric $(\frac{1}{2}p^\alpha(p^\alpha - 1), p^\alpha, 2)$-covering with Hamilton cycle excess.

### 7 Excesses composed of uniform length cycles

In this section we focus on establishing the nonexistence of symmetric $(v, k, \lambda)$-coverings with excesses consisting of a number of cycles of the same length. We begin with some computational results. Table 5 lists cycle types of the form $[nt]$ that can be ruled out by Lemma 2.10 as excesses of symmetric $(v, k, \lambda)$-coverings for $p < 10^4$, $1 \leq \lambda \leq 5$ and $\lambda + 2 < k < 30$. It does not include cycle types ruled out by Theorem 1.1 or those of the form $[v]$ (the latter are listed in Table 4).

As in the previous section, we note that when $1 \leq \lambda \leq 5$ and $\lambda + 2 < k < 30$, more cases can be ruled out using Lemma 2.10 when $v$ is odd than when $v$ is even. Furthermore, Theorem 1.1 has already ruled out a significant portion of the cases when $v$ is even but none of the cases when $v$ is odd. Consequently we investigate the case in which $v$ is odd, and hence both the number of cycles in the excess and the cycle length are odd. Theorem 7.1 treats choices of $p$ that do not divide the cycle length and Theorem 7.3 treats choices of $p$ that do. In Table 5, we mark in boldface the choices of $p$ for which Theorem 7.1 or 7.3 can be used to rule out the case.

**Theorem 7.1.** Let $n, t, k$ and $\lambda$ be positive integers such that $k > \lambda + 2$, $nt = \frac{k(k-1)-2}{\lambda} + 1$ and $nt$ is odd. There does not exist a symmetric $(nt, k, \lambda)$-covering whose excess consists of $t$ cycles of length $n$ if there is an odd prime $p$ such that $k - \lambda + 2 \equiv 0 (\mod p)$, $n \neq 0 (\mod p)$ and
| $v$ | $k$ | $\lambda$ | $[n']$ | $p$ | $v$ | $k$ | $\lambda$ | $[n']$ | $p$ |
|-----|-----|--------|-------|----|-----|-----|--------|-------|----|
| 55  | 8   | 1      | 11$^5$| 43, 307 | 55  | 23   | 2      | 11$^2$| 43   |
| 155 | 13  | 1      | 31$^5$| 2, 7    | 253 | 23   | 2      | 11$^{21}$| 23   |
| 305 | 18  | 1      | 61$^5$| 6709   | 300 | 25   | 2      | 2$^{150}$| 3, 7 |
| 341 | 19  | 1      | 31$^{11}$| 557, 2417 |       |       |       | 6$^{50}$| 3, 7 |
| 505 | 23  | 1      | 5$^{101}$| 2, 3   |       | 2, 3  |       | 10$^{30}$| 3, 7 |
|     |     |        | 101$^5$| 2, 3   |       |       |       | 30$^{10}$| 3, 7 |
|     |     |        |        | 50$^6$| 3, 7 |
| 15  | 6   | 2      | 3$^5$ | 2, 3   | 325 | 26   | 2      | 5$^{65}$| 19, 29 |
| 21  | 7   | 2      | 7$^3$ | 7, 13  | 351 | 27   | 2      | 3$^{117}$| 2, 3 |
| 28  | 8   | 2      | 4$^7$ | 2, 3   |       |       |       | 65$^{5}$| 2, 13, 19, 29 |
| 45  | 10  | 2      | 9$^5$ | 2, 5   |       |       |       | 15$^3$| 2, 71 |
| 55  | 11  | 2      | 11$^5$| 5, 197 |       |       |       | 27$^{13}$| 2, 3, 71 |
| 78  | 13  | 2      | 6$^{13}$| 2, 5  |       |       |       | 39$^9$| 2, 3 |
| 91  | 14  | 2      | 7$^{13}$| 2, 223 |       |       |       | 117$^{3}$| 2, 71, 233, 1637 |
| 105 | 15  | 2      | 15$^7$| 3, 5, 59, 509 | 406 | 29   | 2      | 14$^{29}$| 41, 461 |
|     |     |        | 21$^5$| 3, 5, 419 |       |       |       |       |       |
| 120 | 16  | 2      | 4$^{30}$| 2, 3  |       |       |       | 169 | 23, 3 |
|     |     |        | 12$^{10}$| 2, 3 |       |       |       | 176 | 27, 4 |
|     |     |        | 26$^6$| 2, 3   |       |       |       | 88$^2$| 3, 7 |
|     |     |        | 60$^2$| 2, 3   |       |       |       |       |       |
| 153 | 18  | 2      | 5$^{11}$| 2, 5  |       |       |       | 15$^9$| 2, 3 |
|     |     |        | 9$^{17}$| 5, 71 |       |       |       | 5$^3$| 2, 3 |
|     |     |        | 17$^9$| 101   |       |       |       | 17$^9$| 2, 3 |
|     |     |        | 51$^3$| 2, 5, 101, 2447, 5303 | 27 | 12   | 5      | 9$^3$| 2, 107 |
| 171 | 19  | 2      | 19$^9$| 19, 113, 227 | 55 | 17   | 5      | 5$^{11}$| 2, 7 |
|     |     |        | 57$^4$| 19, 113, 227, 4217 |       |       |       | 11$^5$| 2, 7 |
| 190 | 20  | 2      | 38$^5$| 37, 113, 797 | 93 | 22   | 5      | 3$^{31}$| 991 |
| 231 | 22  | 2      | 3$^{77}$| 2, 11 | 111 | 24   | 5      | 3$^{37}$| 2, 3 |
|     |     |        | 11$^{11}$| 2   |       |       |       | 141 | 27, 5 |
|     |     |        | 21$^{11}$| 2, 11, 41 |       |       |       | 47$^3$| 2, 3 |
|     |     |        | 33$^7$| 11    |       |       |       |       |       |
|     |     |        | 77$^3$| 2      |       |       |       |       |       |

Table 5: Cycle types $[n']$ that are ruled out by Lemma 2.10 as excesses of symmetric $(v, k, \lambda)$-coverings.

- $\alpha$ is even, $\gamma$ is odd and $\left(\frac{s}{p}\right) = -1$; or
- $\alpha$ is odd, $\gamma$ is even and $\left(\frac{(-1)^{(t-1)/2n\lambda}}{p}\right) = -1$; or
- $\alpha$ is odd, $\gamma$ is odd and $\left(\frac{(-1)^{(t+1)/2n\lambda}}{p}\right) = -1$;

where $sp^\alpha$ and $\bar{\lambda}p^\gamma$ are the $p$-factorisations of $k = \lambda + 2$ and $\lambda$ respectively. Furthermore, for any odd prime $p$ such that $k = \lambda + 2 \equiv 0 \pmod{p}$ and $n \not\equiv 0 \pmod{p}$ but $p$ does not satisfy the above hypotheses, $C_p(X_{(v, k, \lambda)}[n']) = 1$. 

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Proof. Let $p$ be a prime satisfying the hypotheses of the theorem. Let $X = X_{(nt,k,\lambda)}[n^t]$. We may assume that $|X|$ is a perfect square for otherwise we are finished by Proposition 2.3. By Lemma 3.2, noting that $n$ and $t$ are odd, we have

$$C_p(X) = C_p(B_n(k - \lambda))(k - \lambda + 2, -1)_p^{(t-1)/2}(-\lambda, k - \lambda + 2)_p.$$  \hspace{1cm} (7.1)

Since $n \not\equiv 0 \pmod p$ and $n$ is odd, Lemma 6.3(ii) implies that $C_p(B_n(k - \lambda)) = (\frac{-n}{p})^\alpha$. Also, using (2.1), $(-\lambda, k - \lambda + 2) = (-\lambda p^\gamma, sp^\alpha)_p = (\frac{-1}{p})^\alpha(\frac{-1}{p})^\gamma$ and $(k - \lambda + 2, -1)_p^{(t-1)/2} = (sp^\alpha, -1)_p^{(t-1)/2} = (\frac{1}{p})^{(t-1)/2}$. So, from (7.1), we have

$$C_p(X) = \left(\frac{(-1)^{(\alpha + \gamma)(t-1)/2}n^\alpha \lambda^\alpha \gamma}{p}\right).$$

The result now follows from Lemma 2.10 by checking cases. \hfill \blacksquare

We remarked after Theorem 6.5 that Lemma 2.10 cannot rule out Hamilton cycle excesses when $n \not\equiv 0 \pmod p$. It follows that Theorem 7.1 never rules out Hamilton cycle excesses.

The following corollary gives one example of an infinite family of symmetric coverings with specified excesses whose existence is ruled out by Theorem 7.1.

Corollary 7.2. Suppose $p$ is prime and $p \equiv 3, 5 \pmod 8$. Then there does not exist a symmetric $(2p^2 - p, 2p, 2)$-covering with excess consisting of $p$ cycles of length $2p - 1$.

Theorem 7.3. Let $n$, $t$, $k$ and $\lambda$ be positive integers such that $k > \lambda + 2$, $nt = \frac{k(k-1)-2}{\lambda} + 1$ and $nt$ is odd. There does not exist a symmetric $(nt, k, \lambda)$-covering whose excess consists of $t$ cycles of length $n$ if there is an odd prime $p$ such that $k - \lambda + 2 \equiv 0 \pmod p$, $n \equiv 0 \pmod p$, $k - \lambda + 2 \equiv 0 \pmod 9$ if $p = 3$, and

- $\alpha$ is even, $\delta$ is odd and $(\frac{-n}{p}) = -1$; or
- $\alpha$ is odd, $\delta$ is even and $\left(\frac{(-1)^{(t-1)/2n}}{p}\right) = -1$; or
- $\alpha$ is odd, $\delta$ is odd and $\left(\frac{(-1)^{(t-1)/2n \bar{n}}}{p}\right) = -1$;

where $sp^\alpha$ and $\bar{n}p^\delta$ are the $p$-factorisations of $k - \lambda + 2$ and $n$ respectively. Furthermore, for any odd prime $p$ such that $k - \lambda + 2 \equiv 0 \pmod p$, $n \equiv 0 \pmod p$, and $k - \lambda + 2 \equiv 0 \pmod 9$ if $p = 3$, but $p$ does not satisfy the above hypotheses, $C_p(X_{(nt,k,\lambda)}[n^t]) = 1$.

Proof. Let $p$ be a prime satisfying the hypotheses of the theorem. Let $X = X_{(nt,k,\lambda)}[n^t]$. We may assume that $|X|$ is a perfect square for otherwise we are finished by Proposition 2.3. By Lemma 3.2, noting that $n$ and $t$ are odd, we have

$$C_p(X) = C_p(B_n(k - \lambda))(k - \lambda + 2, -1)_p^{(t-1)/2}(-\lambda, k - \lambda + 2)_p.$$  \hspace{1cm} (7.2)

Because $n$ is odd, $g_n(a) = \bar{g}p^\delta$ for some integer $\bar{g} \equiv \bar{n} \pmod p$ by Lemma 6.2 and hence Lemma 6.3(i) implies that

$$C_p(B_n(k - \lambda)) = \left(\frac{(-1)^{(\alpha + \alpha + \delta) s^\delta \bar{n}^\alpha}}{p}\right).$$
By Lemma 6.4(ii), \( \lambda \equiv 2 \pmod{p} \). So, using (2.1), \((-\lambda, k - \lambda + 2)_p = (-\lambda, sp^\alpha)_p = (\frac{-2}{p})^\alpha = (\frac{-2}{p})^\alpha \) and \((k - \lambda + 2, -1)_p^{(t-1)/2} = (sp^\alpha, -1)_p^{(t-1)/2} = (\frac{-2}{p})^\alpha(t-1)/2 \). Thus, from (7.2), we have

\[
C_p(X) = \left(\frac{-1}{p}\right)^{\alpha(t-1)/2 + \alpha \delta \gamma} \delta^2 \alpha \beta \gamma.
\]

The result now follows from Lemma 2.10 by checking cases.

Theorem 7.3 with \( t = 1 \) produces identical results to Theorem 6.5. However, we were able to phrase Theorem 6.5 without resorting to Legendre symbols.

Again, we give an example of an infinite family of symmetric coverings with specified excesses whose existence is ruled out by Theorem 7.3.

**Corollary 7.4.** Suppose \( p \) is prime, \( p > 3 \), and \( p \equiv 3, 7 \pmod{8} \). Then there does not exist a symmetric \((\frac{2}{p}(p - 1), p, 2)\)-covering with excess consisting of \( \frac{p-1}{2} \) cycles of length \( p \).

## 8 Excesses composed of 2-cycles and 3-cycles

In this section we focus on establishing the nonexistence of symmetric \((v, k, \lambda)\)-coverings whose excesses consist of 2-cycles and 3-cycles. As mentioned, results of Bose and Connor (see [2]) already cover the cases in which the excess is composed entirely of 2-cycles or entirely of 3-cycles.

Table 6 lists cycle types of the form \([2^t, 3^s]\) that can be ruled out by Lemma 2.10 as excesses of symmetric \((v, k, 1)\)-coverings for \( p < 10 \) and \( 4 \leq k \leq 10 \). Computational results for small values of \( \lambda \) and \( k \) show that taking \( p = 5 \) or \( p = 2 \) often rules out cycle types composed entirely of 2-cycles and 3-cycles. In Theorem 8.2 and Lemma 8.4 we consider the choices \( p = 5 \) and \( p = 2 \) respectively. In Table 6, we mark in boldface the cases for which \( p = 2 \) rules out the case using Theorem 8.4 and for which \( p = 5 \) rules out the case using Theorem 8.2.

| \( v \) | \( k \) | \( \lambda \) | \([2^t, 3^s]\) | \( p \) |
|---|---|---|---|---|
| 11 | 4 | 1 | \( 2^4 \) | 2, 5 |
| 19 | 5 | 1 | \( 2^5 \) | 2, 3 |
| | | | \( 3^2 \) | 2, 3 |
| 29 | 6 | 1 | \( 2^6, 3^3 \) | 2, 3 |
| | | | \( 3^5 \) | 2, 7 |
| | | | \( 5^3 \) | 3, 7 |
| 41 | 7 | 1 | \( 2^3, 3^7, 3, 5, 7, 3, 19 \) | 2, 5 |
| | | | \( 2^5, 3^5, 5, 7, 3, 19 \) | 2, 3 |
| | | | \( 2^5, 3^5, 5, 7, 3, 19 \) | 3, 5 |
| 71 | 9 | 1 | \( 2^9, 3^3, 5, 7, 3, 19 \) | 2, 3 |
| | | | \( 2^9, 3^3, 5, 7, 3, 19 \) | 2, 5 |
| | | | \( 2^9, 3^3, 5, 7, 3, 19 \) | 3, 5 |
| 89 | 10 | 1 | \( 2^3, 3^9, 3^3, 5, 7, 3, 19 \) | 2 |
| | | | \( 2^9, 3^3, 5, 7, 3, 19 \) | 7 |
| | | | \( 2^9, 3^3, 5, 7, 3, 19 \) | 2, 7 |

Table 6: Cycle types of the form \([2^t, 3^s]\) that are ruled out by Lemma 2.10 as excesses of symmetric \((v, k, 1)\)-coverings.
Lemma 8.1 gives a concise expression for \( C_p(B_3(a)) \). We use this to prove Theorem 8.2 and Theorem 8.4.

**Lemma 8.1.** Let \( a \geq 2 \) be a positive integer and let \( p \) be a prime. Then \( C_p(B_3(a)) = (-1, -1)p(-a - 2, a - 1)p \).

**Proof.** Let \( Y' \) be the matrix \( \text{diag}(B_3, -1) \). Note that \( |B_3| = (a + 2)(a - 1)^2 \). Then

\[
C_p(B_3) = C_p(Y')(|Y'|, -|B_3|)_p
\]

by rearranging (2.8)

\[
= C_p(Y')(-|B_3|, -|B_3|)_p
\]

since \( |Y'| = -|B_3| \)

\[
= C_p(Y')(a - 2, -1)_p
\]

by (2.6) since \( |B_3| = (a + 2)(a - 1)^2 \).

Let \( Y'' \) be the matrix obtained from \( Y' \) by adding the last row to all other rows and then adding the last column to all other columns. Note that the 3rd principal minor of \( Y'' \) is \((a - 1)I_3 \) and that by applying (2.9) twice, we get that \( C_p((a - 1)I_3) = (-1, -1)_p \). Using the equation above, we have

\[
C_p(B_3) = C_p(Y'')(a - 2, -1)_p
\]

since \( Y'' \sim Y' \)

\[
= C_p((a - 1)I_3)(|Y''|, 1 - a)_p(-a - 2, -1)_p
\]

by (2.8) since \(|(a - 1)I_3| = (a - 1)^3 \)

\[
= C_p((a - 1)I_3)(-a - 2, 1 - a)_p(-a - 2, -1)_p
\]

since \( |Y''| = |Y'| = -(a + 2)(a - 1)^2 \)

\[
= (-1, -1)_p(-a - 2, 1 - a)_p(-a - 2, -1)_p
\]

since \( C_p((a - 1)I_3) = (-1, -1)_p \)

\[
= (-1, -1)_p(-a - 2, a - 1)_p
\]

by (2.4).

\[ \square \]

**Theorem 8.2.** Let \( t_2, t_3, \lambda \) and \( k \) be positive integers such that \( k - \lambda > 2 \), \( \lambda \) is not divisible by 5 and \( 2t_2 + 3t_3 = \frac{k(k - 1) - 2}{\lambda} + 1 \). There does not exist a symmetric \((2t_2 + 3t_3, k; \lambda)\)-covering whose excess consists of \( t_2 \) cycles of length 2 and \( t_3 \) cycles of length 3 if

(i) \( k - \lambda = 5^\alpha s + 1 \) where \( \alpha \) is odd, \( s \not\equiv 0 \pmod{5} \), and \( t_3 \) is odd; or

(ii) \( k - \lambda = 5^\alpha s + 2 \) where \( \alpha \) is odd, \( s \not\equiv 0 \pmod{5} \), \( t_2 \) is odd and \( \lambda \equiv 1, 4 \pmod{5} \); or

(iii) \( k - \lambda = 5^\alpha s - 2 \) where \( \alpha \) is odd, \( s \not\equiv 0 \pmod{5} \), \( t_2 + t_3 \) is odd and \( \lambda \equiv 1, 4 \pmod{5} \).

**Proof.** Suppose that one of (i), (ii) or (iii) holds. Let \( X = X_{(2t_2 + 3t_3, k; 1)}[2^{t_2}, 3^{t_3}] \). We may assume that \(|X|\) is a perfect square for otherwise we are finished by Proposition 2.3. Let \( a = k - \lambda \). In the rest of the proof we often use the fact that \(-1 \equiv 2^2 \pmod{5} \). By Lemma 3.2,

\[
f_5(a, \lambda, t_2 + t_3, t_2) = (a + 2, a^2 - 4)_5^{t_2 + t_3}(-\lambda, (a + 2)^2 + t_3(a - 2)^2)_5
\]

\[
= (-\lambda, (a + 2)^2 + t_3(a - 2)^2)_5,
\]

where the last equality follows because \((a + 2, a^2 - 4)_5 = (a + 2, -1)_5(a + 2, a - 2)_5 = 1 \), which is derived using (2.4) and (2.6) and by checking each equivalence class of \( a \) modulo 5.

Observe that \( C_p(B_2(a)) = (-a, 4 - a^2)_p \). By Lemma 3.2 and Lemma 8.1,

\[
C_5(X) = (-\lambda, (a + 2)^2 + t_3(a - 2)^2)_5C_5(B_2(a))^2C_5(B_3(a))^3
\]

\[
= (-\lambda, (a + 2)^2 + t_3(a - 2)^2)_5(-a, 4 - a^2)_5^2(-a - 2, a - 1)_5^3.
\]
It is routine to check that, when one of (i), (ii) or (iii) holds, $C_5(X) = -1$ and the result follows from Lemma 2.10.

If none of (i), (ii) or (iii) holds, then $C_5(X) = 1$ and Lemma 2.10 cannot be used to rule out the existence of a $(2t_2 + 3t_3, k, 1)$-covering with excess having cycle type $[2^{t_2}, 3^{t_3}]$.

Observe that Theorem 8.2 rules out every cycle type $[2^{t_2}, 3^{t_3}]$ as a possible excess for a symmetric $(41, 7, 1)$-covering (see Table 6). This generalises to a direct corollary of Theorem 8.2(i), which rules out any excess of cycle type $[2^{t_2}, 3^{t_3}]$ for an infinite family of symmetric $(v, k, 1)$-coverings.

**Corollary 8.3.** If $v$ and $k$ are positive integers such that $k \equiv 7, 12, 17, 22 \pmod{25}$ and $v = k(k-1) - 1$, then there does not exist a symmetric $(v, k, 1)$-covering with excess consisting of 2- and 3-cycles.

Observing the examples in Table 6, we see that a symmetric $(55, 8, 1)$-covering also cannot have excess consisting only of 2- and 3-cycles. However, Theorem 8.2(ii) establishes this only when there is an odd number of cycles of length 2. To rule out the excess types with even number of 2-cycles, we employ Lemma 2.10 with $p = 2$ in Lemma 8.4 below. This enables us to give another infinite family of parameters for which there does not exist a symmetric covering with excess having only 2- and 3-cycles.

**Lemma 8.4.** Let $t_2$, $t_3$ and $k$ be positive integers such that $k > 3$ and $2t_2 + 3t_3 = k(k-1) - 1$. There does not exist a symmetric $(2t_2 + 3t_3, k, 1)$-covering whose excess consists of $t_2$ cycles of length 2 and $t_3$ cycles of length 3 if

(i) $k \equiv 0 \pmod{4}$ and $t_3 \equiv 1 \pmod{4}$; or

(ii) $k \equiv 1 \pmod{4}$ and $t_3 \equiv 5 \pmod{8}$.

**Proof.** Let $X = X_{(2t_2 + 3t_3, k, 1)}[2^{t_2}, 3^{t_3}]$. We may assume that $|X|$ is a perfect square for otherwise we are finished by Proposition 2.3. By Lemma 2.10, it suffices to establish that $C_2(X) = 1$. Suppose that (i) or (ii) holds. Then $t_3 \equiv 1 \pmod{4}$, $2t_2 + 3t_3 = k(k-1) - 1 \equiv 3 \pmod{4}$ and so $t_2$ is even. Thus, by Lemma 3.2, for any prime $p$,

$$
C_p(X) = (k^2 - 2k - 3, -1)^{t_2/2}_p(-1, k + 1)_pC_p(B_3(k-1))
= (k^2 - 2k - 3, -1)^{t_2/2}_p(-1, k + 1)_p(-1, -1)_p(-k - 1, k - 2)_p \quad \text{by Lemma 8.1}
= (k^2 - 2k - 3, -1)^{t_2/2}_p(-k - 1, -1)_p(-k - 1, k - 2)_p \quad \text{by (2.4)}
= (k^2 - 2k - 3, -1)^{t_2/2}_p(-k - 1, -k + 2)_p \quad \text{by (2.4)}.
$$

We can now establish that $C_2(X) = 1$ by specialising this equation to the case $p = 2$ and applying (2.2), considering the cases $k \equiv 0 \pmod{4}$, $k \equiv 1 \pmod{8}$ and $k \equiv 5 \pmod{8}$ separately. In the first case, $(-k - 1, -k + 2)_2 = (k^2 - 2k - 3, -1)_2 = 1$. In the second case $(-k - 1, -k + 2)_2 = 1$ and it follows from $2t_2 + 3t_3 = k(k-1) - 1$ that $t_2 \equiv 0 \pmod{4}$. In the third case $(-k - 1, -k + 2)_2 = (k^2 - 2k - 3, -1)_2 = -1$ and it follows from $2t_2 + 3t_3 = k(k-1) - 1$ that $t_2 \equiv 2 \pmod{4}$. □
Observe that under the hypotheses of Lemma 8.4 if $k \equiv 1 \pmod{4}$ but $t_3 \equiv 1 \pmod{8}$ then $C_2(X[2^{t_2}, 3^{t_3}]) = -1$ and Lemma 2.10 cannot be used to rule out the existence of a $(2t_2 + 3t_3, k, 1)$-covering with such an excess. We also remark that the proof of Lemma 8.4 easily extends to rule out the existence of a $(v, k, 1)$-covering with excess having $v$-feasible cycle type $[2^{t_2}, 3^{t_3}, c_1^{m_1}, \ldots, c_t^{m_t}]$ where $m_i \equiv 0 \pmod{4}$ for all $1 \leq i \leq t$.

In Lemma 8.4, if $k \equiv 0, 1 \pmod{4}$, then $k(k - 1) - 1$ is odd and hence $t_3$ is odd. Therefore, for a fixed $k$, parts (i) and (ii) of the Lemma 8.4 rule out, respectively, about a half and a quarter of the feasible cycle types of the form $[2^{t_2}, 3^{t_3}]$.

The following corollary is a straightforward application of Theorem 8.2(ii) and Lemma 8.4(i) (note that Lemma 8.4(i) applies whenever $k \equiv 0 \pmod{4}$ and $t_2$ is even).

**Corollary 8.5.** If $k$ is a positive integer such that $k \equiv 8, 48, 68, 88 \pmod{100}$ and $v = k(k - 1) - 1$, then there does not exist a symmetric $(v, k, 1)$-covering with excess consisting of 2- and 3-cycles.

### 9 Conclusion

In Sections 5–8, we ruled out the existence of several infinite families of symmetric $(v, k, \lambda)$-coverings with particular types of excesses using Lemma 2.10. Observe that Theorem 1.1 rules out the existence of infinitely many symmetric $(v, k, \lambda)$-coverings with 2-regular excess when $v$ is even. However, when $v$ is odd and $k > \lambda + 2$, the following problem remains open.

**Open Problem.** Are there infinitely many parameter sets $(v, k, \lambda)$ where $3 \leq \lambda + 2 < k < v$, $v = \frac{k(k - 1) - 2}{\lambda} + 1$ and $v$ is odd, for which there is no symmetric $(v, k, \lambda)$-covering?

Since cyclic symmetric coverings are of particular interest and have applications in related fields of study, as mentioned in Section 4, we note that the following problem remains open as well.

**Open Problem.** Are there infinitely many parameter sets $(v, k, \lambda)$ where $3 \leq \lambda + 2 < k < v$, $v = \frac{k(k - 1) - 2}{\lambda} + 1$ and $v$ is odd, for which there is no cyclic symmetric $(v, k, \lambda)$-covering?

Obviously, an affirmative answer to the first question would answer both questions in the affirmative and a negative answer to the second question would answer both questions in the negative.

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