A Theory of Computation Based on Quantum Logic (I)

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Abstract

The (meta)logic underlying classical theory of computation is Boolean (two-valued) logic. Quantum logic was proposed by Birkhoff and von Neumann as a logic of quantum mechanics more than sixty years ago. It is currently understood as a logic whose truth values are taken from an orthomodular lattice. The major difference between Boolean logic and quantum logic is that the latter does not enjoy distributivity in general. The rapid development of quantum computation in recent years stimulates us to establish a theory of computation based on quantum logic. The present paper is the first step toward such a new theory and it focuses on the simplest models of computation, namely finite automata. We introduce the notion of orthomodular lattice-valued (quantum) automaton. Various properties of automata are carefully reexamined in the framework of quantum logic by employing an approach of semantic analysis. We define the class of regular languages accepted by orthomodular lattice-valued automata. The acceptance abilities of orthomodular lattice-valued nondeterministic automata and their various modifications (such as deterministic automata and automata with ε−moves) are compared. The closure properties of orthomodular lattice-valued regular languages are derived. The Kleene theorem about equivalence of regular expressions and finite automata is generalized into quantum logic. We also present a pumping lemma for orthomodular lattice-valued regular languages. It is found that the universal validity of many properties (for example, the Kleene theorem, the equivalence of deterministic and nondeterministic automata) of automata depend heavily upon the distributivity of the underlying logic. This indicates that these properties does not universally hold in the realm of quantum logic. On the other hand, we show that a local validity of them can be recovered by imposing a certain commutativity to the (atomic) statements about the automata under consideration. This reveals an essential difference between the classical theory of computation and the computation theory based on quantum logic.

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1. Introduction

It is well-known that an axiomatization of a mathematical theory consists of a system of fundamental notions as well as a set of axioms about these notions. The mathematical theory is then the set of theorems which can be derived from the axioms. Obviously, one needs a certain logic to provide tools for reasoning in the derivation of these theorems from the axioms. As pointed out by A. Heyting [He63, page 5], in elementary axiomatics logic was used in an unanalyzed form. Afterwards, in the studies for foundations of mathematics beginning in the early of twentieth century, it had been realized that a major part of mathematics has to exploit the full power of classical (Boolean) logic [Ha82], the strongest one in the family of existing logics. For example, group theory is based on first-order logic, and point-set topology is built on a fragment of second-order logic. However, a few mathematicians, including the big names L. E. J. Brouwer, H. Poincare, L. Kronecker and H. Weyl, took some kind of constructive position which is in more or less explicit opposition to certain forms of mathematical reasoning used by the majority of the mathematical community. Some of them even endeavored to establish so-called constructive math-
emematics, the part of mathematics that could be rebuilt on constructivist principles. The logic employed in the development of constructive mathematics is intuitionistic logic [TD88] which is truly weaker than classical logic.

Since many logics different from classical logic and intuitionistic logic have been invented in the last century, one may naturally ask the question whether we are able to establish some mathematical theories based on other nonclassical logics besides intuitionistic logic. Indeed, as early as the first nonclassical logics appeared, the possibility of building mathematics upon them was conceived. As mentioned by A. Mostowski [M65], J. Łukasiewicz hoped that there would be some nonclassical logics which can be properly used in mathematics as non-Euclidean geometry does. In 1952, J. B. Rosser and A. R. Turquette [RT52, page 109] proposed a similar and even more explicit idea:

"The fact that it is thus possible to generalize the ordinary two-valued logic so as not only to cover the case of many-valued statement calculi, but of many-valued quantification theory as well, naturally suggests the possibility of further extending our treatment of many-valued logic to cover the case of many-valued sets, equality, numbers, etc. Since we now have a general theory of many-valued predicate calculi, there is little doubt about the possibility of successfully developing such extended many-valued theories. ... we shall consider their careful study one of the major unsolved problems of many-valued logic."

Unfortunately, the above idea has not attracted much attention in logical community. For such a situation, A. Mostowski [M65] pointed out that most of nonclassical logics invented so far have not been really used in mathematics, and intuitionistic logic seems the unique one of nonclassical logics which still has an opportunity to carry out the Łukasiewicz’s project. A similar opinion was also expressed by J. Dieudonné [Di78], and he said that mathematical logicians have been developing a variety of nonclassical logics such as second-order logic, modal logic and many-valued logic, but these logics are completely useless for mathematicians working in other research areas.

One reason for this situation might be that there is no suitable method to develop mathematics within the framework of nonclassical logics. As was pointed out above, classical logic is applied as the deduction tool in almost all mathematical theories. It should be noted that what is used in these theories is the deductive (proof-theoretical) aspect of classical logic. However, the proof theory of nonclassical logics is much more complicated than that of classical logic, and it is not an easy task to conduct reasoning in the realm of the proof theory of nonclassical logics. It is the case even for the simplest nonclassical logics, three-valued logics. This is explicitly indicated by the following excerpt from H. Hodes [Ho89]:

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"Of course three-valued logics will be somewhat more complicated than classical two-valued logic. In fact, proof-theoretically they are at least twice as complicated: .... But model-theoretically they are only 50 percent more complicated,....

And much worse, some nonclassical logics were introduced only in a semantic way, and the axiomatizations of some among them are still to be found, and some of them may be not (finitely) axiomatizable. Thus, our experience in studying classical mathematics may be not suited, or at least cannot directly apply, to develop mathematics based on nonclassical logics. In the early 1990’s an attempt had been made by the author [Yi91-93; Yi93] to give a partial and elementary answer in the case of point-set topology to the J. B. Rosser and A. R. Turquette’s question raised above. We employed a semantical analysis approach to establish topology based on residuated lattice-valued logic, especially the Lukasiewicz system of continuous-valued logic. Roughly speaking, the semantical analysis approach transforms our intended conclusions in mathematics, which are usually expressed as implication formulas in our logical language, into certain inequalities in the truth-value lattice by truth valuation rules, and then we demonstrate these inequalities in an algebraic way and conclude that the original conclusions are semantically valid. We believe that semantical analysis approach is an effective method to develop mathematics based on nonclassical logics.

A much more essential reason for the situation that few nonclassical logics have been applied in mathematics is absence of appealing from other subjects or applications in the real world. One major exception may be the case of quantum logic. Quantum logic was introduced by G. Birkhoff and J. von Neumann [BN36] in the thirties of the twentieth century as the logic of quantum mechanics. They realized that quantum mechanical systems are not governed by classical logical laws. Their proposed logic stems from von Neumann’s Hilbert space formalism of quantum mechanics. The starting point was explained very well by the following excerpt from G. Birkhoff and J. von Neumann [BN36]:

"what logical structure one may hope to find in physical theories which, like quantum mechanics, do not conform to classical logic. Our main conclusion, based on admittedly heuristic arguments, is that one can reasonably expect to find a calculus of propositions which is formally indistinguishable from the calculus of linear subspaces [of Hilbert space] with respect to set products, linear sums, and orthogonal complements - and resembles the usual calculus of propositions with respect to 'and', 'or', and 'not'."

Thus linear (closed) subspaces of Hilbert space are identified with propositions concerning a quantum mechanical system, and the operations of set product, linear sum and orthogonal complement are treated as connectives. By observing that the set of
linear subspaces of a finite-dimensional Hilbert space together with these operations enjoys Dedekind’s modular law, G. Birkhoff and J. von Neumann [BN36] suggested to use modular lattices as the algebraic version of the logic of quantum mechanics, just like that Boolean algebras act as an algebraic counterpart of classical logic. However, the modular law does not hold in an infinite-dimensional Hilbert space. In 1937, K. Husimi [Hu37] found a new law, called now the orthomodular law, which is valid for the set of linear subspaces of any Hilbert space. Nowadays, what is usually called quantum logic in the mathematical physics literatures refers to the theory of orthomodular lattices. Obviously, this kind of quantum logic is not very logical. Indeed, there is also another much more 'logical' point of view on quantum logic in which quantum logic is seen as a logic whose truth values range over an orthomodular lattice (for an excellent exposition for the latter approach of quantum logic, see M. L. Dalla Chiara [DC86], or J. P. Rawling and S. A. Selesnick [RS00]).

After the invention of quantum logic, quite a few mathematicians have tried to establish mathematics based on quantum logic. Indeed, J. von Neumann [N62] himself proposed the idea of considering a quantum set theory, corresponding to quantum logic, as does classical set theory to classical logic. One important contribution in this direction was made by G. Takeuti [T81]. His main idea was explained, and the nature of mathematics based on quantum logic was analyzed very well by the following citation from the introduction of [T81]:

"Since quantum logic is an intrinsic logic, i.e. the logic of the quantum world, it is an important problem to develop mathematics based on quantum logic, more specifically set theory based on quantum logic. It is also a challenging problem for logicians since quantum logic is drastically different from the classical logic or the intuitionistic logic and consequently mathematics based on quantum logic is extremely difficult. On the other hand, mathematics based on quantum logic has a very rich mathematical content. This is clearly shown by the fact that there are many complete Boolean algebras inside quantum logic. For each complete Boolean algebra B, mathematics based on B has been shown by our work on Boolean valued analysis to have rich mathematical meaning. Since mathematics based on B can be considered as a sub-theory of mathematics based on quantum logic, there is no doubt about the fact that mathematics based on quantum logic is very rich. The situation seems to be the following. Mathematics based on quantum logic is too gigantic to see through clearly."

The main technical result of G. Takeuti [T81] is a construction of orthomodular lattice-valued universe. He built up such an universe in a way similar to Boolean-valued models of ZF + AC, and showed that a reasonable set theory, including some axioms from ZF + AC or their slight modifications, holds in this universe. Recently, K. -G. Schlesinger [Sc99] developed a theory of quantum sets by using a categorical approach in the spirit of topos theory. He started with the category of complex (pre-)Hilbert spaces and linear maps. This category was seen as the (basic) quantum set
universe. Then he was able to introduce the analog of number systems and to deal with the analog of some algebraic structures in quantum set theory. Indeed, K. -G. Schlesinger’s terminal goal is to build a quantum mathematics, i.e., a mathematical theory where all the ingredients (like logic and set theory) adhere to the rules of quantum mechanics. Quantum set theory is the quantization of the mathematical theory of pure objects, and so it is just the first step toward his goal. It is worth noting that the role of quantum logic in such a quantum mathematics is different from that in G. Takeuti’s quantum set theory, and quantum logic appears as an internal logic in the former.

After a careful examination on the development of mathematics based on non-classical logics, we now come to explore the possibility of establishing a theory of computation based on nonclassical logics. A formal formulation of the notion of computation is one of the greatest scientific achievements in the twentieth century. Since the middle of 1930’s, various models of computation have been introduced, such as Turing machines, Post systems, $\lambda$—calculus and $\mu$—recursive functions. In classical computing theory, these models of computations are investigated in the framework of classical logic; more explicitly, all properties of them are deduced by classical logic as a (meta)logical tool. So, it is reasonable to say that classical computing theory is a part of classical mathematics. Knowing the basic idea of mathematics based on nonclassical logics, we may naturally ask the question: is it possible to build a theory of computation based on nonclassical logics, and what are the same of and difference between the properties of the models of computations in classical logic and the corresponding ones in non-classical logics? There has been a very big population of non-classical logics. Of course, it is unnecessary to construct models of computations in each nonclassical logic and to compare them with the ones in classical logic because some nonclassical logics are completely irrelative to behaviors of computations. Nevertheless, as will be explained shortly, it is absolutely worth studying deeply and systematically models of computations based on quantum logic.

It seems that both points of views on quantum logic mentioned above have no obvious links to computations; but appearance of the idea of quantum computers changed dramatically the long-standing situation. The idea of quantum computation came from the studies of connections between physics and computation. The first step toward it was the understanding of the thermodynamics of classical computation. In 1973, C. H. Bennet [Ben73] noted that a logically reversible operation need not dissipate any energy and found that a logically reversible Turing machine is a theoretical possibility. In 1980, further progress was made by P. A. Benioff [Be80] who constructed a quantum mechanical model of Turing machine. His construction is the first quantum mechanical description of computer, but it is not a real quantum computer. It should be noted that in P. A. Benioff’s model between computation steps the machine may exist in an intrinsically quantum state, but at the end of each computation step the tape of the machine always goes back to one of its classical states. Quantum computers were first envisaged by R. P. Feynman [Fe82; Fe86]. In 1982, he [Fe82] conceived that no classical Turing machine could simulate certain quantum phenomena without an exponential slowdown, and so he realized that
quantum mechanical effects should offer something genuinely new to computation. Although R. P. Feynman proposed the idea of universal quantum simulator, he did not give a concrete design of such a simulator. His ideas were elaborated and formalized by D. Deutsch in a seminal paper [De85]. In 1985, D. Deutsch described the first true quantum Turing machine. In his machine, the tape is able to exist in quantum states too. This is different from P. A. Benioff’s machine. In particular, D. Deutsch introduced the technique of quantum parallelism by which quantum Turing machine can encode many inputs on the same tape and perform a calculation on all the inputs simultaneously. Furthermore, he proposed that quantum computers might be able to perform certain types of computations that classical computers can only perform very inefficiently. One of the most striking advances was made by P. W. Shor [S94] in 1994. By exploring the power of quantum parallelism, he discovered a polynomial-time algorithm on quantum computers for prime factorization of which the best known algorithm on classical computers is exponential. In 1996, L. K. Grover [Gr96] offered another apt killer of quantum computation, and he found a quantum algorithm for searching a single item in an unsorted database in square root of the time it would take on a classical computer. Since both prime factorization and database search are central problems in computer science and the quantum algorithms for them are highly faster than the classical ones, P. W. Shor and L. K. Grover’s works stimulated an intensive investigation on quantum computation. After that, quantum computation has been an extremely exciting and rapidly growing field of research.

The studies of quantum computation may be roughly divided into four strata, arranged according increasing order of abstraction degree: (1) physical implementations; (2) physical models; (3) mathematical models; and (4) logical foundations. Almost all pioneer works such as [Be80, F82, D85] in this field were devoted to build physical models of quantum computing. In 1990’s, a great attention was paid to the physical implementation of quantum computation. For example, S. Lloyd [L93] considered the practical implementation by using electromagnetic pulses and J. I. Cirac and P. Zoller [CZ95] used laser manipulations of cold trapped ions to implement quantum computing. The current theoretical concerns in the area of quantum computation have mainly been given to quantum algorithms. But also there have been a few attempts to develop mathematical models of quantum computation and to clarify the relationship between different models. For example, except quantum Turing machines, D. Deutsch [De89] also proposed the quantum circuit model of computation, and A. C. Yao [Ya93] showed that the quantum circuit model is equivalent to the quantum Turing machine in the sense that they can simulate each other in polynomial time. As is well known, in classical computing theory, there are still two important classes of models of computation rather than Turing machines; namely, finite automata and pushdown automata. They are equipped with finite memory or finite memory with stack, respectively, and so have weaker computing power than Turing machines. Recently, J. P. Crutchfield and C. Moore [CM00], A. Kondacs and J. Watrous [KW97], and S. Gudder [Gu00] tried to introduce some quantum devices corresponding to these weaker models of computation. Roughly speaking,
quantum automata may be seen as quantum counterparts of probabilistic automata. In a probabilistic automaton, each transition is equipped with a number in the unit interval to indicate the probability of the occurrence of the transition; by contrast in a quantum automaton we associate with each transition a vector in a Hilbert space which is interpreted as the probability amplitude of the transition. In a sense, these mathematical models of quantum computation can be seen as abstractions of its physical models.

It should be noted that the theoretical models of quantum computation mentioned above, including quantum Turing machines and quantum automata, are still developed in classical (Boolean) logic. Thus, their logical basis is the same as that of classical computation, and we may argue that sometimes these models might be not suitable for quantum computers that obey some logical laws different from that in Boolean logic. Indeed, V. Vedral and M. B. Plenio [VP98] already advocated that quantum computers require quantum logic, something fundamentally different to classical Boolean logic. As stated above, quantum logic has been existing for a long time. So, the point is how to apply quantum logic in the analysis and design of quantum computers. The background exposed above highly motivates us to explore the possibility of establishing a theory of computation based on quantum logic. The purpose of the present paper and its continuations is exactly to develop such a new theory. In a sense, our approach may be thought of as a logical foundation of quantum computation and a further abstraction of its mathematical models. The relation between Crutchfield et al’s studies [CM00, KW97, Gu00] on quantum automata and our automata theory based on quantum logic is quite similar to that between J. von Neumann’s Hilbert space formalism of quantum mechanics and quantum logic.

Since finite automata are the simplest models of computation (with finite memory), in this paper we focus our attention on developing a theory of finite automata based on quantum logic. The present paper is organized as follows. In Section 2, we recall some basic notions and results of quantum logic and its algebraic semantics needed in the subsequent sections from the previous literature. Some new lemmas on implication operators in quantum logic are presented too. They are crucial in the proofs of several main results in this paper. In Section 3, we introduce the notion of orthomodular lattice-valued (quantum) automaton. Then two different orthomodular lattice-valued predicates of regularity on languages are proposed. These two predicates stands indeed for the (orthomodular lattice-valued) class of languages accepted by orthomodular lattice-valued automata. This provides us with a framework in which various properties of automata can be reexamined within quantum logic. The technique employed in this paper is mainly the approach of semantic analysis developed in [Yi91-93; Yi93]. The acceptance ability of orthomodular lattice-valued nondeterministic automata are then compared with that of their two kinds of modifications, namely deterministic automata and automata with \( \varepsilon \)-moves, respectively in Sections 4 and 5. The closure properties of orthomodular lattice-valued regular languages are derived in Section 6. In Section 7, we introduce the notion of orthomodular lattice-valued regular expression, and the Kleene theorem about equivalence of regular expressions and finite automata is generalized into quantum logic. Section
8 is devoted to present a pumping lemma for orthomodular lattice-valued regular languages. Some basic ideas of this paper were announced in [Yi00], and Definitions 3.1 and 3.2, Examples 3.1-4, and Propositions 6.1 and 6.3 were also presented there. For completeness, however, they are included in the present paper.

The most interesting thing is, in the author’s opinion, the discovery that the universal validity of many properties (for example, the Kleene theorem, the equivalence of deterministic and nondeterministic automata) of automata depend heavily upon the distributivity of the underlying logic. It is shown that the universal validity of these properties is equivalent to the requirement that the set of truth values of the meta-logic underlying our theory of automata is a Boolean algebra. This indicates that these properties does not universally hold in the realm of quantum logic, and it is in fact a negative conclusion in our theory of automata based on quantum logic. Furthermore, it implies the fact that an essential difference exists between the classical theory of computation and the computation theory based on quantum logic.

Observing that some important properties of automata cannot be built within quantum logic, one may naturally ask the question whether they may be partially recast without appealing to distributivity of the underlying logic. Fortunately, we are able to show that a local validity of these properties of automata can be recovered by imposing a certain commutativity to the truth values of the (atomic) statements about the automata under consideration. Very surprisingly, almost all results in classical automata theory that are not valid in a non-distributive logic can be revived by a certain commutativity in quantum logic. This further leads us to a new question: why commutativity plays such a key role for quantum automata, and is there any physical interpretation for it? To answer this question, let us first note that all truth values in quantum logic are taken from an orthomodular lattice. The prototype of orthomodular lattice is the set of linear (closed) subspaces of a Hilbert space with the set inclusion as its ordering relation. Suppose that $X$ and $Y$ are two subspaces of a Hilbert space $H$. Moreover, we use $P_X$ and $P_Y$ to denote the projections on $X$ and $Y$ respectively. Then $P_X$ and $P_Y$ are Hermitian operators on $H$, and they may be seen as two (physical) observables $A$ and $B$ in a quantum system whose state space is $H$, according to the Hilbert space formalism of quantum mechanics. If we write $\Delta(A)$ and $\Delta(B)$ for the respective standard deviations of measurement on $A$ and $B$, then the Heisenberg uncertainty principle gives the following inequality:

$$\Delta(A) \cdot \Delta(B) \geq \frac{1}{2} | \langle \psi | [A,B] |\psi \rangle |$$

for all quantum state $|\psi\rangle$ in $H$, where $[A,b] = AB - BA$ is the commutator between $A$ and $B$. We now turn back to the orthomodular lattice of the linear subspaces of $H$. The commutativity of $A$ and $B$ is defined by the condition $X = (X \land Y) \lor (X \land Y^\perp)$, where $\land$, $\lor$ and $\perp$ are respectively the meet, union and orthocomplement. It may be seen that the commutativity between $X$ and $Y$ is equivalent to exactly the fact that $A$ and $B$ commute, i.e., $AB = BA$. In this
case, \[ < \psi | [A, B] | \psi > = 0, \] and \[ \Delta(A) \cdot \Delta(B) \] may vanish; or in other words, \[ \Delta(A) \] and \[ \Delta(B) \] can simultaneously become arbitrarily small. Remember that in our theory of automata based on quantum logic the commutativity is attached to the basic statements describing the considered automata. On the other hand, the basic statements are indeed corresponding to some actions in these automata. Therefore, a potential physical interpretation for the need of commutativity is that some nice properties of automata require the standard deviations of the observables concerning the basic actions in these automata being able to reach simultaneously very small values.

The results gained in our approach may offer some new insights on the theory of computation. As an example, let us consider the Church-Turing thesis. The realization that the intuitive notion of "effective computation" can be identified with the mathematical concept of "computation by the Turing machine" is based on the fact that the Turing machine is computationally equivalent to some vastly dissimilar formalisms for the same purpose, such as Post systems, \( \mu \)-recursive functions, \( \lambda \)-calculus and combinatory logic. As pointed out by J. E. Hopcroft and J. D. Ullman [HU79], another reason for the acceptance of the Turing machine as a general model of a computation is that the Turing machine is equivalent to its many modified versions that would seem off-hand to have increased computing power. We should note that the equivalence between the Turing machine and its various generalizations as well as other formalisms of computation has been reached in classical Boolean logic. In addition, quantum logic is known to be strictly weaker than Boolean logic. Thus, it is reasonable to doubt that the same equivalence can be achieved when our underlying meta-logic is replaced by quantum logic, and the Church-Turing thesis needs to be reexamined in the realm of quantum logic. Indeed, in a continuation of this paper we are going to establish a theory of Turing machines based on quantum logic. The details of such a theory is still to be exploited, but the conclusion concerning the equivalence between deterministic and nondeterministic automata obtained in this paper suggests us to believe that the equivalence between deterministic and nondeterministic Turing machines also depends upon the distributivity of the underlying logic, and a certainty commutativity for the basic actions in Turing machines will guarantee such an equivalence. Keeping this belief in mind, we may assert that a certain commutativity of the observables for some basic actions in the Turing machine is a physical support of the Church-Turing thesis in the framework of quantum logic. Furthermore, with the above physical interpretation for commutativity, this hints that there might be a deep connection between the Heisenberg uncertainty principle and the Church-Turing thesis, two of the greatest scientific discoveries in the twentieth century. It is notable that such a connection could be observed via an argument in a nonclassical logic (and it is impossible to be found if we always work within the classical logic). As early as in 1985, it was argued by D. Deutsch [De84] that underlying the Church-Turing thesis there is an implicit physical assertion. There is certainly no doubt about the existence of such a physical assertion. The true problem here is: what is it? The answer given by D. Deutsch himself is the following physical principle: "every finitely realizable physical system
can be perfectly simulated by a universal model computing machine operating by finite means”. Our above analysis on the role of commutativity in computation theory based on quantum logic perhaps indicates that in order to be simulated by a universal computing machine some observables of the physical system are required to possess a certain commutativity. So, it is fair to say that the observation on commutativity presented above provides a complement to D. Deutsch’s argument from a logical point of view.

2. Quantum Logic

The aim of this section is to recall some basic notions and results about quantum logic needed in the subsequent sections and to fix notations. In this paper, quantum logic is understood as a complete orthomodular lattice-valued logic. This section is mainly concerned with the semantic aspect of such a logic, and it will be divided into four subsections. The first subsection will briefly review some fundamental results on orthomodular lattices; for more details, we refer to [Ka83] and [BH00]. In the second one we will introduce the language of first-order quantum logic. The third will discuss the algebraic semantics of first-order quantum logic. Some useful properties of orthomodular lattice-valued sets are given in the fourth subsection.

2.1. Orthomodular Lattices

The set of truth values of a quantum logic will be taken to be an orthomodular lattice. So we first introduce the notion of orthomodular lattice. An ortholattice is a 7-tuple 

\[ \ell = < L, \leq, \land, \lor, \bot, 0, 1 > \]

where:

(1) \(< L, \leq, \land, \lor, 0, 1 >\) is a bounded lattice, 0, 1 are the least and greatest elements of \( L \), respectively, \( \leq \) is the partial ordering in \( L \), and for any \( a, b \in L \), \( a \land b \), and \( a \lor b \) stand for the greatest lower bound and the least upper bound of \( a \) and \( b \), respectively;

(2) \( \bot \) is a unary operation on \( L \), called orthocomplement, and required to satisfy the following conditions: for any \( a, b \in L \),

(i) \( a \land a^\bot = 0 \), \( a \lor a^\bot = 1 \);

(ii) \( a^\bot \bot = a \); and

(iii) \( a \leq b \) implies \( b^\bot \leq a^\bot \).

It is easy to see that the condition (iii) is equivalent to one of the De Morgan laws: for any \( a, b \in L \),

(iii’) \( (a \land b)^\bot = a^\bot \lor b^\bot \), \( (a \lor b)^\bot = a^\bot \land b^\bot \).
Figure 1: Benzene ring

Let \( \ell = \langle L, \leq, \land, \lor, \bot, 0, 1 \rangle \) be an ortholattice, and let \( a, b \in L \). We say that \( a \) commutes with \( b \), in symbols \( aCBb \), if

\[
a = (a \land b) \lor (a \land b^\bot).
\]

An orthomodular lattice is an ortholattice \( \ell = \langle L, \leq, \land, \lor, \bot, 0, 1 \rangle \) satisfying the orthomodular law: for all \( a, b \in L \),

(iv) \( a \leq b \) implies \( a \lor (a^\bot \land b) = b \).

The orthomodular law can be replaced by the following equation:

(iv') \( a \lor (a^\bot \land (a \lor b)) = a \lor b \) for any \( a, b \in L \).

A Boolean algebra is an ortholattice \( \ell = \langle L, \leq, \land, \lor, \bot, 0, 1 \rangle \) fulfilling the distributive law of join over meet: for all \( a, b, c \in L \),

(v) \( a \lor (b \land c) = (a \lor b) \land (a \lor c) \).

With the De Morgan law it is easy to know that the condition (v) is equivalent to the distributive law of meet over join: for any \( a, b, c \in L \),

(v') \( a \land (b \lor c) = (a \land b) \lor (a \land c) \).

Obviously, the distributive law implies the orthomodular law, and so a Boolean algebra is an orthomodular lattice.

The following lemma gives a characterization of orthomodular lattices and it distinguishes orthomodular lattices from ortholattices.

**Lemma 2.1.** ([BH00], Propositions 2.1 and 2.2) Let \( \ell = \langle L, \leq, \land, \lor, \bot, 0, 1 \rangle \) be an ortholattice. Then the following seven statements are equivalent:

1. \( \ell \) is an orthomodular lattice;
2. For any \( a, b \in L \), if \( a \leq b \) and \( a^\bot \land b = 0 \) then \( a = b \);
3. For any \( a, b \in L \), if \( aCBb \) then \( bCa \);
4. For any \( a, b \in L \), if \( aCBb \) then \( a^\bot CBb \);
5. For any \( a, b \in L \), if \( aCBb \) then \( a \lor (a^\bot \land b) = a \lor b \);
6. The benzene ring \( O_6 \) (see Figure 1) is not a subalgebra of \( \ell \);
7. For any \( a, b \in L \), if \( a \leq b \) then the subalgebra \([a, b]\) of \( \ell \) generated by \( a \) and \( b \) is a Boolean algebra.\( \diamond \)
The set of truth values of classical logic is a Boolean algebra; whereas quantum logic is an orthomodular lattice-valued logic. It is well-known that a Boolean algebra must be an orthomodular lattice, but the inverse is not true. Thus, quantum logic is weaker than classical logic. The major difference between a Boolean algebra and an orthomodular lattice is that distributivity is not valid in the latter. However, many cases still appeal an application of the distributivity even when we manipulate elements in an orthomodular lattice. This requires us to regain a certain (weaker) version of distributivity in the realm of orthomodular lattices. The key technique for this purpose is commutativity which is able to provide a localization of distributivity. The following lemma together with Lemma 2.1(4) indicates that commutativity is preserved by all operations of orthomodular lattice.

**Lemma 2.2.** ([BH00], Proposition 2.4) Let $\ell = \langle L, \leq, \wedge, \vee, \bot, 0, 1 \rangle$ be an orthomodular lattice, and let $a \in L$ and $b_i \in L \ (i \in I)$. If $a C b_i$ for any $i \in I$, then

$$a C \land_{i \in I} b_i$$

and

$$a C \lor_{i \in I} b_i$$

provided $\land_{i \in I} b_i$ and $\lor_{i \in I} b_i$ exist.

The local distributivity implied by commutativity is then given by the following

**Lemma 2.3.** ([BH00], Proposition 2.3) Let $\ell = \langle L, \leq, \wedge, \vee, \bot, 0, 1 \rangle$ be an orthomodular lattice and let $A \subseteq L$. For any $a \in A$ and $b_i \in A \ (i \in I)$, if $a C b_i$ for all $i \in I$, then

$$a \land_{i \in I} b_i = \lor_{i \in I} (a \land b_i)$$

and

$$a \lor_{i \in I} b_i = \land_{i \in I} (a \lor b_i)$$

provided $\land_{i \in I} b_i$ and $\lor_{i \in I} b_i$ exist.

The above lemma is very useful, and it often enables us to recover distributivity in an orthomodular lattice. However, its condition that all elements involved commute each other is quite strong, and not easy to meet. This suggests us to find a way to weaken this condition. One solution was found by G. Takeuti [T81], and he introduced the notion of commutator which can be seen as an index measuring the degree to which the commutativity is valid.

**Definition 2.1.** ([T81], pages 305 and 307) Let $\ell = \langle L, \leq, \wedge, \vee, \bot, 0, 1 \rangle$ be an orthomodular lattice and let $A \subseteq L$.

1. If $A$ is finite, then the commutator $\gamma(A)$ of $A$ is defined by

$$\gamma(A) = \lor\{ \land_{a \in A} a^{f(a)} : f \text{ is a mapping from } A \text{ into } \{1, -1\} \}$$
where $a^1$ denotes $a$ itself and $a^{-1}$ denotes $a^\perp$.

(2) The strong commutator $\Gamma(A)$ of $A$ is defined by

$$\Gamma(A) = \lor \{ b : C(a, b) \text{ for all } a \in A \text{ and } C(a_1 \land b, a_2 \land b) \text{ for all } a_1, a_2 \in A \}. $$

The relation between commutator and strong commutator is clarified by the following lemma. In addition, the third item of the following lemma shows that commutator is a relativization of the notion of commutativity.

**Lemma 2.4.** ([T81], Proposition 4 and its corollary) Let $\ell = < L, \leq, \land, \lor, \perp, 0, 1 >$ be an orthomodular lattice and let $A \subseteq L$. Then

1. $\Gamma(A) \leq \gamma(A)$.
2. If $A$ is finite, then $\Gamma(A) = \gamma(A)$.
3. $\gamma(A) = 1$ if and only if all the members of $A$ are mutually commutable. $\rotatebox{90}{$\heartsuit$}$

We now can present a generalization of Lemma 2.3 by using the tool of commutator. It is easy to see from Lemmas 2.4(2) and (3) that the following lemma degenerates to Lemma 2.3 when $aCb_i$ for all $i \in I$.

**Lemma 2.5.** ([T81], Propositions 5 and 6) Let $\ell = < L, \leq, \land, \lor, \perp, 0, 1 >$ be an orthomodular lattice and let $A \subseteq L$. Then for any $a \in A$ and $b_i \in A$ ($i \in I$),

$$\Gamma(A) \land (a \land \lor_{i \in I} b_i) \leq \lor_{i \in I}(a \land b_i),$$

$$\Gamma(A) \land \lor_{i \in I}(a \lor b_i) \leq a \lor \land_{i \in I} b_i. \rotatebox{90}{$\heartsuit$}$$

Suppose that we want to use the above lemma on a formula of the form $a \land \lor_{i \in I} b_i$ or $a \lor \land_{i \in I} b_i$ in order to get a local distributivity. In many situations, the elements $a$ and $b_i$ ($i \in I$) may be very complicated, and the operations $\perp, \land$ and $\lor$ are involved in them. Then the above lemma cannot be applied directly, and it needs the help of the following

**Lemma 2.6.** Let $\ell = < L, \leq, \land, \lor, \perp, 0, 1 >$ be an orthomodular lattice and let $A \subseteq L$. Then for any $B \subseteq [A]$ we have

$$\Gamma(A) \leq \Gamma(B),$$

where $[A]$ stands for the subalgebra of $\ell$ generated by $A$. 

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Proof. For any \( X \subseteq L \), we write

\[
K(X) = \{b \in L : aCb \text{ and } (a_1 \land b)C(a_2 \land b) \text{ for all } a, a_1, a_2 \in X\}
\]

Furthermore, we set \( A_0 = A \) and

\[
A_{i+1} = A_i \cup \{a^\perp : a \in A_i\} \cup \{a_1 \land a_2 : a_1, a_2 \in A_i\} \quad (i = 0, 1, 2, \ldots)
\]

First, we prove that \( K(A_i) = K(A) \) for all \( i \geq 0 \) by induction on \( i \). It is obvious that \( K(A_{i+1}) \subseteq K(A) \). Conversely, suppose that \( b \in K(A) \) and we want to show that \( b \in K(A_{i+1}) \). It is easy to see that \( aCb \) for any \( a \in A_{i+1} \). Thus, we only need to demonstrate the following

Claim: \((a_1 \land b)C(a_2 \land b)\) for any \( a_1, a_2 \in A_{i+1}\).

The essential part of the proof of the above claim is the following two cases, and the other cases are clear, or can be treated as iterations of them:

Case 1. \( a_1 \in A_i, a_2 = c_1 \land c_2 \) and \( c_1, c_2 \in A_i \). From the induction hypothesis we have

\[
(a_1 \land b)C(c_1 \land b) \text{ and } (a_1 \land b)C(c_2 \land b).
\]

This yields

\[
(a_1 \land b)C(c_1 \land b) \land (c_2 \land b) = (c_1 \land c_2) \land b = a_2 \land b.
\]

Case 2. \( a_1 \in A_i, a_2 = c^\perp \) and \( c \in A_i \). Then from the induction hypothesis we obtain \((a_1 \land b)C(c \land b)\), and further \((a_1 \land b)C(c \land b)^\perp\) by using Lemma 2.1(4). In addition, \((a_1 \land b)Cb\). This together with Lemma 2.2 yields \((a_1 \land b)Cb \land (c \land b)^\perp\). Note that \( cCb \) and so \( b^\perp Cc^\perp \). Then by Lemma 2.3 we assert that

\[
b^\perp \lor (c \land b) = b^\perp \lor c \text{ and } b \land (c \land b)^\perp = b \land c^\perp
\]

Hence, it follows that \((a_1 \land b)Cb \land c^\perp = a_2 \land b\).

We now write

\[
A_\infty = \bigcup_{i=0}^{\infty} A_i.
\]

Then

\[
K(A_\infty) = \bigcap_{i=0}^{\infty} K(A_i) = K(A).
\]

It is easy to see that \( A \subseteq A_\infty \) is a subalgebra of \( \ell \). So, \([A] \subseteq A_\infty\),

\[
K(A) = K(A_\infty) \subseteq K([A]) \subseteq K(B),
\]

and

\[
\Gamma(A) = \lor K(A) \leq \lor K(B) = \Gamma(B).
\]

As stated in the introduction, the aim of this paper is to develop a theory of computation based on quantum logic. The logical language for a theory of computation has to contain the universal and existential quantifiers, and the two quantifiers
are usually interpreted as (infinite) meet and join, respectively. Hence, we should assume that the lattice of the truth values of our quantum logic is complete. A complete orthomodular lattice is an orthomodular lattice $\ell = \langle L, \leq, \wedge, \vee, \bot, 0, 1 \rangle$ in which for any $M \subseteq L$, both the greatest lower bound $\wedge M$ and the least upper bound $\vee M$ exist.

The function of a logic is provide us with a certain reasoning ability, and the implication connective is an intrinsic representative of inference within the logic. Thus each logic should reasonably contain a connective of implication. To make a complete orthomodular lattice available as the set of truth values of quantum logic, we need to define a binary operation, called implication operator, on it such that this operation may serve as the interpretation of implication in this logic. Unfortunately, it is a very vexed problem to define a reasonable implication operator for quantum logic. All implication operators that one can reasonably introduce in an orthomodular lattice are more or less anomalous in the sense that they do not share most of the fundamental properties of the implication in classical logic. This is different from the cases of most weak logics. (For a thorough discussion on the implication problem in quantum logic, see [DC86], Section 3.)

A minimal condition for an implication operator $\rightarrow$ is the requirement proposed by G. Birkhoff and J. von Neumann [BN36]:

$$ a \rightarrow b = 1 \text{ if and only if } a \leq b $$

for any $a, b \in L$. Usually in a logic, there are two ways in which implication is introduced. The first one is to treat implication as a derived connective; that is, implication is explicitly defined in terms of other connectives such as negation, conjunction and disjunction. All implications of this kind were found by G. Kalmbach [Ka74], and they are presented by the following:

**Lemma 2.7.** ([Ka74]; see also [Ka83], Theorem 15.3) The orthomodular lattice freely generated by two elements is isomorphic to $2^4 \times MO2$, where 2 stands for the Boolean algebra of two elements. The elements of $2^4 \times MO2$ satisfying the Birkhoff-von Neumann requirement are exactly the following five polynomials of two variables:

$$ a \rightarrow_1 b = (a^\perp \wedge b) \vee (a^\perp \wedge b^\perp) \vee (a \wedge (a^\perp \vee b)), $$
$$ a \rightarrow_2 b = (a^\perp \wedge b) \vee (a \wedge b) \vee ((a^\perp \vee b) \wedge b^\perp), $$
$$ a \rightarrow_3 b = a^\perp \vee (a \wedge b), $$
$$ a \rightarrow_4 b = b \vee (a^\perp \wedge b^\perp), $$
$$ a \rightarrow_5 b = (a^\perp \wedge b) \vee (a \wedge b) \vee (a^\perp \wedge b^\perp).$$

Obviously, this lemma implies that the above five polynomials are all implication operators definable in orthomodular lattices. It was shown by G. Kalmbach [Ka74,
Ka83] that the orthomodular lattice-valued (propositional) logic can be (finitely) axiomatizable by using the modus ponens with implication \( \rightarrow_1 \) as the only one rule of inference, but the same conclusion does not hold for the other implications \( \rightarrow_i \) \( (2 \leq i \leq 5) \).

We may also define the material conditional \( \rightarrow_0 \) in an orthomodular \( \ell = < L, \leq, \wedge, \vee, \bot, 0, 1 > \) by

\[
a \rightarrow_0 b = a^\perp \vee b
\]

for all \( a, b \in L \). It is easy to see that \( \rightarrow_0 \) does not fulfil the Birkhoff-von Neumann requirement. On the other hand, the following lemma shows that the five implication operators given in Lemma 2.7 degenerate to the material conditional whenever the two operands are compatible.

**Lemma 2.8.** ([DC86], Theorem 3.2) Let \( \ell = < L, \leq, \wedge, \vee, \bot, 0, 1 > \) be an orthomodular lattice. Then for any \( a, b \in L \),

\[
a \rightarrow_i b = a \rightarrow_0 b
\]

if and only if \( aCb \), where \( 1 \leq i \leq 5 \).

The second way of defining an implication is to take its truth function as the adjointor (i.e., residuation) of the truth function of conjunction. Note that in this case the implication is usually not definable from negation, conjunction and disjunction, and it has been treated as a primitive connective. Indeed, L. Herman, E. Marsden and R. Piziak [HMP75] introduced an implication in the style of residuation. Furthermore, the following lemma shows that the five polynomial implication operators \( \rightarrow_i \) \( (1 \leq i \leq 5) \) cannot be defined as the residuation of the conjunction unless \( \ell \) is a Boolean algebra.

**Lemma 2.9.** ([DC86], the revised version, page 25) Let \( \ell = < L, \leq, \wedge, \vee, \bot, 0, 1 > \) be an orthomodular lattice, and let \( 1 \leq i \leq 5 \). Then the following two statements are equivalent:

(i) \( \ell \) is a Boolean algebra.

(ii) the import-export law: for all \( a, b \in L \),

\[
a \wedge b \leq c \text{ if and only if } a \leq b \rightarrow_i c.
\]

Among the five orthomodular polynomial implications, \( \rightarrow_3 \), named the Sasaki-hook, has often been preferred since it enjoys some properties resembling those in intuitionistic logic. The Sasaki-hook was originally introduced by P. D. Finch [Fi70]. For a detailed discussion of the Sasaki-hook, see L. Román and B. Rumbos [RR91] and L. Román and R. E. Zuazua [RZ99]. Here we first point out that the Sasaki-hook possesses a modification of residual characterization although it is defined as
a polynomial in orthomodular lattice. A weakening of the import-export law is the resulting condition, called compatible import-export law, by restricting the import-export law for any \(a, b \in L\) with \(aCb\); that is, if \(aCb\), then \(a \land b \leq c\) if and only if \(a \leq b \rightarrow c\).

**Lemma 2.10.** ([T81], Proposition 1 and its corollary; [DC86], the revised version, page 25) Let \(\ell = \langle L, \leq, \land, \lor, \bot, 0, 1 \rangle\) be an orthomodular lattice, and let \(a, b, c \in L\). Then
\[
a \rightarrow b = \lor\{x : xCa\text{ and } x \land a \leq b\}.
\]
Moreover, among the five implications \(\rightarrow i\) (\(1 \leq i \leq 5\)), the Sasaki-hook \(\rightarrow_3\) is the only one satisfying the compatible import-export law.

Our mathematical reasoning frequently require that implication relation is preserved by conjunction and disjunction. Also, the negation is needed to be compatible with implication in the sense that the negation can reverse the direction of implication. And, to warrant the validity of a chain of inferences, the transitivity of implication is required. However, this is not the case in general if we are working in an orthomodular lattice. Fortunately, if we adopt the Sasaki-hook, then these properties of implication can be recovered by attaching a certain commutator.

**Lemma 2.11.** Let \(\ell = \langle L, \leq, \land, \lor, \bot, 0, 1 \rangle\) be an orthomodular lattice. Then (1) for any \(a_i, b_i \in L\) (\(i = 1, ..., n\)), let \(X = \{a_1, ..., a_n\} \cup \{b_1, ..., b_n\}\),
\[
\Gamma(X) \land \land_{i=1}^n (a_i \rightarrow_3 b_i) \leq \land_{i=1}^n a_i \rightarrow_3 \land_{i=1}^n b_i,
\]
\[
\Gamma(X) \land \land_{i=1}^n (a_i \rightarrow_3 b_i) \leq \lor_{i=1}^n a_i \rightarrow_3 \lor_{i=1}^n b_i.
\]
(2) for any \(a, b \in L\),
\[
\Gamma(a, b) \land (a \rightarrow_3 b) \leq b^\bot \rightarrow_3 a^\bot.
\]
(3) for any \(a, b, c \in L\),
\[
\Gamma(a, b, c) \land (a \rightarrow_3 b) \land (b \rightarrow_3 c) \leq a \rightarrow_3 c.
\]

**Proof.** (1) We only prove the first inequality, and the proof of the second is similar. With Lemmas 2.5 and 2.6 we obtain
\[
\land_{i=1}^n a_i \rightarrow_3 \land_{i=1}^n b_i = (\land_{i=1}^n a_i)^\bot \lor (\land_{i=1}^n a_i \land \land_{i=1}^n b_i)
\]
\[
= \lor_{i=1}^n a_i^\bot \lor (a_i \land b_i)
\]
\[
\geq \Gamma(X) \land \land_{i=1}^n (\lor_{j=1}^n a_j^\bot \lor (a_i \land b_i))
\]

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\[ \Gamma(X) \wedge \land_{i=1}^{n} (a_i ^\perp \lor (a_i \land b_i)) \]
\[ = \Gamma(X) \wedge \land_{i=1}^{n} (a_i \rightarrow b_i). \]

(2) First, we note that \( a \land b, a ^\perp \land b, a ^\perp \land b ^\perp \leq b \lor (a ^\perp \land b ^\perp) = b ^\perp \rightarrow (a ^\perp \land b ^\perp). \) Thus,
\[ \Gamma(a, b) = (a \land b) \lor (a \land b ^\perp) \lor (a ^\perp \land b) \lor (a ^\perp \land b ^\perp) \]
\[ \leq (b ^\perp \rightarrow (a ^\perp) \lor (a \land b ^\perp), \]
and furthermore with Lemmas 2.5 and 2.6 we have
\[ \Gamma(a, b) \land (a \rightarrow b) = \Gamma(a, b) \land (a ^\perp \lor (a \land b)) \]
\[ \leq \Gamma(a, b) \land (a ^\perp \lor b) \]
\[ = \Gamma(a, b) \land \Gamma(a, b) \land (a ^\perp \lor b) \]
\[ \leq \Gamma(a, b) \land [(b ^\perp \rightarrow (a ^\perp) \lor (a \land b ^\perp)] \land (a ^\perp \lor b) \]
\[ \leq [(b ^\perp \rightarrow (a ^\perp) \lor (a ^\perp \land b)] \lor [(a \land b ^\perp) \land (a ^\perp \lor b)] \]
\[ \leq (b ^\perp \rightarrow (a ^\perp) \lor [(a \land b ^\perp) \land (a ^\perp \lor b)]. \]

Note that \( (a \land b ^\perp)^\perp = a ^\perp \lor b \) and \( (a \land b ^\perp) \land (a ^\perp \lor b) = 0. \) Then
\[ \Gamma(a, b) \land (a \rightarrow b) \leq b ^\perp \rightarrow (a ^\perp \land b ^\perp). \]

(3) Again, we use Lemmas 2.5 and 2.6. This enables us to assert that
\[ \Gamma(a, b, c) \land (a \rightarrow b) \land (b \rightarrow c) = \Gamma(a, b, c) \land (a ^\perp \lor (a \land b)) \land (b ^\perp \lor (b \land c)) \]
\[ \leq \Gamma(a, b, c) \land [(a ^\perp \land (b ^\perp \lor (b \land c)) \lor [(a \land b) \land (b ^\perp \lor (b \land c)))] \]
\[ < \Gamma(a, b, c) \land (a ^\perp \lor [(a \land b) \land (b ^\perp \lor (b \land c))]). \]

We note that \( \Gamma(a, b, c)Ca ^\perp \) and
\[ \Gamma(a, b, c)C[(a \land b) \land (b ^\perp \lor (b \land c))]. \]

Then
\[ \Gamma(a, b, c) \land (a \rightarrow b) \land (b \rightarrow c) \leq (\Gamma(a, b, c) \land a ^\perp) \lor (\Gamma(a, b, c) \land [(a \land b) \land (b ^\perp \lor (b \land c))]) \]
\[ \leq a ^\perp \lor (\Gamma(a, b, c) \land [(a \land b) \land (b ^\perp \lor (b \land c))]) \]
\[ \leq a ^\perp \lor [(a \land b) \land (b ^\perp \lor (b \land c))] \]
\[ = a ^\perp \lor [(a \land b) \land (b \land c)] \]
\[ \leq a ^\perp \lor (a \land c) \]
\[ = a \rightarrow (a \land c, \heartsuit) \]

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For simplicity of presentation, we finally introduce an abbreviation. For each implication operator \( \to \), the bi-implication operator on \( \ell \) is defined as follows:

\[
a \leftrightarrow b \overset{\text{def}}{=} (a \to b) \land (b \to a)
\]

for any \( a, b \in L \).

### 2.2. The Language of Quantum Logic

In this subsection we present the syntax of quantum logic. Given a complete orthomodular lattice \( \ell = \langle L, \leq, \land, \lor, \bot, 0, 1 \rangle \). We require that the language of an \( \ell \)-valued (quantum) logic possesses a nullary connective \( a \) for each \( a \in L \) as well as three other primitive connectives: an unary one \( \neg \) (negation) and two binary ones \( \land \) (conjunction), \( \to \) (implication). The language also has a primitive quantifier \( \forall \) (universal quantifier).

It deserves an explanation for our design decision of choosing implication as a primitive connective. In the sequel, many results only need to suppose that the implication operator satisfies the Birkhoff-von Neumann requirement. It is known that there are five polynomials fulfilling the Birkhoff-von Neumann requirement. If we treated implication as a derived connective defined in terms of negation, conjunction and disjunction, then it would be necessary to assume five different connectives of implication in our logical language. This would often complicate our presentation very much. On the other hand, in some cases, the Birkhoff-von Neumann condition is not enough and it requires the implication operator to be the Sasaki-hook. So, we decide to use implication as a primitive connective, and specify it when needed.

The syntax of \( \ell \)-valued logic is defined in a familiar way; we omit its details. To simplify the notations in what follows, it is necessary to introduce several derived formulas:

(i) \( \varphi \lor \psi \overset{\text{def}}{=} \neg(\neg\varphi \land \neg\psi) \);

(ii) \( \varphi \leftrightarrow \psi \overset{\text{def}}{=} (\varphi \to \psi) \land (\psi \to \varphi) \);

(iii) \( (\exists x)\varphi \overset{\text{def}}{=} \neg(\forall x)\neg\varphi \);

(iv) \( A \subseteq B \overset{\text{def}}{=} (\forall x)(x \in A \rightarrow x \in B) \); and

(v) \( A \equiv B \overset{\text{def}}{=} (A \subseteq B) \land (B \subseteq A) \).

Suppose that \( \Delta \) is a finite set of formulas. The commutator of \( \Delta \) is defined to be

\[
\gamma(\Delta) \overset{\text{def}}{=} \lor\{\land_{\varphi \in \Delta} \varphi^f(\varphi) : f \in \{1, -1\}^\Delta\},
\]

where \( \varphi^1, \varphi^{-1} \) express \( \varphi \) and \( \neg\varphi \), respectively. It is obvious that the above formula is the counterpart of Definition 2.1(1) in the language of our quantum logic.

### 2.3. The Algebraic Semantics of Quantum Logic
We now turn to give the semantics of quantum logic. There are several different versions of semantics for quantum logic; for example, quantum logic enjoys a semantics in the Kripke style [DC86; RS00]. What concerns us here is its algebraic semantics. Assume that \( \ell = \langle L, \leq, \land, \lor, \bot, 0, 1 \rangle \) be an orthomodular lattice equipped with additionally a binary operation \( \rightarrow \) over it. The operation \( \rightarrow \) is required to be suited to serve as the truth function of implication connective. According to our explanation of the connective of implication in the last subsection, we leave the operation \( \rightarrow \) unspecified but suppose that it satisfies the Birkhoff-von Neumann requirement. An \( \ell \)-valued interpretation is an interpretation in which every predicate symbol is associated with an \( \ell \)-valued relation, i.e., a mapping from the product of some copies of the discourse universe into \( L \), where the number of copies is exactly the arity of the predicate symbol. The other items in \( \ell \)-valued logical language are interpreted as usual. For every (well-formed) formula \( \varphi \), its truth value \( \lceil \varphi \rceil \) is assumed in \( L \), and the truth valuation rules for logical and set-theoretic formulas are given as follows:

(i) \( \lceil a \rceil = a \);
(ii) \( \lceil \neg \varphi \rceil = \lceil \varphi \rceil^\bot \);
(iii) \( \lceil \varphi \land \psi \rceil = \lceil \varphi \rceil \land \lceil \psi \rceil \);
(iv) \( \lceil \varphi \rightarrow \psi \rceil = \lceil \varphi \rceil \rightarrow \lceil \psi \rceil \);
(v) if \( U \) is the universe of discourse, then
\[
\lceil (\forall x) \varphi(x) \rceil = \land_{\mu \in U} \lceil \varphi(u) \rceil;
\]
and

(vi) \( \lceil x \in A \rceil = A(x) \), where \( A \) is a set constant (unary predicate symbol) and it is interpreted as a mapping, also denoted as \( A \), from the universe into \( L \), i.e., an \( \ell \)-valued set (more exactly, an \( \ell \)-valued subset of the universe).

Note that in the above truth valuation rules \( \land \) and \( \lor \) in the left-hand side are connectives in quantum logic whereas \( \land \) and \( \lor \) in the right-hand side stand for operations in the orthomodular lattice \( \ell \) of truth values. Also, the symbol \( \rightarrow \) in the left-hand side of (iv) is a connective in the language of quantum logic, but the symbol \( \rightarrow \) in the right-hand side of (iv) is the binary operation attached to \( \ell \) that is explained at the beginning of this subsection.

As we claimed in the introduction, quantum logic will act as our meta-logic in the theory of computation developed in this paper. Then we still have to introduce several meta-logical notions for quantum logic. For every orthomodular lattice \( \ell = \langle L, \leq, \land, \lor, \bot, 0, 1 \rangle \), if \( \Gamma \) is a set of formulas and \( \varphi \) a formula, then \( \varphi \) is a semantic consequence of \( \Gamma \) in \( \ell \)-valued logic, written \( \Gamma \llbracket \ell \rceil \models \varphi \), whenever
\[
\land_{\psi \in \Gamma} \lceil \psi \rceil \leq \lceil \varphi \rceil
\]
for all \( \ell \)-valued interpretations. In particular, \( \models \varphi \) means that \( \phi \models \varphi \), i.e., \( \lceil \varphi \rceil = 1 \) always holds for every \( \ell \)-valued interpretation; in other words, 1 is the unique
designated truth value in \( \ell \). Furthermore, if \( \Gamma \models \varphi \) (resp. \( \models \varphi \)) for all orthomodular lattice \( \ell \) then we say that \( \varphi \) is a semantic consequence of \( \Gamma \) (resp. \( \varphi \) is valid) in quantum logic and write \( \Gamma \models \varphi \) (resp. \( \models \varphi \)).

We here are not going to give a detailed exposition on quantum logic, but would like to point out that quantum logic gives rise to many counterexamples to some meta-logical properties which hold for classical logic and for a large class of weaker logics; for example, M. L. Dalla Chiara [DC81] showed that a minimal version of quantum logic fails to enjoy the Lindenbaum property, and J. Malinowski [Ma90] found that the deduction theorem fails in quantum logic and some of its variants.

### 2.4. The Operations of Quantum Sets

Beside the language of quantum logic introduced in Section 2.2, we will also need some notations such as \( \in \) (membership) from set-theoretical language in our study of computing theory based on quantum logic. As mentioned in the introduction, a theory of quantum sets has already been developed by G. Takeuti [T81]. A careful review of quantum set theory is out of the scope of the present paper. What mainly concerned G. Takeuti [T81] is how some axioms of classical set theory could be modified so that they will hold in the framework of quantum logic. In other words, he tried to clarify the relation of quantum set theory with the classical mathematics. Here, we instead propose some operations of \( \ell \)-valued sets and also introduce several notations for \( \ell \)-valued sets. These are needed in the subsequent sections. We write \( L^X \) for the set of all \( \ell \)-valued subsets of \( X \), i.e., all mappings from \( X \) into \( L \). For any non-empty set \( X \), if \( x \in X \) and \( \lambda \in L - \{0\} \), then \( x_\lambda \) is defined to be a mapping from \( X \) into \( L \) such that

\[
x_\lambda(x') = \begin{cases} 
\lambda & \text{if } x' = x, \\
0 & \text{otherwise},
\end{cases}
\]

and it is often called an \( \ell \)-valued point in \( X \). We write \( p_\ell(X) \) for the set of all \( \ell \)-valued points in \( X \); that is,

\[
p_\ell(X) = \{ x_\lambda : x \in X \text{ and } \lambda \in L - \{0\} \}.
\]

For each \( e = x_\lambda \in p_\ell(X) \), \( x \) is called the support of \( e \) and denoted \( s(e) \), and \( \lambda \) is called the height of \( e \) and written \( h(e) \). In particular, an \( \ell \)-valued point of height 1 is always identified with its support. The predicate \( \in \) can be extended to a predicate between \( \ell \)-valued points and \( \ell \)-valued sets in a natural way:

\[
x_\lambda \in A \overset{\text{def}}{=} x_\lambda \subseteq A.
\]

Then it is easy to see that

\[
[x_\lambda \in A] = \lambda \rightarrow A(x)
\]
for any $x \in X$, $\lambda \in L$ and $A \in L^X$, where $\rightarrow$ is the implication operator under consideration. For any $A \subseteq X$, its characteristic function is a mapping from $X$ into the Boolean algebra $2 = \{0, 1\}$ of two elements, and so it can also be seen as a mapping from $X$ into $L$, namely, an $\ell$-valued subset of $X$. We will identify $A$ with its characteristic function. For any $A \subseteq X$, its characteristic function is a mapping from $X$ into the Boolean algebra $2 = \{0, 1\}$ of two elements, and so it can also be seen as a mapping from $X$ into $L$, namely, an $\ell$-valued subset of $X$ and for all $x \in X$,

(i) $x \in aA \overset{def}{=} a \land (x \in A)$;
(ii) $x \in A^c \overset{def}{=} \neg (x \in A)$;
(iii) $x \in A \cap B \overset{def}{=} (x \in A) \land (x \in B)$;
(iv) $x \in A \cup B \overset{def}{=} (x \in A) \lor (x \in B)$.

From the truth valuation rules and the definition of derived formulas in the $\ell$-valued logical and set-theoretical language, we know that for all $x \in X$,

(i') $(aA)(x) = a \land A(x)$;
(ii') $(A^c)(x) = A(x)^\bot$;
(iii') $(A \cap B)(s) = A(s) \land B(s)$; and
(iv') $(A \cup B)(s) = A(s) \lor B(s)$.

It is easy to see that in the domain of $\ell$-valued sets the intersection and union operations are idempotent, commutative and associative, and they have $X$ and $\phi$, respectively as their unit elements. The intersection and union together with the complement satisfy the De Morgan law, but the distributivity of intersection over union or union over intersection is no longer valid. Clearly, the laws for operations of $\ell$-valued sets are essentially determined by the algebraic properties of the lattice $\ell$ of truth values.

Assume that $X$ and $Y$ are two non-empty sets, and $h : X \rightarrow Y$ is a mapping. For any $A \in L^X$, its image $h(A)$ under $h$ is defined by

$$y \in h(A) \overset{def}{=} (\exists x \in X)(y = f(x) \land x \in A),$$

and for any $B \in L^Y$, its pre-image $h^{-1}(B)$ under $h$ is defined by

$$x \in h^{-1}(B) \overset{def}{=} h(x) \in B.$$

The defining equations of $h(A)$ and $h^{-1}(B)$ may be rewritten, respectively, as follows:

$$h(A)(y) = \lor \{A(X) : x \in X \text{ and } f(x) = y\},$$
and

$$h^{-1}(B)(x) = B(h(x)).$$
Lemma 2.12. Let \( \ell = \langle L, \leq, \wedge, \lor, \perp, 0, 1 \rangle \) be an orthomodular lattice, let \( h : X \to Y \) enjoy the Birkhoff-von Neumann requirement, and let \( \ell \) be a mapping. Then for any \( A, B \in L^Y \),
\[
\ell \models A \equiv B \iff h^{-1}(A) \equiv h^{-1}(B).
\]

Proof.
\[
[h^{-1}(A) \equiv h^{-1}(B)] = \wedge_{x \in X} (h^{-1}(A)(x) \iff h^{-1}(B)(x))
\]
\[
= \wedge_{x \in X} (A(h(x)) \iff B(h(x)))
\]
\[
\geq \wedge_{y \in Y} (A(y) \iff B(y))
\]
\[
= [A \equiv B] \circ
\]

To conclude this section, we introduce the notion of \( \ell \)-valued language as well as some operations of \( \ell \)-valued languages. Suppose that \( \Sigma \) is an alphabet; that is, a finite nonempty set (of input symbols). We write \( \Sigma^* \) for the set of strings over \( \Sigma \):
\[
\Sigma^* = \bigcup_{n=0}^{\infty} \Sigma^n.
\]
An \( \ell \)-valued language over \( \Sigma \) is defined to be an \( \ell \)-valued subset of \( \Sigma^* \). Thus, the set of \( \ell \)-valued languages over \( \Sigma \) is exactly \( L_{\Sigma^*} \). Let \( A, B \in L_{\Sigma^*} \) be two \( \ell \)-valued subsets of \( \Sigma^* \). Then we define the concatenation \( A \cdot B \) of \( A \) and \( B \) and the Kleene closure \( A^* \in L_{\Sigma^*} \) of \( A \) as follows: for any \( s \in \Sigma^* \),
\[
(v) \ s \in A \cdot B \overset{\text{def}}{=} (\exists u, v \in \Sigma^*)(s = uv \land u \in A \land v \in B);
\]
\[
(vi) \ s \in A^* \overset{\text{def}}{=} (\exists n \geq 0)(\exists s_1, ..., s_n \in \Sigma^* (s = s_1...s_n \land \wedge_{i=1}^{n} (s_i \in A))).
\]

The above defining equations can also be translated to the following two formulas in the lattice of truth values by employing the truth valuation rules: for every \( s \in \Sigma^* \),
\[
(v') \ (A \cdot B)(s) = \lor\{A(u) \land B(v) : u, v \in \Sigma^* \text{ and } s = uv\};
\]
\[
(vi') \ A^*(s) = \lor\{\wedge_{i=1}^{n} A(s_i) : n \geq 0, s_1, ..., s_n \in \Sigma^* \text{ and } s = s_1...s_n\}.
\]

It is easy to demonstrate that if the meet \( \wedge \) is distributive over the join \( \lor \) in \( \ell \) (in other words, \( \ell \) is a Boolean algebra), then we have
\[
A^* = \bigcup_{n=0}^{\infty} A^n
\]
where
\[
\begin{aligned}
A^0 &= \{ \varepsilon \}, \\
A^{n+1} &= A^n \cdot A \quad \text{for all } n \geq 0.
\end{aligned}
\]

3. Orthomodular Lattice-Valued Automata

For convenience we first recall some basic notions in classical automata theory. Let \( \Sigma \) be a finite input alphabet whose elements are called input symbols or labels. Then a nondeterministic finite automaton (NFA for short) over \( \Sigma \) is a quadruple
\[
\mathcal{R} = \langle Q, I, T, E \rangle
\]
in which:

(i) \( Q \) is a finite set whose elements are called states;
(ii) \( I \subseteq Q \) and states in \( I \) are said to be initial;
(iii) \( T \subseteq Q \) and states in \( T \) are said to be terminal; and
(iv) \( E \subseteq Q \times \Sigma \times E \), and each \( (p, \sigma, q) \in E \) is called a transition in (or an edge of) \( \mathcal{R} \) and it means that input \( \sigma \) makes state \( p \) evolves to \( q \).

An NFA is said to be deterministic if \( I \) is a singleton, and for any \( p \) in \( Q \) and \( \sigma \) in \( \Sigma \), there is exactly one \( q \) in \( Q \) such that \( (p, \sigma, q) \in E \). Thus, the transition relation \( E \) in a deterministic finite automaton (DFA, for short) may be seen as a mapping from \( Q \times \Sigma \) into \( Q \), and it is called the transition function.

A path in \( \mathcal{R} \) is a finite sequence of the form
\[
c = q_0 \sigma_1 q_1 \ldots q_{k-1} \sigma_k q_k
\]
such that \( (q_i, \sigma_{i+1}, q_{i+1}) \in E \) for each \( i < k \). In this case, the sequence \( \sigma_1 \ldots \sigma_k \) is called the label of \( c \). A path \( c = q_0 \sigma_1 q_1 \ldots q_{k-1} \sigma_k q_k \) is said to be successful if \( q_0 \in I \) and \( q_k \in T \). The language accepted by an automaton \( \mathcal{R} \) is the set of labels of all successful paths in \( \mathcal{R} \).

Let
\[
A \subseteq \Sigma^* = \bigcup_{n=0}^{\infty} \Sigma^n.
\]
Then \( A \) is said to be regular if there is an automaton \( \mathcal{R} \) over \( \Sigma \) such that \( A \) is the language accepted by \( \mathcal{R} \).

The notion of orthomodular lattice-valued automata is a natural generalization of NFAs. Let \( \ell = < L, \leq, \wedge, \vee, \perp, 0, 1 > \) be an orthomodular lattice, and let \( \Sigma \) be a finite alphabet. Then an \( \ell \)-valued (quantum) automaton over \( \Sigma \) is a quadruple
\[
\mathcal{R} = \langle Q, I, T, \delta \rangle
\]
where:

(i) \( Q \) is the same as in an NFA;
(ii) \( I \) is an \( \ell \)-valued subset of \( Q \); that is, a mapping from \( Q \) into \( L \). For each \( q \in Q \), \( I(q) \) indicates the truth value (in the underlying quantum logic) of the proposition that \( q \) is an initial state;
(iii) $T$ is also an $\ell$-valued subset of $Q$, and for every $q \in Q$, $T(q)$ expresses the
truth value (in our quantum logic) of the proposition that $q$ is terminal; and

(iv) $\delta$ is an $\ell$-valued subset of $Q \times \Sigma \times Q$; that is, a mapping from $Q \times \Sigma \times Q$ into
$L$, and it is called the $\ell$-valued (quantum) transition relation of $\mathcal{R}$. Intuitively, $\delta$
is an $\ell$-valued (ternary) predicate over $Q$, $\Sigma$ and $Q$, and for any $p, q \in Q$ and $\sigma \in \Sigma$,
$\delta(p, \sigma, q)$ stands for the truth value (in quantum logic) of the proposition that input
$\sigma$ causes state $p$ to become $q$.

The propositions of the form

"$q$ is an initial state", written ",$q \in I$",

"$q$ is a terminal state", written "$q \in T$",

and

"input $\sigma$ causes state $p$ to become $q$, according to the specification

given by $\delta$," written "$p \xrightarrow{\delta,\sigma} q$"

are assumed to be atomic propositions in our logical language designated for de-
scribing $\ell$-valued automata $\mathcal{R}$. The truth values of the above three propositions
are respectively $I(q)$, $T(q)$ and $\delta(p, \sigma, q)$. The set of these atomic propositions
is denoted $\text{atom}(\mathcal{R})$. Formally, we have

$$\text{atom}(\mathcal{R}) = \{"q \in I" : q \in Q\} \cup \{"q \in T" : q \in Q\} \cup \{"p \xrightarrow{\delta,\sigma} q" : p, q \in Q \text{ and } \sigma \in \Sigma\}.$$  

We write $A(\Sigma, \ell)$ for the (proper) class of all $\ell$-valued automata over $\Sigma$.

Before defining the concept of recognizability for $\ell$-valued automata, we need
to introduce some auxiliary notions and notations. We set

$$T(Q, \Sigma) = (Q\Sigma)^*Q = \bigcup_{n=0}^{\infty}((Q\Sigma)^nQ);$$

that is, the set of all alternative sequences of states and labels beginning at a state
and also ending at a state. For any $c = q_0\sigma_1q_1...q_{k-1}\sigma_kq_k \in T(Q, \Sigma)$, the length of $c$
is defined to be $k$ and denoted by $|c|$, $q_0$ is the beginning of $c$ and denoted by $b(c)$,
$q_k$ is the end of $c$ and denoted by $e(c)$, and sequence $s = \sigma_1...\sigma_k$ is called the label
of $c$ and denoted by $lb(c)$.

Let $\mathcal{R} \in A(\Sigma, \ell)$ be an $\ell$-valued automaton over $\Sigma$. Then the $\ell$-valued (unary)
predicate $\text{path}_\mathcal{R}$ on $T(Q, \Sigma)$ is defined as $\text{path}_\mathcal{R} \in L^{T(Q, \Sigma)}$ (the set of all mappings
from $T(Q, \Sigma)$ into $L$):

$$\text{path}_\mathcal{R}(c) \overset{\text{def}}{=} \bigwedge_{i=0}^{k-1}[(q_i, \sigma_{i+1}, q_{i+1}) \in \delta]$$

for every $c = q_0\sigma_1q_1...q_{k-1}\sigma_kq_k \in T(Q, \Sigma)$.

In intuition, the truth value of the proposition that $c = q_0\sigma_1q_1...q_{k-1}\sigma_kq_k$ is a path in $\mathcal{R}$ is

$$[\text{path}_\mathcal{R}(c)] = \bigwedge_{i=0}^{k-1}\delta(q_i, \sigma_{i+1}, q_{i+1}).$$
Note the difference between the symbols $\wedge$ in the above two equations: the former is a logical connective, whereas the latter is an operation on the lattice of truth values.

Now, we are ready to define one of the key notions in this paper, namely, recognizability for $\ell$−valued automata. It will be seen that the defining equation of $\ell$−valued recognizability is the same as that in the classical theory of automata. The essential difference between the quantum recognizability and the corresponding classical notion implicitly resides in their truth values.

**Definition 3.1.** Let $\mathcal{R} \in \mathcal{A}(\Sigma, \ell)$. Then the $\ell$−valued (unary) recognizability predicate $\text{rec}_\mathcal{R}$ on $\Sigma^*$ is defined as $\text{rec}_\mathcal{R} \in L(\Sigma^*)$ : for every $s \in \Sigma^*$,

$$\text{rec}_\mathcal{R}(s) \overset{\text{def}}{=} (\exists c \in T(Q, \Sigma))(b(c) \in I \land e(c) \in T \land lb(c) = s \land path_\mathcal{R}(c)).$$

In other words, the truth value of the proposition that $s$ is recognizable by $\mathcal{R}$ is given by

$$[\text{rec}_\mathcal{R}(s)] = \lor \{I(b(c)) \land T(e(c)) \land [\text{path}_\mathcal{R}(c)] : c \in T(Q, \Sigma) \text{ and } lb(c) = s\}.$$

We note that $\text{rec}_\mathcal{R}$ is defined above as an $\ell$−valued unary predicate on $\Sigma^*$, so it may also be seen as an $\ell$−valued subset of $\Sigma^*$; that is, a mapping $\text{rec}_\mathcal{R} : \Sigma^* \to L$ with $\text{rec}_\mathcal{R}(s) = [\text{rec}_\mathcal{R}(s)]$ for all $s \in \Sigma^*$.

As a straightforward generalization of regular language, we can also define regularity for $\ell$−valued languages.

**Definition 3.2.** The $\ell$−valued (unary) regularity predicate $\text{Reg}_\Sigma$ on $L(\Sigma^*)$ (the set of all $\ell$−valued subsets of $\Sigma^*$ ) is defined as $\text{Reg}_\Sigma \in L(L(\Sigma^*))$ : for each $A \in L(\Sigma^*)$ ,

$$\text{Reg}_\Sigma(A) \overset{\text{def}}{=} (\exists \mathcal{R} \in \mathcal{A}(\Sigma, \ell))(A \equiv \text{rec}_\mathcal{R}).$$

Thus, the truth value of the proposition that $A$ is regular is

$$[\text{Reg}_\Sigma(A)] = \lor \{[A \equiv \text{rec}_\mathcal{R}] : \mathcal{R} \in \mathcal{A}(\Sigma, \ell)\}.$$

It should be noted that the (automaton) variable $\mathcal{R}$ bounded by the existential quantifier in the right-hand side of the defining formula of $\text{Reg}_\Sigma$ ranges over the proper class $\mathcal{A}(\Sigma, \ell)$. Some readers who are familiar with axiomatic set theory may worry about that this definition will cause a certain set-theoretical difficulty, but we stay well away from anything genuinely problematic. Indeed, for any $\ell$−valued automaton $\mathcal{R} = \langle Q, I, T, \delta \rangle$, there is a bijection $\varsigma : Q \to |Q|$ (the cardinality of $Q$) $\in \{0, 1, \ldots, |Q| - 1\}$ and we can construct a new $\ell$−valued automaton

$$\varsigma(\mathcal{R}) = \langle |Q|, \varsigma(I), \varsigma(T), \varsigma(\delta) \rangle.$$
where
\[ \varsigma(\delta)(m, \sigma, n) = \delta(\varsigma^{-1}(m), \sigma, \varsigma^{-1}(n)) \]
for any \( m, n \in |Q| \) and \( \sigma \in \Sigma \). It is easy to see that \( \text{rec}_\mathbb{R} = \text{rec}_\varsigma(\mathbb{R}) \). Then in Definition 3.2 we may only require that the variable \( \mathbb{R} \) bounded by the existential quantifier ranges over all \( \ell \)-valued automata whose state sets are subsets of \( \omega \) (the set of all non-negative integers) and the class of all \( \ell \)-valued automata with subsets of \( \omega \) as state sets is really a set, and in fact it is a subset of \( (2^\omega)^3 \times \cup_{Q \subseteq \omega} L\mathbb{Q} \times \Sigma \times Q \). In most situations, however, the original version of Definition 3.2 is much more convenient and compatible with the corresponding definition in classical automata theory.

Before investigating carefully various properties of regular \( \ell \)-valued languages, we present some interesting examples. The first one indicates that every \( \ell \)-valued language is regular. It is well-known that a similar conclusion holds in classical automata theory.

**Example 3.1.** For any \( A \in L\Sigma^* \), if \( A \) is finite, i.e., \( \text{supp} A = \{ s \in \Sigma^* : A(s) > 0 \} \) is finite, then
\[ \ell \models \text{Reg}_\Sigma(A). \]
Indeed, suppose that \( \text{supp} A = \{ \sigma_{i1}...\sigma_{im_i} : i = 1, ..., k \} \). Then we construct an \( \ell \)-valued automaton \( \mathbb{R}_A = (Q_A, I_A, T_A, \delta_A) \) in the following way:

(i) \( Q_A = \bigcup_{i=1}^{k} \{ q_{i0}, q_{i1}, ..., q_{im_i} \} \);
(ii) \( I_A = \{ q_{i0}, q_{20}, ..., q_{k0} \} \);
(iii) \( T_A = \{ q_{im_1}, q_{2m_2}, ..., q_{km_k} \} \); and
(iv) We define \( \delta_A(q_{ij}, \sigma_{i(j+1)}, q_{i(j+1)}) = A(\sigma_{i1}...\sigma_{im_i}) \)
for any \( 1 \leq i \leq k \) and \( 0 \leq j < m_i \), and we define \( \delta_A(p, \sigma, q) = 0 \) for other \( (p, \sigma, q) \in Q_A \times \Sigma \times Q_A \). Then it is easy to see that \( \text{rec}_{\mathbb{R}_A} = A \) and
\[ [\text{Reg}_\Sigma(A)] \geq [A \equiv \text{rec}_{\mathbb{R}_A}] = 1. \]

The following example may be seen as an extension of Example 3.1, and it shows that the recognizability of a quantum language is not less than the volume of its finite part.

**Example 3.2.** For any \( A \in L\Sigma^* \), we define \( A \downarrow \lambda = \{ s \in \Sigma^* : A(s) \leq \lambda \} \), and
\[ A \uparrow \lambda = \{ s \in \Sigma^* : A(s) \geq \lambda \}. \]
Let \( A \in L\Sigma^* \). Then
(1) \(\models \mu \rightarrow \text{Reg}_\Sigma(A)\), where \(\mu = \vee\{\lambda^\perp : A \downarrow \lambda \text{ is finite}\}\); and

(2) \(\models \theta \rightarrow \text{Reg}_\Sigma(A)\), where \(\theta = \vee\{\lambda : A \uparrow \lambda \text{ is finite}\}\).

Here, \(\rightarrow\) may be interpreted as any implication operator satisfying the Birkhoff-von Neumann requirement. We only prove (1) and (2) may be proven likewise. For any \(\lambda \in L\), if \(A \downarrow \lambda\) is finite, then we define \(A \downarrow \lambda \in L\) as follows: for any \(s \in \Sigma^*\),

\[
(A \downarrow \lambda)(s) = \begin{cases} 
A(s) & \text{if } A(s) \not\leq \lambda, \\
0 & \text{if } A(s) \leq \lambda.
\end{cases}
\]

Clearly, \(A \downarrow \lambda\) is finite. Then from Example 3.1 we know that there is a \(\ell\)-valued automata \(\mathcal{R}[\lambda]\) such that \(\text{rec}_{\mathcal{R}[\lambda]}(s) = A \downarrow \lambda\), i.e., \(\text{rec}_{\mathcal{R}[\lambda]}(s) = A(s)\) if \(A(s) \not\leq \lambda\) and \(\text{rec}_{\mathcal{R}[\lambda]}(s) = 0\) if \(A(s) \leq \lambda\), and

\[
[\text{Rec}_\Sigma(A)] \geq [A \equiv \text{rec}_{\mathcal{R}[\lambda]}] = \wedge\{A(s) \leftrightarrow \text{rec}_{\mathcal{R}[\lambda]}(s) : A(s) \not\leq \lambda\}\wedge\{A(s) \leftrightarrow 0 : A(s) \leq \lambda\}
\]

\[
= \wedge\{A(s) \leftrightarrow 0 : A(s) \leq \lambda\} \geq \lambda^\perp.
\]

The third example gives a simple connection between recognizability in classical automata theory and the \(\ell\)-valued predicate \(\text{Reg}_\Sigma\) introduced above.

**Example 3.3.** Let \(A \subseteq \Sigma^*\) be a regular language (in classical automata theory), \(B \in L\Sigma^*\) and

\[
\text{supp}B = \{s \in \Sigma^* : B(s) > 0\} \subseteq A,
\]

and let

\[
\lambda = \vee\{\wedge_{s \in A}(a \leftrightarrow B(s)) : a \in L\}.
\]

Then

\[
\models \lambda \rightarrow \text{Reg}_\Sigma(B).
\]

In particular, if \(A \subseteq \Sigma^*\) is regular then for every \(a \in L\),

\[
\models \text{Reg}_\Sigma(A[a]),
\]

where \(A[a] \in L\Sigma^*\) is given as

\[
A[a](s) = \begin{cases} 
a & \text{if } s \in A, \\
0 & \text{otherwise}.
\end{cases}
\]

This conclusion is not difficult to prove. In fact, since \(A\) is regular, there must be an automaton \(\mathcal{R} = \langle Q, I, T, E \rangle\) that accepts the language \(A\). Now, for each \(a \in L\), we construct an \(\ell\)-valued automaton \(\mathcal{R}_a = \langle Q, I, T, \delta_a \rangle\) such that

\[
\delta_a(p, \sigma, q) = \begin{cases} 
a & \text{if } (p, \sigma, q) \in E, \\
0 & \text{otherwise}.
\end{cases}
\]
Then it is easy to know that for all \( s \in \Sigma^* \),
\[
[rec_{\mathcal{R}_a}(s)] = \begin{cases} 
a & \text{if } s \in A, \\
0 & \text{otherwise},
\end{cases}
\]
and
\[
[B \equiv rec_{\mathcal{R}_a}] = \land_{s \in A}(a \leftrightarrow B(s)).
\]
Therefore, we have
\[
[Reg_{\Sigma}(B)] \geq \lor \{[B \equiv rec_{\mathcal{R}_a}] : a \in L\} = \lambda \lor
\]

The fourth example demonstrates that the \( \ell \)-valued predicate \( Reg_{\Sigma} \) defined above is not trivial; that is, it does not in general degenerate into a two-valued (Boolean) predicate.

**Example 3.4.** We consider a canonical orthomodular lattice. This lattice has a clear interpretation in quantum physics. One pasts together observables of the spin one-half system. Then he will obtain an orthomodular lattice \( L(x) \oplus L(\overline{x}) \), where
\[
L(x) = \{0, p_-, p_+, 1\}
\]
corresponds to the outcomes of a measurement of the spin states along the \( x \)-axis and
\[
L(\overline{x}) = \{\overline{0} = 1, \overline{p_-}, p_+, \overline{1} = 0\}
\]
is obtained by measuring the spin states along a different spatial direction; and \( L(x) \oplus L(\overline{x}) \) may be visualized as the following "Chinese lantern" (see [Sv98] for a more detailed description of \( L(x) \oplus L(\overline{x}) \) (see Figure 2).

In this example, we set \( \rightarrow = \rightarrow_3 \) (the Sasaki-hook). By a routine calculation we have
\[
p_- \leftrightarrow p_+ = p_- \leftrightarrow \overline{p_-} = p_- \leftrightarrow \overline{p_+} = 0
\]
and \( p_- \leftrightarrow 1 = p_- \). Thus, for each \( \lambda \in L(x) \oplus L(\overline{x}) \), \( \lambda \not\leq p_- \) implies \( p_- \leftrightarrow \lambda \leq p_- \).

Furthermore, let \( \Sigma = \{\sigma, \tau\} \) and \( A = \{\sigma^n \tau^n : n \in \omega\} \), and for any \( t \in L(x) \oplus L(\overline{x}) \), let \( A_t \in L_{\Sigma^*} \) be given as follows:
\[
A_t(s) = \begin{cases} 
1 & \text{if } s \in A, \\
t & \text{otherwise}.
\end{cases}
\]
Then it holds that
\[
\ell \models p_- \leftrightarrow Reg_{\Sigma}(A_{p_-});
\]
that is, \( [Reg_{\Sigma}(A_{p_-})] = p_- \). In fact, we know that \( \Sigma^* \) is regular (see [E74], Example II.2.3), and with Example 3.3 it is easy to see that \( [Reg_{\Sigma}(A_{p_-})] \geq p_- \). Conversely, for any \( \ell \)-valued automaton \( \mathcal{R} = < Q, I, T, \delta > \), if \( |Q| = n \) then
\[
[A_{p_-} \equiv rec_{\mathcal{R}}] \leq [A_{p_-}(\sigma^n \tau^n) \leftrightarrow rec_{\mathcal{R}}(\sigma^n \tau^n)] \land \land_{k,l \in \omega \ s.t. \ k \neq l}[A_{p_-}(\sigma^k \tau^l) \leftrightarrow rec_{\mathcal{R}}(\sigma^k \tau^l)]
\]

If $\text{rec}_R(\sigma^n \tau^n) \leq p_-$, then $\lceil A_{p-} \equiv \text{rec}_R \rceil \leq p_-$. Now, we consider the case of $\text{rec}_R(\sigma^n \tau^n) \not\leq p_-$. For any $c \in T(Q, \Sigma)$, if $b(c) \in I$, $e(c) \in T$ and $lb(c) = \sigma^n \tau^n$, then $c$ must be of the form

$$c = p_0 \sigma p_1 ... p_{n-1} \sigma p_n \tau q_1 ... q_{n-1} \tau q_n.$$ 

Since $|Q| = n$, there are $i, j$ such that $i < j \leq n$ and $p_i = p_j$. We put

$$c^+ = p_0 \sigma p_1 ... p_{j-1} \sigma p_j (= p_i) \sigma p_{i+1} ... p_{j-1} \sigma p_j \sigma p_{j+1} ... p_{n-1} \sigma p_n \tau q_1 ... q_{n-1} \tau q_n.$$ 

Then $b(c^+) \in I$, $e(c^+) \in T$, $lb(c^+) = \sigma^{n+(j-i)} \tau^n$ and $\lceil \text{path}_R(c^+) \rceil = \lceil \text{path}_R(c) \rceil$. Therefore, it holds that

$$\text{rec}_R(\sigma^{n+(j-i)} \tau^n) \geq \vee \{\lceil \text{path}_R(c^+) \rceil : b(c) \in I, e(c) \in T \text{ and } lb(c) = \sigma^n \tau^n\}$$

$$= \vee \{\lceil \text{path}_R(c) \rceil : b(c) \in I, e(c) \in T \text{ and } lb(c) = \sigma^n \tau^n\}$$

$$= \text{rec}_R(\sigma^n \tau^n),$$

and

$$\text{rec}_R(\sigma^{n+(j-i)} \tau^n) \not\leq p_-.$$ 

Furthermore, we have

$$[A_{p-} \equiv \text{rec}_R] \leq p_- \iff \text{rec}_R(\sigma^n \tau^n) \not\leq p_-.$$ 

So, for all $\ell$-valued automata $R$ we have $[A_{p-} \equiv \text{rec}_R] \leq p_-$, and it follows that

$$[\text{Rec}_\Sigma(A_{p-})] = \vee \{[A \equiv \text{rec}_R] : R \in A(\Sigma, \ell)\} \leq p_-.$$ 

This together with $[\text{Reg}_\Sigma(A_{p-})] \geq p_-$ obtained before leads to $[\text{Reg}_\Sigma(A_{p-})] = p_-$. 

Figure 2: "Chinese lantern"
Similarly, we have $[\text{Reg}_\Sigma(A_t)] = t$ for $t = p_+ , \overline{p_-}$ and $\overline{p_+} \lor$

Motivated by the above example, we propose the open problem: how to describe orthomodular lattices $\ell = \langle L, \leq, \wedge, \vee, \bot, 0, 1 \rangle$ which satisfy that

$$\{ [\text{Reg}_\Sigma(A)] : A \in L^{\Sigma^*} \} = L,$$

i.e., the truth values of recognizability traverse all over $L$, or more explicitly, for every $\lambda \in L$, there is $A \in L^{\Sigma^*}$ such that $[\text{Reg}_\Sigma(A)] = \lambda$. It seems that this is a difficult problem.

The $\ell$–valued regularity predicate $\text{Reg}_\Sigma$ in Definition 3.2 is a direct generalization of the notion of regular language in classical automata theory. In what follows, we will see that the predicate $\text{Reg}_\Sigma$ does not work well in many cases. Why this happens? Note that $\text{Reg}_\Sigma$ is merely a simple mimic of the classical concept of regular language, and an essential feature of quantum logic is missing here. In the defining equation of $\text{Reg}_\Sigma$, the language $A$ to be recognized and the automaton $\mathbb{R}$ for recognizing $A$ are left completely irrelevant. In the case of classical logic, this does not causes any difficulty in manipulating regular languages. Nevertheless, the thing changes when we work in quantum logic. After an analysis it was found that a suitable link between $A$ and $\mathbb{R}$ is a commutativity of them. This motivates the following:

**Definition 3.3.** The $\ell$–valued (unary and partial) predicate $C\text{Reg}_\Sigma$ on $L^{\Sigma^*}$ is called commutative regularity and it is defined as $C\text{Reg}_\Sigma \in L(L^{\Sigma^*})$ : for any $A \in L^{\Sigma^*}$ with finite $\text{Range}(A) = \{ A(s) : s \in \Sigma^* \}$,

$$C\text{Reg}_\Sigma(A) \overset{\text{def}}{=} (\exists \mathbb{R} \in A(\Sigma, \ell))(\gamma(\text{atom}(\mathbb{R}) \cup r(A)) \land (A \equiv \text{rec}_\mathbb{R})), $$

where $r(A) = \{ a : a \in \text{Range}(A) \}$.

The exposition concerning the automata variable $\mathbb{R}$ in the defining equation of $\text{Reg}_\Sigma$ in Definition 3.2 also applies to $C\text{Reg}_\Sigma$ in the above definition.

It is obvious that the notion of commutative regularity is stronger than regularity. In other words, we have for any $A \in L^{\Sigma^*}$,

$$\ell \models C\text{Reg}_\Sigma(A) \to \text{Reg}_\Sigma(A).$$

On the other hand, if $\ell$ is a Boolean algebra; that is, the underlying logic is the classical Boolean logic, then these two notions are equivalent; or formally, for all $A \in L^{\Sigma^*}$, it holds that

$$\ell \models C\text{Reg}_\Sigma(A) \leftrightarrow \text{Reg}_\Sigma(A).$$

This is just why the predicate $\text{Reg}_\Sigma$ works very well in classical automata theory but not in the theory of automata based on quantum logic.
4. Orthomodular Lattice-Valued Deterministic Automata

The notion of nondeterminism plays a central role in the theory of computation. The nondeterministic mechanism enables a device to change its states in a way that is only partially determined by the current state and input symbol. Obviously, the concept of \( \ell \)-valued automaton introduced in the last section is a generalization of nondeterministic finite automaton. In classical theory of automata, each nondeterministic finite automaton is equivalent to a deterministic one; more exactly, there exists a deterministic finite automaton which accepts the same language as the originally given nondeterministic one does. The aim of this section is just to see whether this result is still valid in the framework of quantum logic. To this end, we first introduce the concept of deterministic \( \ell \)-valued automaton.

Let \( \mathcal{R} = < Q, I, T, \delta > \in \mathbf{A}(\Sigma, \ell) \) be an \( \ell \)-valued automaton over \( \Sigma \). If

(i) there is a unique \( q_0 \) in \( Q \) with \( I(q_0) > 0 \); and

(ii) for all \( q \) in \( Q \) and \( \sigma \) in \( \Sigma \), there is a unique state \( p \) in \( Q \) such that \( \delta(q, \sigma, p) > 0 \),

then \( M \) is called an \( \ell \)-valued (quantum) deterministic finite automaton (\( \ell \)-valued DFA for short). The \( \ell \)-valued transition relation \( \delta \) in an \( \ell \)-valued DFA may be equivalently represented by a mapping from \( Q \times \Sigma \) into \( Q \times (L - \{0\}) \). For any \( q \) in \( Q \) and \( \sigma \) in \( \Sigma \), if \( p \) is the unique element in \( Q \) with \( \delta(q, \sigma, p) > 0 \), then \( \delta(q, \sigma) = (p, \delta(q, \sigma, p)) \in Q \times (L - \{0\}) \).

The class of \( \ell \)-valued DFAs over \( \Sigma \) is denoted \( \mathbf{DA}(\Sigma, \ell) \).

Suppose that \( \mathcal{R} \) is an \( \ell \)-valued DFA, \( \delta(q_0, \sigma_1) = (q_1, \lambda_1) \) and \( \delta(q_i, \sigma_{i+1}) = (q_{i+1}, \lambda_{i+1}) \) for all \( i = 1, 2, ..., n - 1 \). Then it is easy to see that

\[
[\text{rec}_\mathcal{R}(\sigma_1...\sigma_n)] = I(q_0) \wedge T(q_n) \wedge \wedge_{i=1}^n \lambda_i.
\]

Throughout this section, we always suppose that the lattice \( \ell \) of truth values is finite.

The proof of the equivalence between classical deterministic finite and nondeterministic finite automata is carried out by building the power set construction of a nondeterministic finite automaton that is deterministic and can simulate the given nondeterministic one. The power set construction can be naturally extended into the case of \( \ell \)-valued automata.

Let \( \mathcal{R} = < Q, I, T, \delta > \in \mathbf{A}(\Sigma, \ell) \) be an \( \ell \)-valued automaton over \( \Sigma \). We define the \( \ell \)-valued power set construction of \( \mathcal{R} \) to be \( \ell \)-valued automaton

\[ \ell^\mathcal{R} = < L^Q, I_1, T, \delta > \]

over \( \Sigma \), where:

(i) \( L^Q \) is the set of all \( \ell \)-valued subsets of \( Q \); that is, mappings from \( Q \) into \( L \);
(ii) $I_1$ is an $\ell-$valued point with height 1; that is, $I_1 \in L^{(L^Q)}$ and
\[
I_1(X) = \begin{cases} 
1 & \text{if } X = I, \\
0 & \text{otherwise}
\end{cases}
\]
for all $X \in L^Q$;

(iii) $T \in L^{(L^Q)}$; that is, $T$ is an $\ell-$valued subset of $L^Q$, and
\[
T(X) = \bigvee_{q \in Q} [X(q) \land T(q)]
\]
for any $X \in L^Q$; and

(iv) $\delta$ is a mapping from $L^Q \times \Sigma$ into $L^Q$, and for each $X \in L^Q$, $\delta(X, \sigma) \in L^Q$ and
\[
\delta(X, \sigma)(q) = \bigvee_{p \in Q} [X(p) \land \delta(p, \sigma, q)]
\]
for every $q \in Q$.

Since $L$ is assumed to be finite, $L^Q$ is finite too. Thus, it is easy to see that $\ell^R$ is an $\ell-$valued DFA. Moreover, both the set of the initial states and the transition relation are two-valued, namely, their truth values are either 0 or 1, and only the set of terminal states carries $\ell-$valued information.

The following theorem compares the abilities of an $\ell-$valued automaton and its power set construction according to the $\ell-$valued languages recognized by them.

**Theorem 4.1.** Let $\ell = < L, \leq, \land, \lor, \bot, 0, 1 >$ be a finite orthomodular lattice, and let $\rightarrow$ be an implication operator satisfying the Birkhoff-von Neumann requirement.

1. For any $\mathcal{R} \in A(\Sigma, \ell)$ and $s \in \Sigma^*$,
\[
\models \ell \text{ rec}_{\mathcal{R}}(s) \rightarrow \text{ rec}_{(\ell \mathcal{R})}(s).
\]

2. For any $\mathcal{R} \in A(\Sigma, \ell)$ and $s \in \Sigma^*$,
\[
\models \ell \gamma(\text{atom}(\mathcal{R})) \land \text{ rec}_{(\ell \mathcal{R})}(s) \rightarrow \text{ rec}_{\mathcal{R}}(s),
\]
and in particular if $\rightarrow = \rightarrow_3$, then
\[
\models \ell \gamma(\text{atom}(\mathcal{R})) \rightarrow (\text{ rec}_{(\ell \mathcal{R})}(s) \leftarrow \text{ rec}_{\mathcal{R}}(s)).
\]

3. The following two statements are equivalent to each other:
   (i) $\ell$ is a Boolean algebra.
   (ii) For any $\mathcal{R} \in A(\Sigma, \ell)$ and $s \in \Sigma^*$,
\[
\models \ell \text{ rec}_{\mathcal{R}}(s) \leftarrow \text{ rec}_{(\ell \mathcal{R})}(s).
\]
Proof. The proof of (1) is easy, and we omit it here.

(2) Suppose that $\mathcal{R} = \langle Q, I, T, \delta \rangle \in A(\Sigma, \ell)$ and $\ell^R = \langle I^R, I, \mathcal{T}, \mathcal{F} \rangle$ is the $\ell$-valued power set construction of $\mathcal{R}$. Our aim is to demonstrate that

$$[\gamma(\text{atom}(\mathcal{R}))] \land [\text{rec}(\mathcal{R})(s)] \leq [\text{rec}_\mathcal{R}(s)]$$

for all $s \in \Sigma^*$. To this end, we first prove the following

claim: $[\gamma(\text{atom}(\mathcal{R}))] \land \overline{\mathcal{R}}(I, \sigma_1...\sigma_n)(q_n)$

$$\leq \lor \{I(q_0) \land \land_{i=0}^{n-1} \delta(q_i, \sigma_{i+1}, q_{i+1}) : q_0, q_1, ..., q_{n-1} \in Q\}$$

for any $\sigma_1,...,\sigma_n \in \Sigma$ and $q \in Q$. We proceed by induction on $n$. For $n = 0$, it is clear. The definition of $\overline{\mathcal{R}}$ yields

$$\overline{\mathcal{R}}(I, \sigma_1...\sigma_n)(q_n) = \overline{\mathcal{R}}(I, \sigma_1...\sigma_{n-1}), \sigma_n)(q_n)$$

$$= \lor_{q_{n-1} \in Q} \overline{\mathcal{R}}(I, \sigma_1...\sigma_{n-1})(q_{n-1}) \land \delta(q_{n-1}, \sigma_n, q_n)].$$

We write

$$[\text{atom}(\mathcal{R})] = \{[\varphi] : \varphi \in \text{atom}(\mathcal{R})\}.$$

Then it holds that

$$[\gamma(\text{atom}(\mathcal{R}))] = \gamma([\text{atom}(\mathcal{R})]).$$

Note that the symbol $\gamma$ in the left-hand side applies to a set of logical formulas, whereas the one in the right-hand side applies to a subset of $L$. Furthermore, it is easy to see that $\delta(q_{n-1}, \sigma_n, q_n)$, $\overline{\mathcal{R}}(I, \sigma_1...\sigma_{n-1})(q_{n-1})$ and $[\gamma(\text{atom}(\mathcal{R}))]$ are all in $[[\text{atom}(\mathcal{R})]]$ (the subalgebra of $\ell$ generated by $[\text{atom}(\mathcal{R})]$). Thus, with Lemmas 2.5 and 2.6 and the induction hypothesis we obtain

$$[\gamma(\text{atom}(\mathcal{R}))] \land \overline{\mathcal{R}}(I, \sigma_1...\sigma_n)(q_n) = [\gamma(\text{atom}(\mathcal{R}))] \land [\gamma(\text{atom}(\mathcal{R}))]$$

$$\land \lor_{q_{n-1} \in Q} \overline{\mathcal{R}}(I, \sigma_1...\sigma_{n-1})(q_{n-1}) \land \delta(q_{n-1}, \sigma_n, q_n)]$$

$$\leq \lor_{q_{n-1} \in Q} \{[\gamma(\text{atom}(\mathcal{R}))] \land (\lor \{I(q_0) \land \land_{i=0}^{n-2} \delta(q_i, \sigma_{i+1}, q_{i+1}) : q_0, q_1, ..., q_{n-2} \in Q\} \land \delta(q_{n-1}, \sigma_n, q_n)].$$

Using Lemmas 2.5 and 2.6 again, we complete the proof of the above claim.

Now with this claim, we can use Lemmas 2.5 and 2.6 twice and derive that

$$[\gamma(\text{atom}(\mathcal{R}))] \land [\text{rec}_R(\mathcal{R})(\sigma_1...\sigma_n)] = [\gamma(\text{atom}(\mathcal{R}))] \land \overline{\mathcal{R}}(I, \sigma_1...\sigma_n)$$

$$= [\gamma(\text{atom}(\mathcal{R}))] \land \lor_{q_n \in Q} \overline{\mathcal{R}}(I, \sigma_1...\sigma_n)(q_n) \land T(q_n)$$

$$\leq \lor_{q_n \in Q} \{[\gamma(\text{atom}(\mathcal{R}))] \land [\gamma(\text{atom}(\mathcal{R}))] \land \overline{\mathcal{R}}(I, \sigma_1...\sigma_n)(q_n) \land T(q_n)\}$$

$$\leq \lor_{q_n \in Q} \{[\gamma(\text{atom}(\mathcal{R}))] \land \lor_{i=0}^{n-1} \delta(q_i, \sigma_{i+1}, q_{i+1}) \land T(q_n) : q_0, q_1, ..., q_{n-1} \in Q\}$$

$$\leq \lor \{I(q_0) \land \land_{i=0}^{n-1} \delta(q_i, \sigma_{i+1}, q_{i+1}) \land T(q_n) : q_0, q_1, ..., q_{n-1} \in Q\}.$$
For the case of \( \gamma = \gamma_{\Delta} \), what we want to prove is

\[
\gamma(\text{atom} (\mathcal{R})) \leq \gamma_{\Delta} \rightarrow 3 [\text{rec}_{\mathcal{R}} (s)].
\]

We observe that \( \mathcal{R} \) is a Boolean algebra. Thus, it is proved that (i) implies (ii). We now turn to show that (ii) implies (i). It suffices to show that the meet \( \wedge \) is distributive over the union \( \vee \); that is,

\[
a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)
\]

for all \( a, b, c \in L \). Let \( a, b, c \in L \). We construct an \( \ell \)-valued automaton

\[
\mathcal{R} = \langle \{ u, v, w \}, \{ u, v \}, \{ w \}, \delta >
\]

over \( \Sigma \) which has at least one element \( \sigma \), where \( \delta (u, \sigma, u) = a \), \( \delta (u, \sigma, w) = c \), \( \delta (v, \sigma, u) = b \), and \( \delta \) takes the value 0 for other cases. It may be visualized by Figure 3.

In the automaton \( \mathcal{R} \) we have

\[
[\text{rec}_{\mathcal{R}} (\sigma \sigma)] = \{ I (q_0) \wedge T (q_2) \wedge \delta (q_0, \sigma, q_1) \wedge \delta (q_1, \sigma, q_2) : q_0, q_1, q_2 \in Q \}
\]

\[
= \{ \delta (u, \sigma, q_1) \wedge \delta (q_1, \sigma, w) : q_1 \in Q \} \vee \{ \delta (v, \sigma, q_1) \wedge \delta (q_1, \sigma, v) : q_1 \in Q \}
\]

\[
= [\delta (u, \sigma, u) \wedge \delta (u, \sigma, w)] \vee [\delta (v, \sigma, u) \wedge \delta (u, \sigma, w)]
\]

\[
= (a \wedge c) \vee (b \wedge c).
\]

Consider the \( \ell \)-valued power set construction \( \ell \mathcal{R} \) of \( \mathcal{R} \). Then

\[
\bar{\delta} (I, \sigma) (u) = \vee_{q \in Q} [I (q) \wedge \delta (q, \sigma, u)]
\]

\[
= \delta (u, \sigma, u) \vee \delta (v, \sigma, u)
\]
Similarly, we obtain $\delta(I, \sigma)(v) = 0$ and $\delta(I, \sigma)(w) = c$. It follows that for any $q \in Q$,

$$
\delta(I, \sigma \sigma)(q) = \bigvee_{q' \in Q} \delta(I, \sigma)(q') \land \delta(q', \sigma, q)
$$

$$
= [\delta(I, \sigma)(u) \land \delta(u, \sigma, q)] \lor [\delta(I, \sigma)(w) \land \delta(w, \sigma, q)]
$$

$$
= (a \lor b) \land \delta(u, \sigma, q).
$$

Thus,

$$
\delta(I, \sigma \sigma)(u) = (a \lor b) \land a = a,
$$

$$
\delta(I, \sigma \sigma)(v) = 0
$$

and

$$
\delta(I, \sigma \sigma)(w) = (a \lor b) \land c.
$$

Therefore,

$$
[\text{rec}^R_{(\ell \mathcal{R})}(\sigma \sigma)] = T(\delta(I, \sigma \sigma))
$$

$$
= \bigvee_{q \in Q} [\delta(I, \sigma \sigma)(q) \land T(q)]
$$

$$
= \delta(I, \sigma \sigma)(w)
$$

$$
= (a \lor b) \land c.
$$

Finally, from the assumption (ii) we assert that

$$
(a \land c) \lor (b \land c) = [\text{rec}^R_{(\ell \mathcal{R})}(\sigma \sigma)] = [\text{rec}^L_{(\ell \mathcal{R})}(\sigma \sigma)] = (a \lor b) \land c. \land
$$
Many results in this paper appear in the same scheme as the above theorem. So, we here give a detailed explanation of this theorem. The above theorem points out that the ability of an ℓ−valued automaton for recognizing language is always weaker than that of its power set construction. On the other hand, in order to warrant that an ℓ−valued automaton ℜ and its power set construction have the same ability of accepting language, the condition γ(atom(ℜ)) has to be imposed. The intuitive meaning of this condition is that (the truth values of) any two atomic propositions describing ℜ should commute. (See also the physical interpretation of commutativity presented in the introductory section.) The third part of Theorem 4.1 indicates that the equivalence between a nondeterministic finite automaton and its power set construction is universally valid if and only if the underlying logic degenerates to the classical Boolean logic. In other words, if the meta-logic that we use in our reasoning does not enjoy distributivity, then such a meta-logic is not strong enough to guarantee the universal validity of any nondeterministic finite automaton and its power set construction, and we can always find a nondeterministic finite automaton such that the equivalence between it and its power set construction is not derivable with the mere inference power provided by such a meta-logic.

In Section 3, we introduced the regularity and commutative regularity predicates RegΣ and CRegΣ. They are all given with respect to nondeterministic ℓ−valued automata. Now we propose a restricted version of them based on the smaller class of deterministic ℓ−valued automata.

**Definition 4.1.** Let ℓ =< L, ≤, ∧, ∨, ⊥, 0, 1 > be an orthomodular lattice. Then the ℓ−valued (unary) predicates DRegΣ and (unary and partial) predicate CDRegΣ on LΣ* are called deterministic regularity and commutative deterministic regularity, respectively, and they are defined as DRegΣ, CDRegΣ ∈ L(LΣ*) : for any A ∈ LΣ*,

\[ DRegΣ(A) \overset{\text{def}}{=} (\exists ℜ ∈ DA(Σ, ℓ))(A \equiv rec_ℜ), \]

and for any A ∈ LΣ* with finite Range(A) = {A(s) : s ∈ Σ*},

\[ CDRegΣ(A) \overset{\text{def}}{=} (\exists ℜ ∈ DA(Σ, ℓ))(\gamma(atom(ℜ)) \cup r(A)) \land (A \equiv rec_ℜ)), \]

where r(A) = {a : a ∈ Range(A)}.

It is similar to the relation between RegΣ and CRegΣ that CDRegΣ is stronger than DRegΣ. In other words, it holds that for any A ∈ LΣ*,

\[ \overset{\ell}{\models} CDRegΣ(A) \rightarrow DRegΣ(A). \]

The following corollary shows that a certain commutativity condition guarantees that they are equivalent. Furthermore, if ℓ is a Boolean algebra, then the four notions RegΣ, CRegΣ, DRegΣ and CDRegΣ all coincide.
Corollary 4.2. Let $\ell = <L, \leq, \wedge, \vee, \bot, 0, 1>$ be a finite orthomodular lattice, and let $\rightarrow = \rightarrow_3$. Then for any $A \in L^{\Sigma^*}$,

$$\ell \models \text{CReg}_\Sigma(A) \leftrightarrow \text{CDReg}_\Sigma(A).$$

In particular, if $\ell$ is a Boolean algebra, then for any $A \in L^{\Sigma^*}$,

$$\ell \models \text{Reg}_\Sigma(A) \leftrightarrow \text{DReg}_\Sigma(A).$$

Proof. It is clear that

$$\ell \models \text{CDReg}_\Sigma(A) \rightarrow \text{CDReg}_\Sigma(A).$$

Then we only need to prove that

$$\ell \models \text{CReg}_\Sigma(A) \rightarrow \text{CDReg}_\Sigma(A);$$

that is, for any $\mathfrak{R} \in A(\Sigma, \ell)$,

$$[\gamma(\text{atom}(\mathfrak{R}) \cup r(A))] \land [A \equiv \text{rec}_\mathfrak{R}] \leq \lor \{[\gamma(\text{atom}(\varphi) \cup r(A))] \land [A \equiv \text{rec}_\varphi] : \varphi \in \text{DA}(\Sigma, \ell)\}.$$

First, by using Lemmas 2.5, 2.6 and 2.11(2) we have

$$[\gamma(\text{atom}(\mathfrak{R}) \cup r(A))] \land [A \equiv \text{rec}_\mathfrak{R}] \land [\text{rec}_\mathfrak{R} \equiv \text{rec}_{(\ell \mathfrak{R})}] =
$$

$$[\gamma(\text{atom}(\mathfrak{R}) \cup r(A))] \land \land_{s \in \Sigma^*} (A(s) \leftrightarrow \text{rec}_\mathfrak{R}(s)) \land \land_{s \in \Sigma^*} (\text{rec}_\mathfrak{R}(s) \leftrightarrow \text{rec}_{(\ell \mathfrak{R})}(s))$$

$$= \land_{s \in \Sigma^*} ([\gamma(\text{atom}(\mathfrak{R}) \cup r(A))] \land (A(s) \rightarrow \text{rec}_\mathfrak{R}(s)) \land (\text{rec}_\mathfrak{R}(s) \rightarrow \text{rec}_{(\ell \mathfrak{R})}(s)) \land$$

$$\land_{s \in \Sigma^*} ([\gamma(\text{atom}(\mathfrak{R}) \cup r(A))] \land (\text{rec}_{(\ell \mathfrak{R})}(s) \rightarrow \text{rec}_\mathfrak{R}(s)) \land (\text{rec}_\mathfrak{R}(s) \rightarrow A(s))$$

$$\leq \land_{s \in \Sigma^*} (A(s) \rightarrow \text{rec}_{(\ell \mathfrak{R})}(s)) \land \land_{s \in \Sigma^*} (\text{rec}_{(\ell \mathfrak{R})}(s) \rightarrow A(s))$$

$$= [A \equiv \text{rec}_{(\ell \mathfrak{R})}].$$

Second, from Theorem 4.1(2) we obtain

$$[\gamma(\text{atom}(\mathfrak{R}) \cup r(A))] \leq [\gamma(\text{atom}(\mathfrak{R}))] \leq [\text{rec}_\mathfrak{R} \equiv \text{rec}_{(\ell \mathfrak{R})}],$$

and

$$[\gamma(\text{atom}(\mathfrak{R}) \cup r(A))] \land [A \equiv \text{rec}_\mathfrak{R}] \leq [\gamma(\text{atom}(\mathfrak{R}) \cup r(A))] \land [A \equiv \text{rec}_\mathfrak{R}] \land [\text{rec}_\mathfrak{R} \equiv \text{rec}_{(\ell \mathfrak{R})}]$$

$$\leq [A \equiv \text{rec}_{(\ell \mathfrak{R})}].$$
In addition, it is easy to see that

\[ \lceil \gamma(\text{atom}(R) \cup r(A)) \rceil \leq \lceil \gamma(\text{atom}(\ell R) \cup r(A)) \rceil \]

from Lemma 2.6. Therefore, it follows that

\[ \lceil \gamma(\text{atom}(R) \cup r(A)) \rceil \land [A \equiv \text{rec}_R] \leq \lceil \gamma(\text{atom}(\ell R) \cup r(A)) \rceil \land [A \equiv \text{rec}_{\ell R}] \]

\[ \leq \lor \{ \lceil \gamma(\text{atom}(\varphi) \cup r(A)) \rceil \land [A \equiv \text{rec}_\varphi] : \varphi \in DA(\Sigma, \ell) \}, \]

and we complete the proof. ♥

It should be noted that in the above corollary the second conclusion is obtained from the first one by removing simply the commutativity. The second conclusion is in general not correct. The reason is that an essential application is needed in the derivation of the implication \( CReg_\Sigma \rightarrow CDReg_\Sigma \).

5. Orthomodular Lattice-Valued Automata with \( \varepsilon \)-Moves

Automata with \( \varepsilon \)-moves are nondeterministic automata in which transitions on the empty input \( \varepsilon \) are included, and they have the same power for accepting languages. In the classical theory of automata, automata with \( \varepsilon \)-moves are very convenient tools in building complex automata from simple ones and in proving the closure properties of regular languages. The aim of this section is to introduce an orthomodular lattice-valued extension of automaton with \( \varepsilon \)-moves. Let \( \ell = \langle L, \leq, \land, \lor, 0, 1 \rangle \) be an orthomodular lattice. Then an \( \ell \)-valued automaton with \( \varepsilon \)-moves over \( \Sigma \) is a quadruple \( R = \langle Q, I, T, \delta \rangle \) in which all components are the same as in an \( \ell \)-valued automaton (without \( \varepsilon \)-moves), but the domain of the quantum transition relation \( \delta \) is changed to \( Q \times (\Sigma \cup \{ \varepsilon \}) \times Q \); that is, \( \delta \) is a mapping from \( Q \times (\Sigma \cup \{ \varepsilon \}) \times Q \) into \( L \), where \( \varepsilon \) stands for the empty string of input symbols.

Thus, in an \( \ell \)-valued automaton with \( \varepsilon \)-moves, transitions of the form "\( p \overset{\delta, \varepsilon}{\rightarrow} q \)" are allowed. So, \( \text{atom}(R) \) contains the atomic propositions "\( p \overset{\delta, \varepsilon}{\rightarrow} q \)" and their truth values are given as \( \delta(p, \varepsilon, q) \) for all \( p, q \in Q \).

Now let \( R = \langle Q, I, T, \delta \rangle \) be an \( \ell \)-valued automaton with \( \varepsilon \)-moves. We put

\[ T_\varepsilon(Q, \Sigma) = \left( (Q(\Sigma \cup \{ \varepsilon \}))^*Q = \cup_{n=0}^\infty (Q(\Sigma \cup \{ \varepsilon \}))^nQ \right). \]

The difference between \( T(Q, \Sigma) \) and \( T_\varepsilon(Q, \Sigma) \) is that in the latter the empty string may be used as labels. For any \( c = q_0\sigma_1q_1...q_{k-1}\sigma_kq_k \in T_\varepsilon(Q, \Sigma) \), \( lb_\varepsilon(c) \) is defined to be the sequence \( \sigma_1...\sigma_k \) with all occurrences of \( \varepsilon \) deleted. Note that it is possible that the length of \( lb_\varepsilon(c) \) is strictly smaller than \( k \). Then the recognizability \( \text{rec}_R \) is also defined as an \( \ell \)-valued unary predicate over \( \Sigma^* \), and it is given by

\[ \text{rec}_R(s) \overset{\text{def}}{=} (\exists c \in T_\varepsilon(Q, \Sigma))(b(c) \in I \land e(c) \in T \land lb_\varepsilon(c) = s \land \text{path}_R(c)) \]
for all \( s \in \Sigma^* \), where \( \text{path}_\mathcal{R} \) is defined in the same way as in an \( \ell \)–valued automaton without \( \varepsilon \)–moves. The defining equation of \( \text{rec}_\mathcal{R} \) may be rewritten in terms of truth valued as follows:

\[
[\text{rec}_\mathcal{R}(s)] = \lor \{ I(b(c)) \land T(e(c)) \mid \text{path}_\mathcal{R}(c) \} : c \in T_\ell(Q, \Sigma) \text{ and } lb_\ell(c) = s, \]

where

\[
[\text{path}_\mathcal{R}(c)] = \land_{i=0}^{k-1} \delta(q_i, \sigma_i+1, q_{i+1})
\]

if \( c = q_0\sigma_1q_1...q_{k-1}\sigma_kq_k \).

For any \( \ell \)–valued automaton \( \mathcal{R} = \langle Q, I, T, \delta \rangle \) with \( \varepsilon \)– moves, its \( \varepsilon \)–reduction is defined to be the \( \ell \)–valued automaton \( \mathcal{R}^{-\varepsilon} = \langle Q, I, T', \delta' \rangle \) (without \( \varepsilon \)–moves) in which

(i) for any \( q \in Q \),

\[
q \in T' \overset{def}{=} (\exists q \in Q, m \geq 0)(q \in T \land \delta(q_0, \varepsilon^m, q));
\]

that is,

\[
T'(q) = \lor_{q \in Q, m \geq 0}(T(q) \land \delta(q_0, \varepsilon^m, q));
\]

(ii) for any \( p, q \in Q \) and \( \sigma \in \Sigma \),

\[
\delta'(p, \sigma, q) \overset{def}{=} (\exists m, n \geq 0)\delta(p, \varepsilon^m \sigma\varepsilon^n, q);
\]

that is,

\[
\delta'(p, \sigma, q) = \lor_{m, n \geq 0}\delta(p, \varepsilon^m \sigma\varepsilon^n, q),
\]

where

\[
\delta(q_0, \sigma_1...\sigma_k, q_k) \overset{def}{=} (\exists q_1, ..., q_{k-1} \in Q)(\delta(q_0, \sigma_1, q_1) \land \delta(q_1, \sigma_2, q_2) \land \delta(q_{k-1}, \sigma_k, q_k)).
\]

In other words,

\[
\delta(q_0, \sigma_1...\sigma_k, q_k) = \lor\{(\delta(q_0, \sigma_1, q_1) \land \delta(q_1, \sigma_2, q_2) \land \delta(q_{k-1}, \sigma_k, q_k) : q_1, ..., q_{k-1} \in Q\}
\]

for all \( k \geq 1, q_0, q_k \in Q \) and \( \sigma_1, ..., \sigma_k \in \Sigma \).

The following theorem gives a clear relation between the language accepted by an orthomodular lattice-valued automaton with \( \varepsilon \)–moves and that accepted by its \( \varepsilon \)–reduction. In general, the \( \varepsilon \)–reduction of an automaton with \( \varepsilon \)–moves has a stronger power of acceptance than itself. A certain commutativity between basic actions of the automaton implies the equivalence between an automaton with \( \varepsilon \)–moves and its \( \varepsilon \)–reduction. However, an universal validity of such an equivalence requires that the underlying logic degenerates to the classical Boolean logic.

**Theorem 5.1.** Let \( \ell = \langle L, \leq, \land, \lor, 0, 1 \rangle \) be an orthomodular lattice, and let \( \rightarrow \) be an implication operator satisfying the Birkhoff-von Neumann requirement.
(1) For any $\ell$-valued automaton $\mathcal{R}$ with $\varepsilon$-moves over $\Sigma$, and for any $s \in \Sigma^*$,

$$\models_{\ell} \mathcal{R}(s) \rightarrow \mathcal{R}_{\ell}(s).$$

(2) For any $\ell$-valued automaton $\mathcal{R}$ with $\varepsilon$-moves over $\Sigma$, and for any $s \in \Sigma^*$,

$$\models_{\ell} \gamma(\operatorname{atom}(\mathcal{R})) \land \mathcal{R}_{\ell}(s) \rightarrow \mathcal{R}(s),$$

and in particular if $\rightarrow = \rightarrow_3$ then

$$\models_{\ell} \gamma(\operatorname{atom}(\mathcal{R})) \rightarrow (\mathcal{R}(s) \leftrightarrow \mathcal{R}_{\ell}(s)).$$

(3) The following two statements are equivalent:

(i) $\ell$ is a Boolean algebra;

(ii) For all $\ell$-valued automaton $\mathcal{R}$ with $\varepsilon$-moves over $\Sigma$, and for all $s \in \Sigma^*$,

$$\models_{\ell} \mathcal{R}(s) \leftrightarrow \mathcal{R}_{\ell}(s).$$

**Proof.** The proof of (1) is similar to that of (2), so we omit it. We now prove (2). First, we use induction on the length $|c|$ of $c$ to show that for any $c \in T(Q, \Sigma)$,

claim: $[\gamma(\operatorname{atom}(\mathcal{R})) \land [\operatorname{path}_{\mathcal{R}_{\ell}}(c)] \leq \forall \{(\operatorname{path}_{\mathcal{R}}(c')) : c' \in T_{\ell}(Q, \Sigma), b(c') = b(c),

\quad e(c') = e(c) \text{ and } lb_{\varepsilon}(c') = lb(c)\}.$

For the case of $|c| = 1$, it is immediate from the definition of transition relation $\delta'$ in $\mathcal{R}_{\ell}$. If $c = c'\sigma q$, then with the induction hypothesis and Lemmas 2.5 and 2.6, we have

$$[\gamma(\operatorname{atom}(\mathcal{R}))] \land [\operatorname{path}_{\mathcal{R}_{\ell}}(c')] = [\gamma(\operatorname{atom}(\mathcal{R}))] \land [\operatorname{path}_{\mathcal{R}_{\ell}}(c')] \land \delta'(e(c'), \sigma, q)$$

$$\leq \forall m, n \geq 0 \delta(e(c'), \varepsilon^m \varepsilon^n, q)$$

$$\leq \forall m, n \geq 0 \delta(e(c'), \varepsilon^m \varepsilon^n, q)$$

$$\leq \forall \{(\gamma(\operatorname{atom}(\mathcal{R})) \land [\operatorname{path}_{\mathcal{R}}(d')] : d' \in T_{\ell}(Q, \Sigma), b(d') = b(c), e(d') = e(c),

\quad \text{and } lb_{\varepsilon}(d') = lb(c)\}.$$
\[ \wedge \delta(q_0, \varepsilon, q_{n-1}) \land \ldots \wedge \delta(q_2, \varepsilon, q_1) \wedge \delta(q_1, \varepsilon, q) : p_1, \ldots, p_m, q_1, \ldots, q_n \in Q. \]

Again, we use Lemmas 2.5 and 2.6 and obtain

\[ \gamma(\text{atom}(\mathcal{R})) \land [\text{path}_{\mathcal{R}^{-}}(c)] \leq \forall \{ \text{path}_{\mathcal{R}}(d') \land \delta(e'(c'), \varepsilon, p_1) \land \delta(p_1, \varepsilon, p_2) \land \ldots \land \delta(p_{m-1}, \varepsilon, p_m) \land \delta(p_m, \sigma, q_n) \land \delta(q_n, \varepsilon, q_{n-1}) \land \ldots \land \delta(q_2, \varepsilon, q_1) \land \delta(q_1, \varepsilon, q) : m, n \geq 0, d' \in T_\varepsilon(Q, \Sigma) \text{ with } b(d') = b(c'), e(d') = e(c') \land lb_\varepsilon(d') = lb(c'), p_1, \ldots, p_m, q_1, \ldots, q_n \in Q. \]

We put \( d = d' \varepsilon p_1 \varepsilon p_2 \ldots \varepsilon p_{m-1} \varepsilon q_n \varepsilon q_{n-1} \ldots \varepsilon q_2 \varepsilon q_1 \varepsilon q. \) Then \( b(d) = b(d') = b(c'), e(d) = q = e(c), lb_\varepsilon(d) = lb_\varepsilon(d') = lb(c') = lb(c), \) and

\[ \Delta_2 = \forall \{ \text{path}_{\mathcal{R}}(d) \} = \text{path}_{\mathcal{R}}(d') = \delta(e(c'), \varepsilon, p_1) \land \delta(p_1, \varepsilon, p_2) \land \ldots \land \delta(p_{m-1}, \varepsilon, p_m) \land \delta(p_m, \sigma, q_n) \land \delta(q_n, \varepsilon, q_{n-1}) \land \ldots \land \delta(q_2, \varepsilon, q_1) \land \delta(q_1, \varepsilon, q). \]

Therefore,

\[ \gamma(\text{atom}(\mathcal{R})) \land \Delta_2 = \forall \{ \text{path}_{\mathcal{R}}(d) : d \in T_\varepsilon(Q, \Sigma), b(d) = b(c), e(d) = e(c) \text{ and } lb_\varepsilon(d) = lb(c) \}

and the claim holds for the case of \(|c| = |c'| + 1.\)

Now it follows from the above claim and Lemmas 2.5 and 2.6 that

\[ \gamma(\text{atom}(\mathcal{R})) \land \Delta = \forall \{ \text{path}_{\mathcal{R}}(d) : d \in T_\varepsilon(Q, \Sigma), b(d) = b(c), e(d) = e(c) \text{ and } lb_\varepsilon(d) = lb(c) \}

and \(|c| = |c'| + 1.\)
Thus, \( \ell \) always holds in a Boolean algebra \( a, b, c \). For any \( 1 \in \varepsilon \) with \( 4 \), then it follows from (ii) that 

\[
\ell \text{ shows that } \phi.
\]

By a routine calculation we know that its \( \varepsilon \)-reduction is \( R^{-\varepsilon} =< \{q_0, q_1, ..., q_5\}, \{q_0\}, \{q_5\}, \delta' > \) where \( \delta'(q_0, \sigma, q_1) = \delta'(q_0, \sigma, q_2) = \delta'(q_0, \sigma, q_3) = a, \delta'(q_0, \sigma, q_4) = (a \land b) \lor (a \land c), \delta'(q_1, \sigma, q_5) = b \lor c, \delta'(q_2, \sigma, q_5) = b, \delta'(q_3, \sigma, q_5) = c, \delta'(q_4, \sigma, q_5) = 1, \) and \( \delta \) takes value 0 for other arguments (see Figure 4). Then it follows from (ii) that 

\[
a \land (b \lor c) = [a \land (b \lor c)] \lor (a \land b) \lor (a \land c) \lor [(a \land b) \lor (a \land c)] = [rec_{R^{-\varepsilon}}(\sigma \sigma)] = [rec_{R}(\sigma \sigma)] = (a \land b) \lor (a \land c).
\]

This shows that \( \ell \) enjoys the distributivity of meet over union, and it is a Boolean algebra.\( \triangledown \)

6. Closure Properties of Orthomodular Lattice-Valued Regularity
It was shown in the classical automata theory that the class of regular languages is closed under various operations such as union, intersection, complement, concatenation, the Kleene closure, substitution and homomorphism. In this section, we are going to examine the closure properties of orthomodular lattice-valued languages under these operations. We first consider the inverse of an \( \ell \)-valued language. Let \( A \in L^\Sigma^* \). Then the inverse \( A^{-1} \in L^\Sigma^* \) of \( A \) is defined as follows:

\[
A^{-1}(\sigma_1...\sigma_m) = A(\sigma_m...\sigma_1)
\]

for any \( m \in \omega \) and for any \( \sigma_1, ..., \sigma_m \in \Sigma \).

The following proposition shows that both regularity and commutative regularity are preserved by the inverse operation.

**Proposition 6.1.** Let \( \ell =< L, \leq, \wedge, \vee, \bot, 0, 1 > \) be a complete orthomodular lattice, and let \( \rightarrow \) fulfil the property that \( a \leftrightarrow a = 1 \) for any \( a \in L \). Then for any \( A \in L^\Sigma^* \),

\[
\models^\ell Reg_\Sigma(A) \leftrightarrow Reg_\Sigma(A^{-1}),
\]

and

\[
\models^\ell CReg_\Sigma(A) \leftrightarrow CReg_\Sigma(A^{-1}).
\]

**Proof.** Noting that \( A = (A^{-1})^{-1} \), it suffices to show that

\[
[Reg_\Sigma(A)] \leq [Reg_\Sigma(A^{-1})].
\]

For any \( \ell \)-valued automaton \( \mathfrak{R} = (Q, I, T, \delta) \), we define the inverse of \( \mathfrak{R} \) to be the \( \ell \)-valued automaton \( \mathfrak{R}^{-1} = (Q, T, I, \delta^{-1}) \), where \( \delta^{-1}(p, \sigma, q) = \delta(q, \sigma, p) \) for any
\( p, q \in Q \) and \( \sigma \in \Sigma \). Then it is easy to see that \( \text{rec}_{\mathcal{R}^{-1}} = (\text{rec}_{\mathcal{R}})^{-1} \), and furthermore we have

\[
[\text{Reg}_\Sigma(A)] = \lor \{ [A \equiv \text{rec}_\mathcal{R}] : \mathcal{R} \in A(\Sigma, \ell) \} \\
= \lor \{ [A^{-1} \equiv (\text{rec}_{\mathcal{R}})^{-1}] : \mathcal{R} \in A(\Sigma, \ell) \} \\
= \lor \{ [A^{-1} \equiv \text{rec}_{\mathcal{R}^{-1}}] : \mathcal{R} \in A(\Sigma, \ell) \} \\
\leq \lor \{ [A^{-1} \equiv \text{rec}_\varphi] : \varphi \in A(\Sigma, \ell) \} = [\text{Reg}_\Sigma(A^{-1})].
\]

. The proof for commutative regularity is similar. \( \heartsuit \)

The commutative regularity is preserved by the complement operation, but it is not the case for the regularity predicate.

**Proposition 6.2.** If \( \ell = \langle L_1, \leq, \land, \lor, 0, 1 \rangle \) is a finite orthomodular lattice, and \( \rightarrow = \rightarrow_3 \), then for any \( A \in \mathcal{L}^{\Sigma^*} \),

\[
\ell \models C\text{Reg}_\Sigma(A) \rightarrow C\text{Reg}_\Sigma(A^c).
\]

**Proof.** For any \( \mathcal{R} = \langle Q, I, T, \delta \rangle \in A(\Sigma, \ell) \), we observe that \( \ell^\mathcal{R} = \langle L^Q, I_1, T^\mathcal{R}, \delta^\mathcal{R} \rangle \) is an \( \ell \)–valued deterministic automaton and only \( T^\mathcal{R} \) carries \( \ell \)–valued information. Then we set \( (\ell^\mathcal{R})^c = \langle L^Q, I_1, T^\mathcal{R}, \delta^\mathcal{R} \rangle \), where for any \( X \in L^Q \), \( T^\mathcal{R}(X) = (T(X))^c \). It is easy to see that for all \( s \in \Sigma^* \), \( \text{rec}_{\ell^\mathcal{R}}(s) = (\text{rec}_{\ell^\mathcal{R}}(s))^c \).

Now by using Theorem 4.1 and Lemmas 2.5 and 2.6 we obtain

\[
\gamma(\text{atom}(\mathcal{R}) \cup r(A)) \land [A \equiv \text{rec}_\mathcal{R}] \leq \gamma(\text{atom}(\mathcal{R}) \cup r(A)) \land [A \equiv \text{rec}_\mathcal{R}] \land [\text{rec}_\mathcal{R} \equiv \text{rec}_{\ell^\mathcal{R}}] \\
= \land_{s \in \Sigma^*} \left( \gamma(\text{atom}(\mathcal{R}) \cup r(A)) \land (A(s) \rightarrow \text{rec}_\mathcal{R}(s)) \land (\text{rec}_\mathcal{R}(s) \rightarrow \text{rec}_{\ell^\mathcal{R}}(s)) \right) \land \land_{s \in \Sigma^*} \left( [\text{rec}_\mathcal{R}(s) \rightarrow A(s)] \land [\text{rec}_{\ell^\mathcal{R}}(s) \rightarrow A(s)] \right) \\
\leq \land_{s \in \Sigma^*} \left( \gamma(\text{atom}(\mathcal{R}) \cup r(A)) \land (A(s) \rightarrow \text{rec}_{\ell^\mathcal{R}}(s)) \right) \land \land_{s \in \Sigma^*} \left( [\text{rec}_\mathcal{R}(s) \rightarrow A(s)] \land [\text{rec}_{\ell^\mathcal{R}}(s) \rightarrow A(s)] \right).
\]

Then Lemma 2.11(2) yields

\[
\gamma(\text{atom}(\mathcal{R}) \cup r(A)) \land [A \equiv \text{rec}_\mathcal{R}] \leq \land_{s \in \Sigma^*} (\text{rec}_{\ell^\mathcal{R}}(s) \rightarrow A^c(s)) \land \land_{s \in \Sigma^*} (A^c(s) \rightarrow \text{rec}_{\ell^\mathcal{R}}(s)) \\
= \land_{s \in \Sigma^*} (A^c(s) \rightarrow \text{rec}_{\ell^\mathcal{R}}(s)) \\
= \land_{s \in \Sigma^*} (A^c(s) \rightarrow \text{rec}_{\ell^\mathcal{R}}(s)) \\
= [A^c \equiv \text{rec}_{\ell^\mathcal{R}}].
\]

In addition, we have

\[
\gamma(\text{atom}(\mathcal{R}) \cup r(A)) \leq \gamma(\text{atom}(\mathcal{R}) \cup r(A^c))
\]

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from Lemma 2.5. Therefore,

\[
\begin{align*}
[\gamma(\text{atom}(\mathcal{R}) \cup r(A))] & \land [A \equiv \text{rec}_{\mathcal{R}}] \leq [\gamma(\text{atom}(\mathcal{R}) \cup r(A^c))] \land [A^c \equiv \text{rec}_{\mathcal{R}}^c] \\
& \leq [C\text{Reg}_\Sigma (A^c)].
\end{align*}
\]

Finally, since \(\mathcal{R}\) is allowed to be arbitrary, it follows that

\[
[C\text{Reg}_\Sigma (A)] = \lor_{\mathcal{R} \in A(\Sigma, \ell)} [\gamma(\text{atom}(\mathcal{R}) \cup r(A))] \land [A \equiv \text{rec}_{\mathcal{R}}] \\
\leq [C\text{Reg}_\Sigma (A^c)],
\]

We now turn to deal with the union of two \(\ell\)-valued language. Let \(\mathcal{R} = (Q_A, I_A, T_A, \delta_A)\) and \(\varphi = (Q_B, I_B, T_B, \delta_B) \in A(\Sigma, \ell)\) be two \(\ell\)-valued automata over \(\Sigma\). We assume that \(Q_A \cap Q_B = \emptyset\). Then the (disjoint) union \(\mathcal{R} \cup \varphi\) of \(\mathcal{R}\) and \(\varphi\) is defined to be \(\mathcal{A} = (Q_C, I_C, T_C, \delta_C)\), where:

(i) \(Q_C = Q_A \cup Q_B\);
(ii) \(I_C = I_A \cup I_B\);
(iii) \(T_C = T_A \cup T_B\); and
(iv) \(\delta_C : Q_C \times \Sigma \times Q_C \rightarrow L\) is given as follows: for any \(p, q \in Q_C\) and \(\sigma \in \Sigma\),

\[
\delta_C(p, \sigma, q) = \begin{cases} 
\delta_A(p, \sigma, q) & \text{if } p, q \in Q_A, \\
\delta_B(p, \sigma, q) & \text{if } p, q \in Q_B, \\
0 & \text{otherwise.}
\end{cases}
\]

The following proposition describes the recognizability of the union of two \(\ell\)-valued automata. As in the classical theory, a word \(s\) in \(\Sigma^*\) is recognized by the union of two \(\ell\)-valued automata if and only if \(s\) is recognized by one of them.

**Proposition 6.3.** Let \(\ell = (L, \leq, \land, \lor, \perp, 0, 1)\) be a complete orthomodular lattice. If the implication operator \(\rightarrow\) satisfies that \(a \leftrightarrow a = 1\) for any \(a \in L\), then for any \(\mathcal{R}, \varphi \in A(\Sigma, \ell)\) and for any \(s \in \Sigma^*\),

\[
(\ell, \leq) \models \text{rec}_{\mathcal{R} \cup \varphi}(s) \leftrightarrow \text{rec}_{\mathcal{R}}(s) \lor \text{rec}_\varphi(s).
\]

**Proof.** Let \(s = \sigma_1...\sigma_k\). Then

\[
[\text{rec}_{\mathcal{R} \cup \varphi}(s)] = \lor \{ (I_A \cup I_B)(q_0) \land (T_A \cup T_B)(q_k) \land \land_{i=0}^{k-1} \delta_{A \cup B}(g_i, \sigma_i+1, q_i+1) : q_0, q_1, ..., q_k \in Q_A \cup Q_B \}
\]

\[
= \lor \{ (I_A \cup I_B)(q_0) \land (T_A \cup T_B)(q_k) \land \land_{i=0}^{k-1} \delta_{A \cup B}(g_i, \sigma_i+1, q_i+1) : q_0, q_1, ..., q_k \in Q_A \}
\]

\[
\lor \lor \{ (I_A \cup I_B)(q_0) \land (T_A \cup T_B)(q_k) \land \land_{i=0}^{k-1} \delta_{A \cup B}(g_i, \sigma_i+1, q_i+1) : q_0, q_1, ..., q_k \in Q_B \}
\]

\[
\lor \lor \{ (I_A \cup I_B)(q_0) \land (T_A \cup T_B)(q_k) \land \land_{i=0}^{k-1} \delta_{A \cup B}(g_i, \sigma_i+1, q_i+1) : q_0, q_1, ..., q_k \in Q_A \cup Q_B, \}
\]

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and there are \( i, j \) such that \( 0 \leq i, j \leq k \) and \( q_i \in Q_A \) and \( q_j \in Q_B \).\]

From the definition of \( R \cup \varphi \), we know that for any \( q_0, q_1, ..., q_k \in Q_A \),

\[
(IA \cup IB)(q_0) = IA(q_0),
\]

\[
(TA \cup TB)(q_k) = TA(q_k),
\]

\[
\land_{i=0}^{k-1}\delta_{A∪B}(q_i, \sigma_{i+1}, q_{i+1}) = \land_{i=0}^{k-1}\delta_A(q_i, \sigma_{i+1}, q_{i+1}),
\]

and for any \( q_0, q_1, ..., q_k \in Q_B \),

\[
(IA \cup IB)(q_0) = IB(q_0),
\]

\[
(TA \cup TB)(q_k) = TB(q_k),
\]

\[
\land_{i=0}^{k-1}\delta_{A∪B}(q_i, \sigma_{i+1}, q_{i+1}) = \land_{i=0}^{k-1}\delta_B(q_i, \sigma_{i+1}, q_{i+1}).
\]

If \( q_0, q_1, ..., q_k \in Q_A \cup Q_B \), and there are \( i, j \) such that \( 0 \leq i, j \leq k \) and \( q_i \in Q_A \) and \( q_j \in Q_B \), then we can find some \( m \in \{0, 1, ..., k-1\} \) such that \( q_m \in Q_A \) and \( q_{m+1} \in Q_B \), or \( q_m \in Q_B \) and \( q_{m+1} \in Q_A \). Then \( \delta_{A∪B}(q_m, \sigma_{m+1}, q_{m+1}) = 0 \), and

\[
\land_{i=0}^{k-1}\delta_{A∪B}(q_i, \sigma_{i+1}, q_{i+1}) = 0.
\]

Therefore, it follows that

\[
\text{rec}_{R∪\varphi}(s) = \land_{i=0}^{k-1}\delta_A(q_i, \sigma_{i+1}, q_{i+1}) = 0 \land \text{rec}_{\varphi}(s).
\]

The following corollary slightly generalizes Example 3.1.

\textbf{Corollary 6.4.} If \( \text{Range}(A) = \{A(s) : s \in \Sigma^*\} \) is finite, and \( A_\lambda = \{s \in \Sigma^* : A(s) \geq \lambda\} \) is a regular language (in classical automata theory) for every \( \lambda \in \text{Range}(A) \), then

\[
\mid \text{Reg}_\Sigma(A) \mid = \ell.
\]

\textbf{Proof.} Suppose that \( \text{Range}(A) = \{\lambda_1, ..., \lambda_n\} \). Then it is easy to see that

\[
A = \bigcup_{i=1}^{n} \lambda_i A_{\lambda_i}.
\]

From Example 3.3 we know that there exists an \( \ell \)-valued automaton \( R_i \) such that \( \text{recrec}_i = \lambda_i A_{\lambda_i} \) for each \( i \leq n \). Thus, by proposition 6.3 we obtain

\[
\text{recrec}_{\bigcup_{i=1}^{n} R_i} = \bigcup_{i=1}^{n} \lambda_i A_{\lambda_i} = A
\]
and complete the proof. ♦

We now consider the product of two ℓ-valued automata. Let \( \mathcal{R} = < Q_A, I_A, T_A, \delta_A > \) and \( \mathcal{Q} = < Q_B, I_B, T_B, \delta_B > \in A(\Sigma, \ell) \) be two ℓ-valued automata over \( \Sigma \). Then their product \( \mathcal{R} \times \mathcal{Q} \) is defined to be \( \mathcal{Z} = (Q_C, I_C, T_C, \delta_C) \), where:

(i) \( Q_C = Q_A \times Q_B \);
(ii) \( I_C = I_A \times I_B \);
(iii) \( T_C = T_A \times T_B \); and
(iv) \( \delta_C : Q_C \times \Sigma \times Q_C \rightarrow L \) and for any \( p_a, q_a \in Q_A, p_b, q_b \in Q_B \) and \( \sigma \in \Sigma \),

\[
\delta_C((p_a, p_b), \sigma, (q_a, q_b)) = \delta_A(p_a, \sigma, q_a) \land \delta_B(p_b, \sigma, q_b).
\]

It is well-known in the classical automata theory that the language accepted by the union of two automata is the union of the languages accepted by these two automata, and the language accepted by the product of two automata is the intersection of the languages accepted by the factor automata. Proposition 6.3 shows that the conclusion about the union of two automata can be generalized into the theory of automata based on quantum logic without appealing any additional condition. One may naturally expect that the conclusion for product of automata can also be easily generalized into the framework of quantum logic. However, the case for the product of two automata is much more complicated, and the following proposition tells us that in order to make the above conclusion about product of automata still valid in the automata theory based on quantum logic, a certain commutativity is necessary to be added on the basic actions of the factor automata.

**Proposition 6.5.** Let \( \ell = < L, \leq, \land, \lor, \bot, 0, 1 > \) be a complete orthomodular lattice.

(1) For any \( \mathcal{R}, \mathcal{Q} \in A(\Sigma, \ell) \), and for any \( s \in \Sigma^* \),

\[
\ell \models \text{rec}_{\mathcal{R} \times \mathcal{Q}}(s) \rightarrow \text{rec}_{\mathcal{R}}(s) \land \text{rec}_{\mathcal{Q}}(s).
\]

(2) For any \( \mathcal{R}, \mathcal{Q} \in A(\Sigma, \ell) \), and for any \( s \in \Sigma^* \),

\[
\ell \models \gamma(\text{atom}(\mathcal{R}) \cup \text{atom}(\mathcal{Q})) \land \text{rec}_{\mathcal{R}}(s) \land \text{rec}_{\mathcal{Q}}(s) \rightarrow \text{rec}_{\mathcal{R} \times \mathcal{Q}}(s),
\]

and in particular if \( \rightarrow = \rightarrow_3 \), then

\[
\ell \models \gamma(\text{atom}(\mathcal{R}) \cup \text{atom}(\mathcal{Q})) \rightarrow (\text{rec}_{\mathcal{R}}(s) \land \text{rec}_{\mathcal{Q}}(s) \leftrightarrow \text{rec}_{\mathcal{R} \times \mathcal{Q}}(s)).
\]

(3) The following two statements are equivalent:
(i) $\ell$ is a Boolean algebra.
(ii) for all $\mathcal{R}, \varphi \in \mathbf{A}(\Sigma, \ell)$, and for all $s \in \Sigma^*$,

$$\ell \models \text{rec}_\mathcal{R}(s) \land \text{rec}_\varphi(s) \leftrightarrow \text{rec}_{\mathcal{R} \times \varphi}(s).$$

**Proof.** We have directly

$$[\text{rec}_{\mathcal{R} \times \varphi}(s)] = \bigvee \{(I_A \times I_B)(q_{a0}, q_{b0}) \land (T_A \times T_B)(q_{a_k}, q_{b_k}) \land \bigwedge_{i=0}^{k-1} \delta_A \times B((q_{a_i}, q_{b_i}),$$

$$\sigma_{i+1}, (q_{a(i+1)}, q_{b(i+1)})) : q_{a0}, q_{a1}, \ldots, q_{a_k} \in Q_A \text{ and } q_{b0}, q_{b1}, \ldots, q_{b_k} \in Q_B\}$$

$$= \bigvee \{I_A(q_{a0}) \land I_B(q_{b0}) \land T_A(q_{a_k}) \land T_B(q_{b_k}) \land \bigwedge_{i=0}^{k-1} \delta_B(q_{a_i}, \sigma_{i+1}, q_{a(i+1)}) \land$$

$$\bigwedge_{i=0}^{k-1} \delta_B(q_{b_i}, \sigma_{i+1}, q_{b(i+1)}) : q_{a0}, q_{a1}, \ldots, q_{a_k} \in Q_A \text{ and } q_{b0}, q_{b1}, \ldots, q_{b_k} \in Q_B\},$$

and

$$[\text{rec}_\mathcal{R}(s) \land \text{rec}_\varphi(s)] = [\bigvee \{I_A(q_0) \land T_A(q_k) \land \bigwedge_{i=0}^{k-1} \delta_A(q_i, \sigma_{i+1}, q_{i+1}) : q_0, q_1, \ldots, q_k \in Q_A\}]$$

$$\land [\bigvee \{I_B(q_0) \land T_B(q_k) \land \bigwedge_{i=0}^{k-1} \delta_B(q_i, \sigma_{i+1}, q_{i+1}) : q_0, q_1, \ldots, q_k \in Q_B\}]$$

from the definitions of product and recognizability of $\ell$-valued automata. It is clear that

$$[\text{rec}_{\mathcal{R} \times \varphi}(s)] \leq [\text{rec}_\mathcal{R}(s) \land \text{rec}_\varphi(s)].$$

This indicates that (1) is true. By using Lemmas 2.4(2), 2.5 and 2.6 twice, we can deduce that

$$[\gamma(\text{atom}(\mathcal{R}) \cup \text{atom}(\varphi)) \land \text{rec}_\mathcal{R}(s) \land \text{rec}_\varphi(s)] \leq [\text{rec}_{\mathcal{R} \times \varphi}(s)].$$

Thus, (2) is proved. The first part of (3) that (i) implies (ii) is immediately derivable from (2) because we have $[\gamma(\text{atom}(\mathcal{R}) \cup \text{atom}(\varphi))] = 1$ provided $\ell$ is a Boolean algebra. Conversely, we show that (ii) implies (i) by constructing two $\ell$-valued automata and examining the behavior of their product. For any $a, b, c \in L$, we choose some $\sigma_0 \in \Sigma$ and set

$$\mathcal{R} = (\{p\}, \{p\}, \{p\}, \delta_A),$$

where $\delta_A(p, \sigma, p) = a$ if $\sigma = \sigma_0$ and 0 otherwise, and

$$\varphi = (\{q, r, s\}, \{q\}, \{r, s\}, \delta_B),$$

where $\delta_B(x, \sigma, y) = b$ if $x = q, y = r$, and $\sigma = \sigma_0$; $c$ if $x = q, y = s$, and $\sigma = \sigma_0$, 0 otherwise. Then $\mathcal{R}, \varphi \in \mathbf{A}(\Sigma, \ell)$, and it is easy to see that

$$\mathcal{R} \times \varphi = (\{(p, q), (p, r), (p, s)\}, \{(p, q)\}, \{(p, r), (p, s)\}, \delta_{A \times B}),$$

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where \( \delta_{A \times B}((p, x), \sigma, (p, y)) = a \land b \) if \( x = q, y = r \) and \( \sigma = \sigma_0; a \land c \) if \( x = q, y = s \) and \( \sigma = \sigma_0; \) and 0 otherwise (see Figure 6). Furthermore, by a routine calculation we have

\[
\left\lceil \text{rec}_R(\sigma_0) \right\rceil = a,
\left\lceil \text{rec}_P(\sigma_0) \right\rceil = b \lor c,
\text{and}
\left\lceil \text{rec}_{R \times P}(\sigma_0) \right\rceil = (a \land b) \lor (a \land c).
\]

Therefore, with (ii) we finally obtain

\[
a \land b \lor c = \left\lceil \text{rec}_R(\sigma_0) \right\rceil \land \left\lceil \text{rec}_P(\sigma_0) \right\rceil
= \left\lceil \text{rec}_{R \times P}(\sigma_0) \right\rceil = (a \land b) \lor (a \land c).
\]

To prove the closure property of orthomodular lattice-valued regularity under the concatenation operation of languages, we propose the concept of concatenation of two orthomodular lattice-valued automata. Suppose that \( R_1 = (Q_1, I_1, T_1, \delta_1) \), \( R_2 = (Q_2, I_2, T_2, \delta_2) \) are two \( \ell \)-valued automata, and \( Q_1 \cap Q_2 = \emptyset \). We define the concatenation of \( R_1 \) and \( R_2 \) to be \( \ell \)-valued automaton \( R_1 R_2 = (Q_1 \cup Q_2, I_1, T_2, \delta_2) \) with \( \varepsilon \)-moves, where \( \delta : Q \times (\Sigma \cup \{\varepsilon\}) \times Q \to L \) is given by

\[
\delta(p, \sigma, q) = \begin{cases} 
\delta_1(p, \sigma, q) & \text{if } p, q \in Q_1 \text{ and } \sigma \neq \varepsilon \\
\delta_2(p, \sigma, q) & \text{if } p, q \in Q_2 \text{ and } \sigma \neq \varepsilon \\
T_1(p) \land T_2(q) & \text{if } p \in Q_1, q \in Q_2 \text{ and } \sigma = \varepsilon \\
0 & \text{otherwise.}
\end{cases}
\]

The following proposition clarifies the relation between the language recognized by the concatenation of two orthomodular lattice-valued automata and the concatenation of the languages recognized by the two automata.

**Proposition 6.6.** Let \( \ell = (L, \leq, \land, \lor, \bot, 0, 1) \) be an orthomodular lattice and \( \to \) fulfil the Birkhoff-von Neumann requirement.
(1) For all \( R_1, R_2 \in A(\Sigma, \ell) \), and for each \( s \in \Sigma^* \),

\[ \models^\ell rec_{R_1, R_2}(s) \rightarrow (\exists s_1, s_2 \in \Sigma^*)(s_1 s_2 = s \land rec_{R_1}(s_1) \land rec_{R_2}(s_2)). \]

(2) For all \( R_1, R_2 \in A(\Sigma, \ell) \), and for each \( s \in \Sigma^* \),

\[ \models^\ell \gamma(\text{atom}(R_1) \cup \text{atom}(R_2)) \land (\exists s_1, s_2 \in \Sigma^*)(s_1 s_2 = s \land \text{rec}_{R_1}(s_1) \land \text{rec}_{R_2}(s_2)) \rightarrow \text{rec}_{R_1, R_2}(s), \]

and if \( \models^\ell \gamma \rightarrow \gamma \rightarrow \gamma \) then

\[ \models^\ell (\exists s_1, s_2 \in \Sigma^*)(s_1 s_2 = s \land \text{rec}_{R_1}(s_1) \land \text{rec}_{R_2}(s_2)). \]

(3) The following two statements are equivalent:

(i) \( \ell \) is a Boolean algebra;

(ii) for all \( R_1, R_2 \in A(\Sigma, \ell) \), and for each \( s \in \Sigma^* \),

\[ \models^\ell \text{rec}_{R_1, R_2}(s) \rightarrow (\exists s_1, s_2 \in \Sigma^*)(s_1 s_2 = s \land \text{rec}_{R_1}(s_1) \land \text{rec}_{R_2}(s_2)). \]

Proof. (1) For any \( q_0, q_1, \ldots, q_m \in Q_1 \cup Q_2; \sigma_1, \ldots, \sigma_m \in \Sigma \cup \{\varepsilon\} \) with \( \sigma_1 \ldots \sigma_m = s \) (note that it is possible that \( |s| < m \) since \( \sigma_1, \ldots, \sigma_m \) may contain \( \varepsilon \)'s), if

\[ I_1(q_0) \land T_2(q_m) \land \land_{i=1}^m \delta(q_i-1, \sigma_i, q_i) > 0, \]

then there exists \( j \leq m \) such that \( \sigma_j = \varepsilon, \sigma_i \neq \varepsilon (i \neq j), q_0, \ldots, q_{j-1} \in Q_1, q_j, \ldots, q_m \in Q_2. \) Thus, \( s = \sigma_1 \ldots \sigma_{j-1} \sigma_j \ldots \sigma_n, \) and

\[ \begin{align*}
I_1(q_0) & \land T_2(q_m) \land \land_{i=1}^m \delta(q_i-1, \sigma_i, q_i) = I_1(q_0) \land T_2(q_m) \land \land_{i=1}^{j-1} \delta(q_i-1, \sigma_i, q_i) \\
& \land T_1(q_{j-1}) \land I_2(q_j) \land \land_{i=j+1}^m \delta(q_i-1, \sigma_i, q_i) \\
& = [I_1(q_0) \land T_1(q_{j-1}) \land \land_{i=1}^{j-1} \delta(q_i-1, \sigma_i, q_i)] \land [I_2(q_j) \land T_2(q_m) \land \land_{i=j+1}^m \delta(q_i-1, \sigma_i, q_i)] \\
& \leq rec_{R_1}(\sigma_1 \ldots \sigma_{j-1}) \land rec_{R_2}(\sigma_{j+1} \ldots \sigma_n) \\
& \leq \land_{s_1, s_2} \{ rec_{R_1}(s_1) \land rec_{R_2}(s_2) : s_1 s_2 = s \}.
\end{align*} \]

(2) First, we can use Lemmas 2.5 and 2.6 to derive that

\[ \begin{align*}
[\gamma(\text{atom}(R_1) \cup \text{atom}(R_2)) \land (\exists s_1, s_2 \in \Sigma^*)(s_1 s_2 = s \land \text{rec}_{R_1}(s_1) \land \text{rec}_{R_2}(s_2))] \\
& = [\gamma(\text{atom}(R_1) \cup \text{atom}(R_2))] \land \land_{s_1, s_2} \text{rec}_{R_1}(s_1) \land \text{rec}_{R_2}(s_2) \\
& \leq \land_{s_1, s_2} \{ [\gamma(\text{atom}(R_1) \cup \text{atom}(R_2))] \land \text{rec}_{R_1}(s_1) \land \text{rec}_{R_2}(s_2) \}. \end{align*} \]
For any \( s_1, s_2 \in \Sigma^* \) with \( s_1 s_2 = s \), we use Lemmas 2.5 and 2.6 again, and this yields

\[
\gamma(\text{atom}(\mathcal{R}_1) \cup \text{atom}(\mathcal{R}_2)) \land \text{rec}_{\mathcal{R}_1}(s_1) \land \text{rec}_{\mathcal{R}_2}(s_2) = [\gamma(\text{atom}(\mathcal{R}_1) \cup \text{atom}(\mathcal{R}_2)) \land \\
\lor \text{rec}_{\mathcal{R}_1}(s_1) \lor \text{rec}_{\mathcal{R}_2}(s_2)]
\]

Furthermore, for any \( c_1 = p_0 \sigma_1 p_1 \cdots p_m-1 \sigma_m p_m \) and \( c_2 = q_0 \tau_1 q_1 \cdots q_{n-1} \tau_n q_n \) with \( s_1 = \sigma_1 \cdots \sigma_m \) and \( s_2 = \tau_1 \cdots \tau_n \),

\[
I_1(b(c_1)) \lor T_1(e(c_1)) \lor [\text{path}_{\mathcal{R}_1}(s_1)] \lor I_2(b(c_2)) \lor T_2(e(c_2)) \lor [\text{path}_{\mathcal{R}_2}(s_2)] = \\
I_1(p_0) \lor T_2(q_m) \lor \land_{i=1}^m \delta_1(p_{i-1}, \sigma_i, p_i) \lor T_1(p_m) \lor I_2(q_0) \lor \land_{j=1}^n \delta_2(q_{j-1}, \tau_j, q_j) \\
= I_1(p_0) \lor T_2(q_m) \lor [\text{path}_{\mathcal{R}_1 \mathcal{R}_2}(p_0 \sigma_1 p_1 \cdots p_m-1 \sigma_m p_m \lor q_0 \tau_1 q_1 \cdots q_{n-1} \tau_n q_n)] \\
\leq \text{rec}_{\mathcal{R}_1 \mathcal{R}_2}(s).
\]

(3) The part that (i) implies (ii) is a simple corollary of (2). Conversely, it suffices to show that \( \ell \) enjoys distributivity; that is, for any \( a, b, c \in L \), \( a \land (b \lor c) = (a \land b) \lor (a \land c) \). Let \( \mathcal{R}_1 = \langle p_0, p_1 \rangle, \{ p_1 \}, \{ p_0 \}, \delta_1 > \) and \( \mathcal{R}_2 = \langle q_0, q_1, q_2 \rangle, \{ q_0 \}, \{ q_1, q_2 \}, \delta_2 > \), where \( \delta_1(p_0, \sigma, p_1) = a, \delta_2(q_0, \sigma, q_1) = b, \delta_2(q_0, \sigma, q_2) = c \), and \( \delta_1, \delta_2 \) take value 0 for other arguments (see Figure 7). Then it follows that

\[
a \land (b \lor c) = [(\exists s_1, s_2 \in \Sigma^*) (s_1 s_2 = s \sigma) \land \text{rec}_{\mathcal{R}_1}(s_1) \land \text{rec}_{\mathcal{R}_2}(s_2)] = [\text{rec}_{\mathcal{R}_1 \mathcal{R}_2}(s \sigma)] = (a \land b) \lor (a \land c).
\]
automaton, and let \( q_0 \notin Q \) be a new state. We define the fold of \( \mathcal{R} \) to be \( \ell \)-valued automaton \( \mathcal{R}^* =< Q \cup \{q_0\}, \{q_0\}, T \cup \{q_0\}, \delta^* > \) with \( \varepsilon \)-moves, where

\[
\delta^*: (Q \cup \{q_0\}) \times (\Sigma \cup \{\varepsilon\}) \times (Q \cup \{q_0\}) \rightarrow L
\]
is given by

\[
\delta^*(p, \sigma, q) = \begin{cases} 
I(q) & \text{if } p = q_0 \text{ and } \sigma = \varepsilon, \\
\delta(p, \sigma, q) & \text{if } p, q \in Q \text{ and } \sigma \neq \varepsilon, \\
T(p) \land I(q) & \text{if } p, q \in Q \text{ and } \sigma = \varepsilon, \\
0 & \text{otherwise}.
\end{cases}
\]

The language accepted by the fold of an orthomodular lattice-valued automaton is then clearly presented by the following proposition.

**Proposition 6.7.** Let \( \ell =< L, \leq, \land, \lor, \bot, 0, 1 > \) be an orthomodular lattice, and let \( \rightarrow \) enjoy the Birkhoff-von Neumann requirement.

1. For any \( \mathcal{R} \in \text{A}(\Sigma, \ell) \) and for all \( s \in \Sigma^* \),

\[
\ell \models \gamma(\text{atom}(\mathcal{R})) \land (\exists m \geq 0, s_1, ..., s_m \in \Sigma^*)(s_1...s_m = s \land \land_{i=1}^m \text{rec}_{\mathcal{R}}(s_i)).
\]

2. For any \( \mathcal{R} \in \text{A}(\Sigma, \ell) \) and for each \( s \in \Sigma^* \),

\[
\ell \models \gamma(\text{atom}(\mathcal{R})) \land (\exists m \geq 0, s_1, ..., s_m \in \Sigma^*)(s_1...s_m = s \land \land_{i=1}^m \text{rec}_{\mathcal{R}}(s_i)) \rightarrow \text{rec}_{\mathcal{R}^*}(s),
\]

and in particular if \( \rightarrow = \rightarrow_3 \), then

\[
\ell \models \gamma(\text{atom}(\mathcal{R})) \rightarrow (\text{rec}_{\mathcal{R}^*}(s) \rightarrow (\exists m \geq 0, s_1, ..., s_m \in \Sigma^*)(s_1...s_m = s \land \land_{i=1}^m \text{rec}_{\mathcal{R}}(s_i))).
\]

3. The following two statements are equivalent:

   (i) \( \ell \) is a Boolean algebra;

   (ii) for all \( \mathcal{R} \in \text{A}(\Sigma, \ell) \) and \( s \in \Sigma^* \),

\[
\ell \models \text{rec}_{\mathcal{R}^*}(s) \leftrightarrow (\exists m \geq 0, s_1, ..., s_m \in \Sigma^*)(s_1...s_m = s \land \land_{i=1}^m \text{rec}_{\mathcal{R}}(s_i)).
\]

**Proof.** For (1), (2) and the part from (i) to (ii) of (3), it is similar to the proof of Proposition 6.6, and here we omit the details. To show that (ii) implies (i), we assume that \( a, b, c \in L \) and want to construct an \( \ell \)-valued automaton for which the validity of (ii) leads to \( a \land (b \lor c) = (a \land b) \lor (a \land c) \). Let \( \mathcal{R} =< \{q_1, q_2, ..., q_6\}, \{q_1, q_2, q_3\}, \{q_6\}, \delta > \) in which \( \delta(q_1, \sigma, q_4) = \delta(q_3, \sigma, q_5) = 1 \), \( \delta(q_2, \sigma, q_6) = a \), \( \delta(q_4, \sigma, q_6) = b \), \( \delta(q_5, \sigma, q_6) = c \), and \( \delta \) takes value 0 for the other arguments. Then \( \mathcal{R}^* \) is visualized as Figure 8.
We now have
\[
 a \land (b \lor c) = [(\exists m \geq 0, s_1, \ldots, s_m \in \Sigma^*)(s_1 \ldots s_m = \sigma^3 \land \land_{i=1}^m rec_R(s_i))] \\
= rec_{R^*}(\sigma^3) \\
= (a \land b) \lor (a \land c).
\]

From the above proposition, we are able to demonstrate that the predicate \( CReg_\Sigma \) is preserved by the Kleene closure. The corresponding result for the predicate \( Reg_\Sigma \) is not true in general.

**Corollary 6.8.** Let \( \ell =< L, \leq, \land, \lor, \bot, 0, 1 > \) be an orthomodular lattice, and let \( \rightarrow = \rightarrow_3 \). Then for any \( A \in \mathcal{L}^\Sigma \),

\[
\models CReg_\Sigma(A) \rightarrow CReg_\Sigma(A^*).
\]

**Proof.** It is similar to the proof of Proposition 6.2. The point here is to show the following inequality:

\[
[\gamma(atom(\mathcal{R}) \cup r(A))] \land [A \equiv rec_R] \leq [A^* \equiv rec_{R^*}]
\]

for any \( \mathcal{R} \in \mathcal{A}(\Sigma, \ell) \). In fact, by using Lemma 2.11(1) we have

\[
[\gamma(atom(\mathcal{R}) \cup r(A))] \land [A \equiv rec_R] = [\gamma(atom(\mathcal{R}) \cup r(A))] \land \land_{s \in \Sigma^*}(A(s) \leftrightarrow rec_R(s))
\]
\[
\leq [\gamma(\text{atom}(\mathcal{R}) \cup r(A))] \land s \in \Sigma^* (\bigvee_{s_1 \ldots s_m = s} A(s_i) \leftrightarrow \bigvee_{s_1 \ldots s_m = s} \text{rec}_\mathcal{R}(s_i))
\]

\[
= [\gamma(\text{atom}(\mathcal{R}) \cup r(A))] \land [A^* \equiv (\text{rec}_\mathcal{R})^*].
\]

On the other hand, it follows from Proposition 6.7 that

\[
[\gamma(\text{atom}(\mathcal{R}))] \leq [(\text{rec}_\mathcal{R})^* \equiv \text{rec}_{\mathcal{R}^*}].
\]

Then with Lemma 2.11(3) we obtain

\[
[\gamma(\text{atom}(\mathcal{R}))] \leq [\text{rec}_\mathcal{R}] \leq \bigvee_{s_1 \ldots s_m = s} [A^* \equiv (\text{rec}_\mathcal{R})^*]
\]

\[
\land [(\text{rec}_\mathcal{R})^* \equiv \text{rec}_{\mathcal{R}^*}]
\]

\[
\leq [A^* \equiv \text{rec}_{\mathcal{R}^*}].
\]

To conclude this section, we show that both the predicate \text{Reg}_\Sigma and \text{CReg}_\Sigma are preserved by the pre-image of a homomorphism between two languages. But the closure property of an orthomodular lattice-valued language under homomorphism is left to be examined in the next section, after the notion of orthomodular lattice-valued regular expression is proposed. Let \Sigma and \Gamma be two alphabets of input symbols. Then each mapping \( h : \Sigma \rightarrow \Gamma^* \) can be uniquely extended to a homomorphism \( h : \Sigma^* \rightarrow \Gamma^* \) such that \( h(\varepsilon) = \varepsilon \) and

\[
h(xy) = h(x)h(y)
\]

for all \( x, y \in \Sigma^* \). Furthermore, we may define images of \( \ell \)-valued subsets of \( \Sigma^* \) under \( h \) and pre-images of \( \ell \)-valued subsets of \( \Gamma^* \) under \( h \). Recall that for any \( A \in L_{\Sigma^*} \) and \( B \in L_{\Gamma^*} \), \( h(A) \in L_{\Gamma^*} \) and \( h^{-1}(B) \in L_{\Sigma^*} \) are given as follows:

\[
h(A)(t) = \vee \{A(s) : s \in \Sigma^* \text{ and } h(s) = t\}
\]

for each \( t \in \Gamma^* \), and

\[
h^{-1}(B)(s) = B(h(s))
\]

for each \( s \in \Sigma^* \).

Let \( \mathcal{R} = < Q, I, T, \delta > \in A(\Gamma, \ell) \) be an \( \ell \)-valued automaton over \( \Gamma \). Then the pre-image of \( \mathcal{R} \) under \( h \) is defined to be an \( \ell \)-valued automaton

\[
h^{-1}(\mathcal{R}) = < Q, I, T, h^{-1}(\delta) > \in A(\Sigma, \ell)
\]

over \( \Sigma \), where for any \( p, q \in Q \) and \( \sigma \in \Sigma \),

\[
h^{-1}(\delta)(p, \sigma, q) = \delta(p, h(\sigma), q).
\]

The pre-image of a homomorphism has a very nice compatibility with the predicates \text{reg}_\Sigma and \text{CReg}_\Sigma, and no commutativity is needed here.
Proposition 6.9. Let $\ell = \langle L, \leq, \land, \lor, \bot, 0, 1 \rangle$ be an orthomodular lattice, let $\to$ enjoy the Birkhoff-von Neumann requirement, and let $h : \Sigma \to \Gamma^*$ be a mapping. Then for any $\mathcal{R} \in \mathbf{A}(\Gamma, \ell)$ and for any $s \in \Sigma^*$,
$$\ell \models \text{rec}_{h^{-1}(\mathcal{R})}(s) \iff \text{rec}_{\mathcal{R}}(h(s)).$$

Proof. Suppose that $s = \sigma_1\sigma_2...\sigma_n$. Then
$$[\text{rec}_{h^{-1}(\mathcal{R})}(s)] = \lor\{I(q_0) \land T(q_n) \land \land_{i=0}^{n-1} h^{-1}(\delta)(q_i, \sigma_{i+1}, q_{i+1}) : q_0, q_1, ..., q_n \in Q\}$$
$$= \lor\{I(q_0) \land T(q_n) \land \land_{i=0}^{n-1} \delta(q_i, h(\sigma_{i+1}), q_{i+1}) : q_0, q_1, ..., q_n \in Q\}$$
$$= [\text{rec}_{\mathcal{R}}(h(\sigma_1)h(\sigma_2)...h(\sigma_n))]$$
$$= [\text{rec}_{\mathcal{R}}(h(s))].$$

Corollary 6.10. Let $\ell = \langle L, \leq, \land, \lor, \bot, 0, 1 \rangle$ be an orthomodular lattice, let $\to$ enjoy the Birkhoff-von Neumann requirement, and let $h : \Sigma \to \Gamma^*$ be a mapping. Then for any $B \in L^{\Gamma^*}$,
$$\ell \models \text{Reg}_{\Gamma}(B) \to \text{Reg}_{\Sigma}(h^{-1}(B)),$$
and
$$\ell \models \text{CReg}_{\Gamma}(B) \to \text{CReg}_{\Sigma}(h^{-1}(B)),$$

Proof. From the above proposition we have
$$h^{-1}(\text{rec}_{\mathcal{R}})(s) = \text{rec}_{\mathcal{R}}(h(s)) = \text{rec}_{h^{-1}(\mathcal{R})}(s)$$
for all $s \in \Sigma^*$. Then with Lemma 2.12 we obtain
$$[\text{Reg}_{\Gamma}(B)] = \lor\{[B \equiv \text{rec}_{\mathcal{R}}] : \mathcal{R} \in \mathbf{A}(\Gamma, \ell)\}$$
$$\leq \lor\{[h^{-1}(B) \equiv h^{-1}(\text{rec}_{\mathcal{R}})] : \mathcal{R} \in \mathbf{A}(\Gamma, \ell)\}$$
$$= \lor\{[h^{-1}(B) \equiv \text{rec}_{h^{-1}(\mathcal{R})}] : \mathcal{R} \in \mathbf{A}(\Gamma, \ell)\}$$
$$\leq \lor\{[h^{-1}(B) \equiv \text{rec}_{\varphi}] : \varphi \in \mathbf{A}(\Sigma, \ell)\}$$
$$= [\text{Reg}_{\Sigma}(h^{-1}(B))].$$

It is similar for the case of commutative regularity.$\heartsuit$

7. Orthomodular Lattice-Valued Regular Expressions
One of the most interesting results in classical automata theory is the Kleene theorem which shows the equivalence between finite automata and regular expressions. The main aim of this section is to present an orthomodular lattice-valued generalization of the Kleene theorem. Let \( \ell = \langle L, \leq, \wedge, \vee, \bot, 0, 1 \rangle \) be an orthomodular lattice, and let \( \Sigma \) be a nonempty set of input symbols. Then the language of \( \ell \)-valued regular expressions over \( \Sigma \) has the alphabet \((\Sigma \cup \{\varepsilon, \phi\}) \cup (L \cup \{+, \cdot, \ast\})\).

The symbols in \( \Sigma \cup \{\varepsilon, \phi\} \) will be used to stand for atomic expressions, and the symbols in \( L \cup \{+, \cdot, \ast\} \) will be used to denote operators for building up compound expressions: \( \ast \) and all \( \lambda \in L \) are unary operators, and \(+, \cdot\) are binary ones. We use \( \alpha, \beta, \ldots \) to act as meta-symbols for regular expressions and \( L(\alpha) \) for the language denoted by expression \( \alpha \). More explicitly, \( L(\alpha) \) will be used to denote an \( \ell \)-valued subset of \( \Sigma^* \); that is, \( L(\alpha) \in L(\Sigma^*) \). Orthomodular lattice-valued regular expressions and the orthomodular lattice-valued languages denoted by them are formally defined as follows:

(i) For each \( a \in \Sigma \), \( a \) is a regular expression, and \( L(a) = a \); \( \varepsilon \) and \( \phi \) are regular expressions, and \( L(\varepsilon) = \varepsilon \), \( L(\phi) = \phi \).

(ii) If both \( \alpha \) and \( \beta \) are regular expressions, then for each \( \lambda \in L \), \( \lambda \alpha \) is a regular expression, and \( L(\lambda \alpha) = \lambda L(\alpha) \); and \( \alpha + \beta \), \( \alpha \cdot \beta \), and \( \alpha^* \) are all regular expressions, and

\[
L(\alpha + \beta) = L(\alpha) \cup L(\beta),
\]

\[
L(\alpha \cdot \beta) = L(\alpha) \cot L(\beta),
\]

\[
L(\alpha^*) = L(\alpha)^*.
\]

It is easy to see that the only difference between orthomodular lattice-valued regular expressions and the classical ones is that the additional unary (scalar) operators \( \lambda \in L \) are permitted to occur in the former.

The central part of the Kleene theorem is a mechanism to transform a finite automaton into a regular expression. This mechanism has a straightforward extension in the framework of orthomodular lattice-valued automata. Let \( \mathcal{R} = \langle Q, I, T, \delta \rangle \in \mathcal{A}(\Sigma, \ell) \) be an \( \ell \)-valued automaton over \( \Sigma \). For any \( u, v \in Q \) and \( X \subseteq Q \), \( \alpha_{uv}^X \) is defined by induction on the cardinality \( |X| \) of \( X \):

(1) \[
\alpha_{uv}^\phi = \begin{cases} 
\Sigma_{\sigma \in \Sigma} \delta(u, \sigma, v)\sigma & \text{if } u \neq v, \\
\varepsilon + \Sigma_{\sigma \in \Sigma} \delta(u, \sigma, v)\sigma & \text{if } u = v.
\end{cases}
\]

(2) if \( X \neq \phi \), then we choose any \( q \in X \) and define

\[
\alpha_{uv}^X = \alpha_{uv}^{X-\{q\}} + \alpha_{uq}^{X-\{q\}} \cdot (\alpha_{qq}^{X-\{q\}})^* \cdot \alpha_{qv}^{X-\{q\}}.
\]

Then the \( \ell \)-valued regular expression

\[
k(\mathcal{R}) = \Sigma_{u, v \in Q} (I(u) \wedge T(v)) \alpha_{uv}^Q,
\]

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is called a Kleene representation of $\mathcal{R}$.

The following theorem describes properly the relationship between the language recognized by an orthomodular lattice-valued automaton and the language expressed by its Kleene representation.

**Theorem 7.1.** Let $\ell = <L, \leq, \wedge, \vee, \bot, 0, 1>$ be an orthomodular lattice, and let $\rightarrow$ satisfy the Birkhoff-von Neumann requirement.

1. For any $\mathcal{R} \in \mathbf{A}(\Sigma, \ell)$ and $s \in \Sigma^*$, if $k(\mathcal{R})$ is a Kleene representation of $\mathcal{R}$, then
   \[
   \ell \models rec_\mathcal{R}(s) \rightarrow s \in L(k(\mathcal{R})).
   \]

2. For any $\mathcal{R} \in \mathbf{A}(\Sigma, \ell)$ and $s \in \Sigma^*$, and for any Kleene representation $k(\mathcal{R})$ of $\mathcal{R}$,
   \[
   \ell \models \gamma(atom(\mathcal{R})) \land s \in L(k(\mathcal{R})) \rightarrow rec_\mathcal{R}(s),
   \]
   and especially if $\rightarrow = \rightarrow_3$, then
   \[
   \ell \models \gamma(atom(\mathcal{R})) \rightarrow (rec_\mathcal{R}(s) \leftrightarrow s \in L(k(\mathcal{R}))).
   \]

3. The following two statements are equivalent:
   (i) $\ell$ is a Boolean algebra.
   (ii) For any $\mathcal{R} \in \mathbf{A}(\Sigma, \ell)$ and $s \in \Sigma^*$, and for any Kleene representation $k(\mathcal{R})$ of $\mathcal{R}$,
   \[
   \ell \models rec_\mathcal{R}(s) \iff s \in L(k(\mathcal{R})).
   \]

**Proof.** We prove (1) and (2) together. To this end, we have to demonstrate that for any $u, v \in Q$, $X \subseteq Q$ and $s \in \Sigma^*$,

\[(a) \begin{align*}
\forall \{\text{path}_\mathcal{R}(c) : c \in T(Q, \Sigma), b(c) = u, e(c) = v, M(c) \subseteq X, lb(c) = s\} & \leq L(\alpha^X_{uv})(s), \\
(b) [\gamma(atom(\mathcal{R}))] \land L(\alpha^X_{uv})(s) \leq
\end{align*}
\]

\[\forall \{\text{path}_\mathcal{R}(c) : c \in T(Q, \Sigma), b(c) = u, e(c) = v, M(c) \subseteq X, lb(c) = s\}, \]

where $M(c)$ stands for the set of states along $c$ except $u$ and $v$; more exactly, $M(c) = \{q_1, ..., q_{k-1}\}$ if $c = u\sigma_1 q_1 ... q_{k-1} \sigma_k v$. This claim may be proved by induction on $|X|$. For the case of $X = \phi$, it is easy. We now suppose that $q \in X \neq \phi$ and

\[\alpha^X_{uv} = \alpha^X_{uv} - \{q\} + [\alpha^X_{uq} \{q\} \alpha^X_{qq} - \{q\}] \alpha^X_{qv} - \{q\}.\]

We first show that (a) is valid in this case. From the induction hypothesis we have

\[(c) \begin{align*}
\forall \{\text{path}_\mathcal{R}(c) : c \in T(Q, \Sigma), b(c) = e(c) = q, \\
\end{align*}
\]

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\[ M(c) \subseteq X - \{ q \}, lb(c) = s \leq L(\alpha_{qq}^{X - \{q\}})(s) \]

for each \( s \in \Sigma^* \). Then we assert that for all \( s \in \Sigma^* \),

\[(d) \lor \{ [\text{path}_R(c)] : c \in T(Q, \Sigma), b(c) = e(c) = q, M(c) \subseteq X, lb(c) = s \}\]

\[\leq L((\alpha_{qq}^{X - \{q\}})^*)(s).\]

In fact, for any \( c \in T(Q, \Sigma) \) if \( b(c) = e(c) = q, M(c) \subseteq X \) and \( lb(c) = s \), we write \( c_i \) for the substring of \( c \) beginning with the \( i \)th \( q \) and ending at the \( (i + 1) \)th \( q \). If the number of occurrences of \( q \) in \( c \) is \( m + 1 \), then

\[[\text{path}_R(c)] = \land_{i=1}^m [\text{path}_R(c_i)].\]

Furthermore, by using (c) and noting that \( s = lb(c_1)...lb(c_m) \) we obtain

\[[\text{path}_R(c)] = \land_{i=1}^m L(\alpha_{qq}^{X - \{q\}})(lb(c_i))\]

\[\leq \lor \{ \land_{i=1}^n L(\alpha_{qq}^{X - \{q\}})(s_i) : n \geq 0, s_1, ..., s_n \in \Sigma^*, s = s_1...s_n \} \]

\[= (L(\alpha_{qq}^{X - \{q\}}))^*(s)\]

\[= L((\alpha_{qq}^{X - \{q\}})^*)(s).\]

Let \( c \) range over \( \{ c \in T(Q, \Sigma) : b(c) = e(c) = q, M(c) \subseteq X, lb(c) = s \} \). Then (d) is proved.

Furthermore, from the induction hypothesis and (d) we have

\[(L(\alpha_{uu}^{X - \{q\}})L((\alpha_{qq}^{X - \{q\}})^*)L(\alpha_{qv}^{X - \{q\}}))(s) =\]

\[\lor \{ [L(\alpha_{uu}^{X - \{q\}})L((\alpha_{qq}^{X - \{q\}})^*)](x) \land L(\alpha_{qv}^{X - \{q\}})(y) : s = xy \} \]

\[= \lor \{ L(\alpha_{uu}^{X - \{q\}})(x_1) \land (\alpha_{qq}^{X - \{q\}})^*(x_2) : x = x_1x_2 \} \land L(\alpha_{qv}^{X - \{q\}})(y) : s = xy \} \]

\[\geq \lor \{ L(\alpha_{uu}^{X - \{q\}})(x_1) \land (\alpha_{qq}^{X - \{q\}})^*(x_2) \land L(\alpha_{qv}^{X - \{q\}})(y) : x_1x_2 \} \]

\[\geq \lor \{ [\text{path}_R(c_1)] \land [\text{path}_R(c_2)] \land [\text{path}_R(c_3)] : c_1, c_2, c_3 \in T(Q, \Sigma), \]

\[b(c_1) = u, e(c_1) = b(c_2) = c(c_2) = b(c_3) = q, e(c_3) = v, s = lb(c_1)lb(c_2)lb(c_3) \}

\[= \lor \{ [\text{path}_R(c)] : c \in T(Q, \Sigma), b(c) = u, e(c) = v, q \in M(c) \} \]

This yields further

\[L(\alpha_{uu}^{X})(s) = L(\alpha_{uu}^{X - \{q\}})(s) \lor ([L(\alpha_{uu}^{X - \{q\}})L((\alpha_{qq}^{X - \{q\}})^*)L(\alpha_{qv}^{X - \{q\}}))(s) \]

\[\geq \text{ the left – hand side of (a).} \]

We now turn to consider (b). The induction hypothesis gives

\[(e) \ [\gamma(\text{atom}(R))] \land L(\alpha_{uu}^{X - \{q\}})(s) \leq \{ [\text{path}_R(c)] : c \in T(Q, \Sigma), \]

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\[ b(c) = u, e(c) = v, M(c) \subseteq X - \{q\}, lb(c) = s \].

For any \( n \geq 0 \) and \( s_1, \ldots, s_n \in \Sigma^* \) with \( s = s_1 \ldots s_n \), from (e) we can apply Lemmas 2.5 and 2.6 to obtain

\[
\begin{align*}
\gamma(\text{atom}(\mathbb{R})) &\land \land_{i=1}^n L(\alpha_{qq}^{X-(q)})(s_i) = \left[ \gamma(\text{atom}(\mathbb{R})) \right] \land \land_{i=1}^n [\left[ \gamma(\text{atom}(\mathbb{R})) \right] \land L(\alpha_{qq}^{X-(q)})(s_i)] \\
&\leq \left[ \gamma(\text{atom}(\mathbb{R})) \right] \land \land_{i=1}^n \lor \{ \text{path}_{\mathbb{R}}(c_i) : c_i \in T(Q, \Sigma), b(c_i) = e(c_i) = q, M(c_i) \subseteq X - \{q\}, lb(c_i) = s_i \} \\
&\leq \lor \{ \land_{i=1}^n [\text{path}_{\mathbb{R}}(c_i) ] : c_i \in T(Q, \Sigma), b(c_i) = e(c_i) = q, M(c_i) \subseteq X - \{q\}, \\
&\quad \quad \quad \quad \quad \quad lb(c_i) = s_i \text{ for each } i = 1, 2, \ldots, n \} \\
&\leq \lor \{ \text{path}_{\mathbb{R}}(c_1, c_n) : c_i \in T(Q, \Sigma), b(c_i) = e(c_i) = q, M(c_i) \subseteq X - \{q\}, \\
&\quad \quad \quad \quad \quad \quad lb(c_i) = s_i \text{ for each } i = 1, 2, \ldots, n \},
\end{align*}
\]

where \( c_1 \ldots c_n = c_1 c'_2 \ldots c'_n, c'_i \) is the resulting string after removing the first \( q \) in \( c_i \) for each \( i \geq 2 \). Note that \( lb(c_1, \ldots, c_n) = s_1 \ldots s_n = s \) whenever \( lb(c_i) = s_i \) \((i = 1, 2, \ldots, n)\).

We write

\[ \lambda = \lor \{ \text{path}_{\mathbb{R}}(c) : c \in T(Q, \Sigma), b(c) = e(c) = q, M(c) \subseteq X, lb(c) = s \}. \]

Then it holds that

\[ \left[ \gamma(\text{atom}(\mathbb{R})) \right] \land \land_{i=1}^n L(\alpha_{qq}^{X-(q)})(s_i) \leq \lambda. \]

Moreover, note that \( \left[ \gamma(\text{atom}(\mathbb{R})) \right], L(\alpha_{qq}^{X-(q)})(s_i) \in [\text{atom}(\mathbb{R})] \). It follows that

\[
\begin{align*}
\gamma(\text{atom}(\mathbb{R})) \land L((\alpha_{qq}^{X-(q)}))^{*}(s) &= \gamma(\text{atom}(\mathbb{R})) \land \gamma(\text{atom}(\mathbb{R})) \land \\
&\lor \{ \land_{i=1}^n L(\alpha_{qq}^{X-(q)})(s_i) : n \geq 0, s = s_1 \ldots s_n \} \\
&\leq \lor \{ \gamma(\text{atom}(\mathbb{R})) \land \land_{i=1}^n L(\alpha_{qq}^{X-(q)})(s_i) : n \geq 0, s = s_1 \ldots s_n \} \leq \lambda.
\end{align*}
\]

This enables us to obtain

\[
\begin{align*}
\gamma(\text{atom}(\mathbb{R})) \land [L(\alpha_{qq}^{X-(q)})L((\alpha_{qq}^{X-(q)}))^{*}](x) &= \gamma(\text{atom}(\mathbb{R})) \land \gamma(\text{atom}(\mathbb{R})) \land [L(\alpha_{qq}^{X-(q)})(x_1) \land L((\alpha_{qq}^{X-(q)}))^{*}(x_2) : x = x_1 x_2] \\
&\leq \lor \{ \gamma(\text{atom}(\mathbb{R})) \land L(\alpha_{qq}^{X-(q)})(x_1) \land L((\alpha_{qq}^{X-(q)}))^{*}(x_2) : x = x_1 x_2 \} \\
&= \lor \{ \gamma(\text{atom}(\mathbb{R})) \land [\gamma(\text{atom}(\mathbb{R})) \land L(\alpha_{qq}^{X-(q)})(x_1)] \land \\
&\quad \quad [\gamma(\text{atom}(\mathbb{R})) \land L((\alpha_{qq}^{X-(q)}))^{*}(x_2) : x = x_1 x_2] \\
&\leq \lor \{ \gamma(\text{atom}(\mathbb{R})) \land \lor \{ \text{path}_{\mathbb{R}}(c_1) : c_1 \in T(Q, \Sigma), b(c_1) = u, e(c_1) = q, \\
\quad M(c_1) \subseteq X - \{q\}, lb(c_1) = x_1 \} \land \lor \{ \text{path}_{\mathbb{R}}(c_2) : c_2 \in T(Q, \Sigma), b(c_2) = e(c_2) = q, M(c_2) \subseteq X, lb(c_2) = x_2 \} : x = x_1 x_2 \}
\end{align*}
\]
\[ \leq \forall \{ \text{path}_R(c_1) \land \text{path}_R(c_2) \} : c_1, c_2 \in T(Q, \Sigma), b(c_1) = u, e(c_1) = b(c_2) = e(c_2) = q, M(c_1) \subseteq X - \{ q \}, M(c_2) \subseteq X, x = lb(c_1)lb(c_2) \}. \]

Furthermore, we can derive in a similar way that
\[ [\gamma(\text{atom}(R))] \land ([L(\alpha_{uq}^{-\langle q \rangle})L((\alpha_{qq}^{-\langle q \rangle})^*)L(\alpha_{qv}^{-\langle q \rangle}))(s) \]
\[ \leq \forall \{ \text{path}_R(c_1) \land \text{path}_R(c_2) \land \text{path}_R(c_3) \} : c_1, c_2, c_3 \in T(Q, \Sigma), b(c_1) = u, e(c_1) = b(c_2) = e(c_2) = b(c_3) = q, e(c_3) = v, s = lb(c_1)lb(c_2)lb(c_3) \}
\[ = \forall \{ \text{path}_R(c) \} : c \in T(Q, \Sigma), b(c) = u, e(c) = v, q \in M(c), s = lb(c) \}. \]

Consequently, it holds that
\[ [\gamma(\text{atom}(R))] \land L(\alpha_{uv}^X)(s) = [\gamma(\text{atom}(R))] \land \{ L(\alpha_{uv}^X^{-\langle q \rangle})(s) \} \]
\[ \leq [\gamma(\text{atom}(R))] \land L(\alpha_{uv}^X^{-\langle q \rangle})(s) \lor \{ [\gamma(\text{atom}(R))] \land ([L(\alpha_{uq}^{-\langle q \rangle})L((\alpha_{qq}^{-\langle q \rangle})^*)L(\alpha_{qv}^{-\langle q \rangle}))(s) \}
\[ \leq \text{the right-hand side of (b)}. \]

After proving (a), we can assert that
\[ [s \in L(k(R))] = \forall_{u,v \in Q} [I(u) \land T(v) \land L(\alpha_{uv}^Q)(s)] \]
\[ \geq \forall_{u,v \in Q} (I(u) \land T(v)) \lor \{ \text{path}_R(c) \} : c \in T(Q, \Sigma), b(c) = u, e(c) = v, lb(c) = s \}
\[ = \forall_{u,v \in Q} \{ I(u) \land T(v) \land \text{path}_R(c) \} : c \in T(Q, \Sigma), b(c) = u, e(c) = v, lb(c) = s \}
\[ = [\text{rec}_R(s)]. \]

By using (b) and Lemmas 2.5 and 2.6, we have
\[ [\gamma(\text{atom}(R))] \land [s \in L(k(R))] = [\gamma(\text{atom}(R))] \land \forall_{u,v \in Q} [I(u) \land T(v) \land L(\alpha_{uv}^Q)(s)] \]
\[ \leq \forall_{u,v \in Q} [I(u) \land T(v) \land [\gamma(\text{atom}(R))] \land L(\alpha_{uv}^Q)(s)] \]
\[ \leq \forall_{u,v \in Q} [I(u) \land T(v) \land [\gamma(\text{atom}(R))] \land \{ \text{path}_R(c) \} : c \in T(Q, \Sigma), b(c) = u, e(c) = v, lb(c) = s \}
\[ = \forall_{u,v \in Q} \{ I(u) \land T(v) \land \text{path}_R(c) \} : c \in T(Q, \Sigma), b(c) = u, e(c) = v, lb(c) = s \}
\[ = [\text{rec}_R(s)]. \]

Thus, (1) and (2) are proved, and the part that (i) implies (ii) of (3) is a simple corollary of (2). We now turn to prove that (ii) implies (i). For any \( a, b, c \in L \), we consider the \( \ell \)-valued automaton
\[ \mathcal{R} = \langle \{ u, v \}, \delta, u_a, \{ u, v \} \rangle, \]

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where $\delta(u, \sigma, u) = b$, $\delta(u, \sigma, v) = c$, and $\delta$ takes value 0 for other cases (see Figure 9). Then

$$\left[\text{rec}_R(\sigma)\right] = \lor \{ I(q_0) \land T(q_1) \land \delta(q_0, \sigma, q_1) : q_0, q_1 \in Q \}$$

$$= (a \land b) \lor (a \land c).$$

On the other hand, we have

$$\begin{align*}
\alpha^\phi_{uu} &= \varepsilon + b\sigma, \\
\alpha^\phi_{uv} &= c\sigma, \\
\alpha^\phi_{vu} &= \varepsilon, \\
\alpha^\phi_{vv} &= \phi.
\end{align*}$$

Therefore,

$$\alpha_{uv}^v = \alpha_{uv}^\phi + [\alpha_{uv}^\phi (\alpha_{uv}^\phi)^*] \alpha_{uv}^\phi$$

$$= (\varepsilon + b\sigma) + [c\sigma(\varepsilon)^*]\phi$$

$$= \varepsilon + b\sigma,$$

$$\alpha_{uv}^v = \alpha_{uv}^\phi + [\alpha_{uv}^\phi (\alpha_{uv}^\phi)^*] \alpha_{uv}^\phi$$

$$= c\sigma + [c\sigma(\varepsilon)^*]\varepsilon$$

$$= c\sigma,$$

and

$$\alpha_{uv}^u = \alpha_{uv}^v + [\alpha_{uv}^v (\alpha_{uv}^v)^*] \alpha_{uv}^v$$

$$= \varepsilon + b\sigma + [(\varepsilon + b\sigma) (\varepsilon + b\sigma)^*] (c\sigma).$$

From the assumption (ii) we know that

$$(a \land b) \lor (a \land c) = \left[\text{rec}_R(\sigma)\right]$$

$$= L(k(\mathcal{R}))(\sigma)$$

$$= [L(aa_{uv}^u) \cup L(aa_{uv}^v)](\sigma)$$

$$\geq L(aa_{uv}^u)(\sigma)$$

$$= a \land L(aa_{uv}^v)(\sigma)$$

$$= a \land L(\varepsilon + b\sigma + [(\varepsilon + b\sigma)(\varepsilon + b\sigma)^*] (c\sigma))(\sigma)$$

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\[ \geq a \land (b \lor c). \]

This completes the proof. 

**Corollary 7.2.** Let \( \ell = < L, \leq, \land, \lor, \bot, 0, 1 > \) be an orthomodular lattice, and let \( \rightarrow = \rightarrow_3 \). Then for any \( A \in L^{\Sigma^*} \),

\[
C_{\mathcal{R}\Sigma}(A) \rightarrow (\exists \text{ regular expression } \alpha)(A \equiv L(\alpha)).
\]

**Proof.** It can be derived from Theorem 7.1 in a way similar to the proof of Corollary 4.2. 

We now turn to consider homomorphisms of \( \ell \)-valued regular expressions. Let \( \Sigma \) and \( \Gamma \) be two alphabet, and let \( h : \Sigma \rightarrow \Gamma^* \) be a mapping. Then it can be uniquely extended to a mapping, denoted still by \( h \), from \( \ell \)-valued regular expressions over \( \Sigma \) into \( \ell \)-valued regular expressions over \( \Gamma \). For any \( \ell \)-valued regular expression \( \alpha \) over \( \Sigma \), \( h(\alpha) \) is defined to be the \( \ell \)-valued regular expression over \( \Gamma \) obtained by replacing each letter \( \sigma \in \Sigma \) appearing in \( \alpha \) with the string \( h(\sigma) \in \Gamma^* \). Formally, \( h(\alpha) \) is defined by induction on the length of \( \alpha \):

- \( h(\varepsilon) = \varepsilon \),
- \( h(\phi) = \phi \),
- \( h(\sigma) \) is already given for each \( \sigma \in \Sigma \),
- \( h(\lambda \alpha) = \lambda h(\alpha) \),
- \( h(\alpha_1 + \alpha_2) = h(\alpha_1) + h(\alpha_2) \),
- \( h(\alpha_1 \cdot \alpha_2) = h(\alpha_1) \cdot h(\alpha_2) \),
- \( h(\alpha^*) = (h(\alpha))^* \).

For each \( \ell \)-valued regular expression \( \alpha \) over \( \Sigma \), we write \( \Lambda(\alpha) \) for the set of scalar values \( \lambda \in L \) occurring in \( \alpha \). Indeed, \( \Lambda(\alpha) \) may be formally defined by induction on the length of \( \alpha \) as follows:

- \( \Lambda(\varepsilon) = \Lambda(\phi) = \Lambda(\sigma) = \phi \) for every \( \sigma \in \Sigma \),
- \( \Lambda(\lambda \alpha) = \{ \lambda \} \cup \Lambda(\alpha) \),
- \( \Lambda(\alpha_1 + \alpha_2) = \Lambda(\alpha_1 \cdot \alpha_2) = \Lambda(\alpha_1) \cup \Lambda(\alpha_2) \),
- \( \Lambda(\alpha^*) = \Lambda(\alpha) \).
It is easy to see that $\Lambda(\alpha)$ is a finite subset of $L$. Moreover, we write

$$\Delta(\alpha) = \{ a : a \in \Lambda(\alpha) \}$$

for the set of (constant) propositions in our logical language corresponding to the elements in $\Lambda(\alpha)$.

The following two lemmas are very useful when we are dealing with orthomodular lattice-valued expressions, they evaluate the range of language generated by an orthomodular lattice-valued regular expression. In particular, it will be shown in Lemma 7.4 that this range is a finite set whenever the lattice $\ell$ of truth values is a Boolean algebra.

**Lemma 7.3.** Let $\ell = < L, \leq, \wedge, \vee, \bot, 0, 1 >$ be an orthomodular lattice. Then for any $\ell$–valued regular expression $\alpha$, $\{ L(\alpha)(s) : s \in \Sigma^* \} \subseteq [\Lambda(\alpha)]$, where $[A]$ denotes the subalgebra of $\ell$ generated by $A$ for any $A \subseteq L$.

**Proof.** We use an induction argument on the length of $\alpha$. For simplicity, we only consider the following two cases, and the other cases are easy or similar.

1. From the induction hypothesis we know that

$$L(\lambda.\alpha)(s) = \lambda \wedge L(\alpha)(s) \in \{ \lambda \} \cup \Lambda(\alpha) = \Lambda(\lambda.\alpha)$$

for each $s \in \Sigma^*$.

2. Let $s \in \Sigma^*$. For any $s_1, \ldots, s_n \in \Sigma^*$ with $s_1 \ldots s_n = s$, we suppose that $s_{i_1}, \ldots, s_{i_m} \neq \varepsilon$ and $s_i = \varepsilon$ for every $i \in \{1, \ldots, n\} - \{i_1, \ldots, i_m\}$. Then $s_{i_1} \ldots s_{i_m} = s$ and

$$L(\alpha)(s_1) \wedge \ldots \wedge L(\alpha)(s_n) = \begin{cases} L(\alpha)(s_{i_1}) \wedge \ldots \wedge L(\alpha)(s_{i_m}) \text{ if } m = n, \\ L(\alpha)(s_{i_1}) \wedge \ldots \wedge L(\alpha)(s_{i_m}) \wedge L(\alpha)(\varepsilon) \text{ if } m < n. \end{cases}$$

Furthermore, we note that

$$\{(s_1, \ldots, s_n) : n \geq 0, s_1, \ldots, s_n \in \Sigma^* - \{\varepsilon\} \text{ and } s_1 \ldots s_n = s\}$$

is finite. Therefore,

$$\{ L(\alpha)(s_1) \wedge \ldots \wedge L(\alpha)(s_n) : s_1 \ldots s_n = s \}$$

is also finite, and with the induction hypothesis we have

$$L(\alpha^*)(s) = \vee \{ L(\alpha)(s_1) \wedge \ldots \wedge L(\alpha)(s_n) : s_1 \ldots s_n = s \} \in \Lambda(\alpha), \lor$$

**Lemma 7.4.** If $\ell = < L, \leq, \wedge, \vee, \bot, 0, 1 >$ is a Boolean algebra, then for any $\ell$–valued regular expression $\alpha$, $\{ L(\alpha)(s) : s \in \Sigma^* \}$ is a finite set.
**Proof.** From Lemma 7.3 and the distributivity of $\land$ over $\lor$ we know that for any $s \in \Sigma^*$, there are $\lambda_{ij} \in \Lambda(\alpha)$ ($i = 1, ..., m; j_i = 1, ..., n_i$) such that

$$L(\alpha)(s) = \lor_{i=1}^{m} (\land_{j_i=1}^{n_i} \lambda_{ij}).$$

Since $\Lambda(\alpha)$ is finite, both

$$\Lambda(\alpha)^{(\land)} = \{\lambda_1 \land ... \land \lambda_n : n \geq 0, \lambda_1, ..., \lambda \in \Lambda(\alpha)\}$$

and

$$\Lambda(\alpha)^{(\land)(\lor)} = \{\lor M : M \subseteq \Lambda(\alpha)^{(\land)}\}$$

are also finite. Therefore,

$$\Lambda(\alpha)^{(\land)(\lor)} \supseteq \{L(\alpha)(s) : s \in \Sigma^*\}$$

is a finite set.$\heartsuit$

The following proposition shows that a homomorphism preserves the language generated by an orthomodular lattice-valued regular expression under the condition that all elements in the range of the expression under consideration are commutative.

**Proposition 7.5.** Let $\ell =< L, \leq, \land, \lor, \bot, 0, 1>$ be an orthomodular lattice and $\rightarrow$ fulfil the Birkhoff-von Neumann requirement, and let $\Sigma$ and $\Gamma$ be two alphabets.

(1) For any mapping $h : \Sigma \rightarrow \Gamma^*$, and for any $\ell-$valued regular expression $\alpha$ over $\Sigma$,

$$\ell \models h(\Delta(\alpha)) \subseteq L(h(\alpha)).$$

(2) For any mapping $h : \Sigma \rightarrow \Gamma^*$, for any $\ell-$valued regular expression $\alpha$ over $\Sigma$, and for any $t \in \Gamma^*$,

$$\ell \models \gamma(\Delta(\alpha)) \land t \in L(h(\alpha)) \rightarrow t \in h(L(\alpha)),$$

and if $\rightarrow\rightarrow\rightarrow_3$ then

$$\ell \models \gamma(\Delta(\alpha)) \rightarrow L(h(\alpha)) \equiv h(L(\alpha)).$$

(3) The following two statements are equivalent:

(i) $\ell$ is a Boolean algebra.

(ii) for any mapping $h : \Sigma \rightarrow \Gamma^*$, and for any $\ell-$valued regular expression $\alpha$ over $\Sigma$,

$$\ell \models h(\Delta(\alpha)) \equiv L(h(\alpha)).$$
Proof. We only prove (2) and (3), and (1) can be observed from the proof of (2). The part that (i) implies (ii) of (3) may be derived from (2); and it can also be proved directly by using Lemma 7.4.

Our first aim is to prove that

$$[\gamma(\Delta(\alpha))] \land L(h(\alpha))(t) = h(L(\alpha))(t)$$

for any $t \in \Gamma^*$ and for any $\ell$–valued regular expression $\alpha$ over $\Sigma$. We proceed by induction on the length of $\alpha$.

(a) It is obvious for the case of $\alpha = \varepsilon$ or $\phi$, or $\alpha \in \Sigma$.

(b) With the definitions of $h(\cdot)$ and $L(\cdot)$ and the induction hypothesis we derive that

$$L(h(\lambda \alpha))(t) = L(\lambda h(\alpha))(t)$$
$$= \lambda \land L(h(\alpha))(t)$$
$$= \lambda \land h(L(\alpha))(t)$$
$$= \lambda \land \lor\{L(\alpha)(s) : s \in \Sigma^* \text{ and } h(s) = t\}.$$}

Then from Lemmas 2.5, 2.6 and 7.3, it follows that

$$[\gamma(\Delta(\alpha))] \land L(h(\lambda \alpha))(t) \leq \lor\{\lambda \land L(\alpha)(s) : s \in \Sigma^* \text{ and } h(s) = t\}$$
$$= \lor\{L(\lambda \alpha)(s) : s \in \Sigma^* \text{ and } h(s) = t\}$$
$$= h(L(\lambda \alpha))(t).$$

(c) It is easy to observe that $h(A \cup B) = h(A) \cup h(B)$ for all $A, B \in L^{\Sigma^*}$. This together with the induction hypothesis as well as Lemmas 2.5 and 2.6 yields

$$[\gamma(\Delta(\alpha_1 + \alpha_2))] \land L(h(\alpha_1 + \alpha_2))(t) = [\gamma(\Delta(\alpha_1 + \alpha_2))] \land L(h(\alpha_1) + h(\alpha_2))(t)$$
$$= [\gamma(\Delta(\alpha_1 + \alpha_2))] \land [\gamma(\Delta(\alpha_1 + \alpha_2))] \land [L(h(\alpha_1))(t) \lor L(h(\alpha_2))(t)]$$
$$\leq [\gamma(\Delta(\alpha_1 + \alpha_2))] \land L(h(\alpha_1))(t) \land [\gamma(\Delta(\alpha_1 + \alpha_2))] \land L(h(\alpha_2))(t)$$
$$\leq [\gamma(\Delta(\alpha_1))] \land L(h(\alpha_1))(t) \land [\gamma(\Delta(\alpha_2))] \land L(h(\alpha_2))(t)$$
$$= h(L(\alpha_1) \cup L(\alpha_2))(t)$$
$$= h(L(\alpha_1 + \alpha_2))(t).$$

(d) For any $t \in \Gamma^*$, Lemmas 2.5, 2.6 and 7.3 enable us to assert that

$$[\gamma(\Delta(\alpha_1 \cdot \alpha_2))] \land L(h(\alpha_1 \cdot \alpha_2))(t) = [\gamma(\Delta(\alpha_1 \cdot \alpha_2))] \land L(h(\alpha_1) \cdot h(\alpha_2))(t)$$
$$= [\gamma(\Delta(\alpha_1 \cdot \alpha_2))] \land L(h(\alpha_1)h(\alpha_2))(t)$$
$$= [\gamma(\Delta(\alpha_1 \cdot \alpha_2))] \land \lor\{L(h(\alpha_1))(t_1) \land L(h(\alpha_2))(t_2) : t_1t_2 = t\}$$

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Therefore, it follows that

\[ \forall \{ \gamma(\Delta(\alpha_1 \cdot \alpha_2)) \} \land L(h(\alpha_1))(t_1) \land L(h(\alpha_2))(t_2) : t_1t_2 = t \]

\[ = \forall \{ \gamma(\Delta(\alpha_1 \cdot \alpha_2)) \} \land \{ \gamma(\Delta(\alpha_1)) \land L(h(\alpha_1))(t_1) \} \land \{ \gamma(\Delta(\alpha_2)) \land L(h(\alpha_2))(t_2) \} : t_1t_2 = t \]

\[ \leq \forall \{ \gamma(\Delta(\alpha_1 \cdot \alpha_2)) \} \land h(L(\alpha_1))(t_1) \land h(L(\alpha_2))(t_2) : t_1t_2 = t \}.

Furthermore, by using Lemmas 2.5, 2.6 and 7.3 again we obtain

\[ \gamma(\Delta(\alpha_1 \cdot \alpha_2)) \land h(L(\alpha_1))(t_1) \land h(L(\alpha_2))(t_2) = \gamma(\Delta(\alpha_1 \cdot \alpha_2)) \land \{ \forall \{ L(\alpha_1)(s_1) : h(s_1) = t_1 \} \} \land \{ \forall (L(\alpha_2)(s_2) : h(s_2) = t_2) \}

\[ \leq \forall \{ L(\alpha_1)(s_1) \land L(\alpha_2)(s_2) : h(s_1) = t_1 \text{ and } h(s_2) = t_2 \}.

Therefore, it follows that

\[ \gamma(\Delta(\alpha_1 \cdot \alpha_2)) \land L(h(\alpha_1 \cdot \alpha_2))(t) \leq \forall \{ L(\alpha_1)(s_1) \land L(\alpha_2)(s_2) : h(s_1) = t_1, h(s_2) = t_2 \text{ and } t_1t_2 = t \}

\[ = \forall \{ L(\alpha_1)(s_1) \land L(\alpha_2)(s_2) : h(s_1s_2) = t \}

\[ = h(L(\alpha_1)L(\alpha_2))(t)

\[ = h(L(\alpha_1\alpha_2))(t).

(e) For every \( t \in \Gamma^* \), Lemmas 2.5, 2.6 and 7.3 guarantee that

\[ \gamma(\Delta(\alpha^*)) \land L(h(\alpha^*))(t) = \gamma(\Delta(\alpha^*)) \land L((h(\alpha))^*)(t) \]

\[ = \gamma(\Delta(\alpha^*)) \land (L(h(\alpha))^*)(t) \]

\[ = \gamma(\Delta(\alpha^*)) \land \forall \{ \land_{i=1}^n L(h(\alpha))(t_i) : n \geq 0, t_1, ..., t_n \in \Gamma^*, t_1...t_n = t \}

\[ \leq \forall \{ \gamma(\Delta(\alpha^*)) \land \land_{i=1}^n L(h(\alpha))(t_i) : n \geq 0, t_1, ..., t_n \in \Gamma^*, t_1...t_n = t \}

\[ = \forall \{ \gamma(\Delta(\alpha)) \land \land_{i=1}^n h(L(\alpha))(t_i) : n \geq 0, t_1, ..., t_n \in \Gamma^*, t_1...t_n = t \}

\[ \leq \forall \{ \gamma(\Delta(\alpha)) \land \land_{i=1}^n h(L(\alpha))(t_i) : n \geq 0, t_1, ..., t_n \in \Gamma^*, t_1...t_n = t \}.

On the other hand, we have

\[ \gamma(\Delta(\alpha)) \land \land_{i=1}^n h(L(\alpha))(t_i) = \gamma(\Delta(\alpha)) \land \land_{i=1}^n (\forall \{ L(\alpha)(s_i) : h(s_i) = t_i \})

\[ \leq \forall \{ \land_{i=1}^n L(\alpha)(s_i) : h(s_i) = t_i \ (i = 1, ..., n) \}.

This further yields

\[ \gamma(\Delta(\alpha^*)) \land L(h(\alpha^*))(t) \leq \forall \{ \land_{i=1}^n L(\alpha)(s_i) : n \geq 0, h(s_i) = t_i \ (i = 1, ..., n) \text{ and } t = t_1...t_n \}

\[ = \forall \{ \land_{i=1}^n L(\alpha)(s_i) : n \geq 0, h(s_1...s_n) = t \}

\[ = \forall \{ L(\alpha)^*(s) : h(s) = t \}

\[ = h((L(\alpha))^*)(t)\]
What remains is to prove that (ii) implies (i) in (3). This needs indeed to show that the distributivity of $\land$ over $\lor$ is derivable from the statement (ii). Suppose that $a, b, c \in L$. We choose an symbol $\sigma \in \Sigma$ and an symbol $\gamma \in \Gamma$, and define $h(\sigma) = \varepsilon$ and $h(\sigma') = \gamma$ for every $\sigma' \in \Sigma - \{\sigma\}$. We further set $\alpha_1 = a.\sigma$ and $\alpha_2 = b.\varepsilon + c.\sigma$. Then

$$L(\alpha_1.\alpha_2)(\sigma) = \begin{cases} a \land b & \text{if } n = 1, \\ a \land c & \text{if } n = 2, \\ 0 & \text{otherwise}, \end{cases}$$

and

$$h(L(\alpha_1.\alpha_2))(\varepsilon) = \lor_{n=0}^{\infty}L(\alpha_1.\alpha_2)(\sigma^n) = (a \land b) \lor (a \land c).$$

On the other hand, we have

$$L(h(\alpha_1.\alpha_2))(\varepsilon) = L((a.\varepsilon).((b.\varepsilon + c.\varepsilon)))(\varepsilon) = L(a.\varepsilon)(\varepsilon) \land L(b.\varepsilon + c.\varepsilon)(\varepsilon) = a \land (b \lor c).$$

From (ii) we know that $h(L(\alpha_1.\alpha_2))(\varepsilon) = L(h(\alpha_1.\alpha_2))(\varepsilon)$. This indicates that $(a \land b) \lor (a \land c) = a \land (b \lor c).$ 

8. Pumping Lemma for Orthomodular Lattice-Valued Regular Languages

The pumping lemma in the classical automata theory is a powerful tool to show that certain languages are not regular, and it exposes some limitations of finite automata. The purpose of this section is to establish a generalization of the pumping lemma for orthomodular lattice-valued languages. It is worth noting that the following orthomodular lattice-valued version of pumping lemma is given for the commutative regularity $CReg_{\Sigma}$. In general, the pumping lemma is not valid for the predicate $Reg_{\Sigma}$.

**Theorem 8.1.** (The pumping lemma) Let $\ell = < L, \leq, \land, \lor, \bot, 0, 1 >$ be an orthomodular lattice, and let $\rightarrow = \rightarrow_3$. Then for any $A \in L^{\Sigma^*}$, if $Range(A)$ is finite, then

$$\ell \models CReg_{\Sigma}(A) \rightarrow (\exists n \geq 0)(\forall s \in \Sigma^*)[s \in A \land |s| \geq n \rightarrow$$

$$(\exists u, v, w \in \Sigma^*)(s = uvw \land |uv| \leq n \land |v| \geq 1 \land (\forall i \geq 0)(uv^iw \in A))],$$

where for any word $t = \sigma_1...\sigma_k \in \Sigma^*$, $|t|$ stands for the length $n$ of $t$. 

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Proof. For simplicity, we use $X(s,n)$ to mean the statement that $u,v,w \in \Sigma^*$, $s = uvw$, $|uv| \leq n$, and $|v| \geq 1$ for each $s \in \Sigma^*$ and $n \geq 0$. Then it suffices to show that

$$[CReg_{\Sigma^*}(A)] \leq \forall n \geq 0 \wedge_{s \in \Sigma^*,|s| \geq n} (A(s) \rightarrow \forall X(s,n) \wedge_{i \geq 0} A(uviw)).$$

From Definition 3.3 we know that

$$[CReg_{\Sigma^*}(A)] = \forall_{R \in A(\Sigma, \ell)} ([\gamma(\text{atom}(R) \cup r(A)) \wedge [A \equiv rec_R]).$$

Thus, we only need to prove that for any $R \in A(\Sigma, \ell)$,

$$[\gamma(\text{atom}(R) \cup r(A)) \wedge [A \equiv rec_R] \leq \forall n \geq 0 \wedge_{s \in \Sigma^*,|s| \geq n} (A(s) \rightarrow
\forall X(s,n) \wedge_{i \geq 0} A(uviw)).$$

Let $Q$ be the set of states of $R$. First, it holds that for any $s \in \Sigma^*$ with $|s| \geq |Q|$, \begin{equation}
rec_R(s) \leq \forall X(s,n) \wedge_{i \geq 0} rec_R(uviw).
\end{equation}
In fact, suppose that $s = \sigma_1...\sigma_k$. Then

\begin{equation}
rec_R(s) = \forall_{q_0,q_1,...,q_k} [I(q_0) \wedge T(q_k) \wedge \wedge_{i=0}^{k-1} \delta(q_i,\sigma_{i+1},q_{i+1})].
\end{equation}
Therefore, it amounts to showing that for any $q_0,q_1,...,q_k \in Q$,

\begin{equation}
I(q_0) \wedge T(q_k) \wedge \wedge_{i=0}^{k-1} \delta(q_i,\sigma_{i+1},q_{i+1}) \leq \forall X(s,n) \wedge_{i \geq 0} rec_R(uviw).
\end{equation}

Since $k = |s| \geq |Q|$, there are two identical states among $q_0,q_1,...,q_{|Q|}$; in other words, there are $m \geq 0$ and $n > 0$ such that $m + n \leq |Q|$ and $q_m = q_{m+n}$. We set $u_0 = \sigma_1...\sigma_m$, $v_0 = \sigma_{m+1}...\sigma_{m+n}$, and $w_0 = \sigma_{m+n+1}...\sigma_k$. Then $s = u_0v_0w_0$,

\begin{equation}
\forall X(s,n) \wedge_{i \geq 0} rec_R(uviw) \geq \wedge_{i \geq 0} rec_R(u_0v_iw_0).
\end{equation}

From the definition of $rec_R$, it is easy to see that for all $i \geq 0$,

\begin{equation}
rec_R(u_0v_iw_0) \geq [\text{path}_R(q_0\sigma_1q_1...q_mq_m)
\end{equation}

$$= I(q_0) \wedge T(q_k) \wedge \wedge_{j=0}^{m+n-1} \delta(q_j,\sigma_{j+1},q_{j+1}) \wedge \wedge_{i=1}^{k-1} \delta(q_m+n,\sigma_{m+1},q_{m+1}) \wedge \wedge_{j=m+1}^{m+n-1} \delta(q_j,\sigma_{j+1},q_{j+1}) \wedge \wedge_{j=m+1}^{k-1} \delta(q_j,\sigma_{j+1},q_{j+1})$$

because $q_{m+n} = q_m$ and $\delta(q_{m+n},\sigma_{m+1},q_{m+1}) = \delta(q_m,\sigma_{m+1},q_{m+1})$. Thus, by combining (4) and (5), we obtain (3) which, together with (2), yields (1).
Now we use Lemmas 2.11(1) and (3) and obtain
\[ \forall x(s, |Q|) \land i \geq 0 \text{rec}_R(uv^iw) \rightarrow \forall x(s, |Q|) \land i \geq 0 A(uv^iw) \geq \left[ \gamma(\text{atom}(R) \cup r(A)) \land \right. \\
\left. \land x(s, |Q|) \land i \geq 0 \text{rec}_R(uv^iw) \rightarrow \land i \geq 0 A(uv^iw) \right] \]
\[ \geq \left[ \gamma(\text{atom}(R) \cup r(A)) \land \land x(s, |Q|) \land i \geq 0 (\text{rec}_R(uv^iw) \rightarrow A(uv^iw)) \right] \]
\[ \geq \left[ \gamma(\text{atom}(R) \cup r(A)) \land \land t \in \Sigma^* (\text{rec}_R(t) \rightarrow A(t)) \right] \]
\[ = \left[ \gamma(\text{atom}(R) \cup r(A)) \land [\text{rec}_R \subseteq A] \right] \]
Furthermore, from the above inequality we have
\[ [\gamma(\text{atom}(R) \cup r(A)] \land [\text{rec}_R \equiv A] = [\gamma(\text{atom}(R) \cup r(A)] \land [A \subseteq \text{rec}_R] \land [\text{rec}_R \subseteq A] \]
\[ = [\gamma(\text{atom}(R) \cup r(A)] \land \land s \in \Sigma^*(A(s) \rightarrow \text{rec}_R(s)) \land [\text{rec}_R \subseteq A] \]
\[ \leq [\gamma(\text{atom}(R) \cup r(A)] \land \land s \in \Sigma^*, |s| \geq |Q| (A(s) \rightarrow \text{rec}_R(s)) \land [\text{rec}_R \subseteq A] \]
\[ = \land s \in \Sigma^*, |s| \geq |Q| ([\gamma(\text{atom}(R) \cup r(A)] \land (A(s) \rightarrow \text{rec}_R(s)) \land (\forall x(s, |Q|) \land i \geq 0 \text{rec}_R(uv^iw) \rightarrow \forall x(s, |Q|) \land i \geq 0 A(uv^iw)))]. \]

Then from (1) it follows that
\[ [\gamma(\text{atom}(R) \cup r(A)] \land [\text{rec}_R \equiv A] \leq \land s \in \Sigma^*, |s| \geq |Q| ([\gamma(\text{atom}(R) \cup r(A)] \land \\
(A(s) \rightarrow \forall x(s, |Q|) \land i \geq 0 \text{rec}_R(uv^iw)) \land \\
(\forall x(s, |Q|) \land i \geq 0 \text{rec}_R(uv^iw) \rightarrow \forall x(s, |Q|) \land i \geq 0 A(uv^iw))). \]

By using Lemmas 2.11(1) and (3) we know that
\[ [\gamma(\text{atom}(R) \cup r(A)] \land [\text{rec}_R \equiv A] \leq \land s \in \Sigma^*, |s| \geq |Q| (A(s) \rightarrow \forall x(s, |Q|) \land i \geq 0 A(uv^iw)) \]
\[ \leq \forall n \geq 0 \land s \in \Sigma^*, |s| \geq n (A(s) \rightarrow \forall x(s, n) \land i \geq 0 A(uv^iw)), \]
and this completes the proof.♡

9. Conclusion

It is argued that a theory of computation based on quantum logic has to be established as a logical foundation of quantum computation. This paper is the first one of a series of papers toward such a new theory. Quantum logic is treated as an orthomodular lattice-valued logic in this paper, and the aim of the paper is to develop elementally a theory of finite automata based on such a logic by employing the
semantical analysis approach. The notions of orthomodular lattice-valued finite automaton and regular language are introduced. Some modifications of orthomodular lattice-valued automaton are presented, including the orthomodular lattice-valued generalizations of deterministic and nondeterministic automata and automata with $\varepsilon$-moves, and their equivalence are thoroughly analyzed. We also examine the closure properties of orthomodular lattice-valued regular languages under various operations. The concept of orthomodular lattice-valued regular expressions is proposed, and the Kleene theorem concerning the equivalence between finite automata and regular expressions is generalized within the framework of quantum logic. Also, an orthomodular lattice-valued version of the pumping lemma is found. Furthermore, a theory of pushdown automata or Turing machines based on quantum logic will be developed in the continuations of the present paper.

In the development of automata theory based on quantum logic, some essential differences between the computation theory established by using the classical Boolean logic as the underlying logical tool and that whose meta-logic is quantum logic have been discovered. First, it is found that the proofs of some even very basic properties of automata appeal an essential application of the distributivity for the lattice of truth values of the underlying logic. This indicates that these properties holds only in Boolean logic but not in quantum logic. We believe that there are also many fundamental properties of pushdown automata and Turing machines whose universal validity requires the distributivity of meta-logic. In a sense, this observation provides us with a set of negative results in the theory of computation based on quantum logic. These negative results might hints some limitations of quantum computers. More explicitly, some methods based on certain properties of classical automata maybe have been successfully used in the implementation of classical computer systems, but they do not apply to quantum computers, or at least they are only conditionally effective for quantum computers. On the other hand, although these negative results are found in the computation theory based on quantum logic, it seems that some similar negative results exist in other mathematical theories based on nonclassical logics. This stimulates us to consider the problem of a logical revisit to mathematics. Various classical mathematical results have been established based upon classical logic, and sometimes, their universal validity can only be established by exploiting the full power of classical logic. Mathematicians usually use logic implicitly in their reasoning, and they do not seriously care which logical laws they have employed. But from a logician’s point of view, it is very interesting to determine how strong a logic we need to validate a given mathematical theorem, and which logic guarantees this theorem and which does not among the large population of nonclassical logics. To be more explicit and also for a comparison, let us present a short excerpt from A. Heyting [He63, page 3]:

"It may happen that for the proof of a theorem we do not need all the axioms, but only some of them. Such a theorem is true not only for models of the whole system, but also for those of the smaller system which contains only the axioms used in the
proof. Thus it is important in an axiomatic theory to prove every theorem from the least possible set of axioms.”

We now are in a similar situation. The difference between our case and A. Heyting’s one is that we are concerned with the limitation or redundancy of power of the logic underlying an axiomatic theory, whereas he considered that of axioms themselves. It seems that the semantical analysis approach provides a nice framework for solving this problem, much more suitable than a proof-theoretical approach.

As stated above, some fundamental properties of automata are not universally valid in quantum logic due to lack of distributivity. However, a certain commutativity are able to regain a local distributivity, and to give further a partial validity of these properties in the theory of automata based on quantum logic. One typical example of such properties is the equivalence of automata and their various modifications. It is well-known that one important witness for the Church-Turing thesis which asserts the Turing machine is a general model of computation is that various extensions of the Turing machine are all equivalent to itself. The fact that the equivalence between automata and their modifications depends upon the commutativity of their basic actions suggests us to guess that the equivalence between the Turing machine and some of its extensions may also need a support from a certain commutativity. In the introduction, we already gave a physical interpretation to the role of commutativity based on the Heisenberg uncertainty principle, and pointed out that an interesting connection may reside between the Heisenberg uncertainty principle and the Church-Turing thesis. If this is true, then it will give once again an evidence to the unity of the whole science and to the fact that science is not only a simple union of various subjects.

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