SEMISIMPLECTY OF THE DS FUNCTOR FOR THE ORTHOSYMPLECTIC LIE SUPERALGEBRA

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ABSTRACT. We prove that the Duflo-Serganova functor $DS_x$ attached to an odd nilpotent element $x$ of $osp(m|2n)$ is semisimple, i.e. sends a semisimple representation $M$ of $osp(m|2n)$ to a semisimple representation of $osp(m - 2k|2n - 2k)$ where $k$ is the rank of $x$. We prove a closed formula for $DS_x(L(\lambda))$ in terms of the arc diagram attached to $\lambda$.

1. Introduction

1.1. The Duflo-Serganova functor $DS$. For a finite dimensional complex Lie superalgebra $g$ and an odd nilpotent element $x$ Duflo and Serganova defined a functor $DS_x : Rep(g) \to Rep(g_x)$ where $g_x := ker \ ad(x)/im \ ad(x)$. This functor is given by taking the cohomology of the complex $V \xrightarrow{\rho(x)} V \xrightarrow{\rho(x)} V$ for $(V, \rho) \in Rep(g)$. For $gl(m|n)$ we have $g_x \simeq gl(m - k|n - k)$ and for $osp(m|2n)$ we have $g_x \simeq osp(m - 2k|2n - 2k)$ where $k$ is the so-called rank of $x$. The $DS$-functor is a symmetric monoidal functor which allows to reduce questions about superdimensions or tensor products to lower rank. It is however very complicated to compute $DS(V)$ explicitly. In [HW] the authors derived a closed formula for $DS_x(L(\lambda))$ where $L(\lambda)$ is a finite dimensional irreducible representation of $gl(m|n)$ and arbitrary $x$. In particular $DS_x(L(\lambda))$ is always semisimple.

1.2. Serganova’s conjecture. More generally Serganova conjectured [S2] that $DS_x$ should be semisimple for any basic classical Lie superalgebra (i.e. $DS_x(L)$ is semisimple for irreducible $L$). Since the conjecture is trivial in the exceptional cases (in these cases $g_x$ is a reductive Lie algebra) this leaves the $osp(m|2n)$-case. We prove Serganova’s conjecture in this article and give a closed formula for $DS_x(L(\lambda))$ for any $x$ and any $\lambda$.

1.3. Main steps and results. We work in the full subcategory $\tilde{F}(g)$ of algebraic representations of $osp(m|2n)$. The category $\tilde{F}(g)$ is canonically isomorphic to the category of $SOSp(m|2n)$-modules; this category decomposes into a direct sum:

$$\tilde{F}(g) = F(g) \oplus \Pi(F(g)),$$

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where \( \mathcal{F}(g) \) is the full subcategory of modules \( M \), where the parity is induced from the parity on \( \Lambda_{m|n} \). We view \( DS_x \) as a functor \( \tilde{\mathcal{F}}(g) \rightarrow \tilde{\mathcal{F}}(g_x) \). For fixed \( x \) of rank \( r \) we also write \( DS_r \).

1.3.1. The cases \( t = 0, 1, 2 \). Recall that \( \mathfrak{osp}(M|N) \) consists of two series: the \( B \)-series (when \( M \) is odd) and the \( D \)-series (when \( M \) is even). In the \( \mathfrak{osp} \)-case each block of atypicality \( k \) is equivalent to the principal block of \( \mathfrak{osp}(2k + t|2k) \) where \( t = 1 \) for the \( B \)-series and \( t = 0, 2 \) for the \( D \)-series. The equivalences are described in [GS1] (we recall details in 6.2). We consider these three cases separately and set \( g := \mathfrak{osp}(2m + t|2n) \), where \( t = 0, 1, 2 \) as above. This division is compatible with DS-functors: if \( N \) lies in a block of type \( t \), then \( DS_x(N) \) lies in a block of type \( t \). In Section 6 we reduce the computations of multiplicities \( [DS_x(L(\lambda)) : L_{g_x}(\nu)] \) to the case of principal blocks.

1.3.2. Reduction to the principal block. Any irreducible module can be moved via a series of translation functors to a stable irreducible module (see section 4). The subcategory \( \mathcal{F}^{st}_t \) of a block \( \mathcal{F}^t(g) \) consisting of stable modules (a module stable if all its simple subquotients are stable) is equivalent via a functor \( Res \) to the principal block of \( \mathfrak{osp}(2k + l|2k) \) for \( l = 0, 1, 2 \). Since \( DS \) commutes with both \( Res \) and translation functors (see section 6.3), we can compute it on irreducible modules in the principal block.

1.3.3. Recursive formulae. We then induct on the degree of atypicality. For atypicality 1 (i.e. the principal block of \( \mathfrak{osp}(2|2) \), \( \mathfrak{osp}(3|2) \) and \( \mathfrak{osp}(4|2) \) \( DS(L(\lambda)) \) can be computed easily (see for example [Gor2]). In order to treat the general case we establish recursive formulae for the multiplicities of irreducible constituents in \( DS_x(L(\lambda)) \).

For a category \( C \) denote by \( \text{Irr}(C) \) the set of isomorphism classes of simple modules in \( C \). Let \( V_{st} \) be the natural representation. For blocks \( \tilde{\mathcal{F}}^{g_1}(g), \tilde{\mathcal{F}}^{g_0}(g) \) we denote by \( T^{g_0}_{g_1} \) the translation functor

\[
T^{g_0}_{g_1} : \tilde{\mathcal{F}}^{g_1}(g) \rightarrow \tilde{\mathcal{F}}^{g_0}(g)
\]

which maps \( N \) to the projection of \( N \otimes V_{st} \) to the subcategory \( \tilde{\mathcal{F}}^{g_0}(g) \). Since blocks of atypicality \( n \) for \( g \) and blocks of atypicality \( n - r \) for \( g_x \) (\( rk(x) = r \)) correspond to each other (via so-called core diagrams), we may look at

\[
T^{g_0}_{g_1} : \tilde{\mathcal{F}}(g)^{g_1} \rightarrow \tilde{\mathcal{F}}(g)^{g_0}, \quad T^{g_0}_{g_1} : \tilde{\mathcal{F}}(g_x)^{g_1} \rightarrow \tilde{\mathcal{F}}(g_x)^{g_0}.
\]

For \( N \in \mathcal{F}(g)^{g_1} \) and \( L' \in \text{Irr}(g_x)^{g_0} \) we have

\[
[DS_r(T^{g_0}_{g_1}(N)) : L'] = [T^{g_0}_{g_1}(DS_r(N)) : L'] = \sum_{L_1 \in \text{Irr}(g_x)^{g_1}} [DS_r(N) : L_1][T^{g_0}_{g_1}(L_1) : L'].
\]

This formula allows us in Section 7 to successively reduce the computation of the multiplicity \( [DS_x(L(\lambda)) : L_{g_x}(\nu)] \) to the case where \( g_x \) is 0 or \( \mathbb{C} \).
1.3.4. **Multiplicities for the case** \( \text{rank } x = 1 \). For such \( x \) the condition \( g_x = 0, \mathbb{C} \) gives \( g = \mathfrak{osp}(m|2) \) for \( m = 2, 3, 4 \); for these cases \( DS_x(L) \) can be easily computed. This gives the multiplicity \( [DS_x(L(\lambda)) : L_{g_x}(\nu)] \) for \( x = 1 \). As in the \( \mathfrak{gl}(m|n) \) and \( \mathfrak{p}_n \) cases, treated in [HW], [EAS] respectively, we give the final answer in terms of arc diagrams.

For a weight \( \lambda \) denote by \( \text{howl}(\lambda) \) the corresponding weight in the principal block; to \( \text{howl}(\lambda) \) we attach an arc diagram \( \text{Arc}(\text{howl}(\lambda)) \), see 8.1.

**Theorem A** (see Theorem 8.2 for a more precise statement). Let \( \text{rk}(x) = 1 \). Then \( L(\nu) \) is a subquotient of \( DS(L(\lambda)) \) if and only if \( \text{Arc}(\text{howl}(\nu)) \) is obtained from \( \text{Arc}(\text{howl}(\lambda)) \) by removing a maximal arc and, in addition, in the \( \mathfrak{osp}(2m + 1|2n) \)-case, if \( \nu \) has a sign, then the signs of \( \lambda \) and \( \nu \) are equal. The multiplicity of \( L(\nu) \) is either 1 or 2 (depending on the shape of the Arc diagram).

1.3.5. **Semisimplicity for the case** \( \text{rank } x = 1 \). In [Gor2] (see also Corollary 6.2.2) it is shown that the extension graph of \( \mathcal{F}(g) \) is bipartite; the bipartition is given by a sign function \( \text{dex} : \text{Irr}(\mathcal{F}(g)) \to \{\pm 1\} \) such that \( \text{Ext}^1(L_1, L_2) = 0 \) if \( \text{dex}(L_1) = \text{dex}(L_2) \). Theorem A implies \( [DS_x(L(\lambda)) : L_{g_x}(\nu)] = 0 \) if \( \text{dex}(L(\lambda)) \neq \text{dex}(L(\nu)) \) (and \( x = 1 \)); this shows the semisimplicity of \( DS_x(L(\lambda)) \).

1.4. **The general case.** We denote with \( \mathcal{F}_+(g) \) the full Serre subcategory of \( \mathcal{F}(g) \) generated by the irreducible objects with \( \text{dex}(L) = 1 \). By definition it is a semisimple category. We say that a module \( M \) is **pure** if for any subquotient \( L \) of \( M \), \( \Pi(L) \) is not a subquotient of \( M \).

Serganova’s semisimplicity conjecture follows from the following theorem.

**Theorem B** (Purity and semisimplicity). For each \( x \) one has \( DS_x(\mathcal{F}_+(g)) = \mathcal{F}_+(g_x) \). In particular \( DS_x(L(\lambda)) \) is pure.

For each \( L \in \text{Irr}(\mathcal{F}(g)) \) the module \( DS_x(L) \) only depends on the rank of \( x \). Moreover, \( DS_{x+1}(L) \cong DS_1(DS_x(L)) \), where \( DS_r \) stands for \( DS_x \) with rank \( x = r \).

By above, the statements follow from Theorem A for rank \( x = 1 \); the general case easily follows by induction on rank \( x \).

For \( \mathfrak{gl}(m|n) \) purity and semisimplicity were established in [HW]; for the exceptional cases purity was checked in [Gor1] (and semisimplicity is trivial since \( g_x \) is reductive). For \( \mathfrak{p}_n \) the semisimplicity does not hold (\( DS_1 \) maps the standard representation of \( \mathfrak{p}_2 \) to the standard representation of \( \mathfrak{p}_1 \) which is not semisimple).

Theorems A,B allow us to compute the superdimension of an irreducible \( \mathfrak{osp}(m|2n) \)-module (since DS is symmetric monoidal it preserves the superdimensions); see [HW] and [EAS] for analogous results in the \( \mathfrak{gl}(m|n) \) and \( \mathfrak{p}(n) \)-case. Our main theorem also allows to reduce certain questions about tensor products to lower rank similarly to [HW2], [H].
1.5. Acknowledgments. The authors are grateful to V. Hinich, V. Serganova and C. Stroppel for numerous helpful discussions.

1.6. Index of definitions and notation. Throughout the paper the ground field is \( \mathbb{C} \); \( \mathbb{N} \) stands for the set of non-negative integers. We denote by \( \Pi \) the parity change functor. In Sections 2—8 \( g = \text{osp}(M|2n) \) with \( M, n \geq 0 \) and \( M \) either \( 2m \) or \( 2m + 1 \). One has \( \text{osp}(1|0) = \text{osp}(0|0) = 0 \) and \( \text{osp}(2|0) = \mathbb{C} \).

\[
\begin{align*}
\Omega(N), \Sigma, \sigma, \mathcal{F}(g), \mathcal{F}(g), \text{Irr}, \Lambda_{m|n}^+, L(\lambda), \chi \quad &\text{Section 2} \\
\text{weight diagram, core symbols, core diagram} \quad &3.1 \\
\mathcal{F}(g) \quad &3.2.1 \\
t, \ell, \Lambda_{m+\ell|n}, \text{core-free}, \Theta_k^{(t)} \quad &3.3 \\
g_r, \Sigma_r, S_r \quad &3.4 \\
tail \quad &3.5 \\
howl, \tau \quad &3.6 \\
dex \quad &3.7.8 \\
\text{stable diagrams} \quad &3.8 \\
\text{translation functor } T_{g'} \quad &4.1 \\
\text{DS}_x \quad &5.2 \\
\text{supp}(x), x_s, \text{DS}_s \quad &5.5 \\
g_x \quad &5.6 \\
\text{graded multiplicity} \quad &6.5 \\
\text{arc diagram} \quad &8.1
\end{align*}
\]

2. Notation

2.1. Root lattice. Our notation and a choice of triangular decomposition follow [GS1],[GS2].

We denote by \( \Delta \) the set of roots of \( g \) and by \( \Delta_0 \) (resp., \( \Delta_1 \)) the set of even (resp., odd) roots. In this paper all modules are weight modules (i.e., \( \mathfrak{h} \) acts diagonally) with finite-dimensional weight spaces; for such a module \( N \) we set

\[
\Omega(N) := \{ \nu \in \mathfrak{h}^* \mid N_\nu \neq 0 \}.
\]

The root system \( \Delta \) lies in the lattice \( \Lambda_{m|n} \subset \mathfrak{h}^* \) spanned by \( \{ \varepsilon_i \}_{i=1}^m \cup \{ \delta_i \}_{i=1}^n \). We denote by \( \Lambda \) the lattice spanned by \( \{ \varepsilon_i \}_{i=1}^m \cup \{ \delta_i \}_{i=1}^n \) and view \( \Lambda_{m|n} \) as a subset of \( \Lambda \). We define the parity homomorphism \( p : \Lambda \to \mathbb{Z}/2 \) by \( p(\varepsilon_i) = \overline{0}, p(\delta_j) = \overline{1} \) for all \( i, j \).

2.2. Triangular decomposition. We fix a Cartan subalgebra \( \mathfrak{h} \subset g_0 \). We fix a triangular decomposition corresponding to a ”mixed” base \( \Sigma \), i.e. a base containing maximal
possible number of odd roots. For $\mathfrak{osp}(2m + 1|2n)$ with $m, n > 0$ we take

$$
\Sigma := \begin{cases} 
\delta_1 - \delta_2, \delta_2 - \delta_3, \ldots, \delta_{m-1} - \delta_m, \varepsilon_1 - \delta_m, \delta_m - \delta_{m+1}, \ldots, \varepsilon_m - \delta_n, \delta_n & \text{for } n > m \\
\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \ldots, \varepsilon_{m-n+1} - \delta_1, \delta_1 - \varepsilon_{m-n+2}, \ldots, \varepsilon_m - \delta_n, \delta_n & \text{for } m \geq n
\end{cases}
$$

and for $\mathfrak{osp}(2m|2n)$ with $m, n > 0$ we take

$$
\Sigma := \begin{cases} 
\delta_1 - \delta_2, \delta_2 - \delta_3, \ldots, \delta_{m-1} - \delta_m, \varepsilon_1 - \delta_m, \delta_m - \delta_{m+1}, \ldots, \varepsilon_m - \delta_n, \delta_n \pm \varepsilon_m & \text{for } n \geq m \\
\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \ldots, \varepsilon_{m-n+1} - \delta_1, \delta_1 - \varepsilon_{m-n+2}, \ldots, \varepsilon_m - \delta_n, \delta_n \pm \varepsilon_m & \text{for } m > n
\end{cases}
$$

For the remaining case $mn = 0$ all triangular decompositions are conjugate and we fix a standard base.

Recall that $\mathfrak{osp}(1|0) = \mathfrak{osp}(0|0) = 0$ and $\mathfrak{osp}(2|0) = \mathbb{C}$; in these cases $\mathfrak{h} = \mathfrak{g}$ and $\Delta = \Sigma = 0$. Note that $\Lambda_{00} = 0$ and $\Lambda_{10} = \mathbb{Z}\varepsilon_1$.

We denote by $\rho$ the Weyl vector of $\mathfrak{g}$.

2.2.1. Involution $\sigma$. The superalgebra $\mathfrak{osp}(2m|2n)$ with $m, n > 0$ or $m > 1, n = 0$ admits an involutive automorphism $\sigma$ induced by the automorphism of the Dynkin diagram of $\Sigma$ (the resulting action on $\mathfrak{h}^*$ is given by the reflection $r_{\varepsilon_m}$). The restriction of $\sigma$ to $\mathfrak{so}_{2n} \subset \mathfrak{g}_0$ is the standard involution induced by the automorphism of the Dynkin diagram of $\mathfrak{so}_{2m}$. For $\mathfrak{osp}(2|0) = \mathbb{C}$ we set $\sigma := -\text{Id}$. For $\mathfrak{osp}(0|2n) = \mathfrak{sp}_{2n}$ we set $\sigma := \text{Id}$. Note that for all cases $\sigma(\Sigma) = \Sigma$.

2.3. Category $\mathcal{F}(\mathfrak{g})$. The category $\mathcal{Fin}(\mathfrak{g})$ of finite dimensional representations of $\mathfrak{g}$ with parity preserving morphisms is the direct sum of two categories: $\mathcal{F}(\mathfrak{g})$ with the modules whose weights lie in $\Lambda_{m|n}$ and $\mathcal{F}^+(\mathfrak{g})$ with the modules whose weights lie in $\mathfrak{h}^* \setminus \Lambda_{m|n}$. The category $\mathcal{F}^+(\mathfrak{g})$ is semisimple and $DS_{x}(\mathcal{F}^+(\mathfrak{g})) = 0$ for $x \neq 0$; all simple modules in $\mathcal{F}^+(\mathfrak{g})$ are typical, their characters are given by the Weyl-Kac character formula.

The category $\mathcal{F}(\mathfrak{g})$ is canonically isomorphic to the category of $SOSp(m|n)$-modules; this category decomposes into a direct sum:

$$
\mathcal{F}(\mathfrak{g}) = \mathcal{F}(\mathfrak{g}) \oplus \Pi(\mathcal{F}(\mathfrak{g})),
$$

where $\mathcal{F}(\mathfrak{g})$ is the full subcategory with the modules $M$, where the parity is induced from the parity on $\Lambda_{m|n}$, i.e. $M \in \mathcal{F}(\mathfrak{g})$ if and only if each weight space $p(M_\nu) = p(\nu)$ for all $\nu \in \Lambda_{m|n}$. We will sometimes omit $\mathfrak{g}$ from notation if this does not lead to ambiguity; for instance, we may use $\mathcal{F}$ instead of $\mathcal{F}(\mathfrak{g})$.

2.3.1. For each category $\mathcal{C}$ we denote by $\text{Irr}(\mathcal{C})$ the set of isomorphism classes of irreducible modules in $\mathcal{C}$. For each finite-dimensional $N \in \mathcal{C}$ and $L \in \text{Irr}(\mathcal{C})$ we consider the graded multiplicity: we write $[N : L] = (d_0|d_1)$ if a Jordan-Hölder series of $N$ contains $d_0$ copies of $L$ and $d_1$ copies of $\Pi(L)$.

We denote by $L(\lambda)$ a simple $\mathfrak{g}$-module of the highest weight $\lambda$ We set

$$
\Lambda_{m|n}^+ := \{ \lambda \in \Lambda_{m|n} \mid \dim L(\lambda) < \infty \}.
$$
For $\lambda \in \Lambda_{m|n}^+$ we fix the grading in such a way that $L(\lambda) \in \mathcal{F}(\mathfrak{g})$. Then

$$\text{Irr}(\mathcal{F}(\mathfrak{g})) = \{L(\lambda) | \lambda \in \Lambda_{m|n}^+\}.$$ 

We denote by $L_{\mathfrak{g}}(\nu)$ the simple $\mathfrak{g}$ of the highest weight $\nu$. 

We denote by $\chi_\lambda$ the central character of $L(\lambda)$.

### 3. Weights, roots and diagrams

#### 3.1. Weight diagrams

Many properties of a finite dimensional representation $L(\lambda)$ can be better understood by assigning a weight diagram to the weight $\lambda$ (see e.g. [GS1]). Note that the conventions how to draw these weight diagrams differ. We follow essentially [GS1] and list some differences below. These weight diagrams were first introduced in the $\mathfrak{gl}(m|n)$-case in [BS] in their work on Khovanov algebras of type $A$. The weight diagrams introduced in [ES] for representations of the orthosymplectic supergroup differ considerably from ours (see [ES], Section 6).

Take $\lambda \in \Lambda_{m|n}^+$ and write

$$\lambda + \rho =: \sum_{i=1}^{m} a_i \varepsilon_i + \sum_{j=1}^{n} b_j \delta_j.$$ 

For $\mathfrak{osp}(2m|2n)$-case all coefficients $a_i, b_j$ are integers which are non-negative except for $a_m$; for $\mathfrak{osp}(2m+1|2n)$-case the numbers $a_i + 1/2, b_j - 1/2$ are non-negative integers. Moreover, $a_i \geq a'_i, b_j \geq b'_j$ for $i < i', j < j'$ and the equalities imply $a_i = 0, b_j = 0$ in $\mathfrak{osp}(2m|2n)$-case, $a_i = -\frac{1}{2}, b_i = \frac{1}{2}$ in $\mathfrak{osp}(2m+1|2n)$-case. The weight $\lambda$ is typical if $a_i \neq b_j$ for all $i, j$.

#### 3.1.1. We assign to $\lambda$ the (weight) diagram using the following procedure: we label the numberline $\mathbb{N}$ as follows:

- for $\mathfrak{osp}(2m|2n)$ we put $>$ (resp., $<$) at the position with the coordinate $t$ if $|a_i| = t$ (resp., $|b_i| = t$) for some $i$;
- for $\mathfrak{osp}(2m+1|2n)$ we put $>$ (resp., $<$) at the position with the coordinate $t - 1/2$ if $|a_i| = t$ (resp., $|b_i| = t$).

If $>, <$ occupy the same position we put the sign $\times$ ($\times^s$ stands for $s$ signs $<$ and $s$ signs $>$; $\times^p$ stands for $s$ signs $<$ and $s+1$ signs $>$). We put an empty sign $\circ$ at the non-occupied positions with the coordinates in $\mathbb{N}$. We call the resulting diagram an unsigned weight diagram.

For $\mathfrak{osp}(2m|2n)$ with $m \geq 1$ we add the sign $+$ (resp., $-$) if $a_m > 0$ (resp., $a_m < 0$).

For $\mathfrak{osp}(2m+1|2n)$ we put the sign $+$ (resp., $-$) before the diagram if the zero position is occupied by $\times^p$ for $p > 0$ and $(\lambda + \rho|\varepsilon_i) = \frac{1}{2}$ for some $i$ (resp., $(\lambda + \rho|\varepsilon_i) \neq \frac{1}{2}$ for each $i$).
We call the resulting diagram a weight diagram of \( \lambda \).

Notice that for \( \mathfrak{osp}(2m|2n) \)-case the action of automorphism \( \sigma : \lambda \mapsto \lambda^\sigma \) corresponds to the change of signs of the diagrams; we denote this operation (the change of sign) also by \( \sigma \).

3.1.2. Note that for \( \mathfrak{osp}(2m+1|2n) \)-case our weight diagram is obtained from the diagram used in [GS1] by the shift by \(-1/2\).

3.1.3. Examples: \( \mathfrak{osp}(2m|2n) \). The empty diagram corresponds to \( \mathfrak{osp}(0|0) = 0 \); the diagram \( > \) (respectively, \( -o > \)) corresponds to the weight 0 (respectively, \(-\varepsilon_1\)) for \( \mathfrak{osp}(2|0) = \mathbb{C} \). One has \( L(\emptyset) = \mathbb{C} \) and \( L(>) \) is the trivial \( \mathfrak{osp}(2|0) \)-module.

For \( s > 0 \) the diagram \( \times^s \) is assigned to the weight 0 for \( \mathfrak{osp}(2s|2n) \); the diagrams \( \times^s >, \times^s >> \) are assigned to the \( \mathfrak{osp}(2s+4|2s) \)-weights 0 and \( \varepsilon_1 + \varepsilon_2 \) respectively; the diagrams \( \pm o \times o \times \) are assigned to the \( \mathfrak{osp}(4|4) \)-weights \( 3(\delta_1 + \varepsilon_1) + (\delta_2 \pm \varepsilon_2) \).

3.1.4. Examples: \( \mathfrak{osp}(2m+1|2n) \). The empty diagram corresponds to \( \mathfrak{osp}(1|0) = 0 \); the diagram \( > \) (resp., \( < \)) corresponds to the weight 0 for \( \mathfrak{osp}(3|0) = o_3 \) (resp., for \( \mathfrak{osp}(1|2) \)).

For \( s > 0 \) the diagram \( -\times^s \) is assigned to the weight 0 for \( \mathfrak{osp}(2s+1|2s) \); the diagram \( +\times^s \) is assigned to the weight \( \varepsilon_1 \) for \( \mathfrak{osp}(2s+1|2s) \).

The diagram \( -\times^n >> \) is assigned to the \( \mathfrak{osp}(2n+5|2n) \)-weight 0 and the diagram \( +\times^n << \) is assigned to \( \mathfrak{osp}(2n+1|2n+4) \)-weight \( \varepsilon_1 \).

3.1.5. The above procedure gives a one-to-one correspondence between \( \Lambda^+_{m|n} \) and the diagrams containing \( k \) symbols \( \times \), \( m - k \) symbols \( > \) and \( n - k \) symbols \( < \) (where \( k \leq \min(m,n) \)) with the following additional properties:

(i) the coordinates of the occupied position lie in \( \mathbb{N} \) and each non-zero occupied position contains exactly one of the signs \( \{>, <, \times\} \);

(ii) for \( \mathfrak{osp}(2m|2n) \) with \( m > 0 \) the zero position contains any number of \( \times \), no \( < \) and at most one \( > \); the diagram has a sign if and only if the zero position is empty;

(iii) for \( \mathfrak{osp}(2m+1|2n) \) the zero position contains any number of \( \times \) and at most one of the symbols \( >, < \); the diagram has a sign if and only if the zero position is occupied by \( \times^i \) for \( i \geq 1 \).

3.1.6. The atypicality of \( \lambda \) is equal to the number of symbols \( \times \) in the diagram of \( \lambda \).

3.1.7. Notation. We sometimes identify a dominant weight and its weight diagram; for instance, \( f \in \Lambda^+_{m|n} \) means that \( f \) is a weight diagram assigned to a weight in \( \Lambda^+_{m|n} \).

We always consider \( t \in \{0, 1, 2\} \) to be fixed. For a weight diagram \( f \) we sometimes use the notation \( L(f) \) for the corresponding highest weight module. For instance, \( L(\emptyset) = \mathbb{C} \) and \( L(>) \) is the trivial \( \mathfrak{osp}(2|0) \)-module; \( L(\times^s) \) (resp., \( L(-\times^s) \)) stands for the trivial
\(osp(2s|2s)\) (resp., \(osp(2s+|2s)\)-module) and \(L(+)\) stands for the standard \(osp(2s+1|2s)\)-module.

### 3.1.8. Remark: OSp-modules.

Take \(g = osp(2m|2n)\). By [ES, Proposition 4.11] the simple \(OSp(2m|2n)\)-modules are either of the form \(L(\lambda)\) if \(\lambda \in \Lambda_{m|n}^+\) is \(\sigma\)-invariant or \(L(\lambda) \oplus L(\lambda^\sigma)\). Thus the simple \(OSp(2m|2n)\)-modules are in one-to-one correspondence with the unsigned \(osp(2m|2n)\)-diagrams. For \(osp(2m+1|2n)\) and any \(\lambda \in \Lambda_{m|n}^+\) there are two irreducible \(OSp(2m+1|2n)\)-modules \(L(\lambda, +)\) and \(L(\lambda, -)\) which restrict to \(L(\lambda)\).

### 3.1.9. We denote by \(f_- f_+\) the diagram obtained by "gluing" the diagrams \(f_-\) and \(f_+\) (where \(f_+\) does not have sign); for instance,

\[
\times \circ \times = f_- f_+ \quad \text{where} \quad f_- = \times \circ, \quad f_+ = \times \\
+ x^2 \circ x = f_- f_+ \quad \text{where} \quad f_- = +x^2, \quad f_+ = \circ x
\]

### 3.2. Core diagrams and central characters.

We say that a \(g\)-central character is \textit{dominant} if \(F(g)\) contains modules with this central character. By [GS1], the blocks in \(F(g)\) are parametrized by the dominant central characters and the dominant central characters can be described in terms of typical dominant weights, see below.

#### 3.2.1. We call the symbols \(>, <\) the \textit{core symbols}. A \textit{core diagram} is a weight diagram which does not contain symbols \(\times\) and does not have \(-\) sign.

For a weight diagram \(f\) we denote by \(\text{core}(f)\) the core diagram which is obtained from the diagram of \(f\) by replacing all symbols \(\times\) by \(\circ\) and by adding the sign \(+\) for \(osp(2m|2n)\)-case if the zero position is empty.

For \(\lambda \in \Lambda_{m|n}^+\) we denote by \(\text{core}(\lambda)\) the weight corresponding to \(\text{core}(f)\), where \(f\) is the diagram of \(\lambda\).

For instance, for \(osp(4|2)\) one has

\[
\text{core}(\times >) = \text{core}(\pm \circ > \times) = +\circ >, \quad \text{core}(\varepsilon_1) = \text{core}(\pm \varepsilon_2 + \varepsilon_1 + \delta_1) = \varepsilon_1.
\]

Note that for \(osp(2m|2n)\) one has \(\text{core}(\lambda) = \text{core}(\lambda^\sigma)\).

#### 3.2.2. For \(osp(2m+1|2n)\)-case the dominant central characters are parametrized by the core diagrams, i.e. for \(\lambda, \nu \in \Lambda_{m|n}^+\)

\[
\chi_\lambda = \chi_\nu \implies \text{core}(\lambda) = \text{core}(\nu);
\]

for \(osp(2m|2n), osp(2m+2|2n)\) one has

\[
\chi_\lambda \in \{\chi_\nu, \chi_\nu^\sigma\} \implies \text{core}(\lambda) = \text{core}(\nu).
\]

(Note that the atypical central characters of \(osp(2m|2n)\) are \(\sigma\)-invariant: \(\chi_\nu = \chi_\nu^\sigma\) if \(\nu\) is atypical.)
3.3. **Categories** $\mathcal{F}^g(\mathfrak{g})$. For each core diagram $g$ we denote by $\mathcal{F}^g(\mathfrak{g})$ the Serre subcategory of $\mathcal{F}(\mathfrak{g})$ generated by $M(\lambda)$ for $\lambda$ having diagrams $f$ with $\text{core}(f) = g$.

By above, the categories $\mathcal{F}^g(\mathfrak{osp}(2m+1|2n))$ are the blocks in $\mathcal{F}(\mathfrak{osp}(2m+1|2n))$. For $\mathfrak{g} := \mathfrak{osp}(2m|2n)$ all typical blocks and $\sigma$-invariant typical blocks are of the form $\mathcal{F}^g(\mathfrak{g})$; for a typical block $B$ with $B \neq B'$ we have $\mathcal{F}^g(\mathfrak{g}) = B \oplus B'$ for a suitable diagram $g$.

Similarly we use the notation $\tilde{\mathcal{F}}^g(\mathfrak{g})$ for $\mathcal{F}^g(\mathfrak{g}) \oplus \Pi \mathcal{F}^g(\mathfrak{g})$.

3.3.1. Let $g$ be a core diagram with $m'$ symbols $>$ and $n'$ symbols $<$. The category $\mathcal{F}^g(\mathfrak{osp}(2m+1|2n))$ is non-zero if and only if $m - m' = n - n' \geq 0$ and that $\mathcal{F}^g(\mathfrak{osp}(2m|2n))$ is non-zero if and only if $m - m' = n - n' \geq 0$ and, in addition, $g$ does not have $<\text{ at the zero position for } m > 0$.

The modules in $\mathcal{F}^g(\mathfrak{g})$ have atypicality $m - m' = n - n'$.

3.4. **Cases** $t = 0, 1, 2$. Recall that $\mathfrak{osp}(M|N)$ consists of two series: $B$ (for odd $M$) and $D$. We will distinguish the following cases ($t = 0, 1, 2$):

- for the $B$-series (and any core diagram $g$) we put $t := 1$;
- for the $D$-series and a core diagram $g$ without $>$ at the zero position we put $t := 0$;
- for the $D$-series and a core diagram $g$ with $>$ at the zero position we put $t := 2$.

3.4.1. We say that a block has type $t$ ($t = 0, 1, 2$) if the core diagram of the corresponding central character has type $t$. We say that $\lambda \in \Lambda^+_{m|n}$ has type $t$ if the diagram of core($\lambda$) has type $t$. Then $L(\lambda)$ lies in a block of type $t$ if and only if $\lambda \in \Lambda^+_{m|n}$ has type $t$.

3.4.2. For $t = 0, 1, 2$ we take $\mathfrak{g} = \mathfrak{osp}(2m+t|2n)$; the weights lattice of $\mathfrak{g}$ is $\Lambda_{m+\ell|n}$, where

$$\ell := \begin{cases} 0 & \text{for } t = 0, 1 \\ 1 & \text{for } t = 2. \end{cases}$$

We denote by $\Lambda_{m+\ell|n}^{(t)}$ the set of dominant weights of type $t$ in $\Lambda^+_{m+\ell|n}$:

- $\Lambda_{m+\ell|n}^{(1)} = \Lambda^+_{m|n}$ for the $B$-series;
- $\Lambda_{m+\ell|n}^{(0)}$ is the set of $\mathfrak{osp}(2m|2n)$-dominant weights in $\Lambda^+_{m|n}$ with the diagrams without symbol $>$ at the zero position;
- $\Lambda_{m+\ell|n}^{(2)}$ is the set of $\mathfrak{osp}(2m + 2|2n)$-dominant weights in $\Lambda^+_{m+1|n}$ with the diagrams having $>$ at the zero position.

We call $\lambda \in \Lambda_{m+\ell|n}^{(t)}$ **core-free** if core($\lambda$) = $\emptyset$ for $t = 0, 1$ and core($\lambda$) => for $t = 2$.

---

1by Serre subcategory generated by a set of simple modules we mean the full subcategory consisting of the modules of finite length whose all simple subquotients lie in a given set.
3.4.3. Observe that for \( \lambda \in \Lambda^{(t)}_{m+\ell|n} \) we have

\[
\lambda + \rho =: \sum_{i=1}^{m+\ell} a_i \varepsilon_i + \sum_{j=1}^{n} b_j \delta_j = \sum_{i=1}^{m} a_i \varepsilon_i + \sum_{j=1}^{n} b_j \delta_j,
\]

since for the case \( \ell \neq 0 \) one has \( \ell = 1, a_{m+1} = 0 \).

For \( t = 0 \) the weight \( \lambda \in \Lambda^{(t)}_{m+\ell|n} \) has atypicality \( k \) if and only if \( \text{core}(\lambda) \in \Lambda^+_{m-k|n-k} \).

For \( t = 1 \) the weight \( \lambda \in \Lambda^{(t)}_{m+\ell|n} \) has atypicality \( k \) if and only if \( \text{core}(\lambda) \in \Lambda^+_{m+1-k|n-k} \).

For \( t = 2 \) the weight \( \lambda \in \Lambda^{(t)}_{m+\ell|n} \) has atypicality \( k \) if and only if \( \text{core}(\lambda) \in \Lambda^+_{m-k|n-k} \cap \Lambda^+_{m+1-k|n-k} \).

Hence in all cases

\[
\text{core}(\lambda)
\]

is a typical weight in \( \Lambda^+_{m-k|n-k} \).

3.4.4. For \( t = 0, 1, 2 \) we denote by \( \Theta_k^{(t)} \) the set of dominant central characters of atypicality \( k \) corresponding to the \( t \)-case. By above, the map

\[
\chi \mapsto \text{core}(\chi)
\]

gives a correspondence between \( \Theta_k^{(t)} \) and the set of typical weights \( \Lambda^+_{m-k|n-k} \). For \( t = 1 \) this is a one-to-one correspondence. For \( t = 0, 2 \) the image consists of the typical weights \( \eta \) satisfying \( (\eta|\varepsilon_{m-k}) \neq 0 \); the map is injective except for the case \( t = 0 \) and \( k = 0 \).

3.5. Algebra \( \mathfrak{g}_r \). For \( t = 0, 1, 2 \) and \( r > 0 \) we set

\[
\mathfrak{g}_r := \mathfrak{osp}(2r + t|2r).
\]

Let \( \Sigma \) be the base of simple roots for \( \mathfrak{g} = \mathfrak{osp}(2m+t|2n) \). For \( 0 < r \leq \min(m;n) \) we denote by \( \Sigma_r \) a subset of \( \Sigma \) which is a base of an algebra isomorphic to \( \mathfrak{g}_r \) (such \( \Sigma_r \) is unique):

\[
\Sigma_r := \left\{ \varepsilon_{m-r+1} - \delta_{n-r+1}, \delta_{n-r+1} - \varepsilon_{m-r+2}, \ldots, \varepsilon_m - \delta_n, \delta_n \right\} \quad \text{for } t = 1
\]

\[
\delta_{n-r+1} - \varepsilon_{m-r+1}, \ldots, \varepsilon_{m-r} - \delta_n, \delta_n - \varepsilon_m, \delta_n + \varepsilon_m, \varepsilon_{m-r} - \delta_n, \delta_n - \varepsilon_m, \delta_n + \varepsilon_{m+1} \quad \text{for } t = 2.
\]

For \( r = 0 \) we set \( \Sigma_r = \emptyset \) and \( \rho_r = 0 \). Recall that \( \mathfrak{osp}(0|0) = \mathfrak{osp}(1|0) = 0; \) for \( t = 2 \) we identify \( \mathfrak{osp}(2|0) \) with \( \mathbb{C} \varepsilon_m^* \subset \mathfrak{h} \) (where \( \varepsilon_m^* \in \mathfrak{h} \) is such that \( \mu(\mathfrak{h}) = (\mu|\varepsilon_m) \) for each \( \mu \in \mathfrak{h}^* \)).

We identify \( \mathfrak{g}_r \) with the corresponding subalgebra of \( \mathfrak{osp}(2m+t|2n) \) (then \( \Sigma_r \) is the base of of \( \mathfrak{g}_r \)). We denote by \( \rho_r \) the Weyl vector of \( \mathfrak{g}_r \); one has \( \rho_r = \rho|_{\mathfrak{g}_r \cap \mathfrak{h}} \) and

\[
\rho_r = 0 \quad \text{for } t = 0, 2; \quad \rho_r = \frac{1}{2} \sum_{i=0}^{r-1} (\delta_{n-i} - \varepsilon_{m-i}) \quad \text{for } t = 1.
\]
We denote by $S_r$ the following set: $S_0 = \emptyset$ and
\[
S_r := \begin{cases} 
\{ \delta_{n-i} - \epsilon_{m-i} \}_{i=0}^{r-1} & \text{for } t = 0 \\
\{ \epsilon_{m-i} - \delta_{n-i} \}_{i=0}^{r-1} & \text{for } t = 1, 2.
\end{cases}
\]
Notice that $S_r$ consists of $r$ isotropic mutually orthogonal roots and $S_r \subset \Sigma_r$.

3.6. **Tail.** Take $\lambda \in \Lambda_{m+\ell|n}^{(t)}$ and let $f$ be the diagram assigned to $\lambda$.

Let $s \leq \min(m, n)$ be the maximal number satisfying $(\lambda|\Sigma_s) = 0$, where $\Sigma_s$ as above (notice that $\Sigma_s$ depends on $t$). We call $s$ the *length of the tail* of $\lambda$ and write
\[
|\text{tail}(\lambda)| = |\text{tail}(f)| := s.
\]

For $\mathfrak{osp}(2m|2n)$-case $s$ is equal to the number of symbols $\times$ in the zero position. For $\mathfrak{osp}(2m+1|2n)$ the zero position contains $s$ (resp., $s+1$) symbols $\times$ if the diagram has sign $-$ (resp., $+$); for instance
\[
|\text{tail}((-)\times^m)| = |\text{tail}(0)| = m, \quad |\text{tail}(+(+)\times^m)| = |\text{tail}(\epsilon_1)| = m - 1.
\]

3.7. **Howl.** The block of the trivial module is called the *principal block* (there are two principal blocks which differ by $\Pi$).

Each block of atypicality $k$ is equivalent to the principal block of $\mathfrak{osp}(2k+t|2k)$. The equivalences are described in [GS1]; we give some details below. For a dominant weight $\lambda$ we denote by $\text{howl}(\lambda)$ the corresponding weight in the principal block (roughly speaking, the passage from $\lambda$ to $\text{howl}(\lambda)$ essentially amounts to removing the core symbols $<$, $>$ from the weight diagram, see the details below); note that $\text{howl}(\lambda)$ has core-free diagram.

3.7.1. Let $\lambda \in \Lambda_{m+\ell|n}^{(t)}$ be a weight of atypicality $k$ and let $f$ be the corresponding diagram. For each $i = 1, \ldots, k$ let $s_i(f)$ be the number of the positions to the left of $i$th symbol $\times$, which do not contain core symbols.

3.7.2. **Case $t = 1$.** In this case $\mathfrak{g} = \mathfrak{osp}(2m+1|2n)$ and the diagram $\text{howl}(f) \in \Lambda_{k|k}^{(1)}$ is the diagram without core symbols, where $i$th symbol $\times$ occupies $s_i(f)$th position and the sign of $\text{howl}(f)$ is such that the tail lengths of $\text{howl}(f)$ and $f$ are the same. For instance,
\[
\text{howl}(< \times < \times) = + \times \times; \quad \text{howl}(\pm \times^2 \circ \times) = \pm \times^2 \circ \times;
\]
\[
\text{howl}(\overset{\times}{>} \circ \times \circ \times) = - \times \circ \times; \quad \text{howl}(\overset{\times}{>} \circ \times \circ \times) = + \times^2 \circ \times
\]
and $\text{howl}(f) = \emptyset$ if and only if $k = 0$. 
3.7.3. Case $t = 0$. In this case $\mathfrak{g} = \mathfrak{osp}(2m\mid 2n)$ with no $>$ at the zero position. The diagram $\text{howl}(f) \in \Lambda^{(0)}_{k\mid k}$ is the diagram without core symbols, where the $i$-th symbol $\times$ occupies the $s_i(f)$-th position and the sign of $\text{howl}(f)$ coincides with the sign of $f$ if $\text{howl}(f)$ requires the sign. For instance,

$$\text{howl}(\pm \circ \times \times) = \pm \circ \times； \quad \text{howl}(\times^2 \circ \times) = \times^2 \circ \times； \quad \text{howl}(\pm \circ \circ \times) = \emptyset.$$ 

Notice that $|\text{tail}(f)| = |\text{tail}(\text{howl}(f))|$, so the only case when the diagrams $f$ and $\text{howl}(f)$ do not have the same sign is when $\text{howl}(f) = \emptyset$.

3.7.4. Case $t = 2$. In this case $\mathfrak{g} = \mathfrak{osp}(2m + 2\mid 2n)$ and the zero position of $f$ is occupied by $\times^p$ ($p \geq 0$). The diagram $\text{howl}(f) \in \Lambda^{(2)}_{k+1\mid k}$ has $\times^p$ at the zero position; for $i = p + 1, \ldots, k$ the $i$th symbol $\times$ in $\text{howl}(f)$ occupies the position $s_i + 1$. For instance,

$$\text{howl}(\times^2 \circ \circ \times) = \times^2 \circ \circ \times； \quad \text{howl}(\times \times \times) = \times \times \times； \quad \text{howl}(> <) = >.$$

3.7.5. For $\lambda \in \Lambda^{(t)}_{m+\ell\mid n}$ with a diagram $f$ let $\text{howl}(\lambda) \in \Lambda^+$ be the weight corresponding to $\text{howl}(f)$. Notice that for $\lambda \in \Lambda^{(t)}_{m+\ell\mid n}$ one has $\text{howl}(f) \in \Lambda^{(t)}_{k+\ell\mid k}$, where $k$ is atypicality of $\lambda$.

If $\ell \neq 0$, then $t = 2$ and $\text{howl}(f)$ has $>$ at the zero position, so $\text{howl}(f)$ lies in $\Lambda^+_{k\mid k}$. Hence in all cases $\text{howl}(f) \in \Lambda^+_{k\mid k}$.

3.7.6. Note that $\text{howl}$ preserves the tail length:

$$|\text{tail}(f)| = |\text{tail}(\text{howl}(f))|.$$

3.7.7. Connection between the cases $t = 1$ and $t = 2$. Below we describe a remarkable bijection between the core-free diagrams in $\Lambda^{(1)}_{(n\mid n)}$ and in $\Lambda^{(2)}_{(n\mid n)}$.

To a diagram $\times^p \circ f$ we assign the diagram $- \times^p f$ if $p > 0$ and the diagram $\circ f$ if $p = 0$; to a diagram $\times^p \times f$ we assign the diagram $+ \times^{p+1} f$.

This assignment gives a bijection $\tau$ between the core-free diagrams for $\mathfrak{osp}(2n + 2\mid 2n)$ and for $\mathfrak{osp}(2n + 1\mid 2n)$. Notice that

$$|\text{tail}(\tau(f))| = |\text{tail}(f)|.$$

3.7.8. Map $\text{dex}$. We introduce a map $\text{dex} : \Lambda^{(t)}_{m+\ell\mid n} \to \{\pm 1\}$ by

$$\text{dex}(\lambda) := \begin{cases} (-1)^{p(\text{howl}(\lambda))} & \text{for } t = 0, 1 \\ (-1)^{p(\tau(\text{howl}(\lambda)))} & \text{for } t = 2 \end{cases}$$
(where \( p \) is the parity) and the map \( \text{Irr}(\tilde{\mathcal{F}}(\mathfrak{g})) \to \{\pm 1\} \) by
\[
dex(L(\lambda)) := \text{dex}(\lambda), \quad \text{dex}(\Pi(L(\lambda))) = - \text{dex}(\lambda).
\]

3.8. **Stable diagrams.** For \( \mathfrak{osp}(2m + 1|2n) \) a diagram is called stable if all symbols \( \times \) precede all core symbols \((<, >)\). For \( \mathfrak{osp}(2m|2n) \) a diagram is called stable if all symbols \( \times \) precede all core symbols, except, possibly, the symbol \( > \) at the zero position.

A weight \( \lambda \) is called stable if the corresponding diagram is stable.

3.8.1. Take \( \lambda \in \Lambda^{(t)}_{m+\ell|n} \) and write
\[
\lambda + \rho =: \sum_{i=1}^{m} a_i \varepsilon_i + \sum_{j=1}^{n} b_j \delta_j.
\]
If \( \lambda \) has atypicality \( k > 0 \), then
\[
\lambda \text{ is stable } \iff \text{core}(\lambda) = \sum_{i=1}^{m-k} a_i \varepsilon_i + \sum_{j=1}^{n-k} b_j \delta_j.
\]
(The same holds for \( k = 0 \) with \( t = 1, 2. \))

3.8.2. If \( f \) is a stable diagram for \( \mathfrak{osp}(2m + 1|2n) \) (resp., for \( \mathfrak{osp}(2m|2n) \)) the diagram \( \text{howl}(f) \) is obtained from \( f \) by replacing all core symbols (resp., all core symbols in the non-zero position) by the empty symbols.

In other words, for \( \lambda \) as above, \( \lambda \) is stable of atypicality \( k \) if and only if
\[
\text{howl}(\lambda) + \rho_k = \sum_{i=1}^{k} a_{m-k+i} \varepsilon_i + \sum_{j=1}^{k} b_{n-k+j} \delta_j,
\]
where \( \rho_k \) is the Weyl vector of \( \mathfrak{g}_k \).

4. **Stabilization**

We call a module stable if all its simple subquotients are of the form \( L(\lambda) \) for stable weights \( \lambda \). The aim of this section is to show that any module in \( \mathcal{F}(\mathfrak{g}) \) can be moved with a translation functor to a stable module.

4.1. **Translation functors.** Let \( V_{sl} \) be the natural representation. For a core diagrams \( g, g' \) we denote by \( T_{g'}^g \) the translation functors
\[
T_{g'}^g : \mathcal{F}(\mathfrak{g}) \to \mathcal{F}(\mathfrak{g}'), \quad T_{g'}^g : \tilde{\mathcal{F}}(\mathfrak{g}) \to \tilde{\mathcal{F}}(\mathfrak{g}')
\]
which map \( N \) to the projection of \( N \otimes V_{sl} \) to the subcategory \( \mathcal{F}(\mathfrak{g})_{g'} \) (resp., \( \tilde{\mathcal{F}}(\mathfrak{g}') \)).
We write $T_g^f(f) = f'$ if $T_g^f(L(\lambda)) = L(\lambda')$ and $f, f'$ are the weight diagrams assigned to $\lambda, \lambda'$ respectively; similarly, we write $T_g^{f'}(f) = f'_1 \oplus f'_2$ if $T_g^{f'}(L(\lambda)) = L(\lambda'_1) \oplus L(\lambda'_2)$.

4.2. **Some useful translation functors.** Let $\text{Trans}_a$ be the set of the translation functors $T_g^f$, where $g$ is a core diagram with an occupied position $a$ and an empty position $a + 1$ and $g'$ is the core diagram obtained from $g$ by interchanging the symbols in the positions $a, a + 1$. For example, $\text{Trans}_1$ contains $T >><><$, $T >>><$ and so on.

Each translation functor $T_g^f \in \text{Trans}_a$ is an equivalence of categories except for the case $\text{osp}(2m|2n)$ with $a = 0$. In the $\text{osp}(2m + 1|2n)$-case (resp., in the $\text{osp}(2m|2n)$-case) let $\text{Trans}$ be the functors which can be written as compositions of functors from $\text{Trans}_a$ for all $a$ (resp., $a \neq 0$). All functors in $\text{Trans}$ are equivalences of categories.

4.2.1. **Case $a \neq 0$.** In this case each $T \in \text{Trans}_a$ is an equivalence of categories acting on simple modules by interchanging the symbols in positions $a, a + 1$ in the corresponding diagrams (and preserving the sign); for example,

$$T_{\circ > \circ}^{\circ > \circ}(x > *) = x * >, \quad T_{\circ > \circ}^{\circ > \circ}(+ \circ > *) = + \circ * >$$

where $* \in \{\times, \circ\}$.

4.2.2. **Case $\mathfrak{g} = \text{osp}(2m + 1|2n), a = 0$.** In this case each $T \in \text{Trans}_0$ is an equivalence of categories acting on simple modules by the following rules:

$$> \circ f \mapsto \circ > f \quad \text{ for each } i > 0 \text{ and similar formulae, where } > \text{ is changed by } <.$$

4.3. **Corollary.**

(i) For $T \in \text{Trans}$ one has $\text{hwl}(T(f)) = \text{hwl}(f)$.

(ii) For any $\lambda \in \Lambda^+_m, \lambda' \in \Lambda^+_{m-i|n-i}$ satisfying $\text{core}(\lambda) = \text{core}(\lambda')$

there exists $T \in \text{Trans}$ such that $T(\lambda), T(\lambda')$ are stable.

(iii) For any module $N \in \mathcal{F}^g(\mathfrak{g})$ there exists $T \in \text{Trans}$ such that $T(N)$ is stable.

5. **DS-functor**

In this section we recall the construction of the DS-functor and describe the algebra $\mathfrak{g}_x$. We prove Corollaries 5.9, 5.10 which will be used later. We distinguish the cases $t = 0, 1, 2$ and take $\mathfrak{g} = \text{osp}(2m + t|2n)$. 
5.1. The DS-functor was introduced in [DS]. We recall definitions and some results below. For a \( g \)-module \( M \) and \( g \in g \) we set

\[
M^g := \text{Ker}_M g.
\]

5.2. **Construction.** We set \( g_x := g^x/[x, g] \); note that \( g^x \) and \( g_x \) are Lie superalgebras. For a \( g \)-module \( M \) we set

\[
\text{DS}_x(M) = M^x/xM.
\]

Observe that \( M^x, xM \) are \( g^x \)-invariant and \([x, g]M^x \subset xM\), so \( \text{DS}_x(M) \) is a \( g^{{\text{ad}}^x} \)-module and \( g_x \)-module. Thus \( \text{DS}_x : M \to \text{DS}_x(M) \) is a functor from the category of \( g \)-modules to the category of \( g_x \)-modules.

There are canonical isomorphisms \( \text{DS}_x(\Pi(N)) = \Pi(\text{DS}_x(N)) \) and

\[
\text{DS}_x(M) \otimes \text{DS}_x(N) = \text{DS}_x(M \otimes N).
\]

If \( N \) is a finite-dimensional \( g \)-module, then \( \text{DS}_x(N^*) \cong (\text{DS}_x(N))^* \).

5.2.1. **Algebraic representations.** The DS functor restricts to a functor

\[
\text{DS}_x : \tilde{\mathcal{F}}(g) \to \tilde{\mathcal{F}}(g_x).
\]

It does not however preserve the subcategory \( \mathcal{F}(g) \). As already noted in [CH] it induces a symmetric monoidal functor between the algebraic representations of \( OSp(m|2n) \) and \( OSp(m-2r|2n-2r) \).

5.2.2. **DS and core diagrams.** By [DS], Sect. 7 (see also Thm. 2.1 in [S1]), the DS-functors preserve the core diagrams, i.e. for a core diagram \( g \) one has

\[
\text{DS}_x(\mathcal{F}^g(g)) \subset \tilde{\mathcal{F}}^g(g_x),
\]

where \( g_x := \text{DS}_x(g) \). Warning: \( \text{DS}_x(\mathcal{F}^g(g)) \) is in general not in \( \mathcal{F}^g(g_x) \), since \( \text{DS}_x \) does not preserve \( \mathcal{F}(g) \).

5.2.3. **DS and Translation functors.** Since DS is a symmetric monoidal functor

\[
\text{DS}_x(N \otimes V_{st}) = \text{DS}_x(N) \otimes \text{DS}_x(V_{st}).
\]

Since \( \text{DS}_x(V_{st}) \) is the natural representation of \( g_x \), the translation functors ”commute with the DS-functors”, i.e. the following diagram is commutative

\[
\begin{array}{ccc}
\mathcal{F}^g_1(g) & \xrightarrow{T^g_1} & \mathcal{F}^g_2(g) \\
\text{DS}_x \downarrow & & \text{DS}_x \downarrow \\
\tilde{\mathcal{F}}^g_1(g_x) & \xrightarrow{T^g_1} & \tilde{\mathcal{F}}^g_2(g_x)
\end{array}
\]
5.3. Corollary. For any $\lambda \in \Lambda^+_{m|n}$, $\nu \in \Lambda^+_{m-s|n-s}$ with
\[ \text{core}(\lambda) = \text{core}(\nu) \]
there exist stable weights $\lambda_{st} \in \Lambda^+_{m|n}$, $\nu_{st} \in \Lambda^+_{m-s|n-s}$ such that
\[ \text{core}(\lambda_{st}) = \text{core}(\nu_{st}); \quad \text{howl}(\lambda) = \text{howl}(\lambda_{st}), \quad \text{howl}(\nu) = \text{howl}(\nu_{st}) \]
and
\[ [DS_x(L(\lambda) : L(\nu))] = [DS_x(L(\lambda_{st}) : L(\nu_{st}))] \quad \text{for each} \ x \ \text{of rank} \ s. \]

Proof. This follows from Corollary 4.3 and the fact that the translation functors commute with $DS_x$ (see 5.2.2).

5.4. $DS$ and automorphisms. Let $\phi : g' \to g$ be a homomorphism of Lie superalgebras; for each $g$-module $N$ denote by $N^\phi$ the $g'$-module (the vector space $N$ with the action $g'.v := \phi(g')v$). Each $\phi \in \text{Aut}(g)$ induces an isomorphism $\bar{\phi} : g_x \to g_{\phi(x)}$ and
\[ DS_{\phi(x)}(N^\phi) = (DS_x(N))^\phi. \]

Let $a \in g_0$ be an ad-nilpotent element and $\phi := e^{ad_a}$ be the corresponding inner automorphism of $g$. If $a$ acts nilpotently on a $g$-module $N$, then $e^a : N \to N^\phi$. Therefore
\[ DS_{\phi(x)}(N) = (DS_x(N))^\phi. \]

Let $G_0 = O_{2m+\ell} \times Sp_{2n}$ be the adjoint group of $g_T$, i.e. the subgroup of $\text{Aut} g$ generated by $e^{ad_a}$, where $a \in g_0$ is ad-nilpotent. By above, if $N$ is a finite-dimensional $g$-module, then
\[ DS_{\phi(x)}(N) = (DS_x(N))^\phi \]
for any inner automorphism $\phi \in G_0$.

5.5. Choice of $x$. We call $S \subset \Delta_{1,\ell}$ an isotropic set if $S$ is a basis of an isotropic subspace in $h^*$. Write $g \in g_T$ as
\[ g = \sum_{a \in \text{supp}(g)} g_a, \]
where $g_a \in g_\alpha \setminus \{0\}$.

By [DS], Sect. 5 the $G_0$-orbits in
\[ X_{iso} := \{ x \in g_1 \mid [x, x] = 0 \} \]
are enumerated by $0, 1, \ldots, \min(m + \ell, n)$: the 0-th orbit is $\{0\}$; the $s$-th orbit contains all elements $x$ where $\text{supp}(x)$ is an isotropic set of cardinality $s$. We say that $x \in X_{iso}$ has rank $s$ if $x$ lies in the $s$-th orbit.
The rank of \( x \in X_{\text{gad}} \) is at most \( \min(m + \ell, n) \). Assume that the rank of \( x \) is greater than \( m \). In this case \( g = \mathfrak{osp}(2n|2n) \), \( t = 2 \) and \( x \) has rank \( n \). However blocks of type \( t = 2 \) in \( \mathcal{F}(\mathfrak{osp}(2n|2n)) \) have atypicality at most \( n - 1 \), so \( DS_x \) annihilates the modules in such blocks.

Thus we can (and will) always assume that the rank of \( x \) is \( s \), where
\[
0 < s \leq \min(m, n).
\]

5.5.1. For each \( s \) as in (3) we fix \( x_s \) of rank \( s \) with 
\[
\text{supp}(x_s) := S_s
\]
and set \( DS_s := DS_{x_s} \). Notice that \( x_s \in g_s \).

5.5.2. In general different \( x \) (even of the same rank) give rise to different functors \( DS_x \). For \( \mathfrak{osp}(m|2n) \) any \( x \) of rank 1 induces the same functor \( DS_x : \mathcal{F}(g) \to \mathcal{F}(g) \) by [CH], Lemma 7.7 (this is not true for \( \mathfrak{gl}(m|n) \); this will later imply that \( DS_x(L(\lambda)) \) depends only on the rank of \( x \). By [CH] \( \ker(DS_1) = \text{Proj} \), the thick ideal of projective objects.

5.6. The algebra \( g_x \). Take \( x := x_s \). Set 
\[
\Delta^+ := \{ \alpha \in \Delta^+ | (\alpha|S_s) = 0 \} \setminus S_s
\]
and denote by \( g_x \) the algebra generated by \( g_{\pm \alpha} \) with \( \alpha \in \Delta^+ \). Clearly, \( g_x \) is a subalgebra of \( g^{\text{ad}}_x \). By [DS], \( DS_x(g) = g^{\text{ad}}_x/[x, g] \) can be identified with \( g_x \). One has 
\[
g_x \cong \mathfrak{osp}(2(m - s) + t|2(n - s))
\]
and \( h_x := g_x \cap h \) is a Cartan subalgebra of \( g_x \). The triangular decomposition given by 
\[
\Delta^+(g_x) := \Delta^+_x
\]
is of the same form as the triangular decomposition fixed in 2.2. We denote the corresponding base by \( \Sigma^x \) and the Weyl vector by \( \rho_x \).

5.6.1. Examples. Take \( s := 2 \).

For \( \mathfrak{osp}(11|8) (m = 5, n = 4, t = 1) \) we have \( g_x \cong \mathfrak{osp}(7|4) \) with 
\[
\Sigma = \{ e_1 - e_2, e_2 - d_1, d_1 - e_3, e_3 - d_2, d_2 - e_4, d_4 - d_3, d_3 - e_5, e_5 - d_4, d_4 \}
\]
\[
\Sigma^x = \{ e_1 - e_2, e_2 - d_1, d_1 - e_3, e_3 - d_2, d_2 - e_4, d_4 - d_3, d_3 - e_5, e_5 - e_2 + e_1 \}
\]
\[
2\rho = d_4 + d_3 + d_2 + d_1 - e_5 - e_4 - e_3 - e_2 + e_1,
\]
\[
2\rho_x = d_2 + d_1 - e_3 - e_2 + e_1.
\]

For \( \mathfrak{osp}(12|8) \) we have \( g_x \cong \mathfrak{osp}(8|4) \) and \( \rho = 0, \rho_x = 0 \). In this case 
\[
\Sigma = \{ e_1 - e_2, e_2 - d_1, d_1 - e_3, e_3 - d_2, d_2 - e_4, e_4 - d_3, d_3 - e_5, e_5 - d_4, d_4 \pm e_6 \}
\]
for \( t = 0 \) we have \( m = 6, n = 4 \) and 
\[
\Sigma^x = \{ e_1 - e_2, e_2 - d_1, d_1 - e_3, e_3 - d_2, d_2 \pm e_4 \};
\]

\[
\Sigma^x = \{ e_1 - e_2, e_2 - d_1, d_1 - e_3, e_3 - d_2, d_2 \pm e_4 \};
\]

and for $t = 2$ we have $m = 5, n = 4$ and
\[ \Sigma^x = \{ \varepsilon_1 - \varepsilon_2, \varepsilon_2 - \delta_1, \delta_1 - \varepsilon_3, \varepsilon_3 - \delta_2, \delta_2 \pm \varepsilon_6 \}. \]

5.6.2. Recall that $\mathfrak{h}^*$ has a basis $\{ \varepsilon_i \}_{i=1}^{m+t} \cup \{ \delta_i \}_{i=1}^{n}$ (where $\ell = 0$ for $t = 0, 1$ and $\ell = 1$ for $t = 2$); it is easy to see that $\mathfrak{h}_x^*$ has a basis $\{ \varepsilon_i \}_{i=1}^{m+t} \cup \{ \delta_i \}_{i=1}^{n} \cup \{ \varepsilon_{m+i} \}$. For the restriction map $\gamma \mapsto \gamma|_{h_x}$ we have
\[ \varepsilon_i \mapsto \varepsilon_i \text{ for } i = 1, \ldots, m - s; \quad \delta_i \mapsto \delta_i \text{ for } i = 1, \ldots, n - s; \quad \varepsilon_{m-i} \mapsto 0 \text{ for } i = 0, \ldots, s - 1; \]
and, for $\ell = 1$, $\varepsilon_{m+1} \mapsto \varepsilon_{m+1}$. One has $\rho|_{h_x} = \rho_x$.

Let $\{ \varepsilon'_i \}_{i=1}^{m+t-s} \cup \{ \delta'_i \}_{i=1}^{n-s}$ be the standard basis in $(\mathfrak{h}')^*$, where $\mathfrak{h}'$ is the Cartan subalgebra of $\mathfrak{osp}(2(m - s) + t|2(n - s))$. The isomorphism $\mathfrak{g}_x \xrightarrow{\sim} \mathfrak{osp}(2(m - s) + t|2(n - s))$ gives $\varepsilon_i \mapsto \varepsilon'_i$ for $i = 1, \ldots, m - s$, $\delta_j \mapsto \delta'_j$ for $j = 1, \ldots, n - s$ and $\varepsilon_{m+1} \mapsto \varepsilon_{m+1-s}$ if $\ell = 1$.

For $t = 0, 2$ we denote by $\sigma_x$ the analogue of $\sigma$ for $\mathfrak{g}_x$; note that $\Sigma_x$ is $\sigma_x$-invariant. Retain the notation of 2.3.1.

5.7. Lemma. Take $\mathfrak{g} := \mathfrak{osp}(2(m + \ell)|2n)$.

(i) If $N$ is finite-dimensional, then $\text{DS}_x(N^\sigma) \cong (\text{DS}_x(N))^\sigma$.

(ii) Let $x$ be of rank 1. If $L$ (resp., $L'$) is a simple finite-dimensional $\mathfrak{g}$ (resp., $\mathfrak{g}_x$)-module, then
\[ [\text{DS}_x(L) : L'] = [\text{DS}_x(L) : (L')^\sigma]. \]

Proof. For each $s$ with $0 < s \leq \min(m, n)$ take $y \in X_{iso}$ with
\[ \text{supp}(y) = \{ \varepsilon_{m+\ell-i} - \delta_{n+1-i} \}_{i=1}^{s}. \]
Note that $y$ has rank $s$ and $\sigma(y) = y$. Moreover, $\sigma$ induces the involution $\sigma_y$ on the algebra $\text{DS}_y(\mathfrak{g}) \cong \mathfrak{osp}(2(m + \ell - s)|2(n - s))$. Using 5.4 we obtain (i) for $x := y$. By 5.5, this implies (i) for each $x \in G_0 \mathfrak{g}$ and thus establishes (i) for all $x$.

For (ii) notice that for a simple $\mathfrak{osp}(2m|2n)$-module $L$ one has $L^* \cong L$ if $m$ is even and $L^* \cong L^\sigma$ if $m$ is odd (since $- \text{Id}$ lies in the Weyl group $W(\mathfrak{osp}(2m|2n))$ if and only if $m$ is even).

If $m + \ell$ is even, then $L \cong L^*$, so $\text{DS}_x(L) \cong \text{DS}_x(L)^*$ and $(L')^* \cong (L')^{\sigma_x}$; this gives the required formula. Consider the case when $m + \ell$ is odd. In this case $L^* \cong (L')^{\sigma_x}$ and $(L')^* \cong L'$. Using (i) we get
\[ [\text{DS}_x(L) : (L')^{\sigma_x}] = [\text{DS}_x(L^*) : L'] = [\text{DS}_x(L^*) : L'] = [\text{DS}_x(L)^* : L'] \]
\[ = [\text{DS}_x(L) : (L')^*] = [\text{DS}_x(L) : L'] \]
as required. \qed
5.8. Lemma. Let \( \lambda \in \Lambda_{m+\ell\mid n}^{(t)} \) be a stable weight of atypicality \( k \) and let \( x := x_s \). If \( \nu \in h^* \) satisfies
\[
\nu \leq \lambda; \quad \nu|_{b_x} \text{ is dominant}; \quad \text{core}(\lambda) = \text{core}(\nu|_{b_x}),
\]
then \( \nu|_{b_x} \) is stable.

Proof. Write
\[
\lambda + \rho =: \sum_{i=1}^{m+\ell} a_i \varepsilon_i + \sum_{j=1}^n b_j \delta_j, \quad \nu + \rho =: \sum_{i=1}^{m+\ell} a'_i \varepsilon_i + \sum_{j=1}^n b'_j \delta_j.
\]
Set \( \nu' := \nu|_{b_x} \). Since \( \rho|_{b_x} = \rho_x \) we obtain
\[
\nu' + \rho_x = (\nu + \rho)|_{b_x}.
\]
Since \( \nu' \) is dominant and \( \text{core}(\lambda) = \text{core}(\nu') \) we have \( \nu' \in \Lambda_{m+\ell-s\mid n-s}^{(t)} \). If \( \ell = 1 \) we have \( t = 2 \) and thus \( a'_{m+1} = 0 \). Therefore for all \( t \) we have
\[
\nu' + \rho_x = (\nu + \rho)|_{b_x} = \sum_{i=1}^{m-s} a'_i \varepsilon_i + \sum_{j=1}^{n-s} b'_j \delta_j.
\]

If \( k = 0 \), then \( s = 0 \) and \( \nu' = \nu \) is stable. If \( \nu' \) is typical, it is stable. Thus we assume that \( \lambda, \nu' \) are atypical. Since \( \lambda \) is stable, 3.8.1 gives
\[
(4) \quad \text{core}(\nu') = \text{core}(\lambda) = \sum_{i=1}^{m-k} a_i \varepsilon_i + \sum_{j=1}^{n-k} b_j \delta_j.
\]
Moreover, by 3.8.1, for stability of \( \nu' \) it is enough to verify that \( a_i = a'_i \) for \( i = 1, \ldots, m - k \) and \( b_j = b'_j \) for \( j = 1, \ldots, n - k \). Let \( p \) (resp., \( q \)) be minimal such that \( a_p \neq a'_p \) (resp., \( b_q \neq b'_q \)). By above it suffices to show that
\[
(5) \quad m - p, n - q < k.
\]
Using (4) (and the atypicality of \( \nu' \) for \( t = 0 \) case) we obtain
\[
(6) \quad a_p \in \{ a'_i \}_{i=p}^{m-s} \text{ if } p \leq m - k; \quad b_q \in \{ b'_j \}_{j=q}^{n-s} \text{ if } q \leq n - k.
\]

The assumption \( \nu \leq \lambda \) gives
\[
(7) \quad \lambda - \nu = (a_p - a'_p) \varepsilon_p + (b_q - b'_q) \delta_q + \sum_{i > p} (a_i - a'_i) \varepsilon_i + \sum_{j > q} (b_j - b'_j) \delta_j \in \mathcal{N}_\Sigma.
\]
Consider the case when \( \varepsilon_p - \delta_q \in \Delta^+ \). Then (7) implies \( a_p > a'_p \). Since \( \nu' \) is dominant
\[
a'_i \leq a'_p < a_p \quad \text{for } i = p, p + 1, \ldots, m - s,
\]
so \( p \geq m - k \) by (6). Notice that \( \delta_{n-j} - \varepsilon_{m-i} \in \Delta^+ \) for \( j > i \), so the assumption \( \varepsilon_p - \delta_q \in \Delta^+ \) gives \( n - q \leq m - p \) and thus implies (5). For the remaining case \( \delta_q - \varepsilon_p \in \Delta^+ \) the proof is similar. \( \square \)
Taking $s = 0$ we obtain the following corollary, which is a reformulation of Lemma 6.2 in [S1].

5.9. **Corollary.** Assume that $\lambda, \nu$ are dominant weights, $\lambda$ is stable and $\text{core}(\lambda) = \text{core}(\nu) \quad \nu \leq \lambda$.

Then $\nu$ is stable.

5.10. **Corollary.** If $N \in \mathcal{F}^g$ is stable, then $\text{DS}_s(N)$ is stable.

**Proof.** Assume that $[\text{DS}_s(N) : L_{g_s}(\nu')] \neq 0$. Then $\nu'$ is dominant and $\text{core}(\nu') = g$.

Let $\nu$ be a vector in $\text{DS}_s(N) = N^x/xN$ which has weight $\nu'$. By [Gor1], Lem. 2.3 we can choose a preimage $v$ of $\nu$ in the space

$$\sum_{\mu \in X} N^x_{\mu}, \text{ where } X := \{\nu \in \Omega(N) | \nu|_{g_s} = \nu', \ (\nu|S_s) = 0\}.$$ 

Take $\nu \in X$. Since $\nu \in \Omega(N)$ there exists a stable dominant weight $\lambda$ (a maximal weight in $\Omega(N)$) such that

$$\nu \leq \lambda, \quad \text{core}(\lambda) = g.$$ 

In the light of 5.6.2 the condition $(\nu|S_s) = 0$ implies

$$\text{core}(\nu) = \text{core}(\nu|_{h_s}) = g.$$ 

By Lemma 5.8, $\nu'$ is stable. \qed

6. **Reduction to principal blocks**

In this section we reduce the computation of multiplicities $[\text{DS}_s(L(\lambda)) : L_{g_s}(\nu')]$ to the case of principal blocks.

In this section $g$ stands for a core diagram of type $t$ for $g := \mathfrak{osp}(2m + t|2n)$. Let $\mu \in \Lambda^{(t)}_{m-k|n-k}$ be the typical weight corresponding to $g$.

6.1. **Notation.** We denote by $\mathcal{F}^{pr}(g_k)$ the principal block for $g_k$:

$$\mathcal{F}^{pr}(g_k) = \begin{cases} \mathcal{F}^0(\mathfrak{osp}(2k + t|2k)) & \text{for } t = 0, 1 \\ \mathcal{F}^>(\mathfrak{osp}(2k + 2|2k)) & \text{for } t = 2. \end{cases}$$

We denote by $\mathcal{F}^{st}_n$ the subcategory of $\mathcal{F}(g)$ consisting of stable modules. This category is zero if and only if the zero position of $g$ is non-empty.

We denote by $\Lambda^+(g; i)$ the set of diagrams $f$ with the following properties:

$$\text{core}(f) = g \text{ and all symbols } \times \text{ lie in the positions } 0, \ldots, i.$$
We denote by $\mathcal{F}_s^q(g)$ the Serre subcategory of $\mathcal{F}(g)$ generated by the modules $L(\lambda)$ with $\lambda \in \Lambda^+(g; i) \cap \Lambda_{m/n}$ and denote by $\mathcal{F}_s^q(r_k(g))$ the corresponding subcategory of $\mathcal{F}_s^q(g_k)$. Note that $\mathcal{F}_s^q(g)$ can be viewed as a “limit” of subcategories $\mathcal{F}_s^q(g_k)$.

6.2. The functor $\text{Res}$ for $(t; k) \neq (0; 0)$. We assume that $\mathcal{F}_s^q(g)$ is a non-principal block and $\mathcal{F}_s^q(g_k) \neq 0$. This means that we exclude the case $t = 0, k = 0$ (in this case $\mathcal{F}_s^q(g)$ is a direct sum of two blocks) and that $g$ has a non-empty symbols at a non-zero position. One has

$$\mathcal{F}_s^q(g_k) = \mathcal{F}_s^q(g),$$

where $q + 1$ is the coordinate of the first occupied non-zero position in $g$.

We retain the notation of B.1. Fix $z \in \mathfrak{h}$ such that $\alpha(z) = 0$ for $\alpha \in \Delta(g_k)$ and $\alpha(z) \in \mathbb{R}_{>0}$ for $\alpha \in \Delta^+ \setminus \Delta(g_k)$.

$$\mathfrak{g}^z = g_k + \mathfrak{h} = g_k \times \mathfrak{h}'',$$

where $\mathfrak{h}''$ is the centralizer of $g_k$. Set $a := (\mu - \rho)(z)$ and define the functor $\text{Res} := \text{Res}_a$ using the construction of B.2 for $t := g_k$ and $\mu$ as above.

6.2.1. Proposition. The functor

$$\text{Res} : \mathcal{F}_s^q(g) \xrightarrow{\sim} \mathcal{F}_s^q(g_k)$$

is an equivalence of the categories and

$$\text{Res}(L(f)) = L(\text{hwl}(f))$$

for each stable $f \in \Lambda_{m+\ell/n}^t$ with core($f$) = $g$.

Proof. Take $\mathfrak{h}' := g_k \cap \mathfrak{h}$. Then $\mathfrak{h} = \mathfrak{h}' \times \mathfrak{h}''$. Setting

$$E := \{\varepsilon_i\}_{i=1}^{m+\ell}, \quad D := \{\delta_i\}_{i=1}^{n}, \quad E' := \{\varepsilon_i\}_{i=m+1}^{m+n}, \quad D' := \{\delta_i\}_{n+1-k}^{n},$$

we see that $(\mathfrak{h}')^*$ is spanned by $E' \cup D'$ and $(\mathfrak{h}'')^*$ is spanned by $(E \setminus E') \cup (D \setminus D')$.

Let $\nu : \mathfrak{h}' \rightarrow (\mathfrak{h}')^*$, $\nu'' : \mathfrak{h}^* \rightarrow (\mathfrak{h}'')^*$ be the projections given by the decomposition $\mathfrak{h} = \mathfrak{h}' \oplus \mathfrak{h}''$. Notice that $\mu \in (\mathfrak{h}'')^*$ and set

$$A := \{\lambda \in \Lambda_{m+\ell/t}^+ | \text{core}(\lambda) = g, \nu''(\lambda) = \mu - \nu''(\rho)\}, \quad A' := \nu(A).$$

One has

$$A = \{\lambda \in \Lambda_{m+\ell/t}^+ | \text{core}(\lambda) = g, \nu''(\lambda) = \mu - \nu''(\rho)\} = \{\lambda \in \Lambda_{m+\ell/t}^+ | \text{core}(\lambda) = g, \nu''(\lambda + \rho) = \mu\}.$$

In the light of 3.8.1 we get $A = \Lambda_{m+\ell/t}^+(g; q)$, so $\mathcal{F}_s^q(g) = \mathcal{F}_s^q(g) = \mathcal{F}(A)$. By Corollary 5.9, $\Lambda_{m+\ell/t}^+(g; q)$ satisfies the assumption (20). By B.5, Res provides an equivalence of categories $\mathcal{F}_s^q(g) \xrightarrow{\sim} \mathcal{F}(A')$.

Take $\lambda \in A$. Since $\lambda$ is stable, 3.8.2 implies that

$$\lambda + \rho = \mu + \text{hwl}(\lambda) + \rho_k,$$
where $\rho_k$ is the Weyl vector of $\mathfrak{g}_k$. Note that $\rho_k = v(\rho)$.

Since $\mu \in (\mathfrak{h}^*)_t$ and $\text{howl}(\lambda) \in (\mathfrak{h}^*)_t$, we have $v(\lambda) = \text{howl}(\lambda)$, so

$$A' = v(\Lambda^+_{m+t}^q) = \Lambda^+(g_{pr};q),$$

where $g_{pr}$ is empty for $t = 0,1$ and $g_{pr} \Rightarrow$ for $t = 2$. Hence $F(A') = F^q_{pr}(\mathfrak{g}_k)$. This completes the proof. \hfill \qed

6.2.2. Retain the notation of 3.7.8.

**Corollary.** For each $\lambda, \nu \in \Lambda_m(\mathfrak{sl}_n)$ one has

$$\text{dex}(\lambda) = \text{dex}(\nu) \implies \text{Ext}^1(L(\lambda), L(\nu)) = 0.$$

**Proof.** For the core-free diagrams the assertion is established in [Gor2]. The general case follows from Proposition 6.2.1 (note that $\text{Ext}^1(L(\lambda), L(\nu)) = 0$ if $\lambda$ is typical). \hfill \qed

6.2.3. **Remark.** Consider the case $k = 0$ (and $t \neq 0$). In this case $F^q_{st}(\mathfrak{g}) = F^q(\mathfrak{g})$ is a typical block containing $L(\mu)$; this block is isomorphic to the category of finite-dimensional even vector spaces.

For $t = 1$ one has $\mathfrak{g}_k = 0$ and $F^q_{pr}(\mathfrak{g}_k) = F(\mathfrak{g}_k)$ is the category of finite-dimensional even vector spaces $(\text{Res}(L(\mu)) = L(\emptyset) = \mathbb{C})$.

For $t = 2$ one has $\mathfrak{g}_k = \mathbb{C}$ and $F(\mathfrak{g}_k)$ is the category of finite-dimensional $\mathfrak{g}_k$-module with the zero action of $\mathfrak{g}_k$ (since $t = 2$), i.e. $(\text{Res}(L(\mu)))$ is the trivial $\mathfrak{g}_k$-module).

6.3. **DS** $x$ and Res. Take $x := x_s$. Since $DS_s(F^q) = 0$ for $s > k$ we assume

$$0 < s \leq k.$$

By 5.5.1 one has $x \in \mathfrak{g}_s \subset \mathfrak{g}_k$.

Let $(\mathfrak{g}_x)_i$ be the subalgebra of $\mathfrak{g}_x$ constructed for $\mathfrak{g}_x$ in the same way as $\mathfrak{g}_i$ to $\mathfrak{g}$. Consider the base $\Sigma_{k-s}^s \subset \Delta^+(\mathfrak{g}_x)$. We define the functors Res, Res$_x$ as in 6.2: we take

$$z \in \mathfrak{h} \text{ such that } \alpha(z) = 0 \text{ for } \alpha \in \Sigma_k, \quad \mu(z) = 1 \text{ for } \alpha \in \Sigma \setminus \Sigma_k$$

$$z_x \in \mathfrak{h}_x \text{ such that } \alpha(z_x) = 0 \text{ for } \alpha \in \Sigma_{k-s}^s, \quad \mu(z_x) = 1 \text{ for } \alpha \in (\Sigma_x \setminus \Sigma_{k-s}^s)$$

and set $a := (\mu - \rho)(z)$, $a_x := (\mu - \rho_x)(z_x)$. We obtain $\text{Res} : F^q_{st}(\mathfrak{g}) \longrightarrow F^q_{pr}(\mathfrak{g}_k)$ and $\text{Res}_x : F^q_{st}(\mathfrak{g}_x) \longrightarrow (\mathfrak{g}_{x,k-s})$

given by

$$\text{Res}(N) := \{v \in N | zv = a\}, \quad \text{Res}_x(N) := \{v \in N | z_x v = a_x v\}.$$
6.3.1. Lemma. One has $\text{DS}_z(\mathfrak{g}_k) = (\mathfrak{g}_z)_{k-s}$. In addition, except for the case $t = 0, k = s$, we have $z_x = z, a_x = a$.

Proof. The partial order $\geq$ gives a total order on the standard basis of $\mathfrak{h}^*$, i.e.

$$\{e_i\}_{i=1}^{m+\ell} \cup \{\delta_j\}_{j=1}^n = \{e_i\}_{i=1}^{m+n+\ell}, \quad e_1 < e_2 < \ldots < e_m = 0$$

(where $e_1 = \delta_n$ for $\mathfrak{osp}(2m+1|2n)$ and $e_1 = \epsilon_{m+\ell}$ for $\mathfrak{osp}(2m|2n)$). Let $\{e_i^*\}_{i=1}^{m+n+\ell}$ be the dual basis of $\mathfrak{h}$.

Denote the base of $\text{DS}_z(\mathfrak{g}_k)$ by $(\Sigma^s_k)^z$. For $k = s$ we have $(\Sigma^s_k)^z = \emptyset = (\Sigma^z)^0$.

In the $t = 0$-case one has

$$\Sigma = \{e_1 + e_2, e_2 - e_1, e_3 - e_2, \ldots, e_m - e_{m-1}\},$$

$$\Sigma_k = \{e_1 + e_2, e_2 - e_1, e_2 - e_{2k-1}\},$$

$$\Sigma^z = \{e_{2s+1} + e_{2s+2} - e_{2s+1}, \ldots, e_m - e_{m-1}\}$$

$$(\Sigma^z)^{k-s} = \{e_{2s+1} + e_{2s+2} - e_{2s+1}, \ldots, e_{2k} + e_{2k-1}\} = (\Sigma^z)^0.$$ 

In the $t = 1$-case one has

$$\Sigma = \{e_1, e_2 - e_1, e_3 - e_2, \ldots, e_m - e_{m-1}\},$$

$$\Sigma_k = \{e_1, e_2 - e_1, e_2k - e_{2k-1}\},$$

$$\Sigma^z = \{e_{2s+1} + e_{2s+2} - e_{2s+1}, \ldots, e_m - e_{m-1}\}$$

$$(\Sigma^z)^{k-s} = \{e_{2s+1}, e_{2s+2} - e_{2s+1}, \ldots, e_{2k} - e_{2k-1}\} = (\Sigma^z)^1.$$ 

In both cases

$$\Sigma \setminus \Sigma_k = \{e_{2k+1} - e_2 + e_{2k+2} - e_{2k+1}, \ldots, e_m - e_{m-1}\}$$

and

$$z = e_{2k+1}^* + e_{2k+2}^* + \ldots + (m - n - 2k)e_{m+n}^*.$$ 

One readily sees that $z = z_x$ (except for $t = 0$ and $k = s$).

In the $t = 2$-case one has

$$\Sigma = \{e_1 + e_2, e_2 - e_1, e_3 - e_2, \ldots, e_{m+n} - e_{m+n-1}\},$$

$$\Sigma_k = \{e_1 + e_2, e_2 - e_1, e_{2k-1}, e_{2k+1} - e_{2k}\},$$

$$\Sigma^z = \{e_1 + e_{2s+2} - e_1, e_{2s+3} - e_{2s+2}, \ldots, e_{m+n} - e_{m+n-1}\}$$

$$(\Sigma^z)^{k-s} = \{e_1 + e_{2s+2} - e_{2s+1}, \ldots, e_{2k+1} - e_{2k}\} = (\Sigma^z)^2.$$ 

Therefore

$$z = e_{2k+2}^* + e_{2k+3}^* + \ldots + (m - n - 2k)e_{m+n+1}^* = z_x.$$ 

By 5.6.2, $\rho_x = \rho|_{\mathfrak{h}_x}$; since $z_x \in \mathfrak{h}_x$ we get

$$a_x - a = (\rho_x - \rho)(z_x) = 0.$$ 

Finally, $(\Sigma^z)^{k-s} = (\Sigma^z)^2$ gives $\text{DS}_z(\mathfrak{g}_k) = (\mathfrak{g}_z)_{k-s}$ as required. □
6.3.2. Assume that $0 < s \leq k$ and $s \neq k$ for $t = 0$. Combining Lemma 6.3.1 and 5.10, 6.2.1 we obtain the following diagram

$$
\mathcal{F}_{st}^g(\mathfrak{g}) \xrightarrow{\text{Res}} \mathcal{F}_{q}^{pr}(\mathfrak{g}_k)
$$

(8)

$$
\downarrow \text{DS}_x \downarrow \quad \downarrow \text{DS}_x \\
\tilde{\mathcal{F}}_{st}^g(\mathfrak{g}_x) \xrightarrow{\text{Res}_x} \tilde{\mathcal{F}}_{q}^{pr}((\mathfrak{g}_x)_{k-s})
$$

where Res, Res$_x$ are equivalence of categories.

Let us show that this diagram is commutative. Take $N \in \mathcal{F}_{st}^g(\mathfrak{g})$. Since $z = z_x \in \mathfrak{h}_x$, the spaces $N^x$ and $xN$ are $z$-stable, so

$$
\text{Res}_x(\text{DS}_x(N)) = (\text{DS}_x(N))_a = (N^x)_a/(xN)_a.
$$

On the other hand,

$$
\text{DS}_x(\text{Res}(N)) = \text{DS}_x(N_a) = (N_a)^x/(xN_a).
$$

Since $[x,z] = 0$ one has $(xN)_a = x(N_a)$ and $(N^x)_a = (N_a)^x$. Hence $\text{Res}_x(\text{DS}_x(N)) = \text{DS}_x(\text{Res}(N))$ as required.

6.4. The case $t = 0$ and $k = s$. Consider the case $t = 0$ and $k = s > 0$. By 6.2.1 we have Res : $\mathcal{F}_{st}^g(\mathfrak{g}) \xrightarrow{\sim} \mathcal{F}_{q}^{pr}(\mathfrak{g}_k)$. Note that

$$
\text{Res}(N) := \{v \in N \mid zv = \mu(z)v\},
$$

where $z$ is as in 6.3. From the proof of Lemma 6.3.1 we see that $z \in \mathfrak{h}_x$.

Note that $k-s = 0$, so $(\mathfrak{g}_x)_{k-s} = 0$ and $\mathcal{F}\text{in}((\mathfrak{g}_x)_{k-s})$ is the category of finite-dimensional supervector spaces, which we denote by $s\text{Vect}$. Define

$$
\text{Res}_x : \mathcal{F}\text{in}(\mathfrak{g}_x) \rightarrow s\text{Vect}
$$

by formula (9). Using the arguments of 6.3.2 we obtain the following commutative diagram

$$
\mathcal{F}_{st}^g(\mathfrak{g}) \xrightarrow{\text{Res}} \mathcal{F}_{q}^{pr}(\mathfrak{g}_k) \\
\downarrow \text{DS}_x \downarrow \quad \downarrow \text{DS}_x \\
\tilde{\mathcal{F}}_{st}^g(\mathfrak{g}_x) \xrightarrow{\text{Res}_x} s\text{Vect}
$$

Observe that $\tilde{\mathcal{F}}_{st}^g(\mathfrak{g}_x) = \tilde{\mathcal{F}}^g(\mathfrak{g}_x)$ is a semisimple category and

$$
\text{Irr}(\tilde{\mathcal{F}}^g(\mathfrak{g}_x)) = \{L', \Pi(L'), (L')^{\sigma_z}, \Pi((L')^{\sigma_z})\}, \text{ where } L' := L_{\mathfrak{g}_x}(\mu).
$$

The functor Res$_x$ commutes with $\Pi$; clearly,

$$
\text{Res}_x(L_{\mathfrak{g}_x}(\mu)) = L(\emptyset) = \mathbb{C}.
$$
The eigenvalues of $z$ on $(L')^{\sigma_x} = L_{q_x}(\sigma(\mu))$ lie in the set $\sigma_x(\mu)(z) - N$. Recall that $t = 0$ and $\mu$ has the diagram with the sign $+$. Using the notation of the proof of Lemma 6.3.1 we have $$ (\mu - \sigma_x(\mu))(z) = 2(\mu|e_{2k+1}) > 0, $$ so $$ \text{Res}_x((L')^{\sigma_x}) = 0. $$

6.5. Graded multiplicity. Retain notation of 2.3.1. We fix $r$ and denote the graded multiplicity $[\text{DS}_r(L(\mu)) : L(\mu')]$ by $\frac{f}{f'}$.

Note that $\frac{f}{f'} = 0$ if core($f$) $\neq$ core($f'$) or atyp $f - $ atyp $f' \neq r$ (where atyp stands for the atypicality, i.e. the number of the symbols $\times$ in the diagram).

6.5.1. Case: $\text{osp}(2n+1|2n)$: switch functor. By [GS1], Lemma 19 the translation functor $T^0_\emptyset$ ("switch functor") acts on simple modules $L(\mu) \in \mathcal{F}$ as follows:

$$ T^0_\emptyset(L(\mu)) = L(\mu^{sw}), $$

where the diagram of $\mu^{sw}$ is obtained form the diagram of $\mu$ by sign change ($\mu^{sw} = \mu$ if the diagram of $\mu$ does not have a sign). Since DS commutes with the translation functors, we get

$$ [\text{DS}_r(L(\lambda)) : L(\nu)] = [\text{DS}_r(L(\lambda^{sw})) : L(\nu^{sw})]. $$

6.5.2. Let $\sigma(f)$ be the diagram obtained from $f$ by the change of sign (for $\text{osp}(2n|2n)$ this notation was used before).

6.5.3. Corollary.

(i) $\frac{f}{f'} = \frac{\text{howl}(f)}{\text{howl}(f')}$;

(ii) $\frac{\sigma(f)}{\sigma(f')} = \frac{f}{f'}$;

(iii) for $t = 0$ and $r = 1$ one has $\frac{f}{\sigma(f')} = \frac{\sigma(f)}{\sigma(f')} = \frac{f}{f'}$.

Proof. Combining 6.3.2 and Corollary 5.3 we obtain (i) for all cases except $t = 0$ with $\text{howl}(f') = \emptyset$.

For a diagram $f$ denote by $\lambda(f)$ the corresponding dominant weight. For $t = 2$ one has $\sigma(f) = f$. For $t = 1$ one has $\lambda(\sigma(f)) = (\lambda(f))^{sw}$ and (ii) follows from 6.5.1.

Consider the remaining case $t = 0$. The formulae (ii), (iii) follow from Lemma 5.7. For (i) take $f'$ such that $\text{howl}(f') = \emptyset$. By 6.4, $\frac{f}{f'} = \frac{\text{howl}(f)}{\text{howl}(f')}$ if the diagram of $f'$ has the sign $+$. Assume that $f'$ has the sign $-$. Since $\sigma(f')$ has sign $+$ we have

$$ \frac{\sigma(f)}{\sigma(f')} = \frac{\text{howl}(\sigma(f))}{\text{howl}(\sigma(f'))}. $$
Using (ii) and the formulae \( \text{howl}(\sigma(f)) = \sigma(\text{howl}(f)) \) we get

\[
\frac{f}{f'} = \frac{\sigma(f)}{\sigma(f')} = \frac{\text{howl}(\sigma(f))}{\text{howl}(\sigma(f'))} = \frac{\sigma(\text{howl}(f))}{\sigma(\emptyset)} = \frac{\text{howl}(f)}{\emptyset}
\]

which implies (i).

\[\square\]

7. Recursive formulae for \([DS_x(L(\lambda)) : L_{\mathfrak{g}_x}(\nu)]\)

Using that \(DS\) commutes with translation functors, we establish recursive formulæ for the multiplicities \([DS_x(L(\lambda)) : L_{\mathfrak{g}_x}(\nu)]\). This will ultimately allow us to reduce the computation of the multiplicities to the atypicality 1 case, i.e. to \(\mathfrak{osp}(2|2)\), \(\mathfrak{osp}(3|2)\) and \(\mathfrak{osp}(4|2)\). For a similar reduction in the \(\mathfrak{gl}(m|n)\)-case see [HW].

We fix \(r\) and retain the notation of 6.5.

7.1. Translation functors. Recall that \(DS\) commutes with the translation functors. Let \(g_0, g_1\) be two core diagrams. Consider the translation functors

\[
T_{g_1}^{g_0} : \tilde{\mathcal{F}}(\mathfrak{g})^{g_1} \to \tilde{\mathcal{F}}(\mathfrak{g})^{g_0}, \quad T_{g_1}^{g_0} : \tilde{\mathcal{F}}(\mathfrak{g}_x)^{g_1} \to \tilde{\mathcal{F}}(\mathfrak{g}_x)^{g_0}.
\]

Since translation functors are exact, they induce morphisms on the Grothendieck ring. For \(N \in \mathcal{F}(\mathfrak{g})^{g_1}\) and \(L' \in \text{Irr}(\mathfrak{g}_x)^{g_0}\) we obtain

\[
[DS_r(T_{g_1}^{g_0}(N)) : L'] = [T_{g_1}^{g_0}(DS_r(N)) : L'] = \sum_{L_1 \in \text{Irr}(\mathfrak{g}_x)^{g_1}} [DS_r(N) : L_1][T_{g_1}^{g_0}(L_1) : L'].
\]

7.1.1. Assume that the number of core symbols in \(g_0\) is larger than the number of core symbols in \(g_1\), i.e. the atypicality of \(g_1\) is larger than the atypicality of \(g_0\). We will use the following results of [GS2], Lemmatta 7, 13, 14:

- the image of a simple module \(L \in \text{Irr}(\mathfrak{g}_x)^{g_1}\) is either zero or simple;
- for \(L_1, L_2 \in \text{Irr}(\mathfrak{g}_x)^{g_1}\) with \(T_{g_1}^{g_0}(L_1) \cong T_{g_1}^{g_0}(L_2) \neq 0\) one has \(L_1 \cong L_2\).

7.1.2. These results will be sufficient for us. A complete description of translation functors on irreducible modules and projective covers can be obtained from [ES],[ES2],[ES3].

7.1.3. Take \(g_0, g_1\) satisfying the assumption in 7.1.1. For each \(L' \in \text{Irr}(\mathfrak{g}_x)^{g_0}\) there exists at most one (up to isomorphism) \(L_1 \in \text{Irr}(\mathfrak{g}_x)^{g_1}\) such that \(T_{g_1}^{g_0}(L_1) \cong L'\). Then (10) gives for \(N \in \mathcal{F}(\mathfrak{g})^{g_1}\)

\[
[DS_r(T_{g_1}^{g_0}(N)) : L'] = [T_{g_1}^{g_0}(DS_r(N)) : L'] = [DS_r(N) : L_1].
\]
In particular,
\begin{equation}
[DS_r(N) : L_1] \neq 0 \implies T_{g_1}(N) \neq 0.
\end{equation}

Now take $N := L(\lambda)$ and $\nu$ such that $[DS_r(L(\lambda)) : L_{g_x}(\nu)] \neq 0$ and $T_{g_1}(L_{g_x}(\nu)) \neq 0$. Then, by 7.1.1,
\[ T_{g_1}(L_{g_x}(\nu)) = L_{g_x}(\nu_1) \]
for some $\nu_1$ and $T_{g_1}(L(\lambda))$ is either zero or simple. By above, $T_{g_1}(L(\lambda)) \neq 0$, so
\[ T_{g_1}(L(\lambda)) = L(\lambda_1) \]
for some $\lambda_1$. Using (10) and Corollary 6.5.3 we conclude
\begin{equation}
\frac{f(\lambda)}{f(\nu)} = \frac{\text{howl}(f(\lambda_1))}{\text{howl}(f(\nu_1))},
\end{equation}
where $f(\lambda)$ stands for the weight diagram of $\lambda$.

7.2. Translation functors $T_u$. We describe some translation functors via their effect (called elementary change in [GS2], Section 6.3) on core/weight diagrams. Let $a$ be a non-negative integer. For each diagram $f$ we denote by $\pos_a(f)$ the subdiagram corresponding to the positions $a, a + 1$. For a core diagram $g_1$ with $\pos_a(g_1) = \circ\circ$ we denote by $\phi_{a}'(g_1)$ the core diagram obtained from $g_1$ by changing $\pos_a(g_1) = \circ\circ$ to $\pos_a(g_1) = \rangle\langle$; for instance,
\[ \phi_{1}'(\circ \circ \times \circ) = \times \times \rangle\langle, \]
\[ \phi_{1}'(\times \times \circ) = \circ \circ \rangle\langle. \]

We denote by $T_a$ the functor which acts as $T_{g_1}(\phi_{a}'(g_1))$ on $\tilde{\mathcal{F}}_{g_1}$ with $\pos_a(g_1) = \circ\circ$ and by zero on $\tilde{\mathcal{F}}_{g'}$ with $\pos_a(g') \neq \circ\circ$. Note that $T_a$ reduces the atypicality by 1.

7.2.1. Take $u > 0$. For a diagram $f$ with $\pos_a(f) = \times \circ$ we define two new diagrams:

- the diagram $\phi_u(f)$ obtained from $f$ by changing $\times \circ$ in the positions $(u, u + 1)$ to $\rangle\langle$ (i.e., $f$ and $\phi_u(f)$ have the same signs and the same symbols in all positions except $u, u + 1$ and $\pos_a(\phi_u(f)) = \rangle\langle$);

- the diagram $\text{shr}_u(f)$ obtained from $f$ by “shrinking” $\times \circ$ at the positions $u, u + 1$, i.e. changing the diagram $f_- \times \circ f_+$ to the diagram $f_- \circ f_+$;

For instance, $\phi_1(\times \times \times \circ), \text{shr}_1(\times \times \times \circ)$ are not defined;
\[ \phi_2(\times \times \times \circ) = \times \times \rangle\langle, \]
\[ \text{shr}_2(\times \times \times \circ) = \times \times. \]

7.2.2. By [GS2], for $u > 0$ one has
\[ T_u(L(\mu)) = L(\phi_u(\mu)) \]
if $\phi_u(\mu)$ is defined and $T_u(L(\mu)) = 0$ otherwise.
7.3. Corollary. For $u > 0$ one has
\[
\frac{f_u' \times f'}{f_u \circ f} = \frac{f_u' \circ f'}{f_u \circ f} = \frac{f_u' \circ f'}{f_u \circ f} = 0
\]
(with all possible signs) and
\[
\frac{f_u' \times f'}{f_u \circ f} = \frac{f_u f'}{f_u f}
\]
where $f_u, f_u'$ stands for the subdiagrams corresponding to the positions $0, 1, \ldots, u - 1$.

Proof. Recall from (11) that for irreducible $N = L(\lambda)$
\[
[DS_r(T^{g_0}_{g_1}(L(\lambda)) : T^{g_0}_{g_1}(L_1))] = [DS_r(L(\lambda) : L_1)].
\]
Take $u > 0$. Let $\tilde{f}$ be a diagram with $\operatorname{pos}_u(\tilde{f}) = \times \circ$. Applying $T_u$ sends $f_u \times f$ to $f_u < f$. Combining (11) and 7.2.1 we get $\tilde{f} = 0$ if $\operatorname{pos}_u(f') \neq \times \circ$; this gives the first formula. Combining 7.2.1 and (12) we have
\[
\frac{f_u' \times f'}{f_u \circ f} = \frac{\operatorname{howl}(f_u' \times f')}{\operatorname{howl}(f_u \times f)}.
\]
Using Corollary 6.5.3 and $\operatorname{howl}(f_u' \times f') = \operatorname{howl}(f_u f')$ we obtain the second formula. \[\square\]

7.4. Reduction to the case $\nu = 0$. Let $f$ be a weight diagram and $u$ be the coordinate of the rightmost symbol $\times$ in $f$. Using 6.5.3 (i) and 7.3 we reduce the computation of $f'$ to the situation when $f, f'$ are core-free and $u = 0$; in the light of 6.5.3 (ii) we can assume that $f$ has sign $-$ for $t = 1$ (for $t = 0, 2$ the diagram $f$ does not have sign if $u = 0$). Notice that in this case $\lambda(f') = 0$, i.e. we reduced the problem to the computation of $[DS_r(L(\lambda)) : L_0(0)]$ for $g = \mathfrak{osp}(2n + 2|2n)$.

For $\mathfrak{gl}(m|n)$ a similar reasoning reduces the computation of $[DS_r(L(\lambda)) : L(\nu)]$ to the case when $\nu$ has the empty diagram, i.e. to the case when $g_x = 0$ (see also [HW]). In the $\mathfrak{osp}$-case this is done in Corollaries 7.5.2, 7.6.4 below.

7.5. The case $\mathfrak{osp}(2m + 1|2n)$. Consider the case $g = \mathfrak{osp}(2m + 1|2n)$. We assume, as always, that "the signs disappear" if the zero position is empty, i.e. $\pm \times^{i-1} f$ stands for $o f$ for $i = 1$.

7.5.1. By [GS2], for the translation functor $T_0 := T^{> \circ}_{\emptyset}$ one has $T_0(L(\mu)) \neq 0$ if and only if the diagram of $\mu$ has the sign $+$ and $\operatorname{pos}_0(\mu) = \times^{i \circ}$ for $i > 0$; moreover,
\[
T_0(L(+ \times^i o f)) = L(\times^{i-1} > < f).
\]
Using (11) we obtain for $j \geq 1$
\[
\frac{\tilde{f}}{+ \times^j o f} \neq 0 \implies \tilde{f} = + \times^p o f' \text{ for some } p \geq 1.
\]
7.5.2. **Lemma.** For $p, i \geq 1$ one has
\[
\frac{+x^p \circ f}{+x^i} = -x^{p-1} \frac{f}{x^{i-1}} \quad \quad \frac{+x^p \circ f}{+x^i} = -x^{p-1} \frac{f}{x^{i-1}}
\]

**Proof.** Using (12) for $T_0$ we obtain for $* \in \{\circ, \times\}$
\[
\frac{+x^p \circ f}{+x^i} = \frac{x^{p-1} > < * f}{x^{i-1} > <}
\]
One has
\[
\text{howl}(\frac{x^{p-1} > < * f}{x^{i-1} > <}) = \begin{cases} -x^{p-1} f & \text{if } * = \circ \\ x^p f & \text{if } * = \times \end{cases}
\]
and \text{howl}(\frac{x^{i-1} > <}{x^{i-1} > <}) = -x^{i-1}$. Now (12) gives
\[
\frac{+x^p \circ f}{+x^i} = \frac{x^{p-1} > < * f}{x^{i-1} > <} \quad \quad \frac{+x^p \circ f}{+x^i} = \frac{x^{p} f}{x^{i}}
\]
as required. □

7.5.3. **Corollary.** Take $i \geq 1$. One has
\[
\frac{-x^p \circ \circ \circ \circ \circ \circ f}{-x^i} = \frac{-x^{p-i+1} f}{0} \quad \frac{-x^p \circ \circ \circ \circ \circ \circ f}{-x^i} = \frac{-x^{p-i} \circ \circ \circ \circ \circ \circ f}{0} \quad \text{for } j \geq 2i.
\]
Moreover, \(\frac{\tilde{f}}{-x^i} = 0\) if \(\tilde{f}\) is not as above, i.e. \(\tilde{f} \neq -x^p \circ \circ \circ \circ f\) for some $p \geq i$ and $j \geq 2i - 1$.

**Proof.** The statement follows by induction from Lemma 7.5.2 and Corollary 6.5.3 (ii). □

7.6. **The case osp(2m|2n).** Recall that the simple $OSp(2m|2n)$-modules are in one-to-one correspondence with the unsigned diagram (see 3.1.8). For a non-empty diagram $f$ we will use the sign $\circ f$ for $+ \circ f \oplus - \circ f$, i.e. $L(\circ f)$ is a simple $OSp(2m|2n)$-module which is the direct sum of $osp(2m|2n)$-modules $L(+ \circ f)$ and $L(- \circ f)$.

For an empty diagram we have $L(\emptyset) = C$; notice that
\[
\text{howl}(\pm \circ) = \emptyset; \quad \text{howl}(\circ) \neq \emptyset.
\]

7.6.1. For the translation functor $T_0 := T_0^{> <}$ one has $T_0(L(\mu)) \neq 0$ if and only if $\text{pos}_0(\mu) = x^i \circ$ for $i > 0$ and
\[
T_0(L(x^i \circ f)) = L(> < f).
\]
Using 7.1.3 we obtain for $j \geq 1$
\[
(13) \quad \frac{\tilde{f}}{x^j \circ f} \neq 0 \quad \implies \quad \tilde{f} = x^p \circ f' \quad \text{for some } p > 0.
\]
7.6.2. The translation functor $T_{>}$ is given by

$$T_{>}(L(\times^p \circ f)) = L(\times^p > f), \quad T_{>}(L(\times^p \times f)) = L(\times^{p+1} > f)$$

for each $p \geq 0$ (where, $L(\circ f) = L(\circ f) \oplus L(- \circ f)$).

7.6.3. The translation functor $T_{>>}$ is adjoint to the functor $T_{>}$.

It is given by

$$T_{>>}(L(\pm \circ f)) = L(\circ f), \quad T_{>>}(L(\times^p > f)) = L(\times^p > f) \oplus L(\times^{p-1} \times f)$$

for each $p > 0$ (there is a misprint in [GS2]).

7.6.4. **Lemma.** Take $i, p \geq 1$.

(i) $\times^p \circ f = \times^{p-1} > f = \times^p > f$.

(ii) $\times^{p-1} > \times f = 0$.

(iii) $\times^p > f = \times^{p-1} > \times f = \times^p \emptyset$.

**Proof.** Using (12) for $T_0$ we obtain

$$\times^p \circ f = \times^{p-1} > f = \times^p > f$$

which establishes the first formula. Using 7.1 for $T_{>}$ we obtain

$$\times^p f = \times^p > f = [DS_r(T_{>}(L(\times^p > f)) : L(\times^p > f))] = \sum [DS_r(L(\times^p > f)) : L_1][T_{>}(L) : L(\times^p > f)] = [DS_r(L(\times^p > f)) : L(\times^p > f)] + [DS_r(L(\times^p > f)) : L(\times^{p-1} \times f)] = \times^p > f + \times^p > \times f.$$  

By Corollary 7.3 the second summand in the last formula is zero; this implies the second formula. Similarly, $T_{>}(L(\times^{p-1} \times f)) = L(\times^p > f)$ implies

$$\times^p f = \times^{p-1} \times f + \times^{p-1} \times f.$$

By (13) if $f = \times f'$, then $\times^p f = 0$, so the both summands in the right-hand side are equal to 0; in particular, $\times^p > \times f' = 0.$
If \( f = \circ f' \) we have

\[
\frac{x^p \circ f'}{x^i} = \frac{x^{p-1} \circ f'}{x^{i-1}} + \frac{x^{p-1} \circ f'}{x^i} + \frac{x^{p-1}}{x^{i-1}}.
\]

Using (i) we conclude

\[
\frac{x^{p-1}}{x^i} \circ f' = 0.
\]

This establishes (ii).

Using 7.1 for \( T_{> >} \) we obtain

\[
\frac{x^p f}{> >} = \frac{x^p > f}{> >} = \frac{\text{DS}_r(T_{> >}(L(\circ f)) : L(>) )}{> >} = \frac{x^p}{> >} f
\]

and, similarly,

\[
\frac{x^p f}{>} = \frac{x^{p-1} > f}{>}.
\]

This establishes (iii). \( \square \)

7.6.5. **Corollary.** Take \( i \geq 1 \). One has \( \frac{x^p \circ f}{x^i} = 0 \) and \( \frac{x^p \circ f}{x^i} = \frac{x^{p-1} > f}{> >} \). Moreover,

\[
\frac{x^p \circ \circ f'}{> j \text{ times}} = \frac{x^{p-i} \circ \circ f'}{> j-2i \text{ times}}
\]

and \( \frac{x^p}{> i} \neq 0 \) implies that \( p \geq i \) and the diagram \( f \) is as above (i.e., \( f = \circ \circ f' \) for some \( j \geq 2i \)).

8. **Computation of \( \text{DS}_1(L) \) in terms of arc diagrams**

8.1. **Arc diagrams.** We assign to each core-free diagram an arc diagram \( \text{Arc}(f) \); if \( f \) is contains at most one \( x \) at the zero position ("gl-type"), the corresponding arc diagram coincides with the usual arc diagram introduced in [GS2]. It differs from the arc- or cup diagrams of [ES]. Advantages of our weight and arc diagrams are that they can immediately be read off from the weight \( \lambda \) and describe the effect of \( \text{DS} \) very nicely. On the other they do not connect to Khovanov algebras of type \( D \) and therefore to the Kazhdan-Lusztig combinatoric of parabolic category \( O \).

An arc diagram is the following data: a diagram \( f \), where the symbols \( x \) at the zero position are drawn vertically and a collection of non-intersecting arcs. Each arc
connects one symbol \( \times \) (the left end) with one or two empty symbols according to the following rules:

if the symbol \( \times \) has non-zero coordinate, the arc connects this symbol with one empty symbol;

for \( \ell = 0 \) (i.e., \( t = 0, 1 \)) the lowest symbol \( \times \) in the zero position, is connected by an arc with one empty symbol and the other symbols \( \times \) in the zero position are connected with two empty symbols;

for \( \ell = 1 \) (i.e., \( t = 2 \)) all symbols \( \times \) in the zero position are connected with two empty symbols.

An empty position in \( f \) is called free in the arc diagram if this position is not an end of an arc.

We say that an arc is supported by a symbol \( \times \) if this symbol is the left end of the arc; if the arc is supported by a symbol \( \times \) with the coordinate \( a \) we denote this arc by \( \text{arc}(a; b) \) (resp., \( \text{arc}(0; b_1, b_2) \)) where \( b \) (resp., \( b_1 < b_2 \)) is the coordinate of the right end (resp., right ends) of the arc.

We remark that we can similarly define an arc diagram for any weight diagram by just fixing and ignoring the core symbols.

8.1.1. Partial order. We consider a partial order on the set of arcs by saying that one arc is smaller than another one if the first one is ”below” the second one, that is

\[
\text{arc}(a; b) > \text{arc}(a'; b') \text{ if and only if } a < a' < b < b';
\]

\[
\text{arc}(0; b_1, b_2) > \text{arc}(a'; b') \text{ if and only if } b' < b_2;
\]

\[
\text{arc}(0; b_1, b_2) > \text{arc}(0; b_1', b_2') \text{ if and only if } b_2' < b_2.
\]

Since the arcs do not intersect, one has

\[
\text{arc}(a; b) > \text{arc}(a'; b') \iff a < a' < b;
\]

\[
\text{arc}(0; b_1, b_2) > \text{arc}(a'; b') \iff a' < b_2
\]

and any two distinct arcs of the form \( \text{arc}(0; b_1, b_2), \text{arc}(0; b_1', b_2') \) are comparable: either \( \text{arc}(0; b_1, b_2) > \text{arc}(0; b_1', b_2') \) or \( \text{arc}(0; b_1, b_2) < \text{arc}(0; b_1', b_2') \).

8.1.2. Definition. We assign to a core-free diagram \( f \) the arc diagram \( \text{Arc}(f) \) with the following properties:

each symbol \( \times \) is the left end of exactly one arc;

there are no free positions under the arcs.

8.1.3. Example. The weight diagram below does not have a sign for \( t = 0 \) (\( \mathfrak{g} = \mathfrak{osp}(10|10) \)) or have one of the signs \( \pm \) for \( t = 1 \) (\( \mathfrak{g} = \mathfrak{osp}(11|10) \)).
8.1.4. Examples. In the following diagrams we ignore signs. Note that in case the arc connects $\times$ with two empty symbols, this still counts as one arc (also for Theorem 8.2).

For $\mathfrak{osp}(5|4)$ the arc diagram of $+ \times \circ \circ \times$ is given by

The arc diagrams of $- \times^2 \circ \ldots \circ \times$ are the following:

The arc diagram of $- \times^2 \times \circ$

The arc diagram of $- \times^2 \circ \circ \circ$

The arc diagram of $- \times^2 \circ \circ \circ \circ$
The arc diagram of $- \times^2 \odot \odot \times \odot$

For $\mathfrak{osp}(6|4)$ the arc diagram of $> \times \odot \odot \times$ is

and the arc diagram of $+ \times \odot \odot \times$ is

8.1.5. Description. The arc diagram $\text{Arc}(f)$ is unique. For instance, for $\ell = 0$ the arc diagram $\text{Arc}(f)$ can be constructed as follows:

1. first we consider the diagram $f'$ obtained from $f$ by removing all but one symbol $\times$ from the zero position ($f' = f$ if the zero position contains $\odot$ or $\times$);

2. for $f'$ we construct the arc diagram in the usual way (we connect each symbol $\times$ with the next free empty symbol starting from the rightmost symbol $\times$ and going to the left);

3. if the zero in $f$ is occupied by $\times^p$ with $p > 1$, we connect each of the remaining $p - 1$ symbol $\times$ with next two free positions starting from the lowest symbol $\times$ and going up.

8.1.6. Maximal arcs. Note that an arc $\text{arc}(a; b)$ (resp., $(0; b_1, b_2)$) is maximal if and only if it is supported by the top symbol $\times$ in the zero position. Note that $\text{Arc}(\times^p f)$ is obtained from $\text{Arc}(\times^{p+1} f)$ by ”removal” of the maximal arc supported by the top symbol $\times$ in the zero position. Notice that for $\mathfrak{osp}(2n + 1|2n)$, when we erase a symbol $\times$ from a signed diagram we either obtain a signed diagram with the same sign or an unsigned diagram if the resulting diagram does not have $\times$ in the zero position.

Consider a diagram of the form $f_1 \times f_2$ where the symbol $\times$ occupies a position $a > 0$. This symbol $\times$ supports a maximal arc $\text{arc}(a; b)$ if and only if $\text{Arc}(f_1 \circ f_2)$ is obtained from $\text{Arc}(f_1 \times f_2)$ by ”removal” of the arc $\text{arc}(a; 0)$, i.e.

$$\text{Arc}(f_1 \circ f_2) = \text{Arc}(f_1) \circ \text{Arc}(f_2)$$

(if we remove an arc which is not maximal, the resulting diagram is not an arc diagram).

An arc $\text{arc}(a; b)$ with $a > 0$ is maximal in $\text{Arc}(f)$ if and only if for each $0 < c < a$ one has $\ell_f(c; a) \geq 0$, where $\ell_f(c; a)$ is the number of the symbols $\times$ minus the number of empty positions strictly between the positions $c$ and $a$. 
For instance, in Example 8.1.3, the maximal arcs are \(arc(11; 12)\) and \(arc(0; 4, 9)\).

8.2. **Theorem.** (i) The module \(L(\nu)\) is a subquotient of \(DS_1(L(\lambda))\) if and only if \(Arc(\text{howl}(\nu))\) is obtained from \(Arc(\text{howl}(\lambda))\) by removing a maximal arc and, in addition, in the \(osp(2m + 1|2n)\)-case, if \(\nu\) has a sign, then the signs of \(\lambda\) and \(\nu\) are equal.

(ii) Let \(e\) be the number of free positions in \(Arc(\text{howl}(\lambda))\) which are to the left of the maximal arc.

For \(osp(2m + 1|2n)\) one has

\[
[DS_1(L(\lambda)) : L(\nu)] = \begin{cases} 
(1|0) & \text{if } e = 0; \\
(2|0) & \text{if } e \text{ is even and } e \neq 0; \\
(0|2) & \text{if } e \text{ is odd.}
\end{cases}
\]

The same formula holds for the case \(t = 2\). For \(t = 0\)

\[
[DS_1(L(\lambda)) : L(\nu)] = \begin{cases} 
(1|0) & \text{if } e \text{ is even; } \\
(0|1) & \text{if } e \text{ is odd.}
\end{cases}
\]

8.2.1. **Examples.** Take \(r = 1\) and use the notations of Section 7.

For \(osp(5|4)\) we have

\[
\begin{array}{c}
+ \times \circ \circ \times \\
\circ \circ \circ \times
\end{array} = (1|0) \quad \begin{array}{c}
+ \times \circ \circ \times \\
+ \times
\end{array} = (0|2)
\]

and \(\begin{array}{c}
+ \times \circ \circ \times \\
\end{array} = 0\) in other cases. The multiplicity is

\[
-x^2 \underbrace{\circ \circ \cdots \circ}_j \times = \begin{cases} 
0 & \text{for } j < 3, \\
(1|0) & \text{for } j = 3, \\
(2|0) & \text{if } j > 3 \text{ is odd,} \\
(0|2) & \text{if } j > 3 \text{ is even.}
\end{cases}
\]

For \(osp(4|4)\) we have

\[
\begin{array}{c}
+ \circ \circ \circ \times \\
+ \circ \times
\end{array} = \begin{array}{c}
+ \circ \circ \circ \times \\
- \circ \times
\end{array} = (0|1).
\]

For \(osp(6|4)\) we have

\[
\begin{array}{c}
> \times \circ \circ \times \\
> \circ \circ \circ \times
\end{array} = (1|0) \quad \begin{array}{c}
+ \times \circ \circ \times \\
+ \times
\end{array} = (0|2)
\]

and \(\begin{array}{c}
+ \times \circ \circ \times \\
\end{array} = 0\) in other cases.
8.3. **Low rank cases.** The proof will be a reduction to the atypicality 1 case, so consider the cases $\mathfrak{osp}(2 + t|2)$ for $t = 0, 1, 2$. In these cases $DS(L(\lambda))$ was computed in [Gor2] (note that $x$ is necessarily of rank 1 and we omit it from the notation). For $t = 0, 1$ $\mathfrak{g}_x = 0$, so $\mathfrak{g}_x$-modules are supervector spaces. For $t = 2$ $\mathfrak{g}_x = \mathbb{C}$. For the principal block $\mathcal{F}^>(\mathfrak{g})$ $DS_x(\mathcal{F}^>(\mathfrak{g}))$ is the category of finite-dimensional supervector spaces with trivial action of $\mathfrak{g}_x$. The simple modules in the principal block are of the form $L(\lambda_j)$, $j \in \mathbb{Z}$, for $t = 0$ and $j \in \mathbb{N}$ for $t = 1, 2$.

For $t = 0$

$$DS(L(\lambda_j)) \cong \Pi^j(\mathbb{C}) \quad \forall j \in \mathbb{Z}$$

where $\lambda_j := j\varepsilon_1 + |j|\delta_1$ for $j \in \mathbb{Z}$.

For $t = 1$ we put $\lambda_0 := 0$ and $\lambda_j := j\varepsilon_1 + (j-1)\delta_1$ for $j \geq 1$. For $t = 2$ we put $\lambda_j := j\varepsilon_1 + j\delta_1$ for $j \geq 0$. Then one has the uniform rule [Gor2]

$$DS(L(\lambda_0)) = DS(L(\lambda_1)) \cong \mathbb{C} \quad \text{and}$$

$$DS(L(\lambda_j)) \cong \Pi^{j-1}(\mathbb{C})^\otimes 2 \quad \text{for } j \geq 2.$$

8.4. **Proof.** Set $f := \text{howl}(\nu)$. Denote by $u$ the coordinate of the rightmost symbol $\times$ in $f$.

If $u > 0$, then $\text{Arc}(f)$ has a minimal arc $\text{arc}(u; u+1)$. If $f' \neq 0$, then, by 7.3, $f'$ has a minimal arc $\text{arc}(u; u+1)$ and

$$\frac{f'}{f} = \frac{f'}{f},$$

where $\overline{f}$ (resp., $\overline{f'}$) are obtained from $f'$ (resp., $f$) by “shrinking” the minimal arc $\text{arc}(u; u+1)$. Clearly, $\text{Arc}(f)$ can be obtained from $\text{Arc}(f')$ by deleting a maximal arc if and only if $\text{Arc}(\overline{f})$ can be obtained from $\text{Arc}(\overline{f'})$ by deleting a maximal arc. Each maximal arc in $\text{Arc}(f)$ corresponds to a maximal arc in $\text{Arc}(f')$ and the number of free positions to the left of each maximal arc is the same. This reduces the statement to the case $u = 0$. Corollaries 7.5.3, 7.6.5 reduce the statement to the cases when $f = \emptyset$ or $f = \Rightarrow$. Since $r = 1$, the case $f = \emptyset$ correspond to $\mathfrak{osp}(2|2), \mathfrak{osp}(3|2)$ and the case $f = \Rightarrow$ correspond to the case $\mathfrak{osp}(4|2)$. For these cases the formula was checked in [Gor2] (see 8.3). \qed

We will use the following lemma.

8.4.1. **Lemma.** Take $\tau$ as in 3.7.7. For $r = 1$ one has $\frac{f}{f'} = \frac{\tau(f)}{\tau(f')}$. 

Proof. One has to check the following cases

\[ f = \times^{\nu} f \circ f_1 \times f_2 \quad f' = \times^{\nu} f_1 \circ f_2 \]
\[ f = \times^{\nu} f_1 \times f_2 \quad f' = \times^{\nu} f_1 \circ f_2 \]
\[ f = \times^{\nu} f_1 \quad f' = \times^{\nu} f_1 \]
\[ f = \times^{\nu+1} f_1 \quad f' = \times^{\nu} f_1 \]
\[ f = \times^{\nu+1} f_1 \quad f' = \times^{\nu} f_1 \]

This can be easily done using the properties of maximal arcs discussed in 8.1.6. \hfill \Box

9. Semisimplicity of \( DS_x(L) \)

Retain the notation of 3.7.8. Denote by \( \mathcal{F}_+(g) \) the Serre subcategory of \( \tilde{\mathcal{F}}(g) \) generated by \( L \in \text{Irr}(\tilde{\mathcal{F}}(g)) \) with \( \text{dex}(L) = 1 \). By Corollary 6.2.2, \( \mathcal{F}_+(g) \) is semisimple.

In Theorem 9.3 we will show that \( DS_x(\mathcal{F}_+(g)) \subseteq \mathcal{F}_+(g_x) \). As a result, \( DS_x(L) \) is semisimple for each simple finite-dimensional module \( L \).

9.1. Supercharacters. Recall that besides the usual character we have the supercharacter

\[ \text{sch} \; L(\lambda) = (-1)^{p(\lambda)} \pi(\text{ch} \; L(\lambda)), \]

where \( \pi : \mathbb{Z}[\Lambda_{m|n}] \to \mathbb{Z}[\Lambda_{m|n}] \) is the linear involution given by

\[ \pi(e^\mu) := (-1)^{p(\mu)} e^\mu. \]

The supercharacter ring of \( \mathcal{F}_{\text{in}} \) is the image of the map \( \text{sch} : \mathcal{F}_{\text{in}} \to \mathbb{Z}[\mathfrak{h}^*] \); we denote this ring by \( \mathcal{J}(g) \).

9.2. We will use the following lemma.

Lemma. Let \( M \in \mathcal{F}_{\text{in}}(g) \) and \( N \in \mathcal{F}_+(g) \) be such that \( \text{sch} \; M = \text{sch} \; N \) in the supercharacter ring. If \( \dim M \leq \dim N \), then \( M \cong N \).

Proof. For each \( \nu \in \Lambda_{m+\ell|n}^{(\ell)} \) we set

\[ (d_0(\nu)|d_1(\nu)) := [N : L(\nu)], \quad (d'_0(\nu)|d'_1(\nu)) := [M : L(\nu)]. \]

Since \( N \in \mathcal{F}_+(g) \) we have \( d_0(\nu)d_1(\nu) = 0 \).

Since \( \{\text{sch} \; L(\nu) \mid \nu \in \Lambda_{m+\ell|n}^{(\ell)} \} \) are linearly independent, the equality \( \text{sch} \; M = \text{sch} \; N \) implies

\[ (d'_0(\nu)|d'_1(\nu)) = (d_0(\nu) + j(\nu)|d_1(\nu) + j(\nu)) \quad \text{for some } j(\nu) \in \mathbb{Z}. \]

Combining \( d_0(\nu)d_1(\nu) = 0 \) with \( d'_0(\nu), d'_1(\nu) \geq 0 \), we obtain \( j(\nu) \geq 0 \) for each \( \nu \). Using \( \dim M \leq \dim N \) we get \( j(\nu) = 0 \) for each \( \nu \), that is

\[ \forall \nu \quad [M : L(\nu)] = [N : L(\nu)]. \tag{17} \]
Hence $M \in \mathcal{F}_+(\mathfrak{g})$. Since $\mathcal{F}_+(\mathfrak{g})$ is semisimple, $M$ and $N$ are completely reducible. Thus (17) gives $M \cong N$. \hfill \square

9.3. **Theorem.**

(i) For each $x$ one has $DS_x(\mathcal{F}_+(\mathfrak{g})) = \mathcal{F}_+(\mathfrak{g}_x)$.

(ii) For each $L \in \text{Irr}(\bar{\mathcal{F}}(\mathfrak{g}))$ one has $DS_{r+1}(L) \cong DS_1(DS_r(L))$.

**Proof.** It is enough to consider the case $x := x_r$.

For (i) we have to verify that for each $L(\lambda) \in \text{Irr}(\mathcal{F}_+(\mathfrak{g}))$, and $L_{\mathfrak{g}_x}(\nu) \in \text{Irr}(\mathcal{F}(\mathfrak{g}_x))$ the graded multiplicity

$$(d_0|d_1) := [DS_x(L(\lambda)) : L_{\mathfrak{g}_x}(\nu)]$$

satisfies

$$(18) \begin{cases} d_1 = 0 & \text{if } \text{dex}(\nu) = 1, \\ d_0 = 0 & \text{if } \text{dex}(\nu) = -1. \end{cases}$$

Recall that $\text{dex}(\lambda) = \text{dex}(\text{howl}(\lambda))$. Using Corollary 6.5.3 we reduce (i) to the case when $\lambda, \nu$ are core-free. In this case $\mathfrak{g} = \mathfrak{osp}(2n+1|2n) = \mathfrak{g}_n$ (for some $n$) and $\mathfrak{g}_x = \mathfrak{g}_{n-r}$.

We proceed by induction on $r$. Note that (ii) for $r = 0$ is tautological.

Consider the case $r = 1$ for $t = 0, 1$. For a core-free diagram $f$ denote by $||f||$ the sum of the coordinates of the symbols $\times$ in $f$. One has $\text{dex}(\lambda(f)) = (-1)^{||f||}$, so (18) follows from Theorem 8.2. The case $r = 1$ for $t = 2$ follows from $t = 1$ and Lemma 8.4.1. This establishes (i) for $r = 1$.

Now fix any $t$ and take $r \geq 2$. By induction, $DS_{r-1}(L(\lambda)) \in \mathcal{F}_+(\mathfrak{g}_{n-r+1})$. Using (i) for $r = 1$ we get

$N := DS_1(DS_{r-1}(L(\lambda))) \in \mathcal{F}_+(\mathfrak{g}_x)$.

By [HR], $\text{sch } N = \text{sch } DS_r(L(\lambda))$; by [Gor1], Lem. 2.4.1, $\dim DS_r(L(\lambda)) \leq \dim N$. Using Lemma 9.2 we obtain $N \cong DS_r(L(\lambda))$ as required. \hfill \square

9.4. **Corollary.** Take $\tau : \Lambda^{(2)}_{m+1|n} \to \Lambda^{(1)}_{m|n}$ as in 3.7.7. One has

$$[DS_r(L(\lambda)) : L_{\mathfrak{g}_x}(\nu))] = [DS_r(L(\tau(\lambda))) : L_{\mathfrak{g}_x}(\tau(\nu))]]$$

**Proof.** The case $r = 1$ was treated in Lemma 8.4.1. The general case follows from Theorem 9.3 (ii). \hfill \square

9.4.1. We say that a module $M$ is pure if for any subquotient $L$ of $M$, $\Pi(L)$ is not a subquotient of $M$. Theorem 9.3 implies immediately the following assertion.

9.4.2. **Corollary.** $DS_x(L(\lambda))$ is pure for any $x$ and any $\lambda$. 

9.4.3. **Corollary.** For irreducible $L(\lambda)$

$$DS_1(L(\lambda)) \cong \bigoplus_i m_i \Pi^n L(\lambda_i)$$

where the arc diagram of $L(\lambda_i)$ is obtained by removing the $i$-th maximal arc and the associated $\times$ from the arc diagram of $\lambda$. The multiplicity $m_i$ is 1 or 2 according to the rules of Theorem 8.2 and $n_i = 1 \mod 2$ if and only if the parities of $\lambda$ and $\lambda_i$ differ.

Since $DS_{r+1}(L) \cong DS_1(DS_r(L))$ we can calculate any $DS_{r+1}(L)$ by repeated application of $DS_1$.

9.4.4. **Remark.** In the $\mathfrak{gl}(m|n)$-case $DS(L(\lambda))$ is even multiplicity free [HW].

9.5. **The $OSp$-case.** Using Corollary 9.4.3 and Lemma 5.7 it is easy to describe the effect of $DS_1$ on irreducible $OSp(M|N)$-modules $L_{OSp}(\lambda)$, see below.

Let $\tilde{F}'(M|N)$ denote the category of algebraic representations of $OSp(M|N)$, then we have a commutative diagram

$$\begin{align*}
\tilde{F}'(M|N) & \xrightarrow{Res} \tilde{F}(M|N) \\
\downarrow DS_1 & \quad \quad \downarrow DS_1 \\
\tilde{F}'(M-2|N-2) & \xrightarrow{Res} \tilde{F}(M-2|N-2).
\end{align*}$$

Formula (15) holds for $OSp(2m+1|2n)$-modules if the signs of $L_{OSp}(\lambda, \pm)$ and $L_{OSp}(\nu, \pm)$ are equal; otherwise the multiplicity is zero.

Consider $OSp(2m|2n)$-case. Combining Remark 3.1.8, Corollary 9.4.3 and Lemma 5.7 we conclude that $DS_{t}(L(\lambda))$ has a structure of $OSp(2m|2n)$-module. Let us show that the multiplicity $d_{OSp}(\lambda; \nu) := [DS_1(L_{OSp}(\lambda)) : L_{OSp}(\nu)]$ is given by the formula (16) in the case $\lambda^\sigma = \lambda$, $t = 0$ and by the formula (15) otherwise.

Indeed, combining Remark 3.1.8 and Lemma 5.7 we get

$$d_{OSp}(\lambda; \nu) = [DS_1(L_{OSp}(\lambda)) : L(\nu)] = \begin{cases} [DS_1(L(\lambda)) : L(\nu)] & \text{if } \lambda^\sigma = \lambda \\ 2[DS_1(L(\lambda)) : L(\nu)] & \text{otherwise.} \end{cases}$$

Thus $d_{OSp}(\lambda; \nu)$ is given by the formula (16) (resp., (15)) for the case when $\lambda^\sigma = \lambda$ and $t = 0$ (resp., $t = 2$). For the remaining case $t = 0$ and $\lambda^\sigma \neq \lambda$ we get

$$d_{OSp}(\lambda; \nu) = \begin{cases} (2|0) & \text{if } e \text{ is even} \\ (0|2) & \text{if } e \text{ is odd}, \end{cases}$$

where $e$ as in Theorem 8.2. Notice that $e \neq 0$, since the condition $\lambda^\sigma \neq \lambda$ implies that the zero position of the diagram of $\lambda$ is empty. Hence $d_{OSp}(\lambda; \nu)$ is given by the formula (15).
9.6. Question. For a simple module $L$ in $\tilde{\mathcal{F}}(\mathfrak{osp}(2m|2n))$, Lemma 5.7 and Theorem 9.3 imply that $DS_x(L)$ is an $OSp(2m|2n)$-module for any $x$. Is this still true for an arbitrary module $L$ in $\tilde{\mathcal{F}}(2m|2n)$?

A related question can be asked about finite-dimensional modules over $\mathfrak{g} = D(2|1,a)$, $F(4)$ (for these cases $DS_x(\mathfrak{g})$ admits an involution $\sigma$ and $DS_x(L)$ is $\sigma$-invariant for each simple module $L$ in $\tilde{\mathcal{F}}(\mathfrak{g})$, see [Gor2]).

9.7. Superdimensions. Similarly to [HW] in the $\mathfrak{gl}(m|n)$-case this allows us now to compute the superdimension of any irreducible $L(\lambda)$. Let $\lambda$ be maximal atypical and $x$ of rank equal to the atypicality. Then $\mathfrak{g}_x$ is either an orthogonal or symplectic Lie algebra or $\mathfrak{osp}(1|2r)$ for some $r$. In each case the superdimension of an irreducible module is known. Since $DS$ is a symmetric monoidal functor it preserves the superdimension. So $\text{sdim}(L(\lambda)) = sdim(DS(L(\lambda)))$. So in order to compute $\text{sdim}(L(\lambda))$ it suffices to compute the multiplicity $m(\lambda)$ of the isotypic representation $DS(L(\lambda))$ of $\mathfrak{g}_x$, but this multiplicity is computed exactly by Theorem 8.2 since $DS_r(L(\lambda)) = DS_1(\ldots(DS_1(L(\lambda))))$. Alternatively the multiplicity can also be computed via [GH].

9.7.1. Example. Consider a weight $\lambda$ with the diagram consisting of $m$ maximal arcs (with no other arcs) and $i$ symbols $\times$ in the zero position (since all arcs are maximal $i \leq 1$). Using the results of [HW] for $\mathfrak{gl}$-case we get

$$\text{sdim } L(\lambda) = \begin{cases} m! & \text{for } \mathfrak{g} = \mathfrak{gl}(m|m) \\ 2^{m-1}m! & \text{for } \mathfrak{g} = \mathfrak{osp}(2m|2m) \\ 2^{m-i}m! & \text{for } \mathfrak{g} = \mathfrak{osp}(2m + 1|2m), \mathfrak{osp}(2m + 2|2m). \end{cases}$$

9.7.2. Example. For an $\mathfrak{osp}(2m + 1|2n)$-diagram with the empty zero position the multiplicity is $2^l|F|!/|F!|$ for $l = \min(m|n)$ where $F$ is the forest associated to the arc diagram of $\lambda$ (see [HW]). For $t = 0$ the multiplicity $m(\lambda)$ is given by a forest factorial exactly as for $\mathfrak{gl}(m|n)$, but one has to take into account that in this case a removal of one arc may produce two arc diagrams, which differ by the sign (if the resulting arc diagram has $\times$ in the zero position).

APPENDIX A. THE LANGUAGE OF GRUSON-SERGANOVA AND EHRLICHS-STRUPPEL

Weight and arc diagrams also appear in the work of Ehrig and Stroppel [ES], but our conventions differ considerably. They attach to a weight $\lambda$ a weight diagram $\overset{\uparrow}{\lambda}^\circ$ and a cup diagram $\overset{\downarrow}{\lambda}^\circ$. We shortly explain the differences, mostly following [ES], Section 6.3.

A.1. The associated tailless weight. Gruson-Serganova [GS2], Section 8, associate to every weight $\lambda$ a tailless weight $\bar{\lambda}$.

Assume that $M = 2m + 1$ and consider the core-free weight diagram $f_\lambda$. Remove the tail of $f_\lambda$. Note that in case the diagram has a sign $+$, one symbol $\times$ remains at the
zero position. Form the arc diagram of the new weight diagram (with removed tail) and
number the free vertices ◦ from left to right by 1, 2, 3, . . . . Label the vertices at positions
1, 3, . . . then by an × until 2l − 1 where l is the number of removed tail symbols × (these
× are called coloured in [GS2]). This gives the weight diagram f_\bar{\lambda}. To assign an arc
diagram, first form the arc diagram of the weight diagram with removed tail and then
connect the coloured ×’ in the usual way with free positions. Finally place a dot on each
arc which starts at one of the edges 1, 3, . . . , 2l − 1. For M = 2m even, proceed in the
same way as if f_\bar{\lambda} was a weight diagram for M = 2m + 1 with a + -sign.

A.2. Example. If f_\bar{\lambda} = + × 3 × ◦ ◦ . . . , then removing the tail gives the diagram × × ◦ ◦ . . . .
The weight diagram f_\bar{\lambda} is given by + × × ◦ ◦ × ◦ ◦ . . . . Its dotted arc diagram is given by

\begin{center}
\begin{tikzpicture}
\node (0) at (0,0) {$0$}; \node (1) at (1,0) {$1$}; \node (2) at (2,0) {$2$}; \node (3) at (3,0) {$3$}; \node (4) at (4,0) {$4$}; \node (5) at (5,0) {$5$}; \node (6) at (6,0) {$6$}; \node (7) at (7,0) {$7$};
\draw (0) -- (1) -- (2) -- (3) -- (4) -- (5) -- (6) -- (7);
\end{tikzpicture}
\end{center}

The arc diagrams of [ES] do not have labelings × etc. and contain moreover infinite
rays (undotted and dotted) at all other free positions. Ignoring this, the arc diagram of \bar{\lambda}
agrees with the cup diagram of [ES]: By [ES], Proposition 6.1, the dotted arc diagram of
\bar{\lambda} is the mirror image of the cup diagram \lceil \lambda \rceil. Note that the labelings for weight diagrams
differ also from the ones used in [ES], for instance our × corresponds to the ∨ in loc. cit.

A.2.1. Example. Consider again the weight diagram f_\bar{\lambda} = + × 3 × ◦ ◦ . . . . Its associated
arc diagram is

\begin{center}
\begin{tikzpicture}
\node (0) at (0,0) {$0$}; \node (1) at (1,0) {$1$}; \node (2) at (2,0) {$2$}; \node (3) at (3,0) {$3$}; \node (4) at (4,0) {$4$}; \node (5) at (5,0) {$5$}; \node (6) at (6,0) {$6$}; \node (7) at (7,0) {$7$};
\draw (0) -- (1) -- (2) -- (3) -- (4) -- (5) -- (6) -- (7);
\end{tikzpicture}
\end{center}

By Theorem 8.2 DS_1(L(\lambda)) \cong L(\lambda_1) with f_{\lambda_1} = × 2 × ◦ ◦ . . . . The dotted arc diagram
of \lambda_1 is given by

\begin{center}
\begin{tikzpicture}
\node (0) at (0,0) {$0$}; \node (1) at (1,0) {$1$}; \node (2) at (2,0) {$2$}; \node (3) at (3,0) {$3$}; \node (4) at (4,0) {$4$}; \node (5) at (5,0) {$5$};
\draw (0) -- (1) -- (2) -- (3) -- (4) -- (5);
\end{tikzpicture}
\end{center}

i.e. by removing the rightmost dotted cup.
App. B. Functors $\text{Res}$ and $\text{Ind}$

Let $g$ be a finite-dimensional Lie superalgebra, $h \subset g$ be a subalgebra and $z \in h$ be an element with the following properties:

(H1) $h$ is the centralizer of $h_\mathfrak{f}$ in $g$;

(H2) $h_0$ acts diagonally on $g$;

(H3) all eigenvalues of $\text{ad} \, h$ are real and $g^{\text{ad} \, h} = h$.

In this case we have a usual triangular decomposition

$$g = h \oplus (\bigoplus_{\alpha \in \Delta(g)} g_\alpha) \quad \text{where}$$

$$\Delta(g) \subset h_\mathfrak{f}^*, \quad \Delta(g) = \Delta^+(g) \bigoplus \Delta^-(g)$$

$$g_\alpha := \{g \in g | [h, g] = \alpha(h)g \ \text{for all} \ h \in h_\mathfrak{f}\}$$

$$\Delta^+(g) := \{\alpha \in \Delta(g) | \alpha(h) > 0\}, \quad \Delta^-(g) := \{\alpha \in \Delta(g) | \alpha(h) < 0\}.$$ 

We set $n^\pm := \bigoplus_{\alpha \in \Delta^\pm} g_\alpha$, $b := n \oplus n^+$. We consider the partial order on $h_\mathfrak{f}^*$ given by

$$\lambda > \nu \ \text{if} \ \nu - \lambda \in \mathbb{N} \Delta^-.$$

B.1. We fix $z \in h_\mathfrak{f}^*$ satisfying

$$\alpha(z) \in \mathbb{R}_{\geq 0} \ \text{for all} \ \alpha \in \Delta^+ \ \text{and} \ \alpha(z) \in \mathbb{R}_{\leq 0} \ \text{for all} \ \alpha \in \Delta^-.$$ 

We set $u := \oplus_{\alpha \in \Delta^+} g_\alpha$, $v := g^\pm + b = t \times m$.

The triples $(p(z), h, h)$ and $(t(z), h, h)$ satisfy (H1)–(H3) and

$$\Delta^+(p) = \Delta^+(g), \quad \Delta^+(t) = \{\alpha \in \Delta^+(g) | \alpha(z) = 0\}$$

$$\Delta^-(p) = \Delta^-(t) = \{\alpha \in \Delta^-(g) | \alpha(z) = 0\}.$$ 

B.2. Functors $\text{Res}_a$ and $\text{Ind}$. We denote by $\mathcal{O}$ the full category of finitely generated modules with a diagonal action of $h_\mathfrak{f}^*$ and locally nilpotent action of $n$. It is easy to see that, up to a parity change, the simple $h$-modules are parametrized by $\lambda \in h_\mathfrak{f}^*$; we denote by $C_\lambda$ a simple $h$-module, where $h_\mathfrak{f}$ acts by $\lambda$. We view $C_\lambda$ as a $b$-module with the zero action of $n$ and set

$$M(\lambda) := \text{Ind}_h^g C_\lambda;$$

this module has a unique simple quotients which we denote by $L(\lambda)$. (The module $M(\lambda)$ is a Verma module if $h_\mathfrak{f}^* = 0$).

For each $a \in \mathbb{C}$ we define a functor $\text{Res}_a : \mathcal{O}(g) \to \mathcal{O}(t)$ by

$$\text{Res}_a(N) := \{v \in N | \ zv = av \} \ \text{for} \ N \in \mathcal{O}(g).$$

We assume that

each module in $\mathcal{O}$ admits a unique maximal finite-dimensional quotient.
This holds, for example, if $\mathfrak{g}_0$ is reductive, since in this case all modules in $O$ have finite lengths.

View $V \in O(t)$ as a $p$-module with the trivial action of $m$ and consider the induced $g$-module

$$\text{Ind}(V) = U(g) \otimes_{U(p)} V.$$  

We denote by $\overline{\text{Ind}}(V)$ the maximal finite-dimensional quotient of $\text{Ind}(V)$.

For a semisimple $h_0$-module $N$ we denote by $N_\nu$ the weight space of the weight $\nu$ and by $\Omega(N)$ the set of weights of $N$.

B.3. **Assumption.** We fix a block $B$ in the category $O$ and set

$$P^+(B)_0 := \{ \lambda \in h_0^\perp | \dim L(\lambda) < \infty, \ L(\lambda) \in B \text{ or } \Pi(L(\lambda)) \in B \}.$$  

We fix $h'' \subset h_0^\perp$ such that $z \in h''$, $[h'', t] = 0$

and $\mu \in (h'')^*$ satisfying

(20) \quad $\forall \lambda, \nu \in P^+(B) \ \lambda \geq \nu, \ \lambda|_{h''} = \mu \implies \nu|_{h''} = \mu.$  

We set

$$A := \{ \lambda \in P^+(B) | \lambda|_{h''} = \mu \}.$$  

By above, if $\lambda \in A$ and $\nu \in P^+(B)$ are such that $\lambda \geq \nu$, then $\nu \in A$.

Let $F_g(A)$ (resp., $F_t(A)$) be the Serre subcategory of $O(g)$ (resp., of $O(t)$) generated by the simple modules $\{ L(\lambda), \Pi(L(\lambda)) \}_{\lambda \in A}$ (resp., by $\{ L_t(\lambda), \Pi(L_t(\lambda)) \}_{\lambda \in A}$).

B.4. **Proposition.** Set $a := \mu(z)$ and $\text{Res} := \text{Res}_a$.

(i) For $N \in F_g(A)$ one has $\text{Res}(N) = N^m$. For $\lambda \in A$ one has $\text{Res}(L(\lambda)) = L_t(\lambda)$.

(ii) The restrictions of $\text{Res}$ and $\overline{\text{Ind}}$ give an equivalence of categories $F_g(A)$ and $F_t(A)$.

**Proof.** For each $h$-module $M$ we denote by $\text{Spec}(M)$ the set of $z$-eigenvalues. Recall that $\text{Spec}(\mathfrak{m}) \subset \mathbb{R}_{> 0}$, $\text{Spec}(\mathfrak{n}^-) \subset \mathbb{R}_{\leq 0}$.

Identifying $V$ with $1 \otimes V \subset \text{Ind}(V)$ we obtain

(21) \quad $\text{Res(Ind}(V)) = V$ for $V \in O(t)$.

Take $N \in F_g(A)$. For $\lambda \in A$ one has $\text{Spec}(L(\lambda)) \subset (\mu(z) - \mathbb{R}_{\geq 0}) =: U$; this gives $\text{Spec}(N) \subset U$ and implies

$$\text{Res}(N) \subset N^m.$$
Set \( n' := n \cap t \). Let \( v \in N^m \) be a non-zero vector of weight \( \nu \). The subspace \( \mathcal{U}(n)v \) contains a singular weight vector \( v' \); let \( \lambda' \) be the weight of \( v' \). Since \( N \in \mathcal{F}_g(A) \) we have \( \lambda' \in A \), that is \( \lambda'(z) = \mu(z) \). Since \( mv = 0 \) we have \( \mathcal{U}(n)v = \mathcal{U}(n')v \), so \( v' \in \mathcal{U}(n')v \), that is \( \nu(z) = \lambda'(z) \). Therefore \( v \in \text{Res}(N) \). Hence \( \text{Res}(N) = N^m \).

Take \( \lambda \in A \) and denote by \( v_{\lambda} \) a highest weight vector in \( L(\lambda) \). By above, for each non-zero vector \( v \) the space \( \mathcal{U}(n')v \) contains \( v_{\lambda} \). Hence \( L(\lambda)^m \) is simple, so \( L(\lambda)^m = L(\lambda) \). This establishes (i).

Clearly, \( \text{Res} \) is exact. By (i), \( \text{Res} \) maps simple modules in \( \mathcal{F}_g(A) \) to simple modules in \( \mathcal{F}_t(A) \). Therefore the restriction of \( \text{Res} \) gives an exact functor

\[
\text{Res} : \mathcal{F}_g(A) \rightarrow \mathcal{F}_t(A).
\]

Take a module \( V \in \mathcal{F}_t(A) \). Note that for \( \nu \in A \) the module \( \text{Ind} L_\iota(\nu) \) lies in \( \mathcal{B} \); by (20) each simple finite-dimensional subquotient of \( \text{Ind} L_\iota(\nu) \) is of the form \( L(\nu') \), where \( \nu' \in A \). Since \( \text{Ind} \) is exact, each simple subquotient of \( \text{Ind}(V) \) is \( L(\nu') \) or \( \Pi(L(\nu')) \) for some \( \nu' \in A \), so \( \overline{\text{Ind}}(V) \in \mathcal{F}_g(A) \). Moreover, using the exactness of \( \text{Res} \) and (21) we get

\[
[\text{Ind}(V) : L(\nu)] = [V : L(\iota(\nu))],
\]

that is

\[(22) \quad [\overline{\text{Ind}}(V) : L(\nu)] \leq [V : L(\iota(\nu))].\]

The module \( \text{Ind}(L(\iota(\lambda))) \) is a quotient of \( M(\lambda) \), so \( L(\lambda) \) is a quotient of \( \text{Ind}(L(\iota(\lambda))) \). Using (22) we get

\[\overline{\text{Ind}}(L(\iota(\lambda))) = L(\lambda)\]

For each \( V \in \mathcal{F}_t(A) \) and \( N \in \mathcal{F}_g(A) \) we have

\[\text{Hom}_g(\overline{\text{Ind}}(V), N) = \text{Hom}_g(\text{Ind}(V), N) = \text{Hom}_p(V, N) = \text{Hom}_t(V, N^m)\]

Using (i) we conclude that \( \overline{\text{Ind}} : \mathcal{F}_t(A) \rightarrow \mathcal{F}_g(A) \) is a left adjoint to \( \text{Res} : \mathcal{F}_g(A) \rightarrow \mathcal{F}_t(A) \); by above, these functors map simple modules to simple modules and \( \text{Res} \) is exact.

Take any \( N \in \mathcal{F}_g(A) \) and set \( V := \text{Res}(N) \). Let \( \phi \in \text{Hom}_g(\overline{\text{Ind}}(V), N) \) be the preimage of the identity map

\[V \xrightarrow{\sim} \text{Res}(N)\]

The image of \( \phi \) is the submodule of \( N \) generated by \( \text{Res}(N) \); since \( \text{Res} \) is exact and \( \text{Res}(M) \neq 0 \) for each \( M \in \mathcal{F}(A) \), \( \phi \) is surjective. Moreover, for each \( \nu \in A \) one has

\[ [N : L(\nu)] = [V : L(\iota(\nu))] \]

Combining with (22) we conclude that \( \phi \) is bijective and for each \( \nu \in A \) one has

\[
[\overline{\text{Ind}}(V) : L(\nu)] = [V : L(\iota(\nu))]
\]

which gives

\[(23) \quad [\text{Res}(\overline{\text{Ind}}(V)) : L(\iota(\nu))] = [V : L(\iota(\nu))].\]
Take $V \in \mathcal{F}_t(A)$. Identifying $V$ and $1 \otimes V \subset N$ we have $V = \text{Res}(\text{Ind} V)$. Since Res is an exact functor on $\mathcal{O}$ this gives the natural surjective map $V \rightarrow \text{Res}(\text{Ind}(V))$. By (23) this map is bijective. Hence Res and Ind provide an equivalence of categories $\mathcal{F}_\mathfrak{g}(A)$ and $\mathcal{F}_t(A)$.

B.5. Remark. Consider the case $\mathfrak{h} = \mathfrak{h}_\mathfrak{g}$ and $t = l \times \mathfrak{h}''$. Set $\mathfrak{h}' := l \cap \mathfrak{h}$ and

$$A' := \{\lambda|_{\mathfrak{h}'} \mid \lambda \in A\}.$$  

Since $\mathfrak{h} = \mathfrak{h}' \oplus \mathfrak{h}''$ the map $\lambda \mapsto \lambda|_{\mathfrak{h}'}$ gives a bijection between $A$ and $A'$. Using Proposition B.4 we obtain the equivalence of the category $\mathcal{F}_\mathfrak{g}(A)$ and the Serre category $\mathcal{F}_t(A')$ which is generated by the simple $l$-modules $\{L_l(\lambda'), \Pi(L_l(\lambda'))\}_{\lambda' \in A'}$.

References

[BS] Brundan, Jonathan and Stroppel, Catharina, Highest weight categories arising from Khovanov’s diagram algebra. IV: the general linear supergroup., J. Eur. Math. Soc. (JEMS), 14, 2, (2012).
[CH] Jonathan Comes and Thorsten Heidersdorf, Thick ideals in Deligne’s category $\text{Rep}(O_{\delta})$. J. Algebra, 480, (2017).
[DS] M. Duflo, V. Serganova On associated variety for Lie superalgebras, arXiv:0507198.
[ES] M. Ehrig, C. Stroppel, On the category of finite-dimensional representations of $\text{OSP}(r|2n)$, Krause, Henning (ed.) et al., Representation theory – Current trends and perspectives. EMS Ser. Congr. Rep., 109-170 (2017).
[ES2] M. Ehrig, C. Stroppel, On the category of finite-dimensional representations of $\text{OSP}(r|2n)$. Part II, to appear.
[ES3] M. Ehrig, C. Stroppel, Nazarov-Wenzl algebras, coideal subalgebras and categorified skew Howe duality, Adv. Math., 331, (2018).
[EAS] I. Entova-Aizenbud, V. Serganova Duflo-Serganova functor and superdimension formula for the periplectic Lie superalgebra, arXiv: 1910.02294, (2019).
[Gor1] M. Gorelik, Depths and cores in the light of DS-functors, arXiv: 2010.05721, (2020).
[Gor2] M. Gorelik Bipartite extension graphs and the DS functor, arXiv:2010.12817, (2020).
[GH] M. Gorelik, T. Heidersdorf Gruson-Serganova character formula and the DS functor, to appear.
[GS1] C. Gruson, V. Serganova Cohomology of generalized supergrassmanians and character formulae for basic classical Lie superalgebras, Proc. London Math. Soc., (3), 101 (2010), 852–892.
[GS2] C. Gruson, V. Serganova Bernstein-Gelfand-Gelfand reciprocity and indecomposable projective modules for classical algebraic supergroups, Mosc. Math. J., 13 (2013), no. 2, 281—313.
[H] T. Heidersdorf, On supergroups and their semisimplified representation categories., Algebr. Represent. Theory, 22, 4, (2019).
[HW] T. Heidersdorf, R. Weissauer Cohomological tensor functors on representations of the general linear supergroup, arXiv:1406.0321, to appear in Mem. Am. Math. Soc.
[HW2] T. Heidersdorf, R. Weissauer On classical tensor categories attached to the irreducible representations of the General Linear Supergroups $GL(n|n)$, arXiv:1805.00384, (2018).
[HR] C. Hoyt, S. Reif, Grothendieck rings for Lie superalgebras and the Duflo-Serganova functor, Algebra Number Theory 12 (2018), no. 9, 2167–2184.
[S1] V. Serganova On a superdimension of an irreducible representation of a basic classical Lie superalgebra, in Supersymmetry in mathematics and physics, 253—273, Lecture Notes in Math., 2027, Springer, Heidelberg, 2011.
[S2] V. Serganova, *Finite dimensional representations of algebraic supergroups*, Proceedings of the International Congress of Mathematicians (ICM 2014), Seoul, Korea, August 13–21, 2014. Vol. I: Plenary lectures and ceremonies, (2014)

[Ser] A. Sergeev, *The invariant polynomials on simple Lie superalgebras*, Represent. Theory, 3, (1999).

[W] R. Weissauer, *Model structures, categorical quotients and representations of super commutative Hopf algebras II, The case GL(m|n)*, arXiv:1010.3217.

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