COMPLETE EINSTEIN-KÄHLER METRIC AND
HOLOMORPHIC SECTIONAL CURVATURE ON \( Y_{II}(r, p; K) \)

WEIPING YIN AND LIYOU ZHANG

Abstract. The explicit complete Einstein-Kähler metric on the second type
Cartan-Hartogs domain \( Y_{II}(r, p; K) \) is obtained in this paper when the pa-
parameter \( K \) equals \( \frac{p^2}{2} + \frac{1}{p+1} \). The estimate of holomorphic sectional curvature
under this metric is also given which intervenes between \(-2K\) and \(-\frac{4K}{3} \), and
it is a sharp estimate. In the meantime we also prove that the complete
Einstein-Kähler metric is equivalent to the Bergman metric on \( Y_{II}(r, p; K) \)
when \( K = \frac{p^2}{2} + \frac{1}{p+1} \).

Introduction

It is well known that the Bergman, Carathéodory, Kobayashi and Einstein-
Kähler metrics are four classical invariant metrics in complex analysis. The Bergman
metric was introduced by S.Bergman for one variable \([1]\) in 1921 and for several
variables\([2]\) in 1933.C.Carathéodory introduced the invariant distance in 1926\([3]\) and
H.Reiffen introduced the invariant metric in 1963\([4]\), therefore the Carathéodory
metric is also called Carathéodory-Reiffen metric. The Kobayashi metric was intro-
duced by S.Kobayashi in 1967\([5]\) and by H.Royden in 1970\([6]\). Therefore the
Kobayashi metric is also called Kobayashi-Royden metric. Let \( M \) be a complex
manifold, then a Hermitian metric \( \sum_{i,j} g_{i,j} dz_i \otimes \overline{dz_j} \) defined on \( M \) is said to be
Kähler if the Kähler form \( \Omega = \sqrt{-1} \sum_{i,j} g_{i,j} dz_i \wedge \overline{dz_j} \) is closed. The Ricci form of
this metric is defined to be \( -\frac{\partial}{\partial t} \log \det(g_{i,j}) \). If the Ricci form of the Kähler metric
is proportional to the Kähler form, the metric is called Einstein-Kähler. If the
manifold is not compact, it requires the metric to be complete.According to a fa-
mous article\([7]\) of Wu, one knows that Einstein-Kähler metric is the most difficult to
compute among the four metrics because its existence is proved by complicate non-
constructive methods. If we normalize the metric by requiring the scalar curvature
to be minus one, then the Einstein-Kähler metric is unique.

Cheng and Yau\([8]\) proved that any bounded pseudo convex domain \( D \) with contin-
uous second partial derivatives boundary admits a complete Einstein-Kähler met-
ric. Mok and Yau have extended this result to an arbitrary bounded pseudo convex
domain in \( \mathbb{C}^n \)[9]. If this Einstein-Kähler metric is given by

\[
E_D(z) := \sum \frac{\partial^2 g}{\partial z_i \partial \overline{z_j}} dz_i \overline{dz_j},
\]

2000 Mathematics Subject Classification. 32H15, 32F07, 32F15.

Key words and phrases. Einstein-Kähler metric, holomorphic sectional curvature, holomorphic
automorphism group.

Project supported in part by NSF of China (Grant NO. 10471097) and the Doctoral Programme
Foundation of NEM of China.
then \( g \) is the unique solution to the boundary problem of the Monge-Ampère equation:

\[
\begin{align*}
\det \left( \frac{\partial^2 g}{\partial z_i \partial \overline{z}_j} \right) & = e^{(n+1)g} \quad z \in D, \\
g & = \infty \quad z \in \partial D,
\end{align*}
\]

and \( g \) is called generating function of \( E_D(z) \). Obviously, if one obtains \( g \) in explicit formula, then the Einstein-Kähler metric is also explicit. We knew by far that the explicit formulas for the Einstein-Kähler metric, however, are only known on homogeneous domains, and it is exactly the Bergman metric. We knew little as far as the nonhomogeneous domains concerned.

In this paper, we just consider a class of nonhomogeneous domains introduced by prof. W.Yin\textsuperscript{[10]} and G.Roos in 1998. It is defined as:

\[
Y_{\text{II}}(r, p; K) = \{ w \in \mathbb{C}^r, Z \in R_{\text{II}}(p) : |w|^2 < \det(I - ZZ^\top)^{\frac{-1}{K-1}}, K > 0 \} := Y_{\text{II}},
\]

where \( R_{\text{II}}(p) \) is the second type of symmetric classical domain and \( \det \) is the usual determinant of matrices. \( p > 1 \) is a positive integer. \( w \) is a vector with \( r \) entries and \( |w|^2 = |w_1|^2 + |w_2|^2 + \cdots + |w_r|^2 \). Its Bergman kernel function is given in explicit formula\textsuperscript{[11]} and hence \( Y_{\text{II}} \) is Bergman exhaustion, therefore, \( Y_{\text{II}} \) is a bounded pseudo convex domain which admits a unique complete Einstein-Kähler metric.

We know that the explicit complete Einstein-Kähler metric on \( Y_{\text{II}}(1, p; K) \) has been obtained\textsuperscript{[12]} when the parameter satisfies \( K = \frac{p}{2} + \frac{1}{p+1} \). The corresponding generating function \( g \) has explicit form as follows:

\[
g = \frac{1}{n+1} \log[Y^{n+1} \det(I - ZZ^\top)^{-1} K^{1-n}]
= \log[\frac{1}{1-X} \det(I - ZZ^\top)^{-1} K^{\frac{1}{n+1}}],
\]

where the parameters \( X \) and \( Y \) are \( X = |w|^2[\det(I - ZZ^\top)]^{-\frac{1}{K}}, Y = (1 - X)^{-1} \) and \( n = \frac{p(p+1)}{2} + 1 \) is the dimension of \( Y_{\text{II}}(1, p; K) \).

From this result, we guess when \( r > 1 \), \( K = \frac{p}{2} + \frac{1}{p+1} \), the generating function \( g \) of complete Einstein-Kähler metric on nonhomogeneous domain \( Y_{\text{II}}(r, p; K) \) has the following form:

\[
g = \frac{1}{N+1} \log[Y^{N+1} \det(I - ZZ^\top)^{-1} K^{r-N}]
= \log[\frac{1}{1-X} \det(I - ZZ^\top)^{-1} K^{\frac{1}{N+1}}],
\]

where

\[
X = |w|^2[\det(I - ZZ^\top)]^{-\frac{1}{K}} = ([|w_1|^2 + |w_2|^2 + \cdots + |w_r|^2]) \det(I - ZZ^\top)^{-\frac{1}{K}},
Y = (1 - X)^{-1}
\]

and \( N = \frac{p(p+1)}{2} + r \) is the dimension of \( Y_{\text{II}}(r, p; K) \). We will prove our conjecture correct through verifying \( g \) is the unique solution to the boundary problem of the Monge-Ampère equation.

This paper is organized as follows. Firstly, we present some basal facts and results which we need about \( Y_{\text{II}}(r, p; K) \). Secondly, we will verify \( g \) satisfies Monge-Ampère equation and the Direchlet boundary condition, so it gives the complete Einstein-Kähler metric. Thirdly, an estimate of the holomorphic sectional curvature under this metric is given. Finally, we prove that the complete Einstein-Kähler metric is equivalent to the Bergman metric on \( Y_{\text{II}}(r, p; K) \) in the case \( K = \frac{p}{2} + \frac{1}{p+1} \).
1. Preliminaries

In this section, we give a few lemmas about $Y_{II}(r, p; K)$ which will be needed later.

**Lemma 1.** Aut($Y_{II}$) indicates the holomorphic automorphism group of $Y_{II}(r, p; K)$ consisting of the following mappings:

$$
\begin{align*}
    w_j^* &= w_j \det(I - Z_0\overline{Z}_0)^{-\frac{1}{2}} \det(I - ZZ_0)^{-\frac{1}{2}}, \quad j = 1, 2, \ldots, r. \\
    Z^* &= A(Z - Z_0)(I - \overline{Z}_0Z)^{-1}\overline{A}^{-1}.
\end{align*}
$$

where $\overline{A}$ denotes the conjugation and transpose of $A$ and $\overline{A}A = (I - Z_0\overline{Z}_0)^{-1}, Z_0 \in R_{II}(p)$.

**Proof.** See ref. [11].

Obviously, any of the above mappings maps the point $(w, Z_0)$ onto the point $(w^*, Z_0^*) = A(Z - Z_0)(I - \overline{Z}_0Z)^{-1}\overline{A}^{-1}$ is the holomorphic automorphism of $R_{II}(p)$.

**Lemma 2.** Let $X = X(Z, w) = |w|^2[\det(I - ZZ_0)]^{-1/K}$, then $X$ is invariant under the mapping of Aut($Y_{II}$). That is $X(Z^*, w^*) = X(Z, w)$.

**Proof.** See ref. [11].

Hereafter we write $Z \in R_{II}(p) as$

$$
Z = Z^t = \begin{pmatrix}
    z_{11} & 1 & 1 & \cdots \\
    \frac{1}{\sqrt{2}} & z_{12} & 1 & \cdots \\
    \frac{1}{\sqrt{2}} & 1 & z_{22} & \cdots \\
    \vdots & \vdots & \ddots & \ddots \\
    \frac{1}{\sqrt{2}} & z_{p1} & 1 & z_{pp}
\end{pmatrix}.
$$

and

$$z = (z_{11}, z_{12}, \cdots, z_{1p}, z_{22}, z_{23}, \cdots, z_{2p}, \cdots, z_{pp})$$

is the $1 \times p(p + 1)/2$ matrix. $Z^t$ denotes the transpose of $Z$.

**Lemma 3.** Suppose $(Z^*, w^*) = F(Z, w) \in$ Aut($Y_{II}$) which maps $(Z_0, w)$ onto $(0, w^*)$; let $J_F$ be the Jacobi matrix of $F(Z, w)$, i.e.

$$
J_F = \begin{pmatrix}
    \frac{\partial z^*}{\partial z} & \frac{\partial w^*}{\partial z} \\
    0 & \frac{\partial w^*}{\partial w}
\end{pmatrix}.
$$
Then one has
\[
\frac{\partial z^*}{\partial z} \bigg|_{z_0=Z} = [A' \times A']_z,
\]
\[
\frac{\partial w^*}{\partial z} \bigg|_{z_0=Z} = \frac{1}{K} \det(I - ZZ) - \frac{1}{4} E(Z)' w,
\]
\[
\frac{\partial w^*}{\partial w} \bigg|_{z_0=Z} = I \det(I - ZZ)^{-\frac{1}{4}},
\]
where \( E(Z) \) is the \( 1 \times (p(p + 1))/2 \) matrix
\[
E(Z) = (tr[(I - ZZ)^{-1} I_{11} Z], \ldots, tr[(I - ZZ)^{-1} I_{p1} Z], \ldots, tr[(I - ZZ)^{-1} I_{pp} Z])
\]
and
\[
I_{kl}^* = \begin{cases} \frac{1}{\sqrt{2}}(I_{kl} + I_{lk}), & k < l, \\ I_{kk}, & k = l. \end{cases}
\]
Here \( I_{kl} \) is defined as a \( p \times p \) matrix, the \( (k, l) \)-th entry of \( I_{kl} \), i.e. the entry located at the junction of the \( k \)-th row and \( l \)-th column of \( I_{kl} \) is 1, and others entries of \( I_{kl} \) are zero. The meaning of \([A \times A]_z\) can be found in [13].

**Proof.** It can be got by direct computation.

**Lemma 4.** If \((z^*, w^*) = F(Z, w) \in \text{Aut}(Y_{11})\) and \(T = T[(Z, w), (Z, w)]\) is the metric matrix of the Einstein-Kähler metric on \(Y_{11}(r, p; K)\), then one has
\[
T[(Z, w), (Z, w)] = [J_F \cdot T[(Z^*, w^*), (Z^*, w^*)], J_F] \bigg|_{z_0=Z},
\]
and \( |J_F|^2_{z_0=Z} = \det(I - ZZ)^{-\frac{(p+1)}{2}} \), where \( |J_F| = \det J_F \).

**Proof.** It can be proved by using the invariance of the Einstein-Kähler metric under the holomorphic automorphism of \(Y_{11}\).

**Lemma 5.** Let \( Z \) be a \( p \times p \) symmetric matrix, then the following inequality holds:
\[
tr(Z\overline{Z}Z\overline{Z}) \leq tr(Z\overline{Z})tr(Z\overline{Z}) \leq p[tr(Z\overline{Z}Z\overline{Z})].
\]

**Proof.** The lemma is trivial when \( p = 1 \), so we consider the case \( p > 1 \) only. Let \( Z \) be a non-zero matrix, there exists a unitary matrix \( U \) such that
\[
Z = U^t \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_p \end{pmatrix} U, \quad \lambda_1 \geq \lambda_2 \geq \cdots \lambda_p \geq 0, \quad \lambda_1 > 0.
\]
then
\[
tr(Z\overline{Z}Z\overline{Z}) = \lambda_1^4 + \lambda_2^4 + \cdots + \lambda_p^4, \quad tr(Z\overline{Z}) = \lambda_1^2 + \lambda_2^2 + \cdots + \lambda_p^2.
\]
By using Cauchy-Schwartz inequality we have
\[
\lambda_1^4 + \cdots + \lambda_p^4 \leq (\lambda_1^2 + \cdots + \lambda_p^2)^2 = |(1^2, \ldots, 1^2) \cdot (\lambda_1^2, \ldots, \lambda_p^2)|^2 \leq |(1^2, \ldots, 1^2)|^2 |(\lambda_1^2, \ldots, \lambda_p^2)|^2 = p(\lambda_1^4 + \cdots + \lambda_p^4),
\]
i.e. \( tr(Z\overline{Z}Z\overline{Z}) \leq tr(Z\overline{Z})tr(Z\overline{Z}) \leq p[tr(Z\overline{Z}Z\overline{Z})] \).
It is obvious that $tr(Z\bar{Z}Z\bar{Z}) = tr(Z\bar{Z})tr(Z\bar{Z})$ holds if $\lambda_2 = \cdots = \lambda_p = 0$ and $tr(Z\bar{Z})tr(Z\bar{Z}) = p[tr(Z\bar{Z}Z\bar{Z})]$ holds if and only if $\lambda_1 = \lambda_2 = \cdots = \lambda_p$.

2. Complete Einstein-Kähler metric with explicit formula

Let $Z \in R_{II}(p)$ and $(Z, w) \in Y_{II}(r, p; K)$. Denote $(z, w) = (z_1, z_2, \cdots, z_{1p}, z_{22}, \cdots, z_{2p}, \cdots, z_{pp}, w_1, \cdots, w_r) = (z_1, z_2, \cdots, z_N)$, where $N = \frac{p(p+1)}{2} + r$ is the dimension of $Y_{II}(r, p; K)$.

Note that $g_{\alpha\beta}(z, w) = \frac{\partial^2 g}{\partial z_\alpha \partial \bar{z}_\beta}$, $\alpha, \beta = 1, 2, \cdots, N$, where

$$\frac{\partial g}{\partial z_{p(p+1)/2+j}} = \frac{\partial g}{\partial w_j}, \quad j = 1, \cdots, r.$$ 

In order to prove our conjecture is right, we have to verify that the generating function $g$ is the unique solution to the Dirichlet boundary problem of complex Monge-Ampère equation:

$$\begin{cases}
\det(g_{\alpha\beta}(z, w)) = e^{(N+1)g(z, w)}, & (z, w) \in Y_{II}, \\
g = \infty, & (z, w) \in \partial Y_{II}. 
\end{cases} \quad (1)$$

Let $F \in \text{Aut}(Y_{II})$, $F(Z, w) = F(Z^*, w^*)$. According to lemma 4 we know

$$\det(g_{\alpha\beta}(z, w)) = |J_F|^2 \det(g_{\alpha\beta}(z^*, w^*)),$$

especially, if choose $Z_0 = Z$ and denote holomorphic automorphism $F|_{Z_0=Z}$ by $F_0$, then

$$\det(g_{\alpha\beta}(z, w)) = |J_{F_0}|^2 \det(g_{\alpha\beta}(0, w^*)) \quad (2)$$

Hence we need only to know the value of $\det(g_{\alpha\beta}(z^*, w^*))$ at the point $(0, w^*)$. We substitute $w$ for $w^*$ in the following computation.

Since $Y = (1 - X)^{-1}, X = |w|^2[\det(I - Z\bar{Z})]^{-1/K}$, the generating function $g$ can be rewritten by

$$g = \log Y + \log X - \log |w|^2 + \frac{r - N}{1 + N} \log K.$$ 

Thus

$$\begin{align*}
\frac{\partial g}{\partial z_\alpha} &= (Y + X^{-1}) \frac{\partial X}{\partial z_\alpha}, \quad \alpha = 1, 2, \cdots, N - r, \\
\frac{\partial g}{\partial w_i} &= Y \frac{\partial X}{\partial w_i}, \quad i = 1, 2, \cdots, r.
\end{align*}$$

and

$$\begin{align*}
\frac{\partial^2 g}{\partial z_\alpha \partial \bar{z}_\beta} &= (Y + X^{-1}) \frac{\partial^2 X}{\partial z_\alpha \partial \bar{z}_\beta} + (Y^2 - X^{-2}) \frac{\partial X}{\partial z_\alpha} \frac{\partial X}{\partial \bar{z}_\beta}, \\
\frac{\partial^2 g}{\partial z_\alpha \partial w_j} &= (Y + X^{-1}) \frac{\partial^2 X}{\partial z_\alpha \partial w_j} + (Y^2 - X^{-2}) \frac{\partial X}{\partial z_\alpha} \frac{\partial X}{\partial w_j}, \quad j = 1, 2, \cdots, r; \\
\frac{\partial^2 g}{\partial w_i \partial \bar{z}_\beta} &= Y \frac{\partial^2 X}{\partial w_i \partial \bar{z}_\beta} + YZ \frac{\partial X}{\partial w_i} \frac{\partial X}{\partial \bar{z}_\beta}, \quad i = 1, 2, \cdots, r; \\
\frac{\partial^2 g}{\partial w_i \partial w_j} &= Y \frac{\partial^2 X}{\partial w_i \partial w_j} + YZ \frac{\partial X}{\partial w_i} \frac{\partial X}{\partial w_j}, \quad i, j = 1, 2, \cdots, r.
\end{align*}$$

(3)
According ref.[12] we have the following results:

\[
\frac{\partial X}{\partial w_i} \bigg|_{z=0} = m_i, \quad \frac{\partial X}{\partial w_j} \bigg|_{z=0} = w_j, \\
\frac{\partial X}{\partial z_\alpha} \bigg|_{z=0} = \frac{\partial X}{\partial z_\beta} \bigg|_{z=0} = \frac{\partial X}{\partial z_{kl}} \bigg|_{z=0} = 0, \\
\frac{\partial^2 X}{\partial z_\alpha \partial z_\beta} \bigg|_{z=0} = \frac{\partial^2 X}{\partial w_i \partial w_j} \bigg|_{z=0} = 0, \\
\frac{\partial^2 X}{\partial z_\alpha \partial z_\beta} \bigg|_{z=0} = 0, \\
\frac{\partial^2 X}{\partial z_\alpha \partial w_j} \bigg|_{z=0} = \delta_{ij},
\]

where \( \delta_{mn} = \begin{cases} 1, & m = n \\ 0, & m \neq n \end{cases} \). Applying the above result to formula (3), we obtain

\[
\frac{\partial^2 g}{\partial z_\alpha \partial z_\beta} \bigg|_{z=0} = \frac{Y}{K} \delta_{\alpha\beta}, \quad \frac{\partial^2 g}{\partial z_\alpha \partial w_j} \bigg|_{z=0} = \frac{\partial^2 g}{\partial w_i \partial w_j} \bigg|_{z=0} = 0, \\
\frac{\partial^2 g}{\partial w_i \partial w_j} \bigg|_{z=0} = Y \delta_{ij} + Y^2 m_i w_j.
\]

Therefore

\[
\left( g_{\alpha\beta}(0, w^*) \right) = \left( \frac{Y}{K} I^{(N-r)} \begin{array}{c} 0 \\ Y I^{(r)} + Y^2 m^t w^* \end{array} \right),
\]

where \( I^{(r)} \) denotes \( r \times r \) unit matrix and we write \( w^* \) back instead of \( w \).

Notice that \( w^* = w \det(I - ZZ^T)^{-\frac{1}{2m}} \), when \( Z_0 = Z \), we get

\[
\det(g_{\alpha\beta}(0, w^*)) = \left( \frac{Y}{K} \right)^{N-r} \det(Y I^{(r)} + Y^2 m^t w^*) \\
= \left( \frac{Y}{K} \right)^{N-r} Y^r \det(I^{(r)} + XY[w]^t m) \\
= \left( \frac{Y}{K} \right)^{N-r} Y^r \det(I^{(1)} + XY[w]^t m) \\
= K^{r-N} Y^t (1 + XY) \\
= K^{r-N} (1 - X)^{-N(N+1)}.
\]

Hence

\[
\det(g_{\alpha\beta}(z, w)) = |J_{\tilde{F}_0}|^2 \det(g_{\alpha\beta}(0, w^*)) \\
= K^{r-N} (1 - X)^{-N(N+1)} \det(I - ZZ^T)^{-(p+1+\frac{r}{p})}, \quad (4)
\]

while the other side of Monge–Ampère equation is

\[
e^{(N+1)g(z, w)} = e^{(N+1) \log \left( \frac{1}{K} \det(I - ZZ^T)^{-\frac{1}{2m}} K^{\frac{-r}{p+1}} \right)} \\
= K^{r-N} (1 - X)^{-N(N+1)} \det(I - ZZ^T)^{\frac{-N+1}{p+1}}. \quad (5)
\]

It is easy to obtain that formula (4) and (5) is equal in the case \( K = \frac{1}{2} + \frac{1}{p+1} \). That is the function \( g \) we guess is a solution to complex Monge–Ampère equation. It remains to prove that \( g \) satisfies the Dirichlet boundary condition.

If \((\tilde{z}, \tilde{w}) \in \partial Y_{II} \) and \( \tilde{w} \neq 0 \), when \((z, w) \in Y_{II} \) and \((z, w) \to (\tilde{z}, \tilde{w}) \), we have \( X \to 1^{-} \), so \( \frac{1}{1-Z^2} \to +\infty \), meanwhile \( \det(I - ZZ^T) \to |\tilde{w}| 2K > 0 \). Hence we have \( g(z, w) \to +\infty \), as \((z, w) \to \partial Y_{II} \).
If \((\tilde{z}, \tilde{w}) \in \partial Y_{11}\) and \(\hat{w} = 0\), when \((z, w) \in Y_{11}\) and \((z, w) \to (\tilde{z}, 0)\), we have \(\frac{1}{1 - X} > 1\), \(\det(I - ZZ^\top) \to 0\), \(\det(I - ZZ^\top)^{-\frac{1}{2}} \to +\infty\), we also have \(g(z, w) \to +\infty\), as \((z, w) \to \partial Y_{11}\).

Up to now, we have proved our conjecture, that is the function
\[
g = \log[\frac{1}{1 - X} \det(I - ZZ^\top)^{-\frac{1}{2}} K^{\frac{N}{1 + N}}]
\]
generates a complete Einstein-Kähler metric on \(Y_{11}(r, p; K)\) in the case \(K = \frac{p}{2} + \frac{1}{p + 1}\). In general, \(Y_{11}(r, p; K)\) is a nonhomogeneous domain when \(K = \frac{p}{2} + \frac{1}{p + 1}\) and \(p > 1\).

3. Holomorphic sectional curvature

Since in the case \(K = \frac{p}{2} + \frac{1}{p + 1}\) the complete Einstein-Kähler metric on \(Y_{11}(r, p; K)\) is generated by
\[
g = \log[\frac{1}{1 - X} \det(I - ZZ^\top)^{-\frac{1}{2}} K^{\frac{N}{1 + N}}], \quad N = p(p + 1)/2 + r,
\]
the holomorphic sectional curvature \(\omega[(z, w), d(z, w)]\) on \(Y_{11}(r, p; K)\) under this metric has the following form:
\[
\omega[(z, w), d(z, w)] = \frac{d(z, w)\left[\overline{\partial dT} + dTT^{-1}\overline{dT}\right]d(z, w)}{|d(z, w)Td(z, w)|^2},
\]
where
\[
d = \sum \frac{\partial}{\partial z_\alpha}dz_\alpha, \quad \overline{d} = \sum \frac{\partial}{\partial \overline{z}_\alpha}d\overline{z}_\alpha, \quad \alpha = 1, 2, \ldots, N, \quad T = \left(\frac{\partial^2 g}{\partial z_\alpha \partial \overline{z}_\beta}\right)_{1 \leq \alpha, \beta \leq N},
\]
and \(\overline{d} = \sum \frac{\partial}{\partial \overline{z}_\alpha}d\overline{z}_\alpha\).

Now that holomorphic sectional curvature \(\omega[(z, w), d(z, w)]\) is invariant under the holomorphic automorphism group \(\text{Aut}(Y_{11})\), and due to the Lemma 1, for \(\forall(z, w) \in Y_{11}\) there exists \(F \in \text{Aut}(Y_{11})\) such that \(F(z, w) = (0, w^*)\). So it suffices to calculate the \(\omega[(z, w), d(z, w)]\) on point \((0, w^*)\). By sec.2, the complete Einstein-Kähler metric matrix is \(T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}\), where
\[
T_{11} = K^{-1}Y[A^t A^s \times A^t A^s] + XK^{-2}Y^2E(Z)^{tE(Z)},
T_{12} = K^{-1}Y^2 \det(I - ZZ^\top)^{-\frac{1}{2}} E(Z)^{tE(Z)},
T_{21} = T_{12}^\top,
T_{22} = Y^2 \overline{w} \det(I - ZZ^\top)^{-\frac{1}{2}} + Y^{rE} \det(I - ZZ^\top)^{-\frac{1}{2}}.
\]

\(Y = (1 - X)^{-1}\) and \(A_s[A^t A^s \times A^t A^s], E(Z)\) are the same as before. Note that
\[
dT = \begin{pmatrix} dT_{11} \\ dT_{21} \end{pmatrix}, \quad d\overline{dT} = \begin{pmatrix} d\overline{dT}_{11} \\ d\overline{dT}_{21} \end{pmatrix},
\]
Using some known results\cite{12}
\[
E(Z)|_{z=0} = dE(Z)|_{z=0} = 0, \quad \overline{dE(Z)}|_{z=0} = (\overline{dz})^t, \quad d\overline{E(Z)}|_{z=0} = dz
\]
and \(d[A^t A^s \times A^t A^s] |_{z=0} = 0\), we can obtain
\[
dT_{11}|_{z=0} = K^{-1}Y^2 \overline{wdw} \overline{I}^{(N-r)}, \quad dT_{12}|_{z=0} = 0, \quad dT_{21}|_{z=0} = K^{-1}Y^2 \overline{wdz}, \quad dT_{22}|_{z=0} = (2Y^3 \overline{w} + Y^2 \overline{I}) \overline{wdw} + Y^2 \overline{wdw}.
\]
Furthermore

\[
\overline{ddT}_{11}(z) = K^{-1}Y(Y|dw|^2I + 2Y^2|\bar{w}dw|^2I + K^{-1}XY|dz|^2I + K^{-1}XZ|dz|^2I)
\]

thus the holomorphic sectional curvature on point(0) under the complete Einstein-Kähler metric is

\[
\omega((z, w), (z, w))|_{z=0} = -2 + \frac{2K^{-2}Y|dz|^4 - 2K^{-1}Y|tr(d\bar{Z}d\bar{Z})|}{K^{-1}Y|dz|^2 + Y^2|\bar{w}dw|^2 + Y|dw|^2}.
\]

It is apparent that if let \(p = 1\), then \(K = 1, tr(d\bar{Z}d\bar{Z}) = |dz|^4\). In this case, \(Y_{11}(r, p; K)\) is a unit ball in \(C^{p+1}\) and \(\omega((z, w), (z, w))|_{z=0} = -2\). It is a well-known result. Our work is to give its estimate in the case \(p > 1\).

Since \(|dz|^2 = tr(d\bar{Z}d\bar{Z})\), according to lemma 5, we get

\[
p^{-1}|dz|^4 \leq tr(d\bar{Z}d\bar{Z}) \leq |dz|^4.
\]
Applying it to $\omega((z, w), d(z, w))|_{z=0}$, we have

$$2(1 - K)\frac{Y}{K^2}|dz|^4 \leq \frac{2Y}{K}tr(dZ\overline{dZ}dZ\overline{dZ}) \leq 2(1 - \frac{K}{p})\frac{Y}{K^2}|dz|^4.$$ 

Now that $1 - K < 0$ and $1 - \frac{K}{p} > 0$ hold in the case $p > 1$ and notice that $Y \geq 1(0 \leq X < 1)$, the above inequality can be expanded as

$$2(1 - K)(K^{-1}Y|dz|^2 + Y^2|dw|^2 + Y|dw|^2)^2 \leq 2K^{-2}Y|dz|^4 - 2K^{-1}Ytr(dZ\overline{dZ}dZ\overline{dZ}) \leq 2(1 - K^{-1})Y|dz|^2 + Y^2|dw|^2 + Y|dw|^2)^2.$$ 

Applying it in the expression of $\omega((z, w), d(z, w))|_{z=0}$ one can obtain the following result immediately:

$$-2K \leq \omega((z, w), d(z, w)) \leq -\frac{2K}{p}$$

where $K = \frac{p}{2} + \frac{1}{p+1}$ and $p > 1$. This estimate is the sharp estimate because of the following facts:

At the point $(z, 0)$ and direction $(dz, 0)$ one easily knows that $X = 0$ and $Y = 1$, and

$$\omega((z, w), d(z, w))|_{z=0} = -2 + \frac{2K^{-2}|dz|^4 - 2K^{-1}tr(dZ\overline{dZ}dZ\overline{dZ})}{K^{-2}|dz|^4}.$$ 

If choose

$$dZ = U^t \begin{pmatrix} \lambda_1 & 0 \\ 0 & 0 \end{pmatrix} U, \quad \lambda_1 > 0,$$

then according to lemma 5 one has $tr(Z\overline{Z}Z\overline{Z}) = tr(Z\overline{Z})tr(Z\overline{Z}) = |dz|^4$, which implies $\omega((z, w), d(z, w))|_{z=0} = -2K$.

If choose

$$dZ = U^t \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & \lambda_p \end{pmatrix} U, \quad \lambda_1 = \lambda_2 = \cdots = \lambda_p > 0,$$

then one has $tr(Z\overline{Z}Z\overline{Z}) = p^{-1}tr(Z\overline{Z})tr(Z\overline{Z}) = p^{-1}|dz|^4$ by lemma 5, which implies $\omega((z, w), d(z, w))|_{z=0} = -\frac{2K}{p}$.

Therefore our estimate is the sharp estimate.

4. **Kähler–Einstein metric Is Equivalent to the Bergman metric**

In this section we will prove that the Kähler–Einstein metric is equivalent to the Bergman metric on $Y_{11}(r, p; K)$ in the case $K = \frac{p}{2} + \frac{1}{p+1}$.

**Definition:** Let $B_0$ and $E_0$ be two complete metrics on domain $\Omega$. If there exists two positive constant $a$ and $b$ (with $a \geq b$) such that the following inequality holds:

$$b \leq \frac{B_0}{E_0} \leq a.$$ 

Then we called that the metric $B_0$ is equivalent to the metric $E_0$. 
From sec. 2 we know in the case \( K = \frac{p}{2} + \frac{1}{p+1} \) the complete Kähler–Einstein metric on \( Y_{II}(r,p; K) \) is

\[
\mathcal{E}_{II} := (dz, dw) T_{\mathcal{E}_{II}} (dz, dw) ,
\]

where

\[
T_{\mathcal{E}_{II}} = J_{F_0} T_{\mathcal{E}_{II}}^{(0)} J_{F_0} = J_{F_0} \left( \begin{array}{cc} \frac{Y}{K} I^{(N-r)} & 0 \\ 0 & Y I^{(r)} + Y^2 \overline{w}^t w^* \end{array} \right) J_{F_0}.
\]

From ref.[11], we know the Bergman kernel function on \( Y_{II}(r,p; K) \) is

\[
K_{II}(W,Z; W, \overline{Z}) = K^{-\frac{p(p+1)}{2}} \pi^{-\frac{p(p+1)}{2}} G(Y) \det(I - Z \overline{Z})^{-\frac{1}{p+1}} K
\]

where

\[
G(Y) = \sum_{j=0}^{h} b_j (r+j) Y^{r+j}, \quad h = \frac{p(p+1)}{2} + 1, \quad b_h = 2^{\frac{p(p+1)}{2}+1}, \quad (r+j) = (r+j-1)!.
\]

Thus the Bergman metric on \( Y_{II}(r,p; K) \) has the following form:

\[
B_{II} := (dz, dw) T_{B_{II}} (dz, dw) ,
\]

where the metric matrix

\[
T_{B_{II}} = \left( \frac{\partial^2 \log K_{II}}{\partial z_\alpha \partial \overline{z}_\beta} \right) = J_{F_0} \left( \frac{\partial^2 \log K_{II}}{\partial z_\alpha \partial \overline{z}_\beta} \right)_{Z = \overline{Z}} = J_{F_0} T_{B_{II}}^{(0)} J_{F_0}
\]

and

\[
T_{B_{II}}^{(0)} = \left( \begin{array}{cc} \frac{1}{K} H' X + p + 1 + \frac{r}{K} & 0 \\ 0 & H' I^{(r)} + H'' \overline{w}^t w^* \end{array} \right), \quad H = \log G(Y).
\]

If we denote \((dz, dw) J_{F_0}\) by \((dZ, dW)\), then \(\mathcal{E}_{II}\) and \(B_{II}\) can be rewritten as

\[
\mathcal{E}_{II} = \frac{Y}{K} |dZ|^2 + dW(Y I^{(r)} + Y^2 \overline{w}^t w^*) d\overline{W}^t
\]

\[
B_{II} = \left( \frac{1}{K} H' X + p + 1 + \frac{r}{K} \right) |dZ|^2 + dW(H' I^{(r)} + H'' \overline{w}^t w^*) d\overline{W}^t.
\]

According to ref.[13] one knows that the vector \( w^* = (w_1^*, w_2^*, \ldots, w_r^*) \) can be transformed into

\[
w^* = e^{i\theta} (\lambda, 0, \cdots, 0) U, \quad \lambda \geq 0,
\]

where \( U \) is a unitary matrix. Hence

\[
\overline{w}^t w^* = \overline{U}^t \left( \begin{array}{cc} \lambda^2 & 0 \\ 0 & 0 \end{array} \right) U, \quad H' I^{(r)} + H'' \overline{w}^t w^* = \overline{U}^t \left( \begin{array}{cc} H' + H'' \lambda^2 & 0 \\ 0 & H' I^{(r-1)} \end{array} \right) U
\]

and

\[
Y I^{(r)} + Y^2 \overline{w}^t w^* = \overline{U}^t \left( \begin{array}{cc} Y + Y^2 \lambda^2 & 0 \\ 0 & Y I^{(r-1)} \end{array} \right) U.
\]

Let \(dWd\overline{W} = (dw, dW)\), then \(\mathcal{E}_{II}\) and \(B_{II}\) have the following form:

\[
\mathcal{E}_{II} = \frac{Y}{K} |dZ|^2 + (Y + Y^2 \lambda^2) |dW|^2 + Y |dW|^2, \\
B_{II} = \left( \frac{1}{K} H' X + p + 1 + \frac{r}{K} \right) |dZ|^2 + (H' + H'' \lambda^2) |dW|^2 + H' |dW|^2.
\]

It is known \(T_{\mathcal{E}_{II}}\) and \(T_{B_{II}}\) are positive definite matrices, which implies

\[
\frac{1}{K} H' X + p + 1 + \frac{r}{K} > 0, \quad H' + H'' \lambda^2 > 0, \quad H' > 0, \quad Y > 0.
\]
Therefore, if we denote
\[ \phi(X) = \frac{K^{-1}H'X + p + 1 + rK^{-1}}{YK^{-1}}, \quad \psi(X) = \frac{H' + H''\lambda^2}{Y + Y^2\lambda^2}, \quad \tau(X) = \frac{H'}{Y}, \]
then \( \phi(X), \psi(X) \) and \( \tau(X) \) are all positive continues functions of \( X \) on the interval \([0, 1]\). If
\[ \lim_{X \to 1} \phi(X), \lim_{Y \to \infty} \psi(X), \lim_{Y \to \infty} \tau(X) \]
are existent and positive, then all of \( \phi(X), \psi(X), \tau(X) \) have the positive maximum and the positive minimum on \([0, 1]\) respectively.

We know that
\[ G(Y) = \sum_{j=0}^{h} b_j \Gamma(r + j)Y^{r+j}, \quad h = \frac{p(p + 1)}{2} + 1, \quad b_h = 2^{\frac{p(p+1)}{2}+1}, \]
then
\[ \frac{dG(Y)}{dX} = G'(Y) = \sum_{j=0}^{h} b_j \Gamma(r+j+1)Y^{r+j+1}, \frac{d^2G(Y)}{dX^2} = G''(Y) = \sum_{j=0}^{h} b_j \Gamma(r+j+2)Y^{r+j+2}, \]
and
\[ H' = G'(Y)G^{-1}(Y), \quad H'' = G''(Y)G^{-1}(Y) - G'(Y)^2G^{-2}(Y). \]

We shall compute the limits of \( \phi(X), \psi(X) \) and \( \tau(X) \) when \( X \) tends to 1.

\[ \lim_{X \to 1} \phi(X) = \lim_{Y \to \infty} \phi(X) = \lim_{Y \to \infty} \frac{K^{-1}H'X + p + 1 + rK^{-1}}{YK^{-1}} = \lim_{Y \to \infty} \frac{G'(Y)}{G(Y)Y}, \]
applying formula (7), we have
\[ \lim_{X \to 1} \phi(X) = \lim_{Y \to \infty} \phi(X) = \frac{b_h \Gamma(r + h + 1)Y^{r+h+1}}{b_h \Gamma(r + h)Y^{r+h+1}} = r + h = N + 1, \]
where \( N = \frac{p(p+1)}{2} + r \) is the dimension of \( Y_{11}(r, p; K) \).

Therefore there exists \( 0 < \nu < \mu \) such that
\[ 0 < \nu \leq \phi(X) \leq \mu. \]

Similarly, we can also obtain
\[ \lim_{X \to 1} \psi(X) = \lim_{Y \to \infty} \psi(X) = \lim_{Y \to \infty} \frac{H' + H''\lambda^2}{Y + Y^2\lambda^2} = \lim_{Y \to \infty} \frac{H''}{Y^2} = N + 1 \]
and
\[ \lim_{X \to 1} \tau(X) = \lim_{Y \to \infty} \tau(X) = \lim_{Y \to \infty} \frac{H'}{Y} = N + 1. \]
Therefore there also exist positive constants \( \zeta, \eta, \rho \) and \( \varrho \) such that
\[ \zeta \leq \psi(X) \leq \eta, \quad \rho \leq \tau(X) \leq \varrho. \]

Let \( a = \max\{\mu, \eta, \rho\} \) and \( b = \min\{\nu, \zeta, \rho\} \), then we have
\[ b \leq \frac{E_{11}}{\epsilon_{11}} \leq a. \]

Up to now we can say that the complete Kähler–Einstein metric is equivalent to the Bergman metric on \( Y_{11}(r, p; K) \) in the case \( K = \frac{p}{2} + \frac{1}{p+1} \).
References

[1] S. Bergman, Über die Entwicklung der harmonischen Funktionen der Ebene und Raumes nach Orthogonalfunktionen, Math. Ann. (in German), 96 (1922), 237-271.
[2] S. Bergman, Über die kernfunktion eines Bereiches und ihre Verhalten am Rande, J. Reine Angew. Math. (in German), 169 (1933), 1-42; 172 (1935), 89-128.
[3] C. Carathéodory, Über des Schwarzshe Lemma bei analytischen Funktionen von zwei komplexen Veranderlichen, Math. Ann. (in German), 97 (1926), 76-98.
[4] H.-J. Reiffen, Die Differentialgeometrischen Eigenschaften der invarianten Distanzfunktionen von Carathéodory, Schr. Math. Inst. Münster, Nr. (in German), 26 (1963), 18.
[5] S. Kobayashi, Invariant distances on complex manifolds and holomorphic mappings, J. Math. Soc. Japan, 19 (1967), 460-480.
[6] H. L. Royden, Remarks on the Kobayashi metric, Several Complex Variables II, Lecture Notes in Mathematics, Springer-Verlag, 185 (1971), 125-137.
[7] H. Wu, Old and new invariant metrics on complex manifolds, Several complex variables, Proceedings of the Mittag-Leffler Institute, 1987-1988 (J. E. Fornæss, ed.), Math. Notes, Princeton Univ. Press, Princeton, NJ, 38 (1993), 640-682.
[8] S.Y. Cheng and S.T. Yau, On the existence of a complete Kähler metric on non-compact complex manifolds and the regularity of Pefferman’s equation, Comm. Pure Appl. Math., 33 (1980), 507-544.
[9] N. Mok and S.T. Yau, Completeness of the Kähler-Einstein metric on bounded domain and the characterization of domain of holomorphy by curvature conditions, Proc Symposia Pure Math., 39 (1983), 41-59.
[10] Weiping Yin, The Bergman Kernels on Cartan-Hartogs domains, Chinese Science Bulletin, 4 (1999), 1947-1951. MR 2001g: 32004.
[11] Weiping Yin, The Bergman Kernel function on Cartan-Hartogs domains of the second type (in Chinese), Chinese Annals of Math., 21A (2000), 331-340.
[12] Xiaoxia Zhao, Liyou Zhang and Weiping Yin, Einstein-Kähler metric on Cartan-Hartogs domain of the second type, Progress in Natural Science, 14 (2004), 201-212.
[13] Qikeng Lu, The Classical Manifolds and the Classical Domains (in Chinese), Shanghai Scientific and Technical press, Shanghai, 1963.

W YIN: DEPT. OF MATH., CAPITAL NORMAL UNIV., BEIJING 100037, CHINA
E-mail address: wyin@mail.cnu.edu.cn; wyin@263.net

L ZHANG: DEPT. OF MATH., CAPITAL NORMAL UNIV., BEIJING 100037, CHINA
E-mail address: zhangly@email.cnu.edu.cn