Probabilistic properties of nonextensive thermodynamic

Franck Jedrzejewski
Commissariat à l’énergie atomique
CEA Saclay, INSTN, F-91191 Gif-sur-Yvette, France
franck@arthaud.saclay.cea.fr

Abstract

Based on the Tsallis entropy, the nonextensive thermodynamic properties are studied as a q-deformation of classical statistical results using only probabilistic methods and straightforward calculations. It is shown that the constant in the Tsallis entropy depends on the deformation parameter and must be redefined to recover the usual thermodynamic relations. The notions of variance and covariance are generalised. A partial derivative formula of the entropy is established. It verifies important relations from which most of the nonextensive thermodynamics relations can be recovered. This leads to a new proof of a highly nontrivial conclusion that thermodynamic relations and Maxwell relations in non extensive thermodynamic have the same forms as those in ordinary nonextensive statistical mechanics. Theoretical results are applied to ideal gas. The case of fermions and bosons systems with fractal distributions is also considered.

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1 Introduction

The purpose of this work is to discuss the thermodynamic relations problem in Tsallis thermostatics recently presented in Ref. [1]. It has been pointed out that q-thermostatics could be understood in terms of q-deformations. The Tsallis’ entropy proposed by Constantino Tsallis ten years ago [2] has found many applications in various domains of physics [3], as the turbulence in electron plasmas, phonon-electron thermalization in ion bombarded solids and flux of solar neutrinos. There are also applications to the mechanisms of anomalous diffusion [4]. When systems involve long-range interactions, statistical mechanics presents serious difficulties due to non-markovianity of stochastic processes. Non-extensive thermodynamics may correctly cover some of the known anomalies. The canonical ensemble of non-extensive thermodynamics for quantum mechanical systems with finite number of degrees of freedom has been studied in Ref. [5]. It has been shown that most of thermodynamic relations are preserved [6]. if introducing the entropic index q, we replace the usual means by q-expectations. However, invariant relations involve Boltzmann-Gibbs constant in the canonical ensemble and are related to Tsallis constant in the grand canonical ensemble. The canonical distribution in non-extensive thermodynamic is given by optimal Lagrange multipliers and is discussed in Ref. [7]. Parameter differentiation of fractional powers of operators has been developed in Ref. [8, 9]. Nonextensive statistical mechanics and thermodynamics, as a new stream in the research of the foundations of statistical mechanics, is presented in a recent publication [10].

2 Tsallis Entropy

The Tsallis’ entropy is defined by the formula

$$S_q = k \int \rho \ln_q(\rho^{-1}) \, d\lambda$$

where $\rho$ is the distribution of probability, $\lambda$ is the Lebesgue measure on the phase space and $k$ is a constant, called the Tsallis constant. In this expression the usual logarithmic function is replaced by the q-logarithmic function

$$\ln_q(x) = \frac{x^{1-q} - 1}{1-q}$$
which is the inverse function of the \( q \)-exponential function
\[
e_q(x) = (1 + (1 - q)x)^{1/(1-q)}
\] (3)
defined if \( (1 + (1 - q)x) > 0 \) and which is prolonged by continuity with \( e_q(x) = 0 \) otherwise. When \( q \to 1 \), the \( q \)-exponential and the \( q \)-logarithmic functions converge to the ordinary exponential and logarithmic functions. The entropy of Tsallis reduces to the Boltzmann-Gibbs formula in the extensive limit. The pseudo-additivity of the \( q \)-logarithmic function
\[
\ln_q(xy) = \ln_q(x) + \ln_q(y) + (1 - q) \ln_q(x) \ln_q(y)
\] (4)
leads to the nonextensivity of the Tsallis entropy. For two independent subsystems \( A \) and \( B \) of a total system, in thermal contact with each other, it has been shown that
\[
S_q(\rho_A \otimes \rho_B) = S_q(\rho_A) + S_q(\rho_B) + (1-q)kS_q(\rho_A)S_q(\rho_B)
\] (5)
The following property of the \( q \)-logarithmic function
\[
\ln_q(x^q) = -x^{q-1} \ln_q(x)
\] (6)
gives the usual definition of the Tsallis entropy \[11\]
\[
S_q = -k \int \rho^q \ln_q(\rho) \, d\lambda
\] (7)
which is also defined by the equivalent formula
\[
S_q = \frac{k}{q-1} \left( 1 - \int \rho^q \, d\lambda \right)
\] (8)
If we define the unnormalized \( q \)-expectation of an observable \( A \) by
\[
[A]_q = \int A\rho^q \, d\lambda
\] (9)
and let \( c_q \) be the constant
\[
c_q = \int \rho^q \, d\lambda
\] (10)
the \( q \)-expectation (normalised) could be defined by
\[
\langle A \rangle_q = \frac{[A]_q}{c_q} = \frac{1}{c_q} \int A\rho^q \, d\lambda
\] (11)
The problem posed by unnormalized expectation is studied in Ref. \[14\]. Introducing the \( q \)-Gibbs measure (Ref. \[13\]) in the canonical ensemble
\[
\rho = \frac{e_q(-\beta H)}{Z_q}
\] (12)
where \( H \) is the Hamiltonian of the system and \( \beta \) the Lagrange multiplier \( (1/kT) \). The expression of the density of probability has been discussed in Ref. \[14\]. It is shown in Ref. \[13\] that, for a canonical distribution of a system in contact with heat bath, the entropic index \( q \) is related to the particle number \( N \) and characterizes the bath. The Tsallis constant \( k \) is choosen as
\[
\beta = \frac{Z_q^{1-q}}{e_qk_BT}
\] (13)
where \( k_B \) is the Boltzmann constant. The \( q \)-partition function \( Z_q \) is given by the normalisation of the probability density
\[
Z_q = \int e_q(-\beta H) \, d\lambda
\] (14)
The measure \( \mu \) defined relatively to the Lebesgue measure by \( d\mu = (\rho^q/c_q) \, d\lambda \) is a measure of probability. In the extensive limit \( q \to 1 \), the \( q \)-Gibbs measure converges to the ordinary Boltzmann-Gibbs density. The substitution of the \( q \)-additivity Eq. (8) in the Tsallis entropy Eq. (5) gives
\[
S_q = -k \int (-\beta H + \ln_q(Z_q^{-1}) - (1-q)\beta H \ln_q(Z_q^{-1}))\rho^q \, d\lambda
\] (15)
and using Eq. (10)
\[
S_q = k \int (\beta H + (1 - (1 - q)\beta H)Z_q^{-q-1}\ln_q(Z_q))\rho^q \, d\lambda
\] (16)
we find that the Tsallis entropy
\[
S_q = k \beta [H]_q + k \ln_q(Z_q)
\] (17)
leads to the following formula
\[
S_q = k \beta [H]_q + k \ln_q(Z_q)
\]
Introducing the $q$-internal energy

$$U_q = \langle H \rangle_q$$  \hspace{1cm} (18)

and the $q$-free energy

$$F_q = -\frac{1}{\beta} \ln_q(Z_q)$$ \hspace{1cm} (19)

one finds

$$F_q = c_q U_q - TS_q$$ \hspace{1cm} (20)

This relation is the $q$-generalisation of the ordinary equation of $S = U/T + k\ln(Z)$.

### 3 Mathematical Formalism

In this section, we introduce five important relations for the nonextensive mathematical formalism.

The generalised internal energy $U_q$ defined as the $q$-expectation of the Hamiltonian is given by

$$U_q = -\frac{1}{c_q} \frac{\partial \ln_q(Z_q)}{\partial \beta}$$ \hspace{1cm} (21)

The proof of this equation is easy to compute.

$$-\frac{\partial \ln_q(Z_q)}{\partial \beta} = -\frac{\partial \ln_q(Z_q)}{\partial Z_q} \frac{\partial Z_q}{\partial \beta} = -1 \frac{\partial Z_q}{Z_q^q \frac{\partial Z_q}{\partial \beta}}$$

$$= -1 \frac{\partial}{Z_q^q} \frac{\partial}{\partial \beta} \int e_q(-\beta H) d\lambda$$ \hspace{1cm} (23)

Using the following expression of the derivative of the $q$-exponential function with respect to $\beta$

$$\frac{\partial e_q(-\beta H)}{\partial \beta} = -He_q^2(-\beta H)$$ \hspace{1cm} (24)

leads to

$$-\frac{\partial \ln_q(Z_q)}{\partial \beta} = \int H \frac{e_q^2(-\beta H)}{Z_q^q} d\lambda$$ \hspace{1cm} (25)

$$= \int H \rho^q d\lambda = |H|_q = c_q U_q$$ \hspace{1cm} (26)

Defining the $q$-movariance of two operators $A$ and $B$ by

$$Mov_q(A, B) = [AB]_{2q-1} - [A]_q [B]_q$$ \hspace{1cm} (27)

the equation (Eq. 28) gives the $q$-movariance of $A$ and $H$ in function of the derivative of the $q$-mean of $A$

\[
\frac{\partial [A]_q}{\partial \beta} - \left[ \frac{\partial A}{\partial \beta} \right]_q = -qZ_q^{q-1}Mov_q(A, H) \quad (28)
\]

The movariance converges to the usual covariance in the extensive limit $q \to 1$. Proof of the equation (Eq. 28) is

\[
\frac{\partial \rho}{\partial \beta} = \frac{1}{Z_q} \frac{\partial e_q(-\beta H)}{\partial \beta} - e_q(-\beta H) \frac{\partial Z_q}{\partial \beta} = \frac{-H}{Z_q} e_q(-\beta H) + e_q(-\beta H) \frac{\partial Z_q}{Z_q^2} \int He_q^3(-\beta H) \quad (32)
\]

Substituting the previous equation in Eq. (29) and using that $Z_q$ is a constant leads to

$$\Delta_{q, \beta} A = \frac{\partial [A]_q}{\partial \beta} - \left[ \frac{\partial A}{\partial \beta} \right]_q$$

$$= -qZ_q^{q-1} \int AH \rho^{2q-2} d\lambda + \cdots$$ \hspace{1cm} (33)

$$+ qZ_q^{q-1} \int A \rho^{q-1} \left( \int H \rho^q d\lambda \right) d\lambda$$ \hspace{1cm} (34)

$$= -qZ_q^{q-1} \left( [AH]_{2q-1} - [A]_q [H]_q \right)$$ \hspace{1cm} (35)

$$= -qZ_q^{q-1} Mov_q(A, H)$$ \hspace{1cm} (36)

If the operator $A$ does not depend explicitly of $\beta$, we find that the $q$-movariance is proportional to the derivative of the $q$-mean of $A$ with respect to $\beta$. We define the $q$-variance by

$$Mar_q(A) = [A^2]_{2q-1} - [A]_q^2$$ \hspace{1cm} (37)

This function converges to the usual variance when $q$ tends to 1. Using the equation of the generalised energy Eq. (21), we see that the $q$-variance of the Hamiltonian is given by

$$Mar_q(H) = \frac{1}{qZ_q^{q-1}} \frac{\partial^2 \ln_q(Z_q)}{\partial \beta^2}$$ \hspace{1cm} (38)
Applying Cauchy-Schwarz inequality
\[
\left( \int f g \, d\lambda \right)^2 \leq \left( \int f^2 \, d\lambda \right) \left( \int g^2 \, d\lambda \right)
\]
with \( f = A^{\rho - 1/2} Z_q^{-1} \) and \( g = \rho^{1/2} \) proves that \( Mar_q(A) \) is always positive. In the extensive limit when \( q \) tends to 1, the variance of the hamiltonian \( Var(H) = \partial^2 \ln(Z) / \partial \beta^2 \) is positive. Thus the internal energy as a function of the Lagrange multiplier \( \beta \) is a convex function.

We suppose now that the Hamiltonian \( H \) and the probability density function \( \rho \) depend on a parameter \( \alpha \) distinct of \( \beta \).
\[
\frac{\partial \ln_q(Z_q)}{\partial \alpha} = \beta \left[ \frac{\partial H}{\partial \alpha} \right]_q
\]

Remark that this equation is also valid for \( \alpha = \beta \) (by Eq. (31)). Proof of this equation is easy to compute.
\[
- \frac{\partial \ln_q(Z_q)}{\partial \alpha} = \frac{1}{Z_q} \frac{\partial Z_q}{\partial \alpha} = -1 \frac{\partial}{\partial \alpha} \int e_q(-\beta H) \, d\lambda
\]

Using the expression of the derivative of the \( q \)-exponential function with respect to \( \alpha \)
\[
\frac{\partial e_q(-\beta H)}{\partial \alpha} = -\beta \frac{\partial H}{\partial \alpha} e_q(-\beta H)
\]
leads to
\[
- \frac{\partial \ln_q(Z_q)}{\partial \beta} = \beta \int \frac{\partial H}{\partial \alpha} \rho^q \, d\lambda = \beta \left[ \frac{\partial H}{\partial \alpha} \right]_q
\]
The next fundamental equation gives the expression of the variance of an observable and the derivation of the hamiltonian with respect to a given parameter
\[
\frac{\partial [A]_q}{\partial \alpha} - \left[ \frac{\partial A}{\partial \alpha} \right]_q = -q \beta Z_q^{-1} Mov_q(A, \frac{\partial H}{\partial \alpha})
\]
The proof of this equation is
\[
\frac{\partial [A]_q}{\partial \alpha} = \int \frac{\partial A}{\partial \alpha} \rho^q \, d\lambda + \int \frac{\partial^2}{\partial \alpha} \frac{\partial}{\partial \rho} \, d\lambda = \left[ \frac{\partial A}{\partial \beta} \right] + q \int \rho^q \frac{\partial}{\partial \alpha} \left( \frac{e_q(-\beta H)}{Z_q} \right)
\]
The derivative of the partition function with respect to the parameter is given by
\[
\frac{\partial Z_q}{\partial \alpha} = -\beta \int \frac{\partial H}{\partial \alpha} e_q(-\beta H) \, d\lambda
\]
Let \( \Delta_{q,\alpha} A \) be the quantity
\[
\Delta_{q,\alpha} A = \frac{\partial [A]_q}{\partial \alpha} - \left[ \frac{\partial A}{\partial \alpha} \right]_q
\]
Substituting Eq. (31) and (47) in the previous equation and using that \( Z_q \) is a constant, we get
\[
\Delta_{q,\alpha} A = q \int A \rho^{q-1} \frac{\partial}{\partial \alpha} \left( \frac{e_q(-\beta H)}{Z_q} \right) \, d\lambda = \int \frac{A \partial H}{\partial \alpha} \rho^q \, d\lambda
\]
In the extensive limit when \( q \) tends to 1, this equation is also valid for \( \beta = \alpha \). Proof of this equation is easy to compute.
\[
\Delta_{q,\alpha} A = \frac{\partial [A]_q}{\partial \alpha} - \left[ \frac{\partial A}{\partial \alpha} \right]_q
\]
If the parameter \( \alpha \) is the volume \( V \) and the pressure defined by \( P = -\langle \partial H / \partial V \rangle \), this equation is the usual thermodynamic relation \( T dS = dU + PdV \).
If we define the \( q \)-pressure by
\[
P_q = -\left[ \frac{\partial H}{\partial V} \right]_q = -\frac{1}{\beta c_q} \frac{\partial \ln_q(Z_q)}{\partial V}
\]
Define the following q-quantities

\[
\frac{\partial S_q}{\partial V} = -k\beta c_q \left( \frac{\partial \langle H \rangle_q}{\partial V} - \left\langle \frac{\partial H}{\partial V} \right\rangle_q \right) 
\]

leads to the ordinary equation

\[
dU_q = -c_q T dS_q - P_q dV
\]

Define the following q-quantities

\[
\frac{\partial Q_q}{\partial V} = \frac{\partial \langle H \rangle_q}{\partial V} - \left\langle \frac{\partial H}{\partial V} \right\rangle_q
\]

\[
\frac{\partial W_q}{\partial V} = -P_q = \left\langle \frac{\partial H}{\partial V} \right\rangle_q
\]

leads to the usual equation in the q-world

\[
dU_q = dQ_q + dW_q = dQ_q - P_q dV
\]

Derivating Eq. (8) with respect to parameter \(\alpha\)

\[
\frac{\partial S_q}{\partial \alpha} = k \frac{\partial c_q}{\partial \alpha}
\]

Substituting the following equation

\[
c_q = 1 + \frac{1 - q}{k} S_q
\]

and taking \(A = 1\) in Eq. (14) leads to the differential equation follows by the Tsallis entropy.

\[
\frac{\partial S_q}{\partial \alpha} - \beta Z_q^{-1} \langle H \rangle_q \frac{\partial \langle H \rangle_q}{\partial \alpha} - \frac{\partial \langle H \rangle_q}{\partial \alpha} + \frac{\partial \langle H \rangle_q}{\partial \alpha} = 0
\]

4 Ideal Gas

The classical ideal gas in D-dimensional space has been exhaustively discussed in Ref. [14]. Correlations were analyzed in Ref. [17]. We summarize some results in the case \(0 < q < 1\) and show by straightforward calculation how the partition function is computed. The hamiltonian is

\[
H = \sum_{i=1}^{N} \frac{p_i^2}{2m}
\]

where \(m\) is the particle mass, \(N\) the particle number and \(p_i\) the momentum of the \(i\)th particle. The partition function is given by

\[
Z_q = \int_{\Omega} (1 - a(p_1^2 + \ldots + p_N^2))^{(1/2 - q)} - 1 \frac{d^D p_i d^D r_i}{N! h^D N}
\]

where \(a\) is the positive constant \(a = (1 - q)\beta/2m\) and \(\Omega\) is domain of integration such that \(1 - a(p_1^2 + \ldots + p_N^2) > 0\).

\[
Z_q = \frac{V^N}{N! h^D N} \int_{\Omega} (1 - a(p_1^2 + \ldots + p_N^2))^{(1/2 - q)} - 1 \frac{d^D p_i}{\Gamma(D/2)}
\]

and

\[
Z_q = \frac{V^N}{N! h^D N} \int_{0}^{1} (1 - a x^2)^{(1/2) - q} - 1 \frac{dx}{\Gamma(D/2)}
\]

Let \(x = a r^2\). Using the fact that the beta distribution is a density of probability for two parameters \(u\) and \(v\) greatest or equal 1, that is

\[
\Gamma(u + v) \Gamma(u) \Gamma(v) \int_{0}^{1} x^{u-1} (1-x)^{v-1} dx = 1
\]

After a simple calculation, we find the following expression of the q-partition function

\[
Z_q = \frac{V^N}{N!} \left( \frac{2\pi m}{(1 - q)\beta h^2} \right)^{D/2} \frac{\Gamma(2 - q)}{\Gamma(2 - q + DN/2)}
\]

The same calculation for the constant \(c_q\) leads to

\[
c_q = Z_q^{-1} \left( \frac{1}{\Gamma(1 - q)} + \frac{DN}{2} \right) \frac{\Gamma(2 - q + DN/2)}{\Gamma(2 - q - DN/2)}
\]

Using the following property of the gamma function

\[
\Gamma(z + 1) = z \Gamma(z)
\]

we find

\[
c_q = Z_q^{-1} q (1 + (1 - q) DN/2)
\]

The generalised energy given by Eq. (21) is computed in the same way

\[
U_q = \frac{DN Z_q^{-1} q}{2\beta c_q}
\]
Using the expression of the Lagrange mutiplier given by Eq. (29), we obtain the following expression of the internal energy which is independent of the entropic index.

\[ U_q = \frac{DN}{2} k_B T \]  

(78)

The Tsallis and Boltzmann-Gibbs constants are related to the equation

\[ k = c_q Z_q^{q-1} k_B = \left( 1 - (1 - q) \frac{DN}{2} \right) k_B \]  

(79)

The equation of state

\[ P_q = \frac{1}{\beta c_q Z_q^q} \frac{\partial Z_q}{\partial V} = \frac{N}{V} \frac{Z_q^{1-q}}{\beta c_q} \]  

(80)

leads to the usual expression

\[ P_q V = N k_B T \]  

(81)

In the extensive limit when \( q \) tends to 1, we recover the equation of ideal gas. Remark that if we replace the hamiltonian by the same expression minus the generalized energy, the results are unchanged [18]. The \( q \)-marianace of the hamiltonian

\[ \text{Marq}(H) = \frac{1}{q} k_B T U_q \]  

(82)

converges in the extensive limit to the variance of the hamiltonian \( DN k_B^2 T^2 / 2 \). Simple calculations give the marovariance of the \( q \)-pressure

\[ \text{Mov}_q(H, P_q) = \frac{1 - q}{q} \frac{DN}{2} P_q c_q Z_q^{1-q} \]  

(83)

and

\[ \text{Mov}_q(H, \frac{\partial H}{\partial V}) = -\frac{N k}{q V k_B \beta^2} Z_q^{2-2q} \]  

(84)

and the ratio

\[ \frac{\text{Mov}_q(H, P_q)}{\text{Mov}_q(H, \frac{\partial H}{\partial V})} = (q - 1) \frac{k}{k_B} \frac{DN}{2} \]  

(85)

5 Quantum Free Particle

In this section, a free particle confined in a 3-dimensional box is considered. Denoting \( L_1, L_2, L_3 \) the sides of the box, it is well known that the energy eigenvalues are given by

\[ E = \frac{h^2}{8 m} \left( \frac{n_1^2}{L_1^2} + \frac{n_2^2}{L_2^2} + \frac{n_3^2}{L_3^2} \right) \]  

(86)

where the quantum numbers are non-negative integers \( n_i = 0, 1, 2, \ldots \). The partition function is given by

\[ Z_q = \sum_{n_1, n_2, n_3} c_q (-\beta E) = \sum_{n_1, n_2, n_3} (1 - (1 - q)E)^{\frac{1}{\beta}} \]  

(87)

Let \( \lambda_T \) be the translation temperature

\[ \lambda_T = \frac{h^2}{8 m k L^2} \]  

(88)

where \( L \sim L_1 \) is the length of a side. For sufficiently large \( T \gg \lambda_T \), the sum over \( n_i \) can be replaced by an integral

\[ Z_q = \frac{1}{8} \int (1 - (1 - q)\beta E)^{\frac{1}{\beta}} 1_{\Omega(n_i)} dn_1 dn_2 dn_3 \]  

(89)

where \( \Omega = \{ 1 - (1 - q)\beta E > 0 \} \) and \( 1_A(x) = 1 \) if \( x \in A \) and 0 otherwise. For a volume \( V = L_1 L_2 L_3 \) and a parameter \( b = \beta (1 - q)h^2 / 8m \) the change of variables

\[ Z_q = \frac{V}{8 b^{3/2} \Gamma(3/2)} \int_0^{1/\sqrt{b}} (1 - br^2)^{\frac{1}{2} - q} r^2 dr \]  

(90)

could be written as a beta integral

\[ Z_q = \frac{V}{8 b^{3/2} \Gamma(3/2)} \int_0^1 (1 - x)^{\frac{1}{2} - q} x^2 dx \]  

(91)

thus

\[ Z_q = V \left( \frac{2\pi}{\beta (1 - q)h^2} \right)^{3/2} \frac{\Gamma(\frac{3-q}{2})}{\Gamma(\frac{3-q}{2} + \frac{3}{2})} \]  

(92)

this is the classical formula with \( N = 1 \) and \( D = 3 \).

6 Fermions and Bosons Distributions

In this section, a free particle confined in a 3-dimensional box is considered. Denoting \( L_1, L_2, L_3 \) the sides of the box, it is well known that the energy eigenvalues are given by
distributions have been studied recently \cite{19, 20, 21}. In the grand canonical ensemble, the grand partition function is the sum over the states $j$

$$\Theta_q = \sum_j (1 - \beta(1 - q)(E_j - \mu))^\frac{1}{q-1} \quad (93)$$

where $\mu$ is the chemical potential. The density of states in the phase space is given by

$$D(E) = \frac{\partial}{\partial E} \int \frac{d^3r d^3p}{h^3} = \frac{2\pi V}{h^3} (2m)^{3/2} \sqrt{E} \quad (94)$$

In the thermodynamic limit, the sum is replaced by a continuous integration, thus the average is defined by

$$\langle A \rangle_q = \sum_j A_j \rho_j^q \simeq \int_0^\infty AD(E) f^q(E) dE \quad (95)$$

where $f$ is the function

$$f(E) = \frac{1}{(1 - \beta(1 - q)(E - \mu))^\frac{1}{q-1}} \quad (96)$$

The change of variable $E = x^2/\beta(q - 1)$ gives

$$\langle A \rangle_q = \frac{4\pi V}{h^3} \left( \frac{2mkT}{q - 1} \right)^{3/2} \int_0^\infty A^q x^2 f(x) dx \quad (97)$$

The average total number of particles

$$\bar{N} = \sum_j \frac{1}{(1 - \beta(1 - q)(E_j - \mu))^\frac{1}{q-1}} \quad (98)$$

in the thermodynamic limit

$$\bar{N} \simeq \int_0^\infty D(E) f(E) dE \quad (99)$$

is calculated by

$$\bar{N} = K \int_a^\infty \frac{x^2}{(1 + x^2 - \beta(q - 1)\mu)^\frac{1}{q-1}} dx \quad (100)$$

where $a^2 = -1 + \beta(q - 1)\mu$ and

$$K = \frac{4\pi V}{h^3} \left( \frac{2mkT}{q - 1} \right)^{3/2} \quad (101)$$

This integral could be computed as a beta integral if $1 \leq q \leq 7/5$

$$\bar{N} = \frac{4\pi V}{3(q - 1)h^3} \left( \frac{2mkT}{q - 1} \right)^{3/2} a^{-2u} \frac{\Gamma(u)\Gamma(v)}{\Gamma(u + v)} \quad (102)$$

where $u = (5 - 3q)/(2q - 1)$ and $v = 1/(1 - q)$. Same calculation could be done for the generalised energy in the thermodynamic limit

$$U_q = \frac{2\pi V}{h^3} \int_0^\infty (2mE)^{3/2} f^q(E) dE \quad (103)$$

after simple manipulation

$$U_q = \frac{4\pi V}{2\beta(q - 1)h^3} \left( \frac{2mkT}{q - 1} \right)^{3/2} a^{-2u} \frac{\Gamma(u)\Gamma(v)}{\Gamma(u + v)} \quad (104)$$

The average energy per particle

$$U_q = \frac{3}{2} \bar{N} kT \quad (105)$$

It is worth noting that the generalized energy is in this equation related to the Tsallis constant (and not to the Boltzmann-Gibbs constant).

7 Conclusions

In this paper, we studied the probabilistic properties within the framework of Tsallis thermodynamics. It has been shown that movariance and mariance are suitable functions for generalized covariance and variance in the non-extensive thermodynamic context. The invariance of thermodynamic relations is established if the Tsallis constant depend on the entropic parameter. However, in the canonical ensemble, as illustrated on the ideal gas, the equation of state as the same form as the ordinary equation in the non-extensive formalism with the Bolzmann-Gibbs constant. In the grand canonical ensemble, the dependance of the q-average internal energy is related to the Tsallis constant. In both cases, we recover the usual equations in the extensive limit.

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