A NOTE ON GLOBAL WELL-POSEDNESS AND BLOW-UP OF SOME SEMILINEAR EVOLUTION EQUATIONS

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2092, Tunis, Tunisia

(Communicated by Thierry Cazenave)

Abstract. We investigate the initial value problems for some semilinear wave, heat and Schrödinger equations in two space dimensions, with exponential non-linearities. Using the potential well method based on the concepts of invariant sets, we prove either global well-posedness or finite time blow-up.

1. Introduction. In this paper, we study global well-posedness and finite time blow-up of three different evolution equations. First, we consider the Cauchy problem for the nonlinear Klein-Gordon equation

\[
\begin{aligned}
\ddot{u} - \Delta u + u &= f'(u);
(u, \dot{u})_{t=0} = (u_0, u_1) \in H^1 \times L^2.
\end{aligned}
\] (1.1)

Second, we treat the initial value problem for the semilinear heat equation

\[
\begin{aligned}
\dot{u} - \Delta u + u &= f'(u);
\end{aligned}
\] (u_{t=0} = u_0 \in H^1.\] (1.2)

Finally, we are interested on the Cauchy problem associated to the semilinear Schrödinger equation

\[
\begin{aligned}
i\dot{u} + \Delta u + f'(u) &= 0;
\end{aligned}
\] (u_{t=0} = u_0 \in H^1.\] (1.3)

Here and hereafter \( u := u(t, x) \) is a function of the variable \((t, x) \in \mathbb{R}_+ \times \mathbb{R}^2\), valued in \( \mathbb{R} \) for the case of (1.1) or (1.2) and in \( \mathbb{C} \) for the Schrödinger case. The nonlinearity \( f' \) is a regular real function satisfying the focusing sign \( xf'(x) > 0 \) for any real number \( x \neq 0 \) and an exponential growth at infinity to precise later.

In the Schrödinger context (1.3), we assume that \( f' \) takes the Hamiltonian form \( f'(x) = xg(x^2) \) for some positive real regular function \( g \) and we extend \( f' \) to \( \mathbb{C} \) by \( f'(z) := \frac{x}{|x|} f'(|x|) \).

The above wave problem has various applications in the areas of nonlinear optics, plasma physics and fluid mechanics [34].

2010 Mathematics Subject Classification. Primary: 35Q55.

Key words and phrases. Nonlinear wave equation, nonlinear heat equation, nonlinear Schrödinger equation, global existence, blow-up, ground state.

The author is grateful to the Laboratory of PDE and Applications at the Faculty of Sciences of Tunis.
The heat problem models diffusion or heat transfer in a system out of equilibrium. The function $u(t, x)$ might represent temperature or the concentration of some substance, a quantity which may vary with time [39].

Semilinear Schrödinger equations with various nonlinearities arise as models for diverse physical phenomena, including Bose-Einstein condensates [9, 24] and as a description of the envelope dynamics of a general dispersive wave in a weakly nonlinear medium [37].

Any solution to (1.1) formally satisfies conservation of the wave energy

$$E(t) = E(u(t), \dot{u}(t)) = \frac{1}{2} \| \dot{u}(t) \|^2_{L^2(\mathbb{R}^2)} + \frac{1}{2} \| \nabla u(t) \|^2_{L^2(\mathbb{R}^2)} - \frac{1}{2} \| u(t) \|^2_{L^2(\mathbb{R}^2)} - \int_{\mathbb{R}^2} f(u(t)) \, dx.$$  

A solution to (1.2) formally verifies decay of the heat energy

$$S(t) = S(u(t)) := \frac{1}{2} \| u(t) \|^2_{H^1(\mathbb{R}^2)} - \int_{\mathbb{R}^2} f(u(t)) \, dx, \quad \dot{S}(t) = -\| \dot{u}(t) \|^2_{L^4(\mathbb{R}^2)}.$$  

Any solution to (1.3) formally satisfies respectively conservation of the Hamiltonian, the mass and the Virial identity [16],

$$M(t) = M(u(t)) := \frac{1}{2} \| u(t) \|^2_{L^2(\mathbb{R}^2)};$$

$$H(t) = H(u(t)) := \frac{1}{2} \| \nabla u(t) \|^2_{L^2(\mathbb{R}^2)} - \int_{\mathbb{R}^2} f(\|u(t)\|) \, dx;$$

$$\frac{1}{8} (\| xu(t) \|^2_{L^2(\mathbb{R}^2)})'' = \| \nabla u \|^2_{L^2(\mathbb{R}^2)} - \int_{\mathbb{R}^2} \left( |u| f'(\|u\|) - 2 f(\|u\|) \right) \, dx. \quad (1.4)$$

Before going further, we recall a few historic facts about the previous problems. In the monomial case $f'(u) = u|u|^{p-1}$, local well-posedness in the energy space holds for any $1 < p < \infty$. We refer to [8] in the wave case, [7, 5] in the Schrödinger context and [19] for the heat problem. So it’s natural to consider problems with exponential nonlinearities, which have several applications, as for example the self trapped beams in plasma [14].

Indeed, Nakamura and Ozawa [21] proved global well-posedness and scattering for some defocusing wave equation with exponential type nonlinearity and small Cauchy data. Later on, A. Atallah [3] showed a local existence result to the defocusing 2D wave equation

$$(E_{\alpha}) \quad \ddot{u} - \Delta u + u e^{\alpha u^2} = 0,$$

for $0 < \alpha < 4\pi$ and with radially symmetric initial data $(0, u_1)$ having compact support. Recently, Ibrahim-Majdoub-Masmoudi [10] obtained global well-posedness of $(E_{4\pi})$ for data in the unit ball of the energy space. The author proved unconditional well-posedness for some weaker exponential growth nonlinearity [17, 18]. See [35, 36] for classical solutions.

For the Schrödinger problem, Nakamura and Ozawa [22] proved global well-posedness and scattering for some defocusing equation with exponential type nonlinearity and small Cauchy data. More recently, global well-posedness in the unit energy ball and scattering hold [6, 4], for the equation

$$iu + \Delta u = u(4\pi |u|^2 - 1).$$

Global well-posedness and scattering are established for some lower type exponential nonlinearity [26, 29, 27, 28, 32, 30].
In the case of semilinear heat equation with exponential type nonlinearity, global well-posedness in some Orlicz space with small data is now proved [13]. Moreover, global well-posedness in the energy space holds for the defocusing sign [12].

In the focusing case, local solution to one of the above evolution equations may exist globally or blow up in finite time [15, 31, 33]. It is the goal of this manuscript to prove global and non global existence of solutions to the problems (1.1), (1.2) and (1.3), when the energy is under the ground state one. It is worth pointing out that the present study uses the potential well method based on the concepts of invariant sets suggested by Payne and Sattinger in [23].

The rest of the paper is organized as follows. Section two contains the main results and some tools needed in the sequel. The third section is devoted to prove the existence of a ground state solution to the associated stationary problem. In the last three sections, we prove results about global well-posedness and finite time blow up of solutions to the evolution problems (1.1), (1.2) and (1.3).

In this manuscript, we are interested in the two space dimensions case, so, here and hereafter, we denote $\int \, dx := \int_{\mathbb{R}^2} \, dx$. For $p \geq 1$, $L^p := L^p(\mathbb{R}^2)$ is the Lebesgue space endowed with the norm $\| \cdot \|_p := \| \cdot \|_{L^p}$, $\| \cdot \| := \| \cdot \|_2$ and $H^1$ is the usual Sobolev space endowed with the norm $\| \cdot \|_{H^1} := (\| \cdot \|^2 + \| \nabla \cdot \|^2)^{\frac{1}{2}}$.

We mention that $C$ denotes an absolute positive constant which may vary from line to line and if $A$ and $B$ are nonnegative real numbers, $A \lesssim B$ means that $A \leq CB$. If $X$ is an abstract space, $C_T(X)$ stands for $C([0,T],X)$. Moreover, for $1 \leq r \leq \infty$ and $(s,T) \in [1,\infty) \times (0,\infty)$, we denote

$$\|u\|_{L^r_T(L^r)} := \left( \int_0^T \|u(t)\|_{L^r}^r \, dt \right)^{\frac{1}{r}}, \quad \|u\|_{L^r_T(L^r)} := \left( \int_0^{+\infty} \|u(t)\|_{L^r}^r \, dt \right)^{\frac{1}{r}}.$$

Finally, we define the derivative operator $(Df)(x) := xf'(x)$.

2. Background material. In this section we give the main results and some technical tools needed in the sequel. First, let us fix the set of nonlinearities considered along this paper.

1. Behavior on zero

$$f'(0) = f''(0) = f'''(0) = 0. \quad (2.5)$$

2. Ground state condition

$$\exists \varepsilon_f > 0 \text{ s. t } \min \{ (D - 2 - \varepsilon_f)f, (D - 2)^2 f \} > 0 \text{ on } \mathbb{R}^+. \quad (2.6)$$

3. Strong ground state condition

$$\exists \varepsilon_f > 0 \text{ s. t } \min \{ (D - 4 - \varepsilon_f)f, (D - 2)(D - 4)f \} > 0 \text{ on } \mathbb{R}^+. \quad (2.7)$$

4. Subcritical case: $\forall \alpha > 0, \exists C_\alpha > 0$ such that $\forall x \in \mathbb{R}$,

$$\left\{ \begin{array}{l} |f''(x)| \leq C_\alpha e^{\alpha x^2}, \text{ in the case of (1.1);} \\ |f'''(x)| \leq C_\alpha e^{\alpha x^2}, \text{ in the case of (1.2) or (1.3).} \end{array} \right. \quad (2.8)$$

5. Critical case: $\exists \alpha_0, M_0 > 0$ such that when $x \to +\infty$,

$$\left\{ \begin{array}{l} |f''(x)| \leq M_0 e^{\alpha x^2}, \text{ in the case of (1.1);} \\ |f'''(x)| \leq M_0 e^{\alpha x^2}, \text{ in the case of (1.2) or (1.3).} \end{array} \right. \quad (2.9)$$

We will say that the nonlinearity is subcritical (respectively critical) if $f$ satisfies (2.5), (2.6) and (2.8) (respectively (2.5), (2.6) and (2.9)).
Remark 2.1.
1. For technical difficulty related to the existence of ground state, we assume that some derivatives of the nonlinearity vanish on zero.
2. Existing instability results \[\{11, 6\},\] justify the fact that the critical nonlinearity growth for the above problems is exponential of the form \(e^{ux}\).
3. Local well-posedness results in \[\{10, 6, 12\},\] are given for some explicit nonlinearities. The same arguments give local well-posedness for nonlinearities with the same behavior, which means the same growth at infinity.

We give two explicit examples.

Proposition 2.2.
1. The function \(f(x) := e^{x} - 1 - x - \frac{x^2}{2} \) is subcritical;
2. The function \(g(x) := e^{x^2} - 1 - x^2 - \frac{x^4}{4} \) is critical.

Proof. 1. We have \((D-2-\varepsilon) f(x) = (x-2-\varepsilon)e^x + 2 + \varepsilon + (1+\varepsilon)x + \frac{1}{2}(\varepsilon-1)x^2 \). Taking the derivatives of the last function, we obtain \([(D-2-\varepsilon)^2 f](x) = (x-1-\varepsilon)e^x + 1 + \varepsilon x + \frac{1}{2}(\varepsilon-1)x^2 \) and \([(D-2) f](x) = (x-\varepsilon)e^x + \varepsilon + (\varepsilon-1)x \). Taking \(\varepsilon = 1\), we have \((D-2-\varepsilon) f \geq 0\).

Now \((D-2)^2 f(x) = (x^2 - 3x + 4)e^x - 4 - x - \frac{x^4}{6} \). Taking the derivatives of the last function, we obtain \([(D-2)^2 f](x) = x[(x+1)e^x - 1] \geq 0\) and \([(D-2)^2 f](0) = (D-2)^2 f(0) = 0\). Thus, \((D-2)^2 f \geq 0\). Then, \(\min((D-2-\varepsilon) f(x), (D-2)^2 f(x)) > 0\) if \(x \neq 0\).

2. Now, \((D-2-\varepsilon) g(x) = (2r-2-\varepsilon)e^{r^2} + 2 + \varepsilon + r \) and \(\phi''(r) = (2r - 2 - \varepsilon)e^r + \varepsilon + 2\). Moreover, \(\phi'(r) = (2r - 2 - \varepsilon)e^r + \varepsilon + (\varepsilon - 2)r\).

Taking \(\varepsilon = 2\), yields \((D-2-\varepsilon) g > 0\) on \(\mathbb{R}^*\).\((D-2)^2 g(x) = 4[(r^2 - r + 1)e^{r^2} - 1] \geq 0\). Thus, \(\min((D-2-\varepsilon) g(x), (D-2)^2 g(x)) > 0\) if \(x \neq 0\). The proof is achieved.

We introduce several notations related to the evolution equations to be considered in this note. Here and hereafter, for \(\alpha, \beta \in \mathbb{R}\), \(v \in H^1(\mathbb{R}^2, \mathbb{R})\) and \(w \in H^1(\mathbb{R}^2, \mathbb{C})\), we denote the action

\[
S(v) := \frac{1}{2} \|v\|^2_{L^2} - \int f(v) \, dx, \quad S(w) := \frac{1}{2} \|w\|^2_{L^2} - \int f(|w|) \, dx.
\]

The following quantity will be called constraint

\[
K_{\alpha, \beta}(v) := \int \left[ \alpha |\nabla v|^2 + (\alpha + \beta)|v|^2 - \alpha v f'(v) - 2\beta f(v) \right] \, dx;
\]

\[
K_{\alpha, \beta}(w) := \int \left[ \alpha |\nabla w|^2 + (\alpha + \beta)|w|^2 - \alpha |w| f'(|w|) - 2\beta f(|w|) \right] \, dx.
\]

The quadratic and nonlinear parts of the constraint are

\[
K^Q_{\alpha, \beta}(w) := \int \left[ \alpha |\nabla w|^2 + (\alpha + \beta)|w|^2 \right] \, dx, \quad K^N_{\alpha, \beta} := K_{\alpha, \beta} - K^Q_{\alpha, \beta}.
\]

Take the minimizing problem under constraint

\[
m_{\alpha, \beta} := \inf_{0 \neq v \in H^1} \left\{ S(v), \ s.t \ K_{\alpha, \beta}(v) = 0 \right\}.
\]

If \((0, 0) \neq (\alpha, \beta) \in \mathbb{R}^2\), it is known \[\{28\}\] that \(m := m_{\alpha, \beta} > 0\) is independent of \((\alpha, \beta)\) and is the energy of some solution to the stationary problem associated to one of the above evolution equations. Such a solution is called ground state.
Definition 2.3. We call a ground state solution to (1.1) (or (1.2) or (1.3)) any solution to
\[
\Delta \phi - \phi + f'(\phi) = 0, \quad 0 \neq \phi \in H^1, \quad m_{\alpha,\beta} = S(\phi).
\]

Finally, we denote some stable sets under the flows of the evolution equations, they will play an essential role in our study. The following sets are adapted to study the wave problem.
\[
\mathcal{I}_{\alpha,\beta}^+ := \{(u,v) \in H^1 \times L^2 \text{ s. t. } E(u,v) < m_{\alpha,\beta} \text{ and } K_{\alpha,\beta}(u) \geq 0\};
\]
\[
\mathcal{I}_{\alpha,\beta}^- := \{(u,v) \in H^1 \times L^2 \text{ s. t. } E(u,v) < m_{\alpha,\beta} \text{ and } K_{\alpha,\beta}(u) < 0\}.
\]

The next sets will be considered when treating the heat or Schrödinger problem.
\[
A_{\alpha,\beta}^+ := \{v \in H^1 \text{ s. t. } S(v) < m_{\alpha,\beta} \text{ and } K_{\alpha,\beta}(v) \geq 0\};
\]
\[
A_{\alpha,\beta}^- := \{v \in H^1 \text{ s. t. } S(v) < m_{\alpha,\beta} \text{ and } K_{\alpha,\beta}(v) < 0\}.
\]

Results proved in this paper are listed in the following subsection.

2.1. Main results. Using a classical fixed point method, via Moser-Trudinger inequalities, under the assumption (2.8) or (2.9) on the nonlinearity, local well-posedness hold in the energy space, for the problems (1.1) (see Theorem 1.1 in [18] for the subcritical case and Theorem 1 in [10] for the critical one), (1.2) (see Theorem 2.1 in [12]) and (1.3) (see Theorem 1.2 in [26] for the subcritical case and Theorem 1.10 in [6] for the critical one).

Proposition 2.4. Let \( f' \) be a regular real function vanishing on zero and satisfying (2.8) or (2.9), then (1.1) (respectively (1.2) and (1.3) if \( f \) takes the Hamiltonian form) has a unique maximal solution in the energy space \( u \in C_T.(H^1) \cap C^T_r.(L^2) \) (respectively \( C^T_r(H^1) \)).

Results listed in this subsection deal with discussing if the maximal solution is global or not.

The study of the Schrödinger problem (1.3) is related to the Virial identity (1.4), which involves the quantity \( K_{1,-1} \). This quantity arises also when investigating the wave problem (1.1). So, we are reduced to establish the existence of ground state in the case \( (\alpha, \beta) = (1, -1) \), because the existence of a ground state is known [28] for nonnegative \( \alpha \) and \( \beta \).

Theorem 2.5. Let \( (\alpha, \beta) = (1, -1) \). Assume that \( f \) satisfies (2.5), (2.7) and (2.8) or (2.9). So, there exists a ground state solution to (2.10).

Now, we are concerned with the wave problem (1.1).

Theorem 2.6. Assume that \( f \) satisfies (2.5), (2.7) and (2.8) or (2.9). Let \( (u_0, u_1) \in H^1 \times L^2 \) and \( u \in C_T.(H^1) \cap C^T_r.(L^2) \) be the maximal solution to (1.1). Thus,
1. if there exist \( (0, 0) \neq (\alpha, \beta) \in \mathbb{R}_2^2 \cup \{(1, -1)\} \) and \( t_0 \in [0, T^*) \) such that \( (u(t_0), \dot{u}(t_0)) \in \mathcal{I}_{\alpha,\beta}^- \) and \( \int u(t_0) \dot{u}(t_0) \, dx > 0 \), then \( u \) blows-up in finite time;
2. if there exist \( (0, 0) \neq (\alpha, \beta) \in \mathbb{R}_2^2 \cup \{(1, -1)\} \) and \( t_0 \in [0, T^*) \) such that \( (u(t_0), \dot{u}(t_0)) \in \mathcal{I}_{\alpha,\beta}^+ \), then \( u \) is global.

The next result treats the heat problem (1.2).

Theorem 2.7. Assume that \( f \) satisfies (2.5), (2.6) and (2.8) or (2.9). Let \( u_0 \in H^1 \) and \( u \in C_T.(H^1) \) be the maximal solution to (1.2). Thus,
The independence of Remark 2.11.

The last result is about the Schrödinger problem (1.3).

**Theorem 2.8.** Assume that \( f \) takes the Hamiltonian form and verifies (2.5), (2.7) and [(2.8) or (2.9)]. Let \( u_0 \in H^1 \) and \( u \in C_T(H^1) \) be the maximal solution to (1.3). Thus,

1. if there exist \((0, 0) \neq (\alpha, \beta) \in \mathbb{R}^2_+ \) and \( t_0 \in [0, T^*) \) such that \( u(t_0) \in A^-_{\alpha, \beta} \), then \( u \) blows-up in finite time;
2. if there exist \((0, 0) \neq (\alpha, \beta) \in \mathbb{R}^2_+ \) and \( t_0 \in [0, T^*) \) such that \( u(t_0) \in A^+_{\alpha, \beta} \), then \( u \) is global.

We collect in what follows some auxiliary results.

2.2. **Tools.** This subsection is devoted to give some standard estimates needed along this paper. When \( \alpha \) and \( \beta \) are nonnegative, the existence of a ground state solution to (2.10) is known [28].

**Proposition 2.9.** Let \((0, 0) \neq (\alpha, \beta) \in \mathbb{R}^2_+ \). Assume that \( f \) satisfies (2.5) and (2.6) with [2.8] or [2.9]). So, there is a ground state solution to (2.10). Moreover \( m := m_{\alpha, \beta} \) is nonzero and independent of \((\alpha, \beta)\).

The fact that \( m_{\alpha, \beta} \) is independent of \((\alpha, \beta)\) implies that some sets are also independent of \((\alpha, \beta)\).

**Lemma 2.10.** Let \((0, 0) \neq (\alpha, \beta) \in \mathbb{R}^2_+ \cup \{(1, -1)\}\). Then, the sets \( A^+_{\alpha, \beta}, A^-_{\alpha, \beta}, T^+_{\alpha, \beta} \) and \( T^-_{\alpha, \beta} \) are independent of \((\alpha, \beta)\).

**Proof.** Let \((\alpha, \beta) \) and \((\alpha', \beta') \) in \( \mathbb{R}^2_+ \cup \{(0, 0)\} \cup \{(1, -1)\}\). We denote, for \( \delta \geq 0 \), the sets
\[
A^+_{\alpha, \beta} : = \{ v \in H^1 \quad s.t \quad S(v) < m - \delta \quad \text{and} \quad K_{\alpha, \beta}(v) \geq 0 \};
A^-_{\alpha, \beta} : = \{ v \in H^1 \quad s.t \quad S(v) < m - \delta \quad \text{and} \quad K_{\alpha, \beta}(v) < 0 \}.
\]

By the previous result, the reunion \( A^+_{\alpha, \beta} \cup A^-_{\alpha, \beta} \) is independent of \((\alpha, \delta)\). So, it is sufficient to prove that \( A^+_{\alpha, \beta} \) is independent of \((\alpha, \beta)\). If \( S(v) < m \) and \( K_{\alpha, \beta}(v) = 0 \), then \( v = 0 \). So, \( A^+_{\alpha, \beta} \) is open. The rescaling \( v^\lambda := e^{\alpha \lambda} v(e^{-\beta \lambda}) \) implies that a neighborhood of zero is in \( A^+_{\alpha, \beta} \). Moreover, this rescaling with \( \lambda \to -\infty \) gives that \( A^+_{\alpha, \beta} \) is contracted to zero and so it is connected. Now, write
\[
A^+_{\alpha, \beta} = A^+_{\alpha, \beta} \cap (A^+_{\alpha', \beta} \cup A^-_{\alpha', \beta'}) = (A^+_{\alpha, \beta} \cap A^+_{\alpha', \beta}) \cup (A^+_{\alpha, \beta} \cap A^-_{\alpha', \beta'}).
\]

Since by the definition, \( A^-_{\alpha, \beta} \) is open and \( 0 \in A^+_{\alpha, \beta} \cap A^+_{\alpha', \beta'} \), using a connectivity argument, we have \( A^+_{\alpha, \beta} = A^+_{\alpha', \beta} \). The proof is similar in the case of \( T^+_{\alpha, \beta} \) and \( T^-_{\alpha, \beta} \).

**Remark 2.11.** The independence of \( m_{\alpha, \beta} \) on the real couple \((\alpha, \beta)\) is based on the existence of ground state, which holds only for some particular couples. For this reason, we keep dependence of the above sets on the parameter \((\alpha, \beta)\).

Let us recall the so-called generalized Pohozaev identity [28].
**Proposition 2.12.** If \( \phi \in H^1 \) is a solution to (2.10), then \( K_{\alpha,\beta}(\phi) = 0 \) for any \( \alpha, \beta \in \mathbb{R} \).

In order to control an exponential type nonlinearity in the energy space, we will use the following Moser-Trudinger inequality [1, 20, 38].

**Proposition 2.13.** Let \( \alpha \in (0, 4\pi) \), a constant \( C_{\alpha} \) exists such that for all \( u \in H^1 \) satisfying \( \|\nabla u\| \leq 1 \), we have
\[
\int \left( e^{\alpha |u(x)|^2} - 1 \right) dx \leq C_{\alpha} \|u\|^2.
\]
Moreover, the previous inequality is false if \( \alpha \geq 4\pi \). The limit case \( \alpha = 4\pi \) becomes admissible if we take \( \|u\|_{H^1} \leq 1 \) rather than \( \|\nabla u\| \leq 1 \). In this case
\[
\sup_{\|u\|_{H^1} \leq 1} \int \left( e^{4\pi |u(x)|^2} - 1 \right) dx < \infty
\]
and this is false for \( \alpha > 4\pi \). See [25] for more details.

Denoting \( H^1_{rd} \) the set of radial functions in \( H^1 \), we give some useful Sobolev embeddings [2].

**Proposition 2.14.**
1. \( W^{s,p}(\mathbb{R}^d) \hookrightarrow L^q(\mathbb{R}^d) \), for any \( 1 < p < q < \infty \) and \( s > 0 \) such that \( \frac{1}{p} \leq \frac{1}{q} + \frac{s}{d} \).
2. \( |u(x)| \lesssim \frac{\|u\|_{H^1}}{\sqrt{|x|}} \) for all \( u \in H^1_{rd}(\mathbb{R}^2) \) and almost all \( |x| > 0 \).

We close this subsection with a classical result about ordinary differential equation.

**Proposition 2.15.** Let \( \varepsilon > 0 \). There is no real function \( G \in C^2(\mathbb{R}^+ \) satisfying
\[G(0) > 0, G'(0) > 0 \text{ and } GG'' - (1 + \varepsilon)(G')^2 \geq 0 \text{ on } \mathbb{R}^+.\]

**Proof.** Assume with contradiction, the existence of such a function. Then \((G^{-1}\varepsilon)G')' \geq 0\) and
\[
\frac{G'}{G^1 + \varepsilon} \geq \frac{G'(0)}{G(0)} > 0.
\]
This is a Riccati inequality with blow-up time \( T < \frac{1}{\varepsilon} \frac{G(0)}{G'(0)}. \) This contradiction achieves the proof.

3. **The stationary problem.** It is the aim of this section to prove that (2.10) has a ground state solution in the particular case \( (\alpha, \beta) = (1, -1) \), in the meaning that it has a nontrivial positive radial solution which minimizes the action \( S \) when \( K_{1,1} \) vanishes. Precisely, we establish Theorem 2.5.

**Remark 3.1.** We treat the real case, the complex one follows similarly taking account of the Hamiltonian form of the function \( f \).

In this section we denote, for \( v \in H^1(\mathbb{R}^2, \mathbb{R}) \) and \( \alpha, \beta \in \mathbb{R} \), the quantities
\[
v^\lambda_{\alpha, \beta} := e^{\alpha \lambda v} (e^{-\beta \lambda}), \quad \mathcal{L}_{\alpha, \beta}(v) := (\partial_\lambda v^\lambda_{\alpha, \beta})|_{\lambda = 0};
\]
\[
\mathcal{L} := \mathcal{L}_{1, -1}, \quad m := m_{1, -1}, \quad K := K_{1, -1}, \quad K^Q(v) := \|\nabla v\|^2;
\]
\[
K^N(v) := (K - K^Q)(v) = -\int (D - 2)f(v) \, dx;
\]
\[
T(v) := (2S - K)(v) = \|v\|^2 + \int (D - 4)f(v) \, dx.
\]
Remarks 3.2.

1. We extend the operator $\mathcal{L}_{\alpha,\beta}$ as follows, if $A : H^1 \to \mathbb{R}$, then

$$\mathcal{L}_{\alpha,\beta}A(v) := \partial_\lambda (A(v_{\alpha,\beta}^\lambda))|_{\lambda = 0}.$$ 

2. With a direct computation, we get $K_{\alpha,\beta} = \mathcal{L}_{\alpha,\beta}S$, so $T = (2 - \mathcal{L})S$.

The proof of Theorem 2.5 is based on the following intermediate result.

**Lemma 3.3.** Let $0 \neq \phi_n$ a bounded sequence of $H^1$ such that $\lim_{n} K^Q(\phi_n) = 0$. Then, there exists $n_0 \in \mathbb{N}$ such that $K(\phi_n) > 0$ for all $n \geq n_0$.

**Proof.** There exists some $p > 2$ satisfying $|rf'(r)| + |f(r)| \lesssim r^p(e^{\alpha_0 r^2} - 1)$. In fact, by (2.8) or (2.9), the ratio is bounded at infinity and using the assumptions on zero, the ratio is bounded near zero. Thus, for any $q \geq 1$,

$$K^N(\phi_n) \lesssim \int |\phi_n|^p(e^{\alpha_0 |\phi_n|^2} - 1) dx$$

$$\lesssim \|\phi_n\|_{L^p}^p e^{\alpha_0|\phi_n|^2} - 1\|q'$$

$$\lesssim \|\phi_n\|_{L^p}^p e^{\alpha_0q'|\phi_n|^2} - 1\|q'.

Now, if $\alpha_0q'||\nabla \phi_n||^2 < 2\pi$, by Moser-Trudinger inequality, we get $K^N(\phi_n) \lesssim \|\phi_n\|_{L^p}^p \|
\phi_n\|^{\frac{2}{p}}$. Moreover, using the interpolation inequality on $\mathbb{R}^2$,

$$\|\cdot\| \lesssim \|\cdot\|^\frac{\epsilon}{2}\|\nabla\|^1 - 1\|r, \forall r \in [2, \infty),$$

we obtain $K^N(\phi_n) \lesssim \|\phi_n\|_{L^p}^p \|
\phi_n\|^{\frac{2}{p}} \lesssim \|\phi_n\|^{2}\|\nabla \phi_n\|^{p - \frac{2}{p}}$. Since $\|\nabla \phi_n\|^2 = K^Q(\phi_n)$ and $(\phi_n)$ is bounded in $H^1$, taking $q$ such that $p - \frac{2}{p} > 2$, we get for $n$ going to infinity

$$K(\phi_n) \simeq K^Q(\phi_n) > 0.$$

The proof is achieved. \qed

**Proof of Theorem 2.5.** Let $(\phi_n)$ a minimizing sequence, namely

$$0 \neq \phi_n \in H^1, \ K(\phi_n) = 0 \text{ and } \lim_{n} S(\phi_n) = m. \quad (3.11)$$

With a rearrangement argument, we can assume that $\phi_n$ is radial decreasing and satisfies

$$0 \neq \phi_n \in H^1, \ K(\phi_n) \leq 0 \text{ and } \lim_{n} S(\phi_n) \leq m.$$

Since $K^Q(\lambda \phi_n) \to 0$ as $\lambda \to 0$, taking account of Lemma 3.3, there exists $\lambda \in (0, 1)$ such that $K(\lambda \phi_n) = 0$. So, we can suppose that $\phi_n$ is radial decreasing and satisfies (3.11). We decompose the rest of the proof to four steps.

• **First step.** $(\phi_n)$ is bounded in $H^1$. Using (3.11), we get

$$\|\nabla \phi_n\|^2 = \int (D - 2)f(\phi_n) dx \quad \text{and} \quad \|\phi_n\|^2_{H^1} - 2 \int f(\phi_n) dx \to 2m.$$

So, for any real number $a \neq 0$,

$$\left((1 - a)\|\nabla \phi_n\|^2 + \|\phi_n\|^2 + a \int |D - 2 - \frac{2}{a}||f(\phi_n) dx \right) \to 2m.$$

Taking $\varepsilon := \varepsilon > 0$ and $a := \frac{\varepsilon}{2 + \varepsilon}$, yields

$$\left(\frac{\varepsilon}{2 + \varepsilon} \|\nabla \phi_n\|^2 + \|\phi_n\|^2 + \frac{2}{2 + \varepsilon} \int |D - 4 - \varepsilon||f(\phi_n) dx \right) \to 2m.$$
We conclude, via (2.7) that \((\phi_n)\) is bounded in \(H^1\).

Taking account of the compact injection of the radial Sobolev space \(H^1_{rd}\) on the Lebesgue space \(L^p\) for any \(2 < p < \infty\), we take
\[
\phi_n \to \phi \quad \text{in} \quad H^1 \quad \text{and} \quad \phi_n \to \phi \quad \text{in} \quad L^p, \quad \forall p \in (2, \infty).
\]

**Second step.** \(\phi \neq 0\). Assume, by contradiction, that \(\phi = 0\).

First, we consider the subcritical case (2.8). The conditions (2.5) and (2.8) imply that for some \(p > 2\) and \(a > 0\) small enough, we have
\[
\max\{|f(\phi_n)| dx, |\phi_n f'(\phi_n)| dx\} \lesssim \|\phi_n^p (e^{a|\phi_n|^2} - 1)\|_1
\]
\[
\lesssim \|\phi_n\|_{2p}^p \|e^{2a|\phi_n|^2} - 1\|_1^{\frac{1}{2}}
\]
\[
\lesssim \|\phi_n\|_{2p}^p \|\phi_n\| \to 0,
\]

where we used Moser-Trudinger inequality, via the fact that \((\phi_n)\) is bounded in \(H^1\).

Thus, by Lemma 3.3, \(K(\phi_n) > 0\) for large \(n\) which contradicts (3.11).

Second, take the critical case (2.9). Since \((\phi_n)\) is bounded in \(H^1\), using Moser-Trudinger inequality, via (2.5) and (2.9), there exist some \(p > 2\) and \(a > 0\), such that for any \(\lambda > 0\) small enough and \(\phi_{n,\lambda} := \lambda \phi_n\), we have
\[
\max\{|f(\phi_{n,\lambda})| dx, |\phi_{n,\lambda} f'(\phi_{n,\lambda})| dx\} \lesssim \|\phi_n^p (e^{a\lambda^2|\phi_n|^2} - 1)\|_1
\]
\[
\lesssim \|\phi_n\|_{2p}^p \|e^{2a\lambda^2|\phi_n|^2} - 1\|_1^{\frac{1}{2}}
\]
\[
\lesssim \|\phi_n\|_{2p}^p \|\phi_n\| \to 0.
\]

Now, the identity \(K^Q(\phi_{n,\lambda}) = \lambda^2 K^Q(\phi_n) = -\lambda^2 K^N(\phi_n) \to 0\), via Lemma 3.3, implies that \(K(\phi_{n,\lambda}) > 0\) for large \(n\). This contradicts (3.11). So, in both subcritical and critical cases
\[
\phi \neq 0.
\]

**Third step.** \(m > 0\). Thanks to the lower semi continuity of \(H^1\) norm, we have \(K(\phi) \leq 0\) and \(S(\phi) \leq m\). Arguing as previously, thanks to Lemma 3.3, we can assume that \(K(\phi) = 0\) and \(S(\phi) \leq m\). So, \(\phi\) is a minimizer satisfying
\[
0 \neq \phi \in H^1_{rd}, \quad K(\phi) = 0 \quad \text{and} \quad S(\phi) = m.
\]

This implies, via the assumption (2.7), that
\[
0 < \|\phi\|^2 \leq \|\phi\|^2 + \int (D - 4) f(\phi) \, dx = 2m.
\]

**Last step.** \(\phi\) is a solution to (2.10). Now, there is a Lagrange multiplier \(\eta \in \mathbb{R}\) such that \(S'(\phi) = \eta K'(\phi)\). Then,
\[
0 = K(\phi) = \mathcal{L} S(\phi) = \langle S'(\phi), \mathcal{L}(\phi) \rangle
\]
\[
= \eta \langle K'(\phi), \mathcal{L}(\phi) \rangle
\]
\[
= \eta \mathcal{L} K(\phi) = \eta \mathcal{L}^2 S(\phi).
\]
Taking account of (2.7), a direct computation gives
\[-\mathcal{L}(\mathcal{L} - 2)S(\phi) = \mathcal{L}\left(\|\phi\|^2 + \int (D - 4)f(\phi) \, dx\right)\]
\[= \int (D - 2)(D - 4)f(\phi) \, dx\]
\[> 0.\]
Thus, \(-\mathcal{L}^2 S(\phi) > 0\), so \(\eta = 0\) and \(S'(\phi) = 0\). Finally, \(\phi\) is a ground state solution to (2.10). \(\square\)

4. The wave problem. This section is concerned with the proof of Theorem 2.6 about either global well-posedness or finite time blow-up of the solution to the Klein-Gordon problem (1.1). So, we assume in this section that (2.5) and (2.7) with [(2.8) or (2.9)] are satisfied.

Let us start with an auxiliary result about stable sets under the flow of the wave problem.

**Lemma 4.1.** Let \((0, 0) \neq (\alpha, \beta) \in \mathbb{R}_+^2 \cup \{(1, -1)\}\) and \(m\) be the energy of a ground state solution to (2.10). Then, the sets \(I_{\alpha, \beta}^+\) and \(I_{\alpha, \beta}^-\) are invariant under the flow of (1.1).

**Proof.** Let \((0, 0) \neq (\alpha, \beta) \in \mathbb{R}_+^2 \cup \{(1, -1)\}\) and \(u \in C_T(H^1) \cap C^1_T(L^2)\) be the maximal solution to (1.1). Assume that there exists a time \(t_0 \in (0, T^*)\) such that \((u(t_0), \dot{u}(t_0)) \notin I_{\alpha, \beta}^+\). Then, with conservation of the energy, \(K_{\alpha, \beta}(u(t_0)) < 0\). So, by a continuity argument, there exists a positive time \(t_1 \in (0, t_0)\) such that \(K_{\alpha, \beta}(u(t_1)) = 0\). This contradicts the definition of \(m\). The proof is similar in the case of \(I_{\alpha, \beta}^-\). \(\square\)

Now, we prove the main result about the wave problem (1.1).

**Proof of Theorem 2.6.**

1. With a translation argument, we can assume that \(t_0 = 0\). Thus, \(S(u_0) \leq E(0) < m\) and with Lemma 4.1, \(u(t) \in I_{\alpha, \beta}^-\) for any \(t \in [0, T^*)\). Take the real function
\[J(t) := \|u(t)\|^2, \quad t \in [0, T^*].\]

With Cauchy-Schwarz inequality, for any positive real number \(\gamma\), we have
\[JJ'' - \frac{\gamma}{4}J^2 = JJ'' - \gamma\left(\int u\dot{u} \, dx\right)^2 \geq JJ'' - \gamma\|\dot{u}\|^2 \|u\|^2 = J(J'' - \gamma\|\dot{u}\|^2).\]

Letting \(\varepsilon > 0\) and \(\gamma := 4 + \varepsilon\), we compute using (1.1),
\[JJ''(t) - \gamma\|\dot{u}\|^2 = (2 - \gamma)\|\dot{u}\|^2 - 2\|u\|_{H^1}^2 + 2\int uf'(u) \, dx\]
\[= -(2 + \varepsilon)\left[2E - \|u\|_{H^1}^2 + 2\int f(u) \, dx\right] - 2\|u\|_{H^1}^2 + 2\int uf'(u) \, dx\]
\[= -2(2 + \varepsilon)E + \varepsilon\|u\|_{H^1}^2 + 2\int (D - 2 - \varepsilon)f(u) \, dx.\]
Then, for $\varepsilon > 0$ near to zero, we get
\[ J''(t) - \gamma ||u||^2 \]
\[ = -2(2 + \varepsilon)E + \varepsilon||u||^2_{H^1} + 2 \int (D - 2 - \varepsilon)f(u) \, dx \]
\[ = 2(2 + \varepsilon)(m - E) - \varepsilon m - 2(2 + \frac{\varepsilon}{2})m + \varepsilon||u||^2_{H^1} + 2 \int (D - 2 - \varepsilon)f(u) \, dx \]
\[ > -2(2 + \frac{\varepsilon}{2})m + \varepsilon||u||^2_{H^1} + 2 \int (D - 2 - \varepsilon)f(u) \, dx. \]

Independently, denoting the quantity $H_{\alpha,\beta} := S - \frac{1}{2(\alpha + \beta)}K_{\alpha,\beta}$, by Lemmas 2.10 and 4.1, via the fact that (see [28]),
\[ m = \inf_{0 \neq v \in H^1} \left\{ H_{\alpha,\beta}(v), \text{ s.t. } K_{\alpha,\beta}(v) \leq 0 \right\}, \quad (4.12) \]
we have for any $\lambda > 0$,
\[ H_{1,\lambda}(u) = \frac{1}{2(1 + \lambda)} \left[ \lambda ||\nabla u||^2 + \int (D - 2)f(u) \, dx \right] > m. \]

It follows that
\[ (I) := -2(2 + \frac{\varepsilon}{2})m + \varepsilon||u||^2 + 2 \int (D - 2 - \varepsilon)f(u) \, dx \]
\[ > -2 + \frac{\varepsilon}{1 + \lambda} \left[ \lambda ||\nabla u||^2 + \int (D - 2)f(u) \, dx \right] + \varepsilon||u||^2 + 2 \int (D - 2 - \varepsilon)f(u) \, dx \]
\[ > \frac{1}{1 + \lambda} \left[ (\varepsilon(1 + \lambda) - \lambda(2 + \frac{\varepsilon}{2}))||\nabla u||^2 + \int [(2\lambda - \frac{\varepsilon}{2})(D - 2) - 2\varepsilon(1 + \lambda)]f(u) \, dx \right]. \]

Then,
\[ (I) > \frac{1}{1 + \lambda} \left\{ (2\lambda - \varepsilon(1 + \frac{\lambda}{2}))\left[ \int (D - 2)f(u) \, dx - ||\nabla u||^2 \right] \right. \]
\[ + \left. \int \frac{\varepsilon}{2}(1 + \lambda)(D - 2) - 2\varepsilon(1 + \lambda)]f(u) \, dx \right\} \]
\[ > \frac{1}{1 + \lambda} \left\{ (2\lambda - \varepsilon(1 + \frac{\lambda}{2}))\left[ \int (D - 2)f(u) \, dx - ||\nabla u||^2 \right] + \frac{\varepsilon}{2}(1 + \lambda) \int (D - 4)f(u) \, dx \right\} \]
\[ > \frac{1}{1 + \lambda} \left\{ -(2\lambda - \varepsilon(1 + \frac{\lambda}{2}))K_{1,-1}(u) + \frac{\varepsilon}{2}(1 + \lambda) \int (D - 4)f(u) \, dx \right\}. \]

Finally, taking $\lambda > 0$ and $\varepsilon > 0$ near to zero such that $2\lambda - \varepsilon(1 + \frac{\lambda}{2}) > 0$ and taking account of (2.7), via the fact that $K_{1,-1}(u) < 0$, we get
\[ J J'' - \frac{\gamma}{4} J^2 \geq \varepsilon J^2 \geq 0. \]

The proof is complete using Proposition 2.15.

2. Thanks to Lemmas 2.10 and 4.1, $u(t) \in \mathcal{I}^+_{1,1}$ for any $t \in [0, T^*)$. So
\[ m > E \geq (S - \frac{1}{4}K_{1,1})(u) \]
\[ = H_{1,1}(u) \]
\[ = \frac{1}{4} \left[ ||\nabla u||^2 + \int (D - 2)f(u) \, dx \right]. \]

Thus, $u(t)$ is bounded in $\dot{H}^1$. Precisely
\[ \sup_{t \in (0,T^*)} ||\nabla u(t)||^2 \leq 4m. \]
Independently, since \( K_{0,1}(u) > 0 \), the identity
\[
2m > 2E = \|\dot{u}\|^2 + \|\nabla u\|^2 + K_{0,1}(u)
\]
implies that
\[
\sup_{t \in (0,T^*)} ||\dot{u}(t)||^2 < 2m.
\]
Thus,
\[
\|u(t)\|^2 = \|u_0\|^2 + 2 \int_0^t \int u(s) \dot{u}(s) \, ds \, dx
\leq \|u_0\|^2 + 4 \int_0^t \left( \|u(s)\|^2 + \|\dot{u}(s)\|^2 \right) \, ds
\leq \|u_0\|^2 + 8mt + 4 \int_0^t \|u(s)\|^2 \, ds.
\]
A Gronwall argument gives
\[
\|u(t)\|^2 \leq \phi(t) + \int_0^t \phi(s) \exp(t-s) \, ds, \quad \phi(t) = \|u_0\|^2 + 8mt.
\]
The previous inequality implies that the \( L^2 \) norm of \( u \) does not explode in finite time. So, \( u \) is global because it is bounded in \( H^1 \).

5. The heat problem. In this section, we prove Theorem 2.7 about global and non global existence of solution to (1.2) in the energy space. We suppose in all this section that the nonlinearity of the heat problem (1.2) satisfies (2.5) and (2.6) with [(2.8) or (2.9)]. Let us recall some quantities needed in this section.

\[
S(u) = \frac{1}{2} \|u\|^2_{H^1} - \int f(u) \, dx, \quad K_{1,0}(u) = \|u\|^2_{H^1} - \int uf'(u) \, dx.
\]

First, we give stable sets under the flow of the heat problem.

**Lemma 5.1.** Let \((0,0) \neq (\alpha,\beta) \in \mathbb{R}^2_+ \) and \( m \) be the energy of a ground state solution to (2.10). Then, the sets \( A^+_{\alpha,\beta} \) and \( A^-_{\alpha,\beta} \) are invariant under the flow of (1.2).

**Proof.** Let \( u_0 \in A^+_{\alpha,\beta} \) and \( u \in C_T(H^1) \) be the maximal solution to (1.2). Assume that \( u(t_0) \notin A^-_{\alpha,\beta} \), for some time \( t_0 \in (0,T^*) \). Since the energy is decreasing, we have \( K_{\alpha,\beta}(u(t_0)) < 0 \). So, with a continuity argument, there exists a positive time \( t_1 \in (0,t_0) \) such that \( K_{\alpha,\beta}(u(t_1)) = 0 \). This contradicts the definition of \( m \). The proof is similar in the case of \( A^-_{\alpha,\beta} \).

Now, we prove the main result about the heat problem (1.2).

**Proof of Theorem 2.7.**

1. With a translation argument, we can assume that \( t_0 = 0 \). Thus, \( S(u(t)) \leq S(u_0) < m \). Moreover, with Lemma 5.1, \( u(t) \in A^-_{\alpha,\beta} \) for any \( t \in [0,T^*) \). Take the real function
\[
L(t) := \frac{1}{2} \int_0^t \|u(s)\|^2 \, ds, \quad t \in [0,T^*).
\]
Using the equation (1.2), a direct computation gives
\[ L'(t) = \frac{1}{2} \|u(t)\|^2 \quad \text{and} \quad L''(t) = \int \dot{u}u \, dx = -\|u(t)\|^2_{H^1} + \int uf'(u) \, dx. \]

We discuss two cases.
(a) **First case.** \( S(u_0) > 0. \) By Lemmas 2.10 and 5.1 via (4.12), we get for any \( \lambda > 0, \)
\[ H_{1,\lambda}(u) = \frac{1}{2(1 + \lambda)} \left[ \lambda \|\nabla u\|^2 + \int (D - 2)f(u) \, dx \right] > m. \]

Thus, for any \( \varepsilon > 0, \)
\[ L'' = -\|u\|^2 + \varepsilon \|\nabla u\|^2 - (1 + \varepsilon)\|\nabla u\|^2 + \int uf'(u) \, dx \]
\[ > -\|u\|^2 + \varepsilon \left[ 2(1 + \frac{1}{\lambda})m - \frac{1}{\lambda} \int (D - 2)f(u) \, dx \right] - (1 + \varepsilon)\|\nabla u\|^2 + \int uf'(u) \, dx \]
\[ > \varepsilon \left[ 2(1 + \frac{1}{\lambda})m - \frac{1}{\lambda} \int (D - 2)f(u) \, dx \right] - (1 + \varepsilon)\|\nabla u\|^2_{H^1} + \int uf'(u) \, dx. \]

Taking account of the identity \( S(u) = \frac{1}{2} \|u\|^2_{H^1} - \int f(u) \, dx = -\int_0^t \|\dot{u}(s)\|^2 \, ds + S(u_0), \) we obtain
\[ L'' > \varepsilon \left[ 2(1 + \frac{1}{\lambda})m - \frac{1}{\lambda} \int (D - 2)f(u) \, dx \right] - 2(1 + \varepsilon)\|u_0\|^2_{H^1} - \int_0^t \|\dot{u}(s)\|^2 \, ds \]
\[ + \int f(u) \, dx + \int uf'(u) \, dx \]
\[ > 2\varepsilon \left[ 1 + \frac{1}{\lambda} \right]m - (1 + \varepsilon)\|u_0\|^2_{H^1} + 2(1 + \varepsilon)\int_0^t \|\dot{u}(s)\|^2 \, ds \]
\[ + (1 - \frac{\varepsilon}{\lambda}) \int [D - 2(1 + \frac{1}{a} - \frac{\varepsilon}{\lambda})] f(u) \, dx \]
\[ := (I) + (II) + (III). \]

Taking \( \lambda := a\varepsilon \) for some \( a > 1 \) and \( \gamma := m - S(u_0), \) we get
\[ (I) = 2 \left[ \varepsilon (1 + \frac{1}{\lambda})m - (1 + \varepsilon)(m - \gamma) \right] \]
\[ = 2 \left[ \gamma (1 + \varepsilon) - m(1 - \frac{1}{a}) \right]. \]

On the other hand,
\[ (III) = (1 - \frac{\varepsilon}{\lambda}) \int [D - 2(1 + \frac{a}{1 - \varepsilon})] f(u) \, dx \]
\[ = (1 - \frac{1}{a}) \int [D - 2(1 + \frac{a}{1 - \varepsilon})] f(u) \, dx. \]

Since \( 0 < \gamma < m, \) we choose \( \varepsilon > 0 \) small enough such that
\[ 0 < \frac{2\varepsilon}{\varepsilon_f} < \frac{\gamma}{m}(1 + \varepsilon) < 1. \]

Then, there exists \( a > 1 \) satisfying
\[ \frac{2\varepsilon}{\varepsilon_f} < 1 - \frac{1}{a} < \frac{\gamma}{m}(1 + \varepsilon). \]
This choice, via (2.6), implies that the terms (I) and (III) are non-negative. Thus,

\[ L'' > 2(1 + \varepsilon) \int_0^t \| \dot{u}(s) \|^2 ds. \]

Thanks to Cauchy-Schwarz inequality, it follows that

\[ LL'' > (1 + \varepsilon) \| u \|_{L^2_t(L^2)}^2 \]

\[ > (1 + \varepsilon) \| u \dot{u} \|^2_{L^1_t(L^1)} \]

\[ > (1 + \varepsilon) L^2. \]

In fact, if \( L(t) = 0 \) for some positive time, we get \( K_{1,0}(u(t)) = 0 \), which contradicts Lemma 5.1. Thus

\[ (L^- \varepsilon)^{''} = -\varepsilon L^{''-2} \left[ L'' L - (1 + \varepsilon)(\dot{L})^2 \right] > 0. \]

Taking account of Proposition 2.15, for some finite time \( T > 0 \),

\[ \lim_{t \to T} \int_0^T \| u(s) \|^2 ds = \infty. \]

Thus, \( T^* < \infty \) and \( u \) is not global. This ends the proof.

(b) **Second case.** \( S(u_0) \leq 0 \). Using (2.6), we compute

\[ L'' = -\| u \|_{H^1}^2 + \int u f'(u) \, dx \]

\[ \geq (2 + \varepsilon_f) \left( \int f(u) \, dx - \frac{1}{2} \| u \|_{H^1}^2 \right) \]

\[ \geq -(2 + \varepsilon_f) S(u). \]

So, thanks to the identity \( S(u) = -\| \dot{u} \|^2 \), we get

\[ L'' \geq (2 + \varepsilon_f) \left( \| u \|_{L^2_t(L^2)}^2 - S(u_0) \right). \] (5.13)

Now, the proof goes by contradiction assuming that \( T^* = \infty \).

**Claim 1.** There exists \( t_1 > 0 \) such that \( \int_0^{t_1} \| \dot{u}(s) \|^2 \, ds > 0 \).

Indeed, otherwise \( u(t) = u_0 \) almost everywhere and solves the elliptic stationary equation \(-\Delta u + u = f'(u)\). Therefore, \( \| u \|_{H^1}^2 = \int u f'(u) dx \) and

\[ 0 \leq \varepsilon_f \int f(u_0) \, dx \leq \int (D - 2) f(u_0) \, dx = 2S(u_0) \leq 0. \]

Taking account of the focusing sign of the nonlinearity, we get \( u_0 = 0 \) which contradicts the fact that \( K_{0,1}(u_0) < 0 \).

**Claim 2.** For any \( 0 < \alpha < 1 \), there exists \( t_\alpha > 0 \) such that

\[ (L' - L'(0))^2 \geq \alpha L^2, \quad \text{on} \quad (t_\alpha, \infty). \]

The claim immediately follows from the first one and (5.13) observing that

\[ \lim_{t \to \infty} L(t) = \lim_{t \to \infty} L'(t) = +\infty. \]

**Claim 3.** One can choose \( \alpha = \alpha(\varepsilon) \) such that

\[ LL'' \geq (1 + \alpha) L^2, \quad \text{on} \quad (t_\alpha, \infty). \]
Indeed, we have
\[ LL'' \geq \frac{2 + \varepsilon f}{2} \| u \|^2_{L^2_t(L^2)} \| \dot{u} \|^2_{L^2_t(L^2)} \]
\[ \geq \frac{2 + \varepsilon f}{2} \| u \|^2_{L^1_t(L^1)} \]
\[ \geq \frac{2 + \varepsilon f}{2} (L' - L'(0))^2 \]
\[ \geq \frac{(2 + \varepsilon f)\alpha}{2} L'^2, \]
where we used (5.13) in the first estimate, Cauchy-Schwarz inequality in the second and Claim 2 in the last one. Now choosing \( \alpha \) such that
\[ 1 < \left( \frac{2 + \varepsilon f}{2} \right) : = 1 + \varepsilon, \]
we get
\[ LL'' > (1 + \varepsilon)L'^2, \quad \text{for large time.} \]

Thanks to Proposition 2.15, this ordinary differential inequality blows up in finite time and contradicts our assumption that the solution is global. This ends the proof.

2. By Lemmas 2.10 and 5.1, \( u(t) \in A_{+}^{1,1} \) for any \( t \in [0, T^*) \). So, thanks to the assumption (2.6), we have
\[ m > S(u) \geq (S - \frac{1}{4} K_{1,1})(u) = H_{1,1}(u) = \frac{1}{4} \left[ \| \nabla u \|^2 + \int (D - 2)f(u)\,dx \right]. \]

Thus, \( u(t) \) is bounded in \( \dot{H}^1 \). Precisely
\[ \sup_{t \in (0, T^*)} \| \nabla u(t) \|^2 \leq 4m. \]
Moreover, since \( \partial_t(\| u(t) \|^2) = -K_{1,0}(u) < 0 \), the \( L^2 \) norm of \( u \) is decreasing and so \( \| u(t) \| \leq \| u_0 \|. \) Thus
\[ \sup_{t \in (0, T^*)} \| u(t) \|_{H^1} < \infty. \]

Then, \( u \) is global.

6. The Schrödinger problem. In this section, we prove Theorem 2.8 about global and non global existence of a solution to (1.3). In all this section we denote \( H^1 := H^1(\mathbb{R}^2, \mathbb{C}) \) and we assume that the nonlinearity satisfies the Hamiltonian form and (2.5), (2.7) with \((2.8) \) or \((2.9)\). We keep notations of the previous sections as follows
\[ K := K_{1,-1}, \quad T := 2S - K, \quad m := m_{1,-1}. \]

Let us start with some auxiliary results.

Lemma 6.1.
1. For \( \phi \in H^1 \), the real function \( \lambda \mapsto T(\lambda \phi) \) is increasing on \( \mathbb{R}_+ \).
2. \( m = \inf_{\phi \neq \phi \in H^1} \left\{ \frac{1}{2} T(\phi) \quad s. t \quad K(\phi) \leq 0 \right\}. \)
Proof. 1. Denoting $\phi_\lambda := \lambda \phi$, we compute
\[ T(\phi_\lambda) = \lambda^2 \|\phi\|^2 + \int (D - 4)f(\lambda|\phi|) \, dx; \]
\[ \partial_\lambda T(\phi_\lambda) = 2\lambda \|\phi\|^2 + \frac{1}{\lambda} \int (D^2 - 4D)f(\lambda|\phi|) \, dx. \]

The proof of this point is ended thanks to (2.7) and the equality $D^2 - 4D = (D - 2)(D - 4) + 2(D - 4)$.

2. Denoting $m_1$ the right hand side of the previous equality, it is sufficient to prove that $m \leq m_1$. Take $\phi \in H^1$ such that $K(\phi) < 0$. Then, by Lemma 3.3, the facts that $\lim_{\lambda \to 0} K_Q(\lambda \phi) = 0$ and $\lambda \mapsto T(\lambda \phi)$ is increasing, there exists $\lambda \in (0, 1)$ such that $K(\lambda \phi) = 0$ and $T(\lambda \phi) \leq T(\phi)$. Thus,
\[ m \leq S(\lambda \phi) = T(\lambda \phi) \leq T(\phi). \]

The proof is closed.

Now, we give stable sets under the flow of the Schrödinger problem (1.3).

Lemma 6.2. Let $(0, 0) \neq (\alpha, \beta) \in \mathbb{R}_+^2 \cup \{(1, -1)\}$ and $m$ be the energy of a ground state solution to (2.10). Then, the sets $A_{\alpha, \beta}^+$ and $A_{\alpha, \beta}^-$ are invariant under the flow of (1.3).

Proof. Let $u_0 \in A_{\alpha, \beta}^+$ and $u \in C_T(H^1)$ be the maximal solution to (1.2). Assume that $u(t_0) \notin A_{\alpha, \beta}^+$ for some time $t_0 \in (0, T^*)$. Since the Hamiltonian and the mass are conserved, we get $K_{\alpha, \beta}(u(t_0)) < 0$. So, with a continuity argument, there exists a positive time $t_1 \in (0, t_0)$ such that $K_{\alpha, \beta}(u(t_1)) = 0$. This contradicts the definition of $m$. The proof is similar in the case of $A_{\alpha, \beta}^-$. \qed

Proof of Theorem 2.8.

1. With a translation argument, we can assume that $t_0 = 0$. Thus, using Lemmas 2.10 and 6.2, $u(t) \in A_{\alpha, \beta}^{-1}$ for any $t \in [0, T^*)$. By contradiction, assume that $T^* = \infty$ and take the real function $Q(t) := \|xu(t)\|^2$. Thanks to the Virial identity (1.4), we get
\[ \frac{1}{8}Q''(t) = \|\nabla u(t)\|^2 - \int (D - 2)f(|u|) \, dx = K(u) < 0. \]

We infer that there exists $\delta > 0$ such that $K(u(t)) < -\delta$ for large time. Indeed, otherwise, there exists a sequence of positive real numbers $t_n \to +\infty$ such that $K(u(t_n)) \to 0$. Taking account of Lemma 6.1, it follows that
\[ m \leq (S - \frac{1}{2}K)(u(t_n)) = S(u_0) - \frac{1}{2}K(u(t_n)) \to S(u_0) < m. \]

This contradiction finishes the proof of the claim. Thus, $Q'' < -8\delta$. Integrating twice, $Q$ becomes negative for some positive time. This absurdity closes the proof.
2. By Lemmas 2.10 and 6.2, \( u(t) \in A_{1,1}^+ \) for any \( t \in [0, T^*) \). Then, using (2.7), we obtain

\[
m > (S - \frac{1}{4} K_{1,1})(u) \\
= H_{1,1}(u) \\
= \frac{1}{4} \left[ \| \nabla u \|^2 + \int (D - 2) f(u) \, dx \right] \\
\geq \frac{1}{4} \| \nabla u \|^2.
\]

Thus, \( u(t) \) is bounded in \( \dot{H}^1 \). Precisely

\[
\sup_{t \in (0, T^*)} \| \nabla u(t) \|^2 \leq 4m.
\]

Moreover, since the \( L^2 \) norm of \( u \) is conserved, we have

\[
\sup_{t \in (0, T^*)} \| u(t) \|_{H^1} < \infty.
\]

Thus, \( u \) is global. This ends the proof.

\[ \square \]

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Received January 2015, revised July 2015.

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