A VARIANT OF THE EFFECTIVE ADJUNCTION CONJECTURE WITH APPLICATIONS

ZHAN LI

Abstract. We propose a variant of the effective adjunction conjecture for lc-trivial fibrations. This variant is suitable for inductions and can be used to treat real coefficients.

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1. Introduction

Throughout this paper, we work with varieties defined over complex numbers.

Let \((X, D)\) be a log pair with mild singularities. Suppose that \(f : X \to Z\) is a projective contraction such that \(K_X + D \sim_{Z, \mathbb{Q}} f^*L\) for some \(\mathbb{Q}\)-Cartier divisor \(L\) on \(Z\). The canonical bundle formula (see [Kaw98]) prescribes divisors \(D_{\text{div}}\) and \(D_{\text{mod}}\) on \(Z\). The divisor \(D_{\text{div}}\) is called the discriminant part which is related to the singularities of \((X, D)\), and \(D_{\text{mod}}\) is called the moduli part which is related to a polarization of the “moduli space” of the fibers. Moreover, we have \(K_Z + D_{\text{div}} + D_{\text{mod}} = L\). The canonical bundle formula is pivotal in translating the study of \((X, D)\) to the lower dimensional variety \(Z\).

The most mysterious part in the canonical bundle formula is the moduli part \(D_{\text{mod}}\). After certain birational modification of the base, \(D_{\text{mod}}\) is known to be nef and abundant (see [Amb05]). \(D_{\text{mod}}\) is conjectured to be semi-ample, or even stronger, effectively base-point free (i.e. there exists an effective \(m \in \mathbb{N}\) such that \(mD_{\text{mod}}\) is base-point free) (see [PS09, Effective...].
adjunction Conjecture 7.13.3). The effective adjunction conjecture is wide open with the only known case when $f$ is fibered by curves.

The purpose of this paper is to propose a variant of the effective adjunction conjecture ($\Gamma$-effective adjunction). It needs some preparations to state this conjecture and thus the precise statement is left to Section 3. We just mention that instead of predicting the moduli part to be effectively base-point free, it is predicted to be effectively $\Gamma$-base-point free. That is, there exists a finite set $\Gamma \subset (0, 1]$, such that after a base change,

$$K_X + D = f^*(K_Z + \tilde{D}_{\text{div}} + \tilde{D}_{\text{mod}})$$

with

$$\tilde{D}_{\text{mod}} = \sum r_i \tilde{D}_{\text{mod}, i},$$

where $\sum r_i = 1$ and $r_i \in \Gamma$. There is an effective $m \in \mathbb{N}$ such that each $mD_{\text{mod}, i}$ is base-point free. Besides, $(Z, \tilde{D}_{\text{div}})$ has the same type of singularities as $(X, D)$ in terms of discrepancies.

The $\Gamma$-effective adjunction conjecture is weaker than the effective adjunction conjecture when $(X, D)$ is klt (see Theorem 5.1). However, $\Gamma$-effective adjunction could be used as a substitute for the effective adjunction in many applications (see Section 6). The advantages of the $\Gamma$-effective adjunction lie in at least two aspects.

1. It is more suitable for the induction purpose. In fact, it can be put in the framework of the minimal model program. We show that the $\Gamma$-effective adjunction conjecture can be derived from the $K_X \sim_{Z, \mathbb{Q}} 0$ and the relative Picard number one cases (see Proposition 5.5).

2. It can be used to treat real coefficients. The effective adjunction only makes sense for rational coefficients.

Along the way to show Theorem 5.1, we establish two decomposition theorems (see Theorem 4.1, 4.2). These results are analogies of [Kaw14, Nak16] for a DCC coefficient set instead of a finite set.

Finally, we should point out that [Flo14, FL19] develop inductive approaches towards the effective adjunction conjecture from other perspectives. It is an interesting question to adopt their methods in the current context.

We describe the structure of the paper. In Section 2, we introduce definitions/notation and collect relevant results. The $\Gamma$-effective adjunction conjecture is stated in Section 3. Section 4 is devoted to the decomposition theorems. We study the relations between the effective adjunction conjecture and the $\Gamma$-effective adjunction conjecture in Section 5. In Section 6, we provide applications of the $\Gamma$-effective adjunction conjecture towards the boundedness problem of log canonical models and distributions of the Iitaka volumes studied in [Li20a].

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2. Preliminaries

2.1. Notation and conventions. For a morphism \( f : Y \to X \), we write \( f : Y \to X/U \) if \( f \) is a morphism over \( U \). When \( f \) is birational, and \( B \) is a divisor on \( X \), then \( f^{-1}_*B \) denotes the strict transform of \( B \) on \( Y \). We say that \( f \) is a contraction if \( f_*\mathcal{O}_Y = \mathcal{O}_X \). If \( f \) is a surjective morphism,

we write \( D^h \) (resp. \( D^v \)) to denote the horizontal (resp. vertical) part of \( D \) over \( X \). A subset \( I \subset \mathbb{R} \) is called an ACC (resp. a DCC) set if it satisfies the ascending chain condition (resp. descending chain condition) with respect to the order \( \leq \). We write \( D \in I \) if all the coefficients of \( D \) belong to \( I \).

We use \( \mathbb{N} \) to denote the set of positive integers. For \( k = \mathbb{Z}, \mathbb{Q}, \mathbb{R} \), and two divisors \( A, B \in k \) on a variety \( X \) over \( U \), \( A \sim_U,k B \) means that \( A \) and \( B \) are \( k \)-linearly equivalent over \( U \). When \( k = \mathbb{Z} \) or \( U = \text{Spec} \mathbb{C} \), we omit \( k \) or \( U \).

A \( \equiv \) means that \( A \) and \( B \) are numerically equivalent over \( U \).

Let \( X \) be a normal variety over \( U \) and \( B \) be an \( \mathbb{R} \)-divisor on \( X \), then \( (X, B)/U \) is called a log pair over \( U \). We assume that \( K_X + B \) is an \( \mathbb{R} \)-Cartier divisor for a log pair. Besides, \( U \) is usually omitted if this is clear from the context. For a prime divisor \( D \) over \( X \), if \( f : Y \to X \) is a birational morphism from a normal variety \( Y \) such that \( D \) is a divisor on \( Y \), then the log discrepancy of \( D \) with respect to \( (X, B) \) is defined to be \( \text{mult}_D(K_Y - f^*(K_X + B)) + 1 \). This definition is independent of the choice of \( Y \). A log pair \((X, B)\) is called sub-klt (resp. sub-lc) if the log discrepancy of any divisor over \( X \) is \( > 0 \) (resp. \( \geq 0 \)). If \( B \geq 0 \), then a sub-klt (resp. sub-lc) pair \((X, B)\) is called klt (resp. lc). The set of lc places of \((X, B)\) is denoted by

\[
\text{LCP}(X, B) := \{ D | D \text{ is a divisor over } X \text{ with log discrepancy } 0 \}.
\]

Assume that \( B_i, 1 \leq i \leq q \) are \( \mathbb{R} \)-divisors. For an \( \mathbb{R} \)-divisor \( B \), if there exist \( r_i \geq 0 \) such that \( B = \sum_{i=1}^q r_i B_i \) with \( \sum_{i=1}^q r_i = 1 \), then we write

\[
B \in \text{Conv}(B_1, \ldots, B_q).
\]

In this case, we also write

\[
(X, B) = \sum_{1 \leq i \leq q} r_i (X, B_i).
\]

Let \( X \) be a normal variety. An integral b-divisor over \( X \) is an element:

\[
D \in \text{Div}X = \lim_{Y \to X} \text{Div}Y,
\]

where the projective limit is taken over all birational models \( f : Y \to X \) proper over \( X \), under the push-forward homomorphism \( f_* : \text{Div}Y \to \text{Div}X \).
If $D = \sum d_{\Gamma} \Gamma$ is a b-divisor on $X$, and $Y \to X$ is a birational model of $X$, then the trace of $D$ on $Y$ is the divisor
\[ D_Y := \sum_{\Gamma \text{ is a divisor on } Y} d_{\Gamma} \Gamma. \]
B-divisors with coefficients in $\mathbb{Q}$ or $\mathbb{R}$ are defined similarly. Let $f : Y \to X$ be a birational morphism, we write $D_Y = D$ if the b-divisor $D$ is obtained by the pullback from $D_Y$.

**Definition 2.1.** For a $\mathbb{Q}$-b-divisor $D$, it is called semi-ample (resp. base-point free) if there exists a birational morphism $Y \to X$ such that $D_Y = D$ with $D_Y$ semi-ample (resp. base-point free).

For a b-divisor $D$, the pair $(X, D)$ is called lc (resp. klt) if $D_X \geq 0$ and $(Y, D_Y)$ is sub-lc (resp. sub-klt) for any birational model $Y \to X$ with $Y$ a $\mathbb{Q}$-factorial variety. The sheaf $\mathcal{O}_X(D)$ on $X$ is defined to be $\mathcal{O}_X(D_X)$. For more on b-divisors, see [Sho96, Amb04] and [Cor07, §2.3].

### 2.2. Canonical bundle formula.

Suppose that $(X, B)$ is a log pair with $B$ an $\mathbb{R}$-divisor. The discrepancy b-divisor $A = A(X, B)$ is the $\mathbb{R}$-b-divisor of $X$ with the trace $A_Y$ defined by the formula $K_Y = f^*(K_X + B) + A_Y$, where $f : Y \to X$ is a proper birational morphism of normal varieties. Similarly, we define $A^* = A^*(X, B)$ by
\[ A^*_Y = \sum_{a_i > -1} a_i A_i \]
for $K_Y = f^*(K_X + B) + \sum a_i A_i$. The following definition is slightly different from [Amb04, Definition 2.1] and [FG14, §3].

**Definition 2.2 (Klt-trivial and lc-trivial fibrations).** A klt-trivial (resp. lc-trivial) fibration $f : (X, B) \to Z$ consists of a projective contraction $f : X \to Z$ between quasi-projective normal varieties and a pair $(X, B)$ satisfying the following properties:

1. $(X, B)$ is sub-klt (resp. sub-lc) over the generic point of $Z$,
2. $\text{rank} f_* \mathcal{O}_X([A(X, B)]) = 1$ (resp. $\text{rank} f_* \mathcal{O}_X([A^*(X, B)]) = 1$), and
3. there exists an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor $L$ on $Z$ such that $K_X + B \sim_{\mathbb{R}} f^* L$.

Let $f : (X, B) \to Z$ be an lc-trivial fibration and $P$ be a prime divisor on $Z$. Because $Z$ is normal, after shrinking around $P$, we can assume that $P$ is Cartier. Define
\[ \text{lct}(\eta_P; X, B) := \max\{ t \in \mathbb{R} \mid (X, B + tf^* P) \text{ is sub-lc over the generic point of } P \} \]
and set
\[ D_{\text{div}, Z} := \sum_{P} (1 - \text{lct}(\eta_P; X, B)) P, \quad D_{\text{mod}, Z} := L - (K_Z + D_{\text{div}, Z}). \]
Then the following canonical bundle formula holds
\begin{equation}
K_X + B \sim_Q f^*(K_Z + D_{\text{div}, Z} + D_{\text{mod}, Z}).
\end{equation}

In this formula, $D_{\text{div}, Z}$ is called the discriminant part (or divisorial part) and $D_{\text{mod}, Z}$ is called the moduli part. When $(X, B)$ is lc, there exist $\mathbb{R}$-b-divisors $D_{\text{div}}$ and $D_{\text{mod}}$ of $X$ such that the traces $(D_{\text{div}}) Z = D_{\text{div}, Z}$ and $(D_{\text{mod}}) Z = D_{\text{mod}, Z}$. Moreover, for birational morphisms $p : Z' \to Z$, $q : X' \to X$ and a morphism $f' : X' \to Z'$ such that $f \circ q = p \circ f'$, we have
\begin{equation}
q^*(K_X + B) \sim_{\mathbb{R}} f'^*(K_{Z'} + (D_{\text{div}})_{Z'} + (D_{\text{mod}})_{Z'}).\end{equation}

This is shown in [Amb04, Theorem 0.2, Theorem 2.7] for $\mathbb{Q}$-divisors. But the actual computation does not need this assumption (see [Amb04, Lemma 2.6]).

By inversion of adjunction (see [Amb04, Theorem 3.1]), when $(X, B)$ is klt (resp. lc), then $(Z, D_{\text{div}})$ is also klt (resp. lc).

Moreover, when $K_X + B$ is $\mathbb{Q}$-Cartier, then $D_{\text{mod}}$ is b-nef and b-abundant in the sense that there is a proper birational morphism $h : Z' \to Z$ and a proper surjective morphism $h' : Z' \to W$ between normal varieties such that (1) $(D_{\text{mod}}) Z' \sim_{\mathbb{Q}} h^* H$ for some nef and big $\mathbb{Q}$-divisor $H$ on $W$, and (2) $D_{\text{mod}} = (D_{\text{mod}})_{Z'}$ (i.e. $D_{\text{mod}}$ is the pullback of $(D_{\text{mod}})_{Z'}$). For details, see [Amb05, Theorem 3.3] and [FG14, Theorem 1.1].

Finally, when the coefficients of $B$ belong to a DCC set $I$, then by (2.2.1) and Theorem 2.3 below, the coefficients of $(D_{\text{div}}) Z$ belong to a DCC set $J$ which depends only on $I$ and dim $X$.

2.3. A collection of relevant results.

**Theorem 2.3** (ACC for the log canonical threshold [HMX14, Theorem 1.1]). *Fix a positive integer $d$, $I \subset (0, 1]$ and a subset $J$ of the positive real numbers. If $I$ and $J$ satisfy the DCC, then
\begin{align*}
\{ \text{lct}(X, \Delta; M) | (X, \Delta) & \text{ is lc with dim } X = d, \Delta \in I, M \in J \}
\end{align*}
satisfies the ACC.*

**Theorem 2.4** (ACC for numerically trivial pairs [HMX14, Theorem D]). *Fix a positive integer $d$ and a set $I \subset (0, 1]$, which satisfies the DCC. Then there is a finite subset $I_0 \subset I$ with the following property:

If $(X, \Delta)$ is a log pair such that
\begin{enumerate}
\item $X$ is projective of dimension $d$,
\item the coefficients of $\Delta$ belong to $I$,
\item $(X, \Delta)$ is lc, and
\item $K_X + \Delta$ is numerically trivial,
\end{enumerate}
then the coefficients of $\Delta$ belong to $I_0$.*

The following result is a generalization of [Kaw14] where $X$ is assumed to be a fixed variety.
Theorem 2.5 ([Nak16, Theorem 1.6]). Fix $d \in \mathbb{N}$. Let $r_1, \ldots, r_c$ be positive real numbers, and let $r_0 = 1$. Assume that $r_0, \ldots, r_c$ are $\mathbb{Q}$-linearly independent. Let $s_1, \ldots, s_c : \mathbb{R}^{c+1} \to \mathbb{R}$ be $\mathbb{Q}$-linear functions from $\mathbb{R}^{c+1}$ to $\mathbb{R}$ (that is, the extensions of $\mathbb{Q}$-linear functions from $\mathbb{Q}^{c+1}$ to $\mathbb{Q}$ by taking the tensor product $\otimes \mathbb{Q}$). Assume that $s_i(r_0, \ldots, r_c) \in \mathbb{R} \geq 0$ for each $i$. Then there exists a positive real number $\epsilon > 0$ such that the following holds: For any $\mathbb{Q}$-Gorenstein normal variety $X$ of dimension $d$ and $\mathbb{Q}$-Cartier effective Weil divisors $D_1, \ldots, D_c$ on $X$, if $(X, \sum_{1 \leq i \leq c} s_i(r_0, \ldots, r_c)D_i)$ is lc, then $(X, \sum_{1 \leq i \leq c} s_i(r_0, \ldots, r_c-1, t)D_i)$ is also lc for any $t$ satisfying $|t - r_c| \leq \epsilon$.

A corollary of the above theorem and its proof is the following.

Theorem 2.6. Fix $d \in \mathbb{N}$, $c \in \mathbb{Z}_{\geq 0}$ and $\epsilon > 0$. Suppose that $a_1, \ldots, a_c \in \mathbb{R}_{>0}$ are positive real numbers. Let $\mathcal{S}$ be the set of log pairs $(X, \sum_{1 \leq i \leq c} a_i D_i)$ satisfying

1. $(X, \sum_{1 \leq i \leq c} a_i D_i)$ is lc with $\dim X = d$, and
2. $X$ is $\mathbb{Q}$-factorial, $D_i$ is a Weil divisor for each $1 \leq i \leq c$.

Then there exist $s \in \mathbb{N}$ and $(b_1^1, \ldots, b_s^c) \in \mathbb{Q}^c$, $1 \leq j \leq s$ such that for any $(X, \sum_{1 \leq i \leq c} a_i D_i) \in \mathcal{S}$,

$$ (X, \sum_{1 \leq i \leq c} b_i^j D_i) $$

is lc and

$$ \sum_{1 \leq i \leq c} a_i D_i \in \text{Conv} \left( \sum_{1 \leq i \leq c} b_i^1 D_i, \ldots, \sum_{1 \leq i \leq c} b_i^s D_i \right). $$

Moreover, $\{(b_i^1, \ldots, b_i^s) \mid 1 \leq j \leq s\}$ can be chosen such that $|a_i - b_i^j| < \epsilon$, and the sets of lc places (see (2.1.1)) satisfy

$$ \text{LCT}(X, \sum_{1 \leq i \leq c} b_i^j D_i) \subset \text{LCT}(X, \sum_{1 \leq i \leq c} a_i D_i). $$

Proof. If $a_i \in \mathbb{Q}$ for all $i$, then both claims automatically hold true. In the following, we assume that $\dim \text{Span}_{\mathbb{Q}}\{1, a_1, \ldots, a_c\} > 1$.

Choose $\{r_0 = 1, r_1, \ldots, r_c\}$ as a basis for $\text{Span}_{\mathbb{Q}}\{1, a_1, \ldots, a_c\}$, then there are $\mathbb{Q}$-linear functions $s_i : \mathbb{R}^{c+1} \to \mathbb{R}$ such that $a_i = s_i(r_0, \ldots, r_c)$. By Theorem 2.5, there are $q, p_c \in \mathbb{Q}$ such that $r_c \in (q, p_c)$ with $|r_c - q| < \epsilon, |r_c - p_c| < \epsilon$ and

$$ (X, \sum_{1 \leq i \leq c} s_i(r_0, \ldots, r_c-1, q)D_i), \ (X, \sum_{1 \leq i \leq c} s_i(r_0, \ldots, r_c-1, p)c)D_i $$

are both lc. Notice that $\text{Span}_{\mathbb{Q}}\{(1) \cup \{s_i(r_0, \ldots, r_c-1, q_i) \mid 1 \leq i \leq c\} \cup \{s_i(r_0, \ldots, r_c-1, p_i) \mid 1 \leq i \leq c\}\}$ has dimension smaller than the dimension of $\text{Span}_{\mathbb{Q}}\{1, a_1, \ldots, a_c\}$. Then, an induction on dimensions proves the first claim.
Thus, induction on dimensions as in the first part, we complete the proof.

**Theorem 2.7**

set by lc pairs with coefficients in a finite set.

We only prove the first inclusion as the second one can be shown by the same argument. Suppose that $P \in \text{LCP}(X, \sum_{1 \leq i \leq c} s_i(r_0, \ldots, r_{c-1}, h)D_i))$. Let $\gamma \in (0, 1)$ such that $h = \gamma h^+ + (1 - \gamma)r_c$. Then

$$1 = a(P; X, \sum_{1 \leq i \leq c} s_i(r_0, \ldots, r_{c-1}, h))$$

$$= \gamma a(P; X, \sum_{1 \leq i \leq c} s_i(r_0, \ldots, r_{c-1}, h^+))$$

$$+ (1 - \gamma) a(P; X, \sum_{1 \leq i \leq c} s_i(r_0, \ldots, r_{c-1}, r_c))$$

$$\leq \gamma + (1 - \gamma).$$

Thus $P \in \text{LCP}(X, \sum_{1 \leq i \leq c} s_i(r_0, \ldots, r_{c-1}, r_c)D_i))$.

We can further choose such $h, h' \in \mathbb{Q}$ with $|r_c - h| < \epsilon$, $|r_c - h'| < \epsilon$. By induction on dimensions as in the first part, we complete the proof. \hfill \Box

We need uniform approximations for lc pairs with coefficients in a DCC set by lc pairs with coefficients in a finite set.

**Theorem 2.7** ([FM18, Lemma 3.2], [HLS19, Theorem 5.21]). Let $d \in \mathbb{N}$ and $\tau \in \mathbb{R}_{>0}$ be fixed numbers. Assume that $I, I' \subset (0, 1]$ are DCC sets. Then there exists a finite set $I_0 \subset (0, 1]$ depending only on $d, \tau, I$ and $I'$ satisfying the following property:

If $(X, \Delta + D)$ is a log pair such that

1. $(X, \Delta + D)$ is lc with $\dim X = d$ and $X$ is a $\mathbb{Q}$-factorial variety,
2. $\Delta, D$ do not have common components, and
3. $\Delta \in I', D \in I$.

There exists $\bar{D} \in I_0$ such that $(X, \Delta + \bar{D})$ is lc, $\text{Supp} \bar{D} = \text{Supp} D$, and coefficients of $\bar{D} - D$ belong to $[0, \tau]$. 

...
Remark 2.8. This result is proved in [FM18, Lemma 3.2] and [HLS19, Theorem 5.21] when $\Delta = 0$. Both arguments work without any change in the above setting. Notice that the additional assumption that “$I \subseteq \mathbb{Q}$ and the accumulation points of $I$ are rationals” in [FM18, Lemma 3.2] is not needed in the argument of the above result. One can find a stronger statement in [HLS19, Theorem 5.21]. Notice that even if $I, I' \subseteq \mathbb{Q}$, $I_0$ may not be contained in $\mathbb{Q}$.

The argument for the following result can be found in [FG12, Proof of Theorem 3.1].

Theorem 2.9 (Relative abundance for numerically trivial klt pairs). Let $X \to Z$ be a projective contraction. Assume that $(X,B)$ is klt with $K_X + B$ a $\mathbb{Q}$-Cartier divisor. If $K_X + B \equiv 0/Z$, then $K_X + B \sim_{Z,\mathbb{Q}} 0$.

Lemma 2.10 ([KMM94, Lemma 1.7]). Let $f : X \to Z$ be a contraction between quasi-projective normal varieties. Let $E$ be a $\mathbb{Q}$-Cartier numerically $f$ trivial divisor on $X$, where no component of $E$ dominant $Z$. Suppose that $Z$ is $\mathbb{Q}$-factorial. Then $E$ is the pullback of a unique $\mathbb{Q}$-divisor on $Z$.

The following result is similar to Lemma 2.10. However, $Z$ is not assumed to be $\mathbb{Q}$-factorial in Lemma 2.11.

Lemma 2.11. Let $f : X \to Z$ be a contraction between quasi-projective normal varieties. Suppose that $E$ is an $\mathbb{R}$-Cartier divisor on $X$ such that $\text{Supp} E$ is vertical over $Z$. If $E \sim_{Z,\mathbb{R}} 0$, then there exists a divisor $P$ on $Z$ such that $E = f^*P$.

Proof. First, we show that it is enough to assume that $Z$ is smooth. Let $U$ be the smooth locus of $Z$, $W = f^{-1}(U)$ and $g = f|_W$. If there exists $P'$ on $U$ such that $E|_W = g^*P'$, we claim that $P = P'$ (i.e. the closure of each component of $P'$) is $\mathbb{R}$-Cartier and $E = f^*P$.

Suppose that $E \sim_{\mathbb{R}} f^*L$, then $g^*P' \sim_{\mathbb{R}} g^*(L|_U)$ and thus $g^*(P' - L|_U) \sim_{\mathbb{R}} 0$. Let $\Theta = P' - L|_U$. First, we show that when $g^*\Theta \sim_{\mathbb{R}} 0$, then $\Theta \sim_{\mathbb{R}} 0$ on $U$. By definition,

$$g^*\Theta = \sum_{1 \leq i \leq q} r_i \text{div}(h_i)$$

with $r_i \in \mathbb{R}$ and $h_i \in K(W)$. Write $\Theta = \sum_{1 \leq j \leq q'} \theta_j \Theta_j$ with $\Theta_j$ a prime divisor and $\text{div}(h_i) = \sum_{1 \leq k \leq q_i} a_i^k H_i^k$ with $a_i \in \mathbb{Z}$, then

$$\sum_{1 \leq j \leq q'} \theta_j g^*\Theta_j = \sum_{1 \leq i \leq q} (r_i \sum_{1 \leq k \leq q_i} a_i^k H_i^k).$$

By comparing the coefficients of each prime divisors, we have a system of rational linear equations:

$$f_s(\{x_{ij} \mid 1 \leq j \leq q'\}) = g_s(\{y_i, z_{ik}^l \mid 1 \leq i \leq q, 1 \leq k \leq q_i\}), 1 \leq s \leq l,$$

where $w := (\{\theta_j \mid 1 \leq j \leq q'\}; \{r_i, a_i^k \mid 1 \leq i \leq q, 1 \leq k \leq q_i\})$.
is a solution. Hence, there are rational solutions \( w_t, 1 \leq t \leq w \) of the above equations such that

\[
w = \sum_{1 \leq t \leq w} \gamma_t w_t
\]

with \( \gamma_t > 0 \) and \( \sum_{1 \leq t \leq w} \gamma_t = 1 \). This implies that there are \( \mathbb{Q} \)-divisors \( \Theta^t, 1 \leq t \leq w \) such that

(2.3.1) \[
\Theta = \sum_{1 \leq t \leq w} \gamma_t \Theta^t
\]

with \( g^* \Theta^t \sim_{\mathbb{Q}} 0 \). Therefore, we have \( m_t \in \mathbb{N} \) such that \( m_t \Theta^t \) is Cartier and \( \mathcal{O}_W(g^*(m_t \Theta^t)) \simeq \mathcal{O}_W \). By the projection formula,

\[
\mathcal{O}_U(m_t \Theta^t) = g_* \mathcal{O}_W(g^*(m_t \Theta^t)) \simeq \mathcal{O}_U.
\]

That is, \( \Theta^t \sim_{\mathbb{Q}} 0 \) on \( U \). By (2.3.1), we have \( \Theta \sim_{\mathbb{R}} 0 \) on \( U \).

By \( \Theta = P' - L'|_U \),

\[
P' \sim_{\mathbb{R}} L'|_U.
\]

As \( \text{codim}(X \setminus U, X) \geq 2 \), \( P' \sim_{\mathbb{R}} L \) and thus \( P = P' \) is \( \mathbb{R} \)-Cartier. By assumption, \( E|_W = g^* P' \). If we write

\[
g^* P - E = B^+ - B^-,
\]

where \( B^+ \geq 0, B^- \geq 0 \) without common components, then \( \text{codim} f(B^+) \geq 2, \text{codim} f(B^-) \geq 2 \). Moreover, \( B^+ - B^- \equiv 0/\mathbb{Z} \). By the negativity lemma for very exceptional divisors (for example, see [Bir12, Lemma 3.3]), we have \( B^+ = B^- = 0 \). This shows the claim.

Now, we can assume that \( Z \) is smooth. Then the argument of [KMM94, Lemma 1.7] applies to \( \mathbb{R} \)-divisors without any change. Or, we can use the same argument in the first part to obtain \( E = \sum \gamma_i E_i \), where \( E_i \) is vertical over \( Z \) with \( E_i \sim_{Z, \mathbb{Q}} 0 \), and \( \sum \gamma_i = 1, \gamma_i > 0 \). Applying [KMM94, Lemma 1.7] to each \( E_i \), we obtain the desired result. \( \square \)

3. \( \Gamma \)-effective adjunction conjecture

3.1. Effective adjunction conjecture. [PS09, Conjecture 7.13] proposes a series of conjectures concerning properties of the canonical bundle formula. We only focus on the effective adjunction conjecture in this paper.

Conjecture 3.1 (Effective adjunction, [PS09, Conjecture 7.13.3]). Under the notation in Section 2.2, suppose that \( (X, B) \) is lc with \( K_X + B \) a \( \mathbb{Q} \)-Cartier divisor. Assume that \( f : (X, B) \to Z \) is an lc-trivial fibration. Then there exists a positive integer \( m \) depending only on the dimension of \( X \) and the horizontal multiplicities of \( B \) (a finite set of rational numbers) such that \( mD_{\text{mod}} \) is a base-point free \( b \)-divisor (see Definition 2.1).

The statement in [PS09, Conjecture 7.13.3] is slightly restrictive by assuming additionally that there is a \( \mathbb{Q} \)-divisor \( \Theta \) on \( X \) such that \( K_X + \Theta \sim_{Z, \mathbb{Q}} 0 \).
and \((F, (1 - t)D|_F + t\Theta|_F)\) is klt for any \(0 < t \leq 1\), where \(F\) is the generic fiber of \(f\).

**Remark 3.2.** This conjecture is known when general fibers of \(f\) are curves (see [Kod63a, Kod63b, Fuj86] and [PS09, Theorem 8.1]). [Fuj03, Theorem 1.2] establishes the semi-ample property of the moduli \(b\)-divisors when a general fiber is a K3 surface or an abelian variety.

In the effective adjunction conjecture, only the coefficients of the horizontal part are required to be chosen from a fixed finite set of rationals. There is no restriction on coefficients of the vertical part (in fact, they can even be reals). The reason is given by the following result (at least in the klt case).

**Proposition 3.3.** If we assume that \((X, B)\) is klt in Conjecture 3.1, then for Conjecture 3.1, it suffices to assume that \(B\) does not have vertical components over \(Z\).

**Proof.** Let \(B = B^h + B^v\), then \(K_X + B^h \sim_{Z, \mathbb{R}} B^v\). Let \(U = Z - f(Supp B^v)\), then \((K_X + B^h)|_{f^{-1}(U)} \sim_{U, \mathbb{Q}} 0\). By [HX13, Theorem 1.1], \((X, B^h)\) has a good minimal model \((Y, B^h_Y)\) over \(Z\). Let \(B^h_Y, B^v_Y\) and \(B_Y\) be the strict transforms of \(B^h, B^v\) and \(B\) on \(Y\) respectively. By \(K_X + B \sim_{Z, \mathbb{R}} 0\) and \((X, B)\) klt, \((Y, B_Y)\) is still klt. Let \(p : Y \rightarrow W/Z\) be the morphism associated with \(K_Y + B_Y\), then \(q : W \rightarrow Z\) is a birational morphism. As \(K_Y + B^h_Y + B^v_Y \sim_{Z, \mathbb{R}} 0\), we have \(B^v_Y \sim_{W, \mathbb{R}} 0\). Moreover, \(B^v_Y\) is vertical over \(W\), hence by Lemma 2.11, there exists \(P\) on \(W\) such that \(B^v_Y = p^*P\). Therefore,

\[K_Y + B_Y = K_Y + B^h_Y + p^*P\]

Let \(D_{div}^B\) (resp. \(D_{div}^{B_Y}, D_{div}^h\)) and \(D_{mod}^B\) (resp. \(D_{mod}^{B_Y}, D_{mod}^h\)) be the discriminant \(b\)-divisor and the moduli \(b\)-divisor for the klt-trivial fibration \((X, B) \rightarrow Z\) (resp. \((Y, B_Y) \rightarrow W, (Y, B^h_Y) \rightarrow W\)).

We claim: (1) \(D_{div}^{B_Y} = D_{div}^h + \bar{P}\), hence \(D_{mod}^{B_Y} = D_{mod}^h\), and (2) \(D_{mod}^B = D_{mod}^{B_Y}\) viewing as \(b\)-divisors over \(W\).

For (1), let \(p' : Y' \rightarrow W'\) be a model of \(p : Y \rightarrow W\) with birational morphisms \(\tau : Y' \rightarrow Y\) and \(\theta : W' \rightarrow W\) such that \(p \circ \tau = \theta \circ p'\). Then

\[K_{Y'} + B_Y' = \tau^*(K_Y + B_Y) = \tau^*(K_Y + B_Y) + p'^*\theta^*(P)\]

\[= K_{Y'} + (B_Y')' + p'^*\theta^*(P)\]

Hence for any prime divisor \(\Theta\) on \(W'\),

\[\lct(\eta_{\Theta}; Y', (B_Y')') = \lct(\eta_{\Theta}; Y', B_Y') + \mult_{\Theta}(\theta^*P)\]
Therefore, (1) follows from (2.2.2). Technically speaking, we take $L$ such that $K_Y + B_Y^h \sim_{\R} p^* L$, $K_Y + B_Y \sim_{\R} p^*(L + P)$, and the moduli b-divisors are with respect to $L$ and $L + P$ respectively (see (2.2.2)).

For (2), by $K_X + B \sim_{Z,\R} 0$, for a common resolution $X \xleftarrow{s} T \xrightarrow{t} Y$, we have $K_T + B_T = s^*(K_X + B) = t^*(K_Y + B_Y)$. By the definition of the discriminant part in the canonical bundle formula, $D_{\text{div}}^B = D_{\text{div}}^{B_Y}$ viewing as b-divisors over $W$. Hence (2) holds true.

A combination of the above two claims shows the desired result. □

3.2. $\Gamma$-effective adjunction conjecture. Now we propose a variant of the effective adjunction conjecture. We need the following notion of $\Gamma$-base-point freeness.

**Definition 3.4 (Γ-base-point freeness).** Let $\Gamma \subset (0,1]$. An $\R$-b-divisor $M$ is called $\Gamma$-base-point free if $M = \sum_{i=1}^{l} a_i M_i$ with $\sum_{i=1}^{l} a_i = 1$, $a_i \in \Gamma$ and $M_i$ is a base-point free b-divisor for each $i$.

**Conjecture 3.5 (Γ-effective adjunction).** Suppose that $(X, D)$ is an klt (resp. lc) pair with $\dim X = d$. Let $f : (X, D) \to Z$ be an lc-trivial fibration as in Definition 2.2.

(1) (Strong $\Gamma$-effective adjunction) Assume that the coefficients of the horizontal divisors of $D$ belong to a finite set $I_h$. Then there exist a finite set $\Gamma \subset (0,1]$ and a positive integer $m$ which both depend only on $d$ and $I_h$, such that $mD_{\text{mod}}$ is $\Gamma$-base-point free, where $D_{\text{mod}}$ is the moduli b-divisor.

(2) (Weak $\Gamma$-effective adjunction) Assume that the coefficients of $D$ belong to a DCC set $I \subset (0,1]$, a finite set $\Gamma \subset (0,1]$ and a positive integer $m$ which all depend only on $d$ and $I$ satisfying the following property:

(a) there is a b-divisor $\tilde{D}_{\text{div}}$, such that its trace $\tilde{D}_{\text{div}}|Z \in J$,
(b) there is a b-divisor $\tilde{D}_{\text{mod}}$, such that $m\tilde{D}_{\text{mod}}$ is $\Gamma$-base-point free,
(c) $(Z, \tilde{D}_{\text{div}})$ is klt (resp. lc), and
(d) for birational morphisms $p : Z' \to Z$, $q : X' \to X$ and a morphism $f' : X' \to Z'$ such that $f \circ q = p \circ f'$, we have

$$q^*(K_X + D) \sim_{\R} f'^*(K_{Z'} + \tilde{D}_{\text{div}}|_{Z'} + \tilde{D}_{\text{mod}}|_{Z'}).$$

**Remark 3.6.** The $\tilde{D}_{\text{div}}$ and $\tilde{D}_{\text{mod}}$ in Conjecture 3.5 (2) may not be the discriminant and moduli b-divisors of the original fibration.

**Remark 3.7.** It is possible to state the “$\Gamma$-adjunction conjecture” by removing “the existence of an effective $m \in \mathbb{N}$” in the above $\Gamma$-effective adjunction conjecture. Some of the rest results still hold true in that setting. We left appropriate modifications to interested readers.
4. Decomposition theorems

The following results are generalizations of Theorem 2.6.

**Theorem 4.1.** Let $d \in \mathbb{N}$ be an integer and $I \subset (0,1]$ be a DCC set. Let $S := \{(X,D) \mid (X,D) \text{ lc, } X \text{ a } \mathbb{Q}-\text{factorial variety,}
\dim X = d \text{ and } D \in I\}.$ Then there exists a finite set $J \subset (0,1] \cap \mathbb{Q}$ satisfying the following property:

For any $(X,D) \in S,$ there exist $r_i \in \mathbb{R}_{>0}$ and divisors $D_i, 1 \leq i \leq q$ such that

1. $(X,D_i)$ is lc with $D_i \in J,$
2. $\text{Supp } D = \text{Supp } D_i$ for each $1 \leq i \leq q,$
3. $(X,D) = \sum_{i=1}^{q} r_i (X,D_i)$ with $\sum_{i=1}^{q} r_i = 1.$

In the above, $r_i, q$ depend on the particular pair $(X,D).$

**Theorem 4.2.** Let $d \in \mathbb{N}$ be an integer. Assume that $I \subset (0,1]$ is a DCC set and $I' \subset (0,1]$ is a finite set. Let $\mathcal{F} := \{(X,\Delta + D) \mid (X,\Delta + D) \text{ lc, } X \text{ a } \mathbb{Q}-\text{factorial variety, } \dim X = d, 
\Delta \in I', D \in I \text{ and } \Delta, D \text{ do not have common components}\}.$ Then there exist a finite set $\Gamma \subset (0,1]$, a DCC set $J \subset (0,1]$, and a finite set $J' \subset (0,1] \cap \mathbb{Q}$ satisfying the following property:

For any $(X,\Delta + D) \in \mathcal{F},$ there exist $r_i \in \Gamma$ and divisors $\Delta_i, D_i, 1 \leq i \leq q$ such that

1. $(X,\Delta_i + D_i)$ is lc,
2. $\Delta_i \in J', D_i \in J$ and $\text{Supp } \Delta_i = \text{Supp } \Delta, \text{Supp } D_i = \text{Supp } D,$
3. $(X,\Delta + D) = \sum_{i=1}^{q} r_i (X,\Delta_i + D_i)$ with $\sum_{i=1}^{q} r_i = 1.$

Moreover, the above statement can be strengthened in the following cases.

(i) If a $(X,\Delta + D) \in \mathcal{F}$ is klt, then $(X,\Delta_i + D_i)$ in (1) can be chosen to be klt as well.
(ii) For a $(X,\Delta + D) \in \mathcal{F},$ and any morphism $X \to Z$ such that $K_X + \Delta \equiv 0/Z,$ then we can further assume that $K_X + \Delta_i \equiv 0/Z$ for each $i.$

**Remark 4.3.** The main difference between Theorem 4.1 and Theorem 4.2 is: in Theorem 4.1, $J$ is a finite set while $\Gamma$ may be an infinite set. On the other hand, in Theorem 4.2, $J$ is a DCC set while $\Gamma$ is a finite set, and thus $q$ is bounded above. We will use Theorem 4.2 to study the relation between the effective adjunction conjecture and the $\Gamma$-effective adjunction conjecture.
Theorem 4.1 and Theorem 4.2 can be shown by similar argument.

**Lemma 4.4.** If there is a finite set \( J \) such that \((X, \sum_{1 \leq i \leq k} d_i D_i), d_i \geq d_i \) can be decomposed as in Theorem 4.1 by coefficients in \( J \), then the same thing holds true for \((X, \sum_{1 \leq i \leq k} d_i D_i)\) after enlarging \( J \) (but it is still a finite set).

**Proof.** Fix \( \delta \in \mathbb{Q} > 0 \) such that \( \delta < \min J \). Replace \( J \) by \( J \cup \{ \delta \} \). It is enough to show the claim when there exists \( l \) such that \( \tilde{d}_i \geq d_i \) and \( \tilde{d}_i = d_i \) for \( 1 \leq i \leq k, i \neq l \).

We write \( \beta \) for \((\beta_1, \cdots, \beta_k) \in \mathbb{R}^k\) and \( \beta \cdot D \) for \( \sum_{i=1}^{k} \beta_i D_i \). If \((X, \tilde{d}_1 D_1 + \sum_{1 \leq i \leq k, i \neq 1} d_i D_i)\) has a decomposition \( \sum_{1 \leq j \leq q} r_j (X, \beta_j \cdot D) \) satisfying the property claimed in Theorem 4.1, then

\[
(d_1, \cdots, \tilde{d}_i, \cdots, d_k) \in \text{Conv}(\{\beta_j \mid 1 \leq j \leq q\})
\]

with \( \beta_j \cdot D \in J \) and \((X, \beta_j \cdot D)\) lc for each \( j \). Moreover, \( \text{Supp} \beta_j \cdot D_j = \bigcup_i \text{Supp} D_i \).

For \( \beta \) above, write \( \alpha = (\beta_1, \cdots, \beta_{l-1}, \delta, \beta_{l+1}, \cdots, \beta_k) \). Then

\[
(d_1, \cdots, d_i, \cdots, d_k) \in \text{Conv}(\{\beta_j \mid 1 \leq j \leq q\} \cup \{\alpha_j \mid 1 \leq j \leq q\}).
\]

Thus there is a decomposition of \((X, \sum_{1 \leq i \leq k} d_i D_i)\) using \((X, \beta_i \cdot D)\) and \((X, \alpha_j \cdot D)\). Moreover, \( \alpha_j \cdot D \in J \) and \((X, \alpha_j \cdot D)\) is lc for each \( j \). As \( \delta > 0 \), we still have \( \text{Supp} \alpha_j \cdot D = \bigcup_i \text{Supp} D_i \).

**Proof of Theorem 4.1.** By Theorem 2.7 (take \( \Delta = 0 \)), we have a finite set \( I_0 \), such that for a \((X, D) \in \mathcal{S}\), there exists an lc pair \((X, \bar{D})\) with \( \bar{D} \geq D \) and \( \bar{D} \in I_0 \). Consider the set of lc pairs

\[
\{(X, \bar{D}) \mid (X, D) \in \mathcal{S}\}.
\]

By Theorem 2.6, there exists a finite rational set \( J \) depending only on \( d \) and \( I \) such that

\[
\bar{D} \in \text{Conv}(D_1, \ldots, D_q)
\]

with \( D_i \in J \), and \((X, D_i)\) is lc. Then by Lemma 4.4, we are done.

**Proof of Theorem 4.2.** We use the above notation. Replacing \( I \) by \( I' \cup I \), we can assume that \( I' \subset I \). Take \( \tau = \frac{1}{\delta} \min I \), and let \( I_0 \) be the finite set in Theorem 2.7. Then for any \((X, \Delta + \bar{D}) \in \mathcal{F}\), there exits an lc pair \((X, \Delta + \bar{D})\) with \( \bar{D} \in I_0 \) such that \( \bar{D} - D \in [0, \tau] \). As \( I' \cup I_0 \) is a finite set, by Theorem 2.6, there exist a finite set \( J' \subset \mathbb{Q} \) and a finite set \( \Gamma \subset \mathbb{R}^\geq \) such that

\[
(X, \Delta + \bar{D}) = \sum_{i=1}^{q} r_i (X, \Delta_i + \bar{D}_i), \quad \sum_{i=1}^{q} r_i = 1,
\]

where \((X, \Delta_i + \bar{D}_i)\) is lc with \( \Delta_i, \bar{D}_i \in J', r_i \in \Gamma \), and \( \text{Supp} \Delta_i = \text{Supp} \Delta \), \( \text{Supp} \bar{D}_i = \text{Supp} \bar{D} \). Moreover, we can assume that \( \Lambda - \Lambda_i, \bar{D} - \bar{D}_i \in (-\frac{1}{\delta} \tau, \frac{1}{\delta} \tau) \). In particular, \( \text{Supp} D = \text{Supp} \bar{D} = \text{Supp} \bar{D}_i \), and \( \bar{D}_i \in (\frac{1}{\delta} \tau, 1] \).
Thus

\[(4.0.2) \quad (X, \Delta + D) = \sum_{i=1}^{q} r_i (X, \Delta_i + (\bar{D}_i - (\bar{D} - D))).\]

As \(\bar{D} - D \in [0, \tau]\), we have \(\Delta_i + (\bar{D}_i - (\bar{D} - D)) \geq 0\) with

\[\text{Supp} \Delta_i = \text{Supp} \Delta \text{ and } \text{Supp} (\bar{D}_i - (\bar{D} - D)) = \text{Supp} D.\]

Because \(\Delta_i, \bar{D}_i \in J', \bar{D} \in I_0\) with both \(J', I_0\) are finite sets, and \(D \in I\) which is a DCC set, coefficients of \(D_i := (\bar{D}_i - (\bar{D} - D))\) and \(\Delta_i\) belong to a set \(J\) which is still DCC. Moreover, \((X, \Delta_i + D_i)\) is lc as \(D_i \geq D_i\).

To show (i) in the second part, suppose that \((X, \Delta + D) \in \mathcal{F}\) is klt, and \(P\) is an lc place of \((X, \Delta_i + D_i)\). Notice that by Theorem 2.6,

\[\text{LCP}(X, \Delta_i + \bar{D}_i) \subset \text{LCP}(X, \Delta + \bar{D})\]

for each \(i\) in \((4.0.1)\), thus \(P\) is an lc place of \((X, \Delta + \bar{D})\). Because \(a(P; X, \Delta + D) > 0\) and

\[\Delta_i + D_i = (\Delta_i + \bar{D}_i) - ((\Delta + \bar{D}) - (\Delta + D)),\]

we have \(a(P; \Delta_i + D_i) > 0\). This is a contradiction and thus \((X, \Delta_i + D_i)\) is klt.

To show (ii) in the second part, let \(I' = \{c_1, \ldots, c_l\}\) and the set of coefficients of all the \(D\) be \(I_0 = \{v_1, \ldots, v_t\}\). As in Theorem 2.5, assume that \(c_0 = 1, c_1, \ldots, c_p\) are \(\mathbb{Q}\)-linearly independent, and \(\{c_0, c_1, \ldots, c_p, v_1, \ldots, v_w\}\) is a basis of

\[\text{Span}_\mathbb{Q}(c_0, c_1, \ldots, c_l, v_1, \ldots, v_t).\]

We use \(x_{c_i}\) and \(x_{v_j}\) to denote the variables corresponding to \(c_i\) and \(v_j\) respectively.

There are rational linear functions

\[c_\lambda(x_{c_i}, x_{v_j} \mid 1 \leq i \leq p, 1 \leq j \leq w), \quad \lambda < l, \quad \text{and}\]

\[v_\sigma(x_{c_i}, x_{v_j} \mid 1 \leq i \leq p, 1 \leq j \leq w), \quad \sigma < t, \quad \text{such that}\]

\[c_\lambda = c_\lambda(c_i, v_j \mid 1 \leq i \leq p, 1 \leq j \leq w), \quad \lambda < l, \quad \text{and}\]

\[c_\sigma = c_\sigma(c_i, v_j \mid 1 \leq i \leq p, 1 \leq j \leq w), \quad \sigma < t.\]

Suppose that \(K_X + \Delta \equiv 0/\mathbb{Z}\) with \(\Delta = \sum_{i=1}^{l} c_i \mathbb{Z}_i\) where \(\mathbb{Z}_i\) is a \(\mathbb{Q}\)-Cartier Weil divisor (because \(X\) is \(\mathbb{Q}\)-factorial). Then there are finite rational linear equations

\[(4.0.3) \quad \ell_{(X, \Delta)/\mathbb{Z}, \mu}(x_{c_i} \mid 1 \leq i \leq p) = 0\]

obtained by intersecting \(K_X + \Delta\) with curve classes in \(N^1(X/\mathbb{Z})_\mathbb{Q}\).
Put them together, we have rational equations

\[ e_{\lambda}(x_{c_i}, x_{v_j} \mid 1 \leq i \leq p, 1 \leq j \leq w) - x_{c_{\lambda}} = 0, \quad p < \lambda \leq l \]

\[ v_{\sigma}(x_{c_i}, x_{v_j} \mid 1 \leq i \leq p, 1 \leq j \leq w) - x_{c_{\sigma}} = 0, \quad w < \sigma \leq t \]

\[ \ell_{(X, \Delta)/Z, \mu}(x_{c_i} \mid 1 \leq i \leq p) = 0 \text{ for all } K_X + \Delta \equiv 0/Z. \]

Although the equations may be infinite, they cut out a rational linear subspace \( V \subset \mathbb{R}^{l+t+1} \). Besides, \((c_0, c_1, \ldots, c_p, v_1, \ldots, v_w) \in V\) by definition.

By Theorem 2.6, there exists a rational polytope \( Q \subset \mathbb{R}^{l+t+1} \) such that \((c_0, c_1, \ldots, c_p, v_1, \ldots, v_w) \in Q\) and for any \((c_0', c_1', \ldots, c_p', v_1', \ldots, v_w') \in Q\),

\[ (X, \sum_{i=1}^{l} c_i' \xi_i + \sum_{j=1}^{t} v_j' \Theta_j) \]

is lc, where \( \tilde{D} = \sum_{j=1}^{t} v_j \Theta_j \) with \( \Theta_j \) a \( \mathbb{Q} \)-Cartier Weil divisor for each \( j \).

Because \((c_0, c_1, \ldots, c_p, v_1, \ldots, v_w) \in V \cap Q\), \( V \cap Q \) is a non-empty rational polytope. Assume that

\[(c_0^{(k)}, c_1^{(k)}, \ldots, c_p^{(k)}, v_1^{(k)}, \ldots, v_w^{(k)}) , 1 \leq k \leq q' \]

are the vertices of \( V \cap Q \). Shrinking \( V \cap Q \), we can assume that \( c_i^{(k)} - c_i \) and \( v_j^{(k)} - v_j \) belong to \((-\frac{1}{3}r, \frac{1}{3}r)\) for each \( i, j \) and \( 1 \leq k \leq q' \).

Next, we repeat the argument in the first part. Let

\[ \Delta_k = \sum_{i=1}^{l} c_i^{(k)} \xi_i \text{ and } \tilde{D}_k = \sum_{j=1}^{t} v_j^{(k)} \Theta_j, \]

then \( K_X + \Delta_k \equiv 0/Z \) and \((X, \Delta_k + \tilde{D}_k)\) is lc. The coefficients of \( \Delta_k \) and \( \tilde{D}_k \) belong to a finite set. Moreover, we have a finite set \( \Gamma \), such that

\[ (X, \Delta + \tilde{D}) = \sum_{k=1}^{q'} r_k(X, \Delta_k + \tilde{D}_k), \quad \sum_{k=1}^{q'} r_k = 1 \]

with \( r_k \in \Gamma \). Define \( D_k := \tilde{D}_k - (\tilde{D} - D) \), and let \( J \) be the set of coefficients of \( D_k \) and \( \Delta_k \). Therefore, \( J \) is a DCC set. Besides, \((X, \Delta_i + D_i)\) is lc (klt when \((X, \Delta + D)\) is klt) with \( \text{Supp} \Delta_k = \text{Supp} \Delta \) and \( \text{Supp} D_k = \text{Supp} D \). \( \square \)

**Remark 4.5.** It is possible to use decomposition theorems to study the boundedness problem and effective Iitaka fibration problem in real coefficients. For example, [HX15, Theorem 1.3] can be shown in this setting. However, there are some technical difficulties to generalize [HX15, Theorem 1.4] to real coefficients. To avoid deviating the topic, we omit the discussions on the details.
5. Effective adjunction v.s. $\Gamma$-effective adjunction

**Theorem 5.1.** Conjecture 3.1 implies Conjecture 3.5 (2) when $(X, D)$ is klt.

**Proof of Theorem 5.1.** We use the assumptions and notation in Conjecture 3.5 (2). Taking a $\mathbb{Q}$-factorial dlt modification in $(X, D)$ (for example, see [BCHM10, Corollary 1.4.3]), we can assume that $X$ is a $\mathbb{Q}$-factorial variety. For a general fiber $F$ of $f$, $K_F + D^h_F = (K_X + D)|_F \equiv 0$. By Theorem 2.4, there exists a finite set $I_h$ such that $D^h \in I_h$.

Step 1. Suppose that $D^h = 0$. We claim that there exists $B \in \mathbb{Q}$ such that $B^h = 0$, $K_X + B \sim_{Z, \mathbb{Q}} 0$ and the moduli b-divisors for klt-trivial fibrations $(X, D) \to Z$ and $(X, B) \to Z$ coincide.

Let $D = \sum_{1 \leq i \leq k} d_i D_i$, then

\begin{equation}
(5.0.1) \quad \mathcal{V} := \{(x_1, \ldots, x_k) \in \mathbb{R}^k_{\geq 0} \mid (X, \sum_{i=1}^k x_i D_i) \text{ is lc}\}
\end{equation}

is a rational polytope. Let $[\ell_1], \ldots, [\ell_l] \in N_1(X/Z/\mathbb{Q})$ be a basis, then

\begin{equation}
(5.0.2) \quad \mathcal{W} := \{(x_1, \ldots, x_k) \in \mathbb{R}^k \mid \sum_{i=1}^k x_i (D_i \cdot \ell_j) = -K_X \cdot \ell_j \text{ for all } \ell_j\}
\end{equation}

is a rational affine subspace. By $(d_1, \ldots, d_k) \in \mathcal{V} \cap \mathcal{W}$, we see that $\mathcal{V} \cap \mathcal{W}$ is a rational polytope. By $(X, D)$ klt, there exists a klt pair $(X, B)$ such that $B \in \mathbb{Q}$, $B^h = 0$ and $(X, B) \to Z$ is klt.

Let $D^D_{\text{div}}$ (resp. $D^B_{\text{div}}$) and $D^D_{\text{mod}}$ (resp. $D^B_{\text{mod}}$) be the discriminant b-divisor and the moduli b-divisor for the klt-trivial fibration $(X, D) \to Z$ (resp. $(X, B) \to Z$). As Claim (1) in the proof of Proposition 3.3, we have

\begin{equation}
(5.0.2) \quad D^D_{\text{div}} = D^B_{\text{div}} + \bar{P}, \text{ hence } D^B_{\text{mod}} = D^D_{\text{mod}}.
\end{equation}

By $B \in \mathbb{Q}$, $B^h = 0$ and (5.0.2), applying Conjecture 3.1 to $(X, B)$, we see that $D^D_{\text{mod}}$ is effectively base-point free. The rest of claims in Conjecture 3.5 (2) follow from the property of the canonical bundle formula (see Section 2.2).

Step 2. Now, assuming that $D^h \neq 0$, we will prove the claim by induction on $\dim(X/Z)$. Because $K_X + D^v \equiv -D^h/Z$ is not pseudo-effective over $Z$, we can run a $(K_X + D^v)$-MMP/$Z$ which terminates to a Mori fiber space $p : Y \to W$ over $Z$ (see [BCHM10, Corollary 1.3.3]). Let $q : W \to Z$ be the corresponding morphism and $w = q \circ p$. Let $D_Y$ be the strict transform of $D$ on $Y$. It is enough to show Conjecture 3.5 (2) for the klt-trivial fibration
\( w : (Y, D_Y) \to Z \). In fact, only Conjecture 3.5 (2)(d) needs to be justified. By \( K_X + D \sim_{\mathbb{R}} 0 \), for a common resolution \( X \xleftarrow{s} T \xrightarrow{t} Y \), we have
\[
(5.0.3) \quad s^*(K_X + D) = t^*(K_Y + D_Y).
\]
Suppose that \( \theta : Z' \to Z, h : X' \to X \) and \( f' : X' \to Z' \) with \( f \circ h = \theta \circ f' \). Assume that \( h' : X'' \to X' \) is a birational morphism. Then \( h'^*L \sim_{\mathbb{R}} h'^*\Theta \) implies that \( f'^*L \sim_{\mathbb{R}} \Theta \). Replacing \( X' \) by \( X'' \), we can assume that \( f' \) factors through \( Y' \) where \( g : Y' \to Y \) is a birational morphism and \( \nu : Y' \to Z' \) is a model of \( w : Y \to Z \). Then Conjecture 3.5 (2)(d) follows by taking \( T = X' \) in (5.0.3).

Step 3. When \( \dim(Y/W) < \dim(X/Z) \), by the induction hypothesis (notice that the coefficients of horizontal divisors over \( W \) is contained in the finite set \( I^h \)), there exist a finite set \( \Gamma_1 \subset (0, 1] \), a DCC set \( J'_1 \subset (0, 1] \) and \( m_1 \in \mathbb{N} \) depending only on \( d \) and \( I \), such that
\[
(5.0.4) \quad K_Y + D_Y \sim_{\mathbb{R}} p^*(K_W + (\tilde{D}_{\text{div}})_W + (\tilde{D}_{\text{mod}})_W),
\]
where \( \tilde{D}_{\text{div}} \) and \( \tilde{D}_{\text{mod}} \) are b-divisors. Moreover, \( (W, \tilde{D}_{\text{div}}) \) is klt with \( (\tilde{D}_{\text{div}})_W \in J'_1 \) and \( m_1\tilde{D}_{\text{mod}} \) is \( \Gamma_1 \)-base-point free. We claim that there is a finite set \( J''_1 \) depending only on \( m_1 \) and \( \Gamma_1 \) (which in turn depending only on \( d \) and \( I \) ), and an effective divisor \( \tilde{D}_W \sim_{\mathbb{R}} (\tilde{D}_{\text{mod}})_W \) such that
1. \( \tilde{D}_W \in J''_1 \),
2. \( \text{Supp} \tilde{D}_W \) and \( \text{Supp}(\tilde{D}_{\text{div}})_W \) do not have common components,
3. \( (W, (\tilde{D}_{\text{div}})_W + \tilde{D}_W) \) is klt.

In fact, by the definition of \( \Gamma \)-base-point freeness, there exists a birational morphism \( \pi : W' \to W \) such that
\[
m_1\tilde{D}_{\text{mod}} = \sum_{1 \leq i \leq q'} r_iM_i
\]
with \( (M_i)_{W'} \) base-point free and \( M_i = (\tilde{M}_i)_{W'} \). Replacing \( m_1 \) by \( 2m_1 \), we can assume that \( m_1 \geq 2 \). Taking a general element \( \tilde{M}_{W',i} \in |(M_i)_{W'}| \), define
\[
\tilde{D}_W = \frac{1}{m_1}\pi_*(\sum_{1 \leq i \leq q'} r_i\tilde{M}_{W',i})
\]
and \( J''_1 = \{ \frac{1}{m_1}r_i \mid 1 \leq i \leq q' \} \). As \( (W, (\tilde{D}_{\text{div}})_W) \) is klt,
\[
(W', (\tilde{D}_{\text{div}})_W' + \frac{1}{m_1}(\sum_{1 \leq i \leq q'} r_i\tilde{M}_{W',i}))
\]
is sub-klt, hence \( (W, (\tilde{D}_{\text{div}})_W + \tilde{D}_W) \) is klt. In fact, by Conjecture 3.5 (2)(d), if \( g : Y' \to Y \) is a birational morphism and \( p' : Y' \to W' \) is a morphism with \( p \circ g = \pi \circ p' \), then
\[
(5.0.5) \quad g^*(K_Y + D_Y) \sim_{\mathbb{R}} p'^*(K_{W'} + (\tilde{D}_{\text{div}})_{W'} + (\tilde{D}_{\text{mod}})_{W'}). \]
Hence
\[(5.0.6) \quad \pi^*(K_W + (\tilde{D}_{\text{div}})_W + (\tilde{D}_{\text{mod}})_W) = K_{W'} + (\tilde{D}_{\text{div}})_{W'} + (\tilde{D}_{\text{mod}})_{W'}.
\]
Therefore,
\[(W, (\tilde{D}_{\text{div}})_W + \tilde{D}_W) = (W, \pi^*((\tilde{D}_{\text{div}})_{W'} + \frac{1}{m_1} (\sum_{1 \leq i \leq q'} r_i \tilde{M}_{W', i})))
\]
is klt. In particular, the coefficients of \((\tilde{D}_{\text{div}})_W + \tilde{D}_W\) belong to a DCC set \(J_1 := J'_1 \cup J''_1\). We can apply the induction hypothesis to the klt-trivial fibration
\[q : (W, (\tilde{D}_{\text{div}})_W + \tilde{D}_W) \to Z.
\]
Notice that the coefficients of horizontal divisors over \(Z\) belong to a finite set depending only on \(\dim(W/Z)\) and \(J_1\), and thus depending only on \(d\) and \(I\). We have a DCC set \(J\), a finite set \(\Gamma \subset (0, 1]\) and \(m \in \mathbb{N}\), such that
\[K_W + (\tilde{D}_{\text{div}})_W + \tilde{D}_W \sim_R q^*(K_Z + (\tilde{D}_{\text{div}})_{Z'} + (\tilde{D}_{\text{mod}})_{Z'}),
\]
where \(\tilde{D}_{\text{div}}'\) and \(\tilde{D}_{\text{mod}}'\) are b-divisors such that \((\tilde{D}_{\text{div}})_Z \in J\), \((Z, \tilde{D}_{\text{div}}')\) is klt and \(m \tilde{D}_{\text{mod}}'\) is \(\Gamma\)-base-point free. To see (d) in Conjecture 3.5 (2), suppose that \(\theta : Z' \to Z\), \(g : Y' \to Y\) and \(\nu : Y' \to Z'\) with \(q \circ p \circ g = \theta \circ \nu\). As in Step 2, it is enough to assume that \(\nu\) factors through \(W'\) with \(p' : Y' \to W'\) and \(q' : W' \to Z'\). By the construction of \(\tilde{D}_{\text{div}}\) and \(\tilde{D}_{\text{mod}}\), we have
\[g^*(K_Y + D_Y) \sim_R p'^*(K_{W'} + (\tilde{D}_{\text{div}})_{W'} + (\tilde{D}_{\text{div}})_{W'}).\]
By the construction of \(\tilde{D}_{\text{div}}'\) and \(\tilde{D}_{\text{mod}}'\), we have
\[(5.0.7) \quad \pi^*(K_W + (\tilde{D}_{\text{div}})_W + \tilde{D}_W) \sim_R q'^*(K_{Z'} + (\tilde{D}_{\text{div}})_{Z'} + (\tilde{D}_{\text{mod}})_{Z'}),
\]
where \(\pi : W' \to W\). Pulling back (5.0.7) through \(p'\) and by (5.0.5), (5.0.6), Conjecture 3.5 (2)(d) follows.

Notice that \(J, \Gamma, m\) depending only on \(d\) and \(I\). Hence \(J, \Gamma, m\) and \((Z, \tilde{D}_{\text{div}}' + \tilde{D}_{\text{mod}}')\) satisfy the claim in Conjecture 3.5 (2).

Step 4. When \(\dim(Y/W) = \dim(X/Z)\), then \(W \to Z\) is a birational morphism and thus it suffices to prove the claim for \(p : Y \to W\). By Theorem 4.2, there is a finite set \(\Gamma \subset (0, 1]\), a finite rational set \(J'\), and a DCC set \(J\) such that
\[(Y, D^h + D^v) = \sum_{1 \leq i \leq q} r_i (Y, D^h_i + D^v_i)
\]
with
1. \((Y, D^h_i + D^v_i)\) klt, \(D^h_i \in J', D^v_i \in J,
2. \(r_i \in \Gamma, \sum_{1 \leq i \leq q} r_i = 1,
3. \text{Supp} D^h = \text{Supp} D^h_i, \text{Supp} D^v = \text{Supp} D^v_i,
4. K_Y + D^h_i \equiv 0/W\)
By $\rho(Y/W) = 1$, we have $D_i^r \equiv 0/Z$. By Theorem 2.9 and a similar argument as in Step 1,
\[
K_Y + D_i^h \sim_{Z,\mathbb{Q}} 0, \quad K_Y + D_i^h + D_i^w \sim_{Z,\mathbb{R}} 0.
\]
Thus $D_i^w \sim_{Z,\mathbb{R}} 0$ and $p^*B_i = D_i^w$ for some $\mathbb{R}$-Cartier divisor $B_i$ on $W$ by Lemma 2.11.

Let $D_i = D_i^h + D_i^w$. Suppose that $D_{i,\text{div}}$ (resp. $D_{i,\text{mod}}$) and $D_{i,\text{mod}}$ (resp. $D_{i,\text{mod}}$) are the discriminant b-divisor and the moduli b-divisor for the klt-trivial fibration $(Y, D_i) \to W$ (resp. $(Y, D_i^h) \to W$) respectively. As Claim (1) in the proof of Proposition 3.3, we have
\[
D_{i,\text{div}} = D_{i,\text{div}} + B_i, \quad \text{hence } D_{i,\text{mod}} = D_{i,\text{mod}}.
\]

Now applying Conjecture 3.1 to $(Y, D_i^h) \to W$, there exists $m \in \mathbb{N}$ such that $m\text{D}_{i,\text{mod}}$ is base-point free. By the canonical bundle formula, for any birational morphisms $g : Y' \to Y$, $\pi : W' \to W$ and $p : Y \to W$, $p' : Y' \to W'$ such that $p \circ g = \pi \circ p'$, we have
\[
g^*(K_Y + D_i^h) \sim_{\mathbb{R}} p'^* (K_{W'} + (D_{i,\text{div}})_{W'} + (D_{i,\text{mod}})_{W'}).
\]
Thus
\[
g^*(K_Y + D_i) \sim_{\mathbb{R}} p'^* (K_{W'} + (D_{i,\text{div}})_{W'} + \pi^* B_i + (D_{i,\text{mod}})_{W'})
\sim_{\mathbb{R}} p'^* (K_{W'} + (D_{i,\text{div}})_{W'} + (D_{i,\text{mod}})_{W'}).
\]
Moreover, $(W, D_{i,\text{div}})$ is klt and as $D_i \in J \cup J'$ which is a DCC set, coefficients of $(D_{i,\text{div}})_{W}$ still belong to a DCC set depending only on $\dim Y$ and $J \cup J'$ which in turn depending only on $d$ and $I$. Finally, by
\[
K_Y + D = \sum_{1 \leq i \leq q} r_i (K_Y + D_i),
\]
we can take $D_{\text{div}} := \sum_{1 \leq i \leq q} r_i D_{i,\text{div}}$ and $D_{\text{mod}} := \sum_{1 \leq i \leq q} r_i D_{i,\text{mod}}$. Then $mD_{\text{mod}}$ is $\Gamma$-base-point free, $(X, D_{\text{div}})$ is still klt and as there are only finite possibilities for $r_i$, $(D_{\text{div}})_{W}$ belongs to a DCC set depending only on $d$ and $I$. This completes the proof. \hfill $\square$

Remark 5.2. The technical assumption on $(X, D)$ klt is needed to apply Theorem 2.9. Notice that in the absolute setting, such result has already been known for numerically trivial lc pairs.

Remark 5.3. When $(X, D) = \sum r_i (X, D_i)$ with $(X, D) \to Z$ and $(X, D_i) \to Z$ lc-trivial fibrations, it is not true that the moduli b-divisors satisfy $D_{\text{mod}} = \sum r_i D_{i,\text{mod}}$.

According to Remark 3.2, Theorem 5.1 implies the following corollary.

Corollary 5.4. Conjecture 3.5 (2) holds true for klt pairs when general fibers are curves.
By a similar argument as above, we show that Conjecture 3.5 (2) can be put in the framework of the minimal model program (MMP) and the abundance conjecture.

**Proposition 5.5.** Assuming the MMP and the abundance conjecture for klt pairs in relative dimensions \( \leq \text{dim}(X/Z) \). To show Conjecture 3.5 (2) for klt pairs, it is enough to show the following two cases:

1. \( K_X \sim_{Z, \mathbb{Q}} 0 \),
2. \( \rho(X/Z) = 1 \).

**Proof.** Run a \( K_X \)-MMP/Z. By assumption, it terminates to \( Y/Z \), where either (i) \( K_Y \) is semi-ample/Z, or (ii) there exists a Mori fiber space \( Y \to W/Z \). Let \( D_Y \) be the strict transform of \( D \) on \( Y \). By Step 2 in the proof of Theorem 5.1, it suffices to show the claim for \( (Y, D_Y) \to Z \). In Case (i), let \( Y \to T/Z \) be the morphism induced be \( K_Y \).

In Case (i), because \( K_X \equiv -D/Z \) is pseudo-effective over \( Z \), the horizontal part \( D^h = 0 \). Thus \( h : T \to Z \) is birational, and \( K_Y \sim_{T, \mathbb{Q}} 0 \). By assumption, we have b-divisors \( \tilde{D}_{\text{div}} \) and \( \tilde{D}_{\text{mod}} \) for the klt-trivial fibration \( (Y, D_Y) \to T \) (not for \( Y \to T \)) satisfying the claim in Conjecture 3.5 (2). \( \tilde{D}_{\text{div}} \) and \( \tilde{D}_{\text{mod}} \) can be viewed as b-divisors over \( Z \), and it is straightforward to check (a), (b), (d) in Conjecture 3.5 (2). For (c), suppose that \( \theta : Z' \to Z \) is a birational morphism from a \( \mathbb{Q} \)-factorial variety and \( Z' \to W \to T \) are birational morphisms such that \( \theta \circ r = h \circ s \). Besides, we can assume that \( m(\tilde{D}_{\text{div}})_W \) is \( \Gamma \)-base-point free. Hence there exists \( 0 \leq M \sim_{\mathbb{R}} (\tilde{D}_{\text{mod}})_W \) such that \( (W, (\tilde{D}_{\text{div}})_W + M) \) is sub-klt. By (d), we have

\[
\begin{align*}
    r^*(K_{Z'} + (\tilde{D}_{\text{div}})_{Z'} + (\tilde{D}_{\text{mod}})_{Z'}) &= s^*(K_T + (\tilde{D}_{\text{div}})_T + (\tilde{D}_{\text{mod}})_T) \\
    \sim_{\mathbb{R}} K_W + (\tilde{D}_{\text{div}})_W + M.
\end{align*}
\]

Thus \( K_{Z'} + (\tilde{D}_{\text{div}})_{Z'} + r_*M = r_*(K_W + (\tilde{D}_{\text{div}})_W + M) \), and \((Z', (\tilde{D}_{\text{div}})_{Z'}) + r_*M)\) is sub-klt. By \( r_*M \geq 0 \), \((Z', (\tilde{D}_{\text{div}})_{Z'})\) is still sub-klt. This shows Case (i).

In Case (ii), apply the assumption to the Mori fiber space \( Y \to W \). We have a DCC set \( J'_1 \), a finite set \( \Gamma_1 \subset (0, 1] \) and \( m_1 \in \mathbb{N} \) such that there exist b-divisors \( \tilde{D}_{\text{div}} \) and \( \tilde{D}_{\text{mod}} \) satisfy the claim in Conjecture 3.5 (2). As in the Step 3 in the proof of Theorem 5.1, there exist a finite set \( J''_1 \) depending only on \( \Gamma_1, m_1 \), and an effective divisor \( \tilde{D}_W \sim_{\mathbb{R}} (\tilde{D}_{\text{mod}})_W \) such that

1. \( \tilde{D}_W \in J''_1 \),
2. \( \text{Supp}(\tilde{D}_W) \) and \( \text{Supp}(\tilde{D}_{\text{div}})_W \) do not have common components,
3. \( (W, (\tilde{D}_{\text{div}})_W + \tilde{D}_W) \) is klt.

The coefficients of \((\tilde{D}_{\text{div}})_W + \tilde{D}_W\) belong to a DCC set which depends only on \( \text{dim} X \) and \( I \). Moreover,

\[
(W, (\tilde{D}_{\text{div}})_W + \tilde{D}_W) \to Z
\]
is a klt-trivial fibration with \( \dim(W/Z) < \dim(X/Z) \). Replacing \((X, D)\) by \((W, (\mathcal{D}_{\text{div}})_W + \hat{D}_W)\) and repeating the above argument, we obtain the desired result. □

We should point out that in the “\(K_X \sim_{Z, \mathbb{Q}} 0\)” case, we need Conjecture 3.5 (2) for the pair \((X, D)\) instead of merely for \(X\). The reason is the following. By \(K_X + D \sim_{Z, \mathbb{R}} 0\), we have \(D = f^*B\) for some \(B\) on \(Z\). Applying Conjecture 3.5 (2) to the klt-trivial fibration \(X \to Z\), we have \(\hat{D}_{\text{div}}\) such that \((Z, \hat{D}_{\text{div}})\) is klt. But \((Z, \hat{D}_{\text{div}} + \bar{B})\) is not necessarily klt. However, it is excepted that \(\hat{D}_{\text{div}}\) is exactly the discriminant b-divisor for the klt-trivial fibration \(X \to Z\). If this is the case, then \((Z, \hat{D}_{\text{div}} + \bar{B})\) is still klt by the property of the canonical bundle formula.

6. Applications of the \(\Gamma\)-effective adjunction

Let \(X\) be a projective variety. In [Li20a], we study the boundedness of log canonical models when the Iitaka volumes are fixed (or bounded above), and the distributions of the Iitaka volumes. Recall that the Iitaka volume (see [Li20a, Definition 1.1]) is defined to be

\[
\text{Ivol}(K_X + D) := \limsup_{m \to \infty} \frac{\kappa(K_X + D)! h^0(X, \mathcal{O}_X(\lfloor m(K_X + D) \rfloor))}{m^{\kappa(K_X + D)}}
\]

when \(\kappa(K_X + D) \neq -\infty\) and 0 otherwise.

Notice that for a klt pair \((X, D)\) over \(U\) with \(D \in \mathbb{R}\) and the relative Kodaira dimension \(\kappa(K_X + D/U) \geq 0\), \((X, D)\) admits a unique log canonical model over \(U\) (see [Jia20] and [Li20b, Corollary 1.2]). The following conjectures are [Li20a, Conjecture 1.2] and [Li20a, Conjecture 1.7] in terms of real coefficients.

**Conjecture 6.1** ([Li20a, Conjecture 1.2]). Let \(d \in \mathbb{N}, v \in \mathbb{R}_{>0}\) be fixed numbers, and \(I \subset (0, 1]\) be a DCC set. Let \(S(d, v, I)\) be the set of varieties \(Z\) satisfying the following properties:

1. \((X, D)\) is a projective klt pair with \(\dim X = d, D \in I\),  
2. \(\text{Ivol}(K_X + D) = v\), and  
3. \(f : X \to Z\) is the log canonical model of \((X, D)\).

Then \(S(d, v, I)\) is a bounded family.

**Conjecture 6.2** ([Li20a, Conjecture 1.6]). Let \(d \in \mathbb{N}\) be a fixed number, and \(I \subset (0, 1]\) be a DCC set. Then the set of Iitaka volumes

\[\{\text{Ivol}(K_X + D) \mid (X, D) \text{ is a projective klt pair, } \dim X = d, D \in I\}\]

is a DCC set.

As [Li20a, Proposition 4.1], we have the following result.

**Proposition 6.3.** Assuming Conjecture 3.5 (2) and the existence of good minimal models, then Conjecture 6.1 and Conjecture 6.2 hold true.
Proof. Replacing \((X, D)\) by a good minimal model, we can assume that \(K_X + D\) is semi-ample with \(f: X \to Z\) the morphism induced by \(K_X + D\). By Conjecture 3.5 (2), there exist \(m \in \mathbb{N}\), a DCC set \(J \in (0, 1]\) and a finite set \(\Gamma\) such that

1. there is a b-divisor \(\tilde{D}_{\text{div}}\), such that \((\tilde{D}_{\text{div}})_Z \in J\),
2. there is a b-divisor \(\tilde{D}_{\text{mod}}\), such that \(m\tilde{D}_{\text{mod}}\) is \(\Gamma\)-base-point free,
3. \((Z, \tilde{D}_{\text{div}})\) is lc and \(K_X + D \sim_R f^*(K_Z + (\tilde{D}_{\text{div}})_Z + (\tilde{D}_{\text{mod}})_Z)\).

As in the Step 3 in the proof of Theorem 5.1, there exists an effective divisor \(\tilde{D}_Z \sim_R (\tilde{D}_{\text{mod}})_Z\) such that \((Z, \tilde{D}_{\text{div}})_Z + \tilde{D}_Z\) is klt and the coefficients of \((\tilde{D}_{\text{div}})_Z + \tilde{D}_Z\) belong to a DCC set depending only on \(d\) and \(I\).

Notice that \(K_Z + (\tilde{D}_{\text{div}})_Z + \tilde{D}_Z\) is ample with \(\text{vol}(K_Z + (\tilde{D}_{\text{div}})_Z + \tilde{D}_Z) = \text{Ivol}(K_X + B)\). Then Conjecture 6.1 follows from Theorem 6.4 below and Conjecture 6.2 follows from [HMX14, Theorem 1.3 (1)].

The following result is needed in the above argument. It is essentially [HMX18, Theorem 1.1] for real klt pairs. The argument combines [HMX14, Theorem 1.6] and [HMX18, Theorem 1.1]. For technical reasons, we need to assume the log pairs are klt instead of lc. However, the result is expected to still hold true for lc pairs.

**Theorem 6.4.** Fix an integer \(d\), a positive real number \(v\) and a set \(I \subset (0, 1]\) which satisfies the DCC. Then the set \(\mathfrak{F}(d, v, I)\) of all log pairs \((X, \Delta)\) such that

1. \(X\) is projective of dimension \(d\),
2. \((X, \Delta)\) is klt,
3. the coefficients of \(\Delta\) belong to \(I\),
4. \(K_X + \Delta\) is an ample \(\mathbb{R}\)-Cartier divisor, and
5. \((K_X + \Delta)^d = v,\)

is bounded. Besides, there is a finite set \(I_0\) such that \(\mathfrak{F}(d, v, I) = \mathfrak{F}(d, v, I_0)\).

**Sketch of the Proof.** Following the argument of [HMX18, Proposition 7.3], there is a projective morphism \(Z \to U\) and a log smooth pair \((Z, B)\) over \(U\) such that if \((X, \Delta) \in \mathfrak{F}(d, v, I)\), then there is a closed point \(u \in U\) and a birational map \(f_u : Z_u \dashrightarrow X\) such that

\[\text{vol}(Z_u, K_{Z_u} + \Phi) = v,\]

where \(\Phi \leq B_u\) is the sum of the strict transform of \(\Delta\) and the \(f_u\)-exceptional divisors. [HMX18, Lemma 2.2.2] implies that \(f_u\) is the log canonical model of \((Z_u, \Phi)\). Besides, every stratum of \(B\) has irreducible fibers over \(U\) (see [HMX18, Lemma 7.2]). Notice that the above argument works for real coefficients, rational coefficient assumption is used in [HMX18, Corollary 1.3].

Now we claim that there is a finite set \(I_0\) such that \(\Delta \in I_0\). As every stratum of \(B\) has irreducible fibers over \(U\), we can identify components of \(\Delta\) with
components of $B$. If the claim were false, then by $B$ fixed and $\Phi \in I \cup \{1\}$, we can assume that there are two pairs $(Z_{u_i}, \Phi_k), u_i \in U, i = 1, 2$ such that the coefficients of $f_{1*}(\Phi_1)$ is less or equal to the corresponding coefficients of $f_{2*}(\Phi_2)$, and the inequality holds for at least one pair of coefficients. In other words, let $\Phi_{u_i}^{(2)}$ be the divisor on $Z_{u_i}$ whose coefficients are chosen as $\Phi_2$, then we have $f_{1*}(\Phi_1) < f_{1*}(\Phi_{u_i}^{(2)})$. By invariance of the plurigenera ([HMX18, Corollary 1.4]),

$$(6.0.1) \quad \text{vol}(Z_{u_i}, \Phi_{u_i}^{(2)}) = \text{vol}(Z_{u_2}, \Phi_2) = v.$$ 

In fact, for $m \in \mathbb{N}$ sufficiently large, $(Z_{u_2}, \frac{|m\Phi_2|}{m})$ has the log canonical model $(X, \frac{|mB_2|}{m})$. As $\frac{|m\Phi_2|}{m} \in \mathbb{Q}$, applying [HMX18, Corollary 1.4], we have $h^0(Z_{u_1}, \frac{k|m\Phi_2^{(2)}}{m}) = h^0(Z_{u_2}, \frac{k|m\Phi_2|}{m})$ for $k \in \mathbb{N}$ such that $m \mid k$. Hence, by the definition of volume, (6.0.1) holds true. By [HMX18, Lemma 2.2.2], as $\Phi_{u_i}^{(2)} > \Phi_1$, we see that $f_1 : Z_{u_i} \rightarrow X_1$ is the log canonical model for both $(Z_{u_1}, \Phi_1)$ and $(Z_{u_1}, \Phi_{u_i}^{(2)})$. As $f_{1*}(\Phi_1) < f_{1*}(\Phi_{u_i}^{(2)})$, this contradicts to $(K_{X_1} + f_{1*}(\Phi_1))^d = (K_{X_1} + f_{1*}(\Phi_{u_i}^{(2)}))^d = v$.

Without loss of generality, we can fix the coefficients of $\Delta$. Then for the corresponding $\Phi$, it can be written in two parts $\Phi = \Phi^c + \Phi^e$ where $\Phi^c$ corresponding to $\Delta$ and $\Phi^e$ corresponding to $f_U$-exceptional divisors. By $(X, \Delta)$ klt, we can choose $\epsilon > 0$ sufficiently small such that $(Z_u, \Phi^c + (1 - \epsilon)\Phi^e)$ still has the log canonical model $(X, \Delta)$. We claim that for any other $(X', \Delta')$ which corresponds to the closed point $u' \in U$, the log canonical model of $(Z_{u'}, \Phi^c + (1 - \epsilon)\Phi^e)$ is still $(X', \Delta')$. Notice that by the choice of $u'$, the log canonical model of $(Z_{u'}, \Phi^c + (1 - \epsilon)\Phi^e)$ is $(X', \Delta')$. By $\text{vol}(K_{Z_{u'}} + \Phi^c + (1 - \epsilon)\Phi^e) = \text{vol}(K_{Z_{u'}} + \Phi^c + (1 - \epsilon)\Phi^e) = v$, we know that $(Z_{u'}, \Phi^c + \Phi^e)$ has some log canonical model $Z_{u'} \rightarrow X''$ ([Li20b, Corollary 1.2]). As

$$\text{vol}(K_{Z_{u'}} + \Phi^c + (1 - \epsilon)\Phi^e) = \text{vol}(K_{Z_{u'}} + \Phi^c + \Phi^e) = v,$$

[HMX18, Lemma 2.2.2] implies that $Z_{u'} \rightarrow X''$ is still the log canonical model of $(Z_{u'}, \Phi^c + \Phi^e)$. Hence $X'' = X'$ by the uniqueness of the log canonical model ([BCHM10, Lemma 3.6.6 (1)]), and the claim is proved.

Finally, we proceed as in the proof of [HMX14, Theorem 1.6]. By [HMX14, Lemma 9.1] (this step needs klt as it uses [BCHM10, Corollary 1.1.5]), there are finite morphisms $g_i : Z \rightarrow W_i/U$ such that for $i \in U$ and for any $0 \leq \Psi \leq \Phi^c + (1 - \epsilon)\Phi^e$, the log canonical model of $(Z_t, \Psi)$ is $f_{it}$ for some $i$ (it seems that [HMX14, Lemma 9.1] assume $\Psi \in \mathbb{Q}$ implicitly as it takes the log canonical model to be the Proj of the log canonical ring. However, this also holds for $\Psi \in \mathbb{R}$ as there exists $\Psi' \in \mathbb{Q}$ with $\Psi' \leq \Psi$ such that $(Z_t, \Psi')$ and $(Z_t, \Psi)$ have the same log canonical model). In particular, this shows that $(X, \Delta)$ belongs to a fiber of $(W_i, B_i)/U$, and thus in a bounded family. □
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DEPARTMENT OF MATHEMATICS, SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY, 1088 XUEYUAN RD, SHENZHEN 518055, CHINA

E-mail address: lizhan@sustech.edu.cn