A new proof of existence in the \(L^3\)-setting of solutions to the Navier-Stokes Cauchy problem

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Abstract - We investigate on the existence of solutions with initial datum \(U_0\) in \(L^3\). Our chief goal is to establish the existence interval \((0,T)\) uniquely considering the size and the absolute continuity of \(|U_0(x)|^3\).

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1 Introduction

This note concerns the 3D-Navier-Stokes Cauchy problem:

\[
\begin{align*}
U_t + U \cdot \nabla U + \nabla \pi &= \Delta U, \quad \nabla \cdot U = 0, \quad \text{in } (0,T) \times \mathbb{R}^3, \\
U &= U_0 \text{ on } \{0\} \times \mathbb{R}^3.
\end{align*}
\]

where \(U\) and \(\pi\) stand, respectively, for the unknown kinetic and pressure fields of an incompressible viscous fluid, \(U_t := \frac{\partial}{\partial t} U\) and \(U \cdot \nabla U := U_k \frac{\partial}{\partial x_k} U\).

We look for a result of existence and uniqueness with an initial datum in \(L^3(\mathbb{R}^3)\) and divergence free. It is known that this kind of result is not new. Indeed, there is a wide literature on it, with a first contribution due to T. Kato in [5]. Moreover, the \(L^3\)-metric of the existence class belongs to the set of scaling invariant metrics, this concept is meant as defined in [1].

This note does not aim at giving an original result of existence in the \(L^3\)-class, but its interest is a little more specific, in a sense that we attempt to clarify below.

As far as we know, considering a scaling invariant \(X\)-norm for an initial datum \(U_0\), the existence interval is global (in time) if \(\|U_0\|_X\) is sufficiently small (in this regard we point out a recent contribution in weighted spaces with increasing weight [3]); otherwise, without any restriction, one proves the existence on some interval \((0,T)\), but no connection is given between the size of \(\|U_0\|_X\) and a dimensionless size of \(T\). Actually, the interval \((0,T)\) is determined, by means of different strategies, with the aid of other metrics and, as matter of fact, it is deduced with respect to another metric.

In this connection, the recent paper [2] seems to be an exception. It is employed the dimensionless weighted functional \(\|U_0\|^2_{w,\ell} := \sup_x \int_{\mathbb{R}^3} \frac{U_0^2(y)}{|x-y|^\ell} \, dy\) and, in the set \(L^2_{w,\ell}\), where \(\| \cdot \|_{w,\ell} < \infty\), the subset of the so called Kato class \(K_3\) is considered. This special set of
initial data furnishes, for the first time, the interval of existence with a dimensionless size which involves properties of data, specifically the ones of the Kato class. We do not give further details of the results that arise, as they are a part of the wider set of results of this note, where we consider \( \|U_0\|_{L^3} \) in place of \( \|U_0\|_{\text{wt}} \). In this regard, we recall that \( \| \cdot \|_{\text{wt}} \) is not equivalent to the \( L^3 \)-norm. We stress that we could also consider the \( n \)-dimensional Navier-Stokes Cauchy problem, of course considering the \( L^n \)-setting.

In addition to the purpose of providing a new sufficient condition in \( L^3 \) (scaling invariant norm) to establish the existence interval \((0,T)\), the results of this note are the starting point for a forthcoming paper, concerning the same questions but in the case of the initial boundary value problem in \((0,T) \times \Omega\), where \( \partial \Omega \) is assumed a sufficiently regular compact set, or \( \Omega \) is the half-space.

Here, we simply argue on the first question, and we only give a hint for the second question.

In paper [2], an element of the Kato class enjoys the following property:

\[
\lim_{\rho \to 0} \sup_x \int_{|x-y|<\rho} \frac{|U_0(y)|^2}{|x-y|} dy = 0.
\]

This limit property allows us to state that

\[
\lim_{t \to 0} t^{\frac{1}{3}} |U^0(t)|_\infty = 0,
\]

where \( U^0(t,x) \) is the solution to the heat equation with initial datum \( U_0 \). Actually, setting \( \|U_0\|_{K^\rho} = \sup_x \int_{|x-y|<\rho} |U_0(y)|^2 dy \), uniformly in \( x \) we get

\[
t^{\frac{1}{3}} |U^0(t,x)| \leq \int_{|x-y|<\rho} H(x-y,t)|U_0(y)|dy + \int_{|x-y|>\rho} H(t,x-y)|U_0(y)|dy
\]

\[
\leq c \|U_0\|_{K^\rho} + c \exp \left[ -\frac{\rho^2}{t} \right] \|U_0\|_{\text{wt}}.
\]

This estimate is the key tool to discuss the local existence of the solution to the integral equation given by means of the Oseen tensor (see equation (2.1)), without requiring auxiliary conditions. Thanks to this estimate we are able to avoid a time parameter to make coercive the integral part related to the convective term.

Here the strategy is the same. We replace the property of the Kato class with the absolute continuity of the integral, in particular of \( |U_0(y)|^3 \in L^1(\mathbb{R}^3) \). So that, (1.2) is substituted by the following:

\[
t^{\frac{1}{3}} |U^0(t,x)| \leq c \|U_0\|_{L^3(B(x,\rho))} + c \exp \left[ -\frac{\rho^2}{t} \right] \|U_0\|_{3},
\]

where the first term on the right-hand side, for a suitable \( \rho > 0 \), satisfies the absolute continuity property uniformly in \( x \in \mathbb{R}^3 \). However, in order to complete the relation which ensures the convergence of successive approximations (or the contraction principle) related to the “integral Oseen equation”, we need a more complete metric. For this task, for \( \rho > 0 \), we set

\[
|u|_{3,\rho} := \sup_{x \in \mathbb{R}^3} \int_{B(x,\rho)} |u(y)|^3 dy,
\]
Theorem 1.1 There exist an absolute constant $C > 0$ such that for all $U_0 \in J^3(\mathbb{R}^3)$ there exists $T := T(U_0)$, defined as

$$
T(U_0) := \sup_{\rho > 0} t(\rho), \quad \text{with } t(\rho) := \sup \left\{ t > 0 : \left[ \|U_0\|_{3,\rho} + t^{\frac{2}{3}} \right] \leq c \|U_0\|_3 \right\},
$$


such that problem (1.1) has a solution $(U, \pi)$ on $(0, T) \times \mathbb{R}^3$ enjoying the properties

$$
\text{for all } t \in (0, T(U_0)), \ \theta \in [0, 1),
U \in C^{2,\beta}(\mathbb{R}^3) \text{ and } U_t, D^2U \in C^{0,\frac{\theta}{2}}((\eta, T(U_0)) \times \mathbb{R}^3),
$$

$$
\sup_{(0,T)} t^{\frac{2}{3}} \|U(t)\|_{\infty} \leq c \|U_0\|_3,
$$

with

$$
\lim_{t \to 0} t^{\frac{2}{3}} \|U(t)\|_{\infty} = 0, \quad \lim_{t \to 0} t \|\nabla U(t)\|_{\infty} = 0, \quad \|U(t)\|_3 \leq c \|U_0\|_3 \text{ for all } t \in [0, T(U_0)), \quad \lim_{t \to 0} \|U(\tau) - U_0\|_3 = 0,
$$

$$
t^{\frac{2}{3}} \|\pi(t)\|_3 \leq c \|U_0\|_3, \quad \pi \in C^{1,0}(\mathbb{R}^3) \text{ for all } t \in [0, T(U_0)) \quad (1.7)
$$

Finally, if the norm $\|U_0\|_3$ is suitably small, then the above results hold for all $t > 0$. 

|
Corollary 1.1 Let \((U, \pi)\) the solution to problem \((1.1)\) stated in Theorem 1.1. Then, for any \(q \in (3, +\infty)\), the following properties hold for \(t \to 0\)

\[
\|U(t)\|_q = o(t^{-\mu}), \quad \|
abla U(t)\|_q = o(t^{-\frac{1}{2} - \mu}), \quad \|
abla\nabla U(t)\|_q = o(t^{-1 - \mu}),
\]

(1.8)

with \(\mu := \frac{q - 3}{2q}\), and

\[
\lim_{t \to 0} t^{\frac{3}{2}} \|
abla\nabla U(t)\|_\infty = 0.
\]

(1.9)

Remark 1.1 We like to point out that we get a more detailed estimate than (1.8). Actually, for any \(q \in [3, +\infty)\) and for \(\mu := \frac{q - 3}{2q}\), we have

\[
t^\mu \|U(t)\|_q \leq c\|U_0\|_q^{\frac{3}{2}} A^\frac{q-3}{q} (\rho, t),
\]

\[
t^{\frac{1}{2} + \mu} \|
abla U(t)\|_q \leq c\|U_0\|_q^{\frac{3}{2}} \left[ A^\frac{q-3}{q} (\rho, t) + A^3 (\rho, t) \right],
\]

\[
t^{1 + \mu} \|
abla\nabla U(t)\|_q \leq c\|U_0\|_q^{\frac{3}{2}} \left[ A^\frac{q-3}{q} (\rho, t) + A^6 (\rho, t) \right],
\]

for any \(t \in [0, T)\), with \(A(\rho, t)\) suitable function defined in \((3.7)\).

Theorem 1.2 (Uniqueness) For all \(U_0 \in \mathcal{C}_0(\mathbb{R}^3)\) a solution to problem \((1.1)\) in the class of solutions \(U \in L^\infty(0, T; \mathcal{C}_0(\mathbb{R}^3))\), satisfying

\[
\lim_{t \to 0} t^{\frac{3}{2}} \|U(t)\|_\infty = 0,
\]

\[
\lim_{t \to 0} (U(t), \psi) = (U_0, \psi), \quad \forall \psi \in \mathcal{C}_0(\mathbb{R}^3),
\]

(1.10)

is unique.

This paper is organized as follows. In sect.2 we introduce some preliminary results. In sect.3 we furnish estimates on the approximating sequence of solutions. In sect.4 we prove the existence and uniqueness results (Theorem 1.1 and Theorem 1.2) and the \(L^q\) limit properties of Corollary 1.1.

2 Preliminary results

We look for a solution to the integral equation

\[
U(t, x) = H * U_0(t, x) - \nabla_x E * (U \otimes U)(t, x), \quad \text{for all} \quad (t, x) \in (0, T) \times \mathbb{R}^3,
\]

(2.1)

where \(H(t, z) := (4\pi t)^{-\frac{3}{2}} \exp[-|z|^2/4t]\) is the fundamental solution of the heat equation and \(E(s, z)\) is the Oseen tensor, fundamental solution of the Stokes system, with components

\[
E_{ij}(s, z) := -H(s, z)\delta_{ij} + D_{2z, j} \phi(s, z),
\]

\[
\phi(s, z) := \phi'(z) s^{-\frac{3}{2}} \int_0^{|z|} \exp[-a^2/4s^2] da,
\]
where \( \mathcal{E} \) is the fundamental solution of the Laplace equation. For the Oseen tensor the following estimates hold (see [4], estimates (VI) and (VIII) on pages 215 and 216, or [7]):

\[
\int_{\mathbb{R}^3} D_s^k D_z^b E(s, z) \, dz = 0, \quad \text{for all } s > 0, \tag{2.2}
\]

\[
|D_s^k D_z^b E(s, z)| \leq c(|z| + s^2)^{-3-k}, \quad \text{for all } s > 0 \text{ and } z \in \mathbb{R}^3. \tag{2.3}
\]

For all \( \theta \in (0, 1) \), by the symbol \(|g(t)|_\theta\) we denote the Hölder semi-norm.

We also recall

\[
[D_s^k E(s)]_{\theta} \leq c[(|z| + s^2)^{-(3+k+1)\theta} + (|z| + s^2)^{-(3+k+1)\theta}],
\]

\[
[D_z^b E(z)]_{\frac{1}{2}} \leq c[(|z| + s^2)^{-(3+k+1)\theta} + (|z| + s^2)^{-(3+k+1)\theta}],
\]

\[
\times ((|z| + s^2)^{-(3+k+1)\theta} + (|z| + s^2)^{-(3+k+1)\theta}), \tag{2.4}
\]

where, for \( h = \alpha_1 + \alpha_2 + \alpha_3 \), \( D_s^k \) denotes partial derivatives with respect to \( z \)-variable \( \alpha_i \) times, \( i = 1, 2, 3 \).

**Lemma 2.1** Let \(|a|_{3, \rho} < \infty\). Then for the convolution product \( H \ast a \) we get

\[
\|H \ast a(t)\|_{3, \rho} \leq \|a\|_{3, \rho}, \quad \text{for all } t > 0. \tag{2.5}
\]

**Proof.** The result follows from a direct application of Minkowski’s inequality. \( \square \)

**Lemma 2.2** Let \( a \in L^3(\mathbb{R}^3) \). Then for the convolution product \( H \ast a \) we get

\[
t^{\frac{3}{2}}\|H \ast a(t)\|_{\infty} + t\|\nabla H \ast a(t)\|_{\infty} + t^{\frac{3}{2}}\left[D_t H \ast a(t)\|_{\infty} + \|\nabla \nabla H \ast a(t)\|_{\infty}\right] \geq h_0|a|_{3, \rho} + h_1 e^{-\rho^2/8t}|a|_3, \quad \text{for all } \rho > 0 \text{ and } t > 0,
\]

with \( h_0 \) and \( h_1 \) positive constants.

**Proof.** By the definition of the heat kernel and applying Hölder’s inequality, we get

\[
|H \ast a(t, x)| \leq \int_{B(x, \rho)} H(t, x-y)|a(y)| \, dy + \int_{|x-y|>\rho} H(t, x-y)|a(y)| \, dy
\]

\[
\leq \left[ \int_{B(0, \rho)} e^{-\frac{1}{4\pi t} |z|^2} \, dz \right]^\frac{1}{2} \|a\|_{3, \rho} + e^{-\frac{1}{8\pi t}} \left[ \int_{|z|>\rho} e^{-\frac{1}{4\pi t} |z|^2} \, dz \right]^\frac{1}{2} \|a\|_3
\]

\[
\leq t^{-\frac{1}{2}} [h_0]_{3, \rho} + h_1 e^{-\frac{\rho^2}{2t}} |a|_3,
\]

where \( h_0 \) and \( h_1 \) are positive constants independent of \( t \) and \( \rho \). The other estimates follow by the same calculations, recalling that

\[
|D_s^k D_z^b H(s, z)| \leq cs^{-\frac{3+k}{2}} - k e^{-|z|^2/4t}, \quad \text{for all } s > 0 \text{ and } z \in \mathbb{R}^3.
\]

\( \square \)
Lemma 2.3 Let \( \sup_{(0,T)}[\|a(t)\|_3 + \|b(t)\|_3] < \infty \). Then there exists a constant \( c \) independent of \( a \) and \( b \) such that, for \( k > 0 \),

\[
\int_0^T \int_{\mathbb{R}^3} \frac{|a(\tau, y)| |b(\tau, y)|}{(|x - y| + (t - \tau)\frac{3}{2})^{3+k}} \, dyd\tau \leq ct^{-\frac{k}{2}} \left[ \sup_{(0,t)} \|a(\tau)\|_2 \rho \right. \\
\left. + t\rho^{-2} \sup_{(0,t)} \|a(\tau)\|_2 \right], \quad \text{for all } \rho > 0 \text{ for all } t \in (0,T).
\]

Proof. By Hölder’s inequality and the hypotheses,

\[
\int_0^T \int_{\mathbb{R}^3} \frac{|a(\tau, y)| |b(\tau, y)|}{(|x - y| + (t - \tau)\frac{3}{2})^{3+k}} \, dyd\tau \\
\leq c \int_0^T \int_{B(x,\rho)} \frac{|a(\tau, y)| |b(\tau, y)|}{(|x - y| + (t - \tau)\frac{3}{2})^{3+k}} \, dyd\tau + \int_0^T \int_{|x - y| > \rho} \frac{|a(\tau, y)| |b(\tau, y)|}{(|x - y| + (t - \tau)\frac{3}{2})^{3+k}} \, dyd\tau \\
\leq c \int_0^T (t - \tau)^{-1-\frac{k}{2}} \|a(\tau)\|_{\frac{2}{3},\rho} d\tau + c\rho^{-2} \int_0^T (t - \tau)^{-\frac{k}{2}} \|a(\tau)\|_{\frac{2}{3}} d\tau \\
\leq ct^{-\frac{k}{2}} \sup_{(0,t)} \|a(\tau)\|_{\frac{2}{3},\rho} + ct^{1-\frac{k}{2}} \rho^{-2} \sup_{(0,t)} \|a(\tau)\|_{\frac{2}{3}},
\]

that is (2.7).

\[\square\]

Lemma 2.4 Let \( \sup_{(0,T)}[t^{\frac{k}{2}}\|a(t)\|_\infty + t^{\frac{k}{2}}\|b(t)\|_\infty] < \infty \) and \( \sup_{(0,T)}[\|a(t)\|_3 + \|b(t)\|_3] < \infty \). Then there exists a constant \( c \) independent of \( a \) and \( b \) such that

\[
t^{\frac{k}{2}} \| \nabla E \ast (a \otimes b)(t) \|_\infty \leq c \left[ \sup_{(0,t)} \| a(\tau) \|_{3,\rho} \| b(\tau) \|_{3,\rho} + \rho^{-2} \sup_{(0,t)} \| a(\tau) \|_{2} \| b(\tau) \|_{2} \right],
\]

for all \( \rho > 0 \) for all \( t \in (0,T) \).

Proof. Via formulae (2.8) we get

\[
\| \nabla E \ast (a \otimes b)(t,x) \| \leq t^{\frac{k}{2}} \int_{\mathbb{R}^3} \frac{|a(\tau, y)| |b(\tau, y)|}{(|x - y| + (t - \tau)\frac{3}{2})^{4}} \, dyd\tau + \int_0^T \int_{\mathbb{R}^3} \frac{|a(\tau, y)| |b(\tau, y)|}{(|x - y| + (t - \tau)\frac{3}{2})^{4}} \, dyd\tau \\
=: I_1(t) + I_2(t).
\]

By our hypotheses we get

\[
I_2(t) \leq c \int_0^t \frac{1}{\tau} \sup_{(0,t)} \| a(\tau) \|_{3,\rho} \| b(\tau) \|_{3,\rho} \left[ \int_{\mathbb{R}^3} (|z|^2 + t - \tau)^{-2} \, dzd\tau \right] \leq ct^{-\frac{k}{2}} \sup_{(0,t)} \| a(\tau) \|_{3,\rho} \| b(\tau) \|_{3,\rho}.
\]
For $I_1$ we use estimate (2.7) with $k = 1$, and we get
\[ I_1(t) \leq ct^{-\frac{1}{2}} \sup_{(0,t)} \| a(\tau) \|^2_{3,\rho} + ct^{\frac{1}{2}} \rho^{-2} \sup_{(0,t)} \| a(\tau) \|^2_{3,\rho}. \]

From the previous we arrive at (2.8). \hfill \Box

**Lemma 2.5** Let $\sup_{(0,T)} \left[ \frac{1}{2} \| a(t) \|_{\infty} + \| b(t) \|_{3,\rho} \right] < \infty$. Then there exists a constant $c$ independent of $a(t,x)$ and $b(t,x)$ such that
\[ \| \nabla * (a \otimes b)(t) \|_{3,\rho} \leq c \sup_{(0,t)} \tau^{\frac{2}{3}} \| a(\tau) \|_{\infty} \| b(\tau) \|_{3,\rho}, \quad \text{for all } \rho > 0 \text{ and } t \in (0,T). \] (2.9)

**Proof.** Set $\xi = y - z$ in the convolution product. We have
\[ \| \nabla * (a \otimes b)(t) \|_{3,\rho} = \left[ \int_{B(x,\rho)} \left[ \int_{0}^{t} \int_{\mathbb{R}^3} \nabla E(t - \tau, \xi) \cdot (a(\tau, y - \xi) \otimes b(\tau, y - \xi)) \, d\xi \, d\tau \right]^{\frac{3}{2}} \, dy \right]^{\frac{1}{3}}. \]

Employing Minkowski’s inequality, then our hypotheses and estimate (2.8) for the Oseen tensor, we find
\[ \| \nabla * (a \otimes b)(t) \|_{3,\rho} \leq \int_{0}^{t} \int_{\mathbb{R}^3} |\nabla E(t - \tau, \xi)| \left[ \int_{B(x,\rho)} |a(\tau, y - \xi)| |b(\tau, y - \xi)| \, dy \right]^{\frac{3}{2}} \, d\xi \, d\tau \]
\[ \leq c \sup_{(0,t)} \tau^{\frac{2}{3}} \| a(\tau) \|_{\infty} \| b(\tau) \|_{3,\rho} \int_{0}^{t} (t - \tau)^{-\frac{2}{3} - \frac{2}{3}} d\tau. \]

that gives (2.9). \hfill \Box

**Lemma 2.6** Let $\sup_{(0,T)} \left[ \frac{1}{2} \| a(t) \|_{\infty} + \| b(t) \|_{3} \right] < \infty$. Then there exists a constant $c$ independent of $a(t,x)$ and $b(t,x)$ such that
\[ |\nabla * (a \otimes b)(t)|_{3} \leq c \sup_{(0,t)} \tau^{\frac{2}{3}} |a(\tau)|_{\infty} |b(\tau)|_{3}, \quad \text{for all } t \in (0,T). \] (2.10)

**Proof.** The proof is analogous to the one of the previous lemma. Hence it is omitted. \hfill \Box

**Lemma 2.7** In the hypotheses of Lemma 2.5 and Lemma 2.6 the convolution products $\nabla H \ast a$ and $\nabla E \ast (a \otimes b)$ are Hölder continuous functions in $x \in \mathbb{R}^3$, with exponent $\theta \in [0,1]$. In particular, we get
\[ \frac{|H * a(t,x) - H * a(t,y)|}{|x - y|^\theta} \leq ct^{-\frac{1+\theta}{2}} \left[ h_0 a \right]_{3,\rho} + h_1 e^{-\frac{2}{\rho_0^2}} \left[ a \right]_3, \]
\[ \frac{|\nabla H * a(t,x) - \nabla H * a(t,y)|}{|x - y|^\theta} \leq ct^{-\frac{1-\theta}{2}} \left[ h_0 a \right]_{3,\rho} + h_1 e^{-\frac{2}{\rho_0^2}} \left[ a \right]_3, \]
\[ \frac{|\nabla E * (a \otimes b)(t,x) - \nabla E * (a \otimes b)(t,y)|}{|x - y|^\theta} \leq ct^{-\frac{1+\theta}{2}} \sup_{(0,t)} \|a(\tau)|_{\infty} \|b(\tau)|_{3} + h_1 e^{-\frac{2}{\rho_0^2}} \left[ a \right]_{3,\rho} \]
\[ + \frac{t}{\rho^2} \left[ a(\tau) b(\tau) \right]_{3,\rho}^2, \quad \text{for all } \rho > 0, \]
with $h_0$ and $h_1$ positive constants independent of $t$ and $\rho$. (2.11)
Proof. The first two estimates follow applying the Lagrange theorem and employing the $L^\infty$ estimates of Lemma \ref{lem:2.2} for the convolution products $H \ast a$, $\nabla H \ast a$ and $\nabla \nabla H \ast a$. Hence we limit ourselves to prove estimate \ref{est:2.11}. From properties \ref{est:2.4} for the Oseen tensor $E$, we get

$$ |\nabla E \ast (a \otimes b)(t,x) - \nabla E \ast (a \otimes b)(t,x)| $$

$$ \leq c|x - x|^\theta \int_0^t \int_{\mathbb{R}^3} |a(\tau,y)||b(\tau,y)| \left[ \frac{1}{(|x - y| + (t - \tau)^\frac{1}{2})^{4+\theta}} + \frac{1}{(|x - y| + (t - \tau)^\frac{1}{2})^{4+\theta}} \right] dyd\tau $$

$$ + c|x - x|^\theta \int_0^t \int_{\mathbb{R}^3} |a(\tau,y)||b(\tau,y)| \left[ \frac{1}{(|x - y| + (t - \tau)^\frac{1}{2})^{4+\theta}} + \frac{1}{(|x - y| + (t - \tau)^\frac{1}{2})^{4+\theta}} \right] dyd\tau $$

$$ =: I_1 + I_2 $$

For $I_2$ we get

$$ I_2 \leq c|x - x|^\theta \sup_{(0,t)} \tau \|a(\tau)b(\tau)\|_\infty \int_0^t \int_{\mathbb{R}^3} \left[ \frac{1}{(|x - y| + (t - \tau)^\frac{1}{2})^{4+\theta}} + \frac{1}{(|x - y| + (t - \tau)^\frac{1}{2})^{4+\theta}} \right] dyd\tau $$

$$ \leq c|x - x|^\theta t^{-\frac{1}{2}} \sup_{(0,t)} \tau \|a(\tau)b(\tau)\|_\infty. $$

For $I_1$ we apply estimate \ref{est:2.7} with exponent $k = 1 + \theta$ and we easily obtain

$$ I_1 \leq c|x - x|^\theta t^{-\frac{1}{2}} \sup_{(0,t)} \tau \|a(\tau)b(\tau)\|_\infty. $$

From the previous estimates, for all $t \in (0, T) \times \mathbb{R}^3$ we easily get \ref{est:2.11}.\hfill \Box

Lemma 2.8 In the hypotheses of Lemma \ref{lem:2.2} and Lemma \ref{lem:2.4} the convolution products $\nabla H \ast a$ and $\nabla E \ast (a \otimes b)$ are Hölder continuous functions in $t \in (0, T)$, with exponent $\theta \in [0, \frac{1}{2})$.

Proof. The proof could be obtained arguing as for the Hölder property with respect to the space variable. On the other hand, for our aims we don’t need estimates of the kind \ref{est:2.11}. Hence we omit further details.\hfill \Box

We set $e_i(t,z) := \nabla E_i(t,z)$, $i \in \{1, 2, 3\}$, and, for some tensor field $w$, we set

$$ W(t,x) := \int_0^t \int_{\mathbb{R}^3} e_i(t - \tau, x - y) \cdot w(\tau,y) dyd\tau. $$
Lemma 2.9 Let $w(t, x) \in L^\infty(0, T; L^{\frac{2}{3}}(\mathbb{R}^3))$ and $w(t, x) \in C^{0, \theta}(\mathbb{R}^3)$ for all $t \in (0, T)$ with
\[
\sup_{(t, x)} \tau^{1+\frac{2}{3}}|w(t)|_\theta < \infty, \quad \text{for all } t \in (0, T),
\]
then, for all $\overline{\theta} < \theta$, we get
\[
t\|\nabla W(t)\|_\infty + t^{1+\frac{2}{3}}|\nabla W(t)|_{\overline{\theta}} \leq c \sup_{(t, x)} \tau^{1+\frac{2}{3}}|w(\tau)|_\theta + c \sup_{(0, \overline{\theta})} \|w(\tau)\|_{\frac{\theta}{2}} + \frac{t}{\rho^2} \|w(\tau)\|_{\frac{\theta}{2}} ,
\]
for all $t \in (0, T)$. Moreover, if $\nabla \cdot w \in C^{0, \theta}(\mathbb{R}^3)$ for all $t \in (0, T)$, with
\[
\sup_{(0, \overline{\theta})} \tau^{1+\frac{2}{3}}|\nabla \cdot w(\tau)|_\theta < \infty, \quad \text{for all } t \in (0, T),
\]
then, for all $\overline{\theta} < \theta$, we get
\[
t^{\frac{1}{2}} \|\nabla \nabla W(t)\|_\infty + t^{1+\frac{2}{3}}|\nabla \nabla W(t)|_{\overline{\theta}} \leq c \sup_{(t, x)} \tau^{1+\frac{2}{3}}|\nabla \cdot w(\tau)|_\theta + c \sup_{(0, \overline{\theta})} \|w(\tau)\|_{\frac{\theta}{2}} + \frac{t}{\rho^2} \|w(\tau)\|_{\frac{\theta}{2}} ,
\]
for all $t \in (0, T)$.

Proof. We set
\[
W_\varepsilon(t, x) := \int_0^{t-\varepsilon} \int_{\mathbb{R}^3} e_\varepsilon(t-\tau, x-y) \cdot w(\tau, y) dyd\tau.
\]
By using the Hölder property of $w$, a classical argument, ensures the existence of
\[
\lim_{\varepsilon \to 0} \nabla W_\varepsilon(t, x) = \nabla W(t, x)
\]
with
\[
\nabla W(t, x) := \lim_{\varepsilon \to 0} \int_0^{t-\varepsilon} \int_{\mathbb{R}^3} \nabla e_\varepsilon(t-\tau, x-y) \cdot w(\tau, y) dyd\tau.
\]
Let us write $\nabla W_\varepsilon(t, x)$ as follows
\[
\nabla W_\varepsilon(t, x) = \int_0^{t} \int_{\mathbb{R}^3} \nabla_x e_\varepsilon(t-\tau, x-y) \cdot w(\tau, y) dyd\tau
\]
\[
+ \int_0^{t-\varepsilon} \int_{\mathbb{R}^3} \nabla_x e_\varepsilon(t-\tau, x-y) \cdot (w(\tau, y) - w(\tau, x)) dyd\tau =: I_1 + I_2.
\]
By using property (2.13) and Lemma 2.3 with $k = 2$, we find
\[
|I_1| \leq c \int_0^{t} \frac{|w(\tau, y)|}{(|x-y| + (t-\tau)^{\frac{1}{2}})^5} dyd\tau \leq \frac{c}{t} \sup_{(0, \overline{\theta})} \|w\|_{\frac{\theta}{2}} + \frac{t}{\rho^2} \|w(\tau)\|_{\frac{\theta}{2}} ,
\]
for all $t \in (0, T)$. 

By using property (2.3) and the Hölder property of $w$, for $I_2$ we get

$$|I_2| \leq \frac{c}{t^{1+\frac{\theta}{2}}} \sup_{(\frac{t}{2}, t)} \tau^{1+\frac{\theta}{2}} |w(\tau)|_\theta \int_{0}^{t-\varepsilon} \tau^{-1-\frac{\theta}{2}} \frac{1}{\rho^2} \left[ \frac{1}{|x-y|+\varepsilon} \right] dyd\tau$$

$$\leq \frac{c}{t^{1+\frac{\theta}{2}}} \sup_{(\frac{t}{2}, t)} \tau^{1+\frac{\theta}{2}} |w(\tau)|_\theta \int_{0}^{t-\varepsilon} (t-\tau)^{-1-\frac{\theta}{2}} d\tau$$

$$\leq \frac{c}{t} \sup_{(\frac{t}{2}, t)} \tau^{1+\frac{\theta}{2}} |w(\tau)|_\theta , \text{ for all } t \in (0, T).$$

Hence, uniformly in $\varepsilon > 0$, we arrive at

$$t|\nabla W_\varepsilon(t)|_\infty \leq \frac{c}{t} \sup_{(\frac{t}{2}, t)} \tau^{1+\frac{\theta}{2}} |w(\tau)|_\theta + \frac{c}{t} \sup_{(\frac{t}{2}, t)} \left[ \|w\|_{L^\rho, \rho} + \frac{t}{\rho^2} \|w(\tau)\|_{L^\infty} \right], \text{ for all } t \in (0, T),$$

which leads to the $L^\infty$-estimate enclosed in (2.12).

Now we show the Hölder property of $\nabla W$. We set

$$|\nabla W_\varepsilon(t, x) - \nabla W_\varepsilon(t, \overline{x})|$$

$$= \left| \int_{0}^{t} \int_{\mathbb{R}^3} [\nabla e_i(t-\tau, x-y) - \nabla e_i(t-\tau, \overline{x}-y)] \cdot w(\tau, y) dyd\tau \right|$$

$$+ \int_{0}^{t} \int_{\mathbb{R}^3} [\nabla e_i(t, x-y) - \nabla e_i(t, \overline{x}-y)] \cdot w(\tau, y) dyd\tau.$$

From properties (2.4) for the Oseen tensor $E$ and Hölder’s assumption on $w$, for all $\theta' < \theta$, we easily get

$$|I_1(t)| \leq c|x-\overline{x}|^{\theta} \int_{0}^{t} \tau^{1+\frac{\theta}{2}} |w(\tau)| \left[ \frac{1}{(|x-y|+\varepsilon)^{5+\theta'}} + \frac{1}{(|\overline{x}-y|+\varepsilon)^{5+\theta'}} \right] dyd\tau,$$

$$|I_2(t)| \leq \frac{c}{t^{1+\frac{\theta}{2}}} \sup_{(\frac{t}{2}, t)} \tau^{1+\frac{\theta}{2}} |w(\tau)|_\theta \int_{0}^{t} \left[ \frac{1}{(|x-y|+\varepsilon)^{5+\theta'-\theta}} + \frac{1}{(|\overline{x}-y|+\varepsilon)^{5+\theta'-\theta}} \right] dyd\tau.$$

By using Lemma 2.3 with $k = 2 + \theta'$, for $I_1$ we find

$$|I_1| \leq c|x-\overline{x}|^{\theta} t^{-1-\frac{\theta}{2}} \sup_{(\frac{t}{2}, t)} \|w\|_{L^\rho, \rho} + t \rho^{-2} \sup_{(\frac{t}{2}, t)} \|w(\tau)\|_{L^\infty}.$$

For $I_2$ an integration furnishes

$$|I_2| \leq c|x-\overline{x}|^{\theta} t^{-1+\frac{\theta}{2}} \sup_{(\frac{t}{2}, t)} \tau^{1+\frac{\theta}{2}} |w(t)|_\theta.$$
then it is enough to consider that an integration by parts furnishes
\[|\nabla\nabla W_\varepsilon(t, x)| = \left| \int_0^{\frac{t}{\varepsilon}} \int_{\mathbb{R}^3} \nabla\nabla e_i(t - \tau, x - y) \cdot w(\tau, y) dy d\tau \right. \]
\[-\left. \int_{\frac{t}{\varepsilon}}^t \int_{\mathbb{R}^3} \nabla\nabla E_i(t - \tau, x - y) \cdot \nabla \cdot w(\tau, y) dy d\tau \right|.
\]
After which, one employs the same arguments considered for \(\nabla W\).

\[\text{Lemma 2.10}\]
If \(t^{\frac{3}{2}}w, t\nabla \cdot w(t) \in L^\infty(0, T; L^3(\mathbb{R}^3))\), then, we get
\[t^{\frac{3}{2}}\|\nabla W(t)\|_3 \leq c \left[ \sup_{(0, \frac{t}{2})} \tau^{\frac{3}{2}}\|w(\tau)\|_3 + \sup_{(\frac{t}{2}, t)} \tau \|\nabla \cdot w(\tau)\|_3 \right], \quad \text{for all } t \in (0, T), \tag{2.15}\]
and if we also assume \(t^{\frac{3}{2}}\nabla\nabla \cdot w \in L^\infty(0, T, L^3(\mathbb{R}^3))\), then, we get
\[t\|\nabla\nabla W(t)\|_3 \leq c \left[ \sup_{(0, \frac{t}{2})} \tau^{\frac{3}{2}}\|w(\tau)\|_3 + \sup_{(\frac{t}{2}, t)} \tau \|\nabla\nabla \cdot w(\tau)\|_3 \right], \tag{2.16}\]
\[\text{Proof.} \quad \text{We prove (2.15). Recalling the definition of } e_i, \text{ via an integration by parts, we have}
\begin{align*}
\|\nabla W(t)\|_3 &= \left| \int_{\mathbb{R}^3} \int_0^{\frac{t}{\varepsilon}} \int_{\mathbb{R}^3} \nabla e_i(y - z, t - \tau) \cdot w(z, \tau) dz d\tau dy d\tau \right. \\
&\quad + \left. \int_{\frac{t}{\varepsilon}}^t \int_{\mathbb{R}^3} \nabla E_i(y - z, t - \tau) \cdot \nabla \cdot w(z, \tau) dz d\tau d\tau \right|^{\frac{3}{2}}.
\end{align*}
\tag{2.17}
\]
Applying for both terms Minkowski’s inequality, estimate \(23\) for the Oseen tensor, we find
\[
\|\nabla W(t)\|_3 \leq c \sup_{(0, \frac{t}{2})} \tau^{\frac{3}{2}}\|w(\tau)\|_3 \int_0^{\frac{t}{2}} \tau^{-\frac{1}{2}} \int_{\mathbb{R}^3} |\nabla e_i(t - \tau, \xi)| d\xi d\tau \\
+ c \sup_{(\frac{t}{2}, t)} \tau \|\nabla \cdot w(\tau)\|_3 \int_{\frac{t}{2}}^t \tau^{-1} \int_{\mathbb{R}^3} |e_i(t - \tau, \xi)| d\xi d\tau \\
\leq c t^{-\frac{1}{2}} \left[ \sup_{(0, \frac{t}{2})} \tau\|w(\tau)\|_3 + \sup_{(\frac{t}{2}, t)} \tau \|\nabla \cdot w(\tau)\|_3 \right],
\]
which proves \(2.15\). To prove \(2.16\), it is enough to employ a further integration by parts on \(\left(\frac{t}{2}, t\right) \times \mathbb{R}^3\), after which the argument lines are the same, so that we consider achieved the lemma.

Let us consider the equation
\[
(\pi, \Delta g) = -(a \otimes u, \nabla \nabla g) \text{ for all } g \in C_0^\infty(\mathbb{R}^3), \tag{2.18}
\]
From the theory of singular integrals \([9]\), one immediately gets the following result.
Lemma 2.11 Let $a$ and $u$ be divergence free. If $a \in L^\infty(\mathbb{R}^3)$ and $u \in L^3(\mathbb{R}^3)$, there exist constants $c$ independent of $a$ and $u$ such that for a solution $\pi$ to problem (2.18) the following holds:

$$\|\pi\|_3 \leq c\|a\|_\infty\|u\|_3. \quad (2.19)$$

Further, if $a, u \in C^{1,\theta}(\mathbb{R}^3)$, for some $\theta \in (0, 1)$, then $\pi \in C^{1,\theta}(\mathbb{R}^3)$.

3 Properties of the approximating sequence of solutions

We study the integral relation

$$U^m(t,x) = H * U_0(t,x) - \nabla_x E * (U^{m-1} \otimes U^{m-1})(t,x). \quad (3.1)$$

Lemma 3.1 Let $U_0(x) \in L^2(\mathbb{R}^3)$. Set $U^0(t,x) := H * U_0$. Then there exist constants $h_0, h_1, c_1$, independent of $U_0$ and $m \in \mathbb{N}$, such that for the sequence (3.1) we get

$$\|U^m\|_{(t,\rho)} \leq (h_0 + 1)\|U_0\|_{3,\rho} + (h_1 \epsilon^{-\rho^2/s^2} + \frac{s^2}{\rho})\|U_0\|_3 + c_1 \|U^{m-1}\|_{(t,\rho)}, \quad (3.2)$$

for all $t > 0$ and $\rho > 0$.

Proof. From definition (3.1), by virtue of Lemma 2.2-Lemma 2.6 for all $t > 0$ and $\rho > 0$, we get

$$s^{\frac{1}{2}}\|U^1(s)\|_\infty \leq h_0\|U_0\|_{3,\rho} + h_1 \epsilon^{-\rho^2/s^2}\|U_0\|_3$$

$$+ c\left[ \sup_{(0,t)} \tau^\frac{1}{2} \|U^0(\tau)\|_\infty + \sup_{(0,s)} \|U^0(\tau)\|_{3,\rho} + \frac{s^2}{\rho} \sup_{(0,s)} \|U^0(\tau)\|_3 \right]^2,$$

$$\|U^1(s)\|_{3,\rho} \leq \|U_0\|_{3,\rho} + c\left[ \sup_{(0,s)} \tau^\frac{1}{2} \|U^0(\tau)\|_\infty \|U^0(\tau)\|_{3,\rho} \right]$$

$$\leq \|U_0\|_{3,\rho} + c\left[ \sup_{(0,s)} \tau^\frac{1}{2} \|U^0(\tau)\|_\infty + \sup_{(0,s)} \|U^0(\tau)\|_{3,\rho} + \frac{s^2}{\rho} \sup_{(0,s)} \|U^0(\tau)\|_3 \right]^2,$$

$$\|U^1(s)\|_{3} \leq \|U_0\|_3 + c\sup_{(0,s)} \tau^\frac{1}{2} \|U^0(\tau)\|_\infty \|U^0(\tau)\|_3,$$

where $c$ is a constant independent of $t, \rho$. Multiplying the last estimate for $\frac{s^2}{\rho}$ and then increasing, we get

$$\frac{s^2}{\rho} \|U^1(s)\|_3 \leq \frac{s^2}{\rho} \|U_0\|_3 + c\left[ \sup_{(0,s)} \tau^\frac{1}{2} \|U^0(\tau)\|_\infty + \sup_{(0,s)} \|U^0(\tau)\|_{3,\rho} + \frac{s^2}{\rho} \sup_{(0,s)} \|U^0(\tau)\|_3 \right]^2.$$

Taking $\sup_{(0,t)}$ of the previous trilogy, then summing the first two with the last one, recalling the definition of the functional $\| \cdot \|_{(t,\rho)}$, we arrive at

$$\|U^1\|_{(t,\rho)} \leq (h_0 + 1)\|U_0\|_{3,\rho} + (h_1 \epsilon^{-\rho^2/s^2} + \frac{s^2}{\rho})\|U_0\|_3 + 3c \|U^0\|_{(t,\rho)},$$

for all $\rho > 0$ and $t > 0$,.
with a constant $c$ independent of the datum $U_0$. So that, for $m = 1$, (3.1) is well defined and estimate (3.2) is true. Then by induction one proves the estimate for all $m \in \mathbb{N}$. □

We use the method of successive approximations. We show that the previous lemmas ensure boundedness and convergence of the approximating sequence of velocity fields $\{U^m\}$. Firstly we recall the following result.

**Lemma 3.2** Let $\xi_0 > 0$ and $c > 0$. Let $\{\xi_m\}$ be a non negative sequence of real numbers such that

$$\xi_m \leq \xi_0 + c\xi_{m-1}.$$  

Assume $1 - 4c\xi_0 > 0$ and $\xi_0 \leq \xi$, where $\xi$ is the minimum solution of the algebraic equation $ce^2 - \xi + \xi_0 = 0$. Then $\xi_{m-1} \leq \xi$ for all $m \in \mathbb{N}$.

**Proof.** For the proof we refer to [8]. □

**Lemma 3.3** Let $\{U^m\}$ be the sequence defined in (3.1) corresponding to $U_0 \in J^3(\mathbb{R}^3)$. Then, there exists a $T(U_0) > 0$ such that, for all $\eta$, the sequence strongly converges in $C((\eta, T(U_0)) \times \mathbb{R}^3)$, to a solution $U$ to (2.1), and, for all $t \in (0, T(U_0))$, the sequence converges to $U$ in $J^3(\mathbb{R}^3)$. In particular we get, for a suitable $\rho$ and for all $t \in [0, T(U_0))$,

$$\|U\|_{(t, \rho)} \leq \frac{2[(h_0 + 1)\|U_0\|_{3, \rho} + (h_1e^{-\rho^2/4st} + \frac{h_2}{\rho})\|U_0\|_{3}]}{1 + (1 - 4c_1[(h_0 + 1)\|U_0\|_{3, \rho} + (h_1e^{-\rho^2/4st} + \frac{h_2}{\rho})\|U_0\|_{3}])^{\frac{1}{2}}}.$$  

(3.3)

and

$$\|U(t)\|_{3} \leq c\|U_0\|_{3}.$$  

(3.4)

Further

$$\limsup_{t \to 0^+}(0, t) \tau^\frac{1}{2}\|U(\tau)\|_{\infty} = 0.$$  

(3.5)

**Proof.** Since $U_0 \in L^3(\mathbb{R}^3)$, for any $\varepsilon \in (0, \frac{1}{4c_1(h_0 + 1)}$, there exists $\rho = \rho(U_0, \varepsilon)$ such that $\|U_0\|_{3, \rho} < \varepsilon$. For any such $\rho$, we denote by $t(\rho)$ the supremum of $t > 0$ for which the following inequality holds

$$1 - 4c_1[(h_0 + 1)\|U_0\|_{3, \rho} + (h_1e^{-\rho^2/4st} + \frac{h_2}{\rho})\|U_0\|_{3}] > 0.$$  

(3.6)

We observe that the definition of $t(\rho)$ is well posed, taking into account that, for any fixed $\rho > 0$, the function in round brackets is a monotonic increasing function of $t$ that tends to zero as $t \to 0$. Finally we denote by $T(U_0)$ the supremum of $t(\rho)$ for which (3.6) holds. Then, by virtue of estimate (3.2) and applying Lemma 3.2 for a fixed $\rho$ and for any $t \in [0, T(U_0))$ and uniformly in $m \in \mathbb{N}$ we get

$$\|U^m\|_{(t, \rho)} \leq \frac{2[(h_0 + 1)\|U_0\|_{3, \rho} + (h_1e^{-\rho^2/4st} + \frac{h_2}{\rho})\|U_0\|_{3}]}{1 + (1 - 4c_1[(h_0 + 1)\|U_0\|_{3, \rho} + (h_1e^{-\rho^2/4st} + \frac{h_2}{\rho})\|U_0\|_{3}])^{\frac{1}{2}}} =: A(\rho, t).$$  

(3.7)

Estimate (3.7) ensures that, for all $t \in [0, T(U_0))$, the sequence $\{\|U^m\|_{(t, \rho)}\}$ is bounded.

On the other hand, the validity of estimate (3.2), for any $\rho > 0$ and for any $t > 0$, ensures that the following property holds true:
P: for any sequence \( \{ t_p \} \to 0 \), one can construct a sequence \( \{ \rho_p \} \to 0 \) such that \( (3.6) \) holds, and along these sequences we get \( \lim_{p \to \infty} A(\rho_p, t_p) = 0 \) too.

Therefore, we can again apply Lemma 3.2 and we get, for all \( p \in \mathbb{N} \),

\[
\| U^m \|_{(t_p, \rho_p)} \leq A(\rho_p, t_p), \quad \forall m \in \mathbb{N}, \quad \text{with} \quad \lim_{p \to \infty} A(\rho_p, t_p) = 0.
\]  

We set \( w^m := U^m - U^{m-1} \). Hence from \( (3.3) \) we arrive at \( (m \geq 0 \text{ and } U^{-1} = 0) \)

\[
w^{m+1}(t, x) = -\nabla_x E * (w^m \otimes U^m)(t, x) - \nabla_x E * (U^{m-1} \otimes w^m)(t, x).
\]

Employing the arguments of Lemma 2.4, Lemma 2.1 and Lemma 2.6, and recalling estimate \( (3.7) \), we easily arrive at the sequence of estimates

\[
\| w^1 \|_{(t, \rho)} \leq c_1 A^2(\rho, t), \ldots, \| w^m \|_{(t, \rho)} \leq 2^{m-1} c_1^m A^{m+1}(\rho, t), \ldots.
\]  

Since \( (3.6) \) furnishes \( A(\rho, t) < 1/2c_1 < 1 \) for all \( t \in (0, T(U_0)) \), we get the convergence of the sequence \( \{ U^m \} \) with respect to the functional \( \| \cdot \|_{(t, \rho)} \). The uniform convergence of the sequence of continuous functions \( \{ U^m \} \) on \( (\eta, T) \times \mathbb{R}^3 \) ensures that the limit is a continuous function in \( (t, x) \in C((\eta, T) \times \mathbb{R}^3) \). We denote by \( U \) the limit.

Recalling the definition of the functional \( \| \cdot \|_{(t, \rho)} \), by virtue of estimate \( (3.8) \), we deduce

\[
\text{for all } t_p \to 0, \quad \lim_{p \to \infty} \sup_{(0,t_p)} \| \tau \|_{\infty} \| U(\tau) \|_{\infty} \leq \lim_{p \to \infty} A(\rho_p, t_p) = 0,
\]

that is estimate \( (3.9) \).

Further, again from the definition of the functional \( \| \cdot \|_{(t, \rho)} \) and using estimate \( (3.3) \), we have, for all \( t \in (0, T(U_0)) \),

\[
\frac{t^\frac{1}{2}}{\rho} \sup_{(0,t)} \| U(\tau) \|_3 \leq 2(h_0 + 1)\| U_0 \|_{3, \rho} + 2(h_1e^{-\frac{\rho}{h_1}} + \frac{t^\frac{1}{2}}{\rho})\| U_0 \|_3.
\]

Dividing by \( t^\frac{1}{2}/\rho \) and passing to the limit for \( t \to T^- \), we get

\[
\sup_{(0,T)} \| U(\tau) \|_3 \leq \frac{\rho}{T^\frac{1}{2}}2(h_0 + 1)\| U_0 \|_{3, \rho} + 2(h_1e^{-\frac{\rho}{h_1}} + 1)\| U_0 \|_3.
\]

Hence

\[
\| U(\tau) \|_3 \leq \frac{\rho}{T^\frac{1}{2}}2(h_0 + 1)\| U_0 \|_{3, \rho} + c\| U_0 \|_3, \quad \text{for all } t \in (0, T),
\]

from which, using that, for any \( \rho, \| U_0 \|_{3, \rho} \leq c\| U_0 \|_3 \), we deduce \( (3.4) \).

\[\Box\]

4 Proof of the main results

In the following, by virtue of estimate \( (3.3) \), we consider \( A(\rho, t) \) defined in \( (3.7) \) as a majorant of \( \| U \|_{(t, \rho)} \), that is

\[
\| U \|_{(t, \rho)} \leq A(\rho, t), \quad \text{for all } t \in (0, T(U_0)), \quad (4.1)
\]

\[1\text{It is sufficient to choose the sequence } \{ \rho_p \} \text{ such that } \rho_p \to 0 \text{ and } \frac{t^\frac{1}{2}}{\rho_p} = o(1).\]
hence, using Hölder’s inequality, we get
\[ t\|U(t) \otimes U(t)\|_{\infty} \leq A^2(\rho, t), \text{ for all } t \in (0, T(U_0)), \]
\[ \|U(t) \otimes U(t)\|_{2, \rho} + \frac{t}{\rho} \|U(t) \otimes U(t)\|_{2} \leq A^2(\rho, t), \text{ for all } t \in (0, T(U_0)). \] (4.2)

**Proof of Theorem 3.1.** In the hypothesis of Theorem 3.1 by virtue of Lemma 3.1 and Lemma 3.3, we establish a solution \(U(t, x)\) divergence free to the integral equation (2.1) such that for all \(t \in [0, T(U_0))\), \((U(t, x), x) \in \mathcal{P}^3(\mathbb{R}^3)\) and satisfies inequality (1.6). Thanks to Lemma 2.7, \(U\) satisfies the Hölder properties with
\[ t^\frac{1}{2} + \frac{1}{2}[U(t)]_3 \leq c(A(\rho, t) + A^2(\rho, t)), \text{ for all } t \in (0, T(U_0)). \] (4.3)

Hence, the following holds:
\[ t^\frac{1}{2} [U(t) \otimes U(t)]_3 \leq c(A^2(\rho, t) + A^3(\rho, t)), \text{ for all } t \in (0, T(U_0)). \] (4.4)

As well as, since
\[ \nabla U(t, x) = \nabla H \ast U_0(t, x) + \lim_{\varepsilon \to 0} \nabla W_\varepsilon(t, x) = \nabla H \ast U_0(t, x) + \nabla W(t, x), \]
applying Lemma 2.2 and Lemma 2.9 where we mean \(w = U \otimes U\), we arrive at
\[ \|\nabla U(t)\|_\infty + t^\frac{1}{2} [\nabla U(t)]_3 \leq cA(\rho, t) + c\left[ A^2(\rho, t) + A^3(\rho, t) \right] \leq c \left[ A(\rho, t) + A^3(\rho, t) \right], \] (4.5)
where we employed (4.2) and (4.4). Since, from property \(P\), \(A(\rho, t)\) tends to zero as \(t \to 0\), we get estimate (1.7) for the \(\nabla U\).

Then, we consider \(\pi\) solution to the Poisson equation \(\Delta \pi = -\nabla \cdot \nabla \cdot (U \otimes U)\). We obtain estimates (1.7) by applying Lemma 2.11.

Since \(U\) is solution to the integral equation (2.1), by the couple \((U, \pi)\) one finds the wanted solution to system (1.1). Concerning the initial condition \(U_0\), we firstly observe that the limit property (1.7) trivially holds for \(U^0(t, x)\). Then, via the integral equation (2.1), and Lemma 2.6 for \(\nabla E \ast (U \otimes U)\), we get
\[ \|U(t) - U^0(t)\|_3 \leq c \sup_{(0, t)} \tau^\frac{1}{2} \|U(\tau)\|_\infty \|U(\tau)\|_3 \text{ for all } t \in [0, T(U_0)) \]

Thus, from (3.5) and (3.4), we arrive at the limit property (1.7).

Finally, if we require \(\|U_0\|_3\) sufficiently small and consider \(t^\frac{1}{2}/\rho\) in constant ratio, since \(\|U_0\|_{3, \rho} \leq \|U_0\|_3\) for all \(\rho > 0\), we can satisfy (3.4) for arbitrary \(\rho\) and then arbitrary \(t\). This gives the stated global existence and completes the proof.

By virtue of Theorem 3.1, we get
\[ t^\frac{1}{2} \|U \otimes U\|_3 \leq t^\frac{1}{2} \|U(t)\|_\infty \|U(t)\|_3 \leq c \|U_0\|_3 A(\rho, t) \] (4.6)
for all \(t \in (0, T(U_0))\).

**Proof of Corollary 3.1.** From Theorem 3.1 the solution \(U\) satisfies inequality (1.8), by interpolation. We get (1.8) by interpolation too. Actually, it suffices to show that
that, in turn, employing (4.7), implies
\[ \| \nabla U(t) \|_3 \lesssim c t^{-\frac{3}{2}} \| U_0 \|_3 \left[ 1 + A^3(\rho, t) \right]. \] (4.8)

By interpolating such estimate and estimate (4.4), we get (4.5).

Since \( \nabla \cdot U \otimes U = U \cdot \nabla U \), via the Hölder properties of \( U \) and \( \nabla U \), (4.9) and (4.10), respectively, we get \( \nabla \cdot w = U \cdot \nabla U \). Hence, by means of a trivial computation, we find
\[ \| \nabla U(t) \|_\infty \lesssim c A(\rho, t) + A^4(\rho, t). \] (4.9)

In order to prove that \( \nabla \nabla W(t) \in L^3(\mathbb{R}^3) \), we apply (2.10), taking into account that \( \nabla \nabla \cdot w = \nabla (U \cdot \nabla U) = U \cdot \nabla U + U \cdot \nabla U \). Hence, by means of a trivial computation, we find
\[ t \| \nabla \nabla W(t) \|_3 \lesssim c \left[ \sup_{(0, \frac{3}{2})} \tau^{\frac{3}{2}} \| U(\tau) \|_3^3 + \sup_{(0, \frac{3}{2})} \tau \| \nabla U(\tau) \|_3 \sup_{(0, \frac{3}{2})} \tau^{\frac{3}{2}} \| \nabla U(\tau) \|_3 \right. 
\left. + \sup_{(0, \frac{3}{2})} \tau^{\frac{3}{2}} \| \nabla U(\tau) \|_\infty \| U(\tau) \|_3 \right]. \]

Hence, by applying estimates (4.4), (4.8), (4.9) and (4.10), we get
\[ t \| \nabla \nabla W(t) \|_3 \lesssim c \| U_0 \|_3 \left[ A(\rho, t) + A^6(\rho, t) \right], \]

that, in turn, employing (4.7), implies
\[ t \| \nabla \nabla U(t) \|_3 \lesssim c \| U_0 \|_3 \left[ 1 + A^6(\rho, t) \right]. \]

By interpolating such estimate and estimate (4.9), we get (1.8).
Proof of Theorem 1.2 Let us consider two solutions $U$ and $\overline{U}$ satisfying the assumptions, and set $u := \overline{U} - U$. Then, for all $t > s \geq 0$, $u$ satisfies the following integral equation

\[\int_{s}^{t} (u(\tau), \varphi_{\tau} + \Delta \varphi) \, d\tau + \int_{s}^{t} \left( (u(\tau) \cdot \nabla \varphi, \overline{U}) + (U \cdot \nabla \varphi, u) \right) \, d\tau = (u(t), \varphi(t)) - (u(s), \varphi(s)), \quad \forall \varphi \in C^{1}([0, T]; \mathcal{C}_{0}(\mathbb{R}^{3})).\] (4.10)

We denote by $\psi$ the solution to the Cauchy problem:

\[
\begin{align*}
\psi_{t} - \Delta \psi &= -\nabla \pi_{\psi}, \quad \nabla \cdot \psi = 0 \quad \text{in} \ (0, T) \times \mathbb{R}^{3}, \\
\psi &= \psi_{0} \in \mathcal{C}_{0}(\mathbb{R}^{3}) \quad \text{on} \ \{0\} \times \mathbb{R}^{3}.
\end{align*}
\] (4.11)

It is well known that $\psi$ is a smooth solution with $\psi \in C([0, T); J^{p}(\mathbb{R}^{3}))$, for all $p \in (1, \infty)$, and satisfies the following estimates:

\[
\begin{align*}
q \geq p, \|\psi(t)\|_{q} &\leq c t^{\frac{2p}{q} - \frac{2}{q}} \|\psi_{0}\|_{p}, \quad \text{for all} \ t > 0, \\
\|\nabla \psi(t)\|_{q} &\leq c t^{\frac{2p}{q} - \frac{2}{q}} \|\psi_{0}\|_{p}, \quad \text{for all} \ t > 0.
\end{align*}
\] (4.12)

For $t > 0$, we set $\widehat{\psi}(\tau, x) := \psi(t - \tau, x)$ provided that $(\tau, x) \in (0, t) \times \mathbb{R}^{3}$. It is well known that $\widehat{\psi}$ is a solution backward in time with $\widehat{\psi}(t, x) = \psi_{0}(x)$.

Let us write the integral equation (4.10) with $\widehat{\psi}$ in place of $\varphi$. We get

\[
(u(t), \psi_{0}) = (u(s), \widehat{\psi}(s)) + \int_{s}^{t} \left( (U \cdot \nabla \widehat{\psi}, u) + (u \cdot \nabla \widehat{\psi}, \overline{U}) \right) \, d\tau.
\] (4.13)

Hence

\[
|(u(t), \psi_{0})| \leq |(u(s), \widehat{\psi}(s))| + c \sup_{(s,t)} \left[ \tau^{\frac{2p}{q}} (\|U(\tau)\|_{\infty} + \|\overline{U}(\tau)\|_{\infty}) \right] \sup_{(s,t)} \|u(\tau)\|_{3} \int_{s}^{t} \tau^{-\frac{2}{q}} \|\nabla \psi(t - \tau)\|_{\frac{q}{2}} \, d\tau
\] (4.14)

for all $t \in [0, T(U)) \cap [0, T(\overline{U}))$. Since $\psi_{0}$ is arbitrary, and then letting $s \to 0$, we obtain

\[
\|u(t)\|_{3} \leq c \sup_{(0,t)} \left[ \tau^{\frac{2p}{q}} (\|U(\tau)\|_{\infty} + \|\overline{U}(\tau)\|_{\infty}) \right] \sup_{(0,t)} \|u(\tau)\|_{3}.
\]

From the validity of the limit property (1.10) on both solutions, one easily deduces the uniqueness on some interval $(0, \delta]$. It remains to discuss the uniqueness when $t \geq \delta$. 

\[\Box\]
Writing estimate (4.14) with $s = \delta$, since $\|u(\delta)\|_3 = 0$, we deduce the estimate

$$
\|u(t)\|_3 \leq \int_\delta^t \left[ \|U(\tau)\|_\infty + \|\overline{U}(\tau)\|_\infty \right] \|u(\tau)\|_3 (t - \tau)^{-\frac{1}{2}} \, d\tau
$$

$$
\leq c \delta^{-\frac{1}{2}} \sup_{(\delta,t)} \tau^\frac{1}{2} \left( \|U(\tau)\|_\infty + \|\overline{U}(\tau)\|_\infty \right) \int_\delta^t \|u(\tau)\|_3 (t - \tau)^{-\frac{1}{2}} \, d\tau.
$$

We are in the hypothesis of the logarithmic Gronwall inequality (see Lemma 4 in [4]). Therefore we obtain $\|u(t)\|_3 = 0$, for any $t \in [\delta,T)$, that completes the proof.

□

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