Twisted Toroidal Lie Algebras

Johan van de Leur

Mathematical Institute,
University of Utrecht,
P.O. Box 80010, 3508 TA Utrecht,
The Netherlands
e-mail: vdleur@math.uu.nl

October 23, 2018

Abstract

Using \( n \) finite order automorphisms on a simple complex Lie algebra we construct twisted \( n \)-toroidal Lie algebras. Thus obtaining Lie algebras which have a rootspace decomposition. For the case \( n = 2 \) we list certain simple Lie algebras and their automorphisms, which produce twisted 2-toroidal algebras. In this way we obtain Lie algebras that are related to all Extended Affine Root Systems of K. Saito.

1 Introduction

At the end of the 1960’s Victor Kac \([20]\) and Bob Moody \([25]\) independently realized that one could generalize Serre’s construction of simple Lie algebras to construct certain infinite-dimensional Lie algebras, that possess root systems. A special class of these Kac-Moody (Lie) algebras were related to the affine root systems. These affine Lie algebras could also be constructed in an explicit way. Nowadays this construction is well known. These algebras are the central extensions of so-called twisted and untwisted loop algebras. The untwisted loop algebras \( \tilde{\mathfrak{g}} \) can be obtained as follows. Let \( \mathfrak{g} \) be a finite dimensional simple Lie algebra, then \( \tilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \), where \( \mathbb{C}[t, t^{-1}] \) is the algebra of Laurent polynomials in the variable \( t \). The twisted loop algebras can be obtained as certain subalgebras of the untwisted ones. Let \( \sigma \) be a certain finite order automorphism induced by a diagram automorphism of the Dynkin diagram (see \([21]\) for more details). Then \( \tilde{\mathfrak{g}} \) decomposes into eigenspaces with respect to this automorphism. To be more precise, let \( n \) be the order of the automorphism and \( \epsilon = e^{\frac{2\pi i}{n}} \)

\[
\tilde{\mathfrak{g}} = \bigoplus_{\tau \in \mathbb{Z}/n\mathbb{Z}} \tilde{\mathfrak{g}}_{\tau}, \quad \text{where} \\
\tilde{\mathfrak{g}}_{\tau} = \{ g \in \tilde{\mathfrak{g}} | \sigma(g) = \epsilon^k g \}.
\]

The twisted loop algebra is the following subalgebra of \( \tilde{\mathfrak{g}} \)

\[
\tilde{\mathfrak{g}}(\sigma) = \bigoplus_{k \in \mathbb{Z}} \tilde{\mathfrak{g}}_{\tau} \otimes t^k.
\]
A generalization of this construction, at least of the untwisted ones is clearly obvious. Instead of tensoring by the algebra of Laurent polynomials in one variable, one can take Laurent polynomials in \(N\) variables \(t_j\). Thus obtaining toroidal Lie algebras. Unfortunately, these Lie algebras are not Kac-Moody algebras, but they still are very interesting and obviously related to certain extensions of affine root systems. They appeared in the work of Slodowy [30] as certain intersection matrix algebras.

K. Saito, interested in singularity theory and inspired by the work of Looijenga [23], [24] and Slodowy [29], [30], classified in [27] extended affine root systems, whose radical is 2-dimensional and for which the quotient of the root system modulo a certain 1-dimensional space is reduced. In 1997 Allison, Azam, Berman, Gao and Pianzola [1] had a different approach, they classified these root systems using semilattices. The Lie algebras corresponding to these root systems, so-called extended affine Lie algebras or more precise bi-affine Lie algebras, were constructed in a paper by Hoegh–Krohn and Torresani [14]. However, their construction was not complete. Pollmann gave a complete construction in [26], which was based on the idea of twisting affine Lie algebras by finite order automorphism. This idea was presented in [14] and also in an unpublished paper of Wakimoto [31]. Although one obtains in this way the bi-affine algebras as subalgebras of 2-toroidal Lie algebras, the grading with respect to the variables of the Laurent polynomials is not so nice. The present paper constructs the same Lie algebras also as subalgebras of 2-toroidal Lie algebras, but in a slightly different way. Whereas [14], [26] and [31] use finite order automorphisms of affine Lie algebras to construct the bi-affine Lie algebras, we use two finite order automorphisms of a simple finite dimensional Lie algebra, which commute and thus are simultaneously diagonalizable, to construct them. This construction generalizes in a different way than the construction of [26] the construction of the twisted affine Lie algebras. This is an experimental fact, unfortunately, at this moment there is no general theory or classification which explains this phenomenon. Note that in some cases one uses only inner automorphisms. Our construction also easily generalizes to twisted \(N\)-toroidal Lie algebras, whereas the generalization of the method of [14], [26] and [31] is somewhat more complicated. A general, but different and more abstract, construction of these extended affine Lie algebras is given in the AMS Memoir [1] and in [2].

The theory of vertex operator constructions of untwisted toroidal Lie algebras is well developed [1], [2], [12], [13], [28], [32] and applied to hierarchies of soliton equations [1], [13], [18], [14]. We hope that the construction given in Section 3 can be used to define vertex operator constructions on twisted bi-affine and extended affine algebras.

Untwisted toroidal Lie algebras appear as current algebras of the symmetry of Kähler–Wess–Zumino–Witten models [16], [17]. This is an extension of (2-dimensional) Wess–Zumino–Witten models on a \(2n\)-dimensional Kähler manifold. As such it is one possible candidate of a construction of integrable quantum field theories in more than two dimensions.

The decomposition of some of the exceptional Lie algebras with respect to the automorphisms were checked by Willem de Graaf using GAP – Groups, Algorithms, and Programming [11]. It is a pleasure to thank him and Prof. P. Slodowy. The latter for sending the manuscript [26].

2 Extended Affine Root Systems

The following definition of an extended reduced root system can be found in [27], this definition is different from the one in [1], there also the isotropic roots and 0 are included.
in the definition.

**Definition 2.1** Let \( V \) be a finite dimensional real vector space with a positive semidefinite symmetric bilinear form \( (\cdot, \cdot)_V \). A subset \( R \) of \( V \), with is called an extended reduced root system in \( V \) if \( R \) satisfies the following axioms:

- The additive subgroup \( Q(R) = \sum_{\alpha \in R} \mathbb{Z}\alpha \) of \( V \) is a full lattice of \( V \), i.e. \( R \otimes_{\mathbb{R}} Q(R) \cong V \),
- \( (\alpha, \alpha)_V \neq 0 \) for any \( \alpha \in R \),
- If \( \alpha \in R \), then \( 2\alpha \not\in R \),
- For any \( \alpha \in R \), \( W_{\alpha}(R) = R \), where
  \[ w_{\alpha}(\beta) = \beta - 2\frac{(\beta, \alpha)_V}{(\alpha, \alpha)_V}\alpha, \]
- If \( \alpha, \beta \in R \), then
  \[ 2\frac{(\beta, \alpha)_V}{(\alpha, \alpha)_V} \in \mathbb{Z} \]
- \( R \) cannot be decomposed in a disjoint union \( R_1 \cap R_2 \), where \( R_1, R_2 \subset R \), both nonempty, satisfying \( (R_1, R_2)_V = 0 \),

The dimension \( \nu \) of the radical
\[ V^0 = \{ v \in V | (v, w)_V = 0 \text{ for all } w \in V \}, \]
is called the nullity of the root system \( R \). For \( \nu = 2 \), Saito [27] classified all marked extended affine root systems. Roughly speaking he considered a 1-dimensional marking, which is a linear subspace \( W \subset V^0 \) and considered the induced space \( V/W \) and corresponding induced root system. Now assuming that this induced root system is a reduced (possibly affine) root system, he obtained the following list. See [27] or [26] for a more precise statement. We write \( R(X_\ell) \) for the root system of a finite type \( X_\ell \).

1. \( X^{(1,1)}_\ell \), where \( X_\ell \) is of type \( A_\ell, B_\ell, C_\ell, D_\ell, E_\ell, F_4 \) or \( G_2 \):
\[ R = \{ \alpha + m\delta_0 + n\delta_1 | \alpha \in R(X_\ell), n, m \in \mathbb{Z} \}. \]

2. \( X^{(1,t)}_\ell \), where \( t = 2 \) for \( X_\ell = B_\ell, C_\ell \) and \( F_4 \), and \( t = 3 \) for \( X_\ell = G_2 \):
\[ R = \{ \alpha + m\delta_0 + n\delta_1 | \alpha \in R(X_\ell) \text{ short, } n, m \in \mathbb{Z} \} \]
\[ \cup \{ \alpha + m\delta_0 + tn\delta_1 | \alpha \in R(X_\ell) \text{ long, } n, m \in \mathbb{Z} \}. \]

3. \( X^{(t,t)}_\ell \), where \( t = 2 \) for \( X_\ell = B_\ell, C_\ell \) and \( F_4 \), and \( t = 3 \) for \( X_\ell = G_2 \):
\[ R = \{ \alpha + m\delta_0 + n\delta_1 | \alpha \in R(X_\ell) \text{ short, } n, m \in \mathbb{Z} \} \]
\[ \cup \{ \alpha + tm\delta_0 + tn\delta_1 | \alpha \in R(X_\ell) \text{ long, } n, m \in \mathbb{Z} \}. \]
4. $A^{(1,1)*}_1$:

$$R = \{ \alpha + m\delta_0 + n\delta_1 | \alpha \in R(A_1), \ n, m \in \mathbb{Z}, \ nm \in 2\mathbb{Z} \}.$$ 

5. $B^{(2,2)*}_t$:

$$R = \{ \alpha + m\delta_0 + n\delta_1 | \alpha \in R(B_t) \text{ short, } n, m \in \mathbb{Z}, \ nm \in 2\mathbb{Z} \}
\cup \{ \alpha + 2m\delta_0 + 2n\delta_1 | \alpha \in R(B_t) \text{ long, } n, m \in \mathbb{Z} \}.$$ 

6. $C^{(1,1)*}_t$:

$$R = \{ \alpha + m\delta_0 + n\delta_1 | \alpha \in R(C_t) \text{ long, } n, m \in \mathbb{Z}, \ nm \in 2\mathbb{Z} \}
\cup \{ \alpha + m\delta_0 + n\delta_1 | \alpha \in R(C_t) \text{ short, } n, m \in \mathbb{Z} \}.$$ 

7. $BC^{(2,1)}_t$:

$$R = \{ \alpha + m\delta_0 + n\delta_1 | \alpha \in R(B_t), \ n, m \in \mathbb{Z} \}
\cup \{ \alpha + (2m + 1)\delta_0 + n\delta_1 | \alpha \in R(C_t) \text{ long, } n, m \in \mathbb{Z} \}.$$ 

8. $BC^{(2,2)}(1)_t$:

$$R = \{ \alpha + m\delta_0 + n\delta_1 | \alpha \in R(B_t), \ n, m \in \mathbb{Z} \}
\cup \{ \alpha + (2m + 1)\delta_0 + 2n\delta_1 | \alpha \in R(C_t) \text{ long, } n, m \in \mathbb{Z} \}.$$ 

9. $BC^{(2,2)}(2)_t$:

$$R = \{ \alpha + m\delta_0 + n\delta_1 | \alpha \in R(B_t) \text{ short, } n, m \in \mathbb{Z} \}
\cup \{ \alpha + m\delta_0 + 2n\delta_1 | \alpha \in R(B_t) \text{ long, } n, m \in \mathbb{Z} \}
\cup \{ \alpha + (2m + 1)\delta_0 + 2n\delta_1 | \alpha \in R(C_t) \text{ long, } n, m \in \mathbb{Z} \}.$$ 

10. $BC^{(2,4)}_t$:

$$R = \{ \alpha + m\delta_0 + n\delta_1 | \alpha \in R(B_t) \text{ short, } n, m \in \mathbb{Z} \}
\cup \{ \alpha + m\delta_0 + 2n\delta_1 | \alpha \in R(B_t) \text{ long, } n, m \in \mathbb{Z} \}
\cup \{ \alpha + (2m + 1)\delta_0 + 4n\delta_1 | \alpha \in R(C_t) \text{ long, } n, m \in \mathbb{Z} \}.$$ 

11. $X^{(t,1)}_t$, where $t = 2$ for $X_t = B_t$, $C_t$ and $F_4$, and $t = 3$ for $X_t = G_2$:

$$R = \{ \alpha + m\delta_0 + n\delta_1 | \alpha \in R(X_t) \text{ short, } n, m \in \mathbb{Z} \}
\cup \{ \alpha + tm\delta_0 + n\delta_1 | \alpha \in R(X_t) \text{ long, } n, m \in \mathbb{Z} \}.$$ 

If we forget the markings, the root systems of type $X^{(1,1)}_t$ are isomorphic to the systems of type $X^{(1,1)}_t$. 
3 Toroidal Algebras

Lie algebras corresponding to the extended affine root systems of type $X^{(1,1)}_{\ell}$ can be easily constructed as follows. Let $\hat{g}$ be a simple finite-dimensional complex Lie algebra with $(\cdot, \cdot)$ the symmetric non-degenerate invariant Killing form. Let $\hat{R}$ be its root system. Choose an integer $N \geq 1$ and consider the tensor product $\tilde{g} = \hat{g} \otimes R$ of $\hat{g}$ with the algebra of Laurent polynomials in $N + 1$ variables:

$$\mathcal{R} = \mathbb{C}[t_0^{\pm 1}, t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$$

The toroidal Lie algebra corresponding to $\hat{g}$ is the universal central extension of $\tilde{g}$. The explicit construction of this extension, which we will present now, is known from the papers [22], [13], see also [3]. Let $K = \Omega^1_{\mathcal{R}}/d\mathcal{R}$ be the space of 1-forms modulo the exact forms. We write $f dg$ for the element of $K$ corresponding to the pair of elements $f, g$ from $\mathcal{R}$ and denote $k_i = t_i^{-1} dt_i$. Thus, $K$ is spanned by elements of the form

$$t^m k_i = t_0^{m_0} t_1^{m_1} \cdots t_N^{m_N} k_i, \quad 0 \leq i \leq N,$$

where $m = (m_0, m_1, ..., m_N) \in \mathbb{Z}^{N+1}$. Exactness implies that these elements are related by

$$\sum_{p=0}^{N} m_p t^m k_p = 0, \quad m \in \mathbb{Z}^{N+1}. \quad (3.1)$$

Then the toroidal Lie algebra is the vector space $\hat{g} = \tilde{g} \oplus K$ with Lie bracket:

$$[g_1 \otimes f_1(t), g_2 \otimes f_2(t)] = [g_1, g_2] \otimes f_1(t) f_2(t) + (g_1, g_2) f_2 d(f_1). \quad (3.2)$$

Now, if $N = 1$, this clearly gives a Lie algebra whose root system is of type $X^{(1,1)}_{\ell}$, viz., the root space corresponding to $\delta_0 + \alpha = \hat{\delta}_0 + \delta_1 + \alpha$, with $\alpha \in \hat{R}$ is $\hat{g}_{\delta_0 + \delta_1 + \alpha} = \hat{g}_{\delta_0} \otimes t_1^m t_2^n$. It is sometimes useful to add certain outer derivations to the algebra $\hat{g}$. To do that, we consider the following algebra of derivations:

$$\mathcal{D} = \sum_{p=0}^{N} \mathcal{R} d_p, \quad (3.3)$$

where $d_j = t_j \frac{\partial}{\partial t_j}$. These derivations extend to derivations of the Lie algebra $\hat{g} \otimes \mathcal{R}$. Since, $\hat{g}$ is the universal central extension of $\tilde{g}$, we can lift these derivations to this universal central extension by using a result of [3]. The action of vector fields on functions and the Lie derivative action of vector fields on 1-forms leads to the following action of $\mathcal{D}$ on $\hat{g}$:

$$[t^m d_j, g \otimes t^r] = r_j g \otimes t^{m+r},$$

$$[t^m d_j, t^r k_i] = r_j t^{m+r} k_i + \delta_{ji} \sum_{p=0}^{N} m_p t^{m+r} k_p. \quad (3.4)$$

The formulas (3.4) determine the Lie product on $\mathcal{D}$ up to a $K$-valued 2-cocycle $\tau \in H^2(\mathcal{D}, K)$:

$$[t^m d_i, t^r d_j] = r_i t^{m+r} d_j - m_j t^{m+r} d_i + \tau(t^m d_i, t^r d_j). \quad (3.5)$$
From the results of [10], any cocycle on $\mathcal{D}$ with values in $\mathcal{K}$ is a linear combination of

$$
\tau_1(t^m d_i, t^r d_j) = m_j r_j \sum_{p=0}^{N} r_p t^{m+r} k_p = -m_j r_j \sum_{p=0}^{N} m_p t^{m+r} k_p,
$$

and

$$
\tau_2(t^m d_i, t^r d_j) = m_i r_j \sum_{p=0}^{N} m_0 t^{m+r} k_p.
$$

So we obtain the two-parametric family of algebras of $\mathfrak{g}$:

$$
\mathfrak{g}^D = \mathfrak{g}_\tau = \hat{\mathfrak{g}} \oplus \mathcal{D} = \hat{\mathfrak{g}} \otimes \mathcal{R} \oplus \mathcal{K} \oplus \mathcal{D}, \quad \text{where } \tau = \mu \tau_1 + \nu \tau_2.
$$

We denote by $\mathfrak{g}$ the following subalgebra of $\mathfrak{g}^D$:

$$
\mathfrak{g} = \hat{\mathfrak{g}} \oplus \mathcal{D} \oplus \bigoplus_{j=0}^{N} \mathbb{C} d_j = \hat{\mathfrak{g}} \otimes \mathcal{R} \oplus \mathcal{K} \oplus \bigoplus_{j=0}^{N} \mathbb{C} d_j.
$$

Note that the advantage of this larger Lie algebra is that the center of the algebra $\mathfrak{g}$ and $\mathfrak{g}^D$ is finite-dimensional and is spanned by $k_0, k_1, \ldots, k_N$, whereas the center of $\hat{\mathfrak{g}}$ is infinite dimensional.

Let $\Sigma = \{\sigma_0, \sigma_1, \ldots, \sigma_N\}$ be a collection of finite order automorphisms of $\hat{\mathfrak{g}}$. N.B., we do not assume that all $\sigma_j$ are different and we allow $\sigma_j$ to be the identity. Let $n_j$ be the order of $\sigma_j$, i.e., $\sigma_j^{n_j} = 1$ for the smallest positive integer $n_j$ and denote by $\epsilon_j = \exp \frac{2\pi i}{n_j}$. Then every $\sigma_j$ is diagonalizable and one can decompose $\hat{\mathfrak{g}}$ in eigenspaces for the eigenvalues $\epsilon_j^k$, $k \in \mathbb{Z}/n_j \mathbb{Z}$. Assume from now on that all $\sigma_j$, $0 \leq j \leq N$ are simultaneously diagonalizable, i.e., one has the following eigenspace decomposition of $\hat{\mathfrak{g}}$. Let $\mathbb{Z}$ be the Cartesian product

$$
\mathbb{Z} = \mathbb{Z}/n_0 \mathbb{Z} \times \mathbb{Z}/n_1 \mathbb{Z} \times \mathbb{Z}/n_2 \mathbb{Z} \times \cdots \times \mathbb{Z}/n_N \mathbb{Z},
$$

then

$$
\hat{\mathfrak{g}} = \bigoplus_{(k_0, k_1, \ldots, k_N) \in \mathbb{Z}} \hat{\mathfrak{g}}(k_0, k_1, \ldots, k_N), \quad \text{where}
$$

$$
\hat{\mathfrak{g}}(k_0, k_1, \ldots, k_N) = \{g \in \hat{\mathfrak{g}} | \sigma_j(g) = \epsilon_j^{k_j} g \text{ for all } 0 \leq j \leq N\}.
$$

The Killing form $(\cdot, \cdot)$ is $\text{Aut} \hat{\mathfrak{g}}$-invariant, hence for every $0 \leq j \leq N$ and all $x \in \hat{\mathfrak{g}}(k_0, k_1, \ldots, k_N)$ and $y \in \hat{\mathfrak{g}}(\ell_0, \ell_1, \ldots, \ell_N)$:

$$
(x, y) = (\sigma_j(x), (\sigma_j(y)) = \epsilon_j^{k_j + \ell_j} (x, y),
$$

from which we conclude part (a) of the following Lemma:

Lemma 3.1 (a) Let $(\cdot, \cdot)$ be the Killing form on $\hat{\mathfrak{g}}$, then

$$
(\hat{\mathfrak{g}}(k_0, k_1, \ldots, k_N), \hat{\mathfrak{g}}(\ell_0, \ell_1, \ldots, \ell_N)) = 0 \quad \text{if } (k_0 + \ell_0, k_1 + \ell_1, \ldots, k_N + \ell_N) \neq (0, 0, \ldots, 0) \in \mathbb{Z}.
$$

(b) The subalgebra $\hat{\mathfrak{g}}(0, 0, \ldots, 0)$ is reductive.
Proof Part (b) of the Lemma is a direct consequence of part (a), the fact that the Killing form is nondegenerate and Proposition 5 in §6.4 of [8]. □

This simple observation makes it possible to define twisted toroidal subalgebras of a toroidal algebra. This construction, which we shall give now, is similar to the one that produces the twisted affine Lie algebras (see [21], Chapter 8). But before we can do that, we will first introduce one more notation. Let \( m = (m_0, m_1, \ldots, m_N) \in \mathbb{Z}^{N+1} \), then we write \( \tilde{m} = (m_0 \mod n_0, m_1 \mod n_1, \ldots, m_N \mod n_N) \in \mathbb{Z}^N \).

Fix \( \Sigma \), we define the subalgebra \( \tilde{\mathfrak{g}}(\Sigma) \) of \( \tilde{\mathfrak{g}} \) by
\[
\tilde{\mathfrak{g}}(\Sigma) = \bigoplus_{\mathfrak{m} \in \mathbb{Z}^{N+1}} \mathfrak{g}_{\mathfrak{m}} \otimes t^\mathfrak{m}. \tag{3.9}
\]

Using Lemma [3.1], one easily sees that one gets a subalgebra of \( \hat{\mathfrak{g}} \), which is a central extension of \( \tilde{\mathfrak{g}} \), if we add the subspace \( K(\Sigma) \subset K \), which is spanned by elements of the form
\[
t_0^{m_0} t_1^{m_1} \ldots t_N^{m_N} k_i, \quad 0 \leq i \leq N,
\]
where of course the relation (3.1) still holds. So define the following subalgebra of \( \hat{\mathfrak{g}} \)
\[
\hat{\mathfrak{g}}(\Sigma) = \tilde{\mathfrak{g}}(\Sigma) \oplus K(\Sigma), \tag{3.10}
\]
with Lie bracket on this algebra still defined by (3.2). We can extend this twisted algebra with an algebra of derivations, however, except when all automorphisms are the identity, not with \( D \), but with a subalgebra of \( D \). Let
\[
D(\Sigma) = \sum_{p=0}^N \mathcal{R}(\Sigma) d_p, \quad \text{where} \quad \mathcal{R}(\Sigma) = \mathbb{C}[t_0^{\pm n_0}, t_1^{\pm n_1}, \ldots t_N^{\pm n_N}], \tag{3.11}
\]
define a subalgebra \( \mathfrak{g}^D(\Sigma) \) of \( \mathfrak{g}^D \) and a subalgebra \( \mathfrak{g}(\Sigma) \) of \( \mathfrak{g} \) by
\[
\mathfrak{g}^D(\Sigma) = \mathfrak{g}^D(\Sigma) = \hat{\mathfrak{g}}(\Sigma) \oplus D(\Sigma) = \hat{\mathfrak{g}}(\Sigma) \oplus K(\Sigma) \oplus D(\Sigma),
\]
\[
\mathfrak{g}(\Sigma) = \hat{\mathfrak{g}}(\Sigma) \oplus \bigoplus_{j=0}^N \mathbb{C} d_j = \hat{\mathfrak{g}}(\Sigma) \oplus K(\Sigma) \oplus \bigoplus_{j=0}^N \mathbb{C} d_j,
\]
where the Lie bracket is still defined by (3.2), (3.4) and (3.5).

Let \( \mathfrak{h}_{\mathfrak{g}} \) be the Cartan subalgebra of \( \mathfrak{g}(\overline{\mathfrak{m}}, \ldots, \overline{\mathfrak{m}}) \), then
\[
\mathfrak{h} = \mathfrak{h}_{\mathfrak{g}} \oplus \mathbb{C} d_0 \oplus \mathbb{C} d_1 \oplus \cdots \oplus \mathbb{C} d_N \oplus \mathbb{C} k_0 \oplus \mathbb{C} k_1 \oplus \cdots \oplus \mathbb{C} k_N
\]
is the Cartan subalgebra of \( \mathfrak{g}^D(\Sigma) \) and \( \mathfrak{g}(\Sigma) \). We extend \( \lambda \in \mathfrak{h}_{\mathfrak{g}}^* \) to a linear function on \( \mathfrak{h} \) by setting \( \lambda(d_i) = \lambda(k_i) = 0 \), for all \( 0 \leq i \leq N \). Denote by \( \delta, \kappa \) the linear function on \( \mathfrak{h} \) defined by
\[
\delta_i(\mathfrak{h}_{\mathfrak{g}}) = 0, \quad \delta_i(d_j) = \delta_{ij}, \quad \delta_i(k_j) = 0,
\]
\[
\kappa_i(\mathfrak{h}_{\mathfrak{g}}) = 0, \quad \kappa_i(d_j) = 0, \quad \kappa_i(k_j) = \delta_{ij}.
\]
The Killing form restricted to $\mathfrak{h}_0$ remains nondegenerate and can be extended to a non-degenerate symmetric bilinear form on $\mathfrak{h}$

$$(k_i, k_j) = 0, \quad (d_i, k_j) = \delta_{ij}, \quad (d_i, d_j) = 0, \quad (k_i, k_j) = (d_i, k_j) = 0.$$ 

This form defines an isomorphism $\mathfrak{h} \to \mathfrak{h}^*$ by

$$\nu(h)(h') = (h, h'), \quad h, h' \in \mathfrak{h}_0$$

and hence a bilinear form on $\mathfrak{h}^*$, viz.,

$$(\alpha, \beta) = (\nu^{-1}(\alpha), \nu^{-1}(\beta)).$$

One thus has

$$(\kappa_i, \kappa_j) = 0, \quad (\delta_i, \kappa_j) = \delta_{ij}, \quad (\delta_i, \delta_j) = 0, \quad (\kappa_i, \mathfrak{h}_0^*) = (\delta_i, \mathfrak{h}_0^*) = 0.$$ 

Then, $\mathfrak{g}(\Sigma)$ decomposes with respect to $\mathfrak{h}^*$ into

$$\mathfrak{g}(\Sigma) = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}(\Sigma)_\alpha.$$ 

Let

$$\Delta = \{ \alpha \in \mathfrak{h}^*| \mathfrak{g}(\Sigma)_\alpha \neq \{0\} \},$$

be the set of roots of $\mathfrak{g}(\Sigma)$ then we have the following root space decomposition

$$\mathfrak{g}(\Sigma) = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}(\Sigma)_\alpha.$$ 

The connection with the extended affine root system of Section 2 is as follows. The linear space $V$ in Definition 2.1, is the subspace

$$V = \mathfrak{h}_0^* \oplus \bigoplus_{i=0}^N \mathbb{C}\delta_i,$$

and the bilinear form of the definition is the restriction $(\cdot, \cdot)_V$ of $(\cdot, \cdot)$ to $V$. One can decompose $\mathfrak{g}(\Sigma)$ with respect to $V$ into

$$\mathfrak{g}(\Sigma) = \bigoplus_{\alpha \in V} \mathfrak{g}(\Sigma)_\alpha.$$ 

Let

$$\overline{R} = \{ \alpha \in V| \mathfrak{g}(\Sigma)_\alpha \neq \{0\} \},$$

then

$$\overline{R} = R \cup R^0, \quad \text{where} \quad R = \{ \alpha \in \overline{R}|(\alpha, \alpha)_V \neq 0 \} \quad \text{and} \quad R^0 = \overline{R} \cap V^0$$

and thus

$$\mathfrak{g}(\Sigma) = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}(\Sigma)_\alpha \oplus \bigoplus_{\alpha \in R^0} \mathfrak{g}(\Sigma)_\alpha.$$ 

Note that there is one problem, for general $\Sigma$, it is not clear that the set $R$ satisfies the axioms of Definition 2.1. In the next section we choose $N = 1$ and list pairs $\hat{\mathfrak{g}}$, $\Sigma$ which give the extended affine Lie algebras that correspond to the extended affine root systems of Saito [27], i.e., to the ones that were presented in section 3.
4 Bi-affine Lie Algebras

In this section we construct the bi-affine Lie algebras, i.e., the twisted 2-toroidal Lie algebras corresponding to the extended affine root systems of Saito [27], which were presented in Section 2. So we assume from now on in this section that $N = 1$. In most cases we will explain how we realize $\hat{g}$, this will however not always be the same, e.g. the Lie algebra of type $D_\ell$ will be realized in different ways.

4.1 Type $X^{(1,1)}_\ell$

Bi-affine Lie algebras of type $X^{(1,1)}_\ell$, can be easily constructed. One takes in the construction of section 3 for $\hat{g}$ the simple Lie algebra of type $X^{(1,1)}_\ell$ and as automorphisms $\sigma_0 = \sigma_1 = \text{id}$.

4.2 Type $X^{(1,t)}_\ell$

The description of the bi-affine Lie algebras of type $X^{(1,t)}_\ell$ is also easy. For $X_\ell$ equal to $B_\ell$, $C_\ell$, $F_4$ and $G_2$, one takes as $\hat{g}$ the simple Lie algebra of type $D_{\ell+1}$, $A_{2\ell-1}$, $E_6$, $D_4$, respectively. One chooses for $\sigma_0$ the identity and for $\sigma_1$ the automorphisms, described in §8 of [21], which are induced by a diagram automorphism. The order of $\sigma_2$ is $t$, which is equal to 2, 2, 2, 3, respectively.

4.3 Type $X^{(t,t)}_\ell$

We start this subsection with the $X_\ell = B_\ell$ and $t = 2$. We take as $\hat{g}$ the Lie algebra of type $D_{\ell+2}$. Let $M_n$ be the linear space of all complex $n \times n$-matrices. We realize $\hat{g}$ as

$$\hat{g} = \{ X \in M_{2\ell+4} | X^T = -X \}, \quad (4.1)$$

where $X^T$ stands for the transposed of the matrix $X$. Let $E_{ij}$ be the matrix with a 1 on the $(i,j)$-th entry and zeros elsewhere. Now choose $\Sigma$ as follows

$$\sigma_0 = \text{Ad} \left( -E_{2\ell+3,2\ell+3} - E_{\ell+4,\ell+4} + \sum_{i=1}^{2\ell+2} E_{ii} \right),$$

$$\sigma_1 = \text{Ad} \left( -E_{2\ell+2,2\ell+2} + E_{2\ell+3,2\ell+3} - E_{2\ell+4,2\ell+4} + \sum_{i=1}^{2\ell+1} E_{ii} \right).$$

The subalgebra $\hat{g}(0,0)$ is the simple Lie algebra of type $B_\ell$. All three other spaces consist of the direct sum of a 1-dimensional trivial and a $2\ell+1$-dimensional irreducible representation of $B_\ell$.

Next we take $X_\ell = C_\ell$ and $t = 2$. In this case $\hat{g}$ is the Lie algebra of type $D_{2\ell}$, which we realize as

$$\hat{g} = \left\{ \left( \begin{array}{cc} a & b \\ c & -a^T \end{array} \right) \in M_{4\ell} | b^T = -b, \ c^T = -c \right\}. \quad (4.2)$$
The automorphisms are defined as follows
\[
\sigma_0 = \text{Ad} \left( \sum_{i=1}^{4\ell} (-)^i E_{i, 4\ell+1-i} \right),
\]
\[
\sigma_1 = \text{Ad} \left( \sum_{i=1}^{2\ell} (-)^i E_{i, 4\ell+1-i} - (-)^i E_{2\ell+i, 2\ell+1-i} \right).
\]
(4.3)

Here \( \hat{\mathfrak{g}}_{(\ell)} \) is the simple Lie algebra of type \( C_\ell \). All three other spaces consist of the direct sum of a 1-dimensional trivial and a \( 2\ell^2 - \ell - 1 \)-dimensional irreducible representation of \( B_\ell \).

Assum now that \( X_\ell = F_4 \), then \( t = 2 \). Now \( \hat{\mathfrak{g}} \) is the Lie algebra of type \( E_7 \), and assume that the roots are labeled as "Planche VI" in [1]. Let \( e_i, f_i, 1 \leq i \leq 7 \) be the Chevalley generators corresponding to these roots. We define both automorphisms on these generators.

\[
\sigma_0(e_i) = e_i, \quad \sigma_0(f_i) = f_i \quad \text{for } i = 2, 4,
\]
\[
\sigma_0(e_1) = e_6, \quad \sigma_0(f_1) = f_6,
\]
\[
\sigma_0(e_3) = e_5, \quad \sigma_0(f_3) = f_5,
\]
\[
\sigma_0(e_5) = e_3, \quad \sigma_0(f_5) = f_3,
\]
\[
\sigma_0(e_6) = e_1, \quad \sigma_0(f_6) = f_1,
\]
\[
\sigma_0(e_7) = e_{-\theta} = [f_1 f_3 f_4 f_5 f_6 f_7], \quad \sigma_0(f_7) = e_\theta.
\]

Then with respect to these two automorphisms \( \hat{\mathfrak{g}} \) splits into the following spaces \( \hat{\mathfrak{g}}_{(\ell)} \), \( \hat{\mathfrak{g}}_{(\ell)} \) and \( \hat{\mathfrak{g}}_{(\ell)} \) is the direct sum of a 1-dimensional trivial representation and the 26-dimensional irreducible representation of \( F_4 \). This was checked using \textit{GAP} – Groups, Algorithms, and Programming [1].

Finally, we describe \( G_2^{(3,3)} \). For this case \( \hat{\mathfrak{g}} \) is the Lie algebra of type \( E_6 \). We define the automorphisms again on the Chevalley generators, wherever the roots are numbered as in "Planche V" of [1]. Let \( \theta \) be the highest root of \( E_6 \), now define
\[
\sigma_0(e_1) = e_6, \quad \sigma_0(f_1) = f_6,
\]
\[
\sigma_0(e_2) = e_3, \quad \sigma_0(f_2) = f_3,
\]
\[
\sigma_0(e_3) = e_5, \quad \sigma_0(f_3) = f_5,
\]
\[
\sigma_0(e_4) = e_4, \quad \sigma_0(f_4) = f_4,
\]
\[
\sigma_0(e_5) = e_2, \quad \sigma_0(f_5) = f_2,
\]
\[
\sigma_0(e_6) = e_{-\theta} = [f_2 f_3 f_4 f_5 f_6 f_7], \quad \sigma_0(f_7) = e_\theta,
\]
which is an automorphism of order 3. Let \( \omega = e^{2\pi i/3} \), define \( \sigma_1 \) as follows:
\[
\sigma_1(e_i) = e_i, \quad \sigma_1(f_i) = f_i \quad \text{for } i \neq 1, 6,
\]
\[
\sigma_1(e_i) = \omega e_i, \quad \sigma_1(f_i) = \omega^2 f_i \quad \text{for } i = 1, 6.
\]

The 78-dimensional Lie algebra \( E_6 \) splits with respect to these automorphisms as follows: \( \hat{\mathfrak{g}}_{(\ell)} \) is the simple Lie algebra of type \( G_2 \), all 8 other spaces \( \hat{\mathfrak{g}}_{(i,j)} \), \( 0 \leq i, j \leq 2 \), \( (i, j) \neq (0, 0) \) is the direct sum of the irreducible 7-dimensional representation and a 1-dimensional trivial representation of \( G_2 \). This example was also checked using \textit{GAP} [1].
4.4 Type $A^{(1,1)*}_1$

We will not describe the bi-affine Lie algebra of this type here, but instead we will obtain it as a special case of the construction of the next section.

4.5 Type $B^{(2,2)*}_\ell$

The construction of the bi-affine Lie algebra of type $B^{(2,2)*}_\ell$ goes as follows. The algebra $\dot{g}$ is the simple Lie algebra of type $B_{\ell+1}$, which we realize as the space of anti-symmetric matrices (cf. (4.1)):

$$\dot{g} = \{ X \in M_{2\ell+3} | X^T = -X \}.$$

The automorphisms are taken as follows:

$$\sigma_0 = \text{Ad} \left( -E_{2\ell+3,2\ell+3} + \sum_{i=1}^{2\ell+2} E_{ii} \right),$$
$$\sigma_1 = \text{Ad} \left( -E_{2\ell+2,2\ell+2} + E_{2\ell+3,2\ell+3} + \sum_{i=1}^{2\ell+1} E_{ii} \right).$$

If we take $\ell = 1$, we obtain the bi-affine Lie algebra of type $A^{(1,1)*}_1$.

Here $\dot{g}_{(0,0)}$ is the simple Lie algebra of type $B_\ell$. The spaces $\dot{g}_{(0,T)}$ and $\dot{g}_{(T,0)}$ consist of a $2\ell + 1$-dimensional irreducible representation of $B_\ell$ and $\dot{g}_{(T,T)}$ is one dimensional.

4.6 Type $C^{(1,1)*}_\ell$

To obtain the bi-affine Lie algebra of type $C^{(1,1)*}_\ell$ we choose $\dot{g}$ to be the simple Lie algebra of type $C_{2\ell}$. This Lie algebra is realized as follows (cf. (4.2)):

$$\dot{g} = \left\{ \begin{pmatrix} a & b \\ c & -a^T \end{pmatrix} \in M_{4\ell} | b^T = b, c^T = c \right\}.$$

The automorphisms $\sigma_0$ and $\sigma_1$ are defined by (4.3). Now, $\dot{g}_{(0,0)}$ is the simple Lie algebra of type $C_\ell$. Both spaces $\dot{g}_{(0,T)}$ and $\dot{g}_{(T,0)}$ are the adjoint representation and $\dot{g}_{(T,T)}$ consist of the direct sum of a $2\ell^2 - \ell - 1$-dimensional irreducible representation and a one dimensional one.

4.7 Type $BC^{(2,1)}_\ell$

This type can be described in the same way as the examples of Section 4.2. Here $\dot{g}$ is the Lie algebra of type $A_{2\ell}$. Let $\sigma_0$ be the automorphism, described in §8 of [21], which produces the affine Lie algebra of type $A^{(2)}_{2\ell}$, for $\sigma_1$ we choose the identity.

4.8 Type $BC^{(2,2)}_\ell(1)$

The bi-affine algebra of type $BC^{(2,1)}_\ell$ is constructed by taking as $\dot{g}$ the simple Lie algebra of type $D_{2\ell+1}$, realized as $\dot{g} = \{ X \in M_{4\ell+2} | X^T = -X \}$. 
In this case $\Sigma$ is given by
\[
\sigma_0 = \text{Ad} \left( \sum_{i=1}^{4\ell+2} (-)^i E_{i,4\ell+3-i} \right),
\]
\[
\sigma_1 = \text{Ad} \left( \sum_{i=1}^{2\ell+1} E_{i,i} - E_{2\ell+1+i,2\ell+1+i} \right).
\]
In this example is $\hat{\mathfrak{g}}_{(0,0)}$ the simple Lie algebra of type $B_\ell$. Both spaces $\hat{\mathfrak{g}}_{(0,1)}$ and $\hat{\mathfrak{g}}_{(1,0)}$ are isomorphic to the ones of type $A_\ell$ and a one dimensional trivial one.

4.9 Type $BC^{(2,2)}_\ell(2)$

Type $BC^{(2,2)}_\ell(2)$ can be obtained by choosing $\hat{\mathfrak{g}}$ the simple Lie algebra of type $A_{2\ell+1}$. We realize this Lie algebra in the usual way as the traceless $(2\ell + 2) \times (2\ell + 2)$-matrices. $\sigma_0$ is the Cartan involution
\[
\sigma_0(X) = -X^T \quad \text{and} \quad \sigma_1 = \text{Ad} \left( -E_{2\ell+2,2\ell+2} + \sum_{i=1}^{2\ell+1} E_{i,i} \right).
\]
For the decomposition of this Lie algebra we refer the reader to the second example of Section 5.

4.10 Type $BC^{(2,4)}_\ell$

Finally, the bi-affine Lie algebra of type $BC^{(2,4)}_\ell$ is constructed as follows. The algebra $\hat{\mathfrak{g}}$ is the Lie algebra of type $D_{2\ell+2}$, which we realize as in (4.2), but then with $4\ell$ replaced by $4\ell + 4$. The automorphism $\sigma_0$ is the involution
\[
\sigma_0 = \text{Ad} \left( E_{2\ell+2,4\ell+4} - E_{4\ell+4,2\ell+2} + \sum_{j=1}^{2\ell+1} (-)^j E_{4\ell+4-j,j} - (-)^j E_{2\ell+2-j,2\ell+2+j} \right).
\]
The other automorphism is an automorphism of order 4:
\[
\sigma_1 = \text{Ad} \left( iE_{2\ell+2,2\ell+4} + iE_{2\ell+4,2\ell+2} + \sum_{j=1}^{2\ell+1} E_{j,j} - E_{2\ell+2+j,2\ell+2+j} \right).
\]
The decomposition of $D_{2\ell+2}$ with respect to the automorphisms is the most complicated one. The subalgebra $\hat{\mathfrak{g}}_{(0,0)}$ the simple Lie algebra of type $B_\ell$. The spaces $\hat{\mathfrak{g}}_{(i,j)}$, where $(i,j) = (0,1), (1,1), (0,3)$ or $(1,3)$ are all irreducible $2\ell + 1$ dimensional representations of $B_\ell$. Next, $\hat{\mathfrak{g}}_{(1,0)}$ is the direct sum of a one dimensional and the $2\ell^2 + 3\ell$-dimensional irreducible representation of $B_\ell$; $\hat{\mathfrak{g}}_{(0,1)}$ is the ad-module (the adjoint representation) and $\hat{\mathfrak{g}}_{(1,2)}$ is again the direct sum of two irreducible modules, viz. the ad-module and a one dimensional trivial one.

This produces all bi-affine Lie algebras related to Saito’s list. The ones of type $X^{(t,1)}_\ell$ are isomorphic to the ones of type $X^{(1,t)}_\ell$. The former can be constructed by interchanging the automorphisms $\sigma_1$ and $\sigma_2$ in the construction of latter. Except for two cases, viz., $A^{(1,1)*}_1$ and $C^{(1,1)*}_\ell$, the tier numbers, introduced by Saito, which are the upper indices in $X^{(s,t)}_\ell$, equal the order of the automorphisms used to construct the bi-affine lie algebras.
5 Some quasi-simple Lie algebras

In 1990 Hoegh-Krohn and Torresani [14] classified and constructed certain quasi-simple Lie algebras. These are characterized by the existence of a finite-dimensional Cartan subalgebra, a nondegenerate invariant symmetric bilinear form and nilpotent root spaces attached to non-isotropic roots. They derive a classification for the possible irreducible elliptic quasi-simple root systems. Obviously they were not aware of the existence of the paper of Saito [27], which appeared 5 years earlier. In the case when the nullity is equal to 2, their list lacked some of the cases Saito obtained. According to the introduction of [1], this was caused by the fact that they assumed or erroneously concluded that the theory governing the isotropic roots was based on lattices. Two of their examples were the root systems

\[ R(X_\ell)(t,t,\ldots,t,1,1,\ldots,1) = \{ \alpha + \sum_{j=0}^{n} k_j \delta_j | \alpha \in R(X_\ell) \text{ short}, k_j \in \mathbb{Z} \} \]

\[ \cup \{ \alpha + \sum_{j=0}^{m} tk_j \delta_j + \sum_{j=m+1}^{n} k_j \delta_j | \alpha \in R(X_\ell) \text{ long}, k_j \in \mathbb{Z} \}, \]

with \( X_\ell = B_\ell, C_\ell \) and \( F_4 \), and \( X_\ell = G_2 \) and \( t = 2, 2, 2, 3 \), respectively and \( 0 \leq m \leq n \);

\[ R(BC_\ell)(2,2,\ldots,2,1,\ldots,1)(2) = \{ \alpha + \sum_{j=0}^{n} k_j \delta_j | \alpha \in R(B_\ell) \text{ short}, k_j \in \mathbb{Z} \}

\[ \cup \{ \alpha + k_0 \delta_0 + \sum_{j=1}^{m+1} 2k_j \delta_j + \sum_{j=m+2}^{n} k_j \delta_j | \alpha \in R(B_\ell) \text{ long}, k_j \in \mathbb{Z} \} \]

\[ \cup \{ \alpha + (2k_0 + 1) \delta_0 + \sum_{j=1}^{m+1} 2k_j \delta_j + \sum_{j=m+2}^{n} k_j \delta_j | \alpha \in R(C_\ell) \text{ long}, k_j \in \mathbb{Z} \}, \]

with \( 0 \leq m < n \).

Using the idea’s of Sections 4.2, 4.3, 4.9 we construct the corresponding extended affine algebra related to (5.1) for \( X_\ell = B_\ell \) and to (5.2). This shows that the approach of this paper, not only works for \( N = 1 \), but that it at least also produces some families of \( N \)-affine Lie algebras.

For the example related to (5.1) we take as \( \mathfrak{g} \) the Lie algebra of type \( D_{\ell+2m} \). We again realize this Lie algebra as the complex space of anti-symmetric \((2\ell + 2^{m+1}) \times (2\ell + 2^{m+1})\)-
matrices (cf. [11]). Let $J_0, J_1, \ldots, J_m$ be the following matrices

$$J_0 = \sum_{i=1}^{2\ell} E_{ii} + \sum_{j=1}^{2^m} (E_{2\ell+2j-1,2\ell+2j-1} - E_{2\ell+2j,2\ell+2j}),$$

$$J_1 = \sum_{i=1}^{2\ell} E_{ii} + \sum_{j=1}^{2^{m-1}} (E_{2\ell+4j-3,2\ell+4j-3} + E_{2\ell+4j-2,2\ell+4j-2} - E_{2\ell+4j-1,2\ell+4j-1} + E_{2\ell+4j,2\ell+4j}),$$

$$J_2 = \sum_{i=1}^{2\ell} E_{ii} + \sum_{j=1}^{2^{m-2}} (E_{2\ell+8j-7,2\ell+8j-7} + E_{2\ell+8j-6,2\ell+8j-6} + E_{2\ell+8j-5,2\ell+8j-5} + E_{2\ell+8j-4,2\ell+8j-4}$$

$$- E_{2\ell+8j-3,2\ell+8j-3} - E_{2\ell+8j-2,2\ell+8j-2} - E_{2\ell+8j-1,2\ell+8j-1} - E_{2\ell+8j,2\ell+8j}),$$

$$\vdots$$

$$J_m = \sum_{i=1}^{2\ell+2^m} E_{ii} - \sum_{j=1}^{2^m} E_{2\ell+2^m+j,2\ell+2^m+j},$$

then we define $\Sigma = (\sigma_0, \sigma_1, \ldots, \sigma_n)$ by

$$\sigma_k = \begin{cases} 
\text{Ad}(J_k) & \text{for } 0 \leq k \leq m, \\
\text{id} & \text{for } k > m.
\end{cases}$$

Then $\hat{\mathfrak{g}}_{(0,\overline{0},\ldots,\overline{0})}$ is the simple Lie algebra of type $B_\ell$ and all the $2^{m+1} - 1$ other spaces $\hat{\mathfrak{g}}_{(i_0,i_1,\ldots,i_m,\overline{i}_m,\ldots,\overline{i})}$ consist of the direct sum of the $2\ell+1$-dimensional irreducible representation together with $2^m - 1$ trivial 1-dimensional ones. Clearly the Lie algebra $\hat{\mathfrak{g}}(\Sigma)$ corresponds to the root system for $X_\ell = B_\ell$ which is given in [11].

For the second example, the one related to (5.2), we take as $\hat{\mathfrak{g}}$ the simple Lie algebra of type $A_{2\ell+2^{m+1}-1}$, which can be realized in the usual way as complex traceless $2\ell + 2^{m+1} \times 2\ell + 2^{m+1}$ matrices. For $\sigma_0$ we again take as $\hat{\mathfrak{g}}_{(1,\overline{1},\ldots,\overline{1})}$ the Cartan involution, i.e., $\sigma_0(X) = -X^T$. All the other automorphisms are defined as follows:

$$\sigma_k = \begin{cases} 
\text{Ad}(J_{k-1}) & \text{for } 1 \leq k \leq m + 1, \\
\text{id} & \text{for } k > m + 1.
\end{cases}$$

The algebra $\hat{\mathfrak{g}}_{(1,\overline{1},\ldots,\overline{1})}$ is the simple Lie algebra of type $B_\ell$. The space $\hat{\mathfrak{g}}_{(\overline{1},\overline{1},\ldots,\overline{1})}$ consists of the direct sum of a $2\ell^2 + 3\ell$-dimensional irreducible representation of $B_\ell$ together with $2^{m} - 1$ 1-dimensional trivial representations. All other $2^{m+2} - 2$ spaces $\hat{\mathfrak{g}}_{(\overline{i}_0,\overline{i}_1,\ldots,\overline{i}_{m+1},\overline{i}_m,\ldots,\overline{i})}$ consist of the direct sum of the $2\ell + 1$ irreducible representation of $B_\ell$ and $2^{m} - 1$ trivial 1-dimensional representations. This leads to a Lie algebra $\hat{\mathfrak{g}}(\Sigma)$, whose root system restricted to $V$ is given by (5.2).

References

[1] B.N. Allison, S. Azam, S. Berman, Y. Gao, A. Pianzola Extended Affine Lie Algebras and Their Root Systems, Memoirs of the AMS, 126, 603, 1997.

[2] S. Azam, Construction of Extended Affine Lie Algebras by the Twisting Process, Commun. Algebra, 28(6) (2000) 2753–2781.
REFERENCES

[3] S. Berman, Y. Billig, Irreducible Representations for Toroidal Lie algebras, *J. Algebra*, **221** (1999) 188–231.

[4] S. Berman, Y. Billig, J. Szmigielski, Vertex operator algebras and the representation theory of toroidal algebras, [math.QA/0101094](http://arxiv.org/abs/math.QA/0101094)

[5] G. Benkart, R. Moody, Derivations, central extensions and affine Lie algebras, *Algebras, Groups, Geometries*, **3** (1993) 456-492.

[6] Y. Billig, Principal vertex operator representations for toroidal Lie algebras, *J. of Math.Phys*, **39** (1998) 3844–3864.

[7] Y. Billig, An extension of the Korteweg-de Vries hierarchy arising from a representation of a toroidal Lie algebra, *J. Algebra*, **217**, (1999), no. 1, 40–64.

[8] N. Bourbaki Groupes et algèbres de Lie, Chapitre I, Hermann, Paris 1971.

[9] N. Bourbaki Groupes et algèbres de Lie, Chapitre 4,5 et 6, Hermann, Paris 1968.

[10] A. Dzhumadil’daev, Virasoro type Lie algebras and deformations. *Z. Phys. C*, **72** (1996) 509–517.

[11] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.2*, Aachen, St Andrews, 2000, (\protect\url{http://www-gap.dcs.st-and.ac.uk/~gap}).

[12] S. Eswara Rao, R. V. Moody, Vertex Representations for N-Toroidal Lie Algebras and a Generalization of the Virasoro Algebra. *Commun. Math. Phys.*, **159**(1994)239–264.

[13] S.Eswara Rao, R.V. Moody, T. Yokonuma, Toroidal Lie algebras and vertex representations Geom. Dedicata, **35**(1990)283–307.

[14] R. Hoegh–Krohn, B. Torresani Classification and Construction of Quasisimple Lie Algebras, *J. Funct. Anal.*, **89**, (1990) 106–136.

[15] T. Ikeda, K. Takasaki, Toroidal Lie algebras and Bogoyavlensky’s 2+1-dimensional equation, [nlin.SI/0004013](http://arxiv.org/abs/nlin.SI/0004013).

[16] T. Inami, H. Kanno, T. Ueno, C-S Xiong, Two-toroidal Lie algebra as current algebra of the four-dimensional Khler WZW model. Phys. Lett. B **399** (1997), no. 1-2, 97–104.

[17] T. Inami, H. Kanno, T. Ueno, Higher-dimensional WZW model on Khler manifold and toroidal Lie algebra. Modern Phys. Lett. A **12** (1997), no. 36, 2757–2764.

[18] K. Iohara, Y. Saito, M. Wakimoto, Hirota bilinear forms with 2-toroidal symmetry, *Phys. Lett. A*, **254**, (1999), no. 1-2, 37–46.

[19] K. Iohara, Y. Saito, M. Wakimoto, Notes on differential equations arising from a representation of 2-toroidal Lie algebras. *Gauge theory and integrable models (Kyoto, 1999). Progr. Theoret. Phys. Suppl. No.* **135** (1999), 166–181.

[20] V.G. Kac, Simple irreducible graded Lie algebras of finite growth *Izv. Akad. Nauk. SSSR*, **32**, (1968), 1271–1311.
[21] V.G. Kac, *Infinite-dimensional Lie algebras*, 3rd ed., Cambridge University Press, 1990.

[22] C. Kassel, Kähler differentials and coverings of complex simple Lie algebras extended over a commutative algebra. *Journ. Pure and Appl. Algebra*, 34 (1985) 256–275.

[23] E. Looijenga On the semi-universal deformation of a simple elliptic singularity II, *Topology*, 17 (1978) 23–40.

[24] E. Looijenga, Root systems and elliptic curves, *Inventiones Math.*, 38, (1976), 17–32.

[25] R.V. Moody A new class of Lie algebras, *Journal of Algebra*, 10, (1968), 211-230.

[26] U. Pollmann, Realisation der biaffen Wurzelsysteme von Saito in Lie–Algebren, *Hamburger Beiträge zur Mathematik*, Heft 29 (1994).

[27] K. Saito, Extended Affine Root Systems I (Coxeter transformations) *Publ. RIMS, Kyoto University*, 21 (1985) 75–179.

[28] K. Saito, D. Yoshii Extended Affine Root Systems IV (Simply–Laced Elliptic Lie algebras) *Publ. RIMS, Kyoto University*, 36 (2000) 385–421.

[29] P. Slodowy, A character approach to Looijenga’s invariant theory for generalized root systems, *Compositio Math.*, 55 (1985), no. 1, 3–32.

[30] P. Slodowy, Singularitäten, Kac–Moody–Liealgebren, assoziierte Gruppen und Verallgemeinerungen, *Habilitationsschrift* publication of the Max–Planck–Institut für Mathematik, Bonn (1984).

[31] M.Wakimoto, Extended affine Lie algebras and certain series of Hermitian representations, Preprint (1985).

[32] H. Yamada, Extended Affine Lie Algebras and their Vertex Representations *Publ. RIMS, Kyoto University*, 25 (1989) 587–603.