Continuous-time DC kernel —
a stable generalized first-order spline kernel

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Abstract

The stable spline (SS) kernel and the diagonal correlated (DC) kernel are two kernels that have been applied and studied extensively for kernel-based regularized LTI system identification. In this note, we show that similar to the derivation of the SS kernel, the continuous-time DC kernel can be derived by applying the same “stable” coordinate change to a “generalized” first-order spline kernel, and thus can be interpreted as a stable generalized first-order spline kernel. This interpretation provides new facets to understand the properties of the DC kernel. In particular, we derive a new orthonormal basis expansion of the DC kernel, and the explicit expression of the norm of the RKHS associated with the DC kernel. Moreover, for the non-uniformly sampled DC kernel, we derive its maximum entropy property and show that its kernel matrix has tridiagonal inverse.

Index Terms

System identification, regularization methods, kernel, orthonormal basis expansion, MaxEnt, tridiagonal inverse.

I. INTRODUCTION

Linear time invariant (LTI) system identification is a classical topic in system identification. The current standard solution to this topic is the maximum likelihood/prediction error method (ML/PEM), see e.g.,
A new solution is the kernel-based regularization method that is first proposed in [2] and further studied in [3], [4], [5], see [6] for a survey of this method. Recent progress for this method includes, e.g., the kernel design [7], [8], [9], analysis of the hyper-parameter estimators [10], [11], input design [12], [13] and development of the dual theory in frequency domain [14].

This method uses the impulse response model and solves a regularized least squares problem with a suitably designed and tuned kernel. The kernel plays a similar role as the parametric model structure in ML/PEM: on the one hand, it decides in what space the estimated impulse response is searched and on the other hand, it encodes the prior knowledge regarding the underlying system to be identified. The first two kernels for this method are the stable spline (SS) kernel [2] and the diagonal correlated (DC) kernel [4]. On the one hand, these two kernels are derived in different ways. The SS kernel is obtained by applying a “stable” coordinate change to the second-order spline kernel [cf. (10)], and the DC kernel is obtained by mimicking the behavior of the optimal kernel [4, eq. (56)] for this method. On the other hand, these two kernels share some common features [15], [7], which are fundamental for developing systematic methods to design kernels for this method. In particular, we have shown in [7] that both kernels belong to the class of amplitude modulated locally stationary kernels, and simulation-induced kernels, leading to a machine learning perspective and a system theory perspective to design more general kernels, respectively.

The SS kernel still has some other features inherited from its mother kernel – the spline kernel [cf. (8)], which were previously regarded as unique for the SS kernel. For example, the orthonormal basis expansion of the SS kernel with respect to a suitably chosen measure can be simply derived by applying the “stable” coordinate change to that of the second-order spline kernel [2], which is key for developing efficient implementation algorithms for this method [16]. Moreover, the same technique applies when deriving the norm of the reproducing kernel Hilbert space (RKHS) induced by the SS kernel and the maximum entropy property of the SS kernel [17]. Interestingly, as will be shown shortly, these features in fact also hold for the continuous-time DC kernel. The key lies in to show that the DC kernel can actually be derived by applying the same “stable” coordinate change to a “generalized” first-order spline kernel. The DC kernel can thus be interpreted as a stable generalized first-order spline kernel. This interpretation provides new facets to understand the properties of the DC kernel. In particular, we derive a new orthonormal basis expansion of the DC kernel and the explicit expression of the norm of the RKHS associated with the DC kernel. Moreover, for the non-uniformly sampled DC kernel, we derive its maximum entropy property and show that its kernel matrix has tridiagonal inverse, which extend the corresponding results in [17], [18].
II. SYSTEM IDENTIFICATION WITH KERNEL-BASED REGULARIZATION METHOD

A. Problem Statement

We consider continuous-time LTI stable and causal systems, which are described by

\[ y(t) = (g * u)(t) + v(t), \]

where \( t \geq 0 \) is the time index, \( y(t), u(t), v(t) \in \mathbb{R} \) are the measured output, input and disturbance of the system at time \( t \), respectively, \( g(t) \) is the impulse response of the LTI system, and \( (g * u)(t) \) is the convolution of the impulse response \( g(t) \) with the input \( u(t) \) (evaluated at \( t \)). Since \( g(t) = 0 \) for \( t < 0 \) due to the causality assumption, the convolution \( (g * u)(t) \) takes the form of

\[ (g * u)(t) = \int_{s=0}^{\infty} g(s)u(t-s)ds, \]

where the unknown input \( u(t) \) with \( t < 0 \) is set to zero. The problem is to estimate \( g(t) \) as well as possible based on the measured data \( \{y(t)\}_{t=1}^{N} \) and \( u(t) \) with \( t \geq 0 \).

B. Kernel-based Regularization Method

To estimate the impulse response \( g(t) \) from \( \{y(t)\}_{t=1}^{N} \) and \( u(t) \) with \( t \geq 0 \) is an ill-conditioned problem. To overcome this difficulty, the kernel-based regularization method first introduces a positive semidefinite kernel \( k(t,s;\theta) \) with \( t, s \geq 0 \) and constrain the search for a suitable impulse response within the reproducing kernel Hilbert space (RKHS) \( \mathcal{H}_k \) induced by \( k(t,s;\theta) \), where \( \theta \) is a hyper-parameter vector that contains the parameters used to parameterize the kernel. In particular, the regularized least squares criterion is used to estimate \( g(t) \):

\[ \hat{g}(t) = \arg \min_{g \in \mathcal{H}_k} \sum_{t=1}^{N} (y(t) - (g * u)(t))^2 + \gamma \|g\|_{\mathcal{H}_k}^2, \]

where \( \| \cdot \|_{\mathcal{H}_k} \) is the norm of \( \mathcal{H}_k \) and \( \gamma > 0 \) is a regularization parameter and controls the tradeoff between the data fit and the regularization term.

The performance of the kernel-based regularization method depends on several factors, such as the choice of the kernel \( k(t,s;\theta) \), the estimation of the hyper-parameter \( \theta \), and the input design. Assume

\(^1\)The kernel \( k(t,s;\theta) \) sometimes will be written as \( k(t,s) \) for simplicity. Recall that a function \( k : X \times X \to \mathbb{R} \) with \( X \) being a metric space is called a positive semidefinite kernel, if it is symmetric and satisfies \( \sum_{i,j=1}^{m} a_i a_j k(x_i, x_j) \geq 0 \) for any \( m \in \mathbb{N}, \{x_1, \cdots, x_m\} \subset X \) and \( \{a_1, ..., a_m\} \subset \mathbb{R} \). According to the Moore-Aronszajn Theorem, see e.g., [19], to every positive semidefinite kernel \( k(x,x') \) there exists a unique RKHS \( \mathcal{H}_k \) with \( k(x,x') \) as the reproducing kernel, i.e., with \( k_x \triangleq k(x,\cdot), k(x,x') \) has the reproducing property \( \langle f,k_x \rangle_{\mathcal{H}_k} = f(x) \), for \( f \in \mathcal{H}_k, x \in X \).
that a kernel \( k(t, s; \theta) \) has been chosen, an estimate of \( \theta \) has been found, and an input \( u(t) \) has been designed. Then by setting \( \gamma = \sigma^2 \) and defining \( A \in \mathbb{R}^{N \times N} \) with its \((t, s)\)th element \( A_{t,s} \) as follows

\[
a(t, s) = (k(t, \cdot) * u)(s), \quad A_{t,s} = (a(\cdot, s) * u) (t),
\]

we get the solution to (3) by the representer theorem [6, Theorem 3, page 671]:

\[
\hat{g}(t) = \sum_{s=1}^{N} \hat{c}_s a(t, s),
\]

where \( \hat{c}_s \) is the \( s \)th element of \( \hat{c} = (A + \gamma I_N)^{-1}Y \).

C. Kernels for Regularized Impulse Response Estimation

Many kernels have been introduced, e.g., the stable spline (SS) kernel in [2] and the tuned correlated (TC) kernel and the diagonal correlated (DC) kernel in [4]:

\[
k_{SS}(t, s; \alpha) = e^{-\alpha(t+s)} - \frac{e^{-3\alpha \max\{t,s\}} - 6}{6}, \quad (6a)
\]

\[
k_{TC}(t, s; \beta) = e^{-\beta(t+s)} - \frac{e^{-\beta|t-s|}}{6}, \quad \beta > 0, \quad (6b)
\]

\[
k_{DC}(t, s; \alpha, \beta) = e^{-\alpha(t+s)} - \frac{e^{-\beta|t-s|}}{6}, \quad \alpha > 0, \beta \geq 0, \quad (6c)
\]

where (6b) is a special case of (6c) with \( \alpha = \beta \) [4] and is also called the first-order stable spline (SS-1) kernel.

The SS kernel (6a) and the TC kernel (6b) can be derived [3] by applying a “stable” coordinate change to the second-order and first-order spline kernel, respectively. To be specific, we first recall the Sobolev space and its associated kernel [20], [21], [22]. The Sobolev space is the general term used for a functional space whose norm involves derivatives. The Sobolev space \( W^m_0 \) defined on \([0, 1]\) is defined as the set of functions \( f : [0, 1] \to \mathbb{R} \) such that the next conditions hold:

1) for \( i = 0, \cdots, m-1 \), its \( i \)th order derivative \( f^{(i)} \) is absolutely continuous and moreover, \( f^{(i)}(0) = 0 \);

2) its \( m \)th order derivative \( f^{(m)} \in L^2([0, 1]) \).

The inner product on \( W^m_0 \) over \( \mathbb{R} \) is defined through the classical inner product in \( L^2([0, 1]) \):

\[
\langle f, h \rangle_{W^m_0} = \langle f^{(m)}, h^{(m)} \rangle_{L^2([0,1])} = \int_0^1 f^{(m)}(\tau)h^{(m)}(\tau) d\tau.
\]

\(^{2}\)Let \( X \) be an interval of \( \mathbb{R} \). A function \( f : X \to \mathbb{R} \) is said to be absolutely continuous on \( X \) if for any \( \epsilon > 0 \), there is \( \delta > 0 \) such that when any finite sequence of pairwise disjoint subinterval \((a_i, b_i)\) of \( X \) satisfies \( \sum_i (a_i - b_i) < \delta \), then \( \sum_i |f(a_i) - f(b_i)| < \epsilon \). Moreover, \( f \) is absolutely continuous on a compact interval \([a, b] \) if and only if \( f \) has a derivative \( f^{(1)} \) almost everywhere and \( f^{(1)} \) is Lebesgue integrable, and \( f(x) = f(a) + \int_a^x f^{(1)}(\tau) d\tau \).
It can be verified [20], [21], [22] that $W^0_m$ is a RKHS with the reproducing kernel, often called the spline kernel,

$$w_m^S(\tau, \nu) = \int_0^1 G_m(\tau, s)G_m(\nu, s)ds, \ \tau, \nu \in [0, 1],$$

(8)

where the Green’s function $G_m(\tau, s) = (\tau - s)^{m-1}/(m-1)!$ with $(x)_+ = x$ for $x \geq 0$ and $(x)_- = 0$ for $x < 0$. In particular, the spline kernels (8) with $m = 1, 2$ are called the first-order and second-order spline kernel, respectively:

$$w_1^S(\tau, \nu) = \min\{\tau, \nu\}, \ \tau, \nu \in [0, 1],$$

(9)

$$w_2^S(\tau, \nu) = \frac{1}{2} \tau \nu \min\{\tau, \nu\} - \frac{1}{6} (\min\{\tau, \nu\})^3, \ \tau, \nu \in [0, 1].$$

(10)

Then it is easy to verify the following result.

Proposition 2.1 ([3]): Consider the SS kernel (6a) and the TC kernel (6b). Then the following result holds

$$k^SS(t, s; \alpha) = w_2^S(e^{-\alpha t}, e^{-\alpha s}),$$

(11)

$$k^{TC}(t, s; \beta) = w_1^S(e^{-2\beta t}, e^{-2\beta s}).$$

(12)

It follows from Proposition 2.1 that the SS and TC kernels have several properties inherited from their mother kernel (8), such as the orthonormal basis expansion, the explicit expression of the norm, and the maximum entropy property. Interestingly, the DC kernel can be derived by applying the same “stable” coordinate change to a “generalized” first-order spline kernel so that the DC kernel have similar properties as the SS and TC kernels inherited from the spline kernel (9).

III. STABLE GENERALIZED FIRST-ORDER SPLINE KERNEL

To show this, we recall the generalized Sobolev space and its associated kernel [20], [21], [22]. The generalized Sobolev space is derived by replacing the $i$th order derivatives $i = 1, \ldots, m$, in the definition of $W^0_m$, by more general derivatives. Specifically, the generalized $m$th order derivative of $f$ are defined as follows:

$$D_m f = \frac{d}{d\tau} \frac{1}{a_1} \frac{d}{d\tau} \frac{1}{a_2} \cdots \frac{d}{d\tau} \frac{1}{a_m} f,$$

(13)

where the functions $a_i(\tau)$ with $\tau \in [0, 1], \ i = 1, \ldots, m$ are functions such that $D_m f$ is well-defined, and moreover,

$$M_0 f = f, M_1 f = \frac{d}{d\tau} \frac{1}{a_m} f, M_2 f = \frac{d}{d\tau} \frac{1}{a_{m-1}} \frac{d}{d\tau} \frac{1}{a_m} f,$$

$$\cdots, M_{m-1} f = \frac{d}{d\tau} \frac{1}{a_2} \cdots \frac{d}{d\tau} \frac{1}{a_m} f.$$  

(14)
Then the generalized Sobolev space $\tilde{W}^0_m$ defined on $[0, 1]$ is defined as the set of functions $f : [0, 1] \rightarrow \mathbb{R}$ such that the following conditions hold:

1) for $i = 0, \cdots, m - 1$, $M_i f$ is absolutely continuous and moreover, $M_i f(0) = 0$;
2) $D_m f \in L^2([0,1])$.

Note that the functions $f : [0, 1] \rightarrow \mathbb{R}$ such that $M_i f(0) = 0$, $i = 0, \cdots, m - 1$ can be represented in the following form

\[
    f(\tau) = a_m(\tau) \int_0^\tau (D_m f)(s) ds, \quad m = 1,
\]

\[
    f(\tau) = a_m(\tau) \int_0^\tau a_{m-1}(t_{m-1}) dt_{m-1} \int_0^{t_{m-1}} a_{m-2}(t_{m-2}) dt_{m-2} \cdots \int_0^{t_1} (D_m f)(s) ds, \quad m > 1.
\]

By interchanging the integration order in (15), we have

\[
    f(\tau) = \int_0^\tau \tilde{G}_m(\tau,s)(D_m f)(s) ds
\]

where

\[
    \tilde{G}_m(\tau,s) = \begin{cases} 
        0 & \tau \leq s \\
        a_m(\tau) & \tau > s
\end{cases}, \quad m = 1,
\]

\[
    \tilde{G}_m(\tau,s) = a_m(\tau) \int_s^\tau a_1(t_1) dt_1 \int_{t_1}^\tau a_2(t_2) dt_2 \cdots \int_{t_{m-2}}^\tau a_{m-1}(t_{m-1}) dt_{m-1}, \quad m > 1.
\]

Moreover, the inner product on $\tilde{W}^0_m$ over $\mathbb{R}$ is defined through the classical inner product in $L^2([0,1])$:

\[
    \langle f, h \rangle_{\tilde{W}^0_m} = \langle D_m f, D_m h \rangle_{L^2([0,1])} = \int_0^1 (D_m f)(\tau)(D_m h)(\tau) d\tau.
\]

It can be verified \cite{20, 21, 22} that the generalized Sobolev space $\tilde{W}^0_m$ is a RKHS with the reproducing kernel

\[
    w_{\text{GS}}^m(\tau, \nu) = \int_0^1 \tilde{G}_m(\tau, s) \tilde{G}_m(\nu, s) ds,
\]

which is called the generalized spline kernel here.

Then the following result can be proved for the DC kernel:

\[\text{All proofs of the propositions are deferred to the Appendix.}\]
Proposition 3.1: Consider the DC kernel (6b) and the generalized spline kernel (19). If we take
\[ m = 1, \quad a_1(\tau) = \tau^\rho, \quad \rho > -0.5, \]
then we have
\[ k^{\text{DC}}(t, s; \alpha, \beta) = w_1^{\text{GS}}(e^{-2\beta t}, e^{-2\beta s}; \frac{\alpha - \beta}{2\beta}), \]
where
\[ w_1^{\text{GS}}(\tau, \nu; \rho) = \tau^\rho \nu^\rho \min\{\tau, \nu\}, \quad \tau, \nu \in [0, 1]. \]

The kernel (22) is called the generalized first-order spline kernel here. Recall from Proposition 2.1 that the SS kernel (6a) is called the stable spline kernel, because it can be obtained by applying a “stable” coordinate change to the second-order spline kernel. Now Proposition 3.1 shows that the DC kernel (6c) can be obtained by applying the same “stable” coordinate change to the generalized first-order spline kernel (22). The DC kernel can thus be called a stable generalized first-order spline kernel. As will be shown below, this finding provides new facets to understand the properties of the DC kernel.

IV. Norm and Orthonormal Basis Expansion

In this section, we derive the norm and the orthonormal basis expansion of the generalized first-order spline kernel and the DC kernel from their mother kernel (9).

To this goal, we first recall that when \( X \) is compact and the positive semidefinite kernel \( k(x, x') \) with \( x, x' \in X \) is continuous, \( H_k \) has the following property by Mercer’s Theorem, see e.g., [23], [24, Thm. 17, page 90], [25, Thm. 1, page 34].

Let \( \mu \) be a Borel measure on \( X \) and \( L^2(X, \mu) \) be the space of functions \( f \) for which \( \int_X (f(x))^2 d\mu(x) < \infty \). Then the integral operator \( L_k \phi(x) \) on \( L^2(X, \mu) \):
\[ L_k \phi(x) \triangleq \int_X k(x, x')\phi(x')d\mu(x'), \quad x \in X, \]
has at most countably many positive eigenvalues \( \{\lambda_i\}_{i=1}^{\infty} \) and orthonormal eigenfunctions \( \{\phi_i\}_{i=1}^{\infty} \), and the positive semidefinite kernel \( k(x, x') \) has a series expansion
\[ k(x, x') = \sum_{i=1}^{\infty} \lambda_i \phi_i(x)\phi_i(x'), \]

\(^4\)When \( X \) is a subset of \( \mathbb{R} \) and \( \mu \) is the Lebesgue measure, \( L^2(X, \mu) \) will be written as \( L^2(X) \) for simplicity.

\(^5\)If for some \( \lambda \), the homogenous integral equation
\[ L_k \phi(x) = \lambda \phi(x), \quad x \in X \]
has solutions other than \( \phi(x) = 0 \), \( \lambda \) and the solutions of \( L_k \) are called the eigenvalues and eigenfunctions of the integral operator \( L_k \), respectively.
which converges uniformly and absolutely on $X \times X$. Moreover, $\{\sqrt{\lambda_i} \phi_i\}_{i=0}^\infty$ forms an orthonormal basis of $H_k$, which gives an alternative representation of $H_k$ by [25, Thm. 4]:

$$H_k = \{f : f = \sum_{i=1}^\infty f_i \phi_i, \text{ with } \sum_{i=1}^\infty \frac{f_i^2}{\lambda_i} < \infty\}. \quad (26)$$

Now we recall the eigenvalues and the orthonormal eigenfunctions of the first-order spline kernel (9) on $L_2([0, 1])$, which are well-known, see e.g., [26, equations (119)-(120), page 196]:

$$\int_0^1 \min\{\tau, \nu\} \varphi_i(\nu) d\nu = \lambda_i \varphi_i(\tau), \; i = 1, 2, \ldots$$

where $\lambda_i = \frac{1}{(i - \frac{1}{2})^2 \pi^2}, \varphi_i(\tau) = 2^{\frac{1}{2}} \sin((i - \frac{1}{2}) \pi \tau)$. \quad (27)

Moreover, Mercer’s Theorem [23], [24, Thm. 17, page 90], [25, Thm. 1, page 34] guarantees that the series

$$\min\{\tau, \nu\} = \sum_{i=1}^\infty \lambda_i \varphi_i(\tau) \varphi_i(\nu) \quad (28)$$

can be proved the next result for the generalized first-order spline kernel (22).

**Proposition 4.1:** Consider the generalized first-order spline kernel (22). Let the eigenvalues and orthonormal eigenfunctions of the first-order spline kernel (9) take the form of (27). Then the following results hold:

1) Let $\phi_i(\tau) = \tau^\rho \varphi_i(\tau), \; \tau \in [0, 1]$. Then $\{\lambda_i\}_{i=1}^\infty$ and $\{\phi_i\}_{i=1}^\infty$ are the eigenvalues and orthonormal eigenfunctions of the generalized first-order spline kernel (22) on $L_2([0, 1], \mu(\nu))$ with the measure $\mu(\nu) = \nu^{-2\rho} d\nu$, respectively.

2) The series

$$\tau^\rho \nu^\rho \min\{\tau, \nu\} = \sum_{i=1}^\infty \lambda_i \phi_i(\tau) \phi_i(\nu) \quad (31)$$

converges absolutely and uniformly on $[0, 1] \times [0, 1]$. \quad (31)
3) \( \{\sqrt{\lambda_i} \phi_i\}_{i=1}^{\infty} \) forms an orthonormal basis of \( \tilde{W}_1^0 \), and \( \tilde{W}_1^0 \) has an equivalent representation:

\[
\tilde{W}_1^0 = \{f|f(\tau) = \sum_{i=1}^{\infty} f_i \phi_i(\tau), \quad \tau \in [0, 1], \quad \sum_{i=1}^{\infty} \frac{f_i^2}{\lambda_i} < \infty \}, \tag{32}
\]

and the norm of \( f \) can be computed according to

\[
\|f\|_{\tilde{W}_1^0}^2 = \sum_{i=1}^{\infty} \frac{f_i^2}{\lambda_i} \left( \int_0^1 \left( \frac{d}{d\tau} f(\tau) \right)^2 d\tau \right). \tag{33}
\]

By Proposition 4.1 and noting (21), we can prove the next result for the DC kernel (6c).

**Proposition 4.2:** Consider the DC kernel (6c). Let the eigenvalues and orthonormal eigenfunctions of the generalized first-order spline kernel (22) take the form of (31). Then the following results hold:

1) Let \( \psi_i(t) = \phi_i(e^{-2\beta t}), \quad i = 1, 2, \ldots \). Then \( \{\lambda_i\}_{i=1}^{\infty} \) and \( \{\psi_i\}_{i=1}^{\infty} \) are the eigenvalues and the orthonormal eigenfunctions of the DC kernel (6c) on \( L_2([0, \infty), \rho) \) with the measure \( \rho(t) \) such that \( d\rho(t) = 2\beta e^{2\beta(2\rho-1)t} dt \), respectively.

2) The series

\[
e^{-\alpha(t+s)}e^{-\beta|t-s|} = \sum_{i=1}^{\infty} \lambda_i \psi_i(t) \psi_i(s) \tag{34}
\]

converges absolutely and uniformly on \( Z \times Z' \) with \( Z, Z' \) being any compact subsets of \( [0, \infty) \).

3) \( \{\sqrt{\lambda_i} \psi_i\}_{i=1}^{\infty} \) forms an orthonormal basis of the RKHS \( \mathcal{H}^{DC} \) induced by the DC kernel (6c), and \( \mathcal{H}^{DC} \) has an equivalent representation:

\[
\mathcal{H}^{DC} = \{g|g(t) = \sum_{i=1}^{\infty} g_i \psi_i(t), t \geq 0, \quad \sum_{i=1}^{\infty} \frac{g_i^2}{\lambda_i} < \infty \}, \tag{35}
\]

and the norm of \( g \) can be computed according to

\[
\|g\|^2_{\mathcal{H}^{DC}} = \sum_{i=1}^{\infty} \frac{g_i^2}{\lambda_i}. \tag{36}
\]

4) \( \mathcal{H}^{DC} \) and \( \tilde{W}_1^0 \) are isometrically isomorphic and

\[
\|g\|^2_{\mathcal{H}^{DC}} = \int_0^\infty 2\beta e^{(4\rho+2\beta)t} \left( \frac{1}{2\beta} g^{(1)}(t) + \rho g(t) \right)^2 dt. \tag{37}
\]

**Remark 4.1:** It should be noted that the orthonormal basis expansion of the DC kernel with respect to the Lebesgue measure has been derived before in [27] and is different from (34). Then a natural question is "what is their difference?". It turns out that these orthonormal basis expansions are optimal in some sense with respect to certain kinds of inputs. Due to the limitation of the space, the details cannot be put here but will be in an independent paper.
Remark 4.2: When \( \rho = 0 \), the DC kernel (6c) reduces to the TC kernel (6b). In this case, we have
\[
\|g\|_{H_{DC}}^2 = \int_0^\infty \frac{1}{2\beta} e^{2\beta t} \left( g^{(1)}(t) \right)^2 dt
\]  
(38)
Comparing (38) with (37) shows from another perspective that the DC kernel is more flexible than the TC kernel for regularize impulse response estimation. For illustration, if \( g(t) \) is constrained to be \( g(t) = e^{-\gamma t} \), then a necessary condition for \( g \in H_{TC} \) is \( \gamma > \beta \), but a necessary condition for \( g \in H_{DC} \) is \( \gamma > (2\rho + 1)\beta \), where \( \rho > -0.5 \).

V. MAXIMUM ENTROPY INTERPRETATION OF NON-UNIFORMLY SAMPLED DC KERNEL

In this section, for the non-uniformly sampled DC kernel (6c), we first derive its maximum entropy (MaxEnt) property from its mother kernel (9) using arguments similar to [17] and then show that its kernel matrix has tridiagonal inverse.

First, recall that a real-valued stochastic process \( w(i) \) with \( i = 0, 1, 2, \cdots \), is called a white Gaussian noise if the \( w(i) \)'s are independent identically Gaussian distributed with zero mean and constant variance. Then we construct a Gaussian process \( f(\tau) \) defined on an ordered index set \( \Gamma = \{ \tau_i | 0 \leq \tau_i < \tau_{i+1} \leq 1, i = 0, 1, 2, \cdots \} \) as follows:
\[
\begin{align*}
\tau_0 &= 0, \\
n &\tau_k = \tau_0^{\rho} \sum_{i=1}^{k} w(i-1) \sqrt{\tau_i - \tau_{i-1}}, k = 1, 2, \cdots.
\end{align*}
\]  
(39)
It is easy to verify that \( f(\tau) \) with \( \tau \in \Gamma \) has the generalized first-order spline kernel (22) with \( \tau, \nu \in \Gamma \) as its covariance function. Then we can prove the next result for the kernel (22).

Proposition 5.1: Let \( h(\tau) \) with \( \tau \in \Gamma \) be any stochastic process with \( h(\tau_0) = 0 \) for \( \tau_0 = 0 \). For any \( n \in \mathbb{N} \), the stochastic process \( f(\tau) \) in (39) is the optimal solution to the MaxEnt problem
\[
\begin{align*}
&\text{maximize } \quad H(h(\tau_1), h(\tau_2), \cdots, h(\tau_n)) \\
&\text{subject to } \quad \mathbb{E}(h(\tau_i)) = 0, \quad i = 1, \cdots, n \\
&\quad \forall \left( \frac{h(\tau_i)}{\tau_i^\rho} \right) = \tau_1, \\
&\quad \forall \left( \frac{h(\tau_i)}{\tau_i^\rho} - \frac{h(\tau_{i-1})}{\tau_{i-1}^\rho} \right) = \tau_i - \tau_{i-1}, i = 2, \cdots, n
\end{align*}
\]  
(40)

Without loss of generality, we assume the variance is 1.

Recall that for a real-valued random variable \( X \), the differential entropy \( H(X) \) of \( X \) is defined as
\[
H(X) = -\int_S p(x) \log p(x) dx,
\]
where \( p(x) \) is the probability density function of \( X \) and \( S \) is the support set of \( X \).
where $\mathbb{E}(\cdot)$ and $\mathbb{V}(\cdot)$ represent the expectation and variance, respectively, and for simplicity, $H(h(t_1), h(t_2), \cdots, h(t_n))$ denotes the differential entropy of $[h(t_1) \ h(t_2) \ \cdots \ h(t_n)]^T$.

Based on the stochastic process $f(\tau)$ in (39), we define another Gaussian process $g(t)$ defined on an ordered index set $\mathcal{T} = \{t_i | 0 \leq t_i < t_{i+1} \leq \infty, i = 0, 1, 2, \cdots, \}$ as follows:

$$g(t_k) = e^{-2\beta \rho t_k} \sum_{i=k}^{n-1} w(n-1-i) \sqrt{e^{-2\beta t_i} - e^{-2\beta t_{i+1}}},$$

$$k = 0, \cdots, n-1, g(t_n) = 0 \text{ with } t_n = \infty.$$  \hspace{1cm} (41)

It is easy to verify that $g(t)$ with $t \in \mathcal{T}$ has the DC kernel (6c) with $t, s \in \mathcal{T}$ as its covariance function. Then we can prove the next result for the DC kernel (6c).

**Proposition 5.2**: Let $h(t)$ be any stochastic process with $h(t_n) = 0$ for $t_n = \infty$. For any $n \in \mathbb{N}$, the stochastic process $g(t)$ in (41) is the optimal solution to the MaxEnt problem

$$\text{maximize } h(\cdot) \quad H(h(t_0), h(t_1), \cdots, h(t_{n-1}))$$

subject to $\mathbb{E}(h(t_i)) = 0, i = 0, \cdots, n-1$

$$\mathbb{V}\left( \frac{h(t_{i+1})}{e^{-2\beta \rho t_{i+1}}} - \frac{h(t_i)}{e^{-2\beta \rho t_i}} \right) = e^{-2\beta t_i} - e^{-2\beta t_{i+1}},$$

$$i = 0, 1, \cdots, n-2$$

$$\mathbb{V}\left( \frac{h(t_{n-1})}{e^{-2\beta \rho t_{n-1}}} \right) = e^{-2\beta t_{n-1}}.$$  \hspace{1cm} (42)

Proposition 5.2 leads to an interesting result that, the kernel matrix of the DC kernel (6c) with $t, s \in \mathcal{T}$ has tridiagonal inverse, which is an extension of the result of [17] from the TC kernel (6b) to the DC kernel (6c) and an extension of the result of [18] from the uniformly sampled case to the non-uniformly sampled case.

**Proposition 5.3**: Consider the DC kernel (6c) with $t, s \in \mathcal{T}$. Then its kernel matrix $K^{DC} \in \mathbb{R}^{\bar{n} \times \bar{n}}$ with $\bar{n} \geq 3$ has tridiagonal inverse, where the $(i, j)$th element of $K^{DC}$ is equal to $K^{DC}(t_i, t_j; \alpha, \beta)$ with $t_i, t_j \in \mathcal{T}$.

For illustration, we consider an example.

**Example 5.1**: We consider the inverse of $K^{DC} \in \mathbb{R}^{\bar{n} \times \bar{n}}$ with $\bar{n} = 10$, $\alpha = -0.2$, $\beta = -0.3$ and $t_i, t_j$ with $i, j = 1, \cdots, 10$ take values from the set generated in Matlab with sort(rand(10,1)). By Proposition 5.3, the inverse of $K^{DC}$ should be tridiagonal, which is confirmed by Fig. 1.

VI. CONCLUSION

In this note, we have shown that the continuous-time diagonal correlated (DC) kernel can be interpreted as a stable generalized first-order spline kernel. This interpretation provides new facets to understand the
Fig. 1. Scaled image of \((K^{DC})^{-1}\) in Example [5.1]. The image is drawn by using \texttt{imagesc} in MATLAB, where the colder the color the smaller the element of \((K^{DC})^{-1}\).

properties of the DC kernel. In particular, we derive a new orthonormal basis expansion of the DC kernel and the explicit expression of the norm of the RKHS associated with the DC kernel. Moreover, for the non-uniformly sampled DC kernel, we derive its maximum entropy property and show that its kernel matrix has tridiagonal inverse. There are several interesting works that can be done in the future. For example, it is interesting and possible to derive the explicit expression of the norm of the sampled DC kernel for discrete time linear time invariant system identification, and compare the derived orthonormal basis expansion with the one in [27].

**APPENDIX**

A. **Proof of Proposition 3.1**

On the one hand, it is easy to check that under the assumption (20) the generalized spline kernel (19) takes the form of (22). On the other hand, the DC kernel (6c) can be rewritten as follows:

\[
k^{DC}(t, s; \alpha, \beta) = e^{(\beta-\alpha)(t+s)} k^{TC}(t, s; \beta) = (e^{-2\beta t})^{\frac{2-\alpha}{2\beta}} (e^{-2\beta s})^{\frac{2-\alpha}{2\beta}} \min\{e^{-2\beta t}, e^{-2\beta s}\}.
\]

Then noting the above equation and (22) gives the result.

B. **Proof of Proposition 4.1**

We first consider the proof for 1). For any \(i \in \mathbb{N}\), we have

\[
\int_0^1 \tau^\rho \nu^\rho \min\{\tau, \nu\} \phi_i(\nu) d\mu(\nu)
= \int_0^1 \tau^\rho \nu^\rho \min\{\tau, \nu\} \nu^\rho \varphi_i(\nu) \nu^{-2\rho} d\nu
= \lambda_i \tau^\rho \varphi_i(\tau) = \lambda_i \phi_i(\tau)
\]
and moreover, note that the orthonormality of \( \{\phi_i\}_{i=1}^\infty \) in \( L_2([0, 1], \mu(\nu)) \) follows from that of \( \{\varphi_i\}_{i=1}^\infty \) in \( L_2([0, 1]) \).

For 2) and 3), we first show that the kernel (22) is continuous. By Schwartz inequality, we have

\[
|w_1^{\text{GS}}(\tau+h, \nu+k) - w_1^{\text{GS}}(\tau, \nu)| \\
\leq \{w_1^{\text{GS}}(\tau+h, \tau+h) \times \\
[w_1^{\text{GS}}(\nu+k, \nu+k) - 2w_1^{\text{GS}}(\nu+k, \nu) + w_1^{\text{GS}}(\nu, \nu)]\}^{\frac{1}{2}} \\
+ \{w_1^{\text{GS}}(\nu, \nu) \times \\
[w_1^{\text{GS}}(\tau+h, \tau+h) - 2w_1^{\text{GS}}(\tau+h, \tau) + w_1^{\text{GS}}(\tau, \tau)]\}^{\frac{1}{2}}.
\]

Now we let \( h, k \to 0 \), then \( w_1^{\text{GS}}(\nu+k, \nu+k) \to w_1^{\text{GS}}(\nu, \nu) \) because \( w_1^{\text{GS}}(\tau, \nu) \) is continuous along the diagonal \( \tau = \nu \) and \( w_1^{\text{GS}}(\nu+k, \nu) \to w_1^{\text{GS}}(\nu, \nu) \) because \( w_1^{\text{GS}}(\tau, \nu) \) is continuous in \( \tau \). This means that the right-hand side of the above inequality converges to zero as \( h, k \to 0 \) and thus the kernel (22) is continuous. Then noting that the measure \( \mu(\nu) \) is a Borel measure, the results follow from \( [25] \) Thm. 1, page 34 and \( [25] \) Thm. 4, page 37, respectively. This completes the proof.

C. Proof of Proposition 4.2

The proof 1) to 3) can be done by applying the coordinate change \( \tau = e^{-2\beta t} \) to 1) to 3) of Proposition 4.1. So we only prove 4) below.

Given any \( f \in \tilde{W}_1^0 \), it follows from (32) that there exist \( f_i, i = 1, 2, \cdots \), such that \( f(\tau) = \sum_{i=1}^\infty f_i \phi_i(\tau) \) with \( \tau \in [0, 1] \). Taking \( \tau = e^{-2\beta t} \), we get a unique \( g(t) \triangleq f(e^{-2\beta t}) = \sum_{i=1}^\infty f_i \phi_i(e^{-2\beta t}) = \sum_{i=1}^\infty f_i \psi_i(t) \in \mathcal{H}^{\text{DC}} \) since \( \sum_{i=1}^\infty \frac{f_i^2}{\lambda_i} < \infty \), and moreover, \( \|f\|_{\tilde{W}_1^0}^2 = \|g\|_{\mathcal{H}^{\text{DC}}}^2 = \sum_{i=1}^\infty f_i^2/\lambda_i \). Similarly, given any \( g \in \mathcal{H}^{\text{DC}} \), it follows from (35) that there exist \( g_i, i = 1, 2, \cdots \) such that \( g(t) = \sum_{i=1}^\infty g_i \psi_i(t) \) with \( t \geq 0 \). Taking \( t = \log \tau/(-2\beta) \), we get a unique \( f(\tau) \triangleq g(\log \tau/(-2\beta)) = \sum_{i=1}^\infty g_i \phi_i(\tau) \in \tilde{W}_1^0 \), since \( \sum_{i=1}^\infty g_i^2/\lambda_i < \infty \), and moreover, \( \|g\|_{\mathcal{H}^{\text{DC}}}^2 = \|f\|_{\tilde{W}_1^0}^2 = \sum_{i=1}^\infty g_i^2/\lambda_i \). Therefore, \( \mathcal{H}^{\text{DC}} \) and \( \tilde{W}_1^0 \) are isometrically isomorphic. Given any \( g \in \mathcal{H}^{\text{DC}} \), we have

\[
\|g\|_{\mathcal{H}^{\text{DC}}}^2 = \|f\|_{\tilde{W}_1^0}^2 = \int_0^1 \left( \frac{d}{d\tau} \frac{f(\tau)}{\tau^\rho} \right)^2 d\tau \\
= \int_0^1 \left( \frac{d}{d\tau} \frac{g(\log \tau/(-2\beta))}{\tau^\rho} \right)^2 d\tau \\
= \int_0^1 \frac{1}{\tau^{2\rho+2}} \left( \frac{1}{2\beta} g^{(1)}(\log \tau/(-2\beta)) + \rho g(\log \tau/(-2\beta)) \right)^2 d\tau
\]
Taking the coordinate change $\tau = e^{-2\beta t}$ in the above equation and simple calculation yields (37). This completes the proof.

D. Proof of Propositions 5.1 and 5.2

The proofs of Propositions 5.1 and 5.2 are similar. Due to the limitation of space, we only give the proof for the latter.

Define

$$ l(t_i) = h(t_i) e^{-2\beta \rho t_i}, i = 0, \ldots, n-1 \quad (43) $$

Then let $L = [l(t_0) \ l(t_1) \ \cdots \ l(t_{n-1})]^T$, $V = [h(t_0) \ h(t_1) \ \cdots \ h(t_{n-1})]^T$, and $B = \text{diag}(e^{-2\beta \rho t_0} \ e^{-2\beta \rho t_1} \ \cdots \ e^{-2\beta \rho t_{n-1}})$. Then we have $V = BL$. Since $B$ is nonsingular, it holds that $H(V) = H(L) + \log \det(B)$ according to [28, Corollary to Thm. 8.6.4]).

Moreover, since $B$ is independent of $l(t)$ and $h(t)$, the MaxEnt problem (42) is equivalent to

$$ \text{maximize} \quad H(l(t_0), l(t_1), \ldots, l(t_{n-1})) $$

subject to

$$ \mathbb{E}(l(t_i)) = 0, i = 0, \ldots, n-1, $$

$$ \mathbb{V}(l(t_{i+1}) - l(t_i)) = e^{-2\beta t_i} - e^{-2\beta t_{i+1}}, i = 0, \ldots, n-2 $$

$$ \mathbb{V}(l(t_{n-1})) = e^{-2\beta t_{n-1}}, $$

By [17, Thm. 1], the optimal solution to the above MaxEnt problem is the Gaussian process

$$ l(t_k) = \sum_{i=k}^{n-1} w(n - 1 - i) \sqrt{e^{-2\beta t_i} - e^{-2\beta t_{i+1}}}, \quad (44) $$

$$ k = 0, \ldots, n-1, l(t_n) = 0 \text{ with } t_n = \infty. $$

Clearly, (44) and (43) implies that the Gaussian process $g(t)$ in (41) is the optimal solution to the MaxEnt problem (42). This completes the proof.

E. Proof of Proposition 5.3

It follows from (42) that the Gaussian process $g(t)$ with $t \in \mathcal{T}$ defined in (41) can be rewritten as follows:

$$ g(t_{i+1}) = e^{-2\beta \rho(t_{i+1} - t_i)} g(t_i) $$

$$ + e^{-2\beta \rho t_{i+1}} \sqrt{e^{-2\beta t_i} - e^{-2\beta t_{i+1}}} \bar{w}(i), i = 0, \ldots, n-2, $$

$$ g(t_{n-1}) = e^{-\beta(2\rho+1)t_{n-1}} \bar{w}(n-1) \quad (45) $$
where $\bar{w}(i)$ with $i = 0, 1, \ldots$ is a white Gaussian noise with zero mean and unit variance. It follows from (45) that the Gaussian process $g(t)$ with $t \in \mathcal{T}$ is also a Markov process with order 1. Then by [7 Lem. A.3], we have the kernel matrix $K^{DC}$ has tridiagonal inverse. This completes the proof.

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