Structural Stability of Supersonic Contact Discontinuities in Three-Dimensional Compressible Steady Flows

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Abstract

In this paper, we study the structurally nonlinear stability of supersonic contact discontinuities in three-dimensional compressible isentropic steady flows. Based on the weakly linear stability result and the $L^2$-estimates obtained in [31], for the linearized problems of three-dimensional compressible isentropic steady equations at a supersonic contact discontinuity satisfying certain stability conditions, we first derive tame estimates of solutions to the linearized problem in higher order norms by exploring the behavior of vorticities. Since the supersonic contact discontinuities are only weakly linearly stable, so the tame estimates of solutions to the linearized problems have loss of regularity with respect to both of background states and initial data, so to use the tame estimates to study the nonlinear problem we adapt the Nash-Moser-Hörmander iteration scheme to conclude that supersonic contact discontinuities in three-dimensional compressible steady flows satisfying the stability conditions ([31]) are structurally nonlinearly stable at least locally in space.

Key words. 3-d compressible isentropic steady flows, supersonic contact discontinuities, structurally nonlinear stability, Nash-Moser-Hörmander iteration

AMS subject classifications. 35L65, 35L67, 76E17, 76N10

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1 Introduction

Based on the conservation of density and momentum, the steady compressible isentropic inviscid flows in three space variables can be described by the following equations,

\[
\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0
\]

\[
\frac{\partial}{\partial x}(\rho u^2 + p(\rho)) + \frac{\partial}{\partial y}(\rho uv) + \frac{\partial}{\partial z}(\rho uw) = 0
\]

\[
\frac{\partial}{\partial x}(\rho uv) + \frac{\partial}{\partial y}(\rho v^2 + p(\rho)) + \frac{\partial}{\partial z}(\rho vw) = 0
\]

\[
\frac{\partial}{\partial x}(\rho uv) + \frac{\partial}{\partial y}(\rho vw) + \frac{\partial}{\partial z}(\rho w^2 + p(\rho)) = 0
\]

(1.1)

where \(\rho, p = p(\rho)\) and \((u, v, w) \in \mathbb{R}^3\) denote the density, pressure and velocity of the fluid respectively, with \(p'(\rho) > 0\) for \(\rho > 0\). It is an important model in gas dynamics, aeronautics and astronautics.

Set \(U = (u, v, w, p)^T\). Obviously, the system (1.1) can be rewritten as the following symmetric form

\[
A_1(U)\partial_x U + A_2(U)\partial_y U + A_3(U)\partial_z U = 0,
\]

(1.2)

where

\[
A_1(U) = \begin{pmatrix}
\rho u & 0 & 0 & 1 \\
0 & \rho u & 0 & 0 \\
0 & 0 & \rho u & 0 \\
1 & 0 & 0 & \frac{\rho}{c^2}
\end{pmatrix}, \quad A_2(U) = \begin{pmatrix}
\rho v & 0 & 0 & 0 \\
0 & \rho v & 0 & 1 \\
0 & 0 & \rho v & 0 \\
1 & 0 & 0 & \frac{\rho}{c^2}
\end{pmatrix}, \quad A_3(U) = \begin{pmatrix}
\rho w & 0 & 0 & 0 \\
0 & \rho w & 0 & 0 \\
0 & 0 & \rho w & 1 \\
0 & 0 & 0 & \frac{\rho}{c^2}
\end{pmatrix}
\]

(1.3)

with \(c = \sqrt{\rho'(\rho)}\) being the sonic speed. When the velocity in the \(x\)-direction is supersonic, i.e. \(u > c\), the coefficient matrix \(A_1(U)\) is positively definite, then the system (1.2) is symmetric hyperbolic with \(x\) being regarded as the time-like direction.

As shown in the monographs [15, 21, 3] etc., it is an important and challenging field to study the propagation, interaction and stability of elementary waves such as the shocks, rarefaction waves and contact discontinuities in quasilinear hyperbolic conservation laws. The stability of shocks and rarefaction waves in multi-dimensional gas dynamics has been studied by Majda [21], Metivier et al. [19, 23], and Alinhac [1]. Contact discontinuities occur ubiquitously, such as slip-stream interfaces, lifting of aircrafts, tornadoes (refer to [15, 18, 24, 27] and references therein), so to understand the stability of contact discontinuities is an important step in studying the multi-dimensional Riemann problem, the Mach reflection of shocks, the interface problem of two-phase flow, etc..

In recent years, there are some interesting works on the stability analysis of contact discontinuities. For the Euler equations in two-dimensional isentropic unsteady gas dynamics, in [12, 13] Coulombel and Secchi obtained a rigorous theory on the stability of a supersonic contact discontinuity when the Mach number (the ratio between the relative speed of the fluid with respect to the discontinuity front over the sonic speed) \(M > \sqrt{2}\), which had been investigated already before in [24] and [4] by the mode analysis and the nonlinear geometric optics approach respectively. A weakly linear stability result was obtained in [25] for a two-dimensional contact discontinuity in nonisentropic compressible flow. Some related problems on the stability of vortex sheets in two dimensional steady flow were studied by Chen et al. in [4, 7] by using the Glimm scheme. However, as shown in [26], unsteady compressible vortex sheets in three space dimensions are always violently unstable, one of main factors is that the tangential velocity fields for the three-dimensional vortex sheets are two-dimensional, and this is the main unstable effect on the vortex sheets. Recently, some works showed that magnetic fields have stabilization effect on vortex
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sheets for two and three dimensional compressible MHD, cf. refer to [5, 6, 28, 29, 30] and references therein.

It is an interesting problem to study the stability of contact discontinuities in three dimensional steady flows, as it not only plays a crucial role in studying the structural stability of interaction of elementary waves, such as the multi-dimensional shock reflection-diffraction on an interface, and also shall provide important insight of the really multi-dimensional contact discontinuities, as tangential velocity fields in the three-dimensional steady contact discontinuities are also two-dimensional, this yields complicated stability phenomena of contact discontinuities. In [31], we have obtained the weakly linear stability criteria of contact discontinuities in three-dimensional compressible isentropic supersonic steady flows, by computing the Lopatinskii determinant for the linearized problem at a planar contact discontinuity, roughly speaking, it says that for a supersonic contact discontinuity in the three-dimensional steady flow with the velocity fields being non-parallel on both sides of the discontinuity front, it is weakly linearly stable if and only if the velocity fields restricted to a space-like plane should be also supersonic (see (2.13)). Moreover, we have established the $L^2$-stability estimates of solutions to the linearized problems of the three-dimensional steady Euler equations (1.1) at a non-planar contact discontinuity by constructing the Kreiss symmetrizers through developing the argument from [20, 12, 30], and using the para-differential calculus. These estimates exhibit loss of regularity of solutions with respect to the background contact discontinuity and the initial data, it also means that this supersonic contact discontinuity is only weakly stable.

The main goal of this work is to study the structurally nonlinear stability of the supersonic contact discontinuities in the three-dimensional steady Euler equations (1.1). As mentioned at above, there is loss of regularity of solutions to the corresponding linearized problem, so we shall adapt the Nash-Moser-Hörmander iteration scheme to study nonlinear problems. From this work, we obtain that a supersonic contact discontinuity satisfying the linearly stable criteria (2.13) is also structurally nonlinearly stable at least locally in the propagation direction.

The remainder of this paper is organized as follows. In Section 2, we formulate the nonlinear problem of a contact discontinuity in three dimensional compressible isentropic steady flow, and state the structural stability result of the supersonic contact discontinuities. To study the nonlinear problem, we establish the tame estimates of solutions to the linearized problem in Section 3. First in §3.1, we derive the effective linearized problem, and present the basic $L^2$ stability estimate, then in §3.2 we derive higher order norm estimates for the linearized problem. Estimates of tangential derivatives of solutions shall be obtained by differentiating the problems directly. Noting that the discontinuity front is characteristics, from the equations one only can estimate the normal derivatives of non-characteristic components of unknowns in terms of tangential derivatives of unknowns. To study the characteristic part of unknowns, inspired from the approach of [13], we introduce a linearized version of vorticity field, and observe each component of vorticity satisfies a transport equation tangential to characteristic boundary, so by combining estimates of vorticity and normal derivatives of non-characteristic unknowns, we conclude the higher order estimates of solutions in §3.2. In Section 4, we apply the Nash-Moser-Hörmander iteration to construct the approximate solutions to the nonlinear problem of the supersonic contact discontinuities in the steady Euler equations (1.1). Finally, the error estimates of the iteration scheme, and the convergence of approximate solutions are given in Section 5, which concludes the structural stability of the supersonic contact discontinuities in the three-dimensional steady Euler flow.

2 Formulation of Problems and Main Results

For the compressible isentropic steady Euler equations (1.1) in three space variables, assume that the piecewise smooth function

$$U(x, y, z) = \begin{cases} 
U^+(x, y, z), & y > \psi(x, z) \\
U^-(x, y, z), & y < \psi(x, z)
\end{cases} \tag{2.1}$$
with \( U = (u, v, w, p)^T \), is a weak solution of \((1.1)\) in the distribution sense, then it satisfies the equations \((1.1)\) classically on both sides of \( \Gamma = \{ y = \psi(x, z) \} \), and the Rankine-Hugoniot jump conditions on the front \( \Gamma = \{ y = \psi(x, z) \} \):

\[
\begin{bmatrix}
\rho u \\
\rho v \\
\rho w \\
\rho u^2 + p
\end{bmatrix}
\begin{bmatrix}
\psi_x \\
\psi_y \\
\psi_z \\
\rho u^2 + p
\end{bmatrix}
-
\begin{bmatrix}
\rho v \\
\rho w \\
\rho v^2 + p
\end{bmatrix}
\begin{bmatrix}
\psi_x \\
\psi_y \\
\psi_z \\
\rho v^2 + p
\end{bmatrix}
+ \psi_z
= 0,
\]

(2.2)

with \([\cdot]\) denoting the jump of a related function acrossing the front \( \Gamma \). Let \( m = \rho(u_x \cdot v + \psi_z \cdot w) \) be the mass flux. If \( m^+ = m^- = 0 \) on \( \Gamma \), i.e. without any mass transfer flux acrossing the front \( \Gamma = \{ y = \psi(x, z) \} \), then \((U^+, U^-, \Gamma)\) is called a contact discontinuity of \((1.1)\), in this case, the Rankine-Hugoniot condition \((2.2)\) reads as

\[
\psi_x u^+ - v^+ + \psi_z w^+ = 0, \quad p^+ = p^-.
\]

(2.3)

The first condition given in \((2.3)\) implies that the normal velocities on both sides of \( \Gamma \) vanish, while the tangential velocity fields of \( U \) acrossing \( \Gamma \) may have jump. As the tangential velocity fields on both sides of \( \Gamma \) are two-dimensional, the stability/instability mechanism of the contact discontinuity \((2.1)\) is very challenging, in contrast to the problems of contact discontinuities in the two-dimensional steady or un-steady compressible flows, in which the tangential velocity fields on both sides of front are of one-dimension only.

In this work, we consider the case that the contact discontinuity \((2.1)\) is supersonic in one direction, say in the \( x \)-direction, i.e. \( u^+ > c_0^+ \), then as mentioned in Section one, \( x \) can be regarded as the time-like. In \([31]\), we have studied the linear stability criteria of this supersonic contact discontinuity, and also obtained the \( L^2 \)-estimate of solutions to the problem of the system \((1.2)\) linearized at a background supersonic contact discontinuity. The aim of this work is to study the structural stability of a supersonic contact discontinuity. For a given supersonic contact discontinuity \((U^+, U^-, \Gamma)\) moving from negative \( x \) to positive \( x \), we are going to see whether this contact discontinuity persists in \( \{ x > 0 \} \) even for small \( x \). This problem can be formulated as the following one:

\[\text{(FBP)}: \text{For a given supersonic contact discontinuity } (U_0^+, U_0^-), \Gamma_0 = \{ y = \psi_0(x, z) \} \text{ of } (1.2)-2.3 \text{ with } u_0^+ > c_0^+ \text{ for } \{ x \leq 0 \}, \text{ to determine } U^+, U^- \text{ and a free boundary } \Gamma = \{ y = \psi(x, z) \} \text{ in } \{ x > 0 \} \text{ satisfying} \]

\[
\begin{align*}
A_1(U^+)\partial_x U^+ + A_2(U^+)\partial_y U^+ + A_3(U^+)\partial_z U^+ &= 0, & \pm(y - \psi(x, z)) > 0 \\
\psi_x u^+ - v^+ + \psi_z w^+ &= 0, & p^+ = p^-, & \text{on } \{ y = \psi(x, z) \} \\
U^+|_{x=0} &= U_0^+(x, y, z), & \pm(y - \psi_0(x, z)) > 0 \\
\psi|_{x=0} &= \psi_0(x, z).
\end{align*}
\]

(2.4)

This is a free boundary problem since the front \( \Gamma = \{ y = \psi(x, z) \} \) is also an unknown. To handle this free boundary, as \([23, 20, 29, 31]\) we introduce the following transformation from \( (x, y, z) \) to \( (\tilde{x}, \tilde{y}, \tilde{z}) \),

\[
x = \tilde{x}, \quad y = \Psi^+(\tilde{x}, \tilde{y}, \tilde{z}), \quad z = \tilde{z},
\]

(2.5)

with \( \Psi^+(\tilde{x}, \tilde{y}, \tilde{z}) \) satisfying the constraints

\[
\begin{align*}
\Psi^+(\tilde{x}, 0, \tilde{z}) &= \psi(\tilde{x}, \tilde{z}) \\
\pm \Psi^+_y &\geq k_0 > 0
\end{align*}
\]

(2.6)

for a positive constant \( k_0 \), then the domain \( \{ \pm(y - \psi(x, z)) > 0 \} \) is changed into \( \Omega = \{ \tilde{y} > 0 \} \) with the fixed boundary \( \{ \tilde{y} = 0 \} \).

\[
\psi_x u^+ - v^+ + \psi_z w^+ = 0, \quad p^+ = p^-.
\]
As in [13], inspired by the transport equation of $\psi$ given in (2.4), the natural candidates of $\Psi^\pm(x, y, z)$ are solutions to the following problem in $[y \geq 0]$:

\[
\begin{align*}
\begin{cases}
  u^\pm \partial_x \Psi^\pm - v^\pm + w^\pm \partial_z \Psi^\pm = 0, & \tilde{x} > 0 \\
  \Psi^\pm(0, \tilde{y}, z) = \pm \tilde{y} + \bar{\Psi}^\pm(\tilde{y}, z),
\end{cases}
\end{align*}
\]  

(2.7)

with $\bar{\Psi}^\pm(\tilde{y}, z)$ being a proper extension of $\psi(0, \tilde{z}) = \psi_0(0, \tilde{z})$ in $[\tilde{y} \geq 0]$, such that $\pm \bar{\Psi}^\pm \geq \kappa_0 > 0$ holds.

Set

\[
U^\pm_0(\tilde{y}, z) = U^\pm_0(0, y, z), \quad U^\pm(\tilde{x}, \tilde{y}, z) = U^\pm(x, y, z).
\]

From (2.4), we know that $\tilde{U}^\pm(\tilde{x}, \tilde{y}, z)$ satisfies the following problem,

\[
\begin{align*}
\begin{cases}
  L(U^\pm, \Psi^\pm)U^\pm = 0, & \text{in } [y > 0] \\
  \mathcal{B}(U^+, U^-, \psi) = 0, & \text{on } [y = 0] \\
  U^\pm|_{x=0} = U^\pm_0(y, z), & \psi|_{x=0} = \psi_0(z)
\end{cases}
\]
\]  

(2.8)

where we have dropped the tildes of notations for simplicity, and

\[
L(U^\pm, \Psi^\pm)U^\pm = A_1(U^\pm)\partial_x U^\pm + \frac{1}{\kappa_0}(A_2(U^\pm) - \Psi^\pm A_1(U^\pm) - \Psi^\pm A_3(U^\pm))\partial_x U^\pm + A_3(U^\pm)\partial_x U^\pm,
\]

(2.9)

with $\psi(x, z) = \Psi^\pm(x, 0, z)$ and $\Psi^\pm(x, y, z)$ being given in (2.7).

To study the nonlinear problem (2.8), let us first give a stable background state. Obviously, the following piecewise constant function

\[
\begin{align*}
\overline{U}(x, y, z) = \begin{cases}
  \overline{U}_r = (\bar{u}_r, 0, \bar{w}_r, \bar{\rho}), & y > 0 \\
  \overline{U}_l = (\bar{u}_l, 0, \bar{w}_l, \bar{\rho}), & y < 0
\end{cases}
\]
\]  

(2.10)

satisfying

\[
\bar{u}_r > \bar{c}, \quad \bar{u}_l > \bar{c}, \quad (\bar{u}_r - \bar{u}_l)^2 + (\bar{w}_r - \bar{w}_l)^2 \neq 0,
\]

(2.11)

with $\bar{c}^2 = p'(\bar{\rho})$, is a planar contact discontinuity for the compressible steady Euler equations (1.1). Here $\bar{\rho}$ is the density corresponding to the pressure $\bar{p}$ by the relation $p = p(\bar{\rho})$. As shown in [31], when the tangential velocity fields $(\bar{u}_r, \bar{w}_r)$ and $(\bar{u}_l, \bar{w}_l)$ are parallel, the planar contact discontinuity $(\overline{U}_r, \overline{U}_l)$ is always nonlinearly unstable, thus in this work we shall only consider the case of $(\bar{u}_r, \bar{w}_r)$ and $(\bar{u}_l, \bar{w}_l)$ being non-parallel to each other. As noted in Remark 2.1 of [31], without loss of generality we can assume

\[
\bar{w}_r \bar{w}_l < 0.
\]

(2.12)

We impose the following stability conditions obtained in Theorem 3.1 of [31] on the state $(\overline{U}_r, \overline{U}_l)$ such that the given planar contact discontinuity (2.10) is weakly stable,

\[
\begin{align*}
\left\{ \begin{array}{ll}
  \left( \frac{c^2}{\bar{u}_r^2} + \frac{c^2}{\bar{u}_l^2} \right) < 1, & \bar{w}_r^2 > \bar{c}^2, \quad \bar{w}_l^2 > \bar{c}^2, \\
  \min_{\theta \in [\theta_r, \theta_l]} \left( \begin{array}{cc}
    \left( \frac{c^2}{\bar{u}_r \sin \theta - \bar{v}_r \cos \theta} \right) & \\
    \left( \frac{c^2}{\bar{u}_l \sin \theta - \bar{v}_l \cos \theta} \right)
  \end{array} \right) & < 1,
\end{array} \right.
\end{align*}
\]

(2.13)

where $\theta_r = \max(\arctan \frac{\bar{w}_l}{\bar{u}_l}, \arctan \frac{\bar{w}_r}{\bar{u}_r})$ and $\theta_l = \min(\arctan \frac{\bar{w}_l}{\bar{u}_l}, \arctan \frac{\bar{w}_r}{\bar{u}_r})$.

The main proposal of this work is to prove the following structural stability of vortex sheet $(\overline{U}_r, \overline{U}_l)$. 

Theorem 2.1. Suppose that the planar contact discontinuity (2.10) satisfies the conditions (2.12) and (2.13). Then, for any fixed \( s > \frac{2}{3} \), there is a small quantity \( \delta > 0 \) depending on \((\overline{U}_r, \overline{U}_l)\) such that when the initial data \( U_0^\pm(y, z) \) and \( \psi_0(z) \) given in (2.8) satisfy
\[
||U_0^\pm - \overline{U}_r||_{H^s(\mathbb{R})} + ||\psi_0||_{H^s(\mathbb{R})} \leq \delta \tag{2.14}
\]
and the compatibility condition of the problem (2.8) up to order \( s - 1 \), there is \( X > 0 \) such that the problems (2.8) and (2.7) admit unique solutions
\[
U^\pm \in H^{s-1}([0, X] \times \mathbb{R}^+ \times \mathbb{R}_c), \quad \Psi^\pm \in H^{s-2}([0, X] \times \mathbb{R}^+ \times \mathbb{R}_c). \tag{2.15}
\]

3 Tame Estimates of Linearized Problems

As shown in [31], the supersonic contact discontinuity \((\overline{U}_r, \overline{U}_l)\) given by (2.10) is only weakly linearly stable, as in [13, 5, 29] we shall adapt the Nash-Moser-Hörmander iteration scheme to study the nonlinear problems (2.8) and (2.7). To do this, in this section, first we derive an effective linearized problem and then we estimate the solutions of linearized problem in higher order norms.

3.1 The effective linearized problem and \( L^2 \)-estimate

Suppose that a perturbed non-planar contact discontinuity of (2.10) takes the form
\[
U(x, y, z) = \begin{cases} 
U_r(x, y, z) = \overline{U}_r + V_r(x, y, z), & y > \psi(x, z) \\
U_l(x, y, z) = \overline{U}_l + V_l(x, y, z), & y < \psi(x, z)
\end{cases} \tag{3.1}
\]
satisfying the Rankine-Hugoniot conditions (2.3) on \( y = \psi(x, z) \). To derive the linearized problem of (1.2) and (2.3) at the given contact discontinuity solution (3.1), as in (2.5), we take the transformation,
\[
x = \tilde{x}, \quad y = \Psi_{r,l}(\tilde{x}, \tilde{y}, \tilde{z}), \quad z = \tilde{z}
\]
with \( \Psi_{r,l} \) satisfying
\[
\begin{cases} 
\Psi_{r,l}(\tilde{x}, 0, \tilde{z}) = \psi(\tilde{x}, \tilde{z}) \\
\pm \partial_y \Psi_{r,l} \geq \kappa_0 > 0
\end{cases} \tag{3.2}
\]
for a positive constant \( \kappa_0 \).

Set
\[
\tilde{U}_{r,l}(\tilde{x}, \tilde{y}, \tilde{z}) = U_{r,l}(\tilde{x}, \tilde{x}, \tilde{y}, \tilde{z}, \tilde{y}, \tilde{z}), \quad \tilde{V}_{r,l}(\tilde{x}, \tilde{y}, \tilde{z}) = V_{r,l}(\tilde{x}, \psi_{r,l}(\tilde{x}, \tilde{y}, \tilde{z}), \tilde{y}, \tilde{z})
\]
and drop the tildes of notations \( \tilde{U}_{r,l}(\tilde{x}, \tilde{y}, \tilde{z}), \tilde{V}_{r,l}(\tilde{x}, \tilde{y}, \tilde{z}) \) and \( \Psi_{r,l}(\tilde{x}, \tilde{y}, \tilde{z}) \) for simplicity in the following calculations.

For a fixed \( X > 0 \), denote by \( \Omega_X = \{(x, y, z) \mid 0 \leq x \leq X, y \in \mathbb{R}^+, z \in \mathbb{R}\} \).

For the contact discontinuity (3.1), we impose the following assumptions on the perturbations:
\[
\begin{align*}
V_r, \ V_l, \ \nabla \Psi_r, \ \nabla \tilde{\Psi}_l \in W^{2,\infty}(\Omega_X), \\
V_r, \ V_l, \ \nabla \Psi_r \text{ and } \nabla \tilde{\Psi}_l \text{ have compact support in } (y, z) \in \mathbb{R}_c^2, \\
||V_r||_{W^{2,\infty}(\Omega_X)} + ||V_l||_{W^{2,\infty}(\Omega_X)} \leq K, \text{ for a constant } K > 0,
\end{align*} \tag{3.3}
\]
where
\[
\Psi_r(x, y, z) = \Psi_r(x, y, z) - y, \quad \tilde{\Psi}_l(x, y, z) = \Psi_l(x, y, z) + y. \tag{3.4}
\]
Letting \((U^\pm, \Phi^\pm)\) be the small perturbation of the contact discontinuity \((U_{r,l}(x, y, z), \Psi_{r,l}(x, y, z))\), from (2.8) we get the following linearized equations of \((U^\pm, \Phi^\pm)\) at \((U_{r,l}, \Psi_{r,l})\):

\[
L'(U_{r,l}, \nabla \Psi_{r,l})(U^\pm, \Phi^\pm) = f^\pm, \tag{3.5}
\]

where

\[
L'(U_{r,l}, \nabla \Psi_{r,l})(U^\pm, \Phi^\pm) = L(U_{r,l}, \nabla \Psi_{r,l})U^\pm + C(U_{r,l}, \nabla U_{r,l}, \nabla \Psi_{r,l})U^\pm
\]

\[
- \frac{\partial \Phi^\pm}{\partial y_{r,l}} (A_2(U_{r,l}) - \partial_\Phi \Psi_{r,l} A_1(U_{r,l}) - \partial_\Psi \Psi_{r,l} A_3(U_{r,l})) \partial_\Psi U_{r,l}
\]

\[
- \frac{1}{\partial_\Psi \Psi_{r,l}} (\partial_\Phi \Phi^\pm A_1(U_{r,l}) + \partial_\Phi \Phi^\pm A_3(U_{r,l})) \partial_\Psi U_{r,l}
\]

in which

\[
L(U_{r,l}, \nabla \Psi_{r,l})U = A_1(U_{r,l}) \partial_\Psi U + A_2(U_{r,l}) \nabla \Psi_{r,l} \partial_\Psi U + A_3(U_{r,l}) \partial_\Psi U,
\tag{3.7}
\]

with

\[
A_b(U, \nabla \Psi) = \frac{1}{\partial_\Psi \Psi}(A_2(U) - \partial_\Psi A_1(U) - \partial_\Psi A_3(U))
\]

and

\[
C(U_{r,l}, \nabla U_{r,l}, \nabla \Psi_{r,l}) = (\nabla A_1(U_{r,l})) \partial_\Psi U_{r,l} + (\nabla A_3(U_{r,l})) \partial_\Psi U_{r,l} + \frac{1}{\partial_\Psi \Psi_{r,l}} (\nabla A_2(U_{r,l})) \partial_\Psi \nabla U_{r,l} - (\nabla A_3(U_{r,l})) \partial_\Psi \nabla U_{r,l} \partial_\Psi \nabla U_{r,l}.
\tag{3.8}
\]

When \(\Psi_{r,l}(x, y, z)\) satisfy the eikonal equations

\[
u_{r,l} \partial_\Psi \Psi_{r,l} - v_{r,l} + w_{r,l} \partial_\Psi \Psi_{r,l} = 0\tag{3.9}
\]

in \(y \geq 0\), we know that the boundary matrix

\[
A_b(U_{r,l}, \nabla \Psi_{r,l}) = \frac{1}{\partial_\Psi \Psi_{r,l}} \begin{pmatrix} 0 & 0 & 0 & -\partial_\Psi \Psi_{r,l} \\ 0 & 0 & 0 & 1 \\ -\partial_\Psi \Psi_{r,l} & 1 & -\partial_\Psi \Psi_{r,l} \\ 0 & 0 & 0 & 0 \end{pmatrix}
\]

has a constant rank in the domain \(\Omega_x\).

As the first order derivatives of \(U^\pm\) and \(\Phi^\pm\) are coupled together in the equations (3.5), to deal with this problem, as in [1], by introducing the “good unknowns”

\[
U_+ = U^+ - \frac{\Phi^+}{\partial_\Psi \Psi_{r,l}} \partial_\Psi U_{r,l}, \quad U_- = U^- - \frac{\Phi^-}{\partial_\Psi \Psi_{r,l}} \partial_\Psi U_{r,l},
\tag{3.10}
\]

we obtain the equations for \(U_\pm\),

\[
L(U_{r,l}, \nabla \Psi_{r,l}) U_\pm + \frac{\Phi^\pm}{\partial_\Psi \Psi_{r,l}} \partial_\Psi [L(U_{r,l}, \nabla \Psi_{r,l}) U_{r,l}] + C(U_{r,l}, \nabla U_{r,l}, \nabla \Psi_{r,l}) U_\pm = f^\pm.
\tag{3.11}
\]

in which \(\Phi^\pm\) is appeared only in the zero-th order terms. By shifting these zero-th order terms into the source terms \(f^\pm\), we obtain that \(U_\pm\) satisfy the following effective linear equations,

\[
L'(U_{r,l}, \nabla \Psi_{r,l}) U_\pm = L(U_{r,l}, \nabla \Psi_{r,l}) U_\pm + C(U_{r,l}, \nabla U_{r,l}, \nabla \Psi_{r,l}) U_\pm = f^\pm.
\tag{3.12}
\]

In terms of the good unknowns \(U = (U_+, U_-)^T\), the linearization of the boundary conditions given in (2.8) is given by

\[
B'_e(U, \phi) = \overline{b}(x, z) \nabla \phi + \overline{b}(x, z) \phi + \overline{M}(x, z) U|_{y=0} = g, \quad \text{on} \quad y = 0
\tag{3.13}
\]
with \( \phi = \Phi^+|_{y=0} = \Phi^-|_{y=0} \) and
\[
\begin{align*}
\mathbf{b}(x, z) &= \begin{pmatrix} u_r & w_r \\ u_l & w_l \end{pmatrix}_{|y=0}, & \mathbf{b} &= \begin{pmatrix} \frac{\partial}{\partial t} U_r \\ \frac{\partial}{\partial t} U_l \\ \frac{\partial}{\partial t} W_r \\ \frac{\partial}{\partial t} W_l \end{pmatrix}_{|y=0}, & (3.14)
\end{align*}
\]
\[
\mathbf{M}(x, z) = \begin{pmatrix} \psi_x & -1 & \psi_z & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \psi_x & -1 & \psi_z & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{pmatrix}. & (3.15)
\]

Therefore, the effective linear problem of \( U \) is formulated as
\[
\begin{align*}
\left\{ \begin{array}{ll}
L^*_\gamma U_+ = f^+, & L^*_\gamma U_- = f^-, & \text{in } \Omega_X \\
B^*_\gamma(U, \phi) = g, & \text{on } \{y = 0\}
\end{array} \right. & (3.16)
\end{align*}
\]
where \( U, \Phi, f^+, f^- \) and \( g \) vanish in \( \{x \leq 0\} \).

This problem has been studied by authors in [31] thoroughly. To recall the \( L^2 \) stability estimate given in [31], we first introduce the weighted Sobolev space \( H^s_\gamma \) for \( \gamma \geq 1 \), \( s \in \mathbb{R} \) as
\[
H^s_\gamma(\Omega_X) = \{ u \in D'(\Omega_X) | \ e^{-\gamma s} u \in H^s(\Omega_X) \},
\]
with the norm
\[
||u||_{H^s_\gamma(\Omega_X)} = ||e^{-\gamma s} u||_{H^s(\Omega_X)}.
\]
The space \( L^2(\mathbb{R}^+_\gamma; H^s_\gamma(\Omega_X)) \) defined in the domain \( \Omega_X \), is endowed with the norm
\[
||u||_{L^2_\gamma^r(H^s_\gamma)} = \left( \int_0^{+\infty} ||u(\cdot, y)||^2_{H^s_\gamma(\Omega_X)} dy \right)^{\frac{1}{2}},
\]
where \( \omega_X = \Omega_X \cap \{y = \text{const.}\} = \{(x, z) | 0 \leq x \leq X, z \in \mathbb{R}\} \) and the space \( H^s(\mathbb{R}^+_\gamma; H^s_\gamma(\omega_X)) \) can be defined similarly.

In Theorem 4.1 of [31], by using the paradifferential calculus and constructing the Kreiss symmetrizers we have obtained the following energy estimate for the problem (3.16):

**Theorem 3.1.** ([31]) Let \((\mathbf{U}_r, \mathbf{U}_l)\) defined in (2.10) be the planar supersonic contact discontinuity satisfying the stability assumptions (2.12) and (2.13), and its perturbed non-planar contact discontinuity (3.1) satisfies the condition (3.3) for a constant \( K > 0 \). Then for the linear problem (3.16), there exist constants \( K_0 > 0 \) depending on the contact discontinuity \((\mathbf{U}_r, \mathbf{U}_l)\), and \( \gamma_0 \geq 1 \), \( C_0 > 0 \) depending on \( K_0 \) such that for all \( K \leq K_0 \), \( \gamma \geq \gamma_0 \) and \((U, \phi) \in H^s_\gamma(\Omega_X) \times H^s_\gamma(\omega_X)\), one has
\[
\gamma||U||_{L^2_\gamma^r(\Omega_X)} + ||\mathbb{B} U||_{L^2_\gamma^r(\omega_X)} + ||\phi||_{H^s_\gamma(\omega_X)}^2 \leq C_0 \left( \frac{1}{\gamma^2} ||f||_{L^2_\gamma^r(H^s_\gamma)}^2 + \frac{1}{\gamma^2} ||g||_{H^s_\gamma(\omega_X)}^2 \right), & (3.17)
\]
where \( U = (U_+, U_-)^T \), \( f = (f^+, f^-)^T \) and
\[
\mathbb{B} U = \begin{pmatrix} \psi_x U_{x, 1} - U_{x, 2} + \psi_z U_{x, 3} \\ U_{x, 4} \end{pmatrix}. & (3.18)
\]

### 3.2 Higher order estimates of solutions to the linearized problem

In this section, we are going to derive the higher order estimates of the solution \((U, \phi)\) to the linearized boundary value problem (3.16), this is the key step for studying the nonlinear problem (2.8) by using the Nash-Moser-Hörmander iteration scheme in next section.

Assuming that for a fixed \( s \in \mathbb{N} \), the perturbation \((\nabla r_{ij}, \nabla \tilde{\nabla} r_{ij})\) belongs to \( H^{s+2}_\gamma(\Omega_X) \cap H^s_\gamma(\Omega_X) \), the main result of this section is the following one:
3.2.1 Estimate of tangential derivatives

Let $s \in \mathbb{N}$ and $X > 0$. Assume that the non-planar contact discontinuity given in (3.1) satisfies (2.13), (3.3) and $(V_{r,l}, \nabla \Xi_{r,l}) \in H^s_y(\Omega_X) \cap H^3_y(\Omega_X)$ with

$$
\|\nabla \Xi\|_{H^s_y(\Omega_X)} + \|\Xi\|_{H^s_y(\Omega_X)} \leq K,
$$

(3.19)

with $\Xi = (\Xi_r, \Xi_l)^T$ and $V = (V_r, V_l)^T$. Then, for the problem (3.10), there exist constants $K > 0$ and $\gamma_s \geq 1$ depending on $s$, such that for all $K \leq K_s$, $\gamma \geq \gamma_s$, and $(U, \phi) \in (H^s_y(\Omega_X) \times H^3_y(\omega_X)) \cap (H^1_y(\Omega_X) \times H^3_y(\omega_X))$, one has

$$
\sqrt{\gamma} \|U\|_{H^s_y(\Omega_X)} + \|U_{|\gamma=0}\|_{H^s_y(\omega_X)} + \|\phi\|_{H^{s+1}_y(\omega_X)} \leq C(K) \left\{ \frac{1}{\gamma^{3/2}} \|f\|_{H^s_y(\Omega_X)} + \frac{1}{\gamma} \|g\|_{H^s_y(\omega_X)} + \right. \\
+ \left. \left( \frac{1}{\gamma^{3/2}} \|V, \nabla \Xi\|_{H^{s+2}_y(\Omega_X)} + \frac{1}{\gamma} \|\partial_x V_{|\gamma=0}\|_{H^s_y(\omega_X)} \right) (\|f\|_{H^1_y(\Omega_X)} + \|g\|_{H^s_y(\omega_X)}) \right\},
$$

(3.20)

where $C(K)$ is a positive constant depending on $K$.

In the proof of this theorem, we shall always use $C(K)$ to denote a general positive constant depending on $K$, which may change from line to line, and shall frequently use the following elementary inequalities, which can be found in textbooks, e.g. [16]:

1. The Gagliardo-Nirenberg inequality,

$$
\|\partial^\alpha u\|_{L^p_y(\Omega_X)} \leq C(\|u\|_{L^q(\Omega_X)}^{1-2/p})^{2/p} \|u\|_{H^s_y(\Omega_X)}^{2/p}
$$

with $\frac{2}{p} = \frac{2}{s}$, for all $u \in H^s_y(\Omega_X) \cap L^\infty(\Omega_X)$.

2. Let $F$ be a $C^\infty$ function defined on $\mathbb{R}^s$ and satisfy $F(0) = 0$. Then, for all $u \in H^s_y(\Omega_X) \cap L^\infty(\Omega_X)$, one has

$$
\|F(u)\|_{H^s_y(\Omega_X)} \leq C(\|u\|_{L^\infty(\Omega_X)}) \|u\|_{H^s_y(\Omega_X)}.
$$

(3.21)

To prove the higher order estimate (3.20), first we shall study tangential derivatives by using the $L^2$-estimate given in Theorem 3.1, then from the equations (3.16) we estimate the normal derivatives of the non-characteristic components of unknowns, finally to estimate the normal derivatives of the characteristic components we study the problems of vorticity fields derived from the problem (3.10). These estimates will be given in the following subsections.

3.2.1 Estimate of tangential derivatives

We introduce the following transformations in the problem (3.16) to diagonalize the boundary matrices $A_0(U_{r,l}, \nabla \Xi_{r,l})$ of the effective linear equations (3.12),

$$
W^+ = T(\nabla \Xi_{r,l})U_+, \quad W^- = T(\nabla \Xi_{l,l})U_-,
$$

(3.22)

where

$$
T(\nabla \Xi) = \begin{pmatrix}
1 & 0 & -\partial_x \Psi & -\partial_x \Psi \\
\partial_x \Psi & \partial_x \Psi & 1 & 1 \\
0 & 1 & -\partial_x \Psi & -\partial_x \Psi \\
0 & 0 & \langle \partial \Psi \rangle & -\langle \partial \Psi \rangle
\end{pmatrix}^{-1}
$$

with $\langle \partial \Psi \rangle = \sqrt{1 + (\partial_x \Psi)^2 + (\partial_x \Psi)^2}$, and multiply

$$
A_0(\nabla \Xi_{r,l}) = \text{diag}(1, 1, \frac{\partial_x \Psi_{r,l}}{\langle \partial \Psi_{r,l} \rangle}, \frac{\partial_x \Psi_{r,l}}{\langle \partial \Psi_{r,l} \rangle}).
$$
from the left hand side of the equations of $W^+, W^-$ respectively. It’s easy to get that $W = (W^+, W^-)^T$ satisfies the following problem,

\[
\begin{align*}
A_1^t \partial_\gamma W^+ &+ \Lambda_2 \partial_\gamma W^+ + A_3^t \partial_\gamma W^+ + A_4^t C^t W^+ = F^+ \quad \text{in } \Omega_X \\
A_1^t \partial_\gamma W^- &+ \Lambda_2 \partial_\gamma W^- + A_3^t \partial_\gamma W^- + A_4^t C^t W^- = F^- \quad \text{in } \Omega_X \\
\frac{\partial \nabla \phi + b \phi + M}{T} \begin{pmatrix} T_{r, l}^{-1} & 0 \\ 0 & T_{r, l}^{-1} \end{pmatrix} W|_{y=0} = g,
\end{align*}
\]

where $A_2 = \text{diag}(0, 0, 1, 1)$, $F^+ = A_0^t T_{r, l} f^+$ and $g$ vanish for $x \leq 0$.

\[
A_1^l = A_0^l T_{r, l} A_1 T_{r, l}^{-1}, \quad A_2^l = A_0^l T_{r, l} A_3 T_{r, l}^{-1},
\]

\[
C^l = T_{r, l} A_1 \partial_\beta T_{r, l}^{-1} + T_{r, l} A_3 \partial_\beta T_{r, l}^{-1} + T_{r, l} A_4 \partial_\gamma T_{r, l}^{-1} + T_{r, l} C T_{r, l}^{-1}
\]

with notations $T_{r, l} = T(\nabla \Psi_{r, l})$, $A_0^l = A(0(\nabla \Psi_{r, l}))$.

From (3.24), we know that $A_0^l$, $A_1^l$, and $A_3^l$ are $C^\infty$ functions of $(U_{r, l}, \nabla \Psi_{r, l})$, and $C^l$ are $C^\infty$ functions of $(U_{r, l}, \nabla U_{r, l}, \nabla \Psi_{r, l}, \nabla \partial_\xi \Psi_{r, l})$.

**Lemma 3.3.** For any $s \in \mathbb{N}$, under the assumptions of Theorem 3.2 there exists a constant $C(K) > 0$ such that the following estimate holds for the solution of (3.23),

\[
\sqrt{\gamma} \| W \|_{L^2(H^s_{r, l})} + \| W \|_{L^2(\text{H}^s_{r, l})} + \| \partial_\gamma \|_{H^s_{r, l}} \leq C(K) \left\{ \frac{1}{\gamma^{1/2}} \| F \|_{L^2(H_{r, l}^{s+1})} + \frac{1}{\gamma^{1/2}} \| W \|_{L^2(\text{H}^s_{r, l})} + \| (V, \nabla \Psi) \|_{L^2(H^2_{r, l})} + \| \partial_\gamma \|_{L^2(\text{H}^s_{r, l})} \right\},
\]

where $W^{mc} = (W_3^+, W_4^+, W_3^-, W_4^-)^T$.

**Proof.** There are three steps to obtain the estimate (3.25).

1. Define $l$-th order tangential operator $\partial_\beta^l = \partial_\alpha^l \partial_\xi^l$ with $|\alpha| = \alpha_1 + \alpha_2 = l$, for some $l \leq s$. From (3.23), we know that

\[
A_1^l \partial_\gamma W^+ + \Lambda_2 \partial_\gamma W^+ + A_3^l \partial_\gamma W^+ + A_4^l C^l \partial_\gamma W^+ + [\partial_\gamma^l, A_1^l \partial_\gamma + A_3^l \partial_\beta + A_4^l C^l] W^+ = \partial_\beta^l F^+,
\]

where $[\cdot, \cdot]$ denotes the commutator. We introduce the notation $a^{(l)}$ as an element of the set $\{a^{(l)}: |\alpha| = l\}$ for any function $a$ belonging to $W^{r, \infty}(\Omega_X)$ or $W^{1, \infty}(\omega_X)$, and rewrite the above equations as

\[
A_1^l \partial_\gamma W^+ + \Lambda_2 \partial_\gamma W^+ + A_3^l \partial_\gamma W^+ + A_4^l C^l \partial_\gamma W^+ + \sum_{|\beta|=1} C_{\alpha, \beta} (\partial_\gamma^l A_1^l \partial_\beta^l \partial_\gamma W^+ + \partial_\beta^l A_3^l \partial_\gamma^l \partial_\beta W^+ - [\partial_\gamma^l, A_4^l C^l] W^+).
\]

where $C_{\alpha, \beta}$ are constants depending on $\alpha, \beta$. The equations of $W^{(l)}$ are similar to (3.26). The corresponding boundary conditions of $W^{(l)} = (W_3^{(l)}, W_4^{(l)})^T$ on $y = 0$ are

\[
\begin{align*}
\begin{pmatrix} \partial_\gamma^l \psi + b \psi + M \partial_\gamma^l W^{mc} \end{pmatrix} &\left( \begin{array}{c}
\partial_\gamma^l g - [\partial_\gamma^l, b \nabla \psi + b] \phi - [\partial_\gamma^l, M] W^{mc}
\end{pmatrix}.
\end{align*}
\]

For simplicity of notations, we rewrite the above problem of $W^{(l)} = (W_3^{(l)}, W_4^{(l)})^T$ as

\[
\begin{align*}
\begin{pmatrix} A_1^l \partial_\gamma W^+ + \Lambda_2 \partial_\gamma W^+ + A_3^l \partial_\gamma W^+ + A_4^l C^l W^+ \end{pmatrix} &\left( \begin{array}{c}
\partial_\gamma^l g + \partial_\beta^l g + M W^{mc} \end{array} \right), \quad \text{in } \Omega_X \\
\begin{pmatrix} \partial_\gamma^l \psi + b \psi + M \partial_\gamma^l W^{mc} \end{pmatrix} &\left( \begin{array}{c}
\partial_\gamma^l g - [\partial_\gamma^l, b \nabla \psi + b] \phi - [\partial_\gamma^l, M] W^{mc}
\end{pmatrix}.
\end{align*}
\]
where $\Lambda_4 = \text{diag}(0, 0, 1, 1, 0, 0, 1, 1)$,

$$A_k = \begin{pmatrix} A_k^r & A_k^l \\ l_k & r_k \end{pmatrix} \quad (k = 1, 3), \quad C = \begin{pmatrix} \tilde{C}^r \\ \tilde{C}^l \end{pmatrix}, \quad F^{(l)} = (F^{(l)}_+ , F^{(l)}_-)^T,$$

with $\tilde{C}^r = A^r_0 C^r + \sum_{|\beta| = 1} C_{\alpha, \beta} (\partial_\gamma^\beta \delta_\gamma^\beta A_1^r + \partial_\gamma^\beta \delta_\gamma^\beta A_2^r)$, $F^{(l)}_+$ and $g^{(l)}$ denoting the right hand sides of (3.26) and (3.27) respectively, and $F^{(l)}_+ \overset{\text{def}}{=} F^{(l)}$ defined similar to $F^{(l)}_+$. $M$ in the boundary conditions is the nonzero submatrix of $M = \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix}$.

Noting that the coefficients in the equations given in (3.28) all belong to $W^{2, \infty}(\Omega \chi)$ except for $C \in W^{1, \infty}(\Omega \chi)$, we can apply Theorem 3.1 in the problem (3.28) to get

$$\gamma \| \delta_\gamma^\beta F^{(l)}_+ \|_{L^2_3(H^1)} + \| \delta_\gamma^\beta F^{(l)}_+ \|_{L^2_0(\Omega \chi)} + \| \alpha \|_{H^1_1(\Omega \chi)} \leq \text{C}_0 \left( 1 \| \delta_\gamma^\beta F^{(l)}_+ \|_{L^2_3(H^1)} + \frac{1}{\gamma} \| \delta_\gamma^\beta F^{(l)}_+ \|_{L^2_2(\Omega \chi)} \right). \tag{3.29}$$

(2) To obtain (3.25), it remains to estimate the source terms $F^{(l)}$ and $g^{(l)}$. From (3.26), we get

$$F^{(l)}_+ = \partial_\gamma^\beta \partial_\gamma^\beta F^{(l)}_+ - \sum_{|\beta| \geq 2, |\alpha| \leq 1} C_{\alpha, \beta} (\partial_\gamma^\beta \delta_\gamma^\beta \partial_\gamma^\beta W^{+} + \partial_\gamma^\beta \delta_\gamma^\beta \partial_\gamma^\beta \partial_\gamma^\beta W^{+} - [\partial_\gamma^\beta, A_0^r C^r] W^{+}.)$$

Obviously, we have

$$\| \partial_\gamma^\beta \partial_\gamma^\beta F^{(l)}_+ \|_{L^2_3(H^1)} \leq \| \partial_\gamma^\beta F^{(l)} \|_{L^2_3(H^1)}, \quad |\alpha| = 1. \tag{3.30}$$

Applying Hölder’s and Gagliardo-Nirenberg’s inequalities for $\beta \leq \alpha$ with $|\beta| \geq 2$, $|\alpha| = 1$, one has

$$\| \partial_\gamma^\beta \partial_\gamma^\beta \partial_\gamma^\beta \partial_\gamma^\beta W^{+} \|_{L^2_3(H^1)} \leq \text{C}(K) \left( \| \partial_\gamma^\beta W^{+} \|_{L^2_3(H^1)} + \| \partial_\gamma^\beta \partial_\gamma^\beta W^{+} \|_{L^2_3(H^1)} \right),$$

and

$$\| \partial_\gamma^\beta \partial_\gamma^\beta \partial_\gamma^\beta \partial_\gamma^\beta W^{+} \|_{L^2_3(H^1)} \leq \text{C}(K) \left( \| \partial_\gamma^\beta \partial_\gamma^\beta W^{+} \|_{L^2_3(H^1)} + \| \partial_\gamma^\beta \partial_\gamma^\beta \partial_\gamma^\beta W^{+} \|_{L^2_3(H^1)} \right).$$

Thus, by using the following equivalent relation

$$\| F \|_{L^2_3(H^1)} \approx \gamma \| F \|_{L^2_3} + \| \partial_\gamma^\beta F \|_{L^2_3},$$

we obtain

$$\| \partial_\gamma^\beta \partial_\gamma^\beta \partial_\gamma^\beta \partial_\gamma^\beta W^{+} \|_{L^2_3(H^1)} \leq \text{C}(K) \left( \| W^{+} \|_{L^2_3} + \| \partial_\gamma^\beta \partial_\gamma^\beta W^{+} \|_{L^2_3} \right) \tag{3.31}$$

Similarly, we can deduce that

$$\| \partial_\gamma^\beta \partial_\gamma^\beta \partial_\gamma^\beta \partial_\gamma^\beta W^{+} \|_{L^2_3(H^1)} \leq \text{C}(K) \left( \| W^{+} \|_{L^2_3} + \| \partial_\gamma^\beta \partial_\gamma^\beta W^{+} \|_{L^2_3} \right) \tag{3.32}$$

for $\beta \leq \alpha$ with $|\beta| \geq 2, |\alpha| = 1$, and

$$\| \partial_\gamma^\beta (A_0^r C^r) \partial_\gamma^\beta W^{+} \|_{L^2_3(H^1)} \leq \text{C}(K) \left( \| W^{+} \|_{L^2_3} + \| \partial_\gamma^\beta \partial_\gamma^\beta W^{+} \|_{L^2_3} \right) \tag{3.33}$$

as $\beta \leq \alpha$ with $|\beta| \geq 1$, by noting that $A_0^r C^r$ is a $C^\infty$ function of $(U^r, \nabla U^r, \nabla \Psi^r, \nabla \partial_\gamma^\beta \Psi^r)$ which vanishes at the origin while $A_0^r$ are $C^\infty$ of $(U^r, \nabla \Psi^r)$.

Adding (3.30), (3.31) and (3.32)-(3.33), we have

$$\| F^{(l)}_+ \|_{L^2_3(H^1)} \leq \text{C}(K) \left( \| F^{(l)}_+ \|_{L^2_3(H^1)} + \gamma \| W^{+} \|_{L^2_3} + \left( \| \partial_\gamma^\beta \partial_\gamma^\beta W^{+} \|_{L^2_3} + \| \partial_\gamma^\beta \partial_\gamma^\beta W^{+} \|_{L^2_3} \right) \right). \tag{3.34}$$
One can have a similar estimate for $F^{(l)}$, and conclude the following estimate for the source term of the equation given in (3.28),
\[
\|F^{(l)}\|_{L^2_t(H^1_\omega)} \leq C(K) \left( \|F\|_{L^2_t(H^1_\omega)} + \gamma \|W\|_{L^2_t(H^1_\omega)} + (\|V\|_{L^2_t(H^2_\omega)} + \|\partial_3 V\|_{L^2_t(H^1_\omega)}) \|W\|_{L^\infty(\Omega_\omega)} \right),
\]
(3.35)
where $W = (W^+, W^-)T$, $F = (F^+, F^-)T$, $V = (V_r, V_i)T$ and $\overline{\Psi} = (\overline{\Psi}_r, \overline{\Psi}_i)T$.

The estimate of the term $g^{(l)}$ can be studied similarly. From the right hand side of (3.27), we get
\[
\|g^{(l)}\|_{H^1_\omega} \leq C(K) \left( \|h^{(l)}\|_{H^1_\omega} + \|\partial_3 h^{(l)}\|_{H^1_\omega} + \|\partial_2^2 h^{(l)}\|_{H^1_\omega} \right) + (\|V\|_{L^2_t(H^2_\omega)} + \|\partial_3 V\|_{L^2_t(H^1_\omega)}) \|W\|_{L^\infty(\Omega_\omega)}
\]
(3.36)
(3.37)

Then, multiplying $\gamma^{-l}$ on (3.37) and taking the summation for $l$ from 0 to $s$, we obtain the estimate (3.25) after absorbing the following term
\[
\gamma^{-1/2} \|W\|_{L^2_t(H^1_\omega)} + \gamma^{-1} \|\partial_3 W\|_{H^1_\omega} \|W\|_{L^\infty(\Omega_\omega)}
\]
by the left hand side of (3.25). \hfill \Box

### 3.2.2 Estimate of “vorticities”

Since the boundary $|y = 0|$ in the problem (3.23) is characteristic for the equations, we can not control the normal derivatives $\partial_3 W^+$ and $\partial_3 W^+$ directly from the equations. Here we employ an idea inspired by the approach given in [13] to study “vorticities” from the original equations given in (3.16) of $(U_+, U_-)$, these vorticities are represented by $\partial_t \Omega^1, \partial_t \Omega^2$ and the tangential derivatives $\partial_\gamma W$.

Obviously, the first three equations of $U_+ = (u_+, v_+, w_+, p_+)^T$ given in (3.12) can be formulated as,
\[
\rho_r u_{\gamma x} + w_r u_\gamma + (\partial_\gamma - \frac{\partial_\gamma \Psi}{\partial_\gamma \Psi})p_+ = (f^+ - C(U_r, V_r, \Psi_r)U_+)_1,
\]
\[
\rho_r u_{\gamma x} + w_r v_\gamma + \frac{1}{\partial_\gamma \Psi} \partial_\gamma p_+ = (f^+ - C(U_r, V_r, \Psi_r)U_+)_2,
\]
\[
\rho_r u_{\gamma x} + w_r w_\gamma + (\partial_\gamma - \frac{\partial_\gamma \Psi}{\partial_\gamma \Psi})p_+ = (f^+ - C(U_r, V_r, \Psi_r)U_+)_3,
\]
(3.38)

with $(\cdot)$ denoting the $i$-th component of a vector. If we introduce “vorticities” defined by
\[
\xi_+ = (\partial_\gamma - \frac{\partial_\gamma \Psi}{\partial_\gamma \Psi})v_\gamma - \frac{1}{\partial_\gamma \Psi} \partial_\gamma u_\gamma, \quad \xi_+ = (\partial_\gamma - \frac{\partial_\gamma \Psi}{\partial_\gamma \Psi})v_\gamma - \frac{1}{\partial_\gamma \Psi} \partial_\gamma w_\gamma,
\]
(3.39)

then, from (3.38) we know that $(\xi_+, \xi_+)^T$ satisfy the following transport equations:
\[
\rho_r u_{\gamma x} + w_r \xi_+ = (f^+ - C(U_r, V_r, \Psi_r)U_+)_1,
\]
\[
\rho_r u_{\gamma x} + w_r \xi_+ = (f^+ - C(U_r, V_r, \Psi_r)U_+_3),
\]
(3.40)
\[
(\xi_+, \xi_+)|_{t=0} = 0,
\]
where
\[
\tilde{f}_i^+ = (f^+ - C(U_r, V_r, \Psi_r)U_+)_i, \quad i = 1, 2, 3
\]
(3.41)

and $R_1, R_2$ are $C^\infty$ vector valued functions depending on $(V_r, \nabla V_r, \nabla \overline{\Psi}_r, \nabla^2 \overline{\Psi}_r)$ and vanish at the origin.

For the problem (3.40), we have the estimates of $\xi_+$ and $\xi_+$ as follows,
Lemma 3.4. Let $s > 1$, there exist constants $C(K) > 0$ and $\gamma_s \geq 1$ such that for all $\gamma \geq \gamma_s$, one has
\[
\sqrt{\gamma} (\|\xi_+\|_{H^{s-1}_1(\Omega_\alpha)} + \|\zeta_+\|_{H^{s-1}_1(\Omega_\alpha)}) \leq \frac{C(K)}{\sqrt{\gamma}} \left( \|f^+\|_{H^s_1(\Omega_\alpha)} + \|f^+\|_{L^\infty(\Omega_\alpha)} \sqrt{\psi_r} \|H^s_1(\Omega_\alpha)} + \|U_+\|_{H^s_1(\Omega_\alpha)} + \|(V_r, \nabla V_r, \nabla \psi_r)\|_{H^s_1(\Omega_\alpha)}\|U_+\|_{W^{1,\infty}(\Omega_\alpha)} \right). \tag{3.42}
\]

Proof. From the problem (3.40), we have the following $L^2$-estimates immediately,
\[
\sqrt{\gamma} (\|\xi_+\|_{L^2_1(\Omega_\alpha)} + \sqrt{\gamma} (\|\xi_+\|_{L^2_1(\Omega_\alpha)}) \leq \frac{C(K)}{\sqrt{\gamma}} \left( \|H^s_1\|_{L^2_1(\Omega_\alpha)} + \sqrt{\gamma} (\|\xi_+\|_{L^2_1(\Omega_\alpha)}) \leq \frac{C(K)}{\sqrt{\gamma}} \|H^s_1\|_{L^2_1(\Omega_\alpha)}, \tag{3.43}
\]
with $H^s_1$ and $H^s_2$ denoting the corresponding right hand sides of the two equations given in (3.40).

(1) In order to get the higher order estimates, we take derivatives $\partial^\alpha = \partial_{x_1}^{a_1} \partial_{x_2}^{a_2} \partial_{x_3}^{a_3}$ with $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3 = l \leq s-1$ on both sides of the equation of $\xi_+$, given in (3.40) to get
\[
\rho(y, u, \partial_y \xi_+^l) = \partial^\alpha \xi_+^l = \rho(y, u, \partial_y \xi_+^l), \tag{3.44}
\]
where $\xi_+^l = \partial^\alpha \xi_+$. Using the Gagliardo-Nirenberg inequality, we know that
\[
\|\partial^\alpha \xi_+^l - [\partial^\alpha \rho(y, u, \partial_y \xi_+^l)]_{L^2_1(\Omega_\alpha)}) \leq C(K) \left( \|H^s_1\|_{L^2_1(\Omega_\alpha)} + \|\xi_+\|_{L^2_1(\Omega_\alpha)} + \|\nabla V_r\|_{H^s_1(\Omega_\alpha)} \|\xi_+\|_{L^2_1(\Omega_\alpha)} \right). \tag{3.45}
\]
Applying the same estimates as (3.43) for the equation (3.44), and using the above inequality, we obtain
\[
\frac{\sqrt{\gamma}}{\gamma) (\|\xi_+\|_{H^{s-1}_1(\Omega_\alpha)} \leq \frac{C(K)}{\sqrt{\gamma}} \left( \|H^s_1\|_{H^{s-1}_1(\Omega_\alpha)} + \|\xi_+\|_{L^\infty(\Omega_\alpha)} \|V_r\|_{H^s_1(\Omega_\alpha)} \right) \tag{3.46}
\]
by absorbing the term $\frac{\sqrt{\gamma}}{\gamma) (\|\xi_+\|_{H^{s-1}_1(\Omega_\alpha)}$ in the left hand side.

(2) Estimate of the term $H^s_1$. From the definition of $H^s_1$, we know that
\[
\|H^s_1\|_{H^{s-1}_1(\Omega_\alpha)} \leq \|\partial_y \psi_r\|_{H^{s-1}_1(\Omega_\alpha)} + \|\psi_r^1\|_{H^{s-1}_1(\Omega_\alpha)} + \|\nabla V_r\|_{H^{s-1}_1(\Omega_\alpha)} + \|\psi_r\|_{L^\infty(\Omega_\alpha)} \tag{3.46}
\]
From (3.41), one has
\[
\|\psi_r^1\|_{H^{s-1}_1(\Omega_\alpha)} \leq \frac{C(K)}{\sqrt{\gamma)} (\|\psi_r\|_{H^{s-1}_1(\Omega_\alpha)} + \|\nabla V_r\|_{H^{s-1}_1(\Omega_\alpha)} + \|\psi_r\|_{L^\infty(\Omega_\alpha)} \tag{3.47}
\]
by noting from (3.3) that
\[
\|C(U_r, \nabla U_r, \nabla \psi_r)\|_{H^{s-1}_1(\Omega_\alpha)} \leq C(K) \|(U_r, \nabla U_r, \nabla \psi_r)\|_{H^{s-1}_1(\Omega_\alpha)} \]
Plugging (3.47) into the inequality (3.46), we get the estimate of $H^s_1$ as follows,
\[
\|H^s_1\|_{H^{s-1}_1(\Omega_\alpha)} \leq C(K) \left( \|f^+\|_{H^s_1(\Omega_\alpha)} + \|\nabla \psi_r\|_{H^s_1(\Omega_\alpha)} \|f^+\|_{L^\infty(\Omega_\alpha)} + \|\nabla V_r\|_{H^s_1(\Omega_\alpha)} + \|U_+\|_{H^s_1(\Omega_\alpha)} \|U_+\|_{W^{1,\infty}(\Omega_\alpha)} \right). \tag{3.48}
\]
(3) Obviously, we have.
\[
\|\xi_+\|_{L^\infty(\Omega_\alpha)} \leq C(K) \|U_+\|_{W^{1,\infty}(\Omega_\alpha)}. \tag{3.49}
\]
Combining (3.45), (3.48) and (3.49), we obtain the following estimate of $\xi_+$ in the end,
\[
\sqrt{\gamma} (\|\xi_+\|_{H^{s-1}_1(\Omega_\alpha)} \leq \frac{C(K)}{\sqrt{\gamma)} (\|f^+\|_{H^s_1(\Omega_\alpha)} + \|f^+\|_{L^\infty(\Omega_\alpha)} \|\nabla \psi_r\|_{H^s_1(\Omega_\alpha)} + \|\nabla V_r\|_{H^s_1(\Omega_\alpha)} + \|U_+\|_{H^s_1(\Omega_\alpha)} \|U_+\|_{W^{1,\infty}(\Omega_\alpha)} \right). \tag{3.50}
\]
One can study $\xi_+$ similarly from the problem (3.40), and deduce the same estimate as (3.50) for $\xi_+$. This finishes the proof of this lemma.
\[\square\]
3.2.3 Estimate of normal derivatives

Thus, we obtain

\[
\xi_+ = (\partial_x - \frac{\partial W_1}{\partial \Psi_1} \partial_y - \frac{1}{\partial \Psi_1} \partial_y u_-, \quad \xi_- = (\partial_z - \frac{\partial W_1}{\partial \Psi_1} \partial_y w_-. \quad (3.51)
\]

From the equations given in (3.12), we deduce that \((\xi_-, \xi_-)\) also satisfy two transport equations similar to (3.40), and conclude

**Lemma 3.5.** Let \(s > 1\), there exist constants \(C(K) > 0\) and \(\gamma_s \geq 1\) such that for all \(\gamma \geq \gamma_s\), the following estimate holds,

\[
\sqrt{\chi} ||\xi_-||_{H^1(\Omega_x)} + ||\xi_-||_{H^1(\Omega_x)} \leq \frac{C(K)}{\sqrt{\gamma}} \left(||f^-||_{H^1(\Omega_x)} + ||f^-||_{L^\infty(\Omega_x)}||\nabla \Psi||_{H^1(\Omega_x)}
\]

\[
+ ||U^-||_{H^1(\Omega_x)} + ||(V_1, \nabla V_1, \nabla \Psi)||_{H^1(\Omega_x)}||U^-||_{L^\infty(\Omega_x)} \right).
\]

The estimates (3.42) and (3.52) will be used to study the normal derivatives of \(W^c = (W_1^+, W_2^+)\) in next subsection.

### 3.2.3 Estimate of normal derivatives

After studying the “vorticities” \((\xi_+, \xi_-)^T\) in the previous subsection, we try to represent \(\partial_y W^c = (\partial_y W_1^+, \partial_y W_2^+, \partial_y W_1^+, \partial_y W_2^+)\) in terms of \((\xi_+, \xi_-)^T\) and \(\partial_y W\).

From the transformations defined in (3.22), we get

\[
\begin{align*}
\partial_y u_+ &= \partial_y W_1^+ - \partial_x \Psi_1 (\partial_y W_2^+ + \partial_y W_2^+), \\
\partial_y v_+ &= \partial_x \Psi_1 \partial_y W_1^+ + \partial_x \Psi_2 \partial_y W_3^+ + \partial_y W_4^+ + (\partial_y T_r^{-1} W^+)_1, \\
\partial_y w_+ &= \partial_y W_2^+ - \partial_x \Psi_1 (\partial_y W_4^+ + \partial_y W_3^+), \\
\partial_y v_- &= \partial_y W_2^+ + \partial_x \Psi_1 \partial_y W_1^+ + \partial_x \Psi_2 \partial_y W_2^+ + \partial_y W_4^+ + (\partial_y T_r^{-1} W^+)_2, \\
\partial_y w_- &= \partial_y W_3^+ + \partial_x \Psi_1 \partial_y W_3^+ + \partial_x \Psi_2 \partial_y W_4^+ + (\partial_y T_r^{-1} W^+)_3,
\end{align*}
\]

which implies that

\[
\begin{align*}
(\partial_x v_+ - \xi_+) \partial_y \Psi_1 &= (1 + (\partial_x \Psi_1)^2) \partial_y W_3^+ + \partial_x \Psi_1 \partial_x \Psi_1 \partial_y W_4^+ + (\partial_x T_r^{-1} W^+)_1 + \partial_x \Psi_1 (\partial_y T_r^{-1} W^+)_2, \\
(\partial_y v_- - \xi_-) \partial_y \Psi_1 &= \partial_x \Psi_1 \partial_x \Psi_1 \partial_y W_2^+ + (1 + (\partial_x \Psi_2)^2) \partial_y W_4^+ + (\partial_y T_r^{-1} W^+)_3 + \partial_y \Psi_1 (\partial_y T_r^{-1} W^+)_2, \\
\end{align*}
\]

with \((\partial_y T_r^{-1} W^+)_i, (i = 1, 2, 3)\) representing the \(i\)-th component of the vector \(\partial_y T_r^{-1} W^+\) and \(T_{r,i} = T(\nabla \Psi_{r,i})\).

Thus, we obtain

\[
\begin{align*}
\partial_y W_1^+ &= \frac{1}{(\partial \Psi_1)^2} \left[ \partial_x \Psi_1 ((1 + (\partial_x \Psi_1)^2)(\partial_y v_+ - \xi_+) - \partial_x \Psi_1 \partial_x \Psi_1 (\partial_y v_+ - \xi_+) \right] \\
&- (1 + (\partial_x \Psi_1)^2)(\partial_y T_r^{-1} W^+)_1 - \partial_x \Psi_1 (\partial_y T_r^{-1} W^+)_2 + \partial_x \Psi_1 \partial_x \Psi_1 (\partial_y T_r^{-1} W^+)_3, \\
\partial_y W_2^+ &= \frac{1}{(\partial \Psi_1)^2} \left[ \partial_x \Psi_1 ((1 + (\partial_x \Psi_2)^2)(\partial_y v_- - \xi_-) - \partial_x \Psi_1 \partial_x \Psi_1 (\partial_y v_- - \xi_-) \right] \\
&+ \partial_x \Psi_1 \partial_x \Psi_1 (\partial_y T_r^{-1} W^+)_1 - \partial_x \Psi_1 (\partial_y T_r^{-1} W^+)_2 - (1 + (\partial_x \Psi_2)^2)(\partial_y T_r^{-1} W^+)_3.
\end{align*}
\]

From (3.54), (3.55) and the equations of \(W^+\) given in (3.23), we can represent \(\partial_y W^+\) by \(\xi_+, \xi_-\) and \(\partial_y W^+\) as follows

\[
\begin{align*}
\partial_y W^+ &= \Lambda_2 F^+ + \Lambda_1 T' W_2^+ + \Lambda_0 W^+ + \Lambda_0' W^+ \\
&+ \frac{\partial_x \Psi_1}{(\partial \Psi_1)^2} \begin{pmatrix}
- (1 + (\partial_x \Psi_1)^2) & \partial_x \Psi_1 \partial_x \Psi_1 \\
\partial_x \Psi_1 \partial_x \Psi_1 & -(1 + (\partial_x \Psi_1)^2)
\end{pmatrix} \begin{pmatrix}
\xi_+ \\
\xi_-
\end{pmatrix}.
\end{align*}
\]
where $\tilde{A}_0^l, \tilde{A}_1^l$ and $\tilde{A}_2^l$ are modifications of $A_0^l C', A_1^l$ and $A_2^l$ given in (3.24) after adding equations (3.54) and (3.55). Similarly, from the formulae of $\xi_-, \zeta_-$ and the equations of $W^-$ given in (3.24), one can derive a representation of $\partial_x W^-$ similar to that given in (3.56).

By using the equation (3.56) of $\partial_x W^\pm$, we get the following result of the normal derivative $\partial_x W$:

**Lemma 3.6.** For $s \geq 1$, there exists a constant $C(K) > 0$ such that for any integer $k$ ($1 \leq k \leq s$), the following estimate holds,

\[
||\partial_y^k W||_{L^2(\Omega)} \leq C(K) \left( ||F||_{H^{s-1}(\Omega)} + ||\xi||_{H^{s-1}(\Omega)} + ||\zeta||_{H^{s-1}(\Omega)} + ||W||_{L^2(\Omega)} + ||W||_{H^{s-1}(\Omega)} \right)
\]

\[
+ ||\xi||_{L^\infty(\Omega)} + ||\zeta||_{L^\infty(\Omega)} ||\nabla \Psi||_{H^{s-1}(\Omega)} + ||W||_{L^\infty(\Omega)} ||(V, \nabla \Psi)||_{H^{s-1}(\Omega)}
\]

where $\xi = (\xi_, \xi_-)^T, \zeta = (\zeta_+, \zeta_-)^T$ are defined in (3.39) and (3.51) respectively.

**Proof.** We shall only study the estimate of $W^+$ by induction on $k$, and the estimate of $W^-$ can be derived similarly.

(1) When $k = 1$, we study the estimate of $||\partial_x W^+||_{L^2(\Omega)}$ through the equations (3.56). As in the proof of Lemma 3.5 using Gagliardo-Nirenberg’s inequality we obtain that

\[
||\tilde{A}_0^l \partial_x W^+ + \tilde{A}_1^l \partial_x W^-||_{L^2(\Omega)} \leq C(K) \left( ||W^+||_{L^2(\Omega)} + ||W^-||_{L^2(\Omega)} ||(V, \nabla \Psi)||_{L^2(\Omega)} \right),
\]

\[
||\tilde{A}_0^l W^-||_{L^2(\Omega)} \leq C(K) \left( ||W^+||_{L^2(\Omega)} + ||W^-||_{L^2(\Omega)} ||(V, \nabla \Psi)||_{L^2(\Omega)} \right)
\]

by noting that $\tilde{A}_0^l$ is a $C^\infty$ function of $(V, \nabla V, \nabla \Psi, \nabla^2 \Psi)$ and vanishes at the origin. Moreover, the estimate for the terms of $(\xi_+, \zeta_+)$ appeared in (3.56) is in the following,

\[
\frac{\partial y^\psi}{\partial y^\phi} (\partial_x \Psi, \partial_y \Psi, \xi_- - (1 + (\partial_y \Psi)^2) \xi_+) ||_{L^2(\Omega)}
\]

\[
+ \frac{\partial y^\psi}{\partial y^\phi} (\partial_x \Psi, \partial_y \Psi, \xi_- - (1 + (\partial_y \Psi)^2) \xi_+ ||_{L^2(\Omega)}
\]

\[
\leq C(K) \left( ||\xi_+, \xi_-||_{L^2(\Omega)} + ||\xi_+, \xi_-||_{L^2(\Omega)} ||\nabla \Psi||_{L^2(\Omega)} \right)
\]

Thus, from the equations of $\partial_x W^+$ given in (3.56), we get

\[
||\partial_y W^+||_{L^2(\Omega)} \leq C(K) \left( ||F||_{H^{s-1}(\Omega)} + ||\xi_+||_{H^{s-1}(\Omega)} + ||\zeta_+||_{H^{s-1}(\Omega)} + ||W^+||_{L^2(\Omega)} \right)
\]

\[
+ ||\xi_-||_{L^\infty(\Omega)} ||\zeta_-||_{L^\infty(\Omega)} ||\nabla \Psi||_{H^{s-1}(\Omega)} + ||W^+||_{L^\infty(\Omega)} ||(V, \nabla \Psi)||_{H^{s-1}(\Omega)}
\]

which implies the estimate (3.57) for the case $k = 1$.

(2) Assuming that the estimate (3.57) holds for $k - 1$, we try to prove that it also holds for $k \leq s$. By taking derivatives with respect to $\gamma$ on the equation (3.56), we get

\[
\partial_y^k W^+ = \Lambda_2 \partial_y^k F + \partial_y^k \left[ \tilde{A}_0^l \partial_x W^+ + \tilde{A}_1^l \partial_x W^- + \tilde{A}_2^l W^+ + S \left( \xi_+ \right) \right]
\]

(3.60)

with $S$ denoting the zero-th order coefficient matrix of $(\xi_+, \zeta_+)^T$ given in (3.56). Obviously, we have

\[
||\Lambda_2 \partial_y^k F||_{L^2(\Omega)} \leq ||F||_{H^{s-1}(\Omega)}.
\]

To estimate $||\partial_y^k \tilde{A}_1^l \partial_x W^+||_{L^2(\Omega)}$, we define $\partial_y^k F = \partial_y^\alpha \partial_y^\beta F$ with $\alpha_1 + \alpha_2 = l \leq s - k$, and get

\[
\partial_y^k \partial_y^\alpha \partial_y^\beta \left( \tilde{A}_1^l \partial_x W^+ \right) = \tilde{A}_1^l \partial_y^\alpha \partial_y^\beta \partial_y^k \partial_x W^+ + \partial_y^\alpha \partial_y^\beta \partial_y^k \partial_x W^+
\]

and

\[
||\tilde{A}_1^l \partial_y^k \partial_x W^+||_{L^2(\Omega)} \leq C(K) ||\partial_y^k W^+||_{L^2(\Omega)},
\]

\[
||\partial_y^k \partial_x W^+||_{L^2(\Omega)} \leq C(K) \left( ||W^+||_{H^{s-1}(\Omega)} + ||W^+||_{L^\infty(\Omega)} ||(V, \nabla \Psi)||_{H^{s-1}(\Omega)} \right).
\]
Substituting the above estimates into (3.60) and taking weighted summation from \(l = 0\) to \(l = s - k\), we obtain that
\[
\|\partial_y^{k-1}(A_0^* \partial_y W^*)\|_{L^2_t(H^{s-k})} \leq C(K) \left( \|\partial_y^{k-1} W^*\|_{L^2_t(H^{s-k})} + \|W^*\|_{L^1_t(H^{s-k})} + \|\triangledown W^*\|_{L_t^\infty(H^{s-k})} \right).
\]
(3.61)

In the same way, we deduce that \(\|\partial_y^{k-1}(A_0^* \partial_y W^*)\|_{L^2_t(H^{s-k})}\) is also bounded by the right hand side of (3.61). Similarly, we can get
\[
\|\partial_y^{k-1}(A_0^* \partial_y W^*)\|_{L^2_t(H^{s-k})} \leq C(K) \left( \|W^*\|_{L^1_t(H^{s-k})} + \|\triangledown W^*\|_{L_t^\infty(H^{s-k})} \right),
\]
(3.62)
\[
\|\partial_y^{k-1}(\xi,\zeta)\|_{L^2_t(H^{s-k})} \leq C(K) \left( \|\xi\|_{L^1_t(H^{s-k})} + \|\zeta\|_{L^1_t(H^{s-k})} \right).
\]
(3.63)

Combining the inequalities (3.59), (3.61), (3.62), (3.63) with (3.58), and using the induction assumption \(\partial_y^{k-1} W^*\), we obtain the estimate (3.57) for all \(k \leq s\) and conclude this lemma.

By plugging the estimates of \(\xi, \zeta\) given in (3.42) and (3.52) into the inequality (3.57) and taking summation from \(k = 1\) to \(k = s\), one deduces

**Lemma 3.7.** Let \(s \geq 1\), there exist constants \(C(K)\) and \(\gamma_s \geq 1\) such that for all \(\gamma \geq \gamma_s\), the following inequality holds,
\[
\sqrt{\gamma} \sum_{k=1}^{\gamma} \|\partial_y^{k} W\|_{L^2_t(H^{s-k})} \leq C(K) \left( \sqrt{\gamma} (\|F\|_{H^{s-k}}(\Omega) + \|W\|_{L^2_t(H^s)} + \|W\|_{H^{s-k}(\Omega)}) \right.
\]
\[
\left. + \|W\|_{L^\infty(\Omega)}(\|V\|_{H^{s-k}(\Omega)} + \|\triangledown V\|_{H^{s-k}(\Omega)}) + \frac{1}{\sqrt{\gamma}} (\|f\|_{H^{s-k}(\Omega)} + \|U\|_{H^{s-k}(\Omega)}) \right)
\]
\[
\left. + \|f\|_{L^\infty(\Omega)}(\|\triangledown\psi\|_{H^{s-k}(\Omega)} + \|\triangledown\psi\|_{H^{s-k}(\Omega)}) + \|f\|_{L^1(\Omega)}(\|\triangledown\psi\|_{H^{s-k}(\Omega)}) \right)
\]
(3.64)

**3.2.4 Proof of Theorem 3.2**

After having the estimates given in (3.25) and (3.64) on tangential derivatives and normal derivatives of solutions to the problem (3.23), we are going to prove the estimate (3.20) given in Theorem 3.2.

From the definition of the space \(H^s_t(\Omega)\),
\[
\|W\|_{H^{s}_t(\Omega)} = \sum_{k=0}^{\gamma} \|\partial_y^{k} W\|_{L^2_t(H^{s-k})},
\]
and combining estimates (3.64) and (3.25), we obtain that
\[
\sqrt{\gamma} \|W\|_{H^s_t(\Omega)} + \|W^\infty_{\gamma = 0}\|_{H^{s-k}(\Omega)} + \|\phi\|_{H^{s-k}(\omega)} \leq C(K) \left( \frac{1}{\sqrt{\gamma}} (\|F\|_{H^s_t(\Omega)} + \|W\|_{L^2_t(H^s)} + \|W\|_{H^{s-k}(\Omega)}) \right.
\]
\[
\left. + \frac{1}{\gamma} (\|\phi\|_{H^{s-k}(\omega)} + \frac{1}{\gamma} (\|W\|_{L^\infty(\Omega)}(\|V\|_{H^{s-k}(\Omega)} + \|\triangledown V\|_{H^{s-k}(\Omega)}) + \|f\|_{L^1(\Omega)}(\|\triangledown\psi\|_{H^{s-k}(\Omega)}) + \|f\|_{L^1(\Omega)}(\|\triangledown\psi\|_{H^{s-k}(\Omega)}) \right)
\]
(3.65)
(1) From the definition of $F$ given in (3.23), we have
\[
\frac{1}{\sqrt[3]{Y}}\|F\|_{L^2(Y)} + \frac{1}{\gamma^{3/2}}\|F\|_{L^2(Y)} \leq C(K) \left( \frac{1}{\sqrt[3]{Y}}\|f\|_{L^2(Y)} + \frac{1}{\gamma^{3/2}}\|f\|_{L^2(Y)} \right).
\]  

(3.66)

From the transformation (3.22) between $U$ and $W$, we know that
\[
\|W\|_{L^\infty(\Omega)} \leq C(K)\|U\|_{L^\infty(\Omega)}, \quad \|W\|_{W^{1,\infty}(\Omega)} \leq C(K)\|U\|_{W^{1,\infty}(\Omega)},
\]

(3.67)

\[
\|U\|_{H^1(\Omega)} \leq C(K) \left( \|W\|_{H^1(\Omega)} + \|W\|_{L^\infty(\Omega)}\|
\]

(3.68)

\[
\|\|U\|_{L^2(\Omega)} \leq C(K) \left( \|W\|_{L^2(\Omega)} + \|W\|_{L^\infty(\Omega)}\|\n\]

(3.69)

By plugging the inequalities (3.66), (3.67), (3.68) and (3.69) into the right hand side of (3.65), and absorbing the term $\frac{1}{\sqrt[3]{Y}}\|W\|_{H^1(\Omega)}$ by the left hand side of (3.65), we deduce that
\[
\sqrt[3]{Y}\|U\|_{H^1(\Omega)} + \|\|U\|_{L^2(\Omega)} + \|\Phi\|_{H^1(\Omega)} \leq C(K) \left( \frac{1}{\sqrt[3]{Y}}\|f\|_{H^1(\Omega)} + \frac{1}{\gamma}g\|_{H^1(\Omega)} \right)
\]

(3.70)

\[
\left( \|W\|_{L^\infty(\Omega)} + \|\Phi\|_{W^{1,\infty}(\Omega)} \right) \leq C(K) \left( \|W\|_{L^\infty(\Omega)} + \|W\|_{L^\infty(\Omega)}\|\right).
\]

(3.71)

(2) To conclude the estimate (3.20) from (3.70), the main remaining task is to control the terms with $L^\infty$ norm and $W^{1,\infty}$ norm on the right hand side of (3.70). Obviously, one has
\[
\|f\|_{L^\infty(\Omega)} \leq \frac{C\gamma X}{\sqrt[3]{Y}}\|f\|_{H^1(\Omega)}, \quad \|U\|_{W^{1,\infty}(\Omega)} \leq C\gamma X\|U\|_{H^1(\Omega)},
\]

(3.72)

\[
\|\|U\|_{L^2(\Omega)} \leq \frac{C\gamma X}{\gamma}\|\|U\|_{H^1(\Omega)} + \|\Phi\|_{H^1(\Omega)} \leq C\gamma X\|\Phi\|_{H^1(\Omega)}.
\]

(3.73)

Using the above estimates in (3.70) and setting $s = 3$, we get
\[
\sqrt[3]{Y}\|U\|_{H^1(\Omega)} + \|\|U\|_{L^2(\Omega)} + \|\Phi\|_{H^1(\Omega)} \leq C(K) \left( \frac{1}{\sqrt[3]{Y}}\|f\|_{H^1(\Omega)} + \frac{1}{\gamma}g\|_{H^1(\Omega)} \right)
\]

(3.74)

\[
\|W\|_{L^\infty(\Omega)} \leq K,
\]

(3.75)

one can eliminate the terms $\|U\|_{H^1(\Omega)}$, $\|\|U\|_{L^2(\Omega)}$ and $\|\Phi\|_{H^1(\Omega)}$ on the right hand side of (3.74) by fixing $\gamma$ large enough, and concludes
\[
\sqrt[3]{Y}\|U\|_{H^1(\Omega)} + \|\|U\|_{L^2(\Omega)} + \|\Phi\|_{H^1(\Omega)} \leq C(K) \left( \frac{1}{\sqrt[3]{Y}}\|f\|_{H^1(\Omega)} + \frac{1}{\gamma}g\|_{H^1(\Omega)} + \frac{\gamma X}{\gamma^2}\|\|f\|_{H^1(\Omega)} + \frac{\gamma X}{\gamma^2}\|\Phi\|_{H^1(\Omega)} \right).
\]

(3.76)
which implies
\[ \| U \|_{W^{1,\infty}(|\Omega|_0)} + \| B U \|_{L^\infty(\omega_X)} + \| \phi \|_{W^{1,\infty}(|\omega_X|)} \leq C(K)(\| f \|_{H^s(\Omega_X)} + \| g \|_{H^4(\omega_X)}) \]

by using (3.71), and fixing a large \( \gamma > 0 \) with \( \gamma X \leq 1 \).

Substituting the above inequality into (3.70), it follows
\[ \sqrt{\gamma} \| U \|_{H^s(\Omega_X)} + \| B U \|_{H^s(\omega_X)} + \| \phi \|_{H^{s+1}(\omega_X)} \leq C(K) \left\{ \frac{1}{\gamma^{s/2}} \| f \|_{H^{s+1}(\Omega_X)} + \frac{1}{\gamma} \| g \|_{H^s(\omega_X)} \right\} \]
\[ + \frac{1}{\gamma^{s/2}} \| f \|_{L^\infty(\omega_X)} \| \nabla \phi \|_{H^{s+1}(\omega_X)} + \frac{1}{\gamma} \| f \|_{W^{1,\infty}(\omega_X)} \| \nabla \phi \|_{H^{s+1}(\omega_X)} \]
\[ + \left( \frac{1}{\gamma^{s/2}} \| V \|_{H^{s+1}(\omega_X)} + \frac{1}{\gamma} \| \partial_y V \|_{H^s(\omega_X)} \right) (\| f \|_{H^s(\Omega_X)} + \| g \|_{H^4(\omega_X)}) \] (3.74)

This completes the proof of the same estimate given in Theorem 3.2.

By fixing a large \( \gamma > 0 \) and then \( \gamma X \leq 1 \) in Theorem 3.2, we immediately obtain,

**Corollary 3.8.** For any \( s \geq 0 \), assume that the non-planar contact discontinuity given in (3.1) satisfies (2.13), (2.3) and \((V_{r,l}, \nabla \theta_{r,l}) \in H^{s+2}(\Omega_X) \cap H^5(\Omega_X)\) with
\[ \| \nabla \phi \|_{H^s(\Omega_X)} + \| V \|_{H^s(\Omega_X)} \leq K. \] (3.75)

Then, for the linear problem (3.16), there exists a constant \( K_0 > 0 \), for all \( K \leq K_0 \), there is a constant \( C(K, s) \) depending on \( K \) and \( s \), such that for all \((U, \phi) \in (H^{s+2}(\Omega_X) \times H^{s+2}(\omega_X)) \cap (H^5(\Omega_X) \times H^5(\omega_X))\), one has
\[ \| U \|_{H^s(\Omega_X)} + \| B U \|_{H^s(\omega_X)} + \| \phi \|_{H^{s+1}(\omega_X)} \]
\[ \leq C(K, s) \left\{ \| f \|_{H^{s+1}(\Omega_X)} + \| g \|_{H^{s+1}(\omega_X)} + \| \nabla \phi \|_{H^{s+1}(\Omega_X)} (\| f \|_{H^s(\Omega_X)} + \| g \|_{H^4(\omega_X)}) \right\}. \] (3.76)

### 4 Iteration scheme

The remainder of this work is to obtain the existence of solutions to the nonlinear problem (2.8) by constructing a proper iteration scheme. From the estimate (3.76), we know that there is loss of regularity of solutions to the linearized problem (3.16) with respect to \( f \) and \( g \), so as in [5, 13, 29], we shall adapt the Nash-Moser-Hörmander iteration scheme to study the nonlinear problem (2.8).

#### 4.1 Compatibility conditions and the zero-order approximate solution

Given initial data \((U^0_0(y, z), \psi_0(z))\) on \( x = 0 \) with \( U^0_0(y, z) = \overline{U}_{r,l} + \tilde{U}^\pm_0(y, z) \), \( \psi_0(z) = \psi_0(0, z) \) satisfying \( \tilde{U}^\pm_0 \in H^{s+2}(\mathbb{R}_+^2), \psi_0 \in H^{s+1}(\mathbb{R})\) for a fixed \( s \geq 3 \), and
\[ \text{Supp } \tilde{U}^\pm_0 \subseteq \{ y \geq 0, y^2 + z^2 \leq 1 \}, \quad \text{Supp } \psi_0 \subseteq [-1, 1], \] (4.1)
we first state the compatibility conditions for the existence of a classical solution to the nonlinear problem (2.8).

As in (2.7), first we extend \( \psi_0(z) \) to \( \tilde{\Psi}_0^\pm(y, z) \) supported in \( \{ y \geq 0, \sqrt{y^2 + z^2} \leq 1 + C(X) \} \) with \( C(X) \) being a function of \( X \) such that \( \tilde{\Psi}_0^\pm \in H^{s+2}(\mathbb{R}_+^2) \) with
\[ \| \tilde{\Psi}_0^\pm \|_{H^{s+2}(\mathbb{R}_+^2)} \leq C \| \psi_0 \|_{H^{s+1}(\mathbb{R})}, \] (4.2)
and \( \Psi_0^\pm(y, z) = \pm y + \tilde{\Psi}_0^\pm \) satisfy
\[ \partial_y \Psi_0^\pm \geq \frac{5}{6}, \quad \partial_y \Psi_0^\pm \leq \frac{-5}{6} \quad \text{for all } y > 0. \] (4.3)
Set
\[ \tilde{U}^\pm(x, y, z) = U^\pm(x, y, z) - \overline{U}_r, \quad \tilde{\Psi}^\pm(x, y, z) = \Psi^\pm(x, y, z) - y. \]

From the equations of \( \Psi^\pm \) and \( U^\pm \) given in (2.7) and (2.8), we can determine \( \partial_x^{j+1}\tilde{\Psi}^\pm \) and \( \partial_x^{j+1}\tilde{U}^\pm \) on \( \{x = 0\} \) by induction on \( j \in \mathbb{N} \) in the following,

\[ \partial_x^{j+1}\tilde{\Psi}^\pm = \partial_x^j\left( \frac{1}{u^\pm}(u^\pm - \partial_x\Psi^\pm w^\pm) \right) \]

\[ \text{(4.4)} \]

and
\[ \partial_x^{j+1}U^\pm = \partial_x^j\left( -A_1^{-1}\left( \frac{1}{\Psi_y^\pm}(A_2 - \Psi_x^\pm A_1 - \Psi_z^\pm A_3)\partial_y U^\pm + A_3\partial_z U^\pm \right) \right). \]

\[ \text{(4.5)} \]

Thus, for a fixed \( k \leq s \), the data \( (U_0^\pm, \Psi_0^\pm) \) are compatible up to order \( k \) for the problem (2.7) and (2.8), if
\[ \partial_x^j\Psi^\pm = \partial_x^j\Psi^- , \quad \partial_x^j p^+ = \partial_x^j \tilde{p}^- \]

\[ \text{(4.6)} \]

hold for all \( 0 \leq j \leq k \) at \( \{y = 0\} \cap \{x = 0\} \).

From now on, we assume the following hypothesis:

(H) for a fixed \( s > \frac{3}{2} \), the initial data
\[ U^\pm|_{x=0} = \overline{U}_r + \tilde{U}_0^\pm(y, z), \quad \Psi^\pm|_{x=0} = \pm y + \tilde{\Psi}_0^\pm(y, z) \]

with \( \tilde{U}_0^\pm \in H^{s+\frac{1}{2}}(\mathbb{R}_+^2), \tilde{\Psi}_0^\pm \in H^{s+\frac{1}{2}}(\mathbb{R}_+^2) \), satisfy the compatibility conditions of the problems (2.8) and (2.7) up to order \( s - 1 \).

Set \( \tilde{U}_0^{\pm,j} = \partial_x^j\tilde{U}^\pm|_{x=0}, \tilde{\Psi}_0^{\pm,j} = \partial_x^j\tilde{\Psi}^\pm|_{x=0} (0 \leq j \leq s - 1) \) defined in (4.5) and (4.4) respectively, then by the inverse trace theorem, we can construct functions \( \Psi^{0,\pm}, u^{0,\pm}, w^{0,\pm}, p^{0,\pm} \) in the domain \( \Omega_X \) satisfying

\[ \begin{cases} \left( u^{0,\pm} - \bar{u}^\pm, w^{0,\pm} - \bar{w}^\pm, p^{0,\pm} - \bar{p}^\pm \right) \in H^s(\Omega_X), & \Psi^{0,\pm} \equiv y \in H^{s+1}(\Omega_X), \\ \partial_x^j(u^{0,\pm} - \bar{u}^\pm, w^{0,\pm} - \bar{w}^\pm, p^{0,\pm} - \bar{p}^\pm)|_{x=0} = (\tilde{U}_0^{\pm,j}, \tilde{\Psi}_0^{\pm,j}, \tilde{p}_0^{\pm,j}), & \partial_x^j\tilde{\Psi}^{\pm,j}|_{x=0} = \tilde{\Psi}_0^{\pm,j}, \\ \Psi^{0,+} = \Psi^{0,-}, & p^{0,+} = p^{0,-}, & \text{on } \{y = 0\} \end{cases} \]

\[ \text{(4.7)} \]

for all \( 0 \leq j \leq s - 1 \).

Define
\[ v^{0,\pm} = \partial_x^j\Psi^{0,\pm} u^{0,\pm} + \partial_x^j\Psi^{0,\pm} w^{0,\pm}. \]

\[ \text{(4.8)} \]

By using the above compatibility conditions, we know that the approximate solutions \( U^a = (U^{a,+}, U^{a,-})^T \) with \( U^{0,\pm} = (u^{0,\pm}, v^{0,\pm}, w^{0,\pm}, p^{0,\pm})^T \), and \( \Psi^a = (\Psi^{a,+}, \Psi^{a,-})^T \) satisfy

\[ \begin{cases} \partial_x^jL(U^{a,\pm}, \Psi^{a,\pm})|_{x=0} = 0, & \text{for } j = 0, \ldots, s - 1, \\ \mathcal{B}(U^{a,+}, U^{a,-}, \Psi^a) = 0, & \text{on } \{y = 0\}, \end{cases} \]

\[ \text{(4.9)} \]

where \( \psi^a = \Psi^{a,+}|_{y=0} = \Psi^{a,-}|_{y=0}, L(U^{a,\pm}, \Psi^{a,\pm})U^a \) is defined in (2.9) and \( \mathcal{B}(U^{a,+}, U^{a,-}, \psi^a) \) denotes the boundary conditions given in (2.8), and

\[ \frac{\|U^{a,\pm}\|_{H^s(\Omega_X)} + \|U^{a,-}\|_{H^s(\Omega_X)} + \|\tilde{U}^{a,\pm}\|_{H^{s+1}(\Omega_X)} + \|\tilde{U}^{a,-}\|_{H^{s+1}(\Omega_X)}}{\|U^{0,\pm}\|_{H^s(\mathbb{R}^2)} + \|\tilde{U}_0^{\pm}\|_{H^s(\mathbb{R}^2)}} \leq C \left( \|\tilde{U}_0^{\pm}\|_{H^{s+\frac{1}{2}}(\mathbb{R}^2)} + \|\psi_0\|_{H^{s}(\mathbb{R})} \right), \]

\[ \text{(4.10)} \]

with
\[ U^{a,+} = U^{a,+} - \overline{U}_r, \quad U^{a,-} = U^{a,-} - \overline{U}_r, \quad \Psi^{a,\pm} = \Psi^{a,\pm} \mp y. \]
Moreover, if \( \tilde{U}_{\theta}^\pm, \psi_0 \) are properly small, we have
\[
\partial_y \Psi^a_{\pm} = \frac{2}{3}, \quad \partial_z \Psi^a_{\pm} = -\frac{2}{3}, \quad \forall (x, y, z) \in \Omega_\chi.
\]
Denoting by
\[
\begin{aligned}
V^\pm &= U^\pm - U^a_{\pm}, \\
\Phi^\pm &= \Psi^a - \Psi^a_{\pm},
\end{aligned} \tag{4.11}
\]
then from (4.9), (2.8) and (2.7) we know that \((V^\pm, \Phi^\pm)\) satisfy the following problem:
\[
\begin{aligned}
\mathcal{L}(V^\pm, \Phi^\pm)V^\pm &= f^\pm_a, \quad \text{in } \{x > 0, \ y > 0\} \\
\mathcal{E}(V^\pm, \Phi^\pm) &= 0, \quad \text{in } \{x > 0, \ y > 0\} \\
\mathcal{B}(V^+, V^-, \phi) &= 0, \quad \text{on } \{y = 0\} \\
V^\pm|_{x=0} &= 0, \quad \Phi^\pm|_{x=0} = 0
\end{aligned} \tag{4.12}
\]
where \( \phi = \Phi^+|_{y=0} = \Phi^-|_{y=0} \).

\[
f^\pm_a = \begin{cases} -L(U^a_{\pm}, \Psi^a_{\pm}) U^a_{\pm}, & \text{if } x > 0 \\ 0, & \text{if } x \leq 0 \end{cases}
\]

\[
\mathcal{L}(V^\pm, \Phi^\pm)V^\pm = L(U^a_{\pm} + V^\pm, \Psi^a_{\pm} + \Phi^\pm)(U^a_{\pm} + V^\pm) - L(U^a_{\pm}, \Psi^a_{\pm}) U^a_{\pm},
\]
\[
\mathcal{E}(V^\pm, \Phi^\pm) = \partial_s(\Psi^a_{\pm} + \Phi^\pm)V_1^\pm - V_2^\pm + \partial_s(\Psi^a_{\pm} + \Phi^\pm)V_3^\pm + \partial_\omega(\Phi^\pm \omega^a_{\pm}) + \partial_\omega(\Phi^\pm \omega^a_{\pm}),
\]
and
\[
\mathcal{B}(V^+, V^-, \phi) = \mathcal{B}(U^a_{(\pm)} + V^+, U^a_{(\pm)} + V^-, \psi^a + \phi).
\]

Moreover, from (4.10), we have
\[
\|f^\pm_a\|_{H^s(\Omega_\chi)} \leq C \left( \|\tilde{U}_{\theta}^\pm\|_{H^{s+1}(\mathbb{R}^2 \setminus \{0\})} + \|\psi_0\|_{H^{s+1}(\mathbb{R})} \right). \tag{4.13}
\]

### 4.2 Description of the iteration scheme.

To construct the Nash-Moser-Hörmander iteration scheme for the nonlinear problem (4.12), first let us recall a family of smoothing operators from [1] [6] [13] as follows:
\[
[S_\theta]_{\theta > 0} : L^2(\Omega_\chi) \rightarrow \bigcap_{s \geq 0} H^s(\Omega_\chi) \tag{4.14}
\]
satisfying
\[
\begin{aligned}
\|S_\theta u\|_{H^s(\Omega_\chi)} \leq C\theta^{(s-\alpha)}, & \|u\|_{H^s(\Omega_\chi)}, \quad \text{for all } s, \alpha \geq 0 \\
\|S_\theta u - u\|_{H^s(\Omega_\chi)} \leq C\theta^{(s-\alpha)} \|u\|_{H^s(\Omega_\chi)}, & \text{for all } 0 \leq s \leq \alpha \\
\|\frac{d}{d\theta} S_\theta u\|_{H^s(\Omega_\chi)} \leq C\theta^{s-\alpha-1} \|u\|_{H^s(\Omega_\chi)}, & \text{for all } s, \alpha \geq 0
\end{aligned} \tag{4.15}
\]
and
\[
\|(S_\theta u^+ - S_\theta u^-)\|_{y=0} \leq C\theta^{(s+1-\alpha)} \|(u^+ - u^-)\|_{y=0} \|u\|_{H^s(\omega_\chi)}, \quad \text{for all } s, \alpha \geq 0. \tag{4.16}
\]

Similarly, one has a family of smoothing operators, still denoted by \([S_\theta]_{\theta > 0}\) acting on \(H^s(\omega_\chi)\), and (4.15) holds as well for norms of \(H^s(\omega_\chi)\). Let \(\theta_0 \geq 1, \theta_n = \sqrt{\theta_0^2 + n}\) for any \(n \geq 1\), and \(S_{\theta_n}\) be the associated smoothing operators defined above.
For the problem (4.12), let \( V^\pm_0 = \Phi_0^\pm = 0 \), and suppose that for any fixed \( n \geq 0 \), the approximate solutions \( (V^\pm_k, \Phi^\pm_k) \) of (4.12) have been constructed, satisfying
\[
\begin{align*}
(V^\pm_k, \Phi^\pm_k)|_{t=0} &= 0, \\
(\Phi^\pm_k)|_{y=0} &= \phi_k, \quad (1 \leq k \leq n).
\end{align*}
\] (4.17)

The \((n+1)\)-th approximate solutions \((V^\pm_{n+1}, \Phi^\pm_{n+1})\) of (4.12) is constructed as
\[
V^\pm_{n+1} = V^\pm_n + \delta V^\pm_n, \quad \Phi^\pm_{n+1} = \Phi^\pm_n + \delta \Phi^\pm_n, \quad \phi_{n+1} = \phi_n + \delta \phi_n,
\] (4.18)

where the increments \( \delta V^\pm_n, \delta \Phi^\pm_n, \delta \phi_n \) satisfy the following linear problem
\[
\begin{align*}
L'_c(U^{a,\pm} + V^\pm_{n+\frac{1}{2}}, \Psi^{a,\pm} + S_{\theta a} \Phi^\pm_n)(\delta V^\pm_n) &= f^\pm_n, \quad \text{in } \Omega_X, \\
B'_n(\delta \tilde{V}^\pm_n, \delta \phi_n) &= g_n, \quad \text{on } \{ y = 0 \}
\end{align*}
\] (4.19)

where \( L'_c(U^\pm, \Phi^\pm) V^\pm \) is the effective linearized operator defined in (3.12), \( V^\pm_n \) is a modified state of \( V^\pm_n \) such that the constraint (3.9) holds for \((U^{a,\pm} + V^\pm_{n+\frac{1}{2}}, \Psi^{a,\pm} + S_{\theta a} \Phi^\pm_n)\), which will be given in (5.19)-(5.20),
\[
B'_n = B'_c(U^{a,\pm} + V^\pm_{n+\frac{1}{2}}, \Psi^{a,\pm} + S_{\theta a} \Phi^\pm_n)
\]
is the effective boundary operator defined in (3.13) at the state \((U^a + V^\pm_{n+\frac{1}{2}}, \Psi^a + S_{\theta a} \phi_n)\),
\[
\delta \tilde{V}^\pm_n = \delta V^\pm_n - \delta \Phi^\pm_n \frac{\partial_j(U^{a,\pm} + V^\pm_{n+\frac{1}{2}})}{\partial_j(\Psi^{a,\pm} + S_{\theta a} \Phi^\pm_n)}
\] (4.20)
is the good unknown introduced in (3.10).

To define the source term \( f^\pm_n \) for the equations of (4.19), obviously, we have
\[
\begin{align*}
&L(U^{a,\pm} + V^\pm_{n+\frac{1}{2}}, \Psi^{a,\pm} + S_{\theta a} \Phi^\pm_n)(U^{a,\pm} + V^\pm_{n+\frac{1}{2}}) - L(U^{a,\pm} + V^\pm_{n}, \Psi^{a,\pm} + \Phi^\pm_n)(U^{a,\pm} + V^\pm_{n}) \\
&= L'(U^{a,\pm} + V^\pm_{n}, \Psi^{a,\pm} + \Phi^\pm_n)(\delta V^\pm_n) - L(U^{a,\pm} + V^\pm_{n+\frac{1}{2}}, \Psi^{a,\pm} + S_{\theta a} \Phi^\pm_n)(U^{a,\pm} + V^\pm_{n+\frac{1}{2}}) \\
&= L(U^{a,\pm} + V^\pm_n, \Psi^{a,\pm} + S_{\theta a} \Phi^\pm_n)(\delta V^\pm_n, \delta \Phi^\pm_n) + e_{n}^{+1} + e_{n}^{-1} + e_{n}^{+2} + e_{n}^{-2} + e_{n}^{+3} + e_{n}^{-3} + e_{n}^{+4} + e_{n}^{-4}
\end{align*}
\] (4.21)

with errors \( e_{n}^{+1} \) arising from the Newton iteration, \( e_{n}^{+2} \) and \( e_{n}^{-2} \) arising from the substitutions in the coefficient functions of the linearized operator \( L' \) from \( V^\pm_{n} \) to \( S_{\theta a} V^\pm_{n} \), and from \( S_{\theta a} V^\pm_{n} \) to \( V^\pm_{n+\frac{1}{2}} \) respectively, and
\[
e_{n}^{+4,4} = \frac{\partial \Phi^\pm_n}{\partial_j(\Psi^{a,\pm} + S_{\theta a} \Phi^\pm_n)} \delta_j \left[ L(U^{a,\pm} + V^\pm_{n+\frac{1}{2}}, \Psi^{a,\pm} + S_{\theta a} \Phi^\pm_n)(U^{a,\pm} + V^\pm_{n+\frac{1}{2}}) \right]
\]

arising from the use of good unknown \( \delta \tilde{V}^\pm_n \) from \( \delta V^\pm_n \). To guarantee the limit of \((V^\pm_n, \Phi^\pm_n)\) defined in (4.18)-(4.19) being the solution of the problem (4.12), we define the source term \( f^\pm_n \) to satisfy,
\[
\sum_{j=0}^{n} f^\pm_j + S_{\theta a} E^\pm_n = S_{\theta a} f^\pm_n, \quad \forall n \geq 1
\]

where \( f^\pm_0 = S_{\theta a} f^\pm_a \), and \( E^\pm_n = \sum_{l=0}^{n-1} e^\pm_l \) with \( e^\pm_l = \sum_{j=1}^{l} e_j^{\pm,j} \), i.e.
\[
f^\pm_n = (S_{\theta a} - S_{\theta a-1}) f^\pm_n - (S_{\theta a} - S_{\theta a-1}) E^\pm_{n-1} - S_{\theta a} e^\pm_{n-1}.
\] (4.22)
The source term $g_0$ of the boundary condition given in (4.19) can be defined in a similar way. It is obvious that

$$
\mathcal{B}(U^{0,+} + V^{0,+}_{n+1}, U^{0,-} + V^{0,-}_{n+1}, \psi^0 + \phi_{n+1}) - \mathcal{B}(U^{0,+} + V^+_n, U^{0,-} + V^-_n, \psi^0 + \phi_n)
$$

\begin{equation}
= B'(U^{0,+} + V^{0,+}_n, \phi^0 + \phi_n) \cdot (\delta V^+_n, \delta V^-_n, \delta \phi_n) + \tilde{e}_n^{(1)}
\end{equation}

noting that $\mathcal{B}(\psi^0, \phi_0) = 0$, to guarantee the limit of $(V^+_n, \Phi^+_n)$ satisfies the boundary condition given in (4.19) to satisfy,

$$
\sum_{j=0}^{n} g_j + \delta \Phi^+_n = 0
$$

by induction on $n$, with $g_0 = 0$, and $\tilde{e}_n = \sum_{j=1}^{4} \tilde{e}^{(j)}_n$. The next goal is to construct $\delta \Phi^+_n$ such that $\delta \Phi^+_n = 0$, this will use the idea from [13].

From the first two components of the boundary conditions given in (4.19), we know that $\delta \Phi^+_n$ satisfies

\begin{equation}
(U^{0,+}_1 + V^{0,+}_{n+\frac{1}{2}}) \partial_x(\delta \phi_n) + (U^{0,+}_3 + V^{0,+}_{n+\frac{1}{2}}) \partial_z(\delta \phi_n) - \delta V^+_{n+\frac{1}{2}} = g_{n,1}
\end{equation}

and

\begin{equation}
(U^{0,-}_1 + V^{0,-}_{n+\frac{1}{2}}) \partial_x(\delta \phi_n) + (U^{0,-}_3 + V^{0,-}_{n+\frac{1}{2}}) \partial_z(\delta \phi_n) - \delta V^-_{n+\frac{1}{2}} = g_{n,2}
\end{equation}

on $\{y = 0\}$, this inspires us to define $\delta \Phi^+_n$ by solving the problems

\begin{equation}
\begin{cases}
(U^{0,+}_1 + V^{0,+}_{n+\frac{1}{2}}) \partial_x(\delta \Phi^+_n) + (U^{0,+}_3 + V^{0,+}_{n+\frac{1}{2}}) \partial_z(\delta \Phi^+_n) - \delta V^+_{n+\frac{1}{2}} + \partial_x(\psi^0 + S_{\theta_0} \phi_0) \delta V^+_{n,1} + \partial_z(\psi^0 + S_{\theta_0} \phi_0) \delta V^+_{n,3} = \Xi g_{n,1} + h^+_n
\end{cases}
\end{equation}

\begin{equation}
\delta \Phi^+_n|_{x=0} = 0
\end{equation}

and

\begin{equation}
\begin{cases}
(U^{0,-}_1 + V^{0,-}_{n+\frac{1}{2}}) \partial_x(\delta \Phi^-_n) + (U^{0,-}_3 + V^{0,-}_{n+\frac{1}{2}}) \partial_z(\delta \Phi^-_n) - \delta V^-_{n+\frac{1}{2}} + \partial_x(\psi^0 - S_{\theta_0} \phi_0) \delta V^-_{n,1} + \partial_z(\psi^0 - S_{\theta_0} \phi_0) \delta V^-_{n,3} = \Xi g_{n,2} + h^-_n
\end{cases}
\end{equation}

\begin{equation}
\delta \Phi^-_n|_{x=0} = 0
\end{equation}
where \( \mathbb{B} \) is a proper extension operator from \( H^p(\omega_X) \) to \( H^{p+\frac{1}{2}}(\Omega_X) \), and \( h^n_k \) need to be determined such that \( h^n_k|_{y=0} = h^n_k|_{y=0} = 0 \), and \( \delta \Phi^\pm_n = \delta \Phi^\pm_n \) on \( y = 0 \).

To determine \( h^n_k \), let us study an iteration scheme for the eikonal equation \( E(V^\pm, \Phi^\pm) = 0 \) given in (4.12).

Obviously, we have
\[
E(V^\pm_{n+1}, \Phi^\pm_{n+1}) - E(V^\pm_n, \Phi^\pm_n) = E'(V^\pm_n, \Phi^\pm_n) (\delta V^\pm_n, \delta \Phi^\pm_n) + \mathbb{E}_{x,n} \tag{4.29}
\]
where
\[
E'(V^\pm, \Phi^\pm)(W^\pm, \Theta^\pm) = (\nu^\pm, V^\pm) \partial_x \Theta^\pm + (\nu^\pm, V^\pm) \partial_x \Theta^\pm - W^\pm + \partial_z(\Psi^\pm + \Phi^\pm)W^\pm + \partial_z(\Psi^\pm + \Phi^\pm)W^\pm \tag{4.30}
\]
is the linearized operator of \( E \).

Thus, from (4.27), (4.28), (4.29) and \( E(V^\pm_0, \Phi^\pm_0) = 0 \) we get
\[
E(V^\pm_{n+1}, \Phi^\pm_{n+1}) = \sum_{k=0}^n (\mathbb{E}_k + h^n_k + \mathbb{E}_{x,k}) \tag{4.33}
\]
by using (4.23). Therefore, we define \( h^n_k \) through
\[
\sum_{k=0}^n h^n_k + S_\theta_k \left( \sum_{k=0}^{n-1} (\mathbb{E}_{x,k} - \mathbb{E}_k) + \sum_{k=0}^n h^n_k + \mathbb{E}_{x,k} \right) = 0 \tag{4.34}
\]
by induction on \( k \). Similarly, from the equations of \( E(V^\pm_{n+1}, \Phi^\pm_{n+1}) \) and \( (\mathbb{B}(U^\pm, + V^\pm_{n+1}, U^\pm_{n+1}, \Psi^\pm + \Phi_{n+1}))_1 \) given in (4.29) and (4.23) respectively, we define \( h^n_k \) by
\[
\sum_{k=0}^n h^n_k + S_\theta_k \left( \sum_{k=0}^{n-1} (\mathbb{E}_{x,k} - \mathbb{E}_k) \right) = 0 \tag{4.35}
\]
The steps for determining \( \delta V^\pm_n, \delta \Phi^\pm_n, \delta \phi_n \) are to solve \( \delta V^\pm_n \) from (4.19) first, then to solve \( \delta \Phi^\pm_n \) from (4.27)-(4.28), which yields \( \delta \phi_n = \delta \Phi^\pm_n |_{y=0} = 0 \) satisfying (4.25) and (4.26).
5 Estimate of approximate solutions and convergence

5.1 Convergence of the iteration scheme

For fixed $s_0 > \frac{5}{2}$, $\alpha \geq s_0 + 5$ and $\alpha + 6 \leq s_1 \leq 2\alpha - s_0 + 1$.
Suppose that the first approximate solutions constructed in §4.1 satisfy

$$\begin{cases}
||\mathcal{L}^{\alpha,x}||_{s_1+2,X} + ||\mathcal{W}^{\alpha,x}||_{s_1+3,X} + ||f^a||_{s_1+1,X} \leq \delta, \\
||f^a||_{\alpha+1,X}/\delta \text{ is small, } ||f^a||_{\alpha+2,X}/\delta \text{ is bounded}
\end{cases} \tag{5.1}$$

for a small $\delta > 0$, where and hereafter we shall use $|| \cdot ||_{s,X}$ to denote the norm in the space $H^s(\Omega_X)$ for simplicity.

For the iteration scheme (4.18)-(4.19), we make the following inductive assumption

$$(H_n) \quad \begin{cases}
||\delta V^\pm_k, \partial \Phi^\pm_k||_{s,X} + ||\partial \phi_k||_{H^{s+1}(\omega_X)} \leq \delta \theta^{s+1-1} \Delta_k, & 0 \leq k \leq n - 1, s_0 \leq s \leq s_1 \\
||\mathcal{L}(V^\pm_k, \Phi^\pm_k)V^\pm_k - f^a||_{s,X} \leq \delta \theta^{s+1-1}, & 0 \leq k \leq n, s_0 \leq s \leq s_1 - 2 \\
||\mathcal{R}(V^\pm_k, \Phi^\pm_k, \phi_k)||_{H^{s+1}(\omega_X)} \leq \delta \theta^{s+1-1}, & 0 \leq k \leq n, s_0 \leq s \leq s_1 - 2
\end{cases}$$

with $\Delta_k = \theta_{k+1} - \theta_k$.

Temporarily, we suppose the above inductive assumption being true for all $n \geq 1$, then we can conclude the main result, Theorem 2.1 immediately.

Proof of Theorem 2.1:
From $(H_n)$ for any $n \geq 0$, we get

$$\sum_{k \geq 0} (||\delta V^\pm_k, \partial \Phi^\pm_k||_{s-1,X} + ||\partial \phi_k||_{H^{s}(\omega_X)}) < +\infty \tag{5.2}$$

which implies that there exist $V^\pm, \Phi^\pm$ in $H^{s-1}(\Omega_X)$ and $\phi$ in $H^s(\omega_X)$ such that

$$\begin{cases}
(V^\pm_n, \Phi^\pm_n) \longrightarrow (V^\pm, \Phi^\pm) \text{ in } H^{s-1}(\Omega_X) \text{ as } n \to +\infty, \\
\phi_n \longrightarrow \phi \text{ in } H^s(\omega_X)
\end{cases} \tag{5.3}$$

and $(V^\pm, \Phi^\pm, \phi)$ are solutions to the problem (4.12).

Thus, we conclude

Theorem 5.1. For any fixed $\alpha > \frac{15}{2}$ and $s_1 \geq \alpha + 6$. Suppose that $\psi_0 \in H^{s_1}(\mathbb{R})$, $U^\pm_0 - \overline{U}_{r,l} \in H^{s_1-\frac{3}{2}}(\mathbb{R}^2_\omega)$ satisfy the compatibility conditions of the problem (2.8) up to order $s_1 - 1$, and the conditions (2.13) and (5.7) are satisfied. Then, there exist solutions $(V^\pm, \Phi^\pm) \in H^{s_1-1}(\Omega_X)$ and $\phi \in H^s(\omega_X)$ to the problem (4.12).

The remaining main task is to estimate solutions of problems (4.19) and (4.27)-(4.28) to verify the inductive assumption $(H_n)$ for all $n \geq 1$.

5.2 Estimates of errors and approximate solutions

The main step for verifying $(H_{n+1})$ under the assumption of $(H_n)$ is to estimate errors appeared in the Nash-Moser iteration scheme (4.19) and (4.27)-(4.28), we shall mainly follow the arguments similar to that given in [5] [13].

First, from $(H_n)$, we immediately have
Lemma 5.2. The following estimates hold:

\[
\begin{align*}
\|V_k^+\|_{s_0, \infty} + \|\phi_k\|_{H^{s_0+2}((\omega_k))} & \leq C\delta \theta_{k}^{(s_0-\alpha)}, \quad s_0 \leq s \leq s_1, \ s \neq \alpha \\
\|V_k^+\|_{s_0, \infty} + \|\phi_k\|_{H^{s_0+2}((\omega_k))} & \leq C\delta \log \theta_k \\
\|S_{\theta_k} V_k^+\|_{s_0, \infty} + \|S_{\theta_k} \phi_k\|_{H^{s_0+2}((\omega_k))} & \leq C\delta \theta_k^{(s_0-\alpha)}, \quad s \geq s_0, \ s \neq \alpha \\
\|S_{\theta_k} V_k^+\|_{s_0, \infty} + \|S_{\theta_k} \phi_k\|_{H^{s_0+2}((\omega_k))} & \leq C\delta \log \theta_k \\
\|\Phi_k^+\|_{s_0, \infty} + \|\phi_k\|_{H^{s_0+2}((\omega_k))} & \leq C\delta \theta_k^{(s_0-\alpha)}, \quad s \leq s_1
\end{align*}
\]

for all \(0 \leq k \leq n\).

Lemma 5.3. For the quadratic errors \(e_k^{\pm, 1}, \tilde{e}_k^{(1)}\) and \(\tilde{\varepsilon}_k^{(1)}\) given in (4.21), (4.32) and (4.23) respectively, we have

\[
\begin{align*}
\|e_k^{\pm, 1}\|_{s_0, 0} & \leq C\theta_k^{(s_0-\alpha)} \Delta_k, \quad s_0 - 1 \leq s \leq s_1 - 1 \\
\|\tilde{e}_k^{(1)}\|_{s_0, 0} & \leq C\theta_k^{(s_0-2\alpha-2)} \Delta_k, \quad s_0 \leq s \leq s_1 - 1 \\
\|\tilde{\varepsilon}_k^{(1)}\|_{H^s((\omega_k))} & \leq C\theta_k^{(s_0+2\alpha-2s)} \Delta_k, \quad s_0 \leq s \leq s_1 - \frac{1}{2}
\end{align*}
\]

for all \(k \leq n - 1\), where

\[
L_1(s) = \max((s+1-\alpha)_+, 2(s_0 - \alpha - 1), s + s_0 - 2\alpha - 2).
\]

Proof. We can get estimates of \(\tilde{e}_k^{(1)}\) and \(\tilde{\varepsilon}_k^{(1)}\) much easier than that of \(e_k^{\pm, 1}\) by using their explicit expressions and the inductive assumption \((H_n)\), so we shall only study \(e_k^{\pm, 1}\) in detail. Obviously, we have

\[
e_k^{\pm, 1} = \int_0^1 (1 - \tau) L''(U_{\tau, k,l}, \Phi_1^+, \Phi_2^+, \tau \delta V_k^+, \tau \delta \phi_k^+)((\delta V_k^+, \delta \Phi_1^+, \delta V_k^+, \delta \Phi_2^+)) d\tau
\]

(5.7)

From (5.1), \((H_n)\) and Lemma 5.2 we get

\[
\begin{align*}
\|U_{\tau, k,l} + \delta V_k^+ + \tau \delta V_k^+\|_{s_0, \infty} & \leq C\delta (1 + \theta_k^{(s_0-\alpha)} + \theta_k^{(s_0-2\alpha-2)}), \quad \forall s_0 \leq s \leq s_1, \ s \neq \alpha \\
\|U_{\tau, k,l} + \delta V_k^+ + \tau \delta V_k^+\|_{s_0, \infty} & \leq C\delta (1 + \log \theta_k + \theta_k^{(s_0-2\alpha-2)})
\end{align*}
\]

(5.8)

for all \(k \leq n - 1\) and \(0 \leq \tau \leq 1\), which implies

\[
\sup_{0 \leq \tau \leq 1} \|U_{\tau, k,l} + \delta V_k^+ + \tau \delta V_k^+\|_{W^{1,s}((\omega_k))} \leq C\delta.
\]

(5.9)

On the other hand, obviously we have

\[
\begin{align*}
\|L''(U_{\tau, k,l}, \Phi_1^+, \Phi_2^+, \tau \delta V_k^+, \tau \delta \phi_k^+)\|_{s_0, \infty} & \leq C\|\hat{U}_{\tau, k,l}\|_{s_0+1, \infty} \|\hat{V}_1^+, \Phi_1^+\|_{W^{1,s_0+1}} \|\hat{V}_2^+, \Phi_2^+\|_{W^{1,s_0+1}} \\
+ & \|\hat{V}_1^+, \Phi_1^+\|_{s_0+1, \infty} \|\hat{V}_2^+, \Phi_2^+\|_{W^{1,s_0+1}} \|\hat{V}_2^+, \Phi_2^+\|_{s_0+1, \infty}.
\end{align*}
\]

(5.10)

Therefore, by using (5.8), \((H_n)\) and Lemma 5.2, we have

\[
\begin{align*}
\|e_k^{\pm, 1}\|_{s_0, \infty} & \leq C\delta (\delta \theta_k^{(s_0-\alpha-1)} \Delta_k) \{1 + \theta_k^{(s_0-\alpha-1)} + \theta_k^{(s_0-2\alpha-1)}\} + C\delta \theta_k^{(s_0-2\alpha-2) \Delta_k^2} \\
& \leq C\delta \theta_k^{(s_0-\alpha) \Delta_k}
\end{align*}
\]

(5.11)

as \(s_0 > \frac{5}{2}\), where \(L_1(s) = \max((s + 1 - \alpha)_+, 2(s_0 - \alpha - 1), s + s_0 - 2\alpha - 2)\), for all \(s_0 - 1 \leq s \leq s_1 - 1\) with \(s \neq \alpha - 1\), and

\[
\|e_k^{\pm, 1}\|_{s_0-1, \infty} \leq C\delta (\delta \theta_k^{(s_0-\alpha-1)} \Delta_k) \{1 + \log \theta_k + \theta_k^{(s_0-\alpha-1)}\} + C\delta \theta_k^{(s_0-\alpha-2) \Delta_k^2} \leq C\delta \theta_k^{L_1(\alpha-1) \Delta_k}.
\]

(5.12)

Thus, we conclude the first result given in (5.5).
Lemma 5.4. For the errors \( e_{k}^{\pm,2}, e_{k}^{(2)} \) and \( \tilde{e}_{k}^{(2)} \) given in (4.21), (4.32) and (4.23) respectively, we have

\[
\begin{align*}
\|e_{k}^{\pm,2}\|_{L^{2}} & \leq C\delta_{2}\theta_{k}^{2}(s)\Delta_{k}, \quad s_{0} \leq s \leq s_{1} - 1 \\
\|e_{k}^{(2)}\|_{L^{2}} & \leq C\delta_{2}\theta_{k}^{2+s+\sigma_{2d}}\Delta_{k}, \quad s_{0} \leq s \leq s_{1} - 1 \\
\|\tilde{e}_{k}^{(2)}\|_{H^{2}(\omega,s)} & \leq C\delta_{2}\theta_{k}^{2+s+\sigma_{2d}+\frac{1}{2}}\Delta_{k}, \quad s_{0} \leq s \leq s_{1} - \frac{1}{2}
\end{align*}
\]
for all \( k \leq n - 1 \), where

\[ L_{2}(s) = \max((s + 1 - \alpha)_{+} + 2(s_{0} - \alpha), s + s_{0} - 2\alpha). \]

Proof. As in Lemma 5.3 we shall only study \( e_{k}^{\pm,2} \) in detail, the estimate of \( \tau_{k}^{(2)} \) and \( \tilde{e}_{k}^{(2)} \) can be easily obtained by using \((H_{n})\).

From the definition of \( e_{k}^{\pm,2} \), obviously we have

\[
e_{k}^{\pm,2} = \int_{0}^{1} L''(U_{n+1}^{\pm} + S_{\theta_{k}} V_{k}^{\pm} + \tau(1 - S_{\theta_{k}}) V_{k}^{\pm})W_{1,\infty}(\Omega_{j}) + \|\Psi_{n+1}^{\pm} + S_{\theta_{k}} \Phi_{k}^{\pm} + \tau(1 - S_{\theta_{k}}) \Phi_{k}^{\pm})W_{1,\infty}(\Omega_{j}) \leq C\delta.
\]

As in \((5.9)\), from the assumption \((H_{n})\) we have

\[
\sup_{0 \leq \tau \leq 1} \left( \|U_{n+1}^{\pm} + S_{\theta_{k}} V_{k}^{\pm} + \tau(1 - S_{\theta_{k}}) V_{k}^{\pm}\|_{W_{1,\infty}(\Omega_{j})} + \|\Psi_{n+1}^{\pm} + S_{\theta_{k}} \Phi_{k}^{\pm} + \tau(1 - S_{\theta_{k}}) \Phi_{k}^{\pm})W_{1,\infty}(\Omega_{j}) \right) \leq C\delta.
\]

Therefore, by using \((5.10)\) in \((5.15)\) we obtain

\[
\|e_{k}^{\pm,2}\|_{L^{2}} \leq C \left( \|\delta V_{k}^{\pm}, \delta \Phi_{k}^{\pm})W_{1,\infty}(\Omega_{j}) + (1 - S_{\theta_{k}})(V_{k}^{\pm}, \Phi_{k}^{\pm}))W_{1,\infty}(\Omega_{j}) \right) + \|\delta V_{k}^{\pm}, \delta \Phi_{k}^{\pm})W_{1,\infty}(\Omega_{j}) + (1 - S_{\theta_{k}})(V_{k}^{\pm}, \Phi_{k}^{\pm}))W_{1,\infty}(\Omega_{j})
\]

By using the properties of smoothing operators, the assumption \((H_{n})\) and Lemma 5.2 in \((5.17)\) we conclude the first estimate given in \((5.13)\) when \( s_{0} - 1 \leq s \leq s_{1} - 1 \).

To estimate the error \( e_{k}^{\pm,3} \), let us define the modified state \( V_{n+1}^{\pm} \), first, this will be done in an idea similar to that given in \([13,5,29]\).

To guarantee that the boundary \( \{y = 0\} \) is uniformly characteristic at each step iteration \((4.19)\), we require that

\[
(V_{n+1}^{\pm}, V_{n+1}^{\mp}, S_{\theta_{k}} \phi_{n})_{1}^{t} = 0, \quad \text{on } \{y = 0\}
\]

for \( i = 1, 2 \) and all \( n \in \mathbb{N} \), which leads to define

\[
V_{n+1}^{\pm} = S_{\theta_{k}} V_{n+1}^{\pm}, \quad j \in \{1, 3\}
\]

and

\[
V_{n+1}^{\pm} = \partial_{x}(\Psi_{n+1}^{\pm} + S_{\theta_{k}} \Phi_{n}^{\pm})V_{n+1}^{\pm} + \partial_{x}(\Psi_{n+1}^{\pm} + S_{\theta_{k}} \Phi_{n}^{\pm})V_{n+1}^{\pm} + w^{\alpha} \partial_{x}(S_{\theta_{k}} \Phi_{n}^{\pm})
\]

Lemma 5.5. For the modified state \( V_{n+1}^{\pm} \) defined at above, we have

\[
\|V_{n+1}^{\pm} - S_{\theta_{k}} V_{n+1}^{\pm}\|_{L^{2}} \leq C\delta_{n}^{1 + \alpha}
\]

for any \( s_{0} \leq s \leq s_{1} + 3 \).
We can prove this lemma in the same way as given in [13, §7.4], so we omit it here.

From the definition of the intermediate state $V_{n+\frac{s}{2}}^{\pm}$ given in (5.19)-(5.20), we know

\[ e^{(3)}_{n+\frac{s}{2},n} = e^{(3)}_n \equiv 0 \]  \hspace{1cm} (5.22)

for these two errors given in (4.32) and (4.23) respectively. The representation of the error $e^{+,-}_k$ given in (4.21) is similar to that of $e^{+,-}_k$, so by using Lemma 5.5 and the same argument as the proof of Lemma 5.4 we conclude

**Lemma 5.6.** For the error $e^{+,-}_k$, we have

\[ \|e^{+,-}_k\|_{s,X} \leq C\delta^2 \theta^{L_3(s)}_k \Delta_k, \hspace{0.5cm} s_0 - 1 \leq s \leq s_1 - 1 \]  \hspace{1cm} (5.23)

for all $k \leq n - 1$, where

\[ L_3(s) = \max((s + 1 - \alpha)s, 2(s_0 - \alpha), s + 2s_0 - 3\alpha + 2, s + s_0 - 2\alpha + 1, 2(s_0 - \alpha) + 1). \]  \hspace{1cm} (5.24)

**Lemma 5.7.** For the errors $e^{+,-}_k$, $e^{(4)}_{\pm,n}$ and $\tilde{e}^{(4)}_n$ given in (4.21), (4.32) and (4.23) respectively, we have

\[
\begin{align*}
\|e^{+,-}_k\|_{s,X} &\leq C\delta^2 \theta^{L_4(s)}_k \Delta_k, \hspace{0.5cm} s_0 \leq s \leq s_1 - 2 \\
\|e^{(4)}_{\pm,n}\|_{s,X} &\leq C\delta^2 \theta^{L_5(s)}_k \Delta_k, \hspace{0.5cm} s_0 + \frac{3}{2} \leq s \leq s_1 - \frac{7}{2} \\
\|\tilde{e}^{(4)}_n\|_{\mathcal{H}(\omega X)} &\leq C\delta^2 \theta^{L_6(s)}_k \Delta_k, \hspace{0.5cm} s_0 + 1 \leq s \leq s_1 - 4
\end{align*}
\]

for all $k \leq n - 1$, where

\[
\begin{align*}
L_4(s) &= \max((s + 1 - \alpha)s, 2(s_0 - \alpha + 1), s + s_0 + 2 - 2\alpha), \\
L_5(s) &= \max((s + 2 - \alpha)s, 2(s_0 - \alpha) - 1, s + s_0 - 2\alpha).
\end{align*}
\]  \hspace{1cm} (5.26)

**Proof.** (1) Denote by

\[ R^+_k = \partial_k(L(U^{\alpha,\pm} + V^{\pm}_{k+\frac{s}{2}}, \Psi^{\alpha,\pm} + S_\theta \Phi^+_k)(U^{\alpha,\pm} + V^{\pm}_{k+\frac{s}{2}})). \]  \hspace{1cm} (5.27)

Obviously, we have

\[
\|R^+_k\|_{s,X} \leq \|L(U^{\alpha,\pm} + V^{\pm}_{k+\frac{s}{2}}, \Psi^{\alpha,\pm} + S_\theta \Phi^+_k)(U^{\alpha,\pm} + V^{\pm}_{k+\frac{s}{2}}) - L(U^{\alpha,\pm} + V^{\pm}_{k}, \Psi^{\alpha,\pm} + \Phi^+_k)(U^{\alpha,\pm} + V^{\pm}_{k})\|_{s+1,X} \\
+ \|L(V^{\pm}_{k}, \Phi^+_k) V^{\pm}_{k} - f^{\pm}_{s}\|_{s+1,X}
\]

which implies

\[
\|R^+_k\|_{s,X} \leq C \left( \|V^{\pm}_{k+\frac{s}{2}} - V^{\pm}_{k}\|_{W^{1,\infty}}(\|U^{\alpha,\pm}_{k} + V^{\pm}_{k}\|_{s+1,X} + \|\Psi^{\alpha,\pm}_{k} + \Phi^+_k\|_{s+2,X}) \\
+ \|V^{\pm}_{k+\frac{s}{2}} - V^{\pm}_{k}\|_{s+2,X}(\|U^{\alpha,\pm}_{k} + V^{\pm}_{k}\|_{L^\infty} + \|\Psi^{\alpha,\pm}_{k} + \Phi^+_k\|_{W^{1,\infty}}) \\
+ \|U^{\alpha,\pm}_{k} + V^{\pm}_{k}\|_{W^{1,\infty}}\|V^{\pm}_{k+\frac{s}{2}} - V^{\pm}_{k}\|_{s+1,X} + \|(1 - S_\theta)\Phi^+_k\|_{s+2,X}) \\
+ \|U^{\alpha,\pm}_{k} + V^{\pm}_{k}\|_{s+2,X}(\|V^{\pm}_{k+\frac{s}{2}} - V^{\pm}_{k}\|_{L^\infty} + \|(1 - S_\theta)\Phi^+_k\|_{W^{1,\infty}}) \\
+ \|L(V^{\pm}_{k}, \Phi^+_k) V^{\pm}_{k} - f^{\pm}_{s}\|_{s+1,X}
\)
\]

\[
\leq C \delta^2 \theta^{L_7(s+2-\alpha)_s,s+1-\alpha + \theta^{L_7(s+3-\alpha)} + 2\delta \theta^{L_7(s)}}
\]

for all $s_0 \leq s \leq s_1 - 3$. 
As in [13], as \( s = s_1 - 2 \), we immediately have
\[
\|R_k^\pm\|_{s,X} \leq \|L(U^{a,\pm} + V_{k+\frac{1}{2}}^\pm, \Psi^{a,\pm} + S_\theta \Phi_k^\pm)(U^{a,\pm} + V_{k+\frac{1}{2}}^\pm)\|_{s+1,X} \leq C \delta \theta^{3-\alpha}. \quad (5.29)
\]
Thus, we get that
\[
e^{\pm,4}_k = \frac{R_k^\pm \partial \Phi_k^\pm}{\partial \psi(\Psi^{a,\pm} + S_\theta \Phi_k^\pm)}
\]
satisfy
\[
\|e^{\pm,4}_k\|_{s,X} \leq C(\|R_k^\pm\|_{s_0,X}(\delta \theta^{s-1-\alpha} \Delta_k + \delta \theta^{s_0-1-\alpha} \Delta_k(\delta + \delta \theta^{(s+1-\alpha)})) + \delta \theta^{s_0-1-\alpha} \Delta_k \|R_k^\pm\|_{s,X}), \quad (5.30)
\]
which yields the first estimate given in (5.23) for any \( s_0 \leq s \leq s_1 - 2 \) by using (5.23) and (5.29).

(2) Denote by
\[
R_k^b = \mathcal{B}(U^{a,\pm} + V_{k+\frac{1}{2}}^\pm, \psi^a + S_\theta \phi_k).
\]
Obviously, we have
\[
\|R_k^b\|_{H^s(\omega_X)} \leq \|\mathcal{B}(U^{a,\pm} + V_{k+\frac{1}{2}}^\pm, \psi^a + S_\theta \phi_k) - \mathcal{B}(U^{a,\pm} + V_{k+\frac{1}{2}}^\pm, \psi^a + \phi_k)\|_{H^s(\omega_X)}
\]
\[
+\|\mathcal{B}(V_{k+\frac{1}{2}}^\pm, \phi_k)\|_{H^s(\omega_X)},
\]
which implies
\[
\|(R_k^b)\|_{H^s(\omega_X)} \leq C \left( (\|S_\theta\| - 1)\phi_k \|H^{s_1(\omega_X)}\|\tilde{U}^{a,\pm} + V_{k+\frac{1}{2}}^\pm \|L^{a,\pm} + \|\tilde{S_\theta}\| - 1)\phi_k \|w^{1,\omega} \|\tilde{U}^{a,\pm} + V_{k+\frac{1}{2}}^\pm \|H^{s(\omega_X)} \right.
\]
\[
+\|S_\theta\| - 1)\phi_k \|w^{1,\omega} \|\|\tilde{U}^{\alpha,\pm} + S_\theta \phi_k \|H^{s(\omega_X)} \right)
\]
\[
+\|\mathcal{B}(V_{k+\frac{1}{2}}^\pm, \phi_k)\|_{H^s(\omega_X)} \leq C \delta \theta^{s_0(s_0-\alpha)+s_0-\alpha} \|R_k^b\|_{H^{s-\alpha}(\omega_X)} \quad (5.32)
\]
for all \( s_0 \leq s \leq s_1 - 3 \).

Thus, we get that
\[
\tilde{e}_{k,1}^{(4)} = -\frac{\partial \psi(R_k^b)}{\partial \psi(\Psi^{a,\pm} + S_\theta \Phi_k^\pm)}_{|y=0} \delta \phi_k
\]
satisfies
\[
\|\tilde{e}_{k,1}^{(4)}\|_{H^s(\omega_X)} \leq C \left( (\|R_k^b\|_{H^s(\omega_X)}(\delta \theta^{s-2-\alpha} \Delta_k + \delta \theta^{s_0-2-\alpha} \Delta_k + \delta \theta^{s_0-s-2} \Delta_k) + \delta \theta^{s_0-s-2} \Delta_k ) \|R_k^b\|_{H^{s-\alpha}(\omega_X)} \right), \quad (5.33)
\]
which yields the estimate of \( \tilde{e}_{k,1}^{(4)} \) given in (5.25) for any \( s_0 + 1 \leq s \leq s_1 - 4 \) by using (5.32).

One can obtain the estimate (5.25) of other components of \( \tilde{e}_{k,1}^{(4)} \) similarly.

Noting that the trace of \( \nabla \tilde{e}_{\pm,k}^{(4)} \) on \( \{y = 0\} \) is equal to \( (\tilde{e}_{k,1}^{(4)}, \tilde{e}_{k,2}^{(4)}) \), the estimate of \( \tilde{e}_{\pm,k}^{(4)} \) in (5.25) follows immediately.

Summarizing all results from Lemmas 5.3, 5.4, 5.6 and 5.7, we conclude

**Lemma 5.8.** The errors \( e_k^\pm = \sum_{j=1}^{4} e_{k,j}^\pm, \tilde{e}_k^\pm = \sum_{j=1}^{4} \tilde{e}_{k,j}^{(4)} \) and \( \tilde{e}_k = \sum_{j=1}^{4} \tilde{e}_{k,j} \) satisfy
\[
\begin{align*}
\|e_k^\pm\|_{s,X} &\leq C \delta^2 \theta^d(s) \Delta_k, \quad s_0 \leq s \leq s_1 - 2 \\
\|R_k^\pm\|_{s,X} &\leq C \delta^2 \theta^d(s) \Delta_k, \quad s_0 + \frac{3}{2} \leq s \leq s_1 - \frac{7}{2}
\end{align*}
\]
\[
\|\tilde{e}_k\|_{H^{s}(\omega_X)} \leq C \delta^2 \theta^d(s) \Delta_k, \quad s_0 + 1 \leq s \leq s_1 - 4
\]
for all \( k \leq n - 1 \), where \( L_4(s) \) and \( L_5(s) \) are given in Lemma 6.6, and
\[
L_6(s) = \max((s + \frac{3}{2} - \alpha)_+, 2(s_0 - \alpha) - 1, s + s_0 - 2\alpha).
\]
From Lemma 5.8, we immediately obtain

**Lemma 5.9.** For any fixed $s_0 > \frac{1}{2}$, $\alpha \geq s_0 + 2$ and $s_0 + 3 \leq s_1 \leq 2\alpha - s_0 + 1$, the accumulated errors

\[
E^+_n = \sum_{k=0}^{n-1} e^+_k, \quad \tilde{E}_n = \sum_{k=0}^{n-1} \tilde{e}_k, \quad \tilde{E}^+_n = \sum_{k=0}^{n-1} \tilde{e}^+_k
\]

satisfy the estimates

\[
\begin{align*}
\|E^+_n\|_{s,X} &\leq C\delta^2 \theta_n, \quad s_0 \leq s \leq s_1 - 2, \\
\|\tilde{E}_n\|_{s,X} &\leq C\delta^2 \theta_n, \quad s_0 + \frac{1}{2} \leq s \leq \frac{s_1}{2}, \\
\|\tilde{E}^+_n\|_{H^{(s_0)}} &\leq C\delta^2 \theta_n, \quad s_0 + 1 \leq s \leq s_1 - 4.
\end{align*}
\]

To study problems (4.19) and (4.27)-(4.28), first we have

**Lemma 5.10.** With the same range of $s_0$ and $s_1$ as given in Lemma 5.9 we have

\[
\begin{align*}
\|f^+_n\|_{s,X} &\leq C\Delta_n(\theta_n^{s-s_0-1})\|f^+_n\|_{s_2,X} + \delta^2 \theta_n^{s-s_1} + \delta^2 \theta_n^{s-s_2} + L_4(s_4), \\
\|g_n\|_{H^{(s_0)}} &\leq C\delta^2 \Delta_n(\theta_n^{s-s_5} + \theta_n^{s-s_6}) + L_4(s_6), \\
\|h^+_n\|_{s,X} &\leq C\delta^2 \Delta_n(\theta_n^{s-s_7} + \theta_n^{s-s_6}) + L_4(s_6) + L_3(s_5)\frac{1}{2}
\end{align*}
\]

for all

\[
\begin{align*}
s_2 &\geq 0, & s_0 &\leq s_3, & s_4 &\leq s_1 - 2, \\
s_0 + 1 &\leq s_5, & s_6 &\leq s_1 - 4, \\
s_0 + \frac{1}{2} &\leq s_7, & s_8 &\leq s_1 - \frac{7}{2},
\end{align*}
\]

where $L_4(s)$, and $L_6(s)$, $L_7(s)$ are given in (5.38) and (5.33) respectively.

**Proof.** From the definitions of $f^+_n$, $g_n$ and $h^+_n$ given in (4.22), (4.24) and (4.34)-(4.35) respectively, obviously we have

\[
\begin{align*}
f^+_n &= (S_{\theta_n} - S_{\theta_{n-1}})f^+_n - (S_{\theta_n} - S_{\theta_{n-1}})E^+_n - S_{\theta_n}e^+_n, \\
g_n &= -S_{\theta_n} + S_{\theta_{n-1}} + \tilde{E}_{n-1} - S_{\theta_n} \tilde{e}_{n-1}, \\
h^+_n &= -(S_{\theta_n} - S_{\theta_{n-1}})(\tilde{E}_{n-1} - \mathbb{E}(\tilde{E}_{n-1,1})) - S_{\theta_n}(\mathbb{E}_{n+1} - \mathbb{E}(\tilde{E}_{n-1,2})), \\
h^-_n &= -(S_{\theta_n} - S_{\theta_{n-1}})(\tilde{E}_{n-1} - \mathbb{E}(\tilde{E}_{n-1,2})) - S_{\theta_n}(\mathbb{E}_{n-1} - \mathbb{E}(\tilde{e}_{n-1,2})).
\end{align*}
\]

By using the properties of the smoothing operators, and Lemmas 5.8 and 5.9 we have that for all $s \geq 0$,

\[
\begin{align*}
\|S_{\theta_n} - S_{\theta_{n-1}}\|_{s,X} \leq C\theta_n^{s-1}\Delta_n\|f^+_n\|_{s,X}, &\quad \bar{s} \geq 0, \\
\|S_{\theta_n} - S_{\theta_{n-1}}\|_{s,X} \leq C\theta_n^{s-1}\Delta_n\|E^+_n\|_{s,X} \leq C\delta^2\theta_n^{s-1}\Delta_n, &\quad s_0 \leq \bar{s} \leq s_1 - 2, \\
\|S_{\theta_n}e^+_n\|_{s,X} \leq C\theta_n^{s-3}\Delta_n\|e^+_n\|_{s,X} \leq C\delta^2\theta_n^{s-1}\Delta_n, &\quad s_0 \leq \bar{s} \leq s_1 - 2.
\end{align*}
\]

which implies the first estimate given in (5.38).

Similarly, we have

\[
\begin{align*}
\|S_{\theta_n} - S_{\theta_{n-1}}\|_{H^{(s_0)}} \leq C\theta_n^{s-1}\Delta_n\|\tilde{E}_{n-1}\|_{H^{(s_0)}} \leq C\delta^2\theta_n^{s-1}\Delta_n, \\
\|S_{\theta_n}\tilde{e}_{n-1}\|_{H^{(s_0)}} \leq C\theta_n^{s-3}\Delta_n\|\tilde{e}_{n-1}\|_{H^{(s_0)}} \leq C\delta^2\theta_n^{s-1}\Delta_n.
\end{align*}
\]

for all $s_0 + 1 \leq \bar{s} \leq s_1 - 4$, this follows the estimate of $g_n$ given in (5.38) immediately.

From the definition of $h^+_n$, we have

\[
\|h^+_n\|_{s,X} \leq C\theta_n^{s-1}\Delta_n\|\tilde{E}_{n-1}\|_{s,X} + C\delta^2\theta_n^{s-1}\|\tilde{e}_{n-1}\|_{s,X},
\]

which yields the conclusion given in (5.38) by using Lemmas 5.8 and 5.9. \[\square\]
To close this Nash-Moser iteration scheme, it remains to verify the inductive assumption \((H_n)\) given at the beginning of this section.

**Verification of the assumption \((H_n)\).**

1. The assertion of \((H_0)\) can be easily verified by studying the problem \((4.19)\) with \(n = 0\) of \(\delta \tilde{V}_0^\pm\), and the problem \((4.27)\) with \(n = 0\) of \(\delta \Phi_0^\pm\).

2. Assume that \((H_n)\) holds, let us study \((H_{n+1})\).

To apply Corollary 3.8 in the problem \((4.19)\), first we note that

\[
||U_n^{\pm} + V_n^{\pm} + \nabla (\Psi_n^{\pm} + S_{\theta_n} \Phi_n^{\pm})||_{s_0+2} \leq ||U_n^{\pm} + \nabla \Psi_n^{\pm}||_{s_0+2} + ||V_n^{\pm} + S_{\theta_n} V_n^{\pm} + \nabla (S_{\theta_n} \Phi_n^{\pm})||_{s_0+2} \\
\leq C \delta (1 + \theta_n^{(s_0+3-\alpha)} + \theta_n^{(s_0+3-\alpha)}).
\]

when \(\alpha \geq s_0 + 3\), by using the assumption \((5.1)\) and Lemmas \(5.2\) and \(5.5\).

Thus, we can apply Corollary 3.8 in the problem \((4.19)\) to obtain

\[
||\delta \tilde{V}_n^\pm ||_{s+1, X} + ||\delta \Phi_n||_{H^{s+1}(\omega_X)} \leq C (||f_n||_{s+1, X} + ||g_n||_{H^{s+1}(\omega_X)}) + ||U_n^{\pm} + \nabla (\Psi_n^{\pm} + S_{\theta_n} \Phi_n^{\pm})||_{s+2} (||f_n||_{s_0+2} + ||g_n||_{H^{s_0+2}(\omega_X)}).
\]  \hspace{1cm} (5.40)

When \(\alpha \geq s_0 + 4\) and \(s_1 \geq \alpha + 5\), setting \(s_2 = \alpha + 1\) and \(s_3 = \alpha + 2\) in Lemma \(5.10\) it follows

\[
||f_n^\pm||_{s+1, X} \leq C \delta \theta_n^{(s+1-s_3)} \Delta_n (||f_n^\pm||_{s+1, X} + \delta) + C \delta^2 \Delta_n \theta_n^{(s+1-s_3)}, L_4(s_4).
\]

On the other hand, by setting

\[
s_4 = \begin{cases} 
  s, & s_0 \leq s \leq s_1 - 2 \\
  s_1 - 1, & s_1 - 1 \leq s \leq s_1 
\end{cases}
\]

one has

\[(s + 1 - s_4)_+ + L_4(s_4) \leq s - \alpha - 1\]

for all \(s_0 \leq s \leq s_1\), thus we get

\[
||f_n^\pm||_{s+1, X} \leq C \delta \theta_n^{(s+1-s_3)} \Delta_n (||f_n^\pm||_{s+1, X} + \delta) + C \delta^2 \Delta_n \theta_n^{(s+1-s_3)}. \hspace{1cm} (5.41)
\]

Similarly, as \(s_1 \geq \alpha + 6\), by setting \(s_5 = \alpha + 2\), and

\[
s_6 = \begin{cases} 
  s + 1, & s_0 \leq s \leq s_1 - 5 \\
  s_1 - 4, & s_1 - 4 \leq s \leq s_1 
\end{cases}
\]

in Lemma \(5.10\) we have

\[
||g_n||_{H^{s+1}(\omega_X)} \leq C \delta^2 \Delta_n \theta_n^{(s+1-s_3)} \hspace{1cm} (5.42)
\]

for all \(s_0 \leq s \leq s_1\).

When \(\alpha \geq s_0 + 5\), by letting \(s_2 = \alpha + 2\), \(s_3 = s_5 = \alpha + 3\), \(s_4 = s_6 = \alpha - 3\) in Lemma \(5.10\) we have

\[
||f_n^\pm||_{s+2, X} + ||g_n||_{H^{s_0+2}(\omega_X)} \leq C \delta \theta_n^{(s_0+2-s_3)} \Delta_n
\]

provided that

\[
||f_n^\pm||_{s+2, X} \leq C < +\infty.
\]
On the other hand, from the assumption (5.1) and Lemmas 5.2 and 5.5 we have
\[ ||U^{a,\pm} + V_{n+1}^{\pm, \frac{1}{2}} \nabla (\Psi^{a,\pm} + S_{\theta_n} \Phi_{n}^{\pm})||_{L^2 + C(1 + \theta_n^{a,3-\alpha} + \theta_n^{a,3-\alpha})} \leq C \delta (\theta_n^{a,3-\alpha} + \theta_n^{a,3-\alpha}) \]
(5.43)
which implies
\[ ||U^{a,\pm} + V_{n+1}^{\pm, \frac{1}{2}} \nabla (\Psi^{a,\pm} + S_{\theta_n} \Phi_{n}^{\pm})||_{L^2 + C(1 + \theta_n^{a,3-\alpha} + \theta_n^{a,3-\alpha})} \leq C \delta (\theta_n^{a,3-\alpha} + \theta_n^{a,3-\alpha}) \]
for all \( s_0 \leq s \leq s_1 \).

Thus, plugging (5.41), (5.42) and (5.44) into (5.40) it follows
\[ ||\partial V_n^{\pm}||_{L^2} + ||\partial \Phi_n||_{H^{s+1}} \leq C \delta \theta_n^{a,3-\alpha} \Delta_n (\frac{||f_a^{\pm}||_{L^2 + \frac{1}{2}}}{\delta} + \delta) + C \delta^2 \theta_n^{a,3-\alpha} \Delta_n \]
(5.45)
for all \( s_0 \leq s \leq s_1 \).

For the problems (4.27) and (4.28), one can easily deduce the following estimate
\[ ||\partial \Phi_n||_{L^2} \leq C \left( ||\partial g_n||_{H^{s+1}} + ||h_n^{\pm}||_{L^2} + ||\partial g_n||_{H^{s+1}} + ||h_n^{\pm}||_{L^2} + \right) \]
(5.46)
for all \( s_{0} \leq s \leq s_{1} \).

Similar to the discussion for the estimate (5.42) of \( g_n \), by choosing \( s_{7} \), \( s_{8} \) properly in Lemma 5.10 we can get
\[ ||h_n^{\pm}||_{L^2} \leq C \delta \Delta_n \theta_n^{a,3-\alpha}, \quad s_0 \leq s \leq s_1. \]
Thus, by using (H_{n}), (5.42) and (5.45) in (5.46), it follows
\[ ||\partial \Phi_n||_{L^2} \leq C \delta \theta_n^{a,3-\alpha} \Delta_n (\frac{||f_a^{\pm}||_{L^2 + \frac{1}{2}}}{\delta} + \delta) + C \delta^2 \theta_n^{a,3-\alpha} \Delta_n \]
(5.47)
for all \( s_0 \leq s \leq s_1 \).

Together (5.45) with (5.47), it follows (H_{n+1}) for \( \delta V^{a,n} \) and \( \delta \Phi^{a,n} \) immediately by using
\[ \delta V_n^{a} = \delta \tilde{V}_n^{a} + \frac{\partial_{s}(U^{a,\pm} + V_{n+1}^{a,\pm})}{\partial_{s}(\Psi^{a,\pm} + S_{\theta_n} \Phi_{n}^{\pm})} \delta \Phi_n^{a} \]
and letting both of \( \delta \), \( \frac{\partial_{s}(U^{a,\pm} + V_{n+1}^{a,\pm})}{\partial_{s}(\Psi^{a,\pm} + S_{\theta_n} \Phi_{n}^{\pm})} \) being properly small.

To verify other inequalities in (H_{n+1}), we shall use the idea from [13]. From (4.21), we have
\[ L(V_{n+1}^{a,n}, \Phi_{n+1}^{a,n})V_{n+1}^{a,n} - f_n^{a} = (S_{\theta_n} - I)f_n^{a} + (I - S_{\theta_n})E_n^{a} + e_n^{a}. \]
(5.48)
From Lemma 5.9 we have
\[ ||(I - S_{\theta_n})E_n^{a}||_{L^2} \leq C \theta_n^{a,3-\alpha} \delta^2 \Delta_n \leq C \theta_n^{a,3-\alpha} \delta^2 \Delta_n \]
(5.49)
for all \( s \leq s - 1 \) by choosing \( s = s - 1 \).

As we already have estimates of \( \delta V_{n}^{a,n} \), \( \delta \Phi_{n}^{a,n} \) given in (H_{n+1}), the result of Lemma 5.8 is also true for \( k = n \), thus we have
\[ ||e_n^{a}||_{L^2} \leq C \theta_n^{a,3-\alpha} \delta^2 \Delta_n \leq C \theta_n^{a,3-\alpha} \delta^2 \Delta_n \]
(5.50)
for all \( s_{0} \leq s \leq s_{1} \).

On the other hand, it is easy to have
\[ ||(S_{\theta_n} - I)f_n^{a}||_{L^2} \leq C \theta_n^{a,3-\alpha} ||f_n^{a}||_{L^2}, \quad s \leq s + 1 \]
(5.51)
\[ ||(S_{\theta_n} - I)f_n^{a}||_{L^2} \leq ||S_{\theta_n} f_n^{a}||_{L^2} + ||f_n^{a}||_{L^2} \leq C \theta_n^{a,3-\alpha} ||f_n^{a}||_{L^2}, \quad \alpha + 2 \leq s \leq s + 1. \]
Substituting (5.51), (5.50) and (5.49) into (5.48), it follows that
\[ ||L(V_{n+1}^{a,n}, \Phi_{n+1}^{a,n})V_{n+1}^{a,n} - f_n^{a}||_{L^2} \leq \delta \theta_n^{a,3-\alpha} \delta^2 \Delta_n \]
(5.52)
for all \( s_{0} \leq s \leq s_{1} \).

Similarly, one can verify the last assertion of (H_{n+1}) for the estimate of \( \mathcal{B}(V_{n+1}^{a,n}, V_{n+1}^{a,n}, \Phi_{n+1}^{a,n}). \)
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