Abstract

The semiclassical limit of the algebraic Bethe Ansatz method is used to solve the theory of Gaudin models for the $sl(2|1)^{(2)}$ R-matrix. We find the spectra and eigenvectors of the $N - 1$ independents Gaudin Hamiltonians. We also use the off-shell Bethe Ansatz method to show how the off-shell Gaudin equation solves the associated trigonometric system of Knizhnik-Zamolodchikov equations.

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1 Introduction

Integrable models of classical statistical mechanics [1, 2] and of two-dimensional quantum field theories [3], have a central object: the $R$-matrix $R(u)$, where $u$ is a spectral parameter, acting on the tensor product $V \otimes V$ of a given vector space $V$ and being a solution of the Yang-Baxter (YB) equation

$$R_{12}(u)R_{13}(u+v)R_{23}(v) = R_{23}(v)R_{13}(u+v)R_{12}(u),$$

in $V^1 \otimes V^2 \otimes V^3$, where $R_{12} = R \otimes 1$, $R_{23} = 1 \otimes R$, etc.

The solution $R(u)$ of Eq.(1.1) is said to be semiclassical if $R(u)$ also depends on an additional parameter $\eta$ in such a way that

$$R(u, \eta) = 1 + \eta r(u) + o(\eta^2),$$

where $1$ is the identity operator on the space $V \otimes V$. The “classical $r$-matrix” obeys the equation

$$[r_{12}(u), r_{13}(u+v) + r_{23}(v)] + [r_{13}(u+v), r_{23}(v)] = 0.$$

This equation, called the classical Yang-Baxter equation, plays an important role in the theory of classical completely integrable systems [4].

Nondegenerate solutions of (1.3) in the tensor product of two copies of a simple Lie algebra $g$, $r_{ij}(u) \in g_i \otimes g_j$, $i, j = 1, 2, 3$, were classified by Belavin and Drinfeld [5].

The classical YB equation has an interplay with conformal field theory, shortly described in the following way: in the skew-symmetric case $r_{ji}(-u) + r_{ij}(u) = 0$, it is the compatibility condition for the system of linear differential equations

$$\kappa \frac{\partial \Psi(z_1, ..., z_N)}{\partial z_i} = \sum_{j \neq i} r_{ij}(z_i - z_j)\Psi(z_1, ..., z_N)$$

in $N$ complex variables $z_1, ..., z_N$ for vector-valued functions $\Psi$ with values in the tensor space $V = V^1 \otimes \cdots \otimes V^N$. $\kappa$ is a coupling constant.

In the rational case [5], very simple skew-symmetric solutions are known: $r(u) = C_2/u$, where $C_2 \in g \otimes g$ is a symmetric invariant tensor of a finite dimensional Lie algebra $g$ acting on a representation space $V$. The above system of linear differential equations (1.4) is known as the Knizhnik-Zamolodchikov (KZ) system of equations for the conformal blocks of the Wess-Zumino-Novikov-Witten (WZNW) models of the conformal field theory on the sphere [6].

The algebraic Bethe Ansatz [7] was formulated in a parallel reasoning to the representation theory of the classical Lie algebras and it is a powerful method in the analysis of integrable models. Besides describing the spectra of quantum integrable systems, the algebraic Bethe Ansatz is also used to construct exact and manageable expressions for correlation functions [2]. Various representations of correlators were found by Korepin [8], using this method.
The Babujian and Flume work [9] unveils a link between the algebraic Bethe Ansatz for the theory of the Gaudin models[12] and the conformal field theory of WZWN models. In their approach, the wave vectors of the Bethe Ansatz equation for an inhomogeneous lattice model render, in the semiclassical limit, solutions of the KZ equation for the case of simple Lie algebras. For instance, in the \( su(2) \) example, the algebraic quantum inverse scattering method [7] allows one to write the following equation

\[
\tau(u\mid z)\Phi(u_1,\ldots,u_p) = \Lambda(u,u_1,\ldots,u_p|z)\Phi(u_1,\ldots,u_p) - \sum_{\alpha=1}^{p} \frac{\mathcal{F}_\alpha \Phi^\alpha}{u - u_\alpha}. \tag{1.5}
\]

Here \( \tau(u\mid z) \) denotes the transfer matrix of the rational vertex model in an inhomogeneous lattice acting on an \( N \)-fold tensor product of \( su(2) \) representation spaces. \( \Phi^\alpha \) meaning \( \Phi^\alpha = \Phi(u_1,\ldots,u_{\alpha-1},u,u_{\alpha+1},\ldots,u_p) \) ; \( \mathcal{F}_\alpha(u_1,\ldots,u_p|z) \) and \( \Lambda(u,u_1,\ldots,u_p|z) \) being \( c \) number functions. The vanishing of the so-called unwanted terms, \( \mathcal{F}_\alpha = 0 \), is enforced in the usual procedure of the algebraic Bethe Ansatz by finding the parameters \( u_1,\ldots,u_p \). In this case the wave vector \( \Phi(u_1,\ldots,u_p) \) becomes an eigenvector of the transfer matrix with eigenvalue \( \Lambda(u,u_1,\ldots,u_p|z) \). If we keep all unwanted terms, i.e. \( \mathcal{F}_\alpha \neq 0 \), then the wave vector \( \Phi \) in general satisfies the equation (1.5), named in [10] as off-shell Bethe Ansatz equation (OSBAE).

There is a neat relationship between the wave vector satisfying the OSBAE (1.5) and the vector-valued solutions of the KZ equation (1.4): The general vector valued solution of the KZ equation for an arbitrary simple Lie algebra was found by Schechtman and Varchenko [11]. It can be represented as a multiple contour integral

\[
\Psi(z_1,\ldots,z_N) = \oint \cdots \oint \mathcal{X}(u_1,\ldots,u_p|z)\phi(u_1,\ldots,u_p|z)du_1\cdots du_p. \tag{1.6}
\]

The complex variables \( z_1,\ldots,z_N \) of (1.6) are related with the disorder parameters of the OSBAE . The vector valued function \( \phi(u_1,\ldots,u_p|z) \) is the semiclassical limit of the wave vector \( \Phi(u_1,\ldots,u_p|z) \). In fact, it is the Bethe wave vector for Gaudin magnets [12], but "off-shell". The Bethe Ansatz for the Gaudin model was derived for any simple Lie algebra by Reshetikhin and Varchenko [13]. The scalar function \( \mathcal{X}(u_1,\ldots,u_p|z) \) is constructed from the semiclassical limit of the \( \Lambda(u = z_k; u_1,\ldots,u_p|z) \) and the \( \mathcal{F}_\alpha(u_1,\ldots,u_p|z) \) functions. This representation of the \( N \)-point correlation function shows a deep connection between the inhomogeneous vertex models and the WZNW theory.

In this work we show that this idea also holds for the case of the semiclassical limit that corresponds to the \( sl(2|1)^{(2)} \) trigonometric \( r \)-matrix [15, 16].

The paper is organized as follows. In Section 2 we present the algebraic Bethe Ansatz for the \( sl(2|1)^{(2)} \) vertex model. Here the inhomogeneous Bethe Ansatz is read from the homogeneous case previously derived for the 19-vertex models [14]. We also derive the off-shell Bethe Ansatz equation for this vertex model. In Section 3, taking into account the semiclassical limit of the results presented in the Section 2, we describe the algebraic structure of the corresponding Gaudin model. In Section 4, data of the
off-shell Gaudin equation are used to construct solutions of the trigonometric KZ equation. Conclusions are reserved for Section 5.

2 Inhomogeneous Algebraic Bethe Ansatz

Consider $V = V_0 \oplus V_1$ a $Z_2$-graded vector space where 0 and 1 denote the even and odd parts respectively.

The components of a linear operator $A \hat{\otimes} B$ in the graded tensor product space $V \hat{\otimes} V$ result in matrix elements of the form

$$ (A \hat{\otimes} B)^{\gamma \delta}_{\alpha \beta} = (-)^{p(\beta)(p(\alpha)+p(\gamma))} A^\alpha_\alpha B^\beta_\beta $$ \hspace{1cm} (2.1)

and the action of the permutation operator $\mathcal{P}$ on the vector $|\alpha\rangle \hat{\otimes} |\beta\rangle \in V \hat{\otimes} V$ is given by

$$ \mathcal{P} |\alpha\rangle \hat{\otimes} |\beta\rangle = (-)^{p(\alpha)p(\beta)} |\beta\rangle \hat{\otimes} |\alpha\rangle \implies (\mathcal{P})^{\gamma \delta}_{\alpha \beta} = (-)^{p(\alpha)p(\beta)} \delta^\gamma_\delta \delta^\alpha_\beta, $$ \hspace{1cm} (2.2)

where $p(\alpha) = 1$ (0) if $|\alpha\rangle$ is an odd (even) element.

Besides $\mathcal{R}$, it is usual to consider the matrix $R = \mathcal{P} \mathcal{R}$ which satisfy

$$ R_{12}(u)R_{23}(u+v)R_{12}(v) = R_{23}(v)R_{12}(u+v)R_{23}(u). $$ \hspace{1cm} (2.3)

The regular solution of the graded YB equation of the 19-vertex model $sl(2|1)^{(2)}$ is given by [15, 16]

$$ R(u, \eta) = \begin{pmatrix}
  x_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & x_5 & 0 & x_2 & 0 & 0 & 0 & 0 \\
 0 & 0 & x_7 & 0 & x_6 & 0 & x_3 & 0 \\
 0 & x_2 & 0 & x_5 & 0 & 0 & 0 & 0 \\
 0 & 0 & -x_6 & 0 & -x_4 & 0 & -x_6 & 0 \\
 0 & 0 & 0 & 0 & 0 & x_5 & 0 & x_2 \\
 0 & 0 & x_3 & 0 & x_6 & 0 & x_7 & 0 \\
 0 & 0 & 0 & 0 & x_2 & 0 & x_5 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_1 
\end{pmatrix}, \hspace{1cm} (2.4)
$$

where

$$ x_1(u) = \cosh(u + \eta) \sinh(u + 2\eta), $$

$$ x_2(u) = \sinh u \cosh(u + \eta), $$

$$ x_3(u) = \sinh u \cosh(u - \eta), $$

$$ x_5(u) = \sinh 2\eta \cosh(u + \eta), $$

$$ x_6(u) = \sinh 2\eta \sinh u, $$

$$ x_7(u) = \sinh 2\eta \cosh \eta $$

$$ x_4(u) = x_2(u) - x_7(u) $$ \hspace{1cm} (2.5)
Consider now an inhomogeneous vertex model, where to each vertex we associate two parameters: a global spectral parameter $u$ and a disorder parameter $z$. In this case, the vertex weight matrix $R$ depends on $u - z$ and consequently the monodromy matrix defined below will be a function of the disorder parameters $z_i$.

The graded quantum inverse scattering method is characterized by the monodromy matrix $T(u|z)$ satisfying the equation

$$R(u - v) \left[ T(u|z) \otimes T(v|z) \right] = \left[ T(v|z) \otimes T(u|z) \right] R(u - v),$$

(2.6)

whose consistency is guaranteed by the graded version of the YB equation (1.1). $T(u|z)$ is a matrix in the space $V$ (usually called auxiliary space) whose matrix elements are operators on the states of the quantum system (which will also be in our work the space $V$). The monodromy operator $T(u|z)$ is defined as an ordered product of local operators $L_n$ (Lax operator), on all sites of the lattice:

$$T(u|z) = L_N(u - z_N)L_{N-1}(u - z_{N-1}) \cdots L_1(u - z_1).$$

(2.7)

We normalize the Lax operator on the $n^{th}$ quantum space to be given by:

$$L_n = \frac{1}{x_2} \left( \begin{array}{ccccccccc} x_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & x_2 & 0 & x_5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_3 & 0 & x_6 & 0 & x_7 & 0 & 0 \\ 0 & x_5 & 0 & x_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_6 & 0 & x_4 & 0 & x_6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_2 & 0 & x_5 & 0 \\ 0 & x_7 & 0 & x_6 & 0 & x_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_5 & 0 & x_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_1 \end{array} \right)$$

(2.8)

$L^{(n)}_{\alpha\beta}(u)$, $\alpha, \beta = 1, 2, 3$ are 3 by 3 matrices acting on the $n^{th}$ site of the lattice, so the monodromy matrix has the form

$$T(u|z) = \begin{pmatrix} A_1(u|z) & B_1(u|z) & B_2(u|z) \\ C_1(u|z) & A_2(u|z) & B_3(u|z) \\ C_2(u|z) & C_3(u|z) & A_3(u|z) \end{pmatrix},$$

(2.9)

where

$$T_{ij}(u|z) = \sum_{k_1, \ldots, k_{N-1} = 1}^{3} L^{(N)}_{i k_1}(u - z_N) \otimes L^{(N-1)}_{k_1 k_2}(u - z_{N-1}) \otimes \cdots \otimes L^{(1)}_{k_{N-1} l}(u - z_1).$$

(2.10)
The vector in the quantum space of the monodromy matrix \( T(u|z) \) that is annihilated by the operators \( T_{ij}(u|z), i > j \) (\( C_i(u|z) \) operators, \( i = 1, 2, 3 \)) and it is also an eigenvector for the operators \( T_{ii}(u|z) \) (\( A_i(u|z) \) operators, \( i = 1, 2, 3 \)) is called a highest weight vector of the monodromy matrix \( T(u|z) \).

The transfer matrix \( \tau(u|z) \) of the corresponding integrable spin model is given by the supertrace of the monodromy matrix in the space \( V \)

\[
\tau(u|z) = \sum_{i=1}^{3} (-1)^{p(i)} T_{ii}(u|z) = A_1(u|z) - A_2(u|z) + A_3(u|z).
\] (2.11)

The algebraic Bethe Ansatz solution for the inhomogeneous \( sl(2|1)^{(2)} \) vertex model can be obtained from the homogeneous case \cite{14}. The proper modification is a local shift of the spectral parameter \( u \to u - z_i \).

Here we will define the functions used in our algebraic Bethe Ansatz:

\[
\begin{align*}
z(u) &= \frac{x_1(u)}{x_2(u)} = \frac{\sinh(u + 2\eta)}{\sinh u}, \\
y(u) &= \frac{x_3(u)}{x_6(u)} = \frac{\cosh(u - \eta)}{\sinh 2\eta}, \\
\omega(u) &= -\frac{x_1(u)x_3(u)}{x_4(u)x_3(u) - x_6(u)x_6(u)} = \frac{\sinh(u + 2\eta)\cosh(u - \eta)}{\sinh(u - 2\eta)\cosh(u + \eta)}, \\
Z(u_k - u_j) &= \begin{cases} 
z(u_k - u_j) & \text{if } k > j \\
z(u_k - u_j)\omega(u_j - u_k) & \text{if } k < j \end{cases}.
\end{align*}
\] (2.12)

We start defining the highest weight vector of the monodromy matrix \( T(u|z) \) in a lattice of \( N \) sites as the even (bosonic) completely unoccupied state

\[
|0\rangle = \otimes_{a=1}^{N} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}_a.
\] (2.13)

Using (2.10) we can compute the normalized action of the monodromy matrix entries on this state

\[
\begin{align*}
A_i(u|z)|0\rangle &= X_i(u|z)|0\rangle, \quad C_i(u|z)|0\rangle = 0, \quad B_i(u|z)|0\rangle \neq \{0, |0\rangle\}, \\
X_i(u|z) &= \prod_{a=1}^{N} \frac{x_i(u - z_a)}{x_2(u - z_a)} , \quad i = 1, 2, 3.
\end{align*}
\] (2.14)

The Bethe vectors are defined as normal ordered states \( \Psi_n(u_1, \cdots, u_n) \) which can be written with aid of a recurrence formula \cite{17}:

\[
\Psi_n(u_1, \cdots, u_n|z) = B_1(u_1|z)\Psi_{n-1}(u_2, \cdots, u_n|z)
\]

\[
- B_2(u_1|z) \sum_{j=2}^{n} \frac{X_1(u_j|z)}{y(u_1 - u_j)} \prod_{k=2, k \neq j}^{n} Z(u_k - u_j) \Psi_{n-2}(u_2, \hat{u_j}, \cdots, u_n|z),
\] (2.15)
with the initial condition \( \Psi_0 = |0\rangle \), \( \Psi_1(u_1|z) = B_1(u_1|z)|0\rangle \). Here \( \hat{u}_j \) denotes that the rapidity \( u_j \) is absent: \( \Psi(\hat{u}_j|z) = \Psi(u_1, \ldots, u_{j-1}, u_{j+1}, \ldots, u_n|z) \).

The action of the transfer matrix \( \tau(u|z) \) on these Bethe vectors gives us the following off-shell Bethe Ansatz equation for the \( sl(2|1)(2) \) vertex model

\[
\tau(u|z)\Psi_n(u_1, \ldots, u_n|z) = \Lambda_n \Psi_n(u_1, \ldots, u_n|z) - \sum_{j=1}^{n} F_j^{(n-1)} \Psi_j^{(n-1)} + \sum_{j=2}^{n-1} \sum_{l=1}^{j-1} F_{lj}^{(n-2)} \Psi_{lj}^{(n-2)}. \quad (2.16)
\]

We now briefly describe each term which appear in the right hand side of (2.16) (for more details the reader can see [14]): in the first term the Bethe vectors (2.15) are multiplied by \( c \)-numbers functions \( \Lambda_n = \Lambda_n(u, u_1, \ldots, u_n|z) \) given by

\[
\Lambda_n = X_1(u|z) \prod_{k=1}^{n} z(u_k - u) - (-)^n X_2(u|z) \prod_{k=1}^{n} \frac{z(u - u_k)}{\omega(u - u_k)} + X_3(u|z) \prod_{k=1}^{n} \frac{x_2(u - u_k)}{x_3(u - u_k)}. \quad (2.17)
\]

The second term is a sum of new vectors

\[
\Psi_j^{(n-1)} = \left( \frac{x_5(u_j - u)}{x_2(u_j - u)} B_1(u|z) + \frac{1}{y(u - u_j)} B_3(u|z) \right) \Psi_{n-1}(\hat{u}_j), \quad (2.18)
\]

multiplied by scalar functions \( F_j^{(n-1)} \) given by

\[
F_j^{(n-1)} = X_1(u_j|z) \prod_{k \neq j}^{n} \frac{Z(u_k - u_j) + (-)^n X_2(u_j|z) \prod_{k \neq j}^{n} Z(u_j - u_k)}. \quad (2.19)
\]

Finally, the last term is a coupled sum of a third type of vector-valued functions

\[
\Psi_{lj}^{(n-2)} = B_2(u|z) \Psi_{n-2}(\hat{u}_l, \hat{u}_j), \quad (2.20)
\]

with coefficients

\[
F_{lj}^{(n-2)} = G_{lj} X_1(u_l|z) X_1(u_j|z) \prod_{k=1, k \neq j, l}^{n} Z(u_k - u_l) Z(u_k - u_j)
- (-)^n Y_{lj} X_1(u_l|z) X_2(u_j|z) \prod_{k=1, k \neq j, l}^{n} Z(u_k - u_l) Z(u_j - u_k)
- (-)^n F_{lj} X_1(u_j|z) X_2(u_l|z) \prod_{k=1, k \neq j, l}^{n} Z(u_l - u_k) Z(u_k - u_j)
+ H_{lj} X_2(u_l|z) X_2(u_j|z) \prod_{k=1, k \neq j, l}^{n} Z(u_j - u_k) Z(u_l - u_k). \quad (2.21)
\]

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where \( G_{ij} \), \( Y_{ij} \), \( F_{ij} \) and \( H_{ij} \) are additional functions defined by

\[
G_{ij} = \frac{x_7(u_i - u_j)}{x_3(u_i - u_j) y(u_i - u_j)} + \frac{z(u_i - u_j) x_5(u_i - u_j)}{\omega(u_i - u_j) x_2(u_i - u_j) y(u_i - u_j)},
\]

\[
H_{ij} = \frac{x_7(u_i - u_j)}{x_3(u_i - u_j) y(u_i - u_j)} - \frac{x_5(u_i - u_j)}{x_3(u_i - u_j) y(u_i - u_j)},
\]

\[
Y_{ij} = \frac{1}{y(u_i - u_j)} \left\{ \frac{z(u_i - u_j) x_5(u_i - u_j)}{x_2(u_i - u_j)} - \frac{y_5(u_i - u_j) x_5(u_i - u_j)}{x_2(u_i - u_j) x_2(u_i - u_j)} \right\},
\]

\[
F_{ij} = \frac{x_5(u_i - u_j)}{x_2(u_i - u_j)} \left\{ \frac{x_5(u_i - u_j)}{x_2(u_i - u_j)} \frac{1}{y(u_i - u_j)} + \frac{z(u_i - u_j)}{\omega(u_i - u_j) y(u_i - u_j)} \right\}.
\]

In the usual Bethe Ansatz method, the next step consist in impose the vanishing of the so-called unwanted terms of (2.16) in order to get an eigenvalue problem for the transfer matrix.

We impose \( F_j^{(n-1)} = 0 \) and \( F_j^{(n-2)} = 0 \) into (2.16) to recover the eigenvalue problem. This means that \( \Psi_n(u_1, ..., u_n|z) \) is an eigenstate of \( \tau(u|z) \) with eigenvalue \( \Lambda_n \), provided the rapidities \( u_j \) are solutions of the inhomogeneous Bethe Ansatz equations

\[
\prod_{a=1}^{N} z(u_j - z_a) = (-)^{n+1} \prod_{k=1, k \neq j}^{n} \frac{z(u_j - u_k)}{z(u_k - u_j)} \omega(u_k - u_j), \quad j = 1, 2, ..., n.
\]

### 3 Structure of the \( \text{sl}(2|1)^{(2)} \) Gaudin Model

In this section we will consider the theory of the Gaudin models. To do this we need to calculate the semiclassical limit of the results presented in the previous section.

First, we recall that the superalgebra \( \text{sl}(2|1) \simeq \text{sl}(1|2) \) is the \((N = 2)\) extended supersymmetric version of \( \text{sl}(2) \) and contains four even (bosonic) generators \( H, W, X^\pm \) which form the Lie algebra \( \text{sl}(2) \oplus U(1) \) and four odd (fermionic) generators \( V^\pm, W^\pm \) whose non-vanishing commutation relations read as [18, 19]:

\[
[H, X^\pm] = \pm X^\pm, \quad [X^+, X^-] = 2H, \quad \{V^+, V^-\} = -\frac{1}{2}H.
\]

\[
[H, V^\pm] = \frac{1}{2} V^\pm, \quad [X^\pm, V^\mp] = V^\pm, \quad \{V^\pm, V^\pm\} = \pm\frac{1}{2} X^\pm,
\]

\[
[X^\pm, W^\pm] = 0, \quad [X^\pm, W^\mp] = 0, \quad \{V^\pm, W^\pm\} = 0.
\]

\[
[X^\pm, W^\mp] = X^\pm, \quad [W, W^\pm] = \mp\frac{1}{2} X^\pm, \quad [W, X^\pm] = \frac{1}{2} W^\pm,
\]

\[
\{V^\pm, W^\mp\} = \frac{1}{2} W^\pm, \quad \{W^+, W^-\} = \frac{1}{2} H, \quad \{W^\pm, W^\pm\} = X^\pm. \tag{3.1}
\]
The quadratic Casimir operator is
\[ C_2 = H^2 - W^2 + X^- X^+ + 2W^- W^+ - 2V^- V^+, \]
and the elementary representation is three-dimensional, given by
\[
W = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad H = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad X^+ = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 \end{pmatrix},
\]
\[
X^- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad V^+ = \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad V^- = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},
\]
\[
W^+ = \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad W^- = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 \end{pmatrix}.
\]

Here the basis is \{1\}, \{0\}, \{-1\}\}, where the first and third vectors are considered as even and the second as odd, \textit{i.e.}, the grading is BFB.

In order to expand the matrix elements of \(T(u|z)\), up to an appropriate order in \(\eta\), we will start by expanding the Lax operator entries defined in (2.8):
\[
L^{(n)}_{11} = 1 + 2\eta \frac{2 - W_n + \mathcal{H}_n \cosh(2u - 2z_n)}{\sinh(2u - 2z_n)} + 4\eta^2 \left( \frac{1}{2} \mathcal{H}_n^2 - \frac{1}{4} \mathcal{H}_n^2 - \mathcal{H}_n \right) + o(\eta^3),
\]
\[
L^{(n)}_{22} = 1 + 2\eta \frac{1 - W_n}{\sinh(2u - 2z_n)} + 4\eta^2 \left( \frac{1}{2} \mathcal{H}_n^2 - \frac{1}{4} \mathcal{H}_n^2 \right) + o(\eta^3),
\]
\[
L^{(n)}_{33} = 1 + 2\eta \frac{2 - W_n - \mathcal{H}_n \cosh(2u - 2z_n)}{\sinh(2u - 2z_n)} + 4\eta^2 \left( \frac{1}{2} \mathcal{H}_n^2 - \frac{1}{4} \mathcal{H}_n^2 + \mathcal{H}_n \right) + o(\eta^3).
\]

and for the elements out of the diagonal we have
\[
L^{(n)}_{12} = 2\eta \frac{e^{u - z_n} W^- - e^{-u + z_n} V^-}{\sinh(2u - 2z_n)} + o(\eta^2), \quad L^{(n)}_{21} = 2\eta \frac{e^{-u + z_n} W^+ + e^{u - z_n} V^+}{\sinh(2u - 2z_n)} + o(\eta^2),
\]
\[
L^{(n)}_{23} = 2\eta \frac{e^{u - z_n} W^- + e^{-u + z_n} V^-}{\sinh(2u - 2z_n)} + o(\eta^2), \quad L^{(n)}_{32} = 2\eta \frac{e^{u - z_n} V^+ - e^{-u + z_n} W^+}{\sinh(2u - 2z_n)} + o(\eta^2),
\]
\[
L^{(n)}_{13} = 2\eta \frac{\mathcal{X}^-}{\sinh(2u - 2z_n)} + o(\eta^2), \quad L^{(n)}_{31} = 2\eta \frac{\mathcal{X}^+}{\sinh(2u - 2z_n)} + o(\eta^2).
\]

where \(V^\pm = 2V^\pm\), \(X^\pm = 2X^\pm\), \(W^\pm = 2W^\pm\), \(\mathcal{H} = 2H\) and \(W = 2W\).

Substituting (3.4) and (3.5) into (2.10) we will get the semiclassical expansion for the monodromy matrix entries. For the diagonal entries we get
\[
A_i(u|z) = 1 + 2\eta A_i^{(1)}(u|z) + 4\eta^2 A_i^{(2)}(u|z) + o(\eta^3), \quad i = 1, 2, 3.
\]
where, the first order terms are given by

\[ A_1^{(1)}(u|z) = \sum_{a=1}^{N} \frac{2 - W_a + H_a \cosh(2u - 2z_a)}{\sinh(2u - 2z_a)}, \quad A_2^{(1)}(u|z) = \sum_{a=1}^{N} \frac{2 - 2W_a}{\sinh(2u - 2z_a)}, \]

and the second order terms are

\[ A_1^{(2)}(u|z) = \sum_{a=1}^{N} \frac{1}{2} H_a^2 - \frac{1}{4} \cosh(2u - 2z_a)^2 \sum_{a<b} \coth(2u - 2z_a) \coth(2u - 2z_b) H_a \otimes H_b \]
\[ + \sum_{a<b} \frac{(2 - W_a) \cosh(2u - 2z_a)}{\sinh(2u - 2z_a) \sinh(2u - 2z_b)} + \sum_{a<b} \frac{(2 - W_a) \otimes H_b \cosh(2u - 2z_b)}{\sinh(2u - 2z_a) \sinh(2u - 2z_b)} \]
\[ + \sum_{a<b} \frac{\cosh(2u - 2z_a) H_a \cosh(2u - 2z_b)}{\sinh(2u - 2z_a) \sinh(2u - 2z_b)} + \sum_{a<b} \frac{\Sigma A}{\sinh(2u - 2z_a) \sinh(2u - 2z_b)} \]
\[ + \sum_{a<b} \frac{e^{u - z_a} W_a - e^{-u + z_a} V_a}{\sinh(2u - 2z_a)} \otimes \frac{e^{u - z_a} W_b + e^{-u + z_a} V_b}{\sinh(2u - 2z_b)}, \]

\[ A_3^{(2)}(u|z) = \sum_{a=1}^{N} \frac{1}{2} H_a^2 - \frac{1}{4} \cosh(2u - 2z_a)^2 \sum_{a<b} \coth(2u - 2z_a) \coth(2u - 2z_b) H_a \otimes H_b \]
\[ + \sum_{a<b} \frac{(2 - W_a) \cosh(2u - 2z_a)}{\sinh(2u - 2z_a) \sinh(2u - 2z_b)} - \sum_{a<b} \frac{(2 - W_a) \otimes H_b \cosh(2u - 2z_b)}{\sinh(2u - 2z_a) \sinh(2u - 2z_b)} \]
\[ - \sum_{a<b} \frac{\cosh(2u - 2z_a) H_a \cosh(2u - 2z_b)}{\sinh(2u - 2z_a) \sinh(2u - 2z_b)} + \sum_{a<b} \frac{\Sigma A}{\sinh(2u - 2z_a) \sinh(2u - 2z_b)} \]
\[ + \sum_{a<b} \frac{e^{u - z_a} V_a^+ - e^{-u + z_a} W_a^+}{\sinh(2u - 2z_a)} \otimes \frac{e^{-u - z_a} W_b^+ + e^{u - z_a} V_b^+}{\sinh(2u - 2z_b)}, \]

\[ A_2^{(2)}(u|z) = \sum_{a=1}^{N} \frac{1}{2} \frac{1 - H_a^2}{\cosh(2u - 2z_a)^2} \sum_{a<b} \frac{(2 - W_a) \cosh(2u - 2z_a)}{\sinh(2u - 2z_a) \sinh(2u - 2z_b)} \]
\[ + \sum_{a<b} \frac{e^{u - z_a} W_a^+ + e^{-u + z_a} V_a^+}{\sinh(2u - 2z_a)} \otimes \frac{e^{-u - z_a} W_b^+ - e^{-u + z_a} V_b^+}{\sinh(2u - 2z_b)} \]
\[ + \sum_{a<b} \frac{e^{u - z_a} W_a^+ + e^{-u + z_a} V_a^+}{\sinh(2u - 2z_a)} \otimes \frac{e^{u - z_a} W_b^+ - e^{-u + z_a} V_b^+}{\sinh(2u - 2z_b)}. \]
For off-diagonal elements we only need to expand them up to the first order in \( \eta \)

\[
B_1(u|z) = 2\eta \sum_{a=1}^{N} \frac{e^{u-z_a}W_a^--e^{-u+z_a}V_a^-}{\sinh(2u-2z_a)} + o(\eta^2),
\]

\[
B_2(u|z) = 2\eta \sum_{a=1}^{N} \frac{X_a^-}{\sinh(2u-2z_a)} + o(\eta^2),
\]

\[
B_3(u|z) = 2\eta \sum_{a=1}^{N} e^{-u+z_a}W_a^++e^{-u-z_a}V_a^+ \sinh(2u-2z_a) + o(\eta^2),
\]

\[
C_1(u|z) = 2\eta \sum_{a=1}^{N} \frac{e^{-u+z_a}W_a^++e^{-u-z_a}V_a^+}{\sinh(2u-2z_a)} + o(\eta^2),
\]

\[
C_2(u|z) = 2\eta \sum_{a=1}^{N} \frac{X_a^+}{\sinh(u-z_a)} + o(\eta^2)
\]

\[
C_3(u|z) = 2\eta \sum_{a=1}^{N} \frac{e^{-u+z_a}V_a^+-e^{-u-z_a}W_a^+}{\sinh(2u-2z_a)} + o(\eta^2),
\]

\[(3.9)\]

From these we have the following expansion for the transfer matrix (2.11):

\[
\tau(u|z) = 1 + 2\eta \sum_{a=1}^{N} \frac{2}{\sinh(2u-2z_a)} + 4\eta^2 \left\{ \sum_{a=1}^{N} (\mathcal{H}_a^2 - \frac{1}{2} \cosh(u-z_a)^2) \right. \\
\left. + \sum_{a\neq b} \frac{2}{\sinh(2u-2z_a) \sinh(2u-2z_b)} \left\{ \mathcal{H}_a \otimes \mathcal{H}_b \cosh(2u-2z_a) \cosh(2u-2z_b) \\
- W_a \otimes W_b + \frac{1}{2} (X_a^+ \otimes X_b^- + X_a^- \otimes X_b^+) + 4 \right) \\
+ e^{-z_a+z_b} (W_a^+ \otimes W_b^- + V_a^+ \otimes V_b^-) - e^{-u-z_a} (W_a^+ \otimes W_b^- + V_a^+ \otimes V_b^-) \right\} \right\}
\]

\[
\equiv 1 + 2\eta \tau^{(1)}(u|z) + 4\eta^2 \tau^{(2)}(u|z) + o(\eta^2).
\]

\[(3.10)\]

Now we will consider the second order term of \( \tau(u|z) \):

\[
\tau^{(2)}(u|z) = \sum_{a<b} \mathcal{G}_{ab}(u) + \sum_{a=1}^{N} \left( \mathcal{H}_a^2 - \frac{1}{2} \cosh(u-z_a)^2 \right).
\]

\[(3.11)\]

which, with aid of the identity

\[
\frac{1}{\sinh(2u-2z_a) \sinh(2u-2z_b)} = \frac{1}{\sinh(2z_a-2z_b)} \left( \frac{e^{-2u+2z_a}}{\sinh(2u-2z_a)} - \frac{e^{-2u+2z_b}}{\sinh(2u-2z_b)} \right)
\]

\[(3.12)\]

can be written in the form

\[
\tau^{(2)}(u|z) = \sum_{a=1}^{N} \frac{\mathcal{G}_a(u)}{e^{2u-2z_a} \sinh(2u-2z_a)} + \sum_{a=1}^{N} \left( \mathcal{H}_a^2 - \frac{1}{2} \cosh(u-z_a)^2 \right).
\]

\[(3.13)\]
where

\[
G_a(u) = \sum_{b \neq a} \frac{2}{\sinh(2z_a - 2z_b)} \left\{ 2 + \cosh(2u - 2z_a) \cosh(2u - 2z_b) \mathcal{H}_a \otimes \mathcal{H}_b - W_a \otimes W_b + \frac{1}{2} \left( \mathcal{X}_a^+ \otimes \mathcal{X}_b^- + \mathcal{X}_a^- \otimes \mathcal{X}_b^+ \right) + e^{-z_a + z_b} \left( W_a^- \otimes W_b^+ + V_a^+ \otimes V_b^- \right) - e^{z_a - z_b} \left( W_a^+ \otimes W_b^- + V_a^- \otimes V_b^+ \right) \right\} ,
\]

(3.14)

Here we observe that \( G_a(u) \) is nothing but the sum of semiclassical trigonometric \( r \)-matrices. This fact follows from the construction of the semiclassical \( r \)-matrices out of the quadratic Casimir:

\[
C_2 = H^2 - W^2 + \frac{1}{2} (X^- X^+ + X^+ X^-) + (W^- W^+ + V^+ V^-) - (W^+ W^- + V^- V^+).
\]

(3.15)

The Gaudin Hamiltonians are defined as the residue of \( \tau(u|z) \) at the point \( u = z_a \). This results in \( N \) non-local Hamiltonians

\[
G_a = \sum_{b \neq a}^{N} \frac{1}{\sinh(2z_a - 2z_b)} \left\{ 2 + \cosh(2z_a - 2z_b) \mathcal{H}_a \otimes \mathcal{H}_b - W_a \otimes W_b + \frac{1}{2} \left( \mathcal{X}_a^+ \otimes \mathcal{X}_b^- + \mathcal{X}_a^- \otimes \mathcal{X}_b^+ \right) + e^{-z_a + z_b} \left( W_a^- \otimes W_b^+ + V_a^+ \otimes V_b^- \right) - e^{z_a - z_b} \left( W_a^+ \otimes W_b^- + V_a^- \otimes V_b^+ \right) \right\} ,
\]

(3.16)

satisfying

\[
\sum_{a=1}^{N} G_a = 0, \quad \frac{\partial G_a}{\partial z_b} = \frac{\partial G_b}{\partial z_a}, \quad [G_a, G_b] = 0, \quad \forall a, b.
\]

(3.17)

In the next section we will use the data of the Bethe Ansatz presented in the previous section in order to find the exact spectrum and eigenvectors for each of these \( N - 1 \) independent Hamiltonians.

We complete this section deriving the \( sl(2|1)^{(2)} \) Gaudin algebra from the semiclassical limit of the fundamental commutation relation (2.6): The semiclassical expansions of \( T \) and \( R \) can be written in the following form

\[
T(u|z) = 1 + 2\eta[l(u|z) + \beta(u|z)] + o(\eta^2), \quad R(u) = \mathcal{P} \left[ 1 + 2\eta[r(u) + \beta(u)] + o(\eta^2) \right].
\]

(3.18)

where

\[
\beta(u|z) = \sum_{a=1}^{N} \frac{2}{\sinh(2u - 2z_a)}.
\]

(3.19)
Using (3.8–3.9) one can see that the "classical l-operator" has the form

\[
\begin{pmatrix}
W(u|z) + \mathcal{H}(u|z) & W^-(u|z) - V^-(u|z) & X^-(u|z) \\
W^+(u|z) + V^+(u|z) & 2W(u|z) & W^-(u|z) + V^-(u|z) \\
X^+(u|z) & -W^+(u|z) + V^+(u|z) & W(u|z) - \mathcal{H}(u|z)
\end{pmatrix}
\] (3.20)

where

\[
\begin{align*}
\mathcal{H}(u|z) &= \sum_{a=1}^{N} \coth(2u - 2z_a) \mathcal{H}_a, \\
W^-(u|z) &= \sum_{a=1}^{N} \frac{e^{u-z_a}}{\sinh(2u - 2z_a)} W^-_a, \\
V^-(u|z) &= \sum_{a=1}^{N} \frac{e^{-u+z_a}}{\sinh(2u - 2z_a)} V^-_a, \\
X^-(u|z) &= \sum_{a=1}^{N} \frac{X^-_a}{\sinh(2u - 2z_a)}, \\
X^+(u|z) &= \sum_{a=1}^{N} \frac{X^+_a}{\sinh(2u - 2z_a)}.
\end{align*}
\] (3.21)

The corresponding semiclassical r-matrix has the form

\[
r(u) = \frac{1}{\sinh 2u} \left\{ \cosh 2u \mathcal{H} \otimes \mathcal{H} - \mathcal{W} \otimes \mathcal{W} + \frac{1}{2} \left( \mathcal{X}^+ \otimes \mathcal{X}^- + \mathcal{X}^- \otimes \mathcal{X}^+ \right) \right. \\
+ e^u \left( \mathcal{W}^- \otimes \mathcal{W}^+ + \mathcal{V}^+ \otimes \mathcal{V}^- \right) - e^{-u} \left( \mathcal{W}^- \otimes \mathcal{W}^+ + \mathcal{V}^- \otimes \mathcal{V}^+ \right) \right\}.
\] (3.22)

Here we notice that (3.22) is equivalent to the r-matrix constructed out of the quadratic Casimir (3.15) in a standard way [20].

Substituting (3.22) and (3.20) into (2.6), we have

\[
\mathcal{P} l(u|z) \otimes l(v|z) + \mathcal{P} r(u - v) \left[ l(u|z) \otimes 1 + 1 \otimes l(u|z) \right]
\]

\[= l(v|z) \otimes l(u|z) \mathcal{P} + \left[ l(v|z) \otimes 1 + 1 \otimes l(u|z) \right] \mathcal{P} r(u - v),
\] (3.23)

whose consistence is guaranteed by the graded classical YB equation (1.3).

From (3.23) we can derive (anti-)commutation relations between the matrix elements of \(l(u|z)\). This
A direct consequence of these relations is the commutativity of $\tau^{(2)}(u|z)$

$$[\tau^{(2)}(u|z), \tau^{(2)}(v|z)] = 0, \quad \forall u, v$$  \hspace{1cm} (3.25)

from which the commutativity of the Gaudin Hamiltonians $G_a$ follows immediately.
4 Off-shell Gaudin Equation

In order to get the semiclassical limit of the OSBAE (2.16) we first consider the semiclassical expansions of the Bethe vectors defined in (2.15), (2.18) and (2.20):

\[ \Psi_n(u_1, \ldots, u_n|z) = (2\eta)^n \Phi_n(u_1, \ldots, u_n|z) + o(\eta^{n+1}), \]

\[ \Psi_j^{(n-1)} = -2(2\eta)^n \left[ \frac{W^-(u|z)e^{-u+u_j} - V^-(u|z)e^{u-u_j}}{\sinh(2u-2u_j)} \right] \Phi_{n-1}(u_j|z) + o(\eta^{n+2}), \]

\[ \Psi_j^{(n-2)} = (2\eta)^{n-1} X^-(u|z) \Phi_{n-2}(u, \hat{u}_j|z) + o(\eta^n), \]  

(4.1)

where

\[ \Phi_n(u_1, \ldots, u_n|z) = \left[ W^-(u_1|z) - V^-(u_1|z) \right] \Phi_{n-1}(u_2, \ldots, u_n|z) - X^-(u_1|z) \sum_{j=2}^{n} \frac{(-)^j}{\cosh(u_1 - u_j)} \Phi_{n-2}(u_2, \hat{u}_j, u_n|z), \]

(4.2)

with \( \Phi_0 = |0\rangle \) and \( \Phi_1(u_1|z) = [W^-(u_1|z) - V^-(u_1|z)] \Phi_0. \)

The corresponding expansions of the c-number functions presented in the OSBAE are: (2.16)

\[ \Lambda_n = 1 + 2\eta \Lambda_n^{(1)} + 4\eta^2 \Lambda_n^{(2)} + o(\eta^3), \]

(4.3)

\[ \mathcal{F}_j^{(n-1)} = 2\eta(-)^j f_j^{(n-1)} + o(\eta^2), \]

(4.4)

\[ \mathcal{F}_j^{(n-2)} = 2(2\eta)^3 \left[ \frac{(-)^j}{\cosh(u_l - u_j)} \left\{ f_j^{(n-1)} \frac{f_l^{(n-1)}}{\sinh(2u - 2u_l)} - f_j^{(n-1)} \frac{f_l^{(n-1)}}{\sinh(2u - 2u_j)} \right\} \right] + o(\eta^4), \]

(4.5)

where

\[ \Lambda_n^{(1)} = \sum_{a=1}^{N} \frac{2}{\sinh(2u - 2z_a)} \]

(4.6)
\[ \Lambda_{n}^{(2)} = N + n - \frac{1}{2} \sum_{a=1}^{N} \frac{1}{\cosh(u - z_{a})^2} \]

\[ - \sum_{a=1}^{N} \sum_{j=1}^{n} \{ \coth(u - z_{a}) \coth(u - u_{j}) + \tanh(u - z_{a}) \tanh(u - u_{j}) \} \]

\[ + \sum_{a < b}^{N} \{ \coth(u - z_{a}) \coth(u - z_{b}) + \tanh(u - z_{a}) \tanh(u - z_{b}) \} \]

\[ + \sum_{j < k}^{n} \{ \coth(u - u_{j}) \coth(u - u_{k}) + \tanh(u - u_{j}) \tanh(u - u_{k}) \} \]

\[ + \frac{4}{\sinh(2u - 2u_{j}) \sinh(2u - 2u_{k})} \]  \hspace{1cm} (4.7)

and

\[ f_{j}(n - 1) = \sum_{a=1}^{N} \coth(u_{j} - z_{a}) - \sum_{k \neq j}^{n} \tanh(u_{j} - u_{k}). \]  \hspace{1cm} (4.8)

Substituting these expressions into the (2.16) and comparing the coefficients of the terms \(2(2\eta)^{n+2}\) we get the first non-trivial consequence for the semiclassical limit of the OSBAE:

\[ (u|z) \Phi_{n}(u_{1}, ..., u_{n}|z) = \Lambda_{n}^{(2)} \Phi_{n}(u_{1}, ..., u_{n}|z) - \sum_{j=1}^{n} (-1)^{j} \frac{2f_{j}(n - 1)\Theta_{j}^{(n - 1)}}{\sinh(2u - 2u_{j})}. \]  \hspace{1cm} (4.9)

Note that in this limit the contributions from \(\Psi_{j}^{(n - 1)}\) and \(\Psi_{j}^{(n - 2)}\) are combined to give a new vector valued function

\[ \Theta_{j}^{(n - 1)} = \left[ \mathcal{W}^{-}(u|z) e^{-u_{j} + u_{j} - z_{a}} - \mathcal{V}^{-}(u|z) e^{-u_{j} - u_{j}} \right] \Phi_{n-1}(\hat{u}_{j}|z) \]

\[ - \mathcal{X}^{-}(u|z) \sum_{k=1, k \neq j}^{n} \frac{(-1)^{k'}}{\cosh(u_{j} - u_{k})} \Phi_{n-2}(\hat{u}_{j}, \hat{u}_{k}|z), \]  \hspace{1cm} (4.10)

where \(k' = k + 1\) for \(k < j\) and \(k' = k\) for \(k > j\).

Finally, we take the residue of (4.9) at the point \(u = z_{a}\) to get the off-shell Gaudin equation:

\[ G_{a}\Phi_{n}(u_{1}, ..., u_{n}|z) = g_{a}\Phi_{n}(u_{1}, ..., u_{n}|z) + \sum_{l=1}^{n} (-1)^{l} \frac{2f_{l}^{(n - 1)}\Phi_{l}^{(n - 1)}}{\sinh(2u_{l} - 2z_{a})}, \]

\[ a = 1, 2, ..., N \]  \hspace{1cm} (4.11)

where \(g_{a}\) is the residue of \(\Lambda_{n}^{(2)}\)

\[ g_{a} = \text{res}_{a=z_{a}}\Lambda_{n}^{(2)} = \sum_{b \neq a}^{N} \coth(z_{a} - z_{b}) - \sum_{l=1}^{n} \coth(z_{a} - u_{l}). \]  \hspace{1cm} (4.12)
and $\phi^j_{(n-1)}$ is the residue of $\Theta^j_{(n-1)}$

$$\phi^j_{(n-1)} = \text{res}_{z=2} \Theta^j_{(n-1)} = \frac{1}{2}(W_a e^{u_j - z_a} - V_a e^{-u_j + z_a})\Phi_{n-1}(u_j | z) - \frac{1}{2}X_a \sum_{k \neq j}^n \frac{(-)^{k'}}{\cosh(u_j - u_k)} \Phi_{n-2}(u_k, u_j | z). \tag{4.13}$$

The equation (4.11) allows us to solve one of the main problems of the Gaudin model, i.e., the determination of the eigenvalues and eigenvectors of the commuting Hamiltonians $G_a$ (3.15): $g_a$ is the eigenvalue of $G_a$ with eigenfunction $\Phi_n$ provided $u_l$ are solutions of the following equations $f_j^{(n-1)} = 0$, i.e.:

$$\sum_{k \neq j}^n \tanh(u_j - u_k) = \sum_{a=1}^N \coth(u_j - z_a), \quad j = 1, 2, ..., n. \tag{4.14}$$

Moreover, as we will see in the next section, the off-shell Gaudin equation (4.11) provides solutions for the differential equations known as KZ equations.

## 5 Knizhnik-Zamolodchikov equation

The KZ differential equation

$$\kappa \frac{\partial \Psi(z)}{\partial z_i} = G_i(z) \Psi(z), \tag{5.1}$$

appeared first as a holonomic system of differential equations of conformal blocks in a WZW models of conformal field theory. Here $\Psi(z)$ is a function with values in the tensor product $V_1 \otimes \cdots \otimes V_N$ of representations of a simple Lie algebra, $\kappa = k + g$, where $k$ is the central charge of the associated Kac-Moody algebra, and $g$ is the dual Coxeter number of the simple Lie algebra.

One of the remarkable properties of the KZ system is that the coefficient functions $G_i(z)$ commute and that the form $\omega = \sum_i G_i(z) dz_i$ is closed [13]:

$$\frac{\partial G_j}{\partial z_i} = \frac{\partial G_i}{\partial z_j}, \quad [G_i, G_j] = 0. \tag{5.2}$$

Indeed, it was indicated in [13] that the equations (5.2) are not just a flatness condition for the form $\omega$ but that the KZ connection is actually a commutative family of connections.

In this section we will identify $G_i$ with our $sl(2|1)^{(2)}$ Gaudin Hamiltonians $G_a$ derived in the previous section

$$G_a = \sum_{b \neq a}^N \frac{1}{\sinh(2z_a - 2z_b)} \left\{ \cosh(2z_a - 2z_b) \mathcal{H}_a \otimes \mathcal{H}_b - \mathcal{W}_a \otimes \mathcal{W}_b + \frac{1}{2} \left( \mathcal{X}_a^+ \otimes \mathcal{X}_b^- + \mathcal{X}_a^- \otimes \mathcal{X}_b^+ \right) + e^{-z_a + z_b} (\mathcal{W}_a^- \otimes \mathcal{W}_b^+ + \mathcal{V}_a^- \otimes \mathcal{V}_b^+) - e^{z_a - z_b} (\mathcal{W}_a^+ \otimes \mathcal{W}_b^- + \mathcal{V}_a^+ \otimes \mathcal{V}_b^-) \right\}, \quad a = 1, 2, ..., N. \tag{5.3}$$
and show that the corresponding differential equations (5.1) can be solved via the off-shell Bethe Ansatz method.

Let us now define the vector-valued function \( \Psi(z_1, ..., z_N) \) through multiple contour integrals of the Bethe vectors (4.2)

\[
\Psi(z_1, ..., z_N) = \oint \cdot \cdot \cdot \oint \mathcal{X}(u|z) \Phi_n(u|z) du_1 ... du_n,
\]

where \( \mathcal{X}(u|z) = \mathcal{X}(u_1, ..., u_n, z_1, ..., z_N) \) is a scalar function which in this stage is still undefined.

We assume that \( \Psi(z_1, ..., z_N) \) is a solution of the equations

\[
\kappa \frac{\partial \Psi(z_1, ..., z_N)}{\partial z_a} = G_a \Psi(z_1, ..., z_N), \quad a = 1, 2, ..., N
\]

where \( G_a \) are the Gaudin Hamiltonians (3.15) and \( \kappa \) is a constant.

Substituting (5.4) into (5.5) we have

\[
\kappa \frac{\partial \Psi(z_1, ..., z_N)}{\partial z_a} = \oint \left\{ \kappa \frac{\partial \mathcal{X}(u|z)}{\partial z_a} \Phi_n(u|z) + \kappa \mathcal{X}(u|z) \frac{\partial \Phi_n(u|z)}{\partial z_a} \right\} du,
\]

where we are using a compact notation \( \oint \{ \circ \} du = \oint \{ \circ \} du_1 \cdot \cdot \cdot du_n \).

Using the Gaudin algebra (3.24) one can derive the following non-trivial identity

\[
\frac{\partial \Phi_n}{\partial z_a} = - \sum_{l=1}^{n} (-)^l \frac{\partial}{\partial u_l} \left( \frac{2\phi^{(n-1)}_{l}}{\sinh(2u_l - 2z_n)} \right),
\]

which allows us write (5.6) in the form

\[
\kappa \frac{\partial \Psi}{\partial z_a} = \oint \left\{ \kappa \frac{\partial \mathcal{X}(u|z)}{\partial z_a} \Phi_n(u|z) + \sum_{l=1}^{n} (-)^l \kappa \frac{\partial \mathcal{X}(u|z)}{\partial u_l} \left( \frac{2\phi^{(n-1)}_{l}}{\sinh(2u_l - 2z_n)} \right) \right\} du
\]

\[
- \kappa \sum_{l=1}^{n} (-)^l \oint \frac{\partial}{\partial u_l} \left( \mathcal{X}(u|z) \frac{2\phi^{(n-1)}_{l}}{\sinh(2u_l - 2z_n)} \right) du.
\]

It is evident that the last term of (5.8) vanishes, because the contours are closed. Moreover, if the scalar function \( \mathcal{X}(u|z) \) satisfies the following differential equations

\[
\kappa \frac{\partial \mathcal{X}(u|z)}{\partial z_a} = g_a \mathcal{X}(u|z), \quad \kappa \frac{\partial \mathcal{X}(u|z)}{\partial u_j} = f_j^{(n-1)} \mathcal{X}(u|z),
\]

we are recovering the off-shell Gaudin equation (4.11) from the first term in (5.8).

Taking into account the definition of the scalar functions \( f_j^{(n-1)} \) (4.8) and \( g_a \) (4.12), we can see that the consistency condition of the system (5.9) is insured by the zero curvature conditions \( \partial f_j^{(n-1)} / \partial z_a = \partial g_a / \partial u_j \). Moreover, the solution of (5.9) is easily obtained

\[
\mathcal{X}(u|z) = \prod_{a<b}^{N} \sinh(z_a - z_b)^{1/\kappa} \prod_{j<k}^{n} \cosh(u_j - u_k)^{1/\kappa} \prod_{a=1}^{N} \prod_{j=1}^{n} \sinh(z_a - u_j)^{-1/\kappa}.
\]

This function determines the monodromy of \( \Psi(z_1, ..., z_N) \) as solution of the trigonometric KZ equation (5.5) and these results are in agreement with the Schectman-Varchenko construction for multiple contour integral as solutions of the KZ equation in an arbitrary simple Lie algebra [11].
6 Conclusion

In this paper a graded 19-vertex model was used to generalize previous rational vertex models results connecting the Gaudin magnet models to the semiclassical off-shell Bethe Ansatz of these vertex models.

Using the semiclassical limit of the transfer matrix of the vertex model we derived the trigonometric $sl(2|1)^{(2)}$ Gaudin Hamiltonians. The reduction of the off-shell Gaudin equation to an eigenvalue equation gives us the exact spectra and eigenvectors for these Gaudin magnets. Data of the off-shell Gaudin equation were used to show that a hypergeometric type integral (5.4) is solution of the trigonometric KZ differential equation.

In fact, this method had already been used with success to constructing solutions of trigonometric KZ equations [21, 22, 23] and elliptic KZ-Bernard equations [24], for the six-vertex model and eight-vertex model, respectively.

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