Heavy Traffic Approximation of Equilibria in Resource Sharing Games

Yu Wu, Loc Bui, and Ramesh Johari

Abstract

We consider a model of priced resource sharing that combines both queueing behavior and strategic behavior. We study a priority service model where a single server allocates its capacity to agents in proportion to their payment to the system, and users from different classes act to minimize the sum of their cost for processing delay and payment. As the exact processing time of this system is hard to compute and cannot be characterized in closed form, we introduce the notion of heavy traffic equilibrium as an approximation of the Nash equilibrium, derived by considering the asymptotic regime where the system load approaches capacity. We discuss efficiency and revenue, and in particular provide a bound for the price of anarchy of the heavy traffic equilibrium.

Keywords: resource sharing, discriminatory processor sharing, equilibrium, heavy traffic approximation

I. INTRODUCTION

A range of resource sharing systems, such as computing or communication services, exhibit two distinct characteristics: queueing behavior and strategic behavior. Queueing behavior arises because jobs or flows are served with the limited capacity of system resources. Strategic behavior arises because these jobs or flows are typically generated by self-interested, payoff-maximizing

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A previous version of this paper was presented at the 7th Workshop on Internet & Network Economics (WINE’11).

This work was supported by the National Science Foundation under grants CMMI-0948434, CNS-0904609, CCF-0832820, and CNS-0644114, and by the Defense Advanced Research Projects Agency under the ITMANET program.
users. Analysis of strategic behavior in queueing systems has a long history, dating to the seminal work of Naor [23]; see the book by Hassin and Haviv [9] for a comprehensive survey. The interaction of queueing and strategic behaviors has become especially important recently, with the rise of paid resource sharing systems such as cloud computing platforms. For example, [1] and [5] discussed systems with multiple service providers, modeled as first-come-first-serve queues, that compete in both price and response time for potential buyers.

In this paper we consider a particular queueing model where a single server is shared among multiple jobs, and the service capacity allocated to each job depends on its priority level. The particular scheduling policy we consider is known in the literature as the discriminatory processor sharing (DPS) policy, introduced by Kleinrock [17]. In the DPS model, the server shares its capacity in proportion to the priority level of all jobs currently in the system. This service allocation rule is a special case of a more general scheduling policy for queueing networks known as proportionally fair resource sharing [14], [19]; such scheduling policies have been studied extensively in the context of networked resource sharing (see [13], [27] and references therein). A survey of the DPS literature can also be found in [2].

We consider a DPS system in steady state, and study a job level game where every individual job is a single strategic user. Each user chooses a payment $\beta$, which corresponds to the priority level of that user. The user also incurs a cost proportional to total processing time. The users’ goal is to choose priority levels to minimize the sum of expected processing cost and payment. (We also briefly discuss a class level game, where every class is a single user.)

This game is inspired by resource sharing in real services. For example, in the Amazon EC2 Spot Instances, a user can bid her own price (priority) and enjoy the service as long as the dynamic benchmark price computed by the system is lower than the bid. The service is terminated either upon completion of task or when the system price rises above the bid price. In many file hosting websites, users can purchase premium packages which increase upload/download bandwidth and speed, and allow parallel tasks among other benefits. Note that in these services, the resource is shared among all users that currently request service, and a higher payment leads to higher performance.

A central difficulty in analysis of equilibria arises because exact computation of the steady state processing time of a single job, given the priority choices of other jobs, is not possible in closed form. Since the queueing behavior computation itself involves numerical complexity,
equilibrium characterization in closed form for the strategic behavior is essentially impossible. Thus obtaining structural insight into the games is a significant challenge.

To tackle this problem, we propose an approximate approach to equilibrium characterization that is amenable to analysis, computable in closed form, and provably exact in an appropriate asymptotic regime where the load on the system increases, known as the heavy traffic regime [16], [25]. The heavy traffic asymptotic regime is widely used in analysis of queueing systems and is especially valuable to study systems with many users. Asymptotics yield two benefits. First, they significantly simplify stochastic analysis. The second key benefit of asymptotics is that we are also able to simplify our game theoretic analysis. Informally, an important reason is that when the number of users grows large, no single user has a large impact on the whole system; this effect allows us to simplify calculation of equilibria. (Note that this is similar to the “large market” approximation used to justify competitive price equilibria in economics.)

We conclude with a brief survey of related work. Priority pricing problems in queueing systems with disciplines other than DPS have been investigated, such as design of efficient, incentive compatible pricing for nonpreemptive priority FIFO queues [15], [20]. Besides the service discipline, our work also differs from this work because in our model users choose their own priority levels, while in previous models the service provider fixes the priority levels available. Other studies focus on optimal arrival strategy of users, including the work by Glazer and Hassin [8] and Jain et al. [11]. In our paper, although we briefly discuss the case where arrivals are endogenously generated, in the main discussion we assume that arrival rates exogenously given.

Turning our attention to DPS specifically, we note that most prior work on DPS considers only analysis of the queueing system without any strategic choice of priorities. Haviv and van der Wal [10] consider a DPS system in which users choose their own priority levels to minimize their costs, but only study a model with one class of users. To the best of our knowledge, there is no previous work that considers priority pricing in the multiple class DPS system, because expected waiting time cannot be characterized in closed form.

Our main contributions are as follows.

(1) An approximate notion of equilibrium. Using an approximation to the processing time derived via the heavy traffic asymptotic regime, we suggest a natural corresponding notion of equilibrium that we call heavy traffic equilibrium (HTE). In an HTE, users minimize the sum of their payment and heavy traffic processing time cost, rather than their true expected processing
time cost. We show that under mild conditions, HTE exists and is unique, and that it can be computed in closed form in terms of system parameters. It is thus both simple to compute, and asymptotically accurate when the system approaches heavy traffic.

(2) Economic analysis: parameter sensitivity, efficiency, and revenue. A significant benefit of our approach is that since we can compute the equilibrium in closed form, it is straightforward to carry out analysis of efficiency and revenue. We study how the system behavior changes when cost or arrival rate parameters are scaled, and more importantly, we investigate social efficiency and system revenue of HTE under different system parameters, and give a bound for the price of anarchy of HTE. We obtain some intriguing insights: in particular, we show that within a particular class of pricing schemes, and for a wide range of parameter choices, the incentives of the revenue maximizing service provider become aligned with minimization of total system processing cost.

The remainder of the paper is organized as follows. In Section II we describe the queueing game setup. In Section III we introduce the notion of heavy traffic equilibrium. We then present the results on parameter sensitivity, efficiency, and revenue under the heavy traffic equilibrium in Section IV. Some extensions of the model are discussed in Section V followed by the conclusion. The proofs of the theorems are in the Appendix.

II. RESOURCE SHARING GAME

We consider a queueing game in which $K$ classes of jobs share a single server of unit capacity. Class $i$ ($i = 1, \cdots, K$) jobs arrive according to a Poisson process with arrival rate $\lambda_i$ and have i.i.d. exponentially distributed service requirements (measured in units of service, e.g., processing cycles) with mean $1/\mu_i$. We assume for simplicity that $\mu_i = \mu$ for all classes. Let $\lambda = \sum_k \lambda_k$ denote the total arrival rate to the system. Also, let $\rho_i = \lambda_i/\mu$ be the load of class $i$, and define the system load as $\rho = \sum_k \rho_k = \sum_k \lambda_k/\mu$. To ensure stability, we assume $\rho < 1$. It is well known that under this condition, the resulting queueing system is ergodic and possesses a unique steady state distribution [18]. Waiting and being served in the system induces a cost $c_i$ per unit time for users of class $i$. Without loss of generality we assume $c_1 > c_2 > \cdots > c_K$: if two classes $i$ and $j$ have the same cost $c_i = c_j$, then they can be merged into one class with arrival rate $\lambda_i + \lambda_j$.

We assume that the server allocates its capacity according to the discriminatory processor
sharing (DPS) policy. Under this policy, each job is associated with a priority level. If there are currently \( N \) jobs in the system and job \( \ell \) has chosen priority level \( \beta_\ell \), then the fraction of service capacity allocated to job \( \ell \) is \( \beta_\ell / \sum_{m=1}^{N} \beta_m \).

Upon arrival, without observing the state of the system, each job chooses a priority level \( \beta > 0 \). We consider a family of pricing rules for priority that we refer to as \( \alpha \)-fair pricing rules, where \( \alpha > 0 \). Formally, we assume that if a job chooses priority level \( \beta \), then the system manager charges that job a price \( \beta^\alpha \), where \( \alpha > 0 \). Varying \( \alpha \) allows us to study a range of pricing schemes. In particular, as \( \alpha \to 0 \), jobs face a strongly diminishing marginal cost to higher choices of \( \beta \); while as \( \alpha \to \infty \), jobs face a strongly increasing marginal cost with higher choices of \( \beta \).

The pricing rules we consider are closely related to \( \alpha \)-fair allocation rules studied in the networking literature [21]. In an \( \alpha \)-fair allocation system, one unit of resource is allocated to \( N \) users, whose utility functions are characterized by \( \alpha \): \( U^{(\alpha)}(x) = x^{1-\alpha}/(1-\alpha) \) if \( \alpha \neq 1 \), and \( U^{(\alpha)}(x) = \log(x) \) if \( \alpha = 1 \). Users make payments for use of the system. Let \( w_\ell \) be the payment of user \( \ell \); the payments determine users’ weights in the system. Formally, suppose the payment vector of users is \( w \) and the allocation vector is \( x \); then the resource manager solves the following optimization problem:

\[
\max_x \sum_{\ell=1}^{N} w_\ell U^{(\alpha)}(x_\ell) \quad \text{s.t.} \quad \sum_{\ell=1}^{N} x_\ell \leq 1.
\]

The solution of this problem is \( x_\ell = w_\ell^{1/\alpha} / (\sum_m w_m^{1/\alpha}) \). A well-known example of an \( \alpha \)-fair allocation rule is the proportionally fair allocation rule, obtained when \( \alpha = 1 \) [14]: resource is allocated proportional to payment. Now, suppose that the \( \alpha \)-fair pricing rule is used in our model, so that \( w_\ell = \beta_\ell^\alpha \). Then the \( \alpha \)-fair allocation rule reduces to the discriminatory processor sharing policy described above—i.e., allocation of server capacity in proportion to the priority levels \( \beta_\ell \).

In this paper we will generally be interested in scenarios where all jobs of the same class \( i \) choose the same priority level. In an abuse of notation we denote by \( \beta_i \) the priority level chosen by all class \( i \) jobs, and in this case we succinctly denote \( (\beta_1, \cdots, \beta_K) \) by \( \beta \). We refer to \( \beta \) as the class priority vector. Let \( V(\beta; \beta) \) be the expected processing time for a job with priority \( \beta \) that arrives to the system in steady state, with the class priority vector given by \( \beta \). Observe that with this notation, a class \( i \) job with priority level \( \beta_i \) has expected processing time \( V(\beta_i; \beta) \). For
convenience we define $W_i(\beta) = V(\beta_i; \beta)$. The total cost of a user is $cV(\beta; \beta) + \beta^\alpha$, where $c$ is the user’s unit time cost and $\beta$ is its priority level.

We frequently make use of Little’s law, which provides a relationship between steady state expected processing times and steady state expected queue lengths [18]. In particular, let $N_i$ denote the steady state number of class $i$ jobs in the system. In a system consisting of $K$ classes $(\lambda_i, \beta_i), i = 1, \ldots, K$, Little’s law establishes that in steady state, for every class $i$ we have $E[N_i] = \lambda_i W_i(\beta)$.

A. Nash Equilibrium

We consider two types of games for this system: the **job level game** and the **class level game**. In the job level game, each job is an individual user, aiming to minimize its expected total cost by choosing its own priority level $\beta$. Although jobs from the same class are allowed to choose different priority levels, because jobs of the same class share the same parameters *ex ante*, we restrict our attention only to symmetric equilibria of the resource sharing game; these are equilibria where jobs from the same class choose the same priority levels. Such an equilibrium can be characterized by a class priority vector $(\beta_1, \ldots, \beta_K)$.

**Definition 1:** A **job level Nash equilibrium** consists of a class priority vector $\beta = (\beta_1, \ldots, \beta_K)$ such that for all $i = 1, \ldots, K$,

$$
\beta_i = \arg \min_{\beta \geq 0} \left[ c_i V(\beta; \beta) + \beta^\alpha \right], \forall \ i = 1, \ldots, K.
$$

(1)

In the class level game, each class is regarded as a single user and chooses a priority level for all of its jobs; therefore the equilibrium is again characterized by a class priority vector.

**Definition 2:** A **class level Nash equilibrium** consists of a class priority vector $\beta = (\beta_1, \ldots, \beta_K)$ such that for all $i = 1, \ldots, K$,

$$
\beta_i = \arg \min_{\beta \geq \beta_i} \left[ c_i W_i(\beta_1, \ldots, \beta_{i-1}, \beta, \beta_{i+1}, \ldots, \beta_K) + \beta^\alpha \right], \forall \ i = 1, \ldots, K.
$$

(2)

We emphasize that, although jobs from the same class choose the same priority in both the symmetric job level equilibrium and the class level equilibrium, these two equilibria are not identical. The difference is that in the class level game, changing the priority level of a whole class $i$ causes an externality within the class itself, while by contrast, in the job level game, a single job alters its priority level in isolation. In this paper, we mainly study the job level game, but also briefly discuss how our study can be adapted to the class level game in the Section V.
B. Characterizing Processing Times

Nash equilibria of both the job level and class level require the characterization of processing times $V$ and $W_i$, which is in general quite complex. For the $K$ class DPS model, Fayolle et al. [7] show that the expected steady state processing time $W_i$ for each class $i$ can be determined by solving a linear system.

**Theorem 1:** In a $K$-class DPS model with class priority vector $\beta$, $(W_1(\beta), \cdots, W_K(\beta))$ is the unique solution of the following system of equations:

$$\mu W_k(\beta) - \sum_{i=1}^{K} \frac{\lambda_i \beta_i}{\beta_i + \beta_k} (W_k(\beta) + W_i(\beta)) = 1, \ k = 1, \cdots, K. \quad (3)$$

On the other hand, computing the processing time $V(\beta; \beta)$ for general $\beta$ can be reduced to computing the processing time $W_i(\beta), i = 1, \cdots, K$ as stated by the following theorem.

**Theorem 2:** Let $N_i$ be the steady state number of class $i$ jobs in a $K$-class DPS system with class priority vector $\beta$. Then the steady state processing time of a job with priority $\beta$ is

$$V(\beta; \beta) = U_0(\beta; \beta) + \sum_{i=1}^{K} U_i(\beta; \beta) E[N_i], \quad (4)$$

where

$$U_i(\beta; \beta) = \frac{\beta_i}{\beta_i + \beta} U_0(\beta; \beta), \ i = 1, \cdots, K; \text{ and } U_0(\beta; \beta) = \left[ \mu - \sum_{i=1}^{K} \frac{\lambda_i \beta_i}{\beta_i + \beta} \right]^{-1} \quad (5)$$

The values of $E[N_i]$ can be obtained by applying Little’s law to the solution of the system of linear equations (3). We conclude, therefore, that solving for $V(\beta; \beta)$ in (4) can be reduced to computing $W_i(\beta)$. In general, explicitly solving (3) requires the inversion of a $K \times K$ matrix with complexity $O(K^3)$, and hence, there is no closed form expression for $W_i$ or $V$.

Nevertheless, when $K = 1$ or $K = 2$, we are able to solve for $W_i$ and $V$ in closed form. The solution of (4) with $K = 1$ is first established in [9], [10] as follows

$$V(\beta; \hat{\beta}) = \frac{1}{\mu(1-\rho)} \cdot \frac{\beta(1-\rho) + \hat{\beta}}{\hat{\beta}(1-\rho) + \beta}. \quad (6)$$

When $K = 2$, the solution for $W_i$ is given by [7], and the solution for $V$ directly follows. Both solutions are lengthy and omitted for brevity.
C. Existence of NE

Existence of Nash equilibrium can be guaranteed when $\alpha \geq 1$, by exploiting convexity of the job cost function in (1). When $\alpha < 1$, the payment term $\beta^\alpha$ is strictly concave, therefore the convexity of the objective function is not guaranteed. Although analytically establishing existence of Nash equilibrium in this regime remains an open question, our numerical computation with best response dynamics converges to a Nash equilibrium even when $\alpha < 1$.

Theorem 3: There exists a Nash equilibrium for the job level game when $\alpha \geq 1$.

As usual, this existence result is nonconstructive, since it uses a fixed point theorem. In general, given the implicit equations that define the processing times in (3), there is no closed form characterization of the Nash equilibrium, and no tractable approach for computation is available. Although we could resort to some heuristics (e.g., best response dynamics) to approach NE, each step of such an algorithm requires computing a range of processing times with fixed parameters, and as established above each such computation has complexity $O(K^3)$. Further, there is no theoretical guarantee that such dynamics will converge. (Though we note, numerical computation suggests that the best response dynamics does converge.) Equilibrium computation is therefore not possible in closed form in general; as a result, we are left with essentially no structural insight into the behavior of players in the game.

III. Heavy Traffic Approximation

In the remainder of the paper we consider an alternate approach to the equilibrium analysis, by approximating the processing time. We aim to overcome the complexity of computing the processing times by exploiting a heavy traffic approximation, i.e., an approximation where the load approaches service capacity. Such an approximation is relevant for large systems such as cloud computing services, where providers will typically not want to provision significant excesses of capacity.  

1One such justification for heavy traffic capacity provisioning comes from Nair et al. [22], who study optimal capacity provisioning for online service providers. They find that as the market size becomes large, heavy traffic emerges as a consequence of a profit maximizing strategy for the service provider, with exact scaling depending on the strength of positive externalities among users.
A. Approximating the Processing Time

In heavy traffic, a phenomenon known as state space collapse gives us a simplified solution for the steady state distribution of the system [25]; informally, state space collapse refers to the fact that the numbers of jobs of each class in the system become perfectly correlated when the system is heavily loaded.

In a slight abuse, whenever we write $\rho \to 1$, we mean that we consider a sequence of systems such that $(\rho_1, \cdots, \rho_K)$ converges to some $(\bar{\rho}_1, \cdots, \bar{\rho}_K)$ with $\sum_{i=1}^{K} \bar{\rho}_i = 1$. Moreover, we emphasize that both $V(\beta; \beta)$ and $W_i(\beta)$ depend on $\rho$, though we suppress this dependence for notational brevity. Let $N_i$ denote the steady state number of type $i$ jobs in the system. Then we have the following result on the joint steady state distribution of $(N_1, \cdots, N_K)$ for a DPS system in heavy traffic.

**Theorem 4:** [24] Let $N_i$ be the steady state number of class $i$ jobs in a $K$-class DPS system with class priority vector $\beta$. Then as $\rho \to 1$, we have

$$ (1 - \rho)(N_1, \cdots, N_K) \overset{d.}{\to} Z \cdot \left( \frac{\bar{\rho}_1}{\beta_1}, \cdots, \frac{\bar{\rho}_K}{\beta_K} \right), $$

where “$d.\to$” denotes convergence in distribution, and $Z$ is an exponentially distributed random variable with mean $1/\gamma(\beta)$ where $\gamma(\beta) = \sum_{i=1}^{K} \rho_i/\beta_i$.

Convergence of the joint distribution implies convergence of marginal distributions, so $(1 - \rho)N_i \overset{d.}{\to} Z \bar{\rho}_i/\beta_i$ for each $i$. Moreover, the second moment of $(1 - \rho)N_i$ is shown to be uniformly bounded [24], so the $N_i$’s are uniformly integrable. It follows from [4] that in this case convergence in distribution implies convergence in the mean, and hence,

$$ (1 - \rho)E[N_i] \to E[Z] \frac{\bar{\rho}_i}{\beta_i} = \frac{\bar{\rho}_i}{\beta_i \gamma(\beta)} \quad \text{as} \quad \rho \to 1. $$

Taking advantage of this approximation of $E[N_i]$, we are now able to approximate $V(\beta; \beta)$.

Note that

$$ \lim_{\rho \to 1} U_0(\beta; \beta) = \left[ \mu \left( 1 - \sum_{i=1}^{K} \frac{\bar{\rho}_i \beta_i}{\beta_i + \beta} \right) \right]^{-1} = \left( \mu \beta \sum_{i=1}^{K} \frac{\bar{\rho}_i}{\beta_i + \beta} \right)^{-1} $$

is finite, so substituting (5) and (8) into (4) yields

$$ \lim_{\rho \to 1} (1 - \rho)V(\beta; \beta) = \left( \lim_{\rho \to 1} U_0(\beta; \beta) \right) \left( \sum_{i=1}^{K} \frac{\beta_i}{\beta_i + \beta} \lim_{\rho \to 1} (1 - \rho)E[N_i] \right) = \frac{1}{\mu \beta \gamma(\beta)}. $$

In the light of the above approximation, we have the following definition.
Definition 3: The heavy traffic processing time for a job with priority level $\beta$ in a system with $K$ classes with class priority vector $\beta$ is defined as

$$V^{HT}(\beta; \beta) = \frac{1}{1 - \rho} \cdot \frac{1}{\mu \gamma(\beta)}, \quad \text{where} \quad \gamma(\beta) = \frac{1}{\rho} \sum_{i=1}^{K} \frac{\rho_i}{\beta_i}. \quad (10)$$

We note that $V^{HT}(\beta; \beta)$ has a closed form, and is easy to compute. Moreover, it is asymptotically exact in the heavy traffic regime: it is straightforward to show that as $\rho \to 1$, $\gamma(\beta) \to \overline{\gamma}(\beta)$, and hence, $(1 - \rho)[V^{HT}(\beta; \beta) - V(\beta; \beta)] \to 0$.

We note here that one reason we consider the case where $\mu_i = \mu$ for all $i$ is that in the absence of this assumption, a similar result to Theorem 2 becomes more challenging (in particular, because (16) in the appendix is no longer tractable). However, an appropriate generalization of Theorem 4 holds even for heterogeneous $\mu_i$, and based on this fact we conjecture that the analysis of this paper can be carried out even with heterogeneity of $\mu_i$. We leave this for future work.

B. Heavy Traffic Equilibrium

For general $K$ it is quite hard to solve for pure Nash equilibrium because: (i) computing $V(\beta; \beta)$ requires matrix inversion to solve the linear system (3), which can only be done numerically; and (ii) even if we are able to solve $V(\beta; \beta)$ numerically and obtain optimality conditions for each player (which cannot be done in closed form), we would still need to solve a generally nonlinear system with $K$ equations and $K$ unknowns to compute the Nash equilibrium.

In this section, we propose a novel concept of equilibrium which can be used to approximate the Nash equilibrium, yet can be computed in closed form. We approximate $V(\beta; \beta)$ by $V^{HT}(\beta; \beta)$ in the objective function, and based on this approximation we define a concept of equilibrium that we call heavy traffic equilibrium (HTE) for job level games, as follows.

Definition 4: A heavy traffic equilibrium of the game consists of a set of priorities $\beta = (\beta_1, \cdots, \beta_K)$ such that

$$\beta_i = \arg\min_{\beta > 0} \left( c_i V^{HT}(\beta; \beta) + \beta^\alpha \right), \quad i = 1, \cdots, K.$$  

We can explicitly compute the heavy traffic equilibrium.

Theorem 5: A heavy-traffic equilibrium always exists, and it is unique. Moreover, it can be calculated in closed form:

$$\beta_i = e_i^{\frac{1}{\alpha}} [\alpha(1 - \rho) \rho^{-1} S_1]^{-\frac{1}{\rho}}, \quad (11)$$
where $S_1 = \sum_{i=1}^{K} \lambda_i c_i^{-\frac{1}{\alpha+1}}$.

We have two remarks on this result. First, this closed form expression allows us to carry out analysis of sensitivity, efficiency, and revenue of the HTE (see Section IV). Second, the HTE is easily computable with complexity $O(K)$. In comparison, the complexity for computing the exact processing time with fixed parameters is $O(K^3)$, and as discussed computing exact NE is intractable.

We have observed above that the difference between the heavy traffic processing time and the exact processing time approaches zero as $\rho \to 1$, when scaled by a factor $1 - \rho$. Using this approximation, we can also prove an approximation theorem for the heavy traffic equilibrium: we show that deviating by any constant factor from the HTE is not profitable as $\rho \to 1$.

**Theorem 6**: Consider a sequence of systems indexed by $n$ such that classes have the same service capacity $\mu$, and the loads of the systems $\rho^{(n)} \to 1$ as $n \to \infty$. Let $\beta^{(n)}$ be the unique HTE of the $n$-th system, then for any $\delta \geq 0$,

$$\lim_{n \to \infty} (1 - \rho^{(n)}) \left[ c_i V(n) \left( \beta_i^{(n)}; \beta^{(n)} \right) + \left( \delta \beta_i^{(n)} \right)^\alpha - c_i V(n) \left( \delta \beta_i^{(n)}; \beta^{(n)} \right) - \left( \delta \beta_i^{(n)} \right)^\alpha \right] \leq 0. \quad (12)$$

Here $V$ is subscripted by $(n)$ to indicate that the processing time is computed in system $n$ with load $\rho^{(n)}$.

In the theorem, we consider deviations by a multiplicative constant factor rather than by an additive constant because (11) implies that, as $\rho \to 1$, the heavy traffic equilibrium increases without bound; as a result, it is straightforward to check that any additive constant deviation has no beneficial effect as $\rho$ approaches 1. Note that the processing time is only asymptotically exact up to a $1 - \rho$ scaling, thus the same is true for this approximation theorem as well. Indeed, this is what we give up by studying heavy traffic: while we gain analytical tractability, the “resolution” to which we can study deviations is scaled by $1 - \rho$. This tradeoff is systematic throughout the study of large scale queueing models even without strategic behavior.

C. Numerics: Approximation Error

In this subsection, we numerically study the approximation error between the HTE and the exact NE, with different system parameters $(K, \{c_i\}, \{\lambda_i\}, \alpha, \rho)$; this complements our theoretical analysis above. Given a HTE $\beta^{HT}$ and an NE $\beta^{NE}$, we use relative error as a measure of approximation, i.e., $\max_i (\beta_i^{HT} - \beta_i^{NE}) / \beta_i^{NE}$. We compute NE using best response dynamics;
Fig. 1. Relative error of HTE vs. NE under different parameter choices.

surprisingly we found that best response dynamics converge to NE for all parameter choices below.

Figure 1 shows the impact of changes in system parameters on approximation accuracy. To quantify the heterogeneity of $c_i$’s, we assume $c_i$’s are i.i.d. from a uniform distribution on $[0, 10]$, plus a constant $c_0$. A smaller $c_0$, therefore, induces a potentially larger ratio between the smallest and largest $c_i$. Similarly, we assume $\lambda_i$’s are i.i.d. drawn from uniform$[0, 10] + \lambda_0$. To
illustrate the change, we fix most of these parameters at $K = 10, c_0 = \lambda_0 = 1, \alpha = 1, \rho = 0.9$, but vary one or two of them at a time. For each set of parameters, we have 100 simulation samples (of cost vector and arrival rate vector), and the approximation errors are summarized by the boxplots.

In the upper panel of Figure 1, we see that heterogeneity in the system weakens the approximation. Numerical results show that approximation error is higher with smaller $c_0$ or a larger number of classes; both increase heterogeneity. Heterogeneity in the arrival rates appears to cause less degradation in the approximation, but it can be shown that both arrival rate heterogeneity and significant cost heterogeneity together can amplify approximation errors.

In the lower panel of Figure 1, we vary $\alpha$ and $\rho$. The results suggests that the approximation error is lower with larger $\alpha$. Note that $\alpha$ describes the marginal cost to payment; therefore larger $\alpha$ induces smaller payments and the relative heterogeneity among user decisions diminishes. Regarding $\rho$, the error decreases as we approach heavy traffic ($\rho$ close to 1), as expected.

We conclude by noting that the approximation error can be made arbitrarily large through appropriate parameter choices; one such example is given when $K = 2$, $c_1/c_2 \to \infty$ and $\lambda_1/\lambda_2 \to \infty$. In this case since $\lambda_2$ and $c_2$ are relatively small, the class 2 optimal priority level in the exact NE is also extremely small. However, in the heavy traffic approximation, the processing time $V^{HT}(\beta, \beta)$ is inversely proportional to $\beta$; so in HTE no user will chooses an extremely small $\beta$, leading to arbitrarily large error rate.

IV. Sensitivity, Efficiency and Revenue

The tractability of heavy traffic equilibrium allows us to analytically study parameter sensitivity, as well as efficiency and revenue at the HTE equilibrium. Throughout this section, we let $\beta^*$ denote the HTE.

A. Sensitivity

In this subsection, we analyze the sensitivity of the HTE, i.e., how the equilibrium behaves with respect to changes in system parameters. These observations follow directly from (11).

Sensitivity with respect to $c$. If all $c_i$ are scaled by a constant $\zeta > 0$, then every $\beta_i^*$ is scaled by $\zeta^{1/\alpha}$. This is rather intuitive since the objective function is the sum of expected processing cost and $\beta_i^\alpha$, and the expected processing cost does not change any $V_i$. Therefore the equilibrium
is the same up to a scaling factor. Further more, simple first derivative analysis shows that all equilibrium $\beta_i^*$’s are increasing in any single $c_j$. Note that increasing $c_j$ provides extra incentive for class $j$ users to invest in priority $\beta_j$. In equilibrium, all other $\beta_i^*$’s also increase as a result of priority competition in the server.

**Sensitivity with respect to $\rho$.** The ratio $\beta_i^*/\beta_j^* = (c_i/c_j)^{\alpha+1}$ is independent of $\rho$, i.e., changing $\rho$ will change each $\beta_i^*$ but will not affect $\beta_i^*/\beta_j^*$ for any $i, j$. Therefore the ratio between service capacity allocated to any pair of jobs, as well as the ratio between the heavy traffic processing times of a pair of jobs, are invariant to the load of the system.

**Sensitivity with respect to $\alpha$.** When $\alpha \to 0$, every $\beta_i^* \to \infty$; when $\alpha \to \infty$, every $\beta_i^* \to 1$. This is due to the fact that as $\alpha \to 0$, jobs face a strongly diminishing marginal cost to higher choices of $\beta$, and hence, prefer to choose higher $\beta$ at the equilibrium; while the effect is reversed as $\alpha \to \infty$.

### B. Efficiency

In HTE, efficiency is characterized by the expected total cost incurred to the system in one unit of time:

$$
C = \sum_{i=1}^{K} \lambda_i c_i V^{HT}(\beta_i^*; \beta^*) = \left( \frac{\rho}{1 - \rho} \right) \left( \sum_{i=1}^{K} \lambda_i c_i \frac{\alpha}{\alpha+1} \right) \left( \sum_{i=1}^{K} \lambda_i c_i \frac{1}{\alpha+1} \right)^{-1}.
$$

We call $C$ the system processing cost (a more efficient system has a lower value of $C$). Given fixed $\lambda_i$ and $c_i$ ($i = 1, \cdots, K$), the efficiency depends on the system parameter $\alpha$ and the load $\rho$ as follows.

**Dependence of $C$ on $\rho$.** We note that $C$ is proportional to $\rho/(1 - \rho)$, and hence is increasing in $\rho$. This is because a larger load $\rho$ implies a busier system, and therefore the processing time is longer. (Note that we fixed $\lambda$, so varying $\rho$ is equivalent to varying $\mu$.)

**Dependence of $C$ on $\alpha$.** It is well known that the system optimal scheduling policy is the $c$-$\mu$ rule [6]: classes are given strict priority in descending order of $c_i \mu_i$ (or equivalently in this paper, in descending order of $c_i$, since we assume that all $\mu_i$ are the same). That is, for any $1 \leq i, j \leq K$, class $j$ jobs are preempted by class $i$ jobs if $c_i \mu_i > c_j \mu_j$. Jobs with the same value of $c_j \mu$ are served in first-in-first-out (FIFO) scheme. Since $\beta_i^*/\beta_j^* = (c_i/c_j)^{\alpha+1}$, for $c_i > c_j$, the ratio $\beta_i^*/\beta_j^*$ is higher with smaller $\alpha$, so we expect higher $\alpha$ lead to less efficient equilibria. This intuition is analytically stated in the following theorem.
Proposition 7: The HTE system processing cost $C$ is increasing in $\alpha > 0$.

We note that even when $\alpha$ approaches 0, the HTE does not approach social optimum. In fact, for any $i, j$ such that $c_i > c_j$, we have that $\beta_i^* / \beta_j^* = (c_i / c_j)^{1/\alpha+1}$, and hence $1 < \beta_i^* / \beta_j^* < c_i / c_j$.

On the other hand, with the $c$-$\mu$ rule, if $c_i > c_j$, then class $i$ jobs completely preempt class $j$ jobs, which can be interpreted as the case where $\beta_i^* / \beta_j^* = \infty$. Therefore, it is clear that the HTE can never be as efficient as the $c$-$\mu$ rule, for any choice of $\alpha$. However, we can upper bound the price of anarchy (PoA) of the HTE, as stated in the following theorem. The PoA is the ratio $C / C^{\text{opt}}$, where $C^{\text{opt}}$ is the minimum expected system processing cost (achieved by the $c$-$\mu$ rule).

Theorem 8: The price of anarchy (PoA) of the HTE is upper-bounded by:

$$\frac{C}{C^{\text{opt}}} < \sum_{i=1}^{K-1} (\lambda_i / \lambda_K) (c_i / c_K)^{\alpha/\alpha+1} + 1 < \left(\frac{\lambda - \lambda_K}{\lambda_K}\right) \left(\frac{c_1}{c_K}\right)^{\alpha/\alpha+1} + 1.$$  \hspace{1cm} (14)

Note that the upper bound can be made arbitrarily large through appropriate parameter choices; further, this is tight, in the sense that there exist systems where the PoA of HTE is in fact arbitrarily large. For example, let $\lambda_i = \lambda$ for all $i$, and set $c_K = 1$, $c_i = m$ for $i = 1, \cdots, K-1$, and choose $\mu$ so that $\rho = 1 - m^{-2}$. Then it can be shown that $C / C^{\text{opt}} = \Omega \left((K-1)m^{\alpha/(\alpha+1)}\right)$ as $m \to \infty$ (see the proof of the theorem for details).

We also note that the PoA bound is increasing in $\alpha$, which matches the intuition that a scheme closer to strict priority in descending cost order yields higher social welfare. If we let $\alpha \to 0$, then the PoA is asymptotically bounded by $\lambda / \lambda_K$. In that case, if the arrival rates of all classes are the same, then the PoA is bounded by $K$. We can also let $\lambda_1, \cdots, \lambda_{K-1} \to 0$ to make the PoA approach 1, but this is not surprising since in this case the system essentially consists of only one class.

C. Revenue

The revenue of the server per unit time is the sum of expected payments in one unit of time:

$$R = \sum_{i=1}^{K} \lambda_i (\beta_i^*)^\alpha = \left(\frac{\rho}{\alpha(1-\rho)}\right) \left(\frac{1}{\sum_{i=1}^{K} \lambda_i c_i^{\alpha/\alpha+1}}\right) \left(\frac{1}{\sum_{i=1}^{K} \lambda_i c_i^{-1/\alpha+1}}\right)^{-1}.$$ \hspace{1cm} (15)

Given fixed $\lambda_i$ and $c_i$ ($i = 1, \cdots, K$), the revenue depends on the system parameter $\alpha$ and the load $\rho$ as follows.
Dependence of \( R \) on \( \rho \). The revenue is proportional to \( \rho/(1 - \rho) \), therefore the revenue is increasing in \( \rho \). Heavier traffic will induce greater congestion, and hence, jobs have to invest more in their purchase of priority in order to keep the same performance.

Dependence of \( R \) on \( \alpha \). The revenue depends on \( \alpha \) in three terms, and it seems that in general the effect of changing \( \alpha \) in the last two terms is significantly smaller than that of changing \( \alpha \) in the first term \( \rho/(\alpha(1 - \rho)) \). Hence we would expect that the revenue is in general decreasing in \( \alpha \). The next result shows this intuition holds if \( c_1/c_K \) is not too high.

**Theorem 9:** The revenue \( R \) is decreasing in \( \alpha > 0 \) if \( c_1/c_K < e^4 \).

On the other hand, \( R \) could be increasing in \( \alpha \) in some cases. For instance, if \( K = 2 \) and \( c_1/c_2 \) is large enough, then \( \partial R/\partial \alpha \) is positive around \( \alpha = 1 \) (see the proof of theorem for details). To explain this special scenario where the monotonicity does not hold, we first note that a smaller \( \alpha \) in general induces a higher revenue because jobs have incentive to purchase higher priority (as a response to the stronger diminishing marginal cost effect). However, in the HTE, significant asymmetry in costs will result in significant asymmetry in equilibrium priorities. Therefore when \( c_1/c_K \) is large, the optimal priorities already exhibit significant differences even when \( \alpha \) is not small, and thus in equilibrium at small \( \alpha \), jobs have lower incentive to increase their priorities compared to what they do with mutually comparable costs.

With both (13) and (15), it is quite surprising to see that in the HTE,

\[
\text{total cost of all jobs} = C = \alpha R = \alpha \cdot \text{total revenue of the system}.
\]

Thus, we obtain an interesting insight: another interpretation of \( \alpha \) is the users’ equilibrium cost per unit revenue. We have shown that the user’s total cost is increasing in \( \alpha \), (i.e., the system efficiency is decreasing in \( \alpha \)), and the system revenue is decreasing in \( \alpha \) under some mild conditions. Therefore, in a wide range of regimes, from the standpoint of system manager, smaller \( \alpha \) is more favorable in terms of both efficiency and revenue. Note that smaller \( \alpha \) is somewhat more “unfair,” however, as it approaches a strict priority system.

**V. Discussion and Conclusions**

We believe our work makes significant progress on two fronts. First, the DPS queueing model is important in its own right as a benchmark model for analysis of priority pricing for shared resource services. Our analysis provides extensive insight into this queueing system with strategic
behavior. Second, and perhaps of greater longer term interest, our approximation methodology suggests a broader research program for understanding strategic behavior in queueing systems: by exploiting large system asymptotics, we can simplify both the complexity of the stochastic system, as well as the complexity of the economic system.

We conclude by discussing several extensions and open directions.

**Random order of service.** Consider an alternative prioritized allocation policy, the random order of service (ROS) policy. In the ROS policy, only one job is served at a time and upon completion of this job, a new job starts to be served with probability proportional to its priority level. Therefore, if there are currently $N$ jobs waiting in the system and job $\ell$ has chosen priority level $\beta_\ell$, then in the ROS policy, job $\ell$ is the next job to start service with probability $\beta_\ell / \sum_{m=1}^{N} \beta_m$.

Although ROS and DPS are different in the ways they allocate service among jobs, the ratio of expected service allocated to two jobs is the same as the ratio of their priorities in both schemes. Therefore, we would expect some similarities in the expected processing times of jobs in these two allocation policies. In fact, Ayesta et al. [3] show that in heavy traffic regime, the expected processing time of a class $j$ job in a ROS system is exactly $V^{HT}(\beta_j; \beta)$, the same as in a DPS system. Thus in the heavy traffic ROS system, the expected processing time of a job with priority $\beta$ is $V^{HT}(\beta; \beta)$. Therefore all our results on heavy traffic equilibria of the DPS system also hold for the ROS system.

**Class level games.** Based on the heavy traffic processing time approximation results in Theorem 4, one can also propose a similar heavy traffic equilibrium (HTE) concept for class level games. However, although the processing time approximation allows us to greatly simplify the computation of best response strategies, we are not able to obtain a closed form expression for the class level HTE. More specifically, we can similarly define heavy traffic processing time by analogously defining

$$\gamma_{-i}(\beta; \beta) = \frac{1}{\rho} \left[ \sum_{j=1,j\neq i}^{K} \frac{\rho_j}{\beta_j} + \frac{\rho_i}{\beta} \right].$$

We can obtain a system of non-linear equations to compute this class level heavy traffic equilibrium, but we are not able to get any closed form expressions for it, mainly due to two features of $\gamma_{-i}(\beta; \beta)$ that are different from $\gamma(\beta)$ of a job level game. First, $\gamma_{-i}(\beta; \beta)$ is subscripted by $i$, which implies that different classes face different environments in the system. This is because,
unlike the infinitesimal single user in job level problem, each class as a player is not negligible in the system. Second, $\gamma_i(\beta; \beta)$ explicitly depends on $\beta$, the action of class $i$, due to the *intra-class externality* behavior: in a class level game, a class chooses one priority level for *all* its jobs simultaneously; while in a job level game, a job can choose any priority level regardless of other jobs belonging to its class.

Nevertheless, the intra-class externality effect will become negligible in a regime where the number of classes is large; this is a particularly useful regime for computing services where the number of users grows large, and each user is viewed as a distinct class. This observation motivates us to consider a limiting model in which the number of classes approaches infinity and any single class becomes infinitesimal. Thus, it can be connected to the job level game model. We show that in the large system limit of the class level game, the heavy traffic equilibrium exists, is unique, can be computed in closed form, and is the limit of the finite class equilibrium as the number of class goes to infinity. Detailed discussions and proofs are in the Appendix.

*Networks.* Our model considered a single resource; more generally, we can extend some of our basic results to a network setting. One approach to considering models with more general network structure is as follows. Consider a setting with multiple resources, where each resource runs its own market, and serves according to the DPS policy. Arriving jobs are characterized by a workload vector: completion of service requires simultaneous effort from all resources in this requirement vector, sufficient to complete the corresponding workload at that resource generated by this job. In this case, each resource market operates independently of the others, but they are coupled through the utility functions of the jobs. In particular, we assume job will be sensitive to the *maximum* completion time across all resources it considers. Since the maximum of a collection of convex functions is still convex, if we consider a heavy traffic equilibrium of the network game, the objective function of a single job will remain convex in its own bid vector. Using this insight we can prove existence of HTE in a similar manner to our earlier development. In addition, we can leverage the price of anarchy bounds of Theorem 8 to derive bounds on inefficiency for HTE of the network setting; our approach here is similar to [12], who prove inefficiency bounds for network resource allocation games by reduction to single resource games. Details of the network model are in the appendix.

*Endogenous arrival rates.* Some previous work (e.g., [15], [20]) on priority pricing in queueing systems allows for strategic choice of the arrival rate. In our model, this might mean jobs only
enter if their total cost (cost of waiting plus payment) does not exceed a reservation utility. A significant challenge here is that when arrival rates are endogenized, heavy traffic cannot be exogenously guaranteed. However, we believe approximating the waiting time may still yield valuable insight into this game. Characterizing the quality of approximate equilibria in this regime remains an open direction.

REFERENCES

[1] G. Allon and I. Gurvich. Pricing and dimensioning competing large-scale service providers. *Manufacturing & Service Operations Management*, 12:449–469, 2010.

[2] E. Altman, K. Avrachenkov, and U. Ayesta. A survey on discriminatory processor sharing. *Queueing Systems*, 53(1-2):53–63, 2006.

[3] U. Ayesta, A. Izagirre, and I.M. Verloop. Heavy traffic analysis of the discriminatory random-order-of-service discipline. *Performance Evaluation Review*, 39(2):41–43, September 2011.

[4] P. Billingsley. *Weak Convergence of Measures: Applications in Probability*. Society for Industrial Mathematics, Philadelphia, PA, 1987.

[5] Y. Chen, C. Maglaras, and G. Vulcano. Design of an aggregated marketplace under congestion effects: Asymptotic analysis and equilibrium characterization. *Working Paper*, 2010.

[6] D.R. Cox and W.L. Smith. *Queues*. Methuen and Wiley, London and New York, 1961.

[7] G. Fayolle, I. Mitran, and R. Iasnogorodski. Sharing a processor among many job classes. *Journal of the ACM*, 27(3):519–532, 1980.

[8] A. Glazer and R. Hassin. °/m/1: On the equilibrium distribution of customer arrivals. *European Journal of Operational Research*, 13:146–150, 1983.

[9] R. Hassin and M. Haviv. *To queue or not to queue: Equilibrium behavior in queueing systems*. Kluwer Academic Publishers, 2003.

[10] M. Haviv and J. van der Wal. Equilibrium strategies for processor sharing and queues with relative priorities. *Probability in the Engineering and Informational Sciences*, 11(4):403–412, 1997.

[11] R. Jain, S. Juneja, and N. Shimkin. The concert queueing game: to wait or to be late. *Discrete Event Dynamic Systems*, 21:103–134, 2011.

[12] R. Johari and J. N. Tsitsiklis. Efficiency loss in a network resource allocation game. *Mathematics of Operations Research*, 29(3):407–435, 2004.

[13] W. Kang, F. Kelly, N. Lee, and R. Williams. State space collapse and diffusion approximation for a network operation under a fair bandwidth sharing policy. *The Annals of Applied Probability*, 19(5):1719–1780, 2009.

[14] F.P. Kelly, A. Maulloo, and D. Tan. Rate control in communication networks: shadow prices, proportional fairness and stability. *Journal of the Operational Research Society*, 49(3):237–252, 1998.

[15] Y. J. Kim and M. V. Mannino. Optimal incentive-compatible pricing for m/g/1 queues. *Operations Research Letters*, 31:459–461, 2003.

[16] J.F.C. Kingman. On queues in heavy traffic. *Journal of the Royal Statistical Society. Series B (Methodological)*, 24(2):383–392, 1962.
A. Proof of Theorem 2

We follow the same approach as in [10] (see also Theorem 4.8 in [9]) by decomposing the problem into two parts. Fix a job with priority $\beta > 0$, and fix $\beta = (\beta_1, \cdots, \beta_K) > 0$ where $\beta_i$ is the priority level of class $i$ jobs. Let $U(\beta; \beta, n_1, \cdots, n_K)$ be the expected time in the system if this job has priority $\beta$, and is currently being processed with $n_i$ class $i$ jobs in the system. Then conditional on the first transition, we have the following recursion:

$$U(\beta; \beta, n_1, \cdots, n_K) = \frac{1}{\lambda + \mu} + \sum_{i=1}^{K} \frac{\lambda_i}{\lambda + \mu} U(\beta; \beta, n_1, \cdots, n_i + 1, \cdots, n_K)$$

$$+ \sum_{i=1}^{K} \frac{\mu}{\lambda + \mu} \frac{n_i \beta_i}{\sum_{k=1}^{K} n_k \beta_k + \beta} U(\beta; \beta, n_1, \cdots, n_i - 1, \cdots, n_K).$$

Following the same argument as in [9], we can show that $U(\beta, n_1, \cdots, n_K)$ is linear in each $n_i$. That is, there are functions $U_i(\beta; \beta)$ ($i = 0, \cdots, K$) such that:

$$U(\beta; \beta, n_1, \cdots, n_K) = U_0(\beta; \beta) + \sum_{i=1}^{K} U_i(\beta; \beta)n_i.$$
Substituting (17) into (16) and comparing the coefficients of each \( n_i \) as well as the constant term, we obtain:

\[
\beta_k \left[ 1 + \sum_{i=1}^{K} \lambda_i U_i(\beta; \beta) \right] = \mu (\beta + \beta_k) U_k(\beta; \beta), \quad k = 1, \cdots, K;
\]

\[
1 + \sum_{i=1}^{K} \lambda_i U_i(\beta; \beta) = \mu U_0(\beta; \beta).
\]

Hence the solution to (16) is

\[
U_i(\beta; \beta) = \frac{\beta_i}{\beta_i + \beta} U_0(\beta; \beta), \quad i = 1, \cdots, K; \quad U_0(\beta; \beta) = \left[ \mu - \sum_{i=1}^{K} \frac{\lambda_i \beta_i}{\beta_i + \beta} \right]^{-1}.
\]

The above equation, along with (17) gives the expression for \( V(\beta; \beta) \) in terms of the expected queue lengths.

**B. Proof of Theorem 3**

It is straightforward to show that \( V(\beta; \beta) \leq 1/(\mu (1 - \rho)) \): the expected processing time of any job, regardless of priority, cannot be longer than the expected length of a busy period in an M/M/1 queue with arrival rate \( \lambda \) and service rate \( \mu \) [18]. (This follows because the discriminatory processor sharing policy is work-conserving, so the length of a busy period will be identical to that in an M/M/1 queue.) It then follows that there exists an upper bound \( \overline{\beta} \) such that no job ever has \( \beta > \overline{\beta} \) as an optimal strategy. Therefore, we can restrict the strategy space of every job to the compact set \([\overline{\beta}, \beta]\).

Next we show that \( V(\beta; \beta) \) is convex in \( \beta \). Combining (4) and (5) gives \( V(\beta; \beta) = f(\beta) / g(\beta) \), where \( f(\beta) = 1 + \sum_{i=1}^{K} \frac{\lambda_i W_i}{\beta_i + \beta} \) and \( g(\beta) = \mu - \sum_{i=1}^{K} \frac{\lambda_i \beta_i}{\beta_i + \beta} \). It is easy to check that \( f(\beta) > 0, f'(\beta) < 0, f''(\beta) > 0; \) and \( g(\beta) > 0, g'(\beta) > 0, g''(\beta) < 0 \), thus

\[
\left[ \frac{f(\beta)}{g(\beta)} \right]' = \frac{f''(\beta) g(\beta) - f(\beta) g''(\beta)}{[g(\beta)]^2} = \frac{2 [f'(\beta) g(\beta) - f(\beta) g'(\beta)] g(\beta) g'(\beta)}{[g(\beta)]^4} > 0.
\]

Moreover, \( \beta^\alpha \) is convex in \( \beta \) since \( \alpha \geq 1 \), so \( c_i V(\beta; \beta) + \beta^\alpha \) is convex in \( \beta \). It follows by Rosen’s existence theorem [26] that a pure Nash equilibrium exists for the game in this case.

**C. Proof of Theorem 5**

We note that the best response of a class \( i \) job given that all class \( j \) jobs choose \( \beta_j \ (j = 1, 2, \ldots, K) \) is

\[
\beta_i(\beta) = \arg \min_{\beta \geq 0} \left( c_i V^{HT}(\beta, \beta) + \beta^\alpha \right) = \arg \min_{\beta \geq 0} \left( \frac{c_i}{\mu (1 - \rho) \beta^\alpha(\beta)} + \beta^\alpha \right).
\]
The first order condition of optimality yields a unique solution:

$$\beta^*_i(\beta) = \left(c_i^{-1} \alpha \mu (1 - \rho) \gamma(\beta)\right)^{-\frac{1}{\alpha+1}}.$$  \hspace{1cm} (18)

And the second derivative of the objective function at this point is

$$\frac{2c_i}{\mu(1 - \rho)\gamma(\beta)} \frac{1}{(\beta^*_i)^\beta} + \alpha(\alpha - 1)(\beta^*_i)^{\alpha-2} = \frac{(\alpha + 1)c_i}{\mu(1 - \rho)\gamma(\beta)} \frac{1}{(\beta^*_i)^\beta} > 0.$$  

Therefore, (18) is the unique minimizer of the objective function. Recall that $\gamma(\beta) = \sum_{i=1}^{K} \rho_i / (\rho \beta_i)$.

Thus, at the equilibrium,

$$\gamma(\beta^*) = \sum_{i=1}^{K} \frac{\rho_i}{\rho_i} \left(c_i^{-1} \alpha \mu (1 - \rho) \gamma(\beta^*)\right)^{\frac{1}{\alpha+1}} \Rightarrow \gamma(\beta^*) = \frac{S_1}{\lambda} \left(\alpha \mu (1 - \rho)\right)^{\frac{1}{\alpha}},$$  \hspace{1cm} (19)

where $S_1 = \sum_{i=1}^{K} \lambda_i c_i^{-\frac{1}{\alpha+1}}$. Plugging (19) into (18) yields the result:

$$\beta_i^* = c_i^{\frac{1}{\alpha+1}} \left[\lambda^{-1} \alpha \mu (1 - \rho) S_1\right]^{-\frac{1}{\alpha}} = c_i^{\frac{1}{\alpha+1}} \left[\rho^{-1} \alpha (1 - \rho) S_1\right]^{-\frac{1}{\alpha}}.$$

Therefore, the heavy-traffic equilibrium always exists, is unique, and can be calculated by the above closed form expressions.

**D. Proof of Theorem 6**

First, we observe that for any $\beta > 0$ and $\beta = (\beta_1, \cdots, \beta_K) > 0$, $V(\beta; \beta)$ and $V^{HT}(\beta; \beta)$ depend on $\beta$ and $\beta$ only through the ratios $\beta_i/\beta$ and $\beta_i/\beta_j$ for any $i, j$.

Now, since $\beta^{(n)}$ is the HTE at the $n$-th system, for any $\delta \geq 0$, we have that

$$c_i V^{HT}(\beta_i^{(n)}; \beta^{(n)}) + (\beta_i^{(n)})^\alpha \leq c_i V^{HT}(\delta \beta_i^{(n)}; \beta^{(n)}) + \left(\delta \beta_i^{(n)}\right)^\alpha.$$  \hspace{1cm} (20)

Define $\theta_j = \frac{\beta_j^{(n)}}{\beta_i^{(n)}}$ ($j = 1, \cdots, K$). Then (11) implies that $\theta_j = \frac{\beta_j^{(n)}}{\beta_i^{(n)}}$ is independent of $n$ and the load $\rho^{(n)}$. Therefore, $V^{HT}_n(\beta_i^{(n)}; \beta^{(n)}) = V^{HT}_n(1; \theta)$ and $V_n(\beta^{(n)}; \beta^{(n)}) = V_n(1; \theta)$.

And thus,

$$\lim_{n \to \infty} \left(1 - \rho^{(n)}\right) \left[V^{HT}_n(\beta_i^{(n)}; \beta^{(n)}) - V_n(\beta_i^{(n)}; \beta^{(n)})\right] = 0.$$  \hspace{1cm} (21)

The last equality follows from the asymptotic exactness of $V^{HT}$. Similarly, $\beta_j^{(n)} / (\delta \beta_i^{(n)})$ is also independent of $n$ and system load, so we have that

$$\lim_{n \to \infty} \left(1 - \rho^{(n)}\right) \left[V^{HT}_n\left(\delta \beta_i^{(n)}; \beta^{(n)}\right) - V_n\left(\delta \beta_i^{(n)}; \beta^{(n)}\right)\right] = 0.$$  \hspace{1cm} (22)

Then the claim follows by plugging (21) and (22) into (20).
E. Proof of Theorem 8

Take the first derivative of $C$ with respect to $\alpha$, we have that

$$\frac{\partial C}{\partial \alpha} = \frac{\rho}{(1-\rho)(\alpha + 1)^2} \sum_{i<j} \left[ \lambda_i \lambda_j (c_i^{\alpha+1} c_j^{1+\frac{\alpha}{\alpha+1}} - c_j^{\alpha+1} c_i^{1+\frac{\alpha}{\alpha+1}}) \ln \frac{c_i}{c_j} \right] \left( \sum_i \lambda_i c_i^{1+\frac{1}{\alpha+1}} \right)^{-2}.$$  

Since $c_i^{\alpha+1} c_j^{1+\frac{\alpha}{\alpha+1}} - c_j^{\alpha+1} c_i^{1+\frac{\alpha}{\alpha+1}}$ and $\ln \frac{c_i}{c_j}$ have the same signs, this derivative is positive and $C$ is increasing in $\alpha$.

F. Proof of Theorem 8

To compute the system processing cost of the $c$-$\mu$ rule, consider the following M/M/1 queue models: Class 1 jobs have the highest priority in the system and themselves form an M/M/1 queue with parameter $(\lambda_1, \mu)$, therefore basic M/M/1 queue result [18] implies the expected number of class 1 jobs in the system is $E[N_1] = \rho_1/(1 - \rho_1)$. Then Little’s law implies the expected processing time for class 1 job is $E[N_1]/\lambda_1$. If we further consider both class 1 and class 2 jobs, they are not preempted by any other jobs in the system, therefore they form yet another M/M/1 queue with parameter $(\lambda_1 + \lambda_2, \mu)$, and thus the expected number of class 2 jobs in the system is $E[N_2] = (\rho_1 + \rho_2)/(1 - \rho_1 - \rho_2) - E[N_1]$. In general, the expected number of class $i$ jobs in the system is $E[N_i] = \frac{\sum_{j=1}^{i} \rho_j}{1 - \sum_{j=1}^{i} \rho_j}$. Finally, the system processing cost of the system with $c$-$\mu$ rule is

$$C_{opt} = \sum_{i=1}^{K} c_i \lambda_i (E[N_i]/\lambda_i) = \sum_{i=1}^{K} c_i \left( \frac{\sum_{j=1}^{i} \rho_j}{1 - \sum_{j=1}^{i} \rho_j} - \frac{\sum_{j=1}^{i-1} \rho_j}{1 - \sum_{j=1}^{i-1} \rho_j} \right). \quad (23)$$

To bound the PoA, we first note that $c_i$ is decreasing in $i$ and the expected number of all jobs in the system is $\rho/(1 - \rho)$, therefore $C_{opt} > c_K \sum_{i=1}^{K} E[N_i] = c_K \rho/(1 - \rho)$. On the other hand, let $c_i' = c_i/c_K$ and $\lambda_i' = \lambda_i/\lambda_K$ for $i = 1, \ldots, K - 1$, then

$$\frac{C}{C_{opt}} < \frac{C(1-\rho)}{c_K \rho} = \frac{\sum_{i=1}^{K-1} \lambda_i' c_i^{\alpha+1} + 1}{\sum_{i=1}^{K-1} \lambda_i' c_i^{1+\frac{\alpha}{\alpha+1}}} < \left( \sum_{i=1}^{K-1} \lambda_i' \right) c^{\alpha+1}_{1+\frac{\alpha}{\alpha+1}} + 1. \quad (24)$$

Therefore

$$\frac{C}{C_{opt}} < \frac{\lambda - \lambda_K}{\lambda_K} \left( \frac{c_1}{c_K} \right)^{\frac{\alpha}{\alpha+1}} + 1.$$

We note that there exist some systems in which the PoA of HTE can be made arbitrarily large. Here is an example. Let $\lambda_i$’s all equal, set $c_i = m c_K$ for $i = 1, \ldots, K - 1$ and $\rho = 1 - m^{-2}$. 


Then \( \sum_{j=1}^{i} \rho_j < 1 - \frac{1}{K} \) for \( i < K \) and \( \rho/(1 - \rho) = m^2 - 1 \). We have that
\[
C_{\text{opt}} < c_K(m^2 + mK), \quad C = \frac{c_K \rho}{(1 - \rho)} \frac{(K - 1)m^{-\alpha+1} + 1}{(K - 1)m^{-\alpha+1} + 1}.
\]
Hence,
\[
\frac{C}{C_{\text{opt}}} > \frac{(K - 1)m^{-\alpha+1} + 1}{(K - 1)m^{-\alpha+1} + 1} \frac{m^2 - 1}{m^2 + mK}.
\]
Letting \( m \) go to infinity makes the PoA arbitrarily large. Also, note that in this case the PoA bound is \( (K - 1)m^{-\alpha+1} + 1 \), which means that the PoA bound is “asymptotically tight”.

**G. Proof of Theorem 9**

Take the first derivative of \( R \) with respect to \( \alpha \), we have that
\[
\frac{\partial R}{\partial \alpha} = \frac{\rho}{(1 - \rho)\alpha^2} \left( \sum \lambda_i c_i^{-\alpha+1} \right)^{-2} \times \left[ \frac{\alpha}{(\alpha + 1)^2} \sum_{i<j} \lambda_i \lambda_j (c_i^{-\alpha+1} c_j^{-\alpha+1} - c_j^{-\alpha+1} c_i^{-\alpha+1}) \ln \frac{c_i}{c_j} - \sum_{i,j} \lambda_i \lambda_j c_i^{-\alpha+1} c_j^{-\alpha+1} \right]. \tag{25}
\]
If \( c_1/c_K < e^4 \), then \( \max_{i,j} \ln(c_i/c_j) \leq \frac{(\alpha+1)^2}{\alpha} \leq 4 \). It follows from (25) that \( \partial R/\partial \alpha \) is negative and \( R \) is decreasing in \( \alpha \).

**H. Class Level Game and Approximation**

As what we did in Section III-A, we can define \( W_i^{HT}(\beta) = 1/\mu(1 - \rho)\beta_i \gamma(\beta) \). It then follows from Theorem 4 and Little’s law that
\[
\lim_{\rho \to 1} (1 - \rho)W_i(\beta) = \lim_{\rho \to 1} \frac{E[N_i]}{\lambda_i} = \frac{1}{\lambda_i \beta_i \gamma(\beta)}.
\]
Therefore \( W^{HT} \) is asymptotically exact: as \( \rho \to 1 \),
\[
(1 - \rho)[W_i^{HT}(\beta) - W_i(\beta)] \to 0.
\]
Similarly, we can define \( \beta = (\beta_1, \ldots, \beta_K) \) as a class level heavy traffic equilibrium if \( \forall i = 1, \ldots, K, \)
\[
\beta_i = \arg \min_{\beta > 0} c_i W_i^{HT}(\beta_1, \ldots, \beta_{i-1}, \beta, \beta_{i+1}, \ldots, \beta_K) + \beta^\alpha
\]
\[
= \arg \min_{\beta > 0} \mu(1 - \rho) \frac{1}{\beta} \left[ \sum_{j \neq i} \frac{\rho_j}{\beta_j} + \frac{\rho_i}{\beta} \right]^{-1} + \beta^\alpha.
\]
We can obtain a system of non-linear equations to compute this class level heavy traffic equilibrium, but we are not able to get any closed form expressions for it, mainly due to the existence of intra-class externality: \( \gamma(\beta) \) depends on all priorities of all classes, including the class who is optimizing. This is fundamentally different from the job level game, where for any individual job, its own priority is infinitesimal and does not affect the auxiliary variable \( \gamma(\beta) \).

To eliminate the externality within each class, we consider a limiting model where the number of classes approaches infinity. Then each class as a whole is infinitesimal in the system, and hence, the system stays almost the same even if we exclude a whole class. Next we formalize this idea. We consider a series of systems indexed by \( K \) satisfying the following properties.

**Definition 5:** A **limiting class level game** consists of the following elements.

1) There are \( K \) classes in system \( K \), each with cost \( c^K_k \) arrival rate \( \lambda^K_k \), \( k = 1, \ldots, K \), and service requirement rate \( \mu^K \).
2) The costs \( c^K_k \) are i.i.d. samples from a distribution \( F(\cdot) \) with a positive and closed support \( S_c \).
3) The arrival rates \( \lambda^K_k \) are i.i.d samples from a distribution \( G(\cdot) \) with a positive and closed support \( S_\lambda \) (independent of the costs). Let \( E_G[\lambda] = \int_{S_\lambda} \lambda dG \), the limiting mean arrival rate.
4) The service rate \( \mu^K \) in system \( K \) is chosen to ensure that the system is stable, i.e., \( \rho^K \triangleq \sum_{k=1}^{K} \lambda^K_k / \mu < 1 \).
5) \( \lim_{K \to \infty} \mu^K / K \) exists; denoting this limit by \( \mu \), we assume that \( E_G[\lambda] < \mu \).
6) Feasible priority levels are bounded by a positive and closed support \( S_\beta = [\underline{\beta}, \overline{\beta}] \).

As \( K \to \infty \), the limit of these systems consists of a continuum of classes with independent cost and arrival rate distributions as \( F \) and \( G \), and with limiting per class service rate \( \mu \).

Suppose we are given a limiting class level game. Let \( B : S_c \to \mathbb{R}^+ \) denote a **priority strategy function**, that maps the unit cost of a class (and its jobs) to the priority level chosen by this class (and its jobs). Let \( V(\beta; B, F, G, \mu) \) be the expected steady state processing time of a job with priority \( \beta \), processed with a continuum of classes of jobs characterized by priority strategy function \( B \), cost distribution \( F \), arrival rate distribution \( G \), and per class service rate \( \mu \). Then the symmetric Nash equilibrium for this continuum game is a strategy function \( B(\cdot) \) such that:

\[
B(c) = \arg \min_{\beta \geq 0} c V(\beta; B, F, G) + \beta^\alpha, \quad \forall \ c \in S_c.
\]

Given the complexity of \( V \) in general, this cannot be solved in closed form.
1) Approximate Processing Time: Let us define

\[
\gamma(B) = \int_{s_c} \int_{s_c} \lambda/(\mu \rho B(c)) dF dG = \int_{s_c} B(c)^{-1} dF dG,
\]

where the last equality comes from the independence between \( \lambda \) and \( c \), and define

\[
V^{HT}(\beta; B) = \frac{1}{\mu}(1 - \rho) \frac{1}{\gamma(B) \beta}.
\]

Then we have similar approximation result in heavy traffic for the limiting class level game, which shows that \( V^{HT} \) is asymptotically exact.

**Proposition 10:** Suppose we are given a limiting class level game, and any positive strategy function \( B : S_c \to S_\beta \). Then for all \( \beta > 0 \):

\[
\lim_{\rho \to 1} (1 - \rho)V^{HT}(\beta, B) = \lim_{K \to \infty} \lim_{\rho^K \to 1} (1 - \rho^K)W_1(\beta, B(c_2), \cdots, B(c_K)).
\]

**Proof:**

By definition (26) and (27) we have

\[
\lim_{\rho \to 1} (1 - \rho)\lambda V^{HT}(\beta; B) = \left[ \beta \int_{s_c} B(c)^{-1} dF \right]^{-1}.
\]

On the other hand, it follows from Theorem 4 and Little’s law that

\[
\lim_{\rho^K \to 1} (1 - \rho^K)\lambda^K W_1(\beta, B(c_2), \cdots, B(c_K)) = \frac{1}{\gamma^K(\beta) \beta},
\]

where \( \gamma^K(\beta) = \frac{\rho_1}{\rho \beta} + \sum_{i=2}^{K} \frac{\rho_i}{\rho B(c_i)} \). The strong law of large numbers implies

\[
\lim_{K \to \infty} \gamma^K(\beta) = \lim_{K \to \infty} \left( \frac{\lambda_1}{\lambda^K \beta} + \sum_{i=2}^{K} \frac{\lambda_i}{\lambda^K B(c_i)} \right) = \int_{s_c} B(c)^{-1} dF.
\]

Therefore,

\[
\lim_{K \to \infty} \lim_{\rho^K \to 1} (1 - \rho^K)\lambda^K W_1(\beta, B(c_2), \cdots, B(c_K)) = \left[ \beta \int_{s_c} B(c)^{-1} dF \right]^{-1}.
\]

Equations (28) and (30) together complete the proof.
2) **Class Level Equilibrium:** Inspired by heavy traffic equilibrium for the job level game, we can also define the class level heavy traffic equilibrium for the limiting class level game.

**Definition 6:** Suppose we are given a limiting class level game. A **class level heavy traffic equilibrium** is characterized by a positive priority strategy function \( B \)

\[
B(c) = \arg \min_{\beta \geq 0} \left( c V^{HT}(\beta, B) + \beta^\alpha \right), \quad \forall c \in S_c,
\]

where \( V^{HT}(\beta, B) \) is defined as in (27), with \( \rho = E_G[\lambda]/\mu \).

Comparing this definition to that in the finite case, we note that the summation over all classes is replaced by integration over the support space, and individual strategies are replaced by a strategy function, since now we have a continuum of classes.

This class level heavy traffic equilibrium can also be solved in closed form.

**Proposition 11:** The class level heavy traffic equilibrium for a limiting class level game always exists and is unique. Moreover, it can be calculated in closed form as follows:

\[
B(c) = \frac{c^{\alpha+1}}{\mu \alpha (1 - \rho) S_2} - \frac{1}{\alpha}, \quad (32)
\]

where \( S_2 = \int_{S_c} c^{\frac{1}{\alpha+1}} dF \) is independent of \( \rho \).

The proof is just a similar repetition of the proof for Theorem 5 and is omitted.

Moreover, we can relate the HTE of the class level game to the HTE of the job level game in a series of finite systems: we show that a class level game with infinitely many classes behaves like a job level game, validating that the in-class externality has been mitigated. If \( K \) is fixed, system \( K \) is a finite class system. The optimal strategies in heavy traffic equilibrium are given by (11). Now send \( K \to \infty \), we find that the series of heavy traffic equilibrium strategies converges almost surely and the limit coincides with the corresponding strategy given by the class level heavy traffic equilibrium function \( B^* \), which can be easily verified by applying the strong law of large numbers in (11).

**Proposition 12:** Suppose we are given a limiting class level game. Let \( \beta^K_i \) denote the strategy of a class \( i \) job used in heavy traffic equilibrium in the \( K \)th system, and \( B(\cdot) \) be the equilibrium strategy function used in the limiting class level game, then

\[
\lim_{K \to \infty} \left( \beta^K_i - B(e^K_i) \right) = 0.
\]
I. Network Model

In this section, we provide a description of a network generalization of our model. Consider a setting with \( J \) resources, and \( K \) classes. Jobs of class \( i \) arrive at rate \( \lambda_i \). Each class requires service from a subset of the resources; in particular, let \( r_i \) denote the subset of resources that are used by a job of class \( i \). Each job generates an exponentially distributed workload with mean \( 1/\mu_j \) at resource \( j \); let \( \rho_{ij} = \lambda_i/\mu_j \) be the traffic intensity of class \( i \) at resource \( j \).

We assume that each resource operates as an independent market. In other words, each job bids independently at each resource, and each resource uses the DPS policy to allocate resources to jobs. Let \( \beta_{ij} \) be the bid of a class \( i \) job at resource \( j \); for simplicity in this section, we assume that the payment of the job to that resource is equal to \( \beta_{ij} \) (i.e., that \( \alpha = 1 \) at each resource.

Finally, in the model we consider, we assume that each job simultaneously requires service from each of the resources it demands. We assume that each resource is sensitive to its maximum waiting time across the resources. This might be a reasonable model if, for example, the resources correspond to resources used to farm out parallelized jobs; in that case, the user would be sensitive to the completion time of the slowest job run on the resources they demand.

We have the following equivalent definition of a Nash equilibrium.

**Definition 7:** A Nash equilibrium of the network game consists of a class priority vector \( \beta = (\beta_{ij}, i = 1, \ldots, K; j \in r_i) \) such that

\[
\beta_i = \arg \min_{\beta > 0} \left[ c_i \max_{j \in r_i} \{V_j(\beta_{ij}; \beta^{(j)})\} + \sum_{j \in r_i} \beta_{ij}\right], \forall i = 1, \ldots, K, \tag{33}
\]

where \( \beta^{(j)} = (\beta_{kj}, k \text{ such that } j \in r_k) \) is the class priority vector of jobs that use resource \( j \).

Here \( V_j(\beta; \beta^{(j)}) \) is the waiting time in a DPS system for a job with priority \( \beta \) at resource \( j \), when the class priority vector at resource \( j \) is \( \beta^{(j)} \), as before. We can analogously define a heavy traffic equilibrium (HTE) of the network game, by replacing \( V_j \) by \( V_j^{HT} \). Note that this notion is formally justified if we consider a heavy traffic limit where \( \rho_{ij} \) converges to a limit \( \bar{\rho}_{ij} \), such that \( \sum_{i,j \in r_i} \bar{\rho}_{ij} = 1 \) for all \( j \). (In particular this corresponds to the limit where every resource approaches to heavy traffic simultaneously.)

If we let \( \bar{V}_j \) denote the waiting time of a particular class \( i \) job at resource \( j \), then observe that \( \max_{j \in r_i} \{V_j\} \) is a convex function of \( (\bar{V}_j, j \in r_i) \). As a result, the objective function of user \( i \) in the definition of heavy traffic equilibrium can be shown to be convex, and so by standard
arguments it is straightforward to show that a HTE exists for this game; for brevity we omit the
details. Unfortunately, due to the complexity of the network setting, it is not possible in general
to establish either uniqueness of the equilibrium or compute the equilibrium in closed form.

However, we can use our earlier results to obtain a bound on the price of anarchy in this model.
We require one additional piece of notation. Let $b_j(\bar{V}_j; \beta^{(j)})$ be the value of $\beta$
that ensures that the heavy traffic waiting time of a job at resource $j$ is $\bar{V}_j$, when the class priority vector of other
jobs at resource $j$ is $\beta^{(j)}$. In other words, let $b_j(\bar{V}_j, \beta^{(j)})$ be the solution $\beta$ to:

$$\bar{V}_j = V_{HT}^j(\beta; \beta^{(j)}).$$

Though $b_j(\cdot)$ can be computed in closed form, the solution is tedious and not particularly
insightful. For our purposes, all we require is that it is convex, decreasing, and differentiable in $\bar{V}_j > 0$.

We can then prove the following theorem.

**Theorem 13:** Suppose $\lambda_i = \lambda$ for all $i$. Let $\beta^*$ be an HTE of the network game, and let $V_{ij}^* = V_j(\beta_{ij}^*, \beta^{(j)*})$. Further, define:

$$b'_{ij} = \frac{\partial b_j}{\partial V_j}(V_{ij}^*, \beta^{(j)*}).$$

Then the price of anarchy, i.e., the ratio of the HTE processing cost to the minimal system
processing cost is bounded above by:

$$(K - 1) \max_j \sqrt{\frac{\max_{i,j \in r_i}(-b'_{ij})}{\min_{i,j \in r_i}(-b'_{ij})}} + 1.$$
This cost function is derived by linearizing around the waiting time vector observed in equilibrium. It has two important properties: first, \( C_i(V_i^*) = \hat{C}_i(V_i^*) \); and second, because of convexity of the original cost function, the first order condition for optimality at the equilibrium can be used to show that for any \( V_i \), there holds \( \hat{C}_i(V_i) \leq C_i(V_i) \). In particular, the optimal system processing cost can only be lower under the new cost function.

Next, observe that in equilibrium, since a user minimizes \( C_i(V_i) + \sum_{j \in r_i} b_j(V_{ij}, \beta^{(j)}) \), the directional derivative of \( C_i(V_i^*) \) in the direction of the vector \((1, \ldots, 1)\) must be equal to \( c_i \), since the job must have equalized its waiting times at the different resources in equilibrium. The first order condition can then be used to conclude that \( \sum_{j \in r_i} -b'_{ij} = c_i \). (Note that \(-b'_{ij} \geq 0\) since \( b_j \) is decreasing in waiting time.) This ensures that \( \hat{C}_i(V_i) > 0 \) for all \( i \) and feasible \( V_i \), and in particular that \( c_i \max_j V_{ij}^* - \sum_j (-b'_{ij})(V_{ij}^*) \geq 0 \).

Finally, we note that \( \hat{C} \) is linear in the waiting times at different resources; thus the first order conditions for optimality for job \( i \) decompose across the resources. Thus if we consider now independent games at each resource \( j \), where player \( i \) with \( j \in r_i \) plays with unit time cost \(-b'_{ij} \), then it follows that a HTE for that game would also be \( \beta^{(j)*} \). Since the equilibrium actions are the same in the network game and in the independent single server games, while the optimal social cost is lower in the latter, the result then follows using the same argument as in \([12]\) by using the price of anarchy bound for single resource games established in Theorem 8.