Ising model with spins $S = 1/2$ and 1 on directed and undirected Erdős-Rényi random graphs

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Abstract: Using Monte Carlo simulations we study the Ising model with spin $S = 1/2$ and 1 on directed and undirected Erdős-Rényi (ER) random graphs, with $z$ neighbors for each spin. In the case with spin $S = 1/2$, the undirected and directed ER graphs present a spontaneous magnetization in the universality class of mean field theory, where in both directed and undirected ER graphs the model presents a spontaneous magnetization at $p = z/N$ ($z = 2, 3, ..., N$), but no spontaneous magnetization at $p = 1/N$ which is the percolation threshold. For both directed and undirected ER graphs with spin $S = 1$ we find a first-order phase transition for $z = 4$ and 9 neighbors.

Keywords: Monte Carlo simulation, spins, networks, Ising, graphs.

Introduction

This paper deals with Ising spins on both directed and undirected Erdős-Rényi (ER) graphs. Sumour and Shabat [1, 2] investigated Ising models with spin $S = 1/2$ on directed BA networks [3] using the usual Glauber dynamics. No spontaneous magnetization was found, in contrast to the case of undirected BA networks [4, 5, 6] where a spontaneous magnetization was found below a critical temperature which increases logarithmically with the system size. For $S = 1/2$ systems on undirected Small-World networks (SW) [7] with scale-free hierarchical-lattice, conventional and algebraic (Berezinskii-Kosterlitz-Thouless) ordering, with finite transition temperatures, have been found. Lima and Stauffer [8] simulated directed square, cubic and hypercubic lattices ranging from two to five dimensions with heat bath dynamics in order to separate the network effects from directedness. They also compared different spin-flip algorithms, including cluster flips, for Ising-BA networks. They found a freezing-in of the magnetization similar to the one in Ref. [1, 2], following an Arrhenius law at least in low dimensions. This lack of a spontaneous magnetization (in the usual sense) is consistent with the fact that if
on a directed lattice a spin $S_j$ influences spin $S_i$, then spin $S_i$ in turn does not influence $S_j$, and there may be no well-defined total energy. Thus, they showed that for the same scale-free networks, different algorithms give different results. Lima et al. [9] studied the Ising model for spin $S = 1, 3/2$ and 2 on directed BA network. The Ising model with spin 1, 3/2 and 2 seemed not to show a spontaneous magnetization and their decay time for flipping of the magnetization follows an Arrhenius law for heat bath algorithms that agrees with the results of the Ising model for spin $S = 1/2$ [1, 2] on directed BA network. Sánchez et al. [10] on directed SW obtained a second-order phase transition for values of rewiring probability $p = 0.1$ and a first-order phase transition for $p = 0.9$ with $p_c \approx 0.65$ for the change of phases. The magnetic properties of Ising models defined on the triangular Apollonian network was investigated for Andrade and Herrmann [11] and no evidence of phase transition was found. In this work, we have studied the Ising model with spins $S = 1/2$ and 1 on directed and undirected ER graphs. Undirected ER graphs with spin $S = 1/2$ present a spontaneous magnetization in the universality class of mean field theory and for $S = 1$, we find evidences of first-order phase transition for $z \geq 2$. Directed ER graphs for spin $S = 1/2$ and $S = 1$ present a spontaneous magnetization for $z \geq 2$. Here $z$ is the number of neighbors for each spin.

**Model and Simulation: Ising model on ER graphs**

We consider the spin $S = 1/2$ and 1 Ising models defined by a set of spin variables $S_i$ located on every site $i$, first of directed ER graphs, with $N$ spins taking the values $\pm 1$ and 0 for $S = 1$, and $\pm 1$ for $S = 1/2$, respectively.

The probability for spin $S_i$ to change its state in this directed network is

$$p_i = 1/[1 + \exp(-2E_i/k_BT)], \quad E_i = -J \sum_k S_i S_k$$

and enters the heat bath algorithm; $k$ runs over all nearest neighbors of $S_i$. In this network, each new site added to the network selects with connectivity $z$ already existing sites as neighbors influencing it; the newly added spin does not influence these neighbors.

To study the spin 1/2 and 1 Ising models we start with all spins up, a number of spins equal to 2,000,000 and 4,000,000, and Monte Carlo step (MCS) time up to 200,000 and 2,000,000, respectively. In our simulations,
Figure 1: $1/\ln(\tau)$ versus temperature for different probabilities $1/N$ (sq.), $2/N$ (x), and $3/N$ (+); directed ER with $S = 1/2$.

one MCS is accomplished after all spins are updated, here, with heat bath Monte Carlo algorithm. Then we vary the temperature and study nine samples. The temperature is measured in units of the critical temperature of the square-lattice Ising model. We determine the time $\tau$ after which the magnetization has flipped its sign for the first time, and then take the median value of our nine samples. So we get different values $\tau$ for different temperatures. To study the critical behavior of this Ising model (with spins $1/2$ and $1$) we define the variable $m = \sum_{i=1}^{N} S_i / N$ as normalized magnetization. The Ising model on directed BA networks has no phase transition and agrees with the modified Arrhenius law for relaxation time, $1/\ln(\tau) \propto T + \ldots$, Lima et al. [9].
Figure 2: Squared normalized magnetization versus temperatures, $S = 1/2$, $p = 4/N$, for different sizes $N$ of the undirected ER graph (top) and directed ER graph (bottom).
Figure 3: Reciprocal logarithm of the relaxation times on directed ER networks versus $T$, with $S = 1$ for different probabilities $p = z/N$ with $z = 2(+)\text{ and } 9(x)$, $N = 4,000,000$.

**Results and Discussion**

**Spin 1/2 Ising model**

We take different probabilities for different number of nodes $N = 2,000,000$ with different temperatures in Fig. 1. There we check the first time after which the magnetization changes sign, take the median from nine samples, and plot the reciprocal of the time for three probabilities $p = z/N$ ($z = 1, 2, \text{ and } 3$) in Fig. 1. The figure shows nicely the difference between probability $1/N$ (= percolation threshold) and larger probabilities. This figure shows that there is a spontaneous magnetization at $p = 2/N$ for the left curve and at $p = 3/N$ for the right curve, but no spontaneous magnetization at $p = 1/N$ which is the percolation threshold. In Fig. 2 we show the dependence of the magnetization $M$ on the temperature, obtained for directed
Figure 4: Magnetisation versus temperature for spin $S = 1$ on directed (top) and undirected (bottom) ER graphs.

and undirected ER graphs with $S = 1/2$, we use only one probability equal $p = 4/N$, because it gives a clear answer compatible with the mean-field
universality class, as expected because of the infinite range of the symmetric interaction. For \textit{undirected} ER graphs, if A is a neighbor of B then, in contrast to the directed case, also B is a neighbor of A. From our simulation we see that the undirected version has a spontaneous magnetization, to which the system relaxes similarly to the standard Ising square lattice. Then we plot the square of normalized magnetization versus temperature in Fig. 2. For $T$ below $T_c$ we have a spontaneous magnetization and above $T_c$ we do not have one as we see in Fig. 2 (part (a)). In equilibrium there is a Curie temperature. The squared magnetization vanishes at this $T_c \approx 3.5J/K_B$ linearly in temperature. This behavior corresponds, not unexpectedly, to a mean field critical exponent. Unexpectedly, this same behavior occurs also for \textit{directed} ER graphs (part (b)) that do not present an infinite range of the symmetric interaction as occurs with \textit{undirected} ER graphs. The squared magnetization vanishes at this $T_c \approx 1.2J/K_B$. These results show that the behaviors of $S = 1/2$ Ising model spin on ER graphs are similar, whether these networks are \textit{directed} or \textit{undirected}.

Figure 5: Energetic Binder Cumulant $B_{\text{min}}$ versus $1/N$ for $z = 9$. 
Figure 6: Energy versus temperature on ERD graphs for \( z = 9 \).

**Spin 1 Ising model**

Fig. 3 is analogous to Fig. 1 except that now \( S = 1 \) instead of \( 1/2 \) for \( N = 4,000,000 \) up sites. In Fig. 4 we show magnetisation versus temperature on directed ER networks (part (a)) and also on undirected ER networks (part (b)) for different probabilities \( p = z/N \) with \( z = 4 \) (left) and 9 (right) for system size \( N = 16,000 \) sites. The shapes of these figures show qualitatively that they present evidence of first-order phase transition and also show that the behaviors of magnetisation versus temperature are identical for the same probabilities regardless of whether the networks are directed or undirected. In order to verify the order of the transition, we apply finite-size scaling (FSS) for \( N = 250, 500, 1,000, 2,000, 4,000, 8,000, \) and 16,000 sites. Initially we search for the minima of the energetic fourth-order cumulant:

\[
B = 1 - \left[ \frac{<e^4>}{3(<e^2>)^2} \right]_{av}
\]  

(2)

It is known that this parameter takes a minimum value \( B_{\text{min}} \) at the effective transition temperature \( T_c(N) \). One can show [12] that for a second-order
transition \( \lim_{N \to \infty} (2/3 - B_{\text{min}}) = 0 \), even at \( T_c \), while at a first-order transition the same limit is different from zero \((\neq 0)\). In Fig. 5 we plot the Binder minimum parameter \( B_{\text{min}} \) versus \( 1/N \) (eq. (2)) for \( z = 9 \), and several system sizes. The Binder parameter goes to a value which is different from 2/3. This is a sufficient condition to characterize a first-order transition. The order of transition can be confirmed by plotting the values of energy versus temperature, see Fig. 6, where we present a jump when system sizes increase. This behavior is evidence for a first-order phase transition for \( z = 9 \), this same behavior occurs also for \( z = 4 \).

**Conclusion**

In conclusion, we have presented the Ising model for spins \( S = 1/2 \) and 1 on directed ER and undirected ER graphs, because our main objective in this paper was to verify the existence or not of phase transitions and also the kind of phase transition.

For spin \( S = 1/2 \) Ising models, both directed or undirected ER graphs have a phase transition temperature below which a spontaneous magnetization exists, where ER graphs have a spontaneous magnetization in the universality class of mean field theory. For spin \( S = 1 \) Ising models, on directed and undirected ER graphs the results are identical, i.e., are independent of the nature of the graphs studied here and have both a good evidence of a first-order phase transition different from spin \( S = 1/2 \). Our results agree with the results of nonequilibrium model on directed and undirected ER graphs studied for Pereira et al. [13] and Lima et al. [14].

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**References**

[1] M.A. Sumour and M.M. Shabat, Int. J. Mod. Phys. C 16, 585 (2005) and [cond-mat/0411055](https://arxiv.org/abs/cond-mat/0411055) at www.arXiv.org.

[2] M.A. Sumour, M.M. Shabat and D. Stauffer, Islamic University Journal (Gaza)14,209(2006), and [cond-mat/0504460](https://arxiv.org/abs/cond-mat/0504460) at www.arXiv.org.
[3] R. Albert and A.L. Barabási, Rev. Mod. Phys. 74, 47 (2002).
[4] A. Aleksiejuk, J.A. Holyst and D. Stauffer, Physica A 310, 269 (2002).
[5] J.O. Indekeu, Physica A 333, 461 (2004).
[6] G. Bianconi, Phys. Lett. A 303, 166 (2002).
[7] M. Hinczewski and A.N. Berker, Phys. Rev. E 73, 066126 (2006).
[8] F.W.S. Lima and D. Stauffer, Physica A 359, 423 (2006).
[9] F.W.S. Lima, E.M.S. Luz and R.N. Costa-Filho, Advances and Applications in Statistical Sciences, 2, 185 (2010).
[10] A.D. Sánchez, J.M. Lopes, and M.A. Rodriguez, Phys. Rev. Lett. 88, 048701 (2002). Khalil, Int. J. Mod. Phys. C 18, 53 (2007); cond-mat/0612189.
[11] R.F.S. Andrade and H.J. Herrmann, Phys. Rev. E 71, 056131 (2005).
[12] V. Priman (Ed.), Finite Size Scaling and Numerical Simulation of statistical Systems, World Scientific, Singapore, 1990.
[13] L.F.C. Pereira and F.G. Brady Moreira, Phys. Rev. E 71, 016123 (2005).
[14] F.W.S. Lima, A.O. Sousa, M.A. Sumour, Physica A, 387, 3503 (2008).