LARGE TIME BEHAVIOR OF SOLUTIONS TO SCHRÖDINGER EQUATION WITH COMPLEX-VALUED POTENTIAL

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ABSTRACT. We study the large-time behavior of the solutions to the Schrödinger equation associated with a non-selfadjoint operator having zero energy eigenvalue and real resonances. Our results extend those of Jensen and Kato in the three dimensional selfadjoint case. We consider a model of Schrödinger operator with a quickly decaying potential in dimension three. We assume that the latter has a finite number of real resonances. We are interested in the expansions of the resolvent in the low energy part and near positive resonances. In particular, we discuss, under different conditions, the three situations of zero energy: zero is an eigenvalue or a resonance, or both.

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1. INTRODUCTION

In this work, we are interested in the large-time behavior of the solution \(u(t) = e^{-itH}u_0\) as \(t \to +\infty\) to the Schrödinger equation

\[
\begin{align*}
\frac{i}{\hbar}\partial_t u(t, x) &= Hu(t, x), \quad x \in \mathbb{R}^3, \quad t > 0 \\
u(0, x) &= u_0(x),
\end{align*}
\]

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where $H = -\Delta + V$, is a perturbation of $-\Delta$ by a complex-valued potential supposed to satisfy the decay condition
\begin{equation}
|V(x)| \leq C_v \langle x \rangle^{-\rho}, \quad \forall x \in \mathbb{R}^3,
\end{equation}
where $\rho > 2$ and $\langle x \rangle = (1 + |x|^2)^{1/2}$.

This equation is a fundamental dynamical equation for the wave-function $u(t, x)$ describing the motion of particles in non-relativistic quantum mechanics. An interesting model in nuclear physics where arising this non-selfadjoint operator is the optical nuclear model [8]. This model describes the dynamic of a compound elastic neutron scattering from a heavy nucleus. In this example, the interaction between the neutron and the nucleus is modeled by a complex-valued potential with negative imaginary part.

It is turned out that the behavior of non-selfadjoint Schrödinger operator may differ from selfadjoint ones (see [33, 3]). Previously, Jensen and Kato [15] have studied the three dimensional selfadjoint operator $H = -\Delta + V$ with real potential $V$ satisfying the decay condition (1.2). They have proved if zero is a regular point of $H$ i.e. neither an eigenvalue nor a resonance in the sense of [15, 25], then
\begin{equation}
\|e^{-itH} - \sum_{j=1}^N e^{-it\lambda_j} \Pi_{\lambda_j}\|_{L^2 \to L^2} = O(t^{-3/2}), \quad t \to +\infty
\end{equation}
for $s, s' > 5/2$ and $\rho > 3$, where $\lambda_j$ are the negative eigenvalues of $H$ with the associated eigenprojections $\Pi_{\lambda_j}$. The above estimate requires precisely the low-energy estimate
\[
\lim_{\epsilon \to 0} \| (H - \lambda \mp i\epsilon)^{-1} \|_{L^2 \to L^2} < \infty \text{ as } \lambda \to 0,
\]
which occurs for $s, s' > 3/2$ only if zero is not an eigenvalue or a resonance. However, the time-decay (1.3) is no longer valid when zero is a resonance and it becomes $O(t^{-1/2})$ for $s, s' > 3/2$. In an extension work [18] of the latter, a similar decay result has been obtained for Schrödinger operator with a magnetic potential under the assumption that zero is a regular point. For studies of the non-selfadjoint Schrödinger operator, we refer for example to [39] for Gevrey estimates of the resolvent and sub-exponential time-decay of solutions, to [38, 40] for time-decay of solutions to dissipative Schrödinger equation and to [10, 11] for dispersive estimates.

Our goal is to extend the study in [15] to non-selfadjoint Schrödinger operator with a complex-valued potential $V$ rapidly decaying. Similar to the selfadjoint case and the dissipative one, the time behavior for solutions to the equation (1.1) also depends on the low energy spectral analysis. We are interested in the spectral analysis of the operator $H$ and the large time behavior of $e^{-itH}$ for some operator model having real resonances. Here, we mean by the latter a real number $\lambda_0 \geq 0$ for which the equation $-\Delta u + Vu - \lambda_0 u = 0$ has a non-trivial solution $\psi \in L^{2-s}(\mathbb{R}^3) \setminus L^2(\mathbb{R}^3)$, for all $s > 1/2$. In particular, for $\lambda_0 > 0$ it can be seen that this solution satisfies the Sommerfeld radiation condition
\begin{equation}
\psi(x) = \frac{e^{\pm i\sqrt\lambda_0 |x|}}{|x|} w\left(\frac{x}{|x|}\right) + o\left(\frac{1}{|x|}\right), \quad \text{as } |x| \to +\infty,
\end{equation}
where \( w \in L^2(\mathbb{S}^2), w \neq 0 \), with \( \mathbb{S}^2 = \{ x \in \mathbb{R}^3 : |x| = 1 \} \).

It is well known that selfadjoint Schrödinger operator can have zero resonance only but no positive real resonances (see [1, 13, 18]). In [15] real resonances are absent and intermediate energy does not contribute to the large time behavior of \( e^{-itH} \). While these numbers are the main difficulty to study the spectral properties of non selfadjoint operators near the positive real axis. Wang in [39] has included the presence of real resonances for a compactly supported perturbation of the Schrödinger operator with a slowly decaying potential satisfying a condition of analyticity. However, usually these real numbers are supposed to be absent (see [11]). See [38] for an example of a real resonance of a dissipative operator in dimension three and [27] in dimension one.

Real resonances are responsible for most remarkable physical phenomena and many problems arising from analysis of spectral properties for non selfadjoint operators. Effectively, the zero resonance is responsible for Efimov effect for N-body quantum systems (see [2, 34, 35, 36] and see also the physical revi [24]). In addition, positive resonances are the only points to which complex eigenvalues may eventually accumulate, consequently the boundary values of the resolvent on the cut along the continuous spectrum, known as limiting absorption principle, does not exist globally (cf. [30, 31, 38]). We can mention also [26, 27, 28] on the connection between the decay-rate of the potential and the presence of accumulation points of the eigenvalues for non-selfadjoint Schrödinger operators.

In order to obtain the large-time behavior of \( e^{-itH} \), we establish the expansions of \( R(z) \) at threshold zero and positive resonances in the sense of bounded operators between two suitable weighted Sobolev spaces (see Section 2). Our main novelties - Theorem 2.2 - on the intermediate energy asymptotic of \( R(z) \). We assume that positive real resonances are finite singularities with specific hypothesis \( (H3) \) on the behavior of the solutions defined by [1, 4] (see [6, 33] for more hypotheses in this way) as well as Theorems 2.2-2.4 on the low energy asymptotics of the resolvent for non-selfadjoint operator. We extend the method of Lidskii [19] to study the giant matrix representation that we find, where there exist many Jordan blocks corresponding to the eigenvectors associated with the eigenvalue zero. We get same singularities (negative powers of \( \sqrt{z} \)) that have appeared in [15] because the presence of the zero energy eigenvalue or/and zero resonance. More recent results can be followed in [11, 39]. Moreover, we deduce a limiting absorption principle for the operator \( H \) and high-energy estimates of the derivatives of its resolvent.

The obtained results are useful for the study of the asymptotic behavior in time of wave functions that would be dependent on the nature of the threshold energy and the characteristic of real resonances. Thus, the asymptotic expansions of the resolvent and the semigroup \( e^{-itH} \) as \( t \to +\infty \) have many interests in the scattering theory (see for example [6, 7]).

This paper is organized as follows. In Section 2, we introduce our hypothesis and we state the main results. In Section 3 we study the asymptotic expansions of the resolvent at zero energy. We prove Theorem 2.2 when zero is an eigenvalue of
arbitrary geometrical multiplicity, then we prove Theorem 2.3 in the case of zero resonance and Theorem 2.4 in the more complicated case when zero is both an eigenvalue and a resonance of \( H \). Section 3 is devoted to the study of outgoing positive real resonances, we establish the proof of Theorem 2.3 by assuming that the set of real resonances is finite and their associated eigenvectors satisfy an appropriate assumption. Moreover, we obtain another result (Theorem 2.5) in more general situation. Finally, in Section 5 we establish a representation formula for the semigroup \( e^{-itH} \) as \( t \to +\infty \) which allows us to prove Theorem 2.6.

Notation. Let \( X \) and \( X' \) be two Banach spaces. We denote \( \mathcal{B}(X, X') \) the set of linear bounded operators from \( X \) to \( X' \). For simplicity, \( \mathcal{B}(X) = \mathcal{B}(X, X) \). For all \( m, m', s, s' \in \mathbb{R} \), we denote by \( \mathbb{H}^{m,s} \) the weighted Sobolev space on \( \mathbb{R}^3 \)

\[
\mathbb{H}^{m,s} = \{ u \in \mathcal{S}'(\mathbb{R}^3) : \| u \|_{m,s} = \| (1 - \Delta)^{m/2} u \|_{L^2} < \infty \},
\]

such that for \( m < 0 \), \( \mathbb{H}^{m,s} \) is defined as the dual of \( \mathbb{H}^{-m,-s} \) with dual product identified with the scalar product of \( L^2 \). The index \( s \) is omitted for standard Sobolev spaces, i.e. \( \mathbb{H}^m \) denotes \( \mathbb{H}^{m,0} \). In particular \( \mathbb{H}^0 = L^2 \) with the associated norm \( \| \cdot \|_0 \). Let \( \mathcal{B}(m, s, m', s') = \mathcal{B}(\mathbb{H}^{m,s}, \mathbb{H}^{m',s'}) \). For linear operator \( T \), we denote by \( \text{Ran} \ T \) the range of \( T \) and by \( \text{rank} \ T \) its rank. We also define the following subsets: \( \mathbb{R}_+ = [0, +\infty[ \), \( \mathbb{C}_+ = \{ z \in \mathbb{C} : \pm \text{Im} \ (z) > 0 \} \) and \( \mathbb{C}_\pm = \{ z \in \mathbb{C} : \pm \text{Im} \ (z) \geq 0 \} \).

2. Assumptions and formulation of the main results

2.1. The operator. We consider the Schrödinger operator \( H = -\Delta + V \) in \( \mathbb{R}^3 \), where \( \Delta \) denotes the Laplacian and \( V \) is a complex-valued potential which will be assumed to satisfy the following decay condition

\[
|V(x)| < C_v \langle x \rangle^{-\rho}, \quad \forall x \in \mathbb{R}^3,
\]

where \( \rho > 2 \) throughout the paper and it will depend on the results to obtain. Under the previous assumptions, \( H \) is a closed non-selfadjoint operator on \( L^2 \) with domain the standard Sobolev space \( \mathbb{H}^2 \). Moreover, the condition (2.1) implies that the operator \( V \) of multiplication by \( V(x) \) is relatively compact with respect to \( -\Delta \). It is then known that the essential spectrum of \( H \) denoted by \( \sigma_e(H) \) coincides with that of the non perturbed operator \( -\Delta \) (cf. [12]). Thus \( \sigma_e(H) \) covers the positive real axis \([0, +\infty[ \). In addition, the operator \( H \) has no eigenvalues along the half real axis \([0, +\infty[ \) (cf. [14]). Hence, the spectrum of \( H \) denoted by \( \sigma(H) \) is the disjoint union of \( \sigma_e(H) \) and a countable set denoted by \( \sigma_d(H) \), with

\[
\sigma_d(H) := \{ z \in \mathbb{C} \setminus [0, +\infty[ : \exists 0 \neq u \in D(H), Hu = zu \}
\]

consisting of discrete eigenvalues with finite algebraic multiplicities. For \( z \in \sigma_d(H) \), the associated Riesz projection of \( H \) is defined by

\[
\Pi_z = -\frac{1}{2i\pi} \int_{|w-z|=\epsilon} (H - wId)^{-1}dw,
\]

for \( \epsilon > 0 \) small enough. These eigenvalues can accumulate only on the half axis \([0, +\infty[ \) at zero or at positive real resonances (see Definition 2.1).

It should be noted that in this work it is sufficient to assume the existence of some constants \( \rho > 2 \) and \( C_v, R > 0 \) such that the assumption (2.1) on the potential
and we need to recall some well-known facts about the resolvent. Denote  
\( R_0(z) = (\Delta - z \text{Id})^{-1} \) for \( z \in \mathbb{C} \setminus \mathbb{R}_+ \) and  
\( R(z) = (H - z \text{Id})^{-1} \) for \( z \in \mathbb{C} \setminus \sigma(H) \). In order to obtain the asymptotic expansion of \( R(z) \), we use the following relation between the resolvents

\[
R(z) = (I + R_0(z)V)^{-1} R_0(z), \forall z \notin \sigma(H),
\]

and we need to recall some well-known facts about \( R_0(z) \).

For \( z \in \mathbb{C} \setminus \mathbb{R}_+ \), \( R_0(z) \) is a convolution operator from \( L^2 \) to itself with kernel

\[
R_0(z)(x) = \frac{e^{\pm \sqrt{z}|x|}}{4\pi|x|}, \quad \text{Im} \sqrt{z} > 0.
\]

Here the branch of \( \sqrt{z} \) is holomorphic in \( \mathbb{C} \setminus \mathbb{R}_+ \) such that \( \lim_{\epsilon \to 0^+} \sqrt{\lambda \pm i\epsilon} = \pm \sqrt{\lambda}, \forall \lambda > 0 \). Moreover, the boundary values of the free resolvent on \( \mathbb{R}_+ \) are defined by the following limits

\[
R_0^\pm(\lambda) := s - \lim_{\epsilon \to 0} R_0(\lambda \pm i\epsilon), \text{ for } \lambda > 0,
\]

which exist in the uniform operator topology of \( \mathcal{B}(0, s, 0, -s') \), for \( s, s' > 1/2 \) (see [14, Theorem 4.1]). In addition, it has the following expansion at low-energy

\[
R_0(z) = \sum_{j=0}^{\infty} (iz^{1/2})^j G_j,
\]

where \( G_j \) is an integral operator with kernel \( G_j(x, y) = |x - y|^{s' - 1}/4\pi j! \), \( j = 0, 1, 2, \cdots \), such that

\[
G_0 \in \mathcal{B}(-1, s, 1, -s'), \quad s, s' > 1/2; \quad s + s' > 2,
\]

\[
G_j \in \mathcal{B}(-1, s, 1, -s'), \quad s, s' > j + 1/2; \quad j \geq 1.
\]

In particular, \( G_0 := s - \lim_{z \to 0, z \in \mathbb{C} \setminus \mathbb{R}_+} R_0(z) \) is formally inverse to \(-\Delta\). See [14, Section 2].

Let \( \mathbb{C} \setminus \mathbb{R}_+ \ni z \mapsto K(z) := VR_0(z) : L^2 \to L^2 \) be an analytic operator valued function. As mentioned before, we see that \( K(z) \) is a compact operator for all \( z \in \mathbb{C} \setminus \mathbb{R}_+ \), and that \( \{I + K(z), z \in \mathbb{C} \setminus \mathbb{R}_+\} \) is a holomorphic family of Fredholm operators ([5, Annexe C.2]). By (2.5), the latter can be continuously extended to a family of operators in \( \mathcal{B}(L^{2-s}) \) for \( 1/2 < s < \rho - 1/2 \) in the two closed half-planes \( \mathbb{C}_\pm \). Therefore, applying analytic Fredholm theory with respect to \( z \), it follows that \( (I + R_0(z)V)^{-1} \) is a meromorphic operator valued function in \( \mathbb{C} \setminus \mathbb{R}_+ \) with values in \( \mathcal{B}(L^{2-s}) \), whose poles are discreet eigenvalues of \( H \) in \( \mathbb{C} \setminus \mathbb{R}_+ \). Moreover, for \( \lambda > 0 \), the limits

\[
\lim_{\epsilon \to 0^+} (I + R_0(\lambda \pm i\epsilon)V)^{-1} = (I + R_0^\pm(\lambda)V)^{-1},
\]

exist in \( (0, -s, 0, -s) \), for every \( 1/2 < s < \rho - 1/2 \), if and only if \( I + R_0^\pm(\lambda)V \) is one to one. In other terms, the above limits do not exist if there exists a non
trivial solution $\psi \in H^{1,-s}, \forall s > 1/2$, of $R_0(\lambda \pm i0)Vg = -g$. And, it can be easily proved that $\psi \in H^{1,-s}, \forall s > 1/2$, is a solution of $R_0(\lambda \pm i0)Vg = -g$ if and only if $(H - \lambda)\psi = 0$ and $\psi$ satisfies the radiation condition (2.3). Similarly, in view of (2.5), we see that
\[
\lim_{z \to 0, z \in \mathbb{C} \setminus \mathbb{R}_+} (I + R_0(z)V)^{-1} = (I + G_0V)^{-1},
\]
even exists in $(0, -s, 0, -s)$, for $1/2 < s < \rho - 1/2$, if and only if $I + G_0V$ is one to one. In the following we shall use the notations $K^+ := R_0^+(\lambda)\psi$ and $K_0 := G_0V$.

This work is concerned with the singularities at zero and at the above positive real numbers that will be called real resonances. More precisely, the definition of a real resonance is given below

**Definition 2.1.** A positive real number $\lambda_0 > 0$ is called an outgoing positive real resonance of $H$ if $-1 \in \sigma(K^+(\lambda_0))$ and it is called an incoming positive real resonance if $-1 \in \sigma(K^-(\lambda_0))$. Moreover, if $-1 \in \sigma_d(K_0)$ then zero is said to be a resonance of $H$ if in addition Ker$(I + K_0) \cap L^{2,-s} \setminus L^2 \neq \emptyset, \forall s > 1/2$. Also, zero is said to be an eigenvalue of $H$ if $Hu = 0$ has a non trivial solution $\psi \in L^2$. Let $\sigma^+(H)$ denotes the set of all outgoing positive real resonances.

Note that zero may be an embedded eigenvalue or/and a resonance of the non-selfadjoint operator $H$ if the decay condition $\rho > 2$ is satisfied. The resonance at zero, if it occurs, is geometrically simple, i.e. $\dim \ker (I + K_0) \setminus \ker L^2, H = 1, \forall s > 1/2$. This can be viewed from the following characterization. Let $s = 1/2 + \epsilon, \rho = 2 + \epsilon_0, 0 < \epsilon < \epsilon_0$ and $\psi \in H^{1,-s}$ such that $(1 + G_0V)\psi = 0$. Then $\psi(x)$ behaves as $|x| \to +\infty$ like
\[
\psi(x) \sim \frac{C}{|x|} + \frac{1}{|x|^{1+\epsilon_0}} \phi(x), \quad C = \frac{-1}{4\pi} \int_{\mathbb{R}^3} V(y)\psi(y)dy,
\]
where $\phi$ is some bounded function on $\{x \in \mathbb{R}^3, |x| > 1\}$. Thus, by iterating the same argument, we can deduce that
\[
\int_{\mathbb{R}^3} V(y)\psi(y)dy = 0 \text{ if and only if } \psi \in L^2.
\]
Furthermore, zero is an eigenvalue of $H$ if and only if $-1$ is an eigenvalue of the compact operator $K_0$ on $L^{2,-s}$ and the associated eigenfunctions belong to the orthogonal space of $1$ defined by \{ $\psi \in L^{2,-s} : \langle \psi, J\nu 1 \rangle = 0$ \} (see (2.8)). If this occurs, then their associated eigenspaces coincide. In particular, they have the same geometrical multiplicity, i.e. $\dim \ker_{L^2}(H) = \dim \ker_{L^2}(I + K_0)$ while the algebraic multiplicity of the eigenvalue zero of $H$ is not defined, only that of the eigenvalue $-1$ of $K_0$ is well defined. Let $\Pi_1 : L^{2,-s} \to L^{2,-s}$ (respectively $\Pi_1^+ : L^{2,-s} \to L^{2,-s}$) be the well defined Riesz projection associated with the eigenvalue $-1$ of the compact operator $K_0$ (respectively $K^+(\lambda)$) (see (2.3)). Then denote $m := \text{rank} \Pi_1$ the algebraic multiplicity of $-1$ as eigenvalue of $K_0$.

Our study concerns three different situations of zero energy. We will use a modified form of the terminology introduced in [25]. If zero is an eigenvalue and not a resonance of $H$, zero is said to be a singularity for $H$ of the **first kind**. If zero is a resonance and not an embedded eigenvalue of $H$, zero is said to be a singularity for $H$ of the **second kind**. Finally, if zero is both an embedded eigenvalue and
Hypothesis (H3): A resonance of $H$, zero is said to be a singularity for $H$ of the third kind. The last case is more complicated than the other cases, we should make some additional assumption on the eigenvectors corresponding to the eigenvalue $-1$ of $K_0$.

For examples on the existence of these situations, we can find in [11, 39] an example of a non-selfadjoint Schrödinger operator having zero eigenvalue and in [37] an example of a zero resonance.

2.2. Hypotheses. Our first two hypotheses are about zero singularity:

Hypothesis (H1): If zero is a singularity of the first kind, then there exists a basis $\{\phi_1, \cdots, \phi_k\} \subset L^2$ of $\text{Ker}(I + K_0)$ such that

\begin{equation}
\det((\phi_j, J\phi_i))_{1 \leq i,j \leq k} \neq 0,
\end{equation}
where $J : w \mapsto \bar{w}$ is the complex conjugation and $k$ is the geometrical multiplicity of the eigenvalue zero of $H$.

Hypothesis (H2): If zero is a singularity of the third kind, then there exists a basis $\{\psi_1, \psi_2, \cdots, \psi_k\}$ of $\text{Ker}_{L^2}((I + K_0) \setminus \text{Ker}_{L^2}(I + K_0))$, and $\{\phi_2, \cdots, \phi_k\}$ is a basis of $\text{Ker}_{L^2}(I + K_0)$ such that

\begin{equation}
\det((\phi_j, J\phi_i))_{2 \leq i,j \leq k} \neq 0.
\end{equation}
Before setting our last hypothesis on positive real resonances we define for $\lambda > 0$ the symmetric bilinear form $B_{\lambda}(\cdot, \cdot)$ on $H^{-1,s} \times H^{-1,s}$ by

\begin{equation}
B_{\lambda}(u, w) = \int_{\mathbb{R}^d \times \mathbb{R}^d} e^{i\sqrt{\lambda}(x-y)} u(x)V(x)w(y)V(y) \, dx \, dy.
\end{equation}

Hypothesis (H3): $H$ has a finite number of outgoing positive real resonances, i.e. $\sigma_+(H) = \{\lambda_1, \cdots, \lambda_N\}$. For each $\lambda_j \in \sigma_+(H)$, there exists $N_j \in \mathbb{N}^*$ and a basis $\{\psi_1, \cdots, \psi_{N_j}\}$ in $L^{2-s}$ of $\text{Ker}(I + K^+(\lambda_0))$, such that

\begin{equation}
\det(B_{\lambda}(\psi_r, \psi_l))_{1 \leq r,l \leq N_j} \neq 0.
\end{equation}

In section 3.1 we find some numerical function $d(z)$ such that $(I + K(z))$ is invertible if and only if $d(z) \neq 0$ and if the condition (2.9) (respectively (2.12)) is satisfied, then $-1$ is an eigenvalue of $K_0$ (respectively, $K^+(\lambda_0)$) with geometrical multiplicity $k$ if and only if $0$ (respectively, $\lambda_0$) is a zero of $d(z)$ with multiplicity $k$. In [39] the condition (2.9) was used in the case when $-1$ is a semi-simple eigenvalue of $K_0$ (i.e. geometrical and algebraic multiplicities are equal) to expand the function $d(z)$ in power of $\sqrt{z}$ near $0$. However, in the present work we use this condition also to compute exactly the leading term of the resolvent expansion near $0$ (see proof of Theorem 2.2).

In addition, with the condition (2.12) and (2.1) for $\rho > 2N + 1$, $N \in \mathbb{N}^*$, we can expand the function $d(z)$ in the following form

\begin{equation}
d(z) = \omega_{N_0}(z - \lambda_0)^{N_0} + \omega_{N_0+1}(z - \lambda_0)^{N_0+1} + \cdots + \mathcal{O}((z - \lambda_0)^{N_0+N})
\end{equation}
for $z$ in a neighborhood in $\mathbb{C}_+$ of $\lambda_0 > 0$, with some $\omega_{N_0} \neq 0$, where $N_0$ is given in (H3). Note that the expansion (2.13) can hold with an additional condition on the analyticity of the potential. See [39] Remark 6.1].
It is important here to mention that we can check the condition (2.12) on a resonance state associated with an outgoing resonance for the Schrödinger operator perturbed by compactly supported potential $V$ constructed by Wang in [35, Remark 5.4]. However, in the general case it is not clear if the condition (2.12) can be satisfied. In section 4, we will study the resolvent expansion near $\lambda_0$ with the more general condition (2.13).

2.3. Main results. As first result, we establish asymptotic expansions for $R(z)$ in three situations of zero singularity. For small $\delta > 0$, we denote

\[\Omega_\delta := \{z \in \mathbb{C} \setminus \mathbb{R}_+ : |z| < \delta\}.\]

**Theorem 2.2.** Assume that zero is a singularity of the first kind of $H$ and that (H1) holds. Then, for $\rho > 2l + 1, l + 1/2 < s < \rho - l - 1/2, l = 4, \cdots$, and $z \in \Omega_\delta, \delta > 0$, we have the following expansion:

\[
R(z) = \frac{R^{(1)}_{-2}}{z} + \frac{R^{(1)}_{-1}}{\sqrt{z}} + \sum_{j=0}^{l-4} z^{j/2} R^{(1)}_j + \tilde{R}^{(1)}_{l-4}(z),
\]

as operators in $(-1, s, 1, -s)$, where $R^{(1)}_{-2} = -P^{(1)}_0 : L^2 \to \text{Ker}(I + K_0) \subset L^2$,

\[
P^{(1)}_0 = \sum_{j=1}^k < \cdot, J Z^{(1)}_j > Z^{(1)}_j, < Z^{(1)}_j, J Z^{(1)}_j > = \delta_{ij}, \forall 1 \leq i, j \leq k,
\]

$k$ being the geometrical multiplicity of the eigenvalue $-1$ of $K_0$ and $R^{(1)}_{-1} : L^2 \to \text{Ker}(I + K_0)$. Moreover, the remainder term $\tilde{R}^{(1)}_{l-4}(z)$ is analytic in $\Omega_\delta$ and for $\lambda > 0$ the limits:

\[
\lim_{\varepsilon \to 0^+} \tilde{R}^{(1)}_{l-4}(\lambda \pm i \varepsilon) := \tilde{R}^{(1)}_{l-4}(\lambda \pm i 0)
\]

exist in $(-1, s, 1, -s)$ and satisfy

\[
\text{d}^r \tilde{R}^{(1)}_{l-4}(\lambda \pm i 0)\big|_{(-1, s, 1, -s)} = o(|\lambda|^{r-2}), \forall \lambda \in [0, \delta[, r = 0, 1, \cdots, l - 4.
\]

If $\rho > 5$ and $5/2 < s < \rho - 5/2$, we can obtain $R(z) = z^{-1} R^{(1)}_{-2} + o(|z|^{-1})$. Moreover, if $\rho > 7$ and $7/2 < s < \rho - 7/2$ we can get $R(z) = z^{-1} R^{(1)}_{-2} + z^{-1/2} R^{(1)}_{-1} + o(|z|^{-1/2}).$

**Theorem 2.3.** Assume that zero is a second kind singularity for $H$. Then, for $\rho > 2l + 1, l + 1/2 < s < \rho - l - 1/2, l = 2, 3, \cdots$ and $z \in \Omega_\delta$, the expansion of $R(z)$ in $(-1, s, 1, -s)$ has the following form:

\[
R(z) = \frac{1}{\sqrt{z}} R^{(2)}_{-1} + \sum_{j=0}^{l-2} z^{j/2} R^{(2)}_j + \tilde{R}^{(2)}_{l-2}(z),
\]

where $R^{(2)}_{-1} : L^{2,s} \to L^{2,-s}, u \mapsto i(u, J \phi)\phi,$ with $\phi$ a resonance state satisfying

\[
\frac{1}{2\sqrt{n}} \int_{\mathbb{R}^3} V(x)\phi(x) \, dx = 1.
\]
Furthermore, the remainder term \( \tilde{R}^{(2)}_{l-2}(z) \) is analytic in \( \Omega_3 \) and for \( \lambda > 0 \) the limits
\[ \tilde{R}^{(2)}_{l-2}(\lambda \pm i0) \] (see (2.10)) exist and satisfy
\[ (2.20) \| \frac{d^r}{d\lambda^r} \tilde{R}^{(2)}_{l-2}(\lambda \pm i0) \|_{[-1,s,1,-s]} = o(|\lambda|^2^{-1-r}), \lambda \notin \{0, \delta, r = 0, 1, \ldots, l - 2. \}
\]

If \( \rho > 3 \) and \( 3/2 < s < \rho - 3/2 \), we can obtain \( R(z) = z^{-1/2} \tilde{R}^{(2)}_{l-2} + o(z^{-1/2}) \).

**Theorem 2.4.** Assume that zero is a third kind singularity for \( H \) and that (H2) holds. Then for \( \rho > 2l + 1 \), \( l + 1/2 < s < \rho - l - 1/2 \), \( l = 4, \ldots \) and \( z \in \Omega_3 \), the expansion of \( R(z) \) in \( (-1,s,1,-s) \) has the following form:
\[ (2.21) \quad R(z) = \frac{R^{(3)}_{-2}}{z} + \frac{R^{(3)}_{1}}{\sqrt{z}} + \sum_{j=0}^{l-4} z^{j/2} R^{(3)}_{j} + \tilde{R}^{(3)}_{l-4}(z), \]

where \( R^{(3)}_{-2} = -\mathcal{P}^{(3)}_0 : L^2 \rightarrow \text{Ker}(I + K_0) \subset L^2 \),
\[ \mathcal{P}^{(3)}_0 = \sum_{j=1}^{k-1} <Jz^{(3)}_j > \langle z^{(3)}_j, Jz^{(3)}_j \rangle = \delta_{ij}, \forall 1 \leq i, j \leq k, \]
and
\[ R^{(3)}_{l-4} = i <J\psi > \psi + S^{(3)}_{-1}, \]

such that \( \psi \) is a resonance state satisfying (2.19) and \( S^{(3)}_{-1} : H^{-1,s} \rightarrow \text{Ker}_{l+2}(I + K_0) \), \( k \) being the geometrical multiplicity of the eigenvalue \(-1\) of \( K_0 \) on \( L^{2-s} \).

In addition, the remainder term \( \tilde{R}^{(3)}_{l-4}(z) \) has the same properties as \( \tilde{R}^{(1)}_{l-4}(z) \) in Theorem 2.2.

If \( \rho > 5 \) and \( 5/2 < s < \rho - 5/2 \), we can obtain \( R(z) = z^{-1} \tilde{R}^{(2)}_{l-2} + o(|z|^{-1}) \).

The following theorem gives an expansion for the resolvent \( R(z) \) near the positive real axis viewed from the upper half-plane under the assumption (H3). Before setting the theorem, let us define a neighbourhood \( \Omega_3^+ \) in \( \mathbb{C}_+ \) of such \( \lambda_0 \in \sigma_+^+(H) \):
\[ \Omega_3^+ := \{ z \in \mathbb{C}_+ : 0 < |z - \lambda_0| < \delta \}\]

**Theorem 2.5.** Assume that (H3) holds. Let \( \lambda_0 \in \sigma_+^+(H) \). Then, for \( \rho > 2l + 1 \), \( l + 1/2 < s < \rho - l - 1/2 \), \( l = 2,3,\ldots \) and \( z \in \Omega_3^+ \), the expansion of \( R(z) \) in \( (-1,s,1,-s) \) has the following form
\[ (2.22) \quad R(z) = \frac{\mathcal{P}(\lambda_0)}{z - \lambda_0} + \sum_{n=0}^{l-2} (z - \lambda_0)^n R_n(\lambda_0) + \tilde{R}_{l-2}(z - \lambda_0), \]

where
\[ \mathcal{P}(\lambda_0) = \sum_{j=1}^{N_0} \langle J\psi_j(\lambda_0) \rangle \psi_j(\lambda_0), \quad \frac{1}{i8\pi \sqrt{\lambda_0}} B_{\lambda_0}(\psi_1(\lambda_0), \psi_2(\lambda_0)) = \delta_{ij}, \]

such that \( \{ \psi_1(\lambda_0), \ldots, \psi_{N_0}(\lambda_0) \} \) is a basis of \( \text{Ker}(I + R^{(3)}_0(\lambda_0)V) \), and \( B_{\lambda_0}(\cdot, \cdot) \) is the bilinear form defined in (2.11). The remainder term \( \tilde{R}_{l-2}(z - \lambda_0) \) is analytic in \( \Omega_3^+ \) and for \( \lambda > 0 \) the limit
\[ \lim_{\epsilon \rightarrow 0^+} \tilde{R}_{l-2}(\lambda - \lambda_0 + i\epsilon) = \tilde{R}_{l-2}(\lambda - \lambda_0 + i0) \]
exists in \((-1, s, 1, -s)\) and for \(|\lambda - \lambda_0| < \delta\), it satisfies

\begin{equation}
(2.23) \left\| \frac{d^r}{dx^r} R_{t-2}(\lambda - \lambda_0 + it) \right\|_{H^{-1},s} = o(|\lambda - \lambda_0|^{(-2-r)}), \quad r = 0, 1, \ldots, t - 2.
\end{equation}

If \(\rho > 3\) and \(3/2 < s < \rho - 3/2\), we can obtain \(R(z) = (z - \lambda_0)^{-1} R_{-1}(\lambda_0 + o(|z - \lambda_0|^{-1}))\).

From the preceding results we deduce the limiting absorption principle for \(H\) (under the condition \(\rho > 1\)) on any open interval \(\sigma^+ := \{\lambda + it, \lambda \in \sigma \subset [0, +\infty]\}\) which does not contain any real resonance and eigenvalue. Moreover, under our hypotheses and the condition \(\rho > 3\) we prove that \(H\) has at most a finite set of discrete eigenvalues located in the closed upper half-plane. However, if zero is an eigenvalue of \(H\) then we need the stronger condition \(\rho > 5\). See Proposition 5.1.

Finally, we obtain the large-time behavior of the strongly continuous Schrödinger semigroup \((e^{-itH})_{t \geq 0}\) in two different situations of zero with eventual presence of real resonances.

Our main result is the following:

**Theorem 2.6.** Assume that \((H3)\) holds.

(a) Assume that zero is a is a singularity for \(H\) of the first kind and that \((H1)\) holds. Then for \(\rho > 11\) and \(11/2 < s < \rho - 11/2\), the semigroup \(e^{-itH}\) has an asymptotic expansion in \((0, s, 0 - s)\) as \(t \to +\infty\) of the form

\begin{equation}
(2.24) \quad e^{-itH} - \sum_{j=1}^{r} e^{-itH} \Pi_{z_j} + \sum_{j=1}^{N} e^{-it\lambda_j} R_{-1}(\lambda_j) =
\end{equation}

\(P^{(1)}_0 - i(\pi)^{-1/2} R^{(1)}_1 t^{-\frac{s}{2}} - (4i\pi)^{-1/2} R^{(1)}_1 t^{-\frac{s}{2}} + o(t^{-3/2}),\)

where \(P^{(1)}_0, R^{(1)}_s\) for \(s = -1, 1\) and \(R_{-1}(\lambda_j)\) are given by theorems 2.2 and 2.5.

If \(\rho > 7\) and \(7/2 < s < \rho - 7/2\), then \((2.24)\) holds with the right hand side replaced by \(P^{(1)}_0 - i(\pi)^{-1/2} R^{(1)}_1 t^{-\frac{s}{2}} + o(t^{-1/2}).\)

(b) Assume that zero is a singularity for \(H\) of the second kind. Then, for \(\rho > 9, 9/2 < s < \rho - 9/2\), the expansion at the right hand side of \((2.24)\) has the following form

\(i(\pi)^{-1/2} \langle \cdot, \phi \rangle t^{-\frac{s}{2}} - (4i\pi)^{-1/2} R^{(2)}_1 t^{-\frac{s}{2}} + o(t^{-3/2}),\)

where \(\phi\) and \(R^{(2)}_1\) are given by Theorem 2.8.

If \(\rho > 4, 5/2 < s < \rho - 5/2\), then \((2.24)\) holds with the right hand side replaced by \(i(\pi)^{-1/2} \langle \cdot, \phi \rangle t^{-\frac{s}{2}} + o(t^{-1/2}).\)

In the above expansions \(\{\Pi_{z_j}\}_{j=1}^P\) denote the Riesz projections associated with the eigenvalues \(\{z_j\}_{j=1}^P\) located in the closed upper half-plane.

Here we note that the same statement in (a) holds if zero is both an eigenvalue and a resonance of \(H\). To obtain the above expansions we find some curve \(\Gamma^{\nu}(\eta)\), for some \(\eta, \nu > 0\) small, which does not intersect the real axis at zero or at points in \(\sigma^+(H)\), such that above this curve, \(H\) has a finite number of eigenvalues. Then the expansions are deduced by representing \(e^{-itH}\) as a sum of some residue terms and a Dunford integral of \(R(z)\) on \(\Gamma^{\nu}(\eta)\).
Remark 2.7. With regard to the case zero is a regular point for $H$, i.e. it is not an eigenvalue nor a resonance of $H$, if $\rho > 3$ then we can obtain the following expansion of $R(z)$ in $(-1,s,1,-s)$ for $s > 5/2$ and $z \in \Omega_5$ with $\delta > 0$ small:

$$
(2.25) \quad R(z) = R_0^{(0)} + z^{1/2} R_1^{(0)} + \tilde{R}_1(z),
$$

where

$$
R_0^{(0)} = (I + G_0 V)^{-1},
$$

$$
R_1^{(0)} = (I + G_0 V)^{-1} G_1 (I - V(I + G_0 V)^{-1} G_0).
$$

Moreover, for $0 < \lambda < \delta$

$$
\frac{d^r}{d\lambda^r} \tilde{R}_1(\lambda \pm i0) \Big|_{(-1,s,1,-s)} = o(|\lambda|^{1/2-r}), \quad r = 0, 1, 2.
$$

Here, $G_0, G_1$ are given at (2.15) and $\Omega_5$ is defined at (2.14). The proof is similar to that of [13, Theorem 6.1].

Furthermore, let $\rho > 3$ and $s > 5/2$. If zero is a regular point for $H$ and (H3) holds, then the expansion at (2.25) yields the following asymptotic in time, as $t \to +\infty$, for $e^{-itH}$ in $(0,s,0,-s)$

$$
e^{-itH} - \sum_{j=1}^N e^{-it\lambda_j} \Pi_2 \sum_{j=1}^N e^{-it\lambda_j} R_{-1}(\lambda_j) = -(4i\pi)^{-1/2} R_1^{(0)} t^{-\frac{1}{2}} + o(t^{-3/2}).
$$

(See (2.24)).

3. Expansion of the resolvent around zero energy

First, let us consider $H_0 = -\Delta$ and the perturbed non-selfadjoint operator $H = -\Delta + V$. In the following, we always assume that $V$ satisfies (2.1), where a high order of $\rho$ is needed for an asymptotic expansion of the resolvent $R(z)$ at a high-order expansion in power of $\sqrt{z}$. In this section we will use some tools developed in [39, Section 5.4].

Riesz projection. Set $E = \text{Ran} \, \Pi_1$, where $\Pi_1$ is the Riesz projection associated with the eigenvalue $-1$ of $K_0$ on $L^{2-s}$ for $1/2 < s < \rho - 1/2$ (see Section 2.1).

Let $J: f \to \bar{f}$ be the operation of complex conjugation, and denote $H^* = JHJ$.

We see that $JV K_0 = K_0^* JV$, and then $JV \Pi_1 = \Pi_1^* JV$. Denote

$$
(3.1) \quad u^* = JV u.
$$

Let us look at the case of geometrically simple eigenvalue. One has dim $\text{Ker}(I + K_0) = 1$ and $\text{rank} \, \Pi_1 = m$. In this case, it was proved in (39, Section 5.4) that $JV: \text{Ran} \, \Pi_1 \to \text{Ran} \, \Pi_1^*$ is a bijection and that the bilinear form $\Theta(\cdot, \cdot)$ defined on $\text{Ran} \, \Pi_1$ by

$$
(3.2) \quad \Theta(u, v) = \int_{\mathbb{R}^2} V(x)u(x)v(x) dx = \langle u, v^* \rangle
$$

is non-degenerate. See [39, Lemma 5.13].

In more general statement, when $-1$ has a geometrical multiplicity $k \geq 1$, we use this fact to obtain the following decomposition lemma:
Lemma 3.1. Assume that $-1$ is an eigenvalue of $K_0$ of geometrical multiplicity $k \geq 1$ and algebraic multiplicity $m$. Then there exists $k$ invariant subspaces of $K_0$ denoted by $E_1, \cdots, E_k$ such that

(1) $E = E_1 \oplus \cdots \oplus E_k$, where $\forall i \neq j: E_i \perp E_j$ with respect to the bilinear form $\Theta$.

(2) $\forall 1 \leq i \leq k$, there exists a basis $U_i := \{u_{i1}^{(i)}, \cdots, u_{im_i}^{(i)}\} \subset L^{2,-s}$ of $E_i$ such that

$$(I + K_0)^m u_m^{(i)} = 0 \text{ and } u_r^{(i)} := (I + K_0)^{m - r} u_{m_i}^{(i)} \neq 0, \forall 1 \leq r \leq m_i.$$

(3) $\forall 1 \leq j \leq k$, there exists a dual basis $V_j := \{w_{1j}^{(j)}, \cdots, w_{mj_j}^{(j)}\} \subset L^{2,-s}$ of $E_j$ such that

$$w_r^{(j)} \in \text{Ker}(I + K_0)^{m_j - 1}, \quad \Theta(u_i^{(i)}, w_r^{(j)}) = \delta_{ir},$$

where $\delta_{ij} = 1$ if $l = r, i = j$ and $\delta_{ij} = 0$ otherwise.

(4) $\dim \text{ker}(1 + K_0)|_{E_j} = 1, \forall j = 1, \cdots, k$.

Moreover, the matrix of $\Pi_1(I + K_0)\Pi_1$ in the basis $U := \bigcup_{i=1}^k U_i$ of $E$ is a block diagonal $m \times m$ matrix of the following form

$$J = \text{diag}[J_{m_1}, J_{m_2}, \cdots, J_{m_k}],$$

where

$$J_{m_j} = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & \\
0 & 0 & \ddots & \ddots & 0 \\
& \vdots & \ddots & \ddots & 1 \\
0 & \cdots & \cdots & \cdots & 0
\end{pmatrix}_{m_j \times m_j}$$

is a Jordan block. We have also denoted $m_j = \dim E_j$ for $j = 1, \cdots, k$, such that $m = m_1 + \cdots + m_k$.

The following statement is an immediate consequence of the previous lemma.

Corollary 3.2. The Riesz projection $\Pi_1$ has the following representation:

$$\Pi_1 = \sum_{j=1}^k \sum_{r=1}^{m_j} \langle \cdot, w_r^{(j)*} \rangle u_r^{(j)}.$$

See [39, Corollary 5.16] for the proof of the corollary. We now prove Lemma 3.1.

Proof. We will proceed by induction on $k$. The case $k = 1$ can be treated in the same way as in [39, section 5.4]. Now, we will show the part $[\Pi]$ when $\dim(I + K_0) = 2$. One has that $1 + K_0$ is nilpotent on $\text{Ran} \Pi_1$, then there exists an integer $1 < m_1 \leq m$ and some non zero function $u_{m_1} \in E$ such that $(1 + K_0)^{m_1} u_{m_1} = 0$ and $(1 + K_0)^{m_1 - 1} u_{m_1} \neq 0$. Let

$$u_j = (1 + K_0)^{m_1 - j} u_{m_1}, \quad j = 1, \cdots, m_1 - 1.$$ 

Then, $u_j \in \text{ker}(1 + K_0_j)$ and $u_1, \cdots, u_{m_1}$ are linearly independent. In particular, $u_1$ belongs to the subspace $\text{ker}(1 + K_0)$ and $u_j \in \text{Ran}(I + K_0)$ for $1 \leq j \leq m_1 - 1$, therefore $\Theta(u_1, u_j) = 0$, $\forall 1 \leq j \leq m_1 - 1$. Since $\Theta(\cdot, \cdot)$ is non degenerate on $E_i$. 

(see [39 Lemma 5.13]), then one has necessarily $\Theta(u_1, u_{m_1}) \neq 0$.

Set $E_1 = \text{span}\{u_1, \cdots, u_{m_1}\}$ and
\[ E^\perp_1 = \{v \in E : \Theta(u, v) = 0 \ \forall u \in E_1\}. \]

We show that $E^\perp_1$ is not empty. Indeed, let $\tilde{u} \in \text{Ker}(I + K_0) \setminus \{u_1\}$ and set $\tilde{v} = \Theta(\tilde{u}, u_{m_1})u_1 - \Theta(u_1, u_{m_1})\tilde{u}$. We can easily check that $\tilde{v}$ belongs to $E^\perp_1$. Moreover, we shall show that $E = E_1 \oplus E^\perp_1$ with respect to $\Theta$. Let $v \in E^\perp_1$. One has $\Theta((1 + K_0)v, u_j) = \langle (1 + K_0)v, JVu_j \rangle = \langle v, JV(1 + K_0)u_j \rangle = \langle v, JVu_{j-1} \rangle = 0$, $\forall j \geq 2$. Also, $\Theta((1 + K_0)v, u_1) = \Theta(v, (1 + K_0)u_1) = 0$, thus $(1 + K_0)v \in E^\perp_1$, and this shows that $(1 + K_0)E^\perp_1 \subseteq E^\perp_1$. Let now $\{w_1, \cdots, w_{m_1}\}$ be the dual basis of $\{u_1, \cdots, u_{m_1}\}$ with respect to $\Theta$ (see [39] Lemma 5.15 for the construction). One has
\[ w_j \in \text{ker}(1 + K_0)^{m_1-j+1}, \quad \Theta(u_i, w_j) = \delta_{ij}, \ \forall i, j = 1, \cdots, m_1, \]
where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. Let $u \in E_1 \cap E^\perp_1$, then there exists a sequence of complex numbers $\alpha_1, \cdots, \alpha_{m_1}$ such that $u = \sum_{j=1}^{m_1} \alpha_j u_j$. But $\forall j \geq 1$, one has $\Theta(u, w_j) = 0 = \alpha_j$ because $u \in E^\perp_1$. Thus $E_1 \cap E^\perp_1 = \{0\}$ and the direct decomposition of $E$ in the assertion 1 is proved for $k = 2$ by taking $E_2 = E^\perp_1$.

Now, let us show the assertions 2 and 3. We have constructed above the basis of $E_1$ as well as its dual basis. Furthermore, we have $(1 + K_0)^{m_2-1} = 0$ and $(1 + K_0)^{m_2} = 0$ for $m_2 = m - m_1$. It follows
\[ E_2 = \text{span}\{v_1, \cdots, v_{m_2}\}, \quad v_j = (1 + K_0)^{m_2-j}v_{m_2}, \ j = 1, 2, \cdots, m_2 - 1, \]
for some non zero function $v_{m_2} \in \text{Ker}(1+K_0)^{m_2}\cap E_2$ such that $(1 + K_0)^{m_2-1}v_{m_2} \neq 0$. In particular, $v_1$ belongs to $\text{Ker}(I + K_0)$. In addition, the dual basis of $\{v_1, \cdots, v_{m_2}\}$ with respect to $\Theta(\cdot, \cdot)$ has the same properties as (3.4) and it can be constructed as in [39] Lemma 5.13.

As consequence, the matrix of $(1 + K_0)$ in the basis
\[ \mathcal{B} = \{u_1, \cdots, u_{m_1}, v_1, \cdots, v_{m_2}\} \]
is the desired block diagonal matrix given by (3.3). This proves 4. Finally, if we assume that 1 - 3 hold for all $l \leq k - 1$ when $\dim \text{ker}(1 + K_0) = l$, then the case $\dim \text{ker}(1 + K_0) = k$ can be proved in the same way as the previous one.

**Remark 3.3.** As mentioned in the proof of the above lemma, one has $\Theta(u_i^{(j)}, u_{m_1}^{(j)}) \neq 0, \forall 1 \leq i, j \leq k$ in view of the non-degenerate bilinear form $\Theta(\cdot, \cdot)$ on each subspace $E_j$, $1 \leq j \leq k$, and the definition of the vectors $u_i^{(j)}$, $1 \leq r \leq m_j$. Note that, in Lemma 5.15 in [39] applied here to construct the dual basis $\{u^{(j)}_1, \cdots, u^{(j)}_{m_1}\}$ of $\{u_1^{(j)}, \cdots, u_{m_1}^{(j)}\}$ of the subspace $E_j$, one has $w_i^{(j)} = c_j u_i^{(j)}$, where $c_j$ is chosen so that $\Theta(u_i^{(j)}, w_i^{(j)}) = 1$.

### 3.1. Zero singularity of the first kind.

In this case we assume that $-1$ is an eigenvalue of the operator $K_0$ on $L^2(-s, 1/2 < s < \rho - 1/2$, with geometrical multiplicity $k \geq 1$. Indeed the case $k = 1$ could be treated using the similar method used in [39] to study the situation of geometrically simple zero eigenvalue for a compactly supported perturbation of the Schrödinger operator $H_0 = -\Delta + V_0(x)$, where $V_0$ is a slowly decaying potential. In the latter case, the matrix of $\Pi_1(I + K_0)\Pi_1$ on $\text{Ran} \ \Pi_1$ is consisting of one Jordan block. Therefore, the usual tools can be used to
compute the singularity of the resolvent at threshold. This work is concerned with
the more interesting case if \( k \geq 2 \). Assume from now that \( k \geq 2 \). The basis \( \mathcal{U} \) and
\( \mathcal{W} \) that we have found in the preceding lemma will be fixed in this section and for
\( \delta > 0 \) we denote
\[
\Omega_\delta = \{ z \in \mathbb{C} \setminus \mathbb{R}_+ : |z| < \delta \}.
\]

Set \( \mathcal{M}(z) = I + K(z) \). Given the decomposition we have just established, we can
identify \( E_j \) with \( \mathbb{C}^{m_j} \) and \( \mathbb{C}^m \) with \( \mathbb{C}^{m_1} \oplus \cdots \oplus \mathbb{C}^{m_k} \) to construct the following
Grushin problem for \( \mathcal{M}(z) \).

Grushin problem for \( \mathcal{M}(z) \). We consider
\[
(3.5) \quad \mathcal{P}(z) := \begin{pmatrix} \mathcal{M}(z) & S \\ T & 0 \end{pmatrix},
\]
\[
S : \bigoplus_{j=1}^k \mathbb{C}^{m_j} \hookrightarrow H^{1,-s} ; \quad \zeta = \bigoplus_{j=1}^k (\zeta_1^{(j)}, \ldots, \zeta_{m_j}^{(j)}) \mapsto S\zeta := \sum_{j=1}^k \sum_{l=1}^{m_j} \zeta_l^{(j)} u_l^{(j)},
\]
\[
T : H^{1,-s} \hookrightarrow \bigoplus_{j=1}^k \mathbb{C}^{m_j} ; \quad v \mapsto Tv := \bigoplus_{j=1}^k ((v, u_1^{(j)*}), \ldots, (v, u_{m_j}^{(j)*})).
\]
The operators \( S \) and \( T \) verify \( TS = I_m \) and \( ST = \Pi_1 \) (see Corollary 3.2), where \( S \)
and \( T \) are chosen so that the problem \( \mathcal{P}(z) \) is invertible. Since \( 1 + K_0 \) is injective
on \( \text{Im } \Pi_1 \), where \( \Pi_1' = I - \Pi_1 \), then by the alternative Fredholm theorem we have
that \( 1 + K_0 \) is invertible on \( \text{Im } \Pi_1' \). Then, by using an argument of perturbation,
for \( \delta > 0 \) small enough, \( \Pi_1 \mathcal{M}(z) \Pi_1' \) is also invertible on \( \text{Im } \Pi_1' \) for all \( z \in \Omega_\delta \), with
inverse
\[
E(z) = (\Pi_1' \mathcal{M}(z) \Pi_1')^{-1} \Pi_1'.
\]
In view of (2.6), for \( \rho > 2l+1, l+1/2 < s < \rho - l - 1/2, l = 1, 2, \ldots \) and \( z \in \Omega_\delta, \delta > 0 \)
small, the expansion of \( E(z) \) in \( (1, -s, 1, -s) \) can be written as follows:
\[
(3.6) \quad E(z) = E_0 + \sum_{j=1}^l z^{j/2} E_j + E_l(z),
\]
where \( E_0 = (\Pi_1' (1 + K_0) \Pi_1')^{-1} \Pi_1', \ E_1 = iE_0 G_1 V \Pi_1' E_0 \) and other terms \( E_j, j = 2, 3, \ldots, l \) can be computed directly. Moreover, the remainder term \( E_l(z) \) is analytic
in \( \Omega_\delta \) and continuous up to \( \mathbb{R}_+ \) satisfying
\[
(3.7) \quad \| \frac{d^r}{dz^r} E_l(z) \|_{L(H^{1,-s})} = o(|z|^{l-r}), \quad \forall z \in \Omega_\delta, \quad r = 0, 1, \ldots, l,
\]
and for \( \lambda > 0 \) the limits
\[
(3.8) \quad \lim_{\epsilon \to 0} E_l(\lambda \pm \epsilon) = E_l(\lambda \pm i0)
\]
exists as operators in \( (-1, s, 1, -s) \) and satisfy similar estimates as (3.7). This
implies that the problem \( \mathcal{P}(z) \) is invertible on \( H^{1,-s} \times \mathbb{C}^m \), and one has
\[
(3.9) \quad \begin{pmatrix} \mathcal{M}(z) & S \\ T & 0 \end{pmatrix}^{-1} = \begin{pmatrix} E(z) & E_+ (z) \\ E_- (z) & E_{-+} (z) \end{pmatrix} : H^{1,-s} \times \mathbb{C}^m \mapsto H^{1,-s} \times \mathbb{C}^m,
\]
where
\[ E_+(z) = S - E(z)M(z)S \]
\[ E_-(z) = T - TM(z)E(z) \]
\[ E_{-+}(z) = -TM(z)S + TM(z)E(z)M(z)S \]
(3.10)

Therefore, \( M(z) \) is invertible if and only if \( E_{-+}(z) \) is invertible, with
\[ M(z)^{-1} = E(z) - E_+(z)E_{-+}(z)^{-1}E_-(z) \quad \text{on} \quad \mathbb{H}^{1-s}. \]
(3.11)

In order to prove the theorem 2.2, we will use the resolvent equation \( R(z) = (I + K(z))^{-1}R_0(z) \). We must establish an asymptotic expansion of \( E_{-+}(z)^{-1} \) first to deduce that of \( (I + K(z))^{-1} \) from the above Grushin problem. To study the matrix \( E_{-+}(z) \), we need to partition it according to the matrix \( J \) defined in (3.3).

More precisely, we write
\[ E_{-+}(z) = [(E_{-+})^j_i(z)]_{1 \leq i,j \leq k} \]
(3.12)

where \( (E_{-+})^j_i(z) \) denotes the \( m_i \times m_j \) block of \( E_{-+}(z) \) located in the same row as \( J_{m_i} \) and in the same column as \( J_{m_j} \).

Let \( \delta > 0 \) and \( z \in \Omega_\delta \). Using the basis \( U_j \) and \( W_i \) which exist by Lemma 3.11 and (3.10) (see also [22], Lemma 5.15), we write
\[ (E_{-+})^j_i(z) = ((E_{-+}(z)u^{(j)}_r, JVw^{(i)}_s))_{1 \leq l \leq m_i, 1 \leq r \leq m_j}. \]
(3.13)

Furthermore, if zero is a singularity of \( H \) of the first kind and there exists a basis \( \{u^{(1)}_1, \cdots , u^{(k)}_1\} \) in \( L^2 \) of \( \text{Ker}(I + K) \), then for \( \rho > 2l + 1, l + 1/2 < s < \rho - l - 1/2, \ l = 3, 4, \cdots \) and \( z \in \Omega_\delta, \delta > 0 \) small, we use (2.6), (3.6) and (3.10) to obtain the following expansion of \( E_{-+}(z) \):
\[ E_{-+}(z) = E_{-+,2}(z) + \sum_{j=3}^l z^{j/2}E_{-+,j} + \tilde{E}_{-+,l}(z), \]
(3.14)

where \( (E_{-+,2})^j_i(z) = N_j^i + z^{1/2}A^j_i + z B^j_i \), such that for all \( 1 \leq l \leq m_i, 1 \leq r \leq m_j \) we have
\[ (N^i_j)_r = -< (I + G_0 V)u^{(j)}_r, JVw^{(i)}_s >, \]
\[ (A^j_i)_r = -i < G_1 Vw^{(j)}_r, JVw^{(i)}_s > +i < E_0 G_1 Vw^{(j)}_r, JV(I + G_0 V)w^{(i)}_s >, \]
\[ (B^j_i)_r = + < G_2 Vw^{(j)}_r, JVw^{(i)}_s > - < E_0 G_2 Vw^{(j)}_r, JV(I + G_0 V)w^{(i)}_s > \]
\[ - < G_1 V E_0 G_1 V w^{(j)}_r, JVw^{(i)}_s > +i < E_1 G_1 Vw^{(j)}_r, JV(I + G_0 V)w^{(i)}_s >. \]

Also \( (E_{-+,n})^j_i, n = 3, \cdots , l \) can be computed explicitly. Moreover, the remainder \( \tilde{E}_{-+,l}(z) \) is analytic in \( \Omega_\delta \) and for \( \lambda > 0 \) the limits:
\[ \lim_{\epsilon \to 0^+} \tilde{E}_{-+,l}(\lambda \pm i \epsilon) = \hat{E}_{-+,l}(\lambda \pm i 0) \]
exist and they satisfy
\[ \frac{d^r}{dx^r} \hat{E}_{-+,l}(\lambda \pm i 0) = o(|\lambda|^{1/2 - r}), \ r = 0, 1, \cdots , l. \]
(3.15)

We can further simplify the previous expression of the matrix \( E_{-+,2}(z) \) as the following: since \( u^{(j)}_1 \in \text{ker}(1 + K_0) \), then one has \( (N^i_j)_1 = 0, \forall 1 \leq l \leq m_i \), while
for \( 2 \leq r \leq m_j \) we see from the definition of vectors \( u^{(j)}_r \) that \((N^j_i)_r = -<u^{(j)}_{r-1}, JVw^{(i)}_i>\neq -\delta^{(j)}_{r-1} \). Moreover, since \( w^{(i)}_m = c_i u^{(i)}_i \) for some \( c_i \neq 0, 1 \leq i \leq k \) (see Remark 3.3), where \( u^{(i)}_1 \in \ker(1 + K_0) \subset L^2 \), then \( G_1 V u^{(j)}_1 = 0 = G_1 V w^{(i)}_m \), \( \forall 1 \leq i, j \leq k \), by (2.8). Thus, it follows that \((A^j_i)_l = 0 = (A^j_i)_m, \forall 1 \leq l \leq m_i, 1 \leq r \leq m_j \).

Summing up, we obtain \( E_{-+,2}(z) = N + z^{1/2} A + z B \), with

\[
N^j_i = \begin{pmatrix} 0 & -\delta_{ij} & 0 \\ 0 & 0 & \ddots \\ \vdots & \vdots & \ddots & -\delta_{ij} \\ 0 & 0 & \cdots & 0 \end{pmatrix}_{m_j \times m_j},
\]

\[
A^j_i = \begin{pmatrix} 0 & \tilde{A}^j_i \\ 0 & \ddots \\ \vdots & \ddots & \ddots \\ 0 & 0 & \cdots & 0 \end{pmatrix}_{m_i \times m_j},
B^j_i = \begin{pmatrix} * & * & \cdots & * \\ * & \ddots & \ddots & \ddots \\ \vdots & \vdots & \ddots & * \\ \beta_{ij} & * & \cdots & * \end{pmatrix}_{m_i \times m_j}, \forall 1 \leq i, j \leq k,
\]

where

\[
\beta_{ij} = (B^j_i)_{m_i,1} = -\lim_{z \in \Omega_\delta, z \to 0} \frac{1}{z} \langle (1 + R_0(z)V) u^{(j)}_1, JV w^{(i)}_m \rangle
=-\lim_{z \in \Omega_\delta, z \to 0} \frac{1}{z} \{ \langle (1 + G_0 V) u^{(j)}_1, JV w^{(i)}_m \rangle + z \langle G_0 V u^{(j)}_1, J R_0(z)V w^{(i)}_m \rangle \}
= -c_i \langle u^{(j)}_1, J u^{(i)}_1 \rangle, \quad \forall 1 \leq i, j \leq k.
\]

Moreover, it follows from (2.10) that for \( l + 1/2 < s < \rho - l - 1/2 \), the remainder term \( \hat{E}_{-+,l}(z) \) satisfies

\[
\| \frac{d^r}{dz^r} \hat{E}_{-+,l}(z) \| = o(|z|^{\frac{l}{2} - r}), \quad \forall z \in \Omega_\delta, \quad r = 0, 1, \cdots, l.
\]

In particular, for \( \lambda > 0 \) we have from (2.25) that the limits in (3.19) exist, and taking \( \epsilon \to 0^+ \) in (3.20) shows that the limits satisfy also the estimates.

Unfortunately, we have found a matrix of high dimension \( E_{-+}(z) \) in (3.13), where the usual methods of algebra are no longer practical to calculate its determinant and to explicitly develop its inverse matrix. To address this we propose a method based on that of Lidskii developed in his original paper [19], and used latter in [22] for the problem of eigenvalues of matrices with arbitrary Jordan structure. This method will be used to prove the following proposition:

**Lemma 3.4.** Assume that zero is a singularity of the first kind of \( H \) and that \( (H1) \) holds. Then, for \( \rho > 5 \) and \( 5/2 < s < \rho - 5/2 \), we have the following expansion:

\[
det E_{-+}(z) = \sigma z^k + \mathcal{O}(|z|^{k+\epsilon}), \quad \forall z \in \Omega_\delta,
\]

for some \( 0 < \epsilon < 1/2 \), where \( \sigma = \sigma' \times \det(\langle u^{(j)}_1, J u^{(i)}_1 \rangle)_{1 \leq i, j \leq k} \), for some \( \sigma' \neq 0 \) and \( k = \dim \ker(I + K_0) \).
Remark 3.5. Set

\[ \phi_k = (\beta_{ij})_{1 \leq i, j \leq k}, \quad L_k = ((u_1^{(j)}, J u_1^{(j)}))_{1 \leq i, j \leq k}, \]

where $\beta_{ij}$ are the coefficients at (3.18). Then, it is seen from (3.19) and Remark 3.3 that

\[ \phi_k = -C_k L_k, \quad C_k = \text{diag}(c_1, \ldots, c_k), \quad c_j = \frac{1}{\Theta(u_1^{(j)}, u_1^{(j)})}, \quad \forall 1 \leq j \leq k, \]

Thus $\det \Phi_k = (-1)^k \det C_k \times \det L_k$, where $\det C_k \neq 0$.

Proof. First, we perform the change of variables $\eta = \sqrt{z}$ for $z \in \Omega$, $\delta > 0$ small. Thus, the expansion of the matrix $E_{-\epsilon}(z)$ in (3.14) can be written as follows:

\[ E_{-\epsilon}(\eta) = E_{-\epsilon,2}(\eta) + O(|\eta|^{2(1+\epsilon)}), \]

for some $0 < \epsilon < 1/2$. Let $Z(\eta) = \det E_{-\epsilon,2}(\eta)$. Then, we reduce the computation to that of $Z(\eta)$ close to $\eta = 0$. To do it we introduce the following diagonal matrix $L(\eta)$ partitioned conformally with $E_{-\epsilon}(\eta)$:

\[ L(\eta) = \text{diag}[L_1(\eta), \ldots, L_k(\eta)], \quad L_i(\eta) = \text{diag}(1, \ldots, 1, \eta^{-2}), \quad 1 \leq i \leq k, \]

where $\eta \in \{ z \in \mathbb{C} : |z| < \delta \}$. We now define

\[ \tilde{E}_{-\epsilon,2}(\eta) = L(\eta)E_{-\epsilon,2}(\eta), \quad \tilde{Z}(\eta) = \det \tilde{E}_{-\epsilon,2}(\eta). \]

Then, by regularity of the matrix $L(\eta)$ for $\eta \neq 0$, we see that $\tilde{Z}(\eta) = 0$ if and only if $Z(\eta) = 0$. Also, we can show that $\tilde{Z}(\eta)$ is polynomial in $\eta$. Indeed, we write

\[ \tilde{E}_{-\epsilon,2}(\eta) = L(\eta)(N + \eta A + \eta^2 B) := \tilde{N}(\eta) + \tilde{A}(\eta) + \tilde{B}(\eta), \]

where, by (3.17), (3.18) and (3.24) we see that

\[ \begin{align*}
&\text{(i) } \tilde{N}(\eta) = L(\eta)N = N, \\
&\text{(ii) } \tilde{A}(\eta) = \eta L(\eta)A \quad \text{with} \quad \tilde{A}_i^j(\eta) = \begin{pmatrix} 0 & \cdots & \eta A_i^j & \cdots \\
& \vdots & \ddots & \vdots & \ddots \\
& 0 & \cdots & 0 & \cdots & 0 \\
\end{pmatrix}_{m_i \times m_j}, \\
&\text{(iii) } \tilde{B}(\eta) = \eta^2 L(\eta)B \quad \text{with} \quad \tilde{B}_i^j(\eta) = \begin{pmatrix} \eta^2 & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & 1 \\
\end{pmatrix}_{m_i \times m_j} \quad \text{and} \quad \begin{pmatrix} * & * & \cdots & * \\
& \vdots & \ddots & \vdots \\
& 0 & \cdots & 0 \\
\end{pmatrix}_{m_i \times m_j}.
\end{align*} \]

This shows that there is no negative powers of $\eta$ in $\tilde{E}_{-\epsilon,2}(\eta)$, thus $\tilde{Z}(\eta)$ is polynomial in $\eta$. We will then examine $\tilde{Z}(0)$. It follows from (i) (resp. (ii)) that $\tilde{N}(0) = N$ (resp. $\tilde{A}(0) = 0$). Moreover,

\[ \tilde{B}_i^j(0) = \begin{pmatrix} 0 & 0 & \cdots & 0 \\
& \vdots & \ddots & \vdots \\
& 0 & \cdots & 0 \\
\beta_{ij} & * & \cdots & * \\
\end{pmatrix}_{m_i \times m_j}, \quad \forall 1 \leq i, j \leq k. \]
Thus,

\[(3.27) \quad (\tilde{E}_{-+}^*)^j_0 = \begin{pmatrix}
0 & -\delta_{ij} & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & 0 & 0 \\
\beta_{ij} & \cdots & \cdots & 0 & 0 \\
\end{pmatrix}_{m_i \times m_j}, \quad \forall 1 \leq i, j \leq k.
\]

We can now calculate \(\tilde{Z}(0)\) which is the determinant of the above matrix \(\tilde{E}_{-+}^*(0)\). By expanding the determinant along the rows of \(\tilde{E}_{-+}^*(0)\) that are containing only \(-1\), we obtain the following:

\[(3.28) \quad \tilde{Z}(0) = \det (\beta_{ij})_{1 \leq i, j \leq k},\]

(see proof of Theorem 2.1 in [22] for a specific example with \(12 \times 12\) matrix that illustrates the strategy). Hence, there exist \(\varepsilon, \eta > 0\) small such that for \(\eta \in \{z \in \mathbb{C}_+ : |z| < \delta\}\)

\[(3.29) \quad \tilde{Z}(\eta) = \tilde{Z}(0) + O(|\eta|^{2\epsilon}) = \det \Phi_k + O(|\eta|^{2\epsilon})\]

where \(\Phi_k = (\beta_{ij})_{1 \leq i, j \leq k}\) is defined by (3.21). Then, it follows from (3.25) and (3.29) with \(\det L(\eta) = \eta^{-2k}\):

\[(3.30) \quad Z(\eta) = (\det L(\eta))^{-1} \tilde{Z}(\eta) = \eta^{2k} \det \Phi_k + O(|\eta|^{2(k+\epsilon)}).
\]

Finally (3.28) with the previous equation implies that for \(\eta \in \{\eta \in \mathbb{C}_+, |\eta| < \delta\}\)

\[\det E_{-+}^*(\eta) = \eta^{2k} \det \Phi_k + O(|\eta|^{2(k+\epsilon)}),\]

where \(\det(\Phi_k) = \sigma' \times \det(<u_{1,1}^*, Ju_{1,1}^*>)_{1 \leq i, j \leq k}\) with \(\sigma' \neq 0\) by Remark 3.5. \(\square\)

Now, we are able to prove Theorem 2.2.

Proof of Theorem 2.2. Firstly, it follows from Lemma 3.4 that \(E_{-+}(z)^{-1}\) exists under the hypothesis (H1). Then, we will show that for \(\rho > 2l + 1, l + 1/2 < s < \rho - l - 1/2, l = 4, 5, \ldots\) and \(z \in \Omega_\delta\), \(\delta > 0\) small, the expansion of \(E_{-+}(z)^{-1}\) has the following form:

\[(3.31) \quad E_{-+}(z)^{-1} = \frac{F_{-2}^{(1)}}{z} + \frac{F_{-1}^{(1)}}{\sqrt{z}} + \sum_{j=0}^{l-4} z^{j/2} F_j^{(1)} + E_{-+,-l-4}^{(-1)}(z),\]

where \(F_{-2}^{(1)}\) is a matrix of rank \(k\), whose blocks are of the form

\[(3.32) \quad (F_{-2}^{(1)})^{i,j} = \frac{1}{\det \Phi_k} \begin{pmatrix}
0 & \cdots & 0 & \gamma_{ij} \\
0 & \cdots & 0 & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & 0 \\
\end{pmatrix}_{m_i \times m_j}, \quad \forall 1 \leq i, j \leq k,
\]

for some \(\gamma_{ij}\) that will be determined during this proof and the matrix \(F_{-1}^{(1)} = -F_{-2}^{(1)} E_{-+} F_{-2}^{(1)}\) has rank at most \(k\). Moreover, the remainder term is analytic in \(\Omega_\delta\) continuous up to \(\mathbb{R}_\delta\) and for \(\lambda > 0\) the limits:

\[(3.33) \quad \lim_{\epsilon \to 0^+} E_{-+,-l-4}^{(-1)}(\lambda \pm i\epsilon) = E_{-+,-l-4}^{(-1)}(\lambda \pm i0)\]
In order to prove (3.31) we make the same change of variables \( \eta = \sqrt{z} \) and we consider the same notations that we have just used in the previous proof. We see that for \( \eta \in \{ \eta \in \mathbb{C}_+, |\eta| < \delta \}, \delta > 0 \) small, \( \tilde{E}_{-+2}(\eta) \) can be developed in powers of \( \eta \) as follows:

\[
(3.35) \quad \tilde{E}_{-+2}(\eta) = \tilde{E}_{-+2}(0) + \eta A + \eta^2 B_1 + o(\eta^2),
\]

where \( B_1 = B - \bar{B}(0) \) and the matrices \( A, B \) and \( \bar{B}(0) \) are given in (3.18) and (3.20).

In addition, \( \tilde{E}_{-+2}(0)^{-1} \) exists by (3.23) under the condition (2.9) with

\[
(3.36) \quad \tilde{E}_{-+2}(0)^{-1} = \frac{\text{Com} \tilde{E}_{-+2}}{\det \Phi_k} = F_{-2}^{(1)} + \mathcal{E},
\]

where \( F_{-2}^{(1)} \) is the above matrix and \( \mathcal{E} \) is a matrix whose blocks are of the following form:

\[
(3.37) \quad \mathcal{E}_i^j = \frac{1}{\det \Phi_k} \begin{pmatrix}
\alpha_{(ij)} & \cdots & \cdots & \cdots & \alpha_{(ij)} \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & \cdots \\
\vdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & \cdots 
\end{pmatrix}_{m_i \times m_j} \quad \forall 1 \leq i, j \leq k,
\]

where \( \tilde{\alpha}_{(ij)} = 0 \) if \( i \neq j \) and \( \tilde{\alpha}_{(ij)} = \alpha_{ri} \in \mathbb{C} \) if \( i = j \) (see (3.39)). Here, we have applied the same process used in the previous proof to calculate the minors of order \( m - 1 \) of the matrix \( \tilde{E}_{-+2}(0) \). More precisely, let \( |M_k^r| \) denote the minor of a matrix \( M \), that is the determinant of the resulting matrix when the row \( i \) and column \( j \) of \( M \) are deleted. Then

\[
(3.38) \quad \gamma_{ij} = (-1)^{\mu_{ij}} |[\tilde{E}_{-+2}(0)]_{m_1+\cdots+m_r+m_i+\cdots+m_j-\mu_{ij}}^{l_1+\cdots+l_r} |, \quad \mu_{ij} = 1 + \sum_{r=1}^{i-1} m_l + \sum_{r=1}^{j} m_r
\]

\[
(3.39) \quad \alpha_{ri} = - |[\tilde{E}_{-+2}(0)]_{r_1+\cdots+m_i+\cdots+m_j-\mu_{ij}}^{l_1+\cdots+l_r} |
\]

for \( 1 \leq i, j \leq k \), with \( m_0 = 0 \). Thus, from (3.35) and by Neumann series, for \( \delta \) small enough \( \tilde{E}_{-+}(\eta)^{-1} \) exists. Hence, it follows from (3.36) and the regularity of \( L(\eta) \) for \( \eta \in \mathbb{C}_+, \eta \) near zero, that \( E_{-+2}(\eta) \) is invertible. Moreover, a little computation gives

\[
E_{-+2}(\eta)^{-1} = \tilde{E}_{-+2}(\eta)^{-1} L(\eta)
\]

\[
(3.40) \quad \frac{F_{-2}^{(1)}}{\eta^2} = \frac{F_{-2}^{(1)}}{\eta^2} + \mathcal{O}(|\eta|), \quad \forall \eta \in \Omega_\delta \cap \mathbb{C}_+,
\]

where \( F_{-2}^{(1)} = \mathcal{E} - \mathcal{E} B_1 F_{-2}^{(1)} \).

Let \( E_{-+}^l(\eta) = E_{-+}(\eta) - E_{-+2}(\eta) \). For \( \eta \in \{ \eta \in \mathbb{C}_+, |\eta| < \delta \} \), we see that \( \| E_{-+2}(\eta)^{-1} E_{-+}^l(\eta) \| = \mathcal{O}(|\eta|) \). Consequently, by Neumann series we deduce from
and that $E_+^{-1}(\eta)$ exists for $\eta \in \{\eta \in \mathbb{C}, |\eta| < \delta\}$ if $\delta > 0$ is small enough, with

$$
E_+^{-1}(\eta) = \frac{F^{(1)}_{-2}}{\eta^2} - \frac{F^{(1)}_{-2} E_{-+}^{-1} F^{(1)}_{-2}}{\eta} - F^{(1)}_0 - F^{(1)}_{-2} E_{-+}^{-1} F^{(1)}_{-2} + \sum_{j=1}^{l-4} \eta^2 F^{(j)}_j + E_{-+}^{-1} F^{(1)}_{-4}(\eta),
$$

where the estimates (3.34) follow from (3.20) and see the proof of (3.15) for (3.33). It is remaining to prove that $F^{(1)}_{-2}$ is of rank $k$. To do it, we shall show

$$
\Gamma_k := (\gamma_{ij})_{1 \leq i,j \leq k} = (\det \Phi_k)^{-1}.
$$

Indeed, by (3.38) we can check that

$$
\gamma_{ij} = \begin{cases} 
(-1)^{\mu_{ij}} (-1)^{m_i-1+\cdots+m_j-1} |[\Phi_k]_{ij}| & \text{if } i < j, \\
(-1)^{\mu_{ij}} (-1)^{m_i-1} |[\Phi_k]_{ij}| & \text{if } i = j, \\
(-1)^{\mu_{ij}} |[\Phi_k]_{ij}| & \text{if } i = j + 1, \\
(-1)^{\mu_{ij}} (-1)^{m_j+1-1+\cdots+m_i-1} |[\Phi_k]_{ij}| & \text{if } i > j + 1,
\end{cases}
$$

where $|[\Phi_k]_{ij}|$ is a minor of the invertible matrix $\Phi_k$ defined in (3.21). And then, by substituting $\mu_{ij}$, we obtain

$$
\gamma_{ij} = (-1)^{i+j} |[\Phi_k]_{ij}|, \quad 1 \leq i,j \leq k,
$$

which is the $(j,i)$-th cofactor of the matrix $\Phi_k$.

Secondly, if $\rho > 2l + 1$, then by (3.11), (3.39) and the identity $R(z) = (I - R_0(z) V)^{-1} R_0(z)$ the expansion of the resolvent in $(-1, s, 1, -s)$ for $l + 1/2 < s < \rho - l - 1/2$ has the following form:

$$
R(z) = \frac{R^{(1)}_{-2}}{z} + \frac{R^{(1)}_0}{\sqrt{2}} + \sum_{j=0}^{l-4} z^j R^{(1)}_j + R^{(1)}_{l-4}(z),
$$

where $R^{(1)}_{-2} = -SF^{(1)}_{-2} TG_0$, $R^{(1)}_0 = E_0 G_0 + SF^{(1)}_{-2} TG_2 - SF_0 TG_0$, $S$ and $T$ are defined in (3.5) and for $f \in \mathbb{H}^{-1,s}$

$$
R^{(1)}_{-2} f = -SF^{(1)}_{-2} TG_0 f
$$

$$
= -\frac{1}{\det \Phi_k} \sum_{i=1}^{k} \sum_{j=1}^{k} \gamma_{ij}(f, JG_0 V w_{m_j}^{(2)}(i)) u_1^{(i)}
$$

$$
= \frac{1}{\det \Phi_k} \sum_{i=1}^{k} \sum_{j=1}^{k} c_j \gamma_{ij}(f, J u_1^{(j)}(i)) u_1^{(i)}
$$

$$
= \sum_{i=1}^{k} \langle f, J v_i \rangle u_1^{(i)}.
$$
Let
\[
V = \begin{pmatrix} v_1 \\
\vdots \\
v_k \end{pmatrix}, \quad U = \begin{pmatrix} u_1^{(1)} \\
\vdots \\
u_1^{(k)} \end{pmatrix}, \quad \Gamma_k = (\gamma_{ij})_{1 \leq i,j \leq k}.
\]

Then, we have
\[
(3.46) \quad V = \frac{1}{\det \Phi_k} \Gamma_k C_k U.
\]

Thus, using (3.42) we obtain
\[
(3.47) \quad V = \Phi_k^{-1} C_k U = - (C_k L_k)^{-1} C_k U = - L_k^{-1} U = - Q^T Q U,
\]
where $C_k = \text{diag}(c_1, \ldots, c_k)$, $L_k = (\langle u_1^{(i)}, Ju_1^{(j)} \rangle)_{1 \leq i,j \leq k}$ is an invertible complex symmetric matrix with (3.22) and $Q = (q_{ij})_{1 \leq i,j \leq k}$ is the upper triangular matrix obtained by the Cholesky decomposition of the matrix $L_k^{-1}$ (cf. [25, Proposition 25]). Thus by returning to (3.45), we get
\[
R_{-2}^{(1)} f = - \sum_{i,j=1}^{k} q_{ij} q_{ij} \langle f, Ju_1^{(i)} \rangle u_1^{(i)} = - \mathcal{P}_0^{(1)} f
\]
where
\[
\mathcal{P}_0^{(1)} = \sum_{\ell=1}^{k} \langle \cdot, J Z^{(1)}_\ell \rangle Z^{(1)}_\ell \quad \text{with} \quad Z^{(1)}_\ell = \sum_{i=1}^{k} q_{ii} u_1^{(i)},
\]
and we see that $\mathcal{P}_0^{(1)}$ is a projection of rank $k$ since for all $1 \leq i, j \leq k$ we have
\[
\langle Z^{(1)}_i, J Z^{(1)}_j \rangle = \sum_{\ell,m} q_{i\ell} q_{jm} \langle u_1^{(\ell)}, Ju_1^{(m)} \rangle = \sum_{i=1}^{k} q_{i\ell} (QL_k)_{j\ell} = (QL_k Q^T)_{ji} = \delta_{ij}.
\]

Moreover, other terms $R_{-j}^{(1)}$, $j = 1, \cdots, l - 4$, can be obtained directly. Finally, the estimate (2.11) can be checked from (3.7), (3.34) and the differentiability of the series $(I + R_0(z)V)^{-1}$, also (2.10) follows from (2.5), (5.5) and (5.33). \hfill \Box

3.2. Zero singularity of the second kind. In this section zero will be only a resonance and not an eigenvalue of $H$.

The same construction made in Lemma [3.1] for a single subspace $E_i$ can be done for $E = \text{Ran} H_1$ at the present case. Thus we can find $\mathcal{U} := \{u_1, \cdots, u_m\} \subset L^2_{-s}$ a basis of $E$ and $\mathcal{V} = \{w_1, \cdots, w_m\}$ its dual basis, such that $u_j = (1 + K_0)^{m-j} u_m \in \text{Ker}(1 + K_0)^{l}$, $j = 1, \cdots, m$. In particular, $u_1 \in \text{Ker}_{L^{2-s}}(1 + K_0)$ is a resonance state.

To calculate the singularity of $R(z)$ due to zero resonance, we will consider the same Grushin Problem as in the first case but with $\dim \text{Ker}_{L^{2-s}}(I + K_0) = k = 1$ at the present case.

First, for $l \in \mathbb{N}^*, \rho > 2l + 1$, $l + 1/2 < 2s < 4 - l - 1/2$ and $z \in \Omega_{\delta}, \delta > 0$, similar calculations as for (3.11) give
\[
E_{-+}(z) = N + \sqrt{z} A + \sum_{j=2}^{l} z^{j/2} E_{-+}^{j} + \tilde{E}_{-+}(z),
\]
such that $N = -(δ_{ij-1})_{1\leq i,j \leq m}$ and the first column and last row of the matrix $A = (a_{ij})_{1\leq i,j \leq m}$ are not necessarily zero. In particular

$$a_{m1} = -i(G_1V_{u_1}, JV_{w_m}) = -ic_m(G_1V_{u_1}, JV_{u_1}) = -\frac{i}{4\pi}c_m(u_1, JV1)^2,$$

where we have used $w_m = c_m u_1$ with $c_m = \langle u_1, JV u_m \rangle^{-1}$ (see [39] Lemma 5.15).

See however (3.18) for $E_{-+2}$ and (3.20) for the remainder $E_{-+1}(z)$.

Let $E_{-+1}(z) = N + \sqrt{z}A$. Then for $3/2 < s < \rho - 3/2$, $\rho > 3$, and $z \in \Omega_\delta = \{z \in \mathbb{C}\setminus\mathbb{R}_+, |z| < \delta\}$, $\delta > 0$, by using an argument of perturbation the determinant the determinant of $E_{-+}(z)$ can be written as follows:

$$\det E_{-+}(z) = \det E_{-+1}(z) + o(\sqrt{z}) = \sqrt{z}a_{m1} + o(\sqrt{z})$$

where $a_{m1}$ is the non zero term in (3.48).

**Proof of Theorem 2.3.** It follows from the previous paragraph that $E_{-+1}(z)$ is invertible for $z \in \Omega_\delta, \delta > 0$ small, and we can easily check that

$$E_{-+1}(z)^{-1} = \frac{\text{Com} E_{-+1}(z)}{\det E_{-+1}(z)} = \left(\begin{array}{ccc} 0 & \ldots & 0 & 1 \\ 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \ldots & 0 & 0 \end{array}\right).$$

Then, for $\rho > 2l + 1$, $l + 1/2 < s < \rho - l - 1/2$ and $z \in \Omega_\delta, \delta > 0$ small, by using the Neumann series, it follows that $E_{-+}(z)^{-1}$ exists, with

$$E_{-+}(z)^{-1} = (I + E_{-+1}(z)^{-1}\sum_{j=2}^{l} z^{j/2}E_{-+,j})^{-1} - E_{-+1}(z)^{-1} = \frac{1}{\sqrt{z}} \tilde{A} + \sum_{j=0}^{l-2} z^{j/2} F_j^{(2)} + F_{-+1,l-2}^{(-1)}(z),$$

where $\tilde{A}$ is the above matrix, $F_j^{(2)}, j = 1, \ldots, l - 2$, can be computed explicitly. Moreover, the remainder term $F_{-+1,l-2}^{(-1)}(z)$ is analytic in $\Omega_\delta$ and for $\lambda > 0$ the limits $E_{-+1,l-2}^{(-1)}(\lambda \pm i0)$ exist and satisfy

$$\|\frac{d^r}{d\lambda^r} E_{-+1,l-2}^{(-1)}(\lambda \pm i0)\| = o(|\lambda|^{\frac{1}{2}-1-r}), \quad r = 0, 1, \ldots, l - 2.$$
where
\[
R^{(2)}_{-1} = -T \tilde{A} G_0 = - \frac{1}{a_{m1}} \langle \cdot, JG_0Vw_m \rangle u_1
\]
\[
= \frac{c_m}{a_{m1}} \langle \cdot, Jv_1 \rangle u_1
\]
\[
= i \frac{4\pi}{\langle u_1, JV1 \rangle^2} \langle \cdot, Jv_1 \rangle u_1.
\]
Let
\[
\phi = \frac{2\sqrt{\pi}}{\langle u_1, JV1 \rangle} u_1.
\]
Then \(\phi\) is a resonance state of \(H\) satisfying (2.19) and \(R^{(2)}_{-1} = i \langle \cdot, \phi \rangle \phi\). Moreover, the estimate (2.20) can be obtained from (3.50) and (3.7).

\(\square\)

3.3. Zero singularity of the third kind. We propose in this section to study the case when zero is both an eigenvalue and a resonance of \(H\). In this case, we assume that the hypothesis (H2) holds. Then \(\dim \ker F_{L,z} - (1 + K_0) = k\) and \(\ker F_{L,z} - (1 + K_0)/\ker F_{L,z}(1 + K_0) = 1\). Let \(m = \text{rank} \pi_1\). We will use the decomposition of \(E = \text{Ran} \Pi_1\) in Lemma 3.1 except that in the present case there is only one Jordan block (let us take the first one) corresponding to the resonant state. At the same time, the other blocks should correspond to the eigenvectors that are the solutions in \(L^2\) of \((I + K_0)g = 0\).

We begin by building the same Grushin problem was studied in Section 3.1 for the family of Fredholm operators \(M(z) = I + K(z)\) in \(H^{1,-s}\). Again, let \(U_i = \{u_1^{(i)}, \ldots, u_m^{(i)}\}\) be a basis of \(E_i\) and \(W_i = \{u_1^{(i)}, \ldots, w_m^{(i)}\}\) be its dual with respect to the bilinear form \(\Theta(\cdot, \cdot)\). In particular, \(u_1^{(1)} \in \ker (I + K_0) \cap (L^2(\mathbb{R}^n) \setminus L^2)\) and \(\{u_1^{(2)}, \ldots, u_k^{(1)}\} \subset \ker (I + K_0) \cap L^2.\) Since \(-1\) is an eigenvalue of the operator \(K_0\) of geometrical multiplicity \(k \geq 1\), then the same computations made to develop \(E_{-1}(z)^{-1}\) can be done here. Indeed, (3.14) holds with a slight difference in the block matrix \(A\) as follows:

\[
(A^j_{1})_{m1} = -i \langle G_1Vw_1^{(j)}, JVw_m^{(j)} \rangle = - \frac{ic_1}{4\pi} \langle u_1^{(j)}, JV1 \rangle \langle u_1^{(j)}, JV1 \rangle
\]
which vanishes for all \(i, j\) except that for \(i = j = 1\) (see (2.18)). Then, one has

\[
A_1^1 = \begin{pmatrix}
* & \tilde{A}_1^1 \\
* & \tilde{A}_1 \\
a & 0 & \cdots & 0
\end{pmatrix}_{m1 \times m1},
\]

and the sub-matrices \(A_1^i, i = 2, \ldots, k\), (respectively, \(A_1^j, j = 2, \ldots, k\)) have only zeros at the last row (respectively, at the first column), while

\[
A^i_j = \begin{pmatrix}
0 \\
\vdots & \tilde{A}^i_j \\
0 & 0 & \cdots & 0
\end{pmatrix}_{m \times m},
\]

\(\forall 2 \leq i, j \leq k.\)
We now check the invertibility of \( E_{-+}(z) \) by following the same steps as before. We define
\[
\Phi_{k-1} = (\beta_{ij})_{2 \leq i, j \leq k},
\]
where \( \beta_{ij}, \ 2 \leq i, j \leq k \), are defined by (3.19). Let
\[
L_{k-1} = (\langle u_1^{(i)}, J u_1^{(i)} \rangle)_{2 \leq i, j \leq k}.
\]
Then, using (3.19) we obtain
\[
\Phi_{k-1} = -C_{k-1} L_{k-1}, \quad C_{k-1} = \text{diag}(c_2, \cdots, c_k).
\]

**Lemma 3.6.** Assume that zero is a singularity of the third kind of \( H \) and that (H2) holds. Then, for \( \rho > 5, 5/2 < s < \rho - 5/2 \), the determinant of \( E_{-+}(z) \) can be developed as follows:

\[
det E_{-+}(z) = \sigma_k z^{k-1/2} + O(|z|^k),
\]
for \( z \in \Omega^\delta = \{ z \in \mathbb{C} \setminus \mathbb{R}^+, |z| < \delta \}, \delta > 0 \) small, where \( \sigma_k = a \times \det \Phi_{k-1} \neq 0 \) and \( a \) is given above.

**Proof.** We will proceed in the same way followed to prove Lemma 3.4 First, we perform the change of variables \( \eta = \sqrt{z} \) and then we define
\[
E_{-+}(\eta) = N + \eta A + \eta^2 B, \quad Z(\eta) = \det E_{-+}(\eta),
\]
for \( \eta \in \{ z \in \mathbb{C}^+ : |z| < \delta \} \). Now, we introduce the modified matrix \( L(\eta) = \text{diag}[L_1(\eta), \cdots, L_k(\eta)] \) such that
\[
L_1(\eta) = \text{diag}(1, \cdots, 1, \eta^{-1}) \quad \text{and} \quad L_i(\eta) = \text{diag}(1, \cdots, 1, \eta^{-2}), \quad i = 2, \cdots, k.
\]
We denote
\[
\bar{E}_{-+}(\eta) = L(\eta) E_{-+}(\eta) = N + \bar{A}(\eta) + \bar{B}(\eta) \quad \text{and} \quad \bar{Z}(\eta) = \det \bar{E}_{-+}(\eta).
\]
Then, it is non difficult to see that

(i) \( L(\eta) N = N \).

(ii) \( [\bar{A}(\eta)]^2_{ij} = \left\{
\begin{array}{cl}
0 & \text{if } i, j = 1, 1 \leq j \leq k,
\end{array}
\right. \quad + O(|\eta|) \)

\[
\begin{array}{cl}
\text{if } 2 \leq i \leq k, 1 \leq j \leq k.
\end{array}
\]

(iii) \( [\bar{B}(\eta)]^2_{ij} = \left\{
\begin{array}{cl}
0 & \text{if } i, j = 1, 1 \leq j \leq k,
\end{array}
\right. \quad + O(|\eta|^2) \)

\[
\begin{array}{cl}
\text{if } 2 \leq i \leq k, 1 \leq j \leq k.
\end{array}
\]

Thus, we also see that \( \bar{E}_{-+}(\eta) \) does not contain any negative power of \( \eta \) and \( \bar{Z}(\eta) \) is consequently polynomial in \( \eta \). Thus, we have to calculate \( \bar{Z}(0) \). Before that we
see by (i), (ii) and (iii) that the blocks \( (\tilde{E}_{-2}^{(i)})^j(0) \) and \( (\tilde{E}_{-2}^{(i)})^j(0) \), \( 2 \leq i \leq k \), \( 1 \leq j \leq k \), of the matrix \( \tilde{E}_{-2}(0) \) have the following forms

\[
(\tilde{E}_{-2}^{(i)})^j(0) = \begin{pmatrix}
0 & -\delta_{ij} & 0 \\
\vdots & \ddots & \ddots \\
0 & 0 & -\delta_{ij} \\
a\delta_{ij} & * & \cdots & * \\
b_i & * & \cdots & *
\end{pmatrix}
\]

Therefore, we obtain

\[
\tilde{Z}(0) = \det \begin{pmatrix}
a & 0 & \cdots & 0 \\
b_2 & b_{22} & \cdots & b_{2k} \\
\vdots & \vdots & \ddots & \vdots \\
b_k & b_{k2} & \cdots & b_{kk}
\end{pmatrix}
= a \times \det(b_{ij})_{2 \leq i,j \leq k} = a \times \det(\Phi_{k-1}) \neq 0
\]

under the condition \((2.10)\).

The rest follows from the proof of Lemma 3.4 with \( \det(L) = \eta^{-2k+1} \). Hence, we obtain

\[
\det E_-(\eta) = \det E_{-2}(\eta) + o(\eta^{2k-1}) = \left(a \times \det(\Phi_{k-1})\right) \eta^{2k-1} + o(|\eta|^{2k-1}).
\]

\[\square\]

**Lemma 3.7.** Assume that the hypotheses in the previous lemma hold. Then, for \( \rho > 2l + 1, l + 1/2 < s < \rho - l - 1/2 \) and \( l = 4, 5, \ldots \), we have

\[
E_-(z)^{(-1)} = \frac{1}{z} F_{-2}^{(3)} + \frac{1}{\sqrt{z}} F_{-1}^{(3)} + \sum_{j=0}^{l-4} z^{j/2} F_j^{(3)} + E_{-l-4}^{(-1)}(z),
\]

for \( z \in \Omega_3 \), \( \delta > 0 \) small, where \( F_{-2}^{(3)} \) is a matrix of rank \( k-1 \), whose blocks \( (F_{-2}^{(3)})^j \), \( i, j = 1, \ldots, k \), are zero everywhere except that the \((1, m_j)\)-th entry of each block denoted by \( \alpha_{ij} \) does not necessarily vanish and such that \( \alpha_{11} = \alpha_{j1} = 0 \) for \( i, j = 1, \cdots, k \). Also, \( F_{-1}^{(3)} \) is a matrix of rank at most \( k \), whose blocks \( (F_{-1}^{(3)})^j \), \( i, j = 1, \cdots, k \), are zero everywhere except that the \((1, m_j)\)-th entry of each block denoted by \( \mu_{ij} \) does not necessarily vanish. In particular \( \mu_{11} = a^{-1} \) and \( \mu_{1j} = 0 \) for \( j = 2, \cdots, k \).

Moreover, the remainder term \( E_{-l-4}^{(-1)}(z) \) has the property in \((3.56)\).

**Proof.** We make the same notation for the rest of this section. We have from \((3.55)\) that \( \tilde{E}_{-2}(0) \) is invertible with

\[
\tilde{E}_{-2}(0)^{-1} = \frac{t \text{Com} \tilde{E}_{-2}(0)}{\det \tilde{E}_{-2}(0)} = F_{-2}^{(3)} + \mathcal{E},
\]

\[
E_{-2}(0)^{-1} = \frac{1}{z} F_{-2}^{(3)} + \frac{1}{\sqrt{z}} F_{-1}^{(3)} + \sum_{j=0}^{l-4} z^{j/2} F_j^{(3)} + E_{-l-4}^{(-1)}(z),
\]

for \( z \in \Omega_3 \), \( \delta > 0 \) small, where \( F_{-2}^{(3)} \) is a matrix of rank \( k-1 \), whose blocks \( (F_{-2}^{(3)})^j \), \( i, j = 1, \ldots, k \), are zero everywhere except that the \((1, m_j)\)-th entry of each block denoted by \( \alpha_{ij} \) does not necessarily vanish and such that \( \alpha_{11} = \alpha_{j1} = 0 \) for \( i, j = 1, \cdots, k \). Also, \( F_{-1}^{(3)} \) is a matrix of rank at most \( k \), whose blocks \( (F_{-1}^{(3)})^j \), \( i, j = 1, \cdots, k \), are zero everywhere except that the \((1, m_j)\)-th entry of each block denoted by \( \mu_{ij} \) does not necessarily vanish. In particular \( \mu_{11} = a^{-1} \) and \( \mu_{1j} = 0 \) for \( j = 2, \cdots, k \).

Moreover, the remainder term \( E_{-l-4}^{(-1)}(z) \) has the property in \((3.56)\).
show that that $F$ argument used to prove that the matrix $\eta$ to the matrix $\tilde{F}$ as follows:

$$
(3.58) \quad \tilde{F} = \begin{pmatrix}
0 & \cdots & 0 & \alpha_{ij} \\
0 & \cdots & 0 & 0 \\
\vdots & \cdots & \vdots & \vdots \\
0 & \cdots & 0 & 0
\end{pmatrix}
$$

if $i = j = 1$ or $2 \leq i \leq k$, $1 \leq j \leq k$,

$$
(3.59) \quad (\tilde{F})_{ij} = \begin{pmatrix}
\ast & \tilde{A}_i^j \\
\vdots & \ast \\
0 & b_{1j} & \ast & \cdots & \ast \\
\ast & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0
\end{pmatrix}
$$

if $i = 1$, $2 \leq j \leq k$.

Here $\alpha_{ij}$, $1 \leq i, j \leq k$, are the cofactors of the matrix $\tilde{E}_{-2}(0)$ divided by $a \times \det \Phi_k$. In particular

$$
\alpha_{11} = a^{-1}.
$$

On the other hand, $\forall \eta \in \Omega$, $\delta > 0$ small, the matrix $\tilde{E}_{-2}(\eta)$ can be developed as follows:

$$
(3.58) \quad \tilde{E}_{-2}(\eta) = \tilde{E}_{-2}(0) + \eta\tilde{E}_{-2}(0) + O(|\eta|^2),
$$

where the blocks of the second matrix at the right member are defined as follows:

$$
(3.59) \quad (\tilde{E}_{-2}(0))_{ij} = \begin{pmatrix}
\ast & \tilde{A}_i^j \\
\vdots & \ast \\
0 & b_{1j} & \ast & \cdots & \ast \\
\ast & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0
\end{pmatrix}
$$

elsewhere.

Then, it follows from (3.14), (3.54), (3.57) and the regularity of $L(\eta)$ that for $\eta \in \Omega \cap \mathbb{C}_+, \delta > 0$ small enough, $E_{-2}(\eta)^{-1}$ exists with

$$
(3.60) \quad E_{-2}(\eta)^{-1} = \frac{F_{-2}(3)}{\eta^2} + \frac{F_{-1}(3)}{\eta} + O(1),
$$

where $(F_{-2}(3))^j = (\tilde{F})_{ij}$ if $2 \leq i, j \leq k$ and $(F_{-2}(3))^j = (0)$ otherwise. Also, $F_{-1} = \tilde{F}_{-2} - F_{-2} - \tilde{F}_{-2} + \tilde{F}_{-2}(0)\tilde{F}_{-2}$ is of the same form as $\tilde{F}_{-2}$ given above, with

$$
(3.61) \quad (F_{-1}(3))^j = \begin{pmatrix}
0 & \cdots & 0 & \mu_{1j} \\
0 & \cdots & 0 & 0 \\
\vdots & \cdots & \vdots & \vdots \\
0 & \cdots & 0 & 0
\end{pmatrix}, \quad \forall 1 \leq i, j \leq k,
$$

such that $\mu_{11} = \alpha_{11} = a^{-1}$ and $\mu_{1j} = 0$ for $j = 2, \cdots, k$ (see proof of Theorem 2.2).

The rest of the proof follows directly from (3.14) and (3.60). Moreover, the same argument used to prove that the matrix $F_{-2}$ at (3.31) is of rank $k$ can be applied to the matrix $F_{-2}(3)$ to show that it is of rank $k - 1$, unless in the present case we show that $(\alpha_{ij})_{2\leq i, j \leq k}$ is a $(k - 1) \times (k - 1)$ invertible matrix. Indeed, we can check that

$$
\alpha_{ij} = (\det \Phi_{k-1})^{-1}(1)^{i+j} \times |(\Phi_{k-1})_{j-1,i-1}|, \quad \forall 2 \leq i, j \leq k,
$$
which denotes the \((i-1, j-1)\)-th entry of the invertible matrix \(\Phi_{k-1}\) (see (3.51)). Thus
\[
(\alpha_{ij})_{z \leq i, j \leq k} = \Phi_{k-1}^{-1}.
\]

We end this section by proving Theorem 2.3.

**Proof of Theorem 2.3.** Since the same proof of Theorem 2.2 can be done here, we will omit the details. By the asymptotic expansion of \(E_{\rho}(z)^{-1}\) that is established in the previous Lemma, for \(\rho > 2l + 1, l + 1/2 < s < \rho - l - 1/2\), \(f, g \in \mathbb{H}^{-1,s}\) and \(z \in \Omega_{\delta}, \delta > 0\) small, the expansion of \(R(z)f\) in the norm sense of \(\mathbb{H}^{1,-s}\) is of the following form:
\[
R(z)f = -\frac{\mathcal{P}^{(3)}_0}{z} + \frac{1}{\sqrt{z}}\{i\langle f, J\psi \rangle \psi + \sum_{i=2}^{k} \langle f, Jv_i^{(3)} \rangle u_1^{(i)} \} \\
+ \sum_{j=0}^{l-4} z^{j/2} R^{(3)}_j f + \tilde{R}^{(3)}(z),
\]
where, by the help of the matrices defined in (3.51) and (3.62), we get the projection
\[
\mathcal{P}^{(3)}_0 = \sum_{i=2}^{k} \langle f, JZ^{(3)}_i \rangle Z^{(3)}_i,
\]
\[
Z^{(3)}_i = \sum_{j=2}^{k} q_{ij}u^{(j)}_1, \quad \langle Z^{(3)}_i, JZ^{(3)}_j \rangle = \delta_{ij}, \quad \forall 1 \leq i, j \leq k,
\]
with \(Q_{k-1} := (q_{ij})_{z \leq i, j \leq k}\) is such that \(L^{-1}_{k-1} = Q^T_{k-1}Q_{k-1}\), and
\[
\psi = (-c_1\mu_{11})^{1/2}u_1^{(1)} = \frac{2\sqrt{\pi}}{<u_1^{(1)}, JV1>} u_1^{(1)},
\]
\[
v_i^{(3)} = -\sum_{j=1}^{k} \mu_{ij} G_1Vw_{m_j}^{(j)} - \sum_{j=2}^{k} i\alpha_{ij} G_1Vw_{m_j}^{(j)} = \sum_{j=1}^{k} \mu_{ij} u_1^{(j)}, \quad i = 1, \ldots, k.
\]

where, \(\alpha_{ij}\) and \(\mu_{ij}\) are respectively the non zero terms of the matrices \(F^{(3)}_{-2}\) and \(F^{(3)}_{-1}\) in (3.31). Here we used \(G_1Vw_{m_j}^{(j)} = 0\) for \(j = 2 \cdots k\) (see Remark 3.3 and (2.8)) and \(G_0Vv_1^{(j)} = -u_1^{(j)}, \forall 1 \leq j \leq k\). Moreover, see proof of Theorem 2.2 for the properties of the remainder term \(\tilde{R}^{(3)}(z)\).

### 4. Behavior of the resolvent near outgoing real resonances

In this section, we prove Theorem 2.5. First, we use the condition (2.12) of the hypothesis (H3) to establish the asymptotic expansion of the resolvent \(R(z) = (H - z)^{-1}\) near an outgoing positive real resonance. Note that the study of incoming positive real resonance can be done in a similar way.

Let \(\lambda_0 \in \sigma^+ (H)\), for \(\delta > 0\) we denote
\[
\Omega^\delta_\lambda := \{z \in \mathbb{C}^+: 0 < |z - \lambda_0| < \delta\} \quad \text{and} \quad \tilde{\Omega}^\delta_\lambda := \{z \in \mathbb{C}^+, |z - \lambda_0| < \delta\}.
\]
Let us begin with the known results on the behavior of the free resolvent $R_0(z)$ on the boundary of the right half-plane (the real half-axis $[0, +\infty)$). Taking the analytic continuation of the kernel $R_0(z)(x)$ to $\mathbb{C} \setminus \{0\}$, the expansion of $R_0(z)$ at order $r$ for every $r \in \mathbb{N}$ and for $z \in \Omega_\delta^+, \delta > 0$, is written

\begin{equation}
R_0(z) = \sum_{j=0}^{r} (z - \lambda_0)^j G_j^+ + o(|z - \lambda_0|^r),
\end{equation}

where

\begin{align}
G_j^+: \mathbb{H}^{-1,s} &\longrightarrow \mathbb{H}^{1,s'}; \quad s, s' > 1/2, \\
G_j^+: \mathbb{H}^{-1,s} &\longrightarrow \mathbb{H}^{1,s'}; \quad s, s' > j + 1/2, \quad j = 1, \cdots, r,
\end{align}

are integral operators with corresponding kernels

\[ r_j^+(x, \lambda_0) := \lim_{z \to \lambda_0, z \in \mathbb{C}_+} \frac{d^j}{d\lambda^j} \frac{e^{i\sqrt{\lambda}|x|}}{4\pi|x|}, \quad j = 0, 1, \cdots, r. \]

We denote by $R_0(\lambda + i0)$ the boundary values of the analytic continuation of $R_0(z)$ to $\Omega_\delta^+$.

Let $\lambda_0$ be an outgoing positive real resonance of the operator $H = -\Delta + V$. Note that the two subspaces $\text{Ker}(I + K^+(\lambda_0))$ and $\{\psi \in \text{Ker}(H - \lambda_0); \psi \text{ satisfies the radiation condition } (1.4) \}$ coincide in $H^{1-s}$, $1/2 < s < \rho - 1/2$ (see Section 2.1). We assume in hypothesis (H3) that $\dim \text{Ker}(I + K_+^+(\lambda_0)) = N_0$. Denote $\Pi_1^{\lambda_0}$ the Riesz projection associated with the eigenvalue $-1$ of the operator $K^+(\lambda_0)$

\[ \Pi_1^{\lambda_0} = \frac{1}{2i\pi} \int_{|w+1|=\epsilon} (w - K^+(\lambda_0))^{-1}dw, \quad \epsilon > 0, \]

we also denote $E_0^{+} = \text{Ran} \Pi_1^{\lambda_0}$ and $m = \text{rank} \Pi_1^{\lambda_0}$.

The same strategy used in Section 3 to prove Theorem 2.2 and Theorem 2.3 will be followed in this section. First, note that the decomposition made in lemma 3.1 can be done in the present case for $E_0^{+}$ with just a change of notation $E_j^+$ instead of $E_j$ in (1). Recall that $U_j = \{u_1^{(j)}, \cdots, u_m^{(j)}\}$ denotes a basis of $E_j^+$ and $W_j = \{w_1^{(j)}, \cdots, w_m^{(j)}\}$ its dual with respect to the non-degenerate bilinear form $\Theta(\cdot, \cdot)$ on $E_j^+$. In particular, $\text{Ker}(I + K^+(\lambda_0))$ is the subspace of $L^{2,-s}$ generalized by $\{u_1^{(j)}, \cdots, u_m^{(j)}\}$. Then, we will study the Grushin Problem for the Fredholm operator $I + K(z)$, that we have constructed in Section 3.1.

For the proof of the theorem we start by the following lemma, where we refer to Section 4.1 for details.

**Lemma 4.1.** For $\rho > 2l+1, l+1/2 < s < \rho-l-1/2, l = 2, 3, \cdots$ and $z \in \Omega_\delta^+, \delta > 0$ small, we have the following expansion of $E_{-+}(z)$:

\begin{equation}
E_{-+}(z) = N + (z - \lambda_0)A(\lambda_0) + \sum_{j=2}^{l} (z - \lambda_0)^j E_{-+,j}(\lambda_0) + E_{-+,l}(z - \lambda_0),
\end{equation}
where $N$ is the block matrix as defined in (3.17) and

\begin{equation}
(A(\lambda_0))_l^j = \begin{pmatrix}
\tilde{A}_l^j(\lambda_0) \\
\ast & \cdots & \ast
\end{pmatrix}_{m_l \times m_j}, \forall 1 \leq i, j \leq N_0,
\end{equation}

where

\[a_{ij}(\lambda_0) = -G_1^+V u_1^{(j)} + JV w_1^{(i)}\]

\[= \frac{ic_l}{8\pi \sqrt{\lambda_0}} \int_{\mathbb{R}^d} e^{i\lambda_0 |x-y|} V(x)u_1^{(j)}(x)V(y)u_1^{(i)}(y) \, dxdy,
\]

such that $c_i \neq 0$ by Remark 3.3 and $G_1^+$ is the integral operator defined by (4.3).
Moreover $E_{-+}(z - \lambda_0)$ is analytic in $\Omega_+^\pm$ and continuous up to $\mathbb{R}_+$, such that the limit $\lim_{\epsilon \to 0^+} E_{-+}(\lambda - \lambda_0 + i\epsilon)$ exists and satisfies

\[\|\frac{d^r}{d\lambda^r} E_{-+}(\lambda - \lambda_0 + i0)\| = o(|\lambda - \lambda_0|^{-r}), \forall \lambda \in \Omega_+^\pm \cap \mathbb{R}_+, \quad r = 0, 1, \cdots, l.
\]

The expansion (4.14) can be obtained directly by introducing in (3.10) the expansion (4.1) of $R_0(z)$ in $B(0, s, 0, -s)$ for $l + 1/2 < s < \rho - l - 1/2$.

Before proving Theorem 2.5 we establish the expansion of $E_{-+}(z)^{-1}$. Let $\rho > 2l + 1$, $l + 1/2 < s < \rho - l - 1/2$, $l = 2, 3, \cdots$ and assume that the hypothesis (H3) holds. We make the change of variables $\eta = z - \lambda_0$ and introduce the block diagonal matrix

\begin{equation}
L(\eta) = \text{diag}[L_1(\eta), \cdots, L_{N_0}(\eta)], \quad L_i(\eta) = \text{diag}(1, \cdots, 1, \eta^{-1}), \forall 1 \leq i \leq N_0,
\end{equation}

for $\eta \in \{z \in \mathbb{C}_+ : |z| < \delta\}$. Then, by proceeding in the same way as in the proof of Lemma 3.4 we obtain

\begin{equation}
det (N + \eta A_0(\lambda_0)) = \eta^{N_0} \det (a_{ij}(\lambda_0))_{1 \leq i, j \leq N_0} + O(|\eta|^{N_0+1}),
\end{equation}

where $a_{ij}(\lambda_0), 1 \leq i, j \leq N_0$, are given above and $N_0 = \text{dim ker} (I + K^+(\lambda_0))$. It follows that, if the condition (2.12) is satisfied then (4.14) together with (4.7) gives

**Proposition 4.2.** For $\rho > 2l + 1$ and $l = 1, 2, \cdots$, $\det E_{-+}(z)$ has the following expansion in power of $z$ for $z \in \Omega_+^\pm$:

\[\det E_{-+}(z) = a_0(\lambda_0) (z - \lambda_0)^{N_0} + \sum_{j=1}^{l} a_j(\lambda_0) (z - \lambda_0)^{N_0+j} + o(|z - \lambda_0|^{N_0+1}),\]

where $a_0(\lambda_0) = \det (a_{ij}(\lambda_0))_{1 \leq i, j \leq N_0} \neq 0$.

Since $E_{-+1}(\eta, \lambda_0) := N + \eta A_0(\lambda_0)$ has the same form as $E_{-+2}(\eta)$ defined in the proof of Theorem 2.2 then the same computation can be done here. We obtain

\begin{equation}
E_{-+1}(\eta, \lambda_0)^{-1} = \frac{\text{Com} E_{-+1}(\eta, \lambda_0)}{\det E_{-+1}(\eta, \lambda_0)} = \frac{1}{\eta} F_{-1}(\lambda_0) + F_0(\lambda_0),
\end{equation}
where $F_{-1}(\lambda_0)$ is a matrix of rank $N_0$, whose blocks $(F_{-1}(\lambda_0))_{ij}^2$, $1 \leq i, j \leq N_0$, are of the form

$$
(F_{-1}(\lambda_0))_{ij}^2 = \frac{1}{a_0(\lambda_0)} \begin{pmatrix}
0 & \cdots & 0 & b_{ij}(\lambda_0) \\
0 & \cdots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & 0
\end{pmatrix}_{m_i \times m_j}
$$

(4.9)

with $b_{ij}(\lambda_0)$ is the $(j, i)$-th cofactor of the invertible matrix $(a_{ij}(\lambda_0))_{1 \leq i, j \leq N_0}$ (see Remark 3.4). On the other hand, by (4.9), for $l \in \mathbb{N}^+$, $\rho > 2l + 1, l + 1/2 < s < \rho - l - 1/2$ and $z \in \Omega_\delta^+$ we have

$$
E_{-+}(z) = E_{-+1}(z - \lambda_0) \times 
$$

(4.10)

$$
\begin{bmatrix}
I + (E_{-+1}(z - \lambda_0))^{-1} \left( \sum_{j=1}^{l-2} (z - \lambda_0)^j E_{-+,j}(\lambda_0) + E_{-+,l}(z - \lambda_0) \right)
\end{bmatrix},
$$

and by Neumann series we can show that $E_{-+}(z)^{-1}$ exists for $z \in \Omega_\delta^+$ if $\delta > 0$ is small enough, with

$$
E_{-+}(z)^{-1} = \frac{1}{z - \lambda_0} F_{-1}(\lambda_0) + \sum_{j=0}^{l-2} (z - \lambda_0)^j \tilde{E}_{-+,j}(\lambda_0) + \tilde{E}_{-+,l}(z - \lambda_0),
$$

where $\tilde{E}_{-+,0}(\lambda_0) = F_0(\lambda_0) - F_{-1}(\lambda_0) E_{-+2}(\lambda_0) F_{-1}(\lambda_0)$, the other terms $\tilde{E}_{-+,j}(\lambda_0)$, $j = 1, \cdots, l - 2$, can be also directly found and the remainder $\tilde{E}_{-+,l}(z - \lambda_0)$ is analytic in $\Omega_\delta^+$ and continuous up to $\mathbb{R}_+$ satisfying

$$
\left\| \frac{d^r}{d\lambda^r} \tilde{E}_{-+,l}(\lambda - \lambda_0 + i0) \right\| = o(|\lambda - \lambda_0|^{l-2-r}), \quad \forall \lambda \in \Omega_\delta^+ \cap \mathbb{R}_+, \quad r = 0, 1, \cdots, l - 2.
$$

Remark 4.3. (1) We see that the coefficients $a_{ij}(\lambda_0)$, $1 \leq i, j \leq N_0$, defined in Lemma 4.1 can be expressed in the following matrix:

$$
(a_{ij}(\lambda_0))_{1 \leq i,j \leq N_0} = \frac{-i}{8\pi \sqrt{\lambda_0}} C_{N_0} A_{N_0}(\lambda_0),
$$

(4.11)

where

$$
A_{N_0}(\lambda_0) = \left( B_{\lambda_0}(u^j_1, u^j_1) \right)_{1 \leq i, j \leq N_0}, \quad C_{N_0} = \text{diag}(c_1, \cdots, c_{N_0}),
$$

and the bilinear form $B_{\lambda_0}(\cdot, \cdot)$ is defined in (4.11).

(2) The non-zero terms of the matrix $F_{-1}(\lambda_0)$ in (4.9) can be expressed as follows

$$
(b_{ij}(\lambda_0))_{1 \leq i,j \leq N_0} = i8\pi \sqrt{\lambda_0 a_0(\lambda_0)} A_{N_0}(\lambda_0)^{-1} C_{N_0},
$$

(4.12)

Proof of Theorem 2.4. Applying Grushin problem 3.9 to $\mathcal{M}(z) := I + R_0(z)V$, it follows from

$$
\mathcal{M}(z)^{-1} = E(z) - (I - E(z)\mathcal{M}(z))SE_{-+}(z)^{-1}T(I - \mathcal{M}(z)E(z))
$$

and (4.10) that $\mathcal{M}(z)$ is invertible for $z \in \Omega_\delta^+$ and we have the following expansion in $\mathcal{B}(\mathbb{H}^{1-s})$ for $l + 1/2 < s < \rho - l - 1/2$ if $\rho > 2l + 1$:

$$
(I + R_0(z)V)^{-1} = -\frac{1}{z - \lambda_0} SF_{-1}(\lambda_0) T + \sum_{j=0}^{l-2} (z - \lambda_0)^j \tilde{F}_j(\lambda_0) + \tilde{F}(z - \lambda_0)
$$

(4.13)
where $F_{-1}(\lambda_0)$ is the matrix of rank $N_0$ given in (4.8), so that for $g \in H^{1,-s}$

\[
SF_{-1}(\lambda_0)Tg = \frac{1}{a_0(\lambda_0)} \sum_{j=1}^{k} b_{ij}(\lambda_0) \left\langle g, JVw_{ij}^{(j)} \right\rangle u_1^{(i)},
\]

and $F_0(\lambda_0) = -SE_{-1,0}(\lambda_0)T + E(0)$ with

\[
E(0) = \left( (I - \Pi_1^{\delta}) (I + G_0^+ V) (I - \Pi_1^{\delta}) \right)^{-1} (I - \Pi_1^{\delta}).
\]

Other terms $F_j(\lambda_0)$, $j = 1, \cdots, l - 2$, can be calculated explicitly and $\tilde{F}(z - \lambda_0)$ is analytic in $\Omega_0^+$ and continuous up to $\mathbb{R}_+$ verifying the following estimates:

\[
\| \frac{d^r}{dz^r} \tilde{F}(z - \lambda_0) \|_{B(\mathbb{H}^{1,-r})} = O(|z - \lambda_0|^{|r - 2|}), \ orall z \in \Omega_0^+, r = 0, \cdots, l - 2,
\]

and for $\lambda \in \bar{\Omega}_0^+ \cap \mathbb{R}_+$ the limit

\[
\lim_{\epsilon \to 0^+} \tilde{F}(\lambda - \lambda_0 + i\epsilon) = \tilde{F}(\lambda - \lambda_0 + i0)
\]

exists as operator in $B(\mathbb{H}^{1,-r})$ and it satisfies similar estimates as in (4.15).

Consequently, using $R(z) = (I + R_0(z)V)^{-1} R_0(z)$, and the expansions (2.6), (4.13) together with (4.14) and (4.12), it follows that for $f \in \mathbb{H}^{1,-s}$

\[
R(z)f = \frac{R_{-1}(\lambda_0)f}{z - \lambda_0} + \sum_{n=0}^{l-2} R_j(\lambda_0)f + \tilde{R}_{l-2}(z - \lambda_0)f,
\]

in $\mathbb{H}^{1,-s}$, where

\[
R_{-1}(\lambda_0)f = -\frac{1}{a_0(\lambda_0)} \sum_{i,j=1}^{N_0} b_{ij}(\lambda_0) c_j \left\langle f, JVU_{ij}^{(j)} \right\rangle u_1^{(i)} = \frac{1}{a_0(\lambda_0)} \sum_{i,j=1}^{N_0} b_{ij}(\lambda_0) c_j \left\langle f, JU_{ij}^{(j)} \right\rangle u_1^{(i)} = i8\pi \sqrt{\lambda_0} \sum_{i=1}^{N_0} \left\langle f, J\phi_i(\lambda_0) \right\rangle u_1^{(i)},
\]

such that by Remark 4.3 and a simple computation we have

\[
\begin{pmatrix}
\phi_1(\lambda_0) \\
\vdots \\
\phi_{N_0}(\lambda_0)
\end{pmatrix} = \mathcal{A}_{N_0}(\lambda_0)^{-1} 
\begin{pmatrix}
u_1^{(1)} \\
\vdots \\
u_{N_0}^{(1)}
\end{pmatrix}.
\]

where $\mathcal{A}_{N_0}(\lambda_0)^{-1}$ is defined by (4.11) and (4.12). Thus, to simplify the above sum we decompose

\[
\mathcal{A}_{N_0}(\lambda_0)^{-1} = Q(\lambda_0)^T Q(\lambda_0)
\]

(see (3.47)). Hence, $R_{-1}(\lambda_0)$ can be written in the following form

\[
R_{-1}(\lambda_0)f = \sum_{i=1}^{N_0} \left\langle f, J\psi_i(\lambda_0) \right\rangle \psi_i(\lambda_0),
\]
Moreover, the remainder term \( R_0(\lambda_0) \) is analytic in \( \Omega_0^+ \) and satisfies the estimates

\[
\left. \left\| \frac{d^r}{dz^r} \tilde{R}_q(\lambda - \lambda_0 + i0) \right\|_{-1,s,1,-s} \right. = O(|\lambda - \lambda_0|^{q-1-r}),
\]

for \(|\lambda - \lambda_0| < \delta\) and \(r = 0, 1, \ldots, q-1\).

If \(\rho > 3\) and \(3/2 < s < \rho - 3/2\), then we can obtain

\[
R(z) = \frac{R(\lambda_0)}{(z - \lambda_0)^{\mu_0 - N_0 + 1}} + O(|z - \lambda_0|^{-\mu_0 + N_0}), \quad \forall z \in \Omega_0^+.
\]

Proof. Under the condition (4.16) there exists \(\delta > 0\) such that for \(z \in \Omega_0^+\) the \(m \times m\) matrix \(E_{-+}(z)\) is invertible and

\[
E_{-+}(z)^{-1} = \frac{M(z)}{d(z)}
\]

where \(M(z) = (a_{ij}(z))_{1 \leq i, j \leq m}\) is the transpose of the cofactor matrix of \(E_{-+}(z)\), analytic in \(z \in \Omega_0^+\) and whose entries \(a_{ij}(z)\), \(i \leq i, j \leq m\), are polynomials of the
entries of $E_+(z)$. Moreover, taking the expansion (4.14) for $q \in \mathbb{N}^*$, $l = \mu_0 - N_0 + q$, $\rho > 2l + 1$ and $l + 1/2 < s < \rho - l - 1/2$, we obtain

$$E_+(z)^{-1} = \frac{1}{d(z)} \left( (z - \lambda_0)^{N_0-1} B_0(\lambda_0) + \sum_{j=N_0}^{q-1+\mu_0} (z - \lambda_0)^j B_j(\lambda_0) \right) + \frac{1}{d(z)} E_{-+}^{-1,q}(z - \lambda_0) = \frac{\hat{B}_0(\lambda_0)}{(z - \lambda_0)^\mu_0 - N_0 + 1} + \sum_{j=-\mu_0 + N_0}^{q-1} (z - \lambda_0)^j \hat{B}_j(\lambda_0) + \hat{E}_{-+}^{-1,q}(z - \lambda_0),$$

where $N_0 = \dim \ker (I + K^+(\lambda_0))$,

$$(\hat{B}_0(\lambda_0))_i^j = \begin{pmatrix} 0 & \cdots & 0 & \beta_1^j(\lambda_0) \\ 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \end{pmatrix}, \quad 1 \leq i, j \leq N_0,$$

such that $\beta_1^j(\lambda_0)$ are polynomials of the entries of $A(\lambda_0)$ and the remainder term $E_{-+}^{-1,q}(z - \lambda_0)$ is analytic in $z \in \Omega_0^+$ and continuous up to $\mathbb{R}_+$ satisfying (4.20)

$$\| \frac{d^r}{d\lambda^r} \hat{E}_{-+}^{-1,q}(\lambda - \lambda_0 + i0) \| = o(|\lambda - \lambda_0|^{q-1-r}), \quad |\lambda - \lambda_0| < \delta, \ r = 0, 1, \ldots, q - 1.$$ 

The rest of the proof can be obtained in the same way as in the previous proof. We obtain the leading term

$$\mathcal{R}(\lambda_0) = \sum_{i=1}^{N_0} \langle \cdot, \hat{\psi}_1(\lambda_0) \rangle u_1^i \text{ on } L^{2,-s},$$

where

$$\begin{pmatrix} \hat{\psi}_1(\lambda_0) \\ \vdots \\ \hat{\psi}_{N_0}(\lambda_0) \end{pmatrix} = \mathcal{B}_{N_0}(\lambda_0) \begin{pmatrix} u_1^{(1)} \\ \vdots \\ u_1^{(k)} \end{pmatrix}, \quad \mathcal{B}_{N_0}(\lambda_0) = (\beta_{ij}(\lambda_0))_{1 \leq i,j \leq N_0} C_{N_0},$$

(see (4.11)). Moreover, the estimate (4.18) can be seen from (4.20). \hfill \Box

5. **Large-time behavior for the semigroup $(e^{-\iota t H})_{t \geq 0}$**

To obtain the large-time expansion of solutions to the Schrödinger equation (1.1), we need the preceding results for the behavior of the resolvent on a contour surrounding the positive resonances in the upper half-plane, encircling the origin and down to the lower half-plane.

First, since $V$ satisfies the condition (1.2) for $\rho > 2$, we can check that for arbitrary small $\epsilon > 0$, there exists $R_\epsilon > 0$ large enough such that the numerical range of $H$ denoted by $\mathcal{N}(H)$ is included in an angular sector

$$\mathcal{N}(H) \subseteq \{ z \in \mathbb{C} : \text{Re } z \geq -R_\epsilon, \ |\arg(z + R_\epsilon)| \leq \frac{\epsilon}{2} \}.$$ 

We recall that $\sigma_d(H)$ denotes the set of discrete eigenvalues of finite algebraic multiplicities (see Section 2.1). Denote $\sigma^+_d(H) = \sigma_d(H) \cap \hat{\mathbb{C}}^+$, whose accumulation
points can exist only in $\sigma^+_d(H) \cup \{0\}$ (see Definition 2.1).

At the first time, we deduce from our previous main results that $H$ has a finite number of discrete eigenvalues in the upper half-plane.

**Proposition 5.1.** Assume that $\rho > 3$ and the hypothesis $(H3)$ holds. In addition, if zero is an eigenvalue of $H$ we assume that $\rho > 5$ and $(H1)$ or $(H2)$ holds. Then $\sigma^+_d(H)$ is finite.

**Proof.** Since the eigenvalues located in the closed upper half-plane can only accumulate at points of $\{0\} \cup \sigma^+_d(H)$, then to prove that there is at most a finite number of these eigenvalues it suffices to prove that neither the eigenvalue zero nor the zero resonance nor the outgoing resonances are accumulation points of $\sigma_d(H) \cap \mathbb{C}_+$. Indeed, in view of Theorem 2.3 for each $\lambda_j \in \sigma^+_d(H)$ there is small $\delta > 0$ such that $(x)^{-s}R(z)(x)^{-s}$ is locally uniformly bounded in $x$ in a neighborhood $\Omega^+_\delta \subset \mathbb{C}_+ \cap \mathbb{C}_+$ of $\lambda_j$. Then we have found a neighborhood of $\lambda_j$ in $\mathbb{C}_+$ which does not contain any pole of $(x)^{-s}R(z)(x)^{-s}$. Thus $\Omega^+_\delta \cap \sigma_d(H) = \emptyset$. Hence $\lambda_j$ is not an accumulation point of $\sigma_d(H) \cap \mathbb{C}_+$. Also, if zero is a resonance of $H$ then it is seen from Theorem 2.3 that the same argument can be done. Moreover, if zero is an eigenvalue of $H$ and the hypothesis $(H1)$ is satisfied then it follows from Theorem 2.2 that if $\rho > 5$ then $\sigma_d(H) \cap \Omega^+_S = \emptyset$ for some $\delta > 0$ small enough. We can check similarly that zero is not an accumulation point of $\sigma_d(H) \cap \mathbb{C}_+$ if it is both an eigenvalue and a resonance of $H$ under the hypothesis $(H2)$. □

At a second time, we check the existence of the limiting absorption principle for the non-selfadjoint Schrödinger operator $H$ on each subinterval of $\mathbb{R}_+$ that does not contain any outgoing positive real resonance. In addition, we establish high energy estimates of the derivatives of the resolvent. Note that, it is required here to think about E. Mourre’s method, which was first developed in [23] for a selfadjoint operator, or it is known as the commutators method. This method was adapted by J.Royer [30] to the dissipative Schrödinger operator $H = H_1 - iV$ where $H_1$ is a selfadjoint operator and $V > 0$ (we refer to [30] for the conditions on $V$).

In the following, we denote by $C^l(\Omega, F)$ the set of all functions $f: \Omega \subset E \rightarrow F$ that is of class $C^j$ on $\Omega$, where $E$ and $F$ denote normed vector spaces. And for $a > 0$, we define the open set $\Lambda_a$ and its closure $\bar{\Lambda}_a$ by

$$\Lambda_a = \bigcap_{j=0}^N \{z \in \mathbb{C}_+ : |z - \lambda_j| > a\}, \quad \bar{\Lambda}_a = \bigcap_{j=0}^N \{z \in \mathbb{C}_+ : |z - \lambda_j| > a\},$$

where $\lambda_0 = 0$ and $\lambda_j \in \sigma^+_d(H), \forall 1 \leq j \leq N$.

The following proposition gives the high energy resolvent estimate as $|z| \rightarrow +\infty$, when $R(z)$ is extended through the upper half-plane to $\bar{\Lambda}_a$ for $a > 0$. It can be proved in the same way as in [15] Theorem 9.2 (see also [18] Theorem 3.8).

**Proposition 5.2.** Assume that $(H3)$ holds. Let $l \in \mathbb{N}$. Then, for $\rho > l + 1$, $s > l + \frac{1}{2}$ and $f, g \in L^{2,s}$:

$$\Lambda_a \ni z \mapsto \langle R(z)f, g \rangle$$

can be continuously extended to a function in $C^l(\bar{\Lambda}_a; \mathbb{C})$. 

Moreover, the boundary values \( \langle R(\lambda + i0)f, g \rangle \) satisfy the following estimates:

\[
(5.2) \quad \left| \frac{d}{d\lambda} R(\lambda + i0)f, g \right| \leq \frac{C_a}{|\lambda|^{\frac{r+1}{2}}} \|f\|_{0,s} \|g\|_{0,s}, \ \lambda \in \Lambda_a \cap \mathbb{R}_+, \ \lambda \to +\infty,
\]

for some constant \( C_a > 0 \).

The existence of the above limit is a direct consequence of the two main theorems and the following known results: for \( f \) continuously extended to uniformly bounded functions on \( \bar{\Lambda} \), \( H \) if zero is an eigenvalue of \( \langle (I + R_0(z)V)^{-1}f, h \rangle \) and \( z \to \langle R_0(z)vf, h \rangle \) can be continuously extended to uniformly bounded functions on \( \Lambda_a \) (see [15, Lemma 9.1]). In addition to the following estimates in [15, Theorem 8.1]

\[
\left| \frac{d^r}{d\lambda^r} (R_0(\lambda \pm i0)f, g) \right| \leq \frac{C}{|\lambda|^{\frac{r+1}{2}}} \|f\|_{0,s} \|g\|_{0,s}, \ r \in \mathbb{N}, \ \text{as} \ \lambda \to +\infty.
\]

Then, we are going to prove Theorem 5.3. Before that, we establish a representation formula for the semigroup \( e^{-itH} \) as \( t \to +\infty \) in \( (0, s, 0, -s) \). Note that in the selfadjoint case [15], where \( \{e^{-itH}\}_{t \in \mathbb{R}} \) is an one parameter unitary group, \( e^{-itH} \) can be defined as a function of \( H \), and it may be represented by integrating the unit function \( e^{-it\lambda} \) on \( [0, +\infty] \) with respect to the spectral measure, which unfortunately can not be usually defined on the whole positive real axis. For non-selfadjoint operators (cf. [33]). The most general formula and that is useful for this work is Dunford-Taylor integral (cf. [16, Section IX.1.6]), which is valid for m-sectorial operator, whose numerical range is a subset of a sector \( \{ |\arg z| \leq \theta < \frac{\pi}{2} \} \). More precisely, we will show that there exists a curve \( \Gamma^\nu(\eta) \) such that \( \Gamma^\nu(\eta) \cap (\sigma^\nu_+(H) \cup \sigma^\nu_-(H) \cup \{0\}) = \emptyset \) and \( \Gamma^\nu(\eta) = \Gamma^\nu_0(\eta) \cup \Gamma_0(\eta) \cup \Gamma_1(\eta) \cup \Gamma_+ \), where

\[
\Gamma^\nu_0(\eta) = \{ z = \eta e^{-i\eta} - \lambda e^{i\nu}, \ \lambda \geq 0 \},
\]

\[
\Gamma_0(\eta) = \{ z = \eta e^{i(2\pi - \theta)}, \ \eta \leq \theta < 2\pi \},
\]

\[
\Gamma_1(\eta) = \bigcup_{j=1}^{N} (\sigma_j(\eta) \cup \gamma_j(\eta)), \ \sigma_j(\eta) = \{ z = \lambda + i0, \ \eta \leq \lambda \leq \lambda_j - \eta \},
\]

\[
\gamma_j(\eta) = \{ z = \lambda + \eta e^{i(\pi - \theta)}, \ 0 < \theta < \pi \}, \ j = 1, \cdots, k,
\]

\[
\Gamma_+ = \{ z = \lambda + i0, \ \lambda \geq \lambda_N + \eta \},
\]

for some \( \nu, \mu > 0 \) chosen so that there are no eigenvalues of \( H \) between \( \Gamma_0(\eta) \cup \Gamma_1(\eta) \) and the real axis, nor between \( \Gamma^\nu_-(\eta) \) and the negative real axis.

We now state the intermediate theorem

**Theorem 5.3.** Let \( \rho > 4 \) and \( 5/2 < s < \rho - 3/2 \). Assume that (H3) holds and if zero is an eigenvalue of \( H \) then we assume that \( \rho > 5 \), \( 5/2 < s < \rho - 5/2 \) and (H1) or (H2) holds. Then, for \( f \) and \( g \in L^{2,s} \), we have the following representation formula:

\[
(5.3) \quad \langle e^{-itH}f, g \rangle = \sum_{z_j \in \sigma^\nu_+(H)} \langle e^{-itH} \Pi_z f, g \rangle + \frac{1}{2i\pi} \int_{\Gamma^\nu(\eta)} e^{-itz} \langle (H - z)^{-1}f, g \rangle \, dz, \ \forall t > 0,
\]
where \( \sigma^+_D(H) \) is the finite set of discrete eigenvalues of \( H \) that are included in the closed upper half-plane with associated Riesz projections \( \{ \Pi_{z_j} \} \) and \( \Gamma'(\eta) \) is the curve described above.

**Proof.** We proceed in 3 steps:

First step: Let \( \epsilon > 0 \). Let \( P_\epsilon = i(e^{-it}H - i\epsilon R_\epsilon) \). Then it can be seen from \( (5.1) \) that

\[
N(P_\epsilon) \subseteq \{ i(e^{-it}z - i\epsilon R_\epsilon), \Re z \geq -R_\epsilon, |\arg(z + R_\epsilon)| \leq \frac{\epsilon}{2} \} \subset \bar{S}_\theta,
\]

where \( S_\theta \) denotes the open sector with angle \( \theta \). Let \( \theta_\epsilon = (\pi - \arctan \epsilon)/2 \in ]0, \pi/2[ \).

Moreover, for \( \lambda_\epsilon := e^{i(\pi/2 + \epsilon)}(\lambda + R_\epsilon) \in \mathbb{C} \setminus N(H) \) with \( \lambda > 0 \) large, it can be seen that \( (P_\epsilon + \lambda) = ie^{-it}(H - \lambda_\epsilon) \) is a bijection of \( \mathbb{H}^2 \) into \( L^2 \). This shows that \( P_\epsilon \) is m-sectorial with semi-angle \( \theta_\epsilon \). Hence, \( -P_\epsilon \) is the unique generator of the semigroup \( (e^{-tP_\epsilon})_{t \geq 0} \), which is bounded by \( ||e^{-tP_\epsilon}|| \leq 1 \) (cf. \([16]\) Theorem IX.1.24).

Therefore, there exists a closed curve \( \Gamma \), oriented in the anticlockwise sense, included in the resolvent set of \( -P_\epsilon \) and enclosing the numerical range of \( -P_\epsilon \) in its interior, where \( \Gamma = \{ \lambda e^{-i(\pi - \theta, \delta)} | \lambda \geq 0 \} \cup \{ \lambda e^{i(\pi - \theta, \delta)} | \lambda \geq 0 \} \) for some \( 0 < \delta < \frac{\pi}{2} - \theta_\epsilon \), such that the semigroup integral representation is written:

\[
e^{-tP_\epsilon}u = \frac{1}{2it\pi} \int_{\Gamma} e^{iz}(P_\epsilon + z)^{-1} u dz, \quad \forall u \in L^2, \quad \forall t > 0,
\]

and we have the following estimate

\[
||e^{-ite^{-it}H}u||_0 \leq e^{tR_{\epsilon}} ||u||_0, \quad t \geq 0.
\]

Denote by \( \Pi_{z_j} : L^2 \rightarrow L^2 \) the Riesz Projection \( (2.3) \) associated to such eigenvalue \( z_j \in \sigma^+_D(H), \sigma^+_D(H) \) is a finite set by Proposition \( (5.1) \) then \( H_j := H\Pi_{z_j} \) defines a bounded operator on the finite dimensional subspace \( \text{Ran} \Pi_{z_j} \), and whose spectrum is \( \sigma(H_j) = \{ z_j \} \) (see \([16]\) p. 178-179]). Let \( H_\epsilon = e^{-it}H \) and \( z_\epsilon = e^{-it}z \). By analytic deformation in \( p(H) \) of the curve \( \Gamma \), we can find a set of curves \( \cup_{j=1}^{\infty} \Sigma_j \) around the eigenvalues \( z_1, \ldots, z_p \in \sigma^+_D(H) \) located above a curve \( \Gamma'(\eta, \epsilon) \) (defined below) oriented in the anti-clockwise sense such that

\[
U_\epsilon(t)u := e^{-itH}\Pi_{z_j}u
\]

\[
= -\frac{1}{2it\pi} \sum_{z_j \in \sigma^+_D(H)} \int_{\Sigma_j} e^{-itz}(H_j - z)^{-1} u dz
\]

\[
+ \frac{1}{2it\pi} \int_{\Gamma'(\eta, \epsilon)} e^{-itz}(H - z)^{-1} u dz
\]

\[
= \sum_{z_j \in \sigma^+_D(H)} e^{-itz}(H - z)^{-1} u dz, \quad \forall t > 0,
\]

\[
= \sum_{z_j \in \sigma^+_D(H)} e^{-itz}(H - z)^{-1} u dz, \quad \forall t > 0,
\]

\[
= \sum_{z_j \in \sigma^+_D(H)} e^{-itz}(H - z)^{-1} u dz, \quad \forall t > 0,
\]
where $\Gamma^\epsilon(\eta, \epsilon)$ is a closed curve oriented from $-\infty$ to $+\infty$ and $\Gamma^\nu(\eta, \epsilon) = \Gamma_0(\eta, \epsilon) \cup \Gamma_1(\eta, \epsilon) \cup \Gamma(\epsilon)$:

\[
\Gamma_0(\eta, \epsilon) = \{z = \eta e^{-i\gamma} - \lambda e^{i\nu}, \ \lambda \geq 0\},
\]

\[
\Gamma_1(\eta, \epsilon) = \bigcup_{j=1}^{\nu} (\sigma_j(\eta, \epsilon) \cup \gamma_j(\eta, \epsilon)),
\]

\[
\sigma_1(\eta, \epsilon) = \{z = \lambda + i\epsilon, \ \eta \cos \epsilon \leq \lambda \leq \lambda_1 - \eta\},
\]

\[
\sigma_2(\eta, \epsilon) = \{z = \lambda + i\epsilon, \ \lambda_1 + \eta \leq \lambda \leq \lambda_2 - \eta\}, j = 2, \ldots, \nu,
\]

\[
\gamma_1(\eta, \epsilon) = \{z = \lambda_1 + i\epsilon + \eta e^{i(\pi - \theta)}, \ 0 < \theta < \pi\}, \ j = 1, \ldots, \nu,
\]

\[
\Gamma(\epsilon) = \{z = \lambda + i\epsilon, \ \lambda \geq \lambda_N + \eta\},
\]

for some fixed $0 < \nu < \frac{
u}{2}$ and $\eta, \epsilon > 0$ small chosen so that $\Gamma^\nu(\eta, \epsilon) \cap \sigma(H) = \emptyset$ and there is no eigenvalue of $H$ between $\Gamma_0(\eta, \epsilon) \cup \Gamma_1(\eta, \epsilon) \cup \Gamma(\epsilon)$ and the positive real axis, nor between $\Gamma^- (\eta)$ and the negative real axis.

**Second step:** Let $f, g \in L^2, s$. We define

\[
\langle U(t)f, g \rangle := \sum_{z_j \in \sigma^+_s(H)} \langle e^{-itH} \Pi_{z_j}f, g \rangle + \frac{1}{2i\pi} \int_{\Gamma(\epsilon)} e^{-itz} \langle (H - z)^{-1}f, g \rangle dz.
\]

In this step we will show that

\[
\langle U(t)f, g \rangle = \lim_{\epsilon \to 0^+} \langle U_\epsilon(t)f, g \rangle, \ \forall f, g \in L^2, s, \ \forall t > 0.
\]

Let us show the convergence of the integral at $\eta \to 0^+$ by decomposing it onto two parts: $\Gamma^\nu(\eta, \epsilon) \cap \{|z| \leq R_1\}$ and $\Gamma^\nu(\eta, \epsilon) \cap \{|z| > R_1\}$, where $R_1 > \lambda_N + 1$ with $\lambda_N := \max \sigma_+^s(H)$.

On one hand, if zero is an eigenvalue of $H$, then Theorem 2.2 gives the uniformly boundedness of the resolvent on $\Gamma(\eta)$ in $(0, s, 0, -s)$ for $5/2 < s < \rho - 5/2$. In addition, Theorems 2.3 shows that the resolvent is uniformly bounded in $z$ on each semicircle $\gamma_j(\eta)$ surrounding the singularity $\lambda_j$ on the positive real axis, where better assumption is required, so that $3/2 < s < \rho - 3/2$. Moreover, by Proposition 5.2 the integrand $g_\epsilon(z, t) := e^{-it\epsilon t - i\epsilon \sigma} \langle (H - z)^{-1}f, g \rangle$ is continuously extended to an uniformly bounded function in $z$ on $\bigcup_{j=1}^{\nu} \sigma_j(\eta)$ for $f, g \in L^2, s$, $1/2 < s < \rho - 1/2$.

Then in view of these results

\[
\int_{\Gamma^\nu(\eta, \epsilon) \cap \{|z| \leq R_1\}} g_\epsilon(z, t) dz \xrightarrow{\epsilon \to 0^+} \int_{\Gamma^\nu(\eta) \cap \{|z| \leq R_1\}} e^{-itz} \langle (H - z)^{-1}f, g \rangle dz.
\]

On the other hand, we have

\[
\int_{\Gamma^\nu(\eta, \epsilon) \cap \{|z| > R_1\}} = \int_{\Gamma^\nu(\eta) \cap \{|z| > R_1\}} + \int_{\Gamma(\epsilon) \cap \{|z| > R_1\}}.
\]

Since $g_\epsilon(z, t)$ is uniformly bounded in $\epsilon \in [0, \frac{\nu}{2}]$ on $\Gamma^\nu(\eta) \cap \{|z| > R_1\}$ with

\[
|e^{-i\epsilon \sigma} - e^{-i\nu} e^{it\lambda e^{i(\nu - \epsilon)}}| \langle (H - (\eta e^{-i\gamma} - \lambda e^{i\nu}))^{-1}f, g \rangle | \leq C_{\eta, R_1} \|f\|_2 \|g\|_2 e^{-t\lambda \sin(\nu/2)\lambda^{1/2}},
\]
∀λ ∈ [R_1, +∞], η > 0, t > 0, and the right member function is integrable on [R_1, +∞] for all η > 0 small and t > 0. Then we deduce by Lebesgue’s dominated convergence theorem

\[
(5.11) \quad \lim_{\epsilon \to 0^+} \int_{\Gamma^{\epsilon}(\eta) \cap \{|z| > R_1\}} g_\epsilon(z,t) \, dz = \int_{\Gamma^{\epsilon}(\eta) \cap \{|z| > R_1\}} e^{-itz} \langle (H - z)^{-1} f, g \rangle \, dz.
\]

However, the integrand of the second integral tends to

\[ e^{-it\lambda} \langle (H - (\lambda + i0))^{-1} f, g \rangle \quad \text{as} \quad \epsilon \to 0^+ \]

which by Proposition 5.2 belongs to \( C^2([R_1, +\infty[, \mathbb{C}) \), with the following estimate:

\[
\left| \frac{d^2}{d\lambda^2} \langle (H - (\lambda + i\epsilon))^{-1} f, g \rangle \right| \leq \frac{C_{R_1}}{(\lambda)^{3/2}} \| f \|_0 \| g \|_0, \quad \forall \epsilon > 0,
\]

that requires \( \rho > 3 \) and \( s > 5/2 \). Then, for \( t > 0 \) fixed we integrate twice by parts to obtain the following:

\[
\int_{R_1}^{+\infty} (-ite^{-i\epsilon})^{-2} e^{-it\lambda e^{-i\epsilon}} \frac{d^2}{d\lambda^2} \langle (H - (\lambda + i\epsilon))^{-1} f, g \rangle \, d\lambda
\]

\[ + \mathcal{O}(t^{-2}|e^{-i(tR_1 e^{-i\epsilon})}|) \| f \|_0 \| g \|_0,
\]

\[(5.12) \quad := \int_{R_1}^{+\infty} f_\epsilon(t, \lambda) \, d\lambda + \mathcal{O}(t^{-2}|e^{-i(tR_1 e^{-i\epsilon})}|) \| f \|_0 \| g \|_0.
\]

Since for \( \rho > 3, s > 5/2 \) and \( t > 0 \), \( f_\epsilon(t, \cdot) \) is uniformly bounded in small \( \epsilon > 0 \) as

\[
|f_\epsilon(t, \lambda)| \leq \frac{C_{R_1}}{t^2} \frac{1}{(\lambda)^{3/2}} \| f \|_0 \| g \|_0,
\]

then by Lebesgue’s dominated convergence theorem \( \int_{R_1}^{+\infty} f_\epsilon(t, \lambda) \, d\lambda \) converges as \( \epsilon \to 0 \), also the second term at the right-hand side of (5.12) is uniformly bounded in \( \epsilon > 0 \) by \( \mathcal{O}(t^{-2})\| f \|_0 \| g \|_0 \). This shows that the second integral at the right-hand side of (5.10) is uniformly convergent in \( \epsilon > 0 \) for all \( t > 0 \).

Consequently, this together with (5.9) and (5.11) implies

\[
\int_{\Gamma^{\epsilon}(\eta, \epsilon)} g_\epsilon(z,t) \, dz \to_{\epsilon \to 0^+} \int_{\Gamma^{\epsilon}(\eta)} e^{-itz} < (H - z)^{-1} f, g > \, dz,
\]

which must establish (5.8).

Finally, we will show at the third step that for all \( f \) and \( g \in L^{2,s} \) we have the following convergence

\[
< e^{-itH} f - e^{-itH} f, g > \to_{\epsilon \to 0^+} 0, \quad \forall t > 0.
\]

Third step: Let \( \phi \) be a test function in \( C_0^\infty(\mathbb{R}^3) \). We write

\[
e^{-itH} \phi - e^{-itH} \phi = \int_0^t \frac{d}{dr} \left( e^{-irH} e^{-i(t-r)H} \phi \right) \, dr
\]

\[ = (-i)(e^{-i\epsilon} - 1) \int_0^t e^{-irH} H e^{-i(t-r)H} \phi \, dr.
\]
Finally, by density of \( C^\infty \) and the same curves \( \Gamma_1(1) \) Lemma 5.4. □ which establishes the desired representation formula.

Now let \( t > 0 \) be fixed. By (5.3) and the previous equality, we see that

\[
\|e^{-itH}\phi - e^{-itH}\phi\|_0 \leq Ce^{\|R\epsilon\|}e^{-\epsilon t} - 1 \int_0^t \|e^{-i(t-r)H}\phi\|_0 \, dr \to 0, \]

i.e. \( e^{-itH}\phi \) converges in \( L^2 \) norm to \( e^{-itH}\phi \) as \( \epsilon \to 0 \) for all \( \phi \in C^\infty_0(\mathbb{R}^3) \). This by uniqueness of the weak limit in (5.8) gives

\[
< e^{-itH}\phi, \psi > = \lim_{\epsilon \to 0} < U(t)\phi, \psi >, \quad \forall \phi, \psi \in C^\infty_0(\mathbb{R}^3), \quad \forall t > 0.
\]

Finally, by density of \( C^\infty_0(\mathbb{R}^3) \) in \( L^{2,s} \), we conclude that

\[
U(t)f = e^{-itH}f, \quad \forall f \in L^{2,s}, \quad \forall t > 0,
\]

which establishes the desired representation formula. □

Next we quote a lemma for some generalized integrals given in [9, Section II.2.].

**Lemma 5.4.** (1) The limit of the function \( \lambda \mapsto (\lambda + i\mu)^{-1} \) as \( \mu \to 0^+ \), is the generalized function \( (\lambda + i0)^{-1} \) defined in the following sense:

For every test function \( \phi \in C^1_0(\mathbb{R}) \)

\[
((x + i0)^{-1}, \phi(x)) = \int_{|x| \leq 1} \frac{\phi(x) - \phi(0)}{x} \, dx + \int_{|x| > 1} \frac{\phi(x)}{x} \, dx - i\pi\phi(0).
\]

Moreover, for \( t > 0 \) we have the following generalized integral

\[
\int_\mathbb{R} e^{-i\lambda x} \frac{\lambda}{\lambda + i0} \, d\lambda = -i2\pi.
\]

(2) For \( t > 0 \) and \( j = -1, 0, 1, \ldots \) we have

\[
\int_0^{+\infty} \lambda^{j/2} e^{-it\lambda} \, d\lambda = \Gamma(\frac{j}{2} + 1)(-it)^{-\frac{j}{2} - 1},
\]

where \( \Gamma(\frac{j}{2} + 1) = \int_0^{+\infty} t^{j/2} e^{-t} \, dt \).

Now we are able to prove Theorem 2.6. Before starting the proof, let us rewrite the representation formula at (5.6) as follows:

\[
< e^{-itH}f, g > = \sum_{j=1}^p < e^{-itH}\Pi_j f, g > + \lim_{\eta \to 0} \lim_{\epsilon \to 0} \frac{1}{2i\pi} \int_{\Gamma'_\nu(\eta, \epsilon)} e^{-ite^{-it}} z \, dz < (H - z)^{-1} f, g > \, dz,
\]

after some analytic deformation of the curve \( \Gamma'(\eta, \epsilon) \) in the following sense:

\( \Gamma'(\eta, \epsilon) \cap (\sigma^\nu_\nu(H) \cup \sigma_\nu^\nu(H)) = \emptyset \) and \( \Gamma'(\eta, \epsilon) = \Gamma'_\nu(\epsilon) \cup \sigma'_\nu(\epsilon) \cup C_0(\eta, \epsilon) \cup \Gamma_1(\eta, \epsilon) \cup \Gamma_+(\epsilon) \), where

\[
\Gamma'_\nu(\epsilon) = \{ z = \nu \epsilon - \lambda e^{i\nu}, \, \lambda \geq 0 \}, \quad \nu = \nu - i\epsilon,
\]

\[
C_0(\eta, \epsilon) = \{ z = \eta e^{i(2\pi - \theta)}, \, \epsilon_\eta < \theta < 2\pi - \epsilon_\eta \}, \quad \epsilon_\eta = \arcsin(\epsilon/\eta), \quad \eta(\epsilon) = \eta \cos \epsilon_\eta,
\]

\[
\sigma'_\nu(\epsilon) = \{ z = \lambda - i\epsilon, \, \eta(\epsilon) \leq \lambda \leq \nu \}, \quad \eta(\epsilon) = \eta \cos \epsilon_\eta,
\]

and the same curves \( \Gamma_1(\eta, \epsilon) \) and \( \Gamma_+(\epsilon) \) defined in (5.7).
Proof of Theorem 2.6. Let \( \chi : \mathbb{R} \to \mathbb{R} \) be a cutoff function such that \( \chi(\lambda) = 1 \) for \( |\lambda| \leq \nu/2 \) and \( \chi(\lambda) = 0 \) for \( |\lambda| \geq \nu \). For \( j = 0, 1, \cdots, N \), we define \( \chi_j(\lambda) = \chi(\lambda - \lambda_j) \), where \( \lambda_0 = 0 \) and \( \lambda_j \in \sigma^+_{\nu}(H), \forall j = 1, \cdots, N \). Let \( g_\epsilon(z, t) = e^{-ite^{-iz}}(H - z)^{-1}f, g) \).

First, we prove the part (a) of the theorem. We begin by introducing at the second member of (5.13) the resolvent expansions near zero energy and real resonances that are obtained by Theorems 2.2 and 2.5 respectively.

\[
\int_{\Gamma^\nu(\eta, \epsilon)} g_\epsilon(z, t) \, dz = \int_{\Gamma^\nu(\eta, \epsilon)} \frac{1}{z^{s/2}} e^{-ite^{-iz}}\chi_0(\text{Re} z) \, dz \\
+ \sum_{j=1}^{N} \sum_{s=1}^{l-2} (\langle R_s(\lambda_j) f, g \rangle \int_{\Gamma^\nu(\eta, \epsilon)} (z - \lambda_j)^j e^{-ite^{-iz}}\chi_j(\text{Re} z) \, dz) \\
+ \int_{\Gamma^\nu(\eta, \epsilon)} e^{-ite^{-iz}}(\hat{R}_{1}^{(1)}(z)f, g)\chi_0(\text{Re} z) \, dz \\
+ \sum_{j=1}^{N} \int_{\Gamma^\nu(\eta, \epsilon)} e^{-ite^{-iz}}(\hat{R}_{1}(z - \lambda_j)f, g)\chi_j(\text{Re} z) \, dz \\
+ \int_{\Gamma^\nu(\eta, \epsilon)} e^{-ite^{-iz}}(\hat{R}(z)f, g)\chi_+(\text{Re} z) \, dz \\
:= I_{0}^{\nu, \eta}(t) + \sum_{j=1}^{N} \sum_{s=1}^{l-2} I_{s}^{\nu, \eta}(t, \lambda_j) + J_{0}^{\nu, \eta}(t) \\
+ \sum_{j=1}^{N} J_{s}^{\nu, \eta}(t, \lambda_j) + J_{1}^{\nu, \eta}(t),
\]

where \( \chi_+ = 1 - \sum_{j=0}^{N} \chi_i \).

Let \( t > 0 \) be fixed. We see that

\[
I_{0}^{\nu, \eta}(t) = \langle R_{-2}^{(1)} f, g \rangle \left[ \int_{\omega_0} e^{-ite^{-iz}} \frac{\chi(\text{Re} z)}{z} \, dz - \int_{\nu+ie}^{\nu-ie} e^{-ite^{-iz}} \frac{\chi(\text{Re} z)}{z} \, dz \right] \\
+ \langle R_{-1}^{(1)} f, g \rangle \int_{\eta(e)}^{+\infty} \left( \frac{e^{-ite^{-i(\lambda + ie)}}}{\sqrt{\lambda + i\epsilon}} - \frac{e^{-ite^{-i(\lambda - ie)}}}{\sqrt{\lambda - i\epsilon}} \right) \chi(\lambda) \, d\lambda \\
+ I_{1}^{\nu, \eta}(t),
\]

where \( \omega_0 := [\nu - ie, \eta \cos \epsilon \eta - ie] \cup \mathbb{C}_0(\eta, \epsilon) \cup [\eta \cos \epsilon \eta + ie, \nu + ie] \cup [\nu + ie, \nu - ie] \) is a closed curve enclosing zero traveled in the clockwise sense. Then, we have that

\[
I_{0}^{\nu, \eta}(t) - I_{1}^{\nu, \eta}(t) \text{ converges as } \epsilon \to 0^+, \eta \to 0^+ \text{ respectively, in the sense of generalized functions of } t,
\]

so

\[
-2\pi\langle R_{-2}^{(1)} f, g \rangle + 2\langle R_{-1}^{(1)} f, g \rangle \left[ \int_{0}^{+\infty} \frac{e^{-it\lambda}}{\sqrt{\lambda}} \, d\lambda + \int_{\nu/2}^{+\infty} \frac{e^{-it\lambda}}{\sqrt{\lambda}} (\chi(\lambda) - 1) \, d\lambda \right] \\
= -2\pi\langle R_{-2}^{(1)} f, g \rangle + 2(-i\pi)^{1/2} \langle R_{-1}^{(1)} f, g \rangle t^{-1/2} + O(t^{-2}),
\]
where the decay rate \( t^{-1/2} \) can be seen from Lemma \([5.4]\) as well as the term \( I_1^{(1)}(t) \) which can be easily estimated using Lemma \([5.4]\) as follows:

\[
\lim_{\eta \to 0^+} \lim_{\epsilon \to 0^+} I_1^{(1)}(t) = 2 < R_1^{(1)} f, g > \int_0^{+\infty} \sqrt{\lambda} e^{-it\lambda} \chi(\lambda) \, d\lambda
\]

(5.14)

\[
= -(i\pi)^{1/2} < R_1^{(1)} f, g > t^{-3/2} + O(t^{-2}).
\]

Moreover, for \( j = 1, \ldots, N \), we can show that \( I_j^{(1)}(t, \lambda_j) \) converges as \( \epsilon \to 0^+ \) to the integral

\[
I_j^{(1)}(t, \lambda_j) = \langle R_j^{(1)}(\lambda_j) f, g \rangle e^{-it\lambda_j} \int_{L_0} \frac{e^{-it\xi}}{\xi} \chi(\text{Re}\, \xi) \, d\xi, \quad \xi = z - \lambda_j,
\]

along the contour \( L_n = ]-\infty, \eta[ \cup \{ \xi = \eta e^{i(\pi - \theta)}, \ 0 < \theta < \pi \} \cup [\eta, +\infty[ \) traveled from \(-\infty\) to \(+\infty\). Then, the limit integral \( I_j^{(1)}(t, \lambda_j) \) converges as \( \eta \to 0 \), in the sense of generalized function of \( t \), to

\[
I_j^{(1)}(t, \lambda_j) := \langle R_j^{(1)}(\lambda_j) f, g \rangle e^{-it\lambda_j} \int_{-\infty}^{+\infty} \frac{e^{-it\lambda}}{\lambda + i0} \chi(\lambda) \, d\lambda
\]

\[
= -2i\pi \langle R_j^{(1)}(\lambda_j) f, g \rangle e^{-it\lambda_j} + O(t^{-2}), \quad \forall t > 0.
\]

However, for \( s = 0, 1, \ldots, l - 2 \), we have

\[
J_s^{(1)}(t) \xrightarrow{\eta \to 0^+, \epsilon \to 0^+} \langle R_s(\lambda_j) f, g \rangle e^{-it\lambda_j} \int_{-\infty}^{+\infty} \lambda^s e^{-it\lambda} \chi(\lambda) \, d\lambda, \quad \forall t > 0,
\]

where the right member decays rapidly at infinity as the \( s - \text{th} \) derivative of the Fourier transform of the cut off function \( \chi \) such that the convergence holds in the sense of regularized function.

Let now estimate \( J_s^*(t) \) and \( J_s^{(1)}(t) \) as \( t \to +\infty \). We decompose \( J_s^*(t) \) as follows:

\[
J_s^*(t) = \int_{\Gamma(\eta, \epsilon) \cup \Gamma_-^*(\epsilon)} g_c(z, t) \chi(\text{Re}\, z) \, dz + \int_{\Gamma_-^*(\epsilon)} g_c(z, t) \chi(\text{Re}\, z) \, dz
\]

\[
:= J_+^*(t) + J_-^*(t).
\]

On the one hand, it is seen that \( J_s^*(t) \) has an exponential time-decay as \( O(e^{-t\epsilon \nu}) \) for some \( \epsilon \) independent on \( \epsilon \) and \( t \). On the other hand, it follows from \([6, 12]\) that

\[
\lim_{\epsilon \to 0^+} |J_s^*(t)| = O(t^{-2}) \|f\|_{0, s} \|g\|_{0, s}, \quad \text{as} \ t \to +\infty.
\]

Next, we have to estimate \( J_0^*(t) \). We see from Theorem \([2.2]\) that \( z \mapsto \tilde{R}_1^{(l)}(z) \) can be continuously extended to \( C^2(\{ |z| < \delta, \pm \text{Im}\, z \geq 0\}, \{ -1, 1, s, -1 \}) \) for \( s > 11/2 \) if \( \rho > 11 \), such that

\[
\| \frac{d^r}{d\lambda^r} \tilde{R}_1^{(1)}(\lambda \pm i0) \|_{(-1, 1, s, -1)} = o(\lambda^{\frac{4}{s} - r}), \quad 0 < \lambda < \delta, \quad r = 0, 1, 2.
\]

Then, it follows by Lemma \( 10.2 \) in \([15]\) that

\[
\lim_{\eta \to 0^+} \lim_{\epsilon \to 0^+} J_0^*(t) = o(t^{-\frac{\nu}{2}}) \quad \text{as} \ t \to +\infty.
\]

Finally, let \( j = 1, \ldots, N \). Then, by Theorem \([2.2]\) for \( \rho > 2l + 1, l + 1/2 < s < \rho - l - 1/2 \) and \( l = 2, 3, \ldots \) the remainder term \( \tilde{R}_{l-2}(z - \lambda_j) \) is continuously extended to

\[
\tilde{R}_{l-2}(\lambda - \lambda_j + i0) \in C^{l-2}(\{ \lambda > 0, \ |\lambda - \lambda_j| < \delta\}, \{ -1, s, 1, -1 \})
\]

(5.16)
Moreover, we can easily see that $J^{c,η}(t, λ_j)$ converges as $c, η → 0$ in the sense of regularized functions to the Fourier transform in $t$ of the regular and compactly supported function $λ \mapsto \tilde{R}_{l-2}(λ + i0)χ(λ)$. This in view of (5.16) gives
\[(5.17) \lim_{η→0, t→0} J^{c,η}(t, λ_j) = o(t^{-l+2})∥f∥_{0,s}∥g∥_{0,s'}, as \ t → +∞\]
(see [15] Lemma 10.1). Hence, we have established the proof of the part (a). In this theorem, the strong condition $ρ > 11$ is required to get the remainder $o(t^{-3/2})$ in (5.15), but it can be relaxed to $ρ > 7$ to obtain $\tilde{R}_{l-1}(z) = o(|z|^{-1/2})$ and then to get the remainder $o(t^{-1/2})$.

Now, to proof the part (b) of Theorem 2.6 we have only to compute the integral $I_0^{c,η}(t)$ which does not have the same behavior as in the previous proof. Indeed, when zero is only a resonance and not an eigenvalue, the term $I_0^{c,η}(t)$ is replaced by
\[
I_0^{c,η}(t) = \sum_{j=1}^{N} \langle R^{(2)}_j f, g \rangle \int_{ν(e, ε)} e^{-iεtλ} \tilde{χ}(λ)^{1/2} dλ + \langle R^{(2)}_j f, g \rangle \int_{ν(e)}^{+∞} \frac{e^{-iεt(λ + iε)}}{√λ + iε} - \frac{e^{-iεt(λ - iε)}}{√λ - iε} \tilde{χ}(λ) dλ + I_1(ε, η, t),
\]
where $R^{(2)}_j$ is the one rank operator defined by Theorem 2.3. It is easily to show that the first integral at the right member vanishes as $η → 0$. However, the second integral at the right member tends as $ε → 0^+$, in the sense of generalized functions, to
\[
2 \int_{0}^{+∞} \frac{e^{-iελ}}{√λ} dλ + 2 \int_{ν/2}^{+∞} \frac{e^{-iελ}}{√λ} (χ(λ) - 1) dλ = 2(-iπ)^{1/2}t^{-3/2} + O(t^{-2}).
\]
Also, see (5.14) for the estimate of $I_1(ε, η, t)$.
At the present case, the remainder $o(t^{-3/2})$ in (5.15) requires $ρ > 7$ and this condition can be relaxed to $ρ > 3$ to get the remainder $o(t^{-1/2})$. In addition, $l = 3$ in Theorem 2.5 suffices here to obtain (5.17) with remainder $o(t^{-1})$.

We end the paper by the following remark:

Remark 5.5. If we assume that the condition (2.15) is satisfied instead of (2.12), then using the expansion (4.17) of $R(z)$ near $λ_j ∈ σ^+(H)$, the oscillating term
\[
\sum_{j=1}^{N} e^{-iλ_j} R_{l-1}(λ_j) in (2.24)
\]
will be replaced by
\[
\sum_{j=1}^{N} e^{-iλ_j} R_j(t, λ_j),
\]
where $R_j(t, λ_j)$ is a polynomial of $t$ of degree at most $μ_j - N_j$ with values in $[0, s, 0, -s)$ and with leading term $-(-it)^{μ_j - N_j} R(λ_j).$ See Theorem 2.5. Indeed, for $l = μ_j - N_j, μ_j - N_j - 1, \ldots, 0, (λ + i0)^{-l-1}$ is defined as the $l$-th derivative, in the sense of generalized functions, of $(-it)^{-l}(λ + i0)^{-1}$. Then, by integrating by part
we have
\[
\int_{\mathbb{R}} \frac{e^{-it\lambda}}{(\lambda + i0)^{l+1}} \chi(\lambda) \, d\lambda = \frac{(-it)^l}{l!} \int_{\mathbb{R}} \frac{e^{-it\lambda}}{(\lambda + i0)} \chi(\lambda) d\lambda \\
+ \sum_{j=1}^{l} \frac{(-1)^j}{j!(l-j)!} (it)^{l-j} \int_{\nu < |\lambda| < \nu} \frac{e^{-it\lambda}}{(\lambda + i0)} \, d\lambda \chi(\lambda) \, d\lambda \\
= a_l t^l + a_{l-1} t^{l-1} + \cdots + a_1 t + a_0,
\]

with \( a_l = (−i)^{l+1} \frac{2\pi}{l!} \).

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