Unifying R-symmetry in M-theory

Axel Kleinschmidt

Max Planck Institute for Gravitational Physics, Albert Einstein Institute,
Am Mühlenberg 1, 14476 Potsdam, Germany

Abstract

In this contribution we address the following question: Is there a group with a fermionic presentation which unifies all the physical gravitini and dilatini of the maximal supergravity theories in $D = 10$ and $D = 11$ (without introducing new degrees of freedom)? The affirmative answer relies on a new mathematical object derived from the theory of Kac–Moody algebras, notably $E_{10}$. It can also be shown that in this way not only the spectrum but also dynamical aspects of all supergravity theories can be treated uniformly.

1 Introduction

One of the major themes in string theory has been unification. By this we mean that hitherto unrelated theories and their properties are interpreted as different aspects of a single more general and more fundamental model. In a very broad sense these advances can be called duality relations and typically were first largely conjectural but were substantiated later by computations. Among the most far-reaching of these duality conjectures is the M-theory conjecture [1, 2] which states that all five known superstring theories have

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a common origin which is usually termed M-theory. However, no complete
definition of M-theory is known to date.

It is the aim of this contribution to illustrate how the M-theory picture
can be made more precise by studying a somewhat restricted set-up. More
precisely, we will focus on

- the low energy effective theories with maximal supersymmetry. These
  are the $D = 11$ supergravity theory and the $D = 10$ type IIA and type
  IIB theories.

- the fermionic sectors of these theories. Since all these models have
  maximal supersymmetry they have the same number of physical degrees
  of freedom, equal to 128, in their fermionic (and bosonic) sectors.
  However, these are distributed differently into representations of the
  relevant Lorentz and R-symmetries.

The fermionic spectra can be summarised by the following table.

| Theory | Lorentz & R-symmetry | Representation |
|--------|----------------------|----------------|
| $D = 11$ | $SO(1, 10)$ | Gravitino $\psi_M$
| | | ($320 \oplus 32$) |
| $D = 10$ IIA | $SO_A(1, 9)$ | Two gravitini $\psi^{(1)}_\mu, \psi^{(1)}_\mu$ (achiral)
| | | ($144 \oplus 16 \oplus 144 \oplus 16$) |
| | | Two dilatini $\lambda^{(1)}, \lambda^{(2)}$ (achiral)
| | | $16 \oplus 16$ |
| $D = 10$ IIB | $SO_B(1, 9) \times SO(2)$ | Two gravitini $\psi^{(1)}_\mu, \psi^{(1)}_\mu$ (chiral)
| | | ($144, 2) \oplus (16, 16)$ |
| | | Two dilatini $\lambda^{(1)}, \lambda^{(2)}$ (chiral)
| | | ($16, 2$) |

Table 1: Fermionic representations of the various maximal supergravity theories in $D = 10$ and $D = 11$.

In this table, the relevant irreducible representations of the different
Lorentz groups are indicated. Since a gravitino is a vector-spinor it always
consists of a $\Gamma$-traceless part and a pure $\Gamma$-trace; in the $D = 11$ case these
are the $320$ and $32$ respectively. As is well known, the type IIA theory
employs spinors of both chiralities of the $D = 10$ Lorentz group whereas in type
IIB only one chirality is used. The known relations for the various Lorentz groups following from dualities are:

\[ SO_B(1,9) \cup \cdots \subset SO(1,8) \subset SO_A(1,9) \subset SO(1,10) \]  

(1)

I.e. the type IIA theory is contained in the \( D = 11 \) theory (via dimensional reduction), but the type IIB theory is not. However, after reduction to \( D = 9 \) the IIA and IIB theories agree. The M-theory conjecture now stipulates that there be a unifying structure to this diagram. This is the first question we address here: \textit{Is there a group \( K \) which has subgroups \( SO(1,10) \), \( SO_A(1,9) \) and \( SO_B(1,9) \times SO(2) \) with embedding relations given as in (1) and with a spinor representation which decomposes under these subgroups into the representations of table 1?} This \textit{kinematical} question will be answered in the affirmative in section 2.

The second question addressed in this contribution is: \textit{Is there a dynamical equation with explicit \( K \) symmetry for the \( K \) spinor representation (constructed in the answer to the first question) which reduces to the dynamics of the fermionic fields of the various supergravity theories?} This \textit{dynamical} question will receive a partially affirmative answer in section 3.

The work reported on here is based on the papers [3, 4, 5, 6, 7] which studied the fermionic sectors of maximal supergravity theories and their symmetries. The approach taken there (and also here) arises from known results of unifying symmetries in the corresponding bosonic sector. In particular, it was shown in [8, 9, 10, 11, 12, 13, 14] that the indefinite Kac–Moody algebras \( E_{10} \) and \( E_{11} \) contain the correct spectra at low levels in so-called level decompositions. The Dynkin diagram of \( E_{10} \) is given in figure 1 and the uncanny resemblance of the right end of the Dynkin diagram to the structure in \( \Pi \) is not accidental. \( E_{11} \) contains the correct fields as covariant Lorentz tensor whereas \( E_{10} \) breaks Lorentz symmetry with only manifest spatial Lorentz symmetry.\textsuperscript{2} The bosonic low level spectra correspond to the bosonic version of the first, kinematical question raised above — in order to address the second, dynamical question for bosons further ‘specifications’ are required.

\textsuperscript{2}For this reason the level decomposition of \( E_{10} \) does not contain anti-symmetric ten-form fields for type IIA and type IIB [15] whereas \( E_{11} \) does [14]. That non-propagating ten-forms, as predicted by \( E_{11} \) are compatible with the supersymmetry algebra was verified in [16] [17].
Figure 1: Dynkin diagram of $E_{10}$ with numbering of nodes. $E_{11}$ has an additional node attached with a single line to node 1.

For $E_{11}$, West proposed in [8] that M-theory should be a non-linear realisation of $E_{11}$; if space-time also carries an $E_{11}$ structure it nicely incorporates all central charges of the $D = 11$ supersymmetry algebra [15] but also infinitely many more new coordinates. The same $E_{11}$ structure was found for the bosonic sectors of (massive) type IIA and type IIB in [8, 9, 10]. For $E_{10}$, Damour, Henneaux and Nicolai proposed in [11] a one-dimensional non-linear $\sigma$-model based on an $E_{10}$ coset space and demonstrated that at low levels null geodesic motion on this coset space is equivalent to the $D = 11$ dynamics around a fixed spatial point truncated roughly after first spatial gradients. Higher order spatial gradients were conjectured to arise via the higher levels in the decomposition. This picture was extended to (massive) type IIA and type IIB in [19, 15]. A model combining $E_{11}$ with the null geodesic idea of $E_{10}$ was given in [20, 21].

In this contribution we will work with $E_{10}$ because in this case we can give a more complete answer to the kinematical and dynamical questions raised above. Since $E_{10}$ treats time and space asymmetrically, all necessary requirements for the sought-after ‘M-theory Lorentz group’ $K$ only involve spatial Lorentz groups and their representations. We will comment on the covariant formulation in the final section. In order to convey the main ideas we mostly refrain from introducing intricate notations and outline the logic; more details can be found in references [3, 4, 5, 6].

2 Kinematics

The study of dimensional reduction [22, 23] suggests that the group $K$ we are looking for is $K = K(E_{10})$, the ‘maximal compact subgroup’ of $E_{10}$. In order to see that this is true we first need to understand what $K(E_{10})$ is.
2.1 Definition of $\mathfrak{e}_{10}$ and $K(\mathfrak{e}_{10})$

$K(E_{10})$ is infinite-dimensional and since global issues are somewhat tricky we will restrict our attention here to the Lie algebras. The Lie algebra $K(\mathfrak{e}_{10})$ of $K(E_{10})$ is a subalgebra of the Lie algebra $\mathfrak{e}_{10}$ of $E_{10}$. The Lie algebra $\mathfrak{e}_{10}$ is defined in the Chevalley–Serre presentation by giving 30 simple generators $e_i, f_i, h_i \ (i = 1, \ldots, 10)$ (2)

and their relations (for all $i, j = 1, \ldots, 10$)

\[ [h_i, e_j] = A_{ij} e_j, \quad [h_i, f_j] = -A_{ij} f_j \quad [e_i, f_j] = \delta_{ij} h_i, \]

\[ [h_i, h_j] = 0, \quad (\text{ad } e_i)^{1-A_{ij}} e_j = 0, \quad (\text{ad } f_i)^{1-A_{ij}} f_j = 0, \]  \hspace{1cm} (3)

where $A_{ij}$ is the generalised Cartan matrix which can be read off from fig. 1 as follows: $A_{ii} = 2$ for $i = 1, \ldots, 10$ and if there is a single link between nodes $i$ and $j$ then $A_{ij} = A_{ji} = -1$ and $A_{ij} = 0$ otherwise. $\mathfrak{e}_{10}$ is defined as the Lie algebra with simple generators (2) and relations (3).

On $\mathfrak{e}_{10}$ one can define the Chevalley involution $\theta$ acting by

\[ \theta(e_i) = -f_i, \quad \theta(f_i) = -e_i, \quad \theta(h_i) = -h_i \]  \hspace{1cm} (4)

on the simple generators. The fixed point set of this involution defines the ‘compact subalgebra’ $K(\mathfrak{e}_{10})$:

\[ K(\mathfrak{e}_{10}) = \{ x \in \mathfrak{e}_{10} : \theta(x) = x \}. \]  \hspace{1cm} (5)

This subalgebra is called compact because it has definite Killing norm, generalising the notion of compact algebras in the finite-dimensional case.

It can be shown [24] that $K(\mathfrak{e}_{10})$ is generated by the simple generators

\[ x_i = e_i - f_i \quad (i = 1, \ldots, 10) \]  \hspace{1cm} (6)

which are manifestly invariant under $\theta$ and defining relations of the type

\[ \sum_{k=0}^{1-A_{ij}} C^{(k)}_{ij} (\text{ad } x_i)^k x_j = 0, \]  \hspace{1cm} (7)

where $C^{(k)}_{ij}$ are constant coefficients and can be computed from the Cartan matrix. This defines a presentation of $K(\mathfrak{e}_{10})$ in terms of generators

\[ \text{ad denotes the adjoint action: } (\text{ad } e_i) e_j = [e_i, e_j]. \]
and relations. For both $\mathfrak{e}_{10}$ and $K(\mathfrak{e}_{10})$ this type of presentation is the only known presentation. Whereas $\mathfrak{e}_{10}$ is a Kac–Moody algebra with well-defined structure theory [25], $K(\mathfrak{e}_{10})$ is not a Kac-Moody algebra [26] and its general representation theory is unknown. Nevertheless, the relations (7) are sufficient to establish the consistency of any tentative representation as we will see below. All Lie algebras we consider are over the real numbers, in particular $\mathfrak{e}_{10}$ is in split form.

2.2 Level decompositions for $D = 11$, IIA and IIB

A more economical and physical description of the generators of $\mathfrak{e}_{10}$ can be obtained via a so-called level decomposition [11, 14] where one represents

$$\mathfrak{e}_{10} = \sum_{\ell \in \mathbb{Z}} \mathfrak{e}_{10}^{(\ell)}$$

as a graded sum of (finite-dimensional reducible) representation spaces of a chosen regular subalgebra. The subalgebras of interest are obtained by removing nodes from the $E_{10}$ Dynkin diagram. The integer $\ell$ represents the level (if several nodes are removed it consists of a tuple of integers). Regular subalgebras of $\mathfrak{e}_{10}$ naturally give rise to subalgebras of $K(\mathfrak{e}_{10})$.

The subalgebras relevant for $D = 11$, type IIA and type IIB are displayed in table 2. From the table it is evident that $K(\mathfrak{e}_{10})$ admits subalgebras of the type required by condition (1) and that these satisfy the necessary embedding conditions. (Recall that the time coordinate is treated separately for $E_{10}$ whence we are only dealing with the spatial Lorentz groups here.)

We exemplify the result of the level decomposition for the $D = 11$ case, that is for the case of the depicted $sl(10)$ subalgebra of $\mathfrak{e}_{10}$. At level $\ell = 0$

| Theory | Dynkin diagram | Subalgebra of $\mathfrak{e}_{10}$ | ... of $K(\mathfrak{e}_{10})$ |
|--------|----------------|----------------------------------|-----------------------------|
| $D = 11$ | ![Dynkin diagram](image) | $sl(10)$ | $so(10)$ |
| $D = 10$ IIA | ![Dynkin diagram](image) | $sl_A(9)$ | $so_A(9)$ |
| $D = 10$ IIB | ![Dynkin diagram](image) | $sl_B(9) \oplus sl(2)$ | $so_B(9) \oplus so(2)$ |

Table 2: The subalgebras relevant for the various maximal supergravity theories. Empty nodes are to be deleted.
the reducible representation of \( \mathfrak{sl}(10) \) turns out to be \( \mathfrak{gl}(10) \) with generators \( K^a_b \). Moreover, all higher levels are representations of \( \mathfrak{gl}(10) \). Concretely,

\[
\begin{align*}
\ell = 0 & : K^a_b \\
\ell = 1 & : E^{abc} = E^{[abc]} \\
\ell = 2 & : E^{a_1 \ldots a_6} = E^{[a_1 \ldots a_6]} \\
\ell = 3 & : E^{a_0|a_1 \ldots a_8} = E^{a_0|[a_1 \ldots a_8]}, \ E^{[a_0|a_1 \ldots a_8]} = 0 \\
& \vdots \quad \vdots 
\end{align*}
\]  

(9)

Here, \((a, b = 1, \ldots, 10)\) are \( \mathfrak{sl}(10) \) vector indices and \( \ell = 1, 2, 3 \) are irreducible representations (accidentally). These tensors suggest a relation to the bosonic fields of \( D = 11 \) as follows: \( \ell = 0 \) is related to the spatial part \( e_m^a \) of the vielbein, \( \ell = 1 \) is related to the spatial components of the anti-symmetric three-form gauge potential, \( \ell = 2 \) is related to the Hodge dual of the three-form potential and \( \ell = 3 \) is related to the dual of the vielbein. That this is true in the one-dimensional \( E_{10}/K(E_{10}) \) \( \sigma \)-model was shown in [11].

Our interest here is in \( K(E_{10}) \) and therefore we have to form the invariant combinations of the generators in (9) to obtain

\[
\begin{align*}
\ell = 0 & : J^{ab} = K^a_b + \theta(K^a_b) = K^a_b - K^b_a = J^{[ab]} \\
\ell = 1 & : J^{abc} = E^{abc} + \theta(E^{abc}) \\
\ell = 2 & : J^{a_1 \ldots a_6} = E^{a_1 \ldots a_6} + \theta(E^{a_1 \ldots a_6}) \\
\ell = 3 & : J^{a_0|a_1 \ldots a_8} = E^{a_0|a_1 \ldots a_8} + \theta(E^{a_0|a_1 \ldots a_8}) \\
& \vdots \quad \vdots 
\end{align*}
\]

(10)

For \( K(\mathfrak{e}_{10}) \) the level \( \ell \) has to be taken with a grain of salt since it does no longer define a grading but only a filtered structure. Indeed, examples of \( K(\mathfrak{e}_{10}) \) commutation relations are [3]

\[
\begin{align*}
[J^{ab}, J^{cd}] &= \delta^{bc} J^{ad} - \delta^{bd} J^{ac} - \delta^{ac} J^{bd} + \delta^{ad} J^{bc}, \\
[J^{a_1 a_2 a_3}, J^{a_4 a_5 a_6}] &= J^{a_1 \ldots a_6} - 18 \delta^{[a_1 a_2} J^{a_3]}_{a_4 a_5} J^{a_6]}.
\end{align*}
\]

(11)

We see that the first line is the \( \mathfrak{so}(10) \) subalgebra of \( K(\mathfrak{e}_{10}) \) and the second line gives generators of ‘levels’ \( \ell = 2 \) and \( \ell = 0 \) on the right hand side in accordance with the filtered structure. The \( \mathfrak{so}(10) \) subalgebra introduces the invariant \( \delta^{ab} \) which can be used to raise and lower the tensor indices. There are infinitely many more relations than (11) involving all the other infinitely many generators and no closed form is known for them.
2.3 Representations of $K(\mathfrak{e}_{10})$

By virtue of the presentation of $K(\mathfrak{e}_{10})$ in terms of generators and relations in (6) and (7) it is sufficient to verify a finite number of relations on a tentative representation. Using the level decomposition one can further reduce this number by starting from a representation of the subalgebra (which obviously constitutes a necessary condition). Then the sufficient consistency conditions involve only levels $\ell = 0$ and $\ell = 1$ (basically since there a only single lines in the $E_{10}$ Dynkin diagram). For the $\mathfrak{so}(10)$ subalgebra of $K(\mathfrak{e}_{10})$ the recipe for constructing $K(\mathfrak{e}_{10})$ representations is:

1. Start from an $\mathfrak{so}(10)$ representation which we call the tentative $K(\mathfrak{e}_{10})$ representation. This defines the action of the $J^{ab}$ generators within $K(\mathfrak{e}_{10})$ on the tentative representation.

2. Make a general ansatz for the action of $J^{abc}$ on the tentative representation from $\mathfrak{so}(10)$ representation theory.

3. Verify that the second line of (11) holds for the case when some of indices are identical on the tentative representation. When some indices are identical the term with six anti-symmetric indices drops out. If there is a solution for the general ansatz then the tentative representation gives rise to a full consistent representation of $K(\mathfrak{e}_{10})$.

For the other subalgebras the procedures are similar. Since it involves the tensors arising in the corresponding level decompositions we do not detail them here in order to keep the exposition simple.

We now construct the gravitino (vector-spinor) representation of $K(\mathfrak{e}_{10})$ following the steps above. The vector-spinor of $\mathfrak{so}(10)$ is reducible of dimension 320 and consists of the irreducible pieces $288 \oplus 32$ corresponding to the $\Gamma$-traceless part and the $\Gamma$-trace. We denote the vector-spinor by $\psi_a$ and suppress the spinor index. The $\mathfrak{so}(10)$ generators $J^{ab}$ act on $\psi_a$ by:

\[ J^{ab} \psi_c = \frac{1}{2} \Gamma^{ab} \psi_c + 2 \delta^a_{[b} \psi^{b]}_c \]  
(12)

In the general ansatz for the $J^{abc}$ action there are three terms and the solution to the necessary commutation condition (11) leads to (13):

\[ J^{abc} \psi_d = \frac{1}{2} \Gamma^{abc} \psi_d + 4 \delta^a_{[a} \Gamma^{b]} \psi^{c]}_d - \Gamma^{d}_d [ab] \psi^c ] \]  
(13)

\(^4\)That this is sufficient follows from the precise expressions for the simple $K(\mathfrak{e}_{10})$ generators $x_i$ of (6) in terms of components of the $J^{ab}$ and $J^{abc}$ which can be found in [6].

\(^5\)Here, $\Gamma^a$ are the real $(32 \times 32)$ $SO(10)$ $\Gamma$-matrices and $\Gamma^{ab} = \Gamma^a \Gamma^b$ etc.
That there exists a solution to the consistency condition implies that there
is a representation of \( K(e_{10}) \) of dimension 320. One can check that this is in
fact an irreducible representation since the \( \Gamma \)-trace no longer separates once
\( J^{abc} \) is considered. \textit{We have thus proved that \( K(e_{10}) \) has an irreducible 320
representation.} Under the \( \mathfrak{so}(10) \) subalgebra it decomposes according to

\[
320 \rightarrow 288 \oplus 32
\]

\[
K(e_{10}) \supset \mathfrak{so}(10)
\]

as required. We denote this representation by \( \Psi \) since it can be defined
independently of the \( \mathfrak{so}(10) \) subalgebra under which it is more conveniently
written as \( \psi_a \).

We now turn to the decompositions under the subalgebras relevant for
type IIA and type IIB. They were derived in \cite{5} and we reproduce the results
here as:

\[
320 \rightarrow (128 \oplus 16) \oplus (128 \oplus 16) \oplus 16 \oplus 16
\]

\[
K(e_{10}) \supset \mathfrak{so}_A(9)
\]

for type IIA, where the last two 16s are the dilatini, and

\[
320 \rightarrow ((128, 2) \oplus (16, 2)) \oplus (16, 2)
\]

\[
K(e_{10}) \supset \mathfrak{so}_B(9) \oplus \mathfrak{so}(2)
\]

for type IIB. Here, the last doublet of 16s corresponds to the IIB dilatini.
Since we are only dealing with the spatial Lorentz group \( \mathfrak{so}(9) \) different chiralities
are not properly distinguished. The calculation shows, however, that
the two doublets of 16s arise differently and in the covariant calculation
one can show that indeed all chiralities also fulfil the necessary requirements
to answer the first question raised in the introduction affirmatively: The
group \( K(E_{10}) \) contains the subalgebras required by the M-theory picture
and has a spinorial representation with the correct number of components
which branches correctly to the fermionic fields of the maximal supergravity
theories.

\footnote{In \cite{5} the subalgebra \( \mathfrak{so}(9,9) \) was chosen for type IIA (instead of \( \mathfrak{so}_A(9) \)) since this
more naturally includes the mass term of the massive extension of IIA. That the result
given here is also correct follows immediately from \( \mathfrak{so}_A(9) \subset \mathfrak{so}(10) \) and the branching
rules for these groups.}
3 Dynamics

To further substantiate the significance of $K(E_{10})$ and its 320 representation $\Psi$ for an algebraic approach to M-theory we now turn to studying a dynamical equation for $\Psi$ and its relation to the fermionic dynamics in the various maximal supergravities.

Since time is treated separately in the $E_{10}$ context and all dynamical equations in the bosonic sector are time evolution equations a natural ansatz for the fermionic equation is

$$D_t \Psi = 0.$$  \hspace{1cm} (17)

This is a Dirac equation for the $K(e_{10})$ vector-spinor coupled minimally to a $K(e_{10})$ connection $Q_t$ via the covariant derivative

$$D_t = \partial_t - Q_t,$$  \hspace{1cm} (18)

where $Q_t \in K(e_{10})$ acts on $\Psi$ in the 320 representation. The gauge field $Q_t$ transforms under $t$-dependent local $K(E_{10})$ gauge transformations. As a $K(e_{10})$ element, $Q_t$ can be expanded over so(10) in the generators (10) via

$$Q_t = \frac{1}{2} Q^{(0)}_{ab} J^{ab} + \frac{1}{3!} Q^{(1)}_{abc} J^{abc} + \frac{1}{6!} Q^{(2)}_{a_1...a_6} J^{a_1...a_6}$$

$$+ \frac{1}{9!} Q^{(3)}_{a_0|a_1...a_8} J^{a_0|a_1...a_8} + \ldots.$$  \hspace{1cm} (19)

Since the action of all the $K(e_{10})$ generators can be computed from multiple commutators of (12) and (13) the Dirac equation (17) can be evaluated to arbitrary level. In [3] it was evaluated up to $\mathfrak{sl}(10)$ level three which is the level to which the field content (9) is understood [11]. The resulting expression contains the gauge field components $Q^{(\ell)}$ (for $\ell = 0, \ldots, 3$) contracted with various $\Gamma$-matrices multiplying the so(10) decomposed vector-spinor $\Psi = (\psi_a)$. Explicitly, we find

$$D_t \psi_c = \partial_t \psi_c - \frac{1}{4} Q^{(0)}_{ab} \Gamma^{ab} \psi_c - Q^{(0)}_{ca} \psi^a - \frac{1}{12} Q^{(1)}_{a_1a_2a_3} \Gamma^{a_1a_2a_3} \psi_c$$

$$- \frac{2}{3} Q^{(1)}_{c_{a_1a_2}} \Gamma^{a_1} \psi^{a_2} + \frac{1}{6} Q^{(1)}_{a_1a_2a_3} \Gamma_a \psi^{a_2} \psi^{a_3} - \frac{1}{1440} Q^{(2)}_{a_1...a_6} \Gamma^{a_1...a_6} \psi_c$$

$$+ \frac{1}{72} Q^{(2)}_{c_{a_1...a_5}} \Gamma^{a_1...a_4} \psi^{a_5} - \frac{1}{180} Q^{(2)}_{c_{a_1...a_6}} \Gamma^{a_1...a_5} \psi^{a_6}$$

$$- \frac{2}{3 \cdot 8!} Q^{(3)}_{a_0|a_1...a_8} \Gamma^{a_1...a_8} \psi^{a_0}$$

$$- \frac{2}{3 \cdot 7!} Q^{(3)}_{c|a_1...a_8} \Gamma^{a_1...a_8} \psi^{a_9} - \frac{4}{3 \cdot 8!} Q^{(3)}_{b|b_{a_1...a_7}} \Gamma^{a_1...a_7} \psi^{b_a}$$

$$- \frac{1}{3 \cdot 6!} Q^{(3)}_{b|b_{a_1...a_7}} \Gamma^{a_1...a_6} \psi^{a_7} + \ldots.$$  \hspace{1cm} (20)

\hspace{1cm}
This equation has to be compared with the dynamical equation for the gravitino in $D = 11$ supergravity. From the analysis of the bosonic sector it is to be expected that gauge-fixing is required in order to establish a connection between the $K(e_{10})$ equation \((20)\) and the supergravity equation \([20]\). Indeed it turns out \([3]\) that one has to fix a supersymmetry gauge \((\psi_0 - \Gamma_0 \Gamma^a \psi_a = 0)\) for the fermions, a pseudo-Gaussian gauge \((E_1 \, ^a = 0)\) for the vielbein and a Coulomb gauge \((A_{00} = 0)\) for the gauge potential. In this case the supergravity equation (to lowest fermion order) takes almost the same form as \([20]\) but where the gauge field components take the values \([3, 6]\):

\[
Q_{ab}^{(0)}(t) = -N \omega_{0ab}(t, x_0), \quad Q_{a_1 \ldots a_q}^{(2)}(t) = -\frac{1}{4!} N \epsilon_{a_1 \ldots a_q b_1 \ldots b_q} F_{b_1 \ldots b_q}(t, x_0), \\
Q_{abc}^{(1)}(t) = N F_{0abc}(t, x_0), \quad Q_{a_0 a_1 \ldots a_q}^{(3)}(t) = \frac{3}{2} N \epsilon_{a_1 \ldots a_q b_1 b_2} \omega_{b_1 b_2 a_0}(t, x_0).
\]

Here, $\omega_{0ab}$ and $\omega_{abc}$ are ‘electric’ and ‘magnetic’ components of the spin connection in flat indices; similarly $F_{0abc}$ and $F_{b_1 \ldots b_q}$ are electric and magnetic components of the four-form field strength in flat indices. The lapse $N = E_t \, ^0$ is needed to convert the objects on the right hand sides into components of a world-line tensor $Q_t$.

The equations \((21)\) are valid only at a fixed spatial point $x_0$ and in order to match \((20)\) to the supergravity equation higher spatial gradients of the fields (and the lapse) have to be ignored. Furthermore, the spatial spin connection must have vanishing trace $\omega_{bba} = 0$ at $x_0$. More details can be found in \([3, 6]\).

To summarise, with the use of the ‘dictionary’ \((21)\) we have succeeded in turning a truncated version of the $D = 11$ gravitino equation of motion into a $K(E_{10})$ covariant Dirac-equation of the type \((17)\). Although not explicitly proved in the type IIB case, one can expect that the very same equation \((17)\) also describes the correct fermionic dynamics of type IIA and IIB by using the decompositions of $K(e_{10})$ detailed in \((15)\) and \((16)\).

Thus we arrive at the main result: $K(E_{10})$ is not only a viable candidate for a kinematical unification of the fermionic symmetries of \textit{all} maximal supergravity theories but also can partially be established as a symmetry of the dynamical equations for the fermions.
4 Discussion

4.1 Remarks

Here, we only briefly sketch some related points and comment on a fully covariant reformulation of the above results.

There exists also a 32 representation of \( K(e_{10}) \) which was called ‘Dirac-spinor’ in [27, 3]. This representation is relevant for the supersymmetry parameter \( \epsilon \) and similarly has the correct branching to the various maximal supergravity theories’ Lorentz and R-symmetries [5].

Both the 320 and the 32 representations of \( K(e_{10}) \) are unfaithful since they are finite-dimensional representations of an infinite-dimensional algebra. This implies that \( K(e_{10}) \) is not a simple Lie algebra but has non-trivial quotients. These one arrives at by factoring out the ideals associated with the unfaithful representations [6]. In the case of the 32 Dirac-spinor the quotient is \( so(32) \) which has been conjectured as a generalized holonomy in [28, 29]. Since \( K(E_{10}) \) acts not only on the Dirac-spinor but also on the 320 gravitino (which \( SO(32) \) does not) it is more general than these conjectured holonomies. Furthermore, certain global issues [30] are resolved in \( K(E_{10}) \) [3].

As mentioned in the introduction, the M-theoretic properties of \( K(E_{10}) \) were derived following similar results in the bosonic sector [8, 9, 10, 11, 19, 15]. The bosonic fields are realised via a coset construction \( E_{10}/K(E_{10}) \) where \( K(E_{10}) \) also acts as a local gauge symmetry. It is non-trivial, but true, that the relation between the gauge connection appearing in the bosonic analysis and the one in the fermionic analysis are related in precisely the same way to the supergravity quantities via (21).

It can also be shown that \( K(E_{11}) \) (if equipped with the temporal involution of [20]) allows for a fermionic representation of dimension 352 = 320 + 32 if written over \( SO(1,10) \) [5]. The IIA and IIB decompositions of this fully covariant gravitino give the correct achiral and chiral fermionic spectra in a covariant fashion so that all the results of section 2 carry over to \( K(E_{11}) \). However, it is not clear how to write a \( K(E_{11}) \) covariant and space-time covariant dynamical equation for this gravitino which generalises the Dirac equation (17). An obvious candidate is

\[
P \Psi = 0,
\]

where \( P = \Gamma^{M}D_{M} = \Gamma^{M}(\partial_{M} - Q_{M}) \). There are a number of subtleties with this suggestive notation that need to be clarified. Firstly, \( D_{M} \) should be
$K(E_{11})$ covariant meaning that the gauge fields transform correctly under $K(E_{11})$. By augmenting an $E_{11}/K(E_{11})$ coset construction by a Borisov–Ogievetsky type construction as in [8] this can probably be achieved. The second problematic point is the symbol $\Gamma^M$ used above since $\Gamma^M$ is not an $K(E_{11})$ invariant tensor and so spoils the $K(E_{11})$ covariance of the equation even if $D_M$ transforms correctly. In line with the philosophy of [18] one should probably replace eleven-dimensional space-time indices $M$ by indices taking values in an infinite-dimensional highest weight representation of $E_{11}$ generalising the translation vector to an $E_{11}$ object. It remains to be seen whether one can make sense of (22) in this framework.

4.2 Outlook

From the discussion in the introduction it is clear that in order to complete the M-theory picture a number of things need to be included in the present algebraic framework, the most pressing of which we briefly discuss now.

Firstly, M-theory should also include the non-maximal heterotic $E_8 \times E_8$ and $SO(32)$ string theories as well as the $SO(32)$ type I superstring. At low energies this requires fitting the heterotic $D = 10$ supergravity with gauge groups $SO(32)$ and $E_8 \times E_8$ into the $E_{10}$ $\sigma$-model or some more general model. As a first step it was shown in [19, 31] that the pure type I supergravity (without any vector multiplets) can be interpreted as a subsector of the $E_{10}$ model. It would be gratifying to see a relation between the algebraic approach taken here and the issue of anomaly freedom.

Secondly, M-theory presumably is a theory of strings and other extended objects. The analysis so far only covered point particles since properties of the low energy field theories were studied. It is not clear if the symmetries found need to be modified when extended objects are also considered. Results from U-duality [2, 32] suggest that the continuous symmetry gets broken to some discrete arithmetic group and first ideas in this direction were discussed in [34]. A different route was taken in [33, 35] where string induced higher derivative corrections to the low energy effective action were studied in relation to $E_{10}$ and good agreement between the algebraic structure and conjectured properties of these correction terms was found (see also [36]).

Thirdly, the bosonic fields appear through the infinite-dimensional coset space $E_{10}/K(E_{10})$ whereas the fermionic fields presently are confined to a finite-dimensional, unfaithful representation of $K(E_{10})$. This seems problematic from a supersymmetry point of view. This dichotomy is partly related to the difference in order of the equations of motion for bosons and fermions.
The fermionic field equations are first order whereas the bosonic ones are second order (allowing for dualisations and triggering for example the infinite duality cascade in $D = 2$). It would be nice to overcome this obstacle through the construction of an appropriate faithful fermionic representation of $K(E_{10})$.

Finally, on the purely mathematical side it could be hoped that a proper understanding of the relation between $E_{10}$ and M-theory may lead to a new presentation of the $E_{10}$ structure itself. Since its inception in the late 1960s [37, 38] the theory of indefinite Kac–Moody algebras has produced few results which truly penetrate the structure of these fascinating objects.

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