Chiral Symmetry Breaking in Strongly Coupled Quenched QED\textsubscript{4} Using the Dyson-Schwinger Equation Formalism

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We study chiral symmetry breaking in quenched strong-coupling QED\textsubscript{4} in arbitrary covariant gauge within the Dyson-Schwinger equation formalism. A recently developed numerical renormalization program is fully implemented. Results are compared for three different fermion-photon proper vertex Ansätze: bare $\gamma^\mu$, minimal Ball-Chiu, and Curtis-Pennington. The procedure is straightforward to implement and numerically stable. We discuss the chiral limit and observe that in this limit the renormalized axial current is conserved. A detailed study of residual gauge dependence due to the vertex choice is in progress. The relevance for lattice studies is discussed.

1. INTRODUCTION

Strong coupling QED in three space and one time dimension has been studied within the Dyson-Schwinger Equation (DSE) formalism for some time \[1\]. For a recent review of Dyson-Schwinger equations and their application see for example Ref. \[2\]. DCSB occurs when the fermion propagator develops a nonzero scalar self-energy in the absence of an explicit chiral symmetry breaking (ECSB) fermion mass. We refer to coupling constants strong enough to induce DCSB as supercritical and those weaker are called subcritical. We write the fermion propagator as

\begin{equation}
S(p) = \frac{Z(p^2)}{p^2 - M(p^2)} = \frac{1}{A(p^2)} \frac{1}{p^2 - B(p^2)}
\end{equation}

with $Z(p^2)$ the finite momentum-dependent fermion renormalization, and $B(p^2)$ the scalar self-energy. In the absence of an ECSB bare electron mass, by definition DCSB occurs when $B(p^2) \neq 0$. Note that $A(p^2) \equiv 1/Z(p^2)$ and $M(p^2) \equiv B(p^2)/A(p^2)$.

Many studies, even until quite recently, have used the bare vertex as an Ansatz for the one-particle irreducible (1-PI) vertex $\Gamma^\nu(k, p)$ despite the fact that this violates the Ward-Takahashi Identity (WTI) \[3\]. With any of these Ansätze the resulting fermion propagator is not gauge-covariant, i.e., physical quantities such as the critical coupling for dynamical symmetry breaking, or the mass itself, are gauge-dependent \[3\]. A general form for $\Gamma^\nu(k, p)$ which does satisfy the Ward Identity was given by Ball and Chiu in 1980 \[4\]; it consists of a minimal longitudinally constrained term which satisfies the WTI, and a set of tensors spanning the subspace transverse to the photon momentum $q$. Although the WTI is necessary for gauge-invariance, it is not a sufficient condition; further, with many of these vertex Ansätze the fermion propagator DSE is not multiplicatively renormalizable. There has been much recent research on the use of the transverse parts of the vertex to ensure both gauge-covariant and multiplicatively renormalizable solutions \[5\], some of which will be discussed below.

What was common to essentially all of the previous studies is that the fermion propagator is not in practice subtractively renormalized. Most of these studies have assumed an initially massless theory and have renormalized at the ultraviolet cutoff of the integrations, taking $Z_1 = Z_2 = 1$. We describe here some results \[6\] of a study of subtractive renormalization in the fermion DSE, in quenched strong-coupling QED\textsubscript{4}. (In the con-
text of this study of QED, the term “quenched” means that the bare photon propagator is used in the fermion self-energy DSE, so that $Z_2 = 1$. Virtual fermion loops may still be present, however, within the vertex corrections.) Results are obtained for DSE with three different vertices: the bare $\gamma^\mu$, the minimal Ball-Chiu vertex form [3], and the Curtis-Pennington vertex [8,9,10,11].

2. DSE and Vertex Ansätze

The DSE for the renormalized fermion propagator, in a general covariant gauge, is

$$S^{-1}(p^2) = Z_2(\mu, \Lambda)\frac{\not{q} - m_0(\Lambda)}{iZ_1(\mu, \Lambda)e^2} - iZ_1(\mu, \Lambda)e^2 \times \int^\Lambda \frac{d^4k}{(2\pi)^4}\gamma^\mu S(k)\Gamma^\nu(k, p)D_{\mu\nu}(q);$$

(2)

where $q = k - p$ is the photon momentum, $\mu$ is the renormalization point, and $\Lambda$ is a regularizing parameter (taken here to be an ultraviolet momentum cutoff). We write $m_0(\Lambda)$ for the regularization-parameter-dependent bare mass. The physical charge is $e$ (as opposed to the bare charge $e_0$), and the general form for the photon propagator is

$$D^{\mu\nu}(q) = \left\{ \frac{\not{q}g^{\mu\nu}}{q^2} - \frac{1}{1 - \Pi(q^2)} - \xi \frac{\not{q}q^\nu}{q^2} \right\}$$

with $\xi$ the covariant gauge parameter. Since we will work in the quenched approximation and the Landau gauge we have $e^2 \equiv e_0^2 = 4\pi\alpha_0$ and

$$D^{\mu\nu}(q) \rightarrow D_0^{\mu\nu}(q) = \left\{ \frac{\not{q}g^{\mu\nu}}{q^2} \right\} \frac{1}{q^2},$$

for the photon propagator.

The requirement of gauge invariance in QED leads to the Ward-Takahashi Identities (WTI); the WTI for the fermion-photon vertex is $q_\mu\Gamma^\mu(k, p) = S^{-1}(k) - S^{-1}(p)$, where $q = k - p$. This is a generalization of the original differential Ward identity, which expresses the effect of inserting a zero-momentum photon vertex into the fermion propagator, $\partial S^{-1}(p)/\partial p_\nu = \Gamma^\nu(p, p)$. In particular, it guarantees the equality of the propagator and vertex renormalization constants, $Z_2 = Z_1$. The Ward-Takahashi Identity is easily shown to be satisfied order-by-order in perturbation theory and can also be derived nonperturbatively.

As discussed in [3], this can be thought of as just one of a set of six general requirements on the vertex: (i) the vertex must satisfy the WTI; (ii) it should contain no kinematic singularities; (iii) it should transform under charge conjugation ($C$), parity inversion ($P$), and time reversal ($T$) in the same way as the bare vertex, e.g., $C^{-1}\Gamma_\nu(k, p)C = -\Gamma_\nu^T(-p, -k)$ (where the superscript $T$ indicates the transpose); (iv) it should reduce to the bare vertex in the weak-coupling limit; (v) it should ensure multiplicative renormalizability of the DSE in Eq. (2); (vi) the transverse part of the vertex should be specified to ensure gauge-covariance of the DSE.

Ball and Chiu [3] have given a description of the most general fermion-photon vertex that satisfies the WTI; it consists of a longitudinally-constrained (i.e., “Ball-Chiu”) part $\Gamma^\mu_{BC}$, which is a minimal solution of the WTI, and a basis set of eight transverse vectors $T^\mu_i(k, p)$, which span the hyperplane specified by $q_\mu T^\mu_i(k, p) = 0, q \equiv k - p$. The minimal longitudinally constrained part of the vertex is given by

$$\Gamma^\mu_{BC}(k, p) = \frac{1}{2}[A(k^2) + A(p^2)]\gamma^\mu + \frac{(k + p)^\mu}{k^2 - p^2} \times$$

$$\left\{ [A(k^2) - A(p^2)] \frac{k + p}{2} - [B(k^2) - B(p^2)] \right\}. \quad (3)$$

The transverse vectors can be found for example in Ref. [14]. A general vertex is then written as

$$\Gamma^\mu(k, p) = \Gamma^\mu_{BC}(k, p) + \sum_{i=1}^{8} \tau_i T^\mu_i(k, p), \quad (4)$$

where the $\tau_i(k^2, p^2, q^2)$ are functions which must be chosen to give the correct $C$, $P$, and $T$ invariance properties. Curtis and Pennington [8,9,10,11] eliminate four of the transverse vectors except $T^\mu_8$, with a dynamic coefficient chosen to make the DSE multiplicatively renormalizable. This coefficient has the form $\tau_8(k^2, p^2, q^2) = (1/2)[A(k^2) - A(p^2)]/d(k, p)$, where $d(k, p)$ is a symmetric, singularity-free function of $k$ and $p$, with the limiting behavior $\lim_{k^2, p^2 \rightarrow p^2} d(k, p) = k^2$. 


[Here, $A(p^2) = 1/Z(p^2)$ is their $1/F(p^2)$] For purely massless QED, they find a suitable form, $d(k, p) = (k^2 - p^2)^2/(k^2 + p^2)$. This is generalized to the case with a dynamical mass $M(p^2)$, to give $d(k, p) = [(k^2 - p^2)^2 + [M^2(k^2) + M^2(p^2)]^2]/(k^2 + p^2)$.

3. The Subtractive Renormalization

As discussed in Ref. [4] one first determines a finite, regularized self-energy, which depends on both a regularization parameter and the renormalization point; then one performs a subtraction at the renormalization point, in order to define the renormalization parameters $Z_1$, $Z_2$, $Z_3$. The DSE for the renormalized fermion propagator, $\tilde{S}^{-1}(p) = Z_2(\mu, \Lambda)[\not{p} - m_0(\Lambda)] - \Sigma'(\mu, \Lambda; p)$

$$= \not{p} - m(\mu) - \Sigma(\mu; p), \quad (5)$$

where the (regularized) self-energy is

$$\Sigma'(\mu, \Lambda; p) = iZ_1e^2 \int^\Lambda d^4k/(2\pi)^4 \gamma^\nu \tilde{\Sigma}^{\nu\lambda} \tilde{D}_{\lambda\nu}. \quad (6)$$

To avoid confusion we will follow Ref. [4] and in this section only we will denote regularized quantities with a prime and renormalized ones with a tilde, e.g. $\Sigma'(\mu, \Lambda; p)$ is the regularized self-energy depending on both the renormalization point $\mu$ and regularization parameter $\Lambda$ and $\Sigma(\mu; p)$ is the renormalized self-energy.] The self-energies are decomposed into Dirac and scalar parts, $\Sigma(\mu; p) = \Sigma_d(\mu, \Lambda; p^2)\not{p} + \Sigma_s(\mu, \Lambda; p^2)$ (and similarly for the renormalized quantity, $\tilde{\Sigma}(\mu; p)$).

By imposing the renormalization boundary condition, $\tilde{S}^{-1}(p)|_{p^2=\mu^2} = \not{p} - m(\mu)$, one gets the relations $\tilde{\Sigma}_{d,s}(\mu; p^2) = \Sigma_{d,s}'(\mu, \Lambda; p^2) - \Sigma_{d,s}'(\mu, \Lambda; \mu^2)$ for the self-energy, $Z_2(\mu, \Lambda) = 1 + \Sigma_d'(\mu, \Lambda; \mu^2)$ for the renormalization, and $m_0(\Lambda) = [m(\mu) - \Sigma_s'(\mu, \Lambda; \mu^2)]/Z_2(\mu, \Lambda)$ for the bare mass. The chiral limit is approached by taking $m(\mu) = \Sigma_s'(\mu, \Lambda; \mu^2) = 0$ for fixed $\mu$. This is the limit in which the renormalized axial vector (non-flavor singlet) current is conserved. As $\Lambda \to \infty$ the numerical results are consistent with the bare mass also vanishing as one would expect. The mass renormalization constant is given by $Z_m(\mu, \Lambda) = m_0(\Lambda)/m(\mu)$. The vertex renormalization, $Z_1(\mu, \Lambda)$ is identical to $Z_2(\mu, \Lambda)$ as long as the vertex Ansatz satisfies the Ward Identity; this is how it is recovered for multiplication into $\Sigma'(\mu, \Lambda; p)$ in Eq. (5). Since for the bare vertex case there is no way to determine $Z_1(\mu, \Lambda)$ independently we will also in this case use $Z_1 = Z_2$ for the sake of comparison.

4. Results

Solutions were obtained for the DSE with the Curtis-Pennington and bare vertices, for couplings $\alpha_0$ from 0.1 to 1.75; solutions were also obtained for the minimal Ball-Chiu vertex, with couplings $\alpha_0$ from 0.1 to 0.6 (for larger couplings the DSE with this vertex was susceptible to numerical noise). For the solutions in the subcritical range, the renormalization point $\mu^2 = 100$, and renormalized masses were either $m(\mu) = 10$ or 30. Ultraviolet cutoffs were $1.0 \times 10^{12}$ and $1.0 \times 10^{18}$. The family of solutions for Landau gauge and the Curtis-Pennington vertex with $m(\mu) = 10$ is shown in Fig. 1 a) shows the finite renormalization $A(p^2)$; note that for the solutions with bare $\gamma^\nu$, we would have $A(p^2) \equiv 1$ for all values of the coupling. Fig. 1 b) shows the masses $M(p^2)$. Since the equations have no inherent mass-scale, the cutoff $\Lambda$, renormalization point $\mu$, $m(\mu)$, and units of $M(p^2)$ or $B(p^2)$ all scale multiplicatively, and the units are arbitrary. The results for all vertices are qualitatively similar. For the Landau gauge solutions we find that as the value of $\mu$ increases, $Z_2$ approaches 1, and also that $Z_2$ is almost independent of the cutoff $\Lambda$, at least for $\Lambda \gg \mu$. The solutions are extremely stable as the cutoff is varied.

In all cases the mass and finite renormalization were stable with respect to very large variations in cutoff. The renormalization constants $Z_2(\mu, \Lambda) = Z_1(\mu, \Lambda)$ remain finite and well-behaved with increasing $\Lambda$ in contrast to what happens in perturbation theory.

5. Summary and Conclusions

A full discussion of the Landau gauge results can be found in Ref. [14]. Results have also been
recently obtained in other covariant gauges and this work is currently being prepared for publication. A careful comparison with lattice formulations of gauge-fixed QED can now be attempted, since we are free to choose the renormalization point in this work as the zero-momentum point for the n-point Green’s functions just as is conventionally done in lattice gauge theory studies. Such comparative studies should yield significant benefits.

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