Graph Domination-Saturation

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Abstract. Graph $G$ is $F$-saturated if $G$ contains no copy of graph $F$ but any edge added to $G$ produces at least one copy of $F$. One common variant of saturation is to remove the former restriction: $G$ is $F$-semi-saturated if any edge added to $G$ produces at least one new copy of $F$. In this paper we take this idea one step further. Rather than just allowing edges of $G$ to be in a copy of $F$, we require it: $G$ is $F$-dominated if every edge of $G$ is in a copy of $F$. It turns out that there is smooth interaction between domination and semi-saturation, which opens for investigation a natural analogue to saturation numbers. Therefore we present preliminary domination-saturation theory and structural bounds for the domination-saturation numbers of graphs. We also establish asymptotic domination-saturation densities for cliques and paths, and upper and lower bounds (with small gaps) for cycles and stars.

1. Introduction to Saturation

We begin with a brief introduction to graph saturation before defining the concept of domination and its corresponding saturation variant. Section 2 establishes preliminary theory for domination-saturation, then we prove several structural bounds on domination-saturation numbers in Section 3. Section 4, the final main part of this paper, investigates domination-saturation numbers for specific classes of graphs, including paths, cycles, and stars. We end with a discussion of several open directions for further study. All graphs are assumed to be simple, finite, and undirected.

Definition 1.1. Graph $G$ is $F$-free provided there is no subgraph of $G$ isomorphic to graph $F$. We say $G$ is $F$-saturated if $G$ is $F$-free and $G + e$ is not $F$-free for any edge $e$ in the complement of $G$.

The saturation number of $F$, sat($n$, $F$), is the fewest number of edges in an $F$-saturated graph on $n$ vertices.

The first result in graph saturation was given by Alexander Zykov [10] in Russian in 1949 and independently by Erdős, Hajnal, and Moon [3] in English in 1964. They found the saturation number of a clique:

$$\text{sat}(n, K_r) = (r - 2)n - \binom{r-1}{2}.$$

1.1. Pseudo-Saturation. Perhaps the first variation of saturation to be studied was weak saturation, introduced in 1967 by Béla Bollobás [11]. It is closely related to bootstrap percolation.

Definition 1.2. Graph $G$ is weakly $F$-saturated provided the edges of the complement of $G$ can be ordered $e_1, e_2, \ldots, e_\ell$ so that when we add the edges to $G$ one at a time, $G_0 = G$ and $G_i = G_{i-1} + e_i$ for $1 \leq i \leq \ell$, then the number of copies of graph $F$ in $G_{i+1}$ is strictly greater than the number of copies of $F$ in $G_i$ for all $i < \ell$.

The weak saturation number of $F$, wsat($n$, $F$), is the fewest number of edges in a weakly $F$-saturated graph on $n$ vertices.
The most relevant variant of saturation to the present work is what was originally called strong saturation. However, some authors (e.g., [9] and [10]) have used the phrase “strong saturation” in reference to the usual saturation simply to contrast it with weak saturation. Thus, to avoid ambiguity, we follow the example of Füredi and Kim [6] and use “semi-saturation” in place of “strong saturation.”

Definition 1.3. Graph $G$ is $F$-semi-saturated if for any edge $e$ in the complement of $G$ the graph $G + e$ contains more copies of graph $F$ than are in $G$.

The semi-saturation number of $F$, $\beta(n, F)$, is the fewest number of edges in an $F$-semi-saturated graph on $n$ vertices.

With this definition, we can restate the definition of $F$-saturation simply as: $F$-free and $F$-semi-saturated.

Example 1.4. (a) The Turán graph, or balanced complete $r$-partite graph, $T(n, r)$ is $K_{r+1}$-saturated for every $n \geq r$.

(b) Every graph $F$ is weakly $K_2$-saturated.

(c) The clique $K_n$ is vacuously $F$-semi-saturated for every $F$.

See the 2011 survey by Faudree, Faudree, and Schmitt [5] for a more comprehensive coverage of known results and open problems about saturation and several variations thereof.

1.2. Anti-Saturation. It is also worth mentioning Turán theory, to which saturation is considered a dual or opposite theory. Whereas $\text{sat}(n, F)$ is the minimum number of edges in an $n$-vertex $F$-saturated graph, the extremal number $\text{ex}(n, F)$ is the maximum number of edges in an $n$-vertex $F$-saturated graph. In 1941, Pál Turán [13] proved (with different notation) that $\lim_{n \to \infty} \text{ex}(n, K_r)/(\binom{n}{2}) = \frac{1}{r-1}$. This was generalized for all graphs in 1946 by Erdős and Stone [4], who proved that $\lim_{n \to \infty} \text{ex}(n, F)/(\binom{n}{2}) = \frac{\chi(F) - 2}{\chi(F) - 1}$, where $\chi(F)$ is the chromatic number of $F$.

So the extremal number of a (non-bipartite) graph is on the order of $n^2$, but sat, wsat, and $\beta$ are on the order of $n$ (or bounded) for every graphs. As stated by Zsolt Tuza [15] in 1992: “in contrast with the Turán numbers (in which the chromatic number as a “global parameter” is essential), the growths of [saturation numbers] depend on some local parameters[...].”

1.3. Asymptotics. Since saturation numbers of $F$ grow no faster than some constant multiple of $n$, it is natural to divide by $n$ and take the limit, but that limit is not known to always exist. Zsolt Tuza [14] conjectured in 1988 that $\lim_{n \to \infty} \frac{\text{sat}(n, F)}{n}$ exists for all $n$. For convenience, we will use $\text{sat}(F)$ when this limit exists and $\text{sat}(F)$ and $\text{osat}(F)$ for the limit infimum and limit supremum, respectively. In 1991, Truszczyński and Tuza [12] made the following progress toward the latter’s conjecture: If $\text{sat}(F) < 1$, then $\text{sat}(F) = 1 - \frac{1}{p}$ for some positive integer $p$. They also gave a characterization of all such graphs.

2. Domination and Saturation

The idea of graph semi-saturation was to lift one of the restrictions imposed by saturation: The edges of an $F$-semi-saturated graph are allowed to be in a copy of $F$. Here we consider a concept we call domination, where edges are not only allowed but required to be in a copy of $F$. This leads to a theory of domination-saturation and an analogous saturation number.

Definition 2.1. Graph $G$ is $F$-dominated provided every edge of $G$ is in a subgraph of $G$ isomorphic to graph $F$.

Example 2.2. The clique $K_n$ is $F$-dominated for any graph $F$ with at least one edge and at most $n$ vertices.

Definition 2.3. Graph $G$ is $F$-dom-sat provided $G$ is both $F$-dominated and $F$-semi-saturated.
Example 2.4. Every graph with at least one edge is $K_2$-dom-sat.

Henceforth, to avoid trivial counterexamples, we assume graphs have at least one edge. Before we introduce the dom-sat analogue of saturation numbers of graphs, let us make a few observations about domination and saturation.

Fact 2.5. (a) Domination is a transitive graph relation:

If $G$ is $F$-dominated and $H$ is $G$-dominated, then $H$ is $F$-dominated.
(b) If $G$ is $F$-dominated and $H$ is $G$-semi-saturated then $H$ is $F$-semi-saturated.
(c) From (a, b): If $G$ is $F$-dominated and $H$ is $G$-dom-sat, then $H$ is $F$-dom-sat.
(d) From (c), domination-saturation is a transitive graph relation:

If $G$ is $F$-dom-sat and $H$ is $G$-dom-sat, then $H$ is $F$-dom-sat.

Fact 2.6. Let $\delta(G)$ denote the minimum degree of graph $G$.

(a) If $G$ is $F$-dominated and $\delta(G) \geq 1$, then $\delta(G) \geq \delta(F)$.
(b) If $G$ is $F$-semi-saturated, then $\delta(G) \geq \delta(F) - 1$.

Semi-saturation and domination also relate naturally to connectivity.

Lemma 2.7. If every connected component of $F$ is $k$-connected and $G$ is $F$-semi-saturated, then $G$ is $(k-1)$-connected.

Proof. For any non-adjacent vertices $x$ and $y$ in $G$, $G + xy$ has a copy of $F$ that uses $xy$. Since each component of $F$ is $k$-connected, by Menger’s theorem, there are $k$ vertex-disjoint $x$-$y$-paths in $F$, and thus too in $G + xy$. Hence we have $k-1$ vertex-disjoint $x$-$y$-paths in $G$. Since this is true for arbitrary non-adjacent $x$ and $y$ in $G$, $G$ is $(k-1)$-connected.

Lemma 2.8. If every connected component of $F$ is $k$-edge-connected and $G$ is $F$-dom-sat, then $G$ is $(k-1)$-edge-connected.

Proof. Adjacent vertices in $G$ are connected by $k$ edge-disjoint paths in $G$ because $G$ is $F$-dominated. Non-adjacent vertices in $G$ are dealt with using the fact that $G$ is $F$-semi-saturated, in the fashion of the previous proof.

The natural interaction between domination and semi-saturation demonstrated in Fact 2.5(b) and Lemma 2.8 is part of our motivation for introducing domination into the rich area of graph saturation. We quantify how small of graphs can be $F$-dom-sat, the analogue to sat$(n, F)$, with the following dom-sat values.

Definition 2.9. For $|F| \leq n$, the dom-sat number of $F$ is

$$dsat(n, F) = \min(|V| : |V| = n, G = (V, E) \text{ is } F\text{-dom-sat});$$

and the (asymptotic) dom-sat density of $F$ is

$$dsat(F) = \lim_{n \to \infty} \frac{dsat(n, F)}{n}$$

when the limit exists. We use $dsat(F)$ and $\omega dsat(F)$ for the limit infimum and limit supremum, respectively.

We will refer to graphs that realize that dom-sat number of $F$ as extremal for $F$. Graphs in a family that realizes the dom-sat density of $F$ are asymptotically extremal for $F$, though the individual graphs may not be extremal for $F$.

A few basic bounds on the dom-sat values follow from the above definitions and observations.

Theorem 2.10. If every connected component of $F$ has at least two edges, then $dsat(F) \geq \delta(F)/2$.

Proof. This follows immediately from Fact 2.5 in the case that $\delta(F) \geq 2$. If $\delta(F) = 1$, observe that any $F$-dom-sat graph has at most one isolated vertex, since $F$ has no isolated edges.
Fact 2.11. If $G$ is $F$-dominated then $\text{dsat}(n, F) \leq \text{dsat}(n, G)$ for all $n \geq |G|$.

Proof. Fact 2.5(c)

3. Upper Bounds on $\text{dsat}(F)$

From Fact 2.11 and Examples 1.4(c) and 2.2, we have that $\text{dsat}(F) \leq \text{dsat}(K_r)$ for all graphs $F$ with at most $r$ vertices. Whereas the Zykov-Erdős-Hajnal-Moon theorem gives us that $\text{sat}(K_r) = r - 2$, we find a slightly larger value for $\text{dsat}(K_r)$, which gives us a general upper bound for dom-sat density.

Theorem 3.1. For the clique $K_r$ on $r \geq 3$ vertices,

$$\text{dsat}(K_r) = r - \frac{3}{2}.$$ 

Proof. For two graphs $G$ and $H$, let $G + H$ be the graph formed by taking a disjoint union $G \cup H$ and adding an edge $gh$ for every $(g, h) \in V(G) \times V(H)$. Set $M_{2k} = kK_2$ (the disjoint union of $k$ edges) and $M_{2k+1} = K_3 \cup (k-1)K_2$. Let $D(n, r) = K_{r-2} + M_{n-(r-2)}$, pictured in Figure 1 (This is an homage to the Turán graph, with “D” for “Domination” in place of “T”). First observe that $D(n, r)$ is $K_r$-dominated and $K_r$-semi-saturated. Therefore, the number of edges in $D(n, r)$ gives an upper bound on the dom-sat number of $K_r$:

$$\text{dsat}(K_r, n) \leq \frac{r - 2}{2} + (r - 2)(n - r + 2) + \left\lceil \frac{n - r + 2}{2} \right\rceil + (n - r)\%2,$$

where $a \% b$ is the remainder when $a$ is divided by $b$. Taking the limit of this bound divided by $n$, we have $\text{dsat}(K_r) \leq (r - 2) + 1/2$.

![Figure 1. Asymptotically extremal graphs $D(n, r)$ for $K_r$, with $n \equiv r \pmod{2}$ on the left and $n \equiv r + 1 \pmod{2}$ on the right.](image)

To show this is best possible, we will get a lower-bound count on the number of edges in two different ways. Let $\delta = \delta(G)$ be the minimum degree of a $K_r$-dom-sat graph $G = (V, E)$, $n = |V|$, and $m = |E|$. Clearly $m \geq \delta n/2$.

For any vertex $v \in V$, if $uv \notin E$, then there are at least $r - 2$ edges from $u$ to $N(v)$ since $G$ is $K_r$-semi-saturated. Moreover, since $G$ is $K_r$-dominated, $u$ is incident to at least $r - 1$ edges, but its $(r - 1)$-th edge could be to another vertex outside $N(u)$. (See Figure 2.) Therefore, with $d = d(v)$,

$$m \geq d + (n - 1 - d) \left[ \frac{(r - 2) + 1}{2} \right] = (n - 1) \left( r - \frac{3}{2} \right) - d \left( r - \frac{1}{2} \right).$$

Now we have that

$$m \geq \max \left( \frac{\delta n}{2}, (n - 1) \left( r - \frac{3}{2} \right) - \delta \left( k - \frac{1}{2} \right) \right).$$
For fixed $n$ and $r$, the former is increasing with $\delta$ and the latter is decreasing with $\delta$ so the maximum is minimized when
\[
\frac{\delta n}{2} = (n - 1)\left(r - \frac{3}{2}\right) - \delta \left(r - \frac{1}{2}\right),
\]
which gives
\[
\delta = \frac{(n - 1)(2r - 3)}{n + 2r - 5}.
\]
This approaches $2r - 3$ as $n$ approaches $\infty$, so
\[
m \geq \frac{n(2r - 3 + o(1))}{2}.
\]

**Corollary 3.2.** For graph $F$, $\varnothing_{dsat}(F) \leq |F| - \frac{3}{2}$.

In a graph, a **bridge** is an edge whose removal increases the number of connected components of the graph.

**Theorem 3.3.** If $F$ has a bridge, then $\varnothing_{dsat}(F) \leq \frac{|F| - 1}{2}$.

Moreover, if $F$ has a bridge $b$ such that every component of $F - b$ has at most $r$ vertices, then $\varnothing_{dsat}(F) \leq \frac{r(r - 1) + 1}{2r}$.

**Proof.** This upper bound is demonstrated by $G$ consisting of disjoint copies of the clique $K_1|F|$ (and one clique $K_s$ with $|F| \leq s < 2|F|$). $G$ is $F$-dominated, because each component is $F$-dominated. $G$ is $F$-semi-saturated, because adding an edge between two of the cliques creates at least one copy of $F$ (many, in fact) since $F$ has a bridge.

With a bound of $r$ on the number of vertices in each component of $F - b$, we can improve the upper bound on $dsat(F)$. This is demonstrated by disjoint copies of pairs of the clique $K_r$ where the two cliques in a pair are connected by a single edge. \(\square\)

In the following theorems, we establish upper bounds on $dsat(F)$ for certain cases of $F$ with a small subgraph that has few neighbors in the rest of the graph.

**Theorem 3.4.** Suppose $F$ has edge $uw$ with $|N(u) \cup N(w)| = k + 2$. Then
\[
\varnothing_{dsat}(F) \leq \begin{cases} k + 1/2, & \delta(F) = k + 1; \\ k, & \text{otherwise.} \end{cases}
\]

**Proof.** Let $G_n$ be the $n$-vertex graph attained by fixing $k$ vertices in a copy of $K_1|F|$, and adding an edge from every one of the $k$ vertices to every one of the other $n - |F|$ vertices. That is, $G$ is the result of attaching the complete bipartite graph $K_{n - |F|, k}$ to $k$ vertices in a clique $K_1|F|$. (See the left construction in Figure 3.) We see $G_n$ is $F$-semi-saturated (for all sufficiently large $n$) since
any added edge would connect two of the \( n - |F| \) vertices, and so serve as \( uw \) in a new copy of \( F \), with the rest of the vertices falling in the \( |F| \)-clique.

So long as some vertex in \( F \) has degree at most \( k \), \( G_n \) is also \( F \)-dominated, giving the bound \( ds(F) \leq k \). The only issue is when \( \delta(F) = k + 1 \) (then \( u \) and \( w \) have the same closed neighborhood).

In this case, simply obtain an \( F \)-dominated graph from \( G_n \) by partitioning the \( n - |F| \) vertices into pairs and adding an edge to each pair, thus increasing the asymptotic edge density by \( 1/2 \).

(See the right construction in Figure 3.)

\[ K_{|F|} \quad \text{vs.} \quad K_k \]

Figure 3. Graphs realizing the upper bounds in Theorem 3.4: \( G_n \) for the general case on the left; for the special case when \( \delta(F) = k + 1 \) on the right.

The weaker bound of Theorem 3.4 can be directly generalized to \( F \) with two small disjoint vertex sets that only have one edge between them and together have a bounded number of neighbors in the rest of \( F \).

**Theorem 3.5.** Suppose \( F \) has disjoint vertex sets \( U \) and \( W \) such that \( |U \cup W| = r \), \( |e(U, W)| = 1 \), and \( (N(U) \cup N(W)) \setminus (U \cup W) = k \). Then \( \text{odsat}(F) \leq k + \frac{r-1}{2} \).

**Proof.** Taking \( G_n \) of the previous proof, partition the \( n - |F| \) vertices into \( r \)-sets and add edges to those \( r \)-sets to form \( r \)-cliques. This increases the asymptotic edge density by \( \frac{r}{2}/r = \frac{r-1}{2} \). Since \( |U \cup W| = r \) and \( (N(U) \cup N(W)) \setminus (U \cup W) = k \), this new graph is still \( F \)-dominated. And since \( |e(U, W)| = 1 \), it is also \( F \)-semi-saturated. \( \square \)

## 4. Graph Classes

Having established preliminary theory and various structural bounds, let us investigate the \( \text{dom-sat} \) numbers for some fundamental classes of graphs: paths, cycles, and stars. For paths, we first need the following technical lemma.

**Lemma 4.1.** Let \( T \) be a tree on \( t \geq 3 \) vertices with \( t < 3j \) for some \( j \). Then either \( T \) is a star or there exist distinct, non-adjacent vertices \( u, v \in V(T) \) such that \( T - u - v \) contains no \( j \)-vertex path with an endpoint in \( N(u) \cup N(v) \).

**Proof.** **Case 1:** If \( T \) is a star, there is nothing to prove. Henceforth, we can assume \( T \) has a path on at least 4 vertices. **Case 2:** If there exists a vertex \( u \) so that \( T - u - v \) contains no \( j \)-vertex path, let \( v \) be any vertex not adjacent to \( u \) and we are done.

**Case 3:** For every edge \( e \in E(T) \), if one of the components of \( T - e \) has no \( j \)-vertex path, orient the edge away from that component. (We need not worry about neither component having a \( j \)-vertex path as that was covered in Case 2.) Observe that the non-oriented edges form a connected subgraph \( U \) of \( T \) and that each component of the oriented subgraph has a unique sink that is a leaf of \( U \). Let \( ab \) be an edge in \( U \) for some leaf \( b \) of \( U \). Then the component of \( T - ab \) that contains \( b \) has a \( j \)-vertex path, else \( ab \) would have been oriented. Therefore, since \( t < 3j \), \( U \) has at most
2 leaves. That is, \( U \) is itself a path; call the endpoints \( u \) and \( w \). Since we are beyond Case 2, we can assume \( u \neq w \). Note that \( T - u - w \) contains no \( j \)-vertex path. The only remaining issue is if \( u \) and \( w \) are neighbors.

**Case 3b:** Assumes \( U \) only consists of the edge \( uw \). Without loss of generality, assume the component \( W \) of \( T - uw \) that contains \( w \) has at most \( t/2 < 3j/2 \) vertices. We know that every \( j \)-vertex path in \( W \) contains \( w \). If no \( j \)-vertex path in \( W \) has \( w \) as an endpoint (e.g., the left graph in Figure 4), then let \( v \) be any neighbor of \( w \) in \( W \) and we are done. Otherwise, let \( v \) be such that some \( j \)-vertex path in \( W \) has endpoint \( w \) and final edge \( vw \), so the component of \( T - u - v \) containing \( w \) has at most \( (j + 1)/2 < j \) vertices (e.g., the right graph in Figure 4).

**Figure 4.** For the proof of Lemma 4.1, examples of trees \( T \) with \( t = 14 \) and \( j = 5 \) in which \( U = T \{u, w\} \); on the left, observe how every 5-vertex path in \( T - u \) contains \( w \) but not as an end-point; on the right, observe how the component of \( T - u - v \) that contains \( w \) has \( 3 \leq (j + 1)/2 \) vertices.

Kázmoryi and Tuza [8] found the saturation number for paths for all sufficiently large \( n \). Their result gives, for \( r \geq 3 \),

\[
\text{sat}(P_r) = \begin{cases} 
1 - \frac{1}{2^{j+1}} & \text{if } r = 2j + 1; \\
1 - \frac{j}{2^{j+1}} & \text{if } r = 2j + 2.
\end{cases}
\]

We attain a reminiscent result—parity dependent and approaching 1 monotonically from below as \( j \) grows—for the dom-sat density of paths.

**Theorem 4.2.** For path \( P_r \) on \( r \geq 3 \) vertices,

\[
\text{dsat}(P_r) = \begin{cases} 
1 - \frac{1}{r}, & r = 2j + 1; \\
1 - \frac{j}{r+1}, & r = 2j + 2.
\end{cases}
\]

**Proof.** We will prove the first case, \( r = 2j + 1 \). The other case follows a nearly identical proof.

**Claim:** Disjoint copies of \( P_{3j} \) comprise an extremal graph for \( P_{2j+1} \).

This proposed graph is clearly \( P_r \)-dominated. To check semi-saturation, we need to consider two cases: an edge connecting two copies of \( P_3 \) and an edge connecting two non-neighbors within a single copy of \( P_r \). The first case is trivial. We demonstrate the latter case in Figure 5.

It remains to show that the proposed graph does in fact minimize edge density. A graph with edge density less than \((3j - 1)/(3j)\) would necessarily have acyclic components on \( s \) vertices for some \( r \leq s < 3j \). Take one such component \( T \). Since \( T \) is acyclic, we appeal to Lemma 4.1. If \( T \) is a star, \( P_r \)-domination fails for \( r \geq 4 \), and \( r = 3 \) implies \( j = 1 \) which contradicts Lemma 4.1.

Thus, we have vertices \( u \) and \( v \) such that \( T - u - v \) contains no \( j \)-vertex path with an endpoint in \( N(u) \cup N(v) \). Now if we try to extend edge \( uv \) into a long path in \( T + uv \), we can only possibly make a path on \( 2 + 2(j - 1) < 2j + 1 \) vertices, contradicting \( P_{2j+1} \)-semi-saturation.

Füredi and Kim [6] showed that

\[
1 + \frac{1}{r+2} \leq \text{sat}(C_r) \leq \text{o sat}(C_r) \leq 1 + \frac{1}{r-4}
\]

and conjectured the upper bound to be the true limit. With a similar construction to theirs we gain similar bounds for the dom-sat density of a cycle.

**Theorem 4.3.** For cycle \( C_r \) on \( r \geq 4 \) vertices, \( 1 \leq \text{dsat}(C_r) \leq \text{o dsat}(C_r) \leq 1 + \frac{1}{r-3} \).
Figure 5. Demonstration that $P_3j$ is sufficiently long to be $P_{2j+1}$-semi-saturated: A new copy of $P_{2j+1}$ (blue, bolded) is obtained when an edge (dashed) is added, whether the newly adjacent vertices were far apart (top) or close together (bottom).

**Proof.** The lower bound comes from Theorem 2.10 since $\delta(C_r) = 2$. For the upper bound, considering the following construction on $n$ vertices, pictured in Figure 6. Fix two vertices on an $\ell$-vertex clique with $r \leq \ell < 2r - 3$ and $\ell \equiv n \pmod{r - 3}$. With the remaining $n - \ell$ vertices, take $\frac{n-\ell}{r-3}$ disjoint $(r-3)$-vertex paths. For each path, add a matching between the end-vertices of the path and the two fixed vertices of the clique. This $C_r$-dom-sat graph has, in the limit, edge density $\frac{r-2}{r-3}$.

![Figure 6. A (possibly asymptotically extremal) $C_r$-dom-sat graph.](image)

The upper bound in Theorem 4.3 is realized by a cliques with pendant loops on $r - 3$ vertices. One might try to improve this bound by using longer loops. However, the graph is no longer $C_r$-semi-saturated when the loops have, for examples, $r - 2$ vertices: Then an edge added between the corresponding vertex in two different loops is contained in no cycle shorter than $r + 1$.

For the star $K_{1,r}$ on $r + 1$ vertices, Kászonyi and Tuza [8] also found the saturation number for all $n$, giving

$$\text{sat}(K_{1,r}) = \frac{r - 1}{2},$$

which matches our lower bound for the dom-sat density of a star.

**Theorem 4.4.** For the star $K_{1,r}$ on $r + 1 \geq 3$ vertices,

$$\frac{r - 1}{2} \leq \text{dsat}(K_{1,r}) \leq \text{odsat}(K_{1,r}) \leq \frac{r - 1}{2} + \frac{4r - 3}{8r - 4}.$$
Proof. We get the lower bound from $K_{1,r}$-semi-saturation: Any added edge must be incident to a vertex of degree at least $r$, so a $K_{1,r}$-semi-saturated graph has only a few (fewer than $r$) vertices of degree less than $r - 1$.

Since $K_{1,r}$ has a bridge, an upper bound of $r/2$ follows from Theorem 3.3. However, we can do slightly better using disjoint copies of the complete bipartite graph $K_{r-1,r}$, which has edge density \( \frac{r(r-1)}{2r-1} \).

\[ r \cdot \left( \frac{r-1}{r} \right) = \frac{r^2 - r}{2r-1}. \]

\( \Box \)

Elegantly, both $\text{dsat}(P_r)$ and $\text{dsat}(C_r)$ approach 1 as $r$ approaches infinity. That is, long paths and long cycles are similar with respect to the dom-sat invariant. On the other hand, stars have dom-sat number near the maximum possible (roughly half the number of vertices) for graphs with a bridge. Thus the dom-sat invariant clearly distinguishes the opposite extremes of trees: paths and stars. However, we see in the following theorem that this might not be the best way to view domination-saturation.

**Theorem 4.5.** Let $G_s$ be the $s$-vertex graph formed by appending an edge onto a leaf of the star $K_{1,s-2}$. Then \( 1 - \frac{1}{2s} \leq \text{dsat}(G_s) \leq 1 - \frac{1}{2s-1} \).

**Proof.** For the lower bound, observe that (aside from at most one isolated vertex) every connected component of a $G_s$-dom-sat graph has at least $s$ vertices.

Let $H_s$ be the $(2s-2)$-vertex graph obtained by connecting the centers of two copies of the star $K_{1,s-2}$ (see Figure 7). The upper bound follows from observing that disjoint copies of $H_s$ form a $G_s$-dom-sat graph.

\[ G_7 \quad \text{and} \quad H_7 \]

**Figure 7.** $G_7$ (left) and a $G_7$-dom-sat tree, $H_7$ (right).

5. Conclusion

Domination opens a natural saturation variant with numerous potential avenues of further research (see below). Moreover, domination-saturation give us a novel graph invariant, a sort of connectivity or centrality measure that is monotone with respect to the domination relation. The dom-sat density of a graph $F$ can only be less than 1 if $F$ has an acyclic component. On the other hand, the dom-sat density of $F$ is at most $|F| - 3/2$, a bound realized by cliques.

5.1. Future Directions. The sets of extremal graphs for the clique in general saturation and in Turán theory, $\text{Sat}(n,K_r)$ and $\text{Ex}(n,K_r)$ respectively, each contain a unique graph (see [3] and [13]). Is this the case for the analogous set $\text{Dsat}(n,K_r)$ of graphs realizing $\text{dsat}(n,K_r)$? In particular, is $\text{Dsat}(n,K_r) = \{D(n,r)\}$ with the graph $D(n,r)$ as defined in Theorem 3.1? Even if extremal graphs are not unique, all the present examples are highly symmetric (aside from some small set of vertices). Perhaps something can be said of the automorphism group of extremal graphs.

It remains to investigate the relationship between the dom-sat number of a graph and other standard saturation numbers ($\text{sat}$, $\text{ss}$, or $\text{wsat}$). There may also be connections between dom-sat density and other graph measures, such as Wiener index.

Saturation has been extensively studied for families of graphs. In fact, Oleg Pikhurko [11] identified families $F$ with as few as 4 graphs such that $\text{sat}(F) \neq \text{osat}(F)$. Additionally, many saturation results have been extended to the $k$-uniform hypergraph setting, where saturations
numbers are no longer $O(n)$ but $O(n^{k-1})$. What theory arises when domination-saturation is considered for hypergraphs or families of (hyper)graphs?

In 1991, the first graph saturation game was introduced by Füredi, Reimer, and Seress [7], in which two players take turns adding an edge to an initially empty vertex set, one with the aim of constructing an $F$-saturated graph as quickly as possible, and the other as slowly as possible. Clearly the game saturation number of $F$ (how long the game lasts when both players playing optimally) lies between $\text{sat}(n,F)$ and $\text{ex}(n,F)$. See the 2016 paper by Carraher et al. [2] for a summary of known results on game saturation numbers. There may be interesting games to study when the objective of one or both players involves attaining or avoiding $F$-dominance.

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