Research Article

Sharp Large Deviation for the Energy of $\alpha$-Brownian Bridge

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1. Introduction

We consider the following $\alpha$-Brownian bridge:

$$dX_t = -\alpha T-t X_t \, dt + dW_t, \quad X_0 = 0,$$  

where $W$ is a standard Brownian motion, $t \in [0, T)$, $T \in (0, \infty)$, and the constant $\alpha > 1/2$. Let $P_\alpha$ denote the probability distribution of the solution $\{X_t, t \in [0, T)\}$ of (1). The $\alpha$-Brownian bridge is first used to study the arbitrage profit associated with a given future contract in the absence of transaction costs by Brennan and Schwartz [1].

$\alpha$-Brownian bridge is a time inhomogeneous diffusion process which has been studied by Barczy and Pap [2, 3], Jiang and Zhao [4], and Zhao and Liu [5]. They studied the central limit theorem and the large deviations for parameter estimators and hypothesis testing problem of $\alpha$-Brownian bridge. While the large deviation is not so helpful in some statistics problems since it only gives a logarithmic equivalent for the deviation probability, Bahadur and Ranga Rao [6] overcame this difficulty by the sharp large deviation principle for the empirical mean. Recently, the sharp large deviation principle is widely used in the study of Gaussian quadratic forms, Ornstein-Uhlenbeck model, and fractional Ornstein-Uhlenbeck (cf. Bercu and Rouault [7], Bercu et al. [8], and Bercu et al. [9, 10]).

In this paper we consider the sharp large deviation principle (SLDP) of energy $S_t$, where

$$S_t = \int_0^t X_s^2 \frac{(s-T)^2}{(s-T)^2} \, ds.$$  

Our main results are the following.

**Theorem 1.** Let $\{X_t, t \in [0, T]\}$ be the process given by the stochastic differential equation (1). Then $\{S_t/\lambda_t, t \in [0, T]\}$ satisfies the large deviation principle with speed $\lambda_t$, and good rate function $I(\cdot)$ defined by the following:

$$I(x) = \begin{cases} 1 & \text{if } x > 0; \\ +\infty & \text{if } x \leq 0, \end{cases}$$  

where $\lambda_t = \log(T/(T-t))$.

**Theorem 2.** $\{S_t/\lambda_t, t \in [0, T]\}$ satisfies SLDP; that is, for any $c > 1/(2\alpha - 1)$, there exists a sequence $b_{c,k}$, such that, for any $p > 0$, when $t$ approaches $T$ enough,

$$P(S_t \geq c\lambda_t) = \exp \left( -I(c) \lambda_t + H(a_c) \right) \sqrt{2\pi a_c} \beta_t \times \left( 1 + \sum_{k=1}^\infty \frac{b_{c,k}}{\lambda_t} + O \left( \frac{1}{\lambda_t^p} \right) \right).$$  

(4)
where
\[\sigma^2_c = 4c^2, \quad \beta_c = \sigma_c \sqrt{\lambda_t}, \quad a_c = \frac{(1 - 2\alpha)c^2 - 1}{8c^2}, \quad H(a_c) = -\frac{1}{2} \log \left( \frac{1 - (1 - 2\alpha)c^2}{2} \right).\]

The coefficients \(b_{\alpha,c}\) may be explicitly computed as functions of the derivatives of \(L\) and \(H\) (defined in Lemma 3) at point \(a_c\). For example, \(b_{\alpha,c}\) is given by
\[b_{\alpha,c} = \frac{1}{\sigma_c^2} \left( -\frac{h_2}{2} - \frac{h_1^2}{2} + \frac{l_4}{8\sigma_c^2} + \frac{l_1h_1}{2a_c} \right), \quad \text{with} \quad \lambda_k = L^{(k)}(a_c), \quad \text{and} \quad h_k = H^{(k)}(a_c).\]

2. Large Deviation for Energy

Given \(\alpha > 1/2\), we first consider the following logarithmic moment generating function of \(\lambda_t\); that is,
\[L_t(u) := \log E_u \exp \left\{ u \int_0^t \frac{X_s^2}{(s - T)^2} ds \right\}, \quad \forall \lambda \in \mathbb{R}. \quad (7)\]

And let
\[\mathcal{D}_{L_t} := \{ u \in \mathbb{R}, \ L_t(u) < +\infty \} \quad (8)\]

be the effective domain of \(L_t\). By the same method as in Zhao and Liu [5], we have the following lemma.

Lemma 3. Let \(\mathcal{D}_{L_t}\) be the effective domain of the limit \(L_t\); then for all \(u \in \mathcal{D}_{L_t}\), one has
\[\frac{L_t(u)}{\lambda_t} = L(u) + \frac{H(u)}{\lambda_t} + \frac{R(u)}{\lambda_t}, \quad (9)\]

with
\[L(u) = -\frac{1 - 2\alpha - \varphi(u)}{4}, \quad H(\lambda) = -\frac{1}{2} \log \left\{ \frac{1}{2} \left( 1 + h(u) \right) \right\}, \quad R(u) = -\frac{1}{2} \log \left\{ 1 + \frac{1 - h(u)}{1 + h(u)} \exp \left\{ 2\varphi(u) \lambda_t \right\} \right\}, \quad (10)\]

where \(\varphi(u) = -\sqrt{(1 - 2\alpha)^2 - 8u}\) and \(h(u) = (1 - 2\alpha)/\varphi(u)\). Furthermore, the remainder \(R(u)\) satisfies
\[R(u) = O_{1-T} \left( \exp \left\{ 2\varphi(u) \lambda_t \right\} \right). \quad (11)\]

Proof. By Itô’s formula and Girsanov’s formula (see Jacob and Shiryaev [11]), for all \(u \in \mathcal{D}_{L_t}\) and \(t \in [0, T)\),
\[\log \frac{dP_u}{dP_\beta(u)} \bigg|_{\omega_t} = (\alpha - \beta) \int_0^t \frac{X_s^2}{s - T} dX_s - \frac{\alpha^2 - \beta^2}{2} \int_0^t \frac{X_s^2}{(s - T)^2} ds, \quad (12)\]

Therefore,
\[L_t(u) = \log E_u \left[ \exp \left\{ u \int_0^t \frac{X_s^2}{(s - T)^2} ds \right\} \right] \frac{dP_u}{dP_\beta(u)} \bigg|_{\omega_t} = \log E_u \left[ \frac{\alpha - \beta}{2} X_t^2 \frac{2}{(t - T)} + \frac{1}{2} \left( \beta^2 - \alpha^2 + \alpha - \beta + 2u \right) \int_0^t \frac{X_s^2}{(s - T)^2} ds \right]. \quad (13)\]

If \(4u \leq (1 - 2\alpha)^2\), we can choose \(\beta\) such that \((\beta - 1/2)^2 - (\alpha - 1/2)^2 + 2u = 0\). Then
\[L_t(u) = -\frac{1 - 2\alpha - \varphi(u)}{4} \left( \frac{1}{2} + h(u) \right) + \frac{1}{2} \log \left\{ \frac{1}{2} \left( 1 + h(u) \right) \right\} \left( 1 - \frac{t}{T} \right), \quad (14)\]

Therefore,
\[L_t(u) = -\frac{1 - 2\alpha - \varphi(u)}{4} \left( 1 - \frac{t}{T} \right) + \frac{1}{2} \log \left\{ \frac{1}{2} \left( 1 + h(u) \right) \right\} \left( 1 - \frac{t}{T} \right), \quad (15)\]

Proof of Theorem 1. From Lemma 3, we have
\[L(u) = \lim_{t \to T} \frac{L_t(u)}{\lambda_t} = -\frac{1 - 2\alpha - \varphi(u)}{4}. \quad (16)\]
and $L(\cdot)$ is steep; by the Gärtner-Ellis theorem (Dembo and Zeitouni [12]), $S_t/\lambda_t$ satisfies the large deviation principle with speed $\lambda_t$ and good rate function $I(\cdot)$ defined by the following:

$$I(x) = \begin{cases} \frac{1}{8x}(2\alpha - 1)x - 1)^2, & \text{if } x > 0; \\ +\infty, & \text{if } x \leq 0. \end{cases}$$ (17)

**Remark 4.** Theorem 1 can also be obtained by using Theorem 1 in Zhao and Liu [5].

## 3. Sharp Large Deviation for Energy

For $c > 1/(2\alpha - 1)$, let

$$a_c = \frac{1}{2}(2\alpha - 1)x^2 - 1, \quad \sigma_c^2 = L''(a_c) = 4c^3,$$

$$H(a_c) = -\frac{1}{2} \log(1 - (1 - 2\alpha)c).$$ (18)

Then

$$P(S_t \geq c\lambda_t)$$

$$= \int_{S_t \geq c\lambda_t} \exp \{L(a_c) - ca_c\lambda_t + ca_c\lambda_t - a_cS_t\} dQ_t$$

$$= \exp \{L(a_c) - ca_c\lambda_t\} \mathbb{E}_Q \exp \{-a_c\beta_t U_t |U_t|_{\{U_t \geq 0\}}\} = A_t B_t,$$ (19)

where $\mathbb{E}_Q$ is the expectation after the change of measure

$$\frac{dQ_t}{dP} = \exp \{a_cS_t - L_t(a_c)\},$$

$$U_t = S_t - c\lambda_t, \quad \beta_t = \sigma_c \sqrt{\lambda_t}.$$ (20)

By Lemma 3, we have the following expression of $A_t$.

**Lemma 5.** For all $c > 1/(2\alpha - 1)$, when $t$ approaches $T$ enough,

$$A_t = \exp \{-I(c)\lambda_t + H(a_c)\} (1 + O((T - t)^\delta)).$$ (21)

For $B_t$, one gets the following.

**Lemma 6.** For all $c > 1/(2\alpha - 1)$, the distribution of $U_t$ under $Q_t$ converges to $N(0,1)$ distribution. Furthermore, there exists a sequence $\psi_k$ such that, for any $p > 0$ when $t$ approaches $T$ enough,

$$B_t = \frac{1}{\sigma_c \sqrt{2\pi} \lambda_t} \left(1 + \sum_{k=1}^{p} \frac{\psi_k}{\lambda_t} + O\left(\lambda_t^{-p+1}\right)\right).$$ (22)

**Proof of Theorem 2.** The theorem follows from Lemma 5 and Lemma 6.

It only remains to prove Lemma 6. Let $\Phi_j(\cdot)$ be the characteristic function of $U_t$ under $Q_t$; then we have the following.

**Lemma 7.** When $t$ approaches $T$, $\Phi_j(\cdot)$ belongs to $L^2(\mathbb{R})$ and, for all $u \in \mathbb{R}$,

$$\Phi_j(u) = \exp \left\{ \frac{-iu \sqrt{\lambda_c}}{\sigma_c} \right\}$$

$$\times \exp \left\{ L_t \left(a_c + \frac{iu}{\beta_t}\right) - L_t(a_c) \right\}.$$ (23)

Moreover,

$$B_t = \mathbb{E}_Q \exp \{-a_c\beta_t U_t |U_t|_{\{U_t \geq 0\}}\} = C_t + D_t,$$ (24)

with

$$C_t = \frac{1}{2\pi a_c \beta_t} \int_{|u| \leq s_t} \left(1 + \frac{iu}{a_c \beta_t}\right)^{-1} \Phi_j(u) du,$$

$$D_t = \frac{1}{2\pi a_c \beta_t} \int_{|u| > s_t} \left(1 + \frac{iu}{a_c \beta_t}\right)^{-1} \Phi_j(u) du,$$ (25)

$$|D_t| = O\left(\exp\left\{-D\lambda_t^{1/3}\right\}\right),$$

where $s_t = s\left(\log\left(\frac{T}{T - t}\right)\right)^{1/6}$,

for some positive constant $s$, and $D$ is some positive constant.

**Proof.** For any $u \in \mathbb{R}$,

$$\Phi_j(u) = \mathbb{E}\left(\exp\{iuU_t\} \exp\{a_cS_t - L_t(a_c)\}\right)$$

$$= \exp \left\{ \frac{-iu \sqrt{\lambda_c}}{\sigma_c} \right\}$$

$$\times \exp \left\{ L_t \left(a_c + \frac{iu}{\beta_t}\right) - L_t(a_c) \right\}.$$ (27)

By the same method as in the proof of Lemma 2.2 in [7] by Bercu and Rouault, there exist two positive constants $r$ and $\kappa$ such that

$$|\Phi_j(u)|^2 \leq 1 + \frac{r u^2}{\lambda_t^{(\kappa)/2}}.$$ (28)

therefore, $\Phi_j(\cdot)$ belongs to $L^2(\mathbb{R})$, and by Parseval’s formula, for some positive constant $s$, let

$$s_t = s\left(\log\left(\frac{T}{T - t}\right)\right)^{1/6};$$ (29)

we get

$$B_t = \frac{1}{2\pi a_c \beta_t} \int_{|u| \leq s_t} \left(1 + \frac{iu}{a_c \beta_t}\right)^{-1} \Phi_j(u) du + \frac{1}{2\pi a_c \beta_t}$$

$$\times \int_{|u| > s_t} \left(1 + \frac{iu}{a_c \beta_t}\right)^{-1} \Phi_j(u) du$$

$$=: C_t + D_t,$$ (30)

$$|D_t| = O\left(\exp\left\{-D\lambda_t^{1/3}\right\}\right),$$ (32)

where $D$ is some positive constant. □
Proof of Lemma 6. By Lemma 3, we have
\[
\frac{L_t^{(k)}}{\lambda_t} (a_c) = L_t^{(k)} (a_c) + \frac{H_t^{(k)}}{\lambda_t} (a_c) + O \left( \frac{(T-t)^{2c}}{\lambda_t} \right). \tag{33}
\]
Noting that \( L'(a_c) = 0 \), \( L''(a_c) = \sigma_c^2 \) and
\[
\frac{L''(a_c) (iu)^2}{2} \frac{\lambda_t}{\beta_t} = -\frac{u^2}{2}, \tag{34}
\]
for any \( p > 0 \), by Taylor expansion, we obtain
\[
\log \Phi_t(u) = \frac{-u^2}{2} + \lambda_t \sum_{k=0}^{2p+3} \left( \frac{iu}{\beta_t} \right)^k \frac{L_t^{(k)} (a_c)}{k!}
+ \sum_{k=1}^{2p+1} \left( \frac{iu}{\beta_t} \right)^k \frac{H_t^{(k)} (a_c)}{k!}
+ O \left( \max \left( 1, \frac{|u|^{2p+4}}{\lambda_t^{p+1}} \right) \right); \tag{35}
\]
therefore, there exist integers \( q(p) \), \( r(p) \) and a sequence \( \varphi_{k,l} \) independent of \( p \); when \( t \) approaches \( T \), we get
\[
\Phi_t(u) = \exp \left\{ -\frac{u^2}{2} \right\} \left( 1 + \frac{1}{\sqrt{\lambda_t}} \sum_{k=0}^{2p} \sum_{l=k+1}^{q(p)} \varphi_{k,l} u^l \right)^{\frac{p}{2}}
+ O \left( \max \left( 1, \frac{|u|^{2p+4}}{\lambda_t^{p+1}} \right) \right), \tag{36}
\]
where \( O \) is uniform as soon as \( |u| \leq s \).

Finally, we get the proof of Lemma 6 by Lemma 7 together with standard calculations on the \( N(0,1) \) distribution.

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