AN ALTERNATIVE BASIS FOR THE KAUFFMAN BRACKET SKEIN MODULE OF THE SOLID TORUS VIA BRAIDS

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Abstract. In this paper we give an alternative basis, $B_{ST}$, for the Kauffman bracket skein module of the solid torus, $KBSM(ST)$. The basis $B_{ST}$ is obtained with the use of the Temperley–Lieb algebra of type B and it is appropriate for computing the Kauffman bracket skein module of the lens spaces $L(p,q)$ via braids.

0. Introduction and overview

Skein modules were independently introduced by Przytycki [P] and Turaev [Tu] as generalizations of knot polynomials in $S^3$ to knot polynomials in arbitrary 3-manifolds. The essence is that skein modules are quotients of free modules over ambient isotopy classes of links in 3-manifolds by properly chosen local (skein) relations.

Definition 1. Let $M$ be an oriented 3-manifold and $\mathcal{L}_{fr}$ be the set of isotopy classes of unoriented framed links in $M$. Let $R = \mathbb{Z}[A^{\pm 1}]$ be the Laurent polynomials in $A$ and let $R\mathcal{L}_{fr}$ be the free $R$-module generated by $\mathcal{L}_{fr}$. Let $S$ be the ideal generated by the skein expressions $L - AL_\infty - A^{-1}L_0$ and $L \bigcup O - (-A^2 - A^{-1})L$, where $L_\infty$ and $L_0$ are represented schematically by the illustrations in Figure 1. Note that blackboard framing is assumed.

Figure 1. The links $L$, $L_0$ and $L_\infty$ locally.

Then the Kauffman bracket skein module of $M$, $KBSM(M)$, is defined to be:

$$KBSM(M) = R\mathcal{L}/S.$$ 

In [Tu] the Kauffman bracket skein module of the solid torus, $ST$, is computed using diagrammatic methods by means of the following theorem:

Theorem 1 ([Tu]). The Kauffman bracket skein module of $ST$, $KBSM(ST)$, is freely generated by an infinite set of generators $\{x^n\}_{n=0}^{\infty}$, where $x^n$ denotes a parallel copy of $n$ longitudes of $ST$ and $x^0$ is the affine unknot (see Figure 2).
In [La2] the most generic analogue of the HOMFLYPT polynomial, $X$, for links in the solid torus ST has been derived from the generalized Hecke algebras of type B, $H_{1,n}$, which is related to the knot theory of the solid torus and the Artin group of Coxeter group of type B, $B_{1,n}$, via a unique Markov trace constructed on them. As explained in [La2, DL2], the Lambropoulou invariant $X$ recovers the HOMFLYPT skein module of ST, $S(ST)$, and is appropriate for extending the results to the lens spaces $L(p,q)$, since the combinatorial setting is the same as for ST, only the braid equivalence includes the braid band moves (shorthanded to bbm), which reflect the surgery description of $L(p,q)$. In [FG] the same procedure is applied for the case of Temperley-Lieb algebras of type B and an invariant $V^B$ for knots and links in ST is constructed, via a unique Markov trace constructed on them, and which is the analogue of the Kauffman bracket polynomial for knots and links in ST.

In this paper the Kauffman bracket skein module of ST, $KBSM(ST)$, is computed using braids and algebraic techniques developed in [LR1, LR2, La1, La2, DL1, DL2, DLP, DL4, DL5] and [FG]. The motivation of this work is the computation of $KBSM(L(p,q))$ via algebraic means.

Our main result is the following:

**Theorem 2.** The following set forms a basis for $KBSM(ST)$:

$$B_{ST} = \{t^n, n \in \mathbb{N}\}.$$  

The method for obtaining the basis $B_{ST}$, is the following:

- We start from elements in the standard basis of $KBSM(ST)$, $B'_{ST}$, presented in [La2]. Then, following the technique in [DL2], we express these elements into sums of elements in the $\Lambda$, using conjugation and stabilization moves. As shown in [DL2], the set $\Lambda$ (see Remark 3), forms a basis for the HOMFLYPT skein module of the solid torus.

- We then express elements in $\Lambda$ to sums of elements in $B_{ST}$, using conjugation, stabilization moves and the Kauffman bracket skein relation.

- We relate the two sets $B'_{ST}$ and $B_{ST}$ via an infinite lower triangular matrix and conclude that the set $B_{ST}$ forms a basis for $KBSM(ST)$.

The paper is organized as follows: In §1 we recall the setting and the essential techniques and results from [La1, La2, LR1, LR2, DL1]. More precisely, we present isotopy moves for knots and links in ST and we then describe braid equivalence for knots and links in ST. We also present results from [La2] and [FG] and in particular we present the basis of the Kauffman bracket skein module of ST in terms of braids and braid groups of type B. In §2 we present results from [DL2] that are crucial for this paper, and using these results, in §3 we present a new basis for the Kauffman bracket skein module of the solid torus ST, $B_{ST}$. As explained in the beginning of §2 the importance of the basis $B_{ST}$ lies in the fact that the braid band moves or slide moves (that
reflect isotopy in the lens spaces \( L(p, q) \) are naturally described via \( \mathcal{B}_{ST} \). Finally in [D] and starting from \( \mathcal{B}_{ST} \), the computation of the Kauffman bracket skein module of the lens spaces is presented.

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1. **Preliminaries**

1.1. **Mixed links and isotopy in ST.** We consider ST to be the complement of a solid torus in \( S^3 \). As explained in [LR1, LR2, DL1], an oriented link \( L \) in ST can be represented by an oriented mixed link in \( S^3 \), that is a link in \( S^3 \) consisting of the unknotted fixed part \( \hat{I} \) representing the complementary solid torus in \( S^3 \) and the moving part \( L \) that links with \( \hat{I} \). A mixed link diagram is a diagram \( \hat{I} \cup \tilde{L} \) of \( \hat{I} \cup L \) on the plane of \( \hat{I} \), where this plane is equipped with the top-to-bottom direction of \( I \) (see right hand side of Figure 3).

Consider now an isotopy of an oriented link \( L \) in ST. As the link moves in ST, its corresponding mixed link will change in \( S^3 \) by a sequence of moves that keep the oriented \( \hat{I} \) point-wise fixed. This sequence of moves consists in isotopy in the \( S^3 \) and the mixed Reidemeister moves. In terms of diagrams we have the following result for isotopy in ST:

*The mixed link equivalence in \( S^3 \) includes the classical Reidemeister moves and the mixed Reidemeister moves, which involve the fixed and the standard part of the mixed link, keeping \( \hat{I} \) pointwise fixed.*

1.2. **Mixed braids and braid equivalence for knots and links in ST.** By the Alexander theorem for knots and links in the solid torus (cf. Thm. 1 [La2]), a mixed link diagram \( \hat{I} \cup L \) of \( \hat{I} \cup L \) may be turned into a mixed braid \( I \cup \beta \) with isotopic closure. This is a braid in \( S^3 \) where, without loss of generality, its first strand represents \( \hat{I} \), the fixed part, and the other strands, \( \beta \), represent the moving part \( L \). The subbraid \( \beta \) is called the moving part of \( I \cup \beta \) (see left hand side of Figure 3).

The sets of braids related to ST form groups, which are in fact the Artin braid groups of type B, denoted \( B_{1,n} \), with presentation:
\[ B_{1,n} = \left\langle t, \sigma_1, \ldots, \sigma_{n-1} \mid \begin{array}{l}
\sigma_1 t \sigma_1 t = t \sigma_1 t \sigma_1 \\
\sigma_i = \sigma_i t, \quad i > 1 \\
\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \quad 1 \leq i \leq n-2 \\
\sigma_i \sigma_j = \sigma_j \sigma_i, \quad |i - j| > 1
\end{array} \right\rangle, \]

where the generators \( \sigma_i \) and \( t \) are illustrated in Figure 4(i).

Let now \( \mathcal{L} \) denote the set of oriented knots and links in \( \text{ST} \). Then, isotopy in \( \text{ST} \) is translated on the level of mixed braids by means of the following theorem:

**Theorem 3** (Theorem 5, [LR2]). Let \( L_1, L_2 \) be two oriented links in \( L(p,1) \) and let \( I \cup \beta_1, \ I \cup \beta_2 \) be two corresponding mixed braids in \( S^3 \). Then \( L_1 \) is isotopic to \( L_2 \) in \( L(p,1) \) if and only if \( I \cup \beta_1 \) is equivalent to \( I \cup \beta_2 \) in \( \mathcal{B} \) by the following moves:

1. **Conjugation**: \( \alpha \sim \beta^{-1} \alpha \beta \), if \( \alpha, \beta \in B_{1,n} \).
2. **Stabilization moves**: \( \alpha \sim \sigma \sigma_i \sigma^{-1} \), if \( \alpha \in B_{1,n} \).
3. **Loop conjugation**: \( \alpha \sim t^{\pm 1} \alpha t^{-1} \), if \( \alpha \in B_{1,n} \).

1.3. **The Kauffman bracket skein module of \( \text{ST} \) via braids.** In [La2] the most generic analogue of the HOMFLYPT polynomial, \( X \), for links in the solid torus \( \text{ST} \) has been derived from the generalized Iwahori–Hecke algebras of type B, \( H_{1,n} \), via a unique Markov trace constructed on them. This algebra was defined by Lambropoulou as the quotient of \( \mathbb{C}[q^{\pm 1}] B_{1,n} \) over the quadratic relations \( g_i^2 = (q-1)g_i + q \). Namely:

\[ H_{1,n}(q) = \frac{\mathbb{C}[q^{\pm 1}] B_{1,n}}{(\sigma_i^2 - (q-1)\sigma_i - q)}. \]

It is also shown that the following sets form linear bases for \( H_{1,n}(q) \) ([La2] Proposition 1 & Theorem 1):

\[
\begin{align*}
(2) & \quad \Sigma_n = \{ t^{k_1}_{i_1} \cdots t^{k_r}_{i_r} \cdot \sigma \}, \text{ where } 0 \leq i_1 < \ldots < i_r \leq n - 1, \\
\Sigma'_n = \{ t^{k_1}_{i_1} \cdots t^{k_r}_{i_r} \cdot \sigma \}, \text{ where } 0 \leq i_1 < \ldots < i_r \leq n - 1,
\end{align*}
\]

where \( k_1, \ldots, k_r \in \mathbb{Z} \), \( t_0 = t \), \( t'_i = g_i \ldots g_{i+1}^{-1} \ldots g_1^{-1} \) and \( t_i = g_i \ldots g_{i+1}^{-1} \ldots g_1^{-1} \) are the ‘looping elements’ in \( H_{1,n}(q) \) (see Figure 4(ii)) and \( \sigma \) a basic element in the Iwahori–Hecke algebra of type A, \( H_n(q) \), for example in the form of the elements in the set \( I_{10} \):

\[ S_n = \{ (g_{i_1}g_{i_1-1} \ldots g_{i_1-k_1})(g_{i_2}g_{i_2-1} \ldots g_{i_2-k_2}) \ldots (g_{i_p}g_{i_p-1} \ldots g_{i_p-k_p}) \}, \]

for \( 1 \leq i_1 < \ldots < i_p \leq n - 1 \). In [La2] the bases \( \Sigma'_n \) are used for constructing a Markov trace on \( \mathcal{H} := \bigcup_{n=1}^{\infty} H_{1,n} \), and using this trace, a universal HOMFLYPT-type invariant for oriented links in \( \text{ST} \) is constructed.

**Theorem 4.** ([La2] Theorem 6 & Definition 1) Given \( z, s_k \) with \( k \in \mathbb{Z} \) specified elements in \( R = \mathbb{C}[q^{\pm 1}] \), there exists a unique linear Markov trace function on \( \mathcal{H} \):

\[ \text{tr} : \mathcal{H} \to R(z, s_k), \ k \in \mathbb{Z} \]
determined by the rules:

\[
\begin{align*}
(1) \quad \text{tr}(ab) &= \text{tr}(ba) \quad \text{ for } a, b \in H_{1,n}(q) \\
(2) \quad \text{tr}(1) &= 1 \quad \text{ for all } H_{1,n}(q) \\
(3) \quad \text{tr}(ag_n) &= z \text{tr}(a) \quad \text{ for } a \in H_{1,n}(q) \\
(4) \quad \text{tr}(at^k_n) &= s_k \text{tr}(a) \quad \text{ for } a \in H_{1,n}(q), \ k \in \mathbb{Z}
\end{align*}
\]
Then, the function $X : L \rightarrow R(z, s_k)$

$$X_\hat{\alpha} = \Delta^{n-1} \cdot (\sqrt{\lambda})^e \text{tr} (\pi (\alpha)),$$

is an invariant of oriented links in $ST$, where $\Delta := -\frac{1-\lambda q}{\sqrt{\lambda (1-q)}}$, $\lambda := \frac{z+1-q}{q z}$, $\alpha \in B_{1,n}$ is a word in the $\sigma_i$’s and $t_i$’s, $\hat{\alpha}$ is the closure of $\alpha$, $e$ is the exponent sum of the $\sigma_i$’s in $\alpha$, $\pi$ the canonical map of $B_{1,n}$ on $H_{1,n}(q)$, such that $t \mapsto t$ and $\sigma_i \mapsto g_i$.

**Remark 1.** As shown in [La2, DL2] the invariant $X$ recovers the HOMFLYPT skein module of $ST$. For a survey on the HOMFLYPT skein module of the lens spaces $L(p, 1)$ via braids, the reader is referred to [DL3].

Following the same idea as in [La2, DL2], in [FG] the analogue of the Kauffman bracket polynomial, $V$, for links in the solid torus $ST$ has been derived from the Temperley-Lieb algebra of type B, $TL^n_B$. This algebra is defined as a quotient of the generalized Iwahori-Hecke algebra of type B, $H_{1,n}(q)$, over the ideal generated by the elements:

$$h_{1,2} := 1 + u (\sigma_1 + \sigma_2) + u^2 (\sigma_1 \sigma_2 + \sigma_2 \sigma_1) + u^3 \sigma_1 \sigma_2 \sigma_1, \quad \text{for all } 1 \leq i \leq n - 2$$

$$h_B := 1 + u \sigma_1 + v t + uv (\sigma t + t \sigma) + u^2 v \sigma t \sigma + uv^2 t \sigma t + (uv)^2 \sigma t \sigma t$$

Note that in [FG] a different presentation for $H_{1,n}$ is used, that involves the parameters $u, v$ and the quadratic equations

$$\sigma_i^2 = (u - u^{-1}) \sigma_i + 1.$$

One can switch from one presentation to the other by taking $\sigma_i = u \sigma_i$, $t = vt$ and $q = u^2$.

Since the Temperley-Lieb algebra of type B is a quotient of the Iwahori-Hecke algebra of type B, in [FG] the authors present the necessary and sufficient conditions so as the Markov trace factors through to $TL^n_B$. Indeed:

**Theorem 5.** [FG, Theorem 4] The trace defined in $H_n(1, q)$ factors through to $TL^n_B$ if and only if the trace parameters take the following values:

$$z = -\frac{1}{u(1+u^2)}, \quad s_1 = \frac{-1 + v^2}{(1+u^2)v}.$$

It is worth mentioning that in [FG] more values of the trace parameters that allow the trace to factor through to $TL^n_B$ are presented, but as explained in [FG], only the values in (5) are of topological interest. Moreover, for those values of the parameters one deduces $\lambda = u^4$. We have the following:
Theorem 6. [FG] The following is an invariant for knots and links in \(ST\):

\[
V^B_{\hat{\alpha}}(u, v) := \left( -\frac{1 + u^2}{u} \right)^{n-1} (u)^{2e} \text{tr}(\pi(\alpha)),
\]

where \(\alpha \in B_{1,n}\) is a word in the \(\sigma_i\)'s and \(t'_i\)'s, \(\hat{\alpha}\) is the closure of \(\alpha\), \(e\) is the exponent sum of the \(\sigma_i\)'s in \(\alpha\), \(\pi\) the canonical map of \(B_{1,n}\) on \(TL_n^B\), such that \(t \mapsto t\) and \(\sigma_i \mapsto g_i\).

In the braid setting of [La2], the elements of KBSM(ST) correspond bijectively to the elements of the following set:

\[
B'_{ST} = \{tt'_1 \ldots t'_n, \ n \in \mathbb{N}\}.
\]

The set \(B'_{ST}\) forms a basis of KBSM(ST) in terms of braids (see also [Tu]). Note that \(B'_{ST}\) is a subset of \(H\) and, in particular, \(B'_{ST}\) is a subset of \(\bigcup_n \Sigma_n'\). Note also that in contrast to elements in \(\Sigma'\), the elements in \(B'_{ST}\) have no gaps in the indices, the exponents are all equal to one and there are no 'braiding tails'.

Remark 2. The invariant \(V^B\) recovers KBSM(ST). Indeed, it gives distinct values to distinct elements of \(B'_{ST}\), since \(\text{tr}(tt'_1 \ldots t'_n) = s_1^n\).

2. THE BASIS \(B_{ST}\) OF KBSM(ST)

In this section we prove the main result of this paper, Theorem 2. Before proceeding with the proof we present the motivation that lead to the new basis \(B_{ST}\) of KBSM(ST):

The relation between KBSM \((L(p,1))\) and KBSM(ST) is presented in [P] and it is shown that:

\[
\text{KBSM} (L(p,1)) = \frac{\text{KBSM}(ST)}{<a-bbm(a)>}, \ a \ in \ the \ basis \ of \ KBSM(ST).
\]

In order to extend \(V^B\) to an invariant of links in \(L(p,q)\) we need to solve an infinite system of equations resulting from the braid band moves. Namely we force:

\[
V^B_{\hat{\alpha}} = V^B_{bbm(\alpha)},
\]

for all \(\alpha\) in the basis of KBSM(ST).

The above equations have particularly simple formulations with the use of the new basis, \(B_{ST}\), for the Kauffman bracket skein module of ST. This is a very technical and difficult task and is the subject of a sequel paper.

We now recall results from [DL2] that we will use throughout the paper.

2.1. AN ORDERING IN THE BASES OF \(S(ST)\). In [DL2] an ordering relation is defined on the sets \(\Sigma\) and \(\Sigma'\) which plays a crucial role in this paper. Before presenting this ordering relation, we first introduce the sets \(\Lambda'\) and \(\Lambda\) and the notion of the index of a word \(w\), denoted \(\text{ind}(w)\), in any of these sets.
Definition 2. We define the following subsets of $\Sigma_n$ and $\Sigma'$ respectively:

\[
\Lambda_{(k)} := \{t_0^{k_0}t_1^{k_1} \ldots t_m^{k_m} \mid k_i \geq k_{i+1}, \sum_{i=0}^{m} k_i = k, k_i \in \mathbb{Z} \setminus \{0\}, \forall i\},
\]

\[
\Lambda'_{(k)} := \{t_0^{t_0^{k_0}}t_1^{t_1^{k_1}} \ldots t_m^{t_m^{k_m}} \mid k_i \geq k_{i+1}, \sum_{i=0}^{m} k_i = k, k_i \in \mathbb{Z} \setminus \{0\}, \forall i\},
\]

\[
\Lambda^{\text{aug}}_{(k)} := \{t_0^{k_0}t_1^{k_1} \ldots t_m^{k_m} \mid \sum_{i=0}^{m} k_i = k, k_i \in \mathbb{Z} \setminus \{0\}, \forall i\},
\]

\[
\Lambda'_{(k)}^{\text{aug}} := \{t_0^{t_0^{k_0}}t_1^{t_1^{k_1}} \ldots t_m^{t_m^{k_m}} \mid \sum_{i=0}^{m} k_i = k, k_i \in \mathbb{Z} \setminus \{0\}, \forall i\}.
\]

Figure 5. Elements in the different bases of KBSM(ST).
Note that elements in the set \( \Lambda_{(k)} \) have ordered exponents, while elements in \( \Lambda_{aug}^{(k)} \) have arbitrary exponents. Obviously, \( \Lambda_{(k)} \subseteq \Lambda_{aug}^{(k)} \subseteq \Sigma_n \).

**Remark 3.** In [DL2] the set \( \Lambda := \bigcup_k \Lambda_{(k)} \) is showed to be a basis for the HOMFLYPT skein module of \( ST \).

**Definition 3.** [DL2] Definition 1] Let \( w \) a word in \( \Lambda \). Then, the index of \( w \), \( ind(w) \), is defined to be the highest index of the \( t_i \)'s in \( w \). Similarly, in \( \Sigma' \) or \( \Sigma \), \( ind(w) \) is defined as above by ignoring possible gaps in the indices of the looping generators and by ignoring the braiding parts in the algebras \( H_n(q) \). Moreover, the index of a monomial in \( H_n(q) \) is equal to 0.

We now proceed with presenting an ordering relation in the sets \( \Sigma \) and \( \Sigma' \), which passes to their respective subsets \( \mathcal{B}_{ST} \) and \( \mathcal{B'}_{ST} \).

**Definition 4.** [DL2] Definition 2] Let \( w = t_{i_1}^{k_1} \cdots t_{i_s}^{k_s} \cdot \beta_1 \) and \( u = t_{j_1}^{\lambda_1} \cdots t_{j_t}^{\lambda_t} \cdot \beta_2 \) in \( \Sigma' \), where \( k_t, \lambda_s \in \mathbb{Z} \) for all \( t, s \) and \( \beta_1, \beta_2 \in H_n(q) \). Then, we define the following ordering in \( \Sigma' \):

(a) If \( \sum_{i=0}^{t} k_i < \sum_{i=0}^{u} \lambda_i \), then \( w < u \).

(b) If \( \sum_{i=0}^{t} k_i = \sum_{i=0}^{u} \lambda_i \), then:

(i) if \( ind(w) < ind(u) \), then \( w < u \),

(ii) if \( ind(w) = ind(u) \), then:

(\(\alpha\)) if \( i_1 = j_1, \ldots, i_{s-1} = j_{s-1}, i_s < j_s \), then \( w > u \),

(\(\beta\)) if \( i_t = j_t \) for all \( t \) and \( k_{\mu} = \lambda_{\mu}, k_{\mu-1} = \lambda_{\mu-1}, \ldots, k_{i+1} = \lambda_{i+1}, |k_i| < |\lambda_i| \), then \( w < u \),

(\(\gamma\)) if \( i_t = j_t \) for all \( t \) and \( k_{\mu} = \lambda_{\mu}, k_{\mu-1} = \lambda_{\mu-1}, \ldots, k_{i+1} = \lambda_{i+1}, |k_i| = |\lambda_i| \) and \( k_i > \lambda_i \), then \( w < u \),

(\(\delta\)) if \( i_t = j_t \forall t \) and \( k_i = \lambda_i \forall i \), then \( w = u \).

The ordering in the set \( \Sigma \) is defined as in \( \Sigma' \), where \( t_i' \)'s are replaced by \( t_i \)'s.

2.2. From \( \mathcal{B'}_{ST} \) to \( \Lambda \). In this subsection we recall a series of results from [DL2] in order to convert elements in \( \mathcal{B'}_{ST} \) to elements in \( \Lambda \). In order to simplify the algebraic expressions obtained throughout this procedure and throughout the paper in general, we first introduce the following notation:

**Notation 1.** We set \( \tau_{i,i+m}^{k_i} \cdots t_{i+m}^{k_{i+m}} \in \Sigma \) and \( \tau_{i,i+m}^{k_i} \cdots t_{i+m}^{k_{i+m}} \in \Sigma' \), for \( m \in \mathbb{N} \), \( k_j \neq 0 \) for all \( j \).

**Remark 4.** Using Notation [1] elements in \( \mathcal{B'}_{ST} \) are of the form \( \tau_{0,n}^{t_{i_0}} \cdots t_{n}^{t_{n}} \), for \( n \in \mathbb{N} \), that is \( \mathcal{B'}_{ST} = \{ \tau_{0,n}^{t_{i_0}} \}_{n=0}^{\infty} \). Moreover, we set \( \mathcal{K}_{ST} = \{ \tau_{0,n}^{t_{i_0}} \}_{n=0}^{\infty} \), and so elements in \( \mathcal{K}_{ST} \) are of the form \( \tau_{0,n}^{t_{i_0}} \cdots t_{n}^{t_{n}} \), for \( n \in \mathbb{N} \).

Moreover,

\[
\Lambda'(k) = \left\{ \tau_{0,n}^{k_0,n} \mid k_i \geq k_i-1, \sum_{i=0}^{n} k_i = k, k_i \in \mathbb{Z} \setminus \{0\} \right\}, \quad \Lambda' = \bigoplus_{k \in \mathbb{Z}} \Lambda'(k)
\]

\[
\Lambda(k) = \left\{ \tau_{0,n}^{k_0,n} \mid k_i \geq k_i-1, \sum_{i=0}^{n} k_i = k, k_i \in \mathbb{Z} \setminus \{0\} \right\}, \quad \Lambda = \bigoplus_{k \in \mathbb{Z}} \Lambda(k)
\]
We also introduce the notion of homologous words, which is crucial for relating the sets $B_{ST}'$ and $K_{ST}$ via a triangular matrix.

**Definition 5.** [DL2, Definition 4] We say that two words $w' \in \Sigma'$ and $w \in \Sigma$ are homologous, denoted $w' \sim w$, if $w$ is obtained from $w'$ by changing $t_{i}'$ into $t_{i}$ for all $i$ and ignoring the braiding parts.

We now recall a result from [DL2] in order to convert monomials in the $t_{i}'$'s in general to monomials in the $t_{i}$'s in $\Sigma$. More precisely:

**Theorem 7.** [DL2, Theorem 7] The following relations hold:

$$\tau_{0,n} = \sum_{j} B_{j} \tau_{j} \cdot \beta_{j},$$

where $w, \beta_{j} \in H_{n+1}(q), \tau_{j} \in \Sigma_{n}$, such that $\tau_{j} < \tau_{0,n}, \forall j$ and $A, B_{j}$ coefficients.

Since now we are only interested in converting elements in the set $B_{ST}'$ to sums of monomials in the $t_{i}$'s, we have the following corollary:

**Corollary 1.** The following relations hold:

$$\tau_{0,n} = \sum_{j} B_{j} \tau_{j} \cdot \beta_{j},$$

where $w, \beta_{j} \in H_{n+1}(q), \tau_{j} \in \Sigma_{n}$, such that $\tau_{j} < \tau_{0,n}, \forall j$ and $A, B_{j}$ coefficients.

After expressing an element $\tau_{0,n} \in B_{ST}'$ as sums of elements in $\Sigma_{n}$, we obtain the homologous word $\tau_{0,n}$, the homologous word again followed by a ‘braiding tail’ $w \in TL_{n}$ and elements in $\Sigma_{n}$ with possible ‘gaps’ in the indices. In [DL2], using conjugation, monomials in the $t_{i}$'s with ‘gaps’ in the indices are expressed as sums of monomials in $\Lambda$, followed by ‘braiding tails’. For the expressions that we obtain after appropriate conjugations we shall use the notation $\hat{\cdot}$. We recall the following result from [DL2]:

**Theorem 8.** [DL2, Theorem 8] Let $T$ be a monomial in the $t_{i}$'s with ‘gaps’ in the indices. The following relations hold:

$$T \hat{\cdot} \equiv \sum_{i} A_{i} \cdot T_{i} \cdot w_{i},$$

where $T_{i} \in \Lambda_{(n)},$ such that $T_{i} < T, \forall i$, $w_{i} \in TL_{n+1}, \forall i$, and $A_{i}$ coefficients.

As shown in [DL2], elements in the set $\Lambda$ followed by ‘braiding tails’ can be expressed as sums of elements in $\Lambda_{aug}$ by using conjugation and stabilization moves. For the expressions that we obtain after appropriate conjugations and stabilization moves we shall use the notation $\tilde{\cdot}$. Indeed, we have the following:

**Theorem 9.** [DL2, Theorem 10] Let $\tau \in \Lambda$ and $w \in TL_{n}$. Then, applying conjugation and stabilization moves we have that:

$$\tau \cdot w \tilde{\equiv} \sum_{j} A_{j} \cdot \tau_{j},$$

where $\Lambda_{(n)} \ni \tau_{j} \leq \tau, \forall j$. 

Combining now Theorems 7, 8 & 9 and Corollary 1 we have that an element $\tau' \in B'_ST$ can be expressed as a sum of the homologous word $\tau \in K_{ST}$ and lower order terms in $\Lambda_{(n)}$. More precisely, we have the following:

**Corollary 2.** Let $\tau'_{0,n} \in B'_ST$. The following relations hold:

\[(12)\quad \tau'_{0,n} \hat{\cong} \tau_{0,n} + \sum_{i} A_i \cdot \tau_i,\]

where $\tau_i \in \Lambda$ such that $\tau_i < \tau_{0,n} \sim \tau'_{0,n}$ for all $i$.

From Corollary 2 we have that monomials $\tau'_{0,n} \in B'_ST$ can be expressed as sums of their corresponding homologous word in $K_{ST}$ with invertible coefficients, and elements $\tau_i \in \Lambda$ of lower order than $\tau_{0,n}$. The point now is that the elements $\tau_i$ do not necessarily belong to $K_{ST}$, but using conjugation and stabilization moves, we will show that these elements can be expressed as monomials in $B_{ST}$ of lower order than $\tau_{0,n}$, and thus, $B_{ST}$ spans $KBSM(ST)$. We deal with these elements in the next subsection.

### 2.3. From $\Lambda$ to $B_{ST}$.

As explained in the Introduction, our goal is to relate the sets $B'_ST$ and $B_{ST}$ via an infinite block diagonal, invertible matrix. From Corollary 2 we have that an element in $B'_ST$ can be expressed as a sum of the homologous word in $K_{ST} \subset \Lambda$ and elements in $\Lambda$ of lower order. In this subsection we convert elements in $\Lambda$ to sums of elements in $B_{ST}$. We first deal with the homologous word $\tau_{0,n} \in \Lambda$ of $\tau'_{0,n} \in B'_ST$. We have the following:

**Proposition 1.** Applying conjugation, stabilization moves and relations $\Xi$ the following relations hold:

\[(13)\quad \Lambda \ni \tau_{0,n} \hat{\cong} A \cdot t^{\text{ind}(\tau_{0,n})+1} + \sum_{i=0}^{\text{ind}(\tau_{0,n})} A_i \cdot t^i,\]

where $A_i$ coefficients in the ground ring for all $i$.

**Proof.** We prove Proposition 1 by strong induction on the order of $\tau_{0,n}$.

The base of the induction is the monomial $tt_1 \in \Lambda$ of index 1. We have that:

\[
\begin{align*}
tt_1 &= t\sigma_1 t\sigma_1 = \sigma_1 t\sigma_1 t = \\
&= -\frac{1}{(uv)^2} (1 + u\sigma_1 + vt + uv(\sigma t + t\sigma) + u^2 v\sigma t + uv^2 t\sigma t) \hat{\cong} \\
&\hat{\cong} -\frac{1}{(uv)^2} (1 + uz + v t + 2uvzt + u^2 v t\sigma_1^2 + uv^2 t^2\sigma_1) \hat{\cong} \\
&\hat{\cong} -\frac{1}{(uv)^2} (1 + uz + v t + 2uvzt + u^2 v t + u^2 vz(u - u^{-1}) t + uv^2 z t^2) = \\
&= (-u^{-1}z) t^2 + \sum_{i=0}^{1} A_i \cdot t^i.
\end{align*}
\]

So, Proposition 1 holds for $tt_1$. 

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Assume now that Proposition 1 holds for all monomials \( \tau_i \) of lower order than \( \tau_{0,n} \). Then, we have:

\[
\tau_{0,n} := n t_1 (\tau_{2,n}) = (t \sigma_1 t \sigma_1) (\tau_{2,n}) = (\sigma_1 t \sigma_1 t) (\tau_{2,n}) =
\]

\[
= -\frac{1}{(uv)^2} \left[ 1 + u \sigma_1 + vt + uv(\sigma_1 t + t \sigma_1) + u^2 \sigma_1 t \sigma_1 + uv^2 t \sigma_1 \right] (\tau_{2,n}) \approx
\]

\[
\approx -\frac{1}{(uv)^2} \left[ \tau_{2,n} + n \tau_{2,n} \sigma_1 + vt \tau_{2,n} + 2uv \tau_{2,n} \sigma_1 + u^2 \tau_{2,n} \sigma_1^2 + uv^2 \tau_{2,n} \sigma_1 \right] \approx
\]

\[
\approx -\frac{1}{(uv)^2} t^2 \tau_{2,n} \sigma_1 + \sum_i A_i \cdot \tau_i, \quad \text{where } \tau_i < \tau, \forall i.
\]

According to the ordering relation, on the right hand side of this equation we have the element \( t^2 \tau_{2,n} \sigma_1 \) and a sum of elements of lower order than \( \tau_{0,n} \), since the sums of the exponents in the \( t_i \)'s in these elements are less than \( n + 1 \). Moreover, the monomial \( t^2 \tau_{2,n} \sigma_1 \) contains a gap in the indices, and thus it is of lower order than \( \tau_{0,n} \) (recall Definition 4). Moreover, this monomial is followed by the ‘braiding tail’ \( \sigma_1 \). According now to Theorems 8 & 9, this element can be expressed as a sum of elements in \( \Lambda_{(n)} \) of lower order than \( t^2 \tau_{2,n} \sigma_1 \) and hence, of lower order than \( \tau_{0,n} \). By the induction hypothesis the proof is now concluded. \( \square \)

We now deal with arbitrary elements in \( \Lambda \) and convert them to sums of elements in \( B(ST) \). We will need the following lemmas:

**Lemma 1.** The following relations hold for all \( n \in \mathbb{N} \):

\[
t^n t_1 \approx -\frac{1}{u} z t^{n+1} + \sum_{i=n-1}^{n} A_i t^i,
\]

where \( A_i \) coefficients for all \( i \).

**Proof.** We prove Lemma 1 by induction on \( n \). For \( n = 1 \) we have: \( t t_1 = -\frac{1}{u} z t_1 + 1 A_i t^i \) (relations 3). Assume now that the relation is true for \( n \). Then for \( n + 1 \) we have:

\[
t^{n+1} t_1 = t \cdot (t^n t_1) \quad \text{ind.} \quad -\frac{1}{u} z t^{n+2} + \sum_{i=n-1}^{n} A_i t^{i+1} = -\frac{1}{u} z t^{n+2} + \sum_{i=n}^{n+1} A_i t^i.
\]

The following lemma will serve as a basis for the induction hypothesis applied in the proof of the main result of this section.

**Lemma 2.** The following relations hold for \( n, m \in \mathbb{N} \):

\[
t^n t_1^m \approx A \cdot t^{n+m} + \sum_{i=0}^{n+m-1} A_i t^i,
\]

where \( A, A_i \) coefficients for all \( i \).

**Proof.** We prove Lemma 2 by strong induction on the order of \( t^n t_1^m \in \Lambda^{aug} \). The base of the induction is Lemma 1 for \( n = 1 \). Assume that the relations are true for all elements in \( \Lambda^{aug} \) of lower order than \( t^n t_1^m \). Then, for \( t^n t_1^m \) we have:

\[
t^n t_1^m = t^{n-1}(tt_1) t_1^{m-1} = -\frac{1}{u^2 v^2} t^{n-1}(1 + \sigma_1 + vt + uv(\sigma_1 t + t \sigma_1) + u^2 \sigma_1 t \sigma_1 + uv^2 t \sigma_1) t_1^{m-1} \approx
\]
\[ \equiv - \frac{1}{u^2v^2} t^{n-1}l^{m-1} - \frac{1}{uv} t^{n-1}l^{m-1}\sigma_1 - \frac{1}{u^2v} t^{n-1}l^{m-1} - \frac{2}{u} t^{n-1}l^{m-1}\sigma_1 - \frac{1}{u} l^{n-1}l^{m-1} - \frac{1}{u} l^{n+1}l^{m-1}l^{n} \sigma_1. \]

The sum of the exponents in the elements \( t^{n-1}l^{m-1} \) and \( l^{n-1}l^{m-1} \) on the right hand side of the relation are less than \( n + m \), and thus, these elements are of lower order than \( t^n l^m \) (recall Definition 3). Applying now Theorem 9 on the elements \( t^{n-1}l^{m-1}\sigma_1, t^n l^{m-1}\sigma_1 \) and \( t^{n+1}l^{m-1}\sigma_1 \), we convert them to sums of elements in \( \Lambda^{\text{aug}} \) of lower order than \( t^n l^m \). The proof is concluded by the induction hypothesis.

**Theorem 10.** Let \( \tau \in \Lambda_{(k)}^{\text{aug}} \). The following relations hold:

\[ \tau \equiv \sum_{i=0}^k A_i t^i, \]

where \( A_i \) coefficients.

**Proof.** Consider a monomial \( \tau = t^{k_0}l^{k_1}\ldots t^{k_n} \in \Lambda^{\text{aug}} \). We prove the relations by strong induction on the order of \( \tau \). The basis of the induction is Lemma 2, since it deals with the monomials of type \( t^n l^m \), which are of minimal order among all non-trivial monomials in \( \Lambda^{\text{aug}} \). We assume that the statement of Theorem 10 is true for all elements in \( \Lambda^{\text{aug}} \) of lower order than \( \tau \) and we will show that it is true for \( \tau \). We have that:

\[ \tau = t^{k_0}l^{k_1}\ldots t^{k_n} = t^{k_0-1}(tt)l^{k_1-1}\ldots t^{k_n} = \]

\[ = t^{k_0-1}\left[-\frac{1}{uv} (1 + u\sigma_1 + vt + uv(\sigma_1 t + t\sigma_1) + u^2v t\sigma_1 + uv^2t\sigma_1)\right] l^{k_1-1}\ldots t^{k_n} \equiv \]

\[ \equiv -\frac{1}{uv} t^{k_0-1}l^{k_1-1}\ldots t^{k_n} - \frac{1}{uv} t^{k_0-1}t^{k_1-1}\ldots t^{k_n} + \frac{1}{u} t^{k_0+1}l^{k_1-1}\ldots t^{k_n} - \]

\[ = \frac{1}{uv} t^{k_0}t^{k_1-1}\ldots t^{k_n} - \frac{1}{uv} t^{k_0-1}l^{k_1-1}\ldots t^{k_n} - \frac{1}{u} t^{k_0+1}l^{k_1-1}\ldots t^{k_n} - \]

\[ = \frac{1}{uv} t^{k_0}l^{k_1-1}t^{k_2,n} - \frac{1}{uv} t^{k_0}l^{k_1-1}t^{k_2,n} + \frac{1}{u} t^{k_0+1}l^{k_1-1}t^{k_2,n} - \]

On the right-hand side of this relation we have the following monomials in \( \Lambda^{\text{aug}} \):

\[ t^{k_0-1}l^{k_1-1}t^{k_2,n} \quad t^{k_0}l^{k_1-1}t^{k_2,n} \quad t^{k_0-1}l^{k_1-1}t^{k_2,n} \quad \tau, \]

and the monomials \( t^{k_0-1}l^{k_1-1}t^{k_2,n}\sigma_1, t^{k_0}l^{k_1-1}t^{k_2,n}\sigma_1 \) and \( t^{k_0+1}l^{k_1-1}t^{k_2,n}\sigma_1 \) in the \( H_n(q) \)-module \( \Lambda^{\text{aug}} \). Applying Theorem 9 on these monomials we have that:

\[ t^{k_0-1}l^{k_1-1}t^{k_2,n}\sigma_1 \equiv \sum_i A_i \tau_i, \quad \text{such that } \tau_i < t^{k_0-1}l^{k_1-1}t^{k_2,n} < \tau_{0,n}^{k_0}, \quad \text{for all } i \]

\[ t^{k_0}l^{k_1-1}t^{k_2,n}\sigma_1 \equiv \sum_j B_j \tau_j, \quad \text{such that } \tau_j < t^{k_0}l^{k_1-1}t^{k_2,n} < \tau_{0,n}^{k_0}, \quad \text{for all } j \]

\[ t^{k_0+1}l^{k_1-1}t^{k_2,n}\sigma_1 \equiv \sum_i C_i \tau_m, \quad \text{such that } \tau_m < t^{k_0+1}l^{k_1-1}t^{k_2,n} < \tau_{0,n}^{k_0}, \quad \text{for all } m \]

and thus, from the induction hypothesis the relation hold. □
2.4. Proof of Theorem 2. Let \( \tau_{0,n}' \in B'(ST) \subset \Lambda_{(k)} \subset \Lambda_{(k)}^{aug} \). Then:

\[
\tau_{0,n}' \overset{\text{Cor. 2}}{\simeq} \tau_{0,n} + \sum_{i=0}^{n} A_i \cdot \tau_i \overset{\text{Prop. 1}}{\simeq} A \cdot t^{\text{index}(\tau+1)} + \sum_{i=0}^{\text{ind}(\tau)} A_i \cdot t^i + \sum_{i=0}^{k} A_i \cdot \tau_i \overset{\text{Thm. 10}}{\simeq} A \cdot t^{\text{index}(\tau+1)} + \sum_{i=0}^{\text{ind}(\tau)} A_i \cdot t^i + \sum_{i=0}^{k} B_i \cdot t^i = \sum_{i=0}^{n+1} C_i \cdot t^i,
\]

where \( A_i, B_i, C_i \) coefficients. Thus, elements in \( B'(ST) \) can be expressed as sums of elements in \( B(ST) \), that is:

The set \( B(ST) \) spans the Kauffman bracket skein module of the solid torus.

We now prove linear independence of the set \( B(ST) \):

The \( t^n \)'s geometrically consist of closed loops in the fundamental group of ST. Since \( \pi_1(ST) = \mathbb{Z} \), \( t^n \neq t^m \) for \( n \neq m \) on the level of \( \pi_1(ST) \). This fact factors through to the Kauffman bracket skein module of ST, since the \( t^n \)'s can not be simplified neither by applying braid relations, nor by conjugation and stabilization moves. Moreover, the Temperley-Lieb type crossing switches cannot be applied on the \( t^n \)'s, since they contain no classical crossings in our setting. Thus, the value of the KBSM(ST) on these elements remains the same as the value of the invariant \( V^B \) on these elements.

The proof of Theorem 2 is now concluded.

References

[D] I. Diamantis, The Kauffman bracket skein module of the lens spaces \( L(p, q) \) via braids, in preparation.
[DL1] I. Diamantis, S. Lambropoulou, Braid equivalences in 3-manifolds with rational surgery description, *Topology and its Applications*, 194 (2015), 269-295.
[DL2] I. Diamantis, S. Lambropoulou, A new basis for the HOMFLYPT skein module of the solid torus, *J. Pure Appl. Algebra* 220 Vol. 2 (2016), 577-605.
[DL3] I. Diamantis, S. Lambropoulou, The braid approach to the HOMFLYPT skein module of the lens spaces \( L(p,1) \), Springer Proceedings in Mathematics and Statistics (PROMS), *Algebraic Modeling of Topological and Computational Structures and Application*, (2017) [arXiv:1702.06290v1[math.GT]].
[DL4] I. Diamantis, S. Lambropoulou, An important step for the computation of the HOMFLYPT skein module of the lens spaces \( L(p,1) \) via braids, to appear, arXiv:???[math.GT].
[DL5] I. Diamantis, S. Lambropoulou, The HOMFLYPT skein module of the lens spaces \( L(p,1) \) via braids, in preparation.
[DLP] I. Diamantis, S. Lambropoulou, J. H. Przytycki, Topological steps on the HOMFLYPT skein module of the lens spaces \( L(p,1) \) via braids, *J. Knot Theory and Ramifications, J. Knot Theory and Ramifications*, 25, No. 14, (2016).
[FG] M. Flores, D. Goundaroulis, TFramization of a Temperley-Lieb algebra of type B.
[GM] B. Gabrovšek, M. Mroczkowski, The Homlypt skein module of the lens spaces \( L(p,1) \), *Topology and its Applications*, 175 (2014), 72-80.
[HK] J. Hoste, M. Kidwell, Dichromatic link invariants, *Trans. Amer. Math. Soc.*, 321 (1990), No. 1, 197-229.
[HP] J. Hoste, J.H. Przytycki, A survey of skein modules of 3-manifolds. *Knots 90 (Osaka, 1990)*, de Gruyter, Berlin, (1992) 3637-379.
V. F. R. Jones, A polynomial invariant for links via Neumann algebras, Bull. Amer. Math. Soc. 129, (1985) 103-112.

S. Lambropoulou, Knot theory related to generalized and cyclotomic Hecke algebras of type B, J. Knot Theory and its Ramifications 8, No. 5, (1999) 621-658.

S. Lambropoulou, Knot theory related to generalized and cyclotomic Hecke algebras of type B, J. Knot Theory Ramifications 8, No. 5, (1999) 621-658.

S. Lambropoulou, C.P. Rourke (2006), Markov’s theorem in 3-manifolds, Topology and its Applications 78, (1997) 95-122.

S. Lambropoulou, C. P. Rourke, Algebraic Markov equivalence for links in 3-manifolds, Compositio Math. 142 (2006) 1039-1062.

J. Przytycki, Skein modules of 3-manifolds, Bull. Pol. Acad. Sci.: Math., 39, 1-2 (1991), 91-100.

V.G. Turaev, The Conway and Kauffman modules of the solid torus, Zap. Nauchn. Sem. Lomi 167 (1988), 79-89. English translation: J. Soviet Math. (1990), 2799-2805.

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