Fix a finite dimensional semisimple Lie algebra over $\mathbb{C}$, and a Borel subalgebra. It is a long standing problem to determine the extension groups between Verma modules in the category $\mathcal{O}$ of [BGG]. No general formula, even conjectural, is known.

Let $W$ be the Weyl group and $w_0 \in W$ the longest element. For $w \in W$ write $\Delta_w$ for the Verma module of highest weight $w^{-1}w_0 \cdot 0$ (dot action). The purpose of this note is to give a recursive formula (Corollary 5) for $\dim \text{Ext}^1(\Delta_v, \Delta_w)$ (throughout $'\text{Ext}^\bullet = '\text{Ext}^\bullet_{\mathcal{O}}$).

Write $X$ for the flag variety associated to our Lie algebra. For each $w \in W$, we have a Schubert cell $C_w \subseteq X$ and an opposite Schubert cell $C_w^\circ \subseteq X$. Let $\ell: W \to \mathbb{Z}_{\geq 0}$ denote the length function and $\preceq$ the Bruhat order. If $v \leq w$, then $C_v \cap C_w$ is affine and smooth of dimension $\ell(w) - \ell(v)$ (see [R]). Further, $C_v \cap C_v^\circ = \text{pt}$; and if $v \not\leq w$, then $C_v \cap C_w = \emptyset$.

By [RSW, Proposition 4.2.1] and [BGS, Proposition 3.5.1]):

$$\text{Ext}^\bullet(\Delta_v, \Delta_w) = H^\bullet_{C_w}(C_v \cap C_w),$$

for all $v, w \in W$,

where $H^\bullet_{C_w}$ denotes compactly supported cohomology (with $\mathbb{C}$-coefficients). The cohomology $H^\bullet_{C_w}(C_v \cap C_w)$ comes equipped with a canonical (rational) Hodge structure which is respected by the usual long exact sequences. Consequently, we can and will view the extensions $\text{Ext}^\bullet(\Delta_v, \Delta_w)$ as Hodge structures. Denote the trivial 1-dimensional Hodge structure by $\mathbb{Q}^H$, and write $(n)$ for the $n$-th Tate twist.

The following Proposition is attributed to V. Deodhar in [RSW] (see [RSW, Lemma 4.3.1] for a proof).

**Proposition 1.** Let $v, w \in W$ with $v \leq w$. Let $s \in W$ be a simple reflection such that $ws < w$.

(i) If $vs < v$, then $C^v \cap C_w \simeq C^{vs} \cap C_{ws}$.

(ii) If $vs > v$ and $vs \not\leq ws$, then $C^v \cap C_w \simeq C^v \cap C_{ws} \times \mathbb{C}^*$.

(iii) If $vs > v$ and $vs \leq ws$, then there exists a closed immersion

$$(C^{ws} \cap C_{ws}) \times \mathbb{C} \hookrightarrow C^v \cap C_w$$

with open complement isomorphic to $(C^v \cap C_{ws}) \times \mathbb{C}^*$.

**Corollary 2.** $\text{Hom}(\Delta_v, \Delta_w)$ is pure of weight $0$. 

Theorem 4. We may assume

Proof. So the cohomology long exact sequence and Künneth formula yield (iii).

Corollary 3. Let \( v, w \in W \) with \( v \leq w \). Let \( s \in W \) be a simple reflection such that \( ws > w \).

(i) If \( vs < v \), then \( \text{Ext}^1(\Delta_v, \Delta_{ws}) \simeq \text{Ext}^1(\Delta_v, \Delta_w) \).

(ii) If \( vs > v \) and \( vs \leq w \), then \( \text{Ext}^1(\Delta_v, \Delta_w) \oplus Q^H(-1) \simeq \text{Ext}^1(\Delta_v, \Delta_{ws}) \).

(iii) If \( vs > v \) and \( vs \leq w \), then there is an exact sequence

\[
0 \to Q^H(-1) \to \text{Ext}^1(\Delta_v, \Delta_w) \oplus Q^H(-1) \to \text{Ext}^1(\Delta_v, \Delta_{ws}) \to \text{Ext}^1(\Delta_v, \Delta_w)(-1).
\]

Proof. (i) is clear. Further, \( \dim \text{Hom}(\Delta_x, \Delta_y) = 1 \) if and only if \( x \leq y \). Thus, the Künneth formula yields (ii). In the situation of (iii):

\[
\text{Hom}(\Delta_v, \Delta_w) = \text{Hom}(\Delta_v, \Delta_{ws}) = \text{Hom}(\Delta_{vs}, \Delta_w) = Q^H.
\]

So the cohomology long exact sequence and Künneth formula yield (iii). \( \square \)

Theorem 4. \( \text{Ext}^1(\Delta_v, \Delta_w) \) is pure of weight 2.

Proof. We may assume \( v \leq w \). Proceed by downwards induction on \( w \). If \( w \) is the longest element, this is [M Theorem 32]. If \( w \) is not the longest element, pick a simple reflection \( s \) such that \( ws > w \) and apply Corollary 3. \( \square \)

Corollary 5. Let \( v, w \in W \) with \( v \leq w \). Let \( s \in W \) be a simple reflection such that \( ws < w \). Then

\[
\dim \text{Ext}^1(\Delta_v, \Delta_w) = \begin{cases} 
\dim \text{Ext}^1(\Delta_v, \Delta_{ws}) & \text{if } vs < v; \\
1 + \dim \text{Ext}^1(\Delta_v, \Delta_{ws}) & \text{if } vs > v \text{ and } vs \leq ws; \\
\dim \text{Ext}^1(\Delta_v, \Delta_{ws}) & \text{if } vs > w \text{ and } vs \leq ws.
\end{cases}
\]

Some concluding observations:

(i) \( \dim \text{Ext}^1(\Delta_v, \Delta_w) \) coincides with \((-1)^{\ell(w) - \ell(v)} \) times the coefficient of \( q \) in the corresponding Kazhdan-Lusztig \( R \)-polynomial.

(ii) Theorem 4 implies that the graded algebra \( \bigoplus_{v,w \in W} \text{Ext}^*(\Delta_v, \Delta_w) \) cannot, in general, be generated in degree 0 and 1. As otherwise \( \text{Ext}^1(\Delta_v, \Delta_w) \) would be pure of weight 2i. This would contradict [Boc], since the Kazhdan-Lusztig \( R \)-polynomials are the Hodge-Euler polynomials of the \( C_v \cap C_w \).
(iii) Using Corollary 5 one can check
\[ \dim \text{Ext}^1(\Delta_v, \Delta_w) = \dim \text{Ext}^{\ell(w) - \ell(v) - 1}(\Delta_v, \Delta_w). \]

This upgrades to a canonical isomorphism by combining Theorem 4 with the main result of [BGS].

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