Cyclotron frequency and the quantum clock

S. Mignemi‡

1 Dipartimento di Matematica e Informatica, Università di Cagliari
viale Merello 92, 09123 Cagliari, Italy
2 INFN, Sezione di Cagliari, Cittadella Universitaria, 09042 Monserrato, Italy

Abstract

We discuss the corrections to the orbital period of a particle in a constant magnetic field, driven by the model of noncommutative geometry recently associated to a quantum clock. The effects are extremely small, but in principle detectable.

‡ e-mail: smignemi@unica.it
1. Introduction

In the last years, models of noncommutative geometries have become one of the main areas of research in the field of quantum gravity [1]. They can usually be obtained by a deformation of the commutation relations of standard quantum mechanics, yielding a nontrivial commutator between spacetime coordinates. The possibility that they could give rise to observable effects has also been largely studied [2]. Realistic experiments are based on astrophysical observations, but it is interesting to suggest different effects that at least in principle could be observed in the laboratory.

Here, we describe a new effect, related to the motion of a particle in a constant magnetic field. We shall consider a recently proposed model of noncommutative geometry [3], inspired by the discussion of a quantum clock [4], since in that case the calculations are particularly simple. A quantum clock is a device that measures time through the decay of a sample of radioactive matter. Its nature entails an uncertainty relation between the interval of time measured and the size of the clock [4]. This uncertainty relation can be derived in a relativistic quantum mechanical setting by postulating a deformation of the Heisenberg algebra, in which the time and space coordinates have nontrivial commutation relations [3].

The main characteristic of this model is that only the time variable is deformed, which makes rather difficult to find observable effects. The deformation can for example manifest itself only in time-related measurements in a relativistic context. A possibility of observing the deformation may however come from periodic motion. In [3] were for example considered planetary orbits in a Schwarzschild background and corrections to the period of orbits predicted by general relativity were found. However, it is unlikely that noncommutative geometries are effective in such macroscopic situations.

A more realistic experiment where such effects could be observable is the motion of a particle in a constant magnetic field. It is well known that in special relativity the orbits are circular, with frequency and radius depending on the velocity (and hence the energy) of the particles. As we shall show, in the quantum clock geometry the orbits are still circular, but their radius and frequency are modified with respect to special relativity. The corrections for realistic experiments are extremely small, but in principle might be detectable observing a great number of orbits.

The same effect can also occur in different models of noncommutative geometries, like Snyder [5] or κ-Poincaré [6], but in those cases the calculations are much more involved.

2. The model

Let us review the geometry of the quantum clock [4]. For simplicity, we consider here its classical limit, where commutators are replaced by Poisson brackets. We denote $x^\mu = (t, x^i)$, $p_\mu = (E, p_i)$, with $i = 1, 2, 3$ and signature $(-1, 1, 1, 1)$. The Poisson brackets read [4]

$$\{x_i, x_j\} = 0, \quad \{t, x_i\} = -\frac{\beta x_i}{r}, \quad \{p_\mu, p_\nu\} = 0, \quad \{x_i, p_j\} = \delta_{ij},
\{x_i, E\} = 0, \quad \{t, E\} = 1, \quad \{t, p_i\} = \frac{\beta}{r} \left(p_i - \frac{x_j p_j}{r^2} x_i\right).$$

(1)

with $r = \sqrt{x_i^2}$ and, in the classical limit, $\beta = \frac{G}{c^2} \sim 8 \cdot 10^{-44} \text{ sec}^2 \text{ m kg}$, with $G$ the gravitational constant. In the following we set $c = 1$.

We study the relativistic motion of a charged particle in a static magnetic field. Since the Poincaré algebra is undeformed in the quantum clock geometry [3], we take as Hamiltonian the standard relativistic one,

$$H = \frac{1}{2}m(p_\mu - c A_\mu)^2 = \frac{m}{2}$$

(2)

and consider a time-independent vector potential, $A_\mu = A_\mu(x_i)$, with $A_0 = 0$. Taking into account the Poisson brackets (1), the Hamilton equations read

$$m \ddot{x}_i = p_i - c A_i, \quad \dot{p}_i = \frac{e}{m} \frac{\partial A_j}{\partial x_i} (p_j - mA_j),$$

(3)
$$ml = E + \frac{\beta}{r} \left[ \left( p_i - \frac{x_j p_j}{r^2} x_i - e x_j \frac{\partial A_i}{\partial x_j} \right) (p_i - e A_i) \right], \quad \dot{E} = 0,$$

where a dot denotes a derivative with respect to the evolution parameter \( \tau \). The only equation that differs from special relativity is the one for the derivative of the coordinate time. It follows that \( E \) is constant and can be identified with the energy of the particle. Moreover, eqs. (3) give

$$\dot{p}_i = e \frac{\partial A_j}{\partial x_i} v^j,$$

and then

$$\dot{u}_i = \frac{e}{m} F_{ij} u_j,$$

where we have defined \( u_i = \dot{x}_i \).

Let us now consider a constant magnetic field in the \( z \) direction, namely

$$F_{ab} = B \epsilon_{ab}, \quad A_a = -\frac{B}{2} \epsilon_{ab} x_b, \quad a, b = 1, 2.$$

Then (5) becomes

$$\dot{u}_a = \frac{eB}{m} \epsilon_{ab} u_b, \quad \dot{u}_3 = 0.$$

It follows that \( u^2 \equiv u_i^2 = \text{const.} \) With suitable initial conditions, the solution of (7) is given by

$$u_1 = u \cos \omega_0 \tau, \quad u_2 = u \sin \omega_0 \tau, \quad u_3 = 0,$$

where \( \omega_0 = \frac{eB}{m} \) and \( u \) is an integration constant which coincides with the norm of \( u_i \). After integration, choosing the origin in the center of the orbit,

$$x_1 = \frac{u}{\omega_0} \sin \omega_0 \tau, \quad x_2 = -\frac{u}{\omega_0} \cos \omega_0 \tau, \quad x_3 = 0.$$

It follows that also \( r \) is constant and equals \( \frac{u}{\omega_0} \). Till now, the solutions for the orbits are identical to those of special relativity. However, the relation between the parameter \( \tau \) and the time coordinate \( t \) is now deformed. In fact, from (4) we obtain, using the solutions (8) and (9),

$$\frac{dt}{d\tau} = \frac{E}{m} + \frac{\beta}{r} \left( p_i - \frac{x_j p_j}{r^2} x_i - e x_j \frac{\partial A_i}{\partial x_j} \right) \dot{x}_i = \frac{E}{m} + \frac{\beta}{2} m \omega_0 u.$$

Defining a constant \( \Omega = \frac{m^2 \omega_0}{E} = \frac{meB u}{E} \), we can write

$$t = \frac{E}{m} (1 + \beta \Omega) \tau,$$

and then

$$u_1 = u \cos \omega t, \quad u_2 = u \sin \omega t, \quad u_3 = 0,$$

with

$$\omega = \frac{m \omega_0}{E(1 + \beta \Omega)} = \frac{eB}{E(1 + \beta \Omega)}.$$

Moreover, we can now define the physical velocity as

$$v_i = \frac{dx_i}{dt} = \frac{m u_i}{E(1 + \beta \Omega)},$$

which generalizes to our case the relativistic relation \( v_i = m u_i/E \).

Starting from (14), we can write \( \Omega \) as a function of \( v = \sqrt{v_i^2} \). In fact,

$$u = (1 + \beta \Omega) \frac{Ev}{m},$$
and substituting in the definition of $\Omega$, we obtain $\Omega = eB(1 + \beta \Omega)v$, which can be solved for $\Omega$, giving

$$\Omega = \frac{\beta eBv}{1 - \beta eBv}.$$  \hfill (16)

At this point, it is useful to obtain the relation between the energy and the velocity of the particle. To this end, we write the dispersion relation $E^2 = m^2 + p^2$ as

$$E^2 = m^2(1 + u^2) = m^2 + E^2(1 + \beta \Omega)^2 v^2,$$  \hfill (17)

from which it follows

$$E^2 = \frac{m^2}{1 - (1 + \Omega)^2 v^2} = \frac{m^2(1 - \beta eBv)^2}{(1 - \beta eBv)^2 - v^2}.$$  \hfill (18)

Substituting in (13), we can finally write

$$\omega = \frac{eB}{m} \sqrt{(1 - \beta eBv)^2 - v^2}, \quad r = \frac{v}{\omega} = \frac{mv}{eB \sqrt{(1 - \beta eBv)^2 - v^2}}.$$  \hfill (19)

These equations give the relation of the cyclotron frequency and orbital radius with the velocity of the particle. They contain corrections of order $\beta eB$ to the special relativity formulas. Expanding to first order in $\beta$, one has for example

$$\omega = \omega_{\text{rel}} \left(1 - \frac{\beta eBv}{1 - v^2}\right),$$  \hfill (20)

where $\omega_{\text{rel}} = \frac{eB}{m} \sqrt{1 - v^2}$ is the relativistic frequency. From (17) one sees that the behavior of the frequency is similar to the one occurring in special relativity, but vanishes for a velocity smaller than the speed of light, namely $v = (1 + \beta eB)^{-1}$. The correction does not depend on the mass of the particle.

Eq. (17) can also be inverted to obtain the velocity as a function of the energy,

$$v = \frac{\sqrt{E^2 - m^2}}{E + \beta eB \sqrt{E^2 - m^2}},$$  \hfill (21)

and substituting in (19), one can write the cyclotron frequency as a function of the energy of the particle. The result is

$$\omega = \frac{eB}{E + \beta eB \sqrt{E^2 - m^2}}.$$  \hfill (22)

As an illustrative example, one may consider a unit charge particle in a magnetic field of 8 Tesla with velocity $v \sim 0.999999999c$, as typical of LHC. In this case, the ratio $\omega/\omega_0$ is of order $10^{-46}$. The predicted effect is extremely small, but one might exploit the cumulative effect over several orbits to increase the efficiency of the measurement.

Although it is unlikely that corrections of this size can be detected, they are important from a theoretical point of view. Similar effects can also be obtained in other noncommutative models, but the calculations are much more complicated in those cases.

References

[1] For a review, see P. Aschieri, M. Dimitrijević, P. Kulish, F. Lizzi and J. Wess, "Noncommutative spacetimes", Springer, Berlin 2009.

[2] See for example G. Amelino-Camelia, Symmetry 2, 230 (2010).

[3] S. Mignemi and N. Uras, Phys. Lett. A383, 585 (2019).

[4] L. Burderi, T. Di Salvo and R. Iaria, Phys. Rev. D93, 064017 (2016).

[5] H.S. Snyder, Phys. Rev. 71, 38 (1947).

[6] J. Lukierski, H. Ruegg, A. Novicki and V.N. Tolstoi, Phys. Lett. B264, 331 (1991).