COMMUTATOR SUBGROUPS OF VIRTUAL AND WELDED BRAID GROUPS

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Abstract. Let $V_B^n$, resp. $W_B^n$ denote the virtual, resp. welded, braid group on $n$ strands. We study their commutator subgroups $V_B' = [V_B^n, V_B^n]$ and $W_B' = [W_B^n, W_B^n]$ respectively. We obtain a set of generators and defining relations for these commutator subgroups. In particular, we prove that $V_B' = [V_B^n, V_B^n]$ is finitely generated if and only if $n \geq 4$, and $W_B'$ is finitely generated for $n \geq 3$. Also we prove that $V_B' / V_B'' = Z_3 \oplus Z_3 \oplus Z_3 \oplus Z_3$, $V_B' / V_B'' = Z_3 \oplus Z_3 \oplus Z_3 \oplus Z_3$, $W_B' / W_B'' = Z_3 \oplus Z_3 \oplus Z_3 \oplus Z$, $W_B' / W_B'' = Z_3$, and for $n \geq 5$ the commutator subgroups $V_B'$ and $W_B'$ are perfect, i.e. the commutator subgroup is equal to the second commutator subgroup.

1. Introduction

Virtual braid groups $V_B^n$ on $n$ strands are certain extensions of the classical braid groups. It was introduced by L. Kauffman [10] (see also [14]). Virtual braids play the same role in the virtual knot theory that classical braids played in the classical knot theory. In particular, like closures of classical braids represent classical knots and links, the closure of virtual braids represent the virtual knots and links (see [9], [10]). On connections of virtual braids with the virtual knot theory, see [3], [4]. For a structure of the virtual braid groups, see [2].

The welded braid group $W_B^n$ is a quotient of $V_B^n$. This group is called the group of conjugating automorphisms [13], the braid-permutation group [7] and so on. For several notions of this group and their equivalence, see [5].

The commutator subgroup $B'_n$ of the classical braid group $B_n$ is studied in the paper [8] (see also [12]). The following facts follow from these papers:

— $B'_n$ is finitely presented for all $n \geq 2$;
— $B'_3$ is a free group of rank two;
— $B'_4$ is a semi-direct product of two free groups of rank two;
— for $n > 4$ the second commutator subgroup $B''_n$ of $B_n$ coincides with the first commutator subgroup $B'_n$, i.e. $B'_n$ is perfect.

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In the present paper we investigate the commutator subgroups $VB'_n$ and $WB'_n$. Our main result is the following.

**Theorem 1.1.** The commutator subgroup $VB'_3$ is infinitely generated. For $n \geq 4$ the commutator subgroup $VB'_n$ can be generated by $2n - 3$ elements.

To prove Theorem 1.1 we obtain a presentation of $VB'_n$ using the classical method of Reidemeister-Schreier, and then remove certain generators and relations using Tietze transformations. As a consequence of Theorem 1.1 we further have the following corollaries.

**Corollary 1.2.** (1) The quotient $VB'_3/VB''_3$ is isomorphic to the direct product $\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}^\infty$, where $\mathbb{Z}^\infty$ is the direct product of counting number of $\mathbb{Z}$. (2) The quotient $VB'_4/VB''_4$ is isomorphic to the direct product $\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$. (3) For $n \geq 5$, $VB'_n$ is perfect, that is $VB'_n = VB''_n$.

Recently the commutator subgroup $WB'_n$ of the welded braid group has been investigated by Zaremsky in [15], who proved that $WB'_n$ is finitely presented if and only if $n \geq 4$. Zaremsky proved this result using discrete Morse theory, without constructing explicit finite presentation. Dey and Gongopadhyay [6] also proved that $WB'_n$ is finitely generated for all $n \geq 3$. In the present paper we have found a better bound on the number of generators than in [6]. We prove the following result.

**Theorem 1.3.** (1) The commutator subgroup $WB'_n$ can be generated by $n$ elements for all $n \geq 4$, and $WB'_3$ can be generated by 4 elements. (2) The quotient $WB'_3/WB''_3$ is isomorphic to the direct product $\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}$. (3) The quotient $WB'_4/WB''_4$ is isomorphic to $\mathbb{Z}_3$. (4) For $n \geq 5$, $WB'_n$ is perfect.

The presentation of $WB'_n$ obtained in this paper is slightly different from the one obtained in [6]. We obtain this presentation using the presentation of $VB_n$, while computing that we use successive conjugation rule in the rewriting process, see Lemma 3.1. This simple conjugation trick has given an alternative presentation of $WB'_n$ where the elimination of generators become simpler, and consequently we get a better bound on the number of generators.

We now briefly describe the structure of the paper. We recall the necessary preliminaries in Section 2. Using Reidemeister-Schreier method, we first obtain a general presentation of $VB'_n$, see Theorem 3.9 in Section 3. We prove Theorem 1.1 in Section 4. Because of the differences of the nature of the proofs for $n \geq 4$ and $n = 3$, as the cases $n = 3$ and $n \geq 4$ are different, accordingly, the proof of Theorem 1.1 is divided over two subsections. In Section 4.1 first, we apply Tietze transformations to remove certain generators from the presentation in Theorem 3.9. This gives a finite generating set for $VB'_n$ for $n \geq 4$. In Section 4.2 we show that $VB'_3$ is infinitely generated. Combining these results, Theorem 1.1 is obtained. We prove Theorem 1.3 in Section 5.

Finally, we note the following problems that remain to be answered.
Problem 1. Is it true that the commutator subgroup $V B_n'$ is not finitely presented for $n \geq 4$.

We expect the answer to be yes, but it is not clear how.

Problem 2. Construct explicit finite presentation of $W B_n'$ for $n \geq 3$.

Problem 3. Let $G$ is a group from the set $\{V B_3, V B_4, W B_3, W B_4\}$. Find the quotients $G^{(i)}/G^{(i+1)}$, $i = 2, 3, \ldots$, where $G^{(k)}$ is the $k$-th commutator subgroup:

$$G^{(1)} = G', \ G^{(k+1)} = [G^{(k)}, G^{(k)}], \ k = 1, 2, \ldots.$$

2. Preliminaries

2.1. Group of Virtual Braids. The virtual braid group of $n$ strands $V B_n$ is generated by the classical braid group $B_n = \langle \sigma_1, \ldots, \sigma_{n-1} \rangle$ and the symmetric group $S_n = \langle \rho_1, \ldots, \rho_{n-1} \rangle$. The generators $\sigma_i, i = 1, \ldots, n-1$ satisfy the relations

$$\begin{align*}
\sigma_i \sigma_j = \sigma_j \sigma_i & \quad \text{for } |i-j| \geq 2, \\
\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} & \quad \text{for } i = 1, \ldots, n-2.
\end{align*}$$

The generators $\rho_i, i = 1, \ldots, n-1$ satisfy the relations of symmetric group $S_n$:

$$\begin{align*}
\rho_i^2 = 1 & \quad \text{for } i = 1, 2, \ldots, n-1, \\
\rho_i \rho_j = \rho_j \rho_i & \quad \text{for } |i-j| \geq 2, \\
\rho_i \rho_{i+1} \rho_i = \rho_{i+1} \rho_i \rho_{i+1} & \quad \text{for } i = 1, 2, \ldots, n-2.
\end{align*}$$

Other defining relations of $V B_n$ are mixed and have the form

$$\begin{align*}
\sigma_i \rho_j = \rho_j \sigma_i & \quad \text{for } |i-j| \geq 2, \\
\rho_i \rho_{i+1} \sigma_i = \sigma_{i+1} \rho_i \rho_{i+1} & \quad \text{for } i = 1, 2, \ldots, n-2.
\end{align*}$$

2.2. Reidemeister-Schreier Algorithm. Given a presentation of a group $G$, this algorithm allows one to find a presentation of a subgroup $H \subset G$. To obtain the presentation of $H$, it is necessary to find a Schreier’s set of right coset of the group $G$ over the subgroup $H$. We give a formal description of this process, for more details see [1].

Let $a_1, \ldots, a_n$ be the generators of the group $G$ and $R_1, \ldots, R_m$ be the set of defining relations for the given set of generators. System of words $N = \{K_\alpha, \alpha \in A\}$ on generators $a_1, \ldots, a_n$ defines a Schreier’s system for the subgroup $H \subset G$ relative to the system of generators $a_1, \ldots, a_n$ if the next conditions are satisfied:

1) in every right coset of the group $G$ over $H$ there is only one word from the system $N$;

2) if the word $K_\alpha = a_{i_1}^{\varepsilon_1} \cdots a_{i_p}^{\varepsilon_p-1} a_{i_p}^{\varepsilon_p}, (\varepsilon_j = \pm 1)$ lies in $N$, then the word $a_{i_1}^{\varepsilon_1} \cdots a_{i_p}^{\varepsilon_p-1}$ also lies in $N$.

Suppose that some Schreier’s system $N$ is chosen for the subgroup $H \subset G$ relative to the system generators $a_1, \ldots, a_n$ of $G$. For every word $Q$ on $a_1, \ldots, a_n$, we denote by $\overline{Q}$ the only word from $N$ which lies in the same right coset of $G$ over the subgroup $H$. Denote

$$S_{K_\alpha, a_\nu} = K_\alpha a_\nu \cdot (K_\alpha a_\nu)^{-1}, \quad \alpha \in A, \ \nu = 1, \ldots, n.$$
Theorem of Reidemeister-Schreier states that the elements $S_{K_{\alpha}, a_{\nu}}$ generate subgroup $H$ and the set of defining relations for this set of generators is divided in two parts. First part consists of trivial relations $S_{K_{\alpha}, a_{\nu}} = 1$, where the pair $K_{\alpha}, a_{\nu}$ is such that the word $K_{\alpha} a_{\nu} \cdot (K_{\alpha} a_{\nu})^{-1}$ is freely equivalent to the word 1. Second part consists of all relations of the form $\tau(K_{\alpha} R_{\mu} K_{\nu}^{-1})$, where $\alpha \in A$, $\mu = 1, \ldots, m$, and $\tau$ is Reidemeister’s transformation, which maps every nonempty word $a_{i_1}^{\varepsilon_1} \ldots a_{i_p}^{\varepsilon_p}$, $(\varepsilon_j = \pm 1)$ from symbols $a_1, \ldots, a_n$ to the word from symbols $S_{K_{\alpha}, a_{\nu}}$ by the rule:

$$\tau(a_{i_1}^{\varepsilon_1} \ldots a_{i_p}^{\varepsilon_p}) = S_{K_{i_1}, a_{i_1}}^{\varepsilon_1} \ldots S_{K_{i_p}, a_{i_p}}^{\varepsilon_p},$$

where $K_{i_j} = a_{i_1}^{\sigma_1} \ldots a_{i_{j-1}}^{\varepsilon_{j-1}}$, if $\varepsilon_j = 1$, and $K_{i_j} = a_{i_1}^{\sigma_1} \ldots a_{i_j}^{\varepsilon_j}$, if $\varepsilon_j = -1$.

### 3. Commutator subgroup $VB'_n$

#### 3.1. Generating set of $VB'_n$

From the above relations it follows that the quotient $VB'_n/VB'_n$ is isomorphic to the direct product $\mathbb{Z} \times \mathbb{Z}_2$. One can define the map $\varphi$ from the following short exact sequence:

$$1 \to VB'_n \to VB'_n \xrightarrow{\varphi} \mathbb{Z} \times \mathbb{Z}_2 \to 1$$

where, for $i = 1, \ldots, n - 1$, $\varphi(\sigma_i)$ is the generator of $\mathbb{Z}$ and $\phi(\rho_1)$ is the generator of $\mathbb{Z}_2$ respectively when viewing it as $VB'_n/VB'_n$. The map $\varphi$ does have a section in the above short exact sequence for $n \geq 3$, and $\ker \varphi = VB'_n$.

As a Schreier set of coset representatives of $VB'_n$ by $VB'_n$ take the words

$$\Lambda = \{ \sigma_i^j \rho_1^k \mid i \in \mathbb{Z}, \ v = 0, 1 \}. $$

The commutator subgroup $VB'_n$ is generated by the words

$$S_{\lambda, a} = \lambda a(\lambda a)^{-1}, \quad \lambda \in \Lambda, \quad a \in \{ \sigma_1, \ldots, \sigma_{n-1}, \rho_1, \ldots, \rho_{n-1} \}. $$

Here $w$ is a coset representative in $\Lambda$ of $wVB'_n$. Find the elements $S_{\lambda, a}$. For this put $\lambda = \sigma_1^j \rho_1^k$ and considering different $a$ we will get the following cases:

1) If $a = \sigma_1$, then

$$S_{\lambda, \sigma_1} = \sigma_1^j \rho_1^k \sigma_1 (\sigma_1^{j+1} \rho_1^k)^{-1}. $$

For $\varepsilon = 0$ we have $S_{\lambda, \sigma_1} = 1$ and for $\varepsilon = 1$ we have $S_{\lambda, \sigma_1} = \sigma_1^j (\rho_1 \sigma_1 \rho_1^{-1}) \sigma_1^{-j}$, which we will denote by $a_i$.

2) If $a = \sigma_2$, then

$$S_{\lambda, \sigma_2} = \sigma_1^j (\rho_1^2 \sigma_2 \rho_1 \sigma_1^{-1}) \sigma_1^{-j};$$

and we will denote this element by $b_i \varepsilon$.

3) If $a = \sigma_l$, $l > 2$, then

$$S_{\lambda, \sigma_l} = \sigma_l \sigma_1^{-1};$$

and we will denote this element by $c_l$.

4) If $a = \rho_1$, then

$$S_{\lambda, \rho_1} = 1.$$
5) If \( a = \rho_2 \), then
\[
S_{\lambda, \rho_2} = \sigma_1^i (\rho_i^2 \rho_2^2 \rho_i^{i+1}) \sigma_1^{-i},
\]
and we will denote this element by \( f_{i, \varepsilon} \).

6) If \( a = \rho_l \), \( l > 2 \), then
\[
S_{\lambda, \rho_l} = \sigma_1^i (\rho_l^1 \rho_1^1) \sigma_1^{-i},
\]
and we will denote this element by \( g_{i, i} \).

To find defining relations of \( VB'_n \) we will use the following conjugation rules by elements \( \rho_1 \) and \( \sigma_1^{-m} \).

**Lemma 3.1.** The following formulas hold
\[
\begin{align*}
(1) & \quad a_i^{\sigma_1^{-m}} = a_{i+m}, \quad b_{i, \varepsilon}^{\sigma_1^{-m}} = b_{i+m, \varepsilon}, \quad c_i^{\sigma_1^{-m}} = c_i, \quad f_{i, \varepsilon}^{\sigma_1^{-m}} = f_{i+m, \varepsilon}, \quad g_{i, \varepsilon}^{\sigma_1^{-m}} = g_{i+m, \varepsilon}; \\
(2) & \quad a_0^{\sigma_1} = a_0^{-1}, \quad b_{0,0}^{\sigma_1} = b_{0,0} a_0^{-1}, \quad b_{0,1}^{\sigma_1} = b_{0,0} a_0^{-1}, \quad b_1^{\sigma_1} = a_0 b_{1,1} a_1^{-1} a_0^{-1}, \quad b_1^{\sigma_1} = a_0 a_1 (b_{2,1} a_2^{-1}) a_1 a_0^{-1}; \\
(3) & \quad c_1^{\sigma_1} = c_1 a_0^{-1}, \quad f_0^{\sigma_1} = f_0, \quad f_0^{\sigma_1} = f_0, \quad f_1^{\sigma_1} = f_1 a_0^{-1}, \quad f_1^{\sigma_1} = f_1 a_0^{-1}; \\
(4) & \quad g_{0,i}^{\sigma_1} = g_{0,i}, \quad g_{i,i}^{\sigma_1} = a_0 g_{i,i} a_0^{-1}, \quad i > 2.
\end{align*}
\]

**Proof.** (1) follow from the definition.

For proving (2) note that:
\[
\begin{align*}
\rho_1 a_0 \rho_1 &= \rho_1 \rho_1 \sigma_1 \rho_1 \sigma_1^{-1} \rho_1 = S_1 \rho_1 S_{\rho_1, \rho_1} S_{\rho_1, \rho_1} S_{\rho_1, \rho_1} = a_0^{-1}, \\
\rho_1 b_{0,0} \rho_1 &= \rho_1 \sigma_2 \sigma_1^{-1} \rho_1 = S_1 \rho_1 S_{\rho_1, \rho_1} S_{\rho_1, \rho_1} = b_{0,0} a_0^{-1}, \\
b_{0,1}^{\sigma_1} &= \rho_1 \rho_1 \sigma_2 \rho_1 \sigma_1^{-1} \rho_1 = S_1 \sigma_2 S_{\rho_1, \rho_1} S_{\rho_1, \rho_1} = b_{0,0} a_0^{-1}, \\
\rho_1 b_{1,0} \rho_1 &= \rho_1 \sigma_1 \sigma_2 \sigma_1^{-1} \rho_1 = S_1 \rho_1 S_{\rho_1, \rho_1} S_{\rho_1, \rho_1} S_{\rho_1, \rho_1} = a_0 b_{1,1} a_1^{-1} a_0^{-1}. \\
\rho_1 b_{2,0} \rho_1 &= \rho_1 \sigma_1 \sigma_2 (\sigma_1^{-1})^3 \rho_1 = \\
&= S_1 \rho_1 S_{\rho_1, \rho_1} S_{\rho_1, \rho_1} S_{\rho_1, \rho_1} S_{\rho_1, \rho_1} S_{\rho_1, \rho_1} S_{\rho_1, \rho_1} S_{\rho_1, \rho_1} = a_0 a_1 (b_{2,1} a_2^{-1}) a_1 a_0^{-1}.
\end{align*}
\]

For (3):
\[
\begin{align*}
\rho_1 c_0 \rho_1 &= \rho_1 \sigma_1 \sigma_1^{-1} \rho_1 = S_1 \rho_1 S_{\rho_1, \rho_1} S_{\rho_1, \rho_1} S_{\rho_1, \rho_1} = c_0 a_0^{-1}, \\
f_0^{\sigma_1} &= \rho_1 \rho_1 \rho_1 \rho_1 = \rho_1 \rho_2 = f_0, \\
f_0^{\sigma_1} &= \rho_1 \rho_1 \rho_2 \rho_1 \rho_1 = f_0, \\
f_1^{\sigma_1} &= \rho_1 \rho_1 \rho_2 \rho_1 \sigma_1^{-1} \rho_1 = S_1 \rho_1 S_{\rho_1, \rho_1} S_{\rho_1, \rho_1} S_{\rho_1, \rho_1} S_{\rho_1, \rho_1} = a_0 f_{1,1} a_0^{-1}, \\
f_1^{\sigma_1} &= \rho_1 \sigma_1 \rho_1 \rho_2 \sigma_1^{-1} \rho_1 = S_1 \rho_1 S_{\rho_1, \rho_1} S_{\rho_1, \rho_1} S_{\rho_1, \rho_1} S_{\rho_1, \rho_1} = a_0 f_{1,1} a_0^{-1}.
\end{align*}
\]

(4):
\[
\begin{align*}
g_{0,i}^{\sigma_1} &= \rho_1 \rho_1 \rho_1 \rho_1 = \rho_1 \rho_i = g_{0,i}, \\
g_{i,i}^{\sigma_1} &= \rho_1 \sigma_1 \rho_1 \sigma_1 \rho_1 \sigma_1^{-1} \rho_1 = S_1 \rho_1 S_{\rho_1, \rho_1} S_{\rho_1, \rho_1} S_{\rho_1, \rho_1} S_{\rho_1, \rho_1} S_{\rho_1, \rho_1} = a_0 g_{i,i} a_0^{-1}, \quad i > 2.
\end{align*}
\]

This proves the lemma.

\[\square\]

3.2. **Defining Relations in** \( VB'_n \). In this subsection we will consider the defining relations of \( VB'_n \), rewrite them in the generators of \( VB'_n \), and conjugating by elements \( \lambda \in \Lambda \), we get the defining relations of \( VB'_n \).
3.2.1. Defining relation of $VB'_n$ that follow from the relation $\sigma_i\sigma_j = \sigma_j\sigma_i$. Rewrite this relation in the form

$$r_1 = \sigma_i\sigma_j\sigma_i^{-1}\sigma_j^{-1}, \quad 1 \leq i < j \leq n - 1, \quad i + 1 < j.$$ 

Using the rewriting process we get:

- for $i = 1$: $r_1 = \sigma_1\sigma_1\sigma_1^{-1}\sigma_1^{-1} = S_1, S_{\sigma_1, \sigma_1}S_{\sigma_1^{-1}, \sigma_1} = c_jc_j^{-1} = 1$;
- for $i = 2$: $r_1 = \sigma_2\sigma_2\sigma_2^{-1}\sigma_2^{-1} = S_1, S_{\sigma_2, \sigma_2}S_{\sigma_2^{-1}, \sigma_2} = b_0, b_0^{-1}c_j^{-1}$;
- for $i > 2$: $r_1 = \sigma_i\sigma_i\sigma_i^{-1}\sigma_i^{-1} = S_1, S_{\sigma_i, \sigma_i}S_{\sigma_i^{-1}, \sigma_i} = c_i, c_i^{-1}c_i^{-1}$.

**Lemma 3.2.** The following four types of relations in $VB'_n$ follow from the relation $r_1$ of $VB_n$:

$$b_{m,0}c_jb_{m+1,0}c_j^{-1} = 1, \quad j \geq 4,$$

$$c_i, c_j^{-1}c_j^{-1} = 1, \quad i \geq 3, \quad j > i + 1,$$

$$b_{m,1}a_m^{-1}c_ja_{m+1}b_{m+1,1}c_j^{-1} = 1, \quad j \geq 4,$$

$$c_i a_m^{-1}c_j c_j^{-1} a_m c_j^{-1} = 1, \quad i \geq 3, \quad j > i + 1.$$ 

**Proof.** Conjugating of $r_1$ by $\rho_1$ and using Lemma 3.1 we get

$$\rho_1(b_{0,0}c_jb_{1,0}^{-1}c_j^{-1})\rho_1 = b_0, a_0^{-1}c_ja_1^{-1}b_1, c_j^{-1},$$

$$\rho_1(c_i, c_j^{-1}c_j^{-1})\rho_1 = c_i a_0^{-1}c_j c_j^{-1} a_0 c_j^{-1}.$$ 

Conjugating $r_1$ and $\rho_1r_1\rho_1$ by $\sigma_i^{-1}$, we get the need relations. \hfill \Box

3.2.2. Defining relations of $VB'_n$ that follow from the relation $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$. Rewrite this relation in the form

$$r_2 = \sigma_i\sigma_{i+1}\sigma_i\sigma_{i+1}\sigma_i^{-1}\sigma_{i+1}^{-1}.$$ 

Then

- for $i = 1$: $r_2 = \sigma_1\sigma_2\sigma_1\sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1} = S_1, S_{\sigma_1, \sigma_2}S_{\sigma_2^{-1}, \sigma_2} = b_0, b_0^{-1}c_j^{-1}$;
- for $i = 2$: $r_2 = \sigma_2\sigma_3\sigma_2\sigma_3^{-1}\sigma_2^{-1}\sigma_3^{-1} = S_1, S_{\sigma_2, \sigma_3}S_{\sigma_3^{-1}, \sigma_3} = b_0, b_0^{-1}c_j^{-1}$;
- for $i > 2$: $r_2 = \sigma_i\sigma_{i+1}\sigma_i\sigma_{i+1}\sigma_i^{-1}\sigma_{i+1}^{-1} = S_1, S_{\sigma_i, \sigma_{i+1}}S_{\sigma_{i+1}^{-1}, \sigma_{i+1}} = c_i c_i^{-1}c_i^{-1}.$

**Lemma 3.3.** The following six types of relations in $VB'_n$ follow from the relation $r_2$ of $VB_n$:

$$b_{m+1,0}b_{m+2,0}b_{m,0}^{-1} = 1,$$

$$b_{m,0}c_3b_{m+2,0}c_3^{-1}b_{m+1,0}c_3^{-1} = 1,$$

$$c_i c_i^{-1}c_i^{-1}c_i^{-1}c_i^{-1} = 1, \quad i \geq 3,$$

$$a_m a_{m+1} a_{m+2}b_{m+2,1}^{-1}a_{m+1}b_{m+1,1}^{-1} = 1,$$

$$b_{m,1}a_m^{-1}c_3a_{m+1}b_{m+2,1}^{-1}a_{m+2}a_{m+1}a_{m+1}c_3^{-1}a_m a_{m+1}b_{m+1,1}c_3^{-1} = 1,$$

$$c_i a_m^{-1}c_i^{-1}c_i^{-1}a_m a_i^{-1} a_m c_i^{-1} = 1, \quad i \geq 3.$$
Proof. Conjugating $r_2$ by $\rho_1$ and using Lemma 3.1 we get
\begin{align*}
(b_{1,0}b_2^{-1}b_0^{-1})^{\rho_1} &= a_0b_{1,1}a_2b_2^{-1}a_1^{-1}b_0^{-1}, \\
(b_{0,0}c_3b_2c_3^{-1}b_1^{-1}c_3^{-1})^{\rho_1} &= b_{0,1}c_3a_1b_{2,1}a_2^{-1}c_1^{-1}a_0a_1b_{1,1}c_3^{-1}, \\
(c_i c_{i+1}c_i^{-1}c_{i+1}^{-1})^{\rho_1} &= c_i a_0^{-1}c_{i+1}a_0^{-1}c_{i+1}^{-1}a_0a_{i+1}^{-1}.
\end{align*}
Conjugating $r_2$ and $\rho_1 r_2 \rho_1$ by $\sigma_1^{-m}$, we get the need relations. \hfill \square

3.2.3. Defining relation that follow from the relation $\rho_i^2 = 1$. Rewrite this relation in the form
\[ r_3 = \rho_i^2. \]
Then for $i = 1$: $r_3 = \rho_1 \rho_1 = S_{1,\rho_1} S_{\rho_1, \rho_1} = 1.$
for $i = 2$: $r_3 = \rho_3 \rho_3 = S_{1,\rho_2} S_{\rho_1, \rho_2} = f_{0,0} f_{0,1}.$
for $i > 2$: $r_3 = \rho_i \rho_i = S_{1,\rho_i} S_{\rho_1, \rho_i} = g_{0,i}.$
The following lemma holds

**Lemma 3.4.** From $r_3$ follow two types of relations in $VB'_n$:
\[ f_{m,0} f_{m,1} = g_{m,i}^2 = 1, \quad i > 2. \]

**Proof.** Conjugating $r_3$ by $\rho_1$ and using Lemma 3.1, we get
\[ (f_{0,0} f_{0,1})^{\rho_1} = f_{0,1} f_{0,0}, \]
\[ (g_{0,i}^2)^{\rho_1} = g_{0,i}^2, \quad i > 2. \]
Conjugating $r_3$ and $\rho_1 r_3 \rho_1$ by $\sigma_1^{-m}$, we get the need relations. \hfill \square

3.2.4. Defining relations of $VB'_n$ that follow from the relation $\rho_i \rho_j = \rho_j \rho_i$. Rewrite this relation in the form
\[ r_4 = \rho_i \rho_j \rho_i \rho_j, \quad 1 \leq i < j \leq n - 1, \quad i + 1 < j. \]
Then for $i = 1$: $r_4 = \rho_1 \rho_j \rho_1 \rho_j = S_{1,\rho_1} S_{\rho_1, \rho_j} S_{1,\rho_1} S_{\rho_1, \rho_j} = g_{0,j}^2, \quad j > 2.$
We have got this relation when we considered the relation $r_3$.
for $i = 2$: $r_4 = \rho_2 \rho_k \rho_2 \rho_k = S_{1,\rho_2} S_{\rho_1, \rho_k} S_{1,\rho_2} S_{\rho_1, \rho_k} = (f_{0,0} g_{0,k})^2, \quad k > 3.$
for $i > 2$: $r_4 = \rho_i \rho_j \rho_i \rho_j = S_{1,\rho_i} S_{\rho_1, \rho_j} S_{1,\rho_i} S_{\rho_1, \rho_j} = (g_{0,j} g_{0,j})^2, \quad 3 \leq i < j \leq n - 1, \quad i + 1 < j.$
The following lemma holds

**Lemma 3.5.** From the relation $r_4$ of $VB_n$, the following three types of relations of $VB'_n$ follow:
\[ (f_{m,0} g_{m,k})^2 = (f_{m,1} g_{m,k})^2 = 1, \quad k > 3, \]
\[ (g_{m,i} g_{m,j})^2 = 1, \quad 3 \leq i < j \leq n - 1, \quad i + 1 < j. \]
3.2.6. This relation in the form Conjugating Lemma 3.6. From the relation $r_{3.2.5.}$, we get $r_1(g_0i,g_0j)^2 r_1 = (g_0i,g_0j)^2, \quad 3 \leq i < j \leq n-1, \quad i+1 < j.$

Conjugating $r_4$ and $r_1 r_4 r_1$ by $\sigma_i^{-m}$, we get the need relations.

3.2.5. Defining relations of $VB'_n$ that follow from the relation $\rho_i \rho_{i+1} \rho_i = \rho_{i+1} \rho_i \rho_{i+1}$.

Rewrite this relation in the form $r_5 = \rho_i \rho_{i+1} \rho_i \rho_{i+1} \rho_i.$

Then

for $i = 1$: $r_5 = \rho_1 \rho_2 \rho_1 \rho_2 \rho_1$ is $S_1, \rho_1 S_{\rho_1, \rho_2} S_{\rho_1, \rho_2} S_{\rho_1, \rho_1} S_{\rho_1, \rho_2} = f_3^3.$

for $i = 2$: $r_5 = \rho_3 \rho_2 \rho_3 \rho_2 \rho_3$ is $S_1, \rho_3 S_{\rho_1, \rho_3} S_{\rho_1, \rho_2} S_{\rho_1, \rho_2} S_{\rho_1, \rho_3} = (f_0, 0g_0)^3.$

for $i > 2$: $r_5 = \rho_{i+1} \rho_i \rho_{i+1} \rho_i$ is $S_1, \rho_{i+1} S_{\rho_1, \rho_{i+1}} S_{\rho_1, \rho_{i+1}} S_{\rho_1, \rho_{i+1}} S_{\rho_1, \rho_{i+1}} = g_{i+1} g_0, i+1)^3.$

The following lemma holds

Lemma 3.6. From the relation $r_5$ of $VB_n$, we have the following five types of relations of $VB'_n$:

$f_{m,1}^3 = 1,$

$(f_{m,0}^3)^3 = 1,$

$(g_{m,1}^3)^3 = 1,$

$f_{m,0}^3 = 1,$

$(f_{m,1}^3)^3 = 1.$

Proof. Conjugating $r_5$ by $\rho_1$ and using Lemma 3.1.4, we get

$(f_{0,1}^3)^3 = f_{0,0}^3,$

$((f_0, 0g_0)^3)^3 = (f_0, 190)^3,$

$((g_0, i g_{0, i+1})^3)^3 = (g_0, j g_{0, i+1})^3, \quad i > 2.$

Conjugating $r_5$ and $r_1 r_5 r_1$ by $\sigma_i^{-m}$, we get the need relations.

3.2.6. Defining relations of $VB'_n$ that follow from the relation $\sigma_i \rho_j = \rho_j \sigma_i$.

Rewrite this relation in the form $r_6 = \sigma_i \rho_j \sigma_i^{-1} \rho_j, \quad |i - j| > 1.$

In dependence of $i$ and $j$ we will consider the next cases

a) $r_6 = \sigma_i \rho_i \sigma_i^{-1} \rho_i = S_1, \sigma_i S_{\sigma_i, \rho_i} S_{\rho_i, \sigma_i} S_{\rho_i, \rho_i} = g_{i+1} g_0, i, \quad \text{for } i > 2.$

b) $r_6 = \sigma_2 \rho_2 \sigma_2^{-1} \rho_2 = S_1, \sigma_2 S_{\sigma_1, \rho_2} S_{\rho_1, \sigma_2} S_{\rho_1, \rho_2} = b_{0, 0g_1, 1g_0}, \quad \text{for } j > 3.$

c) $r_6 = \sigma_k \rho_i \sigma_i^{-1} \rho_i = S_1, \sigma_k S_{\sigma_i, \rho_i} S_{\rho_i, \sigma_i} S_{\rho_i, \rho_i} = c_k g_{1, 1} c_k^{-1} g_{1, l}, \quad \text{for } k, l \geq 3, \quad |l - k| > 1.$

d) $r_6 = \sigma_i \rho_1 \sigma_i^{-1} \rho_1 = S_1, \sigma_i S_{\sigma_i, \rho_1} S_{\rho_1, \sigma_i} S_{\rho_1, \rho_1} = c_i c_i^{-1} = 1, \quad \text{for } i > 2.$

e) $r_6 = \sigma_j \rho_2 \sigma_2^{-1} \rho_2 = S_1, \sigma_j S_{\sigma_2, \rho_2} S_{\rho_2, \sigma_i} S_{\rho_2, \rho_2} = c_j f_{1, 0} c_j^{-1} f_{0, 1}, \quad \text{for } j > 3.$

Now, the following lemma holds.
Lemma 3.7. From the relation $r_6$ of $VB_n$, the following seven types of relations of $VB_n'$ follow:

$$g_{m+1,i}a_m^{-1}g_{m,i} = 1, \quad i \geq 3,$$

$$a_m g_{m+1,i}g_{m,i} = 1, \quad i \geq 3,$$

$$b_{m,0}g_{m+1,i}b_{m,1}^{-1}g_{m,j} = 1, \quad j \geq 4,$$

$$b_{m,1}g_{m+1,i}b_{m,0}^{-1}g_{m,j} = 1, \quad j \geq 4,$$

$$c_k g_{m+1,i}c_k^{-1}g_{m,l} = 1, \quad k, l \geq 3, \quad |l - k| > 1,$$

$$c_j f_{m+1,0}c_j^{-1}f_{m,1} = 1, \quad j \geq 4,$$

$$c_j f_{m+1,1}c_j^{-1}f_{m,0} = 1, \quad j \geq 4.$$

Proof. Conjugating $r_6$ by $\rho_1$ and using Lemma 3.1, we get

$$(g_{1,i}a_0^{-1}g_{0,i})^{\rho_1} = a_0 g_{1,i}g_{0,i}, \quad i \geq 3,$$

$$(b_{0,0}a_1^{-1}g_{1,0}b_{0,0})^{\rho_1} = b_{0,0} a_1 b_{0,0}^{-1}g_{1,0}, \quad j \geq 4,$$

$$(c_k g_{1,0}c_k^{-1}g_{0,1})^{\rho_1} = c_k g_{1,0}c_k^{-1}g_{0,1}, \quad k, l \geq 3, \quad |l - k| > 1,$$

$$(c_j f_{1,0}c_j^{-1}f_{0,1})^{\rho_1} = c_j f_{1,0}c_j^{-1}f_{0,1}, \quad j \geq 4.$$

Conjugating $r_6$ and $\rho_1 r_6 \rho_1$ by $\sigma_1^{-m}$, we get the need relations. \qed

3.2.7. Defining relations of $VB_n'$ that follow from the relation $\rho_i \rho_{i+1} \sigma_i = \sigma_{i+1} \rho_i \rho_{i+1}$. Rewrite this relation in the form

$$r_7 = \rho_i \rho_{i+1} \sigma_i \rho_{i+1} \rho_i \sigma_{i+1}^{-1}.$$

Then

for $i = 1$: $r_7 = \rho_1 \rho_2 \sigma_1 \rho_2 \rho_1 \sigma_2^{-1} = S_{1,\rho_1} S_{1,\rho_2} S_{1,\sigma_1} S_{1,\rho_2} S_{1,\rho_1} S_{1,\rho_2} S_{1,\sigma_2}^{-1} = f_{0,1} f_{1,0} b_{0,0}^{-1}$.

for $i = 2$: $r_7 = \rho_2 \rho_3 \sigma_2 \rho_3 \rho_2 \sigma_3^{-1} = S_{1,\rho_1} S_{1,\rho_2} S_{1,\sigma_2} S_{1,\rho_3} S_{1,\rho_2} S_{1,\rho_1} S_{1,\rho_2} S_{1,\sigma_3}^{-1} = f_{0,0} g_{0,3} b_{0,0} g_{1,3} f_{1,1} c_3^{-1}$.

for $i > 2$: $r_7 = \rho_i \rho_{i+1} \sigma_i \rho_{i+1} \rho_i \sigma_{i+1}^{-1} = S_{1,\rho_1} S_{1,\rho_2} S_{1,\sigma_1} S_{1,\rho_2} S_{1,\rho_1} S_{1,\rho_2} S_{1,\sigma_2} S_{1,\sigma_3}^{-1} = g_{0,0} g_{i+1} c_{i+1} g_{1,i+1} g_{1,i} c_{i+1}^{-1}$.

Now the following defining relations of $VB_n'$ follow from the relation $r_7$.

Lemma 3.8. The following five types of relations in $VB_n'$ follow from the relation $r_7$ of $VB_n$:

$$f_{m,1} f_{m+1,0} b_{m,0}^{-1} = 1,$$

$$f_{m,0} a_m f_{m+1,1} b_{m,1}^{-1} = 1,$$

$$f_{m,0} g_{m+1,3} b_{m,0} g_{m+1,3} f_{m+1,1} c_3^{-1} = 1,$$

$$f_{m,1} g_{m+1,3} b_{m,1} g_{m+1,3} f_{m+1,0} c_3^{-1} = 1,$$

$$g_{m,i} g_{m,i+1} g_{m+1,i} g_{m+1,i} c_{i+1}^{-1} = 1, \quad i > 2.$$
Proof. Conjugating $r_7$ by $\rho_1$ and using Lemma \ref{3.1}(2), (4) 5) and 6), we get

\[
(f_{0,1} f_{1,0} b_{0,0}^{-1})^{\rho_1} = f_{0,0} a_0 f_{1,1} b_{0,1}^{-1},
\]
\[
(f_{0,0} g_0 b_{0,0} g_1 f_{1,1} c_3^{-1})^{\rho_1} = f_{0,1} g_0 b_{0,1} g_1 f_{1,0} c_3^{-1},
\]
\[
(g_{0,i+1} g_{1,i+1} g_{1,i+1} c_{i+1}^{-1})^{\rho_1} = g_{0,i+1} g_{1,i+1} g_{1,i+1} c_{i+1}^{-1}, \quad i > 2.
\]
Conjugating $r_7$ and $\rho_1 r_7 \rho_1$ by $\sigma_1^{-m}$, we get the need relations. \hfill \Box

Using the relations

\[
f_{m,0} f_{m,1} = 1, \quad (\text{see Lemma \ref{3.4}}),
\]
we can remove elements $f_{m,1}$, $m \in \mathbb{Z}$, from the generating set and keep only elements

\[
f_{m,0} = f_m, \quad m \in \mathbb{Z}.
\]

The following result gives a presentation of $VB'_n$.

**Theorem 3.9.** The commutator subgroup $VB'_n$ is generated by elements

\[a_m, \ b_{m, \varepsilon}, \ c_i, \ f_m, \ g_{m,i},\]

where $m \in \mathbb{Z}$, $\varepsilon = 0, 1$, $2 < l < n$ and is defined by the relations

\[
b_{m,0} c_j b_{m+1,0} c_j^{-1} = 1, \quad j \geq 4,
\]
\[
c_i c_j c_i^{-1} c_j^{-1} = 1, \quad i \geq 3, \quad j > i + 1,
\]
\[
b_{m,1} a_m c_j a_{m+1} b_{m+1,1} c_j^{-1} = 1, \quad j \geq 4,
\]
\[
c_i a_m c_j c_i^{-1} a_m c_j^{-1} = 1, \quad i \geq 3, \quad j > i + 1.
\]
\[
b_{m+1,0} b_{m+2,0} b_{m,0}^{-1} = 1,
\]
\[
b_{m,0} c_3 b_{m+2,0} b_{m+1,0} c_3^{-1} b_{m+1,0} c_3^{-1} = 1,
\]
\[
c_i c_i+1 c_i c_i^{-1} c_i^{-1} c_i+1 = 1, \quad i \geq 3,
\]
\[
a_m b_{m+1,1} a_m + 2 b_{m+2,1} a_{m+1} b_{m,1} = 1,
\]
\[
b_{m,1} a_m c_3 a_m b_{m+2,1} a_{m+1} b_{m+1,1} c_3^{-1} a_m b_{m+1,1} b_{m,1} c_3^{-1} = 1,
\]
\[
c_i a_m c_i+1 a_m c_i^{-1} c_i+1 a_m c_i^{-1} a_m c_i+1 = 1, \quad i \geq 3.
\]
\[
g_{m,i}^2 = 1, \quad i > 2.
\]
\[
(f_{m} g_{m,k})^2 = 1, \quad k > 3,
\]
\[
(g_{m,i} g_{m,j})^2 = 1, \quad 3 \leq i < j \leq n - 1, \quad i + 1 < j.
\]
\[
f_m^3 = 1,
\]
\[
(f_m g_{m,3})^3 = 1,
\]
\[
(g_{m,i} g_{m,i+1})^3 = 1, \quad i > 2,
\]
4. Proof of Theorem 1.1

4.1. Finite generation of $VB'_n$, $n \geq 4$. For the next calculations we remove the generators $b_{m,1}$ and $b_{m,0}$ from the presentation in Theorem 3.9.

Using the relations

$$b_{m,1} = f_m a_m f_{m+1}^{-1},$$

we can remove generators $b_{m,1}$ from the set of generators.

We have

$$g_{m+1} c_m^{-1} g_{m,i} = 1, \quad i \geq 3,$$

$$b_{m,1} g_{m+1} b_{m,0}^{-1} g_{m,j} = 1, \quad j \geq 4,$$

$$c_k g_{m+1} c_k^{-1} g_{m,l} = 1, \quad k, l \geq 3, \quad |l - k| > 1,$$

$$c_j f_m c_j^{-1} f_m^{-1} = 1, \quad j \geq 4,$$

$$f_m^{-1} f_m b_{m,0}^{-1} = 1,$$

$$f_m a_m f_{m+1}^{-1} b_{m,1}^{-1} = 1,$$

$$f_m g_m 3 b_{m,0} g_{m+1} a_{m+1} f_{m+1}^{-1} c_3^{-1} = 1,$$

$$f_m^{-1} g_m 3 b_{m,1} g_{m+1} a_{m+1} f_{m+1}^{-1} c_3^{-1} = 1,$$

$$g_{m,i} g_{m+1} c_i g_{m+1} a_{m+1} i c_i^{-1} = 1, \quad i > 2.$$

Proof. The theorem is obtained by combining the set of relations we have obtained in Lemma 3.2, Lemma 3.8. \(\square\)
(f_m f_m, k)^2 = 1, \quad k > 3,
(g_m, i g_m, j)^2 = 1, \quad 3 \leq i < j \leq n - 1, \quad i + 1 < j.

\begin{align*}
&f_m^3 = 1, \\
&(f_m g_m, 3)^3 = 1, \\
&(g_m, i g_m, i + 1)^3 = 1, \quad i > 2,
\end{align*}

\begin{align*}
g_{m+1, i} a_{m}^{-1} g_{m, i} &= 1, \quad i \geq 3, \\
f_m a_m f_{m+1}^{-1} g_{m+1, i} b_{m, 0}^{-1} g_{m, j} &= 1, \quad j \geq 4, \\
c_k g_{m+1, i} c_k^{-1} g_{m, l} &= 1, \quad k, l \geq 3, \quad |l - k| > 1, \\
c_j f_m c_j^{-1} f_{m+1}^{-1} &= 1, \quad j \geq 4,
\end{align*}

Using the relations

\begin{align*}
f_m^{-1} f_m &= f_m f_{m+1},
\end{align*}

we can remove the generators \(b_{m, 0}\) from the generating set.

After removing \(b_{m, 0}\) and \(b_{m, 1}\) we have following set of defining relations of \(V B'_n\):

\begin{align*}
f_m^{-1} f_{m+1} c_j &= c_j f_{m+1}^{-1} f_{m+2}, \quad j \geq 4, \\
c_i c_j c_i^{-1} c_j^{-1} &= 1, \quad i \geq 3, \quad j > i + 1, \\
f_m a_m f_{m+1}^{-1} a_m c_j f_{m+2} a_{m+1}^{-1} f_{m+1} c_{j+1}^{-1} &= 1, \quad j \geq 4, \\
c_i a_m^{-1} c_j c_i^{-1} a_m c_j^{-1} &= 1, \quad i \geq 3, \quad j > i + 1.
\end{align*}

\begin{align*}
f_m f_{m+1}^{-1} f_{m+2} &= f_m f_{m+2} f_{m+3}, \\
f_m^{-1} f_{m+1} c_3 f_{m+2}^{-1} &= c_3 f_{m+1}^{-1} f_{m+2} c_3, \\
c_i c_{i+1} c_i^{-1} c_{i+1}^{-1} &= 1, \quad i \geq 3, \\
a_m f_{m+1} a_{m+1} f_{m+2} a_{m+2} &= f_m a_m f_{m+1}^{-1} a_m f_{m+2} a_{m+2} f_{m+3}, \\
f_m a_m f_{m+1}^{-1} a_m c_3 f_{m+2} a_{m+2} f_{m+3}^{-1} &= c_3 f_{m+1} a_m f_{m+2} a_{m+1}^{-1} c_3 a_{m+1}, \\
c_i a_m^{-1} c_{i+1} a_m^{-1} c_i c_{i+1}^{-1} a_m c_i^{-1} &= 1, \quad i \geq 3.
\end{align*}

\begin{align*}
g_{m, j}^2 &= 1, \quad i > 2,
(f_m g_m, k)^2 &= 1, \quad k > 3,
\end{align*}
\[(g_{m,i}g_{m,j})^2 = 1, \quad 3 \leq i < j \leq n - 1, \quad i + 1 < j.\]

\[f_m^3 = 1,\]
\[(f_mg_{m,3})^3 = 1,\]
\[(g_{m,i}g_{m,i+1})^3 = 1, \quad i > 2,\]

\[g_{m+1,i}a_m^{-1} g_{m,i} = 1, \quad i \geq 3,\]
\[g_{m,j}f_m a_m f_{m+1}^{-1} g_{m+1,j} = f_m^{-1} f_{m+1}, \quad j \geq 4,\]
\[c_k g_{m+1,l} = g_{m,l} c_k, \quad k, l \geq 3, \quad |l - k| > 1,\]
\[c_j f_{m+1} = f_m c_j, \quad j \geq 4,\]

\[f_m g_{m,3} f_{m+1}^{-1} g_{m+1,3} f_{m+1}^{-1} = c_3^{-1},\]
\[f_m^{-1} g_{m,3} f_m a_m f_{m+1}^{-1} g_{m+1,3} f_{m+1} = c_3,\]
\[g_{m,i} g_{m,i+1} c_i = c_{i+1} g_{m+1,i} g_{m+1,i+1}, \quad i > 2.\]

We will use this set of relations to prove that $VB'_n$ is finitely generated for all $n \geq 4$.

**Lemma 4.1.** The commutator subgroup $VB'_n$ is finitely generated for all $n \geq 4$. In particular, $VB'_4$ is generated by 5 elements: $c_3, f_0, f_1, f_2, g_{0,3}$, and $VB'_n$, $n \geq 5$, is generated by $2n - 3$ elements: $c_3, \ldots, c_{n-1}, f_0, f_1, f_2, g_{0,3}, \ldots, g_{0,n-1}$.

**Proof.**

1) Using the relations
\[g_{m,j}g_{m+1,i} = a_m, \quad i \geq 3,\]
we will remove the generators $a_m$, $m \in \mathbb{Z}$, and express them by $g_{m,i}$, $m \in \mathbb{Z}$, $i \geq 3$.

2) Using the relations
\[f_m f_{m+1}^{-1} f_{m+2} = f_{m+1} f_{m+2}^{-1} f_{m+3},\]
we can remove the generators $f_m$, for $m \in \mathbb{Z}$, and keep only $f_0, f_1, f_2$.

3) Using the relations
\[f_m g_{m,3} f_{m+1}^{-1} g_{m+1,3} f_{m+1}^{-1} = c_3^{-1},\]
we can remove the generators $g_{m,3}$, for $m \in \mathbb{Z}$, and keep only $g_{0,3}$.

If $n = 4$, then we have only generators $g_{0,3}, f_0, f_1, f_2, c_3$. Hence, $VB'_4$ is finitely generated.

If $n > 4$, then

4) Using the relations
\[g_{m,i} g_{m,i+1} c_i = c_{i+1} g_{m+1,i} g_{m+1,i+1}, \quad i > 2,\]
we can remove the generators $g_{m,i}$, for $m \in \mathbb{Z}$, $i > 3$, and keep only $g_{0,i}$. \qed
4.2. Infinite generation of $V B'_3'$. Consider the case $n = 3$. From Theorem 3.9 follows that $V B'_3'$ is generated by elements

$$a_m, \ b_{m, \varepsilon}, \ f_m, \ m \in \mathbb{Z},$$

and is defined by the relations

(4.1) $b_{m+1,0}b_{m+2,0}^{-1}b_{m,0}^{-1} = 1,$

(4.2) $a_mb_{m+1,1}^{-1}a_{m+2}b_{m+2,1}^{-1}a_{m+1}^{-1}b_{m,1} = 1,$

(4.3) $f_m^3 = 1,$

(4.4) $f_m^{-1}f_{m+1}f_{m,0}^{-1} = 1,$

(4.5) $f_ma_m^{-1}f_{m+1}^{-1}b_{m,1}^{-1} = 1.$

Now we apply Tietze transformations to the presentation of $V B'_3'$. Using relations (4.5) we can remove the generator $b_{m,1} = f_m^{-1}a_m$. Then the modified set of defining relations take the form:

(4.6) $b_{m+1,0}b_{m+2,0}^{-1}b_{m,0}^{-1} = 1,$

(4.7) $a_m f_{m+1}a_{m+1}^{-1}f_{m+2}^{-1}a_{m+2}^{-1}f_{m+3}^{-1}a_{m+2}^{-1}f_{m+2}^{-1}a_{m+1}^{-1}f_m^{-1} = 1,$

(4.8) $f_m^3 = 1,$

(4.9) $f_m^{-1}f_{m+1}b_{m,0}^{-1} = 1,$

Using relations (4.9) we can remove the generator $b_{m,0} = f_m^{-1}f_{m+1}$. Then $V B'_3'$ is generated by elements

$$a_m, \ f_m, \ m \in \mathbb{Z},$$

and is defined by relation:

(4.10) $f_{m+1}^{-1}f_{m+2}^{-1}f_{m+3}^{-1}f_{m+2}^{-1}f_m^{-1} = 1,$

(4.11) $a_m f_{m+1}a_{m+1}^{-1}f_{m+2}^{-1}a_{m+2}^{-1}f_{m+3}^{-1}a_{m+2}^{-1}f_{m+2}^{-1}a_{m+1}^{-1}f_m^{-1} = 1,$

(4.12) $f_m^3 = 1,$
So, we have the following lemma.

**Lemma 4.2.** The group $V B'_3$ has a presentation with \{a_m, f_m, m \in \mathbb{Z}\} as the generating set, and the relations (4.10)–(4.12) as the defining relations.

**Lemma 4.3.** $V B'_3$ is not finitely generated.

*Proof.* If we put $f_m = 1$, for all $m \in \mathbb{Z}$, then all the relations (4.10)–(4.12) will vanish, i.e. the subgroup $\langle a_m | m \in \mathbb{Z} \rangle$ is infinitely generated free group with the set of free generators $a_m, m \in \mathbb{Z}$ and we have an epimorphism

$V B'_3 \rightarrow F_\infty = \langle a_m, m \in \mathbb{Z} \rangle$

with kernel $\langle f_m, m \in \mathbb{Z} \rangle V B'_3$.

\[ \square \]

### 4.3. Proof of Theorem 1.1

Note that $V B_2 = F_2 \times S_2$ and hence $V B'_2$ is infinitely generated. Then the Theorem 1.1 follows by combining Lemma 4.1 and Lemma 4.3.

### 4.4. Proof of Corollary 1.2

In the quotient $V B'_4/V B''_3$ relations have the form

\[
\begin{align*}
f_m f_{m+1} &= f_{m+2} f_{m+3}, \\
2 \quad f_m^3 &= 1.
\end{align*}
\]

In the generators $f_0, f_1, f_2, a_m, m \in \mathbb{Z}$, we have relations

\[
\begin{align*}
f_0^3 &= f_1^3 = f_2^3 = 1.
\end{align*}
\]

Hence, $V B'_4/V B''_3$ is isomorphic to the direct sum

\[
\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_\infty.
\]

(2) Consider the case $n = 4$. Then $V B'_4$ is generated by elements

\[
a_m, \quad c_3, \quad f_m, \quad g_{m,3},
\]

where $m \in \mathbb{Z}$, and the defining relations have the form

\[
\begin{align*}
f_m f_{m+1} f_{m+2} &= f_{m+1} f_{m+2} f_{m+3}, \\
f_m^{-1} f_{m+1} c_3 f_m^{-1} a_m f_{m+1} a_m f_{m+2} a_m f_{m+2} f_{m+3}^{-1} &= c_3 f_m^{-1} f_{m+1} a_m f_{m+2} a_m f_{m+2} f_{m+3}^{-1}, \\
f_m a_m f_m^{-1} a_m^{-1} c_3 a_m^{-1} f_{m+2} a_m f_{m+2} a_m f_{m+2} f_{m+3}^{-1} a_m^{-1} c_3 a_m^{-1} &= c_3 f_m a_m f_{m+1} a_m f_{m+2} a_m f_{m+2} f_{m+3}^{-1} a_m^{-1} c_3 a_m^{-1}, \\
2 \quad g_{m,3}^2 &= 1, \\
3 \quad f_m^3 &= 1, \\
4 \quad (f_m g_{m,3})^3 &= 1, \\
5 \quad g_{m+1,3} a_m^{-1} g_{m,3} &= 1, \\
6 \quad f_m g_{m,3} f_m^{-1} f_{m+1} g_{m+1,3} f_{m+1}^{-1} &= c_3^{-1}, \\
7 \quad f_m^{-1} g_{m,3} f_m a_m f_{m+1} g_{m+1,3} f_{m+1} &= c_3.
\end{align*}
\]
Consider these relations in the quotient $VB'_3/VB'_4$ and for the images of the generators
\[ c_3, \quad a_m, \quad f_m, \quad g_{m,3}, \quad m \in \mathbb{Z}, \]
we will use the same symbols.

From relation $g^2 = (f_mg_{m,3})^3 = 1$ we get $g_{m,3} = 1$.

Then from the relation $g_{m+1,3}a_m^{-1}g_{m,3} = 1$ follows that $a_m = 1$.

The other relations have the form
\[ c_3 = f^3_m = 1, \quad f_mf_{m+1} = f_{m+2}f_{m+3}. \]
Hence, we can keep only generators $f_0, f_1, f_2$ and defining relations $f_0^3 = f_1^3 = f_2^3 = 1$.

Hence the quotient $VB'_3/VB'_4$ is isomorphic to the direct product $\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$ of three cyclic groups $\mathbb{Z}_3$ of order 3.

Consider the case $n > 4$. We will consider relations of $VB'_n$ in the quotient $VB'_n/VB'_3$ and will denote the images of the generators
\[ a_m, \quad b_{m,\varepsilon}, \quad c_l, \quad f_m, \quad g_{m,l}, \]
where $m \in \mathbb{Z}$, $\varepsilon = 0, 1, 2 < l < n$ by the same symbols.

As in the case $n = 4$ we get $g_{m,3} = a_m = 1$.

Then from the relations
\[ g^2_{m,i} = 1, \quad (g_{m,i}g_{m,i+1})^3 = 1, \quad i > 2, \]
follows that $g_{m,i} = 1, i > 2$.

From the relations
\[ f^3_m = 1, \quad (f_mg_{m,k})^2 = 1, \quad k > 3, \]
follows that $f_m = 1$.

Remaining relations have the form
\[ c_i = 1, \quad i \geq 3. \]
This completes the proof.

5. Commutator subgroup of the welded braid group

The welded braid group $WB_n$, $n \geq 2$, is the quotient of $VB_n$ by the relations
\[ \rho_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\rho_{i+1}, \quad i = 1, 2, \ldots, n-2. \]

In this section we will find a presentation of $WB'_3$. We will use the same set of generators that we used for $VB'_n$ and $VB'_n$. Hence to find defining relations for $WB'_n$ we need to add relations that follow from the relation
\[ r_8 = \rho_i\sigma_{i+1}\sigma_i\rho_{i+1}\sigma_i^{-1}\sigma_{i+1}^{-1}. \]

Depending on $i$ we will consider 3 cases:

if $i = 1$, then
\[ r_8 = \rho_1\sigma_2\sigma_1\rho_2^{-1}\sigma_1^{-1}\sigma_2^{-1} = S_{1,\rho_1}S_{\rho_1,\sigma_1}S_{\sigma_1,\rho_1,\sigma_1}S_{\sigma_1,\rho_1,\sigma_1}^{-1}S_{\sigma_1,\sigma_1}^{-1}S_{\sigma_1,\sigma_2} = b_{0,1}a_1f_{2,1}b_0^{-1}; \]

if $i = 2$, then
\[ r_8 = \rho_2\sigma_3\sigma_2\rho_3\sigma_2^{-1}\sigma_3^{-1} = S_{1,\rho_2}S_{\rho_2,\sigma_2}S_{\sigma_1,\rho_1,\sigma_2}S_{\sigma_1,\rho_1,\sigma_2}^{-1}S_{\sigma_1,\sigma_2}^{-1}S_{\sigma_1,\sigma_3} = f_{0,0}c_3b_{1,1}g_{2,3}b_1^{-1}c_3^{-1}; \]
if $i > 2$, then

$$r_8 = \rho_i\sigma_i+1\sigma_i\rho_i+1\sigma_i^{-1}\sigma_i+1$$

$$= S_{1,\rho_i}S_{\rho_i+1,\sigma_i+1}S_{\sigma_i+1,\rho_i}S_{\sigma_i,\sigma_i+1}S_{\rho_i,\rho_i+1}^{-1}S_{\sigma_i,\sigma_i+1}^{-1}S_{\sigma_i+1,\rho_i}^{-1}$$

$$= g_0a_1c_{i+1}g_2a_{i+1}\sigma_i^{-1}c_i^{-1}\sigma_i.$$  

We will use the following conjugation rules

**Lemma 5.1.** In $WB_n$ the following conjugation rules hold:

1. $a_i^{\rho_i} = a_0a_1^{-1}a_0^{-1},$
2. $f_2^{\rho_i} = a_0a_1f_2a_1^{-1}a_0^{-1}.$
3. $g_2^{\rho_i} = a_0a_1g_2a_1^{-1}a_0^{-1}$ for $i > 2.$

**Proof.**

1. Next we have,

$$\rho_1f_2\rho_1 = \rho_1\sigma_1\rho_1\rho_2\sigma_1^{-1}\sigma_1^{-1}\rho_1$$

$$= S_{1,\rho}S_{\rho_1,\sigma_1}S_{\rho_2,\rho_1}S_{\sigma_1,\rho_1}S_{\sigma_1,\sigma_1}^{-1}S_{\rho_1,\sigma_1}^{-1}S_{\rho_1,\rho_1}$$

$$= a_0a_1f_2a_1^{-1}a_0^{-1};$$

2. Finally,

$$\rho_1g_2\rho_1 = \rho_1\sigma_1\rho_1\rho_1\sigma_i^{-1}\sigma_i^{-1}\rho_1$$

$$= S_{1,\rho}S_{\rho_1,\sigma_1}S_{\rho_2,\rho_1}S_{\sigma_1,\rho_1}S_{\sigma_1,\sigma_1}^{-1}S_{\rho_1,\sigma_1}^{-1}S_{\rho_1,\rho_1}$$

$$= a_0a_1g_2a_1^{-1}a_0^{-1}.$$  

This proves the lemma. \qed

**Lemma 5.2.** From the relation $r_8$ of $WB_n$, the following six types of relations of $WB_n'$ follow:

$$b_{m,1}a_{m+1}f_{m+2,1}b_{m,0}^{-1} = 1,$$

$$f_{m,0}c_{3b_{m+1,1}}g_{m+2,1}b_{m,0}^{-1}c_3^{-1} = 1,$$

$$g_{m,1}c_{i+1}g_{m+2,1}c_i^{-1}c_i^{-1} = 1,$$

$$b_{m,0}f_{m+2,0}a_{m+1}b_{m,1}^{-1} = 1,$$

$$f_{m,1}c_{3f_{m+1,0}}g_{m+2,1}b_{m,0}^{-1}c_3^{-1} = 1,$$

$$g_{m,1}c_{i+1}a_{m+1}g_{m+2,1}c_i^{-1}a_m^{-1}c_i^{-1} = 1.$$  

**Proof.** Conjugating relations $r_8$ by $\rho_1$ and using Lemma 5.1 we get 3 relations:

$$\left(b_{0,1}a_{1}f_{2,1}b_{0,0}^{-1}\right)^{\rho_1} = b_{0,0}f_{2,0}a_1^{-1}b_{0,1}^{-1},$$

$$\left(f_{0,0}c_{3b_{1,0}}g_{2,1}b_{1,0}^{-1}c_3^{-1}\right)^{\rho_1} = f_{0,1}c_{3f_{1,0}}a_1g_{2,3}b_{1,1}^{-1}c_3^{-1},$$
\[(g_0,c_{i+1}c_{g_{i+1}}^{-1}c_{i+1}^{-1})^{\rho_1} = g_0,c_{i+1}a_0a_1^{-1}c_{g_{i+1}}^{-1}a_0^{-1}c_{i+1}^{-1}.
\]

Conjugating relations \(r_8\) and \(\rho_1 r_8 \rho_1\) by \(\sigma_1^m\), we get the six relations from the lemma.

Thus we have the following.

**Corollary 5.3.** The commutator subgroup \(WB'_n\) is generated by elements

\[a_m, \quad b_{m,\varepsilon}, \quad c_l, \quad f_m, \quad g_{m,l},\]

where \(m \in \mathbb{Z}\), \(\varepsilon = 0, 1\), \(2 < l < n\) and is defined by the relations in Theorem 3.9 and Lemma 5.2.

5.1. **Presentation of \(WB'_3\).** We have found a presentation of \(VB'_3\). To get a presentation of \(WB'_3\) we need to add two series of relations:

\[(5.1)\]

\[b_{m,1}a_{m+1}f_{m+2,1}b_{m,0}^{-1} = 1,\]

\[(5.2)\]

\[b_{m,0}f_{m+2,0}a_{m+1}^{-1}b_{m,1}^{-1} = 1,\]

that follow from Lemma 5.2.

As in the case of \(VB'_3\) we can remove the generator \(f_{m,1}\), using the relation \(f_{m,0}f_{m,1} = 1\). Then the relations \(5.1) - 5.2\) have the form

\[(5.3)\]

\[b_{m,1}a_{m+1}f_{m+2,1}b_{m,0}^{-1} = 1,\]

\[(5.4)\]

\[b_{m,0}f_{m+2}a_{m+1}^{-1}b_{m,1}^{-1} = 1,\]

where we denote \(f_m = f_{m,0}\).

Using the relations

\[b_{m,1} = f_{m}a_{m}f_{m+1}^{-1},\]

that hold in \(VB'_3\), we can remove \(b_{m,1}\). Then the relations \(5.3) - 5.4\) have the form

\[(5.5)\]

\[f_{m}a_{m}f_{m+1}^{-1}a_{m+1}f_{m+2}b_{m,0}^{-1} = 1,\]

\[(5.6)\]

\[b_{m,0}f_{m+2}a_{m+1}^{-1}f_{m+1}a_{m}^{-1}f_{m}^{-1} = 1,\]

We see that the second relation is inverse to the first one. Hence, we can remove the second relation.

Next, using the relations \(f_{m}^{-1}f_{m+1}b_{m,0}^{-1} = 1\), which hold in \(VB'_3\), we can remove the generator \(b_{m,0}\). Then \(5.5\) has the form

\[(5.7)\]

\[f_{m}a_{m}f_{m+1}^{-1}a_{m+1}f_{m+2}f_{m+1}^{-1}f_{m} = 1.\]

Using the presentation of \(VB'_3\) we get

**Proposition 5.4.** The group \(WB'_3\) is generated by elements

\[a_m, \quad f_m, \quad m \in \mathbb{Z},\]

and is defined by relation:

\[(5.8)\]

\[f_{m+1}f_{m+2}f_{m+3}f_{m+2}f_{m+1}^{-1}f_{m} = 1,\]
We get relations

\[ W B \]

From the set of relations (5.8) we can express the generators $a$.

**Proof.** In the quotient $W B_{3}'$ we can express the generators $a$, where $l \neq 0$, as words in the generators $a_0, f_0, f_1, f_2$.

**Corollary 5.6.** $W B_{3}'/W B_{3}''$ is isomorphic to the direct sum

\[ Z_3 \oplus Z_3 \oplus Z_3 \oplus Z. \]

**Proof.** In the quotient $W B_{3}'/W B_{3}''$ the relations have the form

\[ f_m f_{m+1} = f_m f_{m+3}, \]
\[ f_m^3 = 1, \]
\[ a_m a_{m+1} = f_m f_{m+1}^{-1} f_m + 1. \]

In the generators $a_0, f_0, f_1, f_2$ we have relations

\[ f_0^3 = f_1^3 = f_2^3 = 1. \]

This completes the proof.

**5.2. The commutator subgroup** $W B_{4}'$. In $W B_{4}'$ we have relations of $V B_{4}'$ and the following relations:

\[ b_m, a_m f_{m+1} b_m^{-1} = 1, \]
\[ 0 \]
\[ b_m, 0 f_{m+2} a_m^{-1} b_m^{-1} = 1, \]
\[ f_m, 0 c_3 b_m+1, 1 g_m+2, b_m^{-1} c_3^{-1} = 1, \]
\[ f_m c_3 f_{m+1} a_m+1 g_m+2, b_m^{-1} c_3^{-1} = 1. \]

Excluding the generators

\[ b_m, 0 = f_m^{-1} f_{m+1}, \quad b_m, 1 = f_m a_m f_{m+1}^{-1}, \quad f_m, 1 = f_m^{-1} = f_{m, 0} = f^{-1} \]

from these relations. We get relations

\[ f_m a_m f_{m+1} a_m+1 f_m^{-1} f_{m+1} f_m = 1, \]
\[ f_m^{-1} f_{m+1} a_m+1 f_m+1 a_m^{-1} f_m = 1, \]
\[ f_m c_3 f_{m+1} a_m+1 f_m+2 g_m+2, 3 f_m+2 f_m+1 c_3^{-1} = 1, \]
\[ f_m c_3 f_{m+1} a_m+1 g_m+2, 3 f_m+2 a_m+1 f_m+1 c_3^{-1} = 1. \]
The second relation is inverse of the first relation. Hence, we can keep only the first relation. Rewrite it in the form
\[ a_m f_{m+1}^{-1} a_{m+1} = f_m f_m f_{m+2}. \]

Rewrite the third and the forth relations in the form
\[ c_3 f_{m+1} a_{m+1} f_{m+2}^{-1} g_{m+2} = f_m f_{m+2} f_{m+3}^{-1}, \]
\[ c_3 f_{m+1} a_{m+1} g_{m+2} = f_m f_{m+2} c_3^{-1} f_{m+1}^{-1}. \]

From these relations:
\[ f_{m+1}^{-1} g_{m+2} f_{m+2}^{-1} f_{m+1} c_3^{-1} f_m = g_{m+2} f_{m+2} a_{m+1} f_{m+1} c_3^{-1} f_{m+1}^{-1}. \]

Since in $V B_4'$ holds
\[ (g_{m+2} f_{m+2})^2 = 1, \quad g_{m+2} = 1, \]
then
\[ (g_{m+2} f_{m+2})^{-2} f_{m+2} f_{m+1} c_3^{-1} f_m = a_{m+1} f_{m+1} c_3^{-1} f_{m+1}^{-1} \]
d and
\[ g_{m+2} f_{m+2} f_{m+2} f_{m+1} c_3^{-1} f_m = a_{m+1} f_{m+1} c_3^{-1} f_{m+1}^{-1}. \]

Therefore,
\[ g_{m+2} = a_{m+1} f_{m+1} c_3^{-1} f_m c_3 f_{m+1}. \]

Including this expression of $g_{m+2}$ in the forth relation:
\[ f_m^{-1} c_3 f_{m+1} a_{m+1} a_{m+1}^{-1} f_{m+1} c_3^{-1} f_m c_3 f_{m+1} c_2 a_{m+1} f_{m+1} c_3^{-1} f_{m+1}^{-1} = 1. \]

We get after cancelation
\[ a_{m+1} = f_{m+1} f_{m+2}. \]

Including this expression of $a_{m+1}$ in the expression for $g_{m+2}$, we get
\[ g_{m+2} = f_m^{-1} f_{m+1} c_3^{-1} f_m c_3 f_{m+1}^{-1} \]
or
\[ g_{m+2} = f_m^{-1} f_{m+1} c_3^{-1} f_m c_3 f_{m+1}^{-1}. \]

Next, the relation $a_m f_{m+1}^{-1} a_{m+1} = f_m f_m f_{m+2}$ after substitution $a_m = f_m f_{m+1}$,
\[ a_{m+1} = f_{m+1} f_{m+2} \]
becomes an identity.

Hence, the new relations in $V B_4'$ are equal to relations
\[ g_{m+2} = f_m^{-1} f_{m+1} c_3^{-1} f_m c_3 f_{m+1}^{-1}, \]
\[ a_m = f_m f_{m+1}. \]

The full set of relations in $V B_4'$ has the form:
\[ f_m f_{m+1}^{-1} f_{m+2} = f_m f_{m+2} f_{m+3}, \]
\[ f_m^{-1} f_{m+1} c_3 f_{m+2} f_{m+3} = f_{m+1} f_{m+2} c_3, \]
\[ a_m f_{m+1} a_{m+1} f_{m+2}^{-1} a_{m+2} = f_m a_m f_{m+1} a_{m+1} f_{m+2} a_{m+2} f_{m+3}^{-1}, \]
\[ f_m a_m f_{m+1} a_{m+1} c_3 a_{m+1} f_{m+2} a_{m+2} f_{m+3} a_{m+2} = f_{m+1} a_m f_{m+1} f_{m+2} a_{m+1} a_{m+1} c_3 a_{m+1}, \]
\[ g_{m+2} = 1, \]
\[ f_m = 1, \]
\[(f_{m}g_{m,3})^3 = 1,\]
\[g_{m+1,3}a_{m}^{-1}g_{m,3} = 1,\]
\[f_{m}g_{m,3}f_{m}^{-1}f_{m+1}g_{m+1,3}f_{m+1}^{-1} = c_3^{-1},\]
\[f_{m}^{-1}g_{m,3}f_{m}a_{m}^{-1}g_{m+1,3}f_{m+1} = c_3,\]
\[g_{m+2,3} = f_{m+1}^{-1}f_{m+1}c_{3}^{-1}f_{m}c_{3}f_{m+1}^{-1},\]
\[a_{m} = f_{m}f_{m+1}.\]

Transform these relations, excluding \(a_{m}\) and \(g_{m,3}\).

1) The relation \(a_{m}f_{m+1}a_{m+1}f_{m+2}^{-1}a_{m+2} = f_{m}a_{m}f_{m+1}^{-1}a_{m}f_{m+1}^{-1}a_{m+2}f_{m+3}^{-1}\), after substitution

\[a_{m} = f_{m}f_{m+1},\quad a_{m+1} = f_{m+1}f_{m+2},\quad a_{m+2} = f_{m+2}f_{m+3}\]

has the form

\[f_{m}f_{m+1}f_{m+1}f_{m+1}^{-1}f_{m+2}^{-1}f_{m+2}f_{m+3} = f_{m}f_{m+1}f_{m+1}^{-1}f_{m+1}f_{m+2}f_{m+2}f_{m+2}f_{m+3}f_{m+3}^{-1}.\]

Using the relation \(f_{m}^3 = 1\), we get

\[f_{m}f_{m+1} = f_{m+2}f_{m+3}.\]

2) The relation

\[f_{m}a_{m}f_{m+1}^{-1}a_{m+1}^{-1}c_{3}a_{m+1}f_{m+2}a_{m+2}f_{m+3}^{-1}a_{m+2}^{-1} = c_{3}f_{m+1}f_{m+1}^{-1}a_{m}^{-1}f_{m+1}^{-1}a_{m+1}^{-1}a_{m+1}^{-1}c_{3}a_{m+1},\]

after substitution

\[a_{m} = f_{m}f_{m+1},\quad a_{m+1} = f_{m+1}f_{m+2},\quad a_{m+2} = f_{m+2}f_{m+3}\]

has the form

\[f_{m}f_{m+1}f_{m+1}^{-1}f_{m+1}^{-1}f_{m+2}^{-1}f_{m+2}^{-1}f_{m+3}f_{m+3}^{-1} = c_{3}f_{m+1}f_{m+1}^{-1}f_{m+1}^{-1}f_{m+1}^{-1}f_{m+1}^{-1}c_{3}f_{m+1}f_{m+2};\]

or, after cancelation and using the relation \(f_{m}^3 = 1\) we get

\[f_{m}^{-1}f_{m+1}^{-1}c_{3}f_{m+1}f_{m+1}^{-1}f_{m+2}^{-1}f_{m+1}^{-1}c_{3}f_{m+1} = c_{3}f_{m+1}^{-1}f_{m+1}^{-1}f_{m+1}^{-1}c_{3}f_{m+1}.\]

3) The relation \(g_{m+2,3}^2 = 1\) after substitution

\[g_{m+2,3} = f_{m+1}^{-1}f_{m+1}c_{3}^{-1}f_{m}c_{3}f_{m+1}^{-1},\]

has the form

\[f_{m+1}^{-1}f_{m+1}c_{3}^{-1}f_{m}c_{3}f_{m+1}^{-1}f_{m+1}^{-1}f_{m+1}^{-1}f_{m+1}^{-1}f_{m+1}^{-1}c_{3}f_{m+1} = 1.\]

4) The relation \((f_{m}g_{m,3})^3 = 1\) after substitution

\[g_{m,3} = f_{m}^{-1}f_{m-1}c_{3}^{-1}f_{m-2}c_{3}f_{m-1}^{-1},\]

has the form

\[(f_{m}^{-1}f_{m-1}c_{3}^{-1}f_{m-2}c_{3}f_{m-1}^{-1})^3 = 1\]

and is identity since \(f_{m}^3 = 1.\)

5) The relation \(g_{m+1,3}a_{m}^{-1}g_{m,3} = 1\) after substitution

\[g_{m+1,3} = f_{m+1}^{-1}f_{m}c_{3}^{-1}f_{m-1}c_{3}f_{m}^{-1},\quad g_{m,3} = f_{m}^{-1}f_{m-1}c_{3}^{-1}f_{m-2}c_{3}f_{m-1}^{-1},\quad a_{m} = f_{m}f_{m+1} \]
has the form
\[ f_{m+1} f_m c_3^{-1} f_{m-1} c_3 f_m^{-1} f_{m+1} c_3^{-1} f_{m-1} c_3 f_m^{-1} f_{m-2} c_3 f_m^{-1} = 1. \]

Using the relation \( f_m^3 = 1 \) and changing the index \( m \) on \( m + 1 \), we get
\[ f_{m+1} f_m c_3^{-1} f_m c_3 f_{m-1} c_3 f_m^{-1} f_{m+1} c_3^{-1} f_m c_3 f_{m-1} f_{m-1} c_3 f_m^{-1} = 1. \]

6) The relation \( f_m g_{m,3} f_m^{-1} f_{m+1} g_{m+1,3} f_{m+1}^{-1} = c_3^{-1} \) after substitution
\[ g_{m+1,3} = f_m f_m c_3^{-1} f_m c_3 f_m^{-1}, \quad g_{m,3} = f_m f_m c_3^{-1} f_m c_3 f_m^{-1}, \]
has the form
\[ f_m f_m c_3^{-1} f_m c_3 f_m^{-1} f_m f_m c_3^{-1} f_m c_3 f_m^{-1} f_m c_3 f_m^{-1} c_3 f_m^{-1} f_{m+1} = c_3^{-1} \]
or after cancelation
\[ f_m c_3^{-1} f_m c_3 f_m^{-1} f_m c_3 f_m^{-1} f_m c_3 f_m^{-1} f_{m+1} = c_3^{-1}. \]

7) The relation \( f_m^{-1} g_{m,3} f_m a_m f_m^{-1} g_{m+1} a_m f_m+1 = c_3 \) after substitution
\[ g_{m+1,3} = f_m f_m c_3^{-1} f_m c_3 f_m^{-1}, \quad g_{m,3} = f_m f_m c_3^{-1} f_m c_3 f_m^{-1}, \quad a_m = f_m f_m+1 \]
has the form
\[ f_m f_m c_3^{-1} f_m c_3 f_m^{-1} f_m f_m c_3^{-1} f_m c_3 f_m^{-1} f_m c_3 f_m^{-1} f_m+1 = c_3 \]
or, after cancelation and using the relation \( f_m^3 = 1 \) we get
\[ f_m f_m c_3^{-1} f_m c_3 f_m^{-1} f_m c_3 f_m^{-1} f_m c_3 f_m^{-1} f_m+1 = c_3. \]

Hence, we have proven

**Theorem 5.7.** The group \( WB'_4 \) is generated by \( c_3, f_m, m \in \mathbb{Z} \), and is defined by the relations

\[
\begin{align*}
f_m f_{m+1} f_{m+2} &= f_{m+1} f_{m+2} f_{m+3}, \\
f_m^{-1} f_{m+1} c_3 f_{m+2} f_{m+3} &= c_3 f_{m+1} f_{m+2} c_3, \\
f_m f_{m+1} &= f_{m+2} f_{m+3}, \\
f_m^{-1} f_{m+1} c_3 f_{m+1} f_m f_{m+3} f_{m+2} &= c_3 f_{m+1} f_{m+2} f_{m+1} f_m f_{m+1} c_3, \\
f_{m+2} f_m c_3^{-1} f_{m+1} c_3 f_{m+2} f_m c_3^{-1} f_{m+2} f_m c_3^{-1} f_{m+2} f_m c_3^{-1} f_{m+1} c_3 f_{m+1} &= 1, \\
f_m^3 &= 1, \\
f_m^{-1} f_{m+1} c_3^{-1} f_m c_3 f_{m+1} f_m c_3^{-1} f_m c_3 f_{m+1} &= 1, \\
f_m^{-1} c_3^{-1} f_m c_3 f_{m+1} f_m c_3^{-1} f_m c_3 f_{m+1} &= c_3^{-1}, \\
f_m f_{m-1} c_3^{-1} f_m c_3 f_{m-1} f_m c_3^{-1} f_m c_3 f_{m-1} f_m+1 &= c_3.
\end{align*}
\]

**Corollary 5.8.** The group \( WB'_4 \) is generated by \( c_3, f_0, f_1, f_2 \).

Indeed, using the relations
\[ f_m f_{m+1} f_{m+2} = f_{m+1} f_{m+2} f_{m+3}, \]
we can save from the generators \( f_m, m \in \mathbb{Z} \), only the relations \( f_0, f_1, f_2 \).

**Corollary 5.9.** \( WB'_4 / WB''_4 \cong \mathbb{Z}_3 \).
Indeed, considering relations of $WB'_4$ by modulo $WB''_4$ we see that $f_m f_{m+1} = 1$, $f_3^3 = 1$ and $c_3 = 1$.

5.3. The commutator subgroup $WB'_n$ for $n \geq 5$.

**Theorem 5.10.** The group $WB'_n$, $n \geq 5$, is generated by $n$ elements $f_0, f_1, f_2, c_3, \ldots, c_{n-1}$.

**Proof.** As we proved before, $VB'_n$, $n \geq 5$, is generated by elements $c_3, \ldots, c_{n-1}, f_0, f_1, f_2, g_0, \ldots, g_{0,n-1}$.

The group $WB'_n$, $n \geq 5$, is defined by relations of $VB'_n$ and the relations:

\[ b_{m,1} a_{m+1} f_{m+2,1} b_{m,0}^{-1} = 1, \]
\[ b_{m,0} f_{m+2,0} a_{m+1}^{-1} b_{m,1}^{-1} = 1, \]
\[ f_{m,0} c_{3} b_{m+1,1} g_{m+2,3} b_{m+1,0}^{-1} c_{3}^{-1} = 1, \]
\[ f_{m,1} c_{3} f_{m+1,0} a_{m+1} g_{m+2,3} b_{m+1,1}^{-1} c_{3}^{-1} = 1, \]
\[ g_{m,i} c_{i+1} c_{i} a_{m+2,i+1} c_{i+1}^{-1} = 1, \]
\[ g_{m,i} c_{i+1} a_{m+1}^{-1} c_{i} a_{m+1} c_{i+1}^{-1} = 1. \]

Similar to the group $WB'_4$, the first for relations are equivalent to the relations

\[ g_{m+2,3} = f_{m+2}^{-1} f_{m+1} c_{3}^{-1} f_{m} c_{3} f_{m+1}^{-1}, \quad a_m = f_{m} f_{m+1}. \]

Hence, the additional relations of $WB'_n$, $n \geq 5$ have the form

\[ g_{m+2,3} = f_{m+2}^{-1} f_{m+1} c_{3}^{-1} f_{m} c_{3} f_{m+1}^{-1}, \]
\[ a_m = f_{m} f_{m+1}, \]
\[ g_{m,i} c_{i+1} c_{i} g_{m+2,i+1} c_{i+1}^{-1} = 1, \]
\[ g_{m,i} c_{i+1} f_{m+1}^{-1} f_{m} c_{i} f_{m+1} f_{m+2} g_{m+2,i+1} f_{m+2}^{-1} f_{m}^{-1} c_{i+1}^{-1} f_{m} f_{m+1} c_{i+1}^{-1} = 1. \]

Using the relations $g_{m,i} c_{i+1} c_{i} g_{m+2,i+1} c_{i+1}^{-1} = 1$, we can express the generators $g_{m,i}$, $i \geq 4$, as words in the generators $c_3, \ldots, c_{n-1}, f_m, g_m, m \in \mathbb{Z}$. Also, as in the case of the group $WB'_4$, we can express the generators $f_m, g_m, m \in \mathbb{Z}$, as words in the generators $c_3, f_0, f_1, f_2$. \[\square\]

5.4. Proof of Theorem 1.3.

**Proof.** Parts (1) of Theorem 1.3 follows by combining Corollary 5.5, Corollary 5.8 and Theorem 5.7. Part (2) and (3) follow from Corollary 5.6 and Corollary 5.9.

For $n \geq 5$, note that $WB'_n$ is perfect as a quotient of the perfect group $VB'_n$. This proves (4). \[\square\]
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