Sine-Gordon model coupled with a free scalar field emergent in the low-energy phase dynamics of a mixture of pseudospin-$\frac{1}{2}$ Bose gases with interspecies spin exchange

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Abstract. Using the approach of low-energy effective field theory, the phase diagram is studied for a mixture of two species of pseudospin-$\frac{1}{2}$ Bose atoms with interspecies spin exchange. There are four mean-field regimes on the parameter plane of $g_e$ and $g_z$, where $g_e$ is the interspecies spin-exchange interaction strength, while $g_z$ is the difference between the interaction strength of interspecies scattering without spin exchange of equal spins and that of unequal spins. Two regimes, with $|g_z| > |g_e|$, correspond to ground states with the total spins of the two species parallel or antiparallel along the $z$ direction, and the low-energy excitations are equivalent to those of two-component spinless bosons. The other two regimes, with $|g_e| > |g_z|$, correspond to ground states with the total spins of the two species parallel or antiparallel on the $xy$ plane, and the low-energy excitations are described by a sine-Gordon model coupled with a free scalar field, where the effective fields are combinations of the phases of the original four boson fields. In $(1+1)$-dimension, they are described by Kosterlitz–Thouless renormalization group (RG) equations, and there are three sectors in the phase plane of a scaling dimension and a dimensionless parameter proportional to the strength of the cosine interaction, both depending on the densities. The gaps of
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these elementary excitations are experimental probes of the underlying many-body ground states.

**Keywords:** Bose–Einstein condensation (theory), quantum gases, superfluidity

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1. Introduction

The Sine-Gordon model is important in field theory and statistical mechanics, and its renormalization group equations define the Kosterlitz–Thouless universality class for a class of (1 + 1)-dimensional quantum systems and two-dimensional classical systems [1, 2]. In this paper, we show that a sine-Gordon model coupled with a free scalar field emerges in the phase dynamics of a mixture of two different species of pseudospin-$\frac{1}{2}$ Bose gases with interspecies spin-exchange interaction. Interestingly, here the scalar field described by the sine-Gordon model and the free scalar field are two different combinations of the phases of the four bosonic fields in this system.

A mixture of two different species of pseudospin-$\frac{1}{2}$ Bose gases with interspecies spin-exchange interaction exhibits novel features beyond a single species of spinor Bose gas as well as a mixture of two species without interspecies spin exchange [3]–[10]. In this model, each atom has an internal degree of freedom represented as a pseudospin with basis states $|\uparrow\rangle$ and $|\downarrow\rangle$, while there are two species of atoms, with the atom number of each species conserved. It can be described by the following Hamiltonian density,

$$
\mathcal{H} = \sum_{\alpha\sigma} \psi_{\alpha\sigma}^\dagger \left( -\frac{1}{2m_\alpha} \nabla^2 + V \right) \psi_{\alpha\sigma} + \frac{1}{2} \sum_{\alpha \sigma \sigma'} g^{(aa)}_{\alpha\sigma\sigma'} |\psi_{\alpha\sigma}|^2 |\psi_{\alpha\sigma'}|^2 \\
+ \sum_{\sigma \sigma'} g^{(ab)}_{\sigma \sigma'} |\psi_{a\sigma}|^2 |\psi_{b\sigma'}|^2 + g_e (\psi_{a\uparrow}^\dagger \psi_{b\downarrow}^\dagger \psi_{b\uparrow} \psi_{a\downarrow} + \psi_{a\downarrow}^\dagger \psi_{b\uparrow}^\dagger \psi_{b\downarrow} \psi_{a\uparrow}),
$$

(1)

where $\alpha = a, b$ represents the two species and $\sigma = \uparrow, \downarrow$. $V = V(x)$ is the external potential, $g^{(aa)}_{\sigma \sigma'}, g^{(ab)}_{\sigma \sigma'}$ and $g_e$ are the interaction strengths for intraspecies scattering, interspecies...
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scattering without spin exchange, and interspecies spin-exchange scattering respectively, proportional to the corresponding scattering lengths. For pseudospin-$\frac{1}{2}$ atoms, the intraspecies scattering strengths with and without spin exchange are the same [11]. For simplicity, we assume $g_{\sigma\sigma'}^{(ab)} = g_{\sigma}$ for any $\sigma$ and $\sigma'$, $g_{\downarrow\downarrow}^{(ab)} = g_{\downarrow}$ and $g_{\uparrow\uparrow}^{(ab)} = g_{\downarrow}$. We define $S_{\alpha}(x) = \Psi_{\alpha}^\dagger s^i \Psi_{\alpha}$, where $\Psi_{\alpha}(x) \equiv (\psi_1(x), \psi_1(x))^T$, $s^i = \tau_i/2$, $\tau_i$ being the Pauli matrix, $(i = x, y, z)$. Then $\mathcal{H}$ can be rewritten as

\begin{equation}
\mathcal{H} = \sum_{\alpha} \Psi_{\alpha}^\dagger \left( -\frac{1}{2m_{\alpha}} \nabla^2 + V \right) \Psi_{\alpha} + \frac{g_{\sigma}}{2} |\Psi_{\sigma}|^4 + \frac{g_{\downarrow}}{2} |\Psi_{\downarrow}|^4 + \frac{g_{\sigma\downarrow}}{2} |\Psi_{\sigma}|^2 |\Psi_{\downarrow}|^2
\end{equation}

\begin{equation}
+ 2g_e(S_{\alpha x}S_{\beta x} + S_{\alpha y}S_{\beta y}) + 2g_2S_{\alpha z}S_{\beta z},
\end{equation}

where $g_{\sigma\downarrow} \equiv g_{\sigma} - g_{\downarrow}$, $g_{\uparrow\downarrow} \equiv g_{\uparrow} - g_{\downarrow}$. It can be seen that $g_{\sigma}$, $g_{\downarrow}$, and $g_{\sigma\downarrow}$ characterize the usual density–density interactions, while $g_{\uparrow\downarrow}$ and $g_{\sigma\uparrow\downarrow}$ characterize the spin coupling between the two species.

We make the presumption that $g_{\sigma} > 0$, $g_{\downarrow} > 0$ and $4g_{\sigma}g_{\downarrow} > g_{\sigma\downarrow}^2$, which is needed for the stability of the system and can be naturally satisfied in reality [10]. We study the phase diagram in the space of the parameters $g_e$ and $g_z$, by using the approach of low-energy effective field theory. The regime of $g_e > g_z > 0$ has been discussed previously for higher dimensions, i.e. when the phase fluctuation is suppressed such that the cosine of a phase variable can be approximated up to second order [10]. In this paper, we first extend the discussion to other parameter regimes. In the regime of $g_z > |g_e|$, the ground state is with the total spins of the two species antiparallel along the $z$ direction. In the regime of $g_z < -|g_e|$, the ground state is with the total spins of the two species parallel along the $z$ direction. In the regime of $g_e > |g_z|$, the ground state is with the total spins of the two species antiparallel on the $xy$ plane. In the regime of $g_e < -|g_z|$, the ground state is with the total spins of the two species parallel on the $xy$ plane. Then we focus on the case of $|g_z| > |g_e|$ in (1 + 1)-dimension. Without approximating the cosine interaction term, the low-energy excitations can be described by a sine-Gordon model coupled with a free scalar field, both fields being combinations of the phases of the original four boson fields. It turns out that for given $g_e$ and $g_z$ with $|g_z| > |g_e|$, there are three phases according to a scaling dimension and a dimensionless parameter proportional to $|g_e|$. Both these two parameters depend on the densities of the two species.

2. Phase diagram on the $g_e$–$g_z$ parameter plane

There is a symmetry between parameter points $(g_e, g_z)$ and $(-g_e, g_z)$. Consider the transformation $\psi'_{\alpha} = -\psi_{\alpha}$, $\psi'_{\alpha} = \psi_{\alpha}$, $\psi'_{\beta} = \psi_{\beta}$, $\psi'_{\beta} = \psi_{\beta}$. The Hamiltonian density in terms of the primed operators in the parameter point $(g_e, g_z)$ has the same form as the Hamiltonian density in terms of the unprimed ones in the parameter point $(-g_e, g_z)$.

Now consider the case of $g_e > |g_z|$. Then, from the Hamiltonian density (2), it is easy to see that in the ground state, $S_a$ and $S_b$ must align oppositely in the $z$ direction. We choose the mean-field values in the ground state to be with $\psi_{\alpha}^0 = \sqrt{n_\alpha}$, $\psi_{\alpha}^0 = 0$, $\psi_{\beta}^0 = \sqrt{n_\beta}$, so that $S_a = (n_a/2)\hat{z}$ and $S_b = -(n_b/2)\hat{z}$, where $\hat{z}$ is the unit vector in the $z$ direction and $n_\alpha$ is the total density of species $\alpha$ ($\alpha = a, b$). The low-energy dynamics is dominated by the fluctuations of $\psi_{\alpha}$ and $\psi_{\beta}$, as the fluctuations of $\psi_{\alpha}$ and $\psi_{\beta}$, whose

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mean-field values are zero, must be of the amplitudes rather than the phases, and thus increase the energy.

Similarly, in the case of \( g_z < -|g_e| \), the ground state is that with \( \mathcal{S}_a \) and \( \mathcal{S}_b \) parallel in the \( z \) direction. We can choose the mean-field values in the ground state to be with
\[
\psi_{a1}^0 = \sqrt{n_a}, \quad \psi_{a1}^0 = 0, \quad \psi_{b1}^0 = \sqrt{n_b}, \quad \psi_{b1}^0 = 0,
\]
so that \( \mathcal{S}_a = (n_a/2) \hat{z} \) and \( \mathcal{S}_b = (n_b/2) \hat{z} \). The low-energy dynamics is dominated by the fluctuations of \( \psi_{a1} \) and \( \psi_{b1} \), as the fluctuations of \( \psi_{a1} \) and \( \psi_{b1} \), whose mean-field values are zero, must be of the amplitudes and thus increase the energy.

In these two cases, which can be represented in a unified form as \( |g_z| > |g_e| \), the system behaves like a mixture of two species of spinless Boson gases, with the effective Hamiltonian density
\[
\mathcal{H} = \sum_\alpha \Psi_\alpha^\dagger \left( -\frac{1}{2m_\alpha} \nabla^2 + V \right) \Psi_\alpha + \frac{g_a}{2} |\Psi_\alpha|^4 + \frac{g_b}{2} |\Psi_\beta|^4 + \frac{g_{ab}}{2} \frac{|g_z|}{2} |\Psi_\alpha|^2 |\Psi_\beta|^2, \tag{3}
\]
whose excitation spectra are \([12]\)
\[
\omega^2 = \frac{1}{2} (\varepsilon_a^2 + \varepsilon_b^2) \pm \frac{1}{2} \sqrt{(\varepsilon_a^2 - \varepsilon_b^2)^2 + 4E_aE_bn_an_b(g_{ab} - |g_z|)^2}, \tag{4}
\]
where we have introduced
\[
\varepsilon_a^2 = E_a(2g_an_a + E_a), \tag{5}
\]
with \( \alpha = a, b \), and \( E_a = k^2/2m_\alpha \) being the free particle energy of species \( \alpha \). All spectra are gapless, as in the usual case of phonon-like Goldstone modes. That is, \( \omega \to 0 \) when \( k \to 0 \).

In the case of \( g_e > |g_z| \), which has been discussed previously \([10]\), the ground state is that with \( \mathcal{S}_a \) and \( \mathcal{S}_b \) antiparallel on the \( xy \) plane. One can choose the ground state to be with \( \psi_{a1}^0 = \psi_{a1}^0 = \sqrt{n_a/2}, \psi_{b1}^0 = -\psi_{b1}^0 = \sqrt{n_b/2} \).

Similarly, in the case of \( g_e < -|g_z| \), the ground state is that with \( \mathcal{S}_a \) and \( \mathcal{S}_b \) parallel on the \( xy \) plane. One can choose the ground state to be with \( \psi_{a1}^0 = \psi_{a1}^0 = \sqrt{n_a/2}, \psi_{b1}^0 = \psi_{b1}^0 = \sqrt{n_b/2} \).

In the latter two cases, which can be represented in a unified form as \( |g_e| > |g_z| \), the effective Lagrangian describing the phase fluctuations is, with \( |g_e| \) replacing \( g_e \) in the result for \( g_e > g_z > 0 \) \([10]\),
\[
\mathcal{L}_{\text{eff}} = \frac{1}{2} (\partial_t \Gamma)^T A^{-1}(\partial_t \Gamma) - \frac{1}{2} (\nabla \Gamma)^TM^{-1}(\nabla \Gamma) + \frac{|g_e|}{2} n_an_b \cos(2\gamma_4) \tag{6}
\]
where
\[
\Gamma = \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ \sqrt{2} & \sqrt{2} & 1 & 1 \\ 0 & 0 & 1 & 1 \\ \sqrt{2} & 1 & \sqrt{2} & 1 \\ \sqrt{2} & 1 & -\sqrt{2} & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 \\ 2 & -2 & 2 & -2 \end{pmatrix} \begin{pmatrix} \Phi_{a1} \\ \Phi_{a1} \\ \Phi_{b1} \\ \Phi_{b1} \end{pmatrix}, \tag{7}
\]
with \( \Phi_{\alpha\sigma} \) being the phase of \( \psi_{\alpha\sigma} \),

\[
A \equiv \begin{pmatrix}
2g_a & g_{ab} - |g_e| & 0 & 0 \\
g_{ab} - |g_e| & 2g_b & 0 & 0 \\
0 & 0 & |g_e|\eta_+ + g_z & |g_e|\eta_- \\
0 & 0 & |g_e|\eta_- & |g_e|\eta_+ - g_z
\end{pmatrix},
\]

with

\[
\eta_\pm \equiv \frac{1}{2} \left( \frac{n_b}{n_a} \pm \frac{n_a}{n_b} \right), \quad M^{-1} \equiv \frac{1}{2} \begin{pmatrix}
\frac{n_a}{m_a} & 0 & 0 & 0 \\
0 & \frac{n_b}{m_b} & 0 & 0 \\
0 & 0 & \xi_+ & \xi_- \\
0 & 0 & \xi_- & \xi_+
\end{pmatrix},
\]

with

\[
\xi_\pm \equiv \frac{1}{2} \left( \frac{n_a}{m_a} \pm \frac{n_b}{m_b} \right).
\]

In \((3+1)\)-dimension or \((2+1)\)-dimension, the fluctuation of \( \gamma_4 \) is largely suppressed and we can make the approximation that \( \cos(2\gamma_4) \approx 1 - 2\gamma_4^2 \); subsequently, the four excitation spectra can be obtained as [10],

\[
\omega_{1,II}^2 = \frac{k^2}{2} \left[ g_an_a + \frac{g_bn_b}{m_b} \right] + \left( \frac{g_an_a - \frac{g_bn_b}{m_b}}{m_a} \right)^2 + \left( \frac{g_{ab} - |g_e|^2n_an_b}{m_am_b} \right),
\]

\[
\omega_{III,IV}^2 = \frac{1}{2} \left[ Bk^2 + \Delta^2 \right] \equiv \sqrt{Ck^4 + Dk^2 + \Delta^4},
\]

where \( \Delta^2 = |g_e|^2((n_b/m_a) + (n_a/m_b)) - 2|g_e|g_z(n_an_b), B \equiv (|g_e|^2/4)((n_b/m_a) + (n_a/m_b))^2 + g_z^2(n_an_b/m_am_b), C \equiv |g_e|^2(g_z^2(n_an_b/m_am_b), D \equiv |g_e|^2(g_z^2(n_an_b/m_am_b) - (n_a/m_b)) ((n_b/m_a) - (n_a/m_b)) - 2|g_e|g_z((n_b/m_a) + (n_a/m_b)) + 2g_z^2((n_a/m_a) + (n_b/m_b)) \right)\]. Under the conditions \( g_a > 0, 4g_a g_b > g_{ab}^2 \), and \( |g_e| > |g_z| \), all these excitations have real energies for any \( k \), guaranteeing the stability of the ground state.

It can be seen that \( \omega_{IV} \) has a gap \( \Delta \) while the other three excitations are gapless. That is, as \( k \to 0 \), \( \omega_{1,II,III} \to 0 \), but \( \omega_{IV} \to \Delta \).

Therefore in \((3 + 1)\)-dimension, we obtain the mean-field phase diagram as shown in figure 1.

3. Renormalization group analysis in the case of \( |g_e| > |g_z| \) in \((1 + 1)\)-dimension

Reconsider the case of \( |g_e| > |g_z| \). In \((1+1)\)-dimension, the fluctuation is important and we must take into account the whole effect of the \( \cos(2\gamma_4) \) term, which is the only interaction term in \( \mathcal{L}_{\text{eff}} \), where \( \gamma_1 \) and \( \gamma_2 \) are both free fields and not coupled to \( \gamma_4 \). Hence we can
Then the action can be written as

\[ S = S_0 + S_1, \]

where \( S_0 \equiv \int \mathcal{L}_0 \, d^2 x \), \( S_1 \equiv \int \mathcal{L}_1 \, d^2 x \), with

\[
\mathcal{L}_0 = \frac{1}{2} \begin{pmatrix} \partial_0 \varphi & \partial_0 \chi \end{pmatrix} \begin{pmatrix} f_1 & f_2 \\ f_2 & 1 \end{pmatrix} \begin{pmatrix} \partial_0 \varphi \\ \partial_0 \chi \end{pmatrix} + \frac{1}{2} \begin{pmatrix} \partial_t \varphi & \partial_t \chi \end{pmatrix} \begin{pmatrix} 1 & p \\ p & 1 \end{pmatrix} \begin{pmatrix} \partial_t \varphi \\ \partial_t \chi \end{pmatrix},
\]

\[
\mathcal{L}_1 = \frac{\lambda}{a^2} \cos(\beta \chi),
\]

where \( \mathcal{L}_0 \) and \( \mathcal{L}_1 \) are the zeroth and first-order Lagrangians, respectively. The terms \( f_1 \) and \( f_2 \) are coefficients related to the interaction terms. The action \( S \) is composed of a zeroth-order part \( S_0 \) and a first-order part \( S_1 \). The zeroth-order part \( S_0 \) includes the kinetic terms and interaction terms, while the first-order part \( S_1 \) includes the temporal derivative terms. The parameter \( \lambda \) is related to the coupling strength of the interaction term.
where
\[ f_1 = \frac{|g_e| \eta_+ - g_z}{|g_e| \eta_+ + g_z}, \quad f_2 = \frac{|g_e| \eta_-}{|g_e| \eta_+ + g_z}, \quad p = \frac{\xi_-}{\xi_+}, \]
\[ \beta = 2 \left[ \xi_+ \left( \frac{|g_e| \eta_+ + g_z}{2(g_e^2 - g_z^2)} \right) \right]^{-1/4}, \quad \lambda = \frac{|g_e| n_a n_b a^2}{2v}, \]
and \( a \) is the short-range cut-off, which is the coherence or healing length, and can be estimated to be \( \hbar |m_b(g_a n_a^2 + g_b n_b^2 + (g_s + g_d - g_e) n_a n_b)|^{-1/2} \), assuming \( m_a \geq m_b \). \( \lambda \) is dimensionless. From \( \mathcal{L}_0 \) the free propagator of \( \chi \) is obtained as
\[ G_{\lambda}(k) = \frac{f_1 k_0^2 + k_1^2}{k^2 (f_1 k_0^2 + k_1^2) - (f_2 k_0^2 + p k_1^2)^2} = \frac{1}{k^2} \frac{f_1 \cos^2 \theta + \sin^2 \theta}{f_1 \cos^2 \theta + \sin^2 \theta - (f_2 \cos^2 \theta + p \sin^2 \theta)^2}, \] (16)
where \( k^2 = k_0^2 + k_1^2 \) and \( \tan \theta = (k_1/k_0) \).

Since the interaction term does not involve \( \varphi \) field, we only need to split \( \chi \) into the fast and slow components,
\[ \chi_\Lambda(x) = \chi_{\Lambda'}(x) + h(x), \] (17)
where
\[ \chi_{\Lambda'}(x) \equiv \sum_{k < \Lambda'} e^{ikx} \chi_k, \] (18)
\[ h(x) \equiv \sum_{\Lambda' < k < \Lambda} e^{ikx} \chi_k, \] (19)
and \( \Lambda = 1/a \) is the momentum cut-off, \( \Lambda' = \Lambda - d\Lambda \). The partition function can be written as [1]
\[ Z_\Lambda = \int \mathcal{D}\varphi \mathcal{D}\chi_\Lambda \mathcal{D}h e^{-S_0[\varphi, \chi_\Lambda] - S_0[h] - S_1[\chi_{\Lambda'} + h]} \] (20)
\[ = Z_h \int \mathcal{D}\varphi \mathcal{D}\chi_{\Lambda'} e^{-S_0[\varphi, \chi_{\Lambda'}]} \langle e^{-S_1[\chi_{\Lambda'} + h]} \rangle_h, \] (21)
where \( Z_h = \int \mathcal{D}h e^{-S_0[h]} \langle \cdots \rangle_h \) means taking the average over the fast components of \( \chi \).

The effective action is thus
\[ S[\varphi, \chi_{\Lambda'}] = S_0[\varphi, \chi_{\Lambda'}] - \ln \langle e^{-S_1[\chi_{\Lambda'} + h]} \rangle_h \]
\[ \approx S_0[\varphi, \chi_{\Lambda'}] + \langle S_1[\chi_{\Lambda'} + h] \rangle_h - \frac{1}{2} (\langle S_1^2[\chi_{\Lambda'} + h] \rangle_h - \langle S_1[\chi_{\Lambda'} + h] \rangle_h^2), \] (22)
which allows us to calculate the RG flows of \( \lambda \) and \( \beta \). We have
\[ \langle h(x)h(0) \rangle_h = \int_{\Lambda' < k < \Lambda} \frac{d^2k}{(2\pi)^2} G_{\lambda}(k) e^{ikx} = \frac{\kappa(\Lambda r)}{2\pi} dl, \] (23)
where
\[ dl \equiv \frac{d\Lambda}{\Lambda}, \]
\[ r \equiv |x|, \]
\[ \kappa(\Lambda r) \equiv \int \frac{d\theta}{2\pi} \frac{f_1 \cos^2 \theta + \sin^2 \theta}{f_1 \cos^2 \theta + \sin^2 \theta - (f_2 \cos^2 \theta + p \sin^2 \theta)^2} e^{i\Lambda r \cos \theta}, \] (24)
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which is a measure of correlation of fluctuations.

Then

$$\langle e^{i\beta h(x)} \rangle_h = e^{-(1/2)\beta^2(h^2(x))_h} = 1 - \frac{\beta^2\kappa(0)}{4\pi} \int dl \equiv 1 - D \int dl. \quad (25)$$

It is seen that the coupling between $\varphi$ and $\chi$ modifies the scaling dimension of $e^{i\beta\chi}$ from $D_0 = (\beta^2/4\pi)$ \cite{1} to

$$D = \frac{\beta^2\kappa(0)}{4\pi}, \quad (26)$$

which means the interaction term $\cos(\beta\chi)$ is less relevant than that in the pure SG model, since $\kappa(0) > 1$.

Now we can obtain the renormalized action. Following \cite{1}, we have

$$\langle S_I[\chi_{\Lambda'} + h]\rangle_h = \lambda(1 - D \int dl) \int \frac{d^2x}{a^2} \cos(\beta\chi_{\Lambda'}), \quad (27)$$

$$\langle S^2_I \rangle_h - \langle S_I \rangle_h^2 = -\alpha\lambda^2 D_0^2 \int dl \int d^2x (\nabla \chi_{\Lambda'})^2, \quad (28)$$

where

$$\alpha = \int_0^\infty \frac{dz}{2\pi} z^3 \kappa(z). \quad (29)$$

Therefore, by rescaling $\Lambda' \rightarrow \Lambda$, the purely $\chi$-dependent part of the action is renormalized to

$$S[\chi] = \frac{1}{2}(1 + \alpha\lambda^2 D_0^2 \int dl) \int d^2x (\nabla \chi)^2 + \lambda[1 + (2 - D \int dl)] \int \frac{d^2x}{a^2} \cos(\beta\chi). \quad (30)$$

The overall factor in front of the Gaussian part of the action requires a renormalization of the field $\chi$, as well as the parameter $\beta$, such that $\beta\chi$ is invariant. That is,

$$\chi(x) \rightarrow (1 + \alpha\lambda^2 D_0^2 \int dl)^{1/2} \chi(x), \quad \beta \rightarrow (1 + \alpha\lambda^2 D_0^2 \int dl)^{-1/2} \beta, \quad (31)$$

which gives RG flows of $\lambda$ and $\beta$.

There is another field $\varphi$ coupled to $\chi$. The coupling terms are those proportional to $f_2$ and $p$, respectively, in (14). The above renormalization of $\chi$ and $\beta$ leads to the renormalization of $f_2$ and $p$. As in the pure SG model, we introduce

$$t \equiv D - 2,$$

and assume $t$ and $\lambda$ are small, as the parameter point of $t = 0$ and $\lambda = 0$ is a fixed point of the RG flows. Then, to the order of $\lambda^2$, the RG equations read

$$\frac{dt}{dl} = -\frac{8\alpha\lambda^2}{\kappa^2(0)}, \quad \frac{d\lambda}{dl} = -\lambda t, \quad \frac{df_2}{dl} = -\frac{2\alpha\lambda^2}{\kappa^2(0)} f_2, \quad \frac{dp}{dl} = -\frac{2\alpha\lambda^2}{\kappa^2(0)} p. \quad (32)$$
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Figure 2. Phase diagram of our emergent sine-Gordon model coupled with a free field in $(1+1)$-dimension, with $y \equiv (\sqrt{8\alpha/\kappa(0)})\lambda$, $t \equiv D-2$, $D$ is a scaling dimension. There are two separatrices $t = \pm y$ that divide the phase plane into three sectors: (1) $t \geq y$, the weak coupling (WC) sector; (2) $|t| < y$, the crossover (C) sector; (3) $t \leq -y$, the strong coupling (SC) sector. The RG flows are similar to the pure sine-Gordon model.

4. Phase diagram on the $t-y$ parameter plane and the mass gap in $(1+1)$-dimension

From the RG equations, one obtains

\[ d(t^2) - \frac{8\alpha}{\kappa^2(0)} d(\lambda^2) = 0, \]

which is similar to the equation in the pure SG model [1], except that $\alpha$ and $\kappa$ are not constant here. Moreover, $d\kappa(0)/dl$ and $d\alpha/dl$ are both proportional to $\lambda^2$. Hence to the order of $\lambda^3$, one can replace $(8\alpha/\kappa^2(0)) d(\lambda^2)$ as $d((8\alpha/\kappa^2(0))\lambda^2)$. Thus we arrive at the following equation,

\[ t^2 - y^2 = \mu^2, \]

where

\[ y \equiv \frac{\sqrt{8\alpha}}{\kappa(0)} \lambda, \]

and $\mu$ represents a constant. For a given $g_e$ and $g_z$ with $|g_e| > |g_z|$, equation (34) determines the phase diagram of the model on the plane $(t, y)$ in the regime where $t$ and $y$ are small, as schematically shown in figure 2.

It can be seen that the $(t, y)$ phase space is divided into three sectors, namely, weak coupling, strong coupling and crossover sectors. In the weak coupling sector, the effective theory scales to a Gaussian model, $y(l) \rightarrow 0$ as $l \rightarrow \infty$, and the spectrum is massless, while in the crossover and strong coupling sectors, the coupling constants flow away from the Gaussian fixed line and the spectrum has a mass gap.

Note that both $t$ and $y$ depend not only on the interspecies spin-exchange coupling, but also on the densities. Consequently the phase is dependent not only on the interaction strengths, but also on the densities of the two species, which can be easily adjusted in experiments. To illustrate this explicitly, let us set $n_a = n_b = n$ so that $f_1 = (|g_e| - g_z/|g_e| + g_z)$, $f_2 = 0$, $p = (m_b - m_a)/(m_b + m_a)$ and thus $\kappa$ and $\alpha$ as well, are all independent of...
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$n$, while $\beta \propto n^{-1/4}$ and $\lambda \propto n^{3/2}$. Therefore $D \propto n^{-1/2}$ and $y \propto n^{3/2}$. Then according to figure 2, for small enough $n$, the system is in the weak coupling phase. For large enough $n$, the system is in the strong coupling phase. Therefore, following the change of density, the system goes through phase transitions.

The mass gap in the crossover and strong coupling sectors can be qualitatively obtained. In the strong coupling sector, $\mu$ is real, while in the crossover sector, $\mu$ is purely imaginary. Following [1], it can be found that the mass gap in the strong coupling sector is

$$M = \begin{cases} \Lambda \left( \frac{y_0}{t_0} \right)^{1/\beta}, & -t_0 \gg y_0, \\ \Lambda \exp(-1/y_0), & \mu \ll |y_0|, \end{cases},$$

(36)

where the subscript ‘0’ means the bare values, that is, the values measured in experiments, while in the crossover sector, the mass gap is

$$M = \Lambda \exp(-\pi/2|\mu|),$$

(37)

with $|\mu| \gg |t_0|$. The scaling behavior of the mass gap may be observed in experiments. Taking $M = \Lambda \exp(-1/y_0)$ as an example. If $n_a = n_b = n$, we have $y_0 = (\sqrt{8\alpha}/\kappa(0))\lambda \sim n^{3/2}$, then the relation between $M$ and $n$ may be investigated.

The last two of the RG equation (32) determine the RG flows of the couplings between the fields $\varphi$ and $\chi$. Since $\alpha > 0$, it is easy to find that $f_2 = 0$, $p = 0$ is the only stable fixed point of the two equations, namely, whatever the initial values of $f_2$ and $p$ are, they inevitably flow to 0. Moreover, the larger $\lambda$ is, the more rapidly $f_2$ and $p$ flow to 0. It is almost as though its strong self-interaction ‘traps’ the field $\chi$ and separates it from $\varphi$. If the bare value of $\lambda$ were 0, there would be no RG flows of $f_2$ and $p$.

Also note that if we diagonalize $L_0$ in (14), then $L_1$ in (15) becomes a cosine term of a linear combination of the two fields, for which the RG analysis is quite difficult. Hence we use the above approach instead.

The elementary excitations studied here can be experimentally measured using Bragg spectroscopy. The gap in a collective mode is a novel feature absent in the BEC mixtures previously studied. The two key parameters $g_e$ and $g_z$ both originate from the interspecies spin-dependent scattering, thus they are roughly of the same order of magnitude. We expect that the excitation gaps can be detected in experiments and are indications of the underlying many-body ground states.

5. Summary

We have developed a low-energy effective theory for a mixture of two species of pseudospin-$\frac{1}{2}$ Bose gases and explored the phase transitions in the space of the parameters $g_e$ and $g_z$, where $g_e$ is the interspecies spin-exchange interaction strength, while $g_z$ is the difference between the strengths of equal-spin and unequal-spin interspecies interaction without spin exchange. The phase diagram on the plane of parameters $g_e$ and $g_z$ is shown in figure 1. In the regime of $|g_z| > |g_e|$, the system is effectively described by a two-component model, and the excitation spectra are gapless. In the regime of $|g_e| > |g_z|$, the system is described by a four effective fields, which are combinations of the phases of the four original boson fields.
There is a cosine interaction term of one of the effective fields, which can be approximated as a square in (3 + 1)-dimension. There are three gapless modes and one gapped mode.

In (1 + 1)-dimension, the effective theory in the regime of $|g_e| > |g_z|$ is a novel realization of a sine-Gordon model coupled with a free scalar field, on which we have made a renormalization analysis. Described by Kosterlitz–Thouless equations, the phase space is further divided into three sectors, as shown in figure 2, according to a scaling dimension $t \equiv (1/\pi)[2(g_e^2 - g_z^2)/(|g_e||g_+ + g_z|)]^{1/2}\kappa(0) - 2$ and a dimensionless parameter $y = (\sqrt{8\alpha}/\kappa(0))\lambda$, where $\kappa$ is a correlation function given in (24), $\alpha = \int_0^\infty (dr/2\pi)r^3\kappa(r)$. Both $t$ and $y$ depend on the densities, through $\xi_+ \equiv \frac{1}{2}((n_a/m_a) + (n_b/m_b))$ and $\lambda \equiv |g_e|n_an_ba^2/2v$, respectively. Both the excitation gap in the strong coupling regime and the density-dependent phase transition can be observed in experiments. On the theoretical side, it is interesting to make further studies of the model in the framework of bosonization [2, 13].

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