STANDARD MODEL CONTRIBUTIONS TO THE NEUTRINO INDEX OF REFRACTION IN THE EARLY UNIVERSE

PAUL LANGACKER AND JIANG LIU

Department of Physics, University of Pennsylvania, Philadelphia, PA 19104

ABSTRACT
With the standard electroweak interactions, the lowest-order coherent forward scattering amplitudes of neutrinos in a CP symmetric medium (such as the early universe) are zero, and the index of refraction of a propagating neutrino can only arise from the expansion of gauge boson propagators, from radiative corrections, and from new physics interactions. Motivated by nucleosynthesis constraints on a possible sterile neutrino (suggested by the solar neutrino deficit and a possible 17 keV neutrino), we calculate the standard model contributions to the neutrino index of refraction in the early universe, focusing on the period when the temperature was of the order of a few MeV. We find sizable radiative corrections to the tree level result obtained by the expansion of the gauge boson propagator. For $\nu_e + e(\bar{e}) \to \nu_e + e(\bar{e})$ the leading log correction is about $+10\%$, while for $\nu_e + \nu_e(\bar{\nu}_e) \to \nu_e + \nu_e(\bar{\nu}_e)$ the correction is about $+20\%$. Depending on the family mixing (if any), effects from different family scattering can be dominated by radiative corrections. The result for $\nu + \gamma \to \nu + \gamma$ is zero at one-loop level, even if neutrinos are massive. The cancellation of infrared divergence in a coherent process is also discussed.

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1. Introduction

Neutrinos propagating in a medium could behave differently from in vacuum. The effect of the medium on the neutrino can conveniently be described by the index of refraction, $n$. Under certain conditions, matter effects could be so enormous that neutrino properties are modified drastically. The best known example is the Mikheyev-Smirnov-Wolfenstein (MSW) mechanism\textsuperscript{1} for the solar neutrino deficit,\textsuperscript{2,3} in which the index of refraction of electron neutrinos propagating inside the sun could induce a sufficiently large effective mixing angle for them to oscillate into other types of neutrinos efficiently.

Neutrino oscillations could also be cosmologically important.\textsuperscript{4} One particularly interesting example,\textsuperscript{5} pointed out by one of us (P.L.) some time ago, would be oscillations between a standard neutrino and a light sterile $SU(2)_L$ singlet neutrino, $\nu_s$. If certain conditions are satisfied prior to neutrino decoupling such oscillations can bring $\nu_s$ into equilibrium and thus alter the expansion rate of the universe. In particular, for a wide range of masses and mixings one would have $N_\nu = 4$ neutrino species in equilibrium, in violation of the bound $N_\nu < 3.3$ (95\% CL) from the observed abundance of $^4He$ and other light elements.\textsuperscript{6} Subsequent more detailed studies\textsuperscript{7−9} have reached similar conclusions.

Theoretically, the possibility of having a standard neutrino oscillate to $\nu_s$ has long been of interest. Many theoretical models based on the compactification of string theories and grand unification predict the existence of light sterile neutrinos. A recent study\textsuperscript{10} shows that if the controversial 17-$keV$ neutrino exists, laboratory, astrophysical, and cosmological bounds, unless significantly weakened, requires $m_{\nu_r} = 17$ $keV$ and $m_{\nu_\mu} \geq m_{\nu_r}$. Thus, the MSW solution to the solar neutrino deficit would require $\nu_e - \nu_s$ oscillations. It turns out that the mass and
mixing parameters required for the MSW solution are close to the boundary that is excluded by primordial nucleosynthesis. Moreover, the exact location of that boundary depends sensitively on the index of refraction and other details.

An accurate determination of the neutrino index of refraction is therefore an important step in the study of neutrino oscillations and their applications to solar neutrinos and cosmology. In the zero temperature limit, this can be done easily by a direct calculation of neutrino forward scattering amplitudes. The situation is more complicated if the thermal motion of scatterers becomes important. For that, lowest order calculations have appealed to a more complicated method derived from finite temperature field theory.

In this article, we wish to study the standard electroweak contributions to the neutrino index of refraction in the early universe. The epoch of interest is radiation dominated and has a temperature $T$ of the order of a few $MeV$. At that time, according to the standard Big Bang cosmology, the universe consisted mainly of photons $\gamma$, electrons, positrons, and three types of neutrinos and anti-neutrinos. Motivated by the smallness of the baryon asymmetry, we will assume, for simplicity, that in that era the universe was CP symmetric. As a consequence, there were equal amount of particles and antiparticles, with densities given by

$$N_\gamma = \frac{2\zeta(3)}{\pi^2} T^3, \quad N_e = N_\bar{e} = \frac{3}{4} N_\gamma, \quad N_{\nu_e,\nu_\mu,\nu_\tau} = N_{\bar{\nu}_e,\bar{\nu}_\mu,\bar{\nu}_\tau} = \frac{3}{8} N_\gamma,$$

where $\zeta(3) = \sum_{n=1}^{\infty} n^{-3} \approx 1.202$.

In terms of the standard electroweak interactions, the CP symmetry of the medium implies that the lowest order terms in neutrino coherent forward scatter-
ing amplitudes cancel. This follows because the amplitude of, say, neutrino-electron scattering is equal in amplitude but opposite in sign to that of neutrino-positron scattering. As a result, the neutrino index of refraction is zero to lowest order. Indeed, previous calculations (Ref. 11) have shown that the only nonzero result arises from the expansion of the $W$ and $Z$ boson propagators, and hence is suppressed by a very small factor of order $q^2/M_W^2$, where $q$ is the momentum carried by the $W$ or $Z$. In the absence of family mixing, even such a tiny result vanishes for scatterings that take place between different families.

The smallness of the tree level result calls for an investigation of its radiative corrections. In some places, particularly in resonance regions, theoretical solutions are sensitive to the value of neutrino index of refraction. Loop effects at zero temperature\textsuperscript{14} have been discussed before. One finds that they introduce new interactions proportional to $(\alpha/\pi)(m_{\ell}^2/M_W^2) \ln(m_{\ell}^2/M_W^2)(\ell = e, \mu, \tau)$ that break the universality of the neutral-current. In this paper, this effort will be carried over to the finite temperature case. Due to the aforementioned cancellation, radiative corrections now take a completely different form.

The remainder of this paper is organized as follows. In the next section we reformulate the lowest order calculation by introducing a method that uses standard field theory rather than the finite temperature field theory. Within this framework, higher order calculations are simplified to a standard calculation of radiative corrections to neutrino forward scattering amplitudes. In section 3, we present the leading log result for the $\nu_e + e(\bar{e}) \rightarrow \nu_e + e(\bar{e})$ scattering. There, we begin with a discussion of the compatibility of coherent conditions with the cancellation of infrared divergence. Neutrino-neutrino scattering and its radiative corrections are discussed in section 4. Scattering between different generations of neutrinos
and leptons is discussed in section 5. Neutrino-photon scattering is discussed in section 6. The imaginary part of neutrino index of refraction is given in section 7. Our results are summarized in the final section and some technical details are presented in the appendix. The application of our results to cases of cosmological and astrophysical interest will be discussed elsewhere.\textsuperscript{15}

2. General Formula

At zero temperature the index of refraction, $n$, is given by\textsuperscript{16}

$$n - 1 = \frac{2\pi}{p_0^2} N f(0), \quad (2.1)$$

where $p_0$ is the energy of the propagating neutrino, $N$ is the number of scatterers per unit volume, and $f(0)$ is the neutrino forward scattering amplitude. In the MSW formula, for example, $f(0)$ arises from $\nu_e - e$ scattering via the charged-current interaction, where\textsuperscript{17} in the rest frame of the electron $f(0) = G_F p_0 / \sqrt{2}\pi$, and hence $n = 1 + \sqrt{2}NG_F/p_0$.

The fundamental formula Eq. (2.1) can be generalized to situations in which the medium has a finite temperature $T$. With the usual assumption that the introduction of a finite temperature does not spoil the coherent condition,\textsuperscript{18} the only modification is to take the thermal average

$$n - 1 = \frac{2\pi}{p_0^2} \langle N f(0) \rangle = \frac{2\pi}{p_0^2} \langle N \rangle \langle f(0) \rangle, \quad (2.2)$$

where the last step follows from the usual assumption that scatterers are not cor-
related. More explicitly

\[
\langle N \rangle = \int \frac{d^3 \vec{q}}{(2\pi)^3} n^*(q_0, T),
\]

\[
\langle f(0) \rangle = \langle N \rangle^{-1} \int \frac{d^3 \vec{q}}{(2\pi)^3} n^*(q_0, T) f(0),
\]

where

\[
n^*(q_0, T) = \frac{g^*}{e^{(\mu+q_0)/kT} \pm 1}
\]

is the occupation number density of the scatterer, \( \mu \) is the chemical potential, and \( g^* \) is the number of spin degrees of freedom. Within this simple scheme, physical quantities such as the forward scattering amplitude are calculated in the standard way (at zero temperature), and the final result is obtained by taking the thermal average according to Eq. (2.2) at the end.

This simple method\(^{19} \) makes the lowest order calculation of neutrino index of refraction trivial. As an illustration, consider the scattering \( \nu_e(p) + e(p') \rightarrow \nu_e(p) + e(p') \). The tree level Feynman graphs are shown in Fig. 1. Neglecting the electron and the neutrino mass, the scattering matrix element is given by

\[
M_0(\nu_e \rightarrow \nu_e) = -\left( \frac{ig}{\sqrt{2}} \right)^2 \left( \frac{-i}{(p-p')^2 - M_W^2} \right) [\bar{u}_{\nu_e}(p') \gamma_\alpha L u_{\nu_e}(p)] [\bar{u}_e(p) \gamma_\alpha L u_e(p')]
\]

\[
-\left( \frac{ig}{2 \cos \theta_W} \right)^2 \left( \frac{-i}{0 - M_Z^2} \right) [\bar{u}_{\nu_e}(p') \gamma_\alpha (L - 2 \sin^2 \theta_W) u_e(p')] [\bar{u}_e(p) \gamma_\alpha L u_{\nu_e}(p)],
\]

where \( \theta_W \) is the weak mixing angle, and \( u_e \) and \( u_{\nu_e} \) are the electron and neutrino spinors, respectively. The first term in Eq. (2.6) arises from the \( u \)-channel scattering (Fig. 1a), and the second comes from the \( t \)-channel (Fig. 1b), where due to the coherent condition the momentum transfer must be zero. The \( u \)-channel scattering matrix element has an additional minus sign, which can be understood as a
consequence of fermion line crossing. One finds from Eq. (2.6) that the coherent forward scattering amplitude is

\[ f(0) = \frac{\sqrt{2}G_F p.p'}{2\pi p_0' \left[ \frac{1 + 4 \sin^2 \theta_W}{2} - \frac{2p.p'}{M_W^2} \right] + ...}, \] (2.7)

where the first term in the parentheses is the same as that obtained from a local V − A interaction (i.e., \( p - p' = 0 \)), the second arises from the expansion of \( W \) propagator, and the ellipses refer to the higher order terms. It then follows from Eqs.(2.2) and (2.7) that

\[ (n - 1)_{\nu_e \rightarrow \nu_e} = \frac{\sqrt{2}G_F}{p_0} \left[ \frac{1 + 4 \sin^2 \theta_W}{2} N_e - \frac{8}{3M_W^2} p_0 \langle p_0' \rangle N_e \right], \] (2.8)

where \( N_e \) is the electron number density at temperature \( T \) and \( \langle p_0' \rangle \) is the average electron energy. In reaching Eq. (2.8), we have assumed that the thermal motion of the electrons is isotropic so that \( \langle \vec{p} \rangle = 0 \).

The matrix element for \( \nu_e(p) + \bar{\nu}_e(p') \rightarrow \nu_e(p) + \bar{\nu}(p') \) (Fig. 2) can be obtained from Eq. (2.6) by changing \( u_e \rightarrow v_e = u_e^c \), \( p' \rightarrow -p' \), and introducing an overall minus sign (due to the anticommutation of fermion fields). One finds

\[ (n - 1)_{\nu_{\bar{e}} \rightarrow \nu_{\bar{e}}} = \frac{\sqrt{2}G_F}{p_0} \left[ \frac{1 + 4 \sin^2 \theta_W}{2} N_{\bar{e}} - \frac{8}{3M_W^2} p_0 \langle p_0' \rangle N_{\bar{e}} \right]. \] (2.9)

The sum of Eqs. (2.8) and (2.9) gives

\[ (n - 1)_{\nu_e(e) \rightarrow \nu_e(e)} = \frac{\sqrt{2}G_F}{p_0} \left[ \frac{1 + 4 \sin^2 \theta_W}{2} (N_e - N_{\bar{e}}) - \frac{8p_0}{3M_W^2} (\langle p_0' \rangle e N_e + \langle p_0' \rangle \bar{e} N_{\bar{e}}) \right]. \] (2.10)

This is precisely the result obtained previously by the finite temperature field theory method (Ref. 11), in which one calculates one-loop contributions to neutrino
self-energy in a background field. The present method is considerably simpler, since it only involves a standard tree level graph calculation.\textsuperscript{20}

Our approach also makes the underlying physics more intuitive. That the first term in Eq. (2.10) is given by the difference of the electron and positron densities is a consequence of the CP transformation property of vector currents

\[ \bar{u}_e \gamma_\alpha u_e \rightarrow -\bar{\nu}_e \gamma_\alpha \nu_e = -\bar{u}_e \gamma_\alpha u_e. \]  

(2.11)

Diagrammatically, from Fig. (1) and Fig. (2) the $t$-channel coherent scattering amplitudes are equal in size but opposite in sign, and hence cancel completely. The same would also be true if the $W$’s in Fig. (1a) and Fig. (2a) were not carrying any momenta. In the neutrino-electron scattering the momentum carried by the $W$ is $p' - p$, whereas in the neutrino-positron scattering it changes to $p' + p$. This relative sign difference compensates the over all sign difference discussed in Eq. (2.11), so that the second term of Eq. (2.10) becomes a sum rather than a difference. Finally, contributions from the longitudinal part of the gauge boson propagators are always negligible if the external fermion masses are small. These terms have the same structure as that in Eq. (2.10) times a multiplicative factor given by a ratio like $m_e^2/M_W^2$.

Similar observations can also be made for loop graphs that generate radiative corrections to the tree level result. Basically, all $t$-channel one-particle-reducible graphs will cancel, and hence we will omit them from now on. This includes all zero temperature correction results discussed in Ref. 14. Only the $s$- and $u$-channel one-particle-reducible diagrams and the box diagrams may contribute to the radiative corrections.
3. One-loop Corrections to the $\nu_e + e(e) \to \nu_e + e(e)$

Forward Scattering Amplitude

The order $\alpha/\pi$ correction results are known.\textsuperscript{21–25} However, they have the same structure as the lowest order weak interaction, and thus the sum of the coherent amplitudes for neutrino-electron and neutrino-positron scattering cancels.

Terms which may survive the cancellation must depend on $(p \pm p')^2$ where the two signs refer to neutrino-electron and neutrino-positron scattering, respectively. Comparing with the tree level result ($\sim G_F (p \pm p')^2/M_{W}^2$) arising from the expansion of the gauge boson propagator, loop corrections are of the order $(\alpha/4\pi) G_F (p \pm p')^2/M_{W}^2$ times a yet to be determined factor. Normally, this factor contains large log terms plus some constant terms of order unity. Since the ratio $\alpha/4\pi$ is already small, in the absence of logarithmic enhancement radiative corrections would be negligible. Thus, in what follows we will only keep those leading log terms.

Explicit calculations (see below) show that there are only two scalar functions arising from loop integrals that have a large log

\begin{align}
I_1(m_1, m_2, M; q^2) &\equiv \frac{q^2}{M^2} \int _0 ^1 dx x(x-1) \ln \frac{m_1^2 + x(m_2^2 - m_1^2) + x(x-1)q^2}{M^2}, \\
I_2(m_1, m_2, M; q^2) &\equiv \int _0 ^1 dx \frac{m_1^2 + x(m_2^2 - m_1^2)}{M^2} \ln \frac{m_1^2 + x(m_2^2 - m_1^2) + x(x-1)q^2}{M^2},
\end{align}

(3.1)

where $m_{1,2}$ are two small masses, one for a neutrino and another for a charged lepton. In Eq. (3.1), $M$ is either $M_W$ or $M_Z$, and $q = p \pm p'$ is the momentum of
the virtual $W$ or $Z$. The logarithmic enhancement arises because of the hierarchy:

$$m_{1,2}^2, |q^2| \ll M^2,$$

and hence $\ln(M^2/m_{1,2}^2), \ln(M^2/|p \pm p'|^2) \gg 1$. Consistent with our leading log approximation, terms proportional to $I_2$ can be neglected, because the difference, which one must take in the end when summing over the neutrino-electron and the neutrino-positron scattering amplitudes,

$$I_2(m_1, m_2, M; (p + p')^2) - I_2(m_1, m_2, M; (p - p')^2)$$

$$= \int_0^1 dx \frac{m_1^2 + x(m_2^2 - m_1^2)}{M^2} \ln \frac{m_1^2 + x(m_2^2 - m_1^2) + x(x - 1)(p + p')^2}{m_1^2 + x(m_2^2 - m_1^2) + x(x - 1)(p - p')^2}$$

(3.2b)

does not have a large log. Effectively, this implies that we can neglect all small masses whenever possible in our calculation. A welcome consequence of such a simplification is that all graphs that contain scalars (in the Feynman gauge) can be neglected. As long as we keep only the leading log term, the result will be gauge invariant. We will assume that neutrino masses are completely negligible.

3.1. Cancellation of Infrared Divergences

A question which one might encounter in the study of radiative corrections is how do the infrared divergences\textsuperscript{26} cancel in a coherent process? In normal (incoherent) cases, the complete cancellation of infrared divergence is well understood.\textsuperscript{27} Although some individual Feynman graphs generate infrared divergent terms, they cancel when summed with the bremmstrahlung diagrams, which are also infrared divergent by themselves.
In a coherent process, however, bremsstrahlung cannot be included. Thus, if a process is to remain as coherent at higher order, the cancellation of infrared divergences must take place among themselves. In the MSW formalism, the constraint becomes even more restrictive. There, the neutral- and the charged-current interaction must be infrared finite separately.

The cancellation of infrared divergences for a coherent interaction has not yet received much attention, partly because there are not many places in which the physical process can be coherent. Marciano and Sirlin have investigated this question\textsuperscript{28} in the context of atomic parity violation. By explicit calculations, they show that the required cancellation does occur at one-loop level. In the following, we will show that this happens to all order in perturbation theory.

Consider an electron scattering with an infrared finite but otherwise arbitrary potential and its QED corrections (Fig. 3). Suppose the forward scattering amplitude of the interaction is

\[ M_0(p, p) = \bar{u}_e(p) \Gamma_0(p, p) u_e(p), \]

where \( \Gamma_0(p, p) \) is the interaction vertex, which is assumed to be renormalizable so that it is also free from ultra-violet divergences.

At one-loop level, QED corrections introduce infrared divergences into some of the Feynman graphs. To isolate them, we recall that infrared divergences arise because photons are massless, and thus a slight acceleration of an external charged particle could result in an emission of an infinite number of soft photons. This implies that only those Feynman graphs in which a virtual photon line has both its ends attached to an external line (Fig. 3b to 3d) are infrared divergent (Ref.
27). The infrared singularity is generated by the photon pole

\[
\frac{-i}{k^2 - \lambda^2 + i\epsilon} = P.V. \frac{-i}{k^2 - \lambda^2} - \pi \delta(k^2 - \lambda^2),
\]

(3.4)

where \(\lambda\) is a ficticious photon mass. The one-loop QED corrections can now be written as

\[
\delta M_1(p, p) = \delta M_1^{(b)}(p, p) + \delta M_1^{(c)}(p, p) + \delta M_1^{(d)}(p, p) + (\text{infrared finite terms}),
\]

(3.5)

where \(\delta M_1^{(b)}(p, p)\) is the one-loop vertex correction (Fig. 3b)

\[
\delta M_1^{(b)}(p, p) = \left. -ie^2 \int \frac{d^4k}{(2\pi)^4} \bar{u}_e(p) \gamma_\alpha \frac{1}{p.\gamma + k.\gamma - m_e} \Gamma_0(p + k, p + k) \frac{1}{p.\gamma + k.\gamma - m_e} \gamma^\alpha u_e(p) \frac{1}{k^2 - \lambda^2 + i\epsilon} \right|_{0 \leq k_0 \leq \omega_0},
\]

(3.6)

and \(\delta M_1^{(c)}(p, p)\) and \(\delta M_1^{(d)}(p, p)\) are the one-loop wave function renormalization contributions.

In the on-shell subtraction scheme, one finds easily

\[
\delta M_1^{(c)}(p, p) + \delta M_1^{(d)}(p, p) = -M_0 \frac{\alpha}{2\pi} \ln \frac{\lambda^2}{m_e^2} + (\text{infrared finite terms}).
\]

(3.7)

The infrared divergence in the vertex correction can be separated out by considering the photon pole (Eq.(3.4)) contribution to the matrix element in the infrared region, \(k \to 0\), where by virtue of the finiteness of \(M_0(p, p)\) one finds \(\lim_{k \to 0} \bar{u}_e(p) \Gamma_0(p + k, p + k) u_e(p) = M_0(p, p)\) and thus

\[
\delta M_1^{(b)}(p, p)|_{0 \leq k_0 \leq \omega_0} = -e^2 M_0(p, p) \int \frac{d^4k}{(2\pi)^4} \frac{m_e^2}{(p.k)^2} \pi \delta(k^2 - \lambda^2) + (\text{infrared finite terms})
\]

\[
= -M_0(p, p) \frac{\alpha}{2\pi} \ln \frac{\omega_0^2}{\lambda^2} + (\text{infrared finite terms}),
\]

(3.8)

where \(\omega_0\) is an infrared cut-off often determined by the resolution of the experi-
mental apparatus.

It is now evident that the infrared divergence in the wave function renormalization cancels the infrared divergence in the vertex correction. The net result is infrared finite

\[ M_0(p, p) + \sum_{i=a,b,...} \delta M_1^{(i)}(p, p) = M_0(p, p) \left(1 - \frac{\alpha}{2\pi} \ln \frac{\omega_0^2}{m_e^2} \right) + ..., \quad (3.9) \]

where the ellipses represent the other infrared finite terms.

The factorization in the vertex correction (Eq. (3.8)) is crucial in reaching our final result. Now, by an elementary induction and by employing the same method presented above, one can readily show that at N-loop level, infrared divergences from the effective vertex (infrared finite at N-1 loop level) always cancel those from wave function renormalization. Thus, the forward scattering amplitude is infrared finite to all orders. This is to be expected, for if the electron forward scattering amplitude were not infrared finite, it would have implied that the electric charge is not a well defined quantity.

### 3.2. QED Correction

It is convenient to separate out the conventional photonic correction from the weak interaction corrections. To do so, we employ the standard method\textsuperscript{29} to decompose the photon propagator

\[ \frac{-i}{k^2} = \frac{-i}{k^2 - M_W^2} + \frac{-i}{k^2} \frac{M_W^2}{M_W^2 - k^2}. \quad (3.10) \]

As explained in Ref. 29, the first term gives rise to a massive photon (\( \gamma_> \)) contribution. It’s effect combined with those from \( W \) and \( Z \) will be referred to as the
weak interaction contributions. The second term is the same as for a massless photon \((\gamma_<)\), but with an additional cutoff \(\Lambda = M_W\) as a regulator for ultra-violet divergences. The contribution of \(\gamma_<\) (Fig. 4a) and (Fig. 4b) plus the photonic box (Fig. 4c) (evaluated with the full photon propagator \(-i/k^2\)) gives the one-loop QED correction. The result for \(\nu_e + e \rightarrow \nu_e + e\) is

\[
\delta M_{QED} = -\frac{ig^2}{2M_W^2} \frac{\alpha}{4\pi} \left\{ \left[ \bar{u}_e(p')\gamma_\lambda L u_e(p) \right][\bar{\nu}_\nu(p)\gamma^\lambda L u_\nu(p)] - \frac{q^2}{M_W^2} \left( \frac{2}{3} \ln \frac{M_W^2}{m_e^2} - \frac{13}{9} \right) \right. \\
\left. + \left[ \bar{u}_e(p')\gamma_\lambda \gamma_5 u_e(p') \right][\bar{\nu}_\nu(p)\gamma^\lambda L u_\nu(p)] \left( 1 + \frac{1}{3} \frac{q^2}{M_W^2} \right) \right\},
\]

(3.11)

where \(q \equiv p - p'\). We have omitted all terms directly proportional to \(m_e\).

The result is finite in both the infrared and the ultra-violet region. Only the axial current (the second term in Eq. (3.11)) receives an order \(\alpha\) correction. This is because QED does not introduce charge renormalization. It is known that the axial current term does not contribute to the neutrino coherent forward scattering if the electrons in the medium are not polarized. We will therefore neglect it from now on.

A notable feature of (3.11) is that it contains a mass singular term \(\ln M_W^2/m_e^2\). As we will see below, this mass singularity won’t cancel even after we add the weak interaction corrections. This is not in conflict with the well known Kinoshita-Sirlin-Lee-Naunberg (KSLN) theorem, however. The KSLN theorem follows because of the finiteness of Green’s functions with non-exceptional Euclidean external momenta in a renormalizable theory. The non-exceptional momentum refers to configurations in which no partial sum of momenta vanishes. Here, we clearly have an exceptional external momentum configuration if one allows \(m_e \rightarrow 0\). The mass singular term is multiplied by \(q^2\) which vanishes in the limit \(m_\nu_e, m_e \rightarrow 0\) in the
rest frame of the electron. Therefore, the KSLN theorem does not apply here.

Comparing with the tree level result (taking only the first term of Eq. (2.6))

$$M_0 = \frac{ig^2}{2 M_W^2} \frac{1}{q^2} [\bar{u}_e(p') \gamma_\alpha L u_\nu_e(p)] [\bar{u}_\nu_e(p) \gamma^\alpha L u_e(p')],$$  \hspace{1cm} (3.12)

the leading log QED correction to the ratio $q^2/M_W^2$ that determines the index of refraction (see Eq. (2.10)) is $-(\alpha/6\pi)\ln(M_W^2/m_e^2)$. The dominant contribution is provided by the wave function renormalization. Numerically, it is $-0.9\%$.

3.3. Weak Interaction Correction

For reasons explained above, as far as the one-particle-reducible graphs are concerned only the $u$-channel scattering graphs are of interest. This makes our calculation very similar to that of $\mu$-decay.\textsuperscript{31} Accordingly, we will employ the on-shell renormalization scheme of Sirlin (Ref. 31)

$$\cos^2 \theta_W = \frac{M_W^2}{M_Z^2}. \hspace{1cm} (3.13)$$

For convenience, we organize our results into the self-energy, the vertex correction, and the box-diagram parts

$$\delta M^{Weak} = \delta M_{Self} + \delta M_{Vertex} + \delta M_{Box}, \hspace{1cm} (3.14)$$

where $\delta M_{Self}$ and $\delta M_{Vertex}$ also include the counterterm contributions and the wave function renormalization effect, respectively.
The self-energy and counterterm diagrams are shown in Fig. 5. The result for the sum of these graphs is known (Ref. 31):

\[
\delta M_{\text{Self}} = M_0 \left[ \frac{A_{WW}(q^2) - \text{Re} A_{WW}(M_W^2)}{q^2 - M_W^2} - \frac{2\delta e}{e} + \frac{c^2}{s^2} \text{Re} \left( \frac{A_{ZZ}(M_Z^2)}{M_Z^2} - \frac{A_{WW}(M_W^2)}{M_W^2} \right) \right] ,
\]

where \( c \equiv \cos \theta_W \), \( s \equiv \sin \theta_W \), and \( M_0 \) is given by Eq. (3.12). The functions \( A_{WW} \) and \( A_{ZZ} \) are the coefficients of \( g_{\mu\nu} \) in the \( WW, ZZ \) self-energies. These functions have been calculated before. We will use the results given by Marciano and Sirlin.\(^{32}\)

The vertex graphs are shown in Fig. 6. In evaluating the wave function renormalization contributions we use the massive photon propagator \( \gamma^> \) (the first term in Eq. (3.10)). The result is

\[
\delta M_{6a+6b+6c+6d} = -M_0 \frac{\alpha}{4\pi} \left\{ \left( \frac{2}{\epsilon} + \Gamma'(1) - \frac{1}{2} - \ln \frac{M_Z^2}{4\pi\mu_0^2} \right) \left( \frac{1}{2c^2s^2} - 1 \right) \right. \\
+ \left( \frac{2}{\epsilon} + \Gamma'(1) - \frac{1}{2} - \ln \frac{M_W^2}{4\pi\mu_0^2} \right) \left( \frac{1}{s^2} + 1 \right) \right\} . \quad (3.16)
\]

To isolate the ultra-violet divergence, we have employed the dimensional-regularization\(^{33}\) with \( \mu_0^2 \) the ultra-violent cut-off and \( \Gamma(\epsilon/2) = 2/\epsilon + \Gamma'(1) + O(\epsilon) \).

Corrections to the vertex are obtained by evaluating the three-point functions from Fig. (6e) and Fig. (6f), in which one uses the full photon propagator. We find (details can be found in the appendix)

\[
\delta M_{6e+6f} = M_0 \frac{\alpha}{2\pi} \left\{ \left( \frac{2}{\epsilon} + \Gamma'(1) - \ln \frac{M_W^2}{4\pi\mu_0^2} \right) \left( \frac{3}{s^2} + \frac{s^2 - c^2}{4s^2c^2} \right) \\
+ \frac{5}{2} + \frac{c^2}{s^2} \left( \frac{5}{2} + \frac{3}{s^2} \ln c^2 \right) - \frac{s^2 - c^2}{4s^2c^2} \left( \frac{1}{2} - \ln c^2 \right) \\
- \frac{q^2}{M_W^2} \left[ \ln \frac{M_W^2}{m_c^2} + \left( 1 - \frac{1}{2s^2} \right) F(q^2) + c' \right] \right\} , \quad (3.17)
\]
where
\[
c' = \frac{5}{6} - \frac{4c^2}{3s^2} \left( 1 - \frac{1}{s^2} + \frac{2c^2}{s^4} \right) - \frac{c^4}{s^4} \left( 1 + \frac{8c^2}{3s^4} \right) \ln c^2 + \frac{1}{6s^2},
\]
(3.18)
and
\[
F(q^2) = \frac{1}{0} \int dx 2x(x-1) \ln \frac{xM_w^2 + (1-x)^2m_e^2}{x(x-1)q^2 + (1-x)m_e^2}.
\]
(3.19)

Combining Eqs. (3.16) and (3.17), we obtain
\[
\delta M_{\text{Vertex}} = M_0 \frac{\alpha}{4\pi} \left\{ \frac{4}{s^2} \left( \frac{2}{\epsilon} + \Gamma'(1) - \frac{1}{2} - \ln \frac{M_w^2}{4\pi\mu_0^2} \right) + \frac{c^2}{s^4} (5 + c^2) \ln c^2 + \frac{8}{s^2}
- 2 \frac{q^2}{M_w^2} \left[ \ln \frac{M_w^2}{m_e^2} + \left( 1 - \frac{1}{2s^2} \right) F(q^2) + c' \right] \right\}.
\]
(3.20)

Comparing with the large log terms, one sees from Eq. (3.20) that the constant \( c' \approx 3 \) clearly can be ignored. Since we are interested in regions in which \( q^2 \) is space-like and \( |q^2| \gg m_e^2 \), in the leading log approximation one can make a further simplification
\[
F(q^2) \approx \frac{1}{3} \ln \frac{q^2}{M_Z^2}.
\]
(3.21)

Here, we have neglected the imaginary part that will be discussed in detail in section 7.

There are a total of eight box diagrams (Fig. 7). A straightforward calculation shows that the leading log plus the order \( \alpha \) terms are given by (details can be
In reaching this simple result, we have made the approximation
\[ I_1(m_{\nu_e}, m_e, M_W; q^2) \approx -\frac{1}{6} \frac{q^2}{M_W^2} \ln \frac{q^2}{M_W^2}, \]  
(3.23)
and ignored all non-leading log terms. The box diagrams also generate terms which do not contribute to the neutrino coherent forward scattering amplitudes. They are of the form \([\bar{u}_e \gamma_5 u_e][\bar{u}_\nu_e \gamma^\alpha L u_{\nu_e}]\) and \(q^\alpha q^\beta [\bar{u}_e \gamma_5 u_e][\bar{u}_\nu_e \gamma_\beta L u_{\nu_e}]\) and have been omitted from Eq. (3.22). The result can be written in a more compact form by using the relation
\[ q^\alpha q^\beta [\bar{u}_e \gamma_\alpha u_e][\bar{u}_\nu_e \gamma_\beta L u_{\nu_e}] = \frac{q^2}{2} [\bar{u}_e \gamma_\alpha u_e][\bar{u}_\nu_e \gamma^\alpha L u_{\nu_e}]. \]  
(3.24)
It then follows that
\[ \delta M_{Box} = \frac{ig^2}{2M_W} \frac{\alpha}{4\pi} [\bar{u}_e(p')\gamma_\alpha L u_e(p)][\bar{u}_\nu_e(p)\gamma^\alpha L u_{\nu_e}(p)] \]
\[ \times \left( \frac{5c^4 - 3s^4}{2s^4} \ln c^2 + \frac{15 - 24s^2c^2}{8s^2c^2} + \frac{2}{3s^2}(1 + s^2 + 4s^4) \frac{q^2}{M_W^2} \ln \frac{q^2}{M_W^2} \right) \]
\[ + \ldots, \]  
(3.25)
where the ellipses represent those (incoherent) terms of no interest to us.
Among the eight box diagrams, the first four also appear in \( \mu \)-decay (with appropriate change of external lines). The order \( \alpha \) correction of these graphs is given by the first term of Eq. (3.25). From Eqs. (3.20) and (3.25), one can see that the coefficients of \( q^2/M_W^2 \) in \( \delta M_{\text{vertex}} \) and in \( \delta M_{\text{Box}} \) are almost equal but with the opposite sign. This results in a large cancellation between these two contributions.

### 3.4. Combination of Results

The combination of all weak interaction corrections plus the tree level result (Eq. (3.12)) can be expressed in terms of the weak coupling constant \( G_\mu = (1.16639 \pm 0.00002) \times 10^{-5} \, \text{GeV}^{-2} \) determined very accurately from \( \mu \)-decay by the substitution (Ref.31)

\[
\frac{g^2}{M_W^2} (1 + \Delta r) = \frac{8G_\mu}{\sqrt{2}},
\]

where

\[
\Delta r = \frac{\text{Re} A_{WW}(M_W^2) - A_{WW}(0)}{M_W^2} - \frac{2\delta e}{e} e^2 + \frac{c^2}{s^2} \left( \frac{A_{ZZ}(M_Z^2)}{M_Z^2} - \frac{A_{WW}(M_W^2)}{M_W^2} \right) + \frac{\alpha}{4\pi} \left[ \frac{4}{s^2} \left( \frac{2}{\epsilon} + \Gamma'(1) - \ln \frac{M_Z^2}{4\pi\mu_0^2} \right) + \left( \frac{7}{2} - 6s^2 \right) \ln \frac{c^2}{s^2} + \frac{6}{s^2} \right].
\]

It then follows from Eqs. (3.12), (3.14), (3.20), (3.25) and (3.26) that

\[
M_0 + \delta M^{\text{Weak}} = -i2\sqrt{2}G_\mu[\bar{\nu}_e(p')\gamma_\alpha L u_e(p')][\bar{\nu}_\mu(p)\gamma^\alpha L u_\mu(p)] \left\{ 1 - \frac{\alpha}{4\pi} \left( \frac{15 - 24s^2c^2}{8s^2c^2} \right) \right\} + \frac{q^2}{M_W^2} \left[ 1 + \Delta R(q^2) + \frac{\alpha}{4\pi} \left( \frac{1 + 4s^2 + 8s^4}{3s^2} \ln \frac{M_W^2}{q^2} - 2\ln \frac{M_W^2}{m_e^2} \right) \right],
\]

(3.28)
where

$$\Delta R(q^2) \equiv \frac{M_W^2}{q^2} \left( \frac{ReA_WW(M_W^2) - A_{WW}(q^2)}{M_W^2 - q^2} - \frac{ReA_WW(M_W^2) - A_{WW}(0)}{M_W^2} \right)$$

$$= \frac{A_{WW}(0) - A_{WW}(q^2)}{q^2} + \frac{ReA_WW(M_W^2) - A_{WW}(0)}{M_W^2} + O(M_W^{-4})$$

is the $W$ self-energy contribution, with the understanding that only the real part of the result will be kept.

The residual order $\alpha$ correction (the second term in Eq. (3.28)) arises because we have omitted all the $t$-channel one-particle-reducible graphs. Also, the last four box diagrams do not appear in the calculation of $\mu$-decay. In any case, this term does not contribute to the index of refraction.

Terms in the squared brackets will contribute to the neutrino index of refraction. The first term is the tree level result obtained by the expansion of the $W$ propagator. The second term, $\Delta R(q^2)$, is due to $W$ self-energy and counterterms, and the rest is the combination of vertex and box diagram corrections.

The large cancellation between the vertex and the box diagrams makes their contribution rather small. The sum is about $+2\%$ for $|q^2| = 1$ $MeV^2$ and it becomes negligibly small, $+0.1\%$, for $|q^2| = 400$ $MeV^2$.

The largest contribution comes from $W$ self-energy. For convenience, we will decompose it into the bosonic, the hadronic and the leptonic parts

$$\Delta R(q^2) = \Delta R^{(b)}(q^2) + \Delta R^{(h)}(q^2) + \Delta R^{(l)}(q^2).$$

The bosonic contribution is negligible, because it does not have any large log term.
Explicitly, from Eq. (3.29) and the given self-energy function in Ref. 32 we find

\[
\Delta R^{(b)}(q^2) = \frac{\alpha}{4\pi} \left( \frac{65}{18} - \frac{3c^2}{2s^2} + \frac{1}{s^2} \int_0^1 dx F(x, \xi) \right), \tag{3.31}
\]

where

\[
F(x, \xi) = \left[ \frac{2s^4 - 16c^2 - 10c^4 - 1}{2c^2} + \frac{1 + 4c^4 + 16c^2 x - 20c^2 + 1 x^2}{2c^2} \ln \frac{c^2x^2 - x + 1}{c^2x - x + 1} \right.
\]

\[
+ \frac{2 - \xi + \xi x - x^2}{2} \ln \frac{x^2 + \xi(1 - x)}{x + \xi(1 - x)} + \frac{x(1 - x)}{x + \xi(1 - x)} + \frac{s^4 x(1 - x)}{c^2 x^2 + (1 - x)}, \tag{3.32}
\]

and \(\xi \equiv m_\phi^2/M_W^2\). Here \(m_\phi\) is the Higgs mass.

The hadronic correction seems to provide the biggest contribution, but its calculation is complicated by strong interaction physics. The conventional wisdom here is to employ dispersion relations to relate \(\Delta^{(h)}(q^2)\) to some physically measurable quantities. While a detailed study of this problem lies beyond the scope of the present paper, we will estimate the result by employing the constituent quark mass. We find for \(|q^2| \ll m_i^2\) (here \(m_i\) is a constituent quark mass)

\[
\Delta R^{(h)}(q^2) = \frac{3\alpha}{2\pi s^2} \sum_{i,j} |V_{ij}|^2 \text{Re} \int_0^1 dx \left[ x(1 - x) \ln \frac{m_i^2 x + m_j^2(1 - x) - M_W^2 x(1 - x)}{m_i^2 x + m_j^2(1 - x)} - \frac{m_i^2 x + m_j^2(1 - x)}{2M_W^2} \ln \frac{m_i^2 x + m_j^2(1 - x) - M_W^2 x(1 - x)}{m_i^2 x + m_j^2(1 - x)} - \frac{1}{12} \right], \tag{3.33}
\]

where \(V_{ij}\) is the KM matrix element.\(^{34}\) The sum is over all three families of quarks with \(m_{u,d} \approx 300\ MeV, m_s \approx 450\ MeV\), etc. The second term in Eq. (3.33) is obviously negligible for the first two generations. We present it here is to show that it is also negligible for a heavy quark. Indeed, for \(m_t \gg M_W\), it becomes
\(\frac{x(1-x)}{2}\), which is much smaller than the first leading log term. Numerically, we find that \(\Delta R^{(0)}(q^2)\) is about +5%.

The calculation of the leptonic contribution is much less ambiguous. We find that the leading log contribution is (neglecting possible mixings in the lepton sector)

\[
\Delta R^{(\ell)}(q^2) = \frac{\alpha}{2\pi s^2} \sum_{\ell=e,\mu,\tau} Re \int_0^1 dx x(1-x) \ln \frac{m^2_{\ell} - M^2_W(1-x)}{m^2_{\ell} - q^2(1-x)}.
\] (3.34)

Numerically, it varies from +4% to +3% in the region \(1 MeV^2 \leq |q^2| \leq 400 MeV^2\).

Now, the sum of all weak corrections plus a \(-1\)% QED correction varies from about +10% to about +7% in the region of interest discussed above. Our results are summarized in Table 1.

The result for \(\nu_e + \bar{\nu} \rightarrow \nu_e + \bar{\nu}\) can be obtained from that of neutrino-electron scattering by changing \(\bar{u}_e \gamma_\alpha L u_e\) to \(-\bar{u}_e \gamma_\alpha R u_e\) and \(q^2 \rightarrow -q^2\) (valid only if the neutrino and the lepton mass is negligible). The outcome is that the constant term (\(q^2\) independent) has the opposite sign, whereas the \(q^2/M^2_W\) term remains the same.

To obtain the index of refraction, we now need to take the thermal average of Eq. (3.28). Since the variation of radiative corrections in the region of \(q^2\) of interest is not significant, in the leading log approximation we can write the index of refraction in a \(N_e = N_{\bar{e}}\) medium as

\[
(n - 1)_{\nu_e(\bar{\nu}) \rightarrow \nu_e(\bar{\nu})} = -\frac{16\sqrt{2}}{3M^2_W} G_\mu N_e p_0(p'_0)(1 + \delta_{\nu_e}),
\] (3.35)

where \(\delta_{\nu_e}\) is simply given by the non-averaged result discussed above; it is between about +10% to about +7% (see Table 2 for detail). Results for other values of \(q^2\)
can easily be obtained from the analytic formulae given above and in the appendix, provided the leading log approximation can still be justified.

4. Neutrino-(anti)neutrino Scattering

Fig. 8 shows the tree level Feynman graphs for a diagonal (in family space) neutrino-neutrino and a neutrino-anti-neutrino scattering. Fig. 8a and 8c are the standard $t$-channel forward scattering graphs induced by the neutral current interaction, where the coherent condition requires that the virtual $Z$ cannot carry any momentum. This feature is shared by all $t$-channel one-particle-reducible graphs. As explained before, the sum of coherent amplitudes of these $t$-channel graphs cancels for $\nu + \nu \rightarrow \nu + \nu$ plus $\nu + \bar{\nu} \rightarrow \nu + \bar{\nu}$. Therefore, we will ignore them from now on.

Due to the identical particle property, there is another angle for which the $\nu\nu \rightarrow \nu\nu$ scattering can be coherent. This corresponds to the configuration in which the projectile and the scatterer switch places after scattering (Figs. (8b)). In contrast to the usual $t$-channel scattering, now the coherent condition allows the virtual $Z$ to have an arbitrary momentum $q^2 \equiv (p - p')^2$

Analogous to the Bhabha scattering in QED, there is an $s$-channel graph for the $\nu\bar{\nu} \rightarrow \nu\bar{\nu}$ scattering (Fig. 8d), in which the virtual $Z$ carries a momentum $(p + p')^2 = -q^2$. Again, the momenta of the virtual $Z$s in Fig. 8b and Fig. 8d have the opposite sign, and thus their sum survives the cancellation that occurs in lowest order.
4.1. $\nu_e + \nu_e (\bar{\nu}_e) \rightarrow \nu_e + \nu_e (\bar{\nu}_e)$

For definiteness, we now consider the $\nu_e \nu_e \rightarrow \nu_e \nu_e$ scattering. Of cosmological interest, we consider situations where $|q|$ is of the order of a few $MeV$. The tree level matrix element of Fig. 8b is simply

$$M_0 = - \left( \frac{ig}{2e} \right)^2 \frac{i}{M_Z^2 - q^2} [\bar{u}_{\nu_e} (p') \gamma_\alpha L u_{\nu_e} (p) ][\bar{u}_{\nu_e} (p) \gamma^\alpha L u_{\nu_e} (p')], \quad (4.1)$$

where $p'$ and $p$ refer to the momenta of the scatterer and the projectile, respectively. The overall minus-sign is due to the exchange of identical fermionic particles. Following the discussion of section 2, one finds that the index of refraction arising from neutrino-neutrino and neutrino-anti-neutrino scattering in a medium with equal amount of neutrinos and anti-neutrinos is

$$(n - 1)_{\nu_e + \nu_e (\bar{\nu}_e) \rightarrow \nu_e + \nu_e (\bar{\nu}_e)} = \frac{\cos^2 \theta_W}{4} (n - 1)_{\nu_e (e) \rightarrow \nu_e (\bar{e})}, \quad (4.2)$$

where $(n - 1)_{\nu_e (e) \rightarrow \nu_e (\bar{e})}$ is given by Eq. (2.10).

Because of the absence of QED corrections, the calculation of one-loop radiative corrections to neutrino-neutrino scattering is considerably simpler. However, in the case of $\nu_e e \rightarrow \nu_e e$ scattering we are interested in radiative corrections to the charged current, whereas here the interest is shifted to the neutral current interaction. The result will therefore have a moderate dependence on the top quark mass. For convenience, we again organize our results into the self-energy, the vertex correction, and the box-diagram parts

$$\delta M^{Weak} = \delta M_{Self} + \delta M_{Vertex} + \delta M_{Box}. \quad (4.3)$$

The result for the self-energy and counterterm (Fig. 9) can be obtained from the analysis of neutral current radiative corrections of Marciano and Sirlin (Ref.
\[ \delta M_{\text{Self}} = M_0 \left[ \frac{A_{ZZ}(q^2) - \text{Re}A_{ZZ}(M_Z^2)}{q^2 - M_Z^2} - \frac{2\delta e}{\epsilon} + \left( \frac{c^2}{s^2} - 1 \right) \text{Re} \left( \frac{A_{ZZ}(M_Z^2)}{M_Z^2} - \frac{A_{WW}(M_W^2)}{M_W^2} \right) \right], \]

where \( M_0 \) is given by Eq. (4.1).

Neglecting the neutrino and the electron mass, we find that the leading log vertex correction (Fig. 10) is

\[ \delta M_{\text{Vertex}} = M_0 \frac{\alpha}{4\pi} \left[ \frac{c^2}{s^2} \left( \frac{2}{\epsilon} + \Gamma'(1) - \ln \frac{M_W^2}{4\pi\mu_0^2} \right) + \frac{q^2}{M_W^2} \left( \frac{4s^2 - 1}{3s^2} \right) \ln \frac{M_W^2}{q^2} \right]. \quad (4.5) \]

Here, we have made the approximation indicated by Eq. (3.23) which is valid for our purpose. A more complete result without this constraint can be found in the appendix. In any case, one can see from Eq. (4.5) that the vertex correction to the neutrino index of refraction is completely negligible because \( 4s^2 - 1 \) is close to zero.

Finally, there are a total of six box diagrams (Fig. 11). The sum of their leading log contributions is

\[ \delta M_{\text{Box}} = M_0 \frac{\alpha}{4\pi s^2} \left[ 2 - \frac{3}{2c^2} + \frac{11}{6} \frac{q^2}{M_W^2} \ln \frac{M_W^2}{q^2} \right]. \quad (4.6) \]

In reaching this result we have again made use of the approximation of Eq. (3.23). Also, non-coherent terms similar to those encountered in the above section have been excluded. It then follows that in the renormalization scheme discussed above

\[ M_0 + \delta M_{\text{Weak}} = -i\sqrt{2}G\mu\rho^{(\nu,\nu)}[\bar{\nu}_\alpha(p)\gamma_\alpha L\nu_\alpha(p)][\bar{\nu}_\alpha(p')\gamma^\alpha L\nu_\alpha(p')] \times \left\{ 1 + \frac{q^2}{M_Z^2} \left[ 1 + \Delta R'(q^2) + \frac{\alpha}{4\pi c^2 s^2} \frac{9 + 8s^2}{6} \ln \frac{M_W^2}{q^2} \right] \right\}, \quad (4.7) \]
where $\Delta R'(q^2)$ is the analogue of $\Delta R(q^2)$ in neutrino-electron scattering

$$
\Delta R'(q^2) = \frac{A_{ZZ}(0) - A_{ZZ}(q^2)}{q^2} + \frac{A_{ZZ}(M_Z^2) - A_{ZZ}(0)}{M_Z^2} + O(M_W^{-4}).
$$

Again only the real part will be kept. The overall normalization constant $\rho^{(\nu\nu)}$ can be conveniently related to the constant $\rho^{(\nu\ell)}_{N.C.}$ introduced in the study of $\nu\ell \rightarrow \nu\ell$ scattering (Refs. 32, 25)

$$
\rho^{(\nu\nu)} = \rho^{(\nu\ell)}_{N.C.} - \frac{\alpha}{4\pi} \left( \frac{1}{2s^2c^2} \right) \left( \frac{7}{2} - 10s^2 + 12s^4 \right),
$$

where

$$
\rho^{(\nu\ell)}_{N.C.} = 1 + \frac{\alpha}{4\pi} \left( \frac{3}{4s^4} \ln c^2 - \frac{7}{4s^2} + \frac{2C_Z}{c^2s^2} + G(\xi, c^2) + \frac{3m_t^2}{4s^2M_W^2} \right),
$$

$$
G(\xi, c^2) = \frac{3\xi}{4s^2} \left( \frac{\ln(c^2/\xi)}{c^2 - \xi} + \frac{1}{c^2} \ln \xi \right),
$$

$$
C_Z = \frac{19}{8} - \frac{7}{2}s^2 + 3s^4,
$$

and $\xi = m_\phi^2/M_Z^2$.

Again, the order $\alpha$ correction (the first term in Eq. (4.7)) cancels for the sum of neutrino-neutrino and neutrino-anti-neutrino scattering. Terms surviving the cancellation are grouped in the squared brackets. Among them, the first term is due to the expansion of the $Z$ propagator, the second is due to the $Z$ self-energy and the last term is the sum of vertex and box diagram corrections. The difference between $\rho^{(\nu\nu)}$ and $\rho^{(\nu\ell)}_{N.C.}$ arises because the three exchange box-diagrams (Fig. 11d to Fig. 11f) do not appear in a $\nu\ell \rightarrow \nu\ell$ scattering. Numerically, this difference is very small, and can be ignored in our leading log approximation. The aforementioned top quark mass dependence is included in the parameter $\rho^{(\nu\ell)}_{N.C.}$ (Eq. (4.10)). This gives an additional correction of the order $+1\%$ for a heavy top with mass $150 \text{ GeV} \lesssim m_t \lesssim 200 \text{ GeV}$.

27
In contrast to the $\nu_e e \rightarrow \nu_e e$ scattering, no significant cancellation between the vertex correction and the box diagram correction occurs. As we already pointed out, this is because the vertex correction is suppressed by $4s^2 - 1$. As a result, within the region $1 \text{MeV}^2 \leq |q^2| \leq 400 \text{MeV}^2$, $\delta M_{\text{Vertex}} + \delta M_{\text{Box}}$ is essentially given by $\delta M_{\text{Box}}$. Numerically, we find that it is about $+13\%$ to $+10\%$.

For the self-energy contributions, we again decompose them into the bosonic, hadronic, and leptonic parts

$$\Delta R'(q^2) = \Delta R^{(b)}(Q^2) + \Delta R^{(h)}(q^2) + \Delta R^{(l)}(q^2).$$

(4.11)

The bosonic part does not have any large log terms, and hence its contribution is negligible. Explicitly,

$$\Delta R^{(b)}(q^2) = -\frac{\alpha}{4\pi s^2} \left[ \frac{1}{6} \left( \frac{17c^2}{2} + \frac{1 + s^4}{2c^2} - s^2 + 2s^4c^2 \right) + \frac{1}{c^2} \int_0^1 dx \frac{x(x-1)}{c^2 + x(1 - \xi) + \xi} \right.$$

$$+ \left( \frac{17}{2} + \frac{s^4}{2c^4} - \frac{s^2}{c^2} - 2s^4 + 5c^2 \right) \int_0^1 dx \ln \frac{c^2 + x(x-1)}{c^2}$$

$$+ \left( \frac{23c^2}{2} + \frac{s^4}{2c^2} - 1 \right) \int_0^1 dx x(x-1) \ln \frac{c^2 + x(x-1)}{c^2}$$

$$+ \frac{1}{2c^2} \int_0^1 dx \left( x^2 + (1 - x)\xi - 2 \right) \ln \frac{x^2 + \xi(1 - x)}{x + \xi(1 - x)} \right].$$

(4.12)

Numerically, we find that the terms in the square brackets add up to a value of the order of $-1$ and they are not sensitive to the choice of $\xi$. Thus, $\Delta R^{(b)}(q^2) \lesssim 0.3\%$.

The hadronic part suffers the same ambiguity as $\Delta R^{(h)}(q^2)$ induced by the complication of strong interaction. Within the constituent quark approximation,
we find

\[
\Delta R^{(h)}(q^2) = \frac{3\alpha}{2\pi c^2 s^2} \sum_f \left( \frac{1}{2} - 2s^2 C_3 f + 4s^4 Q_f^2 \right) \text{Re} \int_0^1 dx x(1-x) \ln \frac{m_f^2 - M_Z^2 x(1-x)}{m_f^2 - q^2 x(1-x)}. 
\]

(4.13)

where \( Q_f \) and \( C_3 f \) are the charge and the third component of weak isospin of the quark \( f \) with a constituent mass \( m_f \). Again, it is easy to show that the contribution due to a heavy quark is completely negligible in \( \Delta R^{(h)}(q^2) \). Numerically, one finds from Eq. (4.13) that for \(|q^2| \ll m_t^2\), \( \Delta R^{(h)}(q^2) \) is about +5%.

The leptonic contribution, \( \Delta R^{(l)} \) can be obtained from Eq. (4.13) by an appropriate change of charge and weak isospin quantum numbers \( f \) or the charged leptons and neutrinos, and then dividing the result by 3. The sum is over all the three families of leptons and neutrinos. In the region \( 1 \text{MeV}^2 \leq |q^2| \leq 400 \text{MeV}^2 \), we find that \( \Delta R^{(l)}(q^2) \) is between +7% and +5%.

Thus, the sum of all weak corrections plus a +1% correction if the top quark mass is within the region of 150 GeV to 200 GeV is quite sizable (\( \sim +20\% \)). The details are summarized in Table 1, in which one finds that for \( 1 \text{MeV}^2 \leq |q^2| \leq 400 \text{MeV}^2 \) the total correction is between +25% to +20%.

The result for \( \nu_e \bar{\nu}_e \rightarrow \nu_e \bar{\nu}_e \) can be obtained from that of \( \nu_e(p)\nu_e(p') \rightarrow \nu_e(p)\nu_e(p') \) by a change of \( \bar{u}(p')\gamma_\alpha Lu(p') \rightarrow -\bar{u}(p')\gamma_\alpha Ru(p') \) and replacing \( q^2 \) by \( -q^2 \). While the order \( \alpha \) constant term will cancel for the sum of these two amplitudes, terms proportional to \( q^2/M_Z^2 \) add up. Thus, the index of refraction due to \( \nu_e \nu_e(\bar{\nu}_e) \rightarrow \nu_e \nu_e(\bar{\nu}_e) \) scattering in a medium which contains an equal amount of \( \nu_e \) and \( \bar{\nu}_e \) is

\[
(n - 1)_{\nu_e \nu_e(\bar{\nu}_e) \rightarrow \nu_e \nu_e(\bar{\nu}_e)} = -\frac{8\sqrt{2}}{3M_Z^2} G_{\mu \nu}^{(\nu;\nu)} N_{\nu_e} p_0(p'_0)(1 + \delta_{\nu_e \nu_e}). \tag{4.14}
\]

In the leading log approximation, \( \delta_{\nu_e \nu_e} \) is about +20% to +25% in the region
1 MeV^2 ≤ |q^2| ≤ 400 MeV^2. The details are summarized in Table 2.

4.2. \( \nu_{\mu,\tau} + \nu_{\mu,\tau}(\bar{\nu}_{\mu,\tau}) \rightarrow \nu_{\mu,\tau} + \nu_{\mu,\tau}(\bar{\nu}_{\mu,\tau}) \)

In the same region of \( q^2 \) discussed above, we now have a slightly different hierarchy

\[ M^2_{W,Z} \gg m^2_{\mu,\tau} \gg |q^2|. \]  

(4.15a)

However, the log factor \( \ln(M^2_W/m^2_{\ell}) \) \( (\ell = \mu, \tau) \) is still rather large, and thus a leading log approximation calculation is still meaningful. In this case the function \( I_1 \) defined in Eq. (3.1) can be approximately simplified to

\[ I_1(m_{\nu\ell}, m_{\ell}, M_W, q^2) \approx -\frac{1}{6} \ln \frac{m^2_{\ell}}{M^2_W}. \]  

(4.15b)

As a result, some of the log factors in \( \delta M_{\text{Vertex}} \) and \( \delta M_{\text{Box}} \) in Eqs. (4.5) and (4.6) should be changed from \( \ln(q^2/M^2_W) \) to \( \ln(m^2_{\ell}/M^2_W) \). The rest of the calculation remains the same. Places where such a change should be made are those in which diagrams contain charged leptons.

For the vertex correction we find

\[ \delta M_{\text{Vertex}} = M_0 \frac{\alpha}{4\pi} \left[ \frac{4c^2}{s^2} \left( \frac{2}{\epsilon} + \Gamma'(1) - \ln \frac{M^2_W}{4\pi \mu^2_0} \right) + \frac{q^2}{M^2_W} \left( \frac{4s^2 - 2}{3s^2} \ln \frac{M^2_W}{m^2_{\ell}} + \frac{1}{3s^2} \ln \frac{M^2_Z}{q^2} \right) \right] \]  

(4.16)

where

\[ M_0 = \frac{i g^2}{4c^2 M^2_Z - q^2} \frac{1}{q^2} \frac{1}{\epsilon} \frac{1}{\Gamma'(1)} \frac{1}{\Gamma'(1)} \left[ \bar{u}_{\nu\ell}(p') \gamma^\alpha L u_{\nu\ell}(p) \right] \left[ \bar{u}_{\nu\ell}(p) \gamma^\alpha L u_{\nu\ell}(p') \right]. \]  

(4.17)

Eq. (4.16) reduces to Eq. (4.5) if one substitutes \( m^2_{\ell} \) by \( q^2 \) and ignores the difference (valid in the leading log approximation) between \( M_W \) and \( M_Z \).
Accordingly, the box diagram contribution becomes

\[
\delta M_{\text{Box}} = M_0 \frac{\alpha}{4\pi s^2} \left[ 2 - \frac{3}{2c^2} + \frac{q^2}{M_W^2} \left( \frac{2}{3} \ln \frac{M_W^2}{m_\ell^2} + \frac{7}{6} \ln \frac{M_W^2}{q^2} \right) \right].
\] (4.18)

It then follows from Eqs. (4.16) and (4.18) that for \( \nu_\ell + \nu_\ell \to \nu_\ell + \nu_\ell \) \((\ell = \mu, \tau)\) our leading log result is

\[
M_0 + \delta M^{\text{Weak}} = -i\sqrt{2}G_\mu \rho^{(\nu;\nu)}[\bar{u}_{\nu_\ell}(p)\gamma_\alpha L u_{\nu_\ell}(p)][\bar{u}_{\nu_\ell}(p')\gamma^\alpha L u_{\nu_\ell}(p')]
\times \left\{ 1 + \frac{q^2}{M_Z^2} \left[ 1 + \Delta R'(q^2) + \frac{\alpha}{4\pi s^2 c^2} \left( \frac{9 + 8s^2}{6} \ln \frac{M_W^2}{q^2} + \frac{8s^2}{6} \ln \frac{q^2}{m_\ell^2} \right) \right] \right\}. \] (4.19)

It bares a strong resemblance to Eq. (4.7) for \( \nu_e + \nu_e \to \nu_e + \nu_e \) scattering.

Given the smallness of \( s^2 \), the difference between Eqs. (4.7) and (4.19) is almost purely academic. Thus, qualitatively, we expect that the total correction for the three types of neutrino-neutrino scattering are approximately the same \((\sim +20\%)\). The details are summarized in Table 2.

### 5. Scattering Between Different Families

In the absence of family mixing, lowest order interactions between different families such as \( \nu_{\mu,\tau} + e(\bar{e}) \to \nu_{\mu,\tau} + e(\bar{e}) \) and \( \nu_i + \nu_j(\bar{\nu}_j) \to \nu_i + \nu_j(\bar{\nu}_j)(i \neq j) \) can only occur through the \( t \)-channel. Then, for the reasons explained above, the coherent amplitudes of these interactions will cancel completely, and they will not contribute to the neutrino index of refraction. The situation will change if one goes beyond the lowest order. In this case, radiative corrections become the only contribution.
5.1. $\nu_{\mu, \tau} + e(\bar{e}) \rightarrow \nu_{\mu, \tau} + e(\bar{e})$

It should be clear by now that in the absence of family mixing only those diagrams which are similar to the last four in Fig. 7 contribute to the neutrino index of refraction. The only difference is to change the $\nu_e$ line by a $\nu_{\mu, \tau}$ line, and, accordingly, the internal electron line by a $\mu$ or $\tau$ line. The result is

$$M_{\nu_{\mu, \tau} + e \rightarrow \nu_{\mu, \tau} + e} = M_0 \frac{\alpha}{4\pi s^2} \left\{ \frac{15 - 24s^2e^2}{4c^2} + \frac{q^2}{M_W^2} \left[ \frac{10 - 8s^2 + 16s^4}{3} \ln \frac{q^2}{M_W^2} + \frac{8}{3} \ln \frac{m_{\mu, \tau}^2}{q^2} \right] \right\} + \ldots$$

(5.1)

where the ellipses represent incoherent terms which are of no interest to us, and

$$M_0 = \frac{ig^2}{4M_W^2} \left[ \bar{u}_e(p') \gamma_\alpha L u_e(p') \right] \left[ \bar{u}_{\nu_{\mu, \tau}}(p) \gamma^\alpha L u_{\nu_{\mu, \tau}}(p) \right].$$

(5.2)

In Eq. (5.1) terms proportional to $q^2/M_W^2$ contribute to the neutrino index of refractions. By employing the by-now familiar method, we find

$$(n - 1)_{\nu_e + e(\bar{e}) \rightarrow \nu_e + e(\bar{e})} = -\frac{8\sqrt{2}}{3M_W} G_\mu N_e p_0(p_0') \delta_{\nu_e e}, \quad \ell = \mu, \tau.$$

(5.3)

In the region $1 \text{MeV}^2 \leq |q^2| \leq 400 \text{MeV}^2$, $\delta_{\nu_e e}$ varies from about +11% to +10%, and $\delta_{\nu_\tau}$ is between +7% to +6% (see Table 2). Notice also, $N_e = 2N_\nu$ (see Eq. (1.1)). Such a sizable correction in the diagonal scattering ($\nu_e e \rightarrow \nu_e e$) is cancelled by the vertex correction. There is no vertex correction for scatterings between different families if the corresponding mixing is zero.
5.2. $\nu_i + \nu_j (\bar{\nu}_j) \to \nu_i + \nu_j (\bar{\nu}_j) (i \neq j)$

Leading log contributions are generated from box diagrams similar to the first three graphs in Fig. 11. A straightforward calculation shows that in the zero mixing limit

$$M = \frac{-i g^2}{4 M_W^2} \left[ \bar{u}_{\nu_i} (p) \gamma_\alpha L u_{\nu_i} (p) \right] \left[ \bar{u}_{\nu_j} (p') \gamma^\alpha L u_{\nu_j} (p') \right]$$

\[ \times \frac{\alpha}{4 \pi s^2 c^2} \left[ c^2 - \frac{3}{4} + \frac{q^2}{M_Z^2} \left( \frac{5}{12} \ln \frac{M_Z^2}{m_{\ell_j}^2} + \frac{2}{3} \ln \frac{M_Z^2}{q^2} \right) \right] + ..., \] (5.4)

where the ellipses again refer to the incoherent terms and $m_{\ell_j}$ is the heaviest charged lepton mass in question. It then follows that

$$\langle n - 1 \rangle_{\nu_i + \nu_j (\bar{\nu}_j) \to \nu_i + \nu_j (\bar{\nu}_j)} = - \frac{8\sqrt{2}}{3 M_W^2} G_{\mu \nu} p_0 \langle \delta_{\nu_i \nu_j} \rangle.$$ (5.5)

Numerically, we find that in the region $1 \text{ MeV}^2 \leq |q^2| \leq 400 \text{ MeV}^2$, $\delta_{\nu_i \nu_j}$ is about +6% to +5% for the scattering $\nu_e + \nu_\mu \to \nu_e + \nu_\tau$. The result increases about +1% for $\nu_e + \nu_\mu \to \nu_e + \nu_\mu$. The details are summarized in Table 2.

6. Neutrino-photon Scattering

Since a photon is its own anti-particle, the cancellation discussed above does not apply to the neutrino-photon scattering. However, we will show below that the neutrino-photon forward scattering amplitude is zero at one-loop level in the standard model.
It is known\textsuperscript{35} that to the lowest nonvanishing order the diagonal effective neutrino-two photon interaction is given by

\[ L_{\text{eff}} = a \bar{\nu} \nu F^{\alpha\beta} F_{\alpha\beta} + i a' \bar{\nu} \gamma_5 \nu \tilde{F}^{\alpha\beta} F_{\alpha\beta}, \] \hspace{1cm} (6.1)

where \( \nu \) is a neutrino field, \( F^{\alpha\beta} \) is the electromagnetic tensor with its dual \( \tilde{F}^{\alpha\beta} = \frac{1}{2} \epsilon^{\alpha\beta\sigma\lambda} F_{\sigma\lambda} \). In terms of the standard electroweak interactions, \( a \) and \( a' \) are zero at one-loop level if neutrinos are massless. Assuming neutrinos are massive, one then finds (Ref. 35)

\[ a' = \frac{G_F}{\sqrt{\pi}} m_\nu f, \quad a = 0, \] \hspace{1cm} (6.2)

where \( m_\nu \) is a neutrino mass and \( f \) is a scalar function obtained from loop integrals with its real part given by

\[ \text{Re} f = \frac{-1}{4k_1.k_2} + \frac{m_\tau^2}{4(k_1.k_2)^2} \times \left\{ \begin{array}{ll}
2 \left[ \sin^{-1} \frac{\sqrt{k_1.k_2/2m_\tau^2}}{2} \right]^2, & \text{if } k_1.k_2 < 2m_\tau^2 \\
\frac{\pi^2}{2} - \frac{1}{2} \ln \frac{1 + \sqrt{1 - 2m_\tau^2/k_1.k_2}}{1 - \sqrt{1 - 2m_\tau^2/k_1.k_2}}, & \text{if } k_1.k_2 \geq 2m_\tau^2.
\end{array} \right. \] \hspace{1cm} (6.3)

Here, \( k_1 \) and \( k_2 \) are, respectively, the initial- and the final-photon momentum.

The function \( f \) has a notable feature that it is zero for a forward scattering in which \( k_1.k_2 = 0 \). Thus, in the forward direction \( L_{\text{eff}} \) is zero at one-loop level. Notice also, the matrix element \( \bar{u}(p)\nu\gamma_5 u(p)\nu \) is zero. We then conclude that the neutrino-photon scattering does not contribute to the real part of the neutrino index of refraction at one-loop level. Thus, the photon plasma is essentially transparent to neutrinos.
7. The Imaginary Part of the Index of Refraction

While the real part of the index of refraction describes the coherent interference of propagating neutrinos, the imaginary part characterizes the incoherent depletion of neutrinos from their original coherent state. In normal situations, $Re(n) \gg Im(n)$, because neutrinos only participate in weak interactions and hence the phase shift in the forward scattering amplitude is very small. In the MSW formalism, for example, $Re(n)$ is of the order of $G_F$, whereas $Im(n)$ is of the order of $G_F^2 q^2$. In a CP symmetric plasma such as the early universe, however, the order $G_F$ of $Re(n)$ cancels. As we have already learned, the leading terms of $Re(n)$ are now only of the order of $G_F q^2 / M_W^2$. Since $q^2 \ll M_W^2$, in this situation the imaginary and the real part of index of refraction become interestingly comparable.

The simplest way of calculating the imaginary part of index of refraction is to employ the optical theorem

$$Im(n) = \frac{1}{2p_0} \sum_j N_j \sigma_j,$$  

(7.1)

where $N_j$ is the density of the jth scatterer and $\sigma_j$ is the corresponding total cross section. Table 3 lists all the relevant scattering cross sections normalized by

$$\sigma_0 = \frac{G_F^2 (p + p')^2}{6\pi}.$$  

(7.2)

The result given by neutrino-photon scattering is completely negligible.
8. Summary

In this paper we have systematically evaluated the standard electroweak interaction contributions to the neutrino index of refraction in the early universe. Of cosmological interest, we have concentrated on the period when the temperature of the universe was of the order of a few $MeV$, and the scatterers were photons, electrons, positrons and three types of neutrinos and anti-neutrinos.

Assuming CP invariance, the number of particles and anti-particles is equal, and as a result the lowest order coherent forward scattering amplitudes completely cancel. Hence, the leading nonvanishing result obtained by the expansion of the gauge boson propagator is very small. In the absence of leptonic family mixing, which happens if $\nu_e, \nu_\mu$ and $\nu_\tau$ are degenerate in mass, even such a tiny result vanishes for scatterings between different families. Although the cancellation does not occur for neutrino-photon scattering, we find that its coherent forward scattering amplitude is zero at one-loop level.

The smallness of tree level result motivates us to investigate its radiative corrections. By employing the on-shell renormalization scheme, we have found that depending on the specific scattering process the leading log corrections are typically of the order of 20% to 10% with the same sign. Radiative corrections become the dominant contribution for scatterings that involve different families if mixings in the lepton sector are very small. Our results for neutrino index of refraction are summarized in Table 2 and Table 4. The justification of our leading log approximation relies on the hierarchy $M_{W,Z}^2 \gg |q^2|, m_\ell^2$.

Numerically, these corrections already become significant. They are about two orders of magnitude bigger than the original expectation (Ref. 11) of the order of $\alpha/\pi$. As summarized in Table 2 and Table 4, the radiative correction to the
neutrino index of refraction is about +20% and +50% for $\nu_e$ and $\nu_{\mu,\tau}$, respectively. This generates a sizable effect in locating the exact boundaries from the nucleosynthesis constraint for the aforementioned $\nu_e - \nu_s$ oscillation. A detailed discussion of the application of our results to cases of cosmological and astrophysical interest will be presented in a forthcoming paper.

Besides the numerical significance, we have proposed a theoretical method to evaluate the neutrino index of refraction at finite temperature. This method makes the lowest order calculation much simpler than that derived from the finite temperature field theory, and more importantly allows us to evaluate higher order corrections in a straightforward way. The underlying physics also appears to be more intuitive and transparent.

We also studied the question of the cancellation of infrared divergence in a coherent process. We have shown that such cancellations indeed take place in all order in renormalizable perturbation theory. As a consequence, the cancellation of infrared divergences will not spoil the coherent condition.

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APPENDIX

In this appendix we give some details of our calculation for the three- and four-point functions. As explained in the text, we have ignored all \( t \)-channel one-particle-reducible graphs. In the remaining graphs we keep all terms of order \( \alpha \) and only leading log terms of order \( \alpha (q^2/M_W^2) \). The order \( \alpha \) terms, although they do not contribute to the neutrino index of refraction, provide a check on the calculation. Our calculation is carried out in the Feynman gauge. Due to the cancellation of particle-particle and particle-anti-particle scattering, graphs with scalar lines can be omitted in the leading log approximation. Consequently, only the leading log terms of our result are gauge invariant.

**Three-point Functions**

The three-point function correction to \( \nu_e e \rightarrow \nu_e e \) given by \( \delta M_6 e + 6 f \) in Eq. (3.17) of the text is obtained by evaluating the one-loop vertex diagrams shown in Fig. 12. For convenience, we parameterize it as a sum of individual diagram contributions

\[
\frac{ig}{\sqrt{2}} [\bar{u}_e(p')\gamma_\alpha L u_\nu_\alpha(p)] \frac{\alpha}{4\pi} \sum_i \Gamma_i, \quad (A.1)
\]

where

\[
\Gamma_1(12a) = 3 \left( \frac{2}{e} + \Gamma'(1) + \frac{5}{6} - \ln \frac{M_W^2}{4\pi \mu_0^2} \right) - \frac{q^2}{M_W^2} \left( \ln \frac{M_W^2}{m_e^2} + \frac{7}{6} \right), \quad (A.2)
\]

\[
\Gamma_2(12b) = \left( \frac{1}{2s^2} - 1 \right) \int_0^1 dx \left[ 6x \left( \frac{2}{e} + \Gamma'(1) + \frac{1}{3} - \ln \frac{M_W^2}{4\pi \mu_0^2} - \ln \frac{1 - xs^2}{c^2} - \frac{1}{c^2} \ln \frac{c^2 + xs^2}{x} \right) \right. \\
- \frac{q^2}{M_Z^2} \left( \frac{(x - 1)(8x - 2)}{c^2} + \frac{x(8x - 2s^2)}{c^4} \ln \frac{c^2 + xs^2}{x} \right), \quad (A.3)
\]
\[ \Gamma_3(12c) = \frac{\Gamma_2(12b)}{(1 - 2s^2)}, \quad (A.4) \]

\[ \Gamma_4(12d) = \frac{1}{c^2} \left(1 - \frac{1}{2s^2}\right) \int_0^1 dx \left[ x \left( \frac{2}{\epsilon} + \Gamma'(1) - 1 - \ln \frac{xM_Z^2 + (1 - x)^2m_e^2}{4\pi\mu_0^2} \right) \right. \]
\[ \left. - \frac{q^2}{M_Z^2} \left(2x(x - 1) \ln \frac{xM_Z^2 + (1 - x)^2m_e^2}{(1 - x)m_e^2 + x(x - 1)q^2} - x^2\right) \right]. \quad (A.5) \]

Similarly, for the \(\nu\nu \rightarrow \nu\nu\) scattering (Fig. 13), we have

\[ \frac{ig}{2c} \left[ \bar{u}_\nu(p')\gamma_\alpha L u_\nu(p) \right] \frac{\alpha}{4\pi} \sum_i \Gamma'_i, \quad (A.6) \]

where

\[ \Gamma'_1(13a) = \frac{1}{4s^2c^2} \left[ \frac{2}{\epsilon} + \Gamma'(1) - \frac{1}{2} - \ln \frac{M_Z^2}{4\pi\mu_0^2} + \frac{q^2}{9M_Z^2} + \frac{4q^2}{M_Z^2} \int_0^1 dx x(x - 1) \ln \frac{x(x - 1)q^2 + m^2}{M_Z^2} \right], \quad (A.7) \]

\[ \Gamma'_2(13b) = \left(1 - \frac{1}{2s^2}\right) \left[ \frac{2}{\epsilon} + \Gamma'(1) - \frac{1}{2} - \ln \frac{M_W^2}{4\pi\mu_0^2} + \frac{5q^2}{18M_W^2} \right. \]
\[ \left. + \frac{4q^2}{M_W^2} \int_0^1 dx x(x - 1) \ln \frac{x(x - 1)q^2 + m_t^2}{M_W^2} \right], \quad (A.8) \]

\[ \Gamma'_3(13c) = \frac{c^2}{s^2} \left[ 3 \left( \frac{2}{\epsilon} + \Gamma'(1) - \frac{1}{6} - \ln \frac{M_W^2}{4\pi\mu_0^2} \right) + \frac{q^2}{3M_W^2} \right]. \quad (A.9) \]

**Box Diagrams**
First, the photon box (Fig. 4c) result is

\[
\delta M_{Box}^{QED} = \frac{-ig^2}{2M_W^2 4\pi} \left[ \bar{u}_e(p') \gamma_{\alpha} L u_e(p') \right] \left[ \bar{u}_{\nu_e}(p) \gamma^{\alpha} L u_{\nu_e}(p) \right] \\
\times \left[ \left( 1 + \frac{q^2}{M_W^2} \right) \left( 2 \ln \frac{\lambda^2}{m_e^2} + \ln \frac{M_W^2}{m_e^2} + \frac{7}{2} \right) - \frac{q^2}{M_W^2} \left( \frac{2}{3} \ln \frac{M_W^2}{m_e^2} - \frac{19}{9} \right) \right] \\
\times \frac{-ig^2}{2M_W^2 4\pi} \left[ \bar{u}_e(p') \gamma_{\alpha} R u_e(p') \right] \left[ \bar{u}_{\nu_e}(p) \gamma^{\alpha} L u_{\nu_e}(p) \right] \left( 1 + \frac{1}{3} \frac{q^2}{M_W^2} \right),
\]

(A.10)

where \( \lambda \) is a fictitious photon mass.

In calculating the remaining weak interaction box diagrams, we frequently encounter a scalar integral

\[
F_{\alpha\beta}(m, m_\ell, q^2) \equiv \int \frac{d^4K}{(2\pi)^4} \frac{(K - q)_\alpha K_\beta}{(K^2 - m^2)[(K - q)^2 - m_\ell^2][(K - p)^2 - M_W^2][(K - p')^2 - M_Z^2]},
\]

(A.11a)

An explicit calculation shows

\[
F_{\alpha\beta}(m, m_\ell, q^2) = \frac{-i}{16\pi^2 M_W^2} \left( \frac{c^2}{s^2} \right) \left\{ \frac{\ln c^2}{4} g_{\alpha\beta} + \frac{g_{\alpha\beta}}{2} \left[ K_0(m, m_\ell, M_W, q^2) - K_0(m, m_\ell, M_Z, q^2) \right] \right. \\
+ q_\alpha q_\beta \left[ \frac{1}{M_W^2} K_1(m, m_\ell, M_W, q^2) - \frac{1}{M_Z^2} K_1(m, m_\ell, M_Z, q^2) \right] \left. \right\} \\
+ ..., 
\]

(A.11b)

where the ellipses represent the terms which do not have a large log, and

\[
K_0(m, m_\ell, M, q^2) \equiv \int_0^1 dx \frac{m^2 + x(m_\ell^2 - m^2) + x(x - 1)q^2}{M^2} \ln \frac{m^2 + x(m_\ell^2 - m^2) + x(x - 1)q^2}{M^2},
\]

(A.12a)
They are related to the functions $I_{1,2}$ (Eq. (3.1)) by

\begin{align*}
K_0(m, m_\ell, M, q^2) &= I_1(m, m_\ell, M, q^2) + I_2(m, m_\ell, M, q^2), \\
K_1(m, m_\ell, M, q^2) &= \frac{M^2}{q^2} I_1(m, m_\ell, M, q^2). \tag{A.12b}
\end{align*}

From the analyticity of scalar integrals\textsuperscript{37}, one can show that for our case only those graphs which have an $s$ and/or $u$-channel light intermediate state have a large log. For the $\nu_e e \to \nu_e e$ scattering (Fig. 7), we find

\begin{align*}
\delta M(7a) &= \frac{-ig^2}{2M_W^2} \frac{\alpha}{4\pi} [\bar{u}_e(p')\gamma_\alpha Lu_{\nu_e}(p)][\bar{u}_{\nu_e}(p)\gamma^\alpha Lu_e(p')] \times \left(1 - \frac{1}{2s^2}\right)^2 \left[\ln c^2 + \frac{2q^2}{3M_W^2s^2} \left(1 + \frac{1 + c^2}{2s^2} \ln c^2\right)\right], \tag{A.13}
\end{align*}

\begin{align*}
\delta M(7b) &= \frac{\delta M(7a)}{(2s^2 - 1)^2}. \tag{A.14}
\end{align*}

\begin{align*}
\delta M(7c) &= \delta M(7d) = \frac{ig^4}{32\pi^2M_Z^2c^2s^2} (2s^2 - 1)[\bar{u}_e(p')\gamma_\alpha Lu_{\nu_e}(p)][\bar{u}_{\nu_e}(p)\gamma^\alpha Lu_e(p')] \\
&\quad \times \left\{\ln c^2 - 2\left[K_0(0, m_e, M_W, -q^2) - K_0(0, m_e, M_Z, -q^2)\right] \\
&\quad + q^2 \left[\frac{1}{M_W^2} K_1(0, m_e, M_W, -q^2) - \frac{1}{M_Z^2} K_1(0, m_e, M_Z, -q^2)\right]\right\}, \tag{A.15}
\end{align*}

\begin{align*}
\delta M(7e) &= \frac{ig^4}{64\pi^2M_Z^2c^4} [\bar{u}_e(p')\gamma_\alpha (4s^4 + (1 - 4s^2)L)u_e(p')][\bar{u}_{\nu_e}(p)\gamma^\alpha Lu_{\nu_e}(p)] \\
&\quad \times \left[1 + 2K_0(0, m_e, M_Z, -q^2) - \frac{q^2}{M_Z^2} K_1(0, m_e, M_Z, -q^2)\right], \tag{A.16}
\end{align*}

\(41\)
\[
\delta M(7f) = \frac{-i g^4}{64 \pi^2 M_Z^2 c^4} \left[ \bar{u}_e(p') \gamma_\alpha (4s^4 + (1 - 4s^2)L) u_e(p') \right] [\bar{u}_\nu_e(p) \gamma^\alpha L u_\nu_e(p)] \\
\times \left[ \frac{1}{4} + \frac{1}{2} K_0(0, m_e, M_Z, q^2) \right] \\
- \frac{i g^4}{64 \pi^2 M_Z^4 c^4} [\bar{u}_e(p') \gamma_\alpha (4s^4 + (1 - 4s^2)L) u_e(p')] [\bar{u}_\nu_e(p) \gamma_\beta L u_\nu_e(p)] \\
\times q^\alpha q^\beta K_1(0, m_e, M_Z, q^2),
\] 
(A.17)

\[
\delta M(7g) = \frac{-i g^4}{16 \pi^2 M_W^2} [\bar{u}_e(p') \gamma_\alpha L u_e(p')] [\bar{u}_\nu_e(p) \gamma^\alpha L u_\nu_e(p)] \left[ \frac{1}{4} + \frac{1}{2} K_0(0, m_e, M_W, q^2) \right] \\
- \frac{i g^4}{16 \pi^2 M_W^4} [\bar{u}_e(p') \gamma_\alpha L u_e(p')] [\bar{u}_\nu_e(p) \gamma_\beta L u_\nu_e(p)] q^\alpha q^\beta K_1(0, m_e, M_W, q^2),
\] 
(A.18)

\[
\delta M(7h) = \frac{i g^4}{16 \pi^2 M_W^2} [\bar{u}_e(p') \gamma_\alpha L u_e(p')] [\bar{u}_\nu_e(p) \gamma^\alpha L u_\nu_e(p)] \\
\times \left[ 1 + 2K_0(m_e, 0, M_W, -q^2) - \frac{q^2}{M_W^2} K_1(m_e, 0, M_W, -q^2) \right].
\] 
(A.19)

For the scattering \( \nu \nu \rightarrow \nu \nu \) (Fig. 11), we find

\[
\delta M(11a) = \frac{i g^4}{64 \pi^2 M_Z^2 c^4} [\bar{u}_\nu(p) \gamma_\alpha L u_\nu(p)] [\bar{u}_\nu(p') \gamma^\alpha L u_\nu(p')] \\
\times \left[ 1 + 2K_0(0, 0, M_Z, -q^2) - \frac{q^2}{M_Z^2} K_1(0, 0, M_Z, -q^2) \right],
\] 
(A.20)

\[
\delta M(11b) = \frac{-i g^4}{64 \pi^2 M_Z^2 c^4} [\bar{u}_\nu(p) \gamma_\alpha L u_\nu(p)] [\bar{u}_\nu(p') \gamma^\alpha L u_\nu(p')] \\
\times \left[ \frac{1}{4} + \frac{1}{2} K_0(0, 0, M_Z, q^2) \right] \\
- \frac{i g^4}{64 \pi^2 M_Z^4 c^4} [\bar{u}_\nu(p) \gamma_\alpha L u_\nu(p)] [\bar{u}_\nu(p') \gamma_\beta L u_\nu(p')] \\
\times q^\alpha q^\beta K_1(0, 0, M_Z, q^2),
\] 
(A.21)
\[\delta M(11c) = \frac{-ig^4}{64\pi^2 M_W^2} \left[ \bar{u}_\nu(p)\gamma_\alpha L u_\nu(p) \right] \left[ \bar{u}_\nu(p')\gamma^\alpha L u_\nu(p') \right] \times \left[ 1 + 2K_0(m_\ell, m_\ell, M_W, q^2) \right] \]
\[\delta M(11d) = \frac{-ig^4}{64\pi^2 M_Z^2 c^4} \left[ \bar{u}_\nu(p)\gamma_\alpha L u_\nu(p) \right] \left[ \bar{u}_\nu(p')\gamma^\alpha L u_\nu(p') \right] \times \left[ 1 + 2K_0(m_\ell, m_\ell, M_Z, q^2) \right] \]
\[\delta M(11e) = \frac{ig^4}{64\pi^2 M_Z^2 c^4} \left[ \bar{u}_\nu(p)\gamma_\alpha L u_\nu(p) \right] \left[ \bar{u}_\nu(p')\gamma^\alpha L u_\nu(p') \right] \times \left[ 1 + 2K_0(0, 0, M_Z, -q^2) - \frac{q^2}{M_Z^2} K_1(0, 0, M_Z, -q^2) \right] \]
\[\delta M(11f) = \frac{-ig^4}{64\pi^2 M_W^2} \left[ \bar{u}_\nu(p)\gamma_\alpha L u_\nu(p) \right] \left[ \bar{u}_\nu(p')\gamma^\alpha L u_\nu(p') \right] \times \left[ 1 + 2K_0(m_\ell, m_\ell, M_W, 0) \right].\]
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FIGURE CAPTIONS

Fig.1. Tree level graphs for $\nu_e e \rightarrow \nu_e e$ forward scattering. The coherent condition only allows the $W$ in the $u$-channel (1a) to carry a non-zero momentum.

Fig.2. Tree level graphs for $\nu_e \bar{e} \rightarrow \nu_e \bar{e}$ forward scattering. Again, only the $W$ in the $s$-channel (2a) is allowed to carry a non-zero momentum.

Fig. 3. Feynman graphs for an electron forward scattering with an infrared finite but otherwise arbitrary potential (the blob) and its QED corrections. The ellipses represent other graphs which are infrared finite.

Fig. 4. One-loop QED corrections to the $u$-channel neutrino-electron forward scattering. The photon propagator in (4a) and (4b) is the massless photon propagator (the second term in Eq. (3.10)). The box diagram (4c) is evaluated with the full photon propagator.

Fig. 5. One-loop self-energy and counterterm contributions to the $u$-channel neutrino-electron forward scattering.

Fig. 6. Vertex and wave function renormalization corrections to the $u$-channel neutrino-electron scattering. The photon propagators in (6a) and (6b) are the massive photon propagators (the first term in Eq. (3.10)). The detail of the blob is shown in Fig.12.

Fig. 7. Box diagram contributions to $\nu_e e \rightarrow \nu_e e$ forward scattering.

Fig. 8. Tree level diagrams for $\nu\nu \rightarrow \nu\nu$ and $\nu\bar{\nu} \rightarrow \nu\bar{\nu}$ forward scattering.

Fig. 9. One-loop self-energy and counterterm contributions to $\nu\nu \rightarrow \nu\nu$ forward scattering.

Fig. 10. Vertex corrections to the exchange channel in $\nu\nu \rightarrow \nu\nu$ scattering. The detail of the blob is shown in Fig. 13. The ellipses represent the wave function
renormalization diagrams.

Fig. 11. Box diagram contributions to $\nu \nu \to \nu \nu$ forward scattering.

Fig. 12. One-loop vertex diagrams for $\nu_e e \to \nu_e e$ forward scattering. Here the photon contribution is evaluated with the full propagator.

Fig. 13. One-loop vertex diagrams for $\nu \nu \to \nu \nu$ forward scattering.
TABLE CAPTIONS

Table 1. Radiative corrections to the neutrino index of refraction, assuming the medium is CP symmetric. Here $|q|^2$ varies from 1 $MeV^2$ to 400 $MeV^2$. The definition of the various corrections is given in the text. A 1% correction should be added to the diagonal $\nu\nu \rightarrow \nu\nu$ scattering if the top quark mass is within 150 $GeV$ to 200 $GeV$. Results for scattering between different families in the small mixing limit are given in Eqs. (5.3) and (5.5) of the text, and are summarized in Table 2.

Table 2. Summary of leading log results of radiative corrections. The interaction process also includes the corresponding scattering with the anti-scatterers.

Table 3. The total scattering cross section for neutrinos with the various scatterers in the early universe. The result is calculated in the center of mass system and normalized by $\sigma_0$ (Eq. (7.2)). The plus-sign refers to $i = e$ and the minus-sign corresponds to $i = \mu, \tau$. Also, $i \neq j$.

Table 4. Neutrino index of refraction in the early universe. Here, $n_0$ is given by $n_0 \equiv -(2\sqrt{2}/M_W^2)G_\mu N_\gamma \langle p'_\gamma \rangle$. The densities of the scatterers are normalized in terms of the photon density according to Eq. (1.1). The various radiative correction results, $\delta_{ij}$, are summarized in Table 2.
| Sources | $\nu_e e(\bar{e}) \rightarrow \nu_e e(\bar{e})$ | $\nu_e \nu_e (\bar{\nu}_e) \rightarrow \nu_e \nu_e (\bar{\nu}_e)$ | $\nu_{\mu,\tau} \nu_{\mu,\tau} (\bar{\nu}_{\mu,\tau}) \rightarrow \nu_{\mu,\tau} \nu_{\mu,\tau} (\bar{\nu}_{\mu,\tau})$ |
|---------|-----------------|-----------------|-----------------|
| hadronic self-energy | 5%:5%:5% | leptonic self-energy | 4% to 3%:7% to 5%: 7% to 5% |
| bosonic self-energy | negligible: negligible: negligible | QED | $-1%$: |
| vertex+box | $\leq 2%$:13% to 10%:12% to 10% | total | 10% to 7%:25% to 20%: 24% to 20% |

Table 1
correction: interaction: result ($|q| = 1 \text{ MeV}$): result ($|q| = 20 \text{ MeV}$)

| Interaction | Result 1 | Result 2 |
|-------------|----------|----------|
| $\delta_{\nu_e,\nu_e}: \nu_e \nu_e \rightarrow \nu_e \nu_e$ | 25%:20% | |
| $\delta_{\nu_\mu,\nu_\mu}: \nu_\mu \nu_\mu \rightarrow \nu_\mu \nu_\mu$ | 24%:20% | |
| $\delta_{\nu_\tau,\nu_\tau}: \nu_\tau \nu_\tau \rightarrow \nu_\tau \nu_\tau$ | 24%:20% | |
| $\delta_{\nu_e,\nu_\mu}: \nu_e \nu_\mu \rightarrow \nu_e \nu_\mu$ | 7%:6% | |
| $\delta_{\nu_e,\nu_\tau}: \nu_e \nu_\tau \rightarrow \nu_e \nu_\tau$ | 6%:5% | |
| $\delta_{\nu_\mu,\nu_\tau}: \nu_\mu \nu_\tau \rightarrow \nu_\mu \nu_\tau$ | 6%:5% | |
| $\delta_{\nu_e,\nu_\mu}: \nu_e \nu_\mu \rightarrow \nu_e \nu_\mu$ | 7%:6% | |
| $\delta_{\nu_e,\nu_\tau}: \nu_e \nu_\tau \rightarrow \nu_e \nu_\tau$ | 6%:5% | |

Table 2
interaction-channel: cross-section

\[
\begin{align*}
\nu_i \bar{\nu}_i & \leftrightarrow e\bar{e}: 8s^4 \pm 4s^2 + 1 \\
\nu_i \bar{\nu}_i & \leftrightarrow \nu_j \bar{\nu}_j: 1 \\
\nu_i e & \leftrightarrow \nu_i e: 8s^4 \pm 6s^2 + \frac{3}{2} \\
\nu_i \bar{e} & \leftrightarrow \nu_i \bar{e}: 8s^4 \pm 2s^2 + \frac{1}{2} \\
\nu_i \nu_i & \leftrightarrow \nu_i \nu_i: 6 \\
\nu_i \nu_j & \leftrightarrow \nu_i \nu_j: 3 \\
\nu_i \bar{\nu}_i & \leftrightarrow \nu_i \bar{\nu}_i: 4 \\
\nu_i \bar{\nu}_j & \leftrightarrow \nu_i \bar{\nu}_j: 1 \\
\nu(\bar{\nu})\gamma & \leftrightarrow \nu(\bar{\nu})\gamma: \text{negligible}
\end{align*}
\]

Table 3
| real part of index of refraction: results |
|------------------------------------------|
| :                                        |
| \[(n - 1)\nu_e/n_0 : 2(1 + \delta\nu_e) + \frac{1}{2}(c^2 \rho^{(\nu\nu)})(1 + \delta\nu_e) + \delta\nu_e\nu_e + \delta\nu_e\nu_r) \] |
| :                                        |
| \[(n - 1)\nu_\mu/n_0 : \delta\nu_\mu e + \frac{1}{2}(c^2 \rho^{(\nu_\nu\nu)})(1 + \delta\nu_\mu\nu_\mu) + \delta\nu_\mu\nu_e + \delta\nu_\mu\nu_r) \] |
| :                                        |
| \[(n - 1)\nu_r/n_0 : \delta\nu_r e + \frac{1}{2}(c^2 \rho^{(\nu\nu)})(1 + \delta\nu_r\nu_r) + \delta\nu_r\nu_e + \delta\nu_r\nu_\mu) \] |

Table 4