Modeling stellar-gaseous disks: rows in spiral patterns of galaxies

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Abstract. The work is aimed at studying the formation mechanisms of straightened segments (rows) in spiral galactic arms, which are found in approximately 5% of S-galaxies. We use the Smoothed Particles Hydrodynamics (SPH) numerical algorithm, which is adapted for parallel computing on the GPUs to simulate the gas component. N-body model is used to simulate the dynamics of collisionless stellar disk. Graphic processors allow you to apply the “Particle-Particle” algorithm (each particle interacts with every other particle), which is the most accurate method to account for self-gravity. Our numerical model demonstrates the formation of a rows system with a pronounced transient character.

1. Introduction
The spiral pattern is the most fascinating structure of galaxies, which determines the morphological type of these objects. Several mechanisms are currently being discussed as generators of the observed spirals. We distinguish global modes due to gravitational instability [1, 2], mechanism of swing amplification [3, 4], role of different environments (small merging galaxies and tidal interactions) [5, 6], induced star formation [5, 7]. The central stellar bar can play a significant role. The processes in the cold gas components are important because young stars, gas and dust form spiral galactic patterns above all.

In this work, we study the phenomenon of so-called “rows” (straight segments), which are one of the characteristic details of spiral arms, and they are observed in approximately 5–10% S-galaxies [8, 9]. B. A. Vorontsov-Vel’yaminov was the first to detect the straightened segments in the arms of some galaxies [10], highlighting them in extragalactic nebulae images. The sequences of such straightened segments are also called polygonal structures [11]. Two catalogs of galaxies with rows contain 480 objects [8, 9]. Statistical analysis of the properties of galaxies with rows indicates that such objects are typical spiral galaxies [9].

Figure 1 shows examples of images of galaxies with rows in different spectral bands. Galaxies NGC 1566 (Spitzer Infrared Nearby Galaxies Survey) and NGC 6962 (UV image, GALEX) have a developed polygonal structure of both spiral arms, but in the second case, we see an almost ring-shaped pattern. Galaxies NGC 289 and NGC 1512 demonstrate rows examples in ultraviolet wavelengths in a gas disk outside the stellar component (orbiting space ultraviolet telescope The Galaxy Evolution Explorer, GALEX), where the gaseous medium is of low density and the disk is gravitationally stable. NGC 5156 is a more peculiar object when straightened parts of the spirals can be identified in the stellar disk according to the 2MASS data (The Two
Micron All-Sky Survey, short-wavelength infrared bands) for the main stellar population. We see rows in only one spiral arm of the galaxy IC 1562 according to the Digitized Sky Survey color composite images (DSS colored).

Computer simulation is the only opportunity for a theoretical study of non-stationary multicomponent processes in 3D galactic systems. The conditions for the formation of polygonal structures were investigated in numerical models for gas disks [12, 13, 14] and for stellar collisionless disks separately [9, 15, 16, 17]. In this paper, we describe the first results of numerical simulations of galactic rows, using a self-consistent model of stellar-gaseous galactic disks. The stellar disk dynamics is modeled by a collisionless system of $N_\star$ self-gravitating particles ($N$-body model). We use three-dimensional hydrodynamic equations with self-gravity in Lagrangian variables for the gas disk. The gravitational interaction between the two components (stars + gas) allows us to correctly describe the gravitational instability. We also consider the gravitational influence of a massive dark halo and a stellar bulge.

To calculate the self-gravity forces, we use the direct method of summation of the gravitational force, the so-called “particle-particle” method, which allows us to simulate a self-gravitating continuous medium with the least error due to the quadratic dependence on the particles number $N$ for the algorithm complexity ($O(N^2)$). Such an approach is the most accurate for calculating self-gravity, which is very important for modeling structures in spirals at small scales in the zone of arm break.

Figure 1. Galaxies with rows from the HyperLeda database (http://leda.univ-lyon1.fr/) and NASA/IPAC Extragalactic Database.
2. Mathematical models

The self-consistent model of the galactic stellar-gaseous disk is based on the numerical integration of the gas dynamics equations and the dynamic model of N-body gravitating system. The particles number in the collisionless component is \( N_\star \). We use the Smooth Particle Hydrodynamics (SPH) algorithm with \( N_p \) particles to simulate a gas component \( (N_{tot} = N_\star + N_p) \).

The stellar disk dynamics is determined by the system of motion equations for \( N_\star \) particles \([18, 15, 16]\):
\[
\frac{d^2 \mathbf{r}_i}{dt^2} = \mathbf{F}_c(\mathbf{r}_i) + \sum_{j=1, i \neq j}^{N_{tot}} \mathbf{f}_{ij} \quad (i = 1, \ldots, N_\star),
\]

where \( \mathbf{r}_i(t) \) is the radius vector of the \( i \)-th particle, \( \mathbf{f}_{ij} \) is the specific interaction force between \( i \)-th and \( j \)-th particles, \( \mathbf{F}_c \) is the external specific gravitational force from the spheroidal components (halo and bulge) (\( \mathbf{F}_c = \mathbf{F}_h + \mathbf{F}_b \)). The centrally symmetric volume density distributions in halo \( \rho_h(\mathbf{r}) \) and bulge \( \rho_b(\mathbf{r}) \) are fixed, creating the corresponding stationary gravitational potential.

The collisionlessness of stellar galactic disk is the most important property of this subsystem \([19]\). The small number of particles in the model \( (N_\star) \) compared with the number of real stars is a numerical simulation problem, for which the gravitational force is cut at small distances from the particle center
\[
\mathbf{f}_{ij} = -G \frac{m_j(\mathbf{r}_i - \mathbf{r}_j)}{(|\mathbf{r}_i - \mathbf{r}_j|^2 + r_c^2)^{3/2}},
\]

where \( G \) is the gravitational constant, \( m_j \) is the mass of the \( j \)-th particle, \( r_c \) is the cutoff radius. The small parameter \( r_c \) ensures collisionlessness of the system.

We use the initial equilibrium state for a galactic disk, for which the initial balance of forces is determined by the gravity of matter, the centrifugal force and the contribution of thermal motion. The last factor is characterized by the dispersions of the three velocity components in the cylindrical coordinate system \( c_r, c_p, c_z \) for stellar disk and the sound speed for gas \( c_s \) \([18, 20]\). We calculate the total gravitational force from stellar disk, gas disk, bulge and dark halo. We use the quasi-isothermal model for dark massive halo, for which the gravitational force is expressed by the formula
\[
\mathbf{F}_h(r) = -\frac{4\pi Ga^3 \rho_h}{r^2} \left\{ \frac{r}{a} - \arctg \left( \frac{r}{a} \right) \right\} \frac{\mathbf{r}}{r},
\]

where the spatial scale of halo \( (a) \) and the central volume density \( \rho_h \) determine the mass \( M_h = \rho_h \left\{ 4\pi a^3 \left[ R_h/a - \arctg(R_h/a) \right] \right\} \) inside the sphere with radius \( r \leq R_h \). The gravitational force from the stellar bulge is
\[
\mathbf{F}_b(r) = -\frac{4\pi Gb^3 \rho_b}{r^2} \left\{ \ln \left( \frac{r}{b} + \sqrt{1 + \frac{r^2}{b^2}} \right) - \frac{r/b}{\sqrt{1 + r^2/b^2}} \right\} \frac{\mathbf{r}}{r},
\]

where \( \rho_b \) is the central volume density, \( b \) is the core radius, \( \mathbf{F}_b(r) = -G M_b/r^2 \) for \( r > r_b^{\text{max}} \), \( r_b^{\text{max}} \) is the bulge truncation radius.

The gaseous disk dynamics is determined by the system of hydrodynamic equations in the form:
\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0,
\]
\[
\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \nabla) \cdot \mathbf{u} = \frac{1}{\rho} \nabla p - \nabla \Psi,
\]
\[
\frac{\partial e}{\partial t} + \mathbf{u} \nabla \cdot e = -\frac{p}{\rho} \nabla \mathbf{u},
\]
where $\mathbf{u}$ is the velocity vector, $\varrho$ is the volume density of the gas, $p$ is the pressure, $e$ is the internal energy, $\Psi$ is the gravitational potential. The ideal gas equation of state $e = \frac{p}{\gamma - 1}\varrho$ closes the system (5)—(7) ($\gamma$ is adiabatic index).

The total gravitational potential $\Psi$ is the sum of the contributions from gas $\Psi_g$, stars $\Psi_\star$ and two fixed components ($\Psi_h$ and $\Psi_b$):

$$\Psi = \Psi_g + \Psi_\star + \Psi_h + \Psi_b.$$

(8)

The particle’s circular velocity is $V_c = \sqrt{-r\frac{\partial \Psi}{\partial r}}$, determined by the total potential $\Psi$ (8).

3. Numerical algorithm

The basis of the Smooth Particle Hydrodynamics method is transform

$$S(r) = \int_\Omega S(r')\delta(|r - r'|)dr'$$

(9)

for each function $S = \{\varrho, e, \mathbf{u}\}$, which is included in the system of equations (5)—(7) ($\delta(|r - r'|)$ is Dirac function). The integration domain $\Omega$ is determined by the presence of mass in this space. Let’s rewrite the integral using the smoothing kernel $W$ with the smoothing length $h$ instead of the $\delta$-function, so that the condition

$$\lim_{h\to0} W(|r - r'|, h) = \delta(|r - r'|)$$

(10)

is satisfied.

The kernel function $W$ must satisfy the standard normalization condition:

$$\int_\Omega W(|r - r'|, h)dr' = 1.$$

(11)

The finite number of $N_p$ particles in the computational domain $\Omega$ dictates the transition from integration to summation:

$$\hat{S}(r) = \sum_{j=1}^{N_p} \frac{m_j}{\varrho(r_j)} S(r_j) W(|r - r_j|, h) ,$$

(12)

$$\nabla \hat{S}(r) = \sum_{j=1}^{N_p} \frac{m_j}{\varrho(r_j)} S(r_j) \nabla W(|r - r_j|, h) .$$

(13)

In accordance with the SPH-approach [21], the density of gas associated with the $i$-th gas particle, the equation of motion and the energy conservation equation can be written in the form (details can be found in [22]):

$$\varrho_i = \varrho(r_i) = \sum_{j=1}^{N_p} m_j W(|r_i - r_j|, h_{ij}) ,$$

(14)

$$\frac{d\mathbf{u}_i}{dt} = - \sum_{j=1,j\neq i}^{N_p} m_j \Pi_{ij} \nabla W_p(|r_i - r_j|, h_{ij}) + f_i^h + f_i^b + \sum_{j=1,j\neq i}^{N} f_{ij} ,$$

(15)
\[ \frac{d\psi_i}{dt} = \frac{1}{2} \sum_{j=1,j\neq i}^{N_p} m_j \Pi_{ij} (\mathbf{u}_i - \mathbf{u}_j) \cdot \nabla W_p (|\mathbf{r}_i - \mathbf{r}_j|, h_{ij}), \quad (16) \]

where \( W \) is the Monaghan smoothing kernel [21], and \( W_p \) is the smoothing kernel, used for the approximation of pressure forces [23, 22], and \( h_{ij} = 0.5 (h_i + h_j) \) is the effective smoothing length, where the smoothing length for each particle depends on its mass and density as \( h_i = 1.3 (m_i/\rho_i)^{1/3} \) [24, 22]. Tensor \( \Pi_{ij} = \frac{p_i}{\rho_i^2} + \frac{p_j}{\rho_j^2} + \nu_{ij}^a \) is the symmetric approximation of the pressure forces, and \( \nu_{ij}^a \) is the artificial viscosity [22].

For the numerical integration of the differential equations (15) and (16) we use the predictor-corrector scheme of second order accuracy (the so-called leapfrog method). To calculate the gravitational forces, we use the direct particle-particle algorithm. The leapfrog method allows us to simulate the dynamics of the disk systems even in the cartesian coordinate system. In paper [25] have shown that use of the leapfrog method in double precision simulations conserves the total energy, momentum, and angular momentum of the equilibrium system with \( N \) particles with an accuracy of \( 10^{-15} \) years in 5\( \times \)1020 particles with an accuracy of \( 10^{-5} \), \( 10^{-15} \), and \( 10^{-13} \), respectively. For single-precision simulations the accuracy of conservation of the above mentioned quantities is equal to \( 10^{-3} \), \( 10^{-2} \), and \( 10^{-3} \), respectively [25].

Details of the realization of the predictor-corrector method are described in papers by [22, 25]. We outline here the coordination procedure for simulations of gaseous and stellar disks. For the gaseous disk, the integration time step \( \Delta t_g (t) \) is limited by the stability condition of the SPH-algorithm [22], while for the stellar disk, the time step is limited by the condition of applicability of Newtonian gravity, namely the time step in the collisionless simulations should be greater than the time of light propagation in the region of the simulations \( \Delta t_s \geq \Delta t_{crit} \). Here, \( \Delta t_{crit} \) is the propagation time of light within the region. If we choose \( \Delta t_s = \Delta t_g \), angular momentum conservation is satisfied with an accuracy of \( 10^{-15} \). In this case, however, the condition of applicability of Newtonian gravity fails, and the total integration time of the problem increases by factors of tens to hundreds. We therefore choose the value \( \Delta t_s = \Delta t_{crit} \approx 2 \times 10^5 \) years in our simulations. Calculation of the gravitational interaction between all particles was carried out once for a time interval \( (t, t + \Delta t_s) \) using expression (2). For the gaseous disk, the number of time steps is large \( (n_g \gg 1) \) for each time interval \( (t, t + \Delta t_s) \), so that the gravitational force vector is constant during each time interval, which leads to conservation of angular momentum to \( 10^{-2} \) accuracy. The following correction procedure for the velocities of gaseous particles at the last time step \( \Delta t_g (t + \Delta t_s) \) allows us to increase the accuracy of angular momentum conservation to \( 10^{-8} \):

\[
\begin{align*}
    u_x(t + \Delta t_s) &= u_x(t) + \frac{x\Delta u_r - y\Delta u_\varphi}{R}, \\
    u_y(t + \Delta t_s) &= u_y(t) + \frac{x\Delta u_\varphi + y\Delta u_r}{R}, \\
    u_z(t + \Delta t_s) &= u_z(t) + \tau[F_z(t + \Delta t_s) - F_z(t)],
\end{align*}
\]

where \( R = \sqrt{x^2 + y^2} \), \( \Delta u_r = \tau[F_r(t + \Delta t_s) - F_r(t)] \), \( \Delta u_\varphi = \tau[F_\varphi(t + \Delta t_s) - F_\varphi(t)] \), \( \tau = 0.5(\Delta t_s - \Delta t_g (t + \Delta t_s)) \), \( (F_r, F_\varphi, F_z) \) are the components of the total gravitational force in the cylindrical system of coordinates.

For multiple GPUs the details of a parallel OpenMP-CUDA implementation of SPH and N-body numerical algorithms are presented in the following papers [22, 25]. Finding the nearest neighbors is the most resource-intensive part of the SPH method. Our implementation of finding the nearest neighbors is based on the hierarchical grid method using a cascading sorting algorithm for parallel computation of partial sums in the CUDA block. We use...
Figure 2. Dimensionless radial profiles in the stellar and gas disks at the initial moment of time \((t = 0)\): 1 — \(Q_T(r)\), 2 — \(V_c(r)\), 3 — \(c_r(r)\), 4 — \(c_\varphi(r)\); \(r_d\) — optical radius.

parallel computing software on Nvidia Tesla GPUs (K20, K40, K80) [22]. The main problem of direct summation (1) is the large computational complexity (\(\sim N^2\)), but new features of graphics processors allow calculations with the number of particles \(N > 10^6\) [25].

4. Galactic rows in numerical simulation

Let us discuss some results of modeling the dynamics of the self-consistent stellar-gaseous disk. We use the following dimensionless parameters for our base model, which corresponds to our Galaxy [26]: \(M_d = 1\) is the mass of the stellar disk, \(r_d = 0.25\) is the exponential scale of the stellar disk, \(M_g = 0.15\) is the mass of the gas subsystem, \(M_h = 3\) is the mass of the dark halo, \(a = 0.3\), \(M_b = 0.3\) is the mass of the bulge, \(b = 0.0167\), \(r_\text{max}^b = 0.1\) (See formulas (3) and (4)). Figure 2 shows the radial distributions of the model parameters at the initial time \((t = 0)\). In this model, the characteristic period of rotation of the stellar disk periphery is about \(t_{\text{dyn}} \approx 3\).

The important role in our models is played by gravitational instability, which is traditionally characterized by the Toomre parameter

\[
Q_T = \frac{c_r(r)\kappa(r)}{3.36\sigma(r)},
\]

\(\sigma\) is surface density, \(\kappa\) is epicyclic frequency. The initial distributions of the our model parameters allow gravitational instability, the nonlinear development of which leads to the generation of a spiral pattern. At some time intervals, we can find characteristic fractures of spiral arms with straightened lines that are morphologically close to the observed rows in galaxies.

We have a non-stationary global spiral pattern with the arms number \(m = 3, 4\) in our numerical model (figure 3). The characteristic feature of developed spiral structure (such as grand design) is the formation of the transient rows system that appear only at short time intervals in the stellar and gas subsystems (figure 4). We analyzed the existence duration of polygonal structures in the model for stellar and gas disks separately (figure 5). The time intervals with rows occupy only an insignificant part of the disk evolution (figure 6). To estimate the duration of the existence of rows, we considered a large time interval \(\tau_0\), which is equal to several periods of disk rotation \(t_{\text{dyn}}\), and we determined the frequencies of occurrence of the rows \(\nu_r = \tau_r/\tau_0\) (\(\tau_r\) is the total time of the existence of galactic rows in the interval \(\tau_0\)) for the gas \(\nu_r^{\text{gas}}\) and the stars \(\nu_r^{\text{star}}\).
Figure 3. Time dependences of the amplitudes of the Fourier harmonics with different azimuth number $m$. The solid black line corresponds to the second mode ($m = 2$), the dashed blue line is the three-arm mode ($m = 3$), the dotted red line is $m = 4$, the solid green line is $m = 5$.

The most basic properties of rows in our numerical model are:

(i) The rows are observed most often at the initial stage after the formation of the spiral pattern. The frequencies of occurrence of $\nu_{\text{gas}}$ and $\nu_{\text{star}}$ decrease 2–3 times after $t \approx 25$ (figure 7). This is apparently due to a more powerful spiral pattern after its appearance as a result of gravitational instability. Then the amplitude of the spiral wave decreases more strongly in the stellar disk in comparison with the gas component.

(ii) The frequency of the appearance of rows in the gas is on average 2–4 times higher than in the stellar disk.

(iii) The polygonal structure in the stellar disk consists of 1–3 rows in contrast to the gas-dynamic models in which the global rows system is formed, covering the entire disc [12].

Figure 4. Dynamics of surface density perturbations in stellar disk (top) and gas disk (bottom) at different time $t$. 
Figure 5. Logarithm perturbations of the surface density in the self-consistent stellar-gaseous disk model at time \( t = 9 \) (only the positive part of the perturbation is shown on the right). Top — the perturbation of the gas component density, bottom — the perturbation of the stellar disk density.

(iv) As a rule, the galactic rows formation in the gaseous and stellar components is not simultaneous, although there is an effect of their coordinated appearance. The numerical experiments do not allow us to detect which component is the main one for the birth of rows.

The straight segments may appear first both in the gas disk and in the stellar component.

(v) Typical values of the frequencies are equal to \( \nu_{\text{gas}}^r = 0.07 - 0.13 \), \( \nu_{\text{star}}^r = 0.01 - 0.05 \) in the considered models at the late stages of evolution.

5. Conclusion

The problems of determining the physical mechanisms of generation of the spiral patterns and the various peculiar features of spiral arms are very important for galaxy physics in recent years [1, 2, 5, 15, 16, 27]. We use the new numerical model to simulate the dynamics of a stellar-gaseous galactic disk to study the mechanism of forming the rows in spiral galactic patterns.

Unlike previous studies, our analysis is based on the self-consistent evolution of both the cold gas and the hotter collisionless stellar component, for which the dispersion of radial velocities \( (c_r) \) is large compared to the sound speed in the gas disk \( (c_s) \). We are building a model of a rather large galaxy of our Milky Way type, the initial state of which is equilibrium, but it is gravitationally unstable. This leads to the formation of a spiral structure, both in the gas and in the stellar disks. Although the relative fraction of gas is small (about 15% of stars), the dynamic effect of gas is significant, and it can even be decisive, because the condition \( \sigma_\star \ll \sigma_{\text{gas}} \) is performed for the perturbations of surface densities. As a result, the dynamic influence of gas and stars can be comparable: \( c_s^2 \sigma_{\text{gas}} \sim c_\star^2 \sigma_\star \), despite \( c_r^2 \gg c_s^2 \). This factor leads to additional
destabilization of the two-component system. If each of the two components separately is stable, then the whole system is gravitationally unstable.

Our numerical experiments allow us to trace the evolution of the galactic system over dozens of rotation periods, revealing morphological changes in spiral arms. The rows are formed in both the gas and stellar subsystems. It is important to note that the analysis of the catalogs of galaxies with rows [8, 9] gives for the frequency of occurrence of such objects $\simeq 6\%$, which is close to our estimates of the relative lifetime of rows according to the results of numerical simulation. Apparently, the structures with rows belong to the class of transient formations [14].

We emphasize that all the quantitative estimates given here correspond to certain model of the galaxy and strongly depend on the choice of a large number of free parameters that determine the radial profiles of physical characteristics. The set of our parameters is quite typical, but
their deviations can both enhance the effect of the rows birth and suppress their appearance. Additional complicating factors are the multiphase interstellar medium and the star formation processes in real galactic systems, which are not considered in our approach.

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