COMPLETE NON-COMPACT GRADIENT RICCI SOLITONS WITH NONNEGATIVE RICCI CURVATURE

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Abstract. In this paper, we give a delay estimate of scalar curvature for a complete non-compact expanding (or steady) gradient Ricci soliton with nonnegative Ricci curvature. As an application, we prove that any complete non-compact expanding (or steady) gradient Kähler-Ricci solitons with positively pinched Ricci curvature should be Ricci flat. The result answers a question in case of Kähler-Ricci solitons proposed by Chow, Lu and Ni in a book.

1. Introduction

Ricci soliton plays an important role in the study of Hamilton’s Ricci flow, in particular in the singularities analysis of Ricci flow [15], [3], [21]. In case of shrinking gradient Ricci solitons with positive curvature, Hamilton proved that the solitons should be isometric to a standard sphere in $\mathbb{R}^3$ in two dimensional case [15]. Perelman generalized Hamilton’s result to three dimensional case [21]. Later on, Nabor proved that any four-dimensional shrinking gradient Ricci soliton with positive bounded curvature operator should be a standard sphere in $\mathbb{R}^5$ [17]. On the other hand, Perelman and Brendle proved that any steady gradient Ricci soliton with nonnegative sectional curvature should be a Bryant’s soliton in case of 2-dimension and 3-dimension, respectively [21], [6], [2], [1]. However, to author’s acknowledge, there is rarely understanding in case of expanding gradient Ricci solitons even for lower dimensional manifolds. For example, how to classify complete non-compact gradient expanding (or steady) Ricci solitons under a suitable curvature condition. The purpose of this paper is to give a rigidity theorem for a class of expanding (or steady) gradient Kähler-Ricci solitons with nonnegative Ricci curvature.

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**Definition 1.1.** A complete Riemannian metric $g$ on $M$ is called a gradient Ricci soliton if there exists a smooth function $f$ (which is called a defining function) on $M$ such that

$$R_{ij} + \rho g_{ij} = \nabla_i \nabla_j f,$$

where $\rho \in \mathbb{R}$ is a constant. The gradient Ricci soliton is called expanding, steady and shrinking according to the sign $\rho >, =, < 0$, respectively.

For simplicity, we normalize $\rho = 1, 0, -1$. In addition, $g$ is a Kähler metric on a complex manifold $M$, we call $g$ is a Kähler-Ricci soliton. Since $\bar{\partial} f$ induces a holomorphic vector field on $M$, (1.1) was usually written in a complex version,

$$R_{i\bar{j}} + \rho g_{i\bar{j}} = \nabla_i \nabla_{\bar{j}} f,$$

A gradient soliton $(M, g, f)$ is called complete if $g$ and $\nabla f$ are both complete. It is known that the completeness of $(M, g)$ implies the completeness of $\nabla f$ [20]. Throughout this paper, we always assume the soliton is complete. If there is a point $o \in M$ such that $\nabla f(o) = 0$, we call $o$ an equilibrium point of $(M, g)$. By studying the existence of equilibrium points, we prove the boundedness of scalar curvature of $g$.

**Theorem 1.2.** Let $(M, g)$ be a complete non-compact expanding gradient Ricci soliton with nonnegative Ricci curvature or a complete non-compact steady gradient Kähler-Ricci soliton with nonnegative bisectional curvature and positive Ricci curvature. Then the scalar curvature of $g$ is bounded and it attains the maximum at the unique equilibrium point.

The proof of Theorem 1.2 will be given in case of expanding Ricci solitons in next section. For the steady Ricci solitons, the proof for the existence of equilibrium points is a bit different, although the boundedness of scalar curvature is directly from an identity (1.2). We will use a result of local convergence for Kähler-Ricci flow by Chau and Tam to prove the existence in Section 5 [11].

Theorem 1.2 will be applied to prove the following rigidity theorem for Kähler-Ricci solitons with nonnegative Ricci curvature.

**Theorem 1.3.** Let $(M^n, g)$ be a complete non-compact gradient Kähler-Ricci soliton with non-negative Ricci curvature. Suppose that there exists a point $p \in M$ such that the scalar curvature $R$ of $g$ satisfies

$$\frac{1}{\text{vol}(B_r(p))} \int_{B_r(p)} R \, dr \leq \frac{\varepsilon(r)}{1 + r^2}, \text{ if } g \text{ is expending;}$$
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or

\[
\frac{1}{\text{vol}(B_r(p))} \int_{B_r(p)} R \, dr \leq \frac{\varepsilon(r)}{1 + r}, \text{ if } g \text{ is steady,}
\]

where \( \varepsilon(r) \to 0 \) as \( r \to \infty \). Then \( g \) is Ricci-flat. Moreover, \((M, g)\) is isometric to \( \mathbb{C}^n \) if \( g \) is expanding.

We note that under the condition of nonnegative sectional curvature (or nonnegative holomorphic bisectional curvature for Kähler manifolds) several rigidity theorems were obtained in [15], [19], [22], etc. For Ricci solitons, we are able to use the Ricci flow to weaken the condition of curvature to nonnegative Ricci curvature.

As a corollary, we obtain a version of Theorem 1.3 under the pointed-wise Ricci decay condition.

**Theorem 1.4.** Let \((M^n, g)\) be a complete non-compact gradient Kähler-Ricci soliton with non-negative Ricci curvature. Suppose that \( g \) satisfies

\[
R(x) \leq \frac{\varepsilon(r(x))}{1 + r(x)^2} (\varepsilon(r) \to 0, \text{ as } r \to \infty), \text{ if } g \text{ is expending;}
\]

or

\[
R(x) \leq \frac{C}{1 + r(x)^{2n+\epsilon}} \text{ for some } C, \epsilon > 0, \text{ if } g \text{ is steady.}
\]

Then \( g \) is Ricci-flat. Moreover, \((M, g)\) is isometric to \( \mathbb{C}^n \) if \( g \) is expanding.

In case of steady solitons in Theorem 1.4, if we assume that \((M, g)\) has nonnegative bisectional curvature instead of nonnegative Ricci curvature, then the condition (1.6) can be weakened as

\[
R(x) \leq \frac{C}{1 + r(x)^{1+\epsilon}}.
\]

In fact, we can prove that \((M, g)\) is isometric to \( \mathbb{C}^n \) by Theorem 1.2, see Proposition 5.1. Proposition 5.1 is an analogy of Hamilton’s result for Kähler manifolds [13].

A Riemannian metric is called with property of positively pinched Ricci curvature if there is a uniform constant \( \delta > 0 \) such that \( \text{Ric}(g) \geq \delta Rg \) [13], [15]. It was proved that the scalar curvature of complete non-compact expanding (or steady) gradient Ricci solitons with positively pinched Ricci curvature has exponential decay (cf. Theorem 9.56, [6]). Thus as a direct consequence of Theorem 1.3, we obtain

**Corollary 1.5.** Non-trivial complete non-compact expanding or steady gradient Kähler-Ricci soliton with positively pinched Ricci curvature doesn’t exist for \( n \geq 2 \).
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Corollary 1.5 answers a question in case of Kähler-Ricci solitons proposed by Chow, Lu and Ni in their book [6] (cf. page 390). They asked whether there exists an expanding gradient Ricci soliton with positively pinched Ricci curvature when \( n \geq 3 \).

Theorem 1.3 and 1.4 will be proved in Section 4 and Section 5 according to expanding or steady solitons, respectively.

2. Boundedness of scalar curvature–I

In this section, we prove the boundedness of scalar curvature in case of expanding Ricci solitons. Let \((M^n, g)\) be a Riemannian manifold. In local coordinates \((x^1, x^2, \cdots, x^n)\), curvature tensor \(Rm\) of \(g\) is defined by

\[
Rm(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) \frac{\partial}{\partial x^k} \triangleq \sum R^{l}_{ijk} \frac{\partial}{\partial x^l}
\]

and \(R^{ijkl} \triangleq \sum_{lm} g^{lm} R^m_{ijkl} \). Then the Ricci curvature is given by

\[
R_{jk} = \sum R^i_{ijk}.
\]

Thus by the commutation formula,

\[
(\nabla_i \nabla_j - \nabla_j \nabla_i) \alpha_{k_1 \cdots k_r} = - \sum_{l=1}^{r} R^{m}_{ijkl} \nabla_l f.
\]

we get from the Bianchi identity,

\[
2 \sum \nabla_i R_{ij} = \nabla_i R = \nabla_j R.
\]

Let \((M^n, g, f)\) be an expanding gradient Ricci soliton and \(\phi_t\) be a family of diffeomorphisms generated by \(-\nabla f\). Then the induced metrics \(g(t) = \phi_t^* g\) satisfy

\[
\frac{\partial}{\partial t} g = -2 \text{Ric}(g) - 2g.
\]

\((2.3)\) is equivalent to

\[
R_{ij}(t) + g_{ij}(t) = \nabla_i \nabla_j f(t),
\]

where \(f(t) = \phi_t^* f\) and \(\nabla\) is taken w.r.t \(g(t)\).

Lemma 2.1.

\[
\frac{\partial}{\partial t} R = 2 \text{Ric}(\nabla f(t), \nabla f(t)).
\]

Proof. Differentiating \((2.4)\) on both sides, we have

\[
\nabla_k R_{ij} = \nabla_k \nabla_i \nabla_j f.
\]

It follows from \((2.1)\),

\[
\nabla_i R_{jk} - \nabla_j R_{ik} = - \sum R_{ijkl} \nabla_l f.
\]
Thus by (2.2), we get
\[ \nabla_j R = -2R_{jl} \nabla_l f. \]
(2.5)
Hence
\[ \partial_t R(x, t) = \partial_t R(\phi_t(x), 0) = -\langle \nabla R, \nabla f \rangle = 2\text{Ric}(\nabla f, \nabla f). \]
\[ \square \]

Let \( B_r(o, t) \) be a \( r \)-geodesic ball centered at \( o \in M \) w.r.t \( g(t) \). Then

**Lemma 2.2.** Let \( g(x, t) \) be a solution of (2.3) with nonnegative Ricci curvature for any \( t \in (0, \infty) \). Then for any \( r > 0 \) and \( \delta > 0 \), there exists a \( T_0 = T_0(r, \delta) > 0 \) such that \( B_r(o, 0) \subset B_\delta(o, t) \) for any \( t \geq T_0 \).

**Proof.** By (2.3), it is easy to see that
\[ \frac{d|v|^2}{dt} \leq -2|v|^2, \quad \forall t \geq 0, \]
where \( v \in T_p^{(0)} M \) for any \( p \in M \). Then
\[ |v|^2 \leq e^{-2t}|v|^2_0. \]
Connecting \( o \) and \( p \) by a minimal geodesic curve \( \gamma(s) \) with an arc-parameter \( s \) w.r.t the metric \( g(x, 0) \) in \( B_r(o, 0) \), we get
\[ d_t(o, p) \leq \int_0^1 |\gamma'(s)|_0 \, ds \leq \int_0^1 |\gamma'(s)|_0 e^{-2t} \, ds \leq r e^{-2t}, \]
where \( l \) is the length of \( \gamma(s) \). Therefore, by taking \( t \) large enough. we see that \( B_r(o, 1) \subset B_\delta(o, t) \). \[ \square \]

Taking an integration along a geodesic curve on both sides of (1.1), on can show that \( f(x) \geq \frac{r(x)}{4} \) under the assumption of nonnegative Ricci curvature. This implies that \( f(x) \) attains the minimum at some point \( o \in M \). Thus \( \nabla f(o) = 0 \). Moreover the equilibrium point \( o \) is unique. This is because, if there is another equilibrium point \( p \), then \( \phi_t(o) = o \) and \( \phi_t(p) = p \). In particular \( d_t(o, p) = d_0(o, p) \). On the other hand, by (2.6), \( d_t(o, p) \leq e^{-2t}d_0(o, p), \forall t > 0 \). Hence, \( d_0(o, p) = 0 \), and consequently \( o = p \).

Now we begin to prove Theorem 1.2.

**Proof of Theorem 1.2 (the expanding case).** Let \( o \) be the unique equilibrium point. Then by Lemma 2.2, for any \( r > \delta > 0 \), there exists \( T_0 \) such that
\[ B_r(o, 0) \subset B_\delta(o, t), \quad \forall t \geq T_0. \]
On the other hand, by Lemma 2.1 we see that \( R(x, t) \) is nondecreasing in \( t \). Thus
\[ \sup_{x \in B_r(o, 0)} R(x, 0) \leq \sup_{x \in B_\delta(o, t)} R(x, t), \quad \forall r > 0, \delta > 0. \]
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Note that \( \phi_t : (M^n, g(t)) \to (M^n, g(0)) \) are a family of isometric deformations. It follows

\[
\sup_{x \in B_\delta(o,0)} R(x,0) = \sup_{x \in B_\delta(o,t)} R(x,t), \quad \forall \; \delta > 0.
\]

Hence, combining (2.7) and (2.8), we get

\[
\sup_{x \in B_r(o,0)} R(x,0) = \sup_{x \in B_\delta(o,0)} R(x,0), \quad \forall \; r > \delta > 0.
\]

Let \( \delta \to 0 \), we derive

\[
\sup_{x \in B_r(o,0)} R(x,0) = R(o,0), \quad \forall \; r > 0.
\]

This proves the theorem. \( \square \)

3. Expanding Kähler-Ricci solitons

In this section, we prove both Theorem 1.3 and Theorem 1.4 in case of expanding Kähler-Ricci solitons. Theorem 1.4 is a consequence of Theorem 1.3 by the following lemma.

Lemma 3.1. Let \((M, g)\) be an expanding gradient Kähler-Ricci soliton which satisfies (1.3) in Theorem 1.4. Then there exists a function \( \varepsilon'(r) \) (\( \varepsilon'(r) \to 0 \) as \( r \to \infty \)) such that

\[
\frac{1}{\text{vol}(B_r(p))} \int_{B_r(p)} R \text{dv} \leq \frac{\varepsilon'(r)}{1 + r^2}.
\]

Proof. Note that an expanding Ricci soliton with nonnegative Ricci curvature has maximal volume growth (cf. [7] or [12]). Namely, there exists a uniform constant \( \delta > 0 \) such that

\[
\text{vol}(B_r(p)) \geq \delta r^{2n}.
\]

On the other hand, by the volume comparison theorem, we have

\[
\text{vol}(\partial B_r(p)) \leq n \frac{\text{vol}(B_r(p))}{r} \leq C r^{2n-1},
\]

where \( C \) is a uniform constant. Thus

\[
\frac{1}{\text{vol}(B_r(p))} \int_{B_r(p)} R \text{dv} = \frac{1}{\text{vol}(B_r(p))} \int_0^r \frac{1}{\text{vol}(B_s(p))} \int_{\partial B_s(p)} R \text{d}\sigma \leq \frac{1}{\delta r^{2n}} \int_0^r \frac{\varepsilon(s)}{1 + s^2} \text{vol}(\partial B_s(p)) \text{ds} \leq \frac{1}{\delta r^{2n}} \int_0^r C(s+1)^{2n-3} \varepsilon(s) \text{ds} \leq \frac{\varepsilon'(r)}{1 + r^2},
\]

where \( \varepsilon'(r) \to 0 \) as \( r \to 0 \). \( \square \)
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Proof of Theorem 1.3 (the expanding case). Ricci-Flat: Let $\phi_t$ be a family of diffeomorphisms generated by $-\nabla f$. Let $g(t) = \phi_t^* g$ and $\tilde{g}(\cdot, t) = tg(\cdot, \ln t)$. Then $\tilde{g}(\cdot, t)$ satisfies

$$(3.1) \begin{cases} \frac{\partial}{\partial t} \tilde{g}_{ij}(x, t) = -\tilde{R}_{ij}(x, t) \\ \tilde{g}_{ij}(x, 1) = g_{ij}(x). \end{cases}$$

Let $F(x, t) = \ln \det(g_{ij}(x, t)) - \ln \det(g_{ij}(x, 1))$. By (3.1), it is easy to see $F(x, t) = -\int_1^t \tilde{R}(x, s)ds \leq 0$.

Since $t\tilde{R}(o, t) = R(o, \ln t) = R(o, 0)$, and $t\tilde{R}(x, t) = R(x, \ln t) \leq R(o, 0)$, where $o$ is the equilibrium point of $M$, by Theorem 1.2, $F$ is uniformly bounded on $x$. Moreover, we have

$$(3.2) \quad M(t) \doteq -\inf_{x \in M} F(x, t) = R(o, 0) \ln t.$$ 

In the following, we shall estimate the upper bound of $M(t)$ by using the Green integration as in [24] (also see [18]).

By a direct computation, we have

$$(3.3) \quad \Delta_1 F(x, t) = \tilde{R}(x, 1) - g_{ij}(x, 1)\tilde{R}_{ij}(x, t) \leq \tilde{R}(x, 1) + \frac{\partial}{\partial t} e^{F(x, t)},$$

where the Laplace $\Delta_1$ is w.r.t $\tilde{g}(x, 1)$. Let $G_r(x, y)$ be a positive Green’s function with zero boundary value w.r.t $\tilde{g}(x, 1)$ on $\tilde{B}_r(x, 1)$. Note that

$$\int_{\tilde{B}_r(x_0, 1)} \frac{\partial G_r(x_0, y)}{\partial \nu} dy = -1 \quad \text{and} \quad \frac{\partial G_r(x_0, y)}{\partial \nu} \leq 0.$$ 

By integrating (3.3) on both sides, we have

$$\int_{\tilde{B}_r(x_0, 1)} G_r(x_0, y)(1 - e^{F(x, t)})dy$$

$$\leq \int_1^t ds \int_{\tilde{B}_r(x_0, 1)} G_r(x_0, y) (\Delta_1 F(x, s))dy$$

$$+ t \int_{\tilde{B}_r(x_0, 1)} G_r(x_0, y) \tilde{R}(y, 1)dy$$

$$= \int_1^t (F(x_0, s) + \int_{\tilde{B}_r(x_0, 1)} \frac{\partial G_r(x_0, y)}{\partial \nu} F(x, s)dy) ds$$

$$+ t \int_{\tilde{B}_r(x_0, 1)} G_r(x_0, y) \tilde{R}(y, 1)dy$$

$$(3.4) \quad \leq t \left( M(t) + \int_{\tilde{B}_r(x_0, 1)} G_r(x_0, y) \tilde{R}(y, 1)dy \right).$$
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On the other hand, by the Green function estimate (cf. Lemma 1.1 in [25]),

\[ G_r(x, y) \geq C_1^{-1} \int_{d(x,y)}^r \frac{s}{\text{vol}(B_s(x,1))} \, ds, \quad \forall \, y \in \hat{B}_r(x,1), \]

where \( C_1 \) is a uniform constant, we get

\[ \int_{\hat{B}_r(x_0,1)} G_r(x_0, y)(1 - e^{F(x,t)}) \, dv \]
\[ \geq C_1^{-1} \int_0^{\frac{r}{2}} \left( \int_0^r \frac{s}{\text{vol}(B_s(x_0,1))} \, ds \right) \left( \int_{\partial \hat{B}_r(x_0,1)} (1 - e^{F(x,t)}) \, d\sigma \right) \, d\tau \]
\[ \geq C_1^{-1} \int_0^{\frac{r}{2}} \left( \int_0^r \frac{s}{\text{vol}(B_s(x_0,1))} \, ds \right) \left( \int_{\partial \hat{B}_r(x_0,1)} (1 - e^{F(x,t)}) \, d\sigma \right) \, d\tau \]
\[ \geq \frac{C_2^{-1} r^2}{\text{vol}(B_{\frac{r}{2}}(x_0,1))} \int_{B_{\frac{r}{2}}(x_0,1)} (1 - e^{F(x,t)}) \, dv \]
\[ \geq \frac{eC_2^{-1} r^2}{\text{vol}(B_{\frac{r}{2}}(x_0,1))} \int_{B_{\frac{r}{2}}(x_0,1)} -F(x,t) \, dv. \]

It follows

\[ \frac{r^2}{\text{vol}(B_{\frac{r}{2}}(x_0,1))} \int_{B_{\frac{r}{2}}(x_0,1)} (-F(x,t)) \, dv \]
\[ \leq C_3(1 + M(t)) \int_{B_r(x_0,1)} G_r(x_0, y)(1 - e^{F(x,t)}) \, dv \]

Hence by (3.4), we derive

\[ \frac{r^2}{\text{vol}(B_{\frac{r}{2}}(x_0,1))} \int_{B_{\frac{r}{2}}(x_0,1)} (-F(x,t)) \, dv \]
\[ \leq C_3(1 + M(t)) \left( M(t) + \int_{\hat{B}_r(x_0,1)} G_r(x_0, y)\hat{R}(y,1) \, dv \right). \]

By (3.3), we have \( \Delta_1(-F(x,t)) \geq -\hat{R}(x,1) \). Then by the mean value inequality (cf. Lemma 2.1 of [18]), we see

\[ -F(x_0,t) \leq \frac{C(n)}{\text{vol}(B_{\frac{r}{2}}(x_0,1))} \int_{B_{\frac{r}{2}}(x_0,1)} (-F(x,t)) \, dv \]
\[ + \int_{B_{\frac{r}{2}}(x_0,1)} G_{\frac{r}{2}}(x_0, y)\hat{R}(y,1) \, dv. \]

Hence, to get an upper bound of \(-F(x_0,t)\), by (3.5) and (3.6), we shall estimate the integral \( \int_{B_{\frac{r}{2}}(x_0,1)} G_{\frac{r}{2}}(x_0, y)\hat{R}(y,1) \, dv. \)
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Recall the Li-Yau’s estimate for the Green function: There exits a positive Green’s function $G(x, y)$ such that (cf. Theorem 5.2 in [16])

\begin{equation}
C(n)^{-1} \int_{d^2(x,y)}^{+\infty} \frac{dt}{\text{vol}(\hat{B}_t(x))} \leq G(x, y) \leq C(n) \int_{d^2(x,y)}^{+\infty} \frac{dt}{\text{vol}(\hat{B}_t(x))}.
\end{equation}

Then

\begin{align*}
\int_{B_r(x_0,1)} G_r(x_0, y) \hat{R}(y, 1) dv & \leq \int_0^r ds \int_{\partial B_s(x_0,1)} G(x_0, y) \hat{R}(y, 1) d\sigma \\
& \leq C(n) \int_0^r ds \left( \int_{\partial B_s(x_0,1)} \hat{R}(y, 1) d\sigma \int_{d^2(x,y)}^{+\infty} \frac{dt}{\text{vol}(\hat{B}_t(x))} \right) \\
& = C(n) \left( \int_{r^2}^{+\infty} \frac{dt}{\text{vol}(\hat{B}_t(x))} \int_{B_r(x_0,1)} \hat{R} dv \right) + \int_0^r \frac{s}{\text{vol}(B_s(x_0,1))} \int_{B_s(x_0,1)} \hat{R} dv ds, \quad \forall \ r > 0.
\end{align*}

(3.8)

Since $(M, g)$ has the maximal volume growth, there exists a uniform constant $\delta > 0$ such that

\begin{equation}
\frac{\text{vol}(B_s(x))}{\text{vol}(B_t(x))} \geq \delta \left( \frac{s}{t} \right)^{2n}, \quad \forall \ s \geq t \geq c_0.
\end{equation}

It follows

\begin{align*}
\int_{d^2(x,y)}^{+\infty} \frac{dt}{\text{vol}(\hat{B}_t(x))} & \leq C_4 \frac{d^2(x,y)}{\text{vol}(B(d(x,y))(x))}, \quad \forall \ d(x,y) \geq c_0.
\end{align*}

Hence, we get from (3.8),

\begin{align*}
\int_{B_r(x_0,1)} G_r(x_0, y) \hat{R}(y, 1) dv & \leq C(n) \left( \frac{r^2}{\text{vol}(B_r(x_0,1))} \int_{B_r(x_0,1)} \hat{R} dv \right) + \int_0^r \frac{s}{\text{vol}(B_s(x_0,1))} \int_{B_s(x_0,1)} \hat{R} dv ds, \quad \forall \ r > 0.
\end{align*}

(3.9)

By the volume comparison theorem and the condition (1.5), we have

\begin{align*}
\frac{1}{\text{vol}(B_r(o))} \int_{B_r(o)} R(x) dv & \leq \frac{\text{vol}(B_{r+d}(p))}{\text{vol}(B_r(o))} \cdot \frac{1}{\text{vol}(B_{r+d}(p))} \int_{B_{r+d}(p)} R(x) dv \\
& \leq \frac{\text{vol}(B_{r+2d}(o))}{\text{vol}(B_r(o))} \cdot \frac{\varepsilon(r + d)}{1 + (r + d)^2} \\
& \leq \left( \frac{r + 2d}{r} \right)^{2n} \frac{\varepsilon(r + d)}{1 + (r + d)^2}.
\end{align*}
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where \( d = d(o,p) \). Then there exists another function \( \varepsilon'(r) \) (\( \varepsilon'(r) \to 0 \) as \( r \to \infty \)) such that

\[
\frac{1}{\text{vol}(B_r(o))} \int_{B_r(o)} R(x) \text{dv} \leq \frac{\varepsilon'(r)}{1 + r^2}.
\]

Thus inserting the above inequality into (3.9), we derive at

\[
(3.10) \quad \int_{B_r(o,1)} G_r(o, y) \tilde{R}(y, 1) \text{dv} \leq \varepsilon''(r) + \varepsilon''(r) \ln(1 + r^2),
\]

where \( \varepsilon''(r) \to 0 \) as \( r \to \infty \).

Combining (3.4), (3.5) and (3.10), it is easy to see

\[
-F(o, t) \leq r^{-2}C(n) t(M(t) + 1) \left( M(t) + C_5 \varepsilon''(r) + C_6 \varepsilon''(r) \ln(1 + r^2) \right) + \left( C_7 \varepsilon''(r) + C_8 \varepsilon''(r) \ln(1 + r^2) \right), \quad \forall r > 0.
\]

Note that \( -F(o, t) = M(t) = R(o, 0) \ln t \). Then by taking \( r = t \), we obtain

\[
R(o, 0) \ln t \leq C_9 \varepsilon''(t) + C_{10} \varepsilon''(t) \ln(1 + t) + C_{11} \ln t, \quad \forall t \geq 1.
\]

Dividing by \( \ln t \) on both sides of the above inequality and letting \( t \to \infty \), we deduce \( R(o, 0) = 0 \). Hence we prove that \( g \) is Ricci flat.

**Flatness:** We shall further prove that \( g \) is a flat metric on \( \mathbb{C}^n \). Note that

\[
(3.11) \quad g = \text{hess} f
\]

since \( g \) is Ricci flat. Then \( f \) is strictly convex and \( f \) attains the minimum at \( o \). By a direct computation, we have

\[
\langle \nabla f, X \rangle = (\nabla df)(\nabla f, X) = \langle \nabla_{\nabla f} \nabla f, X \rangle, \quad \forall X \in \Gamma^\infty(TM).
\]

It follows

\[
\nabla_{\nabla f} \nabla f = \nabla f.
\]

Thus

\[
\nabla_{\frac{\nabla f}{|\nabla f|}} \left( \frac{\nabla f}{|\nabla f|} \right) = \frac{1}{|\nabla f|} \left( \frac{\nabla_{\nabla f} \nabla f}{|\nabla f|} - \frac{\langle \nabla_{\nabla f} \nabla f, \nabla f \rangle}{|\nabla f|^3} \nabla f \right) = 0, \quad x \in M \setminus \{o\}.
\]

This implies that any integral curve generated by \( \frac{\nabla f}{|\nabla f|} \) (\( x \in M \setminus \{o\} \)) is geodesic.

Let \( \phi_t \) and \( \varphi_t \) be one-parameter diffeomorphisms groups generated by \( -\nabla f \) and \( -\frac{\nabla f}{|\nabla f|} \), respectively. Then as in the proof of (2.6), we have \( d(\phi_t(x), o) = e^{-t}d(x, o) \). Thus \( \langle \nabla f, \nabla r \rangle = -\frac{d}{dt}d(\phi_t(x), o) > 0 \). This shows that \( \varphi_s(x) \) is a geodesic curve from \( x \) to \( o \). Let \( \gamma(s) = \varphi_{d(x,o)-s}(x) \). Then
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\( \gamma(s) \) is a minimal geodesic curve from \( o \) to \( x \) as long as \( \text{dist}(o, x) \leq r_0 << 1 \). Moreover, we have

\[
\begin{align*}
\left\{ \begin{array}{l}
\frac{d^2}{ds^2} f(\gamma(s)) = 1, \\
\frac{d}{ds} f(\gamma(s)) = |\nabla f| \rightarrow 0, \text{ as } s \rightarrow 0, \\
f(\gamma(0)) = f(o) = 0.
\end{array} \right.
\]

Therefore, we deduce

\[
f(x) = \frac{1}{2} r^2(x), \text{ if } r(x) \leq r_0.
\]

In particular,

\[|\nabla f(x)| = r. \tag{3.12}\]

We claim that \( g \) is flat on \( B_{r_0}(o) \). Since \( \partial B_{r_0}(o) \) is diffeomorphic to \( S^{2n-1} \), we can choose an orthonormal basis \( \{e_1, \ldots, e_{2n-1}\} \) on \( \partial B_{r_0}(o) \).

Let \( X_i(\varphi_t(x)) = (e_i)_{x} \) for \( x \in \partial B_{r_0}(o), 1 \leq i \leq 2n-1 \). Then \( \{\nabla r = \frac{\nabla f}{|\nabla f|}, X_1, \ldots, X_{2n-1}\} \) is a global frame on \( B_{r_0}(o) \setminus \{o\} \). Clearly, \( \nabla r, X_i = 0, 1 \leq i \leq 2n-1 \). Thus by (3.12) and (3.11), it follows

\[
\partial \partial_r \langle \nabla r, X_i \rangle = \nabla_{\partial r} \langle \nabla r, X_i \rangle = \left( \nabla_r g \right) \langle \nabla r, X_i \rangle + \langle \nabla_r \nabla r, X_i \rangle + \langle \nabla_r, \nabla_r X_i \rangle = \frac{2}{r} \text{Hess } f(\nabla r, X_i) = \frac{2}{r} \langle \nabla r, X_i \rangle.
\]

Since \( \langle \nabla r, X_i \rangle |_{r=r_0} = 0 \), we get \( \langle \nabla r, X_i \rangle = 0 \) for any \( x \in B_{r_0}(o) \setminus \{o\} \). Similarly, we have

\[\frac{\partial}{\partial r} \langle X_i, X_j \rangle = \frac{2}{r} \langle X_i, X_j \rangle, \tag{3.13}\]

Consequently, \( \langle X_i, X_j \rangle = \frac{r^2}{2} \delta_{ij} \) for any \( x \in B_{r_0}(o) \setminus \{o\} \). Hence,

\[
g = dr \otimes dr + \frac{r^2}{r_0^2} \sum_{i,j=1}^{2n-1} d\theta^i \otimes d\theta^j,
\]

where \( \{dr, \theta^1, \ldots, \theta^{2n-1}\} \) are the corresponding coframe of \( \{\nabla r = \frac{\nabla f}{|\nabla f|}, X_1, \ldots, X_{2n-1}\} \). This proves that \( g \) is isometric to an Euclidean metric on \( B_{r_0}(o) \). Therefore, \( g \) is flat on \( B_{r_0}(o) \). The claim is true.

At last, we show that \( g \) is globally flat. Since \( \phi_t \) is an isometric diffeomorphism from \( (B_{r_0}(o, t), g(x, t)) \) to \( (B_{r_0}(o, 0), g(x, 0)) \). We see that
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\((B_{r_0}(o,t), g(x,t))\) is flat by the above claim. On the other hand, by the flow \((2.3)\) and the fact that \(g\) is Ricci-flat, we have \(g(x,t) = e^{-2t}g(x,0)\). Hence, \((B_{r_0}(o,t), g(x,0))\) is also flat. Since \(M\) is exhausted by \(B_{r_0}(o,t)\) as \(t \to \infty\) according to Lemma 2.2, we see that \(g\) is globally flat and \(M\) is simply connected. As a consequence, \((M, g)\) is isometric to \(\mathbb{C}^n\).

4. Steady Kähler-Ricci solitons

In this section, we deal with steady gradient Ricci solitons \((M^n, g, f)\). As in Section 3, we let \(\phi_t\) be a family of diffeomorphisms generated by \(-\nabla f\) and \(g(\cdot, t) = \phi_t^* g\). Then \(g(\cdot, t)\) satisfies

\[
\frac{\partial}{\partial t} g_{ij} = -2R_{ij}.
\]

It turns

\[
R_{ij}(t) = \nabla_i \nabla_j f(t),
\]

where \(f(t) = \phi_t^* f\) and \(\nabla\) is taken w.r.t \(g(t)\). Hence by the Bianchi identity \((2.2)\), one can obtain

\[
R + |\nabla f|^2 = \text{const}.
\]

This shows that the scalar curvature of \(g\) is uniformly bounded.

Analogous to Lemma 2.1, we have

Lemma 4.1. \(\frac{\partial}{\partial t} R = 2\text{Ric}(\nabla f(t), \nabla f(t))\).

In general, we do not know whether a steady gradient Ricci soliton admits an equilibrium point. However, we can still prove Rigidity Theorem 1.3 in case of steady gradient Kähler-Ricci solitons by using the fact of boundedness of scalar curvature.

Proof of Theorem 1.3 (the steady case). Since \((M, g)\) is Kählerian, we may rewrite (4.1) as,

\[
\begin{align*}
\frac{\partial}{\partial t} g_{ij}(x,t) &= -R_{ij}(x,t) \\
g_{ij}(x,0) &= g_{ij}(x).
\end{align*}
\]

In order to get the estimate for the Green function as in Section 3, we use a trick in [24] to consider a product space \(\widehat{M} = M \times \mathbb{C}^2\) with a product metric \(\widehat{g} = g + dw^1 \wedge d\bar{w}^1 + dw^2 \wedge d\bar{w}^2\). Then \(\widehat{g}(x,t) = g(x,t) + dw^1 \wedge d\bar{w}^1 + dw^2 \wedge d\bar{w}^2\) is a solution of (4.3) on \(\widehat{M}\) with the initial metric \(\widehat{g}\).

It was proved by Shi that for any \(s > t\) and \(B_s(x) \subset B_t(x) \subset (\widehat{M}, \widehat{g})\) (cf. Section 6 in [24]), it holds

\[
\frac{\text{vol}(B_s(x))}{\text{vol}(B_t(x))} \geq C_0^{-1} \left(\frac{s}{t}\right)^4.
\]
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Then
\[
\int_{d^2(x,y)}^{+\infty} \frac{dt}{\text{vol}(B_{\sqrt{t}}(x))} \leq C_0 \frac{d^2(x,y)}{\text{vol}(B_{d(x,y)}(x))}, \quad \forall \, d(x,y) \geq c_0.
\]

Thus by the Li-Yau estimate [16], there exists a global Green’s function \(G\) on \(\widehat{M}, \widehat{g}\) which satisfies (3.7).

Let \(F(x,t) = \ln \det(\widehat{g}_{ij}(x,t)) - \ln \det(\widehat{g}_{ij}(x,0))\). By (4.3), it is easy to see
\[
F(x,t) = -\int_0^t \widehat{R}(x,s)ds.
\]

Then by Lemma 4.1, we have
\[
\hat{R}(x,0) \leq -F(x,t) \leq C_0 t,
\]
where \(C_0 = \sup \hat{R}(x,t)\). Thus as in Section 3, to prove that \(R \equiv 0\), we shall give a growth estimate of \(-F(x,t)\) on \(t\).

Fix an arbitrary point \(x_0 \in M\). For convenience, we denote a \(r\)-geodesic ball \(B_r(x_0, t)\) of \((\widehat{M}, \widehat{g}(t))\) centered at \((x_0, 0, 0)\). As in Section 3, by using the Green formula, we can estimate
\[
-F(x_0, t) \leq \frac{C_1 M(t)}{r^2} \left( M(t) + \int_{B_r(x_0, 0)} G(x_0, y) \hat{R}(y,0) dv \right) + \int_{B_{r}(x_0,0)} G(x_0, y) \hat{R}(y,0) dv.
\]

Moreover,
\[
\int_{B_r(x_0, 0)} G_r(x_0, y) \hat{R}(y,0) dv \leq C(n) \left( \frac{r^2}{\text{vol}(B_r(x_0, 0))} \int_{B_r(x_0, 1)} \hat{R} dv \right)
+ \int_0^r \frac{s}{\text{vol}(B_s(x_0, 0))} \int_{B_s(x_0, 0)} \hat{R} dv ds, \quad \forall \, r > 0.
\]

On the other hand, by the volume comparison together with (1.4), we have
\[
\frac{1}{\text{vol}(B_r(x_0, 0))} \int_{B_r(x_0, 0)} \hat{R} dv \leq \frac{C}{\text{vol}(B_r(x_0))} \int_{B_r(x_0)} R dv \leq \frac{\varepsilon_1(r)}{1 + r},
\]
where the function \(\varepsilon_1(r) \rightarrow 0\) as \(r \rightarrow \infty\). Thus combining (4.6) and (4.7), we get from (1.5),
\[
-F(x_0, t) \leq r^{-2} C(n)t(C't + 1)(C't + r\varepsilon_2(r)) + r\varepsilon_2(r), \quad \forall \, r > 0.
\]

where \(\varepsilon_2(r) \rightarrow 0\) as \(r \rightarrow \infty\). Consequently,
\[
tR(x_0, 0) \leq r^{-2} C(n)t(C't + 1)(C't + r\varepsilon_2(r)) + r\varepsilon_2(r), \quad \forall \, r > 0.
\]
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Now we choose a monotonic $\varepsilon_3(r)$ such that $\varepsilon_3(r) \to 0$ and $\frac{\varepsilon_2(r)}{\varepsilon_3(r)} \to 0$ as $r \to \infty$. Let $r = t\varepsilon_3^{-1}(t)$. Then by (4.3), we get

$$tR(x_0,0) \leq C_1\varepsilon_3^2(t)(C't + \frac{\varepsilon_2(t\varepsilon_3^{-1}(t))}{\varepsilon_3(t\varepsilon_3^{-1}(t))}) + \frac{\varepsilon_2(t\varepsilon_3^{-1}(t))}{\varepsilon_3(t\varepsilon_3^{-1}(t))}, \quad \forall t \gg 1.$$  

By dividing by $t$ on both sides of the above inequality and then letting $t \to \infty$, it is easy to see that $R(x_0,0) = 0$. Since $x_0$ is an arbitrary point, we prove that $R(x) \equiv 0$. \hfill $\square$

By Theorem 1.3, we can finish the proof of Theorem 1.4.

**Proof of Theorem 1.4 (the steady case).** Since $(M, g)$ is a complete non-compact manifold with nonnegative Ricci curvature, the volume growth of $g$ is at least linear. Then by (1.6), it is easy to see that the average curvature condition (1.4) is satisfied in Theorem 1.3 as in the proof of Lemma 3.1. Hence by Theorem 1.3 we get Theorem 1.4 immediately. \hfill $\square$

5. **Boundedness of scalar curvature–II**

In this section, we prove the existence and uniqueness of equilibrium point for the steady Kähler-Ricci soliton $(M^n, g, f)$ in Theorem 1.2. As a consequence, the maximum of scalar curvature of $g$ can be attained.

**Proof of Theorem 1.2 (the steady case).** **Existence:** Let $g(\cdot, t) = \phi_t^* g$ be a family of steady solitons generated by $-\nabla f$. Then $g(\cdot, t)$ is an eternal solution of (4.3). Since $g(\cdot, t)$ has uniformly positive holomorphic bisectional curvature in space time $M \times (-\infty, \infty)$, we apply Theorem 2.1 in [11] to see that there exists a sequence of solutions $g_{\alpha}(\cdot, t) = g(\cdot, t_\alpha + t)$ on $\Phi_\alpha(D(r)) \subset M$ such that $\Phi_\alpha^*(g_{\alpha}(\cdot, t))$ converge to a smooth solution $h(x, t)$ of (4.3) uniformly and smoothly on a compact subset $D(r)$ for any $t \in (-1, \infty)$, where $D(r)$ is an Euclidean ball centered at the origin with radius $r$ and $\Phi_\alpha$ are local biholomorphisms from $D(r)$ to $M$. Moreover, by using the Cao’s argument in [3], it was proved that $h(x, t)$ is generated by a steady Kähler-Ricci soliton $(D(r), h, f^h)$ with $\nabla f^h(o) = 0$. Namely, $h(x, t)$ satisfies

$$R_{ij}(h(t)) = \nabla_i \nabla_j f^h(t), \quad \nabla_i \nabla_j f^h(t) = 0,$$

where $f^h(t)$ are induced functions of $f^h$ and $\nabla f^h(t)$ vanish at the origin for any $t \in (-1, \infty)$.

On the other hand, similar to (2.5), we have for solitons $(M, g(t), f(t))$,

$$R_{ij}(t) + R_{ij}(t) \nabla_j f(t) = 0.$$

Then $\nabla f(t)$ is determined by the curvature tensor. Define a sequence of a family of holomorphic vector fields $V(t_\alpha)$ on $D(r)$ by

$$\Phi_\alpha^* R_{ij}(t_\alpha) + \Phi_\alpha^* R_{ij}(t_\alpha)V(t_\alpha)_j = 0.$$
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Clearly, \((\Phi_\alpha)_* V(t_\alpha) = \nabla f(t_\alpha)\). By the convergence of \(g_\alpha(\cdot, t)\), holomorphic vector fields \(V(t_\alpha)\) converge to \(\nabla f^h\) in \(C^\infty\)-topology on \(D(r)\) for any \(t \in (-1, \infty)\). Since the eigenvalues of \(\text{Ric}(h(t))\) are positive at 0 by Proposition 2.2 in [11], the integral curves of \(-\nabla^h f^h\) will converge to 0 in \(D(r)\) when \(r\) is sufficiently small by the soliton equation. By the convergence of \(V(t_\alpha)\), the integral curve of \(-V(t_\alpha)\) will also converge to a point \(q\) in \(D(r_1)\) for some \(r_1 < r\) when \(\alpha\) is large enough (cf. Page 9 of [12]). As a consequence, \(q\) is a zero point of \(V(t_\alpha)\) in \(D(r_1)\). This proves that there exists a zero point of \(\nabla f(t_\alpha)\) in \(M\) for each \(\alpha\) since \(\Phi_\alpha^*\) is a local biholomorphism.

**Uniqueness:** Suppose that \(p\) and \(q\) are two equilibrium points. Then \(d_0(p, q) = d_t(p, q)\). Choose \(l > 0\) such that \(q \in B_l(p, 0)\). Note that \(\phi_t : (M^n, g(t)) \to (M^n, g(0))\) are a family of isometric deformations. Thus

\[
C = \inf_{x \in B_l(p, t)} \mu_1(x, t) = \inf_{x \in B_l(p, 0)} \mu_1(x, 0) > 0, \ \forall x \in B_l(p, 0),
\]

where \(\mu_1(x, t)\) is the smallest eigenvalue of \(\text{Ric}(x, t)\) w.r.t \(g(x, t)\). Since the metric is decreasing along the flow, we see that \(B_l(p, 0) \subset B_l(p, t)\). Hence by [10], we get

\[
\frac{d|v_x|^2}{dt} \leq -\mu_1(x, t)|v_x|^2 \leq -C|v_x|^2, \ \forall \ t \geq 0,
\]

where \(x \in B_l(p, 0)\) and \(v_x \in T^{(1,0)}_x M\). Therefore, if we let \(\gamma(s)\) be a minimal geodesic curve connecting \(p\) and \(q\) with an arc-parameter \(s\) w.r.t the metric \(g(x, 0)\) in \(B_l(p, 0)\), we deduce

\[
d_t(p, q) \leq \int_0^d |\gamma'(s)| ds \leq \int_0^d |\gamma'(s)| e^{-Ct} ds = d_0(p, q) e^{-Ct}.
\]

Letting \(t \to \infty\), we see that \(d_t(p, q) = d_0(p, q) = 0\). This proves that \(p = q\).

**Proposition 5.1.** Let \((M^n, g, f)\) be a simply connected complete non-compact steady gradient Kähler-Ricci soliton with nonnegative bisectional curvature. Suppose that \(g\) satisfies

\[
R(x) \leq \frac{C}{1 + r(x)^{1+\epsilon}},
\]

for some \(\epsilon > 0, C\). Then \((M, g)\) is isometric to \(\mathbb{C}^n\).

**Proof.** We suffice to show that \(g\) is Ricci flat. On the contrary, we may assume that the Ricci curvature of \(g(\cdot, t)\) is positive everywhere by a dimension reduction theorem of Cao for Kähler-Ricci flow on a simply connected complete Kähler manifold with nonnegative bisectional curvature [4], where \(g(\cdot, t) = \phi_t^* g\) is the generated solution of (1.3) as in Section 4. Let \(o\) be the unique equilibrium point of \(g\) according to Theorem 1.2. In the following we
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use an argument of Hamilton in [13] to show that there exists a pointedwise backward limit \( g_\infty(x) \) of \( g(x, t) \) on \( M \setminus \{o\} \) and \( (M \setminus \{o\}, g_\infty) \) is a complete flat Riemannian manifold.

Since
\[
R(x) + |\nabla f|^2 = R(o),
\]
by (5.1), we see
\[
\lim_{d(x, o) \to \infty} |\nabla f|^2(x) = R(o) > 0.
\]
Note the equilibrium point is unique. It follows
\[
C_\delta = \inf_{M \setminus B_\delta(o)} |\nabla f|^2 > 0, \quad \forall \delta > 0.
\]
This implies
\[
d(\phi_t(x), o) \geq C_\delta |t| \quad \forall x \in M \setminus B_\delta(o), \ t \leq 0.
\]
Hence by (5.1), we get from equation (4.3),
\[
0 \leq -\frac{\partial}{\partial t} g(x, t) \leq R(g(x, t)) g(x, t)
\]
\[
\leq \frac{C}{1 + d^{1+\epsilon}(\phi_t(x), o)} g(x, t) \leq \frac{C_\delta'}{1 + |t|^{1+\epsilon}} g(x, t).
\]
Therefore, we derive
\[
(5.2) \quad g(x, 0) \leq g(x, t_1) \leq g(x, t_2) \leq C_\delta g(x, 0),
\]
for any \( x \in M \setminus B_\delta(o, 0) \) and \( -\infty < t_2 \leq t_1 \leq 0 \).

By (5.2) and Shi’s higher order estimate for curvatures, we see that \( g(x, t) \) converge locally to a limit Kähler metric \( g_\infty(x) \) on \( M \setminus \{o\} \) as \( t \to -\infty \).

Clearly, \( g_\infty(x) \) is Ricci-flat since
\[
0 = \lim_{t \to -\infty} -\frac{\partial}{\partial t} g(x, t) = \lim_{t \to -\infty} \text{Ric}(g(\cdot, t)) = \text{Ric}(g_\infty).
\]
Consequently, \( g_\infty \) is flat. Moreover, \( g_\infty \) is a complete, because
\[
\lim_{x' \to o} d_{g_\infty}(x, x') = \lim_{t \to -\infty} d_{g(\cdot, t)}(x, o) = \lim_{t \to -\infty} d_{g}(\phi_t(x), o) = \infty,
\]
where \( x \in M \setminus \{o\} \). On the other hand, it was proved by Chau and Tam that \( M \) is biholomorphic to \( \mathbb{C}^n \) since \( M \) is a simply connected complete non-compact steady gradient Kähler-Ricci soliton with positive Ricci curvature [9]. Thus, \( M \setminus \{o\} \) is simply connected. Hence, \( M \setminus \{o\} \) is also biholomorphic to \( \mathbb{C}^n \). This is a contradiction! Therefore, \( g \) is Ricci flat and consequently, \( (M, g) \) is isometric to \( \mathbb{C}^n \).

\[\square\]
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