Privacy-Preserving Federated Learning via Normalized (instead of Clipped) Updates

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Abstract
Differentially private federated learning (FL) entails bounding the sensitivity to each client’s update. The customary approach used in practice for bounding sensitivity is to clip the client updates, which is just projection onto an $\ell_2$ ball of some radius (called the clipping threshold) centered at the origin. However, clipping introduces bias depending on the clipping threshold and its impact on convergence has not been properly analyzed in the FL literature. In this work, we propose a simpler alternative for bounding sensitivity which is normalization, i.e. use only the unit vector along the client updates, completely discarding the magnitude information. We call this algorithm DP-NormFedAvg and show that it has the same order-wise convergence rate as FedAvg on smooth quasar-convex functions (an important class of non-convex functions for modeling optimization of deep neural networks) modulo the noise variance term (due to privacy). Further, assuming that the per-sample client losses obey a strong-growth type of condition, we show that with high probability, the sensitivity reduces by a factor of $O\left(\frac{1}{m}\right)$, where $m$ is the minimum number of samples within a client, compared to its worst-case value. Using this high probability sensitivity value enables us to reduce the iteration complexity of DP-NormFedAvg by a factor of $O\left(\frac{1}{m^2}\right)$, at the expense of an exponentially small degradation in the privacy guarantee. We also corroborate our theory with experiments on neural networks.

1 Introduction
Collaborative machine learning (ML) systems such as federated learning (FL) [26] are growing at an unprecedented rate. These systems enable training powerful, state-of-the-art predictive models from decentralized and heterogeneous data through collaboration of many participants, e.g., mobile devices, each with different data and capabilities. Specifically in FL, there are a large number of clients (e.g., mobile phones or sensors) each with their own data and resources, and there is a central server (i.e., cloud) whose goal is to manage the training of a centralized model using the decentralized client data.

Despite the locality of data storage, information-sharing open the door to the possibility of sabotaging the security of personal data through communication. Hence, it is crucial to devise effective, privacy-preserving communication strategies that ensure the integrity and confidentiality of user data.

Differential privacy (DP) [11] is a mathematical framework that enables a quantitative study of privacy-preserving properties of machine learning algorithms and models. In particular, DP focuses on a learning algorithm’s sensitivity to an individual’s data; a less sensitive algorithm is less likely to leak individuals’ private details through its output. This idea has laid the foundation for designing a simple strategy to ensure privacy by adding random noise, drawn from a continuous distribution such as Gaussian or Laplacian to the output of the learning algorithm where the random noise is scaled according to the algorithm’s sensitivity to an individual’s data.
DP-SGD [1] is perhaps the seminal work on differentially private gradient-based optimization. It is essentially the same as regular SGD, except that Gaussian noise is added to the average of the “clipped” per-sample gradients (or updates) for privacy. Specifically, if the original update is $g$, then its clipped version is $g \min(1, C/\|g\|_2)$, for some threshold $C$; notice that this is the projection of $g$ onto an $\ell_2$ ball of radius $C$ centered at the origin. For functions that are not Lipschitz (i.e., have bounded gradients), clipping is essential for bounding the sensitivity of the average update to individual sample updates. In practice, we cannot ascertain whether the loss that we are minimizing is Lipschitz or not, due to which clipping is needed.

There is a natural extension of DP-SGD to the federated (and distributed) setting, wherein the server receives a noise-perturbed average of the clipped client updates [2,13,14,34]; again, clipping is done to limit the sensitivity of the average to each client’s local update.

While the privacy aspect of DP-SGD and its variants, both in the centralized and federated setting (see Section 3), is typically the main consideration, the optimization aspect – particularly due to clipping – is not given that much attention. Specifically, the clipped update is no longer unbiased which should impact the rate of convergence – this aspect has been somewhat analyzed in the centralized setting but it is relatively unexplored in the more challenging federated setting with multiple local update steps (see Section 3). Further, the clipping threshold ($C$) is another hyper-parameter that needs to be extensively tuned, as there are no principled methods to choose it.

In this paper, we propose a simpler alternative for bounding the sensitivity which is to always normalize the individual client updates; specifically, if the original update is $g$, then its normalized version is $g/\|g\|_2$. More precisely, we propose to send a noise-perturbed average of the unit vector along the client updates to the server, completely discarding their magnitude information. Our normalization approach is backed by rigorous theory – we show that normalizing the client updates does not change the order-wise convergence rate (see Theorem 1 and Remark 1 for details). The key advantage of normalization over clipping is that it helps to improve the signal (which is the update norm) to noise ratio. With clipping, the norm of the added noise (which has constant expectation proportional to the clipping threshold, regardless of the client update norms) can become arbitrarily larger than the client update norms when (and if) they fall below the clipping threshold; this is detrimental to convergence. This issue does not arise with normalization as the noise norm cannot become arbitrarily larger than the normalized update’s norm (= 1), even if the original update’s norm is small.

Another important parameter in setting the noise variance for DP is the sensitivity to the individual client/sample updates (in the federated/centralized setting). Typically, we take the worst-case sensitivity value which results in poor utility (i.e, performance) in practice. If we can obtain high-probability bounds on the sensitivity, the utility can be greatly improved at the expense of slightly deteriorating the privacy guarantee. In this work, we attempt to do so under an assumption on the per-sample client losses; see Theorem 2 and Corollary 2.1 for details.

We now summarize our main contributions:

(a) In Section 4, we present $\text{DP-NormFedAvg}$ (Algorithm 1), wherein we have the clients send zero-mean Gaussian noise (for privacy) plus their respective normalized (instead of clipped) updates, i.e. only the unit vector along their updates, to the server. We provide a privacy and convergence guarantee for $\text{DP-NormFedAvg}$ on smooth quasar-convex functions (an important class of non-convex functions for modeling optimization of deep networks) in Theorem 1. Our result reveals that the non-private version of $\text{DP-NormFedAvg}$ (i.e., $\text{DP-NormFedAvg}$ without any noise added), which we call $\text{NormFedAvg}$, has the same order-wise convergence rate as $\text{FedAvg}$ in the smooth quasar-convex setting; see Remark 1. In contrast, there are no analogous guarantees for clipped methods in the federated setting (see Section 3 and Remark 2). It is also worth mentioning that this is the first work to provide a convergence result for $\text{FedAvg}$ on quasar-convex
functions (see Appendix [3]) to the best of our knowledge; its proof is significantly different from the pure convex case.

(b) Assuming that the per-sample client losses satisfy a strong-growth \[31,35\] type of condition (Assumption [3]), we show that with high probability, the sensitivity reduces by a factor of \(O\left(\frac{1}{m}\right)\), where \(m\) is the minimum number of samples within a client, compared to its worst-case value (see Theorem 2 and Corollary 2.1). Using this high probability sensitivity value allows us to decrease the iteration complexity of \(\text{DP-NormFedAvg}\) by a factor of \(O\left(\frac{1}{m}\right)\), at the expense of an exponentially small deterioration in the privacy guarantee; see Corollary 2.1 and the discussion after that.

(c) Experiments in Section 5 show that our proposed update normalization approach outperforms the standard clipping-based approach by more than 1% for the task of classification on CIFAR-10 and Fashion-MNIST [38] in the federated setting. The same experiments also show that indeed \(\text{NormFedAvg}\) and \(\text{FedAvg}\) have similar convergence and generalization, validating Remark 1.

2 Preliminaries

Federated Learning (FL): In a standard FL setting, there are \(n\) clients, each having its own training data, and a central server that is trying to train a model, parameterized by \(w \in \mathbb{R}^d\), using the clients’ data. Suppose the \(i\)th client has \(n_i\) training examples \(\{x^{(i)}_1, \ldots, x^{(i)}_{n_i}\} := D_i\), drawn from some distribution \(\mathcal{P}_i\). Then the \(i\)th client has an objective function \(f_i(w)\) which is the expected loss, w.r.t. some loss \(\ell\), over its \(n_i\) samples, and the central server tries to optimize the average loss \(f(w)\), over the \(n\) clients, i.e.,

\[
f(w) := \frac{1}{n} \sum_{i=1}^{n} f_i(w), \quad \text{where} \quad f_i(w) := \frac{1}{n_i} \sum_{j=1}^{n_i} \ell(x^{(i)}_j, w).
\]

The setting where the data distributions of all the clients are identical, i.e. \(\mathcal{P}_1 = \ldots = \mathcal{P}_n\), is known as the “homogeneous” setting. Other settings are known as “heterogeneous” settings.

The key algorithmic idea of FL is Federated Averaging commonly abbreviated as \(\text{FedAvg}\) [26]. In \(\text{FedAvg}\), at every round, the server randomly chooses a subset of the clients and sends them the latest global model. These clients then undertake multiple steps of local updates (on the global model received from the server) with their respective data based on (stochastic) gradient descent, and then communicate back their respective updated local models to the server. The server then averages the clients’ local models to update the global model (hence the name \(\text{FedAvg}\)). \(\text{FedAvg}\) forms the basis of more advanced federated optimization algorithms. We summarize \(\text{FedAvg}\) mathematically in Algorithm 2.

Differential Privacy (DP): Suppose we have a collection of datasets \(D_c\) and a query function \(h : D_c \to \mathcal{X}\). Two datasets \(D \in D_c\) and \(D' \in D_c\) are said to be neighboring if they differ in exactly one sample, and we denote this by \(|D - D'| = 1\). A mechanism \(\mathcal{M} : \mathcal{X} \to \mathcal{Y}\) is said to be \((\varepsilon, \delta)\)-DP, if for any two neighboring datasets \(D, D' \in D_c\) and for any measurable subset of outputs \(\mathcal{R} \in \mathcal{Y}\),

\[
\mathbb{P}(\mathcal{M}(h(D)) \in \mathcal{R}) \leq e^{\varepsilon}\mathbb{P}(\mathcal{M}(h(D')) \in \mathcal{R}) + \delta.
\]

When \(\delta = 0\), it is commonly known as pure DP. Otherwise, it is known as approximate DP.

Adding random noise to the output of \(h(.)\) is the customary approach to provide DP. The noise scale is determined by the “sensitivity” \(\Delta_p\) of \(h(.)\) with respect to the \(\ell_p\) norm (for some \(p > 0\) depending on the noise distribution); this is formally defined as \(\Delta_p := \max_{|D - D'| = 1} \|h(D) - h(D')\|_p\).

\[\text{In general this average may be a weighted one, but here we consider the case of uniform weights.}\]
In this paradigm, the two most common mechanisms are the Laplacian and Gaussian mechanisms [12], wherein zero-mean Laplacian and Gaussian noise are added, respectively. The parameters of the Laplacian (respectively, Gaussian) distribution depend on \( \Delta_1 \) (respectively, \( \Delta_2 \)) [12]. We employ the Gaussian mechanism in our algorithm (DP-NormFedAvg) for privacy.

“Aggregate Privacy”: We now describe the exact quantity that we are trying to privatize. Recall the FL setting (around eq. (1)) described above. Suppose \( S \) denotes the set of clients chosen by the server at the current round and the update of client \( c \in S \) is given by \( u_c \). Then, we wish to make the aggregate of the updates at the server, which is their average \( \frac{1}{|S|} \sum_{c \in S} u_c \), differentially private; hence the name “aggregate privacy”. The same has also been considered in [2,13,34]. Of course, this is weaker than making each \( u_c \) private but the latter results in poor performance (or “utility”) because it requires adding more noise. Moreover, we wish to have “sample-level” DP and not “client-level” DP; thus, we measure the sensitivity with respect to the individual samples within the clients and not the client as a whole. However, as we discuss in Section 4, the worst-case sensitivity value with respect to the individual samples (which we also use in our experiments) is actually equal to the sensitivity value with respect to the client as a whole.

It is worth mentioning here that even though we consider the specific privacy setting described above, our convergence results with normalized updates extend to any DP setting.

Notation: Throughout this paper, we denote the \( \ell_2 \) norm simply by \( \| \cdot \| \) (omitting the subscript 2). Vectors and matrices are written in boldface. In Section 4, \( K \) is the number of communication rounds or the number of global updates, \( E \) is the number of local updates per round, and \( r \) is the number of clients that the server accesses in each round. Finally, for a function \( h(\cdot) \), \( x \) is said to be a \( \gamma \)-stationary point if \( \| \nabla h(x) \| \leq \gamma \).

The proofs of all theoretical results are in the Appendix.

3 Related Work

Differentially private optimization and gradient clipping: Significant amount of research has gone into developing differentially private gradient-based optimization algorithms for empirical risk minimization in the centralized setting [1,5,10,21,24,30,33,36,37] as well as in the FL and distributed (without multiple local updates) setting [2,3,13,14,25,28,29,34]. While some works assume that the gradients are bounded, providing bounded sensitivity by default, others consider a more realistic scenario of unbounded gradients and clip the respective updates to have bounded sensitivity. Recall that if the original update is \( g \), then its clipped version is \( g \min(1,C/\|g\|) \) for some threshold \( C \). The central theme of these works is to just add suitable amount of noise to the average of the clipped updates for providing privacy. Clipping is related to the main idea of DP-NormFedAvg, which is to use only the unit vector along the client updates. In the centralized setting, there are some works which do study the impact of gradient clipping on convergence theoretically such as [8,11,42] but it is not clear if they can be extended to the FL setting (due to multiple local update steps). In comparison, theory for clipped methods in the FL setting was only recently analyzed in [43]; we briefly compare their result with ours in Remark 2.

Normalized gradient descent (GD) and related methods: In the centralized setting, [18] propose (Stochastic) Normalized GD. This is based on a similar idea as DP-NormFedAvg – instead of using the (stochastic) gradient, use the unit vector along the (stochastic) gradient for the update. Extensions of this method incorporating momentum [9,39,10] have been shown to significantly improve the training time of very large models such BERT. However, it must be
noted here that the motivation of these works (for the centralized setting) is to improve the rate of convergence (i.e. accelerate training), whereas the primary motivation of DP-NormFedAvg is to render private training without compromising too much on the rate of convergence as compared to FedAvg.

4 DP-NormFedAvg: Private FL via Client-Update Normalization

In this section, we present our main algorithm DP-NormFedAvg (Algorithm 1). The primary difference from FedAvg (summarized in Algorithm 2 in the Appendix) is in lines 9 and 11 (of Algorithm 1). In line 9 – divergent from the standard clipping-based approach (for bounding sensitivity) – we propose to normalize or use only the unit vector along the client update (viz. \( \mathbf{w}_k - \mathbf{w}^{(i)}_{k,E} \)) for client \( i \), dropping the magnitude information altogether. Each client sends its normalized update plus zero-mean Gaussian noise (for privacy) to the server; since Gaussian noise is additive, we can add it at the clients itself. In line 11, the server computes the mean of the noisy normalized client updates (that it received), namely \( \mathbf{a}_k \), and then uses it to update the global model similar to FedAvg, except with a potentially different global learning rate \( \beta_k \) than the local learning rate \( \eta_k \). Since each \( \mathbf{c}_k^{(i)} \) (i.e., noise added at client \( i \)) is \( \mathcal{N}(0, \sigma^2 I_{1000}) \), the average noise at the server is \( \mathcal{N}(0_d, \sigma^2 I_d) \). This differentially private (DP) averaging of Normalized client updates, followed by FedAvg-style aggregation at the server inspires the name of our algorithm.

To motivate the key idea of DP-NormFedAvg, let us revisit the standard approach for bounding sensitivity in differentially private FL (which is the basis of related works discussed in Section 3). This is clipping the client updates; say the \( i \)th client’s update is \( \mathbf{d}_k^{(i)} := (\mathbf{w}_k - \mathbf{w}^{(i)}_{k,E}) \), then its clipped version is \( \mathbf{d}_k^{(i)} \min \left( 1, \frac{C}{\| \mathbf{d}_k^{(i)} \|} \right) \) for some clipping threshold \( C \). But intuitively, clipping has the following issue with respect to optimization - as the client update norms decrease and fall below the clipping threshold, the norm of the added noise (which has constant expectation proportional to the clipping threshold – regardless of the client update norms) can become arbitrarily larger than the client update norms, which should inhibit convergence. This issue is not as grave in DP-NormFedAvg because its update-normalization step (i.e., line 9) ensures that the noise norm cannot become arbitrarily larger than the normalized update’s norm (even if the original update’s norm is small).

To illustrate this intuition, let us consider a synthetic quadratic optimization problem (in the centralized setting) which is the minimization of \( f(\mathbf{x}) = \frac{1}{2} \mathbf{x}^T \mathbf{Q} \mathbf{x} \), where \( \mathbf{x} \in \mathbb{R}^{1000} \) and \( \mathbf{Q} \) is a 1000 \times 1000 PSD matrix, with normalization and clipping in presence of noise. We set \( \mathbf{Q} = \mathbf{A} \mathbf{A}^T \), where \( \mathbf{A} \) is a 1000 \times 1000 matrix whose entries are drawn in i.i.d fashion from \( \mathcal{N}(0, 10^{-6}) \). Suppose the noise variance is given by \( \sigma^2 \). Let the gradient at a point be given by \( g(\mathbf{x}) \); in this case, \( g(\mathbf{x}) = \mathbf{Q} \mathbf{x} \). Then the normalization update rule with step-size \( \eta \) is

\[
\mathbf{x} \leftarrow \mathbf{x} - \eta \left( \frac{g(\mathbf{x})}{\| g(\mathbf{x}) \|} + \mathcal{N}(0, \sigma^2 I_{1000}) \right).
\]

The clipping update rule with step-size \( \eta \) and clipping threshold \( C \) is

\[
\mathbf{x} \leftarrow \mathbf{x} - \eta \left( g(\mathbf{x}) \min \left( 1, \frac{C}{\| g(\mathbf{x}) \|} \right) + C \mathcal{N}(0, \sigma^2 I_{1000}) \right).
\]

We run each algorithm, i.e. normalization and clipping with some threshold \( C \), for 1000 iterations. We choose the initial point as \( \frac{1}{\sqrt{1000}} \mathbf{I}_{1000} \) (where \( \mathbf{I}_n \) denotes the \( n \)-dimensional vector of all ones) and set \( \sigma = \frac{5}{\sqrt{1000}} \). In Figure 1, we plot the function value versus iteration number for normalization and clipping with thresholds \( C = \{1, 0.1, 0.01\} \), each with their best respective step-sizes (details about the step-sizes are in Appendix F). Consistent with our intuition, we see
that normalization has the best performance. Also clipping with a smaller threshold has better performance than that with a higher threshold because smaller threshold corresponds to more aggressive normalization.

This intuition is further validated by experiments in Section 4. Further, clipping is associated with the obvious problem of deciding a good clipping threshold; our normalization-based approach is free of this issue.

It is worth pointing out that clipping can be equivalent to normalization in certain scenarios. Specifically, suppose the client update norms are lower bounded by $C_{\text{low}}$; then, clipping with threshold $C \leq C_{\text{low}}$ is equivalent to normalization with $\beta_k = C$.

Let us now move onto the convergence analysis of DP-NormFedAvg to theoretically justify why our update normalization approach is not detrimental from an optimization perspective. We list some assumptions and definitions first.

**Assumption 1 (Smoothness).** $\ell(x, w)$ is $L$-smooth with respect to $w$, for all $x$. Thus, each $f_i(w)$ ($i \in [n]$) is $L$-smooth, and so is $f(w)$.

**Assumption 2 (Quasar-Convexity).** Each $f_i(w)$ ($i \in [n]$) is $\zeta$-quasar convex w.r.t. $w^*_i \in \arg \min f_i(w)$, where $\zeta \in (0, 1]$. A function $h$ is said to be $\zeta$-quasar convex ($\zeta \in (0, 1]$) w.r.t. $z^* \in \arg \min h(z)$ if for all $z$, it holds that: $h(z^*) \geq h(z) + \frac{1}{\zeta} \langle \nabla h(z), z^* - z \rangle$; also see [15, 20].

In the special case of $\zeta = 1$, quasar convexity is referred to as star convexity [27]. [44] show that the trajectory followed by SGD while optimizing deep neural networks is a “star-convex path”, i.e. the loss function satisfies star convexity with respect to the minima that SGD converges to. A similar assumption has been made in [23] to show how SGD can escape local minima. The degree of non-convexity of the function increases as $\zeta$ becomes smaller [20].
Trivially, we have \( \Delta \) outputs obtained in the two cases are that \( u \) and \( \beta \) change one sample within that client. As discussed in Section 2, recall that we measure the sensitivity value with respect to the client as a whole. Thus, using Definition 1, because we wish to have sample-level DP and not client-level DP.

Sensitivity with respect to the individual samples within the clients and not the client as a whole, per-client sensitivity measures by how much does the normalized client-update change when we change one sample within that client. We also need to introduce the definitions of per-client sensitivities which determine the global sensitivity, which is used to set the value of the noise variance \( \sigma^2 \) in Algorithm 1.

Algorithm 1: DP-NormFedAvg

1. **Input:** Initial point \( w_0 \), number of communication rounds \( K \), number of local updates per round \( E \), local learning rates \( \{\eta_k\}_{k=0}^{K-1} \), global learning rates \( \{\beta_k\}_{k=0}^{K-1} \), number of participating clients in each round \( r \), and noise variance \( \sigma^2 \).

2. for \( k = 0, \ldots, K - 1 \) do
3. Server sends \( w_k \) to a set \( S_k \) of \( r \) clients chosen uniformly at random w/o replacement.
4. for client \( i \in S_k \) do
5. Set \( w_{k,0} = w_k \).
6. for \( \tau = 0, \ldots, E - 1 \) do
7. Update \( w_{k,\tau+1} = w_{k,\tau} - \eta_k \nabla f_i(w_{k,\tau}) \).
8. end for
9. Let \( g_k = \frac{(w_k - w_k^{(i)})}{\|w_k - w_k^{(i)}\|} \). Send \( (g_k + \zeta_k) \) to the server, where \( \zeta_k \sim \mathcal{N}(0, r^2 1_d) \).
10. end for
11. Server updates \( w_{k+1} = w_k - \beta_k a_k \), where \( a_k = \frac{1}{\tau} \sum_{i \in S_k} (g_k + \zeta_k) \).
12. end for

We also need to introduce the definitions of per-client sensitivities which determine the global sensitivity, which is used to set the value of the noise variance \( \sigma^2 \) in Algorithm 1. At a high level, per-client sensitivity measures by how much does the normalized client-update change when we change one sample within that client. As discussed in Section 2, recall that we measure the sensitivity with respect to the individual samples within the clients and not the client as a whole, because we wish to have sample-level DP and not client-level DP.

**Definition 1 (Per-client sensitivities).** Recall that the dataset of the \( i \)th client is \( D_i \). Consider another dataset \( D'_i \) which differs from \( D_i \) in exactly one sample. Let \( \eta \) be a fixed step-size such that \( \eta LE \leq \frac{1}{3} \) where \( L \) is the smoothness constant (Assumption 7). Now consider running \( E \) steps of gradient descent with step-size \( \eta \) starting from \( w \) on the \( i \)th client with \( D_i \) and \( D'_i \); suppose the outputs obtained in the two cases are \( w^{(i)}_E \) and \( w^{(i)}_{E'} \), respectively. Also, let \( u^{(i)}_{(E)} = w - w^{(i)}_E \) and \( u^{(i)}_{(E') \rightarrow (E)} = w - w^{(i)}_{E'} \). We define the sensitivity, \( \Delta_2^{(i)} \), for the \( i \)th client w.r.t. the \( \ell_2 \) norm as:

\[
\Delta_2^{(i)} := \sup_{D_i, D'_i, w, \{\eta \eta LE \leq 0.25\}} \left\| \frac{u^{(i)}_{(E)}}{\|u^{(i)}_{(E)}\|} - \frac{u^{(i)}_{(E')}}{\|u^{(i)}_{(E')}\|} \right\|.
\]

Trivially, we have \( \Delta_2^{(i)} \leq 2 \).

Note that the worst-case per-client sensitivity value of \( \Delta_2^{(i)} = 2 \) is actually equal to the sensitivity value with respect to the client as a whole. Thus, using \( \Delta_2^{(i)} = 2 \) corresponds to client-level DP. Later, we shall provide stronger bounds for \( \Delta_2^{(i)} \) under an additional assumption.

With the above definition, we are ready to provide a convergence and privacy result for DP-NormFedAvg.

**Theorem 1 (Guarantees for DP-NormFedAvg).** Privacy: Define \( \Delta_2 := \frac{1}{r} \max_{i \in [n]} \Delta_2^{(i)} \), where \( \{\Delta_2^{(i)}\}_{i=1}^n \) are as defined in Definition 1 and \( r \) is the number of participating clients in Algorithm 1. Then,

(a) to make each round of DP-NormFedAvg \( (\varepsilon, \delta) \)-DP, set the noise variance in Alg. 4 as \( \sigma^2 = \frac{2 \log(1/\delta^2 \Delta_2^2)}{\varepsilon^2} \), where \( \tilde{\varepsilon} = \log(1 + \frac{r}{\varepsilon}(e^\varepsilon - 1)) \) and \( \tilde{\delta} = \frac{\sigma^2}{\varepsilon} \).
(b) to make all $K$ rounds of $\text{DP-NormFedAvg}(\varepsilon, \delta)$-DP, where $\varepsilon < O\left(\frac{2K}{n^2}\right)$, set $\sigma = O\left(\frac{\sqrt{K \log(1/\delta) \Delta_2}}{n \varepsilon}\right)$.

**Convergence:** Let Assumptions 1 and 2 hold. In Algorithm 1, set $\eta_k = \frac{\zeta}{4L \sqrt{K(1+\sigma_2^2)}}$, $\beta_k = (4L \|w_0 - w^*\| \eta_k)$ and choose $E \leq \frac{\zeta}{2} \sqrt{K(1+\sigma_2^2)}$. Then for any $w^* \in \arg\min_{w \in \mathbb{R}^d} f_i(w)$ and $w_i^*=\arg\min_{w \in \mathbb{R}^d} f_i(w)$, the global iterates $\{w_k\}_{k=0}^{K-1}$ of Algorithm 1 satisfy:

$$
\min_{k<K} \mathbb{E}[\|\nabla f(w_k)\|] \leq \frac{6L \|w_0 - w^*\|}{\zeta \sqrt{K}} \left( \frac{1}{1+\frac{\varepsilon}{\sqrt{K(1+\sigma_2^2)}}} \right) + \frac{6L}{\zeta} \left( \frac{1}{1+\frac{\varepsilon}{2\sqrt{K(1+\sigma_2^2)}}} \right) \left( \frac{1}{n} \sum_{i \in [n]} \|w_i^* - w^*\| \right).
$$

Note that the quantity $\left( \frac{1}{n} \sum_{i \in [n]} \|w_i^* - w^*\| \right)$ is a measure of the degree of heterogeneity of the clients; for highly heterogeneous settings, we expect this quantity to be large.

**Corollary 1.1 (DP-NormFedAvg guarantees with worst case sensitivity):** Let us take the worst case sensitivity values, i.e. $\Delta_2(i) = 2 \forall i \in [n]$. With this, $\Delta_2 = \frac{2}{\sigma}$. In that case, to make each round of $\text{DP-NormFedAvg}(\varepsilon, \delta)$-DP, where $\varepsilon < 1$, we should set $\sigma = O\left(\frac{\sqrt{\log(n/\delta)}}{n \varepsilon}\right)$. This results in a convergence rate of:

$$
\min_{k<K} \mathbb{E}[\|\nabla f(w_k)\|] \leq O\left(\frac{\sqrt{d \log(1/\delta)}}{n \varepsilon \sqrt{K}}\right) + O\left(\frac{L}{n} \sum_{i \in [n]} \|w_i^* - w^*\|\right).
$$

In other words, $\text{DP-NormFedAvg}$ converges to a $(\beta + O\left(\frac{L}{n} \sum_{i \in [n]} \|w_i^* - w^*\|\right))$-stationary point in $O\left(\frac{d \log(1/\delta)}{n \varepsilon \sqrt{K}}\right)$ rounds of communication (in expectation).

Next, we make some remarks to discuss implications of the previous results.

**Remark 1 (Convergence rate comparison between $\text{DP-NormFedAvg}$ and $\text{FedAvg}$):** Define $B := L \max_{i \in [n]} \|w_i^* - w^*\|$. Then the order-wise rate of convergence of $\text{DP-NormFedAvg}$ is:

$$
\min_{k<K} \mathbb{E}[\|\nabla f(w_k)\|] \leq O\left(\frac{\max(1, \sqrt{d \sigma})}{\sqrt{K}}\right) + O(B).
$$

In comparison, in Appendix B we show the order-wise convergence rate of $\text{FedAvg}$ (with the same assumptions) to be:

$$
\min_{k<K} \mathbb{E}[\|\nabla f(w_k)\|^2] \leq O\left(\frac{1}{K}\right) + O(B^2).
$$

Observe that the result for $\text{DP-NormFedAvg}$ is in terms of the gradient norm, whereas the corresponding result for $\text{FedAvg}$ is in terms of the squared gradient norm. Thus, both algorithms essentially have the same order-wise rate of convergence with respect to the number of communication rounds $K$; the $\sqrt{d \sigma}$ penalty in the rate of $\text{DP-NormFedAvg}$ is unavoidable due to privacy. Hence, discarding the magnitude information of the client updates is not detrimental from an optimization perspective. In fact, if we do not add noise for privacy in Algorithm 1 then $\sigma = 0$ in Theorem 1 and the resultant rate is free of the $\sqrt{d \sigma}$ term, thereby exactly matching the rate of $\text{FedAvg}$ order-wise; we call this algorithm $\text{NormFedAvg}$. $\text{NormFedAvg}$ can be viewed as the base algorithm of $\text{DP-NormFedAvg}$.

The similar performance of $\text{NormFedAvg}$ and $\text{FedAvg}$ is corroborated by experiments in Section 3 (see Figure 2). Also, to our knowledge, ours is the first convergence result for $\text{FedAvg}$ on quasar-convex functions; its proof is significantly different from the pure convex case.

**Remark 2 (Comparison with [43]):** As mentioned in Section 3 [43] provide a recent convergence result for differentially private $\text{FedAvg}$ with clipping (Theorem 3.1 in their paper). While
Theorem 1 below; in it, observe that we are lower bounding (I) by analyzing each client separately.

The main challenge is lower bounding (I) in terms of each client’s progress needs to be analyzed separately. More precisely, refer to the proof sketch of or “client-drift” arising from the use of multiple local update steps in the FL setting, due to which the base algorithm of their result indicates that are two bias terms due to clipping which are not zero even when there is no noise added; thus, their result is under different assumptions than us due to which an apples to apples comparison is in the smooth non-convex case. However, as we discuss in Remark 1, not possible, we would like to point out an interesting difference. In their convergence result, there same order-wise convergence rate as FedAvg with clipping will have worse convergence than regular FedAvg in the smooth quasar-convex case.

Remark 3 (Convergence to stationary point). We are only able to prove convergence to a first-order stationary point rather than to the optimal point \( w^* \) in Theorem 2 because of the bias or “client-drift” arising from the use of multiple local update steps in the FL setting, due to which each client’s progress needs to be analyzed separately. More precisely, refer to the proof sketch of Theorem 2 below; in it, observe that we are lower bounding (I) by analyzing each client separately.

We now give a proof sketch of the convergence result of Theorem 1.

Proof. For any client \( i \in [n] \), let

\[
\mathbf{u}_k^{(i)} := \sum_{\tau=0}^{E-1} \nabla f_i(\mathbf{w}_k^{(i)}) = (\mathbf{w}_k - \mathbf{w}_{k,E}^{(i)})/\eta_k.
\]

In words, \( \mathbf{u}_k^{(i)} \) is the change in the \( i \)-th client’s local parameter after \( E \) steps of local GD with step-size \( \eta_k \), divided by \( \eta_k \). Now for any \( w^* \in \Phi \), we have:

\[
E[\|w_{k+1} - w^*\|^2] \leq E[\|w_k - w^*\|^2] - \frac{2c}{n} \sum_{i \in [n]} \langle w_k - w^*, \frac{\eta_k \mathbf{u}_k^{(i)}}{\|\mathbf{u}_k^{(i)}\|}\rangle + c^2 \eta_k^2 (1 + d\sigma^2).
\]

The main challenge is lower bounding (I) in terms of \( \|\nabla f_i(\mathbf{w}_k)\| \).

Note that for any \( w_i^* \in \Phi_i := \arg \min_w f_i(w) \), we have:

\[
(I) = \left( w_k - w_i^*, \frac{\eta_k \mathbf{u}_k^{(i)}}{\|\mathbf{u}_k^{(i)}\|}\right) + \left( w_i^* - w^*, \frac{\eta_k \mathbf{u}_k^{(i)}}{\|\mathbf{u}_k^{(i)}\|}\right) \\
\geq \|w_i^* - w^*\|.
\]

Now, it can be shown that

\[
(II) = \frac{(\|w_k - w_i^*\|^2 - \|w_{k,E}^{(i)} - w_i^*\|^2) + \eta_k^2 \|\mathbf{u}_k^{(i)}\|^2}{2\|\mathbf{u}_k^{(i)}\|}.
\]

Next, by using the quasar-convexity of \( f_i \) and some algebra, we get for \( \eta_k \leq \frac{\zeta}{2L} \):

\[
\|w_k - w_i^*\|^2 - \|w_{k,E}^{(i)} - w_i^*\|^2 \geq \frac{\eta_k \zeta}{2L} \|\mathbf{u}_k^{(i)}\|^2.
\]

Using eq. (8) in eq. (7) and then substituting it back in eq. (6), we get (for \( \eta_k \leq \frac{\zeta}{2L} \)):

\[
(I) \geq \left( \eta_k^2 + \frac{\eta_k \zeta}{2L} \right) \|\mathbf{u}_k^{(i)}\| - \eta_k \|w_i^* - w^*\|.
\]
Now, the non-trivial part is lower bounding $\|u^{(i)}\|$ in terms of $\|\nabla f_i(w_k)\|$: we do this in Lemma 2. Using the result of Lemma 2, we get for $\eta_k \leq \min(\frac{\epsilon}{2L}, \frac{1}{16})$:

$$1 \geq \frac{1}{3} \left( \eta_k^2 E + \frac{\eta_k \zeta}{2L} \right) \|\nabla f_i(w_k)\| - \eta_k \|w_i^* - w^*\|.$$

(10)

This is the desired lower bound on (I) and the crux of the proof. The final result is obtained by using this lower bound in eq. (5) follows by some more algebra and bookkeeping.

Now, we provide a high-probability bound for $\Delta_2(i)$ under smoothness (Assumption 1) and the following strong-growth condition type of assumption:

**Assumption 3.** Define the population loss $f_i(w) := E_{x_i \sim P_i}[\ell(x_i; w)]$. Then for any $x_i$ drawn from $P_i$, $\|\nabla \ell(x_i, w)\| \leq \rho \|\nabla f_i(w)\|$ for an absolute constant $\rho > 1$.

As a simple example satisfying the above assumption, consider linear regression where $\ell(x_i; w) = \frac{1}{2} ((w^*, x_i) - (w, x_i))^2$, with $x_i \sim P_i$. Further, let $\|x_i\| \leq 1$ for all $x_i \sim P_i$ and $E_{x_i \sim P_i}[xx_i^T] = Q > 0$, such that $\lambda_{\min}(Q) = \lambda^2 \leq 1$. In Appendix C we show that this example satisfies Assumption 3 with $\frac{1}{\rho} = \frac{1}{\sqrt{2}}$.

**Theorem 2 (High probability per-client sensitivity bounds).** Suppose Assumptions 1 and 3 hold. Recall that $n_i$ is the number of training examples in the $i$th client. Suppose $n_i \geq \frac{15\epsilon^{1/4}}{2} \rho$ for all $i \in [n]$. Then with probability at least $1 - 2d \exp\left(-\frac{n_i}{16(\rho + 1)^2}\right)$, $\Delta_2(i) \leq \frac{15\epsilon^{1/4}}{2} \rho n_i$.

**Corollary 2.1 (DP-NormFedAvg guarantees with the result of Theorem 2).** Recall that $\Delta_2 := \frac{1}{n} \max_{i \in [n]} \Delta_2(i)$ from Theorem 2. Also, let $m = \min_{i \in [n]} n_i$. Then in the setting of Theorem 2, with probability at least $1 - 2d \sum_{i \in [n]} \exp\left(-\frac{n_i}{16(\rho + 1)^2}\right)$, we have that $\Delta_2 \leq \frac{15\epsilon^{1/4}}{2} \rho nm$.

So, if we take $\Delta_2 = \frac{15\epsilon^{1/4}}{2} \rho nm$ and therefore set $\sigma = O\left(\frac{\sqrt{\log(n/\delta)} \|w^*\|}{nm\epsilon}\right)$, each round of DP-NormFedAvg is $(\varepsilon, \delta + 2d \sum_{i \in [n]} \exp\left(-\frac{n_i}{16(\rho + 1)^2}\right))$-DP. Also, the convergence rate of DP-NormFedAvg in this case is:

$$\min_{k<K} \mathbb{E}[\|\nabla f(w_k)\|] \leq O\left(\frac{\sqrt{d \log(n/\delta)}}{nm\epsilon\sqrt{K}}\right) + O\left(\frac{L \sum_{i \in [n]} \|w_i^* - w^*\|}{n}\right).$$

Thus, in this case, DP-NormFedAvg converges to a $(\beta + O\left(\frac{\|w_i^* - w^*\|}{nm\epsilon}\right))$-stationary point in $O\left(\frac{d \log(n/\delta)}{nm^2 \epsilon^2 \rho^2}\right)$ communication rounds (in expectation).

As per Corollary 2.1, we can bound the sensitivity by $O(\frac{1}{nm})$ ($m$ being the minimum number of samples within a client and $r$ being the number of participating clients) with a probability of at least $1 - 2d \exp\left(-\mathcal{O}(m)\right)$. This high probability bound is smaller than the worst-case sensitivity value that we took in Corollary 1.1 by a factor of $O(\frac{1}{m})$. Using this upper bound as the sensitivity in Theorem 1 allows us to improve the communication-round complexity by a factor of $O(m^2)$ as compared to the result in Corollary 1.1 at the expense of increasing $\delta$ by $2d \exp\left(-\mathcal{O}(m)\right)$.

Finally, as a prospective future work, it is worth noting that the proof of Theorem 2 can be extended to show that (DP-)NormFedAvg is uniformly stable which, by McDiarmid’s inequality, implies small generalization error (see [17]).

### 5 Numerical Experiments

**Federated Setting:**

We consider the task of private multi-class classification. We perform our experiments on two benchmarking datasets, CIFAR-10 and Fashion-MNIST [38] (abbreviated as FMNIST henceforth), each with 10 classes.
For DP, we compare our normalization approach against clipping, where the clients send clipped updates to the server (as described in the second paragraph of Section 4). We call this method Clipped-FedAvg and it summarized in Algorithm 3 (in the Appendix). To recap, we highlight its main differences from DP-NormFedAvg. In line 9, client $i$ sends $d_k^{(i)} \min(1, \frac{C}{\|d_k^{(i)}\|})$ plus noise to the server, where $d_k^{(i)} = (w_k - w_k^{(i)})$ is the $i$th client’s update and $C$ is the clipping threshold that needs to be tuned; in contrast, client $i$ sends $d_k^{(i)} + \sigma$ plus noise in DP-NormFedAvg.

Also, note that the worst-case sensitivity for Clipped-FedAvg is $2C$ instead of $\frac{2}{r}$ (which is the worst-case sensitivity for DP-NormFedAvg as mentioned in Corollary 1.1). We use the worst-case sensitivity values in these experiments.

Also recall the NormFedAvg algorithm described in Remark 1–this is the same as DP-NormFedAvg, except that we do not add any noise, i.e. $\sigma = 0$ in Algorithm 1. One can view NormFedAvg as the base algorithm of DP-NormFedAvg. Thus, FedAvg and NormFedAvg (both with no DP) serve as natural baselines in our experiments. Further, in Remark 1, we discussed how theoretically NormFedAvg has the same order-wise convergence rate as FedAvg for smooth quasar-convex functions. Now, we verify this assertion with these experiments.

(i) Logistic Regression:

We first consider logistic regression with $\ell_2$-regularization; the weight decay value in PyTorch for $\ell_2$-regularization is set to $1e^{-4}$. For CIFAR-10, we use 512-dimensional features extracted from the last layer of a ResNet-18 [19] model pretrained on the ImageNet dataset. For FMNIST, we just flatten each image into a 784-dimensional vector. Similar to [26], we simulate a heterogeneous setting by distributing the data among the clients such that each client can have data from at most five classes. The exact procedure is described in Appendix F. For the CIFAR-10 (respectively, FMNIST) experiment, the number of clients $n$ is set to 5000 (respectively, 1000), with each client having the same number of samples. The number of participating clients in each round is set to $r = 0.4\cdot n$ for both datasets, with 20 local client updates per-round. We also apply the standard momentum of PyTorch with its default value to the local updates.

We make each round of DP-NormFedAvg and Clipped-FedAvg (0.25, $10^{-7}$)-DP. Each algorithm is run for 100 rounds which leads to a total privacy loss of $\epsilon = 13.83$ and $\delta = 2 \times 10^{-5}$ over all the rounds using the strong composition theorem; these values can be tightened by using the moments account method (but that is not the focus of this paper). Figures 2a, 2b and Table 1 show the comparison between DP-NormFedAvg, Clipped-FedAvg with the best tuned threshold, FedAvg and NormFedAvg, aver-

(ii) CNN and MLP:

For FMNIST, we use a small CNN; the exact architecture is described in Appendix F. For CIFAR-10, we use 512 × 256 × 10 fully connected network (or multi-layer perceptron, abbreviated as MLP) with ReLU activation after the hidden layer. As we did in (a) for CIFAR-10, we use 512-dimensional features extracted from the last layer of a ResNet-18 [19] pretrained on ImageNet. The number of clients $n$ for FMNIST and CIFAR-10 are 1000 and 2000, respectively, with each client having the same number of samples. Just like in (a), we distribute the data among the clients so that each client can have data from at most five classes. Also, the number of participating clients in each round is set to $r = 0.4\cdot n$ for both datasets, with 20 local client updates per-round.

Each algorithm is run for 100 rounds and we make each round of DP-NormFedAvg and Clipped-FedAvg (0.5, $10^{-6}$)-DP. Figures 2c, 2d and Table 2 show the comparison between DP-NormFedAvg, Clipped-FedAvg with the best tuned threshold, FedAvg and NormFedAvg, aver-
aged over three different runs.

The details of $\eta_k$, $\beta_k$ (for DP-NormFedAvg and NormFedAvg) and $C$ (for Clipped-FedAvg) for both (i) and (ii) can be found in Appendix F.

The caption of Figure 2 discusses the results and implications in detail. The two main take-away messages from these experiments are:

- DP-NormFedAvg outperforms Clipped-FedAvg.
- NormFedAvg and FedAvg have similar convergence and generalization. Thus, NormFedAvg can potentially serve as a drop-in replacement for FedAvg.

| Algo.          | CIFAR-10       | FMNIST       |
|----------------|----------------|--------------|
| DP-NormFedAvg  | 60.24 ± 0.08   | 26.59 ± 0.10 |
| Clipped-FedAvg | 61.35 ± 0.12   | 27.69 ± 0.12 |
| FedAvg         | 55.92 ± 0.16   | 18.32 ± 0.01 |
| NormFedAvg     | 56.06 ± 0.04   | 18.85 ± 0.09 |

Table 1: Average test error % (± standard deviation) over the last five rounds for the plots in Figures 2a and 2b (corresponding to (i)). For CIFAR-10, the difference between the average test error over the last 5 rounds of DP-NormFedAvg and FedAvg, and Clipped-NormFedAvg and FedAvg is 4.32% and 5.43%, respectively. For FMNIST, the corresponding numbers are 8.27% and 9.37%, respectively.

| Algo.          | CIFAR-10       | FMNIST       |
|----------------|----------------|--------------|
| DP-NormFedAvg  | 64.07 ± 0.20   | 30.49 ± 0.18 |
| Clipped-FedAvg | 65.65 ± 0.11   | 31.54 ± 0.17 |
| FedAvg         | 54.81 ± 0.16   | 18.62 ± 0.32 |
| NormFedAvg     | 55.23 ± 0.10   | 18.06 ± 0.24 |

Table 2: Average test error % (± standard deviation) over the last five rounds for the plots in Figures 2c and 2d (corresponding to (ii)). For CIFAR-10, the difference between the average test error over the last 5 rounds of DP-NormFedAvg and FedAvg, and Clipped-NormFedAvg and FedAvg is 9.26% and 10.84%, respectively. For FMNIST, the corresponding numbers are 11.87% and 12.92%, respectively.

Centralized Setting:

We also compare per-sample gradient normalization against standard gradient clipping used in DP-SGD (in the centralized setting). For that, we consider the task of privately fine-tuning a VGG-13 network [32], pretrained on CIFAR-100, on CIFAR-10. We use a batch size of 500 (so each epoch consists of 100 iterations) and fine-tune for 50 epochs, such that each iteration is (0.05, $10^{-10}$)-DP. The rest of the experimental details are in Appendix F. Figure 3 shows the comparison between gradient normalization, gradient clipping with the best tuned threshold and regular SGD without DP (as baseline), averaged over three runs. We see that normalization does better than clipping even in the centralized setting.
Figure 2: Comparison of DP-NormFedAvg, Clipped-FedAvg, FedAvg and NormFedAvg: Variation of training loss and test error on CIFAR-10 (left) and FMNIST (right) in the federated settings (i) and (ii). The corresponding average test errors over the last 5 rounds is listed in Tables 1 and 2 for (i) and (ii) respectively. In all cases, DP-NormFedAvg outperforms Clipped-FedAvg by more than 1%. As discussed in Section 4 when the client update norms are smaller than the clipping threshold for Clipped-FedAvg, the noise scale is more than the update norms which seems to be inhibiting convergence, thereby explaining the poorer performance of Clipped-FedAvg compared to DP-NormFedAvg. In other words, the signal (which is the update norm) to noise ratio of Clipped-FedAvg eventually becomes lower than that of DP-NormFedAvg, which explains its poorer performance. Also observe that NormFedAvg and FedAvg have similar performance which validates our theoretical assertion in Remark 1.

6 Conclusion and Limitations

We proposed DP-NormFedAvg, a differentially private federated learning algorithm where we propose using normalized, instead of clipped, client updates. We established that DP-NormFedAvg
Figure 3: Comparison between gradient normalization, clipping and regular SGD (w/o DP) in the \textbf{centralized setting}. The corresponding mean test \textbf{errors} over the last 5 epochs are 40.68(±0.10)\%, 41.49(±0.03)\% and 11.85(±0.08)\%, respectively. Thus, normalization outperforms clipping by more than 0.8\% here.

has the same order-wise convergence rate (w.r.t. the number of communication rounds) as \texttt{FedAvg} on smooth quasar-convex functions. We also derived a high probability bound on the sensitivity under Assumption 3, which leads to a big improvement in the convergence rate at the expense of slightly weakening the privacy guarantee. Our experiments demonstrate the benefit of normalization over clipping.

We conclude by discussing some limitations of our work. First, even though we theoretically showed that \texttt{DP-NormFedAvg} has the same order-wise convergence rate as \texttt{FedAvg} on smooth quasar-convex functions, we are unable to do the same for just smooth (non-convex) functions. Also, our high-probability bound on the sensitivity is derived under Assumption 3, which is hard to verify in practice. Deriving good sensitivity bounds that can be used in practice for improving convergence would be a valuable future work.

References

[1] Abadi, M., Chu, A., Goodfellow, I., McMahan, H. B., Mironov, I., Talwar, K., and Zhang, L. Deep learning with differential privacy. In \textit{Proceedings of the 2016 ACM SIGSAC conference on computer and communications security} (2016), pp. 308–318.

[2] Agarwal, N., Suresh, A. T., Yu, F., Kumar, S., and McMahan, H. B. \texttt{cpsgd}: Communication-efficient and differentially-private distributed sgd. \textit{arXiv preprint arXiv:1805.10559} (2018).

[3] Asoodeh, S., and Calmon, F. Differentially private federated learning: An information-theoretic perspective. In \textit{Proc. ICML-FL} (2020).

[4] Balle, B., Barthe, G., and Gaboardi, M. Privacy amplification by subsampling: Tight analyses via couplings and divergences. \textit{arXiv preprint arXiv:1807.01647} (2018).

[5] Bassily, R., Smith, A., and Thakurta, A. Private empirical risk minimization: Efficient algorithms and tight error bounds. In \textit{2014 IEEE 55th Annual Symposium on Foundations of Computer Science} (2014), IEEE, pp. 464–473.
[6] Bottou, L. Stochastic gradient descent tricks. In Neural networks: Tricks of the trade. Springer, 2012, pp. 421–436.

[7] Chaudhuri, K., Monteleoni, C., and Sarwate, A. D. Differentially private empirical risk minimization. Journal of Machine Learning Research 12, 3 (2011).

[8] Chen, X., Wu, S. Z., and Hong, M. Understanding gradient clipping in private sgd: A geometric perspective. Advances in Neural Information Processing Systems 33 (2020).

[9] Cutkosky, A., and Mehta, H. Momentum improves normalized sgd. In International Conference on Machine Learning (2020), PMLR, pp. 2260–2268.

[10] Dvorkin, V., Fioretto, F., Van Hentenryck, P., Kazempour, J., and Pinson, P. Differentially private convex optimization with feasibility guarantees. arXiv preprint arXiv:2006.12338 (2020).

[11] Dwork, C., McSherry, F., Nissim, K., and Smith, A. Calibrating noise to sensitivity in private data analysis. In Theory of cryptography conference (2006), Springer, pp. 265–284.

[12] Dwork, C., Roth, A., et al. The algorithmic foundations of differential privacy. Foundations and Trends in Theoretical Computer Science 9, 3-4 (2014), 211–407.

[13] Geyer, R. C., Klein, T., and Nabi, M. Differentially private federated learning: A client level perspective. arXiv preprint arXiv:1712.07557 (2017).

[14] Girgis, A., Data, D., Diggavi, S., Kairouz, P., and Suresh, A. T. Shuffled model of differential privacy in federated learning. In International Conference on Artificial Intelligence and Statistics (2021), PMLR, pp. 2521–2529.

[15] Gower, R. M., Sebbouh, O., and Loizou, N. Sgd for structured nonconvex functions: Learning rates, minibatching and interpolation. arXiv preprint arXiv:2006.10311 (2020).

[16] Haddadpour, F., Kamani, M. M., Mokhtari, A., and Mahdavi, M. Federated learning with compression: Unified analysis and sharp guarantees. arXiv preprint arXiv:2007.01154 (2020).

[17] Hardt, M., Recht, B., and Singer, Y. Train faster, generalize better: Stability of stochastic gradient descent. In International Conference on Machine Learning (2016), PMLR, pp. 1225–1234.

[18] Hazan, E., Levy, K. Y., and Shalev-Shwartz, S. Beyond convexity: Stochastic quasi-convex optimization. arXiv preprint arXiv:1507.02030 (2015).

[19] He, K., Zhang, X., Ren, S., and Sun, J. Deep residual learning for image recognition. In Proceedings of the IEEE conference on computer vision and pattern recognition (2016), pp. 770–778.

[20] Hinder, O., Sidford, A., and Sohoni, N. Near-optimal methods for minimizing star-convex functions and beyond. In Conference on Learning Theory (2020), PMLR, pp. 1894–1938.

[21] Iyengar, R., Near, J. P., Song, D., Thakkar, O., Thakurta, A., and Wang, L. Towards practical differentially private convex optimization. In 2019 IEEE Symposium on Security and Privacy (SP) (2019), IEEE, pp. 299–316.

[22] Jin, C., Netrapalli, P., Ge, R., Kakade, S. M., and Jordan, M. I. A short note on concentration inequalities for random vectors with subgaussian norm. arXiv preprint arXiv:1902.03736 (2019).
[23] Kleinberg, B., Li, Y., and Yuan, Y. An alternative view: When does sgd escape local minima? In *International Conference on Machine Learning* (2018), PMLR, pp. 2698–2707.

[24] Kuru, N., Birbil, Ş. İ., Gurbuzbalaban, M., and Yildirim, S. Differentially private accelerated optimization algorithms. *arXiv preprint arXiv:2008.01989* (2020).

[25] Li, T., Liu, Z., Sekar, V., and Smith, V. Privacy for free: Communication-efficient learning with differential privacy using sketches. *arXiv preprint arXiv:1911.00972* (2019).

[26] McMahan, B., Moore, E., Ramage, D., Hampson, S., and y Arcas, B. A. Communication-efficient learning of deep networks from decentralized data. In *Artificial Intelligence and Statistics* (2017), PMLR, pp. 1273–1282.

[27] Nesterov, Y., and Polyak, B. T. Cubic regularization of newton method and its global performance. *Mathematical Programming* 108, 1 (2006), 177–205.

[28] Nguyen, T. D., Rieger, P., Yalame, H., Möllering, H., Fereidooni, H., Marchal, S., Miettinen, M., Mirhoseini, A., Saegh, A.-R., Schneider, T., et al. Figuard: Secure and private federated learning. *arXiv preprint arXiv:2101.02281* (2021).

[29] Peterson, D., Kanani, P., and Marathe, V. J. Private federated learning with domain adaptation. *arXiv preprint arXiv:1912.06733* (2019).

[30] Pichapati, V., Suresh, A. T., Yu, F. X., Reddi, S. J., and Kumar, S. Adaclip: Adaptive clipping for private sgd. *arXiv preprint arXiv:1908.07643* (2019).

[31] Schmidt, M., and Roux, N. L. Fast convergence of stochastic gradient descent under a strong growth condition. *arXiv preprint arXiv:1308.6370* (2013).

[32] Simonyan, K., and Zisserman, A. Very deep convolutional networks for large-scale image recognition. *arXiv preprint arXiv:1409.1556* (2014).

[33] Song, S., Chaudhuri, K., and Sarwate, A. D. Stochastic gradient descent with differentially private updates. In *2013 IEEE Global Conference on Signal and Information Processing* (2013), IEEE, pp. 245–248.

[34] Thakkur, O., Andrew, G., and McMahan, H. B. Differentially private learning with adaptive clipping. *arXiv preprint arXiv:1905.03871* (2019).

[35] Vaswani, S., Bach, F., and Schmidt, M. Fast and faster convergence of sgd for over-parameterized models and an accelerated perceptron. In *The 22nd International Conference on Artificial Intelligence and Statistics* (2019), PMLR, pp. 1195–1204.

[36] Wang, P., Lei, Y., Ying, Y., and Zhang, H. Differentially private sgd with non-smooth loss. *arXiv preprint arXiv:2101.08925* (2021).

[37] Wu, X., Li, F., Kumar, A., Chaudhuri, K., Jha, S., and Naughton, J. Bolt-on differential privacy for scalable stochastic gradient descent-based analytics. In *Proceedings of the 2017 ACM International Conference on Management of Data* (2017), pp. 1307–1322.

[38] Xiao, H., Rasul, K., and Vollgraf, R. Fashion-mnist: a novel image dataset for benchmarking machine learning algorithms. *arXiv preprint arXiv:1708.07747* (2017).

[39] You, Y., Gitman, I., and Ginsburg, B. Large batch training of convolutional networks. *arXiv preprint arXiv:1708.03888* (2017).
[40] You, Y., Li, J., Reddi, S., Hseu, J., Kumar, S., Bhojanapalli, S., Song, X., Demmel, J., Keutzer, K., and Hsieh, C.-J. Large batch optimization for deep learning: Training bert in 76 minutes. *arXiv preprint arXiv:1904.00962* (2019).

[41] Zhang, J., He, T., Sra, S., and Jadbabaie, A. Why gradient clipping accelerates training: A theoretical justification for adaptivity. *arXiv preprint arXiv:1905.11881* (2019).

[42] Zhang, J., Karimireddy, S. P., Veit, A., Kim, S., Reddi, S., Kumar, S., and Sra, S. Why are adaptive methods good for attention models? *Advances in Neural Information Processing Systems 33* (2020).

[43] Zhang, X., Chen, X., Hong, M., Wu, Z. S., and Yi, J. Understanding clipping for federated learning: Convergence and client-level differential privacy. *arXiv preprint arXiv:2106.13673* (2021).

[44] Zhou, Y., Yang, J., Zhang, H., Liang, Y., and Tarokh, V. Sgd converges to global minimum in deep learning via star-convex path. *arXiv preprint arXiv:1901.00451* (2019).
Supplementary Material

A Proof of Theorem 1

Proof. Privacy results:

(a) \((\epsilon, \delta)\)-DP in each round: Recall that in each round only \(r\) out of \(n\) clients are participating, and these are sampled uniformly at random without replacement. Thus, we use the privacy amplification by sub-sampling (without replacement) result of \(1\) (specifically Theorem 9 in their paper) along with the standard formula for Gaussian mechanism to set \(\sigma\).

(b) \((\epsilon, \delta)\)-DP over all \(K\) rounds: This follows from Theorem 1 of \(1\).

Convergence result: Define

\[ u_k^{(i)} := \sum_{\tau=0}^{E-1} \nabla f_i(w_{k,\tau}^{(i)}) = \frac{(w_k - w_{k,\epsilon})}{\eta_k} \]  

and

\[ g_k := \frac{1}{r} \sum_{i \in S_k} g_k^{(i)} = \frac{1}{r} \sum_{i \in S_k} \frac{u_k^{(i)}}{\|u_k^{(i)}\|} \]  

Then

\[ a_k = g_k + \zeta_k, \]

where \(\zeta_k = \frac{1}{r} \sum_{i \in S_k} \zeta_k^{(i)} \sim \mathcal{N}(0_d, \sigma^2 I_d)\).

Let us take \(\beta_k = c\eta_k\) for some \(c > 0\). Then, the update rule of the global iterate is:

\[ w_{k+1} = w_k - c\eta_k a_k. \]  

Also:

\[ \mathbb{E}[a_k] = \mathbb{E}_{S_k}[\mathbb{E}_{\zeta_k}[g_k + \zeta_k]] = \mathbb{E}_{S_k}[g_k] = \mathbb{E}_{S_k}\left[\frac{1}{r} \sum_{i \in S_k} \frac{u_k^{(i)}}{\|u_k^{(i)}\|}\right] = \frac{1}{n} \sum_{i \in [n]} \frac{u_k^{(i)}}{\|u_k^{(i)}\|} \]  

and

\[ \mathbb{E}[\|a_k\|^2] = \mathbb{E}_{S_k}[\mathbb{E}_{\zeta_k}[\|g_k + \zeta_k\|^2]] = \mathbb{E}_{S_k}\left[\|g_k\|^2 + d\sigma^2\right] \]  

\[ = \mathbb{E}_{S_k}\left[\left\|\frac{1}{r} \sum_{i \in S_k} \frac{u_k^{(i)}}{\|u_k^{(i)}\|}\right\|^2 + d\sigma^2\right] \]  

\[ \leq 1 + d\sigma^2. \]

We also define \(\Phi := \arg \min_w f(w)\) and \(\Phi_i := \arg \min_w f_i(w)\).

Now for any \(w^* \in \Phi := \arg \min_w f(w)\), we have:

\[ \mathbb{E}[\|w_{k+1} - w^*\|^2] = \mathbb{E}[\|w_k - c\eta_k a_k - w^*\|^2] \]

\[ = \mathbb{E}[\|w_k - w^*\|^2] - 2c\eta_k \mathbb{E}[(w_k - w^*, a_k)] + c^2 \eta_k^2 \mathbb{E}[\|a_k\|^2] \]

\[ \leq \mathbb{E}[\|w_k - w^*\|^2] - \frac{2c}{n} \sum_{i \in [n]} \left(w_k - w^*, \frac{\eta_k u_k^{(i)}}{\|u_k^{(i)}\|}\right) + c^2 \eta_k^2 (1 + d\sigma^2). \]
Equation (20) follows by plugging in the results of eq. (15) and eq. (19).

Let us analyze (I). For any \( w_i^* \in \Phi_i := \arg \min_w f_i(w) \), we have:

\[
(1) = \left( w_k - w_i^*, \frac{\eta_k u_k^{(i)}}{\|u_k^{(i)}\|} \right) + \left( w_i^* - w_i^{*}, \frac{\eta_k u_k^{(i)}}{\|u_k^{(i)}\|} \right) \\
:= (ii)
\]

Let us now deal with (II). Using the fact that for any two vectors \( a \) and \( b \), \( \langle a, b \rangle = \frac{1}{2}(\|a\|^2 + \|b\|^2 - \|a - b\|^2) \), we get:

\[
(II) = \left( w_k - w_i^*, \frac{\eta_k u_k^{(i)}}{\|u_k^{(i)}\|} \right) = \frac{1}{2\|u_k^{(i)}\|} (\|w_k - w_i^*\|^2 + \eta_k^2 \|u_k^{(i)}\|^2 - \|w_k - \eta_k u_k^{(i)} - w_i^*\|^2) \quad (22)
\]

Notice that \( w_k - \eta_k u_k^{(i)} = w_i^* \). We now present a simple lemma to lower bound \( \|w_k - w_i^*\|^2 - \|w_k^{(e)} - w_i^*\|^2 \).

**Lemma 1.** Suppose Assumptions 1 and 2 hold. Set \( \eta_k \leq \frac{\zeta}{2E} \). Then:

\[
\|w_k^{(i)} - w_i^*\|^2 \leq \|w_k - w_i^*\|^2 - \frac{\eta_k}{2E} \sum_{\tau=0}^{E-1} \|\nabla f_i(w_k^{(i), \tau})\|^2.
\]

The proof of Lemma 1 can be found after the end of this proof.

Using Lemma 1 we get:

\[
\|w_k - w_i^*\|^2 - \|w_k^{(i)} - w_i^*\|^2 \geq \frac{\eta_k \zeta}{2E} \sum_{\tau=0}^{E-1} \|\nabla f_i(w_k^{(i), \tau})\|^2,
\]

for \( \eta_k \leq \frac{\zeta}{2E} \). But:

\[
\|u_k^{(i)}\|^2 = \| \sum_{\tau=0}^{E-1} \nabla f_i(w_k^{(i), \tau}) \|^2 \leq E \sum_{\tau=0}^{E-1} \|\nabla f_i(w_k^{(i), \tau})\|^2.
\]

The above follows from the fact that for any \( p > 1 \) vectors \( \{y_1, \ldots, y_p\} \), \( \|\sum_{i=1}^p y_i\|^2 \leq p \sum_{i=1}^p \|y_i\|^2 \).

Using eq. (24) in eq. (23), we get for \( \eta_k \leq \frac{\zeta}{2E} \):

\[
\|w_k - w_i^*\|^2 - \|w_k^{(i)} - w_i^*\|^2 \geq \frac{\eta_k \zeta}{2E} \|u_k^{(i)}\|^2.
\]

Using eq. (25) in eq. (22) gives us:

\[
(II) \geq \left( \eta_k^2 + \frac{\eta_k \zeta}{2E} \right) \frac{\|u_k^{(i)}\|}{2}.
\]

Now using eq. (26) in eq. (21):

\[
(1) \geq \left( \eta_k^2 + \frac{\eta_k \zeta}{2E} \right) \frac{\|u_k^{(i)}\|}{2} - \eta_k \|w_i^* - w_i^{*}\|,
\]

for \( \eta_k \leq \frac{\zeta}{2E} \).

Next, we provide a lemma to lower bound \( \|u_k^{(i)}\| \) in terms of \( \|\nabla f_i(w_k)\| \). Its proof can be also found after the end of this proof.
Lemma 2. Suppose Assumption 2 holds. Set \( \eta_k LE \leq 1/4 \). Then:

\[
\|u_k^{(i)}\| \geq \frac{2E}{3} \|\nabla f_i(w_k)\|,
\]

Using the result of Lemma 2 we have for \( \eta_k LE \leq 1/4 \):

\[
(1) \geq \frac{1}{3} \left( \eta_k^2 E + \frac{\eta_k \zeta}{2L} \right) \|\nabla f_i(w_k)\| - \eta_k \|w_i^* - w^*\|. \tag{28}
\]

Plugging this back in eq. (20), we get:

\[
E[\|w_{k+1} - w^*\|^2] \leq E[\|w_k - w^*\|^2] - \frac{c}{3} \left( \eta_k^2 E + \frac{\eta_k \zeta}{2L} \right) \sum_{i \in [n]} E[\|\nabla f_i(w_k)\|] + \frac{2c\eta_k}{n} \sum_{i \in [n]} \|w_i^* - w^*\| + \epsilon^2 \eta_k^2 (1 + d\sigma^2).
\]

Let us choose \( \eta_k = \eta \) for all \( k \); this must satisfy \( \eta \leq \min \left( \frac{1}{4LE}, \frac{\zeta}{2L} \right) \). Also, recall that \( \|\nabla f(w_k)\| = \left\| \sum_{i \in [n]} \nabla f_i(w_k) \right\| \leq \frac{1}{n} \sum_{i \in [n]} \|\nabla f_i(w_k)\| \). Using these in the above equation, rearranging it a bit and summing up from \( k = 0 \) through to \( k = K - 1 \), we get:

\[
\frac{1}{K} \sum_{k=0}^{K-1} E[\|\nabla f(w_k)\|] \leq \frac{\|w_0 - w^*\|^2}{\eta K} + \frac{3c(1 + d\sigma^2)}{2(\eta E + \frac{\zeta}{2L})} \frac{1}{K} \sum_{i \in [n]} \|w_i^* - w^*\|. \tag{30}
\]

Let us set \( \eta = \frac{\zeta}{4L\sqrt{K(1 + d\sigma^2)}} \) above; it can be checked that for \( E \leq \frac{1}{\zeta} \sqrt{K(1 + d\sigma^2)} \), all constraints on \( \eta \) are met. With that, we get:

\[
\frac{1}{K} \sum_{k=0}^{K-1} E[\|\nabla f(w_k)\|] \leq \left( \frac{12L^2 \|w_0 - w^*\|^2}{c\zeta^2} + \frac{3c}{4} \right) \left( \frac{\sqrt{1 + d\sigma^2}}{1 + \frac{2K}{2\zeta(1 + d\sigma^2)}} \right) \frac{1}{K} \sum_{i \in [n]} \|w_i^* - w^*\|. \tag{31}
\]

Recall that \( w_i^* \) can be any point in \( \Phi_i \). In order to minimize the LHS, let us set \( w_i^* = \arg\min_{w \in \Phi_i} \|w - w^*\| \). The final result follows from eq. (31) by utilizing the fact that

\[
\min_{k \in \{0, \ldots, K-1\}} E[\|\nabla f(w_k)\|] \leq \frac{1}{K} \sum_{k=0}^{K-1} E[\|\nabla f(w_k)\|],
\]

and choosing \( c = \frac{4L\|w_0 - w^*\|}{\zeta} \).

**Proof of Lemma 1**

Proof. For any \( \tau \geq 0 \), we have:

\[
\|w_{k,\tau+1}^{(i)} - w_i^*\|^2 = \|w_{k,\tau}^{(i)} - w_i^*\|^2 - 2\eta_k \langle \nabla f_i(w_{k,\tau}^{(i)}), w_{k,\tau}^{(i)} - w_i^* \rangle + \eta_k^2 \|\nabla f_i(w_{k,\tau}^{(i)})\|^2 \leq \|w_{k,\tau}^{(i)} - w_i^*\|^2 - 2\eta_k \zeta (f_i(w_{k,\tau}^{(i)}) - f_i^*) + \eta_k^2 \|\nabla f_i(w_{k,\tau}^{(i)})\|^2 \tag{32}\]

\[
\leq \|w_{k,\tau}^{(i)} - w_i^*\|^2 - \eta_k \zeta \frac{L}{E} \|\nabla f_i(w_{k,\tau}^{(i)})\|^2 + \eta_k^2 \|\nabla f_i(w_{k,\tau}^{(i)})\|^2 \tag{33}
\]

\[20\]
Equation (32) follows by using the fact that each \( f_i \) is \( \zeta \)-quasar convex with respect to \( w_i^* \in \Phi_I \) due to which \( \langle \nabla f_i(w^{(i)}_{k,\tau}), w^{(i)}_{k,\tau} - w^*_i \rangle \geq \zeta (f_i(w^{(i)}_{k,\tau}) - f_i^*) \). Equation (33) follows from the fact that \( \| \nabla f_i(w^{(i)}_{k,\tau}) \|^2 \leq 2L(f_i(w^{(i)}_{k,\tau}) - f_i^*) \) for \( L \)-smooth \( f_i \).

Now if we set \( \eta_k \leq \frac{\zeta}{2L} \), then we get:

\[
\|w^{(i)}_{k,\tau+1} - w^*_i\|^2 \leq \|w^{(i)}_{k,\tau} - w^*_i\|^2 - \frac{\eta_k \zeta}{2L} \|\nabla f_i(w^{(i)}_{k,\tau})\|^2.
\] (34)

Doing this recursively for \( \tau = 0 \) through to \( \tau = E - 1 \) and adding everything up gives us the desired result.

**Proof of Lemma 2**

Proof. We have:

\[
\|u^{(i)}_k\|^2 = \| \sum_{\tau=0}^{E-1} \nabla f_i(w^{(i)}_{k,\tau}) \|^2
\]

\[
= \sum_{\tau=0}^{E-1} \| \nabla f_i(w^{(i)}_{k,\tau}) \|^2 + 2 \sum_{\tau \neq \tau'} \langle \nabla f_i(w^{(i)}_{k,\tau}), \nabla f_i(w^{(i)}_{k,\tau'}) \rangle
\]

\[
= \sum_{\tau=0}^{E-1} \| \nabla f_i(w^{(i)}_{k,\tau}) \|^2 + \sum_{\tau < \tau'} \{ \| \nabla f_i(w^{(i)}_{k,\tau}) \|^2 + \| \nabla f_i(w^{(i)}_{k,\tau'}) \|^2 - \| \nabla f_i(w^{(i)}_{k,\tau}) - \nabla f_i(w^{(i)}_{k,\tau'}) \|^2 \}
\] (35)

\[
= E \sum_{\tau=0}^{E-1} \| \nabla f_i(w^{(i)}_{k,\tau}) \|^2 - \sum_{\tau < \tau'} \| \nabla f_i(w^{(i)}_{k,\tau}) - \nabla f_i(w^{(i)}_{k,\tau'}) \|^2.
\] (36)

Equation (35) follows from the fact that for any two vectors \( a \) and \( b \), \( \langle a, b \rangle = \frac{1}{2}(\|a\|^2 + \|b\|^2 - \|a - b\|^2) \). Now:

\[
\| \nabla f_i(w^{(i)}_{k,\tau}) - \nabla f_i(w^{(i)}_{k,\tau'}) \|^2 \leq L^2 \|w^{(i)}_{k,\tau} - w^{(i)}_{k,\tau'}\|^2 \\
\leq \eta_k^2 L^2 \| \sum_{t=\tau}^{\tau'-1} \nabla f_i(w^{(i)}_{k,t}) \|^2 \\
\leq \eta_k^2 L^2 (\tau' - \tau) \sum_{t=\tau}^{\tau'-1} \| \nabla f_i(w^{(i)}_{k,t}) \|^2 \\
\leq \eta_k^2 L^2 E \sum_{t=0}^{E-1} \| \nabla f_i(w^{(i)}_{k,t}) \|^2.
\] (37)

Thus:

\[
\sum_{\tau < \tau'} \| \nabla f_i(w^{(i)}_{k,\tau}) - \nabla f_i(w^{(i)}_{k,\tau'}) \|^2 \leq \sum_{\tau < \tau'} \eta_k^2 L^2 E \sum_{t=0}^{E-1} \| \nabla f_i(w^{(i)}_{k,t}) \|^2 \leq \frac{\eta_k^2 L^2 E^3}{2} \sum_{t=0}^{E-1} \| \nabla f_i(w^{(i)}_{k,t}) \|^2.
\] (38)

Putting eq. (38) in eq. (36), we get:

\[
\|u^{(i)}_k\|^2 \geq E \left( 1 - \frac{\eta_k^2 L^2 E^2}{2} \right) \sum_{\tau=0}^{E-1} \| \nabla f_i(w^{(i)}_{k,\tau}) \|^2.
\] (39)
Obviously, we should choose \( \eta_k LE \leq \sqrt{2} \) for this bound to be non-trivial.

Further, for any \( \tau \geq 1 \):

\[
\| \nabla f_i(w_k) \| \leq \| \nabla f_i(w_{k,\tau}) \| + \| \nabla f_i(w_k) - \nabla f_i(w_{k,\tau}) \|
\leq \| \nabla f_i(w_{k,\tau}) \| + L\| w_k - w_{k,\tau} \|
\leq \| \nabla f_i(w_{k,\tau}) \| + 2\eta_k L\tau \| \nabla f_i(w_k) \|
\]

The last step follows by using Lemma 3 (stated and proved after this proof) for \( \eta_k LE \leq 1/2 \).

Now if we set \( \eta_k LE \leq 1/4 \) (which implies \( 2\eta_k L\tau \leq 1/2 \) for all \( \tau \geq 1 \)), then we get:

\[
\| \nabla f_i(w_{k,\tau}) \| \geq \frac{\| \nabla f_i(w_k) \|}{2} \quad \forall \tau \geq 1.
\]

Using eq. (40) in eq. (39), we get:

\[
\| u_k^{(i)} \|^2 \geq \frac{31E^2}{64} \| \nabla f_i(w_k) \|^2 \geq \frac{4E^2}{9} \| \nabla f_i(w_k) \|^2,
\]

for \( \eta_k LE \leq 1/4 \).

This gives us the desired result.

\[\Box\]

Lemma 3. Suppose Assumption \( \Box \) holds. Set \( \eta_k LE \leq \frac{1}{2} \). Then:

\[
\| w_k - w_{k,\tau}^{(i)} \| \leq \eta_k \| \nabla f_i(w_k) \| \quad \forall \tau \geq 1.
\]

Proof.

\[
\| w_k - w_{k,\tau}^{(i)} \| = \| \eta_k \sum_{t=0}^{\tau-1} \nabla f_i(w_{k,t}) \| \leq \eta_k \sum_{t=0}^{\tau-1} \| \nabla f_i(w_{k,t}) \|.
\]

But:

\[
\| \nabla f_i(w_{k,t}) \| = \| \nabla f_i(w_{k,t}) - \nabla f_i(w_k) + \nabla f_i(w_k) \|
\leq \| \nabla f_i(w_k) \| + \| \nabla f_i(w_{k,t}) - \nabla f_i(w_k) \|
\leq \| \nabla f_i(w_k) \| + L\| w_{k,t}^{(i)} - w_k \|.
\]

Putting eq. (43) back in eq. (42), we get:

\[
\| w_k - w_{k,\tau}^{(i)} \| \leq \eta_k \| \nabla f_i(w_k) \| + \eta_k L \sum_{t=0}^{\tau-1} \| w_{k,t}^{(i)} - w_k \|.
\]

We claim that \( \| w_k - w_{k,\tau}^{(i)} \| \leq 2\eta_k \| \nabla f_i(w_k) \| \) for \( \eta_k LE \leq 1/2 \). We shall prove this by induction. Let us first check the base case of \( \tau = 1 \). Observe that:

\[
\| w_k - w_{k,1}^{(i)} \| = \eta_k \| \nabla f_i(w_k) \| \leq 2\eta_k \| \nabla f_i(w_k) \|.
\]

Hence, the base case is true. Assume the hypothesis holds for \( t \in \{0, \ldots, \tau - 1\} \). Let us now put our induction hypothesis into eq. (44) to see if the hypothesis is true for \( \tau \) as well.

\[
\| w_k - w_{k,\tau}^{(i)} \| \leq \eta_k \| \nabla f_i(w_k) \| + \eta_k L \sum_{t=0}^{\tau-1} 2\eta_k t \| \nabla f_i(w_k) \|
\leq \eta_k \| \nabla f_i(w_k) \| + (\eta_k L) \eta_k \tau^2 \| \nabla f_i(w_k) \|
\leq \eta_k \| \nabla f_i(w_k) \| + \eta_k \| \eta_k L\tau \| \| \nabla f_i(w_k) \|
\leq \eta_k \| \nabla f_i(w_k) \| + 0.5\eta_k \| \nabla f_i(w_k) \| \leq 2\eta_k \| \nabla f_i(w_k) \|.
\]

The second last inequality is true because \( \eta_k L\tau \leq \eta_k LE \leq \frac{1}{2} \), per our choice of \( \eta_k \).

Thus, the hypothesis holds for \( \tau \) as well. So by induction, our claim is true.

\[\Box\]
B Convergence of FedAvg in the Quasar-Convex Setting mentioned in Remark 1

Recall that in Remark 1, we compare the convergence of DP-NormFedAvg and FedAvg in the quasar-convex setting; here, we summarize FedAvg in Algorithm 2 state its complete convergence result and then prove the result. To our knowledge, this is the first convergence result for FedAvg on quasar-convex functions. Note that the proof is significantly different from the pure convex case.

Algorithm 2 FedAvg [26]

1: Input: Initial point $w_0$, number of rounds of communication $K$, number of local updates per round $E$, local learning rates $\{\eta_k\}_{k=0}^{K-1}$ and number of participating clients in each round $r$.
2: for $k = 0, \ldots, K-1$ do
3: Server chooses a set $S_k$ of $r$ clients uniformly at random without replacement and sends $w_k$ to them.
4: for client $i \in S_k$ do
5: Set $w_k^{(i)} = w_k$.
6: for $\tau = 0, \ldots, E-1$ do
7: Update $w_k^{(i)} \leftarrow w_k^{(i)} - \eta_k \nabla f_i(w_k^{(i)})$.
8: end for
9: Send $w_k^{(i)}$ to the server.
10: end for
11: Update $w_{k+1} \leftarrow \frac{1}{r} \sum_{i \in S_k} w_k^{(i)}$.
12: end for

Theorem 3 (Convergence of FedAvg). Suppose Assumptions 1 and 2 hold. In FedAvg (Algorithm 2), set $\eta_k = \frac{c}{L \bar{c}E}$ where $\bar{c} = \max \left( 4, 2 + \frac{23(n-r)}{r(n-1)} \cdot 64 \frac{n(r-1)}{r(n-1)} \right)$. As defined in Theorem 1, recall that $\Phi := \arg \min_w f(w)$ and $\Phi_i := \arg \min_{w \in \Phi_i} f_i(w)$. Then for any $w^* \in \Phi$, the global iterates $\{w_k\}_{k=0}^{K-1}$ of Algorithm 2 satisfy:

$$\min_{k \in \{0, \ldots, K-1\}} \mathbb{E}[\|\nabla f(w_k)\|^2] \leq \left( \frac{32cL^2}{\bar{c}^2} \right) \frac{\|w_0 - w^*\|^2}{K} + \left( \frac{32cL^2}{\bar{c}^2} \right) \frac{1}{n} \sum_{i \in [n]} \|w_i^* - w^*\|^2,$$

(45)

where $w_i^* = \arg \min_{w \in \Phi_i} \|w - w^*\|$.

Proof. Recall from the proof of Theorem 1 that:

$$u_k^{(i)} := \sum_{\tau=0}^{E-1} \nabla f_i(w_k^{(i)}) \left( w_k - w_k^{(i)} \right) = \frac{1}{\eta_k} \sum_{\tau=0}^{E-1} \nabla f_i(w_k^{(i)}) \left( w_k - w_k^{(i)} \right).$$

(46)

Then notice that the update rule of the global iterate for FedAvg can be expressed as:

$$w_{k+1} = \frac{1}{r} \sum_{i \in S_k} w_k^{(i)} + w_k - \frac{1}{r} \sum_{i \in S_k} (w_k - w_k^{(i)}) = w_k - \frac{\eta_k}{r} \sum_{i \in S_k} u_k^{(i)}.$$

(47)
For any \( w^* \in \Phi := \arg \min_w f(w) \), we have:

\[
\mathbb{E}[\|w_{k+1} - w^*\|^2] = \mathbb{E}[\|w_k - w^*\|^2] - 2 \mathbb{E} \left[ \left\langle w_k - w^*, \frac{1}{r} \sum_{i \in S_k} \eta_k u_k^{(i)} \right\rangle \right] + \eta_k^2 \mathbb{E} \left[ \left\| \frac{1}{r} \sum_{i \in S_k} u_k^{(i)} \right\|^2 \right].
\]

Let us first simplify (I).

\[ (I) = \frac{1}{n} \sum_{i \in [n]} \langle w_k - w^*, \eta_k u_k^{(i)} \rangle \]

\[ = \frac{1}{n} \sum_{i \in [n]} \langle w_k - w^*_i, \eta_k u_k^{(i)} \rangle - \frac{1}{n} \sum_{i \in [n]} \langle w^*_i, \eta_k u_k^{(i)} \rangle \]

\[ = \frac{1}{2n} \sum_{i \in [n]} \left( \|w_k - w^*_i\|^2 + \|\eta_k u_k^{(i)}\|^2 - \|w_k - \eta_k u_k^{(i)} - w^*_i\|^2 \right) \]

\[ = \frac{1}{2n} \sum_{i \in [n]} \left( \|w^* - w^*_i\|^2 + \|\eta_k u_k^{(i)}\|^2 - \|w^* - \eta_k u_k^{(i)}\|^2 \right) \]

\[ \geq \frac{1}{2n} \sum_{i \in [n]} \left( \|w_k - w^*_i\|^2 - \|u_k^{(i)} - w^*_i\|^2 \right) - \frac{1}{2n} \sum_{i \in [n]} \|w^* - w^*_i\|^2 \]

Note that eq. \[49\] holds for any \( w^*_i \in \Phi_i := \arg \min_w f_i(w) \). Equation \[50\] follows by using the fact that for any two vectors \( a \) and \( b \), \( \langle a, b \rangle = \frac{1}{2}(\|a\|^2 + \|b\|^2 - \|a - b\|^2) \).

From Lemma 1 for \( \eta_k \leq \frac{c}{2L} \), we have that:

\[
\|w_k - w^*\|^2 - \|u_k^{(i)} - w^*_i\|^2 \geq \frac{\eta_k \zeta}{2L} \sum_{i \in S_k} \|\nabla f_i(w_k^{(i)})\|^2.
\]

Also from eq. \[40\] in the proof of Lemma 2 we have for \( \eta_k LE \leq 1/4 \):

\[
\frac{1}{2} \sum_{i \in [n]} \|\nabla f_i(w_k^{(i)})\|^2 \geq \frac{\eta_k LE}{2L} \sum_{i \in S_k} \|\nabla f_i(w_k^{(i)})\|^2.
\]

Using this in eq. \[52\] with \( \eta_k \leq \min\left(\frac{c}{2L}, \frac{1}{4LE}\right) \), we get:

\[
\frac{1}{16L^2n} \sum_{i \in [n]} \|\nabla f_i(w_k^{(i)})\|^2 - \frac{1}{2n} \sum_{i \in [n]} \|w^* - w^*_i\|^2.
\]

Let us now simplify (II).

\[
(II) = \mathbb{E} \left[ \left\| \frac{1}{r} \sum_{i \in S_k} u_k^{(i)} \right\|^2 \right] = \frac{1}{r^2} \sum_{i \in [n]} \mathbb{P}(i \in S_k) \|u_k^{(i)}\|^2 + \frac{1}{r^2} \sum_{i, j \in [n], i \neq j} \mathbb{P}(i, j \in S_k) \langle u_k^{(i)}, u_k^{(j)} \rangle \]

\[ = \frac{1}{r^2} \sum_{i \in [n]} \frac{r}{n} \|u_k^{(i)}\|^2 + \frac{1}{r^2} \sum_{i, j \in [n], i \neq j} \frac{r(r - 1)}{n(n - 1)} \langle u_k^{(i)}, u_k^{(j)} \rangle \]

\[ = \frac{1}{rn} \sum_{i \in [n]} \|u_k^{(i)}\|^2 + \frac{(r - 1)}{rn(n - 1)} \left( \| \sum_{i \in [n]} u_k^{(i)} \|^2 - \sum_{i \in [n]} \|u_k^{(i)}\|^2 \right) \]

\[ = \frac{(r - 1)}{rn(n - 1)} \left( \| \sum_{i \in [n]} u_k^{(i)} \|^2 + \frac{n - r}{rn(n - 1)} \sum_{i \in [n]} \|u_k^{(i)}\|^2 \right). \]
In eq. (57), \( \mathbb{P}(i, j \in S_k) = \frac{r(r-1)}{n(n-1)} \) as we are sampling without replacement. Equation (58) follows from the fact that \( \| \sum_{i \in [n]} u_k^{(i)} \|^2 = \sum_{i \in [n]} \| u_k^{(i)} \|^2 + \sum_{i, j \in [n]: i \neq j} \langle u_k^{(i)}, u_k^{(j)} \rangle \).

But by the triangle inequality:

\[
\left\| \sum_{i \in [n]} u_k^{(i)} \right\| \leq \left\| \sum_{i \in [n]} \left( u_k^{(i)} - E \nabla f_i(w_k) \right) \right\| + nE \| \nabla f(w_k) \| \tag{60}
\]

\[
\leq \sum_{i \in [n]} \left\| u_k^{(i)} - E \nabla f_i(w_k) \right\| + nE \| \nabla f(w_k) \| \tag{61}
\]

Now:

\[
\left\| u_k^{(i)} - E \nabla f_i(w_k) \right\| = \left\| \sum_{\tau=1}^{E-1} (\nabla f_i(w_k^{(\tau,i)}) - \nabla f_i(w_k)) \right\| \tag{62}
\]

\[
\leq \sum_{\tau=1}^{E-1} \left\| \nabla f_i(w_k^{(\tau,i)}) - \nabla f_i(w_k) \right\| \tag{63}
\]

\[
\leq L \sum_{\tau=1}^{E-1} \left\| w_k^{(\tau,i)} - w_k \right\|. \tag{64}
\]

The last equation uses the \( L \)-smoothness of the \( f_i \)’s.

Using Lemma 3 in eq. (64), we get (recall that we already imposed \( \eta_k LE \leq 1/4 \), so we are good):

\[
\left\| u_k^{(i)} - E \nabla f_i(w_k) \right\| \leq 2\eta_k E \sum_{\tau=1}^{E-1} \tau \left\| \nabla f_i(w_k) \right\| \leq \eta_k LE^2 \left\| \nabla f_i(w_k) \right\| \leq \frac{E}{4} \left\| \nabla f_i(w_k) \right\|. \tag{65}
\]

The last step follows because we have imposed \( \eta_k LE \leq 1/4 \).

Thus:

\[
\left\| \sum_{i \in [n]} u_k^{(i)} \right\| \leq \frac{E}{4} \sum_{i \in [n]} \left\| \nabla f_i(w_k) \right\| + nE \left\| \nabla f(w_k) \right\|. \tag{66}
\]

Also:

\[
\left\| u_k^{(i)} \right\| \leq \left\| u_k^{(i)} - E \nabla f_i(w_k) \right\| + E \left\| \nabla f_i(w_k) \right\| \leq \frac{5E}{4} \left\| \nabla f_i(w_k) \right\|. \tag{67}
\]
Now using eq. (66) and eq. (67) in eq. (59), we get:

\[
(II) \leq \frac{2(r-1)E^2}{16rn(n-1)} \left( \sum_{i \in [n]} \| \nabla f_i(w_k) \| \right)^2 + \frac{2n(r-1)E^2}{r(n-1)} \| \nabla f(w_k) \|^2 + \frac{25(n-r)E^2}{16rn(n-1)} \sum_{i \in [n]} \| \nabla f_i(w_k) \|^2 
\]

Next using the fact that:

\[
\| \nabla f_i(w_k) \| \leq \frac{2r}{n} (\sum_{i \in [n]} \| \nabla f_i(w_k) \|)^2 + 2n\| \nabla f(w_k) \| 
\]

we get:

\[
(II) \leq \frac{2(r-1)E^2}{16rn(n-1)} \sum_{i \in [n]} \| \nabla f_i(w_k) \|^2 + \frac{2n(r-1)E^2}{r(n-1)} \| \nabla f(w_k) \|^2 + \frac{25(n-r)E^2}{16rn(n-1)} \sum_{i \in [n]} \| \nabla f_i(w_k) \|^2 
\]

Next, using the bound for (I) obtained from eq. (55) and the bound for (II) obtained from eq. (72) in eq. (48), we get:

\[
E[\| w_{k+1} - w^* \|^2] \leq E[\| w_k - w^* \|^2] - \frac{\eta_k E}{8Ln} \sum_{i \in [n]} \| \nabla f_i(w_k) \|^2 + \frac{1}{n} \sum_{i \in [n]} \| w^* - w^*_i \|^2 
\]

Now for \( \eta_k \leq (\zeta/LE) / (2 + 23(n-r)/r(n-1)) \):

\[
E[\| w_{k+1} - w^* \|^2] \leq E[\| w_k - w^* \|^2] - \frac{\eta_k E}{16Ln} \sum_{i \in [n]} \| \nabla f_i(w_k) \|^2 + \frac{1}{n} \sum_{i \in [n]} \| w^* - w^*_i \|^2 
\]

we can simplify eq. (74) to:

\[
E[\| w_{k+1} - w^* \|^2] \leq E[\| w_k - w^* \|^2] - \eta_k E \left( \frac{\zeta}{16L} - 2\eta_k E \frac{n(r-1)}{r(n-1)} \right) \| \nabla f(w_k) \|^2 + \frac{1}{n} \sum_{i \in [n]} \| w^* - w^*_i \|^2. 
\]

Now for \( \eta_k \leq (\zeta/LE) / (64n(r-1)/r(n-1)) \), eq. (75) can be simplified to:

\[
E[\| w_{k+1} - w^* \|^2] \leq E[\| w_k - w^* \|^2] - \frac{\eta_k E}{32L} \| \nabla f(w_k) \|^2 + \frac{1}{n} \sum_{i \in [n]} \| w^* - w^*_i \|^2. 
\]
Summing up the above equation from \( k = 0 \) through to \( K - 1 \) with \( \eta_k = \eta \), followed by some re-arrangement, we get:

\[
\frac{1}{K} \sum_{k=0}^{K-1} \| \nabla f(w_k) \|^2 \leq \frac{32L}{\eta \xi E K} \| w_0 - w^* \|^2 + \frac{32L}{\eta \xi E n} \sum_{i \in [n]} \| w_i^* - w_i^* \|^2, 
\]

(77)

where \( \eta \leq \min \left( \frac{c}{2 \xi E}, \frac{1}{2 c \xi \xi E}, \frac{c}{(2 + \frac{23(n-r)}{r(1-r)}) LE}, \frac{c}{(64 \frac{r-1}{n-1}) LE} \right) \). Let us choose \( \eta = \frac{c}{c \xi E} \) while \( c = \max \left( 4, 2 + \frac{23(n-r)}{r(1-r)}, 64 \frac{n(r-1)}{r(n-1)} \right) \). Observe that this choice of \( \eta \) satisfies all the constraints on it. Also recall that \( w_i^* \) can be any point in \( \Phi \). In order to minimize the RHS of eq. (77), we set \( w_i^* = \min_{w \in \Phi_i} \| w - w^* \| \). Then using the fact that:

\[
\min_{k \in \{0, \ldots, K-1\}} \mathbb{E}[\| \nabla f(w_k) \|^2] \leq \frac{1}{K} \sum_{k=0}^{K-1} \mathbb{E}[\| \nabla f(w_k) \|^2],
\]

we get the desired result. \( \blacksquare \)

C  Proof of the example of Assumption 3

Proof. We have:

\[
\| \nabla \ell(x_i; w) \| = \| (x_i x_i^T) (w - w^*) \| \leq \| w - w^* \|
\]

(78)

and

\[
\| \nabla \bar{f}(w) \| = \| \mathbb{E}_{x \sim P_i} [x_i x_i^T] (w - w^*) \| = \| Q(w - w^*) \| \geq \lambda^2 \| w - w^* \|.
\]

(79)

Using these two equations, we get:

\[
\| \nabla \ell(x_i; w) \| \leq \frac{1}{\lambda^2} \| \nabla \bar{f}(w) \|.
\]

(80)

So, our example satisfies Assumption 3 with \( \rho = \frac{1}{\lambda^2} \). \( \blacksquare \)

D  Proof of Theorem 2

Proof. Let us focus on the first client. Recall that the distribution of the data in the first client is given by \( P_1 \). Consider a dataset \( D = \{ x_1, \ldots, x_n \} \) consisting of \( n_1 \) i.i.d. data samples drawn from \( P_1 \). Consider another dataset \( D' \) which differs from \( D \) in exactly one sample, i.e., \( D' = D - \{ x_j \} \) for some index \( j \in [n] \).

The objective functions on \( D \) and \( D' \) are

\[
f_1(w) = \frac{1}{n} \sum_{x \in D} \ell(x; w) \quad \text{and} \quad f'_1(w) = \frac{1}{n_1 - 1} \sum_{x' \in D'} \ell(x'; w),
\]

(81)

respectively.

Also recall that \( \bar{f}_1(w) = \mathbb{E}_{x \sim P_1} [\ell(x; w)] \).

Suppose the initial point is \( w_0 \). Let us denote the \( E \) iterates obtained by training with gradient descent using a constant learning rate of \( \eta \) on \( D \) and \( D' \) by \( \{ w(1), \ldots, w(E) \} \) and \( \{ w'_1(1), \ldots, w'_E(1) \} \), respectively. Then for \( 0 \leq \tau \leq E - 1 \):

\[
w(\tau + 1) = w(\tau) - \eta \nabla f_1(w(\tau)) \quad \text{and} \quad w'_1(\tau + 1) = w'_1(\tau) - \eta \nabla f'_1(w'_1(\tau)),
\]

(82)

where \( w(0) = w'_0 \).
Note that we are interested in bounding 
\[
\Delta_2^{(1)} := \sup_{D,D',w(0),\{\eta_n\} \leq 0.25} \| w_0 - w(E) \| - \frac{w(0) - w'(E)}{w(0) - w'(E)}. \tag{83}
\]

The high-level plan will be to first bound \( \Delta_2^{(1)} \) with high probability.

Using the triangle inequality, we get
\[
\| w_{(\tau+1)} - w'_{(\tau+1)} \| \leq \| w_{(\tau)} - w'_{(\tau)} \| + \eta \| \nabla f_1(w_{(\tau)}) - \nabla f'_1(w_{(\tau)}) \| 
\leq \| w_{(\tau)} - w'_{(\tau)} \| + \eta \sum_{t \neq j} \nabla \ell(x_t; w_{(\tau)}) - \nabla \ell(x_t; w'_{(\tau)}) \| 
+ \frac{\eta}{n_1} \| \nabla \ell(x_j; w_{(\tau)}) \| 
\leq \| w_{(\tau)} - w'_{(\tau)} \| + \eta \| \nabla \ell(x_j; w_{(\tau)}) \| 
+ \frac{\eta}{n_1} \sum_{t \neq j} \nabla \ell(x_t; w_{(\tau)}) - \nabla \ell(x_t; w'_{(\tau)}) \| 
\leq \| w_{(\tau)} - w'_{(\tau)} \| + \eta L \| w_{(\tau)} - w'_{(\tau)} \| 
+ \frac{\eta}{n_1} \sum_{t \neq j} \nabla \ell(x_t; w_{(\tau)}) - \nabla \ell(x_t; w'_{(\tau)}) \| 
\leq (1 + \eta L) \| w_{(\tau)} - w'_{(\tau)} \| + \frac{2\eta}{n_1} \| \nabla f_1(w_{(\tau)}) \|. \tag{84}
\]

In the above simplification, we have used the smoothness and growth-condition (Assumption 3) of \( \ell(x,\cdot) \).

So, we get:
\[
\| w_{(\tau+1)} - w'_{(\tau+1)} \| \leq (1 + \eta L) \| w_{(\tau)} - w'_{(\tau)} \| + \frac{2\eta}{n_1} \| \nabla f_1(w_{(\tau)}) \|. \tag{85}
\]

Unfolding the above recursion, we get that:
\[
\| w_{(E)} - w'_{(E)} \| \leq \frac{2\rho \ell_{1/4}}{n_1} \sum_{\tau=0}^{E-1} (1 + \eta L)^{E-1-\tau} \| \nabla f_1(w_{(\tau)}) \| \leq \frac{2\rho \ell_{1/4}}{n_1} \sum_{\tau=0}^{E-1} \| \nabla f_1(w_{(\tau)}) \|. \tag{86}
\]

The last step follows because \((1 + \eta L)^{E-1-\tau} \leq e^{\eta L^{E-1-\tau}} \leq e^{\eta L} \leq e^{1/4} \) as we have \( \eta L \leq 1/4 \).

By the triangle-inequality and \( L \)-smoothness of \( f_1 \), we have that:
\[
\| \nabla f_1(w_{(\tau)}) \| \leq \| \nabla f_1(w_{(0)}) \| + \| \nabla f_1(w_{(\tau)}) - \nabla f_1(w_{(0)}) \| \leq \| \nabla f_1(w_{(0)}) \| + L \| w_{(\tau)} - w_{(0)} \|, \tag{87}
\]

for all \( \tau \geq 0 \). Next, we can use the result of Lemma 3 to conclude that for \( \eta L \leq 1/2 \):
\[
\| w_{(\tau)} - w_{(0)} \| \leq 2\eta \| \nabla f_1(w_{(0)}) \|. \tag{88}
\]

Putting this in eq. (87) gives us:
\[
\| \nabla f_1(w_{(\tau)}) \| \leq \| \nabla f_1(w_{(0)}) \| + 2\eta L \| \nabla f_1(w_{(0)}) \| \leq \| \nabla f_1(w_{(0)}) \| + \frac{1}{2} \| \nabla f_1(w_{(0)}) \|, \tag{89}
\]

for \( 0 \leq \tau < E \). The last step follows because we impose \( \eta L \leq 1/4 \).

Now using eq. (89) in eq. (86) gives us:
\[
\| w_{(E)} - w'_{(E)} \| \leq \frac{2e^{1/4}\rho \ell_{1/4}}{n_1} (\| \nabla f_1(w_{(0)}) \| + \frac{1}{2} \| \nabla f_1(w_{(0)}) \|). \tag{90}
\]

28
We have that the \(-1\) with probability at least \(1 - p\) for all \(t \in [n_1]\), and by Assumption \(3\) and the triangle inequality, \(|E_t| \leq (\rho + 1)\|\nabla f_1(w_0)\|\) (note that this is an upper bound independent of the \(x_t\)'s). Thus, the \(E_t\)'s are norm-subGaussian as per the definition of \(22\). Hence, using Corollary 7 of \(22\), we get:

\[
\left\| \frac{1}{n_1} \sum_{t \in [n_1]} E_t \right\| \leq 2(\rho + 1)\|\nabla f_1(w_0)\| \sqrt{\frac{\log(2d/p)}{n_1}},
\]

with probability at least \(1 - p\). Using this in eq. (92), followed by some rearrangement, we get:

\[
\|\nabla f_1(w_0)\| \leq \frac{\|\nabla f_1(w_0)\|}{1 - 2(\rho + 1)\sqrt{\frac{\log(2d/p)}{n_1}}}
\]

with probability at least \(1 - p\), for \(p \geq 2d \exp(-\frac{n_1}{4(\rho + 1)^2})\).

For the subsequent results, we shall skip mentioning that they hold with probability at least \(1 - p\). Using eq. (94) in eq. (90), we get:

\[
\|w(E) - w'(E)\| \leq \frac{e^{1/4} \rho \left(3 - 2(\rho + 1)\sqrt{\frac{\log(2d/p)}{n_1}}\right)}{n_1 \left(1 - 2(\rho + 1)\sqrt{\frac{\log(2d/p)}{n_1}}\right)} \|\nabla f_1(w_0)\|.
\]

But from the result of Lemma \(2\) we also have for \(\eta LE \leq 1/4\):

\[
\|\nabla f_1(w_0)\| \leq \frac{3}{2E} \|\sum_{t=0}^{E-1} \nabla f_1(w(t))\|.
\]

Now, we are interested in bounding:

\[
\Delta_2^{(1)} = \sup_{D,D',w_0,(\eta LE \leq 0.25)} \left\| \frac{w_0 - w(E)}{\|w_0 - w(E)\|} - \frac{w_0 - w'(E)}{\|w_0 - w'(E)\|} \right\|,
\]

given that eq. (98) holds with \(\kappa < 1\) which is going to be the case for sufficiently large \(n_1\) and \(p\) (we shall get back to this). We first present a useful lemma to do so.
Lemma 4. Suppose $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^d$, with $\|\mathbf{v}_1\| \neq 0$ and $\|\mathbf{v}_2\| \neq 0$, are such that $\|\mathbf{v}_1 - \mathbf{v}_2\| \leq \alpha \|\mathbf{v}_1\|$ where $\alpha < 1$. Then:

$$\left\| \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} - \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} \right\| \leq 2\alpha.$$ 

The proof of Lemma 4 can be found after the end of this proof.

Let us get a lower bound for $n_1$ and $p$ to ensure that $\kappa < 1$. This can be ensured by having:

$$\left( 3 - 2(\rho + 1) \sqrt{\frac{\log(2d/p)}{n_1}} \right) = 5 \text{ and } \frac{3e^{1/4}\rho}{2n_1} \times 5 \leq 1$$

The two conditions are satisfied for:

$$n_1 = 2d \exp\left(- \frac{n_1}{16(\rho + 1)^2} \right) \text{ and } n_1 \geq \frac{15e^{1/4}\rho}{2}.$$ 

Note that the obtained $p$ satisfies our earlier constraint on it, mentioned after eq. (104). We expect the condition on $n_1$ to be satisfied very easily in practice as it is typically very large.

Then using the result of Lemma 4 we get:

$$\left\| \frac{\mathbf{w}(0) - \mathbf{w}(E)}{\|\mathbf{w}(0) - \mathbf{w}(E)\|} - \frac{\mathbf{w}(0) - \mathbf{w}'(E)}{\|\mathbf{w}(0) - \mathbf{w}'(E)\|} \right\| \leq \frac{15e^{1/4}\rho}{n_1} D, D', \mathbf{w}(0), \{\eta : \eta LE \leq \frac{1}{4}\},$$

with probability at least $1 - 2d \exp(-\frac{n_1}{16(\rho + 1)^2})$.

Observe that this bound is independent of the initial point $\mathbf{w}(0)$, datasets $D$ and $D'$, as well as the learning rate $\eta$. Therefore,

$$\Delta_2^{(i)} \leq \frac{15e^{1/4}\rho}{n_1},$$

with probability at least $1 - 2d \exp(-\frac{n_1}{16(\rho + 1)^2})$.

Applying the same procedure as described above for any client $i \in [n]$ gives us the desired result for $\Delta_2^{(i)}$.

Proof of Lemma 4

Proof. Suppose $\|\mathbf{v}_2\| = c\|\mathbf{v}_1\|$ for some $c > 0$. Note that by the triangle inequality:

$$c\|\mathbf{v}_1\| = \|\mathbf{v}_2\| \leq \|\mathbf{v}_1\| + \|\mathbf{v}_2 - \mathbf{v}_1\| \leq (1 + \alpha)\|\mathbf{v}_1\| \implies c \leq 1 + \alpha.$$ 

(104)

Also:

$$\|\mathbf{v}_1\| \leq \|\mathbf{v}_2\| + \|\mathbf{v}_1 - \mathbf{v}_2\| \leq c\|\mathbf{v}_1\| + \alpha\|\mathbf{v}_1\| \implies c \geq 1 - \alpha.$$ 

(105)

So, $1 - \alpha \leq c \leq 1 + \alpha$ or

$$|c - 1| \leq \alpha.$$ 

(106)

Then:

$$\left\| \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} - \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} \right\| = \left\| \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} - \frac{c\|\mathbf{v}_1\|}{\|\mathbf{v}_1\|} \right\|$$

$$= \left\| \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} - \frac{\mathbf{v}_2}{\|\mathbf{v}_1\|} + \frac{\mathbf{v}_2}{\|\mathbf{v}_1\|} - \frac{c\|\mathbf{v}_1\|}{\|\mathbf{v}_1\|} \right\|$$

$$\leq \left\| \frac{\mathbf{v}_1 - \mathbf{v}_2}{\|\mathbf{v}_1\|} + \frac{\mathbf{v}_2}{\|\mathbf{v}_1\|} - \frac{1}{c} \right\|$$

$$\leq \alpha + \frac{c(1 - \frac{1}{c})}{\|\mathbf{v}_1\|} = \alpha + |c - 1| \leq 2\alpha.$$ 

(106)

The last step above follows by using eq. (106).
E Proof of Corollary 2.1

Proof. We have:

\[ P(\max_{i \in [n]} \Delta_2^{(i)} \leq \frac{15e^{1/4} \rho}{m}) = \prod_{i \in [n]} P(\Delta_2^{(i)} \leq \frac{15e^{1/4} \rho}{m}) \]

(107)

\[ \geq \prod_{i \in [n]} P(\Delta_2^{(i)} \leq \frac{15e^{1/4} \rho}{n_i}) \]

(108)

\[ \geq \prod_{i \in [n]} \left(1 - 2d \exp\left(-\frac{n_i}{16(\rho + 1)^2}\right)\right) \]

(109)

\[ \geq 1 - 2d \sum_{i \in [n]} \exp\left(-\frac{n_i}{16(\rho + 1)^2}\right). \]

(110)

Equation (107) follows due to the independence of the data of each client, eq. (108) follows because \( m \leq n_i \) for all \( i \in [n] \) by definition, while eq. (109) follows from the result of Theorem 2.

F Experimental Details

First, we summarize the Clipped-FedAvg as described in the main paper in Algorithm 3.

Algorithm 3 Clipped-FedAvg

1: \textbf{Input:} Initial point \( w_0 \), number of communication rounds \( K \), number of local updates per round \( E \), local learning rates \( \{\eta_k\}_{k=0}^{K-1} \), global learning rates \( \{\beta_k\}_{k=0}^{K-1} \), clipping threshold \( C \), number of participating clients in each round \( r \) and noise variance \( \sigma^2 \).

2: \textbf{for} \( k = 0, \ldots, K - 1 \) \textbf{do}

3: Server sends \( w_k \) to a set \( S_k \) of \( r \) clients chosen uniformly at random w/o replacement.

4: \textbf{for} client \( i \in S_k \) \textbf{do}

5: Set \( w_{k,0}^{(i)} = w_k \).

6: \textbf{for} \( \tau = 0, \ldots, E - 1 \) \textbf{do}

7: Update \( w_{k,\tau+1}^{(i)} \leftarrow w_{k,\tau}^{(i)} - \eta_k \nabla f_i(w_{k,\tau}^{(i)}) \).

8: \textbf{end for}

9: Let \( g_k^{(i)} = (w_k - w_{k,E}^{(i)}) \min\left(1, \frac{C}{\|w_k - w_{k,E}^{(i)}\|}\right) \). Send \( (g_k^{(i)} + \zeta_k^{(i)}) \) to the server, where \( \zeta_k^{(i)} \sim N(0, r\sigma^2 I_d) \).

10: \textbf{end for}

11: Server updates \( w_{k+1} \leftarrow w_k - a_k \), where \( a_k = \frac{1}{r} \sum_{i \in S_k} (g_k^{(i)} + \zeta_k^{(i)}) \).

12: \textbf{end for}

Now, we explain the procedure we have used to generate heterogeneous data distribution (for the FL experiments). First, the training data of both datasets (CIFAR-10 and FMNIST) was sorted based on labels and then divided into \( 5n \) equal data-shards, where \( n \) is the number of clients. Splitting the data in this way ensures that each shard contains data from only one class for both datasets (and because \( n \) was chosen appropriately). Now, each client is assigned \( 5 \) shards chosen uniformly at random without replacement which ensures that each client can have data belonging to at most \( 5 \) distinct classes.

Next, we describe the CNN architecture used for FMNIST in \((ii)\). It consists of two convolutional layers each followed by a max pool layer and ReLU non-linearity. The first (resp., second) convolutional layer has number of input channels \( = 1 \) (resp., \( 32 \)), output channels \( = 32 \)
(resp., 32) and kernel size = 5 (resp., 5). The output of the second convolutional layer after max pooling and applying ReLU goes through a fully connected layer with output dimension = 512, after which we apply ReLU and dropout with probability 0.5. Finally, this is connected to a softmax layer (with output size = 10 as there are 10 classes).

Finally, we list down the hyper-parameter values used in the experiments:

1. Details of $\eta_k$, $\beta_k$ and $C$ for experiments in the federated setting: We use the learning rate scheme suggested in [6] where we decrease the local learning rate by a factor of 0.99 after every round, i.e. $\eta_k = (0.99)^k \eta_0$. Note that this learning rate scheme has been used previously for FL experiments by [16]. We search the best initial local learning rates $\eta_0$ over $\{10^{-3}, 5 \times 10^{-3}, 10^{-2}, 5 \times 10^{-2}, 10^{-1}, 5 \times 10^{-1}\}$. We use a constant $\beta_k = \beta$ for all rounds and search for the best $\beta$ over $\{0.25, 0.5, 1\}$. Finally, we search the clipping threshold $C$ over $\{0.25, 0.5, 1, 2\}$. The specific values of $\eta_0$, $\beta$ and $C$ used in each of our experiments are as follows:

   - Figure 2a (CIFAR-10 logistic regression (i)): For DP-NormFedAvg, we use $\eta_0 = 0.005$ and $\beta = 0.5$. For Clipped-FedAvg, we use $\eta_0 = 0.001$ and $C = 0.25$. For NormFedAvg, we use $\eta_0 = 0.01$ and $\beta = 1$. For FedAvg, we use $\eta_0 = 0.005$.

   - Figure 2b (FMNIST logistic regression (i)): For DP-NormFedAvg, we use $\eta_0 = 0.01$ and $\beta = 0.5$. For Clipped-FedAvg, we use $\eta_0 = 0.001$ and $C = 0.5$. For NormFedAvg, we use $\eta_0 = 0.01$ and $\beta = 1$. For FedAvg, we use $\eta_0 = 0.001$.

   - Figure 2c (CIFAR-10 MLP (ii)): For DP-NormFedAvg, we use $\eta_0 = 0.001$ and $\beta = 0.25$. For Clipped-FedAvg, we use $\eta_0 = 0.001$ and $C = 0.25$. For NormFedAvg, we use $\eta_0 = 0.01$ and $\beta = 0.5$. For FedAvg, we use $\eta_0 = 0.005$.

   - Figure 2d (FMNIST CNN (ii)): For DP-NormFedAvg, we use $\eta_0 = 0.001$ and $\beta = 0.25$. For Clipped-FedAvg, we use $\eta_0 = 0.001$ and $C = 0.25$. For NormFedAvg, we use $\eta_0 = 0.01$ and $\beta = 1$. For FedAvg, we use $\eta_0 = 0.005$.

2. Figure 3 (CIFAR-10 in centralized setting): Refer to eq. (3) and eq. (4) for the updates of normalization and clipping, respectively. We decay the learning rate by 0.5 after 10 epochs, 0.25 after 20 epochs and 0.125 after 30 epochs. For normalization, we use an initial learning rate of 0.05, constant $\beta$ of 0.25 and momentum = 0.7. For clipping, we use an initial learning rate of 0.05, clipping threshold of 0.5 and momentum = 0.7. For regular SGD, we use an initial learning rate of 0.005 and momentum = 0.9.

3. Figure 1 (synthetic experiment in Section 4): For all the cases, we use a constant learning rate $\eta$ which is searched over $\{10^{-3}, 5 \times 10^{-3}, 10^{-2}, 5 \times 10^{-2}, 10^{-1}, 5 \times 10^{-1}, 1\}$. For normalization, we use $\eta = 0.005$. For clipping with $C = 1$, 0.1 and 0.01, we use $\eta = 0.05$, 0.05 and 0.5, respectively.