Abstract. We study a class of holomorphic matrix models. The integrals are taken over middle dimensional cycles in the space of complex square matrices. As the size of the matrices tends to infinity, the distribution of eigenvalues is given by a measure with support on a collection of arcs in the complex planes. We show that the arcs are level sets of the imaginary part of a hyperelliptic integral connecting branch points.

1. Introduction

According to Dijkgraaf and Vafa [1, 2], the effective glueball superpotential of $\mathcal{N} = 1$ supersymmetric $U(n)$ gauge theory has an asymptotic expansion given by the planar part of the topological expansion of a matrix model. To give the effective potential for various vacua of the gauge theory, Dijkgraaf and Vafa propose a formula obtained by saddle point expansion of the matrix integral around different critical points. The filling fractions, i.e., the fraction of the eigenvalues sitting close to each of the critical points of the potential are the parameters selecting the vacua.

In this paper we give a way to define non-perturbatively (i.e., beyond the saddle point expansion) the matrix integrals. Different integrals are integrals over different cycles in the space of normal matrices. In the case of generic polynomial potentials it is possible to construct a cycle for each of the critical points, so that all effective potentials considered by Dijkgraaf and Vafa arise in the asymptotic expansion of our integrals. We note that the idea of integrating over eigenvalues along suitable contours in the complex plane appears in special situations in [3]. David’s contours were also more recently considered in [4], where a (non-holomorphic) modification of the integral was proposed to obtain arbitrary filling factors.

This paper is based on the ETH diploma thesis of the second author.
A second result of this paper is the description of the asymptotic distribution of eigenvalues for each critical point. The eigenvalues, as predicted by Dijkgraaf and Vafa, lie asymptotically along arcs connecting branch points of a two-fold cover of the complex plane. We give a reality condition which specifies the shape of the arcs and the density of eigenvalues.

Before considering the case of general $N \times N$ matrices, it is instructive to consider the case $N = 1$ of ordinary integrals. Consider the Airy integral

$$u(x) = \int e^{\mu (xt + t^3/3)} dt.$$ 

This integral gives a formal solution of the Airy differential equation

$$-\mu^2 u''(x) + xu(x) = 0.$$ 

Indeed the saddle point expansion around the two critical points of the integrand give the two linear independent formal power series solutions. The non-perturbative solutions can be obtained by integrating along paths in the complex plane connecting any two of the three directions at infinity where the integrand goes to zero. More generally, the integrals of the form $\int \exp(\frac{\mu}{\mu}(xt + P(t))) dt$ for a generic polynomial $P(t)$ of degree $n$ are solutions of a differential equation of order $n - 1$. We show that a system of $n - 1$ linearly independent solutions are obtained by choosing $n$ integration contours in the complex plane going to infinity in directions where the integrand decays exponentially. Moreover the contours can be chosen so that each of them passes through exactly one of the $n$ critical points. It follows that the $n - 1$ saddle point expansions at each of the critical points are the asymptotic expansions of true solutions given by convergent integrals.

In the case of matrices we consider integrals of the form

$$(1) \quad \langle F \rangle_N = Z_N^{-1} \int_{\Gamma} F(\Phi) \exp \left( -\frac{N}{\mu} \text{tr} p(\Phi) \right) d\Phi$$

over real $N^2$-dimensional cycles $\Gamma$ in the space of complex $N \times N$ matrices. The potential $p$ is a polynomial with complex coefficients, $d\Phi = \wedge_{j,k} d\Phi_{jk}$ and the normalization factor $Z_N$ is such that $\langle 1 \rangle_N = 1$. The observables $F(\Phi)$ are holomorphic functions invariant under conjugation. In the well-studied case of integrals over hermitian matrices, with a polynomial $p$ with real coefficients bounded below, the saddle point approximation becomes exact in the large $N$ limit with fixed $\mu$ and the relevant critical point is described by an asymptotic density of eigenvalues which has support on a union of intervals on the real axis. If $\mu$ is small, these intervals are small neighborhoods of the minima of the potential $p$. To obtain the densities of eigenvalues needed to make
contact to gauge theory one considers the variational problem for critical points subject to the side condition that the fraction of eigenvalues in the vicinity of each of the critical points (not just minima) are given numbers.

We consider here the case of a generic polynomial $p$ of degree $n$ with complex coefficients, with distinct critical points $z_1, \ldots, z_{n-1}$, and propose to consider integrals over cycles in the space of normal matrices (a matrix is normal if it commutes with its adjoint or, equivalently, if it is conjugated to a diagonal matrix by a unitary matrix). The cycles we consider are parametrized by integers $N_1, \ldots, N_{n-1}$ whose sum is $N$ and are characterized by the condition that $N_k$ eigenvalues belong to a path $\Gamma_k$ in the complex plane going through $z_k$ and going to infinity in a direction where $\text{Re}(p(z)) \to \infty$. In the limit $N \to \infty$ with $n_k = N_k/N$ fixed, the eigenvalues in the saddle point approximation (supposed to be exact in the limit) are distributed along $n-1$ arcs in the complex plane. For small $\mu$ the arcs are close to the critical points and the $k$th arcs contains the fraction $n_k$ of the eigenvalues.

2. The one-dimensional case

Let $p(z) = a_n z^n + \cdots + a_1 z + a_0$ be a polynomial of degree $n \geq 2$ with complex coefficients. We want to consider integrals of the form

$$\int_{\Gamma} q(z) e^{-\frac{1}{\mu} p(z)} \, dz, \quad \mu > 0.$$ 

for polynomials $q(z)$. Before considering the question of integration cycles we may evaluate such integrals as asymptotic series as $\mu \to 0$ by formal application of the saddle point method at each of the critical points $z_1, \ldots, z_{n-1}$, which we assume to be distinct. In this way we get $n-1$ asymptotic series of the form $\exp\left(-p(z_k)/\mu\right)(c_1 + c_2 \mu + \cdots)$ and the question is whether these are asymptotic expansions of our integral for suitable cycles $\Gamma_k$.

The cycles which we should consider here are (linear combinations of) paths for which the integral converges. As the integrand is holomorphic, homotopic paths will give the same answer and what matters is the behavior at infinity. As $z \to \infty$, $p(z) \sim a_n z^n$, so there are $n$ directions in the complex plane for which $\text{Re}(p(z)) \to +\infty$ as $z$ tends to infinity in these directions. Let us call these asymptotic directions valleys as in these directions the integrand decays exponentially. Neighboring valleys are separated by hills, which are directions of exponential increase of the integrand. So the cycles one needs to consider are linear combinations of infinite paths connecting pairs of distinct valleys. As
paths connecting two valleys can be deformed into sums of paths connecting the two valleys with any third one, there are only \( n - 1 \) linearly independent cycles.

Let us assume for simplicity that the critical values \( p(z_k) \) have distinct imaginary parts. Then there is a canonical way to associate to each critical point \( z_k \) a path \( \Gamma_k \) in such a way that the asymptotic expansion of the integral over \( \Gamma_k \) as \( \mu \to 0 \) is obtained by the saddle point expansion at \( z_k \). Namely, we take the \textit{steepest descent} paths (see, e.g., \cite{5} and \cite{6}) emerging from \( z_k \), defined by the condition that the tangent vector at each point points in the direction of the gradient of \( \text{Re}(p(z)) \).

As \( p(z) \) is holomorphic, the gradient of \( \text{Re} p(z) \) is orthogonal to the gradient of \( \text{Im} p(z) \) by the Cauchy–Riemann equations. Thus steepest descent paths are level lines for the imaginary part of \( p(z) \). Each non-degenerate critical point \( z_k \) is at the intersection of two such level lines. One of these two lines, the one along which \( \text{Re}(p(z)) \) takes its minimum at \( z_k \), is the steepest descent path \( \Gamma_k \). Along the other line \( \Gamma_k' \), the real part takes its maximum at \( z_k \). We claim that \( \Gamma_k \) is a smooth path going to infinity in both directions and connecting two valleys. Indeed, \( \Gamma_k \) is (in suitable parametrization) given by a solution of the differential equation \( \dot{z}(t) = \frac{d}{dt} \text{Re}(p(z(t))) = \left| p'(z(t)) \right|^2 \). By our assumption, critical values have distinct imaginary parts, so the steepest descent path passing through \( z_k \) may not come close to any other critical point. Thus \( \left| p'(z(t)) \right| \) is bounded below so that, as we go away from \( z_k \), \( \text{Re} p(z(t)) \) must go to infinity and the path \( \Gamma_k \) connects two valleys. Similarly \( \Gamma_k' \) connects two hills. As \( \Gamma_k \) and \( \Gamma_k' \) cross at \( z_k \), the two valleys connected by \( \Gamma_k' \) are separated by hills and are thus different.

We have thus shown that for each critical point \( z_k \) there is a steepest descent path \( \Gamma_k \) going through \( z_k \) and connecting pairs of different valleys. On \( \Gamma_k \) the real part of \( p(z) \) is minimal at \( z_k \) so that the saddle point expansion at \( z_k \) indeed gives the asymptotic expansion of the integral over \( \Gamma_k \).

3. Matrix integrals

3.1. Integration cycles. We consider matrix integrals of the form (11) for \( p(z) \) a polynomial of degree \( n \). They are integrals of holomorphic differential forms over \( N^2 \) dimensional cycles. For each set of natural numbers \( N_1, \ldots, N_{n-1} \) summing up to \( N \), we have a cycle in the normal matrices, characterized by the condition that \( N_k \) eigenvalues run over the path \( \Gamma_k \) of the previous section. More precisely, the cycle \( \Gamma \) is
parametrized by \( \mathbb{R}^N \otimes U(N)/U(1)^N \):

\[
(t, U) \mapsto U \begin{pmatrix}
\lambda_1(t_1) \\
\vdots \\
\lambda_N(t_N)
\end{pmatrix} U^{-1}.
\]

The first \( N_1 \) diagonal elements \( \lambda_1(t), \ldots, \lambda_{N_1}(t) \) are parametrizations of \( \Gamma_1 \), the next \( N_2 \) are parametrizations of \( \Gamma_2 \) and so on.

The usual argument to reduce the integral to an integral over the eigenvalues (see [9]) gives

\[
\langle F \rangle_N = \frac{1}{Z_N'} \int F(\lambda_1, \ldots, \lambda_N) e^{-\frac{N}{\mu} \sum_{j=1}^n p(\lambda_j) \prod_{j<k} (\lambda_j - \lambda_k)^2} d\lambda_1 \cdots d\lambda_N,
\]

The integral is over \( \Gamma_1^{N_1} \times \cdots \times \Gamma_{n-1}^{N_{n-1}} \) and the function \( F \), a function on matrices invariant under conjugation, is regarded as a symmetric function of the eigenvalues.

### 3.2. The loop equation

To study the large \( N \) limit, it is useful to introduce the trace of the resolvent:

\[
\omega(z) = \frac{1}{N} \frac{1}{z - \Phi} = \frac{1}{N} \sum_{j=1}^N \frac{1}{z - \lambda_j},
\]

as products of such traces are generating functions of polynomial functions invariant under conjugation. The “loop equation” [7, 8] for this quantity is

\[
\mu \langle \omega(z)^2 \rangle_N - p'(z) \langle \omega(z) \rangle_N + f_N(z) = 0,
\]

where

\[
f_N(z) = \frac{1}{N} \left\langle \sum_i \frac{p'(z) - p'((\lambda_i))}{z - \lambda_i} \right\rangle_N
\]

is a polynomial of degree \( n - 2 \) with leading coefficient \( na_n \). This equation can be derived from the identity

\[
0 = \sum_{i=1}^N \int \frac{\partial}{\partial \lambda_i} \left( \frac{1}{z - \lambda_i} e^{\frac{N}{\mu} \sum_j p(\lambda_j) \prod_{j>k} (\lambda_j - \lambda_k)^2} \right) \prod d\lambda_j.
\]

In the limit \( N \to \infty \), the matrix integral is supposed to be dominated by an integral over a region where the eigenvalues are close (for small \( \mu \)) to critical points of \( p \). With our choice of integration cycles and keeping \( n_j = N_j/N \) fixed as \( N \to \infty \), there will be a fraction \( n_j \) of the eigenvalues close to \( z_j \).
In the limit $N \to \infty$ one expects that the saddle point approximation becomes exact and thus $\langle \omega(z) \rangle_N$ converges to

$$\langle \omega(z) \rangle = \int \frac{1}{z - z'} d\nu(z'),$$

for some probability measure $\nu$ with support in regions around the critical points of $p$ and so that $n_k$ is the measure of the region around $z_k$. Technically one assumes that the limit $N \to \infty$ exists and that

$$\lim_{N \to \infty} (\langle \omega(z)^2 \rangle_N - \langle \omega(z) \rangle_N^2) = 0.$$ 

The function $\langle \omega(z) \rangle$ is defined and holomorphic for $z$ outside the support of the measure. Setting

$$y(z) = 2\mu\langle \omega(z) \rangle - p'(z),$$

we finally obtain

$$y(z)^2 = p'(z)^2 - 4\mu f(z),$$

for some polynomial $f(z)$ of degree $n - 2$ with leading coefficient $na_n$. Thus the function $y(z)$, which is a priori defined on the complement of the support of the measure, has an analytic continuation to a two-fold covering of the complex plane. The original function $y$ is the branch of the function defined by (3) which behaves at infinity as $-p'(z)$.

### 3.3. The density of eigenvalues

With the analogy with the case of hermitian matrices in mind, it is reasonable to assume that the measure $\nu$ has support on a collection of arcs $\gamma_1, \ldots, \gamma_{n-1}$ which for small $\mu$ are close to the critical points of $p$, and that $d\nu(t) = \rho(t)dt$ for some density of eigenvalues $\rho$ defined on the arcs: namely, the measure $\nu(U)$ of a set $U$ intersecting one of the arcs, say $\gamma_j$, in a piece $\gamma_j \cap U$ is

$$\nu(U) = \int_{\gamma_j \cap U} \rho(z)dz.$$ 

Note that for the right-hand side to be defined we need to fix an orientation on $\gamma_j$. Then $y(z)$ is a holomorphic function outside on the complement of the arcs and the density on $\gamma_j$ is related to the discontinuity of $y$:

$$\rho(x) = \frac{1}{4\pi i \mu} (y(x^-) - y(x^+)), \quad x \in \gamma_j.$$ 

Here $y(x^+)$ ($y(x^-)$) denotes the limit of $y(z)$ as $z$ tends to $x$ from the left (from the right) of the oriented curve $\gamma_j$.

From this information we deduce that the arcs $\gamma_j$ connect pairs of zeros of $y$, the branch points of the hyperelliptic curve $y^2 = p'(z)^2 -$
4\mu f(z) of genus \( n - 2 \). The measure of the jth arc is then the period

\[ n_j = \frac{1}{4\pi i \mu} \int_{A_j} y(z) dz, \quad j = 1, \ldots, n - 1, \]

over a cycle \( A_j \) enclosing the pair of branch points in counterclockwise direction. It follows from the condition on the leading coefficient of \( f \) that \( \sum n_j = 1 \). We also note that since \( y(x^-) = -y(x^+) \), we have the formula \( \rho(x) = y(x^-)/2\pi i \mu \).

3.4. The shape of the arcs. There remains to determine the precise form of the arcs and the coefficients in \( f \) as functions of the filling fractions \( n_j \). First of all, the relation between the \( n - 2 \) free complex coefficients \( b_0, \ldots, b_{n-3} \) of \( f(z) \) (recall that the leading coefficient is fixed to be \( b_{n-2} = na_n \)) and the periods \( n_j \) (\( 1 \leq j \leq n - 1 \)) subject to \( \sum n_j = 1 \) is, locally around any point where the branch points are distinct, a holomorphic diffeomorphism, since the Jacobian matrix

\[ \frac{\partial n_j}{\partial b_{k-1}} = -\frac{1}{4\pi i \mu} \int_{A_j} \frac{z^{k-1}}{y} dz, \quad j, k = 1, \ldots, n - 2. \]

is non-degenerate, being the matrix of \( a \)-periods of a basis of Abelian differentials on a smooth curve. It follows that locally there exists a real \( (n-1) \)-dimensional submanifold in the complex space \( \mathbb{C}^n \) of coefficients \( b_j \) which maps to real positive \( n_j \).

The condition that fixes the shape of the arcs is the reality and positivity condition for the density: if \( t \mapsto z(t) \) is a parametrization of the arc \( \gamma_j \) respecting its orientation, the condition is

\[ \rho(z(t)) \frac{dz(t)}{dt} \geq 0. \]

Using \( \rho(z) = y(z^-)/2\pi i \mu \), we may parametrize the arcs (away from the endpoints) to be solutions of the differential equation

\[ \dot{z}(t) = iy(z(t)^-), \]

connecting branch points. Alternatively, arcs may be described as level lines of a function: introduce the hyperelliptic integral

\[ F(z) = \frac{1}{4\pi i \mu} \int_{z_0}^z y(z) dz. \]

It is a holomorphic many-valued function on the complement of the support of the measure. As we go around an arc \( \gamma_j \), \( F(z) \) increases by \( n_j \) so \( \text{Im} F(z) \) is single valued. The measure of a piece between two points \( x_1, x_2 \) on a curve \( \gamma_j \) is \( F(x_2^-) - F(x_1^+) \) which is real. Thus the arcs \( \gamma_j \) are level lines of the imaginary part of \( F(z) \). Around a
branch point \( z_0 \) which is a simple zero of \( y^2, F(z) \sim \text{const} + (z - z_0)^{3/2} \). Therefore there are three smooth level lines of \( F \) emerging from every simple branch points. In the most general situation the support of the measure may then be a graph consisting of level lines of \( \text{Im} F \) joining branch points. A more precise description is possible in the case of small \( \mu \) to which we turn.

3.5. \textbf{Small coupling.} For small \( \mu > 0 \) we claim that the arcs and the density of eigenvalues are determined completely by the filling fractions \( n_j \) through (4) and (5). To show this, notice first that as \( \mu \to 0 \), pairs of branch points \( z'_j, z''_j \) converge to the critical points \( z_j \) and the periods \( n_j \) (eq. (11)), regarded as functions of \( \mu \) and the coefficients \( b_j \) of \( f \) are holomorphic at \( \mu = 0 \). We have

\[
\lim_{\mu \to 0} n_j(\mu, b_0, \ldots, b_{n-3}) = \frac{f(z_j)}{4 p''(z_j)}.
\]

Since there is a bijective holomorphic correspondence between values \( f(z_j) \) at the distinct points \( z_j \) and coefficients \( b_j \) of \( f \), we have at \( \mu = 0 \), and by analyticity also for small \( \mu \), a biholomorphic map \( (b_0, \ldots, b_{n-3}) \to (n_1, \ldots, n_{n-1}) \), \( \sum n_j = 1 \). In particular, we can invert this map and find a unique \( f \) for each set of \( n_j \geq 0 \), such that \( \sum n_j = 1 \). It remains to show that for all small \( \mu > 0 \) there is a level line \( \gamma_j \) of \( \text{Im} F(z) \) connecting \( z'_j \) to \( z''_j \). As \( \mu \to 0 \), \( \mu F(z) \to -\frac{1}{4\pi^2} p(z) + \text{const} \).

The level lines of \( \mu \text{Im} F(z) \) at \( \mu = 0 \) are thus the level lines of \( \text{Re} \, p(z) \).

In the neighborhood of a non-degenerate critical point \( z_j \) they look like the left picture in Fig. 1 on any small circle around \( z_j \) each value is taken on at most four times. For positive small \( \mu \) the critical point

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{level_lines.png}
\caption{Level lines of \( \text{Im} F \) for \( \mu = 0 \) (left) and for \( \mu > 0 \) (right)\}
\end{figure}

\( z_j \) splits into two branch points \( z'_j, z''_j \) from each of which three level lines emerge. The condition that the period \( n_j \) is real implies that
\[ \text{Im} F(z_j') = \text{Im} F(z_j'') \] The function \( g(z) = \text{Im} (F(z) - F(z_j')) \) is defined up to a sign in a neighborhood of \( z_j', z_j'' \), so that its zero level line is uniquely defined. It follows that there is a level line \( \gamma_j \) of \( \text{Im} F \), namely the zero set of \( g \), joining \( z_j' \) to \( z_j'' \) as on the right picture in Fig. 1: if none of the level lines emerging from \( z_j' \) and \( z_j'' \) were to join, \( g(z) \) would take the value zero six times on any small circle encircling \( z_j', z_j'' \), which cannot be, as this does not happen at \( \mu = 0 \). Also, a level line cannot go from a point \( z_j' \) or \( z_j'' \) to itself as the real part of \( F \) is monotonic along level lines.

We conclude that for any small \( \mu > 0 \), and any given filling fractions \( n_1, \ldots, n_{n-1} \geq 0 \) summing up to 1, there is a unique polynomial \( f(z) = \sum a_n z^{n-2} + b_n z^{n-3} + \cdots + b_0 \), so that the curve \( y^2 = y'(z) - 4 \mu f(z) \) has \( a \)-periods \( n_j \). The zeros of \( y \) are connected in pairs by arcs \( \gamma_j \) obeying the reality condition \( (2 \pi i \mu)^{-1} y(x^-), x \in \gamma_j \).

4. Concluding remarks

We have given a non-perturbative definition of the matrix integrals that in the large \( N \) limit give the superpotentials considered in \cite{1, 2}. We considered the case of a generic polynomial potential with complex coefficients. For small \( 't \) Hooft coupling \( \mu \), the density of eigenvalues was shown to be given by arcs connecting pairs of branch points of a hyperelliptic curve. The shape of the arcs is uniquely determined by a reality condition. For larger \( \mu \) or for potentials with degenerate critical points, one expect the arcs to combine into graphs in the complex plane. It would be interesting to understand what kind of graphs can arise in this way.

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