Quasi-period Collapse and $GL_n(\mathbb{Z})$-Scissors Congruence in Rational Polytopes

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Abstract. Quasi-period collapse occurs when the Ehrhart quasi-polynomial of a rational polytope has a quasi-period less than the denominator of that polytope. This phenomenon is poorly understood, and all known cases in which it occurs have been proven with ad hoc methods. In this note, we present a conjectural explanation for quasi-period collapse in rational polytopes. We show that this explanation applies to some previous cases appearing in the literature. We also exhibit examples of Ehrhart polynomials of rational polytopes that are not the Ehrhart polynomials of any integral polytope.

Our approach depends on the invariance of the Ehrhart quasi-polynomial under the action of affine unimodular transformations. Motivated by the similarity of this idea to the scissors congruence problem, we explore the development of a Dehn-like invariant for rational polytopes in the lattice setting.

1. Introduction

A convex rational (respectively, integral) polytope $P \subset \mathbb{R}^n$ is the convex hull of finitely many points in $\mathbb{Q}^n$ (respectively, $\mathbb{Z}^n$). The dimension of $P$ is the dimension of the affine subspace of $\mathbb{R}^n$ spanned by $P$. Dilating $P$ by a positive integer factor $k$ yields the polytope $kP = \{x \in \mathbb{R}^n : \frac{1}{k}x \in P\}$. The denominator of $P$ is the minimum positive integer $D$ such that $DP$ is an integral polytope. A seminal result of Ehrhart in 1962 \cite{Ehr62} provides a beautiful description of the counting function giving the number $|kP \cap \mathbb{Z}^n|$ of integer lattice points in $kP$.

Theorem 1.1 \cite{Ehr62}. If $P \subset \mathbb{R}^n$ is a $d$-dimensional rational polytope, then $|kP \cap \mathbb{Z}^n|$ is given by the restriction to the positive integers of a degree-$d$ quasi-polynomial $L_P : \mathbb{Z} \to \mathbb{Z}$. That is, there exist periodic functions $c_0, \ldots, c_d : \mathbb{Z} \to \mathbb{Q}$ such that $c_d$ is not identically zero and

$$|kP \cap \mathbb{Z}^n| = L_P(k) = c_d(k)k^d + \cdots + c_1(k)k + c_0(k), \quad k \in \mathbb{Z}_{>0}.$$ 

We call $L_P$ the Ehrhart quasi-polynomial of $P$. A positive integer $N$ is a quasi-period of $L_P$ (or of $P$) if $N$ is divisible by the periods of all of the coefficient

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functions $c_i$, $0 \leq i \leq d$. (We do not assume that $N$ is the minimum such positive integer.)

When $P$ is an integral polytope, $L_P$ has quasi-period 1; that is, $L_P(k)$ is a polynomial function of $k$. More generally, the denominator $\mathcal{D}$ of a polytope $P$ is a quasi-period of $L_P$ [Ehr62]. It is somewhat surprising that $\mathcal{D}$ is not always the minimum quasi-period of $P$. When the minimum quasi-period of $P$ is less than $\mathcal{D}$, we say that quasi-period collapse has occurred. Several important polyhedra appearing in the representation theory of Lie algebras exhibit period collapse, but the known proofs of these results are not given in terms of the polyhedral geometry [DLM04, DLM06, DW02, KR86].

Quasi-period collapse cannot happen in dimension 1, but there exist families of polygons in $\mathbb{R}^2$ with arbitrarily large denominators whose minimum quasi-periods are 1. This result was originally proved in [MW05], where the proof of polynomiality involved subdividing the polygons into polygonal pieces whose Ehrhart quasi-polynomials could be computed. The periodic parts for these pieces could be seen by inspection to cancel, with the result that the counting function for the entire polygon was a polynomial.

In this paper, we give a new approach to understanding quasi-period collapse in rational polytopes. This approach yields a much simpler explanation for the polynomiality of the Ehrhart quasi-polynomials appearing in [MW05] (see Example 2.1 below). The demonstration again depends upon polyhedral subdivisions. However, instead of explicitly computing the Ehrhart quasi-polynomials of the pieces in this subdivision, we rearrange unimodular images of the pieces to form an integral polytope. Since this rearrangement does not change the number of lattice points in the polytope or in any of its dilations, it follows immediately that the original Ehrhart quasi-polynomial is a polynomial. Thus we avoid computing the Ehrhart quasi-polynomials of the individual pieces.

This approach provides a unified framework for demonstrating quasi-period collapse of rational polytopes. We conjecture that a polytope exhibits quasi-period collapse only when the pieces of some subdivision of the polytope can be rearranged by affine unimodular transformations to form a polyhedral complex with the “right” denominator. See Conjecture 3.2 for a precise statement. This motivates a study of the invariants of rational polyhedra under polyhedral subdivision and piecewise unimodular transformations. This is reminiscent of the scissors congruence problem for the group of rigid motions in $\mathbb{R}^3$. In the classical scissors congruence problem, congruence classes of polyhedra are parameterized by volume and the Dehn invariant [Syd65]. This suggests that an analogous system of invariants might determine when two rational polyhedra are equidecomposable with respect to the group $\text{Aff}_n(\mathbb{Z}) \cong \text{GL}_n(\mathbb{Z}) \ltimes \mathbb{Z}^n$ of affine unimodular transformations.

2. Proving polynomiality of Ehrhart quasi-polynomials

The phenomenon of quasi-period collapse for rational polytopes is in general poorly understood. In this section, we give examples of rational polytopes that can be shown to have quasi-period 1 by subdivision and rearrangement of unimodular images of the pieces. These examples serve to motivate the following section, in which we conjecture that this method applies to all examples of quasi-period collapse among rational polytopes.
Subdivide $T$ into two triangles by the line $x = 1$ (see left of Figure 1). Let $L$ be the “one-third-open” triangle strictly to the left of the line, and let $R$ be the closed triangle to the right. Thus we have

$$L = \text{conv}\{(0,0)^t, (1,0)^t, (1,\frac{D-1}{D})^t\} \setminus [(1,0)^t, (1,\frac{D-1}{D})^t]$$

$$R = \text{conv}\{(1,0)^t, (D,0)^t, (1,\frac{D-1}{D})^t\}$$

Let $U$ be the affine unimodular transformation $\mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$U(x) = \begin{bmatrix} D-1 & -D \\ -1 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Then $U(L)$ and $R$ are disjoint, and their union is the integral triangle

$$T' = \text{conv}\{(1,0)^t, (1,1)^t, (D,0)^t\}$$

(see right of Figure 1). By construction, $\mathcal{L}_{T'} = \mathcal{L}_T$, and so, since $T'$ is integral, $\mathcal{L}_T$ is a polynomial.

The triangle in Example 2.1 first appeared in [MW05], where it was used to establish the following theorem.

**Theorem 2.2.** Given an integer $D \geq 2$, there exists a polygon with denominator $D$ whose Ehrhart quasi-polynomial is a polynomial.

**Example 2.3.** In [Sta97], Stanley gives an example of a 3-dimensional non-integral polyhedron with quasi-period 1. Let $P \subset \mathbb{R}^3$ be the convex hull of the points $(0,0,0)^t, (1,0,0)^t, (1,1,0)^t, (0,1,0)^t, (1/2,0,1/2)^t$. This is the pyramid pictured on the left side of Figure 2. To see that $\mathcal{L}_P$ is a polynomial, dissect $P$ by the plane perpendicular to the vector $w = (-1,1,1)^t$. The intersection of this plane with $P$ is indicated by the dark gray triangle in Figure 2. Let $U$ be the unimodular transformation of $\mathbb{R}^3$ whose matrix with respect to the standard basis is

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & -1 \\ -1 & 1 & 2 \end{bmatrix}.$$

Applying this transformation to the half-space $\{x \in \mathbb{R}^3: w \cdot x \geq 0\}$ maps $P$ to the integral simplex on the right side of Figure 2.

In the preceding examples, we showed that a non-integral polytope had a polynomial Ehrhart quasi-polynomial because it was, in some sense, a disguised integral polytope—it was an integral polytope up to rearrangement and unimodular transformation of its pieces. One might be tempted to conjecture that all polytopes with polynomial Ehrhart quasi-polynomials are disguised integral polytopes in this...
Figure 2. Non-integral polyhedron and its integral image under piecewise unimodular transformation

sense. In particular, this would imply that, for any rational polytope \( Q \), if \( L_Q \) is a polynomial, then \( L_Q = L_P \) for some integral polytope \( P \). However, this turns out not to be the case. There exist Ehrhart polynomials that are not the Ehrhart polynomials of any integral polytope.

**Example 2.4.** Let \( T \) be the triangle from Example 2.1 and let \( Q \) be the quadrilateral that results from the union of \( T \) with its reflection about the \( x \)-axis. Then \( L_Q(k) = 2L_T(k) - Dk - 1 \) (correcting for the double-counting of the points on the \( x \)-axis). Hence, \( L_Q \) is also a polynomial. Yet we claim that \( L_Q \) is not the Ehrhart polynomial of any integral polygon. This is because \( Q \) has only two lattice points on its boundary, so, by [MW05, Theorem 3.1], the coefficient of the linear term of \( L_Q \) is 1. But any integral polygon \( P \) has at least three lattice points on its boundary, so, by Pick’s theorem, the coefficient of the linear term of \( L_P \) is at least 3/2.

3. Conjectures

As seen in the example concluding the previous section, a polytope \( P \) may have quasi-period 1 and yet not be the result of rearranging unimodular images of the pieces of an integral polytope. Therefore, a more flexible formulation of the process carried out in the preceding examples is necessary if we hope to find a general explanation for the phenomenon of quasi-period collapse.

To this end, recall that a *simplex* is the convex hull of a finite set of affinely independent points. An *open simplex* is the interior of a simplex with respect to the affine subspace that it spans. We call an open simplex *integral* if its closure is integral. The function \( L_S \) counting the lattice points in integral dilations of a \( d \)-dimensional open simplex \( S \) satisfies a well-known reciprocity property: \( L_S(k) = (-1)^dL_{\bar{S}}(-k) \), where \( \bar{S} \) is the closure of \( S \) [Ehr67]. In particular, if \( S \) is an integral open simplex, then \( L_S(k) \) is a polynomial function of \( k \).

**Example 2.4 Continued.** The quadrilateral \( Q \) is a disjoint union of \( T \) and the reflection about the \( x \)-axis of those points in \( T \) strictly above the \( x \)-axis. As in Example 2.1, each of these two sets may in turn be partitioned into open simplices that, under suitable rearrangement by unimodular transformations, form a disjoint union of integral open simplices.
Let \( \text{Aff}_n(\mathbb{Z}) \cong \text{GL}_n(\mathbb{Z}) \times \mathbb{Z}^n \) be the group of affine unimodular transformations on \( \mathbb{R}^n \). To make the process employed above precise, we define the notion of \( \text{GL}_n(\mathbb{Z}) \)-equidecomposability. This definition first appeared in \( \text{Kan98} \) §3.1; it is analogous to the classical Euclidean notion of equidecomposability (see, e.g., \( \text{AZ04} \) Chapter 7).

**Definition 3.1.** We say that two subsets \( P, Q \subset \mathbb{R}^n \) are \( \text{GL}_n(\mathbb{Z}) \)-equidecomposable if there are open simplices \( T_1, \ldots, T_r \) and affine unimodular transformations \( U_1, \ldots, U_r \in \text{Aff}_n(\mathbb{Z}) \) such that

\[
P = \bigsqcup_{i=1}^{r} T_i \quad \text{and} \quad Q = \bigsqcup_{i=1}^{r} U_i(T_i).
\]

(Here, \( \bigsqcup \) indicates disjoint union.)

**Conjecture 3.2.** Suppose that \( P \) is a rational polytope with quasi-period 1. Then there exists a disjoint union \( Q \) of integral open simplices such that \( P \) and \( Q \) are \( \text{GL}_n(\mathbb{Z}) \)-equidecomposable.

Conjecture 3.2 has a natural generalization to polytopes whose quasi-periods collapse to values larger than 1: if \( P \) has minimum quasi-period \( N \), we conjecture that \( P \) is \( \text{GL}_n(\mathbb{Z}) \)-equidecomposable with a disjoint union of open simplices whose denominators are at most \( N \).

The decompositions employed in Examples 2.1 and 2.3 were reasonably easy to find. However, a systematic method of finding such decompositions is obviously desirable if we hope to extend this approach to a general technique for proving polynomiality of Ehrhart quasi-polynomials.

**Open Problem 3.3.** Find a systematic and useful technique that, given a rational polytope \( P \) that is \( \text{GL}_n(\mathbb{Z}) \)-equidecomposable with some integral polytope \( Q \), produces a decomposition \( \{T_i\} \) of \( P \) and a set of unimodular maps \( \{U_i\} \) as in Definition 3.1.

4. \( \text{GL}_n(\mathbb{Z}) \)-Scissors Congruence

Another phenomenon that appeared in the examples from Section 2.1 was the equality of the Ehrhart quasi-polynomials of two distinct polytopes. We say that two rational polytopes \( P \) and \( Q \) are Ehrhart equivalent if and only if their Ehrhart quasi-polynomials are equal. Obviously, any two \( \text{GL}_n(\mathbb{Z}) \)-equidecomposable polytopes are Ehrhart equivalent. But what about the converse? Suppose a rational polytope \( Q \) has the same Ehrhart quasi-polynomial as a polytope \( P \). Are \( P \) and \( Q \) \( \text{GL}_n(\mathbb{Z}) \)-equidecomposable?

The answer is known to be “yes” in the case \( d = 2 \) [Gre93 Theorem 1.3]. An analogy with the scissors congruence problem suggests that this is no longer the case for \( d \geq 3 \). Nonetheless, as we prove below, a weak version of the converse direction does hold (Proposition 4.3). We also propose an ansatz for a \( \text{GL}_n(\mathbb{Z}) \)-Dehn invariant, based on a theorem for reflexive polygons.

**Question 4.1.** Are Ehrhart-equivalent rational polytopes always \( \text{GL}_n(\mathbb{Z}) \)-equidecomposable?
4.1. **Weak GL$_n$($\mathbb{Z}$)-scissors congruence.** If we allow more general translations of the pieces in a decomposition of $P$, we get weak scissors congruences.

**Definition 4.2.** Two rational polytopes $P, Q \subset \mathbb{R}^n$ are weakly GL$_n$($\mathbb{Z}$)-equidecomposable if they can be decomposed into rational polytopes $P_1, \ldots, P_r$ and $Q_1, \ldots, Q_r$, respectively, such that $P_i$ is equivalent to $Q_i$ via GL$_n$($\mathbb{Z}$) $\ltimes \mathbb{Q}^n$.

This is equivalent to saying that there is a factor $k \in \mathbb{Z}_{>0}$ such that $kP$ and $kQ$ are (ordinarily) GL$_n$($\mathbb{Z}$)-equidecomposable.

Observe that the weak version of GL$_n$($\mathbb{Z}$)-equidecomposability does not imply that the Ehrhart quasi-polynomials agree everywhere. Nonetheless, they will agree at infinitely many arguments. Therefore, if two integral polytopes are weakly GL$_n$($\mathbb{Z}$)-equidecomposable, then they must be Ehrhart equivalent.

**Proposition 4.3.** Let $P$ and $Q$ be Ehrhart-equivalent rational polytopes. Then $P$ and $Q$ are weakly GL$_n$($\mathbb{Z}$)-equidecomposable.

**Corollary 4.4.** Two integral polytopes are Ehrhart equivalent if and only if they are weakly GL$_n$($\mathbb{Z}$)-equidecomposable.

**Proof of Proposition 4.3.** By a famous theorem of Kempf et al., there is a positive integer $N$ such that $NP$ and $NQ$ are both integral and admit unimodular triangulations—i.e., triangulations whose simplices are Aff$_n$($\mathbb{Z}$)-equivalent to the standard simplex [KKMSD73]. It is well known that the Ehrhart polynomial of a polytope determines the $f$-vector of a unimodular triangulation of that polytope (see, e.g., [Sta80, Corollary 2.5]). Hence, the triangulations of $NP$ and $NQ$ have the same $f$-vector, and all simplices of a given dimension are equivalent under Aff$_n$($\mathbb{Z}$). Therefore, the corresponding simplices of $P$ and $Q$ are equivalent under GL$_n$($\mathbb{Z}$) $\ltimes \mathbb{Q}^d$. The claim follows. $\square$

4.2. **A GL$_n$($\mathbb{Z}$)-Dehn invariant?** For the classical scissors congruence problem in three dimensions, one uses rigid motions rather than lattice preserving transformations. The volume and the Dehn invariant

$$\text{Dehn}(P) = \sum_{e \text{ an edge of } P} \text{length}(e) \otimes \text{angle}(e) \in \mathbb{R} \otimes_{\mathbb{Z}} \mathbb{R}/\mathbb{Z}\pi$$

provide a complete set of invariants. That is, 3-dimensional polytopes $P$ and $Q$ are scissors congruent if and only if they have the same volume and the same Dehn invariant. The “only if” part is relatively easy to see (see [AZ04, Chapter 7]), because the Dehn invariant is additive, and decompositions of polyhedra satisfy the following two properties.

1. A decomposition edge through a two-dimensional face contributes an angle of $\pi$, so it does not contribute to the Dehn invariant.
2. A decomposition edge through the interior contributes an angle of $2\pi$, so it does not contribute to the Dehn invariant.

**Problem 4.5.** Can we manufacture a Dehn-like invariant in the GL$_n$($\mathbb{Z}$) case?

This invariant, once constructed, will likely be more appropriate to detecting when two lattice polytopes are GL$_n$($\mathbb{Z}$)-equidecomposable into lattice polytopes, in particular, when unimodular triangulations exist.

The role of the full circle $2\pi$ should be played by the “12” of Poonen and Rodriguez-Villegas [PRV00].
Theorem 4.6. The sum of the lengths of a reflexive polygon and its dual is 12.

Here, a lattice polygon is reflexive if it contains a unique interior lattice point, and the length is measured with respect to the lattice. The polygon does not need to be convex. In the non-convex case, the definition of the dual is a little harder [PRV00, HS04]. Around a subdivision edge, we see a polygon with a distinguished interior point—the projection of the edge (see Figure 3).

![Figure 3. Projecting a subdivision edge](image-url)

This gives rise in a canonical way to a (possibly non-convex) reflexive polygon. So we could mimic property $(2\pi)$ of the Dehn invariant by mapping to $\mathbb{Z}/12$. Is there a way to incorporate the property $(\pi)$?

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