The binary Gold function and its c-boomerang connectivity table

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Abstract
Here, we give a complete description of the entire c-Boomerang Connectivity Table for the Gold function over finite fields of even characteristic, by using double Weil sums. As a by-product, we generalize a result of Boura and Canteaut (IACR Trans. Symmetric Cryptol. 2018(3) : 290–310, 2018) for the classical boomerang uniformity (see also the extended abstract by Eddahmani and Mesnager at the Boolean Functions and their Applications (BFA 2021) conference).

Keywords Finite fields · Double Weil sums · Boomerang uniformity · c-boomerang uniformity

Mathematics Subject Classification (2010): 12E20 · 11T24 · 11T06 · 94A60

1 Introduction
Let \( \mathbb{F}_q \) be the finite field with \( q = p^n \) elements, where \( p \) is a prime and \( n \) is a positive integer. The multiplicative cyclic group of nonzero elements of the finite field is denoted by \( \mathbb{F}_q^* = \langle g \rangle \), where \( g \) is a primitive element of \( \mathbb{F}_q \). The canonical additive character is a homomorphism \( \chi_1 : \mathbb{F}_q \rightarrow \mathbb{C} \) of the additive group of \( \mathbb{F}_q \) defined as follows

\[
\chi_1(x) = \exp \left( \frac{2\pi i \text{Tr}(x)}{p} \right),
\]

where \( \mathbb{C} \) is the field of complex numbers and \( \text{Tr} : \mathbb{F}_q \rightarrow \mathbb{F}_p \) is the absolute trace defined by \( \text{Tr}(x) = x + x^p + x^{p^2} + \cdots + x^{p^n-1} \) (to emphasize the dimension, we

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sometimes write this as $\text{Tr}_1^c$). We define the relative trace $\text{Tr}_c : \mathbb{F}_{p^n} \rightarrow \mathbb{F}_{p^r}, c \mid n$, by $\text{Tr}_c(x) = x + x^{p^r} + x^{p^{2r}} + \cdots + x^{p^{(n-1)r}}$. Note that all additive characters of $\mathbb{F}_q$ can be expressed in terms of $\chi_1$ [12, Theorem 5.7].

A Weil sum is an important character sum defined as follows

$$
\sum_{x \in \mathbb{F}_q} \chi(F(x)),
$$

where $\chi$ is an additive character of $\mathbb{F}_q$ and $F$ is a polynomial in $\mathbb{F}_q[x]$. It is well-known that a polynomial $F(x)$ over finite field $\mathbb{F}_q$ is a permutation polynomial (PP) if and only if its Weil sum $\sum_{x \in \mathbb{F}_q} \chi(F(x)) = 0$ for all nontrivial additive characters $\chi$ of $\mathbb{F}_q$. Permutation polynomials are a very important class of polynomials as they have applications in coding theory and cryptography, especially in the substitution boxes (S-boxes) of the block ciphers. The security of the S-boxes relies on certain properties of the function $F(x)$, e.g., its differential uniformity, boomerang uniformity, nonlinearity etc.

Recently, Cid et al. [4] introduced a “new tool” for analyzing the boomerang style attack proposed by Wagner [21]. This new tool is usually referred to as Boomerang Connectivity Table (BCT). Boura and Canteaut [2] further studied BCT and coined the term boomerang uniformity, which is essentially the maximum value in the BCT. Li et al. [13] provided new insights in the study of BCT and presented an equivalent technique to compute BCT, which does not require the compositional inverse of the permutation polynomial $F(x)$ at all. In fact, Li et al. [13] also gave a characterization of BCT in terms of Walsh transform and gave a class of permutation polynomial with boomerang uniformity 4.

Recently, Stănică [16] extended the notion of BCT and boomerang uniformity. In fact, he defined what he termed as $c$-BCT and $c$-boomerang uniformity for an arbitrary polynomial function $F$ over $\mathbb{F}_q$ and for any $c \in \mathbb{F}_q^*$, in the following way. For $a, b \in \mathbb{F}_q$, the entry of the $c$-Boomerang Connectivity Table ($c$-BCT) at $(a, b) \in \mathbb{F}_q \times \mathbb{F}_q$, denoted by $cB_F(a, b)$, is the number of solutions in $\mathbb{F}_p \times \mathbb{F}_p$ of the following system

$$
\begin{aligned}
F(x) - cF(y) &= b \\
F(x + a) - c^{-1}F(y + a) &= b.
\end{aligned}
$$

1.1

The $c$-boomerang uniformity of $F$ is defined as

$$
\beta_{F,c} = \max_{a,b \in \mathbb{F}_p} cB_F(a, b).
$$

In yet other recent papers, Stănică [17, 18] further studied the $c$-BCT for the swapped inverse function and also gave an elegant description of the $c$-BCT entries of the power map in terms of double Weil sums. He further simplified his expressions for the Gold function $x^{q+1}$ over $\mathbb{F}_{p^n}$, for all $1 \leq k < n$ and $p$ odd. In this paper, we shall complement the work of [18] to the finite fields of even characteristic ($p = 2$). As argued also in [18], while the Weil sums expressions may seem complicated from a theoretical perspective, they do have the advantage of being easily implementable. In fact, a simple Sage implementation achieved for all taken examples a speed up of more than 10 times for the current approach versus the Walsh transform or solutions counting approaches. Moreover, on the theoretical side, we generalize a result of Nyberg [15, Proposition 3], as well as a result of Boura and Canteaut [2, Proposition 8] on the classical boomerang uniformities of the Gold function (independently, Eddahmani and Mesnager [7] also explicitly determined the $c$-BCT entries of the permutation Gold functions using a different approach in the particular case of $c = 1$).
The paper is structured as follows. Section 2 contains some preliminary results that will be used throughout. Section 3 contains the characterization of $c$-BCT entries in terms of double Weil sums. For $c = 1$, we further simplify this expression in Section 4. In fact, Theorem 4.1 generalizes previously known results of Boura and Canteaut [2]. In Section 5, we consider the case when $c \in \mathbb{F}_2 \setminus \{0, 1\}$, where $e = \gcd(k, n)$. In Section 6, we discuss the general case. Finally, in Section 7, we discuss the affine, extended affine and CCZ-equivalence as it relates to $c$-boomerang uniformity.

2 Preliminaries

We begin this section by first recalling the recent notion of $c$-differentials introduced in [8]. We shall assume that $q = 2^n$ for rest of the paper. For an $(n, n)$-function $F : \mathbb{F}_q \rightarrow \mathbb{F}_q$, and $c \in \mathbb{F}_q$, we define the (multiplicative) $c$-derivative of $F$ with respect to $a \in \mathbb{F}_q$ to be the function

$$cD_a F(x) = F(x + a) - cF(x), \text{ for all } x \in \mathbb{F}_q.$$ 

Further, for $a, b \in \mathbb{F}_q$, we let the entries of the $c$-Difference Distribution Table (c-DDT) be defined by $c\Delta_c(a, b) = \# \{ x \in \mathbb{F}_q : F(x + a) - cF(x) = b \}$. We call the quantity

$$\delta_{F, c} = \max \{ c\Delta_c(a, b) \mid a, b \in \mathbb{F}_q, \text{ and } a \neq 0 \text{ if } c=1 \},$$

the $c$-differential uniformity of $F$. Note that the case $c = 1$ corresponds to the usual notion of differential uniformity. The interested reader may refer to [1, 9, 11, 14, 19, 20, 22] for some recent results concerning $c$-differential uniformity. It has been proved recently [10] that the differential uniformity is not necessarily smaller than the boomerang uniformity (for non-permutations), as it was previously shown for permutations.

The following theorem is a “binary” analogue of [18, Theorem 1], which gives a nice connection between $c$-BCT and $c$-DDT entries of the power map $x^d$ over $\mathbb{F}_{2^n}$.

**Theorem 2.1** Let $F(x) = x^d$ be a power function on $\mathbb{F}_q$, $q = 2^n$ and $c \in \mathbb{F}_q^*$. Then, for fixed $b \in \mathbb{F}_q^*$, the $c$-Boomerang Connectivity Table entry $\mathcal{B}_F(1, b)$ at $(1, b)$ is given by

$$\frac{1}{q} \left( \sum_{w \in \mathbb{F}_q^*} (\Delta_c(w, b) + c^{-1}\Delta_c(w, b)) \right) \quad - \quad \frac{1}{q^2} \sum_{a, \beta \in \mathbb{F}_q, a \neq \beta} \chi_1(b(\alpha + \beta)) S_{a, \beta} S_{ac, \beta^{-1}} \chi_1(\beta(x + 1)^d),$$

with

$$S_{a, \beta} = \sum_{x \in \mathbb{F}_q} \chi_1(ax^d) \chi_1(\beta(x + 1)^d) = \frac{1}{(q - 1)^2} \sum_{j,k=0}^{q-2} G(\overline{\psi}_j, \chi_1) G(\overline{\psi}_k, \chi_1) \sum_{x \in \mathbb{F}_q} \psi_k((ax^d)^j(\beta(x + 1)^d)^k),$$

where $\chi_1$ is the canonical additive character of the additive group of $\mathbb{F}_q$, $\psi_k$ is the $k$-th multiplicative character of the multiplicative group of $\mathbb{F}_q$ and $G(\psi, \chi)$ is the Gauss sum.
We shall now state some lemmas that will be used in the sequel. The following lemma is well-known and has been used in various contexts.

**Lemma 2.2** Let \( e = \gcd(k, n) \). Then

\[
\text{gcd}(2^k + 1, 2^n - 1) = \begin{cases} 
1 & \text{if } n/e \text{ is odd,} \\
2^e + 1 & \text{if } n/e \text{ is even.}
\end{cases}
\]

We shall also use the following lemma, which appeared in [5], describing the number of roots in \( \mathbb{F}_q^* \) of a linearized polynomial \( u^2x^{2^k} + ux \), where \( u \in \mathbb{F}_q^* \).

**Lemma 2.3** [5, Theorem 3.1] Let \( g \) be a primitive element of \( \mathbb{F}_{2^n}^* \) and let \( e = \gcd(n, k) \). For any \( u \in \mathbb{F}_{2^n}^* \) consider the linearized polynomial \( L_u(x) = u^2x^{2^k} + ux \) over \( \mathbb{F}_{2^n} \). Then for the equation \( L_u(x) = 0 \), the following are true:

1. If \( n/e \) is odd, then there are \( 2^e \) solutions to this equation for any choice of \( u \in \mathbb{F}_{2^n}^* \);
2. If \( n/e \) is even and \( u = g^{t(2^e+1)} \) for some \( t \), then there are \( 2^{2e} \) solutions to the equation;
3. If \( n/e \) is even and \( u \neq g^{t(2^e+1)} \) for any \( t \), then \( x = 0 \) is the only solution.

The explicit expression for the Weil sum of the form \( \sum_{x \in \mathbb{F}_q} \chi_1(ux^{2^k+1} + vx) \), where \( u, v \in \mathbb{F}_{2^n}^* \), is obtained in [5]. In what follows, we shall denote the Weil sum \( \sum_{x \in \mathbb{F}_q} \chi(x^{2^k+1} + vx) \) by \( \mathcal{G}(u, v) \). The following lemma gives the explicit expression for \( \mathcal{G}(u, 0) \).

**Lemma 2.4** [5] Let \( \chi \) be any nontrivial additive character of \( \mathbb{F}_q^* \) and \( g \) be the primitive element of the cyclic group \( \mathbb{F}_q^* \). The following hold:

1. If \( n/e \) is odd, then
   \[
   \sum_{x \in \mathbb{F}_q} \chi(ux^{2^k+1}) = \begin{cases} 
   q & \text{if } u = 0, \\
   0 & \text{otherwise.}
   \end{cases}
   \]
2. Let \( n/e \) be even so that \( n = 2m \) for some integer \( m \). Then
   \[
   \sum_{x \in \mathbb{F}_q} \chi(ux^{2^k+1}) = \begin{cases} 
   (-1)^{m/e}2^m & \text{if } u \neq g^{t(2^e+1)} \text{ for any integer } t, \\
   (-1)^{m/e}2^{m+e} & \text{if } u = g^{t(2^e+1)} \text{ for some integer } t.
   \end{cases}
   \]

From Lemma 2.2, it is easy to see that when \( n/e \) is odd, the power map \( x^{2^k+1} \) permutes \( \mathbb{F}_{2^n} \). Therefore if \( u \neq 0 \), there exists a unique element \( y \in \mathbb{F}_q^* \) such that \( y^{2^k+1} = u \) and hence
\[ \mathcal{S}(u, v) = \sum_{x \in \mathbb{F}_q} \chi(ux^{2^t+1} + vx) = \sum_{x \in \mathbb{F}_q} \chi(x^{2^t+1} + v\gamma^{-1}x) = \mathcal{S}(1, v\gamma^{-1}). \]

The following lemma gives the expression for the Weil sum \( \mathcal{S}(1, v) \) for \( v \neq 0 \) and \( n/e \) odd.

**Lemma 2.5** [5, Theorem 4.2] Let \( v \neq 0 \) and \( n/e \) be odd. Then

\[ \mathcal{S}(1, v) = \begin{cases} 0 & \text{if } \text{Tr}_e(v) \neq 1, \\ \left( \frac{2}{n/e} \right) e^{2 \frac{\pi i}{2}} & \text{if } \text{Tr}_e(v) = 1, \end{cases} \]

where \( \left( \frac{2}{n/e} \right) \) is the Jacobi symbol.

In the case when \( u, v \neq 0 \) and \( n/e \) is even, the Weil sum \( \mathcal{S}(u, v) \) depends on whether or not the linearized polynomial \( L_u(x) = u^{2^t}x^{2^t} + ux \) is a permutation of \( \mathbb{F}_{2^n} \). The following lemma gives the expression for Weil sum \( \mathcal{S}(u, v) \) for \( u, v \neq 0 \) and \( n/e \) even.

**Lemma 2.6** [5, Theorem 5.3] Let \( u, v \in \mathbb{F}_q^* \) and \( n/e \) be even so \( n = 2m \) for some integer \( m \). Then

1. If \( u \neq s^{(2^t+1)} \) for any integer \( t \) then \( L_u \) is a PP. Let \( x_u \in \mathbb{F}_q \) be the unique solution of the equation \( L_u(x) = v^{2^t} \). Then
   \[ \mathcal{S}(u, v) = (-1)^{m/e}2^{m}X_1(ux_u^{2^t+1}). \]
2. If \( u = s^{(2^t+1)} \) for some integer \( t \), then \( \mathcal{S}(u, v) = 0 \) unless the equation \( L_u(x) = v^{2^t} \) is solvable. If the equation \( L_u(x) = v^{2^t} \) is solvable with some solution, say \( x_u \), then
   \[ \mathcal{S}(u, v) = \begin{cases} (-1)^{m/e}2^{m}X_1(ux_u^{2^t+1}) & \text{if } \text{Tr}_e(u) \neq 0, \\ (-1)^{m/e}2^{m+1/2}X_1(ux_u^{2^t+1}) & \text{if } \text{Tr}_e(u) = 0. \end{cases} \]

3 The binary Gold function

In this section, we shall give the explicit expression for the \( c \)-BCT entries of the Gold function \( x^{2^t+1} \) over \( \mathbb{F}_{2^n} \), for all \( c \neq 0 \). Recall that the \( c \)-boomerang uniformity of a power function \( F(x) = x^d \) over \( \mathbb{F}_{2^n} \) is given by \( \max_{b \in \mathbb{F}_{2^n}} B_F(1, b) \), where \( B_F(1, b) \) is the number of solutions in \( \mathbb{F}_q \times \mathbb{F}_q, q = 2^n \) of the following system

\[ \begin{aligned} x^d + cy^d &= b, \\ (x + 1)^d + c^{-1}(y + 1)^d &= b. \end{aligned} \]  

(1.2)

As done in [18], for \( b \neq 0 \) and fixed \( c \neq 0 \), the number of solutions \( (x, y) \in \mathbb{F}_q^2 \) of the system (1.2) is given by
\[ e_{B_F}(1, b) = \frac{1}{q^2} \sum_{x,y \in \mathbb{F}_q} \sum_{a \in \mathbb{F}_q} \chi_1(a(x^d + cy^d + b)) \sum_{\beta \in \mathbb{F}_q} \chi_1(\beta((x + 1)^d + c^{-1}(y + 1)^d + b)) \]

\[ = \frac{1}{q^2} \sum_{a,\beta \in \mathbb{F}_q} \chi_1(b(\alpha + \beta)) \sum_{x \in \mathbb{F}_q} \chi_1(\alpha x^d + \beta(x + 1)^d) \sum_{y \in \mathbb{F}_q} \chi_1(c\alpha y^d + c^{-1}\beta(y + 1)^d) \]

\[ = \frac{1}{q^2} \sum_{a,\beta \in \mathbb{F}_q} \chi_1(b(\alpha + \beta)) S_{a,\beta} S_{c,\alpha^{-1}\beta}, \]

where \( S_{a,\beta} = \sum_{x \in \mathbb{F}_q} \chi_1(\alpha x^d + \beta(x + 1)^d) \). Therefore, the problem of computing the \( c \)-BCT entry \( e_{B_F}(1, b) \) is reduced to the computation of the product of the Weil sums \( S_{a,\beta} \) and \( S_{c,\alpha^{-1}\beta} \). Now, in the particular case when \( d = 2^k + 1 \), i.e., for the Gold case, we shall further simplify the expression for \( S_{a,\beta} \) as follows:

\[ S_{a,\beta} = \sum_{x \in \mathbb{F}_q} \chi_1(\alpha x^{2^k+1} + \beta(x + 1)^{2^k+1}) \]

\[ = \chi_1(\beta) \sum_{x \in \mathbb{F}_q} \chi_1((\alpha + \beta)x^{2^k+1}) \chi_1(\beta x^{2^k} + \beta x) \]

\[ = \chi_1(\beta) \sum_{x \in \mathbb{F}_q} \chi_1((\alpha + \beta)x^{2^k+1}) \chi_1((\beta^{2^k})^x + \beta x) \]

\[ = \chi_1(\beta) \sum_{x \in \mathbb{F}_q} \chi_1((\alpha + \beta)x^{2^k+1} + (\beta^{2^k} + \beta)x) \]

\[ = \chi_1(\beta) \sum_{x \in \mathbb{F}_q} \chi_1(Ax^{2^k+1} + Bx), \]

where \( A = \alpha + \beta \) and \( B = \beta^{2^k} + \beta \). Here one may note that \( A = 0 \) if and only if \( \alpha = \beta \). Also, \( B = 0 \) if and only if \( \beta \in \mathbb{F}_{2^k} \), since

\[ B = 0 \iff \beta^{2^k-1} = 1 \]

\[ \iff \beta^{2^k-2} = 1 \] (as \( \gcd(n - k, n) = e \))

\[ \iff \beta \in \mathbb{F}_{2^k}. \]

Now we shall calculate \( S_{a,\beta} \) in two cases, namely, \( n/e \) odd and \( n/e \) even, respectively.

**Case 1: \( n/e \) is odd.**

In this case, if \( \alpha = \beta \) and \( \beta \in \mathbb{F}_{2^k} \), then \( S_{a,\beta} = q \chi_1(\beta) \). If \( \alpha = \beta \) and \( \beta \in \mathbb{F}_{2^n} \setminus \mathbb{F}_{2^k} \), then \( S_{a,\beta} = 0 \). In the event of \( \alpha \neq \beta \) and \( \beta \in \mathbb{F}_{2^k} \), again we have \( S_{a,\beta} = 0 \). Finally, if \( \alpha \neq \beta \) and \( \beta \in \mathbb{F}_{2^n} \setminus \mathbb{F}_{2^k} \), by Lemma 2.5 we have,
\[ S_{\alpha, \beta} = \begin{cases} 0 & \text{if } \text{Tr}_e(B\gamma^{-1}) \neq 1, \\ \left(\frac{2}{n/e}\right)^{\frac{m}{2}} \chi_1(\beta) & \text{if } \text{Tr}_e(B\gamma^{-1}) = 1, \end{cases} \]

where \( \gamma \in \mathbb{F}_q \) is the unique element such that \( \gamma^{2k+1} = A \).

**Case 2:** \( n/e \) is even.

Let us denote \( n = 2m \), for some positive integer \( m \) and \( g \) be a primitive element of the finite field \( \mathbb{F}_q \). Then \( S_{\alpha, \beta} = g x_1(\beta) \). If \( \alpha = \beta \) and \( \beta \in \mathbb{F}_{2^m} \) then again \( S_{\alpha, \beta} = 0 \). In the event of \( \alpha \neq \beta \) and \( \beta \in \mathbb{F}_{2^m} \), by Lemma 2.4 we have

\[ S_{\alpha, \beta} = \begin{cases} (\alpha)^{m/e} 2^n \chi_1(\beta) & \text{if } A \neq g^{-n/2} \text{ for any integer } t, \\ (\alpha)^{m/e} 2^n \chi_1(\beta) & \text{if } A = g^{-n/2} \text{ for some integer } t. \end{cases} \]

Finally, when \( \alpha \neq \beta \) and \( \beta \in \mathbb{F}_{2^m} \), we shall consider two cases depending on whether or not the linearized polynomial \( L_A(x) = A^{2k}x^{2\gamma} + Ax \) is a permutation polynomial. From Lemma 2.3, \( L_A \) is a permutation polynomial if and only if \( n/e \) is even and \( A \neq g^{n/2+1} \) for any integer \( t \). Therefore, when \( n/e \) is even and \( A \neq g^{n/2+1} \) for any integer \( t \), the equation \( L_A(x) = B^{2k} \) will have a unique solution, say \( x_A \). Therefore, by Lemma 2.6, we have

\[ S_{\alpha, \beta} = (\alpha)^{m/e} 2^n \chi_1(\beta) x_1(\alpha A^{2k+1}). \]

Now if the linearized polynomial \( L_A \) is not permutation, i.e, \( n/e \) is even and \( A = g^{n/2+1} \) for some integer \( t \), we again have two cases depending on whether or not the equation \( L_A(x) = B^{2k} \) is solvable. In the case when equation \( L_A(x) = B^{2k} \) is solvable, let \( x_A \) be one of its solution. Therefore, by Lemma 2.6 we have,

\[ S_{\alpha, \beta} = \begin{cases} (\alpha)^{m/e} 2^n \chi_1(\beta) x_1(A x_1^{2k+1}) & \text{if } \text{Tr}_e(A) = 0, \\ (\alpha)^{m/e} 2^n \chi_1(\beta) x_1(A x_1^{2k+1}) & \text{if } \text{Tr}_e(A) \neq 0. \end{cases} \]

If \( L_A(x) = B^{2k} \) is not solvable, again, by Lemma 2.6, \( S_{\alpha, \beta} = 0 \).

Thus we have computed \( S_{\alpha, \beta} \) in all possible cases. Similarly, we can find \( S_{ca,c^{-1}B} \) by putting \( ca \) and \( c^{-1}B \) in place of \( \alpha \) and \( \beta \), respectively. We shall now explicitly compute the \( c \)-BCT entry \( iB_F(1, b) \) for \( c = 1, c \in \mathbb{F}_q \setminus \{0, 1\} \) and \( c \in \mathbb{F}_{2^m} \setminus \mathbb{F}_{2^m} \) in the forthcoming sections.

### 4 The case \( c = 1 \)

When \( c = 1 \), \( S_{\alpha, \beta} \) and \( S_{ca,c^{-1}B} \) coincide, therefore for any fixed \( b \neq 0 \), the \( c \)-BCT entry is given by,

\[ iB_F(1, b) = \frac{1}{q^2} \sum_{\alpha, \beta \in \mathbb{F}_q} \chi_1(b(\alpha + \beta)) S_{\alpha, \beta}^2. \]

Let us denote \( T_b = S_{\alpha, \beta}^2 \). Now we shall consider two cases, namely, \( n/e \) odd and \( n/e \) even, respectively.
Case 1: $n/e$ is odd. We consider the following subcases.

1. If $\alpha = \beta$ and $\beta \in \mathbb{F}_2^*$, then
   \[ T_b^{[1]} = q^2 \chi_1(\beta)^2 = q^2. \]

2. If $\alpha = \beta$ and $\beta \in \mathbb{F}_2 \setminus \mathbb{F}_2^*$, then
   \[ T_b^{[2]} = 0. \]

3. If $\alpha \neq \beta$ and $\beta \in \mathbb{F}_2$, then
   \[ T_b^{[3]} = 0. \]

4. If $\alpha \neq \beta$ and $\beta \in \mathbb{F}_2 \setminus \mathbb{F}_2^*$, then
   \[ T_b^{[4]} = \begin{cases} 
   0 & \text{if } \text{Tr}_e(B^{-1}) \neq 1, \\
   2^{n+e} & \text{if } \text{Tr}_e(B^{-1}) = 1. 
   \end{cases} \]

Nyberg [15, Proposition 3] showed that the differential uniformity of the Gold function $x \mapsto x^{2^k+1}$ over $\mathbb{F}_2$, is $2^e$, where $e = \gcd(k, n)$. Also, from [4], we know that the boomerang uniformity of the APN function equals 2. Boura and Canteaut [2, Proposition 8] proved that when $n/e$ is odd and $n \equiv 2 \pmod{4}$, then the differential uniformity as well as the boomerang uniformity of the Gold function $x \mapsto x^{2^k+1}$ is 4.

Our first theorem in this section generalizes the two previously mentioned results, and gives the boomerang uniformity of the Gold function for any parameters, when $n/e$ is odd. Note that we would require the notion of Walsh-Hadamard transform in the proof of this theorem, which is defined as follows.

For $f : \mathbb{F}_2^n \rightarrow \mathbb{F}_2$ we define the Walsh-Hadamard transform to be the integer-valued function

\[ W_f(u) = \sum_{x \in \mathbb{F}_2^n} (-1)^{f(x)+\text{Tr}(ux)}, u \in \mathbb{F}_2^n. \]

The Walsh transform $W_F(a, b)$ of an $(n, m)$-function $F : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^m$ at $a \in \mathbb{F}_2^n$, $b \in \mathbb{F}_2^m$ is the Walsh-Hadamard transform of its component function $\text{Tr}_1^m(bF(x))$ at $a$, that is,

\[ W_F(a, b) = \sum_{x \in \mathbb{F}_2^n} (-1)^{\text{Tr}_1^m(bF(x))} - \text{Tr}_1^n(ax). \]

**Theorem 4.1** Let $F(x) = x^{2^k+1}$, $1 \leq k < n$, be a function on $\mathbb{F}_q^n$, $q = 2^n$, $n \geq 2$. Let $c = 1$ and $n/e$ be odd, where $e = \gcd(k, n)$. Then the $c$-BCT entry $\mathcal{B}_F(1, b)$ of $F$ at $(1, b)$ is

\[ \mathcal{B}_F(1, b) = 0, \text{ or, } 2^e, \]

if $\text{Tr}_e\left(b^{\frac{1}{2}}\right) = 0$, respectively, $\text{Tr}_e\left(b^{\frac{1}{2}}\right) \neq 0$.

**Proof** For every $\alpha, \beta$, let $A = \alpha + \beta, B = \beta^{2^k} + \beta$, and $\gamma \in \mathbb{F}_q$ be the unique element such that $\gamma^{2^k+1} = A$. Further, let
\[ A = \{(\alpha, \beta) \in \mathbb{F}_q^2 \mid \alpha = \beta \in \mathbb{F}_{2^e}\}, \]
\[ B = \{(\alpha, \beta) \in \mathbb{F}_q^2 \mid \alpha = \beta \in \mathbb{F}_q \setminus \mathbb{F}_{2^e}\}, \]
\[ C = \{(\alpha, \beta) \in \mathbb{F}_q^2 \mid \alpha \neq \beta \text{ and } \beta \in \mathbb{F}_{2^e}\}, \]
\[ D = \{(\alpha, \beta) \in \mathbb{F}_q^2 \mid \alpha \neq \beta \text{ and } \beta \in \mathbb{F}_q \setminus \mathbb{F}_{2^e}\}, \]
\[ E = \{(\alpha, \beta) \in D \mid \text{Tr}_e(B\gamma^{-1}) \neq 1\}, \]
\[ F = \{(\alpha, \beta) \in D \mid \text{Tr}_e(B\gamma^{-1}) = 1\}. \]

Then,
\[
I_{B}(1, b) = \frac{1}{q^2} \left( \sum_{(a, b) \in A} \chi_1(b(a + \beta))T_b^{[1]} + \sum_{(a, b) \in B} \chi_1(b(a + \beta))T_b^{[2]} + \sum_{(a, b) \in C} \chi_1(b(a + \beta))T_b^{[3]} \right)
\]
\[
+ \sum_{(a, b) \in E} \chi_1(b(a + \beta))T_b^{[4]} + \sum_{a, \beta \in F} \chi_1(b(a + \beta))T_b^{[5]} \right)
\]
\[
= \frac{1}{q^2} \left( \sum_{(a, b) \in A} q^2 + \sum_{(a, b) \in F} \chi_1(b(a + \beta))2^{n+e} \right)
\]
\[
= 2^e + \frac{2^e}{2^n} \sum_{(a, b) \in F} \chi_1(b(a + \beta)). \]

As customary, \( t^{-1} = t^{2^e-2} \), rendering \( 0^{-1} = 0 \). For each \( \beta \in \mathbb{F}_{2^e} \setminus \mathbb{F}_{2^e} \), we let (if \( \beta \in \mathbb{F}_{2^e} \), \( Y_\beta = \mathbb{F}_{2^e} \))
\[
Y_\beta = \left\{ \gamma^{-1} \in \mathbb{F}_{2^e} \mid \text{Tr}_e((\beta^{2^{-e}} + \beta)\gamma^{-1}) = 1 \right\},
\]
and
\[
T_\beta = \left\{ d \in \mathbb{F}_{2^e} \mid \text{Tr}_e((\beta^{2^{-e}} + \beta)d) = 0 \right\} = (\beta^{2^{-e}} + \beta)^{-1}.
\]

We shall use below that when \( e \) is odd, then \( \text{Tr}_e(1) = 1 \). We label by \( \langle S \rangle_\beta \) the \( \mathbb{F}_{2^e} \)-linear subspace in \( \mathbb{F}_{2^e} \) generate by \( S \) and we write \( S_\beta \), for the trace orthogonal (via the relative trace \( \text{Tr}_e \)) of the subspace \( \langle S \rangle_\beta \) (if \( e = 1 \), we drop the subscripts). Since \( \text{Tr}_e(1) = 1 \), then, \( (\beta^{2^{-e}} + \beta)^{-1} \in Y_\beta \). If \( \gamma_1^{-1}, \gamma_2^{-1} \in Y_\beta \), then \( \gamma_1^{-1} + \gamma_2^{-1} \in T_\beta \), of cardinality \( |T_\beta| = 2^{n-1} \). Reciprocally, if \( \gamma^{-1} \in Y_\beta \) and \( d \in T_\beta \), it is easy to see that \( \gamma^{-1} + d \in Y_\beta \). Therefore, \( Y_\beta \) is the affine subspace \( Y_\beta = \gamma_\beta + T_\beta \), where \( \gamma_\beta = (\beta^{2^{-e}} + \beta)^{-1} \).

Next, we observe that the kernel of \( \phi : \beta \mapsto \beta^{2^{-e}} + \beta \), say \( \ker(\phi) \), is an \( \mathbb{F}_{2^e} \)-linear space of dimension \( e \) (in fact, it is exactly \( \mathbb{F}_{2^e} \)) and the image of \( \phi \), say \( \text{Im}(\phi) \), is an \( \mathbb{F}_2 \)-linear space of dimension \( n - e \). Further, we show that \( \text{Im}(\phi)^\perp = \ker(\phi) \). We use below the fact that \( \text{Tr}_e(x^{2^e}) = \text{Tr}_e(x) \) and \( e \mid k \). Let \( u \in \text{Im}(\phi)^\perp \), that is, for all \( \beta \in \mathbb{F}_{2^e}, \)
\[
0 = \text{Tr}_e(u(\beta^{2^{-e}} + \beta)) = \text{Tr}_e(u\beta^{2^{-e}}) + \text{Tr}_e(u\beta) = \text{Tr}_e(u\beta^{2^e}) + \text{Tr}_e(u\beta) = \text{Tr}_e((u + u^{2^e})\beta),
\]
and so, \( u^{2^e} + u = 0 \), which shows the claim. For easy referral, if we speak of the dimension of an \( \mathbb{F}_{2^e} \)-linear space \( S \), we shall be using the notation \( \dim_{\mathbb{F}_e} S \) (no subscript if \( e = 1 \)).

We will be using below the Poisson summation formula (see [3, Corollary 8.9] and [6, Theorem 2.15]), which states that if \( f : \mathbb{F}_{2^e} \rightarrow \mathbb{R} \) and \( S \) is a subspace of \( \mathbb{F}_{2^e} \), of dimension \( \dim S \), then

\[ \sum_{\beta \in S} \chi_1(\beta \gamma) = \frac{\dim S}{q^2} \sum_{\beta \in S} \chi_1(\beta) \]
\[
\sum_{u \in \alpha + S} W_j(u)(-1)^{\text{Tr}(\beta u)} = 2^{\dim S}(-1)^{\text{Tr}(\alpha \beta)} \sum_{u \in \beta + S^\perp} f(u)(-1)^{\text{Tr}(au)},
\]

and in particular,
\[
\sum_{u \in S} W_j(u) = 2^{\dim S} \sum_{u \in S^\perp} f(u).
\]

Now, we are able to compute our sum (labelling \(\alpha = \beta + \gamma^{2^k+1}\), and writing \(\phi^{-1}(t) = (\beta : \phi(\beta) = t);\) we also note that when \(\frac{a}{e}\) is odd, \(\gcd(2^k + 1, 2^n - 1) = 1\), and so \(\gamma \mapsto \gamma^{2^k+1}\) is a permutation)

\[
iB_1(1, b) = 2^e + \frac{2^e}{2^n} \sum_{\beta \in \mathbb{F}_{2^n} \setminus \mathbb{F}_{2^n}, \gamma \in \mathbb{F}_{2^n}, \text{Tr}_e((\beta^{2^k} + \beta)\gamma^{-1}) = 1} \chi_1(b \gamma^{2^k+1})
\]

\[
= 2^e + \frac{2^e}{2^n} \sum_{\beta \in \mathbb{F}_{2^n}} \sum_{\gamma^{-1} \in \mathbb{F}_b} \chi_1(b \gamma^{2^k+1})
\]

\[
= 2^e + \frac{2^e}{2^n} \sum_{\beta \in \mathbb{F}_{2^n}} 2^{-\dim S} \sum_{u \in \langle (\beta^{2^k} + \beta)^{-1} \rangle^\perp} W_{g_\beta}(u)(-1)^{\text{Tr}(u(\beta^{2^k} + \beta)^{-1})}
\]

(by Poisson summation with \(S^\perp = \langle \beta^{2^k} + \beta \rangle^\perp\), and \(g_\beta(x) = \chi_1(b x^{2^k-1})\)).

We now analyze the \(\mathbb{F}_{2^n}\)-linear space

\[
\langle \beta^{2^k} + \beta \rangle^\perp = \{ x \in \mathbb{F}_{2^n} : \text{Tr}(dx) = 0, \forall d \text{ with } \text{Tr}_e(d(\beta^{2^k} + \beta)) = 0 \}.
\]

Further, \(\mathbb{F}_{2^n}\) has dimension \(n/e\) as an \(\mathbb{F}_{2^n}\)-linear space and so, \(\dim \langle (\beta^{2^k} + \beta)^{-1} \rangle = \frac{n}{e} - 1\) as an \(\mathbb{F}_{2^n}\)-linear space, and since \(\mathbb{F}_{2^n}\) has dimension \(e\) as an \(\mathbb{F}_{2^n}\)-linear space, then \(\dim \langle (\beta^{2^k} + \beta)^{-1} \rangle = n - e \) as an \(\mathbb{F}_{2^n}\)-linear space. Thus, \(\dim \langle (\beta^{2^k} + \beta)^{-1} \rangle = \frac{n}{e} - 1\). Moreover, \(\text{Tr}_e(\beta^{2^k} + \beta) = 0\) and if \(u \in \mathbb{F}_{2^n}\) then \(\text{Tr}_e(u(\beta^{2^k} + \beta)) = u \text{Tr}_e(\beta^{2^k} + \beta) = 0\), and consequently (since the dimensions match and \((\beta^{2^k} + \beta)\mathbb{F}_{2^n} \subseteq S\))

\[
S = \langle (\beta^{2^k} + \beta)^{-1} \rangle = (\beta^{2^k} + \beta)\mathbb{F}_{2^n}.
\]

We are now ready to continue the computation, thus,
\[ \mathcal{G}_C = 2^e + \frac{2^e}{2^a} \sum_{\beta \in \mathbb{F}_{2^e}} \sum_{u \in \mathbb{U}_{2^e}} \mathcal{W}_B(u) (-1)^{Tr(u(\beta^{2^n} + \beta)^{-1})} \]

\[ = 2^e + \frac{2^e}{2^a} 2^{-e} \sum_{\beta \in \mathbb{F}_{2^e}} \sum_{d' \in \mathbb{F}_{2^e}} \mathcal{W}_B(d' (\beta^{2^n} + \beta)) (-1)^{Tr(d')} \]

\[ = 2^e + \frac{2^e}{2^a} 2^{-e} \sum_{\beta \in \mathbb{F}_{2^e}} \sum_{d' \in \mathbb{F}_{2^e}} \sum_{x \in \mathbb{F}_{2^e}} \chi_1 \left( b x^{2^n-1} + d' x (\beta^{2^n} + \beta) \right) \]

\[ = 2^e + \frac{2^e}{2^a} 2^{-e} \sum_{d' \in \mathbb{F}_{2^e}} \chi_1 \left( b x^{2^n-1} + d' \right) \sum_{\beta \in \mathbb{F}_{2^e}} \chi_1 \left( d' x (\beta^{2^n} + \beta) \right) \]

\[ = 2^e + \frac{2^e}{2^a} 2^{-e} \sum_{d' \in \mathbb{F}_{2^e}, x \in \mathbb{F}_{2^e}} \chi_1 \left( b x^{2^n-1} + d' \right) \]

\[ = 2^e + \frac{2^e}{2^a} 2^{-e} \sum_{d' \in \mathbb{F}_{2^e}, x \in \mathbb{F}_{2^e}} \chi_1 \left( b x^{2^n-1} + d' \right) \]

\[ = 2^e - 2^e \delta_0 \left( \text{Tr}_e \left( b^{\frac{1}{2}} \right) \right), \]

where \( \delta_0 \) is the Dirac symbol, defined by \( \delta_0(c) = 1 \), if \( c = 0 \), and 0, otherwise. Thus, \( \mathcal{B}_C(1, b) \in \{0, 2^e\} \), and the claim of our theorem is shown.

**Remark 4.2** Note that independently, Eddahmani and Mesnager [7] have explicitly determined the \( c \)-BCT entries of the permutation Gold functions via a different approach in the particular case of \( c = 1 \). However, we determine the expressions for \( c \)-BCT entries for an arbitrary value of \( c \).

**Case 2:** \( n/e \) is even.

1. If \( \alpha = \beta \) and \( \beta \in \mathbb{F}_{2^e} \), then
   \[ T^{11}_{\beta} = q^2 \chi_1(\beta)^2 = q^2. \]

2. If \( \alpha = \beta \) and \( \beta \in \mathbb{F}_{2^e} \setminus \mathbb{F}_{2^n} \), then
   \[ T^{12}_{\beta} = 0. \]

3. If \( \alpha \neq \beta \) and \( \beta \in \mathbb{F}_{2^e} \), then
\[ T_{\beta}^{[3]} = \begin{cases} 2^n & \text{if } A \neq g^{n(2^r+1)} \text{ for any integer } t, \\ 2^{n+2e} & \text{if } A = g^{n(2^r+1)} \text{ for some integer } t. \end{cases} \]

(4) If \( \alpha \neq \beta \) and \( \beta \in \mathbb{F}_2 \setminus \mathbb{F}_{2^r} \), then

(a) If \( A \neq g^{n(2^r+1)} \) for any integer \( t \), then
\[ T_{\beta}^{[4(\alpha)]} = 2^n. \]

(b) If \( A = g^{n(2^r+1)} \) for some integer \( t \), then

(i) If the equation \( L_A(x) = B^{2^t} \) is not solvable, where \( L_A(x) = A^{2^t}x^{2^n} + Ax \), then
\[ T_{\beta}^{[4(\beta)(i)]} = 0. \]

(ii) If the equation \( L_A(x) = B^{2^t} \) is solvable, then
\[ T_{\beta}^{[4(\beta)(ii)]} = \begin{cases} 2^n & \text{if } \text{Tr}_e(A) \neq 0, \\ 2^{n+2e} & \text{if } \text{Tr}_e(A) = 0. \end{cases} \]

Now we shall summarize the above discussion in the following theorem.

**Theorem 4.3** Let \( F(x) = x^{2^k+1}, 1 \leq k < n \) be a function on \( \mathbb{F}_{2^n} \), \( n \geq 2 \). Let \( c = 1 \) and \( n/e \) be even, where \( e = \gcd(k, n) \). Then the \( c \)-BCT entry \( B_F(1, b) \) of \( F \) at \( (1, b) \) is given by

\[
2^e + \frac{1}{2^n} \sum_{(a, \beta) \in G \cup H \cup L} \chi_1(b(a + \beta)) + \frac{2^{2e}}{2^n} \sum_{(a, \beta) \in H \cup L} \chi_1(b(a + \beta)),
\]

with \( A = \alpha + \beta, B = \beta^{2^{n-k}} + \beta, L_A(x) = A^{2^t}x^{2^n} + Ax \), and

\[ G = \{(\alpha, \beta) \in C \mid A \neq g^{n(2^r+1)} \text{ for any integer } t\}, \]
\[ H = \{(\alpha, \beta) \in C \mid A = g^{n(2^r+1)} \text{ for some integer } t\}, \]
\[ I = \{(\alpha, \beta) \in D \mid A \neq g^{n(2^r+1)} \text{ for any integer } t\}, \]
\[ K = \{(\alpha, \beta) \in D \mid A = g^{n(2^r+1)} \text{ for some integer } t, \text{ Tr}_e(A) \neq 0, L_A(x) = B^{2^t} \text{ is solvable}\}, \]
\[ L = \{(\alpha, \beta) \in D \mid A = g^{n(2^r+1)} \text{ for some integer } t, \text{ Tr}_e(A) = 0, L_A(x) = B^{2^t} \text{ is solvable}\}. \]

**Proof** For the proof, we need to define

\[ J = \{(\alpha, \beta) \in D \mid A = g^{n(2^r+1)} \text{ for an integer } t, L_A(x) = B^{2^t} \text{ is not solvable}\}. \]

Then
\[ 1 \mathcal{B}_F(1, b) = \frac{1}{q^2} \left( \sum_{(a, \beta) \in \mathcal{A}} \chi_1(b(\alpha + \beta))T_b^{[1]} + \sum_{(a, \beta) \in \mathcal{B}} \chi_1(b(\alpha + \beta))T_b^{[2]} \right. \\
+ \sum_{(a, \beta) \in \mathcal{G}} \chi_1(b(\alpha + \beta))T_b^{[3]} + \sum_{(a, \beta) \in \mathcal{H}} \chi_1(b(\alpha + \beta))T_b^{[4]} \\
+ \sum_{(a, \beta) \in \mathcal{I}} \chi_1(b(\alpha + \beta))T_b^{[5]} + \sum_{(a, \beta) \in \mathcal{J}} \chi_1(b(\alpha + \beta))T_b^{[6]} \\
\left. + \sum_{(a, \beta) \in \mathcal{K}} \chi_1(b(\alpha + \beta))T_b^{[7]} + \sum_{(a, \beta) \in \mathcal{L}} \chi_1(b(\alpha + \beta))T_b^{[8]} \right) \]

\[ = \frac{1}{q^2} \left( \sum_{(a, \beta) \in \mathcal{A}} q^2 + 2^n \sum_{(a, \beta) \in \mathcal{G} \cup \mathcal{J} \cup \mathcal{K}} \chi_1(b(\alpha + \beta)) + 2^{n+2}e \sum_{(a, \beta) \in \mathcal{H} \cup \mathcal{L}} \chi_1(b(\alpha + \beta)) \right) \]

\[ = 2^e + \frac{1}{2^n} \sum_{(a, \beta) \in \mathcal{G} \cup \mathcal{J} \cup \mathcal{K}} \chi_1(b(\alpha + \beta)) + \frac{2^{2e}}{2^n} \sum_{(a, \beta) \in \mathcal{H} \cup \mathcal{L}} \chi_1(b(\alpha + \beta)). \]

This completes the proof.

**Corollary 4.4** Let \( F(x) = x^{2^k+1}, 1 \leq k \leq n \), be a function on \( \mathbb{F}_q \), \( n \geq 2 \). Let \( c = 1 \) and \( n \) be even, where \( e = \gcd(k, n) \). With the notations of the previous theorem, the \( c \)-boomerang uniformity of \( F \) satisfies

\[ \beta_{F, c} \leq 2^e + 2^{-n} |\mathcal{G} \cup \mathcal{J} \cup \mathcal{K}| + 2^{2e-n} |\mathcal{H} \cup \mathcal{L}|. \]

### 5 The case \( c \in \mathbb{F}_2 \setminus \{0, 1\} \)

Since the case \( c = 1 \) has already been considered in the previous section, throughout this section we assume that \( c \neq 1 \). Notice that when \( c \in \mathbb{F}_2 \), \( \beta \in \mathbb{F}_2 \Leftrightarrow \beta c^{-1} \in \mathbb{F}_2 \). Recall that for any fixed \( b \neq 0 \), the \( c \)-BCT entry is given by

\[ c \mathcal{B}_F(1, b) = \frac{1}{q^2} \sum_{a, \beta \in \mathbb{F}_q} \chi_1(b(\alpha + \beta))S_{a, \beta}S_{ca, c^{-1} \beta}. \]

Let us denote \( T_b = S_{a, \beta}S_{ca, c^{-1} \beta} \) (we will use superscripts to point out the case we are in, for its value). Recall that \( A = \alpha + \beta \) and \( B = \beta^{2^{n-1}} + \beta \). Let us denote \( \gamma = A^{\frac{1}{2^{n-1}}} \), \( A' = ca + c^{-1} \beta \) and \( B' = (c^{-1} \beta)^{2^{n-1}} + c^{-1} \beta \). It is easy to observe that the conditions \( B = 0 \) and \( B' = 0 \) are equivalent. Now we shall consider two cases namely, \( \frac{n}{e} \) odd and \( \frac{n}{e} \) even, respectively.

**Case 1:** \( \frac{n}{e} \) is odd.

1. Let \( A = 0, B = 0 \).

   a. If \( A' = 0, B' = 0 \), then

   \[ T_b^{[1(a)]} = q^2 \chi_1((1 + c^{-1}) \beta). \]

   b. If \( A' \neq 0, B' = 0 \), then \( S_{ca, c^{-1} \beta} = 0 \) and hence
(2) Let \( A = 0, B \neq 0 \). In this case \( S_{a,\beta} = 0 \) and hence
\[
T^{[1]}_b = 0.
\]

(3) Let \( A \neq 0, B = 0 \). Again \( S_{a,\beta} = 0 \) and hence
\[
T^{[3]}_b = 0.
\]

(4) Let \( A \neq 0, B \neq 0 \).

(a) Assume \( A' = 0, B' \neq 0 \), then \( S_{c_0, c_1} = 0 \) and hence
\[
T^{[4(a)]}_b = 0.
\]

(b) Assume \( A' \neq 0, B' \neq 0 \). In this case, recall that \( (\gamma')^{2k+1} = A' \).

(i) If \( \text{Tr}_e(BY^{-1}) \neq 1 \), then \( S_{a,\beta} = 0 \) and hence
\[
T^{[4(b)(i)]}_b = 0.
\]

(ii) If \( \text{Tr}_e(BY^{-1}) = 1 \) and \( \text{Tr}_e(B'(\gamma')^{-1}) \neq 1 \), then \( S_{c_0, c_1} = 0 \) and hence
\[
T^{[4(b)(ii)]}_b = 0.
\]

(iii) If \( \text{Tr}_e(BY^{-1}) = 1 \) and \( \text{Tr}_e(B'(\gamma')^{-1}) = 1 \), then
\[
T^{[4(b)(iii)]}_b = 2^{n+e} \chi_1((1 + c^{-1})\beta).
\]

We now use the above discussion in the following theorem.

**Theorem 5.1** Let \( F(x) = x^{2k+1}, 1 \leq k < n \) be a function on \( \mathbb{F}_2^k \), \( n \geq 2 \). Let \( c \in \mathbb{F}_2^k \setminus \{0, 1\} \) and \( n/e \) be odd, where \( e = \gcd(k, n) \). Then the \( c\)-BCT entry \( c \mathcal{B}_F(1, b) \) of \( F \) at \( (1, b) \) is given by

\[
1 + \frac{e}{2^n} \sum_{(a, \beta) \in F \setminus F'} \chi_1(ba + (1 + c^{-1} + b)\beta),
\]

where

\[
F = \{(a, \beta) \in \mathbb{F}_2^2 \ | \ A, B \neq 0 \text{ and } \text{Tr}_e(BY^{-1}) = 1 \},
\]

\[
F' = \{(a, \beta) \in \mathbb{F}_2^2 \ | \ A', B' \neq 0 \text{ and } \text{Tr}_e(B'(\gamma')^{-1}) = 1 \},
\]

and \( A = \alpha + \beta, \quad B = \beta^{2k+n} + \beta, \quad A' = c\alpha + c^{-1} \beta \) and \( B' = (c^{-1}\beta)^{2k+n} + c^{-1} \beta, \quad \gamma = A^\frac{1}{2^{n+1}}, \quad \gamma' = A'^\frac{1}{2^{n+1}}. \)
**Proof** Let

\[ A' = \{ (\alpha, \beta) \in \mathbb{F}_q^2 \mid \alpha = c^{-1} \beta \text{ and } c^{-1} \beta \in \mathbb{F}_{2^n} \}, \]

\[ B' = \{ (\alpha, \beta) \in \mathbb{F}_q^2 \mid \alpha = c^{-1} \beta \text{ and } c^{-1} \beta \in \mathbb{F}_q \setminus \mathbb{F}_{2^n} \}, \]

\[ C' = \{ (\alpha, \beta) \in \mathbb{F}_q^2 \mid \alpha \neq c^{-1} \beta \text{ and } c^{-1} \beta \in \mathbb{F}_{2^n} \}, \]

\[ D' = \{ (\alpha, \beta) \in \mathbb{F}_q^2 \mid \alpha \neq c^{-1} \beta \text{ and } c^{-1} \beta \in \mathbb{F}_q \setminus \mathbb{F}_{2^n} \}, \]

\[ E' = \{ (\alpha, \beta) \in D' \mid \text{Tr}_c(B'(\gamma')^{-1}) \neq 1 \}. \]

Then,

\[
\begin{align*}
_e B_F(1, b) &= \frac{1}{q^2} \left( \sum_{(\alpha, \beta) \in A \cap A'} \chi_1(b(\alpha + \beta)) T_b^{[1]} + \sum_{(\alpha, \beta) \in A \cap C'} \chi_1(b(\alpha + \beta)) T_b^{[3]} \right) \\
&\quad + \sum_{(\alpha, \beta) \in B} \chi_1(b(\alpha + \beta)) T_b^{[2]} + \sum_{(\alpha, \beta) \in C} \chi_1(b(\alpha + \beta)) T_b^{[3]} \\
&\quad + \sum_{(\alpha, \beta) \in D \cap B'} \chi_1(b(\alpha + \beta)) T_b^{[4]} + \sum_{(\alpha, \beta) \in C} \chi_1(b(\alpha + \beta)) T_b^{[4]} \\
&= \sum_{(\alpha, \beta) \in A \cap A'} \chi_1(b\alpha + (1 + c^{-1} + b)\beta) \\
&\quad + \frac{2^e}{2^n} \sum_{(\alpha, \beta) \in F \cap F'} \chi_1(b\alpha + (1 + c^{-1} + b)\beta) \\
&= 1 + \frac{2^e}{2^n} \sum_{(\alpha, \beta) \in F \cap F'} \chi_1(b\alpha + (1 + c^{-1} + b)\beta). 
\end{align*}
\]

This completes the proof.

**Corollary 5.2** Let \( F(x) = x^{2^k + 1}, 1 \leq k < n, \) be a function on \( \mathbb{F}_q \), \( n \geq 2 \). Let \( c \in \mathbb{F}_{2^n} \setminus \{0, 1\} \) and \( n/e \) be odd, where \( e = \gcd(k, n) \). With the notations of the previous theorem, the \( c \)-boomerang uniformity of \( F \) satisfies

\[
\beta_{F,c} \leq 1 + 2^{e-n}|F \cap F'|. 
\]

**Case 2:** \( n/e \) is even.

(1) Let \( A = 0, B = 0 \).

(a) If \( A' = 0, B' = 0 \), then

\[
T_b^{[1]} = \chi_1((1 + c^{-1})\beta) q^2. 
\]

(b) If \( A' \neq 0, B' = 0 \), let

\[
G' = \{ (\alpha, \beta) \in C' \mid A' \neq g^{(2^n + 1)} \text{ for any integer } t \}, 
\]
\[ H' = \{ (\alpha, \beta) \in C' \mid A' = g^{n(2^t + 1)} \text{ for some integer } t \}. \]

Then,
\[
T_{b}^{[1(b)]} = \begin{cases} 
(1) \frac{n}{2} 2^{n+m} \chi_1((1 + c^{-1})\beta) & \text{if } (\alpha, \beta) \in A \cap G', \\
(1) \frac{n+1}{2} 2^{n+m+e} \chi_1((1 + c^{-1})\beta) & \text{if } (\alpha, \beta) \in A \cap H'. 
\end{cases}
\]

(2) Let \( A = 0, B \neq 0 \). In this case \( S_{a,b} = 0 \) and hence \( T_{b}^{[2]} = 0 \).

(3) Let \( A \neq 0, B = 0 \).

(a) If \( A' = 0, B' = 0 \), then \( T_{b}^{[3(a)]} \) is given by
\[
\begin{cases} 
(1) \frac{n}{2} 2^{m+n} \chi_1((1 + c^{-1})\beta) & \text{if } (\alpha, \beta) \in A' \cap G, \\
(1) \frac{n+1}{2} 2^{m+n+e} \chi_1((1 + c^{-1})\beta) & \text{if } (\alpha, \beta) \in A' \cap H. 
\end{cases}
\]

(b) If \( A' \neq 0, B' = 0 \), then
\[
T_{b}^{[3(b)]} = \begin{cases} 
2^{n} \chi_1((1 + c^{-1})\beta) & \text{if } (\alpha, \beta) \in G \cap G', \\
-2^{n+e} \chi_1((1 + c^{-1})\beta) & \text{if } (\alpha, \beta) \in G \cap H', \\
-2^{n+e} \chi_1((1 + c^{-1})\beta) & \text{if } (\alpha, \beta) \in H \cap G', \\
2^{n+2e} \chi_1((1 + c^{-1})\beta) & \text{if } (\alpha, \beta) \in H \cap H'. 
\end{cases}
\]

(4) Let \( A \neq 0, B \neq 0 \).

(a) If \( A' = 0, B' \neq 0 \), then \( S_{c,a,c^{-1}b} = 0 \) and hence
\[
T_{b}^{[4(a)]} = 0.
\]

(b) If \( A' \neq 0, B' \neq 0 \), let
\[
I' = \{ (\alpha, \beta) \in D' \mid A' \neq g^{n(2^t + 1)} \text{ for any integer } t \},
\]
\[
J' = \{ (\alpha, \beta) \in D' \mid A' = g^{n(2^t + 1)} \text{ for some integer } t, \}
\]
\[
L_{A}^{'}(x) = (B')^{2^t} \text{ is not solvable },
\]
\[
K' = \{ (\alpha, \beta) \in D' \mid A' = g^{n(2^t + 1)} \text{ for some integer } t, \}
\]
\[
\text{Tr}_c(A') \neq 0, L_{A}^{'}(x) = (B')^{2^t} \text{ is solvable},
\]
\[
L' = \{ (\alpha, \beta) \in D' \mid A' = g^{n(2^t + 1)} \text{ for some integer } t, \}
\]
\[
\text{Tr}_c(A') = 0, L_{A}^{'}(x) = (B')^{2^t} \text{ is solvable}.\]
Let Theorem 5.3

We now summarize the above discussion in the following theorem.

Theorem 5.3 Let $F(x) = x^{2^k + 1}$, $1 \leq k < n$ be a function on $\mathbb{F}_{2^n}$, $n \geq 2$. Let $c \in \mathbb{F}_{2^n} \setminus \{0, 1\}$ and $n$ be even, where $e = \gcd(k, n)$. With the previous notations, the $c$-BCT entry $cB_F(1, b)$ of $F$ at $(1, b)$ is given by

$$
\frac{1}{q^2} \left( \sum_{(a, \beta) \in A \cap A'} \chi_1(b(a + \beta))T_b^{[1(a \beta)]} + \sum_{(a, \beta) \in A \cap G'} \chi_1(b(a + \beta))T_b^{[1(b \beta)]} \\
+ \sum_{(a, \beta) \in A' \cap H} \chi_1(b(a + \beta))T_b^{[1(b \beta)]} + \sum_{(a, \beta) \in A' \cap G} \chi_1(b(a + \beta))T_b^{[3(a \beta)]} \\
+ \sum_{(a, \beta) \in A' \cap G} \chi_1(b(a + \beta))T_b^{[3(b \beta)]} + \sum_{(a, \beta) \in G \cap G'} \chi_1(b(a + \beta))T_b^{[3(b \beta)]} \\
+ \sum_{(a, \beta) \in G \cap H'} \chi_1(b(a + \beta))T_b^{[3(b \beta)]} + \sum_{(a, \beta) \in H \cap G'} \chi_1(b(a + \beta))T_b^{[3(b \beta)]} \\
+ \sum_{(a, \beta) \in (I \cup K) \cap L'} \chi_1(b(a + \beta))T_b^{[4(b \beta)]} + \sum_{(a, \beta) \in (I' \cup K') \cap L'} \chi_1(b(a + \beta))T_b^{[4(b \beta)]} \\
+ \sum_{(a, \beta) \in (I \cup K) \cap L'} \chi_1(b(a + \beta))T_b^{[4(b \beta)]} + \sum_{(a, \beta) \in L \cap (I' \cup K')} \chi_1(b(a + \beta))T_b^{[4(b \beta)]} \\
+ \sum_{(a, \beta) \in L \cap (I' \cup K')} \chi_1(b(a + \beta))T_b^{[4(b \beta)]} \right).
$$

6 The general case

Since the case $c \in \mathbb{F}_{2^n}$ has already been considered in previous sections, throughout this section we assume that $c \in \mathbb{F}_{2^n} \setminus \mathbb{F}_{2^n}$. Recall that for any fixed $b \neq 0$, the $c$-BCT entry is given by

$$
cB_F(1, b) = \frac{1}{q^2} \sum_{a, \beta \in \mathbb{F}_q} \chi_1(b(a + \beta))S_{a, \beta}S_{ca, c^{-1} \beta}.
$$
Let us denote \( T_b = S_{a,\beta} S_{ca,c^{-1}\beta} \). Recall that \( A = \alpha + \beta \), \( B = \beta^{2^n} + \beta \), \( A' = \alpha + c^{-1} \beta \) and \( B' = (c^{-1} \beta)^{2^n} + c^{-1} \beta \). Notice that, when \( c \in \mathbb{F}_{2^n} \), then \( \beta \in \mathbb{F}_{2^e} \), and so, \( \beta c^{-1} \in \mathbb{F}_{2^n} \setminus \mathbb{F}_{2^e} \), otherwise \( c \in \mathbb{F}_{2^n} \). Thus \( B = 0 = B' \) if and only if \( \beta = 0 \). Also, observe that the conditions \( A = 0 = A' \) if and only if \( \alpha = 0 = \beta \). Now we shall consider two cases namely, \( e \) is odd and \( e \) is even, respectively.

**Case 1:** \( e \) is odd.

1. Let \( A = 0, B = 0 \). Notice that the cases \( A' = 0, B' \neq 0 \), and \( A' \neq 0, B' = 0 \) would not arise, therefore, we shall calculate \( T_b \) in remaining two cases only.

   a. If \( A' = 0, B' = 0 \), then
   \[
   T_b^{[1(a)]} = \chi_1((1 + c^{-1})\beta) q^2.
   \]

   b. If \( A' \neq 0, B' \neq 0 \), then
   \[
   T_b^{[1(b)]} = \begin{cases} 
   0 & \text{if } \operatorname{Tr}_e(B' (\gamma')^{-1}) \neq 1, \\
   \left(\frac{2}{n/e}\right)^e 2^{\frac{n+e}{2}} \chi_1((1 + c^{-1})\beta) & \text{if } \operatorname{Tr}_e(B' (\gamma')^{-1}) = 1.
   \end{cases}
   \]

2. Let \( A = 0, B \neq 0 \). In this case \( S_{a,\beta} = 0 \) and hence
   \[
   T_b^{[2]} = 0.
   \]

3. Let \( A \neq 0, B = 0 \). Again, \( S_{a,\beta} = 0 \) and hence
   \[
   T_b^{[3]} = 0.
   \]

4. Let \( A \neq 0, B \neq 0 \).

   a. If \( A' = 0, B' = 0 \), then
   \[
   T_b^{[4(a)]} = \begin{cases} 
   0 & \text{if } \operatorname{Tr}_e(B (\gamma^{-1}) \neq 1, \\
   \left(\frac{2}{n/e}\right)^e 2^{\frac{n+e}{2}} \chi_1((1 + c^{-1})\beta) & \text{if } \operatorname{Tr}_e(B (\gamma^{-1}) = 1.
   \end{cases}
   \]

   b. If \( A' = 0, B' \neq 0 \), then \( S_{ca,c^{-1}\beta} = 0 \) and hence
   \[
   T_b^{[4(b)]} = 0.
   \]

   c. If \( A' \neq 0, B' = 0 \), then again \( S_{ca,c^{-1}\beta} = 0 \) and hence
   \[
   T_b^{[4(c)]} = 0.
   \]

   d. If \( A' \neq 0, B' \neq 0 \), then the only relevant case is and
   \[
   T_b^{[4(d)]} = \begin{cases} 
   2^{n+e} \chi_1((1 + c^{-1})\beta) & \text{if } (\alpha, \beta) \in F \cap F', \\
   0 & \text{otherwise}.
   \end{cases}
   \]

We now summarize the above discussion in the following theorem.
Theorem 6.1  Let $F(x) = x^{2^k+1}$, $1 \leq k < n$ be a function on $\mathbb{F}_q$, $n \geq 2$. Let $c \in \mathbb{F}_q \setminus \mathbb{F}_2$, and $n/e$ be odd, where $e = \gcd(k, n)$. Then the $c$-BCT entry $B_F(1, b)$ of $F$ at $(1, b)$ is given by

$$1 + \frac{2^e}{2^n} \sum_{(a, \beta) \in \mathbb{F}_q \setminus \mathbb{F}_2} \chi_1(b \alpha + (1 + c^{-1} + b) \beta)$$

$$+ \frac{2^e}{2^n} \sum_{(a, \beta) \in \mathbb{F}_q \setminus \mathbb{F}_2} \chi_1(b \alpha + (1 + c^{-1} + b) \beta).$$

Proof

$$cB_F(1, b) = \frac{1}{q^2} \left( \sum_{(a, \beta) \in \mathbb{F}_q \setminus \mathbb{F}_2} \chi_1(b \alpha + \beta)T_b^{1(\alpha)} + \sum_{(a, \beta) \in \mathbb{F}_q \setminus \mathbb{F}_2} \chi_1(b \alpha + \beta)T_b^{1(\beta)} \right)$$

$$+ \sum_{(a, \beta) \in \mathbb{F}_q \setminus \mathbb{F}_2} \chi_1(b \alpha + \beta)T_b^{4(\alpha)} + \sum_{(a, \beta) \in \mathbb{F}_q \setminus \mathbb{F}_2} \chi_1(b \alpha + \beta)T_b^{4(\beta)}$$

$$= 1 + \left( \frac{2}{n/e} \right)^e \cdot 2^{\frac{e}{2^n}} \sum_{(a, \beta) \in \mathbb{F}_q \setminus \mathbb{F}_2} \chi_1(b \alpha + (1 + c^{-1} + b) \beta)$$

$$+ 2^{e-n} \sum_{(a, \beta) \in \mathbb{F}_q \setminus \mathbb{F}_2} \chi_1(b \alpha + (1 + c^{-1} + b) \beta).$$

Corollary 6.2  Let $F(x) = x^{2^k+1}$, $1 \leq k < n$, be a function on $\mathbb{F}_q$, $n \geq 2$. Let $c \in \mathbb{F}_q \setminus \mathbb{F}_2$, and $n/e$ be odd, where $e = \gcd(k, n)$. With the notations of the previous theorem, the $c$-boomerang uniformity of $F$ satisfies

$$\beta_{F,c} \leq 1 + \left( \frac{2}{n/e} \right)^e \cdot 2^{\frac{e}{2^n}} |(A \cap F') \cup (A' \cap F)| + 2^{e-n}|F \cap F'|.$$  

Case 2: $n/e$ is even.

1. Let $A = 0, B = 0$. Notice that the cases $A' = 0, B' \neq 0$, and $A' \neq 0, B' = 0$ would not arise, therefore, we shall calculate $T_b$ in remaining two cases only.

   a. If $A' = 0, B' = 0$, then

   $$T_b^{1(\alpha)} = \chi_1((1 + c^{-1}) \beta) q^2.$$  

   b. If $A' \neq 0, B' \neq 0$, then

   $$T_b^{1(\beta)} = \begin{cases} (-1)^\frac{e}{2} 2^{m+n} M' & \text{if } (a, \beta) \in A \cap (I' \cup K'), \\ 0 & \text{if } (a, \beta) \in A \cap J', \\ (-1)^\frac{e-1}{2} 2^{m+n+c} M' & \text{if } (a, \beta) \in A \cap L', \end{cases}$$  

   where $M' = \chi_1((1 + c^{-1}) \beta) \chi_1(A' x_{2^k+1}).$

2. Let $A = 0, B \neq 0$. In this case $S_{a, b} = 0$ and hence

   $$T_b^{2(\beta)} = 0.$$  

3. Let $A \neq 0, B = 0$. Notice that the case $A' = 0, B' = 0$ would not arise. Now we shall calculate $T_b$ in the remaining cases.
(a) If $A' = 0, B' \neq 0$, then $S_{ca,c^{-1} \beta} = 0$ and hence
\[ T_b^{[3(a)]} = 0. \]

(b) If $A' \neq 0, B' = 0$, then
\[ T_b^{[3(b)]} = \begin{cases} 
2^n k_1((1 + c^{-1})\beta) & \text{if } (\alpha, \beta) \in G \cap G', \\
-2^{n+e} k_1((1 + c^{-1})\beta) & \text{if } (\alpha, \beta) \in G \cap H', \\
-2^{n+e} k_1((1 + c^{-1})\beta) & \text{if } (\alpha, \beta) \in H \cap G', \\
2^{n+2e} k_1((1 + c^{-1})\beta) & \text{if } (\alpha, \beta) \in H \cap H'.
\end{cases} \]

(c) If $A' \neq 0, B' \neq 0$, then
\[ T_b^{[3(c)]} = \begin{cases} 
2^n M' & \text{if } (\alpha, \beta) \in G \cap (J' \cup K'), \\
0 & \text{if } (\alpha, \beta) \in (G \cup H) \cap J', \\
-2^{n+e} M' & \text{if } (\alpha, \beta) \in G \cap L', \\
-2^{n+e} M' & \text{if } (\alpha, \beta) \in H \cap (J' \cup K'), \\
2^{n+2e} M' & \text{if } (\alpha, \beta) \in H \cap L'.
\end{cases} \]

(4) Let $A \neq 0, B \neq 0$.

(a) If $A' = 0, B' = 0$, then
\[ T_b^{[4(a)]} = \begin{cases} 
(-1)^{\frac{m}{2}} 2^{m+n} M'' & \text{if } (\alpha, \beta) \in A' \cap (I \cup K), \\
0 & \text{if } (\alpha, \beta) \in A' \cap J, \\
(-1)^{\frac{n}{2} + 1} 2^{n+e} M'' & \text{if } (\alpha, \beta) \in A' \cap L,
\end{cases} \]

where $M'' = k_1((1 + c^{-1})\beta) k_1(A x_A^{e+1})$.

(b) If $A' = 0, B' \neq 0$, then $S_{ca,c^{-1} \beta} = 0$ and hence
\[ T_b^{[4(b)]} = 0. \]

(c) If $A' \neq 0, B' = 0$, then
\[ T_b^{[4(c)]} = \begin{cases} 
2^n M'' & \text{if } (\alpha, \beta) \in G' \cap (I \cup K), \\
0 & \text{if } (\alpha, \beta) \in (G' \cup H') \cap J, \\
-2^{n+e} M'' & \text{if } (\alpha, \beta) \in G' \cap L, \\
-2^{n+e} M'' & \text{if } (\alpha, \beta) \in H' \cap (I \cup K), \\
2^{n+2e} M'' & \text{if } (\alpha, \beta) \in H' \cap L.
\end{cases} \]

(d) If $A' \neq 0, B' \neq 0$, then
\[ T_b^{[4(d)]} = \begin{cases} 
2^n M''' & \text{if } (\alpha, \beta) \in (I \cup K) \cap (I' \cup K'), \\
0 & \text{if } (\alpha, \beta) \in (I \cup K \cup L) \cap J', \\
-2^{n+e} M''' & \text{if } (\alpha, \beta) \in (I \cup K) \cap L', \\
0 & \text{if } (\alpha, \beta) \in J \cap (I' \cup J' \cup K' \cup L'), \\
-2^{n+e} M''' & \text{if } (\alpha, \beta) \in L \cap (I' \cup K'), \\
2^{n+2e} M''' & \text{if } (\alpha, \beta) \in L \cap L',
\end{cases} \]
where $M''' = \chi_1((1 + c^{-1})\beta)\chi_1(Ax_A^{2^k+1} + A'x_A^{2^k+1})$.

We now summarize the above discussion in the form of following theorem.

**Theorem 6.3** Let $F(x) = x^{2^k+1}$, $1 \leq k < n$ be a function on $\mathbb{F}_{2^n}$, $n \geq 2$. Let $c \in \mathbb{F}_{2^n} \setminus \mathbb{F}_2$ and $n/2e$ be even, where $e = \gcd(k, n)$. With the prior notations, the $c$-BCT entry $c_bF(1, b)$ of $F$ at $(1, b)$ is given by

\[
\frac{1}{q^2} \left( \sum_{(a, \beta) \in A \cap L'} \chi_1(b(\alpha + \beta)) T_b^{1(a)} + \sum_{(a, \beta) \in A \cap (P \cup K')} \chi_1(b(\alpha + \beta)) T_b^{1(b)} \right) \\
+ \sum_{(a, \beta) \in A \cap L'} \chi_1(b(\alpha + \beta)) T_b^{1(b)} + \sum_{(a, \beta) \in G \cap G'} \chi_1(b(\alpha + \beta)) T_b^{3(b)} \\
+ \sum_{(a, \beta) \in H \cap H'} \chi_1(b(\alpha + \beta)) T_b^{3(b)} + \sum_{(a, \beta) \in G \cap (P \cup K')} \chi_1(b(\alpha + \beta)) T_b^{3(c)} \\
+ \sum_{(a, \beta) \in H \cap H'} \chi_1(b(\alpha + \beta)) T_b^{3(c)} + \sum_{(a, \beta) \in G \cap (P \cup K')} \chi_1(b(\alpha + \beta)) T_b^{3(c)} \\
+ \sum_{(a, \beta) \in H \cap L'} \chi_1(b(\alpha + \beta)) T_b^{4(a)} + \sum_{(a, \beta) \in G \cap (P \cup K')} \chi_1(b(\alpha + \beta)) T_b^{4(a)} \\
+ \sum_{(a, \beta) \in A' \cap L} \chi_1(b(\alpha + \beta)) T_b^{4(c)} + \sum_{(a, \beta) \in G \cap (P \cup K')} \chi_1(b(\alpha + \beta)) T_b^{4(c)} \\
+ \sum_{(a, \beta) \in G \cap L} \chi_1(b(\alpha + \beta)) T_b^{4(c)} + \sum_{(a, \beta) \in H \cap (P \cup K')} \chi_1(b(\alpha + \beta)) T_b^{4(c)} \\
+ \sum_{(a, \beta) \in H \cap L} \chi_1(b(\alpha + \beta)) T_b^{4(c)} + \sum_{(a, \beta) \in (P \cup K') \cap L} \chi_1(b(\alpha + \beta)) T_b^{4(c)} \\
+ \sum_{(a, \beta) \in (P \cup K') \cap L} \chi_1(b(\alpha + \beta)) T_b^{4(d)} \\
+ \sum_{(a, \beta) \in G \cap L} \chi_1(b(\alpha + \beta)) T_b^{4(d)} + \sum_{(a, \beta) \in (P \cup K') \cap L} \chi_1(b(\alpha + \beta)) T_b^{4(d)} \\
+ \sum_{(a, \beta) \in (P \cup K') \cap L} \chi_1(b(\alpha + \beta)) T_b^{4(d)} \\
+ \sum_{(a, \beta) \in L \cap L} \chi_1(b(\alpha + \beta)) T_b^{4(d)} \right) .
\]

**Remark 4.3** It is natural to wonder if expressing the $c$-BCT entries via Weil sums will have an effect on computation. We implemented such a computation for small dimensions and achieved a speed up of at least tenfold over the brute force or even the Walsh-Hadamard characterization of the $c$-BCT [16].

## 7 Discussion on equivalence

Boura and Canteaut [2] showed that the BCT table is preserved under the affine equivalence but not under the extended affine equivalence (and consequently under the CCZ-equivalence). It is quite natural to ask a similar question in the context of $c$-BCT. It is straightforward to see that in the case of even characteristic, $c$-BCT and $c^{-1}$-BCT entries of an $(n, n)$-function $F : \mathbb{F}_{2^n} \to \mathbb{F}_{2^n}$ are the same under the transformations $x \mapsto x + a$ and $y \mapsto y + a$, since the $c$-boomerang system
We consider the binomial
\[ G(x) = x^2 + ux^{2^{e-1}} + 1 \in \mathbb{F}_{2^n}[x], \]
which is a PP if and only if \( \frac{n}{2} \) is odd and \( u \neq g^{(2^e-1)} \), where \( e = \gcd(n, k) = \gcd(n - k, k) \) and \( g \) is the primitive element of \( \mathbb{F}_{2^n} \). Notice that \( G(x) = (L \circ F)(x) \) where \( L(x) = x^2 + ux \) and \( F(x) = x^{2^{e-1}} + 1 \).

Table 1 $c$-BCT entries of $x^{17}$ and $x^3 + gx^{17}$

| $c$   | Set of $c$-BCT entries of $x^{17}$ | Set of $c$-BCT entries of $G(x)$ |
|-------|----------------------------------|----------------------------------|
| $g$   | $\{0, 1, 2, 3, 4\}$             | $\{0, 1, 2, 3, 4, 5\}$          |
| $g^2$ | $\{0, 1, 2, 3, 4\}$             | $\{0, 1, 2, 3, 4, 5\}$          |
| $g^3$ | $\{0, 1, 2, 3, 5\}$             | $\{0, 1, 2, 3, 4, 5\}$          |
| $g^4$ | $\{0, 1, 2, 3, 4\}$             | $\{0, 1, 2, 3, 4, 5\}$          |
| $g^5$ | $\{0, 1, 2, 3, 4\}$             | $\{0, 1, 2, 3, 4, 5, 6\}$       |
| $g^6$ | $\{0, 1, 2, 3, 5\}$             | $\{0, 1, 2, 3, 4, 5, 6\}$       |
| $g^7$ | $\{0, 1, 2, 3, 4\}$             | $\{0, 1, 2, 3, 4, 5, 6\}$       |
| $g^8$ | $\{0, 1, 2, 3, 4\}$             | $\{0, 1, 2, 3, 4, 5\}$          |
| $g^9$ | $\{0, 1, 2, 3, 4\}$             | $\{0, 1, 2, 3, 4, 5, 6\}$       |
| $g^{10}$ | $\{0, 1, 2, 3, 4\}$ | $\{0, 1, 2, 3, 4, 5\}$ |
| $g^{11}$ | $\{0, 1, 2, 3\}$ | $\{0, 1, 2, 3, 4, 5\}$ |
| $g^{12}$ | $\{0, 1, 2, 3, 5\}$ | $\{0, 1, 2, 3, 4, 5\}$ |
| $g^{13}$ | $\{0, 1, 2, 3\}$ | $\{0, 1, 2, 3, 4, 5\}$ |
| $g^{14}$ | $\{0, 1, 2, 3, 4\}$ | $\{0, 1, 2, 3, 4, 5, 6\}$ |
| $g^{15}$ | $\{0, 1, 2, 3\}$ | $\{0, 1, 2, 3, 4, 5\}$ |
| $g^{16}$ | $\{0, 1, 2, 3\}$ | $\{0, 1, 2, 3, 4, 5\}$ |
| $g^{17}$ | $\{0, 1, 2, 3, 4\}$ | $\{0, 1, 2, 3, 4, 5, 6\}$ |
| $g^{18}$ | $\{0, 1, 2, 3\}$ | $\{0, 1, 2, 3, 4, 5\}$ |
| $g^{19}$ | $\{0, 1, 2, 3\}$ | $\{0, 1, 2, 3, 4, 5\}$ |
| $g^{20}$ | $\{0, 1, 2, 3, 4\}$ | $\{0, 1, 2, 3, 4, 5, 6\}$ |
| $g^{21}$ | $\{0, 1, 2, 3\}$ | $\{0, 1, 2, 3, 4, 5\}$ |
| $g^{22}$ | $\{0, 1, 2, 3\}$ | $\{0, 1, 2, 3, 4, 5\}$ |
| $g^{23}$ | $\{0, 1, 2, 3, 4\}$ | $\{0, 1, 2, 3, 4, 5, 6\}$ |
| $g^{24}$ | $\{0, 1, 2, 3\}$ | $\{0, 1, 2, 3, 4, 5\}$ |
| $g^{25}$ | $\{0, 1, 2, 3\}$ | $\{0, 1, 2, 3, 4, 5\}$ |
| $g^{26}$ | $\{0, 1, 2, 3\}$ | $\{0, 1, 2, 3, 4, 5\}$ |
| $g^{27}$ | $\{0, 1, 2, 3, 4\}$ | $\{0, 1, 2, 3, 4, 5, 6\}$ |
| $g^{28}$ | $\{0, 1, 2, 3, 4\}$ | $\{0, 1, 2, 3, 4, 5, 6\}$ |
| $g^{29}$ | $\{0, 1, 2, 3\}$ | $\{0, 1, 2, 3, 4, 5\}$ |
| $g^{30}$ | $\{0, 1, 2, 3, 4\}$ | $\{0, 1, 2, 3, 4, 5, 6\}$ |
| $g^{31}$ | $\{0, 1, 2, 3\}$ | $\{0, 1, 2, 3, 4, 5\}$ |
| $g^{32}$ | $\{0, 1, 2, 3\}$ | $\{0, 1, 2, 3, 4, 5\}$ |

\[
\begin{align*}
\{ F(x) + cF(y) = b \\
F(x + a) + c^{-1}F(y + a) = b
\end{align*}
\]

becomes

\[
\begin{align*}
\{ F(x) + c^{-1}F(y) = b \\
F(x + a) + cF(y + a) = b
\end{align*}
\]

We consider the binomial \( G(x) = x^{2^{e+1}} + ux^{2^{e-1}+1} \in \mathbb{F}_{2^n}[x] \), which is a PP if and only if \( \frac{n}{2} \) is odd and \( u \neq g^{(2^e-1)} \), where \( e = \gcd(n, k) = \gcd(n - k, k) \) and \( g \) is the primitive element of \( \mathbb{F}_{2^n} \). Notice that \( G(x) = (L \circ F)(x) \) where \( L(x) = x^{2^e} + ux \) and \( F(x) = x^{2^{e-1}+1} \). When
n = 6, k = 2 and u = g, where g is a root of the primitive polynomial y^6 + y^4 + y^3 + y + 1 over F_2, then L(x) and G(x) are PP. It is easy to see from the Table 1 in the Appendix 1 that the c-BCT is not preserved under the (output applied) affine equivalence. However, if the affine transformation is applied to the input, that is, G(x) = (F o L)(x), then the c-BCT spectrum is preserved, as was the case for the c-differential uniformity.

Appendix

Let \( G(x) = x^5 + gx^{17} = (x^4 + gx)x^{17} \in F_{25}[x], \) where g is a root of the primitive polynomial \( y^6 + y^4 + y^3 + y + 1 \) over \( \mathbb{F}_2. \) The following Table 1 gives the set of the c-BCT entries for \( x^{17} \) as well as \( G(x) = x^5 + gx^{17} \) for all \( c \in F_2 \setminus \mathbb{F}_2. \) In view of the discussion in Section 7, it is sufficient to compute the set of c-BCT entries for either of \( c \) or \( c^{-1} \) as they are going to be exactly the same.

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References

1. Bartoli, D., Calderini, M.: On construction and (non)existence of c-(almost) perfect nonlinear functions. Finite Fields Appl. 72, 101835 (2021)
2. Boura, C., Canteaut, A.: On the boomerang uniformity of cryptographic Sboxes. IACR Trans. Symmetric Cryptol. 2018(3), 290–310 (2018)
3. Carlet, C.: Boolean Functions for Cryptography and Coding Theory. Cambridge University Press, Cambridge, Cambridge (2021)
4. Cid, C., Huang, T., Peyrin, T., Sasaki, Y., Song, L.: Boomerang connectivity table: a new cryptanalysis tool. In: Nielsen, J., Rijmen, V. (eds.) Advances in Cryptology-EUROCRYPT 2018, LNCS 10821, pp. 683–714. Springer, Cham (2018)
5. Coulter, R.S.: On the evaluation of a class of Weil sums in characteristic 2. New Zealand J. Math. 28, 171–184 (1999)
6. Cusick, T.W., Stănică, P.: Cryptographic boolean functions and applications (Ed. 2), Academic Press, San Diego, CA, (2017)
7. S. Eddahmani, S.: Mesnager, explicit values of the tables DDT, BCT, FBCT, and FBDT of the inverse, the Gold, and the Bracken-Leander functions, Boolean Functions and their Applications (BFA 2021) (2021)
8. Ellingsen, P., Felke, P., Riera, C., Stănică, P., Tkachenko, A.: C-differentials, multiplicative uniformity and (almost) perfect c-linearity. IEEE Trans. Inform. Theory 66(9), 5781–5789 (2020)
9. Hasan, S.U., Pal, M., Riera, C., Stănică, P.: On the c-differential uniformity of certain maps over finite fields. Des. Codes Cryptogr 89(2), 221–239 (2021)
10. Hasan, S.U., Pal, M., Stănică, P.: Boomerang uniformity of a class of power maps. Des. Codes Cryptogr. 89, 2627–2636 (2021)
11. Hasan, S.U., Pal, M., Stănică, P.: The c-differential uniformity and boomerang uniformity of two classes of permutation polynomials. IEEE Trans. Inform. Theory 68(1), 679–691 (2022)
12. Lidl, R., Niederreiter, H.: FiniteFields (Ed. 2), Encycl. Math. Appl., vol.20, Cambridge Univ. Press, Cambridge (1997)
13. Li, K., Qu, L., Sun, B., Li, C.: New results about the boomerang uniformity of permutation polynomials. IEEE Trans. Inform. Theory 65(11), 7542–7553 (2019)
14. Mesnager, S., Riera, C., Stănică, P., Yan, H., Zhou, Z.: Investigations on c-(almost) perfect nonlinear functions. IEEE Trans. Inform. Theory 67(10), 6916–6925 (2021)
15. Nyberg, K.: Differentially uniform mappings for cryptography. In: Helleseth, T. (ed.) Advances in Cryptology-EUROCRYPT 1993, LNCS 765, pp. 55–64. Springer, Berlin, Heidelberg (1994)
16. Stănică, P.: Investigations on \(c\)-boomerang uniformity and perfect nonlinearity. Discrete Appl. Math. 304, 297–314 (2021)
17. Stănică, P.: Low \(c\)-boomerang uniformity of the swapped inverse function. Discrete Math. 344(10), 112543 (2021)
18. Stănică, P.: Using double Weil sums in finding the \(c\)-boomerang connectivity table for monomial functions on finite fields. Appl. Algebra Eng. Commun. Comput (2021). https://doi.org/10.1007/s00200-021-00520-9
19. Stănică, P., Geary, A.: The \(c\)-differential behaviour of the inverse function under the \(c\)-equivalence. Cryptogr. Commun. 13, 295–306 (2021)
20. Stănică, P., Riera, C., Tkachenko, A.: Characters, Weil sums and \(c\)-differential uniformity with an application to the perturbed Gold function, Cryptogr. Commun. 6, 891–907 (2021)
21. Wagner, D.: The boomerang attack, In: Knudsen, L.R. (ed.) Fast Software Encryption-FSE 1999. LNCS 1636, Springer, Berlin, Heidelberg, pp. 156–170 (1999)
22. Zha, Z., Hu, L.: Some classes of power functions with low \(c\)-differential uniformity over finite fields. Des. Codes Cryptogr. 89, 1193–1210 (2021)