Technical Report:

Achievable Rates for the MAC with Correlated Channel-State Information

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Abstract

In this paper we provide an achievable rate region for the discrete memoryless multiple access channel with correlated state information known non-causally at the encoders using a random binning technique. This result is a generalization of the random binning technique used by Gel’fand and Pinsker for the problem with non-causal channel state information at the encoder in point to point communication.

Index Terms

Multi-user information theory, random binning, multiple access channel, dirty paper coding.

I. THE PROBLEM SETUP

We consider a discrete memoryless multiple access channel (MAC) with two correlated states each known by one of the encoders. Specifically, we assume the following model:

\[ P(y|x_1, x_2, s_1, s_2) \text{ and } P(s_1, s_2), \]

where \( s_1 \in S_1 \) and \( s_2 \in S_2 \) are known non-causally at encoder 1 and encoder 2, respectively. The channel inputs are \( x_1 \in X_1 \) and \( x_2 \in X_2 \), and the channel output is \( y \in Y \). The memoryless channel implies that

\[ P(y|x_1, x_2, s_1, s_2) = \prod_{i=1}^{n} P(y_i|x_{1i}, x_{2i}, s_{1i}, s_{2i}). \]

The first user transmits the message \( m_1 \in \{1, \ldots, M_1\} \), and the second user transmits the message \( m_2 \in \{1, \ldots, M_2\} \), where \( m_1 \) and \( m_2 \) are independent random variables with uniform distributions, and \( M_1 = 2^{nR_1} \), \( M_2 = 2^{nR_2} \). The first encoder observes the channel state information \( S_1 \) non-causally and generates the transmitted codeword

\[ \phi_1 : \{1, \ldots, M_1\} \times S_1^n \to X_1^n. \]
In the same way, the second encoder generates the transmitted codeword

$$\phi_2 : \{1, \ldots, M_2\} \times S_2^n \rightarrow X_2^n. \quad (4)$$

The decoder uses the following mapping to reconstruct the transmitted messages

$$\psi : Y^n \rightarrow \{1, \ldots, M_1\} \times \{1, \ldots, M_2\}, \quad (5)$$
i.e., $$(\hat{m}_1, \hat{m}_2) = \psi(y).$$ The error probability is defined as

$$P_e^{(n)} \triangleq \Pr(\psi(y) \neq (m_1, m_2)). \quad (6)$$

II. MAIN RESULT

In the following theorem we provide an inner bound for the capacity region of (1) which is derived using a generalization of the random binning technique [1].

**Theorem 1.** An inner bound for the capacity region of (1) is given by

$$R \triangleq \text{cl conv} \left\{(R_1, R_2) : R_1 \leq I(U; Y|V) - I(U; S_1|V) \right.$$ 

$$R_2 \leq I(V; Y|U) - I(V; S_2|U) \right.$$ 

$$R_1 + R_2 \leq I(U, V; Y) - I(U, V; S_1, S_2) \right\}$$

where the admissible pairs satisfy:

$$P(U, V, X_1, X_2, S_1, S_2, Y) = P(S_1, S_2)P(U, X_1|S_1)P(V, X_2|S_2)P(Y|X_1, X_2, S_1, S_2). \quad (8)$$

The theorem implies that the following two Markov chains are satisfied:

$$(U, X_1) \leftrightarrow S_1 \leftrightarrow S_2 \leftrightarrow (V, X_2) \quad (9)$$

$$(U, V) \leftrightarrow (X_1, X_2, S_1, S_2) \leftrightarrow Y. \quad (10)$$

**Proof:** We denote the set of $\epsilon$-typical of two $n$-sequences $a$ and $b$ where $a_i \in A, b_i \in B$ for $i = 1, \ldots, n$ by $A_e^{(n)}(A,B)$ (we use the same notation as in [2]).

Fix the distributions $P(U, X_1|S_1)$ and $P(V, X_2|S_2)$. Calculate the marginal distributions $P(U)$ and $P(V)$.

- **Codebooks generation:** Let

$$J_1 = 2^{n[I(U; S_1) + 4\epsilon]} \quad (11)$$

$$J_2 = 2^{n[I(V; S_2) + 4\epsilon]}. \quad (12)$$

**Codebook 1:** Generate $2^{n(J_1+R_1)}$ of independent $u_k$ sequences of length $n$, generating each element i.i.d according to distribution $\prod_{i=1}^nP(u_i)$, and distribute these sequences randomly among $M_1$ bins where each
Then by union bound, the error probability is upper bounded by the second term. We define the following error events for specific state sequences.

- **Encoder of user 1**: Given the state sequence \( s_1 \) and the message \( m_1 \), search in bin \( m_1 \) of codebook 1 for a \( u \) sequence such that \( (u, s_1) \in A^n_\epsilon(U, S_1) \). Send \( x_1 \) which is jointly typical with \( u \) and \( s_1 \), i.e., \( (u, s_1, x_1) \in A^n_\epsilon(U, S_1, X_1) \).

- **Encoder of user 2**: Given the state sequence \( s_2 \) and the message \( m_2 \), search in bin \( m_2 \) of codebook 2 for a \( v \) sequence such that \( (v, s_2) \in A^n_\epsilon(V, S_2) \). Send \( x_2 \) which is jointly typical with \( v \) and \( s_2 \), i.e., \( (v, s_2, x_2) \in A^n_\epsilon(U, S_2, X_2) \).

- **Decoder**: Given the received vector \( y \), search for unique sequences \( u \) and \( v \) such that \( (u, v, y) \in A^n_\epsilon(U, V, Y) \).

**Analysis of the error probability**: The error probability is given by

\[
P_e^{(n)} = \sum_{s_1, s_2 \notin A^n_\epsilon(S_1, S_2)} P_{S_1', S_2'}(s_1, s_2) + \sum_{s_1, s_2 \in A^n_\epsilon(S_1, S_2)} P_{S_1', S_2'}(s_1, s_2) P(e|s_1, s_2)
\]

where the inequality follows the asymptotic equipartition property (AEP) [2]. Hence, we need to evaluate only the second term. We define the following error events for specific state sequences \( (s_1, s_2) \):

- \( E_1(s_1, m_1) = \{ \# j_1, 1 \leq j_1 \leq J_1 : (u_{m_1,j_1}, s_1) \in A^n_\epsilon(U, S_1) \} \).

- \( E_2(s_2, m_2) = \{ \# j_2, 1 \leq j_2 \leq J_2 : (v_{m_2,j_2}, s_2) \in A^n_\epsilon(V, S_2) \} \).

- \( E_3(s_1, s_2, m_1, m_2) = \{(u_{m_1,j_1}, s_1, v_{m_2,j_2}, s_2) : (u_{m_1,j_1}, s_1, v_{m_2,j_2}, s_2) \notin A^n_\epsilon(U, V, S_1, S_2) \} \).

- \( E_4(s_1, s_2, m_1, m_2) = \{(u_{m_1,j_1}, s_1, v_{m_2,j_2}, s_2, y) \notin A^n_\epsilon(U, V, Y) \} \).

- \( E_5(s_1, s_2, m_1, m_2) = \{ \exists u_{m_1,j_1} : m_1 \neq m'_1, (u_{m_1,j_1}, v_{m_2,j_2}, y) \in A^n_\epsilon(U, V, Y) \} \).

- \( E_6(s_1, s_2, m_1, m_2) = \{ \exists v_{m_2,j_2} : m_2 \neq m'_2, (u_{m_1,j_1}, v_{m_2,j_2}, y) \in A^n_\epsilon(U, V, Y) \} \).

- \( E_7(s_1, s_2, m_1, m_2) = \{ \exists u_{m_1,j_1}, v_{m_2,j_2} : m_1 \neq m'_1, m_2 \neq m'_2, (u_{m_1,j_1}, v_{m_2,j_2}, y) \in A^n_\epsilon(U, V, Y) \} \).

Then by union bound, the error probability is upper bounded by

\[
P_e^{(n)} \leq \epsilon + \frac{1}{M_1 M_2} \sum_{m_1, m_2} P_{S_1', S_2'}(s_1, s_2) \left[ \Pr(E_1(s_1, m_1)) + \Pr(E_2(s_2, m_2)) \right]
\]

\[
+ \Pr(E_3(s_1, s_2, m_1, m_2)|\overline{E_1(s_1, m_1)}, \overline{E_2(s_2, m_2)})
\]

\[
+ \Pr(E_4(s_1, s_2, m_1, m_2)|\overline{E_3(s_1, s_2, m_1, m_2)})
\]

\[
+ \Pr(E_5(s_1, s_2, m_1, m_2)|\overline{E_3(s_1, s_2, m_1, m_2)})
\]

\[
+ \Pr(E_6(s_1, s_2, m_1, m_2)|\overline{E_3(s_1, s_2, m_1, m_2)})
\]

\[
+ \Pr(E_7(s_1, s_2, m_1, m_2)|\overline{E_3(s_1, s_2, m_1, m_2)})
\]

(13)
We now evaluate the probability of each error events. For independent $u$ and $s_1$ the probability that $(u, s_1) \in A_\epsilon^n(U, S_1)$ is bounded below by

\[
\Pr((u, s_1) \in A_\epsilon^n(U, S_1)) = \sum_{(u, s_1) \in A_\epsilon^n(U, S_1)} P(u)P(s_1) \geq |A_\epsilon^n(U, S_1)|2^{-n[H(U)+\epsilon]}2^{-n[H(S_1)+\epsilon]}
\]

\[
\geq 2^{n[H(U,S_1)-\epsilon]}2^{-n[H(U)+\epsilon]}2^{-n[H(S_1)+\epsilon]}
\]

\[
= 2^{-n[H(U)+H(S_1)-H(U,S_1)+\epsilon]}
\]

\[
= 2^{-n[I(U;S_1)+3\epsilon]}
\]

Hence, we have that

\[
\Pr(E_1(s_1, m_1)) \leq \left[1 - 2^{-n[I(U;S_1)+3\epsilon]}\right]^{J_i} \leq \exp\left(-J_12^{-n[I(U;S_1)+3\epsilon]}\right) = \exp(-2^{n\epsilon}),
\]

where (15) follows since $1 - x \leq \exp(-x)$. Hence, this term decays to zero as $n \to \infty$. In the same way $\Pr(E_2(s_2, m_2))$ goes to zero as $n \to \infty$.

Provided that $E_1(s_1, m_1)$ and $E_2(s_2, m_2)$ have not occurred, i.e., $(u_{m_1,j_1}, s_1) \in A_\epsilon^n(U, S_1)$ and $(v_{m_2,j_2}, s_2) \in A_\epsilon^n(V, S_2)$, from Markov Lemma [2] we have that

\[
\Pr((u, v, s_1, s_2) \in A_\epsilon^n(U, V, S_1, S_2)|(u, s_1) \in A_\epsilon^n(U, S_1), (v, s_2) \in A_\epsilon^n(V, S_2)) \geq 1 - \epsilon
\]

where the typical set $A_\epsilon^n(U, V, S_1, S_2)$ is associated with the joint distribution

\[
P(U, V, S_1, S_2) = P(S_1, S_2)P(U|S_1)P(V|S_2).
\]

Hence, we have that

\[
\Pr(E_3(s_1, s_2, m_1, m_2)|E_1(s_1, m_1), E_2(s_2, m_2) \leq \epsilon
\]

In fact, we have (with high probability) that the sequences $(u_{m_1,j_1}, v_{m_2,j_2}, s_1, s_2)$ generated using the joint distribution (18), which is equivalent to the Markov chain $U \leftrightarrow S_1 \leftrightarrow S_2 \leftrightarrow V$.

Provided that $E_3(s_1, s_2, m_1, m_2)$ has not occurred, from the AEP we have that

\[
\Pr(E_4(s_1, s_2, m_1, m_2)|E_3(s_1, s_2, m_1, m_2)) = \Pr((u, v, y) \notin A_\epsilon^n(U, V, Y)|(u, s_1) \in A_\epsilon^n(U, S_1), (v, s_2) \in A_\epsilon^n(V, S_2)) \leq \epsilon.
\]
Likewise, we have that

$$
\Pr(E_5(s_1, s_2, m_1, m_2)|E_3(s_1, s_2, m_1, m_2)) \leq M_1 M_2 J_1 J_2 \Pr((u_{m_1'}, j_1), v_{m_2', j_2}, y) \in A^{(n)}_\epsilon(U, V, Y))
$$

$$
= M_1 M_2 J_1 J_2 \sum_{(u,v,y) \in A^{(n)}(U,V,Y)} p(u)p(v)y
$$

$$
\leq M_1 M_2 J_1 |A^{(n)}_\epsilon(U, V, Y)|2^{-n[H(U)-\epsilon]}2^{-n[H(V)-\epsilon]}2^{-n[H(Y)-\epsilon]}
$$

$$
\leq 2^{nR_2}2^{n[I(U;S_1)+4\epsilon]}2^{-n[I(U)+H(V,Y)]-H(U,V,Y)-3\epsilon}
$$

$$
= 2^{nR_2}2^{n[I(U;S_1)+4\epsilon]}2^{-n[I(U;V)]-3\epsilon}
$$

$$
= 2^{nR_2}2^{n[I(U;S_1)+4\epsilon]}2^{-n[I(U;V)]-7\epsilon}
$$

$$
= 2^{nR_2}2^{n[I(U;V)]-H(U)+H(V)]-7\epsilon}
$$

where (23) and (24) follow from AEP; (26) follows from the chain rule for mutual information; (29) follows from the Markov chain $U \leftrightarrow S_1 \leftrightarrow V$. In the same way, it can be shown that

$$
\Pr(E_6(s_1, s_2, m_1, m_2)|E_3(s_1, s_2, m_1, m_2)) \leq 2^{nR_2}2^{-n[I(V;Y)]-I(V;S_2[U]-7\epsilon]}
$$

Furthermore,

$$
\Pr(E_7(s_1, s_2, m_1, m_2)|E_3(s_1, s_2, m_1, m_2)) \leq M_1 M_2 J_1 J_2 \Pr((u_{m_1'}, j_1), v_{m_2', j_2}, y) \in A^{(n)}_\epsilon(U, V, Y))
$$

$$
= M_1 M_2 J_1 J_2 \sum_{(u,v,y) \in A^{(n)}(U,V,Y)} p(u)p(v)y
$$

$$
\leq M_1 M_2 J_1 J_2 |A^{(n)}_\epsilon(U, V, Y)|2^{-n[H(U)-\epsilon]}2^{-n[H(V)-\epsilon]}2^{-n[H(Y)-\epsilon]}
$$

$$
\leq 2^{nR_2}2^{n[I(U;S_1)+4\epsilon]}2^{n[I(V;S_2)+4\epsilon]}2^{-n[H(U)]-H(V)]-7\epsilon}
$$

$$
= 2^{nR_2}2^{n[I(U;S_1)+I(V;S_2)+8\epsilon]}2^{-n[I(U;V)]-7\epsilon}
$$

$$
= 2^{nR_2}2^{n[I(U;V)]-I(U;S_1)-I(V;S_2)+I(U;V)]-11\epsilon}
$$

$$
= 2^{nR_2}2^{n[I(U;V)]-H(U)]-I(V;S_2)+I(U;V)]-11\epsilon}
$$

$$
= 2^{nR_2}2^{n[I(U;V)]-H(U)]-I(V;S_2)-11\epsilon}
$$

$$
= 2^{nR_2}2^{n[I(U;V)]-I(U;S_1)+I(V;S_2)]-I(V;S_2)-11\epsilon}
$$

$$
= 2^{nR_2}2^{n[I(U;V)]-I(U;S_1)+I(V;S_2)-11\epsilon}
$$

$$
= 2^{nR_2}2^{n[I(U;V)]-I(U;S_1)+I(V;S_2)-11\epsilon}
$$
where (35) and (36) follow from AEP; (41) follows from the Markov chain $U \leftrightarrow S_1 \leftrightarrow S_2 \leftrightarrow V$; (43) follows from the chain rule for mutual information.

The theorem follows from (30), (31), (43), since for any arbitrary $\epsilon > 0$ the conditions in (7) imply that $P_e^{(n)} \to 0$ as $n \to \infty$.

III. Special Cases

We consider now two special cases of the memoryless MAC with correlated state information known non-causally at the encoders. The first case is for $S_1 = S_2$, i.e., the relation between the states is deterministic. The second case is for independent states.

I. Single state: In this case we have single state which is known to both encoders, i.e., $S = S_1 = S_2$, the achievable rate region is given by

$$
\mathcal{R} \triangleq \text{cl conv } \left\{ (R_1, R_2) : R_1 \leq I(U; Y| V) - I(U; S| V) \\
R_2 \leq I(V; Y| U) - I(V; S| U) \\
R_1 + R_2 \leq I(U, V; Y) - I(U, V; S) \right\}
$$

(44)

where the admissible pairs satisfy: $(U, X_1) \leftrightarrow S \leftrightarrow (V, X_2)$, and $(U, V) \leftrightarrow (X_1, X_2, S) \leftrightarrow Y$.

The Gaussian case of single interference is given by

$$
Y = X_1 + X_2 + S + Z,
$$

(45)

where $Z \sim \mathcal{N}(0, N)$, the interference $S$ is known non-causally to user 1 and user 2, and the power constraints are $P_1$ and $P_2$ for user 1 and user 2, respectively. This model was considered by Gel’fand and Pinsker [3]. It was shown that the capacity region is equal to clean MAC, i.e., for the case that $S = 0$. In this case, the region in (44) concises with the clean MAC region [2].

II. Independent states: for the case that $S_1$ and $S_2$ are independent, the achievable region becomes

$$
\mathcal{R} \triangleq \text{cl conv } \left\{ (R_1, R_2) : R_1 \leq I(U; Y| V) - I(U; S_1) \\
R_2 \leq I(V; Y| U) - I(V; S_2) \\
R_1 + R_2 \leq I(U, V; Y) - I(U, S_1) - I(V, S_2) \right\}
$$

(46)

where the admissible pairs satisfy: $(U, S_1, X_1)$ is independent of $(V, S_2, X_2)$, and $(U, V) \leftrightarrow (X_1, X_2, S_1, S_2) \leftrightarrow Y$. The case with independent channel states was originally considered in [4], which also introduces the rate region in (46).
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