Abstract—Since the early nineties, it has been observed that the Schrödinger bridge problem can be formulated as a stochastic control problem with atypical boundary constraints. This in turn has a fluid dynamic counterpart where the flow of probability densities represents an entropic interpolation between the given initial and final marginals. In the zero noise limit, such entropic interpolation converges in a suitable sense to the displacement interpolation of optimal mass transport (OMT). We consider two absolutely continuous curves in Wasserstein space $W^2$ and study the evolution of the relative entropy on $W^2 \times W^2$ on a finite time interval. Thus, this study differs from previous work in OMT theory concerning relative entropy from a fixed (often equilibrium) distribution (density). We derive a gradient flow on Wasserstein product space. We find the remarkable property that fluxes in the two components are opposite. Plugging in the “steepest descent” into the evolution of the relative entropy we get what appears to be a new formula: The two flows approach each other at a faster rate than that of two solutions of the same Fokker-Planck. We then study the evolution of relative entropy in the case of uncontrolled-controlled diffusions. In two special cases of the Schrödinger bridge problem, we show that relative entropy may be monotonically decreasing or monotonically increasing.

I. INTRODUCTION

In the Schrödinger bridge problem (SBP) [17], one seeks the random evolution (a probability measure on path-space) which is closest in the relative entropy sense to a prior Markov diffusion evolution and has certain prescribed initial and final marginals $\mu$ and $\nu$. As already observed by Schrödinger [34], [35], the problem may be reduced to a static problem which, except for the cost, resembles the Kantorovich relaxed formulation of the optimal mass transport problem (OMT). Considering that since [2] (OMT) also has a dynamic formulation, we have two problems which admit equivalent static and dynamic versions [23]. Moreover, in both cases, the solution entails a flow of one-time marginals joining $\mu$ and $\nu$. The OMT yields a displacement interpolation flow whereas the SBP provides an entropic interpolation flow.

Trough the work of Mikami, Mikami-Thieullen and Leonard [25], [26], [27], [22], [23], we know that the OMT may be viewed as a “zero-noise limit” of SBP when the prior is a sort of uniform measure on path space with vanishing variance. This connection has been extended to more general prior evolutions in [9], [10]. Moreover, we know that, thanks to a very useful intuition by Otto [29], the displacement interpolation flow $\{\mu_t; 0 \leq t \leq 1\}$ may be viewed as a constant-speed geodesic joining $\mu$ and $\nu$ in Wasserstein space [37]. What can be said from this geometric viewpoint of the entropic flow? It cannot be a geodesic, but can it be characterized as a curve minimizing a suitable action? In [9], we showed that this is indeed the case resorting to a time-symmetric fluid dynamic formulation of SBP. The action features an extra term which is a Fisher information functional. Moreover, this characterization of the Schrödinger bridge answers at once a question posed by Carlen [4, pp. 130-131].

It has been observed since the early nineties that SBP can be turned, thanks to Girsanov’s theorem, into a stochastic control problem with atypical boundary constraints, see [12], [3], [13], [31], [15]. The latter has a fluid dynamic counterpart. It is therefore interesting to compare the flow associated to the uncontrolled evolution (prior) to the optimal one. In particular, it is interesting to study the evolution of the relative entropy on the product Wasserstein space on a finite time interval. Thus, this study differs from previous work in OMT theory concerning relative entropy from an equilibrium distribution (density). We derive in Section IV a gradient flow on Wasserstein product space. We find the remarkable property that fluxes in the two components are opposite. Plugging in the “steepest descent” into the evolution of the relative entropy we get what appears to be a new formula (23). The two flows approach each other at a faster rate than that of two solutions of the same Fokker-Planck.

We then study the evolution of relative entropy in the case of uncontrolled-controlled diffusions. We show by one special case of the Schrödinger bridge problem that such relative entropy may even be monotonically increasing.

The paper is outlined as follows. In Section III we recall some fundamental facts and concepts from the theory of optimal transportation. In Section III we review the variational formulation of the Fokker-Planck equation as a gradient flow on Wasserstein space. Section IV we study the evolution of relative entropy on Wasserstein product space. In Section V we recall some basic elements of the Nelson-Föllmer kinematics of finite-energy diffusions. Finally, in Section VI we study the relative entropy change in the case of a controlled evolution. This is then specialized to the Schrödinger bridge.
II. ELEMENTS OF OPTIMAL MASS TRANSPORT THEORY

The literature on this problem is by now so vast and our degree of competence is such that we shall not even attempt here to give a reasonable and/or balanced introduction to the various fascinating aspects of this theory. Fortunately, there exist excellent monographs and survey papers on this topic, see [33], [14], [37], [1], [38], [30], to which we refer the reader. We shall only briefly review some concepts and results which are relevant for the topics of this paper.

A. The static problem

Let \( v_0 \) and \( v_1 \) be probability measures on the measurable spaces \( X \) and \( Y \), respectively. Let \( c : X \times Y \to [0, +\infty) \) be a measurable map with \( c(x,y) \) representing the cost of transporting a unit of mass from location \( x \) to location \( y \). Let \( \mathcal{T}_{v_0,v_1} \) be the family of measurable maps \( T : X \to Y \) such that \( T\#v_0 = v_1 \), namely such that \( v_1 \) is the push-forward of \( v_0 \) under \( T \). Then Monge’s optimal mass transport problem (OMT) is

\[
\inf_{T \in \mathcal{T}_{v_0,v_1}} \int_{X \times Y} c(x,T(x)) dv_0(x).
\]

As is well known, this problem may be unfeasible, namely the family \( \mathcal{T}_{v_0,v_1} \) may be empty. This is never the case for the “relaxed” version of the problem studied by Kantorovich in the 1940’s

\[
\inf_{\pi \in \Pi(v_0,v_1)} \int_{X \times Y} c(x,y) d\pi(x,y)
\]

where \( \Pi(v_0,v_1) \) are “couplings” of \( v_0 \) and \( v_1 \), namely probability distributions on \( X \times Y \) with marginals \( v_0 \) and \( v_1 \). Indeed, \( \Pi(v_0,v_1) \) always contains the product measure \( v_0 \times v_1 \). Let us specialize the Monge-Kantorovich problem to the case \( X = Y = \mathbb{R}^N \) and \( c(x,y) = \|x-y\|^2 \). Then, if \( v_1 \) does not give mass to sets of dimension \( \leq n-1 \), by Brenier’s theorem [37, p.66], there exists a unique optimal transport plan \( \pi \) (Kantorovich) induced by a \( d\nu \) a.e. unique map \( T \) (Monge), \( T = \nabla \varphi \), \( \varphi \) convex, and we have

\[
\pi = \left(I \times \nabla \varphi\right)\#v_0, \quad \nabla \varphi \#v_0 = v_1. \tag{3}
\]

Here \( I \) denotes the identity map. Among the extensions of this result, we mention that to strictly convex, superlinear costs \( c \) by Gangbo and McCann [18]. The optimal transport problem may be used to introduce a useful distance between probability measures. Indeed, let \( P_2(\mathbb{R}^N) \) be the set of probability measures \( \mu \) on \( \mathbb{R}^N \) with finite second moment. For \( v_0, v_1 \in P_2(\mathbb{R}^N) \), the Wasserstein (Vasershtein) quadratic distance, is defined by

\[
W_2(v_0, v_1) = \left( \inf_{\pi \in \Pi(v_0,v_1)} \int_{\mathbb{R}^N \times \mathbb{R}^N} \|x-y\|^2 d\pi(x,y) \right)^{1/2}. \tag{4}
\]

As is well known [37, Theorem 7.3], \( W_2 \) is a bona fide distance. Moreover, it provides a most natural way to “metrize” weak convergence in \( P_2(\mathbb{R}^N) \) [37, Theorem 7.12], [1, Proposition 7.1.5] (the same applies to the case \( p \geq 1 \) replacing 2 with \( p \) everywhere). The Wasserstein space \( \mathcal{W}_2 \) is defined as the metric space \( (P_2(\mathbb{R}^N), W_2) \). It is a Polish space, namely a separable, complete metric space.

B. The dynamic problem

So far, we have dealt with the static optimal transport problem. Nevertheless, in [2, p.378] it is observed that “...a continuum mechanics formulation was already implicitly contained in the original problem addressed by Monge... Eliminating the time variable was just a clever way of reducing the dimension of the problem”. Thus, a dynamic version of the OMT problem was already in fieri in Gaspar Monge’s 1781 “Memoire sur la theorie des debilais et des remblais”! It was elegantly accomplished by Benamou and Brenier in [2] by showing that

\[
W_2^2(v_0, v_1) = \inf_{(\mu, \nu)} \int_0^1 \int_{\mathbb{R}^N} \|v(x, t)\|^2 \mu_t(dx)dt, \tag{5a}
\]

\[
\frac{d\mu}{dt} + \nabla \cdot (\nu \mu) = 0, \tag{5b}
\]

\[
\mu_0 = v_0, \quad \mu_1 = v_1. \tag{5c}
\]

Here the flow \( \{\mu_t; 0 \leq t \leq 1\} \) varies over continuous maps from \( [0,1] \) to \( P_2(\mathbb{R}^N) \) and \( \nu \) over smooth fields. In [38], Villani states at the beginning of Chapter 7 that two main motivations for the time-dependent version of OMT are:

- a time-dependent model gives a more complete description of the transport;
- the richer mathematical structure will be useful later on.

We can add three further reasons:

- it opens the way to establish a connection with the Schrödinger bridge problem, where the latter appears as a regularization of the former [25], [26], [27], [22], [23], [9], [10];
- it allows to view the optimal transport problem as an (atypical) optimal control problem [6]-[10];
- in some applications such as interpolation of images [11] or spectral morphing [20], the interpolating flow is essential!

Let \( \{\mu_t^*; 0 \leq t \leq 1\} \) and \( \{\nu^* (x, t); (x, t) \in \mathbb{R}^N \times [0,1]\} \) be optimal for (5). Then

\[
\mu_t^* = \left( (1-t)I + t\nabla \varphi \right) \#v_0,
\]

with \( T = \nabla \varphi \) solving Monge’s problem, provides, in McCann’s language, the displacement interpolation between \( v_0 \) and \( v_1 \). Then \( \{\mu_t^*; 0 \leq t \leq 1\} \) may be viewed as a constant-speed geodesic joining \( v_0 \) and \( v_1 \) in Wasserstein space (Otto). This formally endows \( \mathcal{W}_2 \) with a “pseudo” Riemannian structure. McCann discovered [24] that certain functionals are displacement convex, namely convex along Wasserstein geodesics. This has led to a variety of applications. Following one of Otto’s main discoveries [21], [29], it turns out that a large class of PDE’s may be viewed as gradient flows on the Wasserstein space \( \mathcal{W}_2 \). This interpretation, because of the displacement convexity of the functionals, is well suited to establish uniqueness and to study energy dissipation and convergence to equilibrium. A rigorous setting in which
to make sense of the Otto calculus has been developed by Ambrosio, Gigli and Savaré [1] for a suitable class of functionals. Convexity along geodesics in $\mathcal{W}_2$ also leads to new proofs of various geometric and functional inequalities [24], [37, Chapter 9]. Finally, we mention that, when the space is not flat, qualitative properties of optimal transport can be quantified in terms of how bounds on the Ricci-Curbastro curvature affect the displacement convexity of certain specific functionals [38, Part II].

The tangent space of $\mathcal{P}_2(\mathbb{R}^N)$ at a probability measure $\mu$, denoted by $T_\mu \mathcal{P}_2(\mathbb{R}^N)$ [1] may be identified with the closure in $L^2_\mu$ of the span of \{ $\nabla \varphi : \varphi \in C^\infty_c$ \}, where $C^\infty_c$ is the family of smooth functions with compact support. It is naturally equipped with the scalar product of $L^2_\mu$.

III. THE FOKKER-PLANCK EQUATION AS A GRADIENT FLOW ON WASSERSTEIN SPACE

Let us review the variational formulation of the Fokker-Planck equation as a gradient flow on Wasserstein space [21], [37], [36]. Consider a physical system with phase space $\mathbb{R}^N$ and with Hamiltonian $\mathcal{H} : x \mapsto H(x) = E_n$. The thermodynamic states of the system are given by the family $\mathcal{P}(\mathbb{R}^N)$ of probability distributions $P$ on $\mathbb{R}^N$ admitting density $\rho$. On $\mathcal{P}(\mathbb{R}^N)$, we define the internal energy as the expected value of the Energy observable in state $P$

$$U(H, \rho) = \mathbb{E}_P(\mathcal{H}) = \int_{\mathbb{R}^N} H(x) \rho(x) dx = \langle H, \rho \rangle. \quad (6)$$

Let us also introduce the (differential) Gibbs entropy

$$S(\rho) = -k \int_{\mathbb{R}^N} \log(\rho(x)) \rho(x) dx, \quad (7)$$

where $k$ is Boltzmann’s constant. $S$ is strictly concave on $\mathcal{P}(\mathbb{R}^N)$. According to the Gibbssian postulate of classical statistical mechanics, the equilibrium state of a microscopic system at constant absolute temperature $T$ and with Hamiltonian function $H$ is necessarily given by the Boltzmann distribution law with density

$$\bar{\rho}(x) = Z^{-1} \exp \left[ -\frac{H(x)}{kT} \right] \quad (8)$$

where $Z$ is the partition function\(^2\)

Let us introduce the Free Energy functional $F$ defined by

$$F(H, \rho, T) := U(H, \rho) - TS(\rho). \quad (9)$$

Since $S$ is strictly concave on $\mathcal{U}$ and $U(\cdot, \cdot)$ is linear, it follows that $F$ is strictly convex on the state space $\mathcal{P}(\mathbb{R}^N)$. By Gibb’s variational principle, the Boltzmann distribution $\bar{\rho}$ is a minimum point of the free energy $F$ on $\mathcal{P}(\mathbb{R}^N)$. Also notice that

$$\mathbb{D}(\rho \| \bar{\rho}) = \int_{\mathbb{R}^N} \log \left( \frac{\rho(x)}{\bar{\rho}(x)} \right) \rho(x) dx = \frac{1}{k} S(\rho) + \log Z + \frac{1}{kT} \int_{\mathbb{R}^N} H(x) \rho(x) dx = \frac{1}{kT} F(H, \rho, T) + \log Z.$$

Since $Z$ does not depend on $\rho$, we conclude that Gibb’s principle is a trivial consequence of the fact that $\bar{\rho}$ minimizes $\mathbb{D}(\rho \| \bar{\rho})$ on $\mathcal{P}(\mathbb{R}^N)$.

Consider now an absolutely continuous curve $\mu_t : [0, t_1] \to \mathcal{P}_2$. Then [1, Chapter 8], there exist “velocity field” $v_t \in L^2_\mu(t)$ such that the following continuity equation holds on $(0, T)$

$$\frac{d}{dt} \mu_t + \nabla \cdot (v_t \mu_t) = 0.$$

Suppose $d\mu_t = \rho_t dx$, so that the continuity equation

$$\frac{\partial \rho_t}{\partial t} + \nabla \cdot (\rho_t v) = 0 \quad (10)$$

holds. We want to study the free energy functional $F(H, \rho_t, T)$ or, equivalently, $\mathbb{D}(\rho_t \| \bar{\rho})$, along the flow $\{ \rho_t ; t_0 \leq t \leq t_1 \}$, using (9), we get

$$\frac{d}{dt} \mathbb{D}(\rho_t \| \bar{\rho}) = \int_{\mathbb{R}^N} \left[ 1 + \log \rho_t + \frac{1}{kT} H(x) \right] \frac{\partial \rho_t}{\partial t} dx = -\int_{\mathbb{R}^N} \left[ 1 + \log \rho_t + \frac{1}{kT} H(x) \right] \nabla \cdot (\rho_t v) dx. \quad (11)$$

Integrating by parts, if the boundary terms at infinity vanish, we get

$$\frac{d}{dt} \mathbb{D}(\rho_t \| \bar{\rho}) = \int_{\mathbb{R}^N} \nabla \left[ \log \rho_t + \frac{1}{kT} \nabla H(x) \right] \cdot \rho_t v dx = \langle \nabla \left. \left( \log \rho_t + \frac{1}{kT} \nabla H(x) \right) \right|, \rho_t \rangle_{L^2_\mu}. \quad (12)$$

Thus, the Wasserstein gradient of $\mathbb{D}(\rho_t \| \bar{\rho})$ is

$$\nabla_{\mathcal{W}_2} \mathbb{D}(\rho_t \| \bar{\rho}) = \nabla \log \rho_t + \frac{1}{kT} \nabla H(x).$$

The corresponding gradient flow is

$$\frac{\partial \rho_t}{\partial t} = \nabla \cdot \left[ \left( \nabla \log \rho_t + \frac{1}{kT} \nabla H(x) \right) \rho_t \right] = \nabla \cdot \left[ \frac{1}{kT} \nabla H(x) \rho_t \right] + \Delta \rho_t. \quad (12)$$

But this is precisely the Fokker-Planck equation corresponding to the diffusion process

$$dX_t = -\frac{1}{kT} \nabla H(X_t) dt + \sqrt{2} dW_t \quad (13)$$

where $W$ is a standard $n$-dimensional Wiener process. The process (13) has the Boltzmann distribution (8) as invariant density. Recall that [1, p.220] $F(H, \rho_t, T)$ or, equivalently, $\mathbb{D}(\rho_t \| \bar{\rho})$ are displacement convex and have therefore a unique minimizer.

Remark 1: It seems worthwhile investigating to what extent the fundamental assumption of statistical mechanics that the variables with longer relaxation time form a vector Markov process having (8) as invariant density is equivalent to the requirement that the flow of one-time densities be a gradient flow in Wasserstein space for the free energy.

Let us finally plug the “steepest descent” (12) into (11). We get, after integrating by parts, the well known formula

\(^2\)The letter $Z$ was chosen by Boltzmann to indicate “zuständige Summe” (pertinent sum- here integral).
The last integral in (14) is sometimes called the \textit{relative Fisher information} of \( \rho \) with respect to \( \tilde{\rho} \) [37, p.278].

IV. RELATIVE ENTROPY AS A FUNCTIONAL ON WASSERSTEIN PRODUCT SPACES

Consider now two absolutely continuous curves \( \mu_t : [0, t_1] \to \mathcal{W}_2 \) and \( \eta_t : [0, t_1] \to \mathcal{W}_2 \) and their velocity fields \( v_t \in L^2\mu_t \) and \( v_t \in L^2\eta_t \). Then, on \( (0, T) \)

\[
\frac{d}{dt} \mu_t + \nabla \cdot (v_t \mu_t) = 0, \quad (15)
\]

\[
\frac{d}{dt} \eta_t + \nabla \cdot (v_t \eta_t) = 0. \quad (16)
\]

Let us suppose that \( d\mu_t = \rho_t(x)dx \) and \( d\eta_t = \tilde{\rho}_t(x)dx \), for all \( t \in [0, t_1] \). Then (15)-(16) become

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (v \rho) = 0, \quad (17)
\]

\[
\frac{\partial \tilde{\rho}}{\partial t} + \nabla \cdot (v \tilde{\rho}) = 0, \quad (18)
\]

where the fields \( v \) and \( \tilde{v} \) satisfy

\[
\int_{\mathbb{R}^N} ||v(x,t)||^2 \rho_t(x)dx < \infty, \quad \int_{\mathbb{R}^N} ||\tilde{v}(x,t)||^2 \tilde{\rho}_t(x)dx < \infty.
\]

The differentiability of the Wasserstein distance \( W_2(\tilde{\rho}_t, \rho_t) \) has been studied [38, Theorem 23.9]. Consider instead the relative entropy functional on \( \mathcal{W}_2 \times \mathcal{W}_2 \)

\[
\mathcal{D}(\tilde{\rho}_t \| \rho_t) = \int_{\mathbb{R}^N} h(\tilde{\rho}_t, \rho_t)dx = \int_{\mathbb{R}^N} \log \left( \frac{\tilde{\rho}_t}{\rho_t} \right) \tilde{\rho}_t dx,
\]

\[
h(\tilde{\rho}, \rho) = \log \left( \frac{\tilde{\rho}}{\rho} \right) \tilde{\rho}.
\]

Relative entropy functionals \( \mathcal{D}(\tilde{\gamma} \| \gamma) \), where \( \gamma \) is a fixed probability measure (density), have been studied as geodesically convex functionals on \( P_L(\mathbb{R}^N) \), see [1, Section 9.4]. Our study of the evolution of \( \mathcal{D}(\tilde{\rho}_t \| \rho_t) \) is motivated by problems on a finite time interval such as the Schrödinger bridge problem and stochastic control problems (Section IV) where it is important to evaluate relative entropy on \textit{two} flows of marginals.

We get

\[
\frac{d}{dt} \mathcal{D}(\tilde{\rho}_t \| \rho_t) = \int_{\mathbb{R}^N} \left[ \frac{\partial h}{\partial \tilde{\rho}} \frac{\partial \tilde{\rho}}{\partial t} + \frac{\partial h}{\partial \rho} \frac{\partial \rho}{\partial t} \right] dx
\]

\[
= \int_{\mathbb{R}^N} \left[ (1 + \log \tilde{\rho}_t - \log \rho_t) (\nabla \cdot (\tilde{v} \tilde{\rho}_t)) \right.
\]

\[
+ \left( \frac{\tilde{\rho}_t}{\rho_t} \right) (\nabla \cdot (v \rho_t)) \right] dx \quad (19)
\]

After an integration by parts, assuming that the boundary terms at infinity vanish, we get

\[
\frac{d}{dt} \mathcal{D}(\tilde{\rho}_t \| \rho_t) = \int_{\mathbb{R}^N} \left[ \nabla \log \left( \frac{\tilde{\rho}_t}{\rho_t} \right) \cdot \tilde{v} \tilde{\rho}_t - \nabla \log \left( \frac{\tilde{\rho}_t}{\rho_t} \right) \cdot v \rho_t \right] dx
\]

\[
= \int_{\mathbb{R}^N} \left[ \nabla \log \left( \frac{\tilde{\rho}_t}{\rho_t} \right) \cdot \tilde{v} \right] \tilde{\rho}_t dx. \quad (20)
\]

Notice that the last expression looks like

\[
\left\langle \nabla \log \left( \frac{\tilde{\rho}_t}{\rho_t} \right), \tilde{v} \right\rangle_{L^2_{\rho_t}, L^2_{\tilde{\rho}_t}}.
\]

Thus, we identify the gradient of the functional \( \mathcal{D}(\tilde{\rho} \| \rho) \) on \( \mathcal{W}_2 \times \mathcal{W}_2 \)

\[
\frac{\nabla_1^{\mathcal{W}_2} \mathcal{D}(\tilde{\rho} \| \rho)}{\nabla_2^{\mathcal{W}_2} \mathcal{D}(\tilde{\rho} \| \rho)} = \left( \nabla \log \frac{\tilde{\rho}}{\rho}, -\nabla \log \frac{\tilde{\rho}}{\rho} \right). \quad (21)
\]

Let us now compute the gradient flow on \( \mathcal{W}_2 \times \mathcal{W}_2 \) corresponding to gradient (21). We get

\[
\frac{\partial}{\partial t} (\tilde{\rho}_t) - \nabla \cdot \left( \nabla \log \left( \frac{\tilde{\rho}_t}{\rho_t} \right) \tilde{v}_t \right) \tilde{\rho}_t = 0. \quad (22)
\]

Since

\[
J_1 = \nabla \log \left( \frac{\tilde{\rho}_t}{\rho_t} \right) \tilde{v}_t = \nabla \left( \frac{\tilde{\rho}_t}{\rho_t} \right) \tilde{v}_t = -J_2,
\]

we observe the remarkable property that in the “steepest descent” (22) on the product Wasserstein space the “fluxes” are \textit{opposite} and, therefore, \( \frac{\partial \tilde{\rho}}{\partial t} = -\frac{\partial \rho}{\partial t} \). If we plug the steepest descent (22) into (19), we get what appears to be a new formula

\[
\frac{d}{dt} \mathcal{D}(\tilde{\rho}_t \| \rho_t) = \int_{\mathbb{R}^N} \left[ (1 + \log \tilde{\rho}_t - \log \rho_t + \frac{\tilde{\rho}_t}{\rho_t}) \frac{\partial \rho}{\partial t} \right] dx
\]

\[
= -\int_{\mathbb{R}^N} \| \nabla \log \left( \frac{\tilde{\rho}_t}{\rho_t} \right) \|^2 \rho_t + \| \nabla \left( \frac{\tilde{\rho}_t}{\rho_t} \right) \|^2 \tilde{\rho}_t \] dx
\]

\[
= -\int_{\mathbb{R}^N} \left[ \left( 1 + \frac{\tilde{\rho}_t}{\rho_t} \right) \| \nabla \log \left( \frac{\tilde{\rho}_t}{\rho_t} \right) \| \tilde{\rho}_t \right] dx. \quad (23)
\]

which should be compared to (14).

Let us return to equation (20). By multiplying and dividing by \( \tilde{\rho}_t \) in the last term of the middle expression, we get

\[
\frac{d}{dt} \mathcal{D}(\tilde{\rho}_t \| \rho_t) = \int_{\mathbb{R}^N} \left[ \nabla \log \left( \frac{\tilde{\rho}_t}{\rho_t} \right) \cdot (\tilde{v} - v) \right] \tilde{\rho}_t dx \quad (24)
\]

which is precisely the expression obtained in [32, Theorem III.1].

V. ELEMENTS OF NELSON-FÖLLMER KINEMATICS OF FINITE-ENERGY DIFFUSION PROCESSES

Let \( (\Omega, \mathcal{F}, \mathbb{P}) \) be a complete probability space. A stochastic process \( \{ \xi(t) : 0 \leq t \leq t_1 \} \) is called a \textit{finite-energy diffusion} with constant diffusion coefficient \( \sigma^2 I_N \) if the paths
\(\xi(t)\) belong to \(C([t_0,t_1];\mathbb{R}^N)\) (\(N\)-dimensional continuous functions) and
\[
\xi(t) - \xi(s) = \int_s^t \beta(t)d\tau + \sigma[W_+(t) - W_+(s)], \quad t_0 \leq s < t \leq t_1,
\] (25)
where \(\beta(t)\) is at each time \(t\) a measurable function of the past \(\{\xi(\tau); t_0 \leq \tau \leq t\}\) and \(W\) is a standard \(N\)-dimensional Wiener process. Moreover, the drift \(\beta\) satisfies the finite energy condition
\[
\mathbb{E}\left\{ \int_0^{t_1} ||\beta||^2d\tau \right\} < \infty.
\]
In [16], Föllmer has shown that a finite-energy diffusion also admits a reverse-time Ito differential. Namely, there exists a measurable function \(\gamma(t)\) of the future \(\{\xi(\tau); t \leq \tau \leq t_1\}\) called backward drift and another Wiener process \(W_-\) such that
\[
\xi(t) - \xi(s) = \int_s^t \gamma(t)d\tau + \sigma[W_-(t) - W_-(s)], \quad t_0 \leq s < t \leq t_1.
\] (26)
Moreover, \(\gamma\) satisfies
\[
\mathbb{E}\left\{ \int_0^{t_1} ||\gamma||^2d\tau \right\} < \infty.
\]
Let us agree that \(dt\) always indicate a strictly positive variable. For any function \(f : [t_0,t_1] \rightarrow \mathbb{R}\) let \(d_+f(t) = f(t+dt) - f(t)\) be the forward increment at time \(t\) and let \(d_-f(t) = f(t) - f(t-dt)\) be the backward increment at time \(t\). For a finite-energy diffusion, Föllmer has also shown in [16] that forward and backward drifts may be obtained as Nelson’s conditional derivatives [28]
\[
\beta(t) = \lim_{dt \searrow 0} \mathbb{E}\left\{ \frac{d_+\xi(t)}{dt}|\xi(t), t_0 \leq \tau \leq t \right\},
\]
\[
\gamma(t) = \lim_{dt \searrow 0} \mathbb{E}\left\{ \frac{d_-\xi(t)}{dt}|\xi(t), t \leq \tau \leq t_1 \right\},
\]
the limits being taken in \(L^2(\Omega,\mathcal{F},\mathbb{P})\). It was finally shown in [16] that the one-time probability density \(\rho_t(\cdot)\) of \(\xi(t)\) (which exists for every \(t > t_0\)) is absolutely continuous on \(\mathbb{R}^N\) and the following duality relation holds \(\forall t > 0\)
\[
\mathbb{E}\{\beta(t) - \gamma(t)|\xi(t)\} = \sigma^2 \nabla \log \rho(t,\xi(t)), \quad \text{a.s.}
\] (27)
Let us introduce the fields
\[
b_+(x,t) = \mathbb{E}\{\beta(t)|\xi(t) = x\}, \quad b_-(x,t) = \mathbb{E}\{\gamma(t)|\xi(t) = x\}.
\]
Then, Ito’s rule for the forward and backward differential of \(\xi\) imply that \(\rho_t\) satisfies the two Fokker-Planck equations
\[
\frac{d\rho}{dt} + \nabla \cdot (b_+\rho) - \frac{\sigma^2}{2} \Delta \rho = 0,
\] (28)
\[
\frac{d\rho}{dt} + \nabla \cdot (b_-\rho) + \frac{\sigma^2}{2} \Delta \rho = 0.
\] (29)
Following Nelson, let us introduce the current and osmotic drift of \(\xi\) by
\[
v(t) = \frac{\beta(t) + \gamma(t)}{2}, \quad u(t) = \frac{\beta(t) - \gamma(t)}{2},
\] (30)
respectively. Clearly \(v\) is similar to the classical velocity, whereas \(u\) is the velocity due to the noise which tends to zero when \(\sigma^2\) tends to zero. Let us also introduce
\[
v(x,t) = \mathbb{E}\{v(t)|\xi(t) = x\} = \frac{b_+(x,t) + b_-(x,t)}{2}.
\]
Then, combining (28) and (29), we get
\[
\frac{d\rho}{dt} + \nabla \cdot (v\rho) = 0,
\] (31)
which has the form of a continuity equation expressing conservation of mass. When \(\xi\) is Markovian with \(\beta(t) = b_+(\xi(t),t)\) and \(\gamma(t) = b_-(\xi(t),t),\) (27) reduces to Nelson’s relation
\[
b_+(x,t) - b_-(x,t) = \sigma^2 \nabla \log \rho_t(x).
\] (32)
Then (31) holds with
\[
v(x,t) = b_+(x,t) - \frac{\sigma^2}{2} \nabla \log \rho_t(x).
\] (33)
VI. RELATIVE ENTROPY PRODUCTION FOR CONTROLLED EVOLUTION
Consider on \([t_0,t_1]\) a finite-energy Markov process taking values in \(\mathbb{R}^N\) with forward Ito differential
\[
d\xi = b_+(\xi(t),t)dt + \sigma dW_+.\]
(34)
Let \(\rho_t(x)\) be the probability density of \(\xi(t)\). Consider also the feedback controlled process \(\xi^u\) with forward differential
\[
d\xi^u = b_+(\xi^u(t),t)dt + u(\xi^u(t),t)dt + \sigma dW_+.
\]
(35)
Here the control \(u\) is adapted to the past and is such that \(\xi^u(t)\) is a finite-energy diffusion. Let \(\rho^u_t(x)\) be the probability density of \(\xi^u(t)\). We are interested in the evolution of \(\mathbb{D}(\rho^u_t||\rho_t)\). By [33]–[35], the densities satisfy
\[
\frac{d\rho^u_t}{dt} + \nabla \cdot (\nu\rho^u_t) = 0,
\]
\[
v(x,t) = b_+(x,t) - \frac{\sigma^2}{2} \nabla \log \rho_t(x)
\]
\[
\frac{d\rho^u_t}{dt} + \nabla \cdot (\nu^u\rho^u_t) = 0,
\]
\[
v^u(x,t) = b_+(x,t) + u(x,t) - \frac{\sigma^2}{2} \nabla \log \rho^u_t(x).
\]
By (24), we now get
\[
\frac{d}{dt} \mathbb{D}(\rho^u_t||\rho_t) = \int_{\mathbb{R}^N} \left( \nabla \log \left( \frac{\rho^u_t}{\rho_t} \right) \cdot \left( \nu - \nu^u \right) \right) \rho^u_t dx.
\] (36)
Suppose now \(\rho^u_t = \rho^0_t\) is also uncontrolled and differs from \(\rho_t\) only because of the initial condition at \(t = t_0\). Then (36) gives the well known formula generalizing (14)
\[
\frac{d}{dt} \mathbb{D}(\rho^0_t||\rho_t) = -\frac{\sigma^2}{2} \int_{\mathbb{R}^N} \left( \nabla \log \left( \frac{\rho^0_t}{\rho_t} \right) \cdot \left( \nu^0 - \nu \right) \right) \rho^0_t dx.
\] (37)
which shows that two solutions of the same Fokker-Plank equation tend to get closer.
Consider now the situation where $\xi(t)$ represents a "prior" evolution on $[0,t_1]$ and the controlled evolution $\xi^u = \xi^*$ is the solution of the Schrödinger bridge problem for a pair of initial and final marginals $\rho_0$ and $\rho_1$ [17], [39]. Then

$$u^*(\xi^*,t) = \sigma^2 \nabla \log \phi(\xi^*,t)$$

and the differential of $\xi^*$ is given by

$$d\xi^* = b^*(\xi^*(t), t) dt + \sigma^2 \nabla \log \phi(\xi^*(t), t) dt + \sigma dW_t.$$  \hspace{1cm} (38)

where $\phi$ is space-time harmonic for the prior evolution, namely it satisfies

$$\frac{\partial \phi}{\partial t} + b^* \cdot \nabla \phi + \frac{\sigma^2}{2} \Delta \phi = 0.$$ \hspace{1cm} (39)

Let $\rho^\phi$ be the density of $\xi^*$. Let us first consider the special case of the Schrödinger bridge problem where relative entropy on path space is minimized under the only constraint that the initial marginal density be $\rho_0 \neq \rho_0$. Then, the optimal control $u^*$ is identically zero and the evolution of the relative entropy is given by (37). Consider instead the case of the problem where only the final marginal density $\rho_1 \neq \rho_1$ is imposed. In such case,

$$\rho^\phi(\xi^*) = \rho_1(\xi^*) \phi(\xi^*,t).$$

Then (40) gives

$$\frac{d}{dt} \mathbb{D}(\rho^\phi_t || \rho_1) = \frac{\sigma^2}{2} \int_{\mathbb{R}^d} |\nabla \log \phi | \cdot |\nabla \log \rho_1^\phi| d\mathbf{x}.$$ \hspace{1cm} (40)

This shows that $\mathbb{D}(\rho^\phi_t || \rho_1)$ increases up to time $t = t_1$. It represents the intuitive fact that the bridge evolution has to be as close as possible to the prior but the final value of the relative entropy must be the positive quantity $\mathbb{D}(\rho_1 || \rho_1)$. Thus, $\mathbb{D}(\rho^\phi_t || \rho_1)$ approaches this positive quantity from below. Result (40) may be viewed as a reverse-time H-theorem, as the bridge and the reference evolution have the same backward drift [17].

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