On behaviours of functional Volterra integro-differential equations with multiple time lags

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ABSTRACT

In this paper, the authors consider a non-linear Volterra integro-differential equation (NVIDE) of first order with multiple constant time lags. They obtain new sufficient conditions on stability (S), boundedness (B), global asymptotic stability (GAS) of solutions, and in addition, every solution $x$ of the considered NVIDE belong to $L^1[0, \infty)$ and $L^2[0, \infty)$. The authors prove five new theorems on $S$, $B$, GAS, $L^1[0, \infty)$ and $L^2[0, \infty)$ properties of solutions. The technique of the proofs involves the construction of suitable Lyapunov functionals. The obtained conditions are nonlinear generalizations and extensions of those of Becker [Uniformly continuous solutions of Volterra equations and global asymptotic stability. Cubo 11(3):2009:1–24], Graef et al. [Behavior of solutions of non-linear functional Volterra integro-differential equations with multiple delays. Dyn J Syst Appl. 25(1–2):2016:39–46] and Tunç [A note on the qualitative behaviors of non-linear Volterra integro-differential equation. J Egyptian Math Soc. 24(2):2016:187–192; New stability and boundedness results to Volterra integro-differential equations with delay. J Egyptian Math Soc. 24(2):2016:210–213] and they improve some results can found in the literature. The results of this paper are new, and they have novelty and complete some results exist in the literature.

1. Introduction

In the relevant literature, a famous mathematical model is known as Volterra integro-differential equation (VIDE), which appeared after its establishment by Vito Volterra, in 1926. Today, VIDEs and non-linear Volterra integro-differential equations (NVIDEs) are very important effective mathematical models to describe many real-world phenomena concerning atomic energy, biology, chemistry, control theory, economy, engineering technique fields, information theory, medicine, population dynamics, physics, etc. (see [1–11]).

In particular, during the last four decades, numerous authors have obtained many interesting results on the qualitative properties of solutions of linear VIDEs and NVIDEs, such as stability (S), boundedness (B), global asymptotic stability (GAS), integrability, square integrability, oscillation, non-oscillation and etc. by means of the fixed point theory, the perturbation methods, the variations of parameters formulas, the Lyapunov’s function or functional method, etc., (see [12–68]).

When we study the above-mentioned results on the S, B, GAS, integrability, square integrability, etc., of the solutions, it can be seen that during the investigations most of the mentioned results are proved by means of LFs in the literature. Indeed, this information shows the effectiveness and applicability of LFs during the investigations and applications. However, to the best of our information from the relevant literature, only in the proofs of a few results, the fixed point method or the perturbation theory or the variations of parameters formulas have been used. These cases can be verified and seen by studying the context of the mentioned papers and those found in their references. Here, we would not like to give the details of the applications of these methods.

Meanwhile, despite its long history, today the Lyapunov’s second (direct) method still seems as a leading basic tool to reduce a complex system into a relatively simpler system and makes available useful applications in the scientific topics just mentioned above. However, the Lyapunov characterizations for retarded NVIDEs with non-smooth functions have still remained as an open problem in the related literature. Here, we try to provide an application of this fact for a NVIDE with multiple constant time lags. By this way, we would like to say that it is worth to investigate qualitative properties of solutions of NVIDEs with multiple constant time lags.

In this direction, we would like to summarize some related papers on the topic.

Becker [15] consider VIDE

$$\frac{dx}{dt} = -a(t)x + \int_0^t b(t, s)x(s)ds. \quad (1)$$
He commented some properties of solutions of VIDE (1) by the Lyapunov’s functions. The author also gave examples to illustrate the results obtained.

Later, Tunç [53] considered NVIDE
\[
\frac{dx}{dt} = -a(t)h(x) + \int_0^t b(t, s)g(x(s))ds. \tag{2}
\]

In [53], sufficient conditions are introduced so that solutions of NVIDE (2) are stable, global asymptotic stable and so on, and it is also shown that the solutions have bounded derivatives by means of LFs.

Later, Tunç [54] studied the following NVIDE with a constant time lag
\[
\frac{dx}{dt} = -a(t)f(x) + \int_{t-\tau}^t B(t, s)g(x(s))ds + p(t). \tag{3}
\]

Tunç [54] investigated the S, B and convergence of bounded solutions of NVIDE (3) when \( t \to \infty \) by constructing suitable LFs.

Finally, in 2016, Graef et al. [29] considered the following NVIDE with multiple constant time lags
\[
\frac{dx}{dt} = -a(t)x + \sum_{i=1}^n \int_{t-\tau_i}^t b_i(t, s)f_i(x(s))ds. \tag{4}
\]

The authors constructed sufficient conditions so that solutions of NVIDE (4) are bounded, belong to \( L^1[0, \infty) \) and \( L^2[0, \infty) \). The authors also discussed S and GAS of the zero solution of NVIDE (4).

In this paper, we consider the following NVIDE with constant multiple time lags
\[
\frac{dx}{dt} = -a(t)g(x) + \sum_{i=1}^n \int_{t-\tau_i}^t K_i(t, s)g_i(s, x(s))ds + \sum_{i=1}^n r_i(t, x, x(t-\tau_i)), \tag{5}
\]

where \( t \geq 0, \ t - \tau_i \geq 0, \ \tau_i \) is the fixed constant time lag, the functions \( a(t) \) and \( g_i(x) \) are continuous with \( g_i(0) = 0, \ g_i \) is also differentiable, \( g_i \) with \( g_i(t, 0) = 0, \ r_i(t, x, x(t-\tau_i)) \) and \( K_i \) are real-valued and continuous functions on \([0, \infty), \ [0, \infty) \times R \), \([0, \infty) \times R \times R \times R \) and \( \Omega \) with \( \Omega := \{(t, s): 0 \leq s \leq t < \infty \} \), respectively.

Let
\[
g_0(x) = \begin{cases} 
  g(x)x^{-1}, & x \neq 0, \\
  g(0), & x = 0.
\end{cases}
\]

Hence, from NVIDE (5), we have
\[
\frac{dx}{dt} = -a(t)g_0(x) + \sum_{i=1}^n \int_{t-\tau_i}^t K_i(t, s)g_i(s, x(s))ds + \sum_{i=1}^n r_i(t, x, x(t-\tau_i)).
\]

Finally, if we compare VIDEs (1)–(4) with NVIDE (5), then the difference of NVIDE (5) and VIDEs (1)–(4) can be is seen, clearly. That is, NVIDE (5) is a new and different mathematical model. In addition, we obtain five new results on the qualitative properties of solutions NVIDE (5). These are the novelty and originality of this paper.

### 1.1. Notations

Let \( \tau = \max_{1 \leq i \leq n} \tau_i \). For \( \phi \in C[t_0 - \tau, t_0] \), it is supposed that \( |\phi_i|_{t_0} := \sup \{ |\phi(t)|: t \in [t_0 - \tau, t_0] \} \). In addition, let \( L^1[0, \infty) \) and \( L^2[0, \infty) \) represent the set of continuous and real-valued functions, for example, such as \( g \) and \( h \), for which \( \int_0^\infty |g(s)|ds < \infty \) and \( \int_0^\infty |h(s)|^2 ds < \infty \), respectively.

It should be noted that through the paper when we need \( x \) will represent \( x(t) \).

### 2. Main problems

#### 2.1. Assumptions

(A1) The functions \( a_i(0, \infty) \to (0, \infty), \ g_i: R \to R, o(0, \infty) \times R \to R, \ b_i: \Omega \to R \) and \( r_i: \Omega \to R \) are continuous for each \( i = 1, \ldots, n \).

(A2) Let \( \alpha \) and \( \beta \) be positive constants such that \( 1 \leq g_0(x) \leq \beta \) and \( |g_i(t, x)| \leq \alpha|x| \) for all \( x \in R \) and each \( i = 1, \ldots, n \).

(A3) \( a(s)g_0(x(s)) - \sum_{i=1}^n \int_{s-\tau}^s a_iK_i(t, u)du \geq 0, t \geq t_0 - \tau \geq 0 \).

(A4) \( a(t)g_0(x) - \sum_{i=1}^n \int_0^t a_iK_i(t, s)ds \geq 0, t \geq 0 \).

(A5) \( |r_i(t, x, x(t-\tau_i))| \leq \frac{1}{\beta} |q_i(t)||x|_i \), where \( q_i \) are continuous functions such that \( |q_i| \in L^1[0, \infty) \).

The first theorem of this paper is given below. Assume that
\[
r_i(\cdot) = 0, \ (i = 1, 2, \ldots, n).
\]

**Theorem 1.** If assumptions (A1)–(A4) hold, then all solutions of NVIDE (5) are bounded and the zero solution of NVIDE (5) is stable.

**Proof.** Let \( t_0 \geq 0, \ \phi \in C[t_0 - \tau, t_0] \) be an initial function and \( x(t) = x(t, t_0, \phi) \) represent the solution of NVIDE (5) on \([t_0 - \tau, \infty) \) such that \( x(t) = \phi(t) \) on \([t_0 - \tau, t_0] \).

We construct a LF
\[
W: (0, \infty) \times C([0, \infty) \to [0, \infty)
\]
by
\[
W(t) = W(t, x(\cdot)) = x^2 + \int_0^t \left( \alpha(s)g_0(x(s)) - \sum_{i=1}^n \int_{t-\tau}^s a_iK_i(t, u)|du| \right) x^2(s)ds.
\]
In view of assumption (A4), from (6) we see that
\[ W(t, x(.)) \geq x^2, \ t \geq t_0 - \tau \geq 0. \]

From the calculation of the time derivative of the Lyapunov functional \( W \), we have
\[
\frac{dW}{dt} = 2x' + a(t)g_0(x)x - 2 \sum_{i=1}^{n} \int_{t_i}^{t} c_i |K(t, s)| x^2(s)ds
- \int_{0}^{t} \sum_{i=1}^{n} a_i |K(t, u)| du
- \int_{0}^{t} \sum_{i=1}^{n} a_i |K(t, u)| x^2(s)ds
- x^2 \sum_{i=1}^{n} \int_{t_i}^{t} a_i |K(t, u)| du - \int_{0}^{t} \sum_{i=1}^{n} a_i |K(t, u)| |x(s)|ds
- x^2 \sum_{i=1}^{n} \int_{t_i}^{t} a_i |K(t, u)| du - \int_{0}^{t} \sum_{i=1}^{n} a_i |K(t, u)| x^2(s)ds
\leq -a(t)g_0(x)^2 + 2x \sum_{i=0}^{n} \int_{t_{i-1}}^{t} a_i |K(t, s)| x^2(s)ds
- x^2 \sum_{i=1}^{n} \int_{t_i}^{t} a_i |K(t, u)| du - \int_{0}^{t} \sum_{i=1}^{n} a_i |K(t, u)| |x(s)|ds
+ \sum_{i=1}^{n} \int_{t_i}^{t} a_i |K(t, s)| x^2(s)ds
- x^2 \sum_{i=1}^{n} \int_{t_i}^{t} a_i |K(t, u)| du
- \int_{0}^{t} \sum_{i=1}^{n} a_i |K(t, u)| x^2(s)ds
= -\left[ a(t)g_0(x) - \sum_{i=1}^{n} \int_{0}^{t} a_i |K(t, s)| ds \right] x^2
- \left[ \sum_{i=1}^{n} \int_{t_{i-1}}^{t} a_i |K(t, u)| du \right] x^2.
\]
Hence, we can obtain that every solution of NVIDE (5) is bounded for all \( t \geq t_0 \geq 0 \).
Indeed, from (8) and
\[
W(t_0) = \phi^2(t_0) + \int_{0}^{t_0} \left\{ a(s)g_0(\phi(s)) - \sum_{i=1}^{n} \int_{s_{i-1}}^{s} a_i |K(t, u)| du \right\} \phi^2(s)ds
\leq |\phi|^2 M(t_0),
\]
we can get
\[
|x| \leq |\phi|^2 \sqrt{M(t_0)}, \ t \geq t_0 \geq s - \tau. \tag{9}
\]
where
\[
M(t_0) := 1 + \int_{0}^{t_0} \left\{ a(s)g_0(\phi(s)) - \sum_{i=1}^{n} \int_{s_{i-1}}^{s} a_i |K(t, u)| du \right\} ds.
\]
The obtained result shows that the zero solution of NVIDE (5) is stable. That is, for any given \( \epsilon > 0 \), let
\[
\delta = \epsilon / \sqrt{M(t_0)}.
\]
Then, for \( \phi \in C[t_0 - \tau, t_0] \) with
\[
|\phi|^2 < \delta,
\]
we have
\[
|x| \leq \delta \sqrt{M(t_0)} = \epsilon, \ t \geq t_0 \geq s - \tau.
\]
Thus, we can reach to the end of the proof of Theorem 1.

### 2.2. Assumptions

Let the following assumptions hold.
\begin{align*}
\text{(H1)} & \quad a(t)g_0(x) - \sum_{i=1}^{n} \int_{t_{i-1}}^{t} a_i |K(t, s)| ds \geq k, \ t_0 \geq t_0, \\
& \quad k \in \mathbb{R}, \ k > 0.
\end{align*}

\begin{align*}
\text{(H2)} & \quad a(s)g_0(x(s)) - \sum_{i=1}^{n} \int_{s_{i-1}}^{s} a_i |K(t, u)| du \geq k, \ t_0 \geq s - \tau \geq t_0.
\end{align*}

**Theorem 2.** If in addition to assumptions (A1)–(A4), either assumption (H1) or (H2) holds, then every solution of NVIDE (5) belongs to \( L^2[0, \infty) \).

**Proof.** It is known from Theorem 1 that any solution \( x(t) \) of NVIDE (5) is bounded and for which (8) and (9) hold. If assumption (H1) holds, then from (7) we have
\[
W(t) \leq -kx^2(t),
\]
for all \( t \geq t_0 \).
Integration of the last inequality from \( t_0 \) to \( t \) gives that:
\[
W(t) - W(t_0) \leq -k \int_{t_0}^{t} x^2(s)ds,
\]
for all \( t \geq t_0 \).

For the case (H2), we have
so that
\[ k \int_{t_i}^{t} x^2(s)ds \leq W(t) - W(t_i). \]

Hence, we see that \( x \in L^2(0, \infty) \).

If assumption (H2) holds, then from the definition of the functional \( W \), we can get
\[ x^2(t) + k \int_{t_i}^{t} x^2(s)ds \leq W(t) \leq W(t_i). \]

Again, we see that \( x \in L^2(0, \infty) \). This proves Theorem 2.

### 2.3. Assumption

The following assumption is needed for GAS of the zero solution of NVIDE (5).

(C1) There exists a positive constant \( K \) such that
\[ a(t)g_0(x) + \sum_{i=1}^{n} \int_{t_i - \tau}^{t} |K(t_i, s)||g(s, x(s))|ds \leq K, \]
for all \( t \geq t_i \).

#### Theorem 3.
If assumptions (A1)–(A4) and (C1), and either assumption (H1) or (H2) hold, then the zero solution of NVIDE (5) is GAS.

**Proof.** From Theorem 2, we have that every solution of NVIDE (5) belongs to \( L^2(0, \infty) \). In addition, from NVIDE (5) and (9), it follows that
\[ |x| \leq a(t)g_0(x)|x| + \sum_{i=1}^{n} \int_{t_i - \tau}^{t} |K(t_i, s)||g(s, x(s))|ds \]
\[ \leq a(t)g_0(x)|x| + \sum_{i=1}^{n} \int_{t_i - \tau}^{t} \alpha|K(t_i, s)||x(s)|ds \]
\[ \leq a(t)g_0(x)|x| + \sum_{i=1}^{n} \int_{t_i - \tau}^{t} \alpha|K(t_i, s)||x(s)|ds \]
\[ + \sum_{i=1}^{n} \int_{t_i - \tau}^{t} \alpha|K(t_i, s)||x(s)|ds \]
\[ = K|\phi_{t_i}| \sqrt{M(t_0)}. \]

Hence, we have showed that \( x(t) \) is bounded. This result together with the fact \( x \in L^2[0, \infty) \) yields that \( x(t) \to 0 \) as \( t \to \infty \). That is, the zero solution of NVIDE (5) is GAS. So, we can conclude that the idea of Theorem 3 is true.

### 2.4. Assumption

(D1) There exist constants \( k_1 > 0 \) and \( \beta_1, 0 \leq \beta_1 < 1 \), such that
\[ a(t) \geq k_1 \beta_1 a(t)g_0(x) - \sum_{i=1}^{n} \int_{t_i - \tau}^{t} \alpha|K(t, u)||u|du \geq 0. \]

#### Theorem 4.
If assumptions (A2) and (A3) hold, then all solutions of NVIDE (5) are bounded and the zero solution of NVIDE (5) is stable. If, in addition, assumption (D1) holds, then all solutions of NVIDE (5) belong to \( L^2(0, \infty) \). Further, if assumption (C1) holds, then \( x \in L^2(0, \infty) \) and the zero solution of NVIDE (5) is GAS.

**Proof.** Define the LF
\[ W_1: \mathbb{R} \times C(0, \infty) \to \mathbb{R}, \]
by
\[ W_1(t, x) = |x| + \int_{0}^{t} \{a(s)g_0(x(s)) - \sum_{i=1}^{n} \int_{s-\tau}^{s} \alpha|K(t, u)||u|du\} x(s)ds. \]

From (10), we find that \( W_1(t, x) \geq |x| \) for all \( t \geq t_0 - \tau \), \( x \neq 0 \).
We know that for a continuously differentiable function \( h(t) \), \( |h(t)| \) has a right derivative, and this right derivative \( D_r h(t) \) is given by
\[ D_r h(t) = \begin{cases} h'(t) \text{sgn}(h(t)), & \text{if } h(t) \neq 0, \\ |h'(t)|, & \text{if } h(t) = 0. \end{cases} \]

Hence, from (10) the right derivative of \( W_1 \) is given by
\[ D_r W_1(t) = D_r |x| + D_r \int_{0}^{t} a(s)g_0(x(s)) \]
\[ - \sum_{i=1}^{n} \int_{s-\tau}^{s} \alpha|K(t, u)||u|du \}
\[ \sum_{i=1}^{n} \int_{s-\tau}^{s} \alpha|K(t, u)||u|du \}
\[ \leq \sum_{i=1}^{n} \int_{s-\tau}^{s} \alpha|K(t, s)||x(s)|ds \]
\[ - \sum_{i=1}^{n} \int_{s-\tau}^{s} \alpha|K(t, u)||u|du \}
\[ \leq \sum_{i=1}^{n} \int_{s-\tau}^{s} \alpha|K(t, u)||u|du \}
\[ \leq \sum_{i=1}^{n} \int_{s-\tau}^{s} \alpha|K(t, u)||u|du \}
\[ \leq \sum_{i=1}^{n} \int_{s-\tau}^{s} \alpha|K(t, u)||u|du \}
\[ \leq \sum_{i=1}^{n} \int_{s-\tau}^{s} \alpha|K(t, u)||u|du \}
\[ \leq 0. \]

Then, we can conclude that
\[ D_r W_1(t) \leq 0. \]
Thus, it can be shown that

\[ |x| \leq W_1(t) \leq W_1(t_0), \quad t \geq t_0 \geq t_0 - \tau \geq 0. \]

Thus, the B of solutions and the S of the zero solution of NVIDE (5) can be seen as in Theorem 1.

To complete the rest of the proof, we re-define the functional \( W_1 \).

Define

\[
W_p(t) = W_p(t, x(\cdot)) = |x| \\
+ \int_0^t \left\{ \beta_1 a(s) g_0(x(s)) - \sum_{i=1}^n \int_{t-\tau}^{t} \alpha[K_i(t,u)] du \right\} |x(s)| ds.
\]

Hence, by assumption (D1), we have

\[ W_p(t) \geq |x|. \]

Further, the right derivative of functional \( W_p \) in (11) is calculated as

\[
D_t W_p(t) = D_t |x| + D_t \int_0^t \left\{ \beta_1 a(s) g_0(x(s)) \\
- \sum_{i=1}^n \int_{t-\tau}^{t} \alpha[K_i(t,u)] du |x(s)| ds \right\} ds
\]

By the assumptions of Theorem 4, it is now clear that

\[ W_p(t) \geq |x|. \]

Therefore, the improper integral

\[ \int_0^\infty |x(s)| ds, \]

converges. That is, \( x \in L^1[t_0, \infty) \). This completes the proof of Theorem 4.

Suppose that

\[ r_i(,) \neq 0, (i = 1, \ldots, n). \]

**Theorem 5.** Let assumptions (A1)–(A5) be hold. Then all solutions of NVIDE (5) are bounded.

**Proof.** From Theorem 1, any solution of NVIDE (5) satisfies (7). To complete the proof of this theorem, we re-consider the Lyapunov functional \( W(t) = W(t, x(\cdot)) \), which is used in the proof of Theorem 1.

Obviously, we have

\[ W(t) \geq x^2. \]

Next, under the light of assumptions (A1)–(A5), the time derivative of the Lyapunov functional \( W(t) = W(t, x(t)) \) can be re-arranged as

\[
W'(t) \leq 2|x| \sum_{i=1}^n |r_i(t, x, x(t-\tau))| \\
\leq x^2 \sum_{i=1}^n |q_i(t)| \\
\leq \sum_{i=1}^n |q_i(t)| W(t).
\]

By integrating this inequality from \( t_0 \) to \( t \), we get

\[ W(t) \leq W(t_0) + \int_{t_0}^t \sum_{i=1}^n |q_i(s)| W(s) ds. \]

Hence, an application of the Gronwall’s inequality
yields that

\[ x^2 \leq W(t) \leq W(t_0) \exp\left[ \int_{t_0}^{t} \sum_{j=1}^{n} |q_j(s)| ds \right]. \]

Consequently, by the assumption \(|q_j| \in L^1[0, \infty)|, one can arrive at the desirable result that every solution of NVIDE (5) is bounded.

3. Conclusion

We consider a functional NVIDE of first order with multiple constant time lags. The qualitative properties of the considered NVIDE such as \( B, S, G, S, L^1[0, \infty) \) and \( L^2[0, \infty) \) are investigated by defining some suitable LFs. The obtained results have a novelty and contribution to the literature, and they improve or generalize the results of Becker [15], Graef et al. [29] and Tunç [35,54] and those can be found in the relevant literature.

Disclosure statement

No potential conflict of interest was reported by the authors.

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