NONSYMMETRIC MACDONALD POLYNOMIALS AND DEMAZURE CHARACTERS

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ABSTRACT. We establish a connection between a specialization of the nonsymmetric Macdonald polynomials and the Demazure characters of the corresponding affine Kac-Moody algebra. This allows us to obtain a representation-theoretical interpretation of the coefficients of the expansion of the specialized symmetric Macdonald polynomials in the basis formed by the irreducible characters of the associated finite Lie algebra.

INTRODUCTION

Generalizing the characters of compact simple Lie groups I.G. Macdonald associated to each irreducible root system a family of orthogonal polynomials $P_\lambda(q,t)$ indexed by anti-dominant weights and which are invariant under the action of the Weyl group. These polynomials depend rationally on parameters $q$ and $t = (t_s, t_l)$ and for particular values of these parameters reduce to familiar objects in representation theory:

1. when $q = t_s = t_l$ they are equal to $\chi_\lambda$ the Weyl characters of the corresponding root system (in particular they are independent of $q$);
2. when $q = 0$ they are the polynomials that give the values of zonal spherical functions on a semisimple $p$-adic Lie group relative to a maximal compact subgroup;
3. when $t_s = q^{k_s}, t_l = q^{k_l}$ and $q$ tends to 1 they are the polynomials that give the values of zonal spherical functions on a real symmetric space $G/K$ that arise from finite dimensional spherical representations of $G$. Here $k_s, k_l$ are the multiplicities of the short, respectively long, restricted roots.

The nonsymmetric Macdonald polynomials $E_\lambda(q,t)$ (indexed this time by the entire weight lattice) were first introduced by E. Opdam in the differential setting and then by I. Cherednik in full generality. Unlike the symmetric polynomials, their representation-theoretical meaning is still unexplored. At present time their main importance consists in the fact that they form the common spectrum of a family of commuting operators (the Cherednik operators) which play a preponderant role in the representation theory of affine Hecke algebras and related harmonic analysis.

It became clear, especially from the work of Cherednik, that we can in fact construct such families of polynomials for every irreducible affine root
system. From this point of view, the objects studied by Cherednik ([C1], [C2], [C3]) are the polynomials attached to reduced twisted affine root systems, and the Koornwinder polynomials, studied by S. Sahi ([S2], [S3]), are the polynomials attached to non-reduced affine root systems.

This paper was inspired by the result of Y. Sanderson [Sa] who established a connection between a specialized version of the nonsymmetric Macdonald polynomials ($E_{\lambda}(q, \infty)$ in our notation) and the characters of a certain Demazure modules $E_{\lambda}(\tilde{\lambda})$ of the irreducible affine Lie algebra (see Section 1 for the definitions of the ingredients) in the case of an irreducible root system of type $A_n$. Extrapolating from [Sa] we establish here the same connection for all irreducible affine root systems for which the affine simple root is short. This condition identifies precisely the polynomials studied by Cherednik and Sahi. The proof rely heavily on the method of intertwiners in double affine Hecke algebras.

**Theorem 1.** For an affine root system as above and any weight $\lambda$ we have,

$$E_{\lambda}(q, \infty) = q^{(\Lambda_{0}, w_{\lambda}(\tilde{\lambda}))} \chi(E_{w_{\lambda}}(\tilde{\lambda})).$$

The remaining cases: $B^{(1)}_n$, $C^{(1)}_n$, $F^{(1)}_4$ and $G^{(1)}_2$ exhibit some special features. For example, the formula of the affine intertwiner as an element of the double affine Hecke algebra takes a different form (see [I]). Computations suggest that the action on the weight lattice of the degeneration of this affine intertwiner at $t = \infty$ does not equal the action of the affine Demazure operator, but a different action with similar properties.

The connection between nonsymmetric Macdonald polynomials and Demazure characters allows a representation-theoretical interpretation of the coefficients of the expansion of the symmetric polynomials in the basis formed by the irreducible characters of the associated finite Lie algebra. Our second result is the following

**Theorem 2.** For an affine root system as above and any anti-dominant weight $\lambda$ the symmetric polynomial $P_{\lambda}(q, \infty)$ can be written as a sum

$$P_{\lambda}(q, \infty) = \sum_{\mu \leq \lambda} d_{\lambda \mu}(q) \chi_{\mu}$$

where $d_{\lambda \mu}(q)$ is a polynomial in $q^{-1}$ with positive integer coefficients.

Let us mention that in the $A_n$ case, as explained in [Sa], the positivity of the above coefficients is closely related to the positivity of the Kostka-Foulkes polynomials via the duality of the two variable Kostka functions. Another consequence of the Theorem 1 is the following

**Theorem 3.** For an affine root system as above and any weight $\lambda$ we have,

$$E_{\lambda}(\infty, \infty) = \chi(E_{\lambda}(\tilde{\lambda})).$$
This relates the specialization of the nonsymmetric Macdonald polynomials
\[ E_\lambda(\infty, \infty) = \lim_{q \to \infty} \lim_{t \to \infty} E_\lambda(q, t) \]
to the Demazure characters of the finite irreducible Lie algebras. The order in which we compute the above limits seems to be irrelevant.

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1. Preliminaries

1.1. The affine Weyl group. For the most part we shall adhere to the notation in [Ka]. Let \( A = (a_{ij})_{0 \leq i, j \leq n} \) be an irreducible affine Cartan matrix, \( S(A) \) the Dynkin diagram and \( (a_0, \ldots, a_n) \) the numerical labels of \( S(A) \) in Table Aff from [Ka], p.48-49. We denote by \( (a_0^\vee, \ldots, a_n^\vee) \) the labels of the Dynkin diagram \( S^{(A)} \) of the dual algebra which is obtained from \( S(A) \) by reversing the direction of all arrows and keeping the same enumeration of the vertices. Let \( (h, R, R^\vee) \) be a realization of \( A \) and let \( (\hat{h}, \hat{R}, \hat{R}^\vee) \) be the associated finite root system (which is a realization of the Cartan matrix \( \hat{A} = (a_{ij})_{1 \leq i, j \leq n} \)). From this data one can construct an affine Kac-Moody algebra \( \hat{g} \), respectively a finite Lie algebra \( \hat{g} \) such that \( h, \hat{h} \) become the corresponding Cartan subalgebras and \( R, \hat{R} \) become the corresponding root systems. Note also that \( \hat{g} \) is a subalgebra of \( g \). We refer to [Ka] for the details of this construction. If we denote by \( \{\alpha_i\}_{0 \leq i \leq n} \) a basis of \( R \) such that \( \{\alpha_i\}_{1 \leq i \leq n} \) is a basis of \( \hat{R} \) we have the following description
\[ h^* = \hat{h}^* + \mathbb{R}\delta + \mathbb{R}\Lambda_0, \]
where \( \delta = \sum_{i=0}^{n} a_i \alpha_i \). The vector space \( h^* \) has a canonical scalar product defined as follows
\[ (\alpha_i, \alpha_j) := d_i^{-1} a_{ij}, \quad (\Lambda_0, \alpha_i) := \delta_i,0 a_0^{-1} \quad \text{and} \quad (\Lambda_0, \Lambda_0) := 0, \]
with \( d_i := a_i a_i^{-1} \) and \( \delta_i,0 \) Kronecker’s delta. As usual, \( \{\alpha_i^\vee := d_i \alpha_i\}_{0 \leq i \leq n}, \{\lambda_i\}_{1 \leq i \leq n} \) and \( \{\lambda_i^\vee\}_{1 \leq i \leq n} \) are the coroots, fundamental weights and fundamental coweights. Denote by \( P = \bigoplus_{i=1}^{n} \mathbb{Z}\lambda_i \) and \( \hat{Q} = \bigoplus_{i=1}^{n} \mathbb{Z}\alpha_i \) the weight lattice, respectively the root lattice and let
\[ \rho := \frac{1}{2} \sum_{\alpha \in \hat{R}_+} \alpha^\vee = \sum_{i=1}^{n} \lambda_i^\vee. \]

Given \( \alpha \in R, x \in h^* \) let
\[ s_\alpha(x) := x - \frac{2(x, \alpha)}{(\alpha, \alpha)} \alpha. \]
The affine Weyl group \( \hat{W} \) is generated by all \( s_\alpha \) (the simple reflections \( s_i = s_{\alpha_i} \) are enough). The finite Weyl group \( W \) is the subgroup generated by
$s_1, \ldots, s_n$. An important role is played by $\theta = \delta - a_0\alpha_0$. Remark that $a_0 = 1$ in all cases except for $A = A_{2n}^{(2)}$, when $a_0 = 2$. For $s$ a real number, $h_s^* = \{ x \in h \mid (x, \delta) = s \}$ is the level $s$ of $h^*$. We have

$$h_s^* = h_0^* + s\Lambda_0 = \tilde{h}^* + \mathbb{R}\delta + s\Lambda_0.$$

The action of $W$ preserves each of the $h_s^*$ and we can identify each of the $h_s^*$ canonically with $h_0^*$ and obtain an (affine) action of $W$ on $h_0^*$. If $s_i \in W$ is a simple reflexion, write $s_i(\cdot)$ for the regular action of $s_i$ on $h_0^*$ and $s_i(\cdot)$ for the affine action of $s_i$ on $h_0^*$ corresponding to the level one action. These actions differ only for $s_0$:

$$s_0(x) = s_\theta(x) + (x, \theta)\delta,$$

$$s_0(\lambda) = s_\theta(\lambda) + a_0^{-1}\lambda - a_0^{-1}\delta.$$

By $s_i \cdot$ we denote the affine action of $W$ on $h_0^*$

$$s_0 \cdot x = s_\theta(x) + a_0^{-1}\lambda.$$

We will be interested in the cases when $\alpha_0$ is a short root. This happens precisely when the affine root system is twisted or simply laced untwisted. Under these conditions define the fundamental alcove as

$$\mathcal{A} := \{ x \in h_0^* \mid (x, \Lambda_0, \alpha_i^\vee) \geq 0 , \ 0 \leq i \leq n \}.$$

The non-zero elements of $\mathcal{O} = P \cap \mathcal{A}$ are the so-called minuscule weights. Let us remark that the orbits of the affine action of $W$ on $P$ contains a unique $\lambda_i \in \mathcal{A}$ (to keep the notation consistent we set $\lambda_0 = 0$).

In all what follows we assume our affine root system to be such that the affine simple root $\alpha_0$ is short (this condition includes of course the case when all roots have the same length).

1.2. The Bruhat order. Let us first establish some notation. For each $w$ in $W$ let $l(w)$ be the length of a reduced (i.e. shortest) decomposition of $w$ in terms of the $s_i$. We have $l(w) = |\Pi(w)|$ where $\Pi(w) = \{ \alpha \in R_+ \mid w(\alpha) \in R_- \}$. If $w = s_{j_p} \cdots s_{j_1}$ is a reduced decomposition, then

$$\Pi(w) = \{ \alpha^{(i)} \mid 1 \leq i \leq p \},$$

with $\alpha^{(i)} = s_{j_p} \cdots s_{j_{i-1}}(\alpha_{j_i})$. For each weight $\lambda$ define $\lambda_-$, respectively $\tilde{\lambda}$, to be the unique element in $\mathcal{W}\lambda$, respectively $W \cdot \lambda$, which is an anti-dominant weight, respectively an element of $\mathcal{O}$ (that is a minuscule weight or zero), and $\tilde{w}_\lambda^{-1} \in W$, $w_\lambda^{-1} \in W$, to be the unique minimal length elements by which this is achieved. Also, for each weight $\lambda$ define $\lambda_+$ to be the unique element in $\mathcal{W}\lambda$ which is dominant and denote by $w_0$ the maximal length element in $W$.

**Lemma 1.1.** With the notation above, we have

i) $\Pi(\tilde{w}_\lambda^{-1}) = \{ \alpha \in R_+ \mid (\lambda, \alpha) > 0 \}$;

ii) $\Pi(w_\lambda^{-1}) = \{ \alpha \in R_+ \mid (\lambda + \Lambda_0, \alpha) < 0 \}$.
Proposition. Straightforward. See Theorem 1.4 of [3] for a full argument.

The Bruhat order is a partial order on any Coxeter group. For its basic properties see Chapter 5 in [1]. Let us list a few of them (the first two properties completely characterize the Bruhat order):

1. For each $\alpha \in R_+$ we have $s_\alpha w < w$ iff $\alpha$ is in $\Pi(w^{-1})$;
2. $w' < w$ iff $w'$ can be obtained by omitting some factors in a fixed reduced decomposition of $w$;
3. if $w' \leq w$ then either $s_i w' \leq w$ or $s_i w' \leq s_i w$ (or both).

We can use the Bruhat order on $W$ to define a partial order on the weight lattice: if $\lambda, \mu \in P$ then by definition $\lambda < \mu$ iff $w_\lambda < w_\mu$.

Lemma 1.2. Let $\lambda$ be a weight such that $s_i \cdot \lambda \neq \lambda$ for some $0 \leq i \leq n$. Then $w_{s_i, \lambda} = s_i w_\lambda$.

Proof. Because $l(s_i w_\lambda) = l(w_\lambda) + 1$ and $l(s_i w_{s_i, \lambda}) = l(w_{s_i, \lambda}) + 1$ we have four possible situations depending on the choice of the signs in the above relations. The choice of a plus sign in both relations translates in $\alpha_i \notin \Pi(w_\lambda^{-1})$ and $\alpha_i \notin \Pi(w_{s_i, \lambda}^{-1})$ which by Lemma 1.1 and our hypothesis implies that $(\alpha_i, \lambda + \Lambda_0) > 0$ and $(\alpha_i, s_i \cdot \lambda + \Lambda_0) > 0$ (contradiction). The same argument shows that the choice of a minus sign in both relations is impossible. Now, we can suppose that $l(s_i w_\lambda) = l(w_\lambda) + 1$ and $l(s_i w_{s_i, \lambda}) = l(w_{s_i, \lambda}) - 1$, the other case being treated similarly. Using the minimal length properties of $w_\lambda$ and $w_{s_i, \lambda}$ we can write

$$l(w_\lambda) + 1 = l(s_i w_\lambda) \geq l(w_{s_i, \lambda}) = l(s_i w_{s_i, \lambda}) + 1 \geq l(w_\lambda) + 1$$

which shows that $l(s_i w_\lambda) = l(w_{s_i, \lambda})$. Our conclusion follows from the uniqueness of $w_{s_i, \lambda}$.

An immediate consequence is the following

Lemma 1.3. Let $\lambda$ be a weight such that $s_i \cdot \lambda \neq \lambda$ for some $0 \leq i \leq n$. Then $s_i \cdot \lambda > \lambda$ iff $(\alpha_i, \lambda + \Lambda_0) > 0$.

Lemma 1.4. Let $\lambda$ be a weight such that $s_i \cdot \lambda \neq \lambda$ for some $0 \leq i \leq n$. Then $w_{s_i, \lambda} = s_i \theta w_\lambda$ if $i \neq 0$ and $w_{s_0, \lambda} = s_0 \tilde{w}_\lambda$.

Proof. We can prove the statement for $i \neq 0$ with the same arguments as in Lemma 1.2. The remaining statement was essentially proved in Lemma 3.3 of [4].

Definition 1.5. If $\lambda$ and $\mu$ are weights such that $\lambda - \mu \in \hat{Q}$, we say that the weight $\nu$ is a convex combination of $\lambda$ and $\mu$ if $\nu = (1 - \tau)\lambda + \tau\mu$ such that $0 \leq \tau \leq 1$ and $\lambda - \nu \in \hat{Q}$.

The following result was proved in Lemma 5.5 of [3] for a particular affine Weyl group, but the proof provided there works in general.
Lemma 1.6. Let \( \lambda \) be a weight such that \( s_i \cdot \lambda \geq \lambda \) for some \( 0 \leq i \leq n \). If \( \nu \) is a convex combination of \( \lambda \) and \( s_i \cdot \lambda \), then \( \nu < \lambda < s_i \cdot \lambda \).

For each weight \( \lambda \) define \( \overline{\lambda} = \lambda + \hat{w}_\lambda(\rho) \). As a consequence of Lemma 1.4 we have the following

Proposition 1.7. Let \( \lambda \) be a weight such that \( s_i \cdot \lambda \neq \lambda \). Then

\[
    s_i \cdot \lambda = \overline{s_i \cdot \lambda}.
\]

1.3. Demazure modules characters. Recall that \( g \) is the Kac-Moody affine Lie algebra associated with the irreducible affine Cartan matrix \( A \). For the results in this section we refer to [Ku]. Let \( \Lambda \) be an integral dominant weight of \( g \) and let \( V = V(\Lambda) \) be the unique irreducible highest weight \( g \)-module with highest weight \( \Lambda \). For each \( w \in W \) the weight space \( V_w(\Lambda) \) is one-dimensional. Consider \( E_w(\Lambda) \), the \( b \)-module generated by \( V_w(\Lambda) \), where \( b \) is the the Borel subalgebra of \( g \). The \( E_w(\Lambda) \), called the Demazure modules, are finite dimensional vector spaces. If \( \lambda \) is an element of \( \mathcal{O} \), then \( \lambda + \Lambda_0 \) is a dominant weight. In such a case we write \( E_w(\lambda) \) for \( E_w(\lambda + \Lambda_0) \). To a Demazure module \( E_w(\Lambda) \) we can associate its character

\[
    \chi(E_w(\Lambda)) = \sum_{\text{weight}} \dim(E_w(\Lambda)_\gamma) \cdot e^\gamma
\]

which can be regarded as an element in \( P := C[q, q^{-1}][e^\mu; \mu \in P] \) after we ignore the \( e^{\Lambda_0} \) factor and after we set \( q = e^{-\delta} \).

Definition 1.8. For each \( 0 \leq i \leq n \) define an operator \( \Delta_i \) acting on \( P \)

\[
    \Delta_i e^\lambda = \frac{e^\lambda - e^{-\alpha_i}e^{s_i(\lambda)}}{1 - e^{-\alpha_i}}.
\]

Let \( w = s_{i_1} \cdots s_{i_j} \) be a reduced decomposition. Then, we can define \( \Delta_w := \Delta_{i_1} \cdots \Delta_{i_j} \) (the definition of \( \Delta_w \) does not depend on the choice of the reduced decomposition).

Theorem 1.9. Let \( \lambda \) be an element of \( \mathcal{O} \). Then

\[
    \chi(E_w(\lambda)) = \Delta_w(e^\lambda).
\]

The above Theorem is a special case of the Demazure character formula for Kac-Moody algebras, which has proved in full generality by S. Kumar and independently by O. Mathieu. We refer to Theorem 3.4 of [Ku] for the proof. The construction of the Demazure modules \( \hat{E}_w(\lambda) \) for the Lie algebra \( \hat{g} \) is completely analogous (the role of \( b \) is played here by \( \hat{b} \) the Borel subalgebra of \( \hat{g} \)).
2. Nonsymmetric Macdonald polynomials

In what follows we consider our root system to be reduced. Recall that in this case \( \alpha_0 = 1 \) and \( \theta \) is the highest short root of the associated finite root system. The case of a non-reduced root system will be treated in Section 3.

2.1. The double affine Hecke algebra. We introduce a field \( \mathbb{F} \) (of parameters) as follows: fix indeterminates \( q \) and \( t_0, \ldots, t_n \) such that \( t_i = t_j \) iff \( d_i = d_j \); let \( m \) be the lowest common denominator of the rational numbers \( \{(\alpha_i, \lambda_j) \mid 1 \leq i, j \leq n\} \), and let \( \mathbb{F} \) denote the field of rational functions in \( q^{1/m} \) and \( t_i^{1/2} \). Because in our case there are at most two different root lengths we will also use the notation \( t_l, t_s \) for \( t_i \) if the corresponding simple root is long, respectively short. The algebra \( R = \mathbb{F}[e^\lambda; \lambda \in P] \) is the group \( \mathbb{F} \)-algebra of the lattice \( P \) and \( S \) is the subalgebra of \( R \) consisting of elements invariant under the finite Weyl group. For further use we also introduce the following group \( \mathbb{F} \)-algebras of the root lattice:

\[ \mathcal{R}_Y := \mathbb{F}[y^\mu; \mu \in \hat{Q}] \]

and

\[ \mathcal{R}_X := \mathbb{F}[x^\beta; \beta \in \hat{Q}] \]. \( S_Y \) is the subalgebra of \( \mathcal{R}_Y \) consisting of elements invariant under the finite Weyl group.

**Definition 2.1.** The affine Hecke algebra \( \mathcal{H} \) is the \( \mathbb{F} \)-algebra generated by elements \( T_0, \ldots, T_n \) with relations

(i) the \( T_i \) satisfy the same braid relations as the \( s_i \);

(ii) for \( 0 \leq i \leq n \) we have

\[ T_i^2 = (t_i^{1/2} - t_i^{-1/2})T_i + 1. \]

The elements \( T_1, \ldots, T_n \) generate the finite Hecke algebra \( \mathcal{H} \). There are natural bases of \( \mathcal{H} \) and \( \mathcal{H}^\prime \): \( \{T_w\}_{w} \) indexed by \( w \) in \( W \) and in \( \hat{W} \) respectively, where \( T_w = T_{s_1} \cdots T_{s_l} \) if \( w = s_{i_l} \cdots s_{i_1} \) is a reduced expression of \( w \) in terms of simple reflections. There is another important description of the affine Hecke algebra due to Lusztig [L2].

**Proposition 2.2.** The affine Hecke algebra \( \mathcal{H} \) is generated by the finite Hecke algebra and the group algebra \( \mathcal{R}_Y \) such that the following relation is satisfied for any \( \mu \) in the root lattice and any \( 1 \leq i \leq n \):

\[ Y_\mu T_i - T_i Y_{s_i(\mu)} = (t_i^{1/2} - t_i^{-1/2}) \frac{Y_\mu - Y_{s_i(\mu)}}{1 - Y_{\alpha_i}}. \]

**Remark 2.3.** In this description \( T_0^{-1} = Y_\theta T_{s_\theta} \).

Following Macdonald [M], we call the family of commuting operators \( \mathcal{R}_Y \subset \mathcal{H} \), Cherednik operators. In order to state the next result we need the following notations: for \( \mu, \beta \in \hat{Q} \) and \( k \in \mathbb{Z} \), \( X_{\beta+k\delta} := q^{-k} X_\beta \) and \( Y_{\mu+k\delta} := q^k Y_\mu \). For the next results we refer to Cherednik [C1, C3].
Theorem 2.5. Define dominant weights and which is completely characterized by the equations following relation is satisfied for any \( \beta \) in the root lattice and any \( 0 \leq i \leq n \):

\[
T_i X_\beta - X_{s_i(\beta)} T_i = \left( t_i^{1/2} - t_i^{-1/2} \right) \frac{X_\beta - X_{s_i(\beta)}}{1 - X_{-\alpha_i}}.
\]

The following formulas define a faithful representation of the affine Hecke algebra \( \mathcal{H} \) on \( \mathcal{R} \):

\[
\pi(T_i)e^\lambda = t_i^{1/2}e^{s_i(\lambda)} + \left( t_i^{1/2} - t_i^{-1/2} \right) \frac{e^\lambda - e^{s_i(\lambda)}}{1 - e^{-\alpha_i}}, \quad 0 \leq i \leq n
\]

\[
\pi(X_\beta)e^\lambda = e^{\lambda+\beta}, \quad \beta \in \check{Q}.
\]

Theorem 2.5. Define \( T_{(0)} = T_0^{-1}X_{\alpha_0} \). Then for all \( \mu \in \check{Q} \) and all \( \lambda \in P \)

\[
Y_\mu T_{(0)} - T_{(0)} Y_{\delta_0(\mu)} = \left( t_0^{1/2} - t_0^{-1/2} \right) \frac{Y_\mu - Y_{\delta_0(\mu)}}{1 - Y_{\alpha_0}},
\]

\[
\pi(T_{(0)})e^\lambda = t_0^{1/2}e^{\delta_0(\lambda)} + \left( t_0^{1/2} - t_0^{-1/2} \right) \frac{e^\lambda - e^{\delta_0(\lambda)}}{1 - e^{-\alpha_0}}.
\]

The irreducible affine root systems for which the affine simple root is long are \( B_n^{(1)}, C_n^{(1)}, F_4^{(1)} \) and \( G_2^{(1)} \). For these root systems the formula of the element of the double affine Hecke algebra which plays the same role as \( Y \) which sends \( \lambda \) to \( \delta \) (see [C1] for details) for which all operators in \( \mathcal{H} \)) Macdonald defined a basis \( \{ P_\lambda(q,t) \} \) of \( \mathcal{S} \) which is indexed by anti-dominant weights and which is completely characterized by the equations

\[
f \cdot P_\lambda = f(\lambda) P_\lambda
\]

2.2. Macdonald polynomials. Cherednik defined a certain scalar product on \( \mathcal{R} \) (see [C1] for details) for which all operators in \( \mathcal{H} \) became unitary operators. In particular the adjoint of \( Y_\mu \) is \( Y_{-\mu} \). By \( q^{(\mu+k,\lambda)} \) we denote the element of \( \mathbb{F} \)

\[
q^{k+(\mu,\lambda)} \prod_{i=1}^n t_i^{-\langle \mu, \omega_\lambda(\lambda') \rangle}.
\]

For each \( \lambda \in P \) we can construct a \( \mathbb{F} \)-algebra morphism \( ev(\lambda) : \mathcal{R}_\lambda \to \mathbb{F} \), which sends \( Y_\mu \) to \( q^{(\mu,\lambda)} \). If \( f \) is an element of \( \mathcal{R}_\lambda \) we will write \( f(\lambda) \) for \( ev(\lambda)(f) \). Macdonald defined a basis \( \{ P_\lambda(q,t) \} \) of \( \mathcal{S} \) which is indexed by anti-dominant weights and which is completely characterized by the equations

\[
f \cdot P_\lambda = f(\lambda) P_\lambda
\]
for any \( f \in S_Y \), and the condition that the coefficient of \( e^\lambda \) in \( P_\lambda(q,t) \) is 1. The elements of this basis are called symmetric Macdonald polynomials.

Recently, a nonsymmetric version of the Macdonald polynomials was introduced by Opdam \([O]\) in the differential case, Macdonald \([M]\) (for \( t_i = q^k, \ k \in \mathbb{Z}_+ \)) and by Cherednik \([C2]\) in the general (reduced) case and some of their properties were studied. For each weight \( \lambda \) there is an unique element \( E_\lambda(q,t) \in R \) satisfying the conditions

\[
E_\lambda = e^\lambda + \text{lower terms; } \tag{2} \\
\langle E_\lambda, e^\mu \rangle = 0 \text{ for all } \mu < \lambda . \tag{3}
\]

They form a \( F \)-basis of \( R \) and they are the common eigenfunctions of the Cherednik operators. In what follows we will find an explicit recursion formula for the nonsymmetric Macdonald polynomials. In the course of doing that we will give a more transparent proof of their existence and uniqueness.

For all \( 0 \leq i \leq n \) let us introduce the following elements of \( \mathcal{H}^d \) called intertwiners

\[
I_i := T_i(1 - Y_{\alpha_i}) - (t_i^{1/2} - t_i^{-1/2}) .
\]

The intertwiners where first introduced by Knop and Sahi \([Kn], [KS], [S1]\) for \( GL_n \) and then by Cherednik \([C3]\) in the general (reduced) case. Their importance is the following: for any \( \mu \) in the root lattice we have

\[
Y_\mu I_i = I_i Y_{s_i(\mu)}. \tag{4}
\]

This easily follows from Proposition 2.2 and Theorem 2.5. The next results can be proved following closely the ideas in \([S2]\) where the non-reduced case was considered. For every weight \( \lambda \) define

\[
\mathcal{R}_\lambda = \{ f \in R \mid Y_\mu f = q^{(\mu,\lambda)} f \text{ for any } \mu \in M \}.
\]

**Theorem 2.7.** Let \( \lambda \) be a weight such that \( s_i \cdot \lambda \neq \lambda \). Then \( I_i : \mathcal{R}_\lambda \rightarrow \mathcal{R}_{s_i \cdot \lambda} \) is a linear isomorphism.

**Proof.** Let \( f \) be any element of \( \mathcal{R}_\lambda \). Using the intertwining relation (4) and the Proposition 1.7 we get

\[
Y_\mu (I_i f) = q^{(\mu,\lambda)} I_i f .
\]

Therefore, \( I_i f \) is an element of \( \mathcal{R}_{s_i \cdot \lambda} \). A short computation shows that

\[
I_i^2 = t_i + t_i^{-1} - (Y_{\alpha_i} + Y_{-\alpha_i}),
\]

therefore \( I_i^2 \) acts as a constant on \( \mathcal{R}_\lambda \). It is easy to see that our hypothesis implies that this constant is nonzero, showing that \( I_i^2 \) and consequently \( I_i \) is an isomorphism.

**Theorem 2.8.** The spaces \( \mathcal{R}_\lambda \) are one-dimensional.
Proof. The proof is very similar with the proof of the corresponding result (Theorem 6.1) in [S2]. The only difference is that we have to use the fact that \( O \) is a set of representatives for the orbits of the affine action of \( W \) on \( P \), and the fact that \( e^\lambda \) is in \( R_\lambda \) for \( \lambda \in O \). From the proof also follows that an element in \( R_\lambda \) is uniquely determined by the coefficient of \( e^\lambda \) in \( f \).

This result makes possible the following definition.

**Definition 2.9.** For any weight \( \lambda \) define the nonsymmetric Macdonald polynomial \( E_\lambda(q,t) \) to be the unique element in \( R_\lambda \) in which the coefficient of \( e^\lambda \) is 1. If \( k \in \mathbb{Z} \) then denote \( E_{\lambda+k\delta}(q,t) = q^{-k}E_\lambda(q,t) \).

Corollary 2.10. The polynomial \( P_\lambda(q,t) \) can be characterized as the unique \( \check{W} \)-invariant element in \( R_\lambda \) for which the coefficient of \( e^\lambda \) equals 1.

Proof. The result follows from the characterization (1).

**Definition 2.11.** Let \( C \) be the element of the finite Hecke algebra defined by
\[
C := \left( \sum_{w \in \check{W}} \chi(T_w) T_w \right) - \sum_{w \in \check{W}} \chi(T_w) T_w^2, \quad \chi(T_i) = \frac{t^{1/2}}{2^{i^2/2}},
\]
where \( \chi \) is the one dimensional representation of \( \check{H} \) defined by \( \chi(T_i) = t^{1/2} \).

Corollary 2.12. \( \pi(C) \) is a projection from \( R_\lambda \) to \( F P_\lambda \).

Proof. An easy calculation as in Lemma 2.5 of [S3] shows that \( T_i C = t_i^{1/2} C \) for any \( 1 \leq i \leq n \), hence \( T_i(C f) = t_i^{1/2} C f \) for all \( f \in R \). This implies that \( C f \) is \( \check{W} \)-invariant, and so it must be a multiple of \( P_\lambda \). Moreover, \( C \) acts as identity on \( S \).

For any weight \( \lambda \) and any \( 0 \leq i \leq n \) define the operator \( G_{i,\lambda}(q,t) \) as follows
\[
G_{i,\lambda} := t_i^{-1/2} T_i^{-1} \quad \text{if } (\lambda + \Lambda_0, \alpha_i) = 0, \quad \text{and}
G_{i,\lambda} := (1 - q^{-\alpha_i}) t_i^{-1/2} T_i^{-1} + q^{-\alpha_i} (1 - t_i^{-1}) \quad \text{if } (\lambda + \Lambda_0, \alpha_i) \neq 0.
\]

**Theorem 2.13.** Let \( \lambda \) be a weight such that \( (\lambda + \Lambda_0, \alpha_i) \geq 0 \). Then
\[
G_{i,\lambda} E_\lambda = (1 - q^{-\alpha_i}) E_{s_i(\lambda)}.
\]

Proof. When \( (\lambda + \Lambda_0, \alpha_i) = 0 \) the statement follows straightforward from [2], [3] and from the Theorem 2.6. For the remaining case, using Theorem 2.7 all we need is to compute the coefficient of \( e^{s_i(\lambda)} \) in \( G_{i,\lambda} E_\lambda \) which by Theorem 2.6 can be shown to be \( (1 - q^{-\alpha_i}) \).

Proof of Theorem 2.13.
2.3. The specialization at \( t = \infty \). Our goal is to define the specialization of the polynomials \( E_\lambda(q, t) \) at \( t = \infty \) (that means \( t^{-1} = 0 \)) and to obtain recursion formulas for them as in Theorem 2.13. In order to do this we have to closely examine the coefficients of the \( E_\lambda \) and make sure that their limit exists. In fact, we can suitably re-normalize the \( E_\lambda \) such that all the coefficients in this re-normalization are polynomials in \( t_i^{-1} \) and the normalizing factor approaches 1 when \( t \) tends to infinity. This will show that the limit of each of the coefficients of the \( E_\lambda \) exists and it is bounded.

Recall \( w_\lambda \) be the unique minimal length element of \( W \) such that \( w_\lambda \cdot \lambda = \lambda \). Let \( w_\lambda = s_{j_1} \cdots s_{j_l} \) be a reduced decomposition. Then,

\[
\Pi(w_\lambda) = \{ \alpha^{(i)} := s_{j_1} \cdots s_{j_{i-1}}(\alpha_{j_i}) \mid 1 \leq i \leq l \} .
\]

This means in particular that \( \alpha^{(j)} \in R_+ \) and \( w_\lambda(\alpha^{(j)}) \in R_- \). Define

\[
\lambda^{(i)} := s_{j_{i-1}} \cdots s_{j_1} \cdot \tilde{\lambda} ,
\]

for any \( 1 \leq i \leq l + 1 \). Therefore, \( \lambda^{(1)} = \tilde{\lambda} \) and \( \lambda^{(l+1)} = \lambda \). The key property of the \( \lambda^{(i)} \) is that

\[
(\lambda^{(i)} + \Lambda_0, \alpha_{j_i}) > 0 .
\]

This easily follows from (3). Moreover, (8) implies that \( \alpha_{j_i} \in \Pi(\tilde{\lambda}_\lambda^{(i)}) \) if \( j_i \neq 0 \), meaning that \( \tilde{\lambda}_\lambda^{(i)}(\alpha_{j_i}) \) is in \( \tilde{R}_- \), respectively that \( \theta \not\in \Pi(\tilde{\lambda}_\lambda^{(i)}) \) if \( j_i = 0 \), meaning that \( \tilde{\lambda}_\lambda^{(i)}(\theta) \) is in \( \tilde{R}_+ \).

Now, for all \( 1 \leq j \leq l \), all the exponents in the monomial \( q^{(\alpha_{j_i}, \lambda^{(i)})} \) are positive integers and at least one of the exponents the \( t_i \) is nonzero. Define the re-normalization of \( E_\lambda(q, t) \) to be

\[
\prod_{i=1}^{l} (1 - q^{- (\alpha_{j_i}, \lambda^{(i)})}) E_\lambda(q, t) .
\]

This formula (modulo a \( q \) factor) is obtained by applying the recursion formula (3) successively, starting with \( e^{\tilde{\lambda}} \). From this description it is clear that the powers of the \( t_i \) appearing the expansion of this re-normalization of \( E_\lambda(q, t) \) are all negative and therefore our desired specialization at \( t = \infty \) is well defined. We denote by \( E_\lambda(q, \infty) \) this specialization. This re-normalization does not depend on the choice of the reduced decomposition of \( w_\lambda \). Remark also that the coefficient of \( e^{\lambda} \) in \( E_\lambda(q, \infty) \) is 1. For each anti-dominant weight \( \lambda \) we write \( R^\lambda(\infty) \) for the linear subspace spanned by \( \{ E_\mu(q, \infty) \mid \mu \in W\lambda \} \). The polynomial \( P_\lambda(q, \infty) \) is defined to be the unique \( W \)-invariant element in \( R^\lambda(\infty) \) for which the coefficient of \( e^{\lambda} \) equals 1.

3. Nonsymmetric Koornwinder polynomials

In this section we will consider the case of a non-reduced root system. Recall that in this case \( A = A^2_{(2n)} \), \( a_0 = 2 \), \( \theta \) is the highest root and \( \mathcal{O} = \{0\} \).
3.1. The recursion relation. The results in this section are due to Sahi \cite{S2}, \cite{S3}. We introduce the field $\mathbb{F}$ as follows: fix indeterminates $q$, $u = (u_0, u_n)$ and $t_0, \cdots, t_n$ identified as before; the field $\mathbb{F}$ is the field of rational functions in their square roots. We also define

$$a = t_n^{1/2}u_n^{1/2}, \quad b = -t_n^{-1/2}u_n^{-1/2}, \quad c = q^{1/2}t_0^{1/2}u_0^{1/2}, \quad d = -q^{1/2}t_0^{-1/2}u_0^{-1/2}.$$ 

Note that in this case we have three different root lengths, therefore $t = (t_s, t_m, t_l)$, where $t_s = t_0$, $t_l = t_n$ and $t_m = t_i$ for any $i \neq 0, n$. As before $\mathcal{R} = \mathbb{F}[e^\lambda; \lambda \in P]$ is the group $\mathbb{F}$-algebra of the lattice $P$ and $S$ is the subalgebra of $\mathcal{R}$ consisting of elements invariant under the finite Weyl group. Also, define $\mathcal{R}_Y := \mathbb{F}[Y_{\mu}; \mu \in P]$ and $\mathcal{R}_X := \mathbb{F}[X_\beta; \beta \in P]$. $S_Y$ is the subalgebra of $\mathcal{R}_Y$ consisting of elements invariant under the finite Weyl group. The lattice $P$ can be identified with $\mathbb{Z}^n$ such that the scalar product we defined in Section 1.1 is the canonical scalar product on $\mathbb{R}^n$. If $\varepsilon_1, \cdots, \varepsilon_n$ are the unit vectors in $\mathbb{Z}^n$, then our choice of the basis for the affine root system is

$$\alpha_0 = \frac{1}{2}\delta + \varepsilon_1, \quad \alpha_i = -\varepsilon_i + \varepsilon_{i+1}, \quad \alpha_n = -2\varepsilon_n.$$ 

The double affine Hecke algebra in this case has a more complicated description (see \cite{S2} for details). We describe here only its action on $\mathcal{R}$:

- $T_0e^\lambda := t_0^{1/2}e^\lambda + t_0^{-1/2}\frac{1}{1-qe^{2\varepsilon_1}}(e^{s_0(\lambda)} - e^\lambda)$,
- $T_{(0)}e^\lambda := T_0^{-1}e^{\lambda_0}$,
- $T_{(i)}e^\lambda := t_i^{1/2}e^\lambda + t_i^{-1/2}\frac{1}{1-qe^{2\varepsilon_1}}(e^{s_i(\lambda)} - e^\lambda)$, $i \neq 0, n$,
- $T_{(n)}e^\lambda := t_n^{1/2}e^\lambda + t_n^{-1/2}\frac{1}{1-qe^{2\varepsilon_n}}(e^{s_n(\lambda)} - e^\lambda)$.

The commutative algebra $\mathcal{R}_Y$ embeds in the Hecke algebra as follows

$$Y_{\varepsilon_i} = (T_i \cdots T_{n-1})(T_n \cdots T_0)(T_1^{-1} \cdots T_{i-1}^{-1}).$$

The action of $\mathcal{R}_Y$ can be simultaneously diagonalized and the nonsymmetric Koornwinder polynomials $E_\lambda(q, t, u)$ are the corresponding eigenbasis. The eigenvalues are given as follows: by $q^{(\mu+\lambda, \lambda)}$ we denote the element of $\mathbb{F}$

$$q^{k+\mu, \lambda}(t_0t_n) - (\mu, \lambda)(\lambda)\prod_{i=1}^{n-1}t_i^{-1}(\mu, \lambda)(\lambda) = f \cdot P_\lambda = f(\lambda)P_\lambda \quad (9)$$

for any $f \in \mathcal{S}_Y$, and the condition that the coefficient of $e^\lambda$ in $P_\lambda(q, t, u)$ equals 1. In the same manner as is Section 2.2 we define for any weight $\lambda$ the vector spaces $\mathcal{R}_\lambda$ and $\mathcal{R}_\lambda^\lambda$. 
Proposition 3.1. The polynomial $P_{\lambda}(q, t, u)$ can be characterized as the unique $W$-invariant element in $R^\lambda$ for which the coefficient of $e^\lambda$ equals 1.

For any weight $\lambda$ and any $0 \leq i \leq n$ such that $(\lambda + \Lambda_0, \alpha_i) = 0$ define the operator $G_{i, \lambda}(q, t)$ as follows

$$G_{i, \lambda} := t_i^{-1/2} T_i.$$ 

If $(\lambda + \Lambda_0, \alpha_i) \neq 0$ we define

$$G_{i, \lambda} := (1 - q^{-(\alpha_i, \alpha_i)}) t_i^{-1/2} T_i + q^{-(\alpha_i, \alpha_i)} (1 - t_i^{-1}) \quad \text{for } i \neq 0 \quad \text{and}$$

$$G_{0, \lambda} := t_0^{-1/2} ((1 - q^{-(\delta, \delta)}) T_0 + q^{-(\alpha_0, \delta)} (u_n^{1/2} - u_n^{-1/2}) + (u_0^{1/2} - u_0^{-1/2})).$$

Theorem 3.2. Let $\lambda$ be a weight such that $(\lambda + \Lambda_0, \alpha_i) \geq 0$. Then

$$G_{i, \lambda} E_\lambda = (1 - q^{-(\alpha_i, \alpha_i) - \delta, i(\alpha_0, \delta)}) E_{s_i(\lambda)}.$$  \hspace{1cm} (10)

3.2. The specialization at $u = (t_0, 1)$, $t = \infty$. First, there is of course no problem in specializing $u_0 := t_0$ and $u_n = 1$. The problem will arise as in Section 2.3 when we want to specialize $t = \infty$. One can follow closely the argument in Section 2.3 to examine the coefficients of the $E_\lambda$. We will just state the corresponding result in this case. Recall that $w_\lambda$ is the unique minimal length element of $W$ such that $w_\lambda \cdot 0 = \lambda$, $\{\alpha^{(i)}\}$ and $\{\lambda^{(i)}\}$ elements defined as in equations (3) and (7).

Define the re-normalization of $E_\lambda(q, t, u)$ to be

$$\prod_{i=1}^l (1 - q^{-(\alpha_i, \lambda^{(i)}) - \delta, i(\alpha_0, \lambda^{(i)})}) E_\lambda(q, t, u).$$

This formula (modulo a $q$ factor) is obtained by applying the recursion formula (10) successively, starting with 1. The powers of the $t_i$ appearing in the expansion of this re-normalization after the substitution $u = (t_0, 1)$ are all negative and the normalizing factor tends to 1 when $t$ approaches infinity. Therefore our desired specialization at $t = \infty$ is well defined. We denote by $E_\lambda(q, \infty)$ this specialization. Note that the coefficient of $e^\lambda$ in $E_\lambda(q, \infty)$ equals 1. For each anti-dominant weight $\lambda$ we write $R^\lambda(\infty)$ for the linear subspace spanned by $\{E_\mu(q, \infty) \mid \mu \in \bar{W}\lambda\}$. The polynomial $P_\lambda(q, \infty)$ is defined to be the unique $W$-invariant element in $R^\lambda(\infty)$ for which the coefficient of $e^\lambda$ equals 1.

4. The representation-theoretical interpretation

In this section we make no more reference to reducibility of the root system in question, but depending on the case we use the notation $E_\lambda(q, \infty)$ to refer to the specialized versions of the nonsymmetric Macdonald polynomials or nonsymmetric Koornwinder polynomials.
4.1. **Proof of the Theorem** [1]. The strategy is to study the degeneration of the recursion formulas (3) and (4) for the polynomials $E_\lambda(q, \infty)$ and then to relate them with the Demazure character formula (Theorem 1.9). The crucial remark is that we are only interested in the action of the operators $G_{i,\lambda}(q, t)$ on the re-normalization of $E_\lambda(q, t)$ when $(\lambda + \Lambda_0, \alpha_i) \geq 0$. We see, after an examination of the operator $G_{i,\lambda}(q, t)$ in this situation, that the powers of $t$ appearing in the description of its action are negative or zero. Because the same is true for the re-normalization of $E_\lambda(q, t)$ we can first make the specialization at $t = \infty$. Moreover, the operators $G_{i,\lambda}(q, \infty)$ do not depend on $\lambda$ anymore. In fact, $G_{i,\lambda}(q, \infty)$ coincide with the Demazure operators $\Delta_i$. We are ready to state the following

**Theorem 4.1.** Let $\lambda$ be a weight such that $(\lambda + \Lambda_0, \alpha_i) \geq 0$. Then

$$\Delta_i E_\lambda(q, \infty) = q^{-(\Lambda_0, s_i(\lambda))} E_{s_i \lambda}(q, \infty).$$

**Proof.** The statement is obvious for $(\lambda + \Lambda_0, \alpha_i) = 0$. Now, we know from Lemma 1.3 that if $(\lambda + \Lambda_0, \alpha_i) > 0$ we have

$$l(w_{s_i \lambda}) = l(s_i w_\lambda) = l(w_\lambda) + 1.$$

Therefore, if $w_\lambda = s_{j_p} \cdots s_{j_1}$ is a reduced decomposition $w_{s_i \lambda} = s_i s_{j_p} \cdots s_{j_1}$ is also reduced. Henceforth, using the definition of $E_\lambda(q, \infty)$ and $E_{s_i \lambda}(q, \infty)$ and the recursion formulas (3), (4) our conclusion follows.

An immediate consequence of the Theorem 4.1 is that

$$\Delta_{w_\lambda} E_\lambda(q, \infty) = q^{-(\Lambda_0, w_\lambda(\lambda))} E_\lambda(q, \infty).$$

Theorem 4.2 follows by comparing this formula with the Theorem 1.9. A simple consequence of Theorem 4.1 is that if we expand $E_\lambda(q, \infty)$ in terms of monomials the coefficients that appear are polynomials in $q^{-1}$ with positive integer coefficients.

4.2. **Proof of the Theorem** [2]. Let us begin with a characterization of $P_\lambda(q, \infty)$. If $\lambda$ is anti-dominant, $(\lambda, \alpha_i) \leq 0$ and the Theorem 4.1 together with $\Delta^2_i = \Delta_i$ shows that

$$\Delta_i E_\lambda(q, \infty) = E_\lambda(q, \infty).$$

This immediately implies that $E_\lambda(q, \infty)$ is $\tilde{W}$-invariant.

**Theorem 4.2.** If $\lambda$ is an anti-dominant weight then

$$P_\lambda(q, \infty) = E_\lambda(q, \infty).$$

Now, because $P_\lambda(q, \infty)$ is essentially the character of the Demazure module $E_{w_\lambda}(\lambda)$ the $\tilde{W}$-invariance of $P_\lambda(q, \infty)$ translates into saying that $E_{w_\lambda}(\lambda)$ decomposes into a direct sum of simple $\hat{g}$-modules. Let us write

$$E_{w_\lambda}(\lambda) = \bigoplus_{j \geq 0} E_{w_\lambda}(\lambda)_j$$
where \( E_{w,\lambda}(\tilde{\lambda})_j \) is the direct sum of weight spaces whose weights are of the form \( \mu + j\delta + (\Lambda_0, w\lambda(\tilde{\lambda}))\delta \) with integer \( j \) and \( \mu \in P \). Since \( \delta \) is \( W \)-invariant each of the \( E_{w,\lambda}(\tilde{\lambda})_j \) decomposes as a direct sum of simple \( \mathfrak{g} \)-modules. If \( \chi_\mu \) is the character of \( V_\mu \) the irreducible \( \mathfrak{g} \)-module with highest weight \( \mu \)

\[
\chi(E_{w,\lambda}(\tilde{\lambda})_j) = q^{-(\Lambda_0, w\lambda(\tilde{\lambda})) - j} \sum_{\mu} c_{\lambda,\mu}^j \chi_\mu.
\]

Here \( c_{\lambda,\mu}^j \) is the multiplicity of \( V_\mu \) in \( E_{w,\lambda}(\tilde{\lambda})_j \). Summing up we find the polynomials in \( q^{-1} \) with positive integer coefficients such that

\[
P_\lambda(q, \infty) = \sum_{\mu \leq \lambda} d_{\lambda,\mu}(q) \chi_\mu.
\]

The restriction on the sum comes from the triangular properties of \( P_\lambda \). Let us remark that the positive integer numbers \( d_{\lambda,\mu}(1) \) are the multiplicities of the irreducible \( \mathfrak{g} \)-modules in the Demazure module \( E_{w,\lambda}(\tilde{\lambda}) \). Also, \( d_{\lambda,\lambda}(q) = 1 \).

### 4.3. Proof of the Theorem \[3\].

On one hand, because the coefficients of the expansion of \( E_\lambda(q, \infty) \) in terms of monomials are polynomials in \( q^{-1} \) with positive integer coefficients their limit at \( q \to \infty \) exists. We will denote by

\[
E_\lambda(\infty, \infty) = \lim_{q \to \infty} E_\lambda(q, \infty).
\]

On the other hand, using Theorem \[1\] we can see that

\[
E_\lambda(\infty, \infty) = \chi(E_{w,\lambda}(\tilde{\lambda})_0)
\]

where \( E_{w,\lambda}(\tilde{\lambda})_0 \) is the direct sum of weight spaces whose weights are of the form \( \mu + (\Lambda_0, w\lambda(\tilde{\lambda}))\delta \) with \( \mu \in P \). It can be easily seen that \( E_{w,\lambda}(\tilde{\lambda})_0 \) is a \( \mathfrak{b} \)-module. Our conclusion follows if we prove that \( E_{w,\lambda}(\tilde{\lambda})_0 \) is isomorphic to \( E_{w,\lambda,w}(\lambda_+) \) as \( \mathfrak{b} \)-modules. As explained in the proof of the Theorem \[2\] the vector space \( E_{w,\lambda_-}(\lambda_0) \) is also a \( \mathfrak{g} \)-module.

**Theorem 4.3.** The \( \mathfrak{g} \)-module \( E_{w,\lambda_-}(\lambda_0) \) is the irreducible representation of \( \mathfrak{g} \) with highest weight \( \lambda_- \). Furthermore, the \( \mathfrak{b} \)-modules \( E_{w,\lambda}(\tilde{\lambda})_0 \) and \( E_{w,\lambda,w}(\lambda_+) \) are isomorphic.

**Proof.** By the Theorem \[2\] we know that the irreducible representation of \( \mathfrak{g} \) with highest weight \( \lambda_+ \) occurs in the decomposition of \( E_{w,\lambda_-}(\lambda_0) \) with multiplicity one. Let us denote by \( \hat{V} \) the copy of the irreducible representation of \( \mathfrak{g} \) with highest weight \( \lambda_+ \) embedded in \( E_{w,\lambda_-}(\lambda_0) \) and by \( V \) the irreducible representation of \( \mathfrak{g} \) with highest weight \( \Lambda = \tilde{\lambda} + \Lambda_0 \). It is easy to see that \( E_{w,\lambda_-}(\lambda_0) \) is the \( \mathfrak{b} \)-module generated by the weight space \( V_{w(\Lambda)} \), where \( w = w^{-1}_{\lambda} w_{\lambda} \). From the fact that the space \( V_{w(\Lambda)} \) is one dimensional and from

\[
w(\Lambda) = \lambda_- + (\Lambda_0, w\lambda(\tilde{\lambda}))
\]
we deduce that $V_{w(\lambda)}$ is the lowest weight space of $\tilde{V}$, and therefore

$$\tilde{V} = E_{w_\lambda}(\tilde{\lambda})_0,$$

both being equal with the $\mathfrak{b}$-module generated by the weight space $V_{w(\lambda)}$. By the same argument the $\mathfrak{b}$-module $E_{w_\lambda}(\tilde{\lambda})_0$ is generated by the one-dimensional weight space

$$V_{w_\lambda}(\lambda) = \tilde{V}_{w_\lambda w_0(\lambda_+)}$$

which also generates $\tilde{E}_{\tilde{w}_\lambda w_0(\lambda_+)}$ as a $\mathfrak{b}$-module. Our conclusion follows. □

The proof of Theorem [3] is now complete.

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