Coupled $S = \frac{1}{2}$ Heisenberg antiferromagnetic chains in an effective staggered field

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We present a systematic study of coupled $S = 1/2$ Heisenberg antiferromagnetic chains in an effective staggered field. We investigate several effects of the staggered field in the higher (two or three) dimensional spin system analytically. In particular, in the case where the staggered field and the inter-chain interaction compete with each other, we predict, using mean-field theory, a characteristic phase transition. The spin-wave theory predicts that the behavior of the gaps induced by the staggered field is different between the competitive case and the non-competitive case. When the inter-chain interactions are sufficiently weak, we can improve the mean-field phase diagram by using chain mean-field theory and the analytical results of field theories. The ordered phase region predicted by the chain mean-field theory is substantially smaller than that by the mean-field theory.

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I. INTRODUCTION

The effects induced by magnetic fields in magnets have been a subject of theoretical research interest for a long time. In particular, recently the magnetization processes of various spin chains and ladders have been investigated intensively. Owing to the progress of the various experimental methods, there has been an increasing connection between the theories and the experiments. In such a context, one of the new attractive subjects in magnetism is the effects of a staggered magnetic field, namely, a magnetic field which changes direction alternatingly. While it may sound unrealistic, there exist at least three mechanisms generating the staggered fields in real magnets, as discussed in Refs. [1] and [2].

The first mechanism is due to the staggered gyromagnetic $(g)$ tensor, which can be present if the crystal structure is not translationally invariant. The staggered $g$ tensor $g_{\alpha\beta}^{st}$ is defined in the coupling between the spin and the external magnetic field (Zeeman term) as

$$\hat{H}_{\text{Zeeman}} = -\mu_B \sum_j H_\alpha [g_{\alpha\beta}^{st} + (-1)^j g_{\alpha\beta}^{st}] S_j^\beta,$$

where $\vec{H} = (H_x, H_y, H_z)$ is an applied uniform magnetic field and $S_j^\beta$ is the spin operator of the local magnetic moment. Here $H_\alpha g_{\alpha\beta}^{st}$ is nothing but an effective staggered field. In addition, the staggered field may also arise from the staggered Dzyaloshinskii-Moriya (DM) interaction

$$\hat{H}_{\text{DM}} = \sum_j (-1)^j \vec{D} \cdot (\vec{S}_j \times \vec{S}_{j+1}),$$

which can be present if the crystal symmetry is sufficiently low. It is shown in Refs. [1] and [2] that in the presence of the staggered DM interaction along the chain, an applied uniform field $\vec{H}$ also generates an effective staggered field $\vec{h} \propto \vec{D} \times \vec{H}$. Several quasi-one-dimensional Heisenberg antiferromagnets are now known to have the staggered field due to the above mechanisms. The well-known examples are Cu-benzoate, [PM·Cu(NO$_3$)$_2$·(H$_2$O)$_2$]$_n$ (PM=pyrimidine) and Yb$_4$As$_3$. All of these have low-symmetry crystal structures which allow a staggered $g$ tensor and a DM interaction along the chain. It is expected in Refs. [1,14] and [15] that the staggered field induces an excitation gap in the $S = 1/2$ Heisenberg antiferromagnetic (HAF) chain, which should be otherwise gapless. The excitation gap caused by the staggered field is indeed found in these materials. Moreover, low-temperature anomalies in physical quantities, such as the susceptibility and the electron spin resonance line-width, are also successfully explained as effects of the staggered field. Thus it is confirmed that they are described by an $S = 1/2$ HAF model with an effective staggered field.

There is another, rather different, mechanism to generate a staggered field. Let us suppose that the system consists of two sublattices, with a weak inter-lattice coupling and strong intra-lattice one. If one of the sublattices is Néel ordered, the inter-lattice coupling, as a mean field, could give an effective staggered field on the other sublattice. The realization of this scenario is in $R_2$BaNiO$_3$ where R is a magnetic rare earth, and the R-ion lattice provides a staggered field for Ni chains ($S = 1$). Actually all the materials discussed above are highly one-dimensional (1D). However, at lower temperature and lower energy, the inter-chain interaction will eventually be dominant. In addition, there are reports on a few materials [CuCl$_2$·2DMSO] (DMSO=dimethylsulphoxide) (Refs. [10,20,21] and BaCu$_2$(Si$_1$-$_x$Ge$_x$)$_2$O$_7$ (Refs. [22]) which seem to have an effective staggered field and also a relatively large inter-chain interaction. Therefore the work including the inter-chain interaction could be relevant for experiments.

Given these backgrounds, in the present paper, we would like to clarify the characteristic roles of staggered fields in higher dimensional spin systems. In this paper, we are concerned with dimensions higher than 1 but still realistic in condensed matter physics, namely, two or three dimensions. However, most of the analyses in this paper apply straightforwardly even to four dimensions or
higher.

Varieties of spin models with effective staggered fields are conceivable. As a simplest model including both the staggered field and the inter-chain coupling, in this paper, we concentrate on the following $S = 1/2$ spatially anisotropic Heisenberg Hamiltonian

$$
\hat{H} = \sum_{\vec{r}} \left( J \hat{S}_{i,j,k} \cdot \hat{S}_{i+1,j,k} + J_{\perp} \hat{S}_{i,j,k} \cdot \hat{S}_{i,j+1,k} \right) + J'_\perp \hat{S}_{i,j,k} \cdot \hat{S}_{i,j,k+1} - H \sum_{\vec{r}} \hat{S}_{i,j,k}^2 - h \sum_{\vec{r}} (-1)^i \hat{S}_{i,j,k}^z,
$$

where $\hat{S}_{i,j,k}$ is the spin $1/2$ operator on $\vec{r} = (i, j, k)$ site. The coupling constants are restricted to $J > |J_{\perp}| \geq |J'_\perp|$, and thus the $i$ direction is the strongly antiferromagnetic (AF) coupled one. The system with $J'_\perp = 0$ is 2D, in which the index $k$ vanishes. The last two terms represent the uniform and staggered Zeeman terms respectively.

In our model, one can immediately find that when the inter-chain interactions are AF, they compete with the staggered Zeeman energy, while in the ferromagnetic (FM) case, both the interactions and the staggered field $h$ jointly make a Néel state stable. Let us refer the AF case as the competitive case, and the FM case as the non-competitive one. As will be explained later on, we predict that the competition brings a second-order phase transition in the competitive case. Its emergence is one of the most characteristic effects of the staggered field in our higher dimensional spin model.

The rest of this paper is organized as follows. In Sec. II we apply mean-field theory to the model. In the competitive case a phase transition is predicted. Since we are primarily interested in the transition, which is characteristic for the higher dimensional system, in the later sections we will mainly discuss the competitive case. The non-competitive case is touched as a comparison to the competitive case. Besides the phase diagrams, the mean-field magnetization curves and critical exponents are derived from the self-consistent equations. In Sec. III using linear spin-wave approximation, we derive the spin-wave dispersions in the the competitive case and in the non-competitive case. As a result, we find that the excitation gap induced by the staggered field behaves differently between the competitive case and the non-competitive case.

In Sec. IV we improve the mean-field phase diagrams by using chain mean-field theory. The latter is expected to be superior, when the inter-chain interactions are weak and the effective 1D model can be solved exactly. The improved diagram shows that in the weakly coupling region of the competitive case, the ordered phase becomes much narrower than the mean-field prediction. In the last section, we summarize those results and discuss future problems. In Appendix, the details of the spin-wave results are given.

II. MEAN-FIELD THEORY APPROACH

In this section, we treat the model within mean-field theory (MFT) framework. We first discuss the competitive case, and then touch the non-competitive case briefly.

In the competitive case, considering the advantage of both the inter-chain energy and the staggered Zeeman energy as well as the intra-chain coupling $J$, we can expect that the spin moment turns as Fig. 1 at sufficiently low temperature and small fields comparable to the inter-chain couplings. The assumed spin moment is

$$
\langle \hat{S}_{i,j,k} \rangle_{\text{MFT}} = \left( \langle -1 \rangle^i m_x, \langle -1 \rangle^i + j + k m_y, m_z \right). \quad (4)
$$

The choice of the mean field is presumably valid as long as $J$ is sufficiently larger than $J_{\perp}$ and $J'_\perp$. On the other hand, for instance, in the limiting case: $J \to 0$, where the model is reduced to a 2D or 1D AF one with a uniform field, there are possibilities that $\langle \hat{S}_{i,j,k}^z \rangle$ are inhomogeneous along the $j$ or $k$ directions, and thus Eq. (4) is invalid. In this paper, the MFT in the competitive case is performed only within the mean field.

The minimal condition for the mean-field free energy gives the following self-consistent equations:

$$
\begin{align*}
    m_x &= \frac{\epsilon_x}{2\epsilon} \tanh(\beta \epsilon), \\
    m^2 &= \frac{1}{4} \tanh^2(\beta \epsilon),
\end{align*}
$$

FIG. 1: Directions of the magnetic fields and the spin moments in the 2D competitive case. The short black arrows are the spin moments projected onto spin $xy$ plane in the ordered phase (spontaneous symmetry breaking phase, see text) expected by the MFT. The gray arrows indicate the direction of the staggered field $h$. The uniform field $H$ is applied perpendicular to this paper.
and $\beta$ and $m$ are, respectively, the inverse temperature $1/k_B T$ and the total magnetization per site. The numerical solutions of Eq. (5b) are given in Fig. 2. They indicate that there is a second-order phase transition, and the corresponding order parameter is the $y$ component of the spin moment $m_y = |\langle S^y_{i,j,k} \rangle|$. Going back to Fig. 1

one sees that the phase with finite $m_y$ breaks the translational symmetry in the weakly coupled direction. In the following, we call this phase as the SSB (spontaneous symmetry breaking) phase. The other phase, in which the spins are aligned to the field with $m_y = 0$, will be called as the symmetric phase. We emphasize that the transition between these two phases occurs only in the high dimensional spin systems and in the presence of the competition. From these results, we can illustrate the variation of the spin moment when the staggered field $h$ is increased gradually at small $T$ and $H$ with Fig. 3.

FIG. 2: Magnetization curves of the competitive case for $(J, J_\perp, J'_\perp, H) = (1, 0.2, 0.2)$. These are obtained by the numerical iterative method for Eq. (5b). The upper two parts (a) and (b) are in $h = 0.05$ and $h = 0.35$, respectively. The lower two parts (c) and (d) are in $k_B T = 0.3$ and $k_B T = 0.7$, respectively.

In the SSB phase, the relations $m_x = \frac{h}{4(J + J_\perp)}$ and $m_z = \frac{H}{4(J + J_\perp)}$ hold within the MFT. Inserting these into Eq. (5b) and taking the limit $m_y \rightarrow 0$, we obtain the mean-field critical surface in the space $(k_B T, H, h)$:

$$\tilde{h}_c^2 + \tilde{H}_c^2 = \frac{1}{4} \tanh^2 \left\{ \beta_c (J + J_\perp) \sqrt{\tilde{h}_c^2 + \tilde{H}_c^2} \right\},$$

where $\tilde{h}_c = \frac{h_c}{4(J + J'_\perp)}$, $\tilde{H}_c = \frac{H_c}{4(J + J'_\perp)}$ and the subscript $c$ represents critical values. It can be simplified in the cases $T = 0$, $h = 0$, and $H = 0$, respectively, as

$$\begin{cases}
\tilde{h}_c^2 + \tilde{H}_c^2 = \frac{1}{4} \\
\tilde{H}_c = \frac{1}{2} \tanh \left( \frac{\beta_c H_c}{4} \right) \\
\tilde{h}_c = \frac{1}{2} \tanh \left( \frac{\beta_c J_\perp + J'_\perp}{4} \right).
\end{cases}$$

Thus the mean-field phase diagram can be represented as Fig. 4. Using the critical condition, one can calculate some critical exponents within the MFT. Near the critical surface in the SSB side, the order parameter, the off-diagonal uniform and staggered susceptibilities: $\chi_u \equiv$
\[ \frac{\partial m_y}{\partial T} \text{ and } \chi_y \equiv \frac{\partial m_y}{\partial h} \text{ behave , respectively, as } m_y \sim (A - A)^\beta, \chi_y \sim -(A - A)^\gamma \text{ and } \chi_x \sim -(A - A)^\gamma \text{ where } A \text{ stands for } T, H \text{ or } h. \] The critical exponent \( \beta \) is found to be the conventional mean-field value 1/2. On the other hand, both \( \gamma \) and \( \gamma' \) turns out to be 1/2, which is different from the standard MFT result 1. This is because \( m_y \) is perpendicular to \( H \) and \( h \), and thus the latter are not the conjugate field as in the standard case. The mean-field energy per site is given as

\[ e_{\text{MFT}} = \left\{ J(m_y^2 + m_y^2 - m_z^2) + Hm_z + hm_x \right\}
+(J_\perp + J'_\perp)(m_y^2 - m_x^2 - m_z^2), \tag{9} \]

From this, one can easily confirm that the critical exponent for the specific heat is zero.

Now, we turn to the non-competitive case. Because of no competitions, no singular phenomena occur when \( h \neq 0 \). Canting of the spins in \( zz \) plane lowers both the inter-chain interactions and the Zeeman energies. Thus the expectation value of the spin moments can be put as

\[ \langle \vec{S}_{i,j,k} \rangle_{\text{MFT}} = ( (-1)^i m_x, 0, m_z ). \tag{10} \]

The MFT in this case gives the self-consistent equations

\[ m_{l}(z) = (\epsilon_x(z)/2e') \tanh(\beta \epsilon') \text{ where } \epsilon_x(z) \equiv \{-J + |J_\perp| + |J'_\perp| \} m_{l}(z) + h(H)/2 \text{ and } \epsilon' \equiv (\epsilon_x^2 + \epsilon_y^2)^{1/2}. \]

At \( h = 0 \), the system reduces to a conventional AF magnet in a uniform magnetic field. Hence, there must be a phase transition which divides the AF and paramagnetic phases, characterized by the order parameter \( m_z \). The critical line is given by

\[ \frac{H_c}{4J} = \frac{1}{2} \tanh \left\{ \beta \epsilon(J + |J_\perp| + |J'_\perp|) \frac{H}{4J} \right\}. \tag{11} \]

The phase diagram and the variation of the spin moment are drawn in Fig. 5. In the three-dimensional parameter space \((k_BT, H, h)\), the AF phase gives a first-order phase transition plane.

### III. LINEAR SPIN-WAVE APPROXIMATION IN \( T = 0 \)

With the MFT described in the preceding section, we investigate the effects of the quantum fluctuations in both the competitive and the non-competitive cases, at \( T = 0 \). The standard linear spin-wave approximation, based on the Holstein-Primakoff transformation (HPT) (Ref. 25) is employed. The detailed results are given in Appendix A.

First we discuss the SSB phase of the competitive case, which is the main subject. In the HPT, we replace the spin operator with a boson annihilation (creation) operator

\[ H \rightarrow \sum_{i,j,k} \vec{S}_{i,j,k} \rightarrow R_x \left( (1)^{i+j+k} \theta \right) \times \left( 1^{i+j+k} \phi \right) \left( 1^{i+j+k} \phi \right) \]
For the resulting quadratic Hamiltonian $\hat{H}_{\text{HP}}$, we perform a Fourier transformation (FT) $c_i^{\dagger} \equiv N^{-d/2} \sum_{\vec{k}} e^{i\vec{k}\cdot\vec{r}}\tilde{c}_{\vec{k}}^{\dagger}$, where $N$ is the linear system size, $d$ is the dimension of the system, $k_{\alpha} (|k_{\alpha}| < \pi/a_{\alpha})$ and $a_{\alpha}$ are, respectively, the wave-number and the lattice constant for the $\alpha$-direction. A four-mode Bogoliubov transformation (BT) [27] which mixes $\tilde{c}_{\vec{k}}, \tilde{c}_{-\vec{k}}, \tilde{c}_{\vec{k}-\pi}^\dagger, \tilde{c}_{-\vec{k}+\pi}$ and their Hermitian conjugates ($\vec{\pi} \equiv (0, \pi/a_y, \pi/a_z)$), leads to the diagonalized form

$$\hat{H}_{\text{HP}} = \sum_{\vec{k}} \omega(\vec{k}) \tilde{c}_{\vec{k}}^{\dagger} \tilde{c}_{\vec{k}} + \text{const}, \quad (16)$$

with a single band in the first Brillouin zone, where $\tilde{c}_{\vec{k}}$ is the magnon annihilation operator and the $k_x$ direction corresponds to the strongly AF one. The explicit results on the dispersion $\omega(\vec{k})$ are lengthy and thus are given in Appendix A. Here we discuss physical implications of our results.

First, the obtained dispersion satisfies $\omega(\vec{k}) \geq 0$ for all values of parameters. This implies that the SSB phase, which appears as the classical ground state, is stable against quantum fluctuations, at least in the lowest order of $1/S$. Some representatives of the dispersion are shown in Fig. 6. At zero field ($H = h = 0$), there are two gapless points $\vec{k}_h = (0, 0, 0)$ and $\vec{k}_H = (0, \pi/a_x, 0)$, with linear dispersions in the neighborhoods. Let us define $\Delta_h = \omega(\vec{k}_h)$ and $\Delta_H = \omega(\vec{k}_H)$. Since our results in Appendix A indicates that $\Delta_h (\Delta_H)$ is non-vanishing only when $h \neq 0$ ($H \neq 0$), we call it as $h$-($H$)-induced gap. The true excitation gap, namely, the minimum excitation energy, is given by $\Delta = \min(\Delta_h, \Delta_H)$. Thus the gap $\Delta$ vanishes exactly as long as either $h$ or $H$ remains zero. This is contrasting to the $S = 1/2$ 1D HAF model, where the staggered field alone induces the gap [9]. The gapless excitations are identified as Nambu-Goldstone (NG) modes. Indeed, when either $H$ or $h$ is zero, the Hamiltonian has a continuous $U(1)$ symmetry, which is broken spontaneously in the SSB phase.

In the non-competitive case, the magnon dispersion relation $\omega(\vec{k})$ is given in Eq. (9) in Appendix A. Now the staggered field alone can open the gap, because the ground state does not break any continuous symmetry spontaneously. On the other hand, the system remains gapless at $\vec{k} = 0$ due to the NG mechanism if $h = 0$ and $H$ is not too large. The $h$-induced gap is thus defined as $\Delta_h = \omega(\vec{k} = 0)$.

While the $h$-induced gap is defined for both cases, there is a characteristic difference in the $h$ dependence of the gaps. In the limit of small $h$, $\Delta_h \sim h^{1/2}$ for the non-competitive case, but $\Delta_h \sim h$ for the competitive case. In other words, one can say that the $h$-induced gap in the competitive case opens more slowly than in the non-competitive case. This is naturally understood because the competition between $J_\perp$ (or $J'_\perp$) and $h$ weakens the effect of the external symmetry breaking by $h$. It also implies that the ground state is more stable in the non-competitive case, against quantum and thermal fluctuations. The opening gaps are drawn in Fig. 8. On the other hand, in the limit of small $H$, $\Delta_H \sim H$. Similarly to the case of $\Delta_h$, this may be interpreted as a result of the competition between the uniform field and the AF couplings.

We expect the spin-wave theory for the gaps to be qualitatively correct even for $S = 1/2$. As discussed in Appendix A, the spin-wave dispersion of the symmetric phase can be obtained by the replacement $(J_{\perp}, J'_{\perp}) \rightarrow (-J_{\perp}, -J'_{\perp})$ in the dispersion of the non-competitive case.

Finally let us discuss the relation of the present results to the spin-wave theory in the 1D model ($J_{\perp} = J'_{\perp} = 0$) with the staggered field $h$ discussed in Ref. [28]. The spin-wave dispersion $\omega(\vec{k})$ for the non-competitive case does approach smoothly to the 1D dispersion [Eq. (3.11) in Ref. [28], when $J_{\perp}, J'_{\perp} \rightarrow 0$. On the other hand, the dispersion $\omega(\vec{k})$ for the competitive case apparently does not reduce to the 1D one in the limit of $J_{\perp}, J'_{\perp} \rightarrow 0$. This is because $\omega(\vec{k})$ is the dispersion of the magnon excitation.
tion in the SSB phase, which is absent in the 1D model. In fact, for any finite $h$, the SSB phase is realized [and hence $\omega(\vec{k})$ is applicable] only when $J_\perp$ and $J'_\perp$ are above the critical values. Thus by decreasing $h$, the system undergoes a phase transition into the symmetric phase. At the transition the dispersion should also change drastically. In the symmetric phase, similarly to the non-competitive case, the dispersion does approach continuously to the 1D dispersion.

IV. CHAIN MEAN-FIELD THEORY APPROACH

In this section, we reconstruct the phase diagram of our model using chain mean-field theory (CMFT). In the CMFT, weak couplings among the chains are treated with a MFT, and the resulting effective 1D problem is analyzed as precisely as possible. If the 1D problem can be treated exactly, the CMFT is expected to be much reliable when the one dimensionality is strong enough as in the case of Cu-benzoate, since it includes the fluctuations in the strongly coupled direction correctly. The usefulness of the CMFT has been demonstrated in several applications.

In Sec. IV A we discuss how the CMFT determines the phase transition for our model. Sec. IV B is a brief overview of susceptibilities of the $S = 1/2$ HAF chain which are necessary to the CMFT. In Sec. IV C we present the CMFT phase diagrams and compare them to the MFT ones.

A. CMFT for Our Model

Let us derive the effective 1D model for our system, within the CMFT.

In the competitive case, we consider the symmetric phase side for convenience. The mean-field procedure for the weak inter-chain couplings replaces them with the effective external fields. Thus the resulting Hamiltonian is

$$
\hat{H} \rightarrow \sum_{j,k} \hat{H}_{j,k} + N^d(J_\perp + J'_\perp)(m_y^2 - m_z^2),
$$

$$
\hat{H}_{j,k} = \sum_i J S_{i,j,k} \cdot S_{i+1,j,k}
$$

$$
- (-1)^{i+j+k} \{ h + 2(R_1^y) |\chi_j^H(H_\perp, h, 0)|^2 \}
$$

$$
- \{ h - 2(R_1^y) |\chi_j^H(H_\perp, h, 0)|^2 \},
$$

where we introduced an infinitesimal staggered field $(-1)^{i+j+k} h'$ parallel to the order parameter. The CMFT requires that the mean fields $m_{x,y,z}$ be treated exactly, the CMFT is expected to be much reliable when the one dimensionality is strong enough as the critical values. Thus by decreasing $h$, the system undergoes a phase transition into the symmetric phase. At the transition the dispersion should also change drastically. In the symmetric phase, similarly to the non-competitive case, the dispersion does approach continuously to the 1D dispersion.

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\[ \chi^{1D}_s(T, H_z, h'_x) \equiv \frac{\partial m'_x}{\partial m'_x} \] where \( m'_{x,z} = |\langle S'^{x,z}_i \rangle| \). Within the linear-response theory, Eq. (21) is reduced to

\[ \chi^{1D}_s(T, H_z, h'_x) \equiv \cos \delta = \cos \delta m_x - \sin \delta m_y. \] (22)

Let us recall that \( \delta \) is defined by \( m_x \) and \( m_y \), and that \( m_{x,z} \) are determined by the CMFT. Consequently, \( m_y \) can be determined as a solution to Eq. (22). The condition that \( m_y \) diverges would determine the phase transition.

### B. Susceptibilities of \( S = 1/2 \) AF chains

In order to solve Eqs. (19) or (22) in terms of \( m_y \), we need the explicit forms of the susceptibilities: \( \chi^{1D}_s \) and \( \chi^{1D}_u \). Here we briefly summarize the known results on these quantities, obtained by the bosonization technique.

In the absence of the staggered field \( h'_x \), the low-energy effective theory of the Heisenberg chain (20) is given by a free boson field theory, and the uniform susceptibility at zero temperature is obtained as

\[ \chi^{1D}_u(T, H, 0) \approx \frac{a}{(2\pi)^2 R(H)^2 c(H)}, \] (23)

where \( H_z = H \), and \( R \) and \( v \), respectively, are the compactification radius of the effective boson field theory, and the spin-wave velocity. The exact values \( v(H) \) and \( R(H) \) as functions of the uniform field \( H \) are given by a solution of a set of the Bethe ansatz integral equations. In the case of \( H = 0 \), they are explicitly given as

\[ v = \pi J a/2, \quad R = 1/\sqrt{2\pi}. \] (24)

For a small uniform field \( H_z \) \( \ll J \), the asymptotic behavior of the radius \( R \) follows

\[ 2\pi R^2 \approx 1 - \frac{1}{2 \ln(J/H)}. \] (25)

The logarithmic temperature correction due to the marginal operator of the HAF chain is discussed in Ref. 15. Although the transverse staggered field \( h'_z \) induces a gap, it is expected to have little effect on the uniform susceptibility \( \chi^{1D}_u \) if \( h'_z \) is small enough.

Next, we turn to the staggered susceptibility \( \chi^{1D}_s \). In the case \( J \gg h'_z \), \( k_B T \), the low-energy effective theory of the Hamiltonian (20) is given by a quantum sine-Gordon field theory. Using the exact solutions of the HAF chain, the scaling arguments and Lukyanov-Zamolodchikov prediction, the staggered susceptibility \( \chi^{1D}_s \) for small \( H_z = H \) and \( h'_x = h \) at \( T = \) is given as

\[ \chi^{1D}_s \approx \begin{cases} \frac{D(1 - 2\pi R^2)^{1/3}}{\pi R^2} & \text{if } J \gg h \approx 0) \\ \frac{D}{\ln(J/h)} & \text{if } J \gg h \approx 0) \end{cases} \] (26)

where \( R = \) is that of the model without the staggered field, and \( D \approx 0.3868 \ldots \). The formula is actually valid for \( H \leq 2J \) (below saturation field), but the second is only so for \( H \ll J \). These formulas are correct in \( k_B T \ll h \ll J \). (More precisely, within \( k_B T \ll \Delta_h \ll J \) where \( \Delta_h \) is the \( h \)-induced gap in the HAF chain.)

In the intermediate temperature regime \( h, H \ll k_B T \ll J \), where the temperature is larger than the induced gap, the staggered susceptibility may be approximated by that for zero staggered field as

\[ \chi^{1D}_s(k_B T \gg h) \approx D \left[ \ln(J/k_B T) \right]^{1/2}. \] (27)

where \( D \approx 0.2779 \ldots \)

### C. Phase diagrams in CMFT

Employing the results of Secs. IV A and IV B, we study the phase diagrams, in particular for the competitive case, within the CMFT. Unfortunately, it can be applied only to several limited regions in the parameter space, where the susceptibilities of the 1D model are obtained exactly.

First, let us consider the region near \( (k_B T_c, 0, 0) \). In the zero-field case, the effective model \( H_{1D} \) has only an infinitesimal staggered mean field \( h'_x \) as long as it is in the symmetric phase. Therefore the formula (27) can give the self-consistent value of \( m_y \), and Eq. (19) is the critical condition which serves the critical temperature:

\[ k_B T_c = 2D (J_\perp + J_\parallel) \left[ \ln \left( \frac{J}{k_B T_c} \right) \right]^{1/2}. \] (28)

which is reasonable when \( k_B T_c, (J_\perp + J_\parallel) \ll J \) and
agrees with the result of Ref. 24. Equation (28) is also
occurs.
Similarly to Fig. 9, the narrowing of the SSB areas
(a) and (b) are, respectively, predicted within the CMFT
and the MFT. It is confirmed that the SSB areas of the CMFT
are considerably smaller than the MFT.

Combining Eqs. (30) and (31), we obtain the critical staggered field:

$$h_c = D'\left[\frac{J_\perp + J'_\perp}{J_\perp} \ln \left(\frac{\frac{5}{2} J_\perp}{m_x}\right)\right]^{1/2} + 2(J_\perp + J'_\perp)$$

where we kept only the leading order of $(J_\perp + J'_\perp)/J$ and
$D' \equiv (2 \times 10^2 D^3 e^{-6})^{1/2} \approx 7.27 \cdots \times 10^{-2}$. This is valid
if $h_c < (J_\perp + J'_\perp) \ll J$, and is compared to the MFT result
$h_c = 2(J_\perp + J'_\perp)$ extracted from Eq. (8) in Fig. 10. The
CMFT correction to the MFT is found as a significant multiplicative factor $D'[\cdots]^{1/2}$.

We consider furthermore the critical line in the $T = 0$
plane in the limit of $h_c \ll H_c \ll J$. In the symmetric phase
near this line, the effective model $\tilde{H}_{1D}$ has three kinds of the external fields. Hence $m_x$
and $m_z$ must be determined concurrently by the CMFT
scheme. However in the present case $h_c \ll H_c$, $m_x$
may be estimated independently by taking an approximation
$\chi_1^{1D}(0, H, h) \approx \chi_1^{1D}(0, H, 0)$ which was given in Eq. (25).
From this approximation, $m_z$ is also fixed as

$$m_z \approx \frac{H}{\pi^2 J(1 - [\ln(J/H)]^{-1}) + 2(J_\perp + J'_\perp)}.$$  (33)

which is justified in $H, (J_\perp + J'_\perp) \ll J$. In the sufficiently small field case, $J \gg H(\gg h)$, the logarithmic part
may be dropped as well. From the second of Eq. (28) and
Eqs. (30), (31) and (32), we obtain the critical line in
$T = 0$

$$h_c = D''\left[\frac{J_\perp + J'_\perp}{J_\perp} \ln \left(\frac{\pi^2 J(1 - [\ln(J/H)]^{-1}) + 2(J_\perp + J'_\perp)}{\pi^2 h_c}\right)^{1/2} + \cdots\right]^{1/2} 2(J_\perp + J'_\perp),$$  (34)

where $D'' \equiv (10^2 D^3 e^{-3d})^{1/2} \approx 5.40 \cdots \times 10^{-2}$. This condition is presumably suitable for $k_BT_c \ll h_c < H_c \ll J$.
Thus, in order to obtain a more precise condition, it
is necessary to adopt the first formula (28) and the exact
values of $R$ and $v$. Figure 11 shows the comparison
between this line and the mean-field prediction.

Finally, we consider the point $(0, H_c, 0)$, where $\tilde{H}_{1D}$
has only the uniform field $H_z = H_c - 2(J_\perp + J'_\perp)m_z$
except for the infinitesimal field $h_c$. It has been known
from Bethe ansatz that the magnetization saturates at the
point $H_z = 2J$ at $T = 0$ in the HAF chain having
only the field $H_z$. The transition point between the SSB
and symmetric phase in this case should be identified with
the saturation of the uniform magnetization. Hence,
within the CMFT, the critical uniform field is given as

$$H_c = 2J + (J_\perp + J'_\perp).$$  (35)

The substitution $J_\perp + J'_\perp \rightarrow -(|J_\perp| + |J'_\perp|)$ gives the critical field of the AF-paramagnetic transition in the
non-competitive case.
From these results, we can compare the CMFT and the MFT phase diagrams. The comparison for the competitive case is summarized in Fig. 12. In the case of weak inter-chain couplings ($J \gg J_{\perp}, J_{\perp}^*$), the SSB phase of the CMFT is much smaller than one of the MFT. Especially there is a significant narrowing in $k_B T$ and $h$ directions. As seen from Eq. 15, this is because the phase transition in the CMFT framework is driven by the divergence of the susceptibility (in the present case, $\chi^{1D}_{s}$) in the effective chain, while temperature and the field $h_x$ (or $m_x$) suppress the divergence. On the other hand, the reduction of the critical uniform field $H_c$ is small in the weakly coupled case. This is because the uniform field competes with the intra-chain AF interaction as well as with the inter-chain interactions.

Finally we review the validity of the two theories. Since the strong one dimensionality is the basis of the CMFT procedure, it is expected that the CMFT is more reliable in the limit of $J \gg J_{\perp}, J_{\perp}^*$. On the other hand, when $J_{\perp}$ and $J_{\perp}^*$ are comparable to $J$, the special treatment of only one direction is unjustified. Therefore the MFT, which treats all couplings equally, is more reasonable for $J \sim J_{\perp}, J_{\perp}^*$. 

V. SUMMARY AND DISCUSSION

We considered the effects of the staggered field $h$ in an $S = 1/2$ Heisenberg antiferromagnet in two or three dimensions. The system behaves quite differently depending on whether the staggered field and the inter-chain couplings are competitive or not. In the competitive case, the appearance of a characteristic ordered (SSB) phase is predicted by the MFT. The SSB phase breaks the translational symmetry of the weakly coupled direction, and therefore it is peculiar to high dimensional systems. We also applied the CMFT to the model, and predicted that the region of the SSB phase becomes narrow in the CMFT scheme. The MFT and the CMFT are valid, respectively, in $J \sim J_{\perp}, J_{\perp}^*$ and in $J \gg J_{\perp}, J_{\perp}^*$. The crossover behavior between these two regions can not be described by the mean-field type approach. It would require a more precise treatment of fluctuations.

Moreover we studied spin-wave theory in both the competitive and non-competitive cases at $T = 0$. When the uniform field $H$ is non-vanishing, the $h$-induced gap opens as $\Delta_h \sim h$ in the SSB phase, while $\Delta_h \sim h^{1/2}$ in the AF phase of the non-competitive case. This difference reflects the partial cancellation of the staggered field effect due to the competitive inter-chain interaction in the SSB phase. The spin-wave dispersion in the SSB phase remains gapless due to NG mechanism even under a non-vanishing $h$. This is in contrast to the case of the 1D model.

Finally we comment on a few recent reports related to our study. In BaCu$_2$Si$_2$O$_7$ reported in Ref. [39], both the staggered field and the inter-chain interactions are expected, as in our models. However, the effect of the exchange anisotropies, which is ignored in the present paper, is argued to be responsible for the observed two spin-flop transitions. Extending the present work to such a system would be an interesting problem in the future. Furthermore, in BaCu$_2$(Si$_{1-x}$Ge$_x$)$_2$O$_7$ [22] the sign of the inter-chain interaction seems to depend on the doping parameter $x$. Thus, it could provide a realization of the competitive and non-competitive cases.

Wang et al. [22] investigated an $S = 1/2$ AF ladder system with a staggered field. They argue that the competition between the staggered field and the rung interaction brings a quantum criticality. It might be interesting to
compare our analysis on the higher dimensional system with theirs.

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**APPENDIX A: DETAILS OF SPIN-WAVE RESULTS IN SEC. III**

Here we supplement the spin-wave results omitted in Sec. III.

1. The competitive case

We write down the details of the competitive case. After the FT of the boson operator $c_{i,j,k}$, the spin-wave Hamiltonian can be expressed as the following matrix form:

$$
\hat{H}_{HP} = \sum_{\vec{k}} \left( \begin{array}{c}
\xi_{\vec{k}}^T \\
\eta_{\vec{k}}^T
\end{array} \right) \left( \begin{array}{c}
\xi_{\vec{k}} \\
\eta_{\vec{k}}
\end{array} \right) \left( \begin{array}{c}
\xi_{\vec{k}}^T \\
\eta_{\vec{k}}^T
\end{array} \right) - 4E_1(\vec{k}) + E_{cl}^{\text{ed}},
$$

(A1)

where $\xi_{\vec{k}}$, $\eta_{\vec{k}}$, $\Omega_{\vec{k}}$, and $\xi_{\vec{k}}^\ast$, $\eta_{\vec{k}}^\ast$, $\Omega_{\vec{k}}^\ast$ are given as

$$
\xi_{\vec{k}} = \begin{pmatrix}
E_1(\vec{k}) & 0 & 0 & iE_4(k_{y,z}) \\
0 & E_1(\vec{k}) & iE_4(k_{y,z}) & 0 \\
0 & -iE_4(k_{y,z}) & F_1(\vec{k}) & 0 \\
-iE_4(k_{y,z}) & 0 & 0 & F_1(\vec{k})
\end{pmatrix},
$$

$$
\eta_{\vec{k}} = \begin{pmatrix}
0 & E_2(\vec{k}) & 0 & iE_3(\vec{k}) \\
E_2(\vec{k}) & 0 & iE_3(\vec{k}) & 0 \\
iE_3(\vec{k}) & 0 & F_2(\vec{k}) & 0 \\
iE_3(\vec{k}) & 0 & F_2(\vec{k}) & 0
\end{pmatrix},
$$

$$
\Omega_{\vec{k}} = \begin{pmatrix}
\Omega_{\vec{k}} & 0 & 0 & 0 \\
0 & \Omega_{\vec{k}} & 0 & 0 \\
0 & 0 & \Omega_{\vec{k}} & 0 \\
0 & 0 & 0 & \Omega_{\vec{k}}
\end{pmatrix},
$$

(A2)

Here $E_{1,2,3,4}$ are defined as

$$
E_1(\vec{k}) = 2SJ(1 - 2 \sin^2 \theta_{cl} \cos^2 \phi_{cl}) + 2S(J_{\perp} + J_{\perp}')\left(2 \cos^2 \theta_{cl} \cos^2 \phi_{cl} - 1\right) - 2SJ \sin^2 \theta_{cl} \cos^2 \phi_{cl} \cos k_x a_x - 2S \sin \theta_{cl} \cos \phi_{cl} + H \sin \theta_{cl} \cos \phi_{cl} + h \sin \phi_{cl} \right)/8,
$$

$$
E_2(\vec{k}) = S(J \cos^2 \theta_{cl} - \sin^2 \theta_{cl} \sin^2 \phi_{cl}) \cos k_x a_x + S \cos \theta_{cl} \sin \phi_{cl} \cos k_x a_x + J_{\perp} \cos k_y a_y + J_{\perp}' \cos k_x a_x)/4,
$$

$$
E_3(k_x) = -SJ \sin 2\theta_{cl} \sin \phi_{cl} \cos k_x a_x/4,
$$

$$
E_4(k_{y,z}) = S \sin 2\theta_{cl} \sin \phi_{cl} \cos k_y a_y + J_{\perp}' \cos k_x a_x)/4,
$$

(A3)

and $F_{1,2}(\vec{k}) = E_{1,2}(\vec{k} - \vec{\pi})$. Thus $\xi_{\vec{k}}$ is Hermitian and $\eta_{\vec{k}}$ is symmetric. Let us suppose that a four-mode BT

$$
\begin{pmatrix}
\xi_{\vec{k}}^T \\
\eta_{\vec{k}}^T
\end{pmatrix} = M_{BT}(\vec{k}) \begin{pmatrix}
\xi_{\vec{k}}^T \\
\eta_{\vec{k}}^T
\end{pmatrix},
$$

(A4)

where $\tilde{\xi}_{\vec{k}} = (\tilde{\xi}_{\vec{k}}, \tilde{\xi}_{\vec{k} - \vec{\pi}}, \tilde{\xi}_{\vec{k} - \vec{\pi}}, \tilde{\xi}_{\vec{k} - 2\vec{\pi}})^T$ is a set of new boson (magnon) operators and $M_{BT}(\vec{k})$ is an $8 \times 8$ matrix, diagonalizes the Hamiltonian as follows:

$$
\hat{H}_{HP} = \sum_{\vec{k}} \left( \begin{array}{c}
\tilde{\xi}_{\vec{k}}^T \\
\tilde{\eta}_{\vec{k}}^T
\end{array} \right) \left( \begin{array}{c}
\Omega_{\vec{k}} & 0 \\
0 & \Omega_{\vec{k}}
\end{array} \right) \left( \begin{array}{c}
\tilde{\xi}_{\vec{k}} \\
\tilde{\eta}_{\vec{k}}
\end{array} \right) - 4E_1(\vec{k}) + E_{cl}^{\text{ed}},
$$

(A5)

where $\Omega_{\vec{k}} = \text{diag}(\omega_1(\vec{k}), \omega_2(\vec{k}), \omega_3(\vec{k}), \omega_4(\vec{k}))$. According to Ref. [20], determining $\Omega_{\vec{k}}$ and $M_{BT}(\vec{k})$ is equivalent to solving an eigenvalue problem

$$
\begin{pmatrix}
\xi_{\vec{k}} & -\eta_{\vec{k}} \\
\eta_{\vec{k}} & -\xi_{\vec{k}}
\end{pmatrix} M_{BT}^4 = M_{BT} \begin{pmatrix}
\Omega_{\vec{k}} & 0 \\
0 & -\Omega_{\vec{k}}
\end{pmatrix}.
$$

(A6)

The eight eigenvalues $\pm \omega_j(\vec{k})$ ($j = 1, 2, 3, 4$) are given by

$$
\lambda_1(\vec{k}) = [1/2 G_{\vec{k}} - 1/2(G_{\vec{k}}^2 - 4G_{\vec{k}}^4)^{1/2}]^{1/2},
$$

$$
\lambda_2(\vec{k}) = [1/2 G_{\vec{k}} - 1/2(G_{\vec{k}}^2 - 4G_{\vec{k}}^4)^{1/2}]^{1/2},
$$

$$
\lambda_3(\vec{k}) = [1/2 G_{\vec{k}} + 1/2(G_{\vec{k}}^2 - 4G_{\vec{k}}^4)^{1/2}]^{1/2},
$$

$$
\lambda_4(\vec{k}) = [1/2 G_{\vec{k}} + 1/2(G_{\vec{k}}^2 - 4G_{\vec{k}}^4)^{1/2}]^{1/2},
$$

(A7)
and $G_k$ and $G_k^\pm$ are defined as
\[
G_k = E_1^2 - E_2^2 + F_1^2 - F_2^2 - 2E_3^2 + 2E_4^2,
\]
\[
G_k^\pm = (E_3^2 - E_2^2)^2 + (E_4^2 - E_3^2)(F_1^2 - F_2^2)
- 2(E_1F_1 - E_2F_2)(E_2^2 + E_4^2)
\pm 4(E_1F_2 - E_2F_1)E_3E_4.
\] (A8)

The physical dispersion $\omega(\vec{k})$ is given by either $4[\lambda_1(\vec{k}) + \lambda_2(\vec{k})]$ or $4[\lambda_3(\vec{k}) + \lambda_4(\vec{k})]$, depending on the value of $\vec{k}$. In the vicinity of the point $\vec{k} = \vec{k}_h$ ($\vec{k}_H$), which is the gapless points when $h = 0$ ($H = 0$), we find
\[
\omega(\vec{k}) = 4[\lambda_1(\vec{k}) + \lambda_2(\vec{k})].
\] (A9)

If only one of the external fields ($H$ or $h$) is non-vanishing, the result is considerably simplified because we have $E_3 = E_4 = 0$. The spin-wave Hamiltonian can be actually diagonalized by a simpler two-mode BT which is the same type as we need in the non-competitive case in Appendix A2. The resulting magnon dispersion is

\[
\omega(\vec{k}) = \left[\left(2SJ(1 - 2\sin^2 \theta_1 \cos^2 \phi_1) + 2S(J_\perp + J_\perp')(2\cos^2 \theta_1 \cos^2 \phi_1 - 1) - 2SJ\sin^2 \theta_1 \cos^2 \phi_1 \cos k_za_x
+ 2S(\cos^2 \theta_1 (1 + \sin^2 \phi_1) - 1)(J_\perp \cos k_ya_y + J_\perp \cos k_za_z) + \Sigma\right)^2
- \left(2SJ(\cos^2 \theta_1 - \sin^2 \theta_1 \sin^2 \phi_1) \cos k_za_x + 2SJ\cos^2 \theta_1 \cos^2 \phi_1(J_\perp \cos k_ya_y + J_\perp \cos k_za_z)\right)^2\right]^{1/2},
\] (A10)

where $\Sigma = H\sin \theta_1 \cos \phi_1$ when $h = 0$, and $\Sigma = h\sin \phi_1$ when $H = 0$. Of course, in these special cases, Eq. (A9) reduces to Eq. (A10) near $\vec{k} = \vec{k}_h$ or $\vec{k}_H$.

From the dispersion (A9), let us estimate how $\Delta_h$ ($\Delta_H$) grows when a small $h$ ($H$) is applied. To estimate $\Delta_h$, it is sufficient to know the coefficients of Taylor expansion of $E_{1,2,3,4}(\vec{k} = \vec{0})$, $G_{\vec{k} = \vec{0}}$ and $G_{\vec{k} = \vec{0}}^\pm$ around $h = 0$. As a result, in the limit of small $h$, the gap behaves as

\[
\Delta_h \approx 2S(J + J_\perp + J_\perp')/2(J_\perp + J_\perp')(1 - \hat{H})^{1/2}
\times \left(1 - \hat{H}^2 + \frac{J_\perp + J_\perp'}{J_\perp + J_\perp'} \hat{H}^2\right)^{-1/2}
\times \left[1 - 4\hat{H}^2 - \frac{(J_\perp + J_\perp')^2}{J_\perp + J_\perp'} \hat{H}^4\right]^{1/2}
\left[1 - \frac{(J_\perp + J_\perp')^2}{J_\perp + J_\perp'} \hat{H}^4\right]^{1/2}
\hat{h} + \cdots,
\] (A11)

where $\hat{H} = 4S(J + J_\perp + J_\perp')$ and $\hat{h} = \frac{h}{4S(J + J_\perp + J_\perp')}$. At $H = 0$, Eq. (A11) is reduced to the exact result $\Delta_h = [1 + J/((J + J_\perp')(1/2)h which is derived from Eq. (A10) and is drawn in Fig. 8. Similarly to $\Delta_h$, $\Delta_H$ can be estimated. The result is

\[
\Delta_H \approx 2S(J + J_\perp + J_\perp')(1 - \hat{H})^{1/2}
\times (J_\perp + J_\perp')^{1/2}(1 - \hat{H}^2)^{-1/2}
\times \left[1 - \frac{(J_\perp + J_\perp')^2}{(J + J_\perp + J_\perp')^2}(J_\perp + J_\perp') \hat{H}^2\right]^{1/2}
\times \hat{H} + \cdots,
\] (A12)

which is reduced to the conventional result $\Delta_H = H$ when $h = 0$. Both gaps have linear field dependence. The results (A11) and (A12) indicate that $\Delta_h$ ($\Delta_H$) is non-vanishing only when $h \neq 0$ ($H \neq 0$). As a consequence, the true gap $\Delta = \min(\Delta_h, \Delta_H)$ is zero when either of $h$ or $H$ is zero.

### 2. The non-competitive case

Here we summarize the spin-wave approximation on the non-competitive case. We find that it can be straightforwardly applied also to the symmetric phase in the competitive case.

According to the MFT and Fig. 8 the spin configuration in the classical ground state can be written as

\[
\hat{S}_{2n,j,k} \sim S R_y(-\theta)\hat{x},
\] (A13)
\[
\hat{S}_{2n+1,j,k} \sim S R_z(\pi)R_y(-\theta)\hat{x},
\] (A14)

where $\hat{x}$ is the unit vector pointing to the $x$ direction and $\theta$ is the canting angle parameter. The minimization of the classical energy determines $\theta = \theta_1$ as

\[
\cos \theta_1 \sin \theta_1 = \frac{H}{4SJ} \cos \theta_1 - \frac{h}{4SJ} \sin \theta_1.
\] (A15)

A standard spin-wave theory on this classical ground state gives the quadratic Hamiltonian in terms of bosons

\[
\hat{H}_{HP2} = \sum_{\vec{k}} \left( a_\vec{k}^\dagger a_{-\vec{k}} \right) \left( \frac{1}{2}A_{\vec{k}}^2 B_{\vec{k}} + \frac{1}{2}A_{\vec{k}}^2 B_{\vec{k}}^2 \right) \left( a_{\vec{k}}^\dagger a_{-\vec{k}} \right)
- \frac{1}{2}A_{\vec{k}}^2 + \text{const},
\] (A16)
where
\[ A_k = A_{-k} = 2SJ \cos 2\theta_{cl} - 2SJ \sin^2 \theta_{cl} \cos k_x a_x \]
\[ + 2SJ \sin \theta_{cl} \cos k_x a_x \]
\[ + 2SJ J_{1}^\perp (1 - \cos k_x a_y) \]
\[ + H \sin \theta_{cl} + h \cos \theta_{cl}, \]
\[ B_{k_x} = B_{-k_x} = -SJ \cos^2 \theta_{cl} \cos k_x a_x. \]

Now we apply the two-mode BT
\[ \left( \begin{array}{c} a_k^\perp \\ a_{-k}^\perp \end{array} \right) = \left( \begin{array}{cc} u_k & v_k \\ v_k & u_k \end{array} \right) \left( \begin{array}{c} \tilde{a}_k^\perp \\ \tilde{a}_{-k}^\perp \end{array} \right), \]
\[ (A18) \]
where \( u_k^2 - v_k^2 = 1 \), to Eq. (A16). The Hamiltonian is then diagonalized as
\[ \hat{H}_{HP2} = \sum_k \tilde{\omega}(k) \tilde{a}_k^\dagger \tilde{a}_k + \text{const.}, \]
\[ (A19) \]
if we choose \( u_k^2 = \frac{1}{2}(1 + \frac{h}{\omega(k)}) \) and \( v_k^2 = \frac{1}{2}(1 - \frac{h}{\omega(k)}) \).
The dispersion relation is given by
\[ \tilde{\omega}(k) = \sqrt{A_k^2 - 4B_{k_x}^2}. \]
\[ (A20) \]
The above derivation and results apply exactly to the symmetric phase in the competitive case, only with the replacement \((|J_L|, |J_0^\perp|) \rightarrow (-J_L, -J_0^\perp)\). The dispersion \[ \tilde{\Delta}_h = \tilde{\omega}(0,0) \] can be regarded as the h-induced gap. At the small field \( h \), the angle variation around \( h = 0 \):
\[ \delta \theta = \sin^{-1}(\frac{h}{4SJ}) - \theta_{cl} \]
is estimated approximately as
\[ \delta \theta \approx -\frac{H}{(4SJ)^2 - H^2} h. \]
\[ (A21) \]

Therefore expanding \( \tilde{\Delta}_h^2(h, \delta \theta) \) around \( h = \delta \theta = 0 \), one sees that the gap grows as
\[ \tilde{\Delta}_h \approx \sqrt{4SJ} \left[ 1 - \left(\frac{H}{4SJ}\right)^2 \right]^{1/4} \left[ 1 + 2\left(\frac{H}{4SJ}\right)^2 \right]^{1/2} h^{1/2}. \]
\[ (A22) \]
In contrast to the SSB phase (in the competitive case), the h-induced gap opens as \( \Delta_h \propto h^{1/2} \). Furthermore, it is remarkable that Eq. (A22) has no dependence on the inter-chain interactions. In fact, it is identical to the 1D result [Eq. (3.17) in Ref. 3]. This is a reflection of the smoothness which was discussed in the final of Sec. III. At \( H = 0 \), we can obtain the simple exact result
\[ \Delta_h = \sqrt{4SJh[1 + h/(4SJ)]^{1/2}} \] from the dispersion (A20).

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