A CÀDLÀG ROUGH PATH FOUNDATION
FOR ROBUST FINANCE

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Abstract. Using rough path theory, we provide a pathwise foundation for stochastic Itô integration, which covers most commonly applied trading strategies and mathematical models of financial markets, including those under Knightian uncertainty. To this end, we introduce the so-called Property (RIE) for càdlàg paths, which is shown to imply the existence of a càdlàg rough path and of quadratic variation in the sense of Föllmer. We prove that the corresponding rough integrals exist as limits of left-point Riemann sums along a suitable sequence of partitions. This allows one to treat integrands of non-gradient type, and gives access to the powerful stability estimates of rough path theory. Additionally, we verify that (path-dependent) functionally generated trading strategies and Cover’s universal portfolio are admissible integrands, and that Property (RIE) is satisfied by both (Young) semimartingales and typical price paths.

Key words: Föllmer integration, model uncertainty, semimartingale, pathwise integration, rough path, functionally generated portfolios, universal portfolio.
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1. Introduction

A fundamental pillar of mathematical finance is the theory of stochastic integration initiated by K. Itô in the 1940s. Itô’s stochastic integration not only allows for a well-posedness theory for most probabilistic models of financial markets, but also comes with invaluable properties, such as having an integration by parts formula and chain rule, and that of being a continuous operator (with respect to suitable spaces of random variables), which is essential for virtually all applications. However, despite the elegance and success of Itô integration, it also admits some significant drawbacks from both theoretical and practical perspectives.

The construction of the Itô integral requires one to fix a probability measure a priori, and is usually based on a limiting procedure of approximating Riemann sums in probability. While in mathematical finance the Itô integral usually represents the capital gain process from continuous-time trading in a financial market, it lacks a robust pathwise meaning. That is, the stochastic Itô integral does not have a well-defined value on a given “state of the world”, e.g. a realized price trajectory of a liquidly traded asset on a stock exchange. This presents a gap between probabilistic models and their financial interpretation. Addressing the pathwise meaning of stochastic integration has led to a stream of literature beginning with the classical works of Bichteler [7] and Willinger and Taqqu [52]; see also Karandikar [30] and Nutz [40].

The requirement of fixing a probability measure to have access to Itô integration becomes an even more severe obstacle when one wants to develop mathematical finance under model risk—also known as Knightian uncertainty. Starting from the seminal works of Avellaneda,
Levy and Parás \cite{5} and Lyons \cite{37}, there has been an enormous and on-going effort to treat the challenges posed by model risk in mathematical finance, that is, the risk stemming from the possible misspecification of an adopted stochastic model, typically represented by a single fixed probability measure. The majority of the existing robust treatments of financial modelling replace the single probability measure by a family of (potentially singular) probability measures, or even take so-called model-free approaches, whereby no probabilistic structure of the underlying price trajectories is assumed; see for example Hobson \cite{27} for classical lecture notes on robust finance. In particular, the latter model-free approaches often require a purely deterministic integration theory sophisticated enough to handle the irregular sample paths of standard continuous-time financial models and commonly employed functionally generated trading strategies.

In the seminal paper \cite{19}, Föllmer provided the first deterministic analogue to stochastic Itô integration which had the desired properties required by financial applications. Indeed, assuming that a càdlàg path $S: [0, T] \to \mathbb{R}^d$ possesses a suitable notion of quadratic variation along a sequence $(\mathcal{P}^n)_{n \in \mathbb{N}}$ of partitions of the interval $[0, T]$, Föllmer proved that the limit

$$\int_0^t Df(S_u) \, dS_u := \lim_{n \to \infty} \sum_{[u, v] \in \mathcal{P}^n} Df(S_u)(S_{u \wedge t} - S_{u \wedge t}), \quad t \in [0, T],$$

where $Df$ denotes the gradient of $f$, exists for all twice continuously differentiable functions $f: \mathbb{R}^d \to \mathbb{R}$. The resulting pathwise integral $\int_0^t Df(S_u) \, dS_u$ is often called the Föllmer integral, and has proved to be a valuable tool in various applications in model-free finance; for some recent examples we refer to Föllmer and Schied \cite{21}, Davis, Obłój and Raval \cite{14}, Schied, Speiser and Voloshchenko \cite{44} and Cuchiero, Schachermayer and Wong \cite{13}. In fact, even classical Riemann–Stieltjes integration has been successfully used as a substitution to Itô integration in model-free finance; see e.g. Dolinsky and Soner \cite{16} or Hou and Obłój \cite{28}.

By now arguably the most general pathwise (stochastic) integration theory is provided by the theory of rough paths, as introduced by Lyons \cite{38}, and its recent extension to càdlàg rough paths by Friz and Shekhar \cite{25}, Friz and Zhang \cite{26} and Chevyrev and Friz \cite{9}. Rough integration can be viewed as a generalization of Young integration which is able to handle paths of lower regularity. While rough integration allows one to treat the sample paths of numerous stochastic processes as integrators and offers powerful pathwise stability estimates, it comes with a pitfall from a financial perspective: the rough integral is defined as a limit of so-called compensated Riemann sums, and thus apparently does not correspond to the canonical financial interpretation as the capital gain process generated by continuous-time trading. Even worse, choosing a rough path without care might lead to an anticipating integral, corresponding e.g. to Stratonovich integration, thus introducing undesired arbitrage when used as a capital process.

We overcome these issues by introducing the so-called Property (RIE) for a càdlàg path $S: [0, T] \to \mathbb{R}^d$ and a sequence $(\mathcal{P}^n)_{n \in \mathbb{N}}$ of partitions of the interval $[0, T]$. This property is very much in the same spirit as Föllmer’s assumption of quadratic variation along a sequence of partitions. Indeed, we show that Property (RIE) implies the existence of quadratic variation in the sense of Föllmer, and even the existence of a càdlàg rough path $S$ above $S$, which, loosely speaking, corresponds to an “Itô” rough path in a probabilistic setting. Assuming Property (RIE), we prove that the corresponding rough integrals exist as limits of left-point Riemann sums along the sequence of partitions $(\mathcal{P}^n)_{n \in \mathbb{N}}$. This result restores the canonical financial interpretation for rough integration, and links it to Föllmer integration for càdlàg paths.
Property (RIE) was previously introduced by Perkowski and Prömel [41] for continuous paths, though we emphasize that the present more general càdlàg setting requires quite different techniques compared to the continuous setting of [41].

Given the aforementioned results, a càdlàg path which satisfies Property (RIE) permits the path-by-path existence of rough integrals with their desired financial interpretation, and moreover maintains access to their powerful stability results which ensure that the integral is a continuous operator. This appears to be a significant advantage compared to the classical notions of pathwise stochastic integration in [7, 52, 30, 40], which do not come with such stability estimates. In particular, the pathwise stability results of rough path theory allow one to prove a model-free version of the so-called fundamental theorem of derivative trading—see Armstrong, Bellani, Brigo and Cass [4]—and may be of interest when investigating discretization errors of continuous-time trading in model-free finance; see Riga [43]. Furthermore, in contrast to Föllmer integration, rough integration allows one to consider general functionally generated integrands $g(S_t)$, where $g$ is a general (sufficiently smooth) function $g: \mathbb{R}^d \to \mathbb{R}^d$, and not necessarily the gradient of another vector field $f: \mathbb{R}^d \to \mathbb{R}$. For instance, model-free portfolio theory constitutes a research direction in which it is beneficial to consider non-gradient trading strategies; see Allan, Cuchiero, Liu and Prömel [1]. Even more generally, rough integration allows one to treat path-dependent functionally generated options in the sense of Dupire [17], and pathwise versions of Cover’s universal portfolio, as discussed in Section 3.

Of course, it remains to verify that Property (RIE) is a reasonable modelling assumption in mathematical finance, in the sense that it is fulfilled almost surely by sample paths of the commonly used probabilistic models of financial markets. Since it seems natural that continuous-time trading takes place when the underlying price process fluctuates, we employ sequences of partitions based on such a “space discretization”. For such sequences of partitions, we show that the sample paths of càdlàg semimartingales almost surely satisfy Property (RIE). This result is then extended to so-called Young semimartingales, which are stochastic possesses given by the sum of a càdlàg local martingale and an adapted càdlàg process of finite $q$-variation for some $q < 2$. Finally, we prove that Property (RIE) is satisfied by typical price paths in the sense of Vovk [48], which correspond to a model-free version of “no unbounded profit with bounded risk”.

Organization of the paper: In Section 2 we introduce Property (RIE) and verify the properties of the associated rough integration as described above. In Section 3 we exhibit functionally generated trading strategies and generalizations thereof which provide valid integrands for rough integration. In Section 4 we prove that (Young) semimartingales and typical price paths satisfy Property (RIE).

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2. Rough integration under Property (RIE)

In this section we develop pathwise integration under Property (RIE). We set up the essential ingredients from rough path theory in Section 2.2 and show in Section 2.3 that paths
satisfying (RIE) serve as suitable integrators in mathematical finance. Finally, in Section 2.4 we connect Property (RIE) with the existence of quadratic variation in the sense of Föllmer.

2.1. Basic notation. Let $(\mathbb{R}^d, |\cdot|)$ denote standard Euclidean space, and let $D([0,T]; \mathbb{R}^d)$ denote the space of all càdlàg (i.e. right-continuous with left-limits) functions from $[0,T] \to \mathbb{R}^d$. A partition $\mathcal{P} = \mathcal{P}([s,t])$ of the interval $[s,t]$ is a set of essentially disjoint intervals covering $[s,t]$, i.e. $\mathcal{P} = \{s = t_0 < t_1 < \cdots < t_N = t\}$ for some $N \in \mathbb{N}$. The mesh size of a partition $\mathcal{P}$ is given by $|\mathcal{P}| := \max\{|t_{i+1} - t_i| : i = 0, \ldots, N - 1\}$, and, for a partition $\mathcal{P}$ of the interval $[s,t]$ and a subinterval $[u,v] \subset [s,t]$, we write $\mathcal{P}([u,v]) := (\mathcal{P} \cup \{u, v\}) \cap [u,v] = \{r \in \mathcal{P} \cup \{u, v\} : u \leq r \leq v\}$, for the restriction of the partition $\mathcal{P}$ to the interval $[u,v]$. A sequence $(\mathcal{P}^n)_{n \in \mathbb{N}}$ of partitions is called nested if $\mathcal{P}^n \subset \mathcal{P}^{n+1}$ for all $n \in \mathbb{N}$.

Setting $\Delta_{[0,T]} := \{(s,t) \in [0,T]^2 : s \leq t\}$, a control function is defined as a function $w: \Delta_{[0,T]} \to [0,\infty)$ which is superadditive, in the sense that $w(s,u) + w(u,t) \leq w(s,t)$ for all $0 \leq s \leq u \leq t \leq T$.

Throughout this section we fix a finite time interval $[0,T]$ and the dimension $d \in \mathbb{N}$. We also adopt the convention that, given a path $A$ defined on $[0,T]$, we will write $A_{s,t} := A_t - A_s$ for the increment of $A$ over the interval $[s,t]$. Note however that whenever $A$ is a two-parameter function defined on $\Delta_{[0,T]}$, then the notation $A_{s,t}$ will simply denote the value of $A$ evaluated at the pair of times $(s,t) \in \Delta_{[0,T]}$.

If $A$ denotes either a path from $[0,T] \to E$ or a two-parameter function from $\Delta_{[0,T]} \to E$ for some normed vector space $E$, then, for any $p \in [1,\infty)$, the $p$-variation of $A$ over the interval $[s,t]$ is defined by

$$\|A\|_{p,[s,t]} := \left( \sup_{\mathcal{P}([s,t])} \sum_{(u,v) \in \mathcal{P}([s,t])} |A_{u,v}|^p \right)^{\frac{1}{p}}$$

where the supremum is taken over all partitions $\mathcal{P}([s,t])$ of the interval $[s,t] \subseteq [0,T]$, and in the case when $A$ is a path we write $A_{u,v} := A_v - A_u$. If $\|A\|_{p,[0,T]} < \infty$ then $A$ is said to have finite $p$-variation.

We write $D^p = D^p([0,T]; E)$ for the space of all càdlàg paths $A: [0,T] \to E$ of finite $p$-variation, and we similarly write $D^p_2 = D^p_2(\Delta_{[0,T]}; E)$ for the space of two-parameter functions $A: \Delta_{[0,T]} \to E$ of finite $p$-variation which are such that the maps $s \mapsto A_{s,t}$ for fixed $t$, and $t \mapsto A_{s,t}$ for fixed $s$ are both càdlàg. Note that $A$ having finite $p$-variation is equivalent to the existence of a control function $w$ such that $|A_{s,t}|^p \leq w(s,t)$ for all $(s,t) \in \Delta_{[0,T]}$. (For instance, one may take $w(s,t) = \|A\|_{p,[s,t]}^p$.)

2.2. Càdlàg rough path theory and Property (RIE). While rough path theory has by now been well studied in the case of continuous paths, as exhibited in a number of books, notably Friz and Hairer [24], its extension to càdlàg paths appeared only recently, starting with Friz and Shekhar [25]. In this section we mainly rely on results regarding forward integration with respect to càdlàg rough paths as presented in Friz and Zhang [26].

In the following we fix $p \in (2,3)$ and $q \geq p$ such that

$$\frac{2}{p} + \frac{1}{q} > 1,$$

and define $r > 1$ by the relation

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q}.$$
This means in particular that $1 < p/2 \leq r < q < \infty$.

Throughout the paper, we will use the symbol $\lesssim$ to denote inequality up to a multiplicative constant which depends only on the numbers $p, q$ and $r$ as chosen above.

We begin by recalling the definition of a càdlàg rough path, as well as the corresponding notion of controlled paths. In the following we will write $A \otimes B$ for the tensor product of two vectors $A, B \in \mathbb{R}^d$, i.e. the $d \times d$-matrix with $(i, j)$-component given by $[A \otimes B]^{ij} = A^i B^j$ for $1 \leq i, j \leq d$.

**Definition 2.1.** We say that a triplet $X = (X, Z, \mathcal{F})$ is a (càdlàg) $p$-rough path (over $\mathbb{R}^d$) if $X \in D^p([0, T]; \mathbb{R}^d)$, $Z \in D^p([0, T]; \mathbb{R}^d)$ and $\mathcal{F} \in D^{p/2}_2(\Delta_{[0, T]}; \mathbb{R}^{d \times d})$, and if Chen's relation:

$$X_{s,t} = X_{s,u} + X_{u,t} + Z_{s,u} \otimes X_{u,t}$$

holds for all times $0 \leq s \leq u \leq t \leq T$. We denote the space of càdlàg rough paths by $\mathcal{F}^p$.

The unfamiliar reader is encouraged to check that, given càdlàg paths $X$ and $Z$ of bounded variation, setting $X_{s,t} = \int_s^t Z_{s,u} \otimes dX_u$, which corresponds to taking $\parallel X \parallel_{p, [s,t]}$ for $(s, t) \in \Delta_{[0, T]}$, with the integral defined as a limit of left-point Riemann sums, gives a $p$-rough path. Although the integral $\int_s^t Z_{s,u} \otimes dX_u$ is not in general well-defined when $X$ and $Z$ are not of bounded variation, given a rough path $(X, Z, \mathcal{F})$, we may think of $\mathcal{F}$ as postulating a “candidate” for the value of such integrals.

**Remark 2.2.** The definition of rough paths we have introduced above looks slightly different to the standard definition, in which one takes $X = Z$. Our definition is slightly more general, but the corresponding theory works in exactly the same way, and turns out to be more convenient in the context of Property (RIE) as we will see later.

More precisely, later the matrix $X_{s,t}$ will represent the (a priori ill-defined) ‘integral’ $\int_s^t S_{s,u} \otimes dS_u$, which will be defined as the limit as $n \to \infty$ of the Riemann sums $(\int_s^t S_{s,u} \otimes dS_u)_{n \in \mathbb{N}}$ appearing in Property (RIE) below. In the continuous (i.e. without jumps) setting of Perkowski and Prömel [11], a linear interpolation is used to provide a continuous approximation of $S^n$, leading to a Stratonovich type integral in the limit, which is subsequently converted back into an Itô type integral. Thanks to the recently developed theory of càdlàg rough paths, here we can use a more direct argument which avoids this detour. This means working directly with the integral $\int_s^t S_{s,u} \otimes dS_u$, which corresponds to taking $X = S$ and $Z = S^n$ in Definition 2.7 thus requiring $X \neq Z$.

For two rough paths, $X = (X, Z, \mathcal{F})$ and $\tilde{X} = (\tilde{X}, \tilde{Z}, \tilde{\mathcal{F}})$, we use the seminorm

$$\|X\|_{p, [s,t]} := \|X\|_{p, [s,t]} + \|Z\|_{p, [s,t]} + \|\tilde{X}\|_{p, [s,t]},$$

and the pseudometric

$$\|X; \tilde{X}\|_{p, [s,t]} := \|X - \tilde{X}\|_{p, [s,t]} + \|Z - \tilde{Z}\|_{p, [s,t]} + \|\tilde{X} - \tilde{\mathcal{F}}\|_{p, [s,t]},$$

for $[s, t] \subseteq [0, T]$.

In the following we write $\mathcal{L}(\mathbb{R}^d; \mathbb{R}^d)$ for the space of linear maps from $\mathbb{R}^d \to \mathbb{R}^d$.

**Definition 2.3.** Let $Z \in D^p([0, T]; \mathbb{R}^d)$. We say that a pair $(F, F')$ is a controlled path (with respect to $Z$), if $F \in D^p([0, T]; \mathbb{R}^d)$, $F' \in D^p([0, T]; \mathcal{L}(\mathbb{R}^d; \mathbb{R}^d))$ and $R^F \in D^p_2(\Delta_{[0, T]}; \mathbb{R}^d)$, where the remainder $R^F$ is defined implicitly by the relation

$$F_{s,t} = F'_{s,t} Z_{s,t} + R^F_{s,t}, \quad (s, t) \in [0, T].$$
We refer to $F'$ as the Gubinelli derivative of $F$ (with respect to $Z$), and denote the space of such controlled paths by $\mathcal{V}_Z^{p,r}$.

Given a path $Z \in D^{p,∞}([0,T]; \mathbb{R}^d)$, the space of controlled paths $\mathcal{V}_Z^{p,r}$ becomes a Banach space when equipped with the norm $(F,F') \mapsto |F_0| + ∥F,F'∥_{\mathcal{V}_Z^{p,r},[0,T]}$, where

$$∥F,F'∥_{\mathcal{V}_Z^{p,r},[0,T]} := |F_0| + ∥F'∥_{q,[0,T]} + ∥R^F∥_{r,[0,T]}.$$  

With the concepts of rough paths and controlled paths at hand we are ready to introduce rough integration. The following result is a straightforward extension of [1, Lemma 2.6], and its proof follows almost verbatim.

**Proposition 2.4.** Let $X = (X, Z, X) ∈ \mathcal{V}^p$ be a càdlàg rough path, and let $(F,F') ∈ \mathcal{V}_Z^{q,r}$ and $(G,G') ∈ \mathcal{V}_X^{q,r}$ be controlled paths with respect to $Z$ and $X$, respectively, with remainders $R^F$ and $R^G$. Then, for each $t ∈ [0,T]$, the limit

$$\int_0^t F_u \, dG_u := \lim_{|P| → 0} \sum_{[u,v] ∈ P} F_u G_{u,v} + F'_u G'_{u,v} X_{u,v}$$  

exists along every sequence of partitions $P$ of the interval $[0, t]$ with mesh size $|P|$ tending to zero. We call this limit the rough integral of $(F,F')$ against $(G,G')$ (relative to the rough path $X$), which moreover comes with the estimate

$$\left| \int_s^t F_u \, dG_u - F_s G_{s,t} - F'_s G'_{s,t} X_{s,t} \right|$$  

$$≤ C \left( ∥F'∥_∞ (∥G'∥_{q,[s,t]} + ∥Z∥^p_{p,[s,t]})^\frac{1}{2} ∥X∥_{p,[s,t]} + ∥F∥_{p,[s,t]} ∥R^G∥_{r,[s,t]} + ∥R^F∥_{r,[s,t]} ∥G'∥_∞ ∥X∥_{p,[s,t]} + ∥F'G'∥_{q,[s,t]} ∥X∥^q_{q,[s,t]} \right),$$  

for all $(s, t) ∈ Δ_{[0,T]}$ where the constant $C$ depends only on $p, q$ and $r$.

For us, the product of vectors $F_u G_{u,v}$ appearing in (2.2) will usually be interpreted as the Euclidean inner product, but in general this product may be interpreted as a matrix of any desired shape and size, consisting of linear combinations of products of the components of the two vectors, and the product $F'_u G'_{u,v} X_{u,v}$ will be a matrix with the same shape.

**Remark 2.5.** In the special case when $G = X$ (so that $G'$ is the identity map and $R^G = 0$), the integral defined in Proposition 2.4 reduces to the more classical notion of the rough integral of the controlled path $(F,F')$ against the rough path $X$, given by

$$\int_0^t F_u \, dX_u = \lim_{|P| → 0} \sum_{[u,v] ∈ P} F_u X_{u,v} + F'_u X_{u,v}.$$  

**Remark 2.6.** It follows from the estimate in (2.3), combined with the relation $G_{s,t} = G'_{s,t} X_{s,t} + R^G_{s,t}$, that the rough integral $\int_0^t F_u \, dG_u$ is itself a controlled path with respect to $X$, with Gubinelli derivative $F G'$, so that $(\int_0^t F_u \, dG_u, FG') ∈ \mathcal{V}_X^{q,r}$.

Notice that the construction of the rough integral in (2.2) is based on so-called compensated Riemann sums $\sum_{[u,v] ∈ P} F_u G_{u,v} + F'_u G'_{u,v} X_{u,v}$ instead of classical left-point Riemann sums $\sum_{[u,v] ∈ P} F_u G_{u,v}$. While the classical Riemann sums come with a natural interpretation as capital gain processes in the context of mathematical finance, the interpretation of compensated
Riemann sums is by no means obvious. However, one advantage of rough integration is that it provides rather powerful stability estimates, for instance as presented in the next proposition.

**Proposition 2.7.** Let $X = (X, Z, X), \tilde{X} = (\tilde{X}, \tilde{Z}, \tilde{X}) \in V^p$ be càdlàg rough paths, and let $(F, F') \in V^q_{\tilde{Z}}$ and $(\tilde{F}, \tilde{F}') \in V^q_{\tilde{Z}}$ be controlled paths with remainders $R^F$ and $R^{\tilde{F}}$ respectively.

(i) We have the estimate

$$
\left\| \int_0^t F_u^r \, dX_u - \int_0^t \tilde{F}_u^r \, d\tilde{X}_u \right\|_{p, [0, T]} \leq C \left( (|F_0| + |F'_0| + |\tilde{F}'_0| + \|F' - \tilde{F}'\|_q, [0, T] + \|R^F - R^{\tilde{F}}\|_{r, [0, T]})(1 + \|Z\|_{p, [0, T]} + \|X; \tilde{X}\|_{p, [0, T]} ) \right),
$$

where the constant $C$ depends only on $p, q, r$ and $M$. 

(ii) Let $(G, G') \in V^q_{\tilde{X}}$ and $(\tilde{G}, \tilde{G}') \in V^q_{\tilde{X}}$ also be controlled paths with remainders $R^G$ and $R^{\tilde{G}}$ respectively. Let $M > 0$ be a constant such that

$$
|F_0|, \|F, F'\|_{V^q_{\tilde{Z}}, [0, T]}, \|G, G'\|_{V^q_{\tilde{X}}, [0, T]}, |\tilde{F}_0|, \|\tilde{F}, \tilde{F}'\|_{V^q_{\tilde{Z}}, [0, T]}, \|\tilde{G}, \tilde{G}'\|_{V^q_{\tilde{X}}, [0, T]} \leq M.
$$

We then have the estimate

$$
\left\| \int_0^t F_u^r \, dG_u - \int_0^t \tilde{F}_u^r \, d\tilde{G}_u \right\|_{p, [0, T]} \leq C \left( |F_0| + |F'_0| + |\tilde{F}'_0| + \|F' - \tilde{F}'\|_q, [0, T] + \|R^F - R^{\tilde{F}}\|_{r, [0, T]} + |G_0' - \tilde{G}_0'| + \|G' - \tilde{G}'\|_{q, [0, T]} + \|R^G - R^{\tilde{G}}\|_{r, [0, T]} + \|X; \tilde{X}\|_{p, [0, T]} \right)
$$

where the new constant $C$ depends on $p, q, r$ and $M$.

**Proof.** We present here only the proof of part (ii), since the proof of part (i) follows almost verbatim. Here, the multiplicative constant implied by the symbol $\lesssim$ will be allowed to depend on the numbers $p, q$ and $r$ as usual, and additionally on the constant $M$. Following the proof of [26, Lemma 3.4], in our more general setting one deduces the estimates

$$
|F - \tilde{F}|_{p, [0, T]} \lesssim |F_0 - \tilde{F}_0| + |F' - \tilde{F}'|_{q, [0, T]} + \|R^F - R^{\tilde{F}}\|_{r, [0, T]} + \|Z - \tilde{Z}\|_{p, [0, T]},
$$

and

$$
\left\| R^{F} \int_0^t F_u \, dG_u - R^{\tilde{F}} \int_0^t \tilde{F}_u \, d\tilde{G}_u \right\|_{r, [0, T]} \lesssim |F_0 - \tilde{F}_0| + |F - \tilde{F}|_{p, [0, T]} + |F'_0 - \tilde{F}'_0| + |F' - \tilde{F}'|_{q, [0, T]} + \|R^F - R^{\tilde{F}}\|_{r, [0, T]} + \|G_0' - \tilde{G}_0'| + \|G' - \tilde{G}'\|_{q, [0, T]} + \|R^G - R^{\tilde{G}}\|_{r, [0, T]} + \|X; \tilde{X}\|_{p, [0, T]}.
$$

Recalling Remark 2.6, we find, using the controlled path structure of the rough integrals, that

$$
\left\| \int_0^t F_u \, dG_u - \int_0^t \tilde{F}_u \, d\tilde{G}_u \right\|_{p, [0, T]} \lesssim |F_0 - \tilde{F}_0| + |F - \tilde{F}|_{p, [0, T]} + |G_0' - \tilde{G}_0'| + \|G' - \tilde{G}'\|_{q, [0, T]} + \|X - \tilde{X}\|_{p, [0, T]} + \|R^{F} \int_0^t F_u \, dG_u - R^{\tilde{F}} \int_0^t \tilde{F}_u \, d\tilde{G}_u \|_{r, [0, T]}.
$$
The result then follows upon substituting the estimates (2.4) and (2.5) into (2.6). □

In the spirit of Föllmer’s assumption of quadratic variation along a sequence of partitions [19], we introduce the following property.

**Property** (RIE). Let \( p \in (2, 3) \) and let \( \mathcal{P}^n = \{0 = t^n_0 < t^n_1 < \cdots < t^n_{N_n} = T\} \), \( n \in \mathbb{N} \), be a sequence of nested partitions of the interval \([0, T]\) such that \( |\mathcal{P}^n| \to 0 \) as \( n \to \infty \). For \( S \in D([0, T]; \mathbb{R}^d) \), we define \( S^n: [0, T] \to \mathbb{R}^d \) by

\[
S^n_t = S_T \mathbf{1}_{\{T\}}(t) + \sum_{k=0}^{N_n-1} S^n_{t_k} \mathbf{1}_{[t_k, t_{k+1})}(t), \quad t \in [0, T],
\]

for each \( n \in \mathbb{N} \). We assume that:

- the sequence of paths \((S^n)_{n \in \mathbb{N}}\) converges uniformly to \( S \) as \( n \to \infty \),
- the Riemann sums \( \int_0^t S^n_u \otimes dS_u := \sum_{k=0}^{N_n-1} S^n_{t_k} \otimes S^n_{t_k \wedge t_{k+1} \wedge t} \) converge uniformly as \( n \to \infty \) to a limit, which we denote by \( \int_0^t S_u \otimes dS_u \), \( t \in [0, T] \),
- and that there exists a control function \( w \) such that

\[
\sup_{(s, t) \in \Delta_{[0, T]}} \frac{|S^n_{s, t}|^p}{w(s, t)} + \sup_{n \in \mathbb{N}} \sup_{0 \leq k < \ell \leq N_n} \frac{|\int_{t_k}^{t_\ell} S^n_u \otimes dS_u - S^n_{t_k} \otimes S^n_{t_\ell \wedge t}|^\frac{p}{2}}{w(t_k, t_\ell)} \leq 1.
\]

In (2.7), and hereafter, we adopt the convention that \( \frac{0}{0} := 0 \).

**Definition 2.8.** A path \( S \in D([0, T]; \mathbb{R}^d) \) is said to satisfy (RIE) with respect to \( p \) and \((\mathcal{P}^n)_{n \in \mathbb{N}}\), if \( p \), \((\mathcal{P}^n)_{n \in \mathbb{N}}\) and \( S \) together satisfy Property (RIE).

The name “RIE” is an abbreviation for “Riemann”, as we assume the convergence of the Riemann sums \( \int S^n_u \otimes dS_u \), instead of the discrete quadratic variations as in [19]. Indeed, Property (RIE) is a stronger assumption than the existence of quadratic variation in the sense of Föllmer, and is even enough to allow us to lift \( S \) in a canonical way to a rough path—see Lemma 2.12 below—giving us access to the powerful stability results of rough path theory, such as those exhibited in Proposition 2.7. Moreover, Property (RIE) can be verified for most typical stochastic processes in mathematical finance, as we will see in Section 4.

**Remark 2.9.** We highlight that, rather than simply being a property of a path, Property (RIE) is a property of a path together with a given sequence of partitions \((\mathcal{P}^n)_{n \in \mathbb{N}}\). Indeed, such a path will in general not satisfy (RIE) with respect to a different sequence of partitions. However, in practice there is often a natural choice for the sequence of partitions; see Remark 4.2. For clarity, hereafter, whenever we claim that a path satisfies Property (RIE), we will always make explicit the partition with respect to which the path satisfies (RIE), in the sense of Definition 2.8.

**Remark 2.10.** In Proposition 2.13 below, it is actually shown that it is sufficient in Property (RIE) to assume that the sequence \((S^n)_{n \in \mathbb{N}}\) converges only pointwise to \( S \), since the uniformity of this convergence then immediately follows.

Next we shall verify that Property (RIE) ensures the existence of a càdlàg rough path. For this purpose, we consider a suitable approximating sequence of the so-called ‘area process’, which is represented by \( X \) in Definition 2.1.
Lemma 2.11. Suppose $S \in D([0,T];\mathbb{R}^d)$ satisfies Property (RIE) with respect to $p$ and $(\mathcal{P}^n)_{n \in \mathbb{N}}$ (as in Definition 2.8). If for each $n \in \mathbb{N}$ we define $A^n: \Delta_{[0,T]} \to \mathbb{R}^{d \times d}$ by

$$A^n_{s,t} := \int_s^t S_{s,u}^n \otimes dS_u = \int_s^t S_u^n \otimes dS_u - S_s^n \otimes S_{s,t}, \quad (s,t) \in \Delta_{[0,T]},$$

where $\int_s^t S_{s,u}^n \otimes dS_u$ is defined as in Property (RIE), then there exists a constant $C$, depending only on $p$, such that

$$\|A^n\|_{\mathbb{F},[0,T]} \leq Cw(0,T)\frac{2}{\mathbb{F}} \quad \text{for every} \quad n \in \mathbb{N}.$$ 

Proof. Let $n \in \mathbb{N}$ and $(s,t) \in \Delta_{[0,T]}$. If there exists a $k$ such that $t_k^n \leq s < t \leq t_{k+1}^n$ then we simply have that $A^n_{s,t} = S_{t_k^n}^n \otimes S_{s,t} - S_{t_k^n}^n \otimes S_{s,t} = 0$. Otherwise, let $k_0$ be the smallest $k$ such that $t_k^n \in (s,t)$, and let $k_1$ be the largest such $k$. It is easy to see that the triplet $(S,S^n,A^n)$ satisfies Chen’s relation (2.1), from which it follows that

$$A^n_{s,t} = A^n_{s,t_{k_0}^n} + A^n_{t_{k_0}^n,t_{k_1}^n} + A^n_{t_{k_1}^n,t} + S_{s,t_{k_0}^n}^n \otimes S_{s,t_{k_1}^n}^n + S_{s,t_{k_1}^n}^n \otimes S_{s,t}.$$ 

As we have already observed, we have that $A^n_{s,t_{k_0}^n} = A^n_{t_{k_1}^n,t} = 0$. By the inequality (2.7), we have

$$|A^n_{t_{k_0}^n,t_{k_1}^n}|_\mathbb{F} \leq w(t_{k_0}^n,t_{k_1}^n) \leq w(t_{k_0}^n-1,t).$$

We estimate the remaining terms as

$$|S_{s,t_{k_0}^n}^n \otimes S_{t_{k_0}^n,t_{k_1}^n}|_\mathbb{F} + |S_{t_{k_1}^n,t}|_\mathbb{F} \leq |S_{s,t_{k_0}^n}^n|^p + |S_{t_{k_0}^n,t_{k_1}^n}^n|^p + |S_{t_{k_1}^n,t}|^p \leq w(t_{k_0}^n-1,t_{k_0}^n) + w(t_{k_0}^n,t_{k_1}^n) + w(t_{k_0}^n-1,t_{k_1}^n) + w(t_{k_1}^n,t) \leq 2w(t_{k_0}^n-1,t),$$

so that, putting this all together, we deduce the existence of a constant $\hat{C} > 0$ such that $|A^n_{s,t}|_\mathbb{F} \leq \hat{C}w(t_{k_0}^n-1,t)$. Taking an arbitrary partition $\mathcal{P}$ of the interval $[0,T]$, it follows that $\sum_{[s,t] \in \mathcal{P}} |A^n_{s,t}|_\mathbb{F} \leq 2\hat{C}w(0,T)$. We thus conclude that (2.9) holds with $C = (2\hat{C})\frac{2}{\mathbb{F}}$. \qed

Lemma 2.12. Suppose that $S \in D([0,T];\mathbb{R}^d)$ satisfies Property (RIE) with respect to $p$ and $(\mathcal{P}^n)_{n \in \mathbb{N}}$. With the natural notation $\int_s^t S_u \otimes dS_u := \int_s^t S_u \otimes dS_u - \int_0^s S_u \otimes dS_u$, we define $A: \Delta_{[0,T]} \to \mathbb{R}^{d \times d}$ by

$$A_{s,t} = \int_s^t S_u \otimes dS_u - S_s \otimes S_{s,t}, \quad (s,t) \in \Delta_{[0,T]}.$$ 

Then, the triplet $S = (S,S,A)$ is a càdlàg $p$-rough path.

Proof. It is straightforward to verify Chen’s relation (2.1), i.e. that

$$A_{s,t} = A_{s,u} + A_{u,t} + S_{s,u} \otimes S_{u,t}, \quad (s,t) \in \Delta_{[0,T]}.$$ 

By Property (RIE), we know that $\lim_{n \to \infty} A^n_{s,t} = A_{s,t}$, where the convergence is uniform in $(s,t)$, and thus, being a uniform limit of càdlàg functions, $A$ is itself càdlàg. By the lower semi-continuity of the $\mathbb{F}$-variation norm, and the result of Lemma 2.11 we have that

$$\|A\|_{\mathbb{F},[0,T]} \leq \liminf_{n \to \infty} \|A^n\|_{\mathbb{F},[0,T]} \leq Cw(0,T)\frac{2}{\mathbb{F}} < \infty.$$ 

It follows that $(S,S,A)$ is a càdlàg $p$-rough path. \qed
2.3. The rough integral as a limit of Riemann sums. While the rough integral in (2.2) is a powerful tool to study various differential equations, it lacks the natural interpretation as the capital gain process in the context of mathematical finance. The aim of this subsection is to restore this interpretation by showing that the rough integral can be obtained as the limit of left-point Riemann sums provided that the integrator satisfies Property (RIE). As preparation we need the following approximation result.

**Proposition 2.13.** Let \( \mathcal{P}^n = \{0 = t^n_0 < t^n_1 < \cdots < t^n_{N^n} = T\} \), \( n \in \mathbb{N} \), be a sequence of nested partitions with vanishing mesh size, so that \( \mathcal{P}^n \subseteq \mathcal{P}^{n+1} \) for all \( n \), and \( |\mathcal{P}^n| \to 0 \) as \( n \to \infty \) (as in the setting of Property (RIE)). Let \( F : [0, T] \to \mathbb{R}^d \) be a càdlàg path, and define

\[
F^n_t = F_T 1_{[T]}(t) + \sum_{k=0}^{N^n-1} F^n_{t_k} 1_{[t_k, t_{k+1})}(t), \quad t \in [0, T].
\]

Let

\[
J_F := \{ t \in (0, T) : F_{t-} \neq 0 \}
\]

be the set of jump times of \( F \). The following are equivalent:

(i) \( J_F \subseteq \bigcup_{n \in \mathbb{N}} \mathcal{P}^n \),

(ii) The sequence \( (F^n)_{n \in \mathbb{N}} \) converges pointwise to \( F \),

(iii) The sequence \( (F^n)_{n \in \mathbb{N}} \) converges uniformly to \( F \).

**Proof.** We first show that conditions (i) and (ii) are equivalent. To this end, suppose that \( J_F \subseteq \bigcup_{n \geq 1} \mathcal{P}^n \) and let \( t \in (0, T] \). If \( t \in J_F \), then there exists \( m \geq 1 \) such that \( t \in \mathcal{P}^n \) for all \( n \geq m \). In this case we then have that \( F^n_t = F_t \) for all \( n \geq m \). If \( t \notin J_F \), then \( F_{t-} = F_t \), and since the mesh size \( |\mathcal{P}^n| \to 0 \), it follows that \( F^n_t \to F_{t-} = F_t \) as \( n \to \infty \).

Now suppose instead that there exists a \( t \in J_F \) such that \( t \notin \bigcup_{n \geq 1} \mathcal{P}^n \). We then observe that \( F^n_t \to F_{t-} \neq F_t \), so that \( F^n_t \not\to F_t \). This establishes the equivalence of (i) and (ii).

Since (iii) clearly implies (ii), it only remains to show that (ii) implies (iii). By [22, Theorem 3.3], it is enough to show that the family of paths \( \{F^n : n \geq 1\} \) is equiregulated in the sense of [22, Definition 3.1].

**Step 1.** Let \( t \in (0, T] \) and \( \varepsilon > 0 \). Since the left limit \( F_{t-} \) exists, there exists \( \delta > 0 \) with \( t - \delta > 0 \), such that

\[
|F_{s,t-}| < \frac{\varepsilon}{2} \quad \text{for all } s \in (t - \delta, t).
\]

Let

\[
m = \min\{n \geq 1 : \exists k \text{ such that } t^n_k \in (t - \delta, t)\}.
\]

Since \( |\mathcal{P}^n| \to 0 \) as \( n \to \infty \), we know that \( m < \infty \). Moreover, since the sequence of partitions is nested, we immediately have that, for all \( n \geq m \), there exists a \( k \) such that \( t^n_k \in (t - \delta, t) \). We define

\[
u = \min\{t^n_k \in \mathcal{P}^m : t^n_k \in (t - \delta, t)\} = \min(\mathcal{P}^m \cap (t - \delta, t)),
\]

and let \( s \in [u, t) \) and \( n \geq 1 \).

If \( n < m \), then there does not exist a \( k \) such that \( t^n_k \in (t - \delta, t) \), which implies that \( F^n \) is constant on the interval \( (t - \delta, t) \), and hence that \( F^n_s = F^n_{t-} \).

Suppose instead that \( n \geq m \). Let \( i = \max\{k : t^n_k \leq s\} \) and \( j = \max\{k : t^n_k < t\} \). By the definition of \( u \), we see that \( t^n_i \in [u, t) \) and \( t^n_j \in [u, t) \). Then

\[
|F^n_s - F^n_{t-}| = |F^n_{t^n_i} - F^n_{t^n_j}| \leq |F^n_{t^n_i,t-}| + |F^n_{t^n_j,t-}| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]
Thus, we have that \( |F_s^n - F_t^n| < \varepsilon \) for all \( s \in [u, t] \) and all \( n \geq 1 \).

**Step 2.** Let \( t \in (J_F \cup \{0\}) \setminus \{T\} \) and \( \varepsilon > 0 \). Since \( F \) is right-continuous, there exists a \( \delta > 0 \) with \( t + \delta < T \), such that

\[
|F_{t,s}| < \varepsilon \quad \text{for all} \quad s \in [t, t + \delta).
\]

By part (i), we know that \( t \in \bigcup_{n \geq 1} \mathcal{P}^n \). Let

\[
m = \min \{ n \geq 1 : \exists k \text{ such that } t_k^n = t \}.
\]

Since \( t \in \bigcup_{n \geq 1} \mathcal{P}^n \), it is clear that \( m < \infty \). We define

\[
u = \min \{ t_k^n \in \mathcal{P}^m : t_k^n > t \} = \min(\mathcal{P}^m \cap (t, T)).
\]

We then let \( v \in (t, u \wedge (t + \delta)) \), \( s \in (t, v) \), and \( n \geq 1 \).

If \( n < m \), then, since \( v < u \), there does not exist a \( k \) such that \( t_k^n \in [t, v] \). Hence, \( F^n \) is constant on the interval \([t, v]\), so that in particular \( F^n_s = F^n_t \).

Suppose instead that \( n \geq m \). By the definition of \( m \), there exists a \( j \) such that \( t_j^n = t \). Let \( i = \max \{ k : t_k^n \leq s \} \). In particular, we then have that \( t = t_j^n \leq t_i^n \leq s \leq v < t + \delta \), and hence that

\[
|F^n_s - F^n_t| = |F^n_{i,t} - F^n_{j,t}| = |F^n_{i,t,v}| < \varepsilon.
\]

Thus, we have that \( |F^n_s - F^n_t| < \varepsilon \) for all \( s \in (t, v) \) and all \( n \geq 1 \).

**Step 3.** Let \( t \in (0, T) \setminus J_F \) and \( \varepsilon > 0 \). Since \( F \) is continuous at time \( t \), there exists a \( \delta > 0 \) with \( 0 < \delta < t - \delta \) and \( t + \delta < T \), such that

\[
|F_{s,t}| < \frac{\varepsilon}{2} \quad \text{for all} \quad s \in (t - \delta, t + \delta).
\]

Let

\[
m = \min \{ n \geq 1 : \exists k \text{ such that } t_k^n \in (t - \delta, t] \}.
\]

Since \( |\mathcal{P}^n| \to 0 \) as \( n \to \infty \), we know that \( m < \infty \). We define

\[
u = \min \{ t_k^n \in \mathcal{P}^m : t_k^n > t \} = \min(\mathcal{P}^m \cap (t, T)).
\]

We then let \( v \in (t, u \wedge (t + \delta)) \), \( s \in (t, v) \), and \( n \geq 1 \).

If \( n < m \), then, since \( v < u \), there does not exist a \( k \) such that \( t_k^n \in (t, v] \). Hence, \( F^n \) is constant on the interval \([t, v]\), so that in particular \( F^n_s = F^n_t \).

Suppose instead that \( n \geq m \). Let \( i = \max \{ k : t_k^n \leq s \} \) and \( j = \max \{ k : t_k^n \leq t \} \). Since, by the definition of \( m \), there exists at least one \( k \) such that \( t_k^n \in (t - \delta, t] \), and since \( t < s \leq v < t + \delta \), it follows that \( t_i^n \in (t - \delta, t + \delta) \) and \( t_j^n \in (t - \delta, t] \). Then

\[
|F^n_s - F^n_t| = |F^n_{i,t} - F^n_{j,t}| = |F^n_{i,t,v}| + |F^n_{j,t,v}| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]

Thus, we have that \( |F^n_s - F^n_t| < \varepsilon \) for all \( s \in (t, v) \) and all \( n \geq 1 \). It follows that the family of paths \( \{ F^n : n \geq 1 \} \) is indeed equi-\( \varepsilon \)-regulated.

The next theorem is the main result of this section, stating that the rough integral can be approximated by left-point Riemann sums along a suitable sequence of partitions, in the spirit of Föllmer’s pathwise integration.

**Theorem 2.14.** Let \( q \geq p \) such that \( \frac{2}{p} + \frac{1}{q} > 1 \), and let \( r > 1 \) such that \( \frac{2}{r} = \frac{1}{p} + \frac{1}{q} \). Suppose that \( S \in D([0, T] ; \mathbb{R}^d) \) satisfies Property (RIE) with respect to \( p \) and \( (\mathcal{P}_n)_{n \in \mathbb{N}} \). Let \( (F, F') \in \mathcal{Y}^{q,r}_S \) and \( (G, G') \in \mathcal{Y}^{q,r}_S \) be controlled paths with respect to \( S \), and assume that \( J_F \subseteq \bigcup_{n \in \mathbb{N}} \mathcal{P}^n \), where
$J_F$ is the set of jump times of $F$, as in (2.11). Then the rough integral of $(F,F')$ against $(G,G')$ relative to the rough path $S = (S,S,A)$, as defined in (2.2), is given by

$$
\int_0^t F_u \, dG_u = \lim_{n \to \infty} \sum_{k=0}^{N_n-1} F_{t_k} G_{t_k}^p t_{k+1}^a \wedge t_k^b, \wedge t_{k+1}^{b+1},
$$

where the convergence is uniform in $t \in [0,T]$.

**Proof.** We recall from Lemma 2.12 that $S = (S,S,A)$ is a $p$-rough path, so that, by Proposition 2.14, the rough integral of $(F,F')$ against $(G,G')$ (relative to $S$) exists. It is also clear that $S^n := (S,S^n,A^n)$ is a $p$-rough path, where $A^n$ was defined in (2.8). Moreover, by Property (RIE), we immediately have that $S^n$ and $A^n$ converge uniformly to $S$ and $A$ respectively as $n \to \infty$.

For each $n \geq 1$, let $F^n$ be the path defined in (2.10). We consider the pair $(F^n,F')$ as a controlled path with respect to $S^n$, defining the remainder term $R^n$ by the usual relation:

$$
F^n_{s,t} = F^n_S s,t + R^n_{s,t}, \quad (s,t) \in \Delta_{[0,T]}.
$$

Since $S^n$ converges uniformly to $S$ and, by Proposition 2.13, $F^n$ converges uniformly to $F$, it follows that $R^n$ also converges uniformly to the remainder term $R$ corresponding to the $S$-controlled path $(F,F')$.

We observe that $\|S^n\|_{p,[0,T]} \leq \|S\|_{p,[0,T]}$ and $\|F^n\|_{p,[0,T]} \leq \|F\|_{p,[0,T]}$, and we have from Lemma 2.11 that $\|A^n\|_{\frac{p}{q},[0,T]} \leq C W(0,T)^\frac{q}{p}$ for every $n \geq 1$. It remains to show that $R^n$ is bounded in $r$-variation, uniformly in $n$.

Let $n \geq 1$ and $(s,t) \in \Delta_{[0,T]}$. If there exists a $k$ such that $t_k^n \leq s < t < t_k^{n+1}$, then

$$
R^n_{s,t} = F^n_S s,t - F^n_S s,t^n + F^n_S s,t^n - F^n_S s,t^n = 0.
$$

If there exists a $k$ such that $t_k^n \leq s < t = t_k^{n+1}$, then

$$
|R^n_{s,t}| = |F^n_S s,t - F^n_S s,t^n| = |F^n_S s,t_k^n - F^n_S s,t_k^{n+1}| = |F^n_S s,t_k^n - F^n_S s,t_k^{n+1}| = |F^n_S s,t_k^n - F^n_S s,t_k^{n+1}| + |F^n_S s,t_k^n - F^n_S s,t_k^{n+1}| = |F^n_S s,t_k^n - F^n_S s,t_k^{n+1}| + |F^n_S s,t_k^n - F^n_S s,t_k^{n+1}|,
$$

where in the last line we used Young’s inequality, recalling that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$.

Otherwise, let $k_0$ be the smallest $k$ such that $t_k^n \in [s,t]$, and let $k_1$ be the largest such $k$. After a short calculation, we find that

$$
R^n_{s,t} = R^n_{s,t_0} + R^n_{t_0,t_{k_1}} + R^n_{t_{k_1},t} + F^n_{s,t_{k_1}} S^n_{t_{k_1}} + F^n_{t_{k_1}} S^n_{t_{k_1}},
$$

and we deal with the terms $R^n_{s,t_0}$ and $R^n_{t_{k_1},t}$ using the above, and we bound $|F^n_{s,t_{k_1}} S^n_{t_{k_1}}| \leq |F^n_{s,t_{k_1}}| + |S^n_{t_{k_1}}|^q$. Putting this all together, we have that

$$
|R^n_{s,t}| \leq C \left( |R^n_{t_{k_1},t_0} + |F^n_{s,t_{k_1}}| + |F^n_{s,t_{k_1}}| + |F^n_{s,t_{k_1}}| + |F^n_{s,t_{k_1}}| \right) + |F^n_{s,t_{k_1}}|^q + |S^n_{t_{k_1}}|^{p+q},
$$

where the constant $C$ depends only on $p,q,r$. Taking an arbitrary partition $\mathcal{P}$ of the interval $[0,T]$, we deduce that $\sum_{[s,t] \in \mathcal{P}} |R^n_{s,t}|r \leq 2C(\|R^n\|_{r,[0,T]}^r + \|F^n\|_{q,[0,T]}^q + \|S^n\|_{p,[0,T]}^p)$. Thus, $\|R^n\|_{r,[0,T]}$ is bounded uniformly in $n \geq 1$.

Let $p' > p$, $q' > q$ and $r' > r$, such that $p' \in (2,3)$, $q' \geq p'$, $q' > q$ and $r' > r$, such that $p' \in (2,3)$, $q' > q$ and $r' > r$. Since the sequence $(S^n)_{n \geq 1}$ has uniformly bounded $p$-variation, and $S^n$ converges uniformly to $S$, we have that $S^n$ converges uniformly to $S$.
S as \( n \to \infty \), it follows by interpolation that \( S^n \) converges to \( S \) with respect to the \( p' \)-variation norm, i.e. \( \| S^n - S \|_{p',[0,T]} \to 0 \) as \( n \to \infty \). It follows similarly that \( \| A^n - A \|_{q',[0,T]} \to 0 \) and \( \| R^n - R \|_{p',[0,T]} \to 0 \), and hence also that \( \| S^n; S \|_{p',[0,T]} \to 0 \) as \( n \to \infty \). It thus follows from part (ii) of Proposition 2.7 that

\[
\int_0^t F_u^n \, dG_u \longrightarrow \int_0^t F_u \, dG_u \quad \text{as} \quad n \longrightarrow \infty,
\]

where the convergence is uniform in \( t \in [0,T] \). Note that in (2.12) the integral \( \int_0^t F_u^n \, dG_u \) is defined relative to the rough path \( S^n = (S, S^n, A^n) \), whilst the limiting rough integral \( \int_0^t F_u \, dG_u \) is defined relative to \( S = (S, S, A) \).

We recall from Proposition 2.4 that the integral of \( (F^n, F') \) against \( (G, G') \) relative to \( S^n = (S, S^n, A^n) \) is given by the limit

\[
\int_0^t F_u^n \, dG_u = \lim_{|P| \to 0} \sum_{[u,v] \in P} F_u^n G_{u,v} + F'_u G'_{u,v} A^n_{u,v},
\]

where the limit is taken over any sequence of partitions of the interval \([0, t]\) with vanishing mesh size. Take any refinement \( \tilde{P} \) of the partition \( (P^n \cup \{t\}) \cap [0, t] \) (where as usual \( P^n \) is the partition given in Property (RIE)), and let \([u, v] \in \tilde{P}\). By the choice of the partition \( \tilde{P} \), there exists a \( k \) such that \( t_k^n \leq u < v \leq t_{k+1}^n \), which, recalling (2.8), implies that \( A^n_{u,v} = 0 \). Since the mesh size of \( \tilde{P} \) may be arbitrarily small, it follows that

\[
\lim_{|\tilde{P}| \to 0} \sum_{[u,v] \in \tilde{P}} F'_u G'_{u,v} A^n_{u,v} = 0.
\]

To conclude, we then simply recall (2.12), and note that

\[
\int_0^t F_u \, dG_u = \lim_{|P| \to 0} \sum_{[u,v] \in \tilde{P}} F_u G_{u,v} = \sum_{k=0}^{N_n-1} F_{t_k} G_{t_k \wedge t, t_{k+1} \wedge t}.
\]

We can actually generalize the result of Theorem 2.14 to a slightly larger class of integrands.

**Corollary 2.15.** Recall the assumptions of Theorem 2.14 and let \( \gamma \in D'([0,T]; \mathbb{R}^d) \). Then, the rough integral of the controlled path \( H = F + \gamma \), given by \( (H, H') := (F + \gamma, F') \in \mathcal{V}_{S}^{q,r} \), against \( (G, G') \) is given by

\[
\int_0^t H_u \, dG_u = \lim_{n \to \infty} \sum_{k=0}^{N_n-1} H_{t_k} G_{t_k \wedge t, t_{k+1} \wedge t}
\]

for every \( t \in [0,T] \).

The point here is that the path \( \gamma \) may have jump times which do not belong to the set \( \cup_{n \in \mathbb{N}} P^n \).

**Proof.** Since \( \gamma \) has finite \( r \)-variation, we immediately have that \( \gamma \) is a controlled path with Gubinelli derivative simply given by \( \gamma' = 0 \). By linearity, it is then clear that \( (H, H') =
\]
\((F,F') + (\gamma,0)\) is indeed a controlled path with respect to \(S\). Since \(\gamma' = 0\), we have from Proposition \([2.4]\) that
\[
\int_0^t \gamma_u \, dG_u = \lim_{|P| \to 0} \sum_{[u,v] \in P} \gamma_u G_{u,v} = \lim_{n \to \infty} \sum_{k=0}^{N_n-1} \gamma_{t_k} G_{t_k/t_{k+1} \wedge t}.
\]
By linearity, we have that \(\int_0^t H_u \, dG_u = \int_0^t F_u \, dG_u + \int_0^t \gamma_u \, dG_u\), and the result then follows from Theorem \([2.14]\).

### 2.4. Link to Föllmer integration.
In his seminal paper \([19]\), Föllmer introduced a notion of pathwise integration based on the concept of quadratic variation, and derived a corresponding pathwise Itô formula, which have proved to be useful tools in robust approaches to mathematical finance.

In the following we will write \(B[0,T]\) for the Borel \(\sigma\)-algebra on \([0,T]\).

**Definition 2.16.** Let \(S \in D([0,T]; \mathbb{R})\) and let \(\mathcal{P}^n = \{0 = t_0^n < t_1^n < \cdots < t_{N^n} = T\}, n \geq 1\), be a sequence of partitions with vanishing mesh size. We say that \(S\) has quadratic variation along \((\mathcal{P}^n)_{n \in \mathbb{N}}\) in the sense of Föllmer if the sequence of measures \((\mu_n)_{n \in \mathbb{N}}\) on \(([0,T], B[0,T])\) defined by

\[
\mu_n := \sum_{k=0}^{N_n-1} |S^n_{t_k,t_{k+1}}|^2 \delta_{t_k},
\]
converges weakly to a measure \(\mu\), such that the map \(t \mapsto [S]^c_t := \mu([0,t]) - \sum_{0 \leq s \leq t} |S_{s^-} - S_s|^2\) is continuous and increasing. In this case we call the function \([S]\), given by \([S]_t = \mu([0,t])\), the quadratic variation of \(S\) along \((\mathcal{P}^n)_{n \in \mathbb{N}}\).

We say that a path \(S \in D([0,T]; \mathbb{R}^d)\) has quadratic variation along \((\mathcal{P}^n)_{n \in \mathbb{N}}\) in the sense of Föllmer if the condition above holds for \(S^i\) and \(S^i + S^j\) for every \((i,j)\), and in this case we write
\[
[S^i, S^j] := \frac{1}{2}((S^i + S^j) - [S^i] - [S^j]).
\]

Assuming that a path \(S \in D([0,T]; \mathbb{R}^d)\) has quadratic variation along \((\mathcal{P}^n)_{n \in \mathbb{N}}\) and \(f \in C^2(\mathbb{R}; \mathbb{R})\), Föllmer showed that the limit
\[
\int_0^T Df(S_u) \, dS_u := \lim_{n \to \infty} \sum_{[s,t] \in \mathcal{P}^n} Df(S_s) S_{s,t}
\]
eexists and the resulting integral \(\int_0^T Df(S_u) \, dS_u\) satisfies a pathwise Itô formula, see \([19]\) THÉORÈME. Let us remark that the Föllmer integral \(\int_0^T Df(S_u) \, dS_u\) is only well-defined for gradients \(Df\) and not for general functions, as its existence is given by the corresponding pathwise Itô formula. This result can also be explained via the language rough path theory; see Friz and Hairer \([23]\) Chapter 5.3.

In the following we relate Property (RIE) to the existence of quadratic variation in the sense of Föllmer. To this end, for each \(i = 1, \ldots, d\), we introduce
\[
S^{n,i}_t = S^i_T \mathbf{1}_{[T]}(t) + \sum_{k=0}^{N_n-1} S^{i}_t \mathbf{1}_{[t_k, t_{k+1})}(T)
\]
and the discrete quadratic variation \( \langle S^i, S^j \rangle \) by

\[
\langle S^i, S^j \rangle_t^n = \sum_{k=0}^{N_n-1} S_{\tau_{k+1}^{i,\Lambda},t_{k+1}^{n,\Lambda}}^i S_{\tau_{k}^{j,\Lambda},t_{k}^{n,\Lambda}}^j - S_{\tau_{k}^{i,\Lambda},t_{k}^{n,\Lambda}}^i S_{\tau_{k+1}^{j,\Lambda},t_{k+1}^{n,\Lambda}}^j,
\]

\( t \in [0, T] \).

**Proposition 2.17.** Let \( S \in D([0, T]; \mathbb{R}^d) \) and let \( \mathcal{P}^n = \{ 0 = t_0^n < t_1^n < \cdots < t_{N_n}^n = T \} \), \( n \in \mathbb{N} \), be a sequence of nested partitions with vanishing mesh size. The following conditions are equivalent:

(i) For every pair \((i, j)\), the Riemann sums \( \int_0^t S_u^{n,i} \, dS_u^j + \int_0^t S_u^{n,j} \, dS_u^i \) converge uniformly to a limit, which we denote by \( \int_0^t S_u^i \, dS_u^j + \int_0^t S_u^j \, dS_u^i \).

(ii) For every pair \((i, j)\), the discrete quadratic variation \( \langle S^i, S^j \rangle \) converges uniformly to a càdlàg path, which we denote by \( \langle S^i, S^j \rangle \).

(iii) The path \( S \) has quadratic variation along \( \langle \mathcal{P}^n \rangle \) in the sense of Föllmer.

Moreover, if these conditions hold then the path \( \langle S^i, S^j \rangle \) has finite total variation, and, for every \((i, j)\), we have that \( \langle S^i, S^j \rangle = \langle S^i, S^j \rangle \) and the equality

\[
S_t^i S_t^j = S_0^i S_0^j + \int_0^t S_u^i dS_u^j + \int_0^t S_u^j dS_u^i + \langle S^i, S^j \rangle_t
\]

holds for every \( t \in [0, T] \).

**Proof.** We have

\[
S_t^i S_t^j - S_0^i S_0^j = \sum_{k=0}^{N_n-1} (S_{\tau_{k+1}^{i,\Lambda},t_{k+1}^{n,\Lambda}}^i S_{\tau_{k}^{j,\Lambda},t_{k}^{n,\Lambda}}^j - S_{\tau_{k}^{i,\Lambda},t_{k}^{n,\Lambda}}^i S_{\tau_{k+1}^{j,\Lambda},t_{k+1}^{n,\Lambda}}^j)
\]

\[
= \sum_{k=0}^{N_n-1} (S_{\tau_{k+1}^{i,\Lambda},t_{k+1}^{n,\Lambda}}^i S_{\tau_{k}^{i,\Lambda},t_{k}^{n,\Lambda}}^j + S_{\tau_{k}^{j,\Lambda},t_{k}^{n,\Lambda}}^j S_{\tau_{k+1}^{j,\Lambda},t_{k+1}^{n,\Lambda}}^i - S_{\tau_{k}^{i,\Lambda},t_{k}^{n,\Lambda}}^i S_{\tau_{k+1}^{j,\Lambda},t_{k+1}^{n,\Lambda}}^j)
\]

\[
= \int_0^t S_u^{n,i} \, dS_u^j + \int_0^t S_u^{n,j} \, dS_u^i + \langle S^i, S^j \rangle_t^n,
\]

from which it follows that conditions (i) and (ii) are equivalent, and that \(2.14\) then also holds. In this case, we also have that

\[
\langle S^i, S^j \rangle_t = \frac{1}{4} \left( \langle S^i + S^j, S^i + S^j \rangle_t - \langle S^i - S^j, S^i - S^j \rangle_t \right)
\]

so that, as the difference of two non-decreasing functions, \( \langle S^i, S^j \rangle \) has finite total variation.

For one-dimensional paths \( S \), the equivalence of conditions (ii) and (iii) follows from \[51\] Propositions 3 and 4]. The extension of this to \( d \)-dimensional paths \( S \) and the equality \( \langle S^i, S^j \rangle = \langle S^i, S^j \rangle \) then follow from the polarization identity

\[
\langle S^i, S^j \rangle_t^n = \frac{1}{2} \left( \langle S^i + S^j, S^i + S^j \rangle_t^n - \langle S^i - S^j, S^i - S^j \rangle_t^n - \langle S^i, S^i \rangle_t^n - \langle S^j, S^j \rangle_t^n \right)
\]

and the definition of \( \langle S^i, S^j \rangle \) in \(2.13\). \(\square\)

**Remark 2.18.** As an immediate consequence of Proposition 2.17, we have that if a path \( S \) satisfies (RIE) along \( \langle \mathcal{P}^n \rangle \), then it has quadratic variation along \( \langle \mathcal{P}^n \rangle \) in the sense of Föllmer, thus allowing one to apply all the known results regarding Föllmer integration.
In particular, if a vector field \( f: \mathbb{R}^d \to \mathbb{R} \) is of class \( C^3 \), then, by Theorem \ref{thm:smooth}, the Föllmer integral \( \int_0^t Df(S_u) \, dS_u \) coincides with the rough integral \( \int_0^t Df(S_u) \, dS_u \). We thus obtain the rough Itô formula:

\[
\begin{align*}
    f(S_t) - f(S_0) &= \int_0^t Df(S_u) \, dS_u + \frac{1}{2} \int_0^t D^2 f(S_u) \, d[S]_u \\
    &\quad + \sum_{0 < u \leq t} \left( f(S_u) - f(S_{u-}) - Df(S_{u-}) \Delta S_u - \frac{1}{2} D^2 f(S_{u-})(\Delta S_u \otimes \Delta S_u) \right),
\end{align*}
\]

which holds for every \( t \in [0, T] \), where \([S] = ([S^i, S^j])_{1 \leq i, j \leq d}\) denotes the quadratic variation matrix, and \( \Delta S_u := \lim_{x \to u, s < u} S_{s, u} \). We note that the formula above is precisely the Itô formula for rough paths derived in Friz and Zhang \cite{FrizZhang}. 

3. **Functionally generated trading strategies and their generalizations**

Given Property (RIE), we can introduce a model-free framework for continuous-time financial markets with a possibly infinite time horizon. In this section we shall verify that most relevant trading strategies from a practical perspective, such as delta-hedging strategies and functionally generated strategies, are admissible integrands for price paths satisfying Property (RIE). Furthermore, the underlying rough integration allows us to deduce stability estimates for admissible strategies.

3.1. **Price paths and admissible strategies.** For a path \( S: [0, \infty) \to \mathbb{R}^d \), we denote by \( S|_{[0,T]} \) the restriction of \( S \) to the interval \([0,T] \).

**Definition 3.1.** For a fixed \( p \in (2, 3) \), we say that a path \( S \in D([0, \infty); \mathbb{R}^d) \) is a price path, if there exists a nested sequence of locally finite partitions \( (\mathcal{P}^n)_{n \in \mathbb{N}} \) of the interval \([0, \infty) \), with vanishing mesh size on compacts, such that, for all \( T > 0 \), the restriction \( S|_{[0,T]} \) satisfies (RIE) with respect to \( p \) and \( (\mathcal{P}^n([0,T]))_{n \in \mathbb{N}} \).

We denote the family of all such price paths by \( \Omega_p \).

Note that the sequence of partitions \( (\mathcal{P}^n)_{n \in \mathbb{N}} \) may depend on the choice of price path \( S \in \Omega_p \), consistent with the stochastic framework where this sequence will naturally be defined in terms of (probabilistic) stopping times.

Having fixed the model-free structure of the underlying price paths, we can introduce the class of admissible strategies and the corresponding capital process.

**Definition 3.2.** Let \( p \in (2, 3) \) and let \( S \in \Omega_p \) be a price path. We say that a path \( \varphi: [0, \infty) \to \mathbb{R}^d \) is an admissible strategy (with respect to \( S \)), if

- there exist \( q \geq p \) and \( r > 1 \) with \( 2/p + 1/q > 1 \) and \( 1/r = 1/p + 1/q \), such that for every \( T > 0 \), there exists a path \( \varphi': [0, T] \to \mathcal{L}(\mathbb{R}^d; \mathbb{R}^d) \) such that the pair \( (\varphi, \varphi') \in \mathcal{V}^p_r \) is a controlled path with respect to \( S \) in the sense of Definition \ref{def:controlled}.

- and \( J_\varphi \subseteq \cup_{n \in \mathbb{N}} \mathcal{P}^n \), where \( J_\varphi \) is the set of jump times of \( \varphi \) in \([0, \infty) \), and \( (\mathcal{P}^n)_{n \in \mathbb{N}} \) is the sequence of partitions associated with the price path \( S \in \Omega_p \).

We denote the space of all admissible strategies (with respect to \( S \)) by \( \mathcal{A}_S \).

We define the capital process associated with \( \varphi \) and \( S \) as the path \( V^\varphi(S): [0, \infty) \to \mathbb{R} \) given by

\[
V^\varphi(t) := \lim_{n \to \infty} \sum_{k=0}^{N_n - 1} \sum_{i=1}^d \varphi^i_{k+1}(S^i_{k+1 \wedge t} - S^i_{k \wedge t}), \quad t \in [0, \infty),
\]
where $\mathcal{P}^n = \{0 = t^n_0 < t^n_1 < \cdots < t^n_N = T\}$ is the sequence of partitions specified in Property (RIE).

**Remark 3.3.** In a semimartingale setting, as often used in classical mathematical finance, one usually considers left-continuous trading strategies $\varphi$. In the present setting this assumption is not necessary, as the corresponding capital process $V^\varphi(S)$, which, as we will see below, may be expressed as a rough integral, does not change when replacing $\varphi$ by its left-continuous modification; see e.g. [25, Theorem 31]. The reason for this is essentially the left-point Riemann sum construction of the integral. Indeed, suppose that $S$ has a jump at a time $t > 0$. The contribution to the capital process $V^\varphi(S)$ at time $t$ is then given by $\lim_{s \to t, s < t} \varphi_s S_{s,t}$, which is invariant to the choice of $\varphi$ or its left-continuous modification. Furthermore, we will see that $V^\varphi(S)$ coincides with the classical stochastic Itô integral, whenever both the rough and stochastic integrals are defined; see Section 4.4 below.

The condition $J_\varphi \subseteq \bigcup_{n \in \mathbb{N}} \mathcal{P}^n$ means that one is allowed to use trading strategies whose jump points are included in the underlying sequence of partitions $(\mathcal{P}^n)_{n \in \mathbb{N}}$. On the one hand, many frequently used trading strategies, such as delta-hedging, satisfy this condition, and further examples will be discussed later in this section. On the other hand, the sequence of partitions $(\mathcal{P}^n)_{n \in \mathbb{N}}$ can be fixed a priori to allow for a desired class of trading strategies, e.g. buy and hold strategies along the sequence of dyadic partitions.

**Proposition 3.4.** Let $p \in (2,3)$, let $S \in \Omega_p$ be a price path, and let $\varphi \in \mathcal{A}_S$ be an admissible strategy (in the sense of Definition 3.2). Then, the capital process $V^\varphi(S)$ as defined in (3.1) exists as a locally uniform limit, and is actually given by

$$V^\varphi_t(S) = \int_0^t \varphi_s \, dS_s, \quad t \in [0, \infty),$$

that is, the rough integral of the controlled path $(\varphi, \varphi') \in \mathcal{V}^\varphi_{S^r}$ against the rough path $S$ defined in Lemma 2.12.

Moreover, given another price path $\tilde{S} \in \Omega_p$ and an admissible strategy $\tilde{\varphi} \in \mathcal{A}_{\tilde{S}}$ with respect to $\tilde{S}$, we have, for every $T > 0$, that

$$|V^\varphi_T(S) - V^\varphi_T(\tilde{S})|$$

$$\leq C \left( (|\varphi_0| + ||\varphi, \varphi'||_{\mathcal{V}^\varphi_{S^r}})(1 + ||S||_{p,[0,T]} + ||\tilde{S}||_{p,[0,T]})||S; \tilde{S}||_{p,[0,T]} + (|\varphi_0 - \tilde{\varphi}_0| + |\varphi' - \tilde{\varphi}'|_{q,[0,T]} + ||R^\varphi - R^{\tilde{\varphi}}||_{r,[0,T]})(1 + ||S||_{p,[0,T]})||S||_{p,[0,T]} \right),$$

where the constant $C$ depends on $p, q$ and $r$.

**Remark 3.5.** Recall from Remark 2.2 that the rough integral in (3.2) is defined by the limit $\int_0^t \varphi_s \, dS_s = \lim_{|\pi| \to 0} \sum_{[u,v] \in \pi} \varphi_u S_{u,v} + \varphi'_u A_{u,v}$, where the limit is taken over any sequence of partitions of the interval $[0,t]$ with vanishing mesh size. Here, $\varphi_u$ and $S_{u,v}$ both take values in $\mathbb{R}^d$, and we interpret their multiplication as the Euclidean inner product. The derivative $\varphi'_u$ takes values in $\mathcal{L}(\mathbb{R}^d; \mathbb{R}^d)$, which we can also identify with $\mathcal{L}(\mathbb{R}^{d \times d}; \mathbb{R})$. Since $A_{u,v} \in \mathbb{R}^{d \times d}$, the product $\varphi'_u A_{u,v}$ also takes values in $\mathbb{R}$.

**Proof of Proposition 3.4.** Let $T > 0$, and let $(\mathcal{P}^n)_{n \in \mathbb{N}}$ be a sequence of nested partitions such that $p$, $(\mathcal{P}^n)_{n \in \mathbb{N}}$ and $S$ satisfy Property (RIE) on the interval $[0,T]$. Recall from Property (RIE) the existence of the limit $\int_0^T S_u \otimes \, dS_u$ for every $t \in [0,T]$. By Lemma 2.12 defining
the function \( A: \Delta[0,T] \to \mathbb{R}^{d \times d} \) by

\[
A_{s,t} := \int_s^t S_u \otimes dS_u - S_s \otimes S_{s,t},
\]

we have that the triplet \( S = (S, S, A) \) is a càdlàg rough path (in the sense of Definition 2.1). Hence, the rough integral in (3.2) is well-defined by Proposition 2.4 (see also Remark 2.5), and satisfies (3.3) as a locally uniform limit by Theorem 2.1.

For the stability estimate we simply note that

\[
|V^f_T(S) - V^f_T(\tilde{S})| = \left| \int_0^T \varphi_s dS_s - \int_0^T \tilde{\varphi}_s d\tilde{S}_s \right| \leq \left\| \int_0^T \varphi_s dS_s - \int_0^T \tilde{\varphi}_s d\tilde{S}_s \right\|_{p,[0,T]}
\]

and apply part (i) of Proposition 2.7.

In the following we show that the most relevant trading strategies from a practical viewpoint belong to the class of admissible strategies in the sense of Definition 3.2.

### 3.2. Functionally generated trading strategies

Having fixed the set \( \Omega_p \) of underlying price paths, we start by introducing functionally generated portfolios. For this purpose, for some \( d_A \in \mathbb{N} \), we fix a càdlàg path \( A: [0, \infty) \to \mathbb{R}^{d_A} \) of locally bounded variation and assume that the jump times of \( A \) belong to the union of the partitions \( (P_n)_{n \in \mathbb{N}} \) appearing in Property (RIE); that is, we assume that \( J_A \subseteq \bigcup_{n \in \mathbb{N}} P_n \), where \( J_A := \{ t \in (0, \infty) : A_{t^{-}, t} \neq 0 \} \).

For the stability estimate we simply note that

\[
|V^f_T(S) - V^f_T(\tilde{S})| = \left| \int_0^T \varphi_s dS_s - \int_0^T \tilde{\varphi}_s d\tilde{S}_s \right| \leq \left\| \int_0^T \varphi_s dS_s - \int_0^T \tilde{\varphi}_s d\tilde{S}_s \right\|_{p,[0,T]}
\]

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The path \( A \) is supposed to include additional information pertaining to the market which a trader would like to include in their trading decisions. For instance, the components of the path \( A = (A^1, \ldots, A^{d_A}) \) could include time \( t \mapsto t \), the running maximum \( t \mapsto \max_{u \in [0,t]} S_u \), or the integral \( t \mapsto \int_0^t S_u \, du \) for some (or all) \( i = 1, \ldots, d \). A more detailed discussion on practical choices of the path \( A \) can be found in Schied, Speiser and Voloshchenko [14].

For \( \ell = d + d_A \), we denote by \( C^2_b(\mathbb{R}^\ell; \mathbb{R}^d) \) the space of twice continuously differentiable (in the Fréchet sense) functions \( f: \mathbb{R}^\ell \to \mathbb{R}^d \) such that \( f \) and its derivatives up to order 2 are uniformly bounded; that is

\[
C^2_b(\mathbb{R}^\ell; \mathbb{R}^d) := \{ f \in C^2(\mathbb{R}^\ell; \mathbb{R}^d) : \|f\|_{C^2} < \infty \}
\]

with

\[
\|f\|_{C^2} := \|f\|_\infty + \|Df\|_\infty + \|D^2f\|_\infty.
\]

For \( S \in \Omega_p \) we introduce the set \( \mathcal{G}^2_p \) of all generalized functionally generated trading strategies \( \varphi^f \), which are all strategies of the form

\[
\varphi^f_t = (f^1_t, \ldots, f^d_t) := f(S_t, A_t), \quad t \in [0, \infty),
\]

for some \( f \in C^2_b(\mathbb{R}^\ell; \mathbb{R}^d) \). For \( \varphi^f \in \mathcal{G}^2_p \) the corresponding capital process is given by

\[
V^f_t(S) = \lim_{n \to \infty} \sum_{k=0}^{N_t-1} \sum_{i=1}^d \left| \int_{k \Delta t}^{k+1 \Delta t} S^i_{S_t \wedge t} - S^i_{S_{s \wedge t}} \right|, \quad t \in [0, \infty).
\]

### Proposition 3.6

Let \( p \in (2, 3) \) and \( S \in \Omega_p \), and let \( \varphi^f, \tilde{\varphi}^f \in \mathcal{G}^2_p \). Then, \( \varphi^f \) is an admissible strategy, and the capital process \( (V^f_t(S))_{t \in [0, \infty)} \) given in (3.4) is well-defined as a locally uniform limit in \( t \in [0, \infty) \). Moreover, for every \( T \in [0, \infty) \), we have the stability estimate

\[
|V^f_T(S) - V^f_T(\tilde{S})| \leq C\|f - \tilde{f}\|_{C^2} (1 + \|S\|_{p,[0,T]}^2 + \|A\|_{1,[0,T]})(1 + \|S\|_{p,[0,T]}^2)\|S\|_{p,[0,T]}.
\]
where the constant $C$ depends only on $p$, and the triplet $S = (S, S, A)$ is the càdlàg rough path defined in Lemma 2.12.

Proof. Admissibility: Let $\varphi = \varphi^f \in \mathcal{G}_S^2$ be a functionally generated strategy $\varphi = (\varphi^1, \ldots, \varphi^d)$ of the form in (3.3) for some $f \in C^2_b(\mathbb{R}^d; \mathbb{R}^d)$. Fix a $T \in [0, \infty)$. We claim that $(\varphi, \varphi') \in \mathcal{V}_S^{p, q}$ is a controlled path with respect to $S$ in the sense of Definition 2.3 (with $q = p$ and $r = p/2$), where

$$\varphi'_t := D_S f(S_t, A_t), \quad t \in [0, T],$$

and $D_S f$ denotes the derivative of $f$ with respect to its first $d$ components. To see this, we first note that

$$|\varphi'_{s,t}| = |D_S f(S_t, A_t) - D_S f(S_s, A_s)| \leq \|f\|_{C^2_b}(|S_{s,t}| + |A_{s,t}|),$$

so that

$$(3.6) \quad \|\varphi'\|_{p,[0,T]} \lesssim \|f\|_{C^2_b}(|S|_{p,[0,T]} + \|A\|_{1, [0,T]}) < \infty,$$

and hence $\varphi' \in D^p([0,T]; \mathcal{L}(\mathbb{R}^d; \mathbb{R}^d))$. Moreover, we have that

$$R_{s,t}^\varphi := \varphi_{s,t} - \varphi'_{s,t} S_{s,t}$$

$$= f(S_t, A_t) - f(S_s, A_s) - D_S f(S_s, A_s) S_{s,t}$$

$$= f(S_t, A_s) - f(S_s, A_s) - D_S f(S_s, A_s) S_{s,t} + f(S_t, A_t) - f(S_t, A_s)$$

$$= \int_0^t \left( D_S f(S_s + \tau S_{s,t}, A_s) - D_S f(S_s, A_s) \right) S_{s,t} d\tau + f(S_t, A_t) - f(S_t, A_s),$$

so that $|R_{s,t}^\varphi| \leq \|f\|_{C^2_b}(|S_{s,t}| + |A_{s,t}|)$. It follows that

$$(3.7) \quad \|R^\varphi\|_{2, [0,T]} \lesssim \|f\|_{C^2_b}(|S|_{2, [0,T]} + \|A\|_{1, [0,T]}) < \infty,$$

so that $R^\varphi \in D^{p/2}((\Delta_{[0,T]}; \mathbb{R}^d)$, and thus the conditions of Definition 2.3 are satisfied. Thus, by Proposition 2.4 (and Remark 2.5), we have the existence for each $t \in [0, T]$ of the $(\mathbb{R}$-valued) rough integral

$$\int_0^t \varphi_s dS_s = \lim_{|p| \to 0} \sum_{[u,v] \in \mathcal{P}} \varphi_u S_{u,v} + \varphi'_u A_{u,v}.$$ 

For a given path $F$, let $J_F = \{t \in (0, T) : F_{t-} \neq F_t\}$ denote the jump times of $F$. It follows from Property (RIE) and Proposition 2.13 that $J_S \subseteq \cup_{n \in \mathbb{N}} \mathcal{P}^m$. Since we also assumed that $J_A \subseteq \cup_{n \in \mathbb{N}} \mathcal{P}^m$, it then follows from (3.3) that $J_\varphi \subseteq \cup_{n \in \mathbb{N}} \mathcal{P}^m$. Thus, by Theorem 2.14 we have that

$$\int_0^t \varphi_s dS_s = \lim_{n \to \infty} \sum_{k=0}^{N_n-1} \varphi^t_k (S^m_{t+1, \Lambda^t} - S^m_{t, \Lambda^t}) = \lim_{n \to \infty} \sum_{k=0}^{N_n-1} \sum_{i=1}^d f^t_{i,k} (S^m_{t+1, \Lambda^t} - S^m_{t, \Lambda^t}) = V^t_f(S),$$

and that this limit is uniform in $t \in [0, T]$. 
Stability estimate: Let \( \varphi \) and \( \tilde{\varphi} \) be the strategies generated by \( f \) and \( \tilde{f} \) respectively, as defined in (3.3). By part (i) of Proposition 2.7, we have the estimate

\[
|V_T^f(S) - V_T^\tilde{f}(S)|
\]

Next, for \([s, t] \subseteq [0, T]\), we compute

\[
|\varphi_0 - \tilde{\varphi}_0| = |f(S_0, A) - \tilde{f}(S_0, A)| \leq \|f - \tilde{f}\|_{\infty} \leq \|f - \tilde{f}\|_{C^0_b},
\]

and similarly

\[
|\varphi'_0 - \tilde{\varphi}'_0| = |D_S f(S_0, A_0) - D_S \tilde{f}(S_0, A_0)| \leq \|D_S f - D_S \tilde{f}\|_{\infty} \leq \|f - \tilde{f}\|_{C^0_b}.
\]

Next, for \([s, t] \subseteq [0, T]\), we compute

\[
(\varphi' - \tilde{\varphi}')_{s,t} = D_S f(S_t, A_t) - D_S f(S_s, A_s) + D_S \tilde{f}(S_t, A_t) - D_S \tilde{f}(S_s, A_s)
\]

so that

\[
|\varphi' - \tilde{\varphi}'|_{p, [0, T]} \leq \|D^2_S f - D^2_S \tilde{f}\|_{\infty} |S_{s,t}| + \|D^2_A f - D^2_A \tilde{f}\|_{\infty} |A_{s,t}| \leq \|f - \tilde{f}\|_{C^0_b} (|S_{s,t}| + |A_{s,t}|),
\]

and thus

\[
\|\varphi' - \tilde{\varphi}'\|_{p, [0, T]} \leq \|f - \tilde{f}\|_{C^0_b} (\|S\|_{p,[0,T]} + \|A\|_{1,[0,T]}).
\]

Finally, we have that

\[
(R^\varphi - R^{\tilde{\varphi}})_{s,t}
\]

Finally, we have that

\[
(\varphi' - \tilde{\varphi}')_{s,t}
\]
so that
\[ |(R^e - R^f)_{s,t}| \leq \|D^2_{SS}f - D^2_{SS}f\|_\infty |S_{s,t}|^2 + \|D_Af - D_Af\|_\infty |A_{s,t}| \leq \|f - \bar{f}\|_{C^2_b}(|S_{s,t}|^2 + |A_{s,t}|), \]
and hence
\[ (3.12) \quad \|R^e - R^f\|_{L^2([0,T])} \lesssim \|f - \bar{f}\|_{C^2_b}(|S|_{p,[0,T]}^2 + \|A\|_{1,[0,T]}). \]
Substituting (3.9), (3.10), (3.11) and (3.12) into (3.8), we deduce that the estimate in (3.5) holds. \( \square \)

### 3.3. Path-dependent functionally generated trading strategies.

The functionally generated trading strategies considered in Section 3.2 could depend on the past prices only through a process of locally finite variation. In some contexts it is beneficial to work with trading strategies possessing a more general path-dependent structure; see e.g. Schied, Speiser and Voloshchenko for more detailed discussions in this direction. A common way to treat path-dependent and non-anticipating trading strategies is the calculus initiated by Dupire and Cont and Fournié for more detailed discussions in this direction. A common way to treat path-dependent and non-anticipating trading strategies is the calculus initiated by Dupire and Cont and Fournié. For the sake of brevity we recall here only the essential definitions and refer to Ananova [3, Section 3.1] and [11] for full details.

A functional \( F: [0,T] \times D([0,T]; \mathbb{R}^d) \rightarrow \mathbb{R}^d \) is called non-anticipative if
\[ F(t, S) = F(t, S_{\Lambda t}) \quad \text{for all} \quad S \in D([0,T]; \mathbb{R}^d). \]

As usual, the non-anticipative functionals are defined on the space of stopped paths, defined as the equivalence class in \([0,T] \times D([0,T]; \mathbb{R}^d)\) with respect to the following equivalence relation:
\[ (t, S) \sim (t', S') \iff t = t' \quad \text{and} \quad S_{\Lambda t} = S'_{\Lambda t}. \]
The resulting space \( \Lambda_T^d \) is equipped with the distance
\[ d_\infty((t, S), (t', S')) := |t - t'| + \sup_{u \in [0,T]} |S_{u \wedge t} - S'_{u \wedge t'}|, \]
which turns \((\Lambda_T^d, d_\infty)\) into a complete metric space. We introduce the following spaces:

- \( \mathcal{C}_{\varepsilon}([0,T], \Lambda_T^d) \) is the space of left-continuous functionals \( F: \Lambda_T^d \rightarrow \mathbb{R}^d \), i.e. for all \((t, S) \in \Lambda_T^d \) and \( \varepsilon > 0 \), there exists \( \nu > 0 \) such that, for all \((t', S') \in \Lambda_T^d \),
  \[ t' < t \quad \text{and} \quad d_\infty((t, S), (t', S')) < \nu \implies |F(t, S) - F(t', S')| < \varepsilon. \]

- \( \mathbb{B}(\Lambda_T^d) \) is the space of all boundedness-preserving functionals \( F: \Lambda_T^d \rightarrow \mathbb{R}^d \), i.e. for every compact subset \( K \subset \mathbb{R}^d \) and for every \( t_0 \in [0,T] \), there exists \( C > 0 \) such that, for all \( t \in [0,t_0] \) and \((t, S) \in \Lambda_T^d \),
  \[ S([0,t]) \subseteq K \implies |F(t, S)| < C. \]

- \( \text{Lip}(\Lambda_T^d, d_\infty) \) is the space of all Lipschitz continuous functionals \( F: \Lambda_T^d \rightarrow \mathbb{R}^d \), i.e. there exists \( C > 0 \) such that, for all \((t, S), (t', S') \in \Lambda_T^d \),
  \[ |F(t, S) - F(t', S')| \leq C d_\infty((t, S), (t', S')). \]

We define \( \mathcal{C}_{\varepsilon}^{1,1}(\Lambda_T^d) \) as the set of non-anticipative functionals \( F: \Lambda_T^d \rightarrow \mathbb{R} \) which are:

- horizontally differentiable, i.e. for all \((t, S) \in \Lambda_T^d \),
  \[ \mathcal{D}F(t, S) = \lim_{h \downarrow 0} \frac{F(t + h, S_{\Lambda t}) - F(t, S_{\Lambda t})}{h} \]
exists, and \( \mathcal{D}F \) is continuous at fixed times,
• and vertically differentiable, i.e. for all \((t, S) \in \Lambda^d_T\), \(\nabla_x F(t, S) = (\partial_i F(t, S))_{i=1,\ldots,d}\), with
\[
\partial_i F(t, S) = \lim_{h \to 0} \frac{F(t, S \wedge h e_i 1_{[t,T]}) - F(t, S \wedge t)}{h},
\]
even exists, where \((e_i)_{i=1,\ldots,d}\) is the canonical basis of \(\mathbb{R}^d\), and \(\nabla_x F \in C^{0,0}(\Lambda^d_T)\).

- and such that \(DF, \nabla_x F \in \mathbb{B}(\Lambda^d_T)\).

**Corollary 3.7.** If \(F \in C^1_{b}(\Lambda^d_T)\) with \(F\) and \(\nabla_x F\) in \(\text{Lip}(\Lambda^d_T, d_\infty)\), and \(S \in \Omega_p\), then the path-dependent functionally generated trading strategy \(F(\cdot, S)\) is an admissible strategy in the sense of Definition 3.2.

**Proof.** That \((F(\cdot, S), \nabla_x F(\cdot, S))\) is a controlled path with respect to \(S\) is an immediate consequence of [2, Lemma 5.12]; see also [3, Lemma 3.7]. The admissibility condition regarding the jump times of \(F(\cdot, S)\) is ensured by the Lipschitz continuity of \(F\).

**Remark 3.8.** The standard examples of sufficiently regular path-dependent functionals are functionals which depend on the running maximum or on a notion of the average of the underlying path; see e.g. Ananova [2]. Further examples include:

(i) \(F(t, S, \Lambda_t) := \int_0^t \psi(s, S_{\Lambda_s-}) \, d[S]_s\),

(ii) \(F(t, S, \Lambda_t) := \int_0^t \nabla_x f(s, S_{\Lambda_s}) \, ds\),

(iii) \(F(t, S, \Lambda_t) := \sum_{i=1}^d \int_0^t (S^i_t - S^i_s) f_i(S^i_s) \, dS^i_s - \int_0^t f_i(S^i_s) \, d[S^i]_s\),

for a twice differentiable function \(f = (f_1, \ldots, f_d) : \mathbb{R}^d \to \mathbb{R}^d\) and a left-continuous and locally bounded function \(\psi : \Lambda^d_T \to \mathbb{R}^{1 \times d}\); see Chiu and Cont [10, Example 4.18].

### 3.4. Cover’s universal portfolio

While functionally generated trading strategies are most prominent in the literature regarding hedging and control problems in mathematical finance, various other trading strategies with desirable properties have been considered. One example coming from portfolio theory is Cover’s universal portfolio, as introduced in [12]. The basic idea is to invest, not according to one specific trading strategy, but according to a mixture of various other trading strategies with desirable properties. Following Cuchiero, Schachermayer and Wong [13], we introduce here a model-free analogue of Cover’s universal portfolio.

Let \(Z\) be a Borel measurable subset of \(C^2_b(\mathbb{R}^d; \mathbb{R}^d)\), and suppose that \(\nu\) is a probability measure on \(Z\). A model-free version of Cover’s universal portfolio \(\varphi_\nu\) is then given by

\[
\varphi_\nu^t := \int_Z \varphi^t \, d\nu(f), \quad t \in [0, \infty),
\]

where \(\varphi^t = f(S)\) is the portfolio generated by \(f\), for some fixed \(S \in \Omega_p\).

**Lemma 3.9.** Let \(S \in \Omega_p\) and let \(\nu\) be a probability measure on \(Z\), as above. If

\[
\int_Z \|f\|_C^2 \, d\nu(f) < \infty,
\]

then Cover’s universal portfolio \(\varphi_\nu\), as defined in (3.13), is an admissible strategy in the sense of Definition 3.2.

**Proof.** Let \(T > 0\). We know from Proposition 3.6 that, for each \(f \in Z\), the corresponding functionally generated portfolio \(\varphi = \varphi^f\) is an admissible strategy. Let \(\varphi^t\) and \(R^\varphi\) denote the
corresponding Gubinelli derivative and remainder term. It follows from the inequalities in (3.6) and (3.7) that
\[
|\varphi_0| + \|\varphi, \varphi'|_\mathcal{V}^{p,2} = |\varphi_0| + |\varphi'|_\mathcal{V}^{p,0} + \|R^\varphi\|_{\mathcal{L}_2[0,T]} \lesssim \|f\|_{C^2_0} (1 + \|S\|_{\mathcal{L}^2_0[0,T]}),
\]
and hence that
\[
(3.14) \quad \int_\mathbb{Z} (|\varphi_0| + \|\varphi, \varphi'|_\mathcal{V}^{p,2}) \, d\nu \lesssim (1 + \|S\|_{\mathcal{L}^2_0[0,T]}) \int_\mathbb{Z} \|f\|_{C^2_0} \, d\nu < \infty.
\]
Recall that the map \((\varphi, \varphi') \mapsto |\varphi_0| + \|\varphi, \varphi'|_\mathcal{V}^{p,2}\) is a norm on the Banach space of controlled paths \(\mathcal{V}^{p,2}_S\). Note moreover that the subset of controlled paths \((\varphi, \varphi') \in \mathcal{V}^{p,2}_S\) satisfying \(J_\varphi \subseteq \cup_{n \in \mathbb{N}} P^n\) is a closed linear subspace of \(\mathcal{V}^{p,2}_S\), and thus is itself a Banach space.

It follows from the integrability condition in (3.14) that the integral in (3.13) exists as a well-defined Bochner integral, and defines a controlled path \((\varphi', (\varphi')') \in \mathcal{V}^{p,2}_S\) satisfying \(J_{\varphi'} \subset \cup_{n \in \mathbb{N}} P^n\).

3.5. **Functionally generated portfolios from stochastic portfolio theory.** In this section we briefly discuss admissible strategies appearing in stochastic portfolio theory, as initiated by Fernholz [18]; see also e.g. Strong [47] and Karatzas and Ruf [33]. Following the model-free framework for stochastic portfolio theory, as introduced in Schied, Speiser and Voloshchenko [41] and Cuchiero, Schachermayer and Wong [13], we consider a \(d\)-dimensional càdlàg path \(S = (S^1, \ldots, S^d)\) such that \(S^i_t > 0\) for all \(t \geq 0\) and \(i = 1, \ldots, d\). The total capitalization \(\Sigma\) is defined by \(\Sigma_t := S^1_t + \cdots + S^d_t\), and the relative market weight process \(\mu\) is given by
\[
\mu^i_t := \frac{S^i_t}{\Sigma_t} = \frac{S^1_t + \cdots + S^d_t}{\Sigma_t}, \quad i = 1, \ldots, d,
\]
which takes values in the open unit simplex \(\Delta^d := \{x \in \mathbb{R}^d : \sum_{i=1}^d x_i = 1, x_i > 0\ \text{for all} \ i\}\).

Here we impose that the market weight process \(\mu = (\mu^1, \ldots, \mu^d)\) satisfies Property (RIE) with respect to some \(p \in (2, 3)\) and a sequence of nested partitions \((P^n)_{n \in \mathbb{N}}\). More precisely, we assume that \(\mu\) is a price path, in the sense of Definition 3.3.

Given a controlled path \(\theta \in \mathcal{V}^{0,\mathbb{R}}_p\), we define its associated value evolution \(V^\theta\) by \(V^\theta_t := \sum_{i=1}^d \theta^i_t \mu^i_t\) for \(t \geq 0\). We also denote \(Q^\theta_t := V^\theta_t - V^\theta_0 - \int_0^t \theta_s \, d\mu_s\), where \(\int_0^t \theta_s \, d\mu_s\) is interpreted as a rough integral, as in Remark 2.5. As in classical mathematical finance (see e.g. [33, Proposition 2.3]) we can show that every controlled path \(\theta \in \mathcal{V}^{0,\mathbb{R}}_p\) induces a self-financing trading strategy \(\varphi\), and that every such strategy \(\varphi\) is itself a controlled path in \(\mathcal{V}^{0,\mathbb{R}}_p\).

**Proposition 3.10.** Given \(\theta \in \mathcal{V}^{0,\mathbb{R}}_p\) and a constant \(C \in \mathbb{R}\), we introduce
\[
\varphi^i_t := \theta^i_t - Q^\theta_t - C, \quad t \geq 0, \quad i = 1, \ldots, d.
\]
Then the resulting path \(\varphi = (\varphi^1, \ldots, \varphi^d)\) is a controlled path in \(\mathcal{V}^{0,\mathbb{R}}_p\), and \(\varphi\) is self-financing, in the sense that
\[
V^\varphi_t - V^\varphi_0 = \int_0^t \varphi_s \, d\mu_s, \quad t \geq 0,
\]
where \(V^\varphi_t = \sum_{i=1}^d \varphi^i_t \mu^i_t\). Moreover, if \(\theta\) is an admissible strategy in the sense of Definition 3.2, then so is \(\varphi\).
Proof. Since $\theta \in \mathcal{V}_{\mu}^{q,r}$ is a controlled path, by (the trivial extension to càdlàg paths of) \cite[Lemma A.1]{1}, the path $V^\theta = \sum_{i=1}^d \theta^i \mu^i$ is also a controlled path. Recalling Remark 2.18 we also have that the rough integral $\int_0^t \theta \, d\mu$ is itself a controlled path (with Gubinelli derivative equal to $\theta$). It is then clear that $Q^\theta = V^\theta - V_0^\theta - \int_0^t \theta_s \, d\mu_s$ is a controlled path, and hence that $\varphi^i = \theta^i - Q^\theta - C$ is as well for every $i = 1, \ldots, d$, so that $\varphi \in \mathcal{V}_{\mu}^{q,r}$. The self-financing property of $\varphi$ can be verified by following the proof of \cite[Proposition 2.3]{1}.

We know from Proposition 2.13 and the fact that $\mu$ satisfies Property (RIE) that the jump times $J_\mu$ of $\mu$ satisfy $J_\mu \subseteq \bigcup_{n \in \mathbb{N}} \mathcal{P}^n$. It is also clear that the jump times of the integral $\int_0^t \theta \, d\mu$ form a subset of $J_\mu$. Thus, under the assumption that $\theta$ is an admissible strategy, so that $J_\theta \subseteq \bigcup_{n \in \mathbb{N}} \mathcal{P}^n$, we also deduce that $J_\varphi \subseteq \bigcup_{n \in \mathbb{N}} \mathcal{P}^n$, so that $\varphi$ is itself an admissible strategy.

The following definition introduces notions which can be seen as the rough path counterparts of the functionally generated strategies induced by regular and Lyapunov functions from stochastic portfolio theory; see \cite[Definitions 3.1 and 3.3]{3}.

**Definition 3.11.** We say that a continuous function $G: \Delta_+^d \to \mathbb{R}$ is regular for the market weight process $\mu$ if

(i) there exists a measurable function $DG = (D_1 G, \ldots, D_d G)$ on $\Delta_+^d$ such that the path $\theta = (\theta^1, \ldots, \theta^d)$, given by $\theta^i = D_i G(\mu)$ for $i = 1, \ldots, d$, is an admissible strategy (in the sense of Definition 3.2), and

(ii) the (càdlàg) path $\Gamma^G$, given by

$$\Gamma^G_t := G(\mu_0) - G(\mu_t) + \int_0^t \theta_s \, d\mu_s, \quad t \geq 0,$$

has locally bounded variation.

We say that a regular function $G: \Delta_+^d \to \mathbb{R}$ is a Lyapunov function for the market weight process $\mu$ if the path $\Gamma^G$ is also nondecreasing.

**Example 3.12.** Suppose that $G: \Delta_+^d \to \mathbb{R}$ is a $C^3$ function. It is then straightforward to see that $\theta = DG(\mu)$ defines a controlled path, with $DG$ here defined as the classical gradient of $G$. It also follows from the Itô formula for rough paths (recall Remark 2.13) that

$$\Gamma^G_t = -\frac{1}{2} \int_0^t D^2 G(\mu_s) \, d[\mu]^c_s - \sum_{s \leq t} (G(\mu_s) - G(\mu_s-) - DG(\mu_s-) \Delta \mu_s),$$

where $[\mu]^c$ is the continuous part of the quadratic variation of $\mu$, and $\Delta \mu_s = \mu_s - \mu_{s-}$ denotes the jump of $\mu$ at time $s$.

If we also assume that $G$ is concave, then we infer from (3.15) that $\Gamma^G$ is nondecreasing. In this case $G$ is a Lyapunov function in the sense of Definition 3.17. Two important examples of such functions $G$ are the Gibbs entropy function, $H(x) := \sum_{i=1}^d x_i \log \frac{1}{x_i}$, and the quadratic function, $Q^c(x) := c - \sum_{i=1}^d x_i^2$ for some $c \in \mathbb{R}$; see \cite[Section 5.1]{3}.

**Remark 3.13.** If $\mu$ is realized by a semimartingale model, then any continuous concave (but not necessarily $C^3$) function $G$ can be a candidate for a Lyapunov function in the sense of Definition 3.3. This is essentially because in stochastic portfolio theory one only needs the “supergradients” $DG$ to be measurable, so that $DG(\mu)$ is integrable with respect to the semimartingale $\mu$. In contrast, in our purely pathwise setup, measurability of DG alone is not enough to ensure that $\theta = DG(\mu)$ is controlled by $\mu$, so we require more regularity of
the generating function $G$. This illustrates a difference between stochastic integration in a probabilistic setting and rough integration when only a single deterministic path is considered; for more detailed discussions on this theme, we refer to [1]. On the other hand, as shown in the previous example, many important Lyapunov functions from stochastic portfolio theory (such as the Gibbs entropy function) are actually smooth, so they do induce controlled paths (and indeed admissible strategies in the sense of Definition 3.2) for almost all trajectories of $\mu$, and the stability results established in Section 3 remain valid for these functionally generated strategies. It would be interesting to explore further Lyapunov functions for rough paths, but this is beyond the scope of the present paper.

Based on the previous observations, we can also extend the following notions from classical stochastic portfolio theory (e.g. [33]) to our rough path setting. The proof is straightforward and is therefore omitted for brevity.

**Corollary 3.14.** Suppose that the market weight process $\mu = (\mu^1, \ldots, \mu^d)$ is a price path in the sense of Definition 3.1, and let $\theta = (\theta^1, \ldots, \theta^d)$ be an admissible strategy for $\mu$ in the sense of Definition 3.2. Let $G$ be a regular function for $\mu$ in the sense of Definition 3.11. Then the following paths are also admissible strategies for $\mu$:

(i) the additive generated strategy: $\varphi^i = \theta^i - Q^i_1 - C_0$, $i = 1, \ldots, d$, where $Q^i_1 = V^\theta_1 - V^\theta_0 - \int_0^t \theta_s \, d\mu_s$ and $C_0 = \sum_{j=1}^d \mu^1_0 D_j G(\mu_0) - G(\mu_0)$,

(ii) the portfolio weights associated with $\varphi$: $\pi^i = \frac{\mu^i_1 \varphi^i}{\sum_{j=1}^d \mu^j_1 \varphi^j}$, $i = 1, \ldots, d$,

(iii) the multiplicative generated strategy: $\Phi^i = \eta^i - Q^i_1 - C_0$, for $\eta^i = \theta^i \exp(\int_0^t \frac{dG}{G(\mu_s)})$, $i = 1, \ldots, d$, and $Q^i_1 = V^\eta_1 - V^\eta_0 - \int_0^t \eta_s \, d\mu_s$, where here the function $G$ is also assumed to be positive and bounded away from zero,

(iv) the portfolio weights associated with $\Phi$: $\Pi^i = \mu^i_1 (1 + \frac{1}{G(\mu_t)} D_1 G(\mu_t) - \sum_{j=1}^d D_j G(\mu_t) \mu^j_1)$, $i = 1, \ldots, d$.

4. SEMIMARTINGALES AND TYPICAL PRICE PATHS SATISFY PROPERTY (RIE)

In this section we show that many stochastic processes commonly used to model the price evolutions on financial markets satisfy Property (RIE) along a suitable sequence of partitions. In particular, we verify that Property (RIE) holds for semimartingales and for typical price paths in the sense of Vovk [50].

4.1. SEMIMARTINGALES. Semimartingales, such as geometric Brownian motion and Markov jump-diffusion processes, serve as the most frequently used stochastic processes to model price evolutions on financial markets. For more details on semimartingales and Itô integration we refer to the standard textbook by Protter [42, Chapter II].

Usually the considered class of semimartingales is restricted to those satisfying the condition “no free lunch with vanishing risk” in classical mathematical finance, e.g. Delbaen and Schachermayer [15], or the condition “no unbounded profit with bounded risk” (NUPBR) in stochastic portfolio theory; see e.g. Karatzas and Kardaras [31]. Such a restriction is not required here. Property (RIE) is fulfilled by general càdlàg semimartingales with respect to any $p \in (2, 3)$ and a suitable (random) sequence of partitions.

Let us fix a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, \infty)}, \mathbb{P})$ and assume that the filtration $(\mathcal{F}_t)_{t \in [0, \infty)}$ satisfies the usual conditions, i.e. completeness and right-continuity. On this probability space we consider a $d$-dimensional càdlàg semimartingale $X = (X_t)_{t \in [0, \infty)}$. 




Thus, we have that
\[ X(4.2) = C \]
An application of the Burkholder–Davis–Gundy inequality and the Borel–Cantelli lemma, as
\[ Q \]
itself. For each \( (4.1) \) so that in particular
\[ \text{Fix a} \]
Proof. Step 1.

Proposition 4.1. Let \( X \) be a \( d \)-dimensional càdlàg semimartingale. Then almost every sample path of \( X \) is a price path in the sense of Definition 3.7.

Proof. Step 1. Fix a \( T > 0 \). Let \( \int_0^T X_u^- \otimes dX_u \) denote the Itô integral of \( X \) with respect to itself. For each \( n \in \mathbb{N} \), we define the discretized process \( X^n = (X^n_t)_{t \in [0,T]} \) by
\[ X^n_t := \sum_{[u,v] \in \mathcal{Q}_X^n([0,T])} X_u 1_{[u,v]}(t), \quad t \in [0,T], \]
so that in particular \( \int_0^T X^n_u^- \otimes dX_u = \sum_{[u,v] \in \mathcal{Q}_X^n([0,T])} X_u \otimes X_{u\wedge t} \wedge v \) for \( t \in [0,T] \). By the definition of the partition \( \mathcal{Q}_X^n \), we have that
\[ \|X^n - X^-\|_{\infty,[0,T]} \leq 2^{1-n} \quad \text{for all} \quad n \geq 1. \]
An application of the Burkholder–Davis–Gundy inequality and the Borel–Cantelli lemma, as in the proof of [31] Proposition 3.4, then yields the existence of a measurable set \( \Omega' \subseteq \Omega \) with full measure such that, for every \( \omega \in \Omega' \) and every \( \varepsilon \in (0,1) \), there exists a constant \( C = C(\varepsilon, \omega) \) such that
\[ \left\| \left( \int_0^T X^n_{u^-} \otimes dX_u - \int_0^T X_{u^-} \otimes dX_u \right)(\omega) \right\|_{\infty,[0,T]} \leq C2^{-n(1-\varepsilon)}, \quad \forall n \geq 1. \]
Thus, we have that \( X^n(\omega) \rightarrow X(\omega) \) and \( \int_0^T X^n_{u^-} \otimes dX_u(\omega) \rightarrow \int_0^T X_{u^-} \otimes dX_u(\omega) \) uniformly as \( n \rightarrow \infty \), for every \( \omega \in \Omega' \).

Step 2. We choose a \( q_0 \in (2,3) \) close enough to 2 and an \( \varepsilon \in (0,1) \) small enough such that
\[ \frac{p}{2} > \max \left\{ \frac{q_0}{2}, q_0 - 1 \right\} \quad \text{and} \quad \frac{p}{2} \geq \frac{q_0 - 1 - \varepsilon}{1 - \varepsilon}. \]
We also fix a control function \( w_{X,q_0} \) such that
\[
|X_{s,t}|^{q_0} \leq w_{X,q_0}(s,t) \quad \text{for all } (s,t) \in \Delta_{[0,T]}.
\]
This is always possible as \( X \) has almost surely finite \( q \)-variation for every \( q > 2 \), and so without loss of generality we may assume that \( X(\omega) \) has finite \( q_0 \)-variation for every \( \omega \in \Omega' \).

Note that by (4.3) we have \( p > q \) and consequently (by increasing the values of \( w_{X,q_0} \) by a multiplicative constant if necessary) we can also assume that
\[
\sup_{(s,t) \in \Delta_{[0,T]}} \frac{|X_{s,t}|^p}{w_{X,q_0}(s,t)} \leq 1.
\]

Let \( 0 \leq s < t \leq T \) be such that \( s = \tau_{k_0}^n \) and \( t = \tau_{k_0+N}^n \) for some \( n \in \mathbb{N} \), \( k_0 \in \mathbb{N} \cup \{0\} \) and \( N \geq 1 \). By the superadditivity of the control function \( w_{X,q_0} \), there must exist an \( \ell \in \{1,2,\ldots,N-1\} \) such that
\[
w_{X,q_0}(\tau_{k_0+l-1}^n,\tau_{k_0+l}^n) \leq \frac{2}{N-1}w_{X,q_0}(s,t).
\]
Thus, by (4.3),
\[
|X_{\tau_{k_0+l-1}^n,\tau_{k_0+l}^n} \otimes X_{\tau_{k_0+l-1}^n,\tau_{k_0+l}^n} - X_s \otimes X_{s,t}| \leq \sum_{j=2}^{\ell} \left( \frac{2}{N-1}w_{X,q_0}(s,t) \right)^{2/q_0}
\]
\[
\leq N^{-2/q_0}w_{X,q_0}(s,t)^{2/q_0}.
\]
By successively removing in this manner all the intermediate points from the partition \( \{s = \tau_{k_0}^n, \tau_{k_0+1}^n, \ldots, \tau_{k_0+N}^n = t\} \), we obtain the estimate
\[
\left| \sum_{i=0}^{N-1} X_{\tau_{k_0+i}^n} \otimes X_{\tau_{k_0+i}^n,\tau_{k_0+i+1}^n} - X_s \otimes X_{s,t} \right| \leq \sum_{i=0}^{N-1} \left( \frac{2}{N-1}w_{X,q_0}(s,t) \right)^{2/q_0}
\]
\[
\leq N^{1-2/q_0}w_{X,q_0}(s,t)^{2/q_0}.
\]
Since \( w_{X,q_0}(s,t) \geq \sum_{i=0}^{N-1} w_{X,q_0}(\tau_{k_0+i}^n,\tau_{k_0+i+1}^n) \geq \sum_{i=0}^{N-1} |X_{\tau_{k_0+i}^n,\tau_{k_0+i+1}^n}|^{q_0} \geq N2^{-nq_0} \), we have that \( N \leq 2^{nq_0}w_{X,q_0}(s,t) \). Substituting this into the above, we have that
\[
\left| \int_s^t X_{u-} \otimes dX_u - X_s \otimes X_{s,t} \right| \lesssim 2^{n(q_0-2)}w_{X,q_0}(s,t),
\]
where here the discretized version \( X^n \) is defined relative to the partition \( \{\tau_0^n, \tau_1^n, \tau_2^n, \ldots\} \).

If, more generally, \( 0 \leq s < t \leq T \) are such that \( s,t \in \mathcal{P}_X^n \), then \( s = \tau_{k_1}^{m_1} \) and \( t = \tau_{k_2}^{m_2} \) for some \( m_1, m_2 \leq n \) and \( k_1, k_2 \in \mathbb{N} \cup \{0\} \). In this case, the number of partition points \( N \) above satisfies \( N \leq \sum_{m=1}^n 2^{mq_0}w_{X,q_0}(s,t) \lesssim 2^{nq_0}w_{X,q_0}(s,t) \), and we thus still obtain the same bound in (4.6). If we further allow the pair of times \( s,t \) to include the times \( \sigma_k^m \) for \( m \leq n \) and \( k \in \mathbb{N} \cup \{0\} \), then this at most doubles the total number of partition points \( N \), so we can again obtain the same bound. Since the points \( \rho_{k,i}^m \) lie inside the interval \( [\sigma_k^m, \sigma_k^{m+1}] \) on which \( X \) is constant, it is clear for instance that \( X_{\rho_{k,i}^m} = 0 \) and \( \int_{\sigma_k^m}^{\sigma_k^{m+1}} X_{u-} \otimes dX_u = 0 \) for every \( i \in \mathbb{N} \), and it follows that these terms will not contribute anything to the bound above.

Thus, the bound in (4.6) actually holds for all \( 0 \leq s < t \leq T \) with \( s,t \in \mathcal{Q}_X^n \), with \( X^n \) defined relative to the partition \( \mathcal{Q}_X^n \), as in (4.1).
Step 3. We first consider the case that \( w_{X,q_0}(s,t)^{2(1-\varepsilon)} \leq 2^{-n} \). In this case it follows from (4.3) and (4.6) that there exists a constant \( C_1 = C_1(\omega, q_0, \varepsilon) \) such that
\[
\left| \int_s^t X^n_{u-} \otimes dX_u - X_s \otimes X_{s,t} \right| \leq C_1 w_{X,q_0}(s,t).
\]

Now we consider the case that \( w_{X,q_0}(s,t)^{2(1-\varepsilon)} \geq 2^{-n} \). Let \( X_{a,t} := \int_s^t X^n_{u-} \otimes dX_u - X_s \otimes X_{s,t} \) be the second level component of the Itô lift of \( X \). By [34] Proposition 3.4, we know that \( X \) possesses finite \( \frac{p}{2} \)-variation; that is, there exists a control function \( w_{X,p} \) such that
\[
\sup_{(s,t) \in \Delta_{[0,T]}} \frac{|X_{s,t}|^p}{w_{X,p}(s,t)} \leq 1.
\]

Then, in view of (4.2), we obtain
\[
\left| \int_s^t X^n_{u-} \otimes dX_u - X_s \otimes X_{s,t} \right| \leq 2 \left( \left| \int_0^t X^n_{u-} \otimes dX_u - \int_0^t X_{u-} \otimes dX_u \right|_{\infty,[0,T]} + |X_{s,t}| \right)
\leq C_2 \left( 2^{-n(1-\varepsilon)} + w_{X,p}(s,t)\frac{2}{p} \right)
\leq C_2 \left( w_{X,q_0}(s,t)\frac{2}{p} + w_{X,p}(s,t)\frac{2}{p} \right)
\]
for some constant \( C_2 = C_2(\varepsilon, \omega) \), and hence
\[
(4.8) \quad \left| \int_s^t X^n_{u-} \otimes dX_u - X_s \otimes X_{s,t} \right| \frac{2}{p} \leq C_3 \left( w_{X,q_0}(s,t) + w_{X,p}(s,t) \right),
\]
where \( C_3 = (2C_2)^\frac{2}{p} \). Letting \( \tilde{w}_{X,p}(s,t) := 2(1 + C_1 + C_3)(w_{X,q_0}(s,t) + w_{X,p}(s,t)) \), and combining (4.5), (4.7), and (4.8), we conclude that for every \( \omega \in \Omega' \),
\[
\sup_{(s,t) \in \Delta_{[0,T]}} \frac{|X_{s,t}|^p}{\tilde{w}_{X,p}(s,t)} + \sup_{n \in \mathbb{N}} \sup_{s,t \in Q^2_X} \frac{\left| \int_s^t X^n_{u-} \otimes dX_u - X_s \otimes X_{s,t} \right|^{\frac{2}{p}}}{\tilde{w}_{X,p}(s,t)} \leq 1,
\]
from which Property (RIE) follows. 

Remark 4.2. The result of Proposition [4.1] implies that almost every sample path of a semimartingale \( X \) satisfies (RIE) with respect to a suitable sequence of partitions (which depends on the choice of sample path), and may therefore be canonically lifted to a rough path. However, we also see from the proof of Proposition [4.1] that the resulting rough path lift \( (s,t) \mapsto \int_s^t X_{s,u} \otimes dX_u \) is nothing but the standard Itô-rough path lift of \( X \). Thus, the rough path lift itself does not actually depend on the choice of sequence of partitions. For (almost) every sample path, the sequence of partitions specified in Property (RIE) is merely a choice of partitions along which the value of the Itô integral may be well approximated by left-point Riemann sums in a pathwise fashion.

This is analogous to Föllmer integration, in which the quadratic variation of a semimartingale is well-defined as a stochastic process, but to make sense of the quadratic variation in a pathwise sense requires a suitable choice of sequence of partitions, which depends on the sample path being considered. Indeed, given any continuous path, it is possible to find a sequence of partitions along which the quadratic variation of the path is equal to zero (see Freedman [23] (70) Proposition). Similarly, it is natural to allow the sequence of partitions specified in
Property (RIE) to depend on the path being considered, but in practice there typically exists a natural choice for this sequence which results in the desired rough path.

Remark 4.3. Proposition (4.1) holds true even if the semimartingale $X$ takes values in an (infinite dimensional) Hilbert space $E$, as long as the norm on $E \otimes E$ is admissible in the sense of Lyons, Caruana and Lévy [39, Definition 1.25]. In particular, an extension of Proposition (4.1) to so-called piecewise semimartingales, which were introduced in Strong [46, Definition 2.2] as generalized semimartingales with an image dimension evolving randomly in time, appears to be straightforward to implement. As discussed in Karatzas and Kim [32, Remark 6.2 and Section 7], piecewise semimartingales provide a realistic framework to model so-called open markets, which are financial markets with an evolving number of traded assets.

4.2. Generalized semimartingales. It is a well observed fact in the empirical literature, see e.g. Lo [35], that price processes appear regularly in financial markets which are not semimartingales. Motivated by this fact, many researchers have proposed and investigated financial models based on fractional Brownian motions; see for instance Jarrow, Protter and Sayit [29], Cheridito [8] or Bender [6]. One example of such models are the so-called mixed Black–Scholes models. In these models the (one-dimensional) price process $S = (S_t)_{t \in [0, \infty)}$ is usually given by

$$S_t := s_0 \exp(\sigma W_t + \eta Y_t + \nu t + \mu t^{2H}), \quad t \in [0, \infty),$$

for constants $s_0, \sigma, \eta > 0$ and $\nu, \mu \in \mathbb{R}$, where $W = (W_t)_{t \in [0, \infty)}$ is standard Brownian motion and $Y = (Y_t)_{t \in [0, \infty)}$ is a fractional Brownian motion with Hurst index $H \in (0, 1)$. Multi-dimensional versions of the mixed Black–Scholes model (4.9) can be obtained by standard modifications. Notice that, while the price process $S$ as defined in (4.9) is not a semimartingale if $H \neq 1/2$, the mixed Black–Scholes model (4.9) is still arbitrage-free when restricting the admissible trading strategies to classes of trading strategies which, roughly speaking, exclude continuous rebalancing of the positions in the underlying market, cf. [29, 8, 6]. In particular, the mixed Black–Scholes model (4.9) is free of simple arbitrage opportunities if $H > 1/2$, as proven in [6, Section 4.1].

In order to demonstrate that Property (RIE) is satisfied by various financial models based on fractional Brownian motion, we consider the following class of generalized semimartingales.

On a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, \infty)}, \mathbb{P})$ with a complete right-continuous filtration $(\mathcal{F}_t)_{t \in [0, \infty)}$, let $Z$ be a $d$-dimensional process admitting the decomposition

$$Z_t = X_t + Y_t, \quad t \in [0, \infty),$$

where $X$ is a semimartingale, and $Y$ is a càdlàg adapted process with finite $q$-variation for some $q \in [1, 2)$. Processes $Z$ of this form are sometimes called Young semimartingales, and belong to the class of càdlàg Dirichlet processes in the sense of Föllmer [20].

We introduce stopping times $(\tau^n_k)$, such that, for each $n \in \mathbb{N}$, $\tau^n_0 = 0$, and

$$\tau^n_k := \inf\{t > \tau^n_{k-1} : |X_t - X_{\tau^n_{k-1}}| \geq 2^{-n} \text{ or } |Y_t - Y_{\tau^n_{k-1}}| \geq 2^{-n}\}$$

for $k \in \mathbb{N}$, and set

$$\mathcal{P}_Z^n := \{\tau^n_k : m \leq n, k \in \mathbb{N} \cup \{0\}\}.$$

As in the previous section, since we insist that the sequence of partitions in Property (RIE) has vanishing mesh size, we also define

$$\mathcal{Q}_Z^n := \mathcal{P}_Z^n \cup \{\sigma^n_m : m \leq n, k \in \mathbb{N} \cup \{0\}\} \cup \{\rho^n_{k,i} : m \leq n, k \in \mathbb{N} \cup \{0\}, i \in \mathbb{N}\},$$
where the times $\sigma^n_k$ and $\rho^n_{k,i}$ are defined analogously as in Section 4.4

**Proposition 4.4.** Let $Z = X + Y$ be a $d$-dimensional process such that $X$ is a càdlàg semimartingale, and $Y$ is a càdlàg process with finite $q$-variation for some $q \in [1, 2)$. Then, for any $p \in (2, 3)$ such that $1/p + 1/q > 1$, almost every sample path of $Z$ is a price path in the sense of Definition 3.1.

**Proof.** Step 1. It is sufficient to prove that almost all sample paths of $Z$ satisfy Property (RIE) along $(Q^n_Z([0, T]))_{n \in \mathbb{N}}$ for an arbitrary $T > 0$. To this end, for each $n \in \mathbb{N}$ we set

$$Z^n_t := \sum_{[u, v) \in Q^n_Z([0, T])} Z_u 1_{[u, v)}(t), \quad t \in [0, T],$$

and define $X^n$ and $Y^n$ in the same way with respect to the partition $Q^n_Z([0, T])$.

Let $X = M + A$ be a decomposition of the semimartingale $X$ such that $M$ is a locally square integrable martingale and $A$ is of bounded variation. By setting $B := A + Y$, we can write $Z = M + B$, where $B$ has finite $q$-variation. We define the Itô integral of $Z$ with respect to itself by

$$\int_0^t Z_u^- \otimes \mathrm{d}Z_u := \int_0^t Z_u^- \otimes \mathrm{d}M_u + \int_0^t Z_u^- \otimes \mathrm{d}B_u,$$

where the first integral on the right-hand side is an Itô integral and the second one is interpreted as a Young integral, which exists since $Z$ has finite $p$-variation and $1/p + 1/q > 1$; see e.g. Friz and Zhang [26]. Then, since $\|Z^n_u - Z_u\|_{\infty,[0,T]} \leq 2^{1-n}$, by the Burkholder–Davis–Gundy inequality and the Borel–Cantelli lemma, we deduce that for almost all $\omega$ and for every $\varepsilon \in (0, 1)$, there exists a constant $C = C(\omega, \varepsilon)$ such that, for all $n \in \mathbb{N},$

$$\left\| \int_0^t Z^n_u^- \otimes \mathrm{d}M_u - \int_0^t Z_u^- \otimes \mathrm{d}M_u \right\|_{\infty,[0,T]} \leq C 2^{-n(1-\varepsilon)}, \tag{4.10}$$

cf. the proof of [34] Proposition 3.4. By a standard bound for Young integrals (e.g. [26] Proposition 2.4), for every $t \in [0, T]$, we also have (noting that $Z^n_0 = Z_0$ for all $n$) that

$$\left| \int_0^t (Z^n_u^- - Z_u^-) \otimes \mathrm{d}B_u \right| \leq C \|Z^n_u - Z_u\|_{p, [0,T]} \|B\|_{q, [0,T]},$$

for some constant $C = C(p, q)$. Since $\|Z^n_u\|_{p_0, [0,T]} \leq \|Z\|_{p_0, [0,T]}$ holds for every $n$ and every $p_0 \in (2, p)$, a routine interpolation argument shows that, for each $n \in \mathbb{N},$

$$\|Z^n_u^- - Z_u^-\|_{p, [0,T]} \leq C \|Z^n_u - Z_u\|_{\infty,[0,T]} \|Z\|_{p_0, [0,T]} \|Z\|_{p_0, [0,T]}$$

for some constant $C = C(p, p_0)$. Hence, since $\|Z^n_u^- - Z_u^-\|_{\infty,[0,T]} \leq 2^{1-n}$ for all $n$, we have

$$\lim_{n \to \infty} \left\| \int_0^t Z^n_u^- \otimes \mathrm{d}B_u - \int_0^t Z_u^- \otimes \mathrm{d}B_u \right\|_{\infty,[0,T]} = 0. \tag{4.11}$$

Combining (4.10) with (4.11), we conclude that, almost surely, the integral $\int_0^t Z^n_u^- \otimes \mathrm{d}Z_u$ converges uniformly to $\int_0^t Z_u^- \otimes \mathrm{d}Z_u$. 


Step 2. For every $n \in \mathbb{N}$ we set

$$Z_{s,t}^n := \int_s^t Z_{s,u-}^n \otimes dZ_u$$

$$= \int_s^t X_{s,u-}^n \otimes dX_u + \int_s^t Y_{s,u-}^n \otimes dY_u + \int_s^t X_{s,u-}^n \otimes dY_u + \int_s^t Y_{s,u-}^n \otimes dX_u,$$

for $(s, t) \in \Delta_{[0, T]}$. Moreover, we define

$$\tilde{X}_{s,t}^n := \int_s^t X_{s,u-}^n \otimes dX_u, \quad \tilde{Y}_{s,t}^n := \int_s^t Y_{s,u-}^n \otimes dY_u,$$

and

$$\tilde{XY}_{s,t}^n := \int_s^t X_{s,u-}^n \otimes dY_u, \quad \tilde{YX}_{s,t}^n := \int_s^t Y_{s,u-}^n \otimes dX_u,$$

for $(s, t) \in \Delta_{[0, T]}$. We seek a control function $c$ such that

$$\sup_{n \in \mathbb{N}} \sup_{(s,t) \in \Delta_{[0, T]}} \frac{|Z_{s,t}^n|}{c(s,t)} \leq 1.$$

Towards this aim, we first construct a control function $c_X$ such that the above bound holds for $X^n$. Since $||X^n - X||_{\infty, [0, T]} \leq 2^{1-n}$, we still have the bound in (4.2). We also choose $q \in (1, 2)$ and $\varepsilon \in (0, 1)$ as in (4.3), and let $w_{X,q_0}$ and $w_{Y,q}$ be control functions dominating the $q_0$-variation and $q$-variation for $X$ and $Y$ respectively. From the definition of the stopping times $\tau^n_{Xq_0} \in \mathcal{P}_2$ and the fact that $q < 2 < q_0$, it is easy to check that, for all $s < t$ with $s, t \in \mathcal{P}_2$, the number $N$ of partition points in $\mathcal{P}_2$ between $s$ and $t$ can be bounded by

$$N \leq \sum_{m=1}^{2^{mq_0}-1} \left( 2^{mq_0}w_{X,q_0}(s, t) + 2^{mq}w_{Y,q}(s, t) \right) \lesssim 2^{mq_0}w_{q_0,q}(s, t),$$

where $w_{q_0,q}(s, t) := w_{X,q_0}(s, t) + w_{Y,q}(s, t)$. By the same argument as in Step 2 of the proof of Proposition 4.1 we deduce that, for all $n \in \mathbb{N}$ and all $s < t$ with $s, t \in \mathcal{Q}_2^N$,

$$\left| \int_s^t X_{s,u-}^n \otimes dX_u - X_s \otimes X_{s,t} \right| \lesssim 2^{n(q_0-2)}w_{q_0,q}(s, t),$$

which, as in Step 3 of the proof of Proposition 4.1 allows us to conclude the existence of a control function $c_X$ such that

$$\sup_{n \in \mathbb{N}} \sup_{(s,t) \in \Delta_{[0, T]}} \frac{|X_{s,t}^n|}{c_X(s,t)} \leq 1.$$
for some constant $C_{p,q}$ depending only on $p$ and $q$. Since $p \in (2,3)$ and $q \in [1,2)$, we have that $p/2q > 1/2$, and thus $p/2q + 1/2 > 1$, which implies that the map $(s,t) \mapsto w_{Y,q}(s,t)^{p/2q}w_{X,p}(s,t)^{1/2}$ is superadditive, and hence is itself a control function. Thus, the control function $c_{XY} := C_{p,q}w_{Y,q}(s,t)^{p/2q}w_{X,p}(s,t)^{1/2}$ gives the desired bound. Similarly, we can find control functions $c_{YX}$ and $c_Y$ for $Y^n$ and $Y^n$, respectively. Hence, our claim follows by noting that

$$|Z^n| \leq |X^n| + |Y^n| + |XY^n| + |YY^n|.$$ 

All together, we deduce that the sample paths of $Z$ almost surely satisfy Property (RIE) along $(\mathcal{Q}^n_\mathcal{Z}([0,T]))_{n \in \mathbb{N}}$.

\section{Typical price paths.} The notion of “typical price paths” was introduced by Vovk, who introduced a model-free hedging-based approach to mathematical finance allowing to investigate the sample path properties of such “typical price paths” based on arbitrage considerations; see for instance Vovk \cite{Vovk1, Vovk2} or Perkowski and Prömel \cite{Perkowski}. Let us briefly recall the basic setting and definitions of Vovk’s approach.

Let $\Omega_+ := D([0, \infty); \mathbb{R}^d_+)$ be the space of all non-negative càdlàg functions $\omega: [0, \infty) \to \mathbb{R}^d_+$. For each $t \in [0, \infty)$, $\mathcal{F}^t_\omega$ is defined to be the smallest $\sigma$-algebra on $\Omega_+$ that makes all functions $\omega \mapsto \omega(s), s \in [0, t]$, measurable and $\mathcal{F}_t$ is defined to be the universal completion of $\mathcal{F}^0_\omega$. Stopping times $\tau: \Omega_+ \to [0, \infty) \cup \{\infty\}$ with respect to the filtration $(\mathcal{F}_t)_{t \in [0, \infty)}$ and the corresponding $\sigma$-algebras $\mathcal{F}_\tau$ are defined as usual. The coordinate process on $\Omega_+$ is denoted by $S$, i.e. $S_t(\omega) := \omega(t)$ for $t \in [0, \infty)$.

A process $H: \Omega_+ \times [0, \infty) \to \mathbb{R}^d$ is a simple (trading) strategy if there exists a sequence of stopping times $0 = \sigma_0 < \sigma_1 < \sigma_2 < \ldots$ such that for every $\omega \in \Omega_+$, there exists an $N(\omega) \in \mathbb{N}$ such that $\sigma_n(\omega) = \sigma_{n+1}(\omega)$ for all $n \geq N(\omega)$, and a sequence of $\mathcal{F}_{\sigma_n}$-measurable bounded functions $h_n: \Omega_+ \to \mathbb{R}^d$, such that $H_t(\omega) = \sum_{n=0}^{\infty} h_n(\omega)1_{[\sigma_n(\omega), \sigma_{n+1}(\omega)]}(t)$ for $t \in [0, \infty)$. For a simple strategy $H$, the corresponding integral process

$$(H \cdot S)_t(\omega) := \sum_{n=0}^{\infty} h_n(\omega)S_{\sigma_n \land t, \sigma_{n+1} \land t}(\omega)$$

is well-defined for all $(t, \omega) \in [0, \infty) \times \Omega_+$. For $\lambda > 0$, we write $\mathcal{H}_\lambda$ for the set of all simple strategies $H$ such that $(H \cdot S)_t(\omega) \geq -\lambda$ for all $(t, \omega) \in [0, \infty) \times \Omega_+$.

\begin{definition}
Vovk’s outer measure $\overline{P}$ of a set $A \subseteq \Omega_+$ is defined as the minimal super-hedging price for $1_A$, that is

$$\overline{P}(A) := \inf \left\{ \lambda > 0 : \exists (H^n)_{n \in \mathbb{N}} \in \mathcal{H}_\lambda \text{ s.t. } \forall \omega \in \Omega_+, \liminf_{n \to \infty} (\lambda + (H^n \cdot S)_T(\omega)) \geq 1_A(\omega) \right\}.$$ 

A given set $A \subseteq \Omega_+$ is called a null set if it has outer measure zero. A property $(P)$ holds for typical price paths if the set $A$ where $(P)$ is violated is a null set.

\begin{remark}
Loosely speaking, the outer measure $\overline{P}$ corresponds to the (model-free) notion of “no unbounded profit with bounded risk”, see \cite{Perkowski} Section 2.2 for a more detailed discussion in this direction. Furthermore, the outer measure $\overline{P}$ dominates all local martingale measures on the space $\Omega_+$, see \cite{Vovk2} Lemma 2.3 and Proposition 2.5. As a consequence, all results proven for typical price paths hold simultaneously under all martingale measures (or in other words quasi-surely with respect to all martingale measures).
\end{remark}
Let us recall that typical price paths are of finite $p$-variation for every $p > 2$ (see Vovk [49, Theorem 1]) and Vovk’s model-free framework allows for setting up a model-free Itô integration, see e.g. Lochowski, Perkowski and Prömel [36].

**Lemma 4.7.** Typical price paths are price paths in the sense of Definition 3.1 with any $p \in (2, 3)$.

The proof of Lemma 4.7 works verbatim as that of Proposition 4.1 keeping in mind [34, Proposition 3.10] and [36, Corollary 4.9], and is therefore omitted for brevity.

### 4.4. Consistency of rough and stochastic integration

In a probabilistic framework when the underlying process is a semimartingale, one can employ either rough or stochastic Itô integration. In this subsection we briefly demonstrate that, under Property (RIE), these two integrals actually coincide almost surely whenever both are defined.

Let us again fix a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, \infty)}, \mathbb{P})$, and assume that the filtration $(\mathcal{F}_t)_{t \in [0, \infty)}$ satisfies the usual conditions.

**Proposition 4.8.** Let $X = (X_t)_{t \in [0, \infty)}$ be a $d$-dimensional càdlàg semimartingale. Let $Y$ be an adapted càdlàg process such that, for almost every $\omega \in \Omega$, the path $Y(\omega) \in \mathcal{A}_X(\omega)$ is an admissible strategy (in the sense of Definition 3.2). Then the rough and Itô integrals of $Y$ against $X$ coincide almost surely. That is,

$$
\int_0^t Y_s(\omega) \, d\mathbf{X}_s(\omega) = \left( \int_0^t Y_s \, d\mathbf{X}_s \right)(\omega), \quad \text{for all } t \in [0, \infty),
$$

for almost every $\omega \in \Omega$, where $\mathbf{X}(\omega)$ is the canonical rough path lift of $X(\omega)$, as defined in Lemma 2.12.

**Proof.** Fix $T > 0$. By Proposition 4.1, we know that, for any $p \in (2, 3)$, almost every sample path of $X$ is a price path, and satisfies Property (RIE) along a nested sequence of adapted partitions $\mathcal{P}_X^n = \{0 = t^n_0 < t^n_1 < \cdots < t^n_{N_n} = T\}$, $n \in \mathbb{N}$, of the interval $[0, T]$ with vanishing mesh size. Since $Y(\omega) \in \mathcal{A}_X(\omega)$, there exists a càdlàg process $Y'$, such that $(Y(\omega), Y'(\omega)) \in \mathcal{V}_{X(\omega)}^{q,r}$ is a controlled path on $[0, T]$, for almost every $\omega \in \Omega$ and some suitable numbers $q, r$.

By [42, Theorem II.21], we have that

$$
\sum_{k=0}^{N_n-1} Y^n_t k X^n_{t_k \wedge t, t_{k+1} \wedge t} \longrightarrow \int_0^t Y_s \, d\mathbf{X}_s \quad \text{as } n \to \infty,
$$

uniformly in probability, for $t \in [0, T]$. By taking a subsequence if necessary, we can then assume that the (uniform) convergence in (4.12) holds almost surely. On the other hand, by Theorem 2.14, we know that, for almost every $\omega \in \Omega$,

$$
\sum_{k=0}^{N_n-1} Y^n_t k(\omega) X^n_{t_k \wedge t, t_{k+1} \wedge t}(\omega) \longrightarrow \int_0^t Y_s(\omega) \, d\mathbf{X}_s(\omega) \quad \text{as } n \to \infty,
$$

uniformly for $t \in [0, T]$. Combining (4.12) and (4.13), we deduce that, almost surely, $\int_0^t Y_s \, d\mathbf{X}_s = \int_0^t Y_s \, d\mathbf{X}_s$ for all $t \in [0, T]$. Since $T > 0$ was arbitrary, the result follows.
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