Stability of homomorphisms and derivations in $C^*$-ternary algebras

M. Bavand Savadkouhi, M. Eshaghi Gordji and N. Ghobadipour

Department of Mathematics, Semnan University,
P. O. Box 35195-363, Semnan, Iran
E-mail: bavand.m@gmail.com, madjid.eshaghi@gmail.com
ghobadipour.n@gmail.com

Abstract. In this paper, we investigate homomorphisms between $C^*$-ternary algebras and derivations on $C^*$-ternary algebras, associated with the following functional equation

$$f\left(\frac{x_2-x_1}{3}\right) + f\left(\frac{x_1-3x_3}{3}\right) + f\left(\frac{3x_1+3x_3-x_2}{3}\right) = f(x_1).$$

Moreover, we prove the generalized Hyers-Ulam -Rassias stability of homomorphisms in $C^*$-ternary algebras and of derivations on $C^*$-ternary algebras.

1. Introduction

Ternary algebraic operations were considered in the 19th century by several mathematicians and physicists such as Cayley [9] who introduced the notions of cubic matrix, which in turn was generalized by Kapranov et al. [23]. The simplest example of such nontrivial ternary operation is given by the following composition rule:

$$\{a, b, c\}_{ijk} = \sum_{l,m,n} a_{nil} b_{ljm} c_{mkn}. \quad (i, j, k, \ldots = 1, 2, \ldots, N)$$

Ternary structures and their generalization, the so-called n-ary structures, raise certain hopes in view of their applications in physics. Some significant physical applications are as follows (see Refs. [24] and [25]):

(i) The algebra of "nonions" generated by two matrices,

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \omega \\ \omega^2 & 0 & 0 \end{pmatrix}, \quad (\omega = e^{2\pi i/3})$$

was introduced by Sylvester as a ternary analog of Hamilton's quaternions (cf. Ref. [1]).

(ii) A natural ternary composition of 4-vectors in the four-dimensional Minkowskian

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space time $M_4$ can be defined as an example of a ternary operation:

$$(X, Y, Z) \mapsto U(X, Y, Z) \in M_4,$$

with the resulting 4-vector $U^\mu$ defined via its components in a given coordinate system as follows:

$$U^\mu(X, Y, Z) = g^{\mu\alpha\eta\lambda\rho}X^{\nu}Y^{\lambda}Z^{\rho}, \quad \mu, \nu, ... = 0, 1, 2, 3,$$

where $g^{\mu\alpha\eta\lambda\rho}$ is the metric tensor and $\eta^{\sigma\nu\lambda\rho}$ is the canonical volume element of $M_4$. [25]

(iii) The quark model inspired a particular brand of ternary algebraic systems. The "Nambu mechanics" is based on such structures (see Refs. [11] and [52]). Quarks apparently couple by packs of 3.

There are also some applications, although still hypothetical, in the fractional quantum Hall effect, nonstandard statistics, supersymmetric theory, Yang-Baxter equation, etc., cf. Refs. [1], [25], and [54]. Following the terminology of Ref. [12], a nonempty set $G$ with a ternary operation $[\ldots] : G^3 \to G$ is called a ternary groupoid and is denoted by $(G, [\ldots])$. The ternary groupoid $(G, [\ldots])$ is called commutative if $[x_1, x_2, x_3] = [x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}]$ for all $x_1, x_2, x_3 \in G$ and all permutations $\sigma$ of $\{1, 2, 3\}$. If a binary operation $\circ$ is defined on $G$ such that $[x, y, z] = (x \circ y) \circ z$ for all $x, y, z \in G$, then we say that $[\ldots]$ is derived from $\circ$. We say that $(G, [\ldots])$ is a ternary semigroup if the operation $[\ldots]$ is associative, i.e., if $[[x, y, z], u, v] = [x, [y, z, u], v] = [x, y, [z, u, v]]$ holds for all $x, y, z, u, v \in G$ (see Ref. [8]).

As it is extensively discussed in [50], the full description of a physical system $S$ implies the knowledge of three basis ingredients: the set of the observables, the set of the states and the dynamics that describes the time evolution of the system by means of the time dependence of the expectation value of a given observable on a given state. Originally the set of the observable was considered to be a $C^*$-algebra [17]. In many applications, however, this was shown not to be the most convenient choice and the $C^*$-algebra was replaced by a von Neumann algebra, because the role of the representation turns out to be crucial mainly when long range interactions are involved (see [6] and references therein). Here we used a different algebraic structure. A $C^*$-ternary algebra is a complex Banach space $A$, equipped with a ternary product $(x, y, z) \mapsto [x, y, z]$ of $A^3$ into $A$, which is $C$-linear in the outer variables, conjugate $C$-linear in the middle variable, and associative in the sense that $[x, y, [z, w, v]] = [x, [w, z, y], v] = [[x, y, z], w, v]$, and satisfies $\| [x, y, z] \| \leq \| x \| \| y \| \| z \|$ and $\| [x, x, x] \| = \| x \|^3$ (see[27]). If a $C^*$-ternary algebra $(A, [\ldots])$ has an identity, i.e., an element $e \in A$ such that $x = [x, e, e] = [e, e, x]$ for all $x \in A$, then it is routine to verify that $A$, endowed with $xoy := [x, e, y]$ and $x^* := [e, x, e]$, is a unital $C^*$-algebra. Conversely, if $(A, \circ)$ is a unital $C^*$-algebra, then $[x, y, z] := xoy^*oz$ makes $A$ into
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A $C^*$-ternary algebra. A $C$-linear mapping $H : A \rightarrow B$ is called a $C^*$-ternary algebra homomorphism if

$$H([x, y, z]) = [H(x), H(y), H(z)]$$

for all $x, y, z \in A$. A $C$-linear mapping $\delta : A \rightarrow A$ is called a $C^*$-ternary algebra derivation if

$$\delta([x, y, z]) = [\delta(x), y, z] + [x, \delta(y), z] + [x, y, \delta(z)]$$

for all $x, y, z \in A$.

Ternary structures and their generalization the so-called $n$-ary structures, raise certain hops in view of their applications in physics (see [2-4], [6], [17], [24,26,27], [30], [50] and [55]).

The study of stability problems originated from a famous talk given by S. M. Ulam [53] in 1940: ”Under what condition dose there exists a homomorphism near an approximate homomorphism?” In the next year 1941, D. H. Hyers [19] was answered affirmatively the question of Ulam and the result can be formulated as follows: if $\epsilon > 0$ and $f : E_1 \rightarrow E_2$ is a map from $E_1$ a normed space, $E_2$ a Banach spaces such that

$$\|f(x + y) - f(x) - f(y)\| \leq \epsilon$$

for all $x, y \in E_1$, then there exists a unique additive map $T : E_1 \rightarrow E_2$ such that

$$\|f(x) - T(x)\| \leq \epsilon$$

for all $x \in E_1$. Moreover, if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E_1$, then $T$ is linear. This stability phenomenon is called the Hyers-Ulam stability of the additive functional equation $g(x+y) = g(x) + g(y)$. A generalized version of the theorem of Hyers for approximately additive maps was given by Th. M. Rassias [46] in 1978 by considering the case when the above inequality is not bounded:

**Theorem 1.1.** Let $f : E_1 \rightarrow E_2$ be a mapping from a normed vector space $E_1$ into a Banach space $E_2$ subject to the inequality

$$\|f(x + y) - f(x) - f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p)$$

for all $x, y \in E_1$, where $\epsilon$ and $p$ are constants with $\epsilon > 0$ and $p < 1$. Then there exists a unique additive mapping $T : E_1 \rightarrow E_2$ such that

$$\|f(x) - T(x)\| \leq \frac{2\epsilon}{2 - 2p}\|x\|^p,$$

for all $x \in E_1$.

The stability phenomenon that was introduced and proved by Th. M. Rassias is called Hyers-Ulam-Rassias stability. And then the stability problems of several functional equations have been extensively investigated by a number of
authors and there are many interesting results concerning this problem. (see [5], [7],[10],[13-16], [18-22], [28,29], [31-49] and [51]). Throughout this paper, we assume that $A$ is a $C^*$-ternary algebra with norm $\|\cdot\|_A$ and that $B$ is a $C^*$-ternary algebra with norm $\|\cdot\|_B$.

2. Superstability of Homomorphisms and derivations on $C^*$-ternary algebras

In this section, first we investigate homomorphisms between $C^*$-ternary algebras. We need the following Lemma in the main results of the paper.

**Lemma 2.1.** Let $f : A \rightarrow B$ be a mapping such that

$$
\| f \left( \frac{x_2-x_1}{3} \right) + f \left( \frac{x_1-3x_3}{3} \right) + f \left( \frac{3x_1+3x_3-x_2}{3} \right) \|_B \leq \| f(x_1) \|_B,
$$

for all $x_1, x_2, x_3 \in A$. Then $f$ is additive.

**Proof.** Letting $x_1 = x_2 = x_3 = 0$ in (2.1), we get

$$
\| 3f(0) \|_B \leq \| f(0) \|_B.
$$

So $f(0) = 0$. Letting $x_1 = x_2 = 0$ in (2.1), we get

$$
\| f(-x_3) + f(x_3) \|_B \leq \| f(0) \|_B = 0
$$

for all $x_3 \in A$. Hence $f(-x_3) = -f(x_3)$ for all $x_3 \in A$. Letting $x_1 = 0$ and $x_2 = 6x_3$ in (2.1), we get

$$
\| f(2x_3) - 2f(x_3) \|_B \leq \| f(0) \|_B = 0
$$

for all $x_3 \in A$. Hence

$$
f(2x_3) = 2f(x_3)
$$

for all $x_3 \in A$. Letting $x_1 = 0$ and $x_2 = 9x_3$ in (2.1), we get

$$
\| f(3x_3) - f(x_3) - 2f(x_3) \|_B \leq \| f(0) \|_B = 0
$$

for all $x_3 \in A$. Hence

$$
f(3x_3) = 3f(x_3)
$$

for all $x_3 \in A$. Letting $x_1 = 0$ in (2.1), we get

$$
\| f \left( \frac{x_2}{3} \right) + f(-x_3) + f(x_3 - \frac{x_2}{3}) \|_B \leq \| f(0) \|_B = 0
$$

for all $x_2, x_3 \in A$. So

$$
f \left( \frac{x_2}{3} \right) + f(-x_3) + f(x_3 - \frac{x_2}{3}) = 0
$$

(2.2)

for all $x_2, x_3 \in A$. Let $t_1 = x_3 - \frac{x_2}{3}$ and $t_2 = \frac{x_2}{3}$ in (2.2). Then

$$
f(t_2) - f(t_1 + t_2) + f(t_1) = 0
$$
for all $t_1, t_2 \in A$ and so $f$ is additive.

**Theorem 2.2.** Let $p \neq 1$ and $\theta$ be nonnegative real numbers, and let $f : A \to B$ be a mapping such that
\begin{equation}
\|f\left(\frac{x_2 - x_1}{3}\right) + f\left(\frac{x_1 - 3\mu x_3}{3}\right) + \mu f\left(\frac{3x_1 + 3x_3 - x_2}{3}\right)\|_B \leq \|f(x_1)\|_B, \tag{2.3}
\end{equation}
\begin{equation}
\|f([x_1, x_2, x_3]) - [f(x_1), f(x_2), f(x_3)]\|_B \leq \theta(\|x_1\|_A^3 + \|x_2\|_A^3 + \|x_3\|_A^3) \tag{2.4}
\end{equation}
for all $\mu \in \mathbb{T}^1 := \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ and all $x_1, x_2, x_3 \in A$. Then the mapping $f : A \to B$ is a $C^\ast$-ternary algebra homomorphism.

**Proof.** Assume $p > 1$.

Let $\mu = 1$ in (2.3). By lemma 2.1, the mapping $f : A \to B$ is additive. Letting $x_1 = x_2 = 0$ in (2.3), we get
\[\|f(-\mu x_3) + \mu f(x_3)\|_B \leq \|f(0)\|_B = 0\]
for all $x_3 \in A$ and $\mu \in \mathbb{T}^1$. So
\[-f(\mu x_3) + \mu f(x_3) = f(-\mu x_3) + \mu f(x_3) = 0\]
for all $x_3 \in A$ and all $\mu \in \mathbb{T}^1$. Hence $f(\mu x_3) = \mu f(x_3)$ for all $x_3 \in A$ and all $\mu \in \mathbb{T}^1$. By the theorem 2.1 of [33], the mapping $f : A \to B$ is $\mathbb{C}$-linear. It follows from (2.4) that
\[\|f([x_1, x_2, x_3]) - [f(x_1), f(x_2), f(x_3)]\|_B \leq \lim_{n \to \infty} \|f\left(\frac{x_1}{2^n}, \frac{x_2}{2^n}, \frac{x_3}{2^n}\right)\|_B - [f(x_1), f(x_2), f(x_3)]\|_B \leq \lim_{n \to \infty} \frac{\theta}{8^n} (\|x_1\|_A^3 + \|x_2\|_A^3 + \|x_3\|_A^3) = 0\]
for all $x_1, x_2, x_3 \in A$. Thus
\[f([x_1, x_2, x_3]) = [f(x_1), f(x_2), f(x_3)]\]
for all $x_1, x_2, x_3 \in A$. Hence the mapping $f : A \to B$ is a $C^\ast$-ternary algebra homomorphism. Similarly, one obtains the result for the case $p < 1$. 

Now we establish the superstability of derivations from a $C^\ast$-ternary algebra into its $C^\ast$-ternary modules as follows.

**Theorem 2.3.** Let $p \neq 1$ and $\theta$ be nonnegative real numbers, and let $f : A \to A$ be a mapping satisfying (2.3) such that
\begin{equation}
\|f([x_1, x_2, x_3]) - [f(x_1), x_2, x_3] - [x_1, f(x_2), x_3] - [x_1, x_2, f(x_3)]\|_A \leq \theta(\|x_1\|_A^3 + \|x_2\|_A^3 + \|x_3\|_A^3) \tag{2.5}
\end{equation}
for all $x_1, x_2, x_3 \in A$. Then the mapping $f : A \to A$ is a $C^\ast$-ternary derivation.
Proof. Assume \( p > 1 \).
By the theorem 2.2, the mapping \( f : A \to A \) is \( \mathbb{C} \)-linear. It follows from (2.5) that

\[
\|f([x_1, x_2, x_3]) - [f(x_1), x_2, x_3] - [x_1, f(x_2), x_3] - [x_1, x_2, f(x_3)]\|_{A}
\]

\[
= \lim_{n \to \infty} 8^n \|f\left(\frac{x_1, x_2, x_3}{8^n}\right) - [f(\frac{x_1}{2^n}, \frac{x_2}{2^n}, \frac{x_3}{2^n}) - [\frac{x_1}{2^n}, f(\frac{x_2}{2^n}, \frac{x_3}{2^n})]\|_{A}
\]

\[
\leq \lim_{n \to \infty} \frac{8^n}{8^n p} (\|x_1\|_{A}^3 + \|x_2\|_{A}^3 + \|x_3\|_{A}^3) = 0
\]

for all \( x_1, x_2, x_3 \in A \). So

\[
f([x_1, x_2, x_3]) = [f(x_1), x_2, x_3] + [x_1, f(x_2), x_3] + [x_1, x_2, f(x_3)]
\]

for all \( x_1, x_2, x_3 \in A \). Thus the mapping \( f : A \to A \) is a \( C^* \)-ternary derivation. Similarly, one obtains the result for the case \( p < 1 \). \( \square \)

3. Stability of homomorphisms and derivations on \( C^* \)-ternary algebras

First we prove the generalized Hyers-Ulam-Rassias stability of homomorphisms in \( C^* \)-ternary algebras.

**Theorem 3.1.** Let \( p > 1 \) and \( \theta \) be nonnegative real numbers, and let \( f : A \to B \) be a mapping such that

\[
\|f\left(\frac{x_2 - x_1}{3}\right) + f\left(\frac{x_1 - 3\mu x_3}{3}\right) + \mu f\left(\frac{3x_1 + 3x_3 - x_2}{3}\right) - f(x_1)\|_B \\
\leq \theta(\|x_1\|_A^p + \|x_2\|_A^p + \|x_3\|_A^p)
\]

(3.1)

and

\[
\|f([x_1, x_2, x_3]) - [f(x_1), f(x_2), f(x_3)]\|_B \leq \theta(\|x_1\|_A^p + \|x_2\|_A^p + \|x_3\|_A^p)
\]

(3.2)

for all \( \mu \in T^1 \) and all \( x_1, x_2, x_3 \in A \). Then there exists a unique \( C^* \)-ternary homomorphism \( \mathcal{H} : A \to B \) such that

\[
\|\mathcal{H}(x_1) - f(x_1)\|_B \leq \frac{\theta(1 + 2^p)\|x_1\|_A^p}{1 - 3^{1-p}}
\]

(3.3)

for all \( x_1 \in A \).

**Proof.** Let us assume \( \mu = 1 \), \( x_2 = 2x_1 \) and \( x_3 = 0 \) in (3.1). Then we get

\[
\|3f\left(\frac{x_1}{3}\right) - f(x_1)\|_B \leq \theta(1 + 2^p)\|x_1\|_A^p
\]

(3.4)
Now, let it follows that the sequence $\{m\}$ for all nonnegative integers $\mu$ for all $x$ for all $x$ have $x$ for all $B$. we get $\|3^n f\left(\frac{x_1}{3^n}\right) - f(x_1)\|_B \leq \theta(1 + 2^p)\|x_1\|_A \sum_{i=0}^{n-1} 3^{(1-p)} \tag{3.5}$ for all $x_1 \in A$. Hence

$$\|3^{n+m} f\left(\frac{x_1}{3^{n+m}}\right) - 3^m f\left(\frac{x_1}{3^m}\right)\|_B \leq \theta(1 + 2^p)\|x_1\|_A \sum_{i=0}^{n-1} 3^{(1+m)(1-p)}$$

$$\leq \theta(1 + 2^p)\|x_1\|_A \sum_{i=m}^{n+m-1} 3^{i(1-p)} \tag{3.6}$$

for all nonnegative integers $m$ and $n$ with $n \geq m$ and all $x_1 \in A$. From this it follows that the sequence $\{3^n f\left(\frac{x_1}{3^n}\right)\}$ is a Cauchy sequence for all $x_1 \in A$. Since $B$ is complete, the sequence $\{3^n f\left(\frac{x_1}{3^n}\right)\}$ converges. Thus one can define the mapping $H : A \to B$ by

$$H(x_1) := \lim_{n \to \infty} 3^n f\left(\frac{x_1}{3^n}\right)$$

for all $x_1 \in A$. Moreover, letting $m = 0$ and passing the limit $n \to \infty$ in (3.6), we get (3.3). It follows from (3.1) that

$$\|H\left(\frac{x_2 - x_1}{3}\right) + H\left(\frac{x_1 - 3\mu x_3}{3}\right) + \mu H\left(\frac{3x_1 + 3x_3 - x_2}{3}\right) - H(x_1)\|_B$$

$$= \lim_{n \to \infty} 3^n \|f\left(\frac{x_2 - x_1}{3^{n+1}}\right) + f\left(\frac{x_1 - 3\mu x_3}{3^{n+1}}\right) + f\left(\frac{3x_1 + 3x_3 - x_2}{3^{n+1}}\right) - f\left(\frac{x_1}{3^n}\right)\|_B$$

$$\leq \lim_{n \to \infty} \frac{3^n \theta}{3^{np}}(\|x_1\|_A^p + \|x_2\|_A^p + \|x_3\|_A^p) = 0$$

for all $\mu \in \mathbb{T}^1$ and all $x_1, x_2, x_3 \in A$. So

$$H\left(\frac{x_2 - x_1}{3}\right) + H\left(\frac{x_1 - 3\mu x_3}{3}\right) + \mu H\left(\frac{3x_1 + 3x_3 - x_2}{3}\right) = H(x_1)$$

for all $\mu \in \mathbb{T}^1$ and all $x_1, x_2, x_3 \in A$. By the Theorem 2.1 of [33], the mapping $H : A \to B$ is $C$-linear.

Now, let $H' : A \to B$ be another additive mapping satisfying (3.3). Then we have

$$\|H(x_1) - H'(x_1)\|_B = 3^n \|H\left(\frac{x_1}{3^n}\right) - H'\left(\frac{x_1}{3^n}\right)\|_B$$

$$\leq 3^n \left(\|H\left(\frac{x_1}{3^n}\right) - f\left(\frac{x_1}{3^n}\right)\|_B + \|H'\left(\frac{x_1}{3^n}\right) - f\left(\frac{x_1}{3^n}\right)\|_B\right)$$

$$\leq \frac{2.3^n \theta(1 + 2^p)}{3^{np}(3 - 3^1 - p)} \|x\|_A^p,$$
which tends to zero as \( n \to \infty \) for all \( x_1 \in A \). So we can conclude that \( H(x_1) = H'(x_1) \) for all \( x_1 \in A \). This proves the uniqueness of \( H \).

It follows from (3.2) that
\[
\|H([x_1, x_2, x_3]) - [H(x_1), H(x_2), H(x_3)]\|_B \\
= \lim_{n \to \infty} 27^n \|f \left( \frac{[x_1, x_2, x_3]}{3^n} \right) - [f \left( \frac{x_1}{3^n} \right), f \left( \frac{x_2}{3^n} \right), f \left( \frac{x_3}{3^n} \right)]\|_B \\
\leq \lim_{n \to \infty} \frac{27^n \theta}{27^{np}} (\|x_1\|_A^3 + \|x_2\|_A^3 + \|x_3\|_A^3) = 0
\]
for all \( x_1, x_2, x_3 \in A \).

Thus the mapping \( H : A \to B \) is a unique \( C^* \)-ternary homomorphism satisfying (3.3).

\[\square\]

**Theorem 3.2.** Let \( p < 1 \) and \( \theta \) be nonnegative real numbers, and let \( f : A \to B \) be a mapping satisfying (3.1) and (3.2). Then there exists a unique \( C^* \)-ternary homomorphism \( H : A \to B \) such that
\[
\|H(x_1) - f(x_1)\|_B \leq \frac{\theta (1 + 2^p)\|x_1\|_A^p}{3^{1-p} - 1}
\]
for all \( x_1 \in A \).

**Proof.** The proof is similar to the proof of Theorem 3.1. \[\square\]

Now we prove the generalized Hyers-Ulam-Rassias stability of derivations from a \( C^* \)-ternary algebra into its \( C^* \)-ternary moduls.

**Theorem 3.3.** Let \( p > 1 \) and \( \theta \) be nonnegative real numbers, and let \( f : A \to A \) be a mapping such that
\[
\|f \left( \frac{x_2 - x_1}{3} \right) + f \left( \frac{x_1 - 3 \mu x_3}{3} \right) + \mu f \left( \frac{3x_1 + 3x_3 - x_2}{3} \right) - f(x_1)\|_A \\
\leq \theta (\|x_1\|_A^p + \|x_2\|_A^p + \|x_3\|_A^p)
\]
and
\[
\|f([x_1, x_2, x_3]) - [f(x_1), x_2, x_3] - [x_1, f(x_2), x_3] - [x_1, x_2, f(x_3)]\|_A \\
\leq \theta (\|x_1\|_A^3p + \|x_2\|_A^3p + \|x_3\|_A^3p)
\]
for all \( \mu \in \mathbb{T}^1 \) and all \( x_1, x_2, x_3 \in A \). Then there exists a unique \( C^* \)-ternary derivation \( D : A \to A \) such that
\[
\|D(x_1) - f(x_1)\|_A \leq \frac{\theta (1 + 2^p)\|x_1\|_A^p}{1 - 3^{1-p}}
\]
for all \( x_1 \in A \).
Proof. By the same reasoning as in the proof of the Theorem 3.1, there exists a unique $C$-linear mapping $D : A \to A$ satisfying (3.10). The mapping $D : A \to A$ is defined by

$$D(x_1) := \lim_{n \to \infty} 3^n f\left(\frac{x_1}{3^n}\right)$$

for all $x_1 \in A$. It follows from (3.9) that

$$\|D([x_1, x_2, x_3]) - [D(x_1), x_2, x_3] - [x_1, D(x_2), x_3] - [x_1, x_2, D(x_3)]\|_A$$

$$= \lim_{n \to \infty} 27^n \|\frac{x_1}{3^n}, \frac{x_2}{3^n}, \frac{x_3}{3^n} - f\left(\frac{x_1}{3^n}\right), \frac{x_2}{3^n}, \frac{x_3}{3^n} - \frac{x_1}{3^n}, \frac{x_2}{3^n}, f\left(\frac{x_3}{3^n}\right)\|_A$$

$$\leq \lim_{n \to \infty} 27^n \theta (\|x_1\|_A^{3p} + \|x_2\|_A^{3p} + \|x_3\|_A^{3p}) = 0$$

for all $x_1, x_2, x_3 \in A$. So

$$D([x_1, x_2, x_3]) = [D(x_1), x_2, x_3] + [x_1, D(x_2), x_3] + [x_1, x_2, D(x_3)]$$

for all $x_1, x_2, x_3 \in A$. Thus the mapping $D : A \to A$ is a unique $C_*$-ternary derivation satisfying (3.10).

Theorem 3.4. Let $p < 1$ and $\theta$ be nonnegative real numbers, and let $f : A \to A$ be a mapping satisfying (3.8) and (3.9). Then there exists a unique $C_*$-ternary derivation $D : A \to A$ such that

$$\|D(x_1) - f(x_1)\|_A \leq \frac{\theta (1 + 2^p)\|x_1\|_A^p}{3^{1-p} - 1}$$

for all $x_1 \in A$.

Proof. The proof is similar to the proof of Theorems 3.1 and 3.3.

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