ADELES IN MATHEMATICAL PHYSICS

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Dedicated to Yakov Valentinovich Radyno
on the occasion of his 60th birthday

Abstract
Application of adeles in modern mathematical physics is briefly reviewed. In particular, some adelic products are presented.

1 Introduction

$p$-Adic numbers are invented by K. Hansel in 1897. Ideles and adeles are introduced by C. Chevalley and A. Weil, respectively, in the 1930s. $p$-Adic numbers and adeles have many applications in mathematics, e.g. representation theory, algebraic geometry and modern number theory. Since 1987, $p$-adic numbers and adeles have been used in construction of many models in modern mathematical physics and related topics. Here we consider applications of adeles in mathematical physics.

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On the field $\mathbb{Q}$ of rational numbers any non-trivial norm is equivalent either to the usual absolute value $| \cdot |_\infty$ or to a $p$-adic absolute value $| \cdot |_p$ (Ostrowski theorem). For a rational number $x = \frac{a}{b}$, where integers $a$ and $b$ are not divisible by prime number $p$, by definition $p$-adic absolute value is $|x|_p = p^{-\nu}$ and $|0|_p = 0$. This $p$-adic norm is a non-Archimedean (ultrametric) one, because $|x+y|_p \leq \max\{|x|_p, |y|_p\}$. As completion of $\mathbb{Q}$ gives the field $\mathbb{Q}_\infty \equiv \mathbb{R}$ of real numbers with respect to the $| \cdot |_\infty$, by the same procedure one get the fields $\mathbb{Q}_p$ of $p$-adic numbers ($p = 2, 3, 5 \cdots$) using $| \cdot |_p$. Any number $x \in \mathbb{Q}_p$ has a unique canonical representation

$$x = p^{\nu(x)} \sum_{n=0}^{+\infty} x_n p^n, \quad \nu(x) \in \mathbb{Z}, \quad x_n \in \{0, 1, \cdots, p-1\}. \quad (1)$$

Real and $p$-adic numbers, as completions of rationals, unify by adeles. An adele $\alpha$ is an infinite sequence

$$\alpha = (\alpha_\infty, \alpha_2, \alpha_3, \cdots, \alpha_p, \cdots), \quad \alpha_\infty \in \mathbb{R}, \quad \alpha_p \in \mathbb{Q}_p, \quad (2)$$

where for all but a finite set $\mathcal{P}$ of primes $p$ one has that $\alpha_p \in \mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}$. $\mathbb{Z}_p$ is the ring of $p$-adic integers. The set $\mathbb{A}_\mathbb{Q}$ of all adeles can be presented as

$$\mathbb{A}_\mathbb{Q} = \bigcup_{\mathcal{P}} A(\mathcal{P}), \quad A(\mathcal{P}) = \mathbb{R} \times \prod_{p \in \mathcal{P}} \mathbb{Q}_p \times \prod_{p \notin \mathcal{P}} \mathbb{Z}_p. \quad (3)$$

Endowed with componentwise addition and multiplication $\mathbb{A}_\mathbb{Q}$ is the adele ring.

The multiplicative group of ideles $\mathbb{A}_\mathbb{Q}^\times$ is a subset of $\mathbb{A}_\mathbb{Q}$ with elements $\eta = (\eta_\infty, \eta_2, \eta_3, \cdots, \eta_p, \cdots)$, where $\eta_\infty \in \mathbb{R}^\times = \mathbb{R} \setminus \{0\}$ and $\eta_p \in \mathbb{Q}_p^\times = \mathbb{Q}_p \setminus \{0\}$ with the restriction that for all but a finite set $\mathcal{P}$ one has that $\eta_p \in U_p = \{x \in \mathbb{Q}_p : |x|_p = 1\}$. Thus the whole set of ideles is

$$\mathbb{A}_\mathbb{Q}^\times = \bigcup_{\mathcal{P}} A^\times(\mathcal{P}), \quad A^\times(\mathcal{P}) = \mathbb{R}^\times \times \prod_{p \in \mathcal{P}} \mathbb{Q}_p^\times \times \prod_{p \notin \mathcal{P}} U_p. \quad (4)$$

A principal adele (idele) is a sequence $(x, x, \cdots, x, \cdots) \in \mathbb{A}_\mathbb{Q}$, where $x \in \mathbb{Q}$ ($x \in \mathbb{Q}^\times$). $\mathbb{Q}$ and $\mathbb{Q}^\times$ are naturally embedded in $\mathbb{A}_\mathbb{Q}$ and $\mathbb{A}_\mathbb{Q}^\times$, respectively.

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Let \( P \) be set of all primes \( p \). Denote by \( P_i, \ i \in \mathbb{N} \), subsets of \( P \). Let us introduce an ordering by \( P_i \prec P_j \) if \( P_i \subset P_j \). It is evident that \( A(P_i) \subset A(P_j) \) when \( P_i \prec P_j \). Adelic topology in \( A \mathbb{Q} \) is introduced by inductive limit:

\[
A \mathbb{Q} = \lim_{\text{ind}} P A(P).
\]

A basis of adelic topology is a collection of open sets of the form

\[
V(P) = V_{\infty} \times \prod_{p \in P} V_p \times \prod_{p \not\in P} \mathbb{Z}_p,
\]

where \( V_{\infty} \) and \( V_p \) are open sets in \( \mathbb{R} \) and \( \mathbb{Q}_p \), respectively.

A sequence of adeles \( \alpha^{(n)} \in A \mathbb{Q} \) converges to an adele \( \alpha \in A \mathbb{Q} \) if (i) it converges to \( \alpha \) componentwise and (ii) if there exist a positive integer \( N \) and a set \( P \) such that \( \alpha^{(n)} \in A(P) \) when \( n \geq N \). In the analogous way, these assertions hold also for idelic spaces \( A^{x}(P) \) and \( A^{x} \mathbb{Q} \). \( A \mathbb{Q} \) and \( A^{x} \mathbb{Q} \) are locally compact topological spaces.

For various mathematical aspects of adeles one can see books [1, 2, 3].

## 3 Adelic models

Recall that results of measurements are rational numbers, and physical models have been treated using real and complex numbers. Since \( \mathbb{Q} \) is dense not only in \( \mathbb{R} \) but also in \( \mathbb{Q}_p \), it has been natural to expect some applications of \( p \)-adic numbers in mathematical modeling of physical systems. First significant employment of \( p \)-adic numbers in physics started in 1987 by successful construction of \( p \)-adic string amplitudes. From the very beginning there has been an opinion that all prime numbers should be equally important and that \( p \)-adic models should be somehow connected with standard ones (over real or complex numbers). According to the Hasse local-global principle an equation has a solution over \( \mathbb{Q} \) if and only if it has solutions over \( \mathbb{R} \) and all \( \mathbb{Q}_p \).

These ideas naturally gave rise to an application of adeles and construction of adelic physical models (for an early review, see [4, 5]).

Especially so-called adelic products have been attracted much attention. They are of the form

\[
\phi_{\infty}(x_1, \cdots, x_n; a_1, \cdots, a_m) \prod_{p \in P} \phi_p(x_1, \cdots, x_n; a_1, \cdots, a_m) = C,
\]

where \( \phi_{\infty} \) and \( \phi_p \) are real or complex valued functions, \( x_i \in \mathbb{Q} \), \( a_j \in \mathbb{C} \), and \( C \) is a constant (often \( C = 1 \)). It is obvious that expressions of the form \( (5) \) connect real and \( p \)-adic characteristics of the same object at the equal footing. Moreover, the real quantity \( \phi_{\infty}(x_1, \cdots, x_n; a_1, \cdots, a_m) \) can be expressed as product of all \( p \)-adic inverses. This can be of practical
importance when functions $\phi_p$ are simpler than $\phi_\infty$, but may also lead to more profound understanding of physical reality.

For the reason of better understanding, let us first present two simple examples:

$$|x|_\infty \times \prod_{p \in \mathbb{P}} |x|_p = 1, \text{ if } x \in \mathbb{Q}^\times, \quad \text{and} \quad \chi_\infty(x) \times \prod_{p \in \mathbb{P}} \chi_p(x) = 1, \text{ if } x \in \mathbb{Q},$$

(6)

where $\chi_\infty(x) = \exp(-2\pi ix)$ and $\chi_p(x) = \exp 2\pi i \{x\}_p$ are real and $p$-adic additive characters, respectively, and $\{x\}_p$ denotes the fractional part of $x$. It follows from (6) that $d_\infty(x, y) = \prod_{p \in \mathbb{P}} d_p^{-1}(x, y)$, where $d_\infty(x, y) = |x - y|_\infty$ and $d_p(x, y) = |x - y|_p$, i.e. the usual distance between any two rational points can be regarded through product of the inverse $p$-adic ones. One can also write $\chi_\infty(ax + bt) = \prod_{p \in \mathbb{P}} \chi_p(−(ax + bt))$ when $a, b, x, t \in \mathbb{Q}$, and consider a real plane wave as composed of $p$-adic plane waves.

Let us also notice some adelic products related to number theory:

$$\lambda_\infty(x) \prod_{p \in \mathbb{P}} \lambda_p(x) = 1, \quad \left(\frac{x, y}{\infty}\right) \prod_{p \in \mathbb{P}} \left(\frac{x, y}{p}\right) = 1,$$

(7)

where $x$ is presented by (1) and

$$\lambda_p(x) = \begin{cases} 1, & \nu(x) = 2k, \quad p \neq 2, \\
\sqrt{-1} \left(\frac{ax}{p}\right), & \nu(x) = 2k + 1, \quad p \neq 2, \\
\exp[\pi i (x_1 + 1/4)], & \nu(x) = 2k, \quad p = 2, \\
\exp[\pi i (x_2 + x_1/2 + 1/4)], & \nu(x) = 2k + 1, \quad p = 2,
\end{cases}$$

(8)

$$\lambda_\infty(x) = \exp \left(-\frac{\pi i}{4} \text{sgn } x\right), \quad \left(\frac{x, y}{\infty}\right) = \begin{cases} -1, & x < 0, \quad y < 0, \\
1, & \text{otherwise},
\end{cases}$$

(9)

$$\left(\frac{x}{p}\right)$$ and $$\left(\frac{x, y}{p}\right)$$ are Legendre and Hilbert symbols [5], respectively.

Gauss integrals satisfy adelic product formula [6]

$$\int_\mathbb{R} \chi_\infty(ax^2 + bx) \, d_\infty x \prod_{p \in \mathbb{P}} \int_{\mathbb{Q}_p} \chi_p(ax^2 + bx) \, d_p x = 1, \quad a \in \mathbb{Q}^\times, \quad b \in \mathbb{Q},$$

(10)

what follows from

$$\int_{\mathbb{Q}_v} \chi_v(ax^2 + bx) \, d_v x = \lambda_v(a) |2a|^{v-1}_v \chi_v \left(-\frac{b^2}{4a}\right), \quad v = \infty, 2, \ldots, p \ldots .$$

(11)
These Gauss integrals apply in evaluation of the Feynman path integrals

$$K_v(x'', t''; x', t') = \int_{x', t'}^{x'', t''} \chi_v\left(-\frac{1}{\hbar} \int_{t'}^{t''} L(\dot{q}, q, t) \, dt\right) D_v q,$$  

(12)

for kernels $K_v(x'', t''; x', t')$ of the evolution operator in adelic quantum mechanics [7] for quadratic Lagrangians. In the case of Lagrangian $L(\dot{q}, q) = \frac{1}{2} \left(-\frac{q^2}{4} - \lambda q + 1\right)$ for the de Sitter cosmological model (what is similar to a particle with constant acceleration $\lambda$) one obtains [8, 9]

$$K_\infty(x'', T; x', 0) \prod_{p \in \mathbb{P}} K_p(x'', T; x', 0) = 1, \quad x'', x' \in \mathbb{Q}, \quad T \in \mathbb{Q}^\times,$$  

(13)

where

$$K_v(x'', T; x', 0) = \lambda_v(-8T) |4T|^{-\frac{1}{2}} \chi_v\left(-\frac{\lambda^2 T^3}{24} + [\lambda(x'' + x') - 2]T + \frac{(x'' - x')^2}{8T}\right).$$  

(14)

The adelic wave function for the simplest ground state has the form

$$\psi_\Lambda(x) = \psi_\infty(x) \prod_{p \in \mathbb{P}} \Omega(|x|_p) = \begin{cases} 
\psi_\infty(x), & x \in \mathbb{Z}, \\
0, & x \in \mathbb{Q} \setminus \mathbb{Z},
\end{cases}$$  

(15)

where $\Omega(|x|_p) = 1$ if $|x|_p \leq 1$ and $\Omega(|x|_p) = 0$ if $|x|_p > 1$. Since this wave function is non-zero only in integer points it can be interpreted as discreteness of the space due to $p$-adic effects in adelic approach.

The Gel’fand-Graev-Tate gamma and beta functions [4, 5] are:

$$\Gamma_\infty(a) = \int_\mathbb{R} |x|_{\infty}^{a-1} \chi_\infty(x) \, d_\infty x = \frac{\zeta(1-a)}{\zeta(a)},$$

$$\Gamma_p(a) = \int_{\mathbb{Q}_p} |x|_p^{a-1} \chi_p(x) \, d_p x = \frac{1 - p^{a-1}}{1 - p^{-a}},$$

(16)

$$B_\infty(a, b) = \int_\mathbb{R} |x|_{\infty}^{a-1} |1 - x|_{\infty}^{b-1} \, d_\infty x = \Gamma_\infty(a) \Gamma_\infty(b) \Gamma_\infty(c),$$

$$B_p(a, b) = \int_{\mathbb{Q}_p} |x|_p^{a-1} |1 - x|_p^{b-1} \, d_p x = \Gamma_p(a) \Gamma_p(b) \Gamma_p(c),$$

(17)

(18)
where \( a, b, c \in \mathbb{C} \) with condition \( a + b + c = 1 \) and \( \zeta(a) \) is the Riemann zeta function. With a regularization of the product of \( p \)-adic gamma functions one has adelic products:

\[
\Gamma_\infty(u) \prod_{p \in \mathbb{P}} \Gamma_p(u) = 1, \quad B_\infty(a, b) \prod_{p \in \mathbb{P}} B_p(a, b) = 1, \quad u \neq 0, 1, \quad u = a, b, c,
\]

where \( a + b + c = 1 \). It is worth noting now that \( B_\infty(a, b) \) and \( B_p(a, b) \) are the crossing symmetric standard and \( p \)-adic Veneziano amplitudes for scattering of two open tachyon strings. There are generalizations of the above product formulas for integration on quadratic extensions of \( \mathbb{R} \) and \( \mathbb{Q}_p \), as well as on algebraic number fields, and they include scattering of closed strings \([4, 10]\).

Introducing real, \( p \)-adic and adelic zeta functions as

\[
\zeta_\infty(a) = \int_\mathbb{R} \exp\left(-\pi x^2\right) |x|^{a-1} d_\infty x = \pi^{-\frac{a}{2}} \Gamma\left(\frac{a}{2}\right),
\]

\[
\zeta_p(a) = \frac{1}{1-p^{-1}} \int_{\mathbb{Q}_p} \Omega(|x|_p) |x|^{a-1} d_p x = \frac{1}{1-p^{-a}}, \quad \text{Re} \ a > 1,
\]

\[
\zeta_\Lambda(a) = \zeta_\infty(a) \prod_{p \in \mathbb{P}} \zeta_p(a) = \zeta_\infty(a) \zeta(a),
\]

one obtains

\[
\zeta_\Lambda(1-a) = \zeta_\Lambda(a),
\]

where \( \zeta_\Lambda(a) \) can be called adelic zeta function. Let us note that \( \exp\left(-\pi x^2\right) \) and \( \Omega(|x|_p) \) are analogous functions in real and \( p \)-adic cases. Adelic harmonic oscillator \([7]\) has connection with the Riemann zeta function. Namely, the simplest vacuum state of the adelic harmonic oscillator is the following Schwartz-Bruhat function:

\[
\psi_\Lambda(x) = 2^{\frac{1}{2}} e^{-\pi x^2} \prod_{p \in \mathbb{P}} \Omega(|x|_p),
\]

whose the Fourier transform

\[
\psi_\Lambda(k) = \int \chi_\Lambda(kx) \psi_\Lambda(x) = 2^{\frac{1}{2}} e^{-\pi k^2} \prod_{p \in \mathbb{P}} \Omega(|k|_p)
\]

has the same form as \( \psi_\Lambda(x) \). The Mellin transform of \( \psi_\Lambda(x) \) is

\[
\Phi_\Lambda(a) = \int \psi_\Lambda(x) |x|^a d_\Lambda^a x
\]
\[ = \int_{\mathbb{R}} \psi_\infty(x) |x|^{a-1} d_\infty x \prod_{p \in \mathbb{P}} \frac{1}{1 - p^{-1}} \int_{\mathbb{Q}_p} \Omega(|x|_p) |x|^{a-1} d_p x = \sqrt{2} \Gamma\left(\frac{a}{2}\right) \pi^{-\frac{a}{2}} \zeta(a) \]

(26)

and the same for \( \psi_A(k) \). Then according to the Tate formula one obtains (23). It is remarkable that such simple physical system as harmonic oscillator is related to so significant mathematical object as the Riemann zeta function.

Recently \cite{11} adelic properties of dynamical systems, which evolution is governed by linear fractional transformations

\[ f(x) = \frac{ax + b}{cx + d}, \quad a, b, c, d, \in \mathbb{Q}, \quad ad - bc = 1 \]

(27)

is investigated. It is shown that rational fixed points are \( p \)-adic indifferent for all but a finite set \( \mathcal{P} \) of primes, i.e. only for finite number of \( p \)-adic cases a rational fixed point may be attractive or repelling.

4 Concluding remarks

We presented a brief review of some important applications of adeles in modern mathematical physics. We considered above simple cases of adeles \( \mathbb{A}_\mathbb{Q} \) consisting of completions of \( \mathbb{Q} \). There is also ring of adeles \( \mathbb{A}_\mathbb{K} \) related to the completions of any global field \( \mathbb{K} \). There is a straightforward generalization of \( \mathbb{A}_\mathbb{Q} \) to the \( n \)-dimensional vector space \( \mathbb{A}_\mathbb{Q}^n = \prod_{i=1}^n \mathbb{A}_\mathbb{Q}^{(i)} \) (see, e.g. \cite{1}). Adelic algebraic group \( G(\mathbb{A}_\mathbb{K}) \) is an adelization of a linear algebraic group \( G \) over completion fields \( \mathbb{K}_v \) of a global field \( \mathbb{K} \) \cite{1, 2, 3}.

For a more detail insight into this attractive and promising field of investigations let us also mention a few additional topics. Adelic quantum cosmology (for a review, see \cite{9}) is an application of adelic quantum mechanics \cite{7} to explore very early evolution of the universe as a whole. Adelic path integral \cite{12} is a suitable extension of the standard Feynman path integral and serves to describe quantum evolution of adelic objects. Conjecture on the adelic universe with real and \( p \)-adic worlds, as well as \( p \)-adic origin of dark matter and dark energy are discussed in \cite{9}. Adelic summability \cite{13} of perturbation series is an approach to summation of divergent series in the real case when they are convergent in all \( p \)-adic cases. Use of effective Lagrangians on real numbers for \( p \)-adic strings has been very efficient in their application to string theory and cosmology. Paper \cite{14} is an attempt
towards effective Lagrangian for adelic strings without tachyons. Further development of adelic analysis and, in particular, adelic generalized functions [6, 15, 16] is one of mathematical opportunities.

One can conclude that there has been a successful application of adeles in modern mathematical physics and that one can expect a growing interest in their further mathematical developments as well as in applications.

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References

[1] I.M. Gel’fand, M.I. Graev and I.I. Pyatetskii-Shapiro, *Representation Theory and Automorphic Functions*, Saunders, London, 1966.

[2] A. Weil, *Adeles and Algebraic Groups*, Birkhauser, Basel, 1982.

[3] V.P. Platonov, A.S. Rapinchuk, *Algebraic groups and Number Theory* (in Russian), Nauka, Moskva, 1991.

[4] L. Brekke, P.G.O. Freund, *p-Adic Numbers in Physics*, Physics Reports 233 (1993) 1-66.

[5] V.S. Vladimirov, I.V. Volovich and E.I. Zelenov, *p-Adic Analysis and Mathematical Physics* (in Russian), Nauka, Moskva, 1994.

[6] B. Dragovich, *On Generalized Functions in Adelic Quantum Mechanics*, Integral Transform. Spec. Funct. 6 (1998) 197 - 203, arXiv:math-ph/0404076.

[7] B. Dragovich, *Adelic Model of Harmonic Oscillator*, Theor. Math. Phys. 101 (1994) 349-359, arXiv:hep-th/0402193; *p-Adic and Adelic Quantum Mechanics*, Proc. V.A. Steklov Inst. Math. 245 (2004) 72-85, arXiv:hep-th/0312046.

[8] G. Djordjević, B. Dragovich, Lj. Nešić, I.V. Volovich, *p-Adic and Adelic Minisuperspace Quantum Cosmology*, Int. J. Mod. Phys. A 17 (2002) 1413-1434, arXiv:gr-qc/0105050.
[9] B. Dragovich, *p-Adic and Adelic Cosmology: p-Adic Origin of Dark Energy and Dark Matter*, in "p-Adic Mathematical Physics", AIP Conference Proceedings 826 (2006) 25-42, arXiv:hep-th/0602044.

[10] V.S. Vladimirov, *Adelic Formulas for Gamma and Beta Functions of One-Class Quadratic Fields: Applications to 4-Particle Scattering String Amplitudes*, Proc. Steklov Math. Inst. 228 (2000) 67 - 80, arXiv:math-ph/0004017.

[11] B. Dragovich, A. Khrennikov, D. Mihajlović, *Linear Fractional p-Adic and Adelic Dynamical Systems*, arXiv:math-ph/0612058, to appear in Reports on Mathematical Physics.

[12] G. Djordjević, B. Dragovich and Lj. Nešić, *Adelic Path Integrals for Quadratic Lagrangians*, Inf. Dim. Anal. Quant. Probab. Rel. Top. 6 (2003) 179 - 195, arXiv:hep-th/0105030.

[13] B. Dragovich, *p-Adic Perturbation Series and Adelic Summability*, Phys. Lett. B 256 (1991) 392 - 396.

[14] B. Dragovich, *Zeta Strings*, arXiv:hep-th/0703008.

[15] B. Dragovich, Ya. Radyno, A. Khrennikov, *Generalized Functions on Adeles*, Proceedings of the Spring Mathematical School (in Russian), Voronezh, 2000, pp. 85-94.

[16] E.M. Radyno, Ya.V. Radyno, *Distributions and mnemofunctions on Adeles. Fourier Transformation*, Proc. V.A. Steklov Inst. Math. 245 (2004) 228-240.