Boundary Hölder Regularity for Fully Nonlinear Elliptic Equations on Reifenberg Flat Domains

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Abstract In this note, we investigate the boundary Hölder regularity for fully nonlinear elliptic equations on Reifenberg flat domains. We will prove that for any $0 < \alpha < 1$, there exists $\delta > 0$ such that the solutions are $C^\alpha$ at $x_0 \in \partial \Omega$ provided that $\Omega$ is $(\delta, R)$-Reifenberg flat at $x_0$ (see Definition 1.1). A similar result for the Poisson equation has been proved by Lemenant and Sire [5], where the Alt-Caffarelli-Friedman's monotonicity formula is used. Besides the generalization to fully nonlinear elliptic equations, our method is simple. In addition, even for the Poisson equation, our result is stronger than that of Lemenant and Sire.

Keywords Boundary regularity · Hölder continuity · Reifenberg flat domain · Fully nonlinear elliptic equation · Viscosity solution

Mathematics Subject Classification (2000) MSC 35B65 · 35J25 · 35J60 · 35D40

1 Introduction

In the regularity theory of partial differential equations, Hölder continuity is a kind of quantitative estimate. It is usually the first smooth regularity for solutions and the beginning for higher regularity. With respect to the interior

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Hölder regularity, De Giorgi [2], Nash [8] and Moser [7] proved it for elliptic equations in divergence form and many generalizations have been made. In particular, Krylov and Safonov [4] proved the interior Hölder regularity for elliptic equations in nondivergence form and fully nonlinear elliptic equations can also be treated. As regards the boundary Hölder continuity, it is well known that if $\Omega$ satisfies the exterior cone condition at $x_0 \in \partial\Omega$, then the solution is Hölder continuous at $x_0$ (see [6] and [3, Theorem 8.29 and Corollary 9.28]). Recently, the second author proved the boundary Hölder regularity on more general domains (e.g. the Reifenberg flat domains) for various elliptic equations [9].

For the Poisson equations, Lemenant and Sire [5] proved that for any $0 < \alpha < 1$, there exists $\delta > 0$ such that the solutions are $C^\alpha$ at $x_0 \in \partial\Omega$ provided that $\Omega$ is $(\delta, R)$-Reifenberg flat at $x_0$. In this note, we will prove an analogue result for the viscosity solutions of fully nonlinear elliptic equations and our method is simple. Moreover, for the Poisson equations, our result has relaxed the requirements imposed in [5]. The main idea in the proof is that a Reifenberg flat domain can be regarded as a perturbation of half balls in different scales. The idea has been motivated by [9].

**Definition 1.1 (Reifenberg flat domain)** We say that $\Omega$ is $(\delta, R)$-Reifenberg flat from exterior at $x_0 \in \partial\Omega$ if for any $0 < r < R$, there exists a coordinate system $\{y_1, \ldots, y_n\}$ such that $x_0 = 0$ in this coordinate system and

$$B_r \cap \Omega \subset B_r \cap \{y_n > -\delta r\}.$$  

(1.1)

In this note, we treat the viscosity solutions for fully nonlinear elliptic equations and use the standard definition of a viscosity solution. For details, we refer to [1].

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $f$ be a function defined on $\bar{\Omega}$. We say that $f$ is $C^\alpha$ $(0 < \alpha < 1)$ at $x_0 \in \bar{\Omega}$ or $f \in C^\alpha(x_0)$ if there exists a constant $C$ such that

$$|f(x) - f(x_0)| \leq C|x - x_0|^{\alpha}, \quad \forall \ x \in \bar{\Omega}.$$  

(1.2)

Then, define $[f]_{C^\alpha(x_0)} = C$ and $\|f\|_{C^\alpha(x_0)} = \|f\|_{L^\infty(\Omega)} + [f]_{C^\alpha(x_0)}$.

Our main result is the following.

**Theorem 1.1** Let $0 < \alpha < 1$ and $u$ be a viscosity solution of

$$\begin{cases}
u \in S(\lambda, \Lambda, f) & \text{in } \Omega; \\
u = g & \text{on } \partial\Omega,
\end{cases}$$

(1.3)

where $0 \in \partial\Omega$, $g \in C^\alpha(0)$ and $f$ satisfies

$$\left(\frac{1}{|B_r \cap \Omega|} \int_{B_r \cap \Omega} |f|^n\right)^{\frac{1}{n}} \leq C f^{\alpha - 2}, \quad \forall \ r > 0.$$  

(1.4)

Then there exists $\delta$ depending only on $n, \lambda, \Lambda$ and $\alpha$ such that if $\Omega$ is $(\delta, R)$-Reifenberg flat from exterior at $0$ for some $R > 0$, $u$ is $C^\alpha$ at $0$ and

$$|u(x) - u(0)| \leq C|x|^\alpha \left(\|u\|_{L^\infty(\Omega \cap B_R)} + C_f [g]_{C^\alpha(0)}\right), \quad \forall \ x \in \Omega \cap B_R,$$

(1.5)

where $C$ depends only on $n, \lambda, \Lambda, \alpha$ and $R$. 

For the Poisson equation, we have the following corresponding result:

**Theorem 1.2** Let $0 < \alpha < 1$ and $u$ satisfy

\[
\begin{cases}
\Delta u = f & \text{in } \Omega; \\
u = g & \text{on } \partial \Omega,
\end{cases}
\]

where $0 \in \partial \Omega$, $g \in C^\alpha(0)$ and $f \in L^p(\Omega)$ with $p = n/(2 - \alpha)$.

Then there exists $\delta$ depending only on $n$ and $\alpha$ such that if $\Omega$ is $(\delta, R)$-Reifenberg flat from exterior at 0 for some $R > 0$, $u$ is $C^\alpha$ at 0 and

\[
|u(x) - u(0)| \leq C|x|^\alpha \left( \|u\|_{L^\infty(\Omega \cap B_R)} + \|f\|_{L^p(\Omega \cap B_R)} + \|g\|_{C^\alpha(0)} \right), \quad \forall x \in \Omega \cap B_R,
\]

where $C$ depends only on $n, \alpha, p$ and $R$.

**Remark 1.1** Theorem 1.2 is more general by comparing with the result of [5].

### 2 Proof of the main result

Since the proofs of Theorem 1.1 and Theorem 1.2 are similar, we only give the detailed proof for Theorem 1.1.

The main idea is the following. A Reifenberg flat domain is regarded as a perturbation of half balls in different scales. On the other hand, solutions on half balls have sufficient regularity (in fact the Lipschitz regularity). Hence, the boundary Hölder regularity of the solution on a Reifenberg flat domain can be obtained from solutions on half balls by approximation and iteration. The following is the detail of the proof.

**Proof of Theorem 1.1.** Without loss of generality, we assume that $R = 1$ and $g(0) = 0$. Let $M = \|u\|_{L^\infty(\Omega \cap B_1)} + C_f + \|g\|_{C^\alpha(0)}$ and $\Omega_r = \Omega \cap B_r$. To prove (1.5), we only need to show the following:

There exist constants $0 < \delta < 1$, $0 < \eta < 1$ and $C'$ depending only on $n, \lambda, \Lambda$ and $\alpha$ such that for all $k \geq 0$,

\[
\|u\|_{L^\infty(\Omega_r)} \leq C' \eta^{k\alpha}.
\]

We prove (2.1) by induction. For $k = 0$, it holds clearly. Suppose that it holds for $k$. We need to prove that it holds for $k + 1$.

Let $r = \eta^k/2$. Then there exists a coordinate system $\{y_1, ..., y_n\}$ such that

\[
B_r \cap \Omega \subset B_r \cap \{y_n > -\delta r\}.
\]

Let $\bar{B}_r^+ = B_r^+ - \delta r e_n$, $\bar{T}_r = T_r - \delta r e_n$ and $\bar{\Omega}_r = \Omega \cap \bar{B}_r^+$ where $e_n = (0, 0, ..., 0, 1)$. Take

\[
\delta \leq 1/4.
\]

Then $\Omega_{r/2} \subset \bar{\Omega}_r \subset \Omega_{\eta^k}$. 


Let \( v \) solve
\[
\begin{cases}
M^+(D^2 v, \lambda, A) = 0 & \text{in } \hat{B}_r^+; \\
v = 0 & \text{on } \hat{T}_r; \\
v = \hat{C} M \eta^{k\alpha} & \text{on } \partial\hat{B}_r^+ \setminus \hat{T}_r.
\end{cases}
\]
Then \( w := u - v \) satisfies (since \( v \geq 0 \))
\[
\begin{cases}
w \in \mathcal{S}(\lambda/n, A, f) & \text{in } \hat{\Omega}_r; \\
w = g - v \leq g & \text{on } \partial\Omega \cap \hat{B}_r^+; \\
w \leq 0 & \text{on } \partial\hat{B}_r^+ \cap \hat{\Omega}.
\end{cases}
\]

Take \( \delta \leq \eta \).
(2.4)

Then by the boundary Lipschitz estimate for \( v \),
\[
\|v\|_{L^\infty(\Omega_{2r})} \leq C \left( \frac{2\eta + \delta}{r} \right)\|v\|_{L^\infty(\hat{B}_r^+)} \leq C_1 \eta \cdot \hat{C} M \eta^{k\alpha} \leq C_1 \eta^{1-\alpha} \cdot \hat{C} M \eta^{(k+1)\alpha},
\]
where \( C_1 \) depends only on \( n, \lambda \) and \( \Lambda \). For \( w \), by the A-B-P maximum principle, we have for \( x \in \hat{\Omega}_r \),
\[
w(x) \leq \|g\|_{L^\infty(\partial\Omega \cap \hat{B}_r^+)} + C_2 r \|f\|_{L^\infty(\hat{\Omega}_r)} \leq 1^\alpha M + C_2 M r^\alpha = \frac{1 + C_2}{2^{\alpha} \eta^\alpha} \cdot M \eta^{(k+1)\alpha},
\]
where \( C_2 \) depends only on \( n, \lambda \) and \( \Lambda \).

Take \( \eta \) small enough such that
\[
C_1 \eta^{1-\alpha} \leq 1/2.
\]
Next, take \( \hat{C} \) large enough such that
\[
\frac{1 + C_2}{2^{\alpha} \eta^\alpha} \leq \hat{C}/2.
\]
Then combining with (2.5) and (2.6), we have
\[
\sup_{\Omega_{\eta^{k+1}}} u \leq \sup_{\Omega_{\eta^{k+1}}} w + \sup_{\Omega_{\eta^{k+1}}} v \leq \hat{C} M \eta^{(k+1)\alpha}.
\]

The proof for
\[
\inf_{\Omega_{\eta^{k+1}}} u \geq -\hat{C} M \eta^{(k+1)\alpha}
\]
is similar and we omit here. Therefore,
\[
\|u\|_{L^\infty(\Omega_{\eta^{k+1}})} \leq \hat{C} M \eta^{(k+1)\alpha}.
\]
By induction, the proof is completed. \( \square \)
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