EXAMPLES OF AREA-MINIMIZING SURFACES IN THE SUBRIEMANNIAN HEISENBERG GROUP $\mathbb{H}^1$ WITH LOW REGULARITY

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ABSTRACT. We give new examples of entire area-minimizing $t$-graphs in the subriemannian Heisenberg group $\mathbb{H}^1$. Most of the examples are locally lipschitz in Euclidean sense. Some regular examples have prescribed singular set consisting of either a horizontal line or a finite number of horizontal halflines extending from a given point. Amongst them, a large family of area-minimizing cones is obtained.

1. INTRODUCTION

Variational problems related to the subriemannian area in the Heisenberg group $\mathbb{H}^1$ have received great attention recently. A major question in this theory is the regularity of minimizers. A related one is the construction of examples with low regularity properties. The study of minimal surfaces in subriemannian geometry was initiated in the paper by Garofalo and Nhieu [22]. Later Pauls [27] constructed minimal surfaces in $\mathbb{H}^1$ as limits of minimal surfaces in Nil manifolds, the riemannian Heisenberg groups. Cheng, Hwang and Yang [9] have studied the weak solutions of the minimal surface equation for $t$-graphs and have proven existence and uniqueness results. Regularity of minimal surfaces, assuming that they are least $C^1$, has been treated in the papers by Pauls [28] and Cheng, Hwang and Yang [10]. We would like also to mention the recently distributed notes by Bigolin and Serra Cassano [5], where they obtain regularity properties of an $H$-regular surface from regularity properties of its horizontal unit normal. Interesting examples of minimal surfaces which are not area-minimizing are obtained in [11]. See also [13]. Smoothness of lipschitz minimal intrinsic graphs in Heisenberg groups $\mathbb{H}^n$, for $n > 1$, has been recently obtained by Capogna, Citti and Manfredini [6].

Characterization in $\mathbb{H}^1$ of solutions of the Bernstein problem for $C^2$ surfaces has been obtained by Cheng, Hwang, Malchiodi and Yang [8], and Ritoré and Rosales [29] for $t$-graphs, and by Barone Adessi, Serra Cassano and Vittone [4] and Garofalo and Pauls [23] for vertical graphs.

Additional contributions concerning variational problems related to the subriemannian area in the Heisenberg groups include [26], [2], [3], [9], [10], [21], [21], [20], [19], [18], [17], [16], [25], [29]. The recent monograph by Capogna, Danielli, Pauls and Tyson [7] gives a recent overview of the subject with an exhaustive list of references. We would like
to stress that, in $H^1$, the condition $H \equiv 0$ is not enough to guarantee that a given surface of class $C^2$ is even a stationary point for the area functional, see Ritoré and Rosales [29], and Cheng, Hwang and Yang [9] for minimizing $t$-graphs.

The aim of this paper is to provide new examples in $H^1$ of Euclidean locally lipschitz area-minimizing entire graphs over the $xy$-plane.

In section 3 we construct the basic examples. We start from a given horizontal line $L$, and a monotone angle function $\alpha : L \to (0, \pi)$ over this line. For each $p \in L$, we consider the two horizontal halflines extending from $p$ making an angle $\pm \alpha(p)$ with $L$. We prove that in this way we always obtain an entire graph over the $xy$-plane which is Euclidean locally lipschitz and area-minimizing. The angle function $\alpha$ is only assumed to be continuous and monotone. Of course, further regularity on $\alpha$ yields more regularity on the graph. In case $\alpha$ is at least $C^2$ we get that the associated surface is $C^{1,1}$. The surfaces in section 3 are the building blocks for our next construction in section 4. We fix a point $p \in H^1$, and a family of counter-clockwise oriented horizontal halflines $R_1, \ldots, R_n$ extending from $p$. We choose the bisector $L_i$ of the wedge determined by $R_{i-1}$ and $R_i$, and we consider angle functions $\alpha_i : L_i \to (0, \pi)$ which are continuous, nonincreasing as a function of the distance to $p$, and such that $\alpha_i(p)$ is equal to the angle between $L_i$ and $R_i$. For every $q \in L_i$ we consider the halflines extending from $q$ with angles $\pm \alpha_i(q)$. In this way we also a family of area-minimizing $t$-graphs which are Euclidean locally lipschitz. In case the obtained surface is regular enough we have that the singular set is precisely $\bigcup_{i=1}^n L_i$. If the angle functions $\alpha_i$ are constant, then we obtain area-minimizing cones (the original motivation of this paper), which are Euclidean locally $C^{1,1}$ minimizers, and $C^\infty$ outside the singular set $\bigcup_{i=1}^n L_i$. For a single halfline $L$ extending from the origin and an angle function $\alpha : L \to (0, \pi)$, continuous and nonincreasing as a function of the distance to 0, we patch the graph obtained over a wedge of the $xy$-plane with the plane $t = 0$ along the halflines extending from 0 making an angle $\alpha(0)$ with $L$. When $\alpha$ is constant we get again an area-minimizing cone which is Euclidean locally lipschitz. These cones are a generalization of the one obtained by Cheng, Hwang and Yang [9, Ex. 7.2].

An interesting consequence of this construction is that we get a large number of Euclidean locally $C^{1,1}$ area-minimizing cones with prescribed singular set consisting on either a horizontal line or a finite number of horizontal halflines extending from a given point. It is an open question to decide if these examples are the only area-minimizing cones, together with vertical halfspaces and the example by Cheng, Hwang and Yang [9, Ex. 7.2] with a singular halfline and its generalizations in the last section. The importance of tangent cones has been recently stressed in [11].

2. Preliminaries

The *Heisenberg group* $H^1$ is the Lie group $(\mathbb{R}^3, *)$, where the product $*$ is defined, for any pair of points $[z, t], [z', t'] \in \mathbb{R}^3 \equiv \mathbb{C} \times \mathbb{R}$, as

$$[z, t] * [z', t'] := [z + z', t + t' + \text{Im}(z z')]$$,

$$z = x + iy$$. 

For $p \in \mathbb{H}^1$, the left translation by $p$ is the diffeomorphism $L_p(q) = p + q$. A basis of left invariant vector fields (i.e., invariant by any left translation) is given by

$$X := \frac{\partial}{\partial x} + y \frac{\partial}{\partial t}, \quad Y := \frac{\partial}{\partial y} - x \frac{\partial}{\partial t}, \quad T := \frac{\partial}{\partial t}.$$ 

The horizontal distribution $\mathcal{H}$ in $\mathbb{H}^1$ is the smooth planar one generated by $X$ and $Y$. The horizontal projection of a vector $U$ onto $\mathcal{H}$ will be denoted by $U_H$. A vector field $U$ is called horizontal if $U = U_H$. A horizontal curve is a $C^1$ curve whose tangent vector lies in the horizontal distribution.

We denote by $[U, V]$ the Lie bracket of two $C^1$ vector fields $U$, $V$ on $\mathbb{H}^1$. Note that $[X, T] = [Y, T] = 0$, while $[X, Y] = -2T$. The last equality implies that $\mathcal{H}$ is a bracket generating distribution. Moreover, by Frobenius Theorem we have that $\mathcal{H}$ is nonintegrable. The vector fields $X$ and $Y$ generate the kernel of the (contact) 1-form $\omega := -y \, dx + x \, dy + dt$.

We shall consider on $\mathbb{H}^1$ the (left invariant) Riemannian metric $g = \langle \cdot, \cdot \rangle$ so that $\{X, Y, T\}$ is an orthonormal basis at every point, and the associated Levi-Civita connection $\nabla$. The modulus of a vector field $U$ will be denoted by $|U|$.

Let $\gamma : I \to \mathbb{H}^1$ be a piecewise $C^1$ curve defined on a compact interval $I \subset \mathbb{R}$. The length of $\gamma$ is the usual Riemannian length $L(\gamma) := \int_{I} |\dot{\gamma}|$, where $\dot{\gamma}$ is the tangent vector of $\gamma$. For two given points in $\mathbb{H}^1$ we can find, by Chow’s connectivity Theorem [24, p. 95], a horizontal curve joining these points. The Carnot-Carathéodory distance $d_{\mathbb{H}}$ between two points in $\mathbb{H}^1$ is defined as the infimum of the length of horizontal curves joining the given points. A geodesic $\gamma : \mathbb{H}^1 \to \mathbb{R}$ is a horizontal curve which is a critical point of length under variations by horizontal curves. They satisfy the equation

$$D\dot{\gamma} + 2\lambda J(\gamma) = 0,$$

where $\lambda \in \mathbb{R}$ is the curvature of the geodesic, and $J$ is the $\pi/2$-degrees oriented rotation in the horizontal distribution. Geodesics in $\mathbb{H}^1$ with $\lambda = 0$ are horizontal straight lines. The reader is referred to the section on geodesics in [29] for further details.

The volume $|\Omega|$ of a Borel set $\Omega \subset \mathbb{H}^1$ is the Riemannian volume of the left invariant metric $g$, which coincides with the Lebesgue measure in $\mathbb{R}^3$. We shall denote this volume element by $dv_g$. The perimeter of $E \subset \mathbb{H}^1$ in an open subset $\Omega \subset \mathbb{H}^1$ is defined as

$$|\partial E|(\Omega) := \sup \left\{ \int_{\Omega} \text{div} \, U \, dv_g : U \text{ horizontal and } C^1, |U| \leq 1, \text{supp}(U) \subset \Omega \right\},$$

where supp$(U)$ is the support of $U$. A set $E \subset \mathbb{H}^1$ is of locally finite perimeter if $P(E, \Omega) < +\infty$ for any bounded open set $\Omega \subset \mathbb{H}^1$. A set of locally finite perimeter has a measurable horizontal unit normal $v_E$, that satisfies the following divergence theorem [17 Corollary 7.6]: if $U$ is a horizontal vector field with compact support, then

$$\int_{E} \text{div} \, U \, dv_g = \int_{\mathbb{H}^1} \langle U, v_E \rangle \, d|\partial E|.$$

If $E \subset \mathbb{H}^1$ has Euclidean lipschitz boundary, then [17 Corollary 7.7]

$$|\partial E|(\Omega) = \int_{\partial E \cap \Omega} |N_H| \, d\mathcal{H}^2,$$
with respect to the Riemannian distance on \( H \) [14, Lemme 1], see also [3, Theorem 1.2], that, for a \( C^{\Sigma} \) empty interior in \( \Sigma \) the tangent plane \( \partial T \) to \( T \) and, taking again limits when \( \epsilon \to 0 \), we have

\[
|\partial E|(\Omega) \leq |\partial F|(\Omega).
\]

The following extension of the divergence theorem will be needed to prove the area-minimizing property of sets of locally finite perimeter

**Theorem 2.1.** Let \( E \subset H^1 \) be a set of locally finite perimeter, \( B \subset H^1 \) a set with piecewise smooth boundary, and \( U \) a \( C^1 \) horizontal vector field in \( \text{int}(B) \) that extends continuously to the boundary of \( B \). Then

\[
\int_{E \cap B} \text{div} U \, d\nu = \int_{B} \langle U, v_E \rangle d|\partial E| + \int_{E \cap B} \langle U, v_B \rangle d|\partial B|.
\]

**Proof.** The proof is modelled on [15, § 5.7]. Let \( s \) denote the riemannian distance function to \( H^1 - B \). For \( \epsilon > 0 \), define

\[
h_\epsilon(p) := \begin{cases} 
1, & \epsilon \leq s(p), \\
\frac{s(p)}{\epsilon}, & 0 \leq s(p) \leq \epsilon,
\end{cases}
\]

Then \( h_\epsilon \) is a lipschitz function (in riemannian sense). For any smooth \( h \) with compact support in \( B \) we have \( \text{div}(hU) = h \text{div}(U) + \langle \nabla h, U \rangle \). By applying the divergence theorem for sets of locally finite perimeter [12] we get

\[
\int_{H^1} h \langle U, v_E \rangle d|\partial E| = \int_{E} h \text{div}(U) + \int_{E \cap B} \langle \nabla h, U \rangle.
\]

By approximation, this formula is also valid for \( h_\epsilon \). Taking limits when \( \epsilon \to 0 \) we have \( H_\epsilon \to \chi_B \). By the coarea formula for lipschitz functions

\[
\frac{1}{\epsilon} \int_{\{0 \leq s \leq \epsilon\}} \chi_E \langle \nabla s, U \rangle = \frac{1}{\epsilon} \int_{0}^{\epsilon} \left\{ \int_{s=r} \chi_E \langle \nabla s, U \rangle d\mathcal{H}^2 \right\} dr,
\]

and, taking again limits when \( \epsilon \to 0 \) and calling \( N_B \) to the riemannian outer unit normal to \( \partial B \) (defined except on a small set), we have

\[
\lim_{\epsilon \to 0} \int_{E} \langle \nabla h_\epsilon, U \rangle = \int_{\partial B} \chi_E \langle N_B, U \rangle d\mathcal{H}^2 = \int_{E \cap B} \langle v_B, U \rangle d|\partial B|.
\]

Hence (2.4) is proved. \( \square \)

For a \( C^1 \) surface \( \Sigma \subset H^1 \) the singular set \( \Sigma_0 \) consists of those points \( p \in \Sigma \) for which the tangent plane \( T_p \Sigma \) coincides with the horizontal distribution. As \( \Sigma_0 \) is closed and has empty interior in \( \Sigma \), the regular set \( \Sigma - \Sigma_0 \) of \( \Sigma \) is open and dense in \( \Sigma \). It was proved in [14, Lemme 1], see also [3, Theorem 1.2], that, for a \( C^2 \) surface, the Hausdorff dimension with respect to the Riemannian distance on \( H^1 \) of \( \Sigma_0 \) is less than two.

If \( \Sigma \) is a \( C^1 \) oriented surface with unit normal vector \( N \), then we can describe the singular set \( \Sigma_0 \subset \Sigma \), in terms of \( N_H \), as \( \Sigma_0 = \{ p \in \Sigma : N_H(p) = 0 \} \). In the regular part \( \Sigma - \Sigma_0 \),
we can define the horizontal unit normal vector $v_H$, as in [12], [30] and [23] by

\begin{equation}
\label{eq:horizontal_unit_normal}
v_H := \frac{N_H}{|N_H|}.
\end{equation}

Consider the characteristic vector field $Z$ on $\Sigma - \Sigma_0$ given by

\begin{equation}
\label{eq:characteristic_vector_field}
Z := J(v_H).
\end{equation}

As $Z$ is horizontal and orthogonal to $v_H$, we conclude that $Z$ is tangent to $\Sigma$. Hence $Z_p$ generates the intersection of $T_p\Sigma$ with the horizontal distribution. The integral curves of $Z$ in $\Sigma - \Sigma_0$ will be called characteristic curves of $\Sigma$. They are both tangent to $\Sigma$ and horizontal. Note that these curves depend on the unit normal $N$ to $\Sigma$. If we define

\begin{equation}
\label{eq:characteristic_curve}
S := \langle N, T \rangle v_H - |N_H| T,
\end{equation}

then $\{Z_p, S_p\}$ is an orthonormal basis of $T_p\Sigma$ whenever $p \in \Sigma - \Sigma_0$.

In the Heisenberg group $H^1$ there is a one-parameter group of dilations $\{\varphi_s\}_{s \in \mathbb{R}}$ generated by the vector field

\begin{equation}
\label{eq:dilation_vector_field}
W := xX + yY + 2tT.
\end{equation}

We may compute $\varphi_s$ in coordinates to obtain

\begin{equation}
\label{eq:conjugation}
\varphi_s(x_0, y_0, t_0) = (e^s x_0, e^s y_0, e^{2s} t_0).
\end{equation}

Conjugating with left translations we get the one-parameter family of dilations $\varphi_{p,s} := L_p \circ \varphi_s \circ L_p^{-1}$ with center at any point $p \in H^1$. A set $E \subset H^1$ is a cone of center $p$ if $\varphi_{p,s}(E) \subset E$ for all $s \in \mathbb{R}$.

Any isometry of $(H^1, g)$ leaving invariant the horizontal distribution preserves the area of surfaces in $H^1$. Examples of such isometries are left translations, which act transitively on $H^1$. The Euclidean rotation of angle $\theta$ about the $t$-axis given by

\[(x, y, t) \mapsto r_\theta(x, y, t) = (\cos \theta x - \sin \theta y, \sin \theta x + \cos \theta y, t),\]

is also an area-preserving isometry in $(H^1, g)$ since it transforms the orthonormal basis $\{X, Y, T\}$ at the point $p$ into the orthonormal basis $\{\cos \theta X + \sin \theta Y, -\sin \theta X + \cos \theta Y, T\}$ at the point $r_\theta(p)$.

3. Examples with one singular line

Consider the $x$-axis in $H^1 = \mathbb{R}^3$ parametrized by $\Gamma(v) := (v, 0, 0)$. Take a non-increasing continuous function $\alpha : \mathbb{R} \to (0, \pi)$. For every $v \in \mathbb{R}$, consider two horizontal halflines $L^+_v, L^-_v$ extending from $\Gamma(v)$ with angles $\alpha(v)$ and $-\alpha(v)$, respectively. The tangent vectors to these curves at $\Gamma(v)$ are given by $\cos \alpha(v) X_{\Gamma(v)} + \sin \alpha(v) Y_{\Gamma(v)}$ and $\cos \alpha(v) X_{\Gamma(v)} - \sin \alpha(v) Y_{\Gamma(v)}$, respectively.

The parametric equations of this surface are given by

\begin{equation}
\label{eq:parametric_equations}
(v, w) \mapsto \begin{cases} (v + w \cos \alpha(v), w \sin \alpha(v), -\cos \alpha(v)), & w \geq 0, \\ (v - |w| \cos \alpha(v), -|w| \sin \alpha(v), v|w| \sin \alpha(v)), & w \leq 0, \end{cases}
\end{equation}
One can eliminate the parameters $v, w$ to get the implicit equation

$$t + xy - y|y| \cot \alpha \left( -\frac{t}{y} \right) = 0.$$ 

Letting $\beta := \cot(\alpha)$, we get that $\beta$ is a continuous non-decreasing function, and that the surface $\Sigma_\beta$ defined by the parametric equations (3.1) is given by the implicit equation

$$(3.2) \quad 0 = f_\beta(x, y, t) := t + xy - y|y| \beta \left( -\frac{t}{y} \right).$$

Observe that, because of the monotonicity condition on $\alpha$, the projection of relative interiors of the open horizontal halflines to the $xy$-plane together with the planar $x$-axis $L_x$ produce a partition of the plane. Since $\Sigma_\beta$ is the union of the horizontal lifting of these planar halflines and the $x$-axis to $\mathbb{H}^1$, it is the graph of a continuous function $u_\beta : \mathbb{R}^2 \to \mathbb{R}$.

For $(x, y) \in \mathbb{R}^2$, the only point in the intersection of $\Sigma_\beta$ with the vertical line passing through $(x, y)$ is precisely $(x, y, u_\beta(x, y))$. Obviously

$$(3.3) \quad f_\beta(x, y, u_\beta(x, y)) = 0.$$ 

For any $(x, y) \in \mathbb{R}^2$, denote by $\xi_\beta(x, y)$ the only value $v \in \mathbb{R}$ so that either $\Gamma(v) = (x, y, 0)$, or $(x, y, u_\beta(x, y))$ is contained in one of the two above described halflines leaving $\Gamma(v)$. Trivially $\xi_\beta(x, 0) = x$. Using (3.1) one checks that

$$(3.4) \quad \xi_\beta(x, y) = -\frac{u_\beta(x, y)}{y}, \quad y \neq 0.$$ 

Recalling that $\alpha = \cot^{-1}(\beta)$, we see that the mapping

$$(v, w) \mapsto \begin{cases} (v + w \cos \alpha(v), w \sin \alpha(v)), & w \geq 0, \\ (v + |w| \cos \alpha(v), -|w| \sin \alpha(v)), & w \leq 0, \end{cases}$$

is an homeomorphism of $\mathbb{R}^2$ whose inverse is given by

$$(x, y) \mapsto (\xi_\beta(x, y), \text{sgn}(y) |(x - \xi_\beta(x, y), y)|),$$

where $\text{sgn}(y) := y/|y|$ for $y \neq 0$. Hence $\xi_\beta : \mathbb{R}^2 \to \mathbb{R}$ is a continuous function. By (3.4), the function $u_\beta(x, y)/y$ admits a continuous extension to $\mathbb{R}^2$.

Let us analyze first the properties of $u_\beta$ for regular $\beta$.

**Lemma 3.1.** Let $\beta \in C^k(\mathbb{R})$, $k \geq 2$, be a non-decreasing function. Then

(i) $u_\beta$ is a $C^k$ function in $\mathbb{R}^2 - L_x$,

(ii) $u_\beta$ is merely $C^{1,1}$ near the $x$-axis when $\beta \neq 0$,

(iii) $u_\beta$ is $C^\infty$ in $\xi^{-1}(1)$ when $\beta \equiv 0$ on any open set $I \subset \mathbb{R}$, and

(iv) $\Sigma_\beta$ is area-minimizing.

(v) The projection of the singular set of $\Sigma_\beta$ to the $xy$-plane is $L_x$.

**Proof.** Along the proof we shall often drop the subscript $\beta$ for $f_\beta$, $u_\beta$, $\xi_\beta$ and $\Sigma_\beta$.

The proof of [1] is just an application of the Implicit Function Theorem since $f_\beta$ is a $C^k$ function for $y \neq 0$ when $\beta$ is $C^k$. 

\[ \text{Proof.} \]
To prove 2 we compute the partial derivatives of $u_\beta$ for $y \neq 0$. They are given by

$$
(u_\beta)_x(x, y) = \frac{-y}{1 + |y| \beta'(\xi_\beta(x, y))},
$$

$$
(u_\beta)_y(x, y) = \frac{-x + |y| \left(2\beta(\xi_\beta(x, y)) - \beta'(\xi_\beta(x, y)) \xi_\beta(x, y)\right)}{1 + |y| \beta'(\xi_\beta(x, y))}.
$$

Since $u_\beta(x, 0) = 0$ for all $x \in \mathbb{R}$ we get $(u_\beta)_x(x, 0) = 0$. On the other hand

$$(u_\beta)_y(x, 0) = \lim_{y \to 0} \frac{u_\beta(x, y)}{y} = - \lim_{y \to 0} \xi_\beta(x, y) = -\xi_\beta(x, 0) = -x.
$$

The limits, when $y \to 0$, of (3.5) and (3.6) can be computed using (3.4). We conclude that the first derivatives of $u_\beta$ are continuous functions and so $u_\beta$ is a $C^1$ function on $\mathbb{R}^2$. To see that $u_\beta$ is merely lipschitz, we get from (3.6) and (3.4)

$$(u_\beta)_{yy}(x, 0) = \lim_{y \to 0^\pm} \frac{(u_\beta)_y(x, y) + x}{y}
= \lim_{y \to 0^\pm} \frac{|y| \left(2\beta(\xi_\beta(x, y)) - \beta'(\xi_\beta(x, y)) \xi_\beta(x, y) + x\beta'(\xi_\beta(x, y))\right)}{y \left(1 + |y| \beta'(\xi_\beta(x, y))\right)}
= \pm 2\beta(x).
$$

Hence side derivatives exist, but they do not coincide unless $\beta(x) = 0$.

As $u_\beta|_{\xi^{-1}(1)} = -xy$, 3 follows easily.

To prove 4 we use a calibration argument. We shall drop the subscript $\beta$ to simplify the notation. Let $F \subset \mathbb{H}^1$ such that $F = E$ outside a Euclidean ball $B$ centered at the origin. Let $H^1 := \{(x, y, t) : y \geq 0\}$, $H^2 := \{(x, y, t) : y \leq 0\}$, $\Pi := \{(x, y, t) : y = 0\}$. Vertical translations of the horizontal unit normal $v_E$, defined outside $\Pi$, provide two vector fields $U^1$ on $H^1$, and $U^2$ on $H^2$. They are $C^2$ in the interior of the halfspaces and extend continuously to the boundary plane $\Pi$. As in the proof of Theorem 5.3 in [29], we see that

$$
\text{div } U^i = 0, \quad i = 1, 2,
$$
in the interior of the halfspaces. Here $\text{div } U$ is the riemannian divergence of the vector field $U$. Observe that the vector field $Y$ is the riemannian unit normal, and also the horizontal unit normal, to the plane $\Pi$. We may apply the divergence theorem to get

$$
0 = \int_{E \cap \text{int}(H^2) \cap B} \text{div } U^i = \int_E \langle U^i, v_{\text{int}(H^2) \cap B} \rangle \, d|\partial(\text{int}(H^2) \cap B)|
+ \int_{\text{int}(H^2) \cap B} \langle U^i, v_E \rangle \, d|\partial E|.
$$

Let $D := \Pi \cap \overline{B}$. Then, for every $p \in D$, we have $v_{\text{int}(H^1) \cap B} = -Y$, $v_{\text{int}(H^2) \cap B} = Y$, and $U^1 = f(v)$, $U^2 = f(w)$, where $v - w$ is proportional to $Y$, by the construction of $\Sigma_\beta$. Hence

$$
\langle U^1, v_{\text{int}(H^1) \cap B} \rangle + \langle U^2, v_{\text{int}(H^2) \cap B} \rangle = \langle v - w, f(Y) \rangle = 0, \quad p \in D.
$$
Adding the above integrals we obtain
\[
0 = \sum_{i=1,2} \int_{E} \langle U^{i}, v_{\beta} \rangle d|\partial B| + \sum_{i=1,2} \int_{B \cap \text{int}(H^{i})} \langle U^{i}, v_{E} \rangle d|\partial E|.
\]
We apply the same arguments to the set \(F\) and, since \(E = F\) on \(\partial B\) we conclude
\[
(3.7) \quad \sum_{i=1,2} \int_{B \cap \text{int}(H^{i})} \langle U^{i}, v_{E} \rangle d|\partial E| = \sum_{i=1,2} \int_{B \cap \text{int}(H^{i})} \langle U^{i}, v_{F} \rangle d|\partial F|.
\]
As \(E\) is a subgraph, \(|\partial E|(\Pi) = 0\) and so
\[
|\partial E|(B) = \sum_{i=1,2} \int_{B \cap \text{int}(H^{i})} \langle U^{i}, v_{\beta} \rangle d|\partial E|.
\]
Cauchy-Schwarz inequality and the fact that \(|\partial F|\) is a positive measure imply
\[
\sum_{i=1,2} \int_{B \cap \text{int}(H^{i})} \langle U^{i}, v_{F} \rangle d|\partial F| \leq |\partial F|(B),
\]
which implies \( \square \)

To prove \( \square \) simply take into account that the projection of the singular set of \( \Sigma_{\beta} \) to the \(xy\)-plane is composed of those points \((x,y)\) such that \((u_{\beta}) - x - y = (u_{\beta})y + x = 0\). From \( (3.5) \) we get that \((u_{\beta})_{y} = 0\) if and only if
\[
y (2 + |y| \beta'(\xi_{\beta}(x,y))) = 0,
\]
i.e., when \(y = 0\). In this case, from \( (3.6) \), we see that equation \((u_{\beta})y + x = 0\) is trivially satisfied. \( \square \)

We now prove the general properties of \( \Sigma_{\beta} \) from Lemma \(3.1\).

**Proposition 3.2.** Let \( \beta : \mathbb{R} \to \mathbb{R} \) be a continuous non-decreasing function. Let \( u_{\beta} \) be the only solution of equation \( (3.5) \), \( \Sigma_{\beta} \) the graph of \( u_{\beta} \), and \( E_{\beta} \) the subgraph of \( u_{\beta} \). Then

(i) \( u_{\beta} \) is locally lipschitz in Euclidean sense,

(ii) \( E_{\beta} \) is a set of locally finite perimeter in \( \mathbb{H}^{1} \), and

(iii) \( \Sigma_{\beta} \) is area-minimizing in \( \mathbb{H}^{1} \).

**Proof.** Let
\[
\beta_{\epsilon}(x) := \int_{\mathbb{R}} \beta(y) \eta_{\epsilon}(x - y) dy
\]
the usual convolution, where \( \eta \) is a Dirac function and \( \eta_{\epsilon}(x) := \eta(x/\epsilon) \), see [15]. Then \( \beta_{\epsilon} \) is a \( C^{\infty} \) non-decreasing function, and \( \beta_{\epsilon} \) converges uniformly, on compact subsets of \( \mathbb{R} \), to \( \beta \). Let \( u = u_{\beta} \), \( u_{\epsilon} = u_{\beta_{\epsilon}} \), \( f = f_{\beta} \), \( f_{\epsilon} = f_{\beta_{\epsilon}} \).

Let \( D \subset \mathbb{R}^{2} \) be a bounded subset. To check that \( u \) is lipschitz on \( D \) it is enough to prove that the first derivatives of \( u_{\epsilon} \) are uniformly bounded on \( D \).

From \( (3.3) \) we get
\[
\xi(x,y) + |y| \beta'\xi(x,y) = x, \quad y \neq 0.
\]
For \( y \) fixed, define the continuous strictly increasing function
\[
p_{y}(x) := x + |y| \beta(x).
\]
Hence we get
\begin{equation}
\xi(x, y) = \rho_y^{-1}(x).
\end{equation}

We can also define \((\rho_\varepsilon)_y(x) := x + |y|\beta_\varepsilon(x)\). Equation (3.8) holds replacing \(u, \beta\) by \(u_\varepsilon, \beta_\varepsilon\).

Since \(\rho_y^{-1}(x) = \xi(x, y)\), we conclude that \(\rho_y^{-1}\) is a continuous function that depends continuously on \(y\).

Let us estimate
\[
|\rho_y^{-1}(x) - \rho_y^{-1}(x)|.
\]
Let \(z_\varepsilon := (\rho_\varepsilon)_y^{-1}(x), z = \rho_y^{-1}(x)\). Then \(x = (\rho_\varepsilon)_y(z_\varepsilon) = \rho_y(z)\) and we have, assuming \(z_\varepsilon \geq z\),
\[
0 = (\rho_\varepsilon)_y(z_\varepsilon) - \rho_y(z) = z_\varepsilon + |y|\beta_\varepsilon(z_\varepsilon) - (z + |y|\beta(z))
\]
\[
= (z_\varepsilon - z) + |y| (\beta_\varepsilon(z_\varepsilon) - \beta_\varepsilon(z)) + |y| (\beta_\varepsilon(z) - \beta(z))
\]
\[
\geq (z_\varepsilon - z) + |y| (\beta_\varepsilon(z) - \beta(z)).
\]
A similar computation can be performed for \(z_\varepsilon \leq z\). The consequence is that
\[
|z_\varepsilon - z| \leq |y| |\beta_\varepsilon(z) - \beta(z)|,
\]
or, equivalently,
\[
|\rho_y^{-1}(x) - \rho_y^{-1}(x)| \leq |y| |\beta_\varepsilon(\rho_y^{-1}(x)) - \beta(\rho_y^{-1}(x))|.
\]

As \(\beta_\varepsilon \rightarrow \beta\) uniformly on compact subsets of \(\mathbb{R}\), we have uniform convergence of \((\rho_\varepsilon)_y^{-1}(x)\) to \(\rho_y^{-1}(x)\) on compact subsets of \(\mathbb{R}^2\). This also implies the uniform convergence of \(\xi_\varepsilon(x, y)\) to \(\xi(x, y)\) on compact subsets. Hence also \(u_\varepsilon(x, y)\) converges uniformly to \(u(x, y)\) on compact subsets of \(\mathbb{R}^2\).

From (3.5) and (3.6) we have
\[
|(u_\varepsilon)_x(x, y)| \leq |y|,
\]
\[
|(u_\varepsilon)_y(x, y)| \leq |x| + 2|y| |\beta_\varepsilon(\xi_\varepsilon(x, y))| + |\xi_\varepsilon(x, y)|.
\]

As \(\beta_\varepsilon \rightarrow \beta\) and \(\xi_\varepsilon(x, y) \rightarrow \xi(x, y)\) uniformly on compact subsets, we have that the first derivatives of \(u_\varepsilon\) are uniformly bounded on compact subsets. Hence \(u\) is locally lipschitz.

The subgraph of \(u_\beta\) is a set of locally finite perimeter in \(\mathbb{H}^1\) since its boundary is locally lipschitz by \(\ddagger\). This follows from \(\ddagger\) and proves \(\ddagger\).

To prove \(\ddagger\) we use approximation and the calibration argument. Let \(F \subset \mathbb{H}^1\) so that \(F = E\) outside a Euclidean ball \(B\) centered at the origin. For the functions \(\beta_\varepsilon\) consider the vector fields \(U^i_\varepsilon\) obtained by translating vertically the horizontal unit normal to the surface \(\Sigma_\varepsilon\). We repeat the arguments on the proof of \(\ddagger\) in Lemma 3.1 to conclude as in (3.7) that
\[
\sum_{i=1,2} \int_{B \cap \text{int}(\mathbb{H}^1)} \langle U^i_\varepsilon, v_E \rangle d|\partial E| = \sum_{i=1,2} \int_{B \cap \text{int}(\mathbb{H}^1)} \langle U^i_\varepsilon, v_F \rangle d|\partial F|.
\]
Trivially we have
\[
\sum_{i=1,2} \int_{B \cap \text{int}(\mathbb{H}^1)} \langle U^i_\varepsilon, v_F \rangle d|\partial F| \leq |\partial F|(B).
\]
On the other hand, $U_i^\epsilon$ converges uniformly, on compact subsets, to $U_i$ by Lemma 3.3. Passing to the limit when $\epsilon \to 0$ and taking into account that $U_i = \nu_e$ we conclude

$$|\partial E|(B) \leq |\partial F|(B),$$

as desired. □

**Lemma 3.3.** Let $\beta$ be a continuous non-decreasing function. Then the horizontal unit normal of $\Sigma_\beta$ is given, in $\{X, Y\}$-coordinates, by

$$\nu_\beta(x, y) = \left(\frac{1}{(1 + \beta^2)^{1/2}}, -\frac{\text{sgn}(y) \beta}{(1 + \beta^2)^{1/2}}(\xi_\beta(x, y)), \ y \neq 0.\right)$$

Moreover, $\nu_\beta$ admits continuous extensions to $y = 0$ from both sides of this line.

**Proof.** Since $u_\beta$ is lipschitz, it is differentiable almost everywhere on $\mathbb{R}^2$. On these points,

$$\nu_\beta(x, y) = ((u_\beta)_x - y, (u_\beta)_y).$$

The function $-u_\beta(x, y)/y$ is constant along the lines $(x_0, 0) + \lambda (1 + \beta^2)^{-1/2}(\beta, \pm 1)(0)$, for $\lambda \geq 0$. Let $y \geq 0$. From (3.2) we have

$$0 = -x_0 + x - y \beta(x_0).$$

Let $v := (1 + \beta^2)^{-1/2}(\beta, 1)(x_0)$. Then $v(-u_\beta(x, y)/y) = 0$. Hence for almost every point on almost every line, we have

$$\beta(x_0)(u_\beta)_x + (u_\beta)_y = -x_0.$$ 

Hence we have

$$(u_\beta)_y + x = -x_0 - \beta(x_0)(u_\beta)_x + x_0 + y \beta(x_0) = \beta(x_0)(-u_\beta)_x + y.$$ 

We conclude that the horizontal unit normal is proportional to $(1, -\beta)$, which implies (3.9). The case $y \leq 0$ is handled similarly. □

**Example 3.4.** Taking $\beta(x) := x$ we get

$$u_\beta(x, y) = -\frac{xy}{1 + |y|},$$

which is a Euclidean $C^{1,1}$ graph.

Another family of interesting examples are the minimal cones obtained by taking the constant function $\beta(x) := \beta_0$. In this case we get

$$u_\beta(x, y) = -xy + \beta_0 y|y|.$$ 

In this case $\Sigma_\beta$ is a $C^{1,1}$ surface which is invariant by the dilations centered at any point of the singular line.

Take now

$$\beta(x) := \begin{cases} 
0, & x \leq 0, \\
x, & x \geq 0.
\end{cases}$$

In this case we obtain the graph

$$u_\beta(x, y) := \begin{cases} 
-xy, & x \leq 0, \\
\frac{-xy}{1 + |y|}, & x \geq 0.
\end{cases}$$
which is simply locally Lipschitz.

This example was mentioned to me by Scott Pauls. Consider now a continuous non-decreasing function \( \beta : \mathbb{R} \to \mathbb{R} \), constant outside the Cantor set \( C \subset [0,1] \) with \( \beta(0) = 0 \), \( \beta(1) = 1 \). Then the associated surface \( \Sigma_\beta \) is an area-minimizing surface in \( H^1 \).

4. EXAMPLES WITH SEVERAL SINGULAR HALFLINES MEETING AT A POINT

Let \( \alpha_1^0, \ldots, \alpha_k^0 \), be a family of positive angles so that
\[
\sum_{i=1}^k \alpha_i^0 = \pi.
\]

Let \( r_\beta \) be the rotation of angle \( \beta \) around the origin in \( \mathbb{R}^2 \). Consider a family of closed halflines \( L_i \subset \mathbb{R}^2, i \in \mathbb{Z}_k \), extending from the origin, so that \( r_{\alpha_i^0+\alpha_{i+1}^0}(L_i) = L_{i+1} \). Finally, define \( R_i := r_{\alpha_i^0}(L_i) \). (An alternative way of defining this configuration is to start from a family of counter-clockwise oriented halflines \( R_i \subset \mathbb{R}^2, i \in \mathbb{Z}_k \), choosing \( L_i, i \in \mathbb{Z}_k \), as the bisector of the angle determined by \( R_{i-1} \) and \( R_i \), and defining \( \alpha_i^0 \) as the angle between \( L_i \) and \( R_i \)). Define \( W_i \) as the closed wedge, containing \( L_i \), bordered by \( R_{i-1} \) and \( R_i \).

\[\begin{align*}
W_1 & \quad W_2 \\
W_2 & \quad W_3 \\
W_3 & \quad W_4
\end{align*}\]

**Figure 1.** The initial configuration with three halflines \( L_1, L_2, L_3 \).

For every \( i \in \mathbb{Z}_k \), let \( \alpha_i : [0, \infty) \to (0, \pi) \) be a continuous nonincreasing function so that \( \alpha_i(0) = \alpha_i^0 \), and define, as in the previous section, \( \beta_i := \text{cot}(\alpha_i) \). Let \( v_i \in S^1, i \in \mathbb{Z}_k \), be such that \( L_i = \{sv_i : s \geq 0\} \). For every \( i \in \mathbb{Z}_k \) and \( s \geq 0 \), we take the two closed halflines \( L_{s,j}^\pm \) in \( \mathbb{R}^2 \) extending from the point \( sv_i \) with tangent vectors \( (\cos \alpha_i(s), \pm \sin \alpha_i(s)) \). In this way we cover all of \( \mathbb{R}^2 \). We shall define \( \alpha := (\alpha_1, \ldots, \alpha_k) \).

Lift \( L_1, \ldots, L_k \) to horizontal halflines \( L_1', \ldots, L_k' \) in \( H^1 \) from the origin, and \( L_{s,j}^\pm \) to horizontal halflines in \( H^1 \) extending from the unique point in \( L_i' \) projecting onto \( sv_i \). In this way we obtain a continuous function \( u_\alpha : \mathbb{R}^2 \to \mathbb{R} \). The graph \( \Sigma_\alpha \) of \( u_\alpha \) is a topological surface in \( H^1 \).
Obviously the angle functions $\alpha_i(s)$ can be extended continuously and preserving the monotonicity, to an angle function $\tilde{\alpha}_i : \mathbb{L}_i \rightarrow (0, \pi)$, where $\mathbb{L}_i$ is the straight line containing the halfline $L_i$. The graph of $u$ restricted to $W_i$ coincides with the Euclidean locally lipschitz area-minimizing surface $u_{\tilde{\beta}_i}$, for $\tilde{\beta}_i := \cot \tilde{\alpha}_i$, constructed in the previous section. So the examples in this section can be seen as pieces of the examples of the previous one patched together.

**Theorem 4.1.** Under the above conditions

(i) The function $u$ is locally lipschitz in the Euclidean sense.

(ii) The surface $\Sigma_u$ is area-minimizing.

**Proof.** It is immediate that $u$ is a graph which is locally lipschitz in Euclidean sense: choose a disk $D \subset \mathbb{R}^2$. Let $p$, $q \in D$. Assume first that $(p,q)$ intersects the halflines $R_1, \ldots, R_k$ transversally at the points $x_1, \ldots, x_n$. Then $[p,x_1]$, $[x_1,x_2], \ldots, [x_n,p]$ are contained in wedges and hence

$$|u(p) - u(q)| \leq |u(p) - u(x_1)| + \cdots + |u(x_n) - u(q)|$$

$$C(|p - x_1| + \cdots + |x_n - q|) = C|p - q|,$$

where $C$ is the supremum of the Lipschitz constants of $u_{\tilde{\beta}_i}$ restricted to $D$. The general case is then obtained by approximating $p$ and $q$ by points in the condition of the assumption.

To prove that $u$ is area minimizing we first approximate $\alpha_i$ by smooth angle functions $(\alpha_i)_s$ with $(\alpha_i)_s(0) = \alpha_i(0)$. In this way we obtain a calibrating vector field which is continuous along the vertical planes passing through $R_i$ by Lemma 3.3. This allows us to apply the calibration argument to prove the area-minimizing property of $\Sigma_u$. \hfill \Box

**Example 4.2** (Minimizing cones). Let $\alpha_i(s) = \alpha_i^0$ be a constant for all $i$. Then the subgraph of $\Sigma_u$ is a minimizing cone with center at 0. Restricted to the interior of the wedges $W_i$, the surface $\Sigma_u$ is $C^{1,1}$. An easy computation shows that, taking $\beta(s) := \beta_0$ in the construction of the first section, the Riemannian normal to $\Sigma$ along the halflines $\beta_0|y| = x$, $x \geq 0$ (that make angle $\pm \cot^{-1}(\beta_0)$ with the positive $x$-axis) is given by

$$N = \frac{-2yX + 2\beta_0|y|Y - T}{\sqrt{1 + 4y^2 + 4\beta_0^2y^2}} = \frac{-2yX + 2xY - T}{\sqrt{1 + 4x^2 + 4y^2}}.$$ 

This vector field is invariant by rotations around the vertical axis. Hence in our construction, the normal vector field to $\Sigma_u$ is continuous. It is straightforward to show that it is locally lipschitz in Euclidean sense.

**Example 4.3** (Area-minimizing surfaces with a singular halfline). These examples are inspired by Example 7.2. We consider a halfline $L$ extending from the origin, and an angle function $\alpha : L \rightarrow (0, \pi)$ continuous and nonincreasing as a function of the distance to the origin. We consider the union of the halflines $L^+_{\alpha(q)}$, $L^-_{\alpha(q)}$, extending from $q \in L$ with angles $\alpha(q)$, $-\alpha(q)$, respectively. We patch the area-minimizing surface defined by $\alpha$ in the wedge delimited by the halflines $L^+_q$, $L^-_q$, with the plane $t = 0$. In this way we get an entire area-minimizing $t$-graph, with lipschitz regularity. In case the angle function $\alpha$ is
constant, we get an area-minimizing cone with center 0, which is defined by the equation

\[ u(x, y) := \begin{cases} 
-xy + \beta_0 y|y|, & -xy + \beta_0 y|y| \geq 0, \\
0, & -xy + \beta_0 y|y| \leq 0.
\end{cases} \]

This surface is composed of two smooth pieces patched together along the halflines \( x = \beta_0 |y| \).

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