THE ASYMPTOTIC BEHAVIOR OF SOLID CLOSURE IN MIXED CHARACTERISTIC

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ABSTRACT. We study how solid closure in mixed characteristic behaves after taking ultraproducts. The ultraproduct will be chosen so that we land in equal characteristic, and therefore can make a comparison with tight closure. As a corollary we get an asymptotic version of the Hochster-Roberts invariant theorem in dimension three: if \( R \) is a mixed characteristic (cyclically) pure 3-dimensional local subring of a regular local ring \( S \), then \( R \) is Cohen-Macaulay, provided the ramification of \( S \) is large with respect to its dimension and residual characteristic, and with respect to the multiplicity of \( R \).

1. ULTRA- VERSUS CATA-

Solid closure was introduced by Hochster in [5, 6] as a potential substitute for tight closure in mixed characteristic. In this note, we comment on some of its properties, but as the title indicates, only ‘asymptotically’, that is to say, after taking an ultraproduct (see §3.7 for an elaboration on the terminology). More precisely, let \( A_w \) be a sequence of (commutative) rings (with identity), indexed by an infinite index set endowed with a non-principal ultrafilter, which, for technical reasons, we also assume to be countably incomplete.\(^1\) The ultraproduct of the \( A_w \) is again a ring \( A \), realized as the quotient of the product of the \( A_w \) modulo the ideal of all sequences almost all of whose entries are zero (with almost all one means for all indices in some member of the ultrafilter). We sometimes refer to the \( A_w \) as components of \( A \), although they are not uniquely defined by \( A \). If \( P \) is a property of rings, then we say that \( A \) has property ultra-\( P \) if almost all \( A_w \) have property \( P \). If a property \( P \) is first-order, then ultra-\( P \) is the same as \( P \) by Łos’ Theorem. For instance, being local is a first-order property,\(^2\) so that ultra-local is the same as local. However, most properties are not first-order (mostly because they require quantification over ideals or involve infinitely many statements). For instance, an ultra-Noetherian local ring is an ultraproduct of Noetherian local rings, and in general is no longer Noetherian (in fact, its prime spectrum is infinite and can be quite complicated; for some instances of this, see [13, 14, 15]). We will only be concerned with a certain subclass of ultra-Noetherian local rings, those of finite embedding dimension. An ultra-Noetherian local ring has embedding dimension \( n \) if, and only if, almost all of its components have embedding dimension \( n \) (because having embedding dimension \( n \) is a first-order property). Ultra-Noetherian local rings of finite embedding dimension already appeared as an essential tool in the earlier

\(^1\) Suffice it here to say that this set-theoretic notion can be realized on any infinite index set and holds automatically when the index set is countable. Moreover, it is consistent with ZFC (= usual set theory) that every ultrafilter is countably incomplete.

\(^2\) We will call a ring \( R \) local if it has a unique maximal ideal \( m \), and we denote this by \( (R, m) \) (in the literature one sometimes uses the term quasi-local in the non-Noetherian case). Note that \( R \) is local if, and only if, the sum of any two non-units is a non-unit, indeed a first-order property.
papers on non-standard tight closure ([2, 21, 19, 20]), and were used in [17, 18] to get some asymptotic versions of the homological conjectures in mixed characteristic. In [23] a systematic study of this class will be carried out, leading to some improved asymptotic versions of the intersection theorems in mixed characteristic. The present note does not require the full development of this theory, and we will review whatever we need.

In the latter papers, an important technique to study local rings of finite embedding dimension is through their completion, since this is always Noetherian. This leads to a second variant of a property $P$: we call a local ring $R$ of finite embedding dimension a cata-$P$ if its completion has property $P$.\(^3\) In case of an ultra-Noetherian local ring $(R, m)$ of finite embedding dimension, because of saturation properties of ultraproducts, its completion equals its separated quotient $R_{\text{sep}}$ (see for instance [23, Lemma 5.1]) defined as the homomorphic image of $R$ modulo its ideal of infinitesimals $\text{Inf}(R) := \cap_n m^n$; we call $R_{\text{sep}}$ the separated ultraproduct of the components $R_w$. (The term ‘cata’ was chosen because of this fact.) The following example of the close connection between the ultra-variant of a property and its cata-variant was already observed in [26, Corollary 1.14].

1.1. Theorem. An ultra-regular local ring of finite embedding dimension is cata-regular.

Proof. Suppose $(R, m)$ is the ultraproduct of regular local rings $(R_w, m_w)$ of dimension $d$ and let $x = (x_{1w}, \ldots, x_{dw})$ be a regular system of parameters in $R_w$. The ultraproduct $x$ of the $x_{iw}$ (that is to say, the image of the sequence $(x_{iw}, w)$ in $R$) gives rise to a $d$-tuple $x := (x_1, \ldots, x_d)$ in $R$ generating $m$, whence generating $mR_{\text{sep}}$. So remains to show that $R_{\text{sep}}$ has dimension $d$. By Krull’s principal ideal theory, its dimension is at most $d$, so suppose towards a contradiction that it were less. Hence after renumbering, $x_d^n \in (x_1, \ldots, x_{d-1})R_{\text{sep}}$, for some $n$. Contracting back to $R$, we get that $x_d^n$ lies in $(x_1, \ldots, x_{d-1})R + \text{Inf}(R) \subseteq (x_1, \ldots, x_{d-1}, x_d^{n+1})R$. Writing out the latter relation, we get that $x_d^n \in (x_1, \ldots, x_{d-1})R$. By Łos’ Theorem, $x_d^n \in (x_{1w}, \ldots, x_{d-1, w})R_w$ for almost all $w$, contradiction. \(\square\)

From the proof it also follows that $R$ has the same ultra-dimension (=dimension of almost all of its components) as cata-dimension (=dimension of its completion). With this additional assumption, the converse of Theorem 1.1 also holds. This is explained in [23, Theorem 8.1], where it is also shown that an ultra-Noetherian local ring has the same ultra-dimension as cata-dimension if, and only if, the parameter degree of its components is bounded. Recall that the parameter degree of a Noetherian local ring $A$ is the least co-length of a parameter ideal of $A$ (the co-length of an ideal $a$ is the length of $A/a$; a parameter ideal is an ideal generated by a system of parameters). Multiplicity is a lower bound for parameter degree (with equality if, and only if, the ring is Cohen-Macaulay, provided the residue field is infinite; see [22, Corollary 3.3]).

2. Closure operations

In this section $R$ is an ultra-Noetherian local ring of finite embedding dimension, say realized as the ultraproduct of Noetherian local rings $R_w$ of bounded embedding dimension. We want to introduce two closure operations on $R$: ultra-solid closure and cata-tight closure.

\(^3\)Because of its versatility, I opted to introduce the prefix cata- instead of using the more traditional adverbially constructions analytically of formally; similarly, the prefix ultra- replaces terms such as non-standard or generic from the older papers.
2.1. **Ultra-solid closure.** For the definition of solid closure, see [5]. To maintain a consistent notational scheme, we will write $sc(a)$ for the solid closure of an ideal $a$, rather than Hochster’s $a^*$. We say that $z \in R$ lies in the **ultra-solid closure** $usc(I)$ of an ideal $I \subseteq R$, if we can find $z_w \in R_w$ and $a_w \subseteq R_w$ with respective ultraproducts $z$ and $a$, such that $a \subseteq I$ and $z_w \in sc(a_w)$, for almost all $w$. In case $I$ is an **ultra-ideal** (sometimes called an *induced* ideal), that is to say, is itself an ultraproduct of ideals $I_w$, then $z$ lies in its ultra-solid closure if, and only if, almost all $z_w$ lie in the solid closure of $I_w$. In other words, the ultra-solid closure of an ultra-ideal is the ultraproduct of the solid closures of its components (and so again an ultra-ideal). Note that a finitely generated ideal, whence in particular an $m$-primary ideal, is an ultra-ideal.

2.2. **Cata-tight closure.** To define cata-tight closure, which will be derived from tight closure, we need to make an assumption on the characteristic, namely that $R$ is *cata-equicharacteristic*, meaning that its completion (or equivalently, its separated quotient) has equal characteristic. Although in positive characteristic the more common notation for the tight closure of an ideal is $a^*$, we will use instead, in either characteristic, the notation $tc(a)$. For the next definition, we assume that $R$ is cata-equicharacteristic, but there is no not need for assuming it is ultra-Noetherian–having finite embedding dimension suffices.

We say that $z$ lies in the **cata-tight closure** $ctc(I)$ of $I$, if the image of $z$ in $\hat{R}$ lies in the tight closure of $I\hat{R}$. In other words, $ctc(I) = tc(\hat{I}\hat{R}) \cap R$. We have a choice in picking a tight closure operation in equal characteristic zero: there are the ‘classical’ ones introduced by Hochster and Huneke in [9], and there are the ‘non-standard’ variants defined in [2, 19]. As a rule, we will use generic tight closure as defined in the latter papers (see also the next paragraph). It has all the properties we want it to have: it is trivial on regular local rings, it ‘captures colons’ and it is ‘persistent’ (for details see [2, §6.19]).

2.3. **More closure operations.** Of course nothing prevents us from switching around these definitions and introduce also ‘cata-solid closure’ and ‘ultra-tight closure’. We only would be able to say something sensible about the former when it actually coincides with cata-tight closure (namely, when $\hat{R}$ has positive characteristic); as for the latter, it is very akin to generic tight closure in the main case where the components have positive characteristic but $\hat{R}$ has zero characteristic, and so will add nothing new. There is one more closure operation in equal characteristic which is even smaller than tight closure (but conjecturally coincides with it), to wit, **plus closure**. In characteristic $p$, it is defined for a Noetherian local domain $A$ as contraction from the *absolute integral closure* $A^+ = \text{integral closure of } A$ in an algebraic closure of its field of fractions; recall that if $A$ is moreover excellent, then $A^+$ is a big Cohen-Macaulay $A$-algebra by [8]). In equal characteristic zero, it is defined by contraction from a canonically defined big Cohen-Macaulay $A$-algebra $\mathcal{B}(A)$ (see [20] for the affine case and [2] for the general). On occasion, we will thus encounter cata-plus closure as the pre-image of plus closure in $\hat{R}$.

We say that an ideal $I \subseteq \hat{R}$ is **ultra-solidly closed** (respectively, **cata-tightly closed**) if $I$ is equal to its ultra-solid closure (respectively, cata-tight closure). We cannot expect cata-tight closure to be a true ‘tight’ closure operation, since it always contains the $m$-adic closure of the ideal (see Lemma 3.1 for a description of the closure of an ideal). However, $m$-primary ideals are $m$-adically closed and hence their cata-tight closure will be the most accessible.

2.4. **From mixed to equal characteristic.** We now turn to the issue of enforcing equal characteristic for the separated ultraproduct, starting from mixed characteristic. One way is by letting the components $R_w$ have unbounded residual characteristic, that is to say, for
each \( n \), the characteristic of the residue field of \( R_w \) is \( \geq n \) for almost all \( w \). This has a consequence that the ultraproduct \( R \) has residual characteristic zero, whence so does \( R_{\text{sep}} = \hat{R} \). However, there is a second way to get an equal characteristic separated ultraproduct:

Let \((A, p)\) be a local ring of residual characteristic \( p \). We define the ramification index of \( A \), as the supremum of all \( n \) for which \( p \in p^n \). We call \( A \) unramified, if its ramification index is one, and infinitely ramified, if it is infinite, that is to say, if \( p \in \text{Inf}(A) \).

(Note: if the residual characteristic is zero, then we will call \( A \) Noetherian, or more generally, separated, and infinitely ramified, then in fact it has equal characteristic zero (in the literature this situation is erroneously referred to as ‘unramified’). However, in general, a local ring can have characteristic zero and be infinitely ramified: if \( R_w \) are mixed characteristic Noetherian local rings of residual characteristic \( p \) and unbounded ramification index (in the sense that for each \( n \), almost all \( R_w \) have ramification index \( \geq n \)), then their ultraproduct \( R \) has characteristic zero and is infinitely ramified. In particular, the separated ultraproduct \( R_{\text{sep}} \) of the \( R_w \) has equal characteristic \( p \).

In summary, \( R \) is cata-equicharacteristic, if either \( R \) itself has equal characteristic, or otherwise, is infinitely ramified. Unfortunately, solid closure does not behave that well in equal characteristic zero (see Example 2.6), so that we can only compare our two new closure operators when the completion has prime characteristic.

2.5. **Proposition.** If \( R \) is an ultra-Noetherian local domain of finite embedding dimension, which has prime characteristic or is infinitely ramified, then \( \text{usc}(I) \subseteq \text{ctc}(I) \), for all ideals \( I \subseteq R \).

**Proof.** Let \( z \in \text{usc}(I) \) and let \( z_w \in R_w \) and \( a_w \subseteq R_w \) have respective ultraproducts \( z \) and \( a \subseteq I \) with almost all \( z_w \) in the solid closure of \( a_w \). By [5, Proposition 5.3], there exists for almost all \( w \) a solid \( R_w \)-algebra \( S_w \) such that \( z_w \in a_w S_w \). Recall that \( S_w \) is solid when there exists a non-zero \( R_w \)-linear map \( \phi_w \colon S_w \to R_w \). Moreover, we may choose \( \phi_w \) so that \( \phi_w(1) \neq 0 \). Let \( S \) be the ultraproduct of the \( S_w \). The ultraproduct \( \phi \) of the \( \phi_w \) is an \( R \)-linear map \( S \to R \), showing that \( S \) is solid as an \( R \)-algebra. Let \( \hat{S} := S / \text{Inf}(R)S \). Hence \( \phi \) induces by base change an \( R_{\text{sep}} \)-linear map \( \hat{S} \to R_{\text{sep}} \) showing that \( \hat{S} \) is a solid \( R_{\text{sep}} \)-algebra. By assumption, \( R_{\text{sep}} \) has characteristic \( p > 0 \). By Łos’ Theorem, \( z \in a\hat{S} \). Applying Frobenius to this equation and then applying \( \phi \), we get \( cz^q \in a[p] R_{\text{sep}} \), for all powers \( q \) of \( p \), with \( c := \phi(1) \neq 0 \). Hence \( z \in \text{tc}(a R_{\text{sep}}) \) and therefore \( z \in ctc(a) \subseteq ctc(I) \) (recall that \( R_{\text{sep}} = \hat{R} \)).

2.6. **Example.** The above inclusion does not hold in general in equal characteristic zero: in [16] Roberts shows that \( f := X^2 Y^2 Z^2 \) lies in the solid closure of \( I := (X^3, Y^3, Z^3) A \) where \( A := K[[X, Y, Z]] \) with \( K \) a field of characteristic zero. Let \( \hat{R} \) be the ultrapower of \( A \) (an ultrapower is an ultraproduct in which all components are the same). It follows that \( f \in \text{usc}(IR) \). On the other hand, \( R_{\text{sep}} \cong K^+[[X, Y, Z]] \) where \( K^+ \) is the ultrapower of \( K \), so that \( IR_{\text{sep}} \) is tightly closed and hence \( IR \) is cata-tightly closed. I do not know of any example of an equal characteristic zero ultra-Noetherian local domain whose components have mixed characteristic, but for which the above inclusion does not hold. There is also the hope that the above result holds without any restriction on the characteristic when we replace solid closure by parasolid closure (promising in that respect is [3, Theorem 4.1] showing that every ideal in a regular local ring is parasolidly closed).

It is an interesting question whether we can have equality in Proposition 2.5. However, since an ultra-ideal cannot be \( m \)-adically closed unless it is \( m \)-primary, whereas cata-tight
closures are always $m$-adically closed, we can only expect equality for $m$-primary ideals. Using a result of Smith, we can prove this in a special case. Let $(R, m)$ be a local ring of finite embedding dimension. A tuple $x$ in $R$ is called a system of cata-parameters (or a generic tuple), if its image in $R_{\text{sep}}$ is a system of parameters. Any ideal generated by a system of cata-parameters will be called a cata-parameter ideal. In particular, a cata-parameter ideal $I$ is $m$-primary and $IR_{\text{sep}}$ is a parameter ideal.

2.7. Theorem. Let $R$ be an ultra-Noetherian local ring of finite embedding dimension, whose separated quotient is an equidimensional, prime characteristic reduced local ring. If $I$ is a cata-parameter ideal in $R$, then $\text{usc}(I) = \text{ctc}(I)$.

Proof. One inclusion is clear from Proposition 2.5. Hence assume that $z$ lies in $\text{ctc}(I)$, whence its image in $\widehat{R} := R_{\text{sep}}$ lies in $\text{tc}(I\widehat{R})$. Let $p$ be a minimal prime of $\widehat{R}$. Since $\widehat{R}$ is equidimensional, $I(\widehat{R}/p)$ is a parameter ideal. By persistence, $z$ lies in $\text{tc}(I(\widehat{R}/p))$, whence in $I(\widehat{R}/p)^+$ by [24], where $(\widehat{R}/p)^+$ is the absolute integral closure of the complete local domain $\widehat{R}/p$ (see §2.3). It follows that there exists a finite extension $\widehat{R}/p \subseteq S(p)$ of local domains such that $z \in IS(p)$. Let $\tilde{S}$ be the direct sum of all $S(p)$, where $p$ runs over all minimal primes of $\widehat{R}$. Since $\widehat{R}$ is reduced, the natural map $\widehat{R} \to \tilde{S}$ is finite and injective. Hence we can lift this to a finite extension $R \subseteq S$, such that $\tilde{S} \cong S/\text{Inf}(\widehat{R})S$. From $z \in I\tilde{S}$ and the fact that $\text{Inf}(\widehat{R}) \subseteq I$, we get $z \in IS$.

Choose finite local extensions $R_w \subseteq S_w$ whose ultraproduct is $R \subseteq S$. Let $z_w$ and $I_w$ be such that their ultraproducts are $z$ and $I$ respectively. By Łos’ Theorem, almost each $z_w$ lies in $I_wS_w$, whence in the solid closure of $I_w$, since finite extensions are formally solid by [5, Remark 1.3]. In conclusion, we showed that $z \in \text{usc}(I)$. \hfill $\Box$

The argument in the proof actually shows that the cata-plus closure (see §2.3) is always contained in the ultra-solid closure for $m$-primary ideals, for $R$ as in the statement. Therefore, if plus-closure equals tight closure in positive characteristic, ultra-solid closure, cata-tight closure and cata-plus closure are all the same on $m$-primary ideals. If $R_{\text{sep}}$ has equal characteristic zero, the above argument does not work since $\mathcal{B}(R_{\text{sep}})$ is no longer integral over $R_{\text{sep}}$ and hence it is not clear how to ‘descend’ to the components.

3. Properties of cata-tight closure

In this section, $(R, m)$ denotes a cata-equicharacteristic local ring of finite embedding dimension. Our goal is to derive some elementary properties of cata-tight closure on $R$. We already observed that the $m$-adic closure is contained in the cata-tight closure.

3.1. Lemma. The $m$-adic closure of an ideal $I \subseteq R$ is equal to $I + \text{Inf}(R)$.

Proof. Since $R_{\text{sep}}$ is Noetherian, the $m$-adic closure of $IR_{\text{sep}}$ is equal to $I$ by Krull’s Intersection theorem, and the assertion follows. (No assumption on the characteristic is needed for this lemma). \hfill $\Box$

Since in a regular local ring, every ideal is tightly closed, we get immediately:

3.2. Proposition. If $R$ is cata-regular and $I$ an ideal in $R$, then the cata-tight closure of $I$ is equal to its $m$-adic closure, that is to say, $\text{ctc}(I) = I + \text{Inf}(R)$. In particular, any $m$-primary ideal is cata-closed.

Colon Capturing and persistence of (generic) tight closure in equal characteristic leads immediately to the analogous results for cata-tight closure:
3.3. Proposition. If \((x_1, \ldots, x_d)\) is a system of cata-parameters in \(R\), then for each \(i \leq d\), we have an inclusion \( \text{ctc}((x_1, \ldots, x_{i-1})R : x_i) \subseteq \text{ctc}((x_1, \ldots, x_{i-1})R) \) (‘Colon Capturing’).

If \(R \to S\) is a local homomorphism and \(I \subseteq R\) an ideal, then \(\text{ctc}(I)S \subseteq \text{ctc}(IS)\) (‘Persistence’).

3.4. Remark. There are actually many stronger versions of Colon Capturing, of which I only will mention one: given integers \(0 \leq a_i < b_i\) for \(i = 1, \ldots, d\), and a system of cata-parameters \((x_1, \ldots, x_d)\), we have an inclusion

\[
(\text{ctc}((x_1^{b_1}, \ldots, x_d^{b_d})R) : x_1^{a_1} \cdots x_d^{a_d}) \subseteq \text{ctc}((x_1^{b_1-a_1}, \ldots, x_d^{b_d-a_d})R).
\]

In positive characteristic, this inclusion follows from the tight closure version of (1) proven in [11, Theorem 9.2]. The latter then also gives the corresponding result in zero characteristic by the techniques of [2].

In the next result, we have written \(\bar{I}\) to denote the integral closure of an ideal \(I\).

3.5. Theorem (Briançon-Skoda). In a cata-equicharacteristic local ring \((R, \mathfrak{m})\) of finite embedding dimension, we have for each ideal \(I \subseteq R\) an inclusion \(\overline{I^d} \subseteq \text{ctc}(I)\), where \(d\) is the cata-dimension of \(R\).

In particular, if \(R\) is cata-regular and \(I\) is \(\mathfrak{m}\)-primary, then \(\overline{I^d} \subseteq I\).

Proof. Let \(\bar{a} := \bar{I R}\). Clearly, \(\overline{I R} \subseteq \bar{a}\). By the tight closure Briançon-Skoda theorem, \(a^d \subset \text{tc}(a)\), so that the first assertion is clear. The second assertion then follows from Proposition 3.2. \(\square\)

3.6. Remark. In fact, the usual Briançon-Skoda theorem gives the following stronger result: if \(h\) denotes the minimal number of generators of \(IR_{\text{sep}}\), then we have for all \(k\) an inclusion \(\overline{I^{k+1}} \subseteq \text{ctc}(I^{k+1})\). Using an improvement by Aberbach and Huneke in equal characteristic in [1], we actually get the following. Assume \(R\) is cata-regular, with infinite residue field, \(I\) is \(\mathfrak{m}\)-primary and \(J \subseteq I\) is a minimal reduction of \(I\), then

\[
\overline{I^{k+1}} \subseteq J^{k+1}a,
\]

for all \(k\), where \(a\) is maximal among all ideals for which \(aJ = aI\) (note that \(a \subseteq (J : I)\) and hence is a proper ideal unless \(I\) is its own minimal reduction).

3.7. Asymptotic properties. The next type of result explains better the term ‘asymptotic’ from the title: a property (often of homological nature) holds ‘asymptotically’ when the characteristic (or the ramification) is large with respect to the other data (in a sense that has to be made more precise). In [17, 18], the lower bound for the characteristic depended on the degrees of the polynomials defining the data. In [23] an improved bound for the intersection theorems will be given only depending on some (more natural) invariants of the data (like dimension and parameter degree). Proofs of these types of results all follow the same outline: if there are counterexamples, their ultraproduct violates the corresponding ultra-version, which holds because its cata-version holds since we are now in equal characteristic. The first assertion in the next result is just included as an example, for it follows already from the general Briançon-Skoda theorem of Lipman and Sathaye in [12], which holds for all regular local rings, regardless of their characteristic.

3.8. Theorem (Asymptotic Briançon-Skoda in mixed characteristic). For each pair \((d, l)\), there exists a bound \(B := B(d, l)\) with the property that if \(S\) is a \(d\)-dimensional mixed
characteristic regular local ring and if \( I \subseteq S \) is an ideal of co-length at most \( t \), then \( \overline{I} \subseteq \overline{I} \), provided the residual characteristic of \( S \), or its ramification index, is at least \( B \).

In fact, under these assumptions, we have an inclusion \( \overline{I} \subseteq aI \), provided \( I \) has a reduction \( J \) of co-length at most \( l \) and \( a \) satisfies \( aJ = aI \).

**Proof.** I will only give the details for the first assertion, following the outline just mentioned. As for the second assertion, it follows along the same lines, using (2) from Remark 3.6 instead. Suppose the first assertion is false for some pair \((d, l)\). This means that for each \( w \in \mathbb{N} \), we can find a mixed characteristic \( d \)-dimensional regular local ring \( R_w \) whose residual characteristic or ramification index \( \geq w \) and an ideal \( I_w \subseteq R_w \) of co-length at most \( l \), such that \( \overline{I} \) is not contained in \( I_w \). Let \( I \) and \( R \) be the respective ultraproducts of \( I_w \) and \( R_w \). It follows that \( \overline{I} \) is cata-equicharacteristic and ultra-regular, whence cata-regular, by Theorem 1.1. By Łos’ Theorem, \( I \) has co-length at most \( l \) and does not contain \( \overline{I} \), contradicting Theorem 3.5 (the ‘cata-Briançon-Skoda theorem’). \( \square \)

3.9. **Remark.** By the same argument and under the same assumptions, there is also a bound \( B' := B'(d, l, m) \) such that (2) holds for all \( k \leq m \) whenever the residual characteristic or the ramification index \( \geq B' \).

4. **Properties of Ultra-solid Closure**

Although we know very little about solid closure in mixed characteristic, it is clear that the inclusion from Proposition 2.5 together with the results from the previous section should tell us a whole lot more about its ultra-variant. Consequently, we may hope to infer some ‘asymptotic’ properties of solid closure itself, at least for \( m \)-primary ideals of bounded co-length.

Combining Propositions 2.5 and 3.2 with Theorem 1.1 yields immediately:

4.1. **Corollary.** If \((R, m)\) is an ultra-regular local ring of finite embedding dimension which has prime characteristic or is infinitely ramified, then every \( m \)-primary ideal is ultra-solidly closed.

4.2. **Corollary.** Let be \((R, m) \to (S, n)\) be a local homomorphism of ultra-Noetherian local rings of finite embedding dimension. If \( S \) is ultra-regular and has prime characteristic or is infinitely ramified, then \( \text{usc}(a) \subseteq aS \cap R \), for every \( m \)-primary ideal \( a \subseteq R \).

**Proof.** Let \( z \in \text{usc}(a) \). Since solid closure is persistent, so is ultra-solid closure, and hence \( z \) lies in \( \text{usc}(aS) \), whence in \( aS \) by Corollary 4.1. \( \square \)

Following Hochster, we call a Noetherian local ring weakly \( S \)-regular (respectively, \( S \)-rational) if every ideal (respectively, every parameter ideal) is solidly closed. For tight closure, if a single parameter ideal is tightly closed, then so is any other parameter ideal, but not so for solid closure. We therefore will call \( R \) weakly \( S \)-rational if it admits at least one solidly closed parameter ideal. The next result shows that this notion serves some purpose.

4.3. **Proposition.** If an analytically irreducible Noetherian local ring is weakly \( S \)-rational and admits a big Cohen-Macaulay algebra, then it is Cohen-Macaulay.

**Proof.** Let \( R \) be a weakly \( S \)-rational Noetherian local ring and let \( B \) be a big Cohen-Macaulay \( R \)-algebra. By [4, Corollary 8.5.3], we may replace \( B \) by its \( m \)-adic completion and assume that it is even a balanced big Cohen-Macaulay algebra. By the argument in [5, Proposition 7.9(c)] we may then replace \( R \) by its completion. Let \( x := (x_1, \ldots, x_d) \) be a
system of parameters generating a solidly closed ideal. Let \( I_i := (x_1, \ldots, x_i)R \). Before we prove the proposition, let us show by downward induction on \( i \leq d \) that \( I_i = I_iB \cap R \). The case \( i = d \) follows from our assumption that \( I_d \) is solidly closed and the fact that a big Cohen-Macaulay algebra over a Noetherian local domain is solid. Hence let \( i < d \) and assume \( I_{i+1} = I_iB \cap R \). Let \( z \) be an element of \( J_i := I_iB \cap R \). By our induction hypothesis, \( z \in I_{i+1} \), say \( z = y + ax_{i+1} \) with \( y \in I_i \) and \( a \in R \). From \( ax_{i+1} = z - y \in I_iB \) and the fact that \( x \) is \( B \)-regular, we get \( a \) lies in \( I_iB \) whence in \( J_i \). In conclusion, we showed that \( J_i = I_i + x_{i+1}J_i \), so that by Nakayama’s lemma, \( J_i = I_i \), as claimed.

To complete the proof, we must show that \( x \) is \( R \)-regular. To this end, suppose \( zx_{i+1} \in I_i \). Since \( x \) is \( B \)-regular, \( z \) lies in \( I_iB \) whence in \( I_i \), by our previous remark.

By a standard argument (see for instance [20, Proposition 5.6]), every ideal generated by part of a system of parameters is then contracted from \( B \). However, this does not yet prove that \( R \) is \( S \)-rational.

Again we can derive some asymptotic versions in mixed characteristic of the previous results. However, in view of the restriction on the characteristic imposed by Proposition 2.5 (namely that its separated quotient have equal characteristic \( p \)), we only get an asymptotic version for large ramification index. In our first application, even the case \( R = S \) leads to new results (although trivially weakly \( S \)-rational, a regular local ring of mixed characteristic is only conjecturally weakly \( S \)-regular or even \( S \)-rational):

### 4.4. Theorem. For each triple \((p, n, l)\) with \( p \) a prime number, there exists a bound \( N := N(p, n, l) \) with the following property. Let \( R \to S \) be a cyclically pure homomorphism of Noetherian local rings of residual characteristic \( p \) and embedding dimension at most \( n \), with \( S \) regular. Let \( a \) be an ideal in \( R \) of co-length at most \( l \).

If \( S \) has ramification index at least \( N \), then \( a \) is solidly closed. In particular, if \( R \) has moreover parameter degree at most \( l \), then it is weakly \( S \)-rational.

**Proof.** We only need to show the first assertion, so suppose it is false for some triple \((p, n, l)\). Hence, there exists for each \( w \), a cyclically pure homomorphism of Noetherian local rings \( R_w \to S_w \) of residual characteristic \( p \) and embedding dimension at most \( n \), with \( S_w \) regular of ramification index at least \( w \), and an ideal \( a_w \subseteq R_w \) of co-length at most \( l \) which is not solidly closed. Let \( a, R, S \) be the respective ultraproducts of the \( a_w \), \( R_w \) and \( S_w \). Therefore, \( S \) is infinitely ramified and cata-regular. Since \( a_w = a_wS_w \cap R_w \), Łos’ Theorem yields that \( a = aS \cap R \). Moreover, \( a \) has co-length at most \( l \) whence in particular is \( m \)-primary. Therefore \( a \) is ultra-solidly closed by Corollary 4.2, and hence almost all \( a_w \) are solidly closed, contradiction. \( \square \)

### 4.5. Remark. In his lists of open problems ([6, Question 20]), Hochster asks—‘in the hope of getting a negative answer—the following: does \( \pi^2X^2Y^2 \) belong to the solid closure of the ideal \( (\pi^3, X^3, Y^3)R \), where \( R := V[[X, Y]] \) and \( V \) is a mixed characteristic discrete valuation ring with uniformizing parameter \( \pi \) and valuation \( v \)? According to our previous result, for fixed residual characteristic \( p \), the answer is indeed no, provided \( v(p) \) is sufficiently large. Ironically, Hochster asked this for \( V \) an unramified discrete valuation ring (in fact, for \( V \) equal to the ring of \( p \)-adic integers), expecting this to be the easiest case to settle, yet, this case remains open.

### 4.6. Theorem (Asymptotic Hochster-Roberts in dimension 3). For each triple \((p, d, l)\) with \( p \) a prime number, there exists a bound \( \rho := \rho(p, d, l) \) with the following property. Let \( R \to S \) be a cyclically pure homomorphism of Noetherian local rings of residual characteristic \( p \). Assume \( S \) is regular and has dimension at most \( d \), and assume \( R \) has
dimension at most three and parameter degree at most \( l \). If the ramification index of \( S \) is at least \( \rho \), then \( R \) is Cohen-Macaulay.

**Proof.** By Theorem 4.4, if the ramification index of \( S \) is at least \( N(p,d,l) \), then \( R \) is weakly \( S \)-rational. Since \( R \) admits a big Cohen-Macaulay algebra by [7], the result follows from Proposition 4.3 (note that a cyclically pure subring of a regular local ring is analytically irreducible). \( \square \)

One expects that this result is true without any restriction on the ramification index of \( S \). To derive this more general result directly from Hochster’s result on the existence of big Cohen-Macaulay algebras in dimension three, one would need to show that the big Cohen-Macaulay \( B \) admits an \( R \)-algebra homomorphism into a big Cohen-Macaulay algebra for \( S \) (equivalently, into some faithfully flat extension of \( S \)) and this is only known if also \( S \) has dimension at most three. In higher dimensions, the existence of big Cohen-Macaulay algebras is still open, and we have to settle for the following much weaker result. Recall that a tuple \((x_1, \ldots, x_n)\) is called independent (in the sense of Lech), if any relation \( a_1x_1 + \cdots + a_dx_d = 0 \) in \( R \) implies that all \( a_i \) lie in the ideal \( I \) generated by \((x_1, \ldots, x_n)\) (equivalently, if \( I/\mathfrak{I}^2 \) is a free \( R/\mathfrak{I} \)-module of rank \( n \)). Any regular sequence is independent, and conversely, if \((x_1^t, \ldots, x_d^t)\) is independent for infinitely many \( t \), then \((x_1, \ldots, x_d)\) is regular. We are interested in the situation that a (non-Cohen-Macaulay) Noetherian local ring has an independent system of parameters. For instance, the local ring \( K[[X,Y]]/(X^2,XY)K[[X,Y]] \) with \( K \) a field, does not admit an independent system of parameters, whereas \( Y \) is independent in the local ring \( K[[X,Y]]/(X^2,XY^2)K[[X,Y]] \).

Before we state this weaker form, we need to introduce one last concept. Namely, in order to apply Theorem 2.7, we have to enforce for the separated ultraprodut to be reduced and equidimensional. It does not suffice do require that the components have the same properties. For instance, the separated ultraprodut of the analytically irreducible one-dimensional domains \( R_w \) associated to the cusps \( X^2 - Y^w \) in \( \mathbb{A}^2_Ke^* \) is the the non-reduced curve \( X^2 = 0 \) in \( \mathbb{A}^2_Ke^* \), where \( K^e \) is the ultraprower of the field \( K \). To control the separated ultraprodut, we use a result of Swanson. Let us say that a local ring \((A, p)\) has \( k \)-bounded multiplication if for all \( n \) and all \( a, b \in A \) we have

\[
\text{ord}(ab) \leq k \max\{\text{ord}(a), \text{ord}(b)\}
\]

where \( \text{ord}(a) \) is the supremum of all integers \( n \) such that \( a \in p^n \) (with the usual convention that \( p^0 = A \) and \( \text{ord}(a) = \infty \) when \( a \in \text{Inf}(A) \)). It is shown in [25, Theorem 3.4] and [10, Proposition 2.2] that for a Noetherian local ring \( A \) the multiplication is \( k \)-bounded for some \( k \) if, and only if, \( A \) is analytically irreducible (see [14, Proposition 5.6] for a characterization in terms of ultraproducts). At present we do not have a good understanding of the smallest such \( k \): in [10] an upper bound in terms of Rees valuations is given. It is not hard to see that having \( k \)-bounded multiplication is preserved under separated ultraproducts (see also the next proof). Therefore, the above example on the cusps \( X^2 - Y^w \) shows that an upper bound on \( k \) cannot be given in terms of dimension and multiplicity alone.

4.7. **Proposition.** For each quadruple \((p,d,l,k)\) with \( p \) a prime number, there exists a bound \( N := N(p,d,l,k) \) with the following property. Let \( R \) be a \( d \)-dimensional mixed characteristic analytically irreducible local domain of residual characteristic \( p \). Assume the multiplication in \( R \) is \( k \)-bounded. Let \( x \) be a system of parameters generating an ideal of co-length at most \( l \). If \( xR \) is solidly closed (so that \( R \) is weakly \( S \)-rational) and the ramification index of \( R \) is at least \( N \), then \( x \) is independent.
Proof. If not, then we get for a fixed quadruple $(p, d, l, k)$ and for each $w \in \mathbb{N}$ a counterexample consisting of a $d$-dimensional mixed characteristic analytically irreducible local domain $R_w$ of residual characteristic $p$, with $k$-bounded multiplication, and a ‘dependent’ system of parameters $x_w = (x_{1w}, \ldots, x_{dw})$ in $R_w$ generating a solidly closed ideal of co-length at most $l$. This means, after renumbering, that there exists $a_w \notin x_w R_w$ such that $a_wx_{dw} \in (x_{1w}, \ldots, x_{d-1w}) R_w$. Let $a, x_1$ and $R$ be the ultraproducts of the $a_w, x_{1w}$ and $R_w$ respectively.

By Los’ Theorem, $I := (x_1, \ldots, x_d) R$ has co-length at most $l$, does not contain $a$ and is ultra-solidly closed. We leave it to the reader to verify that $(x_1, \ldots, x_d)$ is a system of catata-parameters (use for instance the argument in the proof of Theorem 1.1 or [23, Theorem 8.1]). The multiplication in $R$ is again $k$-bounded by Los’ Theorem, and it is not hard to show that this implies the same for its separated quotient $R_{\text{sep}}$. In particular, $R_{\text{sep}}$ is a domain so that we can apply Theorem 2.7 to conclude that the ideal $I$ is cata-tightly closed. On the other hand, Los’ Theorem yields that $ax_d \in (x_1, \ldots, x_{d-1}) R$ so that $a \in \text{ctc}((x_1, \ldots, x_d) R)$ by Proposition 3.3, whence $a \notin \text{ctc}(I) = I$, contradiction. □

Hence combining this result with Theorem 4.4, we may add to the conclusion in the latter theorem that $R$ as above admits an independent system of parameters.

Our last application gives an ‘asymptotic’ affirmative answer to a question posed by Hochster concerning solid closure in mixed characteristic in [5, Remark 10.13]:

4.8. Theorem. For each quintuple $(p, d, l, m, k)$ with $p$ a prime number, there exists a bound $N := N(p, d, l, m, k)$ with the following property. Let $R$ be a $d$-dimensional mixed characteristic analytically irreducible local domain of residual characteristic $p$. Assume the multiplication in $R$ is $k$-bounded. Let $(x_1, \ldots, x_d)$ be a system of parameters generating an ideal of co-length at most $l$ and let $0 \leq a_i < b_i \leq m$. If the ramification index of $R$ is at least $N$, then

$$(\text{sc}((x_1^{b_1}, \ldots, x_d^{b_d}) R) : x_1^{a_1} \cdots x_d^{a_d}) \subseteq \text{sc}((x_1^{b_1-a_1}, \ldots, x_d^{b_d-a_d}) R).$$

Proof. As before, this follows by the same argument from the corresponding ultra-version, which holds in view of Remark 3.4 and Theorem 2.7. We leave the details to the reader. □
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