Cosmological Constant in LQG Vertex Amplitude

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A new q-deformation of the Euclidean EPRL/FK vertex amplitude is proposed by using the evaluation of the Vassiliev invariant associated with a 4-simplex graph (related to two copies of quantum SU(2) group at different roots of unity) embedded in a 3-sphere. We show that the large-j asymptotics of the q-deformed vertex amplitude gives the Regge action with a cosmological constant. In the end we also discuss its relation with a Chern-Simons theory on the boundary of 4-simplex.

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I. INTRODUCTION:

The spinfoam formalism is currently understood as a covariant formulation of Loop Quantum Gravity (LQG) [5]. In LQG community, it was commonly conjectured that one should make a q-deformation of the spinfoam amplitude (with quantum group) in order to implement the cosmological constant term in the theory [3,6][7]. Such a conjecture was suggested by the lesson from 3d gravity and 4d topological field theory. In 3d gravity, the Turaev-Viro model [8] is a deformation of the Ponzano-Regge model [9] by the quantum group theory. In 3d gravity, the Turaev-Viro model is finite, and its large spin asymptotics is the starting point for the asymptotic analysis, and it gives the Regge action with a cosmological constant. In the end we also discuss its relation with a Chern-Simons amplitude gives the Regge action with cosmological constant. This result can be considered as an evidence supporting that the large-j asymptotics of the q-deformed vertex amplitude, to give an idea for obtaining the cosmological constant term in the spinfoam vertex amplitude.

Given a 4-simplex σ, we label by a, b = 1, · · · , 5 the five tetrahedra on the boundary of the 4-simplex, and denote by the pair (a, b) the triangle shared by two tetrahedra a and b. We assume the Barbero-Immirzi parameter 0 < γ < 1. The Euclidean EPRL/FK vertex amplitude can be written in coherent state representation [18] (± stands for the self-dual/anti-self-dual contribution):

\[ A_\alpha(k_{ab}, n_{ab}) := (-1)^\gamma \int \prod_{a=1}^{5} d\mathbf{g}_a \prod_{a<b} P_{ab}^\pm(k_{ab}, \mathbf{g}_a, n_{ab}) \]

where \((-1)^\gamma\) is a sign defined by the diagrammatic calculus of SU(2) spin-network, \(P_{ab}^\pm\) is a coherent propagator

\[ P_{ab}^\pm := \left( j_{ab}^\pm, -n_{ab}(g_{a}^\pm 1 g_{b}^\pm |j_{ab}^\pm, n_{ab}\right) \]

\(g_a (a = 1, \cdots, 5)\) are 2 × 2 SU(2) matrices, and \(|j, n\rangle\) is a coherent state in the spin-j representation of SU(2) [19]. Here \(|k_{ab}, n_{ab}\rangle\) with \(j_{ab}^\pm = \frac{1+y}{2} j_{ab}\) and \(n_{ab} \in \mathbb{S}^2\) is a set of boundary data for a vertex amplitude. The vector \(j_{ab} n_{ab}\) is an oriented area vector of the triangle \((a, b)\) viewed at the tetrahedron a.

The coherent state representation of EPRL/FK vertex amplitude is the starting point for the asymptotic analysis, and it turns out also to be useful in the analysis of quantum group spinfoam vertex.

II. HEURISTIC DEFORMATION:

Before we come to the systematic q-deformation of the amplitude, we first present a heuristic deformation of EPRL/FK vertex amplitude, to give an idea for obtaining the cosmological constant term in the spinfoam vertex amplitude.

Now we make a heuristic modification of the EPRL/FK vertex amplitude: We consider a 4-simplex graph \(\Gamma_5^+\) (FIG. 1). In FIG. 1 we order the 5 nodes on the paper from left to right, and connect the nodes by oriented links. A link oriented from the node a to the node b is denoted by \(l_{ab}\). We notice that there is a crossing between the links \(l_{31}\) and \(l_{42}\), which motivate us to make the following modification of the coherent propagator.

\[ P_{ab}^{\pm} (\gamma, \mathbf{g}_a,\mathbf{g}_b) = \sum_{n_{ab}} \langle j_{ab}^\pm, n_{ab}, \mathbf{g}_a | \mathbf{g}_b \rangle \]

\[ \langle j_{ab}^\pm, n_{ab}, \mathbf{g}_a | \mathbf{g}_b \rangle = \frac{1}{\sqrt{2}} \left( \begin{array}{c} j_{ab}^\pm \cr n_{ab} \end{array} \right) \]

\[ \langle j_{ab}^\pm, n_{ab}, \mathbf{g}_a | \mathbf{g}_b \rangle \]

FIG. 1: The \(\Gamma_5^+\) graph with one crossing between \(l_{31}\) and \(l_{42}\).
We define two operators $R^\pm_\omega$ on the SU(2) tensor representations $V_{j1} \otimes V_{j2}$ respectively:

$$R^\pm_\omega := \exp \left[ \pm \frac{\lambda_\omega i}{(1 \pm \gamma) \omega} \sum_{j=3}^{\infty} X_j^\pm \otimes X_j^\pm \right]$$

(3)

where $\omega$ is a real dimensionless parameter (a deformation parameter), and $X_j^\pm$ are self-dual/anti-self-dual generators of $\text{Spin}(4)$ with commutator $[X_j^+, X_j^-] = i\epsilon_j g \gamma^j$. We formally expand $R^\pm_\omega$, which contributes the power expansion at the order $\omega^3$. We can written by $A^{\omega}_n(k_{ab}, n_{ab})$, which can be expanded into power series of $\omega$, which results in a power expansion of $K_{31,42}$ in terms of the deformation parameter $\omega$. A building block for constructing the power expansion of $K_{31,42}$ is

$$K_{31,42} = \prod_{i \in \omega^+} p_{31}^{\omega+} \prod_{j \in \omega^0} p_{42}^{\omega0}$$

We then expand Eq. (4) into power series of $\omega$, which results in a power expansion of $K_{31,42}$, as its leading large-$\omega$ asymptotics. Moreover Eq. (5) contributes the expansion at the order $\omega^3$, thus $[\omega^3 \times \text{Eq. (5)}]$ doesn’t scale asymptotically if we propose a scaling of $\omega$ by $\omega \rightarrow \omega/\lambda^2$.

From Eq. (13) we see that the asymptotic formula of a coherent state expectation value for $X_{k_i}$ is given by simply replacing each $\tilde{X}$ by $\lambda_j \tilde{g}_j \tilde{h}_j$. Then we find that under the scaling $j_{ab} \rightarrow j_{ab} \lambda$ and $\omega \rightarrow \omega/\lambda^2$, the asymptotic formula for $K_{31,42}$ as $\lambda \rightarrow \infty$ is obtained by considering the product $R^R \cdot R^{R^\circ}$ and replacing each $\tilde{X}$ in $R^\circ R^R$ by $\lambda_j^2 \tilde{g}_j^2 \tilde{h}_j^2$.

We recall that $j_{ab} = \frac{1+\gamma}{2} k_{ab}$. We write

$$\prod_{a \leq b} P_{ab}^{\omega} = e^{S_0} \text{ in the deformed vertex amplitude } A^{\omega}_n \text{ in } \text{Eq. (5)}, \text{ where } S \text{ is a } \text{“spinfoam action” used in the spinfoam asymptotic analysis} \text{ [18]}

$$S = \sum_{a < b} \sum_{\omega < 0} 2 j_{ab} \log \left( -n_{ab} (g_{ab}^{-1})^3 g_{ab}^2 \right)$$

(16)

The spinfoam action $S$ doesn’t depend on $\omega$. Thus under the scaling $k \rightarrow \lambda k$, $\omega \rightarrow \omega/\lambda^2$ and $\lambda \rightarrow \infty$, the $\omega^3$ part of the
integrand is affected only by the scaling of the spins \(k_{ab}\). The critical point of the action \(S\) under \(\lambda \to \infty\) is analyzed in \([13]\). The critical equations from \(S\)

\[
\sum_b k_{ab}n_{ab} = 0, \quad g^a_{b}n_{ab} = -g^b_{a}n_{ba}
\]

imply that (i) the closure of each tetrahedron and (ii) two neighboring tetrahedron are glued with each other at a triangle. Note that the critical equations Eq. (17) from \(S\) are consistent with the critical equations Eq. (11) from \(S_0\). Suppose we fix a set of boundary data \((k_{ab}, n_{ab})\) corresponding to a non-degenerated flat 4-simplex Regge geometry, and also fix the dihedral angles between each pairs of neighboring tetrahedra (e.g. via imposing boundary state [20]), then there is a unique solution \((g^a_{b}, g^b_{a})\) for the above critical equations. The solution specifies uniquely a bivector geometry of the 4-simplex up to an inversion. The bivector (at the center of 4-simplex) for each triangle \((a, b)\) is given by

\[
B_{ab}(\sigma) = (B^{+}_{ab}, B^{-}_{ab}) = \pm k_{ab} (g^{+}_{ba}, g^{-}_{ab})(n_{ab}, n_{ba})
\]

Then one can see immediately the above \(V_{31,42}\) evaluated at the critical point \((g^{+}_{ba}, g^{-}_{ab})\) gives precisely the 4-volume of the 4-simplex \(\sigma\) (up to an overall constant)

\[
V_{31,42} = B^{+}_{31} \cdot B^{-}_{42} - B^{+}_{31} \cdot B^{-}_{42} = V_{\sigma}
\]

For a geometrical 4-simplex, this expression of 4-volume doesn’t depend on the choice of triangle \((3, 1)\) and \((4, 2)\).

The asymptotics of the deformed vertex amplitude \(A^o_{\sigma}\) is given by its integrand \(\mathcal{K}^0_{31,42} e^{i\theta}\) evaluated at the critical point satisfying both Eqs. (11) and (17) from both actions \(S\) and \(S_0\). We have seen that the two critical equations Eqs. (11) and (17) are consistent with each other. The action \(S\) evaluated at the critical point gives the 4-simplex Regge action \(iS_{\text{Regge}} = i\ell^2 \sum_{\text{tetra}} k_{ab}\Theta_{ab}\), without cosmological constant. Eq. (14) gives the asymptotic behavior of \(\mathcal{K}^0_{31,42}\). Therefore we have the following large-j asymptotics

\[
A^o_{\sigma} \sim \left(\frac{2\pi}{\lambda}\right)^{\frac{3}{2}} \frac{e^{i\text{Im} H}}{\sqrt{\det H}} e^{i4\lambda \sum a \Theta_{ab}} \lambda k_{ab}\Theta_{ab} + \lambda^2 V_{\sigma} + o(1/\lambda)
\]

under \(k_{ab} \to \lambda k_{ab}\), \(\omega \to \omega/\lambda^2\) and \(\lambda \to \infty\), where \(H\) is the Hessian matrix of the spinfoam action \(S\) and \(D\) is the dimension of the integral. The above asymptotic formula manifests that the deformation parameter \(\omega\) is proportional to the cosmological constant \(\Lambda\) in Regge gravity. Note that the above Regge action with \(\Lambda\)

\[
S_{\text{Regge}, \Lambda} = \ell^2 \sum_{a \neq b} k_{ab}\Theta_{ab} + \lambda^2 V_{\sigma}
\]

corresponds to the Regge calculus approximation of continuous curved geometry with flat 4-simplices.

We now discuss the physical meaning of the scaling \(k_{ab} \to \lambda k_{ab}\), \(\omega \to \omega/\lambda^2\) and \(\lambda \to \infty\), which leads us to the asymptotic formula Eq. (20). Given a cosmological constant \(\Lambda = 1/\ell_c^2\) where \(\ell_c\) is the cosmological length, the dimensionless parameter \(\omega\) has to be interpreted as \(\omega = \Lambda \ell_p^2 = \ell_p^2 / \ell_c^2\) from the asymptotic formula Eq. (20). The spins \(k_{ab}\) relate to the area \(A_{ab}\) of the triangle shared by tetrahedra \(a\) and \(b\) by the relation \(y_{k_{ab}} = A_{ab}/\ell_p^2\). Then the scaling \(k_{ab} \to \lambda k_{ab}\) can be understood as a scaling of the Planck length by \(\ell_p^2 \to \lambda^{-1}\ell_p^2\) while keeping the area \(A_{ab}\) fixed. The other scaling \(\omega \to \omega/\Lambda^2\) combined with \(\ell_p^2 \to \lambda^{-1}\ell_p^2\) results in the scaling of the cosmological length \(\ell_c^2 \to \lambda \ell_c^2\). As \(\Lambda \to \infty\), we see that the asymptotic formula Eq. (20) is valid in the regime where the area \(A_{ab}\) is much larger than the Planck area \(\ell_p^2\) but much smaller than the cosmological area \(\ell_c^2\). The assumption that the cosmological length \(\ell_c\) is much larger than the physical scale of the 4-simplex is the reason why we can approximate the local geometry with a flat 4-simplex given by the critical equations Eq. (17) and the boundary data \((k_{ab}, n_{ab})\).

### III. Q-DEFORMATION AND VASSILIEV INVARIANTS:

From the above derivation, we have seen that the expected cosmological constant term comes from the insertion of the operator \(R^n\) in the vertex amplitude, which is responsible for the crossing in the spin-network graph \(\Gamma\). Here we present a more systematic deformation of the EPRL/FK vertex amplitude by using the evaluation of Vassiliev invariants \([21]\) (see also \([13]\) for a brief introduction). The resulting q-deformed vertex amplitude gives the same asymptotic behavior as the above heuristic deformation.

Let’s recall Eq. (1) and carry out the integration over \(g^a_{b}\), we obtain

\[
A_{\omega}(k_{ab}, n_{ab}) = \sum_{[15]} \{15\}^{\pm}_{i_2} f_{i_{2}}^{\pm}(j_{ab}^{+}, n_{ab})
\]

where \([15]\) denotes two copies of SU(2) 15j symbol with spins \(j_{ab}^{+}\) and intertwiners \(i_{2}^{+}\), and \(f_{i_{2}}^{\pm}(j_{ab}^{+}, n_{ab})\) denotes two copies of SU(2) intertwiner \(i_{2}^{+}\) in the coherent state representation.

We define a deformation of the vertex amplitude by simply replace the 15j symbols in Eq. (22) by two q-deformed 15j symbols with \(q^{\pm}\) at different roots of unity. Therefore we define the q-deformed EPRL/FK vertex amplitude by

\[
A_{\omega}^{q}(k_{ab}, n_{ab}) := \sum_{[15]} \{15\}^{\pm}_{i_{2}, q^{\pm}} f_{i_{2}}^{\pm}(j_{ab}^{+}, n_{ab})
\]

The q-deformed 15j symbols are obtained from the evaluation of a 4-simplex spin-network with the corresponding Vassiliev invariant. Here we briefly describe the procedure for the construction.

Let \(X\) be a 1-dimensional oriented compact manifold (an oriented graph). A chord diagram with support \(X\) is defined by the union \(C = D \cup X\), where \(D\) (dash lines) is a (non-planar) graph with end points on \(X\), and the graph \(D\) has only univalent and trivalent vertices. The degree of the chord diagram \(C\) is defined by the half of the number of vertices in \(D\). We define a vector space \(A_{\omega}(X)\) generated by all the chord diagrams with degree \(n\), subject to the some relations \([13]\ [21]\ [22]\).
The space of chord diagrams is used to define the universal Vassiliev invariant for the framed links. Given a deformation parameter \( q = e^{ih} \), the Vassiliev invariant \( Z \) assigns to any framed link \( X \) a formal power series \( Z(X) = \sum_{\nu} \omega h^n Z_{\nu}(X) \), where the coefficients \( Z_{\nu}(X) \in \mathcal{A}_\nu(X) \) is a linear combination of degree-\( n \) chord diagrams. Given the link \( X \), we need three types of building blocks to construct \( Z_{\nu}(X) \) to each order: (1) For each crossing in \( X \) we assign a braiding \( R \in \mathcal{P}_2 \); (2) For each maximum or minimum in \( X \) we assign an unknot \( v^{-\frac{1}{2}} \in \mathcal{P}_1 \); (3) There is also an associator \( \Phi \in \mathcal{P}_3 \). Here \( \mathcal{P}_n \) denotes the space of the series of chord diagrams based on \( n \) lines in \( X \). These building blocks are expressed as power series \( R = \exp \left( \frac{ih}{2} \right) \cdots \), \( \mathcal{V} = \left[ - \frac{h^2}{12} \right]^2 + \cdots \), \( \Phi = \left[ \left[ \left[ - \frac{h^2}{12} \right]^2 \right] \right] + \cdots \).

**FIG. 2:** The building blocks for Vassiliev invariant.

Given a compact Lie group \( G \) and a spin-network \( s \) based on the oriented graph \( X \), for each chord diagram based on \( X \), we can define the evaluation map \( \Omega_{G,s} \) given by \( \Omega_{G,s} \). Here \( X_\alpha \) is a basis of the Lie algebra \( \text{Lie}(G) \) with structure constant \( f_{abc} \), and \( x^a X_\alpha x^b \) is the quadratic casimir of \( \text{Lie}(G) \). It turns out that the evaluation \( \Omega_{G,s} \) of links gives the same result as the Reshetikhin-Turaev evaluation of the link associated with the quantum group \( U_q(G) \).

**FIG. 3:** Evaluation of Vassiliev invariant.

For a 4-simplex SU(2) spin-network based on the graph \( \Gamma_4 \), the corresponding 15j symbol \( \{15f\}_{\omega,q} \) is given by the evaluation of \( \Phi \) with appropriate insertions of \( R \)-matrix, associators \( \Phi \), and unknots \( v^{-\frac{1}{2}} \). We evaluate \( \Phi \) for both self-dual and anti-self-dual sector, and insert them in the definition of the q-deformed vertex amplitude \( A^q \). As we did for the heuristic deformation \( A^q \), we expand the q-deformed vertex amplitude \( A^q \) into a power series of \( \omega \). For the braiding \( R \)-matrix responsible for the only crossing in \( \Phi \), its evaluation coincides with Eq. (3) used in the heuristic deformation, if we choose the deformation parameter \( q^\pm = e^{ih^\mp} \) such that

\[
h^\pm = \pm \frac{8}{(1 \pm \gamma^2)^3} \omega. \tag{24}\]

In the following, we show that both the associator \( \Phi \) and unknot \( v \) don’t contribute the leading asymptotic behavior of \( A^q \) under the scaling \( k_{ab} \rightarrow \lambda k_{ab} \), \( \omega \rightarrow \omega/\lambda^2 \) and \( \lambda \rightarrow \infty \). First of all, the SU(2) evaluation of unknot \( v \) can be expanded as a power series of \( h \) by (see e.g. \[21\])

\[
v = \sum_{n=0}^{\infty} q_n(c) h^{2n} \tag{25}\]

where \( c \) is the quadratic casimir of \( \text{su}(2) \), the polynomial function \( q_n \) relates to the Bernoulli polynomial \( B_{2n+1} \) by

\[
q_n \left( \frac{x^2 - 1}{2} \right) = \frac{B_{2n+1} \left[ \frac{1}{2} x + \frac{1}{2} \right]}{x} \tag{26}\]

In the scaling of spins \( k_{ab} \rightarrow \lambda k_{ab} \), the quadratic casimir scales as \( \lambda^2 \). Then \( q_n(c) \) scales as \( \lambda^{2n} \) since \( B_{2n+1} \) \( \lambda \chi \rightarrow \infty \). As a result each term \( q_n(c) h^{2n} \) in Eq. (25) scales as \( \lambda^{-2n} \) by taking into account the scaling \( \omega \rightarrow \omega/\lambda^2 \). Thus the leading asymptotic behavior of \( A^q \) only sees \( \lambda = 1 \) since all the higher order corrections only contribute \( o(1/\lambda) \)-terms in Eq. (20) as \( \lambda \rightarrow \infty \).

The perturbative expansion of the associator \( \Phi \) can be presented in terms of chord diagrams in FIG. 2 where the degree-\( n \) chord diagram at each \( h^\pm \)-order is built by connecting the 3-valent vertices of dashed lines in FIG. 3. There are \( 2n \) vertices in each degree-\( n \) diagram, in which there are \( m \) vertices are attached to the framed links. Thus there is \( m \) number of internal 3-valent vertices and \( 2n - m > 0 \) for a nontrivial chord diagram. When we scale of spins \( k_{ab} \rightarrow \lambda k_{ab} \) and \( \lambda \rightarrow \infty \), the evaluation of each vertex attached to a framed link gives a factor of \( \lambda f_{ab}^n h_{ab}^n \) as its leading asymptotics, since on each link the su(2) generator \( X_a \) is sandwiched by SU(2) coherent states. Thus for each degree-\( n \) diagram in the perturbative expansion of \( \Phi \), the scaling of spins \( k_{ab} \rightarrow \lambda k_{ab} \) leads to a scaling \( \lambda^m \) of the diagram, while the other scaling \( \omega \rightarrow \omega/\lambda^2 \) contributes \( h^\pm \rightarrow \lambda^{-2n} h^\pm \). Thus the overall scaling of each term is \( \lambda^{-2n-m} \) from which we see that the nontrivial diagrams in \( \Phi \) only contributes to the \( o(1/\lambda) \)-terms in the asymptotic formula as \( \lambda \rightarrow \infty \).

The above power-counting shows that we can take \( \Phi = 1 \) and \( \lambda = 1 \) for the asymptotic analysis of the q-deformed vertex amplitude \( A^q \). By the coincidence of the R-matrix between \( A^q \) and \( A^\omega \), the asymptotic analysis of \( A^q \) reduces to

**FIG. 4:** The evaluation of 4-simplex graph via Vassiliev invariant.
the previous analysis of heuristic deformation $A^{q}_{\nu}$, i.e. under the scaling $k_{ab} \rightarrow \lambda k_{ab}$, $\omega \rightarrow \omega / \lambda^2$ and $\lambda \rightarrow \infty$, $A^{q}_{\nu}$ and $A^{q}_{\nu}$ have the same asymptotic behavior. Thus we can write down the asymptotic formula of the q-deformed vertex amplitude with a given Regge boundary data:

$$A^{q}_{\nu} \sim \frac{2\pi}{\lambda} \frac{e^{i m H}}{\sqrt{\Delta H}} \int d^4 \Sigma_{\nu} e^{i \nu \Theta_{\nu}} e^{i m V_{\nu}} [1 + o(1/\lambda)].$$

Before conclusion, we would like to point out an interesting fact: there is another possibility to obtain the same asymptotics from another q-deformation. We use the deformation parameter $h^+ = \frac{1}{\sqrt{\lambda}} \omega$ instead of Eq. (24), but evaluate the the self-dual and anti-self-dual 15j symbols on different graphs, i.e. we evaluate the self-dual sector on the $\Gamma^+\sigma$ graph as before but evaluate the anti-self-dual sector on the $\Gamma^-\sigma$ graph Fig. 5 with the opposite crossing (with braiding $R^{-1}$) to the one in $\Gamma^+\sigma$. Then it is not hard to see that the resulting q-deformed vertex amplitude has the same asymptotic behavior as the above up to higher order in $\lambda^{-1}$.

IV. CONCLUSION AND DISCUSSION:

To summarize, in this paper we propose a new q-deformation of the Euclidean EPRL/FK spinfoam vertex amplitude. The concrete construction uses the evaluation of the Vassiliev invariant from 4-simplex graphs. We also show that the asymptotics of the q-deformed vertex amplitude gives the Regge gravity with a cosmological constant (from Regge calculus using flat 4-simplices) in the regime that the physical scale of the 4-simplex is much greater than the Planck scale $\ell_p$ but much smaller than the cosmological area $\ell_c$.

The Vassiliev invariants of links come from the Feymann diagrams of perturbative Chern-Simons theory, for evaluating the link observables [22, 24]. The q-deformation of the 15j symbol employed above can be viewed as a Chern-Simons expectation value of a 4-simplex spin-network. Moreover we suppose the boundary of the 4-simplex under consideration is a 3-sphere $S^3$, then the q-deformed vertex amplitude for this 4-simplex is given by the following expectation value of a Chern-Simons theory (with gauge group Spin(4) = SU(2) × SU(2)) on the boundary 3-manifold:

$$A^{q}_{\nu} = \Psi[A^\pm] \cdot \frac{e^{i \frac{i}{\hbar} S_{CS}[A^\pm]+i \frac{\alpha}{\hbar} S_{CS}[A]}^{}DA^\pm$$

where $S_{CS}[A]$ is the SU(2) Chern-Simons action, and $\Psi[A^\pm]$ is a projective spin-network functions on Spin(4) holonomies associated with a 4-simplex graph $\Gamma^\pm\sigma$ (or two graphs $\Gamma^\pm\sigma$) imbedded in the boundary 3-sphere. Interestingly, this result also relates to an old idea by L. Smolin et al. (see [6]).

In addition, although all the discuss in this paper concerns only a single 4-simplex, the asymptotic analysis can be done also for a triangulation with arbitrary many 4-simplices, which results in a Regge action with a cosmological constant (from the Regge calculus with flat simplices) on the triangulation. The detailed analysis will be reported in [27].

Finally we note that the scaling $k_{ab} \rightarrow \lambda k_{ab}$, $\omega \rightarrow \omega / \lambda^2$ used in this paper leads us to the Regge calculus with flat 4-simplex, which is an approximation of curved geometry in presence of a cosmological constant. It would be interesting to find the relation between the q-deformed vertex amplitude and a curved 4-simplex with constant curvature, in analogy with the 3d case (see e.g. [28]). We leave this point to the future research.

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