Ostrowski type inequalities involving conformable fractional integrals

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1 Introduction

Let $I \subseteq \mathbb{R}$ be an interval and $I^o$ the interior of $I$. Then the classical Ostrowski inequality [1] states that a real-valued function $f : I \rightarrow \mathbb{R}$ satisfies the inequality

$$\left| f(x) - \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} f(x) \, dx \right| \leq \frac{1}{4} + \frac{(x - a_1 a_2)}{2 (a_2 - a_1)} \| f' \|_\infty$$

with the best possible constant 1/4 if $a_1, a_2 \in I^o$ with $a_1 < a_2$ and $|f'(x)| \leq M$ for all $x \in [a_1, a_2]$.

Recently, the Ostrowski inequality has attracted the attention of many researchers, many remarkable generalizations, extensions, variants and applications can be found in the literature [2–24].

Let $0 < \alpha \leq 1$ and $g$ be a real-valued function defined on $[0, \infty)$. Then the (conformable) fractional derivative $D_\alpha (g)(t)$ [23] of order $\alpha$ of $g$ at $t > 0$ is defined by

$$D_\alpha (g)(t) = \lim_{\epsilon \to 0} \frac{g(t + \epsilon t^{1-\alpha}) - g(t)}{\epsilon}.$$ 

$g$ is said to be $\alpha$-differentiable if the conformable fractional derivative of order $\alpha$ of $g$ exists. In what follows, we write $g^\alpha(t)$ or $\frac{D_\alpha}{D_\alpha t}(g)$ for $D_\alpha (g)(t)$ to denote the conformable fractional derivative of order $\alpha$ of $g$. The conformable fractional derivative at 0 is defined as $g^\alpha(0) = \lim_{t \to 0^+} g^\alpha(t)$.

Let $\alpha \in (0, 1]$ and $0 \leq a < b$. Then the function $h : [a, b] \rightarrow \mathbb{R}$ is said to be $\alpha$-fractional integrable on $[a, b]$ if the integral

$$\int_a^b h(x) \, d_\alpha x := \int_a^b h(x)x^{\alpha-1} \, dx$$

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exists and is finite. All \( \alpha \)-fractional integrable functions on \([a, b]\) are denoted by \( L^1_\alpha([a, b]) \).

**Remark 1.1** Note that the relation between the Riemann integral and the conformable fractional integral is given by

\[
I^\alpha_a(h(t)) = I^\alpha_a(t^{\alpha-1}h) = \int_a^t \frac{h(x)}{x^{1-\alpha}} \, dx.
\]

Let \( \alpha \in (0, 1] \) and \( f, g \) be \( \alpha \)-differentiable at \( t > 0 \). Then it is well known that

\[
(1) \quad \frac{d}{dt}^\alpha (t^n) = nt^{\alpha-1}
\]

for all \( n \in \mathbb{R} \);

\[
(2) \quad \frac{d}{dt}^\alpha (c) = 0
\]

for all constant \( c \in \mathbb{R} \);

\[
(3) \quad \frac{d}{dt}^\alpha (af(t) + bg(t)) = a \frac{d}{dt}^\alpha (f(t)) + b \frac{d}{dt}^\alpha (g(t))
\]

for all \( a, b \in \mathbb{R} \);

\[
(4) \quad \frac{d}{dt}^\alpha (f(t)g(t)) = f(t) \frac{d}{dt}^\alpha (g(t)) + g(t) \frac{d}{dt}^\alpha (f(t));
\]

\[
(5) \quad \frac{d}{dt}^\alpha \left( \frac{f(t)}{g(t)} \right) = \frac{g(t) \frac{d}{dt}^\alpha (f(t)) - f(t) \frac{d}{dt}^\alpha (g(t))}{g^2(t)}
\]

\[
(6) \quad \frac{d}{dt}^\alpha (f'(g(t))) = f'(g(t)) \frac{d}{dt}^\alpha (g(t)),
\]

if \( f \) is differentiable at \( g(t) \).

The main purpose of the article is to find the Ostrowski type inequalities involving the conformable fractional integrals and give their applications in certain bivariate means.

### 2 Main Results

**Lemma 2.1** Let \( 0 < \alpha \leq 1 \), \( 0 \leq a_1 < a_2 \) and \( h : [a_1, a_2] \rightarrow \mathbb{R} \) be an \( \alpha \)-fractional differentiable function. Then the identity

\[
h(x) - \frac{\alpha}{a_2^{\alpha} - a_1^{\alpha}} \int_{a_1}^{a_2} h(s) \, ds = \frac{x - a_1}{a_2^{\alpha} - a_1^{\alpha}} \int_0^1 \left[ \frac{1}{((1-t)a_1 + tx)^{2\alpha-1} - a_1^{\alpha} ((1-t)a_1 + tx)^{\alpha-1}} \right]
\]

\[
\times D_\alpha(h)((1-t)a_1 + tx) t^1-\alpha \, dt
\]

\[
+ \frac{a_2 - x}{a_2^{\alpha} - a_1^{\alpha}} \int_0^1 \left[ \frac{1}{((1-t)a_2 + tx)^{2\alpha-1} - a_2^{\alpha} ((1-t)a_2 + tx)^{\alpha-1}} \right]
\]

\[
\times D_\alpha(h)((1-t)a_2 + tx) t^1-\alpha \, dt
\]

holds if \( D_\alpha(h) \in L^1_\alpha([a_1, a_2]) \).
Proof. It follows from integration by parts that

\[
\frac{x-a_1}{a_2^\alpha-a_1^\alpha} \int_0^1 \left[ ((1-t)a_1 + tx)^{2\alpha-1} - a_1^\alpha ((1-t)a_1 + tx)^{\alpha-1} \right] D_\alpha (h)(1-t)a_1 + tx) t^{1-\alpha} \, dt
\]

\[
+ \frac{a_2-x}{a_2^\alpha-a_1^\alpha} \int_0^1 \left[ ((1-t)a_2 + tx)^{2\alpha-1} - a_2^\alpha ((1-t)a_2 + tx)^{\alpha-1} \right] D_\alpha (h)(1-t)a_2 + tx) t^{1-\alpha} \, dt
\]

\[
\times D_\alpha (h)(1-t)a_2 + tx) t^{1-\alpha} \, dt
\]

\[
= \frac{x-a_1}{a_2^\alpha-a_1^\alpha} \int_0^1 \left[ ((1-t)a_1 + tx)^{\alpha} - a_1^\alpha \right] h'( (1-t)a_1 + tx) \, dt
\]

\[
+ \frac{a_2-x}{a_2^\alpha-a_1^\alpha} \int_0^1 \left[ ((1-t)a_2 + tx)^{\alpha} - a_2^\alpha \right] h'( (1-t)a_2 + tx) \, dt
\]

\[
= \frac{x-a_1}{a_2^\alpha-a_1^\alpha} \left( \left[ ((1-t)a_1 + tx)^{\alpha} - a_1^\alpha \right] h((1-t)a_1 + tx) \right] \left|_0^1 \right.
\]

\[
- \frac{x-a_1}{a_2^\alpha-a_1^\alpha} \int_0^1 \alpha ((1-t)a_1 + tx)^{\alpha-1} h'( (1-t)a_1 + tx) \, dt
\]

\[
+ \frac{a_2-x}{a_2^\alpha-a_1^\alpha} \left( \left[ ((1-t)a_2 + tx)^{\alpha} - a_2^\alpha \right] h((1-t)a_2 + tx) \right] \left|_0^1 \right.
\]

\[
- \frac{a_2-x}{a_2^\alpha-a_1^\alpha} \int_0^1 \alpha ((1-t)a_2 + tx)^{\alpha-1} h'( (1-t)a_2 + tx) \, dt
\]

\[
= \frac{x-a_1}{a_2^\alpha-a_1^\alpha} \left( \frac{x^\alpha - a_1^\alpha}{x-a_1} h(x) - \frac{\alpha}{x-a_1} \int_{a_1}^x h(s) \, ds \right)
\]

\[
+ \frac{a_2-x}{a_2^\alpha-a_1^\alpha} \left( \frac{a_2^\alpha - x^\alpha}{a_2-x} h(x) - \frac{\alpha}{a_2-x} \int_x^{a_2} h(s) \, ds \right)
\]

\[
= h(x) - \frac{\alpha}{a_2^\alpha-a_1^\alpha} \int_{a_1}^{a_2} h(s) \, ds.
\]

\[
\Box
\]

Theorem 2.2 Let \( 0 < \alpha \leq 1, 0 \leq a_1 < a_2, h : [a_1, a_2] \rightarrow \mathbb{R} \) be an \( \alpha \)-fractional differentiable function and \( D_\alpha (h) \in L_1^\alpha ([a_1, a_2]) \). Then the inequality

\[
\left| h(x) - \frac{\alpha}{a_2^\alpha-a_1^\alpha} \int_{a_1}^{a_2} h(s) \, ds \right| \leq \frac{x-a_1}{a_2^\alpha-a_1^\alpha} \Delta_1 + \frac{a_2-x}{a_2^\alpha-a_1^\alpha} \Delta_2
\]

holds if \( |h'(x)| \) is convex, where

\[
\Delta_1 = \frac{1}{6} a_1^\alpha x |h'(a_1)| + \frac{1}{12} x^\alpha a_1 |h'(a_1)| + \frac{1}{12} x^\alpha |h'(a_1)| - \frac{1}{4} a_1^\alpha |h'(a_1)|
\]

\[
+ \frac{1}{12} a_1 |h'(x)| + \frac{1}{12} x^\alpha a_1 |h'(x)| + \frac{1}{4} x^\alpha |h'(x)| - \frac{1}{2} a_2^\alpha |h'(x)|,
\]

\[
\Delta_2 = \frac{1}{6} a_2^\alpha |h'(a_2)| + \frac{1}{6} x^\alpha |h'(a_2)| + \frac{1}{3} a_2^\alpha |h'(a_2)| - \frac{1}{3} x^\alpha |h'(x)|.
\]
Proof. Let $y > 0$, $\varphi_1(y) = y^{\alpha - 1}$ and $\varphi_2(y) = -y^\alpha$. Then we clearly see that the functions $\varphi_1$ and $\varphi_2$ both are convex. It follows from Lemma 2.1 and the convexity of $\varphi_1$, $\varphi_2$ and $|h'|$ that

$$
|h(x) - \frac{\alpha}{a_2^\alpha - a_1^\alpha} \int_{a_1}^{a_2} h(s) \, ds| \\
\leq \frac{x - a_1}{a_2^\alpha - a_1^\alpha} \int_0^1 \left(\left((1 - t)a_1 + tx\right)^\alpha - a_1^\alpha\right) |h'((1 - t)a_1 + tx)| \, dt \\
+ \frac{a_2 - x}{a_2^\alpha - a_1^\alpha} \int_0^1 \left(\left((1 - t)a_2 + tx\right)^\alpha - a_2^\alpha\right) |h'((1 - t)a_2 + tx)| \, dt \\
\leq \frac{x - a_1}{a_2^\alpha - a_1^\alpha} \int_0^1 \left(\left((1 - t)a_1 + tx\right)^{\alpha - 1} - a_1^{\alpha - 1}\right) |h'((1 - t)a_1 + tx)| \, dt \\
+ \frac{a_2 - x}{a_2^\alpha - a_1^\alpha} \int_0^1 \left(\left((1 - t)a_2 + tx\right)^{\alpha - 1} - a_2^{\alpha - 1}\right) |h'((1 - t)a_2 + tx)| \, dt \\
\leq \frac{x - a_1}{a_2^\alpha - a_1^\alpha} \int_0^1 \left(\left((1 - t)a_1 + tx\right)^{\alpha - 1} - a_1^{\alpha - 1}\right) \\
\times \left[(1 - t)|h'(a_1)| + t|h'(x)|\right] \, dt \\
+ \frac{a_2 - x}{a_2^\alpha - a_1^\alpha} \int_0^1 \left(\left((1 - t)a_2 + tx\right)^{\alpha - 1} - a_2^{\alpha - 1}\right) \\
\times \left[(1 - t)|h'(a_2)| + t|h'(x)|\right] \, dt \\
= \frac{x - a_1}{a_2^\alpha - a_1^\alpha} \Delta_1 + \frac{a_2 - x}{a_2^\alpha - a_1^\alpha} \Delta_2.
$$

Corollary 2.3 Let $x = (a_1 + a_2)/2$. Then Theorem 2.2 leads to

$$
|h\left(\frac{a_1 + a_2}{2}\right) - \frac{\alpha}{a_2^\alpha - a_1^\alpha} \int_{a_1}^{a_2} h(s) \, ds| \\
\leq \frac{a_2 - a_1}{2(a_2^\alpha - a_1^\alpha)} \left[\frac{2a_2^{-1}a_2 - 10a_1^{\alpha} + a_1 + a_2}{24}\right] |h'(a_1)| + \frac{a_1 + a_2}{12}\left(\frac{a_1 + a_2}{2}\right)^{\alpha - 1} |h'(a_1)| \\
+ \left(\frac{5a_1 + 3a_2 - 12a_1^{\alpha}}{24}\right) |h'\left(\frac{a_1 + a_2}{2}\right)| + \frac{a_1 + a_2}{12}\left(\frac{a_1 + a_2}{2}\right)^{\alpha - 1} |h'\left(\frac{a_1 + a_2}{2}\right)| \\
+ \frac{1}{6}a_2^{\alpha} |h'(a_2)| + \frac{1}{6}\left(\frac{a_1 + a_2}{2}\right)^{\alpha} |h'(a_2)| \\
+ \frac{a_2^\alpha}{3} |h'\left(\frac{a_1 + a_2}{2}\right)| - \frac{1}{3}\left(\frac{a_1 + a_2}{2}\right)^{\alpha} |h'\left(\frac{a_1 + a_2}{2}\right)|.
$$

Remark 2.4 If $\alpha = 1$, then Corollary 2.3 becomes

$$
|h\left(\frac{a_1 + a_2}{2}\right) - \frac{\alpha}{a_2^\alpha - a_1^\alpha} \int_{a_1}^{a_2} h(s) \, ds| \\
\leq \frac{a_2 - a_1}{24} \left(|h'(a_1)| + 4|h'\left(\frac{a_1 + a_2}{2}\right)| + |h'(a_2)|\right) \\
\leq \frac{a_2 - a_1}{8} \left(|h'(a_1)| + |h'(a_2)|\right),
$$

where the second inequality is obtained by using the convexity of $|h'|$. 
Theorem 2.5 Let $q > 1$, $M > 0$, $0 < \alpha \leq 1$, $0 \leq a_1 < a_2$, $h : [a_1, a_2] \to \mathbb{R}$ be an $\alpha$-fractional differentiable function and $D_\alpha(h) \in L_0^1([a_1, a_2])$. Then the inequality

$$
\left|h(x) - \frac{\alpha}{a_2^\alpha - a_1^\alpha} \int_{a_1}^{a_2} h(s) \, ds\right|
\leq M \frac{x - a_1}{a_2^\alpha - a_1^\alpha} (A_1(\alpha))^{1-1/q} (A_2(\alpha) + A_3(\alpha))^{1/q}
+ M \frac{a_2 - x}{a_2^\alpha - a_1^\alpha} (B_1(\alpha))^{1-1/q} (B_2(\alpha) + B_3(\alpha))^{1/q}
$$

holds if $|h'|^q$ is convex on $[a_1, a_2]$ and $|h'(x)|^q \leq M$, where

$$
A_1(\alpha) = \frac{\alpha}{(\alpha + 1)(x - a_1)} - a_1^\alpha,
B_1(\alpha) = a_2^\alpha - \frac{\alpha}{(\alpha + 1)(a_2 - x)} - a_1^\alpha,
$$

$$
A_2(\alpha) = \frac{\alpha}{(\alpha + 1)(x - a_1)} (2 + \frac{a_2 - a_1}{a_2 - x}) + a_1^\alpha + \frac{\alpha}{(\alpha + 1)(a_2 - x)^2} - \frac{a_1^\alpha}{2},
B_2(\alpha) = \frac{a_2^\alpha}{2} + \frac{\alpha}{(\alpha + 1)(a_2 - x)} \frac{(\alpha + 2)(x - a_1)}{a_2 - x} + \frac{\alpha}{(\alpha + 1)(a_2 - x)^2} + \frac{a_1^\alpha}{2},
$$

$$
A_3(\alpha) = \frac{\alpha}{(\alpha + 1)(x - a_1)} (2 + \frac{a_2 - a_1}{a_2 - x}) - x + \frac{\alpha}{(\alpha + 1)(a_2 - x)^2} - \frac{a_1^\alpha}{2},
B_3(\alpha) = \frac{a_2^\alpha}{2} - \frac{\alpha}{(\alpha + 1)(a_2 - x)} \frac{(\alpha + 2)(x - a_1) - x}{a_2 - x} - \frac{a_1^\alpha}{2},
$$

Proof From Lemma 2.1, power-mean inequality and the convexity of $|h'|^q$ together with the identities

$$
\int_0^1 ((1 - t)a_1 + tx)^\alpha - a_1^\alpha \, dt = \frac{\alpha}{(\alpha + 1)(x - a_1)} - a_1^\alpha
$$

and

$$
\int_0^1 a_2^\alpha - ((1 - t)a_2 + tx)^\alpha \, dt = a_2^\alpha - \frac{\alpha}{(\alpha + 1)(a_2 - x)}
$$

we clearly see that

$$
\left|h(x) - \frac{\alpha}{a_2^\alpha - a_1^\alpha} \int_{a_1}^{a_2} h(s) \, ds\right|
\leq \frac{x - a_1}{a_2^\alpha - a_1^\alpha} \int_0^1 ((1 - t)a_1 + tx)^\alpha - a_1^\alpha \, dt |h'|((1 - t)a_1 + tx) \, dt
+ \frac{a_2 - x}{a_2^\alpha - a_1^\alpha} \int_0^1 (a_2^\alpha - ((1 - t)a_2 + tx)^\alpha) |h'|((1 - t)a_2 + tx) \, dt,
$$

$$
\int_0^1 ((1 - t)a_1 + tx)^\alpha - a_1^\alpha \, dt |h'|((1 - t)a_1 + tx) \, dt \leq \left(\int_0^1 ((1 - t)a_1 + tx)^\alpha - a_1^\alpha \, dt\right)^{1-1/q}
\times \left(\int_0^1 ((1 - t)a_1 + tx)^\alpha - a_1^\alpha \, dt \, |h'|((1 - t)a_1 + tx)^q \, dt\right)^{1/q},
$$
Therefore, Theorem 2.5 follows easily from (2.1)–(2.5).

\[ \int_0^1 (a_2^\alpha - ((1-t)a_2 + tx)^\alpha) \left| h'((1-t)a_2 + tx) \right| dt \]

\[ \leq \left( \int_0^1 (a_2^\alpha - ((1-t)a_2 + tx)^\alpha) dt \right)^{1-1/q} \]

\[ \times \left( \int_0^1 (a_2^\alpha - ((1-t)a_2 + tx)^\alpha) \left| h'((1-t)a_2 + tx) \right|^q dt \right)^{1/q}, \]

\[ \int_0^1 \left( ((1-t)a_1 + tx)^\alpha - a_1^\alpha \right) \left| h'((1-t)a_1 + tx) \right|^q dt \]

\[ \leq \int_0^1 \left( ((1-t)a_1 + tx)^\alpha - a_1^\alpha \right) \left[ (1-t)\left| h'(a_1) \right|^q + t\left| h'(x) \right|^q \right] dt \]

\[ = \left| h'(a) \right|^q \int_0^1 \left( ((1-t)a_1 + tx)^\alpha - a_1^\alpha \right) (1-t) dt + \left| h'(x) \right|^q \int_0^1 \left( ((1-t)a_1 + tx)^\alpha - a_1^\alpha \right) t dt \]

\[ \leq \int_0^1 \left( a_2^\alpha - ((1-t)a_2 + tx)^\alpha \right) \left[ (1-t)\left| h'(a_2) \right|^q + t\left| h'(x) \right|^q \right] dt \]

\[ = \left| h'(a_2) \right|^q \int_0^1 \left( a_2^\alpha - ((1-t)a_2 + tx)^\alpha \right) (1-t) dt \]

\[ + \left| h'(x) \right|^q \int_0^1 \left( a_2^\alpha - ((1-t)a_2 + tx)^\alpha \right) t dt \]

\[ \leq M \left( \frac{a_2^\alpha}{2} + \frac{a_2^\alpha}{2} \right) \frac{(\alpha + 2)(a_2 - x) + a_2}{(\alpha + 2)(a_2 - x) - x} \]

\[ + M \left( \frac{a_2^\alpha}{2} \right) \frac{(\alpha + 2)(a_2 - x) + a_2}{(\alpha + 2)(a_2 - x) - x} \]

\[ \leq M \left( \frac{a_2^\alpha}{2} + \frac{a_2^\alpha}{2} \right) \frac{(\alpha + 2)(a_2 - x) + a_2}{(\alpha + 2)(a_2 - x) - x} \]

\[ + M \left( \frac{a_2^\alpha}{2} \right) \frac{(\alpha + 2)(a_2 - x) + a_2}{(\alpha + 2)(a_2 - x) - x} \]

Therefore, Theorem 2.5 follows easily from (2.1)–(2.5). \[ \square \]

**Remark 2.6** Let \( \alpha = 1 \). Then Theorem 2.5 leads to

\[ \left| h(x) - \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} h(s) ds \right| \leq M \frac{x-a_1}{a_2 - a_1} (A_1(1))^{1-1/q} [A_2(1) + A_3(1)]^{1/q} \]

\[ + M \frac{a_2 - x}{a_2 - a_1} (B_1(1))^{1-1/q} [B_2(1) + B_3(1)]^{1/q}, \]
where

\[
A_1(1) = \frac{x - a_1}{2}, \quad B_1(1) = \frac{a_2 - x}{2},
\]

\[
A_2(1) = \frac{3a_1^2 x + 6a_1^2 + x^3 - 3a_1 x^2 - 3a_1^3}{6(x - a_1)^2}, \quad B_2(1) = \frac{7a_2^3 + 3a_2 x^2 - 9a_2 x - x^3}{6(a_2 - x)^2},
\]

\[
A_3(1) = \frac{2x^3 - 2a_1^3 - 6a_1 x^2 + 6a_1^2 x}{6(x - a_1)^2}, \quad B_3(1) = \frac{2a_2^3 - 6a_2 x + 2x^3}{6(a_2 - x)^2}.
\]

**Theorem 2.7** Let \( q > 1, M > 0, 0 < \alpha \leq 1, 0 \leq a_1 < a_2, h : [a_1, a_2] \to \mathbb{R} \) be an \( \alpha \)-fractional differentiable function and \( D_{\alpha}(h) \in L^q([a_1, a_2]) \). Then the inequality

\[
\left| h(x) - \frac{\alpha}{a_2^\alpha - a_1^\alpha} \int_{a_1}^{a_2} h(s) \, ds \right| \\
\leq M \frac{x - a_1}{a_2^\alpha - a_1^\alpha} (A_1(\alpha))^{1-1/q} \left( \frac{-8a_1^\alpha + 2a_1^{\alpha-1} x + 2x^{\alpha-1} a_1 + 4x^\alpha}{12} \right)^{1/q} \\
+ M \frac{a_2 - x}{a_2^\alpha - a_1^\alpha} (B_1(\alpha))^{1-1/q} \left( \frac{a_2^\alpha - x^\alpha}{2} \right)^{1/q}
\]

holds if \( |h'|^q \) is convex on \([a_1, a_2]\) and \( |h'(x)|^q \leq M \), where

\[
A_1(\alpha) = \frac{2a_1^\alpha + a_1^{\alpha-1} x + x^{\alpha-1} a_1 + 2x^\alpha - 6a_1^\alpha}{6}, \quad B_1(\alpha) = \frac{a_2^\alpha - x^\alpha}{2}.
\]

**Proof** It follows from the proof of Theorem 2.2 that

\[
\left| h(x) - \frac{\alpha}{a_2^\alpha - a_1^\alpha} \int_{a_1}^{a_2} h(s) \, ds \right| \tag{2.6}
\]

\[
\leq \frac{x - a_1}{a_2^\alpha - a_1^\alpha} \int_0^1 \left( (1-t) a_1^{\alpha-1} + tx^{\alpha-1} \right) \left( \left( (1-t)a_1 + tx \right) - a_1^\alpha \right) |h'|((1-t)a_1 + tx) \, dt \\
+ \frac{a_2 - x}{a_2^\alpha - a_1^\alpha} \int_0^1 \left( a_2^\alpha - ((1-t)a_2^\alpha + tx^\alpha) \right) |h'|((1-t)a_2 + tx) \, dt.
\]

From the power-mean inequality and convexity of \(|h'|^q\) together with the identities

\[
\int_0^1 ((1-t)a_1^{\alpha-1} + tx^{\alpha-1}) \left( \left( (1-t)a_1 + tx \right) - a_1^\alpha \right) \, dt = \frac{2a_1^\alpha + a_1^{\alpha-1} x + x^{\alpha-1} a_1 + 2x^\alpha - 6a_1^\alpha}{6}
\]

and

\[
\int_0^1 \left( a_2^\alpha - ((1-t)a_2^\alpha + tx^\alpha) \right) \, dt = \frac{a_2^\alpha - x^\alpha}{2}
\]

we get

\[
\int_0^1 ((1-t)a_1^{\alpha-1} + tx^{\alpha-1}) \left( \left( (1-t)a_1 + tx \right) - a_1^\alpha \right) |h'|((1-t)a_1 + tx) \, dt \tag{2.7}
\]

\[
\leq \left( \int_0^1 ((1-t)a_1^{\alpha-1} + tx^{\alpha-1}) \left( (1-t)a_1 + tx \right) - a_1^\alpha \right) dt \right)^{1-1/q}
\]

\[
\times \left( \int_0^1 ((1-t)a_1^{\alpha-1} + tx^{\alpha-1}) ((1-t)a_1 + tx) - a_1^\alpha \right) |h'(1-t)a_1 + tx|^{\alpha} \, dt \right)^{1/q},
\]
\[\int_0^1 \left| a_2^q - ((1-t)a_2^q + tx^a) \right| h'((1-t)a_2 + tx) \, dt \tag{2.8}\]
\[
\leq \left( \int_0^1 \left| a_2^q - ((1-t)a_2^q + tx^a) \right| dt \right)^{1-1/q} \\
\times \left( \int_0^1 \left| a_2^q - ((1-t)a_2^q + tx^a) \right| h'((1-t)a_2 + tx)^q \, dt \right)^{1/q},
\]
\[
\int_0^1 \left| (1-t)a_1^{-1} + tx^{a-1} \right| ((1-t)a_1 + tx - a_1^{-1}) h'((1-t)a_1 + tx)^q \, dt \tag{2.9}\]
\[
\leq \int_0^1 \left| (1-t)a_1^{-1} + tx^{a-1} \right| ((1-t)a_1 + tx - a_1^{-1}) ((1-t)a_1 + tx - a_1^{-1}) (1-t) dt
\]
\[
+ |h'(x)|^q \int_0^1 ((1-t)a_1^{-1} + tx^{a-1} - a_1^{-1}) (1-t) dt.
\]
\[
= |h'(a_1)|^q \left( \frac{1}{4} a_1^2 + \frac{1}{12} a_1^{-1} x + \frac{1}{12} x^{a-1} a_1 + \frac{1}{12} x^a - \frac{1}{2} a_1^a \right)
\]
\[
+ |h'(x)|^q \left( \frac{1}{12} a_1^2 + \frac{1}{12} a_1^{-1} x + \frac{1}{12} x^{a-1} a_1 + \frac{1}{4} x^a - \frac{1}{2} a_1^a \right)
\]
\[
\leq M^q \left( -8a_1^a + 2a_1^{-1} x + 2x^{a-1} a_1 + 4x^a \right) / 12
\]
\[
\int_0^1 \left| a_2^q - ((1-t)a_2^q + tx^a) \right| h'((1-t)a_2 + tx)^q \, dt \tag{2.10}\]
\[
\leq \int_0^1 \left| a_2^q - ((1-t)a_2^q + tx^a) \right| [(1-t)|h'(a_2)|^q + t|h'(x)|^q] \, dt
\]
\[
= |h'(a_2)|^q \left( \frac{a_2^a - x^a}{6} \right) + |h'(x)|^q \left( \frac{a_2^a - x^a}{3} \right)
\]
\[
\leq M^q \left( \frac{a_2^a - x^a}{2} \right).
\]

Therefore, Theorem 2.7 follows easily from (2.6)–(2.10). □

**Theorem 2.8** Let \( q > 1, 0 < \alpha \leq 1, 0 \leq a_1 < a_2, h : [a_1, a_2] \rightarrow \mathbb{R} \) be an \( \alpha \)-fractional differentiable function and \( D_\alpha(h) \in L^1([a_1, a_2]) \). Then the inequality

\[
\left| h(x) - \frac{\alpha}{a_2^a - a_1^a} \int_{a_1}^{a_2} h(s) d_\alpha s \right|
\]
\[
\leq \frac{x - a_1}{a_2^a - a_1^a} A_1(\alpha) \left| h' \left( \frac{C_1(\alpha)}{A_1(\alpha)} \right) \right| + \frac{a_2 - x}{a_2^a - a_1^a} B_1(\alpha) \left| h' \left( \frac{C_2(\alpha)}{B_1(\alpha)} \right) \right|
\]

holds if \( |h'|^q \) is concave on \([a_1, a_2]\), where

\[
A_1(\alpha) = \frac{x^a + a_1^a}{(\alpha + 1)(x - a_1)} - a_1^a,
\]
\[
B_1(\alpha) = \frac{x^a + a_2^a}{(\alpha + 1)(a_2 - x)}.
\]
\[ C_1(\alpha) = \frac{x^{\alpha+2} - a_1^{\alpha+2}}{(\alpha + 1)(x - a_1)} - \frac{x^{\alpha+3} + a_1^{\alpha+3}}{(\alpha + 1)(x - a_1)^2(\alpha + 2)} \]
\[ + \frac{a_1x}{(\alpha + 1)(x - a_1)^2(\alpha + 2)} \left( x^{\alpha+1} + a_1^{\alpha+1} \right) - a_1 (a_1 + x) \left( \frac{1}{2} \right), \]
\[ C_2(\alpha) = \frac{a_2 (a_2 + x)}{2} + \frac{a_2^{\alpha+3} + x^{\alpha+3}}{(\alpha + 1)(a_2 - x)^2(\alpha + 2)} \]
\[ - \frac{a_2x}{(\alpha + 1)(a_2 - x)^2(\alpha + 2)} \left( x^{\alpha+1} + a_2^{\alpha+1} \right) + \frac{a_2^{\alpha+2} - x^{\alpha+2}}{(\alpha + 1)(a_2 - x)}. \]

**Proof** It is well known that \(|h'|\) is concave due to \(|h'|^{\alpha}\) being concave. It follows from Lemma 2.1 that

\[ \left| h(x) - \frac{\alpha}{a_2^\alpha - a_1^\alpha} \int_{a_1}^{a_2} h(s) \, ds \right| \]
\[ \leq \frac{x - a_1}{a_2^\alpha - a_1^\alpha} \int_{a_1}^{a_2} \left( ((1 - t)a_1 + tx)^\alpha - a_1^\alpha \right) |h'((1 - t)a_1 + tx)| \, dt \]
\[ + \frac{a_2 - x}{a_2^\alpha - a_1^\alpha} \int_{a_1}^{a_2} \left( a_2^\alpha - ((1 - t)a_2 + tx)^\alpha \right) |h'((1 - t)a_2 + tx)| \, dt. \]

Making use of Jensen’s integral inequality, we have

\[ \int_{0}^{1} \left( ((1 - t)a_1 + tx)^\alpha - a_1^\alpha \right) |h'((1 - t)a_1 + tx)| \, dt \]
\[ \leq \int_{0}^{1} \left( ((1 - t)a_1 + tx)^\alpha - a_1^\alpha \right) |h'\left( \frac{\int_{0}^{1} \left( ((1 - t)a_1 + tx)^\alpha - a_1^\alpha \right) ((1 - t)a_1 + tx) \, dt}{\int_{0}^{1} \left( ((1 - t)a_1 + tx)^\alpha - a_1^\alpha \right) \, dt} \right)| \, dt \]
\[ = A_1(\alpha) \left| h' \left( \frac{C_1(\alpha)}{A_1(\alpha)} \right) \right|, \]
\[ \int_{0}^{1} \left( a_2^\alpha - ((1 - t)a_2 + tx)^\alpha \right) |h'((1 - t)a_2 + tx)| \, dt \]
\[ \leq \int_{0}^{1} \left( a_2^\alpha - ((1 - t)a_2 + tx)^\alpha \right) \left| h' \left( \frac{\int_{0}^{1} \left( a_2^\alpha - ((1 - t)a_2 + tx)^\alpha \right) ((1 - t)a_2 + tx) \, dt}{\int_{0}^{1} \left( a_2^\alpha - ((1 - t)a_2 + tx)^\alpha \right) \, dt} \right) \right| \, dt \]
\[ = B_1(\alpha) \left| h' \left( \frac{C_2(\alpha)}{B_1(\alpha)} \right) \right|, \]

where we have used the identities

\[ \int_{0}^{1} \left( ((1 - t)a_1 + tx)^\alpha - a_1^\alpha \right) \, dt = A_1(\alpha), \]
\[ \int_{0}^{1} \left( a_2^\alpha - ((1 - t)a_2 + tx)^\alpha \right) \, dt = B_1(\alpha), \]
\[ \int_{0}^{1} \left( ((1 - t)a_1 + tx)^\alpha - a_1^\alpha \right) ((1 - t)a_1 + tx) \, dt = C_1(\alpha), \]
\[ \int_{0}^{1} \left( a_2^\alpha - ((1 - t)x + ta_2)^\alpha \right) ((1 - t)a_2 + tx) \, dt = C_2(\alpha). \]
Remark 2.9 If $\alpha = 1$, then Theorem 2.8 becomes

$$\left| h(x) - \frac{\alpha}{a_2^\alpha - a_1^\alpha} \int_{a_1}^{a_2} h(s) \, ds \right|$$

$$\leq \left( x - a_1 \right)^2 \frac{2}{2(a_2 - a_1)} \left| h' \left( \frac{2x^4 - 5a_1x^3 + 3a_1^2x^2 + xa_1^3 - a_1^4}{3(x - a_1)} \right) \right|$$

$$+ \left( a_2 - x \right)^2 \frac{2}{2(a_2 - a_1)} \left| h' \left( \frac{4x^4 - a_2x^3 - 3x^2a_2^2 - 7a_2^3x + 7a_2^4}{3(a_2 - x)} \right) \right|.$$  

Theorem 2.10 Let $q > 1$, $0 < \alpha \leq 1$, $0 \leq a_1 < a_2$, $h : [a_1, a_2] \to \mathbb{R}$ be an $\alpha$-fractional differentiable function and $D_\alpha(h) \in L^1_a([a_1, a_2])$. Then the inequality

$$\left| h(x) - \frac{\alpha}{a_2^\alpha - a_1^\alpha} \int_{a_1}^{a_2} h(s) \, ds \right|$$

$$\leq \frac{x - a_1}{a_2^\alpha - a_1^\alpha} A_1(\alpha) \left| h' \left( \frac{C_1(\alpha)}{A_1(\alpha)} \right) \right| + \frac{a_2 - x}{a_2^\alpha - a_1^\alpha} B_1(\alpha) \left| h' \left( \frac{C_2(\alpha)}{B_1(\alpha)} \right) \right|$$

holds if $|h'|^q$ is concave on $[a_1, a_2]$, where

$$A_1(\alpha) = \frac{2a_1^\alpha + a_1^{\alpha-1}x + x^{\alpha-1}a_1 + 2x^\alpha - 6a_1^\alpha}{6}, \quad B_1(\alpha) = \frac{a_2^\alpha - x^\alpha}{2},$$

$$C_1(\alpha) = -3a_1^{\alpha+1} + x + x^{\alpha-1}a_1^{\alpha+1} + x^\alpha a_1 - 5xa_1^\alpha + x^2a_1^{\alpha-1} + xa_1 + 3x^{\alpha+1},$$

$$C_2(\alpha) = \frac{a_2^{\alpha+1} - x^\alpha a_2 + 2xa_2^\alpha - 2x^{\alpha+1}}{6}.$$  

Proof From the concavity of $|h'|^q$ we know that $|h'|$ is also concave, then from Lemma 2.1 we have

$$\left| h(x) - \frac{\alpha}{a_2^\alpha - a_1^\alpha} \int_{a_1}^{a_2} h(s) \, ds \right|$$

$$\leq \frac{x - a_1}{a_2^\alpha - a_1^\alpha} \int_0^1 ((1 - t)a_1^{\alpha-1} + tx^{\alpha-1}) \left| ((1 - t)a_1 + tx) - a_1^\alpha \right| \left| h' \left( (1 - t)a_1 + tx \right) \right| \, dt$$

$$+ \frac{a_2 - x}{a_2^\alpha - a_1^\alpha} \int_0^1 (a_2^{\alpha-1} - (1 - t)a_2^{\alpha-1} + tx^\alpha) \left| (1 - t)a_2 + tx \right| \left| h' \left( (1 - t)a_2 + tx \right) \right| \, dt.$$  

It follows from the Jensen integral inequality that

$$\int_0^1 ((1 - t)a_1^{\alpha-1} + tx^{\alpha-1}) \left| ((1 - t)a_1 + tx) - a_1^\alpha \right| \left| h' \left( (1 - t)a_1 + tx \right) \right| \, dt$$

$$\leq \int_0^1 ((1 - t)a_1^{\alpha-1} + tx^{\alpha-1}) \left| ((1 - t)a_1 + tx) - a_1^\alpha \right|$$

$$\times \left| h' \left( \frac{\int_0^1 ((1 - t)a_1^{\alpha-1} + tx^{\alpha-1}) \left| ((1 - t)a_1 + tx) - a_1^\alpha \right| \, dt}{\int_0^1 ((1 - t)a_1^{\alpha-1} + tx^{\alpha-1}) \, dt} \right) \right| \, dt$$

$$= A_1(\alpha) h' \left( \frac{C_1(\alpha)}{A_1(\alpha)} \right).$$
Theorem 3.1 Let \( r > 1 \) and \( \alpha \in (0, 1] \). Then the inequality

\[
|A'(a_1, a_2) - L_{(a_1, a_2)}'(a_1, a_2)|
\]

\[
\leq \frac{r(a_2 - a_1)}{2(a_2^\alpha - a_1^\alpha)} \left[ \frac{2a_1^{\alpha - 2}a_2^2 - 10a_1^\alpha a_2 + a_1^2 + a_2^2}{24} \right] a_1|^{-1} + \frac{a_1}{12} \left( \frac{a_1 + a_2}{2} \right)^{\alpha - 1}|a_1|^{-1}
\]

\[
+ \left( \frac{5a_1 + 3a_2 - 12a_1^\alpha}{24} \right) \left( \frac{a_1 + a_2}{2} \right) |a_1 + a_2|^{-1} + \frac{a_1}{12} \left( \frac{a_1 + a_2}{2} \right)^{\alpha - 1} \left( \frac{a_1 + a_2}{2} \right)^{-1}
\]

\[
+ \frac{1}{6} a_2^{\alpha - 1} \left( \frac{a_1 + a_2}{2} \right)^{\alpha - 1} - \frac{1}{6} \left( \frac{a_1 + a_2}{2} \right)^{\alpha} |a_2|^{-1} + \frac{a_2}{3} \left( \frac{a_1 + a_2}{2} \right)^{\alpha - 1} \left( \frac{a_1 + a_2}{2} \right)^{-1}
\]

\[
- \frac{1}{3} \left( \frac{a_1 + a_2}{2} \right)^{\alpha} \left( \frac{a_1 + a_2}{2} \right)^{\alpha - 1}
\]

holds for all \( a_1, a_2 > 0 \).
Theorem 3.2 Let \( r > 1 \) and \( \alpha \in (0, 1] \). Then the inequality

\[
\left| A^r(a_1, a_2) - L^r_{(\alpha, r)}(a_1, a_2) \right| \\
\leq \frac{(a_2 - a_1)}{2(a_2^r - a_1^r)} \left[ \frac{2a_1^{\alpha - 1} b - 10a_1^\alpha + a_1 + a_2}{24} \right] |a_1|^{-2} + \frac{a_1 + a_2}{12} \left( \frac{a_1 + a_2}{2} \right)^{\alpha - 1} |a_1|^{-2} \\
+ \frac{5a_1 + 3a_2 - 12a_1^\alpha}{24} \left( \frac{a_1 + a_2}{2} \right) |a_1|^{-2} + \frac{a_1 + a_2}{12} \left( \frac{a_1 + a_2}{2} \right)^{\alpha - 1} |a_1 + a_2|^{-2} \\
+ \frac{1}{6} a_2^\alpha |a_2|^{-2} - \frac{1}{6} \left( \frac{a_1 + a_2}{2} \right)^\alpha |a_2|^{-2} + \frac{a_2^\alpha}{3} \left( \frac{a_1 + a_2}{2} \right) |a_1 |^{-2} \\
- \frac{1}{3} \left( \frac{a_1 + a_2}{2} \right)^\alpha |a_1 + a_2|^{-2}
\]

holds for all \( a_1, a_2 > 0 \).

4 Results and discussion

There are many results devoted to the well-known Ostrowski inequality. This inequality has many applications in the area of numerical analysis. In this paper, we give results for Ostrowski inequality containing conformable fractional integrals and their applications for means. First, we prove an identity associated with the Ostrowski inequality for conformable fractional integrals. By using this identity and convexity of different classes of functions and some well-known inequalities, we obtain several results for the inequality. The inequalities derived here are also pointed out to correspond to some known results, being special cases. At the end, we also present applications for means. The presented idea may stimulate further research in the theory of conformable fractional integrals.

5 Conclusion

In this paper, we prove an identity associated with the Ostrowski inequality for conformable fractional integral, present several Ostrowski type inequalities involving the conformable fractional integrals, and provide the applications in bivariate means theory. The idea and results presented are novel and interesting.

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Competing interests

The authors declare that they have no competing interests.

Authors’ contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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