Comment on: “Relativistic quantum dynamics of a charged particle in cosmic string spacetime in the presence of magnetic field and scalar potential”. Eur. Phys. J. C (2012) 72:2051

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Abstract

We analyze the results of a paper on “Relativistic quantum dynamics of a charged particle in cosmic string spacetime in the presence of magnetic field and scalar potential”. We show that the authors did not obtain the spectrum of the eigenvalue equation but only one eigenvalue for a specific relationship between model parameters. In particular, the existence of allowed cyclotron frequencies conjectured by the authors is a mere artifact of the truncation condition used to obtain exact solutions to the radial eigenvalue equation.

1 Introduction

In a paper published in this journal Figueiredo Medeiros and Becerra de Mello [1] analyze the relativistic quantum motion of charged spin-0 and spin-$\frac{1}{2}$ particles in the presence of a uniform magnetic field and scalar potentials in the cosmic spacetime.

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string spacetime. They derive an eigenvalue equation for the radial coordinate and solve it exactly by means of the Frobenius method. This approach leads to a three-term recurrence relation that enables the authors to truncate the series and obtain eigenfunctions with polynomial factors. They claim to have obtained the energy spectrum of the model and the truncation condition requires that the cyclotron frequency or other model parameters depend on the quantum numbers. In this Comment we analyze the effect of the truncation condition used by the authors on the physical conclusions that they derive in their paper. In section 2 we apply the Frobenius method, derive a three-term recurrence relation for the coefficients and analyze the results obtained in this way. Finally, in section 3 we summarize the main results and draw conclusions.

2 The truncation method

It is not our purpose to discuss the validity of the models but the way in which the authors solve the eigenvalue equation. For this reason we do not show the main equations displayed in their paper and restrict ourselves to what we consider relevant. We just mention that the authors state that they choose natural units such that $\hbar = c = G = 1$. A rigorous way of deriving dimensionless equations, as well as the choice of natural units, is reviewed in a recent pedagogical paper where we criticize such an unclear way of introducing them [2].

Some of the authors’ eigenvalue equations are particular cases of

$$\hat{L} R = \hat{L} e^{-\gamma |l|} e^{-b \xi^2 - \xi^2} P(\xi), \quad P(\xi) = \sum_{j=0}^{\infty} c_j \xi^j,$$

where $\gamma$, $a$ and $b$ are real numbers and $\gamma$ depends on the rotational quantum number $m = 0, \pm 1, \pm 2, \ldots$. By means of the ansatz

$$R(\xi) = \xi^{|\gamma|} e^{-\frac{b \xi^2}{2} - \frac{\xi^2}{2}} P(\xi), \quad P(\xi) = \sum_{j=0}^{\infty} c_j \xi^j,$$

we obtain a three-term recurrence relation for the coefficients $c_j$:

$$c_{j+2} = \frac{b (2 \gamma + 2 j + 3) - 2 a}{2 (j + 2) (2 \gamma + j + 2)} c_{j+1} + \frac{4 (2 \gamma + 2 j - W + 2) - b^2}{4 (j + 2) (2 \gamma + j + 2)} c_j,$$
\( j = -1, 0, 1, \ldots, c_{-1} = 0, c_0 = 1. \) (3)

In order to obtain “a special kind of exact solutions representing bound states” the authors require the termination conditions

\[
W = W^{(n)}_m = 2(\gamma + n + 1) - \frac{b^2}{4}, \quad c_{n+1} = 0, n = 0, 1, \ldots. \quad (4)
\]

Clearly, under such conditions \( c_j = 0 \) for all \( j > n \) and \( P(\xi) = P^{(n)}_m \) reduces to a polynomial of degree \( n \). In this way, they obtain analytical expressions for the eigenvalues \( W^{(n)}_m \) and the radial eigenfunctions \( R^{(n)}_m(\xi) \). For the sake of clarity and generality we will use \( \gamma \) instead of \( m \) as an effective quantum number.

For example, when \( n = 0 \) we have

\[
a_{0,\gamma} = \frac{b(2\gamma + 1)}{2}, \quad \gamma = 2(\gamma + 1) - \frac{b^2}{4}. \quad (5)
\]

When \( n = 1 \) there are two solutions for \( a \)

\[
W^{(1)}_\gamma = 2(\gamma + 2) - \frac{b^2}{4}, \quad a^{(1)}_{1,\gamma} = \frac{2b(\gamma + 1) - \sqrt{b^2 + 8(2\gamma + 1)}}{2},
\]

\[
a^{(2)}_{1,\gamma} = \frac{2b(\gamma + 1) + \sqrt{b^2 + 8(2\gamma + 1)}}{2}, \quad (6)
\]

or, alternatively,

\[
b^{(1)}_{1,\gamma} = \frac{2 \left[ 2a(\gamma + 1) - \sqrt{a^2 + 2(2\gamma + 3)(2\gamma + 1)^2} \right]}{(2\gamma + 1)(2\gamma + 3)},
\]

\[
b^{(2)}_{1,\gamma} = \frac{2 \left[ 2a(\gamma + 1) + \sqrt{a^2 + 2(2\gamma + 3)(2\gamma + 1)^2} \right]}{(2\gamma + 1)(2\gamma + 3)}. \quad (7)
\]

When \( n = 2 \) we obtain a cubic equation for either \( a \) or \( b \), for example,

\[
W^{(2)}_\gamma = 2(\gamma + 3) - \frac{b^2}{4},
\]

\[
4a^3 - 6a^2b(2\gamma + 3) + a \left[ b^2(12\gamma^2 + 36\gamma + 23) - 16(4\gamma + 3) \right]
- b \left[ 2(2\gamma + 1) \left( b^2(2\gamma + 3)(2\gamma + 5) - 16(4\gamma + 7) \right) \right] = 0, \quad (8)
\]

from which we obtain either \( a_{2,\gamma}(b) \) or \( b_{2,\gamma}(a) \); for example, \( a^{(1)}_{2,\gamma}(b) \), \( a^{(2)}_{2,\gamma}(b) \), \( a^{(3)}_{2,\gamma}(b) \). In the general case we will have \( n + 1 \) curves of the form \( a^{(i)}_{n,\gamma}(b) \),...
\[ i = 1, 2, \ldots, n + 1, \text{ labelled in such a way that } a_{n, \gamma}^{(i)}(b) < a_{n, \gamma}^{(i+1)}(b). \] It can be proved that all these roots are real [3,4].

It is obvious to anybody familiar with conditionally solvable (or quasi-solvable) quantum-mechanical models [3,4] (and references therein) that the approach just described does not produce all the eigenvalues of the operator \( \hat{L} \) for a given set of values of \( \gamma, a \) and \( b \) but only those states with a polynomial factor \( P_\gamma^{(n)}(\xi) \). Each of the particular eigenvalues \( W_\gamma^{(n)} = 1, 2, \ldots \) corresponds to a set of particular curves \( a_{n, \gamma}^{(i)}(b) \) in the plane \( a - b \) of physical model parameters. On the other hand, it is obvious that the eigenvalue equation (1) supports an infinite set of eigenvalues \( W_\nu^{(a, b)} = 0, 1, 2, \ldots \) for each set of real values of \( a, b \) and \( \gamma \). The condition that determines these allowed values of \( W \) is that the corresponding radial eigenfunctions \( R(\xi) \) are square integrable

\[
\int_{0}^{\infty} |R(\xi)|^2 \xi \, d\xi < \infty,
\]

as shown in any textbook on quantum mechanics [5,6]. Notice that \( \nu \) is the actual radial quantum number (that labels the eigenvalues in increasing order of magnitude and the number of nodes of the corresponding radial eigenfunctions), whereas \( n \) is just a positive integer that labels some particular solutions with a polynomial factor \( P_\gamma^{(n)}(\xi) \). In other words: \( n \) is a fictitious quantum number given by the truncation condition (4). More precisely, \( W_\gamma^{(n)} \) is an eigenvalue of a given operator \( \hat{L}_{n, \gamma} \) whereas \( W_\gamma^{(n') \nu'} \) is an eigenvalue of a different linear operator \( \hat{L}_{n', \gamma'} \); for this reason one does not obtain the spectrum of a given quantum-mechanical system by means of the truncation condition (4). The situation is even worse if one takes into consideration that \( \hat{L}_{n, \gamma} \) actually means \( \hat{L}_{n, \gamma, i}, \)

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of $a$ and increasing functions of $b$

$$\frac{\partial W}{\partial a} = -\left\langle \frac{1}{\xi} \right\rangle, \quad \frac{\partial W}{\partial b} = \langle \xi \rangle.$$  

(10)

Therefore, for a given value of $b$ and sufficiently large values of $a$ we expect negative values of $W$ that the truncation condition fails to predict. It is not difficult to prove, from straightforward scaling [2], that

$$\lim_{a \to \infty} \frac{W_{\nu,\gamma}}{a^2} = -\frac{1}{(2\nu + 2\gamma + 1)^2},$$  

(11)

for any given value of $b$. What is more, from equation (10) we can conjecture that the pairs $[a_{n,\gamma}^{(i)}(b), W_{\gamma}^{(n)}]$, $i = 1, 2, \ldots, n+1$ are points on the curves $W_{\nu-1,\gamma}(a, b)$ for a given value of $b$.

The eigenvalue equation (1) cannot be solved exactly in the general case (contrary to what the authors appear to believe). In order to obtain sufficiently accurate eigenvalues of the operator $\hat{L}$ we resort to the reliable Rayleigh-Ritz variational method that is well known to yield increasingly accurate upper bounds to all the eigenvalues of the Schrödinger equation [7] (and references therein). For simplicity we choose the basis set of non-orthogonal functions $\{u_j(\xi) = \xi^{\gamma+j}e^{-\frac{\xi^2}{2}}, j = 0, 1, \ldots\}$. We test the accuracy of these results by means of the powerful Riccati-Padé method [8].

As a first example, we choose $n = 2, \gamma = 0$ and $b = 1$ so that $W_{0}^{(2)} = 5.75$ for the three models $[a_{0,0}^{(1)} = -1.940551663, b = 1], [a_{2,0}^{(2)} = 1.190016441, b = 1]$ and $[a_{2,0}^{(3)} = 5.250535221, b = 1]$. The first four eigenvalues for each of these models are

$$a_{2,0}^{(1)} \rightarrow \begin{cases} W_{0,0} = 5.750000000 \\ W_{1,0} = 9.894040660 \\ W_{2,0} = 14.06831985 \\ W_{3,0} = 18.24977457 \end{cases},$$

$$a_{2,0}^{(2)} \rightarrow \begin{cases} W_{0,0} = -0.1664353619 \\ W_{1,0} = 5.750000000 \\ W_{2,0} = 10.52307155 \\ W_{3,0} = 15.06421047 \end{cases}.$$
\[
\begin{align*}
W_{0,0} &= -27.32460313 \\
W_{1,0} &= -0.5108147276 \\
W_{2,0} &= 5.750000000 \\
W_{3,0} &= 10.90599171
\end{align*}
\]

We appreciate that the eigenvalue \(W^{(2)}_0 = 5.75\) coming from the truncation condition (4) is the lowest eigenvalue of the first model, the second lowest eigenvalue of the second model and the third lowest eigenvalue for the third model (in agreement with the conjecture put forward above). The truncation condition misses all the other eigenvalues for each of those models and for this reason it cannot provide the spectrum of the physical model for any set of values of \(\gamma\), \(a\) and \(b\), contrary to what is suggested by Figueiredo Medeiros and Becerra de Mello [1].

In the results shown above we have chosen model parameters on the curves \(a^{(i)}_{2,0}(b)\). In what follows we consider the case \(a = 2\), \(b = 1\) that does not belong to any curve \(a^{(i)}_{n,\gamma}\) (that is to say, it does not stem from the truncation condition). For this set of model parameters, the first five eigenvalues are \(W_{0,0} = -3.230518994\), \(W_{1,0} = 4.510929109\), \(W_{2,0} = 9.532275968\), \(W_{3,0} = 14.19728140\) and \(W_{4,0} = 18.70978427\). As said above: there are square-integrable solutions (actual bound states) for any set of real values of \(a\), \(b\) and \(\gamma\).

The obvious conclusion is that the dependence of the cyclotron frequency \(\omega\) or other parameters, like \(\eta_L\), on the quantum numbers conjectured by Figueiredo Medeiros and Becerra de Mello [1] is just an artifact of the truncation condition (4). Such claims are nonsensical from a physical point of view.

The red circles in figure [1] denote some of the eigenvalues \(W^{(n,i)}_0(a,1)\) given by the truncation condition (4) and the blue lines connect those corresponding to the actual eigenvalues \(W_{\nu,0}(a,1)\). Some eigenvalues calculated numerically by the methods mentioned above are marked by blue squares. The energy spectrum of a model given by a pair of values of \(a\) and \(b\) is determined by all the intersections between a vertical line and the blue ones. Such intersections meet at most one red circle (left green dashed line, for example). The right green
dashed line cuts the blue lines at some of the eigenvalues calculated numerically.

Figure 2 shows three potentials

\[ V(a, b, \xi) = -\frac{a}{\xi} + b\xi + \xi^2 \]

for \( \gamma = 0 \), \( a = 1 \) and \( b = b_n^{(i)} \). They are given by \( n = 0 \) and \( n = 1 \), with \( i = 1, 2 \). Three horizontal lines indicate the eigenvalues \( W_n^{(0)}, W_n^{(1,1)} \) and \( W_n^{(1,2)} \); their purpose being to make clearer that the eigenvalues \( W_{\gamma}^{(n,i)} \) correspond to different models \( V(a, b_n^{(i)}; \xi) \) and do not give the spectrum of a single model.

### 3 Conclusions

The authors make two basic, conceptual errors. The first one is to believe that the only possible bound states are those given by the truncation condition (4) that have polynomial factors \( P_n^{(i)}(\xi) \). We have argued above that there are square-integrable solutions for all real values of \( a \) and \( b \) and calculated some of them outside the curves \( a_n^{(i)}(b) \) associated to these polynomials. The second error, connected with the first one, is the assumption that the spectrum of the problem is given by the eigenvalues \( W_{\gamma}^{(n)} \) stemming from that truncation condition. It is clear that the truncation condition only provides one energy eigenvalue \( W_{\gamma}^{(n)} \) for a particular set of model parameters given by the curves \( a_n^{(i)} \) discussed above. For this reason, the supposedly necessary dependence of the model parameters on the quantum numbers does not have mathematical support. Such unphysical conclusions stem from an arbitrary truncation condition that only produces particular bound states with no special meaning. We have illustrated these points by means of some numerical calculations and two figures that, hopefully, are clear enough to disclose the misunderstanding about the meaning of the results for this conditionally solvable quantum-mechanical model.

### References

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Figure 1: Eigenvalues $W_{\nu,0}(a,1)$, $\nu = 0, 1, 2, 3, 4, 5$ obtained from the truncation condition (red circles) and numerically (blue squares)

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Figure 2: Potentials $V(a, b, \xi) = -a/\xi + b\xi + \xi^2$ for $\gamma = 0$, $a = 1$ and $n = 0, 1$. The horizontal lines are the corresponding energies $W^{(n,i)}_\gamma$. 