LINEAR CHORD DIAGRAMS ON TWO INTERVALS
J. E. ANDERSEN¹, R. C. PENNER¹,², C. M. REIDYS³, R. R. WANG³

¹ CENTER FOR THE QUANTUM GEOMETRY OF MODULI SPACES
AARHUS UNIVERSITY, DK-8000 AARHUS C, DENMARK
² DEPARTMENTS OF MATH AND PHYSICS
CALTECH, PASADENA, CA 91125 USA
³ CENTER FOR COMBINATORICS
LPMC-TJKLC, NANKAI UNIVERSITY, TIANJIN 300071, P.R. CHINA

Abstract. Consider all possible ways of attaching disjoint chords to two ordered and oriented
disjoint intervals so as to produce a connected graph. Taking the intervals to lie in the real axis
with the induced orientation and the chords to lie in the upper half plane canonically determines
a corresponding fatgraph which has some associated genus $g \geq 0$, and we consider the natural
generating function $C²_g(z) = \sum_{n \geq 0} c²_g(n) z^n$ for the number $c²_g(n)$ of distinct such chord diagrams
of fixed genus $g \geq 0$ with a given number $n \geq 0$ of chords. We prove here the surprising fact that
$C²_g(z) = z^{2g+1} R²_g(z)/(1 - 4z)^{3g+2}$ is a rational function, for $g \geq 0$, where the polynomial $R²_g(z)$
with degree at most $g$ has integer coefficients and satisfies $R²_g(1/4) \neq 0$. Earlier work had already
determined that the analogous generating function $C_g(z) = z^{2g} R_g(z)/(1 - 4z)^{3g-1}$ for chords
attached to a single interval is algebraic, for $g \geq 1$, where the polynomial $R_g(z)$ with degree at most
$g - 1$ has integer coefficients and satisfies $R_g(1/4) \neq 0$ in analogy to the generating function $C_0(z)$
for the Catalan numbers. The new results here on $C²_g(z)$ rely on this earlier work, and indeed, we
find that $R²_g(z) = R_{g+1}(z) - z \sum_{g_1=1}^g R_{g_1}(z) R_{g+1-g_1}(z)$, for $g \geq 1$.

Key words. linear chord diagrams, fatgraph, group character, generating function

AMS subject classifications. 05A15, 05E10

1. Introduction. Fix a collection of disjoint oriented intervals called “backbones” in a specified linear ordering and consider all possible ways of attaching a family of unoriented and unordered intervals called “chords” to the backbones by gluing their endpoints to pairwise distinct points in the backbone. These combinatorial structures occur in a number of instances in pure mathematics including finite type invariants of links [6, 14], the geometry of moduli spaces of flat connections on surfaces [4, 5], the representation theory of Lie algebras [10], and the mapping class groups [2]. They also arise in applied mathematics for codifying the pairings among nucleotides in a collection of RNA molecules [20], or more generally in any macromolecular complex [16], and in bioinformatics where they apparently represent the building blocks of grammars of folding algorithms of RNA complexes [1, 9, 12].

This paper is dedicated to enumerative problems associated with connected chord diagrams on two backbones as follows.

Drawing a chord diagram $G$ in the plane with its backbone segments lying in the real axis with the natural orientation and its chords in the upper half-plane determines cyclic orderings on the half-edges of the underlying graph incident on each vertex. This

E-mail addresses: andersen@imf.au.dk (J. E. Andersen), rpenner@imf.au.dk (R. C. Penner), duck@santafe.edu(C. M. Reidys), wangrui@cfc.nankai.edu.cn (R. R. Wang)

JEA and RCP supported by QGM (Center for Quantum Geometry of Moduli Spaces, funded by the Danish National Research Foundation).

CMR and RRW supported by the 973 Project, the PCSIRT Project of the Ministry of Education, the Ministry of Science and Technology, and the National Science Foundation of China.
defines a corresponding “fatgraph” $G$, to which is canonically associated a topological surface $F(G)$ (cf. §2.3) of some genus, cf. Figure 1.1.

Let $c^g_b(n)$ denote the number of distinct chord diagrams on $b \geq 1$ backbones with $n \geq 0$ chords of genus $g \geq 0$ with its natural generating function $C^g_b(z) = \sum_{n \geq 0} c^g_b(n) z^n$, setting in particular $c^g_b(n) = c^1_b(n)$ and likewise $C^g_b(z) = C^1_b(z)$ for convenience. In particular, the Catalan numbers $c^0_b(n)$, i.e., the number of triangulations of a polygon with $n + 2$ sides, enumerate linear chord diagrams of genus zero, and we have $C^0_b(z) = \frac{1}{1 - \sqrt{1 - 4z}}$ with

The one-backbone generating functions

$$C_g(z) = \frac{P_g(z)}{(1 - 4z)^{3g-4}},$$

were computed in [3], where $P_g(z)$ is a polynomial defined over the integers of degree at most $(3g - 1)$ that is divisible by $z^{2g}$ with $P_g(1/4) \neq 0$. In particular, $C_g(z)$ is algebraic over $\mathbb{C}(z)$ for all $g \geq 1$, just as is the Catalan generating function $C_0(z)$, and there are explicit expressions such as

$$c^1_1(n) = \frac{2^{n-2}(2n-1)!!}{3(n-2)!},$$
$$c^1_2(n) = \frac{2^{n-4}(5n-2)(2n-1)!!}{90(n-4)!},$$
$$c^1_3(n) = \frac{2^{n-6}(35n^2 - 77n + 12)(2n-1)!!}{5670(n-6)!},$$

for these “higher” Catalan numbers, where $c^1_g(n) = 0$, for $n < 2g$.

In Theorem 4.2 of this paper, we derive the two-backbone generating functions

$$C^2_g(z) = \frac{P^2_g(z)}{(1 - 4z)^{3g+2}},$$

where $P^2_g(z)$ is an integral polynomial of degree at most $(3g + 1)$ that is divisible by $z^{2g+1}$ with $P^2_g(1/4) > 0$. In particular, these generating functions are rational functions defined over the integers. In fact, these polynomials

$$P^2_g(z) = z^{-1}P_{g+1}(z) - \sum_{g_1=1}^g P_{g_1}(z)P_{g+1-g_1}(z)$$
are computable in terms of the previous ones, for example:

\[ P_0^2(z) = z, \]
\[ P_1^2(z) = z^3(20z + 21), \]
\[ P_2^2(z) = z^5(1696z^2 + 6096z + 1485), \]
\[ P_3^2(z) = z^7(330560z^3 + 2614896z^2 + 1954116z + 225225), \]
\[ P_4^2(z) = z^9(118652416z^4 + 1661701632z^3 + 2532145536z^2 + 851296320z + 59520825), \]
\[ P_5^2(z) = z^{11}(68602726400z^5 + 1495077259776z^4 + 385081696512z^3 + 2561320295136z^2 + 505213089300z + 24325703325). \]

The experimental fact that the coefficients of \( P_g^2 \) are positive, just as are those of \( P_g(z) \), leads us to speculate that they themselves solve an as yet unknown enumerative problem. Furthermore, explicit expressions for the number of two-backbone chord diagrams of fixed genus such as

\[ c_0^2(n) = n 4^{n-1}, \]
\[ c_1^2(n) = \frac{1}{12} (13n + 3) n (n - 1) (n - 2) 4^{n-3}, \]
\[ c_2^2(n) = \frac{1}{180} (445n^2 - 401n - 210) n (n - 1) (n - 2) (n - 3) (n - 4) 4^{n-6}, \]

can also be derived, cf. Corollary 4.3.

2. Background and Notation. We formulate the basic terminology for graphs and chord diagrams, establish notation and recall facts about symmetric groups, recall the fundamental concepts about fatgraphs, and finally recall and extend results from \( \textit{[3]} \) for application in subsequent sections.

2.1. Graphs and chord diagrams on several backbones. A graph \( G \) is a finite one-dimensional CW complex comprised of one-dimensional open edges \( E(G) \) and zero-dimensional vertices \( V(G) \). An edge of the first barycentric subdivision of \( G \) is called a half-edge. A half-edge of \( G \) is incident on \( v \in V(G) \) if \( v \) lies in its closure, and the valence of \( v \) is the number of incident half-edges.

Consider a collection \( \beta_1, \ldots, \beta_b \) of pairwise disjoint closed intervals with integer endpoints in the real axis \( \mathbb{R} \), where \( \beta_i \), lies to the left of \( \beta_{i+1} \), for \( i = 1, \ldots, b-1 \); thus, these backbone intervals \( \{\beta_i\}_1^b \) are oriented and ordered by the orientation on \( \mathbb{R} \). The backbone \( B = \mathbb{R} \cap \bigcup \{\beta_i\} \) itself is regarded as a graph with vertices given by the integer points \( \mathbb{Z} \cap B \) and edges determined by the unit intervals with integer endpoints that it contains. A chord diagram on the backbone \( B \) is a graph \( C \) containing \( B \) so that \( V(C) = V(B) \) where each vertex in \( B - \partial B \) has valence three and each vertex in \( \partial B \) has valence two; in particular, \( \# V(G) \) is even, where \( \# X \) denotes the cardinality of the set \( X \). Edges in \( E(B) \subseteq E(C) \) are called backbone edges, and edges in the complement \( E(C) - E(B) \) are called chords.

2.2. Permutations. The symmetric group \( S_{2n} \) of all permutations on \( 2n \) objects plays a central role in our calculations, and we establish here standard notation and recall fundamental tools. Let \( (i_1, i_2, \ldots, i_k) \) denote the cyclic permutation \( i_1 \mapsto i_2 \mapsto \cdots \mapsto i_k \mapsto i_1 \) on distinct objects \( i_1, \ldots, i_k \). We shall compose permutations \( \pi, \tau \) from right to left, so that \( \pi \circ \tau(i) = \pi(\tau(i)) \). Two cycles are disjoint if they have disjoint supports.
Let \([\pi]\) denote the conjugacy class of \(\pi \in S_{2n}\). Conjugacy classes in \(S_{2n}\) are identified with classes of partitions of \(\{1, \ldots, 2n\}\), where in the standard slightly abusive notation, \(\pi \in [\pi] = [1^{\pi_1} \cdots 2n^{\pi_{2n}}]\) signifies that \(\pi\) is comprised of \(\pi_k \geq 0\) many \(k\) cycles with pairwise disjoint supports, for \(k = 1, \ldots, 2n\). In particular, \(\sum_{k=1}^{2n} k\pi_k = 2n\), and the number of elements in the class \([\pi]\) is given by

\[
|\pi| = |[\pi]| = \frac{(2n)!}{\prod_{k=1}^{2n} (k\pi_k\pi_k)!}.
\]

We shall make use of the fact that the irreducible characters \(\chi_Y\) of \(S_{2n}\) are labeled by Young tableaux \(Y\) and shall slightly abuse notation writing \(\chi_Y([\pi]) = \chi_Y(\pi)\) for the value taken on either a permutation \(\pi\) or its conjugacy class \([\pi]\). In order to evaluate characters, we shall rely heavily on the Murnaghan-Nakayama rule

\[
\chi_Y((i_1, \ldots, i_m) \circ \sigma) = \sum_{Y_\mu \text{ so that } Y - Y_\mu \text{ is a rim hook of length } m} (-1)^{w(Y_\mu)} \chi_{Y_\mu}(\sigma),
\]

(2.1)

where \(w(Y_\mu)\) is one less than the number of rows in the rim hook \(Y - Y_\mu\). See a standard text such as [21] for further details.

2.3. Fatgraphs. If \(G\) is a graph, then a fattening of \(G\) is the specification of a collection of cyclic orderings, one such ordering on the half-edges incident on each vertex of \(G\). A graph \(G\) together with a fattening is called a fatgraph \(\mathcal{G}\).

The key point is that a fatgraph \(\mathcal{G}\) determines an oriented surface \(F(\mathcal{G})\) with boundary as follows. For each \(v \in V(G)\), take an oriented polygon \(P_v\) with \(2k\) sides, and choose a tree in \(P_v\) with a univalent vertex in every other side of \(P_v\) joined to a single \(k\)-valent vertex in the interior. Identify the half-edges incident on \(v\) with the edges of the tree in the natural way so that the induced counter-clockwise cyclic ordering on the boundary of \(P_v\) agrees with the fattening of \(G\) about \(v\). We construct the surface \(F(\mathcal{G})\) as the quotient of \(\cup_{v \in V(G)} P_v\) by identifying pairs of frontier edges of the polygons with an orientation-reversing homeomorphism if the corresponding half-edges lie in a common edge of \(G\). The oriented surface \(F(\mathcal{G})\) is connected if \(G\) is and has some associated genus \(g(\mathcal{G}) \geq 0\) and number \(r(\mathcal{G}) \geq 1\) of boundary components.

Furthermore, the trees in the various polygons combine to give a graph identified with \(G\) embedded in \(F(\mathcal{G})\), and we may thus regard \(G \subseteq F(\mathcal{G})\). By construction, \(G\) is a deformation retraction of \(F(\mathcal{G})\), hence their Euler characteristics agree, namely,

\[
\chi(G) = \#V(G) - \#E(G) = \chi(F(\mathcal{G})) = 2 - 2g(\mathcal{G}) - r(\mathcal{G}),
\]

where the last equality holds provided \(G\) is connected.

A fatgraph \(\mathcal{G}\) is uniquely determined by a pair of permutations on the half-edges of its underlying graph \(G\) as follows. Let \(v_k \geq 0\) denote the number of \(k\)-valent vertices, for each \(k \geq 1, \ldots, K\), where \(K\) is the maximum valence of vertices of \(G\), so that \(\sum_{k=1}^K k v_k = 2n\), and identify the set of half-edges of \(G\) with the set \(\{1, 2, \ldots, 2n\}\) once and for all. The orbits of the cyclic orderings in the fattening on \(G\) naturally give rise to disjoint cycles comprising a permutation \(\tau \in S_{2n}\) with \(\tau \in [1^{\pi_1} 2^{\pi_2} \cdots K^{\pi_K}]\).

The second permutation \(\iota \in [2^n]\) is the product over all edges of \(G\) of the two-cycle permuting the two half-edges it contains.

One important point is that the boundary components of \(F(\mathcal{G})\) are in bijectional correspondence with the cycles of \(\tau \circ \iota\), i.e., \(r(\mathcal{G})\) is the number of disjoint cycles comprising \(\tau \circ \iota\). Furthermore, isomorphism classes of fatgraphs with vertex valencies
Linear chord diagrams on two intervals

$(v_k)_{k=1}^K$ are in one-to-one correspondence with conjugacy classes of pairs $\tau, \iota \in S_{2n}$, where $\tau \in [1^2 2^2 \cdots K^{v_K}]$ and $\iota \in [2^n]$, cf. Proposition 2.1. Thus, fatgraphs are easily stored and manipulated on the computer, and various enumerative problems can be formulated in terms of Young tableaux.

See [17, 19] for more details on fatgraphs and [3, 8, 11, 13, 15, 18] for examples of fatgraph enumerative problems in terms of character theory for the symmetric groups.

### 2.4. Fatgraphs and chord diagrams.

A convenient graphical method to uniquely determine a fatgraph is the specification of a regular planar projection of a graph in 3-space, where the counter-clockwise cyclic orderings in the plane of projection determine the fattening and the crossings of edges in the plane of projection can be arbitrarily resolved into under/over crossings without affecting the resulting isomorphism class. A band about each edge can be added to a neighborhood of the vertex set in the plane of projection respecting orientations in order to give an explicit representation of the associated surface embedded in 3-space as on the top-right of Figure 1.1.

In particular, the standard planar representation of a chord diagram $C$ on two backbones positions the two backbone segments in the real axis respecting their ordering and orientation and places the chords as semi-circles in the upper half-plane as on the bottom right in Figure 1.1. This determines the canonical fattening $\hat{C}$ on $C$ as illustrated on the top right in the same figure. We may furthermore collapse each backbone segment to a distinct point and induce a fattening on the quotient in the natural way to produce a fatgraph $\mathcal{G}_C$ with two vertices and two distinguished half-edges, namely, the ones coming just after the locations of the collapsed backbones. Upon labeling the half-edges, $C$ determines a pair of permutations $\tau_c \in [c, (2n - c)]$ with its two disjoint cycles corresponding to the fattening of the two vertices of $\mathcal{G}_C$ and $\iota \in [2^n]$ corresponding to the edges of $\mathcal{G}_C$. Provided $C$ and hence $\mathcal{G}_C$ is connected, we may define

$$r(C) = r(\mathcal{G}_C) = \text{the number of disjoint cycles comprising } \tau_c \circ \iota,$$

$$g(C) = g(\mathcal{G}_C) = \frac{1}{2} (n - r(C)).$$

Conversely, the specification of $\tau_c \in [c, (2n - c)]$ and $\iota \in [2^n]$ uniquely determines a chord diagram on two backbones with $2n$ edges. In fact, the isomorphism class of the fatgraph clearly corresponds to the conjugacy class of the pair $\tau_c, \iota$.

For example, labeling the half-edges contained in chords from left to right on the chord diagram $C$ on the bottom-right in Figure 1.1 produces the fatgraph on the left of the figure with permutations given by $\tau_3 = (1, 3, 2)(4, 5, 6, 7, 8)$ and $\iota = (1, 3)(2, 5)(4, 8)(6, 7)$ with $r(C) = 4$ and $g(C) = 0$. 
Proposition 2.1. The following sets are in bijective correspondence:

\{\text{isomorphism classes of chord diagrams on two backbones with } n \text{ chords}\}
\approx \{\text{isomorphism classes of fatgraphs with two vertices and } n \text{ edges : each vertex has a distinguished incident half-edge}\}
\approx \{\text{conjugacy classes of pairs } \tau, \iota \in S_{2n} : \iota \in [2^n] \text{ and } \tau \in [c, (2n-c)] \text{ for some } 1 \leq c \leq 2n-1, \text{ with an ordered pair of elements, one from each of the two cycles of } \tau\}

Our first counting results will rely upon the bijection established here between chord diagrams and pairs of permutations.

2.5. The one backbone case. We collect here results from [3] which will be required in the sequel as well as extend from [3] an explicit computation we shall also need.

As a general notational point for any power series \(T(z) = \sum a_i z^i\), we shall write \([z^i]T(z) = a_i\) for the extraction of the coefficient \(a_i\) of \(z^i\).

As mentioned in the introduction, the generating function \(C_g(z) = \sum_{n \geq 0} c_g(n) z^n\) for \(g \geq 1\) satisfies

\[C_g(z) = P_g(z) (1 - 4z)^{-3g},\]

where the polynomial \(P_g(z)\) has integer coefficients, is divisible by \(z^{2g}\) and has degree at most \(3g - 1\). Indeed, \(C_g(z)\) is recursively computed using the ODE

\[z(1 - 4z) \frac{d}{dz} C_g(z) + (1 - 2z) C_g(z) = \Phi_{g-1}(z),\]

where

\[\Phi_{g-1}(z) = z^2 \left( 4z^3 \frac{d^3}{dz^3} C_{g-1}(z) + 24z^2 \frac{d^2}{dz^2} C_{g-1}(z) + 27z \frac{d}{dz} C_{g-1}(z) + 3C_{g-1}(z) \right),\]

with initial condition \(C_g(0) = 0\). This ODE in turn follows from the recursion

\[(n + 1) c_g(n) = 2(2n - 1) c_g(n - 1) + (2n - 1)(n - 1)(2n - 3) c_{g-1}(n - 2),\]

where \(c_g(n) = 0\) for \(2g > n\), which is derived from a fundamental identity first proved in [11]. Namely, the polynomials

\[P(n, N) = \sum_{\{g : 2g \leq n\}} c_g(n) N^{n+1-2g}\]

combine into the generating function

\[1 + 2 \sum_{n \geq 0} \frac{P(n, N)}{(2n-1)!!} z^{n+1} = \left( \frac{1 + z}{1 - z} \right)^N.\]

In addition to the explicit expressions for “higher” Catalan number, one also computes

\[[z^{2g}] P_g(z) = c_g(2g) = \frac{(4g)!}{4^g(2g + 1)!}.\]
In fact, it is also shown in [3] that \( P_g(1/4) \) is non-zero, but here, we shall require its exact numerical value for subsequent estimates:

**Lemma 2.2.** We have for \( g \geq 1 \),

\[
P_g(1/4) = \left( \frac{9}{4} \right)^g \frac{\Gamma(g - 1/6) \Gamma(g + 1/2) \Gamma(g + 1/6)}{6\pi^{3/2} \Gamma(g + 1)}.
\]  

(2.7)

**Proof.** We compute \( P_g(1/4) \) by induction on \( g \), and for the basis step \( g = 1 \),

\[
C_1(z) = \frac{z^2}{(1 - 4z)^3} \sqrt{1 - 4z} = \frac{z^2}{(1 - 4z)^3} (1 - 2zC_0(z))
\]

(2.8)

according to [3], whence \( P_1(1/4) = (1/4)^2 = 1/16 \).

The solution to the ODE (2.2) is given by

\[
C_{g+1}(z) = \left( \int_0^z \frac{\Phi_g(y)}{(1 - 4y)^2} dy + C \right) \frac{\sqrt{1 - 4z}}{z} = \left( \int_0^z \frac{Q_g(y)}{(1 - 4y)^{3g+4}} dy + C \right) \frac{\sqrt{1 - 4z}}{z},
\]

(2.9)

where \( Q_g(z) \) is shown to be a polynomial of degree at most \( 3g + 2 \).

Insofar as \( C_g(z) = P_g(z)/(1 - 4z)^g \), we find

\[
\frac{dC_g(z)}{dz} = \frac{P_{1g}(z)}{(1 - 4z)^g}, \quad \frac{d^2C_g(z)}{dz^2} = \frac{P_{2g}(z)}{(1 - 4z)^g}, \quad \frac{d^3C_g(z)}{dz^3} = \frac{P_{3g}(z)}{(1 - 4z)^g}
\]

(2.10)

Thus,

\[
Q_g(z) = 4z^5P_{3g}(z) + 24z^4(1 - 4z)P_{2g}(z) + 27z^3(1 - 4z)^2P_{1g}(z) + 3z^2(1 - 4z)^3P_g(z),
\]

whence

\[
Q_g(1/4) = 4^{-4}P_{3g}(1/4) = 4^{-4}(12g + 6)(12g + 2)(12g - 2)P_g(1/4) \neq 0.
\]

(2.11)

Since \( Q_g(z) \) has degree at most \( 3g + 2 \), its partial fraction expansion reads

\[
\frac{Q_g(z)}{(1 - 4z)^{3g+4}} = \sum_{j=2}^{3g+4} \frac{A_j}{(1 - 4z)^j}.
\]

(2.12)

where the \( A_j \in \mathbb{Q} \) and \( A_{3g+4} = Q_g(1/4) \). According to [3], \( P_{g+1}(z) \) is given by

\[
P_{g+1}(z) = -\frac{1}{4z} \left( \sum_{j=2}^{3g+4} \frac{-A_j}{j-1} (1 - 4z)^{3g+4-j} + \sum_{j=2}^{3g+4} \frac{A_j}{j-1} (1 - 4z)^{3g+3} \right),
\]

(2.13)

and hence

\[
P_{g+1}(1/4) = 4^{-4}(12g + 6)(12g + 2)(12g - 2)P_g(1/4)/(3g + 3)
\]

\[
= \frac{9(g + 1/2)(g + 1/6)(g - 1/6)}{4(g + 1)} P_g(1/4),
\]

(2.14)

where \( P_1(1/4) = 1/16 \). The lemma follows by checking that the formula in eq. (2.7) is the unique solution of eq. (2.11) with \( P_1(1/4) = 1/16 \).
3. Lemmas on characters. Lemma 3.1. For any two permutations \( \tau, \pi \in S_{2n} \), we have

\[
\sum_{\sigma \in S_{2n}} \delta_{[\sigma],[2^n]} \cdot \delta_{[\tau \sigma],[\pi]} = \frac{(2n - 1)!!}{\prod_j j^{\pi_j} \cdot \pi_j!} \sum_Y \frac{\chi^Y([2^n]) \chi^Y(\pi) \chi^Y(\tau)}{\chi^Y([12n])}. \tag{3.1}
\]

Proof. In order to prove the lemma, we shall apply orthogonality and completeness of the irreducible characters of \( S_{2n} \), that irreducible representations are indexed by Young diagrams \( Y \) containing \( 2n \) squares, and the fact that \( \chi^Y(\sigma) = \chi^Y(\sigma^{-1}) \), for any \( \sigma \in S_{2n} \).

Fix some \( \pi \in S_{2n} \) with \( \pi \in [\pi_1 \cdot \cdots \cdot \pi_{2n}] \). According to the orthogonality relation of the second kind, we have

\[
\sum_{\sigma \in S_{2n}} \delta_{[\sigma],[2^n]} \cdot \delta_{[\tau \sigma],[\pi]} = \sum_{\sigma \in S_{2n}} \left[ \frac{[2^n]}{(2n)!} \sum_Y \chi^Y(\sigma) \chi^Y([2^n]) \right] \cdot \left[ \frac{[\pi]}{(2n)!} \sum_{Y'} \chi^{Y'}(\tau \sigma) \chi^{Y'}(\pi) \right] = \frac{(2n - 1)!!}{\prod_j j^{\pi_j} \cdot \pi_j!} \sum_Y \chi^Y([2^n]) \chi^Y(\tau) \left[ \frac{1}{(2n)!} \sum_{\sigma \in S_{2n}} \chi^Y(\sigma) \chi^Y(\tau \sigma) \right].
\]

The variant

\[
\frac{1}{(2n)!} \sum_{\sigma \in S_{2n}} \chi^Y(\sigma) \chi^Y(\tau \sigma) = \frac{\chi^Y(\tau)}{\chi^Y([12n])} \cdot \delta_{Y,Y'},
\]

of the orthogonality relation of the first kind gives

\[
\sum_{\sigma \in S_{2n}} \delta_{[\sigma],[2^n]} \cdot \delta_{[\tau \sigma],[\pi]} = \frac{(2n - 1)!!}{\prod_j j^{\pi_j} \cdot \pi_j!} \sum_{Y,Y'} \chi^Y([2^n]) \chi^Y(\tau) \left[ \frac{\chi^{Y'}(\tau)}{\chi^{Y'}([12n])} \cdot \delta_{Y,Y'} \right] = \frac{(2n - 1)!!}{\prod_j j^{\pi_j} \cdot \pi_j!} \sum_Y \chi^Y([2^n]) \chi^Y(\pi) \chi^Y(\tau)
\]

as required. \( \Box \)

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig3.1.png}
\caption{The Young tableaux \( Y_{p,q} \) and \( Y_{p_1,p_2}^{q_1,q_2} \).}
\end{figure}

In the course of our analysis, the two shapes of Young diagrams illustrated in Figure 3.1 are of importance, where
• \( p, q \geq 0 \) with \( p + q + 1 = 2n \) determine the shape \( Y_{p,q} \) which is \((p,q)\)-hook, having a single row of length \( q + 1 \geq 1 \) and \( p \) rows of length one with corresponding character \( \chi^{p,q} = \chi^{Y_{p,q}} \).

• \( p_1 \geq p_2 + 1 \geq 1, q_1 \geq q_2 + 1 \geq 1 \) with \( p_1 + q_1 + p_2 + q_2 + 2 = 2n \) determine the shape \( Y_{p_1,q_1}^{q_2,p_2} \) with one row of length \( q_1 + 1 \), one row of length \( q_2 + 2 \), \( p_2 \) rows of length two and \( p_1 - p_2 - 1 \) rows of length one.

**Lemma 3.2.** Suppose \( \tau_c = (1, \ldots, c) \circ (c + 1, \ldots, 2n) \). Then we have

\[
\sum_{c=1}^{2n-1} \chi^Y(\tau_c) = (-1)^{(p)(q-p)} \delta_{Y,Y_{p,q}}
\]

for any Young diagram \( Y \).

**Proof.** In order to prove eq. (3.2), let us first assume \( Y = Y_{p,q} \). Since the only rim-hooks in \( Y \) lie entirely in the first row or column as in Figure 3.2, the Murnaghan-Nakayama rule gives

\[
\sum_{c=1}^{2n-1} \chi^{p,q}(\tau_c) = \sum_{c=1}^{2n-1} \left( \chi^{p,q-c}((c + 1, \ldots, 2n)) + (-1)^{c-1} \chi^{p-c,q}((c + 1, \ldots, 2n)) \right)
\]

\[
= \sum_{c=1}^{q} (-1)^p \sum_{c=1}^{p} (-1)^{c-1} (-1)^{p-c}
\]

\[
= q(-1)^p + p(-1)^{p-1}
\]

\[
= (-1)^p(q - p).
\]

Next, let us assume \( Y \neq Y_{p,q} \). Since \( \tau_c \) is a product of two disjoint cycles, \( \chi^Y(\tau_c) \neq 0 \) implies that at most two removals of rim-hooks exhaust \( Y \) by the Murnaghan-Nakayama rule, so \( Y \) is necessarily of the form \( Y_{q_1,q_2}^{p_1,p_2} \). Since the cycles of \( \tau_c \) are disjoint, we may remove them in any desired order, and we choose to first remove a rim-hook of size \( c \) and second a rim-hook of size \( 2n - c \). There are exactly four scenarios for such removals as illustrated in Figure 3.3, it is worth mentioning that these scenarios can be distinguished by whether the first rim hook contains no squares in the leftmost column or the top row (a), contains squares in one but not both of
Fig. 3.3. The case of $Y_{q_1,q_2}$ for the character value $\chi_{Y_{q_1,q_2}}$: a rim-hook of size $c$ and a hook of size $2n-c$ has to be removed. We illustrate the four scenarios that arise when removing a rim-hook from $Y_{q_1,q_2}$ such that the remaining shape is a hook of length $2n-c$. We distinguish the case $p_2 > 0, q_2 > 0$ (top row), $p_1 = 1, p_2 = 0, q_2 > 0$ (the case $p_2 > 0, q_1 = 1, q_2 = 0$ being analogous) (middle row) and $p_1 = q_1 = 1, p_2 = q_2 = 0$ (bottom row).

them (b or c), or contains squares in both (d). We accordingly derive

$$\sum_{c=1}^{2n-1} \chi_{Y_{p_1,p_2}}(\tau_c) = \sum_{c=1}^{2n-1} \left[ (-1)^{p_2} \delta_{c,p_2+q_2+1} + (-1)^{p_1} \delta_{2n-c,p_1+q_1+1} + (-1)^{p_1-1} \delta_{c,(p_1+1)+q_2} + (-1)^{p_2} \delta_{2n-c,p_2+1+q_1} + (-1)^{p_2+1} \delta_{c,p_2+q_1+1} + (-1)^{p_1} \delta_{2n-c,p_1+1+q_2} + (-1)^{p_1} \delta_{c,(p_1+1)+q_1} + (-1)^{p_2} \delta_{2n-c,p_2+1+q_2} \right]$$

since the first/third and second/fourth terms pairwise cancel.

Thus, for any Young diagram $Y$ other than a $(p, q)$-hook the term $\sum_c \chi_{Y_{p_1,p_2}}(\tau_c)$ is trivial, and Lemma 3.2 follows.

Lemma 3.3. For any Young diagram $Y$ that contains $2n$ squares, we have

$$\frac{1}{(2n)!} \sum_{\pi \in S_{2n}} N^{\sum_i \pi_i} \chi^Y(\pi) = s_Y(1, \ldots, 1), \quad (3.3)$$

where $s_Y(x_1, \ldots, x_N)$ denotes the Schur-polynomial over $N \geq 2n$ indeterminants.

Proof. Rewriting

$$N^{\sum_i \pi_i} = \prod_{i=1}^N \left( \sum_{h=1}^N x_h \right)^{\pi_i},$$

as a product of power sums $p_i(x_1, \ldots, x_N) = \sum_{h=1}^N x_h^i$, we identify eq. (3.3) via the Frobenius Theorem as a particular value of the Schur-polynomial $s_Y(x_1, \ldots, x_N)$ of
Y over $N \geq 2n$ indeterminants:

$$\frac{1}{(2n)!} \sum_{\pi \in S_{2n}} \prod_{\substack{h=1 \atop h \neq i}}^{N} \pi_i \chi^Y(\pi) = \frac{1}{(2n)!} \sum_{\pi \in S_{2n}} \chi^Y(\pi) \prod_{\pi_i} \mu_1(1, \ldots, 1) = s_Y(1, \ldots, 1).$$

(3.4)

4. The generating function. In analogy to the polynomials $P(n, x)$ in eq. (2.5), we define

$$Q(n, x) = \sum_{\{g | 2g \leq n\}} c_g^{[2]}(n) \cdot x^{n-2g}.$$  

(4.1)

**Lemma 4.1.** The polynomial $Q(n, N)$ can be written as

$$Q(n, N) = U(n, N) - V(n, N)$$

(4.2)

where

$$-3 - 4Nz - 2N^2z^2 + 2 \sum_{n \geq 1} \frac{U(n, N)}{(2n-1)!!} z^{n+2} = (z + z^3) \frac{d}{dz} \left( \left( \frac{1 + z}{1-z} \right)^N \right) - 3 \left( \frac{1 + z}{1-z} \right)^N$$

(4.3)

and

$$V(n, N) = \sum_{1 \leq c \leq n-1} P(c, N)P(n-c, N).$$

(4.4)

**Proof.** A connected chord diagram of genus $g$ on two backbones with $n$ chords is described by permutations $\tau_c = (1, \ldots, c) \circ (c+1, \ldots, 2n)$ and $\iota \in [2^n]$ via Proposition 2.1 satisfies $2 - 2g - r = 2 - n$, where the number $r = n - 2g$ of boundary components is given by the number of cycles of $\tau_c \circ \iota$.

On the other hand, if the chord diagram corresponding to $\tau_c$ and $\iota$ is disconnected, then not only does $\tau_c$ decompose into disjoint cycles $\tau_1 = (1, \ldots, c)$ and $\tau_2 = (c+1, \ldots, 2n)$ but also $\iota = \iota_1 \circ \iota_2$ similarly decomposes into disjoint permutations, and the number of boundary components is given by

$$\sum_{i} (\tau_c \circ \iota)_i = \sum_{s} (\tau_1 \circ \iota_1)_s + \sum_{t} (\tau_2 \circ \iota_2)_t,$$

(4.5)

where $(\pi)_i$ denotes the number of cycles of length $i$ in $\pi, \pi \in S_{2n}$.

We proceed by writing $Q(n, N) = U(n, N) - V(n, N)$ as the difference of two terms, the first being the contribution of all chord diagrams, irrespective of being connected, and the second being the contribution of all disconnected chord diagrams.
Thus,
\[
Q(n, N) = \sum_{\{g \mid 2g \leq n\}} c_g^{[2]}(n) N^{n-2g}
\]
\[
= \sum_{1 \leq c \leq 2n-1} \sum_{\tau_c = (1, \ldots, c+1, \ldots, 2n)} N^{\sum_i (\tau_i, i)} U(n, N)
\]
\[- \sum_{1 \leq d \leq n-1} \left( \sum_{\tau_1 \in [2d]} \sum_{\tau_2 = (2d+1, \ldots, 2n)} N^{\sum_i (\tau_1, i)} \right) \left( \sum_{\tau_2 \in [2n-d]} \sum_{\tau_3 = (2d+1, \ldots, 2n)} N^{\sum_i (\tau_2, i)} \right)
\]

since the number of vertices in a chord diagram is necessarily even. In view of
\[
P(d, N) = \sum_{\tau_1 \in [2d]} N^{\sum_i (\tau_1, i)} \quad \text{and} \quad P(n-d, N) = \sum_{\tau_2 \in [2n-d]} N^{\sum_i (\tau_2, i)}
\]
we conclude
\[
V(n, N) = \sum_{1 \leq d \leq n-1} P(d, N) P(n-d, N).
\]

Turning our attention now to \(U(n, N)\), we have
\[
U(n, N) = \sum_{c=1}^{2n-1} \sum_{\tau_c = (1, \ldots, c+1, \ldots, 2n)} N^{\sum_i (\tau_c, i)} = \sum_{[\pi]} \sum_{c=1}^{2n-1} \sum_{\tau_c \in [\pi]} 1. \tag{4.6}
\]

Expressing the right-hand side of eq. \(4.6\) via Kronecker delta-functions, we obtain a sum taken over all permutations
\[
\sum_{[\pi]} N^{\sum_i \pi_i} \sum_{c=1}^{2n-1} \sum_{\tau_c \in [\pi]} 1 = \sum_{[\pi]} N^{\sum_i \pi_i} \sum_{c=1}^{2n-1} \sum_{\sigma \in S_{2n}} \delta[\sigma, [2n]] \cdot \delta[\tau_c, \sigma] [[\pi]],
\]
and application of Lemma 3.2 gives
\[
U(n, N) = (2n-1)!! \cdot \sum_{c=1}^{2n-1} \sum_{Y} \frac{\chi^Y([2n]) \chi^Y(\tau_c)}{\chi^Y([12n])} \frac{1}{(2n)!} \sum_{\pi \in S_{2n}} N^{\sum_i \pi_i} \chi^Y(\pi). \tag{4.7}
\]

Interchanging the order of summations and using Lemma 3.3, we may rewrite this as
\[
U(n, N) = (2n-1)!! \sum_{Y} \chi^Y([2n]) \sum_{c=1}^{2n-1} \chi^Y(\tau_c) \left( \sum_{\pi \in S_{2n}} \chi^Y(\pi) \right) s_Y(1, \ldots, 1).
\]

Now, according to Lemma 3.2, we have \(\sum \chi^Y(\tau_c) = (-1)^p(q-p) \delta_{Y, Y_{pq}}\). This reduces the character sum to the consideration of characters \(\chi^{p,q}\) and Schur-polynomials \(s_{p,q} =\)
$s_{p,q}$ associated to the irreducible representations $Y_{p,q}$. The corresponding expressions computable using the Murnaghan-Nakayama rule, for instance, are given by

$$\chi^{p,q}([1^{2n}]) = \binom{2n-1}{q},$$  \hspace{1cm} (4.8)$$

$$\chi^{p,q}([2^{2n}]) = \begin{cases} 
(-1)^{\frac{s-1}{2}} \binom{s-1}{\frac{s}{2}} ; & \text{for } p \equiv 0 \mod 2, \\
(-1)^{\frac{s+1}{2}} \binom{s-1}{\frac{s-1}{2}} ; & \text{for } p \equiv 1 \mod 2,
\end{cases} \hspace{1cm} (4.9)$$

$$s_{p,q}(1, \ldots, 1) = \binom{N+q}{2n} \binom{2n-1}{q}. \hspace{1cm} (4.10)$$

Consequently, we compute

$$\frac{U(n,N)}{(2n-1)!!} = \sum_{j=0}^{n-1} (-1)^j \binom{n-1}{j} \left[ (2n-4j-1) \binom{N+2n-2j-1}{2n} 
+ (2n-4j-3) \binom{N+2n-2j-2}{2n} \right]$$

$$= \sum_{j=0}^{n-1} (-1)^j \binom{n-1}{j} \frac{1}{2\pi i} \oint \left( (2n-4j-1) \frac{(1+x)^{N+2n-2j-1}}{x^{N-2j}} \right) dx.$$  

Taking the summation into the integral, we obtain

$$= \frac{1}{2\pi i} \oint \frac{(1+x)^N}{x^N} \left( \sum_{j=0}^{n-1} (-1)^j \binom{n-1}{j} (2n-4j-1) x^{2j} (1+x)^{2n-2j-1} 
+ \sum_{j=0}^{n-1} (-1)^j \binom{n-1}{j} (2n-4j-3) x^{2j+1} (1+x)^{2n-2j-2} \right) dx$$

$$= \frac{1}{2\pi i} \oint \frac{(1+x)^N}{x^N} (1+2x)^{n-1} \left( (n-1) (1+2x)^2 + n \right) dx$$

using $1+2x = (1+x)^2 - x^2$. It remains to substitute $z = 1/(1+2x)$ and compute

$$\frac{2U(n,N)}{(2n-1)!!} = \frac{1}{2\pi i} \oint \left( \frac{1+z}{1-z} \right)^N \left( \frac{n-1}{z^{n+3}} + \frac{n}{z^{n+1}} \right) dz$$

$$= (n-1) [z^{n+2}] \left( \frac{1+z}{1-z} \right)^N + n [z^n] \left( \frac{1+z}{1-z} \right)^N$$

$$= [z^{n+1}] \frac{d}{dz} \left( \left( \frac{1+z}{1-z} \right)^N \right) + [z^{n-1}] \frac{d}{dz} \left( \left( \frac{1+z}{1-z} \right)^N \right) - 3 [z^{n+2}] \left( \frac{1+z}{1-z} \right)^N$$

so that

$$-3 - 4Nz - 2N^2 z^2 + 2 \sum_{n \geq 1} \frac{U(n,N)}{(2n-1)!!} z^{n+2} = (z + z^3) \frac{d}{dz} \left( \left( \frac{1+z}{1-z} \right)^N \right) - 3 \left( \frac{1+z}{1-z} \right)^N.$$
completing the proof. □

**Theorem 4.2.** For any \( g \geq 0 \), the generating function \( C_g^{[2]}(z) \) is a rational function with integer coefficients given by

\[
C_g^{[2]}(z) = \frac{P_g^{[2]}(z)}{(1 - 4z)^{3g+2}},
\]

where \( P_g^{[2]}(z) \) is an integral polynomials of degree at most \((3g + 1)\), \( P_g^{[2]}(1/4) > 0 \) and \([z^h] P_g^{[2]}(z) = 0\), for \( 0 \leq h \leq 2g \). Furthermore, we have

\[
P_g^{[2]}(z) = z^{-1} P_{g+1}(z) - \sum_{g_1=1}^{g} P_{g_1}(z) P_{g+1-g_1}(z), \quad (4.11)
\]

\[
[z^{2g+1}] P_g^{[2]}(z) = c_g^{[2]}(2g + 1) = c_{g+1}(2g + 2) = \frac{(4g + 4)!}{4^{g+1}(2g + 3)!}. \quad (4.12)
\]

and the coefficients of \( C_g^{[2]}(z) \) have the asymptotics

\[
[z^n] C_g^{[2]}(z) \sim \frac{P_g^{[2]}(1/4)}{\Gamma(3g + 2)} n^{3g+1} 4^n. \quad (4.13)
\]

**Proof.** Taking the coefficient of \( z^{n+2} \) in eq. (4.11), we find

\[
\frac{U(n, N)}{(2n - 1)!!} = [z^{n+2}](z + z^3) \sum_{n=0}^\infty (n + 1) \frac{P(n, N)(2n - 1)!!}{(2n - 1)!!} z^n - 3 \sum_{n=0}^\infty \frac{P(n, N)(2n - 1)!!}{(2n - 1)!!} z^{n+1})
\]

\[
= (n + 2) \frac{P(n + 1, N)(2n + 1)!!}{(2n + 1 - 1)!!} + n \frac{P(n - 1, N)(2n - 1)!!}{(2n - 1 - 1)!!} - 3 \frac{P(n + 1, N)(2n + 1)!!}{(2n + 1 - 1)!!},
\]

for any \( n \geq 1 \), whence

\[
(2n + 1) U(n, N) = (n - 1) P(n + 1, N) + n(2n + 1)(2n - 1) P(n - 1, N).
\]

Substituting \( U(n, N) = \sum_{2g \leq n} u_g(n) N^{n-2g} \) and \( P(n, N) = \sum_{2g \leq n} c_g(n) N^{n+1-2g} \), we obtain

\[
(2n + 1) \sum_{2g \leq n} u_g(n) N^{n-2g} = (n - 1) \sum_{2g \leq n+1} c_g(n + 1) N^{n+2-2g}
\]

\[
+ n(2n + 1)(2n - 1) \sum_{2g \leq n-1} c_g(n - 1) N^{n-2g},
\]

so

\[
(2n + 1) u_g(n) = (n - 1) c_{g+1}(n + 1) + n(2n + 1)(2n - 1) c_g(n - 1).
\]

Using \((n + 2) c_{g+1}(n + 1) = 2(2n + 1) c_{g+1}(n) + (2n + 1)n(2n - 1) c_g(n - 1)\) from the recursion eq. (2.3), we have

\[
u_g(n) = c_{g+1}(n + 1) - 2 c_{g+1}(n),
\]

or equivalently, setting \( U_g(z) = \sum_n u_g(n) z^n \), it follows that

\[
z U_g(z) = (1 - 2z) C_{g+1}(z). \quad (4.14)
\]
We next consider the term \( V(n, N) \) as a polynomial in \( N \):

\[
V(n, N) = \sum_{1 \leq d \leq n-1} \left( \sum_{g_1} c_{g_1}(d) N^{d+1-2g_1} \right) \left( \sum_{g_2} c_{g_2}(n-d) N^{(n-d)+1-2g_2} \right)
\]

\[
= \sum_{g \geq 0} \sum_{g_1=0}^{g} \sum_{1 \leq d \leq n-1} c_{g_1}(d) c_{g-g_1}(n-d) N^{n+2-2g},
\]

where for \( i = 1, 2 \), \( c_{g_i}(a) = 0 \) if \( 2g_i > a \). According to eq. (4.15) for fixed genus \( g \), the contribution of pairs of chord diagrams, each having one backbone, of genus \( g_1 \) and \( g_2 \) to \( N^{n-2g} \) is given by

\[
\sum_{g_1=0}^{g+1} \sum_{1 \leq d \leq n-1} c_{g_1}(d) c_{g+1-g_1}(n-d) = [z^n] \sum_{g_1=0}^{g+1} C_{g_1}^*(z) C_{g+1-g_1}(z),
\]

where

\[
C_{g_1}^*(z) = \begin{cases} 
C_0(z) - 1; & \text{for } g_1 = 0, \\
C_{g_1}(z); & \text{otherwise.}
\end{cases}
\]

This necessary modification stems from the fact that \( 1 \leq d \leq n-1 \) implies that the coefficient \( c_0(0) = 1 \) does not appear. Using eqs. (4.13) and (4.16), we obtain

\[
C_{g_1}^*(z) = C_g(z) = \frac{P_{g}(z)\sqrt{1 - 4z}}{(1 - 4z)^{3g_1}}
\]

so

\[
\frac{1 - 2z}{z} C_{g+1}(z) - 2C_{g}^*(z) C_{g+1}(z) = \left[ \frac{1 - 2z}{z} - 2\left( \frac{1 - \sqrt{1 - 4z}}{2z} - 1 \right) \right] C_{g+1}(z)
\]

\[
= \frac{P_{g+1}(z)}{z(1 - 4z)^{3g+2}},
\]

and hence the two non-rational terms conveniently cancel. We continue by computing

\[
C_{g_1}(z) C_{g+1-g_1}(z) = \frac{P_{g_1}(z)\sqrt{1 - 4z} P_{g+1-g_1}(z)\sqrt{1 - 4z}}{(1 - 4z)^{3g_1} (1 - 4z)^{3(g+1-g_1)}} = \frac{P_{g_1}(z) P_{g+1-g_1}(z)}{(1 - 4z)^{3g+2}}
\]

for \( g_1 \geq 1 \), so all other terms in the sum are also rational expressions.

Thus,

\[
C_{g_1}^{(2)}(z) = \frac{z^{-1} P_{g+1}(z)}{(1 - 4z)^{3g+2}} - \sum_{g_1=1}^{g} \frac{P_{g_1}(z) P_{g+1-g_1}(z)}{(1 - 4z)^{3g+2}}
\]

and hence

\[
P_g^{(2)}(z) = C_{g_1}^{(2)}(z)(1 - 4z)^{3g+2} = z^{-1} P_{g+1}(z) - \sum_{g_1=1}^{g} P_{g_1}(z) P_{g+1-g_1}(z)
\]

(4.19)
as was asserted. Since the $P_g(z)$ are polynomials of degree at most $(3g-1)$, it follows from eq. (4.19) that the degree of $P_g^{[2]}(z)$ is at most $3g + 1$. Moreover, it follows immediately from $P_g^{[2]}(z) = C_g^{[2]}(z)(1-4z)^{3g+2}$ that the polynomial $P_g^{[2]}(z)$ has integer coefficients.

We next show that $P_g^{[2]}(1/4) > 0$. According to Lemma 2.2, $P_g(1/4)P_{g+1-g_1}(1/4)$ is given by

$$
\frac{\Gamma (g_1 - \frac{1}{6}) \Gamma (g_1 + \frac{1}{6}) \Gamma (g_1 + \frac{5}{6}) \Gamma (g_1 + \frac{7}{6}) \Gamma (g_1 + \frac{9}{6})}{36 \pi^3 4^{g+1} 9^{-(g+1)} \Gamma (g_1 + 1) \Gamma (g_1 + 1)} \leq \frac{g^{g+1}}{36 \pi^3 4^{g+1}} \left( \Gamma \left( g_1 - \frac{1}{6} \right) \Gamma \left( g_1 + \frac{1}{6} \right) \Gamma \left( g_1 + \frac{5}{6} \right) \Gamma \left( g_1 + \frac{7}{6} \right) \right),
$$

where we use the identity $\Gamma (g_1 + 1/2) < \Gamma (g_1 + 1)$, which follows from $\Gamma(x+1) = x\Gamma(x)$. Furthermore

$$
Z_{g_1} = \Gamma \left( g_1 - \frac{1}{6} \right) \Gamma \left( g_1 + \frac{1}{6} \right) \Gamma \left( g_1 + \frac{5}{6} \right) \Gamma \left( g_1 + \frac{7}{6} \right) = \left( g_1 - \frac{7}{6} \right)_{g_1-1} \left( g_1 - \frac{5}{6} \right)_{g_1} \left( g_1 - \frac{1}{6} \right)_{g_1} \left( g_1 + \frac{1}{6} \right)_{g_1+1-g_1} \left( \Gamma \left( \frac{5}{6} \right) \Gamma \left( \frac{1}{6} \right) \right)^2,
$$

where $(x)_n = x(x-1) \cdots (x-n+1)$ denotes the Pochhammer symbol. It follows that $Z_{g_1} = Z_{g_1+1-g_1}$, for $1 \leq g_1 \leq g$, and furthermore,

$$
Z_{g_1} \leq Z_1 = (g - 7/6)_{g-1}(g - 5/6)_g (\Gamma(5/6)\Gamma(1/6))^2 / 6. \quad (4.20)
$$

Thus,

$$
\sum_{g_1=1}^{g} P_{g_1}(1/4)P_{g+1-g_1}(1/4) \leq \frac{g^{g+1}}{36 \pi^3 4^{g+1}} \cdot \frac{g}{6} \cdot (g - 7/6)_{g-1}(g - 5/6)_g (\Gamma(5/6)\Gamma(1/6))^2. \quad (4.21)
$$

We proceed by estimating

$$
4P_{g+1}(1/4) = \frac{g^{g+1}}{6 \pi^{3/2} 4^g} \cdot \frac{\Gamma (g + 5/6) \Gamma (g + 3/2) \Gamma (g + 7/6)}{\Gamma (g + 2)} \geq \frac{g^{g+1}}{6 \pi^{3/2} 4^g} \cdot \frac{\Gamma (g + 5/6) \Gamma (g + 7/6)}{g + 1},
$$

$$
= \frac{g^{g+1}}{6 \pi^{3/2} 4^g} \cdot \frac{(g - 1/6)_g (g + 1/6)_{g+1} \Gamma(5/6)\Gamma(1/6)}{g + 1},
$$

since $\Gamma (g + 3/2) \geq \Gamma (g + 1)$. Thus,

$$
P_g^{[2]}(1/4) = 4P_{g+1}(1/4) - \sum_{g_1=1}^{g} P_{g_1}(1/4)P_{g+1-g_1}(1/4)
$$

$$
\geq \frac{g^{g+1}}{6 \pi^{3/2} 4^g} \cdot \frac{\Gamma(5/6)\Gamma(1/6)(g-7/6)_{g-1}(g-5/6)_g}{g + 1} \times \left( \frac{4(g - 1/6)(g + 1/6)}{g + 1} - \frac{g \Gamma(5/6)\Gamma(1/6)}{36 \pi^{3/2}} \right).$$
Finally,
\[
\frac{4 (g - 1/6) (g + 1/6)}{g + 1} - \frac{g \Gamma(5/6) \Gamma(1/6)}{36 \pi^{3/2}} = \frac{4 g^2 - 1/9 - g^2/(18 \sqrt{\pi}) - g/(18 \sqrt{\pi})}{g + 1} > 0,
\]
(4.22)
for \( g \geq 1 \), so we indeed have \( P_g^2(1/4) > 0 \) as claimed.

To see that \([z^h]P_g^2(z) = 0\), for any \( 0 \leq h \leq 2g \), it follows from eq. (4.18) that
\[
[z^h]P_g^2(z) = [z^{h+1}]P_{g+1}(z) - [z^h] \sum_{g_1=1}^{g} P_{g_1}(z)P_{g+1-g_1}(z).
\]
(4.23)
Since \([z^h]P_{g_1} = 0\), for \( h < 2g_1 \) or \( h > 3g_1-1 \), we conclude from \([z^h] \sum_{g_1=1}^{g} P_{g_1}(z)P_{g+1-g_1}(z) = \sum_{g_1=1}^{g} \sum_{i=0}^{h} [z^i]P_{g_1}(z)[z^{h-i}]P_{g+1-g_1}(z)\) that \([z^i]P_{g_1}(z)[z^{h-i}]P_{g+1-g_1}(z) \neq 0\) implies \( i \geq 2g_1 \) and \( h - i \geq 2(g + 1 - g_1) \). Thus,
\[
[z^h] \sum_{g_1=1}^{g} P_{g_1}(z)P_{g+1-g_1}(z) = 0,
\]
(4.24)
for \( 0 \leq h \leq 2g + 1 \), and consequently,
\[
[z^h]P_g^2(z) = [z^{h+1}]P_{g+1}(z) = 0,
\]
(4.25)
for \( 0 \leq h \leq 2g \), as required.

It remains only to compute the coefficient of \( z^{2g+1} \) in \( P_g^2(z) \). To this end, we have
\[
[z^{2g+1}]P_g^2(z) = [z^{2g+2}]P_{g+1}(z) - [z^{2g+1}] \sum_{g_1=1}^{g} P_{g_1}(z)P_{g+1-g_1}(z) = [z^{2g+2}]P_{g+1}(z)
\]
(4.26)
in light of eq. (4.24). Since \([z^h]P_g^2(z) = 0\), for any \( 0 \leq h \leq 2g \), we conclude from \( P_g^2(z) = 4^g c_g^2(z)(1 - 4z)^{3g+2} \) that \([z^{2g+1}]P_g^2(z) = c_g^2(2g + 1)\), and hence using eq. (2.6)
\[
c_g^2(2g + 1) = [z^{2g+1}]P_g^2(z) = [z^{2g+2}]P_{g+1}(z) = c_{g+1}(2g+2) = \frac{(4g + 4)!}{4^{g+1}(2g + 3)!}.
\]
(4.27)
\[ \square \]

**Corollary 4.3.** We have the explicit expressions
\[
c_0^2(n) = n 4^{n-1},
\]
\[
c_1^2(n) = \frac{1}{12} (13 n + 3) n (n - 1) (n - 2) 4^{n-3},
\]
\[
c_2^2(n) = \frac{1}{180} (445 n^2 - 401 n - 210) n (n - 1) (n - 2) (n - 3) (n - 4) 4^{n-6}.
\]

**Proof.** Using the expression for \( P_1(z) \) in the Introduction, Theorem 4.2 gives
\[
C_0^2(z) = \frac{z^{-1} P_1(z)}{(1 - 4z)^2} = \frac{z}{(1 - 4z)^2},
\]
(4.28)
which immediately implies
\[
c^{[2]}_0(n) = [z^n] C^{[2]}_0(z) = [z^{n-1}] \left( \frac{1}{1-4z} \right)^2 = \sum_{i=0}^{n-1} 4^i 4^{n-1-i} = n 4^{n-1}. \quad (4.29)
\]

In order to derive the expression for \( c^{[2]}_1(n) \), we use both of the expressions \( P_1(z) = z^2 \) and \( P_2(z) = 21z^5 + 21z^4 \). Theorem 4.2 implies
\[
C^{[2]}_1(z) = \frac{z^{-1} P_2(z) - P_1(z)^2}{(1-4z)^5} = \frac{(20z+21)z^3}{(1-4z)^5}, \quad (4.30)
\]
and differentiation gives
\[
(52z^3 - 13z^2) \frac{d^2 C^{[2]}_1(z)}{dz^2} + (168z^2 + 36z) \frac{d C^{[2]}_1(z)}{dz} + (64z - 30) C^{[2]}_1(z) = 0, \quad (4.31)
\]
where \( \frac{d^2 C^{[2]}_1(z)}{dz^2} \mid_{z=0} = 126 \). Straightforward calculation shows that eq. (4.31) implies
\[
c^{[2]}_1(n+1) = \frac{52n^2 + 116n + 64}{13n^2 - 23n - 6} c^{[2]}_1(n), \quad (4.32)
\]
where \( c^{[2]}_1(i) = 0 \), for \( 0 \leq i \leq 2 \) and \( c^{[2]}_1(3) = 21 \). It remains to observe that \( c^{[2]}_1(n) = \frac{1}{12} (13n + 3) n (n-1) (n-2) 4^{n-3} \) satisfies this recursion together with its initial condition.

To compute \( c^{[2]}_2(n) \), we likewise employ \( P_3(z) = 11z^6(158 z^2 + 558 z + 135) \) and Theorem 4.2 to conclude
\[
C^{[2]}_2(z) = \frac{z^{-1} P_3(z) - 2P_1(z)P_2(z)}{(1-4z)^8} = \frac{(1696z^2 + 6096z + 1485)z^5}{(1-4z)^8}. \quad (4.33)
\]
This yields the ODE
\[
(1780z^4 - 445z^3) \frac{d^3 C^{[2]}_2(z)}{dz^3} + (9076z^4 + 2181z^2) \frac{d^2 C^{[2]}_2(z)}{dz^2} + (6808z^2 - 4020z) \frac{d C^{[2]}_2(z)}{dz} - (664z - 3180) C^{[2]}_2(z) = 0,
\]
where \( \frac{d^3 C^{[2]}_2(z)}{dz^3} \mid_{z=0} = 178200 \). Thus,
\[
c^{[2]}_2(n+1) = \frac{(1780n^3 + 3736n^2 + 1292n - 664)}{(445n^3 - 2181n^2 + 1394n + 840)} c^{[2]}_2(n), \quad (4.34)
\]
where \( c^{[2]}_2(i) = 0 \), for \( 0 \leq i \leq 4 \), with \( c^{[2]}_2(5) = 1485 \), and we conclude as before the verity of the asserted formula. \( \square \)

REFERENCES

[1] C. Alkan, E. Karakoc, J. Nadeau, S. Sahinalp and K. Zhang, RNA-RNA interaction prediction and antisense RNA target search, J. Comput. Biol., 13 (2006), pp. 267-282.
[2] J. E. Andersen, A. J. Bene, J.-B. Meilhan, R. C. Penner, Finite type invariants and fatgraphs, Adv. Math., 225 (2010), pp. 2117-2161.
Linear chord diagrams on two intervals

[3] J. E. Andersen, R. C. Penner, C. M. Reidys, and M. S. Waterman, *Enumeration of linear chord diagrams*, (2010), arXiv:1010.5614.

[4] J. E. Andersen, J. Mattes, N. Reshetikhin, *The Poisson Structure on the Moduli Space of Flat Connections and Chord Diagrams*, Topology, 35 (1996), pp. 1069-1083.

[5] J. E. Andersen, J. Mattes, N. Reshetikhin, *Quantization of the Algebra of Chord Diagrams*, Math. Proc. Camb. Phil. Soc., 124 (1998), pp. 451-467.

[6] D. Bar-Natan, *On the Vassiliev knot invariants*, Topology, 34 (1995), pp. 423-475.

[7] D. Bar-Natan, *Lie algebras and the four colour problem*, Combinatorica, 17 (1997), pp. 43-52.

[8] D. Bessis, C. Itzykson, J.-B. Zuber, *Quantum field theory techniques in graphical enumeration*, Adv. Appl. Math., 1 (1980), pp. 109-157.

[9] A. Busch, A. S. Richter, R. Backofen, *IntaRNA: efficient prediction of bacterial sRNA targets incorporating target site accessibility and seed regions*, Bioinformatics, 24 (2008), pp. 2849-2856.

[10] R. Campoamor-Stursberg and V. O. Manturov, *Invariant tensor formulas via chord diagrams*, J. Math. Sci., 108 (2004), pp. 3018-3029.

[11] J. Harer and D. Zagier, *The Euler characteristic of the moduli space of curves*, Invent. Math., 85 (1986), pp. 457-485.

[12] F. W. Huang, J. Qin, C. M. Reidys, P. F. Stadler, *Target prediction and a statistical sampling algorithm for RNA-RNA interaction*, Bioinformatics, 26 (2010), pp. 175-181.

[13] C. Itzykson and J.-B. Zuber, *Matrix integration and combinatorics of modular groups*, Comm. Math. Phys., 134 (1990), pp. 197-207.

[14] M. Kontsevich, *Vassiliev’s Knot Invariants*, Adv. Soviet Math., 16 (1993), pp. 137-150.

[15] R. J. Milgram and R. C. Penner, *Riemann’s moduli space and the symmetric groups*, Mapping class groups and moduli spaces of Riemann surfaces, eds. C.-F. Bödigheimer and R. M. Hain, AMS Contemporary Math, 150 (1993), pp. 247-290.

[16] Kiyoshi Nagai and Iain W. Mattaj, *RNA-protein interactions*, Frontiers in Molecular Biology, Oxford University Press, New York (1994).

[17] R. C. Penner, *Perturbative series and the moduli space of Riemann surfaces*, J. Diff. Geom., 27 (1988), pp. 35-53.

[18] R. C. Penner, *Weil-Petersson volumes*, J. Diff. Geom., 35 (1992), pp. 559-608.

[19] R. C. Penner, M. Knudsen, C. Wiuf, J. Andersen, *Fatgraph model of proteins*, Comm. Pure Appl. Math., 63 (2010), pp. 1249-1297.

[20] Fenix W.D. Huang, Jing Qin, C.M. Reidys and P.F. Stadler, *Partition function and base pairing probabilities for RNA-RNA interaction prediction*, Bioinformatics, 25 (2009), pp. 2646-2654.

[21] B. E. Sagan, *The Symmetric Group: Representations, Combinatorial Algorithms, and Symmetric Functions*, Springer-Verlag, New York, (2001).