Global optimization of factor models and dictionary learning using alternating minimization

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Abstract

Learning new representations in machine learning is often tackled using a factorization of the data. For many such problems, including sparse coding and matrix completion, learning these factorizations can be difficult, in terms of efficiency and to guarantee that the solution is a global minimum. Recently, a general class of objectives have been introduced, called induced regularized factor models (RFMs), which have an induced convex form that enables global optimization. Though attractive theoretically, this induced form is impractical, particularly for large or growing datasets. In this work, we investigate the use of a practical alternating minimization algorithms for induced RFMs, that ensure convergence to global optima. We characterize the stationary points of these models, and, using these insights, highlight practical choices for the objectives. We then provide theoretical and empirical evidence that alternating minimization, from a random initialization, converges to global minima for a large subclass of induced RFMs. Finally, we provide an extensive investigation into practical optimization choices for using alternating minimization for induced RFMs, for both batch and stochastic gradient descent.

Keywords: Matrix factorization, dictionary learning, sparse coding, alternating minimization, biconvex

1. Introduction

Regularized factor models (RFMs) are broadly used for unsupervised learning and representation learning, in the form of dimensionality reduction, sparse coding, dictionary learning and matrix completion, to name a few. The general form consists of factorizing an input into a dictionary and a representation (or basis), potentially with a nonlinear transfer: $X \approx f(DH)$ (Singh and Gordon, 2008; White 2014). For example, in sparse coding, the input data is represented by sparse coefficient in $H$, with the motivation that the approach mimics the representation in the cortex (Olshausen and Field, 1997). In addition to the generality of this formulation of representation learning, RFMs are amenable to incremental estimation and so to large datasets, because the smaller dictionary $D$ can be feasibly maintained incrementally, and sufficiently summarizes the solution. Stochastic gradient descent can be more reactive to non-stationary data, by more highly weighting recent samples, and can even be more effective than batch solutions (Bousquet and Bottou, 2008). Despite these advantages, incremental estimation of RFMs has been under-explored.

The main reason for the restriction is the difficulty in optimal estimation of RFMs, even in the batch setting. The main difficulty arises from the fact that the optimization is over bilinear parameters $Z = DH$, which interact to create a non-convex optimization. There has been work reformulating this non-convex objective into a convex objective, by performing the optimization directly on $Z$ instead of the factors, e.g., relaxed rank exponential family PCA (Bach et al. 2008; Zhang et al. 2011), multi-view learning (White et al. 2012), (semi-)supervised dictionary learning (Goldberg et al. 2010; Zhang et al. 2011), autoregressive moving average models (White et al. 2015). These approaches, however, are not amenable to incremental estimation because $Z$ grows with the size of the data. Recently, there has been quite a bit work illustrating
that alternating minimizations on the factors produces global models, including for a general setting including (rectified) linear neural networks (Haefele and Vidal, 2015; Kawaguchi 2016), low-rank matrix completion with least-squares losses (Mardani et al. 2013; Jain et al. 2013; Gunasekar et al. 2013), a thresholded sparse coding algorithm (Agarwal et al., 2014b), and semi-definite low-rank optimization, where the two factors are the same \( Z = DD^\top \) (d’Aspremont et al. 2004; Bach et al. 2008; Journé et al. 2010; Zhang et al., 2012; Mirzazadeh et al., 2015). These approaches, however, require specialized initialization strategies, typically require a rank-deficient (i.e., low rank) solution and otherwise include only a small subset of induced RFMs. Further, outside of incremental principal components analysis (Warmuth and Kuzmin, 2008; Feng et al., 2013) and partial least-squares (Arora et al., 2012), little is known about how to obtain optimal incremental RFM estimation and several approaches have settled for local solutions (Mairal et al., 2009a b, 2010). Nonetheless, intuitively because this class enables convex reformulations, even though they are not computable, it identifies induced RFMs as a promising class to study to obtain global solutions for representation learning.

In this work, we investigate estimation of RFMs by returning to the potentially sub-optimal biconvex optimization and demonstrate that we can still obtain global solutions. We make the following strong conjecture for induced RFMs, that leads to simple and effective global optimization algorithms.

**Alternating minimization for induced regularized factor models, with a full rank random initialization, converges to a globally optimal solution.**

The key contributions of this paper are to provide evidence for this conjecture and to develop effective optimization techniques based on alternating minimization. We first expand the set of induced RFMs, and propose several practical modifications to previous objectives, particularly to enable incremental estimation. We prove a novel result that a large subclass of induced RFMs satisfy the property that every local minimum is a global minimum, and further demonstrate that a subclass does not have degenerate saddle-points, providing compelling evidence that alternating minimization will successfully find local minima and global minima. The key novelty in our theoretical results is a new approach to characterizing the overcomplete setting, for full rank solutions \( D, H \), as opposed to the more studied rank-deficient setting. We further support the above conjecture empirically, in particular illustrating that small deviations to outside the class of induced RFMs no longer have the property that alternating minimization produces global solutions. With this justification for alternating minimization, we then develop algorithms that converge faster than vanilla implementations of alternating minimization, for both batch and stochastic gradient descent.2

In addition to providing evidence for this conjecture, the theoretical results in this work provide a new methodology for analyzing stationary points for general dictionary learning problems. With analysis conducted under this more general class of models, we automatically obtain novel results for specific settings of interest, including matrix completion, sparse coding and dictionary learning with elastic net regularization. The current assumptions require twice differentiability, which restricts the results to smooth versions of these problems. Nonetheless, it provides first steps towards a unified analysis that, with extensions to non-smooth functions, could address many of these problems of interest simultaneously.

In this paper, we first motivate the types of dictionary learning problems that can be specified as induced RFMs (Section 2) and then discuss how to generalize and improve these objectives (Section 3). Then we prove the main theoretical result (Section 4). We propose a practical batch optimization approach for induced RFMs, and, using this algorithm, provide empirical evidence for the conjecture (Section 5). We then propose several incremental algorithms to optimize induced RFMs, to make it feasible to learn these models for large datasets (Section 6). Finally, we summarize a large body of related work (Section 7) and conclude with a discussion on the novel outcomes as well as the current limitations of this work and important next steps.

## 2. Regularized factor models

In this section, we introduce induced RFMs and provide several examples of objectives that are within the class. We focus on examples that demonstrate how this optimization formalism can be used to obtain different

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1. This terminology is used for this work, but note that stochastic gradient descent is also called stochastic approximation, and batch gradient descent is called sample average approximation.
representation properties as well as more modern uses to help unify recently introduced models under the formalism. Suitable objectives for these models are summarized in Table 1 which include subspace dictionary learning, matrix completion, co-embedding, sparse coding and elastic-net dictionary learning. In Appendix A we include other RFMs, outside the class of induced RFMs, to give a more complete overview of the representational capabilities of the larger class. The goal of this section is to provide intuition for the uses and generality of induced RFMs.

We begin by defining the class of induced RFMs. Let $X \in \mathbb{R}^{n \times T}$ consist of $T$ samples of $n$ features. The goal is to learn a bilinear factorization $Z = DH$ where $D \in \mathbb{R}^{d \times k}$ and $H \in \mathbb{R}^{k \times T}$ for given $d, k$, given any convex loss $L : \mathbb{R}^{d \times T}$. In practice, this convex loss will typically be defined based on a transfer (or activation function) $f : \mathbb{R} \rightarrow \mathbb{R}$ and the corresponding matching loss $L_x(Z_{i:t}, X_{i:t})$ that is convex in the first argument (see [Helmbold et al., 1996]). This loss evaluates the difference between $f(Z_{i:t})$ and $X_{i:t}$, with overloaded definition that $f(Z)$ is $f$ applied to each entry of $Z$. For example, $f$ could be an identity function, resulting in no transformation, and $L_x(Z_{i:t}, X_{i:t}) = \|Z_{i:t} - X_{i:t}\|^2$ or $f$ could be the sigmoid function and $L_x(Z_{i:t}, X_{i:t})$ the cross-entropy. The loss $L(Z)$ could then be defined as $L(Z) = \sum_{t=1}^{T} L_x(Z_{i:t}, X_{i:t})$, or normalized by the number of instances as discussed in Section 3. The general optimization corresponds to

$$\min_{D \in \mathbb{R}^{d \times k}, H \in \mathbb{R}^{k \times T}} L(DH) + \alpha R_D(D) + \alpha R_H(H)$$

(1)

for convex regularizers $R_D : \mathbb{R}^{d \times k} \rightarrow \mathbb{R}$ and $R_H : \mathbb{R}^{k \times T} \rightarrow \mathbb{R}$ and regularization parameter $\alpha \geq 0$. For induced RFMs, these regularizers have a particular formed defined by column and row norm regularizers $\| \cdot \|_c$ and $\| \cdot \|_r$:

$$R_D(D) = \frac{1}{2} \sum_{i=1}^{k} \|D_{:,i}\|^2$$

(2)

$$R_H(H) = \frac{1}{2} \sum_{i=1}^{T} \|H_{i,:}\|^2$$

(3)

The optimization in (1) is not jointly convex in $D$ and $H$, making it more difficult to solve.

Though the optimization is not jointly convex in the factors, induced RFMs admit a convex reformulation (Bach et al., 2008). The convex reformulation involves directly learning the product $Z = DH$. Bach et al. (2008) proved that there exists an induced norm on $Z$

$$R_Z(Z) = \frac{1}{2} \min_{D \in \mathbb{R}^{d \times k}, H \in \mathbb{R}^{k \times T}} \sum_{i=1}^{k} \|D_{:,i}\|^2 + \sum_{j=1}^{T} \|H_{j,:}\|^2$$

(4)

as long as $k$ is large enough (potentially infinite). Since norms are convex, $L(Z) + \alpha R_Z(Z)$ is a convex optimization over $Z$. This is a surprisingly general result, since it is true for any column and row norms $\| \cdot \|_c$ and $\| \cdot \|_r$; however, the difficulty lies in obtaining an explicit form for $R_Z(\cdot)$. For a small class of norms, such an explicit form has been obtained; for most, however, there is no known form for this induced norm.

Below, we discuss examples of induced RFMs, including those that do not have a known form for the induced matrix norm $R_Z(\cdot)$ on $Z$. The purpose of this summary is to unify existing representation learning approaches under induced RFMs, as well as provide examples of how to encode properties on the learned representation $H$ using the relatively simple induced RFM formalism.

2.1 Subspace dictionary learning

A common goal in representation learning is to to obtain a lower-dimensional representation of input $X$. Learning such a low-dimensional representation using an induced RFM objective has typically been formu-

3. Typically $d = n$, such as for principal components analysis; however, in some setting, $d > n$ is useful, such as for autoregressive moving average models (White et al. 2015).
The red column in $H$ corresponds to a low-dimensional re-representation of the first sample (column) in $X$. The dot product of the re-representation $H$ with the dictionary $D$ provides the approximation $\hat{X} = DH$, where the dot product of the red column with $D$ provides the first sample (column) in $\hat{X}$ as the approximation to the first sample (column) in $X$.

Figure 1: Subspace dictionary learning, where the inner dimension $k < d$, results in dimensionality reduction. The red column in $H$ corresponds to a low-dimensional re-representation of the first sample (column) in $X$. The dot product of the re-representation $H$ with the dictionary $D$ provides the approximation $\hat{X} = DH$, where the dot product of the red column with $D$ provides the first sample (column) in $\hat{X}$ as the approximation to the first sample (column) in $X$.

The corresponding induced convex reformulation is

$$
\min_{D \in \mathbb{R}^{d \times k}, H \in \mathbb{R}^{k \times T}} L(DH) + \alpha \frac{1}{2} \|D\|_F^2 + \frac{\alpha}{2} \|H\|_F^2
$$

(5)

where the Frobenius norm is defined as $\|D\|_F^2 = \sum_{i=1}^{k} \|D_{:,i}\|_2^2$ and the trace norm (or nuclear norm) is defined as $\|Z\|_{tr} = \sum_{i=1}^{\min(d,T)} \sigma_i(Z)$. If $\alpha = 0$, $k$ is fixed to some desired rank and $L_x(Z_{:,i}, X_{:,i}) = \|Z_{:,i} - X_{:,i}\|_F^2$, the solution to this objective results in the same solution as principal components analysis (Xu et al., 2009). Here, instead of fixing $k$ to a smaller rank, the rank is implicitly restricted by the regularizers on $D$ and $H$.

To further understand why these chosen column and row norms result in this dimensionality reduction effect, consider the induced norm. The trace norm is known to be a tight convex relaxation of rank. To see why, consider the singular value decomposition of $D = U \Sigma V^T$. We get an equivalent solution with $\ell_2$-norm on $H$:

$$
\min_{D \in \mathbb{R}^{d \times k}, H \in \mathbb{R}^{k \times T}} L(DH) + \alpha \frac{1}{2} \|D\|_F^2 + \frac{\alpha}{2} \|H\|_F^2
$$

(6)

The $(2,1)$-block norm constitutes a group-sparse regularizer on $H$, where the sum of the $\ell_2$ norms encourages entire rows of $H$ to be zero (Argyriou et al., 2008; White, 2014). Once an entire row of $H$ is set to zero, the rank is reduced by one, and the implicit $k$ is actually one less.

These subspace dictionary learning objectives encompass a wide-range of dimensionality reduction approaches, including principal components analysis, canonical correlation analysis, partial least-squares and nonlinear dimensionality reduction approaches such as Isomap and non-linear embeddings (see Appendix A1, where each is defined by different losses $L$. Typically these models do not use regularization to obtain

4. Note that the summed form in (5) does not necessarily enforce zeroed rows of $D$ and $H$, though it does guarantee equivalently low-rank solutions. To see why, consider the singular value decomposition of $D = U \Sigma V^T$. We get an equivalent solution with $DV$ and $V^T H$, where $VV^T = I$ and so $DVV^T = DH$. The regularization values remain unchanged because the Frobenius norm is invariant under orthonormal matrices. The new solution $DV = U \Sigma$ does in fact only have $k$ non-zero columns, because $\Sigma$ only has $k$ non-zero entries on the diagonal. The optimization, however, has no preference to select the form of $D$ with or without $V$ and so may not prefer the solution with zeroed columns in $D$ and zeroed rows in $H$. 

4.
low-rank solutions, but rather fix \( k \) to be smaller than \( d \); however, we can obtain relaxed-rank versions of each of these models by additionally including these Frobenius-norm regularizers. This addition is sensible as the preference for lower-rank solutions is explicitly specified in the objective.

### 2.2 Matrix completion

The matrix completion problem, formulated as a low-rank completion problem (Candes and Recht, 2009), is an instance of subspace dictionary learning. The matrix completion problem is often used for collaborative filtering, where the goal is to infer rankings or information about a user using a small amount of labeled information from other users. For example, for Netflix ratings, the goal is to complete a matrix of ratings, with \( d \) users as rows and \( T \) movies as columns. The idea behind finding a low-rank \( \textbf{D} \) and \( \textbf{H} \) to factorize the known components of \( \textbf{X} \) is that there is some latent structure that explains the ratings.

A common objective for matrix completion is

\[
\sum_{\text{observed } (i,j)} (\textbf{X}_{ij} - \textbf{D}_i \cdot \textbf{H}_j)^2 + \alpha \|\textbf{D}\|_F^2 + \alpha \|\textbf{H}\|_F^2
\]

which is an instance of the subspace objective in (5). There are numerous variations on this basic objective to improve performance. For example, Salakhutdinov and Srebro (2010) use weighted norms for non-uniform sampling, giving

\[
\sum_{\text{observed } (i,j)} (\textbf{X}_{ij} - \textbf{D}_i \cdot \textbf{H}_j)^2 + \alpha \|\Lambda_{\textbf{D}}\textbf{D}\|_F^2 + \alpha \|\Lambda_{\textbf{H}}\textbf{H}\|_F^2
\]

where \( \Lambda_{\textbf{D}} \in \mathbb{R}^{d \times d} \) is a positive diagonal matrix that reweights rows of \( \textbf{D} \) and where \( \Lambda_{\textbf{H}} \in \mathbb{R}^{T \times T} \) is a positive diagonal matrix that reweights columns of \( \textbf{H} \). With a change of variables, this can equivalently be written as a minimization over

\[
\sum_{\text{observed } (i,j)} (\textbf{X}_{ij} - \Lambda_{\textbf{D}}(i,i)^{-1} \Lambda_{\textbf{H}}(j,j)^{-1} \textbf{D}_i \cdot \textbf{H}_j)^2 + \alpha \|\textbf{D}\|_F^2 + \alpha \|\textbf{H}\|_F^2
\]

where

\[
L(\textbf{DH}) = \sum_{\text{observed } (i,j)} (\textbf{X}_{ij} - \Lambda_{\textbf{D}}(i,i)^{-1} \Lambda_{\textbf{H}}(j,j)^{-1} (\textbf{DH})_{ij})^2
\]

\[
= \sum_{\text{observed } (i,j)} (\textbf{X}_{ij} - \Lambda_{\textbf{D}}(i,i)^{-1} \Lambda_{\textbf{H}}(j,j)^{-1} \textbf{D}_i \cdot \textbf{H}_j)^2.
\]

Many such matrix completion objectives similarly fit within the induced RFM formulation.

### 2.3 Co-embedding

Co-embedding enables the use of multi-modal data by embedding multiple modalities into a common Euclidean space. Examples of problems that have been cast as co-embedding include metric learning, link prediction, multi-label classification and multi-class classification (Mirzazadeh et al. 2014). For example, multi-label classification can be cast as a co-embedding problem by embedding the inputs and labels into a common Euclidean space.
The co-embedding optimization can be written as an induced RFM\(^5\), with symmetric factors, \(H = D^\top\):

\[
\min_{D \in \mathbb{R}^{d \times k}} L(DD^\top) + \alpha \frac{1}{2} \|D\|^2_F + \alpha \frac{1}{2} \|D^\top\|^2_F = \min_{D \in \mathbb{R}^{d \times k}} L(DD^\top) + \alpha \text{tr} (DD^\top) = \min_{Z \in \mathbb{R}^{d \times d}, Z \succeq 0} L(Z) + \alpha \|Z\|_{tr}
\]

When optimizing for \(D\), the resulting \(Z = DD^\top\) is guaranteed to be a positive semi-definite matrix. This optimization has been more generally explored as a low-rank optimization over the cone of positive semi-definite matrices (d’Aspremont et al., 2004; Bach et al., 2008; Journé et al., 2010), with some of the seminal results on incrementally generating columns in \(D\) to obtain global solutions.

2.4 Sparse coding

A sparse representation is typically obtained by using an \(\ell_1\) norm on \(H\); the corresponding induced RFM\(^6\) is (Bach et al., 2008; Zhang et al., 2011)

\[
\min_{D \in \mathbb{R}^{d \times k}, H \in \mathbb{R}^{k \times T}} L(DH) + \alpha \sum_{i=1}^{k} \|D_{:,i}\|_q + \frac{\alpha}{2} \sum_{i=1}^{k} \|H_{:,i}\|_1^2
\]

where \(\|Z^\top\|_{q,1} = \sum_i \|Z_{:,i}\|_q\) and \(q \geq 1\) (see (Zhang et al., 2011, Proposition 2)). The resulting global solution, however, requires \(k = T\) and corresponds to memorizing normalized observations: \(D_{:,i} = Z_{:,i}/\|Z_{:,i}\|_q\) and diagonal \(H_{i,i} = \|Z_{:,i}\|_q\). To remedy this issue with induced RFMs and sparse learning, Bach et al. (2008) proposed to combine the subspace and sparse regularizers, which we describe next.

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5. This connection is not exact, as the regularizer on \(H\) should include an indicator function for \(H = D^\top\). However, many of the definitions and results in this paper apply when \(H = D^\top\), and so we include co-embedding here to highlight the similarity.

6. This objective is different from sparse coding where a specified level of sparsity is given, for which alternating minimization has also been explored (Agarwal et al., 2014a).
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2.5 Elastic-net dictionary learning

We can interpolate between both subspace and sparse regularizers (also called elastic net regularization (Zou and Hastie, 2005)), using parameters \( \nu_D, \nu_H \in [0, 1] \)

\[
\min_{D \in \mathbb{R}^{d \times k}, H \in \mathbb{R}^{k \times T}} L(DH) + \frac{\nu}{2} \sum_{i=1}^{k} \left[ \nu_D \| D_{:,i} \|_2^2 + (1 - \nu_D) \| D_{:,i} \|_1^2 + \nu_H \| H_{:,i} \|_2^2 + (1 - \nu_H) \| H_{:,i} \|_1^2 \right]
\]

with elastic net norm, \( \| v \| = \sqrt{\nu \| v \|_2^2 + (1 - \nu) \| v \|_1^2} \). Because this elastic net norm constitutes a valid column and row norm (see Proposition 14 in Appendix B), we know that a corresponding induced norm on \( Z \) exists; however, there is no known efficiently computable form. We will address this regularizer extensively throughout this work, and provide further insight into its properties. In particular, as intended, we will find that it reduces the magnitude of the required \( k \) as compared to sparse coding, while still maintaining sparse solutions.

Though we write the above having the elastic net norm on both factors, this is not necessary. An elastic net norm could be used on one factor, and a completely different norm on another factor. We simply write the above to introduce notation, and because several common settings are obtained with different \( \nu_D \) and \( \nu_H \). For example, for \( \nu_D = 1 \), we have an \( \ell_2 \) regularizer on the columns of \( D \), and an elastic net norm on the rows of \( H \), which is the setting addressed by Bach et al. (2008).

2.6 Supervised dictionary learning

The objectives so far have focused on unsupervised learning; however, for many cases, the goal is to improve supervised learning, which can be elegantly incorporated under RFMs. A typical strategy is to learn a new representation in an unsupervised way, and then use that representation to learn a supervised predictor given labeled samples. These two stages can be combined into one objective, where the new factorized representation is learned for \( X \) while also using it for supervised learning, given some labels \( Y \in \mathbb{R}^{m \times T} \). For simplicity we will assume that we have labels corresponding to each sample (column) in \( X \); this can be relaxed to a semi-supervised setting by ignoring missing entries in the unsupervised loss component (Zhang et al., 2011).

\[ \text{Figure 3: Supervised dictionary learning, with a separate dictionary for the input} \ X \text{and the target} \ Y. \text{This example depicts} \ k < d, \text{giving dimensionality reduction, though other choices such as a higher-dimensional, sparse representation could have also been used.} \]
The supervised dictionary learning RFM is (Goldberg et al., 2010; Zhang et al., 2011; White et al., 2012) 

\[
\arg\min_{D \in \mathbb{R}^{d \times d}, H \in \mathbb{R}^{d \times T}} L \left( \left[ \begin{array}{c} D^{(1)} \\ D^{(2)} \end{array} \right], \left[ \begin{array}{c} X \\ Y \end{array} \right] \right) + \frac{\alpha}{2} \sum_{i=1}^{k} \max \left( \|D_{i}^{(1)}\|^2, \|D_{i}^{(2)}\|^2 \right) + \frac{\eta}{2} \sum_{i=1}^{k} \|H_{i}\|^2. \tag{7}
\]

\(D\) is now partitioned into two components \(D = \left[ D^{(1)} \mid D^{(2)} \right], \) with \(D^{(1)} \in \mathbb{R}^{d \times k}\) and \(D^{(2)} \in \mathbb{R}^{m \times k}\), but there is a shared representation \(H\). The regularizer uses the max, to ensure that each component is regularized separately. Note that \(\|D_{i}\|_c = \max(\|D_{i}^{(1)}\|^2, \|D_{i}^{(2)}\|^2)\) is a valid norm.

For supervised subspace dictionary learning, with the \(\ell_2\) norm, the induced norm on \(Z = DH = \left[ D^{(1)} \mid D^{(2)} \right] H\) has been derived and corresponds to (White et al., 2012):

\[
R_Z(Z) = \frac{1}{2} \min_{DH=Z} \sum_{i=1}^{k} \max \left( \|D_{i}^{(1)}\|_2, \|D_{i}^{(2)}\|_2 \right) + \|H\|^2_F
\]

\[
= \max_{0 \leq \theta \leq 1} \|A_{\theta}Z\|_{1,\theta} \quad \text{where} \quad A_{\theta} = \begin{bmatrix} \sqrt{\theta}I_d & 0 \\ 0 & \sqrt{1-\theta}I_m \end{bmatrix}.
\]

The rows of \(Z\) are reweighted in two blocks for the unsupervised and supervised components \(D^{(1)}\) and \(D^{(2)}\).

### 2.7 Robust objectives

All of the above objectives can accommodate robust alternatives. This includes using robust convex losses, as well as adding an additional optimization over an additional variable \(S\) that represents the noise (Candes et al., 2011; Xu et al., 2010; Zhang et al., 2011). The formulation remains equivalent, with a particular definition of the general convex loss. For sparse noise, for example, such as in robust PCA (Candes et al., 2011), we can impose an \(\ell_1\) regularizer on \(S\) and define the loss

\[
L(DH) = \min_{S \in \mathbb{R}^{d \times T}} \sum_{j=1}^{T} L_x(DH_{ij} + S_{ij}, X_{ij}) + \alpha_s \|S\|_{1,1}
\]

for some regularization parameter \(\alpha_s > 0\) that enforces the level of sparsity in the learned noise \(S\). This loss \(L(\cdot)\) is convex in \(Z = DH\), because the composition of a convex loss and an affine function on two variables (the sum of the two variables \(Z_{ij} + S_{ij}\)) is jointly convex in those two variables. For robust subspace dictionary learning, if the data is corrupted by sparse noise, then this loss can be used to remove the sparse noise and still learn the lower-dimensional latent structure.

We can see that a general set of representation learning problems can be specified as induced RFMs. In the remainder of the paper, we further generalize the specification of induced RFMs, provide a series of results to better understand the stationary points of these problems and demonstrate how to tractably optimize them using alternating minimization.

### 3. Exploring improved objectives

In this section, we propose some practically important modifications to the previously defined objectives and prove equivalence results, in terms of optima and stationary points. These results ameliorate issues with the objective scaling with samples, highlight better optimization choices and provide alternative but equivalent regularizers that are more suitable for out-of-sample prediction and incremental learning.

#### 3.1 Generalized induced form

An important restriction for previous induced RFMs is that the regularizers must be norms. Surprisingly, however, this restriction is in fact not necessary, and we can generalize the regularizers to any non-negative,
centered convex functions $f_c : \mathbb{R}^d \rightarrow \mathbb{R}^+$ and $f_r : \mathbb{R}^T \rightarrow \mathbb{R}^+$. Define

$$R_k(Z) = \min_{D \in \mathbb{R}^{d \times k}, H \in \mathbb{R}^{k \times T}, Z = DH} \sum_{i=1}^{k} \left( f_c^2(D_i) + f_r^2(H_i) \right)$$

(8)

**Theorem 1** Assume $f_c : \mathbb{R}^d \rightarrow \mathbb{R}^+$ and $f_r : \mathbb{R}^T \rightarrow \mathbb{R}^+$ are convex functions, that are non-negative and centered: $f_r(0) = 0 = f_c(0)$. For all $Z \in \mathbb{R}^{d \times T}$, the limit $R_k(Z) = \lim_{k \to \infty} R_k(Z)$ exists and $R_k$ is a convex function.

**Proof:** For a given $Z$, $R_k(Z)$ is non-negative because of the squares on the function $f_c$ and $f_r$. Consequently, it’s value cannot be pushed to negative $\infty$. Because it is a minimum of these non-negative functions, which can only have more flexibility with increasing $k$, $R_k(Z)$ is non-increasing with $k$. Therefore, it has a finite, non-negative limit as $k$ tends to infinity.

We now show $R_k$ is convex. Take any $Z_1, Z_2 \in \mathbb{R}^{d \times T}$, $\eta \in [0, 1]$ and any $\epsilon > 0$. For large enough $k$, there exists $\epsilon$-optimal decompositions $D^{(1)}, H^{(1)}$ and $D^{(2)}, H^{(2)}$ for $Z_1$ and $Z_2$ respectively

$$R_k(Z_1) \geq \sum_{i=1}^{k} \left( f_c^2(D_{1i}) + f_r^2(H_{1i}) \right) - \epsilon$$

$$R_k(Z_2) \geq \sum_{i=1}^{k} \left( f_c^2(D_{2i}) + f_r^2(H_{2i}) \right) - \epsilon$$

Then for $D = [\sqrt{\eta}D^{(1)} \; \sqrt{1-\eta}D^{(2)}]$ and $H = [\sqrt{\eta}H^{(1)} \; \sqrt{1-\eta}H^{(2)}],$

$$Z = \eta Z_1 + (1-\eta) Z_2 = DH.$$

If $f_c$ is non-negative, then for any $d \in \mathbb{R}^k$, because $f_c(0) = 0$,

$$f_c(\eta d) = f_c(\eta d + (1-\eta)0) \leq \eta f_c(d) + (1-\eta) f_c(0) = \eta f_c(d).$$

Because $f_c$ is non-negative, we can square both sides and maintain the inequality: $f_c^2(\eta d) \leq \eta^2 f_c^2(d)$. This is similarly true for $f_r$.

Now we get

$$R_k(Z) \leq \sum_{i=1}^{2k} \left( f_c^2(D_{1i}) + f_r^2(H_{1i}) \right)$$

$$= \sum_{i=1}^{k} \left( f_c^2(\sqrt{\eta}D_{1i}^{(1)}) + f_r^2(\sqrt{\eta}H_{1i}^{(1)}) \right) + \sum_{i=1}^{k} \left( f_c^2(\sqrt{1-\eta}D_{1i}^{(2)}) + f_r^2(\sqrt{1-\eta}H_{1i}^{(2)}) \right)$$

$$\leq \sum_{i=1}^{k} \left( \eta f_c^2(D_{1i}^{(1)}) + f_r^2(H_{1i}^{(1)}) \right) + \sum_{i=1}^{k} \left( (1-\eta) f_c^2(D_{1i}^{(2)}) + (1-\eta) f_r^2(H_{1i}^{(2)}) \right)$$

$$\leq \eta \sum_{i=1}^{k} \left( f_c^2(D_{1i}) + f_r^2(H_{1i}) \right) + (1-\eta) \sum_{i=1}^{k} \left( f_c^2(D_{1i}^{(2)}) + f_r^2(H_{1i}^{(2)}) \right)$$

$$\leq \eta R_k(Z_1) + (1-\eta) R_k(Z_2) + \epsilon$$

Letting $\epsilon$ go to zero gives the desired result that $R_k$ is convex.

This generalized form includes many regularizers within the class of induced RFMs that were previously not possible. Some examples include
1. smoothed approximations to $\ell_1$, that are no longer norms, such as the pseudo-Huber loss (Fountoulakis and Gondzio 2013): $f_\mu(d) = \sum_{i=1}^d \left( \sqrt{\mu^2 + d_i^2} - \mu \right)$ for some $\mu > 0$. The pseudo-Huber loss is twice differentiable and approaches $\ell_1$ as $\mu \to 0$ (see (Fountoulakis and Gondzio, 2013 Figure 1)).

2. the sum of the squares of any non-negative centered convex functions $g_i$, with $f_i^2 = g_i^2 + g_i^2 + \ldots + g_i^2$. This is allowed because $f_i(d) = \sqrt{g_i^2(d)} + \ldots + g_i^2(d)$ is convex (see Proposition 14 in Appendix B).

3. the smoothed elastic net norm, where the $\ell_1$ component is replaced by the pseudo-Huber loss (see Proposition 14 in Appendix B).

4. regularizers that act on partitions of the column of $D$ or row of $H$ (see Corollary 15 in Appendix B). For example, for supervised learning, $D^{(1)}$ and $D^{(2)}$ could now have different regularizers. This is appropriate as they serve different purpose, one for unsupervised recovery and the other for supervised learning.

This generalization is particularly important for the proof that alternating minimization provides optimal solutions, as we will need twice differentiable regularizers. The generalization beyond norms, to any convex function, enables the use of smoothed versions of non-smooth regularizers.

The generalization to any non-negative convex function significantly expands the space of potential regularizers; however, many regularizers are not designed to then also be squared. Future work will investigate how squaring a given non-negative convex function affects the properties intended to be encoded by that regularizer. We provide one insight into the equivalence of stationary results for a squared versus non-squared form, in Proposition 4 for incremental estimation.

### 3.2 Scaling with samples

An important oversight in the specification of these previous objectives has been an explicit normalization by the number of samples. The magnitude of the regularizer on $H$ grows with samples (since $H \in \mathbb{R}^{k \times T}$ grows with samples), whereas the regularizer on $D \in \mathbb{R}^{d \times k}$ does not. As previously specified, however, these regularizers are equally weighted by $\alpha$; preferably, $H$ should be scaled with samples. For example, the loss is commonly an average error, e.g., $L(DH) = \frac{1}{T} \|DH - X\|_F^2$. Similarly, the regularizer on $H$ should be averaged, to give a more balanced optimization

$$\frac{1}{T} \|DH - X\|_F^2 + \frac{\alpha}{2} \|D\|_F^2 + \frac{\alpha}{2T} \|H\|_F^2.$$

In general, it is clearly useful to be able to normalize $H$ separately from $D$. Though this modification seems trivial, it is not immediately obvious from previous formulations. Because this modification is particularly important for stochastic gradient descent, which we address in this work, we characterize this normalization more explicitly and show that the convex reformulation holds under a scaling on $H$.

**Proposition 2** Given any norms $\| \cdot \|_\cdot$, $\| \cdot \|_r$ and scalar $s > 0$, if $k$ is sufficiently large (potentially infinite)

$$\min_{D \in \mathbb{R}^{d \times k}, \, H \in \mathbb{R}^{k \times T}} L(DH) + \frac{\alpha}{2} \sum_{i=1}^k \|D_{:,i}\|_2^2 + \frac{\alpha}{2s^2} \sum_{i=1}^k \|H_{:,!}\|_r^2$$

$$= \min_{Z \in \mathbb{R}^{d \times T}} L(Z) + \frac{\alpha}{s} R_s(Z).$$

**Proof:** We will instead prove that

$$\min_{D \in \mathbb{R}^{d \times k}, \, H \in \mathbb{R}^{k \times T}} L(DH) + \frac{\alpha}{2} \sum_{i=1}^k \|D_{:,i}\|_c^2 + \frac{\alpha}{2} \sum_{i=1}^k \|H_{:,!}\|_r^2$$

$$= \min_{D \in \mathbb{R}^{d \times k}, \, H \in \mathbb{R}^{k \times T}} L(DH) + \frac{\alpha s}{2} \sum_{i=1}^k \|D_{:,i}\|_c^2 + \frac{\alpha}{2s} \sum_{i=1}^k \|H_{:,!}\|_r^2$$

(9)
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where we already know that \( (10) = \min_{Z \in \mathbb{R}^{d \times T}} L(Z) + \alpha R_Z(Z) \). Using this, we can choose regularizer weights \( \frac{\alpha s}{2} \) to get the desired result.

Take any \( D^*, H^* \) that are minimizers of (10). Assume that \( D^*/\sqrt{s} \) and \( \sqrt{s}H^* \) are not minimizers of (9). Then there exists \( D \) and \( H \) such that

\[
L(DH) + \frac{\alpha s}{2} \sum_{i=1}^{k} \|D_{i,:}\|_F^2 + \frac{\alpha}{2s} \sum_{i=1}^{k} \|H_{i,:}\|_F^2 < L(D^*H^*) + \frac{\alpha s}{2} \sum_{i=1}^{k} \|D^*_{i,:}/\sqrt{s}\|_F^2 + \frac{\alpha}{2s} \sum_{i=1}^{k} \|\sqrt{s}H^*_{i,:}\|_F^2
\]

\[
= L(D^*H^*) + \frac{\alpha}{2} \sum_{i=1}^{k} \|D^*_{i,:}\|_F^2 + \frac{\alpha}{2} \sum_{i=1}^{k} \|H^*_{i,:}\|_F^2
\]

(11)

where the second inequality is due to the fact that \( \|d/\sqrt{s}\|_F^2 = \|d\|_F^2/s \) and similarly for \( \|\cdot\|_r \). The strict inequality in (11) is a contradiction of the fact that \( D^* \) and \( H^* \) are the minimizers of (10). Therefore, \( D^*/\sqrt{s} \) and \( \sqrt{s}H^* \) are minimizers of (9).

Similarly, if \( D^*, H^* \) are minimizers of (9), then \( \sqrt{s}D^* \) and \( H^*/\sqrt{s} \) are minimizers of (10). Therefore, the minimum value for (9) and (10) is equal.

This theorem is only stated for norm regularizers, to show the explicit effect of the choice of \( s \) on the regularization on the induced norm regularizer. For example, for subspace dictionary learning or matrix completion, choosing to scale the Frobenius norm on \( H \) with \( s^2 \) corresponds to having normalized the trace norm with \( s \). More generally, for any valid \( f_c, f_r \) chosen, we can scale the regularizer on \( H \) with \( s > 0 \), because the resulting \( f_r/s^2 \) still satisfies the conditions of Theorem 1 and so an induced regularizer \( R_Z \) is guaranteed to exist. However, the choice of \( s \) no longer has such a clear connection to the regularization weight in front of \( R_Z \); rather, \( s \) modifies the resulting (unknown) \( R_Z \).

3.3 Relationships between multiple forms of the induced norm

The induced norm definition on \( Z \) was introduced with multiple forms (Bach et al., 2008; White, 2014), including the summed form

\[
R_k(Z) = \frac{1}{2} \min_{D \in \mathbb{R}^{d \times k}, H \in \mathbb{R}^{k \times T}} \sum_{i=1}^{k} \|D_{i,:}\|_F^2 + \sum_{i=1}^{k} \|H_{i,:}\|_F^2
\]

the producted form

\[
R_k(Z) = \min_{D \in \mathbb{R}^{d \times k}, H \in \mathbb{R}^{k \times T}} \sum_{i=1}^{k} \|D_{i,:}\|_r \|H_{i,:}\|_r
\]

and the constrained forms

\[
R_k(Z) = \min_{D \in \mathbb{R}^{d \times k}, H \in \mathbb{R}^{k \times T}, \|D_{i,:}\| \leq 1} \sum_{i=1}^{k} \|H_{i,:}\|_r
\]

\[
R_k(Z) = \min_{D \in \mathbb{R}^{d \times k}, H \in \mathbb{R}^{k \times T}, \|H_{i,:}\| \leq 1} \sum_{i=1}^{k} \|D_{i,:}\|_c
\]

These equivalent definitions are only at a global minimum of the above objectives. It is possible, however, that the induced RFMs with these different forms will not have the same sets of stationary points. Understanding the relationships between the stationary points for these different forms can be important for understanding the ramifications of selecting one form or the other. If there is an equivalence, for example, then any of the
forms can be selected; in this case, secondary criteria can be used to select the form, including choosing the form that provides the most numerical stability or even the form that enables the simplest computation of gradients.

We prove that the set of stationary points of the squared form and the producted form are in fact equivalent. We further show that the set of stationary points of the constrained form constitute a super-set of those of the squared and producted forms. We define the pairs of stationary points to be equivalent if their product is equivalent, as in the below definition.

Definition 3 The pairs of stationary points \(D_1, H_1\) and \(D_2, H_2\) are equivalent stationary points if \(D_1 H_1 = D_2 H_2\), ensuring that the resulting induced variables \(Z_1 = Z_2\). The sets of stationary points for two objectives are equivalent if for every point in one set, there is an equivalent stationary point in the other set.

Proposition 4 The stationary points are equivalent for the summed form

\[ L(DH) + \frac{\alpha}{2} \sum_{i=1}^{k} \|D_{i:1}\|^2_c + \frac{\beta}{2} \sum_{i=1}^{k} \|H_{i:}\|^2_r \]

and the producted form

\[ L(DH) + \alpha \sum_{i=1}^{k} \|D_{i:}\|_c \|H_{i:}\|_c \]

The stationary points of the summed and producted forms are also stationary points of the constrained form.

Proof Sketch: See Appendix D for the full proof. The proof follows from taking a stationary point from each optimization, and reweighting with a diagonal matrix to obtain a stationary point in the other form.

Finally, there is a natural question about the equivalence of the three forms under the generalizations to non-negative centered functions \(f_c\) and \(f_r\). The generalization given in Theorem 1 used the summed form for the definition of \(R_h\); the theorem, however, could have been similarly proven for the producted or constrained forms. In contrast to the norm case, it is unclear if the resulting induced variables \(Z\) are the same for each of the forms; it is only the case that they each have an induced convex regularizer. In this text, we advocate for the summed form, and so provide the version of Theorem 1 only for the summed form.

3.4 Regularizers decoupled across samples

In this section, we develop induced RFM objectives that enable unbiased stochastic gradient descent. Consider that samples are processed incrementally, and consider a loss \(L(DH) = \frac{1}{T} \sum_{t=1}^{T} L_x(DH, x_t)\). As is standard for using stochastic gradient descent for two variables (Bottou, 1998; Mairal et al. 2009a), we consider the optimal \(H\) and only stochastically update \(D\). This corresponds to updating \(D\) according to the loss

\[ l_t(D) = \left( \min_h L_x(Dh, x_t) + \frac{\alpha}{2} R_H(h) \right) + \frac{\alpha}{2} \sum_{i=1}^{k} f^2_c(D_{i:}) \quad (12) \]

Given that \(l_t(D)\) is an unbiased estimate of the loss, we initialize \(D\) as usual (say randomly) and then incrementally process the samples, updating \(D\) according to \(\nabla l_t(D)\) for the given sample \(x_t\) at iteration \(t\). To compute the gradient, we first find the minimum \(h\), which is a simple convex optimization given the current \(D\). Then we compute the gradient with respect to \(D\) and step in the direction for a decaying step-size (see Bottou 1998).

In general, however, \(l_t(D)\) is not necessarily unbiased. The issue arises from the fact that the norm on \(H\) couples columns (samples). If \(R_H\) decomposes into a sum across columns then \(l_t(D)\) is unbiased. To see why, consider the setting with \(\|\cdot\|_2 = \|\cdot\|_2\), where \(R_H(H) = \|H\|_F^2 = \sum_{i=1}^{k} \|H_{i:}\|_2^2 = \sum_{j=1}^{T} \|H_{j:}\|_2^2\). For
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i.i.d. samples $x_1, \ldots, x_T$, because this regularizer decomposes across columns, $l_t(D)$ is an unbiased estimate of the expected loss, where the expectation is w.r.t. variables $x_t$

$$
\mathbb{E}[\min_h L_x(Dh, x_t) + \frac{\alpha}{2} \| h \|^2] = \mathbb{E} \left[ \frac{1}{T} \sum_{j=1}^T \min_{H,j} \left( L_x(DH_{j}, X_{j}) + \frac{\alpha}{2} \| H_{j} \|^2 \right) \right]
$$

$$
= \mathbb{E} \left[ \min_{H} \frac{1}{T} \sum_{j=1}^T \left( L_x(DH_{j}, X_{j}) + \frac{\alpha}{2} \| H_{j} \|^2 \right) \right]
$$

$$
= \mathbb{E} \left[ \min_{H} \left( L(DH) + \frac{\alpha}{2T} \sum_{j=1}^T \| H_{j} \|^2 \right) \right]
$$

For other regularizers, however, such as $\| \cdot \|_1^2$, which couple the columns of $H$, we cannot swap $i$ and $j$:

$$
\sum_{i=1}^k \| H_i \|^2 = \sum_{i=1}^k \left( \sum_{j=1}^T |H_{ij}| \right)^2 \neq \sum_{j=1}^T \| H_j \|^2
$$

Therefore, $l_t(D)$ would be a biased estimate of the expected loss.

This issue can be mitigated by either using the product form discussed in the previous section, or potentially by using $\ell_1$ without squaring. The set of stationary points is equivalent for the producted form as and the summed form, but the producted form uses $\ell_1$ instead of $\ell_2^1$. Additionally, though not theoretically shown, we have empirically found that using $\ell_1$ instead of $\ell_2^1$ within the summed form also gives global solutions. As discussed later in this document, we hypothesize that because $f_r = \sqrt{\ell_1}$ is close to being a convex function, it enjoys similar global optimality properties as $f_r$ that are convex.

Remark 1: Once $D$ is learned, out-of-sample prediction for unsupervised learning is done using the following, for a new sample $x$.

$$
\min_h L_x(Dh, x) + \alpha R_H(h)
$$

This shows that even for the batch setting, it makes sense to select the objective that decouples the columns (samples) of $H$, for more effective out-of-sample prediction.

Remark 2: In previous work on incremental sparse coding (Mairal et al., 2010), the strategy has been to instead constrain the column norm on $D$

$$
\min_{D \in \mathbb{R}^{d \times k}, H \in \mathbb{R}^{k \times T}} L(DH) + \alpha \sum_{i=1}^k \| H_{i} \|_r.
$$

This formulation removes issues with rescaling the regularizer on $H$ by the number of samples and removes the square on the row regularizer $\| \cdot \|_r$, decoupling the columns of $H$ for the $\ell_1$ regularizer. This strategy, however, will not decouple columns of $H$ for all row regularizers. Further, projections to satisfy the constraint on $D$ can significantly impact computation (Hazan and Kale, 2012), particularly with the generalizations to any non-negative centered regularizers on $D$. For these reasons, we investigate regularizing both factors for incremental estimation.
Main Theorem (informal): conjecture proposed in this work, that alternating minimization for induced RFMs produces global solutions. These theoretical and empirical insights constitute a significant step towards the global optimality result additionally holds for induced RFMs combined with previous results on rank-deficiency, there is compelling evidence that even though the RFM objective is nonconvex, alternating minimization between H and D should converge to a global solution. Later, in Section 5.3 we illustrate empirically that this global optimality result additionally holds for induced RFMs not covered by the theory, but that it does not hold for two slight modifications that take the objective out of the class of induced RFMs. These theoretical and empirical insights constitute a significant step towards the conjecture proposed in this work, that alternating minimization for induced RFMs produces global solutions.

In this section, we discussed how the objective can be specified in multiple ways, with similar or equivalent modeling properties. We summarize what we believe are effective choices in Table 1 for subspace, sparse, elastic-net and supervised dictionary learning, particularly opting for those objectives we found were also more stable empirically. In the next sections, we demonstrate theoretically and empirically that alternating minimization on induced RFM objectives produces global solutions. In particular, we also provide evidence that moving outside the class of induced RFMs loses this property. The result is actually hopeful: we can globally optimize a wide-range of representation learning problems, with an appropriately chosen objective.

### 3.5 Summarized RFM objectives

In this section, we discussed how the objective can be specified in multiple ways, with similar or equivalent modeling properties. We summarize what we believe are effective choices in Table 1 for subspace, sparse, elastic-net and supervised dictionary learning, particularly opting for those objectives we found were also more stable empirically. In the next sections, we demonstrate theoretically and empirically that alternating minimization on induced RFM objectives produces global solutions. In particular, we also provide evidence that moving outside the class of induced RFMs loses this property. The result is actually hopeful: we can globally optimize a wide-range of representation learning problems, with an appropriately chosen objective.

| Setting                  | Batch Loss                                      |
|--------------------------|-------------------------------------------------|
| SUBSPACE                 | $\frac{1}{2}L(DH) + \frac{\nu}{2T}\|H\|_F^2 + \frac{\alpha}{2}\|D\|_F^2$ |
| MATRIX COMPLETION        | $\frac{1}{2}\sum_{(i,j) \in \text{OBSERVED}} L_{ij}(D, H, X_{ij}) + \frac{\nu}{2T}\|H\|_F^2 + \frac{\alpha}{2}\|D\|_F^2$ |
| SPARSE CODING ($q \geq 1$) | $\frac{1}{2}L(DH) + \frac{\nu}{2}\|H\|_F^2 + \alpha\nu_D\|D\|_F^2 + \alpha(1 - \nu_D)\sum_{i=1}^k\|D_{i,i}\|_F^2$ |
| ELASTIC NET ($\nu_H = 1$) | $\frac{1}{2}L(DH) + \frac{\nu_H}{2}\|H\|_F^2 + \alpha\nu_D\|D\|_F^2 + \alpha(1 - \nu_D)\sum_{i=1}^k\|D_{i,i}\|_F^2$ |
| ELASTIC NET ($\nu_H = 0$) | $\frac{1}{2}L(DH) + \frac{\nu_H}{2}\|H\|_F^2 + \alpha\nu_D\|D\|_F^2 + \alpha(1 - \nu_D)\sum_{i=1}^k\|D_{i,i}\|_F^2$ |
| SUPERVISED DICT. LEARNING | $\frac{1}{2}L \left( \left[ \begin{array}{c|c} D^{(1)}_{(i,j)} & \chi \\ \hline \nu_H & \chi \\ \end{array} \right] \right) + \frac{\nu}{2}\|H\|_F^2 + \alpha\sum_{i=1}^k \max \left( \|D^{(1)}_{i,i}\|_2^2, \|D^{(2)}_{i,i}\|_2^2 \right)$ |

Table 1: The objectives for effective batch and stochastic gradient descent, that give global results and that ensure we have a well-defined expected loss for the stochastic gradient descent setting and out-of-sample prediction. For the supervised representation learning objectives, the column norm on D can be any norm, including the elastic net norm: $\nu_D \|D\|_F^2 + (1 - \nu_D)\sum_{i=1}^k\|D_{i,i}\|_F^2$.

### 4. Local minima are global minima for a subclass of induced RFMs

Our main theoretical result is to show that, for an appropriately chosen inner dimension $k$, local minima are actual global minima. The key novelty over previous work is to characterize the overcomplete setting, with full-rank solutions, whereas previous work has generally analyzed rank-deficient solutions. Combined with previous results on rank-deficiency, there is compelling evidence that even though the RFM objective is nonconvex, alternating minimization between H and D should converge to a global solution. Later, in Section 5.3 we illustrate empirically that this global optimality result additionally holds for induced RFMs not covered by the theory, but that it does not hold for two slight modifications that take the objective out of the class of induced RFMs. These theoretical and empirical insights constitute a significant step towards the conjecture proposed in this work, that alternating minimization for induced RFMs produces global solutions.

We first summarize the main theorem (Theorem 9) informally.

**Main Theorem (informal):** For an RFM with convex, non-negative and centered functions $f_r, f_c$, if $k \geq \min(d, T)$ is large enough such that

$$R_k(Z) = \min_{D \in \mathbb{R}^{d \times k}, H \in \mathbb{R}^{k \times T}} \sum_{i=1}^k \left( f_r^2(D_{i,i}) + f_c^2(H_{i,i}) \right)$$

is convex, then all local minima are global minima.

The key insight is that existence of a convex induced regularizer $R_k$ identifies well-behaved RFMs. Previous results only considered the induced regularizer $R_Z$. From Theorem 1, we know that for $k \to \infty$,
A stationary point

Definition 5 of an overcomplete dictionary, which means that it has more dictionary elements than input dimension.

We formally characterize this result in the remainder of this section. We use the following assumptions.

Assumption 1 The loss $L : \mathbb{R}^{d \times T} \to \mathbb{R}$ and the regularizers $f_c : \mathbb{R}^d \to \mathbb{R}^+$, $f_r : \mathbb{R}^T \to \mathbb{R}^+$ are all twice-differentiable convex functions.

This assumption does not allow non-smooth regularizers, such as the $\ell_1$ or elastic net regularizer. However, due to the generalization in Theorem 1 to any non-negative convex $f_c$ and $f_r$, we can use smooth approximations to these regularizers. The theory applies to these smooth approximations even for parameter selections that make them arbitrarily close to the non-smooth regularizer. Intuitively, this suggests that the results extend to the non-smooth setting; we leave this generalization to future work.

To characterize the stationary points, we define three properties related to the choice of the inner dimension $k$. We introduce the term overcomplete to characterize the inner dimension $k$, to parallel the definition of an overcomplete dictionary, which means that it has more dictionary elements than input dimension.

Definition 5 A stationary point $(\bar{D}, \bar{H})$ with $\bar{D} \in \mathbb{R}^{d \times k}$, $\bar{H} \in \mathbb{R}^{k \times T}$ for an induced RFM is rank deficient if $k$ is greater than the rank of both $D$ and $H$.

Definition 6 The inner dimension $k$ is overcomplete if $k \geq d$ or $k \geq T$.

Definition 7 The inner dimension $k$ satisfies the induced rank condition if $k$ is at least as large as the inner dimension $k^*$ for the global solution, $Z$, to the induced optimization where

$$R_Z(Z) = \frac{1}{2} \min_{D \in \mathbb{R}^{d \times k^*}, H \in \mathbb{R}^{k^* \times T}, Z = DH} R_D(D) + R_H(H)$$

The result for $R_k = R_Z$ is addressed in (Haeffele and Vidal, 2015, Theorem 15), for the rank-deficient setting. Because the rank-deficient setting has been well characterized, here we focus instead on the overcomplete setting with full rank factors, which has been less explored. Further, we do not require $R_k = R_Z$, but rather only require $k$ to be large enough for $R_k$ to be convex. We are guaranteed for $R_k$ to be convex for a sufficiently large $k$, because $R_Z$ is convex; however, $R_k$ could be convex before $k = k^*$. Empirically, we find that we obtain this global optimality property for $k$ much less than $k^*$, for the settings where we know $k^*$. This is contrary to common wisdom that for these models we require $k$ to be as large as the induced rank to obtain global solutions (see (Bach et al., 2008; Zhang et al., 2011)). For example, for sparse coding, the true induced rank is $T$, where an optimal solution consists of memorizing the training samples. When using the factored form, we need $k \geq d$ but not necessarily $k = T$ and so we can still obtain meaningful overcomplete solutions and global optimality.

To prove the key result, we will relate the stationary points for the original factored form and the corresponding induced form. This reduces the problem to showing that the stationary points for the induced form are global minima. We first show that this is indeed the case, in Theorem 8, which characterizes the unregularized case with non-differentiable, convex losses. The induced form can be written as a function $g(DH) = L(DH) + R_k(DH)$, which is convex but likely non-differentiable. This non-differentiability is due to the fact that $R_k$ is defined as a minimization over a set. Therefore, to characterize the stationary points of the induced form, we prove the below theorem for general convex, non-differentiable functions $g$.

---

8. (Haeffele and Vidal, 2015) provide a general result for the fact that local minima are global minima, for a broader class of functions than we consider here. However, they require that the solution be rank deficient. For our setting, however, this forces $R_k = R_Z$, because we can almost always further reduce the error by enabling $k$ to be larger. Here, however, we would like to characterize optimality for smaller $k$. Further, though a nice result, the requirement that an entire column and row of $D$ and $H$ be zero is unrealistic in practice. Here, our aim is to better reflect what is observed in practice, where columns and rows need not be zero and $k$ can be much less than the optimal $k^*$ for $R_Z$. 

---
Theorem 8 (Non-differentiable losses) Let $g : \mathbb{R}^{d \times T} \to \mathbb{R}$ be any real-valued convex function, where the lower Hadamard directional derivative exists:

$$D_Z g(Z) [U] = \lim_{t \downarrow 0} \inf_{U' \to U} t^{-1} [f(Z + tU') - f(Z)]$$

Let $(\bar{D}, \bar{H})$ be a stationary point of

$$\min_{D \in \mathbb{R}^{d \times k}, H \in \mathbb{R}^{k \times T}} g(DH).$$

If either

1. $\bar{D}$ or $\bar{H}$ is full rank and $k$ is overcomplete (i.e., $k \geq d$ or $k \geq T$); or
2. $(\bar{D}, \bar{H})$ is a local minimum and $(\bar{D}, \bar{H})$ is rank-deficient

then $(\bar{D}, \bar{H})$ is a global minimum.

Proof: We need to show two cases. For the overcomplete setting, we consider full rank $\bar{D}$, the argument is similar for full rank $\bar{H}$. Notice that

$$D_H g(DH)[dH] = 0$$

by definition, since $\bar{D}, \bar{H}$ is a stationary point. By the chain rule

$$0 = D_H g(DH)[dH] = D_Z g(DH)[DdH].$$

The chain rule follows, because $g(D(H + tdH)) = g(D\bar{H} + tDdH)$. Because $\bar{D}$ is full rank, $DdH$ constitutes all possible directions $dZ$. Therefore, in all directions, $D_Z g(DH)$ is zero; since this is a convex function, $\bar{D} \bar{H}$ must be a global minimum.

For the second setting, where $\bar{D}$ is rank deficient, of rank $r$, we need to construct a descent direction. Notice that we can reorthogonalize $\bar{D} = U\Sigma V^\top$ and obtain equivalent stationary point $DVV^\top H = DH$, where $D = \bar{D}V$ has $k - r$ zero columns and $H = V^\top \bar{H}$. To show that these are still local minima, assuming there exists descent direction $dD, dH$ such that $g((\bar{D} + tdD)(\bar{H} + tdH)) < g(DH)$. Because $g(DH) = g(\bar{D}\bar{H})$,

$$g(\bar{D}\bar{H}) > g((\bar{D} + tdD)(\bar{H} + tdH)) = g((\bar{D}V + tdD)V^\top V(V^\top \bar{H} + tdH)) = g((\bar{D} + tdDV^\top)(\bar{H} + tVdH))$$

which would mean $dDV^\top, VdH$ is a descent direction from $\bar{D}, \bar{H}$. This would contradict the fact that $\bar{D}, \bar{H}$ is a local minima, and so $D, H$ must also be a local minima.

Finally, we can set the bottom $k - r$ rows in $H_0$ to zero, which correspond to the zeroed columns of $D$. Because there is a connected path from $H$ to this new $H_0$ with zero rows, if $D, H_0$ was not a local minima, then it would contradict the fact that $D, H$ is a local minimum. Any descent direction from $D, H_0$ would have to be on the non-zero dimensions, which would then also constitute a descent direction for $D, H$.

Therefore, the stationary point $D, H_0$ is also a local minimum, but now has a zero column and zero row. We can then extend the argument from (Haeffele and Vidal. 2015 Theorem 15), now with the small generalization to lower Hadamard directional derivatives, and obtain that $D, H_0$ is a global minimum. Therefore, $D, H$ is also a global minimum.

This result shows that the stationary points of the induced form are well-behaved. For the overcomplete setting, with full rank stationary point, any stationary point is guaranteed to be a global minimum. For the rank-deficient setting, $D$ could still be full rank if $k \geq d$, and so is already characterized by the first setting. But, if it is not full rank, then the second condition shows that if this point is a local minimum, then it is a
global minimum. This result does not characterize the saddlepoints for the low-rank setting, but otherwise shows that most stationary points for this induced problem are global minima. This theoretical result accurately reflects the nice properties we see for RFMs in practice. However, this characterization is only for the induced form, which we do not directly optimize; rather, we directly optimize the original factored form.

Now we provide a more restricted result for the original factored form. The key idea is to relate the stationary points of the factored form to the induced form \( R_k \), and then use the result in Theorem 8. For this proof, we drop the regularization parameter \( \alpha \) and scale \( s \), because w.l.o.g. we can assume that the regularizers are defined as \( \frac{\alpha}{2} f_r^2 \) and \( \frac{\alpha}{2} f_r^2 \). We restrict \( R(D) = \|D\|_F^2 \); however, this could be extended to other regularizers that are locally convex as in Lemma 17.

**Theorem 9 (General regularizers on H)**  Let \((\bar{D}, \bar{H})\) be a stationary point of

\[
\min_{D \in \mathbb{R}^{d \times k}, H \in \mathbb{R}^{k \times T}} L(DH) + R_D(D) + R_H(H) \tag{13}
\]

where \( R_D(D) = \|D\|_F^2 \) and \( R_H(H) = \sum_{i=1}^k f_r^2(H_i) \). Let \( R_k \) be the induced norm given \( f_c \) and \( f_r \), as defined in Equation (8). If

1. \( R_k \) is convex
2. \( k \) is overcomplete, i.e., \( k \geq d \) or \( k \geq T \)
3. \((\bar{D}, \bar{H})\) is full rank, i.e., \( DH \) is rank \( \min(d, T) \); and
4. \((\bar{D}, \bar{H})\) is a local minimum of (13)

then \((\bar{D}, \bar{H})\) is a globally optimal solution to (13).

**Proof:** We would like to relate \((\bar{D}, \bar{H})\) to the stationary points of the induced problem

\[
\arg\min_{D \in \mathbb{R}^{d \times k}, H \in \mathbb{R}^{k \times T}} L(DH) + R_k(DH). \tag{14}
\]

According to Theorem 8, if we can show that \( \bar{D}, \bar{H} \) are local minima, then we know they are global minima. The optimal solution to (14) corresponds to the optimal solution of (13), and so we would correspondingly know this is a global minima for (13) and we would be done.

To relate these optimizations, we rewrite the optimization in (14) for full rank \( D, H \) using the factored form with an additional optimization over \( Q \in \mathbb{R}^{k \times k} \):

\[
R_k(DH) = \frac{1}{2} \min_{Dk \in \mathbb{R}^{d \times k}, H_k \in \mathbb{R}^{k \times T}} R_D(D_k) + R_H(H_k)
\]

\[
= \frac{1}{2} \min_{Q \in \mathbb{R}^{k \times k}, Q \text{ full rank}} R_D(DQ^{-1}) + R_H(QH)
\]

The second equality follows from the fact that a minimal such pair \( D_k, H_k \) can always be written \( D_k = DQ^{-1} \) and \( H_k = QH \) for some \( Q \in \mathbb{R}^{k \times k} \). This is because \( DH = DQ^{-1}QH \) and because \( D_k \) and \( H_k \) must also be full rank when \( D \) and \( H \) are full rank.

Consequently, we will instead consider

\[
\arg\min_{\bar{D}, \bar{H}} L(DH) + \min_{Q \in \mathbb{R}^{k \times k}, Q \text{ full rank}} R_D(DQ^{-1}) + R_H(QH)
\]

\[
= \arg\min_{\bar{D}, \bar{H}} \min_{Q \in \mathbb{R}^{k \times k}, Q \text{ full rank}} L(DH) + R_D(DQ^{-1}) + R_H(QH). \tag{15}
\]

By showing that \( \bar{D}, \bar{H} \) is a stationary point of (15) for the optimal \( Q \), this shows that \( \bar{D}, \bar{H} \) is a stationary point of (14). This is because in a small ball around \( \bar{D}, \bar{H}, L(DH) + R_k(DH) \) and \( L(DH) + \min_{Q \in \mathbb{R}^{k \times k}. Q \text{ full rank}} R_D(DQ^{-1}) + R_H(QH) \) have the same value, and so the function surface is the same.
Let
\[ Q = \arg\min_{Q \in \mathbb{R}^{k \times k}, Q \text{ full rank}} R_D(\bar{D}Q^{-1}) + R_H(Q\bar{H}). \]

If \( Q \) strictly decreases the regularization component, we know \( R_D(\bar{D}Q^{-1}) + R_H(Q\bar{H}) < R_D(\bar{D}) + R_H(\bar{H}). \)

For a sufficiently small \( t > 0, (1 - t)I + tQ \) is invertible. Because \( R_D(\bar{D}Q^{-1}) \) is locally convex in \( Q \), around \( Q = I \) by Lemma 17, we have that \( R_D(\bar{D}((1 - t)I + tQ)^{-1}) \leq (1 - t)R_D(\bar{D}) + tR_D(\bar{D}Q^{-1}) \). Therefore, for a small step of \( t \) in a direction that keeps \( \bar{D}\bar{H} \) the same, i.e., \( H = ((1 - t)I + tQ)H = (1 - t)H + tQ\bar{H} = H + t(Q - I)\bar{H} \) and using the Woodbury identity, \( \bar{D} = \bar{D}((1 - t)I + tQ)^{-1} = \bar{D} + t(\bar{D}(\bar{D}^{-1}I) - \bar{D}(\bar{D}^{-1}I)^{-1}) \), we have
\[
R_D(\bar{D}((1 - t)I + tQ)^{-1}) + R_H(((1 - t)I + tQ)H)
\leq (1 - t)R_D(\bar{D}) + tR_D(\bar{D}Q^{-1}) + (1 - t)R_H(\bar{H}) + tR_H(Q\bar{H})
= (1 - t)(R_D(\bar{D}) + R_H(\bar{H})) + t(R_D(\bar{D}Q^{-1}) + R_H(Q\bar{H}))
< R_D(\bar{D}) + R_H(\bar{H})
\]

where the first inequality also follows because \( R_H(Q\bar{H}) \) is convex in \( Q \) and the last inequality from the fact that \( R_D(\bar{D}Q^{-1}) + R_H(Q\bar{H}) < R_D(\bar{D}) + R_H(\bar{H}) \). This would be a contradiction, since \( \bar{D}, \bar{H} \) is a local minimum so there cannot be a direction that decreases the objective function. Therefore, \( Q \) cannot decrease the regularization component and \( Q = I \) is a valid solution.

Since \( Q = I \), clearly \( (\bar{D}, \bar{H}) \) is a stationarity point of \((15)\).

More generally, however, we need these points to also be stationary points of to the induced problem. Because the induced problem may not be differentiable, due to \( R_k \), we turn to more general definitions of directional derivatives (Ivanov 2015). These directional derivatives become the standard Frechet derivatives, when those exist (as they do for twice differentiable functions). For a function \( f \), the lower Hadamard directional derivative is defined as (Ivanov 2015)
\[
D_Y f(Y)[U] = \lim_{t_0 \downarrow 0} \inf_{U \to U} t^{-1}[f(Y + tU) - f(Y)]
\]

We will use Danskin’s theorem to characterize the stationary point for the induced problem; we explicitly prove it here, because we use a generalized directional derivative. Let \( Y_0 = [D; H^\top] \in \mathbb{R}^{(d + T) \times k} \) and pick an arbitrary direction \( U = [dD; dH^\top] \neq 0 \) and let \( \{Y_i\}_{i=1}^\infty, Y_i = Y_0 + t_i U, t_i \geq 0 \) be a sequence converging to \( Y_0 \) (so \( t_i \to 0 \)). Let \( S(Y_i) = \arg\min_{Q \in \mathbb{R}^{k \times k}, Q \text{ full rank}} J(Y_i, Q) \) and pick \( Q_i \) such that \( Q_i = \arg\min_{Q \in S(Y_i)} J(Y_i, Q) \). This distinguishes among equivalent \( Q_i \), to provide a nearby sequence to \( Q_0 \). Since \( J(Y_i) = J(Y_i, Q_i) \), we get that
\[
\frac{\bar{J}(Y_i) - \bar{J}(Y_0)}{t_i} = J(Y_i, Q_i) - J(Y_0, Q_0)
\leq J(Y_i, Q_0) - J(Y_0, Q_0)
\leq D_Y J(Y_0 + t_i U, Q_0)[U]
\]

where the first inequality follows from \( J(Y_i, Q_i) \leq J(Y_i, Q_0) \) and the last equality from the mean value theorem, where there is some \( 0 \leq t'_i \leq t_i \) that provides a point between \( Y_i \) and \( Y_0 \). Taking the limit superior, we obtain
\[
\lim_{i \to \infty} \sup_{t_i} \frac{\bar{J}(Y_i) - \bar{J}(Y_0)}{t_i} \leq D_Y J(Y_0, Q_0)[U].
\]
Similarly

\[
\frac{\tilde{J}(Y_i) - \tilde{J}(Y_0)}{t_i} = \frac{J(Y_i, Q_i) - J(Y_0, Q_0)}{t_i} = \frac{J(Y_i, Q_i) - J(Y_0, Q_i)}{t_i} + \frac{J(Y_0, Q_i) - J(Y_0, Q_0)}{t_i} \\
\geq \frac{J(Y_i, Q_i) - J(Y_0, Q_0)}{t_i} = D_Y J(Y_0 + t'_i U, Q_i)[U]
\]

for some \(0 \leq t'_i \leq t_i\). Because \(\tilde{J}\) is continuous in both arguments, \(\lim_{i \to \infty} \inf (D_Y J(Y_0 + t'_i U, Q_i)[U] = D_Y J(Y_0, Q_0)[U].\)

Therefore, the limit inferior and superior converge to the same point, and we have that the directional derivative for \(\tilde{J}\) exists and equals \(D_Y J(Y_0, Q_0)[U]\). Since \(D_Y J(Y_0, Q_0)[U] = 0\), we know therefore that this is a stationary point for \(\tilde{J}\). Since \(\tilde{J}\) corresponds to the objective in (15) for optimal \(Q\), we know \(Y_0\) is also a stationary point for (14), which explicitly has \(R_k\) rather than an optimization over \(Q\). By Theorem 8 (\(\tilde{D}, \tilde{H}\)) is a global minimum of (14) and so must also be a global minimum of (13).

Several of the conditions in the above theorem were required to ensure that \(Q = I\) within the proof. This could occur, however, under many other settings than those that we assumed. To identify this as a key property, we provide the following corollary.

**Corollary 10 (General regularizers on \(D\) and \(H\))** Let \((\tilde{D}, \tilde{H})\) be a stationary point of (13) where \(R_D(D) = \sum_{i=1}^{k} f^2(D_i)\) and \(R_H(H) = \sum_{i=1}^{k} f^2(H_i)\). If

1. \(R_k\) is convex
2. \(k\) is overcomplete, i.e., \(k \geq d\) or \(k \geq T\)
3. \(R_D(D) + R_H(H) = \min_{D_k \in \mathbb{R}^{d \times k}, H_k \in \mathbb{R}^{k \times T}} \{R_D(D_k) + R_H(H_k)\}

then \((\tilde{D}, \tilde{H})\) is a globally optimal solution to (13).

Finally, we provide one setting where \(R_k\) is guaranteed to be convex, stated in the following corollary.

**Corollary 11** Under the same setting as Theorem 9 if \(k\) satisfies the induced rank condition (Definition 7), then \(DH\) is a globally optimal solution to

\[
\arg\min_{Z \in \mathbb{R}^{d \times r}} L(Z) + R_Z(Z).
\]

**Proof:** This follows from the fact that if \(k\) satisfies the induced rank condition, then \(R_k = R_Z\), which is guaranteed to be convex. For this case, a global minimum of (13) is also a global minimum of (16). \(\square\)

5. Batch optimization for induced RFMs

In this section, we develop effective alternating minimization strategies to optimize induced RFMs on a batch of data. To better place the strategies we pursue relative to previous approaches, we summarize the large literature on global optimization of non-convex objectives; we include this summary, however, at the end of this section to avoid the upfront burden. As opposed to some of the more sophisticated strategies proposed
for specific factorization problems, we opt for a basic alternating minimization strategy, without specific initialization or explicit strategies to escape saddle points. In particular, we focus only on making the alternating minimization more effective by improving the convergence rate in two ways: (a) using an incomplete alternating minimization and (b) producing practical minimization strategies for non-smooth optimization. In Section 5.2, we justify that this strategy will find global minima, by using the theoretical results in the previous section and further discussing the fact that a subclass of induced RFMs do not have degenerate saddlepoints.

5.1 Algorithm

The alternating minimization algorithm for RFMs—summarized in Algorithm 1—is a standard block coordinate descent algorithm, with a few specific choices that we found effective. The standard approach involves descending in one variable with the other fixed, and then alternating. The main algorithmic choices are to use inexact updates for the alternating minimization, with proximal gradient updates for non-smooth regularizers. Empirically, we found both these modifications significantly speed convergence. We opt for proximal gradient approaches rather than other approaches, such as alternating direction method of multipliers, because it maintains sparse variables which can be more efficiently stored and because the convergence results for proximal gradient approaches are well understood, even under approximate updates (Machart et al., 2012).

Inexact alternating step. To alternate between \( D \) and \( H \) in the optimization, one can completely solve for each variable with the other fixed (exact) or alternate between single gradient descent steps (inexact). Both approaches converge under general conditions, proven as part of more general results about the convergence of block coordinate descent for multi-convex problems using exact updates (Xu and Yin, 2013) and inexact updates (Tappenden et al., 2014). In our own experiments, we found the exact updates to be marginally less sensitive to parameter choices, such as the step-size, but significantly slower. We therefore adopt the inexact method, which consistently converges and is significantly faster.

Algorithm 1 Alternating Minimization for Regularized Factor Models (AM-RFM)

\[ \text{Input loss components } L, R_D, R_H, \alpha, s, k \]

\[ D, H \leftarrow \text{full-rank random matrices with inner dimension } k \]

\[ \text{prevobj} \leftarrow \infty \]

\[ \text{repeat} \]

\[ \text{Update } D \text{ using one step of gradient descent (such as in Algorithms 2, 3 or 4)} \]

\[ \text{Update } H \text{ using one step of gradient descent (such as in Algorithms 2, 3 or 4)} \]

\[ \text{currentobj} \leftarrow L(DH) + \alpha R_D(D) + \alpha s R_H(H) \]

\[ \text{if } |\text{currentobj} - \text{prevobj}| < \text{tolerance then break} \]

\[ \text{prevobj} \leftarrow \text{currentobj} \]

\[ \text{until convergence within tolerance or reach maximum number of iterations} \]

\[ \text{return } D, H \]

Algorithm 2 Standard gradient descent step for smooth regularizers

\[ D \leftarrow D - \eta_D(\nabla L(DH)H^T + \alpha \nabla R_D(D)) \]

\[ H \leftarrow H - \eta_H(D^T \nabla L(DH) + \alpha s \nabla R_H(H)) \]

Proximal gradient updates. A standard gradient descent step is problematic when the regularizer, or a component of the regularizer, is non-differentiable. For example, \( \ell_1 \) has a non-differentiable point at 0.
Soft thresholding algorithm for updating column vectors of Algorithm 5

where the multiplication \( \oplus \) is element-wise. Proximal operators are well-known for common non-smooth regularizers such as \( \| \cdot \|_1 \) and \( \| \cdot \|_2 \) (Bach et al., 2011). Some regularizers for induced RFMs, however, involve squares of convex functions, which is atypical. For example, \( \| \cdot \|_2^2 \) is used in the elastic net RFM.

Though one could simply choose a subgradient at 0 and apply subgradient descent, in practice for batch optimization, the convergence properties are poor. For alternating minimization on induced RFMs, we found that the descent would converge to a point and then very slowly decrease over a large number of iterations.

Proximal gradient methods, on the other hand, use a proximity operator that avoids computation of the subgradient of the non-differentiable component. For example, for the \( \ell_1 \) regularizer, with Lipschitz constant \( l \) for the gradient of the loss, the proximity operator is the soft-thresholding operator

\[
prox(u) = \arg\min_{u \in \mathbb{R}^T} \frac{1}{2} \| u - h \|_2^2 + \frac{\alpha}{\lambda} \| h \|_1 = \text{sign}(u) \circ \max(|u| - \frac{\alpha}{\lambda}, 0)
\]

where the multiplication \( \circ \) is element-wise. Proximal operators are well-known for common non-smooth regularizers such as \( \| \cdot \|_1 \) and \( \| \cdot \|_2 \) (Bach et al., 2011). Some regularizers for induced RFMs, however, involve squares of convex functions, which is atypical. For example, \( \| \cdot \|_2^2 \) is used in the elastic net RFM.

To the best of our knowledge, proximal operators have not been derived for \( \| \cdot \|_2^2 \). We provide the proximal updates for the elastic net RFM in Algorithm 4 and 5 with proof of the validity of the operator in the following proposition.

Algorithm 5 Soft thresholding algorithm for updating column vectors of \( D \)

\[
\lambda \leftarrow \frac{(1 - \nu_D)\alpha}{l} \quad \triangleright \text{To update } H: \lambda \leftarrow \frac{(1 - \nu_H)\alpha}{s l}
\]

\begin{algorithm}
\begin{algorithmic}
\State \( u \leftarrow D_i - \frac{1}{2}[\nabla L(DH)H_i^T + 2\nu_D\alpha D_i] \quad \triangleright \text{H update: } u \leftarrow H_i - \frac{1}{2}[D_i^\top \nabla L(DH) + 2\nu_H\alpha/s H_i]
\]
\State Sort \( u = [u_1, \ldots, u_k] \) according to \( |u_i| \) in descending order such that \( |u_{m_1}| \geq |u_{m_2}| \geq \ldots \geq |u_{m_k}| \)
\State \( C \leftarrow 0, r \leftarrow 0, \)
\While {\( r < k \) and \( |u_{m_{r+1}}| > \frac{2\lambda C}{1 + 2\lambda r} \)}
\State \( C \leftarrow C + |u_{m_{r+1}}| \)
\State \( r \leftarrow r + 1 \)
\EndWhile
\For {\( j = 1 \) to \( k \)}
\State \( D_{ij} \leftarrow \text{sign}(u_j) \max \left( |u_j| - \frac{2\lambda C}{1 + 2\lambda r}, 0 \right) \quad \triangleright \text{H update: flip indices to update } H_{ji} \)
\EndFor
\Endalgorithmic
\end{algorithm}

Until difference between \( D \) on successive steps within tolerance or reach maximum number of iterations
**Proposition 12** For a given $u = [u_1, \ldots, u_k] \in \mathbb{R}^k$, let the indices $m_1, \ldots, m_k$ indicate a sorted descending order such that $|u_{m_1}| \geq |u_{m_2}| \geq \ldots \geq |u_{m_k}|$. Let $C(u, r) = \sum_{j=1}^r |u_{m_j}|$. Then the following $z^*$ is an optimal solution of $\min_{z \in \mathbb{R}^k} \frac{1}{2} \|u - z\|_2^2 + \lambda \|z\|_1^2$

$$z_i^* = \text{sign}(u_i) \max \left( |u_i| - \frac{2\lambda C(u, r^*)}{1 + 2\lambda r^*}, 0 \right)$$

where

$$r^* = \begin{cases} 
0 & \text{if } u_{m_1} = 0 \\
\ |u| > 2\lambda C(u, r^*) & \text{if } |u_{m_1}| > 2\lambda C(u, r^*) \\
k & \text{such that } |u_{m_1}| > 2\lambda C(u, r^*) \text{ and } |u_{m_{r+1}}| \leq 2\lambda C(u, r^*) 
\end{cases}$$

**Proof:** Let $\partial f(\cdot)$ denote the subdifferential of a function $f(\cdot)$. If $\frac{u-z^*}{\lambda} \in \partial \|z^*\|_1$, then $z^*$ is the optimum solution of $\min_{z \in \mathbb{R}^k} \frac{1}{2} \|u - z\|_2^2 + \lambda \|z\|_1^2$. Because $\partial \|z\|_1^2 = 2\|z\|_1 \partial \|z\|_1$, we need to show that

$$\frac{u - z^*}{2\lambda \|z^*\|_1} \in \partial \|z^*\|_1.$$

Let $\lambda^* = 2\lambda \|z^*\|_1$. Using the known form for the $\ell_1$ proximal operator, if $z_i^* = \text{sign}(u_i) \max(|u_i| - \lambda^*, 0)$, then $\frac{u - z^*}{\lambda^*} \in \partial \|z^*\|_1$. The following cases verify that $z^*$ satisfies $z_i^* = \text{sign}(u_i) \max(|u_i| - \lambda^*, 0)$.

- **Case 1.** If $r^* = 0$, then $u = 0$ and so $z^* = 0$.
- **Case 2.** If $r^* = k$, then for all $i$, $|u_i| > \frac{2\lambda C(u, r^*)}{1 + 2\lambda r^*}$ and so

$$z_i^* = \text{sign}(u_i) \left( |u_i| - \frac{2\lambda C(u, r^*)}{1 + 2\lambda r^*} \right) = \text{sign}(u_i) \left( |u_i| - \frac{2\|u\|_1}{1 + 2\lambda r^*} \right).$$

Hence,

$$\|z^*\|_1 = \sum_i \left( |u_i| - \frac{2\|u\|_1}{1 + 2\lambda r^*} \right)$$

$$= \sum_i \left( |u_i| - \frac{2\|u\|_1}{1 + 2\lambda k^*} \right) > |u_i| > \frac{2\lambda C(u, r^*)}{1 + 2\lambda r^*}.$$

Therefore, $\lambda^* = 2\lambda \|z^*\|_1 = \frac{2\lambda \|u\|_1}{1 + 2\lambda r^*}$, and so $z_i^* = \text{sign}(u_i) \max(|u_i| - \lambda^*, 0)$.

- **Case 3.** If $0 < r^* < k$, then for $i > r^*$, $z_{m_i}^* = 0$ and for $i \leq r^*$, $z_{m_i}^* = \text{sign}(u_{m_i}) \left( |u_{m_i}| - \frac{2\lambda C(u, r^*)}{1 + 2\lambda r^*} \right)$. Similarly to above, $\|z^*\|_1 = C(u, r^*) - \frac{2\lambda r^* C(u, r^*)}{1 + 2\lambda r^*} = \frac{C(u, r^*) - 2\lambda r^* C(u, r^*)}{1 + 2\lambda r^*}$, and so $\lambda^* = \frac{2\lambda C(u, r^*)}{1 + 2\lambda r^*}$. 

**5.2 Avoiding saddlepoints**

To complete the characterization of using alternating minimization for induced RFMs, it is important to understand any issues with saddlepoints. For non-convex optimizations, saddlepoints are problematic, often corresponding to large flat regions and stalling convergence. For the biconvex optimization for induced RFMs, there are clear symmetries that cause multiple equivalent solutions and saddlepoints between solutions, such as in Figure 4. Our empirical investigation in the next section indicates that alternating minimization for induced RFMs does not get stuck in saddle points. Nonetheless, we would like a stronger guarantee of convergence to a local minimum, which then corresponds to a global minimum.
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Figure 4: Optimization surface for loss \( D \) = \( \min_H \sum_{t=1}^{T} \| X_t - DH_t \|_2^2 + \alpha \| H \|_F^2 + \alpha \| D \|_F^2 \) where \( D \in \mathbb{R}^{2 \times 1} \), \( T = 20 \) and \( \alpha = 0.5 \). The first figure illustrates the symmetry of the problem, where there are two equivalent optimal values for \( D \) in terms of the objective. The second figure is zoomed into the top of the optimization surface, indicating the saddle point that exists between these two optimal areas.

To do so, we can take advantage of recent characterization of non-convex problems using the strict-saddle property (Ge et al., 2015; Sun et al., 2015b,a; Lee et al., 2016), with the most general definition given by Lee et al. (2016). The requirement is simply that either the stationary point is a local minimum or the Hessian has at least one strictly negative eigenvalue. Recall that Hessians at saddle points can have both positive and negative eigenvalues, and degenerate saddlepoints are those that have positive semi-definite Hessians with zero eigenvalues. Therefore, another way to state the strict-saddle property is that there are no degenerate saddle points. Lee et al. (2016) recently proved that for twice continuously differentiable functions that satisfy the strict saddle property, gradient descent with a random initialization and a sufficiently small constant step-size converges to a local minimizer. We conjecture that this is the case for induced RFMs, and believe that this is the main reason that empirically induced RFMs converge stably to local minima, and so to global minima, even without additional noise or a particular initialization. Towards theoretical motivation for this conjecture, we show that a subclass of induced RFMs do not have degenerate saddlepoints, in Proposition 13 which relies on Theorem 16 proved in the appendix. Further, as we showed for the induced optimization in (14), the result is even stronger: all stationary points that are full rank for the overcomplete setting are guaranteed to be global minima. Though these two results provide some insight into saddlepoints properties, an important next step is to more fully characterize the saddlepoints for induced RFMs.

Proposition 13 (Non-degenerate saddlepoints) The induced RFM optimization defined in (18) does not have degenerate saddle points.

Proof: Assume there exists a stationary point \((D, H)\) that is a degenerate saddle point. By definition, this means that the Hessian is positive semi-definite, and so by Theorem 16 corresponds to the global minimum.

5.3 Empirical evidence of global optimality of alternating minimization for induced RFMs

To investigate global optimality of induced RFMs, the alternating minimization is started from different initial values and differences in the solution reported. We report both differences in the objective value (Table 2) and in the matrices themselves (Table 3). The normed differences between the solutions \( Z = DH \) are reported to clarify that similarities in objectives are due to similar solutions, rather than because the objective itself
Table 2: The minimum and maximum relative differences between objective values for the solutions found by alternating minimization. The relative difference between a pair of solutions is computed by taking the absolute difference between their objective values divided by the expected value of the objective across the solutions (to normalize the magnitude for changing \(d, k\)). The reported relative difference is the maximal relative difference for any pair of solutions. The parameter settings that resulted in the minimum and maximum relative difference are reported in brackets below the values with the order \((\alpha, d, k)\).

| LOSS       | SUBSPACE | SPARSE | ELASTIC NET | ELASTIC NET | COUPLED COLS | COUPLED COLS |
|------------|----------|--------|-------------|-------------|--------------|--------------|
|            | \(f^2 = \|\cdot\|^2\) | \(f^2 = \|\cdot\|^2\) | \(\nu \|\cdot\|^2 + (1 - \nu)\|\cdot\|^2\) | \(\nu_d = 0.5 = \nu_H\) | \(\sum_{i=1}^d \|D_i\|^2\) | \(\sum_{i=1}^d \|D_i\|^2\) |
| LEAST-SQ (MIN) | 0.0000000 | 0.000000 | 0.0000000 | 0.0000007 | 0.004586 | 0.466634 |
|             | \((0.5, 5, 10)\) | \((0.05, 5, 3)\) | \((0.5, 5, 3)\) | \((0.05, 50, 3)\) | \((0.5, 10, 3)\) | \((0.005, 5, 3)\) |
| LEAST-SQ (MAX) | 0.000785 | 0.000136 | 0.001269 | 1.535013 | 0.096123 | 0.566597 |
|             | \((0.005, 5, 5)\) | \((0.5, 50, 10)\) | \((0.05, 10, 10)\) | \((0.005, 50, 3)\) | \((0.005, 5, 3)\) | \((0.05, 10, 3)\) |

does not change much. The initializations are random but of highly differing magnitudes to better search for different local minima and saddlepoints in which the optimization could get stuck. Small relative differences between objective values suggest that the local minima are in fact the same minimum. Correspondingly, large differences suggest different local minima.

As baselines and to further elucidate if this global optimality property is characteristic of induced RFMs, we also test two modifications that take the objective outside the class of induced RFMs. These two modifications use the non-norm elastic net and a regularizer that couples columns of \(D\). The non-norm elastic net uses regularizer \(f^2_\nu(d) = \nu\|\cdot\|^2 + (1 - \nu)\|\cdot\|^2\) (Zou and Hastie 2005), which does not satisfy the requirements of Theorem 1 for generalized induced RFMs. The second setting couples the columns of \(D\), say with \(\sum_{i=1}^d \|D_i\|^2\), which is clearly no longer separable across columns and so is not a valid regularizer for induced RFMs. Note that the non-norm elastic net almost satisfies the requirements of Theorem 1 For \(\Lambda = I\), \(f_\nu(d) = \sqrt{\nu\|\cdot\|^2 + (1 - \nu)\|\cdot\|^2}\) is a non-convex function but it is centered and non-negative. Further, examining the plot of this loss, for \(\Lambda = I\), \(f_\nu(d)\) is almost a convex function, with a slight bow. We find, in fact, that with \(\Lambda = I\), alternating minimization does provide global solutions. However, with different choices of \(\Lambda\), the \(f_\nu\) become more non-convex and alternating minimization is no longer global. In the following experiments, we choose \(\Lambda = \text{diag}(1, 2, \ldots, d)\).

The results for the differences in objective values and in solution matrices are summarized in Table 2 and Table 3 respectively. The induced RFMs are in columns 1, 2 and 3 and the modified settings, which no longer correspond to induced RFMs, are in columns 4, 5 and 6. The results are reported across many settings, including \(d \in \{5, 10, 50\}\), \(k \in \{3, 5, 10\}\) and \(\alpha \in \{0.005, 0.05, 0.5\}\), with a fixed sample size of \(T = 100\) and least-squares \(\ell_2\) loss. The initial entries in the factors were randomly selected from unit-variance Gaussian distributions with increasing mean values \(\mu \in \{0, 5, 10, 15, \ldots, 45\}\). For objective values the induced RFMs have relative differences within 0.1%, suggesting they are equivalent global optima, whereas the modified objectives have significantly larger relative differences, between 10% to 150%, demonstrating clearly different local minima. The relative differences between two solutions \(Z_1 = D_1 H_1\) and \(Z_2 = D_2 H_2\) are similarly small, though unsurprisingly larger than the objective values. Particularly for the elastic net, the larger the variable, the more opportunity to select which entries in \(Z\) will be zeroed with similar objective values. The trend, however, remains consistent, with smaller relative differences for the induced RFMs.
The optimality of induced RFMs has been predicated on the selection of the inner dimension \( k \) for \( d < k < T \), but this is less well-understood. For example, for subspace dictionary learning, it is critical to understand how to set this parameter. For certain models, the selection of \( k \) is intuitive. For sparse coding, the increase in the \( \alpha \) does not naturally lead to a decrease in \( k \), but rather to an increase in the level of sparsity. Further, the objective decreases as \( k \) increases to \( T \). The elastic net regularizer, however, has less intuition. For the elastic net there is likely a preference for \( d' \approx k \approx T \). We can see from Figure 5 that for sparse coding (i.e., \( \nu = 0 \)), the optimal solution will have an implicit \( k \) that is smaller than \( T \). The results are summarized in Figure 5. To avoid biasing the choice of \( k \) on a synthetic dataset, where we would choose \( k \), these results are reported on the Extended Yale Face Database B (Georghiades et al. 2001).

The goal of these exploratory results is to better elucidate the impact of using the elastic net regularizer on \( k \), and to facilitate practical choices for \( k \). We can see from Figure 5 that for sparse coding (i.e., \( \nu_H = 0 \) and \( \nu_D = 1 \)), the choice of \( k \) does need to be noticeably larger than for the elastic net (i.e., all other settings).

Additionally, the theoretical results indicate that \( k \) can be less than the true induced dimension, and alternating minimization can still give global solutions for that \( k \) if \( R_k \) is convex. We additionally provide a heatmap in Figure 6 demonstrating the optimality of alternating minimization for \( k < T \). Here, we do not compare to the optimal solution to the induced regularizer \( R_k \). Rather, as with the table of results, we measure the differences of solutions when starting from very different initial points. We find that for even very small \( k \), AM-RFM produces global solutions. This behavior suggests that either \( R_k \) is convex for these \( k \), or potentially that \( R_k \) has other nice properties not currently characterized by our theory. This empirical

### Table 3: The minimum and maximum relative norm differences between solutions found by alternating minimization.

| LOSS       | SUBSPACE          | SPARSE             | ELASTIC NET (NON-NORM) | ELASTIC NET (NON-NORM) | COUPLED COLS | COUPLED COLS |
|------------|-------------------|--------------------|------------------------|------------------------|--------------|--------------|
|            | \( f_2^2 = \| \cdot \|^2 \) | \( f_2^2 = \| \cdot \|^2 \) | \( \nu \| \cdot \|^2 + (1 - \nu)\| \cdot \|^2 \) | \( \nu_D = 0.5 = \nu_H \) | \( \Sigma_{i=1}^d \| \mathbf{D}_i \|_2^2 \) | \( \Sigma_{i=1}^d \| \mathbf{D}_i \|_2^2 \) |
| LEAST-SQ (MIN) | 0.0000000         | 0.0000000          | 0.0000000              | 0.0000000              | 0.0078000   | 0.0040000    |
|             | (0.005, 5, 3)     | (0.005, 5, 3)      | (0.5, 5, 3)            | (0.5, 5, 3)            | (0.005, 50, 3) | (0.05, 5, 10) |
| LEAST-SQ (MAX) | 0.0050000         | 0.0234000          | 0.1248000              | 0.9944000              | 0.6310000   | 0.5920000    |
|             | (0.05, 10, 10)    | (0.05, 50, 3)      | (0.5, 50, 10)          | (0.5, 50, 3)          | (0.5, 10, 10) | (0.005, 50, 3) |

5.4 Selecting the inner dimension

The optimality of induced RFMs has been predicated on the selection of the inner dimension \( k \). Because \( k \) can be critical, it is important to understand how to set this parameter. For certain models, the selection of \( k \) is intuitive. For example, for subspace dictionary learning, \( k \) is the size of the desired latent rank, where \( k \) can be chosen smaller for larger values of \( \alpha \). For sparse coding, the increase in the \( \alpha \) does not naturally lead to a decrease in \( k \), but rather to an increase in the level of sparsity. Further, the objective decreases as \( k \) increases to \( T \). The elastic net regularizer, however, has less intuition. For the elastic net there is likely a preference for \( d < k < T \), but this is less well-understood.

There are several strategies that can be pursued to facilitate choice of \( k \). A simple approach is to allow the optimization to select \( k \) by setting \( k = T \). The optimal solution will have an implicit \( k \) that is smaller than \( T \). This approach, however, is typically not computationally feasible. To avoid setting \( k \) too large, a number of algorithms have been developed that iteratively generate columns and rows to add to \( \mathbf{D} \) and \( \mathbf{H} \) respectively (d’Aspremont et al. 2004; Bach et al. 2008; Journée et al. 2010; Zhang et al. 2012; Hsieh and Olsen 2014; Mirzazadeh et al. 2015). Such a strategy could be combined with AM-RFM, particularly if an exact alternating is performed; this approach, however, creates a more complicated algorithm and reduces the simplicity and efficiency of AM-RFM. In practice, moreover, it is common to choose a fixed \( k \), and a user is more likely to pursue this simpler strategy. We therefore supplement these previous algorithmic approaches by empirically providing insights into the choice of \( k \) for the elastic net, with AM-RFM. We compare the objective values for increasing sizes of \( k \), compared to the least constrained setting of \( k = T \). The results are summarized in Figure 5.
result suggests that we could obtain a stronger theoretical result than Theorem 9 with fewer restrictions on \( k \) for sparse coding. We discuss this outcome further in the conclusion.

6. Incremental estimation

In this section, we explore how to effectively learn \( \mathbf{D} \) incrementally. We discuss an incremental algorithm for least-squares losses that summarizes past data (online AM-RFM) and discuss how to use stochastic gradient descent updates, particularly providing insights into step-size selection.

6.1 Online AM-RFM algorithm

For a least-squares loss, a more sample efficient incremental algorithm can be used to optimize induced RFMs. This algorithm is summarized in Algorithm 6 It is a straightforward modification of Mairal et al. (2010, Algorithm 1), which was introduced for sparse coding and for which they proved convergence to stationary points Mairal et al. (2010, Proposition 1). The main modification is to use a regularizer on \( \mathbf{D} \) instead
Figure 6: Standard deviations of the objective value for $k \leq T$ for elastic net RFMs, over 10 runs of different random initializations with the same regularization parameters. The settings of experiments are the same as in Figure 5, where (a) - (c) corresponds to least-squares loss and (d) - (f) to logistic loss, with the goal to identify if the solutions returned for highly different initial points result in different local minima. The standard deviation is very small for all choices of $k$, suggesting that the AM-RFM algorithm obtains a globally optimum solution even for small $k$.

of a projection onto constraint set. Mairal et al. (2010) discuss other regularizers for $h$ and constraint sets for $D$, but do not discuss replacing the constraint sets with regularizers on $D$. Another important difference is that, given the above results about stationary points, if the algorithm converges to a stationary point, then it is likely to be a global solution. Note that Mairal et al. (2010, Section 3.4) discuss several algorithmic improvements, including down-weighting past samples and mini-batches; these ideas extend directly to this algorithm and so we do not repeat them here.

6.2 Stochastic gradient descent for RFMs

Stochastic gradient descent is another approach to incremental estimation of induced RFMs. The stochastic gradient descent approach we found effective for induced RFMs is summarized in Algorithm 7, which is more lightweight and allows more general losses than online AM-RFM but is likely to be less sample efficient than online AM-RFM. As discussed in Section 3.4 only $D$ is updated incrementally, according to the loss

$$l_t(D) = \alpha R_D(D) + \left( \min_h L_x(Dh, x_t) + \frac{\nu}{2} R_H(h) \right).$$

Therefore, on each step, the optimal $h$ is compute for the current $D$ and data point $x_t$, and then a gradient step is performed for $D$ using $l_t(D)$. Mairal et al. (2009a) found that for sparse coding, their online AM-RFM algorithm converged more quickly in terms of samples than stochastic gradient descent. We find, however, that this is not always the case, particularly by taking advantage of the recent understanding of accelerations
cally, a list of strictly decreasing step sizes is predefined, with \( \eta \) is only decreased when the inner-product between two successive gradient estimates are negative. Specifically, a list of strictly decreasing step sizes is predefined, with \( \eta \) initialized to the first step size in this list. If \( \text{tr}(\nabla l_t(D_{t-1})^\top \nabla l_t(D_{t-2})) \geq 0 \), then \( \eta_t = \eta_{t-1} \); otherwise, the step-size is chosen such that \( \eta_t < \eta_{t-1} \), from the list of step-sizes.

**Algorithm 6** Online AM-RFM

\[ D_t = \arg\min_{D \in \mathbb{R}^{d \times k}} \frac{1}{T} \sum_{t=1}^{T} \left( \frac{1}{2} \|D h_t - x_t\|_2^2 + \alpha R_H(h_t) \right) + \alpha R_D(D) \]

\[ = \arg\min_{D \in \mathbb{R}^{d \times k}} \frac{1}{2} \text{tr}(D^\top DA_t) - \text{tr}(D^\top B_t) + \alpha R_D(D) \]

**Algorithm 7** SGD AM-RFM

\[ D_t = D_{t-1} - \eta_t (\nabla l_t(D_{t-1}) + \alpha \nabla R_D(D_{t-1})) \]

\[ \nabla l_t(D_{t-1}) = \nabla L_x(D_{t-1}h, x_t)^\top h_t \]

**Accelerated SG.** The first acceleration involves an aggressive step-size selection strategy that enables a constant step-size for several iterations. This strategy gives a more aggressive step-size, that can speed convergence, but that adaptively is decreased to ensure convergence (Kesten 1958). On each step, the step-size is only decreased when the inner-product between two successive gradient estimates are negative. Specifically, a list of strictly decreasing step sizes is predefined, with \( \eta_0 \) initialized to the first step size in this list. If \( \text{tr}(\nabla l_t(D_{t-1})^\top \nabla l_t(D_{t-2})) \geq 0 \), then \( \eta_t = \eta_{t-1} \); otherwise, the step-size is chosen such that \( \eta_t < \eta_{t-1} \), from the list of step-sizes.
Momentum. The second acceleration uses a momentum term, which is the difference between two successive iterations (see for example the algorithm by Tseng (1998)). The update for $D$ is instead

$$D_t \leftarrow D_{t-1} - \eta_t (\nabla l_t(D_{t-1}) + \alpha \nabla R_D(D_{t-1})) + \beta_t (D_{t-1} - D_{t-2})$$

Notice that without the momentum term, $D_{t-1} - D_{t-2} = -\eta_{t-1} \nabla l(D_{t-2})$. The momentum term is stepping further along the direction of this previous gradient. Momentum’s acceleration effects are mainly supported by theories in the batch optimization. All bets are off in the stochastic settings (Sutskever et al., 2013). However, in practice momentum term works fine in accelerating the convergence of SGD methods. Our experiments in Section 6.3 show that adding momentum term can accelerate SGD AM-RFM, but not always.

Finally, unlike online AM-RFM, we used subgradient descent updates for both $h$ and $D$ for SGD AM-RFM. Using proximal gradient updates for either $h$ or $D$ resulted in $h$ becoming progressively more sparse and the resulting solution was poor. This remains an important open question for future work in using stochastic gradient descent for induced RFMs.

6.3 Experimental results

In this section, we empirically investigate the properties of these incremental estimation strategies for induced RFMs. Previous sections indicated the global optimality of the alternating minimization for the batch setting; our goal in this section is to empirically demonstrate that this result holds for incremental estimation, and provide insight into which strategies are most effective for incremental estimation. In addition to online AM-RFM and the accelerations to SGD AM-RFM, we explore more basic step-size selection strategies, including constant step-sizes and standard decay schedules. We include results for poorly selected step-sizes, and do not necessarily advocate for any one algorithm. Our overall goal is to provide preliminary insights into the incremental properties for induced RFMs, to make it simpler for others to practically select and use these algorithms.

The experiments compare the objective value of the incremental approaches to the global batch solution. At each iteration, the objective value for the current $D_t$ is computed as $\alpha R_D(D_t) + \min_{H \in \mathbb{R}^{k \times T}} L(D_t H) + \frac{\alpha}{2} R_h(H)$. The loss function is a least-squares loss, with $\alpha = 0.05$. The data is synthetically generated from a unit-variance Gaussian, with $d = 50$ and $T = 100$. A smaller $T$ was chosen to make it computationally feasible to run the batch algorithm. The incremental algorithms iterate over the dataset multiple times, with a random reshuffling of data each time. The results are summarized in Figures 7, 8 and 9. Figure 7 investigates properties of fixed step-sizes and decay schedules for SGD AM-RFM. Figure 8 and 9 investigate the properties of the all the algorithms, including SGD AM-RFM with accelerations and online AM-RFM, on synthetic and real data separately.

The results are described more fully in the figure captions, but we provide the overall conclusions here. 1) SGD converges to a good solution much more efficiently than batch AM-RFM; 2) both accelerations to SGD AM-RFM improve the convergence rate and 3) online AM-RFM takes much time (even though when it takes only a few iterations) to converges to a good solution, without the need to sweep any parameters. The selection of step-sizes for SGD is not particularly sensitive, and even a relatively aggressive step size with decay converged after some oscillation. The step-size selection is even less sensitive with acceleration, because a more conservative list of step-sizes can be chosen. In this way, the algorithm can keep a larger step size for a number of steps until the sign changes and then pick a more conservative step-size from the list. However, if one wants to avoid picking a step-size and more memory and computation is available, online AM-RFM is a reasonable alternative.
Figure 7: Comparison of SGD AM-RFM to the global solution for varying $\alpha_0$ and step-size decays. The initial step-size is $\eta_0 \in \{0.05, 0.5, 5.0\}$. There are three time-based decays to get $\eta_t$: $\eta_0$ (type 1), $\eta_0/\sqrt{t}$ (type 2) and $\eta_0/t$ (type 3). In general, we found type 2 to perform the best across $\eta_0$, and found $\eta_0 = 0.5$ to produce the fastest, most stable convergence. We therefore more specifically report results for these two settings, with (a), (b) for $\eta_0 = 0.5$ and varying types. These results indicate that learning can be significantly different across these choices. As expected, an aggressive initial step-size of 5.0 produces oscillations, though it does in fact converge with the type 2 decay schedule. A conservative step-size of 0.05 converges too slowly, and an overly fast decay (type 3) may converge before reaching global optima. Setting $\eta_0 = 0.5$ and using type 2 decay converges to a global solution quickly and smoothly.

7. Related work

There has been a recent focus on understanding alternating minimization for problems in machine learning, and on effective strategies for escaping saddle points, particularly for neural networks. We summarize much of this work here, to better complete the emerging picture for strategies to optimize such biconvex objectives. The most general and relevant related work is the treatise by (Haeffele and Vidal, 2015) on global optimality for a wide class of models, including regularized factor models and neural networks with rectified linear activations. They show that when the solutions are rank-deficient, in their case specifically having a column of all zeros in their factors, that local minima are in fact global minima. This work complements the results in this paper, because we focused on the overcomplete setting, where the solution is not rank deficient. Other more specific global optimality results, that largely fit within the results by (Haeffele and Vidal, 2015), include a setting with a least-squares loss, unweighted trace norm and conditions on the value of the loss at the given stationary point (Mardani et al. 2013); an unregularized setting (Abernethy et al., 2009, Proposition 5); an online robust PCA setting where the true rank is assumed to be known (Feng et al. 2013); and a recent result on local minima for deep linear neural networks (Kawaguchi 2016). A common strategy in previous efforts to obtain global solutions for biconvex problems has been to provide careful initialization strategies and assume sparsity or coherence properties on the matrices. Jain et al. (2013) showed that alternating minimization for matrix completion provides global solutions, assuming
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Figure 8: Comparison of different incremental algorithms for induced RFMs. The plots in (c) and (d) depict early learning for (a) and (b) respectively (i.e., they are zoomed into early learning in those plots). Accelerated SG is SGD AM-RFM with \( \eta_t = \eta_{t-1} \) if \( \text{tr}(\nabla l_t(D_{t-1})^\top \nabla l_t(D_{t-2})) \geq 0 \) and otherwise selects \( \eta_t = \eta_0/\sqrt{t} \) (type 2) or \( \eta_t = \eta_0/t \) (type 3). For momentum, \( \beta_t = 0.01 \). Both of these accelerations outperformed the best setting for standard SGD AM-RFM (with \( \eta_0 = 0.5 \) and type 2 decay), for subspace dictionary learning. For elastic net, however, the accelerations appeared to have little impact, potentially due to the fact that convergence of the standard SGD AM-RFM was already fast. A surprising observation is that online AM-RFM decreases the function value very fast at the beginning, but then slows before converging to the global solution. Finally, we report the runtime of SGD AM-RFM. The two marks in each plots denote convergence and runtime information based on Accelerated SG. The black circle denotes the step where Accelerated SG is within 5% of the batch solution; the black square denotes the step where Accelerated SG spends as much cumulative time as the batch one. If a mark is missing in a plot, then the mark is beyond the maximum iterations. These black marks are similar for the other stochastic approaches. This convergence is within 30% – 40% of the time needed for the global solution. This is a serious underestimate of the true gains of stochastic gradient descent, because \( T = 100 \) was chosen to make the comparison to the global batch solution computationally feasible. In practice, this value is much larger and stochastic gradient descent becomes the obvious choice for estimating induced RFMs if efficiency is a concern. The red circle and square have the same meanings as the black ones except for that they are for online AM-RFM. In contrast to black marks, this convergence is more than 10 times of the time needed for the global solution, demonstrating that online AM-RFM is computationally costly and slow convergence while it is sample efficient and with no need to select step-size explicitly.

that the given matrix and optimal factors are incoherent matrices (Jain et al., 2013, Definition 2.4), which enables them to define a suitable initial point using a singular value decomposition. Gunasekar et al. (2013) and Hardt (2014) further generalize these assumptions, particularly to the noisy setting and Netrapalli et al. (2015) to the setting for phase retrieval. For sparse coding, Spielman et al. (2012) showed how to recover the optimal dictionary exactly, assuming it has full column rank and assuming a maximum sparsity level on the basis. Since then, most work has focused on the overcomplete case, where the dictionary is not full column rank. Agarwal et al. (2014a) showed that alternating minimization for sparse coding with thresholding gives
global solutions, if the dictionary satisfies the restricted isometry property (Candès and Tao, 2005) and using a provable strategy to initialize the dictionary to be near optimal. Previously, both Agarwal et al. (2014b) and Arora et al. (2014) had independently provided a similar algorithm, with incoherence requirements on the matrices, with slightly different requirements on the sparsity level. Arora et al. (2015) built on this work and expanded the level of possible sparsity.

A separate line of work has required fewer assumptions on the true matrices and pursued initialization strategies by generating candidate columns, typically using singular value decompositions (d’Aspremont et al., 2004; Bach et al., 2008; Journé et al., 2010; Zhang et al., 2012; Mirzazadeh et al., 2015). The bulk of this work has been on the semi-definite case, where the two factors are the same, $Z = DD^\top$. The exception is the GCG algorithm (Zhang et al., 2012), which generates columns and rows for $D$ and $H$ for some of the induced RFMs considered in this work; however, even with this initialization, they do not guarantee convergence to global solutions. Most of this work has not characterized the saddle-points of the non-convex optimization, with the exception of a recent work for the semi-definite setting which identified a smoothness condition on the gradient of the loss that ensured convergence to global solutions (Mirzazadeh et al., 2015, Proposition 2).

Another strategy has been to use the convex reformulations obtained for induced RFMs. For example, instead of alternating on $D$ and $H$ in the subspace formulation, $Z$ is solved for directly in trace norm regularized problem. From there, the factors $D$ and $H$ could typically be recovered, usually with iterative methods. The problems tackled with this approach include relaxed rank exponential family PCA (Bach et al., 2008; Zhang et al., 2011), matrix completion (Cai et al., 2010), (semi-)supervised dictionary learning (Goldberg et al., 2010; Zhang et al., 2011), multi-view learning (White et al., 2012), co-embedding (Mirzazadeh et al., 2014) and autoregressive moving average models (White et al., 2015). These approaches, however, are often not as scalable as learning the factors $D$ and $H$, which can be smaller. Further, because $Z$ grows with the size of the data, and the induced regularizer often couples entries across $Z$, the induced convex formulation is typically not amenable to incremental estimation. As an additional problem, for the general class of induced RFMs, the induced convex formulation is only of theoretical value, since there is no known form for the induced norm and so the optimization over $Z$ cannot be performed. There has been some effort to obtain ap-

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**Figure 9**: Comparison of different incremental algorithms for induced RFMs on data randomly drawn from the Extended Yale Face Database B, with $T = 1000$ and $d = 100$. All the settings are the same as in Figure 8 except for that the black marks are for Accelerated SG ($5.0 + \text{type3}$) rather than Accelerated SG ($0.5 + \text{type2}$). In this experiment, setting $\eta_0 = 5.0$ and using type 3 decay converges faster than setting $\eta_0 = 0.5$ and using type 2. Accelerated SG still accelerates SGD AM-RFM in (a), and appears to have little impact in (b) where SGD AM-RFM already converges fast. The methods with momentum term are not shown in the plots, because they are almost the same as or even a bit worse than the ones without momentum, denoting that adding momentum is not guaranteed to improve the convergence of the SGD algorithm. We also notice that online AM-RFM always needs much more time to converge than all the stochastic methods, even though it converges within fewer steps. For example, in (b), the time needed for online AM-RFM to reach the red circle is around 1400% of the time for Accelerated SG ($5.0 + \text{type3}$) to reach the black circle, where the solutions are within 5% of the batch solution, despite that the red circle is within much fewer steps than the black one.

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(a) Subspace

(b) Elastic net ($\nu_H = 1.0, \nu_D = 0.5$)
proximations to the convex form for the elastic net, which does not have a known induced form (Bach et al. 2008); the proposed approach, however, uses a convex lower bound and the relation to the elastic net solution is unclear. The alternating minimization approach avoids the need to have an explicit form for the induced norm, and rather only uses the property that this class has induced forms to identify promising situations for global optimality.

For incremental estimation of induced RFMs, there are significantly fewer approaches. Outside of incremental principal components analysis (Warmuth and Kuzmin, 2008; Feng et al., 2013) and partial least-squares (Arora et al., 2012), little is known about how to obtain optimal incremental RFM estimation and several approaches have settled for local solutions (Mairal et al., 2009a,b. 2010). De Sa et al. (2015) prove convergence to a global solution, with random initialization, using incremental, stochastic gradient descent algorithms for an interesting subset of induced RFMs, including for matrix completion, phase retrieval and subspace tracking. These problems, however, are formulated as semi-definite problems, again by learning $Z = DD^\top$.

8. Conclusion and Discussion

In this work, we addressed tractable estimation of induced regularized factor models (RFMs), which constitute a broad and widely used formalism for unsupervised, semi-supervised and supervised learning problems. We generalized the class of induced RFMs and provided several new objectives and characterizations of stationary points. We proved that local minima are global minima and that there are no degenerate saddlepoints for a subclass of induced RFMs and further empirically demonstrated that alternating minimization results in global minima for a broader class. We applied these insights to develop efficient batch and incremental estimation strategies for RFMs.

A key insight from this paper is that we can identify tractable dictionary learning objectives using the class of induced RFMs. For those dictionary learning methods that can be specified as induced RFMs, we can show that local minima are global minima and, in some cases, that there are no degenerate saddlepoints. Consequently, a simple alternating minimization scheme is effective for obtaining global solutions. To be more precise for specific settings of interest, our general result in Theorem 9 shows that local minima are global minima

1. matrix completion using squared regularizers on the factors, with any invertible weighting on the squared regularizer on $D$, where the inner dimension $k$ can be set less than $d$

2. sparse coding with a pseudo-Huber smooth approximation to the $\ell_1$ regularizer, for overcomplete $k$

3. dictionary learning with the elastic net on either $D$ and $H$, where again the $\ell_1$ regularizer is approximated with the pseudo-Huber function and $k$ is overcomplete.

The result is proved for the more general class of induced RFMs, as opposed to previous results which were proved for each specific setting. Therefore, though the results from this paper provide novel insights for each of these specific settings, the larger contribution is to provide a new proof strategy for characterizing stationary points of dictionary learning methods.

The generality of this result, however, is limited in two important ways. First, to characterize saddlepoints using directional derivatives, we have assumed that the losses and regularizers are twice differentiable. Consequently, the results do not directly apply to standard sparse regularizers, such as $\ell_1$. Nonetheless, they do apply to smooth approximations, such as the pseudo-Huber smooth approximation. The pseudo-Huber function has a parameter $\sigma$; when $\sigma \to 0$, the pseudo-Huber approaches $\ell_1$. The results in this paper apply to arbitrarily small $\sigma$; a limit argument should extend the results to these non-smooth regularizers. An important next step, therefore, is to formally characterize this intuition, so that these results directly apply to settings such as sparse coding.

A second limitation is that the results do not yet provide a precise specification for the inner dimension, $k$. The proof requires $k$ to be large enough, to ensure that $R_k$ is convex. We know that for $k$ at least as large
as the true induced dimension, then $R_k$ is equal to the induced regularizer $R_Z$ which is guaranteed to be convex. For smaller $k$, however, $R_k$ could still be convex, or potentially have nice properties such as pseudo-convexity. The increase of $k$ has a diminishing returns property, and we expect $R_k$ to behave similarly to $R_Z$ before $k$ actually reaches the magnitude of the true induced dimension. In fact, we find that $k$ can be quite a bit smaller than the true induced dimension and alternating minimization still provides global solutions. We hypothesize that this is in fact because $R_k$ even for small $k$ is already close to being a convex function (e.g., it could be pseudoconvex). To provide an even more general result, and to characterize what we see in practice, a useful direction would be to extend the proof to pseudo-convex induced regularizers and to investigate specific choices of regularizers and $k$ that produce pseudo-convex induced regularizers.

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### Appendix A. Summary of regularized factor models

For further justification for why these below problems can be written in this form, see (White, 2014 Section 3.4).

| Algorithm                      | Loss and Constraints |
|--------------------------------|----------------------|
| CCA ≡ orthonormal PLS          | $\| (XX^{-1})^T X - DH \|_F^2$ |
| Information theoretic clustering | $D_F(x \| Dx)$ with $F(x) = \sum_{i=1}^d x(i) \log x(i) - 1^T x \{H \in \{0, 1\}^{k \times T}, 1H = 1\}$ |
| Isomap                         | $\| K - DH \|_F^2$ with $K = -\frac{1}{2}(I - ee^T)S(I - ee^T)$ with $S_{i,j} = \| x_i - x_j \|$ |
| K-means clustering             | $\| X - DH \|_F^1 \{H \in \{0, 1\}^{k \times T}, 1H = 1\}$ |
| K-medians clustering           | $\| X - DH \|_F^1 \{H \in \{0, 1\}^{k \times T}, 1H = 1\}$ |
| Laplacian eigenmaps ≡ Kernel LPP | $\| K - DH \|_F^2$ for $K = L$ |
| Linde-Buzo-Gray                | $D_F(x \| Dx) = \frac{\partial}{\partial x} \log x - 1 \{H \in \{0, 1\}^{k \times T}, 1H = 1\}$ |
| Locally linear embeddings      | $\| K - DH \|_F^2$ with $K = (I - ee^T)K(1 - ee^T)$ with $K = (\lambda_{\max}(M)I - M)$ |
| Maximum margin matrix factorization | $\sum_{i,j}^R \sum_{x_{ij} \neq 0} \text{hinge}(DH - \text{Bias}_{i,r})$ |
| Metric multi-dimensional scaling | $\| K - DH \|_F^2$ for isotropic kernel $K$ |
| Nonnegative matrix factorization | $\| X - DH \|_F^2$ for $D > 0, H > 0$ |
| Normalized-cut                 | $\| (\Lambda^{-1}X - DH)\Lambda^{1/2} \|_F^2$ |
| Partial least squares          | $\| XY^T - DH \|_F^2$ |
| PCA                            | $\| X - DH \|_F^2$ |
| exp. family PCA                | $D_F(\| DH \|_F - (\| X \|_F))$ |
| kernel PCA                     | $\| K - DH \|_F^2$ |
| sparse PCA                     | $\| X - DH \|_F^2$ for $R_H(H) = \ell_0(H)$ |
| Probabilistic latent semantic indexing | $\| X - DH \|_F^2$ for $D > 0, H > 0$ |
| Ratio cut                      | $\| K - DH \|_F^2$ |

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Appendix B. Additional information about regularizers

**Proposition 14** For any $m$ non-negative convex functions $g_i : \mathbb{R}^d \to \mathbb{R}^+$, $g(d) = \sqrt{g_1^2(d) + \ldots + g_m^2(d)}$ is convex. Further, if the $g_i$ are all norms, then $g$ is a norm.

**Proof:** Notice that $g$ can be seen as the composition of the $\ell_2$ norm on the vector of $m$ values of $g_i$ applied to $d$. We can use the convexity properties of both to obtain the result. Take any $\eta \in [0, 1]$ and any $a, b \in \mathbb{R}^d$.

\[
g(\eta a + (1 - \eta)b) = \ell_2([\eta g_1(a) + (1 - \eta)g_1(b), \ldots, \eta g_m(a) + (1 - \eta)g_m(b)])
\leq \ell_2([\eta g_1(a), \ldots, g_m(a)] + (1 - \eta)[g_1(b), \ldots, g_m(b)])
= \ell_2(\eta [g_1(a), \ldots, g_m(a)] + (1 - \eta)[g_1(b), \ldots, g_m(b)])
\leq \eta g_1(a) + (1 - \eta)g_1(b)
\]

The first inequality follows from the fact that $\ell_2$ is monotonically increasing on the set of non-negative values and the $g_i$ are all non-negative, so for any $d \in \mathbb{R}^d$, $b \in \mathbb{R}^+$ such that, $g_i(d) \leq b$ implies $g_i^2(d) \leq b^2$. Therefore, $g$ is convex.

If the $g_i$ are norms, then clearly $g$ satisfies absolute homogeneity $g(\eta d) = \eta g(d)$ and $g(d) = 0$ implies $d = 0$. The triangle inequality follows similarly to above, obtained by setting $\eta = 0.5$ and using the absolute homogeneity of $g$. \hfill \Box

We can use this result to use separate norms on different parts of the columns or rows of $D$ and $H$. This is particularly useful for supervised learning, where $D$ is partitioned into two components, one for the unsupervised recovery and the other for supervised learning.

**Corollary 15** For some $m < d$ with two functions $g_1 : \mathbb{R}^m \to \mathbb{R}^+$ and $g_2 : \mathbb{R}^{d-m} \to \mathbb{R}^+$ which act on the two partitions of $d \in \mathbb{R}^d$, the regularizer $g(d) = \sqrt{g_1^2(d_{1:m}) + g_2^2(d_{(m+1):d})}$ is a convex, non-negative function in $\mathbb{R}^d$.

**Proof:** We can extend both functions $g_1$ and $g_2$ to the domain $\mathbb{R}^d$, simply by having those parts not influence the function value. These extended functions are still non-negative, and so satisfy the conditions of Proposition 14. The function $g$ defined with the extended functions is exactly the same, and so we obtain the desired result. \hfill \Box

Appendix C. Proof of Global Optimality for Weighted Frobenius Norm

**Theorem 16** For invertible $A \in \mathbb{R}^{d \times d}$, $\alpha \geq 0$ and $s > 0$, let $(\hat{D}, \hat{H})$ be a stationary point of

\[
\min_{D \in \mathbb{R}^{d \times k}, H \in \mathbb{R}^{k \times T}} L(DH) + \frac{s}{2} \|DA\|_F^2 + \frac{\alpha}{2s^2} \|H\|_F^2. \tag{18}
\]

If either

1. $\|A^{-T} \nabla L(DH)\|_2 \leq \alpha/s$ (i.e., the maximum singular value is $\alpha/s$)
2. the Hessian is positive semi-definite with $(\hat{D}, \hat{H})$ rank-deficient; or
3. the Hessian is positive semi-definite with $k$ overcomplete

then $Z = DH$ is a globally optimal solution to

\[
\arg\min_{Z \in \mathbb{R}^{d \times T}} L(Z) + \frac{\alpha}{s} \|AZ\|_{tr}. \tag{19}
\]
Proof:
Without loss of generality, we can absorb $s$ into $\Lambda$, and avoid the cluttered additional weighting on $H$. This can be done by setting $\Lambda = s\Lambda_{\text{original}}$ and $\alpha = \frac{\alpha_{\text{original}}}{s}$. Notice that if $\|\Lambda_{\text{original}}^{-1}\nabla L(DH)\|_2 \leq \alpha/s$ then $\|\Lambda^{-1}\nabla L(DH)\|_2 \leq \alpha$.

Since $\bar{D}$ and $\bar{H}$ are stationary points of (18), we know
\[
\nabla_{\bar{D}} = (\nabla L(DH)) \bar{H}^T + \alpha\Lambda^T\Lambda \bar{D} = 0
\]
\[
\nabla_{\bar{H}} = \bar{D}^T (\nabla L(DH)) + \alpha \bar{H} = 0
\]

By multiplying both sides by $\bar{D}^T$ or $\bar{H}^T$ respectively, and taking the trace, we get
\[
\text{tr} ( (\nabla L(DH)) \bar{H}^T \bar{D}^T ) + \alpha \text{tr} (\Lambda^T\Lambda \bar{D} \bar{D}^T ) = 0
\]
\[
\text{tr} (\bar{H}^T \bar{D}^T (\nabla L(DH))) + \alpha \text{tr} (\bar{H}^T \bar{H}) = 0
\]

Now, we know that the trace norm can be characterized as follows (Recht et al., 2010)
\[
\|Z\|_{tr} = \min_{W_1, W_2} \frac{1}{2} \text{tr}(W_1) + \frac{1}{2} \text{tr}(W_2) \text{ s.t. } W = \begin{pmatrix} W_1 & Z \\ Z^T & W_2 \end{pmatrix} \succeq 0
\]
giving weighted norm
\[
\|\Lambda Z\|_{tr} = \min_{W_1, W_2} \frac{1}{2} \text{tr}(W_1) + \frac{1}{2} \text{tr}(W_2) \text{ s.t. } W = \begin{pmatrix} W_1 & \Lambda Z \\ Z^T \Lambda^T & W_2 \end{pmatrix} \succeq 0
\]

A Lagrange multiplier $M \in \mathbb{R}^{(d+T)\times(d+T)}$ can be used to bring the constraint $W \succeq 0$ into the objective and consider the Lagrangian with $M \succeq 0$:
\[
\mathcal{L}(Z, W_1, W_2, M) = L(Z) + \frac{\alpha}{2} \text{tr}(W_1) + \frac{\alpha}{2} \text{tr}(W_2) - \text{tr}(M^T W)
\]

Notice first that, writing $M$ in a block structure,
\[
W^T M = \begin{pmatrix} W_1^T \\ Z^T \Lambda^T \\ W_2^T \end{pmatrix} \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix} = \begin{pmatrix} W_1^T M_1 + \Lambda Z M_3 \\ Z^T \Lambda^T M_1 + W_2^T M_3 \\ W_1^T M_2 + \Lambda Z M_4 \\ Z^T \Lambda^T M_2 + W_2^T M_4 \end{pmatrix}
\]
we get $\text{tr}(M^T W) = \text{tr}(W^T M) = \text{tr}(W_1^T M_1) + \text{tr}(\Lambda Z M_3) + \text{tr}(Z^T \Lambda^T M_2) + \text{tr}(W_2^T M_4)$.

The KKT conditions for this optimization require

Gradient conditions:
\[
\nabla \mathcal{L}(Z, W_1, W_2, M) = \nabla L(Z) - \Lambda^T M_3 - \Lambda^T M_2 = 0
\]
\[
\nabla_{W_1} \mathcal{L}(Z, W_1, W_2, M) = \frac{\alpha}{2} I_d - M_1 = 0
\]
\[
\nabla_{W_2} \mathcal{L}(Z, W_1, W_2, M) = \frac{\alpha}{2} I_T - M_4 = 0
\]

Complementary slackness:
\[
\text{tr}(M^T W) = 0
\]

Primal feasibility:
\[
W \succeq 0
\]

Dual feasibility:
\[
M \succeq 0
\]

We can introduced candidate variables using the stationary points $\bar{D}$ and $\bar{H}$ of the nonconvex objective (18): $Z = DH$, $W_1 = ADD^T \Lambda^T$, $W_2 = H^T H$, $M_1 = \frac{\alpha}{2} I_d$, $M_4 = \frac{\alpha}{2} I_T$, $M_2 = \frac{\alpha}{2} \Lambda^{-1} \nabla L(Z)$ and $M_3 = M^T_2$. 

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These variables clearly satisfy all the gradient conditions. Complementary slackness holds since

\[
\text{tr} (M^T W) = \text{tr} (\Lambda D D^T \Lambda^T + \frac{1}{2} I_d) + \text{tr} (\Lambda D H \nabla L(Z)^T \Lambda^{-1}) + \\
\text{tr} (H^T D^T \Lambda^T \frac{1}{2} \Lambda^{-T} \nabla L(Z)) + \text{tr} (H^T H \frac{1}{2} I_T) \\
= \frac{3}{2} \text{tr} (\Lambda D D^T \Lambda^T) + \frac{1}{2} \text{tr} (\Lambda D H \nabla L(Z)^T) + \frac{1}{2} \text{tr} (H^T D^T \Lambda^{-T} \nabla L(Z)) + \frac{3}{2} \text{tr} (H^T H) \\
= 0
\]

because of (22) and (23) and because the trace satisfies a circular property, \(\text{tr} (ABC) = \text{tr} (BCA) = \text{tr} (CAB)\) and equality under transpose, \(\text{tr} (A) = \text{tr} (A^T)\).

To show primal feasibility, notice that

\[
\begin{align*}
W &= \begin{pmatrix} \Lambda D D^T \Lambda^T & \Lambda D H \\ H^T D^T & H^T H \end{pmatrix} \\
&= \begin{pmatrix} \Lambda & \Lambda \\ H^T & H^T \end{pmatrix} \begin{pmatrix} \Lambda & \Lambda \\ \Lambda & \Lambda \end{pmatrix}^T \\
&\succeq 0
\end{align*}
\]

To show dual feasibility, we can use the Schur complement condition of block matrices: \(\mathbf{M} \succeq 0\) iff \(\mathbf{M}_1 > 0\) and \(\mathbf{M} - \mathbf{M}_1 \mathbf{M}_2^{-1} \mathbf{M}_2 \succeq 0\). Clearly, \(\mathbf{M}_1 = \frac{3}{2} I_d > 0\), since \(\alpha > 0\). Now,

\[
\mathbf{M} - \mathbf{M}_1 \mathbf{M}_2^{-1} \mathbf{M}_2 = \frac{3}{2} I_T - \frac{1}{2} \nabla L(Z)^T \Lambda^{-1} \left(\frac{3}{2} I_d - \frac{1}{2} \Lambda^{-1} \nabla L(Z)\right) \\
= \frac{3}{2} I_T - \frac{1}{2} \nabla L(Z)^T \Lambda^{-1} \nabla L(Z) \\
\]

**Case 1: general stationary point.** For \(\alpha > 0\), if \(\|\Lambda^{-1} \nabla L(Z)\|_2 \leq \alpha\), then

\[
\nabla L(Z)^T \Lambda^{-1} \nabla L(Z) \leq \alpha^2 I \\
\implies \alpha^2 I_T - \nabla L(Z)^T \Lambda^{-1} \nabla L(Z) \succeq 0 \\
\implies \frac{3}{2} I_T - \frac{1}{2} \nabla L(Z)^T \Lambda^{-1} \nabla L(Z) \succeq 0.
\]

satisfying the dual feasibility requirement for this case.

If \(\alpha = 0\), then \(\|\Lambda^{-1} \nabla L(Z)\|_2 \leq 0\), and so \(\nabla L(Z) = 0\), again indicating \(Z = \tilde{D}H\) is a global optimum.

**Case 2: stationary point has a positive semi-definite Hessian. Case 2a: overcomplete \(k\).** If \(k \geq d\) and \(D\) is full rank, then the stationarity condition \(\tilde{D}^T \nabla L(Z) = 0\) implies \(\nabla L(Z) = 0\). This is because \(\tilde{D}^T\) is full column rank, and so has a left inverse \(\tilde{D}^\dagger\), giving \(\tilde{D} \tilde{D}^\dagger = I\). This is similarly the case if \(k \geq T\) and \(H\) is full rank. Otherwise, if \(D\) is not full rank, then it is rank deficient, and is covered by Case 2a below.

**Case 2b: rank deficient stationary point.** First, we note that without loss of generality, we can assume that if the rank of \(D\) is \(r\), then the last \(k-r\) columns of \(D\) are zero. This is because the Frobenius norm is invariant under orthogonal transformations. If \(D = U \Sigma_r V^T\), then we can multiply both \(D\) and \(H\) with \(V\) without affecting the objective. This rotation of \(D\) to give the last \(k-r\) columns equal to zero must similarly make the last \(k-r\) rows of \(H\) zero, because the rotated stationary point \(\tilde{H}\) must satisfy \(\tilde{D}^T (\nabla L(\tilde{D}H)) + \alpha \tilde{H} = 0\) and \(\tilde{D}\) now zeroes out the last \(k-r\) rows of the first component of the gradient. Correspondingly, they have the same rank \(r\), because otherwise another column could be zeroed in one of the variables by rotation, again forcing the other to reduce its dimension to satisfy the stationarity requirement.

To conveniently write the Hessian for matrix-valued variables, we will use the directional derivative

\[
D_Y g(Y_0)[dY] = \lim_{t \to 0} \frac{g(Y_0 + tdY) - g(Y_0)}{t} = \langle \nabla_Y g(Y_0), dY \rangle
\]

where \(\langle A, B \rangle = \text{tr}(A^T B)\). For the Hessian, we get

\[
D_Y^2 g(Y_0)[dY, dY] = D_Y \langle \nabla_Y g(Y_0), dY \rangle[dY] = \langle \nabla_Y \langle \nabla_Y g(Y_0), dY \rangle, dY \rangle, dY \rangle.
\]
In our setting, \( Y_0 = [D; \bar{H}^\top] \in \mathbb{R}^{(d + T) \times k} \) and \( g([D; \bar{H}^\top]) = L(DH) + \frac{\alpha}{2} \|AD\|^2_F + \frac{\alpha}{2} \|\bar{H}\|^2_F \). Let \( G = \nabla L(Z) \) for \( Z = \bar{D}H \), giving
\[
D_Y g(Y_0)[dY] = \left\langle \begin{bmatrix} G\bar{H}^\top + \alpha A^\top AD \\ G^\top \bar{D} + \alpha \bar{H}^\top \end{bmatrix}, \begin{bmatrix} dD \\ dH^\top \end{bmatrix} \right\rangle = \text{tr}(HG^\top dD) + \alpha \text{tr}(D^\top A^\top AdD) + \text{tr}(D^\top GdH^\top) + \alpha \text{tr}(\bar{H}dH^\top).
\]

Similarly, for the Hessian,
\[
D_Y^2 g(Y_0)[dY, dY] = \langle \nabla_Y \left( \text{tr}(HG^\top dD) + \alpha \text{tr}(D^\top A^\top AdD) + \text{tr}(D^\top GdH^\top) + \alpha \text{tr}(\bar{H}dH^\top) \right), dY \rangle = \left\langle \begin{bmatrix} \alpha A^\top AdD + GdH^\top \\ G^\top \bar{D} + \alpha \bar{H}^\top \end{bmatrix}, \begin{bmatrix} dD \\ dH^\top \end{bmatrix} \right\rangle + D_Y^2 L(Z)[\bar{D}dH + dDH, \bar{D}dH + dH\bar{D}]
= 2\text{tr}(dD^\top GdH^\top) + \alpha \text{tr}(dD^\top A^\top AdD) + \alpha \text{tr}(dH^\top dH)
+ D_Y^2 L(Z)[\bar{D}dH + dDH, \bar{D}dH + dH\bar{D}].
\]

Now we can make the last term, the Hessian w.r.t. \( L \), disappear by carefully choosing \( dH \) and \( dD \). Take \( dH = [0, \ldots, 0, h_1, \ldots, h_{k-r}]^\top \) and \( dD = [0, \ldots, 0, d_1, \ldots, d_{k-r}] \), for any vectors \( h_1, d_1 \). These variables, therefore, have the first \( r \) rows or columns respectively equal to zero and the other rows or columns are any vectors. Because \( \bar{D} \) has the last \( k - r \) columns as zeros (by rotation as discussed at the beginning of the proof), \( \bar{D}dH = U\Sigma_r [0, \ldots, 0, h_1, \ldots, h_{k-r}]^\top = 0 \). Similarly, \( dDH = 0 \). Therefore, \( D_Y^2 L(Z)[\bar{D}dH + dDH, \bar{D}dH + dH\bar{D}] = 0 \), because we product with a zero matrix.

Now, for the first term, because \( h_1 \) and \( d_1 \) are arbitrary vectors in the last \( k - r \) columns, and because we have a positive semi-definite Hessian, we get the inequality for any \( h \in \mathbb{R}^T, d \in \mathbb{R}^d \)
\[
2d^\top Gh + \alpha d^\top A^\top Ad + \alpha h^\top h \geq 0.
\]
Since this is true for any \( h \) and \( d \), we can choose \( d = -\Lambda^{-1}A^\top Gv \) and \( h = \alpha v \) for any \( v \in \mathbb{R}^T \). This gives
\[
0 \leq d^\top Gh + 2d^\top A^\top Ad + \alpha h^\top h = \alpha v^\top G^\top \Lambda^{-1}A^\top Gv - \Lambda^{-1}Av \\
= -\alpha v^\top G^\top \Lambda^{-1}A^\top Gv + \frac{\alpha}{2}v^\top G^\top \Lambda^{-1}A^\top Gv + \frac{\alpha^2}{2}v^\top v \\
= v^\top (\frac{\alpha}{2}I - \frac{\alpha}{2}G^\top \Lambda^{-1}A^\top G) v
\]
for all \( v \), which implies \( \frac{\alpha}{2}I - \frac{\alpha}{2}G^\top \Lambda^{-1}A^\top G \succeq 0 \) (by the definition of positive definite matrices). If \( \alpha > 0 \), then by dividing both sides by \( \alpha^2 \) and recalling that \( G = \nabla L(Z) \),
\[
\frac{\alpha}{2}I_T - \frac{1}{2\alpha^2} \nabla L(Z)^\top \Lambda^{-1}A^\top \nabla L(Z) \succeq 0
\]
which completes the proof for dual feasibility for the second case.

If \( \alpha = 0 \), then above we can simply choose \( d = -\Lambda^{-1}A^\top Gv \) and \( h = v \) for any \( v \in \mathbb{R}^T \), to obtain \( \nabla L(Z) = 0 \), again indicating \( Z = DH \) is a global optimum for this second case. \( \square \)

**Lemma 17** For any convex function \( R_H : \mathbb{R}^{k \times T} \) that is separable across rows \( R_H(H) = \sum_{i=1}^{k} f_r(H_{i,:}) \) with non-negative convex function \( f_r : \mathbb{R}^T \to \mathbb{R}^+ \), the following function is convex in \( Q \in \mathbb{R}^{k \times k} \),
\[
\|DQ^{-1}\|_F^2 + R_H(QH)
\]

**Proof:** We will use directional derivatives. Let \( Q(t) = Q + tB \), where we step some amount in the direction of \( B \), which need not be a full rank matrix. Then, using the Taylor series expansion, we get
\[
Q(t)^{-1} = (Q(I + tQ^{-1}B))^{-1} = Q^{-1} - tQ^{-1}BQ^{-1} + t^2Q^{-1}BQ^{-1}BQ^{-1} - \ldots
\]
Proposition 4
The stationary points are equivalent for the summed form

\[ L(DH) + \frac{\alpha}{2} \sum_{i=1}^{k} \|D_{i}c\|_F^2 + \frac{\alpha}{2} \sum_{i=1}^{k} \|H_{i}\|_F^2 \]

and the producted form

\[ L(DH) + \alpha \sum_{i=1}^{k} \|D_{i}\|_c \|H_{i}\|_c \]
The stationary points of the summed and producted forms are also stationary points of the constrained form.

**Proof:** To simplify notation, let

\[
\begin{align*}
\mathbf{h} = [\mathbf{d}_1, \ldots, \mathbf{d}_k] \\
\mathbf{v} = [\mathbf{h}_1; \ldots; \mathbf{d}_k]
\end{align*}
\]

where this notation will always implicitly be from 1 to \(k\). The stationary points \(\mathbf{D}, \mathbf{H}\) of the summed objective satisfy

\[
\nabla L(\mathbf{D}^s \mathbf{H}^s) \mathbf{H}^s + \alpha \mathbf{h} \nabla \|\mathbf{D}^s_i\|_c \|\mathbf{D}^s_i\|_c = 0
\]

\[
\mathbf{D}^s \nabla L(\mathbf{D}^s \mathbf{H}^s) + \alpha \mathbf{v} \nabla \|\mathbf{H}^s_i\|_r \|\mathbf{H}^s_i\|_r = 0
\]

the stationary points \(\mathbf{D}^p, \mathbf{H}^p\) of the producted objective satisfy

\[
\nabla L(\mathbf{D}^p \mathbf{H}^p) \mathbf{H}^p + \alpha \mathbf{h} \nabla \|\mathbf{D}^p_i\|_c \|\mathbf{H}^p_i\|_r = 0
\]

\[
\mathbf{D}^p \nabla L(\mathbf{D}^p \mathbf{H}^p) + \alpha \mathbf{v} \nabla \|\mathbf{H}^p_i\|_c \|\mathbf{H}^p_i\|_r = 0
\]

We can reweight \(\mathbf{D}, \mathbf{H}\) to get a stationary point to the producted form, and vice-versa. Let \(\mathbf{\Gamma} = \text{diag}(\|\mathbf{D}^s_1\|_c^{-1} \|\mathbf{H}^s_1\|_r, \ldots, \|\mathbf{D}^s_k\|_c^{-1} \|\mathbf{H}^s_k\|_r)\). We require \(\mathbf{\Gamma}\) to be invertible. This will always be true unless a row or column are zero in \(\mathbf{H}\) and \(\mathbf{D}\); in this case, we can always simply remove these.

For any stationary point \(\mathbf{D}^s, \mathbf{H}^s\), let \(\mathbf{D}^p = \mathbf{D}^s \mathbf{\Gamma}^{-1}, \mathbf{H}^p = \mathbf{\Gamma} \mathbf{H}^s\). Then \(\mathbf{D}^p \mathbf{H}^p = \mathbf{D}^s \mathbf{H}^s\) and

\[
\nabla L(\mathbf{D}^p \mathbf{H}^p) \mathbf{H}^p + \alpha \mathbf{h} \nabla \|\mathbf{D}^p_i\|_c \|\mathbf{H}^p_i\|_r = 0
\]

\[
\mathbf{D}^p \nabla L(\mathbf{D}^p \mathbf{H}^p) + \alpha \mathbf{v} \nabla \|\mathbf{H}^p_i\|_c \|\mathbf{H}^p_i\|_r = 0
\]

giving

\[
\nabla L(\mathbf{D}^p \mathbf{H}^p) \mathbf{H}^p \mathbf{\Gamma}^{-1} + \alpha \mathbf{h} \nabla \|\mathbf{D}^p_i\|_c \|\mathbf{H}^p_i\|_r \mathbf{\Gamma}^{-1} = 0
\]

\[
\mathbf{D}^p \nabla L(\mathbf{D}^p \mathbf{H}^p) + \alpha \mathbf{v} \nabla \|\mathbf{H}^p_i\|_c \|\mathbf{H}^p_i\|_r = 0
\]

because \(\mathbf{D}^s\) is a stationary point of the summed form. Therefore, because \(\mathbf{\Gamma}\) is invertible, this means that \(\nabla L(\mathbf{D}^p \mathbf{H}^p) \mathbf{H}^p + \alpha \mathbf{h} \nabla \|\mathbf{D}^p_i\|_c \|\mathbf{H}^p_i\|_r = 0\), and so \(\mathbf{D}^p\) is a stationary point of the producted form. For \(\mathbf{H}^p\),

\[
\mathbf{D}^p \nabla L(\mathbf{D}^p \mathbf{H}^p) + \alpha \mathbf{v} \nabla \|\mathbf{H}^p_i\|_r = 0
\]

giving

\[
\mathbf{\Gamma} \mathbf{D}^s \nabla L(\mathbf{D}^s \mathbf{H}^s) + \alpha \mathbf{h} \nabla \|\mathbf{D}^s_i\|_c \|\mathbf{H}^s_i\|_r = 0
\]

\[
\mathbf{D}^s \nabla L(\mathbf{D}^s \mathbf{H}^s) + \alpha \mathbf{v} \nabla \|\mathbf{H}^s_i\|_c \|\mathbf{H}^s_i\|_r = 0
\]

In the other direction, for any stationary point \(\mathbf{D}^p, \mathbf{H}^p\), let

\[
\mathbf{\Gamma} = \text{diag}(\|\mathbf{D}^p_1\|_c^{-1} \|\mathbf{H}^p_1\|_r, \ldots, \|\mathbf{D}^p_k\|_c^{-1} \|\mathbf{H}^p_k\|_r)
\]
and let \( D^* = D^p \Gamma^{1/3} \), \( H^* = \Gamma^{-1/3} H^p \). Then \( D^* H^* = D^p H^p \) and

\[
\nabla L(D^* H^*) H^*^\top + \alpha \text{hstack}(\|D^*;i\|_c \nabla \|D^*;i\|_c)
\]

\[
= \nabla L(D^p H^p) H^p_\top \Gamma^{-1/3} + \alpha \text{hstack}(\|D^p;i\|_c \Gamma^{1/3}_i \Gamma^{1/3}_i \nabla \|D^p;i\|_c)
\]

\[
= \nabla L(D^p H^p) H^p_\top \Gamma^{-1/3} + \alpha \text{hstack}(\|D^p;i\|_c \nabla \|D^p;i\|_c) \Gamma^{2/3}
\]

giving

\[
\nabla L(D^p H^p) H^p_\top \Gamma^{1/3} \Gamma^{1/3} + \alpha \text{hstack}(\|D^p;i\|_c \nabla \|D^p;i\|_c) \Gamma^{2/3} \Gamma^{1/3}
\]

\[
= \nabla L(D^p H^p) H^p_\top + \alpha \text{hstack}(\|D^p;i\|_c \nabla \|D^p;i\|_c) \Gamma
\]

\[
= \nabla L(D^p H^p) H^p_\top + \alpha \text{hstack}(\|D^p;i\|_c \nabla \|D^p;i\|_c \|D^p;i\|^{-1}_c \|H^p;i\|_r)
\]

\[
= \nabla L(D^p H^p) H^p_\top + \alpha \text{hstack}(\nabla \|D^p;i\|_c \|H^p;i\|_r)
\]

\[
= 0
\]

and so \( D^* \) is a stationary point of the summed form, again assuming \( \Gamma \) is full rank. This is similarly the case for \( H^* \).

For the constrained form, to satisfy the KKT conditions, the Lagrange multipliers can be specified to reweight the solution similarly. Because there is a separate Lagrange multiplier for each constraint \( \|D;i\|_c \leq 1 \), there are enough degrees of freedom to easily reweight a stationary from the producted form to produce a stationary point that satisfies the KKT conditions. ■