Frequentist-inspired approach inspired theory of quantum random phenomena predicts signaling

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Different ensembles of the same density matrix are indistinguishable within the modern Kolmogorov probability measure theory of quantum random phenomena. We find that changing the framework from the Kolmogorov one to a frequentist-inspired approach of quantum random phenomena – à la von Mises – would lift the indistinguishability, and potentially cost us the no-signaling principle (i.e., leads to superluminal communication). We believe that this adds to the recent works on the search for a suitable representation of the state of a quantum system. While erstwhile arguments for potential modifications in the representation of the quantum state were restricted to possible variations in the formalism of the quantum theory, we indicate a possible fallout of altering the underlying theory of random processes.

Born’s statistical interpretation of the state vector in quantum mechanics (QM) and hence the density matrix description is based on Kolmogorov’s modern axiomatic, probability-measure theoretic approach to random phenomena [1–5]. We refer to this as Kolmogorov QM (KQM) [6–10]. The circularity in Kolmogorov’s a priori assumption of a constant value for the probability of a random event and its subsequent justification via the strong law of large numbers (LLN), is well known [2, 11]. It is to be noted that the convergence shown by the strong LLN is in terms of probability but not pointwise [2, 11]. This circularity might be a consequence of Gödel’s incompleteness theorem [12]. The parallel and earlier approach by von Mises employs a limiting relative frequency definition of probability, which assumes existence of the limit [13–15], while it (the limit) does not exist [16, 19, 25]. Hence it may be worthwhile to keep the assumptions to the minimum possible (Ockham’s razor [26]), and derive or obtain the rest of the structure or components experimentally (at least, at the conceptual level). In FQM, we suppose that the limiting relative frequency (LRF) of the event \(X = +1\), denoted as \(F(X = +1)\) (this plays the role of \(P(X = +1)\)), is obtained a posteriori via experiment as follows. Let \(X_i\) be the outcome of the \(i\)th trial of \(X\). Then the number of \(+1\) outcomes in \(N\) independent trials of \(X\) is given by \(N_+ = \sum_{i=1}^{N} X_i\). An operationally motivated definition of LRF of the event \(X = +1\) is

\[
F(X = +1) = \lim_{N \to \infty} \sup \frac{N_+}{N} = \lim_{N \to \infty} \left( \sup_{M \geq N} \frac{N_+(X, M)}{M} \right) = \frac{1}{2} + \kappa(X) \quad (1)
\]

where \(\kappa(X)\) is a random variable which takes values in \([-\epsilon, \delta]\) (\(\epsilon > 0, \delta > 0\), depending on the outcomes in a given experiment (\(\epsilon\) may not be equal to \(\delta\), since \(1/2\) cannot be preferred over \(1/2 + c, |c| > 0\), due to fundamental indeterminacy). \(\kappa(X)\) represents an intrinsic or fundamental fluctuation in \(F(X = +1)\). \(\kappa(X)\) is a consequence of Knightian type of ‘true’ uncertainty [19, 20, 32, 33]. It is important to note that this fluctuation in \(F(X = +1)\) is due to an intrinsic random nature of
outcomes of the trials, and not due to varying conditions from one experiment to another, including imperfections in preparing a quantum state which are unavoidable in the real world. We also have \( F(X = 0) = 1/2 - \kappa(X) \), Note that \( F(X = +1) \) is a random variable, whereas \( P(X = +1) \) is a constant. It is important to note that \( \lim_{N \to \infty} N_{+1}(X,N)/N \) cannot converge pointwise (in event space) [28] to 1/2, unlike, say, \( \lim_{N \to \infty} 1/N = 0 \) and \( \lim_{N \to \infty} N_{+1}(X,N)/N^2 \leq \lim_{N \to \infty} 1/N = 0 \) [2, 34]. This is because \( N_{+1}(X,N) \) is a random variable. The fundamental fluctuation in LRF can be considered as a resource within the frequentist-inspired theory of quantum random phenomena, in particular, as we show now, for distinguishing between two different ensemble preparation procedures of the same density matrix. This cannot be obtained within KQM due to a priori assuming constant values for the corresponding probabilities.

**Distinguishing between two ensemble preparation procedures for the same density matrix.** Consider the two following preparation procedures.

**Procedure A:** In a trial of \( X \), if the outcome is \(+1\) (0), then Alice prepares a qubit in the state \(|0\rangle \) (\(|1\rangle \)). She repeats the preceding step \( |\mathcal{M}| \) times independently. She gives this bunch – call it \( \mathcal{E}_A \) – of \( |\mathcal{M}| \) qubits to Bob.

**Procedure B:** This is the same as procedure A, except that \(|0\rangle \) (\(|1\rangle \)) is replaced by \(|+\rangle \) (\(|-\rangle \)). Again, Alice hands over this bunch – call it \( \mathcal{E}_B \) – of \( |\mathcal{M}| \) qubits to Bob.

Bob is aware of the two preparation procedures but unaware of the outcomes of trials of \( X \). Further, Bob is allowed to choose the number \( |\mathcal{M}| \) as large as he decides, carry out any unitary operation on the states, and measure any observable. The question is whether Bob can distinguish between the procedures A and B. The answer, within standard KQM, is in the negative, as the density matrix corresponding to both the procedures is the same i.e., \(|0\rangle \langle 0| + |1\rangle \langle 1|)/2 \). We now consider the solution within FQM.

Instead of representing the states of the bunches, \( \mathcal{E}_A \) and \( \mathcal{E}_B \), in terms of density matrices, one may choose to represent them as

\[
|\psi_A^j\rangle = \bigotimes_{i=1}^{M} |X_i \oplus 1\rangle,

|\psi_B^j\rangle = \bigotimes_{i=1}^{M} |Z_i\rangle,
\]

where \( \oplus \) is addition modulo 2, \( Z_i = +(-) \) if \( X_i = +1(0) \), and \( j \in \{1, 2, ..., 2^{|\mathcal{M}|}\} \) [35]. As particles in a bunch are indistinguishable, Bob can ignore symmetrizing or anti-symmetrizing the total wave function representing the state of \( \mathcal{E}_A/B \) [9, 36].

The state \( |\psi_A^j\rangle \) has all the information which Bob has about the given \( \mathcal{E}_{A/B} \). It may be noted that \( \langle \psi_A^j| \psi_k^j\rangle = \frac{1}{2^{|\mathcal{M}|}} \neq 1, \forall j, k \). See [37] in this respect.

Bob applies

\[
R_x(X^\Theta) = \exp(-iX^\Theta \sigma_x/2)
\]

to each of the qubits, where \( X^\Theta \) is a random variable which outputs \( \theta_1 \) and \( \theta_2 \) with LRFs \( F(X^\Theta = \theta_1) = 1/2 + \kappa(X^\Theta) \) and \( F(X^\Theta = \theta_2) = 1/2 - \kappa(X^\Theta) \) respectively. Then he measures \( \sigma_z \) on the qubit state.

Suppose, unknown to Bob, the bunch that he obtained was created by procedure A. Now, \( R_x(X^\Theta = \theta_0)|0\rangle = |\theta_n, -\pi/2\rangle \), and \( R_x(X^\Theta = \theta_0)|1\rangle = -i|\pi - \theta_n, \pi/2\rangle \), for \( n = 1, 2 \), where \( |\theta, \phi\rangle = \cos \frac{\theta}{2}|0\rangle + e^{i\phi} \sin \frac{\theta}{2}|1\rangle \) in the usual Bloch sphere representation. Let \( X^\Theta \) be the outcome of measuring \( \sigma_z \) on \( |\theta, \phi\rangle \). Then,

\[
F(X^\Theta = +1) = \cos^2(\theta/2) + \kappa(X^\Theta),
\]

which is the modified Born’s statistical interpretation of \(|\theta, \phi\rangle \). And \( F(X^\Theta = -1) = \sin^2(\theta/2) - \kappa(X^\Theta) \), \( \theta \neq 0, \pi \).

Define sample mean as

\[
S(A, M) = \frac{1}{M} \sum_{i=1}^{M} X_i^\Theta,
\]

where \( A = \{X_1^\Theta, X_2^\Theta, X_3^\Theta, ..., X_N^\Theta \} \), \( X_i^\Theta \in \{\theta_1^\Theta, \theta_2^\Theta, \theta_3^\Theta, ..., \theta_{2N}^\Theta \} \). Let \( M = 1 \). Then

\[
F(S(A, M = 1) = +1) = \limsup_{N \to \infty} \frac{N_{+1}(S(A,M = 1),N)}{N}.
\]

We first consider the situation where \( \theta_2 = \theta_1 \). In this case, \( N_{+1}(S(A,M = 1),N) = N_{+1}(X_1^\Theta, N_{+1}(X_1,N)) + N_{-1}(X_1^\Theta, N_{+1}(X_1,N)) \), where \( N_0(X_1, N) = N - N_{+1}(X_1,N) \). We have

\[
\limsup_{N \to \infty} \frac{N_{+1}(X_1^\Theta, N_{+1}(X_1,N))}{N_{+1}(X_1,N)} \leq \limsup_{N \to \infty} \frac{N_{+1}(X_1^\Theta, N_{+1}(X_1,N))}{N_{+1}(X_1,N)} \limsup_{N \to \infty} \frac{N_{+1}(X_1,N)}{N} = (\cos^2(\theta_1/2) + \kappa(X_1^\Theta) + 1(X_1))(1/2 + \kappa(X_1)),
\]

for \( N_{+1}(X_1,N) \to \infty \) \( > 0 \) [27, 38]. Substituting ineq. (7) and a similar result for \( \limsup_{N \to \infty} N_{+1}(X_1^\Theta, N_{0}(X_1,N)) \), into Eq. (6), we get

\[
F(S(A,M = 1) = +1) \leq \frac{1}{2} + \kappa(X_1) \left( \cos \theta + \kappa(X_1, +1(X_1)) - \kappa(X_1^\Theta, 0(X_1)) \right)

+ \kappa(X_1) \left( \kappa(X_1^\Theta, +1(X_1)) + \kappa(X_1^\Theta, 0(X_1)) \right).
\]

(See supplementary material for details.) And \( F(S(A,M = 1) = -1) = 1 - F(S(A,M = 1) = +1) \).

We note here that if we modify KQM initial density matrix into \( \rho_A = (1/2 + \kappa(X)) |0\rangle \langle 0| + (1/2 - \kappa(X)) |1\rangle \langle 1| \), then one can easily verify that \( R_x(X^\Theta = \theta_0) \rho_A R_x(X^\Theta = \theta_1) \) along with the usual KQM Born rule for the subsequent \( \sigma_z \)-measurement do not reproduce the required result consistent with ineq. (8).
Next suppose that the bunch of $\mathcal{M}$ states that Bob obtained from Alice was prepared by procedure B. As before, Bob is oblivious of this choice of Alice. We have

$$R_{\theta}(X^f = \theta_n) = e^{+i\theta_n/2} \mid \pm \rangle, \quad n = 1, 2.$$  

Therefore,

$$S(B, M) = \frac{1}{M} \sum_{i=1}^{M} X_i^{\pi/2},$$  \hspace{1cm} (9)

where $B = \{X^f\}$. Then

$$F(S(B, M = 1) = +1) = \limsup_{N \to \infty} \frac{N_{+1}(S(B, M = 1), N)}{N} = \frac{1}{2} + \kappa(X_1)$$

$$= \frac{1}{2} + \kappa(X_1^f),$$  \hspace{1cm} (10)

since $X^{\pi/2}$, $X$, and $X^f$ differ only in the value assigned to their outcomes. And $F(S(B, M = 1) = -1) = 1 - F(S(B, M = 1) = +1)$.

For $\theta_1 = 0$, $\pi/2$, ineq. (8) reduces to $F(S(A, M = 1) = +1) = 1/2 + \kappa(X_1)$, because $N_{+1}(X_1^{\theta_1} = 0, N_{+1}(X_1, N)) = N_{+1}(X_1, N)$. (See supplementary material for details.)

However, in general the fluctuations corresponding to the expressions in (8) and (10) are different (see Fig. 1). Assuming that there are no further physical restrictions on the observability of the fluctuations, we have therefore shown that our frequentist-inspired approach distinguishes equal density matrices.

**Signaling:** The distinguishing protocol discussed above can be used to provide instantaneous transfer of information between two separated locations. See [23, 39–41] in this respect. Let Alice and Bob share $\mathcal{M}$ singlets $|S_0\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$, and be space-like separated. If Alice measures $\sigma_1$ on her qubits, then on Bob’s side $\mathcal{E}_{A(B)}$ is produced. As Bob can distinguish (at least in principle) between $\mathcal{E}_{A}$ and $\mathcal{E}_{B}$, he can know Alice’s measurement choice superluminally.

**Further aspects:** Consider $S(\sigma_1^A \sigma_2^B, M) = (1/M) \sum_{i=1}^{M} \sigma_{A1} \sigma_{B1}$, where the random variable $\sigma_{A1}$ is the outcome of Alice (Bob) measuring $\sigma_1$ on her (his) $i^{th}$ qubit in the state $\alpha|01\rangle + \beta|10\rangle, \alpha^2 + \beta^2 = 1$. Then in FQM, one can easily show that $S(\sigma_1^A \sigma_2^B, N \to \infty) = -1$.

(See supplementary material for details.)

Hence, even though one may feel that the randomness of $\kappa(\cdots)$ terms will get canceled by an extra randomness in the anti-correlation of the singlet and prevent signaling, such a thing does not happen, simply because such an extra randomness does not exist. Further, one may also feel that the randomness of $\kappa(\cdots)$ terms will get constrained by constraining the extra randomness in the anti-correlation of the singlet. This also does not happen for the same reason.

Let us now briefly mention the case when $\theta_2 \neq \theta_1$ (in Eq. (3)). Let $M = 1$. Let $N_{x_1 X_1^{\theta}}((X_1, X_1^{\theta}), N)$ be the number of $X_1 = x_1$ and $X_1^{\theta} = x_1^{\theta}$ outcomes in $N$ independent trials each of $X_1$ and $X_1^{\theta}$. Then using the identity $N_{x_1 X_1^{\theta}}((X_1, X_1^{\theta}), N) = 0 \iff (X_1, X_1^{\theta})$ (i.e., events are independent) where $N_{x_1^{\theta}}(N)$ is the number of $x_1^{\theta}$ outcomes in $N$ independent trials of $X_1^{\theta}, x_1 = +1, 0; x_1^\theta = \theta_1, \theta_2$; and $N_{\theta_1}(X_1^{\theta}, N) + N_{\theta_2}(X_1^{\theta}, N) = N$, we obtain for the case $\theta_1 = 0, \theta_2 = \pi$,

$$F(S(A, M = 1) = +1) \leq 1/2$$

$$+ \kappa(X_1)(\kappa(X_1, \theta_1(X_1^{\theta}) + \kappa(X_1, \theta_2(X_1^{\theta})))$$

$$+ (\kappa(X_1, \theta_1(X_1^{\theta}) - \kappa(X_1, \theta_2(X_1^{\theta}))))/2,$$  \hspace{1cm} (11)
where \( \limsup_{N \to \infty} N_{+1}(X_1, N_0, N_{+1}(X_1, N)) \) is \( 1/2 + \kappa(X_1, \theta_1(X_1^{+1})) (\kappa(X_1, \theta_2(X_1^{+1})) + \kappa(X_1, \theta_3(X_1^{+1}))) + \kappa(X_1, \theta_4(X_1^{+1})) - \kappa(X_1, \theta_5(X_1^{+1})). \) (12)

Similarly, \( S(B, N \to \infty) = 2k(X_1) + 2k(X_0) \). (See supplementary material for details.) This suggests that fluctuation of \( F(S(A, M = 1) = +1) \) around 1/2 might be appreciably less than that of \( F(S(B, M = 1) = +1) \) (Eq. (10)).

For \( 1 \ll N < \infty \), we obtain expressions which are same as the expressions (8), (10), (11) but with \( \kappa(\cdots) \)'s replaced by the corresponding \( \kappa_N(\cdots) \)'s (which represent fluctuation corresponding to \( 1 \ll N < \infty \) such that \( \limsup_{N \to \infty} \kappa_N(\cdots) = \kappa(\cdots) \)), and inequalities replaced by equalities. This is because, when we take the limit \( N \to \infty \), it turns out that the limit may not exist. Hence we have to consider limit supremum and/or limit infimum which always exists, and they give rise to inequalities. (See supplementary material for details.) Hence Bob can distinguish even when \( 1 \ll N < \infty \).

Note that if we set \( \kappa(\cdots) \)'s to 0 in expressions (8), (10), (11), we obtain the predictions of KQM. In this sense, KQM can be seen as a special case of FQM.

Conclusion: In summary, we found that a frequentist-inspired theory of quantum random phenomena leads to distinguishing between different ensembles of the same density matrix, which in turn leads to signaling (i.e., superluminal communication). This may be seen in the light of previous comments about the possible incompleteness of the density matrix representation, within modern Kolmogorov probability measure theory of quantum random phenomena, of a situation (state) of a physical system in Refs. [35–37, 42, 43]. To our knowledge, preceding discussions on possible modifications of the density matrix representation confined themselves to revisions of the description of the state within the Hilbert space formalism of quantum mechanics. We showed that remaining within the Hilbert space formalism but looking out for possible implications of variations of the underlying theory of random processes may cost us the no-signaling principle.

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Supplementary material:

(1) Evaluating \( \limsup_{N \to \infty} N_{-1} X^{\pi - \theta_1}, N_{0}(X_{1}, N) / N \):

\[
\liminf a_N \liminf b_N \leq \liminf_{N \to \infty} a_N b_N \leq \liminf_{N \to \infty} a_N \lim_{N \to \infty} b_N \leq \limsup_{N \to \infty} a_N \lim_{N \to \infty} b_N
\]

[27, 38]. Then using ineq. (13) we obtain,

\[
\limsup_{N \to \infty} \frac{N_{+1}(X_1^{\pi - \theta_1}, N_0(X_1, N))}{N} = \frac{1}{2} - \kappa(X_1).
\]

(2) Case where \( \theta_1 = \theta_2 = 0, \pi/2 \):

For \( \theta_1 = \theta_2 = 0 \), \( N_{+1}(X_1^{\pi/2}, N_{+1}(X_1, N)) = N_{-1}(X_1, N) \), and \( N_{+1}(X_1^{\pi - \theta_1}, N_0(X_1, N)) = 0 \).

\[
F(S(A, M = 1) = +1) = \limsup_{N \to \infty} \frac{N_{+1}(S(A, M = 1), N)}{N} = \limsup_{N \to \infty} \frac{N_{-1}(X_1, N)}{N} = 1/2 + \kappa(X_1).
\]

For \( \theta_1 = \theta_2 = \pi/2 \),

\[
N_{+1}(S(A, M = 1), N) = N_{+1}(X_1^{\pi/2}, N_{+1}(X_1, N)) + N_{+1}(X_1^{\pi - \theta_1}, N_{0}(X_1, N)) = N_{+1}(X_1^{\pi/2}, N_{+1}(X_1, N) + N_0(X_1, N)) = N_{+1}(X_1^{\pi/2}, N).
\]

\[
F(S(A, M = 1) = +1) = \limsup_{N \to \infty} \frac{N_{+1}(S(A, M = 1), N)}{N} = \limsup_{N \to \infty} \frac{N_{-1}(X_1^{\pi/2}, N)}{N} = 1/2 + \kappa(X_1^{\pi/2}) = 1/2 + \kappa(X_1).
\]

(3) Limit infimum:

If we define

\[
F(S(A, M = 1) = +1) = \liminf_{N \to \infty} \frac{N_{+1}(S(A, M = 1), N)}{N}
\]

then using ineq. (13), we obtain for the case \( \theta_1 = \theta_2 \) in Eq. (3) of main text,

\[
F(S(A, M = 1) = +1) \geq \frac{1}{2} + \kappa'(X_1)(\cos \theta_1 + \kappa'(X_1^{\theta_1}, +1(X_1)) - \kappa'(X_1^{\pi - \theta_1}, 0(X_1))) + \frac{\kappa'(X_1^{\theta_1}, +1(X_1)) + \kappa'(X_1^{\pi - \theta_1}, 0(X_1))}{2}
\]

where \( \kappa'(\cdots) \)'s correspond to limit infimum.

(4) Perfect anti-correlation of singlet in FQM:
We can rewrite as follows,

\[ S(\sigma^2 x^2, N \to \infty) = \limsup_{N \to \infty} \frac{N+1(S(\sigma^2 x^2, M = 1), N) - N-1(S(\sigma^2 x^2, M = 1), N)}{N}. \]

We have

\[ N+1(S(\sigma^2 x^2, M = 1), N) = N+1+(S(\sigma^2 x^2, M = 1), N) \]
\[ +N-1+(S(\sigma^2 x^2, M = 1), N) = 0 + 0. \]

Substituting \( \sigma \), we obtain

\[ \limsup_{N \to \infty} N+1(x(1, X^0, N), N) \]
\[ \leq \limsup_{N \to \infty} \frac{N+1(x(1, X^0, N), N)}{N} \]
\[ \leq \limsup_{N \to \infty} \frac{N+1(x(1, X^0, N), N)}{N} \]
\[ \times \limsup_{N \to \infty} \frac{N+1(x(1, X^0, N))}{N} \]
\[ \times \limsup_{N \to \infty} \frac{N+1(x(1, X^0, N))}{N}. \]

For \( N+1(x(1, N, X^0, N \to \infty)) > 0, N_0(X^0, N \to \infty) > 0 \). Substituting \( \theta_1 = 0, \theta_2 = \pi \) in the above expression, we obtain

\[ \limsup_{N \to \infty} N+1(x(1, X^0, N), N) \]
\[ \leq (1/2 + \kappa(x(1, X^0, N)))\left(1/2 + \kappa(x(1, X^0, N)) \right) \]
\[ \leq \limsup_{N \to \infty} \frac{N+1(x(1, X^0, N), N)}{N} \]
\[ \times \limsup_{N \to \infty} \frac{N+1(x(1, X^0, N))}{N} \times \limsup_{N \to \infty} \frac{N+1(x(1, X^0, N))}{N}. \]

Substituting \( \theta_1 = 0, \theta_2 = \pi \) in the above expression, we obtain

\[ \limsup_{N \to \infty} \frac{N+1(x(1, X^0, N), N)}{N} = 0 \]
\[ \leq \limsup_{N \to \infty} \frac{N+1(x(1, X^0, N), N)}{N} \]
\[ \times \limsup_{N \to \infty} \frac{N+1(x(1, X^0, N))}{N} \times \limsup_{N \to \infty} \frac{N+1(x(1, X^0, N))}{N}. \]

Substituting \( \theta_1 = 0, \theta_2 = \pi \) in the above expression, we obtain

\[ \limsup_{N \to \infty} \frac{N+1(x(1, X^0, N), N)}{N} = 0 \]
\[ \leq \limsup_{N \to \infty} \frac{N+1(x(1, X^0, N), N)}{N} \]
\[ \times \limsup_{N \to \infty} \frac{N+1(x(1, X^0, N))}{N} \times \limsup_{N \to \infty} \frac{N+1(x(1, X^0, N))}{N}. \]

Substituting \( \theta_1 = 0, \theta_2 = \pi \) in the above expression, we obtain

\[ \limsup_{N \to \infty} \frac{N+1(x(1, X^0, N), N)}{N} \]
\[ \leq (1/2 - \kappa(x(1, X^0, N)))\left(1/2 - \kappa(x(1, X^0, N)) \right) \]
\[ \leq \limsup_{N \to \infty} \frac{N+1(x(1, X^0, N), N)}{N} \]
\[ \times \limsup_{N \to \infty} \frac{N+1(x(1, X^0, N))}{N} \times \limsup_{N \to \infty} \frac{N+1(x(1, X^0, N))}{N}. \]

Substituting the above expressions into Eq. (6) of main text, we obtain for the case \( \theta_1 = 0, \theta_2 = \pi \), the expression for \( F(S(A, M = 1) = 1) \).
Further, we can rewrite,

$$S(A, N \to \infty) = \limsup_{N \to \infty} \frac{N_{+1}(S(A, M = 1), N) - (N - N_{+1}(S(A, M = 1), N))}{N}.$$ 

Then substituting expressions (6, 11) of main text into the above equation, we obtain the expression for $S(A, N \to \infty)$ (ineq. (12) of main text). Similarly we can rewrite,

$$S(B, N \to \infty) = \limsup_{N \to \infty} \frac{N_{+1}(S(B, M = 1), N) - (N - N_{+1}(S(B, M = 1), N))}{N} = 2\kappa(X_1) = 2\kappa(X_1^0) \quad (16)$$

where we used Eq. (10) of main text.

(6) The case when $1 \ll N < \infty$:

Consider the case when $1 \ll N < \infty$. Define

$$F_N(X = +1) = \frac{N_{+1}(X, N)}{N} := \frac{1}{2} + \kappa_N(X) \quad (17)$$

where $\kappa_N(X)$ is a random variable which takes values in $[-\epsilon_N, \delta_N]$ $(\epsilon_N > 0, \delta_N > 0)$. Then the expression corresponding to ineq. (7) of main text will be the following,

$$\frac{N_{+1}(X_1^0, N_{+1}(X_1, N))}{N_{+1}(X_1, N)} = \cos^2(\theta_1/2) + \kappa_N(X_1^0, +1(X_1))(1/2 + \kappa_N(X_1)). \quad (18)$$

for $N_{+1}(X_1, N) > 0$. This shows that in all the results derived in the main text, we just have to replace $\kappa(\cdot, \cdot)$’s with the corresponding $\kappa_N(\cdot, \cdot)$’s, and inequalities become equalities. Of course the constraint that the terms in the denominators should be greater than zero should be satisfied (like $N_{+1}(X_1, N) > 0$ in Eq. (18)). Further note that $\limsup_{N \to \infty} F_N(X = +1) = F(X = +1)$ and hence $\limsup_{N \to \infty} \kappa_N(X) = \kappa(X)$ as required. Similarly we obtain $\limsup_{N \to \infty} \kappa_N(\cdot, \cdot) = \kappa(\cdot, \cdot)$.

Further note that when $N$ is very small (say e.g., $1 \leq N \leq 10$), then both $F_N(S(A, M = 1) = +1)$ and $F_N(S(B, M = 1) = +1)$ will easily saturate i.e., will easily take maximum and minimum possible values which are 1 and 0 respectively. Hence Bob cannot distinguish.

Fig. 1 in the main text is helpful in understanding this point.

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likeliness, upon repeating the experiment many times. Hence the requirement for pointwise convergence is not always satisfied (e.g., for $N_{+1}(X, N) = N$).

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