The Schubert normal form of a 3-bridge link 
and the 3-bridge link group

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Abstract

We introduce the Schubert form a 3-bridge link diagram, as a generalization of the Schubert normal form of a 3-bridge link. It consists of a set of six positive integers, written as \((p/n, q/m, s/l)\), with some conditions and it is based on the concept of 3-butterfly. Using the Schubert normal form of a 3-bridge link diagram, we give two presentations of the 3-bridge link group. These presentations are given by concrete formulas that depend on the integers \(\{p, n, q, m, s, l\}\). The construction is a generalization of the form the link group presentation of the 2-bridge link \(p/q\) depends on the integers \(p\) and \(q\).

1 Introduction

In \([5]\) it was introduced the butterfly presentation of a link diagram as a generalization of the 2-bridge Schubert’s notation. Moreover, the particular concept of a 3-butterfly was implemented in order to study 3-bridge links and to obtain a codification of a 3-bridge link diagram. Here, for our purpose, we do not need all the machinery of the butterfly construction presented in \([6]\), so we will take a different approach. We will describe the construction of the codification by a direct and combinatorial approach, using the ideas in \([3]\), where Ferri constructed the crystallization of the double cover of \(S^3\), with a link as ramification set. For any link diagram \(L\) there is a strong relation
between crystallization of the double cover of $S^3$, with $L$ as the ramification set, and the 3-butterfly associated to $L$, that we will explain in [12]. For any $n$-bridge link diagram the construction of an $n$-butterfly is possible, see [6], but in this paper we want to be specific and we will work only with 3-bridge link diagrams.

To a 3-bridge diagram we associate a 3-butterfly that is described by a set of six positive integers $\{p, n, q, m, s, l\}$, with some restrictions, and then we define the Schubert form of the link diagram as $(p/n, q/m, s/l)$, for geometrical reasons that will be explained in Section 1. As each 3-bridge link admits infinitely many different link diagrams, the Schubert normal form for a link $L$ is defined by taking the minimum among all 3-bridge link diagrams of $L$ according to a lexicographical type of order, see [5]. For the purpose of this paper we only need the Schubert form of the link diagram, but in further research and in the compilation of link tables, it will be interesting to consider the Schubert normal form of a link.

In this paper we find formulas for the over and under presentation of the 3-bridge link represented by the Schubert form $(p/n, q/m, s/l)$, that depends on the integers $\{p, n, q, m, s, l\}$. The formula for the under presentation of the 3-bridge link is a natural extension of the formula for the presentation of the 2-bridge link $p/q$, that depends on the integers $p$ and $q$.

The paper is organized as follows: in Section 1 we describe the construction of a 3-butterfly associated to a 3-bridge link diagram $L$ and introduce the Schubert form of $L$, that consists of a set of 6 positive integers, $(p, n, q, m, s, l)$, that captures the relevant information of the 3-butterfly and, therefore, of the 3-bridge diagram $L$. In Section 3 we describe a canonical diagram associate to a 3-butterfly $(p, n, q, m, s, l)$, in a similar way to the canonical diagram of a 2-braid link, see [13]. In Section 5 we will give an orientation to this canonical diagram.

In Section 4 we define two permutations, $\gamma$ and $\phi$, associated to the Schubert form $(p/n, q/m, s/l)$, and study the composition $\mu = \gamma \phi$. The cyclic structure of $\mu$ is the key point in the rest of the paper. A variation of the permutation $\mu$ is very useful for the construction of a Gauss code for the link diagram, and, from there we can find the Dowker code and we are able to compute link invariants, such as the link group, the Seifert matrix, the Alexander, Jones and HOMFLY polynomials.

In Section 6 we present our main result, Theorems 14, 15 and 18, that give explicit presentations for the knot group $\pi(L)$ of a 3-bridge link $L$. These presentation are described by clear algorithms, that are easy to program in
a computer and depend on the integers in the Schubert form of the link diagram. In the last section we propose a special family of links, \((p/n, p/n, p/n)\), that have a strong symmetry that is reflected in the group presentation. This symmetry could be exploited in the study of the representations into \(SL(2, \mathbb{C})\) of the link group.

Some authors allow that any \(n\)-bridge link diagram can be consider as a \(k\)-bridge link diagram, for any \(k > n\), by considering bridges without any undercrossings, see [9]. We neither allow this situation nor consider a split link diagram with more than 3 components, as the one in Fig. 3a, as a 3-bridge link diagram. We work with 3-bridge link diagrams as in [1] and [8].

Remark on notation: In [11] the author uses subindexes and denote a butterfly by \((M_1, N_1, M_2, N_2, M_3, N_3)\). In this paper we avoid the use of subindexes in the Schubert form, and prefer to assign a different role to each integer, in that way we reach simpler formulas.

2 Description of the 3-butterfly of a 3-bridge link diagram

Let \(L\) be a 3-bridge link such that the projection on the \(xy\) plane is a 3-bridge diagram \(D\). Let \(a, b\) and \(c\) the bridge projections. Draw an ellipse around each of the bridges, in such a way that they are disjoint, and they have the bridges as principal axes. Each ellipse will intercept the diagram \(D\) in an even number of points, that will be the vertices. We denote by \(P, Q, S\) the ellipses around the bridges \(a, b\) and \(c\), respectively, and let \(2p\) (resp. \(2q\) and \(2s\)) the number of intersections of \(P\) (resp. \(Q\) and \(S\)) with the diagram \(D\). Following [6], the ellipses \(P, Q, S\) are called butterflies.

Take the graph \(R_1\) formed by the butterflies \(P, Q, S\), the vertices and the bridges. In each butterfly we have the bridge, that divides each butterfly in two halves, that will be the wings. The reflection along the bridges inside the butterflies is called \(\gamma\). The segment of the underarcs that are inside the butterflies are forgotten, but they can be recovered with the reflection \(\gamma\). The edges outside the butterflies will give the information on how the butterflies intercept to each other, see Fig. 1a. Each one of these edges connect two vertices of two butterflies, we identify these vertices, to form a set that will be the vertices of our graph. The identification will give an involution on the vertex of the graph, that we call \(\phi\).
We define the 3-butterfly as the graph $R = R_1/\phi$, formed with the vertices of $R_1$ identified by the involution $\phi$. We draw the graph $R$ in any of the three forms shown in Fig. 1. If we consider that the diagram $L$ is in $S^2 = \partial B$, then the graph $R$ define a polygonalization of $S^2$ formed by three polygons, as shown in Fig. 2. Compare this simple construction with the formal one given in [6] and [5]. When we identify the butterflies, there will appear two new points, that will be denoted 0 and *. These two points are fundamental, but they are not considered vertices of the 3-butterfly. There can be only two basic forms for the graph $R$, that are determined by the way the butterflies $P, Q$ and $S$ intersect.

**Type I:** The three butterflies intersect in the two points 0 and *, see Fig. 2a.

**Type II:** Two of the polygons do not intersect, see Fig. 2b. When a 3-bridge link diagram produces a type II butterfly, there is a wave move, see
that allows us to construct a new 3-bridge diagram with lower crossing number. So, we work only with type I butterflies, such that there are no wave moves.

In order to obtain a canonical way to describe a 3-butterfly, we will always assume that

\[ p \geq q \geq s \geq 2, \]  

the condition \( s \geq 2 \) is to ensure that each bridge has at least one crossing. By rotating the plane and interchanging the points 0 and \(*\), we can always obtain a 3-butterfly diagram with \( P \) at the top, \( Q \) to the left and \( S \) to the right, and we read it in the counterclockwise direction, \( PQS \), as shown in Fig. 1.

Let \( |P \cap Q| = t \) be the number of vertices between \( P \) and \( Q \), \( v = |Q \cap S| \) and \( w = |P \cap S| \). As each butterfly intersects the other two, then \( t, v \) and \( w \) are positive integers that satisfy \( 2p = t + w \), \( 2q = t + v \), and \( 2s = v + w \), therefore

\[ t = p + q - s, \quad v = q + s - p, \quad w = p + s - q. \]  

As we will only consider the link diagrams with \( v \geq 1 \), then

\[ p + 1 \leq s + q. \]  

So the integers \( p, q \) and \( s \) satisfy (1) and (3). Reciprocally, if we have integers satisfying (1) and (3) we can construct the butterflies \( P, Q \) and \( S \).

Now let us describe the positions of the bridges. We orient clockwise each butterfly. From the point 0, following the orientation, we count the number of vertices between 0 and the vertex in which the bridge begins. We call \( n, m \) and \( l \) the initial points of the bridges in \( P, Q \) and \( S \), respectively. Clearly

\[ 1 \leq n \leq p, \quad 1 \leq m \leq q, \quad 1 \leq l \leq s. \]  

We are working only with link diagrams with exactly three bridges and not only two or one, this impose conditions on the integers \( n, m \) and \( l \). In [5] they found the conditions given in the following theorem.
Theorem 1 Every 3-butterfly defines a unique set of integers \( \{p, m, q, n, s, l\} \) such that

\[
p < q < s < 2, \quad 1 \leq n \leq p, \quad 1 \leq m \leq q, \quad 1 \leq l \leq s, \quad p + 1 \leq s + q
\]

\( n + m \neq q + 1, \quad n + l \neq p + 1, \quad \)

if \( m > q + s - p \) then \( n + m \neq 2q + 1 \) and \( n + m \neq 2q - p + 1 \)

if \( l < p - s \) then \( n + l \neq p - s + 1 \)

if \( m \leq q + s - p \) then \( m + l \neq s + 1 \).

Reciprocally, if a set of integers \( \{p, m, q, n, s, l\} \) satisfies conditions (5) then it defines a 3-butterfly.

Note that we may think that inside the butterfly \( P \) (resp. \( Q, S \)) the bridge \( a \) make a \( (n/p) \pi \) rotation, (resp. \( (m/q) \pi, (l/s) \pi \)). For this geometrical reason we want to use the notation \( (p/n, q/m, s/l) \) instead of \( \{p, m, q, n, s, l\} \), but \( p/n \) is not considered as a rational number.

The conditions on the integers \( \{p, n, q, m, s, l\} \) impose in Fig. 3 define a 3-butterfly and a link diagram \( L \), but it is possible that \( L \) is a split link with some trivial components, as the diagram associated to \( \{5, 1, 5, 2, 5, 1\} \) shows, see Fig. 3.

Definition 2 We say that \( (p, n, q, m, s, l) \) is a 3-butterfly if the set of integers \( \{p, n, q, m, s, l\} \) satisfies the conditions (5). We say that a 3-butterfly is reduced if the associated diagram is a 3-bridge diagram without any trivial components. We say that \( (p/n, q/m, s/l) \) is a Schubert form of a 3-bridge link if the 3-butterfly \( (p, n, q, m, s, l) \) is reduced.
We need to be careful with the relative order of the numbers in \((p/n, q/m, s/l)\).

**Example 3** If we change the order, it is possible that we get different Schubert forms. The Schubert form \((4/2, 4/1, 3/1)\) represents a knot and \((4/1, 4/2, 3/1)\) represents a two component link. See Fig. 3.

### 3 Algorithm to draw a canonical 3-bridge link diagram

We associate to each 3-butterfly \((p, n, q, m, s, l)\) a canonical diagram, in a similar way as the canonical diagram of a 2-bridge link is associated to \(p/q\), see [7]. We draw the three bridges as three segments: bridge \(a\) as a vertical segment; bridge \(b\) as a segment forming a 120° angle with the bridge \(a\) and bridge \(c\) as a segment forming a 240° angle with the bridge \(a\).

We divide the bridge \(a\) in \(p\) segments, and we fix two points in each division, one to the left and one to the right, except at the extreme points, where there is only one. Label them with \(A = \{a_0, a_1, \cdots, a_{2p-1}\}\), in a counterclockwise sense, so the extreme bridges are labeled \(a_0\) and \(a_{2p-1}\). For the bridge \(b\) we repeat the process, but we divide the bridge in \(q\) segments and label the points with \(B = \{b_0, \cdots, b_{2q-1}\}\). For the bridge \(c\) the number of segments is \(s\) and the labels are \(C = \{c_0, \cdots, c_{2s-1}\}\). The subscripts of \(A\) (resp. \(B\) and \(C\)) are taken mod \((2p)\), (resp. mod \((2q)\) and mod \((2s)\)).

To draw the link diagram we need to join, with appropriate arcs, the points \(a_i, b_j\) and \(c_k\), \(i \in \mathbb{Z}_{2p}, j \in \mathbb{Z}_{2q}\), and \(k \in \mathbb{Z}_{2s}\), according to the rules given by permutations \(\phi\) and \(\gamma\).

There are \(t = p + q - s\) arcs between the \(a\) and \(b\) bridges, namely

\[
a_{n-1}b_m, a_{n-2}b_{m+1}, a_{n-3}b_{m+2}, \cdots, a_{n-j}b_{m+j-1}, \cdots, a_{n-t}b_{m+t-1},
\]

likewise there are \(v = q + s - p\) arcs between the \(b\) and \(c\) bridges, that are

\[
b_{m-1}c_l, b_{m-2}c_{l+1}, b_{m-3}c_{l+2}, \cdots, b_{m-j}c_{l+j-1}, \cdots, b_{m-v}c_{l+v-1},
\]

and, finally, \(w = p + s - q\) arcs between the \(c\) and \(a\) bridges,

\[
c_{l-1}a_n, c_{l-2}a_{n+1}, c_{l-3}a_{n+2}, \cdots, c_{l-j}a_{n+j-1}, \cdots, c_{l-w}a_{n+w-1}.
\]

It is enough to know how to construct the first arc between each pair of bridges, and the rest of the arcs are "parallel" arcs to them, see Fig. 4.
In the rest of this paper we will refer to the diagram described as the link canonical diagram associated to the Schubert form \((p/n, q/m, s/l)\). Notice that if \((p/n, q/m, s/l)\) does not satisfy the conditions in Theorem 1, we still may use this algorithm to draw a link diagram.

**Lemma 4.** If the Schubert form \((p/n, q/m, s/l)\) is reduced, the 3-bridge diagram has \(p + q + s - 3\) crossings.

## 4 Permutations associated to a Schubert form

The conditions for a 3-butterfly \(\{p, n, q, m, s, l\}\) to be reduced can not be described using simple conditions on the integers in a similar way as the conditions to be a 3-butterfly given in (5). Now we need to go deeper and study in detail the permutations \(\phi\) and \(\gamma\). Given a set \(\{p, n, q, m, s, l\}\) that satisfies (3) we construct explicitly the associated 3-butterfly and then we draw the 3-bridge diagram.

Define the 3-butterfly by labelling the vertices of each of the butterflies: \(P\) have vertices labeled by \(A = \{a_0, \ldots, a_i, \ldots, a_{2p-1}\}, i \in \mathbb{Z}_{2p}; Q\) with vertices \(B = \{b_0, \ldots, b_j, \ldots, b_{2q-1}\}, j \in \mathbb{Z}_{2q};\) and \(S\) with vertices \(C = \{c_0, \ldots, c_l, \ldots, c_{2s-1}\}, l \in \mathbb{Z}_{2s}.\) The bridge ends are labeled by \(a_0\) and \(a_p\) in \(P\) (resp. by \(b_0\) and \(b_q\) in \(Q\) and \(c_0\) and \(c_s\) in \(S\)). See Fig. 5.

We have the permutations \(\gamma\) and \(\phi\) on the set \(A \cup B \cup C\). The permutation \(\gamma\) is the reflection along the bridges. The permutation \(\phi\) is determined by the identification of the vertices of two butterflies, so in the 3-butterfly each vertex has two labels. The proofs of the following lemmas are straightforward computations.
Lemma 5 The function defined in the set $A \cup B \cup C$ by
\begin{align*}
\gamma (a_i) &= a_{2p-i}, \quad 0 \leq i < 2p, \\
\gamma (b_j) &= b_{2q-j}, \quad 0 \leq j < 2q, \\
\gamma (c_h) &= b_{2s-h}, \quad 0 \leq h < 2s
\end{align*}  
(6)
is an order 2 permutation. The set of fixed points is
\[ E = \{ a_0, a_p, b_0, b_q, c_0, c_s \}. \]  
(7)
The set $E = \{ a_0, a_p, b_0, b_q, c_0, c_s \}$ corresponds to the endpoints of the bridges. It will play an important role in the rest of the paper.

Lemma 6 The map $\phi : A \cup B \cup C \to A \cup B \cup C$ defined by
\begin{align*}
a_{n-i} &\leftrightarrow b_{m+i-1}, \quad \text{if } 1 \leq i \leq t, \\
a_{n+j} &\leftrightarrow c_{l-j-1}, \quad \text{if } 0 \leq j \leq w - 1, \\
b_{m-h} &\leftrightarrow c_{l+h-1}, \quad \text{if } 1 \leq h \leq v,
\end{align*}  
(8)
is an order 2 permutation, where $t = p + q - s$, $v = q + s - p$ and $w = p + s - q$.

Note that $\phi$ does not have fixed points, and among the bicycles in $\phi$ there is no a bicycle in the set
\[ \mathcal{F} = \{ (a_0, b_0), (a_0, b_q), (a_0, c_0), (a_0, c_s), (b_0, a_0), (b_0, a_p), (b_0, c_0), \\
(b_0, c_s), (c_0, a_p), (c_0, b_q), (a_p, b_q), (a_p, c_s), (b_q, c_s) \} \]  
(9)
The construction of $\phi$ is well defined for any polygonalization of $S^2$ with 3 polygons, even if they do not satisfy the conditions of Theorem 1. In fact, in terms of the permutation $\phi$, we can rewrite Theorem 1 as follows.

**Theorem 7** A set $\{p, n, q, m, s, l\}$, with $p \geq q \geq s \geq 2$, $1 \leq n \leq p$, $1 \leq m \leq q$, $1 \leq l \leq s$ describes a 3-butterfly if and only if the associated permutation $\phi$ does not have any of the bicycles in the set $F$.

We study the cyclic decomposition of $\mu = \phi \gamma$. The orbit of a vertex $v$ will be denoted by $O_\mu(v)$. For a cycle $\tau = (z_1 z_2 \cdots z_k)$ we will use the same symbol to refer to the cycle, to its orbit $\{z_1, z_2, \cdots, z_k\}$ and to the word $z_1 z_2 \cdots z_k$. The length of $\tau$ will be denoted by $|\tau|$, $\tau(x)$ will denote the cycle that contains $x$ and for a function $\Gamma$, $\Gamma(\tau)$ will be the word (set) formed by applying $\Gamma$ to each element in $\tau$.

**Theorem 8** Let $(p, n, q, m, s, l)$ be a 3-butterfly, $\gamma$ and $\phi$ be its associated permutations, given in Lemmas 5 and 6 and let $\mu = \phi \gamma$. $(p/n, q/m, s/l)$ is a Schubert form for a 3-bridge link if and only if $\mu$ is the product of three disjoint cycles, $\mu = \tau_1 \tau_2 \tau_3$ such that, for $i=1,2,3$, $|\tau_i \cap E| = 2$, where $E$ is given in (7).

**Proof.** Let $\mu = \phi \gamma$ associated to the 3-butterfly $(p, n, q, m, s, l)$. The 3-butterfly $(p, n, q, m, s, l)$ defines a Schubert form $(p/n, q/m, s/l)$ if and only if the 3-butterfly is reduced.

Suppose that the butterfly is reduced. The orbit of $a_p$ under $\mu$, $O_\mu(a_p)$ will describe a path that follows the underarc with initial point in $a_p$, so eventually it will arrive to the endpoint of the underarc, say $e = \mu^k(a_p)$, $e \in E$, $E$ defined in (7). Then $\mu(e) = \phi \gamma(e) = \phi(e) = \phi \phi \gamma \mu^{k-1}(a_p) = \gamma \mu^{k-1}(a_p)$, so the orbit will go back to the same underarc, in opposite direction. We take $\tau_1$ as the cycle formed by the orbit of $a_p$ and $\tau_1 \cap E$ will contain exactly two vertices. We repeat the same process with the other vertices in $E$. Since the butterfly is reduced, all the vertices will be crossed by one of the underarcs, so we have only three orbits.

Reciprocally, if the butterfly is not reduced, there will be a component whose vertex will not be in the orbit of any of the elements in $E$. See Fig. 3a. ■

From now on we will assume that the permutation $\mu$ associated to the Schubert form $(p/n, q/m, s/l)$ is the product of three disjoint cycles, $\mu = \tau_1 \tau_2 \tau_3$. The cyclic decomposition of $\mu$ allows us to determine the number of components of the associated link diagram.
Theorem 9 (Classification) Let \((p/n, q/m, s/l)\) be a Schubert form, \(\gamma\) and \(\phi\) its associated permutations given in \([3]\) and \([2]\) \(\mu = \phi \gamma\). The 3-bridge link diagram \(L\) represented by \((p/n, q/m, s/l)\) satisfies:

(i) \(L\) is a knot if and only if \(a_p \notin \mathcal{O}_\mu(a_0), b_q \notin \mathcal{O}_\mu(b_0)\) and \(c_s \notin \mathcal{O}_\mu(c_0)\).

(ii) \(L\) is a two component link if and only if one, and only one, of the following conditions holds: \(a_p \in \mathcal{O}_\mu(a_0), b_q \in \mathcal{O}_\mu(b_0)\) or \(c_s \in \mathcal{O}_\mu(c_0)\).

(iii) \(L\) is a three component link if and only if \(a_p \in \mathcal{O}_\mu(a_0), b_q \in \mathcal{O}_\mu(b_0)\) and \(c_s \in \mathcal{O}_\mu(c_0)\).

Proof. Take the cyclic decomposition of \(\mu = \tau_1 \tau_2 \tau_3\) and study each one of the cycles, using the interpretation given in the proof of Theorem 8.

5 Orientation of the canonical 3-bridge link diagram \((p/n, q/m, s/l)\)

Until now we have not considered the orientation of the link \(L\), but in order to find a group presentation for the link group \(\pi(L)\) we will give an orientation to the canonical diagram described in Section 3. Let \(\mu = \phi \gamma = \tau_1 \tau_2 \tau_3\), we study in detail these cycles. In each cycle \(\tau_i\) we have two special vertices, that are the fixed points of \(\gamma\) and form the set \(E\) defined in \((7)\). Each cycle describes a path around one of the link diagram underarcs, see Fig. 6, so one of this special vertices corresponds to the arc initial point, denoted \(I_i\); and the other one to the arc endpoint, denoted \(F_i\). So we consider that when we follow the link, we travel it in the order \(\tau_1, \tau_2\) and \(\tau_3\) and the bridge \(a\) in the direction from \(a_0\) to \(a_p\).

Definition 10 We define \(\delta_a\) (resp. \(\delta_b, \delta_c\)), the direction in which we travel the bridge \(a\) (resp. \(b, c\)) as: \(\delta_a = 1\) and

\[
\delta_b = \begin{cases} 
1, & \text{if we go from } b_0 \text{ to } b_q \\
-1, & \text{if we go from } b_q \text{ to } b_0
\end{cases} \quad \delta_c = \begin{cases} 
1, & \text{if we go from } c_0 \text{ to } c_s \\
-1, & \text{if we go from } c_s \text{ to } c_0
\end{cases}
\]

When the condition (i) in Theorem 9 is satisfied, the Schubert form corresponds to a knot diagram, hence the orientation of bridge \(a\) is enough to determine the knot orientation. We take \(\tau_1\) as the cycle that contains \(a_p\) and \(\tau_3\) as the cycle that contains \(a_0\). In the link case we need to determine the orientation of each component. If \(L\) is a 3-component link, the condition (iii)
Lemma 11  a. If $L$ is a knot, Table 1 contains all possibilities for the endpoints of the cycles $\tau_1, \tau_2, \tau_3$ and the knot orientation.

b. If $L$ is a 2-component link, Table 2 contains all possibilities for the endpoints of the cycles $\tau_1, \tau_2, \tau_3$ and the link orientation.

| $I_1$ | $F_1$ | $I_2$ | $F_2$ | $I_3$ | $F_3$ | $\delta_b$ | $\delta_c$ |
|-------|-------|-------|-------|-------|-------|------------|------------|
| $a_p$ | $b_0$ | $b_q$ | $c_0$ | $c_s$ | $a_0$ | 1          | 1          |
| $a_p$ | $b_0$ | $b_q$ | $c_s$ | $c_0$ | $a_0$ | 1          | -1         |
| $a_p$ | $b_q$ | $b_0$ | $c_0$ | $c_s$ | $a_0$ | -1         | 1          |
| $a_p$ | $b_0$ | $c_0$ | $c_s$ | $c_0$ | $a_0$ | -1         | -1         |
| $a_p$ | $c_0$ | $c_s$ | $b_q$ | $b_0$ | $a_0$ | 1          | 1          |
| $a_p$ | $c_0$ | $c_s$ | $b_q$ | $b_0$ | $a_0$ | 1          | -1         |
| $a_p$ | $c_s$ | $b_0$ | $b_q$ | $a_0$ | $a_0$ | -1         | -1         |

Table 1

| $I_1$ | $F_1$ | $I_2$ | $F_2$ | $I_3$ | $F_3$ | $\delta_b$ | $\delta_c$ |
|-------|-------|-------|-------|-------|-------|------------|------------|
| $a_p$ | $b_0$ | $b_q$ | $a_0$ | $c_s$ | $c_0$ | 1          | 1          |
| $a_p$ | $b_0$ | $b_q$ | $a_0$ | $c_s$ | $c_0$ | 1          | -1         |
| $a_p$ | $b_q$ | $b_0$ | $a_0$ | $c_s$ | $c_0$ | -1         | 1          |
| $a_p$ | $c_0$ | $c_s$ | $a_0$ | $b_q$ | $b_0$ | 1          | 1          |
| $a_p$ | $c_0$ | $c_s$ | $a_0$ | $b_q$ | $b_0$ | 1          | -1         |
| $a_p$ | $a_0$ | $b_q$ | $c_s$ | $c_0$ | $b_0$ | 1          | 1          |
| $a_p$ | $a_0$ | $b_q$ | $c_s$ | $c_0$ | $b_0$ | 1          | -1         |

Table 2

To avoid the lack of uniqueness in the cycles, we always write the cycle $\tau_i$ as an ordered set with initial point $I_i$, but to simplify notation we keep the cycle notation. In general this will not generate confusion in our work.

Lemma 12 Let $\mu$ be the permutation associated to the Schubert form $(p/n, q/m, s/l)$, 
$\mu = \tau_1 \tau_2 \tau_3$, for $i=1, 2, 3$ we have:

(i) Each cycle $\tau_i$ is even, with order $|\tau_i|$ greater than 4.

(ii) $\tau_i = \{I_i, z_1, \cdots, z_k, F_i, \gamma(z_k), \cdots, \gamma(z_1)\}$, for $z_j \in A \cup B \cup C$, $j = 1, \cdots, k$, $k \geq 1$.

(iii) $\tau_i^{1/|\tau_i|}$ contains the transposition $(I_i, F_i)$.

Proof. By condition (9) we get $\mu(I_i) = \tau_i(I_i) \neq F_i$, so the length of the cycle $\tau_i$ is greater than 3. Then, there exists $z_j \in A \cup B \cup C$, $j = 1, \cdots, k \geq 1$ such that $z_1 = \mu(I_i), z_2 = \mu(z_1), \cdots, z_k = \mu(z_{k-1})$ and $F_i = \mu(z_k) = \phi \gamma(z_k)$, this yields

$\mu(F_i) = \phi \gamma(F_i) = \phi(F_i) = \phi(\phi \gamma(z_k)) = \gamma(z_k)$. 

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Now,
\[
\mu(\gamma(z_k)) = \phi(\gamma(z_k)) = \phi(z_k) = \phi(\phi(\gamma(z_k))) = \gamma(z_{k-1}),
\]
and then, for \( j = k, \ldots, 2 \), we get
\[
\mu(\gamma(z_j)) = \phi(\gamma(z_j)) = \phi(z_j) = \phi(\phi(\gamma(z_j))) = \gamma(z_{j-1}).
\]

The relevant information on each cycle \( \tau_i \) is contained in the first part of the cycle, we define the initial segment of \( \tau_i \) as
\[
\tilde{\tau}_i = \{I_i, z_1, \ldots, z_k\}
\] (10)

We may summarize the results up to now in an algorithm that allows us to find the cycles \( \tau_1, \tau_2, \tau_3 \), the set \( E \) and therefore the directions \( \delta_b \) and \( \delta_c \) associated to a Schubert form.

Let \( \sigma \) be the permutation in \( A \cup B \cup C \) defined by
\[
\sigma = (a_0 a_p)(b_0 b_q)(c_0 c_s).
\]
This permutation corresponds to "travel the bridges" in the diagram.

**Algorithm 13** Given a Schubert form \((p/n, q/m, s/l)\) the following algorithm finds the orientation of the associated link diagram. It provides the cycles \( \tau_1, \tau_2, \tau_3 \) such that \( \mu = \tau_1 \tau_2 \tau_3 \), where the cycle \( \tau_i \) has the form given in Lemma 12.

1. Take \( I_1 = a_p, \tau_1 = \mathcal{O}_\mu(I_1) \) and \( F_1 = \tau_1^{[\tau_1]/2}(I_1) \).
2. If \( F_1 = I_1 \) take \( I_2 = b_q \) else take \( I_2 = \sigma(F_1) \).
3. Take \( \tau_2 = \mathcal{O}_\mu(I_2) \) and \( F_2 = \tau_2^{[\tau_2]/2}(I_2) \).
4. If \( \sigma(F_2) \notin \{I_1, F_1, I_2, F_2\} \) then take \( I_3 = \sigma(F_2) \) else take \( I_3 \) as the unique element in \( \{b_q, c_s\} - \{I_1, F_1, I_2, F_2\} \).
5. Take \( \tau_3 = \mathcal{O}_\mu(I_3) \) and \( F_3 = \tau_3^{[\tau_3]/2}(I_3) \).

**6 Presentation of the 3-bridge link group**

Let \( L \) be the link diagram with Schubert form \((p/n, q/m, s/l)\). We have an explicit way to find the over and under presentations of the link group of \( L \), see [1] and [2]. This method requires to use the link diagram. We will use this method, but we will replace the explicit use of the diagram by an algorithm that uses the permutations \( \phi, \gamma \) and \( \mu \) and some new functions defined on the
set $A \cup B \cup C$. As the description of the over and under presentations requires an oriented link diagram, we will always refer to the standard link diagram and orientation described in Section 3. It is important to remark that we need the diagram only to explain the construction, but the algorithm to find the presentations do not require to draw the link diagram, it depends only on the permutations $\phi, \gamma$ and $\mu = \phi \gamma$. As $\phi, \gamma$ depend only of the Schubert form $(p/n, q/m, s/l)$, the presentation of the link group will depend only on the integers $\{p, n, q, m, s, l\}$. The algorithm is efficient and easy to implement in a software such as *Mathematica*.

### 6.1 Over presentation of the 3-bridge link $(p/n, q/m, s/l)$

We take meridians around the bridges as group generators, and label them by the same name as the bridges, so we have generators $a, b$ and $c$, see Fig. 6a.

![Figure 6: Generators for the over presentation in a and the under presentation in c. Path around an underarc in b.](image)

We find the relators by traveling the frontier of a neighborhood of the underarcs, as shown in Fig. 6b, so these paths are precisely the orbits $\tau_i$, $i = 1, 2, 3$.

Each relator is a word in $a, b$ and $c$ constructed with the convention that each time we cross the bridge $a$ (resp. $b, c$) we write $a^{\pm 1}$ (resp. $b^{\pm 1}, c^{\pm 1}$) depending of the sign of the crossing, given by the convention $\uparrow^+ \downarrow^-$. We replace this graphic process by defining a function $\Gamma$ that “forgets the index but remembers the direction”. Consider $\Gamma : A \cup B \cup C \rightarrow \{a^{\pm 1}, b^{\pm 1}, c^{\pm 1}\}$
defined by
\[ \Gamma (a_i) = \begin{cases} a & \text{if } 0 < i \leq p \\ a^{-1} & \text{otherwise} \end{cases}, \quad \Gamma (b_i) = \begin{cases} b^{b_i} & \text{if } 0 < i \leq q \\ b^{-b_i} & \text{otherwise}. \end{cases}, \tag{11} \]
\[ \Gamma (c_i) = \begin{cases} c^{c_i} & \text{if } 0 < i \leq s \\ c^{-c_i} & \text{otherwise}. \end{cases} \]

The relators are \( r_1 = \Gamma (\tau_1), r_2 = \Gamma (\tau_2) \) and \( r_3 = \Gamma (\tau_3) \), where \( \Gamma (\tau_i) \) means the word obtained when we apply \( \Gamma \) to each element in the orbit \( \tau_i \).

Thus we have proved the following proposition.

**Proposition 14** The link group of the link \( L \) given by the Schubert form \((p/n, q/m, s/l)\) admits a presentation given by
\[ \pi (L) = \langle a, b, c \mid \Gamma (\tau_1), \Gamma (\tau_2), \Gamma (\tau_3) \rangle \]
were \( \mu = \tau_1 \tau_2 \tau_3 \) is the associated permutation and \( \Gamma \) is given in (11).

By the symmetry of the cycles described in Lemma 12 we may rewrite the relators as the relations. When the Schubert form \((p/n, q/m, s/l)\) defines a knot, using the information in Table 1 we find that
\[ r_1 : aw_a = w_1 b, \quad r_2 : bw_b = w_2 c, \quad r_3 : cw_c = w_3 a, \]
in the first four cases,

or
\[ r_1 : aw_a = w_a c, \quad r_2 : cw_c = w_b b, \quad r_3 : bw_b = w_3 a, \]
in the last four cases.

For the case when it is a link we have similar relations.

At this moment we have lost the geometrical meaning of the generators, so we may rename the generators, if necessary, and unify the two cases, so we have the following proposition.

**Proposition 15** The link \( L \) given by the Schubert form \((p/n, q/m, s/l)\) admits a presentation given by

i. \( \langle a, b, c \mid aw_1 = w_1 b, \quad bw_2 = w_2 c, \quad cw_3 = w_3 a \rangle \) if \( L \) is a knot,

ii. \( \langle a, b, c \mid aw_1 = w_1 b, \quad bw_2 = w_2 a, \quad cw_3 = w_3 c \rangle \) if \( L \) is a 2-component link,

iii. \( \langle a, b, c \mid aw_1 = w_1 a, \quad bw_2 = w_2 b, \quad cw_3 = w_3 c \rangle \) if \( L \) is a 3-component link,

were \( w_i \) is a word in \( a, b, c \) given by \( w_i = \Gamma (\bar{\tau}_i) \) were \( \Gamma \) is defined in (11) and \( \bar{\tau}_i \) is defined in (10).
We know that in the case of knots, one of the relations is redundant, but for practical reasons we prefer to have all of them and, in particular computations, we omit the longest relation, that we do not know in advance which one will be. This contrasts with the under presentation that we will introduce in the next section, in which we know the lengths of the words involved.

**Lemma 16** The sum of the lengths of the words $w_1, w_2$ and $w_3$ is $p+q+s-3$.

**Lemma 17** When $L$ is a knot, the peripheral system of the group is given by $\langle a, l \rangle$ with $l = w_1 w_2 w_3 a^{-k}$ and $k$ is the exponent sum of the word $w_1 w_2 w_3$.

**Proof.** By direct computation we have

$$al = aw_1 w_2 w_3 a^{-k} = w_1 b w_2 w_3 a^{-k} = w_1 w_2 w_3 a^{-k} = w_1 w_2 w_3 a^{-k} = 1a$$

\[\blacksquare\]

### 6.2 Under presentation of the 3-bridge link $(p/n, q/m, s/l)$

The under presentation is the dual presentation of the over presentation. Those dual presentations play a central role in the proofs of properties of the knot group and the Alexander polynomial of knots, see [2]. By studying these presentations we find a similar algorithm to the known one to find the 2-bridge link group, that has an explicit formula depending of $p$ and $q$. Of course, we need a more elaborate algorithm.

We take as generators of $\pi(L)$ the meridians around the underarcs, see [3]: Again, the key point is to use the cyclic decomposition of $\mu$. We call $a$ (resp. $b$ and $c$) the generators corresponding to the underarc described by $\tau_1$ (resp. $\tau_2$ and $\tau_3$).

The relations are given by traveling the boundary of each butterfly, that describe simple closed paths around the bridges.

So the first path is given by $\{a_0, \cdots, a_{2p-1}\}$, the second by $\{b_0, \cdots, a_{2q-1}\}$ and the third by $\{c_0, \cdots, c_{2s-1}\}$. The graphical procedure to find the relators is: Each time we cross the link we encounter one of the vertices in the set $A \cup B \cup C$, we identify the underarc, say $x$, and the sign of the crossing, $sg$, and write $x^{sg}$, with $x \in \{a, b, c\}$ and $sg = \pm 1$.

Again, this procedure will be established by defining a function $\rho$, similar to the one defined in [11], that identifies the underarc that contains the
vertex and the direction of the crossing. Let \( \rho : A \cup B \cup C \to \{a^{\pm 1}, b^{\pm 1}, c^{\pm 1}\} \) defined by

\[
\begin{aligned}
  \text{If } x \in E, \quad & \rho(x) = \\
  & = \begin{cases} \\
    a & \text{if } x \in \tilde{\tau}_1 \\
    a^{-1} & \text{if } x \notin \tilde{\tau}_1 \\
    b^{\delta_b} & \text{if } x \in \tilde{\tau}_2 \\
    b^{-\delta_b} & \text{if } x \notin \tilde{\tau}_2 \\
    c^{\delta_c} & \text{if } x \in \tilde{\tau}_3 \\
    c^{-\delta_c} & \text{if } x \notin \tilde{\tau}_3 \\
  \end{cases} \\
  \text{If } x \notin E, \quad & \rho(x) = \\
  & = \begin{cases} \\
    a & \text{if } x \in \tilde{\tau}_1 \\
    a^{-1} & \text{if } \gamma(x) \in \tilde{\tau}_1 \\
    b^{\delta_b} & \text{if } x \in \tilde{\tau}_2 \\
    b^{-\delta_b} & \text{if } \gamma(x) \in \tilde{\tau}_2 \\
    c^{\delta_c} & \text{if } x \in \tilde{\tau}_3 \\
    c^{-\delta_c} & \text{if } \gamma(x) \in \tilde{\tau}_3 \\
  \end{cases}
\end{aligned}
\]

The relators are

\[
\begin{aligned}
  s_a &= \rho(a_0a_1 \cdots a_{2p-1}) = \rho(a_0) \rho(a_1) \cdots \rho(a_{2p-1}), \\
  s_b &= \rho(b_0b_1 \cdots b_{2q-1}) = \rho(b_0) \rho(b_1) \cdots \rho(b_{2q-1}), \\
  s_c &= \rho(b_0b_1 \cdots b_{2s-1}) = \rho(c_0) \rho(c_1) \cdots \rho(c_{2s-1}).
\end{aligned}
\]

For the symmetry of the functions and the cycles given in Lemma \[\text{[12]}\], we have that if \( \gamma(x) \neq x \), \( \rho(\gamma(x)) = \rho(x)^{-1} \), therefore if we take the words

\[
\begin{aligned}
  u_a &= \rho(a_1) \cdots \rho(a_{p-1}), \\
  u_b &= \rho(b_1) \cdots \rho(b_{q-1}), \\
  u_c &= \rho(c_1) \cdots \rho(c_{s-1}),
\end{aligned}
\]

the relators become the relations

\[
\begin{aligned}
  cu_a &= u_a, \\
  au_b &= u_b, \\
  bu_c &= u_c
\end{aligned}
\]
or

\[
\begin{aligned}
  bu_a &= u_a, \\
  au_c &= u_c, \\
  cu_b &= u_b.
\end{aligned}
\]

Note that the lengths of the words \( u_a, u_b \) and \( u_c \) are \( p - 1, q - 1 \) and \( s - 1 \), respectively. In this case it is not possible to change the variable names, because we want to have the information about the word lengths.

Now the peripheral system is given by \( \langle a, l' \rangle \) where \( l' = u_a u_b u_c f^{-e} \), were \( e \) is the exponent sum of the word \( u_a u_b u_c \). We have proved the following theorem:

**Theorem 18** The link \( L \) given by the butterfly \((p/n, q/m, s/l)\) admits a presentation given by:

1. If \( L \) is a knot

\[
\begin{aligned}
  & \langle a, b, c \mid cu_a = u_a, \\
  & au_b = u_b, \\
  & bu_c = u_c \rangle,
\end{aligned}
\]

or

\[
\begin{aligned}
  & \langle a, b, c \mid bu_a = u_a, \\
  & au_c = u_c, \\
  & cu_b = u_b \rangle.
\end{aligned}
\]

\[17\]
ii. If $L$ is a 3-component link

\[ \langle a, b, c \mid au_a = u_a a, \quad bu_b = u_b b, \quad cu_c = u_c c \rangle, \]

with $u_a$, $u_b$, and $u_c$ words of length $p-1$, $q-1$ and $s-1$, respectively, defined by (12).

If $L$ is a 2-component link there are six possible combination for the presentation, that are the natural variations of $\langle a, b, c \mid au_a = u_a a, \quad cu_b = u_b b, \quad bu_c = u_c c \rangle$.

**Note:** This construction does not depend on the fact that $p \geq q \geq s$, nor that we are working with type I butterfly. So we may use it in a more general way. However, if we take the Schubert form $(p/n, q/m, s/l)$ we know that in the knot case one of the relations is redundant, and in this presentation we know that the longest is the first one, so usually that is the one we eliminate.

7 Special family: $(p/n, p/n, p/n)$

In general we do not have an exact pattern for a 3-bridge link group, as the one we encounter for 2-bridge links, see [3], but there are families of 3-bridge links with a very regular pattern for the fundamental group. One of them is the family of links with Schubert form $(p/n, p/n, p/n)$, for integers $1 \leq n \leq p$. This family contains: Borromean rings $(5/2, 5/2, 5/2)$, the pretzel link $P(p, p, p)$, that corresponds to $(2p/p, 2p/p, 2p/p)$; the toroidal knot $T(3, p)$ and its mirror image $T(3, -p)$, that corresponds to $(p/1, p/1, p/1)$ and to $(p/p, p/p, p/p)$, respectively. The standard diagrams of the links in this family have symmetries of order 2 and 3.

**Proposition 19** For the link $L$ with Schubert form $(p/n, p/n, p/n)$ there exists a word $w(x, y, z)$ in the variables $x$, $y$ and $z$, such that if $w_a = w(a, b, c)$, $w_b = w(b, c, a)$ and $w_c = w(c, a, b)$ then:

i. If $L$ is a knot, the knot group has the presentation

\[ \langle a, b, c \mid aw_a = w_a b, bw_b = w_b c, cw_c = w_c a \rangle. \]

2. If $L$ is a link, it has 3 components and the link group has the presentation

\[ \langle a, b, c \mid aw_a = w_a a, bw_b = w_b b, cw_c = w_c c \rangle. \]

**Proof.** Study the symmetry of the link diagram. ■

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Example 20 1. The Borromean rings have Schubert normal form \((5/2, 5/2, 5/2)\) and \(w(x, y, z) = yz^{-1}y^{-1}z\).

2. The knot \(8_{19}\) in Rolfsen’s table has Schubert normal form \((4/1, 4/1, 4/1)\) and \(w(x, y, z) = zyx\). Note that it is the toroidal knot \(T(3, 4)\). In general, the toroidal link \((p/1, p/1, p/1)\) is a 3-component link if \(p \equiv 1 \mod 3\) and it is a knot in the other cases; and the word \(w\) in Proposition 19 is

\[
\begin{align*}
  w(x, y, z) = \left\{ \\
  (zyx)^{p/3} & \text{ if } p \equiv 1 \mod 3 \\
  (zyx)^{p/3}z & \text{ if } p \equiv 2 \mod 3 \\
  (zyx)^{p/3}z & \text{ if } p \equiv 0 \mod 3,
\end{align*}
\]

were \([m]\) means the integer part of \(m\).

3. The knot \(9_{35}\) in Rolfsen’s table has Schubert normal form \((6/3, 6/3, 6/3)\) and \(w(x, y, z) = z^{-1}xz^{-1}yz^{-1}\). Note that it is the Pretzel knot \((3, 3, 3)\). In general the Pretzel link \((2p/p, 2p/p, 2p/p)\) is a knot if \(p\) is odd and a 3-component link if \(p\) is even and the word \(w\) in Proposition 19 is

\[
\begin{align*}
  w(x, y, z) = \left\{ \\
  (z^{-1}x)^{p/2}z^{-1}(z^{-1}y)^{p/2} & \text{ if } p \text{ is odd} \\
  (z^{-1}x)^{(p-2)/2}z^{-1}xy^{-1}(xy^{-1})^{(p-2)/2} & \text{ if } p \text{ is even}.
\end{align*}
\]

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