A New Class of Skewed Bimodal Distributions

Ricardo S. Ehlers

Department of Applied Mathematics and Statistics
University of São Paulo - Brazil

1 Introduction

Probability distributions that can accommodate the possible presence of heavy tails and skewness in the distribution of a phenomenon have been the focus of interest in recent years. See for example, Azzalini (1985), Fernandez and Steel (1998), Azzalini and Capitanio (2003), Jones and Faddy (2003) and Ferreira and Steel (2006) to name but a few. However, these distributions fail to capture a possible bimodality in the data under study. In this paper, our aim is to introduce a new family of distributions that is flexible enough to support skewness, heavy tail and bimodal shapes.

Recently, Elal-Olivero, Gómez, and Quintana (2009) and Rocha, Loschi, and Arellano-Valle (2012) extended Azzalini’s skew normal family of distributions to accommodate such behaviour in the resulting distribution. These authors propose to disturb the symmetry of the density,

\[ g(x) = \left( \frac{1 + \alpha x^2}{1 + \alpha b} \right) f(x), \quad x \in \mathbb{R}, \quad (1) \]

where \( \alpha \geq 0 \) and \( f(\cdot) \) is symmetric and unimodal with finite second moment \( b \). The parameter \( \alpha \) control the uni or bimodality of \( g(\cdot) \) since the density is unimodal if \( \alpha \in [0, 0.5) \) and bimodal if \( \alpha \geq 0.5 \). Then, they use a cumulative distribution function \( H(\cdot) \) as a skewing mechanism and the proposed skewed version (possibly bimodal) is given by,

\[ s(x|\alpha, \lambda, H) = 2 \left( \frac{1 + \alpha x^2}{1 + \alpha b} \right) f(x) H(\lambda x), \]

where the parameter \( \lambda \in \mathbb{R} \) introduces skewness.

In this paper, we propose a different route. We first obtain the skewed version of a unimodal symmetric density using a skewing mechanism that is not
based on a cumulative distribution function. Then we disturb the unimodality of the resulting skewed density using the same mechanism as in (1). In order to introduce skewness we use the general method proposed in Fernandez and Steel (1998) which transforms any continuous unimodal and symmetric distribution into a skewed one by changing the scale at each side of the mode. They proposed the following class of skewed distributions indexed by a shape parameter $\gamma > 0$, which describes the degree of asymmetry,

$$
 s(x|\gamma) = \frac{2}{\gamma + 1/\gamma} \left\{ f\left(\frac{x}{\gamma}\right) I_{[0,\infty)}(x) + f(x\gamma)I_{(-\infty,0)}(x) \right\},
$$

(2)

where $f(\cdot)$ is a univariate density symmetric around zero and $I_C(\cdot)$ is an indicator function on $C$. Note that $\gamma = 1$ yields the symmetric distribution as $s(x|\gamma = 1) = f(x)$. Right skewness corresponds to $\gamma > 1$ while left skewness corresponds to $\gamma < 1$. Our preference for this skewing mechanism is mainly due to its simplicity and generality. Moments calculation is straightforward if the moments of the underlying symmetric distribution are available and it does not require calculation of cumulative distribution functions, which yields faster computations. Also, it entirely separates the effects of the skewness and tail parameters thus making prior independence between the two a plausible assumption, and hence facilitates the choice of their prior distributions.

**Proposition 1.1.** Let $f$ be a symmetric unimodal density with mode zero. If $s(\cdot|\gamma)$ is as defined in (2) and $b_\gamma = \int_{-\infty}^{\infty} x^2 s(x|\gamma)dx < \infty$ then,

$$
 s(x|\alpha, \gamma) = \frac{2}{\gamma + 1/\gamma} \left\{ f\left(\frac{x}{\gamma}\right) I_{[0,\infty)}(x) + f(x\gamma)I_{(-\infty,0)}(x) \right\}
$$

is a density for any $\alpha \geq 0$ and $\gamma > 0$.

**Proof.** Clearly $s(x|\alpha, \gamma) \geq 0$. Also,

$$
\int_{-\infty}^{\infty} \left(\frac{1 + \alpha x^2}{1 + \alpha b_\gamma}\right) s(x|\gamma)dx = \frac{1}{1 + \alpha b_\gamma} \left[ 1 + \alpha \int_{-\infty}^{\infty} x^2 s(x|\gamma)dx \right] = 1
$$

since the integral on the right hand side is simply $b_\gamma$. \qed

The existence of the moments of (2) depends only on the existence of moments of the symmetric density $f(\cdot)$ and does not depend on $\gamma$. The $r$th moment is given by,

$$
E(X^r|\gamma) = \frac{\gamma^{r+1} + (-1)^r/\gamma^{r+1}}{\gamma + 1/\gamma} m_r,
$$

where

$$
m_r = 2 \int_0^{\infty} x^r f(x)dx
$$
is the \( r \)-th absolute moment of \( f(x) \) on the positive real line. It is not difficult to see that when the original symmetric distribution has mean zero and variance one then \( m_2 = 1 \). In this case, the second moment \( b_γ \) is given by,

\[
b_γ = \frac{γ^3 + 1/γ^3}{γ + 1/γ}.
\]

So, the moments of this bimodal skewed distribution are given by,

\[
E(X^r|α, γ) = \frac{1}{1 + αb_γ} [E(X^r|γ) + αE(X^{r+2}|γ)].
\]

For example, choosing \( f(x) = φ(x) \) in (2), i.e. the density of a standard normal distribution we obtain the bimodal skew normal distribution with parameters \( α \) and \( γ \) and denote \( X ∼ BSN(α, γ) \). This density is given by,

\[
s(x|α, γ) = \left( \frac{1 + αx^2}{1 + αb_γ} \right) \left( \frac{1}{2} \right)^{1/2} \frac{1}{(γ + 1/γ)} \exp \left\{ \frac{x^2}{2} \left( \frac{1}{γ^2} I_{(0,∞)}(x) + γ^2 I_{(-∞,0)}(x) \right) \right\}, \quad x ∈ \mathbb{R},
\]

and is depicted in Figure 1 for varying \( α ∈ \{1, 3, 10\} \) and fixing \( γ > 1 \) (left panels) and \( γ < 1 \) (right panels). For fixed \( α \), the position of the higher mode is controlled by \( γ \). As \( γ > 1 \) (right skewness) increases density values are higher in the right mode than in the left one as the original (unimodal) skewed density puts more probability mass above zero. Actually, the left mode is pushed towards zero as \( γ \) increases above one. Of course the reverse behaviour is observed when \( 0 < γ < 1 \) (left skewness) decreases.

Since they assign low probabilities to rare events, the family of distributions presented above will fail to fit data with heavy tails and we need to consider alternatives. Choosing \( f(·) \) to be the standardized Student \( t \) density (mean zero and variance one) we obtain the bimodal skewed Student distribution with parameters \( α, γ \) and \( ν \) denoted \( BSSTD(α, γ, ν) \) and density function given by,

\[
s(x|α, γ, ν) = \left( \frac{1 + αx^2}{1 + αb_γ} \right) \frac{2Γ(\frac{ν+1}{2})}{Γ(\frac{ν}{2})(γ + 1/γ)[π(ν - 2)]^{1/2}} \left[ 1 + \frac{x^2}{ν - 2} \left( \frac{1}{γ^2} I_{(0,∞)}(x) + γ^2 I_{(-∞,0)}(x) \right) \right]^{-\frac{ν+1}{2}}, \quad x ∈ \mathbb{R}, \quad ν > 2.
\]

for \( x ∈ \mathbb{R} \) and \( ν > 2 \). This density is depicted in Figure 2 for \( ν = 4 \), fixing the value of \( γ \) and varying \( α ∈ \{1, 3, 10\} \). It is clear that, compared to the BSN case, events far apart in the tails will receive higher probabilities under this family. Note also that using this standardized version of the symmetric \( t \) distribution allows us to keep the same expression for \( b_γ \) in both densities (3) and (4) and propose a scale mixture representation as follows.
Figure 1: Bimodal skew normal densities fixing the value of $\gamma$ and varying $\alpha \in \{1, 3, 10\}$.
Figure 2: Bimodal skew $t$ densities with $\nu = 4$ fixing the value of $\gamma$ and varying $\alpha \in \{1, 3, 10\}$.
Proposition 1.2. A random variable \( X \sim \text{BSSTD}(\alpha, \gamma, \nu) \) admits a scale mixture representation of \( \text{BSN}(\alpha, \gamma) \) distributions with scale \( \lambda^{-1/2} \) and mixing distribution \( \lambda \sim \text{Gamma}(\nu/2, (\nu - 2)/2) \).

**Proof.** Let \( X|\lambda \sim \text{BSN}(\alpha, \gamma) \) with scale \( \lambda^{-1/2} \) and density given by,

\[
s(x|\alpha, \gamma, \lambda) = \left( \frac{1 + \alpha x^2}{1 + \alpha b_\gamma} \right) \left( \frac{2}{\pi} \right)^{1/2} \frac{\lambda^{1/2}}{\gamma + 1/\gamma} \exp \left\{ -\frac{\lambda x^2}{2} \left( \frac{1}{\gamma^2} I_{(0,\infty)}(x) + \gamma^2 I_{(-\infty,0)}(x) \right) \right\}, \quad x \in \mathbb{R},
\]

and

\[
f(\lambda) = \frac{[(\nu - 2)/2]^{\nu/2}}{\Gamma(\nu/2)} \lambda^{\nu/2 - 1} \exp(-\lambda(\nu - 2)/2).
\]

So, the marginal density of \( X \) is given by,

\[
s(x|\alpha, \gamma, \nu) = \left( \frac{1 + \alpha x^2}{1 + \alpha b_\gamma} \right) \left( \frac{2}{\pi} \right)^{1/2} \frac{1}{\Gamma(\nu/2)} \frac{[(\nu - 2)/2]^{\nu/2}}{\Gamma(\nu/2)} \times \int_0^\infty \lambda^{(\nu+1)/2-1} \exp \left\{ -\frac{\lambda}{2} \left[ (\nu - 2) + x^2 \gamma^{-2\text{sign}(x)} \right] \right\} d\lambda
\]

\[
= \left( \frac{1 + \alpha x^2}{1 + \alpha b_\gamma} \right) \frac{2\Gamma(\nu/2)}{\Gamma((\nu/2)(\gamma + 1/\gamma)|\pi (\nu - 2)|^{1/2} \left[ 1 + \frac{x^2}{\nu - 2}\gamma^{-2\text{sign}(x)} \right]^{-\nu/4+1}}.
\]

This representation will enable more efficient Bayesian estimation via Markov chain Monte Carlo (MCMC) algorithms using a data augmentation approach. As a by-product, the mixing parameter \( \lambda \) can be used to identify possible outliers.

Proposition 1.3. A random variable \( X \) with a skewed normal distribution with scale \( \lambda^{-1/2} \) admits the following hierarchical form,

\[
X|\gamma, \lambda, u \sim \text{SU} \left( -\lambda^{-1/2}u^{1/2}, \lambda^{-1/2}u^{1/2}, \gamma \right)
\]

\[
U \sim \text{Gamma} \left( \frac{3}{2}, \frac{1}{2} \right)
\]

where \( \text{SU}(a, b, \gamma) \) denotes the skewed version of the Uniform distribution on \((a, b)\).

**Proof.** The density of the skewed version of a uniform distribution on \((-\lambda^{-1/2}u^{1/2}, \lambda^{-1/2}u^{1/2})\) is given by,
\[ s(x|\gamma, \lambda, u) = \frac{(\lambda/u)^{1/2}}{\gamma + 1/\gamma} [I(0 < x < (u/\lambda)^{1/2}\gamma) + I(-(u/\lambda)^{1/2}\gamma < x < 0)] \]

\[ = \frac{(\lambda/u)^{1/2}}{\gamma + 1/\gamma} [I(u > \delta_1)I(x \geq 0) + I(u > \delta_2)I(x < 0)] \]

where \( \delta_1 = \lambda x^2/\gamma^2 \) and \( \delta_2 = \lambda x^2 \gamma^2 \). Now, integrating with respect to \( u \) this density times the density function of \( u \) we obtain,

\[ s(x|\gamma, \lambda) = \frac{\lambda^{1/2}}{\gamma + 1/\gamma} \frac{(1/2)^{3/2}}{\Gamma(3/2)} \]

\[ \left[ \int_{\delta_1}^{\infty} \exp(-u/2)du I_{[0,\infty)}(x) + \int_{\delta_2}^{\infty} \exp(-u/2)du I_{(-\infty,0)}(x) \right] \]

\[ = \frac{\lambda^{1/2}}{\gamma + 1/\gamma} \left( \frac{2}{\pi} \right)^{1/2} \exp \left\{ -\frac{\lambda x^2}{2} \left[ \frac{1}{\gamma^2} I_{[0,\infty)}(x) + \gamma^2 I_{(-\infty,0)}(x) \right] \right\} \]

Propositions 1.2 and 1.3 allow us to rewrite density (4) as the following scale mixture,

\[ s(x|\alpha, \gamma, \nu) = \int_{0}^{\infty} \int_{0}^{\infty} \left( \frac{1 + \alpha x^2}{1 + \alpha b \gamma} \right) s(x|\gamma, \lambda, u) \times f_G \left( u \left| \frac{3}{2}, \frac{1}{2} \right. \right) f_G \left( \lambda \left| \frac{\nu}{2}, \frac{\nu - 2}{2} \right. \right) dud\lambda. \]

where \( f_G(\cdot|a, b) \) denotes the density of a Gamma distributed random variable with mean \( a/b \) and variance \( a/b^2 \).

2 A Wider Class of Distributions

McDonald and Newey (1988) introduced a flexible symmetric and unimodal distribution as another robust alternative to the normal distribution which they called the generalized t distribution. Its density function with location zero and scale one is given by,

\[ f(x) = \frac{p \Gamma \left( q + \frac{1}{p} \right)}{2q^{1/p} \Gamma \left( \frac{1}{p} \right) \Gamma(q)} \left( 1 + \frac{1}{q} |x|^p \right)^{-(q+1/p)} \]

\[ = \frac{p}{2q^{1/p} B \left( 1/p, q \right)} \left( 1 + \frac{1}{q} |x|^p \right)^{-(q+1/p)} \]

\[ x \in \mathbb{R} \]
where \( B(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b) \) is the Beta function, \( p > 0 \) and \( q > 0 \) are two shape parameters. We refer to this distribution as \( GT(p,q) \). Larger values of \( p \) and \( q \) yield a density with thinner tails than the normal while smaller values are associated with thicker tailed densities. Also, it includes other well known symmetric unimodal distributions as special or limiting cases. In particular, the variance exists when \( pq > 2 \) and is given by,

\[
Var(X) = \frac{q^{2/p}\Gamma(3/p)\Gamma(q-2/p)}{\Gamma(1/p)\Gamma(q)} = \frac{q^{2/p}B(3/2,q-2/p)}{B(1/p,q)}.
\]

Therefore, the standardized version of density (5) is given by,

\[
f(x) = \frac{p}{2\delta q^{1/p}B(1/p,q)} \left( 1 + \frac{1}{q} \left| \frac{x}{\delta} \right|^p \right)^{-1/(q+1/p)} x \in \mathbb{R}
\]

where

\[
\delta = \left( \frac{q^{2/p}B(3/2,q-2/p)}{B(1/p,q)} \right)^{-1/2}.
\]

Using this standardized version of the generalized t distribution we obtain the bimodal skewed generalized t distribution with parameters \( \alpha, \gamma, p \) and \( q \) denoted \( BSGT(\alpha, \gamma, p, q) \) and density given by,

\[
s(x|\alpha, \gamma, p, q) = \left( \frac{1 + \alpha x^2}{1 + \alpha b_\gamma} \right)^p \delta(\gamma + 1/\gamma)q^{1/p}B(1/p,q) \left[ 1 + \frac{1}{q} \left| \frac{x}{\delta} \right|^p \left\{ \frac{1}{\gamma^p} I_{[0,\infty)}(x) + \gamma^p I_{(-\infty,0)}(x) \right\} \right]^{-(q+1/p)}.
\]

Again using a standardized version of the original symmetric distribution allows us to keep the same expression for the second moment \( b_\gamma \). It is not difficult to see that setting \( p = 2 \) we recover the bimodal skewed t distribution with tail parameter \( \nu = 2q \). The bimodal skewed normal is then obtained when \( p = 2 \) and \( q \to \infty \). Density (7) is depicted in Figure 3 with \( p = 2.3, q = 2 \), fixing the value of \( \gamma \) and varying \( \alpha \in \{1, 3, 10\} \). Parameter \( p \) has a larger influence on the shape of the density than \( q \), a feature inherited from the symmetric version of the GT distribution. This is illustrated in Figure 4 where we set \( p = 1.7 \).

One feature of the symmetric generalized t distribution is that it can be represented as a scale mixture of an exponential power distribution (Box and Tiao 1973) with a generalized Gamma as the mixing distribution, a result obtained by Arslan and Genç (2003). In the next proposition we extend this representation to the bimodal skewed GT distribution with density (7).
Figure 3: Bimodal skew generalized $t$ densities with $p = 2.3$, $q = 2$, fixing the value of $\gamma$ and varying $\alpha \in \{1, 3, 10\}$.
Figure 4: Bimodal skew generalized $t$ densities with $p = 1.7$, $q = 2$, fixing the value of $\gamma$ and varying $\alpha \in \{1, 3, 10\}$. 
Proposition 2.1. A random variable \( X \sim BSGT(\alpha, \gamma, p, q) \) with density (7) admits a scale mixture representation with the following hierarchical form,

\[
X \mid S = s \sim BSEP \left( 2^{-1/p} s^{-1/2} \left[ \frac{\Gamma(q - 2/p)}{\Gamma(q)} \right]^{-1/2}, p \right)
\]

\[
S \sim GG \left( \frac{p}{2}, 1, q \right),
\]

where \( BSEP(\lambda, p) \) denotes a bimodal skewed exponential power distribution with scale \( \lambda \) and tail parameter \( p \) and \( GG(\cdot, \cdot, \cdot) \) denotes the generalized Gamma distribution.

**Proof.** The density of a (symmetric) standardized exponential power distribution with tail parameter \( p \) is given by,

\[
f(x \mid p) = \frac{p}{\phi \Gamma(1/p)} 2^{1+1/p} \exp \left\{ -\frac{1}{2} \left| \frac{x}{\phi} \right|^p \right\}
\]

where \( \phi = [2^{2/p} \Gamma(3/p) / \Gamma(1/p)]^{-1/2} \). The skewed version of this exponential power distribution with scale given in the proposition, tail parameter \( p \) and noting that \( \phi = 2^{1/p} q^{-1/p} \left[ \frac{\Gamma(q - 2/p)}{\Gamma(q)} \right]^{1/2} \delta \) is then given by,

\[
s(x \mid p, q, \gamma, s) = \frac{2^{1/p} s^{1/2} p \phi^{1/p} \Gamma(1/p) 2^{1+1/p}}{q^{1/p} \Gamma(1/p) 2^{1+1/p}} \exp \left\{ -\frac{1}{2} \left| \frac{x}{2-1/p s^{-1/2}} \right|^p \left( \frac{1}{\gamma q} I_{[0,\infty)}(x) + \gamma p I_{(-\infty,0)}(x) \right) \right\}
\]

while the density of a generalized Gamma distribution with parameters \( 1/p, 1 \) and \( q \) is given by,

\[
f(s) = \frac{p}{2 \Gamma(q)} s^{pq/2 - 1} \exp(-s^{p/2}).
\]

Since the original symmetric density is in its standardized form we have the same expression for the second moment \( b_\gamma \) of the skewed density. It then follows that,

\[
s(x \mid \alpha, \gamma, p, q) = \int_0^\infty \left( \frac{1+\alpha x^2}{1+\alpha b_\gamma} \right) s(x \mid p, q, \gamma, s) f(s) ds
\]

\[
= \left( \frac{1+\alpha x^2}{1+\alpha b_\gamma} \right) \frac{p^2}{\Gamma(1/p) \Gamma(q)} \delta q^{1/p} \Gamma(1/p) \Gamma(q) \gamma^{-1/\gamma} \times
\]

\[
\int_0^\infty s^{pq/2 + 1/2 - 1} \exp \left\{ -s^{p/2} \left[ 1 + \frac{1}{q} \delta \left( \frac{1}{p} \right)^{p-1} \right] (\gamma^{-p})^{\text{sign}(x)} \right\} ds.
\]
Now, setting \( y = s^{p/2} \) the last integral is rewritten as,
\[
\frac{2}{p} \int_{0}^{\infty} y^{(q+1/p)-1} \exp \left\{ -y \left[ 1 + \frac{1}{q} \frac{|x|}{\delta} \left( \gamma^{-p}\text{sign}(x) \right) \right] \right\} dy,
\]
and finally,
\[
s(x|\alpha, \gamma, p, q) = \left( \frac{1 + \alpha x^2}{1 + \alpha b_\gamma} \right)^{-p \Gamma(q + 1/p)} \frac{\delta^{q+1/p}(\gamma + 1/\gamma)\Gamma(1/p)\Gamma(q)}{\left[ 1 + \frac{1}{q} \frac{|x|}{\delta} \left( \gamma^{-p}\text{sign}(x) \right) \right]^{-(q+1/p)}}.
\]

In what follows we propose an alternative representation based on the skewed version of the uniform distribution used in Proposition 1.3. Choy and Chan (2008) had already proposed an alternative representation for the symmetric generalized t distribution based on a scale mixture of (symmetric) uniform distributions. This was latter extended in Ehlers (2015) for a skewed version.

**Proposition 2.2.** A random variable \( X \sim BSGT(\alpha, \gamma, p, q) \) with density (7) admits a scale mixture representation with the following hierarchical form,
\[
X|u, s, \alpha, \gamma \sim BSU(-a, a, \alpha, \gamma)
\]
\[
U \sim Gamma \left( 1 + \frac{1}{p}, 1 \right)
\]
\[
S \sim GG \left( \frac{p}{2}, 1, q \right)
\]
where
\[
a = 2^{-1/p} s^{-1/2} \left[ \Gamma(q - 2/p) / \Gamma(q) \right]^{-1/2} u^{1/p}
\]

### 3 Bayesian Inference

Following Fernandez and Steel (1998), we shall use a Gamma\((a, b)\) prior distribution on \( \phi = \gamma^2 \) which is the ratio of probability masses above and below the mode, i.e. \( \phi = \gamma^2 = Pr(X \geq 0) / Pr(X < 0) \). For observed data \( x = (x_1, \ldots, x_n) \) the likelihood function in the bimodal skewed normal model is given by
\[
\begin{align*}
    s(x|\alpha, \phi) &\propto (1 + \alpha b_\phi)^{-n} \phi^{n/2} (1 + \phi)^{-n} \prod_{i=1}^{n} (1 + \alpha x_i^2) \\
    &\exp \left\{ -\frac{1}{2} \sum_{i=1}^{n} x_i^2 \left( \frac{1}{\gamma^2} I_{[0,\infty)}(x_i) + \gamma^2 I_{(-\infty,0)}(x_i) \right) \right\}
\end{align*}
\]
where
\[ b_\phi = \frac{1 + \phi^3}{\phi(1 + \phi)}. \]
The complete conditional distributions of \( \phi \) and \( \alpha \) are then given by,
\[
f(\phi|\mathbf{x}, \alpha) \propto (1 + \alpha b_\phi)^{-n} \phi^{a+n/2-1}(1 + \phi)^{-n} \exp \left\{ -\frac{1}{2} \sum_{i=1}^{n} x_i^2 \phi^{-\text{sign}(x_i)} - b_\phi \right\}.
\]
\[
f(\alpha|\mathbf{x}, \phi) \propto (1 + \alpha b_\phi)^{-n} \prod_{i=1}^{n} (1 + \alpha x_i^2) f(\alpha).
\]

If we now assume a bimodal skew \( t \) distribution we need to assign a prior distribution for the tail parameter \( \nu \). Here, we follow Deschamps (2006) and use a translated exponential distribution with density,
\[
p(\nu) = \beta \exp\{-\beta(\nu - 2)\} I(\nu > 2).
\]

Using the scale mixture representation, each observation \( X_i \) is associated with the mixing parameter \( \lambda_i \) and we assume that they are a priori independent. The complete conditional densities are given by,
\[
f(\alpha|\mathbf{x}, \mathbf{\lambda}, \gamma, \nu) \propto s(\mathbf{x}|\alpha, \gamma, \mathbf{\lambda}) f(\alpha) \propto (1 + \alpha b_\phi)^{-n} \prod_{i=1}^{n} (1 + \alpha x_i^2) f(\alpha).
\]
\[
f(\phi|\mathbf{x}, \mathbf{\lambda}, \alpha, \nu) \propto s(\mathbf{x}|\alpha, \gamma, \mathbf{\lambda}) f(\phi) \propto (1 + \alpha b_\phi)^{-n} \phi^{a+n/2-1}(1 + \phi)^{-n} \exp \left\{ -\frac{1}{2} \sum_{i=1}^{n} \lambda_i x_i^2 \phi^{-\text{sign}(x_i)} - b_\phi \right\}.
\]
\[
f(\nu|\mathbf{x}, \mathbf{\lambda}, \alpha, \gamma) \propto f(\nu) \prod_{i=1}^{n} f(\lambda_i|\nu) \propto \exp\{-\beta(\nu - 2)\} \prod_{i=1}^{n} \left[ \frac{[(\nu - 2)/2]^{\nu/2}}{\Gamma(\nu/2)} \right]^{\lambda_i^{(\nu-2)/2}} \exp \left( -\frac{\lambda_i(\nu - 2)}{2} \right) \prod_{i=1}^{n} \frac{[(\nu - 2)/2]^{n\nu/2}}{\Gamma^n(\nu/2)} \exp \left\{ -\nu \left( \beta + \frac{1}{2} \sum_{i=1}^{n} (\lambda_i - \log \lambda_i) \right) \right\}.
\]
\begin{align*}
  f(\lambda|x, \alpha, \gamma, \nu) & \propto \prod_{i=1}^{n} s(x_i|\alpha, \gamma, \lambda_i)f(\lambda_i|\nu) \\
  & \propto \prod_{i=1}^{n} \lambda_i^{(\nu+1)/2-1} \exp \left\{ -\frac{\lambda_i}{2}(\nu - 2 + x_i^2\gamma^{-2}\text{sign}(x_i)) \right\}
\end{align*}

so the complete conditional distribution of each mixing parameter is given by,

\[ \lambda_i|x, \lambda_{-i}, \alpha, \gamma, \nu \sim \text{Gamma} \left( \frac{\nu + 1}{2}, \frac{\nu - 2 + x_i^2\gamma^{-2}\text{sign}(x_i)}{2} \right), \]

being easily sampled from. However, the complete conditional distributions of \( \alpha, \phi \) and \( \nu \) are not of any standard form.

References

Arslan, O. and A. I. Genç (2003). Robust location and scale estimation based on the univariate generalized t (GT) distribution. Communications in Statistics: Theory and Methods 32, 1505–1525.

Azzalini, A. (1985). A class of distributions which includes the normal ones. Scandinavian Journal of Statistics 12, 171–178.

Azzalini, A. and A. Capitanio (2003). Distributions generated by perturbations of symmetry with emphasis on a multivariate skew-t distribution. Journal of the Royal Statistical Society B 65, 367–389.

Box, G. E. P. and G. C. Tiao (1973). Bayesian Inference in Statistical Analysis. Addison-Wesley, Publishing Reading, MA.

Choy, S. T. B. and J. S. K. Chan (2008). Scale mixtures distributions in statistical modelling. Australia and New Zealand Journal of Statistics 50(2), 135–146.

Deschamps, P. J. (2006). A flexible prior distribution for Markov switching autoregressions with Student-t errors. Journal of Econometrics 133(1), 153–190.

Ehlers, R. (2015). A study of skewed heavy-tailed distributions as scale mixtures. American Journal of Mathematical and Management Sciences 33, 301–333.

Elal-Olivero, D., H. W. Gómez, and F. A. Quintana (2009). Bayesian modeling using a class of bimodal skew-elliptical distributions. Journal of Statistical Planning and Inference 139, 1484–1492.
Fernandez, C. and M. Steel (1998). On Bayesian modelling of fat tails and skewness. *Journal of the American Statistical Association* 93, 359–371.

Ferreira, J. T. A. S. and M. F. J. Steel (2006). A constructive representation of univariate skewed distributions. *Journal of the American Statistical Association* 101(474), 823–829.

Jones, M. C. and M. J. Faddy (2003). A skew extension of the t-distribution, with applications. *Journal of the Royal Statistical Society B* 65, 159–174.

McDonald, J. B. and W. K. Newey (1988). Partially adaptive estimation of regression models via the generalized t distribution. *Econometric Theory* 4, 428–457.

Rocha, G. H. M. A., R. H. Loschi, and R. B. Arellano-Valle (2012). Inference in flexible families of distributions with normal kernel. *Statistics: A Journal of Theoretical and Applied Statistics*. 