MEROMORPHY OF LOCAL ZETA FUNCTIONS IN SMOOTH MODEL CASES

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Abstract. It is known that local zeta functions associated with real analytic functions can be analytically continued as meromorphic functions to the whole complex plane. But, in the case of general ($C^\infty$) smooth functions, the meromorphic extension problem is not obvious. Indeed, it has been recently shown that there exist specific smooth functions whose local zeta functions have singularities different from poles. In order to understand the situation of the meromorphic extension in the smooth case, we investigate a simple but essentially important case, in which the respective function is expressed as $u(x, y)x^a y^b +$ flat function, where $u(0, 0) \neq 0$ and $a, b$ are nonnegative integers. After classifying flat functions into four types, we precisely investigate the meromorphic extension of local zeta functions in each cases. Our results show new interesting phenomena in one of these cases. Actually, when $a < b$, local zeta functions can be meromorphically extended to the half-plane $\Re(s) > -1/a$ and their poles on the half-plane are contained in the set \{-$k/b$ : $k \in \mathbb{N}$ with $k < b/a$\}.

1. Introduction

Let us consider the integrals of the form

$$Z_f(\varphi)(s) = \int_{\mathbb{R}^2} |f(x, y)|^s \varphi(x, y) dxdy \quad \text{for } s \in \mathbb{C},$$

where $f, \varphi$ are real-valued ($C^\infty$) smooth functions defined on a sufficiently small open neighborhood $U$ of the origin in $\mathbb{R}^2$, and the support of $\varphi$ is contained in $U$. Since the integrals $Z_f(\varphi)(s)$ converge locally uniformly on the half-plane $\Re(s) > 0$, they become holomorphic functions there, which are sometimes called local zeta functions. It has been known in many cases that they can be analytically continued to wider regions. The purpose of this paper is to understand the analytic continuation of local zeta functions.

When $f$ is real analytic, the analytic continuation of local zeta functions have been precisely understood. By using Hironaka’s resolution of singularities [14], it was shown in [4], [3], etc. that $Z_f(\varphi)(s)$ can be analytically continued as meromorphic functions to the whole complex plane and their poles are contained in finitely many arithmetic progressions consist of negative rational numbers. More precisely, Varchenko [27] investigates the exact location of poles of local zeta functions by using the theory of toric varieties based on the Newton polyhedron of $f$ under some nondegeneracy condition. We remark that the above-mentioned results also hold in general dimensional case.

2000 Mathematics Subject Classification. 11S40 (26E10).

Key words and phrases. local zeta functions, meromorphic continuation, flat functions.
On the other hand, the smooth case (without real analyticity assumption) is not so well known. It is known in [19] that the above result due to Varchenko [27] can be naturally generalized when \( f \) belongs to a certain wide class of smooth functions containing the real analytic functions (see Appendix A.4). But the problem of meromorphic extension of \( Z_f(\varphi) \) is not obvious in the general smooth case. Indeed, it was observed in [12], [22] that \( Z_f(\varphi)(s) \) has a singularity different from a pole in the case of a specific non-real analytic \( f \) (see (2.9)).

In this paper, we precisely investigate the case when \( f \) can be expressed in the form:

\[
(1.2) \quad f(x, y) = u(x, y)x^a y^b + \text{(flat function at the origin)},
\]

where \( a, b \in \mathbb{Z}_+ \) and \( u \) is a smooth function defined near the origin with \( u(0,0) \neq 0 \).

(For a smooth function \( g \) defined near the origin, we say \( g \) is flat at the origin if all the derivatives of \( g \) vanish at the origin.) This case is a simple generalization of the above non-real analytic function \( f \) in [12], [22]. Recalling that the monomial case is essentially important in the real analytic case (see [27], [1]) and noticing that the Newton polyhedron of \( f \) in (1.2) is a simple form: \( \{(\alpha, \beta) \in \mathbb{R}^2 : \alpha \geq a, \beta \geq b\} \) (see Figure 1, below), the above case (1.2) might be considered as a natural model in the smooth case.

It has been recognized (c.f. [1]) that the analytic continuation of local zeta functions is deeply related to the behavior at infinity of oscillatory integrals of the form

\[
I_f(\varphi)(\tau) := \int_{\mathbb{R}^2} e^{irf(x,y)} \varphi(x,y)dx\,dy \quad \text{for } \tau > 0,
\]

where \( f, \varphi \) are the same as in (1.1). The investigation of the behavior of oscillatory integrals has similarities to the analytic continuation of local zeta functions. In fact, the case when the phase \( f \) is real analytic is well understood and, moreover, there have been results under some conditions in smooth case, which are direct generalization of those of the real analytic case ([13], [15], [17], [19], [11], etc.). But, it is shown in [18], [21] that when the phase contains a flat function, the behavior of \( I_f(\varphi) \) may have a different pattern from that in the real analytic case. It is also interesting to consider how flat functions in the phase affect the behavior of \( I_f(\varphi)(\tau) \), which is analogous to that of analytic continuation of local zeta functions in this paper.

This paper is organized as follows. After explaining earlier work about analytic continuation of local zeta functions and related issues in Section 2, we state a main result in Section 3. Sections 4–6 are devoted to the proof of the main results. Our investigation of this paper can be explained without language of Newton polyhedra and related words, which are important in singularity theory. The meaning of our analysis is clearer from the viewpoint of the geometry of Newton polyhedra. In the Appendix, the definitions of Newton polyhedra and related important words are given and our studies are explained from these points of view.

**Notation and symbols.**

- We denote \( \mathbb{R}_+ := \{x \in \mathbb{R} : x \geq 0\} \) and \( \mathbb{Z}_+ := \{x \in \mathbb{Z} : x \geq 0\} \).
- In this paper, Newton polyhedra appear in many situations (see Appendix A.1). We use coordinates \((\alpha, \beta)\) for points in the plane containing the Newton polyhedron in order to distinguish this plane from the \((x, y)\)-plane.
2. KNOWN RESULTS AND DESCRIPTION OF THE PROBLEMS

In this paper, we always assume that \( f \) satisfies
\[
(2.1) \quad f(0, 0) = 0 \quad \text{and} \quad \nabla f(0, 0) = (0, 0).
\]
Unless (2.1) is satisfied, every problem treated in this paper is easy. As for \( \varphi \in C_0^\infty(U) \), we sometimes give the following condition:
\[
(2.2) \quad \varphi(0, 0) > 0 \quad \text{and} \quad \varphi \geq 0 \quad \text{on} \ U.
\]
In order to investigate the analytic continuation of local zeta functions, we only use the half-plane of the form: \( \text{Re}(s) > -\rho \) with \( \rho > 0 \). This is the reason why we observe the situation of analytic continuation through the integrability of the integral (1.1). (Of course, many kinds of regions should be considered in the future.)

2.0.1. Holomorphic continuation. First, let us consider the following quantities:
\[
(2.3) \quad h_0(f, \varphi) := \sup \left\{ \rho > 0 : \begin{array}{l}
\text{The domain in which } Z_f(\varphi) \text{ can} \\
\text{be holomorphically continued}
\end{array} \right\},
\]
where
\[
(2.4) \quad h_0(f) := \inf \{ h_0(f, \varphi) : \varphi \in C_0^\infty(U) \}.
\]
We remark that if \( \varphi \) satisfies (2.2), then \( h_0(f, \varphi) = h_0(f) \) holds; but otherwise, this equality does not always hold. Indeed, there exists \( \varphi \in C_0^\infty(U) \) with \( \varphi(0, 0) = 0 \) such that \( h_0(f, \varphi) > h_0(f) \) (see e.g. [20]).

From the integral representation in (1.1), the relationship between the holomorphy and the convergence of the integral implies that the quantity \( h_0(f) \) is deeply related to the following famous index:
\[
(2.5) \quad c_0(f) := \sup \left\{ \mu > 0 : \begin{array}{l}
\text{there exists an open neighborhood } V \text{ of} \\
\text{the origin in } U \text{ such that} \\
|f|^{-\mu} \in L^1(V)
\end{array} \right\},
\]
which is called \textit{log canonical threshold} or \textit{critical integrability index} and has been deeply understood from various kinds of viewpoints. The equality \( h_0(f) = c_0(f) \) always holds. In fact, the inequality \( h_0(f) \geq c_0(f) \) is obvious; while the opposite inequality can be easily seen by Theorem 5.1 in [22]. In the real analytic case, since all the singularities of the extended \( Z_f(\varphi) \) are poles on the real axis, the leading pole exists at \( s = -h_0(f, \varphi) \).

In the seminal work of Varchenko [27], when \( f \) is real analytic and satisfies some conditions, \( h_0(f) \) can be expressed by using the \textit{Newton polyhedron} of \( f \) as
\[
(2.6) \quad h_0(f) = 1/d(f),
\]
where \( d(f) \) is the \textit{Newton distance} of \( f \) (see Appendix A.2). More detailed investigations into meromorphic continuation of \( Z_f(\varphi) \) in various situations are in [8], [9], [7], [24], [2], etc. An interesting work [6] treating the equality \( c_0(f) = 1/d(f) \) is from another approach. We remark that these results treat the general dimensional case. In the same paper [27], Varchenko more deeply investigated the two-dimensional case. Indeed, without any assumption, he shows that the equality (2.6) holds for real analytic \( f \) in \textit{adapted coordinates}. Here the definition of adapted coordinates is given in Appendix A.3, below. These coordinates are important in the study of oscillatory
integrals and their existence is shown in two dimensions in [27], [25], [16], etc. More generally, let us consider the smooth case. M. Greenblatt [12] obtains a sharp result which generalizes Varchenko’s two-dimensional result.

**Theorem 2.1** (Greenblatt [12]). When \( f \) is a nonflat smooth function defined on \( U \), the equation \( c_0(f) = 1/d(f) \) holds in adapted coordinates.

From the above result, holomorphic extension issue is well understood even in the smooth case. On the other hand, the situation of the meromorphic extension is quite different from the holomorphic one and has not been so well known.

2.0.2. **Meromorphic continuation.** Corresponding to (2.3), (2.4) in the holomorphic continuation case, we analogously define the following quantities:

\[
\begin{align*}
(2.7) & \quad \quad m_0(f, \varphi) := \sup \left\{ \rho > 0 : \text{the domain in which } Z_f(\varphi) \text{ can be meromorphically continued contains the half-plane } \Re(s) > -\rho \right\}, \\
(2.8) & \quad \quad m_0(f) := \inf \left\{ m_0(f, \varphi) : \varphi \in C^\infty_0(U) \right\}.
\end{align*}
\]

It is obvious that \( h_0(f) \leq m_0(f) \leq m_0(f, \varphi) \) and \( h_0(f, \varphi) \leq m_0(f, \varphi) \) always holds. As mentioned in the Introduction, if \( f \) is real analytic, then \( m_0(f) = \infty \) holds; while there exist specific non-real analytic functions \( f \) such that \( m_0(f) < \infty \). Indeed, it was shown in [22] (see also [12]) that when

\[
(2.9) \quad f(x, y) = x^a y^b + x^a y^{b-q} e^{-1/|x|^p},
\]

where \( a, b, q \in \mathbb{Z}_+ \) satisfy that \( a < b \), \( b \geq 2 \), \( 1 \leq q \leq b \), \( q \) is even, \( p > 0 \) and \( \varphi \) satisfies the condition (2.2), \( Z_f(\varphi)(s) \) has a non-polar singularity at \( s = -1/b \), which implies \( m_0(f) = 1/b \). Note that \( d(f) = b \) in this case. At present, properties of the singularity at \( s = -1/b \) are not well understood (see [22] for the details).

**Question 1.** For a given smooth functions \( f \), determine (or estimate) the value of \( m_0(f) \).

In this paper, we consider the above question in the case when \( f \) has the form (1.2) which is a natural generalization of (2.9).

3. **Main results**

In this section, let \( f \) be expressed as in (1.2) on some small open neighborhood \( U \) of the origin. Without loss of generality, we assume that \( a, b \in \mathbb{Z}_+ \) in (1.2) satisfy \( a \leq b \) and \( u(0,0) > 0 \). Moreover, we always assume that \( b \geq 1 \). In fact, when \( a = b = 0 \), \( Z_f(\varphi) \) becomes an entire function if the support of \( \varphi \) is sufficiently small.

It is easy to check the following Newton data of \( f \) in (1.2):

- The Newton polyhedron of \( f \): \( \Gamma_+(f) = \{(\alpha, \beta) \in \mathbb{R}_+^n : \alpha \geq a, \beta \geq b\} \).
- The Newton distance of \( f \): \( d(f) = b \).
- The function \( f \) in (1.2) is expressed in an adapted coordinate.

**Lemma 3.1.** If \( U \) is sufficiently small, then \( f \) in (1.2) can be expressed on \( U \) as one of the following four forms.
Proof. For simplicity, we use the following symbol: For \((\alpha, \beta) \in \mathbb{Z}_+^2\),
\[
f^{(\alpha, \beta)}(x, y) := \frac{\partial^{\alpha+\beta} f}{\partial x^\alpha \partial y^\beta}(x, y).
\]
The Taylor formula implies
\[
f(x, y) = \sum_{\beta=0}^{b-1} y^\beta A_\beta(x) + \sum_{\alpha=0}^{a-1} x^\alpha B_\alpha(y) + x^a y^b C(x, y),
\]
where
\[
A_\beta(x) := \frac{x^a}{(a-1)!\beta!} \int_0^1 (1-t)^{a-1} f^{(\alpha, \beta)}(tx, 0) dt \quad \text{for } \beta \in \{0, \ldots, b-1\},
\]
\[
B_\alpha(y) := \frac{y^b}{\alpha!(b-1)!} \int_0^1 (1-t)^{b-1} f^{(\alpha, b)}(0, ty) dt \quad \text{for } \alpha \in \{0, \ldots, a-1\},
\]
\[
C(x, y) := \frac{1}{(a-1)!(b-1)!} \int_0^1 \int_0^1 (1-t_1)^{a-1}(1-t_2)^{b-1} f^{(\alpha, b)}(t_1 x, t_2 y) dt_1 dt_2.
\]
Since \(f^{(\alpha, \beta)}(\cdot, 0)\) is flat at the origin for \(\beta \in \{0, \ldots, b-1\}\), so is \(A_\beta\) for \(\beta \in \{0, \ldots, b-1\}\).
The flatness of \(B_\alpha\) is similarly shown for \(\alpha \in \{0, \ldots, a-1\}\). An easy computation gives \(C(0, 0) = u(0, 0)\). Putting \(g_j(x) := A_j(x), h_j(y) := B_j(y)\) and \(v(x, y) := C(x, y)\), we can obtain the lemma.

**Remark 3.2.**

(1) The example \([2.9]\) mentioned in Section 2 belongs to the case (B).

(2) In Lemma 3.1, \(v\) does not always equal \(u\) in \((1.2)\) on \(U\) and, moreover, in the cases (B), (C), (D), \(v\) cannot always be replaced by \(v \equiv 1\) by using coordinate changes.

(3) It is easy to see the following equivalences.

(i) When \(a = b\), (B) \(\iff\) (C).

(ii) When \(a = 0\), (A) \(\iff\) (C) and (B) \(\iff\) (D).

The equivalence in (i) means that the roles of the \(x\) and \(y\) variables can be switched.

Let us observe the above classification from the viewpoint of the geometry of Newton polyhedra. Of course, the existence of the flat functions \(h, g\) give no influence on the shape of the Newton polyhedron \(\Gamma_+(f)\). But, in Figure 1, we forcibly draw their influence by adding the points \((0, \infty), (\infty, 0)\) (see also Appendix A.4).
Let us investigate the quantities $h_0(f), m_0(f)$ with $f$ in (1.2) in each of the above cases. As explained in the previous section, Theorem 2.1 implies $h_0(f) = 1/b$ in all the cases. Now, let us consider the value of $m_0(f)$. In the case (A), it is easy to see that $m_0(f) = \infty$ (see also Appendix A.5). In the cases (B), (C), (D), it follows from Theorem 2.1 that the estimate $m_0(f) \geq 1/b$ always holds. This estimate is optimal in the case (B). (Recall that $m_0(f) = 1/b$ holds when $f$ is as in (2.9).) In the case (C), we see a new phenomenon of meromorphic continuation, which is a main theorem of this paper.

**Theorem 3.3.** Let $a > 0$ and let $f$ be expressed as in the case (C) in Lemma 3.1 on $U$. If the support of $\varphi$ is sufficiently small, then $Z_f(\varphi)(s)$ can be analytically continued as a meromorphic function to the half-plane $\text{Re}(s) > -1/a$. Moreover, when $a < b$, its poles on the region $\text{Re}(s) > -1/a$ are contained in the set \{-$k/b : k \in \mathbb{N}$ with $k < b/a$\}. In particular, $m_0(f) \geq 1/a$ holds.

**Remark 3.4.** (1) In the forthcoming paper [23], we will show that there exists a specific function $f$ of the form (C) such that $Z_f(\varphi)$ has the non-polar singularity at $s = -1/a$, which implies $m_0(f) = 1/a$. Therefore, the estimate $m_0(f) \geq 1/a$ in Theorem 3.3 is optimal. Moreover, the optimality of $m_0(f) \geq 1/b$ in the case (D) will also be shown in the same paper.

(2) From Lemma 6.2 (i) below playing essential roles in the proof of the above theorem, the readers might wonder if \{-$j/a : j \in \mathbb{N}$\} is also contained in the set of candidate poles of $Z_f(\varphi)$. Since properties of the singularity of $Z_f(\varphi)$ on the line
Re(\(s\)) = -1/a have not been well-understood at present. \(Z_f(\varphi)\) can be regarded as a meromorphic function only on the region Re(\(s\)) > -1/a in general. Noticing that \(-j/a : j \in \mathbb{N}\) and \(-k/b : k \geq b/a\) are outside of the region Re(\(s\)) > -1/a, we see that \(-k/b : 1 \leq k < b/a\) only appears in the theorem.

Figure 2. Meromorphic continuation in the case (C)

| \(h_0(f)\) | (A) | (B) | (C) | (D) |
|-------------|-----|-----|-----|-----|
| \(m_0(f)\)  | \(\infty\) | \(\geq 1/b\) | \(\geq 1/a\) | \(\geq 1/b\) |

Putting the above mentioned results together, one has Table 1. We remark that Table 1 establishes for \(a > 0\). But, by regarding 1/0 as \(\infty\) and recalling Remark 3.2 (ii), this restriction is not needed.

It should be expected that the inequalities “\(\geq\)” in Table 1 can be removed. In other words, the following question is naturally raised.

**Question 2.** Do the following equalities hold?

(i) \(m_0(f) = 1/b\) for all \(f\) satisfying (B) or (D);
(ii) \( m_0(f) = 1/a \) for all \( f \) satisfying (C) with \( a > 0 \).

As mentioned above, some specific cases showing the above equalities are known but they are very special. It seems to be difficult to generally show the equalities from our method in this paper. Indeed, after some kind of approximation, we apply a van der Corput-type lemma (Lemma 4.5, below). Although this process is available for general smooth functions, this approximation is an obstacle to see the situation of behavior of local zeta functions near the line \( \text{Re}(s) = -1/a \) or \( -1/b \).

4. Auxiliary lemmas

4.1. Meromorphy of one-dimensional model. The following lemma is essentially known (see [10], [4], [1], etc.). Since we will use not only the result but also an idea of its proof in the later computation, we give a complete proof here.

Lemma 4.1. Let \( A, B \) be integers with \( A > 0 \), \( B \geq 0 \) and let \( \psi(u; s) \) be a complex-valued function defined on \([0, r] \times \mathbb{C}\), where \( r > 0 \). We assume that

(a) \( \psi(\cdot; s) \) is smooth on \([0, r]\) for all \( s \in \mathbb{C} \);

(b) \( \frac{\partial^\alpha \psi}{\partial u^\alpha}(u; \cdot) \) is an entire function on \( \mathbb{C} \) for all \( u \in [0, r] \) and \( \alpha \in \mathbb{Z}_+ \).

Let

\[
(4.1) \quad L(s) := \int_0^r u^{As+B} \psi(u; s)du.
\]

Then the following hold.

(i) The integral \( L(s) \) becomes a holomorphic function on the half-plane \( \text{Re}(s) > -(B+1)/A \).

(ii) The integral \( L(s) \) can be analytically continued as a meromorphic function to the whole complex plane. Moreover, its poles are simple and they are contained in the set \( \{-(B+j)/A : j \in \mathbb{N}\} \).

Proof. (i) Since \( L(s) \) locally uniformly converges on the half-plane \( \text{Re}(s) > -(B+1)/A \), the assumption and the Lebesgue convergence theorem give that the integral becomes a holomorphic function there.

(ii) Let \( N \) be an arbitrary natural number. The Taylor formula implies

\[
(4.2) \quad \psi(u; s) = \sum_{\alpha=0}^N \frac{1}{\alpha!} \frac{\partial^\alpha \psi}{\partial u^\alpha}(0; s)u^\alpha + u^{N+1}R_N(u; s)
\]

with

\[
R_N(u; s) = \frac{1}{N!} \int_0^1 (1-t)^N \frac{\partial^{N+1} \psi}{\partial u^{N+1}}(tu; s)dt.
\]

Here \( R_N(u; s) \) satisfies the following.

- \( R_N(\cdot; s) \) is smooth on \([0, r]\) for all \( s \in \mathbb{C} \).
- \( R_N(u; \cdot) \) is an entire function on \( \mathbb{C} \) for all \( u \in [0, r] \).
. Substituting (4.2) into the integral (4.1), we have

\[ L(s) = \sum_{\alpha=0}^{N} \frac{r^{A}B^{\alpha+1}}{\alpha!(A + B + \alpha + 1)} \frac{\partial^{\alpha} \psi}{\partial u^{\alpha}}(0; s) + \int_{0}^{r} u^{A + B + N + 1} R_{N}(u; s) du \]

on \( \text{Re}(s) > -(B+1)/A \). From (i), the integral in (4.3) becomes a holomorphic function on the half-plane \( \text{Re}(s) > -(B + N + 2)/A \). Therefore, \( L(s) \) can be analytically continued as a meromorphic function to the half-plane \( \text{Re}(s) > -(B + j)/A : j = 1, \ldots, N + 1 \). Letting \( N \) tend to infinity, we have the assertion. \( \Box \)

Remark 4.2. Let \( \tilde{\psi}(u; s) \) be a complex-valued function defined on \( \mathbb{R} \times \mathbb{C} \) satisfying that \( \tilde{\psi}(\cdot; s) \) is a \( C_{0}^{\infty} \) function on \( \mathbb{R} \) for all \( s \in \mathbb{C} \) and \( \frac{\partial^{\alpha} \tilde{\psi}}{\partial u^{\alpha}}(u; \cdot) \) is an entire function on \( \mathbb{C} \) for all \( u \in \mathbb{R} \) and \( \alpha \in \mathbb{Z}_{+} \). Then the integral

\[ \tilde{L}(s) := \int_{0}^{\infty} u^{A + B} \tilde{\psi}(u; s) du \]

has the same meromorphy properties as those of \( L(s) \) in Lemma 4.1. Indeed, for \( r > 0 \), \( \psi(\cdot; s) := |0, r] \tilde{\psi}(\cdot; s) \) satisfies the assumptions (a), (b) of Lemma 4.1 and the integral \( \int_{r}^{\infty} u^{A + B} \psi(u; s) du \) becomes an entire function.

4.2. Meromorphy of important integrals. The following lemma will play a useful role in the proof of Theorem 3.3.

Lemma 4.3. Let \( a, b \in \mathbb{N} \) with \( a \leq b \) and let

\[ D := \{(u, v) \in \mathbb{R}^{2} : u^{p} < v \leq R, \; v \leq r \}, \]

where \( p \in \mathbb{N} \) and \( r, R > 0 \) with \( r^{p} \leq R \). We assume that \( \psi(u, v; s) \) is a complex-valued function defined on \( D \times \mathbb{C} \) satisfying the following.

(a) \( \psi(\cdot; s) \) can be smoothly extended to \( \overline{D} \) for all \( s \in \mathbb{C} \);

(b) \( \frac{\partial^{\alpha+\beta} \psi}{\partial u^{\alpha} \partial v^{\beta}}(u, v; \cdot) \) is an entire function for all \( (u, v) \in D \) and \( (\alpha, \beta) \in \mathbb{Z}_{+}^{2} \).

Let

\[ H(s) := \int_{D} u^{as} v^{bs} \psi(u, v; s) dudv. \]

Then the following hold.

(i) The integral \( H(s) \) becomes a holomorphic function on the half-plane \( \text{Re}(s) > -1/b \).

(ii) The integral \( H(s) \) can be analytically continued as a meromorphic function to the whole complex plane and its poles are contained in the set

\[ \left\{ -\frac{j}{a}, -\frac{k}{b}, -\frac{p+l}{ap+b} : j, k, l \in \mathbb{N} \right\}. \]

Remark 4.4. (1) In Lemma 4.3, when the set \( D \) is replaced by the set

\[ \tilde{D} = \{(u, v) \in \mathbb{R}^{2} : 0 \leq u \leq v^{p}, 0 \leq v \leq r \}, \]

the same assertions (i), (ii) hold.
When \( j = k = l = 1 \) in (4.6), the following inequalities hold:

\[
-\frac{1}{a} \leq -\frac{p+1}{ap+b} \leq -\frac{1}{b}.
\]

**Proof of Lemma 4.3.** (i) In a similar fashion to the proof of the Lemma 4.1 (i), it can be easily shown that \( H(s) \) becomes a holomorphic function defined on the half-plane \( \text{Re}(s) > \max\{-1/a, -1/b\} = -1/b \).

(ii) Let us consider meromorphic continuation of \( H(s) \). For simplicity, we use the following symbol: For \((\alpha, \beta) \in \mathbb{Z}_+^2\),

\[
\psi^{(\alpha, \beta)}(u, v; s) := \frac{\partial^{\alpha+\beta} \psi}{\partial u^\alpha \partial v^\beta}(u, v; s).
\]

Let \( N \) be an arbitrary natural number. By the Taylor formula,

\[
\psi(u, v; s) = \sum_{(\alpha, \beta) \in \{0, \ldots, N\}^2} \frac{\psi^{(\alpha, \beta)}(0, 0; s)}{\alpha! \beta!} u^\alpha v^\beta + \sum_{\alpha=0}^N u^\alpha v^{N+1} \tilde{A}^{(N)}_\alpha(v; s)
\]

\[
+ \sum_{\beta=0}^N u^{N+1} v^\beta \tilde{B}^{(N)}_\beta(u; s) + u^{N+1} v^{N+1} \tilde{C}^{(N)}(u, v; s),
\]

where

\[
\tilde{A}^{(N)}_\alpha(v; s) := \frac{1}{\alpha! N!} \int_0^1 (1 - t)^N \psi^{(\alpha, N+1)}(0, tv; s) dt \quad \text{for } \alpha \in \{0, \ldots, N\},
\]

\[
\tilde{B}^{(N)}_\beta(u; s) := \frac{1}{\beta! N!} \int_0^1 (1 - t)^N \psi^{(N+1, \beta)}(tu, 0; s) dt \quad \text{for } \beta \in \{0, \ldots, N\},
\]

\[
\tilde{C}^{(N)}(u, v; s) := \frac{1}{(N!)^2} \int_0^1 \int_0^1 (1 - t_1)^N (1 - t_2)^N \psi^{(N+1, N+1)}(t_1 u, t_2 v; s) dt_1 dt_2.
\]

Note that

- \( \tilde{A}^{(N)}_\alpha(\cdot; s), \tilde{B}^{(N)}_\beta(\cdot; s), \tilde{C}^{(N)}(\cdot; s) \) are smooth functions for each \( s \in \mathbb{C} \).
- \( \tilde{A}^{(N)}_\alpha(v; \cdot), \tilde{B}^{(N)}_\beta(u; \cdot), \tilde{C}^{(N)}(u, v; \cdot) \) are entire functions for each \((u, v) \in D\).

Substituting (4.7) into (4.5), we have

\[
H(s) = \sum_{(\alpha, \beta) \in \{0, \ldots, N\}^2} \frac{\psi^{(\alpha, \beta)}(0, 0; s)}{\alpha! \beta!} H_{\alpha, \beta}(s)
\]

\[
+ \sum_{\alpha=0}^N A^{(N)}_\alpha(v; s) + \sum_{\beta=0}^N B^{(N)}_\beta(v; s) + C^{(N)}(s)
\]

(4.9)
with
\[ H_{\alpha,\beta}(s) = \int_D u^{as+\alpha}v^{bs+\beta}dudv \quad \text{for } (\alpha, \beta) \in \{0, \ldots, N\}^2, \]
\[ A^{(N)}_{\alpha}(s) = \int_D u^{as+\alpha}v^{bs+N+1}\tilde{A}^{(N)}_{\alpha}(v; s)dudv \quad \text{for } \alpha \in \{0, \ldots, N\}, \]
\[ B^{(N)}_{\beta}(s) = \int_D u^{as+N+1}v^{bs+\beta}\tilde{B}^{(N)}_{\beta}(u; s)dudv \quad \text{for } \beta \in \{0, \ldots, N\}, \]
\[ C^{(N)}(s) = \int_D u^{as+N+1}v^{bs+N+1}\tilde{C}^{(N)}(u, v; s)dudv. \]

Now let us consider the meromorphy of the above integrals.

The integral \( H_{\alpha,\beta}(s) \).

A simple computation gives that
\[
H_{\alpha,\beta}(s) = \int_0^r v^{bs+\beta} \left( \int_{vp}^R u^{as+\alpha}du \right)dv = \frac{1}{as+\alpha+1} \left( \frac{R^{as+\alpha+1}v^{bs+\beta+1}}{bs+\beta+1} - \frac{v^{(ap+b)s+\alpha p+\beta p+1}}{(ap+b)s+\alpha p + \beta + p + 1} \right),
\]
which implies that every \( H_{\alpha,\beta}(s) \) becomes a meromorphic function on the whole complex plane and its poles are contained in the set
\[
\left\{ \frac{-j}{a}, \frac{-k}{b}, \frac{-p+l}{ap+b} : j, k, l \in \mathbb{N} \right\}.
\]

The integral \( A^{(N)}_{\alpha}(s) \).

A simple computation gives that
\[
A^{(N)}_{\alpha}(s) = \int_0^r \left( \int_{vp}^R u^{as+\alpha}du \right) v^{bs+N+1}\tilde{A}^{(N)}_{\alpha}(v; s)dv = \frac{1}{as+\alpha+1} \left( \frac{R^{as+\alpha+1}v^{bs+N+1}\tilde{A}^{(N)}_{\alpha}(v; s)dv}{R^{as+\alpha+1}} \right.
\]
\[
\left. - \int_0^r v^{(ap+b)s+\alpha p+N+1}\tilde{A}^{(N)}_{\alpha}(v; s)dv \right).
\]

Lemma 4.1 (i) implies that the first (resp. the second) integral in (4.11) becomes a holomorphic function on the half-plane \( \text{Re}(s) > -(N+2)/b \) (resp. \( \text{Re}(s) > -(\alpha p + p + N + 2)/(ap + b) \)). Thus, \( A^{(N)}_{\alpha}(s) \) can be analytically continued as a meromorphic function to the half-plane \( \text{Re}(s) > \max\{-N+2/b, -(\alpha p + p + N + 2)/(ap + b)\} \) and its poles are contained in the set
\[
\left\{ \frac{-j}{a} : j \in \mathbb{N} \right\}.
\]
(When \( N \) is sufficiently large, the above maximum is \( -(\alpha p + p + N + 2)/(ap + b) \).)

The integral \( B^{(N)}_{\beta}(s) \).
A simple computation gives that

\[
B^{(N)}_\beta(s) = \left( \int_0^{r^p} \int_0^{u^{1/p}} + \int_{r^p}^R \int_0^r \right) u^{as+N+1} v^{bs+\beta} \tilde{B}^{(N)}_\beta(u;s) dv du
\]

\[
= \frac{1}{bs + \beta + 1} \left( \int_0^{r^p} u^{\frac{1}{p} \{ (ap+b)s + pN+p+\beta +1 \}} \tilde{B}^{(N)}_\beta(u;s) du \\
+ r^{bs+\beta+1} \int_{r^p}^R u^{as+N+1} \tilde{B}^{(N)}_\beta(u;s) du \right)
\]

(4.13)

Lemma 4.1 (i) implies that the first integral in (4.13) becomes a holomorphic function on the half-plane Re\(s\) > \(-(pN+2p+\beta+1)/(ap+b)\). Moreover, it is easy to check that the second integral is an entire function. Hence, \(B^{(N)}_\beta(s)\) can be analytically continued as a meromorphic function to the half-plane Re\(s\) > \(-(pN+2p+\beta+1)/(ap+b)\) and its poles are contained in the set

\[
\{ -\frac{k}{b} : k \in \mathbb{N} \}.
\]

(4.14)

The integral \(C^{(N)}(s)\).

It follows from the proof of (i) in this lemma that the integral \(C^{(N)}(s)\) converges on the half-plane Re\(s\) > \(\max\{-(N+2)/a, -(N+2)/b\} = -(N+2)/b\), which implies that \(C^{(N)}(s)\) can be analytically continued as a holomorphic function there.

From the above, letting \(N\) to infinity in (4.9), we can see that \(H(s)\) becomes a meromorphic function on the whole complex plane and that the poles of \(H\) is contained in the set \((4.6)\) from \((4.10), (4.12), (4.14)\).

\[\Box\]

4.3. A van der Corput-type lemma.

**Lemma 4.5 \([12]\).** Let \(f\) be a \(C^k\) function on an interval \(I\) in \(\mathbb{R}\). If \(|f^{(k)}| > \eta\) on \(I\), then for \(\sigma \in (-1/k, 0)\) there is a positive constant \(C(\sigma, k)\) depending only on \(\sigma\) and \(k\) such that

\[
\int_I |f(x)|^\sigma dx < C(\sigma, k) \eta^\sigma |I|^{1+k\sigma},
\]

where \(|I|\) is the length of \(I\).

The above van der Corput-type lemma plays the most important role in our analysis. This lemma has been shown in \([12]\), but we give a proof with more detailed explanation for convenience of readers.

**Proof.** Let \(F(\lambda)\) be the distribution function of \(|f|\), that is, \(F(\lambda)\) is the length of the set \(\{x \in I : |f(x)| \leq \lambda\}\). By using \(F(\lambda)\), the integral in (4.15) is represented as

\[
\int_I |f(x)|^\sigma dx = -\sigma \int_0^\infty \lambda^{\sigma-1} F(\lambda) d\lambda.
\]
Indeed, let $E = \{(x, \lambda) \in I \times (0, \infty) : |f(x)| \leq \lambda\}$ and $1_E$ be the characteristic function of $E$, then

\[
\int_I |f(x)|^\sigma dx = -\sigma \int_I \left( \int_{|f(x)|}^\infty \lambda^{\sigma-1} d\lambda \right) dx
\]

\[
= -\sigma \int_I \left( \int_0^\infty \lambda^{\sigma-1} 1_E(x, \lambda) d\lambda \right) dx
\]

\[
= -\sigma \int_0^\infty \lambda^{\sigma-1} \left( \int_I 1_E(x, \lambda) dx \right) d\lambda
\]

\[
= -\sigma \int_0^\infty \lambda^{\sigma-1} F(\lambda) d\lambda.
\]

We remark that the third equality in (4.16) is given by Tonelli’s theorem for nonnegative measurable functions. Here, we decompose the last integral in (4.16) as follows.

\[
\int_0^\infty \lambda^{\sigma-1} F(\lambda) d\lambda = \int_0^{\lambda_0} \lambda^{\sigma-1} F(\lambda) d\lambda + \int_{\lambda_0}^\infty \lambda^{\sigma-1} F(\lambda) d\lambda
\]

\[
=: J_1 + J_2,
\]

where $\lambda_0 = \eta |I|^k$. From Lemma 3.3 in [5], we have

(4.17) \[ F(\lambda) \leq C \frac{\lambda^{1/k}}{\eta^{1/k}}, \]

where $C$ is a positive constant depending only on $k$. Easy computation with the inequality (4.17) gives $J_1 \leq (Ck/(1 + k\sigma))\eta^\sigma |I|^{1+k\sigma}$. On the other hand, we have $J_2 \leq (-1/\sigma)\eta^\sigma |I|^{1+k\sigma}$ since $F(\lambda) \leq |I|$. The proof is completed.

\[ \Box \]

5. Properties of $f$ in (1.2)

Let $f$ be a smooth function expressed as in the case (C) in Lemma 3.1 on a small open neighborhood $U$ of the origin.

5.1. The zero-variety $V(f)$. In order to understand the analytic continuation of local zeta functions, it is essentially important to understand geometric properties of the zero-variety:

\[
V(f) = \{(x, y) \in U : f(x, y) = 0\}.
\]

It follows from an important factorization formula of Rychkov [26] that $f$ can be expressed as follows.

**Lemma 5.1.** There exists a small open neighborhood $U$ of the origin such that

\[
f(x, y) = \tilde{v}(x, y)y^b \prod_{j=1}^a (x - \phi_j(y)) \quad \text{on } U,
\]

where $\phi_j$ are complex-valued continuous functions defined near the origin satisfying that $\phi_j(y) = O(y^N)$ as $y \to 0$ for any $N \in \mathbb{N}$ and $\tilde{v}$ is a smooth function defined on $U$ with $\tilde{v}(0, 0) > 0$. 

Proof. This is an easy case of Proposition 2.1 in [26].

From the above lemma, the zero variety $V(f)$ is composed by at most $a + 1$ components:

$$Z_0 := \{(x, y) \in U : y = 0\},$$

$$Z_j := \{(x, y) \in U : x = \phi_j(y)\} \text{ for } j = 1, \ldots, a.$$  

When $\phi_j$ takes non-real values, $Z_j$ may be equal to $\{(0, 0)\}$ but they are realized in the complex space. It is possible that $Z_j = Z_k$ for some $j, k$.

Roughly speaking, for the meromorphic extension of $Z_f(\varphi)$, the variety $Z_0$ gives good influence, while the varieties $Z_j$ with $\phi_j \not\equiv 0$ for $j = 1, \ldots, a$ give bad one. Therefore, in the analysis of $Z_f(\varphi)$, the integral region is divided into two parts: one is avoided from the bad varieties as large as possible, while the other is its complement. Of course, their shapes must be useful for the computation. Actually, we use the two regions with a parameter $m \in \mathbb{N}$:

$$U_1^{(m)} = \{(x, y) \in U : x > y^m, \ 0 \leq y \leq r_m\},$$

$$U_2^{(m)} = \{(x, y) \in U : 0 < x \leq y^m, \ 0 \leq y \leq r_m\},$$

where $r_m > 0$ will be appropriately decided later in (5.5).

5.2. Two expressions of $f$. In order to understand important properties of $f$, we express $f$ by using the following two functions: $F : U \to \mathbb{R}$ and $\tilde{f} : U \setminus \{x = 0\} \to \mathbb{R}$ defined by

$$F(x, y) = v(x, y)x^a + \sum_{j=0}^{a-1} x^j \tilde{h}_j(y),$$

$$\tilde{f}(x, y) = v(x, y) + \sum_{j=1}^{a} \frac{\tilde{h}_{a-j}(y)}{x^j},$$

where $\tilde{h}_j(y) := h_j(y)/y^b$ for $j = 0, \ldots, a - 1$. Note that each $\tilde{h}_j$ is flat at the origin. Then $f$ can be expressed as

$$f(x, y) = y^b F(x, y) \quad \text{on } U,$$

$$f(x, y) = x^a y^b \tilde{f}(x, y) \quad \text{on } U \setminus \{x = 0\}.$$  

In order to investigate $f$, we use $\tilde{f}$ on $U_1^{(m)}$ and $F$ on $U_2^{(m)}$.

5.3. Properties of $\tilde{f}$. Let $\phi$ be a function defined near the origin as

$$\phi(y) = \max\{|\phi_j(y)| : j = 1, \ldots, a\},$$

where $\phi_j$ is as in Lemma 5.1. Since $\phi_j(y) = O(y^N)$ as $y \to 0$ for any $N \in \mathbb{N}$, for each $m \in \mathbb{N}$ there exists a positive number $r_m$ such that

$$\phi(y) \leq \frac{1}{2} y^m \quad \text{for } y \in [0, r_m].$$

We take the value of $r_m$ in (5.2) such that (5.5) holds.
Lemma 5.2. The function $\tilde{f}$ satisfies the following properties.

(i) There exists a positive number $c$ independent of $m$ such that $\tilde{f}(x,y) \geq c$ on $U^{(m)}_1$.

(ii) $\tilde{f}(x,y)$ can be smoothly extended to $\overline{U^{(m)}_1}$.

Proof. (i) From Lemma 5.1, $\tilde{f}$ takes the following form on $U^{(m)}_1$:

$$\tilde{f}(x,y) = \tilde{v}(x,y) \prod_{j=1}^{a} \left(1 - \frac{\phi_j(y)}{x}\right).$$

From (5.5), we easily see that

$$\left|\frac{\phi_j(y)}{x}\right| \leq \frac{\phi(y)}{x} \leq \frac{1}{2} y^m \leq \frac{1}{2}$$

for $(x,y) \in U^{(m)}_1$,

which implies that

$$\tilde{f}(x,y) \geq \frac{\tilde{v}(0,0)}{2^a}$$

for $(x,y) \in U^{(m)}_1$.

(ii) Since $\tilde{f}$ is a smooth function on $U \setminus \{x = 0\}$, it suffices to show that all partial derivatives of $\tilde{f}$ can be continuously extended to the origin. Moreover, from the second equation in (5.3), it suffices to show that all partial derivatives of $\psi_j(x,y) := \tilde{h}_{a-j}(y)/x^j$ can be continuously extended to the origin.

Let $(\alpha, \beta) \in \mathbb{Z}^2_+$ be arbitrarily given. An easy computation gives that

$$\frac{\partial^{\alpha + \beta} \psi_j}{\partial x^\alpha \partial y^\beta} = (-1)^a \frac{(j+\alpha-1)! \tilde{h}_{a-j}^{(\beta)}(y)}{(j-1)! x^{j+\alpha}}$$

where $\tilde{h}_{a-j}^{(\beta)}$ is the $\beta$-th derivative of $\tilde{h}_{a-j}$. Since $\tilde{h}_{a-j}^{(\beta)}$ is flat at the origin, there exists $r_{m,\alpha,\beta} > 0$ such that $|\tilde{h}_{a-j}^{(\beta)}(y)| \leq \frac{(j-1)!}{(j+\alpha-1)!} y^{m(j+\alpha+1)}$ for $y \in [0, r_{m,\alpha,\beta}]$ and $j = 1, \ldots, a$. Considering the shape of the set $U^{(m)}_1$, we have

$$\left|\frac{\partial^{\alpha + \beta} \psi_j}{\partial x^\alpha \partial y^\beta}\right| \leq x$$

for $U^{(m)}_1 \cap \{0 < y \leq r_{m,\alpha,\beta}\}$ and $j = 1, \ldots, a$.

Therefore,

$$\lim_{U^{(m)}_1 \ni (x,y) \to (0,0)} \frac{\partial^{\alpha + \beta} \psi_j}{\partial x^\alpha \partial y^\beta}(x,y) = 0$$

for $j = 1, \ldots, a$.

In particular, $\frac{\partial^{\alpha + \beta} \psi_j}{\partial x^\alpha \partial y^\beta}$ can be continuous up to the origin for $j = 1, \ldots, a$. □

5.4. Properties of $F$. When a van der Corput-type lemma in Lemma 4.5 is applied in the next section, the following lemma is important.

Lemma 5.3. There exist $R > 0$ and $\mu > 0$ such that

$$\frac{\partial^a}{\partial x^a} F(x,y) \geq \mu$$

on $[-R, R]^2$. 
Proof. A direct computation gives
\[ \frac{\partial^a}{\partial x^a} F(0,0) = a! v(0,0) > 0, \]
which implies the lemma. \qed

6. Proof of Theorem 3.3

Let \( f \) be expressed as in the case (C) in Lemma 3.1 on a small open neighborhood \( U \) of the origin. Let \( U_j^{(m)} (j = 1, 2) \) be as in \( \text{(5.2)} \) and let \( r_m \) be a positive constant determined by \( \text{(5.5)} \). Let \( \alpha > 0 \).

6.1. A decomposition of \( Z_f(\varphi)(s) \).

Using the orthant decomposition, we have

\[ Z_f(\varphi)(s) = \sum_{\theta \in \{1, -1\}^2} \tilde{Z}_{f_\theta}(\varphi_\theta)(s), \]

where

\[ \tilde{Z}_{f_{\theta}}(\varphi)(s) = \int_{\mathbb{R}_+^2} |f(x,y)|^s \varphi(x,y) \chi_m(y) \, dx \, dy, \]

\( f_{\theta}(x,y) = f(\theta_1 x, \theta_2 y) \) and \( \varphi_{\theta}(x,y) = \varphi(\theta_1 x, \theta_2 y) \) with \( \theta = (\theta_1, \theta_2) \). In order to prove the theorem, it suffices to consider the integral \( \tilde{Z}_f(\varphi)(s) \).

Now, let us decompose \( \tilde{Z}_f(\varphi)(s) \) as

\[ \tilde{Z}_f(\varphi)(s) = I_1^{(m)}(s) + I_2^{(m)}(s) + J^{(m)}(s) \]

with

\[ I_j^{(m)}(s) = \int_{U_j^{(m)}} |f(x,y)|^s \varphi(x,y) \chi_m(y) \, dx \, dy \quad \text{for } j = 1, 2, \]

\[ J^{(m)}(s) = \int_{\mathbb{R}_+^2} |f(x,y)|^s \varphi(x,y)(1 - \chi_m(y)) \, dx \, dy, \]

where \( \chi_m : \mathbb{R} \to [0, 1] \) is a cut-off function satisfying that \( \chi_m(y) = 1 \) if \( |y| \leq r_m/2 \) and \( \chi_m(y) = 0 \) if \( |y| \geq r_m \).

The following lemma will play a useful role in the analysis of the integral \( I_1^{(m)}(s) \).

**Lemma 6.1.** Let \( \Psi : U_1^{(m)} \times \mathbb{C} \to \mathbb{C} \) be defined by

\[ \Psi(x,y,s) = \tilde{f}(x,y)^s \varphi(x,y) \chi_m(y), \]

where \( \tilde{f} \) is as in \( \text{(5.3)} \). Then we have

(i) \( \Psi(\cdot, s) \) can be smoothly extended to \( \overline{U_1^{(m)}} \) for all \( s \in \mathbb{C} \).

(ii) \( \frac{\partial^{\alpha+\beta}}{\partial x^\alpha \partial y^\beta}(x,y,\cdot) \) is an entire function for all \( (x,y) \in U_1^{(m)} \) and \( (\alpha,\beta) \in \mathbb{Z}_+^2 \).
Proof. Since $\varphi(x, y)\chi_m(y)$ does not give any essential influence on the properties (i), (ii), it suffices to consider the function $\tilde{f}(x, y)^s$. On the domain $U_1^{(m)} \setminus \{(x, y) : \tilde{f}(x, y) = 0\}$, every partial derivative of $\tilde{f}(x, y)^s$ with respect to $x, y$ can be expressed as the sum of $s(s - 1) \cdots (s - k + 1)\tilde{f}(x, y)^{s-k}$ for $k \in \mathbb{N}$ multiplied by polynomials of the partial derivatives of $\tilde{f}(x, y)$ with respect to $x, y$. Applying Lemma 5.2 to this expression, we can see that $\tilde{f}(x, y)^s$ has the properties in (i), (ii). \hfill □

6.2. Meromorphic continuation of associated integrals. In order to prove Theorem 3.3, it suffices to show the following.

Lemma 6.2. Let $m \in \mathbb{N}$. If the support of $\varphi$ is contained in $[-R, R]^2$ where $R > 0$ is as in Lemma 5.3, then the following hold:

(i) $I_1^{(m)}(s)$ can be analytically continued as a meromorphic function to the whole complex plane. Moreover, its poles are contained in the set

$$\left\{-\frac{i}{a}, -\frac{k}{b}, -\frac{m+l}{am+b} : j, k, l \in \mathbb{N}\right\}.$$

(ii) $I_2^{(m)}(s)$ can be holomorphically continued as a holomorphic function to the half-plane $\text{Re}(s) > -(m + 1)/(am + b) \geq -1/a$.

(iii) $J^{(m)}(s)$ can be holomorphically continued as a holomorphic function to the half-plane $\text{Re}(s) > -1/a$.

Remark 6.3. The restriction of the set of candidate poles in (i) to the region $\text{Re}(s) > -(m + 1)/(am + b)$ is contained in $\{-k/b : k \in \mathbb{N} with k < b/a\}$.

Proof. (i) From (5.4), $I_1^{(m)}(s)$ can be expressed as

$$I_1^{(m)}(s) = \int_{U_1^{(m)}} \Psi(x, y; s) dxdy.$$

Since Lemma 6.1 implies that $\Psi$ satisfies the same properties as those of $\psi$ in Lemma 4.3, the integral $I_1^{(m)}(s)$ can be analytically continued as a meromorphic function to the whole complex plane and, moreover, its poles are contained in the set

$$\left\{-\frac{i}{a}, -\frac{k}{b}, -\frac{m+l}{am+b} : j, k, l \in \mathbb{N}\right\}.$$

(ii) From (5.4), $I_2^{(m)}(s)$ can be expressed as

$$I_2^{(m)}(s) = \int_{U_2^{(m)}} y^{bs} |F(x, y)|^s \varphi(x, y)\chi_m(y) dxdy.$$

It is easy to see that

$$|I_2^{(m)}(s)| \leq C_m \int_0^{r_m} y^{bs Re(s)} \left( \int_0^y |F(x, y)|^{Re(s)} dx \right) dy,$$

(6.5)
where $C_m = \sup_{(x,y) \in U_2^{(m)}}(|\varphi(x,y)\chi_m(y)|)$. Since Lemma 4.5 can be applied to the integral with respect to the variable $x$ in (6.5) from Lemma 5.3, if $\text{Re}(s) > -1/a$, then

$$|I_2^{(m)}(s)| < C_mC(\text{Re}(s), a)\mu^{\text{Re}(s)} \int_0^{r_m} y^{b\text{Re}(s)} (y^m)^{1+a\text{Re}(s)} dy$$

$$= C_mC(\text{Re}(s), a)\mu^{\text{Re}(s)} \int_0^{r_m} y^{(am+b)\text{Re}(s)+m} dy,$$

where $C(\cdot, \cdot)$ is as in Lemma 4.5 and $\mu$ is as in Lemma 5.3. The last integral in (6.6) converges on the half-plane $\text{Re}(s) > -(m+1)/(am+b)$, on which $I_2^{(m)}(s)$ becomes a holomorphic function. We remark that $-(m+1)/(am+b) \geq -1/a$ holds for all $m \in \mathbb{N}$.

(iii) In a similar fashion to the case of integral $I_2^{(m)}(s)$, we have

$$|J^{(m)}(s)| \leq \tilde{C}_m \int_{r_m/2}^R y^{b\text{Re}(s)} \left( \int_0^R |F(x,y)|^{\text{Re}(s)} dx \right) dy,$$

where $\tilde{C}_m := \sup_{(x,y) \in [0,R] \times [r_m/2,R]}(|\varphi(x,y)(1-\chi_m(x))|)$. Applying Lemma 4.5, we have that if $\text{Re}(s) > -1/a$, then

$$|J^{(m)}(s)| \leq \tilde{C}_mC(\text{Re}(s), a)\mu^{\text{Re}(s)} R^{1+a\text{Re}(s)} \int_{r_m/2}^R y^{b\text{Re}(s)} dy.$$  

Since the above integral converges for any $s \in \mathbb{C}$, $J^{(m)}(s)$ can be analytically continued as a holomorphic function to the half-plane $\text{Re}(s) > -1/a$. \hfill \Box

6.3. **Proof of Theorem 3.3.** From (6.1), (6.2), (6.3), (6.4), Lemma 6.2 gives Theorem 3.3 by letting $m$ to infinity.

**APPENDIX A. Newton polyhedra**

After giving the definitions of Newton polyhedra, Newton distances and adapted coordinates and briefly explaining their properties, we observe our study from these points of view.

Let $f$ be a real-valued smooth function defined on an open neighborhood of the origin in $\mathbb{R}^2$.

**A.1. Newton polyhedra.** The Taylor series of $f$ at the origin is

$$(A.1) \quad f(x,y) \sim \sum_{(\alpha, \beta) \in \mathbb{Z}_+^2} c_{\alpha\beta} x^\alpha y^\beta \quad \text{with} \quad c_{\alpha\beta} = \frac{1}{\alpha!\beta!} \frac{\partial^{\alpha+\beta} f}{\partial x^\alpha \partial y^\beta}(0,0).$$

The Newton polyhedron of $f$ is the integral polyhedron:

$$\Gamma_+(f) = \text{the convex hull of the set } \bigcup \{(\alpha, \beta) + \mathbb{R}_+^2 : c_{\alpha\beta} \neq 0\} \text{ in } \mathbb{R}_+^2$$

(i.e., the intersection of all convex sets which contain $\bigcup \{(\alpha, \beta) + \mathbb{R}_+^2 : c_{\alpha\beta} \neq 0\}$). Note that the flatness of $f$ at the origin is the equivalent to the condition: $\Gamma_+(f) = \emptyset$. We say that $f$ is convenient if the Newton polyhedron of $f$ intersects every coordinate axis.
A.2. **Newton distances.** We assume that $f$ is nonflat. The **Newton distance** $d(f)$ of $f$ is defined by
\[ d(f) = \inf\{ \alpha > 0 : (\alpha, \alpha) \in \Gamma_+(f) \}. \]
The minimal face of $\Gamma_+(f)$ containing the point $(d(f), d(f))$ is called principal face of $\Gamma_+(f)$. Since the Newton distance depends on the coordinates system $(x, y)$ on which $f$ is defined, it is sometimes denoted by $d_{(x,y)}(f)$.

A.3. **Adapted coordinates.** A given coordinate system $(x, y)$ is said to be adapted to $f$, if the equality
\[ d_{(x,y)}(f) = \sup_{(u,v)} \{ d_{(u,v)}(f) \} \]
holds, where the supremum is taken over all local smooth coordinate systems $(u, v)$ at the origin. The existence of adapted coordinates is shown in [27], [25], [16], etc. Furthermore, useful necessary and sufficient conditions for the adaptedness have been obtained. It is known in [16] that if the principal face of $\Gamma_+(f)$ is a noncompact face or a vertex of $\Gamma_+(f)$, then the respective coordinate is adapted to $f$. It follows from this fact that the function in (1.2) is defined in an adapted coordinate.

**Remark A.1.** The existence of adapted coordinates is not obvious. The definition of the adapted coordinate can be directly generalized in higher dimensional case. In three-dimensional case, it is known in [27] that there exists a function admitting no adapted coordinate.

A.4. **The $\gamma$-part and the class $\hat{E}(U)$.** Any line in $\mathbb{R}^2$ can be expressed by using some pair $(a, b; l) \in \mathbb{R}^2 \times \mathbb{R}$ as
\[ L(a, b; l) := \{ (\alpha, \beta) \in \mathbb{R}^2 : a\alpha + b\beta = l \}. \]
For any edge $\gamma \subset \Gamma_+(f)$, there exists a unique pair $(a, b; l) \in \mathbb{Z}_+^2 \times \mathbb{Z}_+$ with $\gcd(a, b) = 1$ such that
\[ \gamma = L(a, b; l) \cap \Gamma_+(f). \]
For a given face $\gamma$ of $\Gamma_+(f)$, we say that $f$ admits the $\gamma$-part on an open neighborhood $U$ of the origin if for any $(x, y) \in U$, the limit:
\[ \lim_{t \to 0} \frac{f(tx, ty)}{t^l} \]
exists for the pair $(a, b; l)$ defining $\gamma$ through (A.2). This process produces the function on $U$, which is called the $\gamma$-part of $f$ and denoted by $f_\gamma$. When a face $\gamma$ is compact, $f$ always admits the $\gamma$-part, which can be simply expressed on $U$ as
\[ f_\gamma(x, y) = \sum_{(\alpha, \beta) \in \gamma \cap \mathbb{Z}_+^2} c_{\alpha\beta} x^\alpha y^\beta, \]
where $c_{\alpha\beta}$ are the same as in (A.1).

The class $\hat{E}(U)$ consists of the smooth functions admitting the $\gamma$-part for all the edges $\gamma$ of $\Gamma_+(f)$. This class contains many kinds of smooth functions (see [19]).

- Every real analytic function belongs to $\hat{E}(U)$.
- Every convenient function belongs to $\hat{E}(U)$. 

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**Note:** The above text is a continuation of a previous discussion on local zeta functions, focusing on Newton distances and adapted coordinates, and includes further details on the $\gamma$-part and the class $\hat{E}(U)$.
• The Denjoy-Carleman (quasianalytic) classes are contained in $\hat{E}(U)$.

It is shown in [19] that if $f$ belongs to the class $\hat{E}(U)$, then the results of Varchenko [27] concerning the real analytic case can be directly generalized.

A.5. The case of $f$ in (1.2). Now let us consider the case when $f$ is as in (1.2) with $a, b \in \mathbb{N}$. As mentioned in the beginning of Section 3, the Newton data of $f$ can be easily obtained. Note that $f$ is not convenient. Moreover, the Newton polyhedron $\Gamma_+(f)$ has the two noncompact edges:

$$\gamma_1 = \{ (\alpha, \beta) : \alpha \geq a, \beta = b \}, \quad \gamma_2 = \{ (\alpha, \beta) : \alpha = a, \beta \geq b \}.$$

The classification in Lemma 3.1, which is obtained by the Taylor formula, can be expressed by using the admission of the $\gamma_1, \gamma_2$-parts.

(A) $f$ admits both the $\gamma_1$-part and the $\gamma_2$-part, i.e., $f \in \hat{E}(U)$.
(B) $f$ admits the $\gamma_2$-part but it does not admit the $\gamma_1$-part.
(C) $f$ admits the $\gamma_1$-part but it does not admit the $\gamma_2$-part.
(D) $f$ admits neither the $\gamma_1$-part nor the $\gamma_2$-part.

From [19], the case (A) can be easily treated in a similar fashion to the real analytic case and, in particular, $m_0(f) = \infty$ is shown. In the other cases, since $f$ does not belong to the class $\hat{E}(U)$, the result in [19] cannot be applied.

Acknowledgments. The authors greatly appreciate that the referee carefully read the first version of this paper and gave many valuable comments. This work was supported by JSPS KAKENHI Grant Numbers JP15K04932, JP19K14563, JP15H02057.

References

[1] V. I. Arnold, S. M. Gusein-Zade and A. N. Varchenko: Singularities of Differentiable Maps II, Birkhäuser, 1988.
[2] F. Aroca, M. Gómez-Morales, E. León-Cardenal: On Archimedean zeta functions and Newton polyhedra, J. Math. Anal. Appl. 473 (2019), 1215–1233.
[3] M. F. Atiyah: Resolution of singularities and division of distributions, Comm. Pure Appl. Math. 23 (1970), 145–150.
[4] I. N. Bernstein and S. I. Gel’fand: Meromorphy of the function $P_\lambda$, Funktsional. Anal. Prilozhen. 3 (1969), 84–85.
[5] M. Christ: Hilbert transforms along curves. I. Nilpotent groups, Ann. of Math. (2) 122 (1985), 575–596.
[6] T. C. Collins, A. Greenleaf and M. Pramanik: A multi-dimensional resolution of singularities with applications to analysis, Amer. J. Math. 135 (2013), 1179–1252.
[7] J. Denef, J. Nicaise and P. Sargos: Oscillating integrals and Newton polyhedra, J. Anal. Math. 95 (2005), 147–172.
[8] J. Denef and P. Sargos: Polyèdra de Newton et distribution $f_+^\ast$. I, J. Anal. Math. 53 (1989), 201–218.
[9] ______: Polyèdra de Newton et distribution $f_+^\ast$. II, Math. Ann. 293 (1992), 193–211.
[10] I. M. Gel’fand and G. E. Shilov: Generalized Functions I, Academic Press, New York, 1964.
[11] M. Gilula: Some oscillatory integral estimates via real analysis, Math. Z. 289 (2018), 377–403.
[12] M. Greenblatt: Newton polygons and local integrability of negative powers of smooth functions in the plane, Trans. Amer. Math. Soc. 358 (2006), 657–670.
[13] ______: The asymptotic behavior of degenerate oscillatory integrals in two dimensions, J. Funct. Anal. 257 (2009), 1759–1798.
[14] H. Hironaka: Resolution of singularities of an algebraic variety over a field of characteristic zero I, II, Ann. of Math. 79 (1964), 109–326.
[15] I. A. Ikromov, M. Kempe and D. Müller: Estimates for maximal functions associated with hypersurfaces in $\mathbb{R}^3$ and related problems of harmonic analysis, Acta Math. 204 (2010), 151–271.
[16] I. A. Ikromov and D. Müller: On adapted coordinate systems, Trans. Amer. Math. Soc. 363 (2011), 2821–2848.
[17] ______: Uniform estimates for the Fourier transform of surface carried measures in $\mathbb{R}^3$ and an application to Fourier restriction, J. Fourier Anal. Appl. 17 (2011), 1292–1332.
[18] A. Iosevich and E. Sawyer: Maximal averages over surfaces, Adv. Math. 132 (1997), 46–119.
[19] J. Kamimoto and T. Nose: Toric resolution of singularities in a certain class of $C^\infty$ functions and asymptotic analysis of oscillatory integrals, J. Math. Soc. Univ. Tokyo, 23 (2016), 425–485.
[20] ______: Newton polyhedra and weighted oscillatory integrals with smooth phases, Trans. Amer. Math. Soc., 368 (2016), 5301–5361.
[21] ______: Asymptotic limit of oscillatory integrals with certain smooth phases, RIMS Kökyūroku Bessatsu., B63 (2017), 103–114.
[22] ______: Non-polar singularities of local zeta functions in some smooth case, Trans. Amer. Math. Soc., 372 (2019), 661–676.
[23] ______: Non-polar singularities of local zeta functions in some smooth case II, In preparation.
[24] T. Okada and K. Takeuchi: Coefficients of the poles of local zeta functions and their applications to oscillating integrals, Tohoku Math. J. 65 (2013), 159–178.
[25] D. H. Phong, E. M. Stein and J. A. Sturm: On the growth and stability of real-analytic functions, Amer. J. Math. 121-3 (1999), 519–554.
[26] V. S. Rychkov: Sharp $L^2$ bounds for oscillatory integral operators with $C^\infty$ phases, Math. Z., 236 (2001), 461–489.
[27] A. N. Varchenko: Newton polyhedra and estimation of oscillating integrals, Functional Anal. Appl., 10-3 (1976), 175–196.

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