BLOWUP OF CLASSICAL SOLUTIONS FOR A CLASS OF 3-D QUASILINEAR WAVE EQUATIONS WITH SMALL INITIAL DATA

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Abstract. This paper is concerned with the small smooth data problem for the 3-D nonlinear wave equation
\[ \partial^2_t u - (1 + u + \partial_t u) \Delta u = 0, \]
where \( x_0 = t \) and \( g_{ij}(u, \nabla u) = c_{ij} + d_{ij}u + \sum_{k=0}^3 e_{ijk} \partial_k u + O(|u|^2 + |\nabla u|^2) \) are smooth functions of their arguments, with \( c_{ij}, d_{ij} \) and \( e_{ijk} \) being constants, and \( d_{ij} \neq 0 \) for some \((i, j)\); moreover, \( \sum_{i,j,k=0}^3 e_{ijk} \partial_k u \partial_{ij} u \) does not fulfill the null condition. For the 3-D nonlinear wave equations \( \partial^2_t u - (1 + u) \Delta u = 0 \) and \( \partial^2_t u - (1 + \partial_t u) \Delta u = 0 \), H. Lindblad, S. Alinhac, and F. John proved and disproved, respectively, the global existence of small smooth data solutions. For radial initial data, we show that the small smooth data solution of \( \partial^2_t u - (1 + u + \partial_t u) \Delta u = 0 \) blows up in finite time. The explicit expression of the asymptotic lifespan \( T_{\varepsilon} \) as \( \varepsilon \rightarrow 0^+ \) is also given.

§1. Introduction and main results

We consider the second-order nonlinear wave equation in \([0, \infty) \times \mathbb{R}^n\)
\[
\begin{cases}
\sum_{i,j=0}^n g_{ij}(u, \nabla u) \partial_{ij} u = 0, \\
(u(0, x), \partial_t u(0, x)) = (\varepsilon u_0(x), \varepsilon u_1(x)),
\end{cases}
\] (1.1)
where \( x_0 = t, x = (x_1, ..., x_n), \nabla = (\partial_0, \partial_1, ..., \partial_n), \varepsilon > 0 \) is a sufficiently small constant, \( u_0(x), u_1(x) \in C^\infty_0(\mathbb{R}^n) \), and the \( g_{ij}(u, \nabla u) \) are smooth functions of their arguments which are of the form
\[ g_{ij}(u, \nabla u) = c_{ij} + d_{ij}u + \sum_{k=0}^n e_{ijk} \partial_k u + O(|u|^2 + |\nabla u|^2) \] (1.2)
with \( c_{ij}, d_{ij} \) and \( e_{ijk} \) being constants. We assume that the linear part \( \sum_{i,j=0}^n c_{ij} \partial_{ij} u \) is strictly hyperbolic with respect to time \( t \). From [8-9, 14-16] we have that (1.1) has a global smooth solution when \( n \geq 4 \).

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If \( n = 3 \) and \( d_{ij} = 0 \) for all \( 0 \leq i, j \leq 3 \) in (1.2), then (1.1) has a global smooth solution if the null condition for the the main part \( \sum_{i,j,k=0}^{3} \epsilon_{ij}^{k} \partial_{k} u \partial_{ij} u \) holds, otherwise the solution of (1.1) blows up in finite time. See [4, 8-13, 17, 20-27] and the references therein.

If \( n = 3 \) and \( d_{ij} \neq 0 \) for some \( (i, j) \), but \( \epsilon_{ij}^{k} = 0 \) for all \( 0 \leq i, j, k \leq 3 \) in (1.2), then it follows from the results in [3] and [18-19] that (1.1) has a global smooth solution.

The following interesting problem naturally arises: If \( n = 3 \), \( d_{ij} \neq 0 \) for some \( (i, j) \), and \( \sum_{i,j,k=0}^{3} \epsilon_{ij}^{k} \partial_{k} u \partial_{ij} u \) in (1.2) does not fulfill the null condition, does the smooth solution of (1.1) blow up in finite time or does it exist globally? In this paper, we are concerned with this problem, especially (and without loss of generality) the prototypical equation \( \partial_{t}^{2} u - (1 + u + \partial_{t} u) \Delta u = 0 \) is studied. More specifically, we consider the problem

\[
\begin{align*}
\partial_{t}^{2} u - (1 + u + \partial_{t} u) \Delta u &= 0, \quad (t, x) \in [0, \infty) \times \mathbb{R}^{3}, \\
(u(0, x), \partial_{t} u(0, x)) &= (\varepsilon u_{0}(x), \varepsilon u_{1}(x)),
\end{align*}
\]

(1.3)

where \( u_{0}(x), u_{1}(x) \in C_{0}^{\infty}(\mathbb{R}^{3}) \) are radially symmetric and \( \sup u_{0} \cup \sup u_{1} \subseteq \{ x : |x| \leq M \} \) with \( M > 0 \) a constant. For notational convenience, we write \((u_{0}(r), u_{1}(r)) \) instead of \((u_{0}(x), u_{1}(x)) \) later on and the domains of definition of \( u_{0}(r) \) and \( u_{1}(r) \) are simultaneously extended to \([-M, M] \). This results from the fact that \( u_{0}(r) \) and \( u_{1}(r) \) are actually smooth functions of \( r^{2} \) due to \( u_{0}(x), u_{1}(x) \in C_{0}^{\infty}(\mathbb{R}^{3}) \).

Let \( F_{0}(s) = \frac{1}{2} \left( s u_{0}(s) + \int_{s}^{\infty} s u_{1}(s) ds \right) \) for \( s \in \mathbb{R} \). According to Theorem 6.2.2 and (6.2.12) of [9], we know that the function \( F_{0}(s) \neq 0 \) unless both \( u_{0}(s) \equiv 0 \) and \( u_{1}(s) \equiv 0 \). Moreover, \( F_{0}(s) \equiv 0 \) for \( |s| \geq M \).

Let \( \tau(s) = \frac{2}{F_{0}''(s)} \ln \frac{F_{0}''(s)}{F_{0}'(s)} \) for \( s \in [-M, M] \), \( F_{0}'(s) \neq 0 \), and \( \frac{F_{0}''(s)}{F_{0}'(s)} > 0 \). Further let \( A = \left\{ s \in (-M, M) : F_{0}(s) \neq 0, \frac{F_{0}''(s)}{F_{0}'(s)} > 0, \tau(s) > 0 \right\} \) and \( B = \{ s \in (-M, M) : F_{0}'(s) = 0, F_{0}''(s) > 0 \} \), and denote

\[
\tau_{0} = \min \left\{ \min_{s \in A} \tau(s), \min_{s \in B} \frac{2}{F_{0}''(s)} \right\}.
\]

(1.4)

It can be shown that \( \tau_{0} \) is a finite positive number if \((u_{0}(r), u_{1}(r)) \neq 0 \) holds.

The main result of this paper is:

**Theorem 1.1.** Assume that \( u_{0}(x), u_{1}(x) \in C_{0}^{\infty}(\mathbb{R}^{3}) \) only depend on \( r = \sqrt{x_{1}^{2} + x_{2}^{2} + x_{3}^{2}} \). If \( u_{0}(x) \neq 0 \) or \( u_{1}(x) \neq 0 \), then problem (1.3) has a \( C^{\infty} \) solution \( u(t, x) \) for \( 0 \leq t < T_{\varepsilon} \), where \( T_{\varepsilon} \) stands for the lifespan of the smooth solution \( u(t, x) \), which satisfies

\[
\lim_{\varepsilon \to 0} \varepsilon \ln T_{\varepsilon} = \tau_{0} > 0.
\]

(1.5)

**Remark 1.1.** It follows from Theorem 1.1 that the smooth solution of (1.3) blows up in finite time provided that \((u_{0}(r), u_{1}(r)) \neq 0 \).

**Remark 1.2.** We have asserted that \( \tau_{0} > 0 \) is a finite number as long as \((u_{0}(r), u_{1}(r)) \neq 0 \). To prove this, it suffices to show that \( \tau_{1} = \min_{s \in A} \tau(s) > 0 \) is finite, since \( \tau_{2} = \min_{s \in B} \frac{2}{F_{0}''(s)} > 0 \) obviously holds. We first show \( A \neq \emptyset \). Let \( A_{1} = \{ s \in (-M, M) : F_{0}'(s) > 0 \} \). Then obviously \( A \subseteq A_{1} \). Since \( A_{1} \) is a bounded, open, and nonempty set by \((u_{0}(r), u_{1}(r)) \neq 0 \) and the continuity of \( F_{0}' \), one can write \( A_{1} = \bigcup_{i} (a_{i}, b_{i}) \), where the union is finite or countable infinite and \((a_{l}, b_{l})\) and \((a_{d}, b_{d})\) are disjoint for \( l_{1} \neq l_{2} \). Moreover, \( F_{0}''(a_{l}) = F_{0}''(b_{l}) = 0 \) for all \( l \). Set \( B_{0} = \{ s \in [-M, M] : F_{0}'(s) = 0 \} \). Then \( B_{0} \) is closed and \((a_{l}, b_{l}) \setminus B_{0} \quad \text{for} \quad l \neq l_{0} \)
is an open and nonempty set. Write \((a_l, b_l) \setminus B_0 = \bigcup_{m=1}^n (a_{l,m}, b_{l,m})\), where \(F_0'(a_{l,m}) = F_0'(b_{l,m}) = 0\) and the intervals \((a_{l,m}, b_{l,m})\) are disjoint for different \((l,m)\). For any \((l_0, m_0)\), there exists at least exists one point \(s_0 \in (a_{l_0,m_0}, b_{l_0,m_0})\) such that \(F_0'(s_0) - F_0'(s_0) > 0\). If not, then \(F_0'(s) - F_0'(s) \leq 0\) for all \(s \in (a_{l_0,m_0}, b_{l_0,m_0})\), which means \((e^{-s} F_0'(s))' \leq 0\) and then \(F_0'(s) = 0\) because of \(F_0'(a_{l_1}) = F_0'(b_{l_1}) = 0\). This contradicts the hypothesis \(F_0'(s) > 0\) for \(s \in (a_{l_0,m_0}, b_{l_0,m_0}) \subseteq A_1\). On the other hand, \(\tau(s_0) > 0\) obviously holds, and thus the set \(A\) is nonempty. Next we show \(\tau_1 > 0\). Set \(x = -F_0'(s) / F_0'(s)\) for \(s \in A\). Then \(\tau(s) = \frac{1}{\max_{s \in A} F_0'(s)}\). If \(z \in (0, N)\) for some fixed \(N > 0\), then \(\tau(s) \geq \frac{C_N}{\max_{s \in A} F_0'(s)}\).

Remark 1.3. The result of this paper can be extended to the more general nonlinear wave equation

\[
\begin{aligned}
\sum_{i,j=0}^{3} g_{ij}(u, \nabla u) \partial_{ij} u &= 0, \\
(u(0, x), \partial_t u(0, x)) &= (\varepsilon u_0(x), \varepsilon u_1(x)),
\end{aligned}
\]

with \(c_{ij}, d_{ij}\) and \(e_{ij}^k\) being constants, and \(d_{ij} \neq 0\) for some \((i,j)\), and the null condition for \(\sum_{i,j,k=0}^{3} c_{ij}^k \partial_k u \partial^2_{ij} u\) does not hold. Because a proof of this statement just requires the methods and estimates used in this paper and the ones of [5] and [25] combined with blowup system techniques of [1-2], but it is technical and tedious otherwise, it is omitted here.

Remark 1.4. For the 2-D case of problem (1.1), if the coefficients \(g_{ij}(u, \nabla u)\) are independent of \(u\), then there is a rather complete collection of results on the global existence and the blowup, respectively, of small smooth data solutions, see [1-2, 7] and the references therein. On the contrary, if the coefficients \(g_{ij}(u, \nabla u)\) depend on both \(u\) and \(\nabla u\), there have been no systematic studies so far. Related results will appear in a forthcoming paper of ours.

In order to prove Theorem 1.1, we first derive the lower bound on the lifespan \(T_\varepsilon\) for problem (1.3) when the initial data are radial. By constructing an approximate solution as in [9] or [6], then considering the difference of the exact solution and the approximate solution, applying the Klainerman-Sobolev inequality, and establishing some further energy estimates, we obtain a lower bound on the lifespan \(T_\varepsilon\). Here we point out that although \(\varepsilon \ln T_\varepsilon \geq C > 0\) has already been shown in [17], for the reader’s convenience and to obtain the sharp lower bound \(T_\varepsilon\), we still give a complete proof. On the other hand, it follows from radial symmetry of the initial data \((u_0(x), u_1(x))\) that the solution \(u(t, x)\) is also radially symmetric for \(t < T_\varepsilon\). Based on this, we change (1.3) into a \(2 \times 2\) system of two independent variables \((t, r)\). Then, by using the properties of the approximate solution constructed above and some delicate analysis, and by treating the solution \(u\) accordingly, we obtain the upper bound on \(T_\varepsilon\). Here the derivation is motivated by the methods of [10], where the equation \(\partial_t^2 u - c^2(\partial_t u) \Delta u = 0\) was studied, the coefficient \(c^2(\partial_t u)\) of which only depends
on the gradient of the solution $u$. Compared with [10] and [6], due to the simultaneous appearance of $u$ and $\partial_t u$ in the coefficients of equation (1.3), we have to introduce a few more quantities in order to get a “blowup”-type nonlinear second-order ordinary differential equation with suitable initial data that provides the upper bound on $\varepsilon \ln T_\varepsilon$. Based on the results in the two steps above, we finally complete the proof of Theorem 1.1.

In this paper, we will use the following notation:

$Z$ denotes one of the Klainerman vector fields in the radially symmetric case, i.e.,

$$\partial_r, \partial_t, S = r\partial_t + r\partial_r, H = r\partial_r + t\partial_r,$$

$\partial$ stand for $\partial_r$ or $\partial_t$, and the norm $\| f \|_{L^2}$ means $\| f(t, \cdot) \|_{L^2(\mathbb{R}^3)}$.

§2. The lower bound on the lifespan $T_\varepsilon$

In this section, we establish the lower bound on $T_\varepsilon$ for smooth solutions of the Cauchy problem (1.3). As in the proof of [9, Theorem 6.5.3], by constructing an approximate solution $u_\varepsilon$ of (1.3) and then estimating the difference of $u_\varepsilon$ and the solution $u$, we obtain the lower bound on $T_\varepsilon$ by a continuity induction argument. The new ingredients in this procedure are the construction of the approximate solution and treating the solution $u$ itself that occurs in the equation in (1.3) rather than the derivatives of this solution only as in [9]. Although this procedure is analogous to the one in [6], for the reader’s convenience and also as it is applied to obtain the upper bound on $T_\varepsilon$, we still give a complete proof.

Let the slow time variable be $\tau = \varepsilon \ln(1 + t)$, and assume the solution of (1.3) is approximated by

$$\frac{\varepsilon}{r} V(q, \tau), \quad r > 0,$$

where $q = r - t$ and $V(q, \tau)$ solves the equation

$$\begin{cases}
2\partial_{qq} V + V \partial_{q}^2 V - \partial_{q} V \partial_{q}^2 V = 0, & (q, \tau) \in \mathbb{R} \times [0, \infty), \\
V(q, 0) = F_0(q), & \\
\text{supp} V \subseteq \{q \leq M\}.
\end{cases}$$

(2.1)

Lemma 2.1. (2.1) has a $C^\infty$ solution for $0 \leq \tau < \tau_0$, where $\tau_0$ is given by (1.4).

Proof. Set $w(q, \tau) = \partial_q V(q, \tau)$. Then it follows from (2.1) that

$$\begin{cases}
2\partial_{q\tau} V + (V - w)\partial_q w = 0, & (q, \tau) \in (-\infty, M] \times [0, \tau_0), \\
w(q, 0) = F'_0(q).
\end{cases}$$

(2.2)

The characteristics of (2.2) starting at the point $(0, s)$ is defined by

$$\begin{cases}
\frac{dq}{d\tau}(\tau, s) = \frac{1}{2}(V - w)(q(\tau, s), \tau), \\
q(0, s) = s.
\end{cases}$$

(2.3)

Along this characteristic curve, we have

$$\begin{cases}
\frac{dw}{d\tau}(q(\tau, s), \tau) = 0, \\
w(q(0, s), 0) = F'_0(s),
\end{cases}$$

which yields for $\tau < \tau_0$

$$w(q(\tau, s), \tau) = F'_0(s) = \partial_q V(q(\tau, s), \tau).$$

(2.4)
Lemma 2.2. From (2.3)-(2.4), we obtain
\[
\begin{aligned}
\partial_\tau q(\tau, s) &= \frac{1}{2} F_0'(s) \partial_\tau q(\tau, s) - \frac{1}{2} F_0''(s), \\
\partial_s q(0, s) &= 1.
\end{aligned}
\]
This yields \( \partial_\tau q(\tau, s) = \exp \left( \frac{1}{2} F_0'(s) \tau \right) \left( 1 - \frac{F_0''(s)}{F_0'(s)} \right) > 0 \) if \( F_0'(s) \neq 0 \) and \( \partial_\tau q(\tau, s) = 1 - \frac{1}{2} \tau F_0''(s) > 0 \) if \( F_0'(s) = 0 \) when \( 0 \leq \tau < \tau_0 \), and then
\[
q(\tau, s) = q(\tau, M) + \int_M^s \left( \exp \left( \frac{1}{2} F_0'(\rho) \tau \right) \left( 1 - \frac{F_0''(\rho)}{F_0'(\rho)} \right) + \frac{F_0''(\rho)}{F_0'(\rho)} \right) d\rho, \tag{2.5}
\]
and
\[
V(q(\tau, s), \tau) = 2q\partial_\tau q(\tau, s) + w = 2q\partial_\tau q(\tau, M) + \int_s^M \exp \left( \frac{1}{2} F_0'(\rho) \tau \right) (F_0'(\rho) - F_0''(\rho)) d\rho + F_0'(s), \tag{2.6}
\]
where we have used that \( \lim_{z \to 0} \left( e^{z\tau} \left( 1 - \frac{V}{e^z} \right) + \frac{V}{e^z} \right) = 1 - \tau y. \)

Note that \( q(\tau, M) = M \) such that \( V(q, \tau) \) satisfies the boundary condition \( V|_{q=M} = 0 \). Together with (2.5)-(2.6), this yields
\[
V_q(q(\tau, s), \tau) = \int_M^s \exp \left( \frac{1}{2} F_0'(\rho) \tau \right) (F_0'(\rho) - F_0''(\rho)) d\rho + F_0'(s) \quad \text{and} \quad q(\tau, s) = M + \int_M^s \exp \left( \frac{1}{2} F_0'(\rho) \tau \right) \left( 1 - \frac{F_0''(\rho)}{F_0'(\rho)} \right) + \frac{F_0''(\rho)}{F_0'(\rho)} \right) d\rho.
\]
On the other hand, by the implicit function theorem, we can obtain the smooth function \( s = s(q, \tau) \) for \( \tau < \tau_0 \). Therefore, \( V(q, \tau) = F_0(s(q, \tau)) \) is a smooth solution of (2.1) for \( 0 \leq \tau < \tau_0 \). □

We now start to construct an approximate solution of (1.3) for \( 0 \leq \tau = \varepsilon \ln(1+t) < \tau_0 \).
Let \( w_0 \) be the solution of the linear wave equation
\[
\begin{aligned}
\partial_t^2 w - \Delta w &= 0, \\
w(0, x) &= u_0(x), \\
\partial_t w(0, x) &= u_1(x).
\end{aligned}
\]
Choose a \( C^\infty \) function \( \chi(s) \) such that \( \chi(s) = 1 \) for \( s \leq 1 \) and \( \chi(s) = 0 \) for \( s \geq 2 \). We set, for \( 0 \leq \tau = \varepsilon \ln(1+t) < \tau_0 \),
\[
u_a(t, x) = \varepsilon \chi(\varepsilon t) w_0(t, x) + \frac{\varepsilon}{\tau} (1 - \chi(\varepsilon t)) \chi(-3\varepsilon q) V(q, \tau). \tag{2.7}
\]
By [9, Theorem 6.2.1] and Lemma 2.1, we have \( |Z^\alpha u_a| \leq C_{a,\varepsilon} (1 + t)^{-1} \) for \( \tau \leq \varepsilon < \tau_0 \) and all multi-indices \( \alpha \). Set
\[
J_a = \partial_t^2 u_a - (1 + u_a + \partial_t u_a) \Delta u_a.
\]
Lemma 2.2.
\[
\int_0^{e^{b/\varepsilon} - 1} \| Z^{\alpha} J_a \|_{L^2} \ dt \leq C_{a,\varepsilon} \varepsilon^{3/2} |\ln \varepsilon|.
\]
Proof. We divide the proof procedure into the following three cases.
Case 1. \( \frac{2}{\varepsilon} \leq t \leq e^{b/\varepsilon} - 1 \). In this case, \( \chi(\varepsilon t) = 0 \) and \( u_a(t, x) = \frac{\varepsilon}{\tau} \chi(-3\varepsilon q) V(q, \tau) \). Then
\[
J_a = -\frac{\varepsilon^2}{\tau^2} (\tilde{V} \partial_q^2 \tilde{V} - \partial_q \tilde{V} \partial_q^2 \tilde{V} + 2 \partial_q \tilde{V}) + O \left( \frac{\varepsilon^2}{(1+t)^2} \right).
\]
where \( V(q, \tau) = \chi(-3\varepsilon q)V(q, \tau) \). Since \( \partial_q V \) has compact support, we have

\[
(V \partial^2_q V - \partial_q V \partial^2_q V + 2\partial_qV)(q, \tau) = 9\varepsilon^2 \chi(-3\varepsilon q)\chi''(-3\varepsilon q)V^2(q, \tau) + 27\varepsilon^3 \chi'(-3\varepsilon q)\chi''(-3\varepsilon q)V^2(q, \tau) - 6\varepsilon \chi(-3\varepsilon q)\partial_t V(q, \tau).
\]

Hence \( |Z^o J_a| \leq C_{\alpha, \beta} \varepsilon^2 (1 + t)^{-3} + C_{\alpha, \beta} \varepsilon^3 (1 + t)^{-2} \psi(-3\varepsilon q) \), where \( \psi(s) \) is a cut-off function satisfying \( \psi(s) = 1 \) for \( s \in [1, 2] \), and \( \psi(s) = 0 \) otherwise.

Case 2. \( t \leq \frac{1}{\varepsilon} \). In this case, \( \chi(\varepsilon t) = 1 \) and \( u_a = \varepsilon w_0 \). This gives \( J_a = -\varepsilon^2 (w_0 + \partial_t w_0) \Delta w_0 \). It follows then from a direct computation that

\[
|Z^o J_a| \leq C_{\alpha} \varepsilon^2 (1 + t)^{-2}.
\]

Case 3. \( \frac{1}{\varepsilon} \leq t \leq \frac{2}{\varepsilon} \). A direct computation yields

\[
u_a = \varepsilon w_0 + \varepsilon (1 - \chi(\varepsilon t)) \left(r^{-1} \chi(-3\varepsilon q) V - w_0 \right).
\]

Then

\[
J_a = J_1 + J_2 + J_3 + J_4,
\]

where

\[
J_1 = -(u_a + \partial_t u_a) \Delta u_a,
J_2 = \varepsilon (\partial_t^2 - \Delta) [ (1 - \chi(\varepsilon t)) r^{-1} \chi(-3\varepsilon q) (V - F_0) ],
J_3 = \varepsilon (\partial_t^2 - \Delta) [ (1 - \chi(\varepsilon t)) \chi(-3\varepsilon q) (r^{-1} F_0 - w_0) ],
J_4 = \varepsilon (\partial_t^2 - \Delta) [ (1 - \chi(\varepsilon t)) \chi(-3\varepsilon q) - 1 ] w_0.
\]

It is easy to see \( |Z^o J_1| \leq C_{\alpha, \beta} \varepsilon^2 (1 + t)^{-2} \).

Due to \( (\partial_t^2 - \partial_r^2) (V(q, \tau) - F_0(q)) = (\partial_t - \partial_r) (\partial_r V - \varepsilon) 2 \partial_r^2 V \frac{\varepsilon}{1 + t} + \partial_r^2 V (\frac{\varepsilon}{1 + t})^2 - \partial_r V \frac{\varepsilon}{(1 + t)^2} \),

and \( V(q, 0) = F_0(q) \), we have \( |Z^o J_2| \leq C_{\alpha, \beta} \varepsilon \ln \varepsilon (1 + t)^{-2} \).

Moreover, by [9, Theorem 6.2.1], we have that, for any constant \( t > 0 \), if \( r \geq \varepsilon t \), then

\[
|Z^o (w_0 - r^{-1} F_0)| \leq C (1 + t)^{-2}.
\] (2.8)

On the other hand, from \( \partial_t = \frac{t S - r H}{t^2 - r^2} \) and \( \partial_r = \frac{t H - r S}{t^2 - r^2} \) we obtain \( \Delta = \frac{1}{r + \varepsilon} (S + H) (\partial_t - \partial_r) - \frac{2}{r} \partial_r \).

Therefore, \( |Z^o J_3| \leq C_{\alpha, \beta} \varepsilon (1 + t)^{-3} \leq C_{\alpha, \beta} \varepsilon^2 (1 + t)^{-2} \).

Since the support of \( J_4 \) is in \( q \leq -\frac{1}{3\varepsilon} \), combining the fact that, for any \( \phi(t, r) \in C^1 \),

\[
|\partial \phi| \leq \frac{C}{1 + |t - r|} \sum_{|\beta| = 1} |Z^\beta \phi|,
\] (2.9)

we get the estimate \( |Z^o J_4| \leq C_{\alpha, \beta} \varepsilon^3 (1 + t)^{-1} \leq C_{\alpha, \beta} \varepsilon^2 (1 + t)^{-2} \).

The above analysis yields

\[
|Z^o J_a| \leq C_{\alpha} \varepsilon^2 (1 + t)^{-2} |\ln \varepsilon|.
\]
Collecting the estimates above, we arrive at
\[
\| Z^\alpha J_a \|_{L^2} \leq C_{\alpha,b} \varepsilon^{3/2} (1 + t)^{-1}, \quad \frac{2}{\varepsilon} \leq t \leq e^{b/\varepsilon} - 1,
\]
\[
\| Z^\alpha J_a \|_{L^2} \leq C_{\alpha} \varepsilon^2 |\ln \varepsilon| (1 + t)^{-1/2}, \quad t \leq \frac{2}{\varepsilon}.
\]
Consequently,
\[
\int_{0}^{e^{b/\varepsilon} - 1} \| Z^\alpha u_a \|_{L^2} dt \leq C_{\alpha,b} \varepsilon^{3/2} |\ln \varepsilon|,
\]
and Lemma 2.2 is proved. □

For later reference, we cite a result that was shown in [17].

**Lemma 2.3.** Let \( f(t,x) \in C^1(\mathbb{R}^4) \) only depend on \((t,r)\). Moreover, assume that \( \text{supp} f \subseteq \{ (t,x) : r \leq M + t \} \). Then
\[
\| (1 + |t-r|)^{-1} f \|_{L^2} \leq C \| \partial_r f \|_{L^2}.
\]

Based on these preparations, we next establish:

**Proposition 2.4.** For sufficiently small \( \varepsilon > 0 \) and \( 0 \leq \tau = \varepsilon \ln(1 + t) \leq b < \tau_0 \), (1.3) has a \( C^\infty \) solution \( u(t,x) \) which satisfies, for all \( |\alpha| \leq 2 \),
\[
|Z^\alpha \partial|u - u_a|\| \leq C \varepsilon^{3/2} |\ln \varepsilon|(1 + t)^{-1} (1 + |t-r|)^{-1/2}.
\]

**Proof.** Set \( v = u - u_a \). Then one has
\[
\begin{align*}
\partial_t^2 v - (1 + u + \partial_t u) \Delta v &= -J_a + (v + \partial_t v) \Delta u_a, \\
v(0,x) &= \partial_t v(0,x) = 0.
\end{align*}
\]
We make the induction hypothesis that, for some \( T \leq e^{b/\varepsilon} - 1 \),
\[
|Z^\alpha \partial v| \leq \varepsilon(1 + t)^{-1} (1 + |t-r|)^{-1/2}, \quad |\alpha| \leq 2, \quad t \leq T,
\]
which then further implies that, for \( |\alpha| \leq 2 \) and \( t \leq T \),
\[
|Z^\alpha v| \leq C \varepsilon(1 + t)^{-1} (1 + |t-r|)^{1/2}.
\]
To verify the validity of (2.12), we will prove that, for sufficiently small \( \varepsilon > 0 \),
\[
|Z^\alpha \partial v| \leq \frac{\varepsilon}{2} (1 + t)^{-1} (1 + |t-r|)^{-1/2}, \quad |\alpha| \leq 2, \quad t \leq T,
\]
and apply the continuity method to obtain \( \varepsilon \ln(1 + T) = b \).

Applying \( Z^\alpha \) to both sides of (2.11) and using \([Z^\alpha, \partial_t^2 - \Delta] = \sum_{|\beta| < |\alpha|} C_{\alpha \beta} Z^\beta (\partial_t^2 - \Delta)\) yields, for any \( |\alpha| \leq 4 \),
\[
(\partial_t^2 - \Delta) Z^\alpha v = Z^\alpha G - \sum_{|\beta| < |\alpha|} C_{\alpha \beta} Z^\beta G,
\]
where
\[
G = (u + \partial_t u) \Delta v - J_a + (v + \partial_t v) \Delta u_a.
\]
Since
\[ Z^\alpha G = (u + \partial_t u) Z^\alpha \Delta v + \sum_{\alpha_1 + \alpha_2 = \alpha \atop |\alpha| \geq 1} Z^{\alpha_1} (u + \partial_t u) Z^{\alpha_2} \Delta v + Z^\alpha \left[ -J_a + (v + \partial_t v) \Delta u_a \right], \]
we have from (2.15) that
\[ (\partial_t^2 - (1 + u + \partial_t u) \Delta) Z^\alpha v = F, \quad (2.16) \]
where
\[ F = \sum_{\alpha_1 + \alpha_2 = \alpha \atop |\alpha| \geq 1} Z^{\alpha_1} (u + \partial_t u) Z^{\alpha_2} \Delta v + Z^\alpha \left[ -J_a + (v + \partial_t v) \Delta u_a \right] - \sum_{|\beta| < |\alpha|} C_{\alpha \beta} Z^\beta G + (u + \partial_t u) [Z^\alpha, \Delta] v. \]

Next we derive an estimate of \( \| \partial_t Z^\alpha v \|_{L^2} \) from equation (2.16). Define the energy
\[ E(t) = \frac{1}{2} \int_{\mathbb{R}^3} (|\partial_t Z^\alpha v|^2 + (1 + u + \partial_t u) |\nabla Z^\alpha v|^2) \, dx. \]

Multiplying both sides of (2.16) by \( \partial_t Z^\alpha v \) \((|\alpha| \leq 4)\) and integrating by parts, and noting that \( |\partial^{\beta} u| = |\partial^3 u_a + \partial^3 v| \leq C_b \varepsilon (1 + t)^{-1} \) \((|\beta| = 1, 2)\), which follows from the construction of \( u_a \) and assumption (2.12), we arrive at
\[ E'(t) \leq \frac{C_b \varepsilon}{1 + t} E(t) + \sum_{|\alpha| \leq 4} \int_{\mathbb{R}^3} |F| |\partial_t Z^\alpha v| \, dx. \quad (2.17) \]

We now treat each term arising in the integration of \( \sum_{|\alpha| \leq 4} \int_{\mathbb{R}^3} |F| |\partial_t Z^\alpha v| \, dx \) separately.

(A) The term \( \sum_{\alpha_1 + \alpha_2 = \alpha \atop |\alpha| \geq 1} \int_{\mathbb{R}^3} |Z^{\alpha_1} (u + \partial_t u) Z^{\alpha_2} \Delta v| |\partial_t Z^\alpha v| \, dx \). First, we note that:

(i) By assumption (2.13), for \( |\beta| \leq 2 \), we have
\[ |(1 + |t - r|)^{-1} Z^\beta (u_a + v)| \leq \frac{C_b \varepsilon}{1 + t}. \quad (2.18) \]

(ii) By (2.18) and (2.9), for \( |\alpha_1| + |\alpha_2| = |\alpha| \leq 4 \) with \( |\alpha_1| \geq 1 \), we have
\[ \int_{\mathbb{R}^3} |Z^{\alpha_1} (u + \partial_t u) (Z^{\alpha_2} \Delta v) \partial_t Z^\alpha v| \, dx \]
\[ \leq C_b \int_{\mathbb{R}^3} |(Z^{\alpha_1} u_a + Z^{\alpha_1} \partial_t u_a) (Z^{\alpha_2} \Delta v) \partial_t Z^\alpha v| \, dx + \int_{\mathbb{R}^3} |(Z^{\alpha_1} v + Z^{\alpha_1} \partial_t v) (Z^{\alpha_2} \Delta v) \partial_t Z^\alpha v| \, dx \]
\[ \leq \frac{C_b \varepsilon}{1 + t} E(t) + C \sum_{|\gamma| \leq |\alpha_1| + 1} \int_{\mathbb{R}^3} |(1 + |t - r|)^{-1} Z^{\alpha_1} v| |Z^\gamma \partial v| |\partial_t Z^\alpha v| \, dx \]
\[ + C_b \sum_{|\gamma| \leq |\alpha_2| + 1} \int_{\mathbb{R}^3} |Z^{\alpha_1} \partial_t v| |Z^\gamma \partial v| |\partial_t Z^\alpha v| \, dx. \quad (2.20) \]

Note further that there is at most one number larger than 2 between \( |\alpha_1| \) and \( |\gamma| \). If \( |\alpha_1| > 2 \), then \( |\gamma| \leq 2 \) and, by Lemma 2.3 applied to \( (1 + |r - r'|)^{-1} Z^{\alpha_1} v \) and by assumption (2.12), we arrive at
\[ \int_{\mathbb{R}^3} |(1 + |r - r'|)^{-1} Z^{\alpha_1} v| |Z^\gamma \partial v| |\partial_t Z^\alpha v| \, dx + \int_{\mathbb{R}^3} Z^{\alpha_1} \partial_t v| |Z^\gamma \partial v| |\partial_t Z^\alpha v| \, dx \leq \frac{C_b \varepsilon}{1 + t} E(t). \quad (2.21) \]
If $|\gamma| > 2$, then $|\alpha_1| \leq 2$ and it follows from (2.13) that $|(1 + |t - r|)^{-1}Z^{\alpha_1}v| \leq C_\varepsilon (1 + t)^{-1} (1 + |t - r|)^{-1/2}$ which leads to

\[
\int_{\mathbb{R}^3} |(1 + |t - r|)^{-1}Z^{\alpha_1}v| |Z^\gamma \partial v| |\partial_t Z^\alpha v| \, dx + \int_{\mathbb{R}^3} |Z^{\alpha_1} \partial_t v| |Z^\gamma \partial v| |\partial_t Z^\alpha v| \, dx \leq \frac{C_b \varepsilon}{1 + t} E(t). \tag{2.22}
\]

Inserting (2.21)–(2.22) into (2.20) yields

\[
\sum_{\alpha_1 + \alpha_2 = \alpha} \int_{\mathbb{R}^3} |Z^{\alpha_2}(u + \partial_t u)Z^{\alpha_2} \Delta v| |\partial_t Z^\alpha v| \, dx \leq \frac{C_b \varepsilon}{1 + t} E(t). \tag{2.23}
\]

(B) The terms $\int_{\mathbb{R}^3} |Z^\beta((u + \partial_t u) \Delta v) \cdot \partial_t Z^\alpha v| \, dx$ with $|\beta| < |\alpha|$. We only need to treat the term $\int_{\mathbb{R}^3} |(u + \partial_t u)Z^\beta \Delta v \cdot \partial_t Z^\alpha v| \, dx$, since the other terms have been estimated in (A).

By (2.12), we have

\[
\int_{\mathbb{R}^3} |(u + \partial_t u)Z^\beta \Delta v \cdot \partial_t Z^\alpha v| \, dx
\leq C \sum_{|\gamma| \leq |\beta| + 1 \leq |\alpha|} \int_{\mathbb{R}^3} |(1 + |t - r|)^{-1}u| |Z^\gamma \partial v| |\partial_t Z^\alpha v| \, dx + \int_{\mathbb{R}^3} |\partial_t u| |Z^\beta \Delta v| |\partial_t Z^\alpha v| \, dx
\leq \frac{C_b \varepsilon}{1 + t} E(t). \tag{2.24}
\]

(C) The terms $\int_{\mathbb{R}^3} |\partial_t Z^\alpha v| |Z^\beta J_a| \, dx$ with $|\beta| \leq |\alpha| \leq 4$. In this case, we have

\[
\int_{\mathbb{R}^3} |\partial_t Z^\alpha v| |Z^\beta J_a| \, dx \leq \|Z^\beta J_a\|_{L^2} \sqrt{E(t)}. \tag{2.25}
\]

(D) The terms $\int_{\mathbb{R}^3} |Z^\beta((v + \partial_t v) \Delta u_a) | |\partial_t Z^\alpha v| \, dx$ with $|\beta| \leq |\alpha| \leq 4$. A direct computation yields

\[
\int_{\mathbb{R}^3} |Z^\beta((v + \partial_t v) \Delta u_a) | |\partial_t Z^\alpha v| \, dx
\leq C \sum_{|\beta_1| + |\beta_2| \leq |\beta| + 1 \atop |\beta_1| \leq |\beta|} \int_{\mathbb{R}^3} \left( |(1 + |t - r|)^{-1} (Z^{\beta_1} v)(Z^{\beta_2} \partial u_a)| + |(Z^{\beta_1} \partial_t v)(Z^{\beta_2} \partial u_a)| \right) |\partial_t Z^\alpha v| \, dx
\leq \frac{C_b \varepsilon}{1 + t} E(t). \tag{2.26}
\]

(E) The term $\int_{\mathbb{R}^3} |(u + \partial_t u) [Z^\alpha, \Delta] v| |\partial_t Z^\alpha v| \, dx$. Since $[Z^\alpha, \Delta] = \sum_{|\beta| \leq |\alpha| - 1} C_{\alpha \beta} \partial^2 Z^\beta$, we have

\[
\int_{\mathbb{R}^3} |(u + \partial_t u) [Z^\alpha, \Delta] v| |\partial_t Z^\alpha v| \, dx \leq C \sum_{|\beta| \leq |\alpha| - 1} \int_{\mathbb{R}^3} |u + \partial_t u| |\partial^2 Z^\beta v| |\partial_t Z^\alpha v| \, dx
\leq C \sum_{|\beta| \leq |\alpha| - 1 \atop |\gamma| \leq |\beta| + 1} \int_{\mathbb{R}^3} |(1 + |t - r|)^{-1} (u + \partial_t u)| |\partial Z^\gamma v| |\partial_t Z^\alpha v| \, dx
\leq \frac{C_b \varepsilon}{1 + t} E(t). \tag{2.27}
\]
Substituting (2.23)–(2.27) into (2.17) yields

$$E'(t) \leq \frac{C_0\varepsilon}{1 + t} E(t) + \sum_{|\beta| \leq 4} \|Z^\beta J_\alpha\|_{L^2} \sqrt{E(t)}.$$  

(2.28)

Thus, by Lemma 2.2 and Gronwall’s inequality we obtain

$$\|\partial Z^\alpha v\|_{L^2} \leq C_0\varepsilon^{3/2} |\ln \varepsilon|, \quad |\alpha| \leq 4,$$

and then

$$\|Z^\alpha \partial v\|_{L^2} \leq C_0\varepsilon^{3/2} |\ln \varepsilon|, \quad |\alpha| \leq 4. \quad (2.29)$$

By (2.29) and the Klainerman-Sobolev inequality (see [9] or [14]), we have

$$|Z^\alpha \partial v| \leq C_0\varepsilon^{3/2} (1 + t)^{-1} (1 + |t - r|)^{-1/2}, \quad |\alpha| \leq 2, \ t \leq T. \quad (2.30)$$

This implies that, for $\varepsilon > 0$ small enough,

$$|Z^\alpha \partial v| \leq \frac{\varepsilon}{2} (1 + t)^{-1} (1 + |t - r|)^{-1/2}, \quad |\alpha| \leq 2, \ t \leq T.$$

Therefore, we have completed the proof of (2.12) and, together with (2.30), the proof of (2.10).  

Proposition 2.4 immediately gives

$$\lim_{\varepsilon \to 0} \varepsilon \ln(1 + T_\varepsilon) \geq \tau_0$$

and, therefore,

$$\lim_{\varepsilon \to 0} \varepsilon \ln T_\varepsilon \geq \tau_0. \quad (2.31)$$

Remark 2.1. The analysis of this section can be directly applied to problem (1.6) with general initial data so that a lower bound on the lifespan $T_\varepsilon$ is obtained as in [5].

§3. THE UPPER BOUND ON THE LIFESPAN $T_\varepsilon$ AND PROOF OF THEOREM 1.1.

In this section, we focus on the upper bound on $T_\varepsilon$. Here some ideas are inspired by [10] and [6]. Since equation (1.3) contains the solution $u$ and its derivatives simultaneously rather than the derivatives of $u$ only, as in [10], and the function $u$ only, as in [6], respectively, we have to be more careful in computations and also need to estimate more quantities. Thanks to the estimate of $Z^\alpha (u - u_a)$ with $|\alpha| \leq 2$ in (2.13), we observe that $|Z^\alpha (u - u_a)| \leq C\varepsilon (1 + t)^{-1}$ holds near the light cone which plays an important role in the analysis below.

Let $U = ru$ and $c^2(u) = 1 + u + \partial_t u$. Due to radial symmetry of $u$, (1.3) can be rewritten as

$$\begin{cases}
\partial_t^2 U - c^2 \partial_r^2 U = 0, \\
U(0, r) = \varepsilon r u_0, \quad \partial_t U(0, r) = \varepsilon r u_1.
\end{cases} \quad (3.1)$$

Define two operators

$$L_1 = \partial_t + c \partial_r, \quad L_2 = \partial_t - c \partial_r.$$

We also set

$$w_1 = L_2 \partial_r U = (\partial_t - c \partial_r) \partial_r U, \quad w_2 = L_1 \partial_r U = (\partial_t + c \partial_r) \partial_r U,$$

which yields $\partial_r U = \frac{w_1 + w_2}{2}$ and $\partial_r^2 U = \frac{w_2 - w_1}{2c}$.

Note that

$$L_1 L_2 = \partial_t^2 - c^2 \partial_r^2 - (L_1 c) \partial_r, \quad L_2 L_1 = \partial_t^2 - c^2 \partial_r^2 + (L_2 c) \partial_r.$$
Then
\[ L_1 w_1 = L_1 L_2 \partial_r U = -\frac{1}{4rc} w_1^2 + \frac{w_1}{4rc} (w_2 + \partial_t u + \frac{r}{c} L_2 u) - \frac{w_2}{4rc} (\partial_t u + \frac{r}{c} L_2 u), \]  
\[ L_2 w_2 = L_2 L_1 \partial_r U = \frac{1}{4rc} w_2^2 + \frac{w_2}{4rc} (-w_1 - \partial_t u + \frac{r}{c} L_1 u) + \frac{w_1}{4rc} (\partial_t u - \frac{r}{c} L_1 u). \]  

(3.2)

(3.3)

Due to \[ \partial_r c = \frac{\partial r}{\partial r} = \frac{1}{2c} \partial_r u \] and 
\[ \partial_t u = \frac{1}{2cr} \partial_t u + \frac{1}{4rc} (w_1 + w_2), \]
we have
\[ L_1 w_1 + w_1 \partial_r c = \frac{w_1}{4rc} (2w_2 - \partial_t u + \frac{r}{c} L_1 u) - \frac{w_2}{4rc} (\partial_t u + \frac{r}{c} L_2 u), \]
\[ L_2 w_2 - w_2 \partial_r c = \frac{w_2}{4rc} (-2w_1 + \partial_t u + \frac{r}{c} L_2 u) + \frac{w_1}{4rc} (\partial_t u - \frac{r}{c} L_1 u). \]

and
\[ d(|w_1| (dr - c dt)) = \text{sgn } w_1 (L_1 w_1 + w_1 \partial_r c) dt \wedge dr \]
\[ = \text{sgn } w_1 \left[ \frac{w_1}{4rc} (2w_2 - \partial_t u + \frac{r}{c} L_1 u) - \frac{w_2}{4rc} (\partial_t u + \frac{r}{c} L_2 u) \right] dt \wedge dr, \]
\[ d(|w_2| (dr + c dt)) = \text{sgn } w_2 (L_2 w_2 - w_2 \partial_r c) dt \wedge dr \]
\[ = \text{sgn } w_2 \left[ \frac{w_2}{4rc} (-2w_1 + \partial_t u + \frac{r}{c} L_2 u) + \frac{w_1}{4rc} (\partial_t u - \frac{r}{c} L_1 u) \right] dt \wedge dr. \]

(3.4)

(3.5)

![Figure 1](image_url)

Figure 1.

\text{From } \S 2 \text{ we have that when } \epsilon \ln(1 + T_b) = b < \tau_0 \text{ and } b > 0 \text{ is a fixed constant, (1.3) has a } C^\infty \text{ solution for } t \leq T_b. \text{ Choosing } \epsilon \text{ sufficiently small such that } 1/\epsilon < e^{b/\epsilon} - 1, \text{ we define the characteristics } \Gamma^\pm_\lambda \text{ by}
Lemma 3.1. There exists a constant $\tau_0 > 0$ such that, for any fixed $\delta > 0$ sufficiently small, there exists $\rho_0 \in (-M, M)$ such that $\|u(r, \rho_0)| \leq \rho_0$ or $\rho_0 = 0$. Another possibility is that there exists no point $\rho_0 \in (-M, M)$ such that $\tau(\rho_0) = \tau_0 + \delta$. However, in this case, by a minor modification which uses the fact that, for any fixed $\delta > 0$ sufficiently small, there exists $\rho_0 \in (-M, M)$ such that $\tau(\rho_0) = \tau_0 + \delta$, an analogous proof still works. Obviously, $\Gamma_M^+$ is the straight line $r = t + M$.

Fix a positive constant $R$ satisfying $R > \tau_0$, and assume that $u$ is a solution of (1.3) (at least) for $t \leq T \leq \exp(R/\varepsilon)$. We define

$$A(t) = \sup_{1/\varepsilon \leq s \leq t} \int_{(s,r) \in \mathcal{D}} |w_1(s,r)| dr,$$

$$B(t) = \sup_{1/\varepsilon \leq s \leq t} s |\partial_s u(s,r)|,$$

$$C(t) = \sup_{1/\varepsilon \leq s \leq t} s^{3/2} |w_2(s,r)|,$$

$$D(t) = \sup_{1/\varepsilon \leq s \leq t} s^{3/2} |L_t u(s,r)|.$$

Lemma 3.1. There exists a constant $E > 0$ such that, for $\varepsilon > 0$ sufficiently small,

$$A(1/\varepsilon) \leq \frac{E \varepsilon}{2}, \quad B(1/\varepsilon) \leq E \varepsilon, \quad C(1/\varepsilon) \leq E^2 \varepsilon^2, \quad D(1/\varepsilon) \leq E \varepsilon. \quad (3.6)$$

Proof. The equation $r = r(t)$ of $\Gamma_M^+$ for $\lambda \in [\rho_0 - 1, M]$ is

$$\left\{ \begin{array}{l}
\frac{dr(t)}{dt} = c(u(t,r(t))) \equiv c(t), \\
r(0) = \lambda.
\end{array} \right.$$ 

This gives

$$r(t) - \lambda = \int_0^t (c(s) - 1) ds + t.$$

Because of $|c(t)| \leq C_b(|u(t,r(t))| + |\partial_t u(t,r(t))|) \leq C_b \varepsilon (1 + |t - r(t)|)^{1/2}$ for $0 \leq \tau = \varepsilon \ln(1 + t) \leq b < \tau_0$,

$$|r(t) - t| \leq |\lambda| + C_b \int_0^t \varepsilon (1 + s)^{-1} (1 + |t - r(s)|)^{1/2} ds,$$

$$\leq m_0 + b + C_b \int_0^t \varepsilon (1 + s)^{-1} |t - r(s)| ds, \quad (3.7)$$

where $m_0 = \max \{|\rho_0 - 1|, M\}$. It follows from Gronwall’s inequality that

$$|r(t) - t| \leq C_b.$$ 

Therefore,

$$|t + M - r(t)| \leq C_b, \quad (3.8)$$
which means the distance between $\Gamma_{b_0-1}$ and $\Gamma_M^+$ is bounded.

The equation $\bar{r} = \hat{r}(t)$ of $\Gamma_M^-$ is

$$\begin{cases}
\frac{d\bar{r}(t)}{dt} = -c(u(t, \bar{r}(t))) \equiv -c(t), \\
\bar{r}(0) = \mu,
\end{cases}$$

and then $|\bar{r}(t) + t - \mu| \leq C_b$ can analogously be shown.

For $(t, r), (t', r') \in \Gamma_M^- \cap D$ for $\mu \in \mathbb{R}$ and $(t, r) \in \Gamma_{\lambda^+}^+, (t', r') \in \Gamma_{\lambda'}^+$, we obtain

$$|t - t'| \leq \frac{1}{2} |t + r - \mu| + |t' + r' - \mu| + |t - r - \lambda| + |t' - r' - \lambda'| + |\lambda - \lambda'| \leq C_b.$$  \hspace{1cm} (3.9)

We now start to prove (3.6). Since $w_1 = \partial_t u + r\partial_r u - c\partial_r u - cr\partial_r^2 u$, one has, for $t \leq e^{b/\varepsilon} - 1$,

$$|w_1(t, r)| \leq C_{b\varepsilon} (1 + |t - r|)^{-3/2}.$$  \hspace{1cm} (3.10)

By (3.8)–(3.10), we thus arrive at

$$\int_{(1/\varepsilon, r) \in D} |w_1(s, r)| dr \leq C_{b\varepsilon}.$$  \hspace{1cm} (3.11)

In addition, because of $|\partial_t u(t, r)| \leq C_{b\varepsilon} (1 + t)^{-1} (1 + |t - r|)^{-1/2}$ for $t \leq e^{b/\varepsilon} - 1$, we have, for $(1/\varepsilon, r) \in D$,

$$\frac{1}{\varepsilon} \left| \partial_k u \left( \frac{1}{\varepsilon}, r \right) \right| \leq C_{b\varepsilon}.$$  \hspace{1cm} (3.12)

Note that

$$w_2(t, r) = (\partial_t + c\partial_r)u + \frac{r}{r + t}(S + H)\partial_r u + (c - 1)r\partial_r^2 u$$

which implies $|w_2(t, r)| \leq C_{b\varepsilon}(1 + t)^{-1}$ for $(t, r) \in D$ and $\varepsilon \ln(1 + t) \leq b$ in view of (2.13). Together with (3.3), this yields

$$|L_2 w_2| \leq \frac{C_{b\varepsilon}^2}{(1 + t)^2}.$$  \hspace{1cm} (3.13)

Because of $w_2(t, t + M) = 0$ and (3.9), we obtain from (3.13) that

$$C(1/\varepsilon) \leq C_{b\varepsilon}^{5/2} \leq C_{b\varepsilon}^2.$$  \hspace{1cm} (3.14)

Finally, from $L_1 u = \frac{1}{r + t}(S + H)u + (c - 1)\partial_r u$ we have that $|L_1 u(t, r)| \leq C_{b\varepsilon}(1 + t)^{-2}$ for $t \leq e^{b/\varepsilon} - 1$ and, therefore,

$$D(1/\varepsilon) \leq C_{b\varepsilon}^{3/2} \leq C_{b\varepsilon}.$$  \hspace{1cm} (3.15)

Collecting (3.11)–(3.12) and (3.14) completes the proof of (3.6) \hspace{1cm} $\square$

Based on Lemma 3.1, we will use the continuity method to establish the upper bound on $T_\varepsilon$. To this end, we assume that, for $0 \leq t \leq T'$, $T'$,

$$A(t) \leq E\varepsilon, \quad B(t) \leq 2E\varepsilon, \quad C(t) \leq 3E^2\varepsilon^2, \quad D(t) \leq 2E\varepsilon.$$  \hspace{1cm} (3.15)
We now start to verify:

**Lemma 3.2.** Under the assumption (3.15) and if \( \varepsilon > 0 \) is sufficiently small, then we have, for \( 1/\varepsilon \leq t \leq T' \),

\[
A(t) \leq \frac{2}{3} E\varepsilon, \quad B(t) \leq E\varepsilon, \quad C(t) \leq \frac{5}{2} E^2\varepsilon^2, \quad D(t) \leq E\varepsilon. \quad (3.16)
\]

**Proof.** We first estimate \( u(t, r) \) for \( 1/\varepsilon \leq t \leq T' \) and \( (t, r) \in D \). Let \( (t, r) \in \Gamma^+_1 \). We integrate \( L_1 u \) along \( \Gamma^+_1 \) from time \( 1/\varepsilon \) to \( t \). From assumption (3.15), it is readily seen that \( |u(t, r)| \leq C\varepsilon^{3/2} \). Hence the definition \( c(t, r) = \sqrt{1 + u + \partial_r u} \) is possible and assumes a value close to 1. Therefore, \( t |\partial_r u(t, r)| \leq C\varepsilon \) which yields \( |u(t, r)| \leq C\varepsilon (1 + t)^{-1} (1 + |t - r|) \) for \( 1/\varepsilon \leq t \leq T' \) and \( (t, r) \in D \).

Similar to the proof (3.8) and (3.9) we also have \( |r - t| \leq C \) and \( |t - t'| \leq C \), where \( t \) and \( t' \) are same as in (3.9).

Now we estimate \( A(t) \). For \( 1/\varepsilon \leq t \leq T' \), by equation (3.4) and Green’s formula, we have

\[
\int_{(t, r) \in D} |w_1(t, r)| \, dr \leq \int_{(1/\varepsilon, r) \in D} |w_1(1/\varepsilon, r)| \, dr + \iint_{1/\varepsilon \leq s \leq t} \left| \frac{w_1}{4rc} \left( 2w_2 - \partial_r u + \frac{r}{c} L_1 u \right) - \frac{w_2}{4rc} \left( \partial_r u + \frac{r}{c} L_2 u \right) \right| (s, r) \, dsdr \leq \frac{E\varepsilon}{2} + \int_{1/\varepsilon \leq s \leq t} \left| \frac{w_2}{4rc} \left( \partial_r u + \frac{r}{c} L_2 u \right) \right| (s, r) \, dsdr. \quad (3.17)
\]

By the induction hypothesis (3.15), we have \( |\partial_r u(s, r)| \leq \frac{2E\varepsilon}{s} \), \( |w_2(s, r)| \leq \frac{3E^2\varepsilon^2}{s^{3/2}} \), and \( |L_1 u(s, r)| \leq \frac{2E\varepsilon}{s^{3/2}} \) for \( 1/\varepsilon \leq s \leq T' \) and \( (s, r) \in D \). This immediately gives \( |L_2 u(s, r)| \leq \frac{5E\varepsilon}{t} \). Note also that \( |r - s| \leq C \) holds for \( s \geq 1/\varepsilon \). We then have \( r \geq s/2 \) and

\[
\int_{1/\varepsilon \leq s \leq t} \frac{w_2}{4rc} \left( \partial_r u + \frac{r}{c} L_2 u \right) (s, r) \, dsdr \leq \int_{1/\varepsilon \leq s \leq t} \frac{E^2\varepsilon^2}{s^{3/2}} \left( \frac{2E\varepsilon}{s} + \frac{6E\varepsilon r}{s} \right) \, dsdr \leq 25E^3\varepsilon^3 \int_{1/\varepsilon}^{t} \frac{ds}{s^{5/2}} \int_{(s, r) \in D} ds \leq CE^3\varepsilon^{9/2} \leq \frac{E\varepsilon}{12}. \quad (3.18)
\]

where the generic constant \( C > 0 \) is independent of \( \varepsilon \).

Similarly,

\[
\int_{1/\varepsilon \leq s \leq t} \frac{|w_1|}{4rc} (2w_2 - \partial_r u + \frac{r}{c} L_1 u) (s, r) \, dsdr \leq \int_{1/\varepsilon \leq s \leq t} \frac{|w_1(s, r)|}{3r} \left( \frac{6E^2\varepsilon^2}{s^{3/2}} + \frac{2E\varepsilon}{s} + \frac{2E\varepsilon r}{s^{3/2}c} \right) \, dsdr \leq 4E\varepsilon \int_{1/\varepsilon}^{t} \frac{ds}{s^{5/2}} \int_{(s, r) \in D} |w_1(s, r)| r \, dr \leq CE^2\varepsilon^{5/2}. \quad (3.19)
\]

Substituting (3.18)–(3.19) into (3.17) yields

\[
A(t) \leq \frac{E\varepsilon}{2} + \frac{E\varepsilon}{12} + CE^2\varepsilon^{5/2},
\]
which implies that $A(t) \leq \frac{2}{3} E\varepsilon$ for $\varepsilon > 0$ sufficiently small.

Next we estimate $B(t)$. Note that $u$ satisfies the equation

$$L_2 \partial_t u = -\frac{c}{r} w_1 + c \frac{\partial}{\partial r} u.$$  \hspace{1cm} (3.21)

We integrate equation (3.21) along the characteristics $\Gamma_\mu$ which insects $\Gamma^*_M$ at the point $(t', r')$. For $t' \geq 1/\varepsilon$, we denote by $D_1$ the domain bounded by $\Gamma^-_\mu$, the line $\{t = t\}$, and $\Gamma^*_M$ (see the Figure 2) which are denoted by $\Gamma_1$, $\Gamma_2$ and $\Gamma_3$, respectively. We have

$$\int_{(s,r) \in D_1} \text{sgn}w_1 \left[ \frac{w_1}{4rc} \left( 2w_2 - \partial_t u + \frac{r}{c} L_1 u \right) - \frac{w_2}{4rc} \left( \partial_t u + \frac{r}{c} L_2 u \right) \right] (s,r) \, dsdr = \left( \int_{\Gamma_1} + \int_{\Gamma_2} \right) |w_1| (dr - cdt)$$

which implies

$$\int_{\Gamma_1} |w_1| (dr - cdt)$$

\begin{align*}
& \leq \int_{(s,r) \in D_1} \left[ \frac{w_1}{4rc} \left( 2w_2 - \partial_t u + \frac{r}{c} L_1 u \right) - \frac{w_2}{4rc} \left( \partial_t u + \frac{r}{c} L_2 u \right) \right] (s,r) \, dsdr + \int_{\Gamma_2} |w_1| (dr - cdt) \\
& \leq \int_{1/\varepsilon \leq s \leq t} \left[ \frac{w_1}{4rc} \left( 2w_2 - \partial_t u + \frac{r}{c} L_1 u \right) - \frac{w_2}{4rc} \left( \partial_t u + \frac{r}{c} L_2 u \right) \right] (s,r) \, dsdr + \int_{(t,r) \in D_1} |w_1(t,r)| \, dr \\
& \leq \frac{1}{6} E\varepsilon + \frac{2}{3} E\varepsilon = \frac{5}{6} E\varepsilon.
\end{align*}
This yields
\[
\int_{t'}^t \frac{|w_1(s, r(s))|}{r(s)} \, ds \leq \int_{t'}^t \frac{|w_1(s, r(s))|}{r(t)} \, ds \leq \frac{2}{t} \int_{t'}^t |w_1(s, r(s))| \, ds = \frac{2}{t} \int_{\Gamma_1} \frac{|w_1|}{\sqrt{1 + c^2}} (dr - \, ds) \leq \frac{5}{6t} E \varepsilon.
\]

Note that \( r(s) \geq \frac{s}{2} \geq \frac{1}{2\varepsilon} \) holds. From (3.21) we then have
\[
|\partial_t u(t, r)| \leq \int_{t'}^t \frac{|cw_1|(s, r(s))}{r(s)} \, ds + \int_{t'}^t \frac{|c\partial_t u|(s, r(s))}{r(s)} \, ds \leq \frac{11}{12t} E \varepsilon + 3\varepsilon \int_{t'}^t |\partial_t u(s, r(s))| \, ds. \tag{3.22}
\]
By Gronwall’s inequality, we obtain
\[
|u(t, r(t))| \leq \frac{11}{12t} E \varepsilon e^{C \varepsilon} \leq \frac{E \varepsilon}{t}.
\]
For \( t' \leq 1/\varepsilon \), we have \( t \leq t' + |t - t'| \leq 2/\varepsilon \), and Section 2 tells us \(|t\partial_t u(t, r)| \leq E \varepsilon\). Therefore, \( B(t) \leq E \varepsilon \) holds.

Finally, we estimate \( C(t) \). We write (3.3) as
\[
L_2 w_2 = aw_2 + b, \tag{3.23}
\]
where
\[
a = \frac{1}{4rc} \left( w_2 - w_1 - \partial_t u + \frac{r}{c} L_1 u \right), \quad b = \frac{1}{4rc} w_1 \left( \partial_t u - \frac{r}{c} L_1 u \right).
\]
Integrating (3.3) along \( \Gamma_\mu^- \) as above, we have
\[
|w_2(t, r)| \leq \int_{t'}^t |aw_2 + b| (s, r(s)) \, ds. \tag{3.24}
\]
Noting \( t' \geq t - |t - t'| \geq \frac{2t}{3} \), we obtain
\[
\int_{t'}^t |b(s, r(s))| \, ds \leq \int_{t'}^t \left| \frac{1}{4rc} w_1 \partial_t u \right| (s, r(s)) \, ds + \int_{t'}^t \left| \frac{1}{4c^2} w_1 L_1 u \right| (s, r(s)) \, ds \leq \frac{2E \varepsilon}{t^{1/2}} \int_{t'}^t \frac{|w_1(s, r(s))|}{r(s)} \, ds \leq \frac{5E^2 \varepsilon^2}{3t^{3/2}} \tag{3.25}
\]
and
\[
\int_{t'}^t |aw_2| (s, r(s)) \, ds \leq \frac{E^2 \varepsilon^2}{t^{3/2}} \int_{t'}^t \frac{|w_2 - w_1 - \partial_t u + \frac{r}{c} L_1 u| (s, r(s))}{r(s)} \, ds \leq \frac{CE^3 \varepsilon^4}{t^{3/2}}. \tag{3.26}
\]
Substituting (3.25)–(3.26) into (3.24) yields
\[
|w_2(t, r)| \leq \int_{t'}^t \frac{|aw_2 + b| (s, r(s))}{\, ds} \leq \frac{5E^2 \varepsilon^2}{2t^{3/2}},
\]
We integrate this equality along $\Gamma_{\mu}$ from $t'$ to $t$ and obtain
\[
|L_1u(t,r)| \leq \int_{t'}^{t} \frac{6E \varepsilon}{sr(s)} \, ds + \int_{t'}^{t} \frac{15E \varepsilon^2}{2s^2} \, ds + \int_{t'}^{t} \left( \frac{|w_1(s,r(s))|}{2r(s)} + \frac{E \varepsilon}{sr(s)} \right) 3E \varepsilon - \frac{s}{s} \, ds.
\]
and, therefore, $D(t) \leq E \varepsilon$. □

Based on Lemmas 3.1–3.2, we will use the following blowup result to establish the upper bound on the lifespan $T_\varepsilon$.

**Lemma 3.3.** Let $w$ be a solution in $(0,T)$ of the ordinary differential equation
\[
\frac{dw}{dt} = a_0(t)w^2 + a_1(t)w + a_2(t)
\]
with $a_j$ continuous on $[0,T]$ and $a_0 \geq 0$. Let
\[
K = \left( \int_0^T |a_2(t)| \, dt \right) \exp \left( \int_0^T |a_1(t)| \, dt \right).
\]
If $w(0) > K$, then
\[
\left( \int_0^T a_0(t) \, dt \right) \exp \left( - \int_0^T |a_1(t)| \, dt \right) < (w(0) - K)^{-1}.
\]

**Proof.** The proof can be found in [8, Lemma 1.4.1] or [9, Lemma 1.3.2]. □

Next we show that
\[
\lim_{\varepsilon \to 0} \varepsilon \ln T_\varepsilon \leq \tau_0. \tag{3.27}
\]

**Proof of (3.27).** Along the characteristics $\Gamma_{\mu}^+$, $w_1(t,r(t))$ satisfies
\[
\frac{dw_1}{dt}(t,r(t)) = L_1w_1 = -\frac{1}{4r}w_1^2 + \frac{w_1}{4rc} \left( w_2 + \partial_r u + \frac{r}{c}L_2 u \right) - \frac{w_2}{4rc} \left( \partial_r u + \frac{r}{c}L_2 u \right).
\]
We fix $0 < \mu < 1$ and set $t_\varepsilon = e^{\mu \tau_\varepsilon} - 1$ and $\hat{w}(t,r(t)) = -w_1(t,r(t)) \exp \left( - \int_{t_\varepsilon}^t \frac{L_2 u}{4c^2} (s,r(s)) \, ds \right)$. Then
\[
\frac{d\hat{w}}{dt}(t,r(t)) = a_0(t)\hat{w}^2 + a_1(t)\hat{w} + a_2(t),
\]
where
\[
a_0(t) = \frac{1}{4r(t)c(t)} \exp \left( \int_{t_\varepsilon}^t \frac{L_2 u}{4c^2} (s,r(s)) \, ds \right),
\]
\[
a_1(t) = \frac{w_2 + \partial_r u}{4rc}(t,r(t)),
\]
\[
a_2(t) = -\left( \frac{w_2}{4rc} \left( \partial_r u + \frac{r}{c}L_2 u \right) \exp \left( - \int_{t_\varepsilon}^t \frac{L_2 u}{4c^2} (s,r(s)) \, ds \right) \right)(t,r(t)).
\]
By (3.15), we have that, for $0 \leq t \leq T$,
\[ |a_1| \leq \frac{E \varepsilon}{t^2}, \quad |a_2| \leq \frac{CE^3 \varepsilon^3}{t^{5/2}} \]
which yields
\[ \int_{t_\varepsilon}^T |a_1| ds \leq E \varepsilon^2, \quad \int_{t_\varepsilon}^T |a_2| ds \leq CE^2 \varepsilon^{9/2}. \]

This further yields
\[ K = \left( \int_{t_\varepsilon}^T |a_2(t)| dt \right) \exp \left( \int_{t_\varepsilon}^T |a_1(t)| dt \right) = O(\varepsilon^{9/2}). \quad (3.28) \]

Due to $t_\varepsilon \geq 2/\varepsilon$ and $|3\varepsilon(t_\varepsilon - r(t_\varepsilon))| \leq 1$, by the definition of $u_a$ in (2.7), we have that $u_a(t_\varepsilon) = \frac{\varepsilon}{r(t_\varepsilon)} V(t_\varepsilon) = \frac{\varepsilon}{r(t_\varepsilon)} V(r(t_\varepsilon) - t_\varepsilon, \mu \tau_0)$ holds. In addition, it follows from Proposition 2.4 that
\[ |Z^\alpha(u - u_a)| \leq C_{\alpha, \delta} \varepsilon^{3/2} |\ln \varepsilon| (1 + t)^{-1} (1 + |t - r|)^{1/2}. \]

Therefore,
\[ w_1(t_\varepsilon) = (\partial_t u + r \partial_t r)(t_\varepsilon) - \left[ c(2\partial_t r + r \partial_t^2 u) \right](t_\varepsilon) = (\partial_t u_a + r \partial_t r u_a)(t_\varepsilon) - \left[ c(2\partial_t r u_a + r \partial_t^2 u_a) \right](t_\varepsilon) + o(\varepsilon) \]
\[ = -2 \varepsilon \partial_t^2 V(t_\varepsilon) + \frac{\varepsilon^2}{1 + t} \partial_t r V(t_\varepsilon) + o(\varepsilon) \]
\[ = -2 \varepsilon \partial_t^2 V(t_\varepsilon) + o(\varepsilon) \]

which yields $\hat{w}(t_\varepsilon) = 2 \varepsilon \partial_t^2 V(t_\varepsilon) + o(\varepsilon)$.

By Lemma 3.3, we obtain
\[ \left( \int_{t_\varepsilon}^{T_\varepsilon} a_0(t) dt \right) \exp \left( - \int_{t_\varepsilon}^{T_\varepsilon} |a_1(t)| dt \right) < (\hat{w}(t_\varepsilon) - K)^{-1}, \]
and then
\[ \left( \int_{t_\varepsilon + \varepsilon/2}^{T_\varepsilon} \frac{1}{4r(t)c(t)} \left( \int_{t_\varepsilon}^{t_\varepsilon + \varepsilon/2} L_2 u ds \right) dt \right) \exp \left( - \int_{t_\varepsilon}^{T_\varepsilon} |a_1(t)| dt \right) < (\hat{w}(t_\varepsilon) - K)^{-1}. \]

It follows
\[ (\ln(T_\varepsilon + 1) - \ln(e^{\mu \tau_0}/(1 + \varepsilon/2)) (1 + O(\varepsilon)) < 4(2 \varepsilon \partial_t^2 V(t_\varepsilon) + o(\varepsilon))^{-1} \exp(\varepsilon^2). \]

In order to obtain (3.27), we need to estimate $V(t_\varepsilon)$. Set $\tilde{q}(\tau) = r(t) - t$. Then
\[ \begin{aligned}
\frac{d \tilde{q}}{d \tau} &= \frac{t + 1}{r(t)(1 + c(t))}(V - \partial_q V)(\tilde{q}, \tau) + O(\varepsilon^{1/2} |\ln \varepsilon|), \\
\tilde{q}(0) &= \rho_0.
\end{aligned} \]

Recalling that, in Lemma 2.1, $q(r, \rho_0)$ is the characteristics of the approximate equation which satisfies $q(0, \rho_0) = \rho_0$, and then comparing $\tilde{q}$ with $q$ yields
\[ \begin{aligned}
\frac{d(q - \rho)}{d \tau} &= \left( \frac{2(t + 1)}{r(t)(1 + c(t))} - 1 \right) \left( V - \partial_q V \right)(\tilde{q}, \tau) + \frac{(V - \partial_q V)(\tilde{q}, \tau)}{2} - \frac{(V - \partial_q V)(q, \tau)}{2} + O(\varepsilon^{1/2} |\ln \varepsilon|) \\
&= \frac{(V - \partial_q V)(\tilde{q}, \tau)}{2} - \frac{(V - \partial_q V)(q, \tau)}{2} + O(\varepsilon^{1/2} |\ln \varepsilon|), \\
q(0) - q(0, \rho_0) &= 0.
\end{aligned} \]

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By Gronwall's inequality, we have \( \bar{q}(\tau) - q(\tau; \rho_0) = O(\varepsilon^{1/2} |\ln \varepsilon|) \) when \( \tau \leq R \). Thus, \( \partial_t^2 V(t_\varepsilon) = \bar{q}_\varepsilon^2 V(q(\mu_0; \rho_0), \rho_0) + O(\varepsilon^{1/2} |\ln \varepsilon|) \). From the expression for \( V \) in Lemma 2.1, we have that \( \partial_t^2 V(q(\tau, s), \tau) = \frac{F''_r(\rho)}{\partial_s q(\tau, s)} \), and consequently \( \partial_t^2 V(t_\varepsilon) = \frac{F''_r(\rho_0)}{\partial_s q(\tau_0, \rho_0)} + O\left(\varepsilon^{1/2} |\ln \varepsilon|\right) \). Moreover, one has \( F''_r(\rho_0) > 0 \) by Remark 1.2. Therefore,
\[
\lim_{\varepsilon \to 0} \varepsilon \ln(T_\varepsilon + 1) - \mu \tau_0 \leq \frac{2 \partial_s q(\mu_0, \rho_0)}{F''_r(\rho_0)},
\]
and as \( \mu \to 1 \),
\[
\lim_{\varepsilon \to 0} \varepsilon \ln T_\varepsilon \leq \tau_0 + \frac{2 \partial_s q(\tau_0, \rho_0)}{F''_r(\rho_0)} = \tau_0.
\]
This finishes the proof of (3.27). \( \square \)

**Proof of Theorem 1.1.** Follows now directly from (2.31) and (3.27). \( \square \)

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**REFERENCES**

1. S. Alinhac, *Blow up of small data solutions for a class of quasilinear wave equations in two space dimensions*, Ann. of Math. (2) 149 (1999), 97–127.
2. ———, *Blow up of small data solutions for a class of quasilinear wave equations in two space dimensions. II*, Acta Math. 182 (1999), 1–23.
3. ———, *An example of blowup at infinity for quasilinear wave equations*, Astérisque 284 (2003), 1–91.
4. D. Christodoulou, *Global solutions of nonlinear hyperbolic equations for small initial data*, Comm. Pure Appl. Math. 39 (1986), 267–282.
5. Ding Bingbing, I. Witt, and Yin Huicheng, *On the lifespan of and the blowup mechanism for smooth solutions to a class of 2-D nonlinear wave equations with small initial data* (2012), arXiv:1210.7980.
6. Ding Bingbing and Yin Huicheng, *The blowup mechanism of small data solution for the quasilinear wave equations in three space dimensions*, Acta Math. Sinica, English Series 18 (2001), 35–76.
26. Nagoya Math. J. 175 (2004), 125-164.
27. Zhou Yi and Xu Wei, Almost global existence for quasilinear wave equations with inhomogeneous terms in 3D, Forum Math. 23 (2011), 1113–1134.

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