ON THE EQUIVARIANT REDUCTION OF STRUCTURE GROUP OF A PRINCIPAL BUNDLE TO A LEVI SUBGROUP

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Abstract. Let $M$ be an irreducible projective variety over an algebraically closed field $k$ of characteristic zero equipped with an action of a group $\Gamma$. Let $E_G$ be a principal $G$–bundle over $M$, where $G$ is a connected reductive algebraic group over $k$, equipped with a lift of the action of $\Gamma$ on $M$. We give conditions for $E_G$ to admit a $\Gamma$–equivariant reduction of structure group to $H$, where $H \subset G$ is a Levi subgroup. We show that for $E_G$, there is a naturally associated conjugacy class of Levi subgroup $s$ of $G$. Given a Levi subgroup $H$ in this conjugacy class, $E_G$ admits a $\Gamma$–equivariant reduction of structure group to $H$, and furthermore, such a reduction is unique up to an automorphism of $E_G$ that commutes with the action of $\Gamma$.

1. Introduction

A holomorphic $G$–bundle over $\mathbb{CP}^1$ admits a holomorphic reduction of structure group to a maximal torus of $G$, where $G$ is a complex reductive group [Gr]. In particular, any holomorphic vector bundle over $\mathbb{CP}^1$ splits as a direct sum of line bundles. If $V$ is a holomorphic vector bundle over $\mathbb{CP}^1$ equipped with a lift, as vector bundle automorphisms, of the standard action of the diagonal matrices in $\text{SL}(2, \mathbb{C})$ on $\mathbb{CP}^1$, then $V$ decomposes as a direct sum of holomorphic line subbundles each left invariant by the action of the torus [Ku]. More generally, let $E_G$ be a principal $G$–bundle over an irreducible complex projective variety $M$, where $G$ is a complex reductive group, with both $E_G$ and $M$ equipped with algebraic actions of a connected complex algebraic group $\Gamma$; the action of $\Gamma$ on $E_G$ commutes with the action of $G$ and the projection of $E_G$ to $M$ is $\Gamma$–equivariant. Assume that $E_G$ admits a reduction of structure group to a maximal torus $T$ of $G$. Then $E_G$ admits a $\Gamma$–equivariant reduction of structure group to $T$ if and only if the action of $\Gamma$ on the automorphism group of $E_G$ leaves a maximal torus invariant [BP].

The aim here is to investigate conditions under which a principal $G$–bundle over a projective variety equipped with an action of a group $\Gamma$ admits a $\Gamma$–equivariant reduction of structure group to a Levi subgroup of $G$.

Let $M$ be an irreducible projective variety over an algebraically closed field $k$ of characteristic zero on which a group $\Gamma$ acts as algebraic automorphisms. Let $G$ be a connected reductive linear algebraic group over $k$ and $E_G$ a principal $G$–bundle over $M$. Let $\text{Aut}(E_G)$ denote the group of all automorphisms of $E_G$. Suppose we are given a lift of the action of $\Gamma$ on $M$ to $E_G$ that commutes with the action of $G$. More precisely, the automorphism of
An algebraic automorphism of \( G \)-bundle over the automorphism of \( M \) defined by \( \gamma \). The action of \( \Gamma \) on \( E_G \) induces an action on \( \text{Aut}(E_G) \) through group automorphisms.

A torus is a product of copies of the multiplicative group \( \mathbb{G}_m \). By a Levi subgroup of \( G \) we will mean the centralizer of some torus of \( G \). Let \( E_H \subset E_G \) be a reduction of structure group to a Levi subgroup \( H \). We will denote by \( Z_0(H) \) the connected component of the center of \( H \) containing the identity element. So \( Z_0(H) \) is contained in the automorphism group of \( E_H \) and hence contained in \( \text{Aut}(E_G) \).

We prove that \( E_H \) is left invariant by the action of \( \Gamma \) on \( E_G \) if and only if \( \Gamma \) acts trivially on the subgroup \( Z_0(H) \subset \text{Aut}(E_G) \) (Theorem 2.2).

A torus of \( \text{Aut}(E_G) \) determines a torus, unique up to inner automorphism, of \( G \). The \( G \)-bundle \( E_G \) admits a \( \Gamma \)-equivariant reduction of structure group to the Levi subgroup \( H \) if and only if there is torus \( T \subset \text{Aut}(E_G) \) satisfying the following two conditions:

1. the action of \( \Gamma \) on \( T \) is trivial;
2. there is a subtorus \( T' \subset Z_0(H) \subset G \) in the conjugacy class of tori of \( G \) defined by \( T \) such that the centralizer of \( T' \) in \( G \) coincides with \( H \).

(See Lemma 3.2 and Proposition 3.3.)

Let \( T_0 \subset G \) be a torus in the conjugacy class of tori defined by a maximal torus of \( \text{Aut}(E_G) \). The conjugacy class of \( T_0 \) does not depend on the choice of the maximal torus. Let \( H_0 \subset G \) be the centralizer of \( T_0 \).

In Theorem 4.1 we prove that \( E_G \) admits a \( \Gamma \)-equivariant reduction of structure group \( E_{H_0} \subset E_G \) to \( H_0 \), which is unique in the following sense:

1. for any \( \Gamma \)-equivariant reduction of structure group \( E'_{H_0} \subset E_G \) to \( H_0 \), there is an automorphism \( \tau \in \text{Aut}(E_G) \) such that \( \tau(E_{H_0}) = E'_{H_0} \) as subvarieties of \( E_G \); and
2. if \( E_H \) is a \( \Gamma \)-equivariant reduction of structure group to a Levi subgroup \( H \subset G \), then there is \( g \in G \) and \( \tau \in \text{Aut}(E_G) \) such that \( g^{-1}H_0g \subset H \) and \( E_{H_0}g \subset \tau(E_H) \).

A theorem due to Atiyah says that for an isomorphism of a vector bundle over \( M \) with any direct sum of indecomposable vector bundles, the direct summands are unique up to a permutation of the summands. Theorem 4.1 is an equivariant principal bundle analog of this result of [At].

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2. Equivariant reduction to a Levi subgroup

Let $M$ be an irreducible projective variety over an algebraically closed field $k$ of characteristic zero. Let $\Gamma$ be a group acting on the left on $M$. So we have a map

$$\phi : \Gamma \times M \rightarrow M$$

such that for any $\gamma \in \Gamma$, the map defined by $x \mapsto \phi(\gamma, x)$ is an algebraic automorphism of $M$, and furthermore, $\phi(\gamma_1 \phi(\gamma_2, x)) = \phi(\gamma_1 \gamma_2, x)$ for all $\gamma_1, \gamma_2 \in \Gamma$ and $x \in M$, with $\phi(e, x) = x$, where $e \in \Gamma$ is the identity element.

Let $G$ be a connected reductive linear algebraic group over the field $k$ and $E_G$ a principal $G$-bundle over $M$. Let

$$f : E_G \rightarrow M$$

be the projection. We will denote by $\text{Aut}(E_G)$ the group of all automorphisms of the $G$-bundle $E_G$ (over the identity automorphism of $M$). So, $\tau(z)g = \tau(zg)$ and $f(\tau(z)) = f(z)$ for all $\tau \in \text{Aut}(E_G)$ and $z \in E_G$. Note that $\text{Aut}(E_G)$ is an affine algebraic group over $k$. After fixing a faithful representation of $G$, the group $\text{Aut}(E_G)$ gets identified with a closed subgroup of the automorphism group of the associated vector bundle; the automorphism group of a vector bundle is a Zariski open dense subset in the vector space defined by the space of all global endomorphisms of the vector bundle. The group $\text{Aut}(E_G)$ is, in fact, the space of all global section of the adjoint bundle

$$\text{Ad}(E_G) := E_G \times^G G = (E_G \times G)/G$$

(the action of any $g \in G$ sends any point $(z, g') \in E_G \times G$ to $(zg, g^{-1}g'g)$). Let

$$\text{Aut}^0(E_G) \subset \text{Aut}(E_G)$$

be the connected component containing the identity element. So $\text{Aut}^0(E_G)$ is a connected affine algebraic group over $k$.

Assume that $E_G$ is equipped with a lift of the action of $\Gamma$ on $M$. So the map

$$\Phi : \Gamma \times E_G \rightarrow E_G$$

defining the action has the property that for any $\gamma \in \Gamma$ the map defined by $z \mapsto \Phi(\gamma, z)$ is an algebraic automorphism of $E_G$ that commutes with the action of $G$ on $E_G$, and $f \circ \Phi(\gamma, z) = \Phi(\gamma, f(z))$, where $f$ is as in (2.2). Note that the action of $\Gamma$ on $E_G$ induces an action of $\Gamma$ on $\text{Aut}(E_G)$ through algebraic group automorphisms. More precisely, the action of any $\gamma \in \Gamma$ sends $F \in \text{Aut}(E_G)$ to the automorphism defined by

$$z \mapsto \Phi(\gamma, F(\Phi(\gamma^{-1}, z)))$$

where $\Phi$ is as in (2.3).
The group $\Gamma$ acts on the adjoint bundle $\text{Ad}(E_G)$ as follows: the action of any $\gamma \in \Gamma$ sends $(z, g) \in E_G \times G$ to
\begin{equation}
(\Phi(\gamma, z), g) \in E_G \times G;
\end{equation}
this descends to an action of $\Gamma$ on the quotient $\text{Ad}(E_G) = (E_G \times G)/G$. This descended action of $\Gamma$ on $\text{Ad}(E_G)$ lifts the action of $\Gamma$ on $M$, and it clearly preserves the algebraic group structure of the fibers of $\text{Ad}(E_G)$. The action of $\Gamma$ on $\text{Ad}(E_G)$ induces an action of $\Gamma$ on the space of all global sections of $\text{Ad}(E_G)$, namely $\text{Aut}(E_G)$. It is straightforward to check that this induced action on $\text{Aut}(E_G)$ coincides with the earlier defined action of $\Gamma$ on $\text{Aut}(E_G)$.

Let
\begin{equation}
\text{Aut}(E_G)^\Gamma \subset \text{Aut}(E_G)
\end{equation}
be the subgroup that is fixed pointwise by the action of $\Gamma$ on $\text{Aut}(E_G)$. Since the action of each $\gamma \in \Gamma$ is an algebraic automorphism of $\text{Aut}(E_G)$, the subgroup $\text{Aut}(E_G)^\Gamma$ is Zariski closed.

A reduction of structure group of $E_G$ to a closed subgroup $H \subset G$ is a section of $E_G/H$ over $M$, or equivalently, a closed subvariety $E_H \subset E_G$ closed under the action of $H$ such that the $H$ action on $E_H$ defines a principal $H$-bundle over $M$.

**Definition 2.1.** A reduction of structure group $E_H \subset E_G$ to $H$ is called $\Gamma$-equivariant if the subvariety $E_H$ is left invariant by the action of $\Gamma$ on $E_G$.

Since the actions of $\Gamma$ and $G$ on $E_G$ commute, there is an induced action of $\Gamma$ on $E_G/H$. It is easy to see that $E_H$ is a $\Gamma$-equivariant reduction of structure group if and only if the section over $M$ of the bundle $E_G/H \to M$ defined by $E_H$ is fixed by the action of $\Gamma$ on the space of all sections of $E_G/H$ induced by the action on $E_G/H$.

By a Levi subgroup of $G$ we will mean the centralizer in $G$ of some torus of $G$. Recall that a torus is a product of copies of $\mathbb{G}_m$ or the trivial group. For a Levi subgroup $H \subset G$, the centralizer in $G$ of the connected component of the center of $H$ containing the identity element coincides with $H$ (see [SS §3]). If $H \subset G$ is a Levi subgroup, then there is a parabolic subgroup $H \subset P \subset G$ such that $H$ projects isomorphically to the Levi quotient of $P$. Conversely, if $H$ is a reductive subgroup of a parabolic subgroup $P \subset G$ such that $H$ projects isomorphically to the Levi quotient of $P$, then $H$ is a Levi subgroup of $G$. Note that if we take the torus to be the trivial group, then the corresponding Levi subgroup is $G$ itself, and hence in that case the corresponding parabolic subgroup is $G$.

Take a Levi subgroup $H \subset G$. Let
\begin{equation}
Z_0(H) \subset H
\end{equation}
be the connected component of the center of $H$ containing the identity element. Let
\begin{equation}
E_H \subset E_G
\end{equation}
be a reduction of structure group of $E_G$ to $H$. We have
\[(2.7) \quad Z_0(H) \subset \text{Aut}^0(E_H) \subset \text{Aut}^0(E_G),\]
where $\text{Aut}^0(E_H)$ is the connected component of the group of all automorphisms of the $H$–bundle $E_H$ containing the identity automorphism; the group $Z_0(H)$ acts on $E_H$ as translations (using the action of $H$ on $E_H$), which makes $Z_0(H)$ a subgroup of $\text{Aut}^0(E_H)$.

**Theorem 2.2.** If the reduction $E_H$ in $(2.6)$ is $\Gamma$–equivariant, then the subgroup $Z_0(H) \subset \text{Aut}^0(E_G)$ in $(2.7)$ is contained in $\text{Aut}(E_G)^\Gamma$ (defined in $(2.5)$).

Conversely, if $Z_0(H) \subset \text{Aut}^0(E_G) \cap \text{Aut}(E_G)^\Gamma$, then the reduction $E_H$ in $(2.6)$ is $\Gamma$–equivariant.

**Proof.** Assume that the reduction $E_H$ in $(2.6)$ is $\Gamma$–equivariant. For any $\gamma \in \Gamma$, let $\Phi_\gamma$ be the automorphism of the variety $E_H$ defined by the action of $\gamma$. The automorphism $g \in Z_0(H) \subset \text{Aut}^0(E_G)$ preserves $E_H$, and on $E_H$ it coincides with the map $z \mapsto zg$. Let $S_g$ be the automorphism of the $H$–bundle $E_H$ defined by $z \mapsto zg$. Since the actions of $G$ and $\Gamma$ on $E_G$ commute, we have
\[\Phi_\gamma \circ S_g \circ \Phi_\gamma^{-1} = S_g \circ \Phi_\gamma \circ \Phi_\gamma^{-1} = S_g\]
on $E_H$. Therefore, the two automorphisms, namely $g \in \text{Aut}^0(E_G)$ (in $(2.7)$) and the image of $g$ by the action $\gamma$ on $\text{Aut}(E_G)$, coincide over $E_H \subset E_G$. Consequently, these two automorphisms of $E_G$ coincide. In other words, the action of $\Gamma$ on $\text{Aut}(E_G)$ fixes the subgroup $Z_0(H)$ pointwise. This completes the proof of the first part.

Assume that $\Gamma$ acts trivially on the subgroup $Z_0(H) \subset \text{Aut}(E_G)$ defined in $(2.7)$. Take a closed point $x \in M$. We will show that the evaluation map
\[(2.8) \quad f_x : Z_0(H) \longrightarrow \text{Ad}(E_G)_x\]
is injective, where $\text{Ad}(E_G)_x$ is the fiber of $\text{Ad}(E_G)$ over $x$; the map $f_x$ sends any $s \in Z_0(H)$ to the evaluation at $x$ of the corresponding section (as in $(2.7)$) of $\text{Ad}(E_G)$.

To prove that $f_x$ is injective, fix a finite dimensional faithful $G$–module $V$ over $k$. Let
\[E_V := (E_G \times V)/G\]
be the vector bundle over $M$ associated to $E_G$ for the $G$–module $V$; the action of any $g \in G$ sends $(z,v) \in E_G \times V$ to $(zg,g^{-1}v)$. Take any $\sigma \in Z_0(H) \subset \text{Aut}^0(E_G)$. So $\sigma$ gives an automorphism
\[\sigma' \in H^0(M, \text{Isom}(E_V))\]
of the vector bundle; the automorphism of $E_G \times V$ that sends any $(z,v) \in E_G \times V$ to $(\sigma(z),v)$ descends to an automorphism of $E_V$.

Since $M$ is complete and irreducible, there are no nonconstant functions on it. Therefore, the coefficients of the characteristic polynomial of the endomorphism
\[\sigma'(y) \in \text{End}((E_V)_y),\]
where \( y \in M \) is a closed point, are independent of \( y \). Also, since \( \sigma \) is an element of a torus, namely \( Z_0(H) \), the endomorphism \( \sigma'(y) \) is semisimple.

If \( f_x(\sigma) = \text{Id}_{(E_V)_x} \), where \( f_x \) is defined in (2.8), then clearly \( \sigma'(x) = \text{Id}_{(E_V)_x} \). Therefore, in that case, all the eigenvalues of \( \sigma'(y) \) are 1 for all \( y \in M \). Since all \( \sigma'(y) \) is semisimple with all eigenvalues 1, it follows immediately that \( \sigma'(y) \) is the identity automorphism of \( (E_V)_y \) for each \( y \in M \).

Since \( V \) is a faithful \( G \)–module and \( \sigma' \) is the identity automorphism of \( E_V \), we conclude that \( \sigma \) is the identity automorphism of \( E_G \). This proves that the homomorphism \( f_x \) defined in (2.8) is injective.

Therefore, using the evaluation map, \( M \times Z_0(H) \subset \text{Ad}(E_G) \) is a subgroup–scheme. Since \( Z_0(H) \) is preserved by the action of \( \Gamma \) on \( \text{Aut}(E_G) \), it follows immediately that the action of \( \Gamma \) on \( \text{Ad}(E_G) \) leaves this subgroup–scheme invariant.

Fix an element \( g_0 \in Z_0(H) \) such that the Zariski closure of the group generated by \( g_0 \) coincides with \( Z_0(H) \). Since \( H \) is the centralizer of the subgroup \( Z_0(H) \subset G \), and the algebraic subgroup generated by \( g_0 \) coincides with \( Z_0(H) \), we conclude that \( H \) coincides with the centralizer of \( g_0 \in G \).

Let

\[
(2.9) \quad F : E_G \times G \longrightarrow \text{Ad}(E_G) := (E_G \times G)/G
\]

be the quotient map. Let

\[
(2.10) \quad \hat{F} := F^{-1}(\text{image}(\hat{g}_0)) \subset E_G \times G
\]

be the subvariety, where

\[
(2.11) \quad \hat{g}_0 : M \longrightarrow \text{Ad}(E_G)
\]

is the section defined by the above element \( g_0 \) using the inclusion \( Z_0(H) \hookrightarrow \text{Aut}(E_G) \) in (2.7). Set

\[
(2.12) \quad \hat{E} := \hat{F} \cap (E_G \times \{g_0\}) \subset E_G \times G,
\]

where \( \hat{F} \) is defined in (2.10), and let

\[
(2.13) \quad E' \subset E_G
\]

be the image of \( \hat{E} \) (constructed in (2.12)) by the projection of \( E_G \times G \) to \( E_G \) defined by \((z, g) \mapsto z\).

Since \( \Gamma \) acts trivially on the subgroup \( Z_0(H) \hookrightarrow \text{Aut}(E_G) \), the image of the map \( \hat{g}_0 \) in (2.11) is left invariant by the action of \( \Gamma \) on \( \text{Ad}(E_G) \). Since the action of \( \Gamma \) on \( \text{Ad}(E_G) \) is the descent, by the projection \( F \) in (2.9), of the diagonal action on \( E_G \times G \) with \( \Gamma \) acting trivially on \( G \), it follows that \( E' \) in (2.13) is left invariant by the action of \( \Gamma \) on \( E_G \).

Since \( E' \) is left invariant by \( \Gamma \), the theorem follows once we show that \( E' \) coincides with the subvariety \( E_H \) in (2.6).
To prove that $E' = E_H$, first note that
\[ E_H \times \{g_0\} \subset \hat{F} \subset E_G \times G \]
with $\hat{F}$ defined in (2.11). Indeed, the automorphism of $E_H$ defined by $g_0$ sends any $z \in E_H$ to $zg_0$ (since $g_0$ is in the center of $H$, this commutes with the action of $H$ and hence it is an automorphism of $E_H$). This immediately implies that $E_H \times \{g_0\} \subset \hat{F}$. Consequently, we have $E_H \subset E'$. On the other hand, for any $x \in M$ and $w \in E' \cap (E_G)_x$ it can be shown that the fiber of $E'$ over $x$ is contained in the orbit of $w$ for the action of the centralizer of $g_0$ in $G$. Indeed, if $F(w', g') = F(w'g, g')$, where $g, g' \in G$, $w' \in (E_G)_x$ and $F$ as in (2.9), then $gg'g^{-1} = g'$, this being an immediate consequence of the definition of $F$. Therefore, if $w, wg \in E'$, with $g \in G$, then $g^{-1}g_0g = g_0$.

We already noted that the centralizer of $g_0$ in $G$ coincides with $H$. We also saw that $E_H \subset E'$. Therefore, the above observation that any two points of $E'$ over a point $x \in M$ differ by an element of the centralizer of $g_0$ implies that $E_H = E'$. This completes the proof of the theorem. \qed

**Example 2.3.** It may happen that $\Gamma$ preserves the subgroup $Z_0(H) \subset \text{Aut}^0(E_G)$ in (2.7), but does not preserve $Z_0(H)$ pointwise. We give an example.

Fix a maximal torus $T \subset G$. Take $\Gamma$ to be the normalizer $N(T)$ of $T$ in $G$, and equip $M$ with the trivial action of $\Gamma$; let $G$ be such that $N(T) \neq T$. Set $E_G$ to be the trivial $G$–bundle $M \times G$. The group $N(T)$ acts on $M \times G$ as left translations of $G$. So the induced action of $N(T)$ on $\text{Aut}(E_G) = G$ is the conjugation action. Set $H = T$. The reduction of structure group of $E_G$ to $T$ defined by the inclusion $M \times T \hookrightarrow M \times G$ has the property that the subgroup
\[ Z_0(H) = T \subset G = \text{Aut}(E_G) \]
(defined in (2.7)) is left invariant by the action of $N(T)$ (in this case it is the adjoint action of $N(T)$ on $G$). However no reduction of structure group of $E_G$ to $T$ is left invariant by the action $N(T)$.

The automorphism group of a torus is a discrete group. Therefore, if $\Gamma$ is a connected algebraic group acting algebraically on $E_G$, then $\Gamma$ acts trivially on $Z_0(H)$ provided $Z_0(H)$ is preserved by $\Gamma$.

**Proposition 2.4.** Let $T \subset \text{Aut}^0(E_G) \cap \text{Aut}^0(E_G)^\Gamma$ be a torus such that there is an element $g \in \text{Aut}^0(E_G)$ satisfying the condition that $g^{-1}Tg = Z_0(H)$, with $Z_0(H)$ constructed in (2.7) for the reduction $E_H$ in (2.6). Then $E_G$ admits a $\Gamma$–equivariant reduction of structure group to the Levi subgroup $H$.

**Proof.** Take $T$ and $g$ as above. So, the image
\[ E'_H := g(E_H) \subset E_G \]
is a reduction of structure group of $E_G$ to $H$, where $E_H$ is the reduction in (2.6). Take any automorphism $\tau$ of the principal $H$–bundle $E_H$. Using the reduction $E_H$ in (2.6), the automorphism $\tau$ gives an automorphism $\tau_1$ of the $G$–bundle $E_G$. On the other hand, using the above reduction $E'_H \subset E_G$ together with the isomorphism of $E_H$ with $E'_H$ defined by $z \mapsto g(z)$ the automorphism $\tau$ gives an automorphism $\tau_2$ of $E_G$. It is easy to see that $\tau_2 = g\tau_1 g^{-1}$.

Therefore, if we substitute $E_H$ by $E'_H$, then the subgroup $Z_0(H) \subset \text{Aut}^0(E_G)$ in (2.7) gets replaced by $gZ_0(H)g^{-1}$. Now, the second part of Theorem 2.2 says that $E'_H$ is left invariant by the action of $\Gamma$ on $E_G$. This completes the proof of the proposition.

\[ \square \]

3. Levi reduction from tori in $\text{Aut}(E_G)^{\Gamma}$

Let $T \subset \text{Aut}^0(E_G)$ be a torus. From the proof of Theorem 2.2 it can be deduced that $T$ determines a torus, unique up to an inner automorphism, in $G$. This will be explained below with more details.

Fix a point $x \in M$. We saw in the proof of Theorem 2.2 that the evaluation map
\[ f_x : T \rightarrow \text{Ad}(E_G)_x \]
(3.1) is injective. Since $\text{Ad}(E_G) = (E_G \times G)/G$, if we fix a point $z \in (E_G)_x$, then the quotient map $F$ (defined in (2.9)) gives an isomorphism of $\{z\} \times G$ with $\text{Ad}(E_G)_x$. This identification of $G$ with $\text{Ad}(E_G)_x$ constructed using $z$ is an isomorphism of algebraic groups. Furthermore, if we substitute $z$ by $zg$, $g \in G$, then the corresponding isomorphism of $G$ with $\text{Ad}(E_G)_x$ is the composition of the earlier one with the automorphism of $G$ defined by the conjugation action of $g$. Therefore, $f_x(T)$, with $f_x$ defined in (3.1), gives a torus in $G$ up to conjugation.

This torus of $G$, up to conjugation, defined by $f_x(T)$ actually does not depend on the choice of the point $x$. To prove this, take $z_1, z_2 \in E_G$ with $f(z_i) = x_i, i = 1, 2$, where $f$ is as in (2.2). Consider the evaluation homomorphism
\[ f_{x_i} : T \rightarrow \text{Ad}(E_G)_{x_i} \]
which is injective. Let
\[ h_{z_i} : T \rightarrow G \]
(3.2) be the composition of $f_{x_i}$ with the identification of $\text{Ad}(E_G)_{x_i}$ with $G$ defined by $z_i$. We want to show that the two subgroups, namely image($h_{z_1}$) and image($h_{z_2}$), of $G$ differ by an inner automorphism of $G$.

Fix a point $t_0 \in T$ such that the Zariski closed subgroup of $T$ generated by $t_0$ is $T$ itself. For a finite dimensional $G$–module $V$ over $k$, let $E_V$ be the vector bundle associated to $E_G$ for $V$ and $t_0$ the automorphism of $E_V$ defined by $t_0 \in \text{Aut}(E_G)$. From the definition of the map $h_{z_i}$ it follows that the automorphism $\hat{t}_0(x_i)$ of $(E_V)_{x_i}$ and the automorphism of
V given by $h_{z_i}(t_0) \in G$ are intertwined by the isomorphism of $(E_V)_{x_i}$ with $V$ constructed using $z_i$. (Since $E_V = (E_G \times V)/G$, we have an isomorphism of $(E_V)_{x_i}$ with $V$ that sends any $v \in V$ to the image of $(z_i, v)$.) We saw in the proof of Theorem 2.2 that the characteristic polynomial of $\hat{t}_0(y) \in \text{Isom}((E_V)_y)$ is independent of $y$. Therefore, the automorphisms of $V$ defined the two elements $h_{z_1}(t_0)$ and $h_{z_2}(t_0)$ of $G$ have same characteristic polynomial.

On the other hand, if $T'' \subset G$ is a maximal torus, then the algebra of all functions on the affine variety $T''/W$, where $W := N(T'')/T''$ is the Weyl group with $N(T'')$ the normalizer of $T''$ in $G$, is generated by trace function of finite dimensional $G$–modules over $k$ [St, p. 87, Theorem 2]. Therefore, $h_{z_1}(t_0)$ and $h_{z_2}(t_0)$ differ by an inner automorphism of $G$ (since the characteristic polynomials of $h_{z_1}(t_0)$ and $h_{z_2}(t_0)$ coincide for any $G$–module). Since image($h_{z_i}$) is generated, as a Zariski closed subgroup, by $h_{z_i}(t_0)$, we conclude that the two subgroups image($h_{z_1}$) and image($h_{z_2}$) differ by an inner automorphism of $G$.

**Remark 3.1.** Let $E_H \subset E_G$ be a reduction of structure group to a Levi subgroup $H \subset G$. Consider the torus $Z_0(H) \subset \text{Aut}^0(E_G)$ in (2.7) corresponding to the reduction $E_H$. By substituting a point of $E_H$ for the point $z_i$ in (3.2) we conclude that the map in (3.2) sends any $g \in Z_0(H) \subset H$ to the point $g \in \text{Aut}^0(E_G)$ (in terms of (2.7)). Consequently, the torus $Z_0(H) \subset G$ is in the conjugacy class of tori given by the torus $Z_0(H) \subset \text{Aut}^0(E_G)$ in (2.7).

We have the following lemma:

**Lemma 3.2.** If the $G$–bundle $E_G$ admits a $\Gamma$–equivariant reduction of structure group to a Levi subgroup $H \subset G$, then there is a torus $T \subset \text{Aut}^0(E_G) \cap \text{Aut}(E_G)^F$ that satisfies the condition that $Z_0(H)$ is the torus in $G$ defined, up to conjugation, by $T$.

**Proof.** Let $E_H \subset E_G$ be a $\Gamma$–equivariant reduction of structure group to $H$. The image of $Z_0(H)$ in $\text{Aut}^0(G)$ by the inclusion map in (2.7) will be denoted by $T$. The first part of Theorem 2.2 says that $T \subset \text{Aut}(E_G)^F$.

Fix a point $z \in E_H \subset E_G$. It is easy to see that the torus $h_z(T) \subset G$ coincides with $Z_0(H)$, where $h_z$ is defined as in (3.2) (by composing the evaluation map $T \rightarrow \text{Ad}(E_G)_{f(z)}$, where $f$ is defined in (2.2), with the isomorphism $\text{Ad}(E_G)_{f(z)} \rightarrow G$ defined by $z$). This completes the proof of the lemma. \qed

In the converse direction we have:

**Proposition 3.3.** Let $T' \subset G$ be a torus in the conjugacy class of tori determined by a torus $T \subset \text{Aut}^0(E_G)^F$ and $H$ the centralizer of $T'$ in $G$. Then $E_G$ admits a $\Gamma$–equivariant reduction of structure group to the Levi subgroup $H$. 
Proof. Fix any point \( z \in E_G \) and consider the homomorphism
\[
(3.3) \quad h_z : T \longrightarrow G
\]
as in (3.2), namely it is the composition of the evaluation map with the identification, constructed using \( z \), of \( G \) with \( \text{Ad}(E_G)_{f(z)} \), where \( f \) is defined in (2.2). There is an element \( g \in G \) with \( gh_z(T)g^{-1} = T' \), where \( T' \) is as in the statement of the proposition.

Let
\[
(3.4) \quad H_z \subset G
\]
be the centralizer of \( h_z(T) \), with \( h_z \) defined in (3.3). Since \( gh_z(T)g^{-1} = T' \), and the centralizer of \( T' \subset G \) is \( H \), we conclude that
\[
(3.5) \quad gH_zg^{-1} = H.
\]

Fix an element \( t_0 \in T \) such that the Zariski closure in \( T \) of the subgroup generated by \( t_0 \) is \( T \) itself. As in (2.11), let
\[
\hat{t}_0 : M \longrightarrow \text{Ad}(E_G)
\]
be the section defined by the automorphism \( t_0 \in \text{Aut}(E_G) \). As in (2.10), set
\[
\hat{F} := F^{-1} (\text{image}(\hat{t}_0)) \subset E_G \times G,
\]
where \( F \) is the projection in (2.9). As in (2.12), define
\[
\hat{E} := \hat{F} \cap (E_G \times \{h_z(t_0)\}) \subset E_G \times G,
\]
where \( h_z \) is defined in (3.3). Let
\[
(3.6) \quad E' \subset E_G
\]
be the image of \( \hat{E} \) by the projection of \( E_G \times G \) to \( E_G \) defined by \( (y, \nu) \mapsto y \).

We will show that \( E' \) constructed in (3.6) is a \( \Gamma \)-equivariant reduction of structure group of \( E_G \) to the subgroup \( H_z \) defined in (3.4).

For this, we will first show that \( E' \) is closed under the action of \( H_z \) (for the action of \( G \) on \( E_G \)). Note that \( h_z(t_0) \) is in the center of \( H_z \) (as \( H_z \) is the centralizer of \( h_z(T) \)). Therefore, for the projection \( F \) in (2.9) we have
\[
(3.7) \quad F(z_1, h_z(t_0)) = F(z_1g_1, h_z(t_0))
\]
for all \( z_1 \in E_G \) and and \( g_1 \in H_z \). Indeed, the map \( F \) clearly has the property that for \( g, g' \in G \) and \( w' \in E_G \)
\[
(3.8) \quad F(w', g') = F(w'g, g')
\]
if and only if \( gg'g^{-1} = g' \). From (3.7) it follows immediately that \( E' \) is closed under the action of \( H_z \).

It also follows from (3.8) that for any point \( y \in M \), the centralizer of \( h_z(t_0) \) (in \( G \)) acts transitively on the fiber of \( E' \) over \( y \). Note that since the Zariski closure of the group
generated by $t_0$ is $T$, and $H_z$ is the centralizer (in $G$) of $h_z(T)$, it follows immediately that the centralizer of $h_z(t_0)$ is $H_z$.

We still need to show that the fiber of $E'$ over each point $y \in M$ is nonempty. For this note that there is a point $z' \in f^{-1}(y)$, with $f$ defined in (2.2), such that the corresponding homomorphism

$$h_{z'} : T \rightarrow G$$

defined as in (3.3) by replacing $z$ by $z'$ has the property that $h_{z'}(t_0) = h_z(t_0)$. Indeed, this follows from the combination of the fact that the conjugacy class of the torus $h_z(T) \subset G$ is independent of the choice of the point $z \in E_G$ and the observation that the two homomorphisms $h_{z_1}$ and $h_{z_1g_1}$ from $T$ to $G$, where $z_1 \in E_G$ and $g_1 \in G$, differ by the inner automorphism of $G$ defined by $g_1$. The identity $h_{z'}(t_0) = h_z(t_0)$ immediately implies that $z'$ is in the fiber of $E'$ over $y$.

Consequently, $E' \subset E_G$ constructed in (3.3) is a reduction of structure group to $H_z$.

Since the action of $\Gamma$ on $\text{Aut}(E_G)$ fixes $t_0$, the action of $\Gamma$ on $E_G$ leaves $E'$ invariant.

Finally, from (3.5) it follows immediately that $E'g^{-1} \subset E_G$ is a reduction of structure group to $H$. As $E'$ is left invariant by the action of $\Gamma$ on $E_G$, and the actions of $\Gamma$ and $G$ on $E_G$ commute, the subvariety $E'g^{-1} \subset E_G$ is also left invariant by the action of $\Gamma$. This completes the proof of the proposition. \hfill \Box

4. A canonical equivariant Levi reduction

Let $T \subset \text{Aut}(E_G)\Gamma$ be a connected maximal torus, where $\text{Aut}(E_G)\Gamma$ is defined in (2.5). So $T$ is a torus of $\text{Aut}^0(E_G)$. We saw in the previous section that $T$ determines a torus, unique up to an inner conjugation, in $G$. We will show that this torus in $G$ (up to conjugation) does not depend on the choice of the maximal torus $T$.

To prove this, first note that any two maximal tori of $\text{Aut}(E_G)\Gamma$ differ by an inner automorphism of $\text{Aut}(E_G)\Gamma$. Consider the maximal torus $g_0Tg_0^{-1}$, where $g_0 \in \text{Aut}(E_G)\Gamma$, and fix a point $z \in E_G$. The point $z$ defines two injective homomorphisms

$$h_z : T \rightarrow G$$

and

$$h'_z : g_0Tg_0^{-1} \rightarrow G$$

defined as in (3.2) using the evaluation map and the isomorphism of groups

$$\phi_z : \text{Ad}(E_G)_{f(z)} \rightarrow G$$

collected using $z$, where $f$ is defined in (2.2). From the construction of $h_z$ and $h'_z$ it follows immediately that

$$\phi_z(g_0(f(z)))h_z(T)(\phi_z(g_0(f(z))))^{-1} = h'_z(g_0Tg_0^{-1}).$$
Therefore, \( h_\sharp(T) \) and \( h'_\sharp(g_0Tg_0^{-1}) \) differ by an inner automorphism of \( G \). Consequently, the torus of \( G \) determined by a maximal torus of \( \text{Aut}(E_G)^\Gamma \) does not depend on the choice of the maximal torus.

Fix a torus \( T_0 \subset G \) in the conjugacy class of tori given by a maximal torus in \( \text{Aut}(E_G)^\Gamma \). The centralizer of \( T_0 \) in \( G \) is a Levi subgroup. This Levi subgroup of \( G \) will be denoted by \( H_0 \).

**Theorem 4.1.** The \( G \)-bundle \( E_G \) admits a \( \Gamma \)-equivariant reduction of structure group to the Levi subgroup \( H_0 \) defined above.

If \( H \subsetneq H_0 \) is a Levi subgroup of \( G \) properly contained in \( H_0 \), then \( E_G \) does not admit any \( \Gamma \)-equivariant reduction of structure group to \( H \).

If \( H \subset G \) is a Levi subgroup such that \( E_G \) admits a \( \Gamma \)-equivariant reduction of structure group to \( H \), but \( E_G \) does not admit a \( \Gamma \)-equivariant reduction of structure group to any Levi subgroup properly contained in \( H \), then \( H \) is conjugate to the above defined subgroup \( H_0 \subset G \).

If \( E_{H_0} \subset E_G \) and \( E'_{H_0} \subset E_G \) are two \( \Gamma \)-equivariant reductions of structure group to \( H_0 \), then there is an automorphism \( \tau \in \text{Aut}(E_G)^\Gamma \) of \( E_G \) such that \( \tau(E_{H_0}) = E'_{H_0} \subset E_G \).

**Proof.** That \( E_G \) admits a \( \Gamma \)-equivariant reduction of structure group to \( H_0 \) follows from the construction in Proposition 3.3. Fix a maximal torus \( T \subset \text{Aut}^0(E_G)^\Gamma \) and a point \( t_0 \in T \) such that the Zariski closure of the subgroup of \( T \) generated by \( t_0 \) coincides with \( T \). Let \( t'_0 \in T_0 \) be the element corresponding to \( t_0 \) by an isomorphism of \( T \) with \( T_0 \) constructed using an element of \( E_G \). As in (2.12), consider

\[
\hat{\mathcal{E}} := F^{-1}(\text{image}(t_0)) \cap (E_G \times \{t'_0\}) \subset E_G \times G,
\]

where \( F \) is defined in (2.12) and \( \hat{t}_0 \) is the section of \( \text{Ad}(E_G) \) defined by \( t_0 \). Finally the image of \( \hat{\mathcal{E}} \) by the projection of \( E_G \times G \) to \( E_G \) gives a reduction of structure group of \( E_G \) to \( H_0 \). See the proof of Proposition 3.3 for the details.

To prove the second statement, let \( H \subsetneq H_0 \) be a Levi subgroup of \( G \) properly contained in \( H_0 \). So \( \dim Z_0(H) > \dim T_0 \), where \( Z_0(H) \) is the connected component of the center of \( H \) containing the identity element (note that \( T_0 \) is contained in the center of the bigger Levi subgroup). The first statement in Theorem 2.2 says that if \( E_H \subset E_G \) is a \( \Gamma \)-equivariant reduction of structure group to \( H \), then \( \text{Aut}(E_G)^\Gamma \) contains a torus isomorphic to \( Z_0(H) \). This is impossible, since a smaller dimensional torus, namely \( T_0 \), is isomorphic to the maximal torus \( T \) and any two maximal tori are isomorphic.

Let \( H \subset G \) be a Levi subgroup as in the third statement, and let \( E_H \subset E_G \) be a \( \Gamma \)-equivariant reduction of structure group to \( H \). The condition on \( H \) implies that the torus \( Z_0(H) \subset \text{Aut}(E_G)^\Gamma \) in (2.7) for the reduction \( E_H \) is a maximal torus of \( \text{Aut}(E_G)^\Gamma \). Indeed, that \( Z_0(H) \subset \text{Aut}(E_G)^\Gamma \) follows from Theorem 2.2. That \( Z_0(H) \) is a maximal torus of
Remark 4.2. If we set $\text{Aut}(E_G)^\Gamma$ can be seen as follows. If $T'' \subset \text{Aut}(E_G)^\Gamma$ is a torus with $Z_0(H) \subset T''$, then take a torus $T''_1$ in the conjugacy class of tori of $G$ given by $T''$ such that $Z_0(H) \subset T''_1 \subset G$. Let $H'' \subset G$ be the centralizer of $T''_1$. Since $Z_0(H)$ is the connected component of the center of $H$ containing the identity element and $Z_0(H) \subset T''_1$ is a proper subtorus, we conclude that $H'' \subset H$. Proposition 3.3 says that $E_G$ admits a $\Gamma$–equivariant reduction of structure group to $H''$. Since $H''$ is a Levi subgroup properly contained in $H$, this contradicts the given condition on $H$. Therefore, $Z_0(H) \subset \text{Aut}(E_G)^\Gamma$ is a maximal torus.

Since $T_0$, by definition, is in the conjugacy class of tori of $G$ given by a maximal torus of $\text{Aut}(E_G)^\Gamma$, using Remark 3.1 we conclude that the two tori $T_0$ and $Z_0(H)$ of $G$ are conjugate. Consequently, $H$ and $H_0$ differ by an inner automorphism of $G$.

Let $E_{H_0}$ and $E_{H_0}'$ be as in the fourth statement. Consider the inclusion in (2.7). Let $T_1$ (respectively, $T'_1$) be the image of $T_0$ in $\text{Aut}(E_G)^\Gamma$ for the reduction $E_{H_0}$ (respectively, $E_{H_0}'$) by (2.7). From dimension consideration we know that both $T_1$ and $T'_1$ are maximal tori in $\text{Aut}(E_G)^\Gamma$. Take an element $\tau \in \text{Aut}(E_G)^\Gamma$ such that

$$T'_1 = \tau^{-1} T_1 \tau.$$  

Let $E_H \subset E_G$ be a $\Gamma$–equivariant reduction of structure group to a Levi subgroup $H \subset G$ and $g_0 \in Z_0(H)$ an element in the connected component of the center of $H$ containing the identity element such that $g_0$ generates $Z_0(H)$ as a Zariski closed subgroup of $G$. In the proof of Theorem 2.2 we gave a reconstruction of $E_H$ from $g_0$ and its image in $\text{Aut}^0(E_G)$ by (2.11). (In the notation of the proof of Theorem 2.2, $E' \subset E_G$ was constructed in (2.13) using $g_0$ and its image in $\text{Aut}^0(E_G)$, and it was shown there that $E_H$ coincides with $E'$.)

Fix an element $g_0 \in T_0$ such that Zariski closed subgroup generated by $g_0$ coincides with $T_0$. Let $g_1$ be the image of $g_0$ in $T_1$ for the above isomorphism of $T_0$ with $T_1$ constructed using $E_H$. Set

$$g'_1 = \tau^{-1} g_1 \tau \in T'_1,$$

where $\tau$ is as in (4.1). Let $g'_0$ be the image of $g'_1$ in $T_0$ for the above isomorphism of $T'_1$ with $T_0$ constructed using $E_H'$. Following the construction of $E'$ in (2.13), we can reconstruct $E_H$ (respectively, $E_H'$) using the pair $(g_0, g_1)$ (respectively, $(g'_0, g'_1)$). Using this reconstruction it is easy to see that

$$E'_H = \tau^{-1}(E_H),$$

where $\tau$ is as in (4.1). This completes the proof of the theorem. \hfill $\square$

**Remark 4.2.** If we set $G = \text{GL}(n,k)$ and $\Gamma = \{e\}$, then Theorem 4.1 becomes the following theorem proved in [At]: any vector bundle $V$ over $M$ is isomorphic to a direct
sum $\bigoplus_{i=1}^{k} U_i$ of indecomposable vector bundles, and if

$$V \cong \bigoplus_{j=1}^{l} W_j,$$

where each $W_j$ is indecomposable, then $k = l$ and the collection of vector bundles $\{W_j\}$ is a permutation of $\{U_i\}$.

**Remark 4.3.** Let $E_*$ be a parabolic $G$–bundle over an irreducible smooth projective variety $X$. Corresponding to $E_*$, there is an irreducible smooth projective variety $Y$, a finite subgroup $\Gamma \subset \text{Aut}(Y)$ with $X = Y/\Gamma$, and a principal $G$–bundle $E_G$ over $Y$ equipped with a lift of the action of $\Gamma$. More precisely, there is a bijective correspondence between parabolic $G$–bundles and $G$–bundles with a finite group action on a (ramified) covering (see [BBN] for the details). Therefore, Theorem 4.1 gives a natural reduction of structure group of a parabolic $G$–bundle to a Levi subgroup of $G$. This Levi reduction satisfies all the analogous conditions in Theorem 4.1.

5. **The Levi Quotient of the Automorphism Group**

In this final section, we will assume $\Gamma$ to be a connected algebraic group. We will also assume the action of $\Gamma$ on $E_G$ to be algebraic, that is, the map $\phi$ in (2.1) is algebraic; consequently, the action of $\Gamma$ on $M$ is also algebraic. Since $\Gamma$ is connected, the action of $\Gamma$ on $\text{Aut}(E_G)$ preserves the subgroup $\text{Aut}^0(E_G)$.

Let $U\text{Aut}^0(E_G)$ be the unipotent radical of the algebraic group $\text{Aut}^0(E_G)$ [Hu, p. 125]. So the Levi quotient

$$(5.1) \quad \text{LAut}^0(E_G) := \text{Aut}^0(E_G)/U\text{Aut}^0(E_G)$$

is a connected reductive algebraic group over $k$. Let

$$(5.2) \quad \psi : \text{Aut}^0(E_G) \rightarrow \text{LAut}^0(E_G)$$

be the quotient map.

From the uniqueness of a unipotent radical it follows immediately that the action of $\Gamma$ on $\text{Aut}^0(E_G)$ preserves the subgroup $U\text{Aut}^0(E_G)$. Therefore, we have an induced action of $\Gamma$ on $\text{LAut}^0(E_G)$.

Let $\widehat{T}_0 \subset G$ be a torus in the conjugacy class of tori of $G$ given by a maximal torus in $\text{Aut}^0(E_G)$. Since any two maximal tori are conjugate, the conjugacy class of $\widehat{T}_0$ does not depend on the choice of the maximal torus. Let $\widehat{H}_0$ be the centralizer of $T_0$ in $G$. Setting $\Gamma = \{e\}$ in Proposition we conclude that $E_G$ admits a reduction of structure group to $\widehat{H}_0$.

**Proposition 5.1.** If $E_G$ admits a $\Gamma$–equivariant reduction of structure group to the Levi subgroup $\widehat{H}_0$, then the induced action of $\Gamma$ on $\text{LAut}^0(E_G)$ (defined in (5.1)) factors through an action of a torus quotient of $\Gamma$. 
If \( \Gamma \) is reductive and the induced action of \( \Gamma \) on \( \text{LAut}^0(E_G) \) factors through an action of a torus quotient of \( \Gamma \), then \( E_G \) admits a \( \Gamma \)-equivariant reduction of structure group to the Levi subgroup \( \hat{H}_0 \).

**Proof.** Assume that \( E_G \) admits a \( \Gamma \)-equivariant reduction of structure group to \( \hat{H}_0 \). Theorem 4.1 says that there is a maximal torus \( T_0 \subset \text{Aut}^0(E_G) \) which is left invariant by the action of \( \Gamma \) on \( \text{Aut}^0(E_G) \). Consider \( \psi(T_0) \), with \( \psi \) defined in (5.2), which is a maximal torus in \( \text{LAut}^0(E_G) \). Note that \( \psi(T_0) \) is left invariant by the induced action of \( \Gamma \) on \( \text{LAut}^0(E_G) \), as \( T_0 \) is \( \Gamma \)-invariant.

Let \( Z \text{LAut}^0(E_G) \subset \text{LAut}^0(E_G) \) be the center and \( \text{PLAut}^0(E_G) := \text{LAut}^0(E_G)/Z \text{LAut}^0(E_G) \) the corresponding adjoint group. All the automorphisms of \( \text{LAut}^0(E_G) \) connected to the identity automorphism are parametrized by \( \text{PLAut}^0(E_G) \), with \( \text{PLAut}^0(E_G) \) acting on \( \text{LAut}^0(E_G) \) as conjugations.

Since \( \Gamma \) is connected, we have a homomorphism of algebraic groups
\[
\rho : \Gamma \longrightarrow \text{PLAut}^0(E_G)
\]
such that the action of any \( g \in \Gamma \) on \( \text{LAut}^0(E_G) \) is conjugation by \( \rho(g) \). Since the action of \( \Gamma \) preserves the maximal torus \( \psi(T_0) \subset \text{LAut}^0(E_G) \), and \( q \circ \psi(T_0) \) is a maximal torus in \( \text{PLAut}^0(E_G) \), where
\[
q : \text{LAut}^0(E_G) \longrightarrow \text{PLAut}^0(E_G)
\]
is the projection, we conclude that \( \rho(\Gamma) \subset q \circ \psi(T_0) \), where \( \psi \) is defined in (5.2). (The maximal torus \( q \circ \psi(T_0) \) is a finite index subgroup of its normalizer in \( \text{PLAut}^0(E_G) \).) Therefore, the action of \( \Gamma \) on \( \text{LAut}^0(E_G) \) factors through the conjugation action of the torus \( \rho(\Gamma) \).

To prove the second statement in the proposition, assume that the induced action of \( \Gamma \) on \( \text{LAut}^0(E_G) \) factors through the torus quotient \( \Gamma \longrightarrow T_\Gamma \). We will first show that the action of \( T_\Gamma \) on \( \text{LAut}^0(E_G) \) preserves a maximal torus.

Construct the semi-direct product \( \text{LAut}^0(E_G) \rtimes T_\Gamma \) using the induced action of \( T_\Gamma \) on \( \text{LAut}^0(E_G) \). We take a maximal torus
\[
\hat{T} \subset \text{LAut}^0(E_G) \rtimes T_\Gamma
\]
containing \( T_\Gamma \) (note that \( T_\Gamma \) is naturally a subgroup of \( \text{LAut}^0(E_G) \rtimes T_\Gamma \)). Finally, consider the intersection
\[
T_1 := \hat{T} \cap \text{LAut}^0(E_G)
\]
(note that $L\text{Aut}^0(E_G)$ is a normal subgroup of $L\text{Aut}^0(E_G) \rtimes T_\Gamma$). From its construction it is immediate that $T_1$ is a maximal torus of $L\text{Aut}^0(E_G)$ and $T_1$ is left invariant by the action of $T_\Gamma$ on $L\text{Aut}^0(E_G)$.

Consider
\[ G' := \psi^{-1}(T_1) \subset \text{Aut}^0(E_G), \]
where $\psi$ is the projection in (5.2). Since $\Gamma$ preserves $T_1 \subset L\text{Aut}^0(E_G)$, the action of $\Gamma$ on $\text{Aut}^0(E_G)$ preserves the subgroup $G'$ defined above. Note that $G'$ fits in an exact sequence
\[ e \to U\text{Aut}^0(E_G) \to G' \to T_1 \to e, \]
where $U\text{Aut}^0(E_G)$, as before, is the unipotent radical.

A maximal torus of $G'$ is a maximal torus of $\text{Aut}^0(E_G)$, and since $\Gamma$ is connected, an algebraic action of $\Gamma$ on a torus through automorphisms is trivial. Therefore, in view of Proposition 3.3 to prove the second statement in the proposition it suffices to show that $\Gamma$ preserves some maximal torus in $G'$.

Denote by $g'$ the Lie algebra of $G'$. The action of $\Gamma$ on $G'$ induces an action of $\Gamma$ on $g'$. Let $u$ (respectively, $t_1$) be the Lie algebra of $U\text{Aut}^0(E_G)$ (respectively, $T_1$). So the above exact sequence of groups give an exact sequence
\[ (5.3) \quad 0 \to u \to g' \overset{\beta}{\to} t_1 \to 0 \]
of Lie algebras.

Let
\[ \mathcal{V} \subset g' \]
be the subspace on which $\Gamma$ acts trivially. Note that $\mathcal{V}$ is a Lie subalgebra. The action of $\Gamma$ on $T_1$ is trivial (as the automorphism group of $T_1$ is discrete and $\Gamma$ is connected). Therefore, the induced action of $\Gamma$ on $t_1$ is trivial.

Since $\Gamma$ is reductive, any exact sequence of finite dimensional $\Gamma$–modules over $k$ splits, in particular, (5.3) splits. Since $t_1$ is the trivial $\Gamma$–module, we conclude that the restriction to the subalgebra $\mathcal{V} \subset g'$ of the projection $\beta$ in (5.3) is surjective.

Let $G_2 \subset G'$ be the Zariski closed subgroup generated by the subalgebra $\mathcal{V}$. Since $\Gamma$ acts trivially on $\mathcal{V}$ we conclude that $G_2$ is fixed pointwise by the action of $\Gamma$ on $G'$.

Since the projection of $\mathcal{V}$ to $t_1$ (by $\beta$ in (5.3)) is surjective, the subgroup $G_2$ projects surjectively to $T_1$. Take any maximal torus $T_2 \subset G_2$. Since the projection of $G_2$ to $T_1$ is surjective and the kernel of the projection $G' \to T_1$ is a unipotent group, we conclude that $T_2$ is a maximal torus of $G'$.

In other words, $T_2$ is a $\Gamma$–invariant maximal torus of $G'$. Since a maximal torus in $G'$ is a maximal torus in $\text{Aut}^0(E_G)$, Proposition 3.3 completes the proof of the proposition. □
It is easy to construct examples showing that the second statement in Proposition 5.1 is not valid for arbitrary connected algebraic group $\Gamma$.

Proposition 5.1 has the following corollary:

**Corollary 5.2.** If $\Gamma$ does not have a nontrivial torus quotient (for example, if it is unipotent or semisimple), and the action of $\Gamma$ on $L\text{Aut}^0(E_G)$ is nontrivial, then $E_G$ does not admit any $\Gamma$–equivariant reduction of structure group to $\hat{H}_0$, provided $\hat{H}_0 \neq G$.

**References**

[At] M. F. Atiyah, On the Krull–Schmidt theorem with application to sheaves, *Bull. Soc. Math. Fr.* **84** (1956), 307–317.

[BBN] V. Balaji, I. Biswas, and D. S. Nagaraj, Ramified $G$-bundles as parabolic bundles, *Jour. Ramanujan Math. Soc.* **18** (2003), 123–138.

[BP] I. Biswas and A. J. Parameswaran, Equivariant reduction to torus of a principal bundle, *K–Theory* (to appear).

[Gr] A. Grothendieck, Sur la classification des fibrés holomorphes sur la sphère de Riemann, *Amer. Jour. Math.* **79** (1957), 121–138.

[Hu] J. E. Humphreys, *Linear algebraic groups*, Graduate Texts in Mathematics, Vol. 21, Springer-Verlag, New York–Heidelberg–Berlin, 1987.

[Ku] S. Kumar, Equivariant analogue of Grothendieck’s theorem for vector bundles on $P^1$, *Perspectives in Geometry and Representation Theory — A tribute to C. S. Seshadri*, 500–501, Hindustan Book Agency, 2003.

[SS] T. A. Springer and R. Steinberg, Conjugacy classes. *Seminar on Algebraic Groups and Related Finite Groups*, 167–266, Lecture Notes in Mathematics, Vol. 131, Springer–Verlag, Berlin–Heidelberg–New York, 1970.

[St] R. Steinberg, *Conjugacy classes in algebraic groups*, Lecture Notes in Mathematics, Vol. 366, Springer–Verlag, Berlin–New York, 1974.

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