CONFIGURATION SPACES IN FUNDAMENTAL PHYSICS

Edward Anderson

DAMTP, Centre for Mathematical Sciences, Wilberforce Road, Cambridge CB3 OWA.

Abstract

I consider configuration spaces for \( N \)-body problems, gauge theories and for GR in both geometrodynamical and Ashtekar variables forms, including minisuperspace and inhomogeneous perturbations thereabout in the former case. These include many interesting spaces of shapes (with and without whichever of local or global notions of scale). In considering reduced configuration spaces, stratified manifolds arise. Three strategies to deal with these are ‘excise’, ‘unfold’ and ‘accept’. I show that spaces of triangles arising from various interpretations of 3-body problems already serve as model arena for all three. I furthermore argue in favour of the ‘accept’ strategy on relational grounds. Sheaf methods then become relevant in this case, as does the stratifold construct that pairs some well-behaved stratified manifolds with sheaves. I apply arguing against ‘excise’ and ‘unfold’ to GR’s superspace and thin sandwich, and to the removal of collinear configurations in mechanics. Non-redundant configurations are also useful in providing more accurate names for various spaces and theories. I also cover notions of distance between shapes, that some perturbative midisuperspace configuration spaces are simple and similar to minisuperspace ones, and similarities between CS (conformal superspace) and CS + V (including the global spatial volume).
1 Introduction

Given a physical system, configuration space \( q \) \([1]\) is the space of all its possible configurations \( Q^A \). The corresponding morphisms – the coordinate transformations of \( q \) – form the point transformations. In some settings, these are the scleronomic (time-independent) morphisms of the form \( q^A = f^A(Q^B) \), the space of which I denote by Point. In other settings, these are the rheonomic morphisms – time-dependent in the sense of \( q^A = f^A(Q^B, t) \) but \( t \) itself not transforming – the space of which I denote by Point\(_t\).

1.1 Configurations and configuration spaces in Mechanics

Example 1) Consider first ‘constellations’ of \( N \) labelled (possibly superposed) material point particles in \( \mathbb{R}^d \) with coordinates \( q_i^J \) [Fig 1.a]. These are taken together to form an \( N \)-d-dimensional configuration, the space of possible values of which form the configuration space \( q(N, d) = \mathbb{R}^{Nd} \). The corresponding mass matrix – alias kinetic metric – is

\[
M_{11} := m_1 \delta_{1J} \delta_{1J}.
\]

Example 2) Taking the centre of mass to be meaningless, there arise relative inter-particle separation vectors \( \ell_{IJ} := q_J - q_I \). Lagrange coordinates (Fig 1.b) are then some basis set made from the, which form an \( nd \)-dimensional configuration space \( v(N, d) = \mathbb{R}^{nd} \). However, the kinetic metric is no longer diagonal in these coordinates. This can be rectified by passing to Jacobi coordinates \([2, 3]\) \( \rho_{\underline{A}} \) [Fig 1.c]. These are a basis of relative inter-particle cluster separation vectors.

![Diagram of configurations for 3 particles in each of 1- and 2-d](image)

Figure 1: Coordinate systems for 3 particles in each of 1- and 2-d; Sec 2.1 justifies concentrating on these two particular cases, though the notions presented here for now indeed trivially extends to arbitrary \( N \) and \( d \). I consider 3 particles because relational nontriviality requires for one degrees of freedom to be expressed in terms of another, by which 2 particles are not enough. a) Absolute particle position coordinates \((q_1, q_2, q_3)\) in 1- and 2-d. These are defined with respect to, where they exist, fixed axes \( A \) and a fixed origin \( O \). b) Relative inter-particle (Lagrange) coordinates \( r = (q_J^I, I > J) \). Their relation to the \( q^I \) are obvious: \( r_{IJ} = q^I - q^J \). In the case of 3 particles, any 2 of these form a basis. No fixed origin enters their definition, but they are in no way freed from \( A \). c) Relative particle inter-cluster mass-weighted Jacobi coordinates \( \rho \), which are more convenient but still involve \( A \). \( \times \) denotes the centre of mass of particles 2 and 3.

Example 3) Use of other coordinatization for the configurations \( Q^A \) that form \( q \). For the examples given so far, these are curvilinear coordinates, e.g. spherical polar coordinates for each \( \mathbb{R}^3 \) factor in Example 1.

Example 4) More generally, configuration spaces can be curved manifolds, with configurations then being coordinates \( \theta \) and \( \phi \) that trace out the surface of the configuration space sphere \( S^2 \). This example also illustrates that configuration coordinates are not necessarily globally defined – \( \phi \) is not defined at the poles (\( \theta = 0, \pi \)); indeed it is well known that no other coordinates on spheres are globally defined: at least two different coordinate charts are required to cover the sphere.

Example 5) Constrained system \([1, 4, 5, 6]\). One can in fact view ‘taking out the centre of mass’ as imposing a zero total momentum constraint \( p := \sum I p^I = 0 \) that eschews absolute translations, and the rigid rotor as a particle confined to move on the surface of a sphere.

Example 6) One can furthermore eschew absolute rotations (Fig 1’s A), by far most simply done in the case of zero total angular momentum \( \Omega := \sum I (q^I \times p^I) = 0 \). These leads to relational configuration coordinates \([3, 7]\) as outlined in Sec 2.

Example 7) One can furthermore eschew absolute scale \([8, 9, 10, 7]\), corresponding either to a) separating out scale or b) to rendering scale meaningless by imposing zero dilational momentum \( v := \sum I q^I \cdot p^I = 0 \). Pure-shape configuration

---

1 use underline, and also sometimes lower-case Latin indices, to denote spatial vectors. I use upper-case Latin indices \( I, J, K \) for particle labels ranging from 1 to \( N \). I use lower-case Latin indices \( A, B, C \) for bases of relative separation labels, ranging from 1 to \( n := N - 1 \).

2 One can see from the forms of the given constraints that these are defined on phase space: the space of all possible values of the coordinates and the momenta. On the other hand, at least for the range of examples considered in this Article, reduced configuration spaces make good sense upon having taken into account (some subset of) constraints. Configuration space itself is already useful in a number of contexts (Sec 1.A) and rather under-studied as compared to phase space.
coordinates [8, 7] ensue, as also outlined in Sec 2. From the perspective of Examples 7) and 8), $q(N, d)$ is rather physically redundant as a configuration space.

Example 8) Jointed rods make it clear that eschewing absolute rotation does not get rid of all forms of rotation or of angular momentum, for such models retain meaningful rotation of some rods relative to others, and of corresponding relative angular momenta. This is also clear within Example 6) and 7)’s models, by considering e.g. the base and median parts of a relational triangle of particles, which are free to rotate relative to each other and to possess a corresponding relative angular momentum.

Some settings in which some of the given examples are useful are as follows.

A) The $N$-body problem, studied in particular in the case of Celestial Mechanics [2, 11], e.g. the Earth–Moon–Sun system or the solar system, but also with larger numbers of bodies in modelling globular clusters, and in ‘medium particle number’ Newtonian cosmology [12]. Most of these are usually modelled using point-particle models.

B) Molecular Physics [3] usually starts from a classical point-particle model, which is subsequently quantized. One situation often modelled assumes that the nuclei form a fixed scaled shape, whilst the lighter electrons fluctuate on a faster time-scale (Born–Oppenheimer and adiabatic approximations [13]).

C) Relational Particle Mechanics (RPM) [14, 15, 7, 16]: whole-universe point-models, which interpretation leads to a faster time-scale (Born–Oppenheimer and adiabatic approximations [13]). RPM comes in scaled and pure shape versions, as per Examples 7) and 8) respectively. In the scaled case, only relative angles and relative separations are meaningful, whereas in the pure-shape case only relative angles and ratios of relative separations are. In each case also only relative times are meaningful. These are model arenas for a number of closed universe, quantum cosmological and background independence features.

D) The rigid rotor is but one example of a problem involving rigid bodies. On the other hand, jointed rod problems are a type of non-rigid body problem, which can be used e.g. to model ‘the falling cat’ [17], and have applications to robotics and manufacturing [18].

1.2 Configuration spaces in field theory and GR

The values of a field over space at a given instant of time are another example of configuration.

Example 9) Consider Electromagnetism, and Yang–Mills Theory; configurations here are a 1-form $A_i$ and a set of internally-indexed 1-forms $A_{i\rho}$ respectively. While gauge theories involving extra scalars and/or fermions can be associated with these, Electromagnetism and Yang–Mills theories are already ‘vacuum’ gauge theories in their own right, with $U(1)$ and a more general Lie group gauge invariances respectively. The above configurations are then gauge fields in the Dirac alias data sense of gauge, as opposed to spacetime or the whole dynamical path senses, which involve more extended (spacetime) indices and domain of definition. See Sec 4 for the corresponding configuration spaces. In gauge theories, the gauge field is not only a 1-form but can furthermore be interpreted as a fibre bundle connection. These gauge theories are also examples of constrained theories, well-covered as such in [6]. E.g. in vacuo the Gauss constraint of Electromagnetism is

$$\varrho := \partial_i \pi^i = 0$$

and the Yang–Mills–Gauss constraint is

$$\varrho_P := D_i \pi^i_P = \partial_i \pi^i_P - g C_{RQP} A_Q^P \pi^R = 0$$

One could then additionally consider working with more physical (in this case gauge-invariant) configuration variables after reducing out the corresponding version of Gauss constraint. This amounts to quotienting out the corresponding gauge group; see Sec 8 for resultant configuration spaces (orbit and loop spaces).

Example 10) Major motivation for configuration space study comes from GR [21, 22, 23, 24, 25, 26, 27, 28, 29, 30]. Whereas GR’s spacetimes are pairs $(\mathfrak{M}, g_{\mu\nu})$ of topological spacetime manifolds and indefinite spacetime metrics respectively, GR’s configurations themselves are pairs $(\Sigma, h_{ij})$ for $\Sigma$ the spatial topological manifold and $h_{ij}$ a positive-definite spatial metric thereupon. Topological manifold $\Sigma$, is standardly taken to be fixed, and, in the current article to additionally be compact without boundary on Machian grounds (and often furthermore $\mathbb{S}^3$ in concrete examples). From a configuration space primality perspective [21, 31], the evolving $h_{ij}$ themselves form spacetime: from a spacetime primality perspective, they are a piece of $g_{\mu\nu}$ under the ADM decomposition [32]. The configuration space formed from the $h_{ij}$ on a given $\Sigma$ is then named Riem($\Sigma$) (after Riemann); see Sec 5 for details. This is another example of highly redundant configuration space, since GR is also a constrained theory [32], with

---

3In this Article, I use the convention of straight font for field quantities and slanty font for finite quantities. I also use $\pi^i$ and $\pi^i_P$ for the corresponding conjugate momenta, and, in the Yang–Mills case, coupling constant $g$, structure constants $C_{RQP}$ and covariant derivative $D_i$ in the fibre bundle sense, whose action on a vector is as per (3).

4This article assumes that the reader is familiar with fibre bundle mathematics and its application to Theoretical Physics (see e.g. [19, 20] for details).

5$h_{ij}$ has determinant $h$, inverse $h^{ij}$, covariant derivative $D_i$, Ricci scalar $R$ and conjugate momentum $p^{ij}$. $A$ is the cosmological constant.
GR momentum constraint \( M_i := -2D_j p^j_i = 0 \),

GR Hamiltonian constraint \( \mathcal{H} := N_{abcd} p^{ab} p^{cd} - \sqrt{h} (R - 2\Lambda) = 0 \)

for

DeWitt supermetric \( N_{abcd} := \frac{1}{\sqrt{h}} \{ h^{ac} h^{bd} - \frac{1}{2} h^{ab} h^{cd} \} \)

which is the inverse of the

GR configuration space metric \( M_{abcd} := \sqrt{h} \{ h^{ac} h^{bd} - h^{ab} h^{cd} \} \).

Now the GR momentum constraint can be straightforwardly interpreted in terms of \( \text{Diff}(\Sigma) \) freedom. The information contained in \( h_{ij} \) can then be considered as split into 3 degrees of freedom per space point (dofssp) of unphysical \( \text{Diff}(\Sigma) \) information and a core of 3 dofssp of partly-physical information: the 3-geometry. This is still but partly-physical due to the GR Hamiltonian constraint not yet having been taken into account; moreover, it is far less clear how to take this into account [33, 34].

\[
\text{Superspace}(\Sigma) := \text{Riem}(\Sigma)/\text{Diff}(\Sigma)
\]

is then the space of 3-geometries, which Wheeler greatly encouraged the study of [21]; see Sec 9 for an outline of what has been determined about this configuration space since. As part of that, Wheeler asked the very natural follow-up question of what is “2/3 of superspace”?

Two geometrically natural possibilities for this ‘2/3 of superspace’ subsequently considered by York are as follows.

1) Conformal superspace [25, 29] \( \text{CS}(\Sigma) \) is the space of all conformal 3-geometries on a fixed \( \Sigma \); it corresponds to the maximal slice condition \( p = 0 \) being imposed;\footnote{\text{Conf}(\Sigma) \text{ are the conformal transformations on } \Sigma; \text{VPConf}(\Sigma) \text{ are the global spatial volume-preserving conformal transformations; these still change the local scaling part of the shape. Then } \text{Conf}(\Sigma) = \text{Dil} \times \text{VPConf}(\Sigma) \text{ for } \text{Dil} \text{ the globally-constant scalings (which change the overall volume of space but not the local shapes). Finally, the semidirect product group } \mathfrak{G} = \mathbb{R} \ltimes \mathfrak{H} \text{ is given by } (n_1, h_1) \circ (n_2, h_2) = (n_1 \varphi_{h_1}^{-1} h_2, h_1 h_2) \text{ for } \mathbb{R} \ltimes \mathfrak{H} \text{ a subgroup of } \mathfrak{G} \text{ and } \varphi : \mathfrak{H} \to \text{Aut}(\mathbb{R}) \text{ a group homomorphism.}}

\[
\text{CS}(\Sigma) = \text{Superspace}(\Sigma)/\text{Conf}(\Sigma) = \text{Riem}(\Sigma)/\text{Conf}(\Sigma) \times \text{Diff}(\Sigma).
\]

2) \{CS + V\}(\Sigma) [37, 38] adjoins to this a solitary global degree of freedom – the spatial volume of the universe; it corresponds to the constant mean curvature slicing condition \( p/\sqrt{h} = \text{const} \) being imposed;

\[
\{CS + V\}(\Sigma) = \text{superspace}(\Sigma)/\text{VPConf}(\Sigma) = \text{Riem}(\Sigma)/\text{VPConf}(\Sigma) \times \text{Diff}(\Sigma).
\]

It is also then natural to consider the simpler if more redundant configuration space conformal Riem

\[
\text{CRIem}(\Sigma) := \text{Riem}(\Sigma)/\text{Conf}(\Sigma);
\]

the space of conformal equivalence classes of Riemannian metrics [22], which can be represented e.g. by the unit-determinant metrics

\[
u_{ij} := h_{ij}/h^{1/3}.
\]

For further mathematical detail, see Sec 5.3 concerning \( \text{CRIem}(\Sigma) \), and Sec 9 as regards \( \text{CS}(\Sigma) \) and \{CS + V\}(\Sigma). N.B. also that 1) and 2) bear close ties to by far the most successful methods for approaching the initial-value problem in GR [37, 39]. Finally, bear in mind, that the ‘2/3 of superspace’ picked out by 1) and 2) might not be directly related to the ‘2/3 of superspace’ picked out by \( \mathcal{H} \) itself. Let us call the latter \( \text{True}(\Sigma) \) whilst acknowledging that for now this is but a formal naming rather than a space of known and understood geometry.

Example 11) Homogeneous GR configurations form configuration spaces widely known as minisuperspaces. Relational nontriviality [Fig 1] then dictates that isotropic such solutions contain at least one matter degree of freedom; with quantization in mind this is to be not phenomenological matter but fundamental matter [40] – most straightforwardly a minimally-coupled scalar field. Solutions with anisotropic degrees of freedom [41, 24], however, can be considered in vacuo.

Example 12) Perturbative inhomogeneous GR configurations are required for structure formation – relevant to observational cosmology – and to tap into numerous diffeomorphism-nontrivial matters [42]. A particular such model consists of inhomogeneous perturbations about the spatially-S\(^3\) isotropic minisuperspace with single minimally-coupled scalar field matter; see Sec 7 for an unreduced treatment and Sec 10 for a reduced one. This particular model becomes the Halliwell–Hawking model [43] at the semiclassical level.

Example 13) GR in Ashtekar variables. These are related to the above geometroodynamical variables by a canonical transformation [44] (and various extensions: this is now a complexified GR with degenerate 3-metrics allowed).
new configurational variable is now a $SU(2)$ Yang–Mills 1-form $A_i^{pq}$, again reinterpretable as a connection, and often
then furthermore recast as a holonomy or loop (Sec 8.2). This formulation’s constraints are then
\[ \phi_{pq} := D_i E_{ij}^{pq} := \partial_i E_{ij}^{pq} + \{[E_i, E_j]^{pq}] = 0 , \quad (13) \]
\[ \phi_i := \text{tr}(E_i F_{ij}) = 0 , \quad (14) \]
\[ h_i := \text{tr}(E_i E_j F_{ij}) = 0 . \quad (15) \]

(13) is an $SU(2)$ Yang–Mills–Gauss constraint. (14) and (15) are the polynomial forms now taken by the GR momentum
and Hamiltonian constraints respectively. One can see that (14) is indeed associated with momentum since it is the
condition for a vanishing Yang–Mills–Poynting vector. Again, this formulation’s version of the Hamiltonian constraint
(15) lacks such a clear-cut interpretation; it is simpler due to being polynomial in this approach’s canonical variables.

Here loops and knots [45] are increasingly reduced configurations, taking into account (13) and also (14) respectively.
See Sec 11 for an outline of the corresponding configuration spaces.

### 1.3 Stratified manifolds arise, and three attitudes to them

I next further motivate the paper on a common theme observed in studying the above examples that pertain to Funda-
mental Physics (or useful model arenas thereof). Namely, that stratified manifolds appear (first widely studied in [23]:
superspace is a stratified manifold). Note that this also happens in study of reduced phase spaces; thus much of what
is said here carries over to symplectic stratified manifolds. These are a type of quotient, see Appendix B for quotient
spaces and more specifically for stratified manifolds themselves. Manifolds are Hausdorff, second-countable and locally
Euclidean. In general stratified manifolds are none of these, though this paper mostly focuses on stratified manifolds
that happen to be Hausdorff and second-countable.

Stratified manifolds having appeared, three strategies for dealing with them are as follows.

Strategy A) **Excise.** This is crude and unphysical. 3-d collinearities example. But by removing all bar the principal
stratum, then fibre bundle mathematics applies.

Strategy B) **Unfold.** But is the unfolding physically meaningful, and is it unique?

Strategy C) **Accept,** which points to harder mathematics being required. Prima facie, it is ‘accept’ that is accord
with Leibniz’s Identity of Indiscernibles. If one takes this path, then fibre bundle theory is not general enough due to
heterogeneity amongst what had been homogeneous fibres.

First note that that the 3-body problem already serves as an arena for this, justifying Sec 2 being quite extensive. [Model
arenas do well to i) exhibit the desired feature and ii) elsewise be as simple as possible]. Furthermore, analysing various
works in terms of the above classification, and favouring C) adds useful interpretation to a number of GR results. E.g.
Fischer’s work [26] is an unfold strategy. Also, one of the two locality criteria used in the Thin Sandwich Theorem of
Bartnik and Fodor [46] (Sec 9.5) can be viewed as an excision. Moreover, the Thin Sandwich has further significance as
one of the Problem of Time facet: [33].

Finally, returning to the breakdown of the scope of fibre bundle methods, I note that more general sheaf methods can
be applied (Appendices B.7 and C). These are rather new methods in the range of theories considered in this Article.

### 1.4 Further motivation for configuration space

0) Understanding configurations and configuration spaces, especially kinematically or dynamically non-redundant ones,
is useful in providing more accurate names for various spaces and theories.

I) Addressing very natural questions along the lines of ‘which shapes are more alike than others?’ or ‘how can one
quantify that one space is more inhomogeneous than another?’ These can be approached by notions of distance between
shapes in shape space, or, more generally, by notions of distance on configuration spaces. For an outline, see Sec 3 for
positive-definite $q$ and 9.7 for indefinite $q$.

II) Addressing questions along the lines of ‘how probable are particular (ranges of) shapes?’. These can be approached by
viewing configuration space as a sampling space, or Kolmogorovian probability space, upon which to build theories
of Probability and Statistics. E.g. Kendall’s (pure) shape statistics [8] is a particular instance of geometrical statistics,
based on particular geometries that so happen to be [9, 47] the configuration spaces for Barbour’s pure-shape RPM [15].
Also note that such studies are not just applicable to Theoretical Physics situations; to date many such considerations
have been in the fields of Biology and Archaeology [8, 48].

---

8The typewriter face capital indices here denote spinorial $SU(2)$ indices. Its conjugate momentum $E_{ij}^{pq}$ is now a 3-bein:
related to the 3-metric by $h_{ij} = -\text{tr}(E_i E_j)$. This is now indeed a conjugate momentum, despite its relation to the previous configurational variable $h_{ab}$
because a canonical transformation has been applied. tr denotes the trace over these. $[\cdot, \cdot]$ denotes the classical Yang–Mills-type commutator.
Note that due to the specific form of $A_i$ and $E_a$, $h_{ij}$ is in fact complexified, i.e. pointwise in $GL(3, \mathbb{C})$ rather than in $GL(3, \mathbb{R})$. Real Ashtekar variables have a more complicated form of $H_i$, but loop variables still apply to these.
III) Addressing questions such as ‘how much information is contained in shapes?’, via considering notions of information on configuration spaces [34].

IV) Configuration spaces are useful in timeless approaches for use in whichever of closed-universe or quantum gravitational situations. This is the case firstly in pure solipsism [49, 47], in which the $Q^A$ are all. It is also the case in approaches for which there is no time for the universe as a whole at the primary level [50, 7, 51]. These nevertheless allow for a notion of time to emerge from change in configuration at the secondary level (so $dQ^A$ makes sense alongside the $Q^A$ themselves). Note that in both these cases, it is Point rather than Point, which are appropriate morphisms. In fact, I–III) are how to equip a solipsist worldview with concrete mathematics, though such timeless calculations can also be part of doing Physics within the more extensive worldviews of b) and VI).

V) Reduced configuration variables appear in the configuration space restriction of some notions of observables or beables [52].

VI) Histories [53] – used in a further range of problem of time strategies and foundational approaches to Quantum Cosmology – can be viewed as strings of configurations.

VII) Dynamics can be re-envisaged as a path on configuration space [1, 24, 50, 7] (this was already used historically by Jacobi and extended by Synge). Then by knowing the geometrical meaning of the configuration to configuration space correspondence, one can read off from such a path the sequence of shapes a given evolution goes through.

VIII) QM in fact unfolds on configuration space. With configuration space acquiring a geometrical character, then such as geometric quantization is to be used; indeed this approach can be heavily centred upon configuration space mathematics [55]. In reduced quantization approaches, quotient configuration spaces that are stratified manifolds feature directly prior to quantization. On the other hand in Dirac quantization approaches, quotienting out linear constraints is postponed until after these have been promoted to quantum equations. Choices of operator-ordering have also been tied to priorly understanding the underlying configuration space geometry [56, 24, 34].

IX) Generalized configuration spaces [35, 36, 34] enter into consideration upon letting a wider range of structures than usual be dynamical and consequently quantum-mechanically fluctuate. These have a wider still range of mathematical structures than just geometries (e.g. the space of topological spaces on a fixed set form a lattice). In this way, an even wider range of notions of distance and information, probability and statistics theories, timeless formulations, dynamics and quantum theories arise.

1. **Configuration space geometry: Mechanics**

2.1 Mechanics and RPM configuration spaces

In the simplest (and relationally redundant) approach to Mechanics, one’s incipient notion of space (NoS) is absolute space $\mathfrak{a}(d)$ of dimension $d$. This is usually taken to be $\mathbb{R}^d$ equipped with standard Euclidean inner product alias metric. The corresponding configuration space $q(N, d)$ is then just $\mathbb{R}^{Nd}$, i.e. itself a Euclidean space but now of dimension $Nd$. In this Article, I consider just the case of equal masses (see [7] for discussion of other cases).

As regards less physically redundant presentations, various possibilities for physically-irrelevant groups of transformations $\mathbb{G}$ are as follows. Translations $\text{Tr}(d)$ forming the noncompact Abelian Lie group $(\mathbb{R}^d, +)$ of dimension $d$, rotations $\text{Rot}(d) = \text{SO}(d)$ i.e. the compact non-Abelian special orthogonal group $\text{SO}(d)$ of $d \times d$ matrices of dimension $d(d − 1)/2$, and dilations $\text{Dil}$ forming the $d$-independent noncompact commutative group $(\mathbb{R}^+, \cdot)$ of dimension 1. Particular further combinations of these then include the Euclidean group $\text{Eucl}(d) = \text{Tr}(d) \times \text{Rot}(d)$ and the similarity group $\text{Sim}(d) = \text{Dil} \times \text{Eucl}(d)$. Strictly speaking, $\text{Eucl}(d)$ and $\text{Sim}(d)$ are the ‘proper’ versions of these groups due to not being taken to include the discontinuous reflections. Then $\dim(\text{Eucl}(d)) = d(d + 1)/2$ and $\dim(\text{Sim}(d)) = d(d + 1)/2 + 1$.

The subsequent quotient spaces $q/\mathbb{G}$ are as follows, Absolute position is rendered meaningless by passing to relative space $\mathfrak{r}(N, d) = q(N, d)/\text{Tr}(d) = \mathbb{R}^{Nd}$ for $n = N − 1$. Moreover, this quotienting is devoid of mathematical structure or any extra analogy with GR, so starting from $\mathfrak{r}(N, d)$ makes for a clearer presentation in Fig 2. The diagonal form for the kinetic matrix for this in relative Jacobi coordinates is $\mu_{ijAB} := \mu_A \delta_{ij} \delta_{AB}$, for $\mu_A$ are the corresponding Jacobi inter-particle cluster reduced masses $\mu_A$. E.g. for 3-body case, these take the form

$$\mu_1 = \frac{m_2 m_3}{m_2 + m_3} \quad \text{and} \quad \mu_2 = \frac{m_1(m_2 + m_3)}{m_1 + m_2 + m_3}.$$  \hspace{1cm} (16)

The $\mu_A$ I use are furthermore mass-weighted, so $\sqrt{\mu_A}$ has already been absorbed into each, and so the final kinetic metric is just an identity array with components $\delta_{ij} \delta_{AB}$.

If absolute axes are also to have no meaning, the configuration space one is left with is

$$\mathfrak{R}(N, d) := \mathfrak{r}(N, d)/\text{Rot}(d) = q(N, d)/\text{Eucl}(d).$$  \hspace{1cm} (17)

I denote configuration space dimension by $k$. In the above case, $k = nd − d(d − 1)/2 = d(2n + 1 − d)/2$, i.e. $N − 1$ in 2-d, $2N − 3$ in 2-d and $3N − 6$ in 3-d. If, instead, absolute scale is also to have no meaning, then the configuration

9Polarizations, more generally, are choices of a suitable half-set of phase space variables [54]. Moreover, arguments for $q$ primality [21, 55] may extend to the possibility of configuration space being a privileged polarization.

10See Appendix B for a general outline of quotient spaces.

11In general this refers to naïve or largest dimension, since the outcome of quotienting in general has strata with a variety of dimensions.
space is *preshape space* \[8\]  
\[ \mathcal{P}(N, d) := \mathcal{V}(N, d)/\text{Dil}, \]  
with \( k = nd - 1 \). If both absolute orientation and absolute scale are to have no meaning, then the configuration space is Kendall’s \[8\]  
\[ \mathfrak{s}(N, d) := \mathcal{Q}(N, d)/\text{Sim}(d) \]. \hfill (18)

Now \( k = N d - \{d(d + 1)/2 + 1\} = d(2n + 1 - d)/2 - 1 \), i.e. \( N - 2 \) in 2-\( d \), 2\( N - 4 \) in 2-\( d \) and 3\( N - 7 \) in 3-\( d \). Note also that \( \mathcal{P}(N, 1) = \mathfrak{s}(N, 1) \), since there are no rotations in 1-\( d \). The above quotient spaces are taken to be not just sets but also normed spaces, metric spaces, topological spaces, and, where possible, Riemannian geometries. Their analogy with GR’s configuration spaces is laid out in Fig 2, along with a summary of the specific geometrical forms these take in various simpler cases.

![Fig 2](image)

Figure 2: a) This Sec’s specific sequence of configuration spaces, as a useful model arena for GR’s in b). See c) to e) respectively for the furtherly simplified cases of 1-\( d \), 2-\( d \), and 3 particles in 2-\( d \), whose forms are derived in the next Subsection.

### 2.2 Picking out the triangleland example

I begin with the pure-shape RPM’s, since these are geometrically simpler than scaled RPM’s, and furthermore occur as subproblems within the latter. Fig 3.a)-b) tabulates configuration space dimension \( k \) in , so as to display inconsistency, triviality, and relational triviality by shading. I follow this up identifying tractable topological manifolds and metric geometries in Fig 3.c)-d) \[9\].

Note that \( N \)-stop metrolands already possesses notions of localization, clumping, inhomogeneity, structure and hence structure formation. Contrast with how in GR these only appear in much more complicated Midisuperspace models. \( N \)-a-gonlands have only distance-ratio structure but also relative-angle structure, as well as the further Midisuperspace-like feature of nontrivial linear constraints. Contrast with how in GR, linear constraints on the one hand, and localization, clumping, inhomogeneity and structure on the other, are interlinked. This is because in GR both follow from spatial derivatives being nontrivial. On the other hand, these two sets of notions are separable in RPM’s, with 1-\( d \) RPM’s then serving to study the former in isolation from the latter.

The \( N \)-stop metroland shape space possesses the *hyperspherical metric*  
\[ ds^2 = \sum_{n=1}^{D} \prod_{m=1}^{D} \sin^2 \theta_m d\theta^2_m \]  
\hfill (19)

These \( \theta_m \)’s are related to ratios of the \( \rho_A \) in the usual manner in which hyperspherical coordinates are related to Cartesian ones \[7\].

On the other hand, the \( N \)-a-gonland shape space has the *Fubini-Study metric* \[8\]  
\[ ds^2 = \{(1 + ||Z||^2)|dZ||^2_c - ||(1 + ||Z||^2||^2_c)^2 \]  
for \( c \) here denoting complex inner product norm, with \( Z \) running over \( n - 1 \) copies of the complex plane. \( Z_{\rho} = R_\rho \exp(i\Phi_\rho) \) – a multiple \( C \) plane polar coordinates version of ratios of the \( \rho_c \), with \( \Phi_\rho \) relative angles between \( \rho_A \) and \( \rho_b \) ratios of magnitudes \( \rho_A \) \[7\].

N.B. both of the above metrics are written in coordinates standard to each of these geometries (hyperspherical angles and inhomogeneous coordinates \[20\] respectively). Moreover, in the present RPM setting these coordinates can be traced back to the spatial coordinates describing the particles themselves: see \[7\]. N.B. also that, as mechanical theories, RPM’s have positive-definite kinetic arc elements, which are significantly different from GR’s indefinite one (that is then inherited by this Article’s other principal model arenas: minisuperspace and inhomogeneous perturbations thereabout).

Next, a generalized notion of *cone* over some topological manifold \( \mathfrak{M} \) is denoted by \( C(\mathfrak{M}) \) and takes the form
Figure 3: a) and b) are pure-shape and scaled RPM’s configuration space dimensions $k$ respectively. c) and d) are the corresponding topological manifolds ([7] summarizes further topological results about RPM configuration spaces). Whilst this gives 3 tractable series – see [8], including for the ‘Casson diagonal’ – only the shade two column groups admit tractable metrics as well. I term 1-d RPM universe models $N$-stop metrolands since their configurations look like an underground train line. I term 2-d RPM universe models $N$-a-gonlands since each point in them is a planar $N$-sided polygon. The mathematically highly special $N = 3$ case of this is triangleland, and the first mathematically-generic $N = 4$ case is quadrilateral land [16]. See [7] for the basic Algebraic Topology of basic RPM configuration spaces.

$$C(\mathfrak{M}) = \mathfrak{M} \times [0, \infty)/\sim.$$  

(21)

$\sim$ here means that all points of the form $\{p \in \mathfrak{M}, 0 \in [0, \infty]\}$ are ‘squashed’ or identified to a single point termed the cone point, 0. Then at the metric level, given a manifold $\mathfrak{M}$ with a metric with line element $ds$, the corresponding cone has a natural metric of form

$$ds_{\text{cone}}^2 := d\rho^2 + \rho^2 ds^2.$$  

(22)

Then relational space is just the cone over shape space [10, 7], which cone structure makes clear the scale–shape split formulation of scaled RPM. Furthermore, $C(\mathfrak{S}(N, 1))$ are just $\mathbb{R}^n$.

For triangleland, the additional coincidence $\mathbb{CP}^1 = S^2$ ‘doubles’ the amount of geometry and linear methods available (and the spherical ones are both simpler and better-known than complex-projective ones). Here,

$$ds^2 = d\Theta^2 + \sin^2 \Theta d\Phi^2;$$  

(23)

see Fig 4 for the meanings of these coordinates.

The scaled case is just the cone over the pure-shape case’s configuration space, allowing for that case to be covered also.

$$ds^2 = d\rho^2 + \rho^2 (d\Theta^2 + \sin^2 \Theta d\Phi^2)/4 = \{dI^2 + I^2 (d\Theta^2 + \sin^2 \Theta d\Phi^2)\}/4I;$$  

(24)

Here, the configuration space radius $\rho := \sqrt{\rho_1^2 + \rho_2^2}$ (alias hyperradius in the Molecular Physics literature); this is also the square root of the moment of inertia, $I$. $C(\mathfrak{S}(3, 2))$ is also $\mathbb{R}^3$, albeit not with the flat metric. It is, however, conformally flat [7] – just apply conformal factor $4I$ to the second form of (24).

Another useful observation is the known forms of the corresponding isometry groups, $\text{Isom}(\mathfrak{S}(N, 1)) = \text{Isom}(\mathbb{S}^{n-1}) = SO(n)$, $\text{Isom}(\mathfrak{S}(N, 2)) = \text{Isom}(\mathbb{CP}^{n-1}) = SU(n)/\mathbb{Z}_n$ [among which triangleland is further distinguished by $\text{Isom}(\mathfrak{S}(3, 2)) = \text{Isom}(\mathbb{CP}^1) = SU(2)/\mathbb{Z}_2 = SO(3) = \text{Isom}(\mathbb{S}^2)$] and $\text{Isom}(\mathfrak{T}(N, 1)) = \text{Isom}(\mathbb{R}^n) = \text{Eucl}(n)$. 


Figure 4: a) For 3 particles in 1-d, just use the magnitudes of the two Jacobi coordinates. b) For 3 particles in 2-d, use the magnitudes of the two Jacobi coordinates and define $\Phi$ as the ‘Swiss army knife’ angle $\arccos(\rho_1 \cdot \rho_3 / \rho_1 \rho_3)$. This is a relative angle, so, unlike the $\rho$, these three coordinates do not make reference to absolute axes A. Pure-shape coordinates are then the relative angle $\Phi$ and some function of the ratio $\rho_2 / \rho_1$. In particular, $\Theta = 2\arctan(\rho_2 / \rho_1)$ is then the azimuth to $\Phi$’s polar angle.

Finally, Atomic and Molecular Physics provide a number of useful parallels for the spherical cases [7]. On the other hand, N-a-gonlands can draw from [16] Geometrical Methods, Shape Statistics, and the standard Representation Theory of $SU(N)$.

2.3 3-particle configuration spaces in more detail

Consider first the topological-level configurations. Here the only distinct 3-particle shapes are the double collision D and the generic other configuration. If the particles are labelled, the D can furthermore be distinguished by which particle remains out of the collision. One can also choose whether mirror images are identified provided that the dimension is low enough that the configurations cannot be rotated into each other. These considerations give four topological 3-stop metrolands (Fig 5) and four topological trianglelands (Fig 7).

One can additionally choose whether to model the particles as indistinguishable. Finally, for small enough physical dimension, one additionally has the choice of whether to identify mirror image configurations. These two features account for the quadrupling of configuration space types in Fig 5.

Consider next the metric level configurations. 3 particles on a line now have continua of distinguishable non-D configurations. These include a further distinguished notion of merger M: a configuration in which the third particle coincides with the centre of mass of the other two: Fig 5.a). In configuration space, these sit in the mid-points of the arcs between adjacent D’s, so e.g. the most extensive 3-stop metroland forms a ‘clock-face’.

On the other hand, triangles additionally have i) a notion of collinear configurations C, either side of which the triangle is ordered clockwise or anticlockwise: Fig 5.b). ii) A notion of isosceles configurations I, either side of which the triangle is right or left leaning: Fig 5.c). iii) A notion of regular configurations ($I_1 = I_2$: equality of base and median partial moments of inertia, or equality of base and median themselves in mass-weighted coordinates). Either side of these, the triangle is sharp or flat: Fig 5.d).

Then in triangleland, C is the equator great circle, dividing the shape sphere into hemispheres of clockwise and anticlockwise ordered triangles. There are then 3 notions of each of isosceles I and regular R, corresponding to the 3 ways of labelling ‘base’ and ‘apex’. These are all meridian great circles, alternating between I and R and evenly spaced out to form the pattern of the ‘zodiac or 12-segmented orange’. Each I divides the shape sphere into hemispheres of left and right leaning triangles, and each R into hemispheres of sharp and flat triangles. All 6 of these great circles intersect at the poles, which are equilateral triangles E (I denote the orientation reversed equilateral triangle pole by $\mathbf{E}$). These
Figure 6: Metric level types of configuration for 3 particles in 1- and 2-d. ‘Tight’ is used as in ‘tight binary’.

Figure 7: Triangleland configuration spaces at the metric level. a) The sphere. b) The $S^2/\mathbb{Z}_3$ orbifold. c) The hemisphere with edge: an example of stratified manifold. f) The hemisphere with edge quotiented by $\mathbb{Z}_3$ is a stratified orbifold. The coordinate range involved here is also Kendall’s spherical blackboard [60]. Note that orbifolds and especially stratified manifolds play a significant role in Appendix B and Sec 9, alongside Sec 1.3’s deliberation of whether to excise, unfold or accept strata, which is sketched out here in b), d), e), f), g) and h).

In 2-d, mirror image identification is optional: a) and b) are both viable options. In 3-d, however, rotation out of the plane sends one mirror image to the other, so a) ceases to be a valid option. As regards stratification, 1-d has no capacity for isotropy groups of different dimension, whereas shape spaces for 2-d shapes avoid stratification issues due to only involving $SO(2) = U(1)$, which acts the same on C and non-C configurations. However, in 3-d the C have only an $SO(2)$ subgroup of the $SO(3)$ acting upon them, so there is stratification. In 3-d also, the inertia tensor has zero eigenvalues for the C, causing mathematical complications (these prevent inversion of kinetic metric and lead to curvature singularities). That is one mathematical reason for excision (Fig 7.d), along with a physical reason to not...
want to: the C configurations that are quite clearly physically acceptable.

A second option is to accept the stratification (Fig 7.c).

A third possibility is to unfold the equator, by introducing an extra angular coordinate that parametrizes the hitherto unused rotation about the collinearity axis. At the level of configuration space, this has the effect of blowing up the equator into a torus: the ‘hemisphere with thick edge’ of Fig 7.e). However, within the point-particle model setting, the value of this extra angular coordinate is not physically meaningful, providing physical and philosophical reasons not to take this path.

Moreover, a gap now becomes apparent in the assumption made so far that unfolding is bereft of physical content, due to the following possibility.

Strategy D) unfolding purely by enhanced physical modelling.

This is clear at this stage through contemplating cases in which the point particles are but modelling approximations for more general bodies of finite extent. Then their centres of mass being aligned does not alter the isotropy group in question. However, enhanced physical modelling would not be expected to get round how quotienting in general does not preserve local Euclideaness (or Hausdorffness or second countability). I.e. there is no guarantee that increasing modelling accuracy will be reflected by a successful unfolding of the reduced configuration space stratified manifold into a manifold.

As regards the corresponding relationalspaces, 3-stop metroland’s is trivially \( \mathbb{R}^2 \), indeed N-stop metroland’s is \( \mathbb{R}^n \). In each case it is entirely clear how to represent an \( n \)-sphere within \( \mathbb{R}^n \). The \( n_A \) play the role of Cartesian directions. What plays this role for \( \mathbb{S}^2 \) within \( \mathbb{R}^4 \)? Here there are four components of \( n_A \); how does one relate these to an \( \mathbb{R}^3 \)? It turns out that \( \mathbb{R}^4 \rightarrow \mathbb{S}^3 \rightarrow \mathbb{S}^2 \rightarrow \mathbb{R}^3 \) occurs, where the second step is the Hopf map. Thus the Hopf–Dragt quantities \([61, 3]\) arise (Dragt is the name used in the Molecular Physics literature):

\[
\begin{align*}
    dra_x &= \sin \Theta \cos \Phi = 2n_1 n_2 , \\
    dra_y &= \sin \Theta \sin \Phi = 2n_1 n_2 \sin \Phi = 2n_1 \cdot n_2 , \\
    dra_z &= \cos \Theta = n_2^2 - n_1^2 .
\end{align*}
\]

[The 3 component in the first of these indicates the component in the fictitious third dimension of this cross-product, and \( n_A := \rho_A / \rho \).] These appear as ‘ubiquitous quantities’ \([62]\) in studying the relational triangle, and are indeed Kuchař observables for that problem \([52]\). They can be interpreted as follows.

\( dra_x \) is a measure of ‘anisoscelesness’ \( aniso \): departure from the underlying clustering’s notion of isoscelesness, c.f. anisotropy in Sec 6. It is specifically a measure of anisoscelesness in that Aniso per unit base length in mass-weighted space is the \( l_1 - l_2 \) indicated in Fig 8.a). I.e., it is the amount by which the perpendicular to the base fails to bisect it (which it would do were the triangle isosceles).

\( dra_y \) is a measure of noncollinearity. Moreover this is actually clustering-independent, known in Molecular Physics as a ‘democracy invariant’ \([3]\). It is furthermore equal to \( 4 \times \text{area} \) (the area of the triangle per unit \( I \) in mass-weighted space), which is lucid enough to use as notation for this quantity. In comparison, in the equal-mass case

\[
    \text{physical area} = \frac{l_1 \sqrt{3}}{m \text{area}} .
\]

Finally, \( dra_z \) is an ellipticity, ellip: the difference of the two ‘normalized’ partial moments of inertia involved in the clustering in question, i.e. that of the base and that of the median. In contrast to \( aniso \), this is clearly a function of pure ratio of relative separations rather than of relative angle.

\[\text{Figure 8: a) Setting up the definition of anisoscelesness quantifier, and b)-d) interpreting the three Hopf–Dragt axes in terms of the physical significance of the planes that they are perpendicular to.}\]

Maximal collisions are singular for both 2- and 3-d RPM’s. E.g. for scaled triangleland, the Ricci scalar is \( R = 6/I \).

Finally, pure-shape triangleland has the maximal three Killing vectors, the ‘axial’ \( \partial / \partial \Phi \) now corresponding to invariance under change of relative angle. Scaled triangleland has six conformal Killing vectors: \( 3 \partial / \partial dra^i \) and \( 3 dra^j \partial / \partial dra^i - dra^i \partial / \partial dra^j \).
3 Notions of distance on configuration spaces

One can build various such from \( q \)'s kinetic metric's norm and inner product \( M \) (at least if this is positive definite) [8, 14, 63, 7]

\[
\text{(Kendall Dist)} = (Q, Q)_M, \\
\text{(Barbour Dist)} = ||dQ||_M^2, \\
\text{(DeWitt Dist)} = (dQ, dQ')_M.
\]

One general if formal and indirect method to incorporate \( g \)-invariance involves acting with \( g \) on a given object, and then using a \( g \)-all' operation which involves the whole of \( g \). Perhaps the most well-known example of this two-step procedure is group averaging. Then

\[
\text{(Kendall \( g \)-Dist)} = (Q, \overrightarrow{g}Q')_M, \\
\text{(Barbour \( g \)-Dist)} = ||d_gQ||_M^2 \quad \text{and} \\
\text{(DeWitt \( g \)-Dist)} = (\overrightarrow{d_g}Q, \overrightarrow{d_g}Q')_M.
\]

[Here \( d_gQ := dQ - \overrightarrow{d_g}Q \) is the best-matched derivative [14].] One can then apply a suitable \( g \)-all move to each of these, such as sum, integrate, average, inf, sup or extremum. Note that (32) differs from the other two in using a finite group action to the other two cases’ infinitesimal ones. On the other hand, (32) and (34) compare two distinct inputs whereas (33) works around a single input. Comparers have a further issue: if \( M = M(Q) \), does one use \( Q \) or \( Q' \) in evaluating \( M \) itself? This situation did not arise in Kendall’s context, but it did in DeWitt’s; he resolved it (Sec 9.7) in the symmetric manner, i.e. making used of \( Q_1 \) and \( Q_2 \) to equal extents.

In some cases, one might instead be able to work directly with, or reduce down to, \( q / g \) objects, in which case there is no need for the above indirect construct. One would then make use of the relational or reduced configuration space geometry \( M \) itself.

Note also the dichotomy between direct comparisons of two configurations as per above, and performing intrinsic computations from each configuration piecemeal, e.g.

\[
\iota : q \longrightarrow \mathbb{R}^P,
\]

and then comparing these computations.\(^{12}\) In the latter case, one can consider using norms in the space of computations that is mapped into (the \( p \)-dimensional Euclidean metric in the above example). Note however that the outcome of doing this may well depend on the precise quantity under computation. Also \( \iota \) will in general has a nontrivial kernel, by which the candidate \( \iota \)-Dist would miss out on the separation property of bona fide distances. If this separation fails, one can usually (see e.g. [64]) quotient so as to pass to a notion of distance. [Though sometimes this leaves one with a single object so that the candidate notion of distance has collapsed to a trivial one.] Also it is sometimes limited or inappropriate to use such a distance if it is the originally intended space \( \mathfrak{X} \) and not the quotient that has deeper significance attached to it.

A range of candidate \( \iota \)'s for the GR case are provided in Sec 9.7. \( \iota \)'s can again be directly or indirectly \( g \)-invariant; indeed that is one way to select amongst the vast number of possibilities for \( \iota \)'s. Other selection criteria include extendability to unions of configuration spaces, physical naturality, and recurrence of the structure used in other physical computations E.g. a notion of distance that is, or at least shares structural features in common with, such as a classical action, an entropy or a notion of information, a quantum path integral or a statistical mechanical partition function.

4 Field Theory: unreduced configuration space geometry

Scalar field theory’s configuration space is a space of scalar field values \( \phi(x) \), which space I denote by Sca [implicitly Sca(\( \mathbb{R}^3 \)) in the most standard flat space case].

Electromagnetism’s configuration space is a space of 1-forms \( A_i(x) \), which space I denote by \( \Lambda_1 \) [implicitly \( \Lambda_1(\mathbb{R}^3) \)]. Yang–Mills theory’s configuration space is a larger space of 1-forms \( A_i^P(x) \), which space I also denote by \( \Lambda_1 \) [thus this notation is also dropping reference to the corresponding gauge group].

In modelling the above, one can start off working with \( \mathcal{L}^2 \): the square-integrable functions. One can furthermore pass to e.g. the Fréchet spaces of Appendix A, which are useful in subsequent curved-space and GR-coupled versions (now with \( \Sigma \) in place of \( \mathbb{R}^3 \)). The scalar field version is then useful in cosmological modelling, and easily appended to the GR configuration space, at least in the minimally-coupled case and at the unreduced level.

\(^{12}\)This is motivated e.g. by the preceding comparers failing to give distances when \( M \) is indefinite – losing the non-negativity and separation properties of bona fide distance – which we know will occur for GR.
5 GR: unreduced configuration space geometry

5.1 Topology of \( \text{Riem}(\Sigma) \)

Let us model the space of Riemannian geometries \( \text{Riem}(\Sigma) \) using an open positive convex cone\(^{13}\) in the Fréchet space (see Appendix A) \( \text{Fre}_{\text{sym}(0,2)}(C^\infty) \) for \( \text{Sym}(0, 2) \) the symmetric rank-2 tensors.

\( \text{Riem}(\Sigma) \) can furthermore be equipped with a metric space notion of metric \([23]\), Dist; this can additionally be chosen such that it is preserved under \( \text{Diff}(\Sigma) \). Thus \( \text{Riem}(\Sigma) \) is a metrizable topological space. Consequently it obeys all the separation axioms (including Hausdorffness), and is also paracompact. \( \text{Riem}(\Sigma) \) is additionally second countable \([30]\), and additionally has an infinite-dimensional analogue of the locally Euclidean property, by which a single type of chart suffices for it. Thus \( \text{Riem}(\Sigma) \) is a manifold that is infinite-dimensional in the sense of Fréchet\((C^\infty)\).

5.2 \( \text{Riem}(\Sigma) \) at the level of geometrical metric structure

In studying of GR, \( \text{Riem}(\Sigma) \) is usually taken to carry the infinite-\( d \) indefinite Riemannian metric provided by GR’s kinetic term, i.e. the inverse DeWitt supermetric \( M^{abcd} \) of \((7)\). More generally, one might consider other members of the family of ultralocal supermetrics \([28, 66]\)

\[
M_g^{abcd} := \sqrt{h}\{h^{ac}h^{bd} - \beta h^{ab}h^{cd}\}.
\]  

These split into three cases: \( \beta < 1/3 \) are positive-definite, \( \beta = 1/3 \) is degenerate, and \( \beta > 1/3 \) are indefinite (heuristically \(\{++++++\infty\}\)). Due to ultralocality, it makes sense to study these pointwise; the more problematic degenerate case is usually dropped from the study. Pointwise, then, they arise from positive-definite symmetric matrices (the \( h_{ab} \) at that point), which is diffeomorphic to the homogeneous space \([28]\) \( GL(3, \mathbb{R})/SO(3) \equiv \mathbb{R}^6 \). On the other hand, the \( u_{ab} \) of \((12)\) are pointwise on \( SL(3, \mathbb{R})/SO(3) \); \( h_{ab} \) can be decomposed into this and a scale part taking values in \( \mathbb{R}^+ \). The split corresponds to \( GL(3, \mathbb{R})/SO(3) \equiv SL(3, \mathbb{R})/SO(3) \times \mathbb{R}_+ \). The pointwise structures then uplift to \( \text{Riem}(\Sigma) \) due to ultralocality. The scale-free part gives rise to 8 Killing vectors and the scale part to a homothety \([28]\). The local Riemannian geometry of this was covered by DeWitt \([22]\), including the form of the geodesics. This exhibits various difficulties: curvature singularities and geodesic incompleteness.

5.3 Conformal variants

Expanding on footnote 7, the conformal transformations \( \text{Conf}(\Sigma) \) are smooth positive functions on \( \Sigma \) form . These form an infinite-dimensional Lie group; moreover this is are Abelian under pointwise multiplication.

Then conformal Riem \( \text{CRIem}(\Sigma) := \text{Riem}(\Sigma)/\text{Conf}(\Sigma) = \{SL_3(\mathbb{R})\}^\infty \) heuristically in parallel to the preceding Subsection. This is simpler and better-behaved \([22]\) than \( \text{Riem}(\Sigma) \) at the level of metric geometry. This is firstly in the sense that the natural supermetric thereupon is

\[
U^{abcd} := u^{ac}u^{bd},
\]

which is positive-definite and thus furthermore the basis of a bona fide notion of distance. Note that the part of the GR configuration space metric that causes it to be indefinite is the (local) scale part \([22]\). Secondly, geodesics are better-behaved upon \( \text{CRIem}(\Sigma) \) as compared to \( \text{Riem}(\Sigma) \).

Conformal Riem has also been termed ‘pointwise conformal superspace’ \([29]\) However, this name is confusing in various ways. Firstly, the name can only be understood if conformal superspace itself has already been introduced. Yet \( \text{CRIem}(\Sigma) \) is a simpler space, and a strong case can be made for simpler entities to be introduced on their own terms rather than by reference to more complicated ones. Secondly, Sec 5.2 already made a distinct use of ‘pointwise’, to mean ‘looking at a field at just one point’, which is a very clear use. The current use, on the other hand, would appear to be along the following lines. ‘Take a space that involves quotienting out \( \text{Conf}(\Sigma) \) and \( \text{Diff}(\Sigma) \) [conformal superspace] but now do not quotient out \( \text{Diff}(\Sigma) \) after all [‘pointwise’]. However, this can be contracted to just ‘take a space that involves quotienting out \( \text{Conf}(\Sigma) \)’, i.e. making no mention, rather than two cancellatory implicit mentions, of the concept that is unnecessary for the definition \( \text{Diff}(\Sigma) \). On these grounds, I use instead the name ‘conformal Riem’. I denote this by ‘\( \text{CRIem} \)’, making use of ‘C’ for ‘conformal’ parallelling the habitual use in ‘CS’ for ‘conformal superspace’, noting that ‘C’ standing for ‘conformal’ can just as well be introduced prior to any mention of superspace or the associated \( \text{Diff}(\Sigma) \).

On the other hand, while passing to equivalence classes is mathematically convenient, the equivalence classes themselves can be considered to be more primary. If \( \text{CRIem}(\Sigma) \) were viewed in this way, it would then make more sense for it and \( \text{Riem}(\Sigma) \) to be renamed so that now \( \text{Riem}(\Sigma) \)’s new name derives from \( \text{CRIem}(\Sigma) \)’s by a ‘locally-scaled’ addendum. One might go as far as viewing \( \text{CS}(\Sigma) \) as primary (for all that this is unlikely to be motivated by the final form of the

\(^{13}\)This is a Linear Algebra characterization of a space \( 5 \) \([23, 65]\), that is not itself linear but obeys \( 5 + 5 \subset 5 \) and \( m5 \subset 5 \) for \( m \in \mathbb{R}_+ \). See \([23]\) for more on this and for consideration of why Fréchet spaces are appropriate. Do not confuse this use of ‘cone’ with Sec 2’s topological and geometrical uses.
‘true degrees of freedom’ of the gravitational field. Such primality amounts to assuming not geometrodynamics but conformogeomododynamics. In this case, a good primary name would be shape space, Shape, for the conformal 3-geometry notion of shape. Whence Superspace would be known as ‘locally-scaled Shape’, CRiem as ‘diffeomorphism-redundant Shape’ and Riem as ‘locally-scaled diffeomorphism-redundant Shape.

Moreover, the conventional approach to conformogeomododynamics is that, in solving the Lichnerowicz–York equation [84], one finally passes from VPConf(Σ) to True(Σ) by the solution fixing a particular form of the local scalefactor to be the physically realized one. Whereas traditionally, Conformogeomododynamics is viewed as a convenient decoupling leading to substantial mathematical and numerical tractability, from the relational perspective, one can take \( \mathcal{G} = \text{Conf}(\Sigma) \times \text{Diff}(\Sigma) \) or VPConf(Σ) \( \times \) Diff(Σ).

Next, \( \{\text{CRiem} + V\}(\Sigma) \)’s metric is \( \{- + + ++\}^{\infty} \), which is actually hyperbolic rather than pointwise hyperbolic. The – direction here corresponds to a global scale variable, such as indeed the global spatial volume, or the cosmological scalefactor \( a \) when applicable.

Both for GR and RPM’s, many of the configuration spaces have physically-significant singular points. In particular, \( a = 0 \) is the Big Bang and \( I = 0 \) is the maximal collision, which are furthermore analogous through each involving scale variables.

Finally, quotienting out conversely Dil alone from Riem(\( \Sigma \)) gives a VPRIem(\( \Sigma \)) configuration space (volume-preserving Riem).

### 5.4 GR alongside minimally-coupled matter

This case is taken to include fundamental-field second-order minimally-coupled bosonic matter, covering e.g. minimally-coupled scalars, Electromagnetism, Yang–Mills Theory and scalar gauge theories. Then the redundant configuration space metric splits according to the direct sum [67]

\[
M = M^{\text{grav}} \oplus M^{\text{mcm}} .
\]

In the case of a minimally-coupled scalar field, I denote this configuration space by RIEM(\( \Sigma \)).\(^{14}\) The (undensitized) metric on this takes the blockwise form \( M(h) := \begin{pmatrix} 1 & 0 \\ 0 & M(h) \end{pmatrix} \) and \( M(h) \) the GR configuration space metric itself.

It is usually additionally assumed that \( M \) is independent of the matter fields. This is well-known, and held to secure freedom to ‘add in’ scalar fields in cosmological modelling.

Similar considerations apply throughout to extending CRiem(\( \Sigma \)), \( \{\text{CRiem} + V\}(\Sigma) \) and VPRIem(\( \Sigma \)).

### 6 Minisuperspace: homogeneous GR

The vacuum case of minisuperspace, Mini(\( \Sigma \)) [24], is the space of homogeneous positive-definite 3-metrics on \( \Sigma \). Each corresponds to a notion of space in which every point is the same. Here full GR’s \( M^{ijkl}(h_{mn}(x^i)) \) collapses to an ordinary \( 6 \times 6 \) matrix, \( M_{AB}(h_{ab}) \): an overall – rather than independently per space point – curved \( (-+++++) \) ‘minisupermetric’. A particular simpler subcase are the diagonal minisuperspaces, with \( 3 \times 3 \) matrix \( M_{AB}[24] \).

Minisuperspaces are classified by the isometry groups Isom(\( \Sigma, h \)) of their spatially homogeneous surfaces. This leads to two cases according to whether Isom(\( \Sigma, h \)) acts simply transitively. If this is not the case, it turns out that [68] there is a single case: \( SO(3) \times \mathbb{R} \) acting upon the cylindrical 3-space \( S^2 \times \mathbb{R} \); this gives the Kantowski–Sachs model. The other case gives the family of Bianchi models. The \( I \) being Lie groups, they are in turn characterized by the form of their structure constants. They are subdivided according to

\[
C^k_{ij} = 0 \text{ for type A and } \neq 0 \text{ for type B} .
\]

A finer classification of \( C^k_{ij} \) yields a subdivision into nine kinds of Bianchi models, labelled by I to IX [68]. The general case’s spatial metric can be represented as

\[
ds^2 = h_{ij}d\sigma^i d\sigma^j
\]

for 1-forms \( d\sigma^k \) obeying \( d^2\sigma^k = C^k_{ij}d\sigma^i \wedge d\sigma^j \).

Example 1) The spatially closed \( S^3 \) isotropic case has spatial metric \( ds^2 = a(t)^2 ds^2_{S^3} \).

Example 2) Diagonal Bianchi IX models have spatial metrics

\[
ds^2 = \exp(2\{-\Omega + \beta_+ + 3\sqrt{3}\beta_-\})d\Omega^2 + \exp(2\{-\Omega + \beta_+ - 3\sqrt{3}\beta_-\})d\beta_+^2 + \exp(2\{-\Omega - 2\beta_+\})d\beta_-^2
\]

on \( S^3 \). Diagonal Bianchi IX models are potentially of great importance through being conjectured to be the generic GR behaviour near cosmological singularities [69].

\(^{14}\)In this Article, I make wider use of such a capping convention for versions including a minimally-coupled scalar field.
These also have a nontrivial potential term inherited from the GR Ricci scalar potential term,
\[ V = \exp(-4\Omega)\{V(\beta) - 1\} , \]
for
\[ V(\beta) = \frac{\exp(-8\beta_1)}{3} - \frac{4\exp(-2\beta_1)}{3}\cosh(2\sqrt{3}\beta_1) + 1 + \frac{2\exp(4\beta_1)}{3}\{\cosh(4\sqrt{3}\beta_1) - 1\} , \]
which is an open-ended equilateral triangular cross-section well (Fig 9.c).

\[ \Omega := \ln a , \]
and also Misner’s [41] parametrization of anisotropy – a type of GR shape variable – by writing the tracefree spatial metric \( u_{ab} \) as
\[ \exp(2\beta_{ab}) \text{ for } \beta_{ab} \text{ a tracefree symmetric matrix} . \]
In the case of diagonal \( u_{ab} \) diagonal,
\[ \beta_{ab} = \text{diag}\{\sqrt{3}\beta_2 - \beta_1, \sqrt{3}\beta_2 + \beta_1\}/2 . \]
These are related to \( \beta_\pm \) by \( \beta_1 = \beta_+ + \sqrt{3}\beta_- \) and \( \beta_2 = \beta_+ - \sqrt{3}\beta_- \). Fig 9 provides a simple conceptual outline of the meaning of anisotropy for a 2-d hypersurface.

Then in the homogeneous case the configuration spaces Riem, CRiem + V, Superspace and CS + V coincide as Minisuperspace, Mini and CRiem, VPRiem, Sec 9.6’s VPSuperspace, and CS coincide as Anisotropyspace, Ani. Ani is yet another example of pure shape space.

Then for diagonal Bianchi class A, Mini = \( \mathbb{M}^3 \) with configuration space metric
\[ ds^2 = -d\Omega^2 + d\beta_+^2 + d\beta_-^2 . \]
Ani = \( \mathbb{R}^2 \) with shape metric
\[ ds^2 = d\beta_+^2 + d\beta_-^2 . \]
In the general non-diagonal minisuperspace, full configurations can be represent by elements of \( GL_3(\mathbb{R}, \Sigma) \) and and pure shapes (anisotropies) by elements of \( SL_3(\mathbb{R}, \Sigma) \).

Example 3) Upon inclusion of a single minimally-coupled scalar field, I use the corresponding capitalized notation MINI and ANI. The undensitized configuration space metric on MINI is
\[ ds^2 = -d\Omega^2 + d\phi^2 \]
and the undensitized potential is
\[ V := -\exp(-2\Omega) + V(\phi) + 2\Lambda . \]
for \( V(\phi) \) an unrestricted function.
7 Perturbations about minisuperspace: unreduced formulation

As regards incipient (redundant) configuration variables, the 3-metric and scalar field are expanded as [43]

\[ h_{ij} = \exp(2\Omega(t)) \left( \delta_{ij} + \epsilon_{ij}(x,t) \right), \quad \Phi = \sigma^{-1} \left\{ \phi(t) + \sum_{n,m} \Omega_{nm}(z) \right\}. \]  (51)

Here, \( S_{ij} \) is the standard hyperspherical \( S^3 \) metric. \( \epsilon_{ij} \) are inhomogeneous perturbations of the form

\[ \epsilon_{ij} = \sum_{n,m} \left\{ \sqrt{2} \partial_{nm} S_{ij} + \sqrt{6} b_{nm} \{ P_{ij} \}^{n,m} \right\} \]

\[ + \sqrt{2} \left\{ c_{n,m}^{o} \{ S_{ij}^{o} \}^{n,m} + c_{n,m}^{e} \{ S_{ij}^{e} \}^{n,m} \right\} + 2 \left\{ d_{n,m}^{o} \{ G_{ij}^{o} \}^{n,m} + d_{n,m}^{e} \{ G_{ij}^{e} \}^{n,m} \right\}. \]  (52)

The superscripts ‘o’ and ‘e’ for stand for ‘odd’ and ‘even’ respectively. I subsequently use \( n \) indices as a shorthand for \( n \).

Let \( x_n \) be a collective label for the 6 gravitational modes per \( n \). On the other hand, \( S_{ij}^{o} \) and \( S_{ij}^{e} \) are all functions of just the coordinate time (which is also label time for GR) \( t \). The \( Q_{nm} \) are the \( S^3 \) scalar harmonics, \( S_{ij}^{o}(\mathbf{x}) \) and \( S_{ij}^{e}(\mathbf{x}) \) are the transverse \( S^3 \) vector (V) harmonics, and the \( G_{ij}^{o}(\mathbf{x}) \) are the transverse traceless \( S^3 \) symmetric 2-tensor (T) harmonics. The \( S_{ij}^{o}(\mathbf{x}) \) are then given by \( D_{j} S_{i}^{o}(\mathbf{x}) + D_{i} S_{j}^{o}(\mathbf{x}) \) (for each of the suppressed \( o \) and \( e \) superscripts). The \( P_{ij}^{o}(\mathbf{x}) \) are traceless objects given by \( P_{ij}^{o} := D_{j} D_{i} Q_{n}/(n^2 - 1) + S_{ij} Q_{n}/3 \). \( \sigma := \sqrt{2/3\pi}/m_p \) is a normalization factor.

Then in the vacuum case, the redundant configuration space is \( \text{Riem}_{0,1,2}(S^3) \); the 0, 1 and 2 subscripts refer to the orders in perturbation theory that feature in it. This is the \( 1 + 6 \times \{ \text{countable} \ \infty \} \) space of scale variable alongside the \( x_n \). In the minimally-coupled scalar field case, the redundant configuration space is \( \text{RIEM}_{0,1,2}(S^3) \). This is the \( 2 + 7 \times \{ \text{countable} \ \infty \} \) space of scale variable, homogeneous scalar field mode and the \( y_n \). The first form in Fig 10 displays the latter for one mode to second order overall in \( y_n, \partial y_n \). By the direct sum split of Sec 5.4, \( \text{RIEM}_{0,1,2} = \text{Riem}_{0,1,2}(S^3) \oplus \text{Sca}_{0,1,2}(S^3) \), for \( \text{Sca} \) standing for scalar field configuration space. Thus the former configuration space can readily be read off the figure as a sub-block.

![Figure 10: a) Slightly inhomogeneous cosmology's configuration space metric [42]. The heavy dot denotes 'same as the transposed element' since metrics are symmetric. N.B. this is the blockwise corrections' configuration space metric rather than the full one.](image)

Blockwise-simplifying coordinates can additionally be found. Considering for instance the modewise case,

\[ \Omega_n = \Omega - A_n/3, \]  (53)

removes the gravitational sector’s off-diagonal terms, for \( [70] \)

\[ A_n := -2 \left\{ 4 \frac{a_n^2}{n^2 - 1} b_n + \frac{n^2 - 4}{n^2 - 1} c_n^2 + d_n^2 \right\}, \]  (54)

By this and trivial rescalings \( b_n' := \sqrt{\frac{n^2 - 4}{n^2 - 1}} b_n \) and \( c_n' := \sqrt{\frac{n^2 - 4}{n^2 - 1}} c_n \), one arrives at the second form of the metric in Fig 10.

The above configuration space geometry for slightly inhomogeneous cosmology is curved; nor is it conformally flat. Its Ricci scalar \( R \) has no singularities away from big bang for \( x_n \) small. In the minimally coupled scalar field case, \( \partial/\partial \phi \) and \( \partial/\partial f_n \) are Killing vectors for slightly inhomogeneous cosmology’s configuration space geometry. This corresponds to the ‘adding on’ status of scalar fields at this level. Additionally, \( \partial/\partial \Omega \) is a conformal Killing vector.

Finally consider these model arenas' scale-free spaces of inhomogeneities. I use \( w_n \) as a collective label for the 5 positive-definite gravitational modes \( b_n, c_n, d_n \), and \( z_n \) for the 6 positive-definite modes \( w_n \) and \( f_n \). Then \( \text{CRIEM}_{2}(S^3) \) is the space of the \( w_n \), which is straightforwardly just \( \mathbb{R}^5 \). Thus it has the obvious 15 Killing vectors built from the Cartesian rescalings of the \( w_n \) coordinates: \( 5 \partial/\partial w_n \) and \( 10 w_n \partial/\partial w_n - w_n \partial/\partial w_n^W \). On the other hand, \( \text{CRIEM}_{0,1,2}(S^3) \) – the space of homogeneous scalar field modes \( \phi \) alongside the \( z_n \) – is neither flat nor conformally flat.
8 Reduced configuration spaces for Field Theory

8.1 Gauge Theory’s orbit spaces

For gauge theories, as well as the configuration space of connections Conn itself, there is a gauge group \( \mathfrak{g} \) acting upon Conn. It is a Lie group, and in the more usual cases such as Electromagnetism and Yang–Mills Theory, it acts internally. In this way, \( \Lambda_1/\mathfrak{g} \) arise as reduced configuration spaces (more concretely, gauge orbit spaces).

Fibre bundle mathematics supports this to some extent, as follows.
1) Modelling using principal bundles: with the above \( \mathfrak{g} \) entering as both structure group and fibres.
2) A wider range of associated bundles with \( \mathfrak{g} \) as structure group and distinct fibres, by which coupling of gauge fields to a number of further (gauged) fields can be modelled.

On the other hand, the space of orbits itself is in general heterogeneous, and thus not itself amenable to fibre bundle description. Moreover, due to the group action in question being smooth and proper, orbit spaces are separable – and thus in particular Hausdorff – as well as metrizable, second-countable and paracompact. See e.g. [71, 72, 73] for more description. Moreover, due to the group action in question being smooth and proper, orbit spaces are separable – and thus in particular Hausdorff – as well as metrizable, second-countable and paracompact. See e.g. [71, 72, 73] for more description.

Note that \( L^2 \) (‘square integrable’) mathematics suffices for the above workings, though one can uplift to more general function spaces [72] including so as to attain compatibility with the GR case (see Sec 8.1).

8.2 Loops and loop spaces for gauge theory

Another approach is to make use of Wilson loop variables; these contain an equivalent amount of information to the \( A_i \) variables. Such a formulation is already meaningful for Electromagnetism; it amounts here to modelling the space of the form \( \wedge_i \text{Ad}_A \). 

For gauge theories, as well as the configuration space of connections Conn itself, there is a gauge group \( \mathfrak{g} \) acting upon Conn. It is a Lie group, and in the more usual cases such as Electromagnetism and Yang–Mills Theory, it acts internally. In this way, \( \Lambda_1/\mathfrak{g} \) arise as reduced configuration spaces (more concretely, gauge orbit spaces).

If modelled in this way, the corresponding loop space is a topological group. It is not however a Lie group, though it is compact if \( \Sigma \) is [76]. Finally, since \( I(\Sigma, h) \) comes in multiple sizes, there are multiple dimensions of the corresponding orbits, pointing to the space of orbits not being a manifold.

9 Reduced configuration spaces for GR

9.1 Topology of \( \mathfrak{g} = \text{Diff}(\Sigma) \)

\( \text{Diff}(\Sigma) \) can be – matching with \( \text{Riem}(\Sigma) \) [23] – taken to be modelled by use of \( \text{Fre}_{(1,0)}(\mathbb{C}^\infty) \) \((1, 0)\) are vector fields. Indeed, diffeomorphisms are commonly modelled in terms of Fréchet manifolds, a fortiori as Fréchet Lie groups [75].

Next consider the group action \( \text{Diff}(\Sigma) \times \text{Riem}(\Sigma) \to \text{Riem}(\Sigma) \). The group orbits of this are then \( \text{Orb}(h) := \{ \phi^* h | \phi \in \text{Diff}(\Sigma) \} \). Then metrics [points in \( \text{Riem}(\Sigma) \) lying on the same orbit amounts to these being isometric. Thus the \( \text{Diff}(\Sigma) \)-orbits partition \( \text{Riem}(\Sigma) \) into isometric equivalence classes [23]. The corresponding stabilizers \( \text{Stab}(\Sigma, h) = \{ \phi | \phi \in \text{Diff}(\Sigma) \text{ such that } \phi^* h = h \} \) constitute the isotropy group \( \text{Isot}(\Sigma, h) \). Moreover, \( \text{Isot}(\Sigma, h) \) coincides with [23] \( \text{Isom}(\Sigma, h) \); I mark this by using \( I(\Sigma, h) \) to denote this coincident entity. The Lie algebra corresponding to this is isomorphic to that of the Killing vector fields of \( (\Sigma, h_{\mu \nu}) \). An interesting result is that \( I(\Sigma, h) \) is compact if \( \Sigma \) is [76]. Finally, since \( I(\Sigma, h) \) comes in multiple sizes, there are multiple dimensions of the corresponding orbits, pointing to the space of orbits not being a manifold.
9.2 Topology of Superspace($\Sigma$)

Fischer showed that Superspace($\Sigma$) = Riem($\Sigma$)/Diff($\Sigma$) [23] can be taken to possess the corresponding quotient topology. Superspace($\Sigma$) additionally admits a metric in the metric space sense of the form [23]

$$\text{Dist}([h_1], [h_2]) = \inf_{\phi \in \text{Diff}(\Sigma)} (\text{Dist}(\phi^*h_1, \phi^*h_2)) .$$

(57)

In this manner, Superspace($\Sigma$) is a metrizable topological space and thus obeys all the separation axioms and thus in particular Hausdorffness; it is also second countable [23]. Thus Superspace is ‘2/3rds of a manifold’ in Appendix B’s sense.\(^{15}\)

However, unlike Riem($\Sigma$), Superspace($\Sigma$) fails to possess the infinite-dimensional analogue of the locally-Euclidean property. Wheeler [21] credits Smale with first pointing this out. Fischer [23] then worked out the detailed structure of Superspace($\Sigma$) as a stratified manifold. In particular, the appearance of non-trivial strata occurs for $\Sigma$ that admit metrics with non-trivial $I(\Sigma, h)$. In these cases Diff($\Sigma$) clearly does not act freely upon these metrics. Rather, the Superspace($\Sigma$) quotient space is here a stratified manifold of nested sets of strata ordered by $\dim(I(\Sigma, h))$. (Indeed, Fischer [23] tabulated the allowed isometry groups on various different spatial topologies.) In this way, Superspace($\Sigma$) is not a manifold in the sense of Fréchet (corresponding to the underlying function spaces used in Fischer’s mathematical modelling).

A further useful concept is the degree of symmetry of $\Sigma$,

$$\deg(\Sigma) := \sup_{h \in \text{Riem}(\Sigma)} (\dim(I(\Sigma, h))) .$$

(58)

Fischer [23] listed 3-manifolds with $\deg(\Sigma) > 0$, and further characterized $\deg(\Sigma) = 0$ manifolds in collaboration with Moncrief, [29]. N.B. that for $\deg = 0 \Sigma$, Superspace($\Sigma$) is a manifold.

Ebin [77] established that Diff($\Sigma$) is not compact. However, Ebin and Palais furthermore showed that Diff($\Sigma$) acting on Riem($\Sigma$) is one of the cases for which a slice (Appendix B.4) does none the less exist.

Ebin–Palais Slice Theorem [77]. For each $h \in \text{Riem}(\Sigma)$ \exists a contractible submanifold $\mathcal{S}$ containing $h$ such that

i) For $\phi$ a diffeomorphism \{in Diff($\Sigma$), $\phi \in I(\Sigma, h) \Rightarrow \phi^*\mathcal{S} = \mathcal{S}$.

ii) $\phi \notin I(\Sigma, h) \Rightarrow \phi^*\mathcal{S} \cap \mathcal{S} = \emptyset$.

iii) $\exists$ in Orb($h$) an open set $\mathcal{O}$ itself containing $h$, and a local cross-section $\Gamma : \mathcal{P} \to \text{Diff}(\Sigma)$ such that $\phi(p, s) = \{\Gamma(p)\}^*s$ is a diffeomorphism of $\mathcal{P} \times \mathcal{S}$ onto an open neighbourhood $\mathcal{U}_h$ of $h$.

Here $\mathcal{P}$ is an open neighbourhood of $I(\Sigma, h)$'s identity in the coset space: Diff($\Sigma$)/I(\Sigma, h), and ‘diffeomorphism’ and ‘submanifold’ are in the sense of Fréchet($C^\infty$).

The following is then a ready consequence [23].

Superspace Decomposition Theorem. The decomposition of Superspace($\Sigma$) into orbits is a countable partially-ordered Fréchet($C^\infty$) manifold partition.

Then via the preceding and Appendix B.5’s definition of inverted stratification, the following also holds [23].

Superspace Stratification, Stratum and Strata Theorems. The manifold partition of superspace is an inverted stratification (Appendix B) indexed by symmetry type. Fischer then classifies the superspace topologies and the strata into two large tables. The Stratum Theorem includes [23] that a stratum of superspace is finite dimensional iff the group action on the manifold is transitive (corresponding to a homogeneous space).

See e.g. [23, 26, 78, 27] for further topological studies of superspace, and [30] for further difficulties with putting a Riemannian metric on superspace.

9.3 Comparison between Theories. 1. Theorems.

Let us next further compare the GR, Gauge Theory and Mechanics cases. Firstly, Slice Theorems are known for each. See above for GR, Sec 8.1 for Gauge Theory and e.g. [79] for the case of Mechanics, Secondly, see the same Sections for the GR and Gauge Theory cases of Stratification Theorems; Mechanics also has one, at least in the symplectic setting [80]. The above two results provide further directions in which to take Sec 2’s model arena study. Thirdly, the Decomposition Theorem that we have seen arise for GR also has a Gauge Theory’s version of Decomposition Theorem [72]. This is for orbit spaces and is more mathematically standard (based on a ‘Hodge–de Rham decomposition’).

\(^{15}\)On the other hand, the space of spacetimes is an example of a non-Hausdorff ‘space of spaces’ [23].
9.4 Comparison between Theories. 2. Handling dynamical trajectories exiting a stratum.

Stratification becomes an issue as regards continuations of dynamical trajectories.

In the GR case, Leutwyler and Wheeler [21] appear to have been the first to ask about initial or boundary conditions on superspace. DeWitt, Fischer and Misner then suggested [63, 23, 81] that when the edge of one of the constituent manifolds – i.e. where the next stratum starts – is reached, the path in Superspace that represents the evolution of the 3-geometry could be reflected. Simpler such reflection conditions were also previously considered for Mechanics; Misner’s considerations were in Minisuperspace and DeWitt considered a further simple model arena [63].

A subsequent alternative proposal of Fischer involved extending such motions via working instead with a nonsingular extended space. This no longer encounters the stratified manifold’s issues as regards differential equations for motion becoming questionable at the junctions between strata.

He explicitly built such an extended space [26] by use of an unfolding which permits access to fibre bundle methods. The unfolding involved is parametrized by 1(Σ), as anticipated in the Mechanics case in Sec 2.3 This unfolding improves on previous such constructs by being generally covariant. It provides the right amount of information at each geometry (the space’s notion of point) to make the space of geometries into a manifold. The unfolding attains this by making use of the bundle of linear frames over Σ, F(Σ). Then no nontrivial isometries fix a frame. Thus the group action on the unfolded space Riem(Σ) × F(Σ) is free. In fact, by applying 1-point compactification to the open case, the open and closed cases are closer to each other than might be expected. In particular, Fischer [26] established that SuperspaceF (1-point compactified Σ) is diffeomorphic to Superspace(Σ).

Fischer [26] also pointed to Superspace possessing a ‘natural minimal resolution’ of the resultant singularities. This is based upon using the frame bundle quotient space Riem(Σ) × F(Σ)/ Diff(Σ). In this particular case, one can then regarding Riem(Σ) as a principal fibre bundle P(Superspace(Σ), Diff(F(Σ))). I.e. Diff(F(Σ)) \xrightarrow{\pi} \text{Riem}(\Sigma) \xrightarrow{\iota} \text{Superspace}_F(\Sigma) for i an inclusion map (Appendix C.1) and π the fibre bundle theory’s projection map.

However, the above unfolding runs against relationalism, due to the F(Σ) involved being a mathematical construct that does not correspond to more detailed modelling of physical entities.

Within the alternative ‘accept’ strategy to strata, see firstly item 1) of Sec 12.1 as regards sheaf methods for extending geodesics between strata, for use in both GR and Mechanics. Secondly, I point out here that stratifolds (Appendix B.7) happen to further model a number of configuration spaces of interest. This is firstly via Appendix B.6’s statement about Mechanics configuration spaces, and secondly via Appendix B.7’s statement about infinite-dimensional stratifolds moving toward being able to model GR configuration spaces. On the other hand, the space of spacetimes modulo spacetime diffeomorphisms not being Hausdorff leaves this space outside the scope of stratifolds, as are some loop spaces. Thus the reason I mention the stratifold construct is its applicability to a range of physically interesting examples, rather than as some full resolution of all stratified manifolds that arise in Physics.

It is also worth pointing out that, as Fischer and Moncrief pointed out, the deg(Σ) = 0 case of Superspace(Σ) avoids having strata in the first place, thus not necessitating any boundary conditions or extension procedure thereat. On the one hand, this deg(Σ) = 0 carries connotations of genericity, upon which general relativists place much weight. On the other hand, there is considerable interest in studying the simpler superspaces which are based on spaces with Killing vectors, such as S^3 and T^3, for which reduced approaches do encounter stratification.

Another direction arises if the actual Universe is acknowledged to at most have approximate Killing vectors. This would however come at the price of significant amounts of standard techniques becoming inapplicable. E.g. 1) perturbation theory that is centred about an exact solution with exact Killing vectors might cease to apply. 2) Ab initio averaging issues enter the modelling. 1) would be covered by modelling the universe on some specific deg 0 spatial topology: in this case we know there are no Killing vectors for strata to arise from. This would greatly complicate calculations as compared to those we are accustomed to on e.g. S^3. 2) however would be manifested through us not knowing which deg 0 spatial topology to take; one would now have to average over all plausible such, and quite possibly allow for these to change over evolution. By this stage one would be modelling with ‘big superspace’ and it would be a ‘higher level excision’ to exclude the superspaces with Killing vectors. One might still hope that sufficiently accurate analysis of the dynamical path would reveal it to avoid deg ≠ 0 topologies, or at least the solutions with Killing vectors therein. However, issues remain as regards whether each of the nongeneric structures (deg(Σ) ≠ 0, h_ab possessing Killing vectors) can come to possess a dynamical attractor role, by which generic paths could be forced to have endpoints in, or go arbitrarily close to, non-generic points.

Finally see e.g. [82] for boundary condition considerations for the Gribov regions of Gauge Theory.
9.5 Locality in the Thin Sandwich amounts to excision

The Thin Sandwich Theorem of Bartnik and Fodor is subject to two locality conditions, one of which involves locality in configuration space and amounts to staying away from solutions with Killing vectors. Thus it amounts to an excision.

On the other hand, conformal mathematics theorems associated with the below formulation are global in character [39].

9.6 \( CS(\Sigma) \) and \( \{CS + V\}(\Sigma) \)

\( \text{Conf}(\Sigma) \) and \( \text{Diff}(\Sigma) \) combine according to \( \text{Conf}(\Sigma) \times \text{Diff}(\Sigma) \) \([83]\); similarly \( \text{VPConf}(\Sigma) \) and \( \text{Diff}(\Sigma) \) combine according to \( \text{VPConf}(\Sigma) \times \text{Diff}(\Sigma) \). Note that \( \text{Conf}(\Sigma) \cap \text{Diff}(\Sigma) \neq \emptyset \) due to the existence of conformal isometries. However, quotienting something out twice is clearly the same as quotienting it out once, so this does not unduly affect the implementation. Also note that \( \text{Conf}(\Sigma) \) is contractible, so \( \text{Conf}(\Sigma) \times \text{Diff}(\Sigma) \) has the same topology as \( \text{Diff}(\Sigma) \), \( \text{Conf}(\Sigma) \times \text{Diff}_F(\Sigma) \) as \( \text{Diff}_F(\Sigma) \), \( \text{CS}(\Sigma) \) as Superspace(\( \Sigma \)) and \( \text{CS}_p(\Sigma) \) as Superspace_\( \Sigma \) (see e.g. \[30\]).

Fischer and Marsden \([83]\) extend Ebin’s work by considering the action of the \( \mathcal{C}^\infty \) version of \( \text{Conf}(\Sigma) \) on \( \text{Riem}(\Sigma) \), as motivated by York’s work... They obtain a \( \text{Conf}(\Sigma) \times \text{Diff}(\Sigma) \) analogue of the Ebin–Palais Slice Theorem. They also demonstrate that \( \text{CS}(\Sigma) \) is an infinite-d weak symplectic manifold near those points \( (h, p) \) with no simultaneous conformal Killing vectors. That implies sensible topology other than as regards being stratified. \([84]\) includes a linearized version of the stratification. That stratification occurs carries over from Superspace(\( \Sigma \)) to \( \text{CS}(\Sigma) \), along with many results that follow from contractibility. I fact, Fischer and Marsden \([83]\) already had a \( \text{CS}_p(\Sigma) \) analogue of the stratification theorem. Fischer and Moncrief’s superspace results \([29]\) carry over to \( \text{CS}(\Sigma) \) as well. Thus for the \( \deg(\Sigma) = 0 \) case, one gets each of orbifolds, manifolds and contractible manifolds twice over.

\( \text{CS}(\Sigma) \) must be positive-definite since it is contained within \( \text{CRIem}(\Sigma) \). \( \text{CRIem}(\Sigma) \) is better-behaved than \( \text{Riem}(\Sigma) \) along lines already established by DeWitt \([22]\). One might hope that \( \text{CS}(\Sigma) \) is better-behaved than Superspace(\( \Sigma \)), in parallel to relational space containing a better-behaved shape space.

\( \text{CS}(\Sigma) \) and \( \{\text{CS} + V\}(\Sigma) \) are of further interest through appearing in shape dynamics \([85]\). As regards what structure \( \{\text{CS} + V\}(\Sigma) \) has, it is far more similar to \( \text{CS}(\Sigma) \) than Superspace(\( \Sigma \)) is, and yet already the latter pair of spaces are heavily interlinked by contractibility. This ensures that many results for \( \text{CS}(\Sigma) \) carry over to \( \{\text{CS} + V\}(\Sigma) \) by a simpler ‘global’ instance of contractibility. Beyond these results, to date the Author is not aware of any substantial mathematical differences between these spaces.

Finally note that conversely \( \text{Dil} \) alone can be quotiented out of Superspace(\( \Sigma \)), giving a \( \text{VPSuperspace}(\Sigma) \) configuration space (volume-preserving Superspace).

9.7 Notions of distance for geometrodynamics

Referring back to Sec 3, \( ||M|| \) is not a notion of distance for \( M \) indefinite, e.g. for \( \text{GR} \) or its minisuperspace. The same restriction occurs again for path metrics. Due to this, the Kendall, Barbour and DeWitt comparers do not carry over to \( \text{GR} \) as notions of distance. \([\text{Whereas the DeWitt comparer originates from geometrodynamics, it did not arise there as a distance, but rather as a metric functional from which an indefinite geometry follows by double differentiation.}] \) Four ways out of this situation are as follows.

1) Consider \( \text{CRIem}(\Sigma) \) and \( \text{CS}(\Sigma) \), which are positive-definite so that the Barbour and DeWitt comparers do carry over as notions of distance \([7]\).
2) Use an inf implementation instead \((\text{c.f. 57 and the double-inf Gromov–Hausdorff notion of distance [64]})\).
3) Use inhomogeneity quantifiers; density contrast is a simple such; see e.g. \([86]\) for more complicated ones.
4) Use spectral notions of distance. The basic idea here is to consider the spectrum of some natural differential operator on the manifold. Problems with this include non-uniqueness of such natural operators and the ‘isospectral problem’ that ‘drums of different shapes’ can none the less sound exactly the same, by which another axiom of distance fails.

10 Reduced perturbative Midisuperspace

This can be taken to arise from a particular example of the sandwich, for which the sandwich manoeuvre by itself fails to factor in the \( \text{Diff}(\Sigma) \) content. In these models, \( \text{Diff}(\mathbb{S}^3) \) start to have effect at first order.
10.1 Vacuum case

This turns out to be more straightforward to handle [70]. Now solving the thin sandwich equations gives (for $v_n$ having components $s_n := a_n + b_n$: the scalar mode sum ubiquitous quantity, $d_n^o$ and $d_n^c$)

$$\frac{2}{\exp(3\Omega)} ds^2 = \left\{-1 + A_n d\Omega^2 + \frac{3}{2} d\Omega A_n + ||d v_n||^2\right\}. \quad (59)$$

This is of dimension $4 + 1$: $a_n$ drops out of the line element, so it is only short by 1 in removing the Diff($S^3$) degrees of freedom. Moreover, geometrically this is just flat $\mathbb{M}^5$. Indeed,

$$T_n := \frac{2}{3} \sqrt{A_n - 1} \cosh(\Omega + \frac{1}{3} \ln(A_n - 1)),$$

$$X_n := \frac{2}{3} \sqrt{A_n - 1} \sinh(\Omega + \frac{1}{3} \ln(A_n - 1))$$

cast the line element takes the familiar form

$$\frac{2}{\exp(3\Omega)} ds^2 = -dT_n^2 + dX_n^2 + ||d v_n||^2. \quad (60)$$

One can proceed from here by the $V$ part of $\kappa$ separating out to give an equation

$$5\Phi_n^2 - 16\Phi_n^* \Omega^* + 8\Omega^* + \exp(-2\Omega) = 0 \quad \text{for } \Phi_n := \exp(3\Omega) \{1 + A_n\}/3. \quad (61)$$

for * denoting label-time derivative and $\Phi_n := \exp(3\Omega) \{1 + A_n\}/3$. for the thus only temporarily convenient mixed-SVT variable $A_n$. This leads to a fully Diff($S^3$)-reduced line element of the form

$$\frac{2}{\exp(3\Omega)} ds^2 = \left\{-1 + f_n(\Omega)\right\} d\Omega^2 + ||d v_n||^2, \quad (62)$$

for $f_n(\Omega) := A_n(\Omega) + \frac{3}{2} \frac{dA_n(\Omega)}{d\Omega}$. This is conformally flat. Finally, define a new scale variable $\zeta_n := \int \sqrt{f_n(\Omega) - 1} d\Omega$ to absorb the first term’s prefactor. This leaves, up to a conformal factor,

$$ds^2 = -d\zeta_n^2 + ||d v_n||^2, \quad (63)$$

which is a spatially infinite slab of $\mathbb{M}^4$.

$\partial/\partial v^N$'s components $\partial/\partial s_n, \partial/\partial e_n^o, \partial/\partial e_n^c$ are then among the 10 conformal Killing vectors; the others are $\partial/\partial \zeta$, 3 $v_n^o \partial/\partial v_n^o - v_n^c \partial/\partial v_n^c$ and 3 $v_n^o \partial/\partial \zeta + \zeta \partial/\partial v_n^o$.

Finally the corresponding shape space is also clearly flat, in this case $\mathbb{R}^3$: $ds^2 = ||d v_n||^2$.

10.2 Minimally coupled scalar field case

In the minimally-coupled scalar field case [42], the outcome of the thin sandwich elimination is the undisturbed $ds^2$ of the densitized version of (49) alongside

$$ds_{hm}^2 = \frac{\exp(3\Omega)}{2} \left\{||d v_n||^2 + df_n^2 + \left\{3 d a_n + \sqrt{3 n^2 - 4} d s_n \right\} f_n + 6 a_n d f_n\right\} d\phi + \frac{3}{2} d A_n d\Omega - A_n \{ -d\Omega^2 + d\phi^2 \}. \quad (64)$$

But since this is of dimension $6 + 2$, it is not yet Superspace. In removing the Diff($S^3$) degrees of freedom, the thin-sandwich manoeuvre has fallen short by 2. We do not know for now how to progress from here with the reduction.

However, the geometry of the currently attained ‘halfway house’ has been further explored. Its configuration space block structure can be tidied up by removing as many off-diagonal terms as possible can be done separately in each of the first two blocks. Diagonalize the one by using (53) again, whilst applying

$$\phi_n = \phi - \frac{2}{3} b_n f_n \quad (65)$$

to the other to set the coefficient of $da_n d\phi_n$ to zero.

This example serves to illustrate that the configuration space metric split (38) and consequently the Hamiltonian constraint metric–matter split [70] are not in general preserved by reduction procedures. Thus one has to face the complication that even minimally-coupled matter influences the form of the gravitational sector’s reduced configuration space geometry.

The Ricci scalar for the partly reduced geometry is

$$R = 7 \exp(-3\Omega)/f_n^2,$$  

by which the matter perturbation going to zero – a physically innocuous situation – gives a curvature singularity. I previously commented that [42] this is not an unexpected phenomenon, paralleling for instance the situation with the collinear configurations in the $N$-body reduced configuration space. However, I now furthermore comment that the latter is based on stratification, whereas the latter is not known to be.

Also $\partial/\partial \phi_n$, $\partial/\partial e_n^o$ and $\partial/\partial e_n^c$ are Killing vectors; whereas $\partial/\partial f_n$ has lost this status upon reduction, the tensor mode directions have gained this property. As ever, $\partial/\partial \Omega$ is a conformal Killing vector. However, none of the above respect this model’s potential (see [42] for its form).

Finally, scaled perturbative minisuperspace does not have a bona fide configuration space metric based notion of distance. On the other hand, the space of pure inhomogeneities is positive-definite. The vacuum case admits a 3-d Euclidean metric with the $v_n$ as the corresponding coordinates.
11 Loop and knot spaces for GR

11.1 Loops and loop spaces

The loops of Sec 8.2 are now taken to involve i) paths embedded in GR’s topological notion of space: $\Sigma$, and ii) the specific gauge group $SU(2)$.

The heuristic outline of the loop spaces being loop groups carries over to this case. For more rigorous treatments, see e.g. [87, 88, 89]. The theory of cylindrical measures is usually evoked. The form taken by the stratification of the gauge orbit space in the GR case is covered in [90].

See e.g. [88] for the LQC equivalent of diagonal anisotropy.

11.2 Knots and knotspace

Begin by considering smoothly or piecewise linearly embedded curves on an orientable manifold (this avoids ‘pathological wild knots’). Then a finite collection of such is termed a link, and a link with just a single component is termed a knot. Two links $L_1, L_2$ in a given $\Sigma$ are equivalent if there is an orientation-preserving automorphism under which $\text{Im}(L_1) = L_2$

![Figure 11: Representation of knots link diagrams](image)

Figure 11: Representation of knots link diagrams: planar graphs with an over-and-under crossing designation. The Reidemeister moves that preserve knots are a) twist/untwist, b) pull back/push under, and c) slide left/slide right. However, we do not know an upper bound on how many such moves are needed to bring knots into obvious equivalence, so these moves are not in general used to determine equivalence in practice. Rather, we seek characterization in terms of knot invariants (a subset of topological properties). These include a number of polynomials (e.g. Jones’, Alexander’s and Conway’s [45, 74]). d) The unknot (trivial knot) has no crossings (or can be continuously deformed – a so-called ‘ambient isotopy’ notion – into having none). e) The simplest nontrivial knot is the trefoil.

Note the rather obvious topological manifold level background dependence in this formulation of knots. See e.g. for some applications of knots in Physics: [91, 92, 74, 45].

The mathematical form of the corresponding ‘Knotspace’ remains an open problem. One approach is to view knot space as $\text{Emb}(S^1, S^3)$: embeddings of the circle in the 3-sphere. [We are OK with this out of working on $\Sigma$ and most commonly taking this to be $S^3$.] Then Vassiliev’s work [93] along these lines has close parallels to Arnol’d’s study of the topological properties of $N$ points in the plane (with the points deleted), in the sense of each $\mathfrak{q}$ being a subspace of a more tractable mapping space [94]. This turns the topological problem into one concerning singular maps, which is then aided by these forming a stratified space. Thus study of the space of knots adds further value to the mechanical model arena and the study of stratified spaces (for all that it lies outside the scope of this Article).

11.3 Another naming: not ‘Loop Quantum Gravity’ but ‘Nododynamics’!

There is the following analogy between preshapes and loops. RPM preshapes are arrived at by quotienting out the dilations $\text{Dil}$ but not the more physically significant and mathematically harder rotations $\text{Rot}(d)$. Loop quantum gravity’s loops are arrived at by quotienting out $SU(2)(\Sigma)$ but not the more physically significant and mathematically harder $\text{Diff}(\Sigma)$. Thus, whilst neither are the most redundant configurations of use in their theory, both are still partly redundant. Nor are they even ‘halfway houses’ in each’s passage to non-redundant ‘physical’ kinematical variables, since both are prior to the main part of that passage, both physically and in terms of the remaining parts of each’s passage being far more mathematically complex than the parts already undertaken. This has long been reflected in the former theories long having been named not after preshapes but after the shapes themselves: ‘shape geometry’, ‘shape statistics’, ‘dynamics of pure shape’, one sense of ‘shape dynamics’, [95, 8, 15, 7]. This suggests that it would be clearer to name the latter theory not after loops but after knots. A suggestion then is Nododynamics, from the Latin nodus for ‘knot’. This additionally makes sense at both the classical and quantum levels, just like ‘Geometrodynamics’ does. I end by noting that Geometrodynamics itself is indeed another naming based on identifying the non-redundant ‘physical’ kinematical variables.
12 Extensions

12.1 Some further applications of this Article’s examples

1) The idea of dynamics as a (para)geodesic principle on configuration space runs into global issues – one of the many Global Problems of Time [34] – upon realizing that in general configuration space is not a manifold but rather a stratified manifold. One possible way out of these is stratified manifold geometry’s own notion of geodesic [96] feeding into a notion of geodesic principle thereupon. This approach is based on the use of sheaf methods [96], thus exemplifying that there are benefits from generalizing from fibre bundle methods to sheaf methods as regards the Global Problems of Time.

2) If $q$ is stratified, then so are $\mathcal{T}(q)$ [96] and the symplectic version of $\mathcal{T}^*(q)$ (see e.g. [97] in the case of Mechanics). [Indeed, Whitney himself had already considered symplectic stratified spaces...] Thus passage from a configuration space based approach to a configuration-and-change of configuration space perspective, or to the more habitual phase space perspective, does not affect arguments concerning stratified manifolds arising.

3) Each version of triangleland presented in this Article will have different kinematical quantization, involve different representations in its quantum theory and have mathematically distinct wavefunctions, due to QM’s global sensitivities [55]. Thus Fig 7’s trichotomy has quantum consequences. Some classical consequences can also be expected from differences between the boundary value problems for each case. This provides yet further reasons to study this range of models ([7] already argued for them to be model arenas for affine geometrodynamics and for foundational issues with Ashtekar variables approaches).

4) One can expect further contact between sheaves and Problem of Time aspects [34], due to sheaves being well-suited to handle constraint algebraic structures, observables algebraic structures, records and histories [33]. E.g. in being tools for tracking locally defined entities by attachment to open sets within a topological space, sheaves are well-placed for handling Records Theory.

5) As regards Probability Theory and Statistics based upon configuration spaces in the role of sample or probability spaces, some cases that happen to be of relevance to RPMs [47] were worked out by Kendall [95, 60, 8]. On the other hand, the current paper makes it clear that the known area of Probability Theory and Statistics on $\mathbb{R}^n$ and $\mathbb{M}^n$ [98] in some senses suffice to cover also Probability Theory and Statistics on some examples of Minisuperspace, Anisotropyspace, modewise perturbative Midisuperspace and its pure-shape counterpart. A remaining caveat preventing just uplifting some techniques is that physically these configuration spaces come paired with specific potential functions, whereas traditional roles for Probability and Statistics on $\mathbb{R}^n$ and $\mathbb{M}^n$ solely involve the metric geometry. Finally, modewise perturbative Midisuperspace is a slab of $\mathbb{M}^4$.

6) The simple geometries laid out in 5) also facilitate further quantum models, though the slab condition together with QM’s global sensitivities would be expected to produce a kinematical quantization other than the standard (whole) Minkowski spacetime one.

12.2 Further range of examples of configuration spaces of interest

1) Spaces of beins from approaching GR in first-order form (needed for subsequent incorporation of fermions).

2) The status of configuration spaces in theories including fermions (whether flat-space, curved-space or coupled to GR) are of further interest, due to fermions’ blurring of the configuration–momentum distinction.

3) One could then furthermore consider the status and geometry of configuration spaces in Supergravity.

4) One could also consider configuration spaces for string and brane configurations.

5) The existing discrete Quantum Gravity programs’ configuration spaces remain under-studied.

6) There is a much larger assortment of generalized configuration spaces than 5)’s, as per Sec 1.4’s point IX).

Acknowledgments. I thank those close to me for support. I thank Chris Isham, Jeremy Butterfield, Sean Gryb, Gabriel Herczeg, Tim Koslowski, Matthias Kreck, Flavio Mercati and Przemyslaw Malkiewicz for discussions and references, and the fqXi for a Travel Grant for 2014.
A Outline of supporting Functional Analysis

A.1 From Hilbert to Banach and Fréchet spaces

The purpose of these Appendices is to provide a basic outline of mathematics relevant to stratified configuration spaces, with some mentions of related notions which are more widely familiar in Theoretical Physics. Let us start with a ladder of increasingly general topological vector spaces which are infinite-

dimension function spaces. A Hilbert space $\text{Hilb}$ is a complete inner product space, a Banach space is a complete normed space, and a Fréchet space is a complete metrizable locally convolution topological vector space [75].

Whilst Hilbert Spaces are the most familiar in Theoretical Physics due to their use in QM, Functional Analysis has also been extensively developed for Banach spaces [19]. Major results here are the Hahn–Banach Theorem, the Uniform Boundedness Principle and the Open Mapping Theorem; see [99] for details and proofs. The second and third of these follow from from Baire’s Category Theorem; also the Inverse Function Theorem extends to Banach spaces, following from the Open Mapping Theorem.

However, treatment of GR configuration spaces involves the even more general Fréchet spaces. Let us first explain their definition. A topological vector space is metrizable if its topology can be induced by a metric – in the Analysis metric space sense – and that is furthermore translation-invariant. This qualification is required since for topological vector spaces, one uses a collection of neighbourhoods of the origin (vector space 0). Then from this, translation (by the vector space $+$) establishes the collection of neighbourhoods at each other point. Next, a base in a topological vector space $V$ is a linearly-independent subset $A$ such that $V$ is the closure of the linear subspace with Hamel basis $A$. A Hamel basis itself is a maximal linearly-independent subset of $V$. Finally, a topological vector space $V$ is locally convex if it admits a base that consists of convex sets.

Next, many important results in Functional Analysis – in particular the Hahn–Banach Theorem, the Uniform Boundedness Principle and the Open Mapping Theorem – further carry over from Banach spaces to Fréchet spaces [75]. On the other hand, be warned that there is no longer in general a Inverse Function Theorem here, though the Nash–Moser Theorem [75] is a replacement of this for a subclass of Fréchet spaces. One further consequence of this is that the usual local existence theorem for ODEs does not hold either. See e.g. [75] as regards Calculus on Fréchet spaces.

A.2 Hilbert, Banach and Fréchet Manifolds

Topological manifolds’ local Euclideaness and ensuing $\mathbb{R}^p$-portion charts extend well to infinite-$d$ cases, for which the charts involve portions of Hilbert, Banach and Fréchet spaces. See e.g. [100] for Hilbert manifolds, [19] for Banach manifolds and [75] for Fréchet manifolds. Banach manifolds are the limiting case as regards having a very broad range of analogies with finite manifolds. Fréchet manifolds remain reasonably tractable [19] despite losing in general the Inverse Function Theorem. Freéchet Lie groups can also be contemplated [75].

Finite manifolds’ incorporation of differentiable structure also has an analogue in each of the above cases. So e.g. one can consider differentiable functions and tangent vectors for each, and then apply multilinearity to set up whichever rank $(p, q)$ and symmetry type $S$ of tensor versions. In particular, applying this construction to a Fréchet manifold with tangent space $\text{Fre}(C^\infty)$ produces another Fréchet space $\text{Fre}(p, q)(C^\infty)$. These are used in Sec 5 and 9.

B Quotient spaces and stratified manifolds

B.1 Quotienting out groups

Quotienting has a number of subtleties. For instance, it is well-known that attempting to quotient one group by another does not in general produce yet another. Indeed, quotienting is more generally an operation under which only some mathematical structures and properties are inherited; see the next SubAppendix for further examples.

For $\mathcal{G}$ a group with an element $g$ and a subgroup $\mathcal{H}$, $g\mathcal{H} := \{gh \mid h \in \mathcal{H}\}$ is a (left) coset, and the corresponding (left) coset space is the set of all of these for that particular $\mathcal{G}$ and $\mathcal{H}$. Given a group action $\alpha$ on a set $X$: a map $\alpha : \mathcal{G} \times X \to X$ such that

i) $\{g_1 \circ \{g_2\}x = g_1 \circ \{g_2x\}$ (compatibility) and
ii) $e \cdot x = x$ (identity) $\forall x \in X$,

Then orbits are defined as $\text{Orb}(x) := \{gx \mid g \in \mathcal{G}\}$: the set of images of $x$, and stabilizers alias isotropy groups by $\text{Stab}(x) := \{g \mid gx = x\}$: the set of $g \in \mathcal{G}$ that fix $x$.  

23
Finally, the *quotient of the action of a group* $\mathfrak{g}$ on a space $\mathcal{S}$, $\mathcal{S}/\mathfrak{g}$ is the set of all orbits, which is usually termed *orbit space*.

### B.2 Quotient topologies

Next consider quotienting a topological space by an equivalence relation, $(X, \tau)/\sim$, so as to produce the corresponding *quotient topology*.

This does not in general preserve a number of topological properties, in particular none of the three properties that occur in the definition of manifolds. In the case of Hausdorffness, a simple counterexample to preservation is as follows. Let $X = \{(x, y) \in \mathbb{R}^2 \mid y = 0 \text{ or } 1\}$ with the obvious topology, and $(x, y) \sim (z, w)$ iff either $(x, y) = (z, w)$ or $x = z \neq 0$. Then this quotient produces the line with one point replaced by two points which cannot be separated. As regards non-preservation of dimension, quotienting is capable of decreasing or increasing topological dimension. Whereas the decreasing case is obvious, space-filling curves provide some examples of it increasing. Quotienting can furthermore produce dimension that varies from point to point in its quotient (Sec 2 contains simple examples of this). Moreover, in the physical examples below, however, Hausdorffness and second countability *are* often retained, so quotienting here leads to entities which are ‘2/3 of a manifold’.

On the other hand, quotienting does preserve connectedness and path connectedness albeit not simple connectedness or contractibility; it also preserves compactness. If $\mathcal{S}/\mathfrak{g}$ arises by a group $\mathfrak{g}$ acting on a space $\mathcal{S}$ freely and properly, then $\mathcal{S}/\mathfrak{g}$ is Hausdorff. One application of this is that this protects all 1- and 2-d RPM shape spaces.

### B.3 Orbifolds

*Orbifolds* are locally quotients $\mathcal{M}/\mathfrak{g}$ following from a properly discontinuous action of a finite Lie group $\mathfrak{g}$ on a manifold $\mathcal{M}$. This construction can moreover be applied to equipped manifolds such as (semi-)Riemannian manifolds. Orbifolds are more general than manifolds, since quotients do not in general preserve manifoldness, by which some orbifolds carry singularities. See e.g. [101] for more on orbifolds.

$\mathcal{M}$ itself admits an open cover $\mathcal{U}_i$. Then each constituent $\mathcal{U}_i$ possesses an *orbifold chart*: a continuous surjective map $\phi_i : U_i \to \mathcal{U}_i$ for $\mathcal{U}_i$ open $\subseteq \mathbb{R}^n$ [for $n = \dim(\mathcal{M})$], with $\mathcal{U}_i$ and $\phi_i$ invariant under the action of $\mathfrak{g}$. One can then define a notion of gluing between such charts and finally a notion of orbifold atlas, in close parallel to that for manifolds.

As a simple example of orbifold, the everyday notion of cone can be thought of as a such. On the other hand, Fig 7.b) is a simple instance of orbifold within the subject matter of the current paper: for a 3-body problem configuration space. More generally, orbifolds are common in $N$-body problem configuration spaces, indeed including the generalized sense of cone that applies to relational spaces. The 2-d $N$-body problem’s simplest shape spaces $\mathbb{CP}^{n-1}$ are best thought of as complex manifolds. There is then indeed a notion of complex orbifold as well as of real orbifold, in parallel to how there are real and complex manifolds [20]. Well-known elsewhere in Theoretical Physics, many of the orbifolds which occur in String Theory are also complex; in particular, these occur in the study of Calabi–Yau manifolds [101]. Furthermore, some simpler models of this last example are closely related to the penultimate example, though both being discrete quotients of $\mathbb{CP}^k$ spaces [7].

### B.4 Quotienting by Lie group action, and slices.

For the action of a Lie group $\mathfrak{g}$ on a space $X$, the generalized slice $S_x$ at $x \in X$ is a manifold transverse to the orbit $\text{Orb}(x)$ (see e.g. [102]). This generalizes the fibre bundle notion of local section to the case involving compact transformation groups in place of principal bundles. [The corresponding generalization of the fibre bundle notion of local trivialization is termed a *tube*.]

The slice can be taken to exist in the above compact case. However, in other cases one can sometimes prove Slice Theorems to this effect (Appendices 8.1 and 9.2). Of subsequent relevance below, the Implicit Function Theorem – a close relative of the Inverse Function Theorem – enters these proofs. A slice $S_x$ gives a local chart for $X/\mathfrak{g}$: the space of orbits near $x$, so the slice notion, when available, is an important tool in the study of the corresponding orbit spaces.

Slices carry information concerning the amount of isotropy of points near $x$ [102]. Let us illustrate ‘amount of isotropy’ using Sec 2’s examples. Whereas 2-d mechanics configurations have just the one isotropy group $SO(2)$, 3-d ones have 3 possible isotropy groups: $\text{id}$, $SO(2)$ and $SO(3)$. These have corresponding orbits of the form $SO(3)$, $S^2$ and 0 respectively. This correspondence follows from the isotropy group also being known as the stabilizer group, and there being well-known ties between orbits and stabilizers. *Multiple dimensions of isotropy groups Isot point to multiple dimensions of orbits. Thus to orbit spaces are not manifolds – entities of unique dimension – but rather collections of manifolds that span various dimensions.* This motivates consideration of further generalizations of manifolds as follows.
B.5 Stratified manifolds

Manifolds do not cover enough cases of quotients $\mathcal{M}/\mathcal{G}$ for the purpose of studying physical reduced or relational configuration spaces $\mathcal{Q}/\mathcal{G}$. We saw above that these more generally produce unions of manifolds of in general different dimensions. Moreover, some of these of further physical relevance – such as reduced configuration spaces in Mechanics and GR, spaces of orbits in Gauge Theory – ‘fit together’ according to some fairly benevolent rules. The constituent manifolds here are known as *strata*, and each collection ‘fitted together’ in this manner is known as a *stratified manifold*. Simple examples include the following.

Example 1) A *manifold with boundary* is locally homeomorphic to open sets in the half-space $\{(x_1, ..., x_p) \in \mathbb{R}^p \mid x_p \geq 0\}$. Charts ending on the half-space’s boundaries are describing part of the manifold that is adjacent to its boundary. This can furthermore be interpreted as a simple type of stratified manifold. Here the manifold and its boundary are the two constituent strata, the former possessing the full dimension and the latter codimension-1. Fig 12 illustrates a particular case of this.

![Figure 12: This configuration space had three types of chart. On the other hand, D and C configurations have the same isotropy group, so the D's do not constitute distinct strata.](image)

Historically, the original formulation of stratified manifolds was of differentiable stratified manifolds by Whitney [103] (and reviewed in [104]). Subsequently, Thom formed a theory of stratified topological manifolds as an arena for dealing with singularities [105]. Thom [106] additionally showed that every stratified space in the sense of Whitney was also one of his own stratified spaces with and the same strata.

Here is a brief outline of some basic concepts in the theory of stratified manifolds. Let $\mathcal{V}$ be a topological space that is not presupposed to be a topological manifold. Suppose this can be split according to $\mathcal{V} = \mathcal{V}_p \cup \mathcal{V}_q$ [104]. Here $\mathcal{V}_p := \{p \in \mathcal{V}, p \text{ simple}, \dim_p(\mathcal{V}) = \dim(\mathcal{V})\}$ with ‘simple’ meaning ‘regular’ and ‘ordinary’, and $\mathcal{V}_q := \mathcal{V} - \mathcal{V}_p$. In fact, one considers a recursion of such splittings, so e.g. $\mathcal{V}_q$ can subsequently be split into $\{\mathcal{V}_q\}_{p}$ and $\{\mathcal{V}_q\}_{q}$. Then setting $\mathcal{M}_1 = \mathcal{V}_p$, $\mathcal{M}_2 = \{\mathcal{V}_q\}_{p}$, $\mathcal{M}_3 = \{\{\mathcal{V}_q\}_{q}\}_{p}$ etc gives $\mathcal{V} = \mathcal{M}_1 \cup \mathcal{M}_2 \cup ... \cup \dim(\mathcal{V}) = \dim(\mathcal{M}_1) > \dim(\mathcal{M}_2) > ...$, where each $\mathcal{M}_i$ is a manifold. The point of this procedure is that it provides the partition of $\mathcal{V}$ by dimension. The indeed $\mathcal{V}$ is only a topological manifold is this is a trivial partition: involving a single dimension only. On the other hand, a *strict partition* of a topological space is a (locally finite) partition into strict manifolds. Here the manifold and its boundary are the two constituent strata, the former possessing the full dimension and the latter codimension-1. Fig 12 illustrates a particular case of this.

Example 2) *Manifolds with corners*. These have, in addition to the previous example’s strata, the codimension-2 strata that are the corners themselves.

Next, a set of manifolds in $\mathcal{O}$ has the *frontier property* if, for any two of them, say $\mathcal{M}$, $\mathcal{M}'$ with $\mathcal{M} \neq \mathcal{M}'$ if $\mathcal{M} \cap \mathcal{M}' \neq \emptyset$, then $\mathcal{M}' \subset \overline{\mathcal{M}}$ and $\dim(\mathcal{M}') < \dim(\mathcal{M})$.

One definition of a *stratification* of $\mathcal{V}$ is finally then [104] a strict partition of $\mathcal{V}$ which has the frontier property. The corresponding set of manifolds are then known as the *strata* of the partition.

The variant that in particular Fischer [23] also makes use of the *inverse frontier property*, i.e. (67) with primed and unprimed switched over, which then feeds into the corresponding notion of inverted stratification. Another sometimes useful [72] property is the *regular stratification property*, which involves $X_i \cup X_j \neq \emptyset \Rightarrow X_i \subseteq X_j \forall i, j \in I$.

Whitney [104] also established that a locally finite partial of $\mathcal{V}$ with the frontier property is a stratification. Moreover, for each stratum $\mathcal{M}$, then $\overline{\mathcal{M}} - \mathcal{M}$ is the union of the other closed strata in $\overline{\mathcal{M}}$. Indeed, any strict partition of $\mathcal{V}$ admits a refinement which is a stratification into connected strata. Take this to be a brief indication that the theory of refinements of partitions (a type of ‘graining’) plays a role in the theory of stratified manifolds.
Due to nontrivial stratified manifolds having strata with a range of different dimensions, clearly the locally Euclidean property of manifolds has broken down, and with it the standard notions of charts and how to patch charts together. These notions still exist for stratified manifolds, albeit in a more complicated form (see Fig 12). Also, in general losing Hausdorffness and second-countability leaves stratified manifolds ‘more to the left’ than topological manifolds in the diagram of the levels of structure. Moreover, this Article considers in any detail only Hausdorff second-countable stratified manifolds, i.e. entities which are ‘2/3rds of a manifold’.

Since Whitney, stratified manifolds have additionally been considered in the case furthermore equipped with differentiable structure. Furthermore, individual strata being manifolds, some are metrizable. E.g. Pflaum [96] then considers Riemannian metric structures on stratified spaces (Kendall also makes use [8] of this level of structure). This permits Pflaum to give furthermore a definition of geodesic distance. Pflaum also considers the morphisms corresponding to stratified manifolds.

Next note that stratified manifolds and bundle theory do not fit well together due to stratified manifolds’ local structure varying from point to point. Three distinct strategies to deal with this are outlined in Sec 1.3. Relational considerations point to the strategy of accepting the stratified manifold. This points to seeking a generalization of fibre bundle mathematics, for which Sheaf Theory (Appendix C.3) is a strong candidate.

Finally note that stratified orbifolds also make sense, and indeed occur in configuration space study: the 3-d case of Fig 7.f).

### B.6 Hausdorff second-countable locally compact (H2LC) spaces

$X$ is locally compact if each point in $X$ is contained in a compact neighbourhood lying within $X$.

Then in particular, Hausdorff second-countable compact spaces are included, and the coning construction is also. Thus many of Sec 2’s configuration spaces from Mechanics fall within this remit.

Furthermore, H2LC spaces are rather well-behaved from an Analysis point of view. As well as being at least ‘2/3rds of a manifold’, HLC suffices to have an analogue of Baire’s Category Theorem (c.f. Sec A.1), and various further Analysis results of note hold for HLC or H2LC [107].

### B.7 Stratifolds

A differential space is a pairing $(X, C)$ of a topological space $X$ and a function space $C$ endowed with algebraic structure; the functions act on $X$. The $C$ generalizes the standard use of smooth functions in elementary algebraic topology.

Sikorski spaces [108] are a prominent early example of such a pairing.

However, in this Article I concentrate on the more recent differential spaces of Kreck [109], which are termed stratifolds. These are rather well-behaved through the $X$ half of the pair being HL2C in this case. Moreover, the $C$ half of the stratifold’s pair receives a sheaf interpretation (outlined in Appendix C.3).

As regards modelling with infinite-d stratifolds, work in this direction has started [110], centering around sheaf methods and study of cohomology, in a Hilbert and Fréchet space setting that does extend to manifolds in these senses.

### C (Pre)sheaves

#### C.1 Categories

A very brief outline of those aspects used in setting up the theory of (pre)sheaves is as follows; see e.g. [111, 112, 113] for further details.

**Categories** $\mathcal{C} = (\mathcal{O}, \mathcal{M})$ consist of objects $\mathcal{O}$ and morphisms $\mathcal{M}$ (the maps between the objects, $\mathcal{M}: \mathcal{O} \rightarrow \mathcal{O}$, obeying the axioms of domain and codomain assignment, identity relations, associativity relations and book-keeping relations. **Functors** are then maps $\mathcal{F}: \mathcal{C}_1 \rightarrow \mathcal{C}_2$ that obey various further axioms concerning domain, codomain, identity and action on composite morphisms.

For example, $\mathcal{S}ets$ is the category of sets; per se this is foundationally trivial, though indirectly it plays a repeated role in the below developments. $\mathcal{V}ec$ is the category of vector spaces, and $\mathcal{T}op$ is the category of topological spaces. Finally functor categories are categories of maps between categories.
For $Y \subset X$, the corresponding inclusion map is the injection $i : Y \to X$ with $i(y) = y \forall y \in Y$.

### C.2 Presheaves

Presheaves are then functors $f : \mathcal{C} \to \mathcal{S}$ (or sometimes some other category such as $\mathcal{V}$) such that the following holds.

Presheaf-1) Each inclusion of open sets $U \subseteq \mathcal{U}$ corresponds to a restriction morphism $\text{res}_{\mathcal{U},U} : f(U) \to f(U)$ in $\mathcal{S}$.

Presheaf-2) $\text{res}_{\mathcal{U},U}$ is the identity morphism.

Presheaf-3) $\text{res}_{\mathcal{V},U} \circ \text{res}_{\mathcal{U},U} = \text{res}_{\mathcal{W},U}$ (transitivity).

For $\mathcal{U}$ an open subset of $X$ (upon which the topological space $\langle X, \tau \rangle$ is based), $f(\mathcal{U})$ is the section of $f$ over $\mathcal{U}$. It is a global section if it is over the whole of $X$ itself. Use of the fibre bundle notation $\Gamma$ for sections carries over to presheaves; moreover, we now write $\Gamma(f, \mathcal{U})$, which is a useful notation since the case in which $\mathcal{U}$ rather than $f$ is fixed is common. This notion of section indeed generalizes that of fibre bundles as regards being the gateway to a more general range of global methods.

### C.3 Sheaves

For a presheaf to additionally be a sheaf [113, 114, 116, 117, 96], two further conditions are required.

Sheaf-1) (local condition): let $\{\mathcal{U}_C\}$ be an open cover of an open set $\mathcal{U}$. If $r, s \in f(\mathcal{U})$ obey $r|_{\mathcal{U}_C} = s|_{\mathcal{U}_C}$ for each $\mathcal{U}_C$, then $r = s$.

Sheaf-2) (gluing condition): let $s_C \in f(\mathcal{U}_C)$ be sections that agree on their pairwise overlaps $s_C|_{\mathcal{U}_C \cap \mathcal{U}_D} = s_D|_{\mathcal{U}_C \cap \mathcal{U}_D}$. Then there exists a section $s \in f(\mathcal{U})$ with $s|_{\mathcal{U}_C} = s_C$ for each $i$ in the cover.

Sheaves are the basis for more general patching constructs. One can now attach heterogeneous objects to different base space points rather than attaching homogeneous fibres in the formation of a fibre bundle. A very simple application of this is to the heterogeneous types of chart involved in the study of a given nontrivial stratified manifold as per Fig 12. See e.g. below and [109, 117, 96] for a wider range of applications to stratified configuration spaces and phase spaces.

By possessing the gluing construction, sheaves provide a means of formulating obstructions that generalizes the topological treatment using fibre bundles of a number of obstructions that are already familiar in Theoretical Physics. In each case, the notion of section has an associated notion of cohomology concerning obstructions to the presence of global sections. In the sheaf case, this has the logical name of sheaf cohomology, and indeed turns out to be widely useful from a computational perspective [116]. This ensures the sheaf encodes the topological level of structure of generalized spaces as well as their geometrical structure.

On paracompact Hausdorff spaces, sheaf cohomology and the somewhat more familiar Čech cohomology coincide [115]. However, more generally, sheaf cohomology extends Čech cohomology; indeed, this is how the former was arrived at by Serre [118] and by Grothendieck [119]. [Historically, sheaves originated in the French School of mathematics through the works of Leray, Henri Cartan, Serre and Grothendieck.]

Moreover, for all that sheaves were not originally developed with singular spaces in mind, Whitney and Thom’s work on the latter proved to be a further place to apply sheaf methods. The more recent development of stratifolds by Kreck is a further variation on this theme. The other half of the stratifold pair is an algebraic structure of continuous functions $C$ which can be interpreted as an algebraic structure of global sections in the sheaf-theoretic sense.

[As regards whether there is relation between slices and sheaves, there is, though it involves the theory of étalé spaces, which lies outside of the scope of the current Article.]

As a concluding punchline, configuration spaces are not in general manifolds, nor are fibre bundle methods always applicable to them either. They are more generally stratified manifolds, for which sheaf methods are more natural and more generally applicable.

---

16I subsequently use the standard notation for restriction $s|_{\mathcal{U}}$ to denote $\text{res}_{\mathcal{U}}(s)$.

17Another manner in which sheaves generalize fibre bundles is in possessing a notion of connection.
References

[1] C. Lanczos, The Variational Principles of Mechanics (University of Toronto Press, Toronto 1949).
[2] See e.g. C. Marchal, Celestial Mechanics (Elsevier, Tokyo 1990).
[3] R.G. Littlejohn and M. Reinsch, “Gauge Fields in the Separation of Rotations and Internal Motions in the N-Body Problem”, Rev. Mod. Phys. 69 213 (1997).
[4] H. Goldstein, Classical Mechanics (Addison-Wesley, Reading, Massachusetts 1980).
[5] P.A.M. Dirac, Lectures on Quantum Mechanics (Yeshiva University, New York 1964).
[6] M. Henneaux and C. Teitelboim, Quantization of Gauge Systems (Princeton University Press, Princeton 1992).
[7] E. Anderson, “The Problem of Time and Quantum Cosmology in the Relational Particle Mechanics Arena”, arXiv:1111.1472.
[8] D.G. Kendall, D. Barden, T.K. Carne and H. Le, Shape and Shape Theory (Wiley, Chichester 1999).
[9] E. Anderson, “Foundations of Relational Particle Dynamics”, Class. Quant. Grav. 25 025003 (2008), arXiv:0706.3934.
[10] E. Anderson, “Relational Mechanics of Shape and Scale”, arXiv:1001.1112.
[11] See e.g. R. Moeckel’s Lecture Course, available at http://www.math.umn.edu/ rick/notes/CMNotes.pdf ; F. Diacu, Singularities of the N-body Problem (Les Publications CRM, Montréal 1992).
[12] R.A. Battye, G.W. Gibbons and P.M. Sutcliffe, “Central Configurations in Three Dimensions”, Proc. R. Soc. Lond. A 459 911 (2003), hep-th/0201101.
[13] E. Merzbacher, Quantum Mechanics (Wiley, New York 1998).
[14] J.B. Barbour and B. Bertotti, Proc. Roy. Soc. Lond. A 382 295 (1982).
[15] J.B. Barbour, “Scale-Invariant Gravity: Particle Dynamics”, Class. Quant. Grav. 20 1543 (2003), gr-qc/0211021.
[16] E. Anderson, “Relational Quadrilateralland. I. The Classical Theory”, Int. J. Mod. Phys. D23 1450014 (2014), arXiv:1202.4186.
[17] R. Montgomery, “Gauge Theory of the Falling Cat”, Fields Institute Communications 86 (1995), gr-qc/9301020.
[18] A. Abrams and R. Ghrist. “Finding topology in a factory: configuration spaces.” American Mathematical Monthly 140 (2002); S.M.
[19] C.W. Misner, “Minisuperspace”, in Geometry, Topology and Physics (Institute of Physics Publishing, London 1990).
[20] J.A. Wheeler, in Battelle Rencontres: 1967 Lectures in Mathematics and Physics ed. C. DeWitt and J.A. Wheeler (Benjamin, New York 1968).
[21] B.S. DeWitt, “Quantum Theory of Gravity. I. The Canonical Theory.”, Phys. Rev. 160 1113 (1967).
[22] A.E. Fischer, “The Theory of Superspace”, in Relativity (Proceedings of the Relativity Conference in the Midwest, held at Cincinnati, Ohio June 2-6, 1969), ed. M. Carmeli, S.I. Fickler and L. Witten (Plenum, New York 1970).
[23] C.W. Misner, “Minisuperspace”, in Magic Without Magic: John Archibald Wheeler ed. J. Klauder (Freeman, San Francisco 1972).
[24] J.W. York Jr., “Covariant Decompositions of Symmetric Tensors in the Theory of Gravitation”, Ann. Inst. Henri Poincaré 21 319 (1974).
[25] J.B. Barbour and F. Mercati, “Classical Machian Resolution of the Spacetime Construction Problem”, arXiv:1311.6541.
[26] E. Anderson and F. Mercati, “Classical Machian Resolution of the Spacetime Construction Problem”, arXiv:1311.6541.
[27] A.E. Fischer and V. Moncrief, “A Method of Reduction of Einstein’s Equations of Evolution and a Natural Symplectic Structure on the Space of Gravitational Degrees of Freedom”, Gen. Rel. Grav. 28, 207 (1996).
[28] J.B. Barbour, B.Z. Foster and N. ó Murchadha, “Relativity Without Relativity”, Class. Quant. Grav. 19 3217 (2002), gr-qc/0012089; E. Anderson and F. Mercati, “Classical Machian Resolution of the Spacetime Construction Problem”, arXiv:1311.6541.
[29] R. Arnowitt, S. Deser and C. Misner, “The Dynamics of General Relativity”, in Gravity: An Introduction to Current Research ed. L. Witten (Wiley, New York 1962), arXiv:gr-qc/0405109.
[30] K.V. Kuchaf, “Time and Interpretations of Quantum Gravity”, in Proceedings of the 4th Canadian Conference on General Relativity and Relativistic Astrophysics ed. G. Kunstatter, D. Vincent and J. Williams (World Scientific, Singapore, 1992), reprinted as Int. J. Mod. Phys. Proc. Suppl. D20 3 (2011); C.J. Isham, “Canonical Quantum Gravity and the Problem of Time”, in Integrable Systems, Quantum Groups and Quantum Field Theories ed. L.A. Ibort and M.A. Rodríguez (Kluwer, Dordrecht 1993), gr-qc/9210011; E. Anderson, “The Problem of Time in Quantum Gravity”, in Classical and Quantum Gravity: Theory, Analysis and Applications ed. V.R. Frigmanni (Nova, New York 2012), arXiv:1009.2157; “Problem of Time in Quantum Gravity”, Annalen der Physik, 524 757 (2012), arXiv:1206.2403; “Problem of Time and Background Independence: the Individual Facets”, arXiv:1409.4117.
A. J. Dragt, “Classification of Three-Particle States According to SU(n), J. Math. Phys. 6 533 (1965); T. Iwai, “A Geometric Setting for Internal Motions of the Quantum Three-Body System”, J. Math. Phys. 28 1315 (1987).

E. Anderson, “Triangleland. I. Classical Dynamics with Exchange of Relative Angular Momentum”, Class. Quant. Grav. 26 135020 (2009), arXiv:0809.1168.

B.S. DeWitt, “Spacetime as a Sheaf of Geodesics in Superspace”, in Relativity (Proceedings of the Relativity Conference in the Midwest, held at Cincinnati, Ohio June 2-6, 1969), ed. M. Carmeli, S.I. Fickler and L. Witten (Plenum, New York 1970).

M. Gromov, “Metric Structures for Riemannian and Non-Riemannian Spaces”, (Birkhäuser, Boston 1999).

M. Reed and B. Simon Methods of Modern Mathematical Physics. II. Fourier Analysis, Self-Adjointness (Academic Press, New York 1975).

C. Kiefer, Quantum Gravity (Clarendon, Oxford 2004).

C. Teitelboim, in General Relativity and Gravitation Vol 1 ed. A. Held (Plenum Press, New York 1980).

H. Stephani, D. Kramer, M.A.H. MacCallum, C.A. Hoenselaers, and E. Herlt Exact Solutions of Einstein’s Field Equations 2nd Edition (Cambridge University Press, Cambridge 2003).

V.A. Belinskii, I.M. Khalatnikov and E.M. Lifshitz, “Oscillatory Approach to a Singular Point in the Relativistic Cosmology”, Adv. Phys. 19 525 (1970).

E. Anderson, “Origin of Structure in the Universe: Quantum Cosmology Reconsidered”, arXiv:1501.02443.

W. Kondracki and J. Rogulski, “On the Stratification of the Orbit Space for the Action of Automorphisms on Connections”, Diss. Math. 250 (1986).

G. Rudolph, M. Schmidt and I.P. Volobuev, “On the Gauge Orbit Space Stratification: a Review”, J. Phys. A. Math. Gen. 35 R1 (2002).

M. Schmidt, “How to Study the Physical Relevance of Gauge Orbit Space Singularities?” Rep. Math. Phys. 51 325 (2003).

R. Gambini and J. Pullin Loops, Knots, Gauge Theories and Quantum Gravity (Cambridge University Press, Cambridge 1996).

R.S. Hamilton, “The Inverse Function Theorem of Nash and Moser”, Bull. Amer. Math. Soc. 7 65 (1982).

S.B. Meyers and N.E. Steenrod. “The Group of Isometries of a Riemannian Manifold”, Ann. Math. 40 400 (1939).

D.G. Ebin, “The Manifold of Riemannian Metrics”, Proc. Symp. Pure Math. AMS 15 11 (1970).

D. Giulini, “Properties of 3-Manifolds for Relativists”, Int. J. Theor. Phys. 33 913 (1994), gr-qc/9308008.

P. Ortega and T.S Ratiu, “Optimal Momentum Map”, in Geometry, Mechanics and Dynamics: Volume in Honour of the 60th Birthday of Jerrold Marsden ed P. Newton, P. Holmes and A. Weinstein (Springer–Verlag, New York 2002); T. Schmah, “A Cotangent Bundle Slice Theorem”, Diff. Geom. Appl. 25 101 (2007), math/0409148.

J.E. Marsden and T.S. Ratiu, Introduction to Mechanics and Symmetry (Springer, New York 1999).

C.W. Misner, “Classical and Quantum Dynamics of a Closed Universe”, in Relativity (Proceedings of the Relativity Conference in the Midwest, held at Cincinnati, Ohio June 2-6, 1969) ed. M. Carmeli, S.I. Fickler and L. Witten (Plenum, New York 1970).

D. Dudał, S.P. Sorella, N. Vandersickel and H. Verschelde;“Gribov no-pole condition, Zwanziger horizon function, Kugo-Ojima confinement criterion, boundary conditions, BRST breaking and all that”, Phys. Rev. D79 121701 (2009), arXiv:0904.0641.

A.E. Fischer and J.E. Marsden, “The Manifold of Conformally Equivalent Metrics”, Can. J. Math. 1 193 (1977).

J.W. York Jr., “Conformally Invariant Orthogonal Decomposition of Symmetric Tensors on Riemannian Manifolds and the Initial-Value Problem of General Relativity”, J. Math. Phys. 14 456 (1973).

H. de A. Gomes, S.B. Gryb and T. Koslowski, “Einstein Gravity as a 3-d Conformally Invariant Theory”, Class. Quant. Grav 28 045005 (2011), arXiv:1010.2481; J.B. Barbour, “Shape Dynamics. An Introduction”, for proceedings of the conference Quantum Field Theory and Gravity (Regensburg, 2010) arXiv:1105.0183; F. Mercati, “A Shape Dynamics Tutorial”, arXiv:1409.0105.

R. Zalaletdinov, “Averaging out the Einstein equations and Macroscopic Spacetime Geometry”, Gen. Rel. Grav. 24 1015 (1992); “Towards a Macroscopic Theory of Gravity”, 673 25 (1993).

A. Ashtekar, and J. Lewandowski, J., “Representation Theory of Analytic Holonomy C∗ algebras”, in Knots and Quantum Gravity, Proceedings of Workshop held at UC Riverside on May 14-16, 1993, Oxford Lecture Series in Mathematics and its Applications 1 ed. J.C. Baez (Clarendon, Oxford and OUP, New York, 1994), arXiv:gr-qc/9311010.

J. Brunnermann and C. Fleischhack, “On the Configuration Spaces of Homogeneous Loop Quantum Cosmology and Loop Quantum Gravity”, arXiv:0709.1621.

C. Fleischhacker, “Loop Quantization and Symmetry: Configuration Spaces”, arXiv:1010.0449.

C. Fleischhack, “Stratification of the Generalized Gauge Orbit Space”, Commun. Math. Phys. 214 607 (2000), math-ph/0001006.

E. Witten, “Quantum Field Theory and the Jones Polynomial”, Comm. Math. Phys. 121 351 (1989).

C. Rovelli and L. Smolin, “Knot Theory and Quantum Gravity”, Phys. Rev. Lett. 61 1155 (1988).
[93] V.A. Vassiliev, “Cohomology of Knot Spaces”, in Theory of Singularities (Advances in Soviet Math., vol. 1) (American Math. Society, 1990).

[94] A blog post by Budney in 2014: https://ldtopology.wordpress.com/2014/06/13/spaces-of-knots-and-low-dimensional-topology/

[95] D.G. Kendall, “Shape Manifolds, Procrustean Metrics and Complex Projective Spaces”, Bull. Lond. Math. Soc. 16 81 (1984).

[96] M.I. Pflaum, Analytic and Geometric Study of Stratified Spaces, Lecture Notes in Mathematics 1768 (Springer, Berlin 2001).

[97] T. Iwai and H. Yamaoka, “Stratified Reduction of Classical Many-Body Systems with Symmetry”, J. Phys. A. Math. Gen 38 2415 (2005).

[98] J. Franchi and Y. Le Jan, “Relativistic Diffusions and Schwarzschild Geometry”, Commun. Pure Appl. Math. 60 187 (2007).

[99] N. Dunford and J.T. Schwarz Linear Operators. Part I. General Theory, (Wiley, Hoboken N.J. 1988).

[100] S. Lang, Differential and Riemannian Manifolds (Springer, New York 1995).

[101] J. Davey, A. Hanany, R.-K. Seong, JHEP 06 010 (2010), arXiv:1002.3609; M. Green, J. Schwarz and E. Witten Superstring Theory. Volume 2. Loop Amplitudes, Anomalies and Phenomenology (Cambridge University Press, Cambridge 1987).

[102] J. Isenberg and J.E. Marsden, A Slice Theorem for the Space of Solutions of Einstein’s Equations, Phys. Rep. 89 179 (1982).

[103] R. Thom, “Les Singularités des Applications Différentiables” (Singularities in Differentiable Maps), Ann. Inst. Fourier (Grenoble) 6 43 (1956).

[104] H. Whitney, “Tangents to an Analytic Variety”, Ann. Math. 81 496 (1965).

[105] H. Whitney, “Complexes of Manifolds”, Proc. Nat. Acad. Sci. USA 33 10 (1946).

[106] J. Śniatycki, Differential Geometry of Singular Spaces and Reduction of Symmetry (Cambridge University Press, Cambridge 2013).

[107] G. Bredon, Sheaf Theory (McGraw–Hill, New York 1986).

[108] J.-L. Brylinski, Loop Spaces, Characteristic Classes and Geometric Quantization (Springer–Verlag, New York 2007).

[109] B. Iversen, Cohomology of Sheaves (Springer–Verlag, Berlin 1986).

[110] N. Dunford and J.T. Schwarz Linear Operators. Part I. General Theory, (Wiley, Hoboken N.J. 1988).

[111] J. Franchi and Y. Le Jan, “Relativistic Diffusions and Schwarzschild Geometry”, Commun. Pure Appl. Math. 60 187 (2007).

[112] J.L. Bell, Toposes and Local Set Theories (Dover, New York 2008).

[113] R. Thom, “Ensembles et Morphismes Stratifiés” (Stratified Spaces and Morphisms), Bull. Amer. Math. Soc. (N.S.) 75 240 (1969).

[114] M. Kreck, Differential Algebraic Topology: From Stratified Spaces to Stratified Spaces (American Mathematical Society, Providence 2010).

[115] M. Kreck, Differential Algebraic Topology: From Stratified Spaces to Stratified Spaces (American Mathematical Society, Providence 2010).

[116] H. Whitney, “Tangents to an Analytic Variety”, Ann. Math. 81 496 (1965).

[117] H. Whitney, “Complexes of Manifolds”, Proc. Nat. Acad. Sci. USA 33 10 (1946).

[118] J.-L. Brylinski, Loop Spaces, Characteristic Classes and Geometric Quantization (Springer–Verlag, New York 2007).

[119] J. Franchi and Y. Le Jan, “Relativistic Diffusions and Schwarzschild Geometry”, Commun. Pure Appl. Math. 60 187 (2007).

[120] J. Śniatycki, Differential Geometry of Singular Spaces and Reduction of Symmetry (Cambridge University Press, Cambridge 2013).

[121] G. Bredon, Sheaf Theory (McGraw–Hill, New York 1986).

[122] J.-L. Brylinski, Loop Spaces, Characteristic Classes and Geometric Quantization (Springer–Verlag, New York 2007).

[123] B. Iversen, Cohomology of Sheaves (Springer–Verlag, Berlin 1986).

[124] M. Banagl, Topological Invariants of Stratified Spaces (Springer–Verlag, Berlin 2007).

[125] J.-P. Serre, “Faisceaux Algébriques Coherents” (Coherent Algebraic Sheaves), Ann. Math. 61 197 (1955).

[126] A. Grothendieck, “Sur Quelques Points d’Algèbre Homologique” I. Tohoku Math. J. 9 119 (1957).