Isomorphism property in nonstandard extensions of 
\textbf{ZFC} universe

Vladimir Kanovei \*, Michael Reeken \†

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Abstract

We study models of HST (a nonstandard set theory which includes, in particular, the \textbf{ZFC} Replacement and Separation schemata and Saturation for well-orderable families of internal sets).

This theory admits an adequate formulation of the \textit{isomorphism property} \textbf{IP}: any two elementarily equivalent internally presented structures of a well-orderable language are isomorphic. \textbf{IP} implies, for instance, that all infinite internal sets are equinumerous, and there exists a unique (up to isomorphism) internal elementary extension of the standard reals.

We prove that \textbf{IP} is independent of HST (using the class of all sets constructible from internal sets) and consistent with HST (using generic extensions of HST models by a sufficient number of generic isomorphisms).

\textit{Keywords}: isomorphism property, nonstandard set theory, constructibility, generic extensions.

\* Moscow Transport Engineering Institute \\
\texttt{kanovei@mech.math.msu.su} and \texttt{kanovei@math.uni-wuppertal.de} \\
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\† Bergische Universität – GHS Wuppertal. \texttt{reeken@math.uni-wuppertal.de}

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Introduction

This article is a continuation of the authors’ series of papers KR 95.1, KR 95.2, KR 96 devoted to set theoretic foundations of nonstandard mathematics. The aim of this paper is to accommodate an important nonstandard tool, the isomorphism property of Henson [He 74], in the context of an axiomatic treatment of nonstandard analysis.

Let $\kappa$ be a cardinal in the ZFC universe. A nonstandard model is said to satisfy the $\kappa$-isomorphism property, $\text{IP}_\kappa$ in brief, iff, whenever $\mathcal{L}$ is a first–order language of $\text{card} \mathcal{L} < \kappa$, any two internally presented elementarily equivalent $\mathcal{L}$-structures are isomorphic. (An $\mathcal{L}$-structure $\mathfrak{A} = \langle A; \ldots \rangle$ is internally presented if the base set $A$ and every interpretation under $\mathfrak{A}$ of a symbol in $\mathcal{L}$ are internal in the given nonstandard model.)

Henson [He 75], Jin [J 92, J 92 a], Jin and Shelah [JS 94] (see also Ross [R 90]) demonstrate that $\text{IP}$ implies several strong consequences unavailable in the frameworks of ordinary postulates of nonstandard analysis, for instance the existence of a set of infinite Loeb outer measure which intersects every set of finite Loeb measure by a set of Loeb measure zero, the theorem that any two infinite internal sets have the same external cardinality, etc.

In the course of this paper, we consider the following formulation of $\text{IP}$ with respect to $\text{HST}$, a nonstandard set theory in the $\text{st}-\in$-language, which reasonably models interactions between standard, internal, and external sets (see Section 1).

$\text{IP}$: If $\mathcal{L}$ is a first–order language containing (standard size)–many symbols then any two internally presented elementarily equivalent $\mathcal{L}$-structures are isomorphic.

(Formally, sets of standard size are those equinumerous to a set of the form: $\sigma S = \{ x \in S : \text{st} x \}$, where $S$ is standard and $\text{st} x$ means: $x$ is a standard set [4])

The following is the main result of the paper referred to in the title.

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The question as how one can adequately develop advanced nonstandard tools like $\text{IP}$ in the frameworks of the “axiomatic” setting of foundations of nonstandard analysis, that is, in a reasonable nonstandard set theory, was discussed in the course of a meeting between H. J. Keisler and one of the authors (V. Kanovei, during his visit to Madison in December 1994).

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The following is the main result of the paper referred to in the title.
Theorem 1 \( \text{IP} \) is consistent with and independent of \( \text{HST} \). In addition, let \( T \) be a theory \( \text{HST} + \text{IP} \) or \( \text{HST} + \neg \text{IP} \). Then

(I) \( T \) is equiconsistent with \( \text{ZFC} \).

(II) \( T \) is a conservative extension of \( \text{ZFC} \) in the following sense. Let \( \Phi \) be a closed \( \in \)-formula, \( \Phi^\text{st} \) be the formal relativization to the predicate \( \text{st} \). Then \( \Phi \) is a theorem of \( \text{ZFC} \) iff \( \Phi^\text{st} \) is a theorem of \( T \).

(III) Every countable model \( S \models \text{ZFC} \) can be embedded, as the class of all standard sets, in a model \( H \) of \( T \), satisfying the following additional property (IV):

(IV) If \( \Phi(x_1, \ldots, x_n) \) is a \( \text{st} \)-\( \in \)-formula then there exists an \( \in \)-formula \( \Phi^*(x_1, \ldots, x_n) \) such that, for all sets \( x_1, \ldots, x_n \in S \), \( \Phi(x_1, \ldots, x_n) \) is true in \( H \) iff \( \Phi^*(x_1, \ldots, x_n) \) is true in \( S \).

The \( \text{HST} \) models involved in the proof of Theorem 1 are obtained as the results of several consecutive extensions of an initial model \( S \) of \( \text{ZFC} \); \( S \) becomes the class of all standard sets in the final and intermediate models. The sequence of extensions contains the following steps:

Step 1. We extend \( S \) to a model \( S^+ \) of \( \text{ZFC} \) plus global choice, adjoining a generic wellordering of the universe by an old known method of Felgner [Fe 71]. \( S \) and \( S^+ \) contain the same sets.

Step 2. We extend \( S \) to a model \( I \) of bounded set theory \( \text{BST} \), a nonstandard set theory similar to \( \text{IST} \) of Nelson [N], using a global choice function from \( S^+ \) to define \( I \) as a kind of ultrapower of \( S^+ \). \( S \) is the class of all standard sets in \( I \). This step was described in [KR 95.1].

Step 3. We extend \( I \) to a model \( H \) of Hrbáček set theory \( \text{HST} \), a nonstandard set theory which contains, for instance, Separation and Collection in the \( \text{st} \)-\( \in \)-language, and Saturation for standard size families of internal sets. The universe \( H \) is isomorphic to an inner \( \text{st} \)-\( \in \)-definable structure in \( I \). This step was described in [KR 95.2]. Elements of \( H \) are essentially those sets which can be obtained from sets in \( I \) by the procedure of assembling sets along wellfounded trees definable in \( I \). \( S \) remains the class of all standard sets in \( H \) while \( I \) becomes the class of all elements of standard sets (that is, internal sets by the formal definition) in \( H \).

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5 In other words it is asserted that \( \text{ZFC} \) proves \( \Phi \) iff \( \text{HST} \) proves that \( \Phi \) holds in the standard subuniverse.

6 Thus the truth of \( \text{st} \)-\( \in \)-formulas with standard parameters in \( H \) can be investigated in \( S \). A similar property was called Reduction in [KR 95.1].

7 It is not known whether \( \text{IST} \) itself admits the treatment similar to steps 1 – 6.
We proved in [KR 95.1, KR 95.2] that BST and HST are equiconsistent with ZFC and are conservative extensions of ZFC in the sense of statement (II).

**Step 4.** Given a model \( \mathbb{H} \) of HST, we define, in Section 4, the class \( L[I] \) of all sets constructible in \( \mathbb{H} \) from internal sets (an inner class in \( \mathbb{H} \)). The particular case we consider (constructibility from internal sets) allows to define the constructible sets much easier than in ZFC, because \( I \) contains all ordinals and essentially all patterns of constructions which might be involved in definitions of constructible sets; this allows to avoid any kind of transfinite recursion in the process.

We prove that \( L[I] \) is a model of HST which satisfies certain additional properties, in particular \( I \)-infinite internal sets of different \( I \)-cardinalities remain non–equinumerous in \( L[I] \), so that \( L[I] \) models the negation of IP. This leads to the proof of different parts of the theorem, with respect to the theory HST + ¬IP.

We prove that in addition \( L[I] \) satisfies the following choice–like statement: for any cardinal \( \kappa \), every \( \kappa \)-closed p. o. set is \( \kappa \)-distributive, which is of great importance for the further use of \( L[I] \) as the ground model for generic extensions.

**Step 5** (actually irrelevant to the content of this paper). Given a standard cardinal \( \kappa \), HST admits a subuniverse \( \mathbb{H}_\kappa \) (an inner st-\( \in \)-definable class in \( \mathbb{H} \)) which models a \( \kappa \)-version of HST (the Saturation and standard size Choice suitably restricted by \( \kappa \)) plus the Power Set axiom. The class \( I_\kappa = I \cap \mathbb{H}_\kappa \) of all internal elements in \( \mathbb{H}_\kappa \) is equal to the collection of all \( x \in I \) which belong to a standard set of \( \mathcal{S} \)-cardinality \( \leq \kappa \). There also exist subuniverses \( \mathbb{H}'_\kappa \) which model a weaker \( \kappa \)-version of HST plus Power Set and full Choice. These subuniverses are defined and studied in [KR 96].

**Step 6.** To get a model for HST + IP, we construct a generic extension \( L[I][G] \) of an HST model of the form \( L[I] \) (or any other model \( \mathbb{H} \) of HST satisfying the abovementioned choice–like principle), where \( G \) is a generic class, essentially a collection of generic isomorphisms between suitable internally presented elementarily equivalent structures.

We show how to develop forcing in HST in Section 5. The forcing technique in principle resembles the classical ZFC patterns. However there are important differences. In particular to prevent appearance of new collections of standard sets in generic extensions (which would contradict Standardization, one of the axioms of HST), the forcing notion must be standard size distributive. More differences appear in the class version of forcing introduced in Section 5. In particular, to provide Separation and Collection in
the class generic extensions we consider, a permutation technique is used; in
the ZFC version, this tool is usually applied for different purposes.

We demonstrate in Section 4 how to force a generic isomorphism between
two particular internally presented elementarily equivalent structures \( \mathfrak{A} = \langle A; \ldots \rangle \) and \( \mathfrak{B} = \langle B; \ldots \rangle \) in a model \( \mathbb{H} \) of HST. The forcing notion
consists of internal 1–1 maps from an internal \( A' \subseteq A \) onto an internal
\( B' \subseteq B \), such that every \( a \in A' \) behaves in \( \mathfrak{A} \) completely as \( b = p(a) \)
behaves in \( \mathfrak{B} \). This requirement means, for instance, that \( a \) satisfies an
\( \mathcal{L} \)-formula \( \Phi(a) \) in \( \mathfrak{A} \) iff \( b = p(a) \) satisfies \( \Phi(b) \) in \( \mathfrak{B} \). But not only
this. The main technical problem is how to expand a condition \( p \) on some
\( a \in A \setminus \text{dom} \ p \), in other words, to find an appropriate counterpart \( b \in B \setminus \text{ran} \ p \)
which can be taken as \( p(a) \). To carry out this operation, we require that
conditions \( p \) preserve sentences of a certain type–theoretic extension of \( \mathcal{L} 
\), the original first–order language, rather than merely \( \mathcal{L} \)-formulas.

It is worth noticing that the generic isomorphisms \( H \) obtained by this
forcing satisfy an interesting additional requirement. They are locally internal
in the sense that, unless \( A \) and \( B \) are sets of standard finite number
of elements, for any \( a \in A \) there exists an internal set \( A' \subseteq A \), containing
more than a standard finite number of elements, such that \( H \upharpoonright A' \) is internal.

Section 5 demonstrates how to gather different generic isomorphisms in
a single generic class using product forcing with internal \( \mathbb{I} \)-finite support.
(Fortunately new internally presented structures do not appear, so that the
product rather than iterated forcing can be used here.) This results in a theo-
rem which says that every countable model of HST satisfying the above-
mentioned additional property admits an extension with the same standard
and internal sets, which satisfies HST + IP. This leads to the proof of
different parts of Theorem 1, with respect to the theory HST + IP.

The countability assumption is used here simply as a sufficient condition
for the existence of generic sets. (We shall in fact, for the sake of convenience,
consider wellfounded HST models — those having a well-founded class of
ordinals in the wider universe — but show how one obtains the result in
the general case.) If the ground model is not assumed to be countable, a
Boolean–valued extension is possible, but we shall not proceed in this way.

Section 6 completes the proof of Theorem 1.

We use the ordinary set theoretic notation, with perhaps one exception:

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8 This type of forcing may lead to results related to nonstandard models in the ZFC
universe, like the following: if \( U \) is an \( \aleph_1 \)-saturated nonstandard structure then there
exists a generic extension of the ZFC universe where \( U \) remains \( \aleph_1 \)-saturated and
satisfies IP\(_{\aleph_1}\), in the sense of locally internal isomorphisms. But this is a different story.
the $f$-image of a set $X$ will be denoted by $f"X = \{f(x) : x \in X\}$. The model theoretic notation will be elementary, self-explanatory, and consistent with Chang and Keisler [CK 90]. The reader is assumed to have an acquaintance with forcing and basic ideas and technique of nonstandard mathematics.

**Remark**

It is sometimes put as a reservation by opponents of the nonstandard approach that the nonstandard real numbers are not uniquely defined, as long as one defines them by different constructions like ultrapowers of the “standard” reals. (See Keisler [Ke 94] for a discussion of this matter.)

It is obvious that any typical nonstandard set theory defines the standard reals and a nonstandard extension of the real line (the reals in the internal universe) uniquely. However **HST** does a little bit more: it is a particular property of this theory that the internal universe $\mathbb{I}$ admits an $\in$-definition in the external (bigger) universe $\mathbb{H}$, as a class of all sets $x$ such that there exists $y$ satisfying the property that the set $\{z : x \in z \in y\}$ is linearly ordered but not well-founded. Thus the nonstandard reals are in a sense $\in$-unique, not merely $st$-$\in$-unique, in **HST**. (Without important changes in the set-up a result like this is hardly possible in nonstandard “superstructures”, where one drops all the memberships from the ground level to carry out the construction.)

The isomorphism property **IP**, if it holds in the external universe, makes the uniqueness much more strong: simply all internally presented elementary extensions of the standard reals are mutually isomorphic.

As for the standard reals, they are also $\in$-unique in **HST**, up to isomorphism at least, because the standard subuniverse $\mathbb{S}$ is isomorphic in **HST** to an $\in$-definable class, the class $\mathbb{V}$ of all wellfounded sets. (This property in principle allows to develop mathematics in **HST** in terms of asterisks rather than the standardness predicate, see Subsection 1.5 below.)

On the other hand, Theorem [1] makes it clear that, as long as one is interested in the study of standard mathematical objects, one can legitimately consider things so that the standard universe $\mathbb{S}$ is in fact the standard part of a wider universe $\mathbb{H}$ of **HST + IP**, where different phenomena of “nonstandard” mathematics can be adequately presented.
1 Basic set theory in HST

The development of HST in this section is based in part on ideas in early papers on external set theory of Hrbáček \[\text{Hr} 78, \text{Hr} 79\] and Kawai \[\text{Kaw} 83\].

1.1 The axioms

Hrbáček set theory HST (introduced in \[\text{KR} 95.2\] on the base of an earlier version of Hrbáček \[\text{Hr} 78\]) is a theory in the $\text{st} \in$-language. It deals with three types of sets, standard, internal, and external.

Standard sets are those $x$ satisfying $\text{st} x$. Internal sets are those sets $x$ which satisfy $\text{int} x$, where $\text{int} x$ is the $\text{st} \in$-formula $\exists \text{st} y (x \in y)$ (saying: $x$ belongs to a standard set). Thus the internal sets are precisely all the elements of standard sets. External sets are simply all sets.

Definition 2 $\mathbb{S}$, $\mathbb{I}$, $\mathbb{H}$ will denote the classes of all standard and all internal sets, and the universe of all sets, respectively.

$\sigma X = \{x \in X : \text{st} x\} = X \cap \mathbb{S}$ for any set $X$.

HST includes the following axioms:

Ax.1 Axioms for standard and internal sets:

(a) $\Phi^{\text{st}}$, where $\Phi$ is an arbitrary axiom of ZFC;

(b) Transfer: $\exists \text{int} x \Phi^{\text{int}}(x) \rightarrow \exists \text{st} x \Phi^{\text{int}}(x)$,

where $\Phi$ is an $\in$-formula containing only standard parameters, and $\Phi^{\text{int}}$ denotes relativization of $\Phi$ to the class $\{x : \text{int} x\}$;

(c) $\forall \text{int} x \forall y \in x (\text{int} y)$: transitivity of the internal subuniverse.

Ax.2 Standardization: $\forall X \exists^\text{st} Y (\sigma X = \sigma Y)$.

Ax.3 The ZFC Pair, Union, Extensionality, Infinity axioms, together with Separation, Collection, Replacement for all $\text{st} \in$-formulas.

Ax.4 Weak Regularity: for any nonempty set $X$ there exists $x \in X$ such that $x \cap X$ contains only internal elements.

Ax.5 Saturation: if $\mathcal{X}$ is a set of standard size such that every $X \in \mathcal{X}$ is internal and the intersection $\bigcap \mathcal{X}'$ is nonempty for any finite nonempty $\mathcal{X}' \subseteq \mathcal{X}$, then $\bigcap \mathcal{X}$ is nonempty.

Ax.6 Choice in the case when the domain $X$ of the choice function is a set of standard size (standard size Choice), and Dependent Choice.
1.2 Comments on the axioms

The quantifiers \( \exists^\text{st}, \forall^\text{st}, \exists^\text{int} \) above have the obvious meaning (there exists a standard \( \ldots \), etc.). The first two will be of frequent use below.

Axiom schema \( \text{Ax.1a} \) says that \( S \), the class of all standard sets, models \( \text{ZFC} \). (Of course the \( \text{ZFC} \) Separation, Collection, and Replacement schemata are assumed to be formulated in the \( \in \)-language in this item.) As an immediate corollary, we note that this implies \( S \subseteq I \).

Transfer \( \text{Ax.1b} \) postulates that \( I \), the universe of all internal sets, is an elementary extension of \( S \) in the \( \in \)-language. Axiom \( \text{Ax.1c} \) says that the internal sets form the ground in the \( \in \)-hierarchy in \( H \), the main universe.

We do not include here \( \text{BST}^{\text{int}} \), all the axioms of \( \text{BST} \) (see Subsection 1.7 on the matters of \( \text{BST} \) ) relativized to the subuniverse \( I \) of all internal sets, to the list of axioms, as it was the case in \( [\text{KR 95.3}, \text{KR 96}] \). Only Transfer and \( \text{ZFC} \) are included explicitly. But the rest of the \( \text{BST} \) axioms are more or less simple corollaries of other axioms, see Proposition 20.

Standardization \( \text{Ax.2} \) is very important: it guarantees that \( H \) does not contain collections of standard sets other than those which essentially already exist in \( S \). A simple corollary: a set in \( H \) cannot contain all standard sets. One more application is worth to be mentioned.

**Lemma 3** [Boundedness] *If \( X \subseteq I \) then \( X \subseteq S \) for a standard \( S \).*

**Proof** Each \( x \in X \) is internal, hence belongs to a standard \( s \). By the Collection axiom, there is a set \( B \) such that every \( x \in X \) belongs to a standard \( s \in B \). By Standardization, there exists a standard set \( A \) containing the same standard elements as \( B \) does. We put \( S = \bigcup A \).

Group \( \text{Ax.3} \) misses the Power Set, Choice, and Regularity axioms of \( \text{ZFC} \). Choice and Regularity still are added in weaker forms below. This is not a sort of incompleteness of the system; in fact each of the three mentioned axioms contradicts \( \text{HST} \).

Axiom \( \text{Ax.4} \) says that all sets are well-founded over the universe \( I \) of internal sets, in the same way as in \( \text{ZFC} \) all sets are well-founded over \( \emptyset \). (Take notice that \( I \) itself is not well-founded in \( H \) : e.g. the set of all nonstandard \( I \)-natural numbers does not contain an \( \in \)-minimal element.) There is, indeed, an essential difference with the \( \text{ZFC} \) setting: now \( I \), the ground level, explicitly contains a sufficient amount of information about the ordinals which determine the cumulative construction of \( H \) from \( I \).

Sets of standard size are those of the form \( \{ f(x) : x \in ^{\sigma} X \} \), where
is standard and \( f \) any function. However we shall see that in HST, “standard size” = “well-orderable” = “equinumerous to a well-founded set”. The notion of a finite set in Axiom \( \text{Ax.5} \) will be commented upon below.

It was convenient in [KR 95.2, KR 96] to include one more axiom, Extension, to the list of HST axioms. Here we obtain it as a corollary.

**Lemma 4** [Extension] Suppose that \( S \) is a standard set and \( F \) is a function defined on the set \( \sigma S = \{ x \in S : \text{st} x \} \), and \( F(x) \) contains internal elements for all \( x \in \sigma S \). Then there exists an internal function \( f \) defined on \( S \) and satisfying \( f(x) \in F(x) \) for every \( x \in \sigma S \).

**Proof** We use the standard size Choice to obtain a (perhaps non–internal) function \( g : \sigma S \rightarrow I \) satisfying \( g(x) \in F(x) \) for all standard \( x \in S \). It remains to apply Saturation to the family of (obviously internal) sets \( G_x = \{ f \in I : \text{dom} f = S \& g(x) = f(x) \} \), where \( x \in \sigma S \).

\[\Box\]

### 1.3 Condensation of standard sets to well-founded sets

It is a typical property of nonstandard structures that standard sets reflect a part of the external universe. In the HST setting, this phenomenon appears in an isomorphism between the standard subuniverse \( S \) and a certain transitive subclass of \( \mathbb{H} \) — the class of all well-founded sets.

**Definition 5** We define \( \bar{x} = \{ y : y \in \sigma x \} \) for every \( x \in S \).

We set \( V = \{ \bar{x} : x \in S \} \) (the condensed subuniverse.)

The next lemma shows that this definition is legitimate in HST.

**Lemma 6** The restriction \( \in \upharpoonright S \) is a well-founded relation in \( \mathbb{H} \).

**Proof** Consider a nonempty set \( X \subseteq S \). By Standardization, there exists a standard set \( S \) such that \( X = \sigma S = S \cap S \). Since \( S \) models ZFC, \( S \) contains, in \( S \), an \( \in \)-minimal element \( s \in S \). Then \( s \) is \( \in \)-minimal in \( S \) also in the subuniverse \( I \) by Transfer, and in \( \mathbb{H} \) by the definition of \( I \).

Thus \( \bar{x} \) is well defined in \( \mathbb{H} \) for all \( x \in S \).

**Lemma 7** The map \( x \mapsto \bar{x} \) is an \( \in \)-isomorphism \( S \) onto \( V \). \( V \) is a transitive class in \( \mathbb{H} \), and an \( \in \)-model of ZFC. Every subset of \( V \) belongs to \( V \).
Proof  We have to prove first that \( x = y \) iff \( \overline{x} = \overline{y} \), and \( x \in y \) iff \( \overline{x} \in \overline{y} \), for all \( x, y \in S \). Only the direction \( \leftarrow \) is not obvious. Let \( \overline{x} = \overline{y} \). Then for each standard \( x' \in x \) there exists standard \( y' \in y \) such that \( \overline{x'} = \overline{y'} \), and vice versa. This observation, plus Transfer (to see that standard sets having the same standard elements are equal) provides the proof of the first assertion by the induction on the \( \in \)-rank of \( x, y \) in \( S \) (based on Lemma 3).

To prove the second assertion, let \( \overline{x} \in \overline{y} \). Then by definition \( \overline{x} = \overline{x'} \) for some standard \( x' \in y \). Therefore \( x = x' \in y \), as required.

Let \( W \subseteq \forall \). Then \( X = \{ x : \overline{x} \in W \} \) is a subset of \( S \), so \( X = \sigma S' \) for a standard \( S \) by Standardization. It follows that \( W = S \), as required. \( \square \)

Thus we have a convenient transitive copy \( \forall \) of \( S \) in \( \mathbb{H} \).

Let a well-founded set mean: a set \( x \) which belongs to a transitive set \( X \) such that \( \in \mid X \) is a well-founded relation. (Axioms of group \( \text{Ax.3} \) suffice to prove that every set belongs to a transitive set \( X \), but the membership on \( X \) may be ill-founded. Take e. g. the set of all \( \mathbb{I} \)-natural numbers.)

Lemma 8  \( \forall \) is the class of all well-founded sets in \( \mathbb{H} \).

Proof  Every \( w \in \forall \) is well-founded in \( \mathbb{H} \) by Lemma 4, because this is true for sets in \( S \). Suppose that \( W \in \mathbb{H} \) is well-founded, and prove that \( W \) belongs to \( \forall \). We can assume that \( W \) is transitive and \( \in \mid W \) is a well-founded relation. In this assumption, let us prove that \( w \in \forall \) for all \( w \in W \), by \( \in \)-induction. Suppose that \( w \in W \), and it is already known that each \( w' \in w \) belongs to \( \forall \). Then \( w \in \forall \) by Lemma 4. \( \square \)

Proposition 9  In \( \mathbb{H} \), \( x \in \mathbb{I} \) iff there is a set \( y \) such that the “interval” \( \{ z : x \in z \in y \} \) is linearly ordered by \( \in \) but not well-ordered. \( \square \)

(This will not be used below, so we leave the proof for the reader.)

1.4 Ordinals and cardinals in the external universe

\( \text{ZFC} \) admits several formally different but equivalent definitions of ordinals. Since not all of them remain equivalent in \( \text{HST} \), let us make it clear that an ordinal is a transitive set well-ordered by \( \in \). The following lemma will be used to prove that \( \mathbb{H} \) and \( \forall \) contain the same ordinals.

Lemma 10  Every set \( w \in \forall \) can be well-ordered and has standard size in \( \mathbb{H} \). Conversely if \( z \in \mathbb{H} \) is a set of standard size or can be well-ordered in \( \mathbb{H} \) then \( z \) is equinumerous with some \( w \in \forall \) in \( \mathbb{H} \).
Proof Each set \( w \in V \) is a set of standard size in \( H \) because we have \( w = \mathcal{P} = \{ \mathcal{P} : y \in \mathcal{P}_x \} \) for a standard \( x \). To prove that \( w \) can be well-ordered, it suffices to check that \( \mathcal{P} x = x \cap S \) can be well-ordered in \( H \). Let \( \prec \) be a standard well-ordering of \( x \) in \( S \). Then \( \prec \) may be not a well-ordering of \( x \) in \( H \) since \( x \) obtains new subsets in \( H \). But \( \prec \) still well-orders \( \mathcal{P} x \). (Indeed let \( u' \subseteq \mathcal{P} x \) be a nonempty set. Then \( u' = \mathcal{P} u \) for a standard set \( u \subseteq x \) by Standardization. Take the \( \prec \)-least element of \( u \) in \( S \).

To prove the converse, note that, in \( H \), every set of standard size can be well-ordered — by the previous argument. Thus let \( Z \in H \) be well-ordered in \( H \); let us check that \( Z \) is equinumerous with a set \( W \in V \).

Since the class \( S0rd \) of all \( S \)-ordinals (i.e. standard sets which are ordinals in \( S \)) is well-ordered by Lemma 8, either there exists an order preserving map: \( S0rd \) onto an initial segment of \( Z \) or there exists an order preserving map: \( Z \) onto a proper initial segment of \( S0rd \).

The “either” case is impossible by axioms \( Ax.3 \) and Lemma 8. Thus we have the “or” case. Let \( \lambda \) be the least standard ordinal which does not belong to the proper initial segment. Then we have a \( 1 - 1 \) map from \( \mathcal{P} \lambda \) onto \( Z \). Then \( W = \lambda \in V \) admits a \( 1 - 1 \) map onto \( Z \), as required.

Corollary 11 Universes \( H \) and \( V \) contain the same ordinals.

Proof Suppose that \( \xi \in V \) is an ordinal in \( V \). Then \( \xi \) remains an ordinal in \( H \) because all elements and subsets of \( \xi \) belong to \( V \) by Lemma 7.

Conversely if \( \xi \in H \) is an ordinal in \( H \) then by Lemma 7, \( \xi \) is equinumerous with a set \( w \in V \). Thus \( w \) admits a well-ordering of length \( \xi \) in \( V \) because subsets of sets in \( V \) belong to \( V \) by Lemma 7. Therefore \( \xi \) is order isomorphic to a set \( \xi' \in V \) which is an ordinal in \( V \). This easily implies \( \xi = \xi' \in V \).

Let a cardinal mean: an ordinal not equinumerous to a smaller ordinal.

Corollary 12 Universes \( H \) and \( V \) contain the same cardinals.

The notions of a regular, singular, inaccessible cardinal, and the exponentiation of cardinals, are absolute in \( H \) for \( V \).

Proof If \( \kappa \in V \) is a cardinal in \( V \) then at least \( \kappa \) is an ordinal in \( H \). A possible bijection onto a smaller ordinal in \( H \) is effectively coded by a subset of \( \kappa \times \kappa \), therefore it would belong to \( V \). The absoluteness holds because \( V \) contains all its subsets in \( H \) as elements by Lemma 7. ☐
Definition 13 \( \text{Ord} \) is the class of all ordinals in \( H \) (or in \( V \), which is equivalent by the above). Elements of \( \text{Ord} \) will be called ordinals below. \( \text{Card} \) is the class of all cardinals in \( H \) (or in \( V \)). Elements of \( \text{Card} \) will be called cardinals below. □ Ordinals, cardinals in \( S \) and \( I \) will be called resp. \( S \)-ordinals, \( S \)-cardinals, and \( I \)-ordinals, \( I \)-cardinals.

Since \( V \) models \( \text{ZFC} \), the ordinals satisfy usual theorems, e.g. \( \text{Ord} \) is well-ordered by the relation: \( \alpha < \beta \) iff \( \alpha \in \beta \), an ordinal is the set of all smaller ordinals, \( 0 = \emptyset \) is the least ordinal, there exist limit ordinals, etc. Furthermore the ordinals can be used to define the rank of sets in \( H \) over \( I \), the internal subuniverse.

Definition 14 The rank over \( I \), \( \text{irk} x \in \text{Ord} \), is defined for each set \( x \) in \( H \) as follows: \( \text{irk} x = 0 \) for internal sets \( x \), and \( \text{irk} x = \sup_{y \in x} \text{irk} y \) for \( x \notin I \). □

(For \( O \subseteq \text{Ord} \), \( \sup O \) is the least ordinal strictly bigger than all ordinals in \( 0 \).) This is well defined in \( H \), by Axiom \( \text{Ax.4} \) (Weak Regularity).

1.5 Change of standpoint. Asterisks

It looks quite natural that \( S \) and \( I \), the classes of all resp. standard and internal sets, are the principal objects of consideration, e.g. because \( S \) is naturally identified with the original set universe of “conventional” mathematics while \( I \) with an ultrapower of \( S \). Following this approach, one considers \( H \) as an auxiliary universe and the notions related to \( H \) as auxiliary notions while the notions related to \( S \) or \( I \) as primary notions.

However at this moment it becomes more convenient to treat the notions related to \( H \) and \( V \) as primary notions, as in Definition 13 above. In a sense, \( V \) is a better copy of the “conventional” set universe in \( H \) than \( S \) is, in particular because \( V \), unlike \( S \), is transitive.

This change of standpoint leads to an interesting parallel with the model theoretic version of nonstandard analysis.

Definition 15 Let, for a set \( w \in V \), \( \ast w \) denote the set \( x \in S \) (unique by Lemma 7) which satisfies \( w = \overline{x} \). □

Corollary 16 \( w \mapsto \ast w \) is an \( \in \)-isomorphism \( V \) onto \( S \).
\( \alpha \in V \) is an ordinal (in \( V \) or in \( H \)) iff \( ^\ast \alpha \) is an \( S \)-ordinal.

\( \alpha \in V \) is a cardinal (in \( V \) or in \( H \)) iff \( ^\ast \alpha \) is an \( S \)-cardinal. \( \Box \)

We have approximately the same as what they deal with inside the model theoretic approach: \( H \) corresponds to the basic set universe, \( V \) to a standard model, \( I \) to its ultrapower (or a nonstandard extension of another type); the map \( w \mapsto ^\ast w \) is an elementary embedding \( V \) to \( I \).

It is an advantage of our treatment that the basic relations in both \( V \) and \( I \) are of one and the same nature, namely, restrictions of the basic relations in \( H \), the external universe.

1.6 Finite sets and natural numbers

Let a natural number mean: an ordinal smaller than the least limit ordinal.

Let a finite set mean: a set equinumerous to a natural number (which is, as usual, equal to the set of all smaller numbers).

The notion of finite set is absolute for \( V \) in \( H \) because every subset of \( V \) belongs to \( V \). On the other hand, \( w \in V \) is finite in \( V \) iff \( ^\ast w \) is finite in \( S \), by Corollary 16.

**Definition 17** \( N \) is the set of all natural numbers in \( H \) (or in \( V \), which is equivalent). Elements of \( N \) will be called natural numbers below. A finite set will mean: a set finite in the sense of \( H \) (or \( V \), which is equivalent provided the set belongs to \( V \).) \( \Box \)

Natural numbers in \( S \) and \( I \) will be called resp. \( S \)-natural numbers (this will become obsolete) and \( I \)-natural numbers. The notions of \( S \)-finite set and \( I \)-finite set will have similar meaning.

**Lemma 18** If \( n \in \mathbb{N} \) then \( ^\ast n = n \). Therefore the classes \( H, V, S \) have the same natural numbers. \( \Box \)

**Proof** We prove the equality \( ^\ast n = n \) by induction on \( n \). Suppose that \( ^\ast n = n \) and prove \( ^\ast (n + 1) = n + 1 \). Since \( n \) and \( n + 1 \) are consecutive ordinals, we have \( n + 1 = n \cup \{ n \} \) in \( V \). We conclude that \( ^\ast (n + 1) = n \cup \{ n \} \) in \( S \) by Lemma 7 in \( I \) by Transfer, and finally in \( H \) because \( I \) is transitive in \( H \). Thus \( ^\ast (n + 1) = n + 1 \). \( \Box \)

**Lemma 19** Any standard \( S \)-finite set \( X \) satisfies \( X \subseteq S \). Conversely any finite \( X \subseteq S \) is standard and \( S \)-finite.
Proof Let \( X \in S \) be \( S \)-finite. Then \( X = \{f(k) : k < n\} \), where \( n \) is an \( S \)-natural number and \( f \) is a standard function. Then \( n = \#n, \) and \( k = \#k \in S \) by Lemma \([8]\), so every \( x = f(k) \in X \) is standard by Transfer.

To prove the converse, let \( X \subseteq S \) be a finite set. Then \( Y = \{\pi : x \in X\} \) is a finite subset of \( V \), so that \( Y \in V \) by Lemma \([7]\). The set \( \#Y \in S \) is a standard \( S \)-finite set, therefore \( \#Y \subseteq S \). We observe that \( X \) and \( \#Y \) contain the same standard elements, since \( \#(\pi) = x \). Thus \( X = \#Y \) is a standard \( S \)-finite set, as required. \( \square \)

1.7 Axioms of BST in the internal subuniverse

*Bounded set theory* BST (explicitly introduced by Kanovei \([Kan 91]\), but very close to the “internal part” of a theory in \([Hr 78]\)) is a theory in the \( st \)-\( \in \)-language, which includes all of ZFC (in the \( \in \)-language) together with the following axioms:

- **Bounded Idealization** BI: \( \forall^{st\text{fin}} A \exists x \in X \forall a \in A \Phi(x,a) \iff \exists x \in X \forall^{st} a \Phi(x,a) \);  
- **Standardization** S: \( \forall^{st} X \exists Y \forall^{st} x [x \in Y \iff x \in X \& \Phi(x)] \);  
- **Transfer** T: \( \exists x \Phi(x) \rightarrow \exists^{st} x \Phi(x) \);  
- **Boundedness** B: \( \forall x \exists^{st} X (x \in X) \).

The formula \( \Phi \) must be an \( \in \)-formula in BI and T, and \( \Phi \) may contain only standard sets as parameters in T, but \( \Phi \) can be any \( st \)-\( \in \)-formula in S and contain arbitrary parameters in BI and S. \( \forall^{st\text{fin}} A \) means: *for all standard finite* \( A \). \( X \) is a standard set in BI.

Thus BI is weaker than the Idealization I of internal set theory IST of Nelson \([N 77]\) (I results by replacing in BI the set \( X \) by the universe of all sets), but the Boundedness axiom B is added.

We proved in \([KR 95.2]\) that any model I of BST can be enlarged to a HST model (where it becomes the class of all internal sets) by assembling sets along well-founded trees. The following is the converse.

**Proposition 20** The class I of internal sets in H models BST.

Proof Boundedness in I follows by the definition of the formula \( \text{int} \).

The BST Standardization in I follows from Axiom \( \text{Ax.2} \). Only the BST Bounded Idealization BI needs some care.
Let $\Phi$ be an $\in$-formula with internal sets as parameters, $X$ a standard set. We have to prove that the following is true in $I$:

$$\forall^{\text{st fin}} A \exists x \in X \forall a \in A \Phi(x, a) \iff \exists x \in X \forall^{\text{st}} a \Phi(x, a).$$

We prove only the direction $\rightarrow$; the other direction follows from the fact that standard $S$-finite sets contain only standard elements by Lemma 19.

The main technical problem is to bound the variable $a$ by a standard set. We can assume that $\Phi$ contains only one parameter, an internal set $p_0$, which is, by definition, a member of a standard set $P$. For any $a$, we let $Z_a = \{ (p, x) \in P \times X : \Phi^{\text{int}}(x, a, p) \}$. By the ZFC Collection and Transfer, there exists a standard set $A_0$ such that $\forall a' \exists a \in A_0 (Z_a = Z_a')$. We put $X_a = \{ x \in X : \Phi^{\text{int}}(x, a, p_0) \}$ for all $a$.

We verify that the family $X_a$ of all sets $X_a$, $a \in \sigma A_0 = \{ a \in A_0 : \text{st } a \}$, satisfies the requirements of Axiom $\text{Ax.5 (Saturation)}$. Indeed each set $X_a$ is internal, being defined in $I$ by an $\in$-formula with internal parameters. Let $X' \subseteq X$ be a finite subset of $X$. By Replacement in $\mathbb{H}$, there exists a finite set $A \subseteq A_0$ such that $X' = \{ X_a : a \in A \}$. We observe that $A$ is standard and $S$-finite by Lemma 19. Therefore, by the left-hand side of $\text{BI}$, the intersection $\bigcap X' = \bigcap_{a \in A} X_a$ is nonempty, as required.

Axiom $\text{Ax.5}$ gives an element $x \in \bigcap X$. We prove that $x$ witnesses the right-hand side of $\text{BI}$. It suffices to check $\forall^{\text{st}} a' \exists^{\text{st}} a \in A_0 (X_a = X_a')$. Consider a standard $a'$. Then $Z_a = Z_a'$ for a standard $a \in A_0$ by the choice of $A_0$, so $X_a = \{ x : (p_0, x) \in Z_a \} = \{ x : (p_0, x) \in Z_a' \} = X_{a'}$. \hfill \Box

### 1.8 Elementary external sets

Let an elementary external set mean a (in $I$) $\text{st-}\in$-definable subclass of an internal set. This looks unsound, but fortunately objects of this type admit a sort of uniform description (first discovered in [Kan 94]), given by

$$C_p = \bigcup_{a \in \eta A} \bigcap_{b \in \eta B} \eta(a, b), \quad \text{where } p = (A, B, \eta), \ A, B \text{ are standard sets, } \eta \text{ being an internal function defined on } A \times B.$$  

If $p \in I$ is not of the mentioned form then we set $C_p = \emptyset$.

**Theorem 21** Let $\Phi(x, q)$ be a $\text{st-}\in$-formula. The following is a theorem of $\text{BST}$: $\forall q \forall^{\text{st}} X \exists p (C_p = \{ x \in X : \Phi(x, q) \})$. \hfill \Box

This result (Theorem 16.3 in [Kan 94], Theorem 2.2 in [KR 95.2]) is an easy consequence of a theorem which asserts that every $\text{st-}\in$-formula is provably
equivalent in \textbf{BST} to a $\Sigma^\text{st}_2$ formula ³ (Theorem 1.5 in \cite{KR95}), and a lemma which allows to restrict the two principal quantifiers in a $\Sigma^\text{st}_2$ formula by standard sets (Lemma 1.7 in \cite{KR95}, see a corrected proof in \cite{KR96} or a more complicated earlier proof in \cite{Kan94}, Lemma 15.1).

\footnote{9 We recall that $\Sigma^\text{st}_2$ denotes the class of all formulas $\exists^\text{st} a \forall^\text{st} b \, (\varepsilon\text{-formula}).$}
2 Constructibility from internal sets in HST

Let, as above, \( \mathbb{H} \) be a universe of HST, \( S \subseteq I \) be the classes of all standard and internal sets in \( \mathbb{H} \), \( V \) the condensed subuniverse of \( \mathbb{H} \) introduced in Subsection 1.3.

The aim of this section is to define an inner class in \( \mathbb{H} \) which models HST + \( \neg \)IP, a contribution to the independence part of Theorem 1. We shall use \( L[I] \), the class of all sets constructible from internal sets. The main result (Theorem 28 below) is similar to what might be expected in the ZFC case: \( L[I] \) models HST plus an extra choice–like principle. In addition, the isomorphism property IP fails in \( L[I] \).

HST is obviously much more cumbersome theory than ZFC is; this leads, in principle, to many additional complications which one never meets running the constructibility in ZFC.

On the other hand, it is a certain relief that the initial class \( I \) contains all standard sets. Indeed, since \( S \) models ZFC, one has, in \( I \), an already realized example of the constructible hierarchy, essentially of the same length as we are looking for in \( \mathbb{H} \), because \( S \) and \( \mathbb{H} \) have order isomorphic classes of ordinals by Corollary 16. This allows to use a strategy completely different from that in ZFC, to define constructible sets. We introduce \( L[I] \) as the class of all sets obtainable in \( \mathbb{H} \) via the procedure of assembling sets along wellfounded trees (which starts from sets in \( I \) and involves trees \( \varepsilon \)-\( \varepsilon \)-definable in \( I \)) as they obtain sometimes models of fragments of ZFC from models of 2nd order Peano arithmetic.

To make the exposition self-contained, we give a brief review of the relevant definitions and results in [KR 96].

2.1 Assembling sets along wellfounded trees

Let \( \text{Seq} \) denote the class of all internal sequences, of arbitrary (but internal) sets, of finite length. For \( i \in \text{Seq} \) and every set \( a \), \( t^a \) is the sequence in \( \text{Seq} \) obtained by adjoining \( a \) as the rightmost additional term to \( t \). The notation \( a^t \) is to be understood correspondingly.

A tree is a non-empty (possibly non–internal) set \( T \subseteq \text{Seq} \) such that, whenever \( t', t \in \text{Seq} \) satisfy \( t' \subseteq t \), \( t \in T \) implies \( t' \in T \). Thus every tree \( T \) contains \( \Lambda \), the empty sequence, and satisfies \( T \subseteq I \).

\( \text{Max } T \) is the set of all \( \subseteq \)-maximal in \( T \) elements \( t \in T \).

A tree \( T \) is wellfounded (\textit{wf tree}, in brief) if and only if every non-empty (possibly non–internal) set \( T' \subseteq T \) contains a \( \subseteq \)-maximal element.
Definition 22  Let a \textit{wf pair} be any pair $\langle T, F \rangle$ such that $T$ is a wf tree and $F : \text{Max } T \rightarrow I$. In this case, the family of sets $F_T(t)$, $t \in T$, is defined, using the HST Replacement, as follows:

1) if $t \in \text{Max } T$ then $F_T(t) = F(t)$;
2) if $t \in T \setminus \text{Max } T$ then $F_T(t) = \{ F_T(t^a) : t^a \in T \}$.

We finally set $F[T] = F_T(\Lambda)$. \hfill \Box

Let, for example, $T = \{ \Lambda \}$ and $F(\Lambda) = x \in I$. Then $F[T] = F_T(\Lambda) = x$.

2.2 Class of elementary external sets

In particular we shall be interested to study the construction of Definition 22 from the point of view of the class $E = \{ C_p : p \in I \}$ of all elementary external sets. (See the definition of $C_p$ in Subsection 1.8.)

Proposition 23  $E$ is a transitive subclass of $H$. $E$ models Separation in the \textit{st-$\in$}-language.

Proof  $I$ is a model of BST, see Proposition 21. It follows that every \textit{st-$\in$}-definable in $I$ subclass of a set in $I$ has the form $C_p$ for some $p \in I$ by Theorem 21. \hfill \Box

We observe that $I \subseteq E$, and every set $X \in E$ satisfies $X \subseteq I$.

Definition 24  $H$ is the class of all \textit{wf pairs} $\langle T, F \rangle$ s. t. $T, F \in E$. \hfill \Box

Let $\langle T, F \rangle \in H$. Since all sets in $E$ are subsets of $I$, the set $F[T]$ cannot be a member of $E$. However, one can determine, in $E$, different properties which sets of the form $F[T]$ ($\langle T, F \rangle$ being \textit{wf pairs} in $H$) may have in $H$, using the following proposition, proved in [KR 96].

Proposition 25  $H$ is \textit{st-$\in$}-definable in $E$ as a subclass of $E \times E$. There exist 4-ary \textit{st-$\in$}-predicates $^h= \text{ and } ^h\in$, and a binary \textit{st-$\in$}-predicate $^h\text{st}$, such that the following holds for all \textit{wf pairs} $\langle T, F \rangle$, $\langle R, G \rangle$ in $H$:

\begin{align*}
F[T] &= G[R] \quad \text{iff it is true in } E \text{ that } \langle T, F \rangle \overset{^h=}{=} \langle R, G \rangle; \\
F[T] &\in G[R] \quad \text{iff it is true in } E \text{ that } \langle T, F \rangle \overset{^h\in}{\in} \langle R, G \rangle; \\
\text{st } F[T] &\quad \text{iff it is true in } E \text{ that } \langle T, F \rangle \overset{^h\text{st}}{\in} \langle R, G \rangle.
\end{align*}
Proof (An outline. See a complete proof in [KR 96], Section 3.) To prove the definability of $H$ in $E$, it suffices to check that if $T \in E$ is a wf tree in $E$ then $T$ is a wf tree in the sense of $H$, too. Since $E$ models Separation, the wellfoundedness of $T$ in $E$ allows to define, in $E$, the rank function $\rho$ from $T$ into $S$-ordinals. Such a function proves that $T$ is wellfounded in $H$.

The formula $\langle T, F \rangle \models (R, G)$ expresses the existence of a computation of the truth values of equalities $F_T(t) = G_R(r)$, where $r \in R$ and $t \in T$, which results in true for the equality $F_T(\Lambda) = G_R(\Lambda)$. The other two predicates are simple derivates of $\models$.

2.3 The sets constructible from internal sets

The following definition introduces the sets constructible from internal sets.

Definition 26 $L[I] = \{ F[T] : \langle T, F \rangle \in \mathcal{H} \}$.

In principle this does not look like a definition of constructibility. However it occurs that $L[I]$ is the least class in $H$ which contains all internal sets and satisfies HST, a sort of characterization of what in general the class $L[I]$ should be. Anyway this gives the same result as the ordinary definition of constructible sets, but with much less effort in this particular case.

To formulate the theorem, let us recall some notation related to ordered sets. A subset $Q \subseteq P$ of a p. o. set $P$ is called open dense in $P$ iff 1) $\forall p \in P \exists q \in Q (q \leq p)$ and 2) $\forall p \in P \forall q \in Q (p \leq q \rightarrow p \in Q)$.

Definition 27 Let $\kappa$ be a cardinal. A p. o. set $P$ is $\kappa$-closed iff every decreasing chain $\langle p_\alpha : \alpha < \kappa \rangle$ (i. e. $p_\alpha \leq p_\beta$ whenever $\beta < \alpha < \kappa$) in $P$ has a lower bound in $P$. A p. o. set $P$ is $\kappa$-distributive iff an intersection of $\kappa$-many open dense subsets of $P$ is dense in $P$.

The distributivity is used in the practice of forcing as a condition which prevents new subsets of sets of certain cardinality to appear in generic extensions. We shall use it with the aim to preserve Standardization in the

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10 This approach to the constructibility from internal sets in HST was introduced in [KR 96]. For any infinite cardinal $\kappa$ we defined the class $I_\kappa = I$ of all internal sets which are elements of standard sets of cardinality $\leq \kappa$ in $S$, and then the class $H_\kappa$ (could be denoted by $L[I_\kappa]$) of all sets constructible in this sense from sets in $I_\kappa$. We proved in [KR 96] that $H_\kappa$ models a certain $\kappa$-version of HST; the proof of Theorem 28 below in part copies some arguments from [KR 96]. But we did not prove results similar to statements P–1 and P–2 in [KR 96].
Theorem 28  \( L[I] \) is a transitive subclass of \( H \), \( E \subseteq L[I] \), \( L[I] \) models \( HST \) and models the following two additional postulates:

P–1. For each cardinal \( \kappa \), every \( \kappa \)-closed p. o. set is \( \kappa \)-distributive.

P–2. The isomorphism property \( IP \) (see Introduction) fails.

Take notice that any model \( H \subseteq H \) of \( HST \) transitive in \( H \) and containing all internal sets satisfies \( \forall \subseteq H \) and has the same classes of ordinals and cardinals as \( H \) and \( V \) do, by Corollary 11. Therefore the cardinals and presumed ordinals in item P–1 are those given by Definition 13 in \( HST \) and extensions. In \( ZFC \) a closed set is distributive, of course, but this simple fact is based on Choice. \( HST \) does not include a sufficient amount of Choice, but we can prove the implication to be true in \( L[I] \).

Proof  To prove the transitivity, let \( X = F[T] \in L[I] \), where \( \langle T, H \rangle \in H \). If \( T = \{ \Lambda \} \) then \( X = F(\Lambda) \in I \) by definition, hence all elements of \( X \) are internal. Suppose that \( T \neq \{ \Lambda \} \). Then \( \text{Min} T = \{ a : \langle a \rangle \in T \} \) is a non-empty set. For \( a \in \text{Min} T \), we put \( T^a = \{ t : a \wedge t \in T \} \) and \( F^a(t) = F(a \wedge t) \) for all \( a \wedge t \in \text{Max} T \). Obviously \( \langle T^a, F^a \rangle \) is a wf pair for all \( a \); moreover \( T^a \) and \( F^a \) belong to \( E \) by Proposition 23, so that in fact \( \langle T^a, F^a \rangle \in H \) and \( x_a = T^a[F^a] \in L[I] \). On the other hand, \( X = \{ x_a : a \in \text{Min} T \} \).

To prove \( E \subseteq L[I] \), let \( A \in E \). We define \( T = \{ \Lambda \} \cup \{ \langle a \rangle : a \in A \} \) and set \( F(\langle a \rangle) = a \) for all \( a \in A \). Then \( \langle T, F \rangle \) is a wf pair; furthermore \( \langle T, F \rangle \in H \) by Proposition 23, so that \( A = F[T] \in L[I] \), as required.

Let us prove three auxiliary claims which will be used below.

Claim 1  In \( L[I] \), every set is a functional image of a standard set.

Proof  Suppose that \( \langle T, F \rangle \in H \), \( X = F[T] \in L[I] \). Then \( A = \text{Min} T \) contains only internal elements. By Lemma 3, there exists a standard set \( U \) such that \( A \subseteq U \). For \( a \in A \), we define \( f(a) = x_a \). For \( a \in U \setminus A \), let \( f(a) = f(a_0) \), where \( a_0 \) is a fixed element of \( A \). Then \( f \) maps \( U \) onto \( X \). Proposition 23 allows to transform the given wf pair \( \langle T, F \rangle \) to a wf pair \( \langle R, G \rangle \in H \) such that \( f = G[R] \), which proves \( f \in L[I] \), as required.

Claim 2  Every set \( X \subseteq L[I] \) of standard size in \( H \) belongs to \( L[I] \).
Proof Lemma 2 gives an internal function $f$ defined on a standard set $A$ so that, for every $a \in \mathcal{A}$, $f(a) = \langle \tau_a, \sigma_a \rangle$ is an internal pair, the sets $T_a = \mathcal{C}_{\tau_a}$ and $F_a = \mathcal{C}_{\sigma_a}$ satisfy $(T_a, F_a) \in \mathcal{H}$, and $X = \{F_a[T_a] : a \in \mathcal{A}\}$.

Let $T = \{\Lambda\} \cup \{a^\Lambda t : a \in \mathcal{A} \land t \in T_a\}$ and $F(a^\Lambda t) = F_a(t)$ for all $a \in \mathcal{A}$ and $t \in \text{Max} T_a$. Thus both $T$ and $F$ are $\varepsilon$-definable in $E$ using only the internal $f$ as a parameter. Therefore $F$ and $T$ belong to $E$, by Proposition 22. On the other hand, by definition $(T, F)$ is a wf pair – thus $(T, F) \in \mathcal{H}$ – and $F[T] = \{F_a[T_a] : a \in \mathcal{A}\} = X \in L[I]$.

Claim 3 Every set $Z \in L[I]$, $Z \subseteq I$ belongs to $E$.

Proof Suppose that $Z = F[T]$, where $(T, F) \in \mathcal{H}$, in particular $T, F \in E$. We set $T^x = \{\Lambda\}$ and $F^x(\Lambda) = x$ for all internal $x$; then obviously $(T^x, F^x) \in \mathcal{H}$ and $F^x[T^x] = x$. Then $Z = \{x \in I : F^x[T^x] \in F[T]\}$. Using propositions 22 and 23, we finally obtain $Z \in E$.

Isomorphism property

We prove statement $P_2$ of the theorem. Since any two infinite sets are elementarily equivalent as structures of the language containing nothing except the equality, the following lemma implies the negation of $\text{IP}$.

Lemma Any two internal $I$-infinite sets of different $I$-cardinalities are non–equinumerous in $L[I]$.

Proof Suppose that $\text{card} X < \text{card} Y$ in $I$, and $f \in L[I]$ maps $X$ into $Y$. We have to prove that $\text{ran} f$ is a proper subset of $Y$. The following argument is a part of a more complicated reasoning in Kanovei [Kan 95]. (where the IST case was considered).

It follows from Claim 3 that $f = c_p$ for some $p \in I$, so that there exist standard sets $A, B$ and an internal set $W \subseteq X \times Y \times A \times B$ such that

$$f(x) = y \quad \text{iff} \quad \exists^s a \in A \land^s b \in B \ W(x, y, a, b),$$

for all $x \in X, y \in Y$. Since $I$ models $\text{BST}$, there exists an $I$-finite (but perhaps infinite) set $Z \in I$ containing all standard elements of $A$ and $B$.

We put $F(x, y) = \{(a, b) \in Z \times Z : W(x, y, a, b)\}$ for $x \in X, y \in Y$. Then obviously $f(x) = y$ iff $f(x') = y'$, provided $x, x' \in X$ and $y, y' \in Y$ satisfy $F(x, y) = F(x', y')$.

On the other hand $F$ is an internal function, taking values in an $I$-finite set $\mathcal{P}(Z \times Z)$. Therefore arguing in $I$ as an $\varepsilon$-model of $\text{ZFC}$ we obtain
a set \( Y' \) of \( \mathbb{I} \)-cardinality \( \text{card} Y' \leq \text{card} X \) (here the \( \mathbb{I} \)-infinity of \( X \) is used) such that for each \( x \in Z \) and \( y \) there exists \( y' \in Y' \) satisfying \( F(x, y) = F(x, y') \). In other words \( \text{ran} f \subseteq Y' \). Finally \( \text{card} Y' < \text{card} Y \) in \( \mathbb{I} \), so that \( Y' \) is a proper subset of \( Y \), as required.

Remark. Internal finite sets of different \( \mathbb{I} \)-cardinality in \( \mathbb{I} \) can become equinumerous in \( L[\mathbb{I}] \). For instance if \( n \) is a nonstandard \( \mathbb{I} \)-natural number then clearly sets containing \( n \) and \( n + 1 \) elements are equinumerous in an external universe. It follows from a result of Keisler, Kunen, Miller, and Leth [KKML] that if \( n \leq k \leq sn \) in \( \mathbb{I} \), for a standard natural \( s \), then sets containing \( n \) and \( k \) elements are equinumerous in an external universe.

Verification of HST axioms

We verify the HST axioms in \( L[\mathbb{I}] \). Axioms of group \([\text{Ax.3}]\) in Subsection \([\text{I.3}]\), Standardization, Saturation, and Weak Regularity are simply inherited from \( \mathbb{H} \) because \( \mathbb{I} \subseteq \mathbb{E} \subseteq L[\mathbb{I}] \).

To prove standard size Choice in \( L[\mathbb{I}] \), we first get a choice function in \( \mathbb{H} \). The function is a standard size subset of \( L[\mathbb{I}] \), therefore belongs to \( L[\mathbb{I}] \) by Claim 2. This argument also verifies Dependent Choice in \( L[\mathbb{I}] \).

Thus actually only the ZFC axioms included in group \([\text{Ax.3}]\) need a consideration. Among them, only Separation and Collection do need a serious verification; the rest of the axioms are either inherited from \( \mathbb{H} \) or proved by elementary transformations of wf pairs involved.

Let us check Separation in \( L[\mathbb{I}] \). Let \( X = F[T], \langle T, F \rangle \in \mathcal{H} \), and all parameters in a \( \text{st} \)-\( \in \)-formula \( \Phi(x) \) belong to \( L[\mathbb{I}] \). We have to prove that the set \( X' = \{ x \in X : L[\mathbb{I}] \models \Phi(x) \} \) also belongs to \( L[\mathbb{I}] \).

Suppose that \( T \neq \{ \Lambda \} \). Then the set \( A = \text{Min}T = \{ a : \langle a \rangle \in T \} \) is non-empty. For \( a \in A \), we define a wf pair \( \langle T^a, F^a \rangle \in \mathcal{H} \) as in the proof of transitivity above. Then \( X = \{ F^a[T^a] : a \in A \} \). Let \( A' \) be the set of all \( a \in A \) such that \( \Phi(F^a[T^a]) \) is true in \( L[\mathbb{I}] \). Then \( A' \) is \( \text{st} \)-\( \in \)-definable in \( \mathbb{E} \) by Proposition \([23]\) therefore \( A' \in \mathbb{E} \) by Proposition \([23]\). Let us define \( T' = \{ \Lambda \} \cup \{ a \uparrow t \in T : a \in A' \} \), and \( F'(a \uparrow t) = F(a \uparrow t) \) for each \( a \uparrow t \in \text{Max} T' \). Then \( \langle T', F' \rangle \in \mathcal{H} \) by Proposition \([23]\), and \( X' = F'[T'] \).

Suppose that \( T = \{ \Lambda \} \), so that \( X = F(\Lambda) \) is internal. Then \( X' \) is definable in \( \mathbb{E} \) by Proposition \([23]\), therefore \( X' \in \mathbb{E} \) by Proposition \([23]\). This implies \( X' \subseteq L[\mathbb{I}] \), since \( \mathbb{E} \subseteq L[\mathbb{I}] \), see above.

Let us check Collection in \( L[\mathbb{I}] \). By the HST Collection in \( \mathbb{H} \), it suffices to verify the following: if a set \( X \in \mathbb{H} \) satisfies \( X \subseteq L[\mathbb{I}] \) then
there exists $X' \in \mathbb{L}[\mathbb{I}]$ such that $X \subseteq X'$. Using Collection again and definition of $\mathbb{L}[\mathbb{I}]$, we conclude that there exists a set $P \subseteq \mathbb{I}$ such that

$$\forall x \in X \exists a = \langle p, q \rangle \in P \left( (\mathcal{C}_p, \mathcal{C}_q) \in \mathbb{H} \land x = x_a = \mathcal{C}_q[\mathcal{C}_p] \right).$$

By Lemma 3, there exists a standard set $S$ such that $P \subseteq S$. Let us define $A$ to be the set of all pairs $a = \langle p, q \rangle \in P$ such that $\langle \mathcal{C}_p, \mathcal{C}_q \rangle \in \mathbb{H}$. Then $A \in \mathbb{E}$, as above.

We set $T = \{ \lambda \} \cup \{ a^\land t : a = \langle p, q \rangle \in A \land t \in \mathcal{C}_p \}$ and $F(a^\land t) = \mathcal{C}_q(t)$ whenever $\langle p, q \rangle \in S$ and $t \in \text{Max}\mathcal{C}_p$. Then both $T$ and $F$ belong to $\mathbb{E}$ by Proposition 2. Since obviously $\langle T, F \rangle \in \mathbb{H}$ and $X' = F[T] \in \mathbb{L}[\mathbb{I}]$. On the other hand, $X \subseteq X'$.

### Distributivity

Let $\kappa$ be a cardinal, $P = \langle S; \leq \rangle \in \mathbb{L}[\mathbb{I}]$ be a $\kappa$-closed in $\mathbb{L}[\mathbb{I}]$ partial order, on a standard (Claim 1) set $S$. Consider a family $\langle D_\alpha : \alpha < \kappa \rangle \in \mathbb{L}[\mathbb{I}]$ of open dense subsets of $\langle S; \leq \rangle$. Let us prove that the intersection $\bigcap_{\alpha < \kappa} D_\alpha$ is dense in $P$. Let $x^0 \in S$. We have to find an element $x \in S$, $x \leq x^0$ such that $x \in \bigcap_{\alpha < \kappa} D_\alpha$.

To simplify the task, let us first correct the order. For $x \in S$, let $\alpha(x)$ denote the largest $\alpha \leq \kappa$ such that $x \in D_\beta$ for all $\beta < \alpha$. For $x, y \in S$, we let $x \prec y$ mean: $x \leq y$, and either $\alpha(y) < \alpha(x) < \kappa$ or $\alpha(x) = \alpha(y) = \kappa$.

Now it suffices to obtain a $\prec$-decreasing $\kappa$-sequence $x = \langle x_\alpha : \alpha < \kappa \rangle$ of $x_\alpha \in S$, satisfying $x_0 \leq x^0$. Indeed, then $x_\alpha \in D_\beta$ for all $\beta < \alpha < \kappa$. Furthermore $x \in \mathbb{L}[\mathbb{I}]$ by Claim 2. It follows that some $x \in S$ is $\leq$ each $x_\alpha$ because $P$ is $\kappa$-closed in $\mathbb{L}[\mathbb{I}]$. Then $x \in \bigcap_{\alpha < \kappa} D_\alpha$.

The order relation $\prec$ belongs to $\mathbb{L}[\mathbb{I}]$ by the already verified Separation in $\mathbb{L}[\mathbb{I}]$. It follows that $\prec$ is $\mathcal{C}_p$ for some internal $p$, by Claim 3; in other words, there exist standard sets $A'$, $B'$ and an internal set $Q \subseteq A' \times B' \times S^2$ such that $x \prec y$ iff $\exists^* a \in A' \forall^* b \in B' Q(a, b, x, y)$ — for all $x, y \in S$.

We have $A' = ^*A$ and $B' = ^*B$, for some $A, B \in \mathbb{V}$, by Corollary 10. Let $Q_{ab} = \{ \langle x, y \rangle \in S^2 : Q(^*a, ^*b, x, y) \}$ for $a \in A$, $b \in B$. Then, in $\mathbb{H}$, $x \prec y$ iff $\exists a \in A \forall b \in B Q_{ab}(x, y)$, and $Q_{ab}$ are internal sets for all $a, b$.

The principal idea of the following reasoning can be traced down to the proof of a choice theorem in Nelson [N 88]: we divide the problem into a choice argument in the $\varepsilon$-setting and a saturation argument.

Let us say that $a \in A$ witnesses $x \prec y$ iff we have $\forall b \in B Q_{ab}(x, y)$.

For any $\alpha \leq \kappa$, we let $A_\alpha$ be the family of all functions $a : \alpha \times \alpha \to A$.
such that there exists a function $x: \alpha \to S$ satisfying $x(0) \leq x^0$ and the requirement that $a(\delta, \gamma) \in A$ witnesses $x(\gamma) < x(\delta)$ whenever $\delta < \gamma < \alpha$.

We observe that, by Lemma 4, each function $a \in \bigcup_{\alpha \leq \kappa} A_\alpha$, every set $A_\alpha$, and the sequence $\langle A_\alpha : \alpha \leq \kappa \rangle$ belong to $\mathbb{V}$.

It suffices to prove that $A_\kappa \neq \emptyset$.

Since the sequence of sets $A_\alpha$ ($\alpha \leq \kappa$) belongs to $\mathbb{V}$, a ZFC universe, the following facts 1 and 2 immediately prove $A_\kappa \neq \emptyset$.

**Fact 1** If $\alpha < \kappa$ and $a \in A_\alpha$ then there exists $a' \in A_{\alpha+1}$ extending $a$.

**Proof** By definition there exists an decreasing $\alpha$-chain $x: \alpha \to S$ such that $a(\delta, \gamma)$ witnesses $x(\gamma) < x(\delta)$ whenever $\delta < \gamma < \alpha$. Since $P$ is $\kappa$-closed, some $x \in S$ is $\leq x(\delta)$ for each $\delta < \alpha$. By the density of the sets $D_\beta$, we can assume that in fact $x < x(\delta)$ for all $\delta < \alpha$. Using the standard size Choice in $\mathbb{H}$, we obtain a function $f: \alpha \to A$ such that $f(\delta)$ witnesses $x < x(\delta)$ for each $\delta < \alpha$. We define $a' \in A_{\alpha+1}$ by $a'(\delta, \gamma) = a(\delta, \gamma)$ whenever $\delta < \gamma < \alpha$, and $a'(\delta, \alpha) = f(\delta)$ for $\delta < \alpha$. $\dashv$

**Fact 2** If $\alpha < \kappa$ is a limit ordinal and a function $a: \alpha \times \alpha \to A$ satisfies $a \upharpoonright (\beta \times \beta) \in A_\beta$ for all $\beta < \alpha$ then $a \in A_\alpha$.

**Proof** Suppose that $\delta < \gamma < \alpha$ and $b \in B$. We let $\Xi_{b \delta \gamma}$ be the set of all internal functions $\xi: *\alpha \to S$ such that $Q_{a(\delta, \gamma)} b(\xi(^*\delta), \xi(^*\gamma))$. The sets $\Xi_{b \delta \gamma}$ are internal because so are all $Q_{ab}$.

We assert that the intersection $\Xi_\beta = \bigcap_{b \in B; \delta < \gamma < \beta} \Xi_{b \delta \gamma}$ is non-empty, for any $\beta < \alpha$. Indeed, since $a \upharpoonright (\beta \times \beta) \in A_\beta$, there exists a function $x: \beta \to S$ such that $a(\delta, \gamma)$ witnesses $x(\gamma) < x(\delta)$ whenever $\delta < \gamma < \beta$. By the Extension lemma (Lemma 4) there exists an internal function $\xi$, defined on $*\alpha$ and satisfying $\xi(\gamma) = x(\gamma)$ for all $\gamma < \alpha$. Then $\xi \in \Xi_\beta$.

Then the total intersection $\Xi = \bigcap_{b \in B; \delta < \gamma < \alpha} \Xi_{b \delta \gamma}$ is non-empty by Saturation in $\mathbb{H}$. Let $\xi \in \Xi$. By definition, we have $Q_{a(\delta, \gamma)} b(\xi(\delta), \xi(\gamma))$ whenever $\delta < \gamma < \alpha$ and $b \in B$. Let $x(\delta) = \xi(\delta)$ for all $\delta < \alpha$. Then $Q_{a(\delta, \gamma)} b(x(\delta), x(\gamma))$ holds whenever $\delta < \gamma < \alpha$ and $b \in B$. In other words, $x$ shows that $a \in A_\alpha$, as required. $\dashv$
3 Forcing over models of HST

The proof of the consistency part of Theorem 1 involves forcing. This section shows how in general one can develop forcing for HST models.

It is a serious problem that the membership relation is not well-founded in HST. This does not allow to run forcing over a HST model entirely in the ZFC manner: for instance the induction on the ∈-rank, used to define the forcing relation for atomic formulas, does not work.

However this problem can be solved, using the axiom of Weak Regularity, or well-foundedness over the internal subuniverse $\mathcal{I}$ (Ax.4 in Subsection 1.3).

We shall assume the following.

\[(\dagger)\text{ H is a model of HST in a wider set universe }\mathcal{U}. \ S \subseteq \mathcal{I} \text{ and } \mathcal{V} \text{ are resp. the classes of all standard and internal sets in H, and the condensed subuniverse defined in H as in Subsection 1.3.}\]

\[(\ddagger)\text{ H is well-founded over }\mathcal{I} \text{ in the sense that the ordinals of } H (\text{ = those of } \mathcal{V}) \text{ are well-founded in the wider universe. (Or, equivalently, } \mathcal{V} \text{ is a well-founded }\!\in\text{-model.})\]

In this case, we shall study generic extensions of H viewing H as a sort of ZFC-like model with urelements; internal sets playing the role of urelements. Of course internal sets behave not completely like urelements; in particular they participate in the common membership relation. But at least this gives an idea how to develop forcing in this case: the extension cannot introduce new internal sets (therefore neither new standard sets nor new well-founded sets — members of the condensed subuniverse $\mathcal{V}$.) Thus, in the frameworks of this approach, we can expect to get only new non–internal sets.

One more problem is the Standardization axiom. Since new standard sets cannot appear, a set of standard size, in particular a set in $\mathcal{V}$, cannot acquire new subsets in the extension. To obey this restriction, we apply a classical forcing argument: if the forcing notion is standard size distributive then no new standard size subsets of H appear in the extension.

3.1 The extension

Let $\mathcal{P} = \langle \mathcal{P}; \leq \rangle$ be a partially ordered set in H — the forcing notion, containing the maximal element $1_{\mathcal{P}}$. Elements of $\mathcal{P}$ will be called (forcing)
conditions and denoted, as a rule, by letters \( p, q, r \).

The inequality \( p \leq q \) means that \( p \) is a stronger condition.

Let \( \bar{x} = \langle 0, x \rangle \) for any set \( x \in \mathbb{H} \). \( \bar{x} \) will be the “name” for \( x \). We define \( N_0 = \{ \bar{x} : x \in \mathbb{H} \} \). For \( \alpha > 0 \), we let \( N_\alpha = \{ a : a \subseteq \mathbb{P} \times \bigcup_{\beta < \alpha} N_\beta \} \).

We observe that “names” in \( N_0 \) never appear again at higher levels.

We define, in \( \mathbb{H} \), \( N = N[\mathbb{P}] = \bigcup_{\alpha \in \text{Ord}} N_\alpha \), the class of \( \mathbb{P} \)-“names” for elements in the planned extension \( \mathbb{H}[\mathbb{P}] \). (We recall that the class \( \text{Ord} \) of all ordinals in \( \mathbb{H} \) was introduced in Subsection 1.4. It follows from 1.4 that \( \mathbb{H} \)-ordinals can be identified with an initial segment of the true ordinals in the wider universe.) For \( a \in N \), we let \( \text{nrk} a \) (the name–rank of \( a \)) denote the least ordinal \( \alpha \) such that \( a \in N_\alpha \).

Suppose that \( G \subseteq \mathbb{P} \) (perhaps \( G \notin \mathbb{H} \)). We define a set \( a[G] \) in the wider universe for each “name” \( a \in N \) by induction on \( \text{nrk} a \) as follows.

First of all, we put \( a[G] = x \) in the case when \( a = \bar{x} \in N_0 \).

Suppose that \( \text{nrk} a > 0 \). Following the ZFC approach, we would define

\[
a[G] = \{ b[G] : \exists p \in G \ (\langle p, b \rangle \in a) \}.
\]

However we face a problem: a set \( a[G] \) defined this way may contain the same elements as some \( x \in \mathbb{H} \) \( \in \mathbb{H} \)-contains in \( \mathbb{H} \), so that \( a[G] \) and \( x \) must be somehow identified in \( \mathbb{H}[G] \) in order not to conflict with Extensionality. This problem is settled as follows. We define, as above, for \( a \in N \setminus N_0 \),

\[
a'[G] = \{ b[G] : \exists p \in G \ (\langle p, b \rangle \in a) \}.
\]

If there exists \( x \in \mathbb{H} \) such that \( y \in a'[G] \) iff \( y \in \mathbb{H} \) & \( y \in \mathbb{H} x \) for each \( y \), then we let \( a[G] = x \). Otherwise we put \( a[G] = a'[G] \). (Take notice that if \( \langle p, b \rangle \in a \in N \) for some \( p \) then \( \text{nrk} b < \text{nrk} a \), so that \( a[G] \) is well defined for all \( a \in N \), because \( \mathbb{H} \) is assumed to be well-founded over \( \mathbb{P} \).)

We finally set \( \mathbb{H}[G] = \{ a[G] : a \in N \} \).

We define the membership \( \in_G \) in \( \mathbb{H}[G] \) as follows: \( x \in_G y \) iff either \( x, y \) belong to \( \mathbb{H} \) and \( x \in \mathbb{H} y \) in \( \mathbb{H} \), or \( y \notin \mathbb{H} \) and \( x \in y \) in the sense of the wider universe. We define the standardness in \( \mathbb{H}[G] \) by: \( \text{st} x \) iff \( x \in \mathbb{H} \) and \( x \) is standard in \( \mathbb{H} \).

**Definition 29** A \( \text{st} \)-\( \in \)-structure \( \mathbb{H}' \) is a plain extension of \( \mathbb{H} \) iff \( \mathbb{H} \subseteq \mathbb{H}' \), \( \mathbb{H} \) is an \( \epsilon_{\mathbb{H}'} \)-transitive part of \( \mathbb{H}' \), \( \epsilon_{\mathbb{H}} = \epsilon_{\mathbb{H}' \upharpoonright \mathbb{H}} \), and the standard (then also internal) elements in \( \mathbb{H}' \) and \( \mathbb{H} \) are the same. \( \Box \)

It is perhaps not true that \( \mathbb{H}[G] \) models \( \text{HST} \) independently of the choice of the notion of forcing \( \mathbb{P} \). To guarantee Standardization in the extension,
new subsets of standard size “old” sets cannot appear. Standard size distributivity provides a sufficient condition.

Definition 30 A p. o. set $P$ is standard size closed iff it is $\kappa$-closed for every cardinal $\kappa$. A p. o. set $P$ is standard size distributive iff it is $\kappa$-distributive for every cardinal $\kappa$.

Theorem 31 Let, in the assumptions $\dagger$ and $\ddagger$, $P \in H$ be a p. o. set and $G \subseteq P$ be $P$-generic over $H$. Then $H[G]$ is a plain extension of $H$ containing $G$ and satisfying Extensionality. If in addition the notion of forcing $P$ is standard size distributive in $H$ then $H[G]$ models HST.

Proof $H \subseteq H[G]$ because $\check{x}[G] = x$ by definition. Furthermore putting $G = \{ (p, \check{p}) : p \in P \}$, we get $G[G] = G$ for all $G \subseteq P$, so that $G$ also belongs to $H[G]$. The membership in $H$ is the restriction of the one in $H[G]$ by definition, as well as the $\in_G$-transitivity of $H$ in $H[G]$ and the fact that the standard sets are the same in $H$ and $H[G]$.

To prove Extensionality, let $a[G], b[G] \in H[G]$ $\in_G$-contain the same elements in $H[G]$: we have to prove that $a[G] = b[G]$. If $a[G] = A \in H$ then $a[G] \in_G$-contains the same elements in $H[G]$ as $A \in_H$-contains in $H$, so that $b[G] = A$ by definition. The case $b[G] \subseteq H$ is similar. If $a[G] \notin H$ and $b[G] \notin H$ then by definition $a[G] = a'[G] = b'[G] = b[G]$.

To proceed with the proof of the theorem, we have to define forcing.

3.2 The forcing relation

We argue in the model $H$ of HST in this subsection.

Let $P \in H$ be a p. o. set. The aim is to define the forcing relation $\models = \models_P$, used as $p \models \Phi$, where $p \in P$ while $\Phi$ is a st-$\in$-formula with “names” in $N$ as parameters.

First of all let us consider the case when $\Phi$ is an atomic formula, $b = a$ or $b \in a$, where $a, b \in N$. The definition contains several items.

F-1. We define: $p \models \check{x} = \check{y}$ iff $x = y$, and $p \models \check{x} \in \check{y}$ iff $x \in y$.

Let $a, b \in N$. We introduce the auxiliary relation

$p \text{ forc } b \in a$ iff \begin{cases} \exists y \in x \ (b = \check{y}) & \text{whenever } a = \check{x} \in N_0 \\ \exists q \geq p \ (\langle q, b \rangle \in a) & \text{otherwise} \end{cases}$

Note that $p \text{ forc } b \in a$ implies that either $a, b \in N_0$ or $\text{nrk } b < \text{nrk } a$. 

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F-2. \( p \vdash a = b \) iff for every condition \( q \leq p \) the following holds:

if \( q \) forces \( x \in a \) then \( q \vdash x \in b \); if \( q \) forces \( y \in b \) then \( q \vdash y \in a \).

F-3. \( p \vdash b \in a \) iff \( \forall q \leq p \exists r \leq q \exists z \ (r \) forces \( z \in a \) and \( r \vdash b = z \)).

Items [F-1] through [F-3] define the forcing for formulas \( a = b \) and \( a \in b \) by induction on the ranks \( \text{nrk} a \) and \( \text{nrk} b \) of “names” \( a, b \in \mathbb{N} \). The following items handle the standardness predicate and non–atomic formulas.

F-4. \( p \vdash \text{st} a \) iff \( \forall q \leq p \exists r \leq q \exists s \ (r \vdash a = \check{s}) \).

F-5. \( p \vdash \neg \Phi \) iff none of \( q \leq p \) forces \( \Phi \).

F-6. \( p \vdash (\Phi \& \Psi) \) iff \( p \vdash \Phi \) and \( p \vdash \Psi \).

F-7. \( p \vdash \forall x \Phi(x) \) iff \( p \vdash \Phi(a) \) for every \( a \in \mathbb{N} \).

It is assumed that the other logic connectives are combinations of \( \neg, \& , \forall \).

**Lemma 32** Let \( a, b \) be “names” in \( \mathbb{N} \). If \( p \) forces \( b \in a \) then \( p \vdash b \in a \).

If \( p \) forces \( b \in a \) and \( q \leq p \) then \( q \) forces \( b \in a \).

If \( p \vdash \Phi \) and \( q \leq p \) then \( q \vdash \Phi \).

**Proof** The first two assertions are quite obvious, the third one can be easily proved by induction on the complexity of \( \Phi \). \( \square \)

**Lemma 33** If \( p \in \mathbb{P} \) does not force \( \Phi \), a closed \( \text{st} \)-formula with “names” in \( \mathbb{N} \) as parameters, then there exists \( q \leq p \) such that \( q \vdash \neg \Phi \).

**Proof** Assume \( \neg p \vdash b \in a \). There exists a condition \( q \leq p \) such that \( \neg \exists r \leq q \exists z \ (r \) forces \( z \in a \) and \( r \vdash b = z \)). To see that \( q \vdash \neg b \in a \), let, on the contrary, a condition \( r \leq q \) forces \( b \in a \). Then by definition we have \( r \) forces \( z \in a \) and \( r \vdash b = z \) for a condition \( r \leq q' \) and a “name” \( z \), a contradiction with the choice of \( q \).

Assume that \( \neg p \vdash a = b \). Then by definition there exists \( q' \leq p \) such that e. g. for a “name” \( x \), \( q' \) forces \( x \in a \) but \( \neg q' \vdash x \in b \). It follows, by the above, that a condition \( q \leq q' \) satisfies \( q \vdash \neg x \in b \). We prove that \( q \vdash a \neq b \). Suppose that on the contrary a condition \( r \leq q \) forces \( a = b \). Since \( r \) forces \( x \in a \) by Lemma 32, we have \( r \vdash x \in b \), contradiction.

A similar reasoning proves the result for formulas \( \text{st} a \).

As for non–atomic formulas, the result can be achieved by a simple straightforward induction on the logical complexity of the formula. \( \square \)
3.3 Truth lemma

Suppose that $\Phi$ is a st-$\in$-formula having “names” in $N$ as parameters. We let $\Phi[G]$ denote the formula obtained by replacing occurrences of $\in$ and st in $\Phi$ by $\in_G$ and st$_G$, and every “name” $a \in N$ by $a[G]$; thus $\Phi$ is a formula having sets in $\mathbb{H}[G]$ as parameters.

**Theorem 34** (The truth lemma.) Let $G \subseteq P$ be a $P$-generic set over $\mathbb{H}$. Let $\Phi$ be a st-$\in$-formula having “names” in $N$ as parameters. Then $\Phi[G]$ is true in $\mathbb{H}[G]$ iff $\exists p \in G (p \models \Phi)$.

**Proof** Let us prove the result for the atomic formulas $a = b$ and $b \in a$ by induction on the ranks $\text{nrk}$ of $a$ and $b$. First of all we summarize the definition of the membership in $\mathbb{H}[G]$ as follows: for all $a, b \in N$,

$$b[G] \in_G a[G] \text{ iff } \exists b' \in N \exists p \in G (b'[G] = b[G] \& p \text{ forces } b' \in a). \quad (*)$$

We verify that $a[G] = b[G]$ iff some $p \in G$ satisfies $p \models a = b$. Suppose that none of $p \in G$ forces $a = b$. By the genericity, $G$ contains a condition $q$ such that, say, $q$ forces $x \in a$ but $q \not\models x \notin b$ for some $x \in N$. Then $x[G] \in_G a[G]$ but $x[G] \notin_G b[G]$ by the induction hypothesis.

Suppose now that $a[G] \neq b[G]$. Then, since $\mathbb{H}[G]$ satisfies Extensionality, the sets differ from each other in $\mathbb{H}[G]$ by their elements, say $x[G] \in_G a[G]$ but $x[G] \notin_G b[G]$ for a “name” $x \in N$. By the induction hypothesis and (*) there exist: a condition $p \in G$ and a “name” $x'$ such that $p$ forces $x' \in a$ but $p \not\models x' \notin b$. Then $p \models a \neq b$, because otherwise there exists a condition $q \leq p$ which forces $a = b$, immediately giving a contradiction.

Consider a formula of the form $b \in a$. Let a condition $p \in G$ force $b \in a$. Then by the genericity of $G$ there exists a condition $r \in G$ such that $r$ forces $z \in a$ and $r \models z = b$ for a “name” $z \in N$. This implies $z[G] \in_G a[G]$ by definition and $z[G] = b[G]$ by the induction hypothesis.

Assume now that $b[G] \in_G a[G]$ and prove that a condition $p \in G$ forces $b \in a$. We observe that, by (*), there exist: a condition $p \in G$ and a “name” $b'$ such that $b'[G] = b[G]$ and $p$ forces $b' \in a$. We can assume that $p \models b = b'$, by the induction hypothesis. Then $p \models b \in a$ by definition.

Formulas of the form $\text{st} a$ are considered similarly. Let us proceed with non-atomic formulas by induction on the complexity of the formula involved.

**Negation.** Suppose that $\Phi$ is $\neg \Psi$. If $\Phi[G]$ is true then $\Psi[G]$ is false in $\mathbb{H}[G]$. That none of $p \in G$ can force $\Psi$, by the induction hypothesis.
However the set \( \{ p \in \mathbb{P} : p \) decides \( \Psi \} \) is dense in \( \mathbb{P} \) and belongs to \( \mathbb{H} \). Thus some \( p \in G \) forces \( \Phi \) by the genericity of \( G \).

If \( p \in G \) forces \( \Phi \) then none of \( q \in G \) can force \( \Psi \) because \( G \) is pairwise compatible. Thus \( \Psi[G] \) fails in \( \mathbb{H}[G] \) by the induction hypothesis.

**Conjunction.** Very easy.

**The universal quantifier.** Let \( p \in G \) force \( \forall x \Psi(x) \). By definition we have \( p \models \Psi(a) \) for all \( a \in N \). Then \( \Psi(a)[G] \) holds in \( \mathbb{H}[G] \) by the induction hypothesis, for all \( a \in N \). However \( \Psi(a)[G] \) is \( \Psi[G](a[G]) \), and \( \mathbb{H}[G] = \{ a[G] : a \in N \} \). It follows that \( \mathbb{H}[G] \models \forall x \Psi(x)[G] \).

Assume that \( \forall x \Psi(x)[G] \) is true in \( \mathbb{H}[G] \). By Lemma 33 and the genericity some \( p \in G \) forces either \( \forall x \Psi(x) \) or \( \neg \Psi(a) \) for a particular \( a \in N \). In the “or” case \( \Psi(a)[G] \) is false in \( \mathbb{H}[G] \) by the induction hypothesis, contradiction with the assumption. Thus \( p \models \forall x \Psi(x) \), as required. \( \square \)

### 3.4 The extension models HST

We complete the proof of Theorem 31 in this subsection. Since the standard (therefore also internal) sets in \( \mathbb{H}[G] \) were already proved to be the same as in \( \mathbb{H} \), we have the axioms of group \( \text{Ax.1} \) (see Subsection 3.4) in \( \mathbb{H}[G] \).

Let us verify the ZFC axioms of group \( \text{Ax.3} \) in \( \mathbb{H}[G] \). We concentrate on the axioms of Separation and Collection; the rest of the axioms can be easily proved following the ZFC forcing patterns. (Extensionality has already been proved, see Proposition 33.)

**Separation.** Let \( X \in N \), and \( \Phi(x) \) be a \textit{st}-\( \epsilon \)-formula which may contain sets in \( N \) as parameters. We have to find a “name” \( Y \in N \) satisfying \( Y[G] = \{ x \in X[G] : \Phi[G](x) \} \) in \( \mathbb{H}[G] \). Note that by definition all elements of \( X[G] \) in \( \mathbb{H}[G] \) are of the form \( x[G] \) where \( x \) belongs to the set \( X = \{ x \in N : \exists p ((p,x) \in X) \} \) in \( \mathbb{H} \). (We suppose that \( \text{rank } X > 0 \). The case \( X \in N_0 \) does not differ much.) Now \( Y = \{ (p,x) \in \mathbb{P} \times X : p \models \Phi(x) \} \) is the required “name”. (See Shoenfield [Sh 71] for details.)

**Collection.** Let \( X \in N \), and \( \Phi(x,y) \) be a formula with “names” in \( N \) as parameters. We have to find a “name” \( Y \in N \) such that

\[
\forall x \in X[G] \ (\exists y \Phi[G](x,y)) \rightarrow \exists y \in Y[G] \ \Phi[G](x,y)
\]

is true in \( \mathbb{H}[G] \). Let \( X \in \mathbb{H} \), \( X \subseteq N \) be defined as above, in the proof of Separation. Using Collection in \( \mathbb{H} \), we obtain a set \( Y \subseteq N \), sufficient in the following sense: if \( x \in X \), and \( p \in \mathbb{P} \) forces \( \exists y \Phi(x,y) \), then

\[
\forall q \leq p \exists r \leq q \exists y \in Y \ (r \models \Phi(x,y)).
\]
The set \( Y = \mathbb{P} \times Y \) (then \( Y[G] = Y \)) is as required.

**Weak Regularity.** Let \( X \in \mathbb{N} \). Using an appropriate dense set in \( \mathbb{P} \), we find a condition \( p \in G \) such that \( p \models a \in X \) for a “name” \( a \in \mathbb{N} \), but 1) \( p \models b \notin X \) for any “name” \( b \in \mathbb{N}_\beta \) where \( \beta < \text{nrk} a \) – provided \( \text{nrk} a > 0 \), and 2) \( p \models \bar{y} \notin X \) for any \( y \in \mathbb{H} \) with \( \text{irk} y < \text{irk} x \) – provided \( a = \bar{x} \in \mathbb{N}_0 \). Now, if \( \text{nrk} a > 0 \), or if \( a = \bar{x} \& \text{irk} x > 0 \), then simply \( p \models a \cap X = \emptyset \). If finally \( a = \bar{x} \) for an internal \( x \) then \( x \cap X[G] \) contains only internal elements.

**Standardization.** Let \( X \in \mathbb{N} \). We have to find a standard set \( Y \) which contains in \( \mathbb{H}[G] \) the same standard elements as \( X[G] \) does. It can be easily proved by induction on \( \text{nrk} a \) that, for every “name” \( a \in \mathbb{N} \),

\[
\text{Stan}(a) = \{ s : \text{st } s \ & \exists p \in \mathbb{P} \ (p \models \bar{s} \in a) \}
\]

is a set in \( \mathbb{H} \). (\( \text{Stan}(a) \) contains all standard \( \in_G \)-elements of \( a[G] \).) Thus \( \text{Stan}(X) \subseteq S \) for a standard \( S \), by Lemma \( \text{[3]} \). Since \( \mathbb{P} \) is standard size distributive in \( \mathbb{H} \), \( G \) contains, by the genericity, a condition \( p \) which, for any standard \( s \in S \), decides the statement \( \bar{s} \in X \). Applying Standardization in \( \mathbb{H} \), we get a standard set \( Y \subseteq S \) such that, for each \( s \in S \), \( s \in Y \iff p \models \bar{s} \in X \). The \( Y \) is as required.

**Standard size Choice.** The problem can be reduced to the following form. Let \( S \) be a standard set, \( P \in \mathbb{N} \), and \( P[G] \) is a set of pairs in \( \mathbb{H}[G] \). Find a “name” \( F \) such that the following is true in \( \mathbb{H}[G] \):

\[
F[G] \text{ is a function defined on } ^sS \text{ and satisfying } \\
\exists y P[G](x,y) \implies P[G](x,F[G](x)) \text{ for each standard } x \in S.
\]

Arguing as above and using the standard size Choice in \( \mathbb{H} \), we obtain a condition \( p \in G \) and a function \( f \in \mathbb{H} \), \( f : ^sS \rightarrow \mathbb{N} \), such that, for every \( x \in ^sS \), \( p \) either forces \( \exists \bar{y} P(\bar{x},y) \) or forces \( P(\bar{x},y) \) where \( y_x = f(x) \in \mathbb{N} \). One easily converts \( f \) to a required “name” \( F \).

**Dependent Choice – similar reduction to \( \mathbb{H} \).**

**Saturation.** Using the same argument, one proves that each standard size family of internal sets in \( \mathbb{H}[G] \) already belongs to \( \mathbb{H} \). \( \square \)
4 Generic isomorphisms

Let us consider a particular forcing which leads to a generic isomorphism between two internally presented elementarily equivalent structures.

We continue to consider a model $H \models HST$ satisfying assumptions (†) and (‡) in Section 3. We suppose in addition that

$(\S) \quad \mathcal{L} \in H$ is a first–order language containing (standard size)–many symbols. $\mathfrak{A} = \langle A; \ldots \rangle$ and $\mathfrak{B} = \langle B; \ldots \rangle$ are two internally presented elementarily equivalent $\mathcal{L}$-structures in $H$.

By definition both $A$ and $B$ are internal sets, and the interpretations of each symbol of $\mathcal{L}$ in $A$ and $B$ are internal in $H$.

The final aim is to obtain a generic isomorphism $\mathfrak{A} \rightarrow \mathfrak{B}$. We shall define a notion of forcing $P = P_{\mathcal{L}\mathfrak{A}\mathfrak{B}} \in H$ such that $\mathfrak{B}$ is isomorphic to $\mathfrak{A}$ in every $P$–generic extension of $H$, provided $H$ satisfies a requirement which guarantees the standard size distributivity of $P$.

It is the most natural idea to choose the forcing conditions among partial functions $p$, mapping subsets of $A$ onto subsets of $B$. We have to be careful: the notion of forcing must be a set, thus for instance maps having a standard size domain do not work because even an $I$–finite internal infinite set has a proper class of standard size subsets in $HST$. We overcome this obstacle using internal partial maps $p$, such that each $a \in \text{dom} p$ satisfies in $A$ exactly the same $\mathcal{L}$-formulas as $p(a)$ does in $B$.

Given a condition $p$ and an element $a \in A \setminus \text{dom} p$, we must be able to incorporate $a$ in $p$, i.e. define a stronger condition $p_+$ such that $a \in \text{dom} p_+$. Here we face a problem: to find an element $b \in B$ which, for each $a \in \text{dom} p$, is in the same relations with $p(a)$ in $\mathfrak{B}$ as $a$ is with $a$ in $\mathfrak{A}$. Since $\text{dom} p$ cannot be a set of standard size, it is not immediately clear how a saturation argument can be used to get a required $b$.

We shall develop the idea as follows. Let $\Phi(x, y)$ be an $\mathcal{L}$-formula. We are willing to find $b \in B$ so that $\Phi(a, a)$ in $\mathfrak{A}$ iff $\Phi(b, p(a))$ in $\mathfrak{B}$ for all $a \in \text{dom} p$. The sets $u = \{a \in \text{dom} p : \mathfrak{A} \models \Phi(a, a)\}$ and $v = \text{dom} p \setminus u$ are internal by the choice of $\mathfrak{A}$. We observe that the chosen element $a$ satisfies $\forall a \in u \Phi(a, a)$ and $\forall a \in v \neg \Phi(a, a)$ in $\mathfrak{A}$, so that the sentence

$$\exists x \left( \forall a \in u \Phi(x, a) \& \forall a \in v \neg \Phi(x, a) \right)$$

is true in $\mathfrak{A}$. Suppose that $p$ also preserves sentences of this form, so that

$$\exists y \left( \forall b \in p^u \Phi(y, b) \& \forall b \in p^v \neg \Phi(y, b) \right)$$
is true in \( B \). ( \( p^u = \{p(a) : a \in u\} \) is the \( p \)-image of \( u \).) This gives an
element \( b \in B \) which may be put in correspondence with \( a \).

Thus we have to preserve formulas of the displayed type, i.e. \( L \)-formulas
with some internal subsets of \( A \) as parameters, so in fact a stronger preservation hypothesis is involved than the result achieved. This leads to a sort
of hierarchical extension of the language \( L \).

### 4.1 The extended language

Arguing in \( \mathbb{H} \), we define the notion of type as follows. Let \( D \in \mathbb{I} \).

0 is a type. An object of type 0 over a set \( D \) is an element of \( D \).

Suppose that \( l_1, \ldots, l_k \) are types. Then \( l = \tau(l_1, \ldots, l_k) \) is a type. (Here \( \tau \) is a formal sign.) An object of type \( l \) over \( D \) is an internal set of \( k \)-tuples \( \langle x_1, \ldots, x_k \rangle \) where each \( x_i \) is an object of type \( l_i \) over \( D \).

E.g. objects of type \( \tau(0, 0) \) over \( D \) are internal subsets of \( D \times D \).

We define \( L_\infty \) as the extension of \( L \) by variables \( x^l, y^l, \ldots \) for each type \( l \), which can enter formulas \( \exists \) only through the expressions \( x^l(x_1^l, \ldots, x_k^l) \) (may be written as \( \langle x_1^l, \ldots, x_k^l \rangle \in x^l \)), provided \( l = \tau(l_1, \ldots, l_k) \), and also \( x = x^0 \), where \( x \) is an \( L \)-variable. (We shall formally distinguish variables of type 0 from \( L \)-variables.)

Let \( \mathcal{C} = \langle C; \ldots \rangle \) be an internally presented \( L \)-structure. Given an internal set \( D \subseteq C \), we define a type-theoretic extended structure \( \mathcal{C}[D] \) which includes the ground domain \( C \) with all the \( \mathcal{C} \)-interpretations of \( L \)-symbols, and the domain \( D^l = \{x^l : x^l \text{ is an object of type } l \text{ over } D\} \) for each type \( l \).

We observe that each \( D^l \) is an internal set because the construction of \( D^l \) can be executed in \( \mathbb{I} \). For instance \( D^{\tau(0)} = \mathcal{P}(D) \) in \( \mathbb{I} \). \( D = D^0 \) is an internal subset of \( C \).

Every \( L_\infty \)-formula (perhaps, containing sets in \( \mathcal{C}[D] \) as parameters) can be interpreted in \( \mathcal{C}[D] \) in the obvious way. (Variables of type \( l \) are interpreted in \( D^l \).) This converts \( \mathcal{C}[D] \) to an internally presented \( L_\infty \)-structure.

It will always be supposed that \( D^{l_1} \cap D^{l_2} = \emptyset \) provided \( l_1 \neq l_2 \).

We put \( D_\infty = \bigcup_l D^l \).

\( ^{12} \) By formulas, with respect to the languages \( L \) and \( L_\infty \), we shall understand finite sequences of something satisfying certain known requirements, not only “metamathematical” formulas. Since any finite tuple of internal sets is internal (an easy consequence of Lemma \( \mathbb{I} \)), a formula with internal parameters is formally an internal object, so for instance its truth domain is internal as well.
4.2 The forcing

We recall that a standard size language \( L \) and a pair of internally presented elementarily equivalent \( L \)-models \( \mathcal{A} = \langle A; \ldots \rangle \) and \( \mathcal{B} = \langle B; \ldots \rangle \) are fixed.

Suppose that \( p \) is an internal \( 1 - 1 \) map from an internal set \( D \subseteq A \) onto a set \( E \subseteq B \). We expand \( p \) on all types \( l \) by induction, putting

\[
p^l(x^l) = \{(p^{l_1}(x^{l_1}), \ldots, p^{l_k}(x^{l_k})) : (x^{l_1}, \ldots, x^{l_k}) \in x^l\}
\]

for all \( x^l \in D \), whenever \( l = \tau(l_1, \ldots, l_k) \). Then \( p^l \) internally \( 1 - 1 \) maps \( D^l \) onto \( E^l \).

If \( \Phi \) is an \( L^\infty \)-formula containing parameters in \( D^\infty \) then let \( p\Phi \) be the formula obtained by changing each parameter \( x \in D^l \) in \( \Phi \) to \( p(x) \in E^l \).

**Definition 35** \( \mathbb{P} = \mathbb{P}_{\mathcal{A} \mathcal{B}} \) is the set of all internal \( 1-1 \) maps \( p \) such that \( D = \text{dom} \ p \) is an (internal) subset of \( A \), \( E = \text{ran} \ p \subseteq B \) (also internal), and, for each closed \( L^\infty \)-formula \( \Phi \) having sets in \( D^\infty \) as parameters, we have \( \mathcal{A}[D] \models \Phi \) iff \( \mathcal{B}[E] \models p\Phi \).

We define \( p \leq q \) ( \( p \) is stronger than \( q \)) iff \( q \subseteq p \).

For instance the empty map \( \emptyset \) belongs to \( \mathbb{P} \) because \( \mathcal{A} \) and \( \mathcal{B} \) are elementarily equivalent. (The properly \( L^\infty \)-variables can be eliminated in this case because the domains become finite.) We shall see (Corollary 39, the main result) that the forcing leads to generic isomorphisms \( \mathcal{A} \) onto \( \mathcal{B} \).

This is based on the following two technical properties of this forcing.

**Proposition 36** \( \mathbb{P} = \mathbb{P}_{\mathcal{A} \mathcal{B}} \) is standard size closed in \( \mathbb{H} \).

**Proposition 37** Let \( p \in \mathbb{P}, \ D = \text{dom} \ p, \ E = \text{ran} \ p \). If \( a \in A \setminus D \) then there exists \( b \in B \setminus E \) such that \( p_+ = p \cup \{\langle a, b \rangle\} \in \mathbb{P} \). Conversely, if \( b \in B \setminus E \) then there exists \( a \in A \setminus D \) such that \( p_+ = p \cup \{\langle a, b \rangle\} \in \mathbb{P} \).

**Proof** of Proposition 36. Let \( \kappa \) be a cardinal. Suppose that \( p_\alpha \) \( (\alpha < \kappa) \) are conditions in \( \mathbb{P} \), and \( p_\beta \leq p_\alpha \) whenever \( \alpha < \beta < \kappa \). By definition \( \mathbb{P} \) is a standard size intersection of internal sets (because the structures are internally presented and \( L^\infty \) is a language of standard size), hence so is each of the sets \( P_\alpha = \{p \in \mathbb{P} : p \leq p_\alpha\} \). Furthermore \( P_\alpha \neq \emptyset \) and \( P_\beta \subseteq P_\alpha \) whenever \( \alpha < \beta < \kappa \). Finally \( \kappa \) (as every set in the condensed universe \( \mathcal{V} \)) is a set of standard size by Lemma 30, so \( \bigcap_{\alpha < \kappa} P_\alpha \neq \emptyset \) by Saturation. Thus there exists \( p \in P \) such that \( p \leq p_\alpha \) for all \( \alpha \), as required. □
**Definition 38** A function \( F \) defined on an internal set \( A \) is **locally internal** iff either \( A \) is finite or for any \( a \in A \) there exists an infinite internal set \( A' \subseteq A \) containing \( a \) and such that \( F \restriction A' \) is internal. \( \square \)

**Corollary 39** Suppose that, in addition to (†), (‡), (§), \( H \) satisfies statement \( P \)–1 in Theorem 28. Let \( P = P_{\mathcal{L}_{\infty}} \) in \( H \). Then, every \( P \)-generic extension \( H[G] \) is a model of \( \text{HST}, \) a plain extension of \( H, \) and \( F = \bigcup G \) is a locally internal isomorphism \( \mathfrak{A} \) onto \( \mathfrak{B} \) in \( H[G]. \)

**Proof** of the corollary. Proposition 36 plus the assumed statement \( P \)–1 guarantee that \( P \) is standard size distributive in \( H \) in the sense of Definition 30. Therefore \( H[G] = \text{HST}, \) by Theorem 31. Furthermore \( F \) maps \( A \) onto \( B \) by Proposition 37. The map is an isomorphism because \( F \) is a union of conditions \( p \in P \) which preserve the truth of \( L \)-sentences.

To prove that \( F \) is locally internal, assume that the sets \( A, B \) are infinite, and \( a \in A. \) It can be easily proved by the same reasoning as in the proof of Proposition 36 that \( G \) contains a condition \( p \) such that \( a \in \text{dom} \ p \) and \( \text{dom} \ p \) is infinite (although perhaps \( I \)-finite). \( \square \)

The remainder of this section is devoted to the proof of Proposition 37. By the symmetry, we concentrate on the first part. Let us fix a condition \( p \in \mathbb{P}. \) Let \( D = \text{dom} \ p, \ E = \text{ran} \ p. \) (For instance we may have \( p = D = E = \emptyset \) at the moment.) Consider an arbitrary \( a \in A \setminus D; \) we have to find a counterpart \( b \in B \setminus E \) such that \( p_+ = p \cup \{ \langle a, b \rangle \} \in \mathbb{P}. \)

### 4.3 Adding an element

Let \( \kappa = \text{card} \mathcal{L} \) (or \( \kappa = \aleph_0 \) provided \( \mathcal{L} \) is finite) in \( H. \) We enumerate by \( \Phi_\alpha(x) (\alpha < \kappa) \) all parameter-free \( L_\infty \)-formulas which contain only one \( L \)-variable \( x \) but may contain several variables \( x^l \) for various types \( l \).

Let us consider a particular \( L_\infty \)-formula \( \Phi_\alpha(x) = \varphi(x, x^{l_1}, ..., x^{l_n}). \) Let \( l = \tau(l_1, ..., l_n). \) (Both \( l \) and each of \( l_i \) are types.) We set

\[
X_\alpha = \{ \langle x^{l_1}, ..., x^{l_n} \rangle \in D^{l_1} \times ... \times D^{l_n} : \mathfrak{A}[D] \models \varphi(a, x^{l_1}, ..., x^{l_n}) \};
\]

thus \( X_\alpha \) is internal \( \langle 13 \rangle \) and \( X_\alpha \in D^l. \) Let \( \Psi_\alpha(X_\alpha, x) \) be the \( L_\infty \)-formula

\[
\forall x^{l_1} ... \forall x^{l_n} [X_\alpha(x^{l_1}, ..., x^{l_n}) \iff \varphi(x, x^{l_1}, ..., x^{l_n})],
\]

\( \langle 13 \rangle \) To prove that \( X_\alpha \) is internal, we first note that every finite subset of \( \mathbb{I} \) is internal, which can be easily proved by induction on the number of elements. Therefore, since only finitely many types \( l \) are actually involved in the definition of \( X_\alpha, \) while all the relevant domains and relations are internal, the definition of \( X_\alpha \) can be executed in \( \mathbb{I}. \) (We cannot, of course, appeal to Transfer since \( \varphi \) is not a metamathematical formula here.)
so that by definition $\mathfrak{A}[D] \models \Psi_\alpha(X_\alpha, a)$. Thus we have $\kappa$-many formulas $\Psi_\alpha(X_\alpha, x)$, realized in $\mathfrak{A}[D]$ by one and the same element $x = a \in A$.

We put $Y_\alpha = p^l(X_\alpha)$; so that $Y_\alpha \in E'$.

**Lemma 40** There exists $b \in B$ which realizes (by $y = b$) every formula $\Psi_\alpha(Y_\alpha, y)$ in $\mathfrak{B}[E]$.

**Proof** By Saturation, it suffices to prove that every finite conjunction $\Psi_{\alpha_1}(Y_{\alpha_1}, y) \land \ldots \land \Psi_{\alpha_m}(Y_{\alpha_m}, y)$ can be realized in $\mathfrak{B}[E]$. By definition $a$ witnesses that $\mathfrak{A}[D] \models \exists x [\Psi_{\alpha_1}(X_{\alpha_1}, x) \land \ldots \land \Psi_{\alpha_m}(X_{\alpha_m}, x)]$. Therefore $\mathfrak{B}[E] \models \exists y [\Psi_{\alpha_1}(Y_{\alpha_1}, y) \land \ldots \land \Psi_{\alpha_m}(Y_{\alpha_m}, y)]$, since $p \in \mathbb{P}$. □

Let us fix an element $b \in B$ satisfying $\Psi_\alpha(Y_\alpha, b)$ in $\mathfrak{B}[E]$ for all $\alpha < \kappa$. We set $p_+ = p \cup \{ (a, b) \}$, $D_+ = D \cup \{ a \}$, $E_+ = E \cup \{ b \}$.

### 4.4 Why the choice is correct

It will take some effort to check that $p_+$ is a condition in $\mathbb{P}$. Let us prove first a particular lemma which shows that $p_+$ preserves formulas containing $a$ and sets in $D^\infty$ as parameters.

**Lemma 41** Let $\varphi(x)$ be an $\mathcal{L}^\infty$-formula which may contain sets in $D^\infty$ as parameters. Then $\varphi(a)$ is true in $\mathfrak{A}[D]$ iff $(p_+)(b)$ is true in $\mathfrak{B}[E]$.

**Proof** The formula $\varphi(x)$ is obtained from a parameter-free $\mathcal{L}^\infty$-formula $\Phi(x, \ldots)$ by changing free variables in the list $\ldots$ to appropriate parameters (of the same type) from $D^\infty$. We can assume that in fact the list $\ldots$ does not include $\mathcal{L}$-variables; indeed if such one, say $y$, occurs then we first change $\Phi(x, y, \ldots)$ to $\exists y [\Phi(x, y, \ldots) \land y = y^0]$, where $y^0$, a variable of type 0, is free. In this assumption, $\varphi(x)$ is $\Phi_\alpha(x, x^{l_1}, \ldots, x^{l_n})$ for some $\alpha$ and parameters $x^{l_i} \in D^{l_i}$. Then, since $\Psi_\alpha(Y_\alpha, a)$ is true in $\mathfrak{A}[D]$, we have

$$X_\alpha(x^{l_1}, \ldots, x^{l_n}) \iff \mathfrak{A}[D] \models \Phi_\alpha(a, x^{l_1}, \ldots, x^{l_n}).$$

Note that $X_\alpha(x^{l_1}, \ldots, x^{l_n}) \iff Y_\alpha(y^{l_1}, \ldots, y^{l_n})$, where $y^{l_i} = p^{l_i}(x^{l_i}) \in E^{l_i}$, since $p \in \mathbb{P}$. Finally, $Y_\alpha(y^{l_1}, \ldots, y^{l_n})$ iff $\mathfrak{B}[E] \models \Phi_\alpha(b, y^{l_1}, \ldots, y^{l_n})$, because $\Psi_\alpha(Y_\alpha, b)$ is true in $\mathfrak{B}[E]$ by the choice of $b$. However the formula $\Phi_\alpha(b, y^{l_1}, \ldots, y^{l_n})$ coincides with $(p_+)(b)$. □

Taking the formula $x \not\in D$ as $\varphi(x)$ (then $p_+(x) = x \not\in E$), we obtain $b \not\in E$, so $p_+$ is a $1-1$ internal map. It remains to check that $p_+$ transforms true $\mathcal{L}^\infty$-formulas with parameters in $D^{\infty}_+$ into true $\mathcal{L}^\infty$-formulas.
with parameters in $E_+^\infty$. The idea is to convert a given formula with parameters in $D_+^\infty$ into a $L^\infty$-formula with parameters in $D^\infty$ plus $a$ as an extra parameter, and use Lemma 41.

Fortunately the structure of types over an internal set $C$ depends only on the internal cardinality of $C$ but does not depend on the place $C$ takes within $A$. This allows to “model” $D_+^\infty$ in $D^\infty$ identifying the $a$ with $\emptyset$ and any $a \in D$ with $\{a\}$. To realize this plan, let us define $D = \{\emptyset\} \cup \{\{a\} : a \in D\}$, so that $D \subseteq D^\ell$, where $\ell = \tau(0)$ (the type of subsets of $D$). Furthermore we have $D \in D^\tau(\ell)$ because $D$ is internal.

For each type $l$, we define a type $\delta(l)$ by $\delta(0) = l$ and $\delta(l) = \tau(\delta(l_1), ..., \delta(l_n))$ provided $l = \tau(l_1, ..., l_n)$.

We put $\delta(a) = \emptyset$, and $\delta(a) = \{a\}$ for all $a \in D$, so that $\delta$ is an internal bijection $D_+$ onto $D$. We expand $\delta$ on higher types by $\delta(x) = \{\delta(x_1), ..., \delta(x_n)\} : \langle x_1, ..., x_n \rangle \in x$; thus $\delta(x) \in D^l \subseteq D^\delta(l)$ whenever $x \in D^l$. Take notice that $\delta(D^l_+) = D^l$. Thus $\delta = \delta_{D\!\!a}$ defines a $1-1$ correspondence between $D_+^\infty$ and $D^\infty$.

Let $\psi(x^{i_1}, ..., x^{i_n}, v_1, ..., v_m)$ be a $L^\infty$-formula, containing $L$-variables $v_j$ and properly $L^\infty$-variables $x^{i_j}$. We introduce another $L^\infty$-formula, denoted by $\psi_{D\!\!a}(\xi^{\delta(i_1)}, ..., \xi^{\delta(l_n)}, v_1, ..., v_m)$, containing $a, D$, and finitely many sets $D^l$ as parameters (this is symbolized by the subscript $D$; the involved sets $D^l$ are derivates of $D$ and $a$), as follows.

Each free variable $x^{i_j}$ is changed to some $\xi^{\delta(i_j)}$, a variable of type $\delta(l_j)$. (We use characters $\xi, \eta, \zeta$ for variables intended to be restricted to $D^\infty$).

Each quantifier $Q u^l ... u^l ...$ is changed to $Q \eta^{\delta(l)} \in D^l ... \eta^{\delta(l)} ...$. (Note that $D^l = \delta(D^l_+)$ is an internal subset of $D^\delta(l)$.)

Each occurrence of type $x = \zeta^l$ (which is obtained by the abovementioned transformations from an original equality $x = x^0$) is changed to

$$(x = a \& \zeta^l = \emptyset) \lor (x \in D \& \zeta^l = \{x\})$$

(the equalities $\zeta^l = ...$ can here be converted to correct $L^\infty$-formulas).

**Lemma 42** Let $\psi(x^{i_1}, ..., x^{i_n}, v_1, ..., v_m)$ be an $L^\infty$-formula, $x^{i_j} \in D^l$, and $a_j \in A$ for all $i$ and $j$. Then $A[D_+] \models \psi(x^{i_1}, ..., x^{i_n}, a_1, ..., a_m)$ if $A[D] \models \psi_{D\!\!a}(\delta(x^{i_1}), ..., \delta(x^{i_n}), a_1, ..., a_m)$.

**Proof** Suppose that $\psi$ is an atomic formula. If $\psi$ is in fact an $L$-formula then the equivalence is obvious. Otherwise $\psi$ is either of the form $x = x^0$ or of the form $\langle x^{i_1}, ..., x^{i_n} \rangle \in x^l$, where $l = \tau(l_1, ..., l_n)$. The latter case does not cause a problem: use the definition of $\delta(x^l)$.
Consider a formula of the form $x = z^0$ as $\psi(z^0, x)$, where $x \in A$ and $z^0 \in D_+ = D_+^0$. By definition $\delta(0) = \ell$, $\delta(z^0) = \emptyset$ provided $z^0 = a$ and $\delta(z^0) = \{ z^0 \}$ otherwise, and $\psi_{DA}(\zeta^\ell, x)$ is the formula $(\ast)$. One easily sees that $x = z^0$ iff $\psi_{DA}(\delta(z^0), x)$.

As for the induction step, we consider only the step $\exists u^l$ because the connectives $\neg$ and $\&$ are automatical, as well as $\exists$ in the form $\exists x$, where $x$ is an $\mathcal{L}$-variable.

Let $\psi(...)$ be the formula $\exists u^l \phi(u^l,...)$. We have the following chain:

1. $\mathfrak{A}[D_+] \models \psi(...)$, that is, $\exists u^l \in D_+\upl (\mathfrak{A}[D_+] \models \phi(u^l,...))$
2. $\exists u^l \in D_+\upl (\mathfrak{A}[D] \models \phi_{Da}(\delta(u^l),...))$
3. $\exists \eta^{\delta(l)} \in \mathcal{D}^l (\mathfrak{A}[D] \models \phi_{Da}(\eta^{\delta(l)},...))$
4. $\mathfrak{A}[D] \models \psi_{Da}(...)$

The equivalence $[1] \iff [2]$ holds by induction hypothesis, equivalence $[2] \iff [3]$ follows from the equality $\mathcal{D}^l = \{ \delta(u^l) : u^l \in D_+\upl \} \subseteq \mathcal{D}^{\delta(l)}$, and the equivalence $[3] \iff [4]$ from the fact that $\psi_{Da}(...)$ is the formula $\exists \eta^{\delta(l)} \in \mathcal{D}^l \phi_{Da}(\eta^{\delta(l)},...)$ by definition.

We complete the proof of Proposition 37

Let $\Phi$ be the $\mathcal{L}^\infty$-formula $\phi(x_1,...,x_n)$ containing sets $x_i \in D_+\upl$ as parameters. Let $y_i = p^{l_i}(x_i)$; so that $y_i \in E_+\upl$. Let $\Psi$ be the $\mathcal{L}^\infty$-formula $\phi(y_1,...,y_n))$. We have to prove that $\mathfrak{A}[D_+] \models \Phi$ iff $\mathfrak{B}[E_+] \models \Psi$.

Step 1. $\mathfrak{A}[D_+] \models \Phi$ iff $\mathfrak{A}[D] \models \phi_{Da}(\delta(x_1),...,\delta(x_n))$ (Lemma 12).

Step 2. Let $E = \{ \emptyset \} \cup \{ \{ b \} : b \in E \} = p^{\upl} \mathcal{D}$. We observe that the final statement of step 1 is equivalent, by Lemma 11, to the following one: $\mathfrak{B}[E] \models \phi_{EB}(p(\delta(x_1)),...,p(\delta(x_n)))$.

Step 3. In the last formula, $\delta = \delta_{Da}$ is the transform determined by $D$ and $a$. Let us consider its counterpart, $\varepsilon = \delta_{EB}$. One can easily verify that then $p(\delta(x_i)) = \varepsilon(y_i)$, where, we recall, $y_i = p^l(x_i)$. So the final statement of step 2 is equivalent to $\mathfrak{B}[E] \models \phi_{EB}(\varepsilon(y_1),...,\varepsilon(y_n))$.

Step 4. Using Lemma 13 with respect to the transform $\varepsilon = \delta_{EB}$ and the model $\mathfrak{B}$, we conclude that the final statement of step 3 is equivalent to $\mathfrak{B}[E_+] \models \Psi$.  \qed
5 A model for the isomorphism property

Fortunately the generic extensions of the considered type do not introduce new internal sets and new standard size collections of internal sets. This makes it possible to “kill” all pairs of elementarily equivalent internally presented structures by a product rather than iterated forcing. Following this idea, we prove (Theorem 4.5 below) that a product \( \Pi \) of different forcing notions of the form \( P_{L_{\mathcal{A}\mathcal{B}}} \), with internal \( \mathcal{I} \)-finite support, leads to generic extensions which model \( \text{HST} \) plus the isomorphism property \( \text{IP} \).

The product forcing will be a class forcing in this case because we have class–many pairs to work with; this will cause some technical problems in the course of the proof, in comparison with the exposition in Section 4.

We continue to consider a model \( \mathbb{H} \) of \( \text{HST} \), satisfying requirements \( (\dagger) \) and \( (\ddagger) \) in Section 4. \( S \subseteq \mathbb{I} \) and \( V \) are resp. the classes of all standard and internal sets in \( \mathbb{H} \), and the condensed subuniverse.

5.1 The product forcing notion

Arguing in \( \mathbb{H} \), let us enumerate somehow all relevant triples consisting of a language \( L \) and a pair of \( L \)-structures \( \mathcal{A}, \mathcal{B} \), to be made isomorphic.

Let, in \( \mathbb{H} \), \( \text{Ind} \) be the class of all 5-tuples \( i = \langle w, \kappa, L, A, B \rangle \) such that \( w \) is an internal set, \( \kappa \) is an \( \mathcal{I} \)-cardinal, \( L = \{ s_\alpha : \alpha < \kappa \} \) a first–order internal language \( \mathcal{I} \)-containing \( \leq \kappa \) symbols, and \( A, B \) are internal \( L \)-structures. (Then obviously \( i \) itself is internal.)

We set \( w_i = w, \kappa_i = \kappa, \; L_i = L, \; A_i = A, \; B_i = B \).

It is clear that \( \text{Ind} \) is a class \( \in \)-definable in \( \mathbb{I} \). Elements of \( \text{Ind} \) will be called \textit{indices}.

Suppose that \( i \in \text{Ind} \). Then by definition \( L_i = L = \{ s_\alpha : \alpha < \kappa \} \) is an internal language (with a fixed internal enumeration of the \( L \)-symbols). We define the \textit{restricted} standard size language \( L = L_i = \{ s_\alpha : \alpha < \kappa \; \& \; \text{st} \alpha \} \).

Let \( \mathcal{A}_i \) and \( \mathcal{B}_i \) denote the corresponding restrictions of \( \mathcal{A} \) and \( \mathcal{B} \); then both \( \mathcal{A}_i \) and \( \mathcal{B}_i \) are internally presented \( L_i \)-structures.

On the other hand, if \( L \) is a standard size language and \( \mathcal{A}, \mathcal{B} \) a pair of internally presented \( L \)-structures then there exists an index \( i \in \text{Ind} \) such that \( L = L_i, \; \mathcal{A} = \mathcal{A}_i, \; \text{and} \; \mathcal{B} = \mathcal{B}_i \).

The forcing \( \Pi \) will be defined as a collection of internal functions \( \pi \). Before the exact definition is formulated, let us introduce a useful notation: \( |\pi| = \text{dom} \pi \) (then \( |\pi| \in \mathbb{I} \)) and \( \pi_i = \pi(i) \) for all \( \pi \in \Pi \) and \( i \in |\pi| \).
Definition 43 \( \Pi \) is the collection of all internal functions \( \pi \) such that \( |\pi| \subseteq \text{Ind} \) is an \( I \)-finite (internal) set, and \( \pi_i \in \mathbb{P}_{L_i} \mathfrak{A}, \mathfrak{B} \) for each \( i \in |\pi| \). We set \( \pi \leq \rho \) (i.e. \( \pi \) is stronger than \( \rho \)) iff \( |\rho| \subseteq |\pi| \) and \( \pi_i \leq \rho_i \) (in \( \mathbb{P}_{L_i} \mathfrak{A}, \mathfrak{B} \), in the sense of Definition 35) for all \( i \in |\rho| \).

We set \( \pi \leq \rho \) (i.e. \( \pi \) is stronger than \( \rho \)) iff \( |\rho| \subseteq |\pi| \) and \( \pi_i \leq \rho_i \) (in \( \mathbb{P}_{L_i} \mathfrak{A}, \mathfrak{B} \), in the sense of Definition 35) for all \( i \in |\rho| \). 

Definition 44 \( \text{IP}^S \) is the strong form of \( \text{IP} \) which asserts that any two internally presented elementarily equivalent structures of a first–order language containing (standard size)–many symbols, are isomorphic via a \textit{locally internal} (see Definition 38) isomorphism.

Theorem 45 Suppose that in addition to (†) and (‡) \( \mathbb{H} \) satisfies statement \( P-1 \) in Theorem 28. Let \( \Pi \) be defined as above, in \( \mathbb{H} \). Then every \( \Pi \)-generic extension \( \mathbb{H}[G] \) is a model of \( \text{HST} \), a plain extension of \( \mathbb{H} \), where the “strong” isomorphism property \( \text{IP}^S \) holds.

This is the main result of this section. We begin the proof with several introductory remarks mainly devoted to relationships between the model \( \mathbb{H}[G] \) and its submodels.

We observe that \( \Pi \) is a proper class, not a set in \( \mathbb{H} \). This makes it necessary to change something in the reasoning in Section 3. For instance now \( G = \{ (\pi, \hat{\pi}) : \pi \in \Pi \} \) is not a set in \( \mathbb{H} \), so that one cannot assert that
$G \in \mathbb{H}[G]$. However this is not a problem because we are now interested in certain small parts of $G$, rather than $G$ itself, to be elements of $\mathbb{H}[G]$.

Let $C \in \mathbb{H}$, $C \subseteq \text{Ind}$. Then $\Pi_C = \{\pi \in \Pi : |\pi| \subseteq C\}$ is a set in $\mathbb{H}$. (Use the HST Collection.) We define $G_C = \Pi_C \cap G$, for each $G \subseteq \Pi$.

Let, for $\pi \in \Pi$, $\pi \upharpoonright C$ be the restriction of $\pi$ to the domain $|\pi| \cap C$; $\pi \upharpoonright C \in \Pi$ and $\in \Pi_C$ provided $|\pi| \cap C$ is internal. Furthermore we have $G_C = \{\pi \upharpoonright C : \pi \in G\}$ provided $G \subseteq \Pi$ is generic and $C$ is internal in $\mathbb{H}$.

We define, in $\mathbb{H}$, a set $\|a\| \subseteq \text{Ind}$ for each “name” $a \in N$, by induction on $\text{nrk}a$ as follows. If $a \in N_0$ then $\|a\| = \emptyset$. Otherwise we put $\|a\| = \bigcup_{(\pi,b) \subseteq a}(\|b\| \cup |\pi|)$. We let $N \upharpoonright C = \{a \in N : \|a\| \subseteq C\}$, for each set $C \subseteq \text{Ind}$. Then $N \upharpoonright C$ is precisely the class of all $\Pi_C$-“names”.

For instance $G_C = \{(\pi, \bar{\pi}) : \pi \in \Pi_C\}$ belongs to $N \upharpoonright C$.

**Proposition 46** Let $G \subseteq \Pi$ be $\Pi$-generic over $\mathbb{H}$. Suppose that $C \subseteq \text{Ind}$ is an internal set. Then

1. $G_C = G_C \upharpoonright [G] \in \mathbb{H}[G]$ is $\Pi_C$-generic over $\mathbb{H}$.
2. $\mathbb{H}[G_C] = \{a[G_C] : a \in N \upharpoonright C\}$ is a transitive subclass of $\mathbb{H}[G]$.
3. If $a \in N \upharpoonright C$ then $a[G] = a[G_C]$.

**Proof** An ordinary application of the product forcing technique. \hfill \Box

It follows that $\mathbb{H}[G]$ is a plain extension of $\mathbb{H}$, by Theorem [31].

### 5.3 The product forcing relation

The continuation of the proof of Theorem [33] involves forcing.

There is a problem related to forcing: the definition of $\models$ for the atomic formulas $a = b$ and $b \in a$ in Subsection [3.2] becomes unsound in the case when the notion of forcing is not a set in the ground model $\mathbb{H}$, as in the case we consider now. (This is a problem in the ZFC setting of forcing as well, see [Sh 71].) The solution follows the ZFC patterns: the inductive definition of forcing for atomic formulas can be executed using only set parts $\Pi_C$ of the whole forcing $\Pi$.

For each set $C \subseteq \text{Ind}$, let $\text{forc}_C$ and $\models_C$ be the forcing relations associated in $\mathbb{H}$ with $\Pi_C$ as the forcing notion, as in Subsection [3.2].

Our plan is as follows. We define the $\Pi$-forcing $\models$ for atomic formulas $a = b$ and $b \in a$ ( $a, b$ being “names” in $N$ ) using $\models_C$ for sufficiently
large internal sets $C \subseteq \text{Ind}$. Then we define $\models$ for other formulas following the general construction (items F-4 through F-7 in Subsection 3.2).

To start with, let us describe connections between $\text{forc}_C$ for different $C$, and the relation $\pi \text{ forc } b \in a$ defined for $\Pi$ as the notion of forcing as in Subsection 3.2.

**Proposition 47** Let $a, b \in N$, $\pi \in \Pi$. Suppose that $C$ is an internal set, and $\|a\cup b\| \subseteq C \subseteq \text{Ind}$. Then $\pi \text{ forc } b \in a$ if $\pi \models C \text{ forc } b \in a$.

**Proof** Elementary verification, based on the fact that $C$ and $\rho \|b\|$ set, and $\rho \|b\|$.

The following lemma is of crucial importance.

**Lemma 48** Let $\Phi$ be a formula of the form $a = b$ or $b \in a$, where $a, b \in N$. Suppose that $C, C'$ are internal sets, and $\|a\cup b\| \subseteq C \subseteq C' \subseteq \text{Ind}$. Let finally $\pi' \in \Pi_{C'}$ and $\pi = \pi' \upharpoonright C$. Then $\pi' \models C \Phi$ iff $\pi \models \text{forc } C \Phi$.

**Proof** The proof goes on by induction on the ranks $\text{nrk } a$ and $\text{nrk } b$.

Let $\Phi$ be the formula $b = a$. Let us suppose that $\pi' \models \text{forc } C \Phi = a$, and prove $\pi \models \text{forc } C \Phi = a$. Let $\rho \in \Pi_{C'}$, $\rho \leq \pi$, and $\rho \text{ forc } C \Phi x \in a$. We define $\rho' = \rho \cup (\pi' \upharpoonright (C' \setminus C')) \in \Pi_{C'}$; then $\rho' \models C = \rho$, and $\rho' \leq \pi'$ because $\rho \leq \pi$. Take notice that $||x|| \subseteq ||a|| \subseteq C$, so we have $\rho' \text{ forc } C \Phi x \in a$ by Proposition 47. It follows that $\rho' \models \text{forc } C \Phi x \in b$, since $\pi' \models \text{forc } C \Phi x \in b$ was assumed. We finally have $\rho \models \text{forc } C \Phi x \in b$ by the induction hypothesis.

Conversely, suppose that $\pi \models \text{forc } C \Phi x \in a$ and prove $\pi' \models \text{forc } C \Phi x \in a$. Assume that $\rho' \in \Pi_{C'}$, $\rho' \leq \pi'$, and $\rho' \text{ forc } C \Phi x \in a$. Then $||x|| \subseteq ||a|| \subseteq C$, so that $\rho = \rho' \upharpoonright C$ satisfies $\rho \leq \pi$ and $\rho \text{ forc } C \Phi x \in a$ by Proposition 47. Since $\pi \models \text{forc } C \Phi x \in b$, we have $\rho \models \text{forc } C \Phi x \in b$. We conclude that $\rho' \models \text{forc } C \Phi x \in b$ by the induction hypothesis.

Let $\Phi$ be the formula $b \in a$. Suppose that $\pi' \models \text{forc } C \Phi b \in a$ and prove $\pi \models \text{forc } C \Phi b \in a$. Let $\rho \leq \pi$. Then $\rho' = \rho \cup (\pi' \upharpoonright (C' \setminus C')) \in \Pi_{C'}$, $\rho' \leq \pi'$, so that there exist $\vartheta' \in \Pi_{C'}$, $\vartheta' \leq \rho'$, and some $z$ such that $\vartheta' \text{ forc } C \Phi z \in a$ and $\vartheta' \models \text{forc } C \Phi b = z$. Then $\vartheta = \vartheta' \upharpoonright C \in \Pi_C$ and $\vartheta \leq \rho$. On the other hand, $||z|| \subseteq ||a|| \subseteq C$, so that $\vartheta \text{ forc } C \Phi z \in a$ and $\vartheta \models \text{forc } C \Phi b = z$, as required.

Conversely, suppose that $\pi \models \text{forc } C \Phi b \in a$ and prove $\pi' \models \text{forc } C \Phi b \in a$. Let $\rho' \in \Pi_{C'}$, $\rho' \leq \pi'$. Then $\rho = \rho' \upharpoonright C \in \Pi_C$ and $\rho \leq \pi$, so that $\vartheta \models \text{forc } C \Phi z \in a$ and $\vartheta \models \text{forc } C \Phi b = z$, for some $\vartheta \in \Pi_C$, $\vartheta \leq \rho$, and a “name” $z$. We put $\vartheta' = \vartheta \cup (\rho' \upharpoonright (C' \setminus C'))$; so that $\vartheta' \in \Pi_{C'}$, $\vartheta' \leq \rho'$, and $\vartheta = \vartheta' \upharpoonright C$. Then $\vartheta' \models \text{forc } C \Phi z \in a$, and $\vartheta' \models \text{forc } C \Phi b = z$ by the induction hypothesis. □
This leads to the following definition.

**Definition 49** We introduce the forcing relation \( \models = \models_\Pi \) as follows.

Let \( a, b \in \mathbb{N} \) and \( \pi \in \Pi \). We set \( \pi \models b \in a \) iff \( \pi \models_C b \in a \), and \( \pi \models b = a \) iff \( \pi \models_C b = a \), whenever \( C \subseteq \text{Ind} \) is an internal set satisfying \( |\pi| \cup |a| \cup |b| \subseteq C \). (This does not depend on the choice of \( C \) by Lemma 48.)

The relation \( \models \) expands on the standardness predicate and non–atomic formulas in accordance with items F-4 through F-7 of Subsection 3.2. ◻

In view of this definition, Lemma 48 takes the following form:

**Corollary 50** Let \( \Phi \) be a formula of the form \( a = b \) or \( b \in a \), where \( a, b \) are names in \( \mathbb{N} \). Suppose that \( \pi \in \Pi \), and \( C \subseteq \text{Ind} \) is an internal set, satisfying \( |a| \cup |b| \subseteq C \). Then \( \pi \models \Phi \) iff \( \pi \models_C \models \Phi \). ◻

Furthermore, it occurs that the forcing \( \models \) still obeys the general scheme !

**Corollary 51** The relation \( \models \) formally satisfies requirements of items \( F-2 \) and \( F-3 \) of the definition of forcing in Subsection 3.2, with respect to \( \Pi \) as the forcing notion.

**Proof** An easy verification, with reference to Corollary 50 for large enough internal sets \( C \). ◻

**Remark 52** Corollary 51 guarantees that the results obtained for “set” size forcing in subsections 3.2 and 3.3 remain valid for the forcing \( \models \) associated with \( \Pi \), with more or less the same proofs. ◻

We need to verify two particular properties of the notion of forcing \( \Pi \), before the proof of Theorem 45 starts. One of them deals with the restriction property of the forcing; we would like to prove that \( p \models \Phi \) iff \( p \models_C \models \Phi \) provided \( |\Phi| \subseteq C \), for all, not only atomic formulas \( \Phi \). The other one is the standard size distributivity of \( \Pi \).

### 5.4 Automorphisms and the restriction property

We apply a system of automorphisms of the notion of forcing \( \Pi \) to approach the restriction property.

Let \( D \subseteq \text{Ind} \) be an internal set. An internal bijection \( h : D \) onto \( D \) satisfying the requirement:
(⋆) If \( i = \langle w, \kappa, \mathbf{L}, \mathbf{A}, \mathbf{B} \rangle \in D \) then \( h(i) = \langle w', \kappa, \mathbf{L}, \mathbf{A}, \mathbf{B} \rangle \) for some (internal) \( w' \) and the same \( \kappa, \mathbf{L}, \mathbf{A}, \mathbf{B} \),

will be called a correct bijection. In this case we define \( H(i) = h(i) \) for \( i \in D \), and \( H(i) = i \) for \( i \in \text{Ind} \setminus D \); so that \( H = H_1 \) maps \( \text{Ind} \) onto \( \text{Ind} \). \( H \) obviously inherits property \( (⋆) \).

The bijection \( H \) generates an order automorphism of the notion of forcing \( \Pi \), defined as follows. Let \( \pi \in \Pi \). We define \( H\pi \in \Pi \) so that \( |H\pi| = \{ H(i) : i \in |\pi| \} \) and \( (H\pi)_{H(i)} = \pi_i \) for each \( i \in |\pi| \). It follows from \( (⋆) \) that the map \( \pi \mapsto H\pi \) is an order automorphism of \( \Pi \).

Let us expand the action of \( H \) onto “names”. We define, in \( \mathbb{H} \), \( H[a] \) for each “name” \( a \), by induction on \( \text{nrk} a \). If \( a = \xi \in \mathbb{N}_0 \) then we put \( H[a] = a \). If \( \text{nrk} a > 0 \) then we set \( H[a] = \{ \langle H\pi, H[b] \rangle : \langle \pi, b \rangle \in a \} \). One easily proves that \( H[a] \in \mathbb{N} \) and \( \text{nrk} a = \text{nrk} H[a] \).

For a st-\( \varepsilon \)-formula \( \Phi \) containing “names” in \( \mathbb{N} \), we let \( H\Phi \) denote the formula obtained by changing each “name” \( a \) in \( \Phi \) to \( H[a] \).

**Proposition 53** Let \( h \) be a correct bijection, and \( H = H_h \). For any condition \( \pi \in \Pi \), and any closed formula \( \Phi \) having “names” in \( \mathbb{N} \) as parameters, \( \pi \models \Phi \) iff \( H\pi \models H\Phi \).

**Proof** We omit the routine verification, which can be conducted by induction on the complexity of the formulas involved, following arguments known from the theory of generic extensions of models of \( \text{ZFC} \). \( \square \)

**Corollary 54** (Restriction) Suppose that \( \pi \in \Pi \), \( \Phi \) is a closed formula containing “names” in \( \mathbb{N} \) as parameters, and \( \pi \models \Phi \). Suppose also that \( C \) is an internal set, and \( \|\Phi\| \subseteq C \). Then \( \pi \models C \models \Phi \).

**Proof** It follows from Lemma \( 33 \) that otherwise there exists a pair of conditions \( \pi, \rho \in \Pi \) such that \( \pi \models C = \rho \models \Phi \), but \( \rho \not\models \neg \Phi \). Let \( D = |\pi|, E = |\rho| \). It is clear that there exists an internal set \( W \) satisfying \( C \cup D \cup E \subseteq W \), and an internal correct bijection \( h : W \) onto \( W \) which is the identity on \( C \) and satisfies \( E \cap (h'' D) \subseteq C \). Let \( H = H_h \) be defined, from \( h \), as above. Let \( \pi' = H\pi \). Then \( \pi' \models C = \pi \models C = \rho \models C \) because \( h \models C \) is the identity. Furthermore \( |\pi'| = h'' D \), so that \( |\pi'| \cap |\rho| \subseteq C \). We conclude that \( \pi' \) and \( \rho \) are compatible in \( \Pi \).

On the other hand, \( \pi' \not\models H\Phi \) by Proposition \( 53 \). Thus it suffices to demonstrate that \( \Phi \) coincides with \( H\Phi \). We recall that \( \|\Phi\| \subseteq C \), so that each “name” \( a \) which occurs in \( \Phi \) satisfies \( \|a\| \subseteq C \). However one
can easily prove, by induction on \( nrk \alpha \) in \( \mathbb{H} \), that \( H[a] = a \) whenever \( \|a\| \subseteq C \), using the fact that \( h \restriction C \) is the identity. We conclude that \( H\Phi \) is \( \Phi \), as required. \( \square \)

5.5 Standard size distributivity of the product forcing

We are going to prove that \( \Pi \) is standard size distributive in \( \mathbb{H} \) provided \( \mathbb{H} \) satisfies requirement \( \text{P–1} \) of Theorem \( \text{28} \).

**Proposition 55** \( \Pi \) is standard size closed in \( \mathbb{H} \).

**Proof** In principle the proof copies that of Proposition \( \text{36} \), but we need to take more time to reduce the problem to Saturation. Suppose that \( \lambda \) is a cardinal, \( \pi_\alpha (\alpha < \lambda) \) are conditions in \( \Pi \), and \( \pi_\beta \leq \pi_\alpha \) whenever \( \alpha < \beta < \lambda \). Using the HST Collection and Lemma \( \text{3} \) in \( \mathbb{H} \), we get a standard set \( S \subseteq \text{Ind} \) such that each \( \pi_\alpha \) in fact belongs to \( \Pi_S \). Let us check that the sequence has lower bound already in \( \Pi_S \).

We observe that by the Collection axiom again, there exists a cardinal \( \kappa \) such that \( \kappa_i \leq \kappa \) in \( \mathbb{I} \) whenever \( i \in S \). \( (\kappa_i \) was defined in Subsection \( \text{5.1} \).) Then \( \Pi_S \) is an intersection of \( (\leq \kappa) \)–many internal sets by definition. Therefore every set \( P_\alpha = \{ \pi \in \Pi_S : \pi \leq \pi_\alpha \} \) \( (\alpha < \lambda) \) is, uniformly on \( \alpha < \lambda \), an intersection of \( (\leq \kappa) \)–many internal sets, too. Furthermore the sets \( P_\alpha \) are nonempty and \( P_\beta \subseteq P_\alpha \) whenever \( \alpha < \beta < \lambda \). Since \( \lambda \) and \( \kappa \) are sets of standard size by Lemma \( \text{10} \), we can use Saturation to obtain \( \bigcap_{\alpha \in \lambda} P_\alpha \neq \emptyset \), as required. \( \square \)

**Proposition 56** Assume that the ground model \( \mathbb{H} \) satisfies statement \( \text{P–1} \) of Theorem \( \text{28} \). Then \( \Pi \) is standard size distributive in \( \mathbb{H} \).

**Proof** We cannot directly refer to \( \text{P–1} \) and Proposition \( \text{55} \) because \( \Pi \) is a proper class rather than a set in \( \mathbb{H} \). \( (\text{Being a set is essential in the proof of Theorem \( \text{28} \), by the way.}) \) But of course we shall reduce the problem to the assumption of \( \text{P–1} \), by the choice of a suitable set part of \( \Pi \).

We have to prove the following. Let \( \kappa \) be a cardinal in \( \mathbb{H} \) (in the sense of Subsection \( \text{1.4} \)), and \( D \) be a \( \text{st}-\in \)-definable in \( \mathbb{H} \) subclass of \( \kappa \times \Pi \). Suppose that each class \( D_\alpha = \{ \pi : \langle \alpha, \pi \rangle \in D \} \) is open dense in \( \Pi \). Then the intersection \( \bigcap_{\alpha < \kappa} D_\alpha \) is dense in \( \Pi \) as well.

To prove the assertion, we fix a condition \( \pi_0 \in \Pi \). Let \( S_0 \subseteq \text{Ind} \) be an arbitrary standard set such that \( \pi_0 \in \Pi_{S_0} \). Let \( \kappa^+ \) be the next cardinal. \( (\text{In concern of cardinals, we are in } \mathbb{V}, \text{ a ZFC universe.}) \) Let us define an
increasing sequence of standard sets $S_\alpha \subseteq \text{Ind} \ (\alpha < \kappa^+)$ as follows. $S_0$ already exists. Suppose that $\gamma < \kappa$ and $S_\alpha$ is defined for each $\alpha < \gamma$. We first put $S'_\gamma = \bigcup_{\alpha < \gamma} S_\alpha$. (For instance $S'_\gamma = S_\beta$ provided $\gamma = \beta + 1$.) Using Collection and Lemma 3 in $\mathbb{H}$, and the assumption that every $D_\alpha$ is dense, we obtain a standard set $S$, $S'_\gamma \subseteq S \subseteq \text{Ind}$, such that for any $\pi \in \Pi S'_\gamma$ ($\Pi S'_\gamma$ is a set!) and any $\alpha < \kappa$ there exists $\rho \in \Pi S \cap D_\alpha$ such that $\rho \leq \pi$ in $\Pi$. Let $S_\gamma$ denote the least standard set $S$ of the form $V_\nu \cap \text{Ind}$ (where $V_\nu$ is the $\nu$-th level of the von Neumann hierarchy; $\nu$ being an $\mathbb{S}$-ordinal) satisfying this property.

Let $S = \bigcup_{\alpha < \kappa^+} S_\alpha$. Then $P = \Pi S$ is a set. Furthermore $\pi_0 \in P$, and each intersection $D'_\alpha = D_\alpha \cap P$ is dense in $P$ by the construction, because $P = \bigcup_{\alpha < \kappa^+} \Pi S_\alpha$. (In this argument, we use Saturation and the fact that $|\pi|$ is internal for $\pi \in \Pi$.) It remains to check that $P$ is $\kappa$-closed in $\mathbb{H}$: every decreasing sequence $\langle \pi_\alpha : \alpha < \kappa \rangle$ has a lower bound in $P$. (Indeed then $P$ is $\kappa$-distributive by the assumption of $P$-1, so that the intersection $\bigcap_{\alpha < \kappa} D'_\alpha$ is dense in $P$, etc.) By Proposition 54, the sequence has a lower bound $\pi \in \Pi$. (We cannot run the proof of Proposition 54 for $P$ directly because $P$ is not a standard size intersection of internal sets.) Since the construction of $S_\alpha$ involves all ordinals $\alpha < \kappa^+$, there exists an ordinal $\gamma < \kappa^+$ such that every condition $\pi_\alpha \ (\alpha < \kappa)$ belongs to $\Pi S_\gamma$. Then $\rho = \pi \restriction S_{\gamma+1}$ still satisfies $\rho \leq \pi_\alpha$ for all $\alpha$, but $\rho \in P$, as required. $\square$

5.6 Verification of the axioms

This subsection starts the proof of Theorem 45.

The verification of $\text{HST}$ in $\mathbb{H}[G]$ copies, to some extent, the proof of Theorem 31. (The standard size distributivity of the forcing, assumed in Theorem 31, now follows from Proposition 56.) Only the proofs of Separation and Collection need to be performed anew, because it was essential in Subsection 3.4 that the notion of forcing is a set in the ground model.

Separation. We follow the reasoning in the proof of Theorem 31. Suppose that $X \in N$, and $\Phi(x)$ is a $\text{st-}\varepsilon$-formula which may contain “names” in $N$ as parameters. We have to find a “name” $Y \in N$ satisfying the equality $Y[G] = \{ x \in X[G] : \Phi[G](x) \}$ in $\mathbb{H}[G]$.

We observe that all elements of $X[G]$ in $\mathbb{H}[G]$ are of the form $x[G]$ where $x$ belongs to the set of “names” $X = \{ x \in N : \exists \pi \ (\langle \pi, x \rangle \in X) \} \in \mathbb{H}$. We cannot now define $Y = \{ \langle \pi, x \rangle \in \Pi \times X : \pi \models x \in X \land \Phi(x) \}$, simply because this may be not a set in $\mathbb{H}$. (We recall that $\Pi$ is a proper
class in $\mathbb{H}$.) To overcome this difficulty, we replace $\Pi$, using the restriction theorem, by a suitable $\Pi_C$.

It follows from Lemma[3] that there exists an internal (even standard) set $C \subseteq \text{Ind}$ such that $\|\Phi\| \cup \|X\| \subseteq C$. Then, by the way, $\|x\| \subseteq C$ for every $x \in X$. We set $Y = \{ (\pi_x, x) \in \Pi_C \times \mathcal{X} : \pi_x \models x \in X \& \Phi(x) \}$. One easily proves that $Y$ is the required “name”, using Corollary[4] (the restriction theorem) and following usual patterns. (See e. g. Shoenfield [Sh 71].)

**Collection.** We suppose that $X \in N$, and $\Phi(x, y)$ is a formula with “names” in $N$ as parameters. Let $\mathcal{X} \subseteq N$, $\mathcal{X} \subseteq \mathbb{H}$, be defined in $\mathbb{H}$ as in the proof of Separation. It would suffice to find a set of “names” $\mathcal{Y} \in \mathbb{H}$, $\mathcal{Y} \subseteq N$, such that for every $x \in \mathcal{X}$ and every condition $\vartheta \in \Pi$, if $\vartheta \models \exists y \Phi(x, y)$ then there exist: a “name” $y \in \mathcal{Y}$ and a stronger condition $\rho \leq \vartheta$ which forces $\Phi(x, y)$.

Let us choose an internal set $C_0$ so that $\|\Phi\| \cup \|X\| \subseteq C_0$.

We have to be careful because $\Pi$, the notion of forcing, is a proper class in $\mathbb{H}$. However, since $\Pi_{C_0}$ is a set in $\mathbb{H}$, there exist: a set $P \subseteq \Pi$ of forcing conditions, and a set $Y_0 \in \mathbb{H}$, $Y_0 \subseteq N$, of “names”, satisfying the property: if $x \in \mathcal{X}$, and $\pi_0 \in \Pi_{C_0}$ forces $\exists y \Phi(x, y)$ then there exist: a condition $\pi \in P$, $\pi \leq \pi_0$, and a “name” $y \in Y_0$, such that $\pi \models \Phi(x, y)$.

The set $Y_0$ is not yet the $\mathcal{Y}$ we are looking for. To get $\mathcal{Y}$, we first of all choose an internal set $C$ such that $C_0 \subseteq C$, $|\pi| \subseteq C$ for all $\pi \in P$, $\|y\| \subseteq C$ for all $y \in Y_0$, the difference $C\setminus C_0$ is $\mathbb{I}$-infinite, and moreover, for any $i = \langle w, \kappa, L, A, B \rangle \in C$ there exist $\mathbb{I}$-infinitely many different indices $i' \in C$ of the form $i' = \langle w', \kappa, L, A, B \rangle \in C$ (with $w \neq w'$ but the same $\kappa, L, A, B$). Each internal correct bijection $h : C$ onto $C$ generates an automorphism $H_h$ of $\Pi$, see Subsection 5.4. Let us prove that

$$\mathcal{Y} = \{ H_h[y] : y \in \mathcal{Y} \text{ and } h \in \mathbb{I} \text{ is a correct bijection } C \text{ onto } C \}$$

is a set of “names” satisfying the property we need. (To see that $\mathcal{Y}$ is a set in $\mathbb{H}$ notice the following: all the bijections $h$ considered are internal by definition, so we can use internal power sets in $\mathbb{I}$.)

Let $x \in \mathcal{X}$ and $\vartheta \in \Pi$. Suppose that $\vartheta \models \exists y \Phi(x, y)$. Then the condition $\pi_0 = \vartheta \upharpoonright C_0$ also forces $\exists y \Phi(x, y)$ by Corollary[4]. ($\|\exists y \Phi(x, y)\| \subseteq C_0$ by the choice of $x$ and $C_0$.) Then, by the choice of $P$ and $Y_0$, there exist: a condition $\pi \in P$, $\pi \leq \pi_0$, and a “name” $y \in Y_0$, such that $\pi \models \Phi(x, y)$. Unfortunately $\pi$ may be incompatible with $\vartheta$; otherwise we would immediately consider any condition $\rho$ stronger than both $\pi$ and $\vartheta$. To overcome this obstacle, let us use an argument from the proof of Corollary[5].
Let \( \vartheta' = \vartheta \upharpoonright C \). Take notice that \( E = |\pi| \) and \( D' = |\vartheta'| \) are, by definition, \( \mathbb{I} \)-finite internal subsets of \( C \). There exists, by the choice of \( C \), an internal correct bijection \( h : C \) onto \( C \) such that \( h \upharpoonright C_0 \) is the identity and \( (h \upharpoonright E) \cap D' \subseteq C_0 \). Let \( H = H_h \). Then \( \pi' = H \pi \in \Pi_C \), \( \pi' \upharpoonright C_0 = \pi \upharpoonright C_0 \leq \pi_0 \), and \( |\pi'| \cap |\vartheta'| \subseteq C_0 \), so that \( \vartheta' \) and \( \pi' \) are compatible. Therefore \( \pi' \) is also compatible with \( \vartheta \) because \( \pi' \in \Pi_C \) and \( \vartheta' = \vartheta \upharpoonright C \). Let \( \rho \in \Pi \) be a condition stronger than both \( \pi' \) and \( \vartheta \).

We observe that \( \pi' \models H \Phi(H[x], H[y]) \), by Theorem 53. But, \( \|\Phi\| \subseteq C_0 \) and \( |x| \subseteq C_0 \) by the choice of \( C_0 \), so that \( H \Phi \) coincides with \( \Phi \) and \( H[x] = x \) because \( H \upharpoonright C_0 \) is the identity. We conclude that \( \rho \models \Phi(x, y') \), where \( y' = H[y] \) is a “name” in \( \mathcal{Y} \) by definition, as required.

5.7 Verification of the isomorphism property in the extension

We accomplish the proof of Theorem \([ \mathbb{H} ] \) in this subsection.

Since the standard sets are essentially the same in \( \mathbb{H} \) and \( \mathbb{H}[G] \), the condensed subuniverse \( \mathcal{V} \) is also one and the same in the two universes. Therefore \( \mathbb{H}[G] \) contains the same ordinals as \( \mathbb{H} \) does. (See subsections \([ \mathbb{L}_2^3 \) and \([ \mathbb{L}_4^3 \). Since standard size subsets of \( \mathbb{H} \) in \( \mathbb{H}[G] \) all belong to \( \mathbb{H} \), cardinals in \( \mathbb{H}[G] \) are the same as cardinals in \( \mathbb{H} \). (We recall that by definition cardinals mean: well-orderable cardinals, in \( \mathbb{HST} \).)

This reasoning shows that all the triples: language – structure – structure, to be considered in the frameworks of the isomorphism property in \( \mathbb{H}[G] \), are already in \( \mathbb{H} \). Thus let \( \mathcal{L} \in \mathbb{H} \) be a standard size first–order language, containing \( \kappa \) symbols in \( \mathbb{H} \) (\( \kappa \) being a cardinal in \( \mathbb{H} \)), and \( \mathfrak{A}, \mathfrak{B} \) be a pair of internally presented \( \mathcal{L} \)-structures in \( \mathbb{H} \). We have to prove that \( \mathfrak{A} \) is isomorphic to \( \mathfrak{B} \) in \( \mathbb{H}[G] \).

Using Lemma \([ \mathbb{H} ] \) in \( \mathbb{H} \), we obtain an internal first–order language \( \mathcal{L} = \{ s_\alpha : \alpha < \kappa \} \), containing \( \kappa \) symbols in \( \mathbb{I} \), and internal \( \mathcal{L} \)-structures \( \mathfrak{A} \) and \( \mathfrak{B} \), such that \( \mathcal{L} = \mathcal{G} \mathcal{L} = \{ s_\alpha : \alpha < \kappa \& \mathfrak{A} \in \mathcal{G} \mathcal{L} \} \), and \( \mathfrak{A}, \mathfrak{B} \) are the corresponding restrictions of \( \mathfrak{A}, \mathfrak{B} \). In other words, \( i = (0, \kappa, \mathcal{L}, \mathfrak{A}, \mathfrak{B}) \) belongs to \( \text{Ind} \) and \( \mathcal{L} = \mathcal{L}_i, \mathfrak{A} = \mathfrak{A}_i, \mathfrak{B} = \mathfrak{B}_i \).

We observe that the set \( G_i = \{ \pi_i : \pi \in G \& i \in |\pi| \} \) belongs to \( \mathbb{H}[G] \). (Indeed, since \( \Pi_i = \mathbb{P}_L \mathfrak{A}_i, \mathfrak{B}_i \) is a set in \( \mathbb{H} \), a “name” for \( G_i \) can be defined in \( \mathbb{H} \) as the set of all pairs \( \langle \pi, p \rangle \), where \( p \in \Pi_i \) and \( \pi = \langle i, p \rangle \in \Pi \) – so that \( |\pi| = \{ i \} \) and \( \pi_i = p \).) Furthermore \( G_i \) is \( \mathbb{P}_L \mathfrak{A}_i, \mathfrak{B}_i \)-generic over

\[ \text{Definition } [ \mathbb{H} ] \text{ is used. In fact the proof does not change much if the domains } |\pi| \text{ are restricted to be less than a fixed } 1 \text{-cardinal.} \]

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(An ordinary product forcing argument.) It follows from Theorem 33 in Section 4 that $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic in $\mathbb{H}[G_i]$ via the locally internal isomorphism $F_i = \bigcup G_i$, therefore in $\mathbb{H}[G]$, as required.

This ends the proof of Theorem 43. \qed
6 Proof of the main theorem

In this section, we gather the material of the model constructions above, with some results from [KR 95.1, KR 95.2, KR 96], to accomplish the proof of the main theorem (Theorem 1).

Let us fix a countable model $S$ of $\text{ZFC}$. We shall not assume that $S$ is a transitive model; in particular the membership relation $\in_S$ acting in $S$ may be not equal to the restriction $\in \upharpoonright S$.

**Proposition 57** There exists a countable model $I$ of $\text{BST}$, bounded set theory, such that the class of all standard sets in $I$ coincides with $S$, in particular $\in_S = \in_I \upharpoonright S$.

**Proof** We refer to [KR 95.1], Theorem 2.4. The proof goes on as follows. We first add to $S$ a generic global choice function $G$, using the method of Felgner [Fe 71]. This converts $S$ into a model $\langle S; G \rangle$ of $\text{ZFC} \cup \text{Global Choice}$, but with the same sets as $S$ originally had. (The assumption of countability of $S$ is used to prove the existence of a Felgner-generic extension of $S$.)

The global choice function makes it possible to define, in $\langle S; G \rangle$, a certain increasing sequence of class-many “adequate” ultrafilters. The corresponding ultralimit of $S$ can be taken as $I$. $\square$

**Proposition 58** There exists a countable model $H'$ of $\text{HST}$, such that the classes of all standard and internal sets in $H'$ coincide with resp. $S$ and $I$, in particular $\in_I = \in_{H'} \upharpoonright I$.

**Proof** We refer to [KR 95.2], Theorem 4.11. To get $H'$, we first consider $E$, the class of all elementary external sets (i.e. $\text{st}-\in$-definable in $I$ subclasses of sets in $I$, see Subsection 2.2 above), then define $H$ as the collection of all sets obtainable by the assembling construction, described in Subsection 2.1, from $\text{wf}$ pairs in $E$. Thus in principle the $H$ obtained this way is equal to $L[I]$, but we rather put this as a separate step. $\square$

We let $H$ be $L[I]$, formally defined in $H'$.

**Corollary 59** $H$ is a countable model of $\text{HST}$, such that the classes of all standard and internal sets in $H$ coincide with resp. $S$ and $I$, the isomorphism property $\text{IP}$ fails, and every standard size closed p. o. set is standard size distributive.
Proof We refer to Theorem 28 above.

This ends the proof of items (I), (II), (III) of Theorem 1, with respect to the theory \(HST + \neg IP\). Let us consider the other one, \(HST + IP\).

Corollary 60 There exists a countable model \(H^+\) of \(HST\), such that the classes of all standard and internal sets in \(H^+\) coincide with resp. \(S\) and \(I\), and the strong isomorphism property \(IP^S\) holds.

Proof Let us assume, for a moment, that \(S\), the initial model of \(ZFC\), is a wellfounded model, in the sense that the membership relation \(\in_S\) is wellfounded in the wider universe. In this case, \(H\) is wellfounded over \(I\) in the sense of Section 3 because the ordinals in \(H\) are the same as in \(V\), the condensed subuniverse, and therefore order isomorphic to the ordinals in \(S\). Since \(H\) is countable, there exists a \(\Pi\)-generic extension \(H^+ = H[G]\), of the type considered in Section 5. \(H^+\) is the required model by Theorem 45.

Let us now consider the general case: \(S\) at the beginning, and \(H\) at the end, may be not wellfounded. Then of course one cannot carry out the construction of \(H[G]\) described in Subsection 3.1.

But one can conduct a different construction, also known from manuals on forcing for models of \(ZFC\). This construction goes on as follows. We first define the forcing relation \(\lceil - \rceil\), associated with \(\Pi\) in \(H\), as in Subsection 5.3, which does not need any previous construction of the extension. Then we define, given a generic set \(G \subseteq \Pi\), the relations: \(a =_G b\) iff \(\exists \pi \in G (\pi \lceil a =_G b)\), and similarly \(a \in_G b\) and \(st_G a\), for all “names” \(a, b \in N\). \(=_G\) can be easily proved to be an equivalence relation on \(N\), while the other two relations to be \(=_G\)-invariant. This allows to define \(H[G]\) to be the quotient \(N/=_G\), equipped with the quotients of \(\in_G\) and \(st_G\) as the atomic relations. The map \(x \mapsto (the \_=_G\text{-class of } x)\) is a \(st\)-\(\in\)-isomorphism \(H\) onto an \(\in_G\)-transitive part of \(H[G]\). (We refer to Shoenfield [Sh 71].)

This approach makes it possible to carry out the whole system of reasoning used to prove Theorem 45, with minor changes.

Corollary 60 implies the statements of items (I), (II), (III) of Theorem 1, with respect to the theory \(HST + IP\).

Let us finally demonstrate that the defined above models, \(H\) of the theory \(HST + \neg IP\) and \(H^+\) of the theory \(HST + IP\), satisfy the additional requirement (IV) of Theorem 1.

We fix a \(st\)-\(\in\)-formula \(\Phi(x_1, ..., x_n)\).
Step 1. Let $\Phi_1(x_1, \ldots, x_n)$ be the formula $\emptyset \models \Phi(x_1, \ldots, x_n)$, where $\models$ is the forcing relation $\models_\Pi$, associated with $\Pi$ in $\mathbb{H}$, while $\emptyset$ is the empty set considered as a forcing condition. It is an easy consequence of the restriction theorem (Corollary 54) and the truth lemma (Theorem 34) that

$$\mathbb{H}^+ \models \Phi(x_1, \ldots, x_n) \quad \text{iff} \quad \mathbb{H} \models \Phi_1(x_1, \ldots, x_n) \quad (1)$$

— for all $x_1, \ldots, x_n \in \mathbb{H}$. This reasoning obviously eliminates $\mathbb{H}^+$ from the problem of consideration, and reduces the question to $\mathbb{H}$.

Step 2. We recall that, by the construction, $\mathbb{H}$ is $L[I]$ in a model $\mathbb{H}'$ of HST. (In fact $\mathbb{H} = \mathbb{H}'$, but we shall not use this.) It follows from Proposition 25, applied in $\mathbb{H}'$, that $\mathbb{H}$ has a definable interpretation in $\mathbb{E}$, the collection of all elementary external sets. Therefore for each st-$\in$-formula $\Phi_1(x_1, \ldots, x_n)$ there exists another st-$\in$-formula $\Phi_2(x_1, \ldots, x_n)$ such that, for all $x_1, \ldots, x_n \in \mathbb{E}$,

$$\mathbb{H} \models \Phi_1(x_1, \ldots, x_n) \quad \text{iff} \quad \mathbb{E} \models \Phi_2(x_1, \ldots, x_n). \quad (2)$$

Step 3. By definition sets in $\mathbb{E}$ admit a uniform st-$\in$-definition from the point of view of $I$. This makes it possible to pull things down to $I$: for each st-$\in$-formula $\Phi_2(x_1, \ldots, x_n)$ there exists another st-$\in$-formula $\Phi_3(x_1, \ldots, x_n)$ such that, for all $x_1, \ldots, x_n \in I$,

$$\mathbb{E} \models \Phi_2(x_1, \ldots, x_n) \quad \text{iff} \quad I \models \Phi_3(x_1, \ldots, x_n). \quad (3)$$

Step 4. We finally observe that $I$ admits a reduction to $S$, by a result proved in [KR 95,1] (Corollary 1.6 there), so that for each st-$\in$-formula $\Phi_3(x_1, \ldots, x_n)$ there exists a $\in$-formula $\Phi_4(x_1, \ldots, x_n)$ such that

$$I \models \Phi_3(x_1, \ldots, x_n) \quad \text{iff} \quad S \models \Phi_4(x_1, \ldots, x_n) \quad (4)$$

holds for all $x_1, \ldots, x_n \in S$.

Taking the statements (1) through (4) together we conclude that the models $\mathbb{H}$ and $\mathbb{H}^+$ satisfy the additional requirement [IV] of Theorem 1.

\[\square\]
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