Thermodynamics of micropolar fluids with variable concentration of polar particles

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Abstract. In this paper we derive equations of micropolar fluid with variable concentration of polar particles. Such fluids have non-symmetrical stress tensor and exhibit micro-rotations. To illustrate the derived equations, we consider a pressure driven flow between two parallel planes.

1. Introduction

There are fluids which exhibit microrotational effects and microrotational inertia. Examples are animal blood, cements, cutting transport by drilling mud, and granular flows. Recent examples concern the proppant flow within the hydrofracture [1]. One can tackle such fluids by applying the notion of the Cosserat continua. The theory of micro-polar fluids goes back to [2, 3, 4] where gyration tensor, inertial spin, conservation of micro-inertia and objectivity of micro-deformation rate tensors are derived and discussed. For an overview of developed theories we refer the reader to [1, 5, 6, 7, 8, 9, 10] Within this theory, each material point is treated like a rigid body. In the theory of classical fluids, the local stress state is characterized completely by the Cauchy stress tensor $T$. To characterize stresses in the micro-polar fluid, one should take also into account the couple stress tensor $N$. Local deformations are characterized also by two tensors which allow to calculate the velocity gradient and the spin gradient.

In the present paper, we derive a mathematical model allowing for variable concentration of polar particles. To this end, we apply the basic thermodynamic principles and formulate conservation laws and constitutive equations in such a way that they agree with the non-negative sign of entropy production. Earlier, thermodynamics of the micropolar fluid was studied without effects of convection and inertia [11]. We emphasize that the concentration flux obeys a generalized Fick law, i. e. it depends not only on gradients of the functions like concentration, temperature, and pressure, but on the particle gyration as well.

To test our model, we address the so-called ”tubular pinch effect” of Ségre and Silberberg [12] showing that for the laminar flow of neutrally buoyant dilute suspension of rigid spheres through a circular tube, particles tend to migrate towards a concentric annular region with the mean radius $r_m = 0.6 R$, where $R$ is the tube radius. To this end, we apply the equations derived to study one-dimensional steady flows in a channel driven by a pressure gradient for different constitutive parameters and discover that the Ségre-Silberberg effect is genuinely nonlinear one since it disappears if the gyration coefficient in the generalized Fick law vanishes.

The paper is organized as follows. First, we recall thermodynamics of classical compressible fluids without dissipation. Next, we consider thermodynamics of compressible micropolar fluids...
without and with dissipation. The equations derived are applied to one-dimensional steady flows between two parallel planes. Particularly, we analyze the Sègre-Silberberg effect.

2. Thermodynamics of micropolar fluid

First, we remind classical conservation laws for fluids without dissipation:

\[ \rho_t + \text{div} \mathbf{j} = 0, \quad \mathbf{j}_t + \text{div} \Pi = 0, \quad \eta_t + \text{div} (\eta \mathbf{v}) = 0, \quad (1) \]

where

\[ \mathbf{j} = \rho \mathbf{v}, \quad \Pi = pI + \mathbf{j} \otimes \mathbf{v}. \quad (2) \]

Here, \( \rho \) is the density, \( p \) is the pressure, \( \mathbf{v} \) is the velocity, \( \eta \) is the specific entropy, \( I \) is the identical matrix. The symbol \( \otimes \) means tensor product of two vectors, \( (a \otimes b)_{ij} = a_i b_j \). Given a matrix \( A \), the vector \( \text{div} A \) is defined by the formula \( (\text{div} A)_i = \partial A_{ij} / \partial x_j \).

Let \( E_0 = E_0(\rho, \eta) \) be the internal energy, then we can write the Gibbs law as follows:

\[ dE_0 = \theta d\eta + \mu d\rho. \quad (3) \]

One can define the absolute temperature \( \theta \) and the chemical potential \( \mu \) as derivatives:

\[ \theta = \frac{\partial E_0}{\partial \eta}, \quad \mu = \frac{\partial E_0}{\partial \rho}. \]

The thermodynamic pressure is given by the expression \([13, 14]\)

\[ p = -E_0 + \theta \eta + \mu \rho. \quad (4) \]

It follows from (1) - (4) that total energy \( E = E_0 + \rho \mathbf{v}^2 / 2 \) satisfies the conservation law

\[ E_t + \text{div} \mathbf{Q} = 0, \quad (5) \]

where the energy flux is equal to

\[ \mathbf{Q} = j \frac{\|\mathbf{v}\|^2}{2} + \eta \theta \mathbf{v} + \rho \mu \mathbf{v}. \quad (6) \]

Our goal is to generalize (1)-(6) for micropolar fluids with dissipation.

We remind that in Cosserat’s continuum each material point with Lagrange coordinates \((t, \xi)\) is characterized by the position vector \( \mathbf{x}(t, \xi) \) and three orthogonal director - vectors \( \mathbf{d}_i(t, \xi), i = 1, 2, 3 \) whose orientation is controlled by the orthogonal tensor \( Q(t, \xi) \):

\[ \mathbf{d}_i(\xi, t) = Q(\xi, t)(\mathbf{d}_i(\xi, 0)), \quad \mathbf{Q} \cdot \mathbf{Q}^* = \mathbf{Q}^* \cdot \mathbf{Q} = I. \]

With the use of the material derivative \( \dot{Q}(x, t) \), one can define the skew-symmetric gyration tensor

\[ \Omega(x, t) = \dot{Q}(x, t) \cdot \mathbf{Q}^*(x, t) \equiv \mathbf{Q}(t, \xi) \cdot \mathbf{Q}^*(t, \xi), \]

which enjoys the formula

\[ \Omega(\mathbf{h}) = \mathbf{w} \times \mathbf{h}, \quad \forall \mathbf{h} \in \mathbb{R}^3, \quad (\Omega(\mathbf{h}))_{ij} \equiv \Omega_{ij} h_j, \]

where \( \mathbf{w}(x, t) - \) gyration vector,

\[ 2\mathbf{w} = \mathbf{e}_i \times \Omega(\mathbf{e}_i) = \varepsilon : \Omega^*. \]
Here \( \{ e_i \}_{i=1}^3 \) is any orthogonal basis in \( \mathbb{R}^3 \) and \( \varepsilon \) is third order Levi Chivita tensor
\[
\varepsilon(a, b, c) = a \cdot (b \times c), \quad e_i \times e_j = \varepsilon_{sij} e_s,
\]
\[
\varepsilon_{sij} \equiv \varepsilon(e_s, e_i, e_j), \quad (\varepsilon : \Omega)_i \equiv \varepsilon_{ijk} \Omega_{jk}.
\]
Scalar product \( A : B \) for two \( 3 \times 3 \) matrices \( A \) and \( B \) is defined by \( A : B = A_{ij} B_{ij} \).

We assume that all particles are balls. We denote by \( \Theta \) the microinertia tensor which satisfies the equation \[5\]
\[
d \frac{d}{dt} \Theta = \Omega \Theta - \Theta \Omega.
\]
For balls, \( \Theta = J \cdot I \), where \( J \) is the inertia constant.

Naturally, for Cosserat continuum without dissipation, microrotation of two different particles do not interact. Hence,
\[
\frac{dw}{dt} \equiv w_t + \nabla w \langle v \rangle = 0, \quad (\rho c)_t + \text{div}(\rho c v) = 0, \quad (\nabla w)_{ij} = \frac{\partial w_i}{\partial x_j};
\]
here \( c \) is the mass concentration of polar particles.

We use equations (1) to write system (7) as the conservation laws
\[
J(\rho c w)_t + \text{div} M = 0, \quad (\rho c)_t + \text{div}(\rho c v) = 0,
\]
where \( M = \rho J c w \otimes v \).

Equations (1)-(6), (8) together constitute a model of micropolar fluid without dissipation:
\[
\rho_t + \text{div} j = 0, \quad j_t + \text{div} \Pi = 0, \quad \eta_t + \text{div} (\eta v) = 0,
\]
\[
J(\rho c w)_t + \text{div} M = 0, \quad (\rho c)_t + \text{div}(c j) = 0.
\]

Let \( E_0(\rho, \eta, c) \) be the internal energy. Starting from the Gibbs law
\[
dE_0 = \theta d\eta + \mu d\rho + z d(\rho c).
\]
one can define chemical potentials \( \mu, z \) and temperature \( \theta \) as follows
\[
\theta = \frac{\partial E_0}{\partial \eta}, \quad \mu = \frac{\partial E_0}{\partial \rho}, \quad z = \frac{\partial E_0}{\partial (\rho c)}.
\]
Pressure is given by the formula \[14\]
\[
p = -E_0 + \theta \eta + \mu \rho + z \rho c.
\]
It follows from (9)-(12) that the total energy
\[
E = E_0 + \frac{\rho |v|^2}{2} + \frac{J c \rho |w|^2}{2}
\]
satisfies the energy equation
\[
E_t + \text{div} Q = 0,
\]
here
\[
Q = \rho v \frac{|v|^2}{2} + \rho v J c \frac{|w|^2}{2} + \eta \theta v + \rho \mu v + z \rho c v,
\]
To equip the model (9) - (15) with dissipation, we write the conservation laws

$$\rho_t + \text{div} j = 0, \quad j_t + \text{div} (\Pi + \Pi_1) = 0, \quad \eta_t + \text{div} \left( \eta v + \frac{q}{\theta} \right) = \frac{R}{\theta}, \quad (16)$$

$$J(\rho c w)_t + \text{div} (M + M_1) = f_w, \quad (\rho c)_t + \text{div}(\epsilon + 1) = 0, \quad (17)$$

$$E_t + \text{div} (Q + Q_1) = 0, \quad (18)$$

with unknown functions $\xi = \{\Pi, M, l, q, f_w, Q_1, R\}$. Here $R$ is entropy production and $q$ is the heat flux. Conservation laws (16)-(18) should agree with the thermodynamic relations (11)-(13). We assume as in [14, 15], that the energy equation (18) results from the system (16)-(18), (11)-(13) In this way we can determine the unknowns $\xi$.

We calculate $E_t$ using conservation laws (16)-(17) for elimination the time derivatives $\rho_t, \eta_t$ and etc. We have

$$E_t = \rho_t \frac{|v|^2}{2} + j \cdot v + J(\rho c)_t \frac{|w|^2}{2} + J\rho c w \cdot w_t + \theta \eta_t + \mu \rho_t + z(\rho c)_t =$$

$$- \frac{|v|^2}{2} \text{div}(\rho v) + v \cdot [\text{div} j - \text{div}(\Pi + \Pi_1)] - J \frac{|w|^2}{2} \text{div}(\epsilon + 1) +$$

$$w \cdot [-\text{div}(M + M_1) + Jw \text{div}(\epsilon + 1) + f_w] + \theta \left( \frac{R}{\theta} - \text{div}(\eta v) - \text{div} \frac{q}{\theta} \right) -$$

$$- \mu \text{div} j - z \text{div}(\epsilon + 1).$$

With the help of the equality

$$u \cdot \text{div} A = \text{div} (A^* \langle u \rangle) - A : \nabla u, \quad (\nabla u)_{ij} = \frac{\partial u_i}{\partial x_j},$$

we calculate that

$$E_t + \text{div} \left[ Q + \Pi^* (v) + M_1^* (w) + q + 1 \left( z - \frac{J |w|^2}{2} \right) \right] =$$

$$\Pi_1 : \nabla v + M_1 : \nabla w + 1 \cdot \nabla \left( z - \frac{J |w|^2}{2} \right) + \frac{q \cdot \nabla \theta}{\theta} + w \cdot f_w + R.$$
For any skew-symmetric matrix $A$, one can find unique vector $a$ such that $A(h) = a \times h \quad \forall h$. One can verify easily that $A_1 : A_2 = 2a_1 \cdot a_2$ if the linear maps $A_i, \ i = 1, 2$, are defined by the formulas $A_i(h) = a_i \times h$. Particularly,

\[
(\nabla v - (\nabla v)^* \h) = \text{rot} \ v \times h \quad \forall h \in \mathbb{R}^3, \quad \text{rot} \ v = -\varepsilon : \nabla v.
\]

We introduce two strain rate tensors $A = \nabla w$ and $B = \nabla v - \Omega$. It is clear that one can write skew-symmetric matrix $B_a$ as follows

\[
B_a(h) = \left(\frac{\text{rot} \ v}{2} - w\right) \times h.
\]

We chose $\Pi_1, M_1, q$ and $f_w$ in the form

\[
\Pi_1 = -2\eta_1 B_a^d - 2\eta_2 B_a - 2\eta_3 \text{tr} B \cdot I, \quad q = -\varepsilon \nabla \theta,
\]

\[
M_1 = -2\beta_1 A_a^d - 2\beta_2 \text{tr} A \cdot I - 2\beta_3 A_a + M_2, \quad f_w = 2\eta_2 (\text{rot} v - 2w).
\]

In general, the viscosities $\eta_i$ and $\beta_i$ depend on mass concentration $c$. Hence,

\[
\Pi_1 : \nabla v + M_1 : \nabla w + w \cdot f_w + \frac{q \cdot \nabla \theta}{\theta} = -r^2 + M_2 : \nabla w,
\]

where

\[
\begin{align*}
   r^2 &= 2\eta_1 \left| B_a^d \right|^2 + 2\eta_2 |B_a|^2 + 2\eta_3 \left| \text{tr} B \right|^2 + 2\beta_1 \left| A_a^d \right|^2 + 2\beta_3 |A_a|^2 \\
   &\quad + 2\beta_2 |\text{tr} A|^2 + \varepsilon |\nabla \theta|^2 / \theta.
\end{align*}
\]

We notice that

\[
1 \cdot \nabla w = 2(w \otimes I) : \nabla w,
\]

hence, the entropy production becomes

\[
R = r^2 + \nabla w : (Jw \otimes I - M_2) - 1 \cdot \nabla z.
\]

We denote

\[
b = w - \frac{\text{rot} \ v}{2} \equiv \varepsilon : B,
\]

and choose

\[
\begin{align*}
   \nabla z &= -1/\alpha_1 - \alpha_2 l \times b + \alpha_3 \text{rot} \ w + \alpha_4 \text{rot} \ w \times b, \\
   M_2 &= Jw \otimes I + \alpha_3 \varepsilon : I + \alpha_4 \varepsilon : (1 \times b),
\end{align*}
\]

here $\alpha_i = \alpha_i(c)$ are diffusion coefficients, so

\[
R = r^2 + l^2 / \alpha_1 - \alpha_3 |\nabla w : (\varepsilon : I) + 1 \cdot \text{rot} \ w| \quad \begin{cases}
   \alpha_4 \{\nabla w : [\varepsilon : (1 \times b)] + 1 \cdot (\text{rot} \ w) \} = r^2 + l^2 / \alpha_1.
\end{cases}
\]

Hence, the conditions

\[
\varepsilon \geq 0, \quad \eta_i \geq 0, \quad \beta_i \geq 0, \quad \alpha_1 > 0
\]

insure the inequality $R \geq 0$.

We introduce the matrix

\[
S_b = \alpha_1^{-1} I - \alpha_2 \varepsilon : b,
\]
and write equation (20) in the form
\[ l = S_b^{-1}(-\nabla z + \alpha_3 \text{rot} \ w + \alpha_4 \text{rot} \ w \times b) \]  
(23)

It is a generalization of the classical Fick law [13] in which \( \alpha_2 = \alpha_3 = \alpha_4 = 0. \)

It results from (11) and (12) that
\[ \nabla p = \eta \nabla \theta + \rho \nabla \mu + \rho c \nabla z. \]

We assume that chemical potential \( \mu \) depends on \( p, \theta \) and \( c \). Hence,
\[ \nabla \mu = \mu_p \nabla p + \mu_\theta \nabla \theta + \mu_c \nabla c. \]

And we can write instead of (23) the following equality
\[ l = -S_b^{-1}(\gamma_3 \nabla c + \gamma_1 \nabla p + \gamma_2 \nabla \theta - \alpha_3 \text{rot} \ w - \alpha_4 \text{rot} \ w \times b), \]  
(24)

where
\[ \nabla z = \gamma_1 \nabla p + \gamma_2 \nabla \theta + \gamma_3 \nabla c, \]
\[ \gamma_1 = \frac{1 - \rho \mu_p}{\rho c}, \quad \gamma_2 = -\frac{\eta + \rho \mu_\theta}{\rho c}, \quad \gamma_3 = -\frac{\mu_c}{c}. \]

For classical fluids, when \( \alpha_2 = \alpha_3 = \alpha_4 = 0 \), equation (24) is usually written in the form
\[ l = -\rho D(\nabla c + k_\theta \nabla \theta + k_p \nabla p), \]

here \( D \) is the diffusion coefficient [15]. This is why we denote
\[ \gamma_1 = \frac{\rho D k_p}{\alpha_1}, \quad \gamma_2 = \frac{\rho D k_\theta}{\alpha_1}, \quad \gamma_3 = \frac{\rho D}{\alpha_1}. \]

Let us introduce a stress tensor \( T \) and a couple stress tensor \( N \):
\[ T = -pI + S, \quad S = -\Pi_1, \quad N = -M_1 + Jw \otimes I, \]

where \( S \) is the viscous part of stress tensor. One can verify that rate of entropy production is given by the formula
\[ R = S : B + N : A + \frac{\eta^2}{\alpha_1} + \alpha_3 \text{tr} (\varepsilon : A) + \frac{\alpha_4}{2} (\varepsilon : A) \times (\varepsilon : B). \]  
(25)

We arrive at the following conservation laws:
\[ \rho_t + \text{div} \rho v = 0, \]  
(26)
\[ (\rho v)_t + \text{div}(\rho v \otimes v) = -\nabla p + \text{div} S, \]  
(27)
\[ J\{(\rho cw)_t + \text{div}[w \otimes (\rho c v + 1)]\} = \text{div} N - \varepsilon : S, \]  
(28)
\[ (\rho c)_t + \text{div}(\rho c v) = -\text{div} l. \]  
(29)
\[ \eta_t + \text{div} \left( \eta v + \frac{q}{\theta} \right) = \frac{R}{\theta}. \]  
(30)

We have verified that constitutive laws
\[ S = 2\eta_1 B_a^d + 2\eta_2 B_a + 2\eta_3 \text{tr} B \cdot I, \]  
(31)
\[ N = 2\beta_1 A_1^2 + 2\beta_2 \text{tr} A \cdot I + 2\beta_3 A_1 - \alpha_3 \varepsilon : 1 - \alpha_4 \varepsilon : (1 \times b), \]  
\[ l = -S_b^{-1}(\gamma_3 \nabla c + \gamma_1 \nabla p + \gamma_2 \nabla \theta - \alpha_3 \text{rot } w - \alpha_4 \text{rot } w \times b) \]  

agree with the inequality \( R \geq 0 \) if 
\[ \eta_1 \geq 0, \quad \beta_1 \geq 0, \quad \alpha_1 > 0. \]

**Remark 1.** The micropolar effect disappears if \( c = 0 \) this is why \( \eta_2 |_{c=0} = 0 \). The concentration flux \( l \) also disappears if \( c = 0 \) and \( c = 1 \). Hence \( \alpha_3, \alpha_4 \) and \( k_F \) should be equal to zero if \( c = 0 \) and \( c = 1 \).

To complete the model (26)-(33), we should add the thermodynamic relations
\[ dE_0 = \theta d\eta + \mu d\rho + z d(\rho c), \quad \theta = \frac{\partial E_0}{\partial \eta}, \quad \mu = \frac{\partial E_0}{\partial \rho}, \quad z = \frac{\partial E_0}{\partial (\rho c)}. \]  
Pressure is defined by (12). The heat flux \( q \) and the rate of entropy production \( R \) are defined by Fourier law \( q = -\kappa \nabla \theta \) and the formula 25 respectively.

**Remark 2.** In the theory of micro-polar fluids, the couple stress tensor \( N \) depends on the rate of strain tensors \( B \) and \( A \). In the case of variable concentration, \( N \) depends on \( l \) also. It is proved in Appendix, that the variables \( l, b, \) and \( N \) are objective. Observe, that objectivity of \( \nabla c \) results from objectivity of the scalar function \( c \).

### 3. One-dimensional flows

We consider the flow between two parallel plains in the \( x \) direction under the hypothesis that \( p_x = \text{const} < 0, p_y = p_z = 0 \). Under the assumption of incompressibility, \( \rho = \text{const} \), the mass conservation law takes the form of \( \text{div } v = 0 \) and is identically satisfied in the one-dimensional case. The velocity vector takes the form \( v = (v(y, t), 0, 0)^T \), where \( y \) is the transverse coordinate \( (-H < y < H) \) and \( t \) is the time variable. Here and in what follows, \( a^T \) is the transposition of the vector \( a \). Gyration vector is written as \( w = (0, 0, w(y, t))^T \) and concentration flux is written as \( l = (l_1(y, t), l_2(y, t), t)^T \). In this case we have
\[
\text{rot } v = (0, 0, -v_y)^T, \quad \text{rot } w = (w_y, 0, 0)^T, \quad b = (0, 0, b)^T, \quad b = w + v_y/2, \\
B = \begin{pmatrix} 0 & v_y + w & 0 \\ -w & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & w_y & 0 \end{pmatrix}.
\]

We consider flows without heating. By projecting equation (24) on \( x \) and \( y \) axes we get
\[
\alpha_1^{-1} l_1 + b\alpha_2 l_2 = -\gamma_1 p_x + \alpha_3 w_y, \\
-\alpha_2 l_2 + \alpha_1^{-1} l_1 = -\gamma_3 c_y - \alpha_4 b w_y.
\]

or equivalently
\[
l_1 = \frac{(-\gamma_1 p_x + \alpha_3 w_y)/\alpha_1 + b\alpha_2 (\gamma_3 c_y + \alpha_4 b w_y)}{\alpha_1^{-2} + \beta^2 \alpha_2^2}, \\
l_2 = \frac{-b\alpha_2 (-\gamma_1 p_x + \alpha_3 w_y) + (\gamma_3 c_y + \alpha_4 b w_y)/\alpha_1}{\alpha_1^{-2} + \beta^2 \alpha_2^2}.
\]
In one-dimensional case, the stress tensor and the couple stress tensor take the form

\[
S = \begin{pmatrix}
0 & S_{12} & 0 \\
S_{21} & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad
N = \begin{pmatrix}
0 & 0 & N_{13} \\
0 & 0 & N_{23} \\
N_{31} & N_{32} & 0
\end{pmatrix},
\]

where

\[
S_{12} = (\eta_1 + \eta_2)v_y + 2\eta_2 w, \quad S_{21} = (\eta_1 - \eta_2)v_y - 2\eta_2 w,
\]

\[
N_{13} = -\alpha_3 l_2 + \alpha_4 b_l, \quad N_{31} = \alpha_3 l_2 - \alpha_4 b_l,
\]

\[
N_{23} = (\beta_1 - \beta_3) w_y + \alpha_3 l_1 + \alpha_4 b_l, \quad N_{32} = (\beta_1 + \beta_3) w_y - \alpha_3 l_1 - \alpha_4 b_l.
\]

Thus equations (27)-(29) can be reduced to the system

\[
\rho c_t = -\frac{\partial}{\partial y} l_2, \quad \rho v_t = -p_x + \frac{\partial}{\partial y} S_{12},
\]

\[
J (\rho c w_t + l_2 w_y) = \frac{\partial}{\partial y} N_{32} + S_{21} - S_{12}.
\]

Let us consider steady flows. In this case \( l_2 = \text{const} \). Channel flows are symmetrical relative to the \( x \)-axis, therefore the functions \( v(y) \) and \( c(y) \) are even, and the functions \( w(y) \) and \( b(y) \) are odd. It follows that \( l_2 = 0 \) and equation (35)(2) can be written as

\[
\gamma_3 \frac{d}{dy} c = b[\alpha_1 \alpha_2 (-\gamma_1 p_x + \alpha_3 w_y) - \alpha_4 w_y], \quad b = w + v_y/2.
\]

In the case of steady flows, the equations for \( v \) and \( w \) can be written in the following form:

\[
\frac{d}{dy} [(\eta_1 + \eta_2)v_y + 2\eta_2 w] = p_x, \quad (37)
\]

\[
\frac{d}{dy} [(\beta_1 + \beta_3 - \alpha_1 \alpha_3^2) w_y + \alpha_1 \alpha_3 \gamma_1 p_x] = 4\eta_2 b. \quad (38)
\]

We integrate equation (37) taking into account that the function \( b(y) \) is odd:

\[
(\eta_1 + \eta_2)v_y + 2\eta_2 w = yp_x.
\]

Hence, we arrive at the equation

\[
b = \frac{\eta_1}{\eta_1 + \eta_2} w + \frac{yp_x}{2(\eta_1 + \eta_2)}.
\]

In general, the viscosities \( \beta_i, \eta_i \) and the diffusion coefficients \( \alpha_i \) depend on the concentration \( c(y) \). To simplify our calculations we set \( \beta_1 = \text{const} \) and \( \beta_3 = \text{const} \). We denote \( \eta_1^0 = \eta_i|_{c=0} \) and introduce dimensionless variables and functions:

\[
y' = \frac{y}{H}, \quad w' = \frac{w}{w_0}, \quad \eta_1'(c) = \frac{\eta_1}{\eta_1^0}, \quad \varepsilon(c) = \frac{\eta_2}{\eta_1 + \eta_2}, \quad s_0 = \frac{4\eta_1^0 H^2}{\beta_1 + \beta_3},
\]

\[
w_0 = \frac{H|p_x|}{2\eta_1^0}, \quad n(c) = \frac{H^2 |p_x|^2 \alpha_1 \alpha_2 k_p}{2\eta_1}, \quad m(c) = \frac{H^2 |p_x|^2 \alpha_1 (\alpha_4 - \alpha_1 \alpha_2 \alpha_3)}{4\eta_1^0 \eta_1 \rho D}
\]

With agreement with Remark 1, we use the next hypotheses:

\[
\varepsilon = \varepsilon_0 c, \quad \eta_1' = 1 + \nu_0 c, \quad \alpha_3 = \alpha_3^0 c(1 - c),
\]
\[ k_p = k_p^0 c(1-c), \quad n = n_0 c(1-c)k_p^0, \quad m = m_0 c(1-c)(\alpha_4^0 - \alpha_1 \alpha_2 \alpha_3^0). \]

Let us introduce dimensionless functions:

\[ l(c) = \frac{\alpha_0^0 \rho D k_p^0}{H^2} c(1 - 3c + 2c^2) = l_0 \alpha_3^0 c(1 - 3c + 2c^2), \]

\[ q(c) = \frac{\alpha_1(\alpha_0^0)^2}{2H^2 \eta_1^0} c(1 - 3c + 2c^2) = q_0(\alpha_3^0)^2 c(1 - 3c + 2c^2), \]

\[ r(c) = r_0 [\beta_1 + \beta_3 - \alpha_1(\alpha_3^0)^2 c^2(1 - c)^2], \quad r_0 = \frac{1}{4H^2 \eta_1^0}. \]

In the dimensionless form, equations (36), (38) become

\[ \frac{d}{dy'} c = [1 - \varepsilon(c)] \eta_1'(c) w' - y' \left( n(c) - m(c) \frac{d}{dy'} w' \right) \tag{39} \]

\[ r(c) \frac{d^2}{dy'^2} w' - [l(c) + q(c) \frac{d}{dy'} w'] \frac{d}{dy'} c = \varepsilon(c) \eta_1'(c) w' - y'. \tag{40} \]

Given \( c(y') \) and \( w'(y') \), the dimensionless velocity

\[ v' = \frac{v}{V}, \quad V = w_0 H, \]

can be found from following equation

\[ \frac{d}{dy'} v' + 2\varepsilon(c) w' = - \frac{2y'(1-\varepsilon(c))}{\eta_1'(c)}. \]

4. Numerical results

Because of symmetry, we perform calculations only in the domain \( 0 \leq y' \leq 1 \). As in [9], we use the no-slip conditions at the boundary \( y = 1 \) and the symmetry condition at the center of the channel:

\[ w(1) = v(1) = 0, \quad w(0) = 0 \tag{41} \]

For the concentration, we formulate the condition

\[ \int_{-1}^{1} c(y') dy' = 2 \int_{0}^{1} c(y') dy' = C_0, \]

where \( C_0 \) is the total concentration of micropolar particles.

We use the following data for the calculations: \( \varepsilon_0 = 0.5, n_0 = 3, m_0 = 100, s_0 = 0.7, \nu_0 = 0.03, C_0 = 0.6, q = 1, r_0 = 0.2, l_0 = 1, q_0 = 1, \alpha_1 = 1, \alpha_2 = 0.5. \)

Let us consider the case \( \alpha_3^0 = \alpha_4^0 = 2 \). Fig. 1 depicts profiles of the velocity \( v'(y') \), the concentration \( c(y') \) and the gyration \( w'(y') \) in the domain \( 0 \leq y' \leq 1 \). Observe that concentration attains its maximal value somewhere between the center and the boundary of the channel, in agree with the Segre-Silberberg effect [12].

Fig. 2 depicts concentration for different parameters \( \alpha_3^0 \) and \( \alpha_4^0 \). In the case \( \alpha_3^0 = \alpha_4^0 = 0 \), the diffusion flux \( l \) doesn’t depend on gyration. As a result, concentration attains its maximum in the center \( y' = 0 \). Fig. 2a depicts concentration for different parameters \( \alpha_3^0 \) and \( \alpha_4^0 \). In the case \( \alpha_3 \neq 0 \) (Fig. 2c, 2d), concentration attains its maximum somewhere between the center and the boundary of the channel. Moreover, the greater \( \alpha_3 \) the greater the shift of the point of maximum to the boundary. Fig. 2b concerns the case when the diffusion flux depends on gyration but only linearly and in this case the Segre-Silberberg effect does not occurs. We remind that \( \alpha_3 \) is the weight of linear contribution of gyration and \( \alpha_4 \) is the weight if nonlinear contribution of gyration into the diffusion flux. Thus, our calculations reveal that the Segre-Silberberg effect is genuinely nonlinear one.
Figure 1. From the top down: velocity, concentration and gyration of the particles.

Figure 2. Concentration for different values of parameters $\alpha_3^0$ and $\alpha_4^0$: a) $\alpha_3^0 = \alpha_4^0 = 0$, b) $\alpha_3^0 = 2$, $\alpha_4^0 = 0$, c) $\alpha_3^0 = 0$, $\alpha_4^0 = 2$, d) $\alpha_3^0 = \alpha_4^0 = 2$

5. Conclusions
Conservation laws and constitutive equations for micropolar fluid with variable concentration of micropolar particles were derived. We should notice that mass concentration in this model obeys generalized Fick law and concentration flux depends on gyration of particles. We considered Poiseuille-like flow between two parallel planes and confirm Segre-Silberberg effect for such flow. Also it was shown that the Segre-Silberberg effect occurs if the concentration flux depends on gyration nonlinearly.
Acknowledgments
The paper is supported by Russian Science Fund (Project No. 15-11-20019).

Appendix A.
Here, we discuss questions of objectivity, i.e. frame independence [4].

Claim 1. Given an orthogonal basis \{\mathbf{e}_i\}_1^3 and an orthogonal matrix \(Q\), we have the equality

\[ \epsilon_{sij} \equiv \mathbf{e}_s \cdot (\mathbf{e}_t \times \mathbf{e}_j) = Q(\mathbf{e}_s) \cdot (Q(\mathbf{e}_t) \times Q(\mathbf{e}_j)) \equiv \alpha_{sij}. \]  

(A.1)

Proof. Let \(Q_i^{e}\) be the rotation around the \(x_i\)-axis through the angle \(\varphi_i\). Clearly, \(Q_i^{e} \langle \mathbf{e}_i \rangle = \mathbf{e}_i\) and any orthogonal matrix \(Q\) can be represented as a composition \(Q = Q_1^{e_1} Q_2^{e_2} Q_3^{e_3}\). We have

\[ Q_3^e (\mathbf{e}_1) \cdot (Q_3^e (\mathbf{e}_2) \times Q_3^e (\mathbf{e}_3)) = Q_3^e (\mathbf{e}_1) \times (Q_3^e (\mathbf{e}_2) \times Q_3^e (\mathbf{e}_3)) = \\
= Q_3^e (\mathbf{e}_3) \cdot \mathbf{e}_3 = \mathbf{e}_1 \cdot (\mathbf{e}_2 \times \mathbf{e}_3). \]

Similarly, one can prove that equality (A.1) holds for \(Q_3^e\). Next, one can verify that

\[ Q_1^{e_1} Q_2^{e_2} Q_3^{e_3} (\mathbf{e}_1) \cdot (Q_1^{e_1} Q_2^{e_2} Q_3^{e_3} (\mathbf{e}_2) \times Q_1^{e_1} Q_2^{e_2} Q_3^{e_3} (\mathbf{e}_3)) = \\
Q_1^{e_1} Q_2^{e_2} (\mathbf{e}_1) \cdot (Q_1^{e_1} Q_2^{e_2} (\mathbf{e}_2) \times Q_1^{e_1} Q_2^{e_2} (\mathbf{e}_3)) = \\
Q_1^{e_1} (\mathbf{e}_1) \cdot (Q_1^{e_1} (\mathbf{e}_2) \times Q_1^{e_1} (\mathbf{e}_3)) = \mathbf{e}_1 \cdot (\mathbf{e}_2 \times \mathbf{e}_3). \]

Hence, the claim (A.1) is proved.

Claim 2. Given an orthogonal matrix \(Q\), we have the equality

\[ Q(\mathbf{x}) \times Q(\mathbf{y}) = Q(\mathbf{x} \times \mathbf{y}), \quad \forall \ \mathbf{x}, \mathbf{y}. \]  

(A.2)

Proof. Due to the identity

\[ Q(\mathbf{e}_i) \times Q(\mathbf{e}_j) = \alpha_{sij} Q(\mathbf{e}_s) \]

and the equality (A.1), the claim (A.2) results from the following equalities:

\[ Q(\mathbf{x}) \times Q(\mathbf{y}) = x_i y_j Q(\mathbf{e}_i) \cdot Q(\mathbf{e}_j) = x_i y_j \epsilon_{sij} Q(\mathbf{e}_s) = Q(x_i y_j \epsilon_{sij} \mathbf{e}_s) = Q(\mathbf{x} \times \mathbf{y}). \]

Claim 3. Given an objective vector \(\mathbf{l}\), the matrix \(\epsilon : \mathbf{l}\) is also objective.

Proof. Objectivity of \(\mathbf{l}\) implies that \([4] \ \mathbf{l}^\prime = Q^* (\mathbf{l})\), where \(\mathbf{l}^\prime\) is the value of \(\mathbf{l}\) resulting from the passage to a new frame of references associated with the rotation \(Q\). Given a vector \(\mathbf{a}\), we have

\[ \epsilon : \mathbf{l}^\prime (\mathbf{a}) = Q^* (\mathbf{l}) \times \mathbf{a} = Q^* (\mathbf{l}) \times Q^* Q(\mathbf{a}) = Q^* (\mathbf{l} \times Q(\mathbf{a})) = Q^* \cdot (\epsilon : \mathbf{l}) \cdot Q(\mathbf{a}). \]

Hence,

\[ \epsilon : \mathbf{l}^\prime = Q^* \cdot (\epsilon : \mathbf{l}) \cdot Q, \]

which implies objectivity of the matrix \(\epsilon : \mathbf{l}\) [4].

Claim 4. Given an objective matrix \(B\), the vector \(\mathbf{b} = \mathbf{e}_i \times B(\mathbf{e}_i)\) is also objective.

Proof. First of all, we observe that the vector \(\mathbf{e}_i \times B(\mathbf{e}_i)\) does not depend on the basis \(\{\mathbf{e}_i\}_1^3\).

Due to equality (A.2), we have

\[ \mathbf{b}' \equiv \mathbf{e}_i \times B'(\mathbf{e}_i) = \mathbf{e}_i \times Q^* BQ(\mathbf{e}_i) = Q^* Q(\mathbf{e}_i) \times Q^* BQ(\mathbf{e}_i) = \\
= Q^* (Q(\mathbf{e}_i) \times BQ(\mathbf{e}_i)) = Q^* (\mathbf{e}_i \times B(\mathbf{e}_i)) = Q^* (\mathbf{b}). \]

The claim is proved.

Claim 5. Given the objective vectors \(\mathbf{l}\) and \(\mathbf{b}\), the vector product \(\mathbf{l} \times \mathbf{b}\) is also objective.

Proof. The claim follows from the equality (A.1). Indeed, we have

\[ \mathbf{l}' \times \mathbf{b}' = Q^* (\mathbf{l}) \times Q^* (\mathbf{b}) = Q^* (\mathbf{l} \times \mathbf{b}). \]
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