$U_A(1)$ Anomaly in Hot and Dense QCD and the Critical Surface

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We discuss the chiral phase transition in hot and dense QCD with three light flavors. Inspired by the well-known fact that the $U_A(1)$ anomaly could induce first order phase transitions, we study the effect of the possible restoration of the $U_A(1)$ symmetry at finite density. In particular, we explore the link between the $U_A(1)$ restoration and the recent lattice QCD results of de Forcrand and Philipsen, in which the first order phase transition region near zero chemical potential ($\mu$) shrinks in the quark mass and $\mu$ space when $\mu$ is increased. Starting from the Ginzburg-Landau theory for general discussions, we then use the Nambu–Jona-Lasinio model for quantitative studies. With the partial $U_A(1)$ restoration modeled by the density dependent 't Hooft interaction, we fit the shrinking of the first order region found in de Forcrand and Philipsen’s lattice calculation at low $\mu$. At higher $\mu$, the first order region might shrink or expand, depending on the scenarios. This raises the possibility that despite the shrinking of the first order region at lower $\mu$, the QCD critical end point might still exist due to the expansion at higher $\mu$. In this case, very high precision lattice data will be needed to detect the recently observed back-bending of the critical surface with the currently available analytic continuation or Taylor expansion approaches. Lattice computations could, however, test whether the $U_A(1)$ restoration is responsible for the shrinking of the first order region by computing the $\eta'$ mass or the topological susceptibility at small $\mu$.

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I. INTRODUCTION

The $U_A(1)$ anomaly is an interesting phenomenon of Quantum Chromodynamics (QCD). In the chiral limit with three massless quark flavors, QCD has the chiral $SU_L(3) \otimes SU_R(3)$ symmetry. This symmetry is spontaneously broken in the QCD vacuum, giving rise to eight massless Nambu-Goldstone bosons. The $U_A(1)$ symmetry, on the other hand, is broken not spontaneously but explicitly due to the quantum anomaly. Thus, the $\eta'$ meson is not a Nambu-Goldstone boson. It remains massive in the chiral limit. This is the so-called $U_A(1)$ problem \cite{1} and its consequence \cite{2}. 't Hooft showed that instantons, which are topological configurations of the classical gluon field, are related to the $U_A(1)$ anomaly. He also constructed an effective quark interaction which breaks the $U_A(1)$ symmetry \cite{3} (see also \cite{4}). The coupling constant of this (Kobayashi-Maskawa) ‘t Hooft interaction measures the effective strength of the $U_A(1)$ anomaly relevant to the hadron spectrum.

It is interesting to investigate the effective restoration of the $U_A(1)$ symmetry at finite temperature ($T$) and/or quark chemical potential ($\mu$) induced by the decrease of instantons \cite{5,6}, even though the triangle anomaly is independent of the infrared scale $T$. Consequences of the $U_A(1)$ restoration have been investigated if the restoration is associated with the chiral transition \cite{8}. In Ref. \cite{8}, based on the instanton liquid model, it was speculated that drastic $U_A(1)$ restoration at the chiral transition could be expected because the chiral transition is caused by the rearrangement of the instanton configurations in that model.

An analysis using the Nambu–Jona-Lasinio (NJL) model was given in Ref. \cite{10}, where the topological susceptibility $\chi_t$, a correlator of topological charges, was considered. In the large $N_c$ (number of colors) limit, $\chi_t$ is related to the $\eta'$ mass through the Witten-Veneziano mass formula \cite{11}, $2N_c \chi_t f_{\pi}^2 = m_{\eta'}^2 + m_{\eta'}^2 - 2m_K^2$, so

\begin{equation}
2N_c \chi_t f_{\pi}^2 = m_{\eta'}^2 + m_{\eta'}^2 - 2m_K^2,
\end{equation}
it can be used to probe the $U_A(1)$ anomaly. The NJL model calculation [10] reproduced the lattice data [12] above the critical temperature up to 1.5 times the chiral phase transition temperature with temperature independent ’t Hooft coupling constant. This implies that, at least in the NJL model, the effective $U_A(1)$ restoration does not necessarily take place near the chiral transition.

At finite density, the medium response of the $U_A(1)$ anomaly is even less understood, largely because of the lack of lattice data due to the fermion sign problem [13]. Reliable analyses are available only at asymptotically high density where QCD is perturbative. In Refs. [14,15,16], the $\eta'$ mass in the Color-Flavor Locked (CFL) phase was calculated and shown to decrease as the density increases. This corresponds to the realization of effective $U_A(1)$ restoration. There are efforts to test this restoration hypothesis by probing the decrease in the $\eta$ and $\eta'$ masses [5,18,19,20,21,22] in medium. It is claimed that the decrease can be studied experimentally through observing the possible formation of $\eta$- and $\eta'$-mesic nuclei [23,24,25,26,27,28].

It is interesting to speculate how the anomalous violation and restoration of $U_A(1)$ symmetry could change the QCD phase diagram (see Ref. [29] for example). In this work, we will not discuss the possible changes of phases and phase boundaries in the intermediate density. Instead, we will focus on the issue regarding the existence of the QCD critical end point (CEP). In the conventional three flavor picture [30,31,32], there is a first order boundary of the chiral phase transition separating the hadronic and quark phases starting from a point with zero $T$ but non-zero $\mu$ to a point ($\mu_c,T_c$), then the QCD phase transition becomes a crossover at lower $\mu$. This end point of the first order phase transition, ($\mu_c,T_c$), is called the CEP. This CEP has an second order phase transition despite the finite quark masses. The search of the QCD CEP is a priority in the next phase of the RHIC running.

However, this conventional picture of the QCD phase diagram and the existence of the CEP are challenged by the recent lattice QCD results of de Forcrand and Phillipsen [33]. The assertion in Ref. [32] can be explained by the so-called Columbia plot, where each point in the parameter space of quark masses and $\mu$ is marked by its order of phase transition when $T$ is increased. We will study QCD with three light flavors, $u,d$ and $s$, with isospin symmetry $m_u = m_d = m_d$. When $\mu = 0$, the chiral limit point ($m_{ud} = m_s = 0$) has a first order phase transition when $T$ is increased due to symmetry reasons [3]. Around the chiral limit, there exists a finite area such that each point in this area represents a first order phase transition. The boundary of this area is marked by a critical curve. Each point on this critical curve is of a second order phase transition. Beyond the critical curve, the theory has a crossover until all the quarks become heavy such that the theory is close to a pure Yang-Mills theory and phase transitions can then take place again. In the following, we will just focus on the light quark region.

At $\mu = 0$, the physical point of quark masses is located in the crossover region, meaning that QCD has a crossover at $\mu = 0$. If the CEP exists at finite $\mu$, then the physical point should enter the first order phase transition region at finite $\mu$. However, in Ref. [33], it was found that near $\mu = 0$, the region of the first order phase transition shrinks as $\mu$ increases. Note that this computation is not directly carried out with finite $\mu$ QCD due to the fermion sign problem. Instead, methods with analyti- cal continuation from imaginary $\mu$ and with derivatives computed at $\mu = 0$ are used. Thus, essentially it is the curvature at $\mu = 0$ that was computed. Although that result of Ref. [33] disfavors the existence of the CEP, to give a definite answer to whether the CEP exists or not, computations at higher $\mu$ are necessary (note that other lattice results using different approaches are consistent with the existence of the CEP [34,35,36,37]).

Given that lattice computations at higher $\mu$ are still challenging, it is our hope that model calculations might shed light on this problem.

Inspired by the fact that the $U_A(1)$ anomaly could induce the first order transition in massless three-flavor QCD, we study the effect of the possible restoration of the $U_A(1)$ symmetry at finite density. Starting from general discussions using the Ginzburg-Landau theory, we then use the NJL model for quantitative studies. With the partial $U_A(1)$ restoration modeled by the density dependent ’t Hooft interaction, we fit the negative curvature of the critical surface at $\mu = 0$ obtained in Ref. [33]. Finally, we discuss the behavior at higher $\mu$ within this model.

II. GINZBURG-LANDAU THEORY ANALYSIS

In this section, we use the Ginzburg-Landau (GL) theory to demonstrate the special role of the $U_A(1)$ anomaly in chiral phase transitions, following the framework of Refs. [3,22,38,39,40].

The effective potential $\Omega_{GL}$ for the order parameter (chiral condensate) field $\Phi_{ij} = \langle \bar{q}_i(1-\gamma_5)q_j \rangle$ in QCD with three flavors ($N_f = 3$) is parametrized by a set of operators satisfying the $SU_L(3) \otimes SU_R(3) \otimes U_V(1) \otimes Z(N_i)$ symmetry, where $Z(N_i)$ is the remaining symmetry of $U_A(1)$ after it is broken by quantum anomaly. Here, we will use the mean field approximation and neglect the space time dependence of $\Phi$. This approximation neglects soft-mode fluctuations, which are large near second order phase transitions, but it is still useful to explore the phase structure. Also, we will Taylor expand $\Omega_{GL}$ in analytic functions of the order parameter. The following analysis helps us to understand our NJL result in later discussions. We have

$$\Omega_{GL} = \frac{a_0}{2} Tr\Phi^4 + \frac{b_1}{4!} (Tr\Phi^4)^2 + \frac{b_2}{4!} Tr(\Phi^4)^2 - \frac{c_0}{2} (det\Phi + det\Phi^4) - \frac{1}{2} Tr h_0 (\Phi + \Phi^4), \quad (1)$$
where the last term breaks chiral symmetry explicitly with \( h_0 \propto \text{diag}(m_u, m_d, m_s) \). Adding higher dimensional operators does not change the analysis qualitatively. The determinant term simulates the \( U_A(1) \) anomaly and has the \( Z(N_f) \) symmetry. We have \( c_0 \geq 0 \) at \( T = 0 \) to yield a finite \( \eta' \) mass in the chiral limit.

For simplicity, we restrict ourselves to the flavor \( SU(3) \) symmetric case: \( m_u = m_d = m_s \) (that is, on the diagonal in the Columbia plot). Then we have \( \Phi = \text{diag}(\sigma, \sigma, \sigma) \).

The \( GL \) functional is reduced to

\[
\Omega_{GL} = \frac{3}{2} a_0 \sigma^2 - c_0 \sigma^3 + \left( \frac{9}{4!} b_1 + \frac{3}{4!} b_2 \right) \sigma^4 - 3h_0 \sigma
\]

\[
= \frac{1}{2} a \sigma^2 - \frac{1}{3} c \sigma^3 + \frac{1}{4} b \sigma^4 - h \sigma, \quad (2)
\]

where \( b > 0 \) such that the free energy is bounded from below.

(i) Chiral limit \((h = 0)\)

We first consider the chiral limit case with \( h = 0 \), where interesting results can be obtained.

If \( a > 0 \) and \( c \neq 0 \) at the phase transition temperature, then Eq. (2) has two local minima and the phase transition is of first order. The first order transition persists against an external field \( h \) until it is washed out at sufficiently large \( h \). Thus, the first order region has some finite extent from \( m_{\text{res}} = m_s = 0 \) in the Columbia plot. Going back to \( h = 0 \), the two minima are located at \( \sigma_1 = 0 \) and \( \sigma_2 = (c + \sqrt{c^2 - 4ab})/2b \). The chiral condensate is

\[
\Delta \sigma = \sigma_2 - \sigma_1 = \frac{c + \sqrt{c^2 - 4ab}}{2b}. \quad (3)
\]

Note that as \( c \to 0 \), if there is a phase transition, then \( a \to 0 \) at the phase transition point and the phase transition becomes second order. Here we have assumed as \( c \to 0 \), \( b \) is still positive and that the higher order terms in \( \sigma \) can further be neglected. Otherwise first order phase transition is still possible through the inclusion of the higher order terms such as \( \sigma^6 \). Only in this simple case, the absence of the \( U_A(1) \) anomaly leads to the disappearance of the first order transition.

For simplicity let us assume an (unphysical) extreme case that the \( \mu \) dependence lies only in the anomaly term \( c \) and \( c \to 0 \) at \( \mu = \mu_{\text{res}} \). Then as we go from \( \mu = 0 \) to higher \( \mu \) on the \( \mu \) axis, the strength of the first order transition gets weakened. At \( \mu = \mu_{\text{res}} \), the first order transition disappears completely, and above \( \mu_{\text{res}} \), the transition turns into second order. However, if fluctuations are taken into account, the above mean-field picture is modified. The second order transition above \( \mu_{\text{res}} \) has the \( SU_L(3) \otimes SU_R(3) \otimes U_A(1) \) symmetry. The renormalization group analysis tells us that there is no infrared stable fixed point for this universality class \([3]\).

This means that the fluctuations wash out the critical point and make it a fluctuation induced first order transition. Thus, at \( m_{\text{res}} = m_s = 0 \), the transition remains first order on the \( \mu \) axis even if the \( U_A(1) \) symmetry is restored. This implies that as we go to \( \mu_{\text{res}} \), the first order region around \( m_{\text{res}} = m_s = 0 \) would evaporate, but the critical surface can never touch the \( \mu \) axis but only approaches it asymptotically.

The above discussion gives a clear picture about the eventual shrinking of the first order region if \( U_A(1) \) is completely restored above a certain \( \mu \) (if terms higher order than Eq. (2) can be neglected). For partial restoration with the \( GL \) coefficients depending on \( \mu \), the eventual shrinking does not have to happen. This can be seen from \( \Delta \sigma \) in Eq. (3). If \( c \) decreases, \( \Delta \sigma \) can still increase if the parameter \( b \) or \( a \) decreases sufficiently fast. This means that even if the \( U_A(1) \) symmetry is partially restored when \( \mu \) increases, the first order region may still expand. In fact, this is confirmed in the NJL calculation in the next section. In contrast, even without the \( U_A(1) \) symmetry restoration, the critical surface can still shrink if \( b \) or \( a \) increases due to some mechanism. This is similar to what happens if a repulsive interaction between vector currents is added to the NJL model, as demonstrated in Ref. [41]. Thus, the \( U_A(1) \) restoration is just one possible explanation to the shrinkage of the first order region at small \( \mu \).

(ii) Finite current quark mass \((h \neq 0)\)

The different scenarios mentioned above can also be seen in the following analysis involving finite quark masses. At the critical end point, the \( GL \) functional takes the form of

\[
\Omega_{GL} = \frac{1}{4} b (\sigma - \sigma_0)^4 + d
\]

\[
= \frac{1}{4} b \left( \sigma^4 - 4\sigma_0 \sigma^3 + 6\sigma_0^2 \sigma^2 - 4\sigma_0^3 \sigma + \sigma_0^4 \right) + d(4)
\]

Comparing this with Eq. (2), we have

\[
b\sigma_0 = \frac{1}{3} c, \quad \frac{3}{2} b \sigma_0^2 = \frac{1}{2} a, \quad b \sigma_0^3 = h, \quad \frac{1}{4} b \sigma_0^4 + d = 0. \quad (5)
\]

These are the conditions to determine the critical surface. However, there are six parameters—\( a, b, c, h, \sigma_0 \), and \( d \), while there are only four relations among them in Eq. (5). Thus, as mentioned above, one cannot identify what causes the shrinking of the first order region without extra inputs.

III. NJL MODEL ANALYSIS

In this section, we give a quantitative analysis using the NJL model \([42, 43, 44]\) with the partial \( U_A(1) \) restoration modeled by the density dependent ’t Hooft interaction. We will fit the curvature of the shrinkage of the first order region at \( \mu = 0 \) to the lattice QCD result of Ref. [33]. We then discuss the behaviors at higher \( \mu \).
A. Model Setting

The NJL Lagrangian is given by

\[ \mathcal{L} = \mathcal{L}_0 + \mathcal{L}_4 + \mathcal{L}_6, \]

\[ \mathcal{L}_0 = \bar{q} (i \gamma \cdot \partial - m) q, \]

\[ \mathcal{L}_4 = \frac{g_s}{2} \sum_{a=0}^{8} \left[ (\bar{q} \gamma_a q)^2 + (\bar{q} \gamma_5 \lambda_a q)^2 \right], \]

\[ \mathcal{L}_6 = g_\mu^6 \left[ \det \bar{q}_i (1 - \gamma_5) q_j + \text{h.c.} \right]. \]

The kinetic term \( \mathcal{L}_0 \) includes the current quark mass matrix \( m \) which breaks chiral symmetry explicitly. The sum of the two four Fermion contact interaction terms in \( \mathcal{L}_4 \) is chirally symmetric. It gives the attractive interaction responsible for the spontaneous chiral symmetry breaking when its strength exceeds a certain critical value. In this minimal model, the other types of four Fermion contact interactions of the same mass dimension are not included because they are heavier excitations below the phase transition. The six Fermion determinant term is included because they are heavier excitations below the first order region. One immediately realizes, if \( p \) and as \( g \) monotonically decreases function of \( m \), then the critical curve will bend to the right (corresponding to shrinking or eventual expansion of the first order region), while the lattice result of \( g_\mu^D \) has shown. But, at higher \( m \), whether the critical surface could keep shrinking or rather expands, depends on the functional form of \( g_\mu^D (m) \), i.e., how fast the \( U_\Lambda(1) \) symmetry gets restored.

As a special example, we will use the ansatz,

\[ g_\mu^D (m) = g_\mu^D (0) e^{-\mu^2 / \mu^2}, \]

with one free parameter \( \mu_0 \) to parametrize the \( U_\Lambda(1) \) symmetry restoration. This form is motivated by the Gaussian suppression of the instanton density due to Debye screening \( \mu_0 \). The critical curve, which is the intersect of the \( SU(3) \) symmetric plane, \( m = m_{ud} = m_s \), to the critical surface can be Taylor expanded near \( m = 0 \), in even powers due to CP symmetry:

\[ \frac{m_c(\mu)}{m_c(0)} = 1 - 3.3(3) \left( \frac{\mu}{T_c} \right)^2 - 47(20) \left( \frac{\mu}{T_c} \right)^4 - \cdots \]

It is not clear what the radius of convergence of the above expansion is. By requiring the third term to be smaller than the second term (which are both \( 23\% \) of the first term), we obtain \( \mu \lesssim 90-100 \text{ MeV} \).
The negative signs for the $\mu^2$ and $\mu^4$ terms suggest the shrinking of the first order region at small $\mu$. The sign of the $\mu^6$ term was also asserted to be negative, making the shrinking even more serious, although the value of the coefficient has not yet been determined [33]. However, as we discussed above, we do not expect the critical surface to touch the $\mu$ axis at zero quark masses. Thus, there must be terms at higher order in the $\mu$ expansion with positive prefactors which change the critical surface from the characteristic front bending to back bending at some density.

C. Numerical Results in the NJL Model

Now let us discuss the numerical results of the NJL model. In Fig. 2, we show the result of the critical curve in the $(\mu, m = m_{ud} = m_s)$ plane for various values of the free parameter $\mu_0$ [31]. The shaded region represents the lattice result of Eq. (11). The lattice value of $T_c$ depends on $\mu$. At $\mu = 0$, $T_c = 135$ MeV which gives a typical value for $T_c$.

![Graph showing critical curves in the (μ, m = m_{ud} = m_s) plane](image)

FIG. 2: The critical curves in the $(\mu, m = m_{ud} = m_s)$ plane with $g_0^n(0) = g_0$ for several values of $\mu_0$. The shaded region represents the lattice result of Eq. (11). The lattice value of $T_c$ depends on $\mu$. At $\mu = 0$, $T_c = 135$ MeV which gives a typical value for $T_c$.

The band shows the errors in the expansion coefficients but not from the higher order terms or the systematic errors.

Several curves are plotted in the figure. As we go from the left-hand curve to the right-hand one, $\mu_0$ increases from 31.6 MeV to infinity. The infinite $\mu_0$ corresponds to the density independent $g_0^n$. The critical curves terminate at the critical chemical potentials at $T = 0$, above which first order phase transition ceases to exist, but crossover transition can still happen. We see that the values of $\mu_0$ can be divided into three regions according to the qualitative behaviors of the corresponding critical curves.

(i) The small $\mu_0$ region: In this region, the suppression of the $U_A(1)$ anomaly is so strong that the first order region shrinks quite rapidly. The surface keeps shrinking and approaches the $\mu$ axis asymptotically.

(ii) The intermediate $\mu_0$ region: At low density, the first order region shrinks and then expands again at high density. This back-bending behavior is due to the density effect. In general, stronger first order transition is favored at higher density, as discussed in detail in Ref. [41]. In the language of the GL theory, this means that the higher order terms enter the thermodynamic potential. Thus at finite density, there is a competition between the $U_A(1)$ restoration which makes $c$ smaller and the density effect which generates higher order terms. In this intermediate $\mu_0$ region, the density effect overcomes the $U_A(1)$ restoration at high density, resulting in the back-bending structure.

(iii) The large $\mu_0$ region: When $\mu_0$ is large, the $U_A(1)$ restoration is weak. Thus, the first order region does not shrink and instead expands monotonically. This corresponds to the conventional scenario with constant $g_0$ in which the CEP could exist.

Now let us compare the curves with the lattice data. We see that the curves with $\mu_0 = 253 \sim 379$ MeV describes the lattice data well for $\mu \lesssim 70$ MeV. It is notable that this value of $\mu_0$ is close to a rough estimate by $\mu_0 \sim 1/(\sqrt{N_f} m_0) \sim 380$ MeV, where $N_f = 3$ and $m_0 \simeq 0.3$ fm is the typical instanton size. For 70 MeV $\lesssim \mu \lesssim 100$ MeV, $\mu_0 = 189 \sim 253$ MeV describes the lattice data better.

In our calculation $m_c(0) \sim 1.1$ MeV is obtained while the lattice data gives $m_c(0) \sim 14$ MeV (extracted from Fig. 9 of the first paper in Ref. [33]). One can argue that since $m_{\eta'}$ is heavier than the cutoff $\Lambda$ of the theory, it should not be used to fit the parameters of the theory. Instead, one can use the $m_s(0)$ computed in lattice QCD. To explore the effects of this new set of parameters, we use $g_{L}^n(0) = 3.5 g_0$ and keep the other parameters the same. This parameter set gives $m_c(0) \sim 26$ MeV, not quite the same as the lattice $m_c(0)$, but rather the Columbia plot at $\mu = 0$ is very similar to that of [33]. In Fig. 6 we plot the corresponding critical curves. The qualitative features are the same as those in Fig. 2. There is a range of $\mu_0 (253 \sim 379$ MeV) where the curvature from lattice QCD is reproduced. In the range below $\mu_0 = 253$ MeV, the first order region does not expand substantially while in the range above $\mu_0 = 379$ MeV, it expands beyond the critical mass at $\mu = 0$. Also, the results of Figs. 2 and 3 suggest that the range of $\mu_0$ that corresponds to the curvature from lattice QCD is insensitive to the size of $g_{L}^n(0)$ or $m_c(0)$ in our model.

To further explore the existence of the CEP within this model, one needs to go beyond the $m_{ud} = m_s$ limit. Again, using $g_{L}^n(0) = 3.5 g_0$, we show the critical surface in the $(\mu, m_{ud}, m_s)$ space with $\mu_0$ constrained by lattice data. In the panel (a) of Fig. 4 $\mu_0 = 253$ MeV, the
$U_A(1)$ restoration is strong enough such that the critical surface does not intersect the physical quark mass line which is denoted by the thick dash line in the plot. Thus, CEP does not exist in this case. In the panel (b) ($\mu_0 = 379$ MeV), on the other hand, the critical surface intersects the physical quark mass line. Thus, the CEP exists.

A few comments are in order:

(a) The most interesting feature of our result is the back-bending behavior of the critical surface. In our model we use $\mu_0$ to parameterize the $U_A(1)$ restoration and fit it to lattice data. But similar behavior was also seen in another model with the repulsive vector-vector current interaction added to the NJL model [41]. Thus, the back-bending seems to be a generic feature of the NJL model, and it could even be a property of QCD. This raises the possibility that despite the shrinking of the first order region at lower $\mu$, the QCD critical end point might still exist due to the expansion at higher $\mu$. In this case, it might be challenging for lattice QCD calculations using the analytic continuation and the Taylor expansion to probe the back-bending behavior. It will be worthwhile to investigate how precise the lattice computation should be in order to detect or rule out the back-bending scenario.

(b) As mentioned above, the reason for shrinking of the first order region is uncertain yet. Both partial $U_A(1)$ restoration and the vector-vector repulsive four-fermion interaction can give this effect. Lattice computations could, however, test whether the $U_A(1)$ restoration is responsible for the shrinking by computing $m_{\eta'}$ or the topological susceptibility at small $\mu$.

**IV. SUMMARY**

We have discussed the chiral phase transition in hot and dense QCD with three light flavors. Inspired by the well-known fact that the $U_A(1)$ anomaly could induce first order phase transitions, we have studied the effect of
the possible restoration of the $U_A(1)$ symmetry at finite density. In particular, we explored the link between the $U_A(1)$ restoration and the recent lattice QCD results of de Forcrand and Philipsen, in which the first order phase transition region near zero chemical potential ($\mu$) shrinks in the quark mass and $\mu$ space when $\mu$ is increased. Starting from the Ginzburg-Landau theory for general discussions, we then used the Nambu–Jona-Lasinio model for quantitative studies. With the partial $U_A(1)$ restoration modeled by the density dependent 't Hooft interaction, we have fit the shrinking of the first order region found in de Forcrand and Philipsen’s lattice calculation at low $\mu$. At higher $\mu$, the first order region might shrink or expand, depending on the scenarios. This raises the possibility that despite the shrinking of the first order region at lower $\mu$, the QCD critical end point might still exist due to the expansion at higher $\mu$. In this case, very high precision lattice data will be needed to detect the back-bending of the critical surface with the currently available analytic continuation or Taylor expansion approaches. Finally, since the $\eta'$ mass and the topological susceptibility are sensitive to the strength of the $U_A(1)$ anomaly, lattice computations of these quantities at small $\mu$ could check whether the strength of the $U_A(1)$ anomaly is reduced when $\mu$ is increased. These calculations, however, have to be carried out using either imaginary chemical potential or derivative expansions due to the fermion sign problem with finite $\mu$.

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