A family of the Poisson brackets compatible with the Sklyanin bracket

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Abstract
We introduce a family of compatible Poisson brackets on the space of $2 \times 2$ polynomial matrices, which contains the Sklyanin bracket, and use it to derive a multi-Hamiltonian structure for a set of integrable systems that includes the XXX Heisenberg magnet, the open and periodic Toda lattices, the discrete self-trapping model and the Goryachev–Chaplygin gyrostat.

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1. Introduction

The ingenious discovery by Magri [6, 7] that integrable Hamiltonian systems usually prove to be bi-Hamiltonian, and vice versa, leads us to the following fundamental problem: given a dynamical system which is Hamiltonian with respect to a Poisson bracket $\{\ldots,\}$, how to find another Poisson bracket $\{\ldots,\}_1$ compatible with the initial bracket and such that our system is Hamiltonian with respect to both brackets. This, along with the related problem of classification of compatible Poisson structures, is nowadays a subject of intense research, see e.g. [2, 6, 7, 14] and references therein.

In this paper, we study a class of finite-dimensional Liouville integrable systems described by the representations of the quadratic $r$-matrix Poisson algebra, or the Sklyanin algebra:

$$\{ \frac{1}{2} T(\lambda), \frac{2}{2} T(\mu) \} = [r(\lambda - \mu), \frac{1}{2} T(\lambda), \frac{2}{2} T(\mu)].$$

Here $\frac{1}{2} T(\lambda) = T(\lambda) \otimes I$, $\frac{2}{2} T(\mu) = I \otimes T(\mu)$ and $r(\lambda - \mu)$ is a classical $r$-matrix [8–11].

The main result of the present paper is a family of the Poisson brackets $\{\ldots,\}_k$, which is compatible with the Sklyanin bracket (1.1), in the simplest case of the $4 \times 4$ rational $r$-matrix

$$r(\lambda - \mu) = \frac{\eta}{\lambda - \mu} \Pi,$$

$$\Pi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \eta \in \mathbb{C},$$

$$\frac{1}{2} T(\lambda) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$
and \(2 \times 2\) matrix \(T(\lambda)\), which depends polynomially on the parameter \(\lambda\)

\[
T(\lambda) = \begin{pmatrix}
A(\lambda) & B(\lambda) \\
C(\lambda) & D(\lambda)
\end{pmatrix}
\]

\[
= \begin{pmatrix}
\alpha \lambda^n + A_1 \lambda^{n-1} + \cdots + A_n \\
\gamma \lambda^n + C_1 \lambda^{n-1} + \cdots + C_n
\end{pmatrix}
\begin{pmatrix}
\beta \lambda^n + B_1 \lambda^{n-1} + \cdots + B_n \\
\delta \lambda^n + D_1 \lambda^{n-1} + \cdots + D_n
\end{pmatrix}.
\]

The leading coefficients \(\alpha, \beta, \gamma, \delta\) and the \(2n\) coefficients of the \(\det T(\lambda)\),

\[
d(\lambda) = \det T(\lambda) = (\alpha \delta - \beta \gamma) \lambda^{2n} + Q_1 \lambda^{2n-1} + \cdots + Q_{2n},
\]

are Casimirs of the bracket (1.1). Therefore, we have a \(4n\)-dimensional space of the coefficients \(A_i, B_i, C_i, D_i\) with \(2n\) Casimir operators \(Q_i\), leaving us with \(n\) degrees of freedom.

For so-called open lattices independent Poisson involutive integrals of motion \(H_i^\alpha = A_i, i = 1, \ldots, n\) are given by the coefficients of the entry \(A(\lambda)\):

\[
A(\lambda) = \alpha \lambda^n + H_1^\alpha \lambda^{n-1} + \cdots H_n^\alpha,
\]

\[
\{H_i^\alpha, H_j^\alpha\} = 0.
\]

In generic case, integrals of motion are given by the coefficients of the tr \(T(\lambda)\):

\[
\text{tr} T(\lambda) = (\alpha + \delta) \lambda^n + H_1 \lambda^{n-1} + \cdots H_n,
\]

\[
\{H_i, H_j\} = 0.
\]

These integrals of motion define two Liouville integrable systems, which are our generic models for the whole paper. Bi-Hamiltonian description of these models gives rise to the bi-Hamiltonian description of the Goryachev–Chaplygin gyrostat [8], open and periodic Toda lattice [9], inhomogeneous Heisenberg magnet [11] and the discrete self-trapping (DST) model [5].

2. The compatible bracket

In this section, we describe the Poisson bracket compatible with the Sklyanin bracket. The Poisson brackets \([\ldots]\)_0 and \([\ldots]\)_1 are compatible if every linear combination of them is still a Poisson bracket. The corresponding compatible Poisson tensors \(P_0\) and \(P_1\) satisfy the following equations

\[
[[P_0, P_0]] = [[P_0, P_1]] = [[P_1, P_1]] = 0,
\]

where \([[\ldots]]\) is the Schouten bracket [2, 6, 7]. Remind that on a smooth finite-dimensional manifold \(\mathcal{M}\) the Schouten bracket of two bivectors \(X\) and \(Y\) is an antisymmetric contravariant tensor of rank three and its components in local coordinates \(z_m\) read

\[
[[X, Y]]^{ijk} = -\sum_{m=1}^{\dim \mathcal{M}} \left( X^{mk} \frac{\partial Y^{ij}}{\partial z_m} + Y^{mk} \frac{\partial X^{ij}}{\partial z_m} + \text{cycle(i,j,k)} \right).
\]
2.1. Open lattices

The Sklyanin bracket (1.1) amounts to having the following Poisson brackets between the entries $A(\lambda), B(\lambda), C(\lambda)$ and $D(\lambda)$ of the matrix $T(\lambda)$:

\[\{A(\lambda), A(\mu)\} = \eta \frac{\lambda - \mu}{\lambda - \mu} (B(\lambda)A(\mu) - B(\mu)A(\lambda)),\]
\[\{B(\lambda), A(\mu)\} = \eta \frac{\lambda - \mu}{\lambda - \mu} (C(\lambda)A(\mu) - C(\mu)A(\lambda)),\]
\[\{B(\lambda), C(\mu)\} = \eta \frac{\lambda - \mu}{\lambda - \mu} (D(\lambda)A(\mu) - D(\mu)A(\lambda)).\]

(2.2)

In (1.1) matrix $r(\lambda - \mu)$ satisfies the Yang–Baxter equation, which ensures the Jacobi identity for the brackets (2.2).

Proposition 1. The Sklyanin bracket (1.1), (2.2) is compatible with the following bracket $\{\cdot,\cdot\}$:

\[\{A(\lambda), A(\mu)\} = \{B(\lambda), B(\mu)\} = \{C(\lambda), C(\mu)\} = 0,\]
\[\{B(\lambda), A(\mu)\} = \eta \frac{\lambda - \mu}{\lambda - \mu} (B(\lambda)A(\mu) - B(\mu)A(\lambda)),\]
\[\{C(\lambda), A(\mu)\} = \eta \frac{\lambda - \mu}{\lambda - \mu} (C(\lambda)A(\mu) - C(\mu)A(\lambda)),\]
\[\{B(\lambda), C(\mu)\} = \eta \frac{\lambda - \mu}{\lambda - \mu} (D(\lambda)A(\mu) - D(\mu)A(\lambda)).\]

(2.3)

and

\[\{D(\lambda), D(\mu)\} = \eta \frac{\lambda - \mu}{\lambda - \mu} (B(\lambda)C(\mu) - B(\mu)C(\lambda)).\]

(2.4)

Proof. It is sufficient to check the statement on an open dense subset of the Sklyanin algebra defined by the assumption that $A(\lambda)$ and $B(\lambda)$ are co-prime and all roots of $A(\lambda)$ are distinct.

□

This assumption allows us to construct a separation representation for the Sklyanin algebra (1.1). In this special representation one has $n$ pairs of Darboux variables, $\lambda_i, \mu_i, i = 1, \ldots, n$, having the standard Poisson brackets,

\[\{\lambda_i, \lambda_j\} = \{\mu_i, \mu_j\} = 0, \quad \{\lambda_i, \mu_j\} = \delta_{ij},\]
with the $\lambda$-variables being $n$ zeros of the polynomial $A(\lambda)$ and the $\mu$-variables being values of the polynomial $B(\lambda)$ at those zeros,

$$A(\lambda_i) = 0, \quad \mu_i = \eta^{-1} \ln B(\lambda_i), \quad i = 1, \ldots, n.$$  

(2.6)

The interpolation data (2.6) plus $n$ identities

$$B(\lambda_i)C(\lambda_i) = -d(\lambda_i)$$

allow us to construct the needed separation representation for the whole algebra:

$$A(\lambda) = \alpha(\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n),$$

$$B(\lambda) = A(\lambda) \left( \frac{\beta}{\alpha} + \sum_{i=1}^{n} \frac{a^{ij}}{(\lambda - \lambda_i)A'(\lambda_i)} \right),$$

(2.7)

$$C(\lambda) = A(\lambda) \left( \frac{\gamma}{\alpha} - \sum_{i=1}^{n} \frac{d(\lambda_i) e^{-a^{ij}}}{(\lambda - \lambda_i)A'(\lambda_i)} \right),$$

$$D(\lambda) = \frac{d(\lambda) + B(\lambda)C(\lambda)}{A(\lambda)}.$$ 

The coefficients of the determinant $d(\lambda)$ (1.4) are Casimir elements for the both brackets $\{\ldots\}_0$ and $\{\ldots\}_1$ and, therefore, we can easily calculate the bracket $\{\ldots\}_1$ in (2.3)–(2.4) in $(\lambda, \mu)$-variables

$$\{\lambda_i, \lambda_j\}_1 = [\mu_i, \mu_j]_1 = 0, \quad [\lambda_i, \mu_j]_1 = \lambda_i \delta_{ij}.$$  

(2.8)

In order to complete the proof, we have to check that brackets (2.8) are compatible with the canonical brackets (2.5). The compatibility of the brackets (2.5) and (2.8) implies the compatibility of the brackets (2.2), (2.3) and vice versa.

The $(\lambda, \mu)$-variables (2.6) are so-called special Darboux–Nijenhuis coordinates [2, 6, 7] because

$$\{P_0 \mathrm{T} (\lambda), P_1 \mathrm{T} (\mu)\}_k = r^{[k]}(\lambda, \mu) \frac{\partial}{\partial \lambda} \otimes d\lambda + \frac{\partial}{\partial \mu} \otimes d\mu,$$

and the corresponding recursion operator $N$ takes the diagonal form

$$N = P_1 P_0^{-1} = \sum_{i=1}^{n} \lambda_i \left( \frac{\partial}{\partial \lambda_i} \otimes d\lambda_i + \frac{\partial}{\partial \mu_i} \otimes d\mu_i \right).$$

(2.9)

These Poisson tensors $P_0$ and $P_1$ satisfy equations (2.1) and the Nijenhuis torsion of $N$ vanishes as a consequence of the compatibility between $P_0$ and $P_1$.

**Proposition 2.** Brackets (2.5) and (2.8) between $(\lambda, \mu)$-variables belong to a whole family of compatible Poisson brackets $\{\ldots\}_k$ associated with the Poisson tensors:

$$P_k = N^k P_0 = \begin{pmatrix} 0 & \mathrm{diag}(\lambda_1^k, \ldots, \lambda_n^k) \\ -\mathrm{diag}(\lambda_1^k, \ldots, \lambda_n^k) & 0 \end{pmatrix}, \quad k = 0, \ldots, n.$$ 

In the matrix form, these brackets are equal to

$$\{ T(\lambda), \overline{T}(\mu) \}_k = r^{[k]}(\lambda, \mu) T(\lambda) \overline{T}(\mu) - \overline{T}(\lambda) T^{[k]}(\lambda, \mu) \overline{T}(\mu) - T(\lambda) r^{[k]}(\lambda, \mu) T(\mu) + T(\lambda) \overline{T}(\mu) r^{[k]}(\lambda, \mu) T(\mu).$$

(2.10)
A family of the Poisson brackets compatible with the Sklyanin bracket

Here

\[
\begin{align*}
  r^{[k]}_{12}(\lambda, \mu) &= \eta \left( \begin{array}{cccc}
    1 & 0 & 0 & 0 \\
    0 & 1 - \frac{\lambda^k + \mu^k}{2} & \mu^k & 0 \\
    0 & \lambda^k & 1 - \frac{\lambda^k + \mu^k}{2} & 0 \\
    0 & \rho^{[k]}_C & -\rho^{[k]}_C & 1 
  \end{array} \right), \\
  r^{[k]}_{21}(\lambda, \mu) &= \eta \left( \begin{array}{cccc}
    1 & 0 & 0 & 0 \\
    0 & 1 - \frac{\lambda^k + \mu^k}{2} & \lambda^k & \rho^{[k]}_B \\
    0 & \mu^k & 1 - \frac{\lambda^k + \mu^k}{2} & -\rho^{[k]}_B \\
    0 & 0 & 0 & 0 
  \end{array} \right), \\
  s^{[k]}_{12}(\lambda, \mu) &= \eta \left( \begin{array}{cccc}
    0 & 0 & 0 & 0 \\
    \rho^{[k]}_B & 0 & 0 & \rho^{[k]}_C \\
    \rho^{[k]}_D & \rho^{[k]}_B & 0 & 0 \\
    0 & 0 & 0 & 0 
  \end{array} \right), \\
  s^{[k]}_{21}(\lambda, \mu) &= \eta \left( \begin{array}{cccc}
    0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 
  \end{array} \right) 
\end{align*}
\]

(2.11)

and

\[
\rho^{[k]}_X = \frac{\lambda^k X(\lambda)}{A(\lambda)} - \frac{\mu^k X(\mu)}{A(\mu)}, \quad \text{where} \quad X = B, C, D,
\]

is a difference of two polynomials, which are quotients of polynomials in variables \( \lambda \) and \( \mu \) over a field.

**Proof.** At \( k = 0 \) one has \( \rho^{[0]}_B = 0 \), \( \rho^{[0]}_C = 0 \) and \( \rho^{[0]}_D = 0 \), so the bracket (2.10) coincides with the Sklyanin bracket (1.1). At \( k = 1 \) we have

\[
\begin{align*}
  \rho^{[1]}_B &= \frac{\beta(\lambda - \mu)}{\alpha}, \\
  \rho^{[1]}_C &= \frac{\gamma(\lambda - \mu)}{\alpha}, \\
  \rho^{[1]}_D &= \frac{\delta(\lambda - \mu)}{\alpha}
\end{align*}
\]

and bracket (2.10) coincides with the bracket (2.3).

At \( k > 1 \) one can easily check that \( k \)th brackets (2.10) between polynomials \( A(\lambda), B(\lambda), C(\lambda) \) and \( D(\lambda) \) (2.7) imply the brackets

\[
\{\lambda_i, \lambda_j\}_k = \{\mu_i, \mu_j\}_k = 0, \quad \{\lambda_i, \mu_j\}_k = \lambda_i^k \delta_{ij}, \quad \text{(2.12)}
\]

and vice versa. This completes the proof. \( \square \)

To proceed further, we need to recall that the normalized traces of the powers of \( N \)

\[
J_m = \frac{1}{2m} \text{trace } N^m = \sum_{i=1}^{n} \lambda_i^m, \quad m = 1, \ldots, n, \quad \text{(2.13)}
\]

are integrals of motion satisfying Lenard–Magri recurrent relations [6, 7]:

\[
P_0 d J_1 = 0, \quad X_J = P_0 d J_1 = P_1 d J_{-1}, \quad P_1 d J_n = 0. \quad \text{(2.14)}
\]

By definition (2.7) polynomial

\[
A(\lambda) = \alpha \lambda^n + A_1 \lambda^{n-1} + \cdots + A_n = \alpha \prod_{i=1}^{n} (\lambda - \lambda_i)
\]

is directly proportional to the minimal characteristic polynomial of \( N \) (2.9)

\[
\Delta_N(\lambda) = (\det(N - \lambda I))^{1/2} = \prod_{i=1}^{n} (\lambda - \lambda_i).
\]
The Hamiltonians $H^\alpha_i$ (1.5) are related to integrals of motion $J_m$ (2.13) by the triangular Newton formulae

$$
\alpha J_1 = H^\alpha_1, \quad \alpha J_2 = H^\alpha_2 + \left(\frac{H^\alpha_1}{2}\right)^2, \quad \alpha J_3 = H^\alpha_3 + H^\alpha_2 H^\alpha_1 + \left(\frac{H^\alpha_1}{3}\right)^3, \ldots .
$$

As a consequence of the recursion relations (2.14), the Hamiltonians $H^\alpha_i$, $i = 1, \ldots, n$, satisfy the Frobenius recursion relations

$$
N^* dH^\alpha_i = dH^\alpha_i + 1 - \alpha^{-1} A_i dH^\alpha_1, \quad (2.15)
$$

where $N^* = P_i^{-1} P_1$ and $H^\alpha_{n+1} = 0$. Such as $A_i = H^\alpha_i$ a straightforward computation shows that they are equivalent

$$
N^* dA(\lambda) = \lambda dA(\lambda) + A(\lambda) dA_1.
$$

The special Darboux–Nijenhuis coordinates $\lambda_i, \mu_i$ are variables of separation of the action-angle type [2], i.e. the corresponding separated equations are trivial

$$
\{H^\alpha_i, \lambda_j\} = \{J_i, \lambda_j\} = 0, \quad i, j = 1, \ldots, n.
$$

We can introduce another separated coordinates $u_i, v_i$, which are the so-called Sklyanin variables defined by

$$
B(u_i) = 0, \quad v_i = -\eta^{-1} \ln A(u_i), \quad i = 1, \ldots, n.
$$

The separation representation of the algebra in $(u, v)$-variables has the form

$$
B(\lambda) = \beta(\lambda - u_1)(\lambda - u_2) \cdots (\lambda - u_n),
$$

$$
A(\lambda) = B(\lambda) \left(\frac{\alpha}{\beta} + \sum_{i=1}^n \frac{e^{-\eta v_i}}{(\lambda - u_i) B'(u_i)}\right),
$$

$$
D(\lambda) = B(\lambda) \left(\frac{\delta}{\beta} + \sum_{i=1}^n \frac{d(u_i) e^{\eta v_i}}{(\lambda - u_i) B'(u_i)}\right),
$$

$$
C(\lambda) = \frac{A(\lambda) D(\lambda) - d(\lambda)}{B(\lambda)}.
$$

Substituting matrix $T(\lambda)$ (1.3) with these entries into the brackets $\{., .\}_k$ (2.10) at $k = 0$, 1 one gets that $u_i, v_j$ coordinates are Darboux variables with respect to the Sklyanin bracket

$$
\{u_i, u_j\}_0 = \{v_i, v_j\}_0 = 0, \quad \{u_i, v_j\}_0 = \delta_{ij},
$$

whereas the second brackets look like

$$
\{u_i, u_j\}_1 = 0, \quad \{u_i, v_j\}_1 = u_i \delta_{ij} - \frac{\beta A(u_j)}{\alpha B'(u_j)}, \quad \{v_i, v_j\}_1 = \frac{A'(u_i)}{B'(u_i)} - \frac{A'(u_i)}{B'(u_i)}.
$$

The corresponding separated equations,

$$
\{A(\lambda), u_j\}_k = \lambda^k A(u_j) \prod_{i \neq j} \frac{\lambda - u_i}{u_j - u_i}, \quad j = 1, \ldots, n,
$$

are linearized by the Abel transformation on the algebraic curve defined by $e^{-\eta v_i} = A(u_i)$. The detailed discussion of these separated equations may be found in [3, 9, 12].

The special Darboux–Nijenhuis coordinates are dual to the Sklyanin variables. Namely, $\lambda_i$, $\mu_i$ are roots of polynomial $A(\lambda)$ and values of polynomial $B(\lambda)$ at $\lambda = \lambda_i$, while $u_i$, $v_i$ are roots of polynomial $B(\lambda)$ and values of polynomial $A(\lambda)$ at $\lambda = u_i$. 

2.2. Generic model

There are many other Poisson brackets compatible with the standard one (2.5). The main property of the proposed above bracket \( \{, \} \) is that

\[
\{ A(\lambda), A(\mu) \}_0 = \{ A(\lambda), A(\mu) \}_1 = 0.
\]

It ensures that integrals of motion \( H^\alpha \) for the open lattices are in bi-involution:

\[
\{ H^\alpha, H^\beta \}_0 = \{ H^\alpha, H^\beta \}_1 = 0.
\]

In this subsection, we are looking for bracket \( \{, \} \), which has to guarantee the similar property for generic integrals of motion \( H_i \) from (1.6):

\[
\{ H_i, H_j \}_0 = \{ H_i, H_j \}'_1 = 0, \quad i, j = 1, \ldots, n.
\]

Remind that \( \{, \}_0 \) is the Sklyanin bracket (1.1), which already has the necessary property:

\[
\{ \text{tr} T(\lambda), \text{tr} T(\mu) \} = 0.
\]

The following propositions can be ascertained by means of direct calculations.

**Proposition 3.** If \( \alpha = \delta \) and \( \beta = \gamma = 0 \) in \( T(\lambda) \) (1.3), then

\[
\{ \text{tr} T(\lambda), \text{tr} T(\mu) \}_1 = 0.
\]

So, the desired bracket \( \{, \}'_1 \) may be obtained from the bracket \( \{, \}_1 \) by using special canonical transformations, which are generated by the suitable transformations of the matrix \( T(\lambda) \).

**Proposition 4.** The Sklyanin bracket (1.1) is invariant with respect to transformation

\[
T(\lambda) \rightarrow V_1 T(\lambda) V_2, \quad V_i = \begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{pmatrix}, \quad \alpha_i, \beta_i, \gamma_i, \delta_i \in \mathbb{R},
\]

where \( V_{1,2} \) are numerical matrices. If

\[
\beta_1 \gamma_2 + \delta_1 \delta_2 = 0,
\]

the bracket (2.3)–(2.4) after transformation (2.19) has the necessary property:

\[
\{ \text{tr} T(\lambda), \text{tr} T(\mu) \}'_1 = 0.
\]

We present an explicit form of the bracket \( \{, \}'_1 \) in section 4 devoted to the periodic Toda lattice.

3. The Heisenberg magnet

Another important representation of the quadratic algebra with the generators \( A_i, B_i, C_i \) and \( D_i \) comes as a consequence of the co-multiplication property of the Sklyanin algebra (1.1). Essentially, it means that the matrix \( T(\lambda) \) can be factorized into a product of elementary matrices, each containing only one degree of freedom. In this picture, our main model turns out to be an \( n \)-site Heisenberg magnet, which is an integrable lattice of \( n \) sl(2) spins with nearest-neighbour interaction.

In the lattice representation the matrix \( T(\lambda) \) acquires the following form:

\[
T(\lambda) = L_1(\lambda - c_1) L_2(\lambda - c_2) \cdots L_n(\lambda - c_n),
\]

where \( c_i \) are constant parameters.
with
\[ L_m(\lambda) = \begin{pmatrix} \lambda - s_3^{(m)} & s_1^{(m)} + i s_2^{(m)} \\ s_1^{(m)} - i s_2^{(m)} & \lambda + s_3^{(m)} \end{pmatrix}, \quad m = 1, \ldots, n. \] (3.2)

Here \( s_i^{(m)} \) are dynamical variables, \( c_m \) are arbitrary numbers and \( i = \sqrt{-1} \).

Substituting matrix (3.1) into the Sklyanin bracket (1.1) and brackets (2.3)–(2.4) at \( \eta = i \) one gets canonical brackets on the direct sum of \( \mathfrak{sl}(2) \)
\[ \{s_i^{(m)}, s_j^{(m)}\}_0 = \varepsilon_{ijk} s_k^{(m)}, \quad \{s_i^{(m)}, s_j^{(l)}\}_1 = (P_{1}^{(m\ell)})_{ij}, \quad m \neq \ell, \] (3.3)
and second compatible brackets
\[ \{s_j^{(m)}, s_j^{(m)}\}_1 = \varepsilon_{ijk} k^{(m)}(c_m - s_3^{(m)}), \quad \{s_j^{(m)}, s_j^{(l)}\}_1 = (P_{1}^{(m\ell)})_{ij}, \quad m \neq \ell, \]
where \( \varepsilon_{ijk} \) is the totally skew-symmetric tensor and
\[ P_{1}^{(m\ell)} = \begin{pmatrix} -i(s_1^{(m)} s_1^{(l)} + s_2^{(m)} s_2^{(l)}) & i s_2^{(m)} s_1^{(l)} - s_3^{(m)} s_2^{(l)} & i s_3^{(m)} (s_1^{(l)} - i s_2^{(l)}) \\ i s_1^{(m)} s_2^{(l)} - s_3^{(m)} s_3^{(l)} & (s_3^{(m)} - i s_1^{(m)}) s_2^{(l)} & -s_3^{(m)} (s_1^{(l)} + i s_2^{(l)}) \\ -i(s_1^{(m)} + i s_2^{(m)}) s_3^{(l)} & (s_3^{(m)} + i s_2^{(m)}) s_1^{(l)} & -i(s_1^{(m)} s_3^{(l)} - i s_2^{(l)}) \end{pmatrix}. \]

The corresponding Poisson tensors \( P_0 \) and \( P_1 \) are degenerate and, therefore, the Hamiltonians \( H_0^\eta \) satisfy the Frobenius recurrence relations (2.15) in the following form:
\[ P_i d H_0^\eta = P_0(d H_0^\eta - A_i d H_1^\eta), \quad i = 1, \ldots, n, \] (3.4)
where \( H_0^\eta = 0 \) and \( A_i = H_0^\eta \) are coefficients of the polynomial \( A(\lambda) \). The first integrals of motion are
\[ H_1^\eta = \sum_{m=1}^n (c_m - s_3^{(m)}), \]
\[ H_2^\eta = \sum_{m > \ell} (s_1^{(m)} - i s_2^{(m)}) (s_1^{(l)} + i s_2^{(l)}) - \frac{1}{2} \sum_{m=1}^n (c_m - s_3^{(m)})^2 + \frac{(H_1^\eta)^2}{2}. \]

Such as \( \alpha = \delta = 0 \) and \( \beta = \gamma = 0 \) we can use these brackets for the open and periodic lattices simultaneously. It means that Hamiltonians \( H_i \) (1.6) from \( \operatorname{tr} T(\lambda) \) satisfy the Frobenius equations (3.4) too.

4. The Toda lattices

The Toda lattices appear as a specialization of our basic model when the parameters are fixed as follows:
\[ \beta = \gamma = \delta = 0 \quad \text{and} \quad \det T(\lambda) = 1. \] (4.1)
We also put \( \alpha = 1 \) and \( \eta = -1 \). In the lattice representation, the monodromy matrix \( T(1.3) \) acquires the form
\[ T(\lambda) = L_1(\lambda) \cdots L_{n-1}(\lambda) L_n(\lambda), \quad L_i = \begin{pmatrix} \lambda - p_i & -e^{\psi_i} \\ e^{-\psi_i} & 0 \end{pmatrix}. \] (4.2)

Here \( p_i, q_i \) are dynamical variables.
4.1. Open lattice

Substituting matrix $T(\lambda)$ (4.2) into the brackets $\{\ldots\}_k$ (2.10) at $k = 0, 1$ one gets that the Poisson tensors $P_0$ and $P_1$ in $(p, q)$ variables take the form

$$P_0 = \sum_{i=1}^{n} \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i},$$

$$P_1 = \sum_{i=1}^{n-1} e^{\alpha_i - \phi_i+1} \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial p_j} + \sum_{i=1}^{n} p_i \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_j} + \sum_{i<j} \frac{\partial}{\partial q_j} \wedge \frac{\partial}{\partial q_i}. \quad (4.3)$$

Namely this bi-Hamiltonian structure of the open Toda lattice was obtained in [1].

For the open Toda lattice the Hamiltonians $H_i^0$ from the $A(\lambda) = \lambda^n + H_i^0 \lambda^{n-1} + \ldots + H_n^0$ satisfy the Frobenius relations (2.15). The first integrals of motion are equal to

$$H_1^0 = -\sum_{i=1}^{n} p_i, \quad H_2^0 = \frac{1}{2} \sum_{i=1}^{n} p_i^2 + \sum_{i=1}^{n-1} e^{\alpha_i - \phi_i+1} - \frac{1}{2} \left( \sum_{i=1}^{n} p_i \right)^2. \quad (4.4)$$

The Sklyanin variables $u_i, v_i$ are introduced as before:

$$B(u_i) = 0, \quad v_i = -\eta^{-1} \ln A(u_i), \quad i = 1, \ldots, n-1,$$

the only difference now is that this gives only $n - 1$ instead of $n$ separation pairs. The missing pair of canonical variables is defined as follows:

$$v_n = \ln b_1 = -q_n, \quad u_n = -a_1 = \sum_{i=1}^{n} p_i. \quad (4.6)$$

The separation representation of the algebra in $(u, v)$-variables may be found in [9, 13]. It is easy to prove [13] that $(u, v)$-variables are Darboux variables

$$\omega = P_0^{-1} = \sum_{i=1}^{n} du_i \wedge dv_i,$$

and the only nonzero second Poisson brackets are

$$\{u_j, v_i\}_1 = u_j \delta_{ij}, \quad \{u_n, u_j\}_1 = -e^{-v_i} \frac{A(u_i) B(u_j)}{B(u_i)}, \quad \{u_n, v_i\}_1 = -e^{-v_i} \frac{A(u_i) B(u_j)}{B(u_i)},$$

$$\{v_n, v_i\}_1 = -1, \quad \{u_n, v_i\}_1 = -\sum_{i=1}^{n} u_i.$$

Remark 1. From the factorization (4.2) of the monodromy matrix $T(\lambda)$ one gets

$$B_n(\lambda) = -e^{\delta} A_{n-1}(\lambda) \Rightarrow B_n(u_j) = -e^{\delta} A_{n-1}(\lambda).$$

This implies that for the $(n-1)$-particle chain special Darboux–Nijenhuis variables $\lambda_j$ coincide with the Sklyanin variables $u_j, i = 1, \ldots, n - 1$ for the $n$-particle chain.

4.2. Periodic lattice

For the Toda lattice $\alpha \neq \delta$ and, therefore, in order to get new bracket $\{\ldots\}'_1$ with the necessary property (2.20) we have to apply transformation (2.19) to the initial bracket $\{\ldots\}_1$ (2.3)–(2.4).

If we put

$$V_1 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad V_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

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then one gets the following brackets between the entries of $T(\lambda)$:

\[
\{A(\lambda), A(\mu)\}'_1 = \frac{\eta \lambda}{\lambda - \mu} (C(\lambda)B(\mu) - C(\mu)B(\lambda)),
\]

\[
\{B(\lambda), B(\mu)\}'_1 = \eta(B(\lambda)D(\mu) - B(\mu)D(\lambda)),
\]

\[
\{C(\lambda), C(\mu)\}'_1 = \eta(C(\lambda)D(\mu) - C(\mu)D(\lambda)),
\]

\[
\{D(\lambda), D(\mu)\}'_1 = 0,
\]

\[
\{A(\lambda), D(\mu)\}'_1 = \eta \frac{\lambda - \mu}{\lambda - \mu} (C(\lambda)B(\mu) - C(\mu)B(\lambda)),
\]

\[
\{B(\lambda), C(\mu)\}'_1 = -\eta \frac{\lambda - \mu}{\lambda - \mu} (C(\lambda)D(\mu) - C(\mu)D(\lambda)),
\]

\[
\{D(\lambda), B(\mu)\}'_1 = \eta \frac{\lambda - \mu}{\lambda - \mu} (B(\lambda)D(\mu) - B(\mu)D(\lambda)),
\]

\[
\{D(\lambda), C(\mu)\}'_1 = -\eta \frac{\lambda - \mu}{\lambda - \mu} (C(\lambda)D(\mu) - C(\mu)D(\lambda)),
\]

\[
(4.7)
\]

According to proposition 4 these brackets have the necessary property (2.20).

Substituting matrix $T(\lambda)$ (4.2) into the brackets (4.7)–(4.8) one gets that the Poisson tensor $P'_1$ in $(p, q)$ variables takes the form

\[
P'_1 = \sum_{i=1}^{n-1} e^{\eta - q_i} \frac{\partial}{\partial p_{i+1}} \wedge \frac{\partial}{\partial p_i} + \sum_{i=1}^{n} p_i \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i} + \sum_{i=1}^{n} e^{\eta - q_i} \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i} - \sum_{i=1}^{n} \left( e^{\eta} \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i} + e^{\eta - q_i} \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i} \right).
\]

For the periodic Toda lattice the Hamiltonians $H_1$ and $H_2$ from the tr $T(\lambda) = \lambda^n + H_1 \lambda^{n-1} + \cdots + H_0$ are equal to

\[
H_1 = H'_1 = -\sum_{i=1}^{n} p_i, \quad H_2 = H'_2 + e^{\eta - q_i} = \frac{1}{2} \sum_{i=1}^{n} p_i^2 + \sum_{i=1}^{n} e^{\eta - q_i} - \frac{1}{2} \left( \sum_{i=1}^{n} p_i \right)^2,
\]

\[
(4.9)
\]

where $q_{ni} = q_i$. These Hamiltonians $H_i, i = 1, \ldots, n$, form the Frobenius chain

\[
N^* dH_i = dH_{i+1} + c_i dH_1, \quad \text{with} \quad H_{n+1} = 0.
\]

\[
(4.10)
\]

Here $N^* = P_0^{-1} P'_1$ and $c_i$ are coefficients of the minimal characteristic polynomial of the recursion operator

\[
\Delta_N(\lambda) = (\det(N - \lambda I))^{1/2} = \lambda^n - (c_1 \lambda^{n-1} + \cdots + c_n),
\]

\[
(4.11)
\]

which can be defined directly via the entries of the matrix $T(\lambda)$

\[
\Delta_N(\lambda) = A(\lambda) + B(\lambda) - C(\lambda) - D(\lambda).
\]

\[
(4.12)
\]
Remark 2. Transformations (2.19) of the matrix $T(\lambda)$ give rise to canonical transformations in the phase space. As sequence tensor $P'_1$ (4.9) coincides with tensor $P_1$ (4.3) after the following canonical transformation

$$p_1 \to p_1 + e^{-qi}, \quad p_n \to p_n + e^{qi},$$

which identifies coefficients $c_i$ with integrals of motion for the open Toda lattice $c_i = -H_i''$.

5. Integrable DST model

The integrable case of the DST (discrete self-trapping) model with $n$ degrees of freedom was studied in [5]. It appears as a specialization of our basic model when several parameters vanish:

$$\beta = \gamma = \delta = 0 \quad \text{and} \quad Q_j = 0, \quad j = 1, \ldots, n - 1. \quad (5.1)$$

We also put $\alpha = 1$ and $\eta = -1$. In the lattice representation, the matrix $T(\lambda)$ (1.3) acquires the form

$$T(\lambda) = L_1(\lambda - c_1)L_2(\lambda - c_2) \cdots L_n(\lambda - c_n), \quad \text{with} \quad L_i(\lambda) = \begin{pmatrix} \lambda - q_i p_i & b q_i \\ -p_i & b \end{pmatrix}. \quad (5.2)$$

Here $p_i, q_i$ are dynamical variables, whereas $b$ and $c_i$ are numbers entering into the Casimir function (1.4)

$$d(\lambda) = \det T(\lambda) = b^n(\lambda - c_1)(\lambda - c_2) \cdots (\lambda - c_n).$$

Substituting matrix $T(\lambda)$ (5.2) into the Sklyanin bracket (1.1) and into the brackets (4.7)–(4.8) one gets canonical brackets

$$\{p_i, q_j\}_0 = \delta_{ij}, \quad \{q_i, q_j\}_0 = \{p_i, p_j\}_0 = 0, \quad i, j = 1, \ldots, n,$$

and quadratic brackets

$$\{q_i, q_j\}_1 = -Q_i Q_j, \quad \{q_i, p_j\}_1 = Q_i P_j - c_i \delta_{ij}, \quad i > j \quad \{p_i, p_j\}_1 = -P_i P_j, \quad \{p_i, q_j\}_1 = q_i p_j - b \delta_{i+1,j},$$

where $Q_1 = q_1 + 1$, $P_n = p_n + b$ and $Q_i = q_i$, $P_i = p_i$ for other values of index $i$.

As above the Hamiltonians $H_i, i = 1, \ldots, n$, from the tr $T(\lambda) = \lambda^n + H_1 \lambda^{n-1} + \cdots + H_n$ satisfy the Frobenius relations (4.10). The two first Hamiltonians of the system are

$$H_1 = -\sum_{i=1}^{n} (q_i p_i - c_i), \quad H_2 = \sum_{i>j} (q_i p_i - c_i)(q_j p_j - c_j) - b \sum_{i=1}^{n} q_i p_{i+1}, \quad p_{n+1} \equiv p_1. \quad (5.3)$$

The Sklyanin variables $(u_i, v_i), i = 1, \ldots, n$, are introduced by the same formulae as for the Toda lattice, cf (4.5) and (4.6).

6. The Goryachev–Chaplygin gyrostat

Let us consider the matrix $T(\lambda)$ introduced in [4]

$$T(\lambda) = \begin{pmatrix} \lambda^3 - 2 \lambda J_3 - J_1^2 - J_2^2 \\ (x_1 + ix_2) \lambda - x_3 (J_1 - i J_2) \\ -x_3^2 \end{pmatrix}. \quad (6.1)$$
Substituting matrix (6.1) into the Sklyanin bracket (1.1) and brackets (2.3)–(2.4) at \( \eta = 2i \) one gets canonical Poisson tensor on the dual space of Euclidean algebra \( e(3) \)

\[
P_0 = \begin{pmatrix}
0 & 0 & 0 & x_3 & -x_2 \\
* & 0 & 0 & -x_3 & 0 & x_1 \\
* & * & 0 & x_2 & -x_1 & 0 \\
* & * & * & 0 & J_3 & -J_2 \\
* & * & * & * & 0 & J_1 \\
* & * & * & * & * & 0
\end{pmatrix}
\]  

(6.2)

and the following quadratic tensor

\[
P_1 = \begin{pmatrix}
0 & -x_3^2 & x_3 x_2 & -x_2 J_1 & -x_2 J_2 & x_3 J_2 - 2x_2 J_3 \\
* & 0 & -x_3 x_1 & x_1 J_1 & x_1 J_2 & 2x_1 J_3 - x_3 J_1 \\
* & * & 0 & 0 & 0 & -x_1 J_2 + x_2 J_1 \\
* & * & * & 0 & -J_1^2 - J_2^2 & -J_3 J_2 \\
* & * & * & * & 0 & J_1 J_3 \\
* & * & * & * & * & 0
\end{pmatrix}
\]  

(6.3)

These tensors satisfy equations (2.1) at any values of the Casimir functions

\[
C_1 = x_1^2 + x_2^2 + x_3^2, \quad C_2 = x_1 J_1 + x_2 J_2 + x_3 J_3.
\]

However, in the proposed method coefficients of \( \det T = -C_1 \lambda^2 + x_3 C_2 \lambda \) have to be the Casimir functions and, therefore, we have to put \( C_2 = 0 \). As sequence, we have \( \{A(\lambda), A(\mu)\}_1 = 0 \) at \( C_2 = 0 \) only.

**Remark 3.** Solving equations \( P_0 dH_0^2 = (P_1 + \alpha P_0)H_0^1 \) at arbitrary values of \( C_{1,2} \) one gets

\[
H_0^1 = J_3, \quad H_0^2 = J_1^2 + J_2^2 + 2J_3^2 + \alpha J_3.
\]

Here \( H_0^i \) is a kinetic part of the Hamiltonian for the Kowalevski gyrostat, which may be studied by using 2 × 2 Lax matrix \( L(\lambda) = K_4 T(\lambda)K_4^{-1}(-\lambda) \) [4]. The tensor \( P_1 \) (6.3) differs from the Poisson tensor for the Kowalevski gyrostat, which appears from the linear \( r \)-matrix algebra [14].

The 2 × 2 Lax matrix for the Goryachev–Chaplygin gyrostat looks like [8]

\[
\tilde{T}(\lambda) = \begin{pmatrix}
e^{\lambda} & 0 \\
0 & e^{-\lambda}
\end{pmatrix}
\begin{pmatrix}
\lambda & 2J_3 + p \\
a & 0
\end{pmatrix}
\begin{pmatrix}
e^{-\lambda} & 0 \\
0 & e^{\lambda}
\end{pmatrix} T(\lambda) \begin{pmatrix}
e^{\lambda} & 0 \\
0 & e^{-\lambda}
\end{pmatrix}.
\]  

(6.4)

Here \( p, q \) are additional dynamical variables, \( a \) is an arbitrary number and \( T(\lambda) \) is given by (6.1).

Substituting this matrix into the Sklyanin bracket (1.1) and into the brackets (4.7)–(4.8) one gets the compatible Poisson tensors on the extended phase space \( e^*(3) \times (p, q) \)

\[
\tilde{P}_0 = \begin{pmatrix}
P_0 & W_0 \\
W_0^T & G_0
\end{pmatrix} = \begin{pmatrix}
P_0 & 0 \\
\vdots & \ddots \\
0 & 2i \\
0 & 0
\end{pmatrix}
\]  

(6.5)
and

\[ \tilde{P}_1 \equiv \begin{pmatrix} P_1 & W^T \cr W & G_1 \end{pmatrix} = \begin{pmatrix} \* & \begin{pmatrix} -2x_3J_2 + 2px_2 + 8x_2J_3 \\ 2x_3J_2 - 2pJ_1 - 8x_1J_3 \\ 2x_3J_2 - 2x_2J_1 \\ 0 \end{pmatrix} \\ 0 & 2i((x_1 - ix_2)e^q - 2J_3 - p - ae^q) \end{pmatrix}, \tag{6.6} \]

which satisfy equations (2.1) at \( C_2 = 0 \) only. Here \( \tilde{P}_0 \) and \( \tilde{P}_1 \) are given by (6.2) and (6.3).

The Hamiltonians \( H_i \) from the \( R(\lambda) = \lambda^3 + H_1\lambda + \lambda^2H_2 + H_3 \) are

\[
H_1 = p, \\
H_2 = -\left(J_1^2 + J_2^2 + 4J_3^2 + 2pJ_3 - 2ax_1\right), \\
H_3 = -(2J_3 + p)(J_1^2 + J_2^2) - 2ax_3J_1.
\]

The obtained tensors \( \tilde{P}_0 \) and \( \tilde{P}_1 \) are degenerate and, therefore, the Hamiltonians \( H_i \) reproduce the Frobenius chain in the following form

\[ \tilde{P}_i dH_i = \tilde{P}_0(dH_{i+1} + c_i dH_1), \quad i = 1, 2, 3, \tag{6.7} \]

where \( H_4 = 0 \) and \( c_i \) are coefficients of the polynomial \( \Delta_N(\lambda) = A(\lambda) + B(\lambda) - C(\lambda) - D(\lambda) \)

At \( p = q = 0 \) matrices \( G_0 \) (6.5) and \( G_1 \) (6.6) are (generically) non-degenerate. So, the Dirac procedure can reduce Poisson tensors \( \tilde{P}_{0,1} \) to a new Poisson tensors \( \tilde{P}_{0,1}^D \) on \( e^*(3) \) defined by

\[ \tilde{P}_k^D = P_k + \left(W_kG_k^{-1}W^T_k\right)_{p=0,q=0}, \quad k = 0, 1. \]

Here \( \tilde{P}_0^D \) is canonical Poisson tensor (6.2) and \( P_1 \) is given by (6.3). This reduction procedure preserves equations (6.7) for the reduced integrals of motion. The main problem is that the Dirac procedure destroys the compatibility of the Poisson tensors \( \tilde{P}_{0,1}^D \).

7. Conclusion

We present a family of compatible Poisson brackets (2.10) that includes the Sklyanin bracket, and prove that the Sklyanin variables are dual to the special Darboux–Nijenhuis coordinates associated with these brackets. The application of the \( r \)-matrix formalism is extremely useful here resulting in drastic reduction of the calculations for a whole set of integrable systems.

The construction can be generalized to other \( r \)-matrix algebras. Remind, if one substitutes \( T(\lambda) = 1 + eL(\lambda) + O(e^2) \), \( r = er \) into (1.1) and let \( e \to 0 \) one gets a linear bracket. Then if \( T(\lambda) \) satisfy the Sklyanin bracket (1.1), then the matrix \( T(\lambda) = T(\lambda)K_*T^{-1}(\lambda) \) obeys the reflection equation algebra [10]. The corresponding compatible brackets for the open generalized Toda lattices were considered in [15].

Moreover, the whole construction can immediately be transferred to the quantum case because \( r \)-matrices in (2.10) became dynamical matrices at \( k > 1 \) only.

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