Black hole solutions for scale-dependent couplings: the de Sitter and the Reissner–Nordström case

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Abstract

Allowing for scale dependence of the gravitational couplings leads to a generalization of the corresponding field equations. In this work, these equations are solved for the Einstein–Hilbert and the Einstein–Maxwell case, leading to generalizations of the (Anti)-de Sitter and the Reissner–Nordström black holes. These solutions are discussed and compared to their classical counterparts.

Keywords: functional renormalization group, black hole, Reissner–Nordström

(Some figures may appear in colour only in the online journal)

1. Introduction

Black holes are very fascinating objects. Apart from the fact that they are classical solutions of the equations of general relativity, they also have additional ‘features’, such as the existence of an event horizon and an essential singularity at the origin \(r = 0\), typically behind the event horizon. The existence of this singularity can be interpreted as the breakdown of the validity of the classical theory that predicts such solutions. In this sense the study of black holes gives an interesting opportunity of exploring general relativity in the transition between a regime where the classical solution is known to be valid to high precision, and a regime where corrections to the classical prediction are to be expected. These expected corrections to the classical regime are broadly assumed to be of quantum nature. A famous example for such quantum corrections is the predicted existence of thermal radiation, emitted by massive black holes, and induced by quantum fluctuations [1].

When one takes general relativity as the classical limit of a quantum theory seriously, one has to deal with a lot of conceptual and technical difficulties. Although these difficulties are far from settled, there exist various promising approaches aiming for a full (or partial)
quantum formulation of general relativity [2–18] (for a review see [19]), even without the necessity of going to a very different theory in higher dimensions. Although the different approaches are quite diverse, most of them still have a common feature. As with most other quantum field theories, they show a non-trivial scale dependence. This means that the effective couplings of the theories become actually scale-dependent quantities. For example, Newton’s constant \( G_0 \) acquires a scale dependence \( G_0 \rightarrow G(k) \). This scaling behavior is especially satisfying to see in the realizations of Weinberg’s asymptotic safety program [20–27]. Such a scaling is expected to modify classical observables, such as the black hole background [28–39]. Even though the way in which this scale dependence is calculated and the functional form of the scale dependence itself can vary among the different approaches, the pure existence of such a scale dependence seems to be a solid statement.

In this paper the idea of scale-dependent couplings will be implemented at the level of effective action and the corresponding improved equations of motion. In order to obtain more generic results, no particular functional form will be assumed for the scale-dependent couplings \( G_k \),... . Instead, a symmetry criterion will be imposed on the metric ansatz, which actually allows us to solve the improved equations of motion without knowledge of the functional form of the coupling constants \( G(k) \),... . This technique will be used in order to obtain two new black hole solutions that are, on the one hand, generalizations of the corresponding classical solution and, on the other hand, solutions of the self-consistent field equations in the context of scale-dependent couplings. The corrections with respect to the classical solution will be parametrized in terms of a dimensionful integration constant \( \epsilon \), which for the case of quantum-gravity-induced scale dependence can be expected to be of the order of the Planck mass \( M_{Pl} \).

The paper is organized as follows. In the section 1.1 properties of the classical (Anti)-de Sitter ((A)dS) solution and the classical Reissner–Nordström (RN) will be briefly summarized. In section 1.2 how a scale dependence of the gravitational couplings \( G(k) \),... can be implemented self-consistently at the level of generalized gravitational field equations will be reviewed. In section 2 the generalized (A)dS black hole solution for the generalized Einstein–Hilbert equations will be discussed. This solution is obtained and parametrized in section 2.1, then in section 2.3 the asymptotic behavior of this solution is calculated. A perturbative analysis of the thermodynamic properties of this solution is presented in section 2.4. In section 3 the generalized RN black hole solution for the generalized Einstein–Maxwell equations will be discussed. The generalized RN solution is presented and parametrized in section 3.1, then in section 3.2 the asymptotic behavior of this solution is calculated and the conserved quantity corresponding to the electrical charge is calculated in section 3.3. The horizon structure, the thermodynamic corrections, and the cosmic censorship of the solution are discussed in section 3.4. The results of this work are summarized in section 4 and important features are highlighted.

1.1. The classical (A)dS and RN black hole solutions

In this subsection the key features such as line element, divergent behavior and location of the horizons of the classical (A)dS and RN black hole solutions will be listed. The line element of both of these black hole solutions takes the form

\[
ds^2 = -f(r) \, dt^2 + f(r)^{-1} \, dr^2 + r^2 d\Omega_2^2,
\]

where \( d\Omega_2^2 \) is the volume element of the two-sphere.
For the case of the (A)dS solution, the function \( f(r) \) takes the form

\[
f(r)_{\text{(A)dS}} = 1 - \frac{2 G_0 M_0}{r} - \frac{1}{3} \Lambda_0 r^2.
\] (2)

Here \( G_0 \) and \( \Lambda_0 \) denote the classical Newton constant and the classical cosmological constant. The integration constant \( M_0 \) is the classical mass of the black hole. The sign of \( \Lambda_0 \) describes, a Schwarzschild-AdS \((\Lambda_0 < 0)\), Schwarzschild \((\Lambda_0 = 0)\), or a Schwarzschild-dS \((\Lambda_0 > 0)\) black hole. The previously mentioned space-like singularity at \( r = 0 \) can be seen for the (A)dS solution by computing the invariant square of the Riemann tensor

\[
R_{\mu
u\rho\sigma}R^{\mu
u\rho\sigma}|_{\text{(A)dS}} = \frac{48 G_0^2 M_0^2}{r^8} + \frac{8 \Lambda_0^2}{3}.
\] (3)

This singularity is hidden behind an event horizon. Horizons are found as zeros of the function \( f(r) \), which due to the cubic nature of the function, allows for three solutions

\[
r_0 \big|_{\text{(A)dS}} = -U^{-1/3} - \frac{1}{\Lambda_0} U^{1/3},
\] (4)

and

\[
r_\pm \big|_{\text{(A)dS}} = \frac{1}{2} \left( 1 \pm i \sqrt{3} \right) U^{-1/3} + \frac{1 \mp i \sqrt{3}}{2 \Lambda_0} U^{1/3}.
\] (5)

Where

\[
U = 3 G_0 M_0 \Lambda_0^3 + \sqrt{9 G_0^2 M_0^2 \Lambda_0^4 - \Lambda_0^3}.
\] (6)

was defined as the typical scale of the solution. If the value of these roots is real and positive, they correspond to a physical horizon. For the case of \( \Lambda_0 \leq 0 \) there is only a single horizon, given by (4). For the case of \( \Lambda_0 > 0 \) and \( M_0 > 0 \) the solution has two physical horizons given by (5).

For the case of the classical RN solution, the function \( f(r) \) in the line element (1) takes the form

\[
f(r)_{\text{RN}} = 1 - \frac{2 G_0 M_0}{r} + \frac{4 \pi G_0 Q_0^2}{r^2 e_0^2},
\] (7)

where the constant of integration \( Q_0 \) is the classical electrical charge of the RN black hole and \( 1/e_0^2 \) is the electromagnetic coupling constant. The classical solution for the electromagnetic stress–energy tensor is

\[
F_{\mu\nu} = -F_{\nu\mu} = \frac{Q_0}{r^2}.
\] (8)

The leading singular behavior of the classical RN solution at \( r = 0 \) can be seen by computing the invariant square of the Riemann tensor

\[
R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}|_{\text{RN}} = 8^2 14 G_0^2 \pi^2 Q_0^4 e_0^4 r^8.
\] (9)

Again, this singular behavior can be shielded by an horizon which can be found by solving the condition of vanishing (7)
One observes that these two horizons \((\pm)\) become degenerate if the square root on the right-hand side vanishes. Furthermore, beyond this point the square root turns negative and no physical horizon is present in the solution, which is undesired since it would lead to an unshielded ‘naked’ singularity. Thus, in order not to get into trouble due to the appearance of a naked singularity, one demands a minimal mass for the classical RN black hole.

\[
M_0 \geq 2\sqrt{\frac{\pi}{e_0}} \frac{Q_0}{\sqrt{G_0}}. 
\]

This reasoning is known as the ‘cosmic censorship’ argument \([40]\).
and for the internal $U(1)$ transformations the corresponding relations are

$$\nabla_{[\mu} F_{\nu\rho]} = 0.$$  \hfill (18)

Please note that one has to work with (18) and not with $\nabla_{[\mu} t_{\nu\rho]} = 0$ [42]. In the following sections, two special black hole solutions for this system will be presented and discussed. First, in section 2 a solution for the system (12)–(16) will be presented, where the electromagnetic coupling is omitted $(1/e^2 = 0)$. Then, in section 3, a solution is found for the case of finite electromagnetic coupling and vanishing cosmological coupling ($\Lambda_k = 0$).

2. Black hole solution for the Einstein–Hilbert case

In the Einstein–Hilbert truncation one neglects the electromagnetic contribution to the action of the system (12)–(16) leading to simplified equations of motion for the metric field $g_{\mu\nu}$

$$G_{\mu\nu} = -g_{\mu\nu}\Lambda_k - \Delta t_{\mu\nu},$$ \hfill (19)

and simplified equations of motion for the scale field $k(x)$ [50]

$$R\nabla^\mu\left(\frac{1}{G_k}\right) - 2\nabla^\mu\left(\frac{\Lambda_k}{G_k}\right) = 0.$$ \hfill (20)

The most general line element consistent with spherical symmetry is

$$ds^2 = -f(r) dt^2 + h(r) dr^2 + r^2d\Omega_2^2.$$ \hfill (21)

One notes that for this symmetry, the system (19), (20) has sufficient independent equations in order to solve for the three $r$-dependent functions $f(r)$, $h(r)$, and $k(r)$. This is, however, assuming that the functional form of the scale-dependent couplings $G_k$ and $\Lambda_k$ is known, for example, from a background independent integration of the functional renormalization group [20–27].

Since the aim is to gain some information on scale-dependent black holes, independent of the details of the derivation and integration of the renormalization group or the particular approach to quantum gravity, we will use the following reasoning:

Even if one does not know the functional form of $G_k$ and $\Lambda_k$, one knows that both couplings will inherit some $r$-dependence from $k(r)$ and, therefore, one might treat them as two independent fields $G(r)$ and $\Lambda(r)$. Thus, one has encoded the ignorance (or ambiguity) on the scale-dependent couplings in an additional field variable ($G(r)$ and $\Lambda(r)$ instead of $k(r)$). Of course, now the system (19), (20) with three independent equations is in any case insufficient to solve for the four $r$-dependent fields $f(r)$, $h(r)$, $G(r)$, and $\Lambda(r)$ in full generality. In order to reduce again the number of free fields one has to impose some condition on those functions. In the presented study we will restrict our search to solutions that have only ’standard’ event horizons. By this we mean that, on the one hand, the signature of a $(t, r)$ line will change from minus to plus or vice versa when passing an event horizon (zero of $f(r)$), which suggests that either $f(r) \sim h(r)$ or $f(r) \sim 1/h(r)$. On the other hand, this means that we demand that the radial part of the line element diverges, when the time part of the line element vanishes. These conditions are implemented straightforwardly by imposing

$$f(r) \sim \frac{1}{h(r)}.$$ \hfill (22)
This choice is commonly referred to as the ‘Schwarzschild ansatz’. With the external restriction (22) the number of fields is thus reduced to three: $f(r)$, $G(r)$, and $\Lambda(r)$, which fits the number of independent equations in the system (19), (20).

2.1. Finding the solution

Based on the condition (22), the ansatz for the line element in the Einstein–Hilbert case will be

$$ds^2 = - f(r) c_t^2 dt^2 + \frac{1}{f(r)} dr^2 + r^2 d\Omega^2_2,$$

where the constant $c_t$ implements explicitly the time-reparametrization invariance of the system. The equations (19) (20) have already been solved [43–45] by using the ansatz (23) for $c_t = 1$. However, in the parametrization found in [43–45], the physical meaning of the integration constants and their relation to the classical (A)dS-Schwarzschild metric remained unclear.

Here, a new parametrization (using the labels $G_0, \Lambda_0, M_0$, and $\epsilon, c_1$, and $c_4$) of the solution is presented, where these problems were solved. The solutions for the three functions are

$$G(r) = \frac{G_0}{c r + 1},$$

$$f(r) = 1 + 3G_0M_0\epsilon - \frac{2G_0M_0}{r} - \left(1 + 6\epsilon G_0M_0\right) c r - \frac{\Lambda_0 r^2}{3} + r^2 \epsilon^2 \left(6\epsilon G_0M_0 + 1\right) \ln \left(\frac{c_4 \left(\epsilon r + 1\right)}{r}\right)$$

$$\Lambda(r) = \frac{-72\epsilon^2 r^2 \left(\epsilon r + 1\right) \left(\epsilon + \frac{1}{2} \right) \left( G_0 M_0 \epsilon + \frac{1}{6} \right) \ln \left(\frac{c_4 \left(\epsilon r + 1\right)}{r}\right) + 4\epsilon^3 \Lambda_0 r^2 + \left(12\epsilon^3 + 6\epsilon \right) + 72\epsilon^4 G_0 M_0 \epsilon^2}{2r \left(\epsilon r + 1\right)^2}$$

$$+ \frac{\left(72\epsilon^6 G_0 M_0 + 11\epsilon^4 + 2\Lambda_0\right) r + 6\epsilon^2 G_0 M_0}{2r \left(\epsilon r + 1\right)^2}.$$

The new and intuitive feature of this non-trivial choice of constants of integration is given by the fact that it was possible to isolate a combination of the original constants of integration such that the classical (A)dS solution can be recovered by sending a single constant ($\epsilon$) to zero. For instance, the scale-dependent Newton coupling reduces to the classical Newton constant

$$\lim_{\epsilon \to 0} G(r) = G_0$$

and the scale-dependent cosmological coupling reduces to the cosmological constant

$$\lim_{\epsilon \to 0} \Lambda(r) = \Lambda_0.$$ 

Finally, the labeling of the constant $M_0$ is justified by taking the same limit for the metric component

$$\lim_{\epsilon \to 0} f(r) = - \frac{\Lambda_0 r^2}{3} - \frac{2G_0M_0}{r} + 1,$$

where the classical solution (2) and the mass $M_0$ is recovered. Thus, the limits (27)–(29) justify the choice of constants of integration as $G_0, \Lambda_0, M_0$, and $\epsilon$ in contrast to the
2.2. The link to perturbative quantum gravity?

Please note that $\epsilon$ has the dimension of one over length, and might thus be associated to the inverse of a typical length scale $r_\epsilon$, where the scale dependence becomes relevant. This implies that sending $\epsilon \to 0$ corresponds to sending $r_\epsilon \to \infty$. Thus, for very small $\epsilon$, the radial coordinate $r$ would have to be very large in order to note a deviation from the typical solution without scale dependence. A behavior which is in complete contradiction to the expectation one would have for perturbative quantum corrections, which typically appear at short-distance scales. One might for example try to match (25) to leading corrections of the Newton potential [46, 47] or to leading loop-induced corrections of the Schwarzschild metric [48]. However, one finds that this is simply not possible, because the loop-induced corrections are typically short-ranged $\sim 1/r^2$, whereas the scale dependence in (25) appears to be long-ranged $r^n$ with $n \geq 0$. From this intent one learns that, if one insists on working with a covariant formulation of scale dependence (12) and with the Schwarzschild ansatz (22), the corrections due to scale dependence are intrinsically long ranged, rather than short ranged. The lack of matching to perturbative quantum corrections in quantum gravity could originate in each of the following conceptual differences:

1. The effect of $A(r)$ is crucial for finding the solution (24)--(26), whereas loop corrections to the gravitational potential [46, 47] neglect any effects of the cosmological constant and are thus biased towards short-range corrections.
2. The classical ‘symmetry’ (22) is typically not maintained in perturbative corrections to the Schwarzschild metric [48].
3. Perturbative calculations in quantum gravity tend to produce strongly gauge-dependent results, which make a definition of scale-dependent couplings like $\tilde{G}(k)$, not unique and problem dependent [49]. Both problems are circumvented in this work, since scale dependence is assumed and since the equations (12) are explicitly gauge invariant. Thus, the presented solutions (24)--(26) reveal novel long-range effects of scale dependence and it is likely that the familiar short-range scale dependence would only appear if one would find a way to drop the ansatz (22) and still find physical solutions.

2.3. Asymptotic space-times and causal structure

The asymptotic behavior of this solution for small scales ($r \to 0$) is closely linked to the singularity at the origin. This singularity can be most clearly studied by evaluating geometrical invariants. For example the Ricci scalar for the solution is

$$
R = -\frac{6G_0M_0\epsilon}{r^2} + \frac{6\epsilon + 36G_0M_0\epsilon^2}{r} + 7\epsilon^2 + 42G_0M_0\epsilon^3 \\
+ 4\lambda_0 - 12\epsilon^2(1 + 6G_0M_0\epsilon)\log\left(\frac{\epsilon_0}{r}\right) + \mathcal{O}(r).
$$

One observes that the classical limit of $4\lambda_0$ is modified by a new quadratic divergence of this quantity which is proportional to $\epsilon$. Another invariant quantity that is frequently studied is the higher curvature scalar.
For this invariant one observes that to leading order the singular behavior of this quantity is the same as in the classical case (3) and that modifications due to $\epsilon$ only appear at subleading orders in $1/r$. From these two examples one can already conclude that the elimination of this radial singularity is not possible for the given solution unless one returns to trivial configurations—say with vanishing $M_0$ and vanishing $\epsilon$.

The other regime of asymptotic behavior can be studied in a large radius expansion $r \to \infty$. In order to get a feeling for this behavior it is instructive to plot the radial function $f(r)$ for varying values of $\epsilon$. In figure 1 one observes that even for the case $L = 0$ the non-classical scale dependence $r^2 \ln r$ can mimic the effect of a cosmological constant by generating an asymptotic (Anti-)de Sitter space-time.

By studying the large $r$ behavior of the metric function (25) one finds

$$f(r) = -r^2 \frac{\Delta}{3} + \mathcal{O}(r),$$

where the effective cosmological constant $\Delta$ is actually a shift of the classical value $\Lambda = \Lambda_0 - 3 \epsilon^2 \left(6 \epsilon G_0 M_0 + 1\right) \ln(\epsilon^4 \epsilon)$. (33)

When it is referred to $\Delta$ as ‘effective cosmological constant’, this is done in the sense that for any measurement, say by observing trajectories, the result would be determined by the form of the metric function rather than by the function $\Lambda(r)$ in (26). One observes that the $\epsilon$ induced shift in the cosmological constant $\Delta = \Lambda_0$ is determined by the sign of the logarithm in (25). In the asymptotic limit $r \to \infty$ one finds for example for $\epsilon > -\frac{1}{6G_0 M_0}$ that for
\begin{align*}
c_4 < \frac{1}{\epsilon} \Rightarrow (\bar{\Lambda} - \Lambda_0) > 0, \\
c_4 = \frac{1}{\epsilon} \Rightarrow (\bar{\Lambda} - \Lambda_0) = 0, \\
c_4 > \frac{1}{\epsilon} \Rightarrow (\bar{\Lambda} - \Lambda_0) < 0.
\end{align*}

(34)

For \( \epsilon < -1/(6G_0 M_0) \) the relations (34) get inverted. A very interesting scenario turns out to be the case of \( c_4 = \epsilon \), where the effective cosmological constant agrees with the classical parameter \( \bar{\Lambda} = \Lambda_0 \). However, studying the asymptotics of the metric function (25) is not the only way one might try to extract a notion of an asymptotic cosmological constant. For a comparison one can take the limit of large \( r \) for the scale-dependent quantity (26) which gives

\[
\lim_{r \to \infty} \Lambda(r) = 2\Lambda_0 - 6\epsilon^2 (6G_0 M_0 \epsilon + 1) \ln (c_4 \epsilon),
\]

which is different from the ‘effective cosmological constant’ (33) extracted from the metric solution. This is, however, not concerning, since one can argue that the asymptotic form of a space-time must be read from the metric and not from a function appearing in the equation of motion. Based on this argument one sticks to (33) as the proper definition of \( \bar{\Lambda} \). Still, it is interesting to note that both possible notions of an ‘effective cosmological constant’ (33 and 35) vanish for the same choice of parameters

\[
\text{if } \Lambda_0 = 3\epsilon^2 (6G_0 M_0 + 1) \ln (c_4 \epsilon) \Rightarrow \lim_{r \to \infty} \Lambda(r) = \bar{\Lambda} = 0.
\]

(36)

Furthermore, one observes that the \( \epsilon \) dependence of both notions vanishes for the particular choice \( c_4 = 1/\epsilon \) (or for \( \epsilon = -1/(6G_0 M_0) \), which will be excluded in the next section).

The causal structure of this solution is basically determined by the structure of horizons and divergences. As can be seen, the topology of these structures is not affected by small \( \epsilon \) corrections and \( r < 1/\epsilon \). Thus, in this case, the only possible modifications could appear at very large, distances. More generally, the number of additional horizons, if they appear, depends in non-analytical form on the choice of the parameters and would have to be determined numerically. The maximal number of possible horizons can be read from the different monomials in \( r \) which could dominate (25) at some scale \( r \). These monomials can be ordered for \( c_4 = 1/\epsilon \) and \( \epsilon > 0 \): \( -1/r, +r^{1/\epsilon}, +r^2 \ln(1 + 1/(r \epsilon)), -r, \pm r^2 \), where the \( \pm \) refers to the AdS/dS respectively. The position of the monotonically decreasing logarithm in this ordering is justified by the typical inequalities \( 1/(1 + r \epsilon) < \ln(1 + 1/(r \epsilon)) < 1/(r \epsilon) \). Each sign change in this ordering could possibly correspond to the appearance of an horizon. Thus for \( c_4 = 1/\epsilon \) and \( \epsilon > 0 \) one finds: The number of horizons for the dS case (\( \Lambda_0 > 0 \)) can only be up to two, just like in the classical case. However in the AdS case (\( \Lambda_0 < 0 \)), the number of possible horizons can in some cases be up to three, two more than in the classical case. Note that by allowing arbitrary signs for the integration constants, and ignoring the choice \( c_4 = 1/\epsilon \), the possible number of horizons could be as high as the number of monomials of \( f(r) \), five.

2.4. Perturbative analysis for horizons and thermodynamics

Since scale dependence of coupling constants is generally assumed to be weak, it is reasonable to treat the dimensionful parameter \( \epsilon \) as small with respect to the other scales entering the problem such as \( 1/\sqrt{G_0} \), or \( M_0 \). As can be seen from the relations (27)–(29), this constant encodes the deviation from the classical solution (2) and therefore its absolute value is also experimentally expected to be very small in comparison with other integration constants with
dimensions of energy. In principle $\epsilon$ could take positive or negative values. However, the following short discussion will show that only small positive values give physically viable (real) solutions at the outside of the event horizon.

From the solution (25) one sees that the argument of logarithm in the metric function could become negative for $\epsilon < 0$ and $c_4 > 0$, at very large values of $r$. Compensating for this by making $c_4 < 0$ is also not possible since in this case the logarithm can become negative for somewhat smaller radii $r_0 < r < 1/|\epsilon|$, where $r_0$ is the radius of the horizon which for small $|\epsilon|$ can be approximated by the classical Schwarzschild radius. Thus, the parameter $\epsilon$ has to be positive and small right from the start.

In this context it is instructive to Taylor expand in this small parameter to see the leading corrections due to the scale dependence of the couplings

\[ G(r) = G_0 - \epsilon \cdot G_0 r + \mathcal{O}(\epsilon^2) \]  
\[ f(r) = 1 - \frac{2G_0 M_0}{r} - \frac{\epsilon^2 \Lambda_0}{3} + \epsilon \cdot (3G_0 M_0 - r) + \mathcal{O}(\epsilon^2) \]  
\[ \Lambda(r) = \Lambda_0 + \epsilon \cdot \Lambda_0 r + \mathcal{O}(\epsilon^2). \]

From equation (38) one sees now more clearly why previous attempts to obtain a meaningful physical parametrization failed. The problem was that it was assumed that for vanishing parameter $M_0$ the flat (A)dS solution $f(r) = 1 - \frac{\epsilon^2 \Lambda_0}{3}$ would be recovered. However, looking at (38) one sees that this is actually not possible without completely returning to the classical solution ($\epsilon = 0$). Apparently, the deviations from the classical space-time metric in (38), could be used in a phenomenological context in order to constrain the value of the supposedly very small parameter $\epsilon$. Such a study is, however, beyond the scope of this work. The perturbative analysis is, however, useful for a first understanding of the leading effects on the black hole horizons and the corresponding thermodynamics.

To first order in $\epsilon$, the horizons are defined by the zeros of (38). For $\Lambda_0 > 0$ and for $0 < M_0 < \frac{1}{3G_0\sqrt{\Lambda_0}}$ the two relevant real horizons are found to be

\[ r_+ = \frac{1 \pm i\sqrt{3}}{2p^{1/3}} + \frac{(1 + i\sqrt{3})p^{1/3}}{2\Lambda_0} \]
\[ + \epsilon \left( \frac{1 \pm i\sqrt{3}}{4p^{2/3}} + \frac{(1 \pm i\sqrt{3})(6G_0 M_0 P - \Lambda_0)}{4p^{4/3}} - \frac{1}{\Lambda_0} \right), \]

where $r_+$ is the outer (cosmological) horizon and $r_-$ is the inner Schwarzschild horizon. $P$ is given by

\[ P = 3G_0 M_0 \Lambda_0 \left( 1 + \sqrt{1 - \frac{1}{9G_0^2 M_0^2 \Lambda_0}} \right). \]

In the classical case ($\epsilon = 0$) these two horizons become degenerate for the critical mass

\[ M_{0,\text{crit}} = \frac{1}{3G_0\sqrt{\Lambda_0}}, \]

which corresponds to the Nariai black hole [41], which is the maximal allowed black hole mass before the appearance of a naked singularity. For non-vanishing $\epsilon$ this critical mass value and the corresponding black hole radius get slightly shifted. A comparison of the
horizon structure as a function of the mass parameter $M_0$ is shown in figure 2. One observes that for a given $M_0$ both horizons get shifted by positive $\epsilon$ towards smaller radii. Furthermore, one sees that the same holds true for the radius of the critical Nariai black hole and that the critical mass parameter $M_0$ which is the value where the cosmological and the inner horizon merge $r_+ = r_-$ gets slightly increased with respect to the classical value ($42$).

Given the horizon structure and the functional form of ($38$) one can calculate the temperature of the corresponding black hole. At the inner horizon this temperature is given by

$$T_c = \frac{1}{4\pi} \left. \frac{df}{dr} \right|_{r_-} = \frac{2G_0 M_0}{r_-^2} - \epsilon - \frac{2r_+ \Lambda_0}{3} + \mathcal{O}(\epsilon^2).$$

In figure 3 this temperature is shown as a function of the mass parameter $M_0$. One observes that for a vast range of parameters the modified temperature is indistinguishable from the classical value, and that only for the largest masses, close to the $M_*$, a slight splitting of the curves occurs. This splitting shows a slightly increased temperature for increasing $\epsilon$ values. Thus, in this region, the shift in the horizon radius $r_-$ overcompensates the negative direct

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**Figure 2.** Mass dependence of the black hole horizons $r_{\pm}$ as a function of the mass parameter $\lambda = M_c/M_0$, for $G_0 = 1$ and $\Lambda_0 = 0.01$. The black dashed curve is for $\epsilon = 0$ the red curve is for $\epsilon = 0.002$ and the blue curve is for $\epsilon = 0.004$.

**Figure 3.** $M_0$ dependence of the temperature ($43$) for $G_0 = 1$ and $\Lambda_0 = 0.01$. The black dashed curve corresponds to $\epsilon = 0$, the red curve is for $\epsilon = 0.002$ and the blue curve is for $\epsilon = 0.004$. 
contribution of $\epsilon$ to the temperature in equation (43). Therefore, it takes slightly higher values of $M_0$ in order to reach zero temperature, and thus the critical black hole state, as can be seen from figures 2 and 3(b).

3. Black hole solution for the Einstein–Maxwell case

In this section, a black hole solution for the Einstein–Maxwell case will be constructed without taking into account the cosmological term ($\Lambda_k = 0$). In this case the equations of motion for the metric field (12) simplify to

$$G_{\mu\nu} = -\Delta f_{\mu\nu} + \frac{8\pi G_k}{e_k^2} T_{\mu\nu},$$

while equations of motion (15) for the $U(1)$ gauge field remain unchanged

$$D_{\mu}\left(\frac{1}{e_k^2} F^{\mu\nu}\right) = 0.$$  \hspace{1cm} (45)

The equations of motion for the scale field $k$ (16) simplify to

$$\left[ R\nabla_{\mu}\left(\frac{1}{G_k}\right) - \nabla_{\mu}\left(\frac{4\pi}{e_k^2}\right) F_{\alpha\beta} F^{\alpha\beta}\right] \cdot (\partial^\nu k) = 0.$$ \hspace{1cm} (46)

The invariance equations (17) and (18) remain unchanged.

3.1. Finding the solution

When searching for a solution of the above equations we proceed by imposing spherical symmetry. For spherical symmetry, the most general line element is again (21). Assuming electric and nonmagnetic charge, this symmetry requirement reduces the degrees of freedom of the electromagnetic stress–energy tensor to

$$F_{\mu\nu} = -F_{\mu\nu} = q(r).$$ \hspace{1cm} (47)

Under these assumptions and for a given scale dependence $G_k$ and $1/e_k^2$ the system (44)–(46) contains four unknown functions $f(r), h(r), q(r),$ and $k(r)$.

Now, the reasoning of section 2 will be repeated and the ignorance (at least model dependence) of the coupling flow will be encoded in trading the radial scale dependence $k = k(r)$ for radial coupling dependence $k(r) \rightarrow G(r) \rightarrow 1/e^2(r)$. The increase in unknown functions will be compensated by imposing the ‘standard black hole’ condition (22). Due to this, the system (44)–(46) will have to be solved for the four functions $f(r), q(r), G(r),$ and $1/e^2(r)$. The ansatz for the line element will be

$$ds^2 = -f(r)dr^2 + \frac{1}{f(r)}dr^2 + r^2d\Omega_2^2.$$ \hspace{1cm} (48)

Note that here, in contrast to (23), the constant $c_1$ is set to one, since due to the $F_{\mu\nu}$ contribution one cannot expect to have time-rescaling invariance of the solution.

A good starting point for solving the system is to observe that $f(r)$ actually decouples from the equations for the radial electric field.
\[
\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{q(r)}{e^2(r)} \right) = 0. \tag{49}
\]

This establishes a first relation between \( e(r) \) and \( q(r) \). With this, the remaining equations of motion (44 and 46) are solved by

\[
\begin{align*}
G(r) &= \frac{G_0}{\epsilon r + 1}, \\
\epsilon^2(r) &= \frac{G_0 r^2 e_0^2}{Q_0^2 G_0 (\epsilon r + 1)^3} \\
q(r) &= \frac{Q_0 e^2(r)}{4\pi e_0^2 r^2}. \tag{50}
\end{align*}
\]

Having learned the lesson from section 2 on the subtleties of choosing the constants of integration, the five constants of integration were chosen such that, for the case that the fifth constant vanishes, the four other constants correspond to constants in the classical solution of \( f(r)_{|_{\text{lev}}} \). This allows us to interpret this fifth constant, which again will be labeled \( \epsilon \), as deviation parameter which introduces corrections to the classical solution due to scale dependence. One confirms for Newton’s coupling

\[
\lim_{\epsilon \to 0} G(r) = G_0, \tag{51}
\]

and for the metric function

\[
\lim_{\epsilon \to 0} f(r) = 1 - \frac{2G_0 M_0}{r} + \frac{4\pi G_0 Q_0^2}{r^2 e_0^2}, \tag{52}
\]

which reproduces the classical solution (7). Similarly, one finds for the scale-dependent electrical coupling

\[
\lim_{\epsilon \to 0} \epsilon^2(r) = (4\pi)^2 e_0^2 \tag{53}
\]

and the electrical field strength

\[
\lim_{\epsilon \to 0} q(r) = \frac{4\pi Q_0}{r^2}. \tag{54}
\]

One observes that this is also in agreement with the classical result. The factor \((4\pi)\), that appears as different normalization of \( q(r) \) is a convention which turns out to cancel for the ratios \( q^2(r)/\epsilon^2(r) \) that enter the equations of motion (44 and 46). This convention is further justified when calculating the actual charge of the solution (50). Taking again \( \epsilon \) smaller than any dimensionful scale of the system, one can expand in this parameter. In this expansion the lowest order corrections to the classical solution are
Fortunately, due to the purely polynomial form of the solution, most of the following results can be discussed with the complete solution (50), without the necessity to use the above expansion.

3.2. Asymptotic behavior of the solution

The asymptotic behavior of this generalized Reissner–Nordström solution for \( r \to 0 \) can be studied by evaluating curvature invariants in this limit. For example, already the Ricci scalar shows a quadratic divergence for small radii

\[
R = -6G_0 \frac{M_0 \epsilon}{r^2} + \frac{4\pi Q_0^2 \epsilon^2 e_0^{-2}}{r^2} + \mathcal{O}(1/r).
\]

This is in contrast to the classical solution where the Ricci scalar vanishes in this limit.

Still, the generalized solution incorporates the classical result, since in the classical limit \( \epsilon \to 0 \), the right-hand side of (56) vanishes accordingly. The invariant contraction of two Riemann tensors is also divergent in this limit, but for this invariant, the leading divergence agrees with the classical behavior

\[
R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = 2^7 \frac{G_0^2 \pi^2 Q_0^4}{e_0^4 r^8} + \mathcal{O}(1/r^7).
\]

This confirms that this solution is singular at the origin for the generalized solution, even in the classical limit.

Taking the opposite limit (for \( r \to \infty \)), the asymptotic behavior provides a surprise, since the metric function approaches

\[
f(r) = 1 + \frac{1}{2\epsilon r} + \mathcal{O}(1/r^2).
\]

This result does not resemble the classically expected value 1, not even by approaching \textit{a posteriori} \( \epsilon \to 0 \). The supposed discrepancy can be explained by the fact that all dimensionless terms \( \epsilon r \) are incompatible with first taking the limit of large \( r \) and then the limit of small \( \epsilon \). Clearly, if one takes the limit of \( \epsilon \to 0 \) first, the classical result is recovered. The asymptotic line element corresponding to (58) is

\[
ds^2_{\infty} = -1 -\frac{4}{d^2} + 4dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2.
\]

One can try to cast this in a more familiar form by introducing a rescaled time \( \tau = \frac{1}{2}t \) and a rescaled radial coordinate \( R = 2r \), giving the line element
However, even though now the radial and temporal part of the line element take the familiar form, the angular part suffers a non-trivial scaling, which corresponds to a deficit solid angle. For example, the area of a sphere with very large radius $R$ is not $4\pi R^2$ but rather $R^2\pi$. The remaining factor of $1/4$ might be absorbed in a rescaling of the angles $\theta$ and $\phi$, but this clearly also implies the mentioned deficit angle of the asymptotic geometry.

A very similar asymptotic behavior for very large radial distance is known from so-called global monopoles \cite{52, 53}. Even though the above solution is to our knowledge not present in the literature, its asymptotic behavior for large $r$ can be matched to the monopole in \cite{52} by identifying $G_{3,4}^{1,8}/\pi$ with the monopole parameter $\eta$, and $G_{1,4}^{3,-} \cdot$ with the mass parameter of the monopole $M$. However, for the presented solution, negative values for $\epsilon$ do not allow a well-defined classical limit and, therefore, the asymptotic results discussed here do not apply for the global monopoles in \cite{52} and vice versa.

In order to get an intuition on the radial dependence of the radial function $f(r)$ and the corresponding asymptotic behavior one can also refer to a graphical analysis, which is done in figure 4. One observes that corrections to the black-dashed classical curve become more and more prominent for large radii and increasing $\epsilon$. One also observes that for small $\epsilon$, the metric function $f$ seems to approach the classical value of one, before converging to the limit expressed in \cite{58}, far outside of the shown region.

### 3.3. Total charge and running $\epsilon$

As written in the solution \cite{50}, $Q_0$ is nothing more than an integration parameter which inherits its interpretation as charge due to this classical limit $\epsilon \to 0$. This does not determine the actual charge of the solution for values of $\epsilon \approx 0$. Given the asymptotic deficit angle \cite{59}, one might expect a corresponding effect for the charge. In curved space-time the actual charge corresponding to the Maxwell equation \cite{15} can be evaluated by the integral \cite{51}

$$Q = \int_{\partial \Sigma} d^2z \sqrt{\gamma^\Sigma} n^\nu \sigma^\mu F_{\rho \sigma}^{\mu \nu}/\epsilon^2,$$

(61)
where \( n_p \) is the unitary vector associated to the time coordinate and \( \sigma_r \) is the unitary vector associated to the radial coordinate. The integral over the surface \( \partial S \) will be evaluated at radial infinity such that, according to the asymptotic metric (58), the two-dimensional surface element is

\[
d s^2 = \left( g^{\phi \phi} \right) \, d\phi \, d\rho^2 \sin(\theta),
\]

with \( \theta : 0 \ldots \pi \) and \( \phi : 0 \ldots 2\pi \). From the same asymptotic metric (58) one reads that the properly normalized unitary vector time vector is

\[
n_t = (2, 0, 0, 0)
\]

and that the properly normalized radial vector is

\[
\sigma_r = (0, 1/2, 0, 0).
\]

The function associated to the electric field of the solution is

\[
\frac{F^{tr}}{e^2} = \frac{Q_0}{4\pi e_0^2} \frac{1}{r^2},
\]

which is simply proportional to \( 1/r^2 \), as it can also be read directly from the relation (49). Putting (61)–(65) together, one finds that all unusual factors of the generalized solution and the corresponding asymptotic metric (58) cancel out. The charge is

\[
Q = \frac{Q_0}{e_0^2},
\]

which just resembles the classical value and has no \( \epsilon \) dependence. This is actually a virtue of the ansatz \( g_{tt} = 1/g_{rr} \) and is also true for any radius, not just \( r \to \infty \).

It is tempting to interpret \( e^2(r) \), and \( G(r) \) as a scale-dependent coupling constant in analogy to the treatment of \( G(r) \) and \( \Lambda(r) \) in [43]. Straightforwardly, one can simply plot the radial dependence of the two functions, which can be seen in figure 5. One observes that for \( \epsilon \to 0 \) the constant values are recovered, whereas for non-vanishing \( \epsilon \) deviations occur. One further observes that these modifications appear for large \( r \) rather than for small \( r \), as was to

\footnote{Please note that the calculation of the total black hole mass \( M \) in terms of the classical mass parameter \( M_0 \) is by far less straightforward and will be postponed to future studies.}
be expected from the expansion (55). This long-range scale dependence, instead of the short-range scale dependence is in complete analogy to the same effect in the Einstein–Hilbert case discussed in section 2.2 and would probably be complemented by a short-range correction if one would find a way to obtain non-trivial solutions without the ansatz (22).

3.4. Horizons, temperature, cosmic censorship

Important information on a black hole solution can be gained by studying its horizon structure and the corresponding thermodynamic behavior. The possible horizons, which correspond to zeros of \( f(r) \) in (50) are found to be

\[
\begin{align*}
    r_1 &= -1 - \frac{1 + 2\varepsilon G_0 M_0 - 2\varepsilon \sqrt{G_0^2 M_0^2 - \frac{4G_0 \pi Q_0^2}{\varepsilon_0^2}}}{\varepsilon}, \\
    r_2 &= -1 + \frac{1 + 2\varepsilon G_0 M_0 - 2\varepsilon \sqrt{G_0^2 M_0^2 - \frac{4G_0 \pi Q_0^2}{\varepsilon_0^2}}}{\varepsilon}, \\
    r_3 &= -1 - \frac{1 + 2\varepsilon G_0 M_0 + 2\varepsilon \sqrt{G_0^2 M_0^2 - \frac{4G_0 \pi Q_0^2}{\varepsilon_0^2}}}{\varepsilon}, \\
    r_4 &= -1 + \frac{1 + 2\varepsilon G_0 M_0 + 2\varepsilon \sqrt{G_0^2 M_0^2 - \frac{4G_0 \pi Q_0^2}{\varepsilon_0^2}}}{\varepsilon}.
\end{align*}
\]

(67)

(68)

(69)

(70)

In the analysis of these horizons we will restrict ourselves to the case of \( \varepsilon > 0 \), since it is this case that allows the transition to the classical values for \( \varepsilon \to 0 \). One sees that \( r_1 \) and \( r_3 \) are always negative for positive \( \varepsilon \). Thus, one defines the remaining horizons

\[
    r_5 = -1 + \frac{1 + 2\varepsilon G_0 M_0 \pm 2\varepsilon \sqrt{G_0^2 M_0^2 - \frac{4G_0 \pi Q_0^2}{\varepsilon_0^2}}}{\varepsilon}.
\]

(71)

One confirms that these two horizons coincide with the classical horizons (10) for \( \varepsilon \to 0^+ \). For vanishing charge \( (Q_0 \to 0) \), \( r_5 \) goes to zero and the remaining horizon is

\[
    r_5 \bigg|_{Q_0=0} = \frac{-1 + \sqrt{1 + 4\varepsilon G_0 M_0}}{\varepsilon},
\]

(72)

which gives in the classical limit the expected Schwarzschild value of \( 2G_0 M_0 \).

As was seen in the asymptotic limit of small radii (57), the singularity at zero radius persists for the generalized solution. Therefore, one still needs to invoke the cosmic censorship hypothesis in order to avoid the ‘visibility’ of this naked singularity. For the case of the present solution, this hypothesis can be addressed by studying the critical value for which the inner and outer horizons merge, which is in the classical case given for (11). For the generalized solution the merging occurs when the inner square root in (71) vanishes. This is true for the critical value
which is exactly the classical (ε independent) value given in (11). There might be the possibility of merging horizons other than \( r_+ \), but this possibility can be ignored since one already knows that \( r_+ - r_- \leq 0 \), which at most would allow merging horizons at the origin. The behavior of the physical horizons (71) and their merging at the classical horizon value is shown in figure 6. One observes clearly that the mass-value of the critical horizon is independent of the value of \( \varepsilon \). One further sees that larger values of \( \varepsilon \) tend to suppress mostly the outer radius \( r_+ \) whereas the inner radius \( r_- \) experiences only modest changes. The topology of this horizon structure determines the causal structure of the solution, which for small and positive \( \varepsilon \) is simply the topology as of the usual Reissner–Nordström solution with the corresponding Penrose diagrams [54, 55].

This outer horizon is responsible for the thermodynamic behavior of the black hole. Imposing regularity around this point one obtains the standard temperature by

\[
T = \frac{\partial f (r)}{4\pi r_+}.
\]

Evaluating this for the solution (50) one obtains

\[
T = \varepsilon^2 \left( 2\varepsilon G_0 M_0 \left( \frac{G_0 M_0^2 - 4\pi Q_0^2}{\varepsilon^2} + G_0 M_0 \right) + \frac{\sqrt{\left( G_0 M_0^2 - 4\pi Q_0^2 \right)^2 + 4\varepsilon G_0 M_0^2}}{\varepsilon^2} - 8\pi \varepsilon G_0 Q_0^2 \right) \left( 2\varepsilon \left( \frac{G_0 M_0^2 - 4\pi Q_0^2}{\varepsilon^2} + G_0 M_0 \right) + 1 \right)^{1/2} \left( 2\varepsilon \left( \frac{G_0 M_0^2 - 4\pi Q_0^2}{\varepsilon^2} + G_0 M_0 \right) + 1 \right)^{1/2}.
\]

This temperature in function of the mass parameter \( M_0 \) is shown in figure 7. In order to get somewhat more analytical insight into this cumbersome expression one can expand it for small values of \( \varepsilon \).
The first term of this expansion corresponds to the expected classical limit, which corresponds to the black dotted line in figure 7. The second term of this expansion turns out to not be linear in the expansion parameter but rather to order $\epsilon^2$, indicating that corrections to the classical temperature tend to be suppressed for small $\epsilon$. One further sees from the second term in (76), that first corrections to the temperature are expected to be positive. For a given $G_0, M_0, Q_0$, and $e_0$ this means that the classical temperature is the minimal temperature found under a variation of $\epsilon$. These observations can be readily confirmed by the behavior of the curves in figure 7.

4. Summary and conclusion

This paper presents and studies two black hole solutions of the Einstein–Hilbert and Einstein–Maxwell equations, generalized to the case of scale-dependent couplings. The usual ambiguity due to model dependence of the functional form of these couplings ($\{G_k, \Lambda_k\}$ and...
\{G_{\mu\nu} \equiv 1/e_3^2 \} \text{ respectively) is circumvented by promoting the couplings to fields in the equations of motion \{(G(r), \Lambda(r)) \text{ and } (G(r), 1/e(r)^2) \text{ respectively). The resulting mismatch between unknown functions and independent equations of motion is absorbed by taking the common ansatz for the spherically symmetric metric } g_{00} \sim 1/g_{11}, \text{ which can be motivated by the known form of the classical solutions (solutions with scale-independent couplings).

The findings for the generalized solution of the Einstein–Hilbert case (24)–(26) are:

- The solution presents two additional arbitrary constants with respect to the classical (Anti)-de Sitter-Schwarzschild black hole ($\epsilon$ and $c_3$). These constants can be chosen such that they produce a well-behaved classical limit in the sense that one of the additional constants of integration ($\epsilon$) parametrizes deviations from the classical solution. This implies that in the limit of $\epsilon \to 0$ the classical (Anti)-de Sitter-Schwarzschild black hole is recovered.
- The asymptotic behavior for small radial coordinate shows that the classical singularity persists for the generalized solution.
- The asymptotic behavior for large radial coordinate shows that two of the additional constants of the generalized solution ($c_4$ and $\epsilon$) can produce a shifted value of the classical value of the cosmological constant $\Lambda \to \tilde{\Lambda}$. The shift disappears for $c_4 = 1/\epsilon$.
- The numerical and perturbative study of the horizons of the solution reveals that the scale-dependence parameter $\epsilon$ tends to reduce the value of the inner and of the cosmological horizon. For the de Sitter case, the mass value of the extremal black hole tends to increase with $\epsilon$.
- The numerical and perturbative study of the radiation behavior of the generalized solution reveals that the scale dependence produces a slight increase in the temperature, which is only of relative importance for the largest mass values close to the critical value.

The findings for the generalized solution of the Einstein–Maxwell case (50) are:

- The generalized solution has one additional constant of integration with respect to the classical Reissner–Nordström black hole. The additional constant of integration (again labeled $\epsilon$) can be chosen such that it allows us to recover the classical result for the limit $\epsilon \to 0$.
- The asymptotic behavior for small radii reveals that the singularity at the radial origin persists (actually it becomes even more visible since $R = 0$).
- The asymptotic behavior for large radii shows that the solution does approach a cone-like asymptotic space-time, similar (but not identical) to the asymptotics of known black hole monopoles [52, 53]. By integrating Gauss’s law for this asymptotic (cone-like) space-time one observes that the effects of the monopole cancel and the resulting charge resembles the classical charge parameter $Q = Q_0/e_3^2$, independently of the value of $\epsilon$.
- The study of the horizon structure of the generalized solution can be performed exactly without the need of an expansion in $\epsilon$. One finds that the two physical horizons are shifted towards smaller values with respect to the two classical horizons. This shift turns out to be $\epsilon$ dependent and more important for the outer horizon $r_+$. Surprisingly this $\epsilon$ dependence cancels out when one evaluates the critical black hole mass (73), implying that the classical ‘cosmic censorship’ relation remains unchanged, independent of the scale-dependence parameter $\epsilon$.
- The thermodynamic behavior of the generalized solution is calculated, showing that the scale-dependence parameter produces a slight relative increase of the temperature with
with respect to the classical solution. This observation is confirmed by a numerical and a perturbative analysis.

An important observation, which is valid for both cases (24)–(26) and (50), is that the covariant form of the improved equations, the assumption of scale-dependent couplings, and the ansatz (22) imply necessarily long-range corrections to the usual scale-independent solution. Thus, in order to (maybe) find short-range corrections one would have to drop at least one (or all) of these three assumptions.

In summary, the analysis of the presented solutions reveals that scale dependence of the couplings (Gr, r), (\(L_r\)) and (Gr, er) respectively, cannot be expected to resolve the problem of singularities at the origin, but it can produce important effects on the asymptotic space-time resulting either in a modified cosmological constant (35) or in an asymptotic monopole (59). Effects on the critical masses and thermodynamic behavior are either rather mild or even absent as in the case of (73).

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