Thorup-Zwick Emulators are Universally Optimal Hopsets*

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Abstract

A \((\beta, \epsilon)\)-hopset is, informally, a weighted edge set that, when added to a graph, allows one to get from point \(a\) to point \(b\) using a path with at most \(\beta\) edges (“hops”) and length \((1 + \epsilon) \text{dist}(a,b)\). In this paper we observe that Thorup and Zwick’s sublinear additive emulators are also actually \((O(k/\epsilon)^k, \epsilon)\)-hopsets for every \(\epsilon > 0\), and that with a small change to the Thorup-Zwick construction, the size of the hopset can be made \(O(n^{1 + \frac{1}{k+1}})\). As corollaries, we also shave “\(k\)” factors off the size of Thorup and Zwick’s \([20]\) sublinear additive emulators and the sparsest known \((1 + \epsilon, O(k/\epsilon)^{k-1})\)-spanners, due to Abboud, Bodwin, and Pettie \([1]\).

1 Introduction

Let \(G = (V, E, w)\) be a weighted undirected graph. Define \(\text{dist}_G^{(\beta)}(u,v)\) to be the length of the shortest path from \(u\) to \(v\) in \(G\) that uses at most \(\beta\) edges, or “hops.” Whereas \(\text{dist}_G = \text{dist}_G^{(\infty)}\) is a metric, \(\text{dist}_G^{(\beta)}\) is not in general. A set \(H \subseteq \binom{V}{2}\) of weighted edges is called a \((\beta, \epsilon)\)-hopset if for every \(u, v \in V\),

\[
\text{dist}_G(u, v) \leq \text{dist}_G^{(\beta)}(u,v) \leq (1 + \epsilon) \text{dist}_G(u, v).
\]

Background.  Cohen \([7]\) formally defined the notion of a hopset, but the idea was latent in earlier work \([21, 14, 6, 18]\). Cohen’s \((\beta, \epsilon)\)-hopset had size \(O(n^{1+1/\kappa} \log n)\) and \(\beta = (\epsilon^{-1} \log n)^{O(\log \kappa)}\). Elkin and Neiman \([9]\) showed that a constant hopbound \(\beta\) suffices (when \(\kappa, \epsilon\) are constants). In particular, their hopset has size \(O(n^{1+1/\kappa} \log n \log \kappa)\) and \(\beta = O(\epsilon^{-1} \log \kappa)^{\log \kappa}\). Abboud, Bodwin, and Pettie \([1]\) recently proved that the tradeoffs of \([9]\) are essentially optimal: for any integer \(k\), any hopset of size \(n^{1+\frac{1}{k+1}}\) must have \(\beta = \Omega((c_k/\epsilon)^{k+1})\), where \(c_k\) is a constant depending only on \(k\). \(^1\) There are other constructions of hopsets \([9, 11, 12, 19]\) that are designed for parallel or dynamic environments; their tradeoffs (between hopset size and hopbound) are worse than \([7, 9]\) and the ones presented here. See Table \([3]\).

Hopsets, Emulators, and Spanners.  Recall that \(G\) is an undirected graph, possibly weighted. A spanner is a subgraph of \(G\) such that \(\text{dist}_H(u,v) \leq f(\text{dist}_G(u,v))\) for some nondecreasing stretch function \(f\). An emulator of an unweighted graph \(G\) is a weighted edge set \(H\) such that \(\text{dist}_H(u,v) \in \left[ \text{dist}_G(u,v), f(\text{dist}_G(u,v)) \right]\). Syntactically, the definition of hopsets is closely related to emulators. The difference is that hopsets have a hopbound constraint but are allowed to use original edges in \(G\) whereas emulators must use only \(H\). The purpose of emulators is to compress the graph metric \(\text{dist}_G^\beta\): ideally \(|H| \ll |E(G)|\). Historically, the literature on hopset constructions \([7, 9]\) has been noticeably more complex than those of spanners and emulators, many

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\(^1\)Note that setting \(\kappa = 2^{k+1} - 1\) in the Elkin-Neiman construction gives \(\beta = O(k/\epsilon)^k\), where \(\log \kappa = |\log \kappa| = k\). Thus, saving any \(\delta\) in the exponent of the hopset increases \(\beta\) significantly. In general, the statement of \([9]\) obscures the nature of the tradeoff: there are not distinct tradeoffs for each \(\kappa \in \{1, 2, 3, \ldots\}\), but only for \(\kappa \in \{1, 3, 7, \ldots, 2^{k+1} - 1, \ldots\}\).
of which [9, 2, 8, 20, 4, 15, 1] are quite elegant. Our goal in this work is to demonstrate that there is nothing intrinsically complex about hopsets, and that a very simple construction improves on all prior constructions and matches the Abboud-Bodwin-Pettie lower bound.

New Results. Thorup and Zwick [20] designed their emulator for unweighted graphs, and proved that it has size $O(kn^{1+\frac{1}{2k-1}})$ and a sublinear additive stretch function $f(d) = d + O(kd^{1-1/k})$. In this paper we show that the Thorup-Zwick emulator, when applied to a weighted graph, produces a $(\beta, \epsilon)$-hopset that achieves every point on the Abboud-Bodwin-Pettie [11] lower bound tradeoff curve. Moreover, with two subtle modifications to the construction, we can reduce the size to $O(n^{1+\frac{1}{2k+1-\delta}})$, shaving off a factor $k$. Our technique also applies to other constructions, and as corollaries we improve the size of Thorup and Zwick’s emulator [20] and Abboud, Bodwin, and Pettie’s $(1+\epsilon, \beta)$-spanners.\(^2\)

**Theorem 1.** Fix any weighted graph $G$ and integer $k \geq 1$. There is a $(\beta, \epsilon)$-hopset for $G$ with size $O(n^{1+\frac{1}{2k+1-\delta}})$ and $\beta = 2\left(\frac{(4+o(1))k}{\epsilon}\right)^k$.

**Theorem 2.** (cf. [20]) Fix any unweighted graph $G$ and integer $k \geq 1$. There is a sublinear additive emulator $H$ for $G$ with size $O(n^{1+\frac{1}{2k+1-\delta}})$ and stretch function $f(d) = d + (4 + o(1))kd^{1-1/k}$.

**Theorem 3.** (cf. [11]) Fix any unweighted graph $G$, integer $k \geq 1$, and real $\epsilon > 0$. There is a $(1+\epsilon, (4 + o(1))k/\epsilon^{k-1})$-spanner $H$ for $G$ with size $O((k/\epsilon)^h n^{1+\frac{1}{2k+1-\delta}})$, where $h = \frac{3^{2^{k-1}-(k+2)-1}}{2^{2k+1-1}} < 3/4$.

**Remark 1.** In recent and independent technical report, Elkin and Neiman [10] also observed that Thorup and Zwick’s emulator yields an essentially optimal hopset. They proposed a modification to Thorup and Zwick’s construction that reduces the size to $O(n^{1+\frac{1}{2k+1-\delta}})$ (eliminating a factor $k$), but increases the hopbound $\beta$ from $O(k/\epsilon)^k$ to $O((k+1)/\epsilon)^{k+1}$. For example, their technique does not imply any of the improvements found in Theorems [11, 2] or 3.

### 2 The Hopset Construction

In this section, we present the construction of the hopset based on Thorup and Zwick’s emulator [20], then analyze its size, stretch, and hopbound.

The construction is parameterized by an integer $k \geq 1$ and a set $\{q_i\}$ of sampling probabilities. Let $V = V_0 \supseteq V_1 \supseteq V_2 \supseteq \cdots \supseteq V_k \supseteq V_{k+1} = \emptyset$ be the vertex sets in each layer. For each $i \in [0, k)$, each vertex in $V_i$ is independently promoted to $V_{i+1}$ with probability $q_{i+1}/q_i$. Thus $E[|V_i|] = nq_i$. For each vertex $v \in V$

\(^2\)A $(1+\epsilon, \beta)$-spanner of an unweighted graph is one with stretch function $f(d) = (1+\epsilon)d + \beta$. 

| Authors | Size | Hopbound | Stretch |
|---------|------|----------|---------|
| Klein and Subramanian [14] | $O(n)$ | $O(\sqrt{n} \log n)$ | 1 |
| Thorup and Zwick [19] | $O(kn^{1+1/\kappa})$ | $2$ | $2\kappa - 1$ |
| Cohen [7] | $O(n^{1+\frac{1}{2-\log n}})$ | $((\log n)/\epsilon)^{O(\log \kappa)}$ | $1 + \epsilon$ |
| Elkin and Neiman [9] | $O(n^{1+\frac{1}{2-\log n}})$ | $O((\log \kappa)/\epsilon)^{\log \kappa}$ | $1 + \epsilon$ |
| Abboud, Bodwin, and Pettie [11] | $n^{1+\frac{1}{2k+1-\delta}}$ | $\Omega(c_k/\epsilon^{k+1})$ | $1 + \epsilon$ |
| **New** | $O\left(n^{1+\frac{1}{2k+1-\delta}}\right)$ | $O(k/\epsilon)^k$ | $1 + \epsilon$ |

Table 1: Tradeoffs between size and hopbound of previous hopsets. Fix the parameter $\kappa = 2^{k+1} - 1$ to compare [11, 9] against the lower bound [11] and the new result.
and \( i \in [1,k] \), define \( p_i(v) \) to be any vertex in \( V_i \) such that \( \text{dist}_G(v,p_i(v)) = \text{dist}_G(v,V_i) \). For any vertex \( v \in V_i \setminus V_{i+1} \), define \( B(v) \) to be:

\[
B(v) = \{ u \in V_i \mid \text{dist}_G(v,u) < \text{dist}_G(v,p_{i+1}(v)) \}
\]

Note that \( p_{k+1}(v) \) does not exist; by convention \( \text{dist}_G(v,p_{k+1}(v)) = \infty \). The hopset is defined to be \( H = E_0 \cup E_1 \cup \cdots \cup E_k \), where

\[
E_i = \bigcup_{v \in V_i \setminus V_{i+1}} \{(v,u) \mid u \in B(v) \cup \{p_{i+1}(v)\}\}.
\]

The length of an edge in \( H \) is always the distance between its endpoints. This concludes the description of the construction.

### 2.1 Size Analysis

The expected size of \( E_i \) is at most \( \mathbb{E}[|V_i|](q_i/q_{i+1}) = nq_i^2/q_{i+1} \), for each \( i \in [0,k] \), and is \((nq_k)^2\) if \( i = k \). Following Pettie [17], we choose \( \{q_i\} \) such that the layers of the hopset have geometrically decaying sizes.

Setting \( q_i = n^{-2^{i+1}-1} \cdot 2^{-2^i} \cdot 2^{-i+1} \), the expected size of \( E_i \), for \( i \in [0,k] \), is

\[
(nq_i)^2 = n^2 \left( n^{-2^{i+1}-1} \cdot 2^{-2^i} \right)^2 \leq n^{-1} \cdot 2^{1+2} = n^2 \cdot 2^{-1+2}.
\]

The expected size of \( E_k \) is

\[
(nq_k)^2 = n^2 \left( n^{-2^k-i} \cdot 2^{-2^i} \right)^2 \leq n^{1+2} \cdot 2^{-k+2}.
\]

so the expected size of \( H \) is at most

\[
\sum_{i=0}^{k} \mathbb{E}[|E_i|] \leq n^{1+2} \left( \sum_{i=0}^{k} 2^{-i+2} \right) = O(n^{1+2}).
\]

### 2.2 Stretch and Hopbound Analysis

Let us first give an informal sketch of the analysis. Let \( a, b \) be vertices. Choose an integer \( r \geq 2 \), and imagine dividing up the shortest \( a-b \) path into \( r^k \) intervals of length \( \mu = \text{dist}_G(a,b)/r^k \), where \( \mu \) defines one “unit” of length. Once \( r \) and \( \mu \) are fixed we prove that given any two vertices \( u, v \) at distance at most \( r^i \mu \), there is \( \text{either an} \ h_i\)-hop path \( a \) to \( v \) with additive stretch \( O(ir^{i-1}) \cdot \mu \), or there is an \( h_i\)-hop path from \( u \) to \( v \) with additive stretch \( O(ir^{i-1}) \cdot \mu \). Of course, when \( i = k \) the set \( V_{k+1} = \emptyset \) is empty, so we cannot be in the second case. Since, by definition of \( \mu \), \( \text{dist}_G(a,b) \leq r^k \mu \), there must be an \( h_k\)-hop path with additive stretch \( O(kr^{k-1}) \cdot \mu \). In order for this stretch to be \( \epsilon \text{dist}_G(a,b) \) we must set \( r = \Theta(k/\epsilon) \).

So, to recap, the integer parameter \( r = \Theta(k/\epsilon) \) depends on the desired stretch \( \epsilon \), and \( r \) determines the hopcount sequence \((h_i)\), which is defined inductively as follows.

\[
\begin{align*}
h_0 &= 1, \\
h_i &= (r+1)h_{i-1} + r \\
\end{align*}
\]

for \( i \in [1,k] \).

The parameter \( \beta \) of the hopset is exactly \( h_k \). It is straightforward to show that \( h_k < 2(r+1)^k \). Once \( r \) and \((h_i)\) are fixed, Theorem [4] is proved by induction.
Theorem 4. For any fixed real \( \mu \) (the “unit”), for all \( i \in [0, k] \) and any pair \( u, v \in V \) such that \( \text{dist}_G(u, v) \leq r^i \mu \), at least one of the following statements holds.

\[ \begin{align*}
(i) & \quad \text{dist}^{(h_i)}_{G \cup H}(u,v) \leq \text{dist}_G(u,v) + ((r+4)^i - r^i)\mu, \\
(ii) & \quad \text{There exists } u_{i+1} \in V_{i+1} \text{ such that } \text{dist}^{(h_i)}_{G \cup H}(u,u_{i+1}) \leq (r+4)^i \mu.
\end{align*} \]

Proof. The proof is by induction on \( i \). In the base case \( i = 0 \) and \( h_0 = 1 \). Let \( u, v \in V \) with \( \text{dist}_G(u, v) \leq r^0 \mu = \mu \). If \( (u, v) \in H \) then \( \text{dist}^{(1)}_{G \cup H}(u,v) = \text{dist}_G(u, v) \) so (i) holds. Otherwise, \( (u, v) \notin H \), meaning \( v \notin B(u) \). If \( u \in V_0 \setminus V_1 \) then \( \text{dist}^{(1)}_{G \cup H}(u,p_1(u)) \leq \text{dist}_G(u,v) \leq \mu \), and if \( u \in V_1 \) then \( p_1(u) = u \), so \( \text{dist}^{(1)}_{G \cup H}(u,p_1(u)) = 0 \). In each case, (ii) holds.

Now assume \( i > 0 \). Consider vertices \( u, v \in V \) with \( \text{dist}_G(u, v) \leq r^i \mu \) and let \( P \) be a shortest \( u-v \) path in \( G \). Then, as shown in Figure 1 we partition \( P \) into at most \( 2r-1 \) segments \( (u_0 = u, u_1), (u_1, u_2), \ldots, (u_{2r-1}, u_r = v) \) as follows. Starting at \( u_0 = u \), we pick \( u_1 \) to be the farthest vertex on \( P \) such that \( \text{dist}_G(u_0, u_1) \leq r^i-1 \mu \), and let \( (u_1, u_2) \) be the next edge on the path. Repeat the process until we reach \( u_r = v \), oscillating between selecting segments that have length at most \( r^i-1 \mu \) and single edges.

- **Multi-hop segment:** the shortest path from \( u_s \) to \( u_{s+1} \) satisfies \( \text{dist}_G(u_s,u_{s+1}) \leq r^{i-1} \mu \).
- **Single-hop segment:** the segment is actually an edge \( (u_s, u_{s+1}) \in E \).

By the induction hypothesis, each multi-hop segment satisfies (i) or (ii) within \( h_{i-1} \) hops. Moreover, in each greedy iteration the sum of the lengths from picked multi-hop segment and immediately followed single-hop segment is strictly greater than \( r^i \mu \) except the last one. Therefore, by the pigeonhole principle, there are at most \( r \) multi-hop segments on \( P \) and at most \( r-1 \) single-hop segments on \( P \).

If condition (i) holds for all multi-hop segments, then in at most \( rh_{i-1} + r - 1 \leq h_i \) hops,

\[
\begin{align*}
\text{dist}^{(h_i)}_{G \cup H}(u,v) & \leq \text{dist}_G(u,v) + r((r+4)^i - r^{i-1})\mu \\
& \leq \text{dist}_G(u,v) + ((r+4)^i - r^i)\mu,
\end{align*}
\]

\[\text{Note that if the first edge has length more than } r^{i-1} \mu, \text{ then } u_1 = u_0.\]
and condition (i) holds for $P$

Otherwise, condition (i) does not hold for at least one multi-hop segment. Consider the first multi-hop segment $(u_{j_1},u_{j_1+1})$ and the last multi-hop segment $(u_{j_2-1},u_{j_2})$ that do not satisfy condition (i). By condition (ii), there exist $u'$ and $v' \in V_i$ satisfying

$$\text{dist}_{G_{i+1}}(u_{j_1},u') \leq (r+4)^{i-1} \mu$$

$$\text{dist}_{G_{i+1}}(u_{j_2},v') \leq (r+4)^{i-1} \mu.$$  

Now we have two cases depending on whether $(u',v') \in H$ or not. If $(u',v') \in H$, then by the triangle inequality, we can get from $u_{j_1}$ to $u_{j_2}$ with $2h_{i-1}+1$ hops and additive stretch

$$\text{dist}_{G_{i+1}}(u_{j_1},u_{j_2}) - \text{dist}_{G}(u_{j_1},u_{j_2}) \leq \text{dist}_{G_{i+1}}(u_{j_1},u') + \text{dist}_{G}(u',u_{j_2}) - \text{dist}_{G}(u_{j_1},u_{j_2})$$

$$\leq 2 \text{dist}_{G_{i+1}}(u_{j_1},u') + 2 \text{dist}_{G_{i+1}}(u',u_{j_2})$$

$$\leq 4(r+4)^{i-1} \mu.$$  

We know there are a total of at most $r-1$ multi-hop segments satisfying condition (i). Hence, within at most $(r-1)h_{i-1} + r - 1 + 2h_{i-1} + 1 \leq h_i$ hops, we can get from $u$ to $v$ with additive stretch

$$\text{dist}_{G_{i}}(u,v) \leq (r-1)((r+4)^{i-1} - r^{i-1}) \mu + \text{dist}_{G_{i+1}}(u_{j_1},u_{j_2})$$

$$\leq [(r-1)((r+4)^{i-1} - r^{i-1}) + 4(r+4)^{i-1}] \mu$$

$$\leq (r+3)(r+4)^{i-1} - r^i + r^{i-1}] \mu$$

$$\leq (r+4)^{i-1} \mu$$

and condition (i) holds for $P$ in this case.

On the other hand, suppose that $(u',v') \notin H$. Since both $u', v' \in V_i$ but $(u',v') \notin H$, we know that $u'' = p_{i+1}(u') \in V_{i+1}$ must exist with $\text{dist}_{H}(u',u'') \leq \text{dist}_{G}(u',v')$. Hence, we can get from $u_{j_1}$ to $u''$ via an $(h_{i-1} + 1)$-hop path with length

$$\text{dist}_{G_{i+1}}(u_{j_1},u'') \leq \text{dist}_{G_{i+1}}(u_{j_1},u') + \text{dist}_{H}(u',u'')$$

$$\leq \text{dist}_{G_{i+1}}(u_{j_1},u') + \text{dist}_{G}(u',v')$$

$$\leq 2 \text{dist}_{G_{i+1}}(u_{j_1},u') + \text{dist}_{G}(u_{j_1},u_{j_2}) + \text{dist}_{G_{i+1}}(u_{j_2},v')$$

$$\leq 3(r+4)^{i-1} \mu + \text{dist}_{G}(u_{j_1},u_{j_2}).$$

Similar to the previous case, there are at most $r-1$ multi-hop segments appeared before $u_{j_1}$, and all of them are satisfying condition (i). Hence, the surplus

$$\text{dist}_{G_{i}}(u,u_{j_2}) \leq \text{dist}_{G}(u,u_{j_2}) + (r-1)((r+4)^{i-1} - r^{i-1}) \mu.$$  

Therefore, in at most $(r-1)h_{i-1} + r - 1 + h_{i-1} + 1 \leq h_i$ hops,

$$\text{dist}_{G_{i}}(u,u'') \leq \text{dist}_{G_{i+1}}(u_{j_1},u_{j_2}) + \text{dist}_{G_{i+1}}(u_{j_1},u') + \text{dist}_{G}(u',v')$$

$$\leq [(r-1)((r+4)^{i-1} - r^{i-1}) + 3(r+4)^{i-1}] \mu + \text{dist}_{G}(u,u_{j_2})$$

$$\leq [(r+2)(r+4)^{i-1} - r^i + r^{i-1}] \mu + \text{dist}_{G}(u,u_{j_2})$$

$$\leq [(r+4)^{i-1} - r^i] \mu + \text{dist}_{G}(u,u_{j_2})$$

$$\leq (r+4)^{i-1} \mu$$

$$(\text{dist}_{G}(u,u_{j_2}) \leq \text{dist}_{G}(u,v) \leq r^i \mu)$$

$\Box$
Proof of Theorem 1. Fix \( u, v \in V \) and \( d = \text{dist}_G(u, v) \). Define \( \epsilon' = \ln(1 + \epsilon) \). Notice that \( 1/\epsilon' = (1 + o(1))(1/\epsilon) \). Set \( r = \lceil 4k/\epsilon' \rceil = \Theta(k/\epsilon) \) and \( \mu = d/r^k \). By Theorem 4, since \( V_{k+1} = \emptyset \), condition (i) must hold: within \( h_k < 2(r + 1)^k \) hops we have

\[
d_G^{(h_k)}(u, v) \leq \text{dist}_G(u, v) + ((r + 4)^k - r^k)\mu \\
= d + \left( \frac{4k}{r} + \frac{4^2(r^2)}{r^2} + \frac{4^3(r^3)}{r^3} + \cdots \right) d \\
\leq \left( 1 + \epsilon' + \frac{\epsilon'^2}{2!} + \frac{\epsilon'^3}{3!} + \cdots \right) d \\
\leq \epsilon' d = (1 + \epsilon)d. \]

Observe that if we set \( k = \log \log n - O(1) \) the size becomes linear.

**Corollary 1.** Every \( n \)-vertex graph has an \( O(n) \)-size \((\beta, \epsilon)\)-hopset with \( \beta = 2(4 + o(1))k \) and \( k = \log \log n - O(1) \).

### 3 Conclusion

In this paper our goal was to demonstrate that hopset constructions need not be complex, and that optimal hopsets can be constructed with a simple and elegant algorithm, namely a small modification to Thorup and Zwick’s emulator construction [20]. From a purely quantitative perspective our hopsets also improve on the sparseness and/or hopbound of other constructions [7, 9, 10]. As a happy byproduct of our construction, we also shave small factors off the best sublinear additive emulators [20] and \((1 + \epsilon, \beta)\)-spanners [1].

We now have a good understanding of the tradeoffs available between \( \beta \) and the hopset size when the stretch is fixed at \( 1 + \epsilon \), \( \epsilon > 0 \) being a small real. However, when \( \epsilon = 0 \) or \( \epsilon \) is large, there are still gaps between the best upper and lower bounds. For example, when \( \epsilon = 0 \) a trivial hopset has size \( O(n) \) with \( \beta = O(\sqrt{n} \log n) \). A construction of Hesse [13] (see also [1, §6]) implies that \( \beta \) must be at least \( n^\delta \) for some \( \delta \), but it is open whether \( O(n) \)-size hopsets exist with \( \beta \ll \sqrt{n} \). At the other extreme, Thorup and Zwick’s distance oracles imply that \( O(\kappa n^{1+1/\kappa}) \)-size hopsets exist with \( \beta = 2 \) and stretch \( 2\kappa - 1 \). Is this tradeoff optimal? Are there other tradeoffs available when \( \beta \) is a fixed constant (say 3 or 4), independent of \( \kappa \)?

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