INFINITE DIMENSIONAL CHERN-SIMONS THEORY

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Abstract. We extend finite dimensional Chern-Simons theory to certain infinite dimensional principal bundles with connections, in particular to the frame bundle $FLM \to LM$ over the loop space of a Riemannian manifold $M$. Chern-Simons forms are defined roughly as in finite dimensions with the invariant polynomials replaced by appropriate Wodzicki residues. This produces odd dimensional $\mathbb{R}/\mathbb{Z}$-valued cohomology classes on $LM$ if $M$ is parallelizable. We compute an example of a metric on the loop space of $S^3 \times S^1$ for which the three dimensional Chern-Simons class is nontrivial.

1. Introduction

A principal $G$-bundle $E \to B$ with connection has characteristic classes determined by applying invariant polynomials $P$ on the Lie algebra of $G$ to the curvature $\Omega$ of the connection. The apparently worst possible case, when the form $P(\Omega)$ vanishes pointwise, in fact leads to nontrivial secondary classes, the Chern-Simons classes, in $H_{\text{odd}}(B, \mathbb{R}/\mathbb{Z})$ [3]. For the frame bundle of a Riemannian manifold, these classes contain geometric information.

This theory assumes that $G$ is a finite dimensional Lie group. For interesting infinite dimensional Riemannian manifolds such as spaces of maps $C^\infty(N, M)$ between Riemannian manifolds, and in particular for loop spaces $LM$, the frame bundle has as structure group the gauge group $G$ of an auxiliary finite rank bundle. However, the Levi-Civita connection on mapping spaces takes values in pseudodifferential operators ($\Psi$DOs) on the auxiliary bundle, so it is necessary to expand the structure group to $\text{Ell}^*_0$, the group of zeroth order invertible $\Psi$DOs.

In [10], a theory of characteristic classes was developed for $G$- and $\text{Ell}^*_0$-bundles. The invariant polynomials for classical groups are generated by $A \mapsto \text{Tr}(A^l)$, but the $\Psi$DOs for mapping spaces are often not trace class operators. Thus the invariant polynomials in [10] use alternate traces, namely the leading order symbol or the Wodzicki residue, to construct characteristic forms and classes. While there are examples of nontrivial leading order Chern classes, no nontrivial examples of the more natural Wodzicki-Chern classes are known.

Both the leading order Chern forms (more precisely, Pontrjagin forms) and the Wodzicki-Chern forms vanish on the frame bundle $FLM \to LM$ for loop spaces (except possibly the first Wodzicki-Chern form – see §4), so it is natural to look for Chern-Simons classes. This paper develops a theory of Chern-Simons classes for loop
spaces for the two types of characteristic classes. For the frame bundle, we are unable to produce nonzero leading order Chern-Simons classes. However, the first Wodzicki-Chern-Simons class is nontrivial in $H^3(L(S^3 \times S^1), \mathbb{R}/\mathbb{Z})$ for a wide class of metrics, which shows that this theory is nonvacuous.

The paper is organized as follows. In §2, relevant results on the Riemannian geometry of $FLM$ are collected. In §3, finite dimensional Chern-Simons theory is reviewed, and the theory is extended to infinite dimensional bundles with structure group either $G$ or $\text{Ell}_0^*$. §4 gives an integrality result for leading order Chern classes (this is joint work with S. Paycha) and new vanishing results for these classes. The integrality result and the existence of a classifying space with universal connection for the gauge group are crucial to define Chern-Simons classes for $G$-bundles. Unfortunately, we do not know if these results hold for $\text{Ell}_0^*$-bundles. As a result, we can only define Chern-Simons classes for trivial $\text{Ell}_0^*$-bundles, such as loop spaces of parallelizable manifolds, and the Chern-Simons class apparently depends on the choice of a global frame. This restriction to parallelizable manifolds often occurs even in finite dimensions, as Chern-Simons classes are notoriously difficult to compute. In any case, in §5 a strategy for proving the nontriviality of a Chern-Simons class is given, and in §6 this is applied to produce a nontrivial Wodzicki-Chern-Simons class on $L(S^3 \times S^1)$.

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2. Preliminaries on the Geometry of $LM$

Let $(M, \langle , \rangle)$ be a compact oriented Riemannian $n$-manifold with loop space $LM = \mathcal{C}^\infty(S^1, M)$ of smooth loops. $LM$ is a smooth infinite dimensional manifold, but it is technically simpler to work with the smooth manifold of loops in some Sobolev class $s \gg 0$, as we now recall. For $\gamma \in LM$, the formal tangent space $T_\gamma LM$ is $\Gamma(\gamma^*TM)$, the space of smooth sections of the pullback bundle $\gamma^*TM \rightarrow S^1$. For $s > 1/2$, we complete $\Gamma(\gamma^*TM)$ with respect to the Sobolev inner product

$$\langle X, Y \rangle_s = \frac{1}{2\pi} \int \langle (1 + \Delta)^s X(\alpha), Y(\alpha) \rangle_{\gamma(\alpha)} d\alpha, \ \ X, Y \in \Gamma(\gamma^*TM).$$

Here $\Delta = D^*D$, with $D = D/d\gamma$ the covariant derivative along $\gamma$. We denote this completion by $H^s(\gamma^*TM)$.

A small neighborhood $U_\gamma$ of the zero section in $H^s(\gamma^*TM)$ is a coordinate chart near $\gamma$ in the space of $H^s$ loops via the pointwise exponential map

$$\exp_\gamma : U_\gamma \rightarrow LM, \ X \mapsto (\alpha \mapsto \exp_{\gamma(\alpha)} X(\alpha)).$$

It is easy to check that $\exp_\gamma$ is a bijection onto its image. The differentiability of the transition functions $\exp_{\gamma_1}^{-1} \cdot \exp_{\gamma_2}$ is proved in [4] and [2, Appendix A]. Since $\gamma^*TM$ is (non-canonically) isomorphic to the trivial bundle $\mathbb{R}^n = S^1 \times \mathbb{R}^n \rightarrow S^1$, the model space for $LM$ is the set of $H^s$ sections of this trivial bundle.
2.1. The Tangent Bundle TLM. The tangent bundle $TLM$ has transition functions $d(\exp^{-1}_1 \circ \exp_{\gamma_2})$. Under the isomorphisms $T_{\gamma_1}LM \approx \mathcal{R}^n \approx T_{\gamma_2}LM$, the transition functions are gauge transformations of $\mathcal{R}^n$.

The $H^s$ metric makes $LM$ a Riemannian manifold. The $H^s$ Levi-Civita connection on $LM$ is determined by the six term formula

$$\langle \nabla^s X, Z \rangle_s = X \langle Y, Z \rangle_s + Y \langle X, Z \rangle_s - Z \langle X, Y \rangle_s + \langle [X, Y], Z \rangle_s + \langle [Z, Y], X \rangle_s - \langle Y, [Z, X] \rangle_s,$$

$\nabla^s$ is explicitly computed in [7, Theorem 2.2]:

$$2(\nabla^s_X Y)^a = 2(\delta_X Y)^a + (1 + \Delta)^{-s} \left[ g^{af} \delta_Y g_{ef} ((1 + \Delta)^s Y)^c + (\delta_X (1 + \Delta)^s Y)^a \right]$$

$$+ (1 + \Delta)^{-s} \left[ g^{af} \delta_Y g_{bf} ((1 + \Delta)^s X)^b + (\delta_Y (1 + \Delta)^s X)^a \right]$$

$$- (1 + \Delta)^{-s} \left[ g^{af} \delta_Y g_{bf} ((1 + \Delta)^s X)^b Y^f + g^{af} g_{bf} ((1 + \Delta)^s X)^b Y^f \right],$$

where $\delta_X$ is the variation in the $X$ direction, and the $g$'s and $\Gamma$'s are the metric tensor and Christoffel symbols of $M$ in a local frame $\{e_a(\gamma_1, \alpha) \in T_{\gamma_1(\alpha)}M\}$ for $\gamma_1$ in a neighborhood of $\gamma$. By (2.1), the connection one-form of the operator $Y \mapsto \nabla_X Y$ is a pseudodifferential operator ($\Psi$DO) of order $0$ acting on sections of $\gamma^*TM$. The curvature $\Omega^s(X, Y) = \nabla^s_X \nabla^s_Y - \nabla^s_Y \nabla^s_X - \nabla^s_{[X,Y]}$, as an operator $Z \mapsto \Omega^s(X, Y)Z$, is a $\Psi$DO of order $-1$.

2.2. The Frame Bundle $FLM$. The frame bundle $FLM \rightarrow LM$ is constructed as in the finite dimensional case. The fiber over $\gamma$ is isomorphic to the gauge group $G$ of $\mathcal{R}^n$ and fibers are glued by the transition functions for $TLM$. Thus the frame bundle is topologically a $G$-bundle.

However, the connection and curvature forms take values in $\Psi DO_{\leq 0}$, the set of $\Psi$DOs of order at most zero. These forms should take values in the Lie algebra of the structure group, so we consider the extended structure group $Ell^*_n$, the group of classical invertible (and therefore elliptic) $\Psi$DOs of order $0$ acting on $\mathcal{R}^n$, as $\Psi DO_{\leq 0} = \text{Lie}(Ell^*_n)$. Note that $G$ embeds in $Ell^*_n$ as multiplication operators. Thus $(FLM, \theta^s)$ as a geometric bundle (i.e. as a bundle with connection $\theta^s$ associated to $\nabla^s$, as explained below) is an $Ell^*_n$-bundle.

In summary, we have

$$G \rightarrow FLM \quad \downarrow \quad Ell^*_n \rightarrow (FLM, \theta^s) \quad \downarrow \quad LM \rightarrow LM.$$

2.3. Connections on the Frame Bundle. We summarize the relationship between the Levi-Civita connection $\theta^s$ on the frame bundle and local expressions for the Levi-Civita connection on the tangent bundle. Let $\chi : N \rightarrow FN$ be a local section of the frame bundle of an $n$-manifold $N$. A metric connection $\nabla$ on $TN$ with local connection one-form $\omega$ determines a connection $\theta_{FN} \in \Lambda^1(FN, so(n))$ on $FN$ by (i)
the standard property for $\theta_{FN}$ on vertical vectors, and (ii) $\theta_{FN}(Y_u) = \omega(X_p)$, for $Y_u = \chi_s X_p$, or equivalently
\[ \chi^* \theta_{FN} = \omega. \] (2.2)
For $N = LM$, $\nabla^s$ determines a connection $\theta^s \in \Lambda^1(LM, \Psi DO_{\leq 0})$ on $FLM$, and the computations of the symbols $\sigma_0(\theta^s)$, $\sigma_{-1}(\theta^s)$ reduce to the computations of $\sigma_0(\omega^s)$ and $\sigma_{-1}(\omega^s)$ of the Levi-Civita connection one-form. By [7], for $X \in \Gamma(\gamma^s TM)$, we have
\[ \sigma_0(\omega^s(X))^k_\ell = \frac{1}{2} (\Gamma^k_{\ell p} + g^k_{\ell m} \Gamma^m_{bp}) X^p, \] (2.3)
\[ \sigma_{-1}(\omega^s(X))^k_\ell = 2is\xi^{-1} \left[ \partial_m \Gamma^k_{\ell m} \dot{X}^n X^m + \Gamma^k_{\ell m} \dot{X}^n X^m \dot{X}^n + \Gamma^k_{\ell m} \Gamma^m_{p n} X^m \dot{X}^n - \Gamma^k_{\ell m} \Gamma^m_{p n} X^m \dot{X}^n \right. \]
\[ + \dot{\gamma}^n \partial_n \Gamma^k_{\ell m} X^m + \Gamma^k_{\ell m} \dot{X}^n - \Gamma^k_{\ell m} \Gamma^m_{p n} g^p_{r t} \dot{X}^r X^t g^t_{r n} - \Gamma^p_{\ell m} g^p_{r n} \Gamma^m_{b r} X^r \dot{X}^n \]
\[ + \left( g^{k b} \Gamma^r_{b m} X^m \right) g^p_{r l}, \] (2.4)
where $\dot{} = d/d\alpha$ along $\gamma$. Using a frame $\{\psi_a\}$ dual to $\{e_a\}$, we may express e.g. (2.3) as a form:
\[ \sigma_0(\omega^s)^k_\ell = \frac{1}{2} \left[ \Gamma^k_{\ell p} + g^k_{\ell m} \Gamma^m_{bp} \right] \psi^p \] (2.5)

### 3. Chern-Simons Forms

In this section, the essentials of Chern-Simons theory are reviewed and extended to infinite dimensions. First, we recall the theory of characteristic classes in finite dimensions and the main results of the seminal paper [3].

#### 3.1. Characteristic Classes

Consider a finite dimensional Lie group $G$, a manifold $M$, and a $G$-bundle $E \rightarrow M$. For $\mathfrak{g} = \text{Lie}(G)$, set $\mathfrak{g}^\ell = \mathfrak{g} \otimes \ell$ and
\[ I^\ell(G) = \{ P : \mathfrak{g}^\ell \rightarrow \mathbb{R} \mid \text{$P$ symmetric, multilinear, ad-invariant} \}. \]
For $\phi \in \Lambda^k(E, \mathfrak{g}^\ell)$, $P \in I^\ell(G)$, set $P(\phi) = P \circ \phi \in \Lambda^k(E)$. Two key properties are:

- **(The commutativity property)** For $\phi \in \Lambda^k(E, \mathfrak{g}^\ell)$,
\[ d(P(\phi)) = P(d\phi). \] (3.1)

- **(The infinitesimal invariance property)** For $\psi_i \in \Lambda^k_i(E, \mathfrak{g})$, $\phi \in \Lambda^1(E, \mathfrak{g})$ and $P \in I^\ell(G)$,
\[ \sum_{i=1}^l (-1)^{k_1 + \cdots + k_i} P(\psi_1 \wedge \cdots \wedge [\psi_i, \phi] \wedge \cdots \psi_l) = 0. \] (3.2)

**Remark 3.1.** For classical Lie groups $G$, (3.1) and (3.2) follow trivially from the fact that the polynomials $I^\ell(G)$ is generated by the Newton polynomials $\text{Tr}(A^\ell)$ and properties of the ordinary finite dimensional trace.
Theorem 3.1 (The Chern-Weyl Homomorphism [6]). Let $E \to M$ have a connection $\theta$ with curvature $\Omega_E \in \Lambda^2(E, g)$. For $P \in I^1(G)$, $P(\Omega^l_E)$ is a closed invariant real form on $E$, and so determines a closed form $P(\Omega_M) \in \Lambda^2(M, \mathbb{R})$, a form with lift $P(\Omega^l_E)$. The Chern-Weil map

$$\bigoplus_{t=1}^l I^1(G) \to H^*(M, \mathbb{R}), \quad P \mapsto [P(\Omega_M)]$$

is a well-defined algebra homomorphism.

$[P(\Omega_M)]$ is called the characteristic class of $P$.

3.2. Chern-Simons Theory for Finite Dimensional Bundles. A crucial observation of Chern-Simons is that $P(\Omega^l_E)$ is exact, although in general $P(\Omega_M)$ is not.

Proposition 3.2. [3, Proposition 3.2] Let $G$ be a finite dimensional Lie group. For a $G$-bundle $E \to M$ with connection $\theta$ and curvature $\Omega = \Omega_E$, and for $P \in I^1(G)$, set

$$\phi_t = t\Omega + \frac{1}{2}(t^2 - t)[\theta, \theta]$$

$$TP(\theta) = l \int_0^1 P(\theta \wedge \phi_t^{-1})dt.$$  

Then $dTP(\theta) = P(\Omega) \in \Lambda^2(E)$.

Proof. We recall the proof for later purposes. Set $f(t) = P(\phi_t^l)$, so $P(\Omega^l) = \int_0^1 f'(t)dt$. We show $f'(t) = l \cdot dP(\theta \wedge \phi_t^{-1})$ by computing each side. First, we have

$$f'(t) = \frac{d}{dt} P(\phi_t^l) = P \left( \frac{d}{dt} \phi_t^l \right) = lP \left( \frac{d}{dt} \phi_t \wedge \phi_t^{-1} \right) = lP(\Omega \wedge \phi_t^{-1}) + \frac{l}{2} P(\theta \wedge \phi_t^{-1}),$$

where we have used the commutativity property (3.1). On the other hand, we have

$$l \cdot dP(\theta \wedge \phi_t^{-1}) = lP(d\theta \wedge \phi_t^{-1}) - l(l - 1)P(\theta \wedge d\phi_t \wedge \phi_t^{-2})$$  

$$= lP(\Omega \wedge \phi_t^{-1}) - \frac{l}{2} lP([\theta, \theta] \wedge \phi_t^{-2}) - l(l - 1)P(\theta \wedge d\phi_t \wedge \phi_t^{-2}),$$

by (3.1) and the structural equation $\Omega = d\theta + \frac{1}{2}[\theta, \theta]$. Since $d\phi_t = t[\phi_t, \theta]$, the last term in (3.4) equals

$$l(l - 1)P(\theta \wedge d\phi_t \wedge \phi_t^{-2}) = l(l - 1)P(\theta \wedge t[\phi_t, \theta] \wedge \phi_t^{-2}).$$

Using the invariance property (3.2) with $\phi = \theta$, $\psi_1 = \theta$ and $\psi_k = \psi_t$, $k = 2, \ldots, l - 1$, we obtain

$$l(l - 1)P(\theta \wedge t[\phi_t, \theta] \wedge \phi_t^{-1}) = -ltP([\theta, \theta] \wedge \phi_t^{-1}).$$

This implies (3.4) equals (3.3). \qed
Setting \( M = BG \) in the theorem gives the universal Chern-Weil homomorphism
\[
W : I^l(G) \to H^2(BG, \mathbb{R}).
\]
We write \( P \in I^l_0(G) \) if \( W(P) \in H^2(BG, \mathbb{Z}) \). For this subalgebra of polynomials, we obtain more information on \( TP(\theta) \).

**Theorem 3.3.** [3, Proposition 3.15] Let \( E \to B \) be a \( G \)-bundle with connection \( \theta \). For \( P \in I^l_0(G) \), let \( \tilde{TP}(\theta) \) be the mod \( \mathbb{Z} \) reduction of the real cochain \( TP(\theta) \). Then there exists a cochain \( U \in C^{2l-1}(B, \mathbb{R}/\mathbb{Z}) \) such that
\[
\tilde{TP}(\theta) = \pi^*(U) + \text{coboundary}.
\]

The proof is essentially given in Theorem 3.8 below.

**Corollary 3.4.** [3, Theorem 3.16] Assume \( P \in I^l_0(G) \) and \( P(\Omega^l_E) = 0 \). Then there exists \( CS_P(\theta) \in H^{2l-1}(B, \mathbb{R}/\mathbb{Z}) \) such that
\[
\tilde{TP}(\theta) = \pi^*(CS_P(\theta)).
\]

**Proof.** Choose \( U \in C^{2l-1}(B, \mathbb{R}/\mathbb{Z}) \) as in Theorem 3.3. Since \( P(\Omega^l_E) = 0 \), Proposition 3.2 implies \( \delta\tilde{TP}(\theta) = dTP(\theta) = 0 \). By Theorem 3.3, \( \pi^*U \) and \( TP(\theta) \) are cohomologous. Set \( CS_P(\theta) = [U] \). \( \square \)

Notice that the secondary class or Chern-Simons class \( CS_P(\theta) \), is defined only when the characteristic form \( P(\Omega_E) \) vanishes. The proof of Theorem 3.8 shows that \( CS_P(\theta) \) is independent of the choice of \( U \).

The following corollary will be taken as the definition of Chern-Simons classes for trivial \( \text{Ell}^*_0 \)-bundles (see Definition 3.2).

**Corollary 3.5.** Let \( (E, \theta) \to B \) be a trivial \( G \)-bundle with connection, and let \( \chi \) be a global section. For \( P \in I^l_0(G) \),
\[
CS_P(\theta) = \chi^*\tilde{TP}(\theta).
\]

**Proof.** This follows from Corollary 3.4 and \( \pi\chi = \text{Id} \). \( \square \)

### 3.3 Chern-Simons Theory on Loop Spaces.

In [10], Chern forms are defined on vector bundles with structure group \( \text{Ell}^*_0 \) and with \( \text{Ell}^*_0 \)-connections, or equivalently on principal \( \text{Ell}^*_0 \)-bundles with connections, where the \( \Psi \)DOs act on sections of a finite rank bundle \( E \to N \) over a closed manifold. The key technical point is to find suitable analogs for the polynomials \( P \in I^l(G) \). We single out two analogs of the Newton polynomials \( \text{Tr}(A^l) \): for \( A \in \text{Ell}^*_0 \), define
\[
P^{(0)}_l(A) = k(l) \int_{S^*N} \text{Tr} \left( \sigma_0(A^l)(x, \xi) \right) d\xi dx.
\]

Here \( S^*N \) is the unit cosphere bundle of \( N \) and \( k(l) = (2\pi i)^{-l}(\text{Vol } N)^{-1} \). Note that \( d_l = (2\pi i)^{-l} \) is the normalizing constant such that \( [d_l \text{Tr}(\Omega^l)] \in H^2(BU(n), \mathbb{Z}) \) for...
a connection $\theta^n$ on $EU(n) \to BU(n)$. In [9], $P_l^{(0)}$ is called a Leading Order Symbol Trace.

The second analog is

$$P_l^{(-1)}(A) = k(l)i^n \int_{S^*N} \text{Tr} \left( \sigma_{-n}(A^l)(x,\xi) \right) d\xi dx. \quad (3.6)$$

$P_l^{(-1)}(A)$ is a multiple of the Wodzicki residue of $A^l$. The factor $i^n$ insures that the Wodzicki residue of a real operator is real. As usual, $P_l^{(i)}, i = 0, -1$, determine polynomials by polarization.

For $P_l^{(i)}$, the commutativity and invariance properties hold because (3.5) and (3.6) are tracial [10] (i.e. $\text{Tr}[\sigma_i(AB)] = \text{Tr}[\sigma_i(BA)]$ for $A, B \in \PsiDO_{\leq 0}$). In particular, $P_l^{(i)}$ are in both $I^0(G)$, $I^l(\text{Ell}_0)$ (although trivially $P_l^{(-1)} = 0$ on $\mathcal{G}$).

The proof of Proposition 3.2 carries over to $\text{Ell}_0^*$-bundles, and so to the frame bundle of loop space.

**Proposition 3.6.** For a bundle $\text{Ell}_0^*$-bundle with connection $(\mathcal{E}, \theta) \to B$, and for $P \in I^l(\text{Ell}_0^*)$, set

$$\phi_t = t\Omega + \frac{1}{2}(t^2 - t)[\theta, \theta],$$

$$TP(\theta) = l \int_0^1 P(\theta \wedge \phi_t^{l-1}) dt \quad (3.7)$$

Then $dTP(\theta) = P(\Omega^l)$.

In the Proposition, we can replace $\text{Ell}_0^*$ by $\mathcal{G}$.

**Remark 3.2.** The tracial properties of $P_l^{(i)}$ imply that $P_l^{(i)}(\Omega)$ is a closed form with cohomology class independent of the connection $\theta$. The cohomology classes for $P_l^{(0)}, P_l^{(-1)}$ are the components of the so-called leading order Chern character and the Wodzicki-Chern character. Using Newton’s formulas, the Chern characters define Chern classes $c_k^{(0)}, c_k^{\text{res}}$, as usual. Examples of nontrivial leading order Chern classes are given in [10]. No nonzero examples of Wodzicki-Chern classes are known; see §4.2.

The main goal of this section is to show that Theorem 3.3 extends to the frame bundle $FLM$ for $P = P_l^{(0)}$. For $P_l^{(-1)}$, we only get an extension of Corollary 3.5.

As a first step, we have

**Lemma 3.7.** Let $\text{Ell}_0^*$ be the set of invertible zeroth order $\PsiDO$s acting on sections of the trivial bundle $\mathcal{R}^n$. Then $P_l^{(0)} \in I_0(\mathcal{G})$.

**Proof.** See §4.1. □

As in [3], we have
Lemma 3.7, there exists a geometric classifying map \( P \) and assume \( E \).

The acyclicity of \( B \) vanishes on all cycles in \( P \) we deduce that \( CS \) class \( C \).

Let \( \text{Definition 3.1.} \)

Theorem 3.8. Let \((\mathcal{E}, \theta) \to \mathcal{B}\) be a \( \mathcal{G} \)-bundle with connection \( \theta \) and assume \( P_l(\Omega) = 0 \). Let \( TP(\theta) \) be the mod \( \mathbb{Z} \) reduction of \( TP(\theta) \). Then there exists a cochain \( U \in C^{2l-1}(\mathcal{B}, \mathbb{R}/\mathbb{Z}) \) such that

\[
\widetilde{TP}(\theta) = \pi^*(U) + \text{coboundary}.
\]

Proof. By [9, §4], \( EG \to BG \) has a universal connection \( \tilde{\theta} \) (with curvature \( \hat{\Omega} \)). Thus there exists a geometric classifying map \( \phi : \mathcal{B} \to BG \): i.e. \((\mathcal{E}, \theta) \simeq (\phi^*EG, \phi^*\tilde{\theta})\). By Lemma 3.7 \( P \in I^l_0(\mathcal{G}) \), so its mod \( \mathbb{Z} \) reduction is zero. From the Bockstein sequence

\[
\cdots \to H^i(BG, \mathbb{Z}) \to H^i(BG, \mathbb{R}) \to H^i(BG, \mathbb{R}/\mathbb{Z}) \to H^{i+1}(BG, \mathbb{Z}) \to \cdots
\]

we deduce that \( \widetilde{P}(\hat{\Omega}) \) represents an integral class in \( BG \). Thus \( \widetilde{P}(\hat{\Omega}) \) as a cochain vanishes on all cycles in \( BG \), and hence is an \( \mathbb{R}/\mathbb{Z} \) coboundary, i.e. there exists \( \tilde{u} \in C^{2l-1}(BG, \mathbb{R}/\mathbb{Z}) \) such that \( \delta \tilde{u} = \widetilde{P}(\hat{\Omega}) \). We have

\[
\delta \pi^*(\tilde{u}) = \pi^*(\delta u) = \pi^*(\widetilde{P}(\hat{\Omega})) = d \widetilde{TP}(\theta) = \delta(\widetilde{TP}(\theta)).
\]

The acyclicity of \( EG \) implies \( TP(\theta) = \pi^*(\tilde{u}) + \text{coboundary} \). Now set \( U = \phi^*(\tilde{u}) \). \( \square \)

Definition 3.1. Let \((\mathcal{E}, \theta) \to \mathcal{B}\) be a \( \mathcal{G} \)-bundle with connection \( \theta \) and curvature \( \Omega \), and assume \( P_l(0)(\Omega) = 0 \). In the notation of Theorem 3.8 define the Chern-Simons class \( CS_{2l-1}(\theta) \in H^{2l-1}(\mathcal{B}, \mathbb{R}/\mathbb{Z}) \) by

\[
CS_{2l-1}^{(0)}(\theta) = [U].
\]

If \( \text{Ell}^*_0 \) acts on \( E \to N \), the top order symbol is a homomorphism \( \sigma_0 : \text{Ell}^*_0 \to \mathcal{G} \), where \( \mathcal{G} \) acts on \( \pi^*E \to S^*N \). A \( \text{Ell}^*_0 \)-bundle \( \mathcal{E} \) has an associated \( \mathcal{G} \)-bundle \( \mathcal{E}' \) with transition function \( \sigma_0(A) \), if \( A \) is a transition function of \( \mathcal{E} \). A connection \( \theta \) with curvature \( \Omega \) on \( \mathcal{E} \) gives rise to a connection \( \theta' = \sigma_0(\theta) \) on \( \mathcal{E}' \) with curvature \( \sigma_0(\Omega) \).

Since \( P_l(0)(\Omega) = P_l(0)(\sigma_0(\Omega)) \), we define \( CS_{2l-1}(\theta) = CS_{2l-1}(\theta') \).

This indirect definition is necessary at present, because we do not know if \( E\text{Ell}^*_0 \to B\text{Ell}^*_0 \) admits a universal connection. As a result, we can only extend the classical definition of Chern-Simons classes to \( P_l(-1) \) for trivial \( \text{Ell}^*_0 \)-bundles, using the construction of Corollary 3.3.

Definition 3.2. For trivial \( \text{Ell}^*_0 \)-bundles with connection \((\mathcal{E}, \theta) \to \mathcal{B}\) and global section \( \chi : \mathcal{B} \to \mathcal{E} \), and assume that \( P_l(-1)(\Omega) = 0 \). Then the Chern-Simons class \( CS_{2l-1}(\theta, \chi) \in H^{2l-1}(\mathcal{B}, \mathbb{R}/\mathbb{Z}) \) is defined by

\[
CS_{2l-1}^{(-1)}(\theta, \chi) = \chi^* \left[ \widetilde{TP}(\theta) \right].
\]

Remark 3.3. The Chern-Simons class is independent of the section \( \chi \) for finite dimensional groups and for \( \mathcal{G} \), since it is defined via a universal connection.
4. Properties of Wodzicki-Chern Classes

In this section we give a proof of Lemma 3.7. We also give a vanishing result for Wodzicki-Chern classes on mapping spaces of manifolds generalizing [8].

4.1. Integrality of Leading Order Symbol Characteristic Classes. The goal of this subsection is to show that \( W(P_t^{(0)}) \in H^2(BG, \mathbb{Z}) \).

We do not know if the corresponding result \( W(P_t^{(-1)}) \in H^2(BEll_0, \mathbb{Z}) \) is true. Fortunately, for \( FLM \), we know that \( P_t^{(-1)}(\Omega^*) = 0 \) (Lemma 4.5). This suffices to define the Chern-Simons class for the Levi-Civita connections on \( FLM \), if \( FLM \) is trivial, although the class depends on the choice of global section.

By [1], \( BG = C_{(1)}^s(S^1, BSO(n)) = \{ f : S^1 \to BSO(n) | f^*ESO(n) \simeq \pi^*\mathcal{R}^n \} \). As a more general setup, consider a closed manifold \( N \) and a finite rank real bundle \( E \to M \). Let \( ev : C^\infty(N, M) \times N \to M \) be the evaluation map \( ev(f, n) = f(n) \).

The bundle \( ev^*E \) determines an infinite rank bundle \( \pi_*ev^*E \to C^\infty(N, M) \), where \( \pi_*ev^*E|_f = \Gamma(f^*E \to N) \), with \( \Gamma \) denoting some Sobolev space of sections. (Here \( \pi : C^\infty(N, M) \times N \to C^\infty(N, M) \) is the projection.) For \( n \in N \), define \( ev_n : C^\infty(N, M) \to M \) by \( ev_n(f) = f(n) \).

It is well known that connections push down under \( \pi_* \). For the gauge group case, this gives the following:

**Lemma 4.1.** The universal bundle \( EG \to BG \) is isomorphic to \( \pi_*ev^*ESO(n) \). \( EG \) has a universal connection \( \theta^{EG} \) defined on \( s \in \Gamma(EG) \) by

\[
(\theta^{EG}_s)(\gamma)(\alpha) = ((ev^u)*(\gamma, 0))_{u_s}(\gamma, \alpha).
\]

Here \( \theta^u \) is the universal connection on \( ESO(n) \to BSO(n) \), and \( u_s : C^\infty(N, M) \times N \to ev^*ESO(n) \) is defined by \( u_s(f, n) = s(f)(n) \).

**Proof.** See [9]. \( \square \)

**Corollary 4.2.** The curvature \( \Omega^{EG} \) of \( \theta^{EG} \) satisfies

\[
\Omega^{EG}(Z, W)s(f)(n) = ev^*\Omega^u((Z, 0), (W, 0))u_s(f, n).
\]

**Proof.** This follows from

\[
\Omega^{EG}(Z, W)s(f)(n) = [\nabla^{EG}_Z \nabla^{EG}_W - \nabla^{EG}_W \nabla^{EG}_Z - \nabla^{EG}_{[Z, W]}]s(f)(n)
\]

and the previous lemma. \( \square \)

We now prove that \( P_t^{(0)}(\Omega^{EG}) \in H^2(BG, \mathbb{Z}) \).

Since \( ev_0 = ev_{\alpha_0} \) is homotopy equivalent for every \( \alpha_0 \in S^1 \), the cohomology class

\[
[P_t^{(0)}(ev^*_0 \Omega^u)] \in H^{2k}(BG \times \{n_0\}, \mathbb{R}) \cong H^{2k}(BG, \mathbb{R})
\]


is independent of $\alpha_0$. Thus

$$\left[ \frac{d_l}{4\pi} \int_{S^1} \text{Tr} \sigma_0((\Omega^E)^l) d\xi d\alpha \right] = \frac{d_l}{4\pi} \int_{S^1} \left[ \text{Tr} \sigma_0 ((\Omega^u)^l) \right] d\xi d\alpha,$$

$$= [d_l \text{ev}_0^* \text{Tr} \sigma_0 ((\Omega^u)^l)], \quad (4.1)$$

since $\Omega^u$ is a multiplication operator. By the choice of $d_l$, the last term in (4.1) lies

$$\text{in ev}_0^* H^{2l}(BSO(n), \mathbb{Z}) \subset H^{2l}(BG, \mathbb{Z}).$$

Thus

$$W(P^{(0)}_l) = [P^{(0)}_l (\Omega^E)] \in H^{2l}(BG, \mathbb{Z}),$$

which completes the proof of the Lemma.

The following table summarizes the results.

**Table 1.** Is $P^{(i)}_l \in I_0(G)$?

| $P^{(i)}_l$ | $G$ | $\mathcal{G}$ | $\text{Ell}_0^*$ |
|------------|-----|----------------|------------------|
| $P^{(0)}_l$ | $\text{yes}$ | $?$ |
| $P^{(-1)}_l$ | $\text{yes, trivially}$ | $?$ |

(but see Lemma 4.5)

**Remark 4.1.** Let $(\mathcal{E}, \theta) \to B$ be a $\mathcal{G}$-bundle with connection, where $\mathcal{G}$ is the gauge group of the rank $n$ hermitian bundle $E \to N$, and let $f : B \to BG$ be a geometric classifying map. The argument above easily extends to show that the $l^{th}$ leading order Chern class equals $f^* \text{ev}_0^* c_l(EU(n))$. Thus all leading order Chern classes are pullbacks of finite dimensional Chern classes. (This argument was developed with S. Paycha.)

4.2. A Vanishing Theorem for Wodzicki-Chern Classes.

**Theorem 4.3.** If $\mathcal{E} \to C^\infty(N, M)$ satisfies $\mathcal{E} = \pi_* \text{ev}^* E$ as above, then the Wodzicki-Chern classes $c^{res}_k(\mathcal{E})$ vanish for all $k$.

**Proof.** As in the previous subsection, $\mathcal{E}$ admits a connection whose curvature $\Omega$ is a multiplication operator. $\Omega^l$ is also a multiplication operator, and hence $P_l^{(-1)}(\Omega) = 0$. □

For a real infinite rank bundle, Wodzicki-Pontrjagin classes are defined as in finite dimensions: $p^{res}_k(\mathcal{E}) = (-1)^k c^{res}_{2k}(\mathcal{E} \otimes \mathbb{C})$. 
Corollary 4.4. The Wodzicki-Pontrjagin classes of $TC^\infty(N, M)$ and of all naturally associated bundles vanish.

Proof. Pick an element $f_0$ in a fixed path component $A_0$ of $C^\infty(N, M)$. For $f \in A_0$, $T_fC^\infty(N, M) \simeq \Gamma(f^*TM \to N) \simeq \Gamma(f_0^*TM \to N)$, where the second isomorphism is noncanonical. Thus over each component, $TC^\infty(N, M)$ is of the form $\pi_*ev^*TM$. The vanishing of the Wodzicki-Pontrjagin classes of associated bundles (such as exterior powers of the tangent bundle) follows as infinite dimensions, since there is a universal geometric bundle. \qed

We also have a trivial vanishing result for Wodzicki-Pontrjagin forms for $FLM$.

Lemma 4.5. The forms $P_l^{(-1)}(\Omega^s)$, $l > 1$, vanish on $FLM$.

Proof. This follows from the fact that $(\Omega^s)^l$ is of order $-l$. \qed

Remark 4.2. Similarly, if $E \to B$ is an infinite rank $\Ell_0^*$-bundle, for $\Ell_0^*$-acting on $E \to N^n$, and if $\mathcal{E}$ admits a $\Ell_0^*$-connection whose curvature has order $-k$, then $0 = c_{[n/k]}(\mathcal{E}) = c_{[n/k]+1}(\mathcal{E}) = \ldots$ Thus the Wodzicki-Chern classes are obstructions to the negativity of the order of the curvature.

5. The Chern-Simons Class and Parallelizable Manifolds

Thanks to Chern-Simons formalism, the vanishing of the curvature expressions $P_l^{(-1)}(\Omega^s)$, $l > 1$, in the last lemma is actually an advantage: as in §3, we can define Chern-Simons $CS_{2l-1}(\theta^s, \chi) \in H^{2l-1}(LM, \mathbb{R}/\mathbb{Z})$ provided $LM$ is parallelizable.

In this section, we describe a strategy to detect non-trivial Chern-Simons classes on parallelizable loop spaces.

Lemma 5.1. If $M$ is parallelizable, then $LM$ is parallelizable.

Proof. Let $\phi : TM \to M \times \mathbb{R}^n$ be a trivialization of $TM$. For $X_\gamma \in T_\gamma LM = \Gamma(\gamma^*TM)$, define

$$
\Psi : TLM \to LM \times \Gamma(S^1 \times \mathbb{R}^n \to S^1)
$$

$$
X_\gamma \mapsto (\gamma, \alpha \mapsto \pi_2(\phi(X_\gamma(\alpha)))),
$$

where $\pi_2 : M \times \mathbb{R}^n \to \mathbb{R}^n$ is the projection. It is easy to check that $\alpha$ is a smooth trivialization of $TLM$ in the $H^s$ norm. \qed

Therefore, for parallelizable $M$ there exists a global section $\chi : LM \to FLM$. For $P$ equal either $P_l^{(0)}$ or $P_l^{(-1)}$ and $l > 1$, $\chi^*TP(\theta^s) \in H^{2l-1}(LM, \mathbb{R})$ and $[\chi^*TP(\theta^s)] = CS_{2l-1}^{[\theta^s]}(\chi) = CS_{2l-1}^{[\theta^s]}(\chi)$ for $i = 0, 1$. Thus $CS_{2l-1}^{[\theta^s]}(\chi)$ is nontrivial if there exists $[z] \in H_{2l-1}(LM; \mathbb{Z})$ with

$$
\langle \chi^*TP(\theta^s), [z] \rangle \notin \mathbb{Z}.
$$
Lemma 5.2. $\beta : N \rightarrow L(N \times S^1), \ x \mapsto (\beta(x)(\alpha) = (x, \alpha))$.

Proof. Fix $\alpha_0 \in S^1$ with its associated evaluation map $ev_0 = ev_{\alpha_0} : L(N \times S^1) \rightarrow N \times S^1$. Let $\pi_1 : N \times S^1 \rightarrow N$ be the projection. From

$$N \xrightarrow{\beta} L(N \times S^1) \xrightarrow{ev_0} N \times S^1 \xrightarrow{\pi_1} N,$$

we obtain $\pi_1 \circ ev_0 \circ \beta = Id_N$, which implies that $\beta_*$ is injective.

Set $N = S^3$ and $M = S^3 \times S^1$. By the lemma, $\beta_*[S^3] \in H_3(L(S^3 \times S^1), \mathbb{Z})$ is nontrivial. This class works well for $l = 2$, since $CS_3^{(i)}(\theta^s) \in H^3(LM, \mathbb{R}/\mathbb{Z})$.

Corollary 5.3. $CS_3^{(i)}(\theta^s)$ is nontrivial in $H^3(L(S^3 \times S^1), \mathbb{R}/\mathbb{Z})$ if

$$\langle \chi^*TP(\theta), \beta_*[S^3] \rangle = \int_{S^3} \beta^*\chi^*TP(\theta) \notin \mathbb{Z}. \quad (5.1)$$

Remark 5.1. To compute the integrand in (5.1), it is useful to pick a global frame $\{E_1, E_2, E_3\}$ of $S^3$ and $E_4 = \partial_\alpha$ for $S^1$. Then of course

$$\beta^*\chi^*TP(\theta)(E_1, E_2, E_3) = \chi^*TP(\theta)(\beta E_1, \beta E_2, \beta E_3).$$

It is easy to check that

$$\beta_*(E_1) = (E_1, 0, 0, 0), \beta_*(E_2) = (0, E_2, 0, 0, 0), \beta_*(E_3) = (0, 0, E_3, 0) \quad (5.2)$$

as constant sections of the trivial bundle $T_{\beta(m)}L(S^3 \times S^1)$.

6. Calculations on $S^3 \times S^1$.

In this section we explicitly compute a three dimensional Chern-Simons class. In §6.1, we begin the computations, and show the vanishing of the Chern-Simons class $CS_3^{(0)}(\theta^s) \in H^3(LM, \mathbb{R}/\mathbb{Z})$ associated to the Levi-Civita connection $\theta^s$ for a class of metrics on $M = S^3 \times S^1$. In §6.2, we find a metric on $M$ and a global frame $\chi$ of $FLM$ such that $CS_3^{(-1)}(\theta^s, \chi) \neq 0$.

6.1. Computations on $S^3 \times S^1$ for $P_2^{(0)}$. Setting $l = 2$ and combining (3), with (3.7) for $\theta = \theta^s$ on the frame bundle $FLM$, we obtain

$$\chi^*TP(\theta) = -\frac{d_2}{6 \cdot 4\pi} P^{(0)}(\chi^*\theta \wedge \chi^*\theta \wedge \chi^*\theta)$$

$$= -\frac{d_2}{6 \cdot 4\pi} \int_{S^3 \times S^1} Tr [\sigma_0(\chi^*\theta) \wedge \sigma_0(\chi^*\theta) \wedge \sigma_0(\chi^*\theta)] d\xi d\alpha.$$

To simplify notation, set $\omega = \chi^*\theta$. Once we choose a metric for $M = S^3 \times S^1$ and $\chi$, we can use (2.2) and (2.3) to explicitly compute $\chi^*TP(\theta)$.
Let $E_1, E_2, E_3$ be a frame of orthonormal left invariant vector fields for $S^3$ with the standard metric. Let $E_4 = \partial_\rho$, where $\rho$ is the coordinate on $S^1$ in $S^3 \times S^1$, and impose the usual Lie relations

$$[E_1, E_2] = 2E_3, \ [E_2, E_3] = 2E_1, \ [E_1, E_3] = -2E_2, [E_i, E_4] = 0.$$  

$\chi : LM \rightarrow FLM$ will be the “loopification” of the global frame $(E_i)$ of $S^3 \times S^1$: $\chi(\gamma)(\alpha) = (E_i(\gamma(\alpha)))$, which is identified with a gauge transformation of $T\gamma LM$ under the isomorphism $T\gamma LM \simeq \mathcal{R}^4$.

Fix functions $\lambda = \lambda(\alpha), \mu = \mu(\alpha), \nu = \nu(\alpha)$. Take the metric on $M$ for which $\lambda E_1, \mu E_2, \nu E_3, E_4$ are orthonormal. The non-zero Christoffel coefficients are

$$\Gamma_{12}^3 = \left( \frac{\mu^2 \lambda^2 - \mu^2 \nu^2 + \nu^2 \lambda^2}{\lambda \mu \nu} \right), \quad \Gamma_{13}^2 = \left( \frac{-\mu^2 \lambda^2 - \mu^2 \nu^2 + \nu^2 \lambda^2}{\lambda \mu \nu} \right) = -\Gamma_{23}^1,$$

$$\Gamma_{31}^2 = \left( \frac{\nu^2 \lambda^2 - \lambda^2 \mu^2 + \mu^2 \nu^2}{\lambda \mu \nu} \right) = -\Gamma_{32}^1, \quad \Gamma_{14}^1 = -\Gamma_{41}^1 = -\frac{\dot{\lambda}}{\lambda}, \quad \Gamma_{24}^3 = -\Gamma_{22}^4 = -\frac{\dot{\mu}}{\mu},$$

$$\Gamma_{24}^2 = -\Gamma_{22}^4 = -\frac{\dot{\nu}}{\nu}, \quad \Gamma_{4j}^i = 0 = \Gamma_{44}^4 = \Gamma_{4j}^4 = 0.$$

Set

$$U = \nu^2 \frac{\lambda^2}{\lambda \mu \nu}, \quad V = \mu^2 \frac{\lambda^2}{\lambda \mu \nu}, \quad W = \lambda^2 \frac{\nu^2}{\lambda \mu \nu},$$

$$A = \frac{\dot{\lambda}}{\lambda}, \quad B = \frac{\dot{\mu}}{\mu}, \quad C = \frac{\dot{\nu}}{\nu}. \quad (6.1)$$

A direct calculation gives

$$\sigma_0(\omega^s) = \begin{pmatrix}
-A\psi^4 & U\psi^3 & -V\psi^2 & \frac{1}{3}A\psi^1 \\
U\psi^3 & -B\psi^4 & W\psi^1 & \frac{1}{3}B\psi^2 \\
-V\psi^2 & W\psi^1 & -C\psi^4 & \frac{1}{3}C\psi^3 \\
\frac{1}{2}A\psi^1 & \frac{1}{2}B\psi^2 & \frac{1}{2}C\psi^3 & 0
\end{pmatrix}$$

Here $\{\psi^i\}$ is the frame dual to $\{E_i\}$.

A straightforward computation using (2.5) gives $\text{Tr} [\sigma_0(\omega^s) \wedge \sigma_0(\omega^s) \wedge \sigma_0(\omega^s)] = 0$. Thus $CS_3^{(0)}(\theta^s) = 0$ for this class of metrics, so we turn our attention to $CS_3^{(-1)}(\theta^s)$. 
6.2. Computations on $S^3 \times S^1$ for $P^{(-1)}$. For the case $l = 2$, (6.4) gives

$$
\chi^* TP(\theta) = 2 \int_0^1 P^{(-1)}_2 \left( \chi^* \theta \wedge t \chi^* \Omega + \frac{1}{2} (t^2 - t) [\chi^* \theta, \chi^* \theta] \right) dt
$$

$$
= P^{(-1)}_2 (\chi^* \theta \wedge \chi^* \Omega) - \frac{1}{6} P^{(-1)} (\chi^* \theta \wedge \chi^* \theta \wedge \chi^* \theta)
$$

(6.2)

$$
= - \frac{i}{8\pi^3} \int_{S^* S^1} \text{Tr} [\sigma^{-1} (\chi^* \theta \wedge \chi^* \Omega)] d\xi d\alpha
$$

$$
+ \frac{i}{48\pi^3} \int_{S^* S^1} \text{Tr} [\sigma^{-1} (\chi^* \theta \wedge \chi^* \theta \wedge \chi^* \theta)] d\xi d\alpha,
$$

where $\theta = \theta^s, \Omega = \Omega^s$. By the symbol calculus for $\Psi$DOs and (2.5), we have

$$
\text{Tr} [\sigma^{-1} (\chi^* \theta \wedge \chi^* \theta \wedge \chi^* \theta)] = 3 \text{Tr} [\sigma^{-1} (\chi^* \theta) \wedge \sigma_0 (\chi^* \theta) \wedge \sigma_0 (\chi^* \theta)]
$$

(6.3)

$$
\sigma^{-1} (\chi^* \theta \wedge \chi^* \Omega) = \sigma_0 (\chi^* \theta) \wedge \sigma^{-1} (\chi^* \Omega)
$$

(6.4)

As in §6.1 we may replace $\chi^* \theta$ by $\omega^s$ and $\chi^* \Omega$ by $\Omega^s$.

First, we compute the contribution from (6.3) to (6.2). Recall that we need to compute the terms on the right hand side of (6.3) on $\beta_\gamma (TS^3)$. On the loop $\beta(m)(\alpha) = (m, \alpha)$, we have $\partial_\alpha \gamma = \dot{\gamma} = (0, 0, 0, 1)$. Thus (2.4) becomes

$$
\sigma^{-1} (\omega^s (X))^b_a = 2i \xi^{-1} \left[ \partial_\Gamma b^a + \left( \Gamma^a_k l^k b^4 l^4 - \delta^a_4 \right) b^4 l^4 \Gamma^q (q^4 l^4) - \delta^a_2 \right] X^l
$$

$$
+ 2i \xi^{-1} \left[ \left( \Gamma^a_l b^4 + \delta^a_2 \right) b^4 l^4 \partial_\alpha X^l + \partial_\alpha \left( \Gamma^a l^4 + \delta^a_2 l^4 X^l \right) \right].
$$

(6.5)

Note that $\partial_\alpha X^l = 0$ for $X \in \beta_\gamma (TS^3)$, as $\beta_\gamma (E_4)$ does not depend on $\alpha$ by (5.2). Thus on $\beta_\gamma (TS^3)$, (6.5) reduces to

$$
\sigma^{-1} (\omega^s (X))^b_a = 2i \xi^{-1} \left[ \partial_\Gamma b^4 + \left( \Gamma^a l^4 - \delta^a_2 \right) b^4 l^4 \Gamma^q (q^4 l^4) - \delta^a_2 b^4 l^4 \Gamma^q (q^4 l^4) \right] X^l
$$

$$
+ 2i \xi^{-1} \left[ \partial_\alpha \left( \Gamma^a l^4 + \delta^a_2 l^4 \right) X^l \right]
$$

(6.6)

Combining (6.6) with the values of the Christoffel symbols from §6.1 gives a messy but explicit expression for the contribution of (6.3) to (6.2).

Second, we compute the contribution of (6.4) to (6.2). The $-1$ order symbol of the curvature in our orthonormal frame is

$$
\frac{1}{2i \xi^{-1}} \sigma^{-1} (\Omega^s (X, Y))^k_l = X^p Y^r \left[ \partial_p \Gamma^k_{rl} - \partial_r \Gamma^k_{pl} - \delta^{kb} \delta_{ml} \partial_r \Gamma^m_{bp} \right]
$$

$$
+ X^p Y^r \left[ \partial_p \Gamma^l_{rm} + \delta^{kb} \delta_{ml} \partial_r \Gamma^m_{bp} - \partial_r \Gamma^k_{pl} - \delta^{kb} \delta_{ml} \partial_r \Gamma^m_{bp} \right]
$$

$$
+ X^p Y^r \left[ \partial_p \Gamma^l_{tr} + \delta^{kb} \delta_{ml} \partial_r \Gamma^m_{bp} - \partial_r \Gamma^k_{pl} \right]
$$

(6.7)

Arguing as above (6.6), we see that the first and third terms on the right hand side of (6.7) do not contribute to the Chern-Simons class. In the second term, the only possible contributions come from $X^k = X^4$ or $Y^r = Y^4$, but again by (5.2), these components vanish on $\beta_\gamma (TS^3)$. Thus (6.4) does not contribute to (6.2).
In summary, on this image
\[ \chi^* TP(\theta) = \frac{s}{2\pi^2} \psi_1 \wedge \psi_2 \wedge \psi_3 \int_{S^1} f(\lambda(\alpha), \mu(\alpha), \nu(\alpha)) , \]
where the complicated function \( f \) is determined explicitly by (6.2), (6.3), (6.6). For \( \lambda(\alpha) = 1, \mu(\alpha) = 2 + \frac{1}{a} \cos(a\alpha) \sin(a\alpha), \nu(\alpha) = 2 - \cos(a\alpha), a \in \mathbb{Z}, \)
we compute via Mathematica [2] that
\[ \int_{S^3} \beta^3 \chi^* TP(\theta) = \int_{S^3} \frac{s}{2\pi^2} \psi_1 \wedge \psi_2 \wedge \psi_3 \int_{S^1} f(\lambda, \mu, \nu) = \frac{s}{4} \int_{S^1} f(\lambda, \mu, \nu) \notin \mathbb{Z} \]
for various choices of \( a \). (See Figures 1, 2.) By Corollary 5.3 for these (and many other) choices of \( \lambda, \mu, \nu, \)
\[ CS_{P(-1)}(\theta) \in H^3(L(S^3 \times S^1), \mathbb{R}/\mathbb{Z}) \]
is nontrivial.
Remark 6.1. The Chern-Simons class \( CS_3^{(-1)}(\theta, \chi) \) has a linear dependence on the Sobolev parameter \( s \), and so keeps track of the \( s \)-dependence of the topology of the frame bundle. Alternatively, one can define a regularization/parameter independent Chern-Simons form as \( \frac{1}{s} CS_3^{(-1)}(\theta, \chi) \) and note that this invariant is non-zero in our example.

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