Extended Attractor Mechanism and Non-Renormalization Theorem in 6D (1,0) Supergravity

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ABSTRACT

We compute the macroscopic entropy of the supersymmetric rotating dyonic strings carrying linear momentum in 6D (1,0) supergravity with curvature squared corrections. Our calculation is based on Sen’s entropy function formalism applied to the near horizon geometry of the string solution taking the form of an extremal BTZ×$S^3$. Upon solving the attractor equations, we observe that inclusion of the higher derivative interactions does not modify the $AdS_3$ radius but does reduce the size of the extremal BTZ horizon. The extremized entropy function depends also on the horizon value of the dilaton which is not fixed by the attractor equations. Therefore, derivation of the entropy must utilize the full-fledged asymptotically flat string solution from which the horizon value of the dilaton can be read off. The final entropy formula states that the two independent supersymmetric completions of Riemann tensor squared contribute equally to the entropy. A further $S^3$ compactification of the 6D theory results in a matter coupled 3D supergravity model in which the quantization condition of the SU(2)$_R$ Chern-Simons level implies the horizon value of the dilaton is not modified by higher derivative interactions beyond supersymmetric curvature squared terms.
1 Introduction

The most striking feature hidden in the black hole thermodynamics is the area law of the entropy instead of the usual volume law observed in all local physical systems. Efforts devoted to understanding such an area law of entropy have led to the discovery of holographic principle of quantum gravity realized concretely in the fruitful framework of AdS/CFT correspondence. One precursor of AdS$_3$/CFT$_2$ correspondence is precisely the micro-states counting of the BPS black holes [1] in string theory. In the infinite charge limit, the logarithm of the corresponding 2D CFT density of states successfully recovers the Bekenstein-Hawking formula. Subsequent studies have extended the agreement between the macroscopic and microscopic entropy to large classes of BPS black holes [2,3] in string/M theory (see [4–7] for nice reviews). When the charges are large but finite, the matching between the macroscopic and microscopic entropy implies deviations from the area law that are suppressed by inverse powers of the large charges\(^1\). Astonishingly, for a class of 4D static BPS black holes, it has been demonstrated [9,10] that the supergravity action supplemented by higher derivative terms does provide the right amount of deviations that matches precisely with its CFT counterpart. Specifically, up to leading order higher derivative corrections, the macroscopic entropy of the 4D black holes agrees precisely with the CFT result computed in [3,11]

\[
S_{\text{micro}} = 2\pi \sqrt{\frac{1}{6} \hat{q}_0 \left( C_{ABC} \hat{p}^A \hat{p}^B \hat{p}^C + c_{2A} \hat{p}^A \right)} ,
\]

where the coefficients $C_{ABC}$ and $c_{2A}$ encode the geometric data about the internal manifold on which string/M-theory is compactified. In particular, $c_{2A}$ corresponds to the coefficient in front of the leading higher derivative terms in the low energy effective action. Comparison

\(^1\)There can also be logarithmic corrections to the area law. However, they are not present in $\mathcal{N} = 4$ theory [8] which is the relevant context of this paper.
between the BPS black hole entropy and relies on an implicit map from the conserved charges carried by the black hole to the CFT data $\hat{q}_0, \hat{p}^A$. At leading order, this identification is straightforward. However, when higher derivative interactions are switched on, the relation between the black hole charges and $\hat{q}_0, \hat{p}^A$ becomes less obvious and may be modified by higher derivative interactions as observed in \cite{12, 13} for a class of 5D spinning BPS black holes. Supersymmetric higher derivative terms in $D > 4$ introduce another subtlety—the non-gauge invariant terms like $A_{(1)} \wedge \text{Tr}(R \wedge R)$ in $D = 5$ and $B_{(2)} \wedge \text{Tr}(R \wedge R)$ in $D = 6$ which require a careful treatment for their contributions to the macroscopic entropy. In fact, \cite{12, 13} have both computed the macroscopic entropy for the same class of spinning BPS black holes with curvature squared corrections. However, their results match only in the static limit.

For a subclass of black holes considered in \cite{12, 13}, computing higher derivative corrections to the macroscopic entropy can be performed in a different set-up. Specifically, this subclass of BPS black holes corresponds to M-theory branes wrapped around cycles in $K3 \times T^2$, which amount to strings or branes in IIA string theory wrapped on $K3 \times S^1$. In the latter case, the leading $\alpha'$ corrections arise from string one loop effects \cite{14}. In supersymmetric compactifications of string/M-theory, one can first retain only the lower dimensional supergravity multiplet first and subsequently incorporate matter multiplets. Specific to IIA string theory compactified on $K3$, the 10D string frame action reduces to an effective action of the form $e^{-2\phi} R + R^2 + \cdots$ where the $R^2$ term descends from the $R^4$ term in $D = 10$. Supersymmetrizing this type of action can be readily performed utilizing the dilaton Weyl multiplet within the framework of off-shell $\mathcal{N} = 2$ supergravity. The resulting supersymmetric $R^2$ action is relatively simple, independent of the dilaton field. On the other hand, compactifications of 11D supergravity on Calabi-Yau three-folds leads to a 5D supergravity model of the schematic form $R + f(\phi) R^2 + \cdots$. In this case, the 5D standard Weyl multiplet is the preferable building block. Rescaling the metric in either case inevitably complicates the curvature squared super-invariants. Therefore, neatness of the higher derivative action selects the suitable off-shell supergravity multiplets. There is the view that in higher derivative theories, one can always perform field redefinitions under which the structure of the higher derivative terms are not preserved while physics remains the same. Therefore, the specific form of the higher derivative terms is not crucial. However, this also means that one is free to choose a frame in which the calculation can be done most efficiently. As we will see later, such a convenient frame for studying supersymmetric black holes naturally comes from the off-shell formulation of supergravity, without which the field equations will be much more complicated.

In this work, we shall revisit the BPS black hole entropy in the IIA set-up for reasons below. It is recalled that the $K3$ compactification of IIA string preserves 16 supercharges. The
supersymmetric curvature squared action consistent with the same amount of supersymmetry was obtained only recently in 6D off-shell supergravity (and its dimensional reduction) based on the dilaton Weyl multiplet \cite{15,16}, where the invariance under 8 supercharges is manifest. A further $S^1$ reduction of the 6D model gives rise to the aforementioned 5D model arising from IIA string on $K3 \times S^1$. In practice, we will uplift the 5D black hole solutions along the $S^1$ direction to six-dimensions and readdress the issue of the macroscopic entropy there. We emphasize that it is totally equivalent to calculate the entropy of black holes in the 5D model based on the dilaton Weyl multiplet as circle reduction preserves conserved quantities of the solution. We prefer the 6D model for computational simplicity as the 6D supergravity multiplet consists of less fields. In the 6D set-up, we will also explore whether different supersymmetric completions of the Riemann tensor squared give the same contribution to the black hole entropy, by virtue of the fact that there is a unique off-shell formulation of 6D (1,0) supergravity based on which all curvature squared super-invariants have been constructed \cite{15,16} (all curvature squared super-invariants based on 5D dilaton Weyl multiplet were previously obtained in \cite{17,19}). This is advantageous to the 5D set-up based on standard Weyl multiplet \cite{12,13}, where only one supersymmetric completion of Riemann tensor squared preserving 8 supercharges \cite{21} instead of 16 supercharges is known explicitly. Up to date, there exists no rigorous proof that the other independent supersymmetric completion of the Riemann tensor squared based on standard Weyl multiplet, if exists, contributes equally to the BPS black hole entropy.

In the next section, we introduce the 6D (1,0) supergravity action equipped with curvature squared super-invariants. For simplicity, we restrict our discussions to NS-NS sector fields that are captured by the dilaton Weyl multiplet. Upon circle reduction, the 6D model reduces to the 5D ungauged STU model extended by curvature squared terms. Without higher derivative corrections, black hole solutions in 5D STU model are well studied \cite{23,24}. Amongst them, we single out the 3 charge rotating black holes obtained in \cite{23} and lift them to 6D rotating dyonic strings with linear momentum. The BPS limit of the 6D strings requires equal angular momenta consequently exhibiting an enhanced $U(2) \times \mathbb{R}^2$ isometry, based on which the general ansatz underlying the BPS string solution and compatible with the 6D off-shell supersymmetry can be derived. Following a procedure proposed in \cite{25}, one can then apply the ansatz to construct rotating BPS dyonic string solutions with higher derivative corrections. The entropy of the string solution is determined from its near horizon geometry in the form of an extremal BTZ $\times S^3$. It has been verified in the previous work \cite{15,16} that

\footnote{There exists also standard Weyl multiplet in $D = 6$. However, the 2-derivative supergravity action based on the standard Weyl multiplet is pathological for the lack of a saddle point \cite{20}.}
the $S^3$ radius and the $AdS_3$ radius associated with the BTZ black hole are not modified by the curvature squared super-invariants. However, as we will show in Section 3 that the size of the extremal BTZ horizon does receive corrections. The entropy computed from extremizing Sen’s entropy function also depends on the horizon value of the dilaton that is not solved from the extremization process. In Section 4, we read off the higher derivative corrected value of dilaton at the horizon from the complete asymptotically flat rotating dyonic string solution. Inserting it to the extremized entropy function, we recover precisely the result obtained in [13] specialized to the K3 compactification. It is also observed that the two independent supersymmetric completions of the Riemann tensor squared contribute equally to the BPS string entropy. We conclude with discussions in Section 5, emphasizing the non-renormalization of the horizon value of the dilaton beyond the leading $\alpha'$ correction.

2 The model

In this section, we set up the 6D supergravity model describing the low energy dynamics of IIA string on K3 with leading $\alpha'$ corrections that are generated by string one loop effects. In the low energy effective action, the massless bosonic fields consist of a metric, an anti-symmetric two-form, 24 vectors and 81 scalars. In the framework of 6D (1,0) off-shell supersymmetry, the metric, anti-symmetric two-form and one scalar denoted as $\{g_{\mu\nu}, B_{\mu\nu}, L\}$ belongs to the bosonic sector of the dilaton Weyl multiplet. Henceforth, we will call $L$ dilaton field which is related to the string coupling via $g_\text{s}^2 = 1/(L)_{\text{vev}}$. The black hole solutions to be considered later are fully captured by this field content. We thus write out terms in the effective action composed by the dilaton Weyl multiplet. Vectors and hypermultiplet scalars can be consistently truncated out from the field equations. The leading term in the action takes the form

$$L_{\text{EH}} = \sqrt{-g}L \left( R + L^{-2} \partial_\mu L \partial^\mu L - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} \right).$$

(2.1)

As explained in [16], the string frame action given above can be coupled to curvature squared super-invariants without modifying the supersymmetry transformation rule. There are three independent curvature squared super-invariants corresponding to the supersymmetrization of the pure Riemann tensor squared [26]

$$L_{\text{Riem}^2} = \sqrt{-g} \left[ R_{\mu\nu\alpha\beta}(\omega) R^{\mu\nu\alpha\beta}(\omega) - \frac{1}{4} R^{\mu\nu\rho\sigma} \lambda^\tau B_{\mu\nu} R_{\rho\sigma}^\alpha (\omega) R^\beta (\omega) R_{\alpha (\omega),} \right],$$

(2.2)

the supersymmetric completion of the Gauss-Bonnet combination

$$L_{\text{GB}} = \sqrt{-g} \left[ R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4R_{\mu\nu} R^{\mu\nu} + R^2 + \frac{1}{6} R H^2 - R^{\mu\nu} H_{\mu\nu}^2 + \frac{1}{2} R_{\mu\nu\rho\sigma} H^{\mu\nu\rho} H_{\lambda}^{\sigma\lambda} + \frac{5}{24} H^4 + \frac{1}{144} (H^2)^2 - \frac{1}{8} (H_{\mu\nu}^2)^2 - \frac{1}{4} \epsilon^{\mu\nu\rho\sigma\lambda\tau} B_{\mu\nu} R_{\rho\sigma}^\alpha \beta (\omega) R^\beta (\omega) \right],$$

(2.3)
and the supersymmetric Ricci scalar squared term whose explicit form can be found in \[16,27\]. One can verify that field equations derived from the supersymmetric Ricci scalar squared is proportional to the those from the leading order action \[2.1\] and thus does not modify the solution at the order we are interested in. Moreover, its contribution to the BPS string entropy vanishes \[25\]. Therefore the Ricci scalar squared super-invariant will be omitted from later discussions. In the above supersymmetric curvature squared actions, \(R_{\mu \nu \rho \sigma}\) is the standard Riemann tensor of the metric, while \(R^\alpha_\mu \nu \alpha \beta \) is the curvature defined with respect to the torsionful spin connections \(\omega^\alpha_\mu \nu \beta\).

\[
R^\alpha_\mu \nu \beta (\omega_\pm) = \partial_\mu \omega^\alpha_\nu \beta + \omega^\alpha_\mu \gamma \omega^\gamma_\nu \beta - (\mu \leftrightarrow \nu), \quad \omega^\alpha_\pm \mu \beta = \omega^\alpha_\mu \beta \pm \frac{1}{2} H^\alpha_\mu \beta. \tag{2.4}
\]

The shorthand notations for various contractions of \(H_{\mu \nu \rho}\) are defined as

\[
H^2 = H_{\mu \nu \rho} H^{\mu \nu \rho}, \quad H^2_{\mu \nu} = H_{\mu \nu \sigma} H^{\nu \rho \sigma}, \quad H^4 = H_{\mu \nu \sigma} H^{\rho \lambda \sigma} H^{\mu \rho \delta} H^{\nu \lambda \delta}. \tag{2.5}
\]

In summary, we will study the entropy of BPS black holes using the model

\[
S_{R+R^2} = \frac{1}{16\pi G_6} \int d^6 x \sqrt{-g} \left( L_{\text{EH}} + \frac{\lambda_1}{16} L_{\text{Riem}}^2 + \frac{\lambda_2}{16} L_{\text{GB}} \right), \tag{2.6}
\]

where the combination with \(\lambda_1 = \lambda_2 = \alpha'\) descends from IIA compactified on K3 and enjoys a supersymmetry enhancement. For general values of \(\lambda_s\), the effective action \[2.6\] preserves only 8 supercharges. In later discussions, we keep \(\lambda_s\) general in order to tell apart the contribution to the BPS string entropy from individual curvature squared super-invariant.

## 3 Macroscopic entropy from extremizing entropy function

The 6D rotating dyonic string \[22\] that comes from lifting the 5D 3-charge rotating BPS black hole \[23\] can be put in the form with a manifest \(U(2) \times \mathbb{R}^2\) isometry

\[
ds_6^2 = -a_1^2(r)(dt + \varpi \sigma_3)^2 + a_2^2(r)(dz + A_{(1)})^2 + b(r)^2 dr^2 + \frac{1}{4} c^2(r)(\sigma_3^2 + d \theta^2 + \sin^2 \theta d \phi^2), \quad B_{(2)} = 2 P \omega_2 + d(r)dt \wedge dz + f_1(r) dt \wedge \sigma_3 + f_2(r) dz \wedge \sigma_3, \quad A_{(1)} = A_0(r) dt + A_3(r) \sigma_3, \quad L = L(r), \tag{3.1}
\]

where in our notation \(\sigma_3 = d \psi - \cos \theta d \phi\) and \(d \omega_2 = \text{Vol}(S^3)\). The \(r\)-dependent functions in \[3.1\] are given by

\[
a_1^2 = \frac{r^4}{(r^2 + Q_1)(r^2 + Q_2)}, \quad a_2^2 = \frac{r^2 + Q_2}{r^2 + Q_1}, \quad c^2 = r^2 + P, \quad b^2 = c^2 r^{-2}, \quad \varpi = \frac{\nu}{2 r^2}, \\
d = \frac{r^2}{r^2 + Q_1}, \quad A_0 = \frac{r^2}{r^2 + Q_2}, \quad f_1 = 0, \quad f_2 = -d \varpi, \quad A_3 = A_0 \varpi, \quad L = \frac{r^2}{d(r^2 + P)}. \tag{3.2}
\]
The near horizon limit is attained by zooming in the $r = 0$ region while keeping other parameters fixed. In terms of the new coordinates and parameter defined below,

$$\rho = r^2, \quad \tau = \frac{2t}{\sqrt{PQ_1\rho_+}}, \quad \tilde{\psi} = \psi + \frac{2\nu}{PQ_1}z, \quad \rho^2 = \frac{Q_2}{Q_1}\left(1 - \frac{\nu^2}{PQ_1Q_2}\right) \quad (3.3)$$

the near horizon geometry can be expressed as an extremal BTZ$\times$S$^3$

$$ds^2_{6,H} = \frac{P}{4}(-\rho^2d\tau^2 + \frac{d\rho^2}{\rho^2}) + \rho^2_+(dz + \frac{\sqrt{P}}{2\rho_+d\tau})^2 + \frac{P}{4}(\tilde{\sigma}_3^2 + d\theta^2 + \sin^2\theta d\phi^2),$$

$$B_{(2),H} = -\frac{P}{4}\cos\theta d\phi \wedge d\tilde{\psi} + \frac{\sqrt{P}\rho_+}{2}\rho_+d\tau \wedge dz - \frac{\nu}{2Q_1}dz \wedge d\psi, \quad L_H = \frac{Q_1}{P}, \quad (3.4)$$
in which $\tilde{\sigma}_3 = d\psi - \cos\theta d\phi$ and evidently the last term in $B_{(2),H}$ is a pure gauge. Locally, the near horizon geometry preserves 8 supercharges [28,30] while the asymptotically flat string solution preserves 4 supercharges.

In the following, we will adopt the entropy function formalism [6] to compute the macroscopic entropy of the BPS string solution with higher derivative corrections. As the near horizon geometry is a direct product of BTZ with S$^3$, we first reduce the 6D theory to $D = 3$. For the purpose of this work, we adopt the the minimal ansätz

$$ds^2_{6} = ds^2_{3} + \frac{P}{4}(\sigma^2_3 + d\theta^2 + \sin^2\theta d\phi^2), \quad \tilde{H}_{(3)} = dB_{(2)} + \frac{P}{4}\sin\theta d\theta \wedge d\phi \wedge d\psi, \quad \tilde{L} = L, \quad (3.5)$$

which suffices to capture the near horizon geometry of the BPS string solution even with higher derivative corrections. The fields on the R.H.S of the equations above are independent of the coordinates on S$^3$. Parameter $P$ labels the S$^3$ radius squared and sets up the physical scale relative to which one can discuss corrections to the leading order solution, as the 6D model does not have an intrinsic cosmological constant. The dimension reduction of leading two-derivative Lagrangian yields

$$L_{EH}^{(3)} = \sqrt{-g}L(R - L^{-2}\partial_\mu L\partial^\mu L - \frac{1}{12}|H_{(3)}|^2 + \frac{4}{P}), \quad H_{(3)} = dB_{(2)}. \quad (3.6)$$

It is straightforward to reduce the supersymmetric curvature squared actions and the results are

$$L_{\text{Riem}}^{(3)} = \sqrt{-g}\left(4R_{\mu\nu}R^{\mu\nu} - R^2 - \frac{1}{2}H_{(3)}\square H_{(3)} - \frac{1}{6}R|H_{(3)}|^2 + \frac{1}{24}(|H_{(3)}|^2)^2\right) + L_{\text{CS}}(\omega_-),$$

$$L_{\text{GB}}^{(3)} = \frac{16}{P}\sqrt{-g}(R + \frac{1}{12}|H_{(3)}|^2) + L_{\text{CS}}(\omega_+),$$

$$L_{\text{CS}}(\omega_-) = \frac{8}{\sqrt{P}}\left(-\frac{1}{24}\text{tr}(\Gamma d\Gamma + \frac{2}{3}\Gamma^3) - \frac{1}{24}H_{(3)}|H_{(3)}|^2 + \frac{1}{2}H_{(3)}R\right),$$

$$L_{\text{CS}}(\omega_+) = \frac{8}{\sqrt{P}}\left(-\frac{1}{24}\text{tr}(\Gamma d\Gamma + \frac{2}{3}\Gamma^3) + \frac{1}{24}H_{(3)}|H_{(3)}|^2 - \frac{1}{2}H_{(3)}R\right), \quad (3.7)$$

\[^3\text{In [31], it was shown that a more complete consistent S}^3\text{ reduction of the 2-derivative 6D supergravity coupled to a single chiral tensor multiplet leads to the SO(4)\times\mathbb{R}^6 gauged N = 4 supergravity in D = 3. Based on this result, some black string solutions in D = 6 were obtained by uplifting solutions in the 3D theory [32].}\]
where the parity odd terms are given as 3-forms. Next we parameterise the putative metric and two-form

$$ds_3^2 = \frac{\ell^2}{4} \left( -\rho^2 dt^2 + \frac{d\rho^2}{\rho^2} \right) + r_+^2 \left( dz + \frac{\ell \rho}{2r_+} d\tau \right)^2, \quad r_+ = \frac{\ell}{2e_1}, \quad B_{(2)} = e_2 \rho d\tau \wedge dz. \quad (3.8)$$

Together with the constant dilaton $L := L_H$, we have 4 independent variables $\{\ell, e_1, e_2, L_H\}$. Off-shell supersymmetry implies a model-independent relation between $e_1$ and $e_2$

$$4e_1 e_2 = \ell^2, \quad (3.9)$$

which should be employed after varying the entropy function w.r.t. $\{\ell, e_1, e_2, L_H\}$. One should also be aware that there is no similar off-shell relation between $e_2$ and $L_H$. Although one of the BPS equations relates $L$ to $d$, $e_2$ is defined w.r.t time $\tau$ differing from the original time $t$ by a numerical factor that depends on the detail of the solution.

The entropy function is an effective action for the independent reduced phase space variables $\{\ell, e_1, e_2, L_H\}$. It is obtained by plugging the ansatz (3.8) to the 3D effective action except for the contribution from the Lorentz Chern-Simons term which is ambiguous for its lack of gauge invariance. The ansatz considered here has a non-degenerate Killing vector $\partial_z$, allowing us to reduce the Lorentz Chern-Simons term to $D = 2$ along the $z$-circle. After omitting certain total derivative terms, one obtains a gauge invariant 2D action. The contribution to the entropy function from the Lorentz Chern-Simons term is then evaluated by plugging the circle reduction of the ansatz (3.8) to the gauge invariant 2D action. This is the procedure adopted in [33]. We will not repeat the calculation here but simply use their results [33]. Assembling all the contributions, we write down Sen’s entropy function

$$\mathcal{E} = 2\pi \left( e_1 q_1 + e_2 q_2 - \frac{1}{8G_3} \left( \mathcal{L}^{(3)}_{\text{EH}} + \frac{\lambda_1}{16} \mathcal{L}^{(3)}_{\text{Riem}} + \frac{\lambda_2}{16} \mathcal{L}^{(3)}_{\text{GB}} \right) \bigg|_{\text{ansatz}} \right), \quad \frac{1}{G_3} = \frac{2\pi^2 P^\frac{3}{4}}{G_6}, \quad (3.10)$$

where $q_1$ and $q_2$ are conserved charges conjugate to $e_1$ and $e_2$ respectively. The period of $z$ is $2\pi$ when combining with $1/(16\pi G_3)$ giving rise to the factor $1/(8G_3)$. In string units, $G_6 = \pi^2/2$ [25]. It is also understood that the contribution from the Lorentz Chern-Simons term is derived following the procedure outlined above. Extremization of the entropy function (3.10) for fixed charges $q_1$ and $q_2$ yields

$$\ell = \sqrt{P}, \quad e_1 = \frac{1}{4} \sqrt{\frac{L_H P + \lambda_1 + \lambda_2}{G_3 \ell q_1}}, \quad e_2 = \frac{\ell^2}{4e_1}, \quad q_2 = \frac{L_H P - \lambda_2}{4G_3 \ell^3}, \quad (3.11)$$

on which the entropy function evaluates to

$$\mathcal{E} = \pi \sqrt{\frac{q_1 (L_H P + \lambda_1 + \lambda_2)}{G_3 \ell}}. \quad (3.12)$$
The expression above can be recast in the form of Cardy formula
\[ E = 2\pi \sqrt{\frac{cq}{6}} \], \quad c = 3 \frac{2G_3}{6} (L_H P + \lambda_1 + \lambda_2) . \quad (3.13)
We also see that the value of $L_H$ is not determined by extremizing the entropy function. In the next section, we will extract the value of $L_H$ from the full-fledged asymptotically flat string solution. Different from the conserved charge $q_2$ which is determined from the extremization procedure, the conserved charge $q_1$ is an input to the entropy function and can be inferred from the known solution in the 2-derivative theory using (3.11) without higher derivative contributions. The result is given by
\[ q_1 = Q_2 - \frac{\nu^2}{PQ_1}, \quad (3.14) \]
in which the parameters $P$, $Q_1$, $Q_2$ and $\nu$ are related to conserved charges carried by the rotating dyonic string and are integer valued in string units. Moreover, (3.14) can be nicely interpreted as the effective energy level in the dual 2D CFT [4]. Using (3.11), we can solve $r_+$ in terms of $q_1$
\[ r_+^2 = \frac{4\ell^3 G_3 q_1}{L_H P + \lambda_1 + \lambda_2} . \quad (3.15) \]
As we will see in the next section, the value of $L_H$ is increased by the higher derivative interactions. Therefore the size of the extremal BTZ horizon actually shrinks when the higher derivative interactions are turned on, although the AdS$_3$ radius remains the same.

4 The renormalized horizon value of dilaton

This section is devoted to compute the value of dilaton at the horizon in the presence of supersymmetric curvature squared terms. For this purpose, the ansatz (3.1) for BPS string solution based on the $U(2) \times \mathbb{R}^2$ isometry is still applicable since the higher derivative extended field equations will not generate terms violating this symmetry. Supersymmetry requires the undetermined functions in (3.1) obey certain relations so that the corresponding Killing spinor equations
\begin{align*}
0 &= (\partial_\mu + \frac{1}{4} \omega_{\mu\alpha\beta} \gamma^{\alpha\beta}) \epsilon^i + \frac{1}{8} H_{\mu\nu\rho} \gamma^{\mu\nu} \epsilon^i , \\
0 &= \frac{1}{2\sqrt{2}} L^{\delta ij} \partial_\mu L \epsilon_j - \frac{1}{12\sqrt{2}} L^{\delta ij} \gamma_{\mu\nu\rho} H^{\mu\nu\rho} \epsilon_j , \quad (4.1)
\end{align*}
admit non-trivial solutions for the symplectic Majorana-Weyl spinor $\epsilon^i$. The indices $i, j$ are raised and lowered by $\epsilon^{ij}$ and $\epsilon_{ij}$. We have also set auxiliary fields to 0 consistent with their field equations [25]. For convenience, we introduce the complex Weyl spinor
\[ \epsilon = \epsilon_1 + i \epsilon_2 , \quad (4.2) \]
assume the Killing spinor to have the form
\[
\epsilon = \Pi(r) \epsilon_0 ,
\]
where $\epsilon_0$ is the standard Killing spinor on a round 2-sphere embedded in the 6D spinor obeying the projection conditions
\[
\gamma^{012345} \epsilon_0 = -\epsilon_0 , \quad \gamma^{01} \epsilon_0 = -\epsilon_0 .
\]
Plugging the ansatz for the bosonic fields and Killing spinor to (4.1) we obtain the sufficient and necessary conditions for the existence of a Killing spinor
\[
0 = A_0 \left( a_2^2 \left( c \omega A_0 - c A_3 + 2 A_3 b \right) - c \omega \dot{d} \right) + c \left( a_1^2 \omega + A_3 d - \dot{f}_1 \right) + 2 b \left( f_1 - a_1^2 \omega \right) ,
\]
\[
0 = a_1 \left( a_2^2 \dot{A}_0 + 2 a_2 \left( A_0 \dot{a}_2 + \dot{a}_1 \right) + \dot{d} \right) + A_0 a_2 \left( \dot{d} - a_2^2 \dot{A}_0 \right) ,
\]
\[
0 = a_2^2 \left( c \left( \omega \dot{A}_0 - \dot{A}_3 \right) + 2 A_3 \dot{b} \right) + 2 b f_2 - c \left( \dot{f}_2 + \omega \dot{d} \right) ,
\]
\[
0 = -a_2^2 \dot{A}_0 + 2 a_1 \dot{a}_2 + \dot{d} ,
\]
\[
0 = b \left( 4 A_0 f_2 \omega - 4 A_3 f_2 + c^2 - 4 f_1 \omega - P \right) - c^2 \dot{c} ,
\]
\[
0 = -a_2^2 A_3 \dot{A}_0 + a_2 \left( -A_0 \dot{f}_2 + 2 a_1 \left( \omega \dot{a}_1 + A_3 \dot{a}_2 \right) + a_1^2 \omega + \dot{f}_1 \right) + a_1 a_2^2 \dot{A}_3 - a_1 \dot{f}_2 ,
\]
\[
0 = a_2 \left( f_1 - A_0 f_2 \right) + a_1 \left( a_2^2 A_3 - f_2 \right) + a_1^2 a_2 \omega ,
\]
\[
0 = \frac{8 A_0 b f_2 \omega}{c^3} - \frac{d}{a_1 a_2} - \frac{8 A_3 b f_2}{c^3} - \frac{8 b f_1 \omega}{c^3} - \frac{2 P b}{c^3} + \frac{\dot{L}}{L} ,
\]
\[
0 = \Pi(r) - \sqrt{a_1} ,
\]
where dots denote derivatives with respect to $r$. The third last equation (4.11) is algebraic whose $r$-derivative is preserved on solutions to equations (4.6), (4.8) and (4.10). Eqs. (4.6) and (4.8) together yield two first integrations
\[
A_0 + \frac{a_1}{a_2} = \kappa_1 , \quad d + a_1 a_2 = \kappa_2 ,
\]
in which $\kappa_2$ can be set to 0 using the residual gauge symmetry of $B_{(2)}$. We can also choose $\kappa_1 = 0$ by shifting the $z$-coordinate and redefining $f_2$ \footnote{For the rotating dyonic string solution without linear momentum \cite{25}, $\kappa_1 = 1$ is the more common choice.}. Subsequently, applying the first equation in (4.14) to (4.10), we find another first integration
\[
a_1 a_2 A_3 + a_1^2 \omega + f_1 = \kappa_3 .
\]
The equation above together with (4.11) forces $\kappa_3 = 0$. From (4.9) and (4.12), we also find
\[
|d| L c^2 = r^2 ,
\]
after choosing the r-coordinate so that $c = br$. Using (4.14) and (4.15), we obtain two more first integrations from Eqs. (4.5) and (4.7)

$$f_2 + d\varpi = r^2\kappa_4, \quad A_3 = A_0\varpi + r^2\kappa_5,$$

(4.17)

where $\kappa_4$ and $\kappa_5$ will be set to 0 in order for the solution to obey the asymptotically flat boundary condition. Substituting (4.17) to Eqs. (4.15) and (4.9), we obtain

$$f_1 = 0, \quad c^2 = r^2 + P.$$  

(4.18)

In summary, the off-shell Killing spinor equations for the asymptotically flat rotating dyonic string ansatz are fully solved provided the following relations are satisfied

$$A_0 + \frac{a_1}{a_2} = 0, \quad d + a_1a_2 = 0, \quad c^2 = r^2 + P, \quad c = br,$$

$$|d|Lc^2 = r^2, \quad f_1 = 0, \quad f_2 = -d\varpi, \quad A_3 = A_0\varpi,$$

(4.19)

which indicates that the horizon value of $L$ is determined by the near horizon behaviour of $d$. This is a significant simplification as $d$-equation is associated with a conservation law and already admits a first integration.

In terms of the new radial coordinate $\rho = r^2$ and applying the set of relations (4.19), we read off the $d$-equation from the $(t, z)$ component of the $B(2)$ field equation. It turns out to be independent of other undetermined variables in the ansatz and can be integrated twice so that the resulting equation is only second order in $\rho$-derivative. Imposing the asymptotically flat boundary condition, one can set one of the integration constant to 0 and the $d$-equation reads

$$(\rho - (\rho + Q_1)d) d + \frac{(\lambda_2 - \lambda_1)P^2}{2(P + \rho)^2}d^2 - \frac{2\rho\lambda_1 + (\lambda_1 + \lambda_2)P}{P + \rho}\rho dd' + \lambda_1\rho^2d^2 - \lambda_1\rho^2dd'' = 0,$$

(4.20)

where prime denotes the $\rho$-derivative. The simple form of the $d$-equation has benefited from the precise structure of the off-shell supersymmetric actions. Had we performed a field redefinition, this simplification is immediately lost. In the context of string compactification, $\lambda_1$ and $\lambda_2$ are of the same order much smaller than $P$ or $Q_1$. The equation above can thus be solved order by order algebraically in the small parameters $\lambda$s. Here, only the corrections at the first order in $\lambda$s are meaningful because higher order terms will be modified by $R^n$ ($n > 2$) interactions which are not included in our discussion. To the first order in $\lambda$s, the solution of the $d$-equation is given by

$$d(\rho) = \frac{\rho}{\rho + Q_1} \left[ 1 - \frac{P^2Q_1^2(\lambda_1 + \lambda_2) + 2PQ_1^2(\lambda_1 + \lambda_2)\rho + (Q_1^2(\lambda_1 + \lambda_2) + (P - Q_1)^2(\lambda_1 - \lambda_2))\rho^2}{2(P + \rho)^2(Q_1 + \rho)^3} \right],$$

(4.21)
from which one can see that corrections to the leading order solution stays small within the entire range of $\rho$ as long as $\lambda$s are much less than $P$ or $Q_1$. In the special case $\lambda_1 = \lambda_2$, the combination of the curvature squared super-invariants is consistent with 16 supercharges, and the solution is independent of $P$. Meanwhile the conserved charge $q_2$ (3.11) is not modified by the higher derivative interactions, which is physically more reasonable as the curvature squared terms do not modify the conserved charges carried by the asymptotically flat dyonic string solution. The equal charge case $P = Q_1$ corresponds to BMPV black hole [2], and the solution also looks quite simple depending only on the sum of $\lambda$s. Near the string horizon at $\rho = 0$,

$$d(\rho) = \frac{\rho}{Q_1} \left( 1 - \frac{\lambda_1 + \lambda_2}{2Q_1} \right) + \mathcal{O}(\rho^2).$$

(4.22)

Combining this result with the BPS equation relating $d$ and $L$, we obtain the horizon value of the dilaton with curvature squared corrections

$$L_H = \frac{Q_1 + \frac{1}{2}(\lambda_1 + \lambda_2)}{P}.$$

(4.23)

Substituting the result above to (3.13), we complete the calculation of the macroscopic entropy for the higher derivative corrected rotating BPS dyonic string solution. The final entropy formula takes the form

$$E = 2\pi \sqrt{\frac{cq_1}{6}}, \quad c = \frac{3}{2G_3\ell_s} \left( Q_1 + \frac{3}{2}(\lambda_1 + \lambda_2) \right), \quad q_1 = Q_2 - \frac{\nu^2}{PQ_1}.$$

(4.24)

The sum of $\lambda$s is related to the total coefficient in front of the $B_{(2)} \wedge \text{Tr}(R \wedge R)$ term or the Riemann tensor squared. It also means each independent supersymmetric completion of the Riemann tensor squared contributes equally to the entropy. In the IIA string embedding, $\lambda_1 + \lambda_2$ is given by $2\alpha'$ [14]. Since entropy is a dimensionless quantity, it does not depends on the choice of units. In particular, we can choose the string units in which the string length $\ell_s = 1$. Also, we recall from (3.10) that in the string units $1/G_3 = 4P^2$. Therefore the central charge is equal to $c = 6P(Q_1 + 3)$ indicating that the macroscopic entropy with leading order higher derivative correction is of the form

$$E = 2\pi \sqrt{PQ_1Q_2 - \nu^2} \left( 1 + \frac{3}{2Q_1} \right),$$

(4.25)

which is consistent with the result obtained in [13] specialized to the K3 $\times T^2$ compactification. One should also be aware that in the string units, the parameters $P$, $Q_1$, $Q_2$ and $\nu$ are all integer valued [23]. The results obtained here and [13] agree with the one reported in [12] only in the static case $\nu = 0$. It should be interesting to connect the macroscopic result to the microscopic one in certain asymptotic limits [35][39].
5 Conclusion and discussions

In this work, we computed the macroscopic entropy of the supersymmetric rotating dyonic strings in 6D (1,0) supergravity with curvature squared corrections, adopting Sen’s entropy function formalism applied to the near horizon geometry of the string. Upon solving the attractor equations whose solution extremize the entropy function, we observed that inclusion of the higher derivative interactions does not modify the \( \text{AdS}_3 \) radius of the BTZ black hole but does shrink the size of the extremal horizon. The expression of the string entropy depends on the horizon value of the dilaton which is not determined by the attractor equations but can only be extracted from the complete asymptotically flat string solution. We also showed that although there are two independent supersymmetric completions of the Riemann tensor squared, the entropy is insensitive to their detailed structures but depends only on the coefficient of the Riemann squared term fixed to be \( \alpha'/8 \) upon embedding the 6D supergravity model in the K3 compactification of IIA string.

Via a circle reduction, the rotating dyonic string becomes the 3-charge spinning black hole in 5D ungauged STU model. Thus our result can be compared to previous work [12,13] on the macroscopic entropy of supersymmetric black holes in 5D \( \mathcal{N} = 2 \) supergravity with curvature squared corrections descending from M-theory compactified on K3×\( T^2 \). Our entropy formula is in agreement with the result obtained in [13] but exhibits a different angular momentum dependence from the result reported in [12]. In future, we hope to compute the microscopic entropy of the dual CFT\(_2\) and compare it to the macroscopic result.

String dualities relate IIA string on K3 to heterotic string on \( T^4 \) in which the black hole entropy with \( \alpha' \) corrections was previous obtained in [36,37]. A comparison between the entropy of black holes in the heterotic and IIA strings shows that once \( \alpha' \) corrections are taken into account, the black hole entropy is not invariant under the duality transformation.

Locally the extremal \( \text{BTZ} \times S^3 \) is equivalent to \( \text{AdS}_3 \times S^3 \) around which the spectrum of fluctuations has been analysed in [15,16]. The fact that the spectrum can be organized into representations of \( \text{SU}(1,1|2) \times \text{SU}(1,1) \times \text{SU}(2) \) [15,16] suggests that the \( S^3 \) compactification of the 6D model can be described by the matter coupled 3D supergravity model based on the \( \text{SU}(1,1|2) \times \text{SU}(1,1) \) algebra. Omitting the curvature squared terms, the action for the massless fields in the supergravity multiplet can be formulated as a Chern-Simons gauge

\[^{5}\text{In the recent work [36] revisiting the \( \alpha' \)-corrected black hole in heterotic string, the coefficient in front of the supersymmetric black hole should be \( \alpha'/8 \) instead of \( \alpha' \).}^5\]
theory. After integrating out auxiliary fields, it takes the form

\[ S_{3d} = \int d^3 x \left[ k_G \sqrt{-g} \left( R + \frac{2}{\ell^2} \right) + k_{CS} \Omega(\Gamma) - (\ell k_G - k_{CS}) \Omega(A_R) \right], \]

\[ \Omega(\Gamma) = \frac{1}{2} \text{Tr} (\Gamma d \Gamma + \frac{2}{3} \Gamma^3), \quad \Omega(A_R) = \text{Tr} (A_R d A_R + \frac{2}{3} A_R^3), \] (5.1)

where \( A_R \) is the gauge field associated with the SU(2) inside SU(1,1|2) and various constants are given by

\[ k_G = \frac{P L_H + \alpha'}{16\pi G_3 P} = \sqrt{P} (Q_1 + 2) \frac{4\pi}{4\pi}, \quad k_{CS} = - \frac{\alpha'}{16\pi G_3 \sqrt{P}} = - \frac{P}{4\pi}. \] (5.2)

One feature is that the effective Newton’s constant depends on the value of dilaton at the string horizon that is affected by the higher derivative interaction via equations of motion. The level of the SU(2)_R Chern-Simons gauge field is proportional to the central charge appearing in the entropy formula

\[ k_R = - \frac{P (Q_1 + 3)}{4\pi}. \] (5.3)

The fact that the quantity above obeys the quantization condition \( 4\pi k_R = n, n = 0, \pm 1, \ldots \) implies that the higher derivative corrections to the horizon value of \( L \) can be at most quadratic in \( \alpha' \). For instance, \( \alpha'^2 / P^2 \) may contribute a \( P \) independent integer to \( 4\pi k_R \). Based on dimension analysis, such term would come from a supersymmetric \( R^3 \) action which is forbidden in both heterotic string and type II string. Therefore the value of dilaton on the string horizon does not receive corrections from higher derivative interactions beyond order \( \alpha' \).

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