Quantum transfer matrices for discrete and continuous quasi-exactly solvable problems

A.V. Zabrodin

Institute of Chemical Physics, Kosygina st. 4, 117334, Moscow, Russia, and
ITEP, 117259, Moscow, Russia,
e-mail: zabrodin@vxitep.itep.ru

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Abstract

We clarify the algebraic structure of continuous and discrete quasi-exactly solvable spectral problems by embedding them into the framework of the quantum inverse scattering method. The quasi-exactly solvable Hamiltonians in one dimension are identified with traces of quantum monodromy matrices for specific integrable systems with non-periodic boundary conditions. Applications to the Azbel-Hofstadter problem are outlined.

1 Introduction

At present time the methods related to quantum integrability are highly developed. Originally they were invented and used in the context of quantum field theory. However, the main ingredients proved to be purely algebraic. They can be successfully applied to quantum-mechanical (i.e., one-particle) problems as well.

An example of a new application of this kind is the recent progress [1], [2] in the famous problem of Bloch electrons in magnetic field on a two-dimensional lattice [3] (sometimes called the Azbel-Hofstadter problem). Even the one-particle problem is non-trivial. In a proper gauge it reduces to a one-dimensional quasiperiodic difference equation (Harper’s equation is the most popular example). It has been shown in [1], [2] that some of these equations admit partial exact solutions of the form typical for quantum integrable systems: the eigenfunctions are polynomials with the roots constrained by Bethe equations. These solutions give some specific states (one for each stable band), the energies being expressed through the roots in a simple way. In the paper [4] this result has been generalized to a whole class of second-order difference operators admitting partial algebraization of the spectrum.

The form of the result suggests to ask for a direct connection with quantum integrability. Indeed, such connection does exist.

A unified approach to quantum integrable systems is most elegantly formulated in terms of the Quantum Inverse Scattering Method (QISM) created by the Leningrad School [5] (for a more recent review see [6]). In the paper [2] (Appendix B) it has been shown how to embed the Azbel-Hofstadter problem into the QISM. The Hamiltonian has been identified with a quantum transfer matrix (trace of a quantum monodromy matrix) for a specific integrable system with boundaries. This gives a possibility to apply the powerful machinery of the QISM such as functional Bethe ansatz [7].

This paper may be considered as an extensive comment to Appendix B of the paper [2]. We give a detailed construction of the quantum monodromy matrices for the general family of difference equations considered in [4]. In a more general context, we provide new formal grounds for studying difference or differential operators in one variable having a finite number of polynomial eigenfunctions. The continuum
limit corresponds to models with the rational $R$-matrix. In this case we reproduce a class of second-order differential operators having the property of the partial algebraization of the spectrum. Their eigenvalue equations were considered in the literature some time ago \cite{12, 13} (the idea goes back to the papers \cite{14, 15}). A systematic treatment, based on the hidden dynamical $sl(2)$-symmetry, was given in \cite{16} (see also the reviews \cite{17, 18} and references therein). A different approach was suggested in \cite{19}. These equations are known as "quasi-exactly solvable" problems ("quasi" means that usually only a part of the spectrum can be found in a closed algebraic form). The corresponding hamiltonians are known \cite{20} to be quadratic forms in the standard generators of $sl(2)$ (taken in a finite-dimensional representation). In our approach the generators of $sl(2)$ appear as matrix elements of the universal $2 \times 2$ $L$-operator of $XXZ$-type. To $q$-deform this picture, one should use $XXZ$-type $L$-operators. Their matrix elements are expressed through generators of $U_q(sl(2))$, the $q$-deformation of the universal enveloping algebra of $sl(2)$. This construction yields difference equations\footnote{An attempt to apply quantum algebras for generating quasi-exactly solvable difference equations was made in \cite{21}. However, the authors used another version of the quantum algebra (which does not allow one to construct hermitian hamiltonians; for details see \cite{22} and did not discuss the Bethe ansatz solutions.}. 

In short, we reduce the quasi-exactly solvable spectral problems mentioned above to the problem typical for quantum integrable systems and lattice statistical models, i.e., to diagonalization of a transfer matrix. Besides, we give the QISM interpretation of isospectral transformations of quasi-exactly solvable hamiltonians (in the continuous case) under adjoint $SL(2)$-action.

Here is a more detailed description of the content.

In Sect.2 we describe universal trigonometric $2 \times 2$ $L$-operators depending on spectral parameter. There are two kinds of them: one is related to $U_q(sl(2))$ (it is usually used in integrable $XXZ$ magnets with higher spin), another one is associated to the dual algebra, $A_q(SL(2))$. The matrix elements are expressed through the generators of $U_q(sl(2))$ and $A_q(SL(2))$ respectively. The relevant representations of these algebras are briefly reviewed. The rational limit of these $L$-operators is also discussed. The former turns to the universal $L$-operator of the isotropic $XXX$-type integrable magnet while the latter becomes a $c$-number $2 \times 2$ matrix independent of the spectral parameter.

The necessary extraction from the formalism treating integrable systems with boundaries is given in Sect.3. The starting point is "reflection equations" \cite{23}. Following \cite{24}, we recall the construction of quantum monodromy and transfer matrices for systems with boundaries.

In Sect.4 we apply this general formalism to the elementary $L$-operators and obtain in this way quasi-exactly solvable hamiltonians. Applications to the Azbel-Hofstadter problem are outlined.

The rational (continuum) limit is treated in Sect.5. A comparison with the representation in terms of Gaudin’s magnet (suggested in \cite{23}) is made. In Sect.6 we discuss isospectral transformations of continuous quasi-exactly solvable hamiltonians under adjoint action of $SL(2)$. The QISM interpretation of these transformations in terms of the rational limit of the $L$-operator related to $A_q(SL(2))$ is suggested. Sect.7 contains conclusions and speculations on some open problems.

## 2 Elementary $L$-operators and quantum algebras

The standard basis of the quantum integrability is the Yang-Baxter equation\footnote{\cite{25}} (YBE) with a spectral parameter $u$:

$$R(u/v)T_1(u)T_2(v) = T_2(v)T_1(u)R(u/v),$$

where $T_1(u) = T(u) \otimes 1$, $T_2(u) = 1 \otimes T(u)$. We take $T(u)$ to be a $2 \times 2$ matrix with operator matrix elements and

$$R(u) = \begin{pmatrix}
    qu-q^{-1}u^{-1} & 0 & 0 & 0 \\
    0 & u-u^{-1} & q-q^{-1} & 0 \\
    0 & q-q^{-1} & u-u^{-1} & 0 \\
    0 & 0 & 0 & qu-q^{-1}u^{-1}
\end{pmatrix}$$

is the symmetric trigonometric $R$-matrix with the parameter $q$ (or its rational degeneration). In the trigonometric case we use the multiplicative parametrization.
The equation (1) determines commutation relations for the elements of the quantum monodromy matrix $T(u)$. The elementary solutions of the YBE (i.e., those which can not be decomposed into a product of simpler ones) are of particular importance for us. They are called $L$-operators. In lattice integrable models (or spin chains) an $L$-operator is usually associated to a lattice site. Here are two main examples of $L$-operators.

1). $L$-operators associated to $U_q(sl(2))$.

Consider the $L$-operator

$$L(u) = \begin{pmatrix} uA - u^{-1}D & (q-q^{-1})C \\ (q-q^{-1})B & uD - u^{-1}A \end{pmatrix}.$$  

(3)

It obeys the YBE (1) if and only if $A$, $B$, $C$, $D$ satisfy the commutation relations of the $U_q(gl(2))$ algebra [19], [20], [21], [22], [23] :

$$AB = qBA, \quad BD = qDB,$$

$$DC = qCD, \quad CA = qAC,$$

$$[B,C] = \frac{A^2 - D^2}{q-q^{-1}},$$

$$[A,D] = 0.$$  

(4)

This quadratic algebra has two central elements. One of them is a $q$-analog of the Casimir operator:

$$w = q^{-1}A^2 + qD^2 + (q - q^{-1})^2 BC,$$  

(5)

another one,

$$w_0 = AD,$$  

(6)

for the $U_q(sl(2))$ case should be put equal to 1. If $q$ is a root of unity some additional central elements appear.

Irreducible finite-dimensional representations of dimension $2j+1$ can be expressed in the weight basis, where $A$ and $D$ are diagonal matrices: $A = \text{diag}(q^j, \ldots, q^{-j})$. An integer or halfinteger $j$ is spin of the representation. There exists the following realization [24] by difference operators acting in the linear space of polynomials $F(z)$ of degree $2j$:

$$AF(z) = q^{-j}F(qz),$$

$$BF(z) = -\frac{z}{q-q^{-1}}(q^{-2j}F(qz) - q^{2j}F(q^{-1}z)),$$  

$$CF(z) = \frac{1}{z(q-q^{-1})}(F(qz) - F(q^{-1}z)),$$  

$$DF(z) = q^jF(q^{-1}z).$$  

(7)

Then $F_0(z) = 1$ is the lowest weight vector whereas $F_{2j}(z) = z^{2j}$ is the highest weight vector, i.e., $CF_0(z) = 0$, $BF_{2j}(z) = 0$. The Casimir operator [5] in this realization is equal to the $c$-number $q^{2j+1} + q^{-2j-1}$.

If $q$ is a root of unity there is, in addition, three-parametric family of finite-dimensional representations having, in general, no lowest and no highest weight [20]. Sometimes they are called cyclic representations [25]. The difference quasi-exactly solvable equations corresponding to the cyclic representations of $U_q(sl(2))$ are particularly important [4] in applications to the Azbel-Hofstadter problem. However, in this paper we do not consider this case.

2). $L$-operators associated to $A_q(SL(2))$.

\footnote{We use Koornwinder’s notation [24] for the generators.}
Another important class of $L$-operators is constructed by means of the dual quantum algebra $A_q(SL(2))$, the $q$-deformed algebra of functions on the group $SL(2)$. Consider the operator matrix
\[
\hat{g}(u) = \begin{pmatrix}
\hat{a} & \hat{b} \\
 u^{-1} \hat{c} & \hat{d}
\end{pmatrix}.
\] (8)

It is easily verified that it satisfies the YBE (9),
\[
R(u/v)\hat{g}_1(u)\hat{g}_2(v) = \hat{g}_2(v)\hat{g}_1(u)R(u/v),
\] if and only if $\hat{a}$, $\hat{b}$, $\hat{c}$, $\hat{d}$ obey the algebra
\[
\hat{a}\hat{b} = q\hat{b}\hat{a}, \quad \hat{b}\hat{d} = q\hat{d}\hat{b}, \quad \hat{a}\hat{c} = q\hat{c}\hat{a}, \quad \hat{c}\hat{d} = q\hat{d}\hat{c}, \quad [\hat{a}, \hat{d}] = (q - q^{-1})\hat{b}\hat{c}, \quad [\hat{b}, \hat{c}] = 0.
\] (10)

These are commutation relations for the generators of the dual algebra of $U_q(gl(2))$ \[21\], \[26\]. We denote it $A_q(GL(2))$. The conventional interpretation of $A_q(GL(2))$ identifies it with a $q$-deformed algebra of functions on the group $GL(2)$.

The central element is $\hat{b}\hat{c}^{-1}$ (it belongs to an extended algebra); another one is the $q$-determinant $\hat{a}\hat{d} - q\hat{b}\hat{c}$, which for the $SL(2)$-case should be put equal to 1. The corresponding factorialgebra is denoted $A_q(SL(2))$. Restricting to the compact real form of the quantum group, one obtains the algebra $A_q(SU(2))$, which was extensively studied \[26\].

For completeness, we give the list of irreducible unitary representations of $A_q(SU(2))$ for real $q$ \[21\]. There are two series:

a). One-dimensional representations:
\[
\hat{b} = \hat{c} = 0, \quad \hat{a} = \hat{d}^{-1} = e^{i\varphi},
\] (11)

$0 \leq \varphi < 2\pi$.

b). Infinite-dimensional representations (parametrized by the same "continuous spin" $\varphi$. They can be realized on functions in one variable \[27\], \[28\]:
\[
\hat{a}f(z) = -z^{-1}(f(qz) - f(q^{-1}z)),
\]
\[
\hat{b}f(z) = q^{-1}e^{i\varphi}f(qz),
\]
\[
\hat{c}f(z) = -q^2e^{-i\varphi}f(qz),
\]
\[
\hat{d}f(z) = qzf(qz).
\] (12)

Note that $e^{i\varphi}$ enters (12) in the same way as the spectral parameter $u$ enters (8).

There are also some $c$-number solutions to (1): diagonal, \[14\];
\[
T(u) = \begin{pmatrix}
* & 0 \\
0 & *
\end{pmatrix},
\] (13)

and antidiagonal, \[14\];
\[
T(u) = \begin{pmatrix}
0 & * \\
* & 0
\end{pmatrix},
\] (14)

where $*$ stands for an arbitrary $u$-independent $c$-number. The former come from the $L$-operators $\hat{g}(u)$ taken in the representation of the series a) \[14\], while the latter correspond to the $L$-operator \[4\] and the one-dimensional representation ($A = D = 0, B$ and $C$ are $c$-numbers) of the algebra \[4\].
The trace (in the auxiliary two-dimensional space) of $T(u)$ obeying \([\mathfrak{g}]\) is a generating function of commuting integrals of motion: $t(u) = \text{Tr} T(u)$,

$$[t(u), t(v)] = 0 .$$

In the case of elementary $L$-operators there is only one independent integral of motion (considered as a hamiltonian). Though the commutativity \([\mathfrak{g}]\) is meaningless in this case, the transfer matrix $t(u)$ possesses all necessary formal properties, which allow one (at least, in principle) to apply the technique of the algebraic (or functional) Bethe ansatz.

The rational limit means $q = e^{\hbar}$, $u = e^{\hbar \tilde{u}}$, $\hbar \to 0$, and $\tilde{u}$ becomes an additive spectral parameter. For future reference, let us present some formulas related to the rational limit. In the rest of this section we write simply $u$ instead of $\tilde{u}$.

The $L$-operator \([\mathfrak{g}]\) becomes

$$L(u) = \left( \begin{array}{cc} u + S_0 & S_- \\ S_+ & u - S_0 \end{array} \right) ,$$

where $S_i$ are generators of $\mathfrak{sl}(2)$:

$$[S_\pm, S_0] = \mp S_\pm , \quad [S_+, S_-] = 2 S_0 .$$

The correspondence with $U_q(\mathfrak{sl}(2))$ is as follows: $(A - D)/(2\hbar) \to S_0$, $B \to S_+$, $C \to S_-$. The realization \([\mathfrak{g}]\) is a smooth $q$-deformation of the standard representation of $\mathfrak{sl}(2)$ by first-order differential operators:

$$S_- = \frac{d}{dz} , \quad S_0 = z \left( \frac{d}{dz} - j \right) , \quad S_+ = -z^* \left( \frac{d}{dz} + 2j \right) .$$

This representation has been used to construct and classify linear differential equations having polynomial solutions \([\mathfrak{g}]\). The QISM interpretation is given in Section 5.

The rational limit of \([\mathfrak{g}]\) is simply a ($u$-independent) generic "group element" of $SL(2)$ taken in the fundamental representation:

$$g(u) = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) .$$

Note that $a, b, c, d$ are c-numbers in this case since the algebra \([\mathfrak{g}]\) becomes commutative. In Section 6 this "$L$-operator" is used for a QISM interpretation of global $SL(2)$-rotations of quasi-exactly solvable hamiltonians.

### 3 General properties of monodromy matrices for open integrable spin chains

Here we give a brief summary of the formalism treating integrable systems with boundaries. The boundary conditions consistent with integrability are determined by c-number $2 \times 2$ matrices $K_l(u)$ and $K_r(u)$ (for the left and right boundary respectively) depending on the spectral parameter and satisfying the "reflection equations" (RE) \([\mathfrak{g}]\):

$$R(u/v) (K_l(u) \otimes 1) R(u v^{-1}) (1 \otimes K_l(v)) =$$

$$(1 \otimes K_r(v)) R(u v^{-1}) (K_l(u) \otimes 1) R(u/v) ,$$

$$R(v/u) (K^*_l(u) \otimes 1) R(u^{-1} v^{-1} q^{-1}) (1 \otimes K^*_r(v)) =$$

$$(1 \otimes K^*_r(v)) R(u^{-1} v^{-1} q^{-1}) (K^*_l(u) \otimes 1) R(v/u)$$

($t$ means the transposition) with the $R$-matrix \([\mathfrak{g}]\). Solutions for $K_l$ and $K_r$ are related: if $K_l(u)$ is a solution to \([\mathfrak{g}]\), then $K^*_l(u^{-1})$ is a solution to \([\mathfrak{g}]\). In the scattering picture, the RE’s describe the factorized scattering of a two-state particle on left and right walls respectively.

More generally, one can consider operator solutions to the RE’s \([\mathfrak{g}], [\mathfrak{g}]\) (i.e., matrices $K(u)$ with operator valued matrix elements). Speaking informally, the wall may carry quantum numbers. In this
case the RE’s determine commutation relations of the matrix elements. They generate the "reflection algebra”. In the case of absence of the spectral parameter this algebra was studied in [29].

The QISM approach to integrable systems with boundaries was developed by Sklyanin [18]. The main results of the paper [18] are summarized below in the form of two theorems.

**Theorem 1.** Let $T(u)$ satisfy the YBE [1] with the $R$-matrix (2) and let $K_l(u)$ (resp., $K_r(u)$) satisfy the RE (20) (resp., (21)) with the same $R$-matrix. Then

$$K_l(u) = T(u)K_l(u)T^t(u^{-1})\sigma_2,$$  
**(22)**

$$K_r(u) = (T^t(u)K_r(u)\sigma_2T(u^{-1})\sigma_2)^t.$$  
**(23)**

satisfy (20) and (21) respectively (here and below $\sigma_i$ are Pauli matrices).

**Remark 1.** The theorem holds for both operator and c-number $K$-matrices.

**Remark 2.** One may interpret (22), (23) as a "dressing transformation": $K$ "dressed" by $T$ yields $K$.

**Remark 3.** For unimodular c-number matrices independent of $u$ the operation $\sigma_2T^t\sigma_2$ is simply $T^{-1}$.

**Remark 4.** For a c-number matrix $K$, the (operator) matrix $K$ is called the quantum monodromy matrix for an integrable system with non-periodic boundary conditions.

It is convenient to represent (22), (23) pictorially as follows:

$$K_l(u) = K_l \quad T \quad K_r(u) = K_r$$

**Theorem 2.** Let $K_l(u)$ and $K_r(u)$ be any solutions of (20) and (21) respectively. Then the quantities

$$\tau(u) = \text{Tr}(K_r(u)K_l(u))$$  
**(24)**

form a commutative family:

$$[\tau(u), \tau(v)] = 0.$$  
**(25)**

The quantity $\tau(u)$ is called a quantum transfer matrix. Its diagonalization can be performed by means of the algebraic (or functional) Bethe ansatz technique. To describe integrable open spin chains, one should put $K_r = K_r$ (a c-number solution) in (24) and substitute $K_l$ from (22):

$$\tau(u) = K_l \quad T \quad K_r$$

## 4 Trigonometric case

The boundary matrices for the reflection of a two-state particle on a scalar wall are given by [30]

$$K_l(u) = \begin{pmatrix} 2x_0(q^{-1}s^{-1}u - qu^{-1}) & x_+(q^{-1}u^{-2} - qu^{-2}) \\ x_-(q^{-1}u^{-2} - qu^{-2}) & -2x_0(su - s^{-1}u^{-1}) \end{pmatrix},$$  
**(26)**

\[ K_r(u) = \begin{pmatrix} 2y_0(qtu - q^{-1}tu^{-1}) & y_+(qu^2 - q^{-1}u^{-2}) & -2y_0(t^{-1}u - tu^{-1}) \\ u_-(qu^2 - q^{-1}u^{-2}) & y_+(qu^2 - q^{-1}u^{-2}) & y_+(qu^2 - q^{-1}u^{-2}) \\ 0 & -2y_0(t^{-1}u - tu^{-1}) & 2y_0(t^{-1}u - tu^{-1}) \end{pmatrix}, \]  

where \( x_0, x_\pm, s \) and \( y_0, y_\pm, t \) are arbitrary parameters characterizing the boundary conditions.

The \( L \)-operator \( L(u) \) still satisfies the YBE (1) if \( u \) is multiplied by a constant \( k \). Substituting \( L(\alpha k) \) for \( T(u) \) in (22) we get the quantum monodromy matrix

\[ M(u) = L(u)K(u)\sigma_2L^\dagger(u^{-1}k)\sigma_2 \]  

(in this specific case we denote it \( M(u) \)). The calculation of its matrix elements is straightforward. One should take into account that the quadratic Casimir element \( w \) under any irreducible representation acts as a \( c \)-number. It is convenient to represent the result in the following form.

Consider the operators

\[ H_1 = x_k A + x_- k^{-1} CA + 2(q - q^{-1})^{-1} x_0 s^{-1} A^2, \]  

\[ H_2 = x_+ k^{-1} DB + x_- k CD - 2(q - q^{-1})^{-1} x_0 s D^2, \]  

\[ H_3 = (q - q^{-1})^{-1} x_+ (k^2A^2 + k^{-2} D^2) - (q - q^{-1}) x_- C^2 + 2x_0 (sk^{-1} DC - s^{-1} k AC), \]

\[ \tilde{H}_3 = (q - q^{-1})^{-1} x_+ (k^2A^2 + k^{-2} D^2) - (q - q^{-1}) x_- C^2 + 2x_0 (skBD - s^{-1} k^{-1} BA). \]

We note that \( H_1, H_2, H_3 \) form a simple quadratic algebra (a slightly different version of this algebra was previously studied in [31]):

\[ q^{-1} H_\alpha H_\beta - q H_\beta H_\alpha = g_\gamma H_\gamma + h_\gamma, \]

where \( \{\alpha, \beta, \gamma\} \) stands for any cyclic permutation of \( \{1, 2, 3\} \). The structure constants are:

\[ g_1 = -(1 + q^{-2}) x_+, \quad g_2 = -(1 + q^2) x_+, \quad g_3 = (q + q^{-1}) x_- , \]  

\[ h_1 = -2 \frac{x_0 x_+}{q - q^{-1}} (s (k^2 + k^{-2}) + q^{-1} s^{-1} w) , \]  

\[ h_2 = -2 \frac{x_0 x_+}{q - q^{-1}} (s (k^2 + k^{-2}) + qsw) , \]  

\[ h_3 = \frac{1}{q - q^{-1}} (4x_0^2 - x_+ x_- (k^2 + k^{-2}) w). \]

The operators \( H_1, H_2, \tilde{H}_3 \) form a similar algebra; in particular,

\[ q^{-1} H_2 H_1 - q H_1 H_2 = (q + q^{-1}) x_+ \tilde{H}_3 + h_3. \]

Let us decompose \( M(u) \) into the operator part \( \hat{M}(u) \) and the \( c \)-number part \( M^{(c)}(u) \):

\[ M(u) = \hat{M}(u) + M^{(c)}(u). \]

Then \( \hat{M}(u) \) can be compactly written down in terms of the operators [29]-[32]:

\[ \hat{M}(u) = \frac{q^{-1} u^2 - qu^{-2}}{q - q^{-1}} \begin{pmatrix} -uH_1 + u^{-1} H_2 & H_3 \\ \tilde{H}_3 & q^{-1} u^{-1} H_1 - qu H_2 \end{pmatrix} \]

Note that the boundary parameters do not appear explicitly in \( \hat{M}(u) \) entering only through the structure constants of the algebra [33]. For the \( c \)-number part one has

\[ M_{11}^{(c)}(u) = M_{22}^{(c)}(qu^{-1}) = 2x_0 w(su - s^{-1} u^{-1}) + (k^2 + k^{-2})(q^{-1} s^{-1} u - qu^{-1}) \]  

\[ \frac{(q - q^{-1})^2}{(q - q^{-1})^2} \]

\[ M_{12}^{(c)}(u) = \frac{x_+}{x_-} M_{21}^{(c)}(u) = -x_+ \frac{(u^2 + u^{-2})(q^{-1} u^2 - qu^{-2})}{(q - q^{-1})^2}. \]
It is clear from (31), (32) that the non-diagonal elements of $M(u)$ generally do not have a zero mode (a "false vacuum") independent of $u$. In such a case the general strategy of the functional Bethe ansatz consists essentially in passing to the new basis formed by the eigenvectors of $H_3$ or $H_3$. Note that these operators contain only elements of the lower (resp., upper) Borel subalgebra of $U_q(sl(2))$ (i.e., for example, $H_3$ is a quadratic form in $A$, $D$ and $C$, not $B$). This allows one to find the eigenvectors of $H_3$ and $H_3$ in a quite explicit form (4). Under the representation (3) the eigenfunctions are big $q$-Jacobi polynomials (see e.g. (32)). This fact may be useful for diagonalization of $\tau(u)$ by means of the functional Bethe ansatz.

Disregarding the $c$-number part, we get the quantum transfer matrix:

$$\tau(u) = Tr(K_r(u)\hat{M}(u)) = \frac{(qu^2 - q^{-1}u^{-2})(q^{-1}u^2 - qu^{-2})}{q - q^{-1}}(2y_0(t^{-1}H_2 - tH_1) + y_+\hat{H}_3 + y_-\hat{H}_3). \quad(40)$$

It is clear that in this case the family of commuting integrals of motion generated by $\tau(u)$ contains only one (independent) operator. In terms of $A$, $B$, $C$, $D$ (4) the transfer matrix becomes a generic homogeneous quadratic form in these operators (see (29)-(32)) since it depends on 7 parameters: 3 in each $K$-matrix (the common factor in (26) is inessential) and $k$. Indeed, the total number of coefficients of a general quadratic form is 10 but two of them contribute only to the $c$-number term in (37) due to the two central elements ($AD = 1$ and the Casimir operator); besides, the common multiplier is also inessential.

As it is shown in (1), (3) the hamiltonian $\hat{H}$ of the Bloch particle in a magnetic field is a particular quadratic form in the $U_q(sl(2))$ generators with $|q| = 1$ (the coefficients depend on the gauge and the type of the lattice) and therefore this system can be considered as an integrable model. Here is a list of the most important examples (1), (2), (3).

1). Square lattice, modified Landau gauge: $K_i = \sigma_3, K_r = \sigma_1, k \to \infty$,

$$\hat{H} = -i(q - q^{-1})q^{-1/2}(CA + BD). \quad(41)$$

The flux per plaquette is $\Phi = 2\pi P/Q$ ($P$, $Q$ are coprime integers), $q = e^{i\Phi/2}$.

2). Square lattice, chiral gauge: $K_i = \sigma_3 + (q^{-1/2}u - q^{1/2}u^{-1})\sigma_+, K_r = \sigma_3 + (q^{1/2}u - q^{-1/2}u^{-1})\sigma_-$, $k = 1$,

$$\hat{H} = i(q - q^{-1})q^{-1/2}(CA - BD + qBA - qCD). \quad(42)$$

The flux per plaquette is $4\pi P/Q$ ($P$ odd).

3). Triangular lattice, modified Landau gauge: $x_+ = y_- = 0, x_0 = y_0 = k = 1, t = -s \to \infty, x_- = y_+ \to \infty, y_+/s \to 2AQ^{-1/2}exp(-i\pi(P - 1)/2)$, where $\lambda$ is the hopping amplitude along the third axis, the flux per elementary triangle is $\pi P/Q$ ($P$ odd). The hamiltonian is

$$\hat{H} = \lambda(A^2 + D^2) + e^{i\pi(P - 1)/2}q^{-1/2}(q - q^{-1})(CA + BD). \quad(43)$$

4). Square lattice, chiral gauge (a different version). We include this example only for completeness. It is related to transfer matrix for a closed chain: $\tau(u) = Tr(\sigma_2L(u))$ (see (13)). Here $L(u)$ is the $L$-operator (3) and $\sigma_2$ is a $c$-number solution to (1) of the form (14). The hamiltonian is

$$\hat{H} = i(q - q^{-1})(C - B), \quad(44)$$

($Q$ odd, $P$ even).

Under the representation (5), the transfer matrix (10) becomes a second-order difference operator in $z$. Clearly, this operator has the invariant subspace of polynomials spanned by $1, z, z^2, \ldots, z^{2j}$. The polynomial eigenfunctions lying in this "algebraic" sector and the eigenvalues can be found in the form standard for the algebraic Bethe ansatz technique. A detailed analysis of the equations arising from (40) and their Bethe ansatz solutions is given in (3) (see also Appendix B in [2]).

Here is the explicit form of these equations in terms of the parameters of $K_i$ and $K_r$. By means of (7) and (13) we rewrite the spectral problem for (40) as follows:

$$a(z)\psi(q^2z) + d(z)\psi(q^{-2}z) - v(z)\psi(z) = E\psi(z), \quad(45)$$

where

$$a(z) = (q^{-2j+1}x_+z - 2x_0q^{-j} - x_-k^{-2}z^{-1})^{-1}q^{-2j}y_+z + 2y_0tkq^{-j} + q^{-1}y_-k^2z^{-1}, \quad(46)$$
In this section we consider the limit \( q \to 1 \) (the ”rational”, or continuum limit) of the monodromy matrix \((32)\), providing a basis for embedding the continuous quasi-exactly solvable problems \([12, 13]\) into the quantum inverse scattering approach.

The construction of the previous section may have different continuum limits. Below we consider the most important one, which is directly related to integrable models with the rational \( R \)-matrix. The rule of performing this limit is as follows. Put \( q = e^{i\hbar}, u = e^{i\hbar \bar{s}}, s = e^{i\hbar \bar{t}}, t = e^{i\hbar \bar{t}}, k = e^{i\hbar \bar{k}}, \) and find the \( \hbar \)-expansion of the monodromy matrix \((28)\) as \( \hbar \to 0 \), provided \( x_0, x_\pm, y_0, y_\pm \) are \( \hbar \)-independent constants. In doing so, we will write \( u, s, t, k \) instead of \( \bar{u}, \bar{s}, \bar{t}, \bar{k} \) respectively. Since the trigonometric and rational cases never mix, this convention can not lead to a confusion. Note that \( u \) becomes an additive spectral parameter.

5 Rational limit

In this section we consider the limit \( q \to 1 \) (the ”rational”, or continuum limit) of the monodromy matrix \((32)\), providing a basis for embedding the continuous quasi-exactly solvable problems \([12, 13]\) into the quantum inverse scattering approach.

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This rule is equivalent to repeating the general arguments of Sects. 2 and 3 in the context of the rational \( \mathcal{R} \)-matrix and the \( L \)-operator \( \hat{L} \). The general \( c \)-number solutions to the rational RE are \( \bar{h} \) number solutions to the rational \( \mathcal{R} \)-matrix.

\[
K_j = \begin{pmatrix}
2x_0(u-s-1) & x_+(2u-1) \\
x_-(2u-1) & -2x_0(u+s)
\end{pmatrix},
\]
\[
K_r = \begin{pmatrix}
2y_0(u+t+1) & y_+(2u+1) \\
y_-(2u+1) & -2y_0(u-t)
\end{pmatrix}.
\]

However, it is more convenient to proceed by considering the limits of \( \bar{h} \). The operators \( \bar{h} \) are expanded as

\[
H_1 = h^{-1}x_0 + H_1^{(0)} + hH_1^{(1)} + \mathcal{O}(h^2),
\]
\[
H_2 = -h^{-1}x_0 + H_2^{(0)} + hH_2^{(1)} + \mathcal{O}(h^2),
\]
\[
H_3 = h^{-1}x_+ + 2hH_3^{(1)} + \mathcal{O}(h^2),
\]
\[
H_3 = h^{-1}x_- + 2hH_3^{(1)} + \mathcal{O}(h^2),
\]

where the operator coefficients are expressed through the generators \( \{\mathfrak{l}\} \) of \( \text{sl}(2) \):

\[
H_1^{(0)} = H_2^{(0)} = 2x_0S_0 + x_+S_+ + x_-S_- - x_0s,
\]
\[
H_1^{(1)} = -H_2^{(1)} = 2x_0S_0^2 + x_+S_0S_+ + x_-S_-S_0 -
-2x_0sS_0 + x_+kS_+ - x_-kS_- + \frac{1}{2}x_0(s^2 - \frac{1}{3}),
\]
\[
H_3^{(1)} = x_+S_0^2 - x_-S_0^2 - 2x_0S_0S_- + 2x_+kS_0 + 2x_0(s-k)S_- + x_+(k^2 - \frac{1}{12}),
\]
\[
\tilde{H}_3^{(1)} = x_-S_0^2 - x_+S_0^2 - 2x_0S_0S_+ - 2x_-kS_0 + 2x_0(s+k)S_+ + x_-(k^2 - \frac{1}{12}).
\]

The operator \( \tilde{M}(u) \) acquires a \( c \)-number part as \( \bar{h} \to 0 \), with the leading term being singular \( \sim \bar{h}^{-1} \). In what follows we neglect all next-to-leading \( c \)-number contributions since they are absolutely irrelevant. In particular, we can throw away the \( c \)-number terms in \( \tilde{M}(u) \)\( \sim \bar{h}^{-1} \). Moreover, it is easy to see that the \( \bar{h}^{-1} \)-terms exactly cancel in the sum \( \tilde{M}(u) \). Finally, one obtains the following rational monodromy matrix:

\[
M^{(\text{rational})}(u) = \lim_{\bar{h} \to 0} \frac{1}{2\bar{h}} \tilde{M}(e^{\bar{h}u}) =
\]
\[
(2u-1) \begin{pmatrix}
-H_1^{(1)} - uH_1^{(0)} & H_3^{(1)} \\
\tilde{H}_3^{(1)} & H_1^{(1)} - (u+1)H_1^{(0)}
\end{pmatrix}.
\]

Combining it with \( \tilde{M}(u) \), we get the transfer matrix:

\[
\tau(u) = (4u^2 - 1) \left( y_+\tilde{H}_3^{(1)} + y_-H_3^{(1)} - 2y_0H_1^{(1)} - 2y_0tH_1^{(0)} \right) =
\]
\[
= (4u^2 - 1) \left( (y_+x_+ + y_-x_- + 4y_0x_0)S_0^2 - y_+x_+S_+^2 - y_-x_-S_-^2 -
-(y_+x_0 + y_0x_+)(S_0 + S_0S_+) - (y_-x_0 + y_0x_-)(S_-S_0 + S_0S_-) +
+2(k(y_+x_+ - y_-x_-) + 2(s-t)y_0x_0)S_0 +
+2((s + k + 1/2)y_+x_0 - (t + k + 1/2)y_0x_+)S_+ +
+2((s - k + 1/2)y_-x_0 - (t - k + 1/2)y_0x_-)S_- \right).
\]

It is a generic mixed quadratic-linear form in \( S_i \). The number of independent parameters is the same as in \( (40) \).
The diagonalization of (58) gives (for the spin \( j \) representation (58)) the following differential equation:

\[
-Q(z) \frac{d^2 \Psi(z)}{dz^2} + \left( j - \frac{1}{2} \right) Q'(z) + 2P(z) \frac{d \Psi(z)}{dz} - \left( \frac{1}{3} j(j - \frac{1}{2})Q''(z) + 2jP'(z) \right) \Psi(z) = E \Psi(z),
\]

where

\[
Q(z) = (x_+ z^2 - 2x_0 z - x_-(y_+ z^2 - 2y_0 - y_-),
\]

\[
P(z) = -(s + k + 1/2)y_+ x_0 - (t + k + 1/2)y_0 x_+ ) z^2 + (k(y_- x_+ - y_+ x_-) + 2(s - t)y_0 x_0) z + (s - k + 1/2)y_- x_0 - (t - k + 1/2)y_0 x_- .
\]

Equations of this type are well-studied. If \( Q(z) \) has 4 simple roots, the eigenvalue problem (60) can be reduced to Heun’s equation \([33]\). The transformation of (60) to the Schrödinger form is discussed in detail in the reviews \([13], [14]\).

A relation of (60) to integrable spin chains was already mentioned in the literature \([15]\). However, the known relation is absolutely different: the operator in the l.h.s. of (60) is identified with the hamiltonian of inhomogeneous Gaudin’s magnet \([35]\) on 3 (or 4) sites with periodic boundary conditions. It is known that Gaudin’s magnet is a quasiclassical limit of the integrable spin chain with the rational \( R \)-matrix.

In the present paper we identify (60) with the eigenvalue problem for the transfer matrix of a formal XXX-type “magnet” on only one site \([1]\) but with non-periodic boundary conditions. As it is shown in Sect.4, the trigonometric generalization of this system leads to a class of quasi-exactly solvable difference equations whereas an analog of the Gaudin’s magnet picture for the latter is not known.

### 6 Adjoint action of \( SL(2) \)

There is an obvious group of isospectral transformations of (58). These transformations are induced by the adjoint action of \( SL(2) \): \( S_i \rightarrow g^{-1} S_i g \). One may say that really different spectral problems correspond to \( SL(2) \)-orbits in the space of quadratic forms (58). The generic orbit is 3-dimensional, so the number of independent parameters is reduced to 4. Under the adjoint action of a group element

\[
g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ad - bc = 1,
\]

the generators transform as follows:

\[
S_+ \rightarrow d^2 S_+ + 2cdS_0 - c^2 S_-,
\]

\[
S_0 \rightarrow bdS_+ + (1 + 2bc)S_0 - acS_-,
\]

\[
S_- \rightarrow -b^2 S_+ - 2abS_0 + a^2 S_-.
\]

Making this transformation in (58), one gets an operator having the same spectrum for any spin \( j \). Our aim in this section is to show how this transformation may be interpreted in terms of the QISM.

To do this, recall the \( c \)-number solutions (19) of the rational YBE. Let us consider the group element \([28]\) as such a solution and apply Theorem 1 (see Sect.3) to the pair \( g, K_l \) (or \( g, K_r \)) for \( K_l \) (or \( K_r \)) given by \([60], (51)\). It is convenient to fix common multipliers in (60) and (51) by putting \((2s + 1)x_0 = 1, (2t + 1)y_0 = 1\). Then \( K_l \) (resp., \( K_r \)) depends on a 3-dimensional vector \( x = (x_1, x_2, x_0) \) (resp., \( y = (y_1, y_2, y_0) \)), where \( x_\pm = x_1 \pm x_2, y_\pm = y_1 \pm y_2 \). Working in this normalization, we indicate the dependence on \( x \) and \( y \) explicitly:

\[
K_l(u; x) = (2u - 1)(x_0) - 1,
\]

\(^4\)This "spin chain" has only one spin because \([28]\) includes just one elementary \( L \)-operator.
\[ K_r(u; y) = (2u + 1)(y \sigma) + 1, \quad (66) \]

where \((x \sigma) = x_1 \sigma_1 + x_2 \sigma_2 + x_0 \sigma_3\) denotes the inner product of 3-dimensional vectors (here \(\sigma_i\) are Pauli matrices), and \(1\) is the unit matrix.

Now, according to Theorem 1, we should “dress” \(K\)-matrices using (22), (23). In the simple case at hand the dressing is reduced to the conjugation:

\[ K_l \rightarrow g K_l(u; x) g^{-1} = K_l(u; g x), \quad (67) \]

\[ K_r = g^{-1} K_r(u; y) g = K_r(u; g^{-1} y), \quad (68) \]

where the dashed lines (carrying trivial one-dimensional ”quantum space”) denote the insertions of the ”L-operator" \(g\). The equalities immediately follow from (65), (66). The shorthand \(g x\) means the adjoint action of \(g\) to the 3-component vector \(x\). One concludes from (67), (68) that the dressing in the case at hand is equivalent to a rotation of the vector parameter.

Let us represent the generators \(S_i\) as a vector with operator-valued components:

\[ S = \left( \frac{1}{2}(S_- + S_+), \frac{1}{2i}(S_- - S_+), S_0 \right). \quad (69) \]

The adjoint action \(g^{-1}S\) is given by (64). In these terms the transfer matrix (58) can be written in the form:

\[ \tau(u) = (4u^2 - 1) \left( (y \times S)(x \times S) + ((x \times S)(y \times S)) - (y S)(x S) - (x S)(y S) + 2((y - x)S) + 4ik((y \times x)S) \right) + \text{c-number}, \quad (70) \]

where \(a \times b\) denotes the skew product of 3-vectors: \((a \times b)_\alpha = \epsilon_{\alpha\beta\gamma}a_\beta b_\gamma\). It is clear from (70) that the operator part of \(\tau(u)\) is invariant under simultaneous rotations of all the vectors \(x, y\) and \(S\). In other words, the rotation \(g^{-1}S\) in (70) is equivalent to \(x \rightarrow gx, y \rightarrow gy\) given by the ”dressing" (67), (68). The dressing means the insertion of \(g\) to the left and \(g^{-1}\) to the right of the line corresponding to \(L(u)\).
Schematically,

\[
K_l L(g) = K_r g^{-1} L(g) K_r = K_l L(g) K_r g^{-1}
\]  

(71)

Another way to see this is to observe that

\[
g^{-1} L(u + k) g = (u + k) \mathbf{1} + (\sigma(g^{-1} \mathbf{S}))
\]

for the \(L\)-operator (16), and \(\sigma_2 L^t (-u) \sigma_2 = -L(u)\). Then

\[
\tau(u) = -\text{Tr} (K_r(u; y) L(u + k) K_l(u; x) L(u - k)),
\]

and the transformation \(S \rightarrow g^{-1} S\) in \(L(u \pm k)\) leads to

\[
\tau'(g)(u) = -\text{Tr} (K_r(u; y) g^{-1} L(u + k) g K_l(u; x) g^{-1} L(u - k) g)
\]

\[
= -\text{Tr} (K_r(u; gy) L(u + k) K_l(u; gx) L(u - k)),
\]

(74)

which is equivalent to (71) due to (77), (78).

It is an interesting open problem to find a proper \(q\)-analog of the transformation considered in this section. In particular, it is not known whether there are any isospectral subfamilies among the operators of the form (40) other than the trivial ones (\(B \rightarrow e^{-2i\varphi} B, C \rightarrow e^{2i\varphi} C\)), which correspond to the similar insertion of \(\hat{g}(u)\) (8) taken in the one-dimensional representation (11). We hope that our approach may help to solve this problem.

7 Concluding remarks

We have shown that both discrete and continuous quasi-exactly solvable problems of quantum mechanics are tractable in the framework of the quantum inverse scattering method. Quantum transfer matrices for a peculiar simple integrable system with boundaries yield the complete collection of quasi-exactly solvable hamiltonians. These hamiltonians are quadratic forms in the generators of \(U_q(sl(2))\) (or \(sl(2)\) in the rational limit) taken in a finite-dimensional representation.

This reformulation opens a way to apply the powerful methods specific for quantum integrable systems. For the representations of \(U_q(sl(2))\) having both highest and lowest weights these methods give the results which are either already known or can be obtained by means of more elementary tools [4]. However, if \(q\) is a root of unity there exists a family of cyclic representations having in general neither highest nor lowest weights. This is just the case relevant to the Azbel-Hofstadter problem, where the generic points of bands are described by cyclic representations. The Bethe ansatz in this case is much harder problem. Here the reformulation in terms of the QISM may led to really new outcomes. The appropriate method is the technique of Baxter’s intertwining vectors applied to the chiral Potts model (which is also known to be connected with cyclic representations) in [36]. Recently, this method was applied [34] to the Azbel-Hofstadter problem. Presumably, the method should work for any operator of the form (40) as well.

Let us point out two questions, where the results of the present paper may contribute something to the conceptual understanding.

One of them was already mentioned at the end of Sect.6. It is the question about isospectral subfamilies among operators of the form (10) (or, rather, about a proper \(q\)-analog of them). In the continuous case, they are orbits of the adjoint action of \(SL(2)\) group elements in the space of quadratic forms in \(sl(2)\)-generators. We have seen that the group elements may be considered as \(L\)-operators (obeying the YBE with spectral parameter), the adjoint action being an insertion of them into the monodromy matrix. Remarkably, each ingredient of this picture has its natural \(q\)-deformed counterpart. The notion of a
quantum group-like element was recently discussed [37] (from another point of view) in connection with quantum \( \tau \)-functions. A comparison of these studies with our results may be fruitful for both approaches.

A related question concerns separation of variables. It is known [38] that inhomogeneous \( n \)-site Gaudin’s magnets are in one-to-one correspondence with separated coordinate systems for the Laplace-Beltrami operator on the \((n - 1)\)-dimensional sphere (or hyperboloid). On the other hand, Gaudin’s magnet on 3 sites generates continuous quasi-exactly solvable hamiltonians [15]. Considered as quadratic forms in generators of \( sl(2) \) such a hamiltonian determines a separated coordinate system on the 2-sphere (or hyperboloid). The non-equivalent separated systems correspond [39] to \( SL(2) \)-orbits (under the adjoint action). The quadratic forms in generators of \( U_q(sl(2)) \) might have a similar relation to hypothetical ”separated coordinate systems” on quantum spheres and hyperboloids.

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