On the Waring–Goldbach problem for squares, cubes and higher powers

Min Zhang\textsuperscript{1} · Jinjiang Li\textsuperscript{2}

Received: 28 March 2020 / Accepted: 5 September 2020 / Published online: 6 February 2021
© Springer Science+Business Media, LLC, part of Springer Nature 2021

Abstract
Let $P_r$ denote an almost-prime with at most $r$ prime factors, counted according to multiplicity. In this paper, we generalize the result of Vaughan (Bull Lond Math Soc 17(1):17–20, 1985) for ternary ‘admissible exponent’. Moreover, we use the refined ‘admissible exponent’ to prove that, for $3 \leq k \leq 14$ and for every sufficiently large even integer $n$, the following equation

$$n = x^2 + p_1^2 + p_2^3 + p_3^3 + p_4^3 + p_5^k$$

is solvable with $x$ being an almost-prime $P_{r(k)}$ and the other variables primes, where $r(k)$ is defined in Theorem 1.1. This result constitutes a deepening of previous results.

Keywords Waring–Goldbach problem · Hardy–Littlewood method · Sieve method · Almost-prime

Mathematics Subject Classification 11P05 · 11P32 · 11P55 · 11N36

This work is supported by the National Natural Science Foundation of China (Grant Nos. 11901566, 11971476, 12001047, 12071238), the Fundamental Research Funds for the Central Universities (Grant No. 2019QS02), and the Scientific Research Funds of Beijing Information Science and Technology University (Grant No. 2025035).

© Jinjiang Li
jinjiang.li.math@gmail.com

Min Zhang
min.zhang.math@gmail.com

\textsuperscript{1} School of Applied Science, Beijing Information Science and Technology University, Beijing 100192, People’s Republic of China

\textsuperscript{2} Department of Mathematics, China University of Mining and Technology, Beijing 100083, People’s Republic of China
1 Introduction and main result

The famous Goldbach Conjecture states that every even integer $N \geq 6$ can be written as the sum of two odd primes, i.e.

$$N = p_1 + p_2. \quad (1.1)$$

This conjecture still remains open. Recent developments on Goldbach Conjecture can be found in [22,23,34,37,38] and their references.

In view of Hua’s theorem [14] on five squares of primes and Lagrange’s theorem on four squares, it seems reasonable to conjecture that every sufficiently large integer satisfying some necessary congruence conditions can be written as the sum of four squares of primes, i.e.

$$N = p_1^2 + p_2^2 + p_3^2 + p_4^2. \quad (1.2)$$

However, such a conjecture is out of reach at present. Recent developments on conjecture (1.2) can be found in [12,13,20,33] and their references.

Motivated by Hua’s nine cubes of primes theorem [14], it seems reasonable to conjecture that every sufficiently large even integer is the sum of eight cubes of primes, i.e.

$$N = p_1^3 + p_2^3 + p_3^3 + p_4^3 + p_5^3 + p_6^3 + p_7^3 + p_8^3. \quad (1.3)$$

But unfortunately, such a conjecture (1.3) is still out of reach at present. For the recent developments on conjecture (1.3), one can see [18,19] and its references.

Linnik [27,28] proved that each sufficiently large odd integer $N$ can be written as $N = p + n_1^2 + n_2^2$, which was firstly formulated by Hardy and Littlewood [9], where $n_1$ and $n_2$ are integers. In view of this result, it seems reasonable to conjecture that every sufficiently large even integer satisfying some necessary congruence conditions is a sum of a prime and two squares of primes, i.e.

$$N = p_1 + p_2^2 + p_3^2. \quad (1.4)$$

But current techniques lack the power to solve it. Many authors considered this problem and gave some approaches to approximate (1.4) (See [13,14,21,24,25,29,35,45–47]). Meanwhile, we can regard this problem as the hybrid problem of (1.1) and (1.2).

In [30], Liu considered the hybrid problem of (1.1) and (1.3), i.e.

$$N = p_1 + p_2^3 + p_3^3 + p_4^3 + p_5^3. \quad (1.5)$$

There are some approximations to (1.5). On one hand, as an approach to prove (1.5), Liu and Lü [32] proved that every sufficiently large odd integer can be written as the sum of a prime, four cubes of primes and bounded number of powers of 2, i.e.

$$N = p_1 + p_2^3 + p_3^3 + p_4^3 + p_5^3 + 2^{v_1} + 2^{v_2} + \cdots + 2^{v_{r_1}},$$
and gave an acceptable value of $K_1$. On the other hand, Liu [30] gave another approximation to (1.5). He proved that every sufficiently large odd integer $N$ can be written in the form $N = x + p_1^3 + p_2^3 + p_3^3 + p_4^3$, where $p_1, p_2, p_3, p_4$ are primes and $x$ is an almost-prime $P_2$. As usual, $P_r$ always denotes an almost-prime with at most $r$ prime factors, counted according to multiplicity. In [32], Liu and Lü also considered the hybrid problem of (1.2) and (1.3),

$$N = x^2 + p_1^2 + p_2^2 + p_3^2 + p_4^2 + p_5^2 + p_6^3. \quad (1.6)$$

In their paper, they gave an approximation to (1.6) and proved that every sufficiently large even integer can be written as the sum of two squares of primes, four cubes of primes and 211 powers of 2, i.e.

$$N = p_1^2 + p_2^2 + p_3^3 + p_4^3 + p_5^3 + p_6^3 + 2^{v_1} + 2^{v_2} + \cdots + 2^{v_{211}}. \quad (1.7)$$

Later, in 2017, Liu [31] proved that every sufficiently large even integer can be written as the sum of two squares of primes, three cubes of primes, one fourth power of prime and a bounded number of powers of 2, i.e.

$$N = p_1^2 + p_2^2 + p_3^3 + p_4^3 + p_5^4 + 2^{v_1} + 2^{v_2} + \cdots + 2^{v_{121}}. \quad (1.8)$$

Also, in 2016, Cai [6] gave another approximation to (1.6), and proved that any sufficiently large even integer $N$ can be written in the form $N = x^2 + p_1^2 + p_2^3 + p_3^3 + p_4^3 + p_5^3$, where $p_1, p_2, p_3, p_4, p_5$ are primes and $x$ is an almost-prime $P_3$.

In view of the results (1.7), (1.8) and the result of Cai, in this paper, we shall give some approximations to the generalized cases of (1.6).

**Theorem 1.1** For $3 \leq k \leq 14$, let $R_k(n)$ denote the number of solutions of the equation

$$n = x^2 + p_1^2 + p_2^3 + p_3^3 + p_4^3 + p_5^k \quad (1.9)$$

with $x$ being an almost-prime $P_r(k)$ and the $p_j$’s primes. Then, for every sufficiently large even integer $n$, there holds

$$R_k(n) \gg n^{\frac{17}{18} + \frac{3}{56} \log^{-6} n},$$

where

\[
\begin{align*}
  r(3) &= 3, \quad r(4) = 4, \quad r(5) = 5, \quad r(6) = 5, \quad r(7) = 6, \quad r(8) = 7, \\
  r(9) &= 7, \quad r(10) = 8, \quad r(11) = 9, \quad r(12) = 10, \quad r(13) = 11, \quad r(14) = 13.
\end{align*}
\]

We approach Theorem 1.1 via the Hardy–Littlewood method, and in a certain sense by a unified approach. To be specific, we use the ideas, which were firstly created by Brüdern [1,2] and developed by Brüdern and Kawada [3,4], combining with Hardy–Littlewood method and Iwaniec’s linear sieve method to give the proof of Theorem 1.1.
To treat the minor arcs in the final application of the circle method it is necessary to improve ‘admissible exponents’ (for the definition see Sect. 2) for mixed sums of cubes and $k$th powers. In the proof of Theorem 1.1 we require a result on two cubes and a $k$th power. The main idea is to apply the Hardy–Littlewood method as modified by Vaughan [42] to the mixed situation for one cubes and two $k$th powers and then to combine this with the result of Vaughan [42], by the Cauchy’s inequality. This auxiliary result constitutes the most novel part of the present paper which may perhaps be of interest in its own right. We formulate it precisely as Theorem 2.1 in the following section. Unfortunately Vaughan’s elegant argument in [42] does not carry over very well to mixed problems; a considerable refinement of his method will be necessary. A detailed explanation is given during the proof in Sect. 2.

Notation. Throughout this paper, small italics denote integers when they do not obviously represent a function; $p$, $p_1$, $p_2$ . . . , with or without subscript, always stand for a prime number; $\varepsilon$ always denotes an arbitrary small positive constant, which may not be the same at different occurrences; $\gamma$ denotes Euler’s constant; $f(x) \ll g(x)$ means that $f(x) = O(g(x))$; $f(x) \asymp g(x)$ means that $f(x) \ll g(x) \ll f(x)$; the constants in the $O$-term and $\ll$-symbol depend at most on $\varepsilon$; $P_r$ always denotes an almost-prime with at most $r$ prime factors, counted according to multiplicity. As usual, $\varphi(n)$, $\mu(n)$ and $\tau_j(n)$ denote Euler’s function, Möbius’ function and the $j$-dimensional divisor function, respectively. Especially, we write $\tau(n) = \tau_2(n)$. We denote by $a(m)$ and $b(\ell)$ arithmetical functions satisfying $|a(m)| \ll 1$ and $|b(\ell)| \ll 1$; $(s, t)$ denotes the greatest common divisor of $s$ and $t$, while $(k; \lambda)$ is a pair of admissible exponents (see the next section); $e(\alpha) = e^{2\pi i \alpha}$ for abbreviation.

2 Admissible exponents for cubes and higher powers

The idea of admissible exponents goes back to Hardy and Littlewood [10], but was introduced formally by Davenport and Erdös [7]. Our definition is adapted from Thangasalam [39]. let

$$f_k(\alpha, X) = \sum_{X < x \leq 2X} e(\alpha x^k).$$

Let $k_i \in \mathbb{N}$, $0 < \lambda_i \leq 1$ ($i = 1, 2, \ldots, s$) and $P_i = N^{\lambda_i/k_i}$. Then the pairs

$$(k_1; \lambda_1), (k_2; \lambda_2), \ldots, (k_s; \lambda_s)$$

are said to form admissible exponents if

$$\int_0^1 \left| f_{k_1}(\alpha, P_1) \cdots f_{k_s}(\alpha, P_s) \right|^2 d\alpha \ll P_1 P_2 \cdots P_s N^{\varepsilon}. \quad (2.1)$$
This is equivalent to Thanigasalam’s definition, for the integral in (2.1) is equal to the number of solutions of
\[
x_1^{k_1} + x_2^{k_2} + \cdots + x_s^{k_s} = y_1^{k_1} + y_2^{k_2} + \cdots + y_s^{k_s}; \quad P_i < x_i, y_i \leq 2P_i.
\]

Our aim is to generalize the result of Vaughan [42] and establish the following Theorem.

**Theorem 2.1** For \(k \geq 4\), the pairs \((3; 1), (k; \frac{5}{6}), (k; \frac{5}{6})\) form admissible exponents.

**Proof of Theorem 2.1** Let \(Q = P^\frac{5}{3}\) and let \(S\) denote the number of solutions of
\[
x_1^3 + y_1^k + y_2^k = x_2^3 + y_3^k + y_4^k
\]
with \(P < x_i \leq 2P\) and \(Q < y_i \leq 2Q\). Then we have to show that
\[
S \ll P^{1+\varepsilon} Q^2.
\]
Let \(S_1\) and \(S_2\) denote the number of solutions of (2.2) with \(x_1 = x_2\) and \(x_1 \neq x_2\), respectively. Then, by Hua’s inequality (see Lemma 2.5 of Vaughan [44]), it is easy to see that
\[
S_1 \ll P Q^{2+\varepsilon},
\]
which is acceptable. It remains to estimate \(S_2\). Write \(x_2 = x_1 + h\). Then (2.2) becomes
\[
h(3x_1^2 + 3x_1h + h^2) = y_1^k + y_2^k - y_3^k - y_4^k.
\]
By symmetry it is sufficient to estimate the solutions of (2.4) with \(h > 0\). Since \(y_1^k + y_2^k \leq 2^{k+1} Q^k\) and \(x_1^2 > P^2\), it follows that
\[
h < \frac{2^{k+1}}{3} Q^k P^{-2} < 2^k Q^k P^{-2} = 2^k P^{\frac{1}{2}} = H,
\]
say. Let
\[
G(\alpha) = \sum_{0<h<H} \sum_{P < x \leq 2P} e\left(\alpha h(3x^2 + 3xh + h^2)\right),
\]
then
\[
S_2 \ll \int_0^1 G(\alpha) |f(\alpha)|^4 d\alpha = \int_{\frac{1}{2m}}^{1} G(\alpha) |f(\alpha)|^4 d\alpha,
\]
where \( f(\alpha) = f_k(\alpha, Q) \) for abbreviation. By Dirichlet’s theorem on Diophantine rational approximation (for instance, see Lemma 2.1 of Vaughan [44]), each \( \alpha \in [1/(PH), 1 + 1/(PH)] \) can be written in the form

\[
\alpha = \frac{a}{q} + \lambda, \quad |\lambda| \leq \frac{1}{q PH}
\]

for some integers \( a, q \) with \( 1 \leq a \leq q \leq PH \) and \( (a, q) = 1 \). Then we define the major arcs \( \mathcal{M} \) and minor arcs \( m \) as follows:

\[
\mathcal{M} = \bigcup_{1 \leq q \leq P} \bigcup_{1 \leq a \leq q \atop (a,q)=1} \mathcal{M}(q, a), \quad m = \left[ \frac{1}{PH}, 1 + \frac{1}{PH} \right] \setminus \mathcal{M},
\]

where

\[
\mathcal{M}(q, a) = \left[ \frac{aq - 1}{q PH}, \frac{aq + 1}{q PH} \right].
\]

Then we have

\[
\int_{\frac{1}{PH}}^{1+\frac{1}{PH}} G(\alpha) \left| f(\alpha) \right|^4 \, d\alpha = \left\{ \int_{\mathcal{M}} + \int_{m} \right\} G(\alpha) \left| f(\alpha) \right|^4 \, d\alpha. \tag{2.7}
\]

According to the Lemma on p. 18 of Vaughan [42], we know that

\[
G(\alpha) \ll HP^{1+\varepsilon} (q^{-1} + P^{-1} + qP^{-2}H^{-1})^{\frac{1}{2}}.
\]

As the structure of \( m \), we know that, for \( \alpha \in m \), there holds \( P < q \leq PH \), and thus

\[
G(\alpha) \ll P^{1+\varepsilon},
\]

from which and a simple consequence of Hua’s lemma (Lemma 2.5 of Vaughan [44])

\[
\int_{0}^{1} \left| f(\alpha) \right|^4 \, d\alpha \ll Q^{2+\varepsilon}. \tag{2.8}
\]

we derive that

\[
\int_{m} G(\alpha) \left| f(\alpha) \right|^4 \, d\alpha \ll P^{1+\varepsilon} Q^2. \tag{2.9}
\]

From (2.5), (2.7) and (2.9), we deduce that

\[
S_2 \ll \int_{\mathcal{M}} G(\alpha) \left| f(\alpha) \right|^4 \, d\alpha + P^{1+\varepsilon} Q^2. \tag{2.10}
\]
In order to estimate the integral on the major arcs, we approximate \( G(\alpha) \) by a suitable function \( G_1(\alpha) \). Define

\[
\sigma_h(q, a) = \sum_{x=1}^{q} e\left( \frac{a}{q}(x + h)^3 - x^3 \right),
\]

\[
v_h(\lambda) = \int_{P}^{2P} e\left( \lambda((u + h)^3 - u^3) \right) du,
\]

\[
G_1(\alpha) = \sum_{0 < h < H} q^{-1} \sigma_h(q, a) v_h\left( \alpha - \frac{a}{q} \right).
\]

Then for \( \alpha \in M(q, a) \), \( G_1(\alpha) \) is well defined on \( M \). By (2.13) of Lemma 2 in Vaughan [43] with \( k = 3 \), one has

\[
\sum_{P < x \leq 2P} e\left( \alpha h(3x^2 + 3xh + h^2) \right) = q^{-1} \sigma_h(q, a) v_h(\lambda) + O\left( q^{\frac{1}{2} + \varepsilon} (q, h)^{\frac{1}{2}} \right),
\]

from which we obtain

\[
G(\alpha) = G_1(\alpha) + O\left( q^{\frac{1}{2} + \varepsilon} \sum_{0 < h < H} (q, h)^{\frac{1}{2}} \right). \tag{2.11}
\]

For the \( O \)-term in (2.11), writing \( (q, h) = d \), we see that

\[
q^{\frac{1}{2} + \varepsilon} \sum_{0 < h < H} (q, h)^{\frac{1}{2}} \ll q^{\frac{1}{2} + \varepsilon} \sum_{d | q} d^{\frac{1}{2}} \sum_{h < H/d} d^{\frac{1}{2}} \ll q^{\frac{1}{2} + \varepsilon} \sum_{d | q} d^{\frac{1}{2}} \frac{H}{d}
\]

\[
\ll Hq^{\frac{1}{2} + \varepsilon} r(q) \ll Hq^{\frac{1}{2} + \varepsilon},
\]

from which and (2.11) we derive that

\[
G(\alpha) = G_1(\alpha) + O\left( P^{1+\varepsilon} \right) \tag{2.12}
\]

uniformly for \( \alpha \in M \). Combining (2.8), (2.10) and (2.12), we have

\[
S_2 \ll \int_{\mathfrak{M}} G_1(\alpha) |f(\alpha)|^4 d\alpha + P^{1+\varepsilon} Q^2. \tag{2.13}
\]

In order to give a proper upper bound for the integral on the right-hand side of (2.13), we need to establish the following lemma, which is the crucial ingredient of this section. \( \square \)

**Lemma 2.2** Let \( \mathfrak{M} \) be defined as in (2.6), then for \( k \geq 4 \) and \( X \leq P \), there holds

\[
\int_{\mathfrak{M}} |G_1^2(\alpha) f_k^4(\alpha, X)| d\alpha \ll HP^\varepsilon (PX^2 + X^4).
\]
First of all, we use Lemma 2.2 to give the expected estimate of the integral on the right-hand side of (2.13) and prove it afterwards. Taking \( X = Q \) and \( f(\alpha) = f_k(\alpha, Q) \) in Lemma 2.2, then it follows from (2.8), Lemma 2.2 and Cauchy’s inequality that

\[
\int_{\mathfrak{M}} |G_1(\alpha)| |f(\alpha)|^4 d\alpha \ll \left( \int_{\mathfrak{M}} |G_1^2(\alpha) f^4(\alpha)| d\alpha \right)^{1/2} \left( \int_0^1 |f(\alpha)|^4 d\alpha \right)^{1/2}
\ll \left( HP^\varepsilon (PQ^2 + Q^4) \right)^{1/2} \left( Q^{2+\varepsilon} \right)^{1/2} \ll H^{1/2} P^{1/2} Q + Q^2 \ll P^{1+\varepsilon} Q^2.
\]

(2.14)

From (2.3), (2.13) and (2.14), we derive the conclusion of Theorem 2.1.

**Proof of Lemma 2.2** By Theorem 7.1 of Vaughan [44], it is easy to see that

\[
\sigma_h(q, a) \ll q^{1/2+\varepsilon} (q, h)^{1/2}.
\]

For \( \alpha \in \mathfrak{M}(q, a) \), it follows from Cauchy’s inequality that

\[
|G_1(\alpha)|^2 = \left| \sum_{0<h<H} q^{-1} \sigma_h(q, a) v_h(\lambda) \right|^2 \ll P^\varepsilon \left| \sum_{0<h<H} \frac{(q, h)^{1/2}}{q^{1/2}} |v_h(\lambda)|^2 \right|
\ll P^\varepsilon \left( \sum_{0<h<H} \frac{1}{q} \right) \left( \sum_{0<h<H} (q, h) |v_h(\lambda)|^2 \right)
\ll P^\varepsilon HQ^{-1} \sum_{0<h<H} (q, h) |v_h(\lambda)|^2.
\]

(2.15)

By the standard estimate

\[
v_h(\lambda) \ll \frac{P}{1 + P^2 h |\lambda|},
\]

which combines (2.15) to give

\[
\int_{\mathfrak{M}} |G_1^2(\alpha) f_k^4(\alpha, X)| d\alpha \ll P^\varepsilon H \sum_{0<h<H} \int_{\mathfrak{M}} \frac{(q, h)}{q} |v_h(\lambda)|^2 |f_k^4(\alpha, X)| d\alpha
\ll P^\varepsilon H \sum_{0<h<H} \sum_{1 \leq q \leq P} \sum_{(a, q) = 1}^q \frac{(q, h)}{q} \int_{-\frac{1}{\sqrt{P}}}^{\frac{1}{\sqrt{P}}} |v_h(\lambda)|^2
d \times \sum_{X < x_1, \ldots, x_4 \leq 2X} e\left( \frac{a}{q} + \lambda \right) \left( x_1^k + x_2^k - x_3^k - x_4^k \right) d\lambda.
\]
Setting $u = x_1^k + x_2^k - x_3^k - x_4^k$, then

\[
\int_{\mathfrak{M}} \left| G_7^2(\alpha) f_k^*(\alpha, X) \right| d\alpha
\]

\[
\ll P^{\varepsilon} H \sum_{0 < h < H} \sum_{1 \leq q \leq P} \sum_{a=1 \atop (a, q) = 1}^q \frac{(q, h)}{q} \sum_{u} \phi(u) \int_{\frac{1}{q} P}^{P} \left| v_h(\lambda) \right|^2 e\left(\frac{au}{q} + \lambda \right) d\lambda
\]

\[
\ll P^{\varepsilon} H \sum_{0 < h < H} \sum_{1 \leq q \leq P} \frac{(q, h)}{q} \sum_{u} \phi(u) \left| \sum_{a=1 \atop (a, q) = 1}^q e\left(\frac{au}{q} \right) \right| \int_{|\lambda| \leq \frac{1}{P^h}} \frac{P^2}{(1 + P^2 h |\lambda|)^2} d\lambda
\]

\[
\ll P^{\varepsilon} H \sum_{0 < h < H} \sum_{1 \leq q \leq P} \frac{(q, h)}{qh} \sum_{u} \phi(u) \left| \sum_{a=1 \atop (a, q) = 1}^q e\left(\frac{au}{q} \right) \right| \times \left( \int_{|\lambda| \leq \frac{1}{P^h}} P^2 d\lambda + \int_{P^h < |\lambda| \leq \frac{1}{P^4 h^2 |\lambda|^2}} P^2 d\lambda \right)
\]

\[
\ll P^{\varepsilon} H \sum_{0 < h < H} \sum_{1 \leq q \leq P} \frac{(q, h)}{qh} \sum_{u} \phi(u) \left| \sum_{a=1 \atop (a, q) = 1}^q e\left(\frac{au}{q} \right) \right|,
\]

where $\phi(u)$ denotes the number of solutions of $u = x_1^k + x_2^k - x_3^k - x_4^k$ with $X < x_i \leq 2X (i = 1, 2, 3, 4)$. By Hua’s lemma (Lemma 2.5 of Vaughan [44]), we have $\phi(0) \ll X^{2+\varepsilon}$. For $u \neq 0$, it follows from Theorem 271 of Hardy and Wright [11] that

\[
\sum_{a=1 \atop (a, q) = 1}^q e\left(\frac{au}{q} \right) = \sum_{d|\langle q, u \rangle} \mu\left(\frac{q}{d}\right) d.
\]

Thus, the right-hand side of (2.16) is bounded by

\[
\ll P^{\varepsilon} H \left( X^2 \sum_{0 < h < H} \sum_{1 \leq q \leq P} \frac{(q, h)}{h} + \sum_{0 < h < H} \sum_{1 \leq q \leq P} \frac{(q, h)}{qh} \sum_{u \neq 0} \phi(u) \sum_{d|\langle q, u \rangle} d \right)
\]

\[
\ll P^{\varepsilon} H \left( \Sigma_1 + \Sigma_2 \right),
\]

say. Writing $r = (q, h)$, then $q = rq_1, h = rh_1$ with $(q_1, h_1) = 1$. Thus, we have

\[
\Sigma_1 \ll X^2 \sum_{1 \leq r \leq P} \sum_{1 \leq h_1 < H/r} \sum_{1 \leq q_1 \leq P/r} \frac{1}{h_1^2}
\]

\[ \odot \text{ Springer} \]
\[ \ll X^2 \left( \sum_{1 \leq r \leq P} \frac{P}{r} \right) \left( \sum_{1 \leq h_1 \leq H/r} \frac{1}{h_1} \right) \ll X^2 P \left( \sum_{1 \leq r \leq P} \frac{\log(H/r)}{r} \right) \ll X^2 P^{1+\varepsilon}. \quad (2.18) \]

For \( \Sigma_2 \), by the same transformation, we obtain

\[ \Sigma_2 \ll \sum_{1 \leq r \leq P} \sum_{1 \leq h_1 \leq H/r} \sum_{1 \leq q_1 \leq P/r} \frac{1}{q_1 h_1 r} \sum_{u \neq 0} \frac{\varrho(u)}{d} \sum_{d | (rq_1, u)} d. \]

We first consider the inner double sums over \( u \) and \( d \), and see that \( d | (rq_1, u) \) implies \( d | u \) and \( rq_1 = ds \) for some integer \( s \). Moreover, for fixed \( d \) and \( s \), there exist \( O(P^\varepsilon) \) solutions of \( rq_1 = ds \) in integer variables \( r \) and \( q_1 \). Hence, we deduce that

\[ \Sigma_2 \ll P^\varepsilon \sum_{u \neq 0} \varrho(u) \sum_{d | u} d \sum_{s \leq P} \frac{1}{h_1} ds \ll P^\varepsilon \sum_{u \neq 0} \varrho(u) \ll X^4 P^\varepsilon. \quad (2.19) \]

Combining (2.16), (2.17), (2.18) and (2.19), we get the conclusion of Lemma 2.2.

From Theorem 2.1 and the Theorem of Vaughan [42], we obtain the following corollary.

**Corollary 2.3** For \( k \geq 3 \), the pairs \((3; 1), (3; \frac{5}{6}), (k; \frac{5}{6})\) form admissible exponents.

**Proof of corollary 2.3** For \( k = 3 \), the conclusion follows from the Theorem of Vaughan [42]. For \( k \geq 4 \), by the Theorem of Vaughan [42], Theorem 2.1 and Cauchy’s inequality, we deduce that

\[
\int_0^1 \left| f_3(\alpha, N^{\frac{5}{6}}) f_3(\alpha, N^{\frac{5}{10}}) f_k(\alpha, N^{\frac{5}{6}}) \right|^2 d\alpha \\
\ll \left( \int_0^1 \left| f_3(\alpha, N^{\frac{5}{6}}) f_3(\alpha, N^{\frac{5}{10}}) \right|^2 d\alpha \right)^{\frac{1}{2}} \left( \int_0^1 \left| f_k(\alpha, N^{\frac{5}{6}}) \right|^2 d\alpha \right)^{\frac{1}{2}} \\
\ll (N^{\frac{1}{6} + \frac{5}{10} + \frac{5}{6} + \varepsilon})^{\frac{1}{2}} (N^{\frac{1}{6} + \frac{5}{10} + \frac{5}{6} + \varepsilon})^{\frac{1}{2}} \ll N^{\frac{1}{3} + \frac{5}{10} + \frac{5}{6} + \varepsilon},
\]

which implies \((3; 1), (3; \frac{5}{6}), (k; \frac{5}{6})\) form admissible exponents for \( k \geq 4 \). \( \square \)
3 Proof of Theorem 1.1: Preliminaries

In this section, we shall give some notations and preliminary lemmas. We always denote by \( \chi \) a Dirichlet character \((\text{mod } q)\), and by \( \chi^0 \) the principal Dirichlet character \((\text{mod } q)\). Let

\[
A = 10^{200}, \quad Q_0 = \log^{50} A n, \quad Q_1 = n^5 \frac{5}{6} + 50\epsilon, \quad Q_2 = n^4 + \frac{5}{6} n - 50\epsilon,
\]

\[
D = n^5 \frac{5}{6} - \frac{1}{6} n - 51\epsilon, \quad z = D^{\frac{1}{3}}, \quad X_j = \frac{1}{2} \left( \frac{2n}{3} \right)^{\frac{1}{7}},
\]

\[
X_j^* = \frac{1}{2} \left( \frac{2n}{3} \right)^{\frac{5}{6}}, \quad \mathcal{P} = \prod_{2 < p < z} p,
\]

\[
F_j(\alpha) = \sum_{X_j < m \leq 2X_j} e(m^j \alpha), \quad f_j(\alpha) = \sum_{X_j < p \leq 2X_j} (\log p)e(p^j \alpha),
\]

\[
w_j(\lambda) = \int_{X_j}^{2X_j} e(\lambda u^j) du,
\]

\[
F_j^*(\alpha) = \sum_{X_j^* < m \leq 2X_j^*} e(m^j \alpha), \quad f_j^*(\alpha) = \sum_{X_j^* < p \leq 2X_j^*} (\log p)e(p^j \alpha),
\]

\[
w_j^*(\lambda) = \int_{X_j^*}^{2X_j^*} e(\lambda u^j) du,
\]

\[
G_j(\chi, a) = \sum_{m=1}^{q} \chi(m)e\left( \frac{am^j}{q} \right), \quad S_j^s(q, a) = G_j(\chi^0, a),
\]

\[
S_j(q, a) = \sum_{m=1}^{q} e\left( \frac{am^j}{q} \right),
\]

\[
h(\alpha) = \sum_{m \leq D^{2/3}} a(m) \sum_{s \leq D^{1/3}} b(s) \sum_{X_2 < t \leq 2X_2} e\left( (mst)^2 \alpha \right),
\]

\[
B_d(q, n) = \sum_{(a, q) = 1}^{q} S_2(q, ad^2) S_3^s(q, a) S_3^s(q, a) S_3^s(q, a) e\left(-\frac{an}{q}\right),
\]

\[
B(q, n) = B_1(q, n), \quad A_d(q, n) = \frac{B_d(q, n)}{q \varphi^5(q)}, \quad A(q, n) = A_1(q, n),
\]

\[
\mathcal{G}_d(n) = \sum_{q=1}^{\infty} A_d(q, n), \quad \mathcal{G}(n) = \mathcal{G}_1(n),
\]

\[
\mathcal{J}(n) = \int_{-\infty}^{+\infty} w_2^2(\lambda) w_3^2(\lambda) w_5^2(\lambda) w_k^2(\lambda) e(-n\lambda)d\lambda,
\]

\[
\mathcal{B}_r = \{m : X_2 < m \leq 2X_2, \ m = p_1 p_2 \cdots p_r, \ z \leq p_1 \leq p_2 \leq \cdots \leq p_r\},
\]

\[
\mathcal{N}_r = \{m : m = p_1 p_2 \cdots p_{r-1}, \ z \leq p_1 \leq p_2 \leq \cdots \}
\]
\[
g_r(\alpha) = \sum_{\ell \in \mathbb{N}_r, \ell \leq \log X^2} \log p \frac{\log p}{\log X^2_\ell} e^\left(\alpha(\ell p)^2\right),
\]

\[
\log \Xi = (\log 2X) (\log 2X^2) (\log 2X^*_3) (\log 2X^*_k),
\]

\[
\log \Theta = (\log X^2) (\log X^*_3) (\log X^*_k).
\]

**Lemma 3.1** For \((a, q) = 1\), we have

(i) \(S_j(q, a) \ll q^{1 - \frac{1}{j}}\);

(ii) \(G_j(\chi, a) \ll q^{\frac{1}{2} + \varepsilon}\).

In particular, for \((a, p) = 1\), we have

(iii) \(|S_j(p, a)| \leq (j, p - 1) \sqrt{p}\);

(iv) \(|S_j^*(p, a)| \leq (j, p - 1) \sqrt{p} + 1\);

(v) \(S_j^*(p^k, a) = 0\) for \(\ell \geq \gamma(p)\), where

\[
\gamma(p) = \begin{cases}
\theta + 2, & \text{if } p^\theta \parallel j, \ p \neq 2 \text{ or } p = 2, \ \theta = 0,
\theta + 3, & \text{if } p^\theta \parallel j, \ p = 2, \ \theta > 0.
\end{cases}
\]

**Proof** For (i) and (iii)–(iv), see Theorem 4.2 and Lemma 4.3 of Vaughan [44], respectively. For (ii), see Lemma 8.5 of Hua [15] or the Problem 14 of Chapter VI of Vinogradov [41]. For (v), see Lemma 8.3 of Hua [15]. \(\square\)

**Lemma 3.2** We have

(i) \(\int_0^1 \left| F_2(\alpha) F_3(\alpha) F_3^*(\alpha) F_k^*(\alpha) \right|^2 d\alpha \ll n^{\frac{11}{9} + \frac{5}{27}} (\log n)^c\),

(ii) \(\int_0^1 \left| f_2(\alpha) f_3(\alpha) f_3^*(\alpha) f_k^*(\alpha) \right|^2 d\alpha \ll n^{\frac{11}{9} + \frac{5}{27}} (\log n)^{c+8}\),

where \(c\) is an absolute constant.

**Proof** By Lemma 2.4 of Cai [5], we know that

\[
\int_0^1 \left| F_3(\alpha) F_3^*(\alpha) \right|^4 d\alpha \ll n^{\frac{13}{9}}.
\]

By the above estimate, Cauchy’s inequality and the explicit form of Hua’s inequality (see Theorem 4 on p. 19 of Hua [15]), we deduce that

\[
\int_0^1 \left| F_2(\alpha) F_3(\alpha) F_3^*(\alpha) \right|^2 d\alpha
\]

\(\square\) Springer
\[
\ll \left( \int_0^1 |F_2(\alpha)|^4 d\alpha \right)^{1/2} \left( \int_0^1 |F_3(\alpha)F_3^*(\alpha)|^4 d\alpha \right)^{1/2} \\
\ll \left( X_2^2 (\log X_2)^c \right)^{1/2} \left( n^{\frac{13}{7}} \right)^{1/2} \ll n^{\frac{11}{7}} (\log n)^c.
\]  

(3.1)

Then (i) follows from (3.1) and the trivial estimate \(|F_k^*(\alpha)| \ll n^{5/8} \). Moreover, by considering the number of solutions of the underlying Diophantine equation and the result of (i), we obtain the estimate (ii). This completes the proof of Lemma 3.2.

Lemma 3.3 For \( \alpha = \frac{a}{q} + \lambda \), define

\[
\mathcal{M}(q, a) = \left( \frac{a}{q} - \frac{1}{qQ_0}, \frac{a}{q} + \frac{1}{qQ_0} \right],
\]

(3.2)

\[
\Delta_k(\alpha) = f_k(\alpha) - \frac{S_k^*(q, a)}{\varphi(q)} \sum_{X_k < m \leq 2X_k} e(m^k\lambda),
\]

(3.3)

\[
\mathcal{W}(\alpha) = \sum_{d \leq D} \frac{c(d)}{d q} S_2(q, ad^2) w_2(\lambda),
\]

(3.4)

where

\[
c(d) = \sum_{\substack{d = m \ell \\ell \leq D^{2/3} \\ell \leq D^{1/3} \\ell \leq D^{2/3}}} a(m)b(\ell) \ll \tau(d).
\]

Then we have

\[
\sum_{1 \leq q \leq Q_0} \sum_{a=\pm q \pmod{(a,q)=1}} 2q \int_{\mathcal{M}(q, a)} |\mathcal{W}(\alpha)\Delta_k(\alpha)|^2 d\alpha \ll n^{\frac{2}{7}} \log^{\frac{100}{123}} A n.
\]

Proof See Lemma 2.4 of Li and Cai [26].

Lemma 3.4 For \( \alpha = \frac{a}{q} + \lambda \), define

\[
\mathcal{V}_k(\alpha) = \frac{S_k^*(q, a)}{\varphi(q)} w_k(\lambda).
\]

(3.5)

Then we have

\[
\sum_{1 \leq q \leq Q_0} \sum_{a=\pm q \pmod{(a,q)=1}} 2q \int_{\mathcal{M}(q, a)} |\mathcal{V}_k(\alpha)|^2 d\alpha \ll n^{\frac{2}{7} - 1} \log^{21A} n,
\]
and

\[
\sum_{1 \leq q \leq Q_0} \sum_{a=-q}^{2q} \left| \mathcal{W}(\alpha) \right|^2 d\alpha \ll \log^{21} A n,
\]

where \( \mathcal{M}(q, a) \) and \( \mathcal{W}(\alpha) \) are defined by (3.2) and (3.4), respectively.

**Proof** See Lemma 2.5 of Li and Cai [26].

For \((a, q) = 1, 1 \leq a \leq q \leq Q_2\), define

\[
\mathcal{M}(q, a) = \left[ \frac{a}{q} - \frac{1}{qQ_2}, \frac{a}{q} + \frac{1}{qQ_2} \right], \quad \mathcal{M} = \bigcup_{1 \leq q \leq Q_0} \bigcup_{1 \leq a \leq q} \mathcal{M}(q, a),
\]

\[
\mathcal{M}_0(q, a) = \left[ \frac{a}{q} - \frac{Q_0}{n}, \frac{a}{q} + \frac{Q_0}{n} \right], \quad \mathcal{M}_0 = \bigcup_{1 \leq q \leq Q_0} \bigcup_{1 \leq a \leq q} \mathcal{M}_0(q, a),
\]

\[
\mathcal{I}_0 = \left( -\frac{1}{Q_2}, 1 - \frac{1}{Q_2} \right], \quad m_0 = \mathcal{M} \setminus \mathcal{M}_0,
\]

\[
m_1 = \bigcup_{Q_0 < q \leq Q_1} \bigcup_{1 \leq a \leq q} \mathcal{M}(q, a), \quad m_2 = \mathcal{I}_0 \setminus (\mathcal{M} \cup m_1).
\]

Then we obtain the Farey dissection

\[
\mathcal{I}_0 = \mathcal{M}_0 \cup m_0 \cup m_1 \cup m_2. \tag{3.6}
\]

**Lemma 3.5** For \(\alpha = \frac{a}{q} + \lambda\), define

\[
\mathcal{V}_k^*(\alpha) = \frac{S_k^*(q, a)}{\varphi(q)} w_k^*(\lambda).
\]

Then for \(\alpha = \frac{a}{q} + \lambda \in \mathcal{M}_0\), we have

(i) \(f_j(\alpha) = \mathcal{V}_j(\alpha) + O(X_j \exp(- \log^{1/3} n))\),

(ii) \(f_j^*(\alpha) = \mathcal{V}_j^*(\alpha) + O(X_j^* \exp(- \log^{1/3} n))\),

(iii) \(g_r(\alpha) = \frac{c_r(k) \mathcal{V}_2(\alpha)}{\log X_2} + O(X_2 \exp(- \log^{1/3} n))\),

where \(\mathcal{V}_j(\alpha)\) is defined (3.5), and

\[
c_r(k) = (1 + O(\varepsilon))
\]

\(\Box\)
\[
\times \left( \int_{r_1}^{\frac{37k-15}{15-k}} \frac{dt_1}{t_1} \int_{r_2}^{t_1-1} \frac{dt_2}{t_2} \cdots \int_3^{t_r-4} \frac{dt_{r-3}}{t_{r-3}} \int_2^{t_{r-3}-1} \frac{\log(t_{r-2} - 1)}{t_{r-2}} \right) dt_{r-2}.
\]

(3.7)

**Proof** By some routine arguments and partial summation, (i)–(iii) follow from Siegel–Walfisz theorem and prime number theorem. \(\square\)

**Lemma 3.6** For \(\alpha \in m_2\), we have

\[
h(\alpha) \ll n^{\frac{2}{9} + \frac{5}{12}k - 24\varepsilon}.
\]

**Proof** By the estimate (4.5) of Lemma 4.2 in Brüdern and Kawada [3], we deduce that

\[
h(\alpha) \ll \frac{n^{\frac{1}{2}} \tau^2(q) \log^2 n}{(q + n|q\alpha - a|)^{1/2}} + n^{\frac{1}{4} + \varepsilon} D^{\frac{2}{3}}
\]

\[
\ll n^{\frac{1}{2} + \varepsilon} Q_1^{-\frac{1}{2}} + n^{\frac{1}{4} + \varepsilon} D^{\frac{2}{3}} \ll n^{\frac{2}{9} + \frac{5}{12}k - 24\varepsilon},
\]

which completes the proof of Lemma 3.6. \(\square\)

## 4 Mean value theorems

In this section, we shall prove the mean value theorems for the proof of Theorem 1.1.

**Proposition 4.1** For \(3 \leq k \leq 14\), define

\[
J(n, d) = \sum_{m^2 + p_1^2 + p_2^3 + p_3^3 + p_4^3 + p_5^k = n} \prod_{j=1}^{5} \log p_j.
\]

Then we have

\[
\sum_{m \leq D^{2/3}} a(m) \sum_{\ell \leq D^{1/3}} b(\ell) \left( J(n, m\ell) - \frac{\Theta_{m\ell}(n)}{m\ell} J(n) \right) \ll n^{\frac{17}{18} + \varepsilon} \log^{-A} n.
\]

**Proof** Let

\[
K(\alpha) = h(\alpha) f_2(\alpha) f_3^2(\alpha) f_3^*(\alpha) f_k^*(\alpha) e(-n\alpha).
\]

By the Farey dissection (3.6), we have

\[
\sum_{m \leq D^{2/3}} a(m) \sum_{\ell \leq D^{1/3}} b(\ell) J(n, m\ell)
\]
\[
\int_{I_0} K(\alpha) d\alpha = \left( \int_{M_0} + \int_{m_1} + \int_{m_2} \right) K(\alpha) d\alpha. \tag{4.1}
\]

From Hua’s lemma (see Lemma 2.5 of Vaughan [44]), Corollary 2.3 and Hölder’s inequality, we obtain

\[
\int_0^1 \left| f_2(\alpha) f_3^2(\alpha) f_4^*(\alpha) f_k^*(\alpha) \right| d\alpha
\ll \left( \int_0^1 \left| f_2(\alpha) \right|^4 d\alpha \right)^{\frac{1}{4}} \left( \int_0^1 \left| f_3(\alpha) \right|^4 d\alpha \right)^{\frac{1}{4}} \left( \int_0^1 \left| f_4(\alpha) f_4^*(\alpha) f_k^*(\alpha) \right|^2 d\alpha \right)^{\frac{1}{2}}
\ll \left( X_2^{2+\varepsilon} \right)^{\frac{1}{2}} \left( X_3^{2+\varepsilon} \right)^{\frac{1}{2}} \left( n^{\frac{1}{2} + \frac{5}{12} + \frac{5}{6\varepsilon} + \varepsilon} \right)^{\frac{1}{2}} \ll n^{\frac{12}{19} + \frac{5}{13\varepsilon} + \varepsilon}. \tag{4.2}
\]

By Lemma 3.6 and (4.2), we obtain

\[
\int_{m_2} K(\alpha) d\alpha \ll \sup_{\alpha \in m_2} |h(\alpha)| \times \int_0^1 \left| f_4(\alpha) f_3^2(\alpha) f_4^*(\alpha) f_k^*(\alpha) \right| d\alpha
\ll n^{\frac{3}{2} + \frac{5}{12} - 24\varepsilon} \cdot n^{\frac{13}{19} + \frac{5}{13\varepsilon} + \varepsilon} \ll n^{\frac{17}{19} + \frac{5}{13\varepsilon} - 23\varepsilon}. \tag{4.3}
\]

For \( \alpha \in m_1 \), it follows from Theorem 4.1 in Vaughan [44] that

\[
h(\alpha) = \mathcal{W}(\alpha) + O(DQ_1^{2+\varepsilon}) = \mathcal{W}(\alpha) + O(n^{\frac{17}{19} + \frac{5}{13\varepsilon} - 25\varepsilon}), \tag{4.4}
\]

where \( \mathcal{W}(\alpha) \) is defined by (3.4). Define

\[
\mathcal{K}_1(\alpha) = \mathcal{W}(\alpha) f_2(\alpha) f_3^2(\alpha) f_4^*(\alpha) f_k^*(\alpha) e(-n\alpha).
\]

Then by (4.2) and (4.4) we have

\[
\int_{m_1} K(\alpha) d\alpha = \int_{m_1} \mathcal{K}_1(\alpha) d\alpha + O\left(n^{\frac{23}{29} + \frac{5}{13\varepsilon} - 24\varepsilon}\right). \tag{4.5}
\]

Let

\[
\mathcal{M}_0(q, a) = \left( \frac{a}{q} - \frac{1}{n^{\frac{25}{36} + \frac{5}{13\varepsilon}}}, \frac{a}{q} + \frac{1}{n^{\frac{25}{36} + \frac{5}{13\varepsilon}}} \right], \quad \mathcal{M}_0 = \bigcup_{1 \leq q \leq Q_0} \bigcup_{a = -q}^{2q} \mathcal{M}_0(q, a),
\]

\[
\mathcal{M}_1(q, a) = \mathcal{M}(q, a) \setminus \mathcal{M}_0(q, a), \quad \mathcal{M}_1 = \bigcup_{1 \leq q \leq Q_0} \bigcup_{a = -q}^{2q} \mathcal{M}_1(q, a),
\]

\[
\mathcal{M} = \bigcup_{1 \leq q \leq Q_0} \bigcup_{a = -q}^{2q} \mathcal{M}(q, a).
\]
where \( \mathcal{M}(q, a) \) is defined by (3.2). Then we have \( m_1 \subseteq \mathcal{I}_0 \subseteq \mathcal{M} \). By Dirichlet’s theorem on Diophantine rational approximation, we obtain

\[
\int_{m_1} K_1(\alpha) \, d\alpha \ll \sum_{1 \leq q \leq Q_0} \sum_{\substack{a = -q \\ (a, q) = 1}}^{2q} \int_{m_1 \cap \mathcal{M}_0(q, a)} |K_1(\alpha)| \, d\alpha
\]

\[
+ \sum_{1 \leq q \leq Q_0} \sum_{\substack{a = -q \\ (a, q) = 1}}^{2q} \int_{m_1 \cap \mathcal{M}_1(q, a)} |K_1(\alpha)| \, d\alpha. \tag{4.6}
\]

By Lemma 4.2 of Titchmarsh [40], we have

\[
w_j(\lambda) \ll \frac{X_j}{1 + |\lambda|n},
\]

from which and the trivial estimate \((q, d^2) \leq (q, d)^2\), we deduce that

\[
|\mathcal{W}(\alpha)| \ll \sum_{d \leq D} \frac{\tau(d)}{d} (q, d^2)^1/2 q^{-1/2} |w_2(\lambda)|
\]

\[
\ll \tau_3(q) q^{-1/2} |w_2(\lambda)| \log^2 n \ll \frac{\tau_3(q) X_2 \log^2 n}{q^{1/2} (1 + |\lambda|n)}. \tag{4.7}
\]

Therefore, for \( \alpha \in \mathcal{M}_1(q, a) \), we get

\[
\mathcal{W}(\alpha) \ll n^{\frac{7}{36} + \frac{5}{12}} \log^2 n,
\]

which combines (4.2) to derive that

\[
\sum_{1 \leq q \leq Q_0} \sum_{\substack{a = -q \\ (a, q) = 1}}^{2q} \int_{m_1 \cap \mathcal{M}_1(q, a)} |K_1(\alpha)| \, d\alpha
\]

\[
\ll n^{\frac{7}{36} + \frac{5}{12}} \log^2 n \times \int_{0}^{1} |f_2(\alpha) f_3^2(\alpha) f_3^*(\alpha) f_k^*(\alpha)| \, d\alpha \ll n^{\frac{11}{36} + \frac{5}{12} + \varepsilon}. \tag{4.8}
\]

For \( \alpha \in \mathcal{M}_0(q, a) \), it follows from Lemma 4.8 of Titchmarsh [40] that

\[
f_3(\alpha) = \Delta_3(\alpha) + \mathcal{V}_3(\alpha) + O(1).
\]

Hence, one obtains

\[
\sum_{1 \leq q \leq Q_0} \sum_{\substack{a = -q \\ (a, q) = 1}}^{2q} \int_{m_1 \cap \mathcal{M}_0(q, a)} |K_1(\alpha)| \, d\alpha
\]
\[
\begin{align*}
&\ll \sum_{1 \leq q \leq Q_0} \sum_{a=-q}^{2q} \int_{m_1 \cap M_0(q,a)} \left| \mathcal{W}(\alpha) \Delta_3(\alpha) f_2(\alpha) f_3(\alpha) f_3^*(\alpha) f_k^*(\alpha) \right| d\alpha \\
&+ \sum_{1 \leq q \leq Q_0} \sum_{a=-q}^{2q} \int_{m_1 \cap M_0(q,a)} \left| \mathcal{W}(\alpha) \mathcal{V}_3(\alpha) f_2(\alpha) f_3(\alpha) f_3^*(\alpha) f_k^*(\alpha) \right| d\alpha \\
&+ \sum_{1 \leq q \leq Q_0} \sum_{a=-q}^{2q} \int_{m_1 \cap M_0(q,a)} \left| \mathcal{W}(\alpha) f_2(\alpha) f_3(\alpha) f_3^*(\alpha) f_k^*(\alpha) \right| d\alpha \\
=: \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3,
\end{align*}
\]

where \( \Delta_3(\alpha) \) and \( \mathcal{V}_3(\alpha) \) are defined by (3.3) and (3.5), respectively.

It follows from Cauchy’s inequality, Lemmas 3.2 and 3.3 that

\[
\begin{align*}
\mathcal{J}_1 &\ll \left( \sum_{1 \leq q \leq Q_0} \sum_{a=-q}^{2q} \int_{\mathcal{M}(q,a)} \left| \mathcal{W}(\alpha) \Delta_3(\alpha) \right|^2 d\alpha \right)^{\frac{1}{2}} \\
&\quad \times \left( \int_{0}^{1} \left| f_2(\alpha) f_3(\alpha) f_3^*(\alpha) f_k^*(\alpha) \right|^2 d\alpha \right)^{\frac{1}{2}} \\
&\ll \left( n^2 \log^{-100A} n \right)^{\frac{1}{2}} \left( n^{\frac{11}{9} + \frac{5}{3k}} \log^{c+8} n \right)^{\frac{1}{2}} \ll n^{\frac{17}{18} + \frac{5}{6k}} \log^{-40A} n. \tag{4.10}
\end{align*}
\]

By (4.7), it is easy to see that, for \( \alpha \in m_1 \), there holds

\[
\sup_{\alpha \in m_1} |\mathcal{W}(\alpha)| \ll n^{\frac{1}{2}} \log^{-30A} n. \tag{4.11}
\]

Therefore, by Lemma 3.2, Lemma 3.4, (4.11) and Cauchy’s inequality, we derive that

\[
\begin{align*}
\mathcal{J}_2 &\ll \sup_{\alpha \in m_1} |\mathcal{W}(\alpha)| \cdot \left( \sum_{1 \leq q \leq Q_0} \sum_{a=-q}^{2q} \int_{\mathcal{M}(q,a)} \left| \mathcal{V}_3(\alpha) \right|^2 d\alpha \right)^{\frac{1}{2}} \\
&\quad \times \left( \int_{0}^{1} \left| f_2(\alpha) f_3(\alpha) f_3^*(\alpha) f_k^*(\alpha) \right|^2 d\alpha \right)^{\frac{1}{2}} \\
&\ll \left( n^{\frac{1}{2}} \log^{-30A} n \right) \cdot \left( n^{-\frac{1}{3}} \log^{21A} n \right)^{\frac{1}{2}} \cdot \left( n^{\frac{11}{9} + \frac{5}{3k}} \log^{c+8} n \right)^{\frac{1}{2}} \\
&\ll n^{\frac{17}{18} + \frac{5}{6k}} \log^{-5A} n. \tag{4.12}
\end{align*}
\]
It follows from Lemmas 3.2 and 3.4 that

\[ \mathcal{I}_3 \ll \left( \sum_{1 \leq q \leq Q_0} \sum_{a=-q}^{2q} \int_{\mathcal{M}(q,a)} |\mathcal{W}(\alpha)|^2 d\alpha \right)^{1/2} \left( \int_0^1 \left| f_2(\alpha) f_3(\alpha) f_3^*(\alpha) f_k^*(\alpha) \right|^2 d\alpha \right)^{1/2} \]

\[ \ll (\log^{21A} n)^{1/2} \cdot (n^{11/9 + 5/6 \log^{c+8}} n)^{1/2} \ll n^{11/18 + 5/6 \log^{-A} n}. \]  

Combining (4.9), (4.10), (4.12) and (4.13), we can deduce that

\[ \sum_{1 \leq q \leq Q_0} \sum_{a=-q}^{2q} \int_{M_1 \cap \mathcal{M}_{0}(q,a)} |\mathcal{K}_1(\alpha)| d\alpha \ll n^{11/18 + 5/6 \log^{-5A} n}. \]  

From (4.5), (4.6), (4.8) and (4.14), we deduce that

\[ \int_{M_1} \mathcal{K}(\alpha) d\alpha \ll n^{11/18 + 5/6 \log^{-5A} n}. \]  

Similarly, we obtain

\[ \int_{M_0} \mathcal{K}(\alpha) d\alpha \ll n^{11/18 + 5/6 \log^{-5A} n}. \]  

For \( \alpha \in \mathcal{M}_0 \), define

\[ \mathcal{K}_0(\alpha) = \mathcal{W}(\alpha) \mathcal{V}_2(\alpha) \mathcal{V}_3^2(\alpha) \mathcal{V}_3^*(\alpha) \mathcal{V}_k^*(\alpha)e(-n\alpha). \]

By noticing that (4.4) still holds for \( \alpha \in \mathcal{M}_0 \), it follows from Lemma 3.5 and (4.4) that

\[ \mathcal{K}(\alpha) - \mathcal{K}_0(\alpha) \ll n^{35/18 + 5/6 \log^{-A} n}, \]

which implies that

\[ \int_{\mathcal{M}_0} \mathcal{K}(\alpha) d\alpha = \int_{\mathcal{M}_0} \mathcal{K}_0(\alpha) d\alpha + O(n^{17/18 + 5/6 \log^{-A} n}). \]

By the well-known standard technique in the Hardy–Littlewood method, we deduce that

\[ \int_{\mathcal{M}_0} \mathcal{K}_0(\alpha) d\alpha = \sum_{m \leq D^{2/3}} a(m) \sum_{\ell \leq D^{1/3}} b(\ell) \frac{\overline{\mathcal{S}_m}(n)}{m\ell} \mathcal{J}(n) + O(n^{17/18 + 5/6 \log^{-A} n}), \]

and

\[ \mathcal{J}(n) \asymp n^{17/18 + 5/6}. \]
Finally, Proposition 4.1 follows from (4.1), (4.3) and (4.15)–(4.19). This completes the proof of Proposition 4.1. \(\square\)

By the same method, we have the following proposition.

**Proposition 4.2** For \(3 \leq k \leq 14\), define

\[
J_r(n, d) = \sum_{\ell \in \mathbb{N}_r, \ t \equiv 0 \pmod{d}} \left( \frac{\log p}{\log X_2} \prod_{j=2}^{5} \log p_j \right).
\]

Then we have

\[
\sum_{m \leq D^{2/3}} a(m) \sum_{t \leq D^{1/3}} b(t) \left( J_r(n, mt) - \frac{c_r(k) \Theta_{mt}(n)}{mt \log X_2} J(n) \right) \ll n^{17/18 + \frac{5}{8\pi}} \log^{-A} n,
\]

where \(c_r(k)\) is defined by (3.7).

### 5 On the function \(\omega(d)\)

In this section, we shall investigate the function \(\omega(d)\) which is defined in (5.10) and required in the proof of Theorem 1.1.

**Lemma 5.1** For \(3 \leq k \leq 14\), let \(\mathcal{H}(q, n)\) and \(\mathcal{L}(q, n)\) denote the number of solutions of the congruences

\[
x^2 + u_1^3 + u_2^3 + u_3^3 + u_4^k \equiv n \pmod{q}, \quad 1 \leq x, u_j \leq q, \quad (xu_j, q) = 1,
\]

and

\[
x_1^2 + x_2^3 + u_1^3 + u_2^3 + u_3^3 + u_4^k \equiv n \pmod{q}, \quad 1 \leq x_i, u_j \leq q, \quad (x_2u_j, q) = 1,
\]

respectively. Then, for all \(n \equiv 0 \pmod{2}\), we have \(\mathcal{L}(p, n) > \mathcal{H}(p, n)\) for all primes. Moreover, there holds

\[
\mathcal{L}(p, n) = p^5 + O(p^4),
\]

\[
\mathcal{H}(p, n) = p^4 + O(p^3).
\]

**Proof** Let \(\mathcal{L}^*(q, n)\) denote the number of solutions of the congruence

\[
x_1^2 + x_2^3 + u_1^3 + u_2^3 + u_3^3 + u_4^k \equiv n \pmod{q}, \quad 1 \leq x_i, u_j \leq q, \quad (x_1x_2u_j, q) = 1.
\]
Then by the orthogonality of Dirichlet characters, we have

\[ p \cdot \mathcal{L}^*(p, n) = \sum_{a=1}^{p} S_2^2(p, a)S_3^3(p, a)S_k^*(p, a)e\left(-\frac{an}{p}\right) \]

\[ = (p - 1)^6 + E_p, \tag{5.1} \]

where

\[ E_p = \sum_{a=1}^{p-1} S_2^2(p, a)S_3^3(p, a)S_k^*(p, a)e\left(-\frac{an}{p}\right). \]

By (iv) of Lemma 3.1, we have

\[ |E_p| \leq (p - 1)(\sqrt{p} + 1)^2(2\sqrt{p} + 1)^3(13\sqrt{p} + 1). \tag{5.2} \]

It is easy to check that \(|E_p| < (p - 1)^6\) for \(p \geq 19\). Hence, we get \(\mathcal{L}^*(p, n) > 0\) for \(p \geq 19\). On the other hand, for \(p = 2, 3, 5, 7, 11, 13, 17\), we can check \(\mathcal{L}^*(p, n) > 0\) directly by hand. Therefore, we obtain \(\mathcal{L}^*(p, n) > 0\) for all primes and

\[ \mathcal{L}(p, n) = \mathcal{L}^*(p, n) + \mathcal{K}(p, n) > \mathcal{K}(p, n). \tag{5.3} \]

From (5.1) and (5.2), we derive that

\[ \mathcal{L}^*(p, n) = p^5 + O(p^4). \tag{5.4} \]

By a similar argument of (5.1) and (5.2), we have

\[ \mathcal{K}(p, n) = p^4 + O(p^3). \tag{5.5} \]

Combining (5.3)–(5.5), we obtain the desired results. \(\square\)

**Lemma 5.2** The series \(\mathcal{S}(n)\) is convergent and satisfying \(\mathcal{S}(n) > 0\).

**Proof** From (i) and (ii) of Lemma 3.1, we obtain

\[ |A(q, n)| \ll \frac{|B(q, n)|}{q^{\phi^5(q)}} \ll \frac{q^{2+5\varepsilon}}{\phi^4(q)} \ll \frac{q^{2+5\varepsilon} (\log \log q)^4}{q^4} \ll \frac{1}{q^{3/2}}. \]

Thus, the series

\[ \mathcal{S}(n) = \sum_{q=1}^{\infty} A(q, n) \]
converges absolutely. Noting the fact that $A(q, n)$ is multiplicative in $q$ and by (v) of Lemma 3.1, we get
\[ S(n) = \prod_p \left(1 + A(p, n)\right). \tag{5.6} \]
From (iii) and (iv) of Lemma 3.1, we know that, for $p \geq 29$, we have
\[ |A(p, n)| \leq \frac{(p - 1)\sqrt{p}(\sqrt{p} + 1)(2\sqrt{p} + 1)^3(13\sqrt{p} + 1)}{p(p - 1)^5} \leq \frac{200}{p^2}. \]
Therefore, there holds
\[ \prod_{p \geq 29} \left(1 + A(p, n)\right) \geq \prod_{p \geq 29} \left(1 - \frac{200}{p^2}\right) \geq c_1 > 0. \tag{5.7} \]
On the other hand, it is easy to see that
\[ 1 + A(p, n) = \frac{L(p, n)}{(p - 1)^5}. \tag{5.8} \]
By Lemma 5.1, we have $L(p, n) > 0$ for all $p$ with $n \equiv 0 \pmod{2}$, and thus $1 + A(p, n) > 0$. Consequently, we obtain
\[ \prod_{p < 29} (1 + A(p, n)) \geq c_2 > 0. \tag{5.9} \]
Combining (5.6), (5.7) and (5.9), we conclude that $S(n) > 0$, which completes the proof of Lemma 5.2. \qed

In view of Lemma 5.2, we define
\[ \omega(d) = \frac{\mathcal{S}_d(n)}{\mathcal{S}(n)}. \tag{5.10} \]
Similar to (5.6), we have
\[ \mathcal{S}_d(n) = \prod_p (1 + A_d(p, n)). \tag{5.11} \]
If $(d, q) = 1$, then we have $S_k(q, ad^k) = S_k(q, a)$. Moreover, if $p|d$, then we get $A_d(p, n) = A_p(p, n)$. Therefore, we derive that
\[ \omega(p) = \frac{1 + A_p(p, n)}{1 + A(p, n)}, \quad \omega(d) = \prod_{p|d} \omega(p). \tag{5.12} \]
Also, it is easy to show that

\[ 1 + A_p(p, n) = \frac{p}{(p-1)^5} \mathcal{K}(p, n). \]  

(5.13)

Using (5.8), (5.12) and (5.13), we derive

\[ \omega(p) = \frac{p \cdot \mathcal{K}(p, n)}{\mathcal{L}(p, n)}. \]

from which and Lemma 5.1, we derive the following lemma.

**Lemma 5.3** The function \( \omega(d) \) is multiplicative and satisfies

\[ 0 \leq \omega(p) < p, \quad \omega(p) = 1 + O(p^{-1}). \]  

(5.14)

### 6 Proof of Theorem 1.1

In this section, let \( f(s) \) and \( F(s) \) denote the classical functions in the linear sieve theory. Then it follows from (2.8) and (2.9) of Chapter 8 in [8] that

\[ F(s) = \frac{2e^{\gamma}}{s}, \quad 1 \leq s \leq 3; \quad f(s) = \frac{2e^{\gamma} \log(s - 1)}{s}, \quad 2 \leq s \leq 4. \]

In the proof of Theorem 1.1, let \( \lambda^\pm(d) \) be the lower and upper bounds for Rosser’s weights of level \( D \), hence for any positive integer \( d \) we have

\[ |\lambda^\pm(d)| \leq 1, \quad \lambda^\pm(d) = 0 \text{ if } d > D \text{ or } \mu(d) = 0. \]

For further properties of Rosser’s weights we refer to Iwaniec [16]. Define

\[ \mathcal{W}(z) = \prod_{2<p<z} \left( 1 - \frac{\omega(p)}{p} \right). \]

Then from Lemma 5.3 and Mertens’ prime number theorem (See [36]) we obtain

\[ \mathcal{W}(z) \asymp \frac{1}{\log N}. \]  

(6.1)

In order to prove Theorem 1.1, we need the following lemma.

**Lemma 6.1** Under the condition (5.14), then if \( z \leq D \), there holds

\[ \sum_{d|\mathfrak{p}} \lambda^{-}(d) \omega(d) \geq \mathcal{W}(z) \left( f \left( \frac{\log D}{\log z} \right) + O\left( \log^{-1/3} D \right) \right). \]  

(6.2)
and if \( z \leq D^{1/2} \), there holds

\[
\sum_{d|\mathcal{P}} \frac{\lambda^+(d)\omega(d)}{d} \leq \mathcal{W}(z) \left( F\left(\frac{\log D}{\log z}\right) + O\left(\log^{-1/3} D\right)\right). \tag{6.3}
\]

**Proof** See Iwaniec [17], (12) and (13) of Lemma 3. □

From the definition of \( \mathcal{B}_r \), we know that \( r \leq \left\lfloor \frac{36k}{15-k} \right\rfloor \). Hence, we obtain

\[
\mathcal{R}_k(N) \geq \sum_{m^2+p_1^2+p_2^3+p_3^3+p_5^k=n} 1 - \sum_{r=r(k)+1}^{36k/15-k} \sum_{m\in\mathcal{B}_r} \sum_{k_p < k_p \leq 2X_k} \mathcal{Y}_r.
\]

By the property (6.2) of Rosser’s weight \( \lambda^-(d) \) and Proposition 4.1, we get

\[
\mathcal{Y}_0 \geq \frac{1}{\log \mathfrak{E}} \sum_{m^2+p_1^2+p_2^3+p_3^3+p_5^k=n} \prod_{j=1}^{5} \log p_j = 1 - \sum_{d|m, \mathcal{P}} \mu(d) \prod_{j=1}^{5} \log p_j = \frac{1}{\log \mathfrak{E}} \sum_{m^2+p_1^2+p_2^3+p_3^3+p_5^k=n} \prod_{j=1}^{5} \log p_j \sum_{d|m, \mathcal{P}} \lambda^-(d).
\]
According to simple numerical calculations, we know that
\[ c_4(3) \leq 0.4443636, \quad c_5(3) \leq 0.0578256, \quad c_j(3) \leq 0.0027627 \quad \text{with} \quad 6 \leq j \leq 9; \]
\[ c_5(4) \leq 0.3029445, \quad c_6(4) \leq 0.0459743, \quad c_j(4) \leq 0.00388094 \quad \text{with} \quad 7 \leq j \leq 13; \]
c_6(5) \leq 0.1892887, \ c_j(5) \leq 0.0307123 \text{ with } 7 \leq j \leq 18;
\ c_6(6) \leq 0.4867818, \ c_7(6) \leq 0.1133016, \ c_8(6) \leq 0.01913692,
\ c_j(6) \leq 0.00237244 \text{ with } 9 \leq j \leq 24;
\ c_7(7) \leq 0.2978111, \ c_8(7) \leq 0.0672273, \ c_j(7) \leq 0.0117295 \text{ with } 9 \leq j \leq 31;
\ c_8(8) \leq 0.1830229, \ c_9(8) \leq 0.0407894, \ c_j(8) \leq 0.0073521 \text{ with } 10 \leq j \leq 41;
\ c_8(9) \leq 0.4323101, \ c_9(9) \leq 0.1169923, \ c_{10}(9) \leq 0.02614497,
\ c_{11}(9) \leq 0.0048887, \ c_j(9) \leq 0.000772739 \text{ with } 12 \leq j \leq 54;
\ c_9(10) \leq 0.3023038, \ c_{10}(10) \leq 0.0809431, \ c_{11}(10) \leq 0.0184125,
\ c_j(10) \leq 0.003597861 \text{ with } 12 \leq j \leq 72;
\ c_{10}(11) \leq 0.2360241, \ c_{11}(11) \leq 0.0639155, \ c_{12}(11) \leq 0.01504156,
\ c_j(11) \leq 0.003105002 \text{ with } 13 \leq j \leq 99;
\ c_{11}(12) \leq 0.2231261, \ c_{12}(12) \leq 0.06262236, \ c_{13}(12) \leq 0.01555779,
\ c_{14}(12) \leq 0.00344782, \ c_j(12) \leq 0.0006868855 \text{ with } 15 \leq j \leq 144;
\ c_{12}(13) \leq 0.2976851, \ c_{13}(13) \leq 0.0895433, \ c_{14}(13) \leq 0.0242215,
\ c_{15}(13) \leq 0.005929363, \ c_{16}(13) \leq 0.0013212887,
\ c_j(13) \leq 0.0002694412 \text{ with } 17 \leq j \leq 234;
\ c_{14}(14) \leq 0.2926583, \ c_{15}(14) \leq 0.09172191, \ c_{16}(14) \leq 0.026363835,
\ c_{17}(14) \leq 0.006978431, \ c_{18}(14) \leq 0.001783123,
\ c_j(14) \leq 0.0002510648 \text{ with } 19 \leq j \leq 504.

Therefore, if we write

\[ C(k) = \sum_{r=r(k)+1}^{\frac{36k}{15-k}} c_r(k), \]  

(6.7)

then we have

\[
\begin{align*}
C(3) &< 0.513241, \quad C(4) < 0.376086, \quad C(5) < 0.557837, \quad C(6) < 0.657181, \quad (6.8) \\
C(7) &< 0.634817, \quad C(8) < 0.459081, \quad C(9) < 0.613564, \quad C(10) < 0.621131, \quad (6.9) \\
C(11) &< 0.585117, \quad C(12) < 0.394051, \quad C(13) < 0.477439, \quad C(14) < 0.541523. \quad (6.10)
\end{align*}
\]

From (6.1), (6.4)–(6.10), we derive that

\[ \mathcal{R}_k(N) \geq \left( f(3) - F(3) \sum_{r=r(k)+1}^{\frac{36k}{15-k}} c_r(k) \right) \left( 1 + O\left( \log^{-1/3} D \right) \right) \]
\[
\frac{\mathcal{S}(n) \mathcal{J}(n) \mathcal{W}(z)}{\log \Xi} + O\left(n^{17/18} + \frac{5}{6} \log^{-A} n\right) \\
\geq \frac{2e^{\gamma}}{3} \left(\log 2 - 0.657181 \right) \left(1 + O\left(\log^{-1/3} D\right)\right) \times \\
\frac{\mathcal{S}(n) \mathcal{J}(n) \mathcal{W}(z)}{\log \Xi} + O\left(n^{17/18} + \frac{5}{6} \log^{-A} n\right) \\
\gg n^{17/18} + \frac{5}{6} \log^{-6} n,
\]

which completes the proof of Theorem 1.1.

**Acknowledgements** The authors would like to express their most sincere gratitude to the referee for his/her patience in refereeing this paper.

**References**

1. Brüdern, J.: A sieve approach to the Waring–Goldbach problem I. Sums of four cubes. Ann. Sci. cole Norm. Sup. (4) 28(4), 461–476 (1995)
2. Brüdern, J.: A sieve approach to the Waring–Goldbach problem II. On the seven cubes theorem. Acta Arith. 72(3), 211–227 (1995)
3. Brüdern, J., Kawada, K.: Ternary problems in additive prime number theory. In: Jia, C., Matsumoto, K. (eds.) Analytic Number Theory, Dev. Math. vol. 6, pp. 39–91. Kluwer, Dordrecht (2002)
4. Brüdern, J., Kawada, K.: On the Waring–Goldbach problem for cubes. Glasg. Math. J. 51(3), 703–712 (2009)
5. Cai, Y.C.: The Waring–Goldbach problem: one square and five cubes. Ramanujan J. 34(1), 57–72 (2014)
6. Cai, Y.C.: Waring-Goldbach problem: two squares and higher powers. J. Théor. Nombres Bordeaux 28(3), 791–810 (2016)
7. Davenport, H., Erdös, P.: On sums of positive integral k-th powers. Ann. Math. (2) 40, 533–536 (1939)
8. Halberstam, H., Richert, H.E.: Sieve Methods. Academic Press, London (1974)
9. Hardy, G.H., Littlewood, J.E.: Some problems of ‘Partitio numerorum’ III: on the expression of a number as a sum of primes. Acta Math. 44(1), 1–70 (1923)
10. Hardy, G.H., Littlewood, J.E.: Some problems of ‘Partitio numerorum’ (VI): further researches in Waring’s Problem. Math. Z. 23(1), 1–37 (1925)
11. Hardy, G.H., Wright, E.M.: An Introduction to the Theory of Numbers, 5th edn. Oxford University Press, New York (1979)
12. Harman, G., Kumchev, A.V.: On sums of squares of primes. Math. Proc. Cambridge Philos. Soc. 140(1), 1–13 (2006)
13. Harman, G., Kumchev, A.V.: On sums of squares of primes II. J. Number Theory 130(9), 1969–2002 (2010)
14. Hua, L.K.: Some results in the additive prime number theory. Q. J. Math. Oxf Ser. (2) 9(1), 68–80 (1938)
15. Hua, L.K.: Additive Theory of Prime Numbers. American Mathematical Society, Providence, RI (1965)
16. Iwaniec, H.: Rosser’s sieve. Acta Arith. 36(2), 171–202 (1980)
17. Iwaniec, H.: A new form of the error term in the linear sieve. Acta Arith. 37(1), 307–320 (1980)
18. Kawada, K., Zhao, L.: The Waring-Goldbach problem for cubes with an almost prime. Proc. Lond. Math. Soc. (3) 119(4), 867–898 (2019)
19. Kumchev, A.V., Tolev, D.I.: An invitation to additive prime number theory. Serdica Math. J. 31(1–2), 1–74 (2005)
20. Kumchev, A.V., Zhao, L.: On sums of four squares of primes. Mathematika 62(2), 348–361 (2016)
21. Leung, M.C., Liu, M.C.: On generalized quadratic equations in three prime variables. Monatsh. Math. 115(1–2), 133–167 (1993)
22. Li, H.Z.: The exceptional set of Goldbach numbers. Q. J. Math. Oxf Ser. (2) 50(200), 471–482 (1999)
23. Li, H.Z.: The exceptional set of Goldbach numbers II. Acta Arith. 92(1), 71–88 (2000)
24. Li, H.Z.: Representation of odd integers as the sum of one prime, two squares of primes and powers of 2. Acta Arith. 128(3), 223–233 (2007)
25. Li, H.Z.: Sums of one prime and two prime squares. Acta Arith. 134(1), 1–9 (2008)
26. Li, Y.J., Cai, Y.C.: Waring-Goldbach problem: two squares and some higher powers. J. Number Theory 162, 116–136 (2016)
27. Linnik, YuV: Hardy-Littlewood problem on representation as the sum of a prime and two squares. Dokl. Akad. Nauk SSSR 124, 29–30 (1959)
28. Linnik, YuV: An asymptotic formula in an additive problem of Hardy–Littlewood. Izv. Akad. Nauk SSSR Ser. Mat. 24, 629–706 (1960)
29. Liu, T.: Representation of odd integers as the sum of one prime, two squares of primes and powers of 2. Acta Arith. 115(2), 97–118 (2004)
30. Liu, Z.X.: Cubes of primes and almost prime. J. Number Theory 132(6), 1284–1294 (2012)
31. Liu, Z.X.: Goldbach–Linnik type problems with unequal powers of primes. J. Number Theory 176, 439–448 (2017)
32. Liu, Z.X., Lü, G.S.: Two results on powers of 2 in Waring–Goldbach problem. J. Number Theory 131(4), 716–736 (2011)
33. Liu, J.Y., Zhan, T.: The quadratic Waring–Goldbach problem. J. Shandong Univ. Nat. Sci. 42(2), 1–18 (2007)
34. Lu, W.C.: Exceptional set of Goldbach number. J. Number Theory 130(10), 2359–2392 (2010)
35. Lü, G.S., Sun, H.W.: Integers represented as the sum of one prime, two squares of primes and powers of 2. Proc. Am. Math. Soc. 137(4), 1185–1191 (2009)
36. Mertens, F.: Ein Beitrag zur analytischen Zahlentheorie. J. Reine Angew. Math. 78, 46–62 (1874)
37. Pintz, J.: Landaus problems on primes. J. Théor. Nombres Bordeaux 21(2), 357–404 (2009)
38. Pintz, J.: A new explicit formula in the additive theory of primes with applications II. The exceptional set in Goldbach’s problem. arXiv:1804.09084
39. Thanigasalam, K.: On sums of powers and a related problem. Acta Arith. 36(2), 125–141 (1980)
40. Titchmarsh, E.C.: The Theory of the Riemann Zeta-Function, 2nd edn. Oxford University Press, Oxford (1986)
41. Vinogradov, I.M.: Elements of Number Theory. Dover Publications, New York (1954)
42. Vaughan, R.C.: Sums of three cubes. Bull. Lond. Math. Soc. 17(1), 17–20 (1985)
43. Vaughan, R.C.: On Waring’s problem for smaller exponents. Proc. Lond. Math. Soc. (3) 52(3), 445–463 (1986)
44. Vaughan, R.C.: The Hardy–Littlewood Method, 2nd edn. Cambridge University Press, Cambridge (1997)
45. Wang, M.Q.: On the sum of a prime and two prime squares. Acta Math. Sin. (Chin. Ser.) 47(5), 845–858 (2004)
46. Wang, M.Q., Meng, X.M.: The exceptional set in the two prime squares and a prime problem. Acta Math. Sin. (Engl. Ser.) 22(5), 1329–1342 (2006)
47. Zhao, L.: The additive problem with one prime and two squares of primes. J. Number Theory 135, 8–27 (2014)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.